INTEGRATED HARNACK INEQUALITIES ON LIE GROUPS

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Abstract

We show that the logarithmic derivatives of the convolution heat kernels on a uni-modular Lie group are exponentially integrable. This result is then used to prove an “integrated” Harnack inequality for these heat kernels. It is shown that this integrated Harnack inequality is equivalent to a version of Wang’s Harnack inequality. (A key feature of all of these inequalities is that they are dimension independent.) Finally, we show these inequalities imply quasi-invariance properties of heat kernel measures for two classes of infinite dimensional “Lie” groups.

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1. Introduction

1.1. Basic setup. Let $(M, g)$ be a connected complete Riemannian manifold, $d : M \times M \to [0, \infty)$ be the Riemannian distance function, $dV$ be the Riemannian volume measure on $M$, $\Delta$ be the Laplace–Beltrami operator acting on the space of smooth differential forms, $\Omega(M)$, over $M$, and $\Delta_0 := \Delta|_{\Omega_0^0(M)}$, where $\Omega_0^0(M) := C_0^{\infty}(M)$ is the space of compactly supported smooth functions on $M$. From Gaffney [26], Roelcke [53], Chernoff [11] and Strichartz [60], we know that the $L^2(M, dV)$–closure, $\overline{\Delta_0}$, of $\Delta_0$ is a non-positive self-adjoint operator on $L^2(M, dV)$. Moreover, there exists an associated smooth heat kernel, $(0, \infty) \times M \times M \ni (t, x, y) \rightarrow p_t(x, y) \in (0, \infty)$, such that

$$\int_M p_t(x, y)dV(y) \leq 1 \text{ for all } x \in M, \text{ and}$$

$$\left( e^{t\Delta_0/2} f \right)(x) = \int_M p_t(x, y)f(y)dV(y) \text{ for all } f \in L^2(M).$$

We also let “Ric” denote the Ricci curvature tensor of $(M, g)$. For the bulk of this paper we will be considering the special case, where $M = G$ is a Lie group equipped with a left invariant Riemannian metric as we now describe.

Let $G$ be a connected finite dimensional uni-modular Lie group, $g = \text{Lie}(G)$ be its Lie algebra, and suppose that $g$ is equipped with an inner product, $(\cdot, \cdot) = (\cdot, \cdot)_g$. Let $|A|_g := \sqrt{(A, A)}$ for all $A \in g$. We endow $G$ with the unique left invariant Riemannian metric which agrees with $(\cdot, \cdot)_g$ at $e \in G$, i.e. the unique metric on $G$ such that $L_{g^*} : g \to T_gG$ is isometric for all $g \in G$. The Riemannian distance between $x, y \in G$ will be denoted by $d(x, y)$.

For $A \in g$ let $\tilde{A}$ denote the unique left invariant vector field on $G$ such that $\tilde{A}(e) = A \in g$ and let $L = \sum_{i=1}^{\dim g} \tilde{A}_i^2$, where $\{\tilde{A}_i\}_{i=1}^{\dim g}$ is an orthonormal basis for $g$. As is well-known, since $G$ is uni-modular, $L$ is the Laplace-Beltrami operator (for example, see [22, Remark 2.2] and Lemma 6.1 below) restricted to $C^\infty(G)$. Since $L_g : G \to G$ is an isometry for all $g \in G$, if $p_t(x, y)$ is the heat kernel on $G$, then
$p_t(gx, gy) = p_t(x, y)$ for all $x, y, g \in G$. Taking $g = x^{-1}$ then implies that $p_t(x, y) = p_t(e, x^{-1}y)$. Similarly, $d(gx, gy) = d(x, y)$ for all $x, y, g \in G$ and therefore $d(x, y) = d(e, x^{-1}y)$.

**Notation 1.1.** By a slight abuse of notation, let $p_t(x) := p_t(e, x)$ for $x \in G$. We will refer to $p_t(x)$ as the **convolution heat kernel** on $G$ and to the probability measure, $d\nu_t(x) := p_t(x)$ $dx$, as the **heat kernel measure on** $G$. We also write $dx$ for $dV(x)$ and $|x|$ for $d(e, x)$.

The following lemma is an immediate consequence of the comments above and the basic properties of $p_t(x, y)$.

**Lemma 1.2.** For all $x, y \in G$
1) $d(x, y) = |x^{-1}y|$,  
2) $|x^{-1}| = |x|$,  
3) $p_t(x^{-1}) = p_t(x)$,  
4) $p_t(x, y) = p_t(x^{-1}y) = p_t(y^{-1}x)$,  
5) $dV$ is a bi-invariant Haar measure on $G$,  
6) for $f \in L^2(G, dV)$,  
\[
\left( e^{t\Delta_0/2} f \right)(x) = \int_G p_t(x^{-1}y) f(y) \, dy \\
= \int_G p_t(y^{-1}x) f(y) \, dy \\
= \int_G p_t(yx) f(y^{-1}) \, dy.
\]

1.2. **The main theorems.** We may now state the main theorems of this paper.

**Theorem 1.3.** If $T > 0$ and $A \in \mathfrak{g}$, then
\[
(1.3) \quad \int_G \exp \left( -\frac{\tilde{A} p_T(x)}{p_T(x)} \right) p_T(x) \, dx \leq \exp \left( \frac{k/2}{e^{kT} - 1} |A|^2 \right),
\]
where $k \in \mathbb{R}$ is a lower bound on the Ricci curvature, i.e. $\text{Ric} \geq kI$. (Here and in what follows we will always use the convention that $k/ (e^{kT} - 1) = 1/T$ whenever $k = 0$.)

The proof of this theorem relies on martingale inequalities applied to the probabilistic representation for $\tilde{A} \ln p_T(x)$ in Theorem 6.4. We also have another related integral bound on $\tilde{A} \ln p_T(x)$.

**Theorem 1.4.** Continuing the notation in Theorem 1.5, for any $q \in (1, \infty)$ there is a constant, $C_q < \infty$ such that
\[
(1.4) \quad \left\| \tilde{A} \ln p_T \right\|_{L^q(\nu_T)} \leq C_q \sqrt{\frac{k}{e^{kT} - 1}} |A| \text{ for all } A \in \mathfrak{g}.
\]
These theorems will be proved in Sections 5 and 6 below. Also see [23, Theorem 5.11] for a version of this theorem valid on a general compact Riemannian manifold and Proposition E.1 in Appendix E where we use a Hamilton type inequality to show that an inequality similar to that in Eq. (1.3) holds on any complete Riemannian manifolds whose Ricci curvature is bounded from below. However, as is noted in Remark E.2, in general, we can not choose the constants appearing in Proposition E.1 to be independent of dimension.

The following theorem is a corollary of Theorem 1.3 above and Theorem 2.5 below. The details will be given in Section 3 below.

**Theorem 1.5.** Let $T > 0$ be given and let $k \in \mathbb{R}$ be a lower bound on the Ricci curvature, $\text{Ric} \geq kI$. Then for every $y \in G$ and $q \in [1, \infty)$,

$$
\left( \int_G \left[ \frac{p_T(y{x}^{-1})}{p_T(x)} \right]^q p_T(x) \, dx \right)^{1/q} \leq \exp \left( \frac{(q - 1)k/2}{e^{kT} - 1} |y|^2 \right).
$$

From Theorem 1.5 and Lemma 1.2 we have,

$$
\left( \int_G \left[ \frac{p_T(y, x)}{p_T(z, x)} \right]^q p_T(z, x) \, dx \right)^{1/q} = \left( \int_G \left[ \frac{p_T(y^{-1}x)}{p_T(z^{-1}x)} \right]^q p_T(z^{-1}x) \, dx \right)^{1/q} = \left( \int_G \left[ \frac{p_T(y^{-1}z)}{p_T(z)} \right]^q p_T(z) \, dx \right)^{1/q} \leq \exp \left( \frac{(q - 1)k/2}{e^{kT} - 1} |y^{-1}z|^2 \right)
$$

$$
\text{(1.6)} \quad \exp \left( \frac{(q - 1)k/2}{e^{kT} - 1} d^2(y, z) \right)
$$

for all $y, z \in G$. This form of the integrated Harnack inequality makes sense on any Riemannian manifold. We will show in Corollary D.3 of Appendix D below that Eq. (1.6) does indeed hold when $G$ is replaced by a complete connected Riemannian manifold with $\text{Ric} \geq kI$ for some $k \in \mathbb{R}$. The key point is that the estimate in Eq. (1.6) is equivalent to Wang’s dimension free Harnack inequality, see [66, 67] and Theorem D.2 below. We are grateful to Michael Röckner for pointing out the relationship between Wang’s inequality and the integrated Harnack inequality in Eq. (1.6).

**Remarks 1.6.** Some of the key features of Theorem 1.5 are:

1) As seen in Example 1.1) below, the estimate in Eq. (1.5) is sharp when $G = \mathbb{R}^n$. 
2) For $T$ near zero, $k/ (e^{kT} - 1) \cong 1/T$ and for $T$ large, $k/ (e^{kT} - 1) \cong \max (0, -k)$.

3) The estimate in Eq. (1.5) is dimension independent and therefore has applications to infinite dimensional settings, see Section 7 below.

Let $R_y : G \rightarrow G$ ($L_y : G \rightarrow G$) be the operation of right (left) multiplication by $y \in G$, $\nu_T \circ R_y^{-1}$ ($\nu_T \circ L_y^{-1}$) be $\nu_T$ pushed forward by $R_y$ ($L_y$), and $d (\nu_T \circ R_y^{-1}) / d\nu_T$ denote the Radon-Nikodym derivative of $\nu_T \circ R_y^{-1}$ with respect to $\nu_T$. For the infinite dimensional applications of Section 7, it is convenient to rewrite Eq. (1.5) as

\[
L^q(G, \nu_T) \leq \exp \left( \frac{(q - 1) k/2}{e^{kT} - 1} \right) d^2 (e, y) \quad .
\]

By Lemma 1.2, Eq. (1.5) may be also be expressed as

\[
\left( \int_G \left[ \frac{p_T(xy)}{p_T(x)} \right]^q p_T(x) \, dx \right)^{1/q} \leq \exp \left( \frac{(q - 1) k/2}{e^{kT} - 1} |y|^2 \right)
\]

or as

\[
\left( \int_G \left[ \frac{p_T(y^{-1}x)}{p_T(x)} \right]^p p_T(x) \, dx \right)^{1/q} \leq \exp \left( \frac{(q - 1) k/2}{e^{kT} - 1} |y|^2 \right).
\]

This last equality is equivalent to the left translation analogue of Eq. (1.7), namely

\[
L^q(G, \nu_T) \leq \exp \left( \frac{(q - 1) k/2}{e^{kT} - 1} |y|^2 \right).
\]

1.3. Examples and applications.

Definition 1.7. For $A \in g$ and $T > 0$, let

\[
W^T_A (x) := - \left( \tilde{A} \ln p_T \right) (x) = - \left( \frac{\tilde{A} p_T (x)}{p_T(x)} \right).
\]

The significance of $W^T_A$ in the above definition stems from the following integration by parts identity;

\[
\int_G \tilde{A} f (x) p_T (x) \, dx = \int_G f (x) W^T_A (x) p_T (x) \, dx \quad \forall \ f \in C_c^\infty (G).
\]

Thus $W^T_A$ is the $p_T (x) \, dx$– divergence of $\tilde{A}$ as described in Definition 2.3 below.

Example 1.1. Suppose $G = \mathbb{R}^n$ so that $g \cong \mathbb{R}^n$ which we assume has been equipped with the standard inner product. In this case

\[
p_T (x) = \left( \frac{1}{2\pi T} \right)^{n/2} \exp \left( - \frac{|x|^2}{2T} \right),
\]
where $|x|^2 := \sum_{i=1}^{n} x_i^2$. For $A \in \mathfrak{g}$ and $f \in C_c^1(\mathbb{R}^n)$ we have $\tilde{A} = \partial_A$ and
\[
\int_{\mathbb{R}^n} \tilde{A} f(x) p_T(x) \, dx
\]
\[= - \int_{\mathbb{R}^n} f(x) \partial_A p_T(x) \, dx = \int_{\mathbb{R}^n} f(x) \frac{x \cdot A}{T} p_T(x) \, dx
\]
from which it follows that $W^T_A(x) = \frac{2}{T} A$. By simple Gaussian integrations,
\[
\int_{\mathbb{R}^n} e^{W^T_A(x)} p_T(x) \, dx = \exp\left(\frac{|A|^2}{2T}\right),
\]
\[
\left(\int_{\mathbb{R}^n} \left[ \frac{p_T(x-y)}{p_T(x)} \right]^q p_T(x) \, dx \right)^{1/q}
\]
\[= \exp\left(\frac{(q-1)}{2T} |y|^2\right),
\]
and
\[
(1.13) \int_{\mathbb{R}^n} |W^T_A(x)|^q p_T(x) \, dx = \int_{\mathbb{R}^n} \left| \frac{x \cdot A}{T} \right|^q p_T(x) \, dx = T^{q/2} |A|^q \tilde{C}_q,
\]
where
\[
\tilde{C}_q := \int_{\mathbb{R}^n} |x|^q p_1(x) \, dx.
\]
The first two results show the estimates in Eqs. (1.3) and (1.5) are sharp. The identity in Eq. (1.13) shows the form of Eq. (1.4) is sharp. We do not know if, in general, the constant $C_q$ appearing in Eq. 1.4 can be taken to be $\tilde{C}_q$ defined above.

Our main interest in Theorem 1.5 is in its application to proving that certain “heat kernel measures” on infinite dimensional Lie groups, $G$, are quasi-invariant under left and right translations by elements of a certain subgroup, $G_0$. We will postpone our discussion of this application to Section 7. For now let us give a couple of finite dimensional applications of Theorems 1.5 and 1.4.

**Proposition 1.8.** Suppose that $T > 0$, $q > 1$, and $f \in L^q(\nu_T)$ is a harmonic function, i.e. $\Delta f = 0$. Then
\[
(1.14) \int_G p_T(y, x) f(x) \, dx = f(y) \quad \text{for all} \quad y \in G.
\]
At an informal level we expect
\[
\int_G p_t(y, x) f(x) \, dx = \left( e^{t\Delta_0/2} f \right)(y)
\]
and hence
\[
\frac{d}{dt} \int_G p_t(y, x) f(x) \, dx = \frac{d}{dt} \left( e^{t\Delta_0/2} f \right)(y) = \left( e^{t\Delta_0/2} \bar{\Delta}_0 f \right)(y) = 0.
\]
Therefore it is reasonable to conclude that
\[
\int_G p_T(y, x) f(x) \, dx = \left( e^{T\Delta_0/2} f \right)(y) = \left( e^{\theta\Delta_0/2} f \right)(y) = f(y).
\]
However, this argument is not rigorous as \( f \) is only square-integrable relative to the rapidly decaying measure, \( \nu_T \), rather than to Haar measure on \( G \). The rigorous proof of Proposition 1.8 will be given in Section 7.

The following corollary is a simple consequence of Proposition 1.8, Eq. (7.4) in the proof of this proposition, and Theorem 1.5 in the form of Eq. (1.9).

**Corollary 1.9.** Suppose that \( q \in (1, \infty) \). Under the hypothesis of Theorem 1.5, if \( f \in L^q(\nu_T) \) and \( f \) is harmonic (i.e. \( \Delta f = 0 \)), then
\[
|f(y)| \leq \|f\|_{L^q(\nu_T)} \exp \left( \frac{k/2}{q - 1} e^{kT} - \frac{1}{q - 1} |y|^2 \right).
\]
In particular, if \( G \) is further assumed to be a complex Lie group and \( f \in L^q(\nu_T) \) is assumed to be holomorphic, then the pointwise bound in Eq. (1.15) is still valid.

**Remark 1.10.** When \( f \) is holomorphic, \( q = 2, T = 1/2 \), and \( G = \mathbb{C}^d \), the inequality in Eq. (1.15) is Bargmann’s pointwise bound in [4, (Eq. (1.7))] except that the constant in the exponent is off by a factor of two. More generally, when \( G \) is a general complex Lie group and \( f \) is holomorphic, it has been shown in [22, Corollary 5.4] that
\[
|f(y)| \leq \|f\|_{L^2(\nu_{1/2})} e^{y^2/2t} \text{ for all } y \in G.
\]
The reason for the discrepancy in the coefficients in the exponents between these inequalities is that \( p_{1/2}(x, y) \) is not the reproducing kernel for the holomorphic functions in \( L^2(\nu_{1/2}) \) in that \( y \to p_{1/2}(x, y) \) is not holomorphic. The coefficient in the exponent of Eq. (1.15) is also not sharp since \( y \to p_T(x, y) \) is not harmonic.

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**2. \( L^q \) – Jacobian estimates**

Let \( M \) be a finite dimensional manifold, \( \mu \) be a probability measure on \( M \) with a smooth, strictly positive density in each coordinate chart. For \( r > 0 \), let \( \|f\|_r := (\int_M |f|^r \, d\mu)^{1/r} \) denote the \( L^r(\mu) \) norm of \( f : M \to \mathbb{C} \).
Let \( X_t \) be a time dependent vector field and let \( S_t \) denote its flow, i.e. \( S_t(m) \) solves,

\[
\frac{d}{dt} S_t(m) = X_t \circ S_t(m) \quad \text{with} \quad S_0(m) = m \quad \text{for all} \quad m \in M.
\]

We will assume that \( X_t \) is forward complete, i.e. \( S_t(m) \) exists for all \( t \geq 0 \) and \( m \in M \).

Define

\[
\mu_t = (S_t)_* \mu = \mu \circ S_t^{-1}.
\]

Since \( \mu_t \) also has a strictly positive density in each coordinate chart the Radon-Nikodym derivative

\[
J_t = \frac{d\mu_t}{d\mu}
\]

exists for all \( t \geq 0 \). Our goal of this section is to prove Theorem 2.5 below which gives an upper bound on \( \|J_t\|_q \) for \( q \in (1, \infty) \). This result is a slight extension of part of Theorem 2.14 in Galaz-Fontes, Gross, and Sontz \cite{Galaz-Fontes:2006} to the setting of time dependent vector fields, \( X_t \). For the readers convenience we will sketch the method introduced in \cite[Theorem 2.14]{Galaz-Fontes:2006}. In what follows, \( 0 \ln 0 \) is to always be interpreted to be 0.

**Lemma 2.1.** Suppose that \( (t, m) \in (0, T) \times M \to h_t(m) \in [0, \infty) \) is a smooth bounded function and \( r : (0, T) \to (1, \infty) \) is a \( C^1 \) function. Then

\[
\frac{d}{dt} \ln \|h_t\|_{r(t)} = \frac{\dot{r}(t)}{r(t)} \int_M \frac{h_t^{r(t)}}{\|h_t\|_{r(t)}} \left( \ln \frac{h_t}{\|h_t\|_{r(t)}} \right) d\mu
\]

\[
+ \frac{1}{r(t)} \int_M \frac{d}{ds} \ln \|h_t\|_{r(t)} d\mu.
\]

**Proof.** For the reader’s convenience we will give a formal derivation of this identity and refer the reader to Gross \cite[Lemma 1.1]{Gross:1972} for the technical details. For \( r > 0 \) and any bounded measurable function, \( g : M \to \mathbb{R} \), a straightforward calculation shows

\[
\frac{d}{dr} \ln \|g\|_r = \frac{1}{r} \int_M \frac{|g|^r}{\|g\|_r^r} \left( \ln \frac{|g|}{\|g\|_r} \right) d\mu.
\]

If we further assume that \( r > 1 \) and \( v : M \to \mathbb{R} \) is another bounded measurable function, then

\[
\partial_v \ln \|g\|_r = \partial_v \left[ \frac{1}{r} \ln \left( \int_M |g|^r d\mu \right) \right] = \frac{1}{r} \int_M \partial_v |g|^r d\mu = \int_M \frac{|g|^{r-1} \text{sgn}(g)}{\|g\|_r^r} v \, d\mu.
\]
These two identities along with the chain rule,
\[ \frac{d}{dt} \ln \|h_t\|_r(t) = \frac{d}{ds}\big|_{s=t} \left[ \ln \|h_t\|_r(s) + \ln \|h_s\|_r(t) \right], \]
easily give Eq. (2.2). q.e.d.

**Lemma 2.2.** Let \( W \in L^1(\mu) \) and \( f \geq 0 \) be a bounded measurable function. Then, for all \( s > 0 \),
\[ \int_M Wf d\mu \leq s \int_M f \ln \frac{f}{\mu(f)} d\mu + sB(W/s) \int_M f d\mu, \]
where \( B(W) := \ln (\mu(e^W)) = \ln \left( \int_M e^W d\mu \right). \)

**Proof.** Recall that Young’s inequality states, \( xy \leq e^x + y \ln y - y \) for \( x \in \mathbb{R} \) and \( y \geq 0 \), where \( 0 \ln 0 := 0 \).
Applying Young’s inequality with \( x = W \) and \( y = f \) and then integrating the result gives
\[ \int_M Wf d\mu \leq \int_M e^W d\mu + \int_M [f \ln f - f] d\mu. \]
Replacing \( f \) by \( \lambda f \) with \( \lambda > 0 \) in this inequality then shows
\[ \int_M Wf d\mu \leq \lambda^{-1} \left[ \int_M e^W d\mu + \int_M [\lambda f \ln (\lambda f) - \lambda f] d\mu \right] \]
\[ = \lambda^{-1} \int_M e^W d\mu + \ln \lambda \int_M fd\mu + \int_M [f \ln f - f] d\mu. \]
The minimizer of the right side of this inequality occurs at \( \lambda = \left( \int_M e^W d\mu \right) \cdot \left( \int_M fd\mu \right)^{-1} \) and using this value for \( \lambda \) gives
\[ \int_M Wf d\mu \leq \int_M f \ln \frac{f}{\mu(f)} d\mu + B(W) \int_M f d\mu. \]
(The proof of Eq. (2.4) was predicated on the assumption that \( B(W) < \infty \) but clearly Eq. (2.4) remains valid when \( B(W) = \infty \).) The estimate in Eq. (2.3) follows directly from this by replacing \( W \) by \( W/s \). q.e.d.

**Definition 2.3.** The \( \mu \)-divergence of a smooth vector field, \( X \), on \( M \) is the function \( W = W^\mu_X \) defined by
\[ \int_M X\varphi d\mu = \int_M \varphi W d\mu, \text{ for all } \varphi \in C_c^1(M). \]

**Proposition 2.4.** Let \( X_t \) and \( S_t \) be as in Eq. (2.1), \( W_t := W_{X_t} \)
be the \( \mu \)-divergence of \( X_t \), \( h \in C^1(M,[0,\infty)) \), \( h_t := h \circ S_t^{-1} \), and \( r \in C^1((0,\tau),(1,\infty)). \) Then for any \( s > 0 \) we have
\[ \frac{d}{dt} \ln \|h_t\|_r(t) \geq (\frac{\dot{r}}{r} - s) \int_M \frac{h_t^r}{\|h_t\|_r^r} \left( \ln \frac{h_t}{\|h_t\|_r} \right) d\mu - \frac{s}{r} B\left(s^{-1}W_t\right). \]
Proof. Differentiating the identity \( S_t \circ S_t^{-1}(m) = m \) and making use of the flow Eq. (2.1) implies
\[
X_t(m) + (S_t)_* \frac{d}{dt} S_t^{-1}(m) = 0.
\]
Therefore,
\[
\frac{d}{dt} S_t^{-1}(m) = - (S_t^{-1})_* X_t(m)
\]
or equivalently,
\[
\frac{d}{dt} f(S_t^{-1}(m)) = -X_t(f \circ S_t^{-1})(m) \text{ for all } f \in C^1(M).
\]
Using this identity along with Eq. (2.2) shows
(2.6)
\[
\frac{d}{dt} \ln \| h_t \|_r(t) = \frac{\dot{r}}{r} \int_M \frac{h_t^r}{\| h_t \|_r} \left( \ln \frac{h_t}{\| h_t \|_r} \right) d\mu - \frac{1}{r^2} \int_M \frac{X_t h_t^r}{\| h_t \|_r} d\mu,
\]
where \( r = r(t) \) and \( \dot{r} = \dot{r}(t) \). Combining this identity with the definition of \( W_t \) and the estimate in Eq. (2.3) with \( W = W_t \) and \( f = \frac{h_t^r}{\| h_t \|_r} \), then implies,
\[
\frac{d}{dt} \ln \| h_t \|_r(t) = \frac{\dot{r}}{r} \int_M \frac{h_t^r}{\| h_t \|_r} \left( \ln \frac{h_t}{\| h_t \|_r} \right) d\mu - \frac{1}{r^2} \int_M \frac{X_t h_t^r}{\| h_t \|_r} d\mu \geq - \frac{s}{r} \left[ \int_M \frac{h_t^r}{\| h_t \|_r} \ln \frac{h_t^r}{\| h_t \|_r} d\mu + \mathcal{B}(W_t/s) \right]
\]
which is the same as Eq. (2.5). q.e.d.

The following theorem is the extension of Galaz-Fontes, Gross, and Sontz [28, Theorem 2.14] from time–independent vector fields to time–dependent vector fields. These results generalize the fundamental results of Cruzerio [12] – also see [5, 6, 13, 18, 50, 51] for other related results.

**Theorem 2.5 (Jacobian Estimate).** Let \( q > 1 \) and \( r \in C([0, \tau], [1, \infty)) \cap C^1((0, \tau), (1, \infty)) \) such that \( r(0) = 1, r(\tau) = q \) and \( \dot{r}(t) > 0 \) for \( 0 < t < \tau \), then
(2.7)
\[
\| J_t \|_{q'} \leq e^{\Lambda(r)},
\]
where \( q' := q/(q - 1) \) is the conjugate exponent to \( q \) and
(2.8)
\[
\Lambda(r) = \Lambda_X(r) := \int_0^r \frac{\dot{r}(t)}{r^2(t)} \mathcal{B} \left( \frac{r(t)}{\dot{r}(t)} W_t \right) dt.
\]

Proof. Taking \( s = \dot{r}/r \) in Eq. (2.5) gives
\[
\frac{d}{dt} \ln \| h_t \|_r(t) \geq - \frac{\dot{r}}{r^2} \mathcal{B} \left( \frac{\dot{r}}{r^2} W_t \right)
\]
which integrates to
\[ \| h \circ S^{-1}_\tau \|_q = \| h \|_q \geq \| h \|_1 \exp \left( - \int_0^\tau \frac{\dot{r}(t)}{r^2(t)} B \left( \frac{r(t)}{\dot{r}(t)} \right) \right) dt. \]

Replacing \( h \) by \( h \circ S \) in this inequality implies
(2.9)
\[ \int_M h J_r \, d\mu = \| h \circ S \|_1 \leq \| h \|_q e^{\Lambda(r)}. \]

Let \( L^q(\mu)^+ \) denote the almost everywhere non-negative functions in \( L^q(\mu) \). Since Eq. (2.9) is valid for all \( h \in C^1(M, [0, \infty)) \) and the latter functions are dense in \( L^q(\mu)^+ \) (see the proof of Lemma 2.8 in [28]), it follow that Eq. (2.9) is valid for all \( h \in L^q(\mu)^+ \). Equation (2.7) now follows by the converse to Hölder’s inequality. Indeed, let \( K \subset M \) be a compact set and take \( h = J_r^{1/(q-1)} 1_K = J_r^{1/(q-1)} 1_K \) in Eq. (2.9) to find
\[ \int_M J_r^{1/(q-1)} 1_K \, d\mu \leq \left\| J_r^{1/(q-1)} 1_K \right\|_q e^{\Lambda(r)} = \left( \int_M J_r^{1/(q-1)} 1_K \, d\mu \right)^{1/q} e^{\Lambda(r)}. \]

This inequality is equivalent to
\[ \| J_r 1_K \|_{q'} = \left( \int_M J_r^{1/(q-1)} 1_K \, d\mu \right)^{1-1/q} \leq e^{\Lambda(r)}. \]

Now replacing \( K \) by \( K_n \) with \( K_n \) compact and \( K_n \uparrow M \) and passing to the limit as \( n \to \infty \) in the previous inequality gives the estimate in Eq. (2.7).

q.e.d.

3. Proof of Theorem 1.5

In this section we will give a proof of Theorem 1.5 assuming that Theorem 1.3 holds. We will use the following notation in the proofs.

**Notation 3.1.** Let \( c(t) \) be defined by \( c(0) = 1 \) and
(3.1)
\[ c(t) = \frac{t}{e^t - 1} \text{ for all } t \neq 0. \]

**Proof.** (Proof of Theorem 1.5.) In order to abbreviate the notation, let \( c := c(kT) / T \). Let \( g \in C^1([0, 1], G) \) be such that \( g(0) = e \in G \) and \( g(1) = y \in G \) and define \( A_t := L_{g(t)}^{-1} \dot{g}(t) \in g \). If we now let \( X_t := \tilde{A}_t \in \Gamma(TG) \), then the flow, \( S_t \), of \( X_t \) satisfies, \( S_t(x) = x g(t) \). Indeed, because \( X_t \) is left invariant,
\[ \frac{d}{dt} x g(t) = L_{x*} \dot{g}(t) = L_{x*} L_{g(t)*} A_t = L_{x g(t)*} A_t = X_t(x g(t)). \]
In order to apply the Jacobian estimate in Theorem 2.5, let \( d\mu (x) = dv_T(x) := p_T(x) \, dx \) and observe that
\[
\int_G h(S_1(x)) \, d\mu (x) = \int_G h(xy) \, d\mu (x) = \int_G h(xy) \, p_T(x) \, dx = \int_G h(x) \, p_T(xy^{-1}) \, dx = \int_G h(x) \, \frac{p_T(xy^{-1})}{p_T(x)} \, d\mu (x)
\]
from which it follows that
\[
J_1(x) := \frac{d(S_1)}{d\mu}(x) = \frac{p_T(xy^{-1})}{p_T(x)}.
\]
Moreover, if \( W_t = W_{X_t} \) is the \( \mu = \nu_T \) divergence of \( X_t \), by Theorem 1.3,
\[
\mathcal{B}(\lambda W_t) = \ln \left( \int_G e^{\lambda W_t} \, d\mu \right) \leq \frac{c(kT)}{T} \lambda^2 |A_t|^2.
\]
Hence it follows from Theorem 2.5 that
\[
\left[ \int_G \left( \frac{p_T(xy^{-1})}{p_T(x)} \right)^{q'} p_T(x) \, dx \right]^{1/q'} = \|J_1\|_{q'} \leq e^{\Lambda(r)},
\]
where
\[
\Lambda(r) = \int_0^1 \frac{\dot{r}(t)}{r^2(t)} \mathcal{B} \left( \frac{r(t)}{\dot{r}(t)} W_t \right) dt \\
\leq c \int_0^1 \frac{\dot{r}(t)}{r^2(t)} \frac{r^2(t)}{\dot{r}(t)} |A_t|^2 dt = c \int_0^1 \frac{|A_t|^2}{\dot{r}(t)} dt,
\]
and \( r \in C([0,1], [1,\infty)) \cap C^1((0,1), (1,\infty)) \) such that \( r(0) = 1, r(1) = q \) and \( \dot{r}(t) > 0 \) for \( 0 < t < 1 \). We now want to choose \( r(t) \) so as to minimize \( \Lambda(r) \) subject to the constraints \( \dot{r}(t) > 0, r(0) = 1 \) and \( r(1) = q \). To see how to choose \( r(t) \), let us differentiate \( \Lambda(r) \) in a direction \( v \) such that \( v(0) = 0 = v(1) \) and then require
\[
0 \overset{\text{set}}{=} (\partial_v \Lambda)(r) = -\frac{c}{2} \int_0^1 \frac{|A_t|^2}{\dot{r}^2(t)} \dot{v}(t) \, dt = -\frac{c}{2} \int_0^1 v(t) \frac{d}{dt} \left( \frac{|A_t|^2}{\dot{r}^2(t)} \right) dt.
\]
Since \( v(t) \) is arbitrary, we should require \( \frac{|A_t|^2}{\dot{r}^2(t)} = \kappa^{-2} \), where \( \kappa > 0 \) is a constant, i.e. \( \dot{r}(t) = \kappa |A_t|_g \). Hence we take
\[
r(t) = 1 + \kappa \int_0^t |A_{\tau}|_g \, d\tau,
\]
where
\[
\kappa := (q - 1) \left( \int_0^1 |A_{\tau}|_g \, d\tau \right)^{-1}
\]
has been chosen so that $r(1) = q$. With this choice of $r$,
\[
\Lambda (r) := \frac{c}{2} \int_{0}^{1} \frac{|A(t)|_{g}^{2}}{|A|_{g}} dt = \frac{c}{2(\kappa - 1)} \left( \int_{0}^{1} |A_{l}|_{g} dt \right)^{2}
\]
and using this value for $\Lambda (r)$ in Eq. (3.4) along with the identity,
\[
(q - 1)^{-1} = q' - 1 \implies \left( \int_{G} \left[ \frac{p_{T} (xy^{-1})}{p_{T} (x)} \right]^{q'} p_{T} (x) dx \right)^{1/q'} = \| J_{1} \|_{q'}
\]
\[
\leq \exp \left( \frac{c(q' - 1)}{2} \left( \int_{0}^{1} |A_{l}|_{g} dt \right)^{2} \right)
\]
Upon noting that $q' := q(q - 1)^{-1}$ ranges over $(1, \infty)$ as $q$ ranges over $(1, \infty)$, the proof of Theorem 1.5 is complete.

q.e.d.

4. Properties of the Hodge – de Rham semigroups

This section gathers a number of technical functional analytic results needed to establish the representation formula in Theorem 5.4 below. Let $(M, g)$ be a complete Riemannian manifold, $dV$ denote the volume measure on $M$ associated to $g$, $\nabla$ denote the Levi-Civita covariant derivative, $\Lambda^{k} = \Lambda^{k} (T^{*}M)$, $\Lambda = \bigoplus_{k=0}^{\dim M} \Lambda^{k}$, $\Omega^{k} (M)$ ($\Omega^{k}_{c} (M)$) denote the space of (compactly supported) smooth $k$ – forms over $M$, and $\Omega (M) = \bigoplus_{k=0}^{\dim M} \Omega^{k} (M)$ be the space of all smooth forms over $M$. If $\alpha$ and $\beta$ are measurable $k$ – forms, let
\[
\langle \alpha, \beta \rangle_{m} := \sum_{j_{1}, \ldots, j_{k}=1}^{d} \alpha (e_{j_{1}}, \ldots, e_{j_{k}}) \beta (e_{j_{1}}, \ldots, e_{j_{k}}),
\]
where $\{ e_{j} \}_{j=1}^{d}$ is any orthonormal frame for $T_{m}M$. When $m \to \langle \alpha, \beta \rangle_{m}$ is integrable, let
\[
\langle \alpha, \beta \rangle := \int_{M} \langle \alpha, \beta \rangle dV
\]
and let $L^{2} (\Lambda^{k})$ denote the measurable $k$ – forms, $\alpha$, such that $\langle \alpha, \alpha \rangle < \infty$. Further let
\[
L^{2} (\Lambda) := \bigoplus_{k=0}^{\dim M} L^{2} (\Lambda^{k}).
\]
Two measurable $k$ – forms, $\alpha$ and $\beta$, are take to be equivalent if $\alpha = \beta$ a.e.

Let $d : \Omega (M) \to \Omega (M)$ be the differential operator taking $k$ – forms to $k + 1$ – forms, $\delta$ be the formal $L^{2}$ – adjoint of $-d$,
\[
\Delta := -(\delta d + d\delta) = -(d + \delta)^{2}
\]
be the Hodge-de Rham Laplacian on $\Omega (M)$, and $\Box$ be the Bochner (i.e. flat) Laplacian on $\Omega (M)$. More precisely if $\alpha$ is a $k$-form, $\delta \alpha$ is the $k - 1$ form defined by

$$ (\delta \alpha)_m := \sum_{j=1}^{d} (\nabla_{e_j} \alpha)(e_j, -) $$

and

$$ (\Box \alpha)_m := \sum_{j=1}^{d} \nabla^2_{e_j \otimes e_j} \alpha := \sum_{j=1}^{d} \left( \nabla^2_{E_j} \alpha - \nabla_{\nabla_{E_j} E_j} \alpha \right)_m, $$

where $\{E_j\}_{j=1}^{\dim M}$ is an orthonormal frame for $TM$ defined in a neighborhood of $m$. The next two theorems summarize the properties about these operators that will be needed in this paper.

**Theorem 4.1.** The operators, $d_k := d|_{\Omega^k_c (M)} : \Omega^k_c (M) \rightarrow \Omega^{k+1}_c (M)$ for $k = 0, 1, 2, \ldots, \dim M - 1$ are $L^2 (\Lambda^k)$-closable with closure denoted by $\bar{d}_k$. Let us now further assume that $(M, g)$ is complete. Then:

1) Each of the operators, $\Delta_k := \Delta|_{\Omega^k_c (M)}$ for $k = 0, 1, 2, \ldots, \dim M$ thought of as unbounded operators on $L^2 (\Lambda^k)$, are essentially self-adjoint operators. Let $\bar{\Delta}_k$ denote the (self-adjoint) closure of $\Delta_k$.

2) Each operator, $\bar{\Delta}_k$, is non-negative. Let $e^{t\bar{\Delta}_k}$ denotes the contraction semi-group on $L^2 (\Lambda^k)$ associated to $\bar{\Delta}_k$.

3) For $k \in \{0, 1, \ldots, \dim M - 1\}$ and $t > 0$, $\bar{d}_k e^{t\bar{\Delta}_k} = e^{t\bar{\Delta}_{k+1}} \bar{d}_k$ on the domain of $\bar{d}_k$.

4) $\delta e^{t\bar{\Delta}_k} \omega = e^{t\bar{\Delta}_{k-1}} \delta \omega$ for all $\omega \in \Omega^k_c (M)$ with $k = 1, 2, \ldots, \dim M$.

**Proof.** Let $\delta_k := \delta|_{\Omega^k_c (M)} : \Omega^k_c (M) \rightarrow \Omega^{k-1}_c (M)$. As $-\delta_{k+1} \subset d^*_k$, $d^*_k$ is densely defined and hence $\bar{d}_k$ is closable. For items 1. and 2., see Gaffney [26], Roelecke [53], Chernoff [11], and Strichartz [60]. Item 3. is a simple application of Theorem A.2 of Appendix A below. In applying this theorem, take $W = L^2 (\Lambda^{k-1})$, $X = L^2 (\Lambda^k)$, $Y = L^2 (\Lambda^{k+1})$ and $Z = L^2 (\Lambda^{k+2})$ with $A = \bar{d}_{k-1}$, $B = \bar{d}_k$, and $C := \bar{d}_{k+1}$. By convention $\Omega^{-1} (M) = \{0\} = \Omega^{\dim M+1} (M)$ and $d_{-1} = 0 = d_{\dim M}$. With these assignments, the self-adjoint operators, $L$ and $S$, in Theorem A.2 become

$$ L = \bar{d}_{k-1} d^*_k + d^*_k \bar{d}_k $$

and

$$ S = \bar{d}_k d^*_k + d^*_k \bar{d}_{k+1}. $$

As $\Delta_k|_{\Omega^k_c (M)} \subset -L$ and $-L$ is self-adjoint (see Theorem A.1 below), it follows that $\bar{\Delta}_k = -L$ and similarly, $\bar{\Delta}_{k+1} = -S$. For item 4., let $\omega \in \Omega^k_c (M)$ and $\varphi \in \Omega^{k-1}_c (M)$. Then

$$ (\delta e^{t\bar{\Delta}_k} \omega, \varphi) = - (e^{t\bar{\Delta}_k} \omega, d \varphi) = - (\omega, e^{t\bar{\Delta}_k} d \varphi) = - (\omega, e^{t\bar{\Delta}_{k-1}} \varphi) $$

$$ = (\delta \omega, e^{t\bar{\Delta}_{k-1}} \varphi) = (e^{t\bar{\Delta}_{k-1}} \delta \omega, \varphi). $$

q.e.d.
Remark 4.2. With a little more work it is possible to show that \( \bar{d}_k = -\delta_k^{k+1} \) and that \( \bar{d}_k e^{t \Delta_k} = e^{t \Delta_k - 1} \bar{d}_k \) on the domain of \( \bar{d}_k \). We will omit the proof of these results as they are not needed for this paper.

We are primarily concerned with zero and one forms. A key ingredient in the sequel is the Bochner identity,

\[
\Delta \alpha = \square \alpha - \alpha \circ \text{Ric} \quad \text{for all } \alpha \in L^1(M).
\]

Assumption 1. For the rest of this paper we will assume that \((M, g)\) is a complete Riemannian manifold such that \(\text{Ric} \geq k\) for some \(k \in \mathbb{R}\), i.e. \(\text{Ric}_m \geq k \text{Id}_M\) for all \(m \in M\).

Theorem 4.3 (Semi-group domination). Suppose that \((M, g)\) is a complete Riemannian manifold such that \(\text{Ric} \geq k\) for some \(k \in \mathbb{R}\). Then for all \(f \in L^2(\Lambda^0)\) and \(\alpha \in L^2(\Lambda^1)\),

\[
|e^{t \Delta_0} f| \leq e^{t \Delta_0} |f| \leq \|f\|_\infty \quad \text{a.e.}
\]

and

\[
|e^{t \Delta_1} \alpha| \leq e^{-kt} e^{t \Delta_0} |\alpha| \leq e^{-kt} \|\alpha\|_\infty \quad \text{a.e.,}
\]

where \(\|f\|_\infty\) and \(\|\alpha\|_\infty\) denote the essential supremums of the functions, \(|f|\) and \(m \to |\alpha_m|\) respectively.

Proof. The inequality in Eq. (4.4) is an immediate consequence Eqs. (1.2), (1.1) and the positivity of the heat kernel, \(p_t(x, y)\). This inequality may also be proved using the semi-group domination ideas that will be used below to prove Eq. (4.5). The proof of Eq. (4.5) will be an application of the results in Simon [57, 58] and Hess, Schrader, and Uhlenbrock [35] along with a Kato [40] type inequality. The general Kato inequality we need is given in Theorem B.2 of Appendix B. We apply Theorem B.2 with \(E = \Lambda^1(T^*M)\) to conclude,

\[
(\square \alpha, \varphi \text{sgn}_e(\alpha)) \leq (|\alpha|, \Delta \varphi)
\]

for all \(\alpha \in \Omega^1_c(M)\) and \(\varphi \in C^\infty(M)_+ := C^\infty(M \to [0, \infty))\). In Eq. (4.6),

\[
\text{sgn}_e(\alpha) := 1_{\alpha \neq 0} \frac{\alpha}{|\alpha|} + 1_{\alpha = 0} e,
\]

where \(e\) is any measurable section of \(E\) such that \(\langle \square \alpha, e \rangle = 0\) on \(M\). This inequality and the Bochner identity in Eq. (4.3) shows

\[
(\Delta_1 \alpha, \varphi \text{sgn}_e(\alpha)) = (\square \alpha, \varphi \text{sgn}_e(\alpha)) - (\alpha \circ \text{Ric}, \varphi \text{sgn}_e(\alpha)) \\
\leq (|\alpha|, \Delta \varphi) - (\alpha \circ \text{Ric}, \varphi \text{sgn}_e(\alpha)).
\]
To evaluate the last term, let \( Y \) be the vector field on \( M \) such that 
\[ \alpha = \langle Y, \cdot \rangle. \]
Then \( \alpha \circ \text{Ric} = \langle \text{Ric} Y, \cdot \rangle \) and
\[
\langle \alpha \circ \text{Ric}, \text{sgn}_\varepsilon (\alpha) \rangle = 1_{\alpha \neq 0} \frac{1}{|\alpha|} \langle \alpha \circ \text{Ric}, \alpha \rangle = 1_{\alpha \neq 0} \frac{1}{|\alpha|} \langle \text{Ric} Y, Y \rangle \geq k 1_{\alpha \neq 0} \frac{1}{|\alpha|} |Y|^2 = k |\alpha|. 
\]
Therefore,
\[
(\alpha \circ \text{Ric}, \varphi \text{sgn}_\varepsilon (\alpha)) = \int_M (\alpha \circ \text{Ric}, \text{sgn}_\varepsilon (\alpha)) \varphi dV \geq k (|\alpha|, \varphi)
\]
which combined with Eq. (4.7) implies
\[
(\Delta_1 \alpha, \varphi \text{sgn}_\varepsilon (\alpha)) \leq (|\alpha|, \Delta \varphi) - k (|\alpha|, \varphi)
\]
or equivalently,
\[
(H_0 \alpha, \varphi \text{sgn}_\varepsilon (\alpha)) \geq (|\alpha|, -\Delta \varphi),
\]
where \( H_0 := - (\Delta + k)|\Omega^2(M)\). In particular, if \( g \in C^\infty_c (M)_+ \), \( \lambda > 0 \),
\[ \varphi = (-\Delta_0 + \lambda)^{-1} g, \]
and \( \alpha_1 \in \Omega^1_c (M) \) and we define \( \alpha_2 := \varphi \text{sgn}_\varepsilon (\alpha_1) \in L^2 (\Lambda^1) \), then \( (\alpha_1, \alpha_2)_{L^2 (\Lambda^1)} = (|\alpha_1|, |\alpha_2|)_{L^2 (\Lambda_0)}, |\alpha_2| = \varphi, \)
and
\[
(H_0 \alpha_1, \alpha_2)_{L^2 (\Lambda^1)} \geq (|\alpha_1|, -\Delta_0 \varphi)_{L^2 (\Lambda_0)}. 
\]
Hence we have verified the hypothesis of Proposition 2.14 and Theorem 2.15 in [35] and as a consequence,
\[
|e^{-tH_0} \alpha| \leq e^{-t(-\Delta_0)} |\alpha| \ a.e. \ \text{for all} \ \alpha \in L^2 (\Lambda^1). 
\]
As \( H_0 = -\Delta_1 - k \) and hence, \( e^{-tH_0} = e^{t\Delta_1} e^{tk} \), Eq. (4.9) is equivalent to the first inequality in Eq. (4.5). q.e.d.

5. A path integral derivative formula

5.1. Brownian motion and the divergence formula. We start with a filtered probability space, \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \), satisfying the usual hypothesis. For each \( x \in M \), let \( \{\Sigma_t^x : t < \zeta(x)\} \) be an \( M \)-valued Brownian motion on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \), starting from \( x \), with possibly finite lifetime \( \zeta(x) \). Recall \( \Sigma_t^x \) is said to be an \( M \)-valued Brownian motion provided it is a Markov diffusion process starting at \( x \) with transition semigroup determined by the heat kernel, \( p_t (\cdot, \cdot) \). Because of our standing assumption, \( \text{Ric} \geq k \), it is well–known that \( \int_M p_t (x, y) dy = 1 \) for all \( x \in M \) and consequently that \( \zeta (x) = \infty \), see [3, 27, 68, 16, 42, 29, 30, 32] and the books [63, Theorem 8.62], [37, Chapter 4.] and [14, Theorem 5.2.6]. For our purposes it will be convenient to construct \( \Sigma_t^x \) as a solution to a stochastic differential equation which we will describe shortly.
Notation 5.1. Given two isometric isomorphic real finite-dimensional inner product spaces, \( V \) and \( W \), let \( O (V, W) \) denote the set of linear isometries from \( V \) to \( W \).

Let \( //_t (\sigma) \) denote parallel translation along a curve \( \sigma \) in \( TM \) and all associated bundles. We also introduce the horizontal vector fields on the orthogonal frame bundle over \( M \) as

\[
B_v(u) = \frac{d}{dt} |_{0} //_t (\sigma) u \quad \text{for} \quad v \in \mathbb{R}^d \quad \text{and} \quad u \in O \left( \mathbb{R}^d, T_x M \right),
\]

where \( \sigma (t) \) is a curve in \( M \) such that \( \dot{\sigma} (0) = uv \).

Notation 5.2. Given a semi-martingale, \( Y_t \), we will denote its Itô differential by \( dY_t \) and its Fisk-Stratonovich differential by \( \circ dY_t \).

Let \( b_t \) denote a \( \mathbb{R}^d \)–valued Brownian motion, \( x \in M \), and \( u_0 \in O \left( \mathbb{R}^d, T_x M \right) \), then \( \Sigma^x_t \) may be defined as the unique solution to the stochastic differential equation,

\[
\circ d \Sigma^x_t = u_t \circ db_t \quad \text{with} \quad \Sigma^x_0 = x,
\]

\[
\circ du_t = B_{\circ db_t} (u_t) \quad \text{with} \quad u_0.
\]

The stochastic parallel translation along \( \Sigma^x_t \) up to time \( t \) is taken to be,

\[
//_t := u_t u_0^{-1} \in O \left( T_x M, T_{\Sigma^x_t} M \right).
\]

Suppose that \( f (t, m) \) \((\alpha (t, m))\) is a smooth time dependent function (one form), then the Itô differentials of \( f (t, \Sigma^x_t) \) and \( \alpha (t, \Sigma^x_t) //_t \) are

\[
d [f (t, \Sigma^x_t)] = \left( \frac{\partial}{\partial t} f (t, \Sigma^x_t) + \frac{1}{2} \Delta_0 f (t, \Sigma^x_t) \right) dt + \langle \text{grad} f (t, \cdot), //_t db_t \rangle,
\]

(5.1)

\[
d [\alpha (t, \Sigma^x_t) //_t] = \left( \frac{\partial}{\partial t} \alpha (t, \Sigma^x_t) + \frac{1}{2} \Box \alpha (t, \Sigma^x_t) \right) dt + [\nabla //_t db_t, \alpha (t, \cdot)] //_t.
\]

(5.2)

See (for example) \([24, 46, 63, 37, 20]\) for more on the general background used in this section.

5.2. The divergence formula. Let \( Q_t \) denote the End \((T_x M)\)–valued process satisfying the ordinary differential equation,

\[
\frac{d}{dt} Q_t = -\frac{1}{2} \text{Ric} //_t Q_t \quad \text{with} \quad Q_0 = id_{T_x M},
\]

(5.3)

where

\[
\text{Ric} //_t := //_t^{-1} \text{Ric}_{\Sigma^x_t} //_t.
\]

Lemma 5.3. If \( \text{Ric} \geq k \) for some \( k \in \mathbb{R} \) and \( \| \cdot \|_{op} \) denotes the operator norm on \( T_x M \), then

\[
\| Q_t \|_{op} \leq e^{-kt/2}.
\]

(5.5)
Similarly if \( \text{Ric} \leq K \) for some \( K \in \mathbb{R} \), then
\[
\|Q_t^{-1}\|_{op} \leq e^{Kt/2}.
\]

Proof. For any \( v \in T_xM \), we have
\[
\frac{d}{dt} \|Q_t v\|^2 = \left\langle -\text{Ric}(Q_t v, Q_t v) \right\rangle \leq -k \|Q_t v\|^2
\]
from which Eq. (5.5) easily follows. To prove Eq. (5.6), let \( R_t := (Q_t^{-1})^* \) and observe that
\[
\frac{d}{dt} R_t = -\left( Q_t^{-1}Q_t^{-1} \right)^* = \frac{1}{2} \left( Q_t^{-1} \text{Ric}(Q_t^{-1})^* \right) = \frac{1}{2} \text{Ric}(R_t).
\]
Hence reasoning as above we may conclude that
\[
\|Q_t^{-1}\|_{op} = \|R_t\|_{op} \leq e^{Kt/2}.
\]
q.e.d.

When \( M \) is compact, the following result is Theorem 5.10 of Driver and Thalmaier [23].

**Theorem 5.4** (A divergence formula). Assume the Ricci curvature, \( \text{Ric} \), on \( M \) satisfies, \( k \leq \text{Ric} \leq K \) for some \( -\infty < k \leq K < \infty \). Let \( T > 0 \) and \( \ell \) be a \( C^1 \) – adapted real-valued process such that \( \ell_0 = 0 \), \( \ell_T = 1 \), and
\[
(5.7) \quad \int_0^T \left| \frac{d}{d\tau} \ell_\tau \right| d\tau \leq C,
\]
where \( C < \infty \) is a non-random constant. Then for every \( C^2 \) – vector field, \( Y \), on \( M \) with compact support the following identity holds
\[
(5.8) \quad \mathbb{E} [\nabla \cdot Y (\Sigma^T)] = \mathbb{E} \left[ \left\langle Y (\Sigma^T), \text{//}_{TQ_T} \int_0^T \ell_\tau Q_t^{-1} \right\rangle \right],
\]
where \( \nabla \cdot Y \) is the divergence of \( Y \) and \( \ell_t := \frac{d}{dt} \ell_t \).

Proof. The proof will consist of adding some technical details to the proof of Theorem 5.10 in [23]. Suppose \( a \) is a smooth one form on \( M \) with compact support,
\[
(5.9) \quad a_t := e^{(T-t)\Delta_1/2} a,
\]
\( \ell_\tau \) is an adapted continuously differentiable real–valued process, and \( \ell_0 \) is a fixed vector in \( T_xM \). Then as shown in [23, Theorem 3.4] (and repeated below in Lemma C.1 for the readers convenience) the process, \( Z_t := (a_t (\Sigma^T) \circ //_t) Q_t \left[ \int_0^t Q_t^{-1} \left( \frac{d}{d\tau} \ell_\tau \right) d\tau + \ell_0 \right] - (\delta a_t) (\Sigma^T) \ell_t \)
is a local martingale. From Theorems 4.1 and 4.3 we have

$$|a_t| \leq e^{-(T-t)\Delta/2} \|a\|_{\infty} \leq e^{T\Delta/2} \|a\|_{\infty}$$

and

$$|\delta a_t| = \left| e^{(T-t)\Delta/2} \delta a \right| \leq \|\delta a\|_{\infty}.$$

Making use of these estimates along with Lemma 5.3 and Eq. (5.7) shows that $Z_t$ is a bounded local martingale and hence, by a localization argument, a martingale. In particular, it follows that $t \to \mathbb{E}Z_t$ is constant for $0 \leq t \leq T$ and hence

$$0 = \mathbb{E} \left[ a (\Sigma^T_t) \circ [\mathbb{T}] Q_T \left( \int_0^T Q_{-1}^{-1} \left( \frac{d}{d\tau} \tilde{\ell}_\tau \right) \, db + \ell_0 \right) \right] - \mathbb{E} \left[ \delta a (\Sigma^T_t) \tilde{\ell}_T \right].$$

If we now suppose that $\ell_0 = 0$, $\tilde{\ell}_0 = 0$, and $\tilde{\ell}_T = 1$, the above formula reduces to

$$0 = \mathbb{E} \left[ a (\Sigma^T_t) \circ [\mathbb{T}] Q_T \int_0^T Q^{-1}_{-1} \left( \frac{d}{d\tau} \tilde{\ell}_\tau \right) \, db - \delta a (\Sigma^T_t) \right].$$

This identity is equivalent to the identity in Eq. (5.8) as is seen by taking $a(x) v := \langle Y(x), v \rangle$ for all $x \in M$ and $v \in T_x M$ and recalling that

$$\delta a = \sum_{i=1}^d i_e_i \nabla_{e_i} \langle Y, \cdot \rangle = \sum_{i=1}^d i_e_i \langle \nabla_{e_i} Y, \cdot \rangle = \nabla \cdot Y.$$

q.e.d.

**Example 5.5.** Taking $\tilde{\ell}_t = t/T$ in Eq. (5.8) shows

$$\mathbb{E} \left[ \nabla \cdot Y (\Sigma^T_t) \right] = \frac{1}{T} \mathbb{E} \left[ \left\langle Y(\Sigma^T_t), [\mathbb{T}] Q_T \int_0^T Q_{-1}^{-1} \, db \right\rangle \right].$$

6. **Exponential integrability of $W^T_A$**

In this section and for the remainder of the paper we will again go back to the setting where $M = G$ is a connected uni-modular Lie group equipped with a left–invariant Riemannian metric as described in the introduction. We are now going to use Theorem 5.4 to estimate $W_A := W^T_A$ in Definition 1.7. In order to do this we will use Eq. (5.8) to find a useful path integral expression for $W_A$, see Theorem 6.4 below.

For $A, B \in \mathfrak{g}$, let $D_A B := \nabla_A \tilde{B} \in \mathfrak{g}$ where $\nabla$ is the Levi-Civita covariant derivative on $TG$. Observe that $\nabla_A \tilde{B}$ is a left invariant vector field and $(\nabla_{\tilde{A}} \tilde{B})(e) = \nabla_A \tilde{B} = D_A B$. Hence we have the identity, $\nabla_{\tilde{A}} \tilde{B} = D_A B$. 


Lemma 6.1. Suppose that $\{A_i\}_{i=1}^{\dim \mathfrak{g}}$ is an orthonormal basis for $\mathfrak{g}$ and $G$ is uni-modular. Then

1) $\sum_{i=1}^{\dim \mathfrak{g}} D_{A_i} A_i = 0$ or equivalently $\sum_{i=1}^{\dim \mathfrak{g}} \nabla_{\tilde{A}_i} \tilde{A}_i = 0$.
2) The divergence of $\tilde{B}$, $\nabla \cdot \tilde{B}$, is zero for all $B \in \mathfrak{g}$.
3) $\Delta_0 = \sum_{i=1}^{\dim \mathfrak{g}} \tilde{A}_i^2$ is the Laplace Beltrami operator on $G$.

Proof. 1) The formula for $D_{A_i} A_i$ is

$$D_{A_i} A_i = \frac{1}{2} (ad_{A_i} B - ad^*_A B - ad^*_B A)$$

and hence $D_{A_i} A_i = -ad^*_A A$ and for any $B \in \mathfrak{g}$ we find

$$\left( \sum_{i=1}^{\dim \mathfrak{g}} D_{A_i} A_i, B \right)_{\mathfrak{g}} = \sum_{i=1}^{\dim \mathfrak{g}} (A_i, ad_{A_i} B)_{\mathfrak{g}} = - \sum_{i=1}^{\dim \mathfrak{g}} (A_i, ad_B A_i)_{\mathfrak{g}} = - \text{tr} (ad_B).$$

Since $G$ is uni-modular, $\det (Ad_e t B) = 0$ for all $t$ and therefore $\text{tr} (ad_B) = 0$.

2) The following simple computation shows $\nabla \cdot \tilde{B} = 0$

$$\nabla \cdot \tilde{B} = \sum_{i=1}^{\dim \mathfrak{g}} \left( \nabla_{\tilde{A}_i} \tilde{B}, \tilde{A}_i \right)_{T G} = \sum_{i=1}^{\dim \mathfrak{g}} (D_{A_i} B, A_i)_{\mathfrak{g}} = - \sum_{i=1}^{\dim \mathfrak{g}} (B, D_{A_i} A_i)_{\mathfrak{g}} = 0.$$

3) Observe that $\{\tilde{A}_i\}_{i=1}^{\dim \mathfrak{g}}$ is a globally defined orthonormal frame for $T G$ and that

$$\Delta_0 = \sum_{i=1}^{\dim \mathfrak{g}} \left[ \tilde{A}_i^2 - \nabla_{\tilde{A}_i} \tilde{A}_i \right] = \sum_{i=1}^{\dim \mathfrak{g}} \tilde{A}_i^2.$$ q.e.d.

In Theorem 6.4 below, we will specialize Theorem 5.4 in order to find a probabilistic representation for $W_A$ of Definition 1.7. This representation will then be used to estimate $\int_G e^{W_A} d\nu_T$ for all $A \in \mathfrak{g}$. Let $\{\Sigma_t\}_{t \geq 0}$ be a Brownian motion on $G$ such that $\Sigma_0 = e$, $b_t$ be the $\mathfrak{g}$-valued Brownian motion defined by,

$$b_t := \int_0^t \left/ \left( \Sigma_t \right)^{-1} \right. \circ d\Sigma_t,$$
and $\beta_t$ be the $g$–valued semi-martingale defined by

$$\beta_t := \int_0^t \theta (\circ d\Sigma_t) = \int_0^t L^{-1}_t \circ d\Sigma_t,$$

where $\theta (vg) := L^{-1}_g vg$ for all $vg \in T_g G$. As a reflection of the fact that $\sum_{i=1}^{\dim g} A_i^2$ is the Laplace–Beltrami operator, $\beta_t$ is another $g$–valued Brownian motion. This will also be evident from the following proposition.

**Proposition 6.2.** Fix $T > 0$ and let $U_t \in O(g)$ be the unique solution to the stochastic differential equation

$$(6.1) \quad dU_t + D_{od\beta_t}U_t = 0 \text{ with } U_0 = I.$$

Further define $Y_t := U_t Q_t$, and $V_t := Y_T Y_t^{-1}$. Then

$$(6.2) \quad /t := L\Sigma_t U_t$$

and

$$(6.3) \quad \int_0^t U_t^{-1} \circ d\beta_t = \int_0^t U_t^{-1} d\beta_t = \int_0^t /t^{-1} \circ d\beta_t = b_t.$$

**Proof.** The fact that $/t := L\Sigma_t U_t$ is explained in [19, Theorem 6.6] and hence

$$b_t = \int_0^t U_t^{-1} L^{-1}_t \circ d\Sigma_t = \int_0^t U_t^{-1} \theta (\circ d\Sigma_t) = \int_0^t U_t^{-1} \circ d\beta_t,$$

i.e. $d\beta_t = U_t \circ db_t$. Letting $\{A_i\}_{i=1}^{\dim g}$ be an orthonormal basis for $g$, it follows from Lemma 6.1 and the fact that $\{U_t A_i\}_{i=1}^{\dim g}$ is also an orthonormal basis for $g$ that

$$dU_t db_t = -\frac{1}{2} D_{od\beta_t} U_t db_t = -\frac{1}{2} D_{U_t db_t} U_t db_t$$

$$= -\frac{1}{2} \sum_{i=1}^{\dim g} D_{U_t A_i} U_t A_i dt = 0.$$

This allows us to conclude that $d\beta_t = U_t \circ db_t = U_t db_t$ which completes the proof of the proposition. q.e.d.

**Proposition 6.3.** Let $Y_t := U_t Q_t$ and for fixed $T > 0$ let $V_t := Y_T Y_t^{-1}$ and $G_t := \sigma (\beta_t - \beta_s : t \leq s, \tau \leq T)$ – the completion of the $\sigma$–algebra generated by $\{\beta_t - \beta_s : t \leq s, \tau \leq T\}$. Then

1) $V_t$ is $G_t$–measurable, and

2) $V_t$ is the unique solution to the backwards stochastic differential equation,

$$dV_t = V_t \left( D_{od\beta_t} + \frac{1}{2} Ric_e \, dt \right) \text{ with } V_T = I.$$
Proof. Because $L_{\Sigma_t^*}$ is an isometry of $G$, it follows that
\begin{equation}
\label{6.4}
\text{Ric}^\perp_t = \langle -\text{Ric}_{\Sigma_t} \rangle = U_t^{-1}L_{\Sigma_t^*}^{-1}\text{Ric}_{\Sigma_t}L_{\Sigma_t^*}U_t = U_t^{-1}\text{Ric}_e U_t.
\end{equation}

Using this identity and the definition of $Y_t$ we find, $Y_0 = \text{Id}$ and
\begin{align*}
dY_t &= -D_{d\beta_t}U_t Q_t - \frac{1}{2}U_t \text{Ric}^\perp_t Q_t dt \\
(6.5) &= -D_{d\beta_t}Y_t - \frac{1}{2}U_t \text{Ric}^\perp_t U_t^{-1}Y_t dt \\
(6.6) &= -D_{d\beta_t}Y_t - \frac{1}{2}\text{Ric}_e Y_t dt.
\end{align*}

Since $dY_t^{-1} = -Y_t^{-1}(cdY_t)Y_t^{-1}$, it follows that $Y_t^{-1}$ satisfies,
\begin{equation}
\label{6.7}
dY_t^{-1} = Y_t^{-1}D_{d\beta_t} + \frac{1}{2}Y_t^{-1}\text{Ric}_e dt \text{ with } Y_0^{-1} = \text{Id}.
\end{equation}

For $T \geq t \geq 0$, let $Y_{T,t}$ solve,
\begin{equation}
\label{6.8}
d_TY_{T,t} = -D_{d\beta_t}Y_{T,t} - \frac{1}{2}\text{Ric}_e Y_{T,t} dT \text{ with } Y_{t,t} = \text{Id},
\end{equation}

and observe that $Y_{T,t}$ is $\sigma(\beta_t - \beta_s : t \leq s, \tau \leq T)$ – measurable. By the uniqueness of solutions to linear stochastic differential equations we may conclude
\[
Y_T = Y_{T,t}Y_t \text{ a.s. for all } 0 \leq t \leq T
\]
and hence it follows that $V_t = Y_T Y_t^{-1}$ a.s. $Y_{T,t}$ is also $\sigma(\beta_t - \beta_s : t \leq s, \tau \leq T)$ – measurable. Moreover we have,
\begin{align*}
dV_t &= Y_T d(Y_t^{-1}) = -Y_T Y_t^{-1}(cdY_t)Y_t^{-1} \\
&= -V_t \left(-D_{d\beta_t} - \frac{1}{2}\text{Ric}_e dt \right) \\
&= V_t \left(D_{d\beta_t} + \frac{1}{2}\text{Ric}_e dt \right) \text{ with } V_T = \text{Id}.
\end{align*}

See [19, Section 4.1] for more on the backwards stochastic integral interpretation of this equation.

q.e.d.

**Theorem 6.4.** If $A \in \mathfrak{g}$ and $\ell \in C^1([0,T], \mathbb{R})$ with $\ell(0) = 0$ and $\ell(T) = 1$, then
\begin{equation}
\label{6.9}
W_A(x) = \mathbb{E} \left[ \left(A, \int_0^T \dot{\ell}(\tau) V_\tau d\overline{\beta}_\tau \right) \mid \Sigma_T = x \right],
\end{equation}

where $\int_0^T \dot{\ell}(\tau) V_\tau d\overline{\beta}_\tau$ is a backwards Itô integral and $V_t$ satisfies the (backwards) stochastic differential equation,
\[
dV_t = \frac{1}{2}V_t \text{Ric}_e dt + V_t D_{d\beta_t} \text{ with } V_T = \text{Id}.
\]
Proof. Let \( f \in C^\infty_c (G) \) and
\[
Y(x) := f(x) \tilde{A}(x) = f(x) L_{xA} A.
\]
As shown in Lemma 6.1, \( \nabla \cdot \tilde{A} = 0 \) from which it follows that
\[
\nabla \cdot Y = \left( \text{grad } f, \tilde{A} \right)_T = \tilde{A} f.
\]
Therefore an application of Theorem 5.4 (with \( \tilde{\ell} \) now being denoted by \( \ell(t) \)) shows,
\[
E \left[ \left( \tilde{A} f \right) (\Sigma_T) \right] = E \left[ f (\Sigma_T) \left( \tilde{A} (\Sigma_T), \int_0^T \dot{\ell}(\tau) Q_{\tau}^{-1} d\tau \right) \right]
\]
\[
= E \left[ f (\Sigma_T) \left( A, L_{\Sigma_T^{-1}} \int_0^T \dot{\ell}(\tau) Q_{\tau}^{-1} d\tau \right) \right].
\]
From Eq. (6.3)
\[
\left( A, L_{\Sigma_T^{-1}} \int_0^T \dot{\ell}(\tau) Q_{\tau}^{-1} d\tau \right) = \left( A, Y_T \int_0^T \dot{\ell}(\tau) Y_{\tau}^{-1} d\beta_{\tau} \right)
\]
\[
= \left( A, \int_0^T \dot{\ell}(\tau) V_{\tau} d\beta_{\tau} \right).
\]
Moreover, we may write the last expression as a backwards Itô integral, since
\[
dV_{\tau} d\beta_{\tau} = V_{\tau} D_{\tau} d\beta_{\tau} = V_{\tau} \sum_{A \in ONB(g)} D_A A \cdot dt = 0,
\]
wherein we have used Lemma 6.1 again for the last equality. Hence we now have
\[
\left( A, L_{\Sigma_T^{-1}} \int_0^T \dot{\ell}(\tau) Q_{\tau}^{-1} d\tau \right) = \left( A, \int_0^T \dot{\ell}(\tau) V_{\tau} d\beta_{\tau} \right).
\]
These computations may be justified by the same methods introduced in [19]. This completes the proof because,
\[
E \left[ W_A (\Sigma_T) f (\Sigma_T) \right] = E \left[ \left( \tilde{A} f \right) (\Sigma_T) \right]
\]
\[
= E \left[ f (\Sigma_T) \left( A, \int_0^T \dot{\ell}(\tau) V_{\tau} d\beta_{\tau} \right) \right]
\]
for all \( f \in C^\infty_c (G) \). q.e.d.
Our next goal is to bound \( \int_G e^{W_A} d\nu_T \) for all \( A \in g \). In order to do this it will be necessary to estimate the size of the process \( V_t \).

**Lemma 6.5.** Suppose \( k \in \mathbb{R} \) is chosen so that \( \text{Ric} \geq kI \), then

\[
|V_t^* A|^2 \leq |A|^2 e^{-k(T-t)} \quad \text{for all} \quad A \in g.
\]

*Proof.* Since

\[
dV_t = \frac{1}{2} V_t \text{Ric}_e \, dt + V_t D_{od^3}\,
\]

we have

\[
dV_t^* = \frac{1}{2} \text{Ric}_e V_t^* \, dt - D_{od^3} V_t^*
\]

wherein we have used the fact that \( D_A : g \rightarrow g \) is antisymmetric. In particular it now follows that

\[
d |V_t^* A|^2 = 2 (odV_t^* A, V_t^* A) = 2 \left( \frac{1}{2} \text{Ric}_e V_t^* \, dt - D_{od^3} V_t^* A, V_t^* A \right)\]

\[
= (\text{Ric}_e V_t^* A, V_t^* A) \, dt \geq k |V_t^* A|^2 \, dt \quad \text{with} \quad |V_T^* A|^2 = |A|^2.
\]

We may write this inequality as

\[
\frac{d}{dt} \ln |V_t^* A|^2 \geq k \quad \text{with} \quad |V_T^* A|^2 = |A|^2.
\]

which upon integration gives,

\[
\ln |A|^2 - \ln |V_t^* A|^2 = \ln |V_T^* A|^2 - \ln |V_t^* A|^2 \geq k (T-t).
\]

Hence \( |A|^2 / |V_t^* A|^2 \geq e^{k(T-t)} \) which is equivalent to Eq. (6.12).

*q.e.d.*

**Lemma 6.6.** Let \( k \in \mathbb{R} \) and \( T > 0 \), then

\[
\inf \left\{ \int_0^T \ell^2 (\tau) e^{-k(T-\tau)} d\tau \right\} \leq \frac{k}{e^{kT} - 1},
\]

where the infimum is taken over all \( \ell \in C^1 ([0,T], \mathbb{R}) \) such that \( \ell (0) = 0 \) and \( \ell (T) = 1 \).

*Proof.* By a simple calculus of variation argument, \( \ell \in C^1 ([0,T], \mathbb{R}) \) with \( \ell (0) = 0 \) and \( \ell (T) = 1 \) is a critical point for the function,

\[
K (\ell) := \int_0^T \ell^2 (\tau) e^{-k(T-\tau)} d\tau,
\]

iff \( \ell (\tau) e^{k\tau} \) is constant in \( \tau \). This constraint and the boundary conditions imply that \( K \) has a unique critical point at

\[
\ell_c (\tau) = \frac{e^{-k\tau} - 1}{e^{-kT} - 1}.
\]
Plugging this value of $\ell_c$ into $K$ then shows $K(\ell_c) = k \left( 1 - e^{-kT} \right)^{-1}$ from which Eq. (6.13) follows.

6.1. Proof of Theorems 1.3 and 1.4. With the above results as preparation, we are now in position to complete the proofs of Theorem 1.3 and 1.4.

Proof. Proof of Theorem 1.3. Let $\ell \in C^1([0, T], \mathbb{R})$ such that $\ell(0) = 0$ and $\ell(T) = 1$. From Theorem 6.4, Lemma 6.5, Jensen’s inequality for conditional expectations, and a standard martingale argument (see the proof of Lemma 7.6 and especially Eq. 7.17 in [18]) we have

$$\int_G e^{W^A} d\nu_T = E \left[ \exp \left( \mathbb{E} \left[ \left\langle A, \int_0^T \dot{\ell}(\tau) V_{\tau} d\beta_\tau \right\rangle \right| \sigma(\Sigma_T) \right] \right)$$

$$\leq E \left[ \exp \left( \mathbb{E} \left[ \left\langle A, \int_0^T \dot{\ell}(\tau) V_{\tau} d\beta_\tau \right\rangle \right| \sigma(\Sigma_T) \right] \right)$$

$$= E \left[ \exp \left( \mathbb{E} \left[ \left\langle A, \int_0^T \dot{\ell}(\tau) V_{\tau} d\beta_\tau \right\rangle \right] \right) \right]$$

$$\leq \exp \left( \frac{1}{2} \left\| \int_0^T \dot{\ell}^2(\tau) |V_{\tau}^* A|^2 d\tau \right\|_{L^\infty(P)} \right)$$

$$\leq \exp \left( \frac{|A|^2}{2} \int_0^T \dot{\ell}^2(\tau) e^{-k(T-\tau)} d\tau \right),$$

where $P$ is the underlying probability measure. Since $\ell$ was arbitrary, it follows from Lemma 6.6 that,

$$\int_G e^{W^A} d\nu_T \leq \inf_{\ell} \exp \left( \frac{1}{2} \int_0^T \dot{\ell}^2(\tau) |A|^2 e^{-k(T-\tau)} d\tau \right)$$

$$\leq \exp \left( \frac{1}{2} \frac{k}{e^{kT} - 1} |A|^2 \right) = \exp \left( \frac{1}{2T} c(kT) |A|^2 \right).$$

q.e.d.

Proof. (Proof of Theorem 1.4.) From Theorem 6.4, Lemma 6.5, Jensen’s inequality for conditional expectations, and Burkholder-Davis-Gundy inequality (see for example [61, Corollary 6.3.1a on p.344], [49, Appendix A.2], or [48, p. 212] and [39, Theorem 17.7] for the real case),
there exists $C_q < \infty$ such that

\[
\int_G |W_A|^q \, d\nu_T = E \left[ \left| \mathbb{E} \left[ \left\langle A, \int_0^T \dot{\ell} (\tau) V_{\ell} d\beta_\tau \right\rangle \right| \sigma (\Sigma_T) \right]^q \right]
\]

\[
\leq E \left[ \left| \mathbb{E} \left[ \left\langle A, \int_0^T \dot{\ell} (\tau) V_{\ell} d\beta_\tau \right\rangle \right| \sigma (\Sigma_T) \right]^q \right]
\]

\[
= E \left| \mathbb{E} \left[ \left\langle A, \int_0^T \dot{\ell} (\tau) V_{\ell} d\beta_\tau \right\rangle \right]^q \right|
\]

\[
= E \left| \mathbb{E} \left[ \int_0^T \dot{\ell} (\tau) \left( V_{\ell} A, d\beta_\tau \right) \right|^q \right|
\]

\[
\leq C_q E \left| \int_0^T \dot{\ell}^2 (\tau) \left| V_{\ell} A \right|^2 d\tau \right|^{q/2}
\]

\[
\leq C_q \left( |A|^2 \int_0^T \dot{\ell}^2 (\tau) e^{-k(T-\tau)} d\tau \right)^{q/2}.
\]

Using Lemma 6.6, we may optimize this last estimate over the admissible $\ell$ to find,

\[
\int_G |W_A|^q \, d\nu_T \leq C_q \left( |A|^2 \frac{k}{e^{kT} - 1} \right)^{q/2} = C_q \left( |A|^2 \frac{c(kT)}{T} \right)^{q/2}
\]

which is equivalent to Eq. (1.4).

q.e.d.

7. Applications

Lemma 7.1. Suppose that $T > 0$, $q > 1$, and $f \in L^q (\nu_T) \cap C^2 (G)$ such that $\Delta f \in L^q (\nu_T)$. Then $f, \Delta f \in L^q (\nu_t)$ for $0 < t \leq T$ and

\[
\frac{\partial}{\partial t} \int_G p_t (x, y) f (y) \, dy = \frac{1}{2} \int_G p_t (x, y) \Delta f (y) \, dy \text{ for all } 0 < t < T.
\]

Proof. Since the Ricci curvature is left translation invariant, it is bounded on $G$. Applying the Li – Yau Harnack inequality (see Eq. (D.6 below), we have for any $\gamma > 1/2$ that there exists $K = K (\gamma, T) < \infty$ such that

\[
p_t (x) \leq K \left( \frac{T}{t} \right)^{d_1} p_T (x) \quad \forall \, (x, t) \in G \times (0, T] \quad \text{for all } 0 < t < T.
\]

In particular it follows that

\[
\|f\|_{L^q (\nu_t)} \leq K \left( \frac{T}{t} \right)^{d_1 q} \|f\|_{L^q (\nu_T)} \quad \forall \quad 0 < t \leq T.
\]
Using $q' - 1 = (q - 1)^{-1}$ and Eq. (1.6), it follows that
\[
\int_G p_t (y, x) |f(x)| \, dx = \int_G \frac{p_t (y, x)}{p_t (x)} |f(x)| \, d\nu_t (x)
\leq \left\| \frac{p_t (y, \cdot)}{p_t (\cdot)} \right\|_{L^q (\nu_t)} \cdot \| f \|_{L^q (\nu_t)} \\
\leq \| f \|_{L^q (\nu_t)} \exp \left( \frac{c (kt) (q' - 1)}{2t} |y|^2 \right)
\leq \| f \|_{L^q (\nu_t)} \exp \left( \frac{c (kt)}{2t (q - 1)} |y|^2 \right).
\]
(7.4)
Therefore the integrals in Eq. (7.1) are well defined. Moreover,
\[
\int_G p_t (y, x) f(x) \, dx = \int_G p_t (y^{-1} x) f(x) \, dx = \int_G p_t (x) f (yx) \, dx
\leq \int_G f \circ L_y (x) p_t (x) \, dx
\]
and for any $r \in (1, q)$,
\[
\| f \circ L_y \|_{L^r (\nu_t)} = \int_G |f (yx)|^r p_t (x) \, dx = \int_G |f (x)|^r p_t (y^{-1} x) \, dx
\leq \int_G |f (x)|^r \frac{p_t (y^{-1} x)}{p_t (x)} \, d\nu_t (x)
\leq \| f \|_{L^q (\nu_t)} \exp \left( \frac{c (kt) q (q - r)^{-1}}{2t} |y|^2 \right)
\]
wherein we have used Hölder’s inequality and Eq. (1.9) for the last inequality. From these remarks and the fact that $\Delta (f \circ L_y) = (\Delta f) \circ L_y$, it suffices to prove Eq. (7.1) in the special case when $y = e$. From Eq. (7.2) and the dominated convergence theorem, the function
\[
F(t) = \int_G f (x) \, d\nu_t (x) \quad \text{for all } t \in (0, T]
\]
is continuous. Our goal now is to show $F$ is differentiable and that $F'(t) = \frac{1}{2} \int_G \Delta f (x) \, d\nu_t (x)$ for all $0 < t < T$. To prove this suppose that $h \in C_c^\infty (G)$ and consider,
\[
F_h (t) := \int_G f (x) h (x) p_t (x) \, dx.
\]
To simplify notation in the computation below, let $\{A_i\}_{i=1}^{\dim g}$ be an orthonormal basis for $g$, $\nabla f = \left( \tilde{A}_i f \right)_{i=1}^{\dim g}$, and $\nabla \cdot U = \sum A_i U_i$, where $U = (U_i)_{i=1}^{\dim g}$ with $U_i \in C^\infty (G)$. Using $\frac{\partial}{\partial t} p_t (x) = \frac{1}{2} \Delta p_t (x)$, and a few
integration by parts we find
\[ \hat{F}_h(t) = \frac{1}{2} \int_G f(x) h_t(x) \Delta p_t(x) \, dx \]
\[ = \frac{1}{2} \int_G \Delta (fh) \, p_t \, dV - \frac{1}{2} \int_G (f \Delta h + 2 \nabla f \cdot \nabla h + h \Delta f) \, p_t \, dV \]
\[ = \frac{1}{2} \int_G (f \Delta h + h \Delta f) \, p_t \, dV - \int_G f \nabla \cdot [\nabla h \, p_t] \, dV \]
\[ = \frac{1}{2} \int_G (f \Delta h + h \Delta f) \, p_t \, dV - \int_G f [\Delta h \, p_t + \nabla h \cdot \nabla p_t] \, dV \]
\[ = -\frac{1}{2} \int_G f \Delta h \, dV_t - \int_G f \nabla h \cdot \nabla p_t \, dV_t + \frac{1}{2} \int_G h \Delta f \, dV_t. \]
(7.5)

Therefore,
\[ \hat{F}_h(t) - \frac{1}{2} \int_G \Delta f \, dV_t = -\frac{1}{2} R_h(t) - S_h(t) + \frac{1}{2} U_h(t), \]

where, making use of Eqs. (7.3) and (1.4), we have
\[ |R_h(t)| \leq \int_G |f| |\Delta h| \, dV_t \leq \|f\|_{L^q(\nu_t)} \|\Delta h\|_{L^{q'}(\nu_t)} \leq K^2 \left( \frac{T}{t} \right)^{\frac{d\gamma}{q}} \|f\|_{L^q(\nu_T)} \|\Delta h\|_{L^{q'}(\nu_T)}, \]
\[ (7.6) \]
\[ |S_h(t)| = \sum_i \int_G |f| |\tilde{A}_i h| \|W_{A_i}\| \, dV_t \leq \sum_i \|f \cdot \tilde{A}_i h\|_{L^q(\nu_T)} \|W_{A_i}\|_{L^{q'}(\nu_T)} \]
\[ \leq C_q \sqrt{\frac{e(kt)}{t}} K \left( \frac{T}{t} \right)^{\frac{d\gamma/2}{q}} \sum_i \|f \cdot \tilde{A}_i h\|_{L^q(\nu_T)}, \]
\[ (7.7) \]
and
\[ |U_h(t)| \leq \int_G |\Delta f| |h - 1| \, dV_t \leq \|\Delta f\|_{L^q(\nu_T)} \|1 - h\|_{L^{q'}(\nu_T)} \]
\[ \leq K^2 \left( \frac{T}{t} \right)^{\frac{d\gamma}{q}} \|\Delta f\|_{L^q(\nu_T)} \|1 - h\|_{L^{q'}(\nu_T)}. \]
\[ (7.8) \]

From [22, Lemma 3.6], we may choose \( \{h_n\}_{n=1}^{\infty} \subseteq C^\infty_c (G \times [0, 1]) \) such that \( h_n(x) = 1 \) whenever \(|x| \leq n\) and \( \sup_n \sup_{x \in G} \left| \tilde{A}_{i_1} \ldots \tilde{A}_{i_k} h_n(x) \right| < \infty \) for all \( i_1, \ldots, i_k \in \{1, 2, \ldots, \dim g\} \) and \( k \in \mathbb{N} \). It then follows from Eqs. (7.3), (7.5), (7.6), (7.7), and (7.8) and the dominated convergence theorem that, as \( n \to \infty \),
\[ \left| \hat{F}_{h_n}(t) - \frac{1}{2} \int_G \Delta f \, dV_t \right| \leq \frac{1}{2} |R_{h_n}(t)| + |S_{h_n}(t)| + \frac{1}{2} |U_{h_n}(t)| \to 0 \]
uniformly on compact subsets of $(0,T)$. Moreover, by the dominated convergence theorem, $F_{h_n}(t) \to F(t)$ as $n \to \infty$ and therefore we may conclude that $F(t) = \frac{1}{2} \int_G \Delta f \, d\nu$ for $t \in (0,T)$.

**7.1. The proof of Proposition 1.8.**

**Proof.** Now suppose, as in Proposition 1.8, $T > 0$, $q > 1$, and $f \in L^q(\nu_T)$ such that $\Delta f = 0$. As in the proof of Lemma 7.1, we may reduce the proof to the case where $y = e$. Let $F(t) := \int_G f \, d\nu_t$. By Lemma 7.1 and the mean value theorem, $F(t) = F(t)$ for all $t \in (0,T)$ and in particular, $F(T) = \lim_{t \downarrow 0} F(t)$. We are going to finish the proof by showing $\lim_{t \downarrow 0} F(t) = f(e)$. To do this, let $h \in C^n_c(G, [0,1])$ be chosen so that $h(x) = 1$ if $|x| \leq 1$. Then

$$F(t) = \int_G f(x) h(x) p_t(x) \, dx + r(t),$$

where

$$|r(t)| \leq \int_G |f(x)| |1 - h(x)| p_t(x) \, dx \leq \int_{|x| \geq 1} |f(x)| p_t(x) \, dx$$

(7.9)

$$= \int_{|x| \geq 1} |f(x)| \frac{p_t(x)}{p_T(x)} \, d\nu_T(x) \leq \sup_{|x| \geq 1} \frac{p_t(x)}{p_T(x)} \|f\|_{L^1(\nu_T)}.$$

Since $\lim_{t \downarrow 0} \int_G f(x) h(x) p_t(x) \, dx = f(e) h(e) = f(e)$, it suffices to show $\lim_{t \downarrow 0} |r(t)| = 0$. To estimate $r(t)$ we will make use of some crude upper and lower bounds on the heat kernel, $p_t(x)$, for example see [65, Theorem V.4.4 or Theorem IX.1.2.] for more precise bounds. According to either of these theorems, there exists a constant $c > 0$ such that

$$\frac{p_t(x)}{p_T(x)} \leq \frac{ct^{-d/2} \exp \left(-c|x|^2 / t \right)}{c^{-1} T^{-d/2} \exp \left(-c^{-1} |x|^2 / T \right)} = c^2 \left( \frac{T}{t} \right)^{d/2} \exp \left( \frac{1}{cT} - \frac{c}{t} \right) |x|^2.$$

From this estimate it follows that $\lim_{t \downarrow 0} \sup_{|x| \geq 1} (p_t(x) / p_T(x)) = 0$ which combined with Eq. (7.9) shows $\lim_{t \downarrow 0} |r(t)| = 0$. q.e.d.

**7.2. Applications to infinite–dimensional groups.** For this section, suppose that $G$ is a topological group, $\mathcal{B}$ is the Borel $\sigma$–algebra over $G$, and $G_0$ is a dense subgroup of $G$ which is endowed with the structure of an infinite–dimensional Hilbert Lie group. Further assume that $\mathfrak{g}_0 := \text{Lie} (G_0) = T_e G_0$ is equipped with a Hilbertian inner product, $\langle \cdot , \cdot \rangle_{\mathfrak{g}_0}$. We will also assume that $(G, \mathcal{B})$ is also equipped with a probability measure, $\nu$, to be thought of as the “heat kernel” measure at some time $T > 0$ associated to the given inner product on $\mathfrak{g}_0$. We will now give two theorems which guarantee that $\nu$ is quasi-invariant under both
left and right translations by elements of $G_0$. The two cases considered
are where $G$ can be thought of as either a projective or inductive limit
of finite-dimensional Lie groups.

**Theorem 7.2** (Projective Limits). Suppose that $T > 0$, $A$ is a di-
rected set, $\{G_\alpha\}_{\alpha \in A}$ is a collection of finite dimensional uni-modular
Lie groups, and $\{\pi_\alpha : G \to G_\alpha\}_{\alpha \in A}$ is a collection of continuous group
homomorphisms satisfying the following properties.

1) $\mathcal{B}$ is equal to the $\sigma$– algebra generated by the projections, $\{\pi_\alpha\}_{\alpha \in A}$.
2) $\pi_\alpha|_{G_0} : G_0 \to G_\alpha$ is a smooth surjection. Let $d\pi_\alpha : g_0 \to g_\alpha$ be the
differential of $\pi_\alpha$ at $e$.
3) $\nu_\alpha := (\pi_\alpha)_* \nu = \nu \circ \pi_\alpha^{-1}$ is the time $T$ heat kernel measure on $G_\alpha$
determined by the unique inner product. $(\cdot, \cdot)_\alpha$ on $g_\alpha$ which makes
$$d\pi_\alpha|_{\text{Nul}(\pi_\alpha)} : \text{Nul}(\pi_\alpha) \to g_\alpha$$
an isometric isomorphism of inner product spaces.
4) There exists $k \in \mathbb{R}$ such that $\text{Ric}_\alpha \geq kg_\alpha$ for all $\alpha \in A$, where $\text{Ric}_\alpha$
is the Ricci tensor on $G_\alpha$ equipped with the left invariant metric
determined by $(\cdot, \cdot)_\alpha$.

Under these assumptions, to each $h \in G_0$, $\nu \circ R_h^{-1}$ is absolutely con-
tinuous relative to $\nu$. Moreover, if $J_h := d(\nu \circ R_h^{-1})/d\nu$ is the Radon-
Nikodym derivative of $\nu \circ R_h^{-1}$ with respect to $\nu$ and $1 \leq q < \infty$, then

$$\|J_h\|_{L^q(\nu)} \leq \exp\left(\frac{c(kT)(q - 1)}{2T}d^2_{G_0}(e, h)\right),$$

where $d_{G_0}$ is the Riemannian distance function on $G_0$.

**Proof.** Since the estimate in Eq. (7.10) holds for $q = 1$, we may
assume without loss of generality that $1 < q < \infty$. Let $\mathcal{H}$ denote the
linear space of bounded measurable functions of the form $f = u \circ \pi_\alpha$,\nwhere $\alpha \in A$ and $u : G_\alpha \to \mathbb{R}$ is a bounded measurable function on $G_\alpha$.
Because of assumption 1., $\mathcal{H}$ is dense in $L^q(G, \nu)$. (An easy proof may
be given using a functional form of the monotone class theorem, see for
example [38, Theorem A.1 on p. 309].) By Theorem 1.5 in the form of
Eq. (1.7),

$$J_\alpha(x) := \frac{\nu_\alpha(dx \cdot \pi_\alpha(h^{-1}))}{\nu_\alpha(dx)} \text{ for } x \in G_\alpha,$$

satisfies

$$\|J_\alpha\|_{L^q(G_\alpha, \nu_\alpha)} \leq \exp\left(\frac{c(kT)(q - 1)}{2T}d^2_{G_\alpha}(e, \pi_\alpha(h))\right) \text{ for all } 1 < q < \infty.$$
Using this result and assumption 3, if \( f = u \circ \pi_\alpha \in \mathbb{H} \), then
\[
\int_G |f(xh)| \, d\nu(x) = \int_G |u \circ \pi_\alpha(xh)| \, d\nu(x)
= \int_G |u(\pi_\alpha(x) \pi_\alpha(h))| \, d\nu(x)
= \int_{G_\alpha} |u(y \cdot \pi_\alpha(h))| \, d\nu_\alpha(y)
= \int_{G_\alpha} |u(y) J_\alpha(y) \, d\nu_\alpha(y).
\]
An application of Hölder’s inequality then implies,
\[
\int_G |f(xh)| \, d\nu(x) \leq \|u\|_{L^q(G_\alpha, \nu_\alpha)} \cdot \|J_\alpha\|_{L^{q'}(G_\alpha, \nu_\alpha)}
\leq \|f\|_{L^q(G, \nu)} \exp \left( \frac{c(kT)(q' - 1)}{2T} d_{G_\alpha}^2(e, \pi_\alpha(h)) \right).
\tag{7.11}
\]
Now suppose that \( k \in C^1([0, 1], G_0) \) such that \( k(0) = e \) and \( k(1) = h \). Then the length of \( t \to \pi_\alpha(k(t)) \in G_\alpha \) is given by
\[
\ell_{G_\alpha}(\pi_\alpha \circ k) = \int_0^1 |L_{\pi_\alpha(k(t))^{-1} \pi_\alpha}(\dot{k}(t))| \, dt.
\]
Since
\[
L_{\pi_\alpha(k(t))^{-1} \pi_\alpha}(\dot{k}(t)) = \frac{d}{ds}|_{s=0} \pi_\alpha(k(t))^{-1} \pi_\alpha(k(t + s))
= \frac{d}{ds}|_{s=0} \pi_\alpha(k(t)^{-1} k(t + s)) \cdot d\pi_\alpha(L_{k(t)^{-1}} \dot{k}(t))
\]
and
\[
\left| L_{\pi_\alpha(k(t))^{-1} \pi_\alpha}(\dot{k}(t)) \right|_{G_\alpha} \leq \left| d\pi_\alpha(L_{k(t)^{-1}} \dot{k}(t)) \right|_{G_\alpha} \leq \left| L_{k(t)^{-1}} \dot{k}(t) \right|_{G_0},
\]
it follows that
\[
d_{G_\alpha}(e, \pi_\alpha(h)) \leq \ell_{G_\alpha}(\pi_\alpha \circ k) \leq \int_0^1 |L_{k(t)^{-1}} \dot{k}(t)|_{G_0} \, dt = \ell_{G_0}(k).
\]
Taking the infimum over all such \( k \) implies
\[
d_{G_\alpha}(e, \pi_\alpha(h)) \leq d_{G_0}(e, h).
\]
Combining this inequality with Eq. (7.11) gives the estimate,
\[
\int_G |f(xh)| \, d\nu(x) \leq \|f\|_{L^q(G, \nu)} \exp \left( \frac{c(kT)(q' - 1)}{2T} d_{G_\alpha}^2(e, h) \right).
\tag{7.12}
\]
The afore mentioned density of \( \mathbb{H} \) in \( L^q(G, \nu) \) along with Eq. (7.12) shows the linear functional \( \phi : \mathbb{H} \to \mathbb{R} \), defined by
\[
\phi_h(f) := \int_G f(xh) \, d\nu(x),
\]
extends uniquely to a continuous linear functional, \( \tilde{\varphi}_h \), on \( L^q(G, \nu) \) satisfying
\[
|\tilde{\varphi}_h(f)| \leq \|f\|_{L^q(G, \nu)} \exp \left( \frac{c(kT)(q'-1)}{2T} d_{G_0}^2(e, h) \right) \quad \forall \ f \in L^q(G, \nu).
\]
Since \( L^q(G, \nu)^* \cong L^{q'}(G, \nu) \), there exists \( J_h \in L^{q'}(G, \nu) \) such that
\[
\|J_h\|_{L^{q'}(G, \nu)} \leq \exp \left( \frac{c(kT)(q'-1)}{2T} d_{G_0}^2(e, h) \right)
\]
and
\[
\tilde{\varphi}_h(f) = \int_G f(x) J_h(x) \, d\nu(x) \quad \text{for all} \ f \in L^q(G, \nu).
\]
Restricting this formula to \( \mathbb{H} \) shows,
\[
\int_G f(x) \, d\nu(xh^{-1}) = \int_G f(xh) \, d\nu(x) = \tilde{\varphi}_h(f)
\]
\[(7.13)\]
\[= \int_G f(x) J_h(x) \, d\nu(x) \quad \text{for all} \ f \in \mathbb{H}.
\]
Another monotone class argument (again use [38, Theorem A.1 on p. 309]) shows that Eq. (7.13) remains valid for all bounded measurable functions, \( f : G \to \mathbb{R} \). Therefore, we have shown that \( J_h := d\nu \circ R_h^{-1}/d\nu \) exists and satisfies the bound in Eq. (7.10).

We now turn to the inductive limit quasi-invariance theorem. The following result is an abstraction of the quasi-invariance result in [18]. For related results of this type see, Fang [25] and Airault and Malliavin [1].

**Theorem 7.3 (Inductive Limits).** Again, let \( T > 0, G_0 \subset G \), and \( (G, \mathcal{B}, \nu) \) be as described at the start of this section. Further assume there exists, \( \{G_\alpha\}_{\alpha \in A} \), where \( A \) is a directed set and for each \( \alpha \in A, G_\alpha \) is a finite dimensional uni-modular Lie subgroup of \( G_0 \) such that \( G_\alpha \subset G_\beta \) if \( \alpha < \beta \). Let \( i_\alpha : G_\alpha \to G_0 \) denote the smooth injection map. The following properties are assumed to hold.

1) \( \cup_{\alpha \in A} G_\alpha \) is a dense subgroup of \( G_0 \).
2) For all \( f \in BC(G, \mathbb{R}) \) (the bounded continuous maps from \( G \) to \( \mathbb{R} \)),
\[
\int_G f \, d\nu = \lim_{\alpha \to \infty} \int_{G_\alpha} (f \circ i_\alpha) \, d\nu_\alpha,
\]
where \( \nu_\alpha \) is the time, \( T \), heat kernel measure on \( G_\alpha \) associated to inner product, \( (\cdot, \cdot)_{g_\alpha} \), defined to be the restriction of \( (\cdot, \cdot)_{g_0} \) to \( g_0 \times g_\alpha \).
3) There exists \( k \in \mathbb{R} \) such that \( \text{Ric}_\alpha \geq kg_\alpha \) for all \( \alpha \in A \), where \( \text{Ric}_\alpha \) and \( g_\alpha \) are the left invariant Ricci and the metric tensors on \( G_\alpha \) induced by \( (\cdot, \cdot)_{g_\alpha} \).
4) For each \( \alpha \in A \), there exists a smooth section, \( s_\alpha : G_0 \to G_\alpha \) (i.e. \( s_\alpha \circ \iota_\alpha = id_{G_\alpha} \)) satisfying the following property. Given \( \alpha_0 \in A \), and \( k \in C^1([0,1], G_0) \) with \( k(0) = e \), there exists an increasing sequence, \( \{ \alpha_n \}_{n=1}^\infty \subset A \) (i.e. \( \alpha_0 < \alpha_1 < \alpha_2 \leq \cdots \)), such that

\[
\ell_{G_0}(k(\cdot)) = \lim_{n \to \infty} \ell_{G_\alpha}(s_{\alpha_n} \circ k).
\]

(We do not assume that \( s_\alpha : G_0 \to G_\alpha \) is a homomorphism.)

Under these assumptions, to each \( h \in G_0 \), \( \nu \circ R_h^{-1} \) is absolutely continuous relative to \( \nu \) and the Moreover, the Radon-Nikodym derivative, \( \ell_h := d \left( \nu \circ R_h^{-1} \right) / d\nu \), again satisfies the bounds in Eq. (7.10).

Proof. As in the proof of Theorem 7.2 it suffices to assume \( q \in (1, \infty) \) throughout the proof. Let \( \alpha_0 \in A \), \( h \in G_{\alpha_0} \), and \( \alpha_0 < \alpha_1 < \alpha_2 < \cdots < H_{\alpha_n} < \cdots \) be as in item 4. above. By Theorem 1.5 in the form of Eq. (1.7), the Radon-Nikodym derivative, \( J_{\alpha_n}(x) \), of \( \nu_{\alpha_n}(dx \cdot s_{\alpha_n}(h)^{-1}) = \nu_{\alpha_n}(dx \cdot h^{-1}) \) relative to \( \nu_{\alpha_n}(dx) \) satisfies the estimate,

\[
\|J_{\alpha_n}\|_{L^q(G_{\alpha_n} \nu_{\alpha_n})} \leq \exp \left( \frac{c(kT)(q' - 1)}{2T} d_{G_{\alpha_n}}^2(e, h^{-1}) \right) = \exp \left( \frac{c(kT)(q' - 1)}{2T} d_{G_{\alpha_n}}^2(e, h) \right) \leq \exp \left( \frac{c(kT)(q' - 1)}{2T} d_{G_{\alpha_n}}^2(s_{\alpha_n} \circ \sigma) \right),
\]

where \( \sigma \) is any path in \( C^1([0,1], G_0) \) such that \( \sigma(0) = e \) and \( \sigma(1) = h \). Assuming the \( f \in BC(G) \), by the definition of \( J_{\alpha_n} \) and Hölder’s inequality,

\[
\int_{G_{\alpha_n}} |f(x \cdot h)| d\nu_{\alpha_n}(x) = \int_{G_{\alpha_n}} J_{\alpha_n}(x) |f(x)| d\nu_{\alpha_n}(x)
\leq \|f\|_{L^q(G_{\alpha_n} \nu_{\alpha_n})} \cdot \exp \left( \frac{c(kT)(q' - 1)}{2T} d_{G_{\alpha_n}}^2(s_{\alpha_n} \circ \sigma) \right).
\]

Using the assumptions in items 2. and 4. of the theorem, we may pass to the limit \( (n \to \infty) \) in this inequality to find,

\[
(7.15) \quad \int_G |f(x \cdot h)| d\nu(x) \leq \|f\|_{L^q(G, \nu)} \cdot \exp \left( \frac{c(kT)(q' - 1)}{2T} d_{G_0}(e, h) \right).
\]

Optimizing this inequality over \( \sigma \in C^1([0,1], G_0) \) joining \( e \) to \( h \) gives

\[
(7.16) \quad \int_G |f(x \cdot h)| d\nu(x) \leq \|f\|_{L^q(G, \nu)} \cdot \exp \left( \frac{c(kT)(q' - 1)}{2T} d_{G_0}^2(e, h) \right).
\]

Up to now we have verified Eq. (7.16) for any \( h \in \bigcup_{\alpha \in A} G_\alpha \). As the latter set is dense in \( G_0 \), the dominated convergence theorem along with the continuity of \( d_{G_0}^2(e, h) \) in \( h \) allows us to conclude that the estimate in
Eq. (7.16) is valid for all $h \in G_0$. Since $BC(G, \mathbb{R})$ is dense in $L^q(G, \nu)$ (again use [38, Theorem A.1 on p. 309]) and because of Eq. (7.16), the linear functional, $\varphi_h : BC(G) \to \mathbb{R}$ defined by

\begin{equation}
\varphi_h(f) = \int_G f(xh) d\nu(x),
\end{equation}

has a unique extension to an element, $\overline{\varphi}_h$, of $L^q(G, \nu)^*$. Hence if $f$ and $\varphi$ are linear functional,

\begin{equation}
\overline{\varphi}_h(f) = \int_G f(x) J_h(x) d\nu(x)
\end{equation}

and

\begin{equation}
\|J_h\|_{L^q(G, \nu)} \leq \exp \left( \frac{c(kT)(q' - 1)}{2T} d_{G_0}^2(e, h) \right).
\end{equation}

Furthermore, from Eqs. (7.17) and (7.19) it follows that

\begin{equation}
\int_G f(xh) d\nu(x) = \int_G f(x) J_h(x) d\nu(x)
\end{equation}

for all $f \in BC(G)$.

Another monotone class argument [38, Theorem A.1 on p. 309] then shows Eq. (7.20) is valid for all bounded measurable functions, $f : G \to \mathbb{R}$. Hence $\nu (dxh^{-1}) = J_h(x) \nu(dx)$ and $J_h(x)$ satisfies the estimate in Eq. (7.10).

**Corollary 7.4.** Under the hypothesis of either Theorem 7.2 or 7.3, the heat kernel measure, $\nu$, is quasi-invariant under left translations by elements of $h \in G_0$. Moreover, the Radon-Nikodym derivative, $J_h^1 := d(\nu \circ L_h^{-1}) / d\nu$ satisfies the same bound as $d(\nu \circ R_h^{-1}) / d\nu$ which is given in Eq. (7.10).

**Proof.** Since the heat kernel measures $\{\nu_\alpha\}_{\alpha \in A}$ on the Lie groups, $\{G_\alpha\}_{\alpha \in A}$, are invariant under inversion, $x \to x^{-1}$, it follows that $\nu$ also inherits this property. Hence if $f : G \to \mathbb{R}$ is a bounded measurable function, then

\begin{align*}
\int_G f(hx) d\nu(x) &= \int_G f(hx^{-1}) d\nu(x) = \int_G f((xh^{-1})^{-1}) d\nu(x) \\
&= \int_G f(x^{-1}) J_{h^{-1}}(x) d\nu(x) = \int_G f(x) J_{h^{-1}}(x^{-1}) d\nu(x),
\end{align*}

from which it follows that $J_h^1(x) = J_{h^{-1}}(x^{-1})$ for $\nu$ - a.e. $x$. Therefore,

\begin{equation}
\|J_h^1\|_{L^q(\nu)} = \|J_{h^{-1}}\|_{L^q(\nu)} \leq \exp \left( \frac{c(kT)(q - 1)}{2T} d_{G_0}^2(e, h^{-1}) \right).
\end{equation}
which completes the proof since \( d_{G_0}^2(e, h^{-1}) = d_{G_0}^2(h, e) = d_{G_0}^2(e, h) \).

q.e.d.

See Driver [18] for an explicit application of the projective limit Theorem 7.2 in the setting of loop groups and see Driver and Gordina [21] for an application of the inductive limit Theorem 7.3 to an infinite dimensional Heisenberg group setting.

Appendix A. A commutator theorem

In this section we will develop the abstract functional analytic results which were used in the proofs of Theorems 4.1 and 4.3. Results similar to the next theorem may be found in Br"uning and Lesch [7], Xue-Mei Li [44, 45] and in Bueler [8].

**Theorem A.1.** Let \( W, X, \) and \( Y \) be Hilbert spaces and \( A : W \to X \) and \( B : X \to Y \) be densely defined closed (unbounded) operators such that \( \text{Ran}(A) \subset \text{Nul}(B) \). Let \( Q : X \to W \oplus Y \) be the unbounded linear operator defined by: \( D(Q) = D(A^*) \cap D(B) \) and for \( x \in D(Q) \), \( Qx := (A^*x, Bx) \). Let us also define \( R : W \oplus Y \to X \) by \( D(R) = D(A) \oplus D(B^*) \) and \( R(w, y) := Aw + B^*y \). Then

1) \( \text{Ran}(A) \) and \( \text{Ran}(B^*) \) are orthogonal.
2) \( R \) is closed.
3) \( Q = R^* \) is a closed densely defined operator.
4) The operator, \( L := AA^* + B^*B \), on \( X \) is densely defined, non-negative, and self adjoint operator. Moreover, \( L := Q^*Q \).

**Proof.** We will denote all of the inner products on these Hilbert spaces by \( \langle \cdot, \cdot \rangle \). Let \( w \in D(A) \) and \( y \in D(B^*) \). Since \( \text{Ran}(A) \subset \text{Nul}(B) \), \( 0 = \langle BAw, y \rangle = \langle Aw, B^*y \rangle \) which proves item 1. For item 2, suppose that \( (w_n, y_n) \in D(R) \) are such that there exists \( (w, y) \in W \oplus Y \) and \( x \in X \) such that

\[
(w_n, y_n) \to (w, y) \quad \text{as} \quad n \to \infty \quad \text{and} \quad R(w_n, y_n) \to x \quad \text{as} \quad n \to \infty.
\]

We must show that \( w \in D(A) \), \( y \in D(B^*) \) and that \( x = Aw + B^*y \). We are given that \( Aw_n + B^*y_n \to x \) as \( n \to \infty \). But by the first item and the Cauchy criteria, this implies that both \( \lim_{n \to \infty} Aw_n \) and \( \lim_{n \to \infty} B^*y_n \) exist. Because both \( A \) and \( B^* \) are closed, this implies that \( w \in D(A) \), \( y \in D(B^*) \) and that

\[
Aw + B^*y = \lim_{n \to \infty} Aw_n + \lim_{n \to \infty} B^*y_n = \lim_{n \to \infty} (Aw_n + B^*y_n).
\]

Hence we have proved item 2. To check item 3, note that as \( R \) is closed, it follows that \( R^* \) is densely defined. Therefore we need only show that \( R^* = Q \). For this, let us recall that \( x \in D(R^*) \) and \( R^*x =

Let $w, y$ iff $\langle (w, y), (w', y') \rangle = \langle x, R(w', y') \rangle$ for all $(w', y') \in \mathcal{D}(R)$. This is equivalent to the following statements:

- $\langle w, w' \rangle + \langle y, y' \rangle = \langle x, Aw' + B*y' \rangle$ for all $w' \in \mathcal{D}(A)$ and $y' \in \mathcal{D}(B^*)$.
- $\langle w, w' \rangle = \langle x, Aw' \rangle$ and $\langle y, y' \rangle = \langle x, B^*y' \rangle$ for all $w' \in \mathcal{D}(A)$ and $y' \in \mathcal{D}(B^*)$.
- $x \in \mathcal{D}(A^*)$, $x \in \mathcal{D}(B^*) = \mathcal{D}(B)$, $A^*x = w$ and $Bx = B^*x = y$.
- $x \in \mathcal{D}(Q)$ and $Qx = (w, y)$.

Thus we have proved item 3. of the theorem. Item 4. By a Theorem of Von-Neumann, [52, Theorem X.25], $Q^*Q$ is a non-negative densely defined self adjoint operator on $X$. So it suffices to show that $Q^*Q = AA^* + B*B$. By items 2. and 3., $Q^* = R^* = R$. Therefore, $Q^*Q = RQ$.

Now the following are equivalent:

- $x \in \mathcal{D}(RQ)$ and $RQx = x'$.
- $x \in \mathcal{D}(A^*) \cap \mathcal{D}(B)$, $Qx := (A^*x, Bx) \in \mathcal{D}(R)$, and $R(A^*x, Bx) = x'$.
- $x \in \mathcal{D}(A^*) \cap \mathcal{D}(B)$, $A^*x \in \mathcal{D}(A)$, $Bx \in \mathcal{D}(B^*)$ and $AA^*x + B*Bx = x'$.
- $x \in \mathcal{D}(AA^*) \cap \mathcal{D}(B^*)$ and $AA^*x + B*Bx = x'$.
- $x \in \mathcal{D}(AA^* + B*B)$ and $AA^*x + B*Bx = x'$.

This shows $Q^*Q = AA^* + B*B$ and thus completes the proof.

q.e.d.

**Theorem A.2** (Commutator Theorem). Let $W$, $X$, $Y$, and $Z$ be Hilbert spaces and $A : W \rightarrow X$, $B : X \rightarrow Y$, and $C : Y \rightarrow Z$ be densely defined closed (unbounded) operators such that $\text{Ran}(A) \subset \text{Nul}(B)$ and $\text{Ran}(B) \subset \text{Nul}(C)$. Let $L := AA^* + B*B$ and $S := BB^* + C^*C$. Then $B(e^{-tL}x) = e^{-tS}Bx$ for all $x \in \mathcal{D}(B)$ and any $t \geq 0$.

**Proof.** Let $\lambda > 0$. Observe that $BL = BB^*B$ on $\mathcal{D}(BL) = \mathcal{D}(AA^*) \cap \mathcal{D}(BB^*)$ and that $SB = BB^*B$ on $\mathcal{D}(SB) = \mathcal{D}(BB^*)$. In particular we have shown

$$\text{(A.1)} \quad SB = BB^*B = BL \text{ on } \mathcal{D}(BL) = \mathcal{D}(AA^*) \cap \mathcal{D}(BB^*)$$

and hence,

$$\text{(A.2)} \quad (1 + \lambda S)B = B(1 + \lambda L) \text{ on } \mathcal{D}(BL).$$

If $x \in \mathcal{D}(B)$ and $x' := (1 + \lambda L)^{-1}x$, then $x' \in \mathcal{D}(L) \subset \mathcal{D}(B)$ and

$$Lx' = (1 + \lambda L)x' - \lambda x' = x - \lambda x' \in \mathcal{D}(B).$$

Therefore $x' \in \mathcal{D}(BL)$ and so by Eq. (A.2) applied to $x' = (1 + \lambda L)^{-1}x$ we discover that,

$$\lambda S)B(1 + \lambda L)^{-1}x = B(1 + \lambda L)(1 + \lambda L)^{-1}x = Bx.$$

Applying $(1 + \lambda S)^{-1}$ to both sides of this equation shows

$$\text{(A.3)} \quad B(1 + \lambda L)^{-1} = (1 + \lambda S)^{-1}B \text{ on } \mathcal{D}(B).$$
Multiplying Eq. (A.3) on the right by \((1 + \lambda L)^{-1}\) gives
\[
B(1 + \lambda L)^{-2} = (1 + \lambda S)^{-1}B(1 + \lambda L)^{-1} = (1 + \lambda S)^{-2}B
\]
on \(D(B)\), wherein we have used Eq. (A.3) again in the second equality. Continuing this way inductively allows us to conclude.

(A.4) \(B(1 + \lambda L)^{-n} = (1 + \lambda S)^{-n}B\) on \(D (B)\) for all \(n \in \mathbb{N}\).

To complete the proof the theorem recall \(e^{-tL} = s - \lim_{n \to \infty} \left(1 + \frac{t}{n} L\right)^{-n}\) and that \(e^{-tS} = s - \lim_{n \to \infty} \left(1 + \frac{t}{n} S\right)^{-n}\). Hence, taking \(\lambda = t/n\) in Eq. (A.4) and then passing to the limit allows us to conclude
\[
\lim_{n \to \infty} B(1 + \frac{t}{n} L)^{-n} x = \lim_{n \to \infty} (1 + \frac{t}{n} S)^{-n} Bx = e^{-tS} Bx
\]
for all \(x \in D (B)\).
Since \(B\) is closed, it follows that, for all \(x \in D (B)\), that
\[
e^{-tL} x = \lim_{n \to \infty} \left(1 + \frac{t}{n} L\right)^{-n} x \in D (B)
\]
and
\[
Be^{-tL} x = \lim_{n \to \infty} B \left(1 + \frac{t}{n} L\right)^{-n} x = e^{-tS} Bx.
\]

q.e.d.

Appendix B. A Kato type inequality

Let \(E\) be a real Euclidean vector bundle over a Riemannian manifold, \(M\), \(\Gamma^\infty (E)\) \((\Gamma^\infty_c (E))\) be the smooth (compactly supported) sections of \(E\), and \(H := L^2 (E)\) be the space of square integrable sections of \(E\). Further assume that \(E\) is equipped with a metric compatible connection, \(\nabla^E\), and that \(\Box = \Box^E\) is the associated Bochner Laplacian on \(\Gamma^\infty (E)\).

To be more explicit, if \(\{e_i\}_{i=1}^{\text{rank}(E)}\) is a local orthonormal frame, then
\[
\Box f = \text{tr} \left( \nabla^T \otimes M \otimes E \nabla^E f \right) = \sum_i \left( \nabla_{e_i}^E \nabla_{e_i}^E f - \nabla_{\nabla^T M e_i}^E f \right).
\]

To simplify notation in the computations below, we will drop the superscripts, \(E\) and \(TM\) from the symbols since they can be deduced from the context.

**Notation B.1.** Given a measurable section, \(e : M \to E\), and \(f \in \mathcal{H}\), let
\[
\text{sgn}_e (f) := 1_{f \neq 0} \frac{f}{|f|} + 1_{f = 0} e = \left\{ \begin{array}{ll} \frac{f}{|f|} & \text{if } f \neq 0 \\ e & \text{if } f = 0 \end{array} \right.
\]

With this notation we have the polar decomposition, \(f = |f| \text{ sgn}_e (f)\), which is valid no matter what the choice of \(e\).
Theorem B.2 (Kato’s Inequality). Let $\varepsilon > 0$, $f \in \Gamma^\infty(E)$, $|f|_\varepsilon := \sqrt{|f|^2 + \varepsilon^2}$, and $\hat{f}_\varepsilon := f / |f|_\varepsilon$. Then

$$d|f|_\varepsilon = \left\langle \hat{f}_\varepsilon, \nabla f \right\rangle \quad \text{and}$$

(B.1) $$\Delta_0 |f|_\varepsilon = \frac{1}{|f|_\varepsilon} \sum_i \left( |\nabla_{e_i} f|^2 - \left| \left\langle \hat{f}_\varepsilon, \nabla_{e_i} f \right\rangle \right|^2 + \left\langle \hat{f}_\varepsilon, \Box f \right\rangle \right)$$

(B.2) $$\geq \left\langle \hat{f}_\varepsilon, \Box f \right\rangle.$$

Moreover if $\varphi \in C^\infty(M)_+$ and $f \in C^\infty_c(E)$, then

(B.3) $$\langle \Box f, \varphi \text{sgn}(f) \rangle \leq \left( |f|, \Delta_0 \varphi \right),$$

where $e$ is any measurable section of $E$ such that $\langle \Box f(x), e(x) \rangle_x = 0$ and $|e(x)|_x = 1$ on the set where $f = 0$.

Proof. This theorem is mostly a straightforward computation. (See [35], where a local coordinate version of this calculation is done.) We start by computing the gradient of $|f|_\varepsilon$ as

$$d|f|_\varepsilon = \frac{1}{2\sqrt{|f|^2 + \varepsilon^2}} d|f|^2 = \frac{1}{\sqrt{|f|^2 + \varepsilon^2}} \langle f, \nabla f \rangle.$$

With this in hand we have the following formula for the Hessian of $|f|_\varepsilon$:

$$\nabla d|f|_\varepsilon = - \left( |f|^2 + \varepsilon^2 \right)^{-3/2} \left( (f, \nabla f)^2 \right.$$  

$$+ \frac{1}{\sqrt{|f|^2 + \varepsilon^2}} \left( (\nabla \cdot f, \nabla f) + \left\langle f, \nabla^2 f \right\rangle \right).$$

Taking the trace of this result gives

$$\Delta_0 |f|_\varepsilon = - \left( |f|^2 + \varepsilon^2 \right)^{-3/2} \sum_i |\langle f, \nabla_{e_i} f \rangle|$$  

$$+ \frac{1}{\sqrt{|f|^2 + \varepsilon^2}} \left( \sum_i |\nabla_{e_i} f|^2 + |\langle f, \Box f \rangle| \right),$$

which is equivalent to Eq. (B.1). Equation (B.2) follows by the Cauchy-Schwarz inequality which implies

$$|\nabla_{e_i} f|^2 - \left| \left\langle \hat{f}_\varepsilon, \nabla_{e_i} f \right\rangle \right|^2 \geq |\nabla_{e_i} f|^2 - \left| \hat{f}_\varepsilon \right|^2 \cdot |\nabla_{e_i} f|^2 \geq 0.$$

If we now assume that $f \in \Gamma^\infty_c(E)$ and $\varphi \in C^\infty(M,[0,\infty))$, then multiplying Eq. (B.2) by $\varphi$ and integrating gives,

(B.4) $$\int_M \left\langle \Box f, \frac{f}{|f|_\varepsilon} \right\rangle \varphi dV \leq \int_M \Delta_0 |f|_\varepsilon \varphi dV = \int_M |f|_\varepsilon \Delta_0 \varphi dV.$$
where we have done two integrations by parts to get the last equality. Letting \( \varepsilon \downarrow 0 \) in Eq. (B.4) then implies

\[
\int_M \langle \Box f, \text{sgn}_0 (f) \rangle \varphi dV \leq \int_M |f| \Delta_0 \varphi dV
\]  

which is to say

\[
\langle \Box f, \text{sgn}_0 (f) \rangle \leq \Delta_0 |f| \text{ (in the distributional sense).}
\]  

If we now choose \( e \) to be a measurable section of \( E \) such that \( |e| = 1 \) and \( \langle \Box f, e \rangle = 0 \), then \( \langle \Box f, \text{sgn}_0 (f) \rangle = \langle \Box f, \text{sgn}_e (f) \rangle \) and we may rewrite Eqs. (B.5) and (B.6) as,

\[
\int_M \langle \Box f, \text{sgn}_e (f) \rangle \varphi dV \leq \int_M |f| \Delta_0 \varphi dV
\]  

and

\[
\langle \Box f, \text{sgn}_e (f) \rangle \leq \Delta_0 |f| \text{ (in the distributional sense).}
\]  

These last two equations are equivalent to Eq. (B.3). q.e.d.

**Appendix C. A local martingale**

In this appendix we will continue to use the notation in Section 5.1 unless otherwise stated.

**Lemma C.1** (Local martingale lemma). Let \( \tilde{\ell}_t \in \mathbb{R} \) be an adapted continuously differentiable real valued process, \( \ell_0 \in T_x M \),

\[
\ell_t = Q_t \left[ \int_0^t Q_{\tau}^{-1} \left( \frac{d}{d\tau} \tilde{\ell}_\tau \right) dB_\tau + \ell_0 \right],
\]

\( a \in \Omega^1 (M) \), and

\[
Z_t := (a_t (\Sigma_t) \circ / / t) \ell_t - \delta a_t (\Sigma_t) \tilde{\ell}_t,
\]

be as in Eq. (5.10). Then \( Z_t \) is a local martingale whose Itô differential is given by

\[
dZ_t = (\nabla_{/ / t} a_t) (\Sigma_t) \circ / / t \ell_t + (a_t (\Sigma_t) \circ / / t) \left( \frac{d}{dt} \tilde{\ell}_t \right) dB_t
\]

\[
- (\nabla_{/ / t} a_t) (\Sigma_t) \tilde{\ell}_t.
\]

**Proof.** The proof of this lemma is purely a computation. For the sake of the reader’s understanding we will give a slightly inefficient proof designed to motivate the form of \( Z_t \) in Eq. (5.10). Let \( a_t \) be as in Eq. (5.9) and then set

\[
N_t := a_t (\Sigma_t) \circ / / t.
\]
Then by Itô’s lemma in Eq. (5.2), Theorem 4.1, and Bochner identity in Eq. (4.3), we find

\[dN_t = (\nabla_{/1}a_t) \circ /_{/t} + \frac{1}{2} \left((\Box - \Delta) a_t(\Sigma_t)\right) \circ /_{/t} dt\]

(C.4) \[= (\nabla_{/1}a_t) \circ /_{/t} + \frac{1}{2} \left[a_t(\Sigma_t) \circ \text{Ric} \circ /_{/t}\right] dt.\]

Also by Itô’s lemma in Eq. (5.1) and item 4. of Theorem 4.1,

\[d [\delta a_t(\Sigma_t)] = d \left[\left(e^{(T-t)\Delta_0/2}\delta a\right)(\Sigma_t)\right]
\]

(C.5) \[= (\nabla_{/1}a_t) \left[\left(e^{(T-t)\Delta_0/2}\delta a\right)(\Sigma_t)\right] = (\nabla_{/1}a_t)[\delta a_t](\Sigma_t).\]

Now suppose \(\ell_t \in T_xM\) and \(\tilde{\ell}_t \in \mathbb{R}\) are arbitrary continuous Brownian semi-martingales such that

\[d\ell_t = \alpha_t \, dB_t + \beta_t \, dt\]

and \(d\tilde{\ell}_t = \tilde{\alpha}_t \, dB_t + \tilde{\beta}_t \, dt\)

with \(\alpha_t, \beta_t, \tilde{\alpha}_t,\) and \(\tilde{\beta}_t\) being continuous adapted processes with values in \(\text{End}(T_xM), T_xM, T_xM^*,\) and \(\mathbb{R}\) respectively and let

\[Z_t = N_t \ell_t - (\delta a_t)(\Sigma_t) \tilde{\ell}_t.\]

(C.6)

Making use of Eqs. (C.4) and (C.5), the Itô differential of \(Z\) in Eq. (C.6) is,

\[dZ_t = (\nabla_{/1}a_t) \circ /_{/t} \ell_t + \frac{1}{2} \left[a_t(\Sigma_t) \circ \text{Ric} \circ /_{/t}\right] dt
\]

\[+ (a_t(\Sigma_t) \circ /_{/t}) [\alpha_t dB_t + \beta_t dt] + (\nabla_{/1}\ell_t)(\Sigma_t) \circ /_{/t} \alpha_t dB_t dt
\]

\[+ (\nabla_{/1}\tilde{\ell}_t)(\Sigma_t) \circ /_{/t} \tilde{\alpha}_t dB_t + \tilde{\beta}_t dt
\]

\[+ (\nabla_{/1}\tilde{\ell}_t)(\Sigma_t) \circ /_{/t} \tilde{\alpha}_t dB_t + \tilde{\beta}_t dt
\]

(C.7) \[+ \left(\frac{1}{2} [a_t(\Sigma_t) \circ \text{Ric} \circ /_{/t}] + (a_t(\Sigma_t) \circ /_{/t}) \beta_t
\]

\[+ (\nabla_{/1}\ell_t)(\Sigma_t) \circ /_{/t} \alpha_t dB_t - \delta a_t(\Sigma_t) \tilde{\beta}_t - (\nabla_{/1}\tilde{\ell}_t)(\Sigma_t) \tilde{\alpha}_t dB_t,\right) dt.\]

Our goal is to choose \(\alpha_t, \beta_t, \tilde{\alpha}_t,\) and \(\tilde{\beta}_t\) in such a way that \(Z_t\) is a local martingale. To do this we need to make the term in the parenthesis in Eq. (C.7) vanish. Grouping the terms according to the number of derivatives on \(a_t\), the term in parenthesis in Eq. (C.7) will vanish.
provided
\[
\frac{1}{2} \left[ a_t (\Sigma_t) \circ \text{Ric} \circ \ell_t \right] + \left( a_t (\Sigma_t) \circ \ell_t \right) \beta_t = 0,
\]
\[
\left( \nabla \ell_t \right) (\Sigma_t) \circ \ell_t \alpha_t e_i - \delta a_t (\Sigma_t) \tilde{\beta}_t = 0,
\]
and
\[
\left( \nabla \left[ \delta a_t \right] \right) \tilde{\alpha}_t e_i = 0.
\]
Moreover because of Eq. (4.1), these equations may be satisfied by choosing \( \tilde{\alpha} \equiv 0 \) (so that \( \tilde{\ell}_t \) is differentiable and \( \frac{d\tilde{\ell}_t}{dt} = \tilde{\beta}_t \)),
\[
\beta_t = -\frac{1}{2} \left/ \ell_t^{-1} \text{Ric} \ell_t =: -\frac{1}{2} \text{Ric} \ell_t,
\]
and
\[
\alpha_t = \tilde{\beta}_t I_{T_x M} = \frac{d\tilde{\ell}_t}{dt} I_{T_x M}.
\]
Thus we have shown,
\[
Z_t := \left( a_t (\Sigma_t) \circ \ell_t \right) \ell_t - \delta a_t (\Sigma_t) \tilde{\ell}_t,
\]
is a local martingale provided \( \tilde{\ell}_t \) is an adapted \( C^1 \) process and \( \ell_t \) solves
(C.8)
\[
d\ell_t = \frac{d\tilde{\ell}_t}{dt} db_t - \frac{1}{2} \text{Ric} \ell_t dt.
\]
To solve this equation for \( \ell_t \), let \( Q_t \) solve the ODE in Eq. (5.3) and write \( \ell_t = Q_t k_t \) where \( k_t := Q_t^{-1} \tilde{\ell}_t \). Plugging this expression for \( \ell_t \) into Eq. (C.8) using,
\[
d\ell_t = -\frac{1}{2} \text{Ric} \ell_t dt + Q_t dk_t,
\]
implies,
\[
-\frac{1}{2} \text{Ric} \ell_t dt + Q_t dk_t = \frac{d\tilde{\ell}_t}{dt} db_t - \frac{1}{2} \text{Ric} \ell_t dt
\]
from which we learn, \( dk_t = \frac{1}{Q_t} \frac{d\tilde{\ell}_t}{dt} db_t \). Integrating this equation and multiplying the result on the left by \( Q_t \) gives Eq. (C.1). Equation (C.3) now follows from Eq. (C.7) with \( \tilde{\alpha} = 0 \) and \( \alpha_t = \frac{d\tilde{\ell}_t}{dt} I_{T_x M} \). \( \text{q.e.d.} \)

**Appendix D. Wang’s dimension free Harnack inequality**

Suppose that \( p_T (\cdot, \cdot) > 0 \) is the heat kernel at time \( T > 0 \) on a complete connected Riemannian manifold \( (M) \) and for measurable \( f : M \to [0, \infty) \), let
\[
(P_T f)(x) := \int_M p_T (x, y) f(y) dV(y).
\]
Hence if \( f \in L^2 (V) \), then \( P_T f = e^{T \Delta_0 / 2} f \). The following lemma reflects the fact that \( (L^q)^* \) and \( L^{q'} \) are isometrically isomorphic Banach spaces for \( 1 < q < \infty \) and \( q' = q / (q - 1) \) – the conjugate exponent to \( q \).
**Lemma D.1.** Let $x, y \in M$, $T > 0$, $q \in (1, \infty)$, and $C \in (0, \infty]$. Then
\[
(P_T f) (x)^q \leq C^q (P_T f^q) (y)
\]
for all $f \geq 0$ if and only if
\[
\left( \int_M \left[ \frac{p_T (x, z)}{p_T (y, z)} \right]^q p_T (y, z) \, dV (z) \right)^{1/q'} \leq C.
\]

**Proof.** Since
\[
(P_T f) (x) = \int_M p_T (x, z) f (z) p_T (y, z) \, dV (z),
\]
if $d \mu (z) := p_T (y, z) \, dV (z)$ and $g (x) := \frac{p_T (x, z)}{p_T (y, z)}$, then
\[
(P_T f) (x) = \int_M f (x) g (x) \, d\mu (x).
\]
Since $g \geq 0$ and $L^q (\mu)^* \approx L^{q'} (\mu)^*$ under the pairing in Eq. (D.3), it follows that
\[
\|g\|_{L^{q'} (\mu)} = \sup_{f \geq 0} \int_M f (x) g (x) \, d\mu (x) = \sup_{f \geq 0} \frac{(P_T f) (x)}{[(P_T f^q) (y)]^{1/q}}.
\]
The last equation may be written more explicitly as,
\[
\left( \int_M \left[ \frac{p_T (x, z)}{p_T (y, z)} \right]^q p_T (y, z) \, dV (z) \right)^{1/q'} = \sup_{f \geq 0} \frac{(P_T f) (x)}{[(P_T f^q) (y)]^{1/q}},
\]
and from this equation the lemma easily follows. q.e.d.

The following theorem appears in [66, 67] – see also [2].

**Theorem D.2** (Wang’s Harnack inequality). Suppose that $M$ is a complete connected Riemannian manifold such that $\text{Ric} \geq k I$ for some $k \in \mathbb{R}$. Then for all $q > 1$, $f \geq 0$, $T > 0$, and $x, y \in M$, we have
\[
(P_T f) (y) \leq (P_T f^q) (z) \exp \left( q \frac{k}{e^{kT} - 1} d^2 (y, z) \right),
\]
where $q' = q / (q - 1)$ is the conjugate exponent to $q$.

In applying Wang’s results the reader should use $k = -K$, $V \equiv 0$, and replace $T$ by $T/2$ since Wang’s generator is $\Delta$ rather than $\Delta/2$.

**Corollary D.3.** Let $(M, g)$ be a complete Riemannian manifold such that $\text{Ric} \geq k I$ for some $k \in \mathbb{R}$. Then for every $y, z \in M$ and $q \in [1, \infty)$,
\[
\left( \int_M \left[ \frac{p_T (y, x)}{p_T (z, x)} \right]^q p_T (z, x) \, dV (x) \right)^{1/q} \leq \exp \left( \frac{c (kT) (q - 1)}{2T} d^2 (y, z) \right),
\]
where $c$ is a constant depending on $M$.
where $c(\cdot)$ is defined as in Eq. (3.1), $p_t(x, y)$ is the heat kernel on $M$ and $d(y, z)$ is the Riemannian distance from $x$ to $y$ for $x, y \in M$.

Proof. From Lemma D.1 and Theorem D.2 with

$$C = \exp \left( \frac{q'}{q} \frac{k}{e^{kT} - 1} d^2(y, z) \right) = \exp \left( \frac{1}{q - 1} \frac{k}{e^{kT} - 1} d^2(y, z) \right),$$

it follows that

$$\left( \int_M \frac{p_T(x, z)}{p_T(y, z)} \right)^{q'} \leq \exp \left( \frac{1}{q - 1} \frac{k}{e^{kT} - 1} d^2(y, z) \right).$$

Using $q - 1 = (q' - 1)^{-1}$ and then interchanging the roles of $q$ and $q'$ gives Eq. (D.5). q.e.d.

For comparison sake, recall that the classical Li - Yau Harnack inequality (see Li and Yau [43] and Davies [14, Theorem 5.3.5]) states if $\alpha > 1, s > 0, \text{ and } \text{Ric} \geq -K$ for some $K \geq 0$, then

$$p_t(y, x) \leq \left( \frac{t + s}{t} \right)^{d\alpha/2} \exp \left( \frac{\alpha d^2(y, z)}{2s} + \frac{d \cdot \alpha K s}{8(\alpha - 1)} \right),$$

for all $x, y, z \in M^d$ and $t > 0$. However when $s = 0$, Eq. (D.6) gives no information on $p_t(y, x)/p_t(z, x)$ when $y \neq z$.

Remark D.4. Since our heat equation is determined by $\Delta_0/2$ rather than $\Delta_0$, the reader should replace $t$ and $s$ by $t/2$ and $s/2$ when applying the results in [43, 14].

**Appendix E. Consequences of Hamilton’s estimates**

Let $T \in (0, \infty), M (d = \text{dim}(M))$ be a complete Riemannian manifold with $\text{Ric} \geq -K I$ for some $K \geq 0$, and let $V(x, r) := \text{Vol}(B(x, r))$ be the volume of the ball, $B(x, r)$, centered at $x \in M$ with radius $r > 0$. Suppose, for $0 \leq t \leq t_1$, that $u(t, x)$ is a positive solution to the heat equation, $\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u$. The Hamilton type gradient bounds [34, 59, 41] state that if

$$m := \sup \{ u(t, x) : 0 \leq t \leq t_1, x \in M \},$$

then

$$t |\nabla \log(u(t, x))|^2 \leq 2(1 + Kt) \log(m/u(t, x))$$

for all $(t, x) \in [0, t_1] \times M$.

The standard heat kernel bounds (see for example Theorems 5.6.4, 5.6.6, and 5.4.12 in Saloff-Coste [55] and for more detailed bounds see [43, 14, 54, 15, 31]) state that there exist constants, $c = c(K, d, T)$ and
\[ C = C(K, d, T), \text{ such that,} \]
\[
\frac{c}{V(x, \sqrt{t/2})} \exp \left(-C \frac{d^2(x, y)}{t} \right)
\leq p(t, x, y) \leq \frac{C}{V(x, \sqrt{t/2})} \exp \left(-c \frac{d^2(x, y)}{t} \right),
\]
(E.2)

for all \( x, y \in M \) and \( t \in (0, T] \).

Let \( s \in (0, T], o \in M, t_1 = s/2 \) and \( u(t, x) = p_{s/2+t}(o, x). \) Combining Eqs. (E.1) and (E.2) then shows,
\[
t \left| \nabla_x \log p_{s/2+t}(o, x) \right|^2
\leq 2(1 + K t) \log \left( \frac{CV(0, \sqrt{s/4 + t/2})}{c} \frac{V(o, \sqrt{s/4})}{V(o, \sqrt{t/4})} \exp \left( C \frac{d^2(o, y)}{s/2 + t} \right) \right).
\]
(E.3)

Taking \( t = s/2 \) in Eq. (E.3) and then replacing \( s \) by \( t \) in the resulting inequality implies,
\[
\frac{t}{2} \left| \nabla_x \log p_t(o, x) \right|^2
\leq 2 \left( 1 + K \frac{t}{2} \right) \log \left( \frac{CV(0, \sqrt{t/2})}{c} \frac{V(o, \sqrt{t/4})}{V(o, \sqrt{t/4})} \exp \left( C \frac{d^2(o, y)}{t} \right) \right).
\]
(E.4)

Using the volume estimate (see [10] and [55, Theorem 5.6.4]),
\[
\frac{V(x, \sigma)}{V(x, s)} \leq \left( \frac{s}{\sigma} \right)^d \exp \left( \sqrt{(d-1)K} \right) \forall x \in M \text{ and } 0 \leq s < \sigma,
\]

it follows that
\[
\frac{V(x, \sqrt{t/2})}{V(x, \sqrt{t/4})} \leq 2^{d/2} \exp \left( \sqrt{(d-1)Kt/2} \right) \leq 2^{d/2} \exp \left( \sqrt{(d-1)KT/2} \right).
\]
(E.5)

Combining Eqs. (E.4) and (E.5) then allows us to conclude that there exist constants, \( c_1 \) and \( c_2 \) depending on \( T, K, \) and \( d \) such that
\[
\left| \nabla_x \log p_t(o, x) \right| \leq \left( \frac{c_1}{\sqrt{t}} + c_2 \frac{d(o, x)}{t} \right) \text{ for all } t \in (0, T] \text{ and } o, x \in M.
\]
(E.6)

For this estimate in the compact case with its relations to stochastic analysis, see [17, 47, 62, 64, 36].
**Proposition E.1.** Continuing the notation and assumptions used above, there exist constants, $C_1(d, K)$ and $C_2(d, K, t)$ such that,

$$
\int_M \exp (\lambda |\nabla_x \log p_t(o, x)|) p_t(o, x) \, dx \leq C(d, K, t) \exp \left( C(d, K) \lambda^2 / t \right)
$$

for all $o \in M$ and $t \in (0, T]$.

**Proof.** Let $v(r) := \text{Vol}(B(o, r))$, $\kappa := \sqrt{K/(d-1)}$, $\gamma := (d-1) \kappa = \sqrt{K(d-1)}$, and $\omega_{d-1}$ be the volume of the standard $d-1$ sphere. Using Bishop's comparison theorem (see [9, 56]) which states,

$$
\int_0^\infty v(r) \leq \omega_{d-1} \left( \frac{\sinh \kappa r}{\kappa} \right)^{d-1} \kappa \, dr \leq \left( \frac{\omega_{d-1}}{2\kappa} \right)^{d-1} e^{\gamma(d-1)r} \, dr,
$$

along with the estimates in Eqs. (E.2) and (E.6), we have

$$
\int_M \exp (\lambda |\nabla_x \log p_t(o, x)|) p_t(o, x) \, dx \\
\leq C t^{-d/2} \int_0^\infty \exp \left( \lambda \left( \frac{c_1}{\sqrt{t}} + c_2 \frac{r}{t} \right) \right) \exp \left( -\frac{C}{2t} r^2 \right) \, dv(r)
$$

$$
\leq C \left( \frac{\omega_{d-1}}{2\kappa} \right)^{d-1} t^{-d/2} \int_0^\infty \exp \left( \lambda \left( \frac{c_1}{\sqrt{t}} + c_2 \frac{r}{t} \right) \right) \exp \left( -\frac{C}{2t} r^2 \right) e^{\gamma r} \, dr
$$

$$
= C(d, K, t) t^{-d/2} \exp \left( \frac{\lambda c_1}{\sqrt{t}} \right) \int_0^\infty \exp \left( \left( \gamma + \frac{c_2}{t} \right) r \right) \exp \left( -\frac{C}{2t} r^2 \right) \, dr.
$$

Equation (E.7) follows easily from Eq. (E.10) and the following two estimates

$$
c_1 \frac{\lambda}{\sqrt{t}} \leq \frac{1}{2} \left( c_1^2 + \frac{\lambda^2}{2t} \right)
$$

and

$$
\int_0^\infty \exp \left( \left( \gamma + \frac{c_2}{t} \right) r \right) \exp \left( -\frac{C}{2t} r^2 \right) \, dr
\leq \int_{-\infty}^\infty \exp \left( \left( \gamma + \frac{c_2}{t} \right) r \right) \exp \left( -\frac{C}{2t} r^2 \right) \, dr
\leq \sqrt{2\pi t/C} \exp \left( \frac{t}{2C} \left( \gamma + \frac{c_2}{t} \right)^2 \right).
$$

**Remark E.2.** When $M = \mathbb{R}^d$, using Laplace asymptotics, one may show:

$$
\lim_{d \to \infty} e^{-\frac{\lambda^2}{4t}} \int_{\mathbb{R}^d} \exp (\lambda |\nabla_x \log p_t(o, x)|) p_t(o, x) \, dx = e^{\lambda^2/4t} \forall t, \lambda > 0.
$$
In particular, this implies that we cannot take both \( C(d, 0, t) \) and \( C(d, 0) \) in Eq. (E.7) to be independent of the dimension, \( d = \dim(M) \).

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