Representation Formulas for Contact Type Hamilton-Jacobi Equations

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Abstract

We discuss various kinds of representation formulas for the viscosity solutions of the contact type Hamilton-Jacobi equations by using the Herglotz’ variational principle.

Keywords Hamilton-Jacobi equation · Representation formula · Viscosity solutions

1 Introduction

Let $M$ be a $C^1$ connected and compact manifold without boundary. Let $TM$ and $T^*M$ denote the tangent and cotangent bundles respectively. A point of $TM$ will be denoted by $(x, v)$ with $x \in M$ and $v \in T_x M$, and a point of $T^*M$ by $(x, p)$ with $p \in T^*_x M$ is a linear form on the vector space $T_x M$. With a slight abuse of notation, we shall denote by $|\cdot|_x$ the norm on the fiber $T_x M$ and also the dual norm on $T^*_x M$.

In this paper, we want to discuss the representation formula for the viscosity solutions of the evolutionary Hamilton-Jacobi equation

$$\begin{cases}
D_t u(t, x) + H(t, x, u(t, x), D_x u(t, x)) = 0, \quad (t, x) \in (0, +\infty) \times M \\
u(0, x) = \phi(x), \quad x \in M,
\end{cases} \tag{HJe}$$

and the stationary equation

$$H(x, u(x), Du(x)) = 0, \quad x \in M. \tag{HJs}$$

Here we suppose 0 on the right side of (HJs) belongs to the set of Mañé’s critical values.
Because of the Lagrangian formalism, we endow some suitable conditions on the associated Lagrangian \( L \) with respect to a convex \( H \) defined by

\[
L(s, x, v, u) = \sup_{p \in T^*_s M} \{ p \cdot v - H(s, x, p, u) \}, \quad (s, x, v, u) \in \mathbb{R} \times TM \times \mathbb{R}.
\]

The conditions on \( L \) are imposed at the beginning of Section 2.

The representation formula for the viscosity solutions of the Hamilton-Jacobi equations in various kinds of problems typically connects the solution of PDEs to the value function for the relevant problems from calculus of variations and optimal control. A representation formula provides further information on the underlying dynamical systems, which is important for certain finer analysis of the solutions including qualitative Lipschitz and semiconcavity estimate and some implications in/from dynamical systems. Standard references on the representation formula for classical convex Hamiltonians include \([1,2,11,19,20,22,25]\) and \([12,17,18,21,23,24]\) for weak KAM and geometric aspects respectively.

The representation formula for the viscosity solutions of \((HJ_e)\) and \((HJ_x)\) is known for the discounted systems even in the early period of the theory of viscosity solutions. A systematic approach for the study of equations \((HJ_e)\) and \((HJ_x)\) firstly appears in \([26,27]\) in an implicit way. More precisely, if \( u \) is a viscosity solution of \((HJ_x)\) for a \(C^3 \) Hamiltonian \( H \), then the following representation formula holds:

\[
u(x) = \inf_{y \in M} h_{y,u}(x, t), \quad \forall t > 0,
\]

where \( h_{y,u}(x, t) \) in the implicit action function introduced in \([26]\).

An alternative Lagrangian approach is based on a rigorous treatment of the classical Herglotz’ variational principle (\([8,10]\)). Recall some basic results from \([8,10]\) on the Herglotz’ variational principle and the Hamilton-Jacobi equations of contact type. For any \( t_2 > t_1 \), \( u_0 \in \mathbb{R} \) and \( x, y \in M \), denote the set

\[
\Gamma_{t_1,t_2}^{x,y} = \{ \xi \in W^{1,1}([t_1, t_2], M) : \xi(t_1) = x, \xi(t_2) = y \},
\]

and consider the following Carathéodory equation

\[
\begin{align*}
\dot{u}_\xi(s) &= L(s, \xi(s), \dot{\xi}(s), u_\xi(s)) \quad \text{a.e. } s \in [t_1, t_2], \\
u_\xi(t_1) &= u_0.
\end{align*}
\]

It is clear that equation (1.1) admits a unique solution (see \([15]\)). We define

\[
h_L(t_1, t_2, x, y, u_0) := \inf_{\xi \in \Gamma_{t_1,t_2}^{x,y}} \int_{t_1}^{t_2} L(s, \dot{\xi}(s), \xi(s), u_\xi(s)) \, ds.
\]

As shown in \([8]\), the infimum in the definition of the negative-type fundamental solution \( h_L(t_1, t_2, x, y, u_0) \) can be achieved and any minimizer \( \xi \in \Gamma_{t_1,t_2}^{x,y} \) is as smooth as \( L \). We introduce the associated Lax-Oleinik operator

\[
(T^*_t \phi)(x) = \inf_{y \in M} \{ \phi(y) + h_L(t_1, t_2, y, x, \phi(y)) \}, \quad t_2 > t_1, x \in M,
\]

where \( \phi : M \to [-\infty, +\infty] \) is any function. For any \( t > 0 \) and \( x \in M \), set

\[
u(t, x) := (T^*_0 \phi)(x) = \inf_{y \in M} \{ \phi(y) + h_L(0, t, y, x, \phi(y)) \}.
\]

It is known that \( u(t, x) \) defined in (1.3) is a viscosity solution of \((HJ_e)\) (see Proposition 2.1 for a precise statement).
Comparing to the implicit representation formula in [27], an advantage of Herglotz’ variational principle is that one can obtain various kinds of representation formulas by choosing different ways to solve the Carathéodory equation (1.1). The readers can find more information on the relation between the implicit action function used in [26] and ours from the proof of Theorem 3.1, especially a comparison result therein.

These representation formulas are also useful for many applications. One example is the problem on vanishing discount ([16]) and vanishing contact structure ([14,28]), where such a representation formula plays an important role. Another example is the problem of the propagation of singularities. The regularity properties of the fundamental solution such as quantitative semiconcavity and convexity estimates can be obtained by using Herglotz’ variational principle, as well as the representation formulas for both negative-type and positive-type fundamental solutions, which will be adapted to our intrinsic method developed in [5–7,9].

The paper is organized as follows. We discuss the representation formulas in Section 2 and Section 3 for evolutionary equation (HJE) and stationary equation (HJS) respectively. The last section contains some concluding remarks including the discounted systems comparing to some known results.

2 representation formula of evolutionary equation

We assume that \( L \) is of class \( C^1 \). For the purpose of this paper, we need the following conditions:

(L1) \( L(s, x, v, u) \) is strictly convex on \( T_x M \) for all \( s \in \mathbb{R}, x \in M \) and \( u \in \mathbb{R} \).

(L2) There exist \( c_0 > 0 \) and a superlinear and nondecreasing function \( \theta_0 : [0, +\infty) \to [0, +\infty) \), such that

\[
L(s, x, v, 0) \geq \theta_0(|v|_x) - c_0, \quad (s, x, v) \in \mathbb{R} \times TM.
\]

(L3) There exists \( K > 0 \) such that

\[
|L_u(s, x, v, u)| \leq K, \quad (s, x, v, u) \in \mathbb{R} \times TM \times \mathbb{R}.
\]

(L4) There exist \( C_1, C_2 > 0 \) such that \( |L_t(s, x, v, u)| \leq C_1 + C_2 L(s, x, v, u) \) for all \( (s, x, v, u) \in \mathbb{R} \times TM \times \mathbb{R} \).

(L5) The map \( u \mapsto L(s, x, v, u) \) is concave for all \( (s, x, v) \in \mathbb{R} \times TM \).

(L6) \( L_u(s, x, v, u) < 0 \) for all \( (s, x, v, u) \in \mathbb{R} \times TM \times \mathbb{R} \).

2.1 Herglotz’ variational principle

We recall some known results based on Herglotz’ variational principle and the representation of the viscosity solutions. Set

\[
\mathcal{A}_{t,x} = \{ \xi \in W^{1,1}([0, t], M) : \xi(t) = x \}.
\]

We suppose condition (L1)-(L4) are satisfied.

Proposition 2.1 ([8]) Let \( \phi \) be lower semi-continuous and \((\kappa_1, \kappa_2)\)-Lipschitz in the large\(^1\) and the function \( u(t, x) \) be defined in (1.3).

\(^1\) Let \((X, d)\) be a metric space. A function \( \phi : X \to \mathbb{R} \) is called \((\kappa_1, \kappa_2)\)-Lipschitz in the large if there exist \( \kappa_1, \kappa_2 \geq 0 \) such that \( |\phi(y) - \phi(x)| \leq \kappa_1 + \kappa_2 d(x, y) \), for all \( x, y \in X \)
(1) The function \( u(t, x) \) is finite-valued.
(2) For any \( t > 0 \) and \( x \in M \) the function \( y \mapsto \phi(y) + h_L(0, t, y, x, \phi(y)) \) admits a minimizer.
(3) For any \( t > 0 \) and \( x \in M \), let \( y_{t,x} \) be a minimizer of the function \( y \mapsto \phi(y) + h_L(0, t, y, x, \phi(y)) \). Then, there exists a minimizer \( \xi_{t,x} \in \Gamma_{y_{t,x},x}^0 \) such that
\[
 u(t, x) = \phi(y_{t,x}) + \int_0^t L(s, \xi_{t,x}(s), \dot{\xi}_{t,x}(s), u_{\xi_{t,x}}(s)) \, ds.
\]
(4) Equivalently, for any \( t > 0 \) and \( x \in M \), there exists \( \xi \in \mathcal{A}_{t,x} \) such that
\[
 u(t, x) = \phi(\xi(0)) + \int_0^t L(s, \xi(s), \dot{\xi}(s), u_\xi(s)) \, ds.
\]
(5) Moreover, \( u(\cdot, x) \) is right continuous at \( t = 0 \) for all \( x \in M \), and the extension of \( u \) on \([0, +\infty) \times M\) is the unique solution of \((HJ_e)\) in the sense of viscosity.

### 2.2 Representation formula in general

The representation formula for the viscosity solutions of Hamilton-Jacobi equation \((HJ_e)\) and \((HJ_s)\) was first systematically studied in the papers [26,27] by using an implicit variational principle for \( M \) being compact and \( L \) being time-independent. Equivalently, by using Herglotz’ variational principle (see [8,10]), we have our first representation formula.

**Proposition 2.2** (Representation formula I) Suppose \( L \) satisfies conditions \((L1)-(L4)\) and \( H \) is the associated Hamiltonian. If \( \phi \) is lower semi-continuous and \((\kappa_1, \kappa_2)\)-Lipschitz in the large, then the unique viscosity solution \( u \) of \((HJ_e)\) has the following representation: for any \( t > 0 \) and \( x \in M \),
\[
 u(t, x) = \inf_{\xi \in \mathcal{A}_{t,x}} u_\xi(t) = \inf_{\xi \in \mathcal{A}_{t,x}} \left\{ \phi(\xi(0)) + \int_0^t L(s, \xi(s), \dot{\xi}(s), u_\xi(s)) \, ds \right\}, \quad (2.1)
\]
where \( u_\xi \) is uniquely determined by \((1.1)\) with \( u_0 = \phi(\xi(0)) \).

The second representation formula for the viscosity solutions of Hamilton-Jacobi equation \((HJ_e)\) appears in [28]. Here we give a slight extension for the time-dependent Lagrangian on manifold, with a different proof.

**Proposition 2.3** (Representation formula II) Suppose \( L \) satisfies condition \((L1)-(L4)\) and \( H \) is the associated Hamiltonian. If \( \phi \) is lower semi-continuous and \((\kappa_1, \kappa_2)\)-Lipschitz in the large, then the unique viscosity solution \( u \) of \((HJ_e)\) has the following representation: for any \( t > 0 \) and \( x \in M \),
\[
 u(t, x) = \inf_{\xi \in \mathcal{A}_{t,x}} \left\{ e^{\int^t_0 L_u d\tau} \phi(\xi(0)) + \int_0^t e^{\int^s_0 L_u d\tau} (L - u_\xi L_u) \, ds \right\}, \quad (2.2)
\]
where \( L_u(s) = L_u(s, \xi(s), \dot{\xi}(s), u_\xi(s)) \) and \( u_\xi \) is uniquely determined by \((1.1)\) with \( u_0 = \phi(\xi(0)) \).

**Proof** Adding a term \(-u_\xi(s)L_u\) to the both sides of \((1.1)\), we obtain
\[
 \dot{u}_\xi - u_\xi L_u(s, \xi(s), \dot{\xi}(s), u_\xi(s)) = L(s, \xi(s), \dot{\xi}(s), u_\xi(s)) - u_\xi L_u(s, \xi(s), \dot{\xi}(s), u_\xi(s))
\]
Thus (2.2) follows by integrating both sides from 0 to \( t \). Due to the relation \( u(t, x) = \inf_{\xi \in \mathcal{A}(t,x)} u_\xi(t) \), this completes the proof. \( \square \)

### 2.3 Representation formula for \( u \)-concave Lagrangian

Fix any \( t_2 > t_1, u_0 \in \mathbb{R} \) and \( x, y \in M \). Let \( \xi^* \in \Gamma^{t_1,t_2}_{x,y} \) be a minimizer for the functional \( u_\xi(t) \) where \( u_\xi \) is uniquely determined by (1.1). Consider a new Carathéodory equation

\[
\begin{align*}
\dot{v}_\eta &= L(s, \eta, \dot{\eta}, u_\xi^*) + L_u(s, \eta, \dot{\eta}, u_\xi^*)(v_\eta - u_\xi^*), \quad a.e., \ s \in [t_1, t_2], \\
v_\eta(t_1) &= u_0,
\end{align*}
\]

where \( v_\eta(s) = v_{\xi^*, \eta}(s) \) for \( \eta \in \Gamma^{t_1,t_2}_{x,y} \).

**Lemma 2.4** Suppose \( L \) satisfies condition (L1)-(L5) and \( H \) is the associated Hamiltonian. If \( u_\xi \) and \( v_\eta \) are determined by (1.1) and (2.3) for \( \xi, \eta \in \Gamma^{t_1,t_2}_{x,y} \) respectively, then,

1. we have

\[
\begin{align*}
v_{\xi^*}(t_2) &= \inf_{\xi \in \Gamma^{t_1,t_2}_{x,y}} u_\xi(t_2) = \inf_{\eta \in \Gamma^{t_1,t_2}_{x,y}} v_\eta(t_2);
\end{align*}
\]

2. \( \arg \min_{\eta \in \Gamma^{t_1,t_2}_{x,y}} v_{\xi^*, \eta}(t_2) \subseteq \arg \min_{\xi \in \Gamma^{t_1,t_2}_{x,y}} u_\xi(t_2) \). In particular, if \( \xi^* \) is a unique minimizer for \( h_L(t_1, t_2, x, y, u_0) \), then the relation of subset is indeed an equality and each of two sets is a singleton.

**Proof** For any \( \xi \in \Gamma^{t_1,t_2}_{x,y} \), set \( w_\xi = u_\xi - v_\xi \). Then, by concavity of \( L \) with respect to \( u \), we have that

\[
\begin{align*}
\dot{w}_\xi &= L(s, \xi, \dot{\xi}, \xi) - [L(s, \xi, \dot{\xi}, u_\xi^*) + L_u(s, \xi, \dot{\xi}, u_\xi^*)(v_\xi - u_\xi^*)] \\
&\leq [L_u(s, \xi, \dot{\xi}, u_\xi^*)(u_\xi - u_\xi^*)] - [L_u(s, \xi, \dot{\xi}, u_\xi^*)(v_\xi - u_\xi^*)] \\
&= L_u(s, \xi, \dot{\xi}, u_\xi^*) w_\xi
\end{align*}
\]

with \( w_\xi(t_1) = 0 \). It follows that \( w_\xi(s) \leq 0 \) for all \( s \in [t_1, t_2] \). Therefore, \( u_\xi(t_2) \leq v_\xi(t_2) \) for all \( \xi \in \Gamma^{t_1,t_2}_{x,y} \). Thus,

\[
\inf_{\xi \in \Gamma^{t_1,t_2}_{x,y}} u_\xi(t_2) \leq \inf_{\eta \in \Gamma^{t_1,t_2}_{x,y}} v_\eta(t_2). \tag{2.4}
\]

Now, set \( \xi = \eta = \xi^* \) in (1.1) and (2.3) respectively and \( w = v_{\xi^*} - u_{\xi^*} \). Then

\[
\begin{align*}
\dot{w}(s) &= L_u(s, \xi^*(s), \dot{\xi}^*(s), u_{\xi^*}(s))w(s), \quad a.e., \ s \in [t_1, t_2], \\
w(t_1) &= 0.
\end{align*}
\]

This implies \( w \equiv 0 \) on \([t_1, t_2]\). It follows that

\[
\inf_{\eta \in \Gamma^{t_1,t_2}_{x,y}} v_\eta(t_2) \leq v_{\xi^*}(t_2) = u_{\xi^*}(t_2) = \inf_{\xi \in \Gamma^{t_1,t_2}_{x,y}} u_\xi(t_2). \tag{2.5}
\]

This completes the proof of (1) together with (2.4).
To see (2), we suppose \( \eta' \in \text{arg min}_{\eta \in \Gamma_{t_1, t_2}} v_{\eta}(t_2) \). Then
\[
\eta'(t_2) \leq v_{\eta'}(t_2) = \inf_{\eta \in \Gamma_{t_1, t_2}} v_{\eta}(t_2) = \inf_{\xi \in \Gamma_{t_1, t_2}} u_{\xi}(t_2)
\]
by (1). This implies \( \eta' \in \text{arg min}_{\xi \in \Gamma_{t_1, t_2}} u_{\xi}(t_2) \). Therefore we have \( \text{arg min}_{\xi \in \Gamma_{t_1, t_2}} u_{\xi}(t_2) \subset \text{arg min}_{\eta \in \Gamma_{t_1, t_2}} v_{\eta}(t_2) \).

\[\square\]

Fix \( t_2 > t_1, u_0 \in \mathbb{R} \) and \( x, y \in M \). Let \( \xi^* \) be a minimizer for \( h_L(t_1, t_2, x, y, u_0) \). In light of Lemma 2.4, we define a new Lagrangian
\[
L^\xi(s, x, v, u) := L(s, x, v, u_{\xi^*}(s)) + L_u(s, x, v, u_{\xi^*}(s))(u - u_{\xi^*}(s)). \tag{2.6}
\]
Now we can reformulate our results in Lemma 2.4.

**Proposition 2.5** Fix \( t_2 > t_1, u_0 \in \mathbb{R} \) and \( x, y \in M \). Let \( \xi^* \) be a minimizer for \( h_L(t_1, t_2, x, y, u_0) \). Then,

1. \( h_L(t_1, t_2, x, y, u_0) = h_L^\xi(t_1, t_2, x, y, u_0) \);
2. If \( \xi' \) is a minimizer for \( h_L^\xi(t_1, t_2, x, y, u_0) \), then \( \xi' \) is a minimizer for \( h_L(t_1, t_2, x, y, u_0) \).

**Theorem 2.6** (Representation formula III) Suppose \( L \) satisfies condition (L1)-(L5) and \( H \) is the associated Hamiltonian. If \( f \) is lower semi-continuous and \((\kappa_1, \kappa_2)\)-Lipschitz in the large, then the unique viscosity solution \( u \) of \((HJ_e)\) has the following representation: for any \( t > 0 \) and \( x \in M \), if \( \xi^* \in \mathcal{A}_{t,x} \) be a minimal curve in the definition of \( u(t, x) \) with \( y^* = \xi^*(0) \), then
\[
\begin{align*}
\dot{y}_\eta &= L(s, \eta, \dot{\eta}, u_{\xi^*}) + L_u(s, \eta, \dot{\eta}, u_{\xi^*})(v_{\eta} - u_{\xi^*}), \quad \text{a.e. } s \in [0, t], \\
v_{\eta}(0) &= \phi(y^*),
\end{align*}
\tag{2.8}
\]
where \( L_u(s) := L_u(s, \eta(s), \dot{\eta}(s), u_{\xi^*}(s)) \). Moreover, the right side of (2.7) is independent of the choice of \( \xi^* \).

**Proof** Let \( t > 0, x \in M \) and \( \xi^* \in \mathcal{A}_{t,x} \) be a minimal curve in the definition of \( u(t, x) \) with \( y^* = \xi^*(0) \). Consider the Carathéodory equation respect to \( L^\xi \),
\[
\begin{align*}
\dot{y}_\eta &= L(s, \eta, \dot{\eta}, u_{\xi^*}) + L_u(s, \eta, \dot{\eta}, u_{\xi^*})(v_{\eta} - u_{\xi^*}), \quad \text{a.e. } s \in [0, t], \\
v_{\eta}(0) &= \phi(y^*),
\end{align*}
\tag{2.8}
\]
where \( \eta \in \Gamma_{0, t}^{y^*, x} \). By solving (2.8) we have
\[
v_{\eta}(t) = e_{t_0}^{t} L_u d\tau \phi(\eta(0)) + \int_{0}^{t} e_{s}^{t} L_u d\tau \{L(s, \eta, \dot{\eta}, u_{\xi^*}) - u_{\xi^*}L_u\} ds.
\]
Invoking Lemma 2.4, we have that
\[
\begin{align*}
u(t, x) &= \inf_{\xi \in \Gamma_{0, t}^{y^*, x}} u_{\xi}(t) = \inf_{\eta \in \Gamma_{0, t}^{y^*}} v_{\eta}(t),
\end{align*}
\]
which leads to (2.7).

\[\square\]
3 representation formula of stationary equation

In this section, we will study the representation formula of the unique viscosity solution of \((\text{HJ}_s)\) with \(L\) time-independent. Fix \(x \in M\) and \(t > 0\), denote the sets
\[
\mathcal{A}_{t,x}^u = \{ \xi \in W^{1,1}([-t,0], M) : \xi(0) = x \},
\]
\[
\mathcal{A}_{\infty,x}^u = \{ \xi \in W^{1,0}((-\infty,0], M) : \xi(0) = x \}.
\]
Suppose \(u\) is the unique viscosity solution of \((\text{HJ}_s)\). For any \(\xi \in \mathcal{A}_{t,x}^u\) we consider the Carathéodory equation
\[
\begin{aligned}
\dot{u}_\xi(s) &= L(\xi(s), \dot{\xi}(s), u_\xi(s)), \quad \text{a.e. } s \in [-t,0], \\
u_\xi(-t) &= u(\xi(-t)).
\end{aligned}
\] (3.1)

We know the viscosity solution \(u\) satisfies that property that \(u(x) = (T_0^t u)(x) = u(t, x)\) for all \(t \geq 0\). Then, we rewrite \(u\) as
\[
u(x) = \inf_{\xi \in \mathcal{A}_{\infty,x}^u} \left\{ \liminf_{t \to \infty} \int_0^t e^t \int_{-t}^0 L(\xi(s), \dot{\xi}(s), u_\xi(s))ds \right\}. \] (3.2)

where \(u_\xi\) is uniquely determined by (3.1). It is known that the infimum in (3.2) can be achieved.

**Theorem 3.1 (Representation formula IV)** Suppose \(L\) satisfies condition (L1)-(L3) and (L6) as \(H\) is the associated Hamiltonian, and \((\text{HJ}_s)\) has a Lipschitz viscosity solution \(u(x)\), then the following representation formula holds
\[
u(x) = \inf_{\xi \in \mathcal{A}_{\infty,x}^u} \left\{ \liminf_{t \to \infty} \int_0^t e^t \int_{-t}^0 L(\xi(s), \dot{\xi}(s), u_\xi(s))ds \right\}. \] (3.3)

where \(u_\xi\) satisfies the Carathéodory equation with \(u_\xi(0) = u(x)\) for all \(t > 0\). Moreover, there exists \(\xi^* \in \mathcal{A}_{\infty,x}^u\) such that
\[
u(x) = \int_0^t e^t \int_{-t}^0 L(\xi^*(s), \dot{\xi}^*(s), u_{\xi^*}(s))ds. \]

**Remark 3.2** From (2.2) it is obvious to see that if \(L_u\) satisfies a more restricted condition such that \(-K \leq L_u \leq -\delta < 0\), then one can see the term \(e_t \int_{-t}^0 L_u u_{\xi^*}(s)ds\) in (2.2), in the form on time interval \([-t, 0]\), vanishes as \(t \to \infty\). However, if only assumption (L3) is supposed, we need a priori estimates to ensure the existence of such a positive \(\delta\) for the solution of \((\text{HJ}_s)\).

For any \(x \in M\), if \(\xi^* = \xi_x^* \in \mathcal{A}_{\infty,x}^u\) is such a minimizer, we call \(\xi^*\) a backward calibrated curve from \(x\).

**Proof** Under our assumptions, it is well known that the viscosity solution of \((\text{HJ}_s)\) is unique. Recall that \(0\) on the right side of \((\text{HJ}_s)\) is a critical value.

**I. A comparison result.** We first discuss a comparison result. For any \(\xi \in \mathcal{A}_{t,x}^u\), let \(\tilde{u}_{\xi}(s) = u(\xi(s), u_\xi(s), ...), \quad \text{a.e. } s \in [-t,0],
\]
\[
u \tilde{u}_\xi(0) = u(x).
\] (3.4)
Since $u$ is a viscosity solution of (HJ)$_t$, we obtain that for any $0 \leq s \leq t$,
\[
    u(\xi_t(-s)) \leq u(\xi_t(-t)) + \int_{-t}^{-s} L(\xi_t, \dot{\xi}_t, \tilde{u}_{\xi_t}) \, d\tau = \tilde{u}_{\xi_t}(-s). \tag{3.5}
\]

Denote by $\xi_s$ the restriction of $\xi_t$ on $[-s, 0]$ for $s \in [0, t]$. Then, $\tilde{u}_{\xi_s}$ satisfies (3.1) with $\tilde{u}_{\xi_t}(-s) = u(\xi_t(-s))$. Thus, we also have
\[
    u_{\xi_t}(0) = u(x) = u(\xi_s(0)) \leq u(\xi_s(-s)) + \int_{-s}^{0} L(\xi_s, \dot{\xi}_s, \tilde{u}_{\xi_s}) \, ds = \tilde{u}_{\xi_s}(0).
\]

Invoking Cauchy-Lipschitz theorem for the Carathéodory equation, we conclude $u_{\xi_t}(-\tau) \leq \tilde{u}_{\xi_t}(-\tau)$ for all $\tau \in [0, s]$, which implies $u_{\xi_t}(-s) \leq \tilde{u}_{\xi_t}(-s) = u(\xi_t(-s)) = u(\xi_t(-s))$. Therefore, together with (3.5) we obtain
\[
    u_{\xi_t}(-s) \leq u(\xi_t(-s)) \leq \tilde{u}_{\xi_t}(-s), \quad \forall s \in [0, t]. \tag{3.6}
\]

It is obvious that the equalities in (3.6) hold if $\xi_t$ is a minimizer in (3.2).

**II. $u(x)$ is bounded above by the integral.** Now, suppose $\xi$ is a Lipschitz curve in $A_{\infty,x}^\pi$.

By solving the corresponding Carathéodory equation, we have that, for each $t \geq 0$
\[
    u(x) = e^{\int_{-t}^{0} L_u d\tau} u_{\xi}(-t) + \int_{-t}^{0} e^{\int_{-t}^{\tau} L_u d\tau} (L(\xi, \dot{\xi}, u_{\xi}) - u_{\xi} \cdot L_u(\xi, \dot{\xi}, u_{\xi})) \, d\tau.
\]

We claim that
\[
    u(x) \leq \liminf_{t \to \infty} \int_{-t}^{0} e^{\int_{-t}^{\tau} L_u d\tau} (L(\xi, \dot{\xi}, u_{\xi}) - u_{\xi} \cdot L_u(\xi, \dot{\xi}, u_{\xi})) \, d\tau. \tag{3.7}
\]

It is enough to show
\[
    \limsup_{t \to \infty} e^{\int_{-t}^{0} L_u d\tau} u_{\xi}(-t) \leq 0. \tag{3.8}
\]

Notice that $u_{\xi}(-t) \leq u(\xi(-t)) \leq A$ by (3.6) where $A = \max_{\xi \in M} u(x)$. Set $v = \sup_{t \leq 0, u \in \{-1, A+1\}} |L(\xi(s), \dot{\xi}(s), u)| + 1 < \infty$. If $\limsup_{t \to \infty} u_{\xi}(-t) \leq 0$, then (3.8) obviously holds. Otherwise, we can choose a sequence $t_k \to +\infty$ such that $u_{\xi}(-t_k) > 0$, $t_1 > 1/v$ and $t_{k+1} - t_k > 2/v$. We conclude
\[
    (-t_k - 1/v, -t_k + 1/v) \subset \{ t : -1 \leq u_{\xi}(-t) \leq A + 1 \}, \quad \forall k \in \mathbb{N}. \tag{3.9}
\]

Indeed, let $(-t^-, -t^+)$ be the maximal open interval containing $-t_k$ such that $-1 < u_{\xi}(-t) \leq A + 1$ for all $t \in (-t^-, -t^+)$. Then,
\[
    |u_{\xi}(-t^+) - u_{\xi}(-t_k)| \leq \int_{-t_k}^{-t^+} |L(\xi, \dot{\xi}, u_{\xi})| \, ds \leq v(t_k - t^+),
\]

and it follows
\[
    t_k - t^+ \geq \frac{1}{v} (u_{\xi}(-t_k) - u_{\xi}(-t^+)) > \frac{1}{v}
\]

if $t^+ \neq 0$, because $u_{\xi}(-t^+) = -1$. If $t^+ = 0$, then $t_k - t^+ = t_k > 1/v$. The case for $t^-$ can be treated similarly. This leads to (3.9). Let $\delta = \inf_{t \leq 0, u \in \{-1, A+1\}} |L_u(\xi(t), \dot{\xi}(t), u)|$. Due to (3.9), there holds
\[
    \int_{-t_k}^{0} L_u(\xi, \dot{\xi}, u_{\xi}) \, ds \leq -\frac{(2k - 1)\delta}{v}.
\]
Thus, we have \( \lim_{t \to \infty} \int_{-t}^{0} L_u(\xi, \hat{\xi}, u_\xi) \, ds = -\infty \) and \( \lim_{t \to \infty} e^{\int_{-t}^{0} L_u(\xi, \hat{\xi}, u_\xi) \, ds} = 0 \). Therefore, (3.8) also holds.

**III. The minimizer can be attained.** For all \( t > 0 \) let \( \xi^*_t \in \mathcal{A}^*_{t,x} \) be a minimizer for (3.2), then

\[
u(x) = u(\xi^*_t(\cdot - t)) + \int_{-t}^{0} L(\xi^*_t(s), \dot{\xi}^*_t(s), u_{\xi^*_t}(s)) \, ds
\]

by the equalities in (3.6). Fix \( t > 0 \) and define a functional \( J_t \),

\[
J_t(\xi) = u(\xi(\cdot - t)) + \int_{-t}^{0} L(\xi(s), \dot{\xi}(s), u(\xi(s))) \, ds
\]

for any \( \xi \in \Gamma_{t,x} = \{ \xi \in W^{1,\infty}([-t,0], M) : \xi(0) = x \}. \)

Now, take any sequence \( t_k > T \) such that \( \lim_{k \to \infty} t_k = +\infty \). Recall that \( \{\dot{\xi}^*_t\}_{t > 0} \) are uniformly bounded (see, for instance, [8]). By the Dunford-Pettis theorem there exists a subsequence, which we still denote by \( \{\dot{\xi}^*_t\} \), and a function \( \eta^* \in L^1_{\text{loc}}([-\infty,0], M) \) such that \( \dot{\xi}^*_t \rightarrow \eta^* \) in the weak-\( L^1_{\text{loc}} \) topology. By the Ascoli-Arzelà Theorem, we can also assume that the sequences \( s_{tk}^* \) and \( u(s_{tk}^*) \) converges uniformly on any compact interval to some Lipschitz function \( \xi^* \in \mathcal{A}^*_{\infty,x} \) and \( u(\xi^*) \) respectively. Observe that, for any \( \varphi \in C^1([-t,0], M) \),

\[
\int_{-t}^{0} \varphi \cdot \eta^* = \lim_{k \to \infty} \int_{-t}^{0} \varphi \cdot \dot{\xi}^*_{tk} = - \lim_{k \to \infty} \int_{-t}^{0} \dot{\varphi} \cdot \xi^*_{tk} = - \int_{-t}^{0} \dot{\varphi} \cdot \xi^*.
\]

Thus, we conclude \( \dot{\xi}^* \) \( = \eta^* \) almost everywhere by du Bois-Reymond lemma, i.e., \( \dot{\xi}^*_t \rightarrow \dot{\xi}^* \) in the weak-\( L^1_{\text{loc}} \) topology. Therefore, a classical result (see, for instance, [4, Theorem 3.6] or [3, Section 3.4]) on the sequentially lower semicontinuity implies

\[
\liminf_{k \to \infty} J_t(\xi^*_{tk}) \geq J_t(\xi^*).
\]

It follows

\[
u(x) \leq u(\xi^*_{t}(\cdot - t)) + \int_{-t}^{0} L(\xi^*_{t}(s), \dot{\xi}^*_{t}(s), u(\xi^*_{t}(s))) \, ds, \quad \forall t > 0.
\]

Denote by \( \xi^*_t \) the restriction of \( \xi^* \) on \([-t,0]\). Let \( \tilde{u}_\xi^* \) be defined as above such that \( \tilde{u}_\xi^*(t) = u(\xi^*_t(t)) \). Due to (3.2),(L6) and (3.6), it follows that

\[
u(x) \leq u(\xi^*_t(\cdot - t)) + \int_{-t}^{0} L(\xi^*_t(s), \dot{\xi}^*_t(s), \tilde{u}_\xi^*(s)) \, ds \leq u(\xi^*_{t}(\cdot - t)) + \int_{-t}^{0} L(\xi^*_{t}(s), \dot{\xi}^*_{t}(s), u(\xi^*_{t}(s))) \, ds.
\]

The combination of the two inequalities leads to

\[
u(x) = u(\xi^*_{t}(\cdot - t)) + \int_{-t}^{0} L(\xi^*_{t}(s), \dot{\xi}^*_{t}(s), u(\xi^*_{t}(s))) \, ds, \quad \forall t > 0.
\]

Notice that \( u(\xi^*_{t}(\cdot)) \) satisfies (3.4) on \([-t,0]\) with respect to \( \xi^* \) and it follows \( u_{\xi^*}(\cdot) \equiv u(\xi^*_{t}(\cdot)) \). Therefore,

\[
u(x) = u_{\xi^*}(\cdot - t) + \int_{-t}^{0} L(\xi^*_{t}(s), \dot{\xi}^*_{t}(s), u_{\xi^*}(s)) \, ds, \quad \forall t > 0.
\]
Solving the corresponding Carathéodory equation again, we obtain
\[
u(x) = e^{-\lambda t} L_0(x, \nu) - \lambda u \]
Noticing the fact that \( \xi^- \), \( \dot{\xi}^- \), \( u_x^- \) are uniformly bounded, we can find a constant \( \delta > 0 \) such that
\[
-\lambda \leq L_0(x, \xi, \dot{\xi}, u_x) \leq -\delta.
\]
by taking \( t \to \infty \) and applying Lebesgue’s theorem, we conclude that
\[
u(x) = \int_{-\infty}^{\infty} e^{-\lambda t} L_0(x, \xi, \dot{\xi}, u_x) \, dt.
\]
This completes the proof.

**Corollary 3.3** (Representation formula V) Suppose \( L \) satisfies condition (L1)-(L6) and \( H \) is the associated Hamiltonian. Then the unique viscosity solution \( u \) of \( (HJ_e) \) has the following representation formula: let \( x \in M \) and \( \xi^* \in A_{\infty, x}^* \) be a backward calibrated curve from \( x \) with \( u_{\xi^*} \) satisfying (3.1) with respect to \( \xi^* \) for all \( t > 0 \), then
\[
u(x) = \inf_{\xi \in A_{\infty, x}^*} \left\{ \liminf_{t \to \infty} \int_{-t}^{0} e^{-\lambda s} L_0(x, \dot{\xi}, u_{\xi^*}) \, ds \right\}.
\]
Moreover, there exists \( \xi^* \in A_{\infty, x}^* \) such that
\[
u(x) = \int_{-\infty}^{0} e^{-\lambda s} L_0(x, \dot{\xi}, u_{\xi^*}) \, ds.
\]

**Proof** The conclusion is direct from Theorem 3.1 and Theorem 2.6.

### 4 Concluding remarks

#### 4.1 Discounted system as an example

Throughout this section we set
\[
L(x, v, u) = L_0(x, v) - \lambda u
\]
with \( \lambda \in \mathbb{R} \), where \( L_0 \) is a Tonelli Lagrangian on \( TM \). Let \( H_0 \) be the associated Hamiltonian with respect to \( L_0 \).

**Corollary 4.1** Let \( L \) be a discounted Lagrangian defined in (4.1) with \( \lambda \in \mathbb{R} \) and \( H \) be the associated Hamiltonian. Then the unique viscosity solution \( u \) of \( (HJ_e) \) has the following representation:
\[
u(t, x) = \inf_{\xi} \left\{ e^{-\lambda t} \phi(\xi(-t)) + \int_{-t}^{0} e^{\lambda s} L_0(\xi(s), \dot{\xi}(s)) \, ds \right\}
\]
where the infimum is taken over the set of the absolutely continuous curve \( \xi : [-t, 0] \to M \) such that \( \xi(0) = x \).
Proof Applying Proposition 2.3, we have

\[
  u(t, x) = \inf_{\xi \in A_{t,x}} \left\{ e^{-\lambda t} \phi(\xi(0)) + \int_0^t e^{\lambda(s-t)} L_0(\xi(s), \dot{\xi}(s)) \, ds \right\}. 
\] (4.2)

By using the variable transformation \( \tau = s - t \) and \( \eta(\tau) = \xi(\tau + t) \), we complete our proof. \( \square \)

A consequence of Theorem 3.1 and Remark 3.2 leads to the following result. See also [16].

Corollary 4.2 Let \( L \) be a discounted Lagrangian defined in (4.1) with \( \lambda > 0 \) and \( H \) be the associated Hamiltonian. Then the unique viscosity solution \( u \) of \((HJ_\lambda)\) has the following representation:

\[
  u(x) = \inf_{\xi} \int_{-\infty}^0 e^{\lambda s} L_0(\xi(s), \dot{\xi}(s)) \, ds
\]
where the infimum is taken over the set of the curve \( \xi : (-\infty, 0] \to M \), which is absolutely continuous on each compact interval of \((-\infty, 0]\), such that \( \xi(0) = x \).

Fix \( \lambda \in \mathbb{R} \). From (4.2), we also have that

\[
  u(t, x) = e^{-\lambda t} \inf_{\xi \in A_{t,x}} \left\{ \phi(\xi(0)) + \int_0^t e^{\lambda(s-t)} L_0(\xi(s), \dot{\xi}(s)) \, ds \right\}.
\]

Set \( L_\lambda(t, x, v) = e^{\lambda t} L_0(x, v) \). Then, the associated Hamiltonian has the form \( H_\lambda(t, x, p) = e^{\lambda t} H_0(x, e^{-\lambda t} p) \). Therefore, \( v(t, x) = e^{\lambda t} u(t, x) \) is a viscosity solution of the Hamilton-Jacobi equation

\[
  \begin{cases}
    D_t v + e^{\lambda t} H_0(x, e^{-\lambda t} D_x v) = 0, & (t, x) \in (0, +\infty) \times M, \\
    v(0, x) = \phi(x), & x \in M,
  \end{cases}
\] (4.3)

if and only if \( u(t, x) \) is a viscosity solution of

\[
  \begin{cases}
    D_t u + \lambda u + H_0(x, D_x u) = 0, & (t, x) \in (0, +\infty) \times M, \\
    u(0, x) = \phi(x), & x \in M,
  \end{cases}
\]

Similarly, \( u \) is a viscosity solution of the stationary equation

\[
  \lambda u(x) + H_0(x, Du(x)) = 0, \quad x \in M
\]
if and only if \( v(t, x) = e^{\lambda t} u(x) \) is a viscosity solution of (4.3) with \( \phi = u \).

One can compare with the discussions in [13].

4.2 Concluding remarks

From the solving-ODE method used previously, we should have more comments on the representation formula for the viscosity solutions of the contact type Hamilton-Jacobi equations.

– Consider

\[
  \begin{cases}
    \dot{u}_\xi(s) = L(s, \xi(s), \dot{\xi}(s), u_\xi(s)) \quad a.e. \ s \in [0, t], \\
    u_\xi(0) = u_0.
  \end{cases}
\] (4.4)
Observe that
\[ L(s, \xi(s), \dot{\xi}(s), u_\xi(s)) = L(s, \xi(s), \dot{\xi}(s), 0) + \hat{L}_u(s) \cdot u_\xi(s) \quad (4.5) \]
where
\[ \hat{L}_u(s) = \int_0^1 L_u(s, \xi(s), \dot{\xi}(s), \lambda u_\xi(s)) \, d\lambda, \quad s \in [0, t]. \]
Therefore,
\[ u_\xi(t) = e^f_0 \hat{L}_u \, d\tau \cdot u_\xi(0) + \int_0^t e^f_0 \hat{L}_u \, d\tau \{ L(s, \xi(s), \dot{\xi}(s), u_\xi(s)) - F(s, \xi(s), \dot{\xi}(s), u_\xi(s)) \cdot u_\xi(s) \} \]
\[ + F(s, \xi(s), \dot{\xi}(s), u_\xi(s)) \cdot u_\xi(s), \]
where \( F \) is an arbitrary \( C^1 \) function, we obtain that
\[ u_\xi(t) = e^f_0 F \, d\tau \cdot u_\xi(0) + \int_0^t e^f_0 F \, d\tau \{ L(s, \xi(s), \dot{\xi}(s), u_\xi(s)) - F(s) u_\xi(s) \} \, ds \]
\[ = e^f_0 F \, d\tau \cdot u_\xi(0) + \int_0^t e^f_0 F \, d\tau \{ L(s, \xi(s), \dot{\xi}(s), 0) + (\hat{L}_u - F(s)) u_\xi(s) \} \, ds \]
where \( F(s) := F(s, \xi(s), \dot{\xi}(s), u_\xi(s)) \). This leads to another new representation formula for solutions of (HJ\( \xi \)). But, it is unclear if such a formula has some applications in the case that
\[ \hat{L}_u - F > 0, \quad F < 0, \]
especially studying the solutions of relevant stationary equations when \( L_u > 0 \).

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