Complete combinatorial characterization of greedy-drawable trees

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Abstract

A (Euclidean) greedy drawing of a graph is a drawing in which, for every pair of vertices \( s, t \) (\( s \neq t \)), there is a neighboring vertex of \( s \) that is closer to \( t \) than \( s \) in the Euclidean distance. Greedy drawings are crucial in the context of message routing in networks, and graph classes that admit greedy drawings have been actively investigated. Nölleburg and Prutkin (Discrete Comput. Geom., 58(3), pp. 543–579, 2017) characterized greedy-drawable trees in terms of an inequality system containing a non-linear equation. Using the characterization, they designed a linear-time recognition algorithm for greedy-drawable trees with maximum degree \( \leq 4 \). However, the possibility of a combinatorial characterization of greedy-drawable trees with maximum degree 5 was left open. In this paper, we present a combinatorial characterization of greedy-drawable trees with maximum degree 5, which leads to a complete combinatorial characterization of greedy-drawable trees. Furthermore, we present a characterization of greedy-drawable pseudo-trees.

1 Introduction

Geographic routing (or geometric routing) is a type of routing that uses the geographic coordinates of the nodes as addresses for the purpose of routing. One of the simplest routing algorithms is greedy routing, in which each node simply forwards a message to its neighbor that is closest to the destination. In (pure) greedy routing, however, a message delivery fails if the message is forwarded to a node with no neighbor that is closer to the destination. A graph drawing in which greedy routing is guaranteed to work is called a greedy drawing (or greedy embedding) [18]. That is, a (Euclidean) greedy drawing of a graph \( G \) is a drawing of \( G \) in which for every pair of vertices \( s, t \) (\( s \neq t \)) of \( G \), there is a neighboring vertex \( u \) of \( s \) with \( d(s, t) > d(u, t) \), where \( d \) is the Euclidean distance. This notion is motivated by work of Rao et al. [19], who proposed the idea of applying greedy routing based on virtual coordinates (i.e., a graph drawing) rather than geographic coordinates, which makes greedy routing applicable even in the absence of locational information. Through extensive experiments, the authors showed that their approach makes greedy routing much more reliable.

In [18], Papadimitriou and Ratajczak presented a remarkable conjecture that every 3-connected planar graph admits a greedy drawing. This conjecture gives a theoretical guarantee to the approach proposed by Rao et al. [19], and triggered intense research on the class of graphs that admit greedy drawings (greedy-drawable graphs). The conjecture of Papadimitriou and Ratajczak was first proved for triangulations by Dhandapani [9]. The conjecture itself was proved independently by Leighton and Moitra [15] and Angelini et al. [4]. Subsequently, Da Lozzo et al. [7] proved a stronger version of the conjecture, asserting the existence of a planar greedy drawing. As greedy-drawability is a monotonic graph property (that is, the property is preserved under edge addition), the class of greedy-drawable trees has also gained much attention. In [14], Leighton and Moitra investigated a sufficient condition in which a binary tree does not admit a greedy drawing. Nölleburg and Prutkin [16] characterized greedy-drawable trees in terms of an inequality system containing a non-linear equation. Investigating the feasibility of the inequality system,
they derived an explicit description of the greedy-drawable trees with maximum degree 3 and a linear-time recognition algorithm for greedy-drawable trees with maximum degree \( \leq 4 \). In the case of maximum degree 5, however, testing the feasibility of the inequality system is sometimes difficult, and neither an explicit description nor a recognition algorithm is known for greedy-drawable trees. Indeed, the authors of the paper [16] presented a tree with maximum degree 5 (p. 574, Fig. 21 (d)) whose greedy-drawability was unclear. Meanwhile, Kleinberg [14] proved that every tree has a greedy drawing in the hyperbolic plane.

Our contributions. In this paper, we present a complete combinatorial characterization of (Euclidean) greedy-drawable trees. First, we carefully analyze and clarify the work of Nöllenburg and Prutkin [16], and derive an explicit description of greedy-drawable trees with maximum degree \( \leq 4 \) (Proposition 4.1). Next, we derive an explicit description of greedy-drawable trees with maximum degree 5 (Theorem 5.1) by developing a new technique for determining the feasibility of the inequality system proposed in [16]. As the maximum degree of a greedy-drawable tree must be less than 6 [18], our results represent a complete combinatorial characterization of greedy-drawable trees. Furthermore, we study the greedy-drawability of pseudo-trees, i.e., graphs obtained by adding one edge to a tree. We provide a complete characterization of greedy-drawable pseudo-trees (Theorem 6.3). Using our characterizations of greedy-drawable trees and pseudo-trees, one can verify the greedy-drawability of a tree or pseudo-tree in linear time (Corollaries 7.1 and 7.2).

Related Work. In a greedy drawing of a graph, one can find a path between any two vertices by iteratively selecting a neighboring vertex closer to the destination. In this sense, greedy drawings can be viewed as graph drawings in which one can easily find a path between any two vertices. Such a property is desirable in many applications, and several other types of graph drawings have been proposed for helping path-finding tasks. Alamdari et al. [1] introduced self-approaching drawings, which satisfy a condition stronger than greedy drawings that every two vertices \( s \) and \( t \) are joined by an \( st \)-path such that \( d(b, c) < d(a, c) \) holds for any points (not necessarily vertices) \( a, b, c \) along the path, where \( d \) is the Euclidean distance. Self-approaching drawings have several advantages over greedy drawings. For example, the stretch factor, i.e., the maximum, over vertices \( s \) and \( t \), ratio between the length of a shortest \( st \)-path and the Euclidean distance \( d(s, t) \), is always at most 5.3332 [12] for self-approaching drawings, whereas the ratio can be arbitrarily large for greedy drawings. The authors of [1] characterized the trees that have self-approaching drawings. In terms of our characterization of greedy-drawable trees, those trees correspond to subdivisions of type-\( D_{k,l,n} \) trees with maximum degree 3 (see Section 3), which form a very special case of greedy-drawable trees.

Increasing-chord drawings [1] are an even more restricted class of drawings, in which there is always a path between two vertices that is monotone with respect to some direction. The authors of [1] proved that every biconnected graph and outer planar graph admits a monotone drawing. They also introduced the notion of strongly monotone drawings, which assumes the existence of an \( st \)-path between every pair of vertices \( s \) and \( t \) that is monotone with respect to the direction \( \vec{st} \). Kindermann et al. [13] proved that every tree admits a strongly monotone drawing.

Dehkhori et al. [8] introduced angle-monotone drawings (this name is given in [6]), which require a path between every pair of vertices that is \( x \)- and \( y \)-monotone after some rotation.

## 2 Preliminaries

Let \( G \) be a graph. A drawing \( \Gamma \) of \( G \) is called a straight-line drawing if every node is represented as a point in the plane and every edge as the line segment between its endpoints. We hereinafter assume that
all drawings are plane straight-line drawings, that is, straight-line drawings with no edge crossing. The drawing \( \Gamma \) is said to be greedy if, for every pair of vertices \( s, t (s \neq t) \), there exists a neighboring vertex \( u \) of \( s \) with \( d(u, t) < d(s, t) \), where \( d \) is a distance function. That is, the drawing \( \Gamma \) is greedy if, for every pair of vertices \( s, t (s \neq t) \), there exists a path \( v_0(=s), v_1, \ldots, v_m(=t) \) such that \( d(v_i, t) > d(v_{i+1}, t) \) for \( i = 1, \ldots, m - 1 \). Throughout this paper, we consider the case in which \( d \) is the Euclidean distance. If \( G \) has a greedy drawing, we say that \( G \) is greedy-drawable.

Let \( T \) be a tree. For an edge \( uv \) of \( T \), we consider the subtree \( T'_{uv} \) of \( T \) that contains \( u \) obtained by deleting \( uv \). We let \( \text{axis}(uv) \) be the perpendicular bisector of \( uv \), and consider the open half-plane \( h_{uv}^u \) bounded by \( \text{axis}(uv) \) that contains \( u \). Since the half-plane \( h_{uv}^u \) corresponds to the set of points that are closer to \( u \) than to \( v \), the following proposition holds.

**Proposition 2.1** ([16] Lemma 3.1, [16] Lemma 2.3)
A drawing \( \Gamma \) of a tree \( T \) is greedy if and only if every subtree \( T'_{uv} \) is contained in \( h_{uv}^u \) in \( \Gamma \).

Based on this proposition, Nöllenburg and Prutkin [16] developed the following strategy to determine the greedy-drawability of a tree \( T \). Let \( r \) be a vertex of \( T \), and \( v_0, \ldots, v_{d-1} \) be the neighbors of \( r \). We first decompose \( T \) into \( d \) subtrees \( T'_{rv_i} + rv_i \) \((i = 0, \ldots, d - 1)\). Then, we regard each tree \( T'_{rv_i} + rv_i \) as a rooted tree with root \( r \), and denote it as \( T_i = (V_i, E_i) \). We let \( \text{poly}(T_i) := \bigcap \{ h_{uv}^u \mid uv \in E_i, uv \neq rv_i, d_T(u, v) < d_T(u, r) \} \), where \( d_T \) is the graph distance, and let \( T_i \) be the star induced by \( r \) and its neighbors. Suppose we already have a greedy drawing of each \( T_i \) and want to construct a greedy drawing of \( T \) by combining them. Then, the task is to construct a drawing in which \( T_i \) is drawn greedily and each \( T_j \) \((j \neq i)\) is contained in \( \text{poly}(T_i) \). If \( \text{poly}(T_i) \) is unbounded with respect to the direction \( v_i \), then this task can be simplified; if this condition holds, one can transform the drawing of \( T_i \) into a drawing in which \( T'_{rv_i} \) is drawn infinitesimally small, where \( \text{poly}(T_i) \) is (arbitrarily close to) the open cone spanned by the two unbounded edges (see Figure 1 for an intuition). This observation leads to the notion of opening angles, which plays a fundamental role in classifying greedy-drawable trees.

**Definition 2.2** ([16] Definitions 2.8, 2.10, and 2.12, slightly modified)
Let \( \Gamma \) be a drawing of \( T \).

- If \( \text{poly}(T_i) \) is unbounded, we say that \( T_i \) is drawn with an open angle in \( \Gamma \). Let \( a_1b_1 \) and \( a_2b_2 \) be the edges of \( T \), where \( d_T(a_1, r) < d_T(b_1, r) \), such that \( \text{axis}(a_1b_1) \) and \( \text{axis}(a_2b_2) \) are the supporting lines of the two unbounded edges of \( \text{poly}(T_i) \). We define \( \angle T_i := h_{a_1b_1}^a \cap h_{a_2b_2}^a \) and refer to it as the opening angle of \( T_i \) in \( \Gamma \). We write \( |\angle T_i| = \alpha \) for every greedy drawing of \( T_i \), where the supremum is taken over all greedy drawings of \( T_i \).

- If \( \text{poly}(T_i) \) is bounded, we say that \( T_i \) is drawn with a closed angle in \( \Gamma \) and write \( |\angle T_i| < 0 \). If \( |\angle T_i| < 0 \) for every greedy drawing of \( T_i \), we write \( |\angle T_i| < 0 \).

Slightly perturbing a greedy drawing, one can always assume that \( |\angle T_i| > 0 \) or \( |\angle T_i| < 0 \). In what follows, we do not pay so much attention to the case \( |\angle T_i| = 0 \). Nöllenburg and Prutkin [16] proved that the drawing of \( T'_{rv_i} \) can always be shrunk while preserving greediness of \( T \) if \( |\angle T_i| > 0 \).

**Lemma 2.3** (shrinking lemma [16] Lemma 2.18)
Let \( T = (V, E) \) be a tree and \( T' = T'_{rv} + rv, rv \in E \), be a subtree of \( T \). If \( T \) has a greedy drawing \( \Gamma \) such that \( |\angle T'| > 0 \), then there is a greedy drawing \( \Gamma' \) such that \( T'_{rv} \) is drawn infinitesimally small, and the drawing of \( T'_{rv} \) and the angle \( |\angle T'| \) are the same as those in \( \Gamma \) (and \( \angle T' \) contains the original angle in \( \Gamma \)).

**Remark 2.4** For some technical reasons, the original definition of an opening angle is slightly more complicated. Indeed, Lemma 2.14 in [16], which claims that the apex of the opening angle \( \angle T_i \) of a subtree \( T_i \) is always contained in the opening angle \( \angle T_j \) of another subtree \( T_j \) in every greedy drawing of \( T' \), does not always hold under the present definition. This fact is important because the shrunk drawing
Figure 1: Drawings of $T_i$ (left: original drawing, right: shrunk drawing)

in Lemma 2.3 is constructed in [16] by placing an infinitesimally small drawing of $T_i$ on the apex of $\angle T_i$. However, once we accept Lemma 2.3 and consider such a drawing, the original and present definition coincide (with arbitrary precision). Throughout the remainder of this paper, we always consider such drawings, making the two definitions functionally equivalent.

Suppose that each $T_i$ can be drawn with an open angle, and let $\varphi_i := |\angle T_i|$. Based on Lemma 2.3, Nöllenburg and Prutkin [16, Theorem 4.4] proved that $T$ has a greedy drawing if and only if there is a permutation $\tau$ on $\{0, \ldots, d-1\}$ such that the following inequality system has a solution. For $i = 0, \ldots, d-1$,

\begin{align}
0 < \alpha_i, \beta_i, \gamma_i < 180, \\
\alpha_i + \beta_i + \gamma_i = 180, \alpha_0 + \cdots + \alpha_{d-1} = 360, \\
\sin(\beta_0) \cdots \sin(\beta_{d-1}) = \sin(\gamma_0) \cdots \sin(\gamma_{d-1}), \\
\beta_i < \alpha_i, \gamma_i < \alpha_i, \\
\beta_i + \gamma_{i+1} < \varphi_{\tau(i)} (i \text{ mod } d).
\end{align}

The conditions in the first three lines describe the possible angles $\alpha_i, \beta_i, \gamma_i$ of the wheel graph $W_d$, which appear in the work of Di Battista and Vismara [10]. Together with the conditions in the fourth line, they describe all possible angles in greedy drawings of $\tilde{T}$. Finally, the combined system describes all possible angles of greedy drawings of $T$ in which the subtrees $T_0, \ldots, T_{d-1}$ are drawn infinitesimally small with the order $T_{\tau(0)}, \ldots, T_{\tau(d-1)}$ around $r$. See Figure 2 for an intuition. We will see that, for any greedy-drawable tree $T$, a vertex $r$ can be chosen so that $|\angle T_0| > 0, \ldots, |\angle T_{d-1}| > 0$ (Lemma 3.4). Thus, the classification of greedy-drawable trees can be performed by the following two steps [16]:

1. Determine all rooted trees that can be drawn with open angles, and compute the supremum of opening angles of each tree. This step is discussed in Section 3.

2. Determine all possible combinations of rooted trees that can be glued at a single vertex by considering feasibility of the system (2.1). This step is discussed in Sections 4 and 5.

3 Rooted trees that can be drawn with open angles

Nöllenburg and Prutkin [16] introduced a procedure for computing the supremum of opening angles of a rooted tree. By analyzing their results, we present an explicit description of rooted trees that can be drawn with an open angle.

To describe the result, we introduce some definitions. Let $P$ be a path. A tree $T$ is called a degree-$k$ caterpillar associated with $P$ if $T$ is constructed by attaching at most $k-1$ leaves to each terminal vertex.
of $P$, and at most $k - 2$ leaves to each internal vertex of $P$. We call the number of degree-$k$ vertices the \textit{weight} of $T$. A leaf of $T$ that is adjacent to a terminal vertex of $P$ is called an \textit{end leaf} of $T$.

If a tree $T$ can be drawn with an opening angle $\varphi - \epsilon$ for any $\epsilon > 0$, but not $\varphi$, we indicate this information by writing $|\angle T|_\ast = \varphi^\circ$. As subdividing a tree does not affect the supremum of opening angles, we describe the result under the assumption that a tree does not contain any degree-2 vertices.

Here, we review results in [16]. Let $T$ be a rooted tree with degree-1 root. If all vertices of $T$ are within distance 2 from the root, i.e., if $T$ is a star, it is easy to determine $|\angle T|_\ast$. We have $|\angle T|_\ast = 180^\circ, (120^\circ)^-, (60^\circ)^-$ if $T$ is either a single edge, a triple, or a quadruple respectively. If $T$ contains a vertex of degree greater than 5, $T$ cannot be drawn with an open angle. If there is a vertex of degree greater than 2, one can compute $|\angle T|_\ast$ by applying the following lemma recursively. We remark that if $|\angle T|_\ast \neq 180^\circ$, then it means the tree $T$ contains a vertex of degree $\geq 3$, and thus $|\angle T|_\ast \leq 120^\circ$.

\textbf{Lemma 3.1} ([16] Lemmas 3.1–3.6) Let $T_i$, for $i = 1, 2, 3$, be a rooted tree with a degree-1 common root $r'$ that contains no degree-2 vertices. Suppose that each $T_i$ can be drawn with an open angle, and the supremum $\varphi_i := |\angle T_i|_\ast$ of opening angles is not equal to $180^\circ$.

(I) Let $T := T_1 + ar' + rr'$ be the rooted tree with root $r$, as depicted in Figure 3a. If $90^\circ < \varphi_1 < 120^\circ$, we have $|\angle T|_\ast = (45^\circ + \frac{\varphi_1}{2})^-$. If $\varphi_1 \leq 90^\circ$, we have $|\angle T|_\ast = \varphi_1^\circ$ (see Figures 4a and 4b for an intuition).

(II) Let $T := T_1 + ar' + br' + rr'$ be the rooted tree with root $r$, as depicted in Figure 3b. If $0^\circ < \varphi_1 \leq 120^\circ$, then we have $|\angle T|_\ast = (\frac{\varphi_1}{2})^-$ (see Figure 4c for an intuition).

(III) Let $T := T_1 + T_2 + rr'$ be the rooted tree with root $r$, as depicted in Figure 3c. If $90^\circ < \varphi_1, \varphi_2 \leq 120^\circ$, we have $|\angle T|_\ast = (\varphi_1 + \varphi_2 - 180^\circ)^-$. If $\varphi_1 \leq 90^\circ$ or $\varphi_2 \leq 90^\circ$, then $T$ cannot be drawn with an open angle (see Figure 4d for an intuition).

(IV) Let $T := T_1 + T_2 + ar' + rr'$ be the rooted tree with root $r$, as depicted in Figure 3d. If $90^\circ < \varphi_2 \leq \varphi_1 \leq 120^\circ$, we have $|\angle T|_\ast = (\frac{3}{8}\varphi_1 + \frac{5}{8}\varphi_2 - 112.5^\circ)^-$. If $\varphi_1 \leq 90^\circ$ or $\varphi_2 \leq 90^\circ$, then $T$ cannot be drawn with an open angle (see Figure 4e for an intuition).

(V) Let $T := T_1 + T_2 + T_3 + rr'$ be the rooted tree with root $r$, as depicted in Figure 3e. Then, $T$ cannot be drawn with an open angle.

For more details, see [16]. Using this lemma, we can prove the following proposition.

\textbf{Proposition 3.2} Let $T$ be a rooted tree with degree-1 root $r$ that contains no degree-2 vertices. Then, $T$ has a greedy drawing with an open angle if and only if $T$ satisfies one of the following conditions.

(A) $T$ is a single edge. In this case, we have $|\angle T|_\ast = 180^\circ$ and call $T$ a type-A tree.

(B) $T$ is a degree-3 caterpillar of weight $n \geq 1$ and $r$ is an end leaf of $T$. In this case, we have $|\angle T|_\ast = (90^\circ + 60^\circ \times \frac{1}{n})^-$ and call $T$ a type-$B_n$ tree. See Figure 6.
Figure 3: Rooted trees in Lemma 3.1

Figure 4: Optimal constructions for Cases (II)-(IV). Each subtree $T_{v_i r_i'}$ is drawn infinitesimally small. One can construct proper greedy drawings by perturbing these drawings. For more details, see [16].
(C) $T$ is a degree-4 caterpillar of weight $n \geq 1$ and $r$ is an end leaf of $T$. Let $v$ be the degree-4 vertex farthest from $r$ (with respect to the graph distance), and $k$ be the weight of the degree-3 path formed by the degree-3 vertices that are farther from $r$ than $v$. In this case, we have $|\angle T|_r = (120^\circ - \frac{1}{2\pi})^\times \frac{1}{3\pi}$ if $k = 0$ and $|\angle T|_r = (90^\circ + 60^\circ \times \frac{1}{2\pi})^\times \frac{1}{3\pi}$ otherwise. We call $T$ a type-$C_{k,n}$ tree. See Figure 6.

(D) $T$ is obtained by attaching a degree-4 caterpillar of weight $n \geq 0$ and two degree-3 caterpillars of weight $k,l \geq 1$ ($l \geq k$) to a single vertex $v$, called the joint vertex. $r$ is an end leaf of the attached degree-4 caterpillar that is farthest from $v$ (with respect to the graph distance). In this case, we have $|\angle T|_r = (60^\circ \times \frac{1}{2\pi} + 60^\circ \times \frac{1}{2\pi})^\times \frac{1}{3\pi}$ and call $T$ a type-$D_{k,l,n}$ tree. See Figure 7.

(E) $T$ is obtained by attaching a degree-4 caterpillar of weight $n \geq 0$ and two degree-3 caterpillars of weight $k,l \geq 1$ ($l \geq k$) to a single vertex $v$, called the joint vertex. $r$ is an end leaf of the attached degree-4 caterpillar that is farthest from $v$ (with respect to the graph distance). In this case, we have $|\angle T|_r = (45^\circ \times \frac{1}{2\pi} + 30^\circ \times \frac{1}{2\pi})^\times \frac{1}{3\pi}$ and call $T$ a type-$E_{k,l,n}$ tree. See Figure 8.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{first_example.png}
\caption{Type-$B_{n}$ tree}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{second_example.png}
\caption{Type-$C_{k,n}$ tree}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{third_example.png}
\caption{Type-$D_{k,l,n}$ tree}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fourth_example.png}
\caption{Type-$E_{k,l,n}$ tree}
\end{figure}

Proof. Let $r'$ be the neighboring vertex of $r$, and $T'$ the tree obtained from $T$ by removing $r$ and the leaves adjacent to $r'$. Let $t$ be the (graph) distance between $r$ and a farthest vertex of $T$ from $r$. We prove the following two claims separately.

- Each tree $T$ in Cases (A)–(E) has a greedy drawing with an open angle, and the supremum of opening angles is as described in the proposition.

- If $T$ has a greedy drawing with an open angle, then $T$ must be one of the trees that appear in Cases (A)–(E).

We first prove the first claim by induction on $t$. Suppose $T$ is a tree that appears in one of Cases (A)–(E). If $t = 2$, then $T$ must be a type-$A, B_{1}$, or $C_{0,1}$ tree, and the claim is clearly true. Now we consider the case $t > 2$ by considering each type separately.

Case 1: $T$ is the type-$B_{n}$ tree. In this case, $T'$ is the type-$B_{n-1}$ tree. By induction hypothesis, $T'$ has a greedy drawing with an open angle, and $|\angle T'|_r = (90^\circ + 60^\circ \times \frac{1}{2\pi})^\times \frac{1}{3\pi}$. Therefore, $T$ has a greedy drawing with an open angle, and we have $|\angle T|_r = (90^\circ + 60^\circ \times \frac{1}{2\pi})^\times \frac{1}{3\pi}$ by Lemma 3.1 (I).

Case 2: $T$ is a type-$C_{k,n}$ tree. Let us first assume that the degree of $r'$ is 3. Then, the tree $T'$ is a type-$C_{k,n}$ tree. By induction hypothesis, we have $|\angle T'|_r = (120^\circ - \frac{1}{2\pi})^\times \frac{1}{3\pi}$ if $k = 0$ and $|\angle T'|_r = (90^\circ + 60^\circ \times \frac{1}{2\pi})^\times \frac{1}{3\pi}$ otherwise. By Lemma 3.1 (I), $T$ has a greedy drawing with an open angle, and $|\angle T|_r = |\angle T'|_r$, which implies that $|\angle T|_r$ is as described in the proposition. We next assume that the degree of $r'$ is 4. Then, if $n \geq 2$, $T'$ is a type-$C_{k,n-1}$ tree, and we have $|\angle T'|_r = (120^\circ - \frac{1}{2\pi})^\times \frac{1}{3\pi}$ if $k = 0$, and $|\angle T'|_r = (90^\circ + 60^\circ \times \frac{1}{2\pi})^\times \frac{1}{3\pi}$ otherwise, by induction hypothesis. If $n = 1$, then $T'$ is a type-$B_{k}$
tree, and we have IS* = (90° + 60° × 1 2r ) by induction hypothesis. By Lemma 3.1(II), T has a greedy
drawing with an open angle in every case, and IS = 1 2IS , which implies IS* = (120°) × 1 2r if
k = 0 and IS* = (90° + 60° × 1 2r ) − 1 2r otherwise. Since the degree of r must be either 3 or 4, the
claim is true in Case 2.

Case 3: T is a type-Dk,l,n tree. We first assume that r is the joint vertex (and thus n = 0). Then,
T is the tree in Case (III) where T1 and T2 are the type-Bk and type-Bl trees respectively. By induction
hypothesis, we have |ST| = (90° + 60° × 1 2r ) by induction hypothesis and Lemma 3.1(III). Next, we
case the claim that r is not the joint vertex. Then, T is either a type-Dk,l,n tree (if the degree of r is
3) or a type-Dk,l,n−1 tree (if the degree of r is 4). By induction hypothesis and Lemma 3.1(I) and (II),
in either case, T has a greedy drawing with an open angle, and we have IS* = (60° × 1 2r + 60° × 1 2r ) × 1 2r .
Since the degree of r must be either 3 or 4, the claim is true in Case 3.

Case 4: T is a type-Ek,l,n tree. If r is the joint vertex (and thus n = 0), then T is the tree in Case (IV)
where T1 and T2 are type-Bk and type-Bl trees respectively. Thus, we have IS* = 3 2 (90° + 60° × 1 2r ) + 1 2 × (90° + 60° × 1 2r ) = 45° × 1 2r + 30° × 1 2r by induction hypothesis and Lemma 3.1(IV). Next, we
case the claim that r is not the joint vertex. Then, T is either a type-Ek,l,n tree (if the degree of r is
3) or a type-Ek,l,n−1 tree (if the degree of r is 4). By induction hypothesis and Lemma 3.1(I) and (II),
in either case, T has a greedy drawing with an open angle, and we have IS* = (45° × 1 2r + 30° × 1 2r ) × 1 2r .
Since the degree of r must be either 3 or 4, the claim is true in Case 4.

Therefore, the first claim is proved for all cases. Next, we prove the second claim by induction on t.
Suppose that T has a greedy drawing with an open angle. If t = 2, then T must be a single edge (type-A
tree), a triple (type-B1 tree), or a quadruple (type-C0,1 tree), and hence the claim is true. Now we assume
that t > 2. By Lemma 3.1 T must be one of the forms in Case (I) (IV) in Lemma 3.1. We consider each
case separately.

Case 1: T is of the form in Case (I). By induction hypothesis, T1 must be one of the trees appearing
in Cases (B) (E). Suppose T1 is the type-Bn tree. Then, T is the type-Bn+1 tree. If T1 is a type-Ck,n,
Dk,l,n, or Ek,l,n tree, then T is also a type-Ck,n, Dk,l,n, or Ek,l,n tree respectively. Therefore, in every
case, T is one of the trees appearing in Cases (B) (E).

Case 2: T is of the form in Case (II). By induction hypothesis, T1 must be one of the trees appearing
in Cases (B) (E). Suppose T1 is the type-Bn tree. Then, T is the type-Cn,n tree. If T1 is a type-Ck,n,
Dk,l,n, or Ek,l,n tree, then T is a type-Ck,n+1, Dk,l,n+1, or Ek,l,n+1 tree respectively. Thus, in every case,
T is one of the trees appearing in Cases (B) (E).

Case 3: T is of the form in Case (III) or (IV). Since ST1, ST2 > 90°, subtrees T1 and T2
must be type-Bn and type-Bn0 trees for some n, n0 ≥ 1 by induction hypothesis. Thus, T is a type-
Dn,n0,0 tree in Case (III) and a type-En,n0,0 tree in Case (IV). Thus, in every case, T is one of the trees
appearing in Cases (D) (E).

Therefore, the second claim is proved for all cases.

With this proposition, we can completely determine the rooted trees that have greedy drawings with
opening angles greater than 7.5°. The classification is described in Table

For I ⊂ [0°, 180°], we say that a rooted tree T has angle type I if the supremum of opening angles of
T is in the range I. For any I ⊂ (7.5°, 180°], we can obtain an explicit description of the rooted trees
of angle type I using Table

For example, the root trees having angle type [45°, 90°] are Ck,1 (k ≥ 0),
D0,1,0, D1,2,0. We denote by Φ the set of the suprema of opening angles of rooted trees, and by Φ the
topological closure of Φ.

Nöllenburg and Prutkin [16] Proposition 6.2 implicitly proved that any greedy-drawable tree can be
constructed by adjoining some rooted trees that can be drawn with open angles. Here, we make this
fact more explicit. Let T1, T2 be subtrees of a tree T with degree-1 roots r1, r2, and v1, v2 be the
neighbors of r1, r2 respectively. The subtrees T1 and T2 are said to be independent if T2 \ {v2} ⊂ T1

v1,r1
| $|\angle T|_s$ | $A$ | $B_1$ | $B_2$ | $C_{0,1}, C_{1,1}, D_{1,1,0}$ | $D_{1,2,0}$ | $D_{1,3,0}, E_{1,1,0}$ | $D_{1,1,0} (l \geq 4)$ | $D_{1,1,1} (l \geq 3), D_{1,1,2} E_{2,3,0}, E_{1,2,1}$ | $C_{k,2} (k \geq 2), E_{1,1,0} (l \geq 3)$ | $C_{k,3} (k \geq 3), E_{2,1,0} (l \geq 5), E_{1,1,1} (l \geq 4)$ | $D_{3,1,0} (l \geq 4), D_{2,1,1} (l \geq 3), D_{1,1,2} (l \geq 2), E_{3,3,0}, E_{2,2,1}, E_{1,1,2}$ |
|-----------------|--------------|-------------|-------------|-------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $45^\circ \leq |\angle T|_s \leq 60^\circ$ | | | | | | | | | | | | |
| $30^\circ \leq |\angle T|_s \leq 33.75^\circ$ | | | | | | | | | | | | |
| $22.5^\circ \leq |\angle T|_s \leq 26.25^\circ$ | | | | | | | | | | | | |
| $15^\circ \leq |\angle T|_s \leq 22.5^\circ$ | | | | | | | | | | | | |
| $|\angle T|_s = (13.125)^\circ$ | | | | | | | | | | | | |
| $11.25^\circ \leq |\angle T|_s \leq 13.125^\circ$ | | | | | | | | | | | | |
| $7.5^\circ \leq |\angle T|_s \leq 11.25^\circ$ | | | | | | | | | | | | |

Table 1: Classification of rooted trees with the supremum of opening angles $> 7.5^\circ$

and $T_1 \setminus \{r_1\} \subset T'_2 \setminus v'_1 \setminus v'_2$. (It may be the case that $v'_1 = r'_2$ and $v'_2 = r'_1$.) We use the following fact, proved in [16].

**Lemma 3.3** ([16] Lemma 2.15) Let $T$ be a greedy-drawable tree and $T'_1, T'_2$ be independent subtrees of $T$. In any greedy drawing of $T$, we have $|\angle T'_1| > 0$ or $|\angle T'_2| > 0$.

For an edge $uv$ of a tree $T$, we let $\tilde{T}^u_{uv} := T^u_{uv} + uv$ and regard $\tilde{T}^u_{uv}$ as a rooted tree with root $u$.

**Lemma 3.4** Let $T = (V(T), E(T))$ be a greedy-drawable tree. Then, there exists a vertex $u \in V(T)$ such that $|\angle T^u_{uv}| > 0$ for every neighboring vertex $v$ of $u$.

**Proof.** For contradiction, we assume that, for any vertex $u$ of $T$, there exists a neighboring vertex $v$ of $u$ with $|\angle T^u_{uv}| \leq 0$. Consider a directed graph $D(T)$ with the vertex set $V(T)$ such that $(u, v) \in V(T) \times V(T)$ is an arc if and only if $uv \in E(T)$ and $|\angle T^u_{uv}| \leq 0$. By the assumption, there exists an edge outgoing from $w$ for any vertex $w$ in $D(T)$. Thus, a directed cycle must exist in $D(T)$. Since $T$ is a tree, the directed cycle must contain a directed cycle of length 2. If we denote such a cycle by $a \rightarrow b \rightarrow a$, then we have $|\angle T^u_{ab}| \leq 0$ and $|\angle T^u_{ab}| \leq 0$. By Lemma 3.3, this is a contradiction. \[Q.E.D.\]

## 4 Greedy-drawable trees with maximum degree $\leq 4$

Nöllenburg and Prutkin [16] provided a simple characterization of greedy-drawable trees with maximum degree $\leq 4$ in terms of opening angles and a linear-time recognition algorithm based on the characterization. In this section, we present an explicit description of greedy-drawable trees with maximum degree $\leq 4$, based on the results of Nöllenburg and Prutkin [16] and those in Section 3.

Let $T$ be a greedy-drawable tree with maximum degree $\leq 4$. By Lemma 3.4, there is a vertex $r$ of $T$ with degree $d \leq 4$ such that the subtrees $T_0 := T_{v_0} + r_{v_0}, \ldots, T_{d-1} := T_{v_{d-1}} + r_{v_{d-1}}$ can be drawn with open angles, where $v_0, \ldots, v_{d-1}$ are the neighbors of $r$. Applying Lemma 2.3 to a greedy drawing of $T$, we construct a drawing of $T$ in which each $T_i$ is infinitesimally small and each $\angle T_i$ contains the original angle. In this drawing, each $\angle T_i$ must contain the $d$-gon formed by the vertices $v_0, \ldots, v_{d-1}$, and thus
each apex of $\angle T_j$ ($i \neq j$). Therefore, the following inequality holds:

$$\sum_{i=0}^{d-1} |\angle T_i| > (d - 2)180^\circ. \quad (4.1)$$

Nöllenburg and Prutkin [16] Lemmas 4.5–4.7 proved a beautiful fact that $T$ is greedy-drawable if and only if the above inequality holds. Using Proposition 3.2, we can classify the possible combinations of angle types of $T_i$’s. Combining the results for $d = 2, 3, 4$, we obtain the following proposition.

**Proposition 4.1** A tree $T$ with maximum degree $d \leq 4$ is greedy-drawable if and only if $T$ is a subgraph of a subdivision of a tree obtained by joining four trees $T_0, \ldots, T_3$ that have angle types listed in Table 2 at a single vertex; that is, $T$ is a subgraph of a subdivision of a tree obtained by combining four trees $T_0, \ldots, T_3$ as described in Table 3.

| $T_0$ | $T_1$ | $T_2$ | $T_3$ |
|-------|-------|-------|-------|
| 180°  | 180°  | 0°, 180° | 0°, 180° |
| 180°  | 120°  | 60°   | 0°, 60° |
| 180°  | 120°  | 52.5° | (5.5°, 52.5°) |
| 180°  | 120°  | 48.75° | (11.25°, 48.75°) |
| 180°  | 120°  | (45°, 46.875°) | (15°, 46.875°) |
| 180°  | 120°  | 45°   | (15°, 45°) |
| 180°  | 120°  | 37.5° | (22.5°, 37.5°) |
| 180°  | 120°  | (30°, 33.75°) | (30°, 33.75°) |
| 180°  | (90°, 105°) | (90°, 105°) | (0°, 105°) |
| 180°  | 105°  | 60°   | (15°, 60°) |
| 180°  | 105°  | 52.5° | (22.5°, 52.5°) |
| 180°  | 105°  | (45°, 48.75°) | (30°, 48.75°) |
| 180°  | 105°  | 45°   | (30°, 45°) |
| 180°  | 97.5° | 60°   | (22.5°, 60°) |
| 180°  | 97.5° | 52.5° | (30°, 52.5°) |
| 180°  | 97.5° | (45°, 48.75°) | (37.5°, 48.75°) |

| $T_0$ | $T_1$ | $T_2$ | $T_3$ |
|-------|-------|-------|-------|
| 180°  | (90°, 97.5°) | 45°   | 45°   |
| 180°  | (90°, 93.75°) | 60°   | (30°, 60°) |
| 180°  | (90°, 93.75°) | 62.5° | (37.5°, 62.5°) |
| 180°  | (90°, 93.75°) | 45°   | (45°, 48.75°) |
| 180°  | (90°, 93.75°) | 45°   | (45°, 48.75°) |
| 180°  | (90°, 93.75°) | 45°   | (45°, 48.75°) |
| 180°  | (90°, 93.75°) | 45°   | (45°, 48.75°) |
| 180°  | (90°, 93.75°) | 45°   | (45°, 48.75°) |
| 180°  | (90°, 93.75°) | 45°   | (45°, 48.75°) |
| 180°  | (90°, 93.75°) | 45°   | (45°, 48.75°) |

**Table 2:** Angle types of subtrees $T_0, \ldots, T_3$ of greedy-drawable trees with maximum degree 4
Proof. We give a proof for the case that $|\angle T_0|_* = 180^\circ$ and $|\angle T_1|_* \geq 120^\circ$. Repeating the same discussion, we can prove the proposition. We first assume that $|\angle T_1|_* = 180^\circ$, i.e., $T$ is the type-$A$ tree. Then, the inequality (4.1) holds if and only if $|\angle T_2|_* > 0^\circ$ and $|\angle T_3|_* > 0^\circ$, i.e., each of the trees $T_2$ and $T_3$ is a subgraph of a type-$E_{k,l,n}$ tree for some $k,l,n$. This result corresponds to the result in the first row of Table 3. Next, let us assume that $|\angle T_1|_* < 180^\circ$. Then, we have $|\angle T_1|_* = 120^\circ$ by the result in Table 1 which implies that $T_1$ is the type-$B_1$ tree. We consider several cases separately.

**Case 1:** $60^\circ \leq |\angle T_2|_* \leq 120^\circ$.

In this case, $T_2$ is a subgraph of a type-$C_{1,1}$ or $D_{1,1,0}$ tree. (Here, we note that the type-$B_n$ tree is a subgraph of some type-$C_{1,1}$ tree.) The inequality (4.1) holds if and only if $|\angle T_3|_* > 0^\circ$, i.e., $T_3$ is a subgraph of a type-$E_{k,l,n}$ tree for some $k,l,n$. Thus, we obtain the result in the second row of Table 3.

**Case 2:** $52.5^\circ \leq |\angle T_2|_* < 60^\circ$.

In this case, we have $|\angle T_2|_* = 52.5^\circ$ by the result in Table 1 and thus $T_2$ is a type-$C_{2,1}$ tree. The inequality (4.1) holds if and only if $|\angle T_3|_* > 7.5^\circ$, i.e., $T_3$ is a subgraph of the following type trees: $C_{k,3}, D_{3,l,0}, D_{2,l,1}, D_{1,l,2}, E_{3,3,0}, E_{2,2,1}, E_{1,1,2}, E_{2,1,0}, E_{1,1,1}$. This corresponds to the result in the third row of Table 3 (Note that a type-$D_{k,l,n}$ tree is a subgraph of some type-$E_{k,l,n}$ tree, and a type-$C_{k,n}$ tree is a subgraph of some type-$D_{1,1,1}$ tree.)

**Case 3:** $48.75^\circ \leq |\angle T_2|_* < 52.5^\circ$.

In this case, we have $|\angle T_2|_* = 48.75^\circ$ by the result in Table 1 and thus $T_2$ is a type-$C_{3,1}$ tree. The inequality (4.1) holds if and only if $|\angle T_3|_* > 11.25^\circ$, i.e., $T_3$ is a subgraph of one of the following type

| $T_0$ | $T_1$ | $T_2$ | $T_3$ |
|-------|-------|-------|-------|
| $A$   | $A$   | $E_{k,n}$ | $E_{k,n}$ |
| $A$   | $B_1$ | $C_{2,1}$ | $C_{2,1}$ |
| $A$   | $B_1$ | $C_{3,1}$ | $C_{3,1}$ |
| $A$   | $B_1$ | $C_{k,1}$ | $C_{k,1}$ |
| $A$   | $B_1$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_1$ | $D_{1,2,0}, E_{1,1,0}$ | $D_{1,2,0}, E_{1,1,0}$ |
| $A$   | $B_1$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_1$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_1$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |
| $A$   | $B_n$ | $D_{1,2,0}$ | $D_{1,2,0}$ |

Table 3: Subtrees $T_0, \ldots, T_3$ of maximal greedy-drawable trees with maximum degree 4
trees: $C_{k,3}, D_{3,3,0}, D_{2,2,1}, D_{1,1,2}, E_{2,1,0}, E_{1,1,1}$. This corresponds to the result in the fourth row of Table 3.

**Case 4:** $45^\circ \leq |\angle T_2|_s < 48.75^\circ$.

In this case, we have $45^\circ \leq |\angle T_2|_s < 46.875^\circ$ by the result in Table 1 and thus $T_2$ is a type-$C_{k,1}$ tree for some $k \geq 4$. Since $|\angle T_3|_s > 13.125^\circ$ is satisfied only for trees $T_3$ with $|\angle T_2|_s \geq 15^\circ$ (see Table 1), the inequality (4.1) holds if and only if $|\angle T_3|_s \geq 15^\circ$, i.e., $T_3$ is a subgraph of one of the following type trees: $C_{1,3}, D_{3,3,0}, D_{2,2,1}, D_{1,1,2}, D_{1,1,1}, D_{2,2,0}, E_{2,3,0}, E_{1,2,1}, E_{1,1,0}$. This corresponds to the result in the fifth row of Table 3.

**Case 5:** $37.5^\circ \leq |\angle T_2|_s < 45^\circ$.

In this case, we have $|\angle T_2|_s = 37.5^\circ$ (see Table 1), and thus $T_2$ is a type-$D_{1,1,0}$ or $E_{1,1,0}$ tree. The inequality (4.1) holds if and only if $|\angle T_3|_s > 22.5^\circ$, i.e., $T_3$ is a subgraph of one of the following type trees: $C_{k,2}, D_{2,2,0}, D_{1,1,1}, E_{1,1,0}$. This corresponds to the result in the sixth row of Table 3.

**Case 6:** $30^\circ \leq |\angle T_2|_s < 37.5^\circ$.

In this case, we have $30^\circ \leq |\angle T_2|_s \leq 33.75^\circ$ (see Table 1), and thus $T_2$ is a type-$D_{1,1,0}$ tree for some $l \geq 4$. Since $|\angle T_3|_s > 26.25^\circ$ is satisfied only for trees $T_3$ with $|\angle T_3|_s \geq 30^\circ$, the inequality (4.1) holds if and only if $|\angle T_3|_s \geq 30^\circ$, i.e., $T_3$ is a subgraph of one of the following type trees: $C_{1,2}, D_{2,2,0}, D_{1,1,1}, D_{1,1,0}, E_{1,2,0}$. This corresponds to the result in the seventh row of Table 3.

**Case 7:** $|\angle T_2|_s < 30^\circ$.

Since $|\angle T_3|_s \leq |\angle T_2|_s < 30^\circ$, the inequality (4.1) does not hold for any choice of trees $T_2$ and $T_3$.

By repeating the same discussion, we can prove the proposition.

5 Greedy-drawable trees with maximum degree 5

This section presents an explicit description of greedy-drawable trees with maximum degree 5. Unlike the case of maximum degree $\leq 4$, greedy drawability cannot be characterized by the inequality (4.1) and a complete combinatorial characterization is left open in [13].

Let $T$ be a tree with maximum degree 5. Take a degree-5 vertex $r$, and let $v_0, \ldots, v_4$ be the neighbors of $r$, and consider the rooted trees $T_i := T_i^v + rv_i$ ($i = 0, \ldots, 4$). If $T$ can be drawn greedily, each $T_i$ must satisfy $|\angle T_i|_s > 0$. Indeed, if $|\angle T_j|_s \leq 0$ for some $j$, then the digraph $D(T)$, defined in the proof of Lemma 5.4 contains a directed cycle $v_j \rightarrow v_j \rightarrow r$, which is a contradiction. Thus, $T$ is greedy-drawable if and only if $|\angle T_i|_s > 0$ for each $i = 0, \ldots, 4$, and the inequality system (2.1) is feasible for some permutation $\tau$ of $\{0, 1, \ldots, 4\}$. We prove the following theorem:

**Theorem 5.1** Let $T$ be a tree with degree-5 vertex $r$. Let $v_0, \ldots, v_4$ be the neighboring vertices of $r$, and $T_i := T_i^{v_i} + rv_i$ for $i = 0, \ldots, 4$. The tree $T$ is greedy-drawable if and only if the subtrees $T_0, \ldots, T_4$ are subgraphs of subdivisions of trees listed in Table 4.

| $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $A$   | $A$   | $A$   | $A$   | $A$   | $A$   | $A$   | $A$   | $A$   | $A$   |
| $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
| $B_6$ | $B_7$ | $B_8$ | $B_9$ | $B_{10}$ | $B_6$ | $B_7$ | $B_8$ | $B_9$ | $B_{10}$ |
| $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ |
| $B_6$ | $B_7$ | $B_8$ | $B_9$ | $B_{10}$ | $B_6$ | $B_7$ | $B_8$ | $B_9$ | $B_{10}$ |

Table 4: Subtrees $T_0, \ldots, T_4$ of maximal greedy-drawable trees with maximum degree 5
We prove Theorem 5.1 by distinguishing the following two cases.

5.1 Case 1: $\varphi_0 \neq 180^\circ$ or $\varphi_0 = \varphi_1 = 180^\circ$

In this case, the conditions for greedy drawability are given in [16], which are summarized as follows:

- If $\varphi_0 \neq 180^\circ$, $T$ is greedy-drawable if and only if $\varphi_1, \ldots, \varphi_4 > 0$ and $\sum_{i=0}^{4} \varphi_i > 540^\circ$ ([16 Lemma 4.6]).
- If $\varphi_0 = \varphi_1 = \varphi_2 = 180^\circ$, $T$ is greedy-drawable if and only if $\varphi_3, \varphi_4 > 0$ and $\varphi_3 + \varphi_4 > 120^\circ$ ([16 Lemmas 4.8 and 4.9]).
- If $\varphi_0 = \varphi_1 = 180^\circ$ and $\varphi_2 \neq 180^\circ$, $T$ is greedy-drawable if and only if $\varphi_2, \varphi_3, \varphi_4 > 0$ and $\varphi_2 + \varphi_3 + \varphi_4 > 240^\circ$ ([16 Lemma 4.10]).

Using these results and Proposition 3.2, we can explicitly describe the possible angle types of $T_0, \ldots, T_4$ as listed in Table 5.

| $T_0$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ |
|-------|-------|-------|-------|-------|
| $180^\circ$ | $180^\circ$ | $120^\circ, 180^\circ$ | $120^\circ, 180^\circ$ | $0^\circ, 180^\circ$ |
| $180^\circ$ | $180^\circ$ | $120^\circ, 180^\circ$ | $105^\circ$ | $(15^\circ, 105^\circ)$ |
| $180^\circ$ | $180^\circ$ | $120^\circ, 180^\circ$ | $97.5^\circ$ | $(22.5^\circ, 97.5^\circ)$ |
| $180^\circ$ | $180^\circ$ | $120^\circ, 180^\circ$ | $(90^\circ, 93.75^\circ)$ | $30^\circ, 93.75^\circ$ |
| $180^\circ$ | $180^\circ$ | $105^\circ$ | $105^\circ$ | $(30^\circ, 105^\circ)$ |
| $180^\circ$ | $180^\circ$ | $97.5^\circ$ | $97.5^\circ$ | $(45^\circ, 97.5^\circ)$ |
| $180^\circ$ | $180^\circ$ | $97.5^\circ$ | $(90^\circ, 93.75^\circ)$ | $(52.5^\circ, 93.75^\circ)$ |
| $180^\circ$ | $180^\circ$ | $(90^\circ, 93.75^\circ)$ | $(90^\circ, 93.75^\circ)$ | $(60^\circ, 93.75^\circ)$ |
| $120^\circ$ | $120^\circ$ | $120^\circ$ | $(90^\circ, 120^\circ)$ | $(90^\circ, 120^\circ)$ |
| $120^\circ$ | $120^\circ$ | $105^\circ$ | $105^\circ$ | $(90^\circ, 105^\circ)$ |

Table 5: Possible angle types of $T_0, \ldots, T_4$ ($\varphi_0 = \varphi_1 = 180^\circ$ or $\varphi_0 \leq 120^\circ$)

5.2 Case 2: $\varphi_0 = 180^\circ$ and $\varphi_1, \ldots, \varphi_4 \neq 180^\circ$

Characterizing greedy-drawable trees in this case was left open in [16]. We resolve this case by proving the following proposition.

Proposition 5.2 Suppose that $\varphi_0 = 180^\circ$ and $\varphi_1, \ldots, \varphi_4 \neq 180^\circ$. Then, the tree $T$ is greedy-drawable if and only if $T_0, \ldots, T_4$ have one of the angle types listed in Table 6.

Proof. We prove Proposition 5.2 with assistance of computer. To prove the if-part, it suffices to verify feasibility of the system (2.1) (for some permutation $\tau$) for each vector $(\varphi_0, \ldots, \varphi_4)$ listed as follows:

Case I: $(180,120,120,120,33.75)$, Case II: $(180,120,120,105,45)$,
Case III: $(180,120,120,97.5, 46.875)$, Case IV: $(180,120,120,93.75,48.75)$,
Case V: $(180,120,120,90,52.5)$, Case VI: $(180,120,105,93.75, 60)$,
Case VII: $(180,120,90,90, 90)$, Case VIII: $(180,105,97.5,90, 90)$,
Case IX: $(180,105,93.75,93.75, 90)$, Case X: $(180,105,93.75,91.875, 91.875)$,
Case XI: $(180,97.5, 97.5, 97.5, 90.9375)$. Since the system (2.1) contains a non-linear equation, it is often hard to find a concrete solution. To prove feasibility, we use the intermediate value theorem (as done in [16]). Let $S_{\tau}(\varphi_0, \ldots, \varphi_4)$ be the set
In Rational LP Solver [5], we verified emptiness of vectors \((\beta_0, \ldots, \beta_4, \gamma_0, \ldots, \gamma_4)\) that satisfy the following inequalities:

\[
0 < \beta_i, \gamma_i < 180, \\
2\beta_i + \gamma_i < 180, \beta_i + 2\gamma_i < 180, \\
\beta_i + \gamma_{i+1} < \varphi_{\tau(i)}, \\
\sum_{i=0}^4 (\beta_i + \gamma_i) = 540.
\]  

This system is obtained by eliminating \(\alpha_i\)'s from the system (2.1) and omitting the non-linear equation. When \(\tau\) and \(\varphi_0, \ldots, \varphi_4\) are clear from the context, we simply denote the solution set by \(S\). We then let \(\omega(\beta_0, \ldots, \beta_4, \gamma_0, \ldots, \gamma_4) := \prod_{i=0}^4 \sin(\beta_i) - \prod_{i=0}^4 \sin(\gamma_i)\). To verify feasibility of the original system (2.1), it suffices to find a pair of vectors \((x_+, x_-)\) such that \(x_+, x_- \in \bar{S}\) and \(\omega(x_+) > 0, \omega(x_-) < 0\), where \(\bar{S}\) is the topological closure of \(S\). Vectors \(x_+, x_-\) that prove greedy drawability for Cases I–IX are listed in Table 7. We found those vectors by trial and errors; we computed vertices of the solution set \(S\) (by solving linear programs with certain objective functions) and selected ones that take positive and negative values. To obtain exact solutions to linear programs, we used QSopt-ex Rational LP Solver [5], developed by Applegate, Cook, Dash, and Espinoza. For those vectors, we determined the values of the function \(\omega\) to be positive or negative. Since the function \(\omega\) contains trigonometric functions, it is difficult to compute exact values. To verify that \(\omega\) is negative or positive for the vectors in Table 7, we employed MPFI library [20] for arbitrary precision interval arithmetic through the interface of SageMath [22].

On the other hand, we prove the only-if part by verifying infeasibility of the system (2.1) for the maximal vectors \((\varphi_0, \ldots, \varphi_4) \in \Phi\) (with respect to the lexicographic order) that do not correspond to the angle types listed in Table 6. The maximal vectors are described as follows:

\[
\begin{align*}
(180, 120, 120, 31.875), & \quad (180, 120, 120, 37.5), \quad (180, 120, 120, 97.5, 45.9375), \\
(180, 120, 93.75, 46.875), & \quad (180, 120, 120, 91.875, 48.75), \quad (180, 120, 120, 60, 60), \\
(180, 120, 105, 52.5), & \quad (180, 120, 105, 91.875, 60), \quad (180, 120, 97.5, 97.5, 60), \\
(180, 105, 105, 60), & \quad (180, 105, 93.75, 91.875, 90.9375), \quad (180, 105, 91.875, 91.875, 91.875), \\
(180, 97.5, 97.5, 93.75, 93.75), & \quad (180, 97.5, 97.5, 97.5, 90.46875).
\end{align*}
\]

We must verify infeasibility of the system (2.1) for each permutation \(\tau\) on \(\{0, 1, \ldots, 4\}\) and for each vector \((\varphi_0, \ldots, \varphi_4)\) listed above. This was achieved through the following three steps.

**Step 1.**

If \(S\) is empty, then the system (2.1) is clearly infeasible. Since \(S\) is defined by linear inequalities, emptiness of \(S\) can be checked by linear programming. Using QSopt-ex Rational LP Solver [5], we verified emptiness...
For the 14 cases in Table 8, the set (corresponding to rotations or reflections to those vectors) ω box linear function, it is difficult to verify directly. Therefore, we first compute the axis-aligned bounding ways positive or always negative on Q S (β, τ i − γ i = 0). We determined the signs of ω using MPFI library [20] through SageMath [22]. Computational results are, if ω is always positive or always negative on B(S), the system (2.1) is clearly infeasible (see Figure 9 for an intuition). For the vectors (φ(τ(0), · · · , τ(4))) in Table 8 one can easily verify that B(S) ⊂ [0, 90]10 (see Table 9). Therefore, letting B(S) = [β−, β+] × · · · × [γ−, γ+]14, we have \( \prod_{i=0}^{4} \sin(\beta^{-}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{-}_{i}) \leq \omega(x) \leq \prod_{i=0}^{4} \sin(\beta^{+}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{+}_{i}) \) for all x ∈ B(S). Thus, if \( \prod_{i=0}^{4} \sin(\beta^{-}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{-}_{i}) \) and \( \prod_{i=0}^{4} \sin(\beta^{+}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{+}_{i}) \) have the same sign, then ω is always positive or always negative on S. We determined the signs of \( \prod_{i=0}^{4} \sin(\beta^{-}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{-}_{i}) \) and \( \prod_{i=0}^{4} \sin(\beta^{+}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{+}_{i}) \) using MPFI library [20] through SageMath [22]. Computational results are summarized in Table 9. With the exceptions of Cases (c), (d), (e), and (f), we were able to verify that the original system (2.1) is infeasible.

| Case | 1, 2, 3, 4, 0 | 5, 6, 7, 8, 9 | 10, 11, 12, 13, 14 |
|------|---------------|---------------|-------------------|
| I    | 60 60 60 60 60 | 60 60 60 60 60 | 60 60 60 60 60 |
| II   | 45 60 60 30 15 | 45 60 60 30 15 | 45 60 60 30 15 |
| III  | 30 60 60 10 5   | 30 60 60 10 5   | 30 60 60 10 5   |
| IV   | 15 60 60 30 15  | 15 60 60 30 15  | 15 60 60 30 15  |
| V    | 10 60 60 20 10  | 10 60 60 20 10  | 10 60 60 20 10  |
| VI   | 5 60 60 10 5    | 5 60 60 10 5    | 5 60 60 10 5    |
| VII  | 15 60 60 30 15  | 15 60 60 30 15  | 15 60 60 30 15  |
| VIII | 10 60 60 20 10  | 10 60 60 20 10  | 10 60 60 20 10  |
| IX   | 5 60 60 10 5    | 5 60 60 10 5    | 5 60 60 10 5    |
| X    | 10 60 60 20 10  | 10 60 60 20 10  | 10 60 60 20 10  |
| XI   | 5 60 60 10 5    | 5 60 60 10 5    | 5 60 60 10 5    |

Table 7: \( x_{+} \) and \( x_{-} \) for maximal greedy-drawable trees

| Case | \((\varphi(\tau(0), \ldots , \tau(4)))\) | \((\varphi(\tau(0), \ldots , \tau(4)))\) |
|------|------------------------------------------|------------------------------------------|
| (a)  | (120, 120, 120, 31, 875, 180)           | (h)                                      |
| (b)  | (105, 120, 120, 37, 5, 180)            | (i)                                      |
| (c)  | (97.5, 120, 120, 45, 9375, 180)        | (j)                                      |
| (d)  | (93.75, 120, 120, 46, 875, 180)        | (k)                                      |
| (e)  | (91.875, 120, 120, 48, 75, 180)        | (l)                                      |
| (f)  | (105, 120, 120, 52, 5, 180)           | (m)                                      |
| (g)  | (105, 120, 120, 52, 5, 180)           | (o)                                      |

Table 8: Remaining cases: \( S_{r}(\varphi_{0}, \ldots , \varphi_{4}) \) is not empty

Step 2.

For the 14 cases in Table 8, the set S is non-empty, and we must show that adding the equation ω = 0 makes the system (2.1) infeasible. Since S is connected and ω is continuous on S, it can be done by verifying that the function ω is always positive or always negative on S. However, since ω is a non-linear function, it is difficult to verify directly. Therefore, we first compute the axis-aligned bounding box B(S) of S (by determining the maximum and minimum of \( \beta_{i} \) and \( \gamma_{i} \) over \( i \)) and estimate the value of \( \omega \) on B(S). If \( \omega \) is always positive or always negative on B(S), the system (2.1) is clearly infeasible (see Figure 9 for an intuition). For the vectors (\( \varphi(\tau(0), \ldots , \tau(4)) \)) in Table 8 one can easily verify that B(S) ⊂ [0, 90]10 (see Table 9). Therefore, letting B(S) = [\( \beta^{-}_{0}, \beta^{+}_{0} \)] × · · · × [\( \gamma^{-}_{4}, \gamma^{+}_{4} \)], we have \( \prod_{i=0}^{4} \sin(\beta^{-}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{-}_{i}) \leq \omega(x) \leq \prod_{i=0}^{4} \sin(\beta^{+}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{+}_{i}) \) for all x ∈ B(S). Thus, if \( \prod_{i=0}^{4} \sin(\beta^{-}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{-}_{i}) \) and \( \prod_{i=0}^{4} \sin(\beta^{+}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{+}_{i}) \) have the same sign, then ω is always positive or always negative on S. We determined the signs of \( \prod_{i=0}^{4} \sin(\beta^{-}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{-}_{i}) \) and \( \prod_{i=0}^{4} \sin(\beta^{+}_{i}) - \prod_{i=0}^{4} \sin(\gamma^{+}_{i}) \) using MPFI library [20] through SageMath [22]. Computational results are summarized in Table 9. With the exceptions of Cases (c), (d), (e), and (f), we were able to verify that the original system (2.1) is infeasible.
Step 3.

For Cases (c), (d), (e), and (j) in Table 8 the bounding box $B(S)$ is too large to approximate $S$, and it is therefore impossible to prove that $\omega$ is always positive or always negative on $S$ in Step 2. In those cases, we partition the solution space $S$ into $S_1, \ldots, S_k$ by some parallel hyperplanes $H_1, \ldots, H_k$ and then consider the bounding boxes $B(S_1), \ldots, B(S_k)$. Then, we obtain a finer approximation $B(S_1) \cup \cdots \cup B(S_k)$ of $S$. (See Figure 10 for an intuition.) If $\omega$ always takes the same sign on each $B(S_i)$, the original system (2.1) is infeasible. For Case (c), we considered a partition of $S$ induced by hyperplanes $\gamma_4 = 7$ and $\gamma_4 = 11$, and obtained the results presented in Table 10. For Cases (d), (e), and (j), the obtained results are listed in Tables 11 and 13, respectively.

Combining the results in Steps 1–3, we complete the proof of the only-if part.

Table 9: Bounding box of $S$ and upper and lower bounds of $\omega(x)$

| Case | $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$ | $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ | $\omega(x)$ |
|------|---------------------------------------------|---------------------------------|-----------|
| a    | 315 75 105 145                             | 37.5 52.5 67.5 82.5            | 0         |
| b    | 45 60 60 60                               | 7.5 12.5 17.5 22.5             | 0         |
| c    | 165 105 105 105                            | 65 125 185 245                | 0         |
| d    | 165 105 105 105                            | 65 125 185 245                | 0         |
| e    | 45 60 60 60                               | 7.5 12.5 17.5 22.5             | 0         |
| f    | 45 60 60 60                               | 7.5 12.5 17.5 22.5             | 0         |
| g    | 135 75 135 135                            | 60 90 120 150                | 0         |
| h    | 45 105 135 165                            | 75 121 165 211               | 0         |
| i    | 45 105 135 165                            | 75 121 165 211               | 0         |
| j    | 45 105 135 165                            | 75 121 165 211               | 0         |
| k    | 45 105 135 165                            | 75 121 165 211               | 0         |
| l    | 45 105 135 165                            | 75 121 165 211               | 0         |
| m    | 45 105 135 165                            | 75 121 165 211               | 0         |
| n    | 45 105 135 165                            | 75 121 165 211               | 0         |

Figure 9: Bounding box of $S$  
Figure 10: Bounding boxes of $S_1$ and $S_2$
Table 10: Bounding box of $S_i$ and upper and lower bounds of $\omega(x)$ (Case (c), $S_1 := S \cap \{y \leq 11\}$, $S_3 := S \cap \{y \geq 11\}$)

| $S_i$ | $\omega(x)$ |
|-------|-------------|
| $S_1$ | $(5, 11)$ 152.555 401.672 0.0 | $\geq 0.0931413818384 + 2.82 \times 10^{-15}$ |
| $S_3$ | $(155, 5025) 1845 411 0.0 | $\leq 0.449134318384 + 2.82 \times 10^{-15}$ |
| $S_4$ | $(455, 9845) 1845 411 0.0 | $\leq 0.449134318384 + 2.82 \times 10^{-15}$ |

Table 11: Bounding box of $S_i$ and upper and lower bounds of $\omega(x)$ (Case (d), $S_1 := S \cap \{y \leq 5\}$, $S_2 := S \cap \{y \geq 5\}$)

| $S_i$ | $\omega(x)$ |
|-------|-------------|
| $S_1$ | $(5, 11)$ 152.555 401.672 0.0 | $\geq 0.0931413818384 + 2.82 \times 10^{-15}$ |
| $S_3$ | $(155, 5025) 1845 411 0.0 | $\leq 0.449134318384 + 2.82 \times 10^{-15}$ |
| $S_4$ | $(455, 9845) 1845 411 0.0 | $\leq 0.449134318384 + 2.82 \times 10^{-15}$ |

Table 12: Bounding box of $S_i$ and upper and lower bounds of $\omega(x)$ (Case (e), $S_1 := S \cap \{y \leq 10\}$, $S_3 := S \cap \{y \geq 10\}$)

| $S_i$ | $\omega(x)$ |
|-------|-------------|
| $S_1$ | $(5, 11)$ 152.555 401.672 0.0 | $\geq 0.0931413818384 + 2.82 \times 10^{-15}$ |
| $S_3$ | $(155, 5025) 1845 411 0.0 | $\leq 0.449134318384 + 2.82 \times 10^{-15}$ |
| $S_4$ | $(455, 9845) 1845 411 0.0 | $\leq 0.449134318384 + 2.82 \times 10^{-15}$ |

Table 13: Bounding box of $S_i$ and upper and lower bounds of $\omega(x)$ (Case (j), $S_1 := S \cap \{y \leq 7\}$, $S_2 := S \cap \{y \geq 7\}$)

6 Greedy-drawable pseudo-trees

In this section, we characterize greedy-drawable pseudo-trees, i.e., graphs obtained by adding one edge to a tree. Let $T$ be a pseudo-tree. Then, $T$ contains a single cycle. Let $C = e_0, \ldots, e_{m-1}, e_0$ be the cycle in $T$. If we remove the edges of $C$ from $T$, we obtain $m$ trees. Let $T_i$ be the tree that contains $e_i$. We consider $T_i := T_i + \bar{v}_i\bar{v}_i$, where $\bar{v}_i$ is a new vertex. Then, we consider the new pseudo-tree $\bar{T}$, called auxiliary pseudo-tree of $T$, obtained by joining each $T_i$ to the cycle $C = \bar{v}_0, \ldots, \bar{v}_{m-1}, \bar{v}_0$. We first observe the following fact.

Lemma 6.1 If a pseudo-tree $T$ has a (planar) greedy drawing, then the drawing must be outerplanar.

Proof. Suppose that there is an edge $uv$ that is drawn inside the region bounded by the cycle. Then, the entire cycle must be contained on the same side of axis $(uv)$ by Proposition 2.1. However, since axis $(uv)$ passes through the interior of the region, it is impossible.

Thus, if $T$ has a greedy drawing, then $\bar{T}$ also has a greedy drawing. A greedy drawing of $\bar{T}$ is constructed by drawing each edge $v_i\bar{v}_i$ infinitesimally small. Thus, we can always interpret a greedy drawing...
of $\mathcal{T}$ as a greedy drawing of $\mathcal{T}$ in which each edge $v_i \overline{v_i}$ is drawn infinitesimally small. With this interpretation, we define $\text{polygon}(T_i) := \text{polygon}(\overline{T}_i)$ and $\angle T_i := \angle \overline{T}_i$. Then, we can apply an argument similar to the proof of [16] Lemma 2.16, and obtain the following lemma.

**Lemma 6.2** In any (planar) greedy drawing of the pseudo-tree $\mathcal{T}$, the following inequality holds:

$$\sum_{i=0, \ldots, m-1, |\angle T_i| > 0} |\angle T_i| > 180^\circ(m - 2). \quad (6.1)$$

As we will see in Theorem 6.3, the above inequality also works as a sufficient condition for greedy-drawability of pseudo-trees $\mathcal{T}$ having at least four subtrees $T_i$ with $|\angle T_i| > 180^\circ$. To deal with the other case, we make some preparations. Suppose that there are at most three subtrees $T_i$ with $|\angle T_i| > 180^\circ$ in $\mathcal{T}$. Select three vertices $v_j, v_k, v_l$, called connection vertices, on the cycle $C$ in such a way that $T_j, T_k, T_l$ include all such subtrees. We define a $Y$-transformed tree of $\mathcal{T}$ to be the tree obtained from $T_j, T_k, T_l$ by connecting each of the vertices $v_j, v_k, v_l$ to a new vertex $w$, called the central vertex (See Figure 11).

Now suppose $m = 3$, and $\mathcal{T}$ is greedy-drawable. Then $\overline{T}$ has a greedy drawing in which each subtree $T_i (= \overline{T}_j \setminus \overline{v_i})$ with $|\angle T_i| > 0$ is infinitesimally small. It can be proved by almost the same argument as the proof of Lemma 2.18 in [16]. We give a brief overview here. Since replacing a subpath of a greedy path with a line segment keeps greediness (Lemma 2.5 in [16]), there is a greedy path between every vertex $i$ and every vertex in $\overline{T}_i$ and $\overline{T}_j$ ($i \neq j$) using the edge $v_i \overline{v_j}$. Therefore, $\overline{T}_i$ is contained in the half-plane $h_{v_i \overline{v}_j}$ and the half-plane $h_{uv}$ for every edge $uv$ in $\overline{T}_j$ ($i \neq j$) with $d_{\overline{T}}(\overline{v}_j, w) < d_{\overline{T}}(\overline{v}_j, v)$, where $d_{\overline{T}}$ is the graph distance on $\overline{T}$. Then, applying the same argument as the proof of Lemma 2.14 (a) in [16], we can show that the apex of $\angle T_i$ is also contained in all of those half-planes. Thus, if we draw $T_i$ infinitesimally small at the apex of $\angle T_i$ in such a way that $\angle T_i$ contains the original angle, the constructed drawing is a greedy drawing.

**Figure 11:** Constructing a Y-transformed tree

Now we are ready to present a characterization of greedy-drawable pseudo-trees.

**Theorem 6.3** Let $\mathcal{T}$ be a pseudo-tree with a cycle $C = v_0, \ldots, v_m, v_0$. For $i = 0, \ldots, m$, let $T_i$ be the connected component of $\mathcal{T} \setminus C$, which is a tree, that contains $v_i$. Let $\varphi_0 := |\angle T_0|_* \ldots, \varphi_{m-1} := |\angle T_{m-1}|_*$, where we set $\varphi_i = 0$ if $|\angle T_i|_* \leq 0$. Then, the pseudo-tree $\mathcal{T}$ is greedy-drawable if and only if one of the following conditions holds:

(a) There are at least four angles $\varphi_i$ with $\varphi_i \neq 180^\circ$, and the inequality $\sum_{i=0}^{m-1} \varphi_i > 180^\circ(m - 2)$ holds.

(b) There are at most three angles $\varphi_i$ with $\varphi_i \neq 180^\circ$, and a $Y$-transformed tree of $\mathcal{T}$ is greedy-drawable.

**Proof.** Let $k$ be the number of angles $\varphi_i$ with $\varphi_i \neq 180^\circ$. We first prove the only-if part. Suppose $\mathcal{T}$ is greedy-drawable. Then, we have $\sum_{i=0}^{m-1} \varphi_i > 180^\circ(m - 2)$ by Lemma 6.2. If $k \geq 4$, the condition (a) is already fulfilled, and thus we consider the case that $k \leq 3$. Let $\mathcal{T}$ be a $Y$-transformed tree of $\mathcal{T}$,
and \(v_a, v_b, v_c\) the connection vertices of \(T\). We prove that \(T\) is greedy-drawable. Since \(\varphi_i = 180^\circ\) for all \(i \notin \{a, b, c\}\), the above inequality implies \(\varphi_a + \varphi_b + \varphi_c > 180^\circ\). If \(\varphi_a, \varphi_b, \varphi_c > 0\), then \(T\) is greedy-drawable by Lemma 4.5 in [16]. Thus, we suppose one of \(\varphi_i\), for \(i \in \{a, b, c\}\), satisfies \(\varphi_i = 0\). Without loss of generality, we assume that \(\varphi_c = 0\). Then, in any greedy drawing of \(\mathcal{T}\), we have \(|\angle T_a| \leq 0\), which also implies \(|\angle T_a| > 0\), \(|\angle T_b| > 0\) by Lemma 6.2. Now we consider the auxiliary pseudo-tree \(\mathcal{T}'\) and the graph \(\mathcal{T}'\) obtained from the subtrees \(T_a, T_b, T_c\) by connecting each pair of the vertices \(v_a, v_b, v_c\). To construct a greedy drawing of \(T\), we first construct a greedy drawing of \(\mathcal{T}'\), and then transform it into a greedy drawing of \(T\).

Take a greedy drawing of \(\mathcal{T}'\). Slightly perturbing the drawing if necessary, we assume that \(|\angle T_c| < 0\).

We transform the drawing into a drawing \(\Gamma_{\varphi}\) in which \(T_a (= T_a \setminus v_a)\) and \(T_b (= T_b \setminus v_b)\) are drawn infinitesimally small. Let \(\Gamma_{\varphi}\) be the drawing of \(\mathcal{T}'\) obtained by restricting the drawing \(\Gamma_{\varphi}\) into \(T_a, T_b, T_c\), and connecting each pair of \(v_a, v_b, v_c\) by a line segment. Since replacing a subpath of a greedy path with a line segment keeps greediness (Lemma 2.5 in [16]), \(\Gamma_{\varphi}\) is a greedy drawing of \(\mathcal{T}'\).

Now we transform the drawing \(\Gamma_{\varphi}\) into a greedy drawing of \(T\). In \(\Gamma_{\varphi}\), the tree \(T\) is already drawn greedily, and we have to draw \(T_{ab} := T \setminus T_c (= v_c + wv_a + wv_b + T_a + T_b)\) appropriately. We first observe that the tree \(T_{ab}\) can be drawn greedily. Indeed, since \(\varphi_a + \varphi_b > 180^\circ\) and \(\varphi_a, \varphi_b \leq 120^\circ\), we have \(\varphi_a, \varphi_b > 60^\circ\). By the result in Table 1 this implies \(\varphi_a, \varphi_b > 90^\circ\). By Lemma 3.1 (III), the tree \(T_{ab}\) (with root \(v_c\)) has a greedy drawing with an open angle, and the supremum of opening angles is \(\varphi_a + \varphi_b - 180^\circ\). Then, our task is to place an infinitesimally small drawing of \(T_{ab}\) \(v_c\) at an appropriate place in polygon(\(T_c\)) in such a way that (1) \(\angle T_{ab}\) contains \(T_c\), (2) \(h_{v_c}^{wv_a}\) contains \(T_a\), and (3) \(h_{v_c}^{wv_b}\) contains \(T_b\). Intuitively, these conditions can be satisfied if we draw \(T_{ab}\) \(v_c\) inside of polygon(\(T_c\)) sufficiently far from \(T_c\) in such a way that \(\angle T_{ab}\) is sufficiently large (see Figure 12). Here, polygon(\(T_c\)) is bounded, and the interior of polygon(\(T_c\)) is partitioned into some regions by the edges of \(T_c\). We consider a vertex \(p\) of polygon(\(T_c\)) that lies on the boundary of the region containing \(\mathcal{T}' \setminus T_c\). We show that if we place the vertex \(w\) and an infinitesimally small drawing of \((T_{ab})^{wv_a} v_c\) inside of polygon(\(T_c\)) sufficiently close to \(p\), the conditions (1)–(3) are satisfied.

Let \(l_1\) and \(l_2\) be the supporting lines of the two edges of polygon(\(T_c\)) incident to \(p\). See Figure 12 for an illustration. Let \(u, u', v, v' (d_{T_c}(v, v) < d_{T_c}(v, u') < d_{T_c}(u, u') < d_{T_c}(v, v'))\) be the vertices of \(T_c\), such that \(l_1 = \text{axis}(uv)\) and \(l_2 = \text{axis}(uv')\), where \(d_{T_c}\) is the graph distance on \(T_c\). Let \(l_1^\perp := h_{uv}^{u_1}, l_2^\perp := h_{uv}^{u_2}\). Since the apex of \(\angle T_a\) lies in \(l_1^\perp \cap l_2^\perp\) (recall that \(T_a\) is drawn infinitesimally small in \(\Gamma_{\varphi}\)), and \(\angle T_a\) contains \(v \notin l_1^\perp\) and \(v' \notin l_2^\perp\), one of the rays of \(\angle T_a\) intersects with \(l_1\), and is pointing outward \(l_1^\perp\), and the other ray intersects with \(l_2\), and is pointing outward \(l_2^\perp\). Now we remark that \(\text{axis}(\overline{v_c}v_c)\) separates \(v_a, v_b, \overline{v_a}, \overline{v_b}, \overline{v_c}\) from \(T_c\) by greediness of \(\mathcal{T}'\) (recall Proposition 2.1). Since \(\text{axis}(\overline{v_c}v_c)\) does not cross any edge of \(T_c\) (by Proposition 2.1), and \(v_a, v_b, \overline{v_a}, \overline{v_b}, \overline{v_c}, p\) belong to the same connected component of polygon(\(T_c\)) \(T_c\), the apex \(p\) is also separated from \(T_c\), and thus from \(v\) and \(v'\) by \(\text{axis}(\overline{v_c}v_c)\). Therefore, both of the two unbounded edges of \(\angle T_a \cap \angle T_b\) must intersect with \(\text{axis}(\overline{v_c}v_c)\). One of the unbounded edges is pointed outward \(l_1^\perp\), the other is pointed outward \(l_2^\perp\). Let \(x\) be the intersection of the lines obtained by extending the two unbounded edges of \(\angle T_a \cap \angle T_b\). Then, \(x\) belong to polygon(\(T_c\)). Let \(R\) be the unbounded region bounded by the rays obtained by extending the two unbounded edges to \(x\). Then, a simple calculation shows that the angle between the two extreme rays of \(R\) is less than \(|\angle T_a| + |\angle T_b| - 180^\circ\), and thus less than \(\varphi_a + \varphi_b - 180^\circ\). Therefore, we can construct a drawing of \(T_{ab}\) in which \(\angle T_{ab}\) contains \(R\) by placing \(T_{ab} \setminus v_c\) sufficiently close to \(p\) (see Figure 12). Since \(T_c\) is contained in \(R\), the constructed drawing satisfies the condition (1).

Now we verify the condition (2). See Figure 13. We first construct a finer description of a region (finer than \(R\)) in which \(T_c\) must lie. Let \(v'_c\) be a vertex of \(T_c\) adjacent to \(v_c\). Then, \(v'_c\) must lie outside of the circle \(S(p)\) with radius \(|pv_c|\) centered at \(p\). Otherwise, \(h_{v'_c}^{v_c}\) does not contain \(p\), which contradicts to the assumption that \(p\) is a vertex of polygon(\(T_c\)). Next, consider a vertex \(v''_c \neq v_c\) of \(T_c\) that is adjacent to \(v'_c\). Then, \(v''_c\) must lie outside of the circle with radius \(|pv'_c|\) centered at \(p\). Since \(|pv'_c| < |pv_c|\), this implies that \(v''_c\) must also lie outside of \(S(p)\). Continuing this discussion, we can show that \(T_c\) must lie outside of \(S(p)\). This implies that the region \(R := R \setminus S(p)\) contains \(T_c\). We prove that \(R\) is contained in \(h_{v''_c}^{v'_c}\). Let \(q\) and \(r\) be the intersections of \(S(p)\) with the two extreme rays of \(R\). Let \(M\) be the midpoint of
the line segment $pv_c$, and let $M_q$ and $M_r$ be the intersections of axis($pv_c$) with the line segments $pr$ and $pq$ respectively. Since the angle between the two extreme rays of $R$ is less than $\varphi_a + \varphi_b - 180^\circ$, and thus less than $60^\circ$, we have $\angle qpr < 60^\circ$. This leads to that $|pM_q| < 2|pq|$, and $|pM_r| < 2|pr|$, which implies that $q, r \in h_{pv_c}^v$. Therefore, $R$ is contained in $h_{pv_c}^v$. Since $w$ is placed sufficiently close to $p$, it implies $T_c$ is contained in $h_{wv_c}^v$. Since the condition (3) is clearly satisfied, the constructed drawing is a greedy drawing of $T$. Therefore, the if part is proved for all cases.

Figure 12: Transforming a greedy drawing of $\bar{T}'$ into a greedy drawing of $T$

Figure 13: Proving that $T_{uv}^w$ is contained in $h_{wv_c}^v$

Now we prove the if-part. Assume that $T$ satisfies the condition (a). Then, at most five angles $\varphi_i$ can satisfy $\varphi_i \not= 180^\circ$, i.e., $\varphi_i \leq 120^\circ$. Otherwise, $\sum_{i=0}^{m-1} \varphi_i \leq 180^\circ (m - 6) + 120^\circ \times 6 = 180^\circ (m - 2)$,
which contradicts to the assumption. Let \( \psi_0 := \varphi_{i_0}, \ldots, \psi_{k-1} := \varphi_{i_{k-1}} (i_0 \leq \cdots < i_{k-1}) \) be the angles with \( \varphi_{i_j} \neq 180^{\circ} \), i.e., \( \varphi_{i_j} \leq 120^{\circ} \). Let us denote by \( V_j \), for \( j = 0, \ldots, k-1 \), the set of inner vertices of the \( v_i, v_{i+1} \)-path in \( C \) that contains the edge \( v_i, v_{i+1} \), where the indices are interpreted modulo \( m + 1 \). Based on the above observation, we consider the following three cases.

**Case 1** \( T \) satisfies the condition (a), and there are exactly five angles \( \varphi_i \) satisfying \( \varphi_i \leq 120^{\circ} \).

In this case, no angle \( \varphi_i \) satisfies \( \varphi_i \leq 90^{\circ} \). Suppose otherwise. Then, some angle \( \varphi_j \) satisfies \( \varphi_j \leq 90^{\circ} \), which implies \( \varphi_j \leq 60^{\circ} \) by the result in Table I. This leads to that \( \sum_{i=0}^{m-1} \psi_i \leq 180^{\circ}(m-5) + 120^{\circ} \times 4 + 60^{\circ} = 180^{\circ}(m-2) \), which is a contradiction. Let us consider a pentagon \( \Gamma_1 \) with vertices \( W_0, \ldots, W_4 \), labeled counterclockwise, satisfying

- \( 90^{\circ} < \angle W_j < \psi_j \) for \( j = 0, \ldots, 4 \).
- The bisector axis \( W_j W_{j+1} \) does not pass through \( W_{j+3} \), for \( j = 0, \ldots, 4 \).

Here, the indices of \( W_j \) are interpreted modulo 5. Since \( \sum_{j=0}^{4} \psi_j > 540^{\circ} \), it is easy to construct a pentagon satisfying the first condition. Slightly perturbing this pentagon, we can construct a pentagon \( \Gamma_1 \) satisfying both of the two conditions. Then, we place each vertex \( v_i \) at \( W_j \), and the vertices in \( V_j \) on the line segment \( W_j W_{j+1} \) in the following way. We first note that \( \angle W_j > 90^{\circ} \) and \( \angle W_{j+1} > 90^{\circ} \), and thus the bisector axis \( W_j W_{j+1} \) passes through either the interior of the line segment \( W_{j-1} W_{j-2} \) or that of \( W_j W_{j+1} \). In the former case, we place the vertices in \( V_j \) on \( W_j W_{j+1} \) sufficiently close to \( W_j \). In the latter case, we place those vertices on \( W_j W_{j+1} \) sufficiently close to \( W_{j+1} \). See Figure 14. For \( k = 0, \ldots, 4 \), we call the set of vertices placed sufficiently close to (or placed at) \( W_k \) the \( \epsilon \)-neighborhood of \( W_k \).

Since a greedy path clearly exists between every pair of vertices in \( V_j \), we prove that there is a greedy \( uw \)-path for every vertex \( w \in V_j \) and every vertex \( v \notin V_j \). Let \( l_j := \text{axis}(W_j W_{j+1}) \), and \( b_j \) the perpendicular line to \( W_j W_{j+1} \) at \( W_k \) for \( k = j, j+1 \). Suppose that \( w \) is in the \( \epsilon \)-neighborhood of \( W_j \). Then, the counterclockwise path from \( w \) to a vertex in the \( \epsilon \)-neighborhood of \( W_{j+1}, W_{j+2}, W_{j+3} \) is greedy. Indeed, the perpendicular bisector of each edge on the path is sufficiently close to one of \( l_j, l_{j+1}, l_{j+2}, l_{j+3} \), and one can easily check that the half-plane defined for each edge on the path contains \( v \) (recall Proposition 2.1). On the other hand, the clockwise path from \( w \) to a vertex in the \( \epsilon \)-neighborhood of \( W_{j-1} \) is greedy. Next, suppose that \( w \) is in the \( \epsilon \)-neighborhood of \( W_{j+1} \). Then, the counterclockwise path from \( w \) to a vertex in the \( \epsilon \)-neighborhood of \( W_{j+2} \) is greedy, and the clockwise path from \( w \) to a vertex in the \( \epsilon \)-neighborhood of \( W_j, W_{j-1}, W_{j-2} \) is greedy. Therefore, the constructed drawing of \( C \), denoted hereinafter by \( \Gamma_2 \), is greedy.

Now we replace each line segment \( W_j W_{j+1} \) by a sufficiently flat convex polygonal chain (see Figure 15). Then, the perpendicular bisector of each edge on the polygonal chain is sufficiently close to one of \( l_j, l_{j+1} \). If we draw each \( T_i \) infinitesimally small in such a way that each \( \angle T_i \) contains \( \Gamma_2 \), then the constructed drawing is a greedy drawing of \( T \).

**Case 2** \( T \) satisfies the condition (a), and there are exactly four angles \( \varphi_i \) satisfying \( \varphi_i \leq 120^{\circ} \).

In this case, at most one angle \( \varphi_i \) satisfies \( \varphi_i \leq 90^{\circ} \). Suppose otherwise. Then, at least two angles \( \varphi_j \) satisfy \( \varphi_j \leq 90^{\circ} \), and these two angles satisfy \( \varphi_j \leq 60^{\circ} \) by the result in Table I. Therefore, we have \( \sum_{i=0}^{m-1} \psi_i \leq 180^{\circ}(m-4) + 120^{\circ} \times 2 + 60^{\circ} \times 2 = 180^{\circ}(m-2) \), which is a contradiction. We also remark that no angle \( \varphi_i \) satisfies \( \varphi_i = 0 \) because otherwise we have \( \sum_{i=0}^{m-1} \psi_i \leq 180^{\circ}(m-4) + 120^{\circ} \times 3 + 0 = 180^{\circ}(m-2) \), a contradiction. Without loss of generality, let \( \psi_0 \) be the smallest angle among the four angles \( \psi_0, \ldots, \psi_3 \). Then, we have \( 90^{\circ} < \min(\psi_1, \psi_2, \psi_3) \leq 120^{\circ} \). We consider the following three cases separately.

**Case 2a** \( 0^{\circ} < \psi_0 \leq 45^{\circ} \).

We consider a quadrilateral \( \Gamma_1 \) with vertices \( W_0, \ldots, W_3 \), labeled counterclockwise, satisfying

- \( \angle W_0 = \psi_0 - \epsilon, \angle W_1 = \psi_1 - \epsilon, \angle W_2 = 360^{\circ} - \psi_0 - \psi_1 + 3\epsilon, \angle W_3 = \psi_3 - \psi_1 - 3\epsilon, \angle W_0 W_1 W_3 = 90^{\circ} + \epsilon, \angle W_0 W_2 W_3 = 90^{\circ} - \psi_0, \angle W_2 W_1 W_3 = \psi_1 - 90^{\circ} - 2\epsilon, \angle W_2 W_3 W_1 = \psi_0 + \psi_3 - 90^{\circ} - \epsilon \),
where $\epsilon > 0$ is a sufficiently small number. See Figure 16a. Note that all the angles appearing above are positive. We first show that there is a greedy path between every pair of vertices of the quadrilateral $\Gamma_1'$. First, observe that $\angle W_2W_3W_1 < 45^\circ + 120^\circ - 90^\circ < 75^\circ$ and $\angle W_2 > 360^\circ - 45^\circ - 120^\circ - 120^\circ = 75^\circ$, and thus $\angle W_2W_3W_1 < \angle W_2$. Similarly, we have $\angle W_2W_3W_1 < 30^\circ < \angle W_2$. It follows that $|W_1W_3| > |W_1W_2|$ and $|W_1W_3| > |W_2W_3|$. Moreover, since $\angle W_0W_1W_3 > 90^\circ$, we have $|W_0W_3| > |W_1W_3|$ and $|W_0W_3| > |W_0W_1|$. We also observe that $\angle W_0W_3W_1 > 45^\circ > \angle W_0$, which implies $|W_0W_1| > |W_1W_3|$. Therefore, there is a greedy path between every pair of vertices of $\Gamma_1'$. The greedy routes are summarized as follows:

- $W_0 \rightarrow W_3 \rightarrow W_2 \rightarrow W_1,$
- $W_1 \rightarrow W_0, W_1 \rightarrow W_2 \rightarrow W_3,$
- $W_2 \rightarrow W_1 \rightarrow W_0, W_2 \rightarrow W_3,$
- $W_3 \rightarrow W_2 \rightarrow W_1, W_3 \rightarrow W_0.$

Then, we place the vertices in $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ on the edges of $\Gamma_1'$ sufficiently close to $W_1, W_1, W_3, W_0$ respectively. Since $\angle W_3 > 90^\circ$ and $\angle W_2W_3W_1 < 90^\circ$, the perpendicular bisector of each edge drawn small on $W_2W_3$ passes through the interior of $W_0W_1$. This implies that move along each small edge on $W_2W_3$ toward $W_2$ decreases the distance to $W_1$ and $W_2$. Continuing this discussion, one can prove that the constructed drawing of $C$ is also greedy. Next, we construct drawings of $T_0, \ldots, T_3$. We first remark that the inequality $\sum_{j=0}^3 \psi_j > 360^\circ$ implies $\psi_2 > 360^\circ - \psi_0 - \psi_1 - \psi_3$. This leads to $\angle W_2 < \psi_2$ because $\epsilon$ is sufficiently small. Therefore, we have $\angle W_j < \psi_j$ for all $j = 0, \ldots, 3$. Thus, if we draw each of $T_i$ sufficiently small in such a way that $\angle T_i$ contains $\Gamma'_1$, the constructed drawing is greedy. Finally, we replace the edges of $\Gamma_1'$ by convex polygonal chains, and draw subtrees $T_j$ with $|\angle T_j| = 180^\circ$ as in Case 1. Then, the constructed drawing is a greedy drawing of $T$.

**Case 2b** $45^\circ < \psi_0 \leq 90^\circ$. We first remark that $\psi_0 \leq 60^\circ$ by the result in Table 1. We consider a quadrilateral $\Gamma_1''$ with vertices $W_0, \ldots, W_3$, labeled counterclockwise, satisfying

- $\angle W_0 = \psi_0 - \epsilon, \angle W_1 = \psi_1 - \epsilon, \angle W_2 = 360^\circ - \psi_0 - \psi_1 - \psi_3 + 4\epsilon, \angle W_3 = \psi_3 - 2\epsilon$,
- $\angle W_0W_2W_3 = \psi_0 - 2\epsilon, \angle W_0W_1W_3 = 180^\circ - 2\psi_0 + 3\epsilon, \angle W_2W_3W_1 = \psi_3 - \psi_0$,
- $\angle W_2W_1W_3 = 2\psi_0 + \psi_1 - 180^\circ - 4\epsilon$. 


where $\epsilon > 0$ is a sufficiently small number. See Figure 16b. Note that all the angles appearing above are positive. We first show that there is a greedy path between every pair of vertices of the quadrilateral $\Gamma''_i$. We first observe that $\angle W_0 < 60^\circ$, $\angle W_0 W_1 W_3 < 60^\circ$, $\angle W_0 W_1 W_3 > 60^\circ$, which implies $|W_0 W_1| < |W_0 W_3|$ and $|W_1 W_3| < |W_0 W_3|$. Since $\angle W_2 W_1 W_3 < 2 \times 60^\circ + 120^\circ - 180^\circ = 60^\circ$ and $\angle W_2 > 360^\circ - 120^\circ - 120^\circ - 60^\circ = 60^\circ$, we have $\angle W_2 W_1 W_3 > \angle W_2$. This implies $|W_2 W_3| < |W_1 W_3|$. Moreover, we have $\angle W_3 W_0 W_0 < \angle W_0$, which leads to $|W_1 W_3| > |W_1 W_0|$. Therefore, there is a greedy path between every pair of vertices of $\Gamma''_i$. The greedy routes are summarized as follows:

- $W_0 \rightarrow W_1 \rightarrow W_2 \rightarrow W_3$, 
- $W_1 \rightarrow W_0$, $W_1 \rightarrow W_2 \rightarrow W_3$, 
- $W_2 \rightarrow W_1 \rightarrow W_0$, $W_2 \rightarrow W_3$, 
- $W_3 \rightarrow W_2$, $W_3 \rightarrow W_0 \rightarrow W_1$.

Then, we place the vertices in $V_0, V_1, V_2, V_3$ on the edges of $\Gamma''_i$ sufficiently close to $W_0, W_1, W_3, W_3$ respectively. By a similar discussion to Case 2a, one can prove that the constructed drawing of $C$ is also greedy. Then, replacing each line segment by a convex polygonal chain, and drawing the remaining subtrees as in Case 2a, we obtain a greedy drawing of $T$.

**Case 2c** $90^\circ < \psi_0 \leq 120^\circ$.

We consider a square $\Gamma''_i$ with vertices $W_0, \ldots, W_3$, labeled counterclockwise. Then, we place the vertices in $V_0, V_1, V_2, V_3$ on the edges of $\Gamma''_i$ sufficiently close to $W_0, W_1, W_2, W_3$ respectively. See Figure 16c.

Then, replacing each line segment by a convex polygonal chain, and drawing the remaining subtrees as in Case 2a, we obtain a greedy drawing of $T$.

Therefore, we can always construct a greedy drawing of $T$ in Case 2.

**Figure 16:** Greedy drawings of $C$ in Case 2

**Case 3** There are at most three angles $\varphi_i$ satisfying $\varphi_i \leq 120^\circ$, and $T$ satisfies the condition (b).

Let $T$ be a Y-transformed tree of $T$. Let $v_a, v_b, v_c$ be the connection vertices and $w$ the center vertex. Let $\Gamma_T$ be a greedy drawing of $T$. By Lemma 3.3, at most one of $T_i$ for $i \in \{a, b, c\}$, satisfies $|\angle T_i| \leq 0$ in $\Gamma_T$. Let us first assume that there is a subtree $T_i$ with $|\angle T_i| \leq 0$, and suppose without loss of generality that $T_i$ is such a subtree. Slightly perturbing if necessary, we assume $|\angle T_i| < 0$. Applying the shrinking lemma (Lemma 2.3), we transform the drawing into a drawing in which $T_a$ and $T_b$ are infinitesimally small. Let $p$ be a vertex of polygon($T_c$) that belongs to the same connected component of polygon($T_c$) \ $T_i$ as $T \setminus T_i$. Let $l_1$ and $l_2$ be the supporting lines of the two edges of polygon($T_c$) incident to $p$. We move $w, v_a, v_b$ into places in polygon($T_c$) that are sufficiently close to $p$, and translate $T_a$ and $T_b$ accordingly.
Then, each of $\angle T_a$ and $\angle T_b$ contains the original angle, and thus contains $T_c$. We also note that the transformation keeps planarity. Let $A, B, C$ be the geometric points corresponding to $v_a, v_b, v_c$ respectively. Then, the triangle $ABC$ formed by $A, B, C$ has a property that $|AB|$ is sufficiently small. We construct a greedy drawing of $T$ by placing the remaining vertices of $C$ on the edges of the triangle $ABC$, and placing the remaining subtrees. Without loss of generality, we suppose that $\angle B > \angle A$. Let us consider the following two cases separately.

(Case 3a) $\angle B < 90^\circ$.

In this case, $\angle B$ is sufficiently close to $90^\circ$. Let $Q$ be the point on the line segment $AB$ with $|CQ| = |CB|$, and $R$ the point on the line segment $BC$ with $|AR| = |AB|$. Note that $|BR|$ is sufficiently small, compared to $|AB|$. We place each vertex $v$ in $V_0$ on the line segment $AQ$ sufficiently close to $A$, each vertex $v$ in $V_1$ on the line segment $QR$ sufficiently close to $Q$, and each vertex $v$ in $V_2$ on the line segment $CR$ sufficiently close to $R$. See Figure 18a. Then, it is easy to check that the constructed drawing of $C$ is greedy. Replacing the line segments by convex polygonal chains and placing the remaining subtrees as in Cases 1 and 2, we obtain a greedy drawing of $T$.

(Case 3b) $\angle B \geq 90^\circ$.

We place each vertex in $V_0$ on the line segment $AB$ sufficiently close to $A$, each vertex in $V_1$ on the line segment $BC$ sufficiently close to $B$, and each vertex in $V_2$ on the line segment $CA$ sufficiently close to $C$. Then, the constructed drawing of $C$ is clearly greedy. Replacing the line segments by convex polygonal chains and placing the remaining subtrees as in Cases 1 and 2, we obtain a greedy drawing of $T$.

Therefore, in either case, $T$ has a greedy drawing. Finally, we assume that $|\angle T_i| > 0$ for all $i = a, b, c$. This implies that $\psi_a, \psi_b, \psi_c > 0$. Without loss of generality, we assume that $\psi_a$ is the largest angle among these three angles. Then, we have $\psi_a > 90^\circ$ by the result in Table 1. We consider a triangle with vertices $Z_a, Z_b, Z_c$ satisfying the following two conditions:

- $\angle Z_i < \psi_i$ for $i = a, b, c$,
- $\angle Z_a = \psi_a - \epsilon$,

for sufficiently small $\epsilon > 0$. Since $\psi_a + \psi_b + \psi_c > 180^\circ$, such a triangle clearly exists. Then, one can construct a greedy drawing of $T$ similarly to Case 3b. Therefore, we can always construct a greedy drawing of $T$ in Case 3.

Figure 17: Transforming a greedy drawing of $T$
7 Concluding remarks

In this paper, we have presented a complete combinatorial characterization of greedy-drawable trees (Proposition 4.1 and Theorem 5.1). Our characterization immediately leads to a linear-time recognition algorithm for greedy-drawable trees. That is, one can determine the greedy-drawability of a tree with maximum degree 5 by checking whether the suprema of opening angles of the subtrees around a degree-5 vertex are in the ranges of angles listed in Tables 5 and 6. Since the supremum of opening angles of a tree can be computed in linear time by the algorithm \textit{getOpenAngle} in \cite{16}, this condition can likewise be verified in linear time. Combining this algorithm with the linear-time recognition algorithm for greedy-drawable trees with maximum degree \leq 4 by Nöllenburg and Prutkin \cite{16}, we obtain a linear-time recognition algorithm for greedy-drawable trees in the general case.

\textbf{Corollary 7.1} Greedy drawability of a tree \( T = (V, E) \) can be checked in \( O(|V|) \) time.

As a subsequent step, we have investigated a characterization of greedy-drawable pseudo-trees. Although we did not present an explicit description of greedy-drawable pseudo-trees, such a description can easily be obtained by determining the possible angle types of subtrees using Table 1. Similarly to the case of trees, we obtain the following corollary:

\textbf{Corollary 7.2} Greedy drawability of a pseudo-tree \( T = (V, E) \) can be checked in \( O(|V|) \) time.

Acknowledgement

This work was supported by JSPS KAKENHI Grant Number JP19K20210.

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