PERTURBED BASINS OF ATTRACTION

HAN PETERS

Abstract. Let $F$ be an automorphism of $\mathbb{C}^k$ which has an attracting fixed point. It is well known that the basin of attraction is biholomorphically equivalent to $\mathbb{C}^k$. We will show that the basin of attraction of a sequence of automorphisms $f_1, f_2, \ldots$ is also biholomorphic to $\mathbb{C}^k$ if every $f_n$ is a small perturbation of the original map $F$.

1. Introduction

We are interested in the following conjecture, posed by Eric Bedford:

**Conjecture 1.** Let $F$ be a holomorphic automorphism of a complex manifold $X$, which is hyperbolic on an invariant compact subset $K$. Then for every $p \in K$, the stable manifold through $p$ is biholomorphically equivalent to complex Euclidean space.

It was proved by Jonsson and Varolin [JV] that Conjecture 1 holds almost everywhere with respect to any invariant probability measure supported on $K$.

One could show that Conjecture 1 holds by solving a related problem in non-autonomous dynamics. For $z \in K$, let $\phi_z$ be a biholomorphic mapping that maps the local stable manifold through $z$ onto the local stable tangent space at $z$ (which after scaling we can identify with the unit ball in $\mathbb{C}^k$), where $k$ is the dimension of the stable manifold. We may assume that $\phi'_z(z)$ equals the identity. Let $p_0 = p, p_1, p_2, \ldots$ be the orbit of $p \in K$. Then the maps

$$f_n(z) = \phi_{p_{n+1}} F \phi^{-1}_{p_n},$$

are biholomorphic mappings from the unit ball into the unit ball (we may assume that these maps are really contracting, since otherwise we consider some high iterate of $F$ instead). The maps satisfy the conditions of the main theorem in [FSt], and thus there is a basin of attraction, defined in terms of the tail space. It is clear from the definition that this basin of attraction is biholomorphic to the stable manifolds, and the main theorem of [FSt] says that the basin of attraction is biholomorphic to a basin of attraction of a sequence of global automorphisms, which are uniformly contracting on some neighborhood of the origin. Thus, Conjecture 1 can be answered positively by proving that the following conjecture holds:

**Conjecture 2.** Let $f_1, f_2, \ldots$ be a sequence of automorphisms of $\mathbb{C}^k$ that fix the origin. Suppose that there exist $a, b \in \mathbb{R}$ that satisfy $0 < a < b < 1$, and such that the following holds for every $n \in \mathbb{N}$ and every $z \in B$:

$$a\|z\| \leq \|f_n(z)\| \leq b\|z\|.$$
Then the basin of attraction of the sequence $f_1, f_2, \ldots$ is biholomorphic to $\mathbb{C}^k$.

The basin of attraction of a sequence $f_1, f_2, \ldots$ of automorphisms that all fix the origin is defined as the set of all points $z \in \mathbb{C}^k$ such that $f(n)(z) = f_n \circ \cdots \circ f_1(z) \rightarrow 0$.

If the automorphisms $f_1, f_2, \ldots$ do not all fix the origin, but there does exist a small neighborhood $N$ of 0 such that $f_n(N) \subset \subset N$ for every $n \in \mathbb{N}$, then we define the basin of attraction of the sequence $f_1, f_2, \ldots$ as $\bigcup f_n^{-1}(N)$.

If all the maps $f_n$ are equal then the conjecture follows immediately from the following theorem, which follows from the work of Sternberg [Sg], and was proved independently by Rosay and Rudin [RR]:

**Theorem 3.** Let $F$ be an automorphism of $\mathbb{C}^k$, which has an attracting fixed point at 0, then the basin of attraction of $F$ is biholomorphically equivalent to $\mathbb{C}^k$.

Here, $F$ is attracting at the origin means that all eigenvalues of $F'(0)$ are smaller than 1 in absolute value. We will focus on Conjecture 2. Our main result is the following generalization of Theorem 3:

**Main Theorem.** Let $F$ be an automorphism of $\mathbb{C}^k$ which has an attracting fixed point at 0. Then there exists an $\epsilon > 0$ such that for any sequence $f_1, f_2, \ldots$ of automorphisms which satisfy $\|F(z), f_n(z)\| < \epsilon$ for any $n \in \mathbb{N}$ and all $z$ in the unit ball, one has that the basin of attraction of the sequence $f_1, f_2, \ldots$ is biholomorphic to $\mathbb{C}^k$.

The maps $f_n$ don’t necessarily have 0 as an attracting fixed point. However, if we choose $\epsilon$ small enough then there exists a small neighborhood of $N$ of 0, such that $f_n(N) \subset \subset N$ for all $n \in \mathbb{N}$. Notice that the definition of the basin of attraction is independent of the choice of $N$. Also notice that if the maps $f_n$ do fix the origin, then the two definitions of the basin of attraction are exactly equal.

Hence the Main Theorem gives sufficient conditions for when the basin of attraction of a sequence $f_1, f_2, \ldots$ of automorphisms which have an attracting fixed point at the origin is biholomorphic to $\mathbb{C}^k$. The following theorem can be found in [Wd]:

**Theorem 4.** Let $f_1, f_2, \ldots$ be a sequence of automorphisms of $\mathbb{C}^k$ which all fix the origin, and let $a, b \in \mathbb{R}$ that satisfy $0 < a < b < 1$ and the additional condition $b^2 < a$. Suppose that the following holds for every $n \in \mathbb{N}$ and every $z \in \mathcal{B}$:

$$a\|z\| \leq \|f_n(z)\| \leq b\|z\|.$$ 

Then the basin of attraction of the sequence $f_1, f_2, \ldots$ is biholomorphic to $\mathbb{C}^k$.

The proof of Theorem 4 can be adapted to get the following result which is slightly stronger:

**Theorem 5.** Let $f_1, f_2, \ldots$ be a sequence of automorphisms which all fix the origin, and let $t \in (1, 2)$. Suppose that for every $n \in \mathbb{N}$ there exists $c_1, c_2, \ldots \in (0, 1)$ such that $\prod c_n = 0$ and the following holds for every $z \in \mathcal{B}$:

$$c_n^t\|z\| \leq \|f_n(z)\| \leq c_n\|z\|.$$ 

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**Main Theorem.** Let $F$ be an automorphism of $\mathbb{C}^k$ which has an attracting fixed point at 0. Then there exists an $\epsilon > 0$ such that for any sequence $f_1, f_2, \ldots$ of automorphisms which satisfy $\|F(z), f_n(z)\| < \epsilon$ for any $n \in \mathbb{N}$ and all $z$ in the unit ball, one has that the basin of attraction of the sequence $f_1, f_2, \ldots$ is biholomorphic to $\mathbb{C}^k$.

The maps $f_n$ don’t necessarily have 0 as an attracting fixed point. However, if we choose $\epsilon$ small enough then there exists a small neighborhood of $N$ of 0, such that $f_n(N) \subset \subset N$ for all $n \in \mathbb{N}$. Notice that the definition of the basin of attraction is independent of the choice of $N$. Also notice that if the maps $f_n$ do fix the origin, then the two definitions of the basin of attraction are exactly equal.

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$$c_n^t\|z\| \leq \|f_n(z)\| \leq c_n\|z\|.$$ 

Then the basin of attraction of the sequence $f_1, f_2, \ldots$ is biholomorphic to $\mathbb{C}^k$. 

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The idea in the proof of both Theorem 4 and 5 is the following: one defines the maps $H_n := A(n)^{-1} f(n)$, where $A(n) = A_n \circ \cdots \circ A_1$ and $A_n = f'_n(0)$. As $n \to \infty$, one gets that $H_n$ converges uniformly on compact subsets of the basin of attraction to a biholomorphic mapping from the basin onto $\mathbb{C}^k$.

The following theorem was proved in [PW] and is also a generalization of Theorem [RR]:

**Theorem 6.** Let $f_1, f_2, \ldots$ be a sequence of automorphisms of $\mathbb{C}^k$, which are all attracting at the origin. Then there exist large enough integers $n_1, n_2, \ldots$ such that the basin of attraction of the maps $f_1^{n_1}, f_2^{n_2}, \ldots$ is biholomorphic to $\mathbb{C}^k$.

Here the definition of the basin of attraction must be slightly changed for this theorem to hold. The exact definition is given in [PW], as well as an example that shows that Theorem [BB] does not hold if one uses a more straightforward definition.

In the next section we will set our notation, and introduce the main ingredient in the proof of our Main Theorem, namely Theorem [9]. In the Third section we will prove our Main Theorem using Theorem [9]. In section 4 we give the main idea of the proof of Theorem [9] and we finish up by proving Theorem [9] in the fifth section.

2. Preliminaries

Throughout the paper we write $\text{Aut}_0(\mathbb{C}^k)$ for the set of automorphisms of $\mathbb{C}^k$ that fix the origin. We denote by $B(r)$ the ball or radius $r$ centered at the origin and we write $B$ for the unit ball. For a sequence $F_1, F_2, \ldots$ we will use the notation $F(n)$ for the composition of the first $n$ maps, $F_n \cdot \cdots \cdot F_1$ and we write $F(m,n) = F_n \cdot \cdots \cdot F_m = F(n)F(m)^{-1}$.

It follows from a well known fact from linear algebra (called $QR$ factorization) that for any automorphism $F \in \text{Aut}_0(\mathbb{C}^k)$ we can find a unitary matrix $U \in U_k(\mathbb{C})$ such that $(U \circ F)'(0)$ is lower triangular. The map $U$ is in fact unique up to multiplying rows with complex numbers of unit length. We say that $UF$ is in lower triangular form.

**Remark 7.** Given some sequence $F_1, F_2, \ldots \in \text{Aut}_0(\mathbb{C}^k)$ we can find unitary matrices $U_1, U_2, \ldots$ such that $U_1 F_1, U_2 F_2 U_1^{-1}, U_3 F_3 U_2^{-1}, \ldots$ are all in lower triangular form.

If all the maps $F_n$ are attracting at the origin, then we define $\hat{F}_n = U_n F_n U_n^{-1}$. We have that $\hat{F}(n) = U_n F(n)$, therefore the basin of attraction of the sequence $\{\hat{F}_n\}$ is exactly equal to the basin of the original maps, so we may assume that our original maps were already in lower triangular form.

We will say that $F \in \text{Aut}_0(\mathbb{C}^k)$ which is in lower triangular form is **correctly ordered** if $F'(0)$ has diagonal entries (from upper left to lower right) $\lambda_1, \ldots, \lambda_k$ that satisfy the following condition:

$$|\lambda_j| |\lambda_i| < |\lambda_i|,$$

for $\ell \leq j$ and any $i$. Note in particular that $|\lambda_j| < 1$ for every $j$, and that $|\lambda_j|^2 < |\lambda_i|$ for $j \geq i$. Note also that if the eigenvalues satisfy the ordering $|\lambda_1| \geq \cdots \geq |\lambda_k|$ then $F$ is correctly ordered, but the sequence $|\lambda_1|, \ldots, |\lambda_k|$ may fail to be decreasing by a relatively small amount.
Definition 8. Let $\mathcal{F} \subset \text{Aut}_0(\mathbb{C}^k)$ be a family of correctly ordered automorphisms. We say that $\mathcal{F}$ is uniformly attracting if there exist $a, b \in \mathbb{R}$ with $0 < a < b < 1$ and we have that for every $F \in \mathcal{F}$ and every $z \in B$,

$$a\|z\| \leq \|F(z)\| \leq b\|z\|.$$  

Additionally we require that there exists a uniform $\xi < 1$ such that

\begin{equation}
|\lambda_i|\lambda_j| \leq \xi|\lambda_i|,
\end{equation}

for $l \leq j$ and any $i$.

We will now prove the following theorem:

Theorem 9. Let $F_1, F_2, \ldots \in \text{Aut}_0(\mathbb{C}^k)$ be a uniformly attracting sequence of correctly ordered automorphisms. Then the basin of attraction of $F_1, F_2, \ldots$ is biholomorphic to $\mathbb{C}^k$.

3. Proof of the Main Theorem

We will now prove our main Theorem, using Theorem 9. The proof consists of several (but finitely many) steps, and in each step we may decrease the value of $\epsilon$.

We may assume that $F$ is such that $F'(0)$ is a lower triangular matrix with diagonal entries (from upper left to lower right) $\lambda_1, \ldots, \lambda_k$ that satisfy $|\lambda_1| \geq \ldots \geq |\lambda_k|$. We may also assume that the off-diagonal terms of $F'(0)$ are arbitrarily small.

Now we show that we may assume that all the maps $f_n$ have an attracting fixed point at 0. Since $F'(0)$ is arbitrarily close to a diagonal matrix, we choose the neighborhood of the origin $N$ so small that $F$ is contracting on $N$, i.e. that there exist some $\theta < 1$ such that for all $x, y \in N$ we have that $\|F(x) - F(y)\| < \theta\|x - y\|$. Then we can make $\epsilon$ small enough such that every $f_n$ is also contracting on $N$ (with uniform constant $\theta < 1$ that is slightly increased if necessary). Let $x_0 = 0, x_1, x_2, \ldots$ be the orbit of 0, i.e. $x_n = f_n(x_{n-1})$. We have that $f_n(N) \subset \subset N$ for every $n \in \mathbb{N}$, so that $x_n \in N$ for every $n \in \mathbb{N}$. Let $T_n$ be the translation of $\mathbb{C}^k$ that maps $x_n$ to 0. Then define $\tilde{f}_n = T_n \circ f_n \circ T_n^{-1}$. We have that $\tilde{f}_n(0) = 0$ for all $n$, 0 is an attracting fixed point for every map $f_n$, and the maps $f_n$ are still arbitrarily close to the original map $F$. Since $\tilde{f}(n) = T_n f(n)$, we have that the approximate basin of attraction of the sequence $f_1, f_2, \ldots$ is exactly equal to the basin of attraction of $\tilde{f}_1, \tilde{f}_2, \ldots$. Therefore, we may as well assume that all the maps $f_n$ have 0 as an attracting fixed point, and we only consider the basin of attraction of the sequence $f_1, f_2, \ldots$.

We will now prove the Main Theorem in the case that $k = 2$. We may assume that $|\lambda_2|$ is strictly smaller than $|\lambda_1|$, otherwise the result follows easily from Theorem 3. The fact that $F'(0)$ is lower diagonal means exactly that $(0, 1)$ is an eigenvector of $F'(0)$. Let $\Phi$ be the action on $\mathbb{P}^1$ induced by the mapping $F'(0)$. It follows that $\Phi([0 : 1]) = [0 : 1]$, and the multiplier of $\Phi$ at $[0 : 1]$ is exactly $\frac{\lambda_2}{\lambda_1}$. Hence $[0 : 1]$ is a repelling fixed point, and there exists an arbitrarily small neighborhood of $[0 : 1]$, say $N$ such that $N \subset \subset \Phi(N)$.

Let $\phi_n$ be the action on $\mathbb{P}^1$ induced by $f_n'(0)$. We can make sure that $\phi_n$ is arbitrarily close to $\Phi$ such that $N \subset \subset \phi_n(N)$ for all $n \in \mathbb{N}$. Then we have that $N \supset \supset \phi_n^{-1}(N) \supset \supset \phi(2)^{-1}(N) \cdots$. It follows that

$$\bigcap_{n \in \mathbb{N}} \phi(n)^{-1}(N) \neq \emptyset.$$
Let $v \in \bigcap \phi(n)^{-1}(\mathcal{N})$, and let $v_0 = v, v_1, v_2, \ldots$ be the orbit of $v$, i.e. $\phi_n(v_{n-1}) = v_n$. We have that $v_n \in \mathcal{N}$ for every $n \in \mathbb{N}$.

Let $U_n$ be a unitary $2 \times 2$ matrix that maps some length 1-representative of $v_n$ in $\mathbb{C}^2$ onto $(0, 1)$, and define $\tilde{f}_n := U_n f_n U_n^{-1}$. Then it follows that $(0, 1)$ is an eigenvector of every map $\tilde{f}_n'(0)$, and thus we have that $\tilde{f}_n'(0)$ is lower triangular. The basin of the sequence $\tilde{f}_1, \tilde{f}_2, \ldots$ is exactly equal to $U_0(\Omega)$, so in particular biholomorphic to $\Omega$. Since the unitary matrices $U_n$ are arbitrarily close to the identity matrix, we have that $\tilde{f}_n$ is arbitrarily close to $F$, and it follows that the sequence $\tilde{f}_1, \tilde{f}_2, \ldots$ satisfies the properties in Theorem \ref{perturbthm} and we are done.

We will use a similar argument to that for 2 dimensions to prove the Main Theorem in the general case.

**Lemma 10.** Let $A$ be a lower triangular $k \times k$ matrix whose diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_k$ satisfy $|\lambda_1| \geq \cdots \geq |\lambda_k|$. Suppose that $l \in \{1, \ldots, k-1\}$ is such that $|\lambda_l| > |\lambda_{l+1}|$, and let $L$ be the linear subspace of $\mathbb{P}^{k-1}$ defined by

$$L = \{ [z_1 : \cdots : z_k] \in \mathbb{P}^{k-1} \mid z_1 = \cdots = z_l = 0 \}.$$

Then there exists an arbitrarily small neighborhood $\mathcal{N} \subset G(k-l-1, k-1)$ of $L$ such that $A(\mathcal{N}) \supset \supset \mathcal{N}$.

**Proof.** First assume that $A$ is diagonal. For $X \in G(k-l-1, k-1)$ close enough to $L$ we can write

$$X = \{ [z_1 : \cdots : z_k] \in \mathbb{P}^{k-1} \mid z_1 = \sum_{j \geq l} \epsilon_{1,j} z_j, \ldots, z_l = \sum_{l \geq j} \epsilon_{l,j} z_j \}.$$

We define $d(X, L) = \max \epsilon_{i,j}$.

For $\delta > 0$ small let $\mathcal{N}_\delta = \{ X \in G(k-l-1, k-1) \mid d(X, L) < \delta \}$. Since $A$ is diagonal and $|\lambda_i| > |\lambda_{i+1}|$ for all $i \leq l$ and $j \geq l+1$, we have that

$$A(\mathcal{N}_\delta) \supset \mathcal{N}(\lambda_{l+1}^{\delta}) \supset \mathcal{N}_\delta$$

Fix some $\delta$. Then $\mathcal{N}_\delta$ will also work for arbitrarily small perturbations of the diagonal matrix $A$.

We can conjugate any $A$ with an invertible linear mapping $T$ such that $T^{-1}AT$ is lower triangular and arbitrarily close to a diagonal mapping (whose diagonal entries are exactly those of $A$). Hence the set $T(\mathcal{N}_\delta)$ will suffice for $A$. \hfill $\Box$

Let $l \in \{1, \ldots, k-1\}$ be such that $|\lambda_l| > |\lambda_{l+1}| = |\lambda_k|$, and let $\Phi$ be the action of $F'(0)$ on $\mathbb{P}^{k-1}$ (as in the 2-dimensional case, we may assume that such an $l$ exists, otherwise we are done by Theorem \ref{perturbthm}). Then it follows from Lemma \ref{perturblem} that there exists a small neighborhood $\mathcal{N} \in G(k-l-1, k-1)$ of $L$ (where $L$ is as in Lemma \ref{perturblem}) such that $\Phi(\mathcal{N}) \supset \supset \mathcal{N}$. As in the 2-dimensional case, we denote by $\phi_n$ the action of $f_n$ on $\mathbb{P}^{k-1}$. Then we can decrease the value of $\epsilon$ if necessary to get that $\phi_n(\mathcal{N}) \supset \supset \mathcal{N}$ for every $n \in \mathbb{N}$. Therefore we can find an orbit of linear subspaces $L_0, L_1, \ldots$ in $\mathcal{N}$ with $\phi_n(L_{n-1}) = L_n$ for every $n \in \mathbb{N}$.

We denote by $U_n$ a unitary matrix arbitrarily close to the identity that maps a representative of $L_n$ in the unit sphere in $\mathbb{C}^k$ onto the set

$$T = \{ z \in \mathbb{C}^k \mid \|z\| = 1, z_1 = \cdots = z_l = 0 \}.$$

We define the maps $\tilde{f}_n$ by

$$\tilde{f}_n = U_n f_n U_n^{-1}.$$
Then we have that for every $v \in T$ there exists a $c \in (0, 1)$ such that $\tilde{f}_n'(0)(v) \in cT$, and therefore we have that for any $n \in \mathbb{N}$ the entries in the $p$-th rows and $q$-th columns of the matrix $f_n'(0)$ are equal to 0 for $p \leq l$ and $q \geq l + 1$. Since the matrices $U_n$ are arbitrarily close to the identity, we have that $\tilde{f}_n'(0)$ is arbitrarily close to $F'(0)$.

We may also assume that the $(k-l) \times (k-l)$ blocks in the lower right corner of the matrices $\tilde{f}_n'(0)$ are lower triangular, since we can apply QR-factorizations as in Remark 7 to these $(k-l) \times (k-l)$ blocks. The diagonal entries in these blocks must be arbitrarily close to $|\lambda_k|$, since in this last step have only composed with unitary matrices, and in Step 2 we made sure that $F'(0)$ is arbitrarily close to a diagonal matrix. Also, it follows that the off-diagonal terms in the $(k-l) \times (k-l)$ block in the lower right corner of the matrices $\tilde{f}_n'(0)$ are arbitrarily small.

Since the last $k-l$ columns of the matrices $\tilde{f}_n'(0)$ are already lower triangular, and the corresponding eigenvalues are the smallest in absolute value, we can restrict ourselves to the first $l$ dimensions and apply the same arguments to the next eigenvalues of $F'$ which are equal in absolute value.

In finitely many steps we get that all the maps $\tilde{f}_n'(0)$ are lower triangular matrices. To change from $f$ to $\tilde{f}$, we only compose with unitary matrices that are arbitrarily close to the identity. As a result, we get that the diagonal entries of the matrices $\tilde{f}_n'(0)$ are arbitrarily close to the diagonal entries of $F'(0)$. Therefore, all the maps in the sequence $f_1, f_2, \ldots$ are correctly ordered.

We have constructed unitary matrices $U_0, U_1, \ldots$ such that for every $n \in \mathbb{N}$ we have that $f(n) = U_n f(n) U_0^{-1}$. It follows that the basin of attraction of the sequence $f_1, f_2, \ldots$ is equal to the image of the basin of attraction of the original sequence $f_1, f_2, \ldots$ under the map $U_0^{-1}$, which is a biholomorphism. The sequence $\tilde{f}_1, \tilde{f}_2, \ldots$ satisfies the conditions in Theorem 9, and the Main Theorem.

4. Main Ideas of Theorem 9

In the proof of Theorem 9 we will construct a sequence of maps $\Psi_n := G(n)^{-1} \circ X_n \circ F(n)$ which converges to the Fatou Bieberbach mapping $\Psi : \Omega \to \mathbb{C}^k$. Here the $G_n$’s are lower triangular polynomial mappings (as in [RR]), and the $X_n$’s are polynomial mappings whose linear parts are the identity. To be more specific, we will start with some choice for $X_1$ and then define $X_n = [G_n \circ X_{n-1} \circ F_n^{-1}]_p$. Here $d$ is some large integer and we mean by $[\cdot]_d$ that we discard all terms of degree $d+1$ and higher. It follows immediately that

$$\|G_n^{-1}X_n F_n(z) - X_{n-1}(z)\| = O(\|z\|^{d+1}).$$

The challenge is to choose a bounded sequence of lower triangular polynomial mappings $\{G_n\}$ such that there exists a bounded orbit $\{X_n\}$, and to get the integer $d$ as large as necessary. We first show that we can do this in some simpler cases before we complete the proof of theorem 9.

Recall that a polynomial selfmap $G = (g_1, \ldots, g_k)$ of $\mathbb{C}^k$ with $G(0) = 0$ is called lower triangular if

$$g_j(z) = c_j z_j + h_j(z_1, \ldots, z_{j-1}),$$

for all $j \in \{1, \ldots, k\}$. 

Lemma 11. Let $G_1, G_2, \ldots$ be a sequence of lower triangular polynomial mappings of some fixed degree, whose coefficients are uniformly bounded. Then we have the following:

(a) The degrees of the maps $G(n)$ are bounded, and there is a constant $\beta < \infty$ so that

$$G(n)(B) \subset B(\beta^n)$$

(b) If also $|c_i| < \theta < 1$ for all $G_j$, then $G(n)(z) \to 0$, uniformly on compact subsets of $\mathbb{C}^k$, and for every $R > 0$ and $\epsilon > 0$ there exist an $N \in \mathbb{N}$ such that for $n \geq N$ we have that

$$G(n)(B(R)) \subset B(\epsilon).$$

This lemma is similar to Lemma 1 in the appendix of [RR], and so is its proof.

We now prove the following simple lemma.

Lemma 12. Let $F_1, F_2, \ldots$ be a uniformly bounded sequence of affine maps on $\mathbb{C}$ that are all expanding. Then there exists a bounded orbit $z_0, z_1, z_2, \ldots$.

Let $F_i(z) = az + b$. With uniformly bounded we mean that the constants $a$ and $b$ are bounded from above, and that the constants $a$ are also bounded from below by some $c > 1$.

Proof. Since the sequence is compact, there exists a constant $R$ such that

$$F_i(B(R)) \supset B(R)$$

for all $i \in \mathbb{N}$. Now we have that

$$B(R) \supset F(1)^{-1}B(R) \supset F(2)^{-1}B(R) \supset \ldots,$$

So the intersection $\bigcap F(n)^{-1} \Delta(R)$ contains a point $z_0$. Then we have that the sequence $z_0, z_1, z_2, \ldots$, where $z_n = F_n(z_{n-1})$, is a bounded orbit. Indeed, $|z_n| < R$ for all $n \in \mathbb{N}$.

4.1. Bounded Orbits in $\mathbb{C}$. The argument that shows that there exist bounded sequences $G_n$ and $X_n$ is somewhat complicated. To make it more clear we will first prove the existence in the one dimensional case, in which case it is much easier.

Let $f_1, f_2, \ldots$ be a sequence of polynomials with uniformly bounded coefficients of the form $f_n(z) = \lambda_n z + h.o.t.$, where $|\lambda_n| > \theta > 1$. Let $g_n(z) = \lambda_n^{-1} z$. Let $\mathcal{P}_d$ be the space of polynomials of the form $z + h.o.t.$ of degree at most $d$. Define the map $\phi_n : \mathcal{P}_d \to \mathcal{P}_d$ by $\phi_n(X) = [g_n \circ X \circ f_n]_d$, where we mean by $[\cdot]_d$ that we discard all terms of degree strictly higher than $d$.

Proposition 13. There exists a bounded sequence $X_0, X_1, \ldots \in \mathcal{P}_d$ with $X_n = \phi_n(X_{n-1})$.

Proof. We will use induction on the degree. For degree 1 the statement is clear since we take $X_n(z) = z$ for all $n$.

Suppose we have constructed a bounded orbit $Y_n$ that satisfies condition (2) for some $p = d - 1 \geq 1$. We will add terms of degree $p$ so that the sequence satisfies condition (2) for $p = d$. Say $X_0 = Y_0 + c_0 z^d$ for some $c_0 \in \mathbb{C}$. Then it is easy to see that $X_1 = \phi_1(X_0) = Y_1 + c_1 z^d$, where $c_1$ is equal to $\lambda_1^{d-1} c_0$ plus some constant depending linearly on the coefficients of $Y_0$ and the higher order terms of $f_1$ (which are uniformly bounded by assumption). We see that we get a uniformly bounded
sequence of expanding affine maps $\psi_n : \mathbb{C} \to \mathbb{C}$ that take $c_{n-1}$ to $c_n$. It now follows from Lemma 12 that we can choose $c_0$ such that the sequence $c_0, c_1, \ldots$ is bounded.

It follows by induction that we can find a bounded sequence $X_n$ for any $d \in \mathbb{N}$.

4.2. Degree 2 polynomials in $\mathbb{C}^2$. Next we show that we can get bounded sequences $\{G_n\}$ and $\{X_n\}$ in the two dimensional case for degree 2, where we can explicitly calculate the maps. We write $F_n(x, y) = (\lambda_n x, \mu_n y + a_n x) + \text{h.o.t.}$ and $(F_n^{-1})(x, y) = (\lambda_n^{-1} x, \mu_n^{-1} y + b_n x) + \text{h.o.t.}$ and let $G_n(x, y) = (\lambda_n x, \mu_n y + a_n x + d_n x^2)$ for some constants $d_n \in \mathbb{C}$ to be chosen later. We will also write $X_n(x, y) = (x + \alpha_n y^2 + \beta_n x y + \gamma_n x^2, y + \delta_n y^2 + \epsilon_n x y + \zeta_n x^2)$. We will identify the map $X_n$ with $(\alpha_n, \beta_n, \gamma_n, \delta_n, \epsilon_n, \zeta_n) \in \mathbb{C}^6$. Consider the map $X_{n-1} \mapsto X_n = [G_n \circ X_{n-1} \circ F_n^{-1}]_2$.

We have that
\[
\begin{align*}
\alpha_n &= \lambda_n \mu_n^{-2} \alpha_{n-1} + l_{1,n}, \\
\beta_n &= \mu_n^{-1} \beta_{n-1} + l_{2,n}(\alpha_{n-1}), \\
\gamma_n &= \lambda_n^{-1} \gamma_{n-1} + l_{3,n}(\alpha_{n-1}, \beta_{n-1}), \\
\delta_n &= \mu_n^{-1} \delta_{n-1} + l_{4,n}(\alpha_{n-1}), \\
\epsilon_n &= \lambda_n^{-1} \epsilon_{n-1} + l_{5,n}(\alpha_{n-1}, \beta_{n-1}, \delta_{n-1}), \\
\zeta_n &= \mu_n \lambda_n^{-2} \zeta_{n-1} + l_{6,n}(\alpha_{n-1}, \ldots, \epsilon_{n-1}) + d_n \lambda_n^{-2}.
\end{align*}
\]

Here the $l_{i,n}$ are linear maps that depend only on the coefficients of $F_n^{-1}$ (which are uniformly bounded) and on the given variables.

It follows from equation 1 that $|\lambda_n \mu_n^{-2}| > 1$ for any $n$, and therefore we get a uniformly bounded sequence of expanding affine maps $\alpha_{n-1} \mapsto \alpha_n$. Hence it follows from Lemma 12 that we can find $a_0$ such that the sequence $a_0, a_1, \ldots$ is bounded. Having fixed the $\alpha_n$’s, we can use the same argument for the $\beta_n$’s, since we also have that $|\mu_n^{-1}| > 1$ for all $n$. After we fix the $\beta_n$’s, we can find a bound on the $\gamma_n$’s, then the $\delta_n$’s, and finally the $\epsilon_n$’s.

We can’t use the same argument for the $\zeta_n$’s, since we may not have that $|\mu_n \lambda_n^{-2}| > 1$. However, we can choose $\zeta_0 = 0$, and then choose the constants $d_n$ such that $\zeta_n = 0$ for every $n$. Hence we get the bounded sequences $\{G_n\}$ and $\{X_n\}$.

5. Proof of Theorem 9

The argument that we use to construct bounded sequences $\{X_n\}$ and $\{G_n\}$ for higher dimensions and higher degrees is essentially the same as the argument for degree 2 polynomial mappings in $\mathbb{C}^2$.

We will write $\lambda_{n,1}, \ldots, \lambda_{n,k}$ for the diagonal terms of $F_n'(0)$.

Proposition 14. For any $d \geq 2$ we can find a bounded sequence of polynomial mappings $X_0, X_1, \ldots$, where $X_n = I_k + \text{h.o.t.}$, and a bounded sequence of lower triangular polynomial mappings $G_1, G_2, \ldots$, where $G_n(0) = F_n'(0)$, such that equation (2) holds for any $n \geq 1$.

Proof. We will construct bounded sequences $X_1, X_2, \ldots$ and $G_1, G_2, \ldots$ such that the following equation holds:
\[
X_n = [G_n X_{n-1} F_n^{-1}]_d.
\]
for every $n \in \mathbb{N}$. Notice that this will imply that Equation 2 holds for every $n \in \mathbb{N}$.

We write

$$X_n(z) = (x_{n,1}(z), x_{n,2}(z), \ldots, x_{n,k}(z)),$$

and

$$x_{n,j} = \sum_{\alpha} c_{n,j,\alpha} z^\alpha,$$

where $\alpha$ is a $k$-tuple, and $z^\alpha = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$. We refer to $c_{n,j,\alpha} z^\alpha$ as a term of degree $|\alpha| = \alpha_1 + \cdots + \alpha_k$, index $j$, and power $\alpha$. We similarly write $g_{n,j,\alpha} z^\alpha$ for the term of the mapping $G_n$ of degree $|\alpha|$, index $j$ and power $\alpha$.

For two $k$-tuples $\alpha$ and $\beta$ with $|\alpha| = |\beta|$ we will write $\alpha > \beta$ if $\alpha$ has a higher lexicographical ordering than $\beta$. That is, $\alpha > \beta$ if and only if there is some $j \in \{1, \ldots, k\}$ such that $\alpha_j < \beta_j$ and $\alpha_i = \beta_i$ for $i \in \{1, \ldots, j-1\}$ (so that $z_1^d$ comes first in the alphabet and $z_k^d$ comes last).

We will use induction on the degree and index and reverse induction on the power to fix all the terms of the sequences $\{X_n\}$ and $\{G_n\}$.

Let $\alpha$ be some $k$-tuple, and let $j \in \{1, \ldots, k\}$. Suppose that we have fixed all the terms of degree up to $|\alpha| - 1$ in the sequences $\{X_n\}$ and $\{G_n\}$, such that (3) holds for $d = |\alpha| - 1$, and assume that the corresponding coefficients are uniformly bounded. Also assume that we have fixed all terms of degree $|\alpha|$ and index up to $j - 1$ in the sequences $\{X_n\}$ and $\{G_n\}$ such that (3) holds for index $1, \ldots, j - 1$ and $d = |\alpha|$. Finally, assume that all terms of degree $|\alpha|$, index $j$ and powers $\beta > \alpha$ in the sequences $\{X_n\}$ and $\{G_n\}$ are fixed such that (3) holds for the terms of degree $|\alpha|$, index $j$ and power $\beta$.

We will continue to choose the terms of index $j$ and power $\alpha$ in the sequences $\{X_n\}$ and $\{G_n\}$. It is clear that after fixing the $c_{n,j,\alpha}$’s and $g_{n,j,\alpha}$’s, Equation (3) will still hold for $d = |\alpha| - 1$. Since the linear parts of the $G_n$’s are lower triangular, it also follows that (3) will still hold for degree $d = |\alpha|$ and index $1, \ldots, j - 1$. Furthermore, it follows from the fact that $F_n'(0)$ is lower triangular that (3) will still hold for the terms of degree $d = |\alpha|$, index $j$ and all powers $\beta$ with $\beta > \alpha$.

Once some $c_{n-1,j,\alpha}$ is chosen, it follows from (3) that we must have

$$c_{n,j,\alpha} = \lambda_{n,j} \lambda_n^{\alpha} c_{n-1,j,\alpha} + g_{n,j,\alpha} \lambda_n^{\alpha} + C_{n,j,\alpha}$$

where the constant $C_{n,j,\alpha}$ depends on the terms of $f_n^{-1}$ (which are uniformly bounded for any fixed degree) and the terms of $G_n$ and $X_{n-1}$ that we have already fixed (which are also uniformly bounded per hypothesis).

If the term of index $j$ and power $\alpha$ is lower triangular (i.e. if $\alpha_i = 0$ for $i \in \{j, \ldots, k\}$), then we can choose $c_{n,j,\alpha} = 0$ for all $n \in \mathbb{N}$ and we choose $g_{n,j,\alpha} = -C_{n,j,\alpha} \lambda_n^{\alpha}$ such that (3) holds for every $n \in \mathbb{N}$. It follows that the constants $g_{n,j,\alpha}$’s are bounded.

If the term of index $j$ and power $\alpha$ is not lower triangular, then it follows from the hypothesis that the sequence $\{F_n\}$ is uniformly attracting that $|\lambda_{n,j} \lambda_n^{\alpha}| > \xi > 1$ (where $\xi$ is as in Definition 5). We also have that $g_{n,j,\alpha} = 0$ for all $n \in \mathbb{N}$, and it follows that the sequence of maps given by $c_{n-1,j,\alpha} \mapsto c_{n,j,\alpha}$ is a uniformly bounded sequence of affine maps on $\mathbb{C}$ that are all expanding. It follows from Lemma 12 that there exists a bounded sequence $c_{n,j,\alpha}$.

So whether the term is lower triangular or not, we can always choose bounded sequences $\{x_{n,j,\alpha}\}$ and $\{g_{n,j,\alpha}\}$ such that (3) is satisfied.

The Proposition follows by induction. □
Now let $p$ be so large that for every $n \in \mathbb{N}$ the eigenvalues $\lambda_{n,1}, \ldots, \lambda_{n,k}$ of $F_n'(0)$ satisfy $|\lambda_{n,i}|^p < |\lambda_{n,j}|$ for $i,j \in \{1, \ldots, k\}$, where $\xi > 1$. We can do this because the sequence $\{F_n\}$ is uniformly attracting. We construct sequences $\{G_n\}$ and $\{X_n\}$ as in Proposition $\text{[14]}$ for $d = p$. It follows from part (a) of Lemma $\text{[11]}$ that there exists a $\gamma > 1$ such that

$$
\|G(n)^{-1}(w) - G(n)^{-1}(w')\| \leq \gamma^n \|w - w'\|,
$$

for any $w, w' \in B$. Recall that we assumed that there is a constant $b < 1$ such that $\|F_n(z)\| < b|z|$ for any $z \in B$. Now fix an integer $q$ such that $\gamma b^q < \alpha < 1$.

We change the mappings $X_n$ by adding higher order terms such that $\{2\}$ holds for $d = q + 1$. Since we chose $p \in \mathbb{N}$ such that $|\lambda_i|^p < |\xi \lambda_j|$ holds for all $i,j$, we have that $|\lambda_{n,j} \lambda_{n}^{-\alpha}| > \xi$ in $\{11\}$ even for the terms that are lower triangular. Hence we can make sure that the altered sequence $\{X_n\}$ is bounded without changing the sequence $\{G_n\}$. Therefore we have that

$$
g_{n+1}^{-1}X_n F_n(z) - X_{n-1}(z) \leq C\|z\|^{q+1},
$$

for some $C > 0$ independent of $n \in \mathbb{N}$ and every $z \in B$.

The proof of Theorem $\text{[9]}$ now follows quickly from an argument similar to that of the proof of Theorem $\text{[3]}$ which can be found in the appendix of $\text{[RR]}$.

Define the maps $\Psi_n : \Omega \to \mathbb{C}^k$ by

$$
\Psi_n := G(n)^{-1} \circ X_n \circ F(n).
$$

We will show that the maps $\Psi_n$ converge uniformly on compact subsets of $\Omega$ to a biholomorphic map from $\Omega$ onto $\mathbb{C}^k$.

Since the sequence $\{X_n\}$ is bounded there is a radius $r$ such that all the maps $X_n$ are invertible on $B(r)$.

Let $K$ be a compact set in $\Omega$. Then there is an $m \in \mathbb{N}$ such that $F(m)(K) \subset B(r)$. Let $n \geq m$. Then we have that $\|F(m,n)(z)\| \leq b^n - m \|z\|$, for all $z \in B(r)$.

We notice that

$$
\|\Psi_{n+1}(z) - \Psi_n(z)\| = \|G(n)^{-1}G_{n+1}^{-1}X_{n+1}F_{n+1} - G(n)^{-1}X_n F(n)(z)\|
\leq C \gamma^n (b^n - m \|F(m)(z)\|)^{q+1} < \hat{C} \alpha^n \|F(m)(z)\|^{q+1}.
$$

Since the sequence $\{\alpha^n\}$ is summable, the maps $\Psi_n$ converge uniformly on compact subsets of $\Omega$ to a holomorphic map $\Psi$. Also, for any compact subset $K$ of $\Omega$, there is a large $N \in \mathbb{N}$ such that for $n \geq N$, $\Psi_n$ is biholomorphic on $K$. It is a well known fact that the limit of a convergent sequence of biholomorphic mappings is either injective or degenerate everywhere. We have that $\Psi_n'(0) = I$ for all $n$, therefore we have that $\Psi'(0) = I$ and the limit map $\Psi$ is injective.

To prove surjectivity of the map $\Psi$, we may assume that we have chosen $r$ so small that

$$
r^2 \sum_{n=1}^{\infty} (Ca^n) < \frac{1}{2}.
$$

It follows from estimates $\{3\}$ and $\{4\}$ that for $z \in B(r)$ and for any $n \geq m \geq 1$ we have that

$$
\|G(m,m)^{-1}X_n F(m,n)(z) - z\| \leq \frac{\|z\|}{2}.
$$

Now let $K_m$ be the compact subset of the basin of attraction such that $F(m)(K_m) = B(r)$. Then we have that $G(m,n)^{-1}X_n F(n)(K_m) \supset B(r/2)$ for any $n \geq m$. It
follows from part (b) of Lemma 11 that for every $R > 0$ there exist an $N \in \mathbb{N}$ such that for $m \geq N$ we have that

$$B(R) \subset G(m)^{-1}(B(r/2)).$$

Therefore $B(R) \subset \Psi(K_m)$ for large enough $m$, and thus we have that $\Psi(\Omega)$ contains balls around the origin with arbitrarily large radii. This completes the proof.

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