A class of non-weight modules over the Schrödinger-Virasoro algebras

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Abstract. We construct and classify the free $U(CL_0 \oplus CM_0 \oplus CY_0)$-modules of rank 1 over the Schrödinger-Virasoro algebra $sv(s)$ for $s = 0$. Moreover, we show that the class of free $U(CL_0 \oplus CM_0)$-modules of rank 1 over the Schrödinger-Virasoro algebra $sv(s)$ for $s = \frac{1}{2}$ is nonexistent.

1. Introduction

In the 1960s, physicist Virasoro gave an important infinite dimensional Lie algebra—the Virasoro algebra with a basis $\{L_n, C | n \in \mathbb{Z}\}$ and the following relations

\[
[L_n, L_m] = (m - n)L_{m+n} + \delta_{m+n} \frac{n^3 - n}{12} C, \quad m, n \in \mathbb{Z},
\]

\[
[L_n, C] = 0, \quad n \in \mathbb{Z},
\]

which is a universal central extension of the Witt algebra. The Virasoro algebra is closely related to string theory \cite{17} and conformal field theory \cite{15}. Additionally, the representation theory of the Virasoro algebra is widely applied in many branches of mathematics and physics, such as vertex algebras \cite{7, 19} and quantum physics \cite{10}. Furthermore, the generalizations of the Virasoro algebra are extensively studied, such as the Schrödinger-Virasoro algebras \cite{4, 16, 20}, the Virasoro-like algebras \cite{9, 14}, the affine Virasoro algebras \cite{8} and so on.

The Schrödinger-Virasoro algebra is one of the natural generalizations of the Virasoro algebra, which was introduced by M. Henkel in \cite{11} during his study of the free Schrödinger equations in non-equilibrium statistical physics. The Schrödinger-Virasoro algebra $sv(s)$ for $s = 0$ or $\frac{1}{2}$ (c.f. \cite{11, 13}) is an infinite-dimensional Lie algebra over $\mathbb{C}$ with a basis $\{L_n, M_n, Y_p | n \in \mathbb{Z}, p \in \mathbb{Z} + s\}$ and satisfying the following relations

\[
[L_n, L_m] = (m - n)L_{m+n},
\]

\[
[L_n, Y_p] = (p - \frac{n}{2})Y_{p+n},
\]

\[
[L_n, M_m] = mM_{m+n},
\]

\[
[Y_p, Y_q] = (q - p)M_{p+q},
\]

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\[ [Y_p, M_m] = [M_n, M_m] = 0, \]

where \( m, n \in \mathbb{Z} \) and \( p, q \in \mathbb{Z} + s \). The Schrödinger-Virasoro algebras play an important role in many fields of mathematics and physics. Furthermore, there were a number of works on the Schrödinger-Virasoro algebras and their representations theory (see \([4,16,20,22]\) etc.). Actually, the weight modules of the Schrödinger-Virasoro algebras have been studied extensively. For instance, Li and Su studied the weight modules with finite-dimensional weight spaces over the Schrödinger-Virasoro algebras in \([16]\).

On the other hand, it is well known that the theory of non-weight modules has been extensively studied in the past few years. In \([1]\), Batra and Mazorchuk gave a general setup for studying Whittaker modules, from which Nilsson considered to construct families of simple \( \mathfrak{sl}_{n+1} \)-modules. In \([18]\), Nilsson determined and classified the free \( U(\mathfrak{h}) \)-modules of rank 1 over \( \mathfrak{sl}_{n+1} \), in which \( \mathfrak{h} \) is the standard Cartan subalgebra of \( \mathfrak{sl}_{n+1} \). Furthermore, the idea of Nilsson in \([18]\) has been generalized and applied into many infinite dimensional algebras. The non-weight modules which are free of rank 1 over the Kac-Moody algebras and the classical Lie superalgebras were studied in \([5,6]\) respectively. Moreover, such modules for the Virasoro algebra and its related algebras were investigated in \([2,12]\).

In addition, Chen and Guo constructed the free modules of rank 1 and determined the simplicity over the Heisenberg-Virasoro algebra and the \( W(2,2) \) algebra in \([3]\).

Our goal of the present paper is to focus on the non-weight modules over the Schrödinger-Virasoro algebra \( \mathfrak{sv}(s) \) for \( s = 0 \) or \( \frac{1}{2} \). We construct a class of free \( U(\mathbb{C}L_0 \oplus \mathbb{C}M_0 \oplus \mathbb{C}Y_0) \)-modules of rank 1 over the algebra \( \mathfrak{sv}(0) \), denoted by \( \Phi(\lambda, \alpha) \) for \( \lambda \in \mathbb{C}^* \) and \( \alpha \in \mathbb{C} \). Moreover, we classify all such modules and obtain the main result of the present paper: Let \( \mathfrak{sv}(0) \) be the Schrödinger-Virasoro algebra \( \mathfrak{sv}(s) \) for \( s = 0 \), if \( M \) is a free \( U(\mathbb{C}L_0 \oplus \mathbb{C}M_0 \oplus \mathbb{C}Y_0) \)-module of rank 1 over the algebra \( \mathfrak{sv}(0) \), then there exists some \( \lambda \in \mathbb{C}^* \) and \( \alpha \in \mathbb{C} \) such that \( M \cong \Phi(\lambda, \alpha) \). In addition, we prove that the class of free \( U(\mathbb{C}L_0 \oplus \mathbb{C}M_0) \)-modules of rank 1 over the algebra \( \mathfrak{sv}(\frac{1}{2}) \) is nonexistent.

The paper is organized as follows. In Section 2, we demonstrate the definition and properties of the algebra \( \mathfrak{sv}(0) \). Besides, we construct a class of free \( U(\mathbb{C}L_0 \oplus \mathbb{C}M_0 \oplus \mathbb{C}Y_0) \)-modules of rank 1 over the algebra \( \mathfrak{sv}(0) \). Section 3 is aimed to classify the free \( U(\mathbb{C}L_0 \oplus \mathbb{C}M_0 \oplus \mathbb{C}Y_0) \)-modules of rank 1 over the algebra \( \mathfrak{sv}(0) \). In Section 4, we mainly prove the nonexistence of free \( U(\mathbb{C}L_0 \oplus \mathbb{C}M_0) \)-modules of rank 1 over the algebra \( \mathfrak{sv}(\frac{1}{2}) \).

2. Preliminaries

In this section, we list some basic notations and useful results for our purpose. Throughout the paper, \( \mathbb{N}, \mathbb{C}, \mathbb{C}^*, \mathbb{Z} \) and \( \mathbb{Z}^* \) stand for the set of all natural numbers, complex numbers, nonzero complex numbers, integers and nonzero integers, respectively. Note that we focus on the Schrödinger-Virasoro algebra \( \mathfrak{sv}(s) \) for \( s = 0 \) in Section 2 and Section 3.
The Schrödinger-Virasoro algebra $\mathfrak{sv}(0)$ has a $\mathbb{C}$-basis $\{L_n, Y_n, M_n | n \in \mathbb{Z}\}$ with the following relations

\begin{align}
(2.1) & \quad [L_n, L_m] = (m - n)L_{m+n}, \\
(2.2) & \quad [L_n, Y_m] = (m - \frac{n}{2})Y_{m+n}, \\
(2.3) & \quad [L_n, M_m] = mM_{m+n}, \\
(2.4) & \quad [Y_n, Y_m] = (m - n)M_{m+n}, \\
(2.5) & \quad [Y_n, M_m] = [M_n, M_m] = 0,
\end{align}

where $m \in \mathbb{Z}, n \in \mathbb{Z}$.

From the above relations (2.1)-(2.5), we obtain some important formulas that will be needed later.

**Proposition 2.1.** For $i \in \mathbb{Z}, n \in \mathbb{Z}$ and $i \geq 1$, the following formulas hold:

1. $M_n L^i_0 = (L_0 - n)^i M_n$.
2. $M_n M^i_0 = M^i_0 M_n$.
3. $M_n Y^i_0 = Y^i_0 M_n$.
4. $Y_n L^i_0 = (L_0 - n)^i Y_n$.
5. $Y_n M^i_0 = M^i_0 Y_n$.
6. $Y_n Y^i_0 = Y^i_0 Y_n - ni Y^{i-1}_0 M_n$.
7. $L_n L^i_0 = (L_0 - n)^i L_n$.
8. $L_n M^i_0 = M^i_0 L_n$.
9. $L_n Y^i_0 = Y^i_0 L_n - \frac{ni}{2} Y^{i-1}_0 Y_n + \frac{n^2(i-1)}{4} Y^{i-2}_0 M_n$.

**Proof.** Here we only check formula (6) and formula (9), others can be easily shown by induction on $i$. For formula (6), it is obvious for $i = 1$ according to $Y_n Y^i_0 = Y^i_0 Y_n + [Y_n, Y^i_0] = Y^i_0 Y_n - nM_n$. Now assume that formula (6) holds for $i = k - 1$, that is,

$$Y_n Y^{k-1}_0 = Y^{k-1}_0 Y_n - (k - 1)n Y^{k-2}_0 M_n.$$  

For $i = k$, by the inductive assumption we have

$$Y_n Y^k_0 = Y_n Y^{k-1}_0 Y_0 = Y^{k-1}_0 Y_n Y_0 - (k - 1)n Y^{k-2}_0 M_n Y_0 = Y^{k}_0 Y_n - n Y^{k-1}_0 M_n - (k - 1)n Y^{k-1}_0 M_n = Y^{k}_0 Y_n - kn Y^{k-1}_0 M_n,$$

which implies that formula (6) holds.

For formula (9), we first have $L_n Y^0_0 = Y^0_0 L_n + [L_n, Y^0_0] = Y^0_0 L_n - \frac{1}{2} Y_n$, formula (9) holds for $i = 1$. Next suppose that formula (9) holds for $i = k - 1$, one has

$$L_n Y^{k-1}_0 = Y^{k-1}_0 L_n - \frac{(k - 1)n}{2} Y^{k-2}_0 Y_n + \frac{(k - 1)(k - 2)n^2}{4} Y^{k-3}_0 M_n.$$
For \( i = k \), we immediately obtain that

\[
L_n Y_0^k = L_n Y_0^{k-1} Y_0 - \frac{(k-1)n}{2} Y_0^{k-2} Y_0 + \frac{(k-1)(k-2)n^2}{4} Y_0^{k-3} M_n Y_0
\]

\[
= Y_0^{k-1} L_n Y_0 - \frac{n}{2} Y_0^{k-1} Y_0 - \frac{(k-1)n}{2} Y_0^{k-2} Y_0 + \frac{(k-1)n^2}{2} Y_0^{k-2} M_n
\]

\[
+ \frac{(k-1)(k-2)n^2}{4} Y_0^{k-2} M_n
\]

\[
= Y_0^k L_n - \frac{nk}{2} Y_0^{k-1} Y_0 + \frac{k(k-1)n^2}{4} Y_0^{k-2} M_n.
\]

Therefore formula (9) holds.

\[\square\]

**Remark 2.2.** Note that these formulas in Proposition 2.1 also hold in the case of \( i = 0 \).

**Definition 2.3.** For \( \lambda \in \mathbb{C}^* \) and \( \alpha \in \mathbb{C} \), define the action of a basis of the algebra \( \mathfrak{so}(0) \) on \( \Phi(\lambda, \alpha) := \mathbb{C}[s, t, v] \), in which \( \mathbb{C}[s, t, v] \) is the polynomial algebra in three indeterminates \( s, t \) and \( v \), as follows:

\[
L_m \cdot f(s, t, v) = \lambda^m \left( (s + m\alpha) f(s - m, t, v) - \frac{m}{2} \frac{\partial f(s - m, t, v)}{\partial v} + \frac{m^2}{4} \frac{\partial^2 f(s - m, t, v)}{\partial v^2} \right),
\]

(2.6)

\[
M_m \cdot f(s, t, v) = \lambda^m t f(s - m, t, v),
\]

(2.7)

\[
Y_m \cdot f(s, t, v) = \lambda^m \left( v f(s - m, t, v) - mt \frac{\partial f(s - m, t, v)}{\partial v} \right),
\]

(2.8)

where \( m \in \mathbb{Z} \) and \( f(s, t, v) \in \mathbb{C}[s, t, v] \).

**Proposition 2.4.** For \( \lambda \in \mathbb{C}^* \) and \( \alpha \in \mathbb{C} \), \( \Phi(\lambda, \alpha) \) is a class of free modules of rank 1 over the algebra \( \mathfrak{so}(0) \) under the action defined by (2.6) - (2.8).

**Proof.** For \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z} \), according to (2.6), one has the following

\[
L_n \cdot L_m \cdot f(s, t, v) = L_n \left[ \lambda^m \left( (s + m\alpha) f(s - m, t, v) - \frac{m}{2} \frac{\partial f(s - m, t, v)}{\partial v} + \frac{m^2}{4} \frac{\partial^2 f(s - m, t, v)}{\partial v^2} \right) \right]
\]

\[
= \lambda^{m+n} \left\{ (s + n\alpha)(s - n + m\alpha) f(s - m - n, t, v) + \left[ -\frac{n}{2} v(s - n + m\alpha)
\right.
\right.
\]

\[
- \frac{m}{2} (s + n\alpha) v + \frac{mn}{4} t \frac{\partial f(s - m - n, t, v)}{\partial v}
\]

\[
+ \frac{mn}{4} t v - \frac{mn^2}{8} v t \frac{\partial^2 f(s - m - n, t, v)}{\partial v^2}
\]

\[
+ \left[ \frac{mn^2}{8} v^2 - \frac{mn^2}{8} v t \right] \frac{\partial^3 f(s - m - n, t, v)}{\partial v^3}
\]

\[
+ \frac{m^2 n^2}{16} t^2 \frac{\partial^4 f(s - m - n, t, v)}{\partial v^4} \right\}.
\]
which implies that
\[ L_n \cdot L_m \cdot f(s, t, v) - L_m \cdot L_n \cdot f(s, t, v) \]
\[ = \lambda^{m+n} \left\{ (m-n) \left[ (s+(m+n)\alpha) f(s-m-n, t, v) - \frac{(m+n)}{2} v \frac{\partial f(s-m-n, t, v)}{\partial v} \right] 
\quad + \frac{(m+n)^2}{4} t \frac{\partial^2 f(s-m-n, t, v)}{\partial v^2} \right\} \]
\[ = (m-n)L_{m+n} \cdot f(s, t, v) \]
\[ = [L_n, L_m] \cdot f(s, t, v). \]

By (2.6) and (2.8), one easily gets
\[ L_n \cdot Y_m \cdot f(s, t, v) \]
\[ = L_n \cdot \left\{ \lambda^n \left[ v f(s-m, t, v) - m t \frac{\partial f(s-m, t, v)}{\partial v} \right] \right\} \]
\[ = \lambda^{m+n} \left\{ \left( s+n\alpha \right) v - \frac{n}{2} v \right\} f(s-m, t, v) + \left( -m t - \frac{n^2}{4} t - m(s+n\alpha) t \right) \]
\[ \frac{\partial f(s-m, t, v)}{\partial v} + \left( \frac{n^2}{4} v t + \frac{m n}{2} t \right) \frac{\partial^2 f(s-m, t, v)}{\partial v^2} \]
\[ - \frac{m n^2}{4} t^2 \frac{\partial^3 f(s-m, t, v)}{\partial v^3} \} \right\}. \]

Similarly, it immediately holds that
\[ Y_m \cdot L_n \cdot f(s, t, v) \]
\[ = Y_m \cdot \left\{ \lambda^n \left[ \left( s+n\alpha \right) f(s-m, t, v) - \frac{n}{2} v \frac{\partial f(s-m, t, v)}{\partial v} \right] \right\} \]
\[ = \lambda^{m+n} \left\{ v(s-m+n\alpha) f(s-m, t, v) + \left( -m t(s-m+n\alpha) - \frac{n}{2} v^2 + \frac{n}{2} m t \right) \right\} \]
\[ \frac{\partial f(s-m, t, v)}{\partial v} + \left( \frac{n}{2} m v + \frac{n^2}{4} v t \right) \frac{\partial^2 f(s-m, t, v)}{\partial v^2} \]
\[ - \frac{n^2}{4} m t^2 \frac{\partial^3 f(s-m, t, v)}{\partial v^3} \} \right\}. \]

Subtracting (2.10) from (2.9) gives rise to
\[ L_n \cdot Y_m \cdot f(s, t, v) - Y_m \cdot L_n \cdot f(s, t, v) \]
\[ = \lambda^{m+n} \left\{ (m-n) \left[ v f(s-m, t, v) - (m+n) t \frac{\partial f(s-m, t, v)}{\partial v} \right] \right\} \]
\[ = (m-n)Y_{m+n} \cdot f(s, t, v) \]
\[ = [L_n, Y_m] \cdot f(s, t, v). \]

Using (2.6) and (2.7), one obtains
\[ L_n \cdot M_m \cdot f(s, t, v) \]
\[ = L_n \cdot \left\{ \lambda^m f(s-m, t, v) \right\} \]
\[ = \lambda^{m+n} \left\{ s + n\alpha \right\} f(s-m, t, v) - \frac{n}{2} v t \frac{\partial f(s-m, t, v)}{\partial v} \]
\[ + \frac{n^2}{4} t^2 \frac{\partial^2 f(s-m, t, v)}{\partial v^2} \} \right\}. \]
Similarly, it is easy to see that
\[ M_m \cdot L_n \cdot f(s, t, v) = M_m \cdot \left[ \lambda^n \left( (s + n\alpha)f(s - n, t, v) - \frac{n}{2}v f(s - n, t, v) \right) \right. \]
\[ + \frac{n^2}{4} \frac{\partial^2 f(s - n, t, v)}{\partial v^2} \left. \right] \]
\[ = \lambda^{m+n} \left\{ t(s - m + n\alpha)f(s - m - n, t, v) - \frac{n}{2}tv f(s - m - n, t, v) \right. \]
\[ + \frac{n^2}{4}t^2 \frac{\partial^2 f(s - m - n, t, v)}{\partial v^2} \right\}. \] (2.12)

Considering (2.11) and (2.12), we have
\[ L_n \cdot M_m \cdot f(s, t, v) - M_m \cdot L_n \cdot f(s, t, v) \]
\[ = \lambda^{m+n} mt f(s - m - n, t, v) \]
\[ = m M_{m+n} \cdot f(s, t, v) \]
\[ = [L_n, M_m] \cdot f(s, t, v). \]

It follows from (2.8)
\[ Y_n \cdot Y_m \cdot f(s, t, v) \]
\[ = Y_n \left[ \lambda^m \left( \frac{f(s - n, t, v)}{v} - n \frac{\partial f(s - n, t, v)}{\partial v} \right) \right. \]
\[ = \lambda^{m+n} \left\{ (v^2 - nt) f(s - m - n, t, v) + nmt \frac{\partial^2 f(s - m - n, t, v)}{\partial v^2} \right\}, \]
which implies that
\[ Y_n \cdot Y_m \cdot f(s, t, v) - Y_m \cdot Y_n \cdot f(s, t, v) \]
\[ = \lambda^{m+n} (v^2 - nt - v^2 + nt) f(s - m - n, t, v) \]
\[ = (n - m) M_{m+n} \cdot f(s, t, v) \]
\[ = [Y_n, Y_m] \cdot f(s, t, v). \]

Finally, due to (2.8) and (2.7), it is straightforward to verify that
\[ Y_n \cdot M_m \cdot f(s, t, v) - M_m \cdot Y_n \cdot f(s, t, v) \]
\[ = Y_n \left[ \lambda^n \left( \frac{f(s - n, t, v)}{v} - n \frac{\partial f(s - n, t, v)}{\partial v} \right) \right. \]
\[ - M_m \left[ \lambda^n \left( \frac{f(s - n, t, v)}{v} - n \frac{\partial f(s - n, t, v)}{\partial v} \right) \right. \]
\[ = 0 = [Y_n, M_m] \cdot f(s, t, v) \]
and
\[ M_n \cdot M_m \cdot f(s, t, v) - M_m \cdot M_n \cdot f(s, t, v) \]
\[ = M_n \cdot (\lambda^n f(s - m, t, v)) - M_m \cdot (\lambda^n f(s - n, t, v)) \]
\[ = 0 = [M_n, M_m] \cdot f(s, t, v). \]

This completes the proof. \(\square\)

**Remark 2.5.** For \(\lambda \in \mathbb{C}^*\) and \(\alpha \in \mathbb{C}\), \(\Phi(\lambda, \alpha)\) is reducible over the Schrödinger-Virasoro algebra \(\mathfrak{so}(s)\) for \(s = 0\) as a consequence of Definition 2.3. In fact, it is effortless to get that \(t^i \mathbb{C}[s, t, v]\) is a submodule of \(\Phi(\lambda, \alpha)\) for \(i \in \mathbb{Z}\).
3. Modules over $\mathfrak{sv}(0)$

In this section, we classify the free $U(\mathbb{C}L_0 \oplus \mathbb{C}M_0 \oplus \mathbb{C}Y_0)$-modules of rank 1 over the algebra $\mathfrak{sv}(0)$. Indeed, we get the following main result.

**Theorem 3.1.** Let $\mathfrak{sv}(0)$ be the Schrödinger-Virasoro algebra $\mathfrak{sv}(s)$ for $s = 0$, if there exists a free $U(\mathbb{C}L_0 \oplus \mathbb{C}M_0 \oplus \mathbb{C}Y_0)$-module of rank 1 over the algebra $\mathfrak{sv}(0)$, denoted by $M$, then

$$M \cong \Phi(\lambda, \alpha)$$

for some $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$.

In order to prove Theorem 3.1 we first show several important lemmas.

**Lemma 3.2.** Let $M$ be a free $U(\mathbb{C}L_0 \oplus \mathbb{C}M_0 \oplus \mathbb{C}Y_0)$-module of rank 1 over the algebra $\mathfrak{sv}(0)$. For $m \in \mathbb{Z}$, assume that

$$L_m \cdot 1 = g_m(L_0, M_0, Y_0),$$
$$M_m \cdot 1 = a_m(L_0, M_0, Y_0),$$
$$Y_m \cdot 1 = p_m(L_0, M_0, Y_0),$$

in which $g_m(L_0, M_0, Y_0)$, $a_m(L_0, M_0, Y_0)$, $p_m(L_0, M_0, Y_0) \in M$, then $g_m(L_0, M_0, Y_0)$, $a_m(L_0, M_0, Y_0)$ and $p_m(L_0, M_0, Y_0)$ completely determine the action of $L_m$, $M_m$ and $Y_m$ on $M$.

**Proof.** Now take any $u(L_0, M_0, Y_0) = \sum_{i,j,k \geq 0} a_{i,j,k} L_0^i M_0^j Y_0^k \in M$ and according to formulas (7)-(9) in Proposition 2.1, one has

$$L_m \cdot u(L_0, M_0, Y_0) = L_m \cdot \sum_{i,j,k \geq 0} a_{i,j,k} L_0^i M_0^j Y_0^k$$

$$= \sum_{i,j,k \geq 0} a_{i,j,k} (L_0^i + m)^i M_0^j Y_0^k - Y_0^k g_m(L_0, M_0, Y_0) - \frac{mk}{2} Y_0^{k-1} p_m(L_0, M_0, Y_0)$$

$$+ \frac{k(k-1)m^2}{4} Y_0^{k-2} a_m(L_0, M_0, Y_0).$$

By formulas (4)-(6) in Proposition 2.1 it is easy to deduce that

$$Y_m \cdot u(L_0, M_0, Y_0) = Y_m \cdot \sum_{i,j,k \geq 0} a_{i,j,k} L_0^i M_0^j Y_0^k$$

$$= \sum_{i,j,k \geq 0} a_{i,j,k} (L_0^i + m)^i M_0^j Y_0^k - Y_0^k p_m(L_0, M_0, Y_0) - mk Y_0^{k-1} a_m(L_0, M_0, Y_0)$$

$$= u(L_0 - m, M_0, Y_0) p_m(L_0, M_0, Y_0) - m \frac{\partial u(L_0 - m, M_0, Y_0)}{\partial Y_0} a_m(L_0, M_0, Y_0).$$
Using formulas (11)-(13) in Proposition 2.1, one gets
\[ M_m \cdot u(L_0, M_0, Y_0) = M_m \cdot \sum_{i,j,k \geq 0} a_{i,j,k} L_0^i M_0^j Y_0^k \]
\[ = \sum_{i,j,k \geq 0} a_{i,j,k} (L_0 - m)^i M_0^j Y_0^k a_m(L_0, M_0, Y_0) \]
\[ = u(L_0 - m, M_0, Y_0) a_m(L_0, M_0, Y_0). \]
(3.1)

As a consequence, Lemma 3.2 holds. \(\square\)

**Lemma 3.3.** \(a_m(L_0, M_0, Y_0) \neq 0\) for all \(m \in \mathbb{Z}\).

**Proof.** Suppose that \(a_m(L_0, M_0, Y_0) = 0\) for some \(m \in \mathbb{Z}^*, \) then \(M_m \cdot M = 0\) owing to (3.1). Now according to relation (2.3), one has
\[ mM_0 \cdot M = [L_m, M_m] \cdot M = 0. \]
It suggests that \(M_0 \cdot M = 0\), hence \(a_0(L_0, M_0, Y_0) = 0\), which is a contradiction because of \(a_0(L_0, M_0, Y_0) = M_0\). Therefore, Lemma 3.3 holds. \(\square\)

**Lemma 3.4.** For all \(m \in \mathbb{Z}\), we have \(a_m(L_0, M_0, Y_0) \in \mathbb{C}[Y_0, M_0]\).

**Proof.** For \(m \in \mathbb{Z}\), now assume that \(a_m(L_0, M_0, Y_0) = \sum_{i=0}^{k_m} b_{m,i} L_0^i\), in which \(b_{m,i} \in \mathbb{C}[Y_0, M_0]\), \(k_m \in \mathbb{N}\) and \(b_{m,k_m} \neq 0\). Due to relation (2.5) and Proposition 2.1, one obtains the following
\[ 0 = [M_n, M_m] \cdot 1 \]
\[ = \sum_{i=0}^{k_m} b_{m,i}(L_0 - n)^i \sum_{j=0}^{k_n} b_{n,j} L_0^j - \sum_{j=0}^{k_n} b_{n,j}(L_0 - m)^j \sum_{i=0}^{k_m} b_{m,i} L_0^i \]
\[ \equiv b_{m,k_m} b_{n,k_n} (mk_m - nk_n) L_0^{k_n + k_m - 1} \text{ (mod } \sum_{i=0}^{k_m + k_n - 2} \mathbb{C}[Y_0, M_0] L_0^i), \]
yielding that \(mk_m - nk_n = 0\), thus \(k_m = mk_n\). If \(k_1 > 0\), then it is nonsense for \(k_m\) when \(m\) is negative, which forces \(k_1 = 0\), hence \(k_m = 0\) for \(m \in \mathbb{Z}\), therefore \(a_m(L_0, M_0, Y_0) \in \mathbb{C}[Y_0, M_0]\). \(\square\)

**Lemma 3.5.** For \(m \in \mathbb{Z}\), one has \(p_m(L_0, M_0, Y_0) \in \mathbb{C}[Y_0, M_0]\) and \(a_m(L_0, M_0, Y_0) \in \mathbb{C}[M_0]\).

**Proof.** For all \(m \in \mathbb{Z}\), according to Lemma 3.4, assume that \(a_m(L_0, M_0, Y_0) = \sum_{i=0}^{h_m} l_{m,i} Y_0^i\), where \(l_{m,i} \in \mathbb{C}[M_0]\), \(h_m \in \mathbb{N}\), \(l_{m,h_m} \neq 0\). Moreover, let \(p_m(L_0, M_0, Y_0) = \sum_{i=0}^{t_m} d_{m,i} L_0^i\), in which \(d_{m,i} \in \mathbb{C}[Y_0, M_0]\), \(t_m \in \mathbb{N}\) and \(d_{m,t_m} \neq 0\). Applying relation (2.5) and Proposition 2.1, one has
\[ 0 = [Y_n, M_m] \cdot 1 \]
\[ = \sum_{i=0}^{h_m} l_{m,i} \left( Y_0^i \sum_{j=0}^{t_m} d_{m,j} L_0^j - ni Y_0^{i-1} \sum_{j=0}^{h_m} l_{n,j} Y_0^j \right) \]
\[-\sum_{j=0}^{n} d_{n,j}(L_0 - m)^j \sum_{i=0}^{m} l_{m,i}Y_0^i \]

\[= \sum_{i=0}^{h_m} \sum_{j=0}^{t_n} l_{m,i}d_{n,j}L_0^j Y_0^i - \sum_{i=1}^{h_m} \sum_{j=0}^{t_n} l_{m,i}l_{n,j}n^{i-1}Y_0^j \]

\[-\sum_{i=0}^{h_m} \sum_{j=0}^{t_n} l_{m,i}d_{n,j}(L_0 - m)^j Y_0^i.\]

(3.2)

Observing all terms of $L_0$ in (3.2), one knows that $t_n \leq 1$, that is $t_n = 0$ or 1.

We claim that $t_n = 0$ for all $n \in \mathbb{Z}$. In fact, suppose $t_k = 1$ for $k \in \mathbb{Z}^*$, then (3.2) becomes

\[0 = m \sum_{i=0}^{h_m} l_{m,i}d_{k,1}Y_0^i - k \sum_{i=1}^{h_k} \sum_{j=0}^{h_k} l_{m,i}l_{k,j}Y_0^{i-1}Y_0^j,\]

taking $m = k$ and $m = -k$ respectively, one has

\[0 = k \sum_{i=0}^{h_k} l_{k,i}d_{k,1}Y_0^i - k \sum_{i=1}^{h_k} \sum_{j=0}^{h_k} l_{k,i}l_{k,j}Y_0^{i-1}Y_0^j\]

\[= k \sum_{i=0}^{h_k} l_{k,i}d_{k,1}Y_0^i - k \sum_{j=1}^{h_k} \sum_{i=0}^{h_k} l_{k,j}l_{k,i}Y_0^{j-1}Y_0^i\]

\[= k \sum_{i=0}^{h_k} l_{k,i}Y_0^i \left(d_{k,1} - \sum_{j=1}^{h_k} l_{k,j}Y_0^{j-1}\right)\]

and

\[0 = -k \sum_{i=0}^{h_{-k}} l_{-k,i}d_{k,1}Y_0^i - k \sum_{i=1}^{h_{-k}} \sum_{j=0}^{h_{-k}} l_{-k,i}l_{k,j}Y_0^{i-1}Y_0^j\]

\[= -k \sum_{i=0}^{h_{-k}} l_{-k,i}\left(d_{k,1}Y_0 + \sum_{j=0}^{h_k} l_{k,j}Y_0^j\right)Y_0^{i-1},\]

thus $d_{k,1} = \sum_{j=1}^{h_k} l_{k,j}Y_0^{j-1}$ and $d_{k,1}Y_0 + \sum_{j=0}^{h_k} l_{k,j}h_{-k}Y_0^j = 0$ which suggests $l_{k,0}h_{-k} = 0$

thereby $d_{k,1} = \sum_{j=1}^{h_k} l_{k,j}Y_0^{j-1} = -\sum_{j=1}^{h_k} l_{k,j}h_{-k}Y_0^{j-1} \neq 0$, which is impossible. Therefore, we get $t_n = 0$ for $n \in \mathbb{Z}$.

In addition, for $n \neq 0$ and $t_n = 0$, it is easy to find that (3.2) becomes

\[0 = -n \sum_{i=0}^{h_m} \sum_{j=0}^{h_n} l_{m,i}l_{n,j}Y_0^{i-1}Y_0^j,\]

whence $h_m = 0$ for all $m \in \mathbb{Z}$, then $a_m(L_0, M_0, Y_0) \in \mathbb{C}[M_0]$.

\[\square\]

**Lemma 3.6.** $\deg_{Y_0}p_m(L_0, M_0, Y_0) = 1$ for all $m \in \mathbb{Z}$.

**Proof.** For $m \in \mathbb{Z}$ assume that $p_m(L_0, M_0, Y_0) = \sum_{i=0}^{f_m} q_{m,i}Y_0^i$, $a_m(L_0, M_0, Y_0) = \sum_{i=0}^{f_m} s_{m,i}M_0^i$, in which $q_{m,i} \in \mathbb{C}[M_0], s_{m,i} \in \mathbb{C}$ and $q_{m,l_m} \neq 0, s_{m,0} \neq 0, l_m \in \mathbb{N}$,
\( e_m \in \mathbb{N} \). Due to relation (2.4), one easily finds
\[
(3.3) \quad [Y_n, Y_m] \cdot 1 = (m - n)M_{m+n} \cdot 1.
\]

By straightforward calculations, the left hand side of (3.3) is
\[
Y_nY_m \cdot 1 - Y_nY_m \cdot 1
\]
\[
= \sum_{i=0}^{f_m} q_{n,i} \left( Y_0^i Y_n \cdot 1 - niY_0^{i-1}M_{m} \cdot 1 \right) - \sum_{j=0}^{f_n} q_{m,j} \left( Y_0^j Y_m \cdot 1 - mjY_0^{j-1}M_{m} \cdot 1 \right)
\]
\[
= \sum_{i=0}^{f_m} q_{n,i} \left( Y_0^i \sum_{j=0}^{f_n} q_{n,j}Y_0^j - niY_0^{i-1} \sum_{k=0}^{e\epsilon_m} s_{n,k}M_0^k \right)
\]
\[
- \sum_{j=0}^{f_n} q_{m,j} \left( Y_0^j \sum_{i=0}^{f_m} q_{m,i}Y_0^i - mjY_0^{j-1} \sum_{k=0}^{e\epsilon_m} s_{m,k}M_0^k \right)
\]
\[
= \sum_{j=0}^{f_n} \sum_{k=0}^{e\epsilon_m} s_{m,k}q_{m,i}mY_0^{j-1}M_0^k - \sum_{i=0}^{f_m} \sum_{k=0}^{e\epsilon_m} s_{n,k}q_{n,i}nY_0^{i-1}M_0^k.
\]

While the right hand side of (3.3) does not contain the terms of \( Y_0 \) and is not zero from Lemma 3.5 and Lemma 3.3, thus we have \( f_n = f_m \neq 0 \), thereby one directly obtains \( f_m = f_0 = 1 \).

**Lemma 3.7.** For all \( m \in \mathbb{Z} \), one has \( \deg\, g_m(L_0, M_0, Y_0) = 1 \) and \( g_m(L_0, M_0, Y_0) \in \mathbb{C}[L_0, M_0] \).

**Proof.** Let \( g_m(L_0, M_0, Y_0) = \sum_{i=0}^{r_m} c_{n,i}L_0^i \), in which \( c_{n,i} \in \mathbb{C}[M_0, Y_0] \) and \( c_{n,r_m} \neq 0 \), \( r_n \in \mathbb{N} \). Thanks to Lemma 3.5, for \( m \in \mathbb{Z} \) assume that \( a_m(L_0, M_0, Y_0) = \sum_{i=0}^{e\epsilon_m} s_{m,i}M_0^i \), where \( s_{m,i} \in \mathbb{C} \) and \( s_{m,e_m} \neq 0 \), \( e_m \in \mathbb{N} \). Owing to relation (2.3), it is easy to see that
\[
(3.4) \quad [L_n, M_m] \cdot 1 = mM_{m+n} \cdot 1.
\]

Then the left hand side of (3.4) directly becomes
\[
L_nM_m \cdot 1 - M_mL_n \cdot 1
\]
\[
= \sum_{i=0}^{e\epsilon_m} s_{m,i}L_0^i \sum_{j=0}^{r_n} c_{n,j}L_0^j - \sum_{j=0}^{r_n} c_{n,j}(L_0 - m)^j \sum_{i=0}^{e\epsilon_m} s_{m,i}M_0^i
\]
\[
= - \sum_{i=0}^{e\epsilon_m} \sum_{j=1}^{r_n} s_{m,i}c_{n,j}M_0^i \left( -jmL_0^{j-1} + C^2_j(-m)^2L_0^{j-2} + \cdots + C^2_j(-m)^j \right).
\]

Note that the right hand side of (3.4) does not contain the terms of \( L_0 \), thereby one deduces that \( r_n \leq 1 \). If \( r_k = 0 \) for some \( k \in \mathbb{Z}^* \), then the left hand side of (3.4) vanishes, however the right hand side of (3.4) is not equal to zero if \( m \neq 0 \) as a result of Lemma 3.3, hence \( r_n = 1 \) for \( n \in \mathbb{Z} \).

In addition, it is clearly that the right hand side of (3.4) does not contain the terms of \( Y_0 \) according to Lemma 3.5, hence from (3.3) one knows \( c_{n,1} \in \mathbb{C}[M_0] \).
Next suppose that \(g_n(L_0, M_0, Y_0) = c_{n,1}L_0 + \sum_{i=0}^{\lambda_n} \alpha_{n,i} Y_0^i\), in which \(c_{n,1} \neq 0\) and \(\alpha_{n,i} \in \mathbb{C}[M_0]\), \(\alpha_{n,0} \neq 0\), \(\lambda_n \in \mathbb{N}\). On the one hand,

\[
L_n L_m \cdot 1 - L_m L_n \cdot 1
= L_n (c_{n,1} L_0 + \sum_{i=0}^{\lambda_n} \alpha_{n,i} Y_0^i) - L_m (c_{n,1} L_0 + \sum_{i=0}^{\lambda_n} \alpha_{n,i} Y_0^i)
= -\sum_{i=0}^{\lambda_n} \alpha_{n,i} (Y_0^i L_n \cdot 1 - \frac{mi}{2} Y_0^{i-1} Y_m \cdot 1 + \frac{m^2(i-1)}{4} Y_0^{i-2} M_m \cdot 1)
+ \sum_{i=0}^{\lambda_n} \alpha_{m,i} (Y_0^i L_n \cdot 1 - \frac{ni}{2} Y_0^{i-1} Y_n \cdot 1 + \frac{n^2(i-1)}{4} Y_0^{i-2} M_n \cdot 1)
+ c_{n,1} (L_n - n) L_n \cdot 1 - c_{n,1} (L_0 - m) L_m \cdot 1
= mc_{n,1} \sum_{i=0}^{\lambda_m} \alpha_{m,i} Y_0^i - nc_{m,1} \sum_{i=0}^{\lambda_m} \alpha_{m,i} Y_0^i - \sum_{i=0}^{\lambda_m} \alpha_{m,i} \frac{mi}{2} Y_0^{i-1} Y_n \cdot 1
+ \sum_{i=0}^{\lambda_m} \alpha_{m,i} \frac{ni}{2} Y_0^{i-1} Y_n \cdot 1 + \sum_{i=0}^{\lambda_m} \alpha_{m,i} \frac{n^2(i-1)}{4} Y_0^{i-2} M_n \cdot 1
- \sum_{i=0}^{\lambda_n} \alpha_{n,i} \frac{m^2(i-1)}{4} Y_0^{i-2} M_m \cdot 1 + (m - n) c_{n,1} c_{n,1} L_0,
\]

on the other hand,

\[
(m - n) L_{m+n} \cdot 1
= (m - n) (c_{m+n,1} L_0 + \sum_{i=0}^{\lambda_{m+n}} \alpha_{m+n,i} Y_0^i)
\]

According to relation (2.1), it is obvious that \(\lambda_m = \lambda_n = \lambda_{m+n} = \lambda_0 = 0\), that is, \(g_m(L_0, M_0, Y_0) \in \mathbb{C}[L_0, M_0]\).

From what has been discussed above, now we turn to the proof of Theorem 3.1.

**The Proof of Theorem 3.1.**

Lemma 3.2 suggests that \(g_m(L_0, M_0, Y_0), a_m(L_0, M_0, Y_0)\) and \(p_m(L_0, M_0, Y_0)\) completely determine the action of \(L_m, M_m\) and \(Y_m\) on \(M\). Thus the next task is just to determine \(g_m(L_0, M_0, Y_0), a_m(L_0, M_0, Y_0)\), and \(p_m(L_0, M_0, Y_0)\) for all \(m \in \mathbb{Z}\).

According to Lemma 3.5, Lemma 3.7, and Lemma 3.6 here we need to rewrite the preceding assumptions. For \(m \in \mathbb{Z}\), suppose that

\[
g_m(L_0, M_0, Y_0) = a_m + b_m L_0,
\]

\[
p_m(L_0, M_0, Y_0) = c_m + d_m Y_0,
\]

in which \(a_m, b_m, c_m, d_m \in \mathbb{C}[M_0]\) and \(b_m \neq 0, d_m \neq 0\). In particular, \(a_0 = c_0 = 0\) and \(b_0 = d_0 = 1\). Thanks to relation (2.1), one gets directly

\[
[L_n, L_m \cdot 1] = (m - n) L_{m+n} \cdot 1.
\]

Substituting (3.6) into (3.8), one immediately finds that the left hand side of (3.8) becomes

\[
L_n L_m \cdot 1 - L_m L_n \cdot 1
\]
Comparing the coefficients of \( L_0 \) of (3.9) and (3.10), one has \( b_n b_m = b_{m+n} \), yielding that \( b_m = \lambda^m (\lambda := b_1 \in \mathbb{C}^*) \) for all \( m \in \mathbb{Z} \). Moreover, observing the constant term in (3.9) and (3.10), one obtains that \( ma_m b_n - na_n b_m = (m - n)a_{m+n} \). Choosing \( m = 1 \) in the formula gives \( \lambda^a_1 - n \lambda a_n = (1 - n)a_{n+1} \), from which we have \( a_m = m \lambda^m a \) by induction on \( n \), where \( \alpha = \frac{a_1}{a_0} \) and \( a_1 \in \mathbb{C} \). Therefore, \( g_m(L_0, M_0, Y_0) = \lambda^m (L_0 + m \alpha) \).

Similarly, according to relation (2.2), it is effortless to see that

\[
[L_n, Y_m] \cdot 1 = (m - \frac{n}{2}) Y_{m+n} \cdot 1.
\]

Thanks to (3.6) and (3.7) it is easy to deduce that the left hand side of (3.11) is

\[
L_n Y_m \cdot 1 - Y_m L_n \cdot 1 = L_n (c_m + d_m Y_0) - Y_m (a_n + b_n L_0)
\]

and applying (3.7) one finds that the right hand side of (3.11) becomes

\[
(m - \frac{n}{2}) Y_{m+n} \cdot 1
\]

Hence \( m c_m b_n - \frac{n}{2} d_m c_n = (m - \frac{n}{2}) c_{m+n} \) and \( m b_n d_m - \frac{n}{2} d_m d_n = (m - \frac{n}{2}) d_{m+n} \). Taking \( m = 1 \) in the latter formula one gets \( \lambda^a d_1 - \frac{n}{2} d_1 d_n = (1 - \frac{n}{2}) d_{n+1} \), which implies that \( d_m = \lambda^m \) for \( m \in \mathbb{Z} \). Besides, choosing \( m = \pm 1 \) in the former formula gives

\[
\lambda^a c_1 - \frac{n}{2} \lambda c_n = (1 - \frac{n}{2}) c_{n+1}
\]

and

\[
- c_1 \lambda^n - \frac{n}{2} \lambda^{-1} c_n = (-1 - \frac{n}{2}) c_{n-1}.
\]

Setting \( n = \pm 1 \) in (3.14), one deduces that \( c_2 = \lambda c_1 \) and \( c_{-1} = -2 \lambda^{-2} c_1 \). In addition, taking \( n = 2 \) in (3.15), one easily has

\[
- c_1 \lambda^2 - \lambda^{-1} c_2 = -2 c_1.
\]

It yields that \( c_2 = 4 \lambda c_1 \), whence \( c_1 = 0 \). Thus from (3.14) or (3.15), it is easy to find that \( c_m = 0 \) for \( m \in \mathbb{Z} \). Therefore \( p_m(L_0, M_0, Y_0) = \lambda^m Y_0 \).

Now it remains to consider how to determine \( a_m(L_0, M_0, Y_0) \). For \( m \neq 0 \), continue to use the assumption \( a_m(L_0, M_0, Y_0) = \sum_{i=0}^{m} s_{m,i} M_0^i \), in which \( s_{m,i} \in \mathbb{C} \) and \( s_{m,0} \neq 0 \).

Due to relation (2.3), it is clearly to see that

\[
[L_n, M_m] \cdot 1 = m M_{m+n} \cdot 1.
\]
Now calculating the left hand side of (3.16), one always has
\[
L_n M_m \cdot 1 - M_n L_n \cdot 1
= \sum_{i=0}^{\infty} s_{m,i} \lambda^n (L_0 + n\alpha) M_0^i = \sum_{i=0}^{\infty} s_{m,i} \lambda^n (L_0 - m + n\alpha) M_0^i
= m \sum_{i=0}^{\infty} s_{m,i} \lambda^n M_0^i,
\]
thus one immediately finds that
\[
(3.17) \quad m \sum_{i=0}^{\infty} s_{m,i} \lambda^n M_0^i = m \sum_{i=0}^{\infty} s_{m+n,i} M_0^i.
\]
Whence one knows
\[
e_m = e_{m+n} = e_0 = 1 \quad \text{for all } m \in \mathbb{Z}.
\]
Therefore (3.17) becomes
\[
m \lambda^n (s_{m,0} + s_{m,1} M_0) = m (s_{m+n,0} + s_{m+n,1} M_0).
\]
Hence one always has
\[
s_{m+n,0} = \lambda^n s_{m,0} \quad \text{and} \quad s_{m+n,1} = \lambda^n s_{m,1},
\]
from which it is direct to obtain that for all \( m \in \mathbb{Z} \)
\[
s_{m,0} = \lambda^n s_{0,0} = 0 \quad \text{and} \quad s_{m,1} = \lambda^n s_{0,1} = \lambda^n.
\]
By putting everything together, it can be seen that
\[
g_m(L_0, M_0, Y_0) = \lambda^m (L_0 + m\alpha),
p_m(L_0, M_0, Y_0) = \lambda^m Y_0,
a_m(L_0, M_0, Y_0) = \lambda^m M_0,
\]
for \( \lambda \in \mathbb{C}^*, \alpha \in \mathbb{C} \) and all \( m \in \mathbb{Z} \). Thus according to Lemma 3.2 Theorem 3.1 holds.

4. Modules over \( \mathfrak{sv}(\frac{1}{2}) \)

In this section, we focus on the Schrödinger-Virasoro algebra \( \mathfrak{sv}(s) \) for \( s = \frac{1}{2} \) and show that the class of free \( U(CL_0 \oplus CM_0) \)-modules of rank 1 over the algebra \( \mathfrak{sv}(\frac{1}{2}) \) is nonexistent.

The Schrödinger-Virasoro algebra \( \mathfrak{sv}(\frac{1}{2}) \) is an infinite-dimensional Lie algebra over \( \mathbb{C} \) with a basis
\[
\{L_n, M_n, Y_p | n \in \mathbb{Z}, p \in \mathbb{Z} + \frac{1}{2}\}
\]
and has the following relations
\[
(4.1) \quad [L_n, L_m] = (m - n)L_{m+n},
(4.2) \quad [L_n, Y_p] = (p - \frac{n}{2})Y_{p+n},
(4.3) \quad [L_n, M_m] = mM_{m+n},
(4.4) \quad [Y_p, Y_q] = (q - p)M_{p+q},
(4.5) \quad [Y_p, M_m] = [M_n, M_m] = 0,
\]
where \( m, n \in \mathbb{Z}, p, q \in \mathbb{Z} + \frac{1}{2} \).

From the above relations (4.1)-(4.5), there are some useful formulas that are similar to Proposition 2.7.
\textbf{Proposition 4.1.} For \(i \in \mathbb{Z}, n \in \mathbb{Z}, p \in \mathbb{Z} + \frac{1}{2}\) and \(i \geq 0\), the following formulas hold:

\begin{enumerate}
\item \(M_n L_0^i = (L_0 - n)^i M_n\).
\item \(M_n M_0^i = M_0^i M_n\).
\item \(Y_p L_0^i = (L_0 - p)^i Y_p\).
\item \(Y_p M_0^i = M_0^i Y_p\).
\item \(L_n L_0^i = (L_0 - n)^i L_n\).
\item \(L_n M_0^i = M_0^i L_n\).
\end{enumerate}

It is straightforward to check these formulas by induction on \(i\) for all \(i \geq 0\).

\textbf{Theorem 4.2.} Let \(\mathfrak{sv}(\frac{1}{2})\) be the Schrödinger-Virasoro algebra \(\mathfrak{sv}(s)\) for \(s = \frac{1}{2}\), denote by \(\mathcal{M}\) the set of all free \(U(\mathbb{C}L_0 \oplus \mathbb{C}M_0)\)-modules of rank 1 over \(\mathfrak{sv}(\frac{1}{2})\), then

\[\mathcal{M} = \emptyset.\]

\textbf{Proof.} Assume on the contrary that \(\mathcal{M} \neq \emptyset\) and let \(N \in \mathcal{M}\), so \(N = U(\mathbb{C}L_0 \oplus \mathbb{C}M_0)\). Take any \(w(L_0, M_0) = \sum_{i,j \geq 0} a_{i,j} L_0^i M_0^j \in N\) and suppose that

\[L_m \cdot 1 = g_m(L_0, M_0),\]

\[M_m \cdot 1 = a_m(L_0, M_0),\]

\[Y_p \cdot 1 = h_p(L_0, M_0),\]

where \(m \in \mathbb{Z}, p \in \mathbb{Z} + \frac{1}{2}\) and \(g_m(L_0, M_0), a_m(L_0, M_0), h_p(L_0, M_0) \in N\).

Next we will prove that \(g_m(L_0, M_0), a_m(L_0, M_0)\) and \(h_p(L_0, M_0)\) completely determine the action of \(L_m, M_m\) and \(Y_p\) on \(N\).

It follows from formulas (3)-(6) in Proposition 4.1

\[L_m \cdot w(L_0, M_0) = L_m \cdot \left( \sum_{i,j \geq 0} a_{i,j} L_0^i M_0^j \right) = \sum_{i,j \geq 0} a_{i,j,k}(L_0 - m)^i M_0^j g_m(L_0, M_0) = w(L_0 - m, M_0) g_m(L_0, M_0).\]

According to formulas (1)-(4) in Proposition 4.1 one always has

\[Y_p \cdot w(L_0, M_0) = Y_p \cdot \left( \sum_{i,j \geq 0} a_{i,j} L_0^i M_0^j \right) = \sum_{i,j \geq 0} a_{i,j}(L_0 - p)^i M_0^j h_p(L_0, M_0) = w(L_0 - p, M_0) h_p(L_0, M_0).\]

By formulas (1)-(2) in Proposition 4.1 it is easy to obtain that

\[M_m \cdot w(L_0, M_0) = M_m \cdot \left( \sum_{i,j \geq 0} a_{i,j} L_0^i M_0^j \right).\]
Combining relation (4.5) with Proposition 4.1, one has
\[ a_m(L_0 - m)^i M_0^i a_m(L_0, M_0) \]
(4.8)

hence referring to the proof of Lemma 3.4, therefore □ of Theorem 4.2.

The above three formulas (4.6)-(4.8) suggest that one only needs to consider how to determine \( g_m(L_0, M_0), a_m(L_0, M_0), \) and \( h_p(L_0, M_0) \) for \( m \in \mathbb{Z} \) and \( p \in \mathbb{Z} + \frac{1}{2} \) if one wants to know the action of \( L_m, M_m \) and \( Y_p \) on \( M \). Whence we will discuss how to determine them in the next step.

For \( m \in \mathbb{Z} \), assume that \( a_m(L_0, M_0) = \sum_{i=0}^{k_m} b'_{m,i} L_0^i \), in which \( b'_{m,i} \in \mathbb{C}[M_0] \), \( k_m \in \mathbb{N} \) and \( b'_{m,k_m} \neq 0 \). According to relation (4.5) and Proposition 4.1 one obtains \( k_m = 0 \) referring to the proof of Lemma 4.4 therefore \( a_m(L_0, M_0) \in \mathbb{C}[M_0] \).

In addition, we claim that \( a_m(L_0, M_0) \neq 0 \) for all \( m \in \mathbb{Z} \). In fact, if there exists \( m \in \mathbb{Z}^* \) such that \( a_m(L_0, M_0) = 0 \), then \( M_m \cdot M = 0 \) due to formula (4.8). According to relation (4.3), one immediately deduces that

\[ m M_0 \cdot M = [L_m, M_m] \cdot M = 0, \]

which implies that \( a_0(L_0, M_0) = 0 \). This is a contradiction.

For \( m \in \mathbb{Z}^* \) and \( p \in \mathbb{Z} + \frac{1}{2} \), suppose that \( a_m(L_0, M_0) = \sum_{i=0}^{e_m} s'_{m,i} M_0^i \) and \( h_p(L_0, M_0) = \sum_{i=0}^{t_p} d'_{p,i} L_0^i \), in which \( s'_{m,i} \in \mathbb{C}, e'_m \in \mathbb{N}, s'_{m,e'_m} \neq 0 \) and \( d'_{p,i} \in \mathbb{C}[M_0], t'_p \in \mathbb{N}, d'_{p,t'_p} \neq 0 \).

Combining relation (4.5) with Proposition 4.1 one has

\[ 0 = [Y_p, M_m] \cdot 1 \]
\[ = \sum_{i=0}^{e_m} s'_{m,i} M_0^i \sum_{j=0}^{t'_p} d'_{p,j} L_0^j - \sum_{j=0}^{t'_p} d'_{p,j} (L_0 - m)^j \sum_{i=0}^{e_m} s'_{m,i} M_0^i \]
\[ = - \sum_{i=0}^{e_m} \sum_{j=0}^{t'_p} s'_{m,i} d'_{p,j} M_0^j \left( (-j m L_0^{-1} + C_2 (-m)^2 L_0^{-2} + \cdots + C_j (-m)^j) \right), \]

hence \( t'_p = 0 \), that is \( h_p(L_0, M_0) \in \mathbb{C}[M_0] \). Thus from (4) in Proposition 4.1 and relation (4.4), we always have

\[ [Y_p, Y_q] \cdot 1 = 0 \neq (q - p) M_{p+q} \]

for \( p \neq q \), which is impossible.

From what has been discussed above, we draw a conclusion that the class of free \( U(\mathbb{C}L_0 \oplus \mathbb{C}M_0) \)-modules of rank 1 over the algebra \( \mathfrak{sw}(L_{\frac{1}{2}}) \) is not existent, which completes the proof of Theorem 4.2.

\[ \square \]

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