On regular graphs with four distinct eigenvalues

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Abstract Let $G(4,2)$ be the set of connected regular graphs with four distinct eigenvalues in which exactly two eigenvalues are simple, $G(4,2,-1)$ (resp. $G(4,2,0)$) the set of graphs belonging to $G(4,2)$ with $-1$ (resp. $0$) as an eigenvalue, and $G(4,\geq -1)$ the set of connected regular graphs with four distinct eigenvalues and second least eigenvalue not less than $-1$. In this paper, we prove the non-existence of connected graphs having four distinct eigenvalues in which at least three eigenvalues are simple, and determine all the graphs in $G(4,2,-1)$. As a by-product of this work, we characterize all the graphs belonging to $G(4,\geq -1)$ and $G(4,2,0)$, respectively, and show that all these graphs are determined by their spectra.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph on $n$ vertices with adjacency matrix $A = A(G)$. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_t$ all the distinct eigenvalues of $A$ with multiplicities $m_1, m_2, \ldots, m_t$ ($\sum_{i=1}^{t} m_i = n$), respectively. These eigenvalues are also called the eigenvalues of $G$. All the eigenvalues together with their multiplicities are called the spectrum of $G$ denoted by $\text{Spec}(G) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \ldots, [\lambda_t]^{m_t}\}$. If $G$ is a connected $k$-regular graph, then $\lambda_1$ denotes $k$, and has multiplicity $m_1 = 1$.

A graph $G$ is said to be determined by its spectrum (DS for short) if $G \cong H$ whenever $\text{Spec}_A(G) = \text{Spec}_A(H)$ for any graph $H$. Also, a graph $G$ is called walk-regular if for which the number of walks of length $r$ from a given vertex $x$ to itself (closed walks) is independent of the choice of $x$ for all $r$ (see [16]). Note that a walk-regular graph is always regular, but in general the converse is not true.

Throughout this paper, we denote the neighbourhood of a vertex $v \in V(G)$ by $N_G(v)$, the complete graph on $n$ vertices by $K_n$, the complete multipartite graph with $s$ parts of sizes $n_1, \ldots, n_s$ by $K_{n_1,\ldots,n_s}$, and the graph obtained by removing a perfect matching from...
$K_{n,n}$ by $K_{n,n}^\ast$. Also, the $n \times n$ identity matrix, the $n \times 1$ all-ones vector and the $n \times n$ all-ones matrix will be denoted by $I_n$, $e_n$ and $J_n$, respectively.

Connected graphs with a few eigenvalues have aroused a lot of interest in the past several decades. This problem was perhaps first raised by Doob [15]. It is well known that connected regular graphs having three distinct eigenvalues are strongly regular graphs [21], and connected regular bipartite graphs having four distinct eigenvalues are the incidence graphs of symmetric balanced incomplete block designs [2, 7]. Furthermore, connected non-regular graphs with three distinct eigenvalues and least eigenvalue $-2$ were determined by Van Dam [9]. Very recently, Cioab˘a et al. in [6] (resp. [5]) determined all graphs with at most two eigenvalues (multiplicities included) not equal to $\pm 1$ (resp. $-2$ or $0$). De Lima et al. in [17] determined all connected non-bipartite graphs with all but two eigenvalues in the interval $[-1, 1]$. For more results on graphs with few eigenvalues, we refer the reader to [1, 3, 4, 8, 11–15, 18, 20].

Van Dam in [8, 12] investigated the connected regular graphs with four distinct eigenvalues. He classified such graphs into three classes according to the number of integral eigenvalues (see Lemma 2.1 below). Based on Van Dam’s classification and the number of simple eigenvalues, we can classify such graphs more precisely, that is, if $G$ is a connected $k$-regular graph with four distinct eigenvalues, then

1. $G$ has at least three simple eigenvalues, or
2. $G$ has two simple eigenvalues:
   (2a) $G$ has four integral eigenvalues in which two eigenvalues are simple;
   (2b) $G$ has two integral eigenvalues, which are simple, and two eigenvalues of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b \in \mathbb{Z}, b > 0$, with the same multiplicity, or
3. $G$ has one simple eigenvalue, i.e., its degree $k$:
   (3a) $G$ has four integral eigenvalues;
   (3b) $G$ has two integral eigenvalues, and two eigenvalues of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b \in \mathbb{Z}, b > 0$, with the same multiplicity;
   (3c) $G$ has one integral eigenvalue, its degree $k$, and the other three have the same multiplicity $m = \frac{1}{3}(n-1)$, and $k = m$ or $k = 2m$.

In this paper, we continue to focus on connected regular graphs with four distinct eigenvalues. Concretely, we show that there are no graphs in (1), and give a complete characterization of the graphs belonging to $G(4, 2, -1)$: if $-1$ is a non-simple eigenvalue, we determine all such graphs; if $-1$ is a simple eigenvalue, we prove that such graphs cannot belong to (2a) and (2b), respectively, and so do not exist. In the process, we determine all the graphs in $G(4, \geq -1)$ and $G(4, 2, 0)$, respectively, and show that all these graphs are DS.

## 2 Main tools

In this section, we recall some results from the literature that will be useful in the next section.

**Lemma 2.1.** (See [8, 12].) If $G$ is a connected $k$-regular graph on $n$ vertices with four distinct eigenvalues, then

1. $G$ has four integral eigenvalues, or
Lemma 2.2. (See [2].) If \( G \) is connected and regular with four distinct eigenvalues, then \( G \) is walk-regular.

Let \( G \) be a \( k \)-regular graph. We say that \( G \) admits a regular partition into halves with degrees \((a, b)\) \((a + b = k)\) if we can partition the vertices of \( G \) into two parts of equal size such that every vertex has \( a \) neighbors in its own part and \( b \) neighbors in the other part [8].

Lemma 2.3. (See [8].) Let \( G \) be a connected walk-regular graph on \( n \) vertices and degree \( k \), having distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_t \), of which an eigenvalue unequal to \( k \), say \( \lambda_j \), has multiplicity 1. Then \( n \) is even and \( G \) admits a regular partition into halves with degrees \((\frac{1}{2}(k + \lambda_j), \frac{1}{2}(k - \lambda_j))\). Moreover, \( n \) is a divisor of

\[
\prod_{i \neq j} (k - \lambda_i) + \prod_{i \neq j} (\lambda_j - \lambda_i) \quad \text{and} \quad \prod_{i \neq j} (k - \lambda_i) - \prod_{i \neq j} (\lambda_j - \lambda_i).
\]

From the proof of Lemma 2.3, we obtain the following corollary immediately.

Corollary 2.1. Under the assumption of Lemma 2.3, the eigenvector of \( \lambda_j \) can be written as \( x_j = \frac{1}{\sqrt{n}}(e_{ij} - e_{ij})^T \), and the vertex partition \( V(G) = V_1 \cup V_2 \) with \( V_1 = \{ v \in V \mid x_j(v) = 1 \} \) and \( V_2 = \{ v \in V \mid x_j(v) = -1 \} \) is just the regular partition of \( G \) into halves with degrees \((\frac{1}{2}(k + \lambda_j), \frac{1}{2}(k - \lambda_j))\) described in Lemma 2.3.

A balanced incomplete block design, denoted by \( BIBD \), consists of \( v \) elements and \( b \) subsets of these elements called blocks such that each element is contained in \( t \) blocks, each block contains \( k \) elements, and each pair of elements is simultaneously contained in \( \lambda \) blocks (see [7]). The integers \((v, b, r, k, \lambda)\) are called the parameters of the design. In the case \( r = k \) (and then \( v = b \)) the design is called symmetric with parameters \((v, k, \lambda)\).

The incidence graph of a \( BIBD \) is the bipartite graph on \( b + v \) vertices (correspond to the blocks and elements of the design) with two vertices adjacent if and only if one corresponds to a block and the other corresponds to an element contained in that block. As shown in [7], the incidence graph has spectrum \([\sqrt{kr}]^1, [\sqrt{r - \lambda}]^{v-1}, [0]^{b-v}, [-\sqrt{r - \lambda}]^{v-1}, [-\sqrt{kr}]^1\). In particular, if the design is symmetric, then the incidence graph is a \( k \)-regular bipartite graph with spectrum \([k]^1, [\sqrt{k - \lambda}]^{v-1}, [-\sqrt{k - \lambda}]^{v-1}, [-k]^1\).

The following lemma gives a characterization of regular bipartite graphs with four distinct eigenvalues.

Lemma 2.4. (See [2, 7].) A connected regular bipartite graph \( G \) with four distinct eigenvalues is the incidence graph of a symmetric \( BIBD \).

Denote by \( A(l, m, n) \) and \( B(l, m, n, p) \) \((l, m, n, p \geq 1)\) the two graphs from Fig.1, where the vertices contained in an ellipse form an independent set, and any two ellipses or a vertex and an ellipse joined with one line denote a complete bipartite graph.

Lemma 2.5. (See [22].) The second least eigenvalue of a connected graph \( G \) is greater than \(-1\) if and only if
First of all, we will prove the non-existence of connected regular graphs with four vertices with adjacency matrix $A$ and spectrum $\{k\}^1, \{\lambda_2\}^1, \{\lambda_3\}^1, \{\lambda_4\}^{n-3}$. Then $G$ has minimal polynomial $p(x) = (x - k)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)$. By Lemmas 2.2 and 2.3, $\lambda_2$ and $\lambda_3$ are integers, so $\lambda_4$ is also an integer. Furthermore, by Corollary 2.1, we may assume that $\lambda_2$ and $\lambda_3$, respectively, have orthonormal eigenvectors as follows:

$$x_2 = \frac{1}{\sqrt{n}}(e_{\frac{n}{2}}^T, -e_{\frac{n}{2}}^T)^T \quad \text{and} \quad x_3 = \frac{1}{\sqrt{n}}(e_{\frac{n}{2}}^T, e_{\frac{n}{2}}^T, -e_{\frac{n}{2}}^T)^T.$$  

Taking $f(x) = x - \lambda_4$, by the spectral decomposition of $f(A)$ we get

$$A - \lambda_4 I_n = \frac{1}{n(k - \lambda_4)}e_n e_n^T + (\lambda_2 - \lambda_4)x_2 x_2^T + (\lambda_3 - \lambda_4)x_3 x_3^T,$$

or equivalently,

$$n(A - \lambda_4 I_n) = (k - \lambda_4)J_n + (\lambda_2 - \lambda_4)\begin{pmatrix} J_{\frac{n}{2}} & -J_{\frac{n}{2}} \\ -J_{\frac{n}{2}} & J_{\frac{n}{2}} \end{pmatrix} + (\lambda_3 - \lambda_4)\begin{pmatrix} J_{\frac{n}{2}} & -J_{\frac{n}{2}} & -J_{\frac{n}{2}} \\ -J_{\frac{n}{2}} & J_{\frac{n}{2}} & J_{\frac{n}{2}} \\ -J_{\frac{n}{2}} & -J_{\frac{n}{2}} & J_{\frac{n}{2}} \end{pmatrix}. \quad (1)$$
On the other hand, by considering the traces of $A$ and $A^2$, respectively, we obtain

$$k + \lambda_2 + \lambda_3 + (n - 3)\lambda_4 = 0, \quad (2)$$

$$k^2 + \lambda_2^2 + \lambda_3^2 + (n - 3)\lambda_4^2 = kn. \quad (3)$$

Now we partition $V(G)$ the same way as we partition the matrix $x_i x_i^T$ in (1), and denote by $V_1, V_2, V_3, V_4$ the corresponding vertex subsets, respectively. By considering the block matrix $A(V_1, V_4)$ in (1), we have

$$nA(V_1, V_4) = ((k - \lambda_4) - (\lambda_2 - \lambda_4) - (\lambda_3 - \lambda_4))J_{\frac{n}{4}}. \quad (4)$$

First suppose that $\lambda_4 = 0$. From (2) and (4) we know that $nA(V_1, V_4) = (k - \lambda_2 - \lambda_3)J_{\frac{n}{4}} = 2kJ_{\frac{n}{4}}$. Thus $n = 2k$ and $A(V_1, V_4) = J_{\frac{n}{4}}$ because $k \neq 0$. Putting $n = 2k$ in (3), we get $\lambda_2^2 + \lambda_3^2 = k^2$, and so $\lambda_2 \lambda_3 = 0$ by (2). This implies that $\lambda_2 = 0$ or $\lambda_3 = 0$, which is a contradiction because $\lambda_2$, $\lambda_3$ and $\lambda_4$ are distinct.

Now we can assume that $\lambda_4 \neq 0$. For the block matrix $A(V_1, V_1)$, from (1) and (2) it is seen that $nA(V_1, V_1) - \lambda_4 I_{\frac{n}{4}} = ((k - \lambda_4) + (\lambda_2 - \lambda_4) + (\lambda_3 - \lambda_4))J_{\frac{n}{4}} = -n\lambda_4 J_{\frac{n}{4}}$, that is, $A(V_1, V_1) = -\lambda_4 (J_{\frac{n}{4}} - I_{\frac{n}{4}})$. If $n \geq 8$, then $J_{\frac{n}{4}} - I_{\frac{n}{4}} \neq 0$. Thus $\lambda_4 = -1$ because $\lambda_4 \neq 0$. Putting $\lambda_4 = -1$ in (4), we get $nA(V_1, V_4) = (k - \lambda_2 - \lambda_3 - 1)J_{\frac{n}{4}}$. Then $A(V_1, V_4) = 0$ or $A(V_1, V_4) = J_{\frac{n}{4}}$. If $A(V_1, V_4) = 0$, we have $k - \lambda_2 - \lambda_3 - 1 = 0$. Then from (2) and (3), we get $\lambda_2 = k$ and $\lambda_3 = -1$, or $\lambda_2 = -1$ and $\lambda_3 = k$, a contradiction. Thus $A(V_1, V_4) = J_{\frac{n}{4}}$, and so $k - \lambda_2 - \lambda_3 - 1 = n$. Again from (2), we get $k = n - 1$, which implies that $G$ is a complete graph, a contradiction. If $n < 8$, from the above arguments we see that $\frac{n}{4}$ is an integer, so $n = 4$. Then $G = K_4$ or $G = C_4$ because $G$ is a connected regular graph. In both cases, $G$ has at most three distinct eigenvalues.

We complete the proof. □

Recall that $G(4, 2)$ denotes the set of connected regular graphs with four distinct eigenvalues in which exactly two eigenvalues are simple. The following lemma provides a necessary condition for the graphs belonging to $G(4, 2)$.

**Lemma 3.1.** If $G$ is a connected $k$-regular graph on $n$ vertices with spectrum $\{[k]_1, \lambda_2, \lambda_3, \lambda_4\}$ (2 $\leq m \leq n - 4$), then $G$ admits a regular partition $V(G) = V_1 \cup V_2$ into halves with degrees $\left(\frac{1}{2}(k + \lambda_2), \frac{1}{2}(k - \lambda_2)\right)$ such that

$$|N_G(u) \cap N_G(v)| = \begin{cases} 
\frac{1}{n}(k - \lambda_3)(k - \lambda_4) + (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) & \text{if } u \sim v \text{ in } V_1 \text{ or } V_2; \\
\frac{1}{n}(k - \lambda_3)(k - \lambda_4) + (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) & \text{if } u \nsim v \text{ in } V_1 \text{ or } V_2; \\
\frac{1}{n}(k - \lambda_3)(k - \lambda_4) - (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) & \text{if } u \sim v, u \in V_1, v \in V_2; \\
\frac{1}{n}(k - \lambda_3)(k - \lambda_4) - (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) & \text{if } u \nsim v, u \in V_1, v \in V_2. 
\end{cases}$$

**Proof.** Since $\lambda_2 (\neq k)$ is a simple eigenvalue of $G$, by Lemmas 2.2, 2.3 and Corollary 2.1 we know that $n$ is even, $\lambda_3$ is an integer and $x_2 = \frac{1}{\sqrt{n}}(e_1^T, -e_2^T)^T$ is an eigenvector of $\lambda_2$. Putting $V_1 = \{v \in V \mid x_2(v) = 1\}$ and $V_2 = \{v \in V \mid x_2(v) = -1\}$, again by Corollary 2.1 we see that $V(G) = V_1 \cup V_2$ is a regular partition of $G$ into halves with degrees $\left(\frac{1}{2}(k + \lambda_2), \frac{1}{2}(k - \lambda_2)\right)$. Furthermore, the matrix $(A - \lambda_3 I_n)(A - \lambda_4 I_n)$ has the spectral decomposition

$$(A - \lambda_3 I_n)(A - \lambda_4 I_n) = \frac{1}{n}(k - \lambda_3)(k - \lambda_4)e_n e_n^T + (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)x_2 x_2^T,$$
that is,
\[ A^2 - (\lambda_3 + \lambda_4)A + \lambda_3 \lambda_4 I_n = \frac{1}{n} (k - \lambda_3)(k - \lambda_4)J_n + \frac{1}{n} (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) \begin{pmatrix} J_2 & -J_2 \\ -J_2 & J_2 \end{pmatrix}. \] (5)

Note that the \((u, v)\)-entry of \(A^2\) is equal to \(|N_G(u) \cap N_G(v)|\). Then (5) implies that
\[
|N_G(u) \cap N_G(v)| = \begin{cases} 
\lambda_3 + \lambda_4 + \frac{1}{n} [(k - \lambda_3)(k - \lambda_4) + (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)] & \text{if } u \sim v \text{ in } V_1 \text{ or } V_2; \\
\frac{1}{n} [(k - \lambda_3)(k - \lambda_4) + (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)] & \text{if } u \not\sim v \text{ in } V_1 \text{ or } V_2; \\
\lambda_3 + \lambda_4 + \frac{1}{n} [(k - \lambda_3)(k - \lambda_4) - (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)] & \text{if } u \sim v, u \in V_1, v \in V_2; \\
\frac{1}{n} [(k - \lambda_3)(k - \lambda_4) - (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)] & \text{if } u \not\sim v, u \in V_1, v \in V_2.
\end{cases}
\]

This completes the proof. \(\square\)

The Kronecker product \(A \otimes B\) of matrices \(A = (a_{ij})_{m \times n}\) and \(B = (b_{ij})_{p \times q}\) is the \(mp \times nq\) matrix obtained from \(A\) by replacing each element \(a_{ij}\) with the block \(a_{ij}B\). Given a graph \(G\) on \(n\) vertices with adjacency matrix \(A\), we denote by \(G \otimes J_m\) the graph with adjacency matrix \(A \otimes J_m = (A + I_n) \otimes J_m - I_{nm}\) (see [8]). By the definition, \(G \otimes J_m\) is just the graph obtained from \(G\) by replacing every vertex of \(G\) with a clique \(K_m\), and two such cliques are joined if and only if their corresponding vertices are adjacent in \(G\). It is easy to see that the spectra of \(G\) and \(G \otimes J_m\) are determined by each other, that is,
\[
\text{Spec}(G) = \{[\lambda_1]^{m_1}, \ldots, [\lambda_i]^{m_i}\} \\
\Leftrightarrow \text{Spec}(G \otimes J_m) = \{[m \lambda_1 + m - 1]^{m_1}, \ldots, [m \lambda_i + m - 1]^{m_i}, [-1]^{p n - m}\}. \quad (6)
\]

Recall that \(G(4, 2, -1)\) denotes the set of graphs belonging to \(G(4, 2)\) with \(-1\) as an eigenvalue. The following result gives a partial characterization of the graphs in \(G(4, 2, -1)\).

**Theorem 3.2.** Let \(G \in G(4, 2, -1)\). Then \(-1\) is a non-simple eigenvalue of \(G\) if and only if \(G = K_{s,s} \otimes J_t\) with \(s, t \geq 2\), or \(G = K_{s,s}^- \otimes J_t\) with \(s \geq 3\) and \(t \geq 1\).

**Proof.** By the assumption, let \(G\) be a connected \(k\)-regular graph with \(\text{Spec}(G) = \{[k]^1, [\alpha]^1, [\beta]^{n-2-m}, [-1]^m\} (2 \leq m \leq n - 4)\). By considering the traces of \(A(G)\) and \(A^2(G)\), we get
\[
\begin{cases} 
k + \alpha + (n-2-m)\beta - m = 0, \\
k^2 + \alpha^2 + (n-2-m)\beta^2 + m = kn.
\end{cases} \quad (7)
\]

Owing to \(\beta \neq -1\) and \(k + \alpha + (n-2-m)\beta - m = 0\), we have \(n - k - \alpha - 2 \neq 0\). Then from (7) we can deduce that
\[
\begin{cases} 
\beta = \frac{kn - k^2 - k - \alpha^2 - \alpha}{n - k - \alpha - 2}, \\
m = n - 1 + \frac{(n - k - 1)(n - 2\alpha - 2)}{k^2 + (2 - n)k + \alpha^2 + 2\alpha - n + 2}.
\end{cases} \quad (8)
\]

Since \(\alpha \neq k\) is a simple eigenvalue of \(G\), by Lemma 3.1 we know that \(G\) admits a regular partition \(V(G) = V_1 \cup V_2\) into halves with degree \((\frac{1}{2}(k + \alpha), \frac{1}{2}(k - \alpha))\), and that if \(u, v \in V_i\) \((i = 1, 2)\) are adjacent, then
\[
|N_G(u) \cap N_G(v)| = \beta - 1 + \frac{1}{n} [(k - \beta)(k + 1) + (\alpha - \beta)(\alpha + 1)]. \quad (9)
\]
Combining (8) and (9), we have \(|N_G(u) \cap N_G(v)| = k - 1\) by simple computation, so \(u\) and \(v\) have the same neighbors, that is, \(N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}\). Since \(u\) has exactly \(\frac{k + \alpha}{2}\) neighbors in \(G[V_i]\), each of them has the same neighbors (in \(G\)) as \(u\), we see that such \(\frac{k + \alpha}{2}\) neighbors together with \(u\) induce a clique \(K_{\frac{k+\alpha}{2}}\), which is totally included in \(G[V_i]\) itself. Furthermore, again by \(N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}\), there are no edges between any two such cliques in \(V_i\), and if \(v_1 \in V_1\) is adjacent to \(v_2 \in V_2\), then the clique containing \(v_1\) must be joined with the clique containing \(v_2\). Moreover, both \(G[V_1]\) and \(G[V_2]\) consist of the disjoint union of \(\frac{2n}{k+\alpha+2}\) copies of \(K_{\frac{k+\alpha}{2}}\) because \(|V_1| = |V_2| = \frac{n}{2}\). Hence, there exists a regular bipartite graph \(H\) on \(\frac{2n}{k+\alpha+2}\) vertices such that \(G = H \circledast J_{\frac{n}{k+\alpha+2}}\). Then from (6) we deduce that

\[
\operatorname{Spec}(H) = \left\{ \left[ \frac{k - \alpha}{k + \alpha + 2} \right], \left[ \frac{-k - \alpha}{k + \alpha + 2} \right], \left[ \frac{2\beta - k - \alpha}{k + \alpha + 2} \right]^{n-m-2}, \left[ -1 \right]^{-\frac{k+\alpha}{k+\alpha+2}} \right\}.
\]

Since \(\frac{k - \alpha}{k + \alpha + 2}\) is the maximum eigenvalue of \(H\) which is simple, \(H\) must be a connected \((\frac{n}{k+\alpha+2})\)-regular bipartite graph. Clearly, \(-\frac{k - \alpha}{k + \alpha + 2} \neq -1\) since otherwise we have \(\alpha = -1\), a contradiction. Thus \(-\frac{k - \alpha}{k + \alpha + 2} < -1\) because it is an integer, so \(\alpha < -1\). We consider the following two situations.

**Case 1.** \(m - \frac{k+\alpha}{k+\alpha+2} n = 0\);

Since \(H\) is bipartite, we get \(\frac{2\beta-k-\alpha}{k+\alpha+2} = 0\), and then \(\beta = \frac{k+\alpha}{2}\). Putting \(\beta = \frac{k+\alpha}{2}\) in (8) and considering \(\alpha < k\), we deduce that \(\alpha = -k - n, \beta = k - \frac{n}{2}, m = \frac{2k-n}{2k+n+2}\), and \(\frac{n}{2} + 1 \leq k \leq \frac{3n}{4} - 1\) because \(\alpha \geq -k\) and \(2 \leq m \leq n - 4\). Thus

\[
\operatorname{Spec}(H) = \left\{ \left[ \frac{n}{2k-n+2} \right], \left[ -\frac{n}{2k-n+2} \right], \left[ 0 \right]^{-\frac{2n}{2k-n+2}} \right\}.
\]

Then \(H\) is a connected \((\frac{n}{k+\alpha+2})\)-regular bipartite graph with three distinct eigenvalues. Therefore, \(H = K_{\frac{n}{2k-n+2}} \circledast J_{\frac{n}{2k-n+2}}\) because complete bipartite graphs are the only connected bipartite graphs with three distinct eigenvalues. Hence, \(G = H \circledast J_{\frac{n}{k+\alpha+2}} = K_{\frac{n}{k+\alpha+2}} \circledast J_{\frac{n}{k+\alpha+2}}\). If we set \(s = \frac{n}{n+2k+n+2}\) and \(t = \frac{2k+\alpha}{2k-n+2}\), then \(G = K_{s, s} \circledast J_t\) with \(s, t \geq 2\) from the above arguments.

**Case 2.** \(m - \frac{k+\alpha}{k+\alpha+2} n \geq 1\).

Since \(H\) is bipartite, we have \(\frac{2\beta-k-\alpha}{k+\alpha+2} = (-1) = 1\), and so \(\beta = k + \alpha + 1\). Combining this with (8) we obtain that \((\alpha + 1)(n - 2k - 2) = 0\), which implies that \(k = \frac{n}{4} - 1\) because \(a \neq -1\). Then \(\beta = \frac{n}{2} + \alpha, m = n - 1 - \frac{2n}{n+20+2}\) by (8), and so \(-\frac{n}{6} + 1 \leq \alpha \leq \frac{n}{6} - 1\) because \(2 \leq m \leq n - 4\). Furthermore, \(n - m - 2 = m - \frac{k+\alpha}{k+\alpha+2} n = \frac{n+2\alpha-5}{n+2\alpha+2}\). Thus we get

\[
\operatorname{Spec}(H) = \left\{ \left[ \frac{n-2\alpha-2}{n+2\alpha+2} \right], \left[ -\frac{n-2\alpha-2}{n+2\alpha+2} \right], \left[ 1 \right]^{-\frac{2\alpha+2}{n+2\alpha+2}}, \left[ -1 \right]^{-\frac{2\alpha+2}{n+2\alpha+2}} \right\}.
\]

Then \(H\) is a connected \((\frac{n-2\alpha-2}{n+2\alpha+2})\)-regular bipartite graph with four distinct eigenvalues. By Lemma 2.4, we may conclude that \(H\) is the incidence graph of a symmetric BIBD with parameters \((\frac{n-2\alpha-2}{n+2\alpha+2} + 1, \frac{n-2\alpha-2}{n+2\alpha+2} - 1, \frac{n-2\alpha-2}{n+2\alpha+2} - 1)\). It is well known that such a BIBD is unique (see [8]), and the corresponding incidence graph can be obtained by removing a perfect matching from the complete bipartite graph \(K_{s, s}\), where \(s = \frac{n-2\alpha-2}{n+2\alpha+2} + 1 = \frac{2n}{n+2\alpha+2}\). Hence, \(G = H \circledast J_{\frac{n}{k+\alpha+2}} = K_{s, s} \circledast J_{\frac{n}{k+\alpha+2}}\). If we put \(s = \frac{2n}{n+2\alpha+2}\) and \(t = \frac{n+2\alpha+2}{4}\), then \(G = K_{s, s} \circledast J_t\) with \(s \geq 3\) and \(t \geq 1\) from the above arguments.
Conversely, by simple computation we obtain that
\[
\text{Spec}(K_{s,s} \odot J_t) = \{(st + t - 1)^1, [-st + t - 1]^1, [t - 1]^{2s-2}, [-1]^{2t(s-1)}\};
\]
\[
\text{Spec}(K_{s,s}^- \odot J_t) = \{(st - 1)^1, [-st + 2t - 1]^1, [2t - 1]^{t-1}, [-1]^{2st-s-1}\},
\]
and our result follows. \(\square\)

By Theorem 3.2, we obtain the following corollary immediately.

**Corollary 3.1.** The graphs \(K_{s,s} \odot J_t (s, t \geq 2)\) and \(K_{s,s}^- \odot J_t (s \geq 3, t \geq 1)\) are DS.

**Proof.** Note that any graph cospectral with \(K_{s,s} \odot J_t\) or \(K_{s,s}^- \odot J_t\) must be connected. By Theorem 3.2, it suffices to prove that \(K_{s_1,s_1} \odot J_{t_1} (s_1, t_1 \geq 2)\) and \(K_{s_2,s_2}^- \odot J_{t_2} (s_2 \geq 3, t_2 \geq 1)\) cannot share the same spectrum. On the contrary, assume that \(\text{Spec}(K_{s_1,s_1} \odot J_{t_1}) = \text{Spec}(K_{s_2,s_2}^- \odot J_{t_2})\). Then \(s_1t_1 = s_2t_2\) and \(s_1t_1 + t_1 - 1 = s_2t_2 - 1\) because they are regular graphs which have the same number of vertices and share the common degree. This implies that \(t_1 = 0\), which is impossible because \(t_1 \geq 2\). \(\square\)

**Remark 1.** It is worth mentioning that the DS-properties of \(K_{s,s} \odot J_t\) and \(K_{s,s}^- \odot J_t\) have been mentioned in [10] and [8], respectively.

Recall that \(\mathcal{G}(4, \geq -1)\) denotes the set of connected regular graphs with four distinct eigenvalues and second least eigenvalue not less than \(-1\). Petrović in [19] determined all the connected graphs whose second least eigenvalue is not less than \(-1\). However, it is a difficult work to pick out all the graphs belonging to \(\mathcal{G}(4, \geq -1)\) from their characterization. Here, from Theorem 3.2 and Lemma 2.5, we can easily give a complete characterization of the graphs in \(\mathcal{G}(4, \geq -1)\).

**Theorem 3.3.** A connected graph \(G \in \mathcal{G}(4, \geq -1)\) if and only if \(G = K_{s,s} \odot J_t\) with \(s, t \geq 2\), or \(G = K_{s,s}^- \odot J_t\) with \(s \geq 3\) and \(t \geq 1\).

**Proof.** Suppose that \(G \in \mathcal{G}(4, \geq -1)\), and \(\alpha, \beta\) are the least and second least eigenvalues of \(G\), respectively. If \(\beta > -1\), then \(G = K_{m,n} \odot A(l, m, n)\) or \(B(l, m, n, p)\) with proper parameters \(l, m, n, p\) by Lemma 2.5. It is seen that \(A(l, m, n)\) and \(B(l, m, n, p)\) cannot be regular, so \(G = K_{n,n} \) with \(n \geq 2\). Note that \(K_{n,n}\) has only three distinct eigenvalues, so there are no graphs in \(\mathcal{G}(4, \geq -1)\) with \(\beta > -1\), and thus \(\beta = -1\) because \(G \in \mathcal{G}(4, \geq -1)\). We claim that \(\alpha < \beta = -1\) since otherwise \(G\) will be a complete graph due to the least eigenvalue of \(G\) is \(-1\), and thus \(\alpha\) is a simple eigenvalue of \(G\). As a consequence, \(G\) will belong to \(\mathcal{G}(4, 2, -1)\) and \(-1\) is a non-simple eigenvalue of \(G\) by Theorem 3.1. Hence, \(G = K_{s,s} \odot J_t\) with \(s, t \geq 2\), or \(G = K_{s,s}^- \odot J_t\) with \(s \geq 3\) and \(t \geq 1\) by Theorem 3.2.

Conversely, from (10) we obtain the required result immediately. \(\square\)

Now we continue to consider the graphs in \(\mathcal{G}(4, 2, -1)\). The following result excludes the existence of such graphs belonging to (2b) (see Section 1).

**Theorem 3.4.** There are no connected \(k\)-regular graphs with spectrum \(\{[k]^1, [-1]^1, [\alpha]^m, [\beta]^{n-2-m}\}\), where \(\alpha\) and \(\beta\) are not integers and \(2 \leq m \leq n - 4\).
Proof. On the contrary, assume that $G$ is such a graph. Then $G$ will be a connected $k$-regular graphs having exactly two integral eigenvalues. By Lemma 2.1, the eigenvalues $\alpha$ and $\beta$ are of the form $\frac{1}{2}(a \pm \sqrt{b})$ ($a, b \in \mathbb{Z}, b > 0$) and have the same multiplicity, i.e., $m = \frac{1}{2}(n - 2)$. Then $\alpha$ and $\beta$ satisfy the following two equations:

$$\begin{cases}
k - 1 + \frac{1}{2}(n - 2)\alpha + \frac{1}{2}(n - 2)\beta = 0, \\
k^2 + 1 + \frac{1}{2}(n - 2)\alpha^2 + \frac{1}{2}(n - 2)\beta^2 = kn.
\end{cases}$$

By simple computation, we obtain

$$\begin{cases}
\alpha = -k + 1 + \frac{\sqrt{kn - k - 1}(n - k - 1)}{n - 2}, \\
\beta = -k + 1 - \frac{\sqrt{kn - k - 1}(n - k - 1)}{n - 2}.
\end{cases}$$

Considering that $\alpha$ and $\beta$ are of the form $\frac{1}{2}(a \pm \sqrt{b})$ ($a, b \in \mathbb{Z}$), we claim that $\frac{-k + 1}{n - 2} = \frac{a}{2}$, and so $a = -1$ because $a \in \mathbb{Z}$ and $1 < k < n - 1$. Thus $n = 2k$, and then

$$\alpha = \frac{1}{2}(-1 + \sqrt{2k + 1}), \quad \beta = \frac{1}{2}(-1 - \sqrt{2k + 1}). \quad (11)$$

Furthermore, by Lemma 3.1, the graph $G$ admits a regular partition $V(G) = V_1 \cup V_2$ into halves with degree $(\frac{1}{4}(k - 1), \frac{1}{4}(k + 1))$ such that

$$|N_G(u) \cap N_G(v)| = \begin{cases}
\alpha + \beta + \frac{1}{n}[(k - \alpha)(k - \beta) + (-1 - \alpha)(-1 - \beta)] & \text{if } u \sim v \text{ in } V_i; \\
\frac{1}{n}[(k - \alpha)(k - \beta) + (-1 - \alpha)(-1 - \beta)] & \text{if } u \not\sim v \text{ in } V_i.
\end{cases}$$

Putting (11) and $n = 2k$ in the above equation, we get

$$|N_G(u) \cap N_G(v)| = \begin{cases}
k - 1 & \text{if } u \sim v \text{ in } V_i; \\
\frac{k}{2} & \text{if } u \not\sim v \text{ in } V_i,
\end{cases}$$

which is impossible because $k$ is odd due to $\frac{1}{2}(k - 1)$ is an integer.

This completes the proof. \qed

By Theorems 3.2 and 3.4, in order to determine all the graphs in $\mathcal{G}(4, 2, -1)$, it remains to consider such graphs belonging to (2a) (see Section 1), i.e., the $k$-regular graphs with spectrum $\{[k]^1, [-1]^1, [\alpha]^m, [\beta]^n[2-2m]\}$, where $\alpha$ and $\beta$ are integers and $2 \leq m \leq n - 4$. By Appendix A in [12], we find that there are no such graphs up to 30 vertices. In what follows, we will prove the non-existence of such graphs for arbitrary $n$.

**Theorem 3.5.** There are no connected $k$-regular graphs with spectrum $\{[k]^1, [-1]^1, [\alpha]^m, [eta]^n[2-2m]\}$, where $\alpha$ and $\beta$ are integers and $2 \leq m \leq n - 4$.
**Proof.** On the contrary, assume that $G$ is such a graph with adjacency matrix $A$. Firstly, we assert that $3 \leq k \leq n - 2$. In fact, if $k = 2$, then $G = C_n$, where $n$ is even by Lemmas 2.2 and 2.3, and so $G$ is a bipartite graph, which is impossible because $2 \leq m \leq n - 4$; if $k = n - 1$, then $G$ is a complete graph, which has exactly two distinct eigenvalues, a contradiction.

Now suppose that $\alpha \beta \geq 0$, then $\alpha, \beta \geq 0$ or $\alpha, \beta \leq 0$. In the former case, we see that $G$ is a complete graph because $-1$ is the least eigenvalue of $G$, which is a contradiction. In the later case, we claim that $G$ is a complete multipartite graph because $G$ has only one positive eigenvalue, thus $G = K_{n-k,n-k-\ldots,n-k}$ due to $G$ is $k$-regular. Then $\text{Spec}(G) = \{(k^2, 2), (1, 0)\}$, a contradiction. Thus $\alpha \beta < 0$. Without loss of generality, we may assume that $\alpha \geq 1$ and $\beta \leq -2$. By considering the traces of $A$ and $A^2$, we get
\[
\begin{cases}
  k - 1 + m\alpha + (n - 2 - m)\beta = 0, \\
  k^2 + 1 + m\alpha^2 + (n - 2 - m)\beta^2 = kn.
\end{cases}
\]

Canceling out $m$ in the above two equations, we have
\[
\beta = -\frac{k(n - k) + (k - 1)\alpha - 1}{(n - 2)\alpha + k - 1}. \tag{12}
\]

By Lemmas 2.2 and 2.3, it is seen that $n$ is a divisor of $(k - \alpha)(k - \beta) + (-1 - \alpha)(-1 - \beta)$ and $(k - \alpha)(k - \beta) - (-1 - \alpha)(-1 - \beta)$, and so a divisor of $-2(-1 - \alpha)(-1 - \beta) = -2(1 + \alpha)(1 + \beta) > 0$ because $\alpha \geq 1$ and $\beta \leq -2$ are integers. Thus $c = \frac{-2(1 + \alpha)(1 + \beta)}{n}$ is a positive integer. Combining this with (12), we get
\[
c = \frac{2(k - \alpha)(\alpha + 1)(n - k - 1)}{n(n - 2)\alpha + n(k - 1)} < \frac{2(k - \alpha)(\alpha + 1)(n - k - 1)}{n(n - 2)\alpha} = \frac{2(n - k - 1)}{n \cdot \frac{\alpha - 2}{k - \alpha} \cdot \frac{\alpha}{\alpha + 1}}< 4.
\]

It suffices to consider the following three situations.

**Case 1.** $c = 1$; 

In this case, we get $\frac{2(k - \alpha)(\alpha + 1)(n - k - 1)}{(n - 2)\alpha + k - 1} = n$, that is,
\[
2(n - k - 1)\alpha^2 + (n^2 - 2kn + 2k^2 - 2)\alpha + (2k - n)(k + 1) = 0, \tag{13}
\]

which implies that $n > 2k$ because $n - k - 1 > 0$, $n^2 - 2kn + 2k^2 - 2 > 0$ and $\alpha \geq 1$. Solving (13), we get
\[
\alpha = \frac{-(n^2 - 2kn + 2k^2 - 2) + \sqrt{(n^2 - 2kn + 2k^2 - 2)^2 + 8(n - k - 1)(n - 2k)(k + 1)}}{4(n - k - 1)}. \tag{14}
\]

Again from (12) we obtain
\[
\beta = \frac{-(n^2 - 2kn + 2k^2 - 2) - \sqrt{(n^2 - 2kn + 2k^2 - 2)^2 + 8(n - k - 1)(n - 2k)(k + 1)}}{4(n - k - 1)}. \tag{15}
\]

Then, by Lemma 3.1, $G$ admits a regular partition $V(G) = V_1 \cup V_2$ into halves with degrees $(\frac{1}{2}(k - 1), \frac{1}{2}(k + 1))$ such that
\[
|N_G(u) \cap N_G(v)| = \alpha + \beta + \frac{1}{n}[(k - \alpha)(k - \beta) + (-1 - \alpha)(-1 - \beta)] \text{ for } u \sim v \text{ in } V_i. \tag{16}
\]
Combining (14), (15) and (16), we thus have \(|N_G(u) \cap N_G(v)| = k - \frac{n}{2} - 1 \). This implies that \(n \leq 2k - 2\) because \(|N_G(u) \cap N_G(v)| \geq 0\), contrary to \(n > 2k\).

**Case 2.** \(c = 2\):

In this case, we have \(\frac{(k-\alpha)(\alpha+1)(n-k-1)}{(n-2)\alpha+k-1} = n\). This implies that

\[
\begin{align*}
\alpha &= \frac{-(n^2-(k+1)n+k^2-1)+\sqrt{(n^2-(k+1)n+k^2-1)^2-4(n-k-1)(k^2+k-n)}}{2(n-k-1)}, \\
\beta &= \frac{-(n^2-(k+1)n+k^2-1)-\sqrt{(n^2-(k+1)n+k^2-1)^2-4(n-k-1)(k^2+k-n)}}{2(n-k-1)}.
\end{align*}
\]

Again from (16) we deduce that \(|N_G(u) \cap N_G(v)| = k - n - 1 < 0\) for \(u \sim v\) in \(V_i\) \((i = 1, 2)\), which is a contradiction.

**Case 3.** \(c = 3\):

In this case, we have \(\frac{2(k-\alpha)(\alpha+1)(n-k-1)}{(n-2)\alpha+k-1} = 3n\), that is,

\[
2(n-k-1)\alpha^2 + [3n^2 - (2k+4)n + 2k^2 - 2]\alpha + (k-3)n + 2k^2 + 2k = 0,
\]

which is impossible because \(n-k-1 > 0\), \(3n^2-(2k+4)n+2k^2-2 > 0\), \((k-3)n+2k^2+2k > 0\) due to \(k \geq 3\), and \(\alpha > 0\).

We complete this proof. \(\square\)

Combining Theorems 3.2, 3.4 and 3.5, we obtain the main result of this paper.

**Theorem 3.6.** A connected graph \(G \in \mathcal{G}(4, 2, -1)\) if and only if \(G = K_{s,s} \odot J_i\) with \(s, t \geq 2\), or \(G = K_{s,s}^- \odot J_i\) with \(s \geq 3\) and \(t \geq 1\).

Recall that \(\mathcal{G}(4, 2, 0)\) denotes the set of graphs belonging to \(\mathcal{G}(4, 2)\) with 0 as an eigenvalue. Since the spectrum of a regular graph could be deduced from its complement, we can easily characterize all the graphs in \(\mathcal{G}(4, 2, 0)\) by Theorems 3.2, 3.4 and 3.5.

**Theorem 3.7.** A connected graph \(G \in \mathcal{G}(4, 2, 0)\) if and only if \(G = K_{s,s}^- \odot J_i\) with \(s \geq 3\) and \(t \geq 1\).

**Proof.** Let \(G \in \mathcal{G}(4, 2, 0)\) be a connected \(k\)-regular graph with adjacency matrix \(A\), and let \(\overline{G}\) be the complement of \(G\).

If 0 is a non-simple eigenvalue of \(G\), suppose that \(\text{Spec}(G) = \{[k]^1, [\alpha]^1, [\beta]^{n-2-m}, [0]^m\}\) \((2 \leq m \leq n-4)\). Then \(\overline{G}\) is a \((n-k-1)\)-regular graph with \(\text{Spec}(\overline{G}) = \{[n-k-1]^1, [-1-\alpha]^1, [-\beta]^{n-2-m}, [-\beta]^m\}\). We claim that \(\alpha > 0\) and \(\beta < 0\) or \(\alpha < 0\) and \(\beta > 0\), since otherwise \(G\) will be a regular complete multipartite graph (which has only three distinct eigenvalues) or does not exist. Assume that \(\overline{G}\) is disconnected. If \(\alpha > 0\) and \(\beta < 0\), we have \(n-k-1 = -1-\beta\), i.e., \(\beta = k-n\). By considering the traces of \(A\) and \(A^2\), we obtain

\[
\begin{align*}
k + \alpha + (n-2-m)(k-n) &= 0, \\
k^2 + \alpha^2 + (n-2-m)(k-n)^2 &= kn,
\end{align*}
\]

and so \(\alpha = 0\) or \(\alpha = k-n\), which are impossible due to \(\alpha > 0\). If \(\alpha < 0\) and \(\beta > 0\), similarly, we have \(\alpha = k-n\) because \(\overline{G}\) is disconnected and regular, and so \(n = 2k\), or \(n \neq 2k\) and \(\beta = k-n\) by considering the traces of \(A\) and \(A^2\). In both cases, we can deduce
a contradiction because $G$ cannot be a bipartite graph and $\alpha \neq \beta$. Therefore, $\overline{G}$ must be connected, and thus $\overline{G} \in \mathcal{G}(4, 2, -1)$ with $-1$ as a non-simple eigenvalue. By Theorem 3.2, we may conclude that $\overline{G} = K_{s,s} \circ J_t$ with $s, t \geq 2$, or $\overline{G} = K_{s,s}^- \circ J_t$ with $s \geq 3$ and $t \geq 1$. Hence, $G = K_{s,s}^- \circ J_t$ with $s \geq 3$ and $t \geq 1$ because $G$ is connected.

If $0$ is a simple eigenvalue of $G$, we suppose that $\text{Spec}(G) = \{[k]^1, [0]^1, [\alpha]^m, [\beta]^{n-2-m}\}$. Then $\text{Spec}(\overline{G}) = \{[n-k-1]^1, [-1]^1, [-1-\alpha]^m, [-1-\beta]^{n-2-m}\}$. As above, one can easily deduce that $\overline{G}$ is connected, and so $\overline{G} \in \mathcal{G}(4, 2, -1)$ with $-1$ as a simple eigenvalue. Then, by Theorems 3.4 and 3.5, $G$ does not exist and so is $G$.

Consequently, if $G \in \mathcal{G}(4, 2, 0)$ then $G = K_{s,s}^- \circ J_t$ with $s \geq 3$ and $t \geq 1$. Obviously, $K_{s,s}^- \circ J_t \in \mathcal{G}(4, 2, 0)$ because $\text{Spec}(K_{s,s}^- \circ J_t) = \{[st]^1, [st-2t]^1, [-2t]^{s-1}, [0]^{2st-s-1}\}$. Our result follows.

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