Tight Algorithms for the Submodular Multiple Knapsack Problem

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Abstract

Submodular function maximization has been a central topic in the theoretical computer science community over the last decade. A plenty of well-performed approximation algorithms have been designed for the maximization of monotone/non-monotone submodular functions over a variety of constraints. In this paper, we consider the submodular multiple knapsack problem (SMKP), which is the submodular version of the well-studied multiple knapsack problem (MKP). Roughly speaking, the problem asks to maximize a monotone submodular function over multiple bins (knapsacks). Despite lots of known results in the field of submodular maximization, surprisingly, it remains unknown whether or not this problem enjoys the well-known tight $(1 - 1/e)$-approximation. In this paper, we answer this question affirmatively by proposing tight $(1 - 1/e - \epsilon)$-approximation algorithms for this problem in most cases.

We first considered the case when the number of bins is a constant. Previously a randomized approximation algorithm can obtain approximation ratio $(1 - 1/e - \epsilon)$ based on the involved continuous greedy technique. Here we provide a simple combinatorial deterministic algorithm with ratio $(1 - 1/e)$ by directly applying the greedy technique. We then generalized the result to arbitrary number of bins. When the capacity of bins are identical, we design a combinatorial and deterministic algorithm which can achieve the tight approximation ratio $(1 - 1/e - \epsilon)$. When the ratio between the maximum capacity and the minimum capacity of the bins is bounded by a constant, we provide a $(1/2 - \epsilon)$-approximation algorithm which is also combinatorial and deterministic. We can further boost the approximation ratio to $(1 - 1/e - \epsilon)$ with the help of continuous greedy technique, which gives a tight randomized approximation algorithm for this case.

Keywords: submodular maximization, multiple knapsack problem, capacity constraints

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1 Introduction

The multiple knapsack problem (MKP) is defined as follows. We are given a set $N$ of $n$ items and a set of $m$ bins (knapsacks) such that the $j$-th bin has a capacity $B_j$ and an item $u \in N$ has a size $c(u)$ and a profit $p(u)$. The task is to select a subset of items such that the sum of their profits is maximized and they can be packed into those $m$ bins without exceeding the capacities. It is well-known that the problem admits a PTAS but has no FPTAS assuming $P \neq NP$ [17, 7, 16].

In this paper, we generalize the above problem to the submodular multiple knapsack problem (SMKP), where, instead of specifying a profit for each item, the profit of each subset of items is described by a monotone submodular function $f$ defined over all subsets of $N$. The task is again to select a subset of items which maximizes the value of $f$ and can be packed into those $m$ bins without exceeding the capacities.

By introducing a monotone submodular objective function, SMKP falls into the field of submodular function maximization, which enjoys a long history of more than forty years. As early as in 1978, a simple greedy algorithm was proposed which yields a $(1 - 1/e)$-approximation for the problem of maximizing a monotone submodular function subject to a cardinality constraint $|S| \leq k$ [23], and a $1/2$ approximation for the problem of maximizing a monotone submodular function subject to a matroid constraint [14]. On the other hand, even for the cardinality constraint, submodular maximization problems can not be approximated within a ratio better than $1 - 1/e$ [22]. Since then, it remains a central open question in this field whether or not the problem of maximizing a monotone submodular function admits a $(1 - 1/e)$-approximation when subject to a matroid constraint. Until 2008, a groundbreaking work [25] answers this question affirmatively by proposing the so-called continuous greedy technique. Since then, a plenty of tight or well-performed approximation algorithms have been devised to maximize (monotone or non-monotone) submodular functions over a variety of constraints [2, 4, 5, 9, 12, 13, 15, 20, 21, 26]. Among them, the most relevant constraint to this work is the $m$-knapsack constraint.

The $m$-knapsack constraint assumes that there are $m$ knapsacks and the $j$-th knapsack has budget $B_j$. In contrast to our problem, the constraint assumes that each picked element $u \in N$ incurs a cost $c_j(u)$ in the $j$-th knapsack for all $j \in [m] := \{1, \cdots, m\}$. A set of elements is feasible if it is feasible in each knapsack. In some sense, under the $m$-knapsack constraint, each picked element is packed into all knapsacks simultaneously; while in our model, under the capacity constraint, each picked element is packed into only one knapsack (we call it bin instead of knapsack in the paper). The $m$-knapsack constraint is closely related to the capacity constraint in our problem. Firstly, the submodular maximization problem subject to the 1-knapsack constraint (often known as a knapsack constraint) generalizes the classical knapsack problem, and is exactly SMKP with a single bin as input. Secondly, SMKP can actually be reduced to the submodular maximization problem subject to the $m$-knapsack constraint. The reduction is as follows. For an instance of SMKP, an item $u \in N$ can be divided into $m$ items $u_1, \cdots, u_m$ and for $j \in [m]$, and $u_j$ incurs a cost $c(u_j)$ in the $j$-th bin and a zero cost in other bins. The function value of a set of new items is equal to the value of the set of the original items from which they are generated. And now we have obtained a submodular maximization instance subject to the $m$-knapsack constraint. Actually, SMKP is strictly simpler than the submodular maximization problem subject to the $m$-knapsack constraint. The latter problem is NP-hard to approximate within an $n^{1-\epsilon}$ factor when $m$ is a part of input, while it is not difficult to find a constant approximation algorithm for SMKP for arbitrary $m$ (see the following subsection for details). However, finding a tight approximation algorithm for SMKP is a highly non-trivial task. Such tight algorithms have been the central topic in the field of submodular function maximization.
1.1 Our Results and Techniques

In this work we answer the aforementioned question affirmatively by presenting tight \((1 - 1/e)\) algorithms for SMKP in most cases: 1) constant case where the number of bins \(m\) is a constant; 2) identical-size case where all the bins have the same capacity \(B_j \equiv B\); 3) bounded-size case where the ratio between the maximum capacity and the minimum capacity of the bins is bounded by a constant. The following table summarizes our results in this paper and lists the best known results previously as comparison.

| Case            | Previous result                      | Our result                        |
|-----------------|--------------------------------------|-----------------------------------|
| Constant bins   | \(1 - 1/e - \epsilon\) [19] (randomized, continuous greedy) | \(1 - 1/e\) (deterministic)      |
| Identical-size  | 0.468 (folklore) (deterministic)     | \(1 - 1/e - \epsilon\) (deterministic) |
| Bounded-size    | - (deterministic)                    | \(1/2 - \epsilon\) (deterministic) \(1 - 1/e - \epsilon\) (randomized, continuous greedy) |
| General         | 0.387 (folklore) (deterministic)     | - (randomized, continuous greedy) |
|                 | 0.468 (folklore, [6]) (deterministic)   |                                    |

The main contribution in this paper is the tight \((1 - 1/e - \epsilon)\)-approximation algorithm in the identical-size case and the \(1/2\) approximation algorithm in the bounded-size case. Both algorithms are deterministic and based on the greedy technique with several new insights in the analysis. In the study of submodular maximization under variant constraints, though most state-of-art ratio is achieved by randomized algorithm, the simple deterministic algorithm always takes an important place. However, only very few problems have tight deterministic algorithms now: monotone submodular maximization under cardinality constraint [23], monotone submodular maximization under one knapsack constraint [18, 24], and unconstrained non-monotone submodular maximization [2]. Our results contain new deterministic algorithms on submodular maximization and might inspire other results in the area.

All the algorithms in this paper are built on the greedy algorithm presented in Section 2.2. The solution \(G\) returned by the greedy algorithm enjoys a good approximation. Formally, for any set \(T \subseteq N\),

\[
f(G) \geq \left(1 - e^{-c(G)/c(T)}\right) f(T).
\]

Here \(c(G) = \sum_{u \in G} c(u)\) is the size of \(G\). On the other hand, \(G\) is infeasible in that the capacity of each bin is violated by the last element added into it. These elements are referred to as reserved elements. By plugging \(T = OPT\) into the last inequality and combining the fact that \(c(G) \geq \text{total capacity} \geq c(OPT)\), we obtain an infeasible solution \(G\) which admits \((1 - 1/e)\)-approximation. The algorithms in this paper are all about how one manages to round the greedy solution into a feasible one by dealing with the reserved elements carefully, such that the rounded solution will not lose too much in the approximation ratio.

One widely used technique to handle this problem is the enumeration technique. By enumerating all possible partial solutions, we are able to pack large-valued elements in \(OPT\). Then we can fill the bins by greedily picking small-valued elements, rendering the reserved elements to be small-valued.
This idea provides us a simple polynomial time approximation algorithm when the number of bins is constant.

**Theorem 1.** There is a deterministic combinatorial algorithm for SMKP which admits a $(1 - 1/e)$-approximation and runs in $O((mn)^{em+4})$ time, specially when the number of bins $m$ is a constant, the algorithm runs in polynomial time.

**Remark.** When the number of bins $m$ is a constant, the randomized algorithm for the $m$-knapsack constraint \[19\] already achieves a $(1 - 1/e - \epsilon)$-approximation for SMKP. However, such algorithm adopts the continuous greedy technique and requires an involved rounding procedure.

When applying such idea to arbitrary $m$, the obstacle is that the enumeration step requires exponential time. This is because the “constant-bins” algorithm takes all the bins into consideration simultaneously. A natural idea to resolve it is to pack bins one at a time. For each bin, we manage to pack elements of as large marginal values as possible. This is actually running a greedy algorithm on an exponential size submodular maximization problem (see the reductions in Section 2.1). As a result, to achieve a $(1 - 1/e)$-approximation, we find it suffices, taking the first bin as an example, to find a feasible set $S$ with $c(S) \leq B_1$ and $f(S) \geq \frac{B_1}{c(OPT)} f(OPT)$, where $OPT$ is the optimal solution of the SMKP instance. This is possible by observing that if one packs the first bin by the greedy algorithm, the returned infeasible set $G$ satisfies

$$f(G) \geq \left(1 - e^{-B_1/c(OPT)}\right) f(OPT).$$

The ratio $1 - e^{-B_1/c(OPT)}$ is approximately $B_1/c(OPT)$ when $B_1/c(OPT)$ is small enough. For the identical-size case and the bounded-size case, this is exactly the case when $m$ is large.

Our task is again to handle the reserved element in $G$ carefully. One thus may expect the enumeration technique still works here. However, this is not the case. The reason is simple: once the bin has packed some elements before the greedy process, the remaining capacity of the bin would become smaller, and therefore no longer a desired fraction of $c(OPT)$.

In order to tackle such difficulty, the first trick we use is to enumerate large-sized elements rather than large-valued ones, and to greedily pick small-sized elements rather than small-valued ones. There are several advantages to classify elements according to their sizes instead of values. Firstly, the value function is submodular while the size function is linear. Thus we do not need to handle the difficulties coming from the change of marginal values. Such division is more stable and will not change in the whole process. Secondly, dividing elements according to their sizes allows us to manipulate them in a more flexible way, e.g. small-sized elements can be transferred among bins.

Thirdly, in the analysis of greedy subroutine, we have $f(G) \geq \left(1 - e^{-c(G)/c(T)}\right) f(T)$. Since the ratio mainly depends on the ratio between capacities, the based-on-size division is also useful in the analysis. In the following, we further explain how to achieve the desired ratio in the identical-size case and the bounded-size case, respectively.

For the identical-size case, the basic framework of the algorithm is as follows. For each bin, we adapt the enumeration step by only enumerating all feasible solutions of large-sized elements. Then we fill the remaining part of the bin for each enumerated set by packing only small-sized elements greedily. Finally, the set of the maximum value is returned.

Take the first bin as an example. Let $OPT_l$ and $OPT_s$ be the set of large-sized and small-sized elements in $OPT$, respectively. Let $ALG$ be the set packed into the first bin and let $ALG_l$ and $ALG_s$ be the set returned by the enumeration step and the greedy step, respectively. To prove $f(ALG) \geq \frac{f(OPT)}{m}$, we can show that $f(ALG_l) \geq \frac{f(OPT)}{m}$ and $f(ALG_s | ALG_l) \geq \frac{f(OPT | ALG_l)}{m}$. The first inequality is easy to achieve by the enumeration process, while the second one is satisfied if we
have \( c(ALG_i) \geq \frac{c(OPT_i)}{m} \). Equivalently, we hope the algorithm can find a suitable set \( ALG_i \) such that

\[
c(ALG_i) \leq \frac{c(OPT_i)}{m} \quad \text{and} \quad f(ALG_i) \geq \frac{f(OPT_i)}{m}.
\]

However, this is obviously a far-fetched condition for \( ALG_i \). To assist the analysis, our first key observation is that under mild precondition, we can still prove \( f(ALG) \geq \frac{f(OPT)}{m} \) once the following condition holds for \( ALG_i \):

\[
c(ALG_i) \geq \frac{c(OPT_i)}{m} \quad \text{and} \quad f(ALG_i) \geq \frac{f(OPT_i)}{m}.
\]

To summarize, we find two distinct sufficient conditions for finding a good \( ALG \). Unfortunately, we find it impossible to achieve either conditions by considering only a single bin, even if we only require that they are satisfied approximately. One can imagine that the elements in \( OPT_i \) might be packed into the bins by \( OPT \) in an unbalanced manner. That is, some bins contain elements in \( OPT_i \) with very large values as well as very large sizes, while other bins contain elements in \( OPT_i \) with small values as well as small sizes.

In order to circumvent this difficulty, we introduce the average argument. Let \( OPT_i = (OPT_{i,1}, \cdots, OPT_{i,m}) \) be its partition into the \( m \) identical bins. Intuitively, we want to group constant sets \( OPT_{i,j} \) together such that the average size of each group is always on the one side of \( \frac{c(OPT_i)}{m} \). That is, either all of them are smaller than \( \frac{c(OPT_i)}{m} \), or they are larger than \( \frac{c(OPT_i)}{m} \). This can be approximately achieved by a coupling process: order the sets \( OPT_{i,j} \) by their sizes and group them pairwise such that the largest one is coupled with the smallest one, the second to largest one is coupled with the second to smallest one, etc. After constant rounds of this process, the returned groups will satisfy the desired property. More specifically, let \( E = (E_1, E_2, \cdots, E_k) \) be the returned groups. Let \( b(E_i) \) denote the number of set \( OPT_{i,j} \) in \( E_i \), which means \( E_i \) can be packed into \( b(E_i) \) bins. Then, we have \( a) \forall i, \frac{c(E_i)}{b(E_i)} \leq \frac{c(OPT_i)}{m}, \) or \( b) \forall i, \frac{c(E_i)}{b(E_i)} \geq \frac{c(OPT_i)}{m}. \) In either case, we can find a set \( ALG_i \) which approximately satisfies one of the aforementioned conditions for a good \( ALG \), with a compromise that \( ALG \) needs to be packed into multiple bins. This suggests that the overall algorithm needs to pack multiple (but constant) bins one at a time.

Combining all these techniques, we have the following result for the identical-size case of SMKP.

**Theorem 2.** For the identical-size case of SMKP, there is a polynomial time deterministic algorithm which yields a \((1 - 1/e - O(\epsilon))\)-approximation.

Next, we manage to generalize the above result to the bounded-size case where the ratio between the maximum capacity and the minimum capacity of the bins is bounded by a constant \( \gamma \). Clearly, the identical-size case is a special case of the bounded-size case by setting \( \gamma = 1 \). The algorithm framework is quite similar as the identical-size case, but the analysis uses different ideas. The algorithm pack bins one at a time. For each bin, the algorithm enumerate large-sized elements, greedily pick small-sized elements, and pack the reserved element into the reserved bins. The fact that \( \gamma \) is a constant ensures that the algorithm is polynomial.

For the analysis, we combine the idea of classifying elements based on their sizes and the analysis for the classical greedy algorithm under a matroid constraint [14]. Let \( S_j \) denote the elements packed in the \( j \)-th bin and \( T_j \) denote the elements packed in the first \( j \) bins by an algorithm. In a standard analysis, if one can find a partition \((OPT_1, \cdots, OPT_m)\) of \( OPT \) such that \( f(S_j | T_{j-1}) \geq f(OPT_j | T_{j-1}) \), then the algorithm admits a \( 1/2 \) approximation. We show that we can actually construct such a partition for our algorithm. The key observation is that the small elements in \( OPT \) can be arbitrarily re-partitioned according to our requirements, since small-sized elements can be transferred freely among the bins. To conclude, we have the following result.
Theorem 3. For the bounded-size case of SMKP, there is a polynomial time deterministic algorithm which yields a \(\left(\frac{1}{2} - 2\epsilon\right)\)-approximation.

The ratio 1/2 coincides with the ratio of the classical greedy algorithm for a matroid constraint and therefore is acceptable (the state of art deterministic algorithm \([3]\) achieves a 0.5008 approximation for a matroid constraint). It is also expected that a \((1-1/e)\)-approximation can be achieved by applying the continuous greedy technique. However, the standard implementation \([6]\) can only achieve a \(1 - e^{-(1-1/e)} \approx 0.468\) approximation. Instead, we adopt a modified continuous greedy process in \([1]\) to interpolate our 1/2 algorithm with the continuous greedy process, which results in a tight approximation ratio successfully.

Theorem 4. For the bounded-size case of SMKP, there is a polynomial time randomized algorithm which yields a \(\left(1 - 1/e - O(\epsilon)\right)\)-approximation.

1.2 Related Work

The multiple knapsack problem has been fully studied previously. Kellerer \([17]\) proposed the first PTAS for the identical-size case of the problem. Soon after, Chekuri and Khanna \([7]\) proposed a PTAS for the general case. The result was later improved to an EPTAS by Jansen \([16]\). On the other hand, it is easy to see that the problem does not admit an FPTAS even for the case of \(m = 2\) bins unless \(P = NP\), by reducing the PARTITION problem to it.

For the problem of maximizing a monotone submodular function subject to a knapsack constraint, a tight \((1 - 1/e)\) algorithm was known which runs in \(O(n^5)\) time \([17, 24]\). Later, a fast algorithm was proposed in \([1]\) which achieves a \((1 - 1/e - \epsilon)\)-approximation and runs in \(n^2(\log n/\epsilon)^{O(1/\epsilon^8)}\) time. This was recently improved in \([10]\) by a new algorithm which runs in \((1/\epsilon)^{O(1/\epsilon^8)}n\log^2 n\) time.

For the problem of maximizing a monotone submodular function subject to the \(m\)-knapsack constraint, there is a tight \(1 - 1/e - \epsilon\) algorithm \([19]\) when the number of knapsacks \(m\) is a constant. The problem is NP-hard to approximate within an \(n^{1-\epsilon}\) factor when \(m\) is a part of input, since it contains the INDEPENDENT-SET problem as a special case. For the submodular multiple knapsack problem considered in this paper, to the best of our knowledge, nobody except Moran Feldman has studied this problem. In his Ph. D thesis \([11]\), Feldman proposed a polynomial time \((1/9 - o(1))\)-approximation algorithm and a pseudo polynomial time \(1/4\) approximation algorithm for the problem. For the identical-size case, he improved the results to a polynomial time \((e - 1)/(3e - 1) - o(1))\)-approximation algorithm and a pseudo polynomial time \((1 - 1/e - o(1))\)-approximation algorithm. We note that by generalizing the algorithms for the multiple knapsack problem in \([7]\), there is a 0.468 deterministic algorithm for the identical-size case and a \(1 - 1/e - \epsilon\) \(\approx 0.387\) deterministic algorithm for the general case. The latter ratio can be improved to 0.468 by applying the approximate continuous greedy process in \([6]\).

1.3 Organization

We first present a formal description of the submodular multiple knapsack problem (SMKP) and some notations in Section 2. The greedy algorithm is also presented in this section. Then we present a tight \((1 - 1/e)\) algorithm for SMKP in Section 3 assuming the number of bins \(m\) is a constant. Tight algorithms for the identical-size case and the bounded-size case of SMKP are presented in Section 4 and Section 5, respectively. Algorithms from these two sections assume that the number of bins \(m\) is “large”. Finally, we conclude the paper in Section 6.
2 Preliminaries

Let $f : 2^N \to \mathbb{R}$ be a set function. $f$ is monotone if $f(S) \leq f(T)$ for any $S \subseteq T$. $f$ is submodular if $f(S + u) - f(S) \geq f(T + u) - f(T)$ for any $S \subseteq T$ and $u \notin T$. Here $S + u$ is a shorthand for $S \cup \{u\}$. Throughout this paper, we also use $f(u | S)$ and $f(T | S)$ to denote the marginal gains $f(S + u) - f(S)$ and $f(S \cup T) - f(S)$, respectively. An explicit representation of $f$ requires an exponential space with respect to the number of elements $|N|$. Thus we assume that $f$ is accessed by a value oracle that given a set $S$, returns $f(S)$.

An instance of the submodular multiple knapsack problem (SMKP) is defined as follows. We are given a ground set $N$ of $n$ elements and $m$ bins (knapsacks). For $j \in [m] := \{1, \ldots, m\}$, the $j$-th bin has a positive capacity $B_j$. Each element $u \in N$ has a positive size $c(u)$, and the size of a set $S \subseteq N$ is $c(S) = \sum_{u \in S} c(u)$. In addition, there is a non-negative objective function $f : 2^N \to \mathbb{R}_{\geq 0}$ defined over all subsets of $N$. In this paper, the objective function $f$ is assumed to be monotone and submodular. A set $S \subseteq N$ is feasible if there exists an ordered partition $(S_1, \ldots, S_m)$ of $S$ such that $c(S_j) \leq B_j$ for each $j \in [m]$. In other words, $S$ is feasible if there is a way to pack it into those $m$ bins. The way how $S$ is packed does not change the value of $f(S)$. The goal is to find a feasible set $S$ (as well as the way it is packed) which maximizes the value of objective function $f$.

**Notations.** For SMKP, the identical-size case refers to the case where all bins have the same capacity $B_j \equiv B$. The bounded-size case refers to the case where the ratio between the maximum capacity and the minimum capacity of the bins is bounded by a constant $\gamma \geq 1$. Throughout the paper, when we refer to a set $S \subseteq N$, we assume that some of its partition $(S_1, \ldots, S_m)$ is implicitly given. The partition may be generated from an algorithm, or be specified explicitly if necessary. Depending on the context, sometimes the partition might not be a feasible solution for the problem, that is, $c(S_j) \leq B_j$ might not hold for some $j \in [m]$. We use $|S|$ to denote the number of elements in $S$ and $b(S) = \{|j \in [m] : S_j \neq \emptyset\|$ to denote the number of bins used in its partition $(S_1, \ldots, S_m)$. The term $b(S)$ actually depends on the partition $(S_1, \ldots, S_m)$ of $S$, but we slightly abuse the notation here for ease of representation.

**The multilinear extension.** For a set function $f : 2^N \to \mathbb{R}$, its multilinear extension $F$ : $[0,1]^N \to \mathbb{R}$ is defined as $F(x) = \mathbb{E}[f(R(x))]$, where $R(x)$ is a random set such that $u \in N$ appears in it independently with probability $x_u$. Given $x \in [0,1]^N$, $F(x)$ can be approximately computed within an arbitrary small multiplicative error $\epsilon$ by the Monte Carlo sampling. For ease of representation, we assume in this paper there is an oracle which returns the exact value of $F(x)$ when given $x \in [0,1]^N$.

2.1 An Exponential Size Reduction

In the following two theorems, we show that an instance of SMKP can be reduced to an exponential-size instance of maximizing a monotone submodular function subject to a matroid constraint. This fact help understand how our algorithms perform and why they can achieve the desired ratios. It is also used in the continuous greedy algorithm for the bounded-size case. We note that the reduction is similar to that for generalized assignment problem in \cite{6}.

**Theorem 5.** An instance of SMKP can be reduced to an exponential-size instance of maximizing a monotone submodular function subject to a partition matroid.

**Proof.** Given an instance of SMKP, we construct the reduction as follows. Let $\mathcal{X}_j = \{S_j \mid S_j \subseteq N \text{ and } c(S_j) \leq B_j\}$ for $j \in [m]$ and $\mathcal{X} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_m$. Here $S_{j_1}$ and $S_{j_2}$ from distinct $\mathcal{X}_{j_1}$ and $\mathcal{X}_{j_2}$
may contain the same elements. However, we still regard them as two distinct sets, distinguished by their indexes. Then the capacity constraint can be translated into a partition matroid constraint which requires to find a set \( T \subseteq X \) such that \( |T \cap X_j| \leq 1 \) for \( j \in [m] \). We then define a function \( g : 2^Y \rightarrow \mathbb{R}_+ \) over all subsets of \( X \) such that for a set \( T \subseteq X \), \( g(T) = f(\cup_{S \in T} S) \). Clearly, \( g \) is monotone, since \( f \) does. Besides, for \( R \subseteq T \) and \( S \not\in T \), \( g(S | R) = f(S | \cup_{S' \in R} S') \geq f(S | \cup_{S' \in T} S') = g(S | T) \). Thus \( g \) is also submodular. This completes the proof.

**Theorem 6.** An instance of SMKP in the identical-size case can be reduced to an exponential-size instance of maximizing a monotone submodular function subject to a cardinality constraint.

**Proof.** Given an instance of SMKP, we construct the reduction as follows. Let \( \mathcal{X} = \{S | S \subseteq N \text{ and } c(S) \leq B\} \). Then the capacity constraint can be translated into a cardinality constraint which requires to find a set \( T \subseteq \mathcal{X} \) such that \( |T \cap \mathcal{X}| \leq m \). The definition of the objective function \( g \) is the same as that in the previous theorem.

### 2.2 The Greedy Algorithm

The greedy algorithm is shown as Algorithm 1. It returns a (possibly infeasible) set that admits a \((1 - 1/e)\)-approximation. All the algorithms in this paper are built based on it. It selects elements to be packed in a greedy manner, according to the densities of elements’ current marginal values to their sizes. It packs the selected elements into bins as long as there is a bin whose capacity has not been fully used. As a result, Algorithm 1 might not return a feasible solution. However, it is easy to see that each bin packs at most one additional element. That is, if \( c(S_j) > B_j \) for some \( j \in [m] \) and \( u \) is the last element added into \( S_j \), then \( c(S_j \setminus \{u\}) < B_j \) and hence \( S_j \setminus \{u\} \) is feasible in the \( j \)-th bin. For convenience, we say Algorithm 1 returns an almost feasible solution. For each bin, the last element that violates the capacity constraint is called a reserved element. The following lemma lower bounds the quality of the almost feasible set returned by Algorithm 1.

| Algorithm 1: Greedy |
|---------------------|
| **Input:** ground set \( N \), objective function \( f \), size function \( c \), number of bins \( m \), capacities \((B_1, \ldots, B_m)\). |
| **Output:** An almost feasible set (and the way it is packed). |
| 1 Let \( S = \emptyset \) and \( S = (S_1, \ldots, S_m) \) be its partition into \( m \) bins. |
| 2 while \( N \not\subseteq S \) and there exists \( j \in [m] \) such that \( c(S_j) < B_j \) do |
| 3 \( u^* = \arg \max_{u \in N \setminus S} f(u | S)/c(u) \). |
| 4 \( S = S + u^* \) and \( S_j = S_j + u^* \). |
| 5 end |
| 6 return \( S = (S_1, \ldots, S_m) \). |

**Lemma 1.** Let \( S \) be the set returned by Algorithm 1. For any set \( T \subseteq N \), we have

\[
    f(S) \geq \left(1 - e^{-\frac{c(S)}{\alpha T}}\right) \cdot f(T).
\]

**Proof.** We assume that \( c(S) \geq \sum_{j=1}^m B_j \), since otherwise all elements in \( N \) have been packed into \( S \) and therefore the lemma holds trivially. Assume that \( S = \{u_1, u_2, \ldots, u_l\} \), and for \( i \in [l] \), \( S^i = \{u_1, u_2, \ldots, u_i\} \) denotes the first \( i \) elements picked by the algorithm. Then, by the greedy rule,

\[
    \frac{f(u_i \mid S^{i-1})}{c(u_i)} \geq \frac{f(t \mid S^{i-1})}{c(t)}, \forall t \in T \setminus S^{i-1}.
\]
This gives us
\[
\frac{f(S^i) - f(S^{i-1})}{c(u_i)} \geq \frac{f(T \setminus S^{i-1} \mid S^{i-1})}{c(T \setminus S^{i-1})} \geq \frac{f(T) - f(S^{i-1})}{c(T)}.
\]
The first inequality holds since \( f \) is submodular. The second inequality holds since \( f \) is monotone and \( c(T \setminus S^{i-1}) \leq c(T) \).

We also assume that \( f(T) > f(S^i) \), since otherwise the lemma already holds. Under this assumption, it must holds that \( c(u_i) < c(T) \), since otherwise the inequality
\[
\frac{f(S^i) - f(S^{i-1})}{c(u_i)} \geq \frac{f(T) - f(S^{i-1})}{c(T)}
\]
implies that \( f(T) \leq f(S^i) \leq f(S^j) \). A contradiction!

By rearranging the last inequality, we have
\[
f(T) - f(S^i) \leq \left(1 - \frac{c(u_i)}{c(T)}\right) (f(T) - f(S^{i-1})).
\]

The recurrence gives us
\[
f(T) - f(S^i) \leq \prod_{j=1}^{i} \left(1 - \frac{c(u_j)}{c(T)}\right) \cdot f(T) \leq \prod_{j=1}^{i} e^{-\frac{c(u_j)}{c(T)}} \cdot f(T) = e^{-\frac{c(S^i)}{c(T)}} \cdot f(T).
\]
The second inequality holds due to \( e^x \geq 1 + x \). Hence we have
\[
f(S^i) \geq \left(1 - e^{-\frac{c(S^i)}{c(T)}}\right) \cdot f(T).
\]
The lemma follows immediately from it. \( \square \)

Let \( OPT \) be an optimal solution. The above lemma immediately leads to the following corollary.

**Corollary 1.** The set \( S \) returned by Algorithm 1 satisfies \( f(S) \geq (1 - 1/e) f(OPT) \).

**Proof.** If set \( S \) returned by Algorithm 1 satisfies \( c(S) \geq mB \), then we have \( c(S) \geq c(OPT) \), and Lemma 1 immediately leads to the corollary. If set \( S \) returned by Algorithm 1 satisfies \( c(S) < mB \), this means \( S = N \). Thus \( c(S) = c(OPT) \). The corollary also holds. \( \square \)

### 3 Constant Number of Bins

In this section, we present a tight \( (1 - 1/e) \) algorithm for SMKP, assuming the number of bins \( m \) is a constant. If we use the greedy algorithm in Section 2.2 to pack elements into those \( m \) bins, we are able to obtain a set with the approximation ratio \( (1 - 1/e) \), but this set might be infeasible, with a reserved element in each bin. If the reserved elements are of small values, we can just discard these elements without losing too much. However, the greedy algorithm itself cannot guarantee this property. In light of this, Algorithm 2 manages to first pack elements of large values in an optimal solution by the enumeration technique, and then pack elements of small value by the greedy algorithm. In doing so, it ensures that the values of the reserved elements are small and therefore can be safely discarded.

**Theorem 7.** Algorithm 2 admits a \( (1 - 1/e) \)-approximation and runs in \( O((mn)^{em + 4}) \) time. When \( m \) is a constant, it runs in polynomial time.
Denote by $\text{OPT}$ the optimal solution. Assume w.l.o.g. that $|\text{OPT}| > [1/\delta]$, since otherwise $\text{OPT}$ will be enumerated in the enumeration step. We order elements in $\text{OPT}$ greedily according to their marginal values, i.e. $o_1 = \max_{o \in \text{OPT}} f(o)$, $o_2 = \max_{o \in \text{OPT} \setminus \{o_1\}} f(o \mid o_1)$, etc. In the enumeration step, the solution $E = (E_1, \ldots, E_m)$ must be visited such that $E$ contains exactly the first $[1/\delta]$ elements in $\text{OPT}$ and these elements are packed in the same way as in $\text{OPT}$. In the following analysis, we focus on this solution and show that $S_E = E \cup G_E$ achieves the desired ratio. Since the algorithm returns the solution with the maximum value, this completes the proof.

We claim that $f(o \mid E) \leq \delta f(E)$ for any $o \in \text{OPT} \setminus E$. Let $\text{OPT}_i$ be the first $i$ elements in $\text{OPT}$. Then for $j \leq i$ and any $o \not\in \text{OPT}_i$, $f(o_j \mid \text{OPT}_{j-1}) \geq f(o \mid \text{OPT}_{j-1}) \geq f(o \mid \text{OPT}_i)$. Summing up from $j = 1$ to $i$, we have $f(\text{OPT}_i) \geq i \cdot f(o \mid \text{OPT}_i)$. By plugging $i = [1/\delta]$, $f(E) \geq [1/\delta] f(o \mid E) \geq 1/\delta f(o \mid E)$. The claim holds. This implies that $D \cap (\text{OPT} \setminus E) = \emptyset$. As a result, elements in $\text{OPT} \setminus E$ will not be excluded from the execution of the greedy algorithm. Besides, since elements in $E$ are packed in the same way as in $\text{OPT}$, $\text{OPT} \setminus E$ is actually a feasible (indeed optimal) solution when invoking the greedy algorithm. By Corollary $\mathbb{□}$ the set $G'_E$ returned by $\text{GREEDY}$ satisfies
\[
\frac{f(G'_E \setminus E)}{f(G_E \setminus E)} \leq \frac{1 - 1/e}{f(\text{OPT} \setminus E \mid E)}.
\]
Since $G_E$ is obtained from $G'_E$ by discarding at most $m$ reserved elements in $G''_E$, by the submodularity of $f$,
\[
f(G_E \setminus E) \geq f(G'_E \setminus E) - \sum_{u \in G''_E} f(u \mid E)
\geq (1 - 1/e) f(\text{OPT} \setminus E \mid E) - \delta m f(E).
\]
Hence we have
\[
f(E \cup G_E) \geq (1 - 1/e) f(\text{OPT}) + (1/e - \delta m) f(E) \geq (1 - 1/e) f(\text{OPT}).
\]
The last inequality holds since $1/\delta \geq em$.

Finally, since there are $O((mn)^{1/\delta + 2})$ feasible solutions $E = (E_1, \ldots, E_m)$ with $|E| \leq [1/\delta]$, and $\text{GREEDY}$ costs $O(n^2)$ queries, Algorithm $\mathbb{□}$ uses $O(mn^e + 4)$ queries. Since $m$ is a constant, Algorithm $\mathbb{□}$ runs in polynomial time. $\square$
4 The Identical-size Case

In this section, we present a tight deterministic algorithm for the identical-size case of SMKP where all bins have the same capacity $B_j = B$. The main algorithm is depicted as Algorithm 3. Given a constant $\epsilon > 0$ as input, it requires that the number of bins $m \geq 4/e^8$ and will achieve a $1 - 1/e - O(\epsilon)$ approximation. When $m < 4/e^8$, we can use Algorithm 2 to achieve a tight $(1 - 1/e)$-approximation. Combining these two results, we resolve the identical-size case completely and Theorem 2 follows.

In Algorithm 3 the first $(1 - \epsilon)m$ bins are called working bins. In contrast, the last $em$ bins are called reserved bins. It mainly uses working bins to pack elements. A bin is called empty if no elements have been packed in this bin. Whenever there remains at least $4/e^7$ empty working bins, Algorithm 3 attempts to pack $4/e^7$ empty working bins with the remaining elements by invoking Algorithm 4. Let $S$ be the set returned by an execution of Algorithm 4 and $b(S)$ denote the number of bins that Algorithm 4 used to pack $S$. We will show in Lemma 2 that the marginal value of $S$ w.r.t. the elements already packed is approximately a $b(S)/m$ fraction of the marginal value of $OPT$. This directly leads to the desired ratio (Theorem 5).

Algorithm 3 divides elements into two classes according to their sizes. Given input $\epsilon$, an element $u \in N$ is large if $c(u) > \epsilon B$ and small otherwise. Let $N_l = \{u \in N \mid c(u) > \epsilon B\}$ be the set of large elements in $N$ and $N_s = N \setminus N_l$ be the set of small elements. From a high-leveled viewpoint, Algorithm 4 first enumerates all feasible ways of packing only the large elements. Then for each enumerated solution, small elements are packed into it by invoking the greedy algorithm. Finally, the one with the maximum “average value” will be returned.

Note that if an enumerated solution $E = (E_1, \cdots, E_m)$ only uses $m'$ bins, Algorithm 4 will only add small elements to those $m'$ non-empty bins. The only exception is that when $E = \emptyset$ (or equivalently $m' = 0$), Algorithm 4 will use a single bin to pack small elements. As a result, although Algorithm 3 invokes Algorithm 4 with $4/e^7$ bins, the set $S$ returned by Algorithm 4 might use fewer than $4/e^7$ bins. Besides, the solution $S$ might not be feasible. Due to the greedy algorithm, $S$ might contain a reserved element in each of the bins it uses. Nonetheless, we still let Algorithm 4 returns an infeasible solution and tackle the reserved elements in Algorithm 3. At this point, the reserved bins come to rescue. Observe that all reserved elements must be small elements. Thus the $em$ empty reserved bins can pack at least $m$ reserved elements. On the other hand, there are only $(1 - \epsilon)m$ working bins; each of them contains at most one reserved element. Consequently, all reserved elements can be packed into the reserved bins without exceeding the capacities, thus, line 4 in Algorithm 3 can always be executed.

We remark that though Algorithm 2 and Algorithm 4 are very similar, there are some differences between them. First, in the enumeration step, Algorithm 2 enumerates feasible solutions containing constant number of elements, while Algorithm 4 enumerates feasible solutions of large elements. Second, in the greedy step, Algorithm 2 directly discards the reserved elements, while Algorithm 4 retains the reserved elements and may return an infeasible solution.

We now introduce some notations for sake of analysis. Assume that the while loop in Algorithm 3 has been executed $r \leq (1 - \epsilon)m$ times. For $j \in [r]$, denote by $S_j$ the set returned by Algorithm 4 in the $j$-th loop and by $T_j$ the packed elements after the $j$-th loop. Then $T_j = T_{j-1} \cup S_j$. Recall that $b(S_j)$ is the number of bins used to pack $S_j$ by Algorithm 4. Note that $b(S_j)$ does not count the number of reserved bins used to pack the reserved elements in $S_j$. Let $OPT$ be the optimal solution. The following lemma bounds the marginal value of $S_j$ with respect to $T_{j-1}$ in terms of the marginal value of $OPT$.  


Algorithm 3: The Identical-size Case

**Input:** constant $\epsilon > 0$, ground set $N$, objective function $f$, size function $c$, number of bins $m \geq 4/\epsilon^8$, capacity $B$.

**Output:** A feasible set (and the way it is packed).

1. Let $T$ denote the packed elements so far and initialize it as $T = \emptyset$.
2. while there remains at least $4/\epsilon^7$ empty working bins do
   3. $S = \text{Constant-bins}(\epsilon, N \setminus T, f(\cdot | T), c(\cdot), 4/\epsilon^7, B)$, where $S$ has been packed into $b(S) \leq 4/\epsilon^7$ bins, but with one reserved elements in each bin.
   4. Pack those $b(S)$ reserved elements into reserved bins.
   5. $T = T \cup S$.
3. end
4. return $T$ (and the way it is packed).

Algorithm 4: Constant-bins

**Input:** constant $\epsilon > 0$, ground set $N$, objective function $f$, size function $c$, number of bins $m$ (constant), capacity $B$.

**Output:** An almost feasible set (and the way it is packed).

1. Let $N_l = \{u \in N \mid c(u) > \epsilon B\}$ be the set of large elements and $N_s = N \setminus N_l$ be the set of small elements.
2. Enumerate all feasible solutions $E = (E_1, \ldots, E_m)$ such that $E \subseteq N_l$.
3. foreach feasible solution $E = (E_1, \ldots, E_m)$ do
   4. Reorder $E_j$’s to ensure that there is an integer $m' \leq m$ such that $E_1 \neq \emptyset, \ldots, E_{m'} \neq \emptyset, E_{m'+1} = \emptyset, \ldots, E_m = \emptyset$.
   5. Let $m' = \max\{m', 1\}$.
   6. Use the first $m'$ bins to pack small elements and let $G_E = \text{Greedy}(N_s, f(\cdot | E), c(\cdot), m', (B, \ldots, B))$. Then $S_E = E \cup G_E$ is an almost feasible solution.
4. end
5. return $S = \arg \max_E f(S_E)/b(S_E)$ (and the way it is packed).
**Lemma 2.** For $1 \leq j \leq r$, we have

$$f(S_j \mid T_{j-1}) \geq \frac{(1 - 2\epsilon)b(S_j)}{m} f(OPT \setminus T_{j-1} \mid T_{j-1}).$$

The proof of Lemma 2 is delayed to Section 4.1. For now, we show that it directly implies the desired approximation ratio.

**Theorem 8.** Algorithm 3 admits a $(1 - 1/e - O(\epsilon))$-ratio and runs in $O(mn^2 \left(\frac{\ln n}{\epsilon}\right)^{1/\epsilon})$ time.

*Proof.* By Lemma 2 and the monotonicity of $f$, we have

$$f(T_j) - f(T_{j-1}) \geq \left(1 - \frac{(1 - 2\epsilon)b(S_j)}{m}\right)(f(OPT) - f(T_{j-1})).$$

By rearranging the above inequality, we obtain

$$\left(1 - \frac{(1 - 2\epsilon)b(S_j)}{m}\right)(f(OPT) - f(T_{j-1})) \geq f(OPT) - f(T_j).$$

This gives us

$$f(OPT) - f(T_j) \leq \prod_{i=1}^{j} \left(1 - \frac{(1 - 2\epsilon)b(S_j)}{m}\right) f(OPT)$$

$$\leq \prod_{i=1}^{j} e^{-\frac{(1 - 2\epsilon)b(S_j)}{m}} f(OPT)$$

$$\leq e^{-\frac{2\epsilon}{m} \sum_{i=1}^{j} b(S_j)} f(OPT).$$

On the other hand, by the ending condition of the while loop,

$$\sum_{i=1}^{r} b(S_i) \geq (1 - \epsilon)m - 4/\epsilon^2 \geq (1 - 2\epsilon)m.$$ 

The last inequality holds since $m \geq 4/\epsilon^8$. Combining the above inequalities, we have

$$f(OPT) - f(T_r) \leq e^{-\frac{(1 - 2\epsilon)^2}{m}} f(OPT),$$

which implies that

$$f(T_r) \geq (1 - e^{-\frac{(1 - 2\epsilon)^2}{m}}) f(OPT) = (1 - e^{-1} - O(\epsilon)) f(OPT).$$

The running time follows by observing that the algorithm runs in at most $m$ rounds, the enumeration step costs $O((\frac{\ln n}{\epsilon})^{1/\epsilon})$ time and the greedy algorithm needs $O(n^2)$ time. \qed

### 4.1 Proof of Lemma 2

This section is dedicated to prove Lemma 2. For simplicity, we show that the lemma holds for the first iteration of the while loop in Algorithm 3. A similar argument holds for all remaining loops, in which it suffices to replace a set $S$ by $S \setminus T_{j-1}$ and the function value $f(\cdot)$ by $f(\cdot \mid T_{j-1})$ for the $j$-th loop. Let $OPT_l = \{u \in OPT \mid c(u) > \epsilon B\}$ be the set of large elements in $OPT$ and $OPT_s = OPT \setminus OPT_l$ be the set of small elements in $OPT$. We prove Lemma 2 by a case analysis, based on the densities of the large and small elements in $OPT$ (Lemma 3 and Lemma 5).
Lemma 3. If \( \frac{f_{\text{OPT}}}{c(\text{OPT}_s)} \geq \frac{f(\text{OPT})}{c(\text{OPT})} \) holds, then \( f(S) \geq \frac{(1-2\epsilon) f(S_1)}{m} \).

Proof. If \( c(\text{OPT}_s) \geq emB \), consider the case where \( E = \emptyset \) in the enumeration step of Algorithm 4. Recall that in this case the algorithm uses a single bin to pack small elements. By Lemma 1, this case the lemma holds.

If \( E \neq \emptyset \), consider the case where \( x \geq 0 \). The second inequality holds since \( 1 - e^{-x} \geq x - x^2/2 \) for \( x \geq 0 \). The third inequality holds since \( c(\text{OPT}) \leq mB \) and \( c(\text{OPT}_s) \geq emB \). The second to last inequality holds due to the condition of the lemma and the monotonicity of \( f \). The last inequality holds as long as \( m \geq 1/(2\epsilon^3) \). Thus in this case the lemma holds.

Thus in this case the lemma holds.

Below we assume that \( c(\text{OPT}_s) < emB \) and \( f(\text{OPT}_s) < (1 - e^{-B/c(\text{OPT}_s)})^{-1} \cdot \frac{f(\text{OPT})}{m} \). Combining these two assumptions, we then show the small elements in \( \text{OPT} \) only contributes a negligible fraction of the value function:

\[
f(\text{OPT}_s) < (1 - e^{-1/em})^{-1} \cdot \frac{f(\text{OPT})}{m} \leq \left( \frac{1}{em} - \frac{1}{2\epsilon^2 m^2} \right)^{-1} \cdot \frac{f(\text{OPT})}{m} \leq \left( \frac{1}{2\epsilon m} \right)^{-1} \cdot \frac{f(\text{OPT})}{m} = 2\epsilon f(\text{OPT}).
\]

The first inequality holds since \( (1 - e^{-B/x})^{-1} \) is monotone increasing. The second inequality holds since \( 1 - e^{-x} \geq x - x^2/2 \) for \( x \geq 0 \). The third inequality holds as long as \( m \geq 1/\epsilon \). Hence by the submodularity of \( f \),

\[
f(\text{OPT}_1) \geq f(\text{OPT}) - f(\text{OPT}_s) \geq (1 - 2\epsilon) \cdot f(\text{OPT}).
\]

Let \( E^* = \arg \max \{ f(E) \mid E \subseteq N_l \text{ and } c(E) \leq B \} \). Then in the enumeration step of Algorithm 4, the feasible solution where a single bin is used to pack \( E^* \) will be enumerated, and it holds that

\[
f(E^* \cup G_{E^*}) \geq f(E^*) \geq \frac{1}{m} f(\text{OPT}_1) \geq \frac{1 - 2\epsilon}{m} \cdot f(\text{OPT}).
\]

Thus in this case the lemma holds. This completes the proof. \( \square \)
Lemma 4. If \( \frac{f(OPT)}{c(OPT)} \geq \frac{f(OPT)}{c(OPT)} \) holds, and additionally in the enumeration step in Algorithm \#4 there is a feasible solution \( E \subseteq OPT \) which satisfies

\[
\frac{c(E)}{b(E)} \geq \frac{c(OPT)}{m} \quad \text{and} \quad \frac{f(E)}{c(E)} \geq \frac{f(OPT)}{c(OPT)}
\]

then \( f(S_1) \geq \frac{(1-\epsilon)b(S_1)}{m} f(OPT) \).

Proof. The first condition \( \frac{c(E)}{b(E)} \geq \frac{c(OPT)}{m} \) for the enumerated set \( E \) corresponds to the case where on average the algorithm uses more space to pack large elements than \( OPT \) does. Although this ensures that the algorithm collects an enough value from large elements, it may make the algorithm collect an insufficient value from small elements, since there is no enough space for small elements. The key observation is that under the second condition \( \frac{f(E)}{c(E)} \geq \frac{f(OPT)}{c(OPT)} \), one can use the redundant value collected from large elements to cover the deficit incurred when packing small elements. We now give a formal proof of this lemma.

We first construct a set \( E' \) which is a surrogate of \( OPT \) and satisfies \( c(E)/b(E) = c(E')/m \). Initially, \( E' = OPT \). Let \( OPT = \{u_1, u_2, \ldots\} \) be ordered such that \( u_1 = \arg\min_{u \in OPT} \{f(u | OPT)/c(u)\} \), \( u_2 = \arg\min_{u \in OPT \setminus \{u_1\}} \{f(u | OPT \cup \{u_1\})/c(u)\} \), etc. Add elements in \( OPT \) into \( E' \) in this order until there is a \( u_k \) such that \( c(E')/m \leq c(E')/b(E) \) and \( c(E' + u_k)/m > c(E')/b(E) \). At this point, we split \( u_k \) into two elements \( u_k' \) and \( u_k'' \) such that 1) \( c(u_k') = mc(E')/b(E) - c(E') \) and \( c(u_k'') = c(u') - c(u_k''); \) 2) for any \( S \subseteq N \setminus \{u_k\} \), \( f(u_k' | S) = pf(u_k | S) \) and \( f(u_k'' | S) = f(u_k | S) - f(u_k' | S) \), where \( p = c(u_k')/c(u_k) \). Finally, add \( u_k' \) into \( E' \). The construction of \( E' \) is finished.

Next we prove that \( \frac{f(E')}{b(E')} \geq \frac{f(E')}{m} \). By the construction of \( E' \),

\[
\frac{f(E \setminus OPT) | OPT}{c(E \setminus OPT)} \leq \frac{f(OPT_s) | OPT}{c(OPT_s)}.
\]

On the other hand, by the condition of the lemma,

\[
\frac{f(OPT_s) | OPT}{c(OPT_s)} \leq \frac{f(OPT) | OPT}{c(OPT)} \leq \frac{f(OPT)}{c(OPT)} \leq \frac{f(OPT)}{c(OPT)}.
\]

The above two inequalities together imply that

\[
\frac{f(E') - f(OPT)}{c(E' \setminus OPT)} \leq \frac{f(OPT)}{c(OPT)}.
\]

Note that \( OPT \subseteq E' \), the last inequality is equivalent to \( \frac{f(E')}{c(E')} \leq \frac{f(OPT)}{c(OPT)} \). Since \( \frac{f(E')}{c(E')} \geq \frac{f(OPT)}{c(OPT)} \) by the condition of the lemma, we have \( \frac{f(E)}{c(E)} \geq \frac{f(E')}{c(E')} \). Together with the fact that \( \frac{c(E)}{b(E)} = \frac{c(E')}{m} \), we have shown that \( \frac{f(E)}{b(E)} \geq \frac{f(E')}{m} \).

On the other hand, let \( E'' = OPT_s \setminus E' - u_k + u_k'' \). Clearly, \( f(E' \cup E'') = f(OPT_i \cup OPT_s) = f(OPT) \) and \( c(E' \cup E'') = c(OPT_i \cup OPT_s) = c(OPT) \). We assume that the set \( G_E \) returned by the greedy algorithm satisfies that \( c(G_E) \geq b(E) \cdot B - c(E) \), since otherwise all the remaining
small elements will be packed into $G_E$ and it follows that $f(G_E \mid E) \geq f(E'' \mid E)$. Under this assumption, we have

$$\frac{c(G_E)}{b(E)} = B - \frac{c(E)}{b(E)} = B - \frac{c(E')}{m} \geq \frac{c(OPT) - c(E')}{m} = \frac{c(E'')}{m}. $$

By Lemma 1,

$$ f(G_E \mid E) \geq (1 - e^{-c(G_E)/c(E'')}) \cdot f(E'' \mid E) \geq (1 - e^{-b(E)/m}) \cdot f(E'' \mid E) \geq \left( \frac{b(E)}{m} - \frac{b(E)^2}{2m^2} \right) \cdot f(E'' \mid E) \geq \frac{(1 - e)b(E)}{m} \cdot f(E'' \mid E). $$

The third inequality holds since $1 - e^{-x} \geq x - x^2/2$ for $x \geq 0$. The last inequality holds as long as $b(E) \leq 2\epsilon m$, which is ensured by the facts that $b(E) \leq 4/\epsilon^7$ and $m \geq 4/\epsilon^8$. Finally, by the above argument,

$$ f(E \cup G_E) = f(E) + f(G_E \mid E) \geq \frac{b(E)}{m} f(E') + \frac{(1 - e)b(E)}{m} f(E'' \mid E) \geq \frac{(1 - e)b(E)}{m} f(E' \cup E'') \geq (1 - e)b(E) f(OPT). $$

This completes the proof.

In the enumeration step of Algorithm 4 if there is a feasible solution $E \subseteq OPT_1$ satisfying

$$ \frac{c(E)}{b(E)} \leq \frac{c(OPT_1)}{m} \quad \text{and} \quad \frac{f(E)}{b(E)} \geq \frac{f(OPT_1)}{m},$$

then Lemma 2 holds. Intuitively, this is because the solution $E$ collects a sufficiently large value in $OPT_1$ and there remains enough space to pack small elements using the greedy algorithm. Together with Lemma 4 we find two distinct sufficient conditions for $E \subseteq OPT_1$ in the enumeration step which lead to the proof of Lemma 2. The difficulty lies in how one can prove the existence of such a feasible solution $E$ which satisfies one of the two sufficient conditions. We remark that if Algorithm 4 has only one bin as input, there are instances where no feasible subsets of $OPT_1$ satisfy either conditions.

The second key step of our analysis is to prove a technical lemma by an averaging argument, which shows that the desired feasible solution must exist when Algorithm 4 takes multiple bins as input at a time. Indeed, constant bins suffice for the lemma to work. The technical lemma is presented in Section 4.2. Based on it, the final result is proved below in Lemma 5.

**Lemma 5.** If $\frac{f(OPT)}{c(OPT)} \geq \frac{f(OPT)}{c(OPT)}$ holds, then $f(S_1) \geq \frac{(1-2\epsilon)b(S_1)}{m} f(OPT)$. 
Proof. For technical reasons, we first show that we can assume w.l.o.g. that \( c(\text{OPT}_s) \geq \epsilon m B \) and \( c(\text{OPT}_t) \geq \epsilon m B \), which means that both large and small elements occupy a non-negligible fraction in \( \text{OPT} \).

Consider the case \( c(\text{OPT}_s) \leq \epsilon m B \). We can assume w.l.o.g. that \( c(\text{OPT}) \geq \frac{1}{2} m B \), since we can assume that a half size of each bin is used to pack elements. Under this assumption, \( c(\text{OPT}_s) \leq 2\epsilon \cdot c(\text{OPT}) \). Due to the condition of this lemma and the submodularity of \( f \), we have

\[
\frac{f(\text{OPT}_s)}{c(\text{OPT}_s)} \leq \frac{f(\text{OPT})}{c(\text{OPT})}.
\]

The above inequalities imply that \( f(\text{OPT}_s) \leq 2\epsilon f(\text{OPT}) \). By the submodularity of \( f \),

\[
f(\text{OPT}_t) \geq f(\text{OPT}) - f(\text{OPT}_s) \geq (1 - 2\epsilon) f(\text{OPT}).
\]

Let \( E^* = \arg \max \{ f(E) \mid E \subseteq N_t \text{ and } c(E) \leq B \} \). In the enumeration step of Algorithm 5, the feasible solution where a single bin is used to pack \( E^* \) will be enumerated, and it holds that

\[
f(E^* \cup G_{E^*}) \geq f(E^*) \geq \frac{1}{m} f(\text{OPT}_t) \geq \frac{1 - 2\epsilon}{m} f(\text{OPT}).
\]

Thus the lemma holds in this case.

Consider the case \( c(\text{OPT}_t) \leq \epsilon m B \). Consider the set \( E = \{ u_l \} \), where \( u_l \in \text{OPT}_t \) satisfies \( f(u_l)/c(u_l) \geq f(\text{OPT}_t)/c(\text{OPT}_t) \). It holds that \( c(\text{OPT}_t)/m \leq \epsilon B \leq c(E) \leq B \) and \( E \) can be packed into a single bin. Consequently, \( E \) satisfies the conditions of Lemma 5 which implies the lemma.

In the remaining part of the proof, we assume that \( c(\text{OPT}_s) \geq \epsilon m B \) and \( c(\text{OPT}_t) \geq \epsilon m B \). Assume that in \( \text{OPT} \), \( \text{OPT}_t \) is partitioned into \( (\text{OPT}_{t,1}, \text{OPT}_{t,2}, \ldots, \text{OPT}_{t,m}) \) with \( c(\text{OPT}_{t,j}) \leq B \). We apply the procedure defined in Section 4.2 with \( \text{OPT}_t = (\text{OPT}_{t,1}, \text{OPT}_{t,2}, \ldots, \text{OPT}_{t,m}) \) as the initial partition for \( t = \left\lceil \frac{2\log 1/\epsilon^2}{\log 3/2} \right\rceil \) rounds. By Corollary 2, there is a set \( E \subseteq \text{OPT}_t \) which is the union of at most \( 2^{t+1} \) subsets in the partition \( (\text{OPT}_{t,1}, \text{OPT}_{t,2}, \ldots, \text{OPT}_{t,m}) \) and satisfies one of the following two conditions.

1. \( \frac{c(E)}{b(E)} \leq \frac{c(\text{OPT}_t)}{m} + \epsilon^2 B \) and \( \frac{f(E)}{b(E)} \geq \frac{f(\text{OPT}_t)}{m} \).
2. \( \frac{c(E)}{b(E)} \geq \frac{c(\text{OPT}_t)}{m} - \epsilon^2 B \) and \( \frac{f(E)}{c(E)} \geq \frac{f(\text{OPT}_t)}{c(\text{OPT}_t)} \).

Since \( E \) is the union of at most \( 2^{t+1} \) subsets in the form of \( \text{OPT}_{t,j} \), if \( E \) is packed as in \( \text{OPT}_t \), the number of bins \( b(E) \) used by it satisfies \( b(E) \leq 2^{t+1} \leq 4/\epsilon^7 \), by the value of \( t \). As a result, in the enumeration step, the set \( E \) as well as the way it is packed in \( \text{OPT}_t \) will be enumerated. Below we complete the proof by showing that the lemma holds whenever the set \( E \) satisfies either of the above two conditions.

**Case 1:** \( \frac{c(E)}{b(E)} \leq \frac{c(\text{OPT}_t)}{m} + \epsilon^2 B \) and \( \frac{f(E)}{b(E)} \geq \frac{f(\text{OPT}_t)}{m} \). Recall that \( c(E) + c(G_E) \leq b(E) \cdot B \) and \( c(\text{OPT}_t) + c(\text{OPT}_s) \leq m B \). We have

\[
\frac{c(G_E)}{b(E)} \geq B - \frac{c(E)}{b(E)} \geq B - \frac{c(\text{OPT}_t)}{m} - \epsilon^2 B \geq \frac{c(\text{OPT}_s)}{m} - \frac{\epsilon c(\text{OPT}_s)}{m} = \frac{(1 - \epsilon)c(\text{OPT}_s)}{m}.
\]
The last inequality holds since we have assumed that \( c(OPT_s) \geq \epsilon mB \). By Lemma 11

\[
  f(G_E \mid E) \geq (1 - e^{-(c(E)/c(OPT_s))}) f(OPT_s \mid E)
  \geq (1 - e^{-(1-\epsilon)b(E)/m}) f(OPT_s \mid E)
  \geq (1 - e^{-(1-\epsilon)b(E)/m}) f(OPT_s \mid OPT_i)
  \geq \left(\frac{(1-\epsilon)b(E)}{m} - \frac{(1-\epsilon)^2b(E)^2}{2m^2}\right) (f(OPT) - f(OPT_i))
  \geq \frac{(1 - 2\epsilon)b(E)}{m} f(OPT).
\]

The second to last inequality holds since \( 1 - e^{-x} \geq x - x^2/2 \). The last inequality holds as long as \( b(E) \leq \frac{2m}{(1-\epsilon)^2} \), which follows from the facts that \( b(E) \leq 4/\epsilon^7 \) and \( m \geq 4/\epsilon^8 \). Finally,

\[
  f(E \cup G_E) = f(E) + f(G_E \mid E)
  \geq \frac{b(E)}{m} f(OPT_i) + \frac{(1 - 2\epsilon)b(E)}{m} f(OPT)
  \geq \frac{(1 - 2\epsilon)b(E)}{m} f(OPT) f(OPT_i).
\]

The lemma still holds in this case.

**Case 2:** \( \frac{c(E)}{b(E)} \geq \frac{c(OPT_i)}{m} - e^2 B \) and \( \frac{f(E)}{c(E)} \geq \frac{f(OPT_i)}{c(OPT_i)} \). If \( \frac{c(E)}{b(E)} \geq \frac{c(OPT_i)}{m} \), then the conditions of Lemma 11 holds and therefore the lemma follows. Hence we assume that \( \frac{c(E)}{b(E)} \leq \frac{c(OPT_i)}{m} \) and it follows that

\[
  \frac{c(G_E)}{b(E)} \geq \frac{c(OPT_s)}{m}.
\]

By Lemma 11

\[
  f(G_E \mid E) \geq (1 - e^{-c(G_E)/c(OPT_s)}) \cdot f(OPT_s \mid E)
  \geq (1 - e^{-b(E)/m}) f(OPT_s \mid OPT_i)
  \geq \frac{(1 - \epsilon)b(E)}{m} f(OPT - f(OPT_i)).
\]

The last inequality holds since \( 1 - e^{-x} \geq x - x^2/2 \), \( b(E) \leq 4/\epsilon^7 \) and \( m \geq 4/\epsilon^8 \).

On the other hand, by the condition of this case,

\[
  f(E) \geq \frac{c(E)}{c(OPT_i)} f(OPT_i)
  \geq \left(\frac{b(E)}{m} - \frac{e^2 B b(E)}{c(OPT_i)}\right) f(OPT_i)
  \geq \left(\frac{b(E)}{m} - \frac{eb(E)}{m}\right) f(OPT)
  = \frac{(1 - \epsilon)b(E)}{m} f(OPT_i).
\]

The last inequality holds since we have assumed that \( c(OPT_i) \geq \epsilon mB \). Finally,

\[
  f(E \cup G_E) = f(E) + f(G_E \mid E)
\]
Lemma 6. The above procedure has the following properties.

\[ \frac{(1 - \epsilon)b(E)}{m} f(OPT) + \frac{(1 - \epsilon)b(E)}{m} (f(OPT) - f(OPT_t)) \]
\[ \geq \frac{(1 - \epsilon)b(E)}{m} f(OPT) \]

The lemma still holds in this case. \qed

4.2 A Technical Lemma

Below we define an auxiliary procedure to assist the analysis of Lemma 2. Note that the procedure itself will never be executed in the algorithm.

The procedure starts with a feasible set \( X^0 \) which enjoys a partition \( X^0 = (X^0_1, X^0_2, \ldots, X^0_{m_0}) \) with \( m_0 \) subsets and satisfies \( c(X^0_j) \leq B \) for all \( j \in [m_0] \). Then it runs \( t \) rounds in order to obtain a new partition of \( X^0 \) with desired properties. Assume that at the beginning of the \( k \)-th round, the partition of \( X^0 \) has been updated to \( X^{k-1} = (X^{k-1}_1, X^{k-1}_2, \ldots, X^{k-1}_{m_{k-1}}) \), where each \( X^{k-1}_j \) is a union of \( b^{k-1}_j \) subsets in \( X^0 \). The procedure proceeds by applying the so-called COUPLE operation to construct a new partition \( X^k = (X_k^1, X_k^2, \ldots, X_k^{m_k}) \) as follows.

1. Reorder the subsets in \( X^{k-1} \) according to their average size such that

\[ \frac{c(X^{k-1}_1)}{b^{k-1}_1} \leq \frac{c(X^{k-1}_2)}{b^{k-1}_2} \leq \cdots \leq \frac{c(X^{k-1}_{m_{k-1}})}{b^{k-1}_{m_{k-1}}} \]

2. If \( m_{k-1} \) is odd, let \( c_{k-1} = \left\lfloor \frac{c(X^{k-1}_1)}{b^{k-1}_1} \right\rfloor \) and \( d_{k-1} = \left\lfloor \frac{c(X^{k-1}_1)}{b^{k-1}_1} \right\rfloor \).

If \( c_{k-1} < d_{k-1} \), replace \( X^{k-1}_2 \) by \( X^{k-1}_1 \cup X^{k-1}_2 \), which contains \( b^{k-1}_1 + b^{k-1}_2 \) subsets in \( X^0 \). Let \( X_j = X_{j+1} \) for \( j \in [m_{k-1} - 1] \) and finally let \( m_{k-1} = m_{k-1} - 1 \). If \( c_{k-1} > d_{k-1} \), replace the last two sets \( X^{m_{k-1}-1}_{m_{k-1}-1} \) and \( X^{m_{k-1}}_{m_{k-1}-1} \) in \( X^{k-1} \) by \( X^{k-1}_{m_{k-1}-1} \cup X^{k-1}_{m_{k-1}} \), which contains \( b^{k-1}_{m_{k-1}-1} + b^{k-1}_{m_{k-1}} \) subsets in \( X^0 \), and let \( m_{k-1} = m_{k-1} - 1 \).

3. Let \( X^k_1 = X^{k-1}_1 \cup X^{k-1}_{m_{k-1}-1}, X^k_2 = X^{k-1}_2 \cup X^{k-1}_{m_{k-1}-1}, \ldots, X^k_{m_k} = X^{k-1}_{m_k} \cup X^{k-1}_{m_k+1-1} \).

Clearly, \( m_k = \left\lfloor \frac{m_{k-1}}{2} \right\rfloor \). Let \( u_k = \max_j \frac{c(X^k_j)}{b^k_j} - \frac{c(X^0_j)}{m_0} \) and \( l_k = \frac{c(X^0_j)}{m_0} - \min_j \frac{c(X^k_j)}{b^k_j} \). Since \( \sum_{j=1}^{m_k} b_j^k = m_0 \) and \( \sum_{j=1}^{m_k} c(X^k_j) = c(X^0) \) for any \( 0 \leq k \leq t \), by a simple averaging argument, it holds that \( u_k, l_k \geq 0 \). The following lemma proves several properties about the above procedure.

Lemma 6. The above procedure has the following properties.

1. For any \( 0 \leq k \leq t \) and \( j \in [m_k] \), \( 2^k \leq b^k_j < 2^{k+1} \).

2. Both \( u_k \) and \( l_k \) are monotone non-increasing in \( k \).

3. If in \( X^{k-1} \), \( c_{k-1} \leq d_{k-1} \), then \( u_k \leq \frac{2}{3} u_{k-1} \).

4. If in \( X^{k-1} \), \( c_{k-1} \geq d_{k-1} \), then \( l_k \leq \frac{2}{3} l_{k-1} \).

Proof. 1) Since \( b_j^0 = 1 \) for any \( j \in [m_0] \), property 1 holds trivially in the base case where \( k = 0 \).

Assume that property 1 holds for \( k - 1 \). For any \( j \in [m_k] \), by the construction of \( X^k_j \), there are either two indexes \( j_1 \) and \( j_2 \) such that \( X^k_j = X^{k-1}_{j_1} \cup X^{k-1}_{j_2} \), or three indexes \( j_1, j_2, j_3 \) such that \( X^k_j = X^{k-1}_{j_1} \cup X^{k-1}_{j_2} \cup X^{k-1}_{j_3} \). In either case, \( b^k_j \geq 2^{k-1} + 2^{k-1} = 2^k \).
Next, we prove that $b_j^k < 2^{k+1}$. We first prove by induction that $m_0 - 2^k m_k < 2^k$ for $0 \leq k \leq t$. The claim holds trivially for $k = 0$. Assume it holds for $k - 1$. If $m_{k-1}$ is even, $m_0 - 2^k m_k = m_0 - 2^{k-1} m_{k-1} < 2^{k-1} < 2^k$. If $m_{k-1}$ is odd, $m_0 - 2^k m_k = m_0 - 2^{k-1} m_{k-1} + 2^{k-1} < 2^k$. Thus the claim holds. Let $b_j^k = 2^k + a_j^k$ for any $j \in [m_k]$. Then $m_0 = \sum_{j \in [m_k]} b_j^k = 2^k m_k + \sum_{j \in [m_k]} a_j$. Thus $\sum_{j \in [m_k]} a_j = m_0 - 2^k m_k < 2^k$, which implies $b_j^k < 2^{k+1}$ immediately.

2) For any $j \in [m_k]$, by the construction of $X_j^k$, there are either two indexes $j_1$ and $j_2$ such that $X_j^k = X_{j_1}^{k-1} \cup X_{j_2}^{k-1}$, or three indexes $j_1, j_2, j_3$ such that $X_j^k = X_{j_1}^{k-1} \cup X_{j_2}^{k-1} \cup X_{j_3}^{k-1}$. In the former case, we have

$$c(X_j^k) = \frac{c(X_{j_1}^{k-1}) + c(X_{j_2}^{k-1})}{b_j^k} \leq \max_j \frac{c(X_{j_1}^{k-1})}{b_j^{k-1}}.$$  

The same argument holds for the latter case. Thus we have $\max_j \frac{c(X_j^k)}{b_j^k} \leq \max_j \frac{c(X_{j_1}^{k-1})}{b_j^{k-1}}$ and therefore $u_k \leq u_{k-1}$. The same holds for $l_k$ by a similar argument.

3) When $c_{k-1} \leq d_{k-1}$, a set $X_{j_1}^{k-1}$ with $\frac{c(X_{j_1}^{k-1})}{b_{j_1}^{k-1}} \geq \frac{c(X_0)}{m_0}$ will not merge with another set in step 2 and will be coupled with another (possibly updated) set $X_{j_2}^{k-1}$ with $\frac{c(X_{j_2}^{k-1})}{b_{j_2}^{k-1}} < \frac{c(X_0)}{m_0}$ in step 3 to obtain a new set $X_j^k$. And we have

$$\frac{c(X_j^k)}{b_j^k} = \frac{c(X_{j_1}^{k-1}) + c(X_{j_2}^{k-1})}{b_{j_1}^{k-1} + b_{j_2}^{k-1}} \leq \frac{b_{j_1}^{k-1}}{b_{j_1}^{k-1} + b_{j_2}^{k-1}} \left( \frac{c(X_{j_1}^{k-1})}{b_{j_1}^{k-1}} - \frac{c(X_0)}{m_0} \right) + \frac{b_{j_2}^{k-1}}{b_{j_1}^{k-1} + b_{j_2}^{k-1}} \left( \frac{c(X_{j_2}^{k-1})}{b_{j_2}^{k-1}} - \frac{c(X_0)}{m_0} \right) \leq \frac{2}{3} u_{k-1}.$$

The last inequality holds since $b_{j_1}^{k-1} < 2b_{j_2}^{k-1}$, due to $b_{j_1}^{k-1} < 2^k$ and $b_{j_2} \geq 2^{k-1}$.

4) This is true by a similar argument as in case 3).

Given a monotone submodular function $f$ defined on $X^0$, we have the following corollary.

**Corollary 2.** Assume that $t = \left\lceil \frac{2 \log 1/\epsilon}{\log 3/2} \right\rceil$, then in $X^t$ there is a set $X_j^t$ which contains at most $2^{t+1}$ subsets in $X^0$ and satisfies one of the following two conditions.

1. $\frac{c(X_j^t)}{b_j^t} \leq \frac{c(X_0)}{m_0} + \epsilon B$ and $\frac{f(X_j^t)}{c(X_j^t)} \geq \frac{f(X_0)}{c(X_0)}$.

2. $\frac{c(X_j^t)}{b_j^t} \geq \frac{c(X_0)}{m_0} - \epsilon B$ and $\frac{f(X_j^t)}{c(X_j^t)} \leq \frac{f(X_0)}{c(X_0)}$. 

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Proof. By Lemma 6 any $X^t_j$ contains $b_j^t < 2^{t+1}$ subsets in $X^0$. If the procedure runs $t$ rounds, then either property 3 or property 4 in Lemma 6 holds for at least $t/2$ rounds. If property 3 holds for $t/2$ rounds, since $u_k$ is non-increasing, $u_t \leq (\frac{2}{3})^{t/2}u_0 \leq (\frac{2}{3})^{t/2}B \leq \epsilon B$. Hence all $X^t_j$ satisfy $c(X^t_j) \leq \frac{c(x^0)}{m_0} + \epsilon B$, and the one with the largest $f(X^t_j)/b^t_j$ satisfies $f(X^t_j) \geq f(x^0) / m_0$ simultaneously. Thus condition 1 in this corollary holds. If property 4 in Lemma 6 holds for $t/2$ rounds, by a similar argument, all $X^t_j$ satisfy $c(X^t_j) / b^t_j \geq c(x^0) / m_0 - \epsilon B$, and the one with the largest $f(X^t_j) / c(X^t_j)$ satisfies $f(X^t_j) / c(X^t_j) \geq f(x^0) / c(x^0)$ simultaneously. Thus condition 2 in this corollary holds.

5 The Bounded-size Case

In this section, we consider the bounded-size case of SMKP where the ratio between the maximum capacity and the minimum capacity of the bins is bounded by a constant $\gamma \geq 1$. Equivalently, we assume all capacities $B_j \in [B, \gamma B]$, where $B$ is the size of the smallest bin. In Section 5.1 we present a deterministic algorithm which admits a $(1/2 - 2\epsilon)$-approximation. In Section 5.2 we present a randomized algorithm by applying the continuous greedy technique to enhance the ratio to $1 - 1/\epsilon - O(\epsilon)$. Both algorithms assume that the number of bins $m \geq \max\{1/2\epsilon, 2\gamma \ln(1/\epsilon) + 1\}$. When this is not the case, we can use Algorithm 2 to obtain a tight $(1 - 1/\epsilon)$-approximation. Thus Theorem 3 and Theorem 4 hold.

5.1 A 1/2 Approximation Algorithm

Algorithm 5 first divides bins into three categories as follows. It sorts bins in an increasing order of their sizes. Then the first $\lceil 2em \rceil$ bins are called reserved bins. The following $m'' = \lceil 2\gamma \epsilon \ln(1/\epsilon) \rceil$ bins are called patched bins. The remaining bins are called working bins. We assume there are $m'$ working bins. Algorithm 5 mainly uses working bins and patched bins to pack elements. It packs working bins one at a time by invoking Algorithm 4 with a single bin as input. After all working bins have been packed, it uses patched bins to pack small elements in $N \setminus T_{m'}$ that have not been packed so far by the greedy algorithm.

Algorithm 5 invokes Algorithm 4 as a subroutine, which requires the notions of large elements and small elements as in the identical-size case. We directly adopt these notions in the bounded-size case. That is, given input $\epsilon$, an element $u \in N$ is large if $c(u) > \epsilon B$ and small otherwise. Again, $N_l = \{u \in N \mid c(u) > \epsilon B\}$ denotes the set of large elements in $N$ and $N_s = N \setminus N_l$ denotes the set of small elements. For convenience of analysis, we slightly modify the greedy algorithm invoked in steps 4 and 8 such that each bin is allowed to pack one more element after the last element violating its capacity was added. As a result, each bin might contain two reserved elements. Since there are $\lceil 2em \rceil$ reserved bins, by a similar argument as in the identical-size case we can show that all the reserved elements can be packed into the reserved bins during the execution of the algorithm.

Below we show that the set $T_{m'+1}$ returned by Algorithm 5 admits a $(1/2 - 2\epsilon)$-approximation. Let $OPT^*$ be an optimal solution of the original instance and $OPT$ be an optimal solution when one can only pack elements into working bins. Since $m'' = \lceil 2\gamma \epsilon \ln(1/\epsilon) \rceil$ and $m \geq 2\gamma \ln(1/\epsilon) + 1$, we have $m'' \leq 3m$. Hence the reserved bins and patched bins consist of the first $3em$ bins with the smallest sizes by the construction, and the working bins consist of the remaining $(1 - 3\epsilon)m$ bins with the largest sizes. As a result, $f(OPT) \geq (1 - 3\epsilon)f(OPT^*)$.

From now on, we focus on how to bound the value $f(T_{m'+1})$ in terms of the value $f(OPT)$. Let $OPT_l$ and $OPT_s$ be the set of large elements and small elements in $OPT$, respectively. We first construct a partition $(OPT_1, \ldots, OPT_{m'}, OPT_{m'+1})$ of $OPT$. Note that this partition is not
Algorithm 5: The Bounded-size Case, 1/2-ratio

**Input:** constant $\epsilon \in (0,1)$, ground set $N$, objective function $f$, size function $c$, the number of bins $m$, capacities $(B_1, \ldots, B_m)$ with $B_1 = B$ and $B_j \in [B, \gamma B]$.

**Output:** A feasible set (and the way it is packed).

1. Divide bins into three categories: reserved bins, patched bins and working bins. They are constructed as follows. Assume that bins are ordered such that $B = B_1 \leq B_2 \leq \cdots \leq B_m$. Then the first $\lceil 2\epsilon m \rceil$ bins are reserved bins. The following $m'' = \lceil 2\epsilon \ln(1/\epsilon) \rceil$ bins are patched bins. The remaining bins are working bins. Assume that there are $m'$ working bins, number them from 1 to $m'$ (not necessarily in an increasing order). The patched bins are numbered from $m' + 1$ to $m' + m''$.

2. $T_0 = \emptyset$.

3. for $j = 1$ to $m'$ do

4. \hspace{1em} $S_j = \text{Constant-bins}(\epsilon, N \setminus T_{j-1}, f(\cdot | T_{j-1}), c(\cdot), 1, B_j)$; /* The greedy algorithm in Constant-bins has been modified such that each bin might contain two reserved elements. */

5. \hspace{1em} Pack the reserved elements in $S_j$ into reserved bins.

6. \hspace{1em} $T_j = T_{j-1} \cup S_j$.

7. end

8. $S_{m'+1} = \text{Greedy}(N \setminus T_{m'}, f(\cdot | T_{m'}), c(\cdot), m'', (B_{m'+1}, \ldots, B_{m'+m''}))$; /* The greedy algorithm here has been modified such that each bin might contain two reserved elements. */

9. Pack the reserved elements in $S_{m'+1}$ into reserved bins.

10. $T_{m'+1} = T_{m'} \cup S_{m'+1}$.

11. return $T_{m'+1}$ (and the way it is packed).
the partition where the optimal solution packs the elements in \( m' \) working bins. It is used in the analysis only. For each \( OPT_j \), let \( OPT_{i,j} = OPT_j \cap OPT_i \) and \( OPT_{s,j} = OPT_j \cap OPT_s \), respectively.

For each element in \( OPT_i \), we pack it exactly the same as \( OPT \) does. For the elements in \( OPT_s \), intuitively, we will shuffle them and pack them in the order of increasing marginal value. Assume that we have constructed \( OPT_{s,1}, \ldots, OPT_{s,j-1} \). For the \( j \)-th bin, we then pack the remaining small elements in the order of increasing marginal values with respect to \( OPT_{i,j} \cup T_{j-1} \) until the bin is full. Specifically, define \( O_j = OPT_s \setminus (OPT_1 \cup \cdots \cup OPT_{j-1}) \) which contains the remaining small elements we need to pack. We order elements in \( O_j \) such that \( o_1 = \arg\min_{o \in O_j} f(o | OPT_{i,j} \cup T_{j-1}) / c(o), \) etc. Let \( k \) be the first index such that \( c(\{o_1, \ldots, o_k\}) \geq B_j - c(OPT_{i,j}) \), then \( OPT_{s,j} = \{o_1, \ldots, o_k\} \). Note that \( T_{j-1} \) is the set of elements which are packed by the algorithm in the first \( j-1 \) working bins, thus the way we construct \( OPT_{s,j} \) helps us to compare \( f(S_j | T_{j-1}) \) and \( f(OPT_j | T_{j-1}) \) in the analysis. Due to \( OPT_j = OPT_{i,j} \cup OPT_{s,j} \) and the construction of \( OPT_{s,j} \), we have \( B_j = c(OPT_j) \leq B_j + \epsilon B \). Since each working bin is "full" in the partition, we can of course pack all small elements by repeating the above procedure. But we will stop when for some \( j \)-th bin, we find \( c(O_j) \leq 2\epsilon \gamma B \). Let \( J \) be the index of such bin. Note that it is possible we never stop before we pack all small elements, and in this case, set \( J = m' + 1 \). We put all the elements in \( O_j \) into \( OPT_{m'+1} \) and finish our construction.

Thus, by the above construction, we know for any \( 1 \leq j < J \), we have \( OPT_j = OPT_{i,j} \cup OPT_{s,j} \), \( B_j \leq c(OPT_j) \leq B_j + \epsilon B \) and \( c(O_j) > 2\epsilon \gamma B \). For any \( J \leq j \leq m' \), we have \( OPT_j = OPT_{i,j} \) containing only large elements. For \( OPT_{m'+1} \), it equal to \( O_J = OPT_s \setminus (OPT_1 \cup \cdots \cup OPT_{j-1}) \) where \( c(OPT_{m'+1}) = c(O_J) \leq 2\epsilon B \).

**Lemma 7.** For \( j \in [m'+1] \), \( f(S_j | T_{j-1}) \geq (1-\epsilon) f(OPT_j | T_{j-1}) \).

**Proof.** We first show that \( f(S_j | T_{j-1}) \geq (1-\epsilon) f(OPT_j | T_{j-1}) \) for \( j < J \). Recall that \( OPT_{i,j} \) is the large elements packed in the \( j \)-th working bins by \( OPT \). Thus when Algorithm 4 is invoked, \( OPT_{i,j} \cup T_{j-1} \) will be listed in the enumeration step. Let \( G_j \) be the set returned by the greedy algorithm which begins with \( OPT_{i,j} \cup T_{j-1} \). Assume w.l.o.g. that \( G_j \) contains two reserved elements that violate the capacity constraint (recall that we have modified the greedy algorithm), then \( c(G_j) \geq B_j - c(OPT_{i,j} \cup T_{j-1}) + \epsilon B \). On the other hand, \( c(OPT_{s,j}) \leq B_j - c(OPT_{i,j}) + \epsilon B \) by its construction and the fact that \( OPT_{s,j} \) contains only small elements. Thus \( c(G_j) \geq c(OPT_{s,j}) \).

By Lemma 6 we have

\[
  f(G_j | (OPT_{i,j} \setminus T_{j-1}) \cup T_{j-1}) = f(G_j | OPT_{i,j} \cup T_{j-1})
  \geq (1 - e^{-c(G_j)/c(O_j)}) \cdot f(O_j | OPT_{i,j} \cup T_{j-1})
  \geq (1 - e^{-c(G_j)/c(O_j)}) \cdot f(O_j | OPT_{i,j} \cup T_{j-1})
  \geq \left( \frac{c(G_j)}{c(O_j)} - \frac{c(G_j)^2}{2c(O_j)^2} \right) \cdot f(O_j | OPT_{i,j} \cup T_{j-1})
  \geq \left( \frac{1 - e^{-c(G_j)/c(O_j)}}{c(O_j)} \right) \cdot f(O_j | OPT_{i,j} \cup T_{j-1}).
\]

The second inequality holds since \( 1 - e^{-x} \) is non-decreasing. The third inequality holds since \( 1 - e^{-x} \geq x - x^2/2 \) for \( x \geq 0 \). The last inequality holds as long as \( c(G_j) \leq 2\epsilon \cdot c(O_j) \), which is guaranteed by the fact that \( c(O_j) \geq 2\epsilon \gamma B \) and \( c(G_j) \leq \gamma B \).

On the other hand, by the construct of \( OPT_{s,j} \), it holds that

\[
  \frac{f(O_j | OPT_{i,j} \cup T_{j-1})}{c(O_j)} \geq \frac{f(OPT_{s,j} | OPT_{i,j} \cup T_{j-1})}{c(OPT_{s,j})}.
\]
Combining the above inequalities, we have

\[
f(G_j \mid OPT_{i,j} \cup T_{j-1}) \geq \frac{(1 - \varepsilon)c(G_j)}{c(OPT_{s,j})}f(OPT_{s,j} \mid OPT_{i,j} \cup T_{j-1}) \\
\geq (1 - \varepsilon)f(OPT_{s,j} \mid OPT_{i,j} \cup T_{j-1}).
\]

By adding \(f(OPT_{i,j} \mid T_{j-1})\) on both sides of the above inequality, we obtain

\[
f(S_j \mid T_{j-1}) \geq f(OPT_{i,j} \cup G_j \mid T_{j-1}) \geq (1 - \varepsilon)f(OPT_j \mid T_{j-1}).
\]

Next we show that \(f(S_j \mid T_{j-1}) \geq f(OPT_j \mid T_{j-1})\) for \(J \leq j \leq m'\). Since \(OPT_j = OPT_{i,j}\) and \(OPT_{i,j} \setminus T_{j-1}\) will be listed in the enumeration step, we have

\[
f(S_j \mid T_{j-1}) \geq f(OPT_{i,j} \setminus T_{j-1} \mid T_{j-1}) = f(OPT_{i,j} \mid T_{j-1}) = f(OPT_j \mid T_{j-1}).
\]

Finally, we show that \(f(S_{m'+1} \mid T_{m'}) \geq (1 - \varepsilon)f(OPT_{m'+1} \mid T_{m'})\). Consider the set \(S_{m'+1}\) returned by the greedy algorithm with \(N_s \setminus T_{m'}\) and patched bins as input. Since there are \(m'' = \lceil 2\gamma B \ln(1/\varepsilon) \rceil\) patched bins and each bin has capacity at least \(B\), the total capacity of patched bins is at least \(2\gamma B \ln(1/\varepsilon)\). Assume w.l.o.g. that \(c(S_{m'+1}) \geq 2\gamma B \ln(1/\varepsilon)\), since otherwise \(S_{m'+1} = N_s \setminus T_{m'}\) and the lemma holds immediately. On the other hand, \(c(OPT_{m'+1}) \leq 2\varepsilon B\) by its construction. As a result, \(c(S_{m'+1})/c(OPT_{m'+1}) \geq \ln(1/\varepsilon)\). By Lemma 1 we have

\[
f(S_{m'+1} \mid T_{m'}) \geq (1 - e^{-c(S_{m'+1})/c(OPT_{m'+1})})f(OPT_{m'+1} \setminus T_{m'} \mid T_{m'}) \\
\geq (1 - e^{-c(S_{m'+1})/c(OPT_{m'+1})})f(OPT_{m'+1} \mid T_{m'}) \\
\geq (1 - e^{-\ln(1/\varepsilon)})f(OPT_{m'+1} \mid T_{m'}) \\
= (1 - \varepsilon)f(OPT_{m'+1} \mid T_{m'}).\]

The proof is completed. \(\square\)

The above lemma immediately leads to the following theorem.

**Theorem 9.** Algorithm 5 admits a \((1/2 - 2\varepsilon)\)-approximation and runs in \(O(mn^{\gamma/\varepsilon + 3})\) time. Since \(\gamma\) is a constant, it runs in polynomial time.

**Proof.** By Lemma 7 and the submodularity of \(f\), \(f(S_j \mid T_{j-1}) \geq (1 - \varepsilon)f(OPT_j \mid T_{m'+1})\) for \(j \in [m'+1]\). Summing up these \(m'+1\) inequalities, we have \(f(T_{m'+1}) \geq (1 - \varepsilon)f(OPT \mid T_{m'+1})\) by the submodularity of \(f\). Together with \(f(OPT) \geq (1 - 3\varepsilon)f(OPT^*)\), it holds that

\[
f(T_{m'+1}) \geq \frac{1 - \varepsilon}{2}f(OPT) \geq \frac{1 - 4\varepsilon}{2}f(OPT^*).
\]

For complexity, Algorithm 5 needs to pack \((1 - 3\varepsilon)m\) working bins. For each working bin, there are \(O(n^{\gamma/\varepsilon + 1})\) possible solutions in the enumeration step. Each enumerated solution is augmented by the \(O(n^2)\) greedy algorithm. Thus the total running time is bounded by \(O(mn^{\gamma/\varepsilon + 3})\). \(\square\)

### 5.2 Enhance the Ratio to \(1 - 1/e\)

In this section, we employ the continuous greedy technique to boost the approximation ratio to \(1 - 1/e - O(\varepsilon)\). The main algorithm is depicted as Algorithm 6. As in Algorithm 5, the algorithm divides bins into reserved bins, patched bins and working bins. Unlike Algorithm 5, it merges the patched bins into one bin. We call it the *merged bin*. The algorithm defines new instances (say
\[ I \text{ and } I' \) from the original one, which take as input only the working bins and the merged bin. Instead of dealing with the instance \( I' \) directly, it applies the reduction in Theorem 5 and then applies the continuous greedy process to the exponential size instance generated by the reduction.

The key point of applying the process is to determine a good enough base \( I \) and update \( x \) to \( x + \epsilon 1\tau \) in each iteration. By using the standard update rule, however, the approximate greedy process in [6] only achieves a 0.468 approximation ratio for SMKP. Instead, we use a modified continuous greedy process in [1] to interpolate Algorithm 5 and achieve a better update.

The update procedure is depicted as Algorithm 7. Basically, it behaves almost the same as Algorithm 5. The only difference is that, while Algorithm 5 finds a set \( I \) and \( I' \) and continuous greedy process in [6] only achieves a 0.

Algorithm 5. The only difference is that, while Algorithm 5 finds a set \( I \) where \( \text{the function } f \text{ is applied the continuous greedy process to the exponential size instance generated by the reduction.} \)

The rounding procedure. After the continuous greedy process terminates, we obtains an updated fractional solution \( x \). We need to round it to a base \( S = \{S_1, \cdots, S_{m'+1}\} \) of \( X \). Standard rounding techniques that preserve the approximation ratio for a matroid constraint include the pi-page rounding [6] and the swap rounding [8]. However, for the partition matroid, there is a simple rounding procedure in [6]. Let \( S_{j,1}, \cdots, S_{j,k} \in X_j \) denote the subsets (coordinates) in \( X_j \) such that \( x_{S_{j,i}} > 0 \) for \( i \in [k] \). Then \( \sum_{i=1}^{k} x_{S_{j,i}} \leq 1 \). Consequently, we can interpret \( x_{S_{j,i}} \) as probabilities and sample a subset from \( X_j \) according to it independently for each \( j \in [m'+1] \). The base returned by this procedure will preserve the approximation ratio.

We now give a formal analysis to Algorithm 6 and Algorithm 7.

**Lemma 8.** Let \( \text{OPT} \) be the optimal solution of instance \( I \). There is a partition \( (\text{OPT}_1, \cdots, \text{OPT}_m) \) of \( \text{OPT} \) such that \( h(S_j) \geq (1 - \epsilon)h(\text{OPT}_j) \). In other words, we have \( E[g(S_j \mid R(x + \epsilon 1\tau_{j-1}))] \geq E[g(\text{OPT}_j \mid R(x + \epsilon 1\tau_{j-1}))] \).

The rounding procedure. After the continuous greedy process terminates, we obtains an updated fractional solution \( x \). We need to round it to a base \( S = \{S_1, \cdots, S_{m'+1}\} \) of \( X \). Standard rounding techniques that preserve the approximation ratio for a matroid constraint include the pi-page rounding [6] and the swap rounding [8]. However, for the partition matroid, there is a simple rounding procedure in [6]. Let \( S_{j,1}, \cdots, S_{j,k} \in X_j \) denote the subsets (coordinates) in \( X_j \) such that \( x_{S_{j,i}} > 0 \) for \( i \in [k] \). Then \( \sum_{i=1}^{k} x_{S_{j,i}} \leq 1 \). Consequently, we can interpret \( x_{S_{j,i}} \) as probabilities and sample a subset from \( X_j \) according to it independently for each \( j \in [m'+1] \). The base returned by this procedure will preserve the approximation ratio.

**Lemma 9.** Let \( x \) be the input of Algorithm 7 and \( x' \) be the updated value of \( x \) by the end of the algorithm. Then \( G(x') - G(x) \geq \epsilon(1 - \epsilon)(g(\text{OPT}) - G(x')) \) where \( G(\cdot) \) is the multilinear extension of value function \( g(\cdot) \).

**Proof.** Let \( T_j = \{S_1, \cdots, S_j\} \) for \( j \in [m] \). Then

\[
G(x') - G(x) = G(x + 1\tau) - G(x) \\
= \sum_{j=1}^{m} (G(x + 1\tau_j) - G(x + 1\tau_{j-1})) \\
= \sum_{j=1}^{m} \epsilon \frac{\partial G}{\partial x_{S_j}} |_{x + 1\tau_{j-1}} \\
\geq \sum_{j=1}^{m} \epsilon E[g(S_j \mid R(x + 1\tau_{j-1}))] \\
\geq \sum_{j=1}^{m} \epsilon (1 - \epsilon) E[g(\text{OPT}_j \mid R(x + 1\tau_{j-1}))] \\
\geq \sum_{j=1}^{m} \epsilon (1 - \epsilon) E[g(\text{OPT} \mid R(x'))] \\
\geq \epsilon (1 - \epsilon) E[g(\text{OPT} \mid R(x'))]
\]
Algorithm 6: The Bounded-size Case, \((1 - 1/e)\)-ratio

**Input:** constant \(\epsilon \in (0, 1)\), ground set \(N\), objective function \(f\), size function \(c\), the number of bins \(m\), capacities \((B_1, \cdots, B_m)\) with \(B_1 = B\) and \(B_j \in [B, \gamma B]\).

**Output:** A feasible set (and the way it is packed).

1. Construct reserved bins, patched bins and working bins as in Algorithm 5.
2. Merge the patched bins together into one bin, called the **merged bin**.
3. Construct a new instance \(I\) from the original instance by taking as input only the working bins and the merged bin. Denote by \(m'\) the number of working bins and assume that \(I\) has capacities \((B'_1, \cdots, B'_{m'}, B'_{m'+1})\) as input.
4. Construct a new instance \(I'\) from \(I\) by replacing the capacities with \((B'_1 + 2\epsilon B, \cdots, B'_{m'} + 2\epsilon B, B'_{m'+1} + 2\epsilon B)\).
5. Apply the reduction in Theorem \(5\) to the instance \(I'\), in which \(X\) denotes the ground set and \(g\) denotes the objective function.
6. Let \(x \in [0, 1]^X\) and initialize \(x = 0\).
7. for \(t = \epsilon; t \leq 1; t = t + \epsilon\) do
6. \(T = \text{One-step}(\epsilon, N, g(\cdot), c(\cdot), m' + 1, (B'_1 + 2\epsilon B, \cdots, B'_{m'} + 2\epsilon B, B'_{m'+1} + 2\epsilon B), x)\).
6. \(x = x + \epsilon 1_T\).
10. end
11. \(S = (S_1, \cdots, S_{m'+1}) = \text{Rounding}(x)\).
12. Pack the reserved elements in \(S\) into the reserved bins.
13. return \(S\) (and the way it is packed).

Algorithm 7: One-step

**Input:** constant \(\epsilon \in (0, 1)\), ground set \(N\), objective function \(g\), size function \(c\), the number of bins \(m\), capacities \((B_1, \cdots, B_m)\), \(x \in [0, 1]^X\).

**Output:** A set \(T \subseteq X\) such that \(|T \cap X_j| \leq 1\) for \(j \in [m]\).

1. \(T_0 = \emptyset\).
2. for \(j = 1\) to \(m - 1\) do
3. Define a set function \(h : 2^N \to \mathbb{R}\) such that \(h(S) = \mathbb{E}[g(S \mid R(x))]\) for \(S \subseteq N\).
4. \(S_j = \text{Constant-bins}(\epsilon, N, h(\cdot), c(\cdot), 1, B_j)\).
5. \(T_j = T_{j-1} \cup \{S_j\}\).
6. \(x = x + \epsilon 1_{S_j}\).
7. end
8. \(S_m = \text{Greedy}(N, h(\cdot), c(\cdot), 1, B_m)\).
9. \(T_m = T_{m-1} \cup \{S_m\}\).
10. \(x = x + \epsilon 1_{S_m}\).
11. return \(T_m = \{S_1, \cdots, S_m\}\).
≥ \epsilon(1-\epsilon)(g(OPT) - G(x')).

The second inequality holds by the last lemma. The following inequalities hold due to the submodularity of f.

**Theorem 10.** Algorithm admit a $(1 - 1/e - O(\epsilon))$-approximation and runs in polynomial time.

**Proof.** Denote by $x(t)$ the value of $x$ at time $t$. By Lemma, we have

$$G(x(t+\epsilon)) - G(x(t)) \geq \epsilon(1 - \epsilon)(g(OPT) - G(x(t+\epsilon))).$$

Rearranging the above inequality, we obtain

$$g(OPT) - G(x(t)) \geq (1 + \epsilon(1 - \epsilon))(g(OPT) - G(x(t+\epsilon))).$$

By expanding the above recurrence, it holds that

$$g(OPT) - G(x(t)) \leq \frac{1}{(1 + \epsilon(1 - \epsilon))^{t/\epsilon}} g(OPT).$$

Plugging $t = 1$ into it yields

$$G(x(1)) \geq \left(1 - \frac{1}{(1 + \epsilon(1 - \epsilon))^{1/\epsilon}}\right) g(OPT) = (1 - e^{-1} - O(\epsilon)) g(OPT).$$

**6 Conclusion**

In this paper, we presented tight algorithms for SMKP for identical-size case and bounded-size case. The first algorithm and the core of the second algorithm are both deterministic algorithms. The major open problem is how to close the gap between 0.468 and $1 - 1/e$ for the general case of this problem. For MKP, a PTAS exists even for the most general case. Thus a better than $(1 - 1/e)$ inapproximability result will show an interesting distinction between the submodular version and the linear version of the multiple knapsack problem. It is also interesting to ask for a tight algorithm for the bounded-size case whose running time is polynomial in both $\gamma$ and $n$, or equivalently, a pseudo-polynomial time tight approximation algorithm for the general case of SMKP. Another direction is to design better deterministic algorithm for the bounded-size case and general case, or maybe, on the other hand, show the difference between the tight deterministic approximation ratio and the tight randomized approximation ratio for either case.

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