Analytical Studies of Quasi Steady-State Model in Power System Long-Term Stability Analysis

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Abstract—In this paper, a theoretical foundation for the Quasi Steady-State (QSS) model in power system long-term stability analysis is developed. Sufficient conditions under which the QSS model gives accurate approximations of the long-term stability model in terms of trajectory and $\omega$-limit set are derived. These sufficient conditions provide some physical insights regarding the reason for the failure of the QSS model. Additionally, several numerical examples are presented to illustrate the analytical results derived.

Index Terms—sufficient conditions, quasi steady-state model, power system long-term stability.

I. INTRODUCTION

The ever-increasing loading of transmission networks together with a steady increase in load demands has pushed many power systems ever closer to their stability limit [1]-[3]. Long-term stability has become more and more important for secure operation of power systems. However, the long-term stability model is large and involves different time scales. The time domain simulation approach for the long-term stability model is expensive in terms of computational efforts and data processing. These constraints are even more stringent in the context of on-line stability assessment. The quasi steady-state (QSS) proposed in [4]-[6] tried to reach a good compromise between accuracy and efficiency for long-term stability analysis. The assumptions behind the QSS model that the post-fault transient stability model is stable and the long-term stability model is singularity-free are not necessarily true. There have been some efforts attending to address these issues [7]-[9]. However, less attention has been paid to another critical issue that even these assumptions are satisfied, the QSS model may still provide incorrect approximations for the long-term stability model. Some counter examples in which the QSS model were stable while the long-term stability model underwent long-term instabilities were presented in [10]. Since the QSS model can not consistently provide correct stability analysis of the long-term stability model, there is a great need to identify conditions under which the QSS model works. In this paper, sufficient conditions under which the QSS model can provide correct approximations for the long-term stability model are developed. Briefly speaking, if neither the long-term stability model nor the QSS model meets a singularity, the QSS model provides correct approximations for the long-term stability model in terms of trajectory if the QSS model moves along the stable component of its constraint manifold and the projection of each point on the trajectory of the long-term stability model lies inside the stability region of the corresponding transient stability model. Moreover, if the QSS model converges to a long-term stable equilibrium point (SEP), then the long-term stability model will converge to the same point. Several numerical examples in which the QSS model succeeded or failed are analyzed by the derived analytical results.

This paper is organized as follows. Section II recalls basic concepts of power system models, and Section III introduces mathematical preliminaries in nonlinear system theories. Then sufficient conditions of the QSS model are derived in Section IV, and several numerical examples are analyzed based on the derived theorems in Section V. Conclusions and perspectives are stated in Section VI.

II. POWER SYSTEM MODELS

The long-term stability model, or interchangeably complete dynamic model, for calculating system dynamic response relative to a disturbance can be described as:

\[
\dot{z}_c = \epsilon h_c(z_c, z_d, x, y) \quad (1)
\]

\[
z_d(k+1) = h_d(z_c, z_d(k), x, y) \quad (2)
\]

\[
\dot{x} = f(z_c, z_d, x, y) \quad (3)
\]

\[0 = g(z_c, z_d, x, y) \quad (4)
\]

Equation (3) describes the electrical transmission system and the internal static behaviors of passive devices, and (4) describes the internal dynamics of devices such as generators, their associated control systems, certain loads, and other dynamically modeled components. $f$ and $g$ are continuous functions, and vector $x$ and $y$ are the corresponding short-term state variables and algebraic variables. Besides, Equations (1) and (2) describe long-term dynamics including exponential recovery load, turbine governor, load tap changer (LTC), over excitation limiter (OXL), etc. $z_c$ and $z_d$ are the continuous and discrete long-term state variables respectively, and $1/\epsilon$ is the maximum time constant among devices. Since transient dynamics have much smaller time constants compared with those of long-term dynamics, $z_c$ and $z_d$ are also termed as slow state variables, and $x$ are termed as fast state variables. Detailed power system models and corresponding variables are given in Appendix.
The transient stability model and the QSS model are regarded as two approximations of the long-term stability model in short-term and long-term time scales respectively, and they are believed to offer a good compromise between accuracy and efficiency. In transient stability model, slow variables are considered as constants. While in the QSS model, the dynamic behavior of fast variables is considered as instantaneous fast and thus replaced by its equilibrium equations in the long-term time scale. If we represent the long-term stability model and the QSS model in τ time scale, where τ = tε, and we denote t as \( \frac{dt}{\tau} \), then power system models can be represented as shown in Table I.

### TABLE I

| Type of Model | Mathematical Description |
|---------------|--------------------------|
| Long-term equilibrium point | \( z_t = h_e(z_e, z_d, x, y) \) |
| Long-term asymptotic stability | \( z_d(k + 1) = h_d(z_e, z_d(k), x, y) \) |
| Long-term stability | \( \dot{x} = f(z_e, z_d, x, y) \), \( 0 = g(z_e, z_d, x, y) \) |

The QSS model may fail to capture dynamics of the long-term stability model, thus provide incorrect approximations of the long-term stability model leading to incorrect stability assessment.

### III. MATHEMATICAL PRELIMINARIES

In this section, some relevant stability concepts from nonlinear system theories are briefly reviewed. Knowledge of stability region is required in analyzing the QSS model for long-term stability analysis.

#### A. Stability of Equilibrium Point and Stability Region

We consider the following autonomous nonlinear dynamical system:

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n
\]  

(5)

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies a sufficient condition for the existence and uniqueness of a solution. The solution of (5) starting at initial state \( x \) at time \( t = 0 \) is called the system trajectory and is denoted as \( \phi(t, x) \). \( \bar{x} \in \mathbb{R}^n \) is said to be an equilibrium point of (5) if \( f(\bar{x}) = 0 \). The definition of asymptotic stability is given as below [11]:

**Definition 1: Asymptotic Stability**

An equilibrium point \( \bar{x} \in \mathbb{R}^n \) of (5) is said to be asymptotically stable if, for each open neighborhood \( U \) of \( \bar{x} \in \mathbb{R}^n \), the followings are true: (i) \( \phi(t, x) \in U \) for all \( t > 0 \); (ii) \( \lim_{t \to \infty} ||\phi(t, x) - \bar{x}|| = 0 \).

Without confusion, we use stable equilibrium point (SEP) instead of asymptotically stable equilibrium point in this paper. An equilibrium point is hyperbolic if the corresponding Jacobian matrix has no eigenvalues with zero real parts. And a hyperbolic equilibrium point \( \bar{x} \) is a type-k equilibrium point if there exist \( k \) eigenvalues of \( D_x f(\bar{x}) \) with positive real parts.

The stability region of a SEP \( x_s \) is the set of all points \( x \) such that \( \lim_{t \to \infty} \phi(t, x) \rightarrow x_s \). In other words, the stability region is defined as:

\[
A(x_s) := \{ x \in \mathbb{R}^n : \lim_{t \to \infty} \phi(t, x) = x_s \}
\]

From a topological point of view, the stability region is an open invariant and connected set. Every trajectory in a stability region lies entirely in the stability region and the dimension of the stability region is \( n \).

**Definition 2: \( \omega \)-limit Set**

A point \( p \) is said to be the \( \omega \)-limit point of \( x \) if, corresponding to each \( \epsilon > 0 \) and \( T > 0 \), there is a \( t \geq T \) with the property that \( ||\phi(t, x) - p|| < \epsilon \). Equivalently, there is a sequence \( t_i \in \mathbb{R}, t_i \to +\infty \), with the property that \( p = \lim_{i \to +\infty} \phi(t, x) \). The set of all \( \omega \)-limit points for \( x \) is defined as its \( \omega \)-limit set.

#### B. Singular Perturbed System

We next consider the following general singular perturbed model:

\[
\Sigma_\varepsilon : \dot{z} = f(z, x) \quad z \in \mathbb{R}^n
\]

\[
\dot{\varepsilon} = g(z, x) \quad x \in \mathbb{R}^m
\]

(6)

where \( \varepsilon \) is a small positive parameter, \( z \) is a vector of slow variables while \( x \) is a vector of fast variables. Let \( \phi_\varepsilon(t, 0, x_0) \) denotes the trajectory of model (6) starting at \( (z_0, x_0) \) and \( E \) denotes the set of equilibrium points of it, i.e., \( E = \{(z, x) \in \mathbb{R}^n \times \mathbb{R}^m : f(z, x) = 0, g(z, x) = 0 \} \). If \( (z_s, x_s) \) is a SEP of model (6), then the stability region of \( (z_s, x_s) \) is defined as:

\[
A_s(z_s, x_s) := \{(z, x) \in \mathbb{R}^n \times \mathbb{R}^m : \phi_\varepsilon(t, 0, x_0) \rightarrow (z_s, x_s) \text{ as } t \to \infty \}
\]

The slow model is obtained by setting \( \varepsilon = 0 \) in (6):

\[
\Sigma_0 : \dot{z} = f(z, x) \quad z \in \mathbb{R}^n
\]

\[
0 = g(z, x) \quad x \in \mathbb{R}^m
\]

(7)

The algebraic equation \( 0 = g(z, x) \) constrains the slow dynamics to the following set which is termed as constraint manifold:

\[
\Gamma := \{(z, x) \in \mathbb{R}^n \times \mathbb{R}^m : g(z, x) = 0 \}
\]

(8)

The trajectory of model (7) starting at \( z_0 \) is denoted by \( \phi_0(t, z_0, x_0) \) and the stability region is

\[
A_0(z_s, x_s) := \{(z, x) \in \Gamma : \phi_0(t, z_0, x_0) \rightarrow (z_s, x_s) \text{ as } t \to \infty \}
\]

The singular points of system (7) or singularity \( S \) is defined as:

\[
S := \{(z, x) \in \Gamma : \text{det}(D_{zg}(z, x)) = 0 \}
\]

(9)

Singular points can drastically influence the trajectories of the differential-algebraic equation (DAE) system. Typically, the singular set \( S \) is a stratified set of maximal dimension \( n-1 \) embedded in \( \Gamma \) and \( \Gamma \) is separated by \( S \) into open regions [11] [12].

**Definition 3: Type of Constraint Manifold**

The connected set \( \Gamma_i \subset \Gamma \) is a type-k component of \( \Gamma \) if the matrix \( D_{zg} \), evaluated at every point of \( \Gamma_i \), has \( k \) eigenvalues
that have positive real parts. If all the eigenvalues of $D_x g$ calculated at points of $\Gamma$, have a negative real part, then we call $\Gamma$, a stable component of $\Gamma$; otherwise, it’s an unstable component of $\Gamma$.

We next define the fast model associated with the singularly perturbed model, i.e. boundary layer model. Define the fast time scale $\sigma = t/\epsilon$. In this time scale, model (6) takes the form:

$$
\Pi_f : \frac{dx}{d\sigma} = g(z, x) \quad x \in \mathbb{R}^n \quad (10)
$$

Let $\phi_f(\sigma, z_0, x_0)$ denote the trajectory of model (10) starting at $(z_0, x_0)$.

$$
\Pi_f : \frac{dx}{d\sigma} = g(z, x) \quad (11)
$$

where $z$ is frozen and treated as a parameter. The constraint manifold $\Gamma$ is a set of equilibriums of models (11). For each fixed $z$, a fast dynamical model (11) is defined.

**Definition 4: Uniformly Asymptotically Stable**

Assuming $(z, x) \notin S$, and $x = j(z)$ is an isolated root of equation:

$$
0 = g(z, x) \quad (12)
$$

then $x = j(z)$ is an equilibrium point of system (11), if $x = j(z)$ is a SEP of system (11) for all $z \in Z$, then $j(z)$ is uniformly asymptotically stable with respect to $z \in Z$.

The next Theorem ensures that, that solutions of the singular perturbed model (6) can be, at least for sufficiently small $\epsilon$, approximated by solutions of the slow model (7).

**Theorem 1 (Tikhonov’s Result on Finite Interval)**

Consider the singular perturbation problem (6) and let $x = j(z)$ be an isolated root of system (12). Assume that there exist positive constants $t_0 > 0$, $r$ and $\epsilon_0$, and a compact domain $Z \subset \mathbb{R}^n$ such that the following conditions are satisfied for all $t_0 \leq t \leq t_1$, $z \in Z$, $||x - j(z)|| \leq r$, $0 < \epsilon \leq \epsilon_0$

(a). The functions $f(z, x)$, $g(z, x)$ and $j(z)$ are continuous;

(b). The slow model (7) has a unique solution $z(t)$ with initial condition $z(t_0) = z_0$, defined on $[t_0, t_1]$ and $z_0 \in Z$ for all $t \in [t_0, t_1]$;

(c). The fast model (11) has the uniqueness of the solutions with prescribed initial conditions. Let $\hat{x}(\sigma)$ be the solution of system:

$$
\frac{dx}{d\sigma} = g(z_0, x), \quad x(\sigma_0) = x_0 \quad (13)
$$

(d). The equilibrium point $x = j(z)$ of fast model is uniformly asymptotically stable in $z \in Z$;

(e). The initial condition $x_0$ belongs to the stability region $A(j(z_0))$ of system (13).

Then for every $\delta > 0$, there exists a positive constant $\epsilon^*$ such that for all $0 < \epsilon < \epsilon^*$, every solution $(z(t), x(t))$ of the singular perturbation model (6) exists for all $t \geq t_0$, and satisfies

$$
||z(t) - z_0(t)|| \leq \delta
$$

$$
||x(t) - \hat{x}(t - t_0) - j(z_0(t)) + j(z_0)|| \leq \delta
$$

for all $t \geq t_0$.

Assume the solution of singular perturbation problem (6) $(z(t), x(t), \epsilon(t))$ is unique, then we have (14) (15):

$$
\lim_{\epsilon \to 0} \lim_{t \to +\infty} z(t, \epsilon) = \lim_{t \to +\infty} z_0(t) = z_s \quad (14)
$$

$$
\lim_{\epsilon \to 0} \lim_{t \to +\infty} x(t, \epsilon) = \lim_{t \to +\infty} j(z_0(t)) = x_s \quad (15)
$$

**IV. Analytical Studies of QSS Model**

The long-term stability model of power system can be represented as:

$$
\frac{dz_c}{dt} = h_c(z_c, z_d, x, y), \quad z_c(\tau_0) = z_{c0} \quad (15)
$$

$$
\frac{dz_d}{dt} = h_d(z_c, z_d(k - 1), x, y), \quad z_d(\tau_0) = z_d(0)
$$

$$
0 = g(z_c, z_d, x, y)
$$

where $\tau = t \epsilon$. Note that shunt compensation switching and LTC operation are typical discrete events captured by $z_d(k) = h_d(z_c, z_d(k - 1), x, y)$ and $z_d$ is shunt susceptance and the transformer ratio, respectively. Transitions of $z_d$ depend on system variables, thus $z_d$ change values from $z_d(k - 1)$ to $z_d(k)$ at distinct times $\tau_k$ where $k = 1, 2, 3, \ldots N$, otherwise, these variables remain constants.
Consider the long-term stability model (15), it can be regarded as two decoupled systems (16) and (17) shown as below when $z_d$ jump from $z_d(k-1)$ to $z_d(k)$:

$$z_d(k) = h_d(z_c, z_d(k-1), x, y), \quad z_d(\tau_0) = z_d(k-1) \quad (16)$$

and

$$z_c' = h_c(z_c, z_d(k), x, y), \quad z_c(\tau_0) = z_c(k) \quad (17)$$

$$ex' = f(z_c, z_d(k), x, y), \quad x(\tau_0) = x_k$$

$$0 = g(z_c, z_d(k), x, y)$$

discrete variables $z_d$ are updated first and then system (17) works with fixed parameters $z_d$.

Similarly, when $z_d$ jump from $z_d(k-1)$ to $z_d(k)$, the QSS model

$$z_c' = h_c(z_c, z_d(k), x, y), \quad z_c(\tau_0) = z_c(0) \quad (18)$$

$$z_d(k) = h_d(z_c, z_d(k-1), x, y), \quad z_d(\tau_0) = z_d(0)$$

$$0 = f(z_c, z_d, x, y)$$

$$0 = g(z_c, z_d, x, y)$$

can be decoupled as:

$$z_d(k) = h_d(z_c, z_d(k-1), x, y), \quad z_d(\tau_0) = z_d(k-1) \quad (19)$$

and

$$z_c' = h_c(z_c, z_d(k), x, y), \quad z_c(\tau_0) = z_c(k) \quad (20)$$

$$0 = f(z_c, z_d, x, y)$$

$$0 = g(z_c, z_d, x, y)$$

A. Models in Nonlinear Framework

For the study region $U = D_z \times D_{zd} \times D_z \times D_y$, where $D_z \subseteq \mathbb{R}^p, D_{zd} \subseteq \mathbb{R}^q, D_z \subseteq \mathbb{R}^m, D_y \subseteq \mathbb{R}^n$, both the long-term stability model and the QSS model have the same set of equilibrium points $E = \{ (z_c, z_d, x, y) : f(z_c, z_d, x, y) = 0, (z_c, z_d, x, y) = 0 \}$. Assuming $(z_{cls}, z_{dls}, x_{ls}, y_{ls}) \in E$ is a long-term SEP of both the long-term stability model (13) and the QSS model (18) starting from $(z_{c0}, z_{d0}(0), x_0, y_0)$ and $(z_{c0}, z_{d0}(0), x_0, y_0)$ respectively, and $\phi_{l}(\tau, z_{c0}, z_{d0}(0), x_0, y_0)$ denotes trajectory of the long-term stability model (15) and $\phi_{y}(\tau, z_{c0}, z_{d0}(0), x_0, y_0)$ denotes trajectory of the QSS model (18). Then, the stability region of the long-term stability model (15) is:

$$A_{l}(z_{cls}, z_{dls}, x_{ls}, y_{ls}) := \{ (z_c, z_d, x, y) \in U : \phi_{l}(\tau, z_{c0}, z_{d0}(0), x_0, y_0) \rightarrow (z_{cls}, z_{dls}, x_{ls}, y_{ls}) \text{ as } \tau \rightarrow +\infty \} \quad (21)$$

The stability region of the QSS model (18) is

$$A_{y}(z_{cls}, z_{dls}, x_{ls}, y_{ls}) := \{ (z_c, z_d, x, y) \in \Gamma : \phi_{y}(\tau, z_{c0}, z_{d0}(0), x_0, y_0) \rightarrow (z_{cls}, z_{dls}, x_{ls}, y_{ls}) \text{ as } \tau \rightarrow +\infty \} \quad (22)$$

The singular points of constraint manifold $\Gamma$ are:

$$S := \{ (z_c, z_d, x, y) \in \Gamma : \det \begin{bmatrix} D_x f & D_y f \\ D_x g & D_y g \end{bmatrix} = 0 \} \quad (23)$$

And type-$k$ component of $\Gamma$ where $0 \leq k \leq m + n$ is defined as:

$$\Gamma_k = \{ (z_c, z_d, x, y) \in \Gamma : \text{there are } k \text{ eigenvalues of} \begin{bmatrix} D_x f & D_y f \\ D_x g & D_y g \end{bmatrix} \text{ satisfy } \Re(\lambda) > 0 \} \quad (24)$$

When $z_c \in D_z$ and $z_d \in D_{zd}$ for each fixed $z_c$ and $z_d(k)$, given a point $(z_c, z_d(k), x, y)$ on $\Gamma$, the corresponding transient stability model is defined as:

$$\dot{x} = f(z_c, z_d(k), x, y) \quad (25)$$

$$0 = g(z_c, z_d(k), x, y)$$

If $(z_c, z_d(k), x, y) \not\in S$, then $(z_c, z_d(k), x_{ts}, y_{ts})$ is an equilibrium point of (25), where

$$\begin{bmatrix} x_{ts} \\ y_{ts} \end{bmatrix} = \begin{bmatrix} l_1(z_c, z_d(k)) \\ l_2(z_c, z_d(k)) \end{bmatrix} = l(z_c, z_d(k))$$

If $(z_c, z_d(k), x_{ts}, y_{ts})$ is a SEP of (25), then the stability region of $(z_c, z_d(k), x_{ts}, y_{ts})$ is represented as:

$$A_{t}(z_c, z_d(k), x_{ts}, y_{ts}) := \{ (x, y) \in D_x \times D_y, z_c = z_c, z_d = z_d(k) : \phi_{t}(t, z_c, z_d(k), x, y) \rightarrow (z_c, z_d(k), x_{ts}, y_{ts}) \text{ as } t \rightarrow +\infty \}$$

where $\phi_{t}(t, z_c, z_d(k), x, y)$ denotes the trajectory of the transient stability model (25).

Assuming that $D_y g$ is nonsingular, then transient stability model (25) can be linearized near the equilibrium point as:

$$\dot{x} = (D_x f - D_y f D_y g^{-1} D_x g)x \quad (26)$$

and we can define a subset of the stable component of constraint manifold $\Gamma_s \subset \Gamma_0$:

$$\Gamma_s = \{ (z_c, z_d, x, y) \in \Gamma : \text{all eigenvalues } \lambda \text{ of} \quad (D_x f - D_y f D_y g^{-1} D_x g) \text{ satisfy } \Re(\lambda) < 0, \quad (27)$$

and $D_y g$ is nonsingular$\}$. Such that each point on $\Gamma_s$ is a SEP of the corresponding transient stability model (25) for fixed $z_c$ and $z_d(k)$. A comprehensive theory of stability regions can be found in (12) (16) (17).

We divide the task of establishing a theoretical foundation for the QSS model into two steps as Case I and Case II. Firstly, we analyze the trajectory and $\omega$-limit set relations of the long-term stability model (17) and the QSS model (20), that is we regard discrete variables $z_d$ as fixed parameters. Next, we move one step further to include discrete dynamics $z_d$ and deduce the relations of the long-term stability model (15) and the QSS model (18) in terms of trajectory and $\omega$-limit set.

Before proceeding, we need some important assumptions:

S1. Neither the long-term stability model nor the QSS model meet singularity points.

S2. The trajectories of the long-term stability model, the QSS model and transient stability models with specified initial conditions exist and are unique. Additionally, $D_z$ is compact.
S3. Equilibrium point of transient stability model is continuous in \(z_c\) when \(z_d\) are fixed as parameters.

Note that the uniqueness of solutions is generally satisfied in power system models. Besides, since a power system is a real physical system, the domain of each variable is generally compact. As for S3, if S1 is satisfied, we know that equilibrium point of transient stability model \(l(z_c, z_d(k))\) is at least locally continuous by Implicit Function Theorem. Moreover, as \(z_c\) only varies slowly and subtly, S3 is also generally satisfied. As a result, if S1 is satisfied, we can safely assume that S2 and S3 are satisfied in power system models.

B. Case I: Relations of Trajectory and \(\omega\)-limit Set

Assuming the initial point of the long-term stability model is \((z_{ck}, z_d(k), x_k^q, y_k^q)\), and the initial point of QSS model is \((z_{ck}, z_d(k), x_k^q, y_k^q)\). Then the initial transient stability model can be represented as:

\[
\begin{align*}
\dot{x} &= f(z_{ck}, z_d(k), x, y), \quad x(0) = x_k^q (28) \\
0 &= g(z_{ck}, z_d(k), x, y)
\end{align*}
\]

with the equilibrium point \(\left(\begin{array}{c} x_k^q \\ y_k^q \end{array}\right) = \left(\begin{array}{c} l_1(z_{ck}, z_d(k)) \\ l_2(z_{ck}, z_d(k)) \end{array}\right) = l(z_{ck}, z_d(k))\). Equivalently, system (28) can be represented as

\[
\begin{align*}
\dot{x} &= f(z_{ck}, z_d(k), x, l_2(z_{ck}, z_d(k))) \quad x(0) = x_k^q (29)
\end{align*}
\]

Additionally, the solution of QSS model (20) as \(z_{ck}(\tau) \in D_{z_c}\) and the solution of the initial transient stability model as \(\dot{x}_k(t)\). Besides, denote \(D_{z_c}^k \subset D_z\) to be a set such that for all \(x \in D_{z_c}^k\), \(||x - l_1(z_{ck}, z_d(k))|| \leq r\), and let \(U_r = D_{z_c} \times D_{z_d} \times D_{z_c}^k \times D_y\).

**Theorem 3: (Trajectory Relation):**

Assuming there exist positive constants \(\tau_1 > \tau_0, r \) and \(\epsilon_0\) such that S1-S3 and the following conditions are satisfied for all \(\tau, z_c, z_d, x, y, \epsilon \in [\tau_0, \tau_1] \times U_r \times [0, \epsilon_0]:(\text{a})\) The trajectory \(\phi_{q}(\tau, z_{ck}, z_d(k), x_k^q, y_k^q)\) of the QSS model (20) moves along \(\Gamma_s\);

(b) The projection of initial point \((z_{ck}, z_d(k), x_k^q, y_k^q)\) of the long-term stability model (17) to the subspace of \(z_{ck}\) and \(z_d(k)\) is inside the stability region of the initial transient stability model (29).

Then for every \(\tau > 0\) there exists a positive constant \(\epsilon^*\) such that for all \(0 < \epsilon < \epsilon^*\), the solution \((z_{ck}(\tau), x_k(\tau))\) of the long-term stability model (17) exists at least on \([\tau_0, \tau_1]\), and satisfies:

\[
\begin{align*}
\|z_{ck}(\tau) - z_{ck}(\tau)\| &\leq \delta (30) \\
\|x_k(\tau) - l_1(z_{ck}(\tau), z_d(k))\| &\leq \delta \\
-\hat{x}_k\left(\frac{\tau - \tau_0}{\epsilon}\right) + l_1(z_{ck}, z_d(k)) &\leq \delta
\end{align*}
\]

for all \(\tau \in [\tau_0, \tau_1]\).

Theorem 3 asserts that if the projection of initial point of the long-term stability model lies inside the stability region of the initial transient stability model, and \(\phi_{q}(\tau, z_{ck}, z_d(k), x_k^q, y_k^q)\) moves along \(\Gamma_s\), then for sufficiently small \(\epsilon\), trajectory of the long-term stability model (17) can be approximated by trajectory of the QSS model (20).

**Proof:** If S1 is satisfied, then \(D_{q}g\) and \([D_{x}f, D_{y}f, D_{z}g, D_{y}g]\) are nonsingular, according to the Implicit Function Theorem, \(x, y\) can be solved from:

\[
\begin{align*}
0 &= f(z_c, z_d(k), x, y) \\
0 &= g(z_c, z_d(k), x, y)
\end{align*}
\]

with the solution \(\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} l_1(z_c, z_d(k)) \\ l_2(z_c, z_d(k)) \end{array}\right) = l(z_c, z_d(k))\).

Thus the long-term stability model (17) becomes:

\[
\begin{align*}
z_c' &= h_c(z_c, z_d(k), x, l_2(z_c, z_d(k))) \\
&= H_c(z_c, z_d(k), x), \quad z_c(\tau_0) = z_{ck} \\
\epsilon x' &= f(z_c, z_d(k), x, l_2(z_c, z_d(k))) \\
&= F(z_c, z_d(k), x), \quad x(\tau_0) = x_k^q
\end{align*}
\]

Hence, the long-term stability model is transformed into the standard singular perturbation problem considered in Theorem 1, and the QSS model is the corresponding slow model. Next, from the detailed power system models in Appendix, the following fact follows.

**Fact 1:** These maps \(h_c, f\) and \(g\) that describe slow dynamics, fast dynamics and algebraic constraints respectively are continuous.

With Fact 1 and S3, condition (a) of Theorem 1 is satisfied; with S2, condition (b) and (c) of Theorem 1 are satisfied. Furthermore, if condition (a) of Theorem 3 is satisfied, then \(\Gamma_s\) is a subset of stable component of the constraint manifold, thus each point on \(\Gamma_s\) is a SEP of the corresponding transient stability model. In other words, if \(\phi_{q}(\tau, z_{ck}, z_d(k), x_k^q, y_k^q)\) moves along \(\Gamma_s\), then \(x = l_1(z_{ck}, z_d(k))\) is asymptotically stable uniformly in \(z_{ck}\), hence condition (d) of Theorem 1 is satisfied. Note that if \(x = l_1(z_{ck}, z_d(k))\) is asymptotically stable, \(x = l_1(z_{ck}, z_d(k))\) is necessarily to be isolated by definition. Finally, condition (b) of Theorem 3 ensures the satisfaction of condition (e) in Theorem 1. According to Theorem 1, it follows that for every \(\delta > 0\) there exists a positive constant \(\epsilon^*\) such that for all \(0 < \epsilon < \epsilon^*\), the solution \((z_{ck}(\tau), x_k(\tau))\) of system (32) exists at least on \([\tau_0, \tau_1]\), and satisfies:

\[
\begin{align*}
\|z_{ck}(\tau) - z_{ck}(\tau)\| &\leq \delta, \quad \tau_0 \leq \tau \leq \tau_1 \\
\|x_k(\tau) - l_1(z_{ck}(\tau), z_d(k))\| &\leq \delta \\
\|x_k(\tau) - \hat{x}_k\left(\frac{\tau - \tau_0}{\epsilon}\right) + l_1(z_{ck}, z_d(k))\| &\leq \delta, \quad \tau_0 \leq \tau \leq \tau_1
\end{align*}
\]

This completes the proof of the theorem.

Next we proceed to identify the \(\omega\)-limit set relation between the long-term stability model (17) and the QSS model (20).

**Theorem 4: (\(\omega\)-Limit Set Relation):**

Assuming there exist positive constants \(r\) and \(\epsilon_0\) such that S1-S3 and the following conditions are satisfied for all \([\tau, z_c, z_d, x, y, \epsilon] \in [\tau_0, +\infty) \times U_r \times [0, \epsilon_0]:(\text{a})\) The trajectory \(\phi_{q}(\tau, z_{ck}, z_d(k), x_k^q, y_k^q)\) of the QSS model (20) moves along \(\Gamma_s\);

(b) The projection of initial point \((z_{ck}, z_d(k), x_k^q, y_k^q)\) of the long-term stability model (17) to the subspace of \(z_{ck}\) and \(z_d(k)\)
is inside the stability region of the initial transient stability model \([29]\).

(c). The \(\omega\)-limit set of the QSS model \([20]\) starting from \((z_{ck}, z_d(k), x^q_k, y^q_k)\) is a SEP \((z_{cks}, z_d(k), x_{cks}, y_{cks})\).

Then the solution \((z_{ck}(\tau, \epsilon), x_k(\tau, \epsilon))\) of the long-term stability model \([17]\) exists for all \(\tau \geq \tau_0\), and satisfies the following limit relations:

\[
\lim_{\epsilon \to 0^+} \tau \to +\infty z_{ck}(\tau, \epsilon) = z_{cks} \quad (34)
\]

\[
\lim_{\epsilon \to 0^+} \tau \to +\infty x_k(\tau, \epsilon) = x_{ks} \quad (35)
\]

Theorem 4 asserts that if all conditions of Theorem 3 are satisfied and the QSS model \([20]\) converges to a long-term SEP of the QSS model, then for sufficiently small \(\epsilon\), the long-term stability model \([17]\) will converge to the same point.

**Proof:** If S1 is satisfied, then \(D_y g\) and \(\begin{bmatrix} D_x f & D_y f \\ D_x g & D_y g \end{bmatrix}\) are nonsingular. Likewise, we can transform the long-term stability model \([17]\) to system \([32]\) which is the standard singular perturbation problem considered in Theorem 2.

From the proof of Theorem 3, we have that with S2, S3 and Fact 1, condition (a) and (c) of Theorem 2 are satisfied. Besides, condition (a) and (b) of Theorem 4 ensures the satisfaction of condition (d) and (e) in Theorem 2 respectively. Finally, with S2 and condition (c) of Theorem 4, it follows that the solution of the QSS model \([20]\) exists for all \(\tau \geq \tau_0\) and the \(\omega\)-limit set of the QSS model is a SEP, thus condition (b) of Theorem 2 is satisfied. Therefore all conditions of Theorem 2 are satisfied, it follows that for every \(\delta > 0\), there exists a positive constant \(\epsilon^*\) such that for all \(0 < \epsilon < \epsilon^*\), the solution \((z_{ck}(\tau), x_k(\tau))\) of the long-term stability model \([17]\) exists for all \(\tau \geq \tau_0\), and satisfies

\[
\|z_{ck}(\tau) - \bar{z}_{ck}(\tau)\| \leq \delta \quad (36)
\]

\[
\|x_k(\tau) - l_1(z_{ck}(\tau), z_d(k)) - \hat{x}_k(\frac{\tau - \tau_0}{\epsilon}) + l_1(z_{cks}, z_d(k))\| \leq \delta
\]

for all \(\tau \geq \tau_0\). Since the solution of the long-term stability model \([17]\) \((z_{ck}(\tau, \epsilon), x_k(\tau, \epsilon))\) is unique, we have

\[
\lim_{\epsilon \to 0^+} \tau \to +\infty z_{ck}(\tau, \epsilon) = z_{cks} \quad (37)
\]

\[
\lim_{\epsilon \to 0^+} \tau \to +\infty x_k(\tau, \epsilon) = x_{ks} \quad (38)
\]

This completes the proof of the theorem. And Fig. 1 gives an illustration of Theorem 3 and Theorem 4.

**C. Case II: Relations of Trajectory and \(\omega\)-Limit Set**

Next, we are at the stage to incorporate discrete behaviors of \(z_d\) in the long-term stability model and the QSS model, and explore trajectory and \(\omega\)-limit set relations between them.

Assuming \(z_d = z_d(0)\) initially at \(\tau_0\), and jump from \(z_d(k-1)\) to \(z_d(k)\) at time \(\tau_k\), where \(k = 1, 2, 3, \ldots N\). Similarly, the initial point of the long-term stability model is \((z_{c0}, z_d(0), x^q_0, y^q_0)\), and the initial point of QSS model is \((z_{c0}, z_d(0), x^q_0, y^q_0)\). Then the initial transient stability model can be represented as:

\[
\dot{x} = f(z_{c0}, z_d(0), x, l_z(z_{c0}, z_d(0))) \quad x(t_0) = x^i_0 \quad (39)
\]

\[
\text{with equilibrium point } x^i_0 = l_1(z_{c0}, z_d(0)).
\]

Denote the solution of QSS model \([20]\) as \(z_{ck}(\tau) \in D_{z_c}\), and denote the solution of the initial transient stability model and transient stability models immediately after \(z_d\) jump to \(z_d(k)\) as \(\hat{x}_k(\frac{\tau - \tau_0}{\epsilon})\) for all \(k = 0, 1, 2, \ldots N\).

**Definition 5: Consistent Attraction**

We say that the long-term stability model satisfies the condition of consistent attraction, if whenever long-term discrete variables jump from \(z_d(k-1)\) to \(z_d(k)\), \(k = 1, 2, 3, \ldots N\), the point on the trajectory of the long-term stability model immediately after \(z_d\) jump stays inside the stability region of the corresponding transient stability model.

The following two theorems provide a theoretical foundation for the QSS model in which trajectory and \(\omega\)-limit set relations of the long-term stability model \([15]\) and the QSS model \([18]\) are established.

**Theorem 5: (Trajectory Relation)**

Assuming there exist positive constants \(\tau_1 > \tau_0\), \(r\) and \(\epsilon_0\) such that S1-S3 and the following conditions are satisfied for all \([\tau, z_c, z_d, x, y, \epsilon] \in [\tau_0, \tau_1] \times U_r \times [0, \epsilon_0]\):

(a). The trajectory \(\phi_y(\tau, z_{c0}, z_d(0), x^q_0, y^q_0)\) of the QSS model \([19]\) moves along \(\Gamma_s\);

(b). The projection of initial point \((z_{c0}, z_d(0), x^i_0, y^i_0)\) of the long-term stability model \([15]\) to the subspace of \(z_{c0}\) and \(z_d(0)\) is inside the stability region of the initial transient stability model \([39]\), and the long-term stability model \([15]\) satisfies the condition of consistent attraction.

Then for every \(\delta > 0\) there exists a positive constant \(\epsilon^*\) such that for all \(0 < \epsilon < \epsilon^*\), every solution \((z_{ck}(\tau), x_k(\tau))\) of system \([17]\) exists at least on \([\tau_k, \tau_{k+1}]\), and satisfies:

\[
\|z_{ck}(\tau) - \bar{z}_{ck}(\tau)\| \leq \delta \quad (40)
\]

\[
\|x_k(\tau) - l_1(z_{ck}(\tau), z_d(k)) - \hat{x}_k(\frac{\tau - \tau_k}{\epsilon}) + l_1(z_{cks}, z_d(k))\| \leq \delta
\]

for all \(\tau \in [\tau_k, \tau_{k+1}], k = 0, 1, 2, \ldots N\).
Theorem 5 asserts that if the trajectory of QSS model moves along $\Gamma_s$, and the projection of each point on trajectory of the long-term stability model always lies inside the stability region of the corresponding transient stability model, then for sufficiently small $\epsilon$, trajectory of the long-term stability model can be approximated by trajectory of the QSS model (18). Thus we can apply the conclusions of Theorem 3 for each fixed $z_d(k)$, $k = 0, 1, 2, \ldots, N$. We can thus apply the conclusions of Theorem 3 for each $z_d(k)$. We have that, for every $\delta > 0$ there exists a positive constant $\epsilon_k$ such that for all $0 < \epsilon < \epsilon_k$, the solution $(z_{ck}(\tau), x_k(\tau))$ of system (17) exists at least on $[\tau_k, \tau_{k+1}]$, and satisfies:

$$
\|z_{ck}(\tau) - \bar{z}_{ck}(\tau)\| \leq \delta
$$

$$
\|x_k(\tau) - l_1(\bar{z}_{ck}(\tau), z_d(k)) - \bar{x}(\frac{\tau - \tau_k}{\epsilon}) + l_1(\bar{z}_{ck}(\tau), z_d(k))\| \leq \delta
$$

for all $\tau \in [\tau_k, \tau_{k+1}]$. Let $\epsilon^* = \min(\epsilon_0, \epsilon_1, \ldots, \epsilon_N)$, then for every $\delta > 0$ there exists a positive constant $\epsilon^*$ such that for all $0 < \epsilon < \epsilon^*$, the solution $(z_{ck}(\tau), x_k(\tau))$ of system (17) exists at least on $[\tau_k, \tau_{k+1}]$, and satisfies:

$$
\|z_{ck}(\tau) - \bar{z}_{ck}(\tau)\| \leq \delta
$$

$$
\|x_k(\tau) - l_1(\bar{z}_{ck}(\tau), z_d(k)) - \bar{x}(\frac{\tau - \tau_k}{\epsilon}) + l_1(\bar{z}_{ck}(\tau), z_d(k))\| \leq \delta
$$

for all $\tau \in [\tau_k, \tau_{k+1}]$, where $k \in [0, 1, 2, \ldots, N]$. The proof is complete.

We next show the $\omega$-limit set relation between the long-term stability model (15) and the QSS model (18).

**Theorem 6:** ($\omega$-Limit Set Relation)

Assuming there exist positive constants $r$ and $\epsilon_0$ such that S1-S3 and the following conditions are satisfied for all $\tau, z_c, z_d, x, y, \epsilon \in [\tau_0, +\infty) \times U_x \times [0, \epsilon_0]$:

(a). The trajectory $\phi_\tau(\tau, z_c(0), z_d(0), x_0, y_0)$ of the QSS model (18) moves along $\Gamma_s$;

(b). The projection of initial point $(z_c(0), z_d(0), x_0, y_0)$ of the long-term stability model (15) to the subspace of $z_c$ and $z_d$ is inside the stability region of the initial transient stability model (59), and the long-term stability model (15) satisfies the condition of consistent attraction;

(c). The $\omega$-limit set of the QSS model (18) starting from $(z_c(0), z_d(0), x_0^l, y_0^l)$ is a SEP $(z_{cls}, z_{lds}, x_{ls}, y_{ls})$.

Then the solution $(z_c(\tau), x(\tau))$ of the long-term stability model (15) exists for all $\tau \geq \tau_0$, and satisfies the following limit relations:

$$
\lim_{\epsilon \to 0} \lim_{\tau \to +\infty} z_c(\tau, \epsilon) = \lim_{\tau \to +\infty} z_c(\tau, \epsilon) = z_{cls}
$$

$$
\lim_{\epsilon \to 0} \lim_{\tau \to +\infty} x(\tau, \epsilon) = \lim_{\tau \to +\infty} x(\tau, \epsilon) = x_{ls}
$$

**Proof:** Since $(z_c, \tau) = (z_{cls}, \tau)$, then according to Theorem 4, we have:

$$
\lim_{\epsilon \to 0} \lim_{\tau \to +\infty} z_c(\tau, \epsilon) = z_c = z_{cls}
$$

$$
\lim_{\epsilon \to 0} \lim_{\tau \to +\infty} x(\tau, \epsilon) = x(\tau, \epsilon) = x_{ls}
$$

Next, since the long-term stability model (32) with each fixed parameter $z_d(k)$ has a unique solution for all $k = 0, 1, 2, \ldots, N$, the whole long-term stability model (15) with initial condition $(z_c(0), z_d(0), x_0^l, y_0^l)$ also has a unique solution which is denoted as $(z_c(\tau, \epsilon), x(\tau, \epsilon))$. Hence we have:

$$
\lim_{\epsilon \to 0} \lim_{\tau \to +\infty} z_c(\tau, \epsilon) = \lim_{\epsilon \to 0} \lim_{\tau \to +\infty} x(\tau, \epsilon) = x_{ls}
$$

The proof is complete.
the QSS model. In this system, an exponential recovery load was included at Bus 5 and two turbine governors at Bus 1 and Bus 2 were added respectively. The assumption S1 that neither the long-term stability model and the QSS model meet singularity points was satisfied. And we can safely assume that S2 was also satisfied. Besides, from the trajectory of the QSS model, we can see that S3 was also satisfied in this case. In addition, the trajectory \( \phi_q(\tau, z_c(0), z_d(0), x^0_l, y^0_l) \) of QSS model moved along \( \Gamma_s \). As QSS model was implemented 30s after the contingency, fast dynamics settled down at the time such that the projection of the initial point \((z_d(0), x^0_l, y^0_l)\) of the long-term stability model lied inside \( A_4(z_c(0), z_d(0), l_1(z_c(0), l_2(z_c(0), z_d(0)))) \). And whenever \( z_d \) jumped to \( z_d(k), k = 1, 2, ..., N \), the first point \((z_c, z_d(k), x^1_l, y^1_l)\) on \( \phi_q(\tau, z_c(0), z_d(0), x^0_l, y^0_l) \) after the jump stayed inside the stability region of the corresponding transient stability model such that the long-term stability model satisfied the condition of consistent attraction. Since all conditions of Theorem 5 were satisfied, \( \phi_q(\tau, z_c(0), z_d(0), x^0_l, y^0_l) \) always stayed close to \( \phi_q(\tau, z_c(0), z_d(0), x^0_l, y^0_l) \). Additionally, as the QSS model converged to a long-term SEP, the long-term stability model converged to the same point. Fig. 4 shows the trajectory comparisons of the long-term stability model and the QSS model.

### A. Numerical Example I

The first example was a modified IEEE 14-bus systems [19] in which QSS model gave correct approximations of the long-term stability model. In this system, an exponential recovery load was included at Bus 5 and two turbine governors at Bus 1 and Bus 2 were added respectively. The assumption S1 that neither the long-term stability model and the QSS model meet singularity points was satisfied. And we can safely assume that S2 was also satisfied. Besides, from the trajectory of the QSS model, we can see that S3 was also satisfied in this case. In addition, the trajectory \( \phi_q(\tau, z_c(0), z_d(0), x^0_l, y^0_l) \) of QSS model moved along \( \Gamma_s \). As QSS model was implemented 30s after the contingency, fast dynamics settled down at the time such that the projection of the initial point \((z_d(0), x^0_l, y^0_l)\) of the long-term stability model lied inside \( A_4(z_c(0), z_d(0), l_1(z_c(0), l_2(z_c(0), z_d(0)))) \). And whenever \( z_d \) jumped to \( z_d(k), k = 1, 2, ..., N \), the first point \((z_c, z_d(k), x^1_l, y^1_l)\) on \( \phi_q(\tau, z_c(0), z_d(0), x^0_l, y^0_l) \) after the jump stayed inside the stability region of the corresponding transient stability model such that the long-term stability model satisfied the condition of consistent attraction. Since all conditions of Theorem 5 were satisfied, \( \phi_q(\tau, z_c(0), z_d(0), x^0_l, y^0_l) \) always stayed close to \( \phi_q(\tau, z_c(0), z_d(0), x^0_l, y^0_l) \). Additionally, as the QSS model converged to a long-term SEP, the long-term stability model converged to the same point. Fig. 4 shows the trajectory comparisons of the long-term stability model and the QSS model.

To check the condition of consistent attraction, we did the following simulations. When the QSS model was implemented and the ratio of the LTC firstly jumped at 30s, the intersection of the stability region in the subspace of two fast variables is plotted in Fig. 5a, and the first point \((z_c(1), z_d(1), x_1, y_1)\) on \( \phi_q(\tau, z_c(0), z_d(0), x_0, y_0) \) when \( z_d \) jumped to \( z_d(1) \) was marked. Additionally, trajectories of two fast variables in the corresponding transient stability model are shown in Fig. 5b. It can be seen that the trajectory starting from \((z_c(1), z_d(1), x_1, y_1)\) settled down to the SEP of the transient stability model which further confirmed that \((z_c(1), z_d(1), x_1, y_1)\) did lie inside the stability region \( A_4(x_1, y_1, z_d(1)), l_1(z_c(1), l_2(z_c(1), z_d(1))) \) of the corresponding transient stability model.

Fig. 5a shows stability region of the transient stability model in the subspace of the same fast variables when \( z_d \) jumped from \( z_d(2) \) to \( z_d(3) \). Likewise, this procedure can be done successively to verify that the long-term stability model satisfied the condition of consistent attraction.

### B. Numerical Example II

This was also a 14-bus system, while the QSS model did not give correct approximations of the long-term stability model due to the violation of condition (b) in Theorem 5. The trajectory comparisons are shown in Fig. 7.

In this case, S1-S3 was also satisfied, and trajectory \( \phi_q(\tau, z_c(0), z_d(0), x^0_l, y^0_l) \) of QSS model moved along \( \Gamma_s \).
However, condition (b) of Theorem 5 was violated. When $z_d$ jumped from $z_d(0)$ to $z_d(1)$ at 30s, the long-term stability model fixed at $z_d(1)$ was stable, the trajectory comparison is plotted in Fig. 8a in which both the long-term stability model and the QSS model converged to the same long-term SEP. (b) Trajectory of the transient stability model when $z_d$ changed to $z_d(1)$ at 30s. The trajectory starting from the first point of the long-term stability model converged to the SEP of the transient stability model.

Although when $z_d$ jumped from $z_d(1)$ to $z_d(2)$ at 40s, the long-term stability model no longer stable which can be seen from the trajectory comparison in Fig. 10a. The transient variables were excited due to the evolution of discrete variables $z_d$ and the trajectory of the long-term stability model was trapped in a stable limit cycle. From a physical viewpoint, the OXL of the generator at Bus 2 reached its limit while the LTC between Bus 2 and Bus 4 tried to restore the voltage at Bus 4 thus required more power support from the generator at Bus 2. The conflict between the OXL and the LTC resulted in the limit cycle shown in Fig. 10a.

Similarly, Fig. 10b shows the trajectory of a fast variable in the transient stability model, and Fig. 2b shows the stability region of the corresponding transient stability model in the subspace of two fast variables. From these two figures it can be seen that the first point $(z_{c2}, z_{d2}(2), x_2, y_2)$ of $\varphi_1(\tau, z_{c2}, z_{d2}(0), x_0, y_0)$ after $z_d$ jumped to $z_d(2)$ lied outside the stability region $A_1(z_{c2}, z_{d2}(2), x_1(z_{c2}, z_{d2}(2)), l_2(z_{c2}, z_{d2}(2)))$ of the corresponding transient stability model. As a result, the long-term stability model did not satisfy the condition of consistent attraction.

In summary, fast dynamics were excited by the evolution of long-term discrete dynamics $z_d$ such that the condition of consistent attraction was violated. As a result, the QSS model did not provide correct approximations of the long-term stability model in terms of trajectories and presented incorrect stability assessment in concluding that the long-term stability model was stable while the long-term stability model was long-term unstable.

We provide some physical explanation behind sufficient conditions of the QSS model to explain when the QSS model may fail. In long-term time scale, LTCs are to restore the load-side voltages and hence the corresponding load powers, while OXLs restrict the power support from generators. The counter effects between LTCs and OXLs further introduce large changes on exciters, leading to long-term instabilities. However, the QSS model assumes that variables of exciters are stable and converge instantaneously fast as LTCs and OXLs evolve, therefore, large changes occurring in the variables of exciters are not reflected in the QSS model. As a result, when the described physical mechanism of long-term instability occurs, the QSS model can fail to provide correct approximations of the long-term stability model.
VI. CONCLUSIONS AND PERSPECTIVES

A theoretical foundation for the QSS model intended for power system long-term stability analysis has been developed. Sufficient conditions for the QSS model to approximate the long-term stability model are derived and relations of trajectory as well as \( \omega \)-limit point between the long-term stability model and the QSS model are established. Several numerical examples in which the QSS model either succeeds or fails to provide accurate approximations are analyzed using the derived analytical results.

The analytical results derived also point to a research direction for improving the QSS model. It has been shown that the QSS model will provide accurate approximations if the trajectory of QSS model moves along the stable component of the constraint manifold and fast dynamics are not excited by the slow variables. All conditions in Theorem 5 are easy to check except the condition of consistent attraction. If an efficient numerical scheme can be developed to check this condition, then the QSS model can be improved based on the theoretical foundation. It’s our intent to develop an improved QSS model to accurately approximate the long-term stability model.

APPENDIX A

DETAILED POWER SYSTEM MODELS [4] [18]

A. Generator (GEN):

Notations are in Table II. Dynamic Equations:

\[
\dot{\delta} = \Omega_0(\omega - 1) \quad (46)
\]
\[
\dot{\omega} = (p_m - p_e - D\omega)/M 
\quad (47)
\]
\[
\dot{e}_q = -f_s(e'_q) - (x_d - x'_d)i_d + v'_f)/T_{d\theta} 
\quad (48)
\]
\[
\dot{e}'_d = -e'_d + (x_q - x'_q)i_q)/T'_{d\theta} 
\quad (49)
\]

\( f_s(e'_q) \) is a function for saturation and

\[
p_e = (v_q + r_a)q + (v_d + r_a)i_d 
\quad (50)
\]
\[
v'_f = v_f + K\omega(\omega - 1) - K_p(p - p_0) 
\quad (51)
\]

besides \( v_p \) and \( v_q \) are defined as \( v_d = v\sin(\delta - \theta) \), \( v_q = v\cos(\delta - \theta) \), and following equations describe the relation between the voltage and current:

\[
0 = v_q + r_aq - e'_q + x'_d i_d, \quad 0 = v_d + r_ai_d - e'_d - x'_q i_q. 
\]

Algebraic Equations:

\[
0 = v_d i_d + v_q i_q - p 
0 = v_q i_d - v_d i_q - q 
0 = p^0_m - p_m 
0 = v'_f - v_f 
\quad (52)
\]

B. Automatic Voltage Regulator (AVR):

Notations are in Table III. Dynamic Equations:

\[
v'_m = (v - v_m)/T_r 
\quad (53)
\]
\[
v'_{r1} = (K_a(v_{ref} - v_m - v_{r2}) - K_f v_f - v_{r1})/T_a 
\quad (54)
\]
\[
v'_{r2} = -(K_f v_f + v_{r2})/T_f 
\quad (55)
\]
\[
v'_f = -(v_f(K_e + S_e(v_f)) - v_r)/T_e 
\quad (56)
\]

\[ v_f = \begin{cases} 
  v_{r1} & \text{if } v_{r1} < v_{r1} \\
  v_{r2} & \text{if } v_{r2} > v_{r2} \\
  v_{r1} & \text{if } v_{r1} < v_{r1} 
\end{cases} 
\quad (57)
\]

and \( S_e \) is the ceiling function: \( S_e(v_f) = A_ee^{B_0|v_f|} \).

Algebraic Equations:

\[
0 = v_f - v_f^{ref} 
0 = v_{r1} - v_{r1} 
\quad (58)\quad (59)
\]

TABLE II

SYNCHRONOUS MACHINE VARIABLES

| Variable | Description |
|----------|-------------|
| \( \delta \) | generator rotor angle |
| \( \omega \) | generator rotor speed |
| \( e'_q \) | q-axis transient voltage |
| \( e_d \) | d-axis transient voltage |
| \( p_m \) | mechanical power |
| \( p_{in} \) | initial mechanical power |
| \( v_f \) | field voltage |
| \( v'_{f0} \) | initial field voltage |
| \( T'_{q0} \) | q-axis open circuit transient time constant |
| \( T'_{d0} \) | d-axis open circuit transient time constant |
| \( x'_q \) | q-axis synchronous reactance |
| \( x'_d \) | q-axis transient reactance |
| \( r_a \) | armature resistance |
| \( M = 2H \) | mechanical starting time (2x inertia constant) |
| \( D \) | damping coefficient |
| \( K_w \) | speed feedback gain |
| \( K_p \) | active power feedback gain |
| \( U_b \) | base frequency |
| \( \rho_e \) | electrical power |

TABLE III

EXCITER VARIABLES

| Variable | Description |
|----------|-------------|
| \( v'_{r1} \) | maximum regulator voltage |
| \( v'_{r2} \) | minimum regulator voltage |
| \( K_a \) | amplifier gain |
| \( T_a \) | amplifier time constant |
| \( K_p \) | stabilizer gain |
| \( T_f \) | stabilizer time constant |
| \( K_e \) | field circuit integral deviation |
| \( T_e \) | field circuit time constant |
| \( T_{ce} \) | measurement time constant |
| \( A_e \) | \( 1^{st} \) ceiling coefficient |
| \( B_e \) | \( 2^{nd} \) ceiling coefficient |
| \( v_{ref} \) | the reference voltage (or initial) |
| \( v_{i1, i2, i3} \) | state variables |

C. Turbine Governor (TG):

Notations are in Table IV. Dynamic Equations:

\[
\dot{x}_{g1} = (p_{in} - x_{g1})/T_s 
\quad (60)
\]
\[
\dot{x}_{g2} = ((1 - T_3/T_c)x_{g1} - x_{g2})/T_c 
\quad (61)
\]
\[
\dot{x}_{g3} = ((1 - T_4/T_5)(x_{g2} + T_3/T_c x_{g1}) - x_{g3})/T_5 
\quad (62)
\]
where
\[ p_{in}^* = p_{order} + \frac{1}{R} (\omega_{ref} - \omega) \]
\[ p_{in} = \begin{cases} 
  p_{in}^* & \text{if } p_{min} \leq p_{in}^* \leq p_{max} \\
  p_{max} & \text{if } p_{in}^* > p_{max} \\
  p_{min} & \text{if } p_{in}^* < p_{min}
\end{cases} \]
\[ p_m = \frac{T_4}{T_5} (x_{g2} + \frac{T_3}{T_c} x_{g1}) \]

Algebraic Equations:
\[ 0 = p_m - p_{m^{syn}} - \omega_{ref}^0 \]

**TABLE IV**

| Variable  | Description             |
|-----------|-------------------------|
| \( \omega_{ref}^0 \) | reference speed         |
| \( R \) | droop                   |
| \( p_{max} \) | maximum turbine output  |
| \( p_{min} \) | minimum turbine output  |
| \( x \) | governor time constant  |
| \( k_x \) | servom time constant    |
| \( k_h \) | transient gain time constant |
| \( T_4 \) | power fraction time constant |
| \( T_5 \) | reheat time constant    |
| \( x_{g1} \) | state variables (i=1,2,3) |

**D. Over Excitation Limiter (OXL):**

Notations are in table [VI]

Dynamic Equations:
\[ \dot{v}_{OXL} = \begin{cases} 
  (i_f - i_{f^{lim}})/T_0 & \text{if } i_f > i_{f^{OXL}} \\
  0 & \text{if } i_f \leq i_{f^{OXL}}
\end{cases} \]

Algebraic Equations:
\[ 0 = \sqrt{(v + \gamma_q) + p^2} + \frac{x_d}{x_q} \]
\[ 0 = \frac{\gamma_q (v + \gamma_q) + \gamma_p}{\sqrt{(v + \gamma_q)^2 + p^2}} - i_f \]

with \( \gamma_p = x_q p/v, \gamma_q = x_q q/v \). And the over excitation limiter starts to work after a fixed delay \( T_0 \) regardless of the field current overload.

**TABLE V**

| Variable | Description         |
|----------|---------------------|
| \( x_d \) | d-axis estimated generator reactance |
| \( x_q \) | q-axis estimated generator reactance |
| \( i_f \) | synchronous machine field current |
| \( i_{f^{lim}} \) | maximum field current |
| \( v_{ref}^0 \) | the reference voltage of automatic voltage regulator |
| \( T_{i}^0 \) | integrator time constant |
| \( p_{(or) q} \) | active (or reactive) power of generator |
| \( K_0 \) | fixed time delay |
| \( v_{OXL} \) | state variable         |

**E. Exponential Recovery Load (ERL):**

Notations are in table [VI]

Dynamic Equations:
\[ \dot{x}_p = -x_p/T_p + p_s - p_t \]
\[ \dot{x}_q = -x_q/T_q + q_s - q_t \]

where \( p_s \) and \( p_t \) are the static and transient real power absorptions, similar definition for \( q_s \) and \( q_t \). \( p^0 \) and \( q^0 \) are PQ load power from power flow solutions. Besides, \( p^0 = \frac{k_p}{100} q^0, q^0 = \frac{k_q}{100} q^0 \), \( p_s = p^0 (v/v^0)^{\alpha_s}, p_t = p^0 (v/v^0)^{\alpha_t}, q_s = p^0 (v/v^0)^{\beta_s}, q_t = p^0 (v/v^0)^{\beta_t} \).

Algebraic Equations:
\[ p = x_p/T_p + p_t \]
\[ q = x_q/T_q + q_t \]

**TABLE VI**

| Variable | Description |
|----------|-------------|
| \( k_p \) | active power percentage |
| \( k_q \) | reactive power percentage |
| \( T_p \) | active power time constant |
| \( T_q \) | reactive power time constant |
| \( \alpha_s \) | static active power exponent |
| \( \alpha_t \) | active power exponent |
| \( \beta_s \) | static reactive power exponent |
| \( \beta_t \) | reactive power exponent |
| \( x_{p,q} \) | state variables |

**F. Load Tap Changer (LTC):**

\[ m_{k+1} = \begin{cases} 
  m_k + \triangle m & \text{if } v > v_0 + d \text{ and } m_k < m_{max} \\
  m_k - \triangle m & \text{if } v < v_0 + d \text{ and } m_k > m_{min} \\
  m_k & \text{otherwise}
\end{cases} \]

The tap changing delay are assumed to be independent of \( V \), but larger for first tap change than for the subsequent ones while without the inverse time characteristic. Refer to [H] for more details.

**APPENDIX B**

**DETAILED AND GENERIC LONG-TERM STABILITY MODELS**

The detailed and generic long-term stability model are shown in Table [VII]. Moreover, the detailed variables and their corresponding generic variables \( z_c, z_d, x \) and \( y \) are also indicated.

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**TABLE VII**

**LONG-TERM STABILITY MODEL AND CORRESPONDING GENERIC VARIABLES**

| Detailed Long-Term Stability Model | Generic Long-Term Stability Model | Detailed Variables |
|------------------------------------|-----------------------------------|--------------------|
| TG: [60]-[62], OXL: [67].         | $z_c = ch_c(z_c, z_d, x, y)$     | slow continuous variables $z_c$: |
| ERL: [70]-[71].                    | $z_d(k + 1) = h_d(z_c, z_d(k), x, y)$ | TG: $x_{g1}, x_{g2}, x_{g3}$, OXL: $v_{OXL}$, ERL: $x_p, x_q$. |
| LTC: [74].                         | $x = f(z_c, z_d, x, y)$          | fast continuous variables $x$: |
| GEN: [46]-[49], AVR: [53]-[56].   | $0 = g(z_c, z_d, x, y)$          | slow discrete variables $z_d$: $m_k$. |
| TG: [66], OXL: [68]-[69], ERL: [72]-[73], GEN: [52], AVR: [58]-[59]. | power relations.               | algebraic variables $y$: |

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