BILLEY-POSTNIKOV DECOMPOSITIONS AND THE FIBRE BUNDLE STRUCTURE OF SCHUBERT VARIETIES

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ABSTRACT. A theorem of Ryan and Wolper states that a type $A$ Schubert variety is smooth if and only if it is an iterated fibre bundle of Grassmannians. We extend this theorem to arbitrary finite type, showing that a Schubert variety in a generalized flag variety is rationally smooth if and only if it is an iterated fibre bundle of rationally smooth Grassmannian Schubert varieties. The proof depends on deep combinatorial results of Billey-Postnikov on Weyl groups. We determine all smooth and rationally smooth Grassmannian Schubert varieties, and give a new proof of Peterson’s theorem that all simply-laced rationally smooth Schubert varieties are smooth. Taken together, our results give a fairly complete geometric description of smooth and rationally smooth Schubert varieties using primarily combinatorial methods.

We also give some partial results for Schubert varieties in Kac-Moody flag varieties. In particular, we show that rationally smooth Schubert varieties of affine type $\tilde{A}$ are also iterated fibre bundles of rationally smooth Grassmannian Schubert varieties. As a consequence, we finish a conjecture of Billey-Crites that a Schubert variety in affine type $\tilde{A}$ is smooth if and only if the corresponding affine permutation avoids the patterns 4231 and 3412.

1. Introduction

Let $G$ be a connected semisimple Lie group over an algebraically closed field $k$. Let $W$ denote the Weyl of $G$ with simple generating set $S$. For any subset $J \subseteq S$, let $P_J \subseteq G$ denote the corresponding parabolic subgroup. Let $J \subseteq K \subseteq S$ and consider the projection between flag varieties

$$
\pi : G/P_J \to G/P_K
$$

(1)

In this paper, we study the map $\pi$ restricted to a Schubert variety of $G/P_J$. Let $W_J$ denote the subgroup of $W$ generated by $J$. The Schubert varieties of $G/P_J$ are indexed by the set of minimal length coset representatives $W^J \simeq W/W_J$. Let $X^J(w)$ denote the Schubert subvariety of $G/P_J$ corresponding to $w \in W^J$. There exists a unique decomposition $w = vu$ where $v \in W^K$ and $u \in W^J \cap W_K$. We call this decomposition the parabolic decomposition of $w$ with respect to $K$. Restricting $\pi$ to $X^J(w)$, we get a projection between Schubert varieties

$$
\pi : X^J(w) \to X^K(v)
$$

(2)

with generic fibre isomorphic to the Schubert variety $X^J(u)$. While the projection in equation (1) is a $P_K/P_J$-fibre bundle, the restricted projection in equation (2) is not necessarily a fibre bundle.
In type $A_{n-1}$ with $k = \mathbb{C}$, the group $G$ is the special linear group $\text{SL}_n(\mathbb{C})$. If $J = \emptyset$, then $B = P_J$ is a Borel subgroup and $G/B$ is the variety of complete flags on the vector space $\mathbb{C}^n$. In this case, Ryan proved that smooth Schubert varieties are iterated fibre bundles of Grassmannians \cite{Rya87}. This result was extended to partial flag varieties $G/P_J$ of type $A$ over any algebraically closed field $k$ of characteristic zero by Wolper \cite{Wol89}.

The main result of this paper is an extension of Ryan and Wolper’s theorems to flag varieties of all finite types as well as affine type $\tilde{A}$. We consider both the class of smooth Schubert varieties, and the larger class of rationally smooth Schubert varieties. A variety $Y$ over $k$ is rationally smooth if, for $y \in Y$, the étale cohomology $H^*_y(Y; \mathbb{Q}_l)$ with support in $y$ is the same as the cohomology of a smooth variety, i.e. one-dimensional and concentrated in top degree ($l$ is a prime not equal to the characteristic of $k$). Define the Poincaré polynomial of $X^J(w)$ as

$$P^J_w(t) := \sum_{x \in [e, w] \cap W^J} t^{\ell(x)},$$

where $[e, w]$ denotes the interval between the identity $e$ and $w$ in the Bruhat partial order and $\ell$ denotes the length function on $W$. A theorem of Carrell-Peterson states that a Schubert variety $X^J(w)$ is rationally smooth if and only if $P^J_w(t)$ is a palindromic polynomial, meaning $t^{\ell(w)} P^J_w(t^{-1}) = P^J_w(t)$ \cite{Car94}. A theorem of Peterson, proved in full generality by Carrell-Kuttler, states that if $G$ is simply-laced then $X^J(w)$ is rationally smooth if and only if it is smooth \cite{CK03}.

Our extension of Ryan and Wolper’s theorem rests on two results. In the first result, we characterize when the projection in equation (2) is a fibre bundle. Notice that if this projection is an $X^J(u)$-fibre bundle, then by the Leray-Hirsch theorem the Poincaré polynomial of $X^J(w)$ as

$$P^J_w(t) = P^K_v(t) \cdot P^J_u(t).$$

In addition, the singular cohomology ring $H^*(X^J(w))$ is a free $H^*(X^K(u))$-module over $H^*(X^J(u))$. We prove that the factorization of Poincaré polynomials in equation (3) is also a sufficient condition for the projection in equation (2) to be a fibre bundle. This result is stated in Theorem 2.3 and holds for Schubert varieties of any Kac-Moody group. We remark that the presentation of the cohomology algebra of Schubert varieties has been studied in \cite{GR02}. The factoring of Poincaré polynomials has been studied in \cite{Gas98, Las98, Bil98, BP05, BC12} and plays an important role in the pattern avoidance characterization of rationally smooth Schubert varieties in \cite{Bil98, BP05, BC12}.

In the second result, we establish the existence of certain Poincaré polynomial factorizations for rationally smooth Schubert varieties. For this result, we consider only finite type Schubert varieties, and Schubert varieties of the affine type $\tilde{A}$. Billey and Postnikov show that if $X^w(w)$ is a rationally smooth Schubert variety of finite type, then there exists a set $K \subseteq S$ for which either $w$ or $w^{-1}$ has a parabolic decomposition $vu$ with respect to $K$ such that the Poincaré polynomial factors as $P^w_w(t) = P^K_v(t) \cdot P^u_u(t)$ \cite{BP05}. Moreover, $K$ can be chosen such that $S \setminus K = \{s\}$ for some leaf $s$ in the Dynkin diagram of $G$. In \cite{OY10}, Oh and Yoo call such a parabolic
decomposition a Billey-Postnikov decomposition. Given $w \in W^J$, in this paper we say that a parabolic decomposition $w = vu$ with respect to $K$ is Billey-Postnikov (BP) if the polynomial $P^J_w(t)$ factors as in equation (3) (dropping the condition that $S \setminus K = \{s\}$ for some leaf $s$). This definition is also used by the authors in [RS14], except that here we also consider relative Poincaré polynomials $P^J_w(t)$ where $J$ is not necessarily empty. In this paper we prove that if $X^J(w)$ is rationally smooth (and of finite type or affine type $A$), then there exists a nontrivial $K$ containing $J$ for which $w$ has a BP decomposition. This result is stated in Theorem 2.5. The proof uses the existence theorem in [BP05] combined with an inductive argument. For Schubert varieties of affine type $\tilde{A}$, we use a similar existence theorem for BP decompositions due to Billey-Crites [BC12].

Using these two results, we prove the following extension of Ryan and Wolper’s theorems: a Schubert variety (in finite type or affine type $\tilde{A}$) is (rationally) smooth if and only if it is an iterated fibre bundle of (rationally) smooth Grassmannian Schubert varieties. A Schubert variety $X^J(w)$ is Grassmannian if $|S \setminus J| = 1$ (i.e. $P^J_w(t)$ is a maximal parabolic of $G$). Note that we cannot apply the existence theorem for BP decompositions stated in [BP05] to construct fibre bundle structures of Schubert varieties directly. While the Bruhat intervals $[e, w]$ and $[e, w^{-1}]$ are order isomorphic, the Schubert varieties $X^\emptyset(w)$ and $X^\emptyset(w^{-1})$ are not necessarily isomorphic. Hence a fibre bundle structure on $X^\emptyset(w^{-1})$ may not yield a fibre bundle structure on $X^\emptyset(w)$.

We also classify all finite type (rationally) smooth Grassmannian Schubert varieties. This result, stated in Theorem 2.7, gives a fairly complete geometric description of (rationally) smooth Schubert varieties in finite type. In particular, we get a new proof of Peterson’s theorem that $X^J(w)$ is rationally smooth if and only if it is smooth when $G$ is simply-laced. Peterson’s theorem was originally proved in [CK03], and other proofs have been given by Dyer [Dye] and Juteau-Williamson [JW12]. Finally, in affine type $\tilde{A}$, we complete a conjecture of Billey-Crites that an affine type $\tilde{A}$ Schubert variety is smooth if and only if the corresponding affine permutation avoids the patterns 4231 and 3412.

Our results make it possible to list all (rationally) smooth Schubert varieties. With some additional work, it is possible to list these varieties bijectively, and thus enumerate (rationally) smooth Schubert varieties in finite type. This will appear in a later paper.

1.1. **Background and terminology.** We use the notation from the introduction throughout the paper. In particular, we work over a fixed algebraically closed field $k$ of arbitrary characteristic. $G$ will denote a semisimple Lie group over $k$ or, as long as $k = \mathbb{C}$, a Kac-Moody group. When working with Kac-Moody groups, we take $G$ to be the minimum Kac-Moody group $G^{\min}$ as defined in [Kum02]. Given $G$, we fix a choice of maximal torus and Borel $T \subset B \subset G$. Let $W = N(T)/T$ denote the Weyl group of $G$ and fix a simple generating set $S$ for $W$. We choose a representative in the normalizer $N(T)$ of $T$ for each element $w \in W$. 

We now recall some basic facts about Schubert varieties. The group $G$ has Bruhat decomposition

$$G = \coprod_{w \in W} BwB,$$

and the double cosets $BwB$ satisfy the multiplication relation

$$BsB \cdot BwB = \begin{cases} 
BswB & \text{if } \ell(sw) = \ell(w) + 1, \\
BswB \cup BwB & \text{if } \ell(sw) = \ell(w) - 1
\end{cases}$$

for every $s \in S$. Consequently, given $J \subseteq S$, the set $P_J := BW_JB$ is a parabolic subgroup of $G$. The group $G$ also has relative Bruhat decomposition

$$G = \coprod_{w \in W_J} BwP_J,$$

and the flag variety $G/P_J$ is a disjoint union

$$G/P_J = \coprod_{w \in W_J} BwP_J/P_J,$$

The set $BwP_J/P_J$ is called the Schubert cell indexed by $w$, and the Schubert variety is defined as the closure

$$X^J(w) := \overline{BwP_J/P_J}$$

in $G/P_J$. If $w \in W_J$, then the dimension of $X^J(w)$ is $\ell(w)$, the length of $w$. Note that for Kac-Moody groups $G$, the flag variety $G/P_J$ can be an infinite-dimensional ind-variety, but the Schubert varieties $X^J(w)$ are always finite-dimensional.

Bruhat order on $W$ can be defined by setting $x \leq w$ if and only if $X^\emptyset(x) \subseteq X^\emptyset(w)$. It follows that

$$X^J(w) = \coprod_{x \in [e,w] \cap W_J} BxP_J/P_J$$

where $[e,w]$ denotes the interval in Bruhat order between $e$ and $w$. If $W_J$ is finite and $w \in W_J$, then the preimage of $X^J(w)$ in $G/B$ is $X^\emptyset(w')$, where $w'$ is the maximal length representative of the coset $wW_J$. Alternatively, $w' = wu_0$, where $u_0$ is the longest element of $W_J$. If $w_1$ and $w_2$ are in $W_J$, then $w_1 \leq w_2$ if and only if $w_1' \leq w_2'$, where $w_i'$ is the longest element in the coset $w_iW_J$. Geometrically, $w_1 \leq w_2$ if and only if $X^J(w_1) \subseteq X^J(w_2)$.

The support of an element $w \in W$ is defined as

$$S(w) := \{s \in S \mid s \leq w\}.$$ 

Equivalently, $S(w)$ is the set of simple reflections appearing in some reduced decomposition of $w$. The left and right descents set of $w$ are

$$D_L(w) := \{s \in S \mid \ell(sw) \leq \ell(w)\},$$

$$D_R(w) := \{s \in S \mid \ell(ws) \leq \ell(w)\}.$$ 

A generalized Grassmannian is a flag variety $G/P_J$, where $|S \setminus J| = 1$. In other words, $P_J$ is a maximal parabolic subgroup of $G$. A Schubert variety $X^J(w)$ of a generalized Grassmannian is called a Grassmannian Schubert variety. An element $w \in W$ is called a Grassmannian element if $w \in W_J$ for some generalized Grassmannian $G/P_J$. 

Equivalently, \( w \in W \) is Grassmannian if and only if \( w \) has a unique right descent. By a Grassmannian parabolic decomposition, we mean a parabolic decomposition \( w = vu \) with respect to a set \( K \) such that \( |K \cap S(w)| = |S(w)| - 1 \). Finally, we say \( w = w_1 \cdots w_k \) is a reduced decomposition if \( \ell(w) = \sum \ell(w_i) \). Note that all parabolic decompositions are reduced.

For more information about Schubert varieties over an arbitrary field, we point to [Bor91] and [BK04].

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1.3. Organization. The main results are stated in Section 2. Section 3 is concerned with characterizations of Billey-Postnikov decompositions, including our geometric characterization. In Section 4, we restate the results of Billey-Postnikov and others on the existence of Billey-Postnikov decompositions, and classify rationally smooth Grassmannian Schubert varieties. In Section 5 we prove our main theorem on the existence of Billey-Postnikov decompositions. In Section 5.2 we finish the proof of the Ryan-Wolper theorem. Finally, there are two appendices where we list Dynkin diagrams and give additional examples.

2. Main results

As stated in the introduction, we define Billey-Postnikov decompositions as follows:

**Definition 2.1.** Let \( w \in W^J \), and let \( w = vu \) be a parabolic decomposition with respect to \( K \), where \( J \subseteq K \subseteq S \). We say that \( w = vu \) is a Billey-Postnikov (BP) decomposition with respect to \((J, K)\) if

\[
P_w^J(t) = P_v^K(t) \cdot P_u^J(t).
\]

When \( J = \emptyset \), we simply say that \( w = vu \) is a BP decomposition with respect to \( K \).

Some elementary equivalent definitions of BP decompositions are given in Proposition 3.2. In particular, checking whether or not a given parabolic decomposition is a BP decomposition is computationally easy, and does not require working with Bruhat order.

**Example 2.2.** Let \( G = SL_4(k) \), with Weyl group \( W \) generated by \( S = \{s_1, s_2, s_3\} \) according to the type A Dynkin diagram in Section 6. If \( J = \{s_1, s_3\} \), then the parabolic decomposition \( w = vu = (s_1 s_3 s_2)(s_3 s_1) \) is a BP decomposition with respect to \( J \) since

\[
P_w^J(t) = P_v^J(t) \cdot P_u^J(t) = (t^3 + 2t^2 + t + 1)(t^2 + 2t + 1)
= t^5 + 4t^4 + 6t^3 + 5t^2 + 3t + 1.
\]

The parabolic decomposition \( w = vu = (s_1 s_3 s_2)(s_1) \) is not a BP decomposition with respect to \( J \) since

\[
P_w^J(t) = t^4 + 3t^3 + 4t^2 + 3t + 1
\]

and

\[
P_v^J(t) \cdot P_u^J(t) = (t^3 + 2t^2 + t + 1)(t + 1) = t^4 + 3t^3 + 5t^2 + 2t + 1.
\]
Our first main theorem is a geometric characterization of BP decompositions. Note that this theorem holds when $G$ is a general Kac-Moody group.

**Theorem 2.3.** Let $w \in W^J$ and $w = vu$ be a parabolic decomposition with respect to $K$. Then the following are equivalent:

(a) The decomposition $w = vu$ is a BP decomposition with respect to $(J, K)$.
(b) The projection $\pi: X^J(w) \to X^K(v)$ is Zariski-locally trivial with fibre $X^J(u)$.

Consequently, if $w = vu$ is a BP decomposition then:

1. $X^J(w)$ is (rationally) smooth if and only if $X^J(u)$ and $X^K(v)$ are (rationally) smooth.
2. The projection $\pi: X^J(w) \to X^K(v)$ is smooth if and only if $X^J(u)$ is smooth.

We give some examples illustrating Theorem 2.3 in Section 7. If $Y$ is a variety over $k$, let $H^*(Y)$ denote either etale cohomology, or, when $k = \mathbb{C}$, singular cohomology. For etale cohomology, we take coefficients in $\mathbb{Q}_l$, where $l$ is a prime not equal to the characteristic of $k$, while for singular cohomology we take coefficients in $\mathbb{C}$.

**Corollary 2.4.** Let $w = vu$ be a parabolic decomposition with respect to $K$. Then the following are equivalent:

(a) The decomposition $w = vu$ is a BP decomposition with respect to $(J, K)$.
(b) There is an isomorphism $H^*(X^J(w)) \cong H^*(X^K(v)) \otimes H^*(X^J(u))$ as $H^*(X^K(v))$-modules.

It is well known that the Poincaré polynomial $P_w(t^2) = \sum H^i(X^J(w))t^i$, so the $(b) \Rightarrow (a)$ direction of Corollary 2.4 follows immediately from the definition. The $(a) \Rightarrow (b)$ direction of Corollary 2.4 and Theorem 2.3 will be proved in Section 3.2.

Our second main theorem concerns the existence of BP decompositions when $G$ is semisimple, or equivalently when $W$ is finite.

**Theorem 2.5.** Let $w \in W^J$, where $W$ is finite, and suppose $|S(v) \setminus J| \geq 2$. If $X^J(w)$ is rationally smooth, then $w$ has a Grassmannian BP decomposition with respect to $(J, K)$ for some maximal proper $K$ containing $J$.

As an application of our main theorems, we get the following extension of the Ryan-Wolper theorem to arbitrary finite type.

**Corollary 2.6.** Let $w \in W^J$, where $W$ is finite, and set $m = |S(v) \setminus J|$. Then $X^J(w)$ is rationally smooth if and only if there is a sequence

$$X^J(w) = X_0 \to X_1 \to \cdots \to X_{m-1} \to X_m = \text{Spec } k,$$

where each morphism is a Zariski locally-trivial fibre bundle, and the fibres are rationally smooth Grassmannian Schubert varieties.

Similarly, $X^J(w)$ is smooth if and only if there is a sequence as in (1) where all the fibres, or equivalently, all the morphisms, are smooth.
Each projection $X_i \rightarrow X_{i+1}$ in Corollary 2.6 corresponds to a BP decomposition. However, these BP decompositions are not usually Grassmannian, since the fibre of the projection is Grassmannian rather than the base. To deduce Corollary 2.6 from Theorem 2.5 we start with the morphisms $X_i \rightarrow X_{m-1}$ (which do correspond to Grassmannian BP decompositions), and then apply a certain associativity property (stated in Lemma 3.3) for BP decompositions. Theorem 2.5 and Corollary 2.6 are proved in Section 5.

To complete the description of rationally smooth Schubert varieties, we list all rationally smooth Grassmannian Schubert varieties of finite type.

**Theorem 2.7.** Let $W$ be a finite Weyl group. Suppose $w \in W^J$ for some $J = S \setminus \{s\}$, and that $S(w) = S$. Then $X^J(w)$ is rationally smooth if and only if either

1. $w$ is the maximal element of $W^J$, in which case $X^J(w)$ is smooth.
2. $w$ is one of the following elements:

| $W$ | $s$ | $w$ | index set | $X^J(w)$ smooth? |
|-----|-----|-----|-----------|-----------------|
| $B_n$ | $s_1$ | $s_k s_{k+1} \cdots s_n s_{n-1} \cdots s_1$ | $1 < k \leq n$ | no |
| $B_n$ | $s_k$ | $u_{n,k+1} s_1 \cdots s_k$ | $1 < k < n$ | no |
| $B_n$ | $s_n$ | $s_1 \cdots s_n$ | $n \geq 2$ | yes |
| $C_n$ | $s_1$ | $s_k s_{k+1} \cdots s_n s_{n-1} \cdots s_1$ | $1 < k \leq n$ | yes |
| $C_n$ | $s_k$ | $u_{n,k+1} s_1 \cdots s_k$ | $1 < k < n$ | yes |
| $C_n$ | $s_n$ | $s_1 \cdots s_n$ | $n \geq 2$ | no |
| $F_4$ | $s_1$ | $s_4 s_3 s_2 s_1$ | n/a | no |
| $F_4$ | $s_2$ | $s_3 s_2 s_4 s_3 s_4 s_2 s_3 s_1 s_2$ | n/a | no |
| $F_4$ | $s_3$ | $s_2 s_3 s_1 s_2 s_1 s_3 s_2 s_4 s_3$ | n/a | yes |
| $F_4$ | $s_4$ | $s_1 s_2 s_3 s_4$ | n/a | yes |
| $G_2$ | $s_1$ | $s_2 s_1, s_1 s_2 s_1, s_2 s_1 s_2 s_1$ | n/a | no |
| $G_2$ | $s_2$ | $s_1 s_2$ | n/a | yes |
| $G_2$ | $s_2$ | $s_2 s_1 s_2, s_1 s_2 s_1 s_2$ | n/a | no |

The simple generators $\{s_i\}$ are the simple reflections corresponding to the labelled Dynkin diagrams listed in Section 7. When $W$ has type $B_n$ or $C_n$, we let $u_{n,k}$ be the is maximal element in $W^{S\{s_1,s_k\}} \cap W_{S\{s_1\}}$. In each case, the set $J = S \setminus \{s\}$, where $s$ is listed in the table.

**Remark 2.8.** All the elements listed in part (2) of Theorem 2.7 satisfy a Coxeter-theoretic property which we term almost maximality. This property is defined in Definition 4.2, and plays an important role in the proof of Theorem 2.5.

**Remark 2.9.** The assumption in Theorem 2.7 that $S(w) = S$ does not weaken the characterization. If $S(w)$ is a strict subset of $S$, then $X^J(w)$ is isomorphic to the Schubert variety indexed by $w$ in the smaller flag variety $G_{S(w)}/P_{S(w) \cap J}$ corresponding to the algebraic subgroup $G_{S(w)} \subset G$ with Weyl group $W_{S(w)}$. For a precise statement, see Lemma 3.8.
The singular locus of a Schubert variety in a generalized Grassmannian has been extensively studied, in particular by Lakshmibai-Weyman [LW90] in the minuscle case, and by Brion-Polo [BP99] in the minuscle and cominuscule cases. A summary can be found in [BL00]. More recently, smooth Grassmannian Schubert varieties have been studied in the context of homological rigidity [Rob14] [HM13]. From this work, the list of smooth Schubert varieties in a generalized Grassmannian is known in many cases. In particular, Hong and Mok show that if $X^J(w)$ is a smooth Schubert variety in a generalized Grassmannian corresponding to a long root, then $w$ must be the maximal element of $W^J_{S(w)}$ (in the cominuscule case this also follows from the earlier work of Brion-Polo). The smooth Schubert varieties in $C_n$ with $s = s_k$ with $1 < k < n$ arise as “odd symplectic manifolds”, and have been studied by Mihai [Mih07]. To the best of the authors’ knowledge, the cases $F_4$, $s = s_3$ and $s = s_4$ are not covered by previous work, and the completeness of the above list has not been addressed outside of the cases mentioned above.

It is well known that a Schubert variety $X^J(w)$ is rationally smooth if and only if the corresponding Kazhdan-Lusztig polynomials are trivial. While there are a number of explicit formulas for Kazhdan-Lusztig polynomials of the Schubert varieties of minuscule and cominuscule generalized Grassmannians (see sections 9.1 and 9.2 of [BL00] for a summary), the authors’ are not aware of any complete list of rationally smooth Grassmannian Schubert varieties of finite type in previous work.

Combining Theorem 2.7 with our main results, we get a new proof of Peterson’s theorem:

**Corollary 2.10.** Suppose $W$ is simply-laced. Then $X^J(w)$ is rationally smooth if and only if it is smooth.

**Proof.** If $W$ is simply-laced, then all rationally smooth Grassmannian Schubert varieties are maximal, and hence smooth, by Theorem 2.7. So if $X^J(w)$ is rationally smooth, then it is smooth by Corollary 2.6. □

Corollary 2.10 is a consequence of a more general theorem proved by Peterson, which states that if $W$ is simply-laced then the rationally smooth and smooth locus of any Schubert variety coincide. Our methods do not seem to apply to this more general situation.

We also prove an analogue of Theorem 2.5 and Corollary 2.6 for Schubert varieties in the full flag variety of affine type $\tilde{A}_n$.

**Theorem 2.11.** Let $G$ be of type $\tilde{A}_n$. If $X^J(w)$ is rationally smooth, then $w$ has a Grassmanian $BP$ decomposition relative to $J$. Moreover, $X^J(w)$ is (rationally) smooth if and only if $X^J(w)$ is an iterated fibre bundle as in Corollary 2.6.

Billey-Crites make the following conjecture in [BC12]. We refer to their paper for the definitions of affine permutation and pattern avoidance.

**Conjecture 2.12.** ([BC12 Conjecture 1]) A Schubert variety $X^\emptyset(w)$ of type $\tilde{A}_n$ is smooth if and only if $w$ avoids the patterns $3412$ and $4231$ as an affine permutation.

As an application of Theorem 2.11 we can complete the proof of this conjecture:
Corollary 2.13. Conjecture 2.12 is true.

The proof of Corollary 2.13 is given in Section 5. Smooth and rationally smooth Grassmannian Schubert varieties of type $\tilde{A}_n$ have been classified in [BM10] Theorem 1.4 and Corollary 1.6.

3. Characterization of Billey-Postnikov decompositions

The goal of this section is to prove Theorem 2.3 and Corollary 2.4. We first give several equivalent combinatorial characterizations of BP decompositions, and then apply these characterizations to the geometry of Schubert varieties.

3.1. Combinatorial characterizations. In this section, we can assume $W$ is an arbitrary Coxeter group with simple generating set $S$. Note that Definition 2.1 still makes sense for arbitrary Coxeter groups. We restrict to the finite or crystallographic case only when necessary. We start by proving some important facts about BP decompositions based on known facts from the case $J = \emptyset$. For notational simplicity, define

$$W^J_K := W^J \cap W_K$$

for any $J \subseteq K \subseteq S$.

Lemma 3.1. For every element $w \in W^J$ and subset $K \subseteq S$ containing $J$, there is a unique maximal element $\bar{u}$ in $[e, w] \cap W^J_K$ with respect to Bruhat order. If $w = vu$ is the parabolic decomposition of $w$ with respect to $K$, then $\bar{u}$ has a reduced decomposition $\bar{u} = v \bar{u}$, where $\bar{v} \in [e, v] \cap W_K$.

Proof. When $J = \emptyset$, the existence of $\bar{u}$ is proved in [vdH74] Lemma 7]. Let $u'$ denote the maximal element of $[e, w] \cap W_K$ and let $u'' = \bar{u}u''$ be the parabolic decomposition of $u'$ with respect to $J$, so $\bar{u} \in W^J_K$ and $u'' \in W_J$. If $u_0 \in [e, w] \cap W^J_K$, then $u_0 \leq u$ and hence $u_0 \leq \bar{u}$.

For the second part of the lemma, if we take $J = \emptyset$ then it is easy to prove by induction on $e(u)$ that $u'$ above has a reduced decomposition $u' = v'u$, where $v' \in [e, v] \cap W_K$. For arbitrary $J$, observe that if $u \in W^J$ and $s \in K$ such that $\ell(su) = \ell(u) + 1$, then either $su \in W^J_K$, or $su = ut$ for some $t \in J$. Indeed, $su = ut$ where $u_1 \in W_J$ and $t \in W_J$. Since $u \leq su$, we get that $u \leq u_1$, and if $t \neq e$ then we must have $u_1 = u$ and $\ell(t) = 1$. Taking a reduced decomposition $s_1 \cdots s_k$ for $u'$ and considering the products $s_ku$, $s_{k-1}s_ku$, $\ldots$, we eventually conclude that $\bar{u}$ has reduced decomposition $s_{i_1} \cdots s_{i_m}u$, where $1 \leq i_1 < \ldots < i_m \leq k$. □

Recall that Bruhat order on $W^J$ induces a relative Bruhat order $\leq_J$ on the coset space $W/W_J$. By definition,

$$w_1W_J \leq_J w_2W_J$$

if and only if $\bar{w}_1 \leq \bar{w}_2$ in the usual Bruhat order, where $\bar{w}_j$ is the minimal length coset representative of $w_jW_J$. Note that if $w_1 \leq w_2$ in Bruhat order, then $w_1W_J \leq w_2W_J$ even if $w_1, w_2 \notin W^J$. Define the descent set relative to $J$ to be

$$D^J_J(w) := \{ s \in S \mid swW_J \leq_J wW_J \}$$
Proposition 3.2. Let \( w = vu \in W^J \) be a parabolic decomposition with respect to \( K \), so \( v \in W^K \), \( u \in W^J_K \). The following are equivalent:

(a) \( w = vu \) is a BP decomposition with respect to \((J, K)\).
(b) The multiplication map
\[
([e, v] \cap W^K) \times ([e, u] \cap W^J_K) \to [e, w] \cap W^J
\]
is surjective.
(c) The element \( u \) is the maximal element of \([e, w] \cap W^J_K\).
(d) \( S(v) \cap K \subseteq D^J_L(u) \).

Furthermore, if \( W_J \) is a finite Coxeter group and \( u' \) the maximal element of coset \( uW_J \), then the following are equivalent to parts (a)-(d).

(e) \( S(v) \cap K \subseteq D^L(u') \).
(f) The element \( u' \) has reduced decomposition \( u_0u_1 \), where \( u_0 \) is the maximal element of \( W_{S(v) \cap K} \).
(g) \( w = vu' \) is a BP decomposition with respect to \( K \).

Proof. Note that the multiplication map in part (b) is always injective. Hence part (b) is equivalent to part (c). The multiplication map is also length preserving, so part (b) is equivalent to part (a).

To show that parts (c) and (d) are equivalent, suppose \( u \) is the maximal element of \([e, w] \cap W^J_K\). If \( s \in S(v) \cap K \) then \( su \in W_K \) and hence \( suW_J \leq_J uW_J \). For the converse, suppose \( \bar{u} \) is the maximal element of \([e, w] \cap W^J_K\), so \( u \leq \bar{u} \leq w \). By Lemma 3.1 if \( u < \bar{u} \) then there is a simple reflection \( s \in S(v) \cap K \) such that \( u < su \leq \bar{u} \) and \( su \in W^J \). Hence \( S(v) \cap K \) is not a subset of \( D^J_L(u) \).

The equivalence of parts (e) and (f) is immediate. Let \( w' \) be the maximal element in the coset \( uW_J \). Then \( u \) is maximal in \([e, w] \cap W^J_K \) if and only if \( w' \) is maximal in \([e, w'] \cap W^J_K \). This implies part (f) is equivalent to part (c).

Finally, \( u' \) is the maximal element of \([e, w'] \cap W_K \) if and only if \( su' \leq u' \) for all \( s \in S(v) \cap K \). Hence part (g) is equivalent to part (e). This completes the proof.

In the case when \( J = \emptyset \), the equivalence of Proposition 3.2 parts (a)-(c) is proved in [BP05, Theorem 6.4], (a) \( \Rightarrow \) (d) in [OY10, Lemma 10], and (a) \( \Leftarrow \) (d) in [RS14, Lemma 2.2]. We now show that BP decompositions are associative, in the same way that parabolic decompositions are associative.

Lemma 3.3. Let \( I \subseteq J \subseteq K \subseteq S \) and \( w \in W^I \). Write \( w = xyz \) where \( x \in W^K \), \( y \in W^I_K \), and \( z \in W^J_I \). Then the following are equivalent:

(a) \( x(yz) \) is a BP decomposition with respect to \((I, K)\) and \( yz \) is a BP decomposition with respect to \((I, J)\).
(b) \( (xy)z \) is a BP decomposition with respect to \((I, J)\) and \( xy \) is a BP decomposition with respect to \((J, K)\).

Proof. The multiplication map
\[
([e, x] \cap W^K) \times ([e, y] \cap W^I_K) \times ([e, z] \cap W^J_I) \to [e, w] \cap W^I
\]
factors through the diagram

\[
([e,x] \cap W^K) \times ([e,y] \cap W_J^I) \times ([e,z] \cap W_J^I)
\]

\[
([e,xy] \cap W_J^I) \times ([e,z] \cap W_I^J)
\]

\[
([e,x] \cap W^K) \times ([e,yz] \cap W_I^K)
\]

Since all the maps above are injective, the lemma follows from part (b) of Proposition 3.2.

The last combinatorial property concerns Poincaré polynomials.

**Lemma 3.4.** Let \( W \) be a crystallographic Coxeter group and \( w \in W \). Let \( w = vu \in W^J \) be a parabolic decomposition with respect to \( K \). If \( w = vu \) is a BP decomposition with respect to \((J, K)\), then \( P^J_w(t) \) is palindromic if and only if \( P^K_v(t) \) and \( P^J_u(t) \) are palindromic.

**Proof.** Let \( P_1(t) \) and \( P_2(t) \) be polynomials of degree \( d_1 \) and \( d_2 \) respectively. Suppose \( P_j(t) = \sum_i c_i^j t^i \) has the property that \( c_i^j \leq c_{d_j-i}^j \) for all \( i \leq \lfloor d_j/2 \rfloor \) and \( j = 1, 2 \). Then it is easy to check that \( P_1 \cdot P_2 \) is palindromic if and only if \( P_1 \) and \( P_2 \) are palindromic. By [BE09, Theorem A], the relative Poincaré polynomials \( P^J_w \) of elements in crystallographic Coxeter groups have this property.

**Remark 3.5.** For arbitrary Coxeter groups, Lemma 3.4 holds in the case that \( J = \emptyset \). Indeed, it suffices to prove that if \( P^J_w(t) \) is palindromic, then \( P^J_u(t) \) is palindromic. This follows from [BP05, Lemma 6.6] and [Car94, Theorem B] along with the recent result in [EW14] that the Kazhdan-Lusztig polynomials of arbitrary Coxeter groups have nonnegative coefficients.

### 3.2. Geometric characterizations

In this section we give some geometric properties of BP decompositions, finishing with the proof of Theorem 2.3. We return to the assumption that \( W \) is the Weyl group of some Kac-Moody group \( G \), and hence is crystallographic.

For the remainder of the section, we fix \( J \subseteq K \subseteq S \) and the corresponding parabolic subgroups \( P_J \subseteq P_K \subseteq G \). For any \( g \in G \), let \([g] \in G/P_K \) denote the image of \( g \) under the projection \( G \to G/P_K \).

**Lemma 3.6.** Let \( w = vu \in W^J \) be a parabolic decomposition with respect to \( K \) and recall the projection \( \pi : X^J(w) \to X^K(v) \). Let \([b_0v_0] \) be a point of \( X^K(v) \), where \( b_0 \in B \) and \( v_0 \in [e,v] \cap W^K \). Then

\[
\pi^{-1}([b_0v_0]) = b_0v_0 \cup B u' P_J/P_I
\]

where the union is over \( u' \in W_K^J \) such that \( v_0 u' \leq w \).
Proof. Using the parabolic decomposition of $W_K$, we get that
\[ P_K = BW_KB = \bigcup_{u' \in W_K'} Bu'P_J. \]
So the fibre of $G \to G/P_K$ over $[b_0v_0]$ is
\[ b_0v_0P_K = \bigcup_{u' \in W_K'} b_0v_0Bu'P_J, \]
Since $\ell(v_0u') = \ell(v_0) + \ell(u')$, we have $b_0v_0Bu'P_J$ is a subset of $Bv_0u'P_J$. The image of the set $Bv_0u'P_J$ under the projection $G \to G/P_J$ lies outside of $X^J(w)$ unless $v_0u' \leq w$, in which case the image is contained in $X^J(w)$. \(\square\)

Lemma 3.6 yields the following intermediate criterion for BP decompositions.

**Proposition 3.7.** Let $w = vu \in W^J$ be a parabolic decomposition with respect to $K$. Then $w = vu$ is a BP decomposition with respect to $(J,K)$ if and only if the fibres of the projection $\pi : X^J(w) \to X^K(v)$ are equidimensional.

Proof. Suppose $w = vu$ is a BP decomposition and take $b_0 \in B$, $v_0 \in [e,v] \cap W^K$. Then the fibre $\pi^{-1}([b_0v_0]) = b_0v_0X^J(u)$, since if $u' \in [e,w] \cap W_K^I$ then $u' \leq u$ by Proposition 3.2 part (c).

Conversely, the fibre $\pi^{-1}([e]) = X^J(\bar{u})$, where $\bar{u}$ is the maximal element of $[e,w] \cap W_K^I$, while the fibre $\pi^{-1}([v]) = vX^J(u)$. Hence if $\pi : X^J(w) \to X^K(v)$ is equidimensional, then we must have $\ell(u) = \ell(\bar{u})$. But $u \leq \bar{u}$, so this implies $u = \bar{u}$. \(\square\)

To finish the proof of Theorem 2.3, we need two standard lemmas.

**Lemma 3.8.** Let $v \in W^K$ and $I = S(v)$. Let $G_I$ be the reductive subgroup of $P_I$, and let $P_I, I \cap K := G_I \cap P_K$ be the parabolic subgroup of $G_I$ generated by $I \cap K$. Finally, let $X_I^{I \cap K}(v) \subseteq G_I/P_I, I \cap K$ be the Schubert variety indexed by $v \in W_I^K$. Then the map $G_I/P_I, I \cap K \to G/P_K$ induces an isomorphism $X_I^{I \cap K}(v) \to X^K(v)$.

Proof. It suffices to show that the induced map $X_I^{I \cap K}(v) \to X^K(v)$ is surjective. Write $P_I = G_I N_I$ where $N_I$ is the unipotent subgroup of $P_I$ and let $B_I = G_I \cap B$ denote the Borel of $G_I$. If $v' \in W_I$, then $N_I$ is stable under conjugation by $v'^{-1}$. Write $B = B_I N_I$. Then for any $v' \in W_I$, the Schubert cell
\[ Bv'P_K/P_K = B_1 N_I v'P_K/P_K = B_1 (v'v'^{-1}) N_I v'P_K/P_K = B_1 v'P_K/P_K. \]
Since $v \in W_I^K \subseteq W_I$, we have that $X_I^{I \cap K}(v) \to X^K(v)$ is surjective. \(\square\)

**Lemma 3.9.** If $u \in W^J$, then $X^J(u)$ is closed under the action of $P_{D_L^J(u)}$, the parabolic subgroup generated by the left descent set $D_L^J(u)$ of $u$ relative to $J$.

Proof. Let $Z$ be the inverse image of $X^J(u)$ under the projection $G \to G/P_J$. Then
\[ Z = \bigcup_{u' \leq u} Bu'BW_JB = \bigcup_{u' \leq u} Bu'W_JB. \]
If $s \in D_L^J(u)$ and $u'W_J \leq_J uW_J$, then $su'W_J \leq_J uW_J$. Thus
\[ sBu'W_JB \subseteq BsW_JB \cup Bu'W_JB \subseteq Z. \]
So $Z$ is closed under $D_L^J(u)$, and therefore $X^J(u)$ is closed under $P_{D_L^J(u)}$. □

Proof of Theorem 2.3. If $\pi : X^J(w) \to X^K(v)$ is locally-trivial then the fibres of $\pi$ are equidimensional. Thus $w = vu$ is a BP decomposition with respect to $(J, K)$ by Proposition 3.7.

Conversely, suppose that $w = vu$ is a BP decomposition with respect to $(J, K)$, and let $I = S(v)$ as in Lemma 3.8. Recall from the proof of Proposition 3.7 that if $b_0 \in B$, $v_0 \in [e, v] \cap W^K$, then the fibre $\pi^{-1}([b_0v_0]) = b_0v_0X^J(u)$. Suppose $g \in G_I$ maps to $[b_0v_0]$ in $G/P_K$. Then we can write $g = b_1v_0p$, where $b_1 \in B_I$ and $p \in P_{I,I \cap K}$. By Proposition 3.2 part (d), $I \cap K \subseteq D_L^J(u)$. Hence by Lemma 3.9, $pX^J(u) = X^J(u)$, and we conclude that $gX^J(u) = b_1v_0X^J(u)$ is the fibre of $\pi$ over $[g] = [b_1v_0]$.

By [Kum02], Corollary 7.4.15 and Exercise 7.4.5 the projection $G_I \to G_I/P_{I,I \cap K}$ is locally trivial, and thus has local sections. Given $x \in X^K(v)$, there is a Zariski open neighbourhood $U_x \subseteq X^K(v)$ of $x$ with a local section $s : U_x \to G_I \subseteq G$ of the projection $G \to G/P_K$. Let $m : U_x \times X^J(u) \to G/P_J$ denote the multiplication map $(u, y) \mapsto s(u) \cdot y$. The image of $m$ is contained in $X^J(w)$, and thus we get a commuting square

$$
\begin{array}{ccc}
U_x \times X^J(u) & \xrightarrow{m} & X^J(w) \\
\downarrow & & \downarrow \\
U_x & \xrightarrow{\pi} & X^K(v)
\end{array}
$$

in which the fibres of projection $U_x \times X^J(u) \to U_x$ are mapped bijectively onto the fibres of $\pi$. If $z \in \pi^{-1}(U_x)$ and $g = s(\pi(z))$, then $z \in gX^J(u)$. So $m$ maps bijectively onto $\pi^{-1}(U_x)$, and we can define an inverse $\pi^{-1}(U_x) \to U_x \times X^J(u)$ by $z \mapsto (\pi(z), g^{-1}z)$ where $g = s(\pi(z))$. We conclude that $m$ is an isomorphism, and ultimately that $\pi : X^J(w) \to X^K(v)$ is locally trivial.

Now the projection $U_x \times X^J(u) \to U_x$ is smooth if and only if $X^J(u)$ is smooth, and thus the projection $\pi : X^J(w) \to X^K(v)$ is smooth if and only if $X^J(u)$ is smooth. In particular, if $X^J(u)$ and $X^K(v)$ are both smooth, then $X^J(w)$ is smooth. Conversely, if $X^J(w)$ is smooth then the product $U_x \times X^J(u)$ must be smooth whenever the projection $G_I \to X^K(v)$ has a local section over $U$. Looking at Zariski tangent spaces, we conclude that $\dim T_xX^K(v) + \dim T_yX^J(u) \leq \ell(w)$ for all $x \in X^K(v)$, $y \in X^J(u)$. Since $\ell(w) = \ell(u) + \ell(v)$, both $X^K(v)$ and $X^J(u)$ must be smooth.

The Schubert variety $X^J(w)$ is rationally smooth if and only if $X^J(u)$ and $X^K(v)$ are rationally smooth by Lemma 3.4. □

We finish the section by proving Corollary 2.4.

Proof of Corollary 2.4. For singular cohomology, the proof follows easily from the Leray-Hirsch theorem. For etale cohomology, we use the Leray-Serre spectral sequence

$$
E_2^{s,s} = H^s_{et}(X^K(v), Q_l) \Rightarrow H^{s+s}_{et}(X^J(w), Q_l)
$$

for the projection $\pi : X^J(w) \to X^K(v)$ (see, e.g. [Tam94]). Since $\pi$ is locally trivial, the sheaf $R^s\pi_*Q_l$ is locally constant, and by the proper base change theorem we see that it is in fact isomorphic to $H^s_{et}(X^J(u), Q_l)$. Since $H^s_{et}(X^K(v), Q_l)$
and $H^*_\text{et}(X^J(u), \mathbb{Q}_l)$ are concentrated in even dimensions, the Leray-Serre spectral sequence collapses at the $E_2$-term, and the spectral sequence converges to $H^*_\text{et}(X^K(v)) \otimes H^*_\text{et}(X^J(u))$ as an algebra. Using the action of $H^*_\text{et}(X^K(v))$ on $H^*_\text{et}(X^J(w))$, we can solve the lifting problem to get an isomorphism

$$H^*_\text{et}(X^K(v), \mathbb{Q}_l) \otimes H^*_\text{et}(X^J(u), \mathbb{Q}_l) \rightarrow H^*_\text{et}(X^J(w), \mathbb{Q}_l)$$

of $H^*_\text{et}(X^K(v))$-modules. \hfill $\square$

4. Rationally smooth Grassmannian Schubert varieties

In this section we define almost maximal elements of a Weyl group and prove Theorem 2.7. We take $G$ to be a simple Lie group of finite type, and hence the Weyl group $W$ is a finite Coxeter group. It is well known that simple Lie groups are classified into four classical families $A_n, B_n, C_n, D_n$ and exceptional types $E_6, E_7, E_8, F_4$ and $G_2$. We use Dynkin diagrams given in Section 6 to describe the relations between simple generators of the Weyl groups $W$.

Let $G^0$ be rationally smooth Schubert variety with $|S(w)| \geq 2$. Then there is a leaf $s \in S(w)$ of the Dynkin diagram of $W_{S(w)}$ such that either $w$ or $w^{-1}$ has a BP decomposition $wuv$ with respect to $J = S \setminus \{s\}$.

Furthermore, $s$ can be chosen so that $v$ is either the maximal length element in $W^J$, or one of the following holds:

(a) $W_{S(v)}$ is of type $B_n$ or $C_n$, with either

(1) $J = S \setminus \{s_1\}$, and $v = s_k s_{k+2} \ldots s_{n-1} s_1$, for some $1 < k \leq n$.

(2) $J = S \setminus \{s_n\}$ with $n \geq 2$, and $v = s_1 \ldots s_n$.

(b) $W_{S(v)}$ is of type $F_4$, with either

(1) $J = S \setminus \{s_1\}$ and $v = s_4 s_3 s_2 s_1$.

(2) $J = S \setminus \{s_4\}$ and $v = s_1 s_2 s_3 s_4$.

(c) $W_{S(v)}$ is of type $G_2$, and $v$ is one of the elements

$s_2 s_1, s_1 s_2 s_1, s_2 s_1 s_2 s_1, s_1 s_2, s_2 s_1 s_2, s_1 s_2 s_1 s_2$.

Note that the elements listed in parts (a)-(c) of Theorem 4.1 correspond to the elements listed in part (2) of Theorem 2.7 for which $s$ is a leaf of the Dynkin diagram. For the classical types, Theorem 4.1 is due to Billey [Bi98] Theorem 3.3 and Proposition 6.3]. The type $A$ case was also proved by Gasharov [Ga98] and Lascoux [Las98]. In the exceptional types, the result that $w$ or $w^{-1}$ has a BP decomposition with respect to a leaf is due to Billey-Postnikov [BP05]. The proof uses an exhaustive computer verification. Again in the exceptional types, if $s$ is a leaf then $W^\setminus \{s\}$ is small enough that the list of rationally smooth elements in $W^\setminus \{s\}$ can be easily determined by computer. In particular, Oh and Yoo have shown for type $E$ that only the maximal length elements of $W^\setminus \{s\}$ are rationally smooth when $s$ is a leaf [OY10]. In type $F_4$, there are two rationally smooth elements in addition to

---

1. For instance, in $E_8$ if $s = s_8$ then $|W^\setminus \{s\}| = 240$. If $s = s_3$ then $|W^\setminus \{s\}| = 17280$.

2. This also follows from the geometric results of [HM13] together with Peterson’s theorem that all rationally smooth elements in type $E$ are smooth.
the maximal elements. Finally, in type $G_2$ every Grassmannian element is rationally smooth.

Note that the condition that either $w$ or $w^{-1}$ has a BP decomposition in Theorem 4.1 can be rephrased as $w$ having “left” or “right” sided BP decompositions. For any $J \subseteq S$, let $J^w \simeq W_J \backslash W$ denote the set of minimal length left sided coset representatives. Any $w \in W$ has unique left sided parabolic decomposition $w = wv$ with respect to $J$ where $u \in W_J$ and $v \in J^w$. We say a left sided parabolic decomposition $w = wv$ is a left sided $BP$ decomposition with respect to $J$ if

$$P_w(t) = P_u(t) \cdot J_P_v(t)$$

where

$$J_P_v(t) := \sum_{x \in [e,v] \cap J^w} t^{(x)}.$$ 

By a right sided parabolic or BP decomposition $w = vu$ with respect to $J$, we simply mean a usual parabolic or BP decomposition where $v \in W^J$ and $u \in W_J$. With this terminology, $w = vu$ is a right sided BP decomposition if and only if $w^{-1} = u^{-1}v^{-1}$ is a left sided BP decomposition. The combinatorial characterizations given in Proposition 3.2 have left sided BP decomposition analogues. In particular, for Proposition 3.2 part (e) we let $u'$ be the maximal element of the coset $(W_J)u$ and replace the left descents $D_L(u')$ with right descents $D_R(u')$.

The elements listed in Theorem 4.1 parts (a)-(c) share an important property we term “almost maximality”. Recall that an element $w \in W$ is the maximal element in $W_S$ if and only if $D_L(w) = S(w)$, or equivalently if $D_R(w) = S(w)$. An element $w \in W^J$ is the maximal element in $W_S^J$ if and only if the longest element of $wW_S$ is in fact the longest element of $W_S$. Based on these properties of maximal elements, we make the following definition.

**Definition 4.2.** Given $w \in W^J$, let $w'$ be the longest element in $wW_S$. We say that $w$ is almost-maximal in $W_S^J$ if all of the following are true.

(a) There are elements $s, t \in S(w)$ (not necessarily distinct) such that

$$D_R(w') = S(w') \setminus \{s\} \quad \text{and} \quad D_L(w') = S(w') \setminus \{t\}.$$ 

(b) If $w' = vu$ is the right sided parabolic decomposition with respect to $D_R(w')$, then $S(v) = S(w')$.

(c) If $w' = uv$ is the left sided parabolic decomposition with respect to $D_L(w')$, then $S(u) = S(w')$.

Similarly, if $w \in J^w$, we say $w$ is almost-maximal in $W_S$ if parts (a)-(c) are true with $w'$ the longest element in the coset $(W_S)w$.

Note that an almost-maximal element is not maximal in $W_S^J$ by definition. If $W_S^J$ is clear from context, we will omit it. By the following lemma, the most interesting case is when $J = S \setminus \{s\}$ for some $s \in S$.

**Lemma 4.3.** Let $w \in W^J$ and assume $J \subseteq S = S(w)$. If $w$ is almost maximal, then there exists $s \notin J$ and parabolic decomposition $w = vu$ with respect to $K = S \setminus \{s\}$...
such that \( v \) is an almost-maximal element of \( W^K \) and \( u \) is the maximal element of \( W^J \).

**Proof.** Assume that \( w \in W^J \) is almost-maximal. Let \( w' \) be the longest element of \( wW_J \). Let \( s \) be the unique element of \( S \setminus D_R(w') \) and consider the parabolic decomposition \( w' = vu' \) with respect to \( K = S \setminus \{s\} = D_R(w') \). Note that \( S(v) = S(w') = S \) and \( u' \) is maximal in \( W_K \) since \( J \subseteq K \). Write \( u' = uu_0 \) where \( u \) and \( u_0 \) are maximal in \( W^K \) and \( W_J \) respectively. Then

\[
w' = uu_0 = v\left(uu_0\right)_{wu'}
\]

which implies that \( w = vu \). Since \( w \) is almost-maximal and \( w' \) is the maximal element of \( vW_K \), we have that \( v \) is almost maximal. \( \square \)

### 4.1. Proof of Theorem 2.7

We begin with the following proposition.

**Proposition 4.4.** Let \( J = S \setminus \{s\} \) for some \( s \) and suppose \( v \in W^J \) such that \( S(v) = S \) and \( X^J(v) \) is rationally smooth. Then the following are equivalent.

1. \( v \) is not maximal in \( W^J \)
2. \( v \) is almost-maximal in \( W^J \)
3. \( v \) appears on the list in Theorem 2.7 part (2).

**Remark 4.5.** If \( v \in W^J \) is almost maximal, then \( X^J(v) \) is not necessarily rationally smooth. For example, let \( W \) be of type \( D_4 \) with \( J = S \setminus \{s_4\} \) according to the Dynkin diagram in Section 6. Then \( v = s_1s_2s_3s_4 \in W^J \) is almost maximal but \( X^J(v) \) is not rationally smooth.

Most of Theorem 2.7 follows from Proposition 4.4, leaving only the determination of which Schubert varieties listed in Theorem 2.7 part (2) are smooth. Before we prove Proposition 4.4, we first analyze the elements arising in parts (a) and (b) of Theorem 4.1.

**Lemma 4.6.** Let \( W = B_n \) or \( C_n \) and \( J = \{s_2, \ldots, s_n\} \). Let \( v = s_k \cdots s_{n-1} s_n s_{n-1} \cdots s_1 \) where \( 1 < k \leq n \) and \( w' \) be the longest element of \( vW_J \). Then the following are true:

1. \( D_L(w') = S \setminus \{s_{k-1}\} \).
2. The minimal length representative of \( WD_L(w')w' \) is \( s_{k-1} \cdots s_1 u_{n,k}^{-1} \), where \( u_{n,k} \) is the maximal length element of \( W^J \setminus \{s_k\} \).

**Proof.** Partition \( S \) into \( S_1 = \{s_1, \ldots, s_{k-1}\} \) and \( S_2 = \{s_k, \ldots, s_n\} \). If \( w_0 \) denotes the maximal element of \( W_J \) then \( w' = vu_0 \). Since the elements of \( W_{S_1} \) and \( W_{S_2 \setminus \{s_k\}} \) commute, we can write

\[
w_0 = u_0 u_1 u_{n,k}^{-1},
\]

where \( u_0 \) is maximal in \( W_{S_1 \cap J} \) and \( u_1 \) is maximal in \( W_{S_2 \setminus \{s_k\}} \). Then

\[
w' = (s_k \cdots s_n \cdots s_1)(u_0 u_1 u_{n,k}^{-1})
\]

\[
= (s_k \cdots s_n \cdots s_k u_1)(s_{k-1} \cdots s_1 u_0)u_{n,k}^{-1}.
\]
Since \((s_k \cdots s_n \cdots s_k)\) is maximal in \(W_{S_2 \setminus \{s_k\}}\), we have that \((s_k \cdots s_n \cdots s_k u_1)\) is a maximal element in \(W_{S_2}\). In particular, \(S_2 \subseteq D_L(w')\). Similarly \((s_{k-1} \cdots s_1)\) is maximal in \(W_{S_1 \setminus \{s_1\}}\), and hence \((s_{k-1} \cdots s_1 u_0)\) is maximal in \(W_{S_1}\). Consequently
\[
s_{k-1} \cdots s_1 u_0 = u_2 s_{k-1} \cdots s_1,
\]
where \(u_2\) is the maximal element in \(W_{S_1 \setminus \{s_{k-1}\}}\). Now we have
\[
w' = (s_k \cdots s_n \cdots s_k u_1) \cdot (u_2 s_{k-1} \cdots s_1 u_{n,k}^{-1})
\]
\[
= (u_2 s_k \cdots s_n \cdots s_k u_1) \cdot (s_{k-1} \cdots s_1 u_{n,k}^{-1}).
\]
Thus \(S_1 \setminus \{s_{k-1}\} \subseteq D_L(w')\) and hence \(S \setminus \{s_{k-1}\} \subseteq D_L(w')\). Since \(w'\) is not maximal in \(W\), the element \(s_{k-1} \notin D_L(w')\). This proves part (1), and part (2) follows from the fact that \((u_2 s_k \cdots s_n \cdots s_k u_1)\) is maximal in \(W_{D_L(w')}\).

Note that if \(k = 2\) in Lemma 4.6, then \(D_L(w') = J\), and the minimal length representative of \(W_J w'\) is \(s_1 u_{n,k}^{-1} = w^{-1}\).

**Lemma 4.7.** Let \(W = B_n\) or \(C_n\) and \(J = \{s_1, \ldots, s_{n-1}\}\) where \(n \geq 2\). Let \(v = s_1 \cdots s_n\) and \(w'\) be the longest element of \(vW_J\). Then the following are true:

1. \(D_L(w') = J\)
2. The minimal length representative of \(W_{D_L(w')} w'\) is \(s_n \cdots s_1\).

**Proof.** If \(w_0\) denotes the longest element of \(W_J\), then \(w' = vw_0\), and we can write
\[
w_0 = u_0 s_{n-1} \cdots s_1,
\]
where \(u_0\) is the longest element of \(W_J \setminus \{s_{n-1}\}\). Hence
\[
w' = (s_1 \cdots s_{n-1} s_n)(u_0 s_{n-1} \cdots s_1) = (s_1 \cdots s_{n-1} u_0)(s_n s_{n-1} \cdots s_1).
\]
The lemma now follows from the fact that \(s_1 \cdots s_{n-1} u_0 = w_0\). \(\Box\)

**Lemma 4.8.** Let \(W = F_4\) and \(J = \{s_2, s_3, s_4\}\). Let \(v = s_4 s_3 s_2 s_1\) and \(w'\) be the longest element of \(vW_J\). Then the following are true:

1. \(D_L(w') = \{s_1, s_3, s_4\}\)
2. The minimal length representative of \(W_{D_L(w')} w'\) is \(s_2 s_1 s_3 s_2 s_4 s_3 s_4 s_2 s_3\).

**Proof.** Computation in \(F_4\) shows that
\[
w' = (s_4 s_3 s_2 s_1)(s_4 s_3 s_2 s_3 s_4 s_2 s_3 s_2)
\]
\[
= (s_1 s_3 s_4 s_3)(s_2 s_1 s_3 s_2 s_4 s_3 s_4 s_2).
\]
\(\Box\)

Note that Lemma 4.8 also applies to \(v = s_1 s_3 s_2 s_4\) in \(F_4\), since \(F_4\) has an automorphism sending the simple generator \(s_k \mapsto s_{n-k}\) for \(k \leq 4\). (This automorphism is not a diagram automorphism, and hence is not defined on the root system, but it is defined for the Coxeter group).

**Lemma 4.9.** Let \(v\) be almost-maximal in \(W_{S(v) \cap J}\) and let \(w'\) be the longest element in \(vW_{S(v) \cap J}\). Let \(w' = u_1 v_1\) be the left sided parabolic decomposition of \(w'\) with respect to \(J' := D_L(w')\). Then the following are true:
(1) $v_1^{-1}$ is almost-maximal in $W_{S(v_1)}$

(2) $X^J(v)$ is rationally smooth if and only $X^J(v_1^{-1})$ is rationally smooth.

Proof. Part (1) of the lemma is immediate from Definition 4.2 of almost-maximal. For the second part, let $u_0$ be the maximal element of $W_{S(v)} \cap J$, so $w' = vu_0$. By Proposition 3.2, $w' = vu_0$ is a BP decomposition and hence by Lemma 3.3, we have $X^J(v)$ is rationally smooth if and only if $X^0(w')$ is rationally smooth. But

$$P_w(t) = P_{(w')^{-1}}(t),$$

and so $X^0(w')$ is rationally smooth if and only if $X^0((w')^{-1})$ is rationally smooth.

Finally, since $u_1$ is the maximal element of $W_J$, we have $w' = u_1v_1$ is a left-sided BP decomposition. Thus $X^0((w')^{-1})$ is rationally smooth if and only if $X^J(v_1^{-1})$ is rationally smooth.

Proof of Proposition 4.4. Clearly (ii) $\Rightarrow$ (i) in the proposition. We will show (iii) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). We start with the proof of (iii) $\Rightarrow$ (ii). Suppose $v \in W_J$ is an element listed in parts (a)-(c) of Theorem 4.1. If $v$ is of type $G_2$, then it is easy to see that $v$ is almost-maximal. For types $B, C$ and $F$, it follows from Lemmas 4.6, 4.7, and 4.8 that all elements listed in parts (a)-(c) are almost-maximal. As mentioned previously, the elements listed in parts (a)-(c) of Theorem 4.1 are precisely the elements listed in the table of Theorem 2.7 for which $s$ is a leaf of the Dynkin diagram. By the second parts of Lemmas 4.6, 4.7, and 4.8 if $v'$ is any other element listed in Theorem 2.7, then there is a leaf $s$ of the Dynkin diagram, and a rationally smooth element $v \in W_{S \setminus \{s\}}$ listed in Theorem 4.1 such that the longest element $w'$ in $vW_{S(v) \cap J}$ has left-sided parabolic decomposition $w' = u_1v_1$, where $v' = v_1^{-1}$. Lemma 4.9 now implies that all the elements listed in Theorem 2.7 are almost-maximal and the corresponding Schubert varieties are rationally smooth. This proves (iii) $\Rightarrow$ (ii) of the proposition.

Now we prove (i) $\Rightarrow$ (iii). Suppose $v \in W_J$ is not maximal and let $w'$ be the longest element of $vW_J$. Theorem 4.1 implies there exists a leaf $s' \in S$ such that either $w'$ or $(w')^{-1}$ has a BP decomposition $v'u'$ such that $v'$ appears on listed in parts (a)-(c) of Theorem 4.1. If $w' = v'u'$, then the fact that $D_R(w') = J$ and $w'$ is not maximal implies that $s' = s$ and hence, $w' = u$ and $v' = v$. Now suppose that $(w')^{-1} = v'u'$ and let $J' = S \setminus \{s'\}$. By Proposition 3.2, part (e) we have $S(v') \cap J' = J' \subseteq D_L(u')$. Since $w'$ is not maximal, we must have $J' = D_L(u')$ and that $u'$ is the longest element of $W'_{J'}$. Thus $w'$ is almost-maximal. Lemma 4.9 together with Lemmas 4.6, 4.7, and 4.8 imply that $v$ is an element listed in Theorem 2.7.

The equivalence of Proposition 4.4 parts (ii) and (iii) gives the following rephrasing Theorem 4.1.

Corollary 4.10. Let $X^0(w)$ be rationally smooth, where $|S(w)| \geq 2$. Then there is a leaf $s \in S(w)$ of the Dynkin diagram of $W_{S(w)}$ such that either $w$ or $w^{-1}$ has a BP decomposition $vu$ with respect to $J = S \setminus \{s\}$. Furthermore, $s$ can be chosen so that $v$ is either maximal or almost-maximal in $W_{S(v) \cap J}$. 

Corollary 4.10 plays an important role in the proof of Theorem 2.5 in the next section.

We finish the proof of Theorem 2.7 by determining which Schubert varieties listed in Theorem 2.7, part (2) are smooth.

Lemma 4.11. Let $v \in W^{\langle v \rangle \cap J}$ and let $w'$ be the longest element in $vW^{\langle v \rangle \cap J}$. Let $w' = u_1v_1$ be the left sided parabolic decomposition of $w'$ with respect to $J' := D_L(w')$. Then $X^J(v)$ is smooth if and only if $X^{J'}(v_1^{-1})$ is smooth.

Proof. Let $u_0$ be maximal in $W_{\langle v \rangle} \cap J$, so that $w' = vu_0$, and let

$$Z = \bigcup_{x \leq w'} BxB$$

be the inverse image of $X^J(v)$ in $G$. Then $Z$ is a principal $P_J$-bundle over $X^J(w)$, so $Z$ is smooth if and only if $X^J(w)$ is smooth. But $Z$ is isomorphic to the inverse image

$$\bigcup_{x \leq (w')^{-1}} BxB$$

of $X^{J'}(v_1^{-1})$ in $G$, so the lemma follows. □

By Lemma 4.11 and Lemmas 4.6, 4.7, and 4.8 it suffices to determine when $X^J(w)$ is smooth for $J = S \setminus \{s\}$, $s$ a leaf. Note that Kumar has given a general criterion for smoothness of Schubert varieties, and by this criterion the smoothness of Schubert varieties is independent of characteristic [Kum96].

If $W$ is of type $B_n$ or $C_n$, then the singular locus of $X^J(w)$ when $J = S \setminus \{s\}$, $s$ a leaf, is well-known (see pages 138-142 of [BL00]). In type $G_2$, the smooth Schubert varieties are also well-known (see the exercise on page 464 of [Kum02]). Finally, for type $F_4$ we use a computer program to apply Kumar’s criterion to the Schubert varieties in question.

Remark 4.12. Note that we only use prior results on smoothness for the rationally smooth almost-maximal elements, which do not occur in the simply-laced case. Hence the proof of Peterson’s theorem (Corollary 2.10) depends only on Theorem 4.1.

5. Existence of Billey-Postnikov decompositions

In this section, we prove Theorem 2.5 and Corollary 2.6. The main theorem, stated below, is an extension of Theorem 4.1, wherein we show that if $X^0(w)$ is rationally smooth then $w$ has a right-sided Grassmannian BP decomposition. In Section 5.1 we examine affine type $\tilde{A}$, proving Theorem 2.11 and Corollary 2.13.

Theorem 5.1. Let $W$ be a finite Weyl group. Suppose $X^0(w)$ is rationally smooth for some $w \in W$. Then one of the following is true:

(a) The element $w$ is the maximal element of $W_{\langle v \rangle}$.
(b) There exists $s \in S(w) \setminus D_R(w)$ such that $w$ has a right sided BP decomposition $w = vu$ with respect to $J = S \setminus \{s\}$, where $v$ is either the maximal or an almost-maximal element of $W_{\langle v \rangle \cap J}$.
Note that if \( w \) is almost-maximal (relative to \( J = \emptyset \)) then \( w \) satisfies part (b) of Theorem 5.1 by definition. The requirement that \( s \) not belong to \( D_R(w) \) in part (2) of the theorem is critical both for the inductive proof of the theorem, and for showing that BP decompositions exist in the relative case.

We introduce some terminology for subsets of the generating set \( S \). The Dynkin diagrams of the finite type Weyl groups are listed for reference in Section 6.

**Definition 5.2.** A subset \( T \subseteq S \) is connected if the Dynkin diagram of \( W_T \) is connected. The connected components of a subset \( T \subseteq S \) are the maximal connected subsets of \( T \).

**Definition 5.3.** We say that \( s \) and \( t \) in \( S \) are adjacent if \( s \) and \( t \) are adjacent in the Dynkin diagram. We also say that \( s \) is adjacent to a subset \( T \) if \( s \) is adjacent to some \( t \in T \), and that two subsets \( T_1, T_2 \) are adjacent if there is some element of \( T_1 \) adjacent to \( T_2 \).

We use this terminology for the following lemma:

**Lemma 5.4.** Let \( w = vu \) be a right sided parabolic decomposition with respect to some \( J \subseteq S \). If \( s \in S \setminus S(v) \) is adjacent to \( S(v) \), then \( s \notin D_L(w) \).

Similarly, let \( w = vw \) be a left sided parabolic decomposition with respect to some \( J \subseteq S \). If \( s \in S \setminus S(v) \) is adjacent to \( S(v) \), then \( s \notin D_R(w) \).

**Proof.** Clearly the second statement of the lemma follows from the first by considering \( w^{-1} \). We proceed by induction on the length of \( v \). If \( \ell(v) = 1 \) then the proof is obvious. Otherwise take a reduced decomposition \( v = v_1tv_0 \), where \( s \) is adjacent to \( t \), but not adjacent to \( S(v_0) \). (Note that \( v_0 \) could be equal to the identity.) Then \( tv_0 \in W_J \) and since \( s \notin S(v) \), we have \( s \in D_L(w) \) only if \( s \in D_L(u) \). Assume that \( s \in D_L(u) \) and write

\[
w = (v_1tv_0)(u) = (v_1tsv_0)(su).
\]

Once again, we have \( s \in D_L(w) \) only if \( s \in D_L(tsv_0(su)) \). But since \( t \) and \( s \) are adjacent, we have \( s \in D_L(tsv_0(su)) \) only if \( t \in D_L(v_0(su)) \). If \( t \notin J \), then \( t \notin S(v_0(su)) \), and we are done. Otherwise, if \( t \in J \), then \( t \) must be adjacent to \( S(v_0) \) since \( tv_0 \in W_J \). But \( v_0 \in W_J \), so by induction, we get \( t \notin D_L(v_0(su)) \). \( \square \)

We now proceed to the proof of Theorem 5.1. The proof uses multiple reduced decompositions for the same element, so we have included certain schematic diagrams to aid the reader in keeping track of the reduced decompositions under consideration. For example, if \( w = w_1w_2w_3 \) is a reduced decomposition with \( v = w_1w_2 \) and \( u = w_2w_3 \), then we diagram this relation by

```
  w
 /\  \\
 v /  \\
/  \\
 w1 \  \\
 \ \\
\ w1 w2 w3
```

**Proof of Theorem 5.1.** We proceed by induction on \( |S(w)| \). It is easy to see that the theorem is true if \( |S(w)| = 1, 2 \). Hence we can assume that \( |S(w)| > 2 \). We can also
assume without loss of generality that the Dynkin diagram of \( W_{S(w)} \) is connected. Since \( w \) is rationally smooth, we can apply Corollary \( 4.10 \) to get that \( w \) has either a right sided BP decomposition \( w = vu \), or a left sided BP decomposition \( w = uv \), with respect to \( J = S \setminus \{ s \} \) where \( s \in S \) is a leaf of the Dynkin diagram of \( W_{S(w)} \). Note that in both cases, \( X^\theta(u) \) is rationally smooth by Lemma \( 3.4 \). We now consider four cases, depending on whether we get a right or left BP decomposition from Corollary \( 4.10 \) and depending on whether or not \( u \) is maximal in \( W_{S(w)} \).

Case 1: \( w \) has a right sided BP decomposition \( w = vu \) as in Corollary \( 4.10 \) where \( u \) is maximal in \( W_{S(w)} \). If \( s \in D_R(w) \), then \( w \) is the maximal element of \( W \). Otherwise, if \( s \notin D_R(w) \), then the decomposition \( w = vu \) satisfies condition (b) in Theorem \( 5.1 \), since \( v \) is maximal or almost-maximal in \( W_{S(v) \cap J} \) by Corollary \( 4.10 \).

Case 2: \( w \) has a left sided BP decomposition \( w = uv \) as in Corollary \( 4.10 \) where \( u \) is maximal in \( W_{S(u)} \), and \( v \) is either maximal or almost-maximal in \( W_{S(v) \cap J} \) by Corollary \( 4.10 \). Since \( S(u) = S(v) \cap J \), then \( w \) is either maximal or almost-maximal respectively, in which case we are done.

Since \( S(w) \) is connected, if \( S(v) \cap J \subseteq S(v) \) then we can choose \( s' \in S(u) \setminus S(v) \) adjacent to \( S(v) \). Since \( u \) is maximal in \( W_{S(u)} \), the parabolic decomposition \( u = v'u' \) with respect to \( J' := S(u) \setminus \{ s' \} \) is a BP decomposition with \( v' \) is maximal in \( W_{S(u)} \). Moreover, \( s' \notin D_R(w) \) by Lemma \( 5.4 \). Hence the decomposition \( w = v'(u'v) \) satisfies condition (b) in Theorem \( 5.1 \).

Case 3: \( w \) has a left sided BP decomposition \( w = uv \) as in Corollary \( 4.10 \) where \( u \) is not maximal in \( W_{S(u)} \). Then by induction, we have a right BP decomposition \( u = v'u' \) with respect to \( J' := S(u) \setminus \{ s' \} \) for some \( s' \in S(u) \), satisfying the conditions of Theorem \( 5.1 \). Since \( s' \notin D_R(u) \), we must have \( s' \notin S(v) \setminus \{ s \} \subseteq D_R(u) \), so \( w = v'(u'v) \) is a parabolic decomposition, and \( s' \notin D_R(w) \). But

\[
S(v') \cap J' \subseteq D_L(u') \subseteq D_L(u'v),
\]

and thus \( w = v'(u'v) \) is a BP decomposition with respect to \( S(w) \setminus \{ s' \} \), by Proposition \( 3.2 \) (d). Since \( v' \) is either maximal or almost-maximal by the inductive hypothesis, the decomposition \( w = v'(u'v) \) satisfies condition (b) of Theorem \( 5.1 \).

![Figure 1](attachment:image.png)

Figure 1: \( w = uv \) in Cases 2 and 3.

Case 4: \( w \) has a right BP decomposition \( w = vu \) as in Corollary \( 4.10 \) where \( u \) is not maximal in \( W_{S(w)} \). Note that if \( s \notin D_R(w) \) then condition (b) of Theorem \( 5.1 \) is satisfied immediately, and we would be done. We consider several subcases where we either prove \( s \notin D_R(w) \) or we find \( s' \notin D_R(w) \) that satisfies condition (b) of Theorem \( 5.1 \).
First, suppose that $v$ is almost maximal in $W_{S(v)\cap J}$, then conclude that pairwise commute with the elements of $S$.

Let $w = u_0 u_1$, where $u_0$ is the maximal element in $W_{S(v)\cap J}$. Then $S(v) \cap J \subseteq D_R(vu_0) \subseteq S(v)$, implying that $s \notin D_R(vu_0)$. It follows that $s \notin D_R(w)$.

![Figure 2: $w = vu$ with $u$ not maximal and $v$ almost maximal.](image)

Now assume $v$ is maximal in $W_{S(v)\cap J}$, and apply induction to get a BP decomposition $u = v'u'$ with respect to $S(u) \setminus \{s\}$, satisfying condition (b) of Theorem 5.1. By Proposition 3.2, $S(v') \setminus \{s\} \subseteq D_R(u)$, and hence if $t \in S(v) \setminus \{s\}$ is adjacent to $S(v')$, then $t \in S(v')$ by Lemma 5.4. Now since $v$ is Grassmannian, the set $S(v)$ is connected, and since $s$ is a leaf, the set $S(v) \setminus \{s\}$ is also connected. We conclude that if $S(v) \setminus \{s\}$ is adjacent to $S(v')$, then $S(v) \setminus \{s\} \subseteq S(v')$. Furthermore $s$ is adjacent to $S(v) \setminus \{s\}$, and hence $s$ is adjacent to $S(v')$. Conversely, if $S(v) \setminus \{s\}$ is non-empty and $s$ is adjacent to some element of $S(v')$, then this element must be contained in $S(v) \setminus \{s\}$, since $S(v)$ is connected and $s$ is adjacent to a unique element of $S(w)$. We conclude that either $S(v)$ is not adjacent to $S(v')$, or $S(v) \setminus \{s\} \subseteq S(v')$ with $s$ adjacent to $S(v')$.

In the former case, when $S(v)$ is not adjacent to $S(v')$, the elements of $S(v)$ pairwise commute with the elements of $S(v')$. Hence the decomposition $w = v'(vu')$ is a BP decomposition with respect to $S(w) \setminus \{s\}$. The element $v'$ is either maximal or almost-maximal in $W_{S(v')}^{S(v') \setminus \{s\}}$ by induction. Since $s' \notin D_R(u)$ and $s' \in J$, we conclude that $s' \notin D_R(w)$.

![Figure 3: $w = vu$ with $u$ not maximal, $v$ maximal, and $S(v), S(v')$ not adjacent.](image)

This leaves the case that $s$ is adjacent to $S(v')$ and $S(v) \setminus \{s\} \subseteq S(v')$. Take a reduced decomposition $u' = u'_0 u'_1$, where $u'_0$ is maximal in $W_{S(v') \setminus \{s'\}}$. Suppose first that $v'$ is maximal, so that $v'u'_0$ is maximal in $W_{S(v')}$. If $S(v) \setminus \{s\} = S(v')$, then $v'u'_0$ is the maximal element of $W_{S(v)}$, and hence there is a BP decomposition $vu'u'_0 = x_1 x_0$, where $x_1$ is the maximal element of $W_{S(v') \setminus \{s'\}}$ and $x_0$ is the maximal element of $W_{S(v) \setminus \{s'\}}$. Since $s' \notin D_R(u)$ and $s' \in J$, we have $s' \notin D_R(w)$, and thus $w = x_1(x_0 u'_1)$ is a BP decomposition with respect to $S(w) \setminus \{s'\}$ satisfying...
the condition of Theorem 5.1. If $S(v) \setminus \{s\}$ is a strict subset of $S(v')$, then find $t \in S(v')$ such that $t \notin S(v) \setminus \{s\}$. Consider the parabolic decomposition $v'u'_0 = y_0y_1$, where $y_0 \in W_{S(v') \setminus \{t\}}$ and $y_1 \in S(v') \setminus \{t\} W_{S(v')}$. Since $t \notin S(v)$ and $s$ is adjacent to $S(v') = S(y_1)$, we conclude from Lemma 5.4 that $s \notin D_R(vv'u'_0)$. Since $s \notin S(u'_1)$, we get that $s \notin D_R(w)$.

\[\begin{array}{c|c|c|c|c}
v & v' & u' & u'_0 & u'_1 \\
\hline \end{array}\]

Figure 4: $w = vv'$ with $u$ not maximal, $v$ maximal, and $S(v), S(v')$ adjacent.

We use a similar argument for the case that $v'$ is almost-maximal in $W_{S(v')}$. By definition $v'u'_0$ is almost maximal in $W_{S(v')}$, meaning that $D_L(v'u'_0) = S(v') \setminus \{t\}$ for some $t \in S(v')$. Since $s' \notin S(u'_1)$ and $u'_0$ is maximal in $W_{S(v')}$, the element $u'_1$ belongs to $S(v') W_{S(u')}. Thus t does not belong to $D_L(u)$, since otherwise $t \in D_L(v'u'_0)$, and we conclude that $t \notin S(v)$. Take the unique left sided BP decomposition $v'u'_0 = y_0y_1$, where $y_0$ is the maximal element in $W_{D_L(v')}$, and $y_1 \in S(v') \setminus \{t\} W_{S(v')}$. Then $S(y_1) = S(v')$ by the definition of almost-maximal, and since $s$ is adjacent to $S(y_1)$, we conclude from Lemma 5.4 that $s \notin D_R(vv'u'_0)$. Consequently $s \notin D_R(w)$.

Using Theorem 5.1 we can prove Theorem 2.5 for relative Schubert varieties.

Proof of Theorem 2.5. Suppose $X^J(w)$ is rationally smooth and that $|S(w) \setminus J| \geq 2$. Let $v_0$ be maximal in $W_J$, so $w' = wW_J$ is the longest element of $wW_J$. Then $w'$ is rationally smooth by Lemma 3.4 so we can apply Theorem 5.1. If $w'$ is maximal, then $w$ is the maximal element in $W_{S(w)}$, and hence by choosing any $s \in S(w)$ we get a Grassmannian BP decomposition of $w$ with respect to $K = S \setminus \{s\}$ as required.

If $w'$ is not maximal, then there exists $s \in S(w) \setminus D_R(w)$ such that $w'$ has a BP decomposition $w' = vv'$ with respect to $K = S \setminus \{s\}$. Since $s \notin D_R(w)$, $s$ must not be in $J$, so $u$ has parabolic decomposition $u = u_1w_0$ with respect to $W_J$, and $w$ has parabolic decomposition $w = vu_1$ with respect to $K$. This latter decomposition is a BP decomposition by Proposition 3.5.

5.1. BP decompositions in affine type $\tilde{A}$. In this section we prove an analogue of Theorem 5.1 for affine type $\tilde{A}$, and use it to prove Corollary 2.13. We let $W = \tilde{A}_n$ throughout, and say $w \in W$ avoids a pattern if it avoids the pattern as an affine permutation. We do not use pattern avoidance or the affine permutation structure directly in any of the proofs in this section, so we refer the reader to \cite{BCT2} for details on these properties. The following is the main result of \cite{BCT2}.
Theorem 5.5. ([BC12 Theorem 1.1, Corollary 1.2 and Remark 2.16]) The Schubert variety $X^\emptyset(w)$ is rationally smooth if and only if either $w$ avoids the permutation patterns 3412 and 4231, or $w$ is a twisted spiral permutation.

Moreover if $X^\emptyset(w)$ is smooth then $w$ avoids the permutation patterns 3412 and 4231. If $w$ is a twisted spiral permutation, then $X^\emptyset(w)$ is not smooth.

If $w$ is a twisted spiral permutation (defined in [BC12 Section 2.5]), then $w$ is in fact almost-maximal, and in particular has a right-sided Grassmannian BP decomposition $w = vu$ with respect to $J = S \setminus \{s_i\}$ for some $i$, where $S(v) = S$ and $u$ is the maximal element of $W_J$. The varieties $X^J(v)$ were introduced by Mitchell, and are called spiral Schubert varieties [Mit86] [BM10]. Reduced expressions for the elements $v$ can be found alongside the definition of twisted spiral permutations in [BC12].

When $w$ avoids the patterns 3412 and 4231, Billey and Crites prove the following analogue of Theorem 1.1.

Theorem 5.6. ([BC12]) If $w \in W$ avoids 3412 and 4231 then either $w$ or $w^{-1}$ has a Grassmannian BP decomposition, where both factors belong to a proper standard parabolic subgroup of $\tilde{A}_n$.

Theorem 5.6 is proved in [BC12]. In particular see the proof of Theorem 3.1, and the discussion before Corollary 7.1 in [BC12]. We use this result to prove the following analogue of Theorem 5.1.

Proposition 5.7. Suppose $X^\emptyset(w)$ is rationally smooth for some $w \in W$ with $S(w) = S$. Then there exists $s \in S(w) \setminus D_R(w)$ such that $w$ has a right-sided BP decomposition $w = vu$ with respect to $J = S \setminus \{s\}$. Furthermore, one of the following is true:

(a) $X^J(v)$ is a spiral Schubert variety, and $w$ is a twisted spiral permutation.

(b) $S(v)$ and $S(u)$ are proper subsets of $S$, and $v$ is the maximal element of $W_{S(u) \cap J}^{S(v) \cap J}$.

Proof. Since twisted spiral permutations are almost-maximal, the proposition holds if $w$ is a twisted spiral permutation. By Theorem 5.5 we can assume that $w$ avoids 3412 and 4231. The proof then follows along the same lines as the proof of Theorem 5.1 with Proposition 5.6 used in place of Corollary 4.10. In fact, the proof is somewhat simpler in this case: if $w = vu$ is a right BP decomposition with respect to $J$, and $S(v)$ is a proper subset of $S$, then $X^J(v)$ is a rationally smooth Schubert variety of finite type $A$, and $v$ is maximal in $W_{S(v) \cap J}^{S(v) \cap J}$. Thus we do not have to consider almost-maximal elements. However, there is one additional complication: the unique element $s$ of $S(v) \setminus J$ cannot be a leaf of the Dynkin diagram of $S$, since the Dynkin diagram is a cycle.

The only place the leaf assumption is used in the proof of Theorem 5.1 is in Case 4. Recall that in this case, $w$ has a right BP decomposition $w = vu$ (where now $S(v)$ and $S(u)$ are proper subsets of $S$), and $u$ is not maximal in $W_{S(u)}$. We then take a right BP decomposition $u = v'u'$ with respect to $S(u) \setminus \{s'\}$, where $s' \not\in D_R(u)$. If $S(v)$ is not adjacent to $S(v')$ then the proof proceeds as before. If $S(v)$ meets...
or is adjacent to \( S(v') \), then \( s \) must be adjacent to \( S(v') \). What we cannot assume is that \( S(v) \setminus \{ s \} \subseteq S(v') \) — it is possible that there is a connected component of \( S(v) \setminus \{ s \} \) which is not adjacent to \( S(v') \). However, this is not a serious difficulty: if \( S(v') \subseteq S(v) \), then \( w = (vv')u' \) is a BP decomposition, where \( vv' \) is maximal in \( W_{S(v)}^J \) for \( J = S \setminus \{ s', s'' \} \), and we can construct a BP decomposition of \( w \) with respect to \( S \setminus \{ s' \} \). On the other hand if \( S(v') \) is not contained in \( S(v) \), then we can show that \( s \not\in D_R(w) \) as before.

**Proof of Corollary 2.13.** If \( X^0(w) \) is smooth, then \( w \) avoids the permutation patterns 3412 and 4231 by Theorem 5.5. Suppose \( w \) avoids the permutation patterns 3412 and 4231. By Proposition 5.7, \( w \) has a Grassmannian BP decomposition \( w = vu \) with respect to some \( J = S \setminus \{ s \} \). Moreover, we have \( S(v) \subseteq S(w) \) and \( S(u) \subseteq S(w) \). Theorem 5.5 implies \( X^0(w) \) is rationally smooth, and hence \( X^J(v) \) and \( X^0(u) \) are rationally smooth by Lemma 3.4. But since \( S(v) \) and \( S(u) \) are proper in \( S \), \( X^J(v) \) and \( X^0(u) \) are isomorphic to Schubert varieties in finite type \( A \) and hence are smooth by Corollary 2.10. Theorem 2.3 now implies that \( X^0(w) \) is smooth. □

### 5.2. The Ryan-Wolper theorem.

In this section we use our results to prove Corollary 2.6 and Theorem 2.11 on iterated fibre bundle structures of smooth and rationally smooth Schubert varieties. We assume that \( X^J(w) \) is a rationally smooth Schubert variety of finite type or of affine type \( A_n \).

**Proof of Corollary 2.6 and Theorem 2.11.** First suppose that \( W \) is a finite Weyl group, and that \( X^J(w) \) is rationally smooth. By repeatedly applying Theorems 2.3 and 2.5 we can write

\[
w = v_m \cdots v_1,
\]

where \( v_i \in W_{J_i}^{J_i-1} \) for \( J_i := J \cup \bigcup_{j \leq i} S(v_j) \), and \( v_i(v_{i-1} \cdots v_1) \) is a Grassmannian BP decomposition. Let \( w_i := v_m \cdots v_{i+1} \in W_{J_i} \), so \( w_0 = w \) and \( w_m \) is the identity. By Lemma 3.3 \( w_i = w_{i+1}v_{i+1} \) is a BP decomposition with respect to \( (J_i, J_{i+1}) \), so the morphism

\[
X^J(w_i) \to X^{J_{i+1}}(w_{i+1})
\]

is a locally-trivial fibre bundle with fibre \( X^J(w_{i+1}) \). Hence the sequence

\[
X^J(w) = X^{J_0}(w_0) \to X^{J_1}(w_1) \to \cdots \to X^{J_{m-1}}(w_{m-1}) = X^{J_{m-1}}(v_m) \to \text{Spec } k
\]

meets the required conditions in Corollary 2.6. If \( X^J(w) \) is smooth, then by Theorem 2.3 all the fibres \( X^J(v_i) \) are smooth, and hence so are the morphisms.

Conversely, given a locally-trivial morphism \( X \to Y \) with fibre \( F \) such that the cohomology \( H^*(Y) \) of the base and the cohomology \( H^*(F) \) of the fibre are concentrated in even degrees, we can argue as in the proof of Corollary 2.4 that \( H^*(X) \) is isomorphic to \( H^*(Y) \otimes H^*(F) \). Thus \( H^*(X) \) will be concentrated in even degrees, and if \( H^*(Y) \) and \( H^*(F) \) satisfy Poincaré duality, then so does \( H^*(X) \). If there is a sequence

\[
X^J(w) = X_0 \to X_1 \to \cdots \to X_m = \text{Spec } k
\]
in which all the morphisms are locally trivial, and all the fibres are rationally smooth Schubert varieties, then $H^*(X_i)$ satisfies Poincaré duality for all $i = 0, \ldots, m$. In particular, $X^J(w)$ will be rationally smooth by the Carrell-Peterson theorem [Car94].

In the affine setting, the proof of the first part of Theorem 2.11 (concerning the existence of Grassmannian BP decompositions) is identical to the proof of Theorem 2.5 using Proposition 5.7 in place of Theorem 5.1. The analogue of the Ryan-Wolper theorem which forms the second part of Theorem 2.11 then follows immediately using the argument from the finite case above.

Only limited results are known about BP decompositions outside of finite type. In [RS14], the authors show that right-sided BP decompositions exist for rationally smooth Schubert varieties in the full flag varieties of a large class of non-finite Weyl groups. Hence the Ryan-Wolper theorem holds for Schubert varieties $X^\theta(w)$ in this class via the application of Theorem 2.3. However, with the exception of $\tilde{A}_3$, all of the Coxeter groups in this class are of indefinite type. It is an open problem to prove the existence of BP decompositions for this class when $J$ is non-empty.

Based on this evidence, the following conjecture seems plausible:

**Conjecture 5.8.** If $W$ is any Coxeter group, and $w$ belongs to $W^J$ with $P^J_{\theta}$ palindromic, then $w$ has a Grassmannian BP decomposition. As a result, the Ryan-Wolper theorem holds in any Kac-Moody flag variety.

### 6. Appendix: Dynkin diagrams

In this paper we use the following labellings for the Dynkin diagrams of the finite Weyl groups and the affine Weyl group $\tilde{A}_n$:

- **$A_n$:** 1 $\rightarrow$ 2 $\rightarrow$ $\cdots$ $\rightarrow$ $(n-1) \rightarrow n$ $\ n \geq 1$
- **$B_n$:** 1 $\rightarrow$ 2 $\rightarrow$ $\cdots$ $\rightarrow$ $(n-1) \rightarrow n$ $\ n \geq 2$
- **$C_n$:** 1 $\leftarrow$ 2 $\leftarrow$ $\cdots$ $\leftarrow$ $(n-1) \leftarrow n$ $\ n \geq 3$
- **$D_n$:** 1 $\rightarrow$ 3 $\rightarrow$ 4 $\rightarrow$ $\cdots$ $\rightarrow$ $(n-1) \rightarrow n$ $\ n \geq 4$
- **$E_n$:** 1 $\rightarrow$ 2 $\rightarrow$ 4 $\rightarrow$ $\cdots$ $\rightarrow$ $(n-1) \rightarrow n$ $\ n = 6, 7, 8$
Note that the Weyl groups of type $B_n$ and $C_n$ are the same. However, the root system, flag variety, and Schubert varieties are different. Since rational smoothness can be characterized in terms of the Weyl group, the Schubert variety of type $B$ indexed by $w$ is rationally smooth if and only if the Schubert variety of type $C$ indexed by $w$ is rationally smooth. The same is not true for smoothness.

7. Appendix: Examples

In this section we give several examples of the projection map between Schubert varieties

$$
\pi : X^J(w) \to X^K(v)
$$

as in equation (2). In all of the following examples, let $G = \text{SL}_4(k)$ with $X := G/B$ and $X^J := G/P_J$. Geometrically, we have

$$
X = \{V_\bullet = (V_1 \subset V_2 \subset V_3 \subset k^4) \mid \dim V_i = i\}
$$

Here we will denote Schubert varieties $X^\emptyset(w)$ in $X$ by simply $X(w)$. Let $E_\bullet$ denote the unique flag in $X$ stabilized by $B$.

7.1. Example 1. Let $w = s_1 s_2 s_3 s_2 s_1$. Then geometrically,

$$
X(w) = \{V_\bullet \mid \dim(V_2 \cap E_2) \geq 1\}.
$$

If $J = \{s_1, s_3\}$, then $X^J$ is the Grassmannian $\text{Gr}(2,4)$ and the map $\pi : X \to X^J$ is simply the projection $V_\bullet \mapsto V_2$. It is easy to see that $w = vu = (s_1 s_3 s_2)(s_3 s_1)$ is a BP decomposition with respect to $J$. In particular, the Schubert variety

$$
X^J(v) = \{V_2 \mid \dim(V_2 \cap E_2) \geq 1\}
$$

and the fiber over $V_2 \in X^J(v)$ in the projection map $X_w \mapsto X^J(v)$ is

$$
\pi^{-1}(V_2) = \{(V_1, V_3) \mid V_1 \subset V_2 \subset V_3\} \cong X(u) \cong \mathbb{P}^1 \times \mathbb{P}^1.
$$

In this case, the uniform fibre $X(u)$ is smooth. However since $X^J(v)$ is singular, we have that $X(w)$ is singular. In fact, in one-line permutation notation we have $w = (4231)$. 

\begin{align*}
F_4: & \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \\
G_2: & \quad 1 \rightarrow 2 \\
\tilde{A}_n: & \quad 1 \rightarrow 2 \rightarrow \cdots \rightarrow (n-1) \rightarrow n \quad n \geq 1
\end{align*}
If \( J = \{s_1, s_2\} \), then \( w = vu = (s_1s_2s_3)(s_2s_1) \) is not a BP decomposition. In this case, \( X^J \) is the Grassmannian \( \text{Gr}(3, 4) \) and the map \( \pi : X(w) \to X^J(v) \) is the projection \( V_0 \to V_3 \) where \( X^J(v) = X^J \). The fiber over \( V_3 \) is given by

\[
\pi^{-1}(V_3) = \{(V_1, V_2) \mid V_1 \subset V_2 \subset V_3 \text{ and } \dim(V_2 \cap E_2) \geq 1\}
\]

\[
\cong \begin{cases} 
X(s_2s_1) & \text{if } \dim(V_3 \cap E_2) = 1 \\
X(s_1s_2s_1) & \text{if } E_2 \subset V_3
\end{cases}
\]

Note that the fibers are not equidimensional.

7.2. Example 2. Let \( w = s_1s_2s_3s_1 \). Then

\[
X(w) = \{V_0 \mid V_2 \subset E_3\}.
\]

We consider all sets \( J \) where \( |J| = 2 \). Let \( \pi : X \to X^J \) denote the projection map in each case.

If \( J = \{s_1, s_2\} \), then \( w = vu = (s_1s_2s_3)(s_1) \) is not a BP decomposition. In this case, \( X^J(v) = \text{Gr}(3, 4) \) with \( \pi(V_0) = V_3 \). The fiber

\[
\pi^{-1}(V_3) = \{(V_1, V_2) \mid V_1 \subset V_2 \subseteq V_3 \cap E_3\}
\]

\[
\cong \begin{cases} 
X(s_1) & \text{if } \dim(V_3 \cap E_3) = 2 \\
X(s_1s_2s_1) & \text{if } V_3 = E_3
\end{cases}
\]

Here we have that both \( X^J(v) \) and \( X(w) \) are smooth, but the fibers are not equidimensional.

If \( J = \{s_1, s_3\} \), then \( w = vu = (s_1s_2)(s_3s_1) \) is a BP decomposition. In this case, \( X^J(v) = \{V_2 \mid V_2 \subset E_3\} \) with \( \pi(V_0) = V_2 \). The fiber

\[
\pi^{-1}(V_2) = \{(V_1, V_3) \mid V_1 \subset V_2 \subset V_3\} \cong X(u) \cong \mathbb{P}^1 \times \mathbb{P}^1.
\]

If \( J = \{s_2, s_3\} \), then \( w = vu = (s_2s_1)(s_2s_3) \) is a BP decomposition. In this case, \( X^J(v) = \{V_1 \mid V_1 \subset E_3\} \) with \( \pi(V_0) = V_1 \). The fiber

\[
\pi^{-1}(V_1) = \{(V_2, V_3) \mid V_1 \subset V_2 \subset V_3 \text{ and } V_2 \subset E_3\} \cong X(u).
\]

7.3. Example 3. Let \( w = s_1s_2s_2s_1 \). Then

\[
X(w) = \{V_0 \mid E_2 \subset V_3\}.
\]

We consider all sets \( J \) where \( |J| = 2 \). Let \( \pi : X \to X^J \) denote the projection map in each case.

If \( J = \{s_1, s_2\} \), then \( w = vu = (s_3)(s_1s_2s_1) \) is a BP decomposition. In this case, \( X^J(v) = \{V_3 \mid E_2 \subset V_3\} \) with \( \pi(V_0) = V_3 \). The fiber

\[
\pi^{-1}(V_3) = \{(V_1, V_2) \mid V_1 \subset V_2 \subset V_3\} \cong X(u),
\]

which is isomorphic to the variety of complete flags on \( k^3 \).
If \( J = \{ s_1, s_3 \} \), then \( w = vu = (s_1s_3s_2)(s_1) \) is not a BP decomposition. In this case, \( X^J(v) = \{ V_2 \mid \dim(V_2 \cap E_2) \geq 1 \} \) with \( \pi(V_2) = V_2 \). The fibre

\[
\pi^{-1}(V_2) = \{ (V_1, V_3) \mid V_1 \subseteq V_2 \subseteq V_3 \text{ and } E_2 \subseteq V_3 \}
\]

\[
\cong \begin{cases} 
X(s_1) & \text{if } \dim(V_2 \cap E_2) = 1 \\
X(s_1s_3) & \text{if } V_2 = E_2 
\end{cases}
\]

Note that \( X^J(v) \) is singular even though \( X(w) \) is smooth.

If \( J = \{ s_2, s_3 \} \), then \( w = vu = (s_3s_2s_1)(s_2) \) is not a BP decomposition. In this case, \( X^J(v) = \Gr(1, 4) \) with \( \pi(V_2) = V_1 \). The fibre

\[
\pi^{-1}(V_1) = \{ (V_2, V_3) \mid V_1 \subset V_2 \subseteq V_3 \text{ and } E_2 \subseteq V_3 \}
\]

\[
\cong \begin{cases} 
X(s_2) & \text{if } \dim(V_1 \cap E_2) = 0 \\
X(s_3s_2) & \text{if } V_1 \subset E_2 
\end{cases}
\]

Here we have that both \( X^J(v) \) and \( X(w) \) are smooth, but the fibres are not equidi-dimensional.

References

[BC12] Sara Billey and Andrew Crites, Pattern characterization of rationally smooth affine Schubert varieties of type A, J. Algebra 361 (2012), 107–133.
[BE09] Anders Björner and Torsten Ekedahl, On the shape of Bruhat intervals, Ann. of Math. (2) 170 (2009), no. 2, 799–817.
[Bil98] Sara C. Billey, Pattern avoidance and rational smoothness of Schubert varieties, Adv. Math. 139 (1998), no. 1, 141–156.
[BK04] M. Brion and S. Kumar, Frobenius splitting methods in geometry and representation theory, Progress in Mathematics, Birkhäuser Boston, 2004.
[BL00] Sara Billey and V. Lakshmibai, Singular loci of Schubert varieties, Progress in Mathematics, vol. 182, Birkhäuser Boston Inc., Boston, MA, 2000.
[BM10] Sara C. Billey and Stephen A. Mitchell, Smooth and palindromic Schubert varieties in affine Grassmannians, J. Algebraic Combin. 31 (2010), no. 2, 169–216.
[Bor91] A. Borel, Linear algebraic groups, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, 1991.
[BP99] Michel Brion and Patrick Polo, Generic singularities of certain Schubert varieties, Math. Z. 231 (1999), no. 2, 301–324.
[BP05] Sara Billey and Alexander Postnikov, Smoothness of Schubert varieties via patterns in root subsystems, Adv. in Appl. Math. 34 (2005), no. 3, 447–466.
[Car94] James B. Carrell, The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties, Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, pp. 53–61.
[CK03] James B. Carrell and Jochen Kuttler, Smooth points of T-stable varieties in G/B and the Peterson map, Inventiones Mathematicae 151 (2003), no. 2, 353–379.
[Dye] Matt Dyer, Rank two detection of singularities of Schubert varieties, preprint.
[EW14] Ben Elias and Goerdtle Williamson, The Hodge theory of Soergel bimodules, Annals of Mathematics 180 (2014), no. 3, 1089–1136.
[Gas98] Vesselin Gasharov, *Factoring the Poincaré polynomials for the Bruhat order on $S_n$*, J. Combin. Theory Ser. A 83 (1998), no. 1, 159–164.

[GR02] V. Gasharov and V. Reiner, *Cohomology of smooth Schubert varieties in partial flag manifolds*, J. London Math. Soc. (2) 66 (2002), no. 3, 550–562.

[HM13] Jaehyun Hong and Ngaiming Mok, *Characterization of smooth Schubert varieties in rational homogeneous manifolds of Picard number 1*, Journal of Algebraic Geometry 22 (2013), 333–362.

[JW12] Daniel Juteau and Geordie Williamson, *Kumar’s criterion modulo $p$*, to appear in Duke Math. Journal (2012), arXiv:1201.5341.

[Kum96] Shrawan Kumar, *The nil Hecke ring and singularity of Schubert varieties*, Invent. Math. 123 (1996), no. 3, 471–506.

[Kum02], *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[Las98] Alain Lascoux, *Ordonner le groupe symétrique: pourquoi utiliser l’algèbre de Iwahori-Hecke?*, Documenta Mathematica Extra Volume ICM III (1998), 355–364.

[LW90] V. Lakshmibai and J. Weyman, *Multiplicities of points on a Schubert variety in a minuscule $G/P$*, Adv. Math. 84 (1990), no. 2, 179–208.

[Mih07] Ion Alexandru Mihai, *Odd symplectic flag manifolds*, Transformation Groups 12 (2007), no. 3, 573–599.

[Mit86] Stephen A. Mitchell, *A filtration of the loops on SU($n$) by Schubert varieties*, Math. Z. 193 (1986), no. 3, 347–362.

[OY10] Suho Oh and Hwanchul Yoo, *Bruhat order, rationally smooth Schubert varieties, and hyperplane arrangements*, DMTCS Proceedings FPSAC (2010).

[Rob14] C. Robles, *Singular loci of cominuscule Schubert varieties*, Journal of Pure and Applied Algebra 218 (2014), no. 4, 745–759.

[RS14] Edward Richmond and William Slofstra, *Rationally smooth elements of Coxeter groups and triangle group avoidance*, Journal of Algebraic Combinatorics 39 (2014), no. 3, 659–681.

[Rya87] Kevin M. Ryan, *On Schubert varieties in the flag manifold of $SL(n, \mathbb{C})$*, Mathematische Annalen 276 (1987), no. 2, 205–224.

[Tam94] Günter Tamme, *Introduction to étale cohomology*, Universitext, Springer-Verlag, Berlin, 1994, Translated from the German by Manfred Kolster.

[vdH74] A van den Hombergh, *About the automorphisms of the bruhat-ordering in a coxeter group*, Indagationes Mathematicae (Proceedings) 77 (1974), no. 2, 125–131.

[Wol89] James S Wolpert, *A combinatorial approach to the singularities of Schubert varieties*, Advances in Mathematics 76 (1989), no. 2, 184–193.

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