Remarks on the Fuchs indices and the first integrals for nonlinear ordinary differential equations

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Abstract. The Painlevé analysis of nonlinear ordinary differential equations is used to
construct the first integrals. The connection between the Fuchs indices and the first integrals
of nonlinear differential equations is discussed. Some simple propositions are presented. Some
first integrals for nonlinear ordinary differential are found taking into account the values of the
Fuchs indexes.

1. Introduction

It is well known that many processes in physics, biology, economics and in other disciplines are
described by nonlinear differential equations and there is the important problem for finding the
analytical solutions of them. Recently many methods were suggested for finding exact solutions
of nonlinear differential equations.

At the present we have a long list of nonlinear partial differential equations for which the
Cauchy problem can be solved by the Inverse scattering transform. However many processes are
described by nonlinear nonintegrable partial differential equations and we need to have exact
solutions for these differential equations too.

In fact, the first attempt of finding exact solutions for nonlinear nonintegrable differential
equations were made in works [1, 2, 3] almost thirty years ago and later many methods have been
suggested to find exact solutions of various nonlinear differential equations (see, for example
papers [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). However we should note here that some of the
methods are not effective and some of them lead to many redundant exact solutions (see papers
[14, 15, 16, 17, 18]).

In this paper we study the connection between analytical properties of nonlinear ordinary
differential equations and first integrals for them. This approach allows us to look for the
general solutions for some nonlinear differential equations. Let us consider the nonlinear ordinary
differential equation in the general form

$$E(w, w_x, w_{xx}, \ldots, w_{n,x}, z) = 0, \quad w_{n,x} = \frac{d^n w}{dz^n}. \quad (1)$$

Let us assume that the general solution $w(z)$ has pole $p$. It means that $w(z)$ takes the form

$$w(z) = \frac{a_0}{(z - z_0)^p} \quad (2)$$
or has the expansion in the Laurent series in the form

$$w(z) = \frac{a_0}{(z - z_0)^p} + \ldots,$$

where $z_0$ is a movable singular point.

There are some simple prepositions.

**Preposition 1.** The derivative $w_{k,z}$ in $z$ of order $k$ has the order of pole $k + p$.

**Proof.** The proof follows from the following calculation

$$w_{k,z} = \frac{(-1)^p p (p - 1) \ldots (p - k) a_0}{(z - z_0)^{p+k}} + \ldots,$$

**Preposition 2.** The power $m$ of function $w(z)$ has the order of pole $m p$.

**Proof.** The proof follows from the formula

$$w^m = \frac{a_0^m}{(z - z_0)^{mp}} + \ldots,$$

2. The Fuchs indices and the first integrals for nonlinear ordinary differential equations

For nonlinear ordinary differential equations one can suggest some additional properties. To understand these properties let us consider the nonlinear ordinary differential equation in the form

$$E_k = w_k, z - bw^m = 0.$$  

**Preposition 3.** The order of pole for the general solution of equation (6) is determined by formula

$$p = \frac{k}{m - 1}.$$  

**Proof.** The proof follows from the formula

$$p + k = mp.$$  

**Preposition 4.** The derivative of $E_k$ in $z$ does not change the order of pole for the general solution of equation (6).

**Proof.** The differentiation of equation (6) leads to multiplying formula (8) on integer.

**Example 1.** Let us take the first order equation in the from

$$E_1 = w_z + w^2 = 0.$$  

The general solution of equation (9) takes the pole of the first order $p = 1$ and takes the form

$$w = \frac{1}{z - z_0}.$$  

Differentiating (9) in $z$ we obtain the second-order differential equation in the form

$$E_{1a} = w_{zz} + 2w w_z = 0.$$  

We can see that the order of the pole for the general solution of equation (11) is not changed and solution (10) satisfies equation (11).
**Example 2.** Let us consider the second-order equation in the form

\[ E_2 = w_{zz} - 6w^2 = 0. \quad (12) \]

The general solution of equation (12) has the second order pole \( p = 2 \) and is expressed via the Weierstrass elliptic function

\[ w = \wp(z - z_0, 0, g_3), \quad (13) \]

where \( z_0 \) and \( g_3 \) are arbitrary constants.

At investigation of the analytical properties for nonlinear differential equation appears the important notion – the Fuchs indices. The discussion this notion can be found in many books. For equation (9) we obtain the one Fuchs index which is equal to \( j_1 = -1 \). For equation (12) we have two Fuchs indices \( j_1 = -1 \) and \( j_2 = 6 \).

**Remark 1.** It is obviously that differentiation of equation leads to the change of the general solution of equation.

We can see that the equation of the second order (11) that is found by differentiation of equation (8) in \( z \) has the general solution with the first order of pole in the form

\[ w(z) = \sqrt{C_1} \tanh \left( \sqrt{C_1}(z - z_0) \right). \quad (14) \]

because equation (11) has the first integral with arbitrary constant \( C_1 \) in the form

\[ E_{1a} = w_z + w^2 - C_1 = 0. \quad (15) \]

It is well known that the quantity of the Fuchs indices is equal to the order of nonlinear differential equation. So, differentiating equation (11) in \( z \) we have the third-order equation

\[ E_{3a} = w_{zzz} + 2w w_{zz} + 2w_z^2 = 0 \]

which have three Fuchs indices in the form

\[ j_1 = -1, \quad j_2 = 2, \quad j_3 = 3. \quad (17) \]

Equation (17) has the general solution that was found from the irreducible equation in the form

\[ E_{1aa} = w_z + w^2 - C_1 z - C_2 = 0 \quad (18) \]

and that is expressed via the Airy functions.

Let us note that the the order of pole for leading members of equation (18) \( w_z \) and \( w^2 \) is equal to 2 and correspond to the Fuchs index \( j_2 = 2 \). Arbitrary constant \( C_2 \) also corresponds to the Fuchs index \( j_2 = 2 \). At the same time the arbitrary constant \( C_1 \) in equation (18) correspond to the Fuchs index \( j_3 = 3 \) because multiplying on \( z \) leads to the second order of pole.

So, the values of the Fuchs indexes point out us the leading members of the First integrals in the polynomial form for nonlinear ordinary differential equations. These ideas can be used for finding first integrals of nonlinear differential equations. Let us demonstrate our approach taking into account three examples.

**Example 3.** Consider the nonlinear ordinary differential equation in the form

\[ y_{zzz} + \delta y_{zz} + \epsilon y_z + \beta y y_z + \alpha y^2 - C_0 y + C_1 = 0. \quad (19) \]

Equation (19) can be obtained from the nonlinear evolution equation for description of nonlinear waves in a convecting fluid

\[ u_t + \alpha u u_x + \beta (u u_x)_x + \epsilon u_{xx} + \delta u_{xxx} + u_{xxxx} = 0 \quad (20) \]
if we look for exact solution using the traveling wave
\[ u(x, t) = y(z), \quad z = x - C_0 t. \] (21)

Equation (20) is quite popular and was studied many times. It was obtained in works [19, 20] for description of long shallow wave evolution in a convecting fluid in case when the Rayleigh number slightly exceeds its critical value.

Nonlinear waves described by equation (20) were studied in paper [21]. Some exact solutions of the nonlinear dissipative equation (20) for surface waves in a convecting liquid layer were obtained in works [22, 23, 24, 25]. The study of equation (20) was presented in [26] including symmetries (nonlocal and contact transformations), similarity reductions and the application of the Ablowitz–Ramani–Segur algorithm to the reductions.

Various methods were applied to solve equation (20) in [27]. Some explicit traveling wave solutions of equation (20) were obtained in paper [28] through factorization techniques when coefficients of the equation fulfill a certain condition. Some exact solutions of equation (20) were found in paper [29] using the tanh-expansion method. Using some kinds of standard and nonstandard truncation approaches for extended singular manifolds to an equation of convecting fluid in paper [30] solitary wave structures were obtained.

Equation (19) has the following Fuchs indices at the expansion of the general solution in the Laurent series
\[ y_1 = -1, \quad j_2 = 4, \quad j_3 = 6. \] (22)

Taking into account the Fuchs index \( j_2 = 4 \) we can look for the First integral in the form
\[ y_{zz} + ay^2 + by + cy_z - S(z) = 0, \] (23)
where expressions \( y_{zz} \) and \( y^2 \) are taken because they has the fourth order of poles, parameters \( a, b \) and the function \( S(z) \) can be found.

Substituting \( y_{zz} \) from (23) into equation (19) and equating various expressions of powers \( y_z \) and \( y \) to zero we obtain the following conditions
\[ a = -\frac{\beta}{2}, \quad b = \varepsilon - \frac{2\alpha\delta}{\beta} - \frac{4\alpha^2}{\beta^2}, \quad c = \delta + \frac{2\alpha}{\beta}, \] (24)
\[ C_0 = \frac{2\alpha\varepsilon}{\beta} - \frac{8\alpha^3}{\beta^3} - \frac{4\alpha^2\delta}{\beta^2}, \] (25)
\[ S(z) = \frac{C_1\beta}{2\alpha} + C_2 e^{-2\frac{\alpha z}{\beta}}. \] (26)

Using conditions (24) and 26 we have that nonlinear ordinary differential equation (19) at \( C_0 \) determined by formula (25) has the first integral in the form
\[ y_{zz} - \frac{\beta}{2} y^2 + \left( \varepsilon - \frac{2\alpha\delta}{\beta} - \frac{4\alpha^2}{\beta^2} \right) y + \left( \delta + \frac{2\alpha}{\beta} \right) y_z - \frac{C_1\beta}{2\alpha} + C_2 e^{-2\frac{\alpha z}{\beta}} = 0, \] (27)
where \( C_2 \) is an arbitrary constant.

Taking into consideration the first integral (27) we can look for exact solutions of equation (19).

**Example 4.** As the second example of nonlinear differential equation let us consider the nonlinear ordinary differential equation
\[ y_{zzz} - \mu y_{zz} + \alpha y y_z - \beta y^2 - C_0 y_z - \varepsilon y = 0. \] (28)
Equation (28) is found from the generalized Korteweg-de Vries equation with source in the form

\[ u_t + \alpha u u_x + u_{xxx} = \mu u_{xx} + \beta u^2 + \varepsilon y \] (29)

if we look for exact solution using the traveling wave (21).

Equation (28) has the following Fuchs indices at the expansion of the general solution in the Laurent series

\[ y_1 = -1, \quad j_2 = 4, \quad j_3 = 6. \] (30)

Taking into account the Fuchs index \( j_2 = 4 \) we can look for the First integral again in the form

\[ y_{zz} - a y^2 + b y_z + d y - S(z) = 0, \] (31)

where parameters \( a, b, d \) and the function \( S(z) \) can be found.

Substituting \( y_{zz} \) from (31) into equation (28) and equating various expressions of powers \( y_z \) and \( y \) to zero we obtain

\[ a = -\frac{\alpha}{2}, \quad b = \mu - 2\frac{\beta}{\alpha}, \quad d = -\frac{\alpha \varepsilon}{2\beta}, \] (32)

\[ C_0 = \frac{4\beta^2}{\alpha^2} - 2\frac{\beta \mu}{\alpha} \frac{\alpha \varepsilon}{2\beta}, \] (33)

\[ S(z) = C_1 e^{-\frac{2\beta z}{\alpha}}. \] (34)

Using conditions (32) and (34) we obtain that nonlinear ordinary differential equation (28) at (33) has the first integral in the form

\[ y_{zz} + \frac{\alpha}{2} y^2 - \left( \mu - 2\frac{\beta}{\alpha} \right) y_z + \frac{\alpha \varepsilon}{2\beta} y + C_1 e^{-\frac{2\beta z}{\alpha}} = 0, \] (35)

where \( C_1 \) is an arbitrary constant.

**Example 5.** As the third example of nonlinear differential let us consider the equation in the form

\[ y_{zzz} - \mu y_{zz} + \alpha y^2 y_z - \beta y^3 - C_0 y_z - \varepsilon y = 0. \] (36)

Equation (36) can be found from the generalized modified Korteweg-de Vries equation with source in the form

\[ u_t + \alpha u^2 u_x + u_{xxx} = \mu u_{xx} + \beta u^3 + \varepsilon y \] (37)

if we look for exact solution using the traveling wave (21).

Equation (36) has the following Fuchs indices at the expansion of the general solution in the Laurent series

\[ y_1 = -1, \quad j_2 = 3, \quad j_3 = 4. \] (38)

Taking into account the Fuchs index \( j_2 = 3 \) we can look for the First integral in the form

\[ y_{zz} - a y_z - b y^3 - h y + S(z) = 0, \] (39)

where parameters \( a, b, h \) and the function \( S(z) \) can be found.

Substituting \( y_{zz} \) from (39) into equation (36) and equating various expressions of powers \( y_z \) and \( y \) to zero we obtain

\[ a = \mu - \frac{3\beta}{\alpha}, \quad b = -\frac{\alpha}{3}, \quad h = -\frac{\alpha \varepsilon}{3\beta}, \] (40)

\[ C_0 = \frac{9\beta^2}{\alpha^2} - 3\mu \frac{\beta}{\alpha} - \frac{\alpha \varepsilon}{3\beta}. \] (41)
Using conditions (40) and (42) we obtain that nonlinear ordinary differential equation (36) at condition (41) has the first integral in the form

\[ y_{zz} - \alpha y^3 - \left( \mu - \frac{3}{\alpha} \right) y_z - \alpha \varepsilon y - C_1 e^{\frac{3 \beta z}{\alpha}} = 0, \]  

(43)

where \( C_1 \) is an arbitrary constant.

These examples illustrate that the values of the Fuchs indices allows us to see the form of the first integrals for nonlinear ordinary differential equations. The open question is the connection of the Fuchs indices with the first integrals for the Painlevé equations and higher-order Painlevé analogies [32, 33, 34]. We know that these equations have arbitrary constants at expansion of the general solution in the Laurent series but all these equations do not have any first integrals in the polynomial form.

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