Another (wrong) construction of $\pi$

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Abstract. A simple way is shown to construct the length $\pi$ from the unit length with 4 digits accuracy.

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1 Introduction

It is well-known that accurate construction of the ratio between the perimeter and the diameter of a circle is theoretically impossible [1, 2]. Before Lindemann’s result (and even thereafter), however, several attempts were recorded to geometrically construct the number $\pi$ with various means, in most cases by using a compass and a straightedge. One of the most successful attempts is Kochański’s work [3] that produces an approximation of $\pi$ by 4 digits, $\sqrt{\frac{40}{3}} - 2\sqrt{3} \approx 3.14153333\ldots$

Kochański’s construction is relatively easy, it requires just a little amount of steps, and can be discussed in the school curriculum as well. In this note the same approximation is given, by using—at least geometrically—an even simpler approach.

2 The construction

The proposed way to construct $\pi$ is shown in Fig. 1 (see also [4]).

A proof that $|RS| = \sqrt{\frac{40}{3} - 2\sqrt{3}}$ is as follows. We assume that $A_0 = (0, 0)$ and $A_1 = (1, 0)$ (see Fig. 2). By using that $360^\circ/12 = 30^\circ$, $\cos(30^\circ) = \sqrt{3}/2$ and $\sin(30^\circ) = 1/2$, the exact coordinates of the appearing vertices in the construction are $A_3 = (3/2 + \sqrt{3}/2, 1/2 + \sqrt{3}/2)$, $A_6 = (1, 2 + \sqrt{3})$, $A_7 = (0, 2 + \sqrt{3})$, $A_8 = (-\sqrt{3}/2, 3/2 + \sqrt{3})$, $A_9 = (-1/2 - \sqrt{3}/2, 3/2 + \sqrt{3}/2)$, $A_{11} = (-\sqrt{3}/2, 1/2)$.

To find the coordinates of $R$ we can compute the equation of the line $A_{11}A_6$. By substituting the coordinates of $A_{11}$ and $A_6$ it can be verified that the equation is

$$(3/2 + \sqrt{3})x - (1 + \sqrt{3}/2)y = -2 - \sqrt{3},$$
Fig. 1. A new way to construct $\pi$ approximately

Fig. 2. Explanation for the proof
and solving this for $y = 0$ we obtain the exact coordinates $R = (-2/\sqrt{3},0)$. (Alternatively, it can be shown that $|A_1 R| = 1 + 2/\sqrt{3}$, because it is the shorter cathetus of the triangle $RA_1 A_6$ which is a half of an equilateral triangle—this holds because the angle $A_1 A_6 A_{11}$ is an inscribed angle of the circumcircle of the regular 12-gon, and it must be $60^\circ/2 = 30^\circ$.)

Now, $A_3 A_8 \perp A_6 A_{11}$, so we are searching for the equation of line $A_3 A_8$ in form

$$(1 + \sqrt{3}/2)x + (3/2 + \sqrt{3})y = c.$$

After substituting the coordinates of $A_3$ in this, we obtain that $c = (9 + 5\sqrt{3})/2$. Because of symmetry, it is clear that the equation of line $A_7 A_9$ is of form

$$y = x + d.$$  

By using the coordinates of $A_7$, we immediately obtain that $d = 2 + \sqrt{3}$.

To find the coordinates of $S$ we solve the equation system

\begin{align*}
(1 + \sqrt{3}/2)x + (3/2 + \sqrt{3})y &= (9 + 5\sqrt{3})/2, \quad (1) \\
y &= x + 2 + \sqrt{3} \quad (2)
\end{align*}

now, which produces the coordinates $x = (\sqrt{3} - 3)/2, \ y = (3\sqrt{3} + 1)/2$.

Finally we compute the length of $RS$:

$$|RS| = \sqrt{((\sqrt{3} - 3)/2 + (2/\sqrt{3}))^2 + ((3\sqrt{3} + 1)/2)^2} = \sqrt{40/3 - 2\sqrt{3}}.$$

### 3  Remarks

This result has been found by the software tool [5] in an automated way by considering all possible configurations of distances between intersections of diagonals in regular polygons. It seems very likely that the construction described above is the simplest and most accurate one among the considered cases. Another construction which is based on a regular star-12-gon (see [6]) can produce the same approximation.

Regular polygons with less sides have already been completely studied for the constructible cases for $n < 12$ with less accurate results (see [4] for details). Checking cases $n = 15, 16, 17, 20$ (all are constructible) is an on-going project.

While the software tool [5] gave a machine assisted proof by using Gröbner bases and elimination (see [7,8] for more details), the proof above was compiled by the author manually.

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References

1. Wantzel, P.: Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas. Journal de Mathématiques Pures et Appliquées 1 (1837) 366–372
2. Lindemann, F.: Über die Zahl \( \pi \). Mathematische Annalen 20 (1882) 213–225
3. Kochański, A.A.: Observationes cyclometricae adfacilitandam praxin accomodatae. Acta Eruditorum 4 (1685) 394–398
4. Kovács, Z.: Constructing \( \pi \) from a regular 12-gon. GeoGebra (2018) https://www.geogebra.org/m/qFvttny2G
5. Kovács, Z.: RegularNGons. A GitHub project (2018) https://github.com/kovzol/RegularNGons
6. Kovács, Z.: Constructing \( \pi \) from a regular star-12-gon. GeoGebra (2018) https://www.geogebra.org/m/jnZSeBnq
7. Cox, D., Little, J., O’Shea, D.: Ideals Varieties, and Algorithms. Springer New York (2007)
8. Recio, T., Vélez, M.P.: Automatic discovery of theorems in elementary geometry. Journal of Automated Reasoning 23 (1999) 63–82