Compactness of $H(i)$, $U(i)$, $R(i)$ spaces via filters

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Abstract

Recently, filters were applied to give affirmative answers to two long-standing questions [1]. It was proved that a topological space is compact if and only if each closed subset is Hausdorff-closed (Urysohn-closed) [regular-closed]. As an improvement, it was established that a Hausdorff-closed (Urysohn-closed) [regular-closed] space is compact if and only if each closed subset is an $H$-set (a $U$-set) [an $R$-set] [2] (See AMS Mathematical Reviews MR 3191275, MR 3112925). Stone, in 1937 [3], using Boolean rings and transfinite induction, proved that a Hausdorff space $X$ is compact if and only if each closed subset of $X$ is Hausdorff-closed. In 1940, Katětov [4] gave a topological proof. Topological methods are used in this paper to generalize these results to non-Hausdorff (non-Urysohn) [non-regular] spaces. Stephenson (Scarboorouh and Stone) established in [5] ([6]) that a countable minimal Urysohn (minimal regular) space is compact. It is shown here that every countable Urysohn-closed (regular-closed) space is compact. It is shown in [7] that a Hausdorff-closed (Urysohn-closed) [regular-closed] space is compact if and only if each closed subset is paracompact (metacompact). It is proved here that a Hausdorff-closed (Urysohn-closed) [regular-closed] space is compact if and only if it is Lindelöf (countably compact) [normal], and filters are utilized to generalize all of these results to $H(i)$ ($U(i)$ [$R(i)$] spaces.

Keywords: Filters; compact, $H(i)$, $U(i)$, $R(i)$.

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1. Introduction and Preliminaries

All spaces are topological. If $X$ is a space and $A \subset X$, let $clA$ be the closure of $A$. If $A$ is a subset of a space $X$, denote by $\sum A$ the collection of open $V$ such that $A \subset V$. A Hausdorff-closed (Urysohn-closed) [regular-closed] space is one which is a closed subset of any Hausdorff (Urysohn) [regular] space in which it is embedded. In [8], Herrington called a family $\mathcal{G}x$ of open sets about $x$, a shrinkable family of open sets about $x$ if for each $V \in \mathcal{G}x$ there exists $U \in \mathcal{G}x$ such that $cl(U) \subset V$.

Theorem 1.1. If $\mathcal{B}$ is a filter base on a space $X$ and $C = \{C$ open in $X : B \subset C$ for some $B \in \mathcal{B}\}$, then $C$ is an open ultrafilter on $X$.

Proof. Order the collection of $C$ by inclusion and let $\mathcal{D}$ be the union of the elements of this collection. Then $\mathcal{D}$ is an upper bound for this collection. So, by Zorn, the proof is complete.

The following theorem is found in [2]. The proof is included here for completeness.

Theorem 1.2. If $\mathcal{U}$ is an ultrafilter on a space $X$ and $O = \{\text{Open in } X : U \subset O$ for some $U \in \mathcal{U}\}$, $O$ is an open ultrafilter, $O \subset \mathcal{U}$, $adh_0\mathcal{U} = adh_0O(adh_0\mathcal{U} = adh_0O)[adh_0\mathcal{U} = adh_0O]$.

Proof. It is clear that $O$ is an open ultrafilter on $X$ from Theorem 1.1 and that $O \subset \mathcal{U}$ from the property of ultrafilters; $O \subset \mathcal{U}$, so that $adh_0\mathcal{U} \subset adh_0O(adh_0\mathcal{U} \subset adh_0O)[adh_0\mathcal{U} \subset adh_0O]$.

For the reverse inclusion, if $x \notin adh_0\mathcal{U}(x \notin adh_0\mathcal{U})[x \notin adh_0\mathcal{U}], then x \notin cl_0\mathcal{U}(x \notin cl_0\mathcal{U}[x \notin cl_0\mathcal{U}] for some $U \in \mathcal{U}$. Therefore, there exist $U \in \mathcal{U}$ and $V \in \sum [x]$ such that $cl_0\mathcal{U} \cap cl_0\mathcal{U} \cap cl_0\mathcal{U} \cap cl_0\mathcal{U} = \emptyset$. Therefore, $U \subset X - cl_0\mathcal{U}(U \subset X - cl_0\mathcal{U})[U \subset X - cl_0\mathcal{U}]. Hence, $X - cl_0\mathcal{U}(X - cl_0\mathcal{U})[X - cl_0\mathcal{U}] \in \mathcal{O}$, and $cl_0\mathcal{U} \cap (X - cl_0\mathcal{U})(cl_0\mathcal{U} \cap (X - cl_0\mathcal{U})[cl_0\mathcal{U} \cap (X - cl_0\mathcal{U}) = \emptyset$. So $x \notin adh_0O(x \notin adh_0O)[x \notin adh_0O]$.

Corollary 1.3. If $\mathcal{U}$ is an ultrafilter on an $H(i)$ (a $U(i)$) [an $R(i)$] space $X$, then $adh_0\mathcal{U}(adh_0\mathcal{U})(adh_0\mathcal{U})[adh_0\mathcal{U}] = cl_0\mathcal{U}(cl_0\mathcal{U})(cl_0\mathcal{U})(cl_0\mathcal{U})[cl_0\mathcal{U}]$ for some $x \in X$.

Proof. If $O$ is an open ultrafilter, then $adh_0O = cl_0\mathcal{U}[adh_0O = cl_0\mathcal{U}][adh_0O = cl_0\mathcal{U}$, for some $x \in X$.

The proof of the next theorem is omitted. It will be used often in the sequel.

Theorem 1.4. If $X$ is a space and $\emptyset \neq A \subset X$, an ultrafilter $\mathcal{U}$ converges to $A(\mathcal{U} \to A)$ if and only if $\sum A \subset \mathcal{U}$ (If $A = \{x\}$, use the notation $\mathcal{U} \to x$).

Definition 1.5. In a space $X$, $A \subset X$ is called an $H(i)$-set (a $U(i)$-set) $[R(i)$-set] if $A \neq \emptyset$ or each filter base $\Omega$ on $X$ satisfies $A \cap adh_0\Omega \neq \emptyset(A \cap adh_0\Omega \neq \emptyset)[A \cap adh_0\Omega \neq \emptyset]$. An $H(i)$-set (A $U(i)$-set) $[R(i)$-set] in a Hausdorff (Urysohn) [regular] space is called an $H$-set (a $U$-set) [an $R$-set].

If $X$ is a space and $A \subset X$, $bdA$ will represent the boundary of $A$ in $X$.

Definition 1.6. A space $X$ is called rim $H(i)$ (rim $U(i)$) [rim $R(i)$] if each point has a base of open sets with $H(i)$-set (U(i)-set) [R(i)-set] boundaries.
Definition 1.7. A space $X$ is called rim H-set (rim U-set) [rim R-set] if each point has a base of open sets with H-set (U-set) [R-set] boundaries.

Definition 1.8. A space $X$ is called rim Hausdorff-closed (rim Urysohn-closed) [rim regular-closed] if each point has a base of open sets with Hausdorff-closed (Urysohn-closed) [regular-closed] boundaries.

Definition 1.9. A space is called rim $\theta$-closed (rim $u$-closed) [rim $s$-closed] if each point has a base of open sets with $\theta$-closed ($u$-closed) [$s$-closed] boundaries.

2. Main results

Theorem 2.1. If $X$ is a space, $\emptyset \neq A$ is compact and $\Omega$ is a filter on $X$ such that $A \cap F \neq \emptyset$ for every $F \in \Omega$, there exists $x \in A$ such that $F \cap V \neq \emptyset$, $F \in \Omega$, $V \in \sum \{x\}$.

Proof. Suppose that for each $x \in A, F_x \cap V_x = \emptyset, F_x \in \Omega, V_x \in \sum \{x\}$. Then, there exists finite $B \subset A$ such that $A \subset \bigcup B V_x = V, F = \bigcap B F_x \in \Omega, V \in \sum A, F \cap V = \emptyset$. Therefore, there exists $x \in A$ such that $F \cap V \neq \emptyset, F \in \Omega, V \in \sum \{x\}$.

Corollary 2.2. If $X$ is a space, $\emptyset \neq A$ is a compact subset of $X$ and $\mathcal{U}$ is an ultrafilter on $X$ such that $A \cap \mathcal{U} \neq \emptyset$ for every $\mathcal{U} \in \mathcal{U}$ there exists $x \in A$ such that $\mathcal{U} \rightarrow x$.

It has been shown in [11] ([12]) ([7]) that a Hausdorff (Urysohn) [regular] space is Hausdorff-closed (Urysohn-closed) [regular-closed] if and only if each filter base with at most one $\theta$-adherent ($u$-adherent) [$s$-adherent] is $\theta$convergent ($u$convergent) [$s$convergent].

Theorem 2.3. A space is $H(i)$ ($U(i)$) [$R(i)$] if and only if each filter base with at most one $\theta$-adherent ($u$-adherent) [$s$-adherent] is $\theta$convergent ($u$convergent) [$s$convergent].

Proof. If each filter base $\Omega$ has $\text{adh}_\theta \Omega \text{adh}_u \Omega \subset \{x\}$, then $\Omega \rightarrow x (\Omega \rightarrow u x) [\Omega \rightarrow s x]$ and thus $\text{adh}_\theta \Omega [\text{adh}_u \Omega] [\text{adh}_s \Omega] \neq \emptyset$. So $X$ is $H(i)$ ($U(i)$) [$R(i)$]. For the converse, let $\Omega$ be a filter base on an $H(i)$ ($U(i)$) [$R(i)$] space $X$ and let $x \in X, \text{adh}_\theta \Omega \cap (X - \{x\}) = \emptyset, \text{adh}_u \Omega \cap (X - \{x\}) = \emptyset$). Then $x \in \text{adh}_\theta \Omega [\text{adh}_u \Omega]$ and hence $\Omega \rightarrow x (\Omega \rightarrow u x) [\Omega \rightarrow s x]$.

Recall that a Hausdorff (Urysohn) [regular] space is minimal Hausdorff (minimal Urysohn) [minimal regular] if and only if each open (Urysohn) [regular] filter base with at most one adherent point is convergent [1]. Herrington and Long [13] have shown that a Hausdorff space $X$ is minimal Hausdorff if and only if each filter on $X$ with at most one $\theta$-adherent point is convergent. It may be shown that a (Urysohn space) [regular space] $X$ is (minimal Urysohn) [minimal regular] if and only if each filter on $X$ with at most one ($u$-adherent) [$s$-adherent] point is convergent. This suggests the following definition which is used to offer different proofs of several theorems in [1], as is shown below.

Definition 2.4. A space $X$ is minimal $H(i)$ (minimal $U(i)$) [minimal $R(i)$] if and only if each filter on $X$ with at most one $\theta$-adherent ($u$-adherent) [$s$-adherent] point is convergent.

Theorem 2.5. A minimal Hausdorff Urysohn space $X$ is compact.

Proof. Let $\mathcal{U}$ be an ultrafilter on $X$. Then $\mathcal{U} \rightarrow\theta$ to a single point because the space is Urysohn. Thus $\mathcal{U}$ is convergent.

Theorem 2.6. A minimal Urysohn Hausdorff-closed space $X$ is compact.

Proof. Let $\mathcal{U}$ be an ultrafilter on $X$. Then $\mathcal{U} \rightarrow u$ to a single point because the space is Urysohn. Thus $\mathcal{U}$ is convergent.
Theorem 2.7. A minimal regular Urysohn-closed space is compact.

Proof. Let \( \mathcal{U} \) be an ultrafilter on \( X \). Then \( \mathcal{U} \to_n \) to a single point because the space is regular. Thus \( \mathcal{U} \) is convergent.

Theorem 2.8. A minimal Urysohn regular space is compact.

Proof. Let \( \mathcal{U} \) be an ultrafilter on \( X \). Then \( \mathcal{U} \to_n \) to a single point because the space is regular. Thus \( \mathcal{U} \) is convergent.

The next theorem is immediate.

Theorem 2.9. Each minimal \( H(i) \) (minimal \( U(i) \)) [minimal \( R(i) \)] is \( H(i) \) (\( U(i) \)) [\( R(i) \)].

The next result characterizes compactness of \( H(i) \) (\( U(i) \)) [\( R(i) \)] spaces.

Theorem 2.10. The following statements are equivalent for an \( H(i) \) (\( U(i) \)) [\( R(i) \)] space \( X \):

1. \( X \) is compact.
2. Each closed \( A \subset X \) is \( H(i) \) (\( U(i) \)) [\( R(i) \)] and \( A \cap \text{adh}_0 \mathcal{U}(\text{adh}_u \mathcal{U})[\text{adh}_s \mathcal{U}] \) is compact for each ultrafilter \( \mathcal{U} \) on \( X \).
3. Each closed \( A \subset X \) is \( H(i) \) (\( U(i) \)) [\( R(i) \)] and \( \text{adh}_0 \mathcal{U}(\text{adh}_u \mathcal{U})[\text{adh}_s \mathcal{U}] \) is compact for each ultrafilter \( \mathcal{U} \) on \( X \).
4. Each closed \( A \subset X \) is minimal \( H(i) \) (minimal \( U(i) \)) [minimal \( R(i) \)] and \( \text{adh}_0 \mathcal{U}(\text{adh}_u \mathcal{U})[\text{adh}_s \mathcal{U}] \) is compact for each ultrafilter \( \mathcal{U} \) on \( X \).
5. The boundary \( bdV \) is an \( H(i) \)-set (a \( U(i) \)-set) [an \( R(i) \)-set] for each open \( V \) and \( \text{adh}_0 \mathcal{U}(\text{adh}_u \mathcal{U})[\text{adh}_s \mathcal{U}] \) is compact for each ultrafilter \( \mathcal{U} \) on \( X \).
6. \( X \) is rim\( H(i) \) (rim\( U(i) \)) [rim\( R(i) \)] and \( \text{adh}_0 \mathcal{U}(\text{adh}_u \mathcal{U})[\text{adh}_s \mathcal{U}] \) is compact for each ultrafilter \( \mathcal{U} \) on \( X \).
7. \( X \) is rim\( H(i) \) (rim\( U(i) \)) [rim\( R(i) \)] and \( cl_\theta[x](cl_u[x])[cl_s[x]] \) is compact for each \( x \in X \), and ultrafilter \( \mathcal{U} \) on \( X \).

Proof. (1)\( \Rightarrow \)(2): Obvious, since \( A \) and \( \text{adh}_0 \mathcal{U}(\text{adh}_u \mathcal{U})[\text{adh}_s \mathcal{U}] \) are closed.

(2)\( \Rightarrow \)(3): Obvious.

(3)\( \Rightarrow \)(4): The boundary \( bdV \) is closed.

(4)\( \Rightarrow \)(5): Definition.

(5)\( \Rightarrow \)(1): Let \( \mathcal{U} \) be an ultrafilter on \( X \). By Theorem 2.1, there is an \( x \in \text{adh}_0 \mathcal{U}(\text{adh}_u \mathcal{U})[\text{adh}_s \mathcal{U}] \) such that \( F \cap V \neq \emptyset \) for all \( F \in \mathcal{U}, V \in \sum [x] \); for such \( x \), there exists \( F \in \mathcal{U}, V \in \sum [x] \) with \( F \cap bdV = \emptyset \). If not, \( V, bdV \in \mathcal{U} \) for some \( x \in \text{adh}_0 \mathcal{U}(\text{adh}_u \mathcal{U})[\text{adh}_s \mathcal{U}], V \in \sum [x] \). Thus, for some \( x \in \text{adh}_0 \mathcal{U}(\text{adh}_u \mathcal{U})[\text{adh}_s \mathcal{U}], \sum [x] \subset \mathcal{U}, \mathcal{U} \to x \).

(5)\( \Rightarrow \)(6)\( \iff \)(7): Theorem 1.2, Corollary 1.3 and the fact that \( cl_\theta[x](cl_u[x])[cl_s[x]] \) is closed in any topological space.

Theorem 2.11. The following statements hold:

1. An \( H(i) \) (\( U(i) \)) [\( R(i) \)] space \( X \) is compact if each closed subset \( A \) is \( \theta \)-closed (\( u \)-closed) [\( s \)-closed].
2. In any compact space space \( X \), each closed subset \( A \) satisfies \( A \cap cl_\theta[x](A \cap cl_u[x])[A \cap cl_s[x]] \neq \emptyset \) for every \( x \in cl_\theta A(x \in cl_u A)[x \in cl_s A] \).

Proof. (1) If \( \Omega \) is a filter base on \( X \), then

\[
\emptyset \neq \text{adh}_0 \Omega(\text{adh}_u \Omega)\text{adh}_s \Omega \subset \text{adh}_0 \Omega \cap \text{clF}(\text{adh}_u \Omega)\text{adh}_s \Omega \cap \text{clF} = \text{adh} \Omega.
\]

So if each closed subset is \( \theta \)-closed (\( u \)-closed) [\( s \)-closed ], the space is compact.

(2) If \( X \) is compact and \( A \subset X \) is closed, \( A \) is compact, \( x \in cl_\theta A(x \in cl_u A)[x \in cl_s A], A \cap \bigcap_{V \in \sum [x]} cl_\theta V(A \cap \bigcap_{V \in \sum [x]} cl_u V) \neq \emptyset \). So \( A \cap cl_\theta[x] \neq \emptyset(A \cap cl_u[x] \neq \emptyset)[A \cap cl_s[x] \neq \emptyset] \).
**Corollary 2.12.** A Hausdorff-closed (A Urysohn-closed) [A regular-closed] space $X$ is compact if and only if each closed subset $A$ is $\theta$-closed (u-closed) [s-closed].

**Proof.** In a Hausdorff space (Urysohn space) [regular space] $cl_\theta(x) = \{x\}(cl_u(x) = \{x\})[cl_s(x) = \{x\}]$ for each $x \in X$.

**Corollary 2.13.** Let $X$ be a Hausdorff-closed (Urysohn-closed) [regular-closed] space. The following are equivalent [2, 7]:

1. $X$ is compact.
2. Each closed subset of $X$ is Hausdorff-closed (Urysohn-closed) [regular-closed].
3. Each nonempty closed subset of $X$ is an H-set (a U-set) [an R-set].
4. The boundary $bdV$ is an H-set (a U-set) [an R-set] for each open $V$ with $bdV$ nonempty.
5. The boundary $bdV$ is Hausdorff-closed (Urysohn-closed) [regular-closed] for each open subset $V$ of $X$.
6. $X$ is rim H-set (rim U-set) [rim R-set].
7. Each closed subset of $X$ is $\theta$-closed (u-closed) [s-closed].
8. The boundary $bdV$ is $\theta$-closed (u-closed) [s-closed].
9. Each H-set (U-set) [R-set] in $X$ is $\theta$-closed (u-closed) [s-closed].
10. $X$ is rim Hausdorff-closed (rim Urysohn-closed) [rim regular-closed].

**Proof.** In a Hausdorff-closed (Urysohn-closed) [regular-closed] space, $adh_\theta U(adh_u U)[adh_s U]$ is a singleton for an ultrafilter $U$.

The following theorem gives a proof of compactness for $H(i) (U(i))[R(i)]$ spaces.

**Theorem 2.14.** An $H(i)$ (U(i) [An R(i)]) space $X$ is compact if and only if $adh_\theta U(adh_u U)[adh_s U]$ is compact and each closed subset is an $H(i)$ set (a U(i)-set) [R(i)-set].

**Proof.** One direction is clear. For the other direction, choose an ultrafilter $U$ on $X; adh_\theta U(adh_u U)[adh_s U] = cl_\theta(x)(cl_u(x))cl_s(x)$, for some $x \in X$. Show first, for such an $x$, that $\mathcal{U} \rightarrow cl_\theta(x)(cl_u(x))cl_s(x)$. Let $\mathcal{V} \in \sum cl_\theta(x)(\sum cl_u(x))cl_s(x)$. Then $F - V = \emptyset$ for some $F \in \mathcal{U}$ since $adh_\theta U(adh_u U)[adh_s U] \cap (X - V) = \emptyset$, because $X - V$ is an $H(i)$-set (a U(i)-set) [R(i)-set] $\mathcal{U} \rightarrow cl_\theta(x)(cl_u(x))cl_s(x)$. Now, since $cl_\theta(x)(cl_u(x))cl_s(x)$ is closed and hence compact, from Theorem 2.1, $\mathcal{U} \rightarrow z \in cl_\theta(x)(cl_u(x))cl_s(x)$.

The next corollary is readily obtained for Hausdorff-closed spaces, Urysohn-closed spaces, and regular-closed spaces, H-sets, U-sets, and R-sets.

**Corollary 2.15.** $X$ is compact if and only if each closed subset is Hausdorff-closed (Urysohn-closed) [regular-closed] (an H set (a U set) [an R set]).

**Proof.** $X$ is an $H(i)$ (U(i)) [an R(i)] space and $adh_\theta U(adh_u U)[adh_s U]$ is a singleton for each ultrafilter $U$ on $X$.

**Definition 2.16.** A space is called locally $H(i)$ (LH(i)) (locally $U(i)$ (LU(i))) [locally $R(i)$ (LR(i)))] if each point has a $H(i)$ (U(i)) [R(i)] neighborhood.

The next theorem is clear.

**Theorem 2.17.** An $H(i)$ (A U(i) [An R(i)]) space is LH(i) (LU(i)) [LR(i)].

**Theorem 2.18.** The following are equivalent for a space $X$:

1. $X$ is compact.
2. $X$ is LH(i) (LU(i)) [LR(i)] and $adh_\theta U(adh_u U)[adh_s U]$ is compact for each ultrafilter $U$ on $X$. 
(3) $X$ is $H(i)$ ($U(i)$) [R(i)] and $\text{adh}_\theta U(\text{adh}_\theta U)[\text{adh}_s U]$ is compact for each ultrafilter $U$ on $X$.

**Proof.** (1)$\Rightarrow$ (2): A compact space is an $H(i)$ (a $U(i)$) [an $R(i)$] space and $\text{adh}_\theta U(\text{adh}_\theta U)[\text{adh}_s U]$ is closed for each ultrafilter $U$.

(2)$\Rightarrow$ (3): There is an $x \in \text{adh}_\theta U(\text{adh}_\theta U)[\text{adh}_s U]$ such that all $F \in U$ satisfy $F \cap \text{cl}_\theta V(\text{cl}_\theta V)[\text{cl}_s V] \neq \emptyset$, $V \in \sum$, so $F - \text{cl}_\theta V(F - \text{cl}_\theta V)[F - \text{cl}_\theta V] = \emptyset$ for some $V \in \sum$. If not, $\text{cl}_\theta V(\text{cl}_\theta V)[\text{cl}_s V], X - \text{cl}_\theta V(X - \text{cl}_\theta V)[X - \text{cl}_s V] \in U$. Hence, $U \rightarrow_\theta x_U \rightarrow_\theta x \rightarrow_\theta x$ and $X$ is $H(i)$ ($U(i)$) [R(i)].

(3)$\Rightarrow$ (1): From (2)$\Rightarrow$ (3), $U \rightarrow \text{cl}_\theta x(\text{cl}_\theta x)[\text{cl}_s x]$. Since $\text{cl}_\theta x(\text{cl}_\theta x)[\text{cl}_s x]$ is compact, there is, from Theorem 4, a $z \in \text{cl}_\theta x(\text{cl}_\theta x)[\text{cl}_s x]$ such that $U \rightarrow z$.

**Definition 2.19.** A space is called locally Hausdorff-closed [4] (LHC) (locally Urysohn-closed (LUC) [locally regular-closed (LRC)] if each point has a Hausdorff-closed (Urysohn-closed) [regular-closed] neighborhood.

**Corollary 2.20.** A LHC (LUC) [LRC] space $X$ is compact if and only if each closed set is $\theta$-closed (u-closed) [s-closed].

**Theorem 2.21.** A Lindelöf space is compact if and only if each countable open filter base $\Omega$ on each closed subset satisfies $\text{adh}_\theta \Omega(\text{adh}_\theta \Omega)[\text{adh}_s \Omega] \neq \emptyset$ and $\text{adh}_\theta U(\text{adh}_\theta U)[\text{adh}_s U]$ is compact for each ultrafilter $U$.

**Proof.** The necessity is clear. For the sufficiency, let $U$ be an ultra filter on $X$. Every closed subset is $H(i)$ ($U(i)$) [R(i)] and thus the space is compact.

**Theorem 2.22.** A Lindelöf space is $H(i)$ ($U(i)$) [R(i)] if and only if each countable open filter base has nonempty adherence (u-adherence) [s-adherence].

**Proof.** Every open filter base $\Omega$ has nonempty $\theta$-adherence (u-adherence) [s-adherence].

**Corollary 2.23.** A Lindelöf space is compact if and only if each countable open filter base has nonempty adherence (u-adherence) [s-adherence] and each closed set is $H(i)$ ($U(i)$) [R(i)].

**Corollary 2.24.** A Lindelöf space is compact if and only if each countable open filter base has nonempty adherence (u-adherence) [s-adherence] and each closed set is Hausdorff-closed (Urysohn-closed) [regular-closed].

**Proof.** The set $\text{adh}_\theta U(\text{adh}_\theta U)[\text{adh}_s U]$ is a singleton, for each ultrafilter $U$ on $X$. Each closed subset of a Lindelöf (normal) [countably compact] space is Lindelöf (normal) [countably compact]. So the converse holds.

Stephenson [5] proved that a countable minimal Urysohn space is compact. A generalization of this result is offered next.

**Theorem 2.25.** If a Urysohn space $X$ has a dense set of isolated points, $X$ is compact.

**Proof.** Let $A \subset X$ be closed and let $D \subset X$ the dense set, $y \in X - A$. Then, since $X$ is Urysohn, there is an open set $V, y \in V \subset cl_V \subset cl_V \subset X - A, D \cap V \neq \emptyset, y \notin X - cl_\theta A$. So each closed set is $\theta$-closed and hence the space is compact [15].

**Corollary 2.26.** A countable Urysohn-closed (regular-closed) space is compact.

**Proof.** The space has a dense collection of isolated points [1].

**Corollary 2.27.** (Stephenson [5] (Scaborough and Stone [6]) A countable minimal-Urysohn (minimal-regular) space is compact.
The following is well known.

**Corollary 2.28.** (Stephenson [5]) A countable Urysohn-closed space is absolutely closed.

**Definition 2.29.** A filter base $\Omega$ on a set is point dominating (p.d.) [12] if each point is a member of all but finitely many elements of $\Omega$; a filter base $\Omega$ on a space is neighborhood dominating (n.d.) [16] if each point has a neighborhood contained in all but finitely many elements of $\Omega$. A filter $\Omega$ is called an M-filter (P-filter) if each p.d. (n.d.) subcollection of $\Omega$ has nonempty adherence.

The concepts of (p.d.) and (n.d.) families were used to give the following characterizations. A space $X$ is Lindelöf [12] (paracompact [16]) if each $M$-filter (P-filter) on $X$ has nonempty adherence. The proof of the next result is left to the reader.

**Theorem 2.30.** If $X$ is a space and $\Omega$ is an $M$-filter (P-filter) on $X$, and $\emptyset \subset \Omega \subset X$, then $\{F \cap \Omega : F \in \Omega\}$ is an $M$-filter (P-filter) on $\Omega$.

**Theorem 2.31.** An $H(i)$ (A $U(i)$) [An $R(i)$] space is compact if and only if each closed subset is metacompact (paracompact), and $\text{adh}_0 \mathcal{U}(\text{adh}_0 \mathcal{U}) \cap \mathcal{U}$ is compact for each ultrafilter $\mathcal{U}$ on $X$.

**Proof.** One direction is clear. For the other direction, let $\mathcal{U}$ be an ultrafilter on $X$. Choose

$$V \in \sum_{x \in X} \text{cl}_0(x)(\sum_{u \in X} \text{cl}_u(x))[\sum_{s \in X} \text{cl}_s(x)], x \in X.$$  

There is an $F \in \mathcal{U}$, $F \cap X - V = \emptyset$. Otherwise, $\{F \cap (X - V) : F \in \mathcal{U}\}$ is an ultrafilter on $X - V$ and then $V, X - V \notin \mathcal{U}$, a contradiction. Thus, $\mathcal{U} \rightarrow \text{cl}_0(x)(\text{cl}_u(x))[\text{cl}_s(x)]$. Hence by Theorem 2.1, there exists $z \in \text{cl}_0(x)(\text{cl}_u(x))[\text{cl}_s(x)]$ such that $\mathcal{U} \rightarrow z$.

**Corollary 2.32.** A Hausdorff-closed (Urysohn-closed) (regular-closed) space $X$ is compact if and only if $X$ is metacompact (paracompact).

**Proof.** The set $\text{adh}_0 \mathcal{U}(\text{adh}_0 \mathcal{U}) \cap \mathcal{U}$ is a singleton. For the converse, every closed subset of a metacompact (paracompact) is metacompact (paracompact).

**Theorem 2.33.** An $H(i)$ (A $U(i)$) [An $R(i)$] space is compact if and only if each closed subset is Lindelöf (countably compact) [normal], and $\text{adh}_0 \mathcal{U}(\text{adh}_0 \mathcal{U}) \cap \mathcal{U}$ is compact for each ultrafilter $\mathcal{U}$ on $X$.

**Proof.** One direction is clear. For the other direction, let $\mathcal{U}$ be an ultrafilter on $X$. Choose

$$V \in \sum_{x \in X} \text{cl}_0(x)(\sum_{u \in X} \text{cl}_u(x))[\sum_{s \in X} \text{cl}_s(x)].$$

Then there is an $F \in \mathcal{U}$, $F \cap (X - V) = \emptyset$, and $\{F \cap (X - V) : F \in \mathcal{U}\}$ is not a filter on $X - V$, since both $V, X - V \notin \mathcal{U}$. Thus, $\mathcal{U} \rightarrow \text{cl}_0(x)(\text{cl}_u(x))[\text{cl}_s(x)]$, by Theorem 2.1, $z \in \text{cl}_0(x)(\text{cl}_u(x))[\text{cl}_s(x)]$ such that $\mathcal{U} \rightarrow z$.

**Corollary 2.34.** A Hausdorff-closed (Urysohn-closed) (regular-closed) space $X$ is compact if and only if $X$ is Lindelöf (countably compact) [normal].

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