1. Introduction

1.1. Context. There is a wide network of conjectures, including the Birch–Swinnerton-Dyer and Bloch–Kato conjectures, that describe important arithmetic invariants in terms of special values of (complex) $L$-functions. One of the most successful tools in tackling these conjectures has been Iwasawa theory, in which one seeks to formulate $p$-adic analogues of these conjectures, replacing the complex analytic $L$-function with a $p$-adic $L$-function. The resulting $p$-adic Iwasawa main conjectures are often more tractable than their complex counterparts, and provide beautiful, deep connections between analysis and arithmetic.

A crucial launching point for Iwasawa theory is proving existence of $p$-adic $L$-functions. Coates and Perrin-Riou conjectured the existence of a $p$-adic $L$-function attached to every motive $M$ over $\mathbb{Q}$ that is ordinary with good reduction at $p$, and whose $L$-function has at least one critical value [CPR89, Coa89]. This itself rests on Deligne’s period conjecture [Del79]. One may formulate automorphic realisations of these (motivic) conjectures, but proving these remains very difficult.

To illustrate this, consider the fundamental case where $\Pi$ is a regular algebraic cuspidal automorphic representation of $\text{GL}_3(A_{\mathbb{Q}})$. Then Coates–Perrin-Riou predict that when $\Pi$ is unramified and ordinary at $p$, there exists a $p$-adic measure on $\mathbb{Z}_p^\times$ — the $p$-adic $L$-function of $\Pi$ — interpolating all its critical $L$-values at $p$. We make this precise in Conjecture B.

For $n = 1, 2$, existence of $p$-adic $L$-functions has been known for decades (starting from e.g. [KL64, MSD74]). For $n \geq 3$, however, our understanding of the conjecture remains poor, and it is known only in very special cases, e.g. symmetric squares of classical modular forms [Sch88, Hid90, DD97], RACARs of $\text{GL}_{2n}$ with Shalika models [AG94, DJR20], or Rankin–Selberg transfers from $\text{GL}_n \times \text{GL}_{n+1}$ to $\text{GL}_{n(n+1)}$ [Sch93, KMS00, Jan]. In particular, in all known cases, $\Pi$ arises as a functorial lift from a group whose $L$-group is a proper subgroup of $\text{GL}_n$: beyond the classical cases of $n = 1, 2$, there are no examples when $\Pi$ has ‘general type’, i.e. doesn’t arise from a smaller group in this way.

In this paper, we prove existence of $p$-adic $L$-functions for $p$-ordinary RACARs of $\text{GL}_3(A_{\mathbb{Q}})$, making no self-duality or functorial lift assumptions. This provides the first examples of $p$-adic $L$-functions for general type RACARs of $\text{GL}_n(A_{\mathbb{Q}})$ for any $n \geq 3$.

We briefly highlight some further strengths of our construction:

- We work in arbitrary cohomological weight, and prove the so-called ‘Manin relations’: that is, we construct a single $p$-adic measure that sees $L$-values at all critical integers (rather than a separate measure for each different critical integer).

- We actually prove a more general conjecture of Panchishkin [Pan94] that refines Coates–Perrin-Riou: rather than Coates–Perrin-Riou’s assumption of Borel-ordinarity at $p$, we impose $p$-ordinarity only along the maximal standard parabolic $P_1$ with Levi $\text{GL}_1 \times \text{GL}_2$. 

\textbf{Abstract.} Let $\Pi$ be a regular algebraic cuspidal automorphic representation of $\text{GL}_3(A_{\mathbb{Q}})$. When $\Pi$ is $p$-ordinary for the maximal standard parabolic with Levi $\text{GL}_1 \times \text{GL}_2$, we construct a $p$-adic $L$-function for $\Pi$. More precisely, we construct a (single) bounded measure $\eta_{p}(\Pi)$ on $\mathbb{Z}_p^\times$ attached to $\Pi$, and show it interpolates all the critical values $L(\Pi \times \eta, j)$ at $p$ in the left-half of the critical strip for $\Pi$ (for varying $\eta$ and $j$). This proves conjectures of Coates–Perrin-Riou and Panchishkin in this case. Our construction uses the theory of spherical varieties to build a “Betti Euler system”, a norm-compatible system of classes in the Betti cohomology of a locally symmetric space for $\text{GL}_3$. We work in arbitrary cohomological weight, allow arbitrary ramification at $p$ along the other maximal standard parabolic, and make no self-duality assumption.

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To illustrate this, consider the fundamental case where $\Pi$ is a regular algebraic cuspidal automorphic representation (RACAR) of $\text{GL}_3(A_{\mathbb{Q}})$. Then Coates–Perrin-Riou predict that when $\Pi$ is unramified and ordinary at $p$, there exists a $p$-adic measure on $\mathbb{Z}_p^\times$ — the $p$-adic $L$-function of $\Pi$ — interpolating all its critical $L$-values at $p$. We make this precise in Conjecture B.

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We allow arbitrary ramification at $p$. As a result, our construction applies even to a class of RACARs that have infinite slope at $p$ for the Borel subgroup (those that are ordinary along $P_1$, but that have infinite slope along the other maximal parabolic subgroup $P_2$).

We believe that our result is “optimal”, in the sense that $p$-adic $L$-functions for RACARs of $GL_3$ should only exist as bounded measures when these conditions hold. (Our methods can be adapted to construct finite-order tempered distributions interpolating $L$-values of RACARs which have positive, but sufficiently small, slope for $P_1$. This will be pursued elsewhere.)

1.2. Our results. To motivate the precise form of our main result, we first state Panchishkin’s refinement of the Coates–Perrin-Riou conjecture for $GL_n$.

Let $n = 2m + 1$ be odd. Let $\Pi$ be a (unitary) RACAR of $GL_n(\mathbb{A}_Q)$ with central character $\omega_\Pi$ and weight $\lambda = (\lambda_1, \ldots, \lambda_{2m+1})$. Note $\lambda_1 \geq \lambda_{i+1}$, $\lambda_m = 0$, and $\lambda_i = -\lambda_{2m+1-i}$. Let

$$\text{Crit}^-(\Pi) = \{(j, \eta) : -\lambda_m - 1 \leq j \leq 0, \ \eta \omega_\Pi(-1) = (-1)^j\},$$

$$\text{Crit}^+ (\Pi) = \{(j, \eta) : 1 \leq j \leq 1 + \lambda_m - 1, \ \eta \omega_\Pi(-1) = (-1)^{j+1}\},$$

where $j \in \mathbb{Z}$ and $\eta$ runs over Dirichlet characters (see Proposition 2.4). These are the critical values of $L(\Pi, s)$ in the left and right halves of the critical strip respectively. Let $e_{\infty}(\Pi, j)$ be the modified Euler factor at infinity of $[\text{Coa}89, \text{§}1]$, which is an explicit rational multiple of a power of $2\pi i$ (see §2.5), depending only on $j$ and $\lambda$.

Algebraicity. As a prerequisite for $p$-adic interpolation, we first need an algebraicity result for $L$-values. Let $E$ denote the rationality field of $\Pi$. If $\eta$ is a Dirichlet character, let $E[\eta]$ denote the extension of $E$ obtained by the values of $\eta$. The following is $[\text{Coa}89, \text{Period Conjecture}]$, a reformulation of the conjectures of $[\text{Del}79]$ better suited to $p$-adic interpolation:

**Conjecture A.** There exist complex periods $\Omega^\pm_\Pi \subset \mathbb{C}^\times$ such that for all $(j, \eta) \in \text{Crit}^\pm(\Pi)$, we have

$$e_{\infty}(\Pi, j) \cdot \frac{L(\Pi \times \eta, j)}{\Omega^\pm_\Pi} \cdot G(\eta^{-1})^{(3\pm1)/2} \in E[\eta],$$

where $G(\eta)$ is the Gauss sum. Moreover, this ratio depends $\text{Gal}(E[\eta]/E)$-equivariantly on $\eta$.

Note that it suffices to prove the conjecture for one choice of the sign $\pm$, since the functional equation interchanges $\text{Crit}^\pm(\Pi)$ and $\text{Crit}^\mp(\Pi')$. A partial result towards this conjecture is known for $n = 3$, by work of Mahnkopf and Kasten–Schmidt (see $[\text{RS}17]$ for an overview): they show that the formula holds with $e_{\infty}(\Pi, j)$ replaced by some other (inexplicit) constants $\hat{e}_{\infty}(\Pi, j) \in \mathbb{C}^\times$.

**P-adic interpolation.** Now suppose $p$ is a prime, and let $P_m$ be the block-upper-triangular parabolic subgroup with Levi subgroup $GL_m \times GL_{m+1}$. The Hecke operator $U_p$ associated to the double coset of $\left\{ \begin{smallmatrix} p^m \\ 1 \\ 1^{m+1} \end{smallmatrix} \right\}$ acts on the $N_m(\mathbb{Z}_p)$-invariants of $\Pi_p$; and we say $\Pi_p$ is $P_m$-ordinary if there exists a vector in this space which is a $U_p$-eigenvector with $p$-adic unit eigenvalue (see §2.7). Let $\text{Crit}^\pm_p(\Pi) \subset \text{Crit}^\pm(\Pi)$ be the subset of $(j, \eta)$ where $\eta$ has $p$-power conductor. For such $(j, \eta)$, let $e_p(\Pi, j, \eta) \in E[\eta, G(\eta)]$ denote the modified Euler factor at $p$ of $[\text{Coa}89, \text{§}2]$ (see §2.10).

We work with $C^\pm_p(\Pi)$, i.e. with $(j, \eta)$ where $j \leq 0$. We later prefer to work with non-negative integers, so henceforth we exchange $j$ and $-j$. Let $L(p)(\pi \times \eta, s)$ be the $L$-function without its Euler factor at $p$. Then the conjectures of Coates–Perrin-Riou and Panchishkin predict:

**Conjecture B.** Suppose $\Pi$ admits a $P_m$-ordinary $p$-refinement $\tilde{\Pi}$. Then there exists a $p$-adic measure $L(p)(\Pi)$ on $\mathbb{Z}_p^\times$ such that for all $(-j, \eta) \in \text{Crit}^\pm_p(\Pi)$, we have the interpolation

$$\int_{\mathbb{Z}_p^\times} \eta(x) x^j \cdot dL(p)(\tilde{\Pi})(x) = e_{\infty}(\Pi, -j) \cdot e_p(\tilde{\Pi}, \eta, -j) \cdot \frac{L(p)(\Pi \times \eta, -j)}{\Omega^p_{\tilde{\Pi}}}.$$

We call $L(p)(\tilde{\Pi})$ the $p$-adic $L$-function of $\tilde{\Pi}$. If $L(p)(\tilde{\Pi})$ exists, then because it is a measure (i.e. a bounded distribution) it is uniquely determined by its interpolation property.

If instead $\Pi$ admits a $P_{m+1}$-ordinary $p$-refinement, then there should be a $p$-adic measure on $\mathbb{Z}_p^\times$ interpolating the values in $\text{Crit}^\pm_p(\Pi)$. If $\Pi$ is $P_{m+1}$-ordinary, then $\Pi^\vee$ is $P_{m}$-ordinary, so this

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1. The case $n = 2m$ is similar, but in many ways simpler: only the $(m, m)$ parabolic subgroup is relevant; and $L(\Pi \times \eta, j)$ is critical for all $j$ in a certain interval, independently of the parity of $\eta$, so we do not need to split the critical interval into the two halves $\text{Crit}^\pm$. 

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\[ \text{David Loeffler and Chris Williams} \]
follows from Conjecture B for $\Pi^\vee$ and the functional equation (which exchanges $\text{Crit}^+_{\mathfrak{p}}(\Pi^\vee)$ and $\text{Crit}^+_0(\Pi)$).

In this paper, we prove:

**Theorem C.** Conjectures A and B hold in full when $n = 3$. □

1.3. Mahnkopf’s work on algebraicity. The starting point of our proof is Mahnkopf’s work on (1.1). He uses the Rankin–Selberg integral for $GL_3 \times GL_2$: taken with $\Pi$ on $GL_3$ and an Eisenstein series on $GL_2$, this integral computes a product of (twisted) $L$-functions for $\Pi$.

Mahnkopf [Mah98] gave a cohomological interpretation of this integral, as we now sketch. We take $\Pi$ to have weight $\lambda = (a, 0, -a)$, and let $V_\lambda$ be the $GL_3$-representation of highest weight $\lambda$. For $j \geq 0$, let $Y_j^{GL_3}$ be the $GL_2$-representation of highest weight $(x, y) \mapsto y^{-j}$. Let $Y_j^{GL_3}(U)$ be the locally symmetric space for $GL_3$ (of some level $U$), and let $Y_j^{GL_3}(p^n)$ be the locally symmetric space (of level $\Gamma_1(p^n)$) for $GL_2$. For a Dirichlet character $\eta$ of conductor $p^j$, one has:

- a compactly supported Betti class $\phi_\eta \in H_2^0(Y_j^{GL_3}(U), \mathbb{C}V_\eta(\mathbb{C}))$ attached to any $\varphi \in \Pi$, and
- and Harder’s weight $j$ Eisenstein class $Eis^j(\Pi, \eta) \in H^j(Y_j^{GL_3}(p^n), \mathbb{C}V_\eta(\mathbb{C}))$.

We have the following crucial branching law (cf. [Mah00, Lem. 3.1]):

\begin{equation}
(1.3) \quad \text{If } (-j, \eta) \in \text{Crit}^+_{\mathfrak{p}}(\Pi), \text{ then } Y_j^{(0, -j)} \subset V_\lambda|_{GL_2} \text{ with multiplicity one.}
\end{equation}

By pushing forward $Eis^j(\Pi, \eta)$ to $Y_j^{GL_3}$, and using (1.3), one can then define a pairing

\begin{equation}
(1.4) \quad \langle - , - \rangle : H^2(Y_j^{GL_3}(U), V_\eta) \times H^1(Y_j^{GL_3}(p^n), V_j^{GL_3}) \to \mathbb{C}
\end{equation}

such that $\langle \phi_\eta, Eis^j(\Pi, \eta) \rangle$ is a Rankin–Selberg integral; and for appropriate $\varphi$, one of the $L$-values it computes is $L(\Pi \times \eta, -j)$. Mahnkopf and Kasten–Schmidt used this in [Mah98, Mah00, KS13] to prove that for each fixed $-j$, (1.1) holds up to replacing $e_{\infty}(\Pi, -j)$ with some (implicit) $e_{\infty}(\Pi, -j) \in \mathbb{C}^\times$.

1.4. $p$-adic interpolation. Given Mahnkopf’s results, we first show Conjecture B holds for the inexplicit values $e_{\infty}(\Pi, -j)$. There are two major aspects to the proof:

(I) $p$-adically interpolate the pushforward of $Eis^j(\Pi, \eta)$ to $Y_j^{GL_3}$ as $\eta$ (hence $n$) varies;

(II) $p$-adically interpolate the pushforward of $Eis^j(\Pi, \eta)$ to $Y_j^{GL_3}$ as $j$ varies.

The classes $Eis^j(\Pi)$ have algebraic analogues, and are themselves known to vary $p$-adically as $n$ and $j$ vary. Conjecture B has remained unknown in this case, however, because it is hard to control their denominators and much more difficult to $p$-adically vary their pushforwards to $Y_j^{GL_3}$.

Our key innovations in solving (I) and (II) are:

- the use of Beilinson’s motivic Eisenstein classes $Eis^{j}_{\text{mot}, \Phi_{n}}$ (see §5.3), and

- the systematic variation of a third parameter: the definition of the pairing (1.4).

The motivic classes are norm compatible, so their Betti realisations – the Betti–Eisenstein classes $Eis^{j}_{\Phi_{n}}$ – form a tower as $n$ varies. Kings has shown they interpolate well as $j$ varies (see §5.3.4).

Whilst these classes are not integral, crucially for an auxiliary integer $c$ one can define ‘$c$-smoothed’ modifications $Eis^{j}_{\Phi_{n}, c}$, that are integral and which remain norm-compatible (see §5.3.3).

By varying the pairing, we build a machine that ports these compatibility and integrality properties from $GL_2$ to $GL_3$. To make this more precise, we elaborate on our approach, which is somewhat different to [Mah98] from the outset. Let $H := GL_2 \times GL_1$, and let $\iota : H \hookrightarrow GL_3$ be the map $(\gamma, z) \mapsto (\gamma, z)$. Our construction involves pulling back Eisenstein classes under the natural projection $H \to GL_2$, applying the branching law (1.3), pushing forward under $\iota : H \to GL_3$, and then twisting by a certain operator $u\tau^n \in GL_3(\mathbb{Q}_p)$, where $u = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$ and $\tau = \text{diag}(p, 1, 1)$. Varying the parameters in this process varies the pairing (1.4). Ultimately, by ‘spreading out’ this twisted pushforward map over a group algebra, we construct a machine that is, at level $n$, a map

$$H^1\left(Y_j^{GL_3}(p^n), V_{(0, -j)}(\mathbb{Z}_p)\right) \to H^3\left(Y_j^{GL_3}(U), V_\lambda(\mathbb{Z}_p)\right) \otimes \mathbb{Z}_p[[u]][Z/p^n]^\times, \quad cEis^{j}_{\Phi_{n}, c} \mapsto cEis^{j}_{\Phi_{n}, c}.$$ 

Note that since $Y_j^{GL_3}$ is 5-dimensional, $H^2(Y_j^{GL_3}(U), V_\lambda)$ is Poincaré dual to $H_2^1(Y_j^{GL_3}(U), V_\lambda)$.

Importantly, $\mathcal{P}^1(\mathbb{Z}_p) \cdot u \cdot H(\mathbb{Z}_p)$ is open and dense in $GL_3(\mathbb{Z}_p)$, where we recall the parabolic $P_1 \subset GL_3$ from above. This provides a link with the theory of spherical varieties, via $GL_3/H$, as
explored in [Loc]. Exploiting this link, in Theorem 6.11 we show that for each fixed $0 \leq j \leq a$ the machine preserves norm-compatibility, but not on the nose: rather, for the Hecke operator $U_{p,1}$ at $P_1$, we prove that

$$\text{Norm}^{n+1}_n(\xi^{[j]}_{n+1}) = U_{p,1}(\xi^{[j]}_n).$$

Now let $\varphi \in \Pi$ be a suitable $U_{p,1}$-eigenform with an associated class $\phi_\varphi \in H^2_c(Y^{\text{GL}(3)}(\mathbb{H}), V_{\gamma}^j(\mathcal{O}))$, where $\mathcal{O}$ is a finite extension of $\mathbb{Z}_p$, and write $\tilde{\Pi} = (\Pi, \varphi)$ for the corresponding refinement. Let $\alpha_{p,1}$ be the $U_{p,1}$-eigenvalue of $\varphi$. Suppose $\varphi$ is $P_1$-ordinary, i.e. $\alpha_{p,1} \in \mathcal{O}^\times$; then for each fixed $j$, pairing $\alpha_{p,1} \varphi$ with $\xi^{[j]}_n$ (under Poincaré duality (3.6)) yields an element $\xi^{[j]}_n \in \lim \mathcal{O}[\{(\mathbb{Z}/p^n)^\times\} \cong \mathcal{O}[\mathbb{Z}_p^\times)].$ In §7, we show that if $(-j, \eta) \in \text{Crit}_F(\Pi)$, then integrating this measure against $\eta$ on $\mathbb{Z}_p^\times$ interpolates a relevant Rankin–Selberg integral, and in §§8 and 9 we show that this integral computes (a given multiple of) $L(\pi \otimes \eta, -j)$. In particular, this solves (I).

It remains to prove (II), the compatibility between the $a+1$ measures $\{\xi^{[j]}_n\}_{n=0}$. Again we exploit the connection to spherical varieties: our set-up puts us in the framework of [LRZ], which we apply to prove

$$\int_{\mathbb{Z}_p^\times} x^j f(x) \cdot d_x \xi^{[0]}(x) = \int_{\mathbb{Z}_p^\times} f(x) \cdot d_x \xi^{[j]}(x),$$

i.e. the $\xi^{[j]}$ are Tate twists of each other. Thus the measure $\xi L_p(\tilde{\Pi}) := \xi^{[0]}$ on $\mathbb{Z}_p^\times$ satisfies

$$\int_{\mathbb{Z}_p^\times} \eta(x) x^j \cdot \xi L_p(\tilde{\Pi}) = \int_{\mathbb{Z}_p^\times} \eta(x) \cdot \xi^{[j]} = (\ast) \cdot L^{(p)}(\Pi \times \eta, -j),$$

where the term $(\ast)$ is a product of a global period, certain local zeta integrals, and a ‘correction’ factor for the smoothing integer $c$. We can normalise away this correction term, leaving a measure $L_p(\tilde{\Pi})$ on $\mathbb{Z}_p^\times$ that is independent of $c$. Away from infinity, for a good choice of $\varphi$, these local zeta factors are explicitly computed in §8, and shown to compute the correct interpolation factors.

This leaves the local zeta integral at $\infty$. By definition, for each $j$ this Rankin–Selberg integral is the inexplicit factor $\tilde{e}_\infty(\Pi, -j)$ from [Mah00], which is non-zero by [KS13,Sun17]. In particular, $L_p(\tilde{\Pi})$ satisfies (1.2) up to replacing $e_{\infty}(\Pi, -j)$ with $\tilde{e}_\infty(\Pi, -j)$.

1.5. The factor at $\infty$. It remains to prove $\tilde{e}_\infty(\Pi, -j) = e_\infty(\Pi, -j)$, i.e. that the factor at infinity has the expected form.

By construction, the factor $\tilde{e}_\infty(\Pi, -j)$ depends only on $\Pi$ and $\Pi_{\infty}$, and in turn $\Pi_{\infty}$ depends only on $\lambda$ and $\omega_{\Pi}$. Whilst we do not evaluate this integral directly, we do know that it is non-vanishing (for all $\Pi_{\infty}$ and $j$) by [KS13]. To get the correct interpolation factor $e_\infty$, we exploit the fact that Theorem C is already known in full when $\Pi$ is a (twist of a) symmetric square from GL2. In this case, we compute the ratio of our measure $L_p(\tilde{\Pi})$ with the symmetric square $p$-adic $L$-function, and show this ratio is constant. By (II), we thus deduce that the $\tilde{e}_\infty(\Pi, -j)$‘s satisfy the expected compatibility as $j$ varies. Up to a global renormalisation, we thus deduce $\tilde{e}_\infty(\Pi, -j) = e_\infty(\Pi, -j)$ for all $j$ in the left-half of the critical strip. Combining with §1.4, we deduce $L_p(\tilde{\Pi})$ satisfies (1.2), completing the proof of Conjecture B when $n = 3$.

We emphasise finally that our main result (and main motivation) is the $p$-adic interpolation of $\lambda$-values, that is, the proof of Conjecture B for $n = 3$. However, we also get the following cute application. Since our constructions towards Conjecture B do not depend on prior knowledge of Conjecture A at any point, and $e_p(\Pi, \eta, -j) \in G(\eta^{-1}) \cdot E[\eta]$, we obtain a full proof of Conjecture A when $n = 3$ as an immediate corollary of our main theorem.

1.6. Relation to previous literature. The most notable partial results towards Theorem C came in works of Mahnkopf [Mah00] (for trivial weight) and Geroldinger [Ger15] (comological weight), who gave constructions of algebraic $p$-adic distributions satisfying a partial form of the interpolation (1.2). More precisely, for each critical integer $j$ for $\Pi$, they constructed a separate algebraic distribution $L^\mu_p(\tilde{\Pi})$ (denoted $\mu_{\Pi,j}$ in op. cit.) satisfying a formula similar to (1.2) for that fixed $j$ and all sufficiently ramified $\eta$.

However, using their methods, they were not able to sufficiently control the denominators of their distributions, or obtain any kind of compatibility for varying $j$. In particular, their

\footnote{By considering $\Pi^G$ and reflecting in the functional equation, we also obtain this in the right-half.}
distributions only defined functions on the set of \( \text{locally-algebraic} \) characters of \( \mathbb{Z}_p^\times \) of degree \( \leq a \); they could not prove that their distributions had any uniquely-determined extension from this discrete set to the whole weight space of continuous characters of \( \mathbb{Z}_p^\times \). Thus their methods did not give a \( p \)-adic \( L \)-function in any usual sense (notwithstanding the titles of these papers). This is the fundamental breakthrough in the present work: our methods give uniform control over the denominators of Eisenstein classes, allowing us to construct a uniquely-determined, bounded measure interpolating all critical values.

(Whilst our \( p \)-adic methods differ totally from those of [Mah00] and [Ger15], these references do contain excellent accounts of a number of the automorphic aspects we require, particularly the local zeta integral at infinity.)

In the special case when \( \Pi \) is a symmetric square lift of a \( \text{RACAR} \) of \( \text{GL}_2 \), a different construction of the \( p \)-adic \( L \)-function is possible, using Rankin–Selberg convolutions of the \( \text{GL}_2 \) cusp form with half-integer weight theta-series; see e.g. [Sch88, Hid90, DD97]). However, these methods are completely specific to the case of symmetric-square lifts.

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## 2. Preliminaries: automorphic representations

### 2.1. Characters.

If \( \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \) is a Dirichlet character, there is a unique finite-order Hecke character \( \tilde{\chi} : \mathbb{Q}^\times \backslash \mathbb{A}_F^\times / \mathbb{A}_F^\times \to \mathbb{C}^\times \) such that \( \tilde{\chi}(\varpi_l) = \chi(l) \) for all primes \( l \nmid N \), where \( \varpi_l \) is a uniformiser at \( l \); conversely, every finite-order Hecke character is \( \tilde{\chi} \) for a unique primitive Dirichlet character \( \chi \).

Note that the restriction of \( \tilde{\chi} \) to \( \tilde{\mathbb{Z}}^\times \subseteq \mathbb{A}_F^\times \) is the inverse of the composition \( \tilde{\mathbb{Z}}^\times \to (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times \). Moreover, if \( l \mid N \), then \( \tilde{\chi}_l = \chi^{(l)}(l) \), where we have written \( \chi = \chi^{(l)} a \) product of characters of \( l \)-power and prime-to-\( l \) conductor. Given a representation \( \Pi \) of \( \text{GL}_3(\mathbb{A}_F) \), we write \( \Pi \times \chi \) for the representation \( \Pi \times \tilde{\chi} \).

We let \( \psi \) be the unique character of the additive group \( \mathbb{A}/\mathbb{Q} \) such that \( \psi(x) = \exp(-2\pi i x) \) for \( x \in \mathbb{R} \). Note that the restriction of \( \psi \) to \( \mathbb{Q}_l \), for a finite prime \( l \), maps \( 1/l^n \) to \( \exp(2\pi i / l^n) \) for all \( n \in \mathbb{Z} \). Let \( \varepsilon(\tilde{\chi}, \psi) \) be the local \( \varepsilon \)-factor (with respect to the unramified Haar measure \( dx \)), as defined in [Tat79] for example. Then we have

\[
\prod_{l \mid N} \varepsilon(\tilde{\chi}_l, \psi_l) = G(\chi) := \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) \exp(2\pi i a/N)
\]

for \( \chi \) primitive of conductor \( N \). (We do not include the Archimedean root number here.)

### 2.2. Algebraic representations.

If \( J \) is a split reductive group, and \( \mu \) a \( B_J \)-dominant weight for some choice of Borel \( B_J \subset J \), then we write \( V_\mu^J \) for the unique irreducible algebraic representation of \( J \) of highest weight \( \mu \). When \( J = \text{GL}_3 \), we will drop the \( J \) and just write \( V_\mu \).

Let \( T \) denote the diagonal torus of \( \text{GL}_3 \) (identified with \( (\mathbb{G}_m)^3 \) in the obvious fashion), and \( B = T \ltimes N \) the upper-triangular Borel subgroup. The \( B \)-dominant weights for \( \text{GL}_3 \) are of the form \( \lambda = (a, b, c) \in \mathbb{Z}^3 \), with \( a \geq b \geq c \). If \( E \) is any \( \mathbb{Q} \)-algebra, then we can realise \( V_\lambda(E) \) as a space of polynomial functions on \( \text{GL}_3 \), via

\[
V_\lambda(E) = \{ f : \text{GL}_3(E) \to E : f \text{ algebraic, } f(n^{-1} t g) = \lambda(t) f(g) \forall n^{-1} \in N^{-}(E), t \in T(E) \},
\]

where \( N^{-}(E) \) is the unipotent radical of the opposite Borel. We get a natural left-action of \( \gamma \in \text{GL}_3(E) \) on \( V_\lambda(E) \) by \( (\gamma \cdot f)(g) = f(\gamma g) \). Let \( V_\lambda^\vee(E) \) denote the \( E \)-linear dual, with the dual left action \( (\gamma \cdot \mu)(f) = \mu(\gamma^{-1} \cdot f) \).

A weight \( \lambda = (a, b, c) \) is pure if \( a + c = 2b \). These are precisely the weights such that \( V_\lambda^\vee \) is isomorphic to a twist of \( V_\lambda \).

### 2.3. Automorphic representations for \( \text{GL}_3 \).

We recall some standard facts about automorphic representations of \( \text{GL}_3 \) (for a fuller account, see [Mah05, §3.1], summarising [Clo90, §3]). Let
II be a cuspidal automorphic representation of \( \GL_3(\mathbb{A}) \), with central character \( \omega_\Pi \). We identify II with its realisation in \( L^1_\nu(\GL_3(\mathbb{Q}) \backslash \GL_3(\mathbb{A})) \), considering any \( \varphi \in \Pi \) as a function on \( \GL_3(\mathbb{A}) \).

Let \( \mathfrak{g}_3 = \text{Lie}(\GL_3) \). Recall that the centre of the universal enveloping algebra at \( \infty \) acts on \( \Pi_\infty \) via a ring homomorphism \( Z(U(\mathfrak{g}_3)_\mathbb{C}) \to \mathbb{C} \) (the infinitesimal character of \( \Pi_\infty \)). We say II is regular algebraic of weight \( \lambda \) if \( \Pi_\infty \) has the same infinitesimal character as the irreducible algebraic representation \( V_\lambda \), for some dominant integral weight \( \lambda \). (This determines \( \lambda \) uniquely.) We use the abbreviation “RACAR” for “regular algebraic cuspidal automorphic representation”.

2.3.1. Whittaker models. We denote the standard Whittaker model of \( \Pi \) by

\[
W_\psi : \Pi \cong W_\psi(\Pi) \subset \text{Ind}_{\mathbb{A}(\mathbb{A})}^{\GL_3(\mathbb{A})} \psi, \quad \varphi \mapsto W_\psi(\varphi)(g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \varphi(ng) \psi^{-1}(n) \, dn.
\]

where \( \psi \left( \begin{pmatrix} 1 & \ast & \ast \\ & 1 & \ast \\ & & 1 \end{pmatrix} \right) := \psi(x + y) \). We denote cusp forms in \( \Pi \) by \( \varphi \), and elements of \( W_\psi(\Pi) \) by \( \psi \).

2.3.2. Cohomological automorphic representations. Let \( K_{\infty} \) be a maximal compact subgroup of \( \GL_3(\mathbb{R}) \), \( Z_{\GL_3, \infty} = Z_{\GL_3}(\mathbb{R}) \), and \((-)^o\) denote the identity component. Write \( K_{3, \infty} = K_{\GL_3, \infty} Z_{\GL_3, \infty} \) for shorthand. We say \( \Pi \) is cohomological with coefficients in an algebraic representation \( W \) if

\[
\Pi^* = \text{Ind}_{K_{3, \infty}}^{\GL_3(\mathbb{A})} \omega(\mathbb{C}) \otimes W(\mathbb{C}) \neq 0.
\]

**Proposition 2.1.** [Clo90, Lem. 4.9]. *If II is a RACAR of weight \( \lambda \), then it is cohomological with coefficients in \( W = V_{\lambda}^\vee \) (and this is the unique irreducible representation for which \( II \) is cohomological). Moreover, \( \lambda \) is necessarily pure (as in §2.2).*

This cohomology is then supported in degrees 2 and 3 [Clo90, Lem. 3.14], and in each of these degrees (2.1) is necessarily one-dimensional [Mah05, (3.2), (3.4)]. We will consider throughout only the lowest degree \( i = 2 \); exactly as in [Mah00, §3.1] (where it is denoted \( \omega_\infty \)) we choose a generator

\[
\zeta_{\infty} \in H^2(\mathfrak{g}_3, K_{3, \infty}^o; II \otimes V_{\lambda}^\vee(\mathbb{C}))
\]

**Convention:** Let II be a RACAR of weight \( \lambda \). As in [Mah00, §1],3 without loss of generality we may normalise so that \( b = 0 \), so (by purity) \( \lambda = (a, 0, -a) \) for some \( a \geq 0 \). In this case, we see that

\[
\Pi_{\infty} \cong \text{Ind}_{P_3(\mathbb{R})}^{\GL_3(\mathbb{R})}(D_{2a+3}, \text{id}) \quad \text{or} \quad \text{Ind}_{P_3(\mathbb{R})}^{\GL_3(\mathbb{R})}(D_{2a+3}, \text{sgn}),
\]

where \( P_3 \) is the parabolic with Levi \( \GL_2 \times \GL_1 \), \( D_{2a+3} \) is the discrete series representation of \( \GL_2(\mathbb{R}) \) of lowest weight \( 2a + 3 \), and sgn is the sign character. In particular, this implies that its central character \( \tilde{\omega}_\Pi \) has finite order (i.e. it is the adelic character associated to a Dirichlet character \( \omega_\Pi \)), and hence II is unitary.

**Remark 2.2:** If \( \omega_\Pi \) is odd, then \( \Pi_{\infty} \cong \text{Ind}_{P_3(\mathbb{R})}^{\GL_3(\mathbb{R})}(D_{2a+3}, \text{id}) \) and \( \Pi_{\infty} \) is the twist of this by sgn when \( \omega_\Pi \) is even. In [Mah00], only the case of \( \omega_\Pi \) odd is considered, but we allow both signs here.

2.3.3. Conductors. It is known that, for any II as above, there exists an \( N \) such that II has non-zero invariants under the subgroup

\[
U_{\nu}^{\GL_3}(N) := \left\{ g \in \GL_3(\mathbb{Z}) : g \equiv \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \pmod{N} \right\}.
\]

Moreover, if \( N \) is the smallest such integer, then the invariants are one-dimensional [JPSS81, §5]; we denote this \( N_\Pi \) and call it the conductor of II. It is a standard result that there are only finitely many RACARs II of weight \( (a, 0, -a) \) and conductor \( N \), for any given \( a \) and \( N \).

2.3.4. Self-duality. We say II is self-dual if \( \Pi^* \cong II \), and more generally essentially self-dual if II is isomorphic to a twist of \( \Pi^* \). A theorem of Ramakrishnan [Ram14] shows that if II is an essentially self-dual RACAR of \( \GL(3) \), then there exists a non-CM-type cuspidal modular newform \( f \) of weight \( a + 2 \), and a character \( \nu \), such that \( II = \text{Sym}^2(f) \otimes \nu \).

**Remark 2.3:** Ash and Pollack have conjectured [AP08] that all level 1 RACARs of \( \GL_3 \) are self-dual, and arise as symmetric squares of level 1 cuspidal eigenforms for \( \GL_2 \). Examples which are not essentially self-dual do exist in higher levels; see the tables of [GKTV97] for examples.

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3Mahnkopf’s \( \ell_0 \) is our \( 2a + 2 \).
2.4. The L-function of \( \Pi \). We let \( L(\Pi, s) = \prod_{\ell < \infty} L(\Pi_{\ell}, s) \) denote the standard L-function of \( \Pi \) (without its Archimedean \( \Gamma \)-factors). We use the analytic normalisations here, so the Euler product defining \( L(\Pi, s) \) converges for \( \Re(s) > 1 \).

Since \( \Pi \) is cohomological, it is \( C \)-algebraic in the sense of [BG14], i.e. there exists a number field \( E \) such that \( \Pi_{\ell} \) is definable as an \( E \)-linear representation. Since half the sum of the positive roots is in the weight lattice for \( \text{GL}_3 \), it is also \( L \)-algebraic: that is, if \( E \) is any number field over which \( \Pi_{\ell} \) is definable, then for primes \( \ell \) such that \( \Pi_{\ell} \) is unramified, we have

\[
L(\Pi_{\ell}, s) = P_{\ell} \left( \varepsilon^{\ell s} \right)^{-1}, \quad P_{\ell}(X) = (1 - \alpha_{\ell}X)(1 - \beta_{\ell}X)(1 - \gamma_{\ell}X) \in E[X],
\]

where \( \alpha_{\ell}, \beta_{\ell}, \gamma_{\ell} \) are units outside \( \ell \), and have valuation \( \geq -1 - a \) at \( \ell \). If \( \ell \) is a ramified prime, then we still have \( L(\Pi_{\ell}, s) = P_{\ell}(\varepsilon^{\ell s}) \) for some polynomial \( P_{\ell} \in 1 + X E[X] \), but of degree \( < 3 \).

As \( \Pi_{\infty} \) is given by \((2a + 2) \oplus (\pm, 0)\) in the notation of [Kna94, §3], at infinity we have

\[
L_{\infty}(\Pi_{\infty}, s) = \Gamma_{C}(s + 1 - \kappa)\Gamma_{C}(s + a + 1)
\]

where \( \kappa = 0 \) if \( \omega_{\Pi} \) is even or \( \kappa = 1 \) if \( \omega_{\Pi} \) is odd, \( \Gamma_{C}(s) = (2\pi)^{-s}/2\Gamma(s/2) \), and \( \Gamma_{C}(s) = 2(2\pi)^{-s}\Gamma(s) \).

Proposition 2.4. Let \( \eta \) be a Dirichlet character. Then the critical values of \( L(\Pi \times \eta, s) \) are at \( s = j \) for integers \( j \) satisfying

\[
\{-a \leq j \leq 0 \text{ and } (-1)^j = \omega_{\Pi}(\eta(-1))\} \quad \text{or} \quad \{1 \leq j \leq 1 + a \text{ and } (-1)^j = -\omega_{\Pi}(\eta(-1))\}.
\]

In particular, the near-central values \( s = 0 \) and \( s = 1 \) of \( L(\Pi, s) \) are critical if and only if \( \omega_{\Pi} \) is even. Note that none of these critical values can be zero (since \( L(\Pi, s) \neq 0 \) for \( \Re(s) \geq 1 \)).

Remark 2.5: An important example of \( \Pi \) satisfying our conditions is the (normalised) symmetric square lift of a modular form \( f \) of weight \( k = a + 2 \geq 2 \) and character \( \varepsilon_{f} \). Then we have \( L(\Pi, s) = L(\text{Sym}^{2} f, s + a + 1) \) and \( \omega_{\Pi}(-1) = (-1)^{a} \), so the above is consistent with the fact that \( L(\text{Sym}^{2}(f), 1) \) and \( L(\text{Sym}^{2}(f) \times \varepsilon_{f}^{-1}, k - 1) \) are always critical values (independent of \( a \)).

2.4.1. Galois representations. By [HLTT16], for each prime \( p \) and embedding \( \iota : E \hookrightarrow \overline{\mathbb{Q}}_{p} \), there is a Galois representation \( \rho_{\Pi p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_{3}(\overline{\mathbb{Q}}_{p}) \), uniquely determined up to semisimplification, such that for all primes \( \ell \neq p \) such that \( \Pi_{\ell} \) is unramified, we have

\[
\det(1 - X \rho_{\Pi_{\ell},i}(\text{Frob}_{\ell}^{-1})) = P_{\ell}(X).
\]

Here \( \text{Frob}_{\ell} \) is an arithmetic, and hence \( \text{Frob}_{\ell}^{-1} \) a geometric, Frobenius.

Conjecturally \( \rho_{\Pi_{p}} \) should be de Rham at \( p \), with Hodge–Tate weights \( (-1 - a, 0, 1 + a) \); and if \( \Pi_{p} \) is unramified, then it should be crystalline at \( p \), and the eigenvalues of \( \varphi \) on \( D_{\text{cris}}(\rho_{\Pi_{p}}) \) should be \((\alpha_{p}, \beta_{p}, \gamma_{p}) \). More generally, even if \( \Pi_{p} \) is ramified, the Weil–Deligne representation \( \rho_{\Pi_{p}} \) should be related to \( \Pi_{p} \) via the local Langlands correspondence. This conjecture is “local-global compatibility for \( \ell = p \)”:

it is known if \( \Pi \) is essentially self-dual. None of our results will logically rely on this conjecture, or indeed on the existence of \( \rho_{\Pi_{p}} \); but it serves as important motivation to explain why the statements are natural ones.

2.5. Rationality of L-values. Recall Proposition 2.4, and let

\[
\text{Crit}^{-}(\Pi) = \{(j, \eta) : -a \leq j \leq 0, \ (-1)^{j} = \omega_{\Pi}(\eta(-1))\}
\]

\[
\text{Crit}^{+}(\Pi) = \{(j, \eta) : 1 \leq j \leq 1 + a, \ (-1)^{j} = -\omega_{\Pi}(\eta(-1))\}
\]

so \( L(\Pi \times \eta, j) \) is critical if and only if \( (j, \eta) \in \text{Crit}^{-}(\Pi) \cup \text{Crit}^{+}(\Pi) \). For later use we write \( \text{Crit}^{\pm}(\Pi) = \{(j, \eta) \in \text{Crit}(\Pi) : \eta \text{ has } p\text{-power conductor}\} \).

Convention: We will concentrate on \( \text{Crit}^{-}(\Pi) \), and replace \( j \) with \(-j \), with \( 0 \leq j \leq a \).

If \( \Pi \) is a RACAR of \( \text{GL}_{3} \), then conjecturally it has an attached motive \( M_{\Pi} \) of weight \( 0 \). We note that if \((j, \eta) \in \text{Crit}^{-}(\Pi) \), then the Hodge decomposition is \( H_{\text{dR}}(M_{\Pi}(-j)(\eta)) \otimes C = H^{-a-1,j+1}\oplus H^{j-1} \oplus H^{j+1} \oplus H^{j-1} \oplus H^{j-1} \oplus H^{j-1} \), with each summand 1-dimensional over \( C \). This motivates the following modified Euler factor at infinity (denoted \( L^{(i)}_{\infty}(M_{\Pi}(-j)(\eta)) \) in [Coa89, §1]):

Definition 2.6. For \( j \in \mathbb{Z}_{\geq 0} \), let

\[
e_{\infty}(\Pi, -j) := i(-a-1) \cdot \Gamma_{C}(a + 1 - j) = 2 \cdot (2\pi i)^{-a-1} \cdot \Gamma(a + 1 - j).
\]
Conjecture 2.7. There exists a complex period $\Omega_{\Pi} \in \mathbb{C}^\times$ such that if $(-j, \eta) \in \text{Crit}^-(\Pi)$, then
\begin{equation}
\epsilon_{\infty}(\Pi, -j) \cdot \frac{L(\Pi \times \eta, -j)}{\Omega_{\Pi}} \in E[\eta, G(\eta)].
\end{equation}

There are partial results towards this conjecture (see [RS17] for an overview):

**Theorem 2.8** (Mahnkopf [Mah98, Mah00], Kasten–Schmidt [KS13]). Conjecture 2.7 holds up to replacing $\epsilon_{\infty}(\Pi, -j)$ with an inexplicit scalar $\epsilon_{\infty}(\Pi, -j)$ in $\mathbb{C}^\times$.

**Remark 2.9:** We shall recall in §3.3.1 the definition of a cohomological (Whittaker) period $\Theta_{\Pi}$ (2.5) since it is far from the centre $e_{\Pi}$.

**Theorem 2.8** (Mahnkopf [Mah98, Mah00], Kasten–Schmidt [KS13]). Conjecture 2.7 holds up to replacing $\epsilon_{\infty}(\Pi, -j)$ with an inexplicit scalar $\epsilon_{\infty}(\Pi, -j)$ in $\mathbb{C}^\times$.

**Remark 2.9:** We shall recall in §3.3.1 the definition of a cohomological (Whittaker) period $\Theta_{\Pi} \in \mathbb{C}^\times$ associated to $\Pi$ (well defined up to $E^\times$). Note $(\omega_{\Pi}^{-1}, 1) \in \text{Crit}^+(\Pi)$, and $L(\Pi \times \omega_{\Pi}^{-1}, 1) \neq 0$ (since it is far from the centre $e_{\Pi}$ of the critical strip); then, precisely, we take $\Omega_{\Pi}$ to be an algebraic multiple of $\Theta_{\Pi}/L(\Pi \times \omega_{\Pi}^{-1}, 1)$. (Analogously, we define $\Omega_{\Pi}$ by scaling instead by $L(\Pi \times \omega_{\Pi}^{-1}, 0)$). Given Conjecture 2.7, the analogous algebraicity for $(j, \eta) \in \text{Crit}^+(\Pi)$, with $\Omega_{\Pi}$, would follow by reflecting Conjecture 2.7 for $\Pi^\vee$ in the functional equation).

**2.6. Parabolics and Hecke operators at $p$.** Let $p$ be a prime. Consider the two maximal parabolic subgroups
\[ P_1 = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} * & * & * \\ 0 & 0 & 1 \\ 0 & 1 & * \end{pmatrix}. \]

Denote the parahoric subgroup for $P_i$ by
\begin{equation}
J_{P_i} := \{ g \in GL_3(\mathbb{Z}_p) : g \text{ (mod $p$)} \in P_i(\mathbb{F}_p) \}.
\end{equation}

Let $N_{P_i}$ be the unipotent radical of $P_i$. If $K_p \subseteq J_{P,i}$ is any open subgroup containing $N_{P_i}(\mathbb{Z}_p)$ and having an Iwahori decomposition with respect to $P_i$, then we have a normalised Hecke operator
\begin{equation}
U_{p,i} = p^k [K_p \tau_i K_p], \quad \tau_1 = \begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} p & 1 \\ 1 & p \end{pmatrix} \in GL_2(\mathbb{Z}_p).
\end{equation}

**2.7. $P_1$-refinements.** We may consider the (unitarily normalised) Jacquet module $J_{P_1}(\Pi_p)$, which is an admissible smooth representation of the Levi subgroup $M_i$ of $P_i$.

**Definition 2.10.** A $P_1$-refinement $\alpha_p$ of $\Pi_p$ is a choice of one of the finitely many characters of $\mathbb{Q}_p^\times$ appearing in the action of \left\{ $\begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix} : x \in \mathbb{Q}_p^\times \right\}$ on $J_{P_1}(\Pi_p)$. Similarly a $P_2$-refinement is a character $\alpha_p'$ by which \left\{ $\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} : x \in \mathbb{Q}_p^\times \right\}$ acts on $J_{P_2}(\Pi_p)$.

**Remark 2.11:** If $\Pi_p$ is unramified, then its $P_1$-refinements are the unramified characters mapping $p$ to the Satake parameters of $\Pi_p$, so our notation $\alpha_p$ is consistent with §2.4. (We shall repeatedly blur the distinction between an unramified character of $\mathbb{Q}_p^\times$ and its value at $p$.)

**Definition 2.12.** By a $P_1$-refined automorphic representation $\hat{\Pi}$, we mean a pair $(\Pi, \alpha_p)$, where $\alpha_p$ is a choice of $P_1$-refinement of $\Pi_p$.

**Slopes.** Fix an embedding $\mathcal{E} \hookrightarrow \mathbb{Q}_p$, defining an extension of the $p$-adic valuation $v_p$ to $\mathcal{E}$.

**Definition 2.13.** The slope of a $P_1$-refinement (resp. $P_2$-refinement) is $v_p(\alpha_p(p))$ (resp. $v_p(\alpha_p'(p))$).

One can show that if $\Pi$ is a RACAR of weight $(a, 0, -a)$, then the slope of any $P_1$-refinement must lie in the interval $[-1 - a, 1 + a]$. 

8
Definition 2.14. We say that $\Pi$ is $P_1$-nearly-ordinary if there exists a $P_1$-refinement $\alpha_p$ of slope $-1 + a$. If there exists such a refinement with $\alpha_p$ unramified, we say $\Pi$ is $P_1$-ordinary.

For a $P_1$-nearly-ordinary representation, there is in fact a unique $P_1$-refinement of this slope, and we describe it as the nearly-ordinary (resp. ordinary) refinement.

Example 2.15: We briefly recall the classification of generic representations of $GL_3$, and explain the conditions under which these are (nearly) ordinary.

- If $\Pi_p$ is supercuspidal, it admits no $P_1$-refinements or $P_2$-refinements, and hence is never nearly-ordinary for any parabolic.
- If $\Pi_p = St_3 \otimes \lambda$ is a twist of the $GL_3$ Steinberg representation by a character $\lambda$ (necessarily of finite order), then it has a unique $P_1$-refinement and a unique $P_2$-refinement, both of which have slope $-1$. Thus $\Pi_p$ is $B$-nearly-ordinary for $a = 0$ (and $B$-ordinary if $\lambda$ is unramified), but for $a > 0$ it is not nearly-ordinary for either $P_1$ or $P_2$.
- If $\Pi_p$ is parabolically induced from a representation of $GL_1 \times GL_2$ of the form $\theta \boxtimes \sigma$, where $\theta$ is a character and $\sigma$ is supercuspidal, then its unique $P_1$-refinement is $\theta$, and its unique $P_2$-refinement is $\nu_\sigma = \omega_{P_2} \sigma^{-1}$. So it is $P_1$-nearly-ordinary if and only if $\nu_p(\theta(p)) = -1 - a$, and $P_2$-nearly-ordinary if and only if $\nu_p(\theta(p)) = 1 + a$. It is $P_1$-ordinary (resp. $P_2$-ordinary) if furthermore $\theta$ (resp. $\nu_\sigma$) is unramified. It is never $B$-nearly-ordinary.
- If $\Pi_p$ is (irreducibly) induced from $\theta \boxtimes (St_2 \otimes \lambda)$, where $St_2$ is the $GL_2$ Steinberg representation (and hence $\hat{\omega}_{P_2} = \lambda^2 \theta$), then it has two $P_1$-refinements, namely $\theta$ and $\lambda \cdot \chi_{\Pi_p}$; note that $\nu_p(\theta) \leq 1 + a$ implies $\nu_p(\lambda \cdot \chi_{\Pi_p}) \geq 1 - \frac{a}{2}$. Thus $\Pi_p$ is $P_1$-nearly-ordinary in either of two (mutually exclusive) cases: if $\nu_p(\theta(p)) = -1 - a$ and $a$ is arbitrary; or if $a = 0$ and $\nu_p(\theta(p)) = 1$. It is $P_1$-ordinary if $\theta$ is unramified in the former case, and if $\lambda$ is unramified in the latter. (There is a similar criterion for $P_2$-ordinarity, which we leave to the reader.)
- If $\Pi_p$ is an irreducible principal series representation, induced from a character $\chi_1 \times \chi_2 \times \chi_3$ of the diagonal torus, then the possible $P_1$-refinements are exactly the $\chi_1$, and the $P_2$-refinements are the pairs $\{\chi_1 \chi_2, \chi_2 \chi_3, \chi_3 \chi_1\}$. We can assume without loss of generality that $\nu_p(\chi_1(p)) \leq \cdots \leq \nu_p(\chi_3(p))$; then $\Pi_p$ is $P_1$-nearly-ordinary if $\chi_1(p)$ has valuation $-1 - a$, and $P_1$-ordinary if in addition $\chi_1$ is unramified; it is $P_2$-nearly-ordinary if $\chi_1 \chi_2(p)$ has valuation $-1 - a$, and $P_2$-ordinary if in addition $\chi_1 \chi_2$ is unramified.

Galois representations. As noted above, if $\alpha_p$ is a $P_1$-refinement of $\Pi_p$, then $\Pi_p$ is the unique generic constituent of $\text{Ind}^G_{P_1}(\alpha_p \boxtimes \sigma_p)$ for some generic $GL_2$-representation $\sigma_p$. Via compatibility of the local Langlands correspondence with parabolic induction, the Langlands parameter $\phi_{\Pi_p}$ has the form

\[
\begin{pmatrix}
\alpha_p & * & * \\
0 & \phi_{\sigma_p}
\end{pmatrix}
\]

where $\phi_{\sigma_p}$ is the Langlands parameter of $\sigma_p$, and we regard $\alpha_p$ as a 1-dimensional Weil–Deligne representation via class field theory.

If the Galois representation $\rho_{\sigma_p}$ satisfies local-global compatibility, this gives a 1-dimensional $(\varphi, N, G_{\sigma_p})$-stable subspace of $D_{\text{stab}}(\rho_{\sigma_p}, G_{\sigma_p})$ isomorphic to $\alpha_p$. In general this does not arise from a subrepresentation of the Galois representation, since it may not be weakly admissible. However, the following is a straightforward check:

Proposition 2.16. Assuming the local-global compatibility conjecture, $\Pi_p$ is $P_1$-nearly-ordinary if and only if $\rho_{\sigma_p} \in \text{Crit}^+_{GL_2}$ preserves a 1-dimensional subrepresentation of Hodge–Tate weight $(1 + a)$.

This is exactly the Panchishkin condition formulated in [Pan94] for the existence of a $p$-adic $L$-function interpolating the $L$-values $L(\Pi \times \eta, -j)$, for $(-j, \eta) \in \text{Crit}^+_{GL_2}$. Similarly, $P_2$-ordinarity corresponds precisely to the Panchishkin condition for the twists in $\text{Crit}^+_{GL_2}$.

2.8. $P_1$-refined newforms. Let $(\Pi_p, \alpha_p)$ be a $P_1$-refined representation, with $\alpha_p$ unramified (but not necessarily of minimal slope). For $r \geq 0$, consider the subgroup

\[
U^{(P_1)}_r(p^r) = \left\{ g : g = 1 \mod \left( \begin{smallmatrix} p^r \cdots p^r \\ p^r \cdots p^r \end{smallmatrix} \right) \right\} \subset GL_3(\mathbb{Z}_p),
\]
where $R = \max(r, 1)$. This is the intersection of the parahoric $J_{p,1}$ for $P_1$ with the factor at $p$ in the subgroup $U_{1,3}^p(p^r)$ above. It has an Iwahori decomposition with respect to $P_1$, and its intersection with the $GL_2$ factor of $M_{P_1}$ is the level $p^r$ mirabolic subgroup $U_{2,3}^p(p^r)$.

**Proposition 2.17.** For all sufficiently large $r$, there exists a vector $\varphi_p$ in $\Pi_p$ which is invariant under $U_{1,3}^p(p^r)$, and on which $U_{p,1}$ acts via $p^{r+1}\alpha_p(p)$. The minimal such $r$, denoted $r(\Pi)$, is equal to the conductor of the $GL_2$-representation $\sigma_p$ such that $\alpha_p \otimes \sigma_p$ is a subrepresentation of $J_{P_1}(V)$; and for this $r$, the space of such vectors is 1-dimensional.

This follows from the more general fact that the subspace of the $N_{P_1}(\mathbb{Z}_p)$-invariants on which the double coset of $\left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right)$ acts invertibly is a canonical lifting of $J_{P_1}(\Pi_p)$ to a subspace of $\Pi_p$, equivariant for the action of $M_{P_1}(\mathbb{Z}_p)$. (This result originates in an unpublished note of Casselmann; see [Eme06] for an account.) Hence the above space, for general $r$, is a canonical lifting of the $\left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ invariants of $\sigma_p$, and the result follows from new-vector theory.

We define the $P_1$-refined local Whittaker newvector $W^\alpha$ to be the unique basis of the above 1-dimensional space in the Whittaker model of $\Pi_p$, normalised to be 1 at the identity. By comparing Hecke eigenvalues (using the Hecke-equivariance of Casselmann’s lifting), we have the following formula for its values along the torus:

**Proposition 2.18.** We have $W^\alpha \left( \begin{smallmatrix} 1 & \alpha & \cdot \\ & 1 & \cdot \\ & & 1 \end{smallmatrix} \right) = 0$ if $m < 0$ or $n < 0$, and for $m, n \geq 0$ its values are given by

$$W_{\sigma}^{\text{new}} \left( \begin{smallmatrix} 1 & \alpha & \cdot \\ & 1 & \cdot \\ & & 1 \end{smallmatrix} \right)$$

where $W_{\sigma}^{\text{new}}$ is the normalised new-vector of the $GL_2$ representation $\sigma_p$.

### 2.9. Ordinarity for unramified primes.

We briefly explain what the above definitions give in the (important!) special case when $\Pi_p$ is unramified. The parabolics $P_i$ correspond to two normalised Hecke operators in the spherical Hecke algebra (cf. (2.7))

$$(2.9) \quad T_{p,i} = p^n [GL_3(\mathbb{Z}_p) \tau_i GL_3(\mathbb{Z}_p)], \quad T_{p,2} = p^n [GL_3(\mathbb{Z}_p) \tau_2 GL_3(\mathbb{Z}_p)],$$

for $\tau_i$ as in (2.7). Let $a_{p,i}(\Pi)$ be the eigenvalue of $T_{p,i}$ on $\Pi_p^{GL_3(\mathbb{Z}_p)}$; as in §2.6, these are algebraic integers. Recall the Satake parameters $\alpha_p, \beta_p$ and $\gamma_p$ from §2.4: we have

$$a_{p,1}(\Pi) = p^{a_1+1}(\alpha_p + \beta_p + \gamma_p), \quad a_{p,2}(\Pi) = p^{a_2+1}(\alpha_p \beta_p + \beta_p \gamma_p + \gamma_p \alpha_p).$$

Moreover, $a_{p,2} = \omega(\tau) \cdot a_{p,1}$. From these formulae, the following is immediate:

**Lemma 2.19.** $\Pi_p$ is $P_1$-ordinary (with respect to $\tau$) if the eigenvalue $a_{p,i}(\Pi)$ of $T_{p,i}$ acting on $\Pi_p$ is $a$-adic unit.

Hence, if we order the Satake parameters so that $v_p(\alpha_p) < v_p(\beta_p) < v_p(\gamma_p)$, then $\Pi_p$ is $P_1$-ordinary if and only if $v_p(\alpha_p) = -1 - a$ (the smallest possible value), whence the unique $P_1$-ordinary refinement is $\Pi_p(\alpha_p)$, identifying $\alpha_p$ with the unique unramified character sending $p \mapsto \alpha_p \in \mathbb{C}^\times$. It is $P_2$-ordinary if and only if $v_p(\gamma_p) = 1 + a$ (the largest possible value), giving unique $P_2$-ordinary refinement $\Pi_p(\alpha_p \beta_p)$.

**Remark 2.20.** Note that $P_1$-ordinarity and $P_2$-ordinarity are equivalent for essentially self-dual representations (of prime-to-$p$ level), but not for general RACARs. Explicit examples which are $P_1$-ordinary but not $P_2$-ordinary can be found in the computations of [GKT97].

### 2.10. The Coates–Perrin-Riou–Panchishkin conjecture.

Coates–Perrin-Riou and Panchishkin have conjectured the existence of $a$-adic $L$-functions attached to suitable motives. We now formulate an automorphic realisation of this conjecture for the (conjectural) motive $M_{\Pi}$ (from §2.5), which is totally independent of the existence of $M_{\Pi}$. At $p$, we define:

**Definition 2.21.** (cf. [Coa89, Lem. 3]). For $(-j, \eta) \in \text{Crit}_{p}^{-}(\Pi)$, let

$$e_p \left( \Pi, \eta, -j \right) := \begin{cases} G(\eta^{-1}) \cdot (p^{i + 1} \alpha(p))^{-n} \cdot \text{cond}(\eta) = p^a > 1, \\ (1 - p^{-j} \alpha(p)^n) \quad \eta = 1. \end{cases}$$
Coates–Perrin-Riou–Panchishkin then predict: if \( \bar{\Pi} \) is a \( p \)-ordinary refined RACAR for \( \GL_3 \), there exists a \( p \)-adic measure \( L_p(\bar{\Pi}) \) on \( \mathbb{Z}_p^\times \) such that for all \((-j, \eta) \in \text{Crit}_p(\Pi)\) we have

\[
(\ref{eq:2.10}) \quad \int_{\mathbb{Z}_p^\times} \eta(x)x^{-j} \cdot dL_p(\bar{\Pi})(x) = e_\infty(\Pi, -j) \cdot \varepsilon_p(\bar{\Pi}, \eta, -j) \cdot \frac{L^{(p)}(\Pi \times \eta, -j)}{\Omega_{\bar{\Pi}}},
\]

where \( L^{(p)} \) denotes the \( L \)-function without its Euler factor at \( p \) (cf. Conjecture B).

3. Symmetric spaces and Betti cohomology

3.1. Symmetric spaces. For any split reductive group \( J \) over \( \mathbb{Q} \), we define a symmetric space for \( J \) by \( \mathcal{H}_J = J(\mathbb{R})/K_{J,\infty}Z_{J,\infty} \), where \((-)^\circ \) denotes the identity component, \( K_{J,\infty} \) is a maximal compact subgroup of \( J(\mathbb{R}) \), and \( Z_{J,\infty} = Z_J(\mathbb{R}) \). For a neat open compact subgroup \( \mathcal{U} \subset J(\mathbb{A}_f) \), we define

\[
Y^J(\mathcal{U}) = J(\mathbb{Q})\backslash [J(\mathbb{A}_f)/\mathcal{H}_J].
\]

Note that for \( J = \GL_n \), the components of \( Y^J(\mathcal{U}) \) are indexed by the double quotient \( \mathcal{Y}^J(\mathcal{U}) = \mathcal{H}_J(\mathcal{U})/\mathcal{H}_J \)

Each component of \( Y^J(\mathcal{U}) \) is the quotient of \( \mathcal{H}_J^2 \) by an arithmetic subgroup of \( \GL_n(\mathbb{Q}) \).

3.2. Betti cohomology and Hecke operators. Let \( J, \mathcal{U} \) be as above. Given an algebraic representation \( V \) of \( J(\overline{\mathbb{Q}}) \), we have three possible constructions of local systems on \( Y^J(\mathcal{U}) \):

- a local system \( \mathcal{Y}_Q \) of \( \mathbb{Q} \)-vector spaces, given by the locally constant sections of the projection
  \[
  J(\mathbb{Q})\backslash [[J(\mathbb{A}_f) \times \mathcal{H}_J) \times V(\mathbb{Q})]/\mathcal{U} \rightarrow Y^J(\mathcal{U}),
  \]
  with action \( \gamma \cdot [(g, z), v] \cdot u = [\gamma gu, \gamma z], \gamma \cdot v \). (The functor \( V \mapsto \mathcal{Y}_Q \) is Pink’s “canonical construction” functor, [Pin90]).
- a local system \( \mathcal{Y}_\infty \) of \( \R \)-vector spaces, given by the locally constant sections of
  \[
  J(\mathbb{Q})\backslash [[J(\mathbb{A}_f) \times J(\mathbb{R})) \times V(\mathbb{R})]/\mathcal{U}K_{J,\infty}Z_{J,\infty} \rightarrow Y^J(\mathcal{U}),
  \]
  with action \( \gamma \cdot [(g, z), v] \cdot u_{K,\infty} = [\gamma gu, \gamma z], u_{K,\infty}^{-1} \cdot v \).
- a local system of \( \mathbb{Q}_p \)-vector spaces \( \mathcal{Y}_p \) for any finite prime \( p \), given by the locally constant sections of
  \[
  J(\mathbb{Q})\backslash [[J(\mathbb{A}_f) \times \mathcal{H}_J) \times V(\mathbb{Q}_p)]/\mathcal{U} \rightarrow Y^J(\mathcal{U}),
  \]
  with action \( \gamma [(g, z), v] u = [\gamma gu, \gamma z], u_{p}^{-1} \cdot v \).

If \( \mathcal{U} \subset \mathcal{U} \), then the formation of \( \mathcal{Y}_P \) is compatible with the natural projection map \( Y^J(\mathcal{U}) \rightarrow Y^J(\mathcal{U}) \), and we get natural maps \( \mathcal{H}^\bullet(Y^J(\mathcal{U}), \mathcal{Y}_P) \rightarrow \mathcal{H}^\bullet(Y^J(\mathcal{U}), \mathcal{Y}_P) \), where \( \mathcal{H}^\bullet \) denotes Betti cohomology. We write

\[
\mathcal{H}^\bullet(Y^J, \mathcal{Y}_P) = \lim_{\mathcal{U} \rightarrow \mathcal{U}} \mathcal{H}^\bullet(Y^J(\mathcal{U}), \mathcal{Y}_P)
\]

where \( \mathcal{U} \) varies over open compact subgroups of \( J(\mathbb{A}_f) \). This direct limit is naturally a representation of \( J(\mathbb{A}_f) \), and \( \mathcal{H}^\bullet(Y^J(\mathcal{U}), \mathcal{Y}_P) = \mathcal{H}^\bullet(Y^J, \mathcal{Y}_P)_{\mathcal{U}} \). We have direct analogues of these statements for compactly-supported cohomology \( \mathcal{H}^\bullet_{\text{c}} \).

It is standard that there are canonical isomorphisms of local systems

\[
(\ref{eq:3.2}) \quad \mathcal{Y}_Q \otimes \mathbb{Q}_p \cong \mathcal{Y}_p, \quad \mathcal{Y}_Q \otimes \mathbb{R} \cong \mathcal{Y}_\infty.
\]

For instance, the first of these two is given on sections by \([[(g, z), v] \mapsto [(g, z), g_{p}^{-1} \cdot v] \). Details on all of this can be found in [Urb11, §1.2].

The Betti cohomology of \( Y^J(\mathcal{U}) \) is equipped with a natural action of the Hecke algebra \( \mathcal{H}(\mathcal{U}) \). The isomorphisms on cohomology groups induced by the isomorphisms (3.2) of local systems are all equivariant under the Hecke operators [Urb11, §1.2].
For later use, we describe the $P_1$-Hecke operator at $p$ in more detail. Let $U = U^{(p)} U_p \subset GL_3(\mathbb{A}_F)$ be an open compact subgroup with $N_{P_1}(Z_p) \subset U_p \subset J_{p,1}$, for notation as in §2.6. Recall $\tau_i$ from (2.7); for ease of notation, we set $\tau = \tau_1$. Then we have maps

$$Y^{GL_3}(U) \xrightarrow{\text{pr}_{GL_3}} Y^{GL_3}(U \cap \tau U \tau^{-1}) \xrightarrow{\tau} Y^{GL_3}(\tau^{-1} U \tau \cap U) \xrightarrow{\text{pr}_{GL_3}} Y^{GL_3}(U),$$

where the middle map is induced by right-translation of $\tau$ on $GL_3(\mathbb{A}_F)$ and the outside maps are the natural projection maps. Passing from left-to-right and right-to-left respectively, we get associated (normalised) Hecke operators

$$U_{p,1} := p^a \cdot \left( \text{pr}_{GL_3}^{\tau^{-1} U \tau \cap U} \right)^* \circ \tau \circ \left( \text{pr}_{GL_3}^{\tau^{-1} U \tau \cap U} \right)^*,$$

$$U_{p,1} := p^a \cdot \left( \text{pr}_{GL_3}^{U \cap \tau U \tau^{-1}} \right)^* \circ \tau^* \circ \left( \text{pr}_{GL_3}^{U \cap \tau U \tau^{-1}} \right)^*,$$

on the cohomology $H^*\left( Y^{GL_3}(U), \mathbb{A}_F \right)$, for $* = \varnothing$ or $c$ and $\mathbb{A}_F$ as in §3.2. As in (2.9), the scalars $p^a$ are for integral normalisation. We also define ordinary projectors $e_{\text{ord},1} := \lim_{n \to \infty} U_{p,1}^n$ and $e_{\text{ord},1} := \lim_{n \to \infty} (U_{p,1}^n)^*$; by definition, $U_{p,1}$ is invertible on the image of $e_{\text{ord},1}$.

### 3.3. Automorphic cohomology classes

Let $\Pi$ be a RACAR of $GL_3(\mathbb{A})$. We now realise $\Pi$ in the compactly-supported Betti cohomology of $Y^{GL_3}$. Let $U \subset GL_3$ be any open neat compact such that $\Pi_U \neq 0$, and write $\mathbb{A}_F$ for the $\mathbb{Q}$-local system attached to $Y^{GL_3}_\mathbb{A}$. Via the cuspidal cohomology, and the natural map $H^*_{\text{cusp}} \hookrightarrow H^*_{\text{c}}$ of [Clo90, p.123], there is an injection

$$H^* \left( GL_3, K_{3,\infty} \otimes V^{\vee}_\mathbb{A}(\mathbb{C}) \right) \otimes W_0((\Pi)) \hookrightarrow H^*_{\text{c}} \left( Y^{GL_3}(U), \mathbb{A}_F \right),$$

compatible with the Hecke action of $\mathbb{C}[U \setminus GL_3(\mathbb{A})]/U$. Via the choice of $\zeta_\infty$ from (2.2), this yields a map

$$\phi_{\Pi} : W(\Pi) \otimes W_0((\Pi)) \hookrightarrow H^2_{\text{c}} \left( Y^{GL_3}(U), \mathbb{A}_F \right).$$

The $\mathbb{Q}$-local systems, and (3.3), are compatible with varying $U$, giving a $GL_3(\mathbb{A}_E)$-equivariant map

$$\phi_{\Pi} : W(\Pi) \hookrightarrow H^2_{\text{c}} \left( Y^{GL_3}, \mathbb{A}_F \right) = \lim_{U \to \mathbb{A}_F} H^2_{\text{c}} \left( Y^{GL_3}(U), \mathbb{A}_F \right).$$

We shall denote the image of this map by $H^2_{\text{c}}(\Pi, \mathbb{C})$.

#### 3.3.1. Periods and rationality

The representation $H^2_{\text{c}}(Y^{GL_3}, \mathbb{A}_F)$ has a natural $\mathbb{Q}$-structure given by the cohomology of $Y^{GL_3}_F$. Since the complex representation is admissible and contains $\Pi$ with multiplicity 1, it follows that there is a number field $E$, the field of definition of $\Pi$, such that the $E$-linear representation

$$H^2_{\text{c}}(\Pi, E) = H^2_{\text{c}}(\Pi, \mathbb{C}) \cap H^2_{\text{c}} \left( Y^{GL_3}, \mathbb{A}_F \right)$$

is non-zero and gives an $E$-structure on $H^2_{\text{c}}(\Pi, \mathbb{C})$. (This is the same field $E$ as from §2.4. By the strong multiplicity one theorem for $GL_3$, we may take $E$ to be the field generated by the Hecke eigenvalues of $\Pi$ at the unramified primes, although we shall not need this.)

From the uniqueness of Whittaker models, $W(\Pi)$ also has a canonical $E$-structure $W(\Pi, E)$, given by the functions $W \in W(\Pi)$ which take values in $E^{\text{ab}}$ and satisfy

$$W(g)^{\sigma} = W \left( \kappa(\sigma)^2 \kappa(\sigma)^{-1} g \right) \quad \text{for all } \sigma \in \text{Gal}(E^{\text{ab}}/E),$$

where $\kappa : \text{Gal}(E^{\text{ab}}/E) \to \tilde{\mathbb{Z}}^\times$ is the cyclotomic character. Hence there exists $\Theta_{\Pi} \in \mathbb{C}^\times$ such that

$$\phi_{\Pi} \left( W(\Pi), E \right) = \Theta_{\Pi} \cdot H^2_{\text{c}}(\Pi, E).$$

**Remark 3.2:** Having fixed $\zeta_\infty$, the period $\Theta_{\Pi}$ is uniquely determined modulo $E^\times$. We may, however, rescale $\zeta_\infty$ by an arbitrary non-zero complex scalar, then also rescales $\Theta_{\Pi}$. Note that for $GL_n$ with $n$ odd, we obtain only a single Whittaker period, since (2.1) is 1-dimensional. This differs from the case of $GL_n$ for $n$ even, where (2.1) is 2-dimensional and there are two Whittaker periods (one for each choice of sign at $\infty$).

#### 3.3.2. Integral lattices

Let $V^{\mu}_J$ be an algebraic representation of a split reductive group $J$, with highest weight vector $v_\mu \in V^{\mu}_J$. An admissible lattice is a lattice $L \subset V^{\mu}$ such that: (1) the map
\[ J/\mathbb{Q} \to \text{GL}(V_p) \text{ extends to a map of } \mathbb{Z}\text{-group schemes } J/\mathbb{Z} \to \text{GL}(\mathcal{L}), \text{ and (2) the intersection of the highest weight space in } V_p \text{ with } L \text{ is } \mathbb{Z} \cdot v_\lambda. \] There are finitely many such lattices (cf. [LSZb, §4.2]).

We adopt the convenient (but somewhat misleading) notation that \( V_{\mu,Z}^\text{GL} \) denotes the minimal such lattice in \( V_{\mu,Z}^\text{GL} \), whilst \( V_{\mu,Z}^\text{GL} \) denotes the maximal such lattice in \( V_{\lambda,Z}^\text{GL} \). This ensures that under the branching laws of the next section, \( V_{\ast,Z}^\text{GL} \) is always mapped into \( V_{\ast,Z}^\text{GL} \).

### 3.3.3. Integral structures.

Now let \( p \) be a prime, and \( v \mid p \) a prime of \( E \). Completing \( E \) at \( v \), we may take \( \phi_{\Pi} \) to be valued in \( H_\varphi^2(Y^\text{GL}_3, \langle - \rangle^\vee(\mathfrak{E}_v)) \). We want to specify a canonical normalisation for \( \Theta_{\Pi} \) up to \( \mathcal{O}_{E,v}^\times \). We use the conductor \( N_{\Pi} \) from §2.3.3, recalling that the invariants \( \mathcal{U}^\text{GL}_1(N_{\Pi}) \) are 1-dimensional. Moreover, there is a unique Whittaker function \( W_{\Pi}^{\text{new}} \in \mathcal{W}(H_{\Pi}, E) \), the normalised Whittaker newform of \( \Pi \), which is stable under \( \mathcal{U}^\text{GL}_1(N_{\Pi}) \) and satisfies \( W_{\Pi}^{\text{new}}(1) = 1 \).

Note as above that \( \langle - \rangle^\vee_{\lambda,Q}(\mathfrak{E}_v) \cong \langle - \rangle^\vee_{\lambda,p}(\mathfrak{E}_v) \) on \( Y^\text{GL}_3(N) = Y^\text{GL}_3(\mathcal{U}^\text{GL}_1(N)) \). We can define an integral version using the lattice \( V_{\lambda,Z} \) above (or, more properly, the analogue \( V_{\lambda,Z}^\varphi \), which can be described similarly); the \( \mathcal{O}_{E,v} \)-points \( \mathcal{V}_{\lambda,Z}(\mathcal{O}_{E,v}) \) of this representation carry an action of \( \text{GL}_3(\mathbb{Z}_p) \), so we get an associated local system \( \mathcal{V}^\vee_{\lambda,p}(\mathcal{O}_{E,v}) \) of \( \mathbb{Z}_p \)-modules (defined exactly as in (3.1)), and where we henceforth drop the \( Z \) for convenience.

The \( \mathcal{O}_{E,v} \)-module

\[ H_\varphi^2(\Pi, \mathcal{O}_{E,v})^{\text{new}} := H_\varphi^2(\Pi, \mathcal{E}_v)\mathcal{U}^\text{GL}_1(N_{\Pi}) \cap \text{image } H_\varphi^2 \left( Y^\text{GL}_3(\mathcal{U}^\text{GL}_1(N_{\Pi})), \langle - \rangle^\vee_{\lambda,p}(\mathcal{O}_{E,v}) \right) \]

is free of rank 1, and we renormalise \( \Theta_{\Pi} \) so that \( \frac{1}{\delta_{\Pi}} \cdot \phi_{\Pi}^{\text{GL}_3(N_{\Pi})}(W_{\Pi}^{\text{new}}) \) spans this module.

**Remark 3.3:** Henceforth we will always take \( L/\mathbb{Q}_p \), a finite extension containing all embeddings of \( E_v \) for \( v \mid p \), and only use the local systems \( \mathcal{V}^\vee_{\lambda,p}(L) \) or \( \mathcal{V}^\vee_{\lambda,p}(\mathcal{O}_L) \) constructed from the action of \( \text{GL}_3(\mathbb{Z}_p) \). To ease notation we will typically drop the subscript \( p \) and write \( \mathcal{V}^\vee_{\lambda} \). Chaining together all of the various maps in this section, we obtain a (non-canonical) map

\[ \phi_{\Pi}/\Theta_{\Pi} : \mathcal{W}(\Pi, E) \longrightarrow H_\varphi^2(Y^\text{GL}_3, \langle - \rangle^\vee(\mathcal{O}_L)), \]

depending on the choices of \( \zeta_\infty \) (well-defined up to \( C^\times \) and the period \( \Theta_{\Pi} \) (which, given the choice of \( \zeta_\infty \), is then well-defined up to \( \mathcal{O}_L^\times \)).

### 3.4. The cup product pairing.

Let \( \mathcal{U} = \mathcal{U}^{(p)} \mathcal{U}_p \subset \text{GL}_3(\mathbb{A}_f) \) be a neat open compact subgroup and let \( R \) be a \( \mathbb{Z}_p \)-algebra. There is a (perfect) Poincaré duality pairing

\[ \langle - , - \rangle_\mathcal{U} : H_\varphi^2(Y^\text{GL}_3(\mathcal{U}), \langle - \rangle^\vee(\mathcal{O}_L)) \times \overset{(\text{torsion})}{H^3(Y^\text{GL}_3(\mathcal{U}), \langle - \rangle^\vee(\mathcal{O}_L))} \longrightarrow R \]

given by composing cup product, the natural pairing \( V^\vee_{\lambda}(R) \otimes V_{\lambda}(R) \to R \), and integration over the smooth (5-dimensional) real manifold \( Y^\text{GL}_3(\mathcal{U}) \). If \( N_{p_i}(\mathbb{Z}_p) \subset \mathcal{U}_p \subset \mathcal{J}_{p_i,1} \), then by adjointness of pullback and pushforward, the Hecke operators \( U_{p,1} \) and \( U_{p,1}' \) from §3.2 are adjoint under \( \langle - , - \rangle_\mathcal{U} \).

### 4. The subgroup \( H \)

**Definition 4.1.** Let \( H = \text{GL}_2 \times \text{GL}_1 \), and let \( \iota : H \hookrightarrow \text{GL}_3 \) be the embedding

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

**Definition 4.2.** Let \( \nu_1, \nu_2 : H \to \text{GL}_1 \) be the homomorphisms given by

\[ \nu_1(\gamma, z) = \frac{\text{det} \gamma}{z} \quad \nu_2(\gamma, z) = z. \]

Note \((\nu_1, \nu_2)\) gives an isomorphism \( H/\text{H}^\text{der} \cong \text{GL}_1 \times \text{GL}_1 \) (this parametrisation of \( H/\text{H}^\text{der} \) may seem slightly unnatural, but will give us nice formulae later), and \( \text{det} \circ \iota = \nu_1 \nu_2^2 \).

### 4.1. Symmetric spaces.

It is important to note that the embedding \( \iota : H \to \text{GL}_3 \) does not induce a map on symmetric spaces, since the inclusion \( Z_{\text{GL}_3} \subset \iota(Z_H) \) is strict. To transfer cohomology classes from \( Y^H \) to \( Y^\text{GL}_3 \), we instead define

\[ \tilde{H}^H = H(\mathbb{Q})/ \left( K^\varphi_{H,\infty} \cdot \iota^{-1}(Z_{G,\infty}) \right), \]

which maps naturally to both \( H_H \) and \( H_G \). If \( \mathcal{U} \subset H(\mathbb{A}_f) \) is open compact, let

\[ \tilde{Y}^H(\mathcal{U}) := H(\mathbb{Q}) \backslash \left( H(\mathbb{A}_f) \times \tilde{H}^H \right). \]
For any open compact \( V \subset \text{GL}_3(\mathbb{A}_f) \), we then have a diagram of maps
\[
(4.3) \quad Y^H(V \cap H) \leftarrow \widetilde{Y}^H(V \cap H) \xrightarrow{\iota} Y^G(V),
\]
where the right-hand map is induced by \( \iota \). Pulling back under the leftward arrow and pushing forward under the rightward arrow gives a map from the cohomology \( Y^H \) to that of \( Y^{GL_2} \).

If \( V \) is small enough, the left arrow is a fibre bundle with fibres isomorphic to \( Z_{H,\infty}^{-1}(Z_{G,\infty}^2) \cong \mathbb{R} \). The spaces \( Y^H \) (resp. \( \widetilde{Y}^H \)) have dimension 2 (resp. 3) as real manifolds.

We get local systems on the modified spaces \( Y^H(U) \) from (4.2), defined identically to §3.2.

### 4.2. Branching laws.

Note \( V^H_{(r,s,t)} = V^{GL_2}_{(r,s)} \otimes V^{GL_1}_{(t)} \) is the \( H \)-representation of highest weight \( \langle (\ast, y, z) \rangle \mapsto x^r \cdot y^s \cdot z^t \). We let \( V^H_{(r,s,t);Z} \) denote the minimal admissible lattice in \( V^H_{(r,s,t)} \). The following is equivalent to the well-known branching law from \( \text{GL}_3 \) to \( \text{GL}_2 \), e.g. [GW09, §8].

**Proposition 4.3.** Let \( \lambda = (a,0,-a) \). The restriction of \( V_\lambda \) to \( H \) (embedded via \( \iota \)) is given by
\[
\iota^* (V_\lambda) \cong \bigoplus_{0 \leq i,j \leq a} V^H_{(j,-i,j)},
\]

for each \( \lambda \), we shall fix choices of non-zero morphisms of \( H \)-representations over \( \mathbb{Q} \)
\[
\text{br}^{[a,j]} : V^H_{(0,0,-j)} \rightarrow \iota^* (V_\lambda)
\]
for each \( j \) as above. Note \( V^H_{(j,0,-j)} = V^H_{(0,j,0)} \otimes \|v_2\| \), hence our choice yields a pairing
\[
(4.4) \quad \langle -,- \rangle_{a,j} : V^\vee \times V^{GL_2}_{(0,-j)} \rightarrow \mathbb{G}_a, \quad (\mu, v) \mapsto \mu(\text{br}^{[a,j]}([\|v_2\| \otimes [v \otimes 1]])).
\]

As in [LSZb, Prop 4.3.5], \( \text{br}^{[a,j]} \) maps the (minimal) admissible lattice \( V^H_{(j,0,-j);Z} \) into the (maximal) admissible lattice \( V_\lambda;Z \). Thus the pairing \( \langle -,- \rangle_{a,j} \) also makes sense integrally.

### 5. Eisenstein series and classes for \( \text{GL}_2 \)

We recall the theory of Eisenstein series and classes attached to adelic Schwartz functions. In particular, we recall the motivic Eisenstein classes of Beilinson (see e.g. [LSZb, §7]) and describe their Betti realisations via adelic Eisenstein series. In this section all symmetric spaces will be for \( \text{GL}_2 \), so we write simply \( Y(U) \) (resp. \( Y^{GL_2}(U) \)) (resp. \( Y^{GL_2} \)).

If \( j \geq 0 \), recall \( V^{GL_2}_{(0,-j)} \) denotes the \( GL_2 \)-representation of highest weight \( (0,-j) \). Similarly, in this section only we will drop the superscript and denote this simply \( V_{(0,-j)} \).

#### 5.1. Schwartz functions.

For a field \( K/\mathbb{Q} \), write \( \mathcal{S}(\mathbb{A}_f^2, K) \) for the Schwartz space of locally-constant, compactly-supported functions on \( \mathbb{A}_f^2 \) with values in \( K \), and \( \mathcal{S}_0(\mathbb{A}_f^2, K) \) for the subspace of \( \Phi \) with \( \Phi(0,0) = 0 \). We also let \( \mathcal{S}(\mathbb{R}^2, K) \) be the usual space of Schwartz functions on \( \mathbb{R}^2 \), write \( \mathcal{S}_0(\mathbb{R}^2, K) \) for the subspace with \( \Phi(0,0) = 0 \), and let \( \mathcal{S}(\mathbb{A}_f^2, K) = \mathcal{S}(\mathbb{A}_f^2, K) \times \mathcal{S}(\mathbb{R}^2, K) \) (and similarly for \( \mathcal{S}_0(\mathbb{A}_f^2, K) \)). We will make specific choices at infinity, depending on an integer \( j \geq 0 \), and use the notation \( \mathcal{S}_0(\mathbb{A}_f^2, K) \) to mean \( \mathcal{S}_0(\mathbb{A}_f^2, K) \) if \( j = 0 \) or \( \mathcal{S}(\mathbb{A}_f^2, K) \) otherwise.

Let \( \chi \) be a Dirichlet character, corresponding to a finite-order Hecke character \( \hat{\chi} \). If \( \Phi_t \in \mathcal{S}(\mathbb{A}_f^2, K) \), let \( R_\chi(\Phi_t) \in \mathcal{S}(\hat{\mathbb{A}}_f^2, K(\chi)) \) be its projection to the \( \hat{\chi}^{-1} \)-isotypical component, given by
\[
(5.1) \quad R_\chi(\Phi_t)(x,y) = \int_{a \in \mathbb{Z}^2} \hat{\chi}(a) \Phi_t(ax, ay) \, d^\times a.
\]

We emphasise that this is a projection operator, not a twisting operator: if \( \chi_1, \chi_2 \) are distinct primitive Dirichlet characters, then \( R_{\chi_1} R_{\chi_2}(\Phi_t) = 0 \).

#### 5.2. Eisenstein series.

We now review some standard definitions of Eisenstein series from Schwartz functions, both adelicly and classically, and relate these definitions.

##### 5.2.1. Adelic Eisenstein series.

Let \( \Phi = \Phi_\infty \cdot \Phi_f \in \mathcal{S}(\mathbb{A}_f^2, \mathbb{C}) \) and \( \hat{\chi} : \mathbb{Q}^\times \backslash \mathbb{A}_f^2 \rightarrow \mathbb{C}^\times \) be a Hecke character. For \( g \in \text{GL}_2(\mathbb{A}) \) and \( \Re(s) > 0 \) we define a **Siegel section**
\[
(5.2) \quad f_\Phi(g; \hat{\chi}, s) := \| \det g \|^s \int_{\mathbb{A}_f^2} \Phi\left( (0,a) g \right) \hat{\chi}(a) \|a\|^{2s} \, d^\times a.
\]
This admits meromorphic continuation to $\mathbb{C}$ and gives an element of the family of GL$_2$-principal series representations $l(||·||^{-1/2}, \hat{x}^{-1}||·||^{-1/2})$, that is, $f_\Phi (\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) g; \hat{x}, s) = \hat{x}^{-1}(d)/|a/d|^s f_\Phi (g; \hat{x}s)$. It thus defines an element of $C^\infty (B_2(\mathbb{Q})\backslash GL_2(\hat{A}), \mathbb{C})$, for $B_2 \subset GL_2$ the upper-triangular Borel.

Define (cf. [Jac72, §19])

\[(5.3) \quad E_\Phi(g; \hat{x}, s) := \sum_{\gamma \in B_2(\mathbb{Q}) \backslash GL_2(\mathbb{Q})} f_\Phi(\gamma g; \hat{x}, s), \]

which converges absolutely and locally uniformly on some right half-plane (for $\Re(s) > 1$ if $\hat{x}$ is unitary), defining a function on the quotient GL$_2(\mathbb{Q})\backslash GL_2(\hat{A})$ that transforms under the centre by $\hat{x}^{-1}$. It has meromorphic continuation in $s$, analytic if $\Phi(0, 0) = \Phi(0, 0) = 0$ or if $\hat{x}$ is ramified at some finite place.

### 5.2.2. Classical Eisenstein series

We introduce classical Eisenstein series.

**Definition 5.1.**

(i) If $\Phi_t \in S_{(0)}(A_2, K)$ and $j \geq 0$, define

\[E_{\Phi_t}^{j+2}(\tau; s) := \frac{\Gamma(s + \frac{j+2}{2})}{(-2\pi i)^{j+2} \pi^{j+2} 2^{2j+2}} \sum_{\Phi_t(m,n) \in \mathbb{C}(\tau, 0, 0)} \Phi_t(m,n) y^{s-\frac{j+2}{2}} |m\tau + n|^2 |\tau|^{2j+2}, \]

where $s \in \mathbb{C}$ and $\tau \in \mathcal{H}_{GL_2}$. This is a classical real-analytic weight $j + 2$ Eisenstein series.

(ii) We extend this to a function on $GL_2(\mathbb{A}_f) \times \mathcal{H}$ by setting

\[E_{\Phi_t}^{j+2}(g, \tau; s) := E_{\Phi_t}^{j+2}(\tau; s), \]

(iii) Finally, for a Dirichlet character $\chi$ we also define

\[E_{\Phi_t}^{j+2}(\tau; \chi, s) := E_{\Phi_t}^{j+2}(\tau; s), \quad E_{\Phi_t}^{j+2}(g, \tau; \chi, s) := E_{\Phi_t}^{j+2}(g, \tau; s). \]

We write $E_{\Phi_t}^{j+2}$ for each of these functions, but it will be clear from context which we use.

**Remarks:** The function defined in (i) is denoted $E^{(j+2, \Phi_t)}(\tau; s)$ in [LPSZ, Def. 7.1]. These series always converge absolutely for $\Re(s) \geq 1$ unless $j = 0$ and $\chi$ is trivial, and even in this case they converge absolutely if $\Phi(0, 0) = 0$.

One has the functional equation $E_{\Phi_t}^{j+2}(\tau; s) = E_{\Phi_t}^{j+2}(\tau; 1 - s)$, where $\hat{\Phi}_t$ is the Fourier transform (normalised as in [LPSZ, §8.1]); this explains the compatibility between our conventions and those of [LSZb, §7]. The definition in (ii) ensures the association $\Phi_t \mapsto E_{\Phi_t}^{j+2}$ equivariant under $GL_2(\mathbb{A}_f)$.

Via the standard procedure (see [Wei71, §I]), we now extend $E_{\Phi_t}^{j+2}$ to a function on $g = g g_\infty \in GL_2(\hat{A})$. By Iwasawa decomposition, $g_\infty \in GL_2(\mathbb{R})$ can be written uniquely in the form $(\begin{smallmatrix} z & \theta \\ 0 & 1 \end{smallmatrix}) r(\theta)$, where $z \in \mathbb{R}^\times$ and $r(\theta) = (\begin{smallmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{smallmatrix}) \in SO_2(\mathbb{R})$. Then we define

\[(5.4) \quad E_{\Phi_t}^{j+2}(-; \chi, -) : GL_2(\hat{A}) \times \mathbb{C} \to \mathbb{C}, \]

\[\quad (g, s) \mapsto \hat{\chi}_{\Phi_t}^{-1}(z) \cdot |\det(g_\infty)|^{-s} \cdot y^{\frac{j+2}{2}} \cdot \exp[i(j + 2)\theta] \cdot E_{\Phi_t}^{j+2}(g, x + iy; \chi, s). \]

This yields an automorphic form on $GL_2(\hat{A})$: the $y^{\frac{j+2}{2}}$ and $e^{i(j+2)\theta}$ are always present in extending from $\mathcal{H}_{GL_2}$ to $GL_2(\mathbb{R})$ [Wei71], the $\hat{\chi}_{\Phi_t}^{-1}(z)$ ensures the central character is correct, and $|\det(g_\infty)|^{-s}$ ensures $E_{\Phi_t}^{j+2}$ factors through $GL_2(\mathbb{Q}) \backslash GL_2(\hat{A})$.

### 5.2.3. Comparison of classical and adelic

At infinity, for $j \geq 0$ we shall henceforth take

\[(5.5) \quad \Phi_\infty = \Phi_\infty^{j+2}(x, y) = 2^{-1-j}(x + iy)^{j+2} \exp(-\pi(x^2 + y^2)). \]

For this choice, by [LPSZ, Prop. 10.1] we have:

**Proposition 5.2.** If $\Phi = \Phi_\infty^{j+2}, \Phi_t \in S(A_2, \mathbb{C})$, for some $j \geq 0$ and $\Phi_t \in S(A_2, \mathbb{C})$, and $\chi$ is a Dirichlet character, then the adelic and classical Eisenstein series are related by

\[E_\Phi(g_\hat{t}; \hat{x}, s) = y^{\frac{j+2}{2}} \| g_\hat{t} \|^s \cdot E_{\Phi_t}^{j+2}(g_\hat{t}, x + iy; \chi, s) \]

for $g_\hat{t} \in GL_2(\mathbb{A}_f)$ and $x + iy \in \mathcal{H}_{GL_2}$.
Corollary 5.3. When \( \Phi_\infty = \Phi_{\chi}^{j+2} \), for \( g \in GL_2(\mathbb{A}) \) we have
\[
E_\Phi(g; \overline{\chi}, s) = \| \det(g) \|^s E_\Phi^{j+2}(g; \chi, s).
\]

Proof. We have \( E_\Phi(g \phi(\theta), \overline{\chi}, s) = e^{j+2} E_\Phi(g, \overline{\chi}, s) \) (cf. [Bum97, §3.(7.35)] and \( E_\Phi \) left-translates as \( \overline{\chi}_{\infty}(z) \) under \( (z, z) \in GL_2(\mathbb{Z}) \). The result follows from Proposition 5.2.

5.2.4. The special value
\[ s = -j/2. \]

We shall be briefly interested in the special value \( s = -j/2 \) (cf. Theorem 5.5 below). Classically, this has the particularly nice form
\[
E_{\Phi_{\chi}}^{j+2}(\tau, -\frac{1}{2}) = \frac{(j+1)!}{(-2\pi i)^{j+2}} \sum_{(m,n) \in \mathbb{Z}^2} \tilde{\Phi}_f(m,n),
\]
where (as in §5.2.2) \( \tilde{\Phi}_f \) is the Fourier transform. This function is denoted \( F_{\Phi_{\chi}}^{j+2} \) in [LSZb]. Passing to the adeles, this motivates the definitions
\[
E_{\Phi_{\chi}}^{j+2} : GL_2(\mathbb{A}) \to \mathbb{C}, \quad g \mapsto E_{\Phi_{\chi}}^{j+2}(g; -\frac{1}{2}).
\]

Via Corollary 5.3, we are led to also consider
\[
E_{\Phi_{\chi}}^{j+2}: GL_2(\mathbb{A}) \to \mathbb{C}, \quad g \mapsto \| \det(g) \|^{j/2} E_\Phi(g; \overline{\chi}, -\frac{1}{2}).
\]

Then by Corollary 5.3, we have
\[
E_{\Phi_{\chi}}^{j+2}(g) = E_{\Phi_{\chi}}^{j+2}(g).
\]

Remark 5.4: The functions \( E_{\Phi_{\chi}}^{j+2}(g) \) depend \( GL_2(\mathbb{A}) \)-equivariantly on \( \Phi \) and transform as elements of the global principal series representation
\[
I_f(\overline{\chi}) := I(\| \cdot \|^{-\frac{1}{2}}, \overline{\chi}^{-1} \| \cdot \|^{j+\frac{1}{2}}) = \bigotimes_{v<\infty} I_f(\overline{\chi}_v).
\]

Note \( I_f(\overline{\chi}) \) is irreducible if \( j > 0 \). For \( j = 0 \) it does not even have finite length, since infinitely many of the local factors \( I_0(\overline{\chi}_v) \) are reducible.

5.3. Betti–Eisenstein classes.

5.3.1. Local systems and realisation maps.

Betti local systems: For \( j \geq 0 \), let \( \mathcal{V}_{(0,-j)} = \mathcal{V}_{GL_2, (0,-j)} \) denote the local system on the \( GL_2 \) symmetric space associated to the \( GL_2 \)-representation \( V_{(0,-j)} \) of highest weight \( (0, -j) \). Note that if \( a \in \mathbb{Q}_{>0}, \left( \frac{a}{1} \right) \) acts on \( H^1(Y, V_{(0,-j)}) \) as multiplication by \( a^{-j} \).

De Rham local systems: To compare our (Betti)-Eisenstein classes to the classical Eisenstein classes of Harder, we go through a comparison to de Rham cohomology. The local system \( \mathcal{V}_{(0,-j)}(\mathbb{C}) \) comprises the flat sections of a vector bundle \( \mathcal{V}_{(0,-j), dr} \) with respect to the Gauss–Manin connection. This is defined over \( \mathbb{Q} \), and we have a comparison isomorphism
\[
C_{B, dr} : H^1(Y(U), \mathcal{V}_{(0,-j)}(\mathbb{Q})) \otimes \mathbb{C} \to H^1_{dr}(Y(U), \mathcal{V}_{(0,-j), dr}) \otimes \mathbb{C}.
\]

The pullback of \( \mathcal{V}_{(0,-j), dr} \) to the upper half-plane is \( \text{Sym}^j \otimes \det^{-j} \) of the relative de Rham cohomology of \( \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \), so has a canonical section \( (2\pi i \tau)^{j+1} \), for \( \tau \) a co-ordinate on \( \mathbb{C} \).

Motivic local systems: Our Betti–Eisenstein classes will be the Betti realisation of Bellimson’s motivic Eisenstein classes. For any level \( U \), there is a \( GL_2(\mathbb{A}) \)-equivariant relative Chow motive \( \mathcal{V}_{(0,-j), mot}(\mathbb{Q}) \) over \( Y(U) \) attached to the representation \( V_{(0,-j)} \) of \( GL_2/\mathbb{Q} \); this gives a coefficient system for motivic cohomology. Then we have realisations
\[
\begin{align*}
\tau_{B} : H^1_{mot}(Y(U), \mathcal{V}_{(0,-j), mot}(1)) & \to H^1(Y(U), \mathcal{V}_{(0,-j)}(\mathbb{Q})(1)) \to H^1(Y(U), \mathcal{V}_{(0,-j)}(\mathbb{Q})), \\
\tau_{dr} : H^1_{mot}(Y(U), \mathcal{V}_{(0,-j), mot}(1)) & \to H^1_{dr}(Y(U), \mathcal{V}_{(0,-j), dr}).
\end{align*}
\]

Base-changing to \( \mathbb{C} \)-coefficients, these are related by
\[
C_{B, dr} \circ \tau_{B} = (2\pi i)^{-j-1} \cdot \tau_{dr}
\]
(see e.g. [KLZ20, §2.2] or [LW20, pf. of Prop. 5.2], noting \( T_j(j+1) \) op. cit. is \( \mathcal{V}_{(0,-j)}(1) \) here).

5.3.2. Betti–Eisenstein classes. The main input in our construction is Bellimson’s family of motivic Eisenstein classes [Bei86], which we briefly summarise (cf. [LSZb, Thm. 7.2.2]).
acts via for each $j$, $\chi$ the Iwahori subgroup $\mathcal{U}$.

Corollary 5.6. For any $j \geq 0$, and any level $\mathcal{U}$, there is a canonical $GL_2(\mathbb{A}_F)$-equivariant map, the Betti–Eisenstein symbol,

$$\mathcal{S}_j(\mathbb{A}^2, \mathcal{Q})^\mathcal{U} \to H^1(\mathcal{U}(\mathcal{Q}), (\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)})),$$

compatible with Hecke operators and changing $\mathcal{U}$, such that the pullback to the upper half-plane of $r_{\mathfrak{A}}(\mathfrak{Eis}_j^{\mathcal{U}})$ is the differential form $-\mathfrak{Eis}_j^{\mathcal{U}}(\tau; -\frac{1}{2}) \cdot (2\pi i d\tau)^{\otimes 2} \cdot 2\pi i d\tau$ (cf. §5.3.1).

Proof. By (5.11), the pullback to the upper half-plane of $C_{\mathfrak{A}, \mathfrak{U}}(\mathfrak{Eis}_j^{\mathcal{U}})$ is $-\mathfrak{Eis}_j^{\mathcal{U}}(\tau; -\frac{1}{2}) \cdot (d\tau)^{\otimes 2}$, $d\tau$. The extension of $\mathfrak{Eis}_j^{\mathcal{U}}$ to $GL_2(\mathbb{A}_F)$ is the unique one preserving $GL_2(\mathbb{A}_F)$-equivariance, and the further extension of this to $GL_2(\mathbb{A}_F)$ in (5.4) is the unique one that is automorphic; it follows that the extension of the differential to $Y(\mathcal{U})$ has form $-\mathfrak{Eis}_j^{\mathcal{U}}(\tau; -\frac{1}{2}) \otimes (d\tau)^{\otimes 2}$ by definition.

5.3.3. Integrality. In general these Eisenstein symbols do not take values in the integral cohomology. However, we can work around this as follows. Let $c$ be coprime to 6, and let $\mathcal{U}$ be a level of the form $\mathcal{U}^{(c)} \times \prod_{\ell \mid c} GL_2(\mathbb{Z}_\ell)$. We write $\mathcal{E}(\mathfrak{Eis}_j^{(c)}, \mathcal{X})$ for the $\mathbb{Z}_\ell$-valued Schwartz functions of the form $\Phi_j^{(c)} \times \prod_{\ell \mid c} \mathcal{E}(\mathfrak{Eis}_j^{(\ell)}, \mathcal{Z})$, and similarly $\mathcal{E}(\mathfrak{Eis}_j^{(c)}, \mathcal{X})$.

Theorem 5.7. There exist homomorphisms

$$\mathcal{E}(\mathfrak{Eis}_j^{(c)}, \mathcal{X}) \to H^1(\mathcal{U}(\mathcal{Q}), (\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)})),$$

compatible with Hecke operators and changing $\mathcal{U}$, such that after base-extending to $\mathbb{Q}$ we have

$$\mathcal{E}(\mathfrak{Eis}_j^{(c)}, \mathcal{X}) \to \mathbb{Z}_\ell$$

where $(c_0, c_1, c_2)$ is understood as an element of $GL_2(\mathbb{Z}_\ell)$.

5.3.4. p-adic interpolation. Let us now fix a prime $p$, an open compact $\mathcal{U}^{(p)} \subseteq GL_2(\mathbb{A}_F^{(p)})$, and a prime-to-$p$ Schwartz function $\Phi(p) \in \mathcal{S}(\mathbb{A}_F^{(p)}, \mathcal{U}^{(p)})$. We suppose $c$ is coprime to $p$, and that $\mathcal{U}$ and $\Phi(p)$ are unramified at the primes dividing $c$. After tensoring with $\mathbb{Q}_p$, we can take the local systems $\mathcal{Y}^{GL_2(\mathfrak{g}_{p, j})}$ (and the class $\mathcal{E}(\mathfrak{Eis}_j^{(p)}, \mathcal{X})$) to have $\mathbb{Z}_p$-coefficients.

For $t \geq 0$, let $\Phi_{p,t} := \mathcal{X}(0, 1) + p^t \mathcal{X}(2)$, a Schwartz function at $p$. The Schwartz functions $\Phi_{p,t} := \Phi^{(p)}, \Phi_{p,t}$, for $t \geq 1$, are stable under the group $\mathcal{U}(p^t) = \mathcal{U}^{(p)} \cdot \{ (\gamma, 1) \text{ mod } p^t \}$, and compatible with the trace maps for varying $t$. So we obtain an Eisenstein–Iwasawa class

$$\mathcal{E}(\mathfrak{Eis}_j^{(p)}, \mathcal{X}) \to H^1_{t\ell}(Y(U_1(p^\infty)), \mathbb{Z}_p) = \lim_{t \to \infty} H^1_{t\ell}(Y(U_1(p^t)), \mathbb{Z}_p)$$

As explained in [LSZb, §9.1], there are natural “moment” maps

$$\text{mom}_j : H^1_{t\ell}(Y(U_1(p^\infty)), \mathbb{Z}_p) \to H^1(Y(U_1(p^t)), \mathfrak{g}_{p, j}(\mathbb{Z}_p))$$

for each $j \geq 0$ and $t \geq 1$; and if $t \geq 1$, we have

$$\text{mom}_j(\mathcal{E}(\mathfrak{Eis}_j^{(p)})) = \mathcal{E}(\mathfrak{Eis}_j^{(p)}, \mathcal{X})$$

This is true by definition for $j = 0$; that it also holds for $j > 0$ is a deep theorem due to Kings. There is a similar statement for $t = 0$, but we need to replace the group $\mathcal{U}(p^t)$ with the GL$_2$ Iwahori subgroup $\mathcal{U}_0(p)$. All of the above structures are compatible with the action of $GL_2(\mathbb{A}_F^{(p+c)})$.

Note that the Iwasawa cohomology group containing $\mathcal{E}(\mathfrak{Eis}_j^{(p)}, \mathcal{X})$ is a module over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[p^\infty]]$, and the moment map $\text{mom}_j$ factors through the quotient where $(1 + p^t \mathcal{X}(p^t)) \cdot x$ acts via $x \mapsto x^t$. If $\Phi^{(p)}$ belongs to the $\chi$-eigenspace for the action of the centre, for some Dirichlet character $\chi$ of prime-to-$p$ conductor (taking values in a finite extension $\mathcal{O}$ of $\mathbb{Z}_p$), then the element

$$\mathcal{E}(\mathfrak{Eis}_j^{(p)}(\mathfrak{A}, \mathfrak{g}, \chi)) = (c^2 - c^{-1} \chi(c))^{-1} \otimes \mathcal{E}(\mathfrak{Eis}_j^{(p)}(\mathfrak{A}, \mathfrak{g}, \chi)) \in H^1_{t\ell} \otimes \text{Frac}(\Lambda)$$

17
is independent of \(c\), where \(j\) is the universal character \(\mathbb{Z}_p^\times \hookrightarrow \Lambda^\times\). In particular, if \(\chi\) is non-trivial, then we can choose \(c\) so that the above factor is invertible in \(\Lambda \otimes \mathbb{Q}_p\), and hence define \(\mathcal{E}L_{\phi(p)}\) as an element of \(H^1_{\text{pro}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\). (We can even work integrally if \(\chi \neq 1 \mod p\)).

6. Constructing the measure

We fix henceforth a prime \(p\), allowing \(p = 2\). We now construct a \(p\)-adic measure interpolating pushforwards (to \(GL_3\)) of Betti–Eisenstein classes. To do so, we use the norm compatibility of these classes to systematically control their denominators after pushforward. Crucially, the measure we construct will be valued in the dual to degree 2 cuspidal cohomology for \(GL_3\).

We work with a fixed RACAR \(\Pi\) in mind. Our \(p\)-adic \(L\)-function will interpolate the critical \(L\)-values in the left half \(\text{Crit}^+_p(\Pi)\) of the critical strip, so in line with \(\S 2.7\), we will privilege the parabolic \(P_1 \subset GL_3\), with associated Hecke operator \(U_{p,1}\). Let \(\Pi\) be a \(P_1\)-refinement of \(\Pi\), and let

\[
K_p := \mathcal{U}_{1,p}^{(p)}(p^{r(\Pi)}) \subset GL_3(\mathbb{Z}_p)
\]

(see (2.8), for \(r(\Pi)\) as in Proposition 2.17).

Denote the weight of \(\Pi\) by \(\lambda = (a, 0, -a)\), with \(a \in \mathbb{Z}_{\geq 0}\), and fix a tame level \(U^{(p)} \subset GL_3(\mathbb{A}_f^{(p)})\) such that \(\Pi\) admits a \(U^{(p)}\)-fixed vector. We also fix a Schwartz function \(\Phi^{(p)} \in \mathcal{S}(k^{(p)}_1, \mathbb{Z}_p)\), and an integer \(c > 1\) coprime to \(6p\). We suppose that \(\Phi^{(p)}\) is stable under \(U^{(p)} \cap GL_2(k^{(p)}_1)\), and that \(c\) is coprime to the levels of \(U^{(p)}\) and \(\Phi^{(p)}\). Let \(U = U^{(p)}K_p\).

**Remark 6.1:** We could perform similar constructions privileging \(P_2\) and \(U_{p,2}\), ultimately interpolating values in \(\text{Crit}^+_p(\Pi)\). We could also work with the Borel \(B\) and \(U_p\), giving a two-variable measure and leading to a \(2\)-variable \(p\)-adic \(L\)-function, where the first variable interpolates \(\text{Crit}^+_p(\Pi)\) and the second \(\text{Crit}^+_{p,2}(\Pi)\), but this requires the stronger condition of \(U_p\)-ordinarity.

6.1. Sketch. Our goal is to define a vector-valued measure \(\Xi^{(a)}\) on \(\mathbb{Z}_p^\times\), taking values in the finitely-generated \(\mathbb{Z}_p\)-module \(c_{\text{ord},1}H^2(\mathcal{Y}^{GL_3}(U), f_\lambda(Z_p))\). The values of this measure at locally-algebraic characters of \(\mathbb{Z}_p^\times\) (of degree at most \(a\)) will be pushforwards of \(GL_2\) Eisenstein classes.

We sketch the construction of \(\Xi^{(a)}\) in the case \(a = 0\), so \(V_\lambda\) is the trivial representation. Let \(\Delta_n := (\mathbb{Z}/p^n)^\times\), so \(\lim_{n \to \infty} \Delta_n = \mathbb{Z}_p^\times\). For \(n \geq 1\), by pushing forward Betti–Eisenstein classes of level \(U_{t_p}(p^{2n})\) one aims to build a class

\[
\xi_n \in H^3(\mathcal{Y}^{GL_3}(U), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta_n],
\]

satisfying the norm relation

\[
\text{norm}_{n+1}^{n+1}(\xi_{n+1}) = U_{t_p}^{(p)} \cdot \xi_n,
\]

where \(\text{norm}_{n+1}^{n+1}: \mathbb{Z}_p[\Delta_{n+1}] \to \mathbb{Z}_p[\Delta_n]\) is the natural projection map. Since \(U_{t_p}^{(p)}\) becomes invertible on the image of the idempotent \(c_{\text{ord}}^{(p)}\), we then define

\[
\Xi^{[0]} = (((U_{t_p}^{(p)} \cdot n_{\text{ord}} \cdot \xi_n))_{n \geq 1}) \in H^3(\mathcal{Y}^{GL_3}(U), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mathbb{Z}_p^\times].
\]

Any finite-order character \(\eta\) of \(\mathbb{Z}_p^\times\) factors through \(\Delta_n\), for some \(n\); then by definition the image of \(\Xi^{[0]}\) under evaluation at \(\eta\) is given by a linear combination of the values of \(\xi_n\). By construction these are pushforwards of Betti–Eisenstein classes, giving the interpolation property.

For general \(a\) the construction is more intricate. For each \(j\) with \(0 \leq j \leq a\), we shall construct a family of classes

\[
\xi^{[a,j]}_n \in H^3(\mathcal{Y}^{GL_3}(U), f_\lambda(Z_p)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta_n]
\]

satisfying (6.2) as \(n\) varies, hence obtaining a family of \(a + 1\) measures \(\Xi^{[a,j]}\) on \(\mathbb{Z}_p^\times\) via the process above. Again, the values of the measure \(\Xi^{[a,j]}\) at finite-order characters will be linear combinations of pushforwards of (weight \(j\)) Betti–Eisenstein classes. We will show that \(\Xi^{[a,j]}\) is the twist of \(\Xi^{[a,0]}\) by \(\chi_{\text{cycl}}\), where \(\chi_{\text{cycl}}\) is the cyclotomic character. This gives the desired output: a single measure with an interpolating property at all locally-algebraic characters of degree up to \(a\).

**Remark 6.2:** We actually prove a slightly different result. It is convenient to introduce a tower of levels \(\cdots \subset Y_{n+1} \subset Y_n \subset \cdots \subset GL_3(A)\), and then construct \(\xi^{[a,j]}_n \in H^3(\mathcal{Y}^{GL_3}(Y_n), \mathcal{V}_n(Z_p)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p\)

18
where the first map is from (6.2) (see Theorem 6.11). We then obtain (scalar-valued) measures by pairing with \( p \)-ordinary classes in \( H^2(Y_{\GL(3)}) \) (see §6.3).

### 6.2. Definition of the classes.

#### 6.2.1. Branching laws and Eisenstein classes.
Recall \( H = \GL_2 \times \GL_1 \). For numerology purposes, our pushforward maps will be from \( \bell_H \) to \( Y_{\GL(3)} \), so we now transfer our Eisenstein classes from \( Y_{\GL(2)} \) to \( \bell_H \). If \( \bell_H \subset H(\Af) \) is open compact, we have a natural composition

\[
pr_{\GL_2} : \bell_H(\bell_H) \to Y_{\GL_2}(\bell_H) \to \bell_H(\bell_H) \cap \GL_2,
\]

where the first map is from (4.3) and the second map is induced by projection \( H \to \GL_2 \). Pullback gives a map

\[
pr_0^{\GL_2} : H^1 \left( Y_{\GL_2}(\bell_H), Y_{\GL_2}(0_{(0,0,0)}) \right) \to H^1 \left( \bell_H(\bell_H), \bell_H(0_{(0,j,0)}) \right).
\]

For appropriate \( \bell_H \) we freely identify \( \iota ! \Eis_{\theta_j} \) with its image under this map.

Let \( j \in \mathbb{Z} \) with \( 0 \leq j \leq a \). Recall the characters \( \nu_1, \nu_2 \) on \( H \) from Definition 4.2. We have an isomorphism

\[
V^{H}_{(0_{(0,0,0)})} \otimes ||\nu_1^j|| \cong V^{H}_{(0_{(0,j,0)})}
\]

of \( H \)-representations. Passing this twist outside the cohomology inverts it, so for appropriate \( \bell_H \) we may thus consider

\[
tw_j(\iota ! \Eis_{\theta_j}) := \iota ! \Eis_{\theta_j} \otimes ||\nu_1^j|| \in H^1 \left( \bell_H(\bell_H), \bell_H(0_{(0,j,0)}) \right) \cong H^1 \left( \bell_H(\bell_H), \bell_H(0_{(0,j,0)}) \right).
\]

Recall the morphisms \( {\text{br}}_{[a,j]} : V^{H}_{(0_{(0,j,0)})} \to \text{I}^*(V_{\lambda}) \) defined in §4.2. We may (and do) suppose that these are integrally defined, and pushing forward we obtain an \( H(\Af) \)-equivariant map

\[
{\text{br}}_{[a,j]} : H^1 \left( \bell_H(\bell_H), \bell_H(0_{(0,j,0)}) \right) \to H^1 \left( \bell_H(\bell_H), \text{I}^*(\lambda)(\Z_p) \right).
\]

#### 6.2.2. Towers of level groups.
We now pushforward under \( \iota \). Let \( r = r(\bellp) \) from Proposition 2.17.

**Definition 6.3.** For \( n \geq 1 \), let \( \U_n \) denote the group

\[
U^0 \times \left\{ g \in \GL_3(\Z_p) : g = 1 \mod \left( \begin{smallmatrix} * & p^n & p^n \\ p^n & * & p^n \\ p^n & p^n & *
\end{smallmatrix} \right) \right\}.
\]

Write \( \nu \) for the unipotent element \( \left( \begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & * \end{smallmatrix} \right) \in \GL_3(\Z_p) \).

We shall let \( t \geq \max(n, r) \) and consider the morphism

\[
\iota_{n,t} : \bell_H(\bell_H^0(p^t) \cap \U_n u^{-1}) \to Y_{\GL_3}(\U_n u^{-1}) \to Y_{\GL_3}(\U_n),
\]

where

\[
\bell_H^0(p^t) = \left( \bell_H^0 \cap \Af(\bell_H^0(p^t)) \right) \times \left( \begin{smallmatrix} * & p^n & p^n \\ p^n & * & p^n \\ p^n & p^n & *
\end{smallmatrix} \right) \mod p^t.
\]

**Proposition 6.4.** If \( t \geq \max(n, r) \), then \( \bell_H^0(p^t) \cap \U_n u^{-1} \) consists of all \( \left( \begin{smallmatrix} a & b & c \\ c & d & 0 \\ 0 & 0 & z \end{smallmatrix} \right) \in \bell_H^0(p^t) \) with \( b = 0 \mod p^n, a = z \mod p^r \). Its index in \( \bell_H^0(p^t) \) is \( p^{2n-1}(p^r - 1) \).

**Proof.** Let \( g = \left( \begin{smallmatrix} A & B & C \\ D & E & F \\ G & H & I \end{smallmatrix} \right) \in \U_n \). Then suppose there exists \( \left( \begin{smallmatrix} a & b & c \\ c & d & 0 \\ 0 & 0 & z \end{smallmatrix} \right) \in \bell_H^0(p^t) \) with

\[
\left( \begin{smallmatrix} a & b & c \\ c & d & 0 \\ 0 & 0 & z \end{smallmatrix} \right) = \iota \left( \begin{smallmatrix} a & b & c \\ c & d & 0 \\ 0 & 0 & z \end{smallmatrix} \right) = \text{agu}^{-1} = \left( \begin{smallmatrix} A+D & -C-E & (B+C) \\ -F & -D & -E \\ G & H & I \end{smallmatrix} \right).
\]

Then \( D = E = 0 \), so \( b = -(C + F) = -C \equiv 0 \mod p^n \), and \( z = -D + E = E \). Moreover \( (B + E) - (A + D) = 0 \) implies \( a = A + D = A + E = z \mod p^n \), as required. Since \( t \geq r \), the congruence conditions \( \mod p^n \) satisfied by \( G, H \) and \( I \) in the definition of \( \U_n \) impose no further restrictions on \( c \equiv 0 \mod p^r \) and \( d \equiv 1 \mod p^r \). The restrictions on \( b \) and \( z \) introduce \( p^r \) and \( p^{r-1}(p^r - 1) \) to the index respectively. \( \square \)

#### 6.2.3. Construction at fixed level \( n \).
Recall the map \( \nu = (\nu_1, \nu_2) : H \to \GL_1 \times \GL_1 \) given by \( (\gamma, z) \mapsto (\frac{\gamma - 1}{\gamma z - 1}, z) \). By Proposition 6.4, we have \( \iota_{t}(h_p) = 1 \mod p^n \) for all \( h \in \bell_H^0(p^t) \cap \U_n u^{-1} \). So \( \nu_1 \) induces a locally constant map

\[
\nu_{1, \mu} : \bell_H^0(p^t) \cap \U_n u^{-1} \to \Delta_n.
\]

David Loeffler and Chris Williams

\[19\]
function notations. Let 

\[ \eta_2 \] be an even Dirichlet character of prime-to-

 conductor, which (for notational convenience) we suppose to be \( \mathbb{Z}_p \)-valued. Shrinking \( \mathcal{U}^{(p)} \) if necessary, we may suppose

\[ \mathcal{U}^{(p)}(p^t) \cap \mathcal{U}_n \cap u \mathcal{U}_n^{-1} \subseteq \ker(\eta_2) \], and thus regard \( \eta_2 \circ \nu_2 \) as a class in \( H^0 \left( \widetilde{Y}^H \left( \mathcal{U}^{(p)}(p^t) \cap \mathcal{U}_n \cap u \mathcal{U}_n^{-1} \right), \mathbb{Z}_p \right) \).

**Remark 6.8:** Philosophically, this process formally ‘spreads out’ the cohomology over (unions of) connected components. More precisely, let \( D = \text{cond}(\eta_2) \). Then we have a map

\[ \nu_{1,(n)} \times \nu_{2,D} : \widetilde{Y}^H \left( \mathcal{U}^{(p)}(p^t) \cap \mathcal{U}_n \cap u \mathcal{U}_n^{-1} \right) \rightarrow \Delta_n \times (\mathbb{Z}/D)^{\times}, \]

and

\[ \widetilde{Y}^H \left( \mathcal{U}^{(p)}(p^t) \cap \mathcal{U}_n \cap u \mathcal{U}_n^{-1} \right) = \bigsqcup \limits_{(x,y) \in \Delta_n \times (\mathbb{Z}/D)^{\times}} \widetilde{Y}^H_{x,y}, \]

where \( \widetilde{Y}^H_{x,y} : \nu_{1,(n)} \times \nu_{2,D} \) is a union of connected components. Let \( i_{x,y} : \widetilde{Y}^H_{x,y} \hookrightarrow \widetilde{Y}^H \left( \mathcal{U}^{(p)}(p^t) \cap \mathcal{U}_n \cap u \mathcal{U}_n^{-1} \right) \) be the inclusion. Then

\[ \mathfrak{c} \text{Eis}_{\Phi_{x,t}} \cup [\nu_{1,(n)}] \cup (\eta_2 \circ \nu_2) = \sum \limits_{x \in \Delta_n} \left( \sum \limits_{y \in (\mathbb{Z}/D)^{\times}} \eta_2(y) \cdot i^*_{x,y} \left( \mathfrak{c} \text{Eis}_{\Phi_{x,t}} \right) \right) \cdot [x]. \]

Our chief interest is in a modified version of these classes.

**Definition 6.9.** Let \( \tau = \tau_1 \in \text{GL}_3(\mathbb{Z}_p) \) from (2.7), and let

\[ V_n := \tau^{-n} \mathcal{U}_n \tau^n = \mathcal{U}^{(p)} \times \left\{ g \in \text{GL}_3(\mathbb{Z}_p) : g = 1 \mod \left( \mathfrak{p}^{n+\tau} \mathfrak{p}^\tau \right) \right\}. \]

Translation by \( \tau^n \) gives a map \( Y^G(\mathcal{U}_n) \rightarrow Y^G(\mathcal{V}_n) \). Crucially, \( \mathcal{V}_n \subset K_p \) from (6.1). We use this to define a pushforward map \( (\tau^n)^* \), on cohomology with coefficients in \( \mathcal{Y}_\lambda \), scaling by \( p^n \) (as in our definition of Hecke operators) in order to obtain a map on the integral lattices \( \mathcal{Y}_\lambda(\mathbb{Z}_p) \).

**Definition 6.10.** We set

\[ c_{\mathcal{V}_n}^{(a,j)}(-) = [\tau^n \cdot 1] \cdot \left( c_{\mathcal{V}_n}^{(a,j)}(-) \right) \in H^3(Y^G(\mathcal{V}_n), \mathcal{Y}_\lambda(\mathbb{Z}_p)) \otimes \mathbb{Z}_p \mathcal{Z}_p[\Delta_n]. \]

Similarly, define \( c_n^{(a,j)}(-) \) (with \( \mathbb{Q}_p \) coefficients) to be the analogue using \( z_n^{(a,j)}(-) \).

The dependence on \( c \in \mathbb{Z} \) is given by

\[ c_n^{(a,j)}(\mathcal{U}^{(p)}, \Phi^{(p)}) = (c^2 - c^{-j} \eta_2(c^{-1}) \langle c \rangle \otimes [c]^{-1}) \cdot \mathfrak{c}^{(a,j)} \left( \mathcal{U}^{(p)}, \Phi^{(p)} \right), \]

where \( [c] \) is the class of \( c \) in \( \Delta_n = (\mathbb{Z}/p^n)^{\times} \), and \( \langle c \rangle \) denotes pullback by \( \text{diag}(c,c,c)^{-1} \in \text{GL}_3(\mathbb{Z}[[c]]) \).

20
6.2.4. Summary. The following diagram summarises the construction of \( \varepsilon^{[a,j]}_{\mu} \). For notational convenience, we write \( U^H_n = U^H(p^n) \cap uU_nu^{-1} \).

\[
\begin{array}{rcl}
\varepsilon^{\text{Eis}}_{p,t,a,j} & \in & H^1 \left( Y^{GL_2}(U^H_n \cap GL_2), \mathcal{F}^{GL_2}_{(0,-1)}(\mathbb{Z}_p) \right) \\
\downarrow & & \downarrow \mathcal{F}^{GL_2} \\
H^1 \left( \tilde{Y}^H(U^H_n), \mathcal{F}^{H}_{(0,-j,0)}(\mathbb{Z}_p) \right) & \cong & H^1 \left( \tilde{Y}^H(U^H_n), \mathcal{F}^{H}_{(j,0,-j)}(\mathbb{Z}_p) \right) \\
\downarrow & & \downarrow \mathcal{F}^H \\
H^1 \left( \tilde{Y}^H(U^H_n), \mathcal{F}^{H}_{(j,0,-j)}(\mathbb{Z}_p) \right) & \cong & Z_p[\Delta_n] \\
\downarrow & & \downarrow \mathcal{F}^H \\
H^1 \left( \tilde{Y}^H(U^H_n), \mathcal{F}_{\lambda}(\mathbb{Z}_p) \right) & \otimes & Z_p[\Delta_n] \\
\downarrow & & \downarrow \mathcal{F}_{\lambda} \\
\mathcal{P}_{\text{a,n}} \mathcal{F}^{[a,j]}_{\mu} & \in & H^3 \left( Y^{GL_3}(U_n), \mathcal{F}_{\lambda}(\mathbb{Z}_p) \right) \otimes Z_p[\Delta_n].
\end{array}
\]

6.2.5. Varying \( n \). If \( U' \subset U \subset GL_3(A_f) \), let \( \mathcal{P}^{U'}_U \) denote the natural projection map \( Y^{GL_3}(U') \to Y^{GL_3}(U) \). We get associated pullback/pushforward maps on cohomology.

**Theorem 6.11 (Norm relation).** For any \( n \geq 1 \) we have
\[
\left( \left( \mathcal{P}^{U^H_n}_{V^H_n} \right)_* \otimes \text{norm}_{\Delta_n}^{\Delta_{n+1}} \right) \left( \varepsilon^{[a,j]}_{\mu}(-) \right) = \left[ \mathcal{P}^{U^H_{n+1}}_{U^H_n} \otimes 1 \right] \varepsilon^{[a,j]}_{\mu}(-)
\]
as elements of \( H^3(Y^{GL_3}(V_n), \mathcal{F}_{\lambda}(\mathbb{Z}_p)) \otimes Z_p[\Delta_n] \).

**Proof.** This is an instance of [Loe, Prop. 4.5.2], elaborated in §4.6 ‘The Betti setting’ and §5.2.3 op. cit. It is an analogue of [LW20, Thm. 3.13]. As this result is key to our construction, for the convenience of the reader we sketch the proof in this special case, translating the notation of [Loe] into our setting. We shall drop the indices \( a, j \) and \( c \) here for brevity.

We first prove an analogue for the elements \( z_n \) (as in [Loe, Prop. 4.5.1]). We claim that
\[
\left[ 1 \otimes \text{norm}_{\Delta_n}^{\Delta_{n+1}} \right] (z_{n+1}) = \mathcal{P}^{U^H_n}_{U^H_{n+1}} \left[ \left( \mathcal{P}^{U^H_{n+1}}_{U^H_n} \right)_* \otimes 1 \right] (z_n)
\]
as elements of \( H^3(Y^{GL_3}(U_{n+1}), \mathcal{F}_{\lambda}(\mathbb{Z}_p)) \otimes Z_p[\Delta_n] \). To prove this, one fixes \( t \geq n+1 \) and checks that the horizontal maps in the diagram
\[
\begin{array}{ccc}
\tilde{Y}^H(U^H_n) & \cap & uU_{n+1}u^{-1} \\
\downarrow & & \downarrow \mathcal{F}^H \\
\tilde{Y}^H(U^H_{n+1}) & \cap & uU_{n}u^{-1}
\end{array}
\]
are injective \(^4\), whence the diagram is Cartesian as both vertical maps have degree \( p^2 \) (using Proposition 6.4). Now, from the definitions we can write
\[
\left[ 1 \otimes \text{norm}_{\Delta_n}^{\Delta_{n+1}} \right] (z_{n+1}) = \mathcal{P}^{(n+1)} \left[ (\mathcal{E}^{[n+1]}_{\text{Eis}})_{p,t,a,j} \cup [\nu(n)] \right].
\]
The class \( \varepsilon^{[a,n]}_{\text{Eis};p,t,a,j} \cup [\nu(n)] \) (at the top left corner) is a pullback from level \( U^H_1(p^t) \cap uU_nu^{-1} \) (that is, the bottom left corner). Thus the right-hand side of (6.8) is obtained by passing from the bottom-left to top right of (6.7) along the left and top arrows, and (6.6) (where the right-hand side is obtained along the bottom and right arrows) follows from the compatibility of pushforward and pullback in Cartesian diagrams.

\(^4\)This injectivity is the crucial reason for introducing the twisting map \( u \) (cf. [LW20, Rem. 4.12]).
Now recall that on cohomology at level $U_n$, we have $U'_{p,1} = \{pr_{U_n}^{-1}(U_n \cap \tau U_n \tau^{-1})\}_{\tau \in \tau_n} \otimes (pr_{U_n}^{-1}(U_n \cap \tau U_n \tau^{-1}))^*$. One easily sees $U_{n+1} = U_n \cap \tau U_n \tau^{-1}$. Applying $\left((pr_{U_n}^{-1}(U_n \cap \tau U_n \tau^{-1}))_{\tau} \otimes 1\right)$ to (6.6), we see

$$
\left((pr_{U_n}^{-1}(U_n \cap \tau U_n \tau^{-1}))_{\tau} \otimes \text{norm}_{\Delta_{n+1}}\right) (z_{n+1}) = \left[U'_{p,1} \otimes 1\right] : z_n
$$

as elements of $H^3(YGL^3(U_n), \mathcal{Y}_X(\mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta_n]$. One checks that

$$
\left((\tau_n) \otimes (pr_{U_n}^{-1}(U_n \cap \tau U_n \tau^{-1}))_{\tau} \otimes 1\right) (z_{n+1}) = \left((pr_{U_n}^{-1}(U_n \cap \tau U_n \tau^{-1}))_{\tau} \otimes (\tau_n)\right) (z_{n}),
$$

whilst a simple check on single coset representatives shows that $(\tau_n) \circ U'_{p,1} = U'_{p,1} \circ (\tau_n)$, and $(\tau_n) \circ \xi_n = (\tau_n) \circ (\tau_n) \circ z_n$. Since $\xi_n = (\tau_n) \circ (\tau_n) \circ z_n$, the result follows by applying $\left((\tau_n) \otimes 1\right)$ to (6.9).

6.3. Pairing with a cuspidal form. Let $\tilde{\Pi} = (\Pi, \alpha_p)$ be a $P_1$-refined RACAR of $GL_3(A)$, let $K_p$ be as in (6.1), and let $\varphi_i \in \Pi^K_p$. Assume that $\varphi_i$ is normalised so that $W_{\varphi_i} \subseteq W_{\varphi_i}(\Pi, E)$, for $E$ a sufficiently large number field, and let $L/Q_p$ contain $E$. Then we have

$$
\phi_{\varphi_i} := \tilde{\xi}_n(\varphi_i) \in H^3(YGL^3, \mathcal{Y}_X(L))^K_p,
$$

where $\tilde{\xi}_n$ is the map from (3.4). Up to renormalising $\varphi_i$, we may assume $\phi_{\varphi_i} \in H^3(YGL^3, \mathcal{Y}_X(\mathcal{O}_L))$, as in §3.3.3. Fix $\Phi^{(p)} \in S(A^{(p)})^2$, and $U^{(p)} \subseteq GL_3(A^{(p)})$ such that $\varphi_i$ (resp. $\Phi^{(p)}$) is fixed by $U^{(p)}$ (resp. $U^{(p)} \cap H$). Let $U_n$ be as in Definition 6.3, and $\mathcal{Y}_n = \tau^{-n}U_n \tau^n$. Let $(\phi_{\varphi_i}, \Phi^{(p)})$, be the projection of $\phi_{\varphi_i}$ to the cohomology of $YGL_n^3(\mathbb{Z}_p)$. Recall the pairing $\langle \cdot, \cdot \rangle_{\mathcal{Y}_n}$ from §3.4. Let $dh_t$ denote the unramified Haar measure on $H(A_t)$, which defines a volume $(U^{(p)} \cap H)$. Proposition 6.12. The element of $\mathcal{O}_L[\Delta_n]$ defined by

$$
\mathcal{E}^{[a,j]}(\varphi_i, \Phi^{(p)}) := \langle \varphi_i, \Phi^{(p)} \rangle \in (U^{(p)} \cap H) \cdot \langle (\phi_{\varphi_i})_n, (\epsilon_{\text{ord},1} \otimes 1)_{\xi_n} \rangle \in \mathcal{Y}_n[\Delta_n],
$$

is independent of $U^{(p)}$ (for suitably chosen $c$).

Proof. If $U' \subseteq U \subseteq GL_3(A^{(p)})$ are two levels, with $U \cap H$ fixing $\Phi^{(p)}$, then one may check that

$$
\left((pr_{U'}^{(p)}), (\xi_n^{[a,j]}(U, \Phi^{(p)})) = \left[(U \cap H) : U' \cap H\right] \cdot \xi_n^{[a,j]}(U', \Phi^{(p)}).
$$

Let $U_1^{(p)}, U_2^{(p)}$ be arbitrary. Then their intersection $U_3^{(p)}$ also satisfies the conditions, and we see the result by applying (6.11) to $U_3^{(p)} \subseteq U_1^{(p)}, U_2^{(p)}$.

We now vary $n$. Let $\varphi_i \in \Pi^K_p[U_{p,1} - \alpha^{p+1}_p(p)]$ (which is possible by Proposition 2.17).

Corollary 6.13. Let $\alpha_p(p) \neq 0$. Then

$$
\mathcal{E}^{[a,j]}(\tilde{\Pi}, \Phi^{(p)}) := \left(\left[p^{a+1} \alpha_p(p)\right]^{-n} \cdot \mathcal{E}^{[a,j]}(\varphi_i, \Phi^{(p)})\right)_{n \geq 1} \in L[Z_p^+]
$$

is a well-defined distribution on $Z_p^+$. If $\tilde{\Pi}$ is $P_1$-ordinary, $\mathcal{E}^{[a,j]}(\tilde{\Pi}, \Phi^{(p)}) \in \mathcal{O}_L[Z_p^+]$ is a measure.

Proof. By the above remarks, it suffices to show that

$$
\mathcal{E}^{[a,j]}(\varphi_i, \Phi^{(p)}) = \left[p^{a+1} \alpha_p(p)\right]^{-n} \mathcal{E}^{[a,j]}(\varphi_i, \Phi^{(p)}) = \left[p^{a+1} \alpha_p(p)\right]^{-n} \cdot \mathcal{E}^{[a,j]}(\varphi_i, \Phi^{(p)})
$$

By definition, we have $(\phi_{\varphi_i})_{n+1} = (pr_{U_n}^{-1}(\varphi_i))_{n+1}$. Since $c, [a,j], \Phi^{(p)}$ and $U^{(p)}$ are fixed, we drop them from notation for clarity, and see

$$
\mathcal{E}^{[a,j]} = \left[p^{a+1} \alpha_p(p)\right]^{-n} \left((\phi_{\varphi_i})_{n+1}, \xi_n \right)_{V_{n+1}} = \left[p^{a+1} \alpha_p(p)\right]^{-n} \left((\phi_{\varphi_i})_{n+1}, \xi_n \right)_{V_{n+1}} = \left[p^{a+1} \alpha_p(p)\right]^{-n} \left((\phi_{\varphi_i})_{n+1}, \xi_n \right)_{V_{n+1}} = \left[p^{a+1} \alpha_p(p)\right]^{-n} \left((\phi_{\varphi_i})_{n}, \xi_n \right)_{V_{n}}.
$$

In the penultimate step we use Theorem 6.11, and in the final step use that $U_{p,1}$ and $U'_{p,1}$ are adjoint under $\langle \cdot, \cdot \rangle$ (and $U_{p,1} \phi_{\varphi_i} = p^{a+1} \alpha_p(p) \phi_{\varphi_i}$). We conclude after rescaling by $\mathcal{E}(U^{(p)} \cap H)$. □
6.4. Getting rid of \( c \). We fix \( \varphi, \Phi^{(p)} \) and \( \mathcal{U}^{(p)} \) and, for now, drop them from notation.

Define ‘non-smoothed’ analogues \( \Xi_{\mathcal{O}}^{[a,j]} \) and \( \Xi_p^{[a,j]} \) of \( \Xi_{\mathcal{O}}^{[a,j]} \) and \( \Xi_p^{[a,j]} \) by using \( \xi_{\mathcal{O}}^{[a,j]} \) instead of \( \xi_p^{[a,j]} \). Since the non-smoothed Eisenstein classes are \( \mathcal{O} \)-rather than \( \mathcal{O}_p \)-valued, these are ostensibly only distributions in \( L[\mathbb{Z}^+_{\mathcal{O}}] \). We now show that they actually lie in \( \mathcal{O}_p[\mathbb{Z}^+_{\mathcal{O}}] \).

**Proposition 6.14.**

(i) We have \( \Xi_{\mathcal{O}}^{[a,j]} = (c^2 - c - j \eta_2(c)^{-1}\omega_{\mathcal{O}}(c)^{-1}[c^{-1}])\Xi_p^{[a,j]} \), for each \( n \).

(ii) If \( \eta_2 \) is chosen so that \( \eta_2 \omega_{\mathcal{O}} \) is congruent mod \( p \) to any character of \( p \)-power conductor, then we have \( \Xi_{\mathcal{O}}^{[a,j]} \in \mathcal{O}_p[\mathbb{Z}^+_{\mathcal{O}}] \), i.e. it is a measure.

**Proof.** (i) This follows from the formula (6.4) above relating \( \xi_{\mathcal{O}}^{[a,j]} \) and the non-\( c \)-version, since the transpose of \( (c) \) with respect to Poincaré duality will be \( (c^{-1}) \), which acts on any vector in \( \Pi \) as multiplication by \( \omega_{\mathcal{O}}(c)^{-1} \).

(ii) Choose \( c \) with \( c = 1 \mod p^n \) and \( \omega_{\mathcal{O}}(c) \neq 1 \mod p \). It follows that the factor relating \( \xi_{\mathcal{O}}^{[a,j]} \) and \( \Xi_p^{[a,j]} \) is invertible in \( \mathbb{Z}_p[\Delta_n] \). Thus \( \Xi_p^{[a,j]} \) is integral for all \( n \) and \( \Xi_{\mathcal{O}}^{[a,j]} \) is a measure.

The auxiliary character \( \eta_2 \) was introduced solely for Proposition 6.14. We must now carry it through all notation, but its only contribution in the rest of the paper will be to the period \( \Omega_{\mathcal{O}} \).

6.5. The Manin relations: compatibility in varying \( j \).

**Theorem 6.15.** For \( 0 \leq j \leq a \) and \( f: \mathbb{Z}^+_{\mathcal{O}} \to \mathbb{Q}_p \), we have a compatibility

\[
\int_{\mathbb{Z}^+_{\mathcal{O}}} f(x) \cdot d\Xi_p^{[a,j]}(\Pi)(x) = \int_{\mathbb{Z}^+_{\mathcal{O}}} f(x) x^j \cdot d\Xi_p^{[a,0]}(\Pi)(x).
\]

**Proof.** We shall prove this by adapting the methods of [LRZ]. However, the result we seek is not quite a direct consequence of the main theorem of op.cit., since the weight \( \lambda = (a, 0, -a) \) of our coefficient sheaf is not induced from a 1-dimensional character of the Levi \( L_1 \) of \( P_1 \), and our level group does not always contain \( L_1(\mathbb{Z}_p) \). So we shall briefly indicate the modifications needed to the theory of op.cit. in order to prove the theorem at hand.

Since our classes are built up from the norm-compatible families \( \xi_p^{[a,j]} \), it suffices to prove a compatibility for \( \xi_{\mathcal{O}}^{[a,j]} \) and \( \xi_p^{[a,0]} \) modulo \( p^n \), regarded as elements of the module

\[
H^3(Y\text{-GL}_1(V_{\lambda}), \mathcal{F}_\lambda(\mathbb{Z}/p^n)) \otimes (\mathbb{Z}/p^n)^{[\Delta_n]}.
\]

More precisely, we shall prove the following: the map \( \text{mom}_\Delta : (\mathbb{Z}/p^n)^{[\Delta_n]} \to (\mathbb{Z}/p^n)^{[\Delta_n]} \), defined on group elements by sending \( [a] \) to \( a^3[a] \), maps \( \xi_{\mathcal{O}}^{[a,0]} \) to \( \xi_{\mathcal{O}}^{[a,j]} \).

We have \( \xi_{\mathcal{O}}^{[a,j]} = [\tau^n \ast \tau^n_j \ast \Xi_{\mathcal{O}}^{[a,j]}] \), where \( \tau \) denotes \( (\begin{smallmatrix} p & 1 \\ 1 & 1 \end{smallmatrix}) \). The action of \( \tau \) on our coefficients is given by a map of \( U_n \)-representations

\[
(\tau^n)_j : V_{\lambda}(\mathbb{Z}_p) \to \tau^n \ast V_{\lambda}(\mathbb{Z}_p)
\]

which is just \( p^n \) times the action of \( \tau^{-n} \) on \( V_{\lambda}(\mathbb{Q}_p) \); the factor \( p^n \) is the image of \( \tau \) under the character \( \lambda \) (compare Definition 2.5.1 of [LRZ]).

Since \( V_{\lambda}(\mathbb{Z}_p) \) is an admissible lattice, it is a direct sum of eigenspaces for the diagonal torus, corresponding to the weights of \( V_{\lambda} \). The map \( (\tau^n)_j \) acts on each of these by a non-negative power of \( p^n \); so on the mod \( p^n \) coefficient sheaf \( \mathcal{F}_\lambda(\mathbb{Z}/p^n) \), it is zero on all eigenspaces except the highest relative weight space for the torus \( Z(L_1) \). So it suffices to prove that the classes \( \xi_{\mathcal{O}}^{[a,j]} \) and \( \text{mom}_\Delta \left( \xi_{\mathcal{O}}^{[a,0]} \right) \) have the same image in the highest relative weight space.

Exactly as in [LRZ], the homomorphism \( br^{[a,j]}_\mathcal{O} \) is determined by the image of the highest-weight vector of \( V^H_{\mathcal{O}}(\lambda(0, -j)) \); let us call this vector \( f^{a,j} \). If we model \( V_{\lambda} \) as a space of functions on \( \mathcal{N}_1(\lambda) \) taking values in the weight \( \lambda \) representation of \( L_1 \), and satisfying \( f(nqg) = \xi \cdot f(q) \), then the value \( f(u) \) determines \( f^{a,j} \) uniquely on \( \mathcal{F}_{\mathcal{O}}^{u^{-1}q^{n-1}} \), which is open in \( G \); so \( f^{a,j} \) is uniquely determined by \( f^{a,j}(u^{-1}) \), and this value lies in a one-dimensional subspace of the weight \( \lambda \) representation which is independent of \( j \) (in fact it is the lowest weight space). So we may suppose that the \( f^{a,j} \) are normalised so that \( f^{0,j}(u^{-1}) \) is independent of \( j \).

Then, exactly as in [LRZ], we obtain a compatibility between moment maps for \( G \) and for \( H \) under pushforward, where \( H = \text{GL}_2 \times \text{GL}_1 \) and \( G = \text{GL}_3 \times \text{GL}_1 \). The moment map for \( G \) gives the twisting operator \( \text{mom}_\Delta \); and the twisting operator on \( H \) maps the weight 0 Eisenstein class to
its weight \( j \) equivalent, by Kings’ theory of \( \Lambda \)-adic Eisenstein classes (see [KLZ17] for a summary in our present notations).

\[ \square \]

### 7. Values of the measure as global Rankin–Selberg integrals

Let \( \Pi \) be a RACAR of \( \GL_3(\A) \). We now show, in Proposition 7.3, that the measures of the previous section compute global \( \GL_3 \times \GL_2 \) Rankin–Selberg integrals for \( \Pi \) with Eisenstein series.

Let \( \tilde{\Pi} = (\Pi, \alpha_{\varphi}) \) be a \( P \)-ordinary refinement (as in §2.7). Let \( \varphi_\ell = \otimes \varphi_\ell \in \Pi \) such that:

(a) \( \varphi_\ell \) is fixed by \( U_{\ell}^{(\Pi)}(p^r(\Pi)) \) and \( U_{p+1, \varphi_\ell} = p^{a_+} \alpha_{\varphi}(p) \cdot \varphi_\ell \) (cf. Proposition 2.17); and

(b) \( \varphi_\ell \) is normalised so that \( \phi_\varphi = \varphi_\Pi(\varphi_\ell)/\Theta_{\Pi} \in H^2(Y_{\GL_3}, \mathcal{F}_{\chi}(\mathcal{O}_L)) \), for \( L/\mathbb{Q}_p \) sufficiently large and \( \varphi_{\Pi} \) as in (3.4) (cf. §3.3.3).

We fix tame data \( \Phi^{(p)} \in S(\A_{\mathbb{Q}}^{(p)})^2 \) and \( \tilde{U}^{(p)} \subset \GL_3(\A_{\mathbb{Q}}^{(p)}) \) such that \( \varphi_\ell \) (resp. \( \Phi^{(p)} \)) is fixed by \( \tilde{U}^{(p)} \) (resp. \( \tilde{U}^{(p)} \cap H \)). These choices fix \( \Phi_{\ell,n} \) and \( \varphi_{\ell,n} \), but we otherwise drop them from notation.

#### 7.1. Values of the measure

Let \( \eta \) be a Dirichlet character of conductor \( p^n \), and let \( n = \max(n_1, 1) \). Consider \( \eta \) as a character on \((\mathbb{Z}/p^n \mathbb{Z})^\times\), and lift it to a character on \( \mathbb{Z}_p^\times \) under the natural surjection. We integrate this against our measure. This factors through \( \Delta_n = (\mathbb{Z}/p^n \mathbb{Z})^\times \), and writing \( x \) for the variable on \( \mathbb{Z}_p^\times \), we have

\[
\int_{\mathbb{Z}_p^\times} \eta(x) \cdot d\zeta_{\ell,n}(\Pi)(x) = \int_{\Delta_n} \eta(x) \cdot d\zeta_{\ell,n}(U_{p+1, \varphi_\ell})(x)
\]

\[
= (p^{a+1} \alpha_{\varphi}(p))^{-n} \left[ 1 \otimes \eta \right] \left( \text{vol}(U^{(p)} \cap H) \cdot \left( \langle (\phi_{\varphi_\ell})_n, \zeta^{[a,j]}_{\ell,n} \rangle_{\Theta_{\Pi}} \right) \right).
\]

We consider this value in a finite extension of \( \mathbb{Q}_p \), but by construction, it is finite at a variable field. For the rest of §7, we will forget this algebraicity and consider it just as a complex number.

To study (7.1), we use the description and notation of Remark 6.8. Set \( t = n \), and for \( x \in \Delta_n \), let \( i_x : Y^H \to Y^H(U^{(p)} \cap H) \) be the natural inclusion. By the remark, the coefficient of \([x]\) in \( \zeta_{\ell,n}^{(a,j)} \) is

\[
\sum_{y \in (\mathbb{Z}/D)^\times} \eta_2(y) \cdot p^{-a} \tau^n \circ u_* \circ i_{x,y} \circ \text{br}^{[a,j]} \circ \text{tw}_j \circ i_{x,y}^*(\text{Eis}_{\Phi^{(p)}}).
\]

Substituting, we see that (7.1) is equal to

\[
\frac{p^{a} \cdot \text{vol}(U^{(p)} \cap H)}{(p^{a+1} \alpha_{\varphi}(p))^{n}} \sum_{x \in \Delta_n} \eta(x) \sum_{y \in (\mathbb{Z}/D)^\times} \eta_2(y) \cdot \left( \langle (\phi_{\varphi_\ell})_n, \tau^n \circ u_* \circ \text{br}^{[a,j]} \circ \text{tw}_j \circ i_{x,y}^*(\text{Eis}_{\Phi^{(p)}}) \rangle_{\Theta_{\Pi}} \right)
\]

via adjointness of pushforward and pullback under the Poincaré duality pairing.

#### 7.2. Cup products as integrals

We now express these cup products as integrals. This follows a standard, if technical, procedure (cf. [Mah98, Lem. 1.3], [Mah00, Lem. 3.2]). This technicality will manifest itself exclusively in a local zeta integral at infinity.

##### 7.2.1. The differential form for \( \Pi \)

Recall \( \zeta_{\infty} \in H^2(\mathfrak{q}_3, K_{3,\infty}^2 ; \Pi_{\infty} \otimes V_{\lambda}^\vee(\mathbb{C})) \) from (2.2). We have

\[
H^1(\mathfrak{g}_3, K_{3,\infty}^2 ; \Pi_{\infty} \otimes V_{\lambda}^\vee(\mathbb{C})) \subset \bigwedge^i (\mathfrak{g}_3/\mathfrak{r}_3^\infty)^\vee \otimes \Pi_{\infty} \otimes V_{\lambda}^\vee(\mathbb{C}),
\]

where \( \mathfrak{r}_3^\infty \subset \text{Lie}(K_{3,\infty}) \). Choose (arbitrary) bases \( \{ \delta_1, \ldots, \delta_5 \} \) of \( (\mathfrak{g}_3/\mathfrak{r}_3^\infty)^\vee \) and \( \{ v_\alpha \} \) of \( V_{\lambda}^\vee(\mathbb{C}) \). Given these choices, there exist vectors \( \varphi_{\infty, r, s, \alpha} \in \Pi_{\infty} \) such that

\[
\zeta_{\infty} = \sum_{r, s, \alpha = 1, \ldots} [\delta_r' \wedge \delta_s'] \otimes \varphi_{\infty, r, s, \alpha} \otimes v_\alpha.
\]

Letting \( \varphi_{r, s, \alpha} := \varphi_{\infty, r, s, \alpha} \otimes \varphi_\ell \in \Pi \), we see the differential form associated to \( \phi_{\varphi_\ell} = \varphi_{\Pi}(\varphi_\ell)/\Theta_{\Pi} \) is

\[
\Theta_{\Pi}^{-1} \sum_{r, s, \alpha = 1, \ldots} [\delta_r' \wedge \delta_s'] \otimes \varphi_{r, s, \alpha} \otimes v_\alpha \in \bigwedge^i (\mathfrak{g}_3/\mathfrak{r}_3^\infty)^\vee \otimes \Pi \otimes V_{\lambda}^\vee(\mathbb{C})
\]
7.2.2. The Eisenstein differential. By Corollary 5.6, the differential on $Y^{GL_2}(U_n^{H} \cap GL_2)$ associated to $Eis_{\psi^1}$ is $-\mathcal{E}_{\psi^1}^{j+2}(g) \cdot (dz)^{\otimes j} \cdot dr$, recalling $\mathcal{E}_{\psi^1}^{j+2}(g) \in I(\| \cdot \|^{-1/2}, \| \cdot \|^{j+1/2})$ (5.6).

For comparison with the GL2 setting, and with [Mah98,Mah00], it is convenient to rephrase this in the (implicit) language of §7.2.1. Combining the above discussion, we see we may choose bases $\{\delta_1, \delta_2\}$ of $(gl_2/\mathcal{R}_2^{\infty})^\vee$ (corresponding to $dr$) and $\{w_{\beta}^{[j]}\}$ of $V^{GL_2}_{(0,-j)}(\mathbb{C})$ (corresponding to $(dz)^{\otimes j}$) such that as a differential form on $Y^{GL_2}$, we may describe $Eis_{\psi^1}$ as

$$\mathcal{E}_{\psi^1}^{j+2}(g) \in I(\| \cdot \|^{-1/2}, \| \cdot \|^{j+1/2}) \quad \text{by Corollary 5.3 and (5.9), with } \eta = 1).$$

7.2.3. The cup product on components. We pass to $H$. We identify the basis elements $\{\delta_1, \delta_2\}$ with their image under the natural pullback $(gl_2/\mathcal{R}_2^{\infty})^\vee \rightarrow (\text{Lie}(H(\mathbb{R}))/\text{Lie}(K_0^{H,\infty}l^{-1}(Z^{\infty}_{G,\infty}))))^\vee$, and then extend to a basis $\{\delta_1, \delta_2, \delta_3\}$ of this latter space. By construction, the pullback of $Eis_{\psi^1}$ to $\tilde{Y}^H$ corresponds to a differential in which only $\delta_1$ and $\delta_2$ appear.

Since the basis $\{\delta_1^\prime\}$ of $(gl_2/\mathcal{R}_2^{\infty})^\vee$ was arbitrary, we may rescale so that under the map

$$i^*: (gl_2/\mathcal{R}_2^{\infty})^\vee \rightarrow (\text{Lie}(H(\mathbb{R}))/\text{Lie}(K_0^{H,\infty}l^{-1}(Z^{\infty}_{G,\infty}))))^\vee,$$

we have $i^*(\delta_1^\prime) = \delta_i$ for $i = 1, 2, 3$ and $i^*(\delta_2^\prime) = 0$ for $i = 4, 5$ (cf. [Mah00, p.277]).

Recall $\langle -,- \rangle_{a,j}$ from (4.4). By definition,

$$\left\langle \text{br}_{[n,j]} \ast u^* \tau^n \ast (\varphi_{r,s,\alpha}), \text{tw}_j i^* Eis_{\psi^1}^{j+2}(\tilde{Y}^H) \right\rangle_{\tilde{Y}^H_{x,y}} = \int_{\tilde{Y}^H_{x,y}} \left[ \Theta_{\Pi}^{-1} \sum_{a,\beta} \sum_{r,s,t=1}^{3} \langle v_{\alpha}, w_{\beta}^{[j]} \rangle_{a,j} \cdot \varphi_{r,s,\alpha} \left( \left( i(h)u \tau^n \right) Eis_{\psi^1}^{j+2}(pr_1(h)) \cdot \|v^{-1}_i(h)\| \cdot [\delta_r \land \delta_s \land \delta_t] \right) \right].$$

Since only $\delta_1$ and $\delta_2$ arise in the Eisenstein differential, the only non-zero terms in the sum over $r,s,t$ are $rst = 132, 231$. As $H_1$ is 3-dimensional, we identify $\delta_1 \land \delta_2 \land \delta_3$ with a fixed choice of Haar measure $d\mu$ on $H(\mathbb{R})$. The expression becomes

$$\left( \Theta_{\Pi}^{-1} \sum_{r,s,t} \varepsilon_{rst} \sum_{a,\beta} \langle v_{\alpha}, w_{\beta}^{[j]} \rangle_{a,j} \int_{\tilde{Y}^H_{x,y}} \varphi_{r,s,\alpha} \left( \left( i(h)u \tau^n \right) Eis_{\psi^1}^{j+2}(pr_1(h)) \cdot \|v^{-1}_i(h)\| \cdot \right) dh, \right.$$

where $\varepsilon_{231} = 1$, $\varepsilon_{132} = -1$, and $\varepsilon_{rst} = 0$ otherwise; and $dh = d\mu d\mu_\infty$, recalling $d\mu$ from §6.3.

7.3. Passing to the Rankin–Selberg integral. By definition, on $\tilde{Y}^{H}_{x,y}$, we have $x = \nu_1(h), y = \nu_2(h)$. Moreover, the characters $\eta$ on $Z^*_p$ and $\eta_2$ on $(\mathbb{Z}/D)^\times$ lift to the (finite order) Hecke characters $\tilde{\eta}$ and $\tilde{\eta}_2$ (as in §2.1). Combining (7.1), (7.2) and (7.5) now gives

$$\int_{Z^*_p} \eta(x) \cdot d\nu^{[a,j]}(\Pi)(x,y) \frac{\text{vol}(U(p) \cap H)}{p^{a+j} \nu_1(\nu_2)} \sum_{r,s,t} \varepsilon_{rst} \sum_{a,\beta} \langle v_{\alpha}, w_{\beta}^{[j]} \rangle_{a,j} \int_{\tilde{Y}^H_{x,y}} \varphi_{r,s,\alpha} \left( \left( i(h)u \tau^n \right) Eis_{\psi^1}^{j+2}(pr_1(h)) \cdot \tilde{\eta} \cdot \|^{-j}(\nu_1(h)) \cdot \tilde{\eta}_2(\nu_2(h)) dh. \right.$$

Up to renormalising our choices at infinity (and hence $d\mu_\infty$), we may take vol($K_0^{H,\infty}$) = 1. Also cancelling the vol($U(p) \cap H$) – which, at $p$, introduces vol($U(p)^H$) – we thus obtain

$$= \frac{\text{vol}(U(p)^H)^{1-1}}{p^{a+j} \nu_1(\nu_2)} \sum_{r,s,t} \varepsilon_{rst} \sum_{a,\beta} \langle v_{\alpha}, w_{\beta}^{[j]} \rangle_{a,j} \int_{H(Q) \setminus H(\mathbb{R})/\mathbb{R}^*} \varphi_{r,s,\alpha} \left( \left( i(h)u \tau^n \right) \right. \times Eis_{\psi^1}^{j+2}(pr_1(h)) \cdot \tilde{\eta} \cdot \|^{-j}(\nu_1(h)) \cdot \tilde{\eta}_2(\nu_2(h)) dh,$$
where $z \in \mathbb{R}_{>0}$ embeds as $[1, z)$. Write $h = (\gamma, z) \in GL_2(A) \times GL_1(A)$, so $\nu_1(h) = \frac{\det(\gamma)}{z}$, and

$$
= \frac{\text{vol}(U^0)}{p^n \alpha_p^* \beta_n} \sum_{r,s,t} e_{rst} \int_{H(Q) \setminus H(A) / \mathbb{R}_{>0}} \varphi_{r,s,a} \left( (\gamma, z) u r^n \right) \times \mathcal{E}_{\Phi}^{j+2}(g) \cdot \eta_j ||^{-j} \frac{\det(\gamma)}{z} \cdot \eta_2(d\gamma, z)
$$

By translation-invariance of Haar measures, we have $dg = d\gamma$. As $||z||^{-j} = 1$ for $z \in \mathbb{Z}^x$, we get

$$
= \frac{\text{vol}(U^0)}{p^n \alpha_p^* \beta_n} \sum_{r,s,t} e_{rst} \int_{GL_2(Q) \setminus GL_2(A) / \mathbb{Z}^x} \varphi_{r,s,a} \left( (\gamma, z) u r^n \right) \times \mathcal{E}_{\Phi}^{j+2}(g) \cdot \eta_j ||^{-j} \frac{\det(\gamma)}{z} \cdot \eta_2(d\gamma, z),
$$

(7.6)

**Lemma 7.1.** Let $k \geq 0$ and $\chi$ be a Dirichlet character. Then we have

$$
\int_{\mathbb{Z}^x} \mathcal{E}_{\Phi}^{j+2}(g) \chi(z) dz = E_{\Phi}^{j+2} \chi(g) = ||g||^{\frac{k}{2}} E_{\Phi}(g; \chi, -\frac{i}{2}).
$$

Proof. By definition, for $z \in \mathbb{Z}^x$ we have $\mathcal{E}_{\Phi}^{j+2}(g) = E_{\Phi}^{j+2} \chi(g)$, where $z$ acts as $(z \cdot)$. Substituting the definitions, we see that the left-hand side is $E_{\Phi}^{j+2} \chi(g) = E_{\Phi}^{j+2} \chi(g)$, where the equality follows from Definition 5.1(iii). We conclude by (5.8) (giving the first equality) and (5.7) (the second).

Taking $\chi = \omega \eta_2$, and collapsing (7.6) with Lemma 7.1, we conclude

$$
\int_{\mathbb{Z}^x} \eta(x) \cdot d\Xi(a, j)(\Pi)(x) = \frac{\text{vol}(U^0)}{p^n \alpha_p^* \beta_n} \sum_{r,s,t} e_{rst} \int_{GL_2(Q) / GL_2(A)} \varphi_{r,s,a} \left( (g, 1) u r^n \right) \cdot E_{\Phi}(g; \tilde{\eta}_2, -\frac{j}{2}) \cdot \eta(dg) ||^{\frac{k}{2}} dg.
$$

**Definition 7.2.** Let $\Pi$ be a unitary automorphic representation of $GL_3(A)$, and $\chi_1$ and $\chi_2$ be Dirichlet characters. For $\varphi \in \Pi$ and $\Phi \in S(A^2, \mathcal{C})$, define

$$
\mathcal{Z}(\varphi, \Phi; \chi_1, \chi_2, s_1, s_2) := \int_{GL_2(Q) / GL_2(A)} \varphi \left( (g, 1) \right) \cdot E_{\Phi}(g; \tilde{\eta}_2, s_2) \cdot \tilde{\eta}_1(dg) ||^{s_1 - \frac{i}{2}} dg.
$$

**Proposition 7.3.** Let $\eta$ be a Dirichlet character of conductor $p^n$. Then

$$
\int_{\mathbb{Z}^x} \eta(x) \cdot d\Xi(a, j)(\Pi)(x) = \frac{\text{vol}(U^0)}{p^n \alpha_p^* \beta_n} \sum_{r,s,t} e_{rst} \int_{GL_2(Q) / GL_2(A)} \varphi \left( (g, 1) \right) \cdot E_{\Phi}(g; \tilde{\eta}_2, -\frac{j}{2}) \cdot \eta(dg) ||^{s_1 - \frac{i}{2}} dg.
$$

Exactly as in [Mah98, Prop. 3.1] (following the proof of [JS81, Prop. 3.3]), for $\Re(s) > 0$ the integral $\mathcal{Z}(\cdot)$ has an Eulerian factorisation in terms of local zeta integrals, namely

$$
(7.7) \quad \mathcal{Z}(\varphi, \Phi; \chi_1, \chi_2, s_1, s_2) = \prod_{v} Z_v \left( W_{\varphi_v, v} \Phi_v; \chi_1, \chi_2, s_1, s_2 \right).
$$

We will define and study the local integrals $Z_v(\cdot)$ in the next section (§8).

We emphasise that in Proposition 7.3, $\varphi_v$ is the finite part of each of the $\varphi_{r,s,a}$ (i.e. the $\varphi_{r,s,a}$ differ only at infinity). This will allow us to move both sums into the local zeta integral at infinity.

## 8. Local zeta integrals

Throughout this section we work locally at a place $v$ of $\mathbb{Q}$, and largely drop $v$ from notation.
8.1. GL₂ principal series. Given \( \Phi \in \mathcal{S}(\mathbb{Q}_p^2) \) and \( g \in \text{GL}_2(\mathbb{Q}_p) \), recall \( f_\Phi(g; \chi, s) \) from §5.2.1. We let \( W_\Phi(g; \chi, s) \) be its Whittaker transform as in [LPSZ, §8.1], defined by analytic continuation of an integral convergent for \( \Re(s) > 0 \); the function \( W_\Phi \) is entire, while \( f_\Phi \) has the same poles as \( \Phi(0, 0) \cdot \chi_2(\chi, s) \). Note the maps \( \Phi \to f_\Phi, \Phi \to W_\Phi \) have the equivariance property
\[
f_{\Phi}(g; \chi, s) = |\det h|^{-s} f_{\Phi}(gh; \chi, s), \quad W_{\Phi}(g; \chi, s) = |\det h|^{-s} W_{\Phi}(gh; \chi, s).
\]

For a fixed \( s \) with \( L(\chi, 2s) \neq \infty \), the space of functions \( f_\Phi(-; \chi, s) \) for varying \( \Phi \) is exactly the induced representation \( \mathcal{I}(\cdot |^{1-s} \chi, \cdot |^{2-s} \chi^{-1}) \), while the space of \( W_\Phi(-; \chi, s) \) is the Whittaker model of this representation (with respect to \( \psi^{-1} \), not \( \psi \)).

If \( L(\chi, 2s) = \infty \), then the \( f_\Phi \) may have poles, and \( \mathcal{I}(\cdot |^{1-s} \chi, \cdot |^{2-s} \chi^{-1}) \) is non-generic: it has a 1-dimensional subrepresentation spanned by the residues of the \( f_\Phi \), and the \( W_\Phi \) lie in the Whittaker model of \( \mathcal{I}(\cdot |^{1-s} \chi^{-1}, \cdot |^{1-s} \chi^{-1}) \) instead.

8.2. A two-parameter zeta integral. Let \( \pi \) be a generic representation of \( \text{GL}_3(\mathbb{Q}_p) \), and let \( \chi_1, \chi_2 \) be two smooth characters of \( \mathbb{Q}_p^* \).

**Definition 8.1.** For \( s_1, s_2 \) complex numbers, \( W \in \mathcal{W}_\pi(\pi) \), and \( \Phi \in \mathcal{S}(\mathbb{Q}_p^2) \), then we set
\[
Z(W, \Phi; \chi_1, \chi_2, s_1, s_2) = \int \left( \mathcal{N}_2 \backslash \text{GL}_2(\mathbb{Q}_p) \right) W(t(g, 1)) W_{\Phi}(g; \chi_2, s_2) \chi_1(\det g) |\det g|^{s_1 - \frac{1}{2}} dg,
\]
which is convergent for \( \Re(s_1) > 0 \) (for fixed \( s_2 \)) and has meromorphic continuation to all \( s_1 \) and \( s_2 \), as a rational function in \( \ell^{(s_1 \pm s_2)} \) if \( v = \ell \) is a finite place.

**Theorem 8.2** (Jacquet, Piatetski-Shapiro, Shalika). The function
\[
Z(W, \Phi; \chi_1, \chi_2, s_1, s_2) := \frac{Z(W, \Phi; \chi_1, \chi_2, s_1, s_2)}{L(\pi \times \chi_1, s_1 + s_2 - \frac{1}{2}) L(\pi \times \chi_1 \chi_2^{-1}, s_1 - s_2 + \frac{1}{2})}
\]
is entire as a function of the \( s_i \), and is a polynomial in \( \ell^{(s_1 \pm s_2)} \) if \( v = \ell \) is a finite place. The ideal generated by these functions for varying \( (W, \Phi) \) is the unit ideal.

If \( v \) is a finite place, there exist \( W \) and \( \Phi \) defined over \( \mathbb{Q} \) such that \( Z(W, \Phi; \chi_1, \chi_2, s_1, s_2) = 1 \) for all \( s_1, s_2 \); and if \( \pi \) and the \( \chi_i \) are unramified, this is true when \( W \) and \( \Phi \) are the normalised spherical data.

In particular, for \( W \) and \( \Phi \) as in the theorem, we have \( Z(W, \Phi; \chi_1 \theta_1, \chi_2 \theta_2, s_1, s_2) = 1 \) for all unramified characters \( \theta_1, \theta_2 \) (since we can move such \( \theta_i \) into the \( s_i \)).

**Proof.** We know that, for each \( s_2 \), the functions \( W_{\Phi}(-; \chi_2, s_2) \) for varying \( \Phi \) form the Whittaker model of the representation \( \Pi' = I(\cdot |^{s_2-1/2} \chi_2^{-1}, \cdot |^{2-s_2} \chi_2^{-1}) \); and evidently the \( W(g) \chi_1(\det g) \) span the Whittaker model of \( \pi \times \chi_1 \). So, by the results of [JPSS83], the greatest common divisor of the \( Z(-; \chi_1, \chi_2, s_1, s_2) \), as functions of \( s_1 \), is the \( L \)-factor
\[
L(\pi \times \Pi' \times \chi_1, s_1) = L(\pi \times \chi_1, s_1 + s_2 - \frac{1}{2}) L(\pi \times \chi_1 \chi_2^{-1}, s_2 - s_2 + \frac{1}{2}),
\]
where the latter equality follows from the compatibility of Rankin–Selberg \( L \)-factors with parabolic induction (also proved in op. cit.).

So, for each fixed \( s_2 \), the normalised integrals \( \tilde{Z}(-; \chi_1, \chi_2, s_1, s_2) \) generate the unit ideal. As these functions are meromorphic in \( s_1 \pm s_2 \), they in fact generate the unit ideal in \( \mathbb{C}[\ell^{s_1 \pm s_2}] \).

Note that
\[
Z(W, \Phi; \chi_1, \chi_2, s_1, 1 - s_2) = Z(W, \tilde{\Phi}; \chi_1 \chi_2^{-1}, \chi_2^{-1}, s_1, s_2).
\]
This is immediate from the functional equation of \( W_\Phi \), cf. [LPSZ, Eq. (8.1)]. The denominator in the definition of \( \tilde{Z} \) is the same for both sides (the factors get swapped) so this relation also holds for \( \tilde{Z} \) in place of \( Z \). There is also a functional equation in \( s_1 \) (for fixed \( s_2 \)), but this is more complicated to state, and we shall not use it here.

8.3. A second zeta-integral. We consider a second zeta-integral, studied in [LSZa] (following a series of earlier works). In this section, we shall assume \( v \) is a finite place.
Definition 8.3. For \((\pi, \chi_1, \chi_2, W, \Phi)\) as before, we define
\[
Y(W, \Phi; \chi_1, \chi_2, s_1, s_2) = \int_{(X_2)(\mathbb{Q}_p)} W \left[ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right] f_\Phi(g; \chi_2, s_2) \left| \det g \right|^{s_1 + \frac{1}{2}} \chi_1(\det g) dg.
\]

This integral converges for \(\Re(s_2) > 0\) for any fixed \(s_1\) [sic!] and extends to a meromorphic function of \(s_1\) and \(s_2\), which is a rational function of \(\ell^{s_1 + s_2}\) if \(v = \ell\) is finite.

8.3.1. Relation to the \(Z\) integral. We will prove the following theorem in an appendix below.

Theorem 8.4. We have the identity
\[
Y(W, \Phi; \chi_1, \chi_2, s_1, s_2) = \gamma(\pi \times \chi_1/\chi_2, s_1 - s_2 + \frac{1}{2}) \cdot Z(W, \Phi; \chi_1, \chi_2, s_1, s_2),
\]
where
\[
\gamma(\pi, s) = \frac{\varepsilon(\pi, s)L(\pi^\vee, 1 - s)}{L(\pi, s)}
\]
is the local \(\gamma\)-factor. In particular, the greatest common divisor of the \(Y(-)\) as \((W, \Phi)\) vary is the principal ideal generated by the product of \(L\)-factors
\[
L(\pi \times \chi_1, s_1 + s_2 - \frac{1}{2}) \cdot L(\pi^\vee \times \chi_1^{-1} \chi_2, s_2 - s_1 + \frac{1}{2}).
\]

8.3.2. Torus integrals. We now give an alternative, more “computable” formula for \(Y(-)\).

Definition 8.5. For \(W \in \mathcal{W}(\pi)\), define a function on \(GL_2(\mathbb{Q}_p)\) by
\[
y(W; \chi_1, \chi_2, s_1, s_2)(g) = \chi_1(\det g) \left| \det g \right|^{s_1 + \frac{1}{2}} \chi_2(\det g) \left| g \right|^{s_2 - s_1 + \frac{1}{2}} \chi^{\vee}(y) d^2 x d^2 y.
\]

The integral converges for \(\Re(s_2) > 0\) (for fixed \(s_1\)), as before; more precisely, it converges in some quadrant \(\Re(s_2 - s_1) > C, \Re(s_2 + s_1) > C\). A computation shows that it transforms as an element of \(I(1/2 - s_2, \chi_1^{-1} \chi_2)\), which is the dual space of the representation in which \(f_\Phi(-; \chi_2, s_2)\) lies, and the duality pairing recovers \(Y\); that is, we have
\[
Y(W, \Phi; \chi_1, \chi_2, s_1, s_2) = \left( y(W; \chi_1, \chi_2, s_1, s_2), f_\Phi(-; \chi_2, s_2) \right).
\]

8.4. Parahoric level test data. We now take \(\pi = \Pi_p\), and evaluate (in Theorem 8.7) the local zeta-integral directly for certain specific test data; the integral we consider here is a local factor of the global integral in Proposition 7.3. We suppose that \(v\) is a finite place, and denote it by \(p\) (for compatibility with our applications below).

Suppose \(\Pi\) has an unramified \(P_1\)-refinement \(\tilde{\Pi} = (\Pi, \alpha_p)\), so there is an unramified character \(\alpha_p\) and GL2-representation \(\sigma_p\) such that \(\alpha_p \boxtimes \sigma_p \mapsto J_{P_1}(\Pi_p)\). We let \(n\) be the conductor of \(\sigma_p\).

Henceforth, we fix the following characters and test data. Recall \(r(\tilde{\Pi})\) from Proposition 2.17.

Notation 8.6.
- Let \(\chi_1 = \hat{\omega}_{\Pi_p, \eta_1, \Pi_2, p}\), where \(\eta_1\) is a Dirichlet character of conductor \(p^{n_1}\); for \(n_1 \in \mathbb{Z}_{\geq 0}\) (so that \(\chi_1(p) = 1\) and \(\chi_1|_{\mathbb{Z}_p^\times} = \eta_1^{-1}\)). Let \(n = \max(1, n_1) \geq 1\).
- Let \(\chi_2 = \hat{\omega}_{\Pi_p, \eta_1, \Pi_2, p}\), where \(\eta_2\) is the auxiliary Dirichlet character from Notation 6.6.
- Let \(W^\alpha = \Pi_p[U_{p, 1} - p^{s_1 + 1} \alpha_p(p)]\) be the \(P_1\)-stabilised newvector of \(\Pi\) as defined in §2.8.
- Let \(u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}_p)\) as before, and take for \(W\) the element \(u^{-1} \cdot W^\alpha\).
- Let \(\Phi\) be the characteristic function \(\Phi_R = \chi((0, 1) + p^R \mathbb{Z}_p^2), \) for \(R = \max(n, r(\tilde{\Pi}))\).

Recall the Coates–Perrin-Riou factor \(e_p(\tilde{\Pi}, \eta, -j)\) from Definition 2.21.

Theorem 8.7. We have
\[
Y(W, \Phi; \chi_1, \chi_2, \frac{1}{2} - i, \frac{1}{2} - i) = \frac{\alpha^n_p}{p^{2R(n + 1) - 1}} \cdot e_p(\tilde{\Pi}, \eta_1, -j) \cdot \varepsilon_0 \cdot L(\chi_1^{-1} \chi_2, s_1 + s_2, 0),
\]
where \(\varepsilon_0\) is an explicit \(p\)-adic unit. In particular, we have
\[
Z(W, \Phi; \chi_1, \chi_2, \frac{1}{2} - i, \frac{1}{2} - i) = \frac{\alpha^n_p}{p^{2R(n + 1) - 1}} \cdot e_p(\tilde{\Pi}, \eta_1, -j) \cdot \frac{\varepsilon_0 \cdot L(\chi_1^{-1} \chi_2, s_1 + s_2, 0)}{\varepsilon(\Pi_p \times \omega_{\Pi_p, \eta_1, \Pi_2, p}^{-1} \chi_2, s_1 + s_2, 0)}.
\]
The factor $\alpha_p^{\sigma}$, and the denominator term (which has an interpretation as the index of a subgroup of $H(\mathbb{Z}_p)$), will naturally cancel out in our $p$-adic interpolation computations. The local $L$-factor will later contribute to the period $\Omega_\Pi$ (see (9.1)).

**Proof.** The statement for $Z$ follows immediately from the statement for $Y$ and Theorem 8.4.

We focus, then, on $Y$. To save notation we give the proof supposing $\eta_2,p$ is trivial (the general case can be reduced to this by twisting). Since $R \geq 1$, one computes that $f_{\eta_2,p}(\chi, \omega_1, \omega_2)$ has support in $B_\beta(\mathbb{Q}_p)K_\alpha(p^R)$, whose measure in the quotient $B_\beta(\mathbb{Q}_p)\backslash GL_2(\mathbb{Q}_p)$ is $\frac{1}{p^{R(1+1/p)}}$; and its value at the identity is $\frac{1}{p^{R(1-1/p)}}$. Hence for all sufficiently large $R$ (depending on $W$) we have

$$Y(W; \Phi_R; \chi_1, \chi_2, s_1, s_2) = \frac{1}{p^{R(1-p^{-1})}} \eta(W; \chi_1, \chi_2, s_1, s_2)(1).$$

Thus the Whittaker function in the integral $y(W; \chi_1, \chi_2, s_1, s_2)(1)$ is given by

$$W_\sigma \left( \begin{pmatrix} s & \gamma \\ \eta & 1 \end{pmatrix} \right) = \psi(x) \omega_\Pi(y)^{-1} W_\sigma \left( \begin{pmatrix} s & y \\ \eta & 1 \end{pmatrix} \right).$$

Hence the integral is given by

$$\sum_{a,b \in \mathbb{R}^2} W_\sigma \left( \begin{pmatrix} s & a \gamma + b \\ \eta & 1 \end{pmatrix} \right) p^{-a_1 + s_2 - \frac{3}{2}} p^{-b(s_2 - s_1 - \frac{1}{2})} \int_{\mathbb{P}_p} \psi(p^s x) \chi_1(x) \, dx.$$

The values of the Whittaker function $W_\sigma$ along the torus are given by Proposition 2.18. The integral over $\mathbb{Z}_p$ is standard: if $\eta$ is ramified, then it is $G(\eta^{-1}) \psi(1 - p^{-1})$ when $a = -\alpha$, and vanishes otherwise; if $\eta$ is unramified (so $n = 1$), then it vanishes for $a < -1$, takes the value $\frac{1}{p^s}$ for $a = -1$, and is 1 for $a \geq 0$.

For non-trivial $\eta$, we thus compute that $Y(W; \Phi_R; \chi_1, \chi_2, s_1, s_2)$ is equal to

$$= \frac{G(\eta^{-1})}{p^{2R + n}(1 - p^{-1})(1 - p^{-2})} \sum_{b \geq 0} W_\sigma \left( \begin{pmatrix} s & b \\ \eta & 1 \end{pmatrix} \right) p^{-b(s_2 - s_1 - \frac{1}{2})} L(\sigma \times \alpha_p, s_2 - s_1 + 1/2),$$

where $\sigma$ is the $GL_2$ representation such that $\Pi_p$ is a subquotient of $\alpha \boxtimes \sigma$. So

$$Y(W; \Phi_R; \chi_1, \chi_2, s_1, s_2) = \frac{G(\eta^{-1})}{p^{2R + n}(1 - p^{-1})(1 - p^{-2})} \cdot L(\sigma \times \alpha_p, s_2 - s_1 + \frac{1}{2}).$$

If $\eta$ is trivial, we obtain instead

$$= \frac{\alpha_p(p)}{p^{2R + n}(1 - p^{-1})(1 - p^{-2})} \cdot \frac{(1 - \alpha_p(p)^{-1} p^{-\frac{s_1 - s_2}{2}})}{(1 - \alpha_p(p)^{-1}) \cdot L(\sigma \times \alpha_p, s_2 - s_1 + \frac{1}{2})}.$$

We see that when $s_1 = \frac{1}{2}$ and $s_2 = -\frac{1}{2}$, then in either case we obtain

$$Y(W; \Phi_R; \chi_1, \chi_2, s_1, s_2) = \frac{\alpha_p^2}{p^{2R + n}(1 - p^{-1})(1 - p^{-2})} \cdot L(\sigma \times \alpha_p, 0).$$

To remove the $L$-factor $L(\sigma \times \alpha_p, 0)$, let

$$E_s = \frac{L(\sigma \times \alpha_p, s)}{L(\Pi_p, \omega_{\Pi_p, s})}.$$

The statement for $Y$ in the theorem follows by replacing $L(\sigma \times \alpha_p, 0)$ with $E_s \cdot L(\Pi_p, \omega_{\Pi_p, 0})$. It only remains to show $E_0$ is a $p$-adic unit, which we do in Lemma 8.8 below. □

**Lemma 8.8.** If $\Pi_p$ is irreducibly induced from a representation $\alpha \boxtimes \sigma$ of $GL_1 \times GL_2$, then we have

$$L(\sigma \times \alpha_p, s) = \frac{1}{L(\Pi_p, \omega_{\Pi_p, s})} = 1 - \omega_{\Pi_p}(\alpha_p^{-1})(p).$$

If the induction of $\alpha_p \boxtimes \sigma_p$ is reducible, then this ratio is identically 1.

In particular, if $\Pi$ is $P_1$-ordinary, then $E_0$ is a $p$-adic unit.

(We have taken $\eta_2,p = 1$ again, but clearly the analogous statement with $\eta_2,p \neq 1$ holds).
Proof. For GL₃, we have $L(\Pi_v^\vee \times \omega_{\Pi_p}, s) = L(\lambda^2 \Pi_p, s)$ (see e.g. [Kim03, §1]). If $\Pi_p$ is irreducibly induced, then we have $L(\lambda^2 \Pi_p, s) = L(\sigma_p \times \alpha_p, s)L(\lambda^2 \sigma_p, s)$, and we have $\lambda^2 \sigma_p = \omega_{\Pi_p} = \omega_{\Pi_p}^{-1}$.

In the remaining cases, one can write down the Langlands parameter as in Example 2.15 and argue similarly to find that the ratio is 1. For the final claim, we note that $E_0$ is either 1, or $1 -\omega_p(p)\alpha_p(p)^{-1}$; and $\alpha_p(p)^{-1}$ has valuation 1 + $a$ with respect to our choice of embeddings into $\mathbb{Q}_p$ (while $\omega_p(p)$ is a root of unity), so $E_0$ must be a $p$-adic unit. □

8.5. Local zeta integrals at infinity. First suppose $v = \infty$. The following is directly analogous to the discussion after [Mah00, Lem. 1.1]. Recall $\Phi^{3+2}$ from (5.5), and let $W_{\infty} \in W_\psi(\pi_{\infty})$.

Lemma 8.9. [JPSS79]. There exists a polynomial $P_j(W_{\infty}; T) \in \mathbb{C}[T]$ such that

$$Z(W_{\infty}, \Phi^{3+2}; \chi_1, \chi_2, s - \frac{1}{2}) = P_j(W_{\infty}; s + \frac{1}{2}) \cdot L(\pi \times \chi_1, s - \frac{1}{2}) \cdot L(\pi \times \chi_1 \chi_2^{-1}, s + \frac{1}{2}).$$

Remark 8.10: Our $\Phi^{3+2}$ is analogous to the discussion after [Mah00, Lem. 1.1]. Recall $\Phi^{3+2}$ from Notation 8.6, and let $\omega_p$ be test data so that $W_\psi(\varphi_p) = W_{\infty}$.

9. The $p$-adic $L$-function

We collect our constructions and prove the interpolation formula, completing the proof of Theorem C from the introduction. As a corollary, in §9.5 we give an alternative proof of the compatibility of Mahnkopf's Archimedean periods for GL₃ (i.e., the final step in Conjecture 2.7).

Fix $\Pi = (\Pi, \alpha_p)$ a $P_1$-ordinary $P_1$-refined RACAR of $GL_3(\mathbb{A})$ of weight $\lambda = (a, 0, -a)$. Recall the auxiliary character $\eta_2$ from Notation 6.6.

9.1. Main result. Fix test data $\varphi = \otimes \varphi_v \in \Pi^{\alpha_1}[U_{p,1} - \alpha_2 + 1; \alpha_p(p)]$ and $\Phi = \otimes \Phi_v \in S(\pi_a^2, E)$ (for a sufficiently large number field $E$) as follows.

Notation 9.1: At $v \nmid p\infty$, let $W_v$ and $\Phi_v$ be test data so that $\tilde{Z}(W_v, \varphi_v; 1, \omega_{\Pi_v}, \eta_{2,v}, s_1, s_2) = 1$ as in Theorem 8.2, and let $W_\psi$ be such that $W_\psi(\varphi_v) = W_v$.

- At $v = p$, we take $\Phi_\psi = \Phi^{3+2}_\psi$, let $\varphi_\psi$ be arbitrary, and define $W_{\infty} = W_\psi(\varphi_\psi)$.

- At $v = \infty$, we take $\varphi_\psi$ such that $W_\psi(\varphi_\psi) = W_\psi$.

We may (and do) normalise so that, as in §7, we have $\phi_\Pi := \phi_{\Pi_{\psi}}(\varphi_\Pi)/\Theta_{\Pi} \in \mathbb{H}^2(\psi_{\mathbb{C}} \chi_{\mathbb{C}}(\mathcal{O}_L))$, for $L/\mathbb{Q}_p$ finite (containing $E$). Recall $\Xi_{\psi}^{[a,b]}(\Pi, \Phi_\psi) \in \mathcal{O}_{\mathbb{L}}[\mathbb{Z}_p^\vee]$ from §§6.3 and 6.4.

Definition 9.2. We define the $p$-adic $L$-function of $\Pi$ to be $L_p(\Pi) := \Xi_{\psi}^{[a,b]}(\Pi, \Phi_\psi) \in \mathcal{O}_{\mathbb{L}}[\mathbb{Z}_p^\vee]$.

Recall $E_0 \neq 0$ from (8.2) and Lemma 8.8, and define a period

$$\Theta_{\Pi} \cdot \varphi \cdot \omega_{\Pi_\psi} \cdot (1 + \Theta_{\Pi} \cdot \varphi \cdot \omega_{\Pi_\psi}^{-1}) \in \mathbb{C}^\times.$$ (9.1)

The following interpolation, which proves Conjecture B of the introduction for $n = 3$, is our main theorem. The proof will occupy the rest of §9.

Theorem 9.3. For $(-j, \eta) \in \text{Crit}_{\Pi_p}^\vee(\Pi)$, we have

$$\int_{\mathbb{Z}_p^\vee} \eta(x) x^\nu \cdot \tilde{d}L_p(\Pi)(x) = e_{\infty}(\Pi, -j)e_\psi(\Pi, \eta, -j) \cdot \frac{L(p)(\Pi \times \eta_{2,j})}{\Omega_{\Pi}}.$$

9.2. Interpolation of $L$-values, part I. Recall from §7.2: our choices of $\zeta_\infty$ (in (2.2)) and bases $v_\alpha, u_{\beta}^{[j]}$ determined forms $\varphi_{\infty_{\alpha}, s, \alpha} \in \Pi_{\infty}$, and we set $\varphi_{\alpha, s, \alpha} = \varphi_{\infty_{\alpha}, s, \alpha} \cdot \varphi_\eta$. By Theorem 6.15 and Proposition 7.3, the left-hand side of (9.2) is then

$$\int_{\mathbb{Z}_p^\vee} \eta(x) x^\nu \cdot \tilde{d}\Xi_{\psi}^{[a,j]}(\Pi)(x) \left(\prod_{\alpha, \beta} \sum_{\nu_{\alpha, \beta}^{[j]} = \nu_{\alpha, \beta}^{[j]}} (v_{\alpha, \beta}^{[j]} \cdot \varphi_{\alpha, \beta, \alpha}, \Phi, \eta, \omega_{\Pi_\psi} \eta_2, 1 - \frac{1}{2}, -\frac{1}{2}) \right).$$ (9.3)
Each \( Z(-) \) is a product of local integrals \( Z_v(-) \) by (7.7). As the finite parts of the \( \varphi_{r,s,\alpha} \) are the same, for each \( v \nmid \infty \) the integral \( Z_v(-) \) is the same in each summand. Such \( Z_v(-) \) is computed in Theorem 8.2 \((v \nmid \infty)\) and Theorem 8.7 \((v = p, \text{noting } W = uw_1^* \cdot W^*\)). We conclude

\[
Z\left( w_1^* \cdot \varphi_{r,s,\alpha}, \Phi; \eta, \omega_\infty \eta_2^*, \frac{1}{2}, -\frac{1}{2} \right) = Z_\infty\left( W_{\infty,r,s,\alpha}, \Phi^{j+2}_\infty; \tilde{\omega}_\infty, \tilde{\omega}_\infty, \frac{1}{2}, -\frac{1}{2} \right)
\]

\[
\times \frac{\alpha_0^{(p)} \cdot E_0}{p^{2R+n}(1-p^{-1})(1-p^{-2})} \cdot e_p(\tilde{\Pi}, \eta, -j) \cdot L^{(p)}(\Pi \times \eta, -j) \cdot \frac{L(\Pi \times (\omega_\infty \eta_2)^{-1}, 1)}{\varepsilon(\Pi_p \times (\omega_\infty \eta_2 p_2, p_2)^{-1}, 1)},
\]

where \( W_{\infty,r,s,\alpha} = W_{\psi}(\varphi_{r,s,\alpha}) \in W_{\psi}(\Pi_\infty) \) (recalling that \( \eta_2 \) is even, so \( \tilde{\omega}_{\infty,2} = 1 \)).

We now sweep the sums into the local zeta integral at \( \infty \). Let

\[
\tilde{c}_\infty(\tilde{\omega}_\infty, -j) := \sum_{r,s,t} \epsilon_{r,s,t} \cdot \sum_{\alpha,\beta} \left\langle v_\alpha, w_\beta^{[j]} \right\rangle \cdot Z_\infty\left( W_{\infty,r,s,\alpha}, \Phi^{j+2}_\infty; \tilde{\omega}_\infty, \tilde{\omega}_\infty, \frac{1}{2}, -\frac{1}{2} \right).
\]

(Note \( \eta_\infty \) is determined by \( j \) and \( \Pi_\infty \), so we suppress it from notation). Then

\[
(9.4) \quad (9.3) = \frac{1}{p^n \text{vol}(U_{n,p}) \cdot \alpha_0^{(p)} \cdot E_0} \cdot \frac{\alpha_0^{(p)} \cdot E_0}{p^{2R+n}(1-p^{-1})(1-p^{-2})} \cdot \tilde{c}_\infty(\tilde{\omega}_\infty, -j) \cdot e_p(\tilde{\Pi}, \eta, -j) \cdot L^{(p)}(\Pi \times \eta, -j) \cdot \frac{L(\Pi \times (\omega_\infty \eta_2)^{-1}, 1)}{\varepsilon(\Pi_p \times (\omega_\infty \eta_2 p_2, p_2)^{-1}, 1)}.
\]

We compute \( \text{vol}(U_{n,p})^{-1} = p^{2R+2n}(1-p^{-1})(1-p^{-2}) \) via Proposition 6.4, whence by (9.1) we find

\[
\int_{\mathbb{Z}_p^n} \eta(x)x^j \cdot dL_p(\tilde{\Pi})(x) = \tilde{c}_\infty(\tilde{\omega}_\infty, -j) \cdot e_p(\tilde{\Pi}, \eta, -j) \cdot \frac{L^{(p)}(\Pi \times \eta, -j)}{\Omega_\Pi}.
\]

9.3. Non-vanishing at infinity. It remains to evaluate \( \tilde{c}_\infty(\tilde{\omega}_\infty, -j) \). We first show it is non-zero. For each \( Z_\infty(W_{\infty,r,s,\alpha}, -j) \) in its definition, by Lemma 8.9 we get an associated polynomial \( P_j(W_{\infty,r,s,\alpha}, T) \in \mathbb{C}[T] \). Define

\[
P_j(\tilde{\omega}_\infty; s + \frac{1}{2}) := \sum_{r,s,t} \epsilon_{r,s,t} \cdot \sum_{\alpha,\beta} \left\langle v_\alpha, w_\beta^{[j]} \right\rangle \cdot P_j(W_{\infty,r,s,\alpha}; s + \frac{1}{2}).
\]

This is (an explicit non-zero multiple of) the analogue of \( P(s) \) in [Mah00, (3.1)] (cf. Remark 8.10). Combining with Lemma 8.9, we see

\[
\tilde{c}_\infty(\tilde{\omega}_\infty, -j) = P_j(\tilde{\omega}_\infty; \frac{1}{2}) \cdot L(\Pi_\infty \times \eta_\infty, -j) \cdot L(\Pi_\infty \times \omega_\infty^{-1}, 1).
\]

By [KS13], we know \( P_j(\tilde{\omega}_\infty; \frac{1}{2}) \neq 0 \) (cf. [Ger15, Thm. 2.1]), and hence \( \tilde{c}_\infty(\tilde{\omega}_\infty, -j) \neq 0 \).

9.4. Symmetric square \( p \)-adic L-functions. To pin the term at infinity down more precisely, we exploit the fact that Theorem 9.3 is known in full when \( \Pi \) is essentially self-dual, i.e. \( \Pi \) is a (twist of a) symmetric square lift. We recall this result.

Let \( f \) be a classical cuspidal \( p \)-ordinary newform of weight \( a + 2 \), and \( \theta \) a Hecke character over \( \mathbb{Q} \) of prime-to-\( p \) conductor. Let \( \pi' = \text{Sym}^2(f) \times \theta \) be the symmetric square lift to \( \text{GL}(4) \), which has weight \((2a, a, 0)\), and let \( \pi = \pi' \times || \cdot ||^{-a} \) (which has weight \((a, 0, -a)\)). Note that if \( A_p(f), B_p(f) \) are the Satake parameters of \( f \) at \( p \), then the Satake parameters of \( \pi \) at \( p \) are

\[
p^a \alpha_p(\pi) = \theta(p) A_p(f)^2, \quad p^a \beta_p(\pi) = \theta(p) A_p(f) B_p(f), \quad p^a \gamma_p(\pi) = \theta(p) B_p(f)^2.
\]

Let \( \pi \) the (unique) \( P_1 \)-ordinary refinement of \( \pi \). If \( \omega_\pi(-1) = -1 \), let \( b = 0 \); otherwise let \( b = 1 \).

Theorem 9.4 (Schmidt, Hida, Darbrowski–Delbourgo). There exists \( \mathcal{L}_p(\pi) \in \mathbb{C}_p \otimes \mathbb{Z}_p \mathbb{Z}_p[\mathbb{Z}_p^*] \) such that for finite-order characters \( \eta \) of \( \mathbb{Z}_p^* \), and \( 0 \leq j \leq a \) with \((-1)^j = \omega_\eta(-1)\), we have

\[
9.7 \quad \int_{\mathbb{Z}_p} \eta(x)x^j \cdot d\mathcal{L}_p(\pi)(x) = \frac{\omega_\pi(-1) \cdot \Gamma(-j + a + 1)}{2^{2a+4} \cdot 5^b \cdot (2\pi i)^{-j}} \cdot e_p(\tilde{\pi}, \eta, -j) \cdot \frac{L^{(p)}(\pi \times \eta, -j)}{\pi^{a+1}(f, f)}.
\]

Proof. This is summarised in [LZ19, Thm. 2.3.2(i)]. Since \( L(\pi, s) = L(\text{Sym}^2(f) \times \eta, s + a + 1) \) we renormalise, first defining \( \mathcal{L}_p'(\pi) \) so that

\[
\int_{\mathbb{Z}_p} f(x) \cdot d\mathcal{L}_p'(\pi) = \int_{\mathbb{Z}_p} x^{a+1} f(x) \cdot dL_p(\text{Sym}^2 f \otimes \theta)
\]
(in the notation op. cit.), and making the substitution $-j = s - a - 1 = s - k + 1$ (with $k = a + 2$). Translating between the Satake parameters of $f$ and $\pi$ via (9.6) then equates $G(\eta) \cdot E_p(\pi, \eta)$ op. cit. with $c_p(\tilde{\pi}, \eta, j)$ here. The parity condition op. cit. is $(-1)^{a+1-j} \eta(-1) = -\theta(-1)$; since $\omega_\tau(-1) = \omega_{Sym^2(f)}(-1) \theta(-1) = (-1)^a \theta(-1)$, this is equivalent to $-(-1)^j = \omega_\tau \eta(-1)$. Combining all of this shows that $\int_{\mathbb{Z}_p} \eta^{-1}(x)x^{-j} \cdot L_p'(\tilde{\pi})(x)$ is equal to the right-hand side of (9.7).

Finally, to translate into our normalisations, we renormalise by the inverse, and define $L_p(\tilde{\pi})(x) = L_p'(\tilde{\pi})(x^{-1})$, which replaces $\eta^{-1}(x)x^{-j}$ with $\eta(x)x^j$, giving the theorem. $\square$

**Remark 9.5:** The proof of Theorem 9.4 is rather circuitous, owing to complications with local Euler factors at the bad primes. Taking $\theta = 1$ for simplicity, the method initially interpolates values of the “imprimitive” symmetric square $L$-function
\[
L^{\text{imp}}(\text{Sym}^2(f), s) = L^{(N_f)}(2s - 2a - 2, \omega_\tau^2) \cdot \sum_{n \geq 1} a_n(f) n^{-s},
\]
which differs from the “true” symmetric square $L$-function by a product of local error terms at the primes dividing $N_f$. Having constructed an imprimitive $p$-adic $L$-function, one can attempt to define a primitive $p$-adic $L$-function by dividing out by the error term; however, it remains to be shown that the resulting function does not have poles at the zeroes of the error term, and that it has the expected interpolation property at all $(j, \eta)$ in the interpolation range (even if the error term vanishes there, which can occur). This requires rather lengthy case-by-case analysis according to the local factors of $f$ (see [Hid90, §6] and [DD97, §3.1]).

However, for our present purpose of identifying the factors $\tilde{e}_\infty(\zeta_\infty, -j)$, it suffices to know that there is a (possibly meromorphic) $p$-adic $L$-function which satisfies the conclusion of Theorem 10.3 for almost all $(j, \eta)$ in the appropriate range. This is much more straightforward to prove using the methods of [Sch88]. Combining this partial result towards Theorem 9.4 with the output of our present construction, we obtain the full strength of Theorem 9.4 as a consequence, yielding an alternative proof not requiring the intricate local computations involved in the previous approach.

**9.5. Interpolation of $L$-values, part II.** The term $\tilde{e}_\infty(\zeta_\infty, -j)$ depends only on data attached to $\Pi_\infty$ and $j$. From (2.3), for fixed $a$ there are only two possibilities for $\Pi_\infty$. Moreover, both of these arise as the infinite component of a RACAR $\pi$ on $\GL_3$ of the form $\text{Sym}^2(f) \otimes \theta \| \cdot \|^{-a}$, as in §9.4; and we may always choose $\pi$ to be $P_1$-ordinary. Let $\tilde{\pi}$ be a $P_1$-ordinary refinement, and let $L_p(\tilde{\pi})$ and $L_p(\tilde{\pi})$ be the $p$-adic $L$-functions of Definition 9.2 and Theorem 9.4 respectively.

Note that $L_p(\tilde{\pi})$ and $L_p(\tilde{\pi})$ interpolate the same $L$-values $L(\pi \times \eta, -j)$, so when considered as (bounded) rigid analytic functions on weight space $\mathcal{W}$, they are supported on the same half of $\mathcal{W}$. In particular, we can make sense of $L_p(\tilde{\pi})/L_p(\tilde{\pi}) \in \text{Frac}(\mathbb{C}_p \otimes \mathbb{Z}_p[\mathbb{Z}_p^\times])$ as a well-defined meromorphic function on $\mathcal{W}$. This quotient is uniquely determined by its integral against finite-order characters $\eta$ of $\mathbb{Z}_p^\times$ such that $\eta(-1) = \omega_\tau(-1)$ (and vanishes when $\eta(-1) = -\omega_\tau(-1)$). By considering $j = 0$ in (9.5) and Theorem 9.4, we deduce that
\[
\int_{\mathbb{Z}_p} \eta(x) \cdot \frac{L_p(\tilde{\pi})}{L_p(\tilde{\pi})} = \tilde{e}_\infty(\zeta_\infty, 0) \cdot \left[ \frac{\omega_\tau(-1) \cdot \Gamma(a + 1)}{2^{2a+1} \cdot \pi^a \cdot \pi^{a+1}} \cdot (\Theta_\infty \cdot (2\pi i)^{-j} \cdot \pi^{a+1}) \right]^{-1} \cdot \frac{(f, f)}{\Omega_\Pi},
\]
As this is independent of $\eta$, the quotient $L_p(\tilde{\pi})/L_p(\tilde{\pi})$ is constant, say equal to $C \in \mathbb{C}_p^\times$.

Now let $0 \leq j \leq a$, and $\eta$ such that $(-1)^j = \omega_\tau \eta(-1)$. By constancy, this is
\[
C = \int_{\mathbb{Z}_p} \eta(x)x^j \cdot \frac{d}{d \tilde{\pi}} \left( \frac{L_p(\tilde{\pi})}{L_p(\tilde{\pi})} \right) = \tilde{e}_\infty(\zeta_\infty, -j) \cdot \left[ \frac{\omega_\tau(-1) \cdot \Gamma(-j + a + 1)}{2^{2a+1} \cdot \pi^a \cdot (2\pi i)^{-j} \cdot \pi^{a+1}} \right]^{-1} \cdot \frac{(f, f)}{\Omega_\Pi},
\]
where the second equality is the interpolation formula; and hence we have
\[
\tilde{e}_\infty(\zeta_\infty, -j) = (2\pi i)^j \cdot \frac{\Gamma(-j + a + 1)}{\Gamma(a + 1)} \cdot \tilde{e}_\infty(\zeta_\infty, 0).
\]

Now, we are free to renormalise $\zeta_\infty$ by any element of $\mathbb{C}_p^\times$; this then rescales $\Theta_\Pi$, $\Omega_\Pi$, $P_j(\zeta_\infty; s)$, and hence also $\tilde{e}_\infty(\zeta_\infty, -j)$. We renormalise it so that $\tilde{e}_\infty(\zeta_\infty, 0) = e_\infty(\Pi, 0)$, and hence also $\tilde{e}_\infty(\zeta_\infty, -j)$. We define $\epsilon_\infty(\Pi, -j) = (2\pi i)^j \cdot \frac{\Gamma(-j + a + 1)}{\Gamma(a + 1)} \cdot e_\infty(\Pi, 0)$, by (9.8) we deduce
\[
(9.9) \quad \tilde{e}_\infty(\zeta_\infty, -j) = e_\infty(\Pi, -j).
\]

Theorem 9.3 follows by combining this with (9.5).
Remark 9.6: The equality (9.9) also completes the proof of Deligne’s (automorphic) period conjecture (Conjecture A) for $n = 3$, and hence of Theorem C.

A. Proof of Theorem 8.4

Replacing $(\pi, \chi_1, \chi_2)$ with $(\pi \times \chi_1, \text{id}, \chi_2)$, we may suppose without loss of generality that $\chi_1 = \text{id}.$

We begin by recalling some GL$_3 \times$ GL$_1$ zeta-integrals introduced by Jacquet et al. We let $F = \mathbb{Q}_p^\times$ (in fact any nonarchimedean local field would work here).

Definition A.1. For $W \in W(\Pi)$ we consider the integrals

\[
\Psi_0(W, \chi, s) = \int_{F \times} W \left( \begin{pmatrix} a & \theta & \mu \\ 1 & 1 & 1 \end{pmatrix} \right) \chi(a)|a|^{s-1} \, d^\gamma a
\]

and

\[
\Psi_1(W, \chi, s) = \int_{F \times F \times} W \left( \begin{pmatrix} a & \theta & \mu \\ 1 & 1 & 1 \end{pmatrix} \right) \chi(a)|a|^{s-1} \, dx \, d^\gamma a.
\]

By [JPSS83, Thm. 2.7(i),(ii)], both integrals converge for $\Re(s) > 0$ and have meromorphic continuation to all $s$, and the greatest common divisor of the values of either integral is $L(\pi \times \chi, s)$.

By part (iii) of the cited theorem, we also have a functional equation for $\Psi_j$: let us write

\[
w_r = \begin{pmatrix} 1 \\ & \ddots \\ & & 1 \end{pmatrix} \in G_{\mathbb{R}}, \quad w_{r,t} = \begin{pmatrix} \text{id} & w_{r-t} \\ \end{pmatrix} \quad (0 \leq t \leq r).
\]

For $W \in W(\pi)$, we write $\hat{W}(g) = W(w_{r,t}g^{-1})$; the functions $\{\hat{W} : W \in W(\pi, \psi)\}$ form the Whittaker model $W(\pi^\vee, \psi^{-1}).$ Then for $j = 0, 1$ we have the functional equation

\[
\Psi((-1)^j)(w_{3,1} \cdot W, \chi, -1 - 1 - s) = \gamma(\pi \times \chi, \psi, s) \cdot \Psi_j(W, \chi, s).
\]

We now write the torus integral $g(\cdots)$ (from Definition 8.5) in terms of $\Psi_0$:

\[
y(W, \text{id}, \chi_2, s_1, s_2)(g) := |\det g|^{s_1 - \frac{1}{2}} \int_a b W \left( \begin{pmatrix} a & 1 & \theta \\ -b & 1 & 1 \end{pmatrix} w_{3,1} g \right) |a|^{s_1} |b|^{-s_2 - s_1 - 1/2} \chi_2(b) \, d^\gamma a \, d^\gamma b
\]

\[
= |\det g|^{s_1 - \frac{1}{2}} \int_a |a|^{s_1} |b|^{-s_2 - 3/2} \left[ \int_b W \left( \begin{pmatrix} w_{3,1} g \theta \\ -b & 1 \end{pmatrix} w_{3,1} g \right) \right] \chi_2(b) \, d^\gamma b
\]

\[
= |\det g|^{s_1 - \frac{1}{2}} \int_a |a|^{s_1} \left[ \int_b w_{3,1} \cdot \left[ w_{3,1} g \theta \left( \begin{pmatrix} a & 1 \\ 1 & 1 \end{pmatrix} w_{3,1} g \cdot W \right) \right] \chi_2(b) \, d^\gamma b \right] \chi_2(1) |\det g|^{s_1 - \frac{1}{2}} \int_a |a|^{s_1} \left[ \int_b w_{3,1} \cdot \left[ w_{3,1} g \theta \left( \begin{pmatrix} a & 1 \\ 1 & 1 \end{pmatrix} w_{3,1} g \cdot W \right) \right] \chi_2(1) \right] \, d^\gamma a,
\]

with the $\chi_2(-1)$ arising from the change of variables $b \leftrightarrow -b$. Applying the functional equation (A.1) to the inner term, we can write this as

\[
y(W, \text{id}, \chi_2, s_1, s_2)(g) = \gamma(\pi \times \chi_2^{-1}, \psi, s_1 - s_2 + \frac{1}{2}) \times \chi_2(-1) |\det g|^{s_1 - \frac{1}{2}} \int_a |a|^{s_1 + s_2 - 3/2} \Psi_1 \left( \begin{pmatrix} a & 1 \\ 1 & 1 \end{pmatrix} w_{3,1} g \cdot W, \chi_2^{-1}, s_1 - s_2 - \frac{1}{2} \right) \, d^\gamma a.
\]

The integral expands to

\[
\int_a |a|^{s_1 + s_2 - 3/2} \int_b |b|^{-s_2 - 1/2} \chi_2(b) \, d^\gamma b \left[ \int_{b,x} |a|^{s_1} \chi_2(b) \, d^\gamma a \, d^\gamma b \right]
\]

\[
= \chi_2(-1) |\det g|^{s_1 - \frac{1}{2}} \int_a |a|^{s_1 + s_2 - 3/2} \Psi_1 \left( \begin{pmatrix} a & 1 \\ 1 & 1 \end{pmatrix} w_{3,1} g \cdot W, \chi_2^{-1}, s_1 - s_2 - \frac{1}{2} \right) \, d^\gamma a,
\]

By (8.1), integrating this against $f_\Phi$ gives $Y(\cdots)$. In this pairing, we obtain an inner integral over $B_2$ and an outer integral over $B_2 \backslash \text{GL}_2$; so this rearranges into

\[
Y(W, \Phi; \text{id}, \chi_2, s_1, s_2) = \chi_2(-1) \gamma(\pi \times \chi_2^{-1}, \psi, s_1 - s_2 + \frac{1}{2}) \int_{\text{GL}_2(F)} W(\phi(g)) f_\Phi(w_2 g) |\det g|^{s_1 - \frac{1}{2}} \, dg.
\]
Splitting into an integral over $N_2$ and an integral over $N_2 \setminus \text{GL}_2$, the $N_2$ factor acts on $W$ via $\psi$, so the integral over $\text{GL}_2$ equals

$$\int_{N_2 \setminus \text{GL}_2(F)} W(\varepsilon(g)) \det g^{n_1 + \frac{1}{2}} \left( \int_F f_\Phi(w_2 (\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}) g; \chi_2, s_2) \psi(x) \, dx \right) \, dg.$$ 

Since $w_2 = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$, and $f_\Phi$ transforms as $\chi(1)$ under $\text{GL}_2$, by [LPSZ, §8.1] the second integral is precisely $\chi_2(-1)W_\Phi(g; \chi_2, s_2)$. The $\chi_2(-1)$'s cancel; we conclude since

$$Y(W; \Phi; \text{id}, \chi_2, s_1, s_2) = \gamma(\pi \times \chi_2^{-1}, \psi, s_1 - s_2 + \frac{1}{2}) \int_{N_2 \setminus \text{GL}_2(F)} W(\varepsilon(g)) W_\Phi(g; \chi_2, s_2) \det g^{n_1 - \frac{1}{2}} \, dg$$

$$= \gamma(\pi \times \chi_2^{-1}, \psi, s_1 - s_2 + \frac{1}{2}) Z(W; \Phi; \text{id}, \chi_2, s_1, s_2).$$

\[ \square \]

Glossary of notation/terminology.

- $\alpha_p$ integer $\geq 0$; weight of $\Pi$ (§2.3.2)
- $\beta_p$ Satake parameter (§2.4)
- $B \subseteq \text{GL}_3$ upper-triangular Borel
- $\beta_0$ Satake parameter (§2.4)
- $b_{ij}^{[a,j]}$ branching law (§4.2)
- $\text{Crit}(\Pi)$ critical values (§2.5)
- $c \in \mathbb{Z}$ aux. integer prime to $6pN_1$
- $\chi$ Dirichlet character (§2.1)
- $\chi$ Hecke character (§2.1)
- $\Delta_n (\mathbb{Z})$ $p^n$-adic ring
- $E$ sufficient large number field (§3.1)
- $E_\Phi$ adelic Eisenstein series (§2.5.1)
- $E_\Phi^{1+2}$ classical Eisenstein series (§2.5.2)
- $E_\Phi^{2+2}$ classical Eisenstein series (§2.5.2)
- $E_\Phi^{k+2}$ Betti–Eisenstein class (Cor. 5.6)
- $E_{k+2}$ integral $E_{k+2}$ (Thm. 5.7)
- $\varepsilon_{\chi_2}(\Pi, \gamma)$ mod. Euler factor at $\infty$ (§2.5)
- $\varepsilon_\infty(\mathbb{C}, -J)$ Mahlfout factor at $\infty$ (§2.5)
- $\varepsilon_p(\mathbb{F}, \eta, -j)$ mod. Euler factor at $p$ (§2.10)
- $\eta$ Dirichlet char. of cond. $p^n$
- $\nu$ Dirich. char., cond. $D$ prime to $p$ (Not. 6.6)
- $G(\eta)$ Gauss sum (§2.1)
- $\gamma_\Phi$ Satake parameter (§2.4)
- $H$ GL$_2 \times$ GL$_2$
- $I_2(\chi)$ principal series for $\text{GL}_2$ (§2.2.1)
- $J_1: H \to \text{GL}_3$ symmetric space for $J$ (§3.1)
- $J_1 \subseteq \text{GL}_3(\mathbb{Z}_p)$ general reductive group
- $J_{p,1} \subseteq \text{GL}_3(\mathbb{Z}_p)$ parahoric for $P_1$ (2.6)
- $\mathcal{J}_j$ integer $0 \leq j \leq a$
- $K_{\text{GL}_n, \infty} \subseteq \text{GL}_n(\mathbb{R})$ max. cpt subgroup
- $K_{n, \infty} \subseteq \text{GL}_n(\mathbb{R})$ $n$-fold centraliser of the $\text{GL}_n(\mathbb{R})$
- $L = (a, 0, -a)$ weight of $\Pi$
- $N \subseteq \text{GL}_3$ upper-triangular unipotent
- $\Pi_{\text{r}}$ conductor of $\Pi$ (§2.3.3)
- $v_{1,2} : H \to \text{GL}_3$ $(\gamma, z) \to \det(\gamma)/z$
- $v_{1,3} : \mathbb{R}^n \to \Delta_n$ (6.3)
- $\omega_{\Pi}$ central char. of $\Pi$
- $P_1, P_2 \subseteq \text{GL}_3$ max. standard parabolics (§2.7)
- $\Pi$ $\Pi_0$ RACAR of $\text{GL}_3$
- $\Phi, \Phi_\Pi$ Schwartz functions (§5.1)
- $\Phi_\Pi$ specific Schwartz function (§5.3.4)
- $\phi_{\Pi,1}$ map $W_\nu(\Pi_1) \to H(2)$ (§3.3)
- $p_{\text{GL}_2}$ projection $H \to \text{GL}_2$
- $\varphi, \varphi_\Pi$ suff. large number field (§3.1)
- $\tau_{\Pi, n, r, \infty}$ vectors at $\infty$ (§7.2.1)
- $\text{U}_{\Pi, 1, \text{U}_p}$ Hecke ops. at $p$ (§2.5.2)
- $\Theta_{\Pi, n, r, \infty}$ cohomological period (§3.1.1)
- $U_{\Pi, 1, \text{U}_p}$ adjoint Hecke ops. (§3.2)
- $U_{\Pi, 1, \text{U}_p}$ open compact level
- $U(\mathbb{g})$ prime-to-$p$ level
- $U_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}$ Def. 6.3
- $U_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}(N)$ Whittaker new lvls. for $\Pi$ (§2.3.3)
- $U_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}^H$ Whittaker transform (§3.3.1)
- $W_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}$ element in $W_\Pi(\Pi)$
- $Y_\lambda$ highest weight rep. of $\text{GL}_3$ (§2.2)
- $Y_{\nu, n, \text{GL}_3(\mathbb{Z}_p)}$ highest weight rep. of $\Pi$
- $Y_{\nu, n, \text{GL}_3(\mathbb{Z}_p)}$ Def. 6.9
- $Y_{\nu, n, \text{GL}_3(\mathbb{Z}_p)}$ local system attached to $V_{\nu, n, \text{GL}_3(\mathbb{Z}_p)}$ (§3.2)
- $Y_{\nu, n, \text{GL}_3(\mathbb{Z}_p)}$ Whittaker transform (§3.3.1)
- $Y_{\nu, n, \text{GL}_3(\mathbb{Z}_p)}$ $\mathbb{Z}_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}$ valued measures (§6.3)
- $Z_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}$ measures (§6.4)
- $Z_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}$ Def. 6.10
- $Y_\nu$ loc. sym. space for $J$ (§3.1)
- $Y_{\nu, n, \text{GL}_3(\mathbb{Z}_p)}$ local zeta integral (Def. 8.3)
- $Y_{\nu, n, \text{GL}_3(\mathbb{Z}_p)}$ modified space for $H$ (4.2)
- $Z_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}$ centre of $\text{GL}_3$ (§8.3)
- $Z_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}$ local zeta integral (Def. 8.1)
- $Z_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}$ global zeta integral (Def. 7.2)
- $\langle - \rangle_{\mathbb{Z}_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}}$ Poincaré duality pairing (§3.4)
- $\langle - \rangle_{\mathbb{Z}_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}}$ branching law pairing (4.4)
- $\langle - \rangle_{\mathbb{Z}_{\Pi, n, \text{GL}_3(\mathbb{Z}_p)}}$ identity component
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35
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