Single reflection quarter-wave device design

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Abstract. If a monochromatic linearly polarized plane wave is incident on an isotropic-uniaxial or isotropic-biaxial interface and if the incidence plane coincides with a crystal principal plane then circularly polarized light can be obtained by means of a single reflection. This result can be achieved using materials of refractive indices which are not as high as those required in isotropic devices. As is well known, in isotropic-anisotropic interfaces, there are two different incidence angles that yield circularly polarized reflected light. These angles depend on characteristics of both media (refractive indices and angle between the optical axis and the interface) and also on wavelength. In this paper a single reflection retarder is designed. Additionally, its efficiency as quarter wave retarder for non collimated or polychromatic beams is analyzed studying how the amplitudes and phases of both orthogonal components of the reflected electric field vary when the incidence angle and/or the wavelength varies.

1. Introduction
Optical retarders are fundamental devices to control polarization not only in the optical components industry but also in other fields such as solar polarimetry, optical communications, ellipsometry and metrology. Retarders are contained in many testing and measurement instruments and in devices such as spectrometers, photopolarimeters, optical isolators, attenuators, beam splitters, lasers, liquid crystal displays, etc. The selection of the most convenient retarder for a certain device depends on its cost and also on many other factors such as energy density, bandwidth, available space, strictness of polarization control, etc. Retarders can contain either isotropic-anisotropic or isotropic-isotropic interfaces, two classical retarders being respectively waveplates and Fresnel Rhombs.

The waveplate [1] is the most widely used retarder and it is a parallel plane plate made of an anisotropic material (polymer caused by deformations, inorganic natural or manufactured material, etc.). There are zero order waveplates (true or compound) and multiple order waveplates. A true zero order waveplate is made of a single anisotropic material and its thickness can be so small as to limit its manipulation though this inconvenience can be overcome by the use of low birefringence polymers yielding thicker and more robust waveplates [2-5]. A compound zero order waveplate is constructed combining multiple order rotated waveplates.
On the other hand, when linearly polarized light is incident on an interface between two isotropic materials, a single total reflection can yield circularly polarized light though, since a medium of very high refraction index is required, this device is impractical from a technological point of view. Thus, when isotropic media are employed, several total reflections are coupled. For example, the Fresnel Rhomb is a rhomb made of an isotropic material and, if light undergoes two total reflections, circularly polarized emergent light can be attained. This rhomb is very robust though its size makes it inadequate for compact devices. Varying the rhomb material, it can be used not only for light but also for invisible radiation corresponding to a wide range of the electromagnetic spectrum and, to improve its performance, various designs combining different materials have been proposed [6-10]. Furthermore, the rhomb is usually considered as an achromatic device though its behavior as a retarder depends not only on the incidence angle and on constructive parameters but also on wavelength. In figure 1 the phase difference between both components of the electric field that emerges from a Fresnel Rhomb is plotted as a function of the incidence angle for three different wavelengths and it results that the variation of phase difference with wavelength is less than one degree.

![Figure 1: Fresnel Rhomb: Phase difference as a function of incidence angle for three wavelengths: 532.8nm (green line); 632.8nm (orange line) and 732.8nm (red line). The constructive angles for which CP light is obtained for \( \lambda = 632.8 \text{nm} \) are \( \beta_1 = 48.05^\circ \), and \( \beta_2 = 55.08^\circ \).](image)

The requirements of high index materials and/or of several total reflections needed when isotropic materials are used to construct retarders are avoided when anisotropic materials are employed. In this paper we design a new uniaxial crystal-glass device, which we denominate waveprism, that yields circularly polarized light in a single reflection (total or partial) and that can be easily and cheaply constructed. To design it, we first consider an isotropic-anisotropic interface such that the crystal optical axis is contained in the incidence plane subtending an arbitrary angle with the interface and we analyze in detail different possibilities for this interface to act as a retarder. To improve this design, we study the influence of wavelength and of the incident beam collimation.

2. Isotropic-uniaxial crystal interfaces: interface and reflected light features
As shown in previous papers [11,12], circularly polarized light (CP) can be obtained in interfaces constituted by an isotropic media (of refractive index \( n \)) and a uniaxial crystal (of principal ordinary and extraordinary indices \( n_o \) and \( n_e \) respectively). This can be achieved using incident linearly polarized light (LP) contained in one of the principal planes of the crystal. If, for example, birefringence is negative (this is, if \( n_e > n_o \)) and the optical axis \( z_1 \) is contained in the incidence plane forming an angle \( \theta \) with the interface then reflected light can be CP under either partial or total reflection conditions. These conditions depend on the value of the incidence angle (term \( \alpha \)) and on the orientation of the optical axis so, for a certain \( \theta \) and if suffixes \( e \) and \( o \) indicate magnitudes related to the extraordinary and ordinary beams respectively, we may have the two following situations.
a) Partial reflection conditions (PR): The incidence angle and the orientation of the optical axis are such that only the extraordinary ray suffers total reflection (ETR).
b) Total reflection conditions (TR): The incidence angle is greater than the limiting angle \( \alpha_o \) (which is greater than \( \alpha_e \)) so both the extraordinary and the ordinary rays suffer total reflection (OTR).

As is well known, the necessary condition for the existence of \( \alpha_o \) is \( n_o > n_e \) for any kind of negative crystal and optical axis orientation. On the other hand, the condition for the existence of \( \alpha_e \) (grazing refracted extraordinary ray) depends not only on the isotropic media refractive index and on crystal type but also on the orientation of the optical axis relative both to the interface and to the incidence plane. Under the conditions here considered, \( \alpha_e \) results to be

\[
\sin \alpha_e = \left( \frac{n_o^2 \cos^2 \theta + n_e^2 \sin^2 \theta}{n} \right)^{1/2}
\]

Since the optical axis is contained in the incidence plane, parallel (extraordinary) and perpendicular (ordinary) modes are separated from each other. Thus, the reflection coefficients, \( R_p \) and \( R_s \), are

\[
R_p = \frac{n \left( n_o^2 \cos^2 \theta + n_e^2 \sin^2 \theta - n^2 \sin^2 \alpha \right)^{1/2} - n_o n_e \cos \alpha}{n \left( n_o^2 \cos^2 \theta + n_e^2 \sin^2 \theta - n^2 \sin^2 \alpha \right)^{1/2} + n_o n_e \cos \alpha}
\]

\[
R_s = \frac{\cos \alpha - \left( (n_o / n)^2 - \sin^2 \alpha \right)^{1/2}}{\cos \alpha + \left( (n_o / n)^2 - \sin^2 \alpha \right)^{1/2}}
\]

When conditions are such that ETR occurs, the modulus of \( R_p \) is unitary and its phase \( \phi_p \) is given by [16]

\[
\phi_p = -2 \arctan \left( \frac{n \left( n_o^2 \sin^2 \alpha - n_o^2 \cos^2 \theta - n_e^2 \sin^2 \theta \right)^{1/2}}{n o n_e \cos \alpha} \right)
\]

Similarly, when conditions are such that OTR occurs, the modulus of \( R_s \) is unitary and its phase \( \phi_s \) is given by

\[
\phi_s = -2 \arctan \left( \frac{n \sin^2 \alpha - n_e^2 \left( \frac{1}{n \cos \alpha} \right)^{1/2}}{n \cos \alpha} \right)
\]

In figure 2 we plot the modulus ratio between \( R_p \) and \( R_s \) (in (a)) and \( \phi_p \), \( \phi_s \) and its difference \( \Delta \phi \) (in (b)) as a function of \( \alpha \) considering, as an example, the case \( \lambda = 632.8 \text{ nm} \), \( n = 1.755 \), \( n_o = 1.65579 \), \( n_e = 1.48521 \) and \( \theta = 70^\circ \). In this case, both \( \alpha_o \) and \( \alpha_e \) exist and a \( \pi / 2 \) phase difference (corresponding to total reflection) is attained for two incidence angles, the first \( (\alpha_o = 66.35^\circ) \) is located between both total reflection angles (PR conditions) while the second \( (\alpha_e = 71.08^\circ) \) is greater than both of them (TR conditions).
Figure 2. (a) Modulus ratio of the reflection coefficients. (b) Phases for parallel mode (pink dotted line) and for the perpendicular mode (green dotted line) and phase difference between both modes (black continuous line) as a function of the incidence angle for a dense Flint glass and calcite interface ($n = 1.755$, $n_o = 1.65579$, $n_e = 1.48521$, $\theta = 70^\circ$) ($\lambda = 632.8$ nm). The angles $\alpha_e$ and $\alpha_o$ are the ETR and OTR angles, respectively.

3. Waveprism analysis

The device we propose as retarder, the waveprism, is very simple and consists in a prism made of isotropic material (glass) optically sticked to a piece of uniaxial material (only one plane and polished face is needed). The waveprism is shown in figure 3 in a bidimensional representation, the crystal optical axis is contained in the incidence plane which is a plane containing the normals to both prism faces and the normal to the interface. In what follows, we analyze the waveprism leaving aside the air-glass and glass-air refractions, since, on the one hand, this consideration does not significantly affect results principally because there are no phase changes in isotropic transmissions and, on the other hand, the corrections due to these refractions in the values of amplitudes and angles could be easily calculated using Snell’s laws and Fresnel’s relations.

Furthermore, we take into account the well-known fact that, to obtain CP emergent light from LP incident light, two conditions are required: the phase difference between both orthogonal reflected field components must be $\pi/2$ and they must have the same amplitude.

Regarding the first condition, the possibility of existence of one or two incidence angles yielding a $\pi/2$ phase difference between both reflected field components depends on the prism refraction index, on the crystal principal refraction indices and on the optical axis orientation. For negative crystals ($n_o > n_e$), the angles $\theta$ for which these two incidence angles exist are such that [12]

$$\theta > \arcsin \left( \frac{n_o n_e}{n^2} \sqrt{\frac{n^2 - n_o^2}{n^2 - n_e^2}} \right)$$

If $\theta$ is equal to the arcsin of the product appearing in equation (6), a single possible angle exists while if $\theta$ is less than this quantity, there is no incidence angle yielding a $\pi/2$ phase difference. Moreover, equation (6) can be fulfilled only if the isotropic media refractive...
index satisfies the condition

\[ n \leq \frac{n_e^2 n_o}{2(n_o^2 - n_e^2)} \left( n_e - \left( 5n_o^2 - 4n_e^2 \right)^{1/2} \right) \]  

(7)

Equation (7) can be fulfilled only for some negative crystals since \( n \) must be a real number so the crystal refraction indices must verify \( n_o \leq \sqrt{1.25} n_e \). Since calcite \( (n_e = 1.65579 \text{ and } n_o = 1.48521) \) satisfies this condition for any isotropic material of refraction index greater than \( n_o \) and for almost every wavelength in the transparency range, it can be used as the anisotropic component of the wave prism. For example, for the mean wavelength of an HeNe laser \( (\lambda = 632.8 \text{ nm}) \) and a prism made of dense Flint glass \( (n = 1.755) \), the optical axis must form an angle with the interface greater than 38° for the two incidence angles yielding a \( \pi/2 \) phase difference between field components to exist.

From the second condition for the emergent light to be CP, we can find the polarization angle, \( \gamma \), that the linearly polarized incident electric field vector must form with the incidence plane. If the incidence angle is between the ETR and OTR angles, we must have \( \tan \gamma = 1/|R_p| \) since, under these conditions, \(|R_p| = 1\) and taking into account equations (2) and (3), we have

\[ \frac{E^*_r}{E^*_s} = \frac{R_p}{R_s} \frac{1}{\tan \gamma} = 1 \]  

(8)

where \( E^*_p \) and \( E^*_s \) the \( p \) and \( s \) components of the reflected field (figure 3). On the other hand, if the incidence condition corresponds to TR, the incident electric field components must be equal, i.e. \( \gamma_2 = 45^\circ \).

To show that the wave prism can yield CP emergent light in a single reflection, we consider, as in the previous section, \( \lambda = 632.8 \text{ nm} \); \( n_e = 1.65579 \); \( n_o = 1.48521 \) and \( n = 1.755 \) [17]. Though in that section we choose \( \theta = 70^\circ \) almost arbitrarily, the variation with incidence angle of the phase difference is qualitatively the same for other optical axis orientations satisfying equation (6).

**Table 1**: Possible incidence and polarization incident Light angles required to obtain CP reflected light, corresponding to different \( \theta \) optical axis orientations. The angle \( \alpha_1 \) corresponds to PR zone and \( \alpha_2 \) to TR zone.

| \( \theta \) | Incidence angle | Linear polarization angle |
|----------------|----------------|--------------------------|
| 45°            | \( \alpha_1 = 69.71^\circ \) | \( \gamma_1 = 61.38^\circ \) |
|                | \( \alpha_2 = 70.67^\circ \) | \( \gamma_2 = 45.00^\circ \) |
| 60°            | \( \alpha_1 = 67.47^\circ \) | \( \gamma_1 = 71.64^\circ \) |
|                | \( \alpha_2 = 70.92^\circ \) | \( \gamma_2 = 45.00^\circ \) |
| 70°            | \( \alpha_1 = 66.35^\circ \) | \( \gamma_1 = 74.36^\circ \) |
|                | \( \alpha_2 = 71.08^\circ \) | \( \gamma_2 = 45.00^\circ \) |
| 80°            | \( \alpha_1 = 65.64^\circ \) | \( \gamma_1 = 75.76^\circ \) |
|                | \( \alpha_2 = 71.19^\circ \) | \( \gamma_2 = 45.00^\circ \) |
| 90°            | \( \alpha_1 = 65.39^\circ \) | \( \gamma_1 = 76.19^\circ \) |
|                | \( \alpha_2 = 71.23^\circ \) | \( \gamma_2 = 45.00^\circ \) |

In Table 1, for various values of \( \theta \) (associated to various orientations of the crystal optical axis), we indicate the values of the angles \( \alpha_1 \) and \( \alpha_2 \) that can yield CP emergent light and also the corresponding polarization angles \( \gamma_1 \) and \( \gamma_2 \) for the incident light. In the PR situation, when the...
optical axis orientation is varied, the incidence angles required to obtain CP light suffer a minor change at 5°, whereas the required polarization angle for incident light varies approximately 15 degrees. On the contrary, in the TR situation, the incidence angle slightly depends on the optical axis orientation (the variation being less than 1%).

4. Effects of beam collimation and chromaticity

As we have shown in previous sections, it is possible to obtain CP reflected light considering the waveprism for a perfectly plane (and, as a consequence, monochromatic) incident wave. To determine its versatility for non-collimated or polychromatic beams, we will study the characteristics of emerging light when the incident beam comes from different types of real sources.

Regarding the incident beam collimation, perfect collimation never exists though, under several circumstances, the beam can be considered to be almost collimated. If the diameter of a laser beam is assumed to be 2mm then, since the composing waves that effectively contribute to the intensity have an aperture equivalent to their divergence [18], the waves aperture can be considered to be less than one minute. As it can be seen in figure 4, this aperture would not modify (in an experimentally significant quantity) the polarization of the light emerging from the waveprism. If, alternatively, the incident beam is such that its component waves aperture is half a degree for an mean incidence angle close to that corresponding to CP conditions, the phase difference between both components of the reflected electric field notably differs from \( \pi/2 \) and the resulting polarization is elliptic.

Figure 4. Zoom of figure 1(b) showing in more detail both possible angles for light to be CP.

![Zoom of figure 1(b) showing in more detail both possible angles for light to be CP.](image)

Figure 5. Eccentricity of the polarization ellipse under conditions close to those required for CP reflected light as a function of the incidence angle. (a) Partial reflection conditions. (b) Total reflection conditions.

(a) ![Eccentricity of the polarization ellipse under conditions close to those required for CP reflected light as a function of the incidence angle. (a) Partial reflection conditions.](image)

(b) ![Eccentricity of the polarization ellipse under conditions close to those required for CP reflected light as a function of the incidence angle. (b) Total reflection conditions.](image)

In order to appraise this effect, we analyze the variation of the eccentricity of the reflected light polarization ellipse when the incidence angle varies. In figure 5 (a) the eccentricity is plotted for incidence conditions near to those corresponding to reflected CP light under partial reflection.
conditions and in figure 5(b) under total reflection conditions. According to figure 5(a), in the case of a 2mm diameter beam, the eccentricity slightly differs from zero, its maximum value being approximately 0.07. On the other hand, in the case of a very thin beam (for example a 0.1mm diameter beam corresponding to a divergence of approximately 12 minutes), the eccentricity slightly differs from zero, its maximum value being approximately 0.3. As expected, the impact of lack of collimation is greater in the TR situation (figure 5(b)) than in the PR one since, in the former situation, although the field modulus ratio does not vary, the phase difference strongly depends on the incidence angle.

Regarding the chromatic effect, frequently it cannot be avoided. To estimate this effect, we plot the phase difference between both components of the reflected electric field for various wavelengths as a function of the incidence angle. For example, in figure 6 we consider that the isotropic material composing the waveprism is Schott N-SF11 glass and we take into account three typical wavelengths (532.8nm, 632.8nm and 732.8nm) associated to refractive indices evaluated using Sellmeier formulas [17,19,20]. According to figure 6, to obtain CP light using polychromatic incident radiation, the PR condition is better than the TR one. This is because, in the former case, the phase difference varies in approximately one degree and the incident polarization angle differs in approximately 3 degrees from the value corresponding to the mean wavelength for the extreme considered wavelengths.

5. Conclusions
To obtain circularly polarized light resulting from linearly polarized monochromatic plane waves, various well known retarders can be employed. The use of materials with high refractive indices or with several total reflections is necessary if isotropic retarders (such as Fresnel’s Rhomb) are employed while robustness can be poor if some anisotropic retarders (such as true zero order waveplates) are used In this paper we have presented a new type of quarter wave retarder, which we denominate waveprism, that lacks these drawbacks and consists in an isotropic prism optically sticked to a piece of uniaxial crystal. This device is comparable to Fresnel’s Rhomb regarding its robustness but is much more compact since a single total reflection is required. Moreover, a piece of adequate anisotropic material is easily obtained since its quality is not critical and a single plane and polished surface is necessary (the one in contact with the prism).

As a preliminary design, we chose materials available in our laboratory, a glass prism with refraction index \(n=1.755\) and calcite as uniaxial crystal and an angle between the crystal optical axis and the interface equal to 70°, this choice being arbitrary though appropriate. We analytically calculated the waveprism sensitivity concerning the incidence angle and our results suggest that, if common laser beams are used as sources illuminating the waveprism, the waves aperture do not significantly modify the polarization of the emergent beam (mostly when PR conditions are considered). Hence the device could be an inexpensive alternative to the fragile retarding waveplates and to the not only expensive but also bulky Fresnel’s Rhombs.

The preliminary results presented in this paper, encourage us to perform the experimental study of the proposed device in the short term. Also, we will optimize the design considering the possibility of
using other type of material combination, and different cuts for the crystal. We will include in future studies the possibility of using positive uniaxial materials (which requires an adequate choice of the isotropic material so that the required phase difference conditions are possible). The last correction to perform would be to consider the transmissivity in the isotropic interfaces, which according to previous calculus will not significantly affect the necessary polarization angle for incident light.

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