Weakly Approximately Quasi-Prime Submodules And Related Concepts

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Article history: Received, 21, April, 2020, Accepted 21, June, 2020, Published in April 2021

Doi: 10.30526/34.2.2613

Abstract

Let R be commutative Ring, and let T be unitary left R–module. In this paper, WAPP-quasi prime submodules are introduced as a new generalization of Weakly quasi prime submodules, where proper submodule C of an R-module T is called WAPP–quasi prime submodule of T, if whenever 0≠rst, for r, s ∈ R, t ∈ T, implies that either rt ∈ C +soc(T) or st ∈ C +soc(T). Many examples of characterizations and basic properties are given. Furthermore, several characterizations of WAPP-quasi prime submodules in the class of multiplication modules are established.

Keywords: Weakly quasi prime submodules, WAPP-quasi prime submodules, Socle of modules, Z-Regular modules, Projective modules.

1. Introduction

Throughout this paper, all rings are commutative with identity, and all modules are left unitary R-modules. Weakly quasi prime submodules were first introduced and studied in 2013 by [1] as a generalization of a weakly prime submodule, where proper submodule C of R-module T was called weakly prime submodule of C, if whenever 0≠atC, for a ∈ R, t ∈ T, implies that either tC or aT ⊆ C [2], and a proper submodule C of R-module T is called weakly quasi prime submodule of T, if whenever 0≠abtC, for a, b ∈ R, t ∈ T, implies that either atC or btC. Recently many generalization of weakly quasi prime submodules were introduced see [3, 4, 5]. In this research we introduced another generalization of weakly quasi prime submodule, where proper submodule C of R-module T is called WAPP-quasi prime submodule of T, if whenever 0≠abtC for a, b ∈ R, t ∈ T implies that either atC+soc(T) or btC+soc(T). Soc(T) is the socle of a module T, defined by the intersection of all essential submodule of T [6], where a nonzero submodule A of an R-module T is called essential if A∩B ≠ (0) for each nonzero submodule B of T [6]. Recall that R-module T is
multiplication if every submodule $C$ of $T$ is of the form $IT$ for some ideal $I$ of $R$, in particular $C=[C_R \ T] \ T$ [7]. Let $A$ and $B$ be a submodule of multiplication module $T$ with $A=IM$ and $B=JT$ for some ideals $I, J$ of $R$, then $AB=IJT=IB$. In particular $AT=ITT=IT=A$. Also for any $t \in T$, $At=A< t >= It$ [8]. Recall that an $R$-module $T$ is faithful, if $ann(T)=\{0\}$ [7]. A $R$-module $T$ is a projective if for any epimorphism $f$ from $R$-module $X$ into $X$, and for any homomorphism $g$ from $T$ in to $X$ there exists a homomorphism $h$ from $T$ in to $X$ such that $f \circ h=g$ [7]. Recall that an $R$-module $T$ is a $Z$-regular, if for each $t \in T$ there exists $f \in T^* = Hom(T, R)$ such that $t=f(t)t$ [10]

2. Basic Properties of WAPP-Quasi Prime Submodule

In this section, we introduced the definition of WAPP-quasi prime submodules and established some of its basic properties, characterization and examples.

Definition(1)

A proper submodule $C$ of an $R$-module $T$ is called Weakly approximaitly quasi prime submodule of $T$ (for short WAPP-quasi prime submodule), if whenever $0 \neq abt \in C$, for $a, b \in R$, $tcT$, implies that either $at \in C + Soc(T)$ or $btc + Soc(T)$.

And an ideal $J$ of ring $R$ is called WAPP-quasi prime ideal of $R$ if $J$ is WAPP- quasi prime submodule of $R$-module $T$.

Examples and Remarks(2)

1. The submodule $C=< \bar{12} >$ of the $Z$-module $Z_{24}$ is a WAPP-quasi prime submodule of $Z_{24}$, since $Soc(Z_{24})=< \bar{4} >$, and for $0 \neq abt \in C$, for $a, b \in \mathbb{Z}, t \in Z_{24}$, implies that either $at \in C + Soc(T)$ or $bt \in 12 + Soc(Z_{24})$. That is either $at \in \bar{12} + Soc(Z_{24})$ or $bt \in 12 + Soc(Z_{24})$. Thus $0 \neq 2, 3 \in \bar{12}$ for $2, 3 \in \mathbb{Z}$, $\bar{2} \in Z_{24}$ implies that $2, \bar{2} = 4 \in \bar{12} + Soc(Z_{24}) = \{0, \bar{4}, 8, 12, 16, 20\}$.

2. The submodule $12Z$ of the $Z$-module $Z$ is not WAPP-quasi prime submodule, since $Soc(Z) = \{0\}$ and whenever $0 \neq 3.4.1 \in 12Z, 3.4 \in \mathbb{Z}$, implies that $3.1 \in 12Z + Soc(Z)$ and $4.1 \in 12Z + Soc(Z)$.

3. It is clear that every weakly quasi prime submodule of an $R$-module $T$ is WAPP-quasi prime but not conversely.

The following example explains that:

Consider the $Z$-module $Z_{24}$, and the submodule $C=< \bar{6} > = \{0, \bar{6}, \bar{12}, \bar{18}\}$, $C$ is not weakly quasi prime submodule of $Z_{24}$ since $2.3 \bar{1} \in C \neq \bar{6}$ for $2, 3 \in \mathbb{Z}, \bar{1} \in Z_{24}$, implies that $2. \bar{1} = 2 \neq \bar{6}$ and $3. \bar{1} = 3 \neq \bar{6}$. But $C$ is a WAPP-quasi prime submodule of $Z_{24}$, since $Soc(Z_{24}) = \{\bar{4}\}$, and whenever $0 \neq ab \in C = \bar{6}$ for $a, b \in Z, t \in Z_{24}$ implies that either $at \in C + Soc(Z_{24}) = \bar{6}$ or $bt \in 12 + Soc(Z_{24}) = \bar{6} + \{0, \bar{4}, 8, 12, 16, 20\}$.

That is $0 \neq 2, 3, \bar{1} \in C$ for $2, 3 \in Z, \bar{1} \in Z_{24}$, implies that $2, \bar{1} \in C + Soc(Z_{24}) = \{\bar{2}\}$.

4. It is clear that every weakly prime submodule of an $R$-module $T$ is a WAPP-quasi prime but not conversely.

The following example explains that:
Consider the $Z$-module $Z_{24}$ and the submodule $C=<\bar{12}>=\{0, \bar{12}\}$. From (1), $C$ is WAPP-quasi prime submodule of $Z_{24}$. But $C$ is not weakly prime submodule of $Z_{24}$. Since if $0\neq 3, 4 \in C$, for $3 \in Z$, $4 \in Z_{24}$, but $4 \notin C$ and $3 \notin [C:Z_{24}]=6Z$.

5. The residual of WAPP-quasi prime submodule $C$ of an $R$-module $T$ needs not to be WAPP-quasi prime ideal of $R$.
The following example explains that:

We have seen in (1) that the submodule $C=<\bar{12}>=\{0, \bar{12}\}$ of the $Z$-module $Z_{24}$ is a WAPP-quasi prime submodule of $Z_{24}$ but $C$ is not weakly prime submodule of $Z_{24}$. Since if $0 \neq 3 \in Z_{24}$, for $3 \in Z$, $4 \in Z_{24}$, but $4 \notin C$ and $3 \notin [C:Z_{24}]=6Z$.

6. The submodules $PZ$ of a $Z$-module $Z$ is a WAPP-quasi prime if and only if $P$ is prime number

7. The intersection of two WAPP-quasi prime submodule of an $R$-module, $T$ need not to be WAPP-quasi prime submodule of $T$ for example:

The submodule $2Z$ and $5Z$ of the $Z$-module $Z$ are WAPP-quasi prime submodule by (6).

But $2Z \cap 5Z=10Z$ is not WAPP-quasi prime submodule of the $Z$-module $Z$, since $0 \neq 2.5.1 \in 10Z$, for $2, 5, 1 \in Z$ but $2.1 \notin 10Z+Soc(Z)$ and $5.1 \notin 10Z+Soc(Z)$

The following proposition are characterizations of WAPP-quasi prime submodules.

**Proposition (3)**

Let $T$ be an $R$-module and $C$ be proper submodule of $T$, then $C$ is WAPP-quasi prime submodule of $T$ if and only if, whenever $0 \neq rsB \subseteq C$, for $r, s \in R$, $B$ is submodule of $T$, implies that either $rB \subseteq C + Soc(T)$ or $sB \subseteq C + Soc(T)$.

**Proof:**

$\Rightarrow$ Assume that $C$ is AWPP-quasi prime submodule of $T$ and $0 \neq rsB \subseteq C$. For $r, s \in R$, $B$ is a submodule of $T$, with $rB \not\subseteq C + Soc(T)$ and $sB \not\subseteq C + Soc(T)$, that is there exists a nonzero elements $b_1, b_2 \in B$ such that $rb_1 \notin C + Soc(T)$ and $sb_2 \notin C + Soc(T)$. Now $0 \neq rsb_1 \in C$, and $C$ is WAPP-quasi prime submodule of $T$ and $rb_1 \notin C + Soc(T)$, implies that $s b_1 \in C + Soc(T)$. Also $0 \neq rsb_2 \in C$, and $C$ is a WAPP-quasi prime submodule of $T$, and $sb_2 \notin C + Soc(T)$, implies that $rb_2 \in C + Soc(T)$. Again since $0 \neq r(s(b_1+ b_2)) \in C$ and $C$ is WAPP-quasi prime submodule of $T$, implies that either $r(b_1+ b_2) \in C + Soc(T)$ or $s(b_1+ b_2) \in C + Soc(T)$. If $r(b_1+ b_2) \in C + Soc(T)$, that is $rb_1 + rb_2 \in C + Soc(T)$, and since $rb_2 \in C + Soc(T)$, it follows that $rb_1 + rb_2 \in C + Soc(T)$ which is contradiction. If $s(b_1+ b_2) \in C + Soc(T)$, that is $sb_1 + sb_2 \in C + Soc(T)$ and since $sb_1 \in C + Soc(T)$, it follows that $sb_2 \in C + Soc(T)$ which is contradiction. Hence $r B \subseteq C + Soc(T)$ or $s B \subseteq C + Soc(T)$.

$\Leftarrow$

Let $0 \neq rstc \subseteq C$, for $r, s \in R$, $t \in T$, it follows that $0 \neq rs<t> \subseteq C$, so by hypothesis either $r<t> \subseteq C + Soc(T)$ or $s<t> \subseteq C + Soc(T)$. That is either $r \in C + Soc(T)$ or $s \in C + Soc(T)$. Hence $C$ is WAPP-quasi prime submodule of $T$. 

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Proposition(4)

Let $T$ be $R -$ module and $C$ be proper submodule of $T$, then $C$ is WAPP - quasi prime submodule of $T$ if and only if whenever $0 \neq IJ B \subseteq C$, for $I, J$ are ideals of $R$ and $B$ is a submodule of $T$, implies that either $IB \subseteq C + Soc(T)$ or $JB \subseteq C + Soc(T)$.

Proof:

$(\Rightarrow)$ Assume that $0 \neq IJ B \subseteq C$. For $I, J$ are ideal of $R$, $B$ is a submodule of $T$, with $IB \subseteq C + Soc(T)$ and $J B \subseteq C + Soc(T)$, so there exists a nonzero elements $b_1, b_2 \in B$ and a nonzero elements $r \in I, s \in J$ such that $rb_1 \notin C + Soc(T)$ and $sb_2 \notin C + Soc(T)$. Now $0 \neq rsb_1 \in C$ and $C$ is a WAPP-quasi prime submodule and $rb_1 \notin C + Soc(T)$, implies that $sb_1 \notin C + Soc(T)$. Also $0 \neq rsb_2 \in C$, and $C$ is a WAPP-quasi prime submodule of $T$, and $sb_2 \notin C + Soc(T)$, implies that $rb_2 \notin C + Soc(T)$. Again $0 \neq rs(b_1 + b_2) \in C$ and $C$ is WAPP-quasi prime submodule of $T$, implies that either $r(b_1 + b_2) \in C + Soc(T)$ and $rb_2 \notin C + Soc(T)$, $sb_1 \notin C + Soc(T)$ and $s(b_1 + b_2) \in C + Soc(T)$ or $sb_1 \notin C + Soc(T)$, which is contradiction. Hence $IB \subseteq C + Soc(T)$ or $JB \subseteq C + Soc(T)$.

$(\Leftarrow)$

Suppose that $0 \neq rst \subseteq C$, for $r, s \in R, t \in T$, that is $0 \neq <r><s><t> \subseteq C$, so by our assumption either $(r)(t) \subseteq C + Soc(T)$ or $(r)(t) \subseteq C + Soc(T)$. That is either $rt \in C + Soc(T)$ or $st \in C + Soc(T)$. Hence $C$ is WAPP-quasi prime submodule of $T$.

As a direct consequence of the above propositions, we get the following corollaries.

Corollary(5)

Let $T$ be $R -$ module and $C$ be proper submodule of $T$, then $C$ is WAPP - quasi prime submodule of $T$ iff whenever $0 \neq rt \subseteq C$, for $r \in R, I$ is an ideal of $R$ and $t \in T$, implies that either $rt \in C + Soc(T)$ or $t \in C + Soc(T)$.

Corollary(6)

Let $T$ be $R -$ module and $C$ be proper submodule of $T$. Then $C$ is WAPP - quasi prime submodule of $T$ iff whenever $0 \neq rt \subseteq C$, for $J, I$ is an ideal of $R$ and $t \in T$, implies that either $Jt \subseteq C + Soc(T)$ or $It \subseteq C + Soc(T)$.

Corollary(7)

Let $T$ be $R -$ module and $C$ be proper submodule of $T$. Then $C$ is WAPP - quasi prime submodule of $T$ iff whenever $0 \neq rB \subseteq C$, for each $r \in R$ and every ideal $I$ of $R$, implies that either $rB \subseteq C + Soc(T)$ or $IB \subseteq C + Soc(T)$.

Proposition(8)
Let $T$ be $R$ module and $C$ be proper submodule of $T$. Then $C$ is WAPP – quasi prime submodule of $T$ if and only if for each $r, s \in R$, $([C:rs] \cup [C+ \text{Soc}(T):T]) \cup U(C+ \text{Soc}(T):T)$. 

**Proof:**

$\Rightarrow$ Let $t \in [C:T]$, implies that $rst \in C$. If $rst = 0$, then $t \in [0:T]$, and hence $t \in [0:T] \cup [C+ \text{Soc}(T):T] \cup U(C+ \text{Soc}(T):T)$. Suppose that $0 \neq rst \in C$, and since $C$ is WAPP-quasi prime submodule of $T$, it follows that either $rt \in C + \text{Soc}(T)$ or $st \in C + \text{Soc}(T)$, implies that $t \in [C+ \text{Soc}(T):T]$. Hence, $[C:T] \subseteq [0:T] \cup [C+ \text{Soc}(T):T]$. 

$\Leftarrow$ Assume that $0 \neq rst \in C$, for $r, s \in R$, $t \in T$, implies that $t \in [C:T] \subseteq [0:T] \cup [C+ \text{Soc}(T):T]$. But $0 \neq rst$, then $t \not\in [0:T]$, hence $t \in [C+ \text{Soc}(T):T]$. That is $t \in [0:T] \cup [C+ \text{Soc}(T):T]$. Hence, $C$ is WAPP-quasi prime submodule of $T$. 

**Proposition(9)**

Let $T$ be $R$ module and $C$ be proper submodule of $T$. Then $C$ is WAPP – quasi prime submodule of $T$ iff for every $r \in R$, and $t \in T$, $([C+ \text{Soc}(T):R] \cup [C+ \text{Soc}(T):R]) \cup U(C+ \text{Soc}(T):R)$. 

**Proof:**

$\Rightarrow$ Suppose that $C$ is WAPP-quasi, and let $s \in [C+ \text{Soc}(T):R]$. If $rst = 0$, then $s \in [0:R]$, hence $s \in [0:R] \cup [C+ \text{Soc}(T):R]$. If $0 \neq rst \in C$, and $C$ is a WAPP-quasi prime submodule of $T$, and $r \in C + \text{Soc}(T)$, then $st \in C + \text{Soc}(T)$, implies that $s \in [C+ \text{Soc}(T):R]$. Hence $s \in [0:R] \cup [C+ \text{Soc}(T):R]$. Thus, $[C+ \text{Soc}(T):R] \subseteq [0:R] \cup [C+ \text{Soc}(T):R]$. 

As a direct consequence of proposition (9) and proposition (3), we get the following corollary:

**Corollary(10)**

Let $T$ be $R$ module and $C$ be proper submodule of $T$. Then $C$ is WAPP – quasi prime submodule of $T$ iff for every $r \in R$, and any submodule $B$ of $T$, $([C+ \text{Soc}(T):R] \cup [C+ \text{Soc}(T):R]) \cup U(C+ \text{Soc}(T):R)$.

As a direct consequence of proposition (9) and proposition (4), we get the following corollary:

**Corollary(11)**

Let $T$ be $R$ module and $C$ be proper submodule of $T$. Then $C$ is WAPP – quasi prime submodule of $T$ iff for every ideal $I$ of $R$, and any submodule $B$ of $T$, $([I+ \text{Soc}(T):R] \cup [I+ \text{Soc}(T):R]) \cup U(C+ \text{Soc}(T):R)$.

**Proposition(12)**
Let $T$ be $R$ module and $C$ be proper submodule of $T$. Then for every $s, r \in R$, and $t \in T$, 
$\{s : s \in [C : R \ r, t] \cup [C + \text{Soc}(T) : R \ r, t] \cup [C + \text{Soc}(T) : R \ s, t]$. 

**Proof:** 
Suppose that $e \in [C : R \ rst]$ implies that $rs(e) \in C$. If $rs(e) = 0$, implies that $e \in [0 : R \ rst]$ and hence $e \in [0 : R \ rst, t] \cup [C + \text{Soc}(T) : R \ r, t] \cup [C + \text{Soc}(T) : R \ s, t].$ If $rs(e) \neq 0$, then $C$ is a WAPP quasi prime submodule of $T$, then either $r(s)C + \text{Soc}(T)$ or $s(e)C + \text{Soc}(T)$. That is either $e \in [0 : R \ rst] \cup [C + \text{Soc}(T) : R \ rst]$ or $e \in [C : R \ rst, t] \cup [C + \text{Soc}(T) : R \ r, t] \cup [C + \text{Soc}(T) : R \ s, t].$ Therefore, $[C : R \ rst] \subseteq [0 : R \ rst, t] \cup [C + \text{Soc}(T) : R \ r, t] \cup [C + \text{Soc}(T) : R \ s, t]$. 

The following are characterizations in the multiplication module.

**Proposition (13)** 
Let $T$ be multiplication $R$ module and $C$ be proper submodule of $T$. Then $C$ is a WAPP quasi prime submodule of $T$ iff $0 \neq K_1K_2 \subseteq C$, for some submodules $K_1, K_2$ of $T$, and $t \in T$. 

**Proof:** 
($\Rightarrow$) Suppose that $C$ is WAPP quasi prime submodule of $T$, and $0 \neq K_1K_2 \subseteq C$ for some submodules $K_1, K_2$ of $T$, and $t \in T$. Since $T$ is a multiplication, then $K_1 = IT$ and $K_2 = JT$ for some ideals $I, J$ of $R$. Thus $0 \neq K_1K_2 = IT \subseteq C$. Since $C$ is a WAPP quasi prime submodule of $T$ then by corollary (6) either $It \subseteq C + \text{Soc}(T)$ or $Jt \subseteq C + \text{Soc}(T)$. Hence either $K_1t \subseteq C + \text{Soc}(T)$ or $K_2t \subseteq C + \text{Soc}(T)$.

($\Leftarrow$) Assume that $0 \neq IJt \subseteq C$, for some ideals $I, J$ of $R$, and $t \in T$. That is $0 \neq K_1K_2t \subseteq C$ for $K_1 = IT$ and $K_2 = JT$. It follows that either $K_1t \subseteq C + \text{Soc}(T)$ or $K_2t \subseteq C + \text{Soc}(T)$; that is $It \subseteq C + \text{Soc}(T)$ or $Jt \subseteq C + \text{Soc}(T)$. Hence $C$ is a WAPP quasi prime submodule of $T$ by corollary (6).

**Proposition (14)**
Let $T$ be multiplication $R$ module and $C$ be proper submodule of $T$. Then $C$ is WAPP quasi prime submodule of $T$ iff $0 \neq K_1K_2H \subseteq C$, for some submodules $K_1, K_2$ and $H$ of $T$, implies that either $K_1H \subseteq C + \text{Soc}(T)$ or $K_2H \subseteq C + \text{Soc}(T)$.

**Proof:**
($\Rightarrow$) Assume that $0 \neq K_1K_2H \subseteq C$ for some submodules $K_1, K_2$ and $H$ of $T$. Since $T$ is a multiplication, then $K_1 = IT, K_2 = JT$ for some ideals $I, J$ of $R$. Hence $0 \neq K_1K_2H = IJH \subseteq C$. But $C$ is WAPP quasi prime submodule of $T$ then by proposition (4) either $IH \subseteq C + \text{Soc}(T)$ or $JH \subseteq C + \text{Soc}(T)$. Hence either $K_1H \subseteq C + \text{Soc}(T)$ or $K_2H \subseteq C + \text{Soc}(T)$.

($\Leftarrow$) Let $0 \neq IJH \subseteq C$, where $I, J$ are ideals of $R$, and $H$ is a submodule of $T$. Since $T$ is multiplication, then $0 \neq IJH = K_1K_2H \subseteq C$, hence by assumption either $K_1H \subseteq C + \text{Soc}(T)$ or $K_2H \subseteq C + \text{Soc}(T)$. That is either $IH \subseteq C + \text{Soc}(T)$ or $JH \subseteq C + \text{Soc}(T)$. Thus by proposition (4) $C$ is WAPP quasi prime submodule of $T$. 

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It is well-known that if $T$ is a $Z$-regular module, then $\text{Soc}(T) = \text{Soc}(R)T$ [11; prop.(3-25)]

**Proposition (15)**

Let $T$ be a $Z$-regular multiplication $R$-module and $C$ be a proper submodule of $T$. Then $C$ is WAPP-quasi prime submodule of $T$ if and only if $[C : R]T$ is WAPP-quasi prime ideal of $R$.

**Proof:**

($\Rightarrow$) Suppose that $C$ is a WAPP-quasi prime submodule of $T$, and let $0 \neq aI \subseteq [C : R]T$, for $a, b \in R$, $I$ is an ideal of $R$. It follows that $0 \neq abIT \subseteq C$. Since $C$ is a WAPP-quasi prime submodule of $T$, then by proposition (3) either $aIT \subseteq C + \text{Soc}(T)$ or $bIT \subseteq C + \text{Soc}(T)$. But $T$ is a $Z$-regular module, then $\text{Soc}(T) = \text{Soc}(R)T$, and since $T$ is a multiplication, then $C = [C : R]T$. Hence either $aIT \subseteq [C : R]T + \text{Soc}(R)T$ or $bIT \subseteq [C : R]T + \text{Soc}(R)T$. Thus either $aI \subseteq [C : R]T + \text{Soc}(R)$ or $bI \subseteq [C : R]T + \text{Soc}(R)$. Hence by proposition $[C : R]T$ is a WAPP-quasi prime ideal of $R$.

($\Leftarrow$) Suppose that $[C : R]T$ is a WAPP-quasi prime ideal of $R$, and $0 \neq rsB \subseteq C$, for $r, s \in R$, and $B$ is a submodule of $T$. Since $T$ is a multiplication, then $B = IT$, for some ideal $I$ of $R$, that is $0 \neq slI \subseteq [C : R]T$. For $[C : R]T$ is a WAPP-quasi prime ideal, then by proposition (3) either $rC \subseteq [C : R]T + \text{Soc}(R)$ or $sC \subseteq [C : R]T + \text{Soc}(R)$, it follows that either $rIT \subseteq [C : R]T + \text{Soc}(R)T$ or $sIT \subseteq [C : R]T + \text{Soc}(R)T$. But $T$ is a $Z$-regular module, then $\text{Soc}(T) = \text{Soc}(R)T$, and since $T$ is a multiplication, then $[C : R]T = C$. Thus either $rB \subseteq C + \text{Soc}(T)$ or $sB \subseteq C + \text{Soc}(T)$. Hence by proposition (3) $C$ is a WAPP-quasi prime submodule of $T$.

It is well-known that if an $R$-module $T$ is projective, then $\text{Soc}(T) = \text{Soc}(R)T$ [11; prop.(3-24)]

**Proposition (16)**

Let $T$ be a projective multiplication $R$-module and $C$ be a proper submodule of $T$. Then $C$ is WAPP-quasi prime submodule of $T$ if and only if $[C : R]T$ is a WAPP-quasi prime ideal of $R$.

**Proof:**

($\Rightarrow$) Let $0 \neq rIJ \subseteq [C : R]T$, for $r, I, J$ are ideal of $R$, then $0 \neq rIJT \subseteq C$. Since $C$ is a WAPP-quasi prime submodule of $T$, then by corollary (7) either $rIJ \subseteq C + \text{Soc}(T)$ or $IJ \subseteq C + \text{Soc}(T)$. Now since $T$ is a projective module, then $\text{Soc}(T) = \text{Soc}(R)T$ and since $T$ is multiplication, then $C = [C : R]T$. Hence either $rIJ \subseteq [C : R]T + \text{Soc}(R)T$ or $IJ \subseteq [C : R]T + \text{Soc}(R)T$. It follows that, either $rJ \subseteq [C : R]T + \text{Soc}(R)$ or $IJ \subseteq [C : R]T + \text{Soc}(R)$. Hence by corollary (7) $[C : R]T$ is a WAPP-quasi prime ideal of $R$.

($\Leftarrow$) Let $T$ be a projective multiplication $R$-module, and $C$ is a submodule of $T$. Since $T$ is a multiplication, then $B = IT$, for some ideal $J$ of $R$. Thus, $0 \neq rJ \subseteq C$, implies that $0 \neq rJ \subseteq [C : R]T$. But $[C : R]T$ is a WAPP-quasi prime ideal, then by corollary (7) either $rJ \subseteq [C : R]T + \text{Soc}(R)$ or $IJ \subseteq [C : R]T + \text{Soc}(R)$, that is either $rJT \subseteq [C : R]T + \text{Soc}(R)T$ or $IJ \subseteq [C : R]T + \text{Soc}(R)T$. Since $T$ is a projective then $\text{Soc}(T) = \text{Soc}(R)T$ and since $T$ is a multiplication, then $[C : R]T = C$. Thus
either \( rB \subseteq C + \text{Soc}(T) \) or \( IB \subseteq C + \text{Soc}(T) \). Hence by corollary (7) \( C \) is a WAPP-quasi prime submodule of \( T \).

We need to recall the following lemma before we introduce the next proposition.

**Lemma (17)**[12, coro, of theo, (9)]

Let \( T \) be a finitely generated multiplication \( R \)-module and \( I, J \) are ideals of \( R \). Then \( IT \subseteq JT \) if and only if \( I \subseteq J + \text{ann}_R(T) \).

**Proposition (18)**

Let \( T \) be a finitely generated multiplication \( \mathbb{Z} \)-regular \( R \)-module and \( I \) is a WAPP-quasi prime ideal of \( R \) with \( \text{ann}_R(T) \subseteq I \). Then \( IT \) is a WAPP-quasi prime submodule of \( T \).

**Proof:**

Let \( 0 \neq I_1, I_2 \subseteq IT \), for \( I_1, I_2 \) are ideals of \( R \), and \( B \) is submodule of \( T \). Since \( T \) is a multiplication then \( B = JT \) for some ideal \( J \) of \( R \). That is \( 0 \neq I_1, I_2 (J T) \subseteq IT \), it follows by lemma (17) \( 0 \neq I_1, I_2 \subseteq I + \text{ann}_R(T). \) But \( \text{ann}_R(T) \subseteq I \), implies that \( I + \text{ann}_R(T) = I \). That is \( 0 \neq I_1, I_2 \subseteq I \). But \( I \) is a WAPP-quasi prime ideal of \( R \), then by proposition (4) either \( 0 \neq I_1, J \subseteq I + \text{Soc}(R) \) or \( 0 \neq I_2, J \subseteq I + \text{Soc}(R) \). It follows that \( 0 \neq J \subseteq IT + \text{Soc}(R)T \) or \( 0 \neq J \subseteq IT + \text{Soc}(R)T \). But \( T \) is a \( \mathbb{Z} \)-regular then \( \text{soc}(R)T = \text{Soc}(T) \). Hence either \( 0 \neq I_1, B \subseteq IT + \text{Soc}(T) \) or \( 0 \neq I_2, B \subseteq IT + \text{Soc}(T) \). Thus by proposition (4) \( IT \) is a WAPP-quasi prime submodule of \( T \).

**Proposition (19)**

Let \( T \) be a finitely generated multiplication projective \( R \)-module and \( I \) is a WAPP-quasi prime ideal of \( R \) with \( \text{ann}_R(T) \subseteq I \). Then \( IT \) is a WAPP-quasi prime submodule of \( T \).

**Proof:**

Let \( 0 \neq I_1, B \subseteq IT \), for \( r \in R \), \( I_1 \) is an ideal of \( R \), and \( B \) is submodule of \( T \). Since \( T \) is multiplication then \( B = JT \) for some ideal \( J \) of \( R \). That is \( 0 \neq I_1, (J T) \subseteq IT \), it follows by lemma (17) \( 0 \neq I_1, J \subseteq I + \text{ann}_R(T). \) But \( \text{ann}_R(T) \subseteq I \), implies that \( I + \text{ann}_R(T) = I \). Hence \( 0 \neq I_1, J \subseteq I \), and since \( I \) is a WAPP-quasi prime ideal of \( R \), then by corollary (7) either \( 0 \neq I_1, J \subseteq I + \text{Soc}(R) \) or \( 0 \neq r J \subseteq I + \text{Soc}(R) \). That is \( 0 \neq J \subseteq IT + \text{Soc}(R)T \) or \( 0 \neq J \subseteq IT + \text{Soc}(R)T \). But \( T \) is a projective then \( \text{soc}(R)T = \text{Soc}(T) \). Thus either \( 0 \neq I_1, B \subseteq IT + \text{Soc}(T) \) or \( 0 \neq r B \subseteq IT + \text{Soc}(T) \). Hence by corollary (7) \( IT \) is a WAPP-quasi prime submodule of \( T \).

It is well-known that cyclic \( R \)-module is multiplication [13], and since cyclic \( R \)-module is a finitely generated, we get the following corollaries:

**Corollary (20)**

Let \( T \) be a cyclic \( \mathbb{Z} \)-regular \( R \)-module and \( I \) is a WAPP-quasi prime ideal of \( R \) with \( \text{ann}_R(T) \subseteq I \). Then \( IT \) is an WAPP-quasi prime submodule of \( T \).

**Corollary (21)**
Let $T$ be a cyclic projective $R$-module and $I$ is an WAPP-quasi prime ideal of $R$ with $\text{ann}_{R}(T) \subseteq I$. Then $IT$ is an WAPP-quasi prim submodule of $T$.

It is well-known that if a submodule $C$ of an $R$-module $T$ is essential in $T$, then $\text{Soc}(C)=\text{Soc}(T)$ [6, P.29].

**Proposition (22)**

Let $T$ be $R$-module ,and $A,B$ are submodules of $T$ with $A \not\subseteq B$ and $B$ is an essential in $T$. If $A$ is an WAPP-quasi prime submodule of $T$, then $A$ is a WAPP-quasi prime submodule of $B$.

**Proof:**

Let $0 \neq rst \in A$, for $r,s \in R$ , $t \in B$, that is $t \in T$. Since $A$ is a WAPP-quasi prime submodule of $T$, then either $rt \in A + \text{Soc}(T)$ or $st \in A + \text{Soc}(T)$. But $B$ is essential in $T$, then $\text{soc}(B)=\text{Soc}(T)$. That is either $rt \in A + \text{Soc}(B)$ or $st \in A + \text{Soc}(B)$. Hence $A$ is an WAPP-quasi prime submodule of $B$.

**Corollary (23)**

Let $T$ be $R$-module ,and $A,B$ are submodules of $T$ with $A \not\subseteq B$ and $\text{Soc}(T) \subseteq \text{Soc}(B)$. Then $A$ is a WAPP-quasi prime submodule of $B$.

It well-known that if $A$ is a submodule of an $R$-module $T$, then $\text{Soc}(A)=A \cap \text{Soc}(T)$ [9, lema 2.3.15]

**Proposition (24)**

Let $T$ be $R$-module ,and $A,B$ are submodules of $T$ with $B$ not contain in $A$ ,and $\text{Soc}(T) \subseteq B$. If $A$ is a WAPP-quasi prime submodule of $T$, then $A \cap B$ is a WAPP-quasi prime submodule of $B$.

**Proof:**

It is clear that $A \cap B$ is an proper submodule of $B$. Now ,let $0 \neq rst \in A \cap B$, for $r,s \in R$ , $t \in B$, implies that $0 \neq rt \in A$, since $A$ is a WAPP-quasi prime submodule of $T$, then either $rt \in A + \text{Soc}(T)$ or $st \in A + \text{Soc}(T)$, hence either $rt \in (A + \text{Soc}(T)) \cap B$ or $st \in (A + \text{Soc}(T)) \cap B$. Since $\text{Soc}(T) \subseteq B$, then by module law either $rt \in (A \cap B) + (B \cap \text{Soc}(T))$ or $st \in (A \cap B) + (B \cap \text{Soc}(T))$. That is either $rt \in (A \cap B) + \text{Soc}(B)$ or $st \in (A \cap B) + \text{Soc}(B)$. Thus $A \cap B$ is a WAPP-quasi prime submodule of $B$.

It well-known that for each submodule $A$ of an $R$-module $T$, then $\text{Soc}(A)=A$, then $A \subseteq \text{Soc}(T)[9, \text{theo.}(9.1.4)(c)]$.

**Proposition (25)**

Let $T$ be an $R$-module , and $A,B$ are submodules of $T$ with $B$ not contain in $A$, with $\text{Soc}(A)=A$ and $\text{soc}(B)=B$. Then $A \cap B$ is a WAPP-quasi prime submodule of $T$.

**Proof:**
Let $0 \neq rsL \subseteq A \cap B$, for $r, s \in R$, $L$ is submodule of $T$, then $0 \neq rs L \subseteq A$, and $0 \neq rs L \subseteq B$. But both $A$ and $B$ are WAPP-quasi prime submodule of $T$, then either $rL \subseteq A + \text{Soc}(T)$ or $sL \subseteq B + \text{Soc}(T)$, and $rL \subseteq B + \text{Soc}(T)$ or $sL \subseteq A + \text{Soc}(T)$. But $\text{Soc}(A) = A$ and $\text{soc}(B) = B$, then $A \subseteq \text{Soc}(T)$ and $B \subseteq \text{Soc}(T)$, hence $A + \text{Soc}(T) = \text{Soc}(T)$ and $B + \text{Soc}(T) = \text{Soc}(T)$, $A \cap B \subseteq \text{Soc}(T)$, implies that $A \cap B + \text{Soc}(T) = \text{Soc}(T)$, so either $rL \subseteq \text{Soc}(T) = A \cap B + \text{Soc}(T)$ or $sL \subseteq \text{Soc}(T) = A \cap B + \text{Soc}(T)$. Hence $A \cap B$ is WAPP-quasi prime submodule of $T$.

**Proposition (26)**

Let $f: T \to T'$ be an $R$-epimorphism, and $C$ be a WAPP-quasi prime submodule of $T$ with $\ker f \subseteq C$. Then $f(C)$ is WAPP-quasi prime submodule of $T'$.

**Proof:**

Let $f: T \to T'$ be an $R$-epimorphism, and $C$ be a WAPP-quasi prime submodule of $T$ with $\ker f \subseteq C$, let $0 \neq rst' \in f(C)$, for $r, s \in R$, $t' \in T$. Since $f$ is onto, then $f(t) = t'$, for some $t \in T$, it follows that $0 \neq rsf(t) \in C$, $0 \neq f(rst') \in C$, so there exists a nonzero $x \in C$ such that, $0 \neq f(rst) = f(x)$. That is $f(rst - x) = 0$, implies that $rst - x \in \ker f \subseteq C$. But $C$ is a WAPP-quasi prime submodule of $T$, then either $rst \in C + \text{Soc}(T)$ or $s t \in C + \text{Soc}(T)$. That is either $r f(t) \in C + \text{Soc}(T')$ or $s f(t) \in C + \text{Soc}(T')$. Thus either $rst' \in f(C) + \text{Soc}(T')$ or $st' \in f(C) + \text{Soc}(T')$. Hence $f(C)$ is an WAPP-quasi prime submodule of $T'$.

**Proposition (27)**

Let $f: T \to T'$ be an $R$-epimorphism, and $C$ be a WAPP-quasi prime submodule of $T$. Then $f^{-1}(C)$ is a WAPP-quasi prime submodule of $T$.

**Proof:**

It is clearly that $f^{-1}(C)$ is proper submodule of $T$. Let $0 \neq rst \in f^{-1}(C)$, for $r, s \in R$, $t \in T$, it follows that then $0 \neq rsf(t) \in C$, but $C$ is a WAPP-quasi prime submodule of $T'$, then either $r f(t) \in C + \text{Soc}(T)$ or $s f(t) \in C + \text{Soc}(T')$. Thus either $r f^{-1}(C) + f^{-1}(\text{Soc}(T')) \subseteq f^{-1}(C) + \text{Soc}(T)$ or $s f^{-1}(C) + f^{-1}(\text{Soc}(T)) \subseteq f^{-1}(C) + \text{Soc}(T)$. Hence $f^{-1}(C)$ is WAPP-quasi prime submodule of $T$.

**Proposition (28)**

Let $T$ be a $Z$-regular finitely generated multiplication $R -$ module, and $C$ be a proper submodule of $T$. Then the following statements are equivalent:

1. $C$ is WAPP-quasi prime submodule of $T$.
2. $[C:R]$ is WAPP-quasi prime ideal of $R$.
3. $C = \text{I}_T$ for some WAPP-quasi prime ideal $\text{I}$ of $R$ with $\text{ann}_R(T) \leq \text{I}$.

**Proof:**

$(1) \Rightarrow (2)$ Follows by proposition [15]
(2) \(\Rightarrow\) (3) Follows directly.

(3) \(\Rightarrow\) (2) Suppose that \(C=IT\) for some \(\text{a WAPP-quasi prime ideal of } R\). Since \(T\) is multiplication, then \(C=[C:_RT]T=IT\) and since \(M\) is finitely generated multiplication, then \([C:_RT]=I+\text{ann}_R(T)\). But \(\text{ann}_R(T)\subseteq I\) it follows that \(I+\text{ann}_R(T)=I\). Thus \([C:_RT]=I\) is a WAPP-quasi prime ideal of \(R\). Hence \([C:_RT]\) is WAPP-quasi prime ideal of \(R\).

The following corollary is a direct consequence of proposition (28)

**Corollary (29)**

Let \(T\) be a cyclic \(Z\)-regular \(R\)-module, and \(C\) be proper submodule of \(T\). Then the following statements are equipollent:

1. \(C\) is WAPP-quasi prime submodule of \(T\).
2. \([C:_RT]\) is WAPP-quasi prime ideal of \(R\).
3. \(C=IT\) for some WAPP-quasi prime ideal \(I\) of \(R\) with \(\text{ann}_R(T)\subseteq I\).

**Proposition (30)**

Let \(T\) be a finitely generated multiplication projective \(R\)-module, and \(C\) be a proper submodule of \(T\). Then the following statements are equipollent:

1. \(C\) is a WAPP-quasi prime submodule of \(T\).
2. \([C:_RT]\) is WAPP-quasi prime ideal of \(R\).
3. \(C=IT\) for some WAPP-quasi prime ideal \(I\) of \(R\) with \(\text{ann}_R(T)\subseteq I\).

**Proof:**

(1) \(\Rightarrow\) (2) Follows by proposition (16)

(2) \(\Rightarrow\) (3) Follows directly.

(3) \(\Rightarrow\) (2) Follows as in proposition (28).

As a direct consequence of proposition (30), we get the following corollary:

**Corollary (31)**

Let \(T\) be cyclic projective \(R\)-module, and \(C\) be proper submodule of \(T\), and \(C\) be a proper submodule of \(T\). Then the following statements are equipollent:

1. \(C\) is WAPP-quasi prime submodule of \(T\).
2. \([C:_RT]\) is WAPP-quasi prime ideal of \(R\).
3. \(C=IT\) for some WAPP-quasi prime ideal \(I\) of \(R\) with \(\text{ann}_R(T)\subseteq I\).

It is well-known that if \(T\) is faithful multiplication \(R\)-module, then \(\text{Soc}(T)=\text{Soc}(R)T\) \([7,\text{CORO.}(2.14)(1)]\).

**Proposition (32)**

Let \(T\) be a faithful multiplication \(R\)-module and \(C\) be a proper submodule of \(T\). Then \(C\) is a WAPP-quasi prime submodule of \(T\) iff \([C:_RT]\) is a WAPP-quasi prime ideal of \(R\).
Proof:

(⇒) Let \(0 \neq IJk \subseteq [C :_RT \ T] T\) , where \(I, J\) and \(k\) are ideals of \(R\) . Then \(0 \neq IJ(kT) \subseteq C\) . Since \(C\) is WAPP-quasi prime submodule of \(T\) , then by proposition\((4)\) either \(J(kT) \subseteq C + \text{Soc}(T)\) or \(I(kT) \subseteq C + \text{Soc}(T)\) . But \(T\) is a faithful multiplication , it follows that \(C = [C :_RT \ T] T\) and \(\text{Soc}(T) = \text{Soc}(R) T\) . Thus either \(I(kT) \subseteq [C :_RT \ T] T + \text{Soc}(R) T\) or \(J(kT) \subseteq [C :_RT \ T] T + \text{Soc}(R) T\) . Hence either \(IK \subseteq [C :_RT \ T] T + \text{Soc}(R)\) or \(JK \subseteq [C :_RT \ T] T + \text{Soc}(R)\) . Thus by proposition\((4)\) \([C :_RT \ T] T\) is WAPP-quasi prime ideal of \(R\).

(⇐) Let \(T 0 \neq abB \subseteq C\) , for \(a, b \in R\) , and \(B\) is submodule of \(T\) . Since \(T\) is multiplication , then \(B = JT\) for some ideal \(J\) of \(R\) . Thus \(0 \neq abJT \subseteq C\) , it follows that \(0 \neq abJ \subseteq [C :_RT \ T] T\) . But \([C :_RT \ T] T\) is WAPP-quasi prime ideal of \(R\) , then by proposition\((3)\) either \(aJ \subseteq [C :_RT \ T] T + \text{Soc}(R)\) or \(bJ \subseteq [C :_RT \ T] T + \text{Soc}(R)\) , it follows that either \(aJT \subseteq [C :_RT \ T] T + \text{Soc}(R) T\) or \(bJT \subseteq [C :_RT \ T] T + \text{Soc}(R) T\) . But \(T\) is a faithful multiplication \(R\)-module then either \(aB \subseteq C + \text{Soc}(T)\) or \(bB \subseteq C + \text{Soc}(T)\) . Thus by proposition \((3)\) \(C\) is a WAPP-quasi prime submodule of \(T\).

The following corollary is a direct consequence of proposition\((32)\)

**Corollary**\((33)\)

Let \(T\) be a faithful cyclic \(R\)-module and \(C\) be a proper submodule of \(T\) . Then \(C\) is WAPP-quasi prime submodule of \(T\) if and only if \([C :_RT \ T] T\) is a WAPP-quasi prime ideal of \(R\).

3. Conclusion

In this proper, we introduced and studied the concept WAPP-quasi prime submodule , and we established several examples , characterizations and basic properties of this concept . WAPP-quasi prime submodule is generalization of a Weakly quasi prime submodule so we give example for converse .

Among \(C\) , the main results of this paper are the following:

1. Proper submodule \(C\) of \(R\)-module \(T\) is WAPP-quasi prime submodule of \(T\) iff whenever \((0) \neq rsB \subseteq C\) , for \(r, s \in R\) , \(B\) is a submodule of \(T\) , implies that either \(rB \subseteq C + \text{Soc}(T)\) or \(sB \subseteq C + \text{Soc}(T)\) .

2. Proper submodule \(C\) of \(R\)-module \(T\) is WAPP-quasi prime submodule of \(T\) iff whenever \((0) \neq IJb \subseteq C\) , for \(I, J\) are ideals of \(R\) , and \(B\) is submodule of \(T\) , implies that either \(IB \subseteq C + \text{Soc}(T)\) or \(JB \subseteq C + \text{Soc}(T)\) .

3. Proper submodule \(C\) of \(R\)-module \(T\) is WAPP-quasi prime submodule of \(T\) iff for all \(r, s \in R\) , \([C :_RT] \subseteq [0 :_RT] \cup [C :_RT] \cup [C :_RT]\)
4. Proper submodule C of R-module T is WAPP-quasi prime submodule of T iff for all r∈R, t∈T with rt∉C + Soc(T) , [C;T]rt⊂[0;T]rt∪[ C + Soc(T);T t].

5. Proper submodule C of multiplication R-module T is WAPP-quasi prime submodule of T iff whenever (0)∉K1K2⊂C , for some submodules K1,K2 of T and t∈T, implies that either K1t⊂C+Soc(T) or K2t⊂C+Soc(T).

6. Proper submodule C of Z-regular multiplication R-module T is a WAPP-quasi prime submodule of T iff [C;R T] is WAPP-quasi prime ideal of R.

7. Proper submodule C of projective multiplication R-module T is WAPP-quasi prime submodule of T iff [C;R T] is WAPP-quasi prime ideal of R.

8. If T is a cyclic a Z-regular R-module and I is WAPP-quasi prime ideal of R with annR(T)⊂I. Then IT is WAPP-quasi submodule of T.

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