IMPROVING THE LIEB-ROBINSON BOUND FOR LONG-RANGE INTERACTIONS

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Abstract. We improve the Lieb-Robinson bound for a wide class of quantum many-body systems with long-range interactions decaying by power law. As an application, we show that the group velocity of information propagation grows by power law in time for such systems, whereas systems with short-range interactions exhibit a finite group velocity as shown by Lieb and Robinson.

1. Introduction

Lieb and Robinson [6] proved that the group velocity of information propagation is bounded by a finite constant in time for quantum many-body systems with short-range interactions (see also [3, 9]). The Lieb-Robinson bound was extended to systems with long-range interactions decaying by power law [4]. However, the resulting upper bound for the group velocity grows exponentially in time. If the upper bound is optimal and gives the true behavior of the group velocity in time, the information must spread to the space with such a fast-growing speed, which is unnatural physically [1]. Actually, the group velocity growing by power law in time was claimed by [2] for a quantum spin system with two-body interactions decaying by power law. In the present paper, we extend, in a mathematically rigorous manner, their argument to a more general class of quantum many-body systems with long-range interactions decaying by power law. As a result, we prove that the group velocity of information propagation grows by power law in time.

The present paper is organized as follows: In the next section, we give the precise definition of the models which we consider, and describe our main result, Theorem 2.1. The strategy for proving Theorem 2.1 is developed in Sec. 3. The proof is given in Sec. 4. Appendices A and B are devoted to technical estimates.

2. Models and main result

Let $\Omega$ be a countable set with a metric $d(\cdot, \cdot)$, and we suppose this metric induces the discrete topology, i.e., each point $x \in \Omega$ is open and closed. We suppose there is a monotone increasing function $g(r)$ on $[0, \infty)$ and a constant $D > 0$ such that

\begin{equation}
\# \{ y \in \Omega | d(x, y) \leq r \} \leq g(r) \leq C(1 + r)^D, \quad r \geq 0, x \in \Omega
\end{equation}

with some $C > 0$. We may consider $D$ as an analogue of the spatial dimension.

We consider quantum spin systems on the point set $\Omega$. We assign a Hilbert space $H_x$ to each site $x \in \Omega$. Let $\Lambda$ be a finite subset of $\Omega$. Then,
the configuration space of spin states on $\Lambda$ is given by the tensor product $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$, and the algebra $\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} B(\mathcal{H}_x)$ of the observables on $\Lambda$ acts on the Hilbert space $\mathcal{H}_\Lambda$, where $B(\mathcal{H}_x)$ denotes the Banach space of the bounded operators on $\mathcal{H}_x$. For $X \subset Y \subset \Omega$, we embed the algebra $\mathcal{A}_X$ on $X$ into $\mathcal{A}_Y$ on $Y$ by identifying $A \in \mathcal{A}_X$ with $A \otimes I \in \mathcal{A}_X \otimes \mathcal{A}(Y \setminus X) \cong \mathcal{A}_Y$.

The algebra of observables on $\Omega$ is defined as the completion of the local algebra $\mathcal{A}_{\text{loc}} = \bigcup \{ \mathcal{A}_X | X \subset \Omega, |X| < \infty \}$ in the sense of the operator-norm topology. Here, $|X|$ stands for the number of the elements in the set $X$.

Let $\Lambda$ be a finite subset of $\Omega$. Then, the Hamiltonian of a quantum spin system on $\Lambda$ is given by

$$H_\Lambda = \sum_{X \subset \Lambda} h_X,$$

where $h_X \in \mathcal{A}_X$ is the local Hamiltonian, i.e., a self-adjoint operator on $\mathcal{H}_X$, $X \subset \Omega$. The time evolution of the local observable $A \in \mathcal{A}_\Lambda$ by the generator $H_\Lambda$ is given by

$$\tau_{t\Lambda}(A) = e^{itH_\Lambda} A e^{-itH_\Lambda},$$

for the time $t \in \mathbb{R}$.

We write $\text{diam}(Z)$ for the diameter of the set $Z \subset \Omega$ which is given by $\text{diam}(Z) := \max\{d(x, y) | x, y \in Z\}$. If $|Z| = +\infty$, then we define $\text{diam}(Z) = +\infty$. Although we will consider general long-range interactions $h_X$ which include an arbitrary many-body interaction, we require the following assumption for the local Hamiltonian $h_X$:

**Assumption A.**

(i) There is a decreasing function $f(R)$ on $[0, \infty)$ such that

$$\sup_{x \in \Omega} \sum_{Z \ni x: \text{diam}(Z) \geq R} \|h_Z\| \leq f(R), \quad R \geq 0.\tag{2.4}$$

(ii) $C_0 = \sup_{x \in \Omega} \sum_{y \in \Omega} \sum_{Z \ni x,y} \|h_Z\| < \infty.\tag{2.5}$$

A typical example is a spin system on $\Omega = \mathbb{Z}^D$, and we let $d(\cdot, \cdot)$ be the graph distance. We suppose it has only two-body interactions $h_{\{x,y\}}$ for $x, y \in \mathbb{Z}^D$, and the following power-law decay condition with $\alpha > 0$:

$$\|h_X\| \leq \frac{C_1}{[1 + d(x, y)]^{\alpha + D}}, \quad \text{for } X = \{x, y\},\tag{2.6}$$

and $h_X = 0$, otherwise, where $C_1$ is some positive constant which is independent of the pair $\{x, y\}$ of the two sites. Then $\{h_X\}$ satisfies Assumption A with $f(R) = C'(1 + R)^{-\alpha}$. This is nothing but the case treated in [2].

Our result is a mathematical justification of the argument in [2].

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1 For an attempt for extending the Lieb-Robinson bounds to systems with an unbounded Hamiltonian, see, e.g., [8] and references therein.
Theorem 2.1. Let \( A \in \mathcal{A}_X \) on \( X \subset \Lambda \), and \( B \in \mathcal{A}_Y \) on \( Y \subset \Lambda \). Let \( R \geq 1 \), and write \( r = d(X, Y) \). Then,

\[
\| \tau_{t, \Lambda}(A), B \| \leq 2\| A \| \| B \| \| X \| e^{vt \cdot r/R} + 4\| A \| \| B \| \| X \| t g(r) f(R) \tag{2.7}
\]

\[+ 2C_2\| A \| \| B \| \| X \|^2 t R(r \vee R)^D f(R) e^{vt \cdot r/R}.
\]

for any \( t \geq 0 \), where \( r \vee R := \max\{r, R\} \), and \( v \) and \( C_2 \) are positive constants independent of \( \Lambda, t, R, X, Y, A \) and \( B \).

Let us explain the physical meaning of the resulting bound (2.7) for the case of the hypercubic lattice \( \mathbb{Z}^D \), with an additional assumption \( \alpha > D \) and (2.6). We recall \( g(r) = C(1 + r)^D \) and \( f(R) = C'(1 + R)^{-\alpha} \). But, in this case, the factor \( (r \vee R)^D \) in the third term in the right-hand side of (2.7) can be replaced with \( (r \vee R)^{D-1} \) by carefully calculating the bound in the proof of Lemma [5, 11] in Appendix B. Let \( r \geq 1 \) and \( t > 0 \). We choose the parameter \( R \) as

\[ R = r^\kappa \quad \text{with} \quad \kappa = \frac{D + 1}{\alpha + 1}.
\]

Substituting these into the right-hand side of the bound (2.7), we obtain

\[
\| \tau_{t, \Lambda}(A), B \| \leq 2\| A \| \| B \| | X | \exp[vt \cdot r^\eta] + C_3\| A \| \| B \| \| X \| | X | \frac{t}{r^\eta} + C_4\| A \| \| B \| \| X \|^2 \frac{t}{r^\eta} \exp[vt \cdot r^\eta],
\]

with \( \eta = (\alpha - D)/(\alpha + 1) \), where \( C_3 \) and \( C_4 \) are some positive constants, and we have used \( \kappa < 1 \) which is derived from the assumption \( \alpha > D \). We define the upper bound of the propagation distance by \( r_{\text{max}}(t) = (\lambda v t)^{1/\eta} \) as a function of \( t \) with the scale parameter \( \lambda > 1 \). Substituting \( r = r_{\text{max}}(t) \) into the above upper bound, one has

\[
\| \tau_{t, \Lambda}(A), B \| \leq 2\| A \| \| B \| \| X \| \exp(- (\lambda - 1) vt) + C_3\| A \| \| B \| \| X \| \frac{1}{\lambda v}
\]

\[+ C_4\| A \| \| B \| \| X \|^2 \frac{1}{(\lambda v)^2} \exp(- (\lambda - 1) vt) \]

\[+ C_3\| A \| \| B \| \| X \| \frac{1}{\lambda v}
\]

for a large \( t \). Clearly, for a large \( \lambda \), the right-hand side becomes small, while it gives the order of 1 for \( \lambda \) of the order of 1. Thus, the quantity \( r_{\text{max}}(t) \) gives the upper bound of the propagation distance as a function of \( t \). From the definition, it obeys the power law as

\[ r_{\text{max}}(t) = (\lambda v)^{1/\eta} t^{1+\gamma} \quad \text{for} \quad t > 0,
\]

with \( \gamma = (D + 1)/(\alpha - D) \). The corresponding group velocity \( v_g(t) \) behaves as

\[ v_g(t) := \frac{d}{dt} r_{\text{max}}(t) = (\lambda v)^{1/\eta} t^{\gamma}.
\]

This exactly coincides with the behavior obtained in [2]. For related recent numerical computations for quantum spin systems with long-range interactions, see, e.g., [5, 12]. For an overview of results and applications on the Lieb-Robinson bounds for quantum many-body systems, see, e.g., [11] and references therein.
3. Decomposition of the Hamiltonian

In this section, we describe our strategy for proving Theorem 2.1. The idea of decomposing the Hamiltonian was introduced in [2].

Let $R$ be a positive number. We decompose the Hamiltonian $H$ of (2.2) into two parts as

$$H = H_{<R} + H_{\geq R}$$

with

$$(3.1) \quad H_{<R} = \sum_{Z \subset \Lambda} h_{<R}^Z,$$

and

$$(3.2) \quad H_{\geq R} = \sum_{Z \subset \Lambda} h_{\geq R}^Z,$$

where the two local Hamiltonians, $h_{<R}^Z$ and $h_{\geq R}^Z$, are given by

$$h_{<R}^Z := \begin{cases} h_Z, & \text{if } \text{diam}(Z) < R, \\ 0, & \text{otherwise}, \end{cases}$$

and $h_{\geq R}^Z := h_Z - h_{<R}^Z$. The Hamiltonian $H_{<R}$ is the short-range part with the interaction range $R$, and $H_{\geq R}$ is the long-range part. Clearly, from Assumption A one has

$$(3.3) \quad \sup_{x \in \Lambda} \sum_{Z \ni x} \|h_{\geq R}^Z\| \leq f(R)$$

for $R \geq 1$.

The time evolution by the short-range Hamiltonian $H_{<R}$ is given by

$$\tau_{t,\Lambda}^{<R}(A) = e^{itH_{<R}} A e^{-itH_{<R}}$$

for a local observable $A \in \mathcal{A}_\Lambda$. We also introduce a unitary operator,

$$\mathcal{U}_\Lambda^R(t) = e^{itH_{<R}} e^{-itH_\Lambda},$$

which satisfies the Schrödinger equation of the interaction picture,

$$i \frac{d}{dt} \mathcal{U}_\Lambda^R(t) = H_{\geq R}(t) \mathcal{U}_\Lambda^R(t),$$

with the initial condition $\mathcal{U}_\Lambda^R(0) = 1$, where we have written

$$(3.5) \quad H_{\geq R}(t) := \tau_{t,\Lambda}^{<R} \left( H_{\geq R} \right)$$

for short. Then, as is well known, the time evolution of the observable $A \in \mathcal{A}_\Lambda$ by the total Hamiltonian $H$ of (2.2) is given by

$$\tau_{t,\Lambda}(A) = \left[ \mathcal{U}_\Lambda^R(t) \right]^* \tau_{t,\Lambda}^{<R}(A) \mathcal{U}_\Lambda^R(t).$$

From

$$[\tau_{t,\Lambda}(A), B] = [\mathcal{U}_\Lambda^R(t)^* \tau_{t,\Lambda}^{<R}(A) \mathcal{U}_\Lambda^R(t), B]$$

$$= \mathcal{U}_\Lambda^R(t)^* [\tau_{t,\Lambda}^{<R}(A), \mathcal{U}_\Lambda^R(t) B \mathcal{U}_\Lambda^R(t)^*] \mathcal{U}_\Lambda^R(t),$$
By using the Schrödinger equation (3.4) and the Jacobi identity, we have

\[ A \]

where

\[ d \]

for any

\[ B \]

one has

\[ \| \tau_{t, \Lambda}(A), B \| = \left\| \tau_{t, \Lambda}^{(<R)}(A), \mathcal{U}_\Lambda^R(t)B\mathcal{U}_\Lambda^R(t)^* \right\| \]

for two local observables \( A \) and \( B \).

For \( r > 0 \) and \( X \subset \Lambda \), we define

\[ \widetilde{X}_r := \{ x \in A \mid d(x, X) \leq r \}, \]

where \( d(x,X) := \min\{d(x,y) \mid y \in X \} \). The set \( \widetilde{X}_r \) is the \( r \)-neighborhood of \( X \).

**Lemma 3.1.** Let \( A \in \mathcal{A}_X \) on a finite subset \( X \) of \( \Omega \), and let \( r > 0 \). Then,

\[ \| \tau_{t, \Lambda}(A), B \| \leq \left\| \tau_{t, \Lambda}^{(<R)}(A), B \right\| + 2\| B \| \sum_{Z \cap \widetilde{X}_r \neq \emptyset} \int_0^t \left\| \tau_{t-s, \Lambda}(A), h_Z^{(\geq R)} \right\| ds \]

\[ + 2\| B \| \sum_{Z \cap \widetilde{X}_r = \emptyset} \int_0^t \left\| \tau_{t-s, \Lambda}(A), h_Z^{(\geq R)} \right\| ds \]

for any \( B \in \mathcal{A}_\Lambda \) and any \( t \geq 0 \).

**Proof.** We introduce a \( \mathcal{A}_\Lambda \)-valued function \( f(t,s) \) for \( s, t > 0 \) by

\[ f(t,s) = [\tau_{t, \Lambda}^{(<R)}(A), \mathcal{U}_\Lambda^R(s)B\mathcal{U}_\Lambda^R(s)^*] \]

Clearly, from (3.6), one has

\[ \| \tau_{t, \Lambda}(A), B \| = \| f(t,t) \| \]

By using the Schrödinger equation (3.4) and the Jacobi identity, we have

\[ \frac{d}{ds} f(t,s) = -i[\tau_{t, \Lambda}^{(<R)}(A), [H^{(\geq R)}_\Lambda(s), \mathcal{U}_\Lambda^R(s)B\mathcal{U}_\Lambda^R(s)^*]] \]

\[ = -i[H^{(\geq R)}_\Lambda(s), f(t,s)] + i[\mathcal{U}_\Lambda^R(s)B\mathcal{U}_\Lambda^R(s)^*, [\tau_{t, \Lambda}^{(<R)}(A), H^{(\geq R)}_\Lambda(s)]] \]

Here, we have used the notation of (3.5). Since the first term in the right-hand side in the second equality is a generator of a unitary evolution, we can apply a variation of the Duhamel principle (see, e.g., Appendix of [7]). Therefore, by integrating both sides with respect to \( s \) from 0 to \( t \), we obtain

\[ \| f(t,t) \| \leq \| f(t,0) \| + 2\| B \| \int_0^t ds \left\| \tau_{t, \Lambda}^{(<R)}(A), H^{(\geq R)}_\Lambda(s) \right\| \]

\[ = \| [\tau_{t, \Lambda}^{(<R)}(A), B] \| + 2\| B \| \int_0^t ds \left\| [\tau_{t-s, \Lambda}(A), H^{(\geq R)}_\Lambda(s)] \right\|. \]

Substituting the expression (3.2) of \( H^{(\geq R)}_\Lambda(s) \) into the integrand of the last integral, we obtain the desired bound (3.7). \( \square \)
4. Proof of Theorem 2.1

Concerning the first term in the right-hand side of the inequality (3.7), we have

$$
\left\| \left[ \tau_{t,\Lambda}^{(<R)} (A), B \right] \right\| \leq 2 \| A \| \| B \| |X| \exp[vt - d(X,Y)/R]
$$

for $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, from Theorem A.1 in Appendix A.

For the second term, by using the inequality (3.3), we obtain

$$
\sum_{Z \cap \tilde{X}_r \neq \emptyset} \int_0^t \left\| \left[ \tau_{t-s,\Lambda}^{(<R)} (A), h_Z^{(\geq R)} \right] \right\| \, ds \leq 2t \| A \| \sum_{Z \cap \tilde{X}_r \neq \emptyset} \sum_{x \in \tilde{X}_r \cap Z} \| h_Z^{(\geq R)} \|
$$

$$
\leq 2t \| A \| \| X \| f(R) \leq 2t \| A \| \| X \| g(r) f(R).
$$

The third term is estimated by Lemma B.1 in Appendix B as

$$
\sum_{Z \cap \tilde{X}_r = \emptyset} \int_0^t \left\| \left[ \tau_{t-s,\Lambda}^{(<R)} (A), h_Z^{(\geq R)} \right] \right\| \, ds \leq C_2 t \| A \| \| X \| ^2 (r \vee R)^D R f(R) e^{vt-r/R},
$$

where $C_2$ is some positive constant.

We set $r = d(X,Y)$. Combining these with Lemma B.1, we have

$$
\left\| \left[ \tau_{t,\Lambda} (A), B \right] \right\| \leq 2 \| A \| \| B \| \| X \| ^2 e^{vt-r/R} + 4t \| A \| \| B \| \| X \| g(r) f(R)
$$

$$
+ 2C_2 t \| A \| \| B \| \| X \| ^2 (r \vee R)^D R f(R) e^{vt-r/R}.
$$

Appendix A. The Lieb-Robinson bound for finite-range interactions

In this appendix, we derive a Lieb-Robinson bound for the Hamiltonian $H_{\Lambda}^{(<R)}$ of (3.5) with finite-range interactions. The Lieb-Robinson bound for finite-range interactions is given by:

**Theorem A.1.** Let $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $X, Y \subset \Lambda$, and let $R > 0$. Then, we have

$$
\left\| \left[ \tau_{t,\Lambda}^{(<R)} (A), B \right] \right\| \leq 2 \| A \| \| B \| \| X \| \exp[vt - d(X,Y)/R]
$$

for any $t \geq 0$ with some positive constant $v$, under Assumption A(iii).

**Proof.** We essentially follow the proof of the Lieb-Robinson bound in [10], with explicit control of the constants. We set

$$
C_B(Z,t) = \sup_{A \in \mathcal{A}_Z} \frac{\| \left[ \tau_{t,\Lambda}^{(<R)} (A), B \right] \|}{\| A \|}, \quad \text{for } Z \subset \Lambda.
$$
By computations similar to the proof of Lemma 3.1 we learn

\[
\frac{d}{dt}[\tau_{t,\Lambda^+}(A), B] = i \sum_{Z \cap X \neq \emptyset} [\tau_{t,\Lambda^+}(h_{\langle R \rangle}^Z), [\tau_{t,\Lambda^+}(A), B]]
\]

\[
- i \sum_{Z \cap X \neq \emptyset} [\tau_{t,\Lambda^+}(A), [\tau_{t,\Lambda^+}(h_{\langle R \rangle}^Z), B]].
\]

Since the first term in the right-hand side is a generator of norm-preserving evolution, we have

\[
\| [\tau_{t,\Lambda^+}(A), B] \| \leq \| [A, B] \| + 2 \| A \| \sum_{Z \cap X \neq \emptyset} \int_0^t \| [\tau_{s,\Lambda^+}(h_{\langle R \rangle}^Z), B] \| ds.
\]

in the same way as in Lemma 3.1. Consequently, we obtain

\[ C_B(X, t) \leq C_B(X, 0) + 2 \sum_{Z \cap X \neq \emptyset} \| h_{\langle R \rangle}^Z \| \int_0^t C_B(Z, s) ds. \]

Iterations of this inequality yield

(A.2) \[ C_B(X, t) \leq C_B(X, 0) + 2 \| B \| \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} a_n, \]

where

\[ a_n = \sum_{Z_1 \cap X \neq \emptyset} \sum_{Z_2 \cap Z_1 \neq \emptyset} \cdots \sum_{Z_n \cap \Lambda \neq \emptyset} \prod_{i=1}^n \| h_{\langle R \rangle}^Z \|. \]

Using Assumption (A)-(ii), we have

\[ a_1 \leq \sum_{x \in X} \sum_{y \in Y} \sum_{Z \ni x, y} \| h_{\langle R \rangle}^Z \| \leq \sum_{x \in X} C_0 \leq C_0 |X|. \]

Similarly, we have

\[ a_2 \leq \sum_{x \in X} \sum_{y \in Y} \sum_{z \in Z_1} \sum_{z \in Z_2} \sum_{z \in \Lambda} \| h_{\langle R \rangle}^Z \| \| h_{\langle R \rangle}^Z \| \leq C_0^2 |X|. \]

Repeating this procedure, we obtain

(A.3) \[ a_n \leq C_0^n |X|, \text{ for } n \geq 1. \]

On the other hand, we have

(A.4) \[ a_n = 0 \text{ if } nR < d(X, Y) \]

because \( h_{\langle R \rangle}^Z = 0 \) for \( \text{diam}(Z) \geq R \). Combining (A.2), (A.3) and (A.4), we have

\[ C_B(X, t) \leq 2 \| B \||X| \sum_{n \geq d(X, Y)/R} \frac{(2C_0t)^n}{n!} e^{-n \cdot d(X, Y)/R} \]

\[ \leq 2 \| B \||X| \sum_{n \geq d(X, Y)/R} \frac{(2C_0t)^n}{n!} e^{-d(X, Y)/R} \]

\[ \leq 2 \| B \||X| \sum_n \frac{(2C_0t)^n}{n!} e^{-d(X, Y)/R} = 2 \| B \||X| e^{st-d(X, Y)/R} \]
for $d(X,Y) > 0$. Here, the group velocity $v$ is given by $v = 2\epsilon C_0$. This completes the proof.

**APPENDIX B. DERIVATION OF THE INEQUALITY (4.2)**

The third term in the right-hand side of (3.7) in Lemma 3.1 is estimated as follows.

**Lemma B.1.** Let $A \in \mathcal{O}_X$, $X \subset \Lambda$. Then

$$\sum_{Z \cap X_r = \emptyset} \| [r_{t,A}^P(A), h_Z^{(\geq R)}] \| \leq \mathcal{C}_2 \| X \|^2 \| (r \vee R)^D R f(R) e^{vt-r/R}$$

for $t \geq 0$, $r > 0$ and $R \geq 1$, where $r \vee R := \max\{r, R\}$, and $\mathcal{C}_2$ is some positive constant.

**Proof.** From Theorem A.1, we have

$$\sum_{Z \cap X_r = \emptyset} \| [r_{t,A}^P(A), h_Z^{(\geq R)}] \| \leq 2 \| A \| \| X \| \sum_{Z \cap X_r = \emptyset} \| h_Z^{(\geq R)} \| e^{vt-d(X,Z)/R}$$

$$\leq 2 \| A \| \| X \| \sum_{k=0}^{\infty} \sum_{r+k<d(X,Z)} \| h_Z^{(\geq R)} \| e^{-(r+k)/R}$$

$$\leq 2 \| A \| \| X \| \sum_{k=0}^{\infty} \sum_{r+k<d(X,Z)} \| h_Z^{(\geq R)} \| e^{-r/k}$$

$$\leq 2 \| A \| \| X \| \sum_{x \in X} \sum_{k=0}^{\infty} \sum_{d(x,z) \leq r+k+1} e^{-r/k}$$

where we have used the inequality (4.3) to show the third inequality. Elementary computations yield

$$\sum_{k=0}^{\infty} \sum_{d(x,z) \leq r+k+1} e^{-r/k} \leq C_1 \int_r^{\infty} (y+1)e^{-y/R} dy$$

$$\leq C_2 R^{D+1} \int_r^{\infty} y^D e^{-y} dy$$

$$\leq C_3 R^{D+1} (r/R + 1)^D e^{-r/R}$$

$$\leq C_3 (r \vee R)^D e^{-r/R}$$

(B.1)

for each $x \in X$, where the constants $C_1, C_2, C_3$ depend only on the constants in (2.1). Combining these, we conclude the assertion.

Finally, we remark on the typical example in Section 2: The first inequality in (B.1) can be replaced by a sum in $z : r + k < d(x,z) \leq r + k + 1$ in general, and hence the factor $g(y+1)$ is replaced by $Cy^{D-1}$ in the case...
of $\mathbb{Z}^D$ lattice. Therefore, the factor $(r \lor R)^D$ in the right-hand side of the fourth inequality can be replaced with $(r \lor R)^{D-1}$.

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