ON MOD $p$ LOCAL-GLOBAL COMPATIBILITY FOR $\text{GL}_n(\mathbb{Q}_p)$ IN THE ORDINARY CASE

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Abstract. Let $p$ be a prime number, $n > 2$ an integer, and $F$ a CM field in which $p$ splits completely. Assume that a continuous automorphic Galois representation $\pi : \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_n(\overline{\mathbb{F}}_p)$ is upper-triangular and satisfies certain genericity conditions at a place $w$ above $p$, and that every subquotient of $\pi|_{\text{Gal}(\mathbb{Q}_p/F_w)}$ of dimension $> 2$ is Fontaine–Laffaille generic. In this paper, we show that the isomorphism class of $\pi|_{\text{Gal}(\mathbb{Q}_p/F_w)}$ is determined by $\text{GL}_n(\mathbb{F}_w)$-action on a space of mod $p$ algebraic automorphic forms cut out by the maximal ideal of a Hecke algebra associated to $\pi$, assuming a weight elimination result which is a theorem of Bao V. Le Hung in his forthcoming paper [LeH]. In particular, we show that the wildly ramified part of $\pi|_{\text{Gal}(\mathbb{Q}_p/F_w)}$ is determined by the action of Jacobi sum operators (seen as elements of $\mathbb{F}_p[\text{GL}_n(\mathbb{F}_p)]$) on this space.

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1. Introduction

It is believed that one can attach a smooth $\overline{\mathbb{F}}_p$-representation of $GL_n(K)$ (or a packet of such representations) to a continuous Galois representation $Gal(\overline{\mathbb{Q}}_p/K) \to GL_n(\overline{\mathbb{F}}_p)$ in a natural way, that is called mod $p$ Langlands program for $GL_n(K)$, where $K$ is a finite extension of $\mathbb{Q}_p$. This conjecture is well-understood for $GL_2(\mathbb{Q}_p)$ (BL94, Ber10, Bre03a, Bre03b, Col10, Pas13, CDP, Eme). Beyond the $GL_2(\mathbb{Q}_p)$-case, for instance $GL_n(\mathbb{Q}_p)$ for $n > 2$ or even $GL_2(\mathbb{Q}_p^f)$ for an unramified extension $\mathbb{Q}_p^f$ of $\mathbb{Q}_p$ of degree $f > 1$, the situation is still quite far from being understood. One of the main difficulties is that there is no classification of such smooth representations of $GL_n(K)$ unless $K = \mathbb{Q}_p$ and $n = 2$: in particular, we barely understand the supercuspidal representations. Some of the difficulties in classifying the supercuspidal representations are illustrated in [BP12], [Hu10] and [Sclar13].

Let $F$ be a CM field in which $p$ is unramified, and $\mathfrak{f} : Gal(\overline{\mathbb{Q}}/F) \to GL_n(\overline{\mathbb{F}}_p)$ an automorphic Galois representation. Although there is no precise statement of mod $p$ Langland correspondence for $GL_n(K)$ unless $K = \mathbb{Q}_p$ and $n = 2$, one can define smooth representations $\Pi(\mathfrak{f})$ of $GL_n(F_w)$ in the spaces of mod $p$ automorphic forms on a definite unitary group cut out by the maximal ideal of a Hecke algebra associated to $\mathfrak{f}$, where $w$ is a place of $F$ above $p$. A precise definition of $\Pi(\mathfrak{f})$ when $p$ splits completely in $F$, which is our context, will be given in Section 1.3 (See also Section 5.7). One wishes that $\Pi(\mathfrak{f})$ is a candidate on the automorphic side corresponding to $\mathfrak{f}|_{Gal(\overline{\mathbb{Q}}_p/F_w)}$ for a mod $p$ Langlands correspondence in the spirit of Emerton [Eme]. However, we barely understand the structure of $\Pi(\mathfrak{f})$ as a representation of $GL_n(F_w)$, though the ordinary part of $\Pi(\mathfrak{f})$ is described in [BH15] when $p$ splits completely in $F$ and $\mathfrak{f}|_{Gal(\overline{\mathbb{Q}}_p/F_w)}$ is ordinary. In particular, it is not known whether $\Pi(\mathfrak{f})$ and $\mathfrak{f}|_{Gal(\overline{\mathbb{Q}}_p/F_w)}$ determine each other. But we have the following conjecture:

**Conjecture 1.0.1.** The local Galois representation $\mathfrak{f}|_{Gal(\overline{\mathbb{Q}}_p/F_w)}$ is determined by $\Pi(\mathfrak{f})$.

This conjecture is widely expected to be true by experts but not explicitly written down before. The case $GL_2(\mathbb{Q}_p^f)$ was treated by Breuil–Diamond [BD14], Herzig–Le–Morra [HLM] considered the case $GL_3(\mathbb{Q}_p)$ when $\mathfrak{f}|_{Gal(\overline{\mathbb{Q}}_p/F_w)}$ is upper-triangular and Fontaine–Laffaille, while the case $GL_3(\mathbb{Q}_p)$ when $\mathfrak{f}|_{Gal(\overline{\mathbb{Q}}_p/F_w)}$ is an extension of a two dimensional irreducible representation by a character was considered by Le–Morra–Park [LMP]. We are informed that John Enns from the University of Toronto has worked on this conjecture for the group $GL_3(\mathbb{Q}_p^f)$. All of the results above are under certain generic assumptions on the tamely ramified part of $\mathfrak{f}|_{Gal(\overline{\mathbb{Q}}_p/F_w)}$.

From another point of view, to a smooth admissible $\overline{\mathbb{F}}_p$-representation $\Pi$ of $GL_n(K)$ for a finite extension $K$ of $\mathbb{Q}_p$, Scholze [Sch13] attaches a smooth admissible $\overline{\mathbb{F}}_p$-representation $S(\Pi)$ of $D^\times$ for a division algebra $D$ over $K$ with center $K$ and invariant $\frac{1}{n}$, which also has a continuous action of $Gal(\overline{\mathbb{Q}}_p/K)$, via the mod $p$ cohomology of the Lubin–Tate tower. Using this construction, it was possible for Scholze to prove Conjecture 1.0.1 in full generality for $GL_2(K)$ (cf. Sch15, Theorem 1.5).
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On the other hand, the proof of Theorem 1.5 of [Sch15] does not tell us where the invariants that determine \( S(\Pi) \) lie. We do not know if there is any relation between these two different methods.

The weight part of Serre’s conjecture already gives part of the information of \( \Pi(\mathfrak{r}) \): the local Serre weights of \( \mathfrak{r} \) at \( w \) determine the socle of \( \Pi(\mathfrak{r})|_{\text{GL}_n(O_{F_w})} \) at least up to possible multiplicities, where \( O_{F_w} \) is the ring of integers of \( F_w \). If \( \mathfrak{r}|_{\text{Gal}(\mathbb{Q}_p/F_w)} \) is semisimple, then it is believed that the Serre weights of \( \mathfrak{r} \) at \( w \) determine \( \mathfrak{r}|_{\text{Gal}(\mathbb{Q}_p/F_w)} \) up to twisting by unramified characters, but this is no longer the case if it is not semisimple: the Serre weights are not enough to determine the wildly ramified part of \( \mathfrak{r}|_{\text{Gal}(\mathbb{Q}_p/F_w)} \), so that we need to understand a deeper structure of \( \Pi(\mathfrak{r}) \) than just its \( \text{GL}_n(O_{F_w}) \)-socle.

In this paper, we show that Conjecture 1.0.1 is true when \( p \) splits completely in \( F \) and \( \mathfrak{r}|_{\text{Gal}(\mathbb{Q}_p/F_w)} \) is upper-triangular and sufficiently generic in a precise sense. Moreover, we describe the invariants in \( \Pi(\mathfrak{r}) \) that determine the wildly ramified part of \( \mathfrak{r}|_{\text{Gal}(\mathbb{Q}_p/F_w)} \). The generic assumptions on \( \mathfrak{r}|_{\text{Gal}(\mathbb{Q}_p/F_w)} \) ensure that very few Serre weights of \( \mathfrak{r} \) at \( w \) will occur, which we call the weight elimination conjecture, Conjecture 1.1.2. The weight elimination results are significant for our method to prove Conjecture 1.0.1. But thanks to Bao V. Le Hung, this weight elimination conjecture is known to be true in his forthcoming paper [LeH].

We follow the basic strategy in [BD14, HLM]: we define Fontaine–Laffaille parameters on the Galois side using Fontaine–Laffaille modules as well as automorphic parameters on the automorphic side using the actions of Jacobi sum operators, and then identify them via the classical local Langlands correspondence. However, there are many new difficulties that didn’t occur in [BD14] or in [HLM]. For instance, the classification of semi-linear algebraic objects of rank \( n > 3 \) on the Galois side is much more complicated. Moreover, failing of the multiplicity one property of the Jordan–Hölder factors of mod \( p \) reduction of Deligne–Lusztig representations of \( \text{GL}_n(\mathbb{Z}_p) \) for \( n > 3 \) implies that new ideas are required to show crucial non-vanishing of the automorphic parameters.

In the rest of the introduction, we explain our ideas and results in more detail.

1.1. Local Galois side. Let \( E \) be a (sufficiently large) finite extension of \( \mathbb{Q}_p \) with ring of integers \( O_E \), a uniformizer \( \varpi_E \), and residue field \( F \), and let \( I_{\mathbb{Q}_p} \) be the inertia subgroup of \( \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \) and \( \omega \) the fundamental character of niveau \( 1 \). We also let \( \overline{\rho}_0 : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \text{GL}_n(F) \) be a continuous (Fontaine–Laffaille) ordinary generic Galois representation. Namely, there exists a basis \( \underline{e} := (e_{n-1}, e_{n-2}, \ldots, e_0) \) for \( \overline{\rho}_0 \) such that with respect to \( \underline{e} \) the matrix form of \( \overline{\rho}_0 \) is written as follows:

\[
\overline{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \begin{pmatrix}
\omega^{c_{n-1}+(-1)} & *_{n-1} & \cdots & * & * \\
0 & \omega^{c_{n-2}+(n-2)} & *_{n-2} & \cdots & * \\
0 & 0 & \omega^{c_{n-3}+(n-3)} & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega^{c_1+1} \\
0 & 0 & 0 & \cdots & \omega^{c_0}
\end{pmatrix}
\]

for some integers \( c_i \) satisfying some genericity conditions (cf. Definition 3.1.2). We also assume that \( \overline{\rho}_0 \) is maximally non-split, i.e., \( *_i \neq 0 \) for all \( i \in \{1, 2, \ldots, n-1\} \).

Our goal on the Galois side is to show that the Frobenius eigenvalues of certain potentially crystalline lifts of \( \overline{\rho}_0 \) determine the Fontaine–Laffaille parameters of \( \overline{\rho}_0 \), which parameterizes the wildly ramified part of \( \overline{\rho}_0 \). When the unramified part and the tamely ramified part of \( \overline{\rho}_0 \) are fixed, we define the Fontaine–Laffaille parameters via the Fontaine–Laffaille modules corresponding to \( \overline{\rho}_0 \) (cf. Definition 3.2.2). These parameters vary over the space of \( \frac{(n-1)(n-2)}{2} \) copies of the projective line \( \mathbb{P}^1(F) \), and we write \( \text{FL}^{(i_0,j_0)}(\overline{\rho}_0) \in \mathbb{P}^1(F) \) for each pair of integers \( (i_0, j_0) \) with \( 0 \leq j_0 < j_0 + 1 < i_0 \leq n-1 \). For each such pair \( (i_0, j_0) \), the Fontaine–Laffaille parameter \( \text{FL}^{(i_0,j_0)}(\overline{\rho}_0) \) is determined by the subquotient \( \overline{\rho}_{i_0,j_0} \) of \( \overline{\rho}_0 \) which is determined by the subset \( (e_{i_0}, e_{i_0-1}, \ldots, e_{j_0}) \) of \( \underline{e} \) (cf. Definition 3.1.2): in fact, we have the identity \( \text{FL}^{(i_0,j_0)}(\overline{\rho}_0) = \text{FL}^{(i_0-j_0+1,j_0)}(\overline{\rho}_{i_0,j_0}) \) (cf. Lemma 3.2.0).

Since potentially crystalline lifts of \( \overline{\rho}_0 \) are not Fontaine–Laffaille in general, we are no longer able to use Fontaine–Laffaille theory to study such lifts of \( \overline{\rho}_0 \); we use Breuil modules and strongly divisible modules for their lifts. It is obvious that any lift of \( \overline{\rho}_0 \) determines the Fontaine–Laffaille parameters,
but it is not obvious how one can explicitly visualize the information that determines \( \mathfrak{p}_0 \) in those lifts.

Motivated by the automorphic side, we believe that for each pair \((i_0, j_0)\) as above the Fontaine–Laffaille parameter \( \text{FL}_{i_0,j_0}(\mathfrak{p}_0) \) is determined by a certain product of Frobenius eigenvalues of the potentially crystalline lifts of \( \mathfrak{p}_0 \) with Hodge–Tate weights \( \{-n-1, \cdots, -1, 0\} \) and Galois type \( \bigoplus_{i=0}^{n-1} \omega^{k_i,j_0} \) where \( \omega \) is the Teichmüller lift of the fundamental character \( \omega \) of niveau 1 and

\[
(1.1.2) \quad k_{i,j}^{i_0,j_0} = \left\{ \begin{array}{l}
c_i + i_0 - j_0 - 1 & \text{for } i = i_0; \\
c_j - (i_0 - j_0 - 1) & \text{for } i = j_0; \\
c_i & \text{otherwise}
\end{array} \right.
\]

modulo \((p-1)\). Here, \( c_i \) are the integers determining the tamely ramified part of \( \mathfrak{p}_0 \) and our normalization of the Hodge–Tate weight of the cyclotomic character \( \varepsilon \) is \(-1\).

Our main result on the Galois side is the following:

**Theorem 1.1.3** (Theorem [3.7.3]). Fix \( i_0, j_0 \) \( \in \mathbb{Z} \) with \( 0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1 \). Assume that \( \mathfrak{p}_0 \) is generic (cf. Definition [3.0.3]) and that \( \mathfrak{p}_{i_0,j_0} \) is Fontaine–Laffaille generic (cf. Definition [3.2.7]), and let \( (\lambda_{n-1}^{i_0,j_0}, \lambda_{n-2}^{i_0,j_0}, \cdots, \lambda_0^{i_0,j_0}) \in (\mathcal{O}_E)^n \) be the Frobenius eigenvalues on the \((\omega^{k_{-1}-1}, \omega^{k_{-2}}, \cdots, \omega^{k_{i_0,j_0}-1})\)-isotypic components of \( \text{D}^{\sigma,n-1}_{\phi}((\rho_0)) \) where \( \rho_0 \) is a potentially crystalline lift of \( \mathfrak{p}_0 \) with Hodge–Tate weights \( \{-n-1, -(n-2), \cdots, -1, 0\} \) and Galois type \( \bigoplus_{i=0}^{n-1} \rho^{k_{i,j_0}} \).

Then the Fontaine–Laffaille parameter \( \text{FL}_{i_0,j_0}(\mathfrak{p}_0) \) associated to \( \mathfrak{p}_0 \) is computed as follows:

\[
\text{FL}_{i_0,j_0}(\mathfrak{p}_0) = \left[ 1 : \frac{\prod_{i=i_0}^{n-1} (\lambda_i^{i_0,j_0} - (i_0 - j_0 - 1))}{\prod_{i=j_0+1}^{n-1} \lambda_i^{i_0,j_0}} \right] \in \mathbb{P}^1(F).
\]

Note that by \( \mathfrak{p} \in F \) in the theorem above we mean the image of \( \ast \in \mathcal{O}_E \) under the natural surjection \( \mathcal{O}_E \to F \). We also note that \( \mathfrak{p}_{i_0,j_0} \) being Fontaine–Laffaille generic implies \( \text{FL}_{i_0,j_0}(\mathfrak{p}_0) \neq 0, \infty \) for all \( i_0, j_0 \) as in Theorem 1.1.3 but is a strictly stronger assumption if \( i_0 - j_0 \geq 3 \).

Let us briefly discuss our strategy for the proof of Theorem 1.1.3. Recall that the Fontaine–Laffaille parameter \( \text{FL}_{i,j}^{i_0,j_0}(\mathfrak{p}_0) \) is defined in terms of the Fontaine–Laffaille module corresponding to \( \mathfrak{p}_0 \). Thus we need to describe \( \text{FL}_{i,j}^{i_0,j_0}(\mathfrak{p}_0) \) by the data of the Breuil modules of inertial type \( \bigoplus_{i=0}^{n-1} \omega^{k_{i,j_0}} \) corresponding to \( \mathfrak{p}_0 \), and we do this via étale \( \phi \)-modules, which requires classification of such Breuil modules. If the filtration of the Breuil modules is of a certain shape, then a certain product of the Frobenius eigenvalues of the Breuil modules determines a Fontaine–Laffaille parameter (cf. Proposition 3.4.3). In order to get such a filtration, we need to assume that \( \mathfrak{p}_{i_0,j_0} \) is Fontaine–Laffaille generic (cf. Definition 3.2.7). Then we determine the structure of the filtration of the strongly divisible modules lifting the Breuil modules by direct computation, which immediately gives enough properties of Frobenius eigenvalues of the potentially crystalline representations we consider. But this whole process is subtle for general \( i_0, j_0 \). To resolve this issue we prove that any potentially crystalline lift of \( \mathfrak{p}_0 \) with Hodge–Tate weights \( \{-n-1, -(n-2), \cdots, 0\} \) and Galois type \( \bigoplus_{i=0}^{n-1} \omega^{k_{i,j_0}} \) has a potentially crystalline subquotient \( \rho_{i_0,j_0} \) of Hodge–Tate weights \( \{-i_0, \cdots, -j_0\} \) and of Galois type \( \bigoplus_{i=j_0}^{i_0} \omega^{k_{i,j_0}} \) lifting \( \mathfrak{p}_{i_0,j_0} \). More precisely,

**Theorem 1.1.4** (Corollary 3.6.4). Every potentially crystalline lift \( \rho_0 \) of \( \mathfrak{p}_0 \) with Hodge–Tate weights \( \{-n-1, -(n-2), \cdots, 0\} \) and Galois type \( \bigoplus_{i=0}^{n-1} \omega^{k_{i,j_0}} \) is a successive extension

\[
\rho_0 \cong \begin{pmatrix}
\rho_{n-1,n-1} & \cdots & * & * & * & \cdots & * \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{i_0+1,i_0+1} & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{i_0,j_0} & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\rho_{j_0-1,j_0-1} & * & \cdots & * \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\rho_{0,0} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
where

- for $n - 1 \geq i > i_0$ and $j_0 > i \geq 0$, $\rho_{i,i}$ is a 1-dimensional potentially crystalline lift of $\tilde{\pi}_{i,i}$ with Hodge–Tate weight $-i$ and Galois type $\tilde{\omega}^{k_{i,i}}$;
- $\rho_{i_0,j_0}$ is a $(i_0 - j_0 + 1)$-dimensional potentially crystalline lift of $\tilde{\pi}_{i_0,j_0}$ with Hodge–Tate weights $\{-i_0, -i_0 + 1, \ldots, -j_0\}$ and Galois type $\bigoplus_{i=j_0}^{i_0} \tilde{\omega}^{k_{i,j_0}}$.

Note that we actually prove the niveau $f$ version of Theorem 1.1.3 since it adds only little more extra work (cf. Corollary 3.6.4).

The representation $\rho_{i_0,j_0} \otimes \varepsilon^{-j_0}$ is a $(i_0 - j_0 + 1)$-dimensional potentially crystalline lift of $\tilde{\pi}_{i_0,j_0}$ with Hodge–Tate weights $\{-i_0 - j_0, -i_0 - j_0 - 1, \ldots, 0\}$ and Galois type $\bigoplus_{i=j_0}^{i_0} \tilde{\omega}^{k_{i,j_0}}$, so that, by Theorem 1.1.3, Theorem 1.1.4 reduces to the case $(i_0, j_0) = (n - 1, 0)$: we prove Theorem 1.1.4 when $(i_0, j_0) = (n - 1, 0)$, and then use the fact $\text{FL}_{i_0,j_0}^{i_0,j_0}(\pi_{0}) = \text{FL}_{i_0,j_0}^{i_0,j_0+1}(\pi_{i_0,j_0})$ to get the result for general $i_0, j_0$.

The Weil–Deligne representation $\text{WD}(\rho_0)$ associated to $\rho_0$ (as in Theorem 1.1.3) contains those Frobenius eigenvalues of $\rho_0$. We then use the classical local Langlands correspondence for $\text{GL}_n$ to transport the Frobenius eigenvalues of $\rho_0$ (and so the Fontaine–Laffaille parameters of $\pi_0$) as well by Theorem 1.1.3 to the automorphic side (cf. Corollary 3.7.3).

1.2. Local automorphic side. We start by introducing the Jacobi sum operators in characteristic $p$. Let $T$ (resp. $B$) be the maximal torus (resp. the maximal Borel subgroup) consisting of diagonal matrices (resp. of upper-triangular matrices) of $\text{GL}_n$. We let $X(T) := \text{Hom}(T, G_m)$ be the group of characters of $T$ and $\Phi^+$ be the set of positive roots with respect to $(B, T)$. We define $\epsilon_i \in X(T)$ as the projection of $T \cong G_m^n$ onto the $i$-th factor. Then the elements $\{\epsilon_i \mid 1 \leq i \leq n\}$ forms a $\mathbb{Z}$-basis of the free abelian group $X(T)$. We will use the notation $(d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n$ for the element $\sum_{k=1}^n d_k \epsilon_k \in X(T)$. Note that the group of characters of the finite group $T(F_p) \cong (F_p^\times)^n$ can be identified with $X(T)/(p - 1)X(T)$, and therefore we sometimes abuse the notation $(d_1, d_2, \ldots, d_n)$ for its image in $X(T)/(p - 1)X(T)$. We define $\Delta := \{\alpha_k := \epsilon_k - \epsilon_{k+1} \mid 1 \leq k \leq n - 1\} \subseteq \Phi^+$ as the set of positive simple roots. Note that we write $s_k$ for the reflection of the simple root $\alpha_k$. For an element $w$ in the Weyl group $W$, we define $\Phi^+_w = \{\alpha \in \Phi^+ \mid w(\alpha) \in -\Phi^+\} \subseteq \Phi^+$ and $U_w = \prod_{\alpha \in \Phi^+_w} U_{\alpha}$, where $U_{\alpha}$ is a subgroup of $U$ whose only non-zero off-diagonal entry corresponds to $\alpha$. Note in particular that $\Phi^+_w = \Phi^+_{w_0}$, where $w_0$ is the longest element in $W$. For $w \in W$ and for a tuple of integers $k = (k_{\alpha})_{\alpha \in \Phi^+_w} \in \{0, 1, \ldots, p - 1\}^{\mid \Phi^+_w \mid}$, we define the Jacobi sum operator

$$S_{k,w} := \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi^+_w} A_{\alpha}^{k_{\alpha}} \right) A \cdot w \in F_p[\text{GL}_n(F_p)]$$

where $A_{\alpha}$ is the entry of $A$ corresponding to $\alpha \in \Phi^+_w$. In Section 4 we establish many technical results, both conceptual and computational, around these Jacobi sum operators. The use of these Jacobi sum operators can be traced back to at least [CL76], and are widely used for $\text{GL}_2$ in [BL22] and [Hu10] for instance. But systematic computation with these operators seems to be limited to $\text{GL}_2$ or $\text{GL}_3$. In this paper, we need to do some specific but technical computation on some special Jacobi sum operators for $\text{GL}_n(F_p)$, which is enough for our application to Theorem 1.1.4 below.

By the discussion on the local Galois side, our target on the local automorphic side is to capture the Frobenius eigenvalues coming from the local Galois side. By the classical local Langlands correspondence, the Frobenius eigenvalues of $\rho_0$ are transported to the unramified part of $\chi$ in the tamely ramified principal series $\text{Ind}^G_{B(Q_p)}(\chi)$ corresponding to the Weil–Deligne representation $\text{WD}(\rho_0)$ attached to $\rho_0$ in Theorem 1.1.4, and it is standard to use $U^p$-operators to capture the information in the unramified part of $\chi$.

The normalizer of the Iwahori subgroup $I$ in $\text{GL}_n(Q_p)$ is cyclic modulo $I$, and this cyclic quotient group is generated by an element $\Xi_n \in \text{GL}_n(Q_p)$ that is explicitly defined in (4.7.7). One of our goals is to translate the eigenvalue of $U^p$-operators into the action of $\Xi_n$ on the space $(\text{Ind}^G_{B(Q_p)}(\chi))_{\text{GL}_n(Z_p)}$. 
This is firstly done for $\text{GL}_2(\mathbb{Q}_p)$ in [BD14], and then the method is generalized to $\text{GL}_3(\mathbb{Q}_p)$ in the ordinary case by [HLM]. Both [BD14] and [HLM] need a pair of group algebra operators: for instance, group algebra operators $\hat{S}, \hat{S}' \in \mathbb{Q}_p[\text{GL}_3(\mathbb{Q}_p)]$ are defined in [HLM] and the authors prove an intertwining identity of the form $\hat{S}' \circ \hat{S} = cS'$ on a certain $I(1)$-fixed subspace of $\text{Ind}_{\hat{B}}^{\text{GL}_3(\mathbb{Q}_p)} \chi$ with $\chi$ assumed to be tamely ramified, where $I(1)$ is the maximal pro-$p$ subgroup of $I$. Here, the constant $c \in \mathcal{O}_E^\times$ captures the eigenvalues of $U_p$-operators. This is the first technical point on the local automorphic side, and we generalize the results in [BD14] and [HLM] by the following theorem.

For an $n$-tuple of integers $(a_{n-1}, a_{n-2}, \cdots, a_0) \in \mathbb{Z}^n$, we write $\mathcal{S}_n$ and $\mathcal{S}'_n$ for $\mathcal{S}_{\sum a_0}$ with $\hat{k}_1 = (k_{1,i,j})$ and $\mathcal{S}_{\sum a_0}$ with $\hat{k}_1 = (k_{1,i,j}')$ respectively, where $k_{1,i+1} = [a_0 - a_{n-i}] + n - 2, k_{1,i+1}' = [a_{n-i} - a_0]$ for $1 \leq i \leq n - 1$, and $k_{1,i} = k_{1,i}' = 0$ otherwise. Here, $(i,j)$ is the entry corresponding to $\alpha$ if $\alpha = \epsilon_i - \epsilon_j \in \Phi^+$ and by $[x]_1$ for $x \in \mathbb{Z}$ we mean the integer in $[0, p - 1)$ such that $x \equiv [x]_1$ modulo $(p - 1)$. We define $\mathcal{S}_n \in \mathbb{Z}_p[\text{GL}_n(\mathbb{Z}_p)]$ (resp. $\mathcal{S}'_n \in \mathbb{Z}_p[\text{GL}_n(\mathbb{Z}_p)]$) by taking the Teichmüller lifts of the coefficients and the entries of the matrices of $\mathcal{S}_n \in \mathbb{F}_p[\text{GL}_n(\mathbb{F}_p)]$ (resp. $\mathcal{S}'_n \in \mathbb{F}_p[\text{GL}_n(\mathbb{F}_p)]$).

We use the notation $\bullet$ for the composition of maps or group operators to distinguish from the notation $\circ$ for a $\mathcal{O}_E$-lattice inside a representation.

**Theorem 1.2.2 (Theorem 4.3).** Assume that the $n$-tuple of integers $(a_{n-1}, a_{n-2}, \cdots, a_0)$ is $n$-generic in the lowest alcove (cf. Definition 4.1.1), and let

$$\Pi_p = \text{Ind}_{\hat{B}}^{\text{GL}_n(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \cdots \otimes \chi_{n-2} \otimes \chi_{n-1} \otimes \chi_0)$$

be a tamely ramified principal series representation with the smooth characters $\chi_k : \mathbb{Q}_p^\times \to E^\times$ satisfying $\chi_k|_{\mathbb{Z}_p^\times} = \hat{c}^{\alpha_k}$ for $0 \leq k \leq n - 1$.

On the 1-dimensional subspace $\Pi_p(I^{(1)}(a_1, a_2, \cdots, a_{n-1}, a_0))$ we have the identity:

$$\mathcal{S}_n \cdot (\Xi_n)^{n-2} = p^{n-2} \kappa_n \left( \prod_{k=1}^{n-2} \chi_k(p) \right) \mathcal{S}_n$$

for $\kappa_n \in \mathbb{Z}_p^\times$ satisfying $\kappa_n \equiv \varepsilon^* \cdot \rho_n(a_{n-1}, \cdots, a_0) \mod (\varpi_E)$ where

$$\varepsilon^* = \prod_{k=1}^{n-2} (-1)^{a_0 - a_k}$$

and

$$\rho_n(a_{n-1}, \cdots, a_0) = \prod_{k=1}^{n-2} \prod_{j=0}^{a_k - a_{n-1} + j} a_k - a_{n-1} + j \in \mathbb{Z}_p^\times.$$

In fact, there are many identities similar to the one in (1.2.3) for each operator $U_i$ for $1 \leq i \leq n - 1$ that is defined in (4.7.2), but it is clear from the proof of Theorem 1.2.2 in Section 4.7 that we need to choose $U^{n-2}$ for the $U_p$-operator acting on $\Pi_p(I^{(1)}(a_1, a_2, \cdots, a_{n-1}, a_0))$, motivated from the local Galois side via Theorem 1.4.3. The crucial point here is that the constant $p^{n-2} \kappa_n \left( \prod_{k=1}^{n-2} \chi_k(p) \right)$, which is closely related to $\text{FL}_{n-1,0}(\mathbb{F}_p)$ via Theorem 1.4.3 and classical local Langlands correspondence, should lie in $\mathcal{O}_E^\times$ for each $\Pi_p$ appearing in our application of Theorem 1.2.2 to Theorem 1.4.1.

The next step is to consider the mod $p$ reduction of the identity (1.2.3), which is effective to capture $p^{n-2} \prod_{k=1}^{n-2} \chi_k(p) \mod (\varpi_E)$ only if $\mathcal{S}_n \not\equiv 0 \mod (\varpi_E)$ for $\varpi \in \Pi_p(I^{(1)}(a_1, a_2, \cdots, a_{n-1}, a_0))$. It turns out that this non-vanishing property is very technical to prove for general $\text{GL}_n(\mathbb{Q}_p)$. Before we state our non-vanishing result, we fix a little more notation: let

$$\mu^* := (a_{n-1} - n + 2, a_{n-2}, \cdots, a_1, a_0 + n - 2);$$
$$\mu_1 := (a_1, a_2, \cdots, a_{n-3}, a_{n-2}, a_{n-1}, a_0);$$
$$\mu_1' := (a_{n-1}, a_0, a_1, a_2, \cdots, a_{n-3}, a_{n-2})$$
be three characters of $T(F_p)$, and write $\pi_1$ and $\pi'_1$ for two characteristic $p$ principal series induced by the characters $\mu_1$ and $\mu'_1$ respectively (cf. \ref{1.2.1}). Note that we can attach an irreducible representation $F(\lambda)$ of $GL_n(F_p)$ to each $\lambda \in X(T)/(p - 1)X(T)$ satisfying some regular conditions (cf. the beginning of Section 3). If we assume that $(a_{n - 1}, \cdots, a_0) \in Z$ is $n$-generic in the lowest alcove, the characters $\mu^\ast$, $\mu_1$ and $\mu'_1$ do satisfy the regular condition and thus we have three irreducible representations $F(\mu^\ast)$, $F(\mu_1)$ and $F(\mu'_1)$ of $GL_n(F_p)$. There is a unique quotient $V$ (resp. $V'$) (up to isomorphism) of $\pi_1$ (resp. of $\pi'_1$) whose socle is isomorphic to $F(\mu^\ast)$, since $F(\mu^\ast)$ has multiplicity one in $\pi_1$ (resp. in $\pi'_1$) by Theorem 4.2.6.

We are now ready to state the non-vanishing theorem.

**Theorem 1.2.4** (Corollary 4.2.7). Assume that the $n$-tuple of integers $(a_{n - 1}, a_{n - 2}, \cdots, a_0)$ is $2n$-generic in the lowest alcove (cf. Definition 1.1.1).

Then we have

$$0 \neq S_n \left(V^{U(F_p),\mu_1}\right) \subseteq V \text{ and } 0 \neq S'_n \left(V'^{U(F_p),\mu'_1}\right) \subseteq V'.$$

The definition of $\mu_1$, $\mu'_1$ and $\mu^\ast$ is motivated by our application of Theorem 1.2.4 to Theorem 1.4.1 and is closely related to the Galois types we choose in Theorem 1.3. We emphasize that, technically speaking, it is crucial that $F(\mu^\ast)$ has multiplicity one in $\pi_1$ and $\pi'_1$. The proof of Theorem 1.2.4 is technical and makes full use of the results in Sections 1.1, 1.5, and 1.6.

### 1.3. Weight elimination and automorphy of a Serre weight.

The weight part of Serre’s conjecture is considered as a first step towards mod $p$ Langlands program, since it gives a description of the socles of $\Pi(\overline{\tau})|_{GL_n(Z_p)}$ up to possible multiplicities. Substantial progress has been made for the groups $GL_2(O_K)$, where $O_K$ is the ring of integers of a finite extension $K$ of $Q_p$ (BDJ10, GHH1, GK14, GLS14, GLS15). For groups in higher semisimple rank, we also have a detailed description. (See EGH12, HLM, LMP, MP, LLMHM for $GL_3$; Her09, GG10, BLCG, LLL, GHS for general $n$.)

Weight elimination results are significant for the proof of our main global application, Theorem 1.4.1. For the purpose of this introduction, we quickly review some notation. Let $F^\ast$ be the maximal totally real subfield of a CM field $F$, and assume that $p$ splits completely in $F$. Fix a place $w$ of $F$ above $p$ and set $v := w|_{F^\ast}$. We assume that $\overline{\tau}$ is automorphic: this means that there exist a totally definite unitary group $G_n$ defined over $F^\ast$ that is an outer form of $GL_n/F$ and split at places above $p$, an integral model $G_n$ of $G_n$ such that $G_n \times O_{F^\ast}$ is reductive if $v' \in \{\text{finite place of } F^\ast \text{ that splits in } F\}$, a compact open subgroup $U = G_n(O_{F^\ast}) \times U^\ast \subseteq G_n(O_{F^\ast}) \times G_n(A_{F^\ast}^{\infty})$ that is sufficiently small and unramified above $p$, a Serre weight $V = \otimes_{v \neq p} V_v$ that is an irreducible smooth $\overline{F}_p$-representation of $G_n(O_{F^\ast,p})$, and a maximal ideal $m_{\overline{\tau}}$ associated to $\overline{\tau}$ in the Hecke algebra acting on the space $S(U, V)$ of mod $p$ algebraic automorphic forms such that

\[ \langle S(U, V)|m_{\overline{\tau}} \rangle \neq 0. \]

We write $W(\overline{\tau})$ for the set of Serre weights $V$ satisfying (1.3.1) for some $U$, and $W_w(\overline{\tau})$ for the set of local Serre weights $V_w$ that is irreducible smooth representations of $G_n(O_{F^\ast}) \cong GL_n(O_{F_w}) \cong GL_n(Z_p)$, such that $V_w \otimes (\otimes_{v \neq v'} V_{v'}) \in W(\overline{\tau})$ for an irreducible smooth representation $\otimes_{v \neq v'} V_{v'}$ of $\prod_{v \neq v'} G_n(O_{F^\ast,v})$. The local Serre weights $V_w$ have an explicit description as representations of $GL_n(F_p)$: there exists a $p$-restricted (i.e. $0 \leq a_{i - 1} - a_i \leq p - 1$ for all $1 \leq i \leq n - 1$) weight $\overline{\omega} := (a_{n - 1}, a_{n - 2}, \cdots, a_0) \in X(T)$ such that $F(\overline{\omega}) \cong V_w$ where $F(\overline{\omega})$ is the irreducible socle of the dual Weyl module associated to $\overline{\omega}$ (cf. Section 5.2 as well as the beginning of Section 4).

Assume that $\overline{\tau}_{\text{Gal}}(Q_p/F_w) \cong \widehat{\rho}_0$, where $\widehat{\rho}_0$ is defined as in (1.1.1). We define certain characters $\mu^{\square}$ and $\mu^{[\lambda_{i_1},j_1]}$ of $T(F_p)$ and principal series $\pi_{\lambda_{i_1},j_1}$ of $GL_n(F_p)$ at the beginning of Section 5.3. Our main conjecture for weight elimination is
Conjecture 1.3.2 (Conjecture [LLHM]). Assume that $\mathcal{P}_{i_0,j_0}$ is Fontaine–Laffaille generic and that $\mu^\square,i_1,j_1$ is $2n$-generic. Then we have an inclusion

$$W_w(\mathcal{T}) \cap \text{JH}(\pi^{i_1,j_1}) \subseteq \{ F(\mu^\square)^\vee, F(\mu^\square,i_1,j_1)^\vee \}. \tag{1.3.3}$$

We emphasize that the condition $\mathcal{P}_{i_0,j_0}$ is Fontaine–Laffaille generic is crucial in Conjecture 1.3.2. For example, if $n = 4$ and $(i_0,j_0) = (3,0)$ and we assume merely $\text{FI}_3^{[3]}(\mathcal{P}_0) \neq 0$, $\infty$ (which is strictly weaker than Fontaine–Laffaille generic), then we expect that an extra Serre weight can possibly appear in $W_w(\mathcal{T}) \cap \text{JH}(\pi^{i_1,j_1})^\vee$.

The Conjecture 1.3.2 is motivated by the proof of Theorem 1.1.3 and the theory of shape in [LLHM]. The special case $n = 3$ of Conjecture 1.3.2 was firstly proven in [HLM] and can also be deduced from the computations of Galois deformation rings in [LLHM].

Remark 1.3.4. In an earlier version of this paper, we prove Conjecture 1.3.2 completely in his forthcoming paper [LeH]. Therefore, Conjecture 1.3.2 becomes a theorem based on the results in [LeH].

Finally, we also show the automorphy of the Serre weight $F(\mu^\square)^\vee$. In other words,

$$F(\mu^\square)^\vee \in W_w(\mathcal{T}) \cap \text{JH}(\pi^{i_1,j_1})^\vee. \tag{1.3.5}$$

1.4. Mod $p$ local-global compatibility. We now give a sketch of our ideas towards our main results on mod $p$ local-global compatibility. As discussed at the beginning of this introduction, we prove that $\Pi(\mathcal{T})$ determines the ordinary representation $\mathcal{T}_0$. We first recall the definition of $\Pi(\mathcal{T})$.

Keep the notation of the previous sections, and write $d_i = -c_{n-1-i}$ for all $0 \leq i \leq n - 1$, with $c_i$ as in (1.1.1). We fix a place $w$ of $F$ above $p$ and write $v := w|_{F^+}$, and we let $\mathcal{T} : G_F \to \text{GL}_n(F)$ be an irreducible automorphic representation, of a Serre weight $V \cong \bigotimes_{\nu} V_{\nu}$ (cf. Section 1.3), with $\mathcal{T}|_{G_{F_w}} \cong \mathcal{T}_0$.

Let $V' := \bigotimes_{\nu} V_{\nu}$ and set $S(U^v, V') := \lim_{\longrightarrow} S(U^v \cdot U_{v', V'})$ where the direct limit runs over compact open subgroups $U_v \subset G_{F_v}$ of $G_{F_v}$. This space $S(U^v, V')$ has a natural smooth action of $G_n(F_w^+ \cap \mathcal{T}_0 \cong \text{GL}_n(F_w) \cong \text{GL}_n(Q_p)$ by right translation as well as an action of a Hecke algebra that commutes with the action of $G_n(F_v^+)$.

We define

$$\Pi(\mathcal{T}) := S(U^v, V') |_{\mathfrak{m_{\mathcal{T}}}}$$

where $\mathfrak{m_{\mathcal{T}}}$ is the maximal ideal of the Hecke algebra associated to $\mathcal{T}$. In the spirit of [Linc], this is a candidate on the automorphic side for a mod $p$ Langlands correspondence corresponding to $\mathcal{T}_0$. Note that the definition of $\Pi(\mathcal{T})$ relies on $U^v$ and $V'$ as well as choice of a Hecke algebra, but we suppress them in the notation.

Fix $n - 1 \geq i_0 > j_0 + 1 > j_0 \geq 0$, and define $i_1$ and $j_1$ by the equation $i_1 + i_0 = j_1 + j_0 = n - 1$. Let $P_{i_1,j_1} \supset B$ be the standard parabolic subgroup of $\text{GL}_n(Q_p)$ corresponding to the subset $\{ \alpha_k \mid n - j_1 \leq k \leq n - 1 - i_1 \}$ of the set $\Delta$ of positive simple roots with respect to $(B,T)$. In particular, we have $P_{0,n-1} = \text{GL}_n$. We denote the unipotent radical of $P_{i_1,j_1}$ by $N_{i_1,j_1}$. We also denote the opposite parabolic subgroup and its unipotent radical by $P_{i_1,j_1}^-$ and $N_{i_1,j_1}^-$. We fix a standard choice of Levi subgroup $L_{i_1,j_1} := P_{i_1,j_1} \cap P_{i_1,j_1}^-$. We can embeds $\text{GL}_{j_1-i_1+1}$ into $\text{GL}_n$ with image denoted by $G_{i_1,j_1}$, such that $L_{i_1,j_1} = G_{i_1,j_1}T$.

Recall that $S_n$ and $S'_n$ are completely determined by fixing the data $n$ and $(a_{n-1}, \cdots, a_0)$. We define $S_{i_1,j_1} \in F_p[\text{GL}_{j_1-i_1+1}(F_p)]$ (resp. $S'_{i_1,j_1} \in F_p[\text{GL}_{j_1-i_1+1}(F_p)]$) by replacing $n$ and $(a_{n-1}, \cdots, a_1, a_0)$ by $j_1 - i_1 + 1$ and $(b_j, j + j_1 - i_1 - 1, b_{j_1-1}, \cdots, b_{j_1-i_1+1}, b_{i_1-j_1-i_1+1} = j_1 - i_1 + 1)$, and then we define $S^{i_1,j_1}$ (resp. $S'^{i_1,j_1}$) to be the image of $S_{i_1,j_1}$ (resp. $S'_{i_1,j_1}$) in $F_p[\text{GL}_n(F_p)]$ via the embedding $\text{GL}_{j_1-i_1+1} \cong G_{i_1,j_1} \hookrightarrow \text{GL}_n$. 


We now state the main results in this paper.

**Theorem 1.4.1** (Theorem 6.7.6). Fix a pair of integers \((i_0, j_0)\) satisfying \(0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1\), and let \(\pi : G_F \rightarrow \text{GL}_n(F)\) be an irreducible automorphic representation with \(\tilde{\pi}|_{G_{F_v}} \cong \tilde{\pi}_0\). Assume that

1. \(\mu^{[n],i,j}\) is 2n-generic;
2. \(\tilde{\pi}_{i_0,j_0}\) is Fontaine–Laffaille generic.

Assume further that

\[
\{F(\mu^{[n]}), \ldots, F(\mu^{[n],i,j})\} \subseteq W_w(\pi) \cap JH((\pi^{[n],i,j})^V) \subseteq \{F(\mu^{[n]}), F(\mu^{[n],i,j})^V\}.
\]

Then there exists a primitive vector (c.f. Definition 5.7.2) in \(\Pi((\pi^{[n],i,j})^V)\). Moreover, for each primitive vector \(v^{i_1,j_1} \in \Pi((\pi^{[n],i,j})^{V^*})\), there exists a connected (c.f. Definition 5.7.2) vector \(\tilde{v}^{i_1,j_1,r} \in \Pi((\pi^{[n],i,j})^{V^*})\) to \(v^{i_1,j_1}\) such that \(S^{i_1,j_1,r}v^{i_1,j_1} \neq 0\) and

\[
\tilde{S}^{i_1,j_1,r}v^{i_1,j_1} = \varepsilon^{i_1,j_1}P_{i_1,j_1}(\delta_{i_1,j_1} - b_0) \cdot \tilde{E}_{i_1,j_1}^{\tilde{v}^{i_1,j_1},r} \cdot S^{i_1,j_1,r}v^{i_1,j_1}
\]

where

\[
\varepsilon^{i_1,j_1} = \prod_{k=i_1+1}^{j_1-1} (-1)^{b_{k-j_1+i_1+1}}
\]

and

\[
P_{i_1,j_1}(\delta_{i_1,j_1} - b_0) = \prod_{k=i_1+1}^{j_1-1} \prod_{j=1}^{j_1-i_1-1} \frac{b_k - b_{j_1-j}}{b_{i_1} - b_k-j} \in \mathbb{Z}_p^\times.
\]

Note that the conditions in 1.4.2 can be removed under some standard Taylor–Wiles conditions (c.f. Remark 1.3.1 and 1.3.3).

Theorem 1.4.1 relies on the choice of a principal series type (the niveau 1 Galois type \(\bigoplus_{i=0}^{n-1} k^{i_0,j_0})\). But this choice is somehow the unique one that could possibly make our strategy of the proof of Theorem 1.4.1 work.

Our proof of Theorem 1.4.1 is a bit different from the one of [HLM]: there are at least two new inputs. Firstly, in [HLM], they require the freeness of a certain module over a Hecke algebra, proved by patching argument, to kill a certain shadow weight (which corresponds to the weight \(F(\mu^{[n],i,j})\) in our context if we fix \((i_0,j_0)\)). In our proof, we use purely modular representation theoretic arguments. Secondly, we cannot apply Theorem 1.2.2 and Theorem 1.2.4 directly to our local global-compatibility for general \((i_1,j_1)\). We need intermediate steps, for example Proposition 5.6.3 to use the results of Theorem 1.2.3.

We quickly review the main strategy of the proof of Theorem 1.4.1. The idea of the proof is essentially the combination of the proof of the \((i_0,j_0)\) case and the fact the general \((i_0,j_0)\)-case comes from parabolic induction, whose accurate meaning will be clear in the following. We let \(V\) be a lift of \(V\) defined in 5.7.1, assuming that each local factor of \(V\) is in the lowest alcove. Then we consider

\[
\tilde{\Pi}(\pi) := S(U^\nu, \tilde{V}^\nu)_{\varpi^\nu}.
\]

Note that \(\tilde{\Pi}(\pi) \otimes_{O_E} E\) is a smooth \(E\)-representation of \(\text{GL}_n(O_p)\) which also depends on \(U^\nu\) and \(\tilde{V}^\nu\), but we omit them from the notation.

We consider the natural surjection onto the coinvariant space

\[
\text{Pr} : \tilde{\Pi}(\pi) \otimes_{O_E} E \twoheadrightarrow (\tilde{\Pi}(\pi) \otimes_{O_E} E)_{N_{i_1,j_1}(O_p)}.
\]

Now we fix a pair of vectors \(v^{i_1,j_1} \in \Pi((\pi^{[n],i,j})^{V^*})\) and \(\tilde{v}^{i_1,j_1,r} \in \Pi((\pi^{[n],i,j})^{V^*})\) that have lifts \(\tilde{v}^{i_1,j_1} \in \Pi((\pi^{[n],i,j})^{V^*})\) and \(\tilde{\tilde{v}}^{i_1,j_1,r} \in \Pi((\pi^{[n],i,j})^{V^*})\), respectively, such that

\[
(\text{GL}_n(O_p)\tilde{v}^{i_1,j_1})_E = (\text{GL}_n(O_p)\tilde{\tilde{v}}^{i_1,j_1,r})_E
\]

and

\[
\text{Pr}(\tilde{v}^{i_1,j_1,r}) = \Xi_{i_1,j_1} \cdot \text{Pr}(\tilde{v}^{i_1,j_1}).
\]
where \( \langle \text{GL}_n(Z_p)^{+} \rangle_E \) is the \( E \)-subrepresentation generated by \( * \) in \( \tilde{\Pi}^{\pi}(\tau) \otimes \mathcal{O}_E E \) as a representation of \( \text{GL}_n(Z_p) \) and \( \Xi_{i_1, j_1} \in \text{GL}_{j_1 - i_1 + 1}(Q_p) \mapsto L_{i_1, j_1}(Q_p) \) is defined in (5.7.2). Note in particular that \( \Xi_{i_1, j_1} \) lies in the normalizer of the standard Iwahori subgroup of \( \text{GL}_{j_1 - i_1 + 1}(Q_p) \) in \( \text{GL}_{j_1 - i_1 + 1}(Q_p) \).

We further define the following subspaces:

\[
\begin{align*}
\Pi^{i_1, j_1} &:= \langle \text{GL}_n(Q_p)^{\pi}(V_{i_1, j_1}) \rangle E = \langle \text{GL}_n(Q_p)^{\pi}(V_{i_1, j_1}) \rangle E \subseteq \tilde{\Pi}^{\pi}(\tau) \otimes \mathcal{O}_E E; \\
\tilde{\pi}^{i_1, j_1} &:= \langle \text{GL}_n(Z_p)^{\pi}(V_{i_1, j_1}) \rangle E = \langle \text{GL}_n(Z_p)^{\pi}(V_{i_1, j_1}) \rangle E \subseteq \Pi^{i_1, j_1}; \\
(\tilde{\pi}^{i_1, j_1})^0 &:= \tilde{\pi}^{i_1, j_1} \cap \tilde{\Pi}^{\pi}(\tau); \\
(\tilde{\pi}^{i_1, j_1})^0 &:= (\tilde{\pi}^{i_1, j_1})^0 \otimes \mathcal{O}_E F.
\end{align*}
\]

We need to assume that \( \pi^{i_1, j_1} \) is primitive (c.f. Definition 5.7.3), which is a technical condition ensuring that we can pick the lift \( \tilde{\pi}^{i_1, j_1} \) such that \( \Pi^{i_1, j_1} \) is an irreducible smooth representation of \( \text{GL}_n(Q_p) \). We show that primitive vectors always exist (and thus this technical assumption is harmless). Note that \( \Pi^{i_1, j_1} \) is semisimple with finite length without this assumption.

The strategy of the proof of Theorem 1.4.1 is summarized in the following two diagrams:

\[
\begin{array}{cccccc}
(\tilde{\pi}^{i_1, j_1})^0 & \hookrightarrow & \Pi^{i_1, j_1} & \twoheadrightarrow & \text{Pr}(\Pi^{i_1, j_1}) \\
\pi^{i_1, j_1} & \hookrightarrow & \tilde{\pi}^{i_1, j_1} & \twoheadrightarrow & \text{Pr}(\tilde{\pi}^{i_1, j_1}) \\
\end{array}
\]

and

\[
\begin{array}{cccccc}
\mathbf{F}[S^{i_1, j_1} \pi^{i_1, j_1}] & = & \mathbf{F}[S^{i_1, j_1, f} \pi^{i_1, j_1, f}] & \twoheadrightarrow & \Xi_{i_1, j_1} \\
\mathbf{F}[S^{i_1, j_1} \pi^{i_1, j_1} \pi^{i_1, j_1, f}] & \twoheadrightarrow & \Xi_{i_1, j_1} \\
\end{array}
\]

In the second diagram, by \( \mathbf{F}[v] \) for a non-zero vector \( v \) in a \( \mathbf{F} \)-vector space we mean the \( \mathbf{F} \)-line generated by \( v \). Theorem 1.4.1 says that \( S^{i_1, j_1} \pi^{i_1, j_1} \) and \( S^{i_1, j_1, f} \pi^{i_1, j_1, f} \) are non-zero and differ by a scalar in \( \mathbf{F}^\times \) that captures the Fontaine–Laffaille parameter \( \text{FL}^{l_0, j_0}(\tau_0) \).

We note that the two leftmost diagonal arrows in the second picture above are where we apply Theorem 1.2.4 together with Proposition 5.6.13. The rightmost vertical arrow in the second picture above is where we apply Theorem 1.2.4 inside the smooth Jacquet module \( \text{Pr}(\Pi^{i_1, j_1}) \) seen as a representation of \( \text{GL}_{j_1 - i_1 + 1}(Q_p) \mapsto L_{i_1, j_1}(Q_p) \).

One of the crucial points of the proof is that we deduce from Morita theory (recalled in Section 5.2) that there exists an \( \mathcal{O}_E \)-representation \( (\tilde{\pi}^{i_1, j_1})^0 \circ \text{L}_{i_1, j_1}(F_p) \) such that

\[
(\tilde{\pi}^{i_1, j_1})^0 = \text{Ind}_{\text{P}^{i_1, j_1}(F_p)}^{\text{GL}_n(F_p)}(\tilde{\pi}^{i_1, j_1, -1}, L_{i_1, j_1})^0.
\]

Namely, the lattice \( (\tilde{\pi}^{i_1, j_1})^0 \) in the principal series type \( \tilde{\pi}^{i_1, j_1} \) comes from the parabolic induction from \( L_{i_1, j_1}(F_p) \). This fact essentially follows from the assumption (1.4.2) together with some elementary arguments from Morita theory in Section 5.2.

**Corollary 1.4.3.** Keep the notation of Theorem 1.4.1 and assume that each assumption in Theorem 1.4.1 holds for all \( (i_0, j_0) \) such that \( 0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1 \).

Then the Galois representation \( \mathcal{P}_0 \) is determined by \( \Pi(\tau) \) in the sense of Remark 5.7.30.

Remark 5.7.20 roughly says the following: although we give an explicit strategy to recover the invariants \( \text{FL}^{l_0, j_0}(\tau_0) \) inside the representation \( \Pi(\tau) \), this strategy does not say that \( \text{FL}^{l_0, j_0}(\tau_0) \) can be recovered from an explicit formula which depends only the structure of \( \Pi(\tau) \) as a smooth admissible representation of \( \text{GL}_n(Q_p) \). In fact, \( \Pi(\tau) \) has many natural restrictions coming from its definition. For example, \( \Pi(\tau) \) admits natural lifts in characteristic zero (smooth or Banach) that satisfy various conditions. Our strategy to recover \( \text{FL}^{l_0, j_0}(\tau_0) \) relies on the existence of these restrictions on \( \Pi(\tau) \). Assuming these restrictions on \( \Pi(\tau) \), our construction of \( \pi^{i_1, j_1, f} \) from \( \pi^{i_1, j_1} \) is canonical and
is independent of various choices of lifts into characteristic zero as discussed in Remark 5.7.30. If \((i_0, j_0) = (n - 1, 0)\), then we simply have \(e^{0,n-1} = \Xi_{0,n-1}e^{0,n-1}\) and \(FL_{n-1}^{n-1}(\overline{\eta}_0)\) can actually be recovered from \(\Pi(\overline{\eta})\) through an explicit formula regardless of the restrictions on \(\Pi(\overline{\eta})\) mentioned above.

Finally, we note that if \(\mu^\square\) is \(3n\)-generic, then \(\mu^\square_{i_1,j_1}\) is \(2n\)-generic for each \((i_1, j_1)\) such that \(0 \leq i_1 < i_1 + 1 < j_1 \leq n - 1\).

1.5. Notation. Much of the notation introduced in this section will also be (or have already been) introduced in the text, but we try to collect together various definitions here for ease of reading.

We let \(E\) be a (sufficiently large) extension of \(\mathbb{Q}_p\) with ring of integers \(\mathcal{O}_E\), a uniformizer \(\varpi_E\), and residue field \(F\). We will use these rings \(E, \mathcal{O}_E,\) and \(F\) for the coefficients of our representations. We also let \(K\) be a finite extension of \(\mathbb{Q}_p\) with ring of integers \(\mathcal{O}_K\), a uniformizer \(\varpi\), and residue field \(k\). Let \(W(k)\) be the ring of Witt vectors over \(k\) and write \(K_0\) for \(W(k)[\frac{1}{p}]\). \((K_0)\) is the maximal absolutely unramified subextension of \(K\). In this paper, we are interested only in the fields \(K\) that are tamely ramified extension of \(\mathbb{Q}_p\), in which case we let \(e := [K : K_0] = p^l - 1\) where \(f = [k : \mathbb{F}_p]\).

For a field \(F\), we write \(G_F\) for \(\text{Gal}(\overline{F}/F)\) where \(\overline{F}\) is a separable closure of \(F\). For instance, we are mainly interested in \(G_{\mathbb{Q}_p}\) as well as \(G_{K_0}\) in this paper. The choice of a uniformizer \(\varpi \in K\) provides us with a map:

\[
\tilde{\omega}_{\varpi} : G_{\mathbb{Q}_p} \rightarrow W(k) : g \mapsto \frac{g(\varpi)}{\varpi}
\]

whose reduction mod \((\varpi)\) will be denoted as \(\omega_{\varpi}\). This map factors through \(\text{Gal}(K/\mathbb{Q}_p)\) and \(\tilde{\omega}_{\varpi}|_{G_{K_0}}\) becomes a homomorphism. Note that the choice of the embedding \(\sigma_0 : k \hookrightarrow \mathbb{F}_p\) provides us with a fundamental character of niveau \(f\), namely \(\omega := \sigma_0 \circ \omega_{\varpi}|_{\text{Gal}(K/K_0)}\), and we fix the embedding in this paper.

For \(a \in k\), we write \(\tilde{a}\) for its Teichmüller lift in \(W(k)\). We also use the notation \([a]\) for \(\tilde{a}\), in particular, in Section 4.7. When the notation for an element \(\bullet\) in \(k\) is quite long, we prefer \([\bullet]\) to \(\bullet\).

For instance, if \(a, b, c, d \in k\) then we write

\[
[(a - b)(a - c)(a - d)(b - c)(b - d)]\text{ for } (a - b)(a - c)(a - d)(b - c)(b - d).
\]

Note that \(\tilde{\omega}_{\varpi}\) is the Teichmüller lift of \(\omega_{\varpi}\).

We normalize the Hodge–Tate weight of the cyclotomic character \(\varepsilon\) to be \(-1\). Our normalization on class field theory sends the geometric Frobenius to the uniformizers. If \(a \in \mathbb{F}_p^\times\) or \(a \in \mathcal{O}_E\) then we write \(U_a\) for the unramified character sending the geometric Frobenius to \(a\). We may regard a character of \(G_{\mathbb{Q}_p}\) as a character of \(\mathbb{Q}_p^\times\) via our normalization of class field theory.

As usual, we write \(S\) for the \(p\)-adic completion of \(W(k)[u, u^{-1}]\), and let \(S_{\mathcal{O}_E} := S \otimes_{\mathbb{Z}_p} \mathcal{O}_E\) and \(S_E := S_{\mathcal{O}_E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\). We also let \(\mathfrak{S}_E := S_{\mathcal{O}_E}/(\varpi E, \text{Fil}^pS_{\mathcal{O}_E}) \cong (k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^p\). Choose a uniformizer \(\varpi\) of \(K\) and let \(E(u) \in W(k)[u]\) be the monic minimal polynomial of \(\varpi\). The group Gal(K/K_0) acts on \(S\) via the character \(\tilde{\omega}_{\varpi}\), and we write \((S_{\mathcal{O}_E})_{\omega}\) for the \(\omega\)-isotypical component of \(S\) for \(m \in \mathbb{Z}\). We define \((\mathfrak{S}_E)_{\omega}\) in a similar fashion. If \(\mathcal{O}_E\) or \(\mathfrak{S}_E\) are clear, we often omit them, i.e., we write \(S_{\omega}\) and \(\mathfrak{S}_{\omega}\) for \((S_{\mathcal{O}_E})_{\omega}\) and \((\mathfrak{S}_E)_{\omega}\) respectively. In particular, \(\mathfrak{S}_0 := \mathfrak{S}_{\omega} \cong (k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^p\) and

\[
S_0 := S_{\varpi} := \left\{ \sum_{i=0}^{\infty} a_i E(u)^i \mid a_i \in W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E \text{ and } a_i \to 0 \text{ \(p\)-adically} \right\}.
\]

The association \(a \otimes b \rightarrow (\sigma(a)b)\sigma\) gives rise to an isomorphism \(k \otimes_{\mathbb{F}_p} \mathfrak{S}_E \cong \prod_{\sigma,k} \mathfrak{S}_F\), and we write \(e_\sigma\) for the idempotent element in \(k \otimes_{\mathbb{F}_p} \mathfrak{S}_E\) that corresponds to the idempotent element in \(\prod_{\sigma,k} \mathfrak{S}_F\) whose only non-zero entry is 1 at the position of \(\sigma\).

To lighten the notation, we often write \(G\) for \(\text{GL}_n/\mathbb{Z}_p\). (By \(G_n\), we mean an outer form of \(\text{GL}_n\) defined in Section 5.3.) We let \(B\) be the Borel subgroup of \(G\) consisting of upper-triangular matrices of \(G\), \(U\) the unipotent subgroup of \(B\), and \(T\) the torus of diagonal matrices of \(\text{GL}_n\). We also write \(B^-\) and \(U^\mp\) for the opposite Borel of \(B\) and the unipotent subgroup of \(B^\mp\), respectively. Let \(\Phi^+\) denote the set of positive roots with respect to \((B,T)\), and \(\Delta = \{\alpha_k\}_{1 \leq k \leq n-1}\) the subset of positive simple roots.
roots. We also let \( W \) be the Weyl group of \( \text{GL}_n \), which is often considered as a subgroup of \( \text{GL}_n \), and let \( s_\alpha \) be the simple reflection corresponding to \( \alpha \). We write \( w_0 \) for the longest Weyl element in \( W \), and we hope that the reader is not confused with places \( w \) or \( w' \) of \( F \).

We often write \( K \) for \( \text{GL}_n(\mathbb{Z}_p) \) for brevity. (Note that we use \( K \) for a tamely ramified extension of \( \mathbb{Q} \), as well, and hope that it does not confuse the reader.) We will often use the following three open compact subgroups of \( \text{GL}_n(\mathbb{Z}_p) \): if we let \( \text{red} : \text{GL}_n(\mathbb{Z}_p) \to \text{GL}_n(\mathbb{F}_p) \) be the natural mod \( p \) reduction map, then

\[
K(1) := \text{Ker(red)} \subset I(1) := \text{red}^{-1}(U(\mathbb{F}_p)) \subset I := \text{red}^{-1}(B(\mathbb{F}_p)) \subset K.
\]

If \( M \) is a free \( \mathbb{F}_p \)-module with a smooth action of \( K \), then \( I \) acts on the pro \( p \) Iwahori fixed subspace \( M^{I(1)} \) via \( I/I(1) \cong T(\mathbb{F}_p) \). We write \( M^{I(1),\mu} \) for the eigenspace with respect to a character \( \mu : T(\mathbb{F}_p) \to \mathbb{F}_p^\times \). \( M^{I(1)} \) decomposes as

\[
M^{I(1)} \cong \bigoplus M^{I(1),\mu}
\]

as \( T(\mathbb{F}_p) \)-representations, where the direct sum runs over the characters \( \mu \) of \( T(\mathbb{F}_p) \). In the obvious similar fashion, we define \( M^{I(1),\mu} \) when \( M \) is a free \( \mathcal{O}_E \)-module or a free \( E \)-module.

By \([m]_f\) for a rational number \( m \in \mathbb{Z}_{(p)}^\times \subset \mathbb{Q} \) we mean the unique integer in \([0, e)\) congruent to \( m \mod (e) \) via the natural surjection \( \mathbb{Z}_{(p)}^\times \to \mathbb{Z}/e\mathbb{Z} \). By \([y] \) for \( y \in \mathbb{R} \) we mean the floor function of \( y \), i.e., the biggest integer less than or equal to \( y \). For a set \( A \), we write \( |A| \) for the cardinality of \( A \). If \( V \) is a finite-dimensional \( \mathbb{F} \)-representation of a group \( H \), then we write \( \text{soc}_H V \) and \( \text{cosoc}_H V \) for the socle of \( V \) and the cosocle of \( V \), respectively. If \( v \) is a non-zero vector in a free \( \mathcal{O}_E \)-module over \( \mathbb{F} \) (resp. over \( \mathcal{O}_E \), resp. over \( E \)), then we write \( \mathcal{O}_E[v] \) (resp. \( \mathcal{O}_E[v] \), resp. \( E[v] \)) for the \( \mathcal{O}_E \)-line (resp. the \( \mathcal{O}_E \)-line, resp. the \( E \)-line) generated by \( v \).

We write \( \mathcal{T} \) for the image of \( x \in \mathcal{O}_E \) under the natural surjection \( \mathcal{O}_E \twoheadrightarrow \mathbb{F} \). We also have a natural surjection \( \mathbb{P}^1(\mathcal{O}_E) \to \mathbb{P}^1(\mathbb{F}) \) defined by letting \( [x : y] \in \mathbb{P}^1(\mathcal{O}_E) \) be the image of \( [x : y] \in \mathbb{P}^1(\mathcal{O}_E) \) where

\[
[x : y] = \begin{cases} 
[1 : (\frac{1}{x})] \quad &\text{if } \frac{x}{y} \in \mathcal{O}_E; \\
[\frac{x}{y} : 1] \quad &\text{if } \frac{x}{y} \not\in \mathcal{O}_E.
\end{cases}
\]

We often write \( \frac{x}{y} \) for \( [x : y] \in \mathbb{P}^1(\mathbb{F}) \) if \( x \neq 0 \).

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## 2. Integral p-adic Hodge theory: preliminary

In this section, we do a quick review of some (integral) \( p \)-adic Hodge theory which will be needed later. We note that all of the results in this section are already known or easy generalization of known results. We closely follow [EGH15] as well as [HLM] in this section.

### 2.1. Filtered \((\phi, N)\)-modules with descent data.

In this section, we review potentially semi-stable representations and their corresponding linear algebra objects, admissible filtered \((\phi, N)\)-modules with descent data.

Let \( K \) be a finite extension of \( \mathbb{Q}_p \), and \( K_0 \) the maximal unramified subfield of \( K \), so that \( K_0 = W(k) \otimes \mathbb{Z}_p, \mathbb{Q}_p \) where \( k \) is the residue field of \( K \). We fix the uniformizer \( p \in \mathbb{Q}_p \), so that we fix an embedding \( \mathbb{B}_k \hookrightarrow \mathbb{B}_\mathbb{R} \). We also let \( K' \) be a subextension of \( K \) with \( K/K' \) Galois, and write \( \phi \in \text{Gal}(K_0/\mathbb{Q}_p) \) for the arithmetic Frobenius.
A $p$-adic Galois representation $\rho : G_{K'} \to \text{GL}_n(E)$ is potentially semi-stable if there is a finite extension $L$ of $K'$ such that $\rho|_{G_L}$ is semi-stable, i.e., $\text{rank}_{L_0 \otimes E} \mathcal{D}^n_{st}(V) = \dim_EV$, where $V$ is an underlying vector space of $\rho$ and $\mathcal{D}^n_{st}(V) := (\mathcal{B}_{st} \otimes \mathbb{Q}_p)G_L$. We often write $\mathcal{D}^n_{st}(\rho)$ for $\mathcal{D}^n_{st}(V)$. If $K$ is the Galois closure of $L$ over $K'$, then $\rho|_{G_K}$ is semi-stable, provided that $\rho|_{G_L}$ is semi-stable.

**Definition 2.1.1.** A filtered $(\phi, N, K/K', E)$-module of rank $n$ is a free $K_0 \otimes E$-module $D$ of rank $n$ together with

- a $\phi \otimes 1$-automorphism $\phi$ on $D$;
- a nilpotent $K_0 \otimes E$-linear endomorphism $N$ on $D$;
- a decreasing filtration $\{\text{Fil}^iD_k\}_{i \in \mathbb{Z}}$ on $D_K = K \otimes_{K_0} D$ consisting of $K \otimes \mathbb{Q}_p$-submodules of $D_K$, which is exhaustive and separated;
- a $K_0$-semilinear, $E$-linear action of $\text{Gal}(K/K')$ which commutes with $\phi$ and $N$ and preserves the filtration on $D_K$.

We say that $D$ is (weakly) admissible if the underlying filtered $(\phi, N, K/K, E)$-module is weakly admissible in the sense of [For94]. The action of $\text{Gal}(K/K')$ on $D$ is often called descent data action. If $V$ is potentially semi-stable, then $\mathcal{D}^n_{st}(V)$ is a typical example of an admissible filtered $(\phi, N, K/K', E)$-module of rank $n$.

**Theorem 2.1.2** ([CF], Theorem 4.3). There is an equivalence of categories between the category of weakly admissible filtered $(\phi, N, K/K', E)$-modules of rank $n$ and the category of $n$-dimensional potentially semi-stable $E$-representations of $G_K$ that become semi-stable upon restriction to $G_K$.

Note that Theorem 2.1.2 is proved in [CF] in the case $K = K'$, and that [Sav03] gives a generalization to the statement with non-trivial descent data in a formal nature.

If $V$ is potentially semi-stable, then so is its dual $V^\vee$. We define $\mathcal{D}^n_{st}(K') := \mathcal{D}^n_{st}(V^\vee)$. Then $\mathcal{D}^n_{st}(K')$ gives an anti-equivalence of categories from the category of $n$-dimensional potentially semi-stable $E$-representations of $G_{K'}$ that become semi-stable upon restriction to $G_K$ to the category of weakly admissible filtered $(\phi, N, K/K', E)$-modules of rank $n$, with quasi-inverse

$$V^*_{st,K'}(D) := \text{Hom}_{\phi, N} (D, B_{st}) \cap \text{Hom}_{\text{Fil}} (D_K, B_{dR}).$$

It will often be convenient to use covariant functors. We define an equivalence of categories: for each $r \in \mathbb{Z}$

$$V^*_{st,K'}(D) \coloneqq V^*_{st,K'}(D)^\vee \otimes \varepsilon^r.$$

The functor $\mathcal{D}^{n,r}_{st}$ defined by $\mathcal{D}^{n,r}_{st}(V) := \mathcal{D}^n_{st}(V \otimes \varepsilon^{-r})$ is a quasi-inverse of $\mathcal{D}^{n,r}_{st}$.

For a given potentially semi-stable representation $\rho : G_{K'} \to \text{GL}_n(E)$, one can attach a Weil–Deligne representation $\text{WD}(\rho)$ to $\rho$, as in [CDT99], Appendix B.1. We refer to $\text{WD}(\rho)|_{\mathbb{Q}_p}$ as to the Galois type associated to $\rho$. Note that $\text{WD}(\rho)$ is defined via the filtered $(\phi, N, K/K', E)$-module $\mathcal{D}^n_{st}(\rho)$ and that $\text{WD}(\rho)|_{k^{\text{irr}}} \cong \text{WD}(\rho \otimes \varepsilon^r)|_{k^{\text{irr}}}$ for all $r \in \mathbb{Z}$.

Finally, we say that a potentially semi-stable representation $\rho$ is potentially crystalline if the monodromy operator $N$ on $\mathcal{D}^n_{st}(\rho)$ is trivial.

### 2.2. Strongly divisible modules with descent data

In this section, we review strongly divisible modules that correspond to Galois stable lattices in potentially semi-stable representations. We keep the notation of Section 2.2.

From now on, we assume that $K/K'$ is a tamely ramified Galois extension with ramification index $e(K/K')$. We fix a uniformizer $\varpi \in K$ with $\varpi^{e(K/K')} \in K'$. Let $e$ be the absolute ramification index of $K$ and $E(u) \in W(k)[u]$ the minimal polynomial of $\varpi$ over $K_0$.

Let $S$ be the $p$-adic completion of $W(k)[u, \varpi^{ie}]_{i \in \mathbb{N}}$. The ring $S$ has additional structures:

- a continuous, $\phi$-semilinear map $\phi : S \to S$ with $\phi(u) = u^p$ and $\phi(\varpi^{ie}) = \varpi^{ie}$;
- a continuous, $W(k)$-linear derivation of $S$ with $N(u) = -u$ and $N(\varpi^{ie}) = -ie \varpi^{ie}$.
a decreasing filtration \( \{ \text{Fil}^i S \}_{i \in \mathbb{Z}_{\geq 0}} \) of \( S \) given by letting \( \text{Fil}^0 S \) be the \( p \)-adic completion of the ideal \( \sum_{j \geq 1} \frac{E(w_j)}{p^j} S \);

- a group action of \( \text{Gal}(K/K') \) on \( S \) defined for each \( g \in \text{Gal}(K/K') \) by the continuous ring isomorphism \( \hat{g} : S \to S \) with \( \hat{g}(w_i \frac{p^j}{(1+j)!}) = g(w_i) h_g \frac{p^j}{(1+j)!} \) for \( w_i \in W(k) \), where \( h_g \in W(k) \) satisfies \( g(\varpi) = h_g \varpi \).

Note that \( \hat{\phi} \) and \( N \) satisfies \( N \hat{\phi} = p \hat{\phi} N \) and that \( \hat{g}(E(u)) = E(u) \) for all \( g \in \text{Gal}(K/K') \) since we assume \( \varpi^{r(K/K')} \in K' \). We write \( \phi \) for \( \frac{1}{p^r} \hat{\phi} \) on \( \text{Fil}^i S \). For \( i \leq p - 1 \) we have \( \hat{\phi}(\text{Fil}^i S) \subseteq p^i S \).

Let \( S_{O_E} := S \otimes_{\mathbb{Z}_p} O_E \) and \( S_E := S_{O_E} \otimes_{\mathbb{Z}_p} Q_p \). We extend the definitions of \( \phi, N \), \( \text{Fil}^i S \), and the action of \( \text{Gal}(K/K') \) to \( S_{O_E} \) (resp. to \( S_E \)) \( O_E \)-linearly (resp. \( E \)-linearly).

**Definition 2.2.1.** Fix a positive integer \( r < p - 1 \). A strongly divisible \( O_E \)-module with descent data of weight \( r \) is a free \( S_{O_E} \)-module \( \hat{M} \) of finite rank together with

- a \( S_{O_E} \)-submodule \( \text{Fil}^r \hat{M} \);
- additive maps \( \phi, N : \hat{M} \to \hat{M} \);
- \( S_{O_E} \)-semilinear bijections \( \hat{g} : \hat{M} \to \hat{M} \) for each \( g \in \text{Gal}(K/K') \)

such that

- \( \text{Fil}^i S_{O_E} \cdot \hat{M} \subseteq \text{Fil}^i \hat{M} \);
- \( \text{Fil}^r \hat{M} \cap I \hat{M} = \text{Fil}^r \hat{M} \) for all ideals \( I \) in \( O_E \);
- \( \phi(sx) = \phi(s) \phi(x) \) for all \( s \in S_{O_E} \) and for all \( x \in \hat{M} \);
- \( \phi(\text{Fil}^r \hat{M}) \) is contained in \( p^r \hat{M} \) and generates it over \( S_{O_E} \);
- \( N(sx) = N(s)x + sN(x) \) for all \( s \in S_{O_E} \) and for all \( x \in \hat{M} \);
- \( N \phi = p \phi N \);
- \( E(u)N(\text{Fil}^r \hat{M}) \subseteq \text{Fil}^r \hat{M} \);
- for all \( g \in \text{Gal}(K/K') \) \( \hat{g} \) commutes with \( \phi \) and \( N \), and preserves \( \text{Fil}^i \hat{M} \);
- \( \hat{g}_1 \circ \hat{g}_2 = \hat{g}_1 \hat{g}_2 \) for all \( g_1, g_2 \in \text{Gal}(K/K') \).

We write \( O_E \text{-Mod}^{dd}_{st} \) for the category of strongly divisible \( O_E \)-modules with descent data of weight \( r \). It is easy to see that the map \( \phi_r = \frac{1}{p^r} \phi : \text{Fil}^r \hat{M} \to \hat{M} \) satisfies \( cN\phi_r(x) = \phi_r(E(u)N(x)) \) for all \( x \in \text{Fil}^r \hat{M} \) where \( c := \frac{\phi(E(u))}{p^r} \in S^K \).

For a strongly divisible \( O_E \)-module \( \hat{M} \) with descent data of weight \( r \), we define a \( G_{K'} \)-module \( T^\circ_{st,K'}(\hat{M}) \) as follows (cf. [EGH15], Section 3.1.1):

\[
T^\circ_{st,K'}(\hat{M}) := \text{Hom}_{F_{[K/K']}},\phi,N(\hat{M}, \hat{A}_{st}).
\]

**Proposition 2.2.2** ([EGH15], Proposition 3.1.4). The functor \( T^\circ_{st,K'} \) provides an anti-equivalence of categories from the category \( O_E \text{-Mod}^{dd}_{st} \) to the category of \( G_{K'} \)-stable \( O_E \)-lattices in finite-dimensional \( E \)-representations of \( G_{K'} \) which become semi-stable over \( K \) with Hodge–Tate weights lying in \([-r, 0]) \), when \( 0 < r < p - 1 \).

Note that the case \( K = K' \) and \( E = Q_p \) is proved by Liu [Liu18].

In this paper, we will be mainly interested in covariant functors \( T^\circ_{st,K'} \) from the category \( O_E \text{-Mod}^{dd}_{st} \) to the category \( \text{Rep}_{O_E}^{K-\text{st},[-r,0]} G_{K'} \) of \( G_{K'} \)-stable \( O_E \)-lattices in finite-dimensional \( E \)-representations of \( G_{K'} \) which become semi-stable over \( K \) with Hodge–Tate weights lying in \([-r, 0]) \) defined by

\[
T^\circ_{st,K'}(\hat{M}) := (T^\circ_{st,K'}(\hat{M})^\vee)^\vee.
\]

Let \( \hat{M} \) in \( O_E \text{-Mod}^{dd}_{st} \), and define a free \( S_E \)-module \( D := \hat{M} \otimes_{Z_p} Q_p \). We extend \( \phi \) and \( N \) on \( D \), and define a filtration on \( D \) as follows: \( \text{Fil}^r D = \text{Fil}^r \hat{M} |_{Z_p} \) and

\[
\text{Fil}^i D := \begin{cases} D & \text{if } i \leq 0; \\ \{ x \in D \mid E(u)^{-i} x \in \text{Fil}^i D \} & \text{if } 0 \leq i \leq r; \\ \sum_{j=0}^{r-i} \text{Fil}^{-j} S_{Q_p} \left( \text{Fil}^j D \right) & \text{if } i > r, \text{inductively.}\end{cases}
\]

(2.2.3)
We let $D := \mathcal{D} \otimes_{S_{Q_p}} K_0$ and $D_K := \mathcal{D} \otimes_{S_{Q_p}} K$, where $s_0 : S_{Q_p} \to K_0$ and $s_\varpi : S_{Q_p} \to K$ are defined by $u \mapsto 0$ and $u \mapsto \varpi$ respectively, which induce $\phi$ and $N$ on $D$ and the filtration on $D_K$ by taking $s_\varpi(\text{Fil}^rD)$. The $K_0$-vector space $D$ also inherits an $E$-linear action and a semi-linear action of $\text{Gal}(K/K')$. Then it turns out that $D$ is a weakly admissible filtered $(\phi, N, K/K', E)$-module with $\text{Fil}^{r+1}D = 0$. Moreover, there is a compatibility (cf. \cite{EGH15}, Proof of Proposition 3.1.4.): if $D$ corresponds to $\mathcal{D} = \mathcal{M}(\frac{1}{p})$, then

$$
T_{\text{st}}^{K', r}(\mathcal{M})[\frac{1}{p}] \cong V_{\text{st}}^{K', r}(D).
$$

2.3. Breuil modules with descent data. In this section, we review Breuil modules with descent data. We keep the notation of Section 2.2, and assume further that in which we will be more interested in this paper.

Definition 2.3.1. Fix a positive integer $r < p - 1$. A Breuil modules with descent data of weight $r$ is a free $S$-module $M$ of finite rank together with

- a $\overline{S}$-submodule $\text{Fil}^rM$ of $M$;
- maps $\phi_r : \text{Fil}^rM \to M$ and $N : M \to M$;
- additive bijections $\hat{g} : M \to M$ for all $g \in \text{Gal}(K/K')$

such that

- $\phi_r$ is $F$-linear and $\phi$-semilinear (where $\phi : k[u]/u^{sp} \to k[u]/u^{sp}$ is the $p$-th power map) with image generating $M$ as $\overline{S}$-module;
- $N$ is $k \otimes_{F_p} F$-linear and satisfies
  - $N(ux) = uN(x) - ux$ for all $x \in M$,
  - $u^cN(\text{Fil}^rM) \subseteq \text{Fil}^rM$, and
  - $\phi_r(u^cN(x)) = cN(\phi_r(x))$ for all $x \in \text{Fil}^rM$, where $c \in (k[u]/u^{sp})^\times$ is the image of $\frac{1}{p}\phi(E(u))$ under the natural map $S \to k[u]/u^{sp}$.
- $\hat{g}$ preserves $\text{Fil}^rM$ and commutes with the $\phi_r$ and $N$, and the action satisfies $\hat{g}_1 \circ \hat{g}_2 = \hat{g}_1 \circ \hat{g}_2$ for all $g_1, g_2 \in \text{Gal}(K/K')$. Furthermore, if $a \in k \otimes_{F_p} F$ and $m \in M$ then $\hat{g}(au'm) = g(a)((\frac{u}{u^{sp}})^{-1} \otimes 1)u(\hat{g}(m))$.

We write $F\text{-BrMod}_{\text{dd}}^r$ for the category of Breuil modules with descent data of weight $r$. For $\mathcal{M} \in F\text{-BrMod}_{\text{dd}}^r$, we define a $G_{K'}$-module as follows (cf. \cite{EGH15}, Section 3.2):

$$
T_{\text{st}}^r(\mathcal{M}) := \text{Hom}_{F\text{-BrMod}}(\mathcal{M}, \hat{\mathcal{A}}).
$$

This gives an exact faithful contravariant functor from the category $F\text{-BrMod}_{\text{dd}}^r$ to the category $\text{Rep}_F G_{K'}$ of finite dimensional $F$-representations of $G_{K'}$. We also define a covariant functor as follows: for each $r \in \mathbb{Z}$

$$
T_{\text{st}}^r(\mathcal{M}) := T_{\text{st}}^r(\mathcal{M})^\vee \otimes \omega^r,
$$

in which we will be more interested in this paper.

If $\hat{M}$ is a strongly divisible module with descent data, then

$$
\mathcal{M} := \hat{M}/(\varpi E, \text{Fil}^p S)
$$

is naturally an object in $F\text{-BrMod}_{\text{dd}}^r$ ($\text{Fil}^r\hat{M}$ is the image of $\text{Fil}^r\hat{M}$ in $\mathcal{M}$, the map $\phi_r$ is induced by $\frac{1}{p}\varphi|_{\text{Fil}^r\hat{M}}$, and $N$ and $\hat{g}$ are those coming from $\hat{M}$). Moreover, there is a compatibility: if $\hat{M} \in O_E\text{-Mod}_{\text{dd}}$ and we let $\mathcal{M} = \hat{M}/(\varpi E, \text{Fil}^p S)$ then

$$
T_{\text{st}}^{K', r}(\hat{M}) \otimes_{O_E} F \cong T_{\text{st}}^r(\mathcal{M}).
$$

(See \cite{EGH15}, Lemma 3.2.2 for detail.)

There is a notion of duality of Breuil modules, which will be convenient for our computation of Breuil modules as we will see later.
Lemma 2.3.5 \(\subseteq [\text{MP}], \text{Lemma 3.1} \)

in the following lemma by integers

Definition 2.3.3.

Moreover, one has

Lemma 2.3.6 \(\subseteq [\text{MP}], \text{Lemma 3.2} \)

Definition 2.3.2. Let \(M \in \mathbf{F} \text{-BrMod}_{\text{dd}}^r \). We define \(M^* \) as follows:

\(\begin{align*}
\circ M^p := \text{Hom}_{k[u]/u^{p^0} \text{-mod}}(M,k[u]/u^{p^0}); \\
\circ \text{Fil}^r M^p := \{ f \in M^p | f(\text{Fil}^r M) \subseteq u^n k[u]/u^{p^0} \}; \\
\circ \phi_r(f) = \text{def}\text{ined by } \phi_r(f)(x) = \phi_r(f)(x) \text{ for all } x \in \text{Fil}^r M \text{ and } f \in \text{Fil}^r M^p, \text{ where } \\
\phi_r : u^n k[u]/u^{p^0} \to k[u]/u^{p^0} \text{ is the unique semilinear map sending } u^n \text{ to } c^n; \\
\circ N(f) := N \circ f \cap N, \text{ where } N : k[u]/u^{p^0} \to k[u]/u^{p^0} \text{ is the unique } k\text{-linear derivation such that } \\
N(u) = -u; \\
\circ (\tilde{g}(f))(x) = g(f(\tilde{g}^{-1}(x))) \text{ for all } x \in M \text{ and } g \in \text{Gal}(K/K'), \text{ where } \\
\text{Gal}(K/K') \text{ acts on } k[u]/u^{p^0} \text{ by } \\
g(a u^{p^0}) = g(a)(\frac{g(\omega)}{\omega})^i u^{i} \text{ for } a \in k.
\end{align*}\)

If \(M \) is an object of \(\mathbf{F} \text{-BrMod}_{\text{dd}}^r \) then so is \(M^* \). Moreover, we have \(M \cong M^{**} \) and

\(T^*_M(M^p) \cong T^*_M(M).\)

(cf. \[\text{Car11}], Section 2.1.)

Finally, we review the notion of Breuil submodules developed mainly by \[\text{Car11}\]. See also \[\text{HLM}\], Section 2.3.

Definition 2.3.3. Let \(M \) be an object of \(\mathbf{F} \text{-BrMod}_{\text{dd}}^r \). A Breuil submodule of \(M \) is an \(S\)-submodule \(N \) of \(M \) if \(N \) satisfies

\(\begin{align*}
\circ N \text{ is a } k[u]/u^{p^0} \text{-direct summand of } M; \\
\circ N(N) \subseteq N \text{ and } \tilde{g}(N) \subseteq N \text{ for all } g \in \text{Gal}(K/K'); \\
\circ \phi_r(N \cap \text{Fil}^r M) \subseteq N.
\end{align*}\)

If \(N \) is a Breuil submodule of \(M \), then \(N \text{ and } M/N \) are also objects of \(\mathbf{F} \text{-BrMod}_{\text{dd}}^r \). We now state a crucial result we will use later.

Proposition 2.3.4 \[\text{[HLM], Proposition 2.3.5}\]. Let \(M \) be an object in \(\mathbf{F} \text{-BrMod}_{\text{dd}}^r \).

Then there is a natural inclusion preserving bijection

\(\Theta : \{ \text{Breuil submodules in } M \} \to \{ G_{K'}\text{-subrepresentations of } T^*_M(M) \}\)

sending \(N \subseteq M \) to the image of \(T^*_M(N) \hookrightarrow T^*_M(M) \). Moreover, if \(M_2 \subseteq M_1 \) are Breuil submodules of \(M \), then \(\Theta(M_1)/\Theta(M_2) \cong T^*_M(M_1/M_2) \).

We will also need classification of Breuil modules of rank 1 as follows. We denote the Breuil modules in the following lemma by \(M(a,s, \lambda) \).

Lemma 2.3.5 \[\text{[MP], Lemma 3.1}\]. Let \(k := F_{p^f}, e := p^f - 1, \omega := \sqrt[p^f]{-p}, \text{ and } K' = Q_p. \) We also let \(M \) be a rank-one object in \(\mathbf{F} \text{-BrMod}_{\text{dd}}^r \).

Then there exists a generator \(m \in M \) such that:

\(\begin{align*}
(i) & \; M = \mathcal{F}_{F_{p^f}} \cdot m; \\
(ii) & \; \text{Fil}^r M = u^s m \text{ where } 0 \leq s \leq \frac{r e}{p-1}; \\
(iii) & \; \varphi_r(u^{s(p-1)} m) = \lambda m \text{ for some } \lambda \in (F_{p^f} \otimes F_p F)^\times; \\
(iv) & \; \tilde{g}(m) = (\omega f(g)^a \otimes 1) m \text{ for all } g \in \text{Gal}(K/K_0) \text{ where } a \text{ is an integer such that } a + ps \equiv 0 \mod \left(\frac{r e}{p-1}\right); \\
(v) & \; N(m) = 0.
\end{align*}\)

Moreover, one has

\(T^*_M(M)|_{Q_p} = \omega_f^{a+ps}.\)

The following lemma will be used to determine if the Breuil modules violate the maximal nonsplitness.

Lemma 2.3.6 \[\text{[MP], Lemma 3.2}\]. Let \(k := F_{p^f}, e := p^f - 1, \omega := \sqrt[p^f]{-p}, \text{ and } K' = Q_p. \) We also let \(M_x := M(k_x, s_x, \lambda_x) \) and \(M_y := M(k_y, s_y, \lambda_y) \) be rank-one objects in \(\mathbf{F} \text{-BrMod}_{\text{dd}}^r \). Assume that the integers \(k_x, k_y, s_x, s_y \in \mathbb{Z} \) satisfy

\(\begin{align*}
2(p s_y - s_x) + |k_y - k_x|_f > 0.
\end{align*}\)
Assume further that $f < p$ and let
\[ 0 \to M_x \to M \to M_y \to 0 \]
be an extension in $\mathbf{F}$-$\text{BrMod}^\tau_{\text{id}}$, with $T^*_{\text{st}}(M)$ being Fontaine–Laffaille.

If the exact sequence of $\mathbf{F}$-$\text{Mod}$-modules
\begin{equation}
0 \to \text{Fil}^r M_x \to \text{Fil}^r M \to \text{Fil}^r M_y \to 0
\end{equation}
splits, then the $G_{\mathbf{F}_p}$-representation $T^*_{\text{st}}(M)$ splits as a direct sum of two characters.

In particular, provided that $p k_y \not\equiv k_x$ modulo $e$ and that $s_y(p - 1) < re$ if $f > 1$, the representation $T^*_{\text{st}}(M)$ splits as a direct sum of two characters if the element $j_0 \in \mathbb{Z}$ uniquely defined by
\begin{equation}
j_0 e + [p^{-1}k_y - k_x]_f < s_x(p - 1) \leq (j_0 + 1)e + [p^{-1}k_y - k_x]_f
\end{equation}
satisfies
\begin{equation}
(r + j_0)e + [p^{-1}k_y - k_x]_f < (s_x + s_y)(p - 1).
\end{equation}

2.4. Linear algebra with descent data. In this section, we introduce the notion of framed basis for a Breuil module $M$ and framed system of generators for $\text{Fil}^r M$. Throughout this section, we assume that $K_0 = K'$ and continue to assume that $K$ is a tamely ramified Galois extension of $K'$. We also fix a positive integer $r < p - 1$.

**Definition 2.4.1.** Let $n \in \mathbb{N}$ and let $(k_{n-1}, k_{n-2}, \ldots, k_0) \in \mathbb{Z}^n$ be an $n$-tuple. A rank $n$ Breuil module $M \in \mathbf{F}$-$\text{BrMod}^\tau_{\text{id}}$ is of (inertial) type $\omega_n^{k_{n-1}} \ominus \cdots \ominus \omega_n^{k_0}$ if $M$ has an $\mathcal{S}$-basis $(e_{n-1}, \ldots, e_0)$ such that $\bar{g} e_i = (\omega_n^{k_i}(g) \otimes 1) e_i$ for all $i$ and all $g \in \text{Gal}(K/K_0)$. We call such a basis a framed basis of $M$.

We also say that $f := (f_{n-1}, f_{n-2}, \ldots, f_0)$ is a framed system of generators of $\text{Fil}^r M$ if $f$ is a system of $\mathcal{S}$-generators for $\text{Fil}^r M$ and $\bar{g} f_i = (\omega_n^{k_i-1}(g) \otimes 1) f_i$ for all $i$ and all $g \in \text{Gal}(K/K_0)$.

The existence of a framed basis and a framed system of generators for a given Breuil module $M \in \mathbf{F}$-$\text{BrMod}^\tau_{\text{id}}$ is proved in \cite{HLM}, Section 2.2.2.

Let $M \in \mathbf{F}$-$\text{BrMod}^\tau_{\text{id}}$ be of inertial type $\bigoplus_{i=0}^{n-1} \omega_n^{k_i}$, and let $e := (e_{n-1}, \ldots, e_0)$ be a framed basis for $M$ and $f := (f_{n-1}, \ldots, f_0)$ be a framed system of generators for $\text{Fil}^r M$. The **matrix of the filtration**, with respect to $e$, $f$, is the matrix $\text{Mat}_e^f(\text{Fil}^r M) \in M_n(\mathcal{S})$ such that
\[ f = e \cdot \text{Mat}_e^f(\text{Fil}^r M). \]

Similarly, we define the **matrix of the Frobenius** with respect to $e$, $f$ as the matrix $\text{Mat}_e^f(\phi_r) \in GL_n(\mathcal{S})$ characterized by
\[ (\phi_r(f_{n-1}), \ldots, \phi_r(f_0)) = e \cdot \text{Mat}_e^f(\phi_r). \]

As we require $e$, $f$ to be compatible with the filtration, the entries in the matrix of the filtration satisfy the important additional properties:
\[ \text{Mat}_e^f(\text{Fil}^r M)_{i,j} \in \mathcal{S}_{\omega_n^{p^{-1}k_j - k_i}}. \]

More precisely, $\text{Mat}_e^f(\text{Fil}^r M)_{i,j} = u^{p^{-1}k_j - k_i} s_{i,j}$, where $s_{i,j} \in \mathcal{S}_{\omega_n^{k_i}} = k \otimes_{\mathbf{F}_p} \mathbf{F}[u^p]/(u^p)$.

We can therefore introduce the subspace $M_n^\mathcal{S} := \bigoplus_{i=0}^{n-1} \omega_n^{k_i}$ as
\[ M_n^\mathcal{S} := \{ V \in M_n(\mathcal{S}), V_{i,j} \in \mathcal{S}_{\omega_n^{p^{-1} k_j - k_i}} \text{ for all } 0 \leq i, j \leq n - 1 \}. \]

We also define
\[ GL_n^\mathcal{S} := GL_n(\mathcal{S}) \cap M_n^\mathcal{S}. \]

Similarly, we define
\[ M_n^\mathcal{S}/(\mathcal{S}) := \{ V \in M_n(\mathcal{S}), V_{i,j} \in \mathcal{S}_{\omega_n^{p^{-1} (k_j - k_i)}} \text{ for all } 0 \leq i, j \leq n - 1 \} \]
and
\[ GL_n^\mathcal{S}/(\mathcal{S}) := \{ V \in GL_n(\mathcal{S}), V_{i,j} \in \mathcal{S}_{\omega_n^{p^{-1} (k_j - k_i)}} \text{ for all } 0 \leq i, j \leq n - 1 \}. \]
As $\varphi_r(f_i)$ is a $\omega_f^{k_i}$-eigenvector for the action of $\Gal(K/K_0)$ we deduce that

$$\Mat_{\mathcal{L}}(\Fil^r\mathcal{M}) \in M_n^{\mathcal{O}_f/(S)} \quad \text{and} \quad \Mat_{\mathcal{L}}(\varphi_r) \in \GL_n^{\mathcal{O}_f/(S)}.$$

We use similar terminologies for strongly divisible modules $\widehat{\mathcal{M}} \in \mathcal{O}_E:\Mod_{\mathcal{F}}^{\mathcal{O}_f}$.

**Definition 2.4.2.** Let $n \in \mathbb{N}$ and let $(k_{n-1},k_{n-2},\ldots,k_0) \in \mathbb{Z}^n$ be an $n$-tuple. A rank $n$ strongly divisible module $\widehat{\mathcal{M}} \in \mathcal{O}_E:\Mod_{\mathcal{F}}^{\mathcal{O}_f}$ is of (inertial) type $\omega^{k_{n-1}}_\infty \oplus \cdots \oplus \omega^{k_0}_\infty$ if $\mathcal{M}$ has an $S_{\mathcal{O}_E}$-basis $\widehat{e} := (\widehat{e}_{n-1},\cdots,\widehat{e}_0)$ such that $\widehat{\varrho} \widehat{e}_i = (\omega^{k_i}_\infty(g) \otimes 1)\widehat{e}_i$ for all $i$ and all $g \in \Gal(K/K_0)$. We call such a basis a framed basis for $\widehat{\mathcal{M}}$.

We also say that $\widehat{f} := (\widehat{f}_{n-1},\widehat{f}_{n-2},\ldots,\widehat{f}_0)$ is a framed system of generators for $\Fil^r\widehat{\mathcal{M}}$ if $\widehat{f}$ is a system of $S$-generators for $\Fil^r\widehat{\mathcal{M}}/\Fil^{r+1}\widehat{\mathcal{M}}$ and $\widehat{g} \widehat{f}_i = (\omega^{p^{-1}k_i}_\infty(g) \otimes 1)\widehat{f}_i$ for all $i$ and all $g \in \Gal(K/K_0)$.

One can readily check the existence of a framed basis for $\widehat{\mathcal{M}}$ and a framed system of generators for $\Fil^r\widehat{\mathcal{M}}$, by Nakayama Lemma. We also define

$$\Mat_{\mathcal{L}}(\Fil^r\widehat{\mathcal{M}}) \quad \text{and} \quad \Mat_{\mathcal{L}}(\varphi_r)$$

each of whose entries satisfies

$$\Mat_{\mathcal{L}}(\Fil^r\widehat{\mathcal{M}})_{i,j} \in S_{\omega^{p^{r+1}k_j-k_i}_\infty} \quad \text{and} \quad \Mat_{\mathcal{L}}(\varphi_r)_{i,j} \in S_{\omega^{k_j-k_i}_\infty},$$
in the similar fashion to Breuil modules. In particular,

$$\Mat_{\mathcal{L}}(\Fil^r\widehat{\mathcal{M}}) \in M_n^{\mathcal{O}_f/(S)} \quad \text{and} \quad \Mat_{\mathcal{L}}(\varphi_r) \in \GL_n^{\mathcal{O}_f/(S)}$$

where $M_n^{\mathcal{O}_f/(S)}$ and $\GL_n^{\mathcal{O}_f/(S)}$ are defined in the similar way to Breuil modules. We also define $\GL_n^{\mathcal{O}_f/(S)}$ in the similar way to Breuil modules again.

The inertial types on a Breuil module $\mathcal{M}$ and on a strongly divisible modules are closely related to the Weil–Deligne representation associated to a potentially crystalline lift of $T_{st}^r(M)$.

**Proposition 2.4.3 [LMP], Proposition 2.12.** Let $\widehat{\mathcal{M}}$ be an object in $\mathcal{O}_E:\Mod_{\mathcal{F}}^{\mathcal{O}_f}$ and let $\mathcal{M} := \widehat{\mathcal{M}} \otimes_{S} S/((\omega_E,\Fil^p S)$ be the Breuil module corresponding to the mod $p$ reduction of $\widehat{\mathcal{M}}$.

If $T_{\mathcal{F}}^{\mathcal{O}_f}(\mathcal{M})[1/p]$ has Galois type $\bigoplus_{i=0}^{n-1} \omega^{k_i}_f$ for some integers $k_i$, then $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1} \omega^{k_i}_\infty$ (resp. $\bigoplus_{i=0}^{n-1} \omega^{k_i}_\infty$).

Finally, we need a technical result on change of basis of Breuil modules with descent data.

**Lemma 2.4.4 [HLM, Lemma 2.2.8].** Let $\mathcal{M} \in \mathcal{F}:\BrMod_{\mathcal{F}}^{\mathcal{O}_f}$ be of type $\bigoplus_{i=0}^{n-1} \omega^{k_i}_f$, and let $\varphi, f$ be a framed basis for $\mathcal{M}$ and a framed system of generators for $\Fil^r\mathcal{M}$ respectively. Write $V := \Mat_{\mathcal{L}}(\Fil^r\mathcal{M}) \in M_n^{\mathcal{O}_f/(S)}$ and $A := \Mat_{\mathcal{L}}(\varphi_r) \in \GL_n^{\mathcal{O}_f/(S)}$, and assume that there are invertible matrices $R \in \GL_n^{\mathcal{O}_f/(S)}$ and $C \in \GL_n^{\mathcal{O}_f/(S)}$ such that

$$R \cdot V \cdot C \equiv V' \mod (u^{r+1}),$$

for some $V' \in M_n^{\mathcal{O}_f/(S)}$.

Then $\varphi' := \varphi \cdot R^{-1}$ forms another framed basis for $\mathcal{M}$ and $f' := f' \cdot V'$ forms another framed system of generators for $\Fil^r\mathcal{M}$ such that

$$\Mat_{\mathcal{L}}(\Fil^r\mathcal{M}) = V' \in M_n^{\mathcal{O}_f/(S)} \quad \text{and} \quad \Mat_{\mathcal{L}}(\varphi_r) = R \cdot A \cdot \phi(C) \in \GL_n^{\mathcal{O}_f/(S)}.$$
2.5. Fontaine–Laffaille modules. In this section, we briefly recall the theory of Fontaine–Laffaille modules over \( \mathbf{F} \), and we continue to assume that \( K_0 = K' \) and that \( K \) is a tamely ramified Galois extension of \( K' \).

**Definition 2.5.1.** A Fontaine–Laffaille module over \( k \otimes_{\mathbf{F}_p} \mathbf{F} \) is the datum \( (M, \text{Fil}^iM, \phi_\bullet) \) of

- a free \( k \otimes_{\mathbf{F}_p} \mathbf{F} \)-module \( M \) of finite rank;
- a decreasing, exhaustive and separated filtration \( \{\text{Fil}^iM\}_{j \in \mathbb{Z}} \) on \( M \) by \( k \otimes_{\mathbf{F}_p} \mathbf{F} \)-submodules;
- a \( \varphi \)-semilinear isomorphism \( \phi_\bullet : \text{gr}^iM \to M \), where \( \text{gr}^iM := \bigoplus_{j \in \mathbb{Z}} \text{Fil}^iM_{\varphi^j} \).

We write \( \mathbf{F} \cdot \text{FLMod}_k \) for the category of Fontaine–Laffaille modules over \( k \otimes_{\mathbf{F}_p} \mathbf{F} \), which is abelian. If the field \( k \) is clear from the context, we simply write \( \mathbf{F} \cdot \text{FLMod} \) to lighten the notation.

Given a Fontaine–Laffaille module \( M \), the set of its Hodge–Tate weights in the direction of \( \sigma \in \text{Gal}(k/\mathbf{F}_p) \) is defined as \( \text{HT}_\sigma := \{ i \in \mathbb{N} \mid e_\sigma \text{Fil}^iM \neq e_\sigma \text{Fil}^{i+1}M \} \). In the remainder of this paper we will be focused on Fontaine–Laffaille modules with parallel Hodge–Tate weights, i.e. we will assume that for all \( i \in \mathbb{N} \), the submodules \( \text{Fil}^iM \) are free over \( k \otimes_{\mathbf{F}_p} \mathbf{F} \).

**Definition 2.5.2.** Let \( M \) be a Fontaine–Laffaille module with parallel Hodge–Tate weights. A \( k \otimes_{\mathbf{F}_p} \mathbf{F} \) basis \( f = (f_0, f_1, \ldots, f_{n-1}) \) on \( M \) is compatible with the filtration if for all \( i \in \mathbb{Z}_{\geq 0} \) there exists \( j_i \in \mathbb{Z}_{\geq 0} \) such that \( \text{Fil}^iM = \sum\limits_{j_i=j_i}^n k \otimes_{\mathbf{F}_p} \mathbf{F} \cdot f_j. \) In particular, the principal symbols \( (\text{gr}(f_0), \ldots, \text{gr}(f_{n-1})) \) provide a \( k \otimes_{\mathbf{F}_p} \mathbf{F} \) basis for \( \text{gr}^0M \).

Note that if the graded pieces of the Hodge filtration have rank at most one then any two compatible basis on \( M \) are related by a lower-triangular matrix in \( \text{GL}_n(k \otimes_{\mathbf{F}_p} \mathbf{F}) \). Given a Fontaine–Laffaille module and a compatible basis \( f \), it is convenient to describe the Frobenius action via a matrix \( \text{Mat}_f(\phi_\bullet) \in \text{GL}_n(k \otimes_{\mathbf{F}_p} \mathbf{F}) \), defined in the obvious way using the principal symbols \( (\text{gr}(f_0), \ldots, \text{gr}(f_{n-1})) \) as a basis on \( \text{gr}^0M \).

It is customary to write \( \mathbf{F} \cdot \text{FLMod}^{[0,p-2]} \) to denote the full subcategory of \( \mathbf{F} \cdot \text{FLMod} \) formed by those \( M \) verifying \( \text{Fil}^0M = M \) and \( \text{Fil}^{p-1}M = 0 \) (it is again an abelian category). We have the following description of mod \( p \) Galois representations of \( G_{K_0} \) via Fontaine–Laffaille modules:

**Proposition 2.5.3 (FLS2, Theorem 6.1).** There is an exact fully faithful contravariant functor

\[
T_{\text{cris},K_0} : \mathbf{F} \cdot \text{FLMod}^{[0,p-2]} \to \text{Rep}_\mathbf{F}(G_{K_0})
\]

which is moreover compatible with the restriction over unramified extensions: if \( L_0/K_0 \) is unramified with residue field \( l/k \) and if \( M \) is an object in \( \mathbf{F} \cdot \text{FLMod}^{[0,p-2]}_{K_0} \), then \( l \otimes_k M \) is naturally regarded as an object in \( \mathbf{F} \cdot \text{FLMod}^{[0,p-2]}_{L_0} \) and

\[
T_{\text{cris},L_0}(l \otimes_k M) \cong T_{\text{cris},K_0}(M)|_{G_{L_0}}.
\]

We will often write \( T_{\text{cris}} \) for \( T_{\text{cris},K_0} \) if the base field \( K_0 \) is clear from the context.

**Definition 2.5.4.** We say that \( \overline{\rho} \in \text{Rep}_\mathbf{F}G_{K_0} \) is Fontaine–Laffaille if \( T_{\text{cris}}(M) \cong \overline{\rho} \) for some \( M \in \mathbf{F} \cdot \text{FLMod}^{[0,p-2]} \).

2.6. Étale \( \phi \)-modules. In this section, we review the theory of étale \( \phi \)-modules, first introduced by Fontaine [For90], and its connection with Breuil modules and Fontaine–Laffaille modules. Throughout this section, we continue to assume that \( K_0 = K' \) and that \( K \) is a tamely ramified Galois extension of \( K' \).

Let \( p_0 := -p \), and let \( p \) be identified with a sequence \( (p_n)_{n \in (\mathbb{Q}_p^\times)}^N \) verifying \( p_n^0 = p_{n-1} \) for all \( n \). We also fix \( \varpi := \sqrt{-p} \in K \), and let \( \omega_0 = \varpi \). We fix a sequence \( (\varpi_n)_{n \in (\mathbb{Q}_p^\times)}^N \) such that \( \varpi_n^\sigma = p_n \) and \( \varpi_n^\sigma = \varpi_{n-1} \) for all \( n \in \mathbb{N} \), and which is compatible with the norm maps \( K(\varpi_{n+1}) \to K(\varpi_n) \) (cf. [Bre14], Appendix A). By letting \( K_{\infty} := \bigcup_{n \in \mathbb{N}} K(\varpi_n) \) and \( (K_0)_{\infty} := \bigcup_{n \in \mathbb{N}} K_0(p_n) \), we have a canonical isomorphism \( \text{Gal}(K_{\infty}/(K_0)_{\infty}) \to \text{Gal}(K/K_0) \) and we will identify \( \omega_{\varpi} \) as a character of \( \text{Gal}(K_{\infty}/(K_0)_{\infty}) \). The field of norms \( k((\varpi)) \) associated to \( (K, \varpi) \) is then endowed with a residual action of \( \text{Gal}(K_{\infty}/(K_0)_{\infty}) \), which is completely determined by \( \overline{g}(\varpi) = \omega_{\varpi}(g) \varpi \).
We define the category \((\phi, F \otimes_{F_p} k((p)))_{\text{-Mod}}\) of \(\text{étale}\) \((\phi, F \otimes_{F_p} k((p)))\)-modules as the category of free \(F \otimes_{F_p} k((p))\)-modules of finite rank \(\mathcal{M}\) endowed with a semilinear map \(\phi : \mathcal{M} \to \mathcal{M}\) with respect to the Frobenius on \(k((p))\) and inducing an isomorphism \(\phi^* \mathcal{M} \to \mathcal{M}\) (with obvious morphisms between objects). We also define the category \((\phi, F \otimes_{F_p} k((\varpi)))_{\text{-Mod}_{\text{dd}}}\) of \(\text{étale}\) \((\phi, F \otimes_{F_p} k((\varpi)))\)-modules with descent data: an object \(\mathcal{M}\) is defined as for the category \((\phi, F \otimes_{F_p} k((p)))_{\text{-Mod}}\) but we moreover require that \(\mathcal{M}\) is endowed with a semilinear action of \(\text{Gal}(\overline{K}/(K_0)_{\infty})\) (semilinear with respect to the residual action on \(F \otimes_{F_p} k((\varpi))\)) where \(F\) is endowed with the trivial \(\text{Gal}(K_{\infty}/(K_0)_{\infty})\)-action commuting with \(\phi\).

By work of Fontaine [Fon90], there are anti-equivalences
\[
(\phi, F \otimes_{F_p} k((p)))_{\text{-Mod}} \xrightarrow{\sim} \text{Rep}_F(G_{(K_0)_{\infty}})
\]
and
\[
(\phi, F \otimes_{F_p} k((\varpi)))_{\text{-Mod}_{\text{dd}}} \xrightarrow{\sim} \text{Rep}_F(G_{(K_0)_{\infty}})
\]
given by
\[
\mathcal{M} \mapsto \text{Hom}(\mathcal{M}, k((p))^{\text{sep}})
\]
and
\[
\mathcal{M} \mapsto \text{Hom}(\mathcal{M}, k((\varpi))^{\text{sep}})
\]
respectively. See also [HLM], Appendix A.2.

The following proposition summarizes the relation between the various categories and functors we introduced above.

**Proposition 2.6.1** ([HLM], Proposition 2.2.9). There exist faithful functors
\[
M_{k((\varpi))} : F_{-}\text{-BrMod}_{\text{dd}} \to (\phi, F \otimes_{F_p} k((\varpi)))_{\text{-Mod}_{\text{dd}}}
\]
and
\[
\mathcal{F} : F_{-}\text{-FLMod}^{[0, p-2]} \to (\phi, F \otimes_{F_p} k((p)))_{\text{-Mod}}
\]
fitting in the following commutative diagram:

\[
\begin{array}{ccc}
F_{-}\text{-BrMod}_{\text{dd}} & \xrightarrow{M_{k((\varpi))}} & (\phi, F \otimes_{F_p} k((\varpi)))_{\text{-Mod}_{\text{dd}}} \\
\text{T}^*_\text{st} & \downarrow & \\
\text{Res} & \downarrow & \\
\text{Rep}_F(G_{K_0}) & \xrightarrow{\text{Res}} & \text{Rep}_F(G_{(K_0)_{\infty}}) \\
\text{T}^*_\text{cris} & \downarrow & \\
F_{-}\text{-FLMod}^{[0, p-2]} & \xrightarrow{\mathcal{F}} & (\phi, F \otimes_{F_p} k((p)))_{\text{-Mod}}
\end{array}
\]

where the descent data is relative to \(K_0\) and the functor \(\text{Res} \circ \text{T}^*_\text{cris}\) is fully faithful.

Note that the functors \(M_{k((\varpi))}\) and \(\mathcal{F}\) are defined in [BD14]. (See also [HLM], Appendix A). The following is an immediate consequence of Proposition 2.6.1 which is also stated in [LMP], Corollary 2.14.

**Corollary 2.6.2.** Let \(0 \leq r \leq p-2\), and let \(\mathcal{M}\) (resp. \(\mathcal{M}\)) be an object in \(F_{-}\text{-BrMod}_{\text{dd}}\) (resp. in \(F_{-}\text{-FLMod}^{[0, p-2]}\)). Assume that \(\text{T}^*_\text{st}(\mathcal{M})\) is Fontaine–Laffaille. If
\[
M_{k((\varpi))}(\mathcal{M}) \cong \mathcal{F}(\mathcal{M}) \otimes_{k((p))} k((\varpi))
\]
then one has an isomorphism of $G_{K_0}$-representations

$$T^*_n(M) \cong T^*_{\text{cris}}(M).$$

The following two lemmas are very crucial in this paper, as we will see later, which describe the functors $M_{k((\varpi))]})$ and $F$ respectively.

**Lemma 2.6.3** ([HLM], Lemma 2.2.6). Let $M$ be a Breuil module of inertial type $\bigoplus_{i=0}^{n-1} \omega^{k_i}$ with a framed basis $\xi$ for $M$ and a framed system of generators $f$ for $\text{Fil}^i M$, and write $M^*$ for its dual as defined in Definition 2.7.2. Let $V = \text{Mat}_{n}([HLM], \text{Lemma } 2.2.6)$ and $A = \text{Mat}_{n}(\phi_r) \in \text{GL}_{n}^*(\mathcal{F})$.

Then there exists a basis $e$ for $M_{k((\varpi))]}(\mathcal{M}^*)$ with $\gamma \varepsilon_i = (\omega^{-p^{-1}k_i}(g) \otimes 1)\varepsilon_i$ for all $i \in \{0, 1, \cdots, n-1\}$ and $g \in \text{Gal}(K/K_0)$, such that the Frobenius $\phi$ on $M_{k((\varpi))]}(\mathcal{M}^*)$ is described by

$$\text{Mat}_{n}(\phi) = \tilde{V}^t \left( \hat{A}^{-1} \right)^t \in M_n(F \otimes_{F_p} k[[\varpi]])$$

where $\tilde{V}$, $\hat{A}$ are lifts of $V$, $A$ in $M_n(F \otimes_{F_p} k[[\varpi]])$ via the reduction morphism $F \otimes_{F_p} k[[\varpi]] \to \mathcal{S}$ induced by $\varpi \mapsto u$ and $\text{Mat}_{n}(\phi)_{i,j} \in (F \otimes_{F_p} k[[\varpi]])_{\omega^{k_i-k_j}}$.

**Lemma 2.6.4** ([HLM], Lemma 2.2.7). Let $M \in F$-\text{FLMod}^{[0,p-2]} be a rank $n$ Fontaine–Laffaille module with parallel Hodge–Tate weights $0 \leq m_0 \leq \cdots \leq m_{n-1} \leq p-2$ (counted with multiplicity). Let $\varepsilon = (\varepsilon_0, \cdots, \varepsilon_{n-1})$ be a $k \otimes_{F_p} F$ basis for $M$, compatible with the Hodge filtration $\text{Fil}^i M$ and let $F \in M_n(k \otimes_{F_p} F)$ be the associated matrix of the Frobenius $\phi: \text{gl}^* M \to M$.

Then there exists a basis $\underline{\varepsilon}$ for $\mathfrak{m} := F(M)$ such that the Frobenius $\phi$ on $\mathfrak{m}$ is described by

$$\text{Mat}_{n}(\phi) = \text{Diag}(\underline{\varepsilon}_m, \cdots, \underline{\varepsilon}_{m-1}) \cdot F \in M_n(F \otimes_{F_p} k[[\varpi]])$$

### 3. Local Galois side

In this section, we study ordinary Galois representations and their potentially crystalline lifts. In particular, we prove that the Frobenius eigenvalues of certain potentially crystalline lifts preserve the information of the wildly ramified part of ordinary representations.

Throughout this section, we let $f$ be a positive integer, $K' = \mathbb{Q}_p$, $e = p^f - 1$, and $K = \mathbb{Q}_p(\sqrt[p^f]{u})$. We also fix $\varpi := \sqrt[p^f]{u}$, and let $\mathcal{S} = (F_{p^f} \otimes_{F_p} F)[\varpi]/u^{e_p}$ and $\mathcal{S}_0 := \mathcal{S}_{\omega} = (F_{p^f} \otimes_{F_p} F)[\varpi]/u^{e_p} \subseteq \mathcal{S}$. Recall that by $[m]_f$ for a rational number $m \in \mathbb{Z}/[p]$ we mean the unique integer in $[0, e]$ congruent to $m$ mod ($e$).

We say that a representation $\overline{\rho}_0 : \text{G}_{\mathbb{Q}_p} \to \text{GL}_n(F)$ is **ordinary** if it is isomorphic to a representation whose image is contained in the Borel subgroup of upper-triangular matrices. Namely, an ordinary representation has a basis $\underline{\varepsilon} := (\varepsilon_{n-1}, \varepsilon_{n-2}, \cdots, \varepsilon_0)$ that gives rise to a matrix form as follows:

$$\begin{pmatrix}
U_{\mu_{n-1}} \omega^{c_{n-1} + (n-1)} & \cdots & \ast_{n-1} & \ast & \ast \\
0 & U_{\mu_{n-2}} \omega^{c_{n-2} + (n-2)} & \cdots & \ast & \ast \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & U_{\mu_1} \omega^{c_1 + 1} & \ast_1 \\
0 & 0 & \cdots & 0 & U_{\mu_0} \omega^{c_0}
\end{pmatrix}_{(3.0.1)}$$

Here, $U_{\mu}$ is the unramified character sending the geometric Frobenius to $\mu \in F_{p^f}$ and $c_i$ are integers.

By $\overline{\rho}_0$, we always mean an $n$-dimensional ordinary representation that is written as in (3.0.1). For $n - 1 \geq i \geq j \geq 0$, we write

$$(3.0.2) \quad \overline{\rho}_{i,j}$$

for the $(i-j+1)$-dimensional subquotient of $\overline{\rho}_0$ determined by the subset $(\varepsilon_i, \varepsilon_{i-1}, \cdots, \varepsilon_j)$ of the basis $\underline{\varepsilon}$. For instance, $\overline{\rho}_{i,i} = U_{\mu_i} \omega^{c_i + 1}$ and $\overline{\rho}_{n-1,0} = \overline{\rho}_0$.

An ordinary representation $\text{G}_{\mathbb{Q}_p} \to \text{GL}_n(F)$ is **maximally non-split** if its socle filtration has length $n$. For instance, $\overline{\rho}_0$ in (3.0.1) is maximally non-split if and only if $s_i \neq 0$ for all $i = 1, 2, \cdots, n - 1$. 
In this paper, we are interested in ordinary maximally non-split representations satisfying a certain genericity condition.

**Definition 3.0.3.** We say that $\mathbf{p}_0$ is generic if

$$c_{i+1} - c_i > n - 1 \text{ for all } i \in \{0, 1, \ldots, n - 2\} \text{ and } c_{n-1} - c_0 < (p - 1) - (n - 1).$$

We say that $\mathbf{p}_0$ is strongly generic if $\mathbf{p}_0$ is generic and

$$c_{n-1} - c_0 < (p - 1) - (3n - 5).$$

Note that this strongly generic condition implies $p > n^2 + 2(n - 3)$.

We describe a rough shape of the Breuil modules with descent data from $K$ to $K' = \mathbb{Q}_p$ corresponding to $\mathbf{p}_0$. Let $r$ be a positive integer with $p - 1 > r \geq n - 1$, and let $\mathcal{M} \in \mathbf{FBrMod}^u_{\text{id}}$ be a Breuil module of inertial type $\bigoplus_{i=0}^{n-1} \omega_f^{k_i}$ such that $T^r_\mathcal{M}(\mathcal{M}) \cong \mathbf{p}_0$, for some $k_i \in \mathbb{Z}$. By Proposition [2.3.4] we note that $\mathcal{M}$ is a successive extension of $\mathcal{M}_i$, where $\mathcal{M}_i := \mathcal{M}(k_i, r_i, \nu_i)$ (cf. Lemma [2.3.6]) is a rank one Breuil module of inertial type $\omega_f^{k_i}$ such that

$$\omega_f^{k_i + pr_i} \cong T^r_\mathcal{M}(\mathcal{M}_i)|_{\mathbb{Q}_p} \cong \omega_f^{\gamma_i + i}$$

for each $i \in \{0, 1, \ldots, n - 1\}$. More precisely, there exist a framed basis $\mathbf{e} = (e_{n-1}, e_{n-2}, \ldots, e_0)$ for $\mathcal{M}$ and a framed system of generators $\mathbf{f} = (f_{n-1}, f_{n-2}, \ldots, f_0)$ for $\text{Fil}^r \mathcal{M}$ such that

$$\text{Mat}_{\mathbf{F}^r}(\mathcal{M}) = \left(\begin{array}{cccc}
u_{n-1} & u_{k-1, n-1}/\nu_{n-1, n-2} & \cdots & u_{k, n-1}/\nu_{n-1, 0} \\
0 & u_{\nu_{n-2}} & \cdots & u_{\nu_{n-2}/\nu_{n-2, 0}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \nu_0 \end{array}\right),$$

and

$$\text{Mat}_{\mathbf{F}^r}(\phi_r) = \left(\begin{array}{cccc}
u_{n-1} & u_{k-1, n-1}/\nu_{n-1, n-2} & \cdots & u_{k, n-1}/\nu_{n-1, 0} \\
0 & u_{\nu_{n-2}} & \cdots & u_{\nu_{n-2}/\nu_{n-2, 0}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \nu_0 \end{array}\right),$$

for some $\nu_i \in (\mathbf{F}_p)^\times$ and for some $\nu_{i,j}, w_{i,j}, \gamma_{i,j} \in \mathbb{F}_p$.

Fix $0 \leq j \leq i \leq n - 1$. We define the Breuil submodules

$$\mathcal{M}_{i,j}$$

that is a subquotient of $\mathcal{M}$ determined by the basis $(e_i, e_{i-1}, \ldots, e_j)$. For instance, $\mathcal{M}_{i,i} \cong \mathcal{M}_i$ for all $0 \leq i \leq n - 1$. We note that $T^r_{\mathcal{M}_i}(\mathcal{M}_{i,j}) \cong \mathbf{p}_{i,j}$ by Proposition [2.3.4].

We will keep these notation and assumptions for $\mathcal{M}$ throughout this paper.

### 3.1. Elimination of Galois types

In this section, we find out the possible Galois types of niveau 1 for potentially semi-stable lifts of $\mathbf{p}_0$ with Hodge–Tate weights $\{- (n - 1), -(n - 2), \ldots, 0\}$.

We start this section with the following elementary lemma.

**Lemma 3.1.1.** Let $\rho : G_{\mathbb{Q}_p} \to \text{GL}_n(E)$ be a potentially semi-stable representation with Hodge–Tate weights $\{- (n - 1), \ldots, -2, -1, 0\}$ and of Galois type $\bigoplus_{i=0}^{n-1} \omega_f^{k_i}$.

Then

$$\det(\rho)|_{\mathbb{Q}_p} = \varepsilon^{\frac{n(n-1)}{2}} \cdot \frac{1}{\omega_f^2} \cdot \prod_{i=0}^{n-1} k_i,$$
where $\varepsilon$ is the cyclotomic character.

Proof. det$(\rho)$ is a potentially crystalline character of $G_{\mathbf{Q}_p}$ with Hodge–Tate weight $-(\sum_{i=0}^{n-1} i)$ and of Galois type $\omega_f^{\sum_{i=0}^{n-1} k_i}$, i.e., det$(\rho) \cdot \omega_f^{-\sum_{i=0}^{n-1} k_i}$ is a crystalline character with Hodge–Tate weight $-(\sum_{i=0}^{n-1} i) = -\frac{n(n-1)}{2}$ so that det$(\rho)|_{\mathbf{Q}_p} \cdot \omega_f^{\sum_{i=0}^{n-1} k_i} \cong \varepsilon^{\frac{n(n-1)}{2}}$.

We will only consider the Breuil modules $M$ corresponding to the mod $p$ reduction of the strongly divisible modules that corresponds to the Galois stable lattices in potentially semi-stable lifts of $\mathcal{P}_0$ with Hodge–Tate weights $\{- (n - 1), -(n - 2), \cdots, -1, 0\}$, so that we may assume that $r = n - 1$, i.e., $M \in \overline{\text{F-BrMod}}^{n-1}_{\text{id}}$.

**Lemma 3.1.2.** Let $f = 1$. Assume that $M \in \overline{\text{F-BrMod}}^{n-1}_{\text{id}}$ corresponds to the mod $p$ reduction of a strongly divisible module $\hat{\mathcal{M}}$ such that $T^{n-1}_{st}(M) \cong \mathcal{P}_0$ and $T^{\tilde{\Omega}^{n-1}}_{st}(\hat{\mathcal{M}})$ is a Galois stable lattice in a potentially semi-stable lift of $\mathcal{P}_0$ with Hodge–Tate weights $\{- (n - 1), -(n - 2), \cdots, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \omega^{k_i}$ for some integers $k_i$.

Then there exists a framed basis $\underline{e}$ for $M$ and a framed system of generators $f$ for $\text{Fil}^{n-1}M$ such that $\text{Mat}_{\underline{e}}(\text{Fil}^{n-1}M), \text{Mat}_{\underline{e}}(\phi_{n-1}),$ and $\text{Mat}_{\underline{e}}(N)$ are as in (3.0.6), (3.0.6), and (5.0.7) respectively. Moreover, the $(k_i, r_i)$ satisfy the following properties:

(i) $k_i \equiv c_i + i - r_i \mod(e)$ for all $i \in \{0, 1, \cdots, n - 1\}$;
(ii) $0 \leq r_i \leq n - 1$ for all $i \in \{0, 1, \cdots, n - 1\}$;
(iii) $\sum_{i=0}^{n-1} r_i = \frac{(n-1)e}{2}$.

Proof. Note that the inertial type of $M$ is $\bigoplus_{i=0}^{n-1} \omega^{k_i}$ by Proposition 2.4.3. The first part of the Lemma is obvious from the discussion at the beginning of Section 3.

We now prove the second part of the Lemma. We may assume that the rank-one Breuil modules $M_i$ are of weight $n - 1$, so that $0 \leq r_i \leq n - 1$ for $i \in \{0, 1, \cdots, n - 1\}$ by Lemma 2.3.5. By the equation (3.0.1), we have $k_i \equiv c_i + i - r_i \mod(e)$, as $e = p - 1$. By looking at the determinant of $\mathcal{P}_0$ we deduce the conditions

$$\omega^{\frac{n(n-1)}{2}+k_{n-1}+k_{n-2}+\cdots+k_0} = \det T^{n-1}_{st}(M)|_{\mathbf{Q}_p} = \det \mathcal{P}_0|_{\mathbf{Q}_p} = \omega^{r_{n-1}+r_{n-2}+\cdots+r_0} = \omega^{\frac{n(n-1)}{2}+k_{n-1}+k_{n-2}+\cdots+k_0}$$

from Lemma 3.1.4. and hence we have $r_{n-1} + r_{n-2} + \cdots + r_0 = \frac{n(n-1)}{2}$ (as $p > n^2 + 2(n - 3)$ due to the genericity of $\mathcal{P}_0$).

One can further eliminate Galois types of niveau 1 if $\mathcal{P}_0$ is maximally non-split.

**Proposition 3.1.3.** Keep the assumptions and notation of Lemma 3.1.2. If the tuple $(k_i, r_i)$ further satisfy one of the following conditions

- $r_i = n - 1$ for some $i \in \{0, 1, 2, \cdots, n - 2\}$;
- $r_i = 0$ for some $i \in \{1, 2, 3, \cdots, n - 1\}$,

then $\mathcal{P}_0$ is not maximally non-split.

Proof. The main ingredient is Lemma 2.4.3. We fix $i \in \{0, 1, 2, \cdots, n - 2\}$ and identify $x = i + 1$ and $y = i$ and all the other following. From the results in Lemma 3.1.2 it is easy to compute that $|k_i - k_{i+1}| = e - (c_{i+1} - c_i + 1) + (r_{i+1} - r_i)$. By the genericity conditions in Definition 3.0.3 and by part (ii) of Lemma 3.1.2 we see that $0 < |k_i - k_{i+1}| < e$ so that if $r_i \geq r_{i+1}$ then the equation (2.3.7) in Lemma 2.3.6 holds.

If $r_{i+1} \leq |k_i - k_{i+1}|$ and $r_i \geq r_{i+1}$, then $\ast_{i+1} = 0$ by Lemma 2.3.6. Since $0 < |k_i - k_{i+1}| < e$, we have $r_{i+1} \leq |k_i - k_{i+1}|$ if and only if $r_{i+1} = 0$, in which case $\mathcal{P}_0$ is not maximally non-split.

We now apply the second part of Lemma 2.3.6. It is easy to check that $j_0 = r_{i+1} - 1$. One can again readily check that the equation (2.3.10) is equivalent to $r_i = n - 1$, in which case $\ast_{i+1} = 0$ so that $\mathcal{P}_0$ is not maximally non-split.

Note that all of the Galois types that will appear later in this section will satisfy the conditions in Lemma 3.1.2 and Proposition 3.1.3 as well if we further assume that $\mathcal{P}_0$ is maximally non-split.
3.2. Fontaine–Laffaille parameters. In this section, we parameterize the wildly ramified part of
generic and maximally non-split ordinary representations using Fontaine–Laffaille theory.

We start this section by recalling that if \( \overline{\rho}_0 \) is generic then \( \overline{\rho}_0 \otimes \omega^{-c_0} \) is Fontaine–Laffaille (cf. [GG10], Lemma 3.1.5), so that there is a Fontaine–Laffaille module \( M \) with Hodge–Tate weights
\( \{0, c_1 - c_0 + 1, \cdots , c_{n-1} - c_0 + (n-1)\} \) such that \( T_{\text{cris}}^r(M) \cong \overline{\rho}_0 \otimes \omega^{-c_0} \) (if we assume that \( \overline{\rho}_0 \) is generic).

**Lemma 3.2.1.** Assume that \( \overline{\rho}_0 \) is generic, and let \( M \in \mathbf{F}\text{-FLMod}_{\mathbf{F}_p}^{[0,p-2]} \) be a Fontaine–Laffaille
module such that \( T_{\text{cris}}^r(M) \cong \overline{\rho}_0 \otimes \omega^{-c_0} \).

Then there exists a basis \( \underline{e} = (e_0, e_1, \cdots , e_{n-1}) \) for \( M \) such that
\[
\text{Fil}^j M = \begin{cases} M & \text{if } j \leq 0; \\ \mathbf{F}(e_1, \cdots , e_{n-1}) & \text{if } c_{i-1} - c_0 + i - 1 < j \leq c_i - c_0 + i; \\ 0 & \text{if } c_{n-1} - c_0 + n - 1 < j. \end{cases}
\]

\[
\text{Mat}_{\underline{e}}(\phi^r) = \begin{pmatrix} \mu_0^{-1} & \alpha_{0,1} & \cdots & \alpha_{0,n-2} & \alpha_{0,n-1} \\ 0 & \mu_1^{-1} & \cdots & \alpha_{1,n-2} & \alpha_{1,n-1} \\ 0 & 0 & \mu_2^{-1} & \cdots & \alpha_{2,n-2} & \alpha_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{n-2}^{-1} & \alpha_{n-2,n-1} \\ 0 & 0 & 0 & \cdots & 0 & \mu_{n-1}^{-1} \end{pmatrix}
\]

where \( \alpha_{i,j} \in \mathbf{F} \).

Note that the basis \( \underline{e} \) on \( M \) in Lemma 3.2.1 is compatible with the filtration.

**Proof.** This is an immediate generalization of [HLM], Lemma 2.1.7. \( \square \)

For \( i \geq j \), the subset \( (e_j, \cdots , e_i) \) of \( \underline{e} \) determines a subquotient \( M_{i,j} \) of the Fontaine–Laffaille module \( M \), which is also a Fontaine–Laffaille module with the filtration induced from \( \text{Fil}^r M \) in the obvious way and with Frobenius described as follows:
\[
A_{i,j} := \begin{pmatrix} \mu_j^{-1} & \alpha_{j,j+1} & \cdots & \alpha_{j,i-1} & \alpha_{j,i} \\ 0 & \mu_{j+1}^{-1} & \cdots & \alpha_{j+1,i-1} & \alpha_{j+1,i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_{i-1}^{-1} & \alpha_{i-1,i} \\ 0 & 0 & \cdots & 0 & \mu_i^{-1} \end{pmatrix}.
\]

Note that \( T_{\text{cris}}^r(M_{i,j}) \otimes \omega^{c_0} \cong \overline{\rho}_{i,j} \). We let \( A'_{i,j} \) be the \((i-j) \times (i-j)\)-submatrix of \( A_{i,j} \) obtained by deleting the left-most column and the lowest row of \( A_{i,j} \).

**Lemma 3.2.3.** Keep the assumptions and notation of Lemma 3.2.1 and let \( 0 \leq j < j + 1 < i \leq n-1 \).
Assume further that \( \overline{\rho}_0 \) is maximally non-split.

If \( \det A'_{i,j} \neq (-1)^{i-j+1} \mu_{j+1}^{-1} \cdots \mu_{i-1}^{-1} \alpha_{j,i} \), then \( [\alpha_{i,j} : \det A'_{i,j}] \in \mathbb{P}^1(\mathbf{F}) \) does not depend on the choice of basis \( \underline{e} \) compatible with the filtration.

**Proof.** This is an immediate generalization of [HLM], Lemma 2.1.9. \( \square \)

**Definition 3.2.4.** Keep the assumptions and notation of Lemma 3.2.3 and assume further that \( \overline{\rho}_0 \) satisfies
\[
\det A'_{i,j} \neq (-1)^{i-j+1} \mu_{j+1}^{-1} \cdots \mu_{i-1}^{-1} \alpha_{j,i}
\]
for all \( i,j \in \mathbb{Z} \) with \( 0 \leq j < j + 1 < i \leq n-1 \).

The Fontaine–Laffaille parameter associated to \( \overline{\rho}_0 \) is defined as
\[
\text{FL}_n(\overline{\rho}_0) := (\text{FL}_n^i(\overline{\rho}_0))_{i,j} \in [\mathbb{P}^1(\mathbf{F})]^{(n-2)(n-1)\choose 2}.
\]
where
\[
\text{FL}_{i,j}^+(\rho_0) := \left[ \alpha_{j,i} : (-1)^{i-j+1} \cdot \det A'_{i,j} \right] \in \mathbb{P}^1(F)
\]
for all \(i, j \in \mathbb{Z}\) such that \(0 \leq j < j + 1 < i \leq n - 1\).

We often write \(\frac{\mathbb{P}^1}{F} [x : y] \in \mathbb{P}^1(F)\) if \(x \neq 0\). The conditions in \((3.2.5)\) for \(i, j\) guarantee the well-definedness of \(\text{FL}_{i,j}^+(\rho_0)\) in \(\mathbb{P}^1(F)\). We also point out that \(\text{FL}_{i,j}^+(\rho_0) \neq (-1)^{-j} \mu_{j+1} \cdots \mu_{i-1}\) in \(\mathbb{P}^1(F)\).

One can define the inverses of the elements in \(\mathbb{P}^1(F)\) in a natural way: for \([x_1 : x_2] \in \mathbb{P}^1(F)\), \([x_1 : x_2]^{-1} := [x_2 : x_1] \in \mathbb{P}^1(F)\).

**Lemma 3.2.6.** Assume that \(\rho_0\) is generic. Then
\[
\begin{align*}
\text{(i)} & \text{ } \rho_0' \text{ is generic;} \\
\text{(ii)} & \text{ if } \rho_0 \text{ is strongly non-split, then so is } \rho_0'; \\
\text{(iii)} & \text{ if } \rho_0 \text{ is maximally non-split, then so is } \rho_0'; \\
\text{(iv)} & \text{ if } \rho_0 \text{ is maximally non-split, then the conditions in } (3.2.3) \text{ are stable under } \rho_0 \mapsto \rho_0'.
\end{align*}
\]

Assume further that \(\rho_0\) is maximally non-split and satisfies the conditions in \((3.2.3)\).

\[
\begin{align*}
\text{(v)} & \text{ for all } i, j \in \mathbb{Z} \text{ with } 0 \leq j < j + 1 < i \leq n - 1, \text{ FL}_{i,j}^+(\rho_0) = \text{FL}_{i,j}^+(\rho_0 \otimes \omega^b) \text{ for any } b \in \mathbb{Z}; \\
\text{(vi)} & \text{ for all } i, j \in \mathbb{Z} \text{ with } 0 \leq j < j + 1 < i \leq n - 1, \text{ FL}_{i,j}^+(\rho_0) = \text{FL}_{i-j,0}^+(\rho_0^i); \\
\text{(vii)} & \text{ for all } i, j \in \mathbb{Z} \text{ with } 0 \leq j < j + 1 < i \leq n - 1, \text{ FL}_{i,j}^+(\rho_0^{-1}) = \text{FL}_{i,j}^+(\rho_0)^{-1}.
\end{align*}
\]

**Proof.** (i), (ii) and (iii) are easy to check. We leave them for the reader.

The only effect on Fontaine–Laffaille module by twisting \(\omega^b\) is shifting the jumps of the filtration. Thus (v) and (vi) are obvious.

For (iv) and (vii), one can check that the Frobenius of the Fontaine–Laffaille module associated to \(\rho_0'\) is described by
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]
where \(\text{Mat}_{\mathbb{Z}}(\rho_0')\) is as in \((3.2.2)\). Now one can check them by direct computation.

We end this section by defining certain numerical conditions on Fontaine–Laffaille parameters. We consider the matrix \((1, n)w_0 \text{Mat}_{\mathbb{Z}}(\phi_\bullet)^t\), where \(\text{Mat}_{\mathbb{Z}}(\phi_\bullet)\) is the upper-triangular matrix in \((3.2.2)\).

Here, \(w_0\) is the longest element of the Weyl group \(\tilde{W}\) associated to \(T\) and \((1, n)\) is a permutation in \(W\). Note that the anti-diagonal matrix displayed in the proof of Lemma 3.2.6 is \(w_0\) seen as an element in \(\text{GL}_n(F)\). For \(1 \leq i \leq n - 1\) we let \(B_i\) be the square matrix of size \(i\) that is the left-bottom corner of \((1, n)w_0 \text{Mat}_{\mathbb{Z}}(\phi_\bullet)^t\).

**Definition 3.2.7.** Keep the notation and assumptions of Definition \((3.2.4)\). We say that \(\rho_0\) is Fontaine–Laffaille generic if moreover \(\det B_i \neq 0\) for all \(1 \leq i \leq n - 1\) and \(\rho_0\) is strongly generic.

We emphasize that by an ordinary representation \(\rho_0\) being Fontaine–Laffaille generic, we always mean that \(\rho_0\) satisfies the maximally non-splitness and the conditions in \((3.2.3)\) as well as \(\det B_i \neq 0\) for all \(1 \leq i \leq n - 1\) and the strongly generic assumption (cf. Definition \((3.2.8)\).

Although the Frobenius matrix of a Fontaine–Laffaille module depends on the choice of basis, it is easy to see that the non-vanishing of the determinants above is independent of the choice of basis compatible with the filtration. Note that the conditions in Definition \((3.2.7)\) are necessary and sufficient conditions for
\[
(1, n)w_0 \text{Mat}_{\mathbb{Z}}(\phi_\bullet)^t \in B(F)w_0B(F)
\]
in the Bruhat decomposition, which will significantly reduce the size of the paper (cf. Remark \((3.2.8)\).

We also note that
o det $B_1 \neq 0$ if and only if $\text{Fil}_{n-1,0}(\overline{\rho}_0) \neq \infty$;
o det $B_{n-1} \neq 0$ if and only if $\text{Fil}_{n-1,0}(\overline{\rho}_0) \neq 0$.

Finally, we point out that the locus of Fontaine–Laffaille generic ordinary Galois representations $\overline{\rho}_0$ forms a (Zariski) open subset in $[P^1(F)]$.

**Remark 3.2.8.** Definition 3.2.4 comes from the fact that the list of Serre weights of $\overline{\rho}_0$ is then minimal in the sense of Conjecture 5.3.1. It is very crucial in the proof of Theorem 5.7.6 as it is more difficult to track the Fontaine–Laffaille parameters on the automorphic side if we have too many Serre weights. Moreover, these conditions simplify our proof for Theorem 3.7.3.

### 3.3. Breuil modules of certain inertial types of niveau 1

In this section, we classify the Breuil modules with certain inertial types, corresponding to the ordinary Galois representations $\overline{\rho}_0$ as in (3.0.1), and we also study their corresponding Fontaine–Laffaille parameters.

Throughout this section, we always assume that $\overline{\rho}_0$ is strongly generic. Since we are only interested in inertial types of niveau 1, we let $f = 1$, $e = p - 1$, and $\varepsilon = \sqrt{-p}$. We define the following integers for $0 \leq i \leq n - 1$:

$$r_i^{(0)} := \begin{cases} 1 & \text{if } i = n - 1; \\ i & \text{if } 0 < i < n - 1; \\ n - 2 & \text{if } i = 0. \end{cases}$$

(3.3.1) We also set

$$k_i^{(0)} := c_i + i - r_i^{(0)}$$

for all $i \in \{0, 1, \ldots, n - 1\}$.

We first classify the Breuil modules of inertial types described as above.

**Lemma 3.3.2.** Assume that $\overline{\rho}_0$ is strongly generic and that $\mathcal{M} \in \text{F-BrMod}_{n-1}^n$ corresponds to the mod $p$ reduction of a strongly divisible modules $\widehat{\mathcal{M}}$ such that $\mathcal{M} \equiv \overline{\rho}_0$ (mod $p$) is a Galois stable lattice in a potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-n, -(n-2), \ldots, 0\}$ and Galois type $\bigoplus_{i=0}^{n-1} \omega_i k_i^{(0)}$.

Then $\mathcal{M} \in \text{F-BrMod}_{n-1}^n$ can be described as follows: there exist a framed basis $\underline{c}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\text{Fil}^{n-1} \mathcal{M}$ such that

$$\text{Mat}_{\underline{f},\underline{c}}(\text{Fil}^{n-1} \mathcal{M}) = \begin{pmatrix} u^{r_0^{(0)}} & \beta_{n-1,n-2} u^{r_0^{(0)} - k_0^{(0)}} & \cdots & \beta_{n-1,0} u^{r_0^{(0)} - k_{n-1,0}^{(0)}} \\ 0 & u^{r_0^{(0)}} & \cdots & \beta_{n-2,0} u^{r_0^{(0)} - k_{n-2,0}^{(0)}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u^{r_0^{(0)}} \end{pmatrix}$$

and

$$\text{Mat}_{\underline{f},\underline{c}}(\phi_{n-1}) = \text{Diag}(\nu_{n-1}, \nu_{n-2}, \ldots, \nu_0)$$

where $k_{i,j} := k_i^{(0)} - k_j^{(0)}$, $\nu_i \in \text{F}^\times$ and $\beta_{i,j} \in \text{F}$. Moreover,

$$\text{Mat}_{\underline{c}}(N) = \begin{pmatrix} \gamma_{i,j} \cdot u^{k_{i,j}^{(0)} - k_i^{(0)}}; i \end{pmatrix}$$

where $\gamma_{i,j} = 0$ if $i \leq j$ and $\gamma_{i,j} \in \text{F}^{k_{i,j}^{(0)} - k_i^{(0)}}$, $\nu_0$ if $i > j$.

Note that $\underline{c}$ and $\underline{f}$ in Lemma 3.3.2 are not necessarily the same as the ones in Lemma 3.1.2.

**Proof.** We keep the notation in (3.0.3), (3.0.6), and (3.1.2). That is, there exist a framed basis $\underline{c}$ for $\mathcal{M}$ and a framed system of generators $\underline{f}$ for $\text{Fil}^{n-1} \mathcal{M}$ such that $\text{Mat}_{\underline{f},\underline{c}}(\text{Fil}^{n-1} \mathcal{M})$, $\text{Mat}_{\underline{f},\underline{c}}(\phi_{n-1})$, $\text{Mat}_{\underline{c}}(N)$ are given as in (3.0.5), (3.0.6), and (3.0.7) respectively. Since $k_i \equiv k_i^{(0)} \text{ mod } (p - 1)$, we have $r_i = r_i^{(0)}$ for all $i \in \{0, 1, \ldots, n - 1\}$ by Lemma 3.1.2, following the notation of Lemma 3.1.2.
We start to prove the following claim: if \( n - 1 \geq i > j \geq 0 \) then

\[
e - (k_{i}^{(0)} - k_{j}^{(0)}) \geq n.
\]

Indeed, by the strongly generic assumption, Definition 3.0.3

\[
e - (k_{i}^{(0)} - k_{j}^{(0)}) = (p - 1) - (c_{i} + i - r_{i}^{(0)}) + (c_{j} + j - r_{j}^{(0)})
\]

\[
= (p - 1) - (c_{i} - c_{j}) - (i - j) + (r_{i}^{(0)} - r_{j}^{(0)})
\]

\[
\geq (p - 1) - (c_{n-1} - c_{0}) - (n - 1 - 0) + (1 - (n - 2))
\]

\[
\geq 3n - 4 - 2n + 4 = n.
\]

Note that this claim will be often used during the proof later.

We now diagonalize \( \text{Mat}_{\mathcal{L}^{\mathbb{F}}}(\phi_{n-1}) \) with some restriction on the powers of the entries of \( \text{Mat}_{\mathcal{L}^{\mathbb{F}}}(\text{Fil}^{n-1}\mathcal{M}) \).

Let \( V_{0} = \text{Mat}_{\mathcal{L}^{\mathbb{F}}}(\text{Fil}^{n-1}\mathcal{M}) \in \mathbb{M}_{n}^{\mathbb{F}}(\mathcal{S}) \) and \( A_{0} = \text{Mat}_{\mathcal{L}^{\mathbb{F}}}(\phi_{n-1}) \in \text{GL}_{n}^{\mathbb{F}}(\mathcal{S}) \). We also let \( V_{1} \in \mathbb{M}_{n}^{\mathbb{F}}(\mathcal{S}) \) be the matrix obtained from \( V_{0} \) by replacing \( u_{i,j} \) by \( u_{i,j}' \in \mathbb{S}^{0} \), and \( B_{1} \in \text{GL}_{n}^{\mathbb{F}}(\mathcal{S}) \) the matrix obtained from \( A_{0} \) by replacing \( w_{i,j} \) by \( w_{i,j}' \in \mathbb{S}^{0} \). It is straightforward to check that \( A_{0} \cdot V_{1} = V_{0} \cdot B_{1} \) if and only if for all \( i > j \)

\[
(3.3.4) \quad v_{i}v_{i,j}u_{i,j}^{[k_{i}^{(0)} - k_{j}^{(0)}]} + \sum_{s=j+1}^{i-1} w_{i,s}v_{s,j}u_{s,j}^{[k_{s}^{(0)} - k_{i}^{(0)}]_{1} + [k_{j}^{(0)} - k_{s}^{(0)}]_{1}} + w_{i,j}u_{i,j}^{[v_{i,j}^{(0)} + [k_{i}^{(0)} - k_{j}^{(0)}]]_{1}} = u_{i,j}'u_{i,j}^{[v_{i,j}'^{(0)} + [k_{j}^{(0)} - k_{i}^{(0)}]]_{1}} + \sum_{s=j+1}^{i-1} v_{s,i}w_{s,j}u_{s,j}^{[k_{s}^{(0)} - k_{i}^{(0)}]_{1} + [k_{j}^{(0)} - k_{s}^{(0)}]_{1}} + \nu_{j}v_{i,j}u_{i,j}^{[k_{i}^{(0)} - k_{j}^{(0)}]}.
\]

Note that the power of \( u \) in each term of \( (3.3.4) \) is congruent to \( [k_{j}^{(0)} - k_{i}^{(0)}]]_{1} \) modulo \( (e) \). It is immediate that for all \( i > j \) there exist \( v_{i,j}' \in \mathbb{S}^{0} \) and \( w_{i,j}' \in \mathbb{S}^{0} \) satisfying the equation \( (3.3.4) \) with the following additional properties: for all \( i > j \)

\[
(3.3.5) \quad \deg v_{i,j}' < r_{i,j}^{(0)} \epsilon.
\]

Letting \( \mathcal{L}' := \mathcal{L}A_{0} \), we have

\[
\text{Mat}_{\mathcal{L}'^{\mathbb{F}}}(\text{Fil}^{n-1}\mathcal{M}) = V_{1} \quad \text{and} \quad \text{Mat}_{\mathcal{L}'^{\mathbb{F}}}(\phi_{n-1}) = \phi(B_{1})
\]

where \( \mathcal{L}' = \mathcal{L}'V_{1} \), by Lemma 2.3.22. Note that \( \phi(B_{1}) \) is congruent to a diagonal matrix modulo \( (u^{ne}) \) by \( (3.3.3) \). We repeat this process one more time. We may assume that \( u_{i,j} \in u^{ne} \mathbb{S}^{0} \), i.e., that \( A_{0} \equiv B_{1} \) modulo \( (u^{ne}) \) where \( B_{1} \) is assumed to be a diagonal matrix. It is obvious that there exists an upper-triangular matrix \( V_{1} = (v_{i,j}u^{[p-1]k_{j}^{(0)} - k_{i}^{(0)}]}_{1}) \) whose entries have bounded degrees as in \( (3.3.5) \), satisfying the equation \( A_{0}V_{1} = V_{0}B_{1} \equiv \text{modulo } (u^{ne}) \). By Lemma 2.3.22 we get \( \text{Mat}_{\mathcal{L}'^{\mathbb{F}}}(\phi_{n-1}) = \phi(B_{1}) \) is diagonal. Hence, we may assume that \( \text{Mat}_{\mathcal{L}'^{\mathbb{F}}}(\phi_{n-1}) \) is diagonal and that \( \deg v_{i,j} \) in \( \text{Mat}_{\mathcal{L}'^{\mathbb{F}}}(\text{Fil}^{n-1}\mathcal{M}) \) is bounded as in \( (3.3.5) \), and we do so. Moreover, this change of basis do not change the shape of \( \text{Mat}_{\mathcal{L}}(N) \), so that we also assume that \( \text{Mat}_{\mathcal{L}}(N) \) is still as in \( (3.0.7) \).

We now prove that

\[
(3.3.6) \quad v_{i,j}u^{[k_{i}^{(0)} - k_{j}^{(0)}]} = \beta_{i,j}u^{r_{i,j}^{(0)} - (k_{i}^{(0)} - k_{j}^{(0)})}
\]

for all \( n - 1 \geq i > j \geq 0 \), where \( \beta_{i,j} \in \mathbb{F} \). Note that this is immediate for \( i = n - 1 \) and \( i = 1 \), since \( u_{i,j}^{(0)} = 1 \) if \( i = n - 1 \) or \( i = 1 \). To prove \( (3.3.6) \), we induct on \( i \). The case \( i = 1 \) is done as above. Fix \( p_{0} \in \{2, 3, \cdots, n-2\} \), and assume that \( (3.3.6) \) holds for all \( i \in \{1, 2, \cdots, p_{0} - 1\} \) and for all \( j < i \). We consider the subquotient \( \mathcal{M}_{p_{0},0} \) of \( \mathcal{M} \) defined in \( (3.0.8) \). By abuse of notation, we write \( \mathcal{L} = (e_{p_{0}}, \cdots, e_{0}) \) for the induced framed basis for \( \mathcal{M}_{p_{0},0} \) and \( f = (f_{p_{0}}, \cdots, f_{0}) \) for the induced framed system of generators for \( \text{Fil}^{n-1}\mathcal{M}_{p_{0},0} \).
We claim that for $p_0 \geq j \geq 0$

$$u^c N(f_j) \in \mathcal{S}_0 u^c f_j + \sum_{t=j+1}^{p_0} \mathcal{S}_0 u^{[k_j^{(0)} - k_t^{(0)}]} f_t,$$

Consider $N(f_j) = N(f_j - u^{r_j^{(0)}} e_j) + N(u^{r_j^{(0)}} e_j)$. It is easy to check that $N(f_j - u^{r_j^{(0)}} e_j)$ and $N(u^{r_j^{(0)}} e_j) + r_j^{(0)} e f_j$ are $\mathcal{S}_0$-linear combinations of $e_{n-1}, \ldots, e_{j+1}$, and they are, in fact, $\mathcal{S}_0$-linear combinations of $u^{[k_j^{(0)} - e_{n-1}]} e_{n-1}, \ldots, u^{[k_j^{(0)} - e_{j+1}]} e_{j+1}$ since they are $\omega^{k_j^{(0)}}$-invariant. Since $u^c N(f_j) \in \text{Fil}^{n-1}\mathcal{M} \supset u^{(n-1)c} \mathcal{M}$ and $u^c N(f_j) + r_j^{(0)} e u^c f_j = [N(f_j - u^{r_j^{(0)}} e_j)] + [N(u^{r_j^{(0)}} e_j) + r_j^{(0)} e f_j]$, we conclude that

$$u^c N(f_j) + r_j^{(0)} e u^c f_j \in \sum_{t=j+1}^{p_0} \mathcal{S}_0 u^{[k_j^{(0)} - k_t^{(0)}]} f_t,$$

which completes the claim.

Let $\text{Mat}_{\mathcal{L}}(N|_{\mathcal{M}_{p_0}}, 0) = \left(\gamma_{i,j}, u^{[k_j^{(0)} - k_i^{(0)}]}\right)$ where $\gamma_{i,j} = 0$ if $i \leq j$ and $\gamma_{i,j} \in \mathcal{S}_0$ if $i > j$. We also claim that

$$\gamma_{i,j} \in u^{[k_j^{(0)} - k_i^{(0)}]} \mathcal{S}_0$$

for $p_0 \geq i > j \geq 0$, which can be readily checked from the equation $cN\phi_{n-1}(f_j) = \phi_{n-1}(u^c N(f_j))$. (Note that $c = 1 \in \mathcal{S}$ as $E(u) = u^c + p$.) Indeed, we have

$$cN\phi_{n-1}(f_j) = (\nu_j e_j) = \nu_j \sum_{i=j+1}^{p_0} \gamma_{i,j} u^{[k_j^{(0)} - k_i^{(0)}]} e_i.$$  

On the other hand, since $\text{Mat}_{\mathcal{L}}(\phi_{n-1}|_{\mathcal{M}_{p_0}}, 0)$ is diagonal, the previous claim immediately implies that

$$\phi_r(u^c N(f_j)) \in \sum_{t=j+1}^{p_0} \mathcal{S}_0 u^{[k_j^{(0)} - k_t^{(0)}]} e_t.$$

Hence, we conclude the claim.

We now finish the proof of \[3.3.3\] by inducting on $p_0 - j$ as well. Write $v_{i,j} = \sum_{t=0}^{r^{(0)}_{j}} x_{i,j}^{(t)} u^c e$ for $x_{i,j}^{(t)} \in \mathcal{F}$. We need to prove $x_{p_0,j}^{(t)} = 0$ for $t \in \{0, 1, \ldots, r^{(0)}_{p_0} - 2\}$. Assume first $j = p_0 - 1$, and we compute $N(f_j)$ as follows:

$$N(f_{p_0-1}) = - \sum_{t=0}^{r^{(0)}_{p_0-1}} x_{p_0-1}^{(t)} [e(t+1) - (k_{p_0}^{(0)} - k_{p_0-1}^{(0)})] u^{(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})} e_{p_0}$$

$$\quad \quad \quad \quad \quad \quad \quad + \gamma_{p_0,p_0-1} u^{(r^{(0)}_{p_0-1}+1)e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})} e_{p_0} - r^{(0)}_{p_0-1} u^{r^{(0)}_{p_0-1}e} e_{p_0},$$

which immediately implies

$$N(f_{p_0-1}) \equiv \sum_{t=0}^{r^{(0)}_{p_0-1}} x_{p_0-1}^{(t)} [e(t+1) - (k_{p_0}^{(0)} - k_{p_0-1}^{(0)})] u^{(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})} e_{p_0}$$

$$\quad \quad \quad \quad \quad \quad \quad + \gamma_{p_0,p_0-1} u^{(r^{(0)}_{p_0-1}+1)e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})} e_{p_0}$$

modulo $\text{Fil}^{n-1}\mathcal{M}_{p_0}$. Since $\gamma_{p_0,p_0-1} \in u^{e-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})} \mathcal{S}_0$ and $e - (k_{p_0}^{(0)} - k_{p_0-1}^{(0)}) \geq n$ by \[3.3.3\], we get

$$N(f_{p_0-1}) \equiv \sum_{t=0}^{r^{(0)}_{p_0-1}} x_{p_0-1}^{(t)} [e(t+1) - (k_{p_0}^{(0)} - k_{p_0-1}^{(0)})] u^{(t+1)-(k_{p_0}^{(0)}-k_{p_0-1}^{(0)})} e_{p_0}$$
modulo $\text{Fil}^{n-1} M_{p_0,0}$, so that

$$u^e N(f_{p_0-1}) = \sum_{t=0}^{r_{p_0}^{(0)}-2} x_{p_0,p_0-1}^{(t)} (e^t \rho_{p_0-1}) + \sum_{t=0}^{r_{p_0}^{(0)}-1} (k_{p_0}^{(0)} - k_{p_0-1}^{(0)}) u^e (t+1) \rho_{p_0-1} \equiv 0 \pmod{\text{Fil}^{n-1} M_{p_0,0}}.$$ 

It is easy to check that

$$(3.3.7) \quad e^t \rho_{p_0-1} \equiv 0 \pmod{\text{Fil}^{n-1} M_{p_0,0}}.$$ 

modulo $(p)$ for all $0 \leq t \leq r_{p_0}^{(0)} - 2$. Indeed, $e^t \rho_{p_0-1} \equiv 0 \pmod{\text{Fil}^{n-1} M_{p_0,0}}$. This completes the proof of $3.3.6$ for $j = p_0 - 1$. Assume that $3.3.6$ holds for $i = p_0$ and $j \in \{p_0 - 1, p_0 - 2, \ldots, s + 1\}$. We compute $N(f_i)$ for $p_0 - 1 > s \geq 0$ as follows: using the induction hypothesis on $i \in \{1, 2, \ldots, p_0 - 1\}$

$$N(f_s) = - \sum_{t=0}^{r_{p_0}^{(0)} - 1} x_{p_0,s}^{(t)} (e(t+1) - (k_{p_0}^{(0)} - k_s^{(0)})) u^e (t+1) \equiv 0 \pmod{\text{Fil}^{n-1} M_{p_0,0}},$$

modulo $\text{Fil}^{n-1} M_{p_0,0}$, which immediately implies

$$N(f_s) = \sum_{t=0}^{r_{p_0}^{(0)} - 1} x_{p_0,s}^{(t)} (e^t - e(t+1) + (k_{p_0}^{(0)} - k_s^{(0)})) u^e (t+1) \equiv 0 \pmod{\text{Fil}^{n-1} M_{p_0,0}},$$

modulo $\text{Fil}^{n-1} M_{p_0,0}$. Now, from the induction hypothesis on $j \in \{p_0 - 1, p_0 - 2, \ldots, s + 1\}$,

$$u^e \sum_{i=s+1}^{r_{p_0}^{(0)} - 1} (k_{p_0}^{(0)} - k_s^{(0)}) u^e (k_{p_0}^{(0)} - k_s^{(0)}) \rho_{p_0-1} \equiv 0 \pmod{\text{Fil}^{n-1} M_{p_0,0}}.$$
and so we have

\[ u^s N(f_s) = \sum_{t=0}^{r_p(0)-2} x_p^{(0)} [r_p^{(0)} e - e(t + 1) + (k_p^{(0)} - k_{j}^{(0)}) u^{e(t+2)-(r_p^{(0)}-k_{j}^{(0)})} e_p] \]

modulo Fil^{n-1} \mathcal{M}_{p_0,0}. By the same argument as (3.3.7), one can readily check that \( r_p^{(0)} e - e(t + 1) + (k_p^{(0)} - k_{j}^{(0)}) \neq 0 \) modulo \((p)\) for all \(0 \leq t \leq r_p^{(0)} - 2\). Hence, we conclude that \( x_p^{(0)} e = 0 \) for all \(0 \leq t \leq r_p^{(0)} - 2\) as \( u^s N(f_s) \in \text{Fil}^{n-1} \mathcal{M}_{p_0,0}\), which completes the proof. \( \square \)

**Proposition 3.3.8.** Keep the assumptions and notation of Lemma 3.3.2. Assume further that \( \overline{p}_0 \) is maximally non-split and satisfies the conditions in (3.2.4).

Then \( \beta_{i,i-1} \in F^* \) for \( i \in \{1, 2, \ldots, n-1\} \) and we have the following identities: for \( 0 \leq j < j + 1 < i \leq n - 1 \)

\[
\text{FL}_{p}^{i,j}(\overline{p}_0) = \left[ \begin{array}{c} \beta_{j+1,j} \nu_{j+1} \cdots \nu_{i-1} : (-1)^{i-j+1} \det A'_{i,j} \end{array} \right] \in P^{1}(F)
\]

where

\[
A'_{i,j} = \begin{pmatrix}
\beta_{j+1,j} & \beta_{j+2,j} & \beta_{j+3,j} & \cdots & \beta_{i-1,j} & \beta_{i,j} \\
1 & \beta_{j+2,j+1} & \beta_{j+3,j+1} & \cdots & \beta_{i-1,j+1} & \beta_{i,j+1} \\
0 & 1 & \beta_{j+3,j+2} & \cdots & \beta_{i-1,j+2} & \beta_{i,j+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \beta_{i-1,i-2} & \beta_{i,i-2} \\
0 & 0 & 0 & \cdots & 1 & \beta_{i,i-1} \\
\end{pmatrix}
\]

**Proof.** We may assume \( c_0 = 0 \) by Lemma 3.2.8. We let \( V := \text{Mat}_{\mathcal{E}_F}(\text{Fil}^{n-1} \mathcal{M}) \) and \( A := \text{Mat}_{\mathcal{E}_F}(\phi_{n-1}) \) be as in the statement of Lemma 3.3.2. By Lemma 2.6.3 the \( \phi \)-module over \( F \otimes_{F_p} F_p((\overline{p})) \) defined by \( \mathfrak{M} := M_{F_p,((\overline{p}))}(\mathcal{M}_\ast) \) is described as follows:

\[
\text{Mat}_{\mathcal{E}_F}(\phi) = (U_{i,j})
\]

where

\[
U_{i,j} = \begin{cases} 
\nu_{j}^{-1} \cdot \overline{p}_0^{(0)} e & \text{if } i = j; \\
0 & \text{if } i > j; \\
\nu_{j}^{-1} \cdot \beta_{j,i} \cdot \overline{p}_0^{r_1(0)} e - (k_p^{(0)} - k_{j}^{(0)}) & \text{if } i < j 
\end{cases}
\]

in a framed basis \( \xi = (\xi_{n-1}, \xi_{n-2}, \cdots, \xi_0) \) with dual type \( \omega^{-k_p^{(0)}, n-1} \oplus \omega^{-k_p^{(0)}, n-2} \cdots \oplus \omega^{-k_p^{(0)}}. \)

By considering the change of basis \( \xi' = (\overline{p}_0^{k_p^{(0)}, n-1}, \overline{p}_0^{k_p^{(0)}, n-2}, \cdots, \overline{p}_0^{k_p^{(0)}}, \xi_0) \), \( \text{Mat}_{\mathcal{E}_F}(\phi) \) is described as follows:

\[
\text{Mat}_{\mathcal{E}_F}(\phi) = (V_{i,j})
\]

where

\[
V_{i,j} = \begin{cases} 
\nu_{j}^{-1} \cdot \overline{p}_0^{(0)} e(k_p^{(0)}) & \text{if } i = j; \\
0 & \text{if } i > j; \\
\nu_{j}^{-1} \cdot \beta_{j,i} \cdot \overline{p}_0^{r_1(0)} e(k_p^{(0)}) & \text{if } i < j. 
\end{cases}
\]

Since \( k_{i}^{(0)} = c_i + r_1^{(0)} \) for each \( n - 1 \geq i \geq 0 \), we easily see that the \( \phi \)-module \( \mathfrak{M}_0 \) is the base change via \( F \otimes_{F_p} F_p((\overline{p})) \to F \otimes_{F_p} F_p((\overline{p})) \) of the \( \phi \)-module \( \mathfrak{M}_0 \) over \( F \otimes_{F_p} F_p((\overline{p})) \) described by

\[
\text{Mat}_{\mathcal{E}_F}(\phi) = \begin{pmatrix}
\nu_{n-1}^{-1} p_{n-1}^{(n-1)+1} & 0 & \cdots & 0 \\
\nu_{n-1}^{-1} \beta_{n-1,n-2} p_{n-1}^{(n-1)+2} & \nu_{n-2}^{-1} p_{n-2}^{(n-2)+1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\nu_{n-1}^{-1} \beta_{n-1,0} p_{n-1}^{(n-1)+2} & \nu_{n-2}^{-1} \beta_{n-2,0} p_{n-2}^{(n-2)+2} & \cdots & \nu_{0}^{-1} p_{0}^{(0)} 
\end{pmatrix}
\]
in an appropriate basis \( \xi'' = (\xi''_{n-1}, \xi''_{n-2}, \ldots, \xi''_0) \), which can be rewritten as

\[
\text{Mat}_{\xi''}(\phi) = \begin{pmatrix}
\nu^{-1}_{n-1} & 0 & \cdots & 0 \\
\nu^{-1}_{n-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\nu^{-1}_{n-1} & \gamma_{n-1,0} & \gamma_{n-2,2} & \nu^{-1}_0
\end{pmatrix} \cdot \text{Diag} \left( \xi''_{c-1+n-1}, \xi''_{c+1}, \xi''_c \right) =: H'.
\]

By considering the change of basis \( \xi''' = \xi'' \cdot H' \) and then reversing the order of the basis \( \xi''' \), the Frobenius \( \phi \) of \( \mathfrak{M}_0 \) with respect to this new basis is described as follows:

\[
(3.3.9) \quad \text{Mat}(\phi) = \text{Diag} \left( \xi''_{c_0}, \xi''_{c+1}, \ldots, \xi''_{c-1+n-1} \right) \cdot \begin{pmatrix}
\nu_0^{-1} & \nu_1^{-1} \beta_{1,0} & \cdots & \nu_{n-1}^{-1} \beta_{n-1,0} \\
0 & \nu_1^{-1} & \cdots & \nu_{n-1}^{-1} \beta_{n-1,1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \nu_{n-1}^{-1}
\end{pmatrix}
= := H
\]

with respect to the new basis described as above.

The last displayed upper-triangular matrix \( H \) is the Frobenius of the Fontaine–Laffaille module \( M \) such that \( T^*_{\text{crys}}(M) \cong \mathfrak{p}_0 \cong T^*_{\text{st}}(M) \), by Lemma 2.6.4. Hence, we get the desired results (cf. Definition 3.2.4).

**Remark 3.3.10.** We emphasize that the matrix \( H \) is the Frobenius of the Fontaine–Laffaille module \( M \), with respect to a basis \( (e_0, e_1, \cdots, e_{n-1}) \) compatible with the filtration, such that \( T^*_{\text{crys}}(M) \cong \mathfrak{p}_0 \cong T^*_{\text{st}}(M) \), so that we can now apply the conditions in (3.2.2) as well as Definition 3.2.7 to the Breuil modules in Lemma 3.3.2. Moreover, \( H \) can be written as

\[
H = \begin{pmatrix}
1 & \beta_{1,0} & \cdots & \beta_{n-1,0} \\
0 & 1 & \cdots & \beta_{n-1,1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1
\end{pmatrix} \cdot \text{Diag} \left( \nu_0^{-1}, \nu_1^{-1}, \cdots, \nu_{n-1}^{-1} \right),
\]

so that we have \( (1, n)\mathfrak{p}_0 H^t \in B(\mathbf{F})\mathfrak{p}_0 B(\mathbf{F}) \) if and only if \( (1, n)\mathfrak{p}_0 (H^t)^t \in B(\mathbf{F})\mathfrak{p}_0 B(\mathbf{F}) \). Hence, \( \mathfrak{p}_0 \) being Fontaine–Laffaille generic is a matter only of the entries of the filtration of the Breuil modules if the Breuil modules are written as in Lemma 3.3.2.

### 3.4. Fontaine–Laffaille parameters vs Frobenius eigenvalues

In this section, we study further the Breuil modules of Lemma 3.3.2. We show that if the filtration is of a certain shape then a certain product of Frobenius eigenvalues (of the Breuil modules) corresponds to the newest Fontaine–Laffaille parameter, \( \text{FL}^{n-1.0}_n(\mathfrak{p}_0) \). To get such a shape of the filtration, we assume further that \( \mathfrak{p}_0 \) is Fontaine–Laffaille generic.

**Lemma 3.4.1.** Keep the assumptions and notation of Lemma 3.3.2. Assume further that \( \mathfrak{p}_0 \) is Fontaine–Laffaille generic (c.f. Definition 3.2.7).

Then \( \mathcal{M} \in \mathbf{F} \text{-BrMod}_{\text{id}}^{n-1} \) can be described as follows: there exist a framed basis \( \xi \) for \( \mathcal{M} \) and a framed system of generators \( \mathbf{f} \) for \( \text{Fil}^{n-1} \mathcal{M} \) such that

\[
\text{Mat}_{\xi}(\phi_{n-1}) = \text{Diag} \left( \mu_{n-1}, \mu_{n-2}, \cdots, \mu_0 \right)
\]

and

\[
\text{Mat}_{\xi}(\text{Fil}^{n-1} \mathcal{M}) = (U_{i,j})
\]
where

\[
U_{i,j} = \begin{cases} 
0 & \text{if } i = n - 1 \text{ and } j = 0; \\
0 & \text{if } 0 < i = j < n - 1; \\
x_{i,j} \cdot u_i^{(0)} e - (k_i^{(0)} - k_j^{(0)}) & \text{if } n - 1 > i > j; \\
u_i^{(0)} e + (k_i^{(0)} - k_j^{(0)}) & \text{if } i = 0 \text{ and } j = n - 1; \\
x_{0,j} \cdot u_i^{(0)} e + (k_j^{(0)} - k_i^{(0)}) & \text{if } i = 0 \leq j < n - 1; \\
0 & \text{otherwise.}
\end{cases}
\]

Here, \( \mu_i \in \mathbb{F}^\times \) and \( x_{i,j} \in \mathbb{F} \).

Moreover, we have the following identity:

\[
\Phi \Gamma_n^{1,0}(\mathcal{P}_0) = \prod_{i=1}^{n-2} \mu_i^{-1}.
\]

Due to the size of the matrix, we decide to describe the matrix \( \text{Mat} \phi_n \mathcal{J} (\Phi \Gamma_n^{1,0}) \) as (3.4.2). But for the reader we visualize the matrix \( \text{Mat} \phi_n \mathcal{J} (\Phi \Gamma_n^{1,0}) \) below, although it is less accurate:

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & u_i^{(0)} e - k_i^{(0)} \\
0 & u_{i-2}^{(0)} & \ldots & x_{n-2,1} u_{n-2}^{(0)} e - k_{n-2,1}^{(0)} & x_{n-2,0} u_{n-2}^{(0)} e - k_{n-2,0}^{(0)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
u_0^{(0)} e + k_{n-1,0}^{(0)} & x_{0,n-2} u_0^{(0)} e + k_{n-2,0} & \ldots & u_1^{(0)} e - k_1^{(0)} & x_{1,0} u_1^{(0)} e - k_{1,0}^{(0)} \\
x_{0,0} u_0^{(0)} e & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

where \( k_{i,j}^{(0)} := k_i^{(0)} - k_j^{(0)} \).

**Proof.** Let \( \mathfrak{m}_0 \) be a framed basis for \( \mathcal{M} \) and \( f_0 \) a framed system of generators for \( \Phi \Gamma_n^{1,0} \) such that \( V_0 := \text{Mat} \phi_n \mathcal{J} (\Phi \Gamma_n^{1,0}) \) and \( A_0 := \text{Mat} \phi_n \mathcal{J} (\phi_{n-1}) \) are given as in Lemma 3.3.2. So, in particular, \( V_0 \) is upper-triangular and \( A_0 \) is diagonal.

By Proposition 3.3.3, the upper-triangular matrix \( H \) in (3.3.9) is the Frobenius of the Fontaine–Laffaille module corresponding to \( \mathcal{P}_0 \), as in Definition 3.2.4. Since we assume that \( \mathcal{P}_0 \) is Fontaine–Laffaille generic, we have \( (1,n) v_0 H' \in B(\mathbb{F}) v_0 B(\mathbb{F}) \) as discussed right after Definition 3.2.4 so that we have \( u_0 H' v_0 \in (1,n) B(\mathbb{F}) v_0 B(\mathbb{F}) v_0 \). Equivalently, \( v_0 (H') v_0 \in (1,n) B(\mathbb{F}) v_0 B(\mathbb{F}) v_0 \) by Remark 3.3.10 where \( H' \) is defined in Remark 3.3.10. Hence, comparing \( V_0 \) with \( u_0 H' v_0 \), there exists a lower-triangular matrix \( C \in \text{GL}_n(\mathbb{S}) \) such that

\[
V_0 \cdot C = V_1 := (U_{i,j})_{0 \leq i,j \leq n-1}
\]

where \( U_{i,j} \) is described as in (3.4.2), since any matrix in \( u_0 B(\mathbb{F}) v_0 \) is lower-triangular. From the identity \( V_0 \cdot C = V_1 \), we have \( V_1 = \text{Mat} \phi_n \mathcal{J} (\Phi \Gamma_n^{1,0}) \) and \( A_1 := \text{Mat} \phi_n \mathcal{J} (\phi_{n-1}) = A_0 \cdot \phi(C) \) by Lemma 2.4.4 where \( e_1 := e_0 \) and \( f_1 := e_1 V_1 \). If \( i < j \), then \( (k_i^{(0)} - k_j^{(0)})_1 = k_i^{(0)} - k_j^{(0)} \geq n \) as \( \mathcal{P}_0 \) is strongly generic, so that \( A_1 \) is congruent to a diagonal matrix \( B'_2 \in \text{GL}_n(\mathbb{F}) \) modulo \( (u^{ne}) \) as \( C = (e_1 \cdot u(k_1^{(0)} - k_2^{(0)})) \) is a lower-triangular and \( A_0 \) is diagonal.

Let \( V_2 \) be the matrix obtained from \( V_1 \) by replacing \( x_{i,j} \) in (3.4.2) by \( y_{i,j} \), and \( B_2 = (b_{i,j}) \) is the diagonal matrix defined by taking \( b_{i,i} = b'_i \) if \( 1 \leq i \leq n - 2 \) and \( b_{i,i} = b'_{n-1-i,n-1-i} \) otherwise, where \( B'_2 = (b'_{i,j}) \). Then it is obvious that there exist \( y_{i,j} \in \mathbb{F} \) such that

\[
A_1 \cdot V_2 = V_1 \cdot B_2
\]

modulo \( (u^{ne}) \). Letting \( \mathfrak{e}_1 := e_1 \cdot A_1 \), we have \( V_2 = \text{Mat} \phi_n \mathcal{J} (\Phi \Gamma_n^{1,0}) \) and \( \text{Mat} \phi_n \mathcal{J} (\phi_{n-1}) = \phi(B_2) \) by Lemma 2.4.4. Note that \( A_2 := \text{Mat} \phi_n \mathcal{J} (\phi_{n-1}) \) is diagonal. Hence, there exist a framed basis for
\( \mathcal{M} \) and a framed system of generators for \( \text{Fil}^{n-1} \mathcal{M} \) such that \( \text{Mat}_{\mathcal{M}}(\phi_{n-1}) \) and \( \text{Mat}_{\mathcal{M}}(\text{Fil}^{n-1} \mathcal{M}) \) are described as in the statement.

We now prove the second part of the lemma. It is harmless to assume \( c_0 = 0 \) by Lemma 3.2.6. Let \( V := \text{Mat}_{\mathcal{M}}(\text{Fil}^{n-1} \mathcal{M}) \) and \( A := \text{Mat}_{\mathcal{M}}(\phi_{n-1}) \) be as in the first part of the lemma. By Lemma 2.6.3, the \( \phi \)-module over \( F \otimes_{F_p} F_p((\mathbb{F}_p)) \) defined by \( \mathfrak{M}_0 := M_{F_p((\mathbb{F}_p))}(\mathcal{M}^*) \) is described as follows: there exists a basis \( \mathcal{e} = (e_{n-1}, e_{n-2}, \cdots, e_0) \), compatible with decent data, such that \( \text{Mat}_{\mathcal{e}}(\phi) = (A^{-1} \tilde{V})^t \) where \( \tilde{V} \) and \( (A^{-1})^t \) are computed as follows:

\[
\tilde{V} = \begin{pmatrix}
0 & 0 & \cdots & 0 & e_0^{(0)} e_{n-1,0}^{(0)} \\
0 & e_0^{(0)} e_{n-2,0}^{(0)} & \cdots & 0 & x_{0,n-2} e_0^{(0)} e_{n-2,0}^{(0)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n-2,0} e_0^{(0)} e_{n-2,0}^{(0)} & \cdots & \cdots & e_0^{(0)} e_{n-2,0}^{(0)} & x_{0,0} e_0^{(0)} e_{n-2,0}^{(0)}
\end{pmatrix}
\]

and

\[
A^{-1} = \text{Diag}(\mu_{n-1}^{-1}, \mu_{n-2}^{-1}, \cdots, \mu_0^{-1}).
\]

By considering the change of basis \( \mathcal{e}' = (\mathcal{e}_{n-1}, \mathcal{e}_{n-2}, \cdots, \mathcal{e}_1, \mathcal{e}_0) \), we have

\[
\text{Mat}_{\mathcal{e}'}(\phi) = (\tilde{V}^t)' \cdot \text{Diag}(\mu_{n-1}^{-1}, \mu_{n-2}^{-1}, \cdots, \mu_0^{-1})
\]

where

\[
(\tilde{V}^t)' = \begin{pmatrix}
0 & 0 & \cdots & 0 & e^{(k_0)+r_0^{(0)}} \\
0 & e^{(k_0)+r_0^{(0)}} & \cdots & 0 & x_{0,n-2} e^{(k_0)+r_0^{(0)}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n-2,0} e^{(k_0)+r_0^{(0)}} & \cdots & \cdots & e^{(k_0)+r_0^{(0)}} & x_{0,0} e^{(k_0)+r_0^{(0)}}
\end{pmatrix}
\]

Since \( k_j^{(0)} + r_j^{(0)} = c_j + j \) for all \( j \), it is immediate that the \( \phi \)-module \( \mathfrak{M} \) over \( F \otimes_{F_p} F_p((\mathbb{F}_p)) \) is the base change via \( F \otimes_{F_p} F_p((\mathbb{F}_p)) \to F \otimes_{F_p} F_p((\mathbb{F}_p)) \) of the \( \phi \)-module \( \mathfrak{M}_0 \) over \( F \otimes_{F_p} F_p((\mathbb{F}_p)) \) described by

\[
\text{Mat}_{\mathcal{e}''}(\phi) = F'' \cdot \text{Diag}(e_{n-1} e_{n-2} e_{n-3} \cdots, e_0)
\]

where

\[
F'' = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & \mu_{n-2}^{-1} & \cdots & 0 & \mu_0^{-1} \\
0 & \mu_{n-2}^{-1} e_{n-2,0} & \cdots & 0 & \mu_0^{-1} x_{0,n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \mu_{n-2}^{-1} x_{n-2,0} & \cdots & \mu_{n-3}^{-1} x_{n-3,0} & \mu_1^{-1} x_{1,0} \\
0 & \mu_{n-2}^{-1} x_{n-1,0} & \cdots & \mu_{n-3}^{-1} x_{n-3,0} & \mu_0^{-1} x_{0,0}
\end{pmatrix}
\]

in an appropriate basis \( \mathcal{e}'' \).

Now, consider the change of basis \( \mathcal{e}''' = \mathcal{e}'' \cdot F'' \) and then reverse the order of the basis \( e''' \). Then the matrix of the Frobenius \( \phi \) for \( \mathfrak{M}_0 \) with respect to this new basis is given by

\[
\text{Diag}(e_{n-1} e_{n-2} e_{n-3} \cdots, e_0) \cdot F
\]
where

\[
F = \begin{pmatrix}
\mu_0^{-1}x_{0,0} & \mu_1^{-1}x_{1,0} & \mu_2^{-1}x_{2,0} & \cdots & \mu_{n-2}^{-1}x_{n-2,0} & \mu_{n-1}^{-1} \\
\mu_0^{-1}x_{0,1} & \mu_1^{-1} & \mu_2^{-1}x_{2,1} & \cdots & \mu_{n-2}^{-1}x_{n-2,1} & 0 \\
\mu_0^{-1}x_{0,2} & 0 & \mu_2^{-1} & \cdots & \mu_{n-2}^{-1}x_{n-2,2} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\mu_0^{-1}x_{0,n-2} & 0 & 0 & \cdots & \mu_{n-2}^{-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

By Lemma 2.6.4, there exists a Fontaine–Laffaille module \( M \) such that \( F(M) = \mathfrak{M}_0 \) with Hodge–Tate weights \((c_0, c_1 + 1, \cdots, c_{n-1} + n - 1)\) and \( \text{Mat}_{\mathfrak{M}}(\phi_s) = F \) for some basis \( \mathfrak{M} \) of \( M \) compatible with the Hodge filtration on \( M \). On the other hand, since \( \text{T}^{\text{cris}}_*(M) \cong \mathfrak{T}_0 \), there exists a basis \( \mathfrak{M}' \) of \( M \) compatible with the Hodge filtration on \( M \) such that

\[
\text{Mat}_{\mathfrak{M}}(\phi_s) = \begin{pmatrix}
w_0 & w_{0,1} & \cdots & w_{0,n-2} & w_{0,n-1} \\
0 & w_1 & \cdots & w_{1,n-2} & w_{1,n-1} \\
0 & 0 & \cdots & w_{n-2} & w_{n-2,n-1} \\
0 & 0 & \cdots & 0 & w_{n-1}
\end{pmatrix}. \tag{3.3.1}
\]

where \( w_{i,j} \in \mathbf{F} \) and \( w_i \in \mathbf{F}^\times \) by Lemma 3.2.4. Since both \( \mathfrak{M} \) and \( \mathfrak{M}' \) are compatible with the Hodge filtration on \( M \), there exists a unipotent lower-triangular \( n \times n \)-matrix \( U \) such that

\[
U \cdot F = G.
\]

Note that we have \( w_{0,n-1} = \mu_{n-1}^{-1} \) by direct computation.

Let \( U' \) be the \((n-1) \times (n-1)\)-matrix obtained from \( U \) by deleting the right-most column and the lowest row, and \( F'(\text{resp. } G') \) the \((n-1) \times (n-1)\)-matrix obtained from \( F \) (resp. \( G \)) by deleting the left-most column and the lowest row. Then they still satisfy \( G' = U' \cdot F' \) as \( U \) is a lower-triangular unipotent matrix, so that

\[
\text{FL}^{n-1,0}_{\mathfrak{T}_0} = \left[w_{0,n-1} : (-1)^n \det G' \right] = \left[ \mu_{n-1}^{-1} : (-1)^n \det F' \right] = \left[ 1 : \prod_{i=1}^{n-2} \mu_i^{-1} \right],
\]

which completes the proof. \( \square \)

**Proposition 3.4.3.** Keep the assumptions and notation of Lemma 3.3.1.

Then \( \mathcal{M} \in \mathbf{F} \text{-BrMod}^{\text{cris}}_{\mathcal{D}_d} \) can be described as follows: there exist a framed basis \( \mathfrak{M} \) for \( \mathcal{M} \) and a framed system of generators \( f \) for \( \text{Fil}^{n-1}_{\mathcal{M}} \) such that

\[
\text{Mat}_{\mathfrak{M}}(\text{Fil}^{n-1}_{\mathcal{M}}) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & u^{-(k_{n-1}^{(0)} - k_{0}^{(0)})} \\
0 & u^{(n-2)e} & 0 & \cdots & 0 & 0 \\
0 & 0 & u^{(n-3)e} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & u^{e} & 0 \\
u^{(n-2)e+(k_{n-1}^{(0)} - k_{0}^{(0)})} & 0 & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

Moreover, if we let

\[
\text{Mat}_{\mathfrak{M}}(\phi_{n-1}) = \left( \alpha_{i,j} u^{k_{j}^{(0)} - k_{i}^{(0)}} \right)
\]

for \( \alpha_{i,i} \in \mathfrak{S}^\times_0 \) and \( \alpha_{i,j} \in \mathfrak{S}_0 \) if \( i \neq j \) then we have the following identity:

\[
\text{FL}^{n-1,0}_{\mathfrak{T}_0} = \prod_{i=1}^{n-2} (\alpha_{i,i}^{(0)})^{-1} = \prod_{i=1}^{n-2} \mu_i^{-1}
\]
where \( \alpha_{ij}^{(0)} \in \mathbb{F} \) is determined by \( \alpha_{ij}^{(0)} \equiv \alpha_{i,j} \text{ modulo } (u^f) \).

Note that \( \text{Mat}_{S,F}(\phi_{n-1}) \) always belong to \( GL_n(\overline{\mathbb{S}}) \) as \( S \) and \( F \) are framed.

**Proof.** We let \( \underline{\omega} \) (resp. \( \underline{\eta} \)) be a framed basis for \( M \) and \( f_0 \) (resp. \( f_i \)) be a framed system of generators for \( \text{Fil}_{i-1}^n M \) such that \( \text{Mat}_{S,F}(\text{Fil}_{i-1}^n M) \) and \( \text{Mat}_{S,F}(\text{Fil}_{i-1}^n M) \) are given as in the statement of Lemma 3.3.1 (resp. in the statement of Proposition 3.3.3). We also let \( V_0 = \text{Mat}_{S,F}(\text{Fil}_{i-1}^n M) \) and \( A_0 = \text{Mat}_{S,F}(\phi_{n-1}) \) as well as \( \overline{V_1} = \text{Mat}_{S,F}(\text{Fil}_{i-1}^n M) \) and \( \overline{A}_1 = \text{Mat}_{S,F}(\phi_{n-1}). \)

It is obvious that there exist \( R = (r_{i,j}^{(0)} u^{k_{ij}^{(0)} - k_{ij}^{(0)}}) \) and \( C = (c_{i,j} u^{k_{ij}^{(0)} - k_{ij}^{(0)}}) \) in \( GL_n(\overline{\mathbb{S}}) \) such that

\[
R \cdot V_0 \cdot C = V_1 \text{ and } \underline{\omega} = \underline{\eta} R^{-1}
\]

for \( r_{i,j} \) and \( c_{i,j} \) in \( \overline{S}_0 \). From the first equation above, we immediately get the identities:

\[
r_{i,j}^{(0)} \cdot c_{0,0}^{(0)} = 1 = r_{0,0}^{(0)}, c_{i,n-1}^{(0)} \text{ and } r_{i,i}^{(0)} \cdot c_{i,i}^{(0)} = 1
\]

for \( 0 < i < n - 1 \), where \( r_{i,j}^{(0)} \in \mathbb{F} \) (resp. \( c_{i,j}^{(0)} \in \mathbb{F} \)) is determined by \( r_{i,j}^{(0)} = r_{i,j} \text{ modulo } (u^f) \) (resp. \( c_{i,j}^{(0)} \equiv c_{i,j} \text{ modulo } (u^f) \)). By Lemma 2.4.3, we see that \( A_1 = R \cdot A_0 \cdot \phi(C) \).

Hence, if we let \( A_1 = \left( \alpha_{i,j}^{(0)} u^{k_{ij}^{(0)} - k_{ij}^{(0)}} \right) \) then

\[
r_{i,i}^{(0)} \cdot \mu_{i,i}^{(0)} = \alpha_{i,i}^{(0)}
\]

for each \( 0 < i < n - 1 \) since \( R \) and \( C \) are diagonal modulo \( (u) \), so that we have

\[
\prod_{i=1}^{n-2} \mu_i = \prod_{i=1}^{n-2} \phi^{(0)}_{i,i}
\]

which completes its proof. \( \square \)

Note that the matrix in the statement of Proposition 3.3.3 gives rise to the elementary divisors of \( M/\text{Fil}^{n-1} M \).

### 3.5. Filtration of strongly divisible modules

In this section, we describe the filtration of the strongly divisible modules lifting the Breuil modules described in Proposition 3.3.3. Throughout this section, we keep the notation \( r_{i,j}^{(0)} \) as in (3.3.1) as well as \( k_{ij}^{(0)} \).

We start to recall the following lemma, which is easy to prove but very useful.

**Lemma 3.5.1.** Let \( 0 < f \leq n \) be an integer, and let \( \mathcal{M} \in \mathcal{O}_E \cdot \text{Mod}_{st}^{n-1} \) be a strongly divisible module corresponding to a lattice in a potentially semi-stable representation \( \rho : G_{\mathbb{Q}_p} \rightarrow \text{GL}_n(E) \) with Hodge–Tate weights \( \{ -(n-1), -(n-2), \ldots, 0 \} \) and Galois type of niveau \( f \) such that \( T_{st}^{Q_p, n-1}(\mathcal{M}) \otimes_{\mathcal{O}_E} F \cong \mathfrak{m}_n \).

If we let

\[
X^{(i)} := \left( \frac{\text{Fil}_{n-1}^i M \cap \text{Fil}^i S \cdot \mathcal{M}}{\text{Fil}^{n-1} S \cdot \mathcal{M}} \right) \otimes_{\mathcal{O}_E} E
\]

for \( i \in \{ 0, 1, \ldots, n-1 \} \), then for any character \( \xi : \text{Gal}(K/K_0) \rightarrow K^\times \) we have that the \( \xi \)-isotypical component \( X^{(i)}_\xi \) of \( X^{(i)} \) is a free \( K_0 \otimes E \)-module of finite rank

\[
\text{rank}_{K_0 \otimes \mathbb{Q}_p} E X^{(i)}_\xi \cong \frac{n(n-1)}{2} - \frac{i(i+1)}{2}.
\]

Moreover, multiplication by \( u \in S \) induces an isomorphism \( X^{(i)}_\xi \xrightarrow{\sim} X^{(i)}_\xi \).

\[\text{□}\]
Proof. Since $\rho$ has Hodge–Tate weights $\{-(n-1), -(n-2), \ldots, 0\}$, by the analogue with $E$-coefficients of [Bre97], Proposition A.4, we deduce that

$$\text{Fil}^{n-1}D = \text{Fil}^{n-1}S_E\hat{f}_{n-1} \oplus \text{Fil}^{n-2}S_E\hat{f}_{n-2} \oplus \cdots \oplus \text{Fil}^1S_E\hat{f}_1 \oplus S_E\hat{f}_0$$

for some $S_E$-basis $\hat{f}_0, \ldots, \hat{f}_{n-1}$ of $D$, where $D := \hat{M}(\frac{1}{p}) \cong S_E \oplus E_1^{Q_{p,n-1}}(V)$, so that we also have

$$\text{Fil}^{n-1}D \cap \text{Fil}^1S_ED = \text{Fil}^{n-1}S_E\hat{f}_{n-1} \oplus \text{Fil}^{n-2}S_E\hat{f}_{n-2} \oplus \cdots \oplus \text{Fil}^1S_E\hat{f}_1 \oplus S_E\hat{f}_0.$$

Since $\rho \cong T^{Q_{p,n-1}}(\hat{M}) \otimes E$ is a $G_{Q_p}$-representation, $\text{Fil}^i(K \otimes \hat{M}_D^{Q_{p,n-1}}(\rho)) \cong K \otimes Q_p \text{Fil}^iD_{dr}(\rho \otimes \varepsilon^{1-n})$, so that $X^{(i)} \cong \text{Fil}^{n-1}D/\text{Fil}^iSD$ is a free $K \otimes Q_p$-$E$-module. Since $\frac{S_E}{\text{Fil}^{n-1}SD} \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{1-n}(K \otimes Q_p)E)^{u^jE(u^i)}$, we have rank$_{K \otimes Q_p}E X^{(i)} = \left[\frac{n(n-1)}{2} - \frac{i(i+1)}{2}\right] e$. We note that $\text{Gal}(K/K_0)$ acts semi-simply and that multiplication by $u$ gives rise to an $K \otimes Q_p$-$E$-linear isomorphism on $S_E/\text{Fil}S_E$ which cyclically permutes the isotypical components, which completes the proof. \(\square\)

Note that Lemma 3.5.1 immediately implies that

$$\text{rank}_{K \otimes Q_p}E X^{(i)} - \text{rank}_{K \otimes Q_p}E X^{(i+1)} = i + 1.$$ (3.5.2)

We will use this fact frequently to prove the main result, Proposition 3.5.4, in this subsection.

To describe the filtration of strongly divisible modules, we need to analyze the $\text{Fil}^{n-1}M$ of the Breuil modules $M$ we consider.

**Lemma 3.5.3.** Keep the notation and assumptions of Lemma 3.3.3.

(i) If $u^e$ is an elementary divisor of $M/\text{Fil}^{n-1}M$ then

$$e - (k_{n-1}^{(0)} - k_0^{(0)}) \leq a \leq (n-2)e + (k_{n-1}^{(0)} - k_0^{(0)}).$$

Moreover, $\text{FL}^{n-1,0}(\mathfrak{p}_0) \neq \infty$ (resp. $\text{FL}^{n-1,0}(\mathfrak{p}_0) \neq 0$) if and only if $u^{e-(k_{n-1}^{(0)} - k_0^{(0)})}$ (resp. $u^{(n-2)e+(k_{n-1}^{(0)} - k_0^{(0)})}$) is an elementary divisor of $M/\text{Fil}^{n-1}M$.

(ii) If we further assume that $\mathfrak{p}_0$ is Fontaine–Laffaille generic, then

$$\{u^{(n-2)e+(k_{n-1}^{(0)} - k_0^{(0)})}, u^{(n-2)e}, u^{(n-3)e}, \ldots, u^e, u^{-(k_{n-1}^{(0)} - k_0^{(0)})}\}$$

are the elementary divisors of $M/\text{Fil}^{n-1}M$.

**Proof.** The first part of (i) is obvious since one can obtain the Smith normal form of $\text{Mat}_{\mathbb{Z}/p}\text{Fil}^{n-1}M$ by elementary row and column operations. By Proposition 3.3.8, we know that $\text{FL}^{n-1,0}(\mathfrak{p}_0) \neq \infty$ if and only if $\beta_{n-1,0} \neq 0$. Since $u^{e-(k_{n-1}^{(0)} - k_0^{(0)})}$ has the minimal degree among the entries of $\text{Mat}_{\mathbb{Z}/p}\text{Fil}^{n-1}M$, we conclude the equivalence statement for $\text{FL}^{n-1,0}(\mathfrak{p}_0) \neq \infty$ holds. The last part of (i) is immediate from the other equivalence statement, $\text{FL}^{n-1,0}(\mathfrak{p}_0) \neq \infty$ if and only if $\beta_{n-1,0} \neq 0$, by considering $M^*$ and using Lemma 3.2.6 (vi).

Part (ii) is obvious from Proposition 3.4.3. \(\square\)

**Proposition 3.5.4.** Assume that $\mathfrak{p}_0$ is Fontaine–Laffaille generic and keep the notation $\mathfrak{r}_i^{(0)}$ as in 3.3.1 as well as $k_i^{(0)}$. Let $\hat{M} \in \mathcal{O}_E\text{-Mod}_{d}^{n-1}$ be a strongly divisible module corresponding to a lattice in a potentially semi-stable representation $\rho : G_{Q_p} \to \text{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1} \omega_i^{\mathfrak{r}_i^{(0)}}$ and Hodge–Tate weights $\{-(n-1), -(n-2), \ldots, 0\}$ such that $T_{st}^{Q_{p,n-1}}(\hat{M}) \otimes_{\mathcal{O}_E} F \cong \mathfrak{p}_0$. 

Then there exists a framed basis \((\hat{e}_{n-1}, \hat{e}_{n-2}, \ldots, \hat{e}_0)\) for \(\hat{\mathcal{M}}\) and a framed system of generators \((\hat{f}_{n-1}, \hat{f}_{n-2}, \ldots, \hat{f}_0)\) for \(\text{Fil}^{n-1} \hat{\mathcal{M}}\) modulo \(\text{Fil}^{n-1} S \cdot \mathcal{M}\) such that \(\text{Mat}_{\hat{\mathcal{E}}} \text{Fil}^{n-1} \hat{\mathcal{M}}\) is described as follows:

\[
\begin{pmatrix}
\frac{-p^{n-1}}{\alpha} & 0 & 0 & \ldots & 0 & u^{e-(k^{0})_{n-1}-k^{0})} \\
0 & E(u)^{n-2} & 0 & \ldots & 0 & 0 \\
0 & 0 & E(u)^{n-3} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & E(u) & 0 \\
1^{(0)}_{n-1}-k^{0} & \sum_{i=0}^{n-2} p^{n-2-i} E(u)^{i} & 0 & 0 & \ldots & 0 & \alpha
\end{pmatrix}
\]

where \(\alpha \in \mathcal{O}_E\) with \(0 < v_p(\alpha) < n - 1\).

Proof. Note that we write the elements of \(\hat{\mathcal{M}}\) in terms of coordinates with respect to a framed basis \(\hat{\mathcal{E}} := (\hat{e}_{n-1}, \hat{e}_{n-2}, \ldots, \hat{e}_0)\). We let \(\mathcal{M} := \hat{\mathcal{M}} \otimes_{\mathcal{F}} \mathcal{S}\), which is a Breuil module of weight \(n - 1\) and of type \(\bigoplus_{i=0}^{n-1} \omega^{k^{i}}\) by Proposition 2.4.3. Note also that \(\mathcal{M}\) can be described as in Proposition 3.4.3 and we assume that \(\mathcal{M}\) has such a framed basis for \(\mathcal{M}\) and such a framed system of generators for \(\text{Fil}^{n-1} \mathcal{M}\).

Let \(\hat{f}_0 = \left(\begin{array}{c}
u^{e-(k^{0})_{n-1}-k^{0})} \sum_{k=0}^{n-2} x_{n-1,k} E(u)^{k} \\
u^{e-(k^{0})_{n-2}-k^{0})} \sum_{k=0}^{n-2} x_{n-2,k} E(u)^{k} \\
\vdots \\
u^{e-(k^{0})_{1}-k^{0})} \sum_{k=0}^{n-2} x_{1,k} E(u)^{k} \\
\sum_{k=0}^{n-2} x_{0,k} E(u)^{k}
\end{array}\right) \in \left(\frac{\text{Fil}^{n-1} \hat{\mathcal{M}}}{\text{Fil}^{n-1} S \mathcal{M}}\right) \omega^{k^{0}}
\]

where \(x_{i,j} \in \mathcal{O}_E\). The vector \(\hat{f}_0\) can be written as follows:

\[
\hat{f}_0 = \left(\begin{array}{c}
u^{e-(k^{0})_{n-1}-k^{0})} \sum_{k=0}^{n-2} x_{n-1,k} E(u)^{k} \\
u^{e-(k^{0})_{n-2}-k^{0})} \sum_{k=0}^{n-2} x_{n-2,k} E(u)^{k} \\
\vdots \\
u^{e-(k^{0})_{1}-k^{0})} \sum_{k=0}^{n-2} x_{1,k} E(u)^{k} \\
\sum_{k=0}^{n-2} x_{0,k} E(u)^{k}
\end{array}\right) + \left(\begin{array}{c}0 \\
0 \\
\vdots \\
0 \\
x_{0,0} + \sum_{k=1}^{n-2} x_{0,k} p^{k}
\end{array}\right).
\]

By (ii) of Lemma 4.5.2, we know that \(\nu^{e-(k^{0})_{n-1}-k^{0})}\) is an elementary divisor of \(\mathcal{M}/\text{Fil}^{n-1} \mathcal{M}\) and all other elementary divisors have bigger powers, so that we may assume \(v_p(x_{n-1,0}) = 0\). Since \(\text{Fil}^{n-1} \mathcal{M} \subseteq \nu^{e-(k^{0})_{n-1}-k^{0})} \mathcal{M}\), we must have \(v_p(x_{0,0}) > 0\). So \(\hat{\mathcal{E}} := (\hat{e}_{n-1}, \hat{e}_{n-2}, \ldots, \hat{e}_0)\) is a framed basis for \(\hat{\mathcal{M}}\) by Nakayama lemma and we have the following coordinates of \(\hat{f}_0\) with respect to \(\hat{\mathcal{E}}:\)

\[
\hat{f}_0 = \left(\begin{array}{c}
u^{e-(k^{0})_{n-1}-k^{0})} \\
0 \\
\vdots \\
0 \\
\alpha
\end{array}\right) \in \left(\frac{\text{Fil}^{n-1} \hat{\mathcal{M}}}{\text{Fil}^{n-1} S \mathcal{M}}\right) \omega^{k^{0}}
\]

for \(\alpha \in \mathcal{O}_E\) with \(v_p(\alpha) > 0\).
Since $u^{k_0} f_0 \in \left( \frac{\Fil^{n-1} M}{\Fil^{n-1} SM} \right) \omega^{k_1}$, there exists $\hat{f}_1$ such that

$$\hat{f}_1 = \left( \begin{array}{c} u^{-(k_0 - k_1)} \\ u^{-(k_0 - k_1)} \sum_{k=0}^{n-2} y_{n-2,k} E(u)^k \\ \vdots \\ u^{k_0 - k_0} \sum_{k=0}^{n-2} y_{k,0} E(u)^k \\ \end{array} \right) \in \left( \frac{\Fil^{n-1} M}{\Fil^{n-1} SM} \right) \omega^{k_1},$$

where $y_{i,j} \in \mathcal{O}_E$. By Lemma 3.5.1 we have $y_{i,0} = 0$ for all $i$: otherwise, both $u^{k_0 - k_0} f_0$ and $\hat{f}_1$ belong to $X^{(0)} \omega^{k_1}$ which violates Lemma 3.5.2. Since $u^e$ is an elementary divisor of $M/\Fil^{n-1} M$ by (ii) of Lemma 3.5.3 we may also assume $y_{1,1} = 1$. Hence, by the obvious change of basis we get $\hat{f}_1$ as follows:

$$\hat{f}_1 = E(u) \left( \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{array} \right) \in \left( \frac{\Fil^{n-1} M}{\Fil^{n-1} SM} \right) \omega^{k_1}.$$

By the same arguments, we get $\hat{f}_i \in \left( \frac{\Fil^{n-1} M}{\Fil^{n-1} SM} \right) \omega^{k_1}$ for $i = 1, 2, \ldots, n - 2$ as in the statement.

Note that the elements in the set

$$\{ u^{k_0} f_0, E(u)u^{k_0} f_0, \ldots, E(u)^{n-2} u^{k_0} f_0 \} \cup \{ u^{k_1} f_1, E(u)u^{k_1} f_1, \ldots, E(u)^{n-3} u^{k_1} f_1 \} \cup \cdots \cup \{ u^{k_{n-1}} f_{n-1} \}$$

are linearly independent in $X^{(0)} \omega^{k_{n-1}}$ over $E$, so that the set forms a basis for $X^{(0)} \omega^{k_{n-1}}$ by Lemma 3.5.1. Hence, $\hat{f}_{n-1}$ is a linear combination of those elements over $E$. We have

$$u^{k_0} \left( \sum_{i=0}^{n-2} p^{n-2-i} E(u)^i \right) \hat{f}_0 = \left( \begin{array}{c} -p^{n-1} \\ 0 \\ \vdots \\ 0 \end{array} \right) \alpha u^{k_0} \left( \sum_{i=0}^{n-2} p^{n-2-i} E(u)^i \right) \hat{f}_0 \in \left( \frac{\Fil^{n-1} M}{\Fil^{n-1} SM} \right) \omega^{k_{n-1}}.$$

Hence, we may let

$$\hat{f}_{n-1} := \frac{1}{\alpha} u^{k_0} \left( \sum_{i=0}^{n-2} p^{n-2-i} E(u)^i \right) \hat{f}_0 \in \left( \frac{\Fil^{n-1} M}{\Fil^{n-1} SM} \right) \omega^{k_{n-1}},$$

since $u^{(n-2)e+(k_0 - k_1)}$ is an elementary divisor for $M/\Fil^{n-1} M$ by (ii) of Lemma 3.5.3. Moreover, $v_p \left( \frac{2^{n-1}}{\alpha} \right) > 0$ since $\Fil^{n-1} M \subseteq u^{-(k_0 - k_1)} M \subseteq u M$ by Proposition 3.3.8.

It is obvious that $\hat{f}_i$ mod $(\omega_E, \Fil^n S)$ generate $M/\Fil^{n-1} M$ for $M$ written as in Proposition 3.3.8. By Nakayama Lemma, we conclude that the $\hat{f}_i$ generate $\hat{M}/\Fil^{n-1} \hat{M}$, which completes the proof.

\textbf{Corollary 3.5.5.} Keep the notation and assumptions of Proposition 3.5.4 and let $(\lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_0) \in (\mathcal{O}_E)^n$
be the Frobenius eigenvalues on the \((\bar{\omega}^{k_{n-1}}, \bar{\omega}^{k_{n-2}}, \cdots, \bar{\omega}^{k_0})\)-isotypic component of \(D^{\mathbb{Q}_p,n-1}(\rho)\). Then
\[
v_p(\lambda_i) = \begin{cases} 
  v_p(\alpha) & \text{if } i = n - 1 \\
  (n - 1) - i & \text{if } n - 1 > i > 0 \\
  (n - 1) - v_p(\alpha) & \text{if } i = 0.
\end{cases}
\]

**Proof.** The proof goes parallel to the proof of [HLM], Corollary 2.4.11. \qed

### 3.6. Reducibility of certain lifts

In this section, we let \(1 \leq f \leq n\) and \(e = p^f - 1\), and we prove that every potentially semi-stable lift of \(\mathcal{E}_0\) with Hodge–Tate weights \(\{-(n - 1), -(n - 2), \cdots, 0\}\) and certain prescribed Galois types \(\bigoplus_{i=0}^{n-1} \bar{\omega}^{k_i}\) is reducible. We emphasize that we only assume that \(\mathcal{E}_0\) is generic (cf. Definition 3.0.3) for the results in this section.

**Proposition 3.6.1.** Assume that \(\mathcal{E}_0\) is generic, and let \((k_{n-1}, k_{n-2}, \cdots, k_0)\) be an \(n\)-tuple of integers. Assume further that \(k_0 \equiv (p^f - 1 + p^{f-2} + \cdots + p + 1)\) modulo \(e\) and that \(k_i\) are pairwise distinct modulo \(e\).

Then every potentially semi-stable lift of \(\mathcal{E}_0\) with Hodge–Tate weights \(\{-(n - 1), -(n - 2), \cdots, 0\}\) and Galois types \(\bigoplus_{i=0}^{n-1} \bar{\omega}^{k_i}\) is an extension of a 1-dimensional potentially semi-stable lift of \(\mathcal{E}_0\) with Hodge–Tate weight 0 and Galois type \(\bar{\omega}^{k_0}\) by an \((n - 1)\)-dimensional potentially semi-stable lift of \(\mathcal{E}_0\) with Hodge–Tate weights \(\{-(n - 1), -(n - 2), \cdots, 1\}\) and Galois types \(\bigoplus_{i=1}^{n-1} \bar{\omega}^{k_i}\).

Note that if \(f = 1\) then the assumption that \(\mathcal{E}_0\) is generic implies that \(k_i\) are pairwise distinct modulo \(e\) by Lemma 3.1.2. In fact, we believe that this is true for any \(1 \leq f \leq n\), but this requires extra works as we did in Lemma 3.1.2. Since we will need the results in this section only when \(f = 1\), we will add the assumption that \(k_i\) are pairwise distinct modulo \(e\) in the proposition.

**Proof.** Let \(\hat{\mathcal{M}} \in \mathcal{O}_E\text{-Mod}^{n-1}_{\text{dd}}\) be a strongly divisible module corresponding to a Galois stable lattice in a potentially semi-stable representation \(\rho: G_{\mathbb{Q}_p} \to \text{GL}_n(E)\) with Galois type \(\bigoplus_{i=0}^{n-1} \bar{\omega}^{k_i}\) and Hodge–Tate weights \(\{-(n-1), -(n-2), \cdots, 0\}\) such that \(T^{\mathbb{Q}_p,n-1}(\hat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbb{F} \cong \mathcal{E}_0\). We also let \(\mathcal{M}\) be the Breuil module corresponding to the mod \(p\) reduction of \(\hat{\mathcal{M}}\). \(\hat{\mathcal{M}}\) (resp. \(\mathcal{M}\)) is of inertial type \(\bigoplus_{i=0}^{n-1} \bar{\omega}^{k_i}\) (resp. \(\bigoplus_{i=0}^{n-1} \bar{\omega}^{k_i}\)) by Proposition 2.3.4.

We let \(\mathcal{E} = (f_{n-1}, f_{n-2}, \cdots, f_0)\) (resp. \(\tilde{\mathcal{E}} = (\tilde{f}_{n-1}, \tilde{f}_{n-2}, \cdots, \tilde{f}_0)\)) be a framed system of generators for \(\text{Fil}^{n-1}\mathcal{M}\) (resp. for \(\text{Fil}^{n-1}\hat{\mathcal{M}}\)). We also let \(\tilde{\mathcal{E}} = (\tilde{e}_{n-1}, e_{n-2}, \cdots, e_0)\) (resp. \(\tilde{\mathcal{E}} = (\tilde{e}_{n-1}, \tilde{e}_{n-2}, \cdots, \tilde{e}_0)\)) be a framed basis for \(\mathcal{M}\) (resp. for \(\hat{\mathcal{M}}\)). If \(x = a_{n-1}e_{n-1} + \cdots + a_0e_0 \in \mathcal{M}\), we will write \([x]_{\mathcal{E}_i}\) for \(a_i\) for \(i \in \{0, 1, \cdots, n - 1\}\). We define \([x]_{\tilde{\mathcal{E}}}\) for \(x \in \hat{\mathcal{M}}\) in the obvious similar way. We may assume that \(\text{Mat}_{\mathcal{E}}(\text{Fil}^{n-1}\mathcal{M}), \text{Mat}_{\tilde{\mathcal{E}}}(\phi_{n-1}),\) and \(\text{Mat}_{\tilde{\mathcal{E}}}(N)\) are written as in \([3.0.4.], [3.0.5.],\), and \([3.0.6.]\) respectively, and we do so.

By the equation \([3.0.1.]\), we deduce \(r_0 \equiv 0\) modulo \(e\) from our assumption on \(k_0\). Recall that \(p > n^2 + 2(n - 3)\) by the generic condition. Since \(0 \leq r_0 \leq (n-1)(p^f-1)/(p-1)\) by (ii) of Lemma 2.3.5, we conclude that \(r_0 = 0\). Thus, we may let \(f_0\) satisfy that \([f_0]_{\mathcal{E}_i} = 0 \text{ if } 0 < i \leq n - 1 \text{ and } [f_0]_{\mathcal{E}_0} = 1\), so that we can also let
\[
\hat{f}_0 = \begin{pmatrix} 0 \\
  \vdots \\
  0 \\
  1 \end{pmatrix}
\]

Hence, we can also assume that \([\tilde{f}_j]_{\tilde{\mathcal{E}}} = 0\) for \(0 < j \leq n - 1\). We let \(V_0 = \text{Mat}_{\tilde{\mathcal{E}}}^{\tilde{f}_0}(\text{Fil}^{n-1}\hat{\mathcal{M}}) \in M^{\tilde{\mathcal{E}}_{\text{dd}}}(\mathcal{O}_E)\) and \(A_0 = \text{Mat}_{\mathcal{E}}^{f_0}(\phi_{n-1}) \in \text{GL}_{n}^{\Box}(\mathcal{O}_E)\).

We construct a sequence of framed bases \(\mathcal{E}(m)\) for \(\hat{\mathcal{M}}\) by change of basis, satisfying that
\[
\text{Mat}_{\mathcal{E}(m)}^{\tilde{f}_0}(\text{Fil}^{n-1}\hat{\mathcal{M}}) \in M_{\tilde{\mathcal{E}}_{\text{dd}}}(\mathcal{O}_E) \text{ and } \text{Mat}_{\mathcal{E}(m)}^{f_0}(\phi_{n-1}) \in \text{GL}_{n}^{\Box}(\mathcal{O}_E)
\]
converge to certain desired forms as \( m \) goes to \( \infty \). We let \( V^{(m)} \in M_n^{\square}(S\Omega) \) and \( A^{(m)} \in GL_n^{\square}(S\Omega) \) for a non-negative integer \( m \). We may write
\[
(x_{n-1}^{(m+1)} u_{[k_n-1-k_0]} f, x_{n-2}^{(m+1)} u_{[k_n-2-k_0]} f, \ldots, x_{m+1}^{(1)} u_{[k_n+1-k_0]} f, x_0^{(m+1)})
\]
for the last row of \((A^{(m)})^{-1}\), where \( x_0^{(m+1)} \in (S\Omega)_0 \) and \( x_j^{(m+1)} \in (S\Omega)_0 \) for \( 0 < j < n - 1 \). We define an \( n \times n \)-matrix \( R^{(m+1)} \) as follows:
\[
R^{(m+1)} = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}
\]
We also define
\[
C^{(m+1)} = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}
\]
by the equation
\[
R^{(m+1)} \cdot V^{(m)} \cdot C^{(m+1)} = V^{(m)}
\]
where \( y_j^{(m+1)} \in (S\Omega)_0 \) for \( 0 < j < n - 1 \). Note that the existence of such a matrix \( C^{(m+1)} \) is obvious, since \( p^{-1}_k \equiv 0 \mod (\varepsilon) \) by our assumption on \( k_0 \) immediately implies \( [p^{-1}_k k_0] f \leq \lfloor k_0 \rfloor [k_0 k_0] f + [p^{-1}_k k_0 k_0] f \). We also note that \( R^{(m+1)} \in GL_n (S\Omega) \) and \( C^{(m+1)} \in GL_2^{\square}(S\Omega) \).

Let \( V^{(m+1)} = V^{(m)} \) for all \( m \geq 0 \). Assume that \( V^{(m)} = \text{Mat}^{\varepsilon^{(m)}} \cdot \text{Fil}^{n-1} \mathcal{M} \) and \( A^{(m)} = \text{Mat}^{\varepsilon^{(m)}}(\phi^{(m-1)}) \), with respect to a framed basis \( \varepsilon^{(m)} \) and a framed system of generators \( \varepsilon^{(m)} \). If we let \( \varepsilon^{(m+1)} = \varepsilon^{(m)} \cdot (R^{(m+1)})^{-1} \), then
\[
\phi^{(m)} \cdot \varepsilon^{(m+1)} \]

Hence, we get
\[
V^{(m+1)} = \text{Mat}^{\varepsilon^{(m+1)}} \cdot \varepsilon^{(m+1)} \cdot \text{Fil}^{n-1} \mathcal{M} \text{ and } R^{(m+1)} \cdot A^{(m)} \cdot \phi^{(m+1)} = \text{Mat}^{\varepsilon^{(m+1)}} \cdot \varepsilon^{(m+1)} \cdot (\phi^{(m-1)}),
\]
where \( \varepsilon^{(m+1)} := \varepsilon^{(m+1)} \cdot V^{(m+1)} \).

We compute the matrix product \( A^{(m+1)} := R^{(m+1)} \cdot A^{(m)} \cdot \phi^{(m+1)} \) as it follows. If we let \( A^{(m)} = (\alpha_{i,j}^{(m)} u_{[k_i-k_j]} f)_{0 \leq i,j \leq n-1} \) for \( \alpha_{i,j}^{(m)} \in (S\Omega)_0 \) if \( i \neq j \) and \( \alpha_{i,i}^{(m)} \in (S\Omega)_0 \), then
\[
A^{(m+1)} = (\alpha_{i,j}^{(m+1)} u_{[k_i-k_j]} f)_{0 \leq i,j \leq n-1} \in GL_n^{\square}(S\Omega)
\]
where $\alpha_{i,j}^{(m+1)} u[k_j-k_i]/$ is described as follows:

\[
\begin{cases}
\alpha_{i,j}^{(m)} u[k_j-k_i]/ + \alpha_{i,0}^{(m)} u[k_0-k_i]/ \phi(y_j^{(m+1)}) u[p^{-1}(k_j-k_0)]/ & \text{if } i > 0 \text{ and } j > 0; \\
\alpha_{i,0}^{(m)} u[k_0-k_i]/ & \text{if } i > 0 \text{ and } j = 0; \\
\alpha_{i,j}^{(m+1)} u[k_j-k_i]/ & \text{if } i = 0 \text{ and } j > 0; \\
\frac{1}{x_i^{(m+1)}} \phi(y_j^{(m+1)}) u[p^{-1}(k_j-k_0)]/ & \text{if } i = 0 \text{ and } j = 0.
\end{cases}
\]

Let $V^{(0)} = V_0$ and $A^{(0)} = A_0$. We apply the algorithm above to $V^{(0)}$ and $A^{(0)}$. By the algorithm above, we have two matrices $V^{(m)}$ and $A^{(m)}$ for each $m \geq 0$. We claim that

\[
\begin{cases}
\alpha_{i,j}^{(m+1)} - \alpha_{i,j}^{(m)} \in u^{(1+p+\cdots+p^m)} e S_{\mathcal{O}_E} & \text{if } i > 0 \text{ and } j > 0; \\
\alpha_{i,j}^{(m+1)} - \alpha_{i,j}^{(m)} \in u^{(1+p+\cdots+p^m)} e S_{\mathcal{O}_E} & \text{if } i > 0 \text{ and } j = 0; \\
\alpha_{i,j}^{(m+1)} - \alpha_{i,j}^{(m)} \in u^{(1+p+\cdots+p^m)} e S_{\mathcal{O}_E} & \text{if } i = 0 \text{ and } j > 0; \\
\alpha_{i,j}^{(m+1)} - \alpha_{i,j}^{(m)} \in u^{(1+p+\cdots+p^m)} e S_{\mathcal{O}_E} & \text{if } i = 0 \text{ and } j = 0.
\end{cases}
\]

It is obvious that the case $i > 0$ and $j = 0$ from the computation \[3.6.2\]. For the case $i = 0$ and $j > 0$ we induct on $m$. Note that $p[p^{-1}(k_j-k_0)] f - [k_j-k_0]/ = p[p^{-1}(k_j-k_0)] f - (k_j-k_0) \geq e$ if $j > 0$. From the computation \[3.6.2\] again, it is obvious that it is true for $m = 0$. Assume that it holds for $m$. This implies that $y_j^{(m+1)} \in u^{(1+p+\cdots+p^m)} e S_{\mathcal{O}_E}$ for $0 < j \leq n - 1$ and so $y_j^{(m+1)} \in u^{(1+p+\cdots+p^m)} e S_{\mathcal{O}_E}$. Since $\phi(y_j^{(m+1)}) u[p^{-1}(k_j-k_0)] f - [k_j-k_0]/ \in u^{(1+p+\cdots+p^m)} e S_{\mathcal{O}_E}$, by the computation \[3.6.2\] we conclude that the case $i = 0$ and $j > 0$ holds. The case $i > 0$ and $j > 0$ follows easily from the case $i = 0$ and $j > 0$, since $p[p^{-1}(k_j-k_0)] f - [k_j-k_0]/ = p[p^{-1}(k_j-k_0)] f - (k_j-k_0) + e + k_0 - k_i - [k_j-k_0]/ = p[p^{-1}(k_j-k_0)] f - (p-1)k_0 \geq e$. Finally, we check the case $i = 0$ and $j = 0$. We also induct on $m$ for this case. It is obvious that it holds for $m = 0$. Note that $R^{(m+1)} \equiv I_n$ modulo $u^{(1+p+\cdots+p^m)} e S_{\mathcal{O}_E}$. Since $A^{(m+1)} = R^{(m+1)}, A^{(m)} \cdot \phi(C^{(m+1)})$, we conclude that the case $i = 0$ and $j = 0$ holds.

The previous claim says the limit of $A^{(m)}$ exists (entrywise), say $A^{(\infty)}$. By definition, we have $V^{(\infty)} = V^{(m)}$ for all $m \geq 0$. In other words, there exist a framed basis $\hat{e}^{(\infty)}$ for $\hat{\mathcal{M}}$ and a framed system of generators $\hat{f}^{(\infty)}$ for Fil$^{-1}\hat{\mathcal{M}}$ such that

\[
\text{Mat}_{\hat{e}^{(\infty)} \cdot \hat{f}^{(\infty)}}(\text{Fil}^{-1}\hat{\mathcal{M}}) = V^{(\infty)} \in M_{n,\hat{\mathcal{M}}}^\infty(S_{\mathcal{O}_E})
\]

and

\[
\text{Mat}_{\hat{e}^{(\infty)} \cdot \hat{f}^{(\infty)}}(\phi e_n-1) = A^{(\infty)} \in GL_n(S_{\mathcal{O}_E}).
\]

Note that $(V^{(\infty)})_{i,j} = 0$ if either $i = 0$ and $j > 0$ or $i > 0$ and $j = 0$, and that $(A^{(\infty)})_{i,j} = 0$ if $i = 0$ and $j > 0$.

Since $\hat{e}^{(\infty)}$ is a framed basis for $\hat{\mathcal{M}}$, we may write

\[
\text{Mat}_{\hat{e}^{(\infty)}}(N) = (\gamma_{i,j} u[k_j-k_i]/)_{0 \leq i,j \leq n-1} \in M_n^\infty(S_{\mathcal{O}_E})
\]

for the matrix of the monodromy operator of $\hat{\mathcal{M}}$ where $\gamma_{i,j} \in (S_{\mathcal{O}_E})_0$, and let

\[
A^{(\infty)} = (\alpha_{i,j}^{(\infty)} u[k_j-k_i]/)_{0 \leq i,j \leq n-1} \in GL_n^\infty(S_{\mathcal{O}_E}).
\]

We claim that $\gamma_{n,j} = 0$ for $n - 1 \geq j > 0$. Recall that $\alpha_{0,j}^{(\infty)} = 0$ for $j > 0$, and write $\hat{f}^{(\infty)} = (\hat{f}^{(\infty)}_{n-1}, \hat{f}^{(\infty)}_{n-2}, \cdots, \hat{f}^{(\infty)}_0)$ and $\hat{e}^{(\infty)} = (\hat{e}^{(\infty)}_{n-1}, \hat{e}^{(\infty)}_{n-2}, \cdots, \hat{e}^{(\infty)}_0)$. We also write

\[
\hat{f}^{(\infty)}_{j} = \sum_{i=1}^{n-1} \hat{g}_{i,j}^{(\infty)} u[p^{-1}k_j-k_i] \hat{e}^{(\infty)}_i.
\]
where $\beta_{i,j}^{(\infty)} \in (S_{O_E})_0$, for each $0 < j \leq n - 1$. From the equation
\[
[cN \phi_{n-1}(\tilde{F}_j^{(\infty)})]_{\mathfrak{p}(\infty)} = [\phi_{n-1}(E(u)N(\tilde{F}_j^{(\infty)}))]_{\mathfrak{p}(\infty)}
\]
for $n - 1 \geq j > 0$, we have the identity
\[
(3.6.3) \quad \sum_{i=1}^{n-1} \beta_{i,j}^{(\infty)} u^{[k_i-k_j]+[k_i-k_0]} \gamma_{0,i} = p \sum_{i=1}^{n-1} \beta_{i,j}^{(\infty)} u^{p[k_i-k_j]+p[k_i-k_0]} \phi(\gamma_{0,i}) \alpha^{(\infty)}
\]
for each $n - 1 \geq j > 0$. Choose an integer $s$ such that $\text{ord}_u(\gamma_{0,s} u^{[k_i-k_j]+[k_i-k_0]})$ for all $n - 1 \geq i > 0$, and consider the identity (3.6.3) for $j = s$. Then the identity (3.6.3) induces
\[
\alpha^{(\infty)} \gamma_{0,s} \equiv 0 \pmod{u^{\text{ord}_u(\gamma_{0,s} u^{[k_i-k_j]+[k_i-k_0]})+1}}.
\]
Note that $\alpha^{(\infty)} \in S_{O_E}^\infty$, so that we get $\gamma_{0,s} = 0$. Recursively, we conclude that $\gamma_{0,j} = 0$ for all $0 < j \leq n - 1$. Finally, it is now easy to check that $(\tilde{c}_1^{(\infty)}, \tilde{c}_2^{(\infty)}, \ldots, \tilde{c}_n^{(\infty)})$ determines a strongly divisible module of rank $n - 1$, that is a submodule of $\tilde{M}$. This completes the proof. 

**Corollary 3.6.4.** Fix a pair of integers $(i_0, j_0)$ with $0 \leq j_0 \leq i_0 \leq n - 1$. Assume that $\varpi_0$ is generic, and let $(k_{n-1}, k_{n-2}, \ldots, k_0)$ be an $n$-tuple of integers. Assume further that
\[
k_i = (p^{f-1} + p^{f-2} + \cdots + p + 1)n_i
\]
for $i > i_0$ and for $i < j_0$ and that the $k_i$ are pairwise distinct modulo $(e)$. Then every potentially semi-stable lift $\rho$ of $\varpi_0$ with Hodge–Tate weights $\{-n-1, -n, \ldots, 0\}$ and Galois types $\bigoplus_{i=0}^{n-1-k_i} \tilde{w}_j$ is a successive extension
\[
\rho \cong \begin{pmatrix}
\rho_{n-1,n-1} & \cdots & * & * & * & \cdots & * \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\rho_{i_0+1,i_0+1} & * & * & \cdots & * \\
\rho_{i_0,i_0} & * & * & \cdots & * \\
\rho_{j_0-1,j_0-1} & \cdots & * \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\rho_{0,0}
\end{pmatrix}
\]
where

- $\rho_{i,i}$ is a 1-dimensional potentially semi-stable lift of $\varpi_{i,i}$ with Hodge–Tate weights $-i$ and Galois type $\tilde{w}_j^{k_{i-1}}$ for $n - 1 \geq i > i_0$ and for $j_0 > i > 0$;
- $\rho_{i_0,j_0}$ is a $(i_0-j_0+1)$-dimensional potentially semi-stable lift of $\varpi_{i_0,j_0}$ with Hodge–Tate weights $\{-i_0, -i_0+1, \ldots, -j_0\}$ and Galois types $\bigoplus_{i=j_0}^{i_0} \tilde{w}_j^{k_{i_0}}$.

**Proof.** Proposition 3.6.1 implies this corollary recursively. Let $\mathcal{M} \in \mathbf{F}^{\text{-BrMod}_{n-1}^d}$ be a Breuil module corresponding to the mod $p$ reduction of a strongly divisible module $\tilde{M} \in \mathcal{O}_E^{\text{-Mod}_{n-1}^d}$ corresponding to a Galois stable lattice in a potentially semi-stable representation $\rho : G_{Q_p} \to \text{GL}_n(E)$ with Galois type $\bigoplus_{i=0}^{n-1} \tilde{w}_j^{k_i}$ and Hodge–Tate weights $\{-n-1, -n, \ldots, 0\}$ such that $\mathbf{T}_{\text{st}}^{n-1} \tilde{M} \otimes_{O_E} F \cong \varpi_0$. Both $\tilde{M}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1} \tilde{w}_j^{k_i}$ (resp. $\bigoplus_{i=0}^{n-1} \tilde{w}_j^{k_i}$) by Proposition 2.4.3. We may assume that $\text{Mat}_{\tilde{M}}(\text{Fil}^{n-1} \mathcal{M})$, $\text{Mat}_{\tilde{M}}(\phi_{n-1})$, and $\text{Mat}_{\mathcal{M}}(N)$ are written as in (3.0.5), (3.0.6), and (3.0.7) respectively, and we do so.

By the equation (3.3.4), it is easy to see that $r_1 = (p^{f-1} + p^{f-2} + \cdots + p + 1)n$ for $i > i_0$ and for $i < j_0$, by our assumption on $k_i$. By Proposition 3.6.1 there exists an $(n-1)$-dimensional subrepresentation $\rho_{n-1,1}$ of $\rho$ whose quotient is $\rho_{0,0}$ which is a potentially semi-stable lift of $\varpi_0$ with Hodge–Tate weight 0 and Galois type $\tilde{w}_j^{k_0}$. Now consider $\rho_{n-1,1} \otimes \varepsilon^{-1}$. Apply Proposition 3.6.1 to $\rho_{n-1,1} \otimes \varepsilon^{-1}$.  


Theorem 3.7.3. Let $\rho$ has subquotients $\rho_{i,j}$ for $0 \leq i \leq j_0 - 1$. Considering $\rho^v \otimes \varepsilon^{n-1}$, one can also readily check that $\rho$ has subquotients $\rho_{i,j}$ lifting $\overline{\rho}_{i,j}$ for $n - 1 \geq i \geq i_0 + 1$. 

The results in Corollary 3.6.4 reduce many of our computations for the main results on the Galois side.

3.7. Main results on the Galois side. In this section, we state and prove the main local results on the Galois side, that connects the Fontaine–Laffaille parameters of $\overline{\rho}_0$ with the Frobenius eigenvalues of certain potentially semi-stable lifts of $\overline{\rho}_0$. Throughout this section, we assume that $\overline{\rho}_0$ is Fontaine–Laffaille generic. We also fix $f = 1$ and $c = p - 1$.

Fix $i_0, j_0 \in \mathbb{Z}$ with $0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1$, and define the $n$-tuple of integers

$$(r^{i_0,j_0}_n, r^{i_0,j_0}_{n-2}, \ldots, r^{i_0,j_0}_0)$$

as follows:

$$(3.7.1) r^{i_0,j_0}_i := \begin{cases} 
    i & \text{if } i_0 \neq i \neq j_0; \\
    j_0 + 1 & \text{if } i = i_0; \\
    i_0 - 1 & \text{if } i = j_0.
\end{cases}$$

We note that if we replace $i_0 - j_0 + 1$ in the definition of $r^{(0)}_i$ in (3.3.1), then we have the identities:

$$(3.7.2) r^{i_0,j_0}_{i_0} = j_0 + i^{(0)}_i$$

for all $0 \leq i \leq i_0 - j_0$. In particular, $r^{n-1,0}_i = r^{(0)}_i$ for all $0 \leq i \leq n - 1$.

From the equation $k^{i_0,j_0}_i \equiv c_i + i - r^{i_0,j_0}_i \mod (c)$ (cf. Lemma 3.1.2 (i)), this tuple immediately determines an $n$-tuple $(k^{i_0,j_0}_{n-1}, k^{i_0,j_0}_{n-2}, \ldots, k^{i_0,j_0}_0)$ of integers mod $(c)$, which will determine the Galois types of our representations. We set

$k^{i_0,j_0}_i := c_i + i - r^{i_0,j_0}_i$

for all $i \in \{0, 1, \ldots, n - 1\}$.

The following is the main result on the Galois side.

**Theorem 3.7.3.** Let $i_0, j_0$ be integers with $0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1$. Assume that $\overline{\rho}_0$ is generic and that $\overline{\rho}_{i_0,j_0}$ is Fontaine–Laffaille generic. Let $(\lambda^{i_0,j_0}_{n-1}, \lambda^{i_0,j_0}_{n-2}, \ldots, \lambda^{i_0,j_0}_0) \in (\mathcal{O}_E)^n$ be the Frobenius eigenvalues on the $(\overline{\omega}^{k^{i_0,j_0}_{n-1}}, \overline{\omega}^{k^{i_0,j_0}_{n-2}}, \ldots, \overline{\omega}^{k^{i_0,j_0}_0})$-isotypic components of $D_{st, n}^{-1}(\rho_0)$ where $\rho_0$ is a potentially semi-stable lift of $\overline{\rho}_0$ with Hodge–Tate weights $\{-n + 1, -n - 1, \ldots, -1, 0\}$ and Galois types $\bigoplus_{i = 0}^{n-1} \omega^{k^{i_0,j_0}_i}$.

Then the Fontaine–Laffaille parameter $FL^{i_0,j_0}_{n}(\overline{\rho}_0)$ is computed as follows:

$FL^{i_0,j_0}_{n}(\overline{\rho}_0) = \left( \frac{p(n-1)^{\frac{p-2}{2}}(i_0 - j_0 - 1)}{\prod_{i=j_0+1}^{i_0} \lambda^{i_0,j_0}_i} \right) \in \mathbb{P}^1(F)$.

We first prove Theorem 3.7.3 for the case $(i_0, j_0) = (n - 1, 0)$ in the following proposition, which is the key first step to prove Theorem 3.7.3.

**Proposition 3.7.4.** Keep the assumptions and notation of Theorem 3.7.3, and assume further $(i_0, j_0) = (n - 1, 0)$. Then Theorem 3.7.3 holds.

Recall that $(\lambda^{n-1,0}_{n-1}, \ldots, \lambda^{n-1,0}_0)$ in Proposition 3.7.4 is the same as $(\lambda^{(0)}_{n-1}, \ldots, \lambda^{(0)}_0)$ in (3.3.1). To lighten the notation, we let $k_i = k^{n-1,0}_i$ and $\lambda_i = \lambda^{n-1,0}_i$ during the proof of Proposition 3.7.4. We heavily use the results in Sections 3.3, 3.4 and 3.5 to prove this proposition.

**Proof.** Let $\tilde{\mathcal{M}} \in \mathcal{O}_E\text{-Mod}^{n-1}$ be a strongly divisible module corresponding to a Galois stable lattice in a potentially semi-stable representation $\rho_0 : G_{\mathbb{Q}_p} \to GL_n(E)$ with Galois type $\bigoplus_{i = 0}^{n-1} \omega^{k_i}$ and Hodge–Tate weights $\{-n + 1, -n - 1, \ldots, 0\}$ such that $T_{st, n}^{-1}(\tilde{\mathcal{M}}) \otimes_{\mathcal{O}_E} F \cong \overline{\rho}_0$. We also let $\mathcal{M}$ be the
Breuil module corresponding to the mod $p$ reduction of $\tilde{\mathcal{M}}$. $\tilde{\mathcal{M}}$ (resp. $\mathcal{M}$) is of inertial type $\bigoplus_{i=0}^{n-1} \omega^{k_{i}}$ (resp. $\bigoplus_{i=0}^{n-1} \omega^{k_{i}}$) by Proposition 3.3.3

We let $f = (f_{n-1}, f_{n-2}, \cdots, f_{1}, f_{0})$ be a framed system of generators for $\text{Fil}^{n-1} \tilde{\mathcal{M}}$, and $\hat{e} = (\hat{e}_{n-1}, \hat{e}_{n-2}, \cdots, \hat{e}_{1}, \hat{e}_{0})$ be a framed basis for $\tilde{\mathcal{M}}$. We may assume that $\text{Mat}_{\mathfrak{e}}(\text{Fil}^{n-1} \tilde{\mathcal{M}})$ is described as in Proposition 3.5.4 and we do so.

Define $\alpha_{i} \in \mathbf{F}^{\times}$ by the condition $\phi_{n-1}(f_{i}) \equiv \alpha_{i} \hat{e}_{i}$ modulo $(\varpi_{E}, u)$ for all $i \in \{0, 1, \cdots, n-1\}$. There exists a framed basis $e = (e_{n-1}, e_{n-2}, \cdots, e_{0})$ for $\mathcal{M}$ and a framed system of generators $f = (f_{n-1}, f_{n-2}, \cdots, f_{0})$ for $\text{Fil}^{n-1} \mathcal{M}$ such that $\text{Mat}_{\mathfrak{e}}(\text{Fil}^{n-1} \mathcal{M})$ is given as in Proposition 3.3.3 and

$$\text{Mat}_{\mathfrak{e}}(\phi_{n-1}) = \left( \alpha_{i,j} u^{[k_{j-1}, k_{i}]} \right) \in \text{GL}_{n}(\overline{\mathbb{Q}})$$

for some $\alpha_{i,j} \in \overline{\mathbb{Z}}$ with $\alpha_{i,i} \equiv \alpha_{i} \mod (u^{e})$.

Recall that $\hat{f}_{i} = E(u)^{e} \hat{e}_{i}$ for $i \in \{1, 2, \cdots, n-2\}$ by Proposition 3.5.4. Write $\phi_{n-1}(\hat{f}_{j}) = \sum_{i=0}^{n-1} \alpha_{i,j} u^{[k_{j-1}, k_{i}]} \hat{e}_{i}$ for some $\alpha_{i,j} \in S_{0}$. Then we get

$$s_{0}(\hat{e}_{i}) \equiv \frac{p \lambda_{i}}{p^{n-1}} \text{ (mod } \varpi_{E})$$

for $i \in \{1, 2, \cdots, n-2\}$ since $\phi_{n-1} = \frac{1}{p^{n-1}} \text{ for the Frobenius } \phi \text{ on } \text{D}^{Q_{p}, n-1}_{\text{st}}(\rho_{0})$, so that we have

$$\prod_{i=1}^{n-2} \lambda_{i} \equiv \prod_{i=1}^{n-2} \frac{\lambda_{i}}{p^{n-1-i}} \text{ (mod } \varpi_{E}).$$

(Note that $\lambda_{i} \in \mathcal{O}_{E}^{\times}$ by Corollary 3.5.5) This completes the proof, by applying the results in Proposition 3.3.4.

We now prove Theorem 3.7.3 which is the main result on the Galois side.

**Proof of Theorem 3.7.3**. Recall from the identities in (3.7.2) that

$$(i_{0}, r_{0}^{i_{0}-1}, \cdots, r_{0}^{j_{0}}) = j_{0} + (1, n'-2, n'-3, \cdots, 1, n'-2)$$

where $n' := i_{0} - j_{0} + 1$. Recall also that $\rho_{0}$ has a subquotient $\rho_{i_{0}, j_{0}}$ that is a potentially semi-stable lift of $\mathfrak{p}_{i_{0}, j_{0}}$ with Hodge–Tate weights $\{-i_{0}, -(i_{0} - 1), \cdots, -j_{0}\}$ and of Galois type $\bigoplus_{i_{0}}^{i_{0}, j_{0}}$ by Corollary 3.6.4.

It is immediate that $\rho_{i_{0}, j_{0}} := \rho_{i_{0}, j_{0}} e^{-j_{0}\varpi_{j_{0}}}$ is another potentially semi-stable lift of $\mathfrak{p}_{i_{0}, j_{0}}$ with Hodge–Tate weights $\{-i_{0} - j_{0}, -(i_{0} - j_{0} - 1), \cdots, 0\}$ and of Galois type $\bigoplus_{i_{0}}^{i_{0}, j_{0}}$ $\mathfrak{p}_{i_{0}, j_{0}}$-isotypic component of $\text{D}^{Q_{p}, i_{0}, j_{0}}_{\text{st}}(\rho_{i_{0}, j_{0}})$ (resp. on the $\mathfrak{p}_{i_{0}, j_{0}}$-isotypic component of $\text{D}^{Q_{p}, i_{0}, j_{0}}_{\text{st}}(\rho_{i_{0}, j_{0}})$). Then we have

$$p^{-j_{0}} \delta_{i} = \eta_{i}$$

for all $i \in \{j_{0}, j_{0}+1, \cdots, i_{0}\}$ and, by Proposition 3.7.4

$$\text{FL}_{i_{0}, j_{0}+1}(\mathfrak{p}_{i_{0}, j_{0}}) = \left[ \left( \prod_{i=j_{0}+1}^{i_{0}-1} \delta_{i} \right) : p^{-j_{0}(i_{0}-j_{0})-1} \right] \in \mathbb{F}^{1}(\mathbb{F}).$$

But we also have that

$$p^{n-1-(i_{0}-j_{0})} \eta_{i} = \lambda_{i}^{i_{0}, j_{0}}$$
with respect to \( \text{torus consisting of diagonal matrices. \( B \)}}

\[ \text{Proof.} \]

Notice that \( \phi \) is diagonal on \( D := D^Q_{st}(\rho_0) \) with respect to a framed basis \( e := (e_{n-1}, e_{n-2}, \ldots, e_0) \) (which satisfies \( g e_i = \omega^{k_i}g \) for all \( i \) and all \( g \in \text{Gal}(K/Q_p) \)) since \( \omega^{k_i} \) are all distinct. Hence, we have \( \text{WD}(\rho_0) = \text{WD}(\rho_0)^{F-sa} \). Similarly, since the descent data action on \( D \) commutes with the monodromy operator \( N \), it is immediate that \( N = 0 \).

From the definition of \( \text{WD}(\rho_0) \) (cf. [CDT99]), the action of \( W_{Q_p} \) on \( D \) can be described as follows: let \( \alpha(g) \in Z \) be determined by \( \tilde{g} = \phi^\alpha(g) \), where \( \phi \) is the arithmetic Frobenius in \( G_{\mathbb{F}_p} \) and \( \tilde{g} \) is the image under the surjection \( W_{Q_p} \to \text{Gal}(K/Q_p) \). Then

\[ \text{WD}(\rho_0)(g) \cdot e_i = \left( \frac{\lambda^{i_0,j_0}}{p^{n-1}} \right)^{-\alpha(g)} \cdot \omega^{k_i}g \cdot e_i \]

for all \( i \). (Recall that \( D^{Q_{p,n-1}}_{st}(\rho_0) = D^{Q_{st}}_{st}(\rho_0) \otimes \varepsilon^{-(n-1)} \), so that the \( \lambda^{i_0,j_0} \) are the Frobenius eigenvalues of the Frobenius on \( D \).) Write \( \Omega_i \) for the eigen-character with respect to \( e_i \).

Recall that we identify the geometric Frobenius with the uniformizers in \( Q_p^\times \) (by our normalization of class field theory), so that \( \Omega_i(p) = \frac{p^{n-1}}{\lambda^{i_0,j_0}} \) which completes the proof by applying Theorem 3.7.3.

4. Local automorphic side

In this section, we establish several results concerning representation theory of \( GL_n \), that will be applied to the proof of our main results on mod \( p \) local-global compatibility, Theorem 5.7.6. The main results in this section are the non-vanishing result, Corollary 4.2.7, as well as the intertwining identity in characteristic 0, Theorem 4.7.3.

We start this section by fixing some notation. Let \( G := \text{GL}_{n}/Z_n \) and \( T \) be the maximal split torus consisting of diagonal matrices. We fix a Borel subgroup \( B \subseteq G \) consisting of upper-triangular matrices, and let \( U \subseteq B \) be the maximal unipotent subgroup. Let \( \Phi^+ \) denote the set of positive roots with respect to \( (B, T) \), and \( \Delta = \{ \alpha_k \}_{1 \leq k \leq n-1} \) the subset of positive simple roots. Let \( X(T) \) and \( X^\vee(T) \) denote the abelian group of characters and cocharacters respectively. We often say a weight for an element in \( X(T) \), and write \( X(T)_+ \) for the set of dominant weights. The set \( \Phi^+ \) induces a partial order on \( X(T) \): for \( \lambda, \mu \in X(T) \) we say that \( \lambda \leq \mu \) if \( \mu - \lambda \in \sum_{\alpha \in \Phi^+} Z \geq 0 \alpha \).
We use the \( n \)-tuple of integers \( \lambda = (d_1, d_2, \cdots, d_n) \) to denote the character associated to the weight \( \sum_{k=1}^{n} d_k \varepsilon_k \) of \( T \) where for each \( 1 \leq i \leq n \) \( \varepsilon_i \) is a weight of \( T \) defined by
\[
\text{diag}(x_1, x_2, \cdots, x_n) \mapsto x_i.
\]
We will often use the following weight
\[
\eta := (n-1, n-2, \cdots, 1, 0).
\]

We let \( \Gamma, \overline{B}, \cdots \) be the base change to \( F_p \) of \( G, B, \cdots \) respectively. The Weyl group of \( G \), denoted by \( W \), has a standard lifting inside \( G \) as the group of permutation matrix, likewise with \( \overline{G} \). We denote the longest Weyl element by \( w_0 \) which is lifted to the antidiagonal permutation matrix in \( G \) or \( \overline{G} \). We use the notation \( s_i \) for the simple reflection corresponding to \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \) for \( 1 \leq i \leq n-1 \). We define the length \( \ell(w) \) of \( w \in W \) to be its minimal length of decomposition into product of \( s_i \) for \( 1 \leq i \leq n-1 \). Given \( A \in U(F_p) \), we use \( A_\alpha \) or equivalently \( A_{i,j} \) to denote the \((i,j)\)-entry of \( A \), where \( \alpha = \varepsilon_i - \varepsilon_j \) is the positive root corresponding to the pair \((i,j)\) with \( 1 \leq i < j \leq n \).

Given a representation \( \pi \) of \( G(F_p) \), we use the notation \( \pi^w \) for the \( T(F_p) \)-eigenspace with the eigencharacter \( \mu \). Given an algebraic representation \( V \) of \( G \) or \( \overline{G} \), we use the notation \( \pi^\lambda \) for the weight space of \( V \) associated to the weight \( \lambda \). For any representation \( V \) of \( \overline{G} \) or \( G(F_p) \) with coefficient in \( F_p \), we define
\[
V_p := V \otimes_{F_p} F
\]
to be the extension of coefficient of \( V \) from \( F_p \) to \( F \). Similarly, we write \( V_{\overline{p}} \) for \( V \otimes_{F_p} \overline{F_p} \).

It is easy to observe that we can identify the character group of \( T(F_p) \) with \( X(T)/(p-1)X(T) \). The natural action of the Weyl group \( W \) on \( T \) and thus on \( T(F_p) \) induces an action of \( W \) on the character group \( X(T) \) and \( X(T)/(p-1)X(T) \). We carefully distinguish the notation between them. We use the notation \( w\lambda \) (resp. \( \mu^w \)) for the action of \( W \) on \( X(T) \) (resp. \( X(T)/(p-1)X(T) \)) satisfying
\[
w\lambda(x) = \lambda(w^{-1}xw) \quad \text{for all } x \in T
\]
and
\[
\mu^w(x) = \mu(w^{-1}xw) \quad \text{for all } x \in T(F_p).
\]
As a result, without further comments, the notation \( w\lambda \) is a weight but \( \mu^w \) is just a character of \( T(F_p) \). There is another dot action of \( W \) on \( X(T) \) defined by
\[
w \cdot \lambda = w(\lambda + \eta) - \eta \quad \text{for all } \lambda \in X(T) \text{ and } w \in W.
\]
The affine Weyl group \( \widetilde{W} \) of \( G \) is defined as the semi-direct product of \( W \) and \( X(T) \) with respect to the natural action of \( W \) on \( X(T) \). We denote the image of \( \lambda \in X(T) \) in \( \widetilde{W} \) by \( t_\lambda \). Then the dot action of \( W \) on \( X(T) \) extends to the dot action of \( \widetilde{W} \) on \( X(T) \) through the following formula
\[
\tilde{w} \cdot \lambda = w \cdot (\lambda + p\mu)
\]
if \( \tilde{w} = wt_\mu \). We use the notation \( \lambda \uparrow \mu \) for \( \lambda, \mu \in X(T) \) if \( \lambda \leq \mu \) and \( \lambda \in \widetilde{W} \cdot \mu \). We define a specific element of \( \widetilde{W} \) by
\[
\tilde{w}_h := w_0t_{-\eta}
\]
following Section 4 of [LLL].

We usually write \( K \) for \( \text{GL}_n(Z_p) \) for brevity. We will also often use the following three open compact subgroups of \( \text{GL}_n(Z_p) \): if we let \( \text{red} : \text{GL}_n(Z_p) \rightarrow \text{GL}_n(F_p) \) be the natural mod \( p \) reduction map, then
\[
K(1) := \ker(\text{red}) \subset I(1) := \text{red}^{-1}(U(F_p)) \subset I := \text{red}^{-1}(B(F_p)) \subset K.
\]
For each \( \alpha \in \Phi^+ \), there exists a subgroup \( U_\alpha \) of \( G \) such that \( xu_\alpha(t)x^{-1} = u_\alpha(x)t \) where \( x \in T \) and \( u_\alpha : G_\alpha \rightarrow U_\alpha \) is an isomorphism sending 1 to 1 in the entry corresponding to \( \alpha \). For each \( \alpha \in \Phi^+ \), we have the following equalities by [Jan03] II.1.19 (5) and (6):
\[
(4.0.1)
\]
\[
u_\alpha(t) = \sum_{m \geq 0} t^m (X_{\alpha,m}^\text{alg}).
\]
where $X_{a,m}^{\text{alg}}$ is an element in the algebra of distributions on $G$ supported at the origin $1 \in G$. The equation (4.0.1) is actually just the Taylor expansion with respect to $t$ of the operation $u_w(t)$ at the origin $1$. We use the same notation $X_{a,m}^{\text{alg}}$ if $G$ is replaced by $G$.

We define the subset of $p$-restricted weights as
\[ X_1(T) := \{ \lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha \rangle \leq p - 1 \text{ for all } \alpha \in \Delta \} \]
and the set of central weights as
\[ X_0(T) := \{ \lambda \in X(T) \mid \langle \lambda, \alpha \rangle = 0 \text{ for all } \alpha \in \Delta \}. \]

We also define the set of $p$-regular weights as
\[ X_1^{\text{reg}}(T) := \{ \lambda \in X(T) \mid 1 \leq \langle \lambda, \alpha \rangle \leq p - 2 \text{ for all } \alpha \in \Delta \}, \]
and in particular we have $X_1^{\text{reg}}(T) \subseteq X_1(T)$. We say that $\lambda \in X(T)$ lies in the lowest $p$-restricted alcove if
\[ 0 < \langle \lambda + \eta, \alpha \rangle < p \text{ for all } \alpha \in \Phi^+. \]

We define a subset $\tilde{W}^+$ of $\tilde{W}$ as
\[ \tilde{W}^+ := \{ \tilde{w} \in \tilde{W} \mid \tilde{w} \cdot \lambda \in X_+(T) \text{ for each } \lambda \text{ in the lowest } p \text{-restricted alcove} \}. \]

We define another subset $\tilde{W}^{\text{res}}$ of $\tilde{W}$ as
\[ \tilde{W}^{\text{res}} := \{ \tilde{w} \in \tilde{W} \mid \tilde{w} \cdot \lambda \in X_1(T) \text{ for each } \lambda \text{ in the lowest } p \text{-restricted alcove} \}. \]

In particular, we have the inclusion
\[ \tilde{W}^{\text{res}} \subseteq \tilde{W}^+. \]

For any weight $\lambda \in X(T)$, we let
\[ H^0(\lambda) := \left( \text{Ind}_{B_{U}}^{\tilde{G}} \right)^{\text{alg}}_{B_{p}} \]
be the associated dual Weyl module. Note by [Jan03], Proposition II.2.6 that $H^0(\lambda) \neq 0$ if and only if $\lambda \in X(T)_+$. Assume that $\lambda \in X(T)_+$, we write $F(\lambda) := \text{soc}_{\tilde{G}}(H^0(\lambda))$ for its irreducible socle as an algebraic representation (cf. [Jan03] part II, section 2). When $\lambda$ is running through $X_1(T)$, the $F(\lambda)$ exhaust all the irreducible representations of $G(F_p)$. On the other hand, two weights $\lambda, \lambda' \in X_1(T)$ satisfies
\[ F(\lambda) \cong F(\lambda') \]
as $G(F_p)$-representation if and only if
\[ \lambda - \lambda' \in (p - 1)X_0(T). \]

If $\lambda \in X_1^{\text{reg}}(T)$, then the structure of $F(\lambda)$ as a $G(F_p)$-representation depends only on the image of $\lambda$ in $X(T)/(p - 1)X(T)$, namely as a character of $T(F_p)$. Conversely, given a character $\mu$ of $T(F_p)$ which lies in the image of
\[ X_1^{\text{reg}}(T) \to X(T)/(p - 1)X(T), \]
its inverse image in $X_1^{\text{reg}}(T)$ is uniquely determined up to a translation of $(p - 1)X_0(T)$. In this case, we say that $\mu$ is $p$-regular. Whenever it is necessary for us to lift an element in $X(T)/(p - 1)X(T)$ back into $X_1(T)$ (or maybe $X_1^{\text{reg}}(T)$), we will clarify the choice of the lift.

Consider the standard Bruhat decomposition
\[ G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} U_w w B = \bigsqcup_{w \in W} B w U_{w^{-1}}. \]
where $U_w$ is the unique subgroup of $U$ satisfying $BwB = U_w w B$ as schemes over $\mathbb{Z}_p$. The group $U_w$ has a unique form of $\prod_{\alpha \in \Phi_w} U_{\alpha}$ for the subset $\Phi_w$ of $\Phi^+$ defined by $\Phi_w^+ = \{ \alpha \in \Phi^+, w(\alpha) \in -\Phi^+ \}$. 


We also have by Theorem 4.4 and Proposition 5.6 of [CL76] that
\[ \dim F \]
\[ w \]
\[ \mu \]
and we also have a unique non-zero morphism up to scalar
\[ \text{dim}_F W \]
By Bruhat decomposition we can deduce that
\[ \text{dim}_F W \]
\[ m \]
\[ \alpha \]
\[ w \]
\[ G(F_p) = \bigcup_{w \in W} B(F_p)wB(F_p) = \bigcup_{w \in W} U_w(F_p)wB(F_p) = \bigcup_{w \in W} B(F_p)wU_{w-1}(F_p). \]

4.1. Jacobi sums in characteristic \( p \). In this section we establish several fundamental properties of Jacobi sum operators on mod \( p \) principal series representations.

Definition 4.1.1. A weight \( \lambda \in X(T) \) is called \( k \)-generic for \( k \in \mathbb{Z}_{>0} \) if for each \( \alpha \in \Phi^+ \) there exists \( m_\alpha \in \mathbb{Z} \) such that
\[ m_\alpha p + k < \langle \lambda, \alpha^\vee \rangle < (m_\alpha + 1)p - k. \]
In particular, the \( n \)-tuple of integers \( (a_{n-1}, \ldots, a_1, a_0) \) is called \( k \)-generic in the lowest alcove if
\[ a_i - a_{i-1} > k \quad \forall 1 \leq i \leq n - 1 \text{ and } a_{n-1} - a_0 < p - k. \]

Note that \( (a_{n-1}, \ldots, a_0) - \nu \) lies the lowest restricted alcove in the sense of (4.0.2) if \( (a_{n-1}, \ldots, a_0) \) is \( k \)-generic in the lowest alcove for some \( k > 0 \). Note also that the existence of a \( n \)-tuple of integers satisfying \( k \)-generic in the lowest alcove implies \( p > n(k + 1) - 1 \).

We use the notation \( \pi \) for a general principal series representation:
\[ \pi := \text{Ind}_{B(F_p)}^{GL_n(F_p)} \mu_{\pi} = \{ f : G(F_p) \to F_p \mid f(bg) = \mu_{\pi}(b)f(g) \quad \forall (b, g) \in B(F_p) \times GL_n(F_p) \} \]
where \( \mu_{\pi} \) is a mod \( p \) character of \( T(F_p) \). The action of \( GL_n(F_p) \) on \( \pi \) is given by \( (g \cdot f)(g') = f(g'g) \).

We will assume throughout this article that \( \mu_{\pi} \) is \( p \)-regular. By definition we have
\[ \text{cosoc}_{G(F_p)}(\pi) = F(\mu_{\pi}) \text{ and } \text{soc}_{G(F_p)}(\pi) = F(\mu_{\pi}^{w_0}). \]

By Bruhat decomposition we can deduce that
\[ \text{dim}_F \pi U(F_p) \mu_{\pi}^{w_0} = 1 \]
for each \( w \in W \). We denote by \( v_{\pi} \) a non-zero fixed vector in \( \pi U(F_p) \cdot \mu_{\pi} \).

Given an element \( w \in W \), we let \( \mu_{\pi}^{\prime} := \mu_{\pi}^{w_0} \) and consider the principal series
\[ \pi' := \text{Ind}_{B(F_p)}^{GL_n(F_p)} \mu_{\pi}^{\prime}. \]
As \( \text{dim}_F (\pi')U(F_p) \cdot \mu_{\pi} = 1 \), by Frobenius reciprocity we have a unique non-zero morphism up to scalar
\[ \mathcal{T}_{\pi} : \pi \to \pi'. \]
Given an element \( w' \in W \), we also let \( \mu_{\pi}^{\prime'} := \mu_{\pi}^{w_0} = \mu_{\pi}^{w_0} \) and consider the principal series
\[ \pi'' := \text{Ind}_{B(F_p)}^{GL_n(F_p)} \mu_{\pi}^{\prime''}, \]
and we also have a unique non-zero morphism up to scalar
\[ \mathcal{T}_{\pi''} : \pi' \to \pi''. \]

In particular, we have
\[ \mathcal{T}_{w} (\pi U(F_p) \cdot \mu_{\pi}^{w_0}) = (\pi')U(F_p) \cdot \mu_{\pi}. \]
We also have by Theorem 4.4 and Proposition 5.6 of [CL76] that
\[ \mathcal{T}_{w'} \circ \mathcal{T}_{w} = \begin{cases} c \mathcal{T}_{w'w} & \text{if } \ell(w) + \ell(w') = \ell(w'w) \\ 0 & \text{if } \ell(w) + \ell(w') > \ell(w'w) \end{cases} \]
for some \( c \in F_p^\times. \)
Given integers $0 \leq k_\alpha \leq p - 1$ indexed by $\alpha \in \Phi_w^+$ for a certain $1 \neq w \in W$, we define the Jacobi sum operators

$$S_{\mathbf{k},w} := \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi_w^+} A_{\alpha}^{k_{\alpha}} \right) A \cdot w \in F_p[G(F_p)]$$

where $\mathbf{k} := (k_\alpha)_{\alpha \in \Phi_w^+}$. These Jacobi sum operators play a main role on the local automorphic side as a crucial computation tool. These operators already appeared in [CL76] for example.

For each $\alpha \in \Phi^+$ and each integer $m$ satisfying $0 \leq m \leq p - 2$, we define the operator

$$X_{\alpha,m} := \sum_{t \in F_p} t^{p-1-m} u_\alpha(t) \in F_p[U(F_p)] \subseteq F_p[G(F_p)].$$

The transition matrix between $\{u_\alpha(t) \mid t \in F_p^*\}$ and $\{X_{\alpha,m} \mid 0 \leq m \leq p - 2\}$ is a Vandermonde matrix

$$(t^k)_{t \in F_p^*, 1 \leq k \leq p - 1}$$

which has a non-zero determinant. Hence, we also have a converse formula

$$u_\alpha(t) = -\sum_{m=0}^{p-2} t^m X_{\alpha,m} \text{ for all } t \in F_p.$$  

By the equation (4.0.1), we note that we have the equality

$$X_{\alpha,m} = -\sum_{k \geq 0} X_{\alpha,m+(p-1)k}.$$  

We also define

$$X_{m_1,\ldots,m_{n-1}} := X_{\alpha_1,m_1} \circ \cdots \circ X_{\alpha_{n-1},m_{n-1}} \in F_p[U(F_p)] \subseteq F_p[G(F_p)]$$

for each tuple of integers $(m_1,\ldots,m_{n-1})$ satisfying $0 \leq m_i \leq p - 2$ for each $1 \leq i \leq n - 1$.

**Lemma 4.1.10.** Fix $w \in W$ and $\alpha_0 = (i_0,j_0) \in \Phi_w^+$ . Given a tuple of integers $\mathbf{k} = (k_{i,j}) \in \{0,1,\ldots,p-1\}^{\Phi_w^+}$ satisfying

$$k_{i,j} = 0 \text{ for all } (i,j) \in \Phi_w^+ \text{ with } j \geq j_0 + 1,$$

we have

$$X_{\alpha_0,m} \cdot S_{\mathbf{k},w} = \left\{ \begin{array}{ll} (-1)^{m+1} c_{k_{\alpha_0},m} S_{\mathbf{k'},w} & \text{if } m \leq k_{\alpha_0} \\ 0 & \text{if } m > k_{\alpha_0} \end{array} \right.$$  

where $\mathbf{k'} = (k'_{\alpha})_{\alpha \in \Phi_w}$ satisfies

$$k_{\alpha} = \left\{ \begin{array}{ll} k_{\alpha_0} - m & \text{if } \alpha = \alpha_0; \\ k_{\alpha} & \text{otherwise.} \end{array} \right.$$  

**Proof.** We prove this lemma by direct computation.

$$X_{\alpha,m} \cdot S_{\mathbf{k},w} = \sum_{t \in F_p} t^{p-1-m} \left( \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi_w^+} A_{\alpha}^{k_{\alpha}} \right) u_\alpha(t) A w \right)$$

$$= \sum_{t \in F_p} t^{p-1-m} \left( \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi_w^+, \alpha \neq \alpha_0} A_{\alpha}^{k_{\alpha}} \right) (A_{\alpha_0} - t)^{k_{\alpha_0}} A w \right)$$

$$= \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi_w^+, \alpha \neq \alpha_0} A_{\alpha}^{k_{\alpha}} \right) \left( \sum_{t \in F_p} t^{p-1-m}(A_{\alpha_0} - t)^{k_{\alpha_0}} A w \right)$$

where the second equality follows from the change of variable $A \leftrightarrow u_{\alpha_0}(t) A$ and the assumption (4.1.11).
Notice that
\[
\sum_{t \in \mathbb{F}_p} t^{p-1-m}(A_{\alpha_0} - t)^{k_{\alpha_0}} = \sum_{t \in \mathbb{F}_p} t^{p-1-m} \left( \sum_{j=0}^{k_{\alpha_0}} (-1)^j c_{k_{\alpha_0},j} A_{k_{\alpha_0}}^{-j} t^j \right)
= \sum_{j=0}^{k_{\alpha_0}} (-1)^j c_{k_{\alpha_0},j} A_{k_{\alpha_0}}^{-j} \left( \sum_{t \in \mathbb{F}_p} t^{p-1-m+j} \right),
\]
which can be easily seen to be
\[(4.1.13) \quad \begin{cases} (-1)^{m+1} c_{k_{\alpha_0},m} A_{k_{\alpha_0}}^{-m} & \text{if } m \leq k_{\alpha_0} \\ 0 & \text{if } m > k_{\alpha_0}. \end{cases} \]
The last computation (4.1.14) follows from the fact that
\[
\sum_{t \in \mathbb{F}_p} t^\ell = \begin{cases} 0 & \text{if } 1 \leq \ell \leq p-2 \\ -1 & \text{if } \ell = p-1 \end{cases}
\]
Applying (4.1.13) back to (4.1.12) gives us the result. \(\square\)

**Lemma 4.1.14.** Fix \(w \in W\) and \(\alpha_0 = (i_0,j_0) \in \Phi_w^+.\) Given a tuple of integers \(\underline{k} = (k_{i,j}) \in \{0,1,\cdots,p-1\}^{\Phi_w^+}\) satisfying
\[k_{i,j} = 0 \text{ for all } (i,j) \in \Phi_w^+ \text{ with } j \geq j_0,
\]
we have
\[u_{\alpha_0}(t) \cdot S_{\underline{k},w} = S_{\underline{k},w}.
\]
**Proof.** By Lemma 4.1.10 we deduce that
\[X_{\alpha_0,m} \cdot S_{\underline{k},w} = \begin{cases} -S_{\underline{k},w} & \text{if } m = 0 \\ 0 & \text{if } 1 \leq m \leq p-2 \end{cases}
\]
Therefore we conclude this lemma from (4.1.7). \(\square\)

**Lemma 4.1.15.** Let \(m_i\) be integers in \(\{0,1,\cdots,p-2\}\) for all \(1 \leq i \leq n-1\), and \(\underline{k} = (k_{i,j}) \in \{0,\cdots,p-1\}^{\Phi_w^+}\) with \(k_{i,j} = 0\) for all \(1 \leq i < i+1 < j \leq n\).

If \(m_i \leq k_{i,i+1}\) for all \(1 \leq i \leq n-1\), then
\[X_{m_1,\cdots,m_{n-1}} \cdot S_{\underline{k'},w_0} = \prod_{i=1}^{n-1} \left((-1)^{m_{i+1}+1} c_{k_{i,i+1},m_i}\right) S_{\underline{k'},w_0} \in \mathbb{F}_p[G(\mathbb{F}_p)]
\]
where \(\underline{k}' = (k_{i,j}')\) satisfies
\[k_{i,j}' = \begin{cases} k_{i,j} - m_i & \text{if } j = i+1; \\ 0 & \text{otherwise}. \end{cases}
\]
Otherwise, \(X_{m_1,\cdots,m_{n-1}} \circ S_{\underline{k},w_0} = 0.\)

**Proof.** This lemma follows directly from Lemma 4.1.10 and the definition in (4.1.9). In fact, we only need to apply Lemma 4.1.10 to the operators \(X_{\alpha_0,m_i}\) for \(i = n-1,\cdots,1\) inductively. \(\square\)

By the definition of principal series representations, we have the decomposition
\[(4.1.16) \quad \pi = \oplus_{w \in W} \pi_w
\]
where \(\pi_w \subset \pi|_{B(\mathbb{F}_p)}\) consists of the functions supported on the Bruhat cell \(B(\mathbb{F}_p)w^{-1}B(\mathbb{F}_p) = B(\mathbb{F}_p)w^{-1}U_w(\mathbb{F}_p).\)
Proposition 4.1.17. Fix a non-zero vector \( v_\pi \in \pi^U(F_p).\mu_\pi \). For each \( w \in W \) with \( w \neq 1 \), the set
\[
\left\{ S_{k,w}v_\pi \mid k = (k_\alpha)_{\alpha \in \Phi^+_w} \in \{0, 1, \cdots, p - 1\}^{\Phi^+_w} \right\}
\]
forms a \( T(F_p) \)-eigenbasis of \( \pi_w \).

Proof. We have a decomposition \( \pi_w = \bigoplus_{A \in U_w(F_p)} \pi_{w,A} \) where \( \pi_{w,A} \) is the subspace of \( \pi_w \) consisting of functions supported on \( B(F_p)w^{-1}A^{-1} \). It is easy to observe by the definition of parabolic induction that \( \dim \pi_{w,A} = 1 \) and \( \pi_{w,A} \) is generated by \( A w \).

We claim that the set of Jacobi sums with the Weyl element \( w \), after being applied to \( v_\pi \), differs from the set \( \{ A w v_\pi, A \in U_w(F_p) \} \) by an invertible matrix. More precisely, for a fixed \( w \in W \), the set of vectors
\[
\{ S_{k,w}v_\pi \mid S = ((k_\alpha)_{\alpha \in \Phi^+_w}, w), 0 \leq k_\alpha \leq p - 1 \quad \forall \alpha \in \Phi^+_w \}
\]
can be linearly represented by the set of vectors \( \{ A w v_\pi, A \in U_w(F_p) \} \) through the matrix \( (m_{k,A}) \) where
\[
k = (k_\alpha)_{\alpha \in \Phi^+_w} \in \{0, 1, \cdots, p - 1\}^{\Phi^+_w}, \quad A \in U_w(F_p)
\]
and \( m_{k,A} := \prod_{\alpha \in \Phi^+_w} A_{k_\alpha} \). Note that this matrix is the \( |\Phi^+_w| \)-times tensor of the Vandermonde matrix
\[
(\lambda^k)_{\lambda \in \mathbb{F}_p, 0 \leq k \leq p - 1}
\]
and therefore has a non-zero determinant. As a result, the matrix \( (m_{k,A}) \) is invertible and \( \{ S_{k,w}v_\pi \mid 0 \leq k_\alpha \leq p - 1 \quad \forall \alpha \in \Phi^+_w \} \) forms a basis of \( \pi_w \).

The fact that this basis is a \( T(F_p) \)-eigenbasis is immediate by the following calculation:
\[
x \cdot S_{k,w}v_\pi = x \cdot \left( \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi^+_w} A_{k_\alpha}^{k_\alpha} \right) A w \right) v_\pi
\]
\[
= \left( \sum_{A \in U_w(F_p)} \left( \prod_{(i,j) \in \Phi^+_w} A_{k_{i,j}}^{k_{i,j}} \right) x A x^{-1} w \right) \left( w^{-1} x w \right) v_\pi
\]
\[
= \left( \sum_{B = x A x^{-1} \in U_w(F_p)} \left( \prod_{(i,j) \in \Phi^+_w} (B_{i,j} x_j x_i^{-1})^{k_{i,j}} \right) B w \right) \left( w^{-1} x w \right) v_\pi
\]
\[
= \mu_\pi(x^{-1} w) \left( \prod_{(i,j) \in \Phi^+_w} (x_j x_i^{-1})^{k_{i,j}} \right) \left( \sum_{A \in U_w(F_p)} \prod_{\alpha \in \Phi^+_w} A_{k_\alpha}^{k_\alpha} A w \right) v_\pi
\]
\[
= (\mu_\pi \lambda)(x) S_{k,w}v_\pi,
\]
where \( x := \text{diag}(x_1, x_2, \cdots, x_n) \), \( \lambda(x) = \prod_{1 \leq i < j \leq n} (x_j x_i^{-1})^{k_{i,j}} \), and \( B_{i,j} = A_{i,j} x_i x_j^{-1} \) for \( 1 \leq i < j \leq n \).

We can further describe the action of \( T(F_p) \) on \( S_{k,w}v_\pi \). By \( \lfloor y \rfloor \) for \( y \in \mathbb{R} \) we mean the floor function of \( y \), i.e., the biggest integer less than or equal to \( y \).

Lemma 4.1.18. Let \( \mu_\pi = (d_1, d_2, \cdots, d_{n-1}, d_n) \). If we write \( (\ell_1, \ell_2, \cdots, \ell_{n-1}, \ell_n) \) for the \( T(F_p) \)-eigencharacter of \( S_{k,w}v_\pi \), then we have
\[
\ell_r \equiv d_{w^{-1}(r)} + \sum_{1 \leq i < r} k_{i,r} - \sum_{r < j \leq n} k_{r,j} \pmod{p - 1}
\]
for all \( 1 \leq r \leq n \), where \( k_{i,j} = k_\alpha \) if \( \alpha \in \Phi^+_w \setminus \Delta \), and for all \( 1 \leq r \leq n \)
\[
\ell_r \equiv d_{w^{-1}(r)} + (1 - \lfloor 1/r \rfloor)k_{r-1,r} - (1 - \lfloor 1/(n + 1 - r) \rfloor)k_{r,r+1} \pmod{p - 1};
\]
(ii) if \( w = w_0 \) and \( k_{i,j} = 0 \) for any \( 2 \leq i < j \leq n \), then

\[
\ell_r \equiv \begin{cases} 
\frac{d_n - \sum_{j=2}^{n} k_{1,j}}{\text{mod } p - 1} & \text{if } r = 1; \\
\frac{d_{n+1-r} + k_{1,r}}{\text{mod } p - 1} & \text{if } 2 \leq r \leq n.
\end{cases}
\]

**Proof.** The first part of the Lemma is a direct calculation as shown at the end of the proof of Proposition \( \text{(4.1.17)} \). The second part follows trivially from the first part. \( \square \)

Given any \( w \in W \), we write \( S_{\underline{w}} \) for \( S_{\underline{w}}^w \) with \( k_\alpha = 0 \) for all \( \alpha \in \Phi^+_w \).

**Lemma 4.1.19.** \( \mathbb{F}_p[S_{\underline{w}}v_\pi] = \pi^U(\mathbb{F}_p),\mu^*_w \).  

**Proof.** Pick an arbitrary positive root \( \alpha \). If \( \alpha \in \Phi^+_w \), then we have (since \( u_\alpha(t) \in U_w(\mathbb{F}_p) \))

\[
u_\alpha(t) \left( \sum_{A \in U_w(\mathbb{F}_p)} A \right) = \left( \sum_{A \in U_w(\mathbb{F}_p)} A \right)
\]

and therefore \( u_\alpha(t)S_{\underline{w}}v_\pi = S_{\underline{w}}v_\pi \) for any \( t \in \mathbb{F}_p \). On the other hand, if \( \alpha \notin \Phi^+_w \), then

\[
u_\alpha(t) \left( \sum_{A \in U_w(\mathbb{F}_p)} A \right) = \left( \sum_{A \in U_w(\mathbb{F}_p)} A \right) u'_\alpha(t)
\]

and

\[u'_\alpha(t)wv_\pi = wu'_\alpha(t)v_\pi = wv_\pi \]

where \( u'_\alpha(t) = \prod_{\alpha \notin \Phi^+_w} \overline{\U}_\alpha(\mathbb{F}_p) \) and \( u'_\alpha(t) \in U(\mathbb{F}_p) \) are elements depending on \( x, w \) and \( \alpha \). Hence, \( u_\alpha(t)S_{\underline{w}}v_\pi = S_{\underline{w}}v_\pi \) for any \( t \in \mathbb{F}_p \) and any \( \alpha \in \Phi^+ \). So we conclude that \( S_{\underline{w}}v_\pi \) is \( U(\mathbb{F}_p) \)-invariant as \( \{u_\alpha(t)\}_{\alpha \in \Phi^+, t \in \mathbb{F}_p} \) generate \( U(\mathbb{F}_p) \).

Finally, we check that \( x \cdot S_{\underline{w}}v_\pi = \mu^w(x)S_{\underline{w}}v_\pi \) for \( x \in T(\mathbb{F}_p) \). But this is immediate from the following two easy computations:

\[
x \cdot \left( \sum_{A \in U_w(\mathbb{F}_p)} A \right) = \left( \sum_{A \in U_w(\mathbb{F}_p)} A \right) \cdot x \in \mathbb{F}_p[G(\mathbb{F}_p)]
\]

and \( xwv_\pi = w(\mu^w(x))v_\pi = \mu^w(x)wv_\pi \).

This completes the proof. \( \square \)

Note that Proposition 4.1.17 and Lemma 4.1.18 are very elementary and have essentially appeared in [CL76]. In this article, we formulate them and give quick proofs of them for the convenience.

**Definition 4.1.20.** Given \( \alpha, \alpha' \in \Phi^+ \), we say that \( \alpha \) is strongly smaller than \( \alpha' \) with the notation

\[\alpha \prec \alpha' \]

if there exist \( 1 \leq i \leq j \leq k \leq n - 1 \) such that

\[
\alpha = \sum_{r=i}^{j} \alpha_r \text{ and } \alpha' = \sum_{r=i}^{k} \alpha_r.
\]

We call a subset \( \Phi' \) of \( \Phi^+ \) good if it satisfies the following:

(i) if \( \alpha, \alpha' \in \Phi' \), then \( \alpha + \alpha' \in \Phi' \);

(ii) if \( \alpha \in \Phi' \) and \( \alpha \prec \alpha' \), then \( \alpha' \in \Phi' \).

We associate a subgroup of \( U \) to \( \Phi' \) by

\[(4.1.21) \quad \Phi'(U) := \langle U_\alpha \mid \alpha \in \Phi' \rangle \]

and denote its reduction mod \( p \) by \( \overline{U}' \). We define \( U_1 \) to be the subgroup scheme of \( U \) generated by \( U_\alpha \), for \( 2 \leq r \leq n - 1 \), and denote its reduction mod \( p \) by \( \overline{U}_1 \).
Example 4.1.22. The following are two examples of good subsets of $\Phi^+$, that will be important for us:

$$\left\{ \sum_{r=1}^{j} \alpha_r \mid 1 \leq i < j \leq n - 1 \right\} \text{ and } \left\{ \sum_{r=1}^{j} \alpha_r \mid 2 \leq i < j \leq n - 1 \right\}.$$ 

The subgroups of $U$ associated with the two good subsets via $(4.1.21)$ are $[U, U]$ and $U_1$ respectively.

We recall that we have defined $\pi_w \subseteq \pi$ in $(4.1.10)$ for each $w \in W$.

Proposition 4.1.23. Let $\Phi' \subseteq \Phi^+$ be good. Pick an element $w \in W$ with $w \neq 1$. The following set of vectors

$$(4.1.24) \quad \left\{ S_{\kappa, w} \pi \mid \kappa = (k_\alpha)_{\alpha \in \Phi^+_w} \in \{0, 1, \ldots, p - 1\}^{\Phi^+_w} \text{ with } k_\alpha = 0 \forall \alpha \in \Phi' \cap \Phi^+_w \right\}$$

forms a basis of the subspace $\pi^{U_{w'/(F_p)}}$ of $\pi_w$.

Proof. By Proposition 4.1.17 the Jacobi sums with the Weyl element $w$, after being applied to $\pi$, form a $T(F_p)$-eigenbasis of $\pi_w$, and so we can firstly write any $U_{w'/(F_p)}$-invariant vector $v$ in $\pi_w$ as a unique linear combination of Jacobi sums with the Weyl element $w$, namely

$$v = \sum_{\kappa \in \{0, \ldots, p - 1\}^{\Phi^+_w}} C_{\kappa, w} S_{\kappa, w} \pi \text{ for some } C_{\kappa, w} \in F_p.$$ 

Assume that $C_{\kappa, w} \neq 0$ for certain tuple of integers $\kappa = (k_\alpha)_{\alpha \in \Phi^+_w}$ such that $k_\alpha > 0$ for some $\alpha \in \Phi' \cap \Phi^+_w$. We choose $\alpha_0$ such that it is maximal with respect to the partial order $\prec$ on $\Phi^+$ for the property

$$(4.1.25) \quad C_{\kappa, w} \neq 0, \quad k_{\alpha_0} > 0, \quad \text{ and } \alpha_0 \in \Phi' \cap \Phi^+_w.$$ 

We may write $v$ as follows:

$$(4.1.26) \quad v = \sum_{\kappa \in \{0, \ldots, p - 1\}^{\Phi^+_w}} C_{\kappa, w} S_{\kappa, w} \pi + \sum_{\substack{\kappa \in \{0, \ldots, p - 1\}^{\Phi^+_w} \kappa \neq 0 \text{ and } k_{\alpha_0} > 0}} C_{\kappa, w} S_{\kappa, w} \pi.$$ 

By the maximal assumption on $\alpha_0$ we know that if $C_{\kappa, w} \neq 0$ and $\alpha_0 \prec \alpha$, then $k_\alpha = 0$. As a result, we deduce from Lemma 4.1.14 that

$$(4.1.27) \quad u_{\alpha_0}(t) \sum_{\kappa \in \{0, \ldots, p - 1\}^{\Phi^+_w} k_{\alpha_0} = 0} C_{\kappa, w} S_{\kappa, w} \pi = \sum_{\kappa \in \{0, \ldots, p - 1\}^{\Phi^+_w} k_{\alpha_0} = 0} C_{\kappa, w} S_{\kappa, w} \pi$$

for all $t \in F_p$.

We define

$$\Phi^{\alpha_0, +}_{w} := \{ \alpha \in \Phi^+_w \mid \alpha_0 \prec \alpha \} \text{ and } \Phi^{\alpha_0, -}_{w} := \Phi^+_w \setminus \Phi^{\alpha_0, +}_{w},$$

and we use the notation

$$\ell := (\ell_\alpha)_{\alpha \in \Phi^{\alpha_0, -}_{w}} \in \{0, \ldots, p - 1\}^{\Phi^{\alpha_0, -}_{w}}$$

for a tuple of integers indexed by $\Phi^{\alpha_0, -}_{w}$. Given a tuple $\underline{\ell}$, we can define

$$\Lambda(\underline{\ell}, \alpha_0) := \begin{cases} \kappa & \text{if } \alpha \in \Phi^{\alpha_0, +}_{w} \setminus \{\alpha_0\}; \\ k_\alpha = 0 & \text{if } \alpha \in \Phi^{\alpha_0, -}_{w} \setminus \{\alpha_0\}; \\ k_\alpha > 0 & \text{if } \alpha = \alpha_0; \\ k_\alpha = \ell_\alpha & \text{if } \alpha \in \Phi^{\alpha_0, -}_{w}. \end{cases}$$

Now we can define a polynomial

$$f(\underline{\ell}, \alpha_0)(x) = \sum_{\kappa \in \Lambda(\underline{\ell}, \alpha_0)} C_{\kappa, w} x^{k_\alpha} \in F_p[x]$$
for each tuple of integers \( \ell \). By definition, we have

\[
\sum_{\ell \in \{0, \ldots, p-1\}^{\Phi_w^+}} C_{\ell, w} S_{\ell, w} \psi_\pi = \sum_{\ell \in \{0, \ldots, p-1\}^{\Phi_w^+}} \left( \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi_w^0} A_\alpha^\ell \right) f(\ell, \alpha_0)(A_{\alpha_0} A) \right) \psi_\pi.
\]

By the assumption on \( v \) we know that \( u_{\alpha_0}(t) v = v \) for all \( t \in F_p \). Using (4.1.27) and (4.1.28) we have

\[
u_{\alpha_0}(t) \sum_{\ell \in \{0, \ldots, p-1\}^{\Phi_w^+}} C_{\ell, w} S_{\ell, w} \psi_\pi = \sum_{\ell \in \{0, \ldots, p-1\}^{\Phi_w^+}} C_{\ell, w} S_{\ell, w} \psi_\pi
\]

and so

\[
\sum_{\ell \in \{0, \ldots, p-1\}^{\Phi_w^+}} \left( \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi_w^0} A_\alpha^\ell \right) f(\ell, \alpha_0)(A_{\alpha_0} A) \right) \psi_\pi
\]

\[
= \sum_{\ell \in \{0, \ldots, p-1\}^{\Phi_w^+}} \left( \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi_w^0} A_\alpha^\ell \right) f(\ell, \alpha_0)(A_{\alpha_0} A - t A) \right) \psi_\pi
\]

where the last equality follows from a change of variable \( A \leftrightarrow u_{\alpha_0}(t) A \).

By the linear independence of Jacobi sums from Proposition 4.1.17 we deduce an equality

\[
\sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi_w^0} A_\alpha^t \right) f(\ell, \alpha_0)(A_{\alpha_0} A) \psi_\pi = \sum_{A \in U_w(F_p)} \left( \prod_{\alpha \in \Phi_w^0} A_\alpha^t \right) f(\ell, \alpha_0)(A_{\alpha_0} - t A) \psi_\pi
\]

for each fixed tuple \( \ell \).

Therefore, again by the linear independence of Jacobi sum operators in Proposition 4.1.17 we deduce that

\[
f(\ell, \alpha_0)(A_{\alpha_0} - t A) = f(\ell, \alpha_0)(A_{\alpha_0})
\]

for all \( t \in F_p \) and each \( (\ell, \alpha_0) \). We know that if \( f \in U_{w}(F_p) \) satisfies \( \deg f \leq p - 1 \), \( f(0) = 0 \) and \( f(x - t) = f(x) \) for all \( t \in F_p \) then \( f = 0 \). Thus we deduce that

\[
f(\ell, \alpha_0) = 0
\]

for each tuple of integers \( \ell \), which is a contradiction to (4.1.25) and so we have \( k_\alpha = 0 \) for any \( \alpha \in \Phi' \) for each tuple of integers \( \ell \) such that \( C_{\ell, w} \neq 0 \).

As a result, we have shown that each vector in \( \pi_{U_{w}^0(F_p)} \) can be written as certain linear combination of vectors in \( \pi_{U_{w}^0(F_p)} \). On the other hand, by Proposition 4.1.17 we know that vectors in \( \pi_{U_{w}^0(F_p)} \) are linear independent, and therefore they actually form a basis of \( \pi_{U_{w}^0(F_p)} \).

**Corollary 4.1.28.** Let \( \mu_{\pi} = (d_1, \cdots, d_n) \) and fix a non-zero vector \( v_\pi \in \pi_{U(F_p)} \). Given a weight \( \mu = (\ell_1, \cdots, \ell_n) \in X_1(T) \) the space

\[
\pi_{U_{w_0}(F_p), U(F_p)} \mu \psi_\pi
\]

has a basis whose elements are of the form

\[
S_{\ell, w_0} \psi_\pi
\]
where \( \underline{k} = (k_\alpha) \) satisfies
\[
\ell_r \equiv d_{n+1-r} + (1 - \lfloor 1/r \rfloor)k_{r-1,r} - (1 - \lfloor 1/(n+1-r) \rfloor)k_{r,r+1} \mod (p-1)
\]
for all \( 1 \leq r \leq n \) and \( k_\alpha = 0 \) if \( \alpha \in \Phi^+ \setminus \Delta \).

**Proof.** By a special case of Proposition 4.1.24 when \( \Phi' = \{ \sum_{r=i}^j \alpha_r \mid 1 \leq i < j \leq n-1 \} \), we know that
\[
\{ S_{\underline{k},w_0} v_\pi \mid k_\alpha = 0 \text{ if } \alpha \in \Phi^+ \setminus \Delta \}
\]
forms a basis of \( \pi_{U(F_\rho)}^{(U(F_\rho),\mu)} \). On the other hand, we know from Proposition 4.1.17 that the above basis is actually an \( T(F_\rho) \)-eigenbasis. Therefore the vectors in this basis with a fixed eigencharacter \( \mu \) form a basis of the eigensubspace \( \pi_{U(F_\rho),\mu}^{(U(F_\rho),\mu)} \). Finally, using (i) of the second part of Lemma 4.1.18 we conclude this lemma. \( \square \)

**Corollary 4.1.29.** Let \( \mu_\pi = (d_1, d_2, \cdots, d_n) \) and fix a non-zero vector \( v_\pi \in \pi_{U(F_\rho),\mu}^{(U(F_\rho),\mu)} \). Given a weight \( \mu = (\ell_1, \cdots, \ell_n) \in X_1(T) \), the space
\[
\pi_{U(F_\rho),\mu}^{(U(F_\rho),\mu)}
\]
has a basis whose elements are of the form
\[
S_{\underline{k},w_0} v_\pi
\]
where \( \underline{k} = (k_{i,j})_{i,j} \) satisfies
\[
k_{i,j} \equiv \ell_j - d_{n+1-j} \mod (p-1)
\]
for \( 2 \leq j \leq n \) and \( k_{i,j} = 0 \) for all \( 2 \leq i < j \leq n \).

**Proof.** By a special case of Proposition 4.1.24 when \( \Phi' = \{ \sum_{r=i}^j \alpha_r \mid 2 \leq i < j \leq n-1 \} \), we know that
\[
\{ S_{\underline{k},w_0} v_\pi \mid k_{i,j} = 0 \text{ if } 2 \leq i < j \leq n \}
\]
forms a basis of \( \pi_{U(F_\rho)}^{(U(F_\rho),\mu)} \). On the other hand, we know from Proposition 4.1.17 that the above basis is actually an \( T(F_\rho) \)-eigenbasis. Therefore the vectors in this basis with a fixed eigencharacter \( \mu \) form a basis of the eigensubspace \( \pi_{U(F_\rho),\mu}^{(U(F_\rho),\mu)} \). Finally, using (ii) of the second part of Lemma 4.1.18 we conclude this lemma. \( \square \)

4.2. Main results in characteristic \( p \). In this section, we state our main results on certain Jacobi sum operators in characteristic \( p \). From now on we fix an \( n \)-tuple of integers \( (a_{n-1}, \cdots, a_0) \) which is assumed to be \( 2n \)-generic in the lowest alcove (cf. Definition 4.1.1).

We let
\[
\mu_1 := (a_1, a_2, \cdots, a_{n-3}, a_{n-2}, a_{n-1}, a_0);
\mu'_1 := (a_{n-1}, a_0, a_1, a_2, \cdots, a_{n-3}, a_{n-2}).
\]

We denote their corresponding principal series representations by
\( \pi_1 \) and \( \pi'_1 \) respectively and their non-zero fixed vectors by
\[
v_1 \in \pi_1^{(U(F_\rho),\mu_1)} \quad \text{and} \quad v'_1 \in (\pi'_1)^{U(F_\rho),\mu'_1}.
\]

Finally, we define one more specific weight
\[
\mu^* := (a_{n-1} - n + 2, a_{n-2}, a_{n-3}, \cdots, a_2, a_1, a_0 + n - 2)
\]
which will play a central role in Corollary 4.2.7.

We let \( \underline{k}^1 = (k_{1,j}^1) \) and \( \underline{k}^{1'} = (k_{1,j}^{1'}) \), where
\[
\begin{align*}
k_{1,i+1}^1 & = [a_0 - a_{n-1}] + n - 2; \\
k_{1,i+1}^{1'} & = [a_{n-i-1} - a_{n-1}] + n - 2
\end{align*}
\]
for $1 \leq i \leq n - 1$ and $k_{i,j}^{1} = k_{i,j}^{1\prime} = 0$ otherwise, and define two most important Jacobi sum operators $S_n$ and $S_n'$ to be

\begin{equation}
S_n := S_{k_1^{1}, w_0} \quad \text{and} \quad S_n' := S_{k_1^{1\prime}, w_0}.
\end{equation}

We also let $V_1$ (resp. $V_1'$) denote the sub-representation of $\pi_1$ (resp. of $\pi_1'$) generated by $S_n v_1$ (resp. by $S_n' v_1'$).

The following theorem, which we usually call the non-vanishing theorem, is a technical heart on the local automorphic side. The proofs of this non-vanishing theorem as well as the next theorem, which we usually call the multiplicity one theorem, will occupy the following sections.

**Theorem 4.2.5.** Assume that $(a_{n-1}, \cdots, a_0)$ is $n$-generic in the lowest alcove.

Then we have

$$F(\mu^*) \in JH(V_1) \cap JH(V_1').$$

**Proof.** This is an immediate consequence of Proposition 4.5.11 and Theorem 4.6.39. □

We also have the following multiplicity one result.

**Theorem 4.2.6.** Assume that $(a_{n-1}, \cdots, a_0)$ is $2n$-generic in the lowest alcove.

Then $F(\mu^*)$ has multiplicity one in $\pi_1$ (or equivalently in $\pi_1'$).

**Proof.** This is a special case of Corollary 4.4.9 replace $u_{a_0, n-1}$ with $\mu^*$.

By Theorem 4.2.6 we can find a unique quotient $\mathcal{V}$ of $\pi_1$ (resp., a unique quotient $\mathcal{V}'$ of $\pi_1'$) such that $\mathcal{V}$ (resp., $\mathcal{V}'$) has $F(\mu^*)$ as socle and $F(\mu_1)$ (resp., $F(\mu_1')$) as cosocle. Here, by our choice of $F$, $\mathcal{V}$ and $\mathcal{V}'$ always exist.

Now we can state one of our main results on the local automorphic side, and prove it under the theorems above.

**Corollary 4.2.7.** Assume that $(a_{n-1}, \cdots, a_0)$ is $2n$-generic in the lowest alcove.

Then we have

$$0 \neq S_n \left( \mathcal{V}^{U(\mathbf{F}_p), \mu_1} \right) \subseteq \mathcal{V} \quad \text{and} \quad 0 \neq S_n' \left( (\mathcal{V}')^{U(\mathbf{F}_p), \mu_1'} \right) \subseteq (\mathcal{V}').$$

**Proof.** On one hand, by definition of $\mathcal{V}$ and $\mathcal{V}'$, we have two natural morphisms

$$V_1 \hookrightarrow \pi_1 \twoheadrightarrow \mathcal{V} \quad \text{and} \quad V_1' \hookrightarrow \pi_1' \twoheadrightarrow (\mathcal{V}').$$

Then by Theorem 4.2.6 and Theorem 4.2.5 we know that $V_1$ (resp. $V_1'$) has $F(\mu^*)$ as a Jordan–Hölder factor and the image of $V_1$ (resp. $V_1'$) in $\mathcal{V}$ (resp. $\mathcal{V}'$) is non-zero. □

**Remark 4.2.8.** It is known by [HLM] (Proposition 3.1.2 and its proof) that both $\mathcal{V}$ and $\mathcal{V}'$ are uniserial of length 2 if $n = 3$. In general, we can prove that the cosocle filtration of both $\mathcal{V}$ and $\mathcal{V}'$ has length $\frac{(n-1)(n-2)}{2} + 1$. In other words, $\mathcal{V}$ and $\mathcal{V}'$ are very big in general. It is a bit surprising at first sight that even though these two representations are big, we can still prove some accurate non-vanishing theorem (Theorem 4.2.7) for Jacobi sums.

### 4.3. Summary of results on Deligne–Lusztig representations

In this section, we recall standard facts on Deligne–Lusztig representations and fix their notation that will be used throughout this paper. We closely follow [Her09]. Throughout this article we will only focus the group $G(\mathbf{F}_p) = \text{GL}_n(\mathbf{F}_p)$, which is the fixed point set of the standard $(p$-power) Frobenius $F$ inside $\text{GL}_n(\mathbf{F}_p)$. We will identify a variety over $\overline{\mathbf{F}}_p$ with the set of its $\overline{\mathbf{F}}_p$-rational points for simplicity. Then our fixed maximal torus $\mathbb{T}$ is $F$-stable and split.

To each pair $(T, \theta)$ consisting of an $F$-stable maximal torus $T$ and a homomorphism $\theta : T^F \to \overline{\mathbf{F}}_p^\times$, Deligne–Lusztig [DL76] associate a virtual representation $R^\theta_{\mathcal{V}}$ of $\text{GL}_n(\mathbf{F}_p)$. (We restrict ourself to $\text{GL}_n(\mathbf{F}_p)$ although the result in [DL76] is much more general.) On the other hand, given a pair $(w, \mu) \in W \times X(T)$, one can construct a pair $(T_w, \theta_{w, \mu})$ by the method in the third paragraph of [Her09] Section 4.1. Then we denote by $R_w(\mu)$ the representation corresponding to $R^\theta_{\mathcal{V}_{w, \mu}}$ after multiplying a sign. This is the so-called Jantzen parametrization in [Jan81] 3.1.
The representations $R_w^p$ (resp. $R_w(\mu)$) can be isomorphic for different pairs $(T, \theta)$ (resp. $(w, \mu)$), and the explicit relation between is summarized in [Her09] Lemma 4.2. As each $p$-regular character $\mu \in X(T)/(p-1)X(T)$ of $T(F_p)$ can be lift to an element in $X^\res_1(T)$ which is unique up to $(p-1)X_0(T)$, the representation $R_w(\mu)$ is well defined for each $w \in W$ and such a $\mu$.

We recall the notation $\Theta(\theta)$ for a cuspidal representation for $\GL_m(F_p)$ from Section 2.1 of [Her06] where $\theta$ is a primitive character of $F_p^\times$ as defined in [Her09], Section 4.2. We refer further discussion about the basic properties and references of $\Theta(\theta)$ to Section 2.1 of [Her06]. The relation between the notation $R_w(\mu)$ and the notation $\Theta(\theta)$ is summarized in the Lemma 4.7 of [Her09]. In this paper, we will use the notation $\Theta_m(\theta_m)$ for a cuspidal representation for $\GL_m(F_p)$ where $\theta$ is a primitive character of $F_p^\times$.

We emphasize that, as a special case of Lemma 4.7 of [Her09], we have the natural isomorphism
\[
\text{Ind}_{B(F_p)}^G(\hat{\mu}) \cong R_1(\mu)
\]
for a $p$-regular character $\mu$ of $T(F_p)$, where $\hat{\mu}$ is the Teichmüller lift of $\mu$.

4.4. Proof of Theorem 4.2.6. The main target of this section is to prove Theorem 4.2.6. In fact, we prove Corollary 4.4.9 which is a generalization of Theorem 4.2.6.

We recall some notation from [Jan03]. We use the notation $\bar{\mathcal{G}}_r$ for the $r$-th Frobenius kernel defined in Chapter I.9 as kernel of $r$-th iteration of Frobenius morphism on the group scheme $\mathcal{G}$ over $F_p$. We will consider the subgroup scheme $\bar{\mathcal{G}}_r \bar{T}_r \bar{\mathcal{G}}_r \bar{B}$, $\bar{\mathcal{G}}_r \bar{B}^+$ of $\bar{\mathcal{G}}$ in the following. Note that our $\bar{B}$ (resp. $\bar{B}^+$) is denoted by $B^+$ (resp. $B$) in [Jan03] Chapter II. 9. We define
\[
\hat{Z}_r'(*) := \text{ind}_{\bar{B}^r}^{\bar{\mathcal{G}}_r \bar{T}_r \bar{\mathcal{G}}_r \bar{B}} (*) \quad \text{and} \quad \hat{Z}_r(*) := \text{coind}_{\bar{B}^r}^{\bar{\mathcal{G}}_r \bar{T}_r \bar{\mathcal{G}}_r \bar{B}^+} (*)
\]
where ind and coind are defined in I.3.3 and I.8.20 of [Jan03] respectively. By [Jan03] Proposition II.9.6 we know that there exists a simple $\bar{\mathcal{G}}_r \bar{T}_r$-module $L_r(\lambda)$ satisfying
\[
\text{soc}_{\bar{\mathcal{G}}_r} \left( \hat{Z}_r'(1) \right) \cong \hat{L}_r(\lambda) \cong \text{cosoc}_{\bar{\mathcal{G}}_r} \left( \hat{Z}_r(*) \right).
\]
The properties of $\hat{Z}_r'(\lambda)$ and $\hat{Z}_r(\lambda)$ are systematically summarized in [Jan03] II.9, and therefore we will frequently refer to results over there.

From now on we assume $r = 1$ in this section.

Now we recall several well-known results from [JanS1], [JanS4] and [Jan03]. We recall the definition of $\hat{W}^\res$ from [H03].

**Theorem 4.4.1** ([JanS1], Satz 4.3). Assume that $\mu + \eta$ is in the lowest restricted alcove and $2n$-generic (Definition 2.1.7). Then we have
\[
\hat{R}_w(\mu + \eta) = \sum_{\hat{w} \in \hat{W}^\res \atop \hat{w} \in \hat{W}^\res} [\hat{Z}_1(\mu - \nu \cdot \eta) : \hat{L}_1(\hat{w} \cdot \mu)] F(\hat{w} \cdot (\mu + \nu \cdot \eta)).
\]

**Proposition 4.4.2.** Let $\lambda \in X(T)_+$. Suppose $\mu \in X(T)$ is maximal for $\mu \uparrow \lambda$ and $\mu \neq \lambda$. If $\mu \in X(T)_+$ and if $\mu \neq \lambda - \alpha \cdot \nu$ for all $\alpha \in \Phi^+$, then
\[
[H^0(\lambda) : F(\mu)] = 1.
\]

**Proof.** This is the Corollary II 6.24 in [Jan03].

If $M$ is an arbitrary $\bar{G}$-module, we use the notation $M^{[1]}$ for the Frobenius twist of $M$ as defined in [Jan03], I.9.10.

**Proposition 4.4.3** ([Jan03], Proposition II. 9.14). Let $\lambda \in X(T)_+$. Suppose each composition factor of $\hat{Z}_1(\lambda)$ has the form $L_1(\mu_0 + \mu_1)$ with $\mu_0 \in X_1(T)$ and $\mu_1 \in X(T)$ such that
\[
\langle \mu_1 + \eta, \beta^\vee \rangle \geq 0
\]
for all $\beta \in \Delta$. Then $H^0(\lambda)$ has a filtration with factors of the form $F(\mu_0) \otimes H^0(\mu_1)[[1]]$. Each such module occurs as often as $\hat{L}_1(\mu_0 + p\mu_1)$ occurs in a composition series of $\hat{Z}_1(\lambda)$.

Remark 4.4.4. Note that if $\mu_1$ is in the lowest restricted alcove, then $F(\mu_0) \otimes H^0(\mu_1)[[1]] = F(\mu)$.

Lemma 4.4.5 ([Jan03], Lemma II. 9.18 (a)). Let $\hat{L}_1(\mu)$ be a composition factor of $\hat{Z}_1(\lambda)$, and write

$$\lambda + \eta = p\lambda_1 + \lambda_0$$

with $\lambda_0, \mu_0 \in X_1(T)$ and $\lambda_1, \mu_1 \in X(T)$.

If

\[(\lambda, \alpha^\vee) \geq n - 2\]

for all $\alpha \in \Phi^+$, then

\[(\mu_1 + \eta, \beta^\vee) \geq 0\]

for all $\beta \in \Phi^+$.

Proof. We only need to mention that $h_\alpha = n$ for all $\alpha \in \Phi^+$ and for our group $G = GL_{n/F_p}$, where $h_\alpha$ is defined in [Jan03], II.9.18. \qed

We define an element $s_{\alpha, m} \in \hat{W}$ by

$$s_{\alpha, m} \cdot \lambda = s_\alpha \cdot \lambda + mp\alpha$$

for each $\alpha \in \Phi^+$ and $m \in \mathbb{Z}$.

Theorem 4.4.7. Let $\lambda, \mu \in X(T)$ such that

\[(\lambda, \alpha^\vee) \geq n - 2\]

Assume further that there exists $\nu \in X(T)$ such that $\lambda + \nu$ satisfies the condition (4.4.6) and that $\nu$ and $\mu_1 + \nu$ are in the lowest restricted alcove.

Then we have

$$[\hat{Z}_1(\lambda) : \hat{L}_1(\mu)] = 1.$$

Proof. The condition (4.4.6) ensures that for any fixed $\nu \in X(T)$, $\mu + \nu$ is maximal for $\lambda + \nu$ and $\mu + \nu \neq \lambda + \nu$. Notice that we have

$$[\hat{Z}_1(\lambda) : \hat{L}_1(\mu)] = [\hat{Z}_1(\lambda) : \hat{L}_1(\mu)]$$

by II 9.2(3) in [Jan03], as the character of a $G, T$-module determine its Jordan–Hölder factors with multiplicities (or equivalently, determine the semisimplification of the $G, T$-module).

By II 9.2(5) and II 9.6(6) in [Jan03] we have

$$[\hat{Z}_1(\lambda) : \hat{L}_1(\mu)] = [\hat{Z}_1(\lambda) \otimes \nu : \hat{L}_1(\mu) \otimes \nu] = [\hat{Z}_1(\lambda + \nu) : \hat{L}_1(\mu + \nu)],$$

and thus we may assume that

$$\langle \lambda, \alpha^\vee \rangle \geq n - 2$$

for all $\alpha \in \Phi^+$ by choosing appropriate $\nu$ (which exists by our assumption) and replacing $\lambda$ by $\lambda + \nu$ and $\mu$ by $\mu + \nu$. Then by Lemma 4.4.5 we know that

$$\langle \mu_1' + \eta, \beta^\vee \rangle \geq 0$$

for any $\mu' = p\mu_1' + \mu_0'$ such that $\hat{L}_1(\mu')$ is a factor of $\hat{Z}_1(\lambda)$.

Thus by Proposition 4.4.3, Proposition 4.4.2 and Remark 4.4.4 we know that

$$[\hat{Z}_1(\lambda) : \hat{L}_1(\mu)] = [H^0(\lambda) : F(\mu_0) \otimes H^0(\mu_1)[[1]]] = [H^0(\lambda) : F(\mu)] = 1$$

which finishes the proof. \qed
We pick an arbitrary principal series \( \pi \) and write
\[
\mu_\pi = (d_1, \cdots, d_n)
\]
For each pair of integers \((i_1, j_1)\) satisfying \(0 \leq i_1 < i_1 + 1 < j_1 \leq n - 1\), we define
\[
\mu_{i_1, j_1}^\pi := (d_1^{i_1, j_1}, \cdots, d_n^{i_1, j_1})
\]
where
\[
d_k^{i_1, j_1} = \begin{cases} 
  d_k & \text{if } k \neq n - j_1 \text{ and } k \neq n - i_1; \\
  d_{n-i_1} + j_1 - i_1 - 1 & \text{if } k = n - i_1; \\
  d_{n-j_1} - j_1 + i_1 + 1 & \text{if } k = n - j_1.
\end{cases}
\]

**Corollary 4.4.9.** Assume that \( \mu_\pi \) is 2n-generic in the lowest alcove (cf. Definition 4.1.1). Then \( F(\mu_{i_1, j_1}^\pi) \) has multiplicity one in \( \pi \).

**Proof.** We only need to apply Theorem 4.4.7 and Theorem 4.4.10 to these explicit examples. We will follow the notation of Theorem 4.4.10. We fix \( w = 1 \) in Theorem 4.4.1 and take
\[
\mu + \eta := \mu_\pi = \mu_{i_1, j_1}^\pi + (j_1 - i_1 - 1) \left( \sum_{r=n-j_1}^{n-1-i_1} \alpha_r \right).
\]
We are considering the multiplicity of \( F(\mu_{i_1, j_1}^\pi) \) in \( \pi = R_1(\mu + \eta) \) and therefore we take \( \tilde{w}' := 1 \in \tilde{W}_{\text{res}} \) and
\[
\nu := \eta - (j_1 - i_1 - 1) \left( \sum_{r=n-j_1}^{n-1-i_1} \alpha_r \right).
\]
By II. 9.2(4) and II.9.16 (4) in [Jan03] we know that
\[
[\tilde{Z}_1(\mu - pv' + p\eta) : \tilde{L}_1(\mu)] = [\tilde{Z}_1((n - j_1, n - i_1) \cdot (\mu - pv) + p\eta) : \tilde{L}_1(\mu)].
\]
We observe that
\[
(n - j_1, n - i_1) \cdot (\mu - pv) + p\eta = (n - j_1, n - i_1) \cdot \mu + p \left( \eta - (n - j_1, n - i_1) \eta - (j_1 - i_1 - 1) \left( \sum_{r=n-j_1}^{n-1-i_1} \alpha_r \right) \right) = (n - j_1, n - i_1) \cdot \mu + p \left( \sum_{r=n-j_1}^{n-1-i_1} \alpha_r \right).
\]
Therefore we have
\[
p < \left( (n - j_1, n - i_1) \cdot (\mu - pv) + p\eta, \sum_{r=n-j_1}^{n-1-i_1} \alpha_r \right) < 2p
\]
and that
\[
\mu = s_{\sum_{r=n-j_1}^{n-1-i_1} \alpha_r} \cdot ((n - j_1, n - i_1) \cdot (\mu - pv) + p\eta).
\]
Moreover, it is easy to see that
\[
(n - j_1, n - i_1) \cdot (\mu - pv) + p\eta + p\eta = (n - j_1, n - i_1) \cdot \mu + p \left( \sum_{r=n-j_1}^{n-1-i_1} \alpha_r \right) + p\eta
\]
satisfies 4.4.6.
Hence, replacing the \( \lambda \) and \( \mu \) in Theorem 4.4.7 by \((n - j_1, n - i_1) \cdot (\mu - pv) + p\eta \) and \( \mu \) respectively, we conclude that
\[
[\tilde{Z}_1((n - j_1, n - i_1) \cdot (\mu - pv) + p\eta) : \tilde{L}_1(\mu)] = 1
\]
which finishes the proof by Theorem 4.4.1 and 4.4.10. \(\square\)
4.5. Some technical formula. In this section, we prove several technical formula that will be used in Section 4.6. The main results of this section are Lemma 4.5.8, Proposition 4.5.11 and Proposition 4.5.13.

We define
\[
\varepsilon_k := (-1)^{\frac{k(k-1)}{2}}.
\]

Let \( R \) be a \( \mathbb{F}_p \)-algebra, and \( A \in \overline{G}(R) \) a matrix. For \( J_1, J_2 \subseteq \{1, \ldots, n-1, n\} \), we write \( A_{J_1, J_2} \) for the submatrix of \( A \) consisting of the entries of \( A \) at the \((i, j)\)-position for \( i \in J_1, j \in J_2 \). For \( 1 \leq i \leq n-1 \), we define
\[
\begin{align*}
J_i^1 & := \{ n, \ldots, n-i+2, n-i+1 \}; \\
J_i^2 & := \{ 1, 2, \ldots, i-1, i \}; \\
J_i^3 & := \{ 1, 2, \ldots, i-2, i-1, i+1 \}; \\
J_i^4 & := \{ 2, 3, \ldots, i, i+1 \}.
\end{align*}
\]

Note that \( |J_i^1| = |J_i^2| = |J_i^3| = |J_i^4| = i \), so that for \( 1 \leq i \leq n-1 \)
\[
D_i := \varepsilon_i \det(A_{J_i^2, J_i^3}), \quad D'_i := \varepsilon_i \det(A_{J_i^1, J_i^3}), \quad \text{and} \quad D''_i := \varepsilon_i \det(A_{J_i^1, J_i^4})
\]
are well-defined. We also set \( D_n := \varepsilon_n \det(A) \). Hence, \( D_i, D'_i \) and \( D''_i \) are polynomials over the entries of \( A \).

Given a weight \( \lambda \in X_+(T) \), we now introduce an explicit model for the representation \( H^\lambda(T) \), and then start some explicit calculation. Consider the space of polynomials on \( \mathbb{C}[T]/\mathbb{F}_p \), which is denoted by \( \mathcal{O}(\overline{G}) \). The space \( \mathcal{O}(\overline{G}) \) has both a left action and a right action of \( \overline{G} \) induced by right translation and left translation by \( \overline{T} \) on \( \overline{G} \) respectively. The fraction field of \( \mathcal{O}(\overline{G}) \) is denoted by \( \mathcal{M}(\overline{G}) \).

Consider the subspace
\[
\mathcal{O}(\lambda) := \{ f \in \mathcal{O}(\overline{G}) \mid f \cdot b = w_0^\lambda(b) f \quad \forall b \in \overline{T} \},
\]
which has a natural left \( \overline{G} \)-action by right translation. As the right action of \( \overline{T} \) on \( \mathcal{O}(\overline{G}) \) is semisimple (and normalizes \( \overline{U} \)), we have a decomposition of algebraic representations of \( \overline{G} \):
\[
\mathcal{O}(\overline{G})^{\overline{T}} := \{ f \in \mathcal{O}(\overline{G}) \mid f \cdot u = f \quad \forall u \in \overline{T} \} = \bigoplus_{\lambda \in X(\overline{T})} \mathcal{O}(\lambda).
\]
It follows from the definition of the dual Weyl module as an algebraic induction that we have a natural isomorphism
\[
H^\lambda(T) \cong \mathcal{O}(\lambda).
\]

Note by [Jan03, Proposition II.2.6] that \( H^\lambda(T) \neq 0 \) if and only if \( \lambda \in X(T)_+ \).

We often write the weight \( \lambda \) explicitly as \((d_1, d_2, \ldots, d_n)\) where \( d_i \in \mathbb{Z} \) for \( 1 \leq i \leq n \). We will restrict our attention to a \( p \)-restricted and dominant \( \lambda \), i.e., \( d_1 \geq d_2 \geq \ldots \geq d_n \) and \( d_{i-1} - d_i < p \) for \( 2 \leq i \leq n \). We recall from the beginning of Section 4 the notation \( \langle \cdot \rangle_\lambda \) for a weight space with respect to the weight \( \lambda' \).

Lemma 4.5.4. Let \( \lambda = (d_1, d_2, \ldots, d_n) \in X_+(T) \). For \( \lambda' \in X(T) \), we have
\[
\dim_{\mathbb{F}_p} H^\lambda(T)_{\lambda'} = 1.
\]

Moreover, the set of \( \lambda' \) such that the above space is nontrivial is described explicitly as follows: consider the set \( \Sigma' \) of \((n-1)\)-tuple of integers \( m = (m_1, \ldots, m_{n-1}) \) satisfying \( 0 \leq m_i \leq d_i - d_{i+1} \) for \( 1 \leq i \leq n-1 \), and let
\[
v^{\text{alg}, \lambda}_{m} := D_n^{d_n} \prod_{i=1}^{n-1} D_i^{d_i - d_{i+1} - m_i} (D'_i)^{m_i}.
\]
Then the set
\[
\{ v^{\text{alg}, \lambda}_{m} \mid m \in \Sigma' \}
\]
forms a basis for the space \( H^\lambda(T)_{\lambda'} \), and the weight of the vector \( v^{\text{alg}, \lambda}_{m} \) is
\[
(d_1 - m_1, d_2 - m_1 - m_2, \ldots, d_{n-1} + m_{n-2} - m_{n-1}, d_n + m_{n-1})
\].
Proof. We define
\[
[U,U] \mathcal{O}(G)^{\overline{U}} := \{ f \in \mathcal{O}(G) \mid u_1 \cdot f = f \cdot u = f \quad \forall u \in \overline{U} \quad \forall u_1 \in [U,U] \}
\]
and
\[
[U,U] \mathcal{M}(G)^{\overline{U}} := \{ f \in \mathcal{M}(G) \mid u_1 \cdot f = f \cdot u = f \quad \forall u \in \overline{U} \quad \forall u_1 \in [U,U] \}.
\]
We consider a matrix \( A \) such that its entries \( A_{i,j} \) are indefinite variables. Then we can write
\[
A = A^{(1)} A^{(2)} A^{(3)}
\]
such that the entries of \( A^{(1)}, A^{(2)}, A^{(3)} \) are rational functions of \( A_{i,j} \) satisfying
\[
A^{(1)}_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i > j, \end{cases}
\]
\[
A^{(2)}_{i,j} = \begin{cases} D_j(A) & \text{if } i + j = n + 1; \\ D_{j-1}^-(A) & \text{if } i + j = n + 2; \\ 0 & \text{if } i + j \neq n + 1, n + 2, \end{cases}
\]
\[
A^{(3)}_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i > j \text{ or } i = j - 1. \end{cases}
\]

For each rational function \( f \in [U,U] \mathcal{M}(G)^{\overline{U}} \), we notice that \( f \) only depends on \( A^{(2)} \), which means that \( f \) is rational function of \( D_i \) for \( 1 \leq i \leq n \) and \( D_i^- \) for \( 1 \leq i \leq n - 1 \). In other word, we have
\[
[U,U] \mathcal{M}(G)^{\overline{U}} = F_p (D_1, \ldots, D_n, D_1^-, \ldots, D_{n-1}^-) \subseteq \mathcal{M}(G).
\]
Then we define
\[
[U,U] \mathcal{O}(G)^{\overline{U},\lambda'} := \{ f \in [U,U] \mathcal{O}(G)^{\overline{U}} \mid x \cdot f = \lambda'(x)f, \text{ and } f \cdot x = \lambda(x)f \quad \forall x \in T \}
\]
and
\[
[U,U] \mathcal{M}(G)^{\overline{U},\lambda'} := \{ f \in [U,U] \mathcal{M}(G)^{\overline{U}} \mid x \cdot f = \lambda'(x)f, \text{ and } f \cdot x = \lambda(x)f \quad \forall x \in T \}.
\]
Note that we have and an obvious inclusion
\[
[U,U] \mathcal{O}(G)^{\overline{U},\lambda'} \subseteq [U,U] \mathcal{M}(G)^{\overline{U},\lambda'}.
\]
We can also identify \( [U,U] \mathcal{O}(G)^{\overline{U},\lambda'} \) with \( H^0(\lambda)_{[U,U]} \) via the isomorphism \( \mathcal{O}(G) \to \mathcal{M}(G) \). By definition of \( D_i \) (resp. \( D_i^- \)) we know that they are \( T \)-eigenvector with eigencharacter \( \sum_{k=1}^i \epsilon_k \) (resp. \( \epsilon_{i+1} + \sum_{k=1}^{i-1} \epsilon_k \)) for \( 1 \leq i \leq n \) (resp. for \( 1 \leq i \leq n - 1 \)). Therefore we observe that \( [U,U] \mathcal{M}(G)^{\overline{U},\lambda} \) is one dimensional for any \( \lambda, \lambda' \in X(T) \) and is spanned by
\[
D_n^{d_n} \prod_{i=1}^{n-1} D_i^{d_i-d_{i+1}+m_i} (D_i^-)^{m_i}
\]
where \( \lambda = (d_1, \ldots, d_n) \) and \( \lambda' = (d_1 - m_1, d_2 + m_1 - m_2, \ldots, d_{n-1} + m_{n-2} - m_{n-1}, d_n + m_{n-1}) \). As \( \mathcal{O}(G) \) is a UFD and \( D_i, D_i^- \) are irreducible, we deduce that
\[
D_n^{d_n} \prod_{i=1}^{n-1} D_i^{d_i-d_{i+1}+m_i} (D_i^-)^{m_i} \in \mathcal{O}(G)
\]
if and only if
\[
0 \leq m_i \leq d_i - d_{i+1} \quad \text{for all } 1 \leq i \leq n - 1
\]
if and only if
\[
H^0(\lambda)_{[U,U]} \neq 0
\]
which finishes the proof. \( \square \)

We recall from Example 4.15.22 the definition of \( U_1 \) and \( U_1 \).
Lemma 4.5.5. Let \( \lambda = (d_1, d_2, \cdots, d_n) \in X_1(T) \). For \( \lambda' \in X(T) \), we have

\[
\dim_{\mathbb{F}_p} H^0(\lambda)_{\lambda'} \leq 1.
\]

Moreover, the set of \( \lambda' \) such that the space above is nontrivial is described explicitly as follows: consider the set \( \Sigma'' \) of \((n - 1)\)-tuple of integers \( \underline{m} = (m_1, \ldots, m_{n-1}) \) satisfying \( m_i \leq d_i - d_{i+1} \) for \( 1 \leq i \leq n - 1 \), and let

\[
v^{\text{alg},''}_{\underline{m}} := D_n^{d_n} \prod_{i=1}^{n-1} D_i^{d_i - d_{i+1} - m_i} (D_i'')^{m_i}.
\]

Then the set

\[
\{v^{\text{alg},''}_{\underline{m}} \mid \underline{m} \in \Sigma''\}
\]

forms a basis of the space \( H^0(\lambda)_{\lambda'} \), and the weight of the vector \( v^{\text{alg},''}_{\underline{m}} \) is

\[
(d_1 - \sum_{i=1}^{n-1} m_i, d_2 + m_1, \ldots, d_{n-1} + m_{n-2}, d_n + m_{n-1}).
\]

Proof. Replacing \( [\overline{U}, \overline{U}] \) by \( \overline{U}_1 \) in the proof of Lemma 4.5.4, we can define the following objects

\[
\overline{U} : \mathcal{O}(\overline{G})^{\overline{U}}, \quad \overline{U} : \mathcal{M}(\overline{G})^{\overline{U}}
\]

and

\[
\overline{U}, \lambda' \mathcal{O}(\overline{G})^{\overline{U}}, \lambda, \quad \overline{U}, \lambda' \mathcal{M}(\overline{G})^{\overline{U}}
\]

for each \( \lambda, \lambda' \in X(T) \). Note that we have an obvious inclusion

\[
\overline{U}, \lambda' \mathcal{O}(\overline{G})^{\overline{U}}, \lambda \subseteq \overline{U}, \lambda' \mathcal{M}(\overline{G})^{\overline{U}}, \lambda.
\]

We can also identify \( \overline{U}, \lambda' \mathcal{O}(\overline{G})^{\overline{U}}, \lambda \) with \( H^0(\lambda)_{\lambda'} \) via the isomorphism 4.5.3.

We consider a matrix \( A \) such that its entries \( A_{i,j} \) are indefinite variables. Then we can write

\[
A = A^{(1)} A^{(2)} A^{(3)}
\]

such that the entries of \( A^{(1)} \), \( A^{(2)} \), \( A^{(3)} \) are rational functions of \( A_{i,j} \) satisfying

\[
A^{(1)}_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i > j; \end{cases}
\]

\[
A^{(2)}_{i,j} = \begin{cases} D_j(A) & \text{if } i + j = n + 1; \\ D''_{j-1}(A) & \text{if } i = n, j > 1; \\ 0 & \text{if } i + j \neq n + 1, i < n, \end{cases}
\]

\[
A^{(3)}_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i > j \text{ or } i = 1 < j. \end{cases}
\]

For each rational function \( f \in \overline{U} : \mathcal{M}(\overline{G})^{\overline{U}} \), we notice that \( f \) only depends on \( A^{(2)} \), which means that \( f \) is rational function of \( D_i \) for \( 1 \leq i \leq n \) and \( D''_i \) for \( 1 \leq i \leq n - 1 \). In other word, we have

\[
\overline{U} : \mathcal{M}(\overline{G})^{\overline{U}} = \mathbb{F}_p (D_1, \ldots, D_n, D'_1, \ldots, D''_{n-1}) \subseteq \mathcal{M}(\overline{G}).
\]

By definition of \( D_i \) (resp. \( D'_i \)) we know that they are \( \overline{U} \)-eigenvector with eigencharacter \( \sum_{k=2}^{i+1} \epsilon_k \) (resp. \( \sum_{k=2}^{i+1} \epsilon_k \)) for \( 1 \leq i \leq n \) (resp. for \( 1 \leq i \leq n - 1 \)). Therefore we observe that \( \overline{U}, \lambda' \mathcal{M}(\overline{G})^{\overline{U}}, \lambda \) is one dimensional for any \( \lambda, \lambda' \in X(T) \) and is spanned by

\[
D_n^{d_n} \prod_{i=1}^{n-1} D_i^{d_i - d_{i+1} - m_i} (D'_i)^{m_i}.
\]
where \( \lambda = (d_1, \ldots, d_n) \) and \( \lambda' = (d_1 - \sum_{i=1}^{n-1} m_i, d_2 + m_1, \ldots, d_{n-1} + m_{n-2}, d_n + m_{n-1}) \). As \( \mathcal{O}(G) \) is a UFD and \( D_i, D_i'' \) are irreducible, we deduce that

\[
D_n^{d_i} \prod_{i=1}^{n-1} D_i^{d_i-d_{i+1}-m_i} \in \mathcal{O}(G)
\]

if and only if

\[
0 \leq m_i \leq d_i - d_{i+1} \quad \text{for all} \quad 1 \leq i \leq n-1
\]

if and only if

\[
H^0(\lambda_{\lambda'}) \neq 0
\]

which finishes the proof. \( \square \)

**Remark 4.5.6.** Lemma 4.5.3 essentially describes the decomposition of an irreducible algebraic representation of \( \text{GL}_n \) after restricting to a maximal Levi subgroup which is isomorphic to \( \text{GL}_1 \times \text{GL}_{n-1} \). This classical result is crucial in the proof of Theorem 4.6.39.

Given a principal series \( \pi \) and an integer \( r \) satisfying \( 1 \leq r \leq n-1 \), we consider the morphism \( \mathcal{T}_{s_r} : \pi \rightarrow \pi' \) defined in \( \text{[4.1.2]} \). We fix a vector \( v_{\pi'} \in \pi'|_{U(F_p), \nu^p} \) such that

\[
\mathcal{T}_{s_r}(v_{\pi}) = S_{d_0, v_{\pi'}}.
\]

**Lemma 4.5.7.** Let \( 1 \leq r \leq n-1 \), and let \( k = (k_{i,j}) \in \{0, 1, \cdots, p-1\}^{\Phi_{\nu_0}} \) such that \( k_{n-r,n+1-r} < p-1 \) and \( k_{i,j} = 0 \) for all \( 1 \leq i < i+1 < j \leq n \).

Then we have

\[
\mathcal{T}_{s_r}(S_{d_0, v_{\pi}}) = \begin{cases} 
0 & \text{if } k_{n-r,n+1-r} \geq |d_{r+1} - d_r|, \\
c_{k_{n-r,n+1-r},[d_{r+1} - d_r]} S_{d_0, v_{\pi'}} & \text{if } k_{n-r,n+1-r} < |d_{r+1} - d_r|,
\end{cases}
\]

where \( s'(k_{i,j}) \) is defined by

\[
s'_{k_{i,j}} = \begin{cases} 
k_{n-r,n+1-r} - |d_{r+1} - d_r| & \text{if } (i, j) = (n-r, n+1-r), \\
k_{i,j} & \text{otherwise}.
\end{cases}
\]

**Proof.** Note that we have

\[
\mathcal{T}_{s_r}(S_{d_0, v_{\pi}}) = S_{d_0, S_{d_0, v_{\pi}}},
\]

and

\[
S_{d_0, S_{d_0, v_{\pi}}} = \sum_{\mathcal{A} \in U(F_p), t \in F_p} \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{k_{i,j}} \right) A_{w_0} u_{\alpha_r}(t)s_r.
\]

We also have the Bruhat decompositions: if \( t = 0 \)

\[
A_{w_0} u_{\alpha_r}(0) s_r = A(w_0 s_r) = A''_{w_0 s_r} u_{\alpha_r} (A_{n-r,n+1-r}),
\]

and if \( t \neq 0 \)

\[
A_{w_0} u_{\alpha_r}(t)s_k = A_{w_0} u_{\alpha_r} (t^{-1}) w_0 \text{diag}(1, \cdots, t, -t^{-1}, \cdots, 1) u_{\alpha_r}(t^{-1}).
\]

Therefore, we have

\[
S_{d_0, S_{d_0, v_{\pi}}} = \sum_{\mathcal{A} \in U(F_p), t \in F_p} \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{k_{i,j}} \right) t^{d_{r+1} - d_r} A_{w_0} u_{\alpha_r} (t) w_0 v_{\pi'} + \sum_{\mathcal{A} \in U(F_p)} \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{k_{i,j}} \right) A_{w_0} s_r v_{\pi'}.
\]

The summation

\[
\sum_{\mathcal{A} \in U(F_p), t \in F_p} \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{k_{i,j}} \right) A_{w_0} s_r v_{\pi'}
\]
can be rewritten as
\[ \sum_{A'' \in U_{w_{0 \pi'}}(F_p)} \left( \prod_{1 \leq i < j \leq n, (i,j) \neq (n-r, n+1-r)} A_{i,j}^{k_{i,j}} \right) \left( \sum_{A_{n-r,n+1-r} \in F_p} A_{n-r,n+1-r}^{k_{n-r,n+1-r}} \right) A'' w_0 s_{\pi'} \]
which is 0 as we assume \( k_{n-r,n+1-r} < p - 1 \). Hence, we have
\[ S_{k,w_0} \cdot S_{0,s_{\pi'},\pi'} = \sum_{A \in U(F_p), \pi \in F^\times_p} \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{k_{i,j}} \right) t^{d_{r+1} - d_r} A u_{\alpha_{n-r}} (t^{-1}) w_0 \pi'. \]
On the other hand, after setting \( A' = A u_{\alpha_{n-r}}(t^{-1}) \) we have
\[ (4.5.8) \sum_{A' \in U(F_p), \pi \in F^\times_p} \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{k_{i,j}} \right) t^{d_{r+1} - d_r} A u_{\alpha_{n-r}} (t^{-1}) w_0 \pi' \]
where \( k_{i,j} = 0 \) for all \( 1 \leq i < i+1 < j \leq n \). Note that for \( \ell \neq 0 \) we have
\[ \sum_{t \in F_p} t^{\ell} = \begin{cases} 0 & \text{if } p - 1 \nmid \ell; \\ -1 & \text{if } p - 1 \mid \ell. \end{cases} \]
One can easily check that
\[ \sum_{t \in F_p} (A'_{n-r,n+1-r} - t^{-1})^{k_{n-r,n+1-r}} t^{d_{r+1} - d_r} \]
\[ = \sum_{t \in F_p} \left( \sum_{s=0}^{k_{n-r,n+1-r}} c_{k_{n-r,n+1-r},s} (-1)^s (A'_{n-r,n+1-r})^{k_{n-r,n+1-r}-s} \right) t^{d_{r+1} - d_r} \]
\[ = \sum_{s=0}^{k_{n-r,n+1-r}} c_{k_{n-r,n+1-r},s} (-1)^s \left( \sum_{t \in F_p} t^{d_{r+1} - d_r-s} \right) (A'_{n-r,n+1-r})^{k_{n-r,n+1-r}-s}, \]
which can be rewritten as follows: if \( k_{n-r,n+1-r} \geq |d_{r+1} - d_r| \) then it is
\[ (-1)^{|d_{r+1} - d_r|+1} c_{k_{n-r,n+1-r},|d_{r+1} - d_r|} (A'_{n-r,n+1-r})^{k_{n-r,n+1-r}-|d_{r+1} - d_r|}, \]
and if \( k_{n-r,n+1-r} < |d_{r+1} - d_r| \) then it is 0. Combining these computations with (4.5.8) finishes the proof.

Recall the definition of \( \mu_1 \) and \( \mu'_1 \) from (4.2.1). We recursively define sequences of elements in the Weyl group \( W \) by
\[ \begin{cases} w_1 = 1, & w_m = s_{n-m} w_{m-1}; \\ w'_1 = 1, & w'_m = s_{n} w'_{m-1} \end{cases} \]
for all \( 2 \leq m \leq n - 1 \), where \( s_m \) are the reflection of the simple roots \( \alpha_m \). We also define sequences of characters of \( T(F_p) \)
\[ \mu_m = \mu_1^{w_m} \text{ and } \mu'_m = \mu'_1^{w'_m}, \]
for all \( 1 \leq m \leq n - 1 \), and thus we have sequences of principal series representations
\[ \pi_m := \text{Ind}_{B(F_p)}^{G(F_p)} \mu_m \text{ and } \pi'_m := \text{Ind}_{B(F_p)}^{G(F_p)} \mu'_m \]
for all \( 1 \leq m \leq n - 1 \). Moreover, we have the following sequences of non-zero morphisms by Frobenius reciprocity:
\[ \varphi_{s_{n-m}}^m : \pi_m \to \pi_{m+1} \text{ and } \varphi'_{s_{n+1}}^m : \pi'_m \to \pi'_{m+1} \]
for all $1 \leq m \leq n - 2$. We fix sequences of non-zero vectors

$$v_m \in \pi_m^{U(p)}\sigma_m \quad \text{and} \quad v'_m \in (\pi'_m)^{U(p)}\sigma'_m$$

such that

$$\mathcal{T}^{\pi_m}_{s_n-1-n}(v_m) = S_{\mathbb{Q}_p,s_n-1-n}v_{m+1} \quad \text{and} \quad \mathcal{T}^{\pi'_m}_{s_n+1}(v'_m) = S_{\mathbb{Q}_p,s_n+1}v'_{m+1}.$$ 

We also define several families of Jacobi sums:

$$S_{k^m,0} \quad \text{and} \quad S_{k^{m'},0}$$

for all integers $m$ with $1 \leq m \leq n - 1$, where $k^m = (k^m_{i,j})$ satisfies

$$k^m_{i,j} = \begin{cases} 
    n - 2 + [a_0 - a_{n-1}] & \text{if } 1 \leq i = j - 1 \leq m; \\
    n - 2 + [a_0 - a_{n-1}] & \text{if } m + 1 \leq i = j - 1 \leq n - 1; \\
    0 & \text{otherwise}.
\end{cases}$$

and $k^{m'} = (k^{m'}_{i,j})$ satisfies

$$k^{m'}_{i,j} = \begin{cases} 
    n - 2 + [a_{n-i-1} - a_{n-1}] & \text{if } 1 \leq i = j - 1 \leq n - m - 1; \\
    n - 2 + [a_0 - a_{n-1}] & \text{if } n - m \leq i = j - 1 \leq n - 1; \\
    0 & \text{otherwise}.
\end{cases}$$

Finally, set

$$(4.5.9) \quad \begin{cases} 
    \mu_0 := \mu_{n-1} = \mu'_{n-1}; \\
    \pi_0 := \pi_{n-1} = \pi'_{n-1}; \\
    k^0 := k^{n-1} = k^{n-1}'.
\end{cases}$$

**Lemma 4.5.10.** Assume that $(a_{n-1}, \ldots, a_0)$ is $n$-generic (Definition 4.1.1). Then we have non-zero scalars $c^m, c^{m'} \in \mathbb{F}_p^*$ such that

$$\mathcal{T}^{\pi_m}_{s_{n-1-m}}(S_{k^m,0}v_m) = c^m S_{k^{m+1},0}v_{m+1}$$

and

$$\mathcal{T}^{\pi'_m}_{s_{n+1}}(S_{k^{m'},0}v'_m) = c^{m'} S_{k^{m+1'},0}v'_{m+1}$$

for all $1 \leq m \leq n - 2$.

**Proof.** This is a direct corollary of Lemma 4.5.7. If we apply Lemma 4.5.7 to $S_{k^m,0}v_m$ and $r = n - 1 - m$, we note that

$$k_{n-r,n-1-r} = k_{n+1,m+2}^m = [a_0 - a_{n-1-m}] + n - 2 > (a_{n-1} - a_{n-1-m}) = [d_{r+1} - d_r],$$

and therefore the conclusion follows and we pick

$$c^m = c_{k_{m+1,m+2},a_{n-1} - a_{n-1-m}}.$$

Similarly, if we apply Lemma 4.5.7 to $S_{k^{m'},0}v'_m$ and $r = m + 1$, we note that

$$k_{n-r,n-1-r} = k_{n+1,m,n-m}^{m'} = [a_m - a_{n-1}] + n - 2 > (a_m - a_0) = [d_{r+1} - d_r],$$

and therefore the conclusion follows by picking $c^{m'} = c_{k_{n-1,m-1-n-m},a_m - a_0}$. \hfill \Box

We define $V_m$ (resp. $V'_m$) to be the sub-representation of $\pi_m$ (resp. of $\pi'_m$) generated by $S_{k^m,0}v_m$ (resp. by $S_{k^{m'},0}v'_m$). By definition we know that $S_{k^0,0}v_0 = S_{k^{0},0}v_0 = S_{k^{n-1},0}v_0$ and therefore $V_0 = V_{n-1} = V_{n-1}'$. It follows easily from the definition that

$$S_{k^m,0}v_m \in \pi_m^{U(p),U(F_p)},v^{*} \quad \text{and} \quad S_{k^{m'},0}v'_m \in (\pi'_m)^{U(p),U(F_p)},v'^{*}$$

for $1 \leq m \leq n - 1$.

**Proposition 4.5.11.** Assume that $(a_{n-1}, \ldots, a_0)$ is $n$-generic (cf. Definition 4.1.1). If $F(\mu^*) \in \text{JH}(V_0)$, then the statement of Theorem 4.2.7 is true.
Proof. By Lemma 3.10, we know that there are surjections
\[ V_m \to V_{m+1} \text{ and } V'_m \to V'_{m+1} \]
for each \( 1 \leq m \leq n - 2 \). Therefore we know that
\[ \text{JH}(V_{m+1}) \subseteq \text{JH}(V_m) \] and \[ \text{JH}(V'_{m+1}) \subseteq \text{JH}(V'_m) \]
As we have an identification \( V_{n-1} = V'_{n-1} = V_0 \), we deduce that
\[ F(\mu') \in \text{JH}(V_0) \subseteq (\text{JH}(V_1) \cap \text{JH}(V'_1)) \]
which completes the proof. \( \square \)

From now on, we assume that \((a_{n-1}, \cdots, a_0)\) is \( n \)-generic in the lowest alcove (cf. Definition 4.1.1).

We need to do some elementary calculation of Jacobi sums. For this purpose we need to define the following group operators for \( 2 \leq r \leq n - 1 \):
\[
X^+_r := \sum_{t \in F_p} t^{p-2} u_{\sum_{i=r}^{n-1} \alpha_i} (t) \in F_p[G(F_p)],
\]
and similarly
\[
X^-_r := \sum_{t \in F_p} t^{p-2} u_{\sum_{i=r}^{n-1} \alpha_i} (t) w_0 \in F_p[G(F_p)].
\]

We notice that by definition we have the identification \( X^+_r = X_{\sum_{i=r}^{n-1} \alpha_i, 1}^+ \), where \( X_{\sum_{i=r}^{n-1} \alpha_i, 1}^+ \) is defined in (3.1.6).

**Lemma 4.5.12.** For a tuple of integers \( k = (k_{i,j}) \in \{0, 1, \cdots, p-1\}^{n \times n} \), we have
\[ X^+_r \cdot S_{k, w_0} = k_r w_0 \cdot S_{k_r, n-r, w_0} \]
where \( k_r, n-r \) satisfies \( k_r, n-r = k_r, n-1 \), and \( k_{i,j} = k_{i,j} \) if \( (i, j) \neq (r, n) \).

**Proof.** This is just a special case of Lemma 4.1.10 when \( \alpha_0 = \sum_{i=r}^{n} \alpha_i \) and \( m = 1 \). \( \square \)

For the following lemma, we set
\[ I := \{(i_1, i_2, \cdots, i_s) \mid 1 \leq i_1 < i_2 < \cdots < i_s = n \text{ for some } 1 \leq s \leq n - 1\}. \]

to lighten the notation.

**Lemma 4.5.13.** Let \( X = (X_{i,j})_{1 \leq i, j \leq n} \) be a matrix satisfying
\[ X_{i,j} = 0 \text{ if } 1 \leq j < i \leq n - 1. \]

Then the determinant of \( X \) is
\[
\det(X) = \sum_{(i_1, \cdots, i_s) \in I} (-1)^{s-1} X_{n,i_1} \left( \prod_{j \neq i_k, 1 \leq k \leq s} X_{j,j} \right) \left( \prod_{k=1}^{n} X_{i_k, i_{k+1}} \right).
\]

**Proof.** By the definition of determinant we know that
\[
\det(X) = \sum_{w \in W} (-1)^{\ell(w)} \prod_{k=1}^{n} X_{k,w(k)}.
\]

From the assumption on \( X \), we know that each \( w \) that appears in the sum satisfies
\[
\ell(w) = \ell(w(k)) < k
\]
for all \( 2 \leq k \leq n - 1 \).

Assume that \( w \) has the decomposition into disjoint cycles
\[
w = (i_1^1, i_2^1, \cdots, i_{n_1}^1) \cdots (i_1^m, i_2^m, \cdots, i_{n_m}^m)
\]
where \( m \) is the number of disjoint cycles and \( n_k \geq 2 \) is the length for the \( k \)-th cycle appearing in the decomposition.
We observe that the largest integer in \( \{i_j^k \mid 1 \leq j \leq n_k\} \) must be \( n \) for each \( 1 \leq k \leq m \) by condition (4.5.15). Therefore we must have \( m = 1 \) and we can assume without loss of generality that \( i_{n_1}^1 = n \). It follows from the condition (4.5.15) that
\[
1 \leq j < i_{j+1}^1 
\]
for all \( 1 \leq j \leq n_1 - 1 \). Hence we can set
\[
s := n_1, \quad i_1 := i_{i_1}^1, \ldots, i_s := i_{i_s}^1.
\]
We observe that \( \ell(w) = s - 1 \) and the formula (4.5.14) follows.  \( \square \)

Recall from the beginning of Section 4.5 that we use the notation \( A_{J_1,J_2} \) for the submatrix of \( A \) consisting of the entries at the \( (i,j) \)-position with \( i \in J_1, j \in J_2 \), where \( J_1, J_2 \) are two subsets of \( \{1, 2, \ldots, n\} \) with the same cardinality. For a pair of integers \((m, r)\) with \( 1 \leq m \leq r - 1 \leq n - 2 \), we let
\[
J_{m,r} := \{n, r + 1, r + 2, \ldots, n\}.
\]
We also recall from (4.5.1) that \( \varepsilon_k = (-1)^{\binom{k+1}{2}} \).

For a matrix \( A \in U(F_p) \), an element \( t \in F_p \), and a triple of integers \((m, r, \ell)\) satisfying \( 1 \leq m \leq r - 1 \leq n - 2 \) and \( 1 \leq \ell \leq n - 1 \), we define some polynomials as follows:
\[
D_{m,r}(A, t) = \varepsilon_{n+1-r} \det \begin{pmatrix} w_0 u_{\sum_{i=1}^{r-1} \alpha_i + 1} (t) w_0 Aw_0 \end{pmatrix}_{i_{m-r}, i_{n+1-r} + 1} \quad \text{when } 1 \leq m \leq r - 1;
\]
\[
D_{r}^{(\ell)}(A, t) = \varepsilon_\ell \det \begin{pmatrix} w_0 u_{\sum_{i=1}^{n-1} \alpha_i + 1} (t) w_0 Aw_0 \end{pmatrix}_{i_{1}, i_{n} + 1} \quad \text{when } 1 \leq \ell \leq n - r.
\]

We define the following subsets of \( I \): for each \( 1 \leq \ell \leq n - 1 \)
\[
I_\ell := \{(i_1, i_2, \ldots, i_s) \mid n - \ell + 1 \leq i_1 < i_2 < \cdots < i_s = n \text{ for some } 1 \leq s \leq \ell\},
\]
Note that we have natural inclusions
\[
I_\ell \subseteq I_{\ell'} \subseteq I
\]
if \( 1 \leq \ell \leq \ell' \leq n - 1 \). In particular, \( I_1 \) has a unique element \( (n) \). Similarly, for each \( 1 \leq \ell' \leq n - 1 \) we define
\[
I'_{\ell'} := \{(i_1, i_2, \ldots, i_s) \mid 1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq n - \ell' < i_s = n \text{ for some } 1 \leq s \leq \ell'\},
\]
and we set
\[
I'_{\ell'} := I_\ell \cap I'_{\ell'}
\]
for all \( 1 \leq \ell' \leq \ell - 1 \leq n - 2 \). We often write \( \underline{i} = (i_1, \ldots, i_s) \) for an arbitrary element of \( I \), and define the sign of \( \underline{i} \) by
\[
\varepsilon(\underline{i}) := (-1)^s.
\]
We emphasize that all the matrices \( w_0 u_{\sum_{i=1}^{r-1} \alpha_i + 1} (t) w_0 Aw_0 \) for \( 1 \leq m \leq r - 1 \), and all the matrices \( w_0 u_{\sum_{i=1}^{n-1} \alpha_i + 1} (t) w_0 Aw_0 \) for \( 1 \leq \ell \leq n - r \), after multiplying a permutation matrix, satisfy the conditions on the matrix \( X \) in Lemma 4.5.13. Hence, by Lemma 4.5.13 we notice that
\[
\begin{cases}
D_{m,r}(A, t) = A_{m,r} + tf_{m,r}(A) \quad \text{when } 1 \leq m \leq r - 1; \\
D_{r}^{(\ell)}(A, t) = 1 - tf_{r,n-\ell+1}(A) \quad \text{when } 1 \leq \ell \leq n - r;
\end{cases}
\]
where for all \( 1 \leq m \leq r - 1 \)
\[
f_{m,r}(A) := \sum_{\underline{i} \in I_{{n_1-r}+1}} \left( \varepsilon(\underline{i}) A_{m,i_1} \prod_{j=2}^{s} A_{i_{j-1},i_j} \right).
\]
Let \( (m, r) \) be a tuple of integers with \( 1 \leq m \leq r - 1 \leq n - 2 \). Given a tuple of integers \( \underline{i} \in \{0, 1, \ldots, p-1\}^{\Phi_{w_0}} \), \( \underline{i} = (i_1, i_2, \ldots, i_s) \in I_{{n_1-r}+1} \), and an integer \( r' \) satisfying \( 1 \leq r' \leq r \), we define two tuples of integers
\[
\underline{i}_{m,r}^r = (i_{m,r}^r) \in \{0, 1, \ldots, p-1\}^{\Phi_{w_0}}
\]
and
\[ \mathbf{k}^{m,r,r'} = (k_{i,j}^{m,r,r'}) \in \{0, 1, \cdots, p-1\}^{\Phi(w)} \]
as follows:
\[
k_{i,j}^{m,r} = \begin{cases} 
  k_{m,r} - 1 & \text{if } (i, j) = (m, r) \text{ and } i_1 > r; \\
  k_{m,r} & \text{if } (i, j) = (m, r) \text{ and } i_1 = r; \\
  k_{i,j} + 1 & \text{if } (i, j) = (t_h, t_{h+1}) \text{ for } 1 \leq h \leq s - 1; \\
  k_{i,j} & \text{otherwise.}
\end{cases}
\]

and
\[
k_{i,j}^{r,m,r'} = \begin{cases} 
  k_{r,m}^{m,r} - 1 & \text{if } (i, j) = (r', n); \\
  k_{i,j}^{m,r} & \text{otherwise.}
\end{cases}
\]

Finally, we define one more tuple of integers \( \mathbf{k}^{r,+} = (k_{i,j}^{r,+}) \in \{0, 1, \cdots, p-1\}^{\Phi(w)} \) by
\[
k_{i,j}^{r,+} := \begin{cases} 
  k_{r,n} + 1 & \text{if } (i, j) = (r, n); \\
  k_{i,j} & \text{otherwise.}
\end{cases}
\]

Lemma 4.5.20. Fix two integers \( r \) and \( m \) such that \( 1 \leq m \leq r - 1 \leq n - 2 \), and let \( \mathbf{k} = (k_{i,j}) \in \{0, 1, \cdots, p-1\}^{\Phi(w)} \). Assume that \( k_{i,j} = 0 \) for \( r + 1 \leq j \leq n - 1 \) and that \( k_{i,r} = 0 \) for \( i \neq m \), and assume further that
\[ a_{n-r} - a_1 + [a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}] + k_{m,r} < p. \]

Then we have
\[
X_r \bullet S_{k,w_0}v_0 = k_{m,r} \sum_{i \in I_{n-r}} \varepsilon(i)S_{k}w_{0}v_0 \\
+ (a_{n-r} - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}] + k_{m,r})S_{\mathbf{k}^{r,+},w_0}v_0 \\
- \sum_{\ell=2}^{n-r} (a_{n-r} - a_{\ell-1} + k_{m,r}) \left( \sum_{i \in I_{n-\ell-1}} \varepsilon(i)S_{k}w_{n-\ell+1,w_0}v_0 \right).
\]

Proof. By the definition of \( X_r \), we have
\[(4.5.21) \quad X_r \bullet S_{k,w_0}v_0 = \sum_{A \in U(F_p), t \in F_p} \left( p^{p-2} \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{k_{i,j}} \right) w_0 u_{\sum_{h=r}^{n-1} \alpha_h} (t) w_0 A w_0 \right) v_0.
\]

For an element \( w \in W \), we use \( E_w \) to denote the subset of \( U(F_p) \times F_p \) consisting of all \((A, t)\) such that
\[ w_0 u_{\sum_{h=r}^{n-1} \alpha_h} (t) w_0 A w_0 \in B(F_p)wB(F_p).
\]

It is not difficult to see that if \( E_w \neq \emptyset \) then \( w_0 w(t) = \hat{i} \) for all \( 1 \leq i \leq r - 1 \).

We define \( M_w \) to be
\[
M_w := \sum_{(A, t) \in E_w} \left( p^{p-2} \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{k_{i,j}} \right) w_0 u_{\sum_{h=r}^{n-1} \alpha_h} (t) w_0 A w_0 \right) v_0.
\]

By the definition of \( E_w \), we deduce that there exist \( A' \in U_w(F_p) \), \( A'' \in U(F_p) \), and \( T \in T(F_p) \) for each given \((A, t) \in E_w\) such that
\[(4.5.22) \quad w_0 u_{\sum_{h=r}^{n-1} \alpha_h} (t) w_0 A w_0 = A' w T A''.
\]

Here, the entries of \( A' \), \( T \) and \( A'' \) are rational functions of \( t \) and the entries of \( A \). We can rewrite (4.5.22) as
\[(4.5.23) \quad w_0 u_{\sum_{h=r}^{n-1} \alpha_h} (-t) w_0 A' w = A w T^{-1} (T(A'')^{-1} T^{-1})
\]
In other words, the right side of (4.5.23) can also be viewed as the Bruhat decomposition of the left side. Therefore the entries of \( A, T, A' \) can also be expressed as rational functions of the entries of \( A' \).

For each \( A' \in U_w(F_p) \) and \( w \in W \), we define
\[
(4.5.24) \quad D_{m,l}^{w}(A', t) := \varepsilon_{n+1-r} \det \left( \left( w_0 u_{\sum_{i=r}^{n-1} \alpha_i(t)} w_0 A' w \right)_{j_i^m, r - r + 1} \right) \text{ when } 1 \leq m \leq r - 1;
\]
\[
D_{r}^{w,(t)}(A', t) := \varepsilon \det \left( \left( w_0 u_{\sum_{i=r}^{n-1} \alpha_i(t)} w_0 A' w \right)_{j_l^r, j_r^r} \right) \text{ when } 1 \leq \ell \leq n - r.
\]

Note that if \( w = w_0 \), then the definition in (4.5.24) specializes to (4.5.17). We notice that for a given matrix \( A' \in U_w(F_p) \), the equality (4.5.23) exists if and only if
\[
(4.5.25) \quad D_{r}^{w,(t)}(A', t) \neq 0 \text{ for all } 1 \leq \ell \leq n - r.
\]

On the other hand, we also notice that given a matrix \( A \in U(F_p) \), the equality (4.5.28) exists if and only if (4.5.25) holds.

By the Bruhat decomposition in (4.5.23), we have
\[
(4.5.26) \quad T^{-1} = \text{diag} \left( D_{r}^{w,(1)}, D_{r}^{w,(2)}, \ldots, D_{r}^{w,(n-1-r)}, \frac{1}{D_{r}^{w,(n-r)}}, 1, \ldots, 1 \right)
\]
in which we write \( D_{r}^{w,(i)} \) for \( D_{r}^{w,(i)}(A', -t) \) for brevity. We also have
\[
(4.5.27) \quad A_{i,j} = \begin{cases} A'_{i,j} & \text{if } 1 \leq i < j \leq n \text{ and } j \leq r - 1; \\ D_{m,r}^{w}(A', -t) & \text{if } (i, j) = (m, r); \\ \frac{D_{m,r}^{w,(1)}(A', -t)}{D_{m,r}^{w,(i)}(A', -t)} & \text{if } 1 \leq i \leq n - 1 \text{ and } j = n. \end{cases}
\]

We apply (4.5.22), (4.5.27) and (4.5.26) to \( M_w \) and get
\[
(4.5.28) \quad M_w = \sum_{(A, t) \in E_w} \left( F(A', w, t) \left( \prod_{1 \leq i < j \leq n \text{ or } j = n} (A'_{i,j})^{k_{i,j}} \right) A' w_0 \right) v_0
\]
where
\[
F(A', w, t) := t^{p-2} \left( D_{r}^{w,(1)} \right)^{k_{m,r}} \left( D_{r}^{w,(2)} \right)^{a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}} \prod_{s=2}^{n-r} \left( D_{r}^{w,(s)} \right)^{a_s - a_{s-1}}
\]
in which we let \( D_{m,r}^{w} := D_{m,r}^{w}(A', -t) \) and \( D_{r}^{w,(s)} := D_{r}^{w,(s)}(A', -t) \) for brevity. We have discussed in (4.5.25) that \( (A, t) \in E_w \) is equivalent to \( (A', t) \in U_w(F_p) \times F_p \) satisfying the conditions in (4.5.25).

As each \( D_{r}^{w,(s)}(A', -t) \) appears in \( F(A', w, t) \) with a positive power, we can automatically drop the condition (4.5.25) and reach
\[
(4.5.29) \quad M_w = \sum_{(A, t) \in U_w(F_p) \times F_p} \left( F(A', w, t) \left( \prod_{1 \leq i < j \leq n \text{ or } j = n} (A'_{i,j})^{k_{i,j}} \right) A' w_0 \right) v_0.
\]

If \( w \neq w_0 \), then there exist a unique integer \( i_0 \) satisfying \( r \leq i_0 \leq n \) such that \( w w_0(i_0) < i_0 \) but \( w w_0(i) = i \) for all \( i_0 + 1 \leq i \leq n \).

By applying Lemma 4.5.13 to \( D_{r}^{w,(n+1-i_0)}(A', -t) \) (as \( (w_0 u_{\sum_{i=r}^{n-1} \alpha_i(t)} w_0 A' w)_{j_1^n, r - r + 1} \) satisfy the condition of Lemma 4.5.13 after multiplying a permutation matrix), we deduce that
\[
D_{r}^{w,(n+1-i_0)}(A', -t) = tf(A')
\]
where \( f(A') \) is certain polynomial of entries of \( A' \).
Now we consider $F(A', w, t)$ as a polynomial of $t$. The minimal degree of monomials of $t$ appearing in $F(A', w, t)$ is at least

$$p - 2 + a_{n+1-i_0} - a_{n-i_0} > p - 1,$$

and the maximal degree of monomials of $t$ appearing in $F(A', w, t)$ is

$$p - 2 + k_{m,r} + [a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}]_1 + \sum_{s=2}^{n-r} a_s - a_{s-1}$$

$$= p - 2 + k_{m,r} + [a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}]_1 + a_{n-r} - a_1$$

$$< 2(p - 1).$$

As a result, the degree of each monomials of $t$ in $F(A', w, t)$ is not divisible by $p - 1$. Hence, we conclude that

$$M_w = 0$$

as we know that $\sum_{t \in \mathbb{F}_p} t^k \neq 0$ if and only if $p - 1 \mid k$ and $k \neq 0$.

Finally, we calculate $M_{w_0}$ explicitly using (4.5.20). Indeed, by applying (4.5.18), the monomials of $t$ appearing in $F(A', w_0, t)$ is nothing else than

$$t^{p-1} \left(-k_{m,r} f_{m,r}(A') + [a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}] f_{r,n}(A') + \sum_{s=2}^{n-r} (a_s - a_{s-1}) f_{r,n+1-s}(A')\right).$$

As $\sum_{t \in \mathbb{F}_p} t^{p-1} = -1$, we conclude that

$$(4.5.30) \quad X_r^{-} \cdot S_{k,w_0} v_0 = M_{w_0} = \sum_{A' \in U(\mathbb{F}_p)} F_0(A') \left( \prod_{1 \leq i < j \leq n} (A'_{i,j})^{k_{i,j}} \right) A' w_0$$

where

$$F_0(A') := k_{m,r} f_{m,r}(A') - [a_1 - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}] f_{r,n}(A') - \sum_{s=2}^{n-r} (a_s - a_{s-1}) f_{r,n+1-s}(A').$$

Recalling the explicit formula of $f_{m,r}$ and $f_{r,n+1-s}$ for $1 \leq s \leq n-r$ from (4.5.19) and then rewriting (4.5.30) as a sum of distinct monomials of entries of $A'$ finishes the proof. \qed

**Proposition 4.5.31.** Keep the assumptions and the notation of Lemma 4.5.20

Then we have

$$X_r^+ \cdot X_r^- \cdot S_{k,w_0} v_0 = k_{r,n} k_{r,n} \sum_{I \in I_{n-r}} \varepsilon(I) S_{k_{r,n-r},w_0} v_0$$

$$+(k_{r,n} + 1)([a_{n-r} - a_{n-1} - \sum_{i=1}^{n-1} k_{i,n}] + k_{m,r}) S_{k_{r,n-r},w_0} v_0$$

$$- k_{r,n} \sum_{\ell=2}^{n-r} (a_{n-r} - a_{\ell-1} + k_{m,r}) \left( \sum_{I \in I_{\ell} \setminus I_{\ell-1}} \varepsilon(I) S_{k_{r,n-r+\ell-1},w_0} v_0 \right).$$

**Proof.** This is just a direct combination of Lemma 4.5.20 and Lemma 4.5.12 \qed
4.6. Proof of Theorem [4.6.39]. The main target of this section is to prove Theorem 4.6.39 which immediately implies Theorem 4.2.5 by Proposition 4.5.11. We start this section by introducing some notation. We first define a subset $\Lambda_{w_0}$ of $\{0, \cdots, p-1\}^{[n]}$ consisting of the tuples $\underline{k} = (k_{i,j})_{i,j}$ such that for each $1 \leq r \leq n-1$

$$\sum_{1 \leq i \leq r, r \leq j \leq n} k_{i,j} = [a_0 - a_{n-1}]_1 + n - 2. \tag{4.6.2}$$

Note that the set $\Lambda_{w_0}$ embeds into $\pi_0$ by sending $\underline{k}$ to $S_{k_{0},w_0}v_0$. Moreover, this family of vectors \{\(S_{k_{0},w_0}v_0 \mid \underline{k} \in \Lambda_{w_0}\)\} shares the same eigencharacter by Lemma 4.1.15.

We define $\underline{k}^s \in \Lambda_{w_0}$ where $\underline{k}^s = (k_{i,j}^s)$ is defined by $k_{1,n}^s = [a_0 - a_{n-1}] + n - 2$ and $k_{i,j}^s = 0$ otherwise. We define $V^s$ to be the subrepresentation of $\pi_0$ generated by $S_{\underline{k}^s,w_0}v_0$. We also need to define several useful elements and subsets of $\Lambda_{w_0}$. For each $1 \leq r \leq n-1$, we define $\underline{k}^s, r \in \Lambda_{w_0}$ where $\underline{k}^s, r = (k_{i,j}^s, r)$ is defined by

$$k_{i,j}^s, r := \begin{cases} n - 2 + [a_0 - a_{n-1}]_1 & \text{if } 2 \leq j = i + 1 \leq r; \\ n - 2 + [a_0 - a_{n-1}]_1 & \text{if } (i, j) = (r, n); \\ 0 & \text{otherwise.} \end{cases} \tag{4.6.3}$$

In particular, we have

$$\underline{k}^{s,1} = \underline{k}^s \text{ and } \underline{k}^{s,n-1} = \underline{k}^0 \tag{4.6.1}$$

where $\underline{k}^0$ is defined in \(4.5.9\).

For each $1 \leq r \leq n - 2$ and $0 \leq s \leq [a_0 - a_{n-1}] + n - 2$, we define a tuple $\underline{k}^{s, r, s} \in \Lambda_{w_0}$ as follows:

$$k_{i,j}^{s, r, s} := \begin{cases} n - 2 + [a_0 - a_{n-1}] & \text{if } 2 \leq j = i + 1 \leq r; \\ n - 2 + [a_0 - a_{n-1}]_1 - s & \text{if } (i, j) = (r, r + 1); \\ s & \text{if } (i, j) = (r, n); \\ n - 2 + [a_0 - a_{n-1}]_1 - s & \text{if } (i, j) = (r + 1, n); \\ 0 & \text{otherwise.} \end{cases} \tag{4.6.2}$$

In particular, we have

$$\underline{k}^{s, r, 0} = \underline{k}^{s, r, 1} \text{ and } \underline{k}^{s, r, [a_0 - a_{n-1}] + n - 2} = \underline{k}^{s, r} \tag{4.6.3}$$

for each $1 \leq r \leq n - 2$.

We now introduce the rough idea of the proof of Theorem 4.6.39. The first obstacle to generalize the method of Proposition 3.1.2 in [HLM] is that $V_0$ does not admit a structure as $\overline{G}$-representation in general. Our method to resolve this difficulty is to replace $S_{\underline{k}^0,w_0}v_0$ by $S_{\underline{k},w_0}v_0$. We prove in Proposition 4.6.33 that $V^s$ admits a structure as $\overline{G}$-representation and actually can be identified with a dual Weyl module $H^0(\mu^w_0)$. (The notation $\mu^w_0$ will be clear in the following.) Now it remains to prove that

$$S_{\underline{k},w_0}v_0 \in V_0 \tag{4.6.3}$$

to deduce Theorem 4.6.39. We will prove in Proposition 4.6.23 that

$$S_{\underline{k}^{s, r-1},w_0}v_0 \in V_0 \implies S_{\underline{k}^{s, r},w_0}v_0 \in V_0 \tag{4.6.3}$$

for each $1 \leq r \leq n - 2$ and $1 \leq s \leq [a_0 - a_{n-1}] + n - 2$. As a result, we can thus pass from $S_{\underline{k}^0,w_0}v_0 \in V_0$ to $S_{\underline{k},w_0}v_0 \in V_0$ for $r = n - 2, n - 3, \cdots, 1$. The identification $\underline{k}^s = \underline{k}^{s,1}$ thus gives us 4.6.3.

We firstly state three direct Corollaries of Proposition 4.5.31. It is easy to check that each tuple $\underline{k}$ appearing in the following Corollaries satisfies the assumption in Proposition 4.5.31.
Corollary 4.6.4. For each $2 \leq r \leq n - 1$ and $0 \leq s \leq [a_0 - a_{n-1}] + n - 3$, we have

\[
X_r^+ \bullet X_r^- \bullet S_{\vec{L}, r, 1, s, w_0} v_0 = ([a_0 - a_{n-1}] + n - 2 - s) \sum_{\ell \in \mathbb{I}_{n-r}} \varepsilon(i) S_{\vec{L}, r, 1, s, \ell, m, r, w_0} v_0
\]

\[
+ ([a_1 - a_{n-1}] - s)([a_0 - a_{n-1}] + n - 1 - s) S_{\vec{L}, r, 1, s, w_0} v_0
\]

\[
- ([a_0 - a_{n-1}] + n - 2 - s) \sum_{\ell = 2}^{n-r} (a_{n-r} - a_{\ell-1} + [a_0 - a_{n-1}] + n - 2 - s)
\]

\[
\cdot \left( \sum_{i \in \mathbb{I}_1 \setminus \mathbb{I}_{\ell-1}} \varepsilon(i) S_{\vec{L}, r, 1, s, \ell, n-\ell+1, r, w_0} v_0 \right).
\]

Corollary 4.6.5. Fix two integers $r$ and $m$ such that $1 \leq m \leq r - 1 \leq n - 2$, and let $\vec{k} = (k_{i,j})$ be a tuple of integers in $\Lambda_{w_0}$ such that

\[
k_{i,j} = \begin{cases} 
0 & \text{if } r + 1 \leq j \leq n - 1; \\
0 & \text{if } i \neq m \text{ and } j = r; \\
0 & \text{if } r + 1 \leq i \leq n - 1 \text{ and } j = n; \\
1 & \text{if } (i, j) = (m, r); \\
1 & \text{if } (i, j) = (r, n).
\end{cases}
\]

Then we have

\[
X_r^+ \bullet X_r^- \bullet S_{\vec{k}, w_0} v_0 = \sum_{\ell \in \mathbb{I}_{n-r}} \varepsilon(i) S_{\vec{k}, m, r, \ell, w_0} v_0 + 2(a_{n-r} - a_0 - n + 3) S_{\vec{k}, w_0} v_0
\]

\[
- \sum_{\ell = 2}^{n-r} (a_{n-r} - a_{\ell-1} + 1) \left( \sum_{i \in \mathbb{I}_1 \setminus \mathbb{I}_{\ell-1}} \varepsilon(i) S_{\vec{k}, n-\ell+1, r, \ell, w_0} v_0 \right).
\]

Corollary 4.6.6. Fix two integers $r$ and $m$ such that $1 \leq m \leq r - 1 \leq n - 2$, and let $\vec{k} = (k_{i,j})$ be a tuple of integers in $\Lambda_{w_0}$ such that

\[
k_{i,j} = \begin{cases} 
0 & \text{if } r + 1 \leq j \leq n - 1; \\
0 & \text{if } r + 1 \leq i \leq n - 1 \text{ and } j = n;
\end{cases}
\]

Then we have

\[
X_r^+ \bullet X_r^- \bullet S_{\vec{k}, w_0} v_0 = (a_{n-r} - a_0 - n + 2) S_{\vec{k}, w_0} v_0
\]

\[
- \sum_{\ell = 2}^{n-r} (a_{n-r} - a_{\ell-1} + 1) \left( \sum_{i \in \mathbb{I}_1 \setminus \mathbb{I}_{\ell-1}} \varepsilon(i) S_{\vec{k}, n-\ell+1, r, \ell, w_0} v_0 \right).
\]

We now define the following constants in $\mathbb{F}_p$:

\[
\begin{align*}
\{ c_\ell & := \prod_{k=1}^{\ell-1} (a_k - a_0 - n + 2 + k)^{\ell-k-1} ; \\
\ell \leq \ell & := (a_\ell - a_0 - n + 3 + \ell) c_\ell'
\end{align*}
\]

for all $1 \leq \ell \leq n - 1$ where we understand $c_1$ to be 1. As the tuple $(a_0, a_1, \ldots, a_0)$ is $n$-generic in the lowest alcove, we notice that $c_\ell \neq 0 \neq c_\ell'$ for all $1 \leq \ell \leq n - 1$. By definition of $c_k$ and $c_k'$ one can also easily check that

\[
(4.6.7) \prod_{k=1}^{\ell-1} (c_k' - c_k) = c_\ell.
\]

We also define inductively the constants: for each $1 \leq \ell \leq n - 1$

\[
d_{\ell, \ell'} := \begin{cases} 
2(a_\ell - a_0 - n + 3) & \text{if } \ell' = 0; \\
(c_\ell' d_{\ell, \ell'-1} - (a_\ell - a_{\ell'} + 1) c_\ell' \prod_{k=1}^{\ell'-1} (c_k' - c_k) & \text{if } 1 \leq \ell' \leq \ell - 1.
\end{cases}
\]
We further define inductively a sequence of group operators $Z_\ell$ as follows:
\[
Z_1 := d_{1,0} \text{Id} - X_{n-1}^+ \bullet X_{n-1}^- \in F_p[G(F_p)]
\]
and
\[
Z_\ell := d_{\ell,\ell-1} \text{Id} - (Z_{\ell-1} \bullet \cdots \bullet Z_1 \bullet X_{n-1}^+ \bullet X_{n-1}^-) \in F_p[G(F_p)]
\]
for each $2 \leq \ell \leq n - 2$.

**Lemma 4.6.8.** For $1 \leq \ell \leq n - 1$, we have the identity
\[
d_{\ell,\ell-1} = (a_\ell - a_0 - n + 2) \left( \prod_{k=1}^{\ell-1} c'_k \right) + c'_\ell.
\]

**Proof.** During the proof of this lemma, we will keep using the following obvious identity with two variables
\[
(4.6.9) \quad ab = (a + 1)(b - 1) + a - b + 1
\]
By definition of $d_{\ell,\ell-1}$ we know that
\[
d_{\ell,\ell-1} = 2(a_\ell - a_0 - n + 3) \prod_{k=1}^{\ell-1} c'_k - \sum_{\ell' = 1}^{\ell-1} (a_\ell - a_{\ell'}) + 1)c_{\ell'} \left( \prod_{k=1}^{\ell' - 1} (c'_k - c_k) \right) \left( \prod_{k=\ell' + 1}^{\ell-1} c'_k \right)
\]
and therefore
\[
d_{\ell,\ell-1} - (a_\ell - a_0 - n + 2) \left( \prod_{k=1}^{\ell-1} c'_k \right) = (a_\ell - a_0 - n + 4) \prod_{k=1}^{\ell-1} c'_k
\]
\[
- \sum_{\ell' = 1}^{\ell-1} (a_\ell - a_{\ell'}) + 1)c_{\ell'} \left( \prod_{k=1}^{\ell' - 1} (c'_k - c_k) \right) \left( \prod_{k=\ell' + 1}^{\ell-1} c'_k \right).
\]
Now we prove inductively that for each $1 \leq j \leq \ell - 1$
\[
(4.6.10) \quad d_{\ell,\ell-1} - (a_\ell - a_0 - n + 2) \left( \prod_{k=1}^{\ell-1} c'_k \right) = (a_\ell - a_0 - n + 3 + j)(a_j - a_0 - n + 3 + j) - (a_\ell - a_j + 1)c_j
\]
\[
- \sum_{\ell' = j}^{\ell-1} (a_\ell - a_{\ell'}) + 1)c_{\ell'} \left( \prod_{k=1}^{\ell' - 1} (c'_k - c_k) \right) \left( \prod_{k=\ell' + 1}^{\ell-1} c'_k \right).
\]
By the identity (4.6.9), one can easily deduce that
\[
(a_\ell - a_0 - n + 3 + j)c'_j - (a_\ell - a_j + 1)c_j
\]
\[
= [(a_\ell - a_0 - n + 3 + j)(a_j - a_0 - n + 3 + j) - (a_\ell - a_j + 1)]c_j
\]
\[
= (a_\ell - a_0 - n + 4 + j)(a_j - a_0 - n + 2 + j)c_j
\]
\[
= (a_\ell - a_0 - n + 4 + j)(c'_j - c_j).
\]
Hence, we get the identity:
\[
(4.6.11) \quad [(a_\ell - a_0 - n + 3 + j)c'_j - (a_\ell - a_j + 1)c_j] \left( \prod_{k=j+1}^{\ell-1} c'_k \right) \left( \prod_{k=1}^{j} (c'_k - c_k) \right)
\]
\[
= (a_\ell - a_0 - n + 4 + j)( \prod_{k=1}^{j} (c'_k - c_k) \right) \left( \prod_{k=j+1}^{\ell-1} c_k \right).
\]
Thus, if the equation (4.6.10) holds for $j$, we can deduce that it also holds for $j+1$. By taking $j = \ell - 1$ and using the equation (4.6.11) once more, we can deduce that

\[ d_{\ell, \ell-1} - (a_\ell - a_0 - n + 2) \left( \prod_{k=1}^{\ell-1} c_k^\ell \right) = (a_\ell - a_0 - n + 3 + \ell) \left( \prod_{k=1}^{\ell-1} (c_k^\ell - c_k) \right). \]

Hence, by the equation (4.6.7), one finishes the proof. \qed

**Proposition 4.6.12.** Fix two integers $r$ and $m$ such that $1 \leq m \leq r - 1 \leq n - 2$.

(i) Let $k = (k_{i,j})$ be as in Corollary 4.6.5. Then we have

\[ Z_{n-r} \cdot S_{k_{i,j}} = c_{n-r} S_{k'}_{i,j}, \]

where $k' = (k'_{i,j})$ is defined as follows:

\[ k'_{i,j} := \begin{cases} 0 & \text{if } (i, j) = (m, r) \text{ or } (i, j) = (r, n); \\ 1 & \text{if } (i, j) = (m, n); \\ k_{i,j} & \text{otherwise.} \end{cases} \]

(ii) Let $k = (k_{i,j})$ be as in Corollary 4.6.6. Then we have

\[ Z_{n-r} \cdot S_{k_{i,j}} = c'_{n-r} S_{k'_{i,j}}. \]

We prove this proposition by a series of lemmas.

**Lemma 4.6.15.** Proposition 4.6.12 is true for $r = n - 1$.

*Proof.* For part (i) of Proposition 4.6.12 by applying Corollary 4.6.5 to the case $r = n - 1$ we deduce that

\[ X_{n-1}^+ \cdot X_{n-1}^- \cdot S_{k_{i,j}} = 2(a_1 - a_0 - n + 3)S_{k_{i,j}} v_0 - S_{k_{i,j}} v_0 = S_{k_{i,j}} v_0 \]

where $S_0 = \{ n-1, n \}$. Hence, part (i) of the proposition follows directly from the definition of $Z_1$ and $c_1$.

For part (ii) of Proposition 4.6.12 again by Corollary 4.6.6 to the case $r = n - 1$ we deduce that

\[ X_{n-1}^+ \cdot X_{n-1}^- \cdot S_{k_{i,j}} v_0 = (a_1 - a_0 - n + 2)S_{k_{i,j}} v_0. \]

Then we have

\[ Z_1 \cdot S_{k_{i,j}} v_0 = (a_1 - a_0 - n + 4)S_{k_{i,j}} v_0 \]

and part (ii) of the proposition follows directly from the definition of $c'_1$. \qed

**Lemma 4.6.16.** Let $\ell$ be an integer with $2 \leq \ell \leq n - 1$. If Proposition 4.6.12 is true for $r \geq n - \ell + 1$, then it is true for $r = n - \ell$.

*Proof.* We prove part (ii) first. Assume that 4.6.12 holds for $r \geq n - \ell + 1$. In fact, for a Jacobi sum $S_{k_{i,j}}$ satisfying the conditions in the (4.6.14) for $r = n - \ell$, we have

\[ X_{n-\ell}^+ \cdot X_{n-\ell}^- \cdot S_{k_{i,j}} v_0 = (a_\ell - a_0 - n + 2)S_{k_{i,j}} v_0 \]

by Corollary 4.6.6. Then we can deduce

\[ Z_{\ell-1} \cdot \cdots \cdot Z_1 \cdot X_{n-\ell}^+ \cdot X_{n-\ell}^- \cdot S_{k_{i,j}} v_0 = (a_\ell - a_0 - n + 2) \left( \prod_{s=1}^{\ell-1} c_s \right) S_{k_{i,j}} v_0 \]

from the assumption of the Lemma. Hence, by definition of $Z_\ell$, we have

\[ Z_\ell \cdot S_{k_{i,j}} v_0 = d_{\ell, \ell-1} S_{k_{i,j}} v_0 - Z_{\ell-1} \cdot \cdots \cdot Z_1 \cdot X_{n-\ell}^+ \cdot X_{n-\ell}^- \cdot S_{k_{i,j}} v_0 \]

\[ = \left( d_{\ell, \ell-1} - (a_\ell - a_0 - n + 2) \left( \prod_{s=1}^{\ell-1} c_s \right) \right) S_{k_{i,j}} v_0 \]

\[ = c'_\ell S_{k_{i,j}} v_0 \]

where the last equality follows from Lemma 4.6.8.
Now we turn to part (i). Assume that (4.6.13) holds for \( r \geq n - \ell + 1 \). We will prove inductively that for each \( \ell' \) satisfying \( 1 \leq \ell' \leq \ell - 1 \), we have

\[
Z_{\ell'} \cdot \cdots \cdot Z_1 \cdot X_{n+\ell}^+ \cdot X_{n-\ell}^- \cdot S_{\mathbb{L}^w_0} v_0
\]

\[
= d_{\ell', \ell} S_{\mathbb{L}^w_0} v_0 + \left( \prod_{s=1}^{\ell'} (c_s' - c_s) \right) \left( \sum_{\mathbb{I}_h^\ell} \epsilon(\hat{z}) S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} \right)
\]

\[
+ \left( \prod_{s=1}^{\ell'} (c_s' - c_s) \right) \left( \sum_{h=\ell'+1}^{\ell-1} (a_\ell - a_h + 1) \sum_{\mathbb{I}_h} \epsilon(\hat{z}) S_{\mathbb{L}^{n-\ell,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} \right)
\]

We begin with studying some basic properties of the index sets \( \mathbb{I}_h^\ell \). First of all, the set \( \mathbb{I}_h^\ell \setminus \mathbb{I}_h^{\ell'+1} \) has a unique element, which is precisely \( \hat{\mathbb{I}} = \{ n - \ell' - 1, n \} \). Furthermore, there is a natural map of sets

\[
\text{res}_{\ell'} : \mathbb{I}_h^\ell \rightarrow \mathbb{I}_h^{\ell'+1}
\]

for all \( \ell' + 2 \leq h \leq \ell \) defined by eliminating the element \( n - \ell' \) from \( \hat{\mathbb{I}} \in \mathbb{I}_h^\ell \) if \( n - \ell' \in \hat{\mathbb{I}} \). In other words, for each \( \hat{\mathbb{I}} \in \mathbb{I}_h^{\ell'+1} \), we have

\[
\text{res}_{\ell'}^{-1}(\hat{\mathbb{I}}) = \{ \hat{\mathbb{I}} \cup \{ n - \ell' \} \} \subseteq \mathbb{I}_h^\ell.
\]

We use the shorten notation

\[
\hat{\mathbb{I}}^\ell := \hat{\mathbb{I}} \cup \{ n - \ell' \}
\]

for each \( \hat{\mathbb{I}} \in \mathbb{I}_h^{\ell'+1} \). Note in particular that \( \epsilon(\hat{\mathbb{I}}) = -\epsilon(\hat{\mathbb{I}}^\ell) \).

Given an arbitrary \( \hat{\mathbb{I}} \in \mathbb{I}_h^{\ell'+1} \) for \( \ell' + 2 \leq h \leq \ell - 1 \), then \( S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} \) (resp. \( S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} \)) satisfies the conditions before the equation (4.6.14) (resp. (4.6.13)). As a result, by the assumption that Proposition 4.6.12 is true for \( r = n - \ell - 1 \), we deduce that

\[
Z_{\ell'+1} \cdot \left( S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} - S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} \right) = (c_{\ell'+1} - c_{\ell'+1}) S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0}.
\]

Similarly, we have

\[
Z_{\ell'+1} \cdot \left( S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} - S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} \right) = (c_{\ell'+1} - c_{\ell'+1}) S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0}
\]

for each \( \hat{\mathbb{I}} \in \mathbb{I}_h^{\ell'+1} \). We also have

\[
Z_{\ell'+1} \cdot S_{\mathbb{L}^w_0 v_0} = c_{\ell'+1} S_{\mathbb{L}^w_0 v_0}
\]

by (4.6.14) for \( r = n - \ell' - 1 \), and

\[
Z_{\ell'+1} \cdot S_{\mathbb{L}^w_0 v_0} = c_{\ell'+1} S_{\mathbb{L}^w_0 v_0}
\]

by (4.6.13) for \( r = n - \ell' - 1 \) where \( \hat{\mathbb{I}}_n = \{ n - \ell' - 1, n \} \).

Now assume that (4.6.17) is true for some \( 1 \leq \ell' \leq \ell - 2 \). Then by combining (4.6.18), (4.6.19), (4.6.20) and (4.6.21), we have

\[
Z_{\ell'+1} \cdot \cdots \cdot Z_1 \cdot X_{n+\ell}^+ \cdot X_{n-\ell}^- \cdot S_{\mathbb{L}^w_0 v_0}
\]

\[
= d_{\ell', \ell} Z_{\ell'+1} \cdot S_{\mathbb{L}^w_0 v_0} + \left( \prod_{s=1}^{\ell'} (c_s' - c_s) \right) Z_{\ell'+1} \cdot \left( \sum_{\mathbb{I}_h^\ell} \epsilon(\hat{z}) S_{\mathbb{L}^{m,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} \right)
\]

\[
+ \left( \prod_{s=1}^{\ell'} (c_s' - c_s) \right) Z_{\ell'+1} \cdot \left( \sum_{h=\ell'+1}^{\ell-1} (a_\ell - a_h + 1) \sum_{\mathbb{I}_h} \epsilon(\hat{z}) S_{\mathbb{L}^{n-\ell,n-\ell,n-\ell,w_0} \mathbb{L}^w_0 v_0} \right)
\]
which is the same as

\[(4.6.22) \quad c'_\ell d_{\ell, \ell'} S_{k, w_0} v_0 + \left( \prod_{s=1}^{\ell'} (c'_s - c_s) \right) (X + Y + Z)\]

where

\[X = (a_\ell - a_{\ell'} + 1) Z_{\ell + 1} \cdot S_{k, w_0, n - \ell, n - \ell' - 1, n - \ell, w_0} v_0,\]

\[Y = \sum_{i \in t'_{h+1}} \varepsilon(i) Z_{\ell + 1} \cdot \left( S_{k, m, n - \ell, n - \ell, w_0} v_0 - S_{k, m', n - \ell, n - \ell, w_0} v_0 \right),\]

and

\[Z = \sum_{h = \ell' + 2}^{\ell - 1} (a_\ell - a_h + 1) \sum_{i \in t'_{h+1} \setminus t'_{h+1}} \varepsilon(i) Z_{\ell + 1} \cdot \left( S_{k, m, n - h, n - h, w_0} v_0 - S_{k, m', n - h, n - h, w_0} v_0 \right).\]

One can also readily check that \((4.6.22)\) is also the same as

\[
\left( c_{\ell' + 1} d_{\ell', \ell'} + c_{\ell' + 1} \prod_{s=1}^{\ell'} (c'_s - c_s) (a_\ell - a_{\ell'} + 1) \right) S_{k, w_0} v_0 \\
+ \left( \prod_{s=1}^{\ell'} (c'_s - c_s) \right) \left( \sum_{i \in t'_{h+1}} \varepsilon(i) S_{k, m, n - \ell, n - \ell, w_0} v_0 \right) \\
+ \left( \prod_{s=1}^{\ell'} (c'_s - c_s) \right) \left( \sum_{h = \ell' + 2}^{\ell - 1} (a_\ell - a_h + 1) \sum_{i \in t'_{h+1} \setminus t'_{h+1}} \varepsilon(i) S_{k, m, n - h, n - h, w_0} v_0 \right),
\]

which finishes the proof of \((4.6.17)\), as we have

\[d_{\ell, \ell' + 1} = c'_{\ell' + 1} d_{\ell', \ell'} + c_{\ell' + 1} \prod_{s=1}^{\ell'} (c'_s - c_s) (a_\ell - a_{\ell'} + 1)\]

by definition.

Note that \((4.6.17)\) for each \(1 \leq \ell' \leq \ell - 1\) then follows from Corollary \((4.6.12)\) for \(r = n - \ell\). Note that the case \(\ell' = \ell - 1\) for \((4.6.11)\) is just the following

\[Z_{\ell - 1} \cdot Z_1 \cdot X_{n - \ell}^+ \cdot X_{n - \ell}^- \cdot S_{k, w_0} v_0 = d_{\ell, \ell - 1} S_{k, w_0} v_0 - \left( \prod_{s=1}^{\ell - 1} (c'_s - c_s) \right) S_{k, m, n - \ell, n - \ell, w_0} v_0\]

where \(\mathcal{I}_1 = \{n\}\).

Finally, \((4.6.13)\) for \(r = n - \ell\) follows from the equation above together with the definition of \(Z_\ell\) and the identity \((4.6.7)\).

**Proof of Proposition \((4.6.12)\)** Proposition \((4.6.12)\) follows easily from Lemma \((4.6.15)\) and Lemma \((4.6.16)\).\(\square\)

**Proposition \((4.6.23)\).** For each \(1 \leq r \leq n - 2\) and \(1 \leq s \leq [a_0 - a_{n-1}] + n - 2\), if \(S_{k, r, r-1, w_0} v_0 \in V_0\), then \(S_{k, r, r-1, w_0} v_0 \in V_0\).

**Proof.** By Proposition \((4.6.12)\) and its proof, we can deduce the following equalities

\[Z_{n - 2 - r} \cdot \cdots \cdot Z_1 \cdot S_{k, r, r-1, w_0} v_0 = \left( \prod_{\ell=1}^{n-2-r} c'_\ell \right) S_{k, r, r-1, w_0} v_0,\]
Lemma 4.6.26. For 1 \leq r \leq n - 1, we have the following equalities on \(H^0(\mu_0^{w_0})\mu^*\):

\[
X_{\beta,k} = -X_{\overline{\beta},k}
\]

for all 1 \leq k \leq p - 1.
Lemma 4.6.28. We have

\[ \mu_0^w - (\mu^s + k\beta) = ([a_0 - a_{n-1}] + n - 2 - k, 0, \ldots, 0, k - ([a_0 - a_{n-1}] + n - 2)). \]

Therefore \( \mu_0^w - (\mu^s + k\beta) \notin \sum_{\alpha \in \Phi^+} Z_{\geq 0} \alpha \) as long as \( k > [a_0 - a_{n-1}] + n - 2 \). As \( (a_{n-1}, \ldots, a_0) \) is assumed to be \( n \)-generic in the lowest alcove throughout this section, we deduce that

\[ (4.6.27) \quad \mu_0^w - (\mu^s + k\beta) \notin \sum_{\alpha \in \Phi^+} Z_{\geq 0} \alpha \text{ for all } k \geq p - 1. \]

On the other hand, by the definition \((4.0.1)\), the image of \( X_{\beta, k}^{alg} \) lies inside \( H^0(\mu_0^w)_{\mu^s + k\beta} \), which is zero by \((4.6.27)\) assuming \( k \geq p - 1 \). Hence we deduce that

\[ X_{\beta, k}^{alg} = 0 \text{ on } H^0(\mu_0^w)_{\mu^s} \text{ for all } k \geq p - 1. \]

Then the conclusion of this lemma follows from the equality \((4.1.8)\).

Proof. On one hand, by Lemma \((4.5.5)\) we know that

\[ \dim_{\mathbb{F}_p} H^0(\mu_0^w)_{\mu^s} = 1, \]

and this space is generated by \( v^{alg, \mu}_{m^t} \) where

\[ (4.6.30) \quad m^t = (m_1^t, \ldots, m_{n-1}^t) := (0, \ldots, 0, [a_0 - a_{n-1}] + n - 2). \]

We now need to identify the vector \( v^{alg, \mu}_{m^t} \) with certain linear combination of Jacobi sums. Note that by Lemma \((4.5.5)\) we have

\[ v^{alg, \mu}_{m^t} = D_n^{d_n} D_{n-1}^{a_0 - a_{n-1} + n - 2} (D_{n-1}^{d_{n-1}})^{[a_0 - a_{n-1}] + n - 2} \prod_{i=1}^{n-2} D_i^{d_i - d_{i+1}}. \]

Given a matrix \( A \in G(\mathbb{F}_p) \), then \( D_i(A) \neq 0 \) for all \( 1 \leq i \leq n - 1 \) if and only if

\[ A \in B(\mathbb{F}_p) w_0 B(\mathbb{F}_p), \]

and thus the support of \( v^{alg, \mu}_{m^t} \) is contained in \( B(\mathbb{F}_p) w_0 B(\mathbb{F}_p) \). As a result, by the proof of Proposition \((4.1.17)\) we know that \( v^{alg, \mu}_{m^t} \) is a linear combination of vectors of the form

\[ S_k w_0 v_0, \]

As \( v^{alg, \mu}_{m^t} \) is \( U_1 \)-invariant, and in particular \( U_1(\mathbb{F}_p) \)-invariant, then by Proposition \((4.1.29)\) we know that it is a linear combination of vectors of the form

\[ (4.6.31) \quad S_k w_0 v_0 \]

such that \( k_{1,n} = [a_0 - a_{n-1}] + n - 2, k_{1,j} = 0 \) or \( p - 1 \) for \( 2 \leq j \leq n - 1 \) and \( k_{i,j} = 0 \) for all \( 2 \leq i < j \leq n \).

Finally, note that

\[ u_\beta(t) v^{alg, \mu}_{m^t} = D_n^{d_n} D_{n-1}^{a_0 - a_{n-1} + n - 2} (D_{n-1}^{d_{n-1}}) \prod_{i=1}^{n-2} D_i^{d_i - d_{i+1}} \]

is a polynomial of \( t \) with degree \( [a_0 - a_{n-1}] + n - 2 \), we conclude that

\[ X_{\beta, [a_0 - a_{n-1}] + n - 2}^{alg, \mu} = v_0^{alg, \mu}. \]
where $\emptyset$ is the $(n-1)$-tuple with all entries zero.

By Lemma 4.6.26 and the fact that
\[
F_{p[\mathcal{H}^w_{\emptyset}]} = F_{p[S_{\emptyset,w_0}]} = \pi_0^{U(F_p),\mu_0^w},
\]
we deduce that
\[
X_{\beta,[a_0-a_{n-1}]+n-2}v^w_{\mathcal{H}^w_{\emptyset}} = c' S_{\emptyset,w_0}v_0
\]
for some non-zero constant $c'$. By Lemma 4.6.25 and the linear independence of Jacobi sums proved in Proposition 4.1.17, we know that only $S_{\emptyset,w_0}v_0$ can appear in the linear combination $4.6.31$. In other words, we have shown that
\[
v^w_{\mathcal{H}^w_{\emptyset}} = c'' S_{\emptyset,w_0}v_0
\]
for some non-zero constant $c''$, and thus we finish the proof. □

Lemma 4.6.32. The dual Weyl module $H^0(\mu_0^w)$ is uniserial with length two with socle $F(\mu_0^w)$ and cosocle $F(\mu^*)$.

Proof. By [Jan03] Proposition II.2.2 we know that $\text{soc}_K(H^0(\mu_0^w))$ is irreducible and can be identified with $F(\mu_0^w)$ (which is in fact the definition of $F(\mu_0^w)$). Therefore it suffices to show that $H^0(\mu_0^w)$ has only two Jordan–Hölder factors $F(\mu_0^w)$ and $F(\mu^*)$, each of which has multiplicity one.

By [Jan03] II.2.13 (2) it is harmless for us to replace $H^0(\mu_0^w)$ by the Weyl module $V(\mu_0^w)$ (defined in [Jan03] II.2.13 ) and show that $V(\mu_0^w)$ has only two Jordan–Hölder factors $F(\mu_0^w)$ and $F(\mu^*)$ and each of them has multiplicity one. As
\[
\begin{align*}
p &< \left\langle \mu_0^w, \left(\sum_{i=1}^{n-1} \alpha_i\right)^\vee \right\rangle < 2p; \\
0 &< \left\langle \mu_0^w, \left(\sum_{i=1}^{n-2} \alpha_i\right)^\vee \right\rangle < p; \\
0 &< \left\langle \mu_0^w, \left(\sum_{i=1}^{n-1} \alpha_i\right)^\vee \right\rangle < p,
\end{align*}
\]
we deduce that the only dominant alcove lying below the one $\mu_0^w$ lies in is the lowest $p$-restricted alcove. In particular, the only dominant weight which is linked to and strictly smaller than $\mu_0^w$ is $\mu^*$. By [Jan03] Proposition II. 8.19, we know the existence of a filtration of subrepresentation
\[
V(\mu_0^w) \supseteq V_1(\mu_0^w) \supseteq \cdots
\]
such that the following equality in Grothendieck group holds
\[
\sum_{i>0} V_i(\mu_0^w) = F(\mu^*).
\]
This equality implies that
\[
V_1(\mu_0^w) = F(\mu^*)
\]
and
\[
V_i(\mu_0^w) = 0 \text{ for all } i \geq 2.
\]
By [Jan03] II.8.19 (2) we also know that
\[
V(\mu_0^w)/V_1(\mu_0^w) \cong F(\mu_0^w),
\]
and thus we have shown that
\[
V(\mu_0^w) = F(\mu_0^w) + F(\mu^*)
\]
in the Grothendieck group. □

Proposition 4.6.33. We have $V^\sharp = H^0(\mu_0^w)$. 
Proof. By Lemma 4.6.32, we have the natural surjection

\[ H^0(\mu_0^{w_0}) \twoheadrightarrow F(\mu^*) \]

which induces a morphism

\[ H^0(\mu_0^{w_0})_{\mu_*} \rightarrow F(\mu^*)_{\mu_*}. \]

(4.6.34)

Now we consider \( H^0(\mu_0^{w_0}) \) as a \( \mathcal{T} \)-representation where \( L \cong \mathbb{G}_m \times \text{GL}_{n-1} \) is the standard Levi subgroup of \( G \) which contains \( U_1 \) as a maximal unipotent subgroup. For any \( \lambda \in X_L(T)_+ \) (cf. (5.0.1)) we use the notation \( H^0_L(\lambda) \) for the \( \mathcal{T} \)-dual Weyl module defined at the beginning of Section 4. The dual Weyl module \( H^0(\mu_0^{w_0}) \) is the mod \( p \) reduction of a lattice \( V_{\mathbb{Q}_p} \) in the unique irreducible algebraic representation \( V_{\mathbb{Q}_p} \) of \( G \) such that \( \left( V_{\mathbb{Q}_p}^{U_1} \right)^{\mu_0^{w_0}} \neq 0 \). As the category of finite dimensional algebraic representations of \( L \) in characteristic 0 is semisimple, \( V \) decomposes into a direct sum of characteristic 0 irreducible representations of \( L \). More precisely, we have the decomposition

\[ V_{\mathbb{Q}_p}|_{L} = \bigoplus_{\lambda \in X_L(T)_+} m_\lambda V_\lambda(L) \]

where \( V_\lambda(L) \) is the unique (up to isomorphism) irreducible algebraic representation of \( L \) such that \( (V_\lambda(L))^{U_1} \not\equiv 0 \) and

\[ m_\lambda := \dim_{\mathbb{Q}_p} V_{\mathbb{Q}_p}^{U_1}(\lambda). \]

Therefore in the Grothendieck group of algebraic representations of \( \mathcal{T} \) over \( \mathbb{F}_p \), we have

\[ [H^0(\mu_0^{w_0})]|_{\mathcal{T}} = \bigoplus_{\lambda \in X_L(T)_+} m_\lambda[H_\lambda(L)] \]

as by Lemma 4.5.5 we know that \( H^0(\mu_0^{w_0})^{U_1} \) is the mod \( p \) reduction of \( V_{\mathbb{Q}_p}^{U_1} \) and that \( V_{\mathbb{Q}_p}^{U_1} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p = V_{\mathbb{Q}_p}^{U_1} \).

We say that

\[ \mu^* \uparrow_L \lambda \]

if there exists \( \tilde{w} \in \tilde{W}^L \) (see the beginning of Section 5) such that

\[ \lambda = \tilde{w} \cdot \mu^* \text{ and } \mu^* \leq \lambda. \]

Assume that there exists a \( \lambda \in X_L(T)_+ \) such that \( \mu^* \uparrow_L \lambda \) and that \( H^0(\mu_0^{w_0})^{U_1} \neq 0 \). We denote by \( v_m^{\lambda \\mu^*} \) the vector in \( H^0(\mu_0^{w_0})^{U_1} \) \( \not\equiv 0 \) given by Lemma 4.5.5. We note that by Lemma 4.5.5, the vector in \( H^0(\mu_0^{w_0})^{U_1} \) is \( v_m^{\lambda \\mu^*} \) (see (4.6.30)). As \( \mu^* \uparrow_L \lambda \), we must firstly have \( \sum_{i=1}^{n-1} m_i = |a_0 - a_{n-1}| + n - 2 \). By the last statement in Lemma 4.5.5 we have

\[ \lambda = \left( a_0 + p - 1 - \sum_{i=1}^{n-1} m_i, a_{n-2} + m_1, \ldots, a_1 + m_{n-2}, a_{n-1} - p + 1 + m_{n-1} \right) \]

\[ = (a_{n-1} - n + 2, a_{n-2} + m_1, \ldots, a_1 + m_{n-2}, a_{n-1} - p + 1 + m_{n-1}). \]

Recall \( \eta = (n-1, n-2, \ldots, 1, 0) \). We notice that \( \mu^* - \eta \) lies in the lowest restricted \( \mathcal{T} \)-alcove in the sense that

\[ \langle \mu^* - \eta, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Phi^+_L \]

where \( \Phi^+_L \) is the positive roots of \( L \) defined at the beginning of Section 5.

As we assume that \( (a_{n-1}, \ldots, a_0) \) is \( n \)-generic, it is easy to see the following

\[
\begin{align*}
\{ & a_{n-2} + m_1 - (a_{n-1} - p + 1 + m_{n-1}) \leq p + 1 + a_{n-2} - a_{n-1} + m_1 < 2p; \\
& a_{n-2} + m_1 - (a_1 + m_{n-2}) \leq a_{n-2} + m_1 - a_1 \leq |a_0 - a_1| < p; \\
& a_{n-3} + m_2 - (a_{n-1} - p + 1 + m_{n-1}) \leq [a_{n-3} - a_{n-1}] + m_2 \leq |a_{n-2} - a_{n-1}| < p, \}
\end{align*}
\]
so that we know that $\lambda - \eta$ lies in either the lowest $\mathcal{T}$-alcove in the sense of (4.6.37) (if we replace $\mu^*$ by $\lambda$) or the $p$-restricted $\mathcal{T}$-alcove described by the conditions

$$\begin{align*}
&\begin{cases} p < \left\langle \lambda, \left( \sum_{i=2}^{n-1} \alpha_i \right) \right\rangle < 2p \\
0 < \left\langle \lambda, \left( \sum_{i=2}^{n-1} \alpha_i \right) \right\rangle < p \\
0 < \left\langle \lambda, \left( \sum_{i=2}^{n-1} \alpha_i \right) \right\rangle < p
\end{cases}
\end{align*}$$

and

$$0 < \langle \lambda, \alpha \rangle < p$$

for all $\alpha \in \Delta_L$

where $\Delta_L := \{ \alpha_i \mid 2 \leq i \leq n - 1 \}$ is the positive simple roots in $\Phi^+_L$.

In the first case, if $\lambda - \eta$ lies in the lowest $\mathcal{T}$-alcove, as we assume that $\mu^* \uparrow_L \lambda$, we must have $\lambda = \mu^*$; in the second case, we must have

$$\lambda = (2, n) \cdot \mu^* + p \left( \sum_{i=2}^{n-1} \alpha_i \right) = (a_{n-1} - n + 2, a_0 + p, a_{n-3}, \cdots, a_1, a_{n-2} + n - 2 - p)$$

which means by (4.6.36) that

$$\underline{m} = (m_1, \cdots, m_{n-1}) = ([a_0 - a_{n-2}]_1 + 1, 0, \cdots, 0, a_{n-2} - a_{n-1} + n - 3).$$

This implies $a_{n-2} - a_{n-1} + n - 1 = m_{n-1} \geq 0$, which is a contradiction to the $n$-generic assumption on $(a_{n-1}, \cdots, a_0)$. Therefore we must have $\lambda = \mu^*$. Hence we deduce by (4.6.35) and the strong linkage principle [Jan02] II.2.12 (1) that $F^L(\mu^*)$ (see the beginning of Section 5 for notation) has multiplicity one in $\text{JH}_\mathcal{T}(H^0(\mu_0^{\text{w}})_{\mathcal{T}})$ and is actually a direct summand.

On the other hand, as $F^L(\mu^*)$ is obviously an $\mathcal{T}$-subrepresentation of $F(\mu^*)$, we know that the surjection of $G$-representation $H^0(\mu_0^{\text{w}}) \to F(\mu^*)$ induces an isomorphism of $L$-representation on the direct summand $F^L(\mu^*)$ on both sides with multiplicity one, by restriction from $G$ to $L$. In particular, we know that the map

$$H^0(\mu_0^{\text{w}})_{\mu^*} \to F(\mu^*)$$

is a bijection, and therefore the composition

$$V^2 \hookrightarrow H^0(\mu_0^{\text{w}}) \to F(\mu^*)$$

is non-zero as

$$H^0(\mu_0^{\text{w}})_{\mu^*} = F_p[v_0^{\text{w}}, \eta] = F_p[S_{\underline{m}, w_0, v_0}]$$

by Lemma 4.6.28. Hence, we have a surjection

(4.6.38)

$$V^2 \twoheadrightarrow F(\mu^*).$$

Combining the surjection (4.6.38) with the injection

$$V^2 \hookrightarrow H^0(\mu_0^{\text{w}}),$$

we finish the proof by Lemma 4.6.32.

\begin{flushright}
\Box
\end{flushright}

**Theorem 4.6.39.** Assume that $(a_{n-1}, \cdots, a_0)$ is $n$-generic in the lowest alcove (cf. Definition 4.7.1). Then $H^0(\mu_0^{\text{w}}) \subseteq V_0$. In particular, we have

$$F(\mu^*) \in \text{JH}(V_0).$$

**Proof.** The first inclusion is a direct consequence of Proposition 4.6.33 together with Corollary 4.6.24. The second inclusion follows from the first as we have $F(\mu^*) \in \text{JH}(H^0(\mu_0^{\text{w}}))$.

Before we end this section, we need several remarks to summarize the proof, and to clarify the necessity for all the constructions.
Remark 4.6.40. If we assume that for all \(2 \leq k \leq n-2\)
\begin{equation}
[a_0 - a_{n-1}]_1 + n - 2 < a_k - a_{k-1}, \tag{4.6.41}
\end{equation}
then we can actually show that
\[ S_{k-w_0}v_0 \in H^0(\mu_0^{w_0})_{\mu^*} \]
using Corollary 4.1.28 and Lemma 4.5.4 and thus
\[ V_0 = H^0(\mu_0^{w_0}). \]
Moreover, under the condition (4.6.41), we can even prove that the set
\[ \{ S_{k-w_0}v_0 \mid k \in \Lambda_{w_0} \} \]
forms a basis for \(H^0(\mu_0^{w_0})_{\mu^*}\).

On the other hand, if we have
\[ [a_0 - a_{n-1}]_1 + n - 2 \geq a_k - a_{k-1} \]
for some \(2 \leq k \leq n-2\), then we can use Lemma 4.5.7 to prove that
\[ F(\mu_0^{w_0}) \in \text{JH}(V_0) \]
which means that the inclusion
\[ H^0(\mu_0^{w_0}) \subseteq V_0 \]
is actually strict.

In fact, through the proof of Proposition 4.6.23 the subrepresentation of \(\pi_0\) generated by \(S_{k,w_0}v_0\) is shrinking if \(r\) is fixed and \(s\) is growing. Therefore the subrepresentation of \(\pi_0\) generated by \(S_{k,w_0}v_0\) shrinks as \(r\) decreases. Finally, we succeeded in shrinking from \(V_0\) to \(V^\sharp\) which can be identified with \(H^0(\mu_0^{w_0})\).

Remark 4.6.42. We need to emphasize that the choice of the operators \(X^+_r\) and \(X^-_r\) for \(1 \leq r \leq n-1\) are crucial. For example, the operator
\[ \sum_{t \in \mathbb{F}_p} tp^{-2}w_0u_{\alpha_N}(t)w_0 \in \mathbb{F}_p[G(\mathbb{F}_p)] \]
for some \(2 \leq r \leq n-2\) does not work in general. The reason is that, as one can check by explicit computation, applying such operator to \(S_{k,w_0}v_0\) for some \(k \in \Lambda_{w_0}\) will generally give us a huge linear combination of Jacobi sum operators. From our point of view, it is basically impossible to compute such a huge linear combination explicitly and systematically. Instead, as stated in Proposition 4.5.31 our operators \(X^+_r\) and \(X^-_r\) can be computed systematically, even though the computation is still complicated.

The motivation of the choice of operators \(X^+_r\) and \(X^-_r\) can be roughly explained as follows. First of all, we need one ‘weight raising operator’ \(X^+\) and one ‘weight lowering operator’ \(X^-\). These are two operators lying in a subalgebra \(\mathbb{F}_p(\pi^+,\pi^-)\) of \(\mathbb{F}_p[G(\mathbb{F}_p)]\) such that
\[ \mathbb{F}_p(\pi^+,\pi^-) \cong \mathbb{F}_p[\text{GL}_2(\mathbb{F}_p)]. \]

We start with the vector \(S_{k,w_0}v_0\) for some \(k \in \Lambda_{w_0}\). We apply the operator \(X^-\) once and then \(X^+\) once, the result is a vector with the same \(T(\mathbb{F}_p)\)-eigencharacter \(\mu^*\). We observe that \(S_{k,w_0}v_0\) is in general not an eigenvector of the operator \(X^+\) once and \(X^-\) once because the representation \(\pi_0\), after restricting from \(\mathbb{F}_p[G(\mathbb{F}_p)]\) to \(\mathbb{F}_p(\pi^+,\pi^-)\), is highly non-semisimple. The naive expectation is that we just take the difference
\[ X^+ \cdot X^- \cdot S_{k,w_0}v_0 - cS_{k,w_0}v_0 \]
for some constant \(c \in \mathbb{F}_p\), and then repeat the procedure by applying some other operators similar to \(X^+\) and \(X^-\).

The case \(n = 3\) is easy. In the case \(n = 4\), the operator
\[ \sum_{t \in \mathbb{F}_p} tp^{-2}w_0u_{\alpha_2}(t)w_0 \in \mathbb{F}_p[\text{GL}_4(\mathbb{F}_p)] \]
is not well behaved as we explained in this remark, and therefore we are forced to use our $X_2^-$ to replace $\sum_{t \in \mathbb{F}_p} f^{p-2w_0w_0}(t)w_0$.

Now we consider the general case, and it is possible for us to carry on an induction step. We have a sequence of growing subgroups of $G$

$$\mathcal{T}_{(n-1)} \subseteq \mathcal{T}_{(n-2,n-1)} \subseteq \cdots \subseteq \mathcal{T}_{(2,\ldots,n-1)}$$

and

$$\mathcal{T}_{(n-1)} \subseteq \mathcal{T}_{(n-2,n-1)} \subseteq \cdots \subseteq \mathcal{T}_{(2,\ldots,n-1)}$$

where $\mathcal{T}_{(r,\ldots,n-1)}$ is the standard parabolic subgroup corresponding to the simple roots $\alpha_k$ for $r \leq k \leq n-1$ and $\mathcal{L}_{(r,\ldots,n-1)}$ is its standard Levi subgroup. Technically speaking, constructing the vector $S_{k+r,\ldots,n}v_0$ (for some $1 \leq r \leq n-2$) from $S_{k,r}v_0$ should be reduced to Corollary 4.6.24 when we replace $G$ by its Levi subgroup $\mathcal{L}_{(r+1,\ldots,n-1)}$. In other words, to construct $S_{k+r,\ldots,n}v_0$ from $S_{k,\ldots,n}v_0$ we only need the operators

$$X_k^+, X_k^- \in \mathbb{F}_p[\mathcal{L}_{(r+2,\ldots,n-1)}(\mathbb{F}_p)] \subseteq \mathbb{F}_p[\mathcal{L}_{(r+1,\ldots,n-1)}(\mathbb{F}_p)]$$

for all $r + 2 \leq k \leq n - 1$.

In order to construct $S_{k+r,\ldots,n}v_0$ from $S_{k+r,\ldots,n}v_0$, we only need to prove Proposition 4.6.24. Then we summarize the proof of Proposition 4.6.24 as the following: for some $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$

$$X_{r+1}^+ \ast X_{r+1}^- \ast S_{k+r,\ldots,n}v_0 \equiv aS_{k+r,\ldots,n}v_0 + bS_{k+r,\ldots,n}v_0 + \text{error terms}$$

and the error terms can be killed by combinations of the operators $X_k^+, X_k^-$ for $r + 2 \leq k \leq n - 1$.

4.7. Jacobi sums in characteristic 0. In this section, we establish an intertwining identity for lifts of Jacobi sums in characteristic 0 in Theorem 4.7.3 which is one of the main ingredients of the proof of Theorem 5.7.6. All of our calculations here are in the setting of $G(\mathbb{Q}_p) = \text{GL}_n(\mathbb{Q}_p)$. We first fix some notation.

Let $A \in G(\mathbb{F}_p)$. By $[A]$ we mean the matrix in $G(\mathbb{Q}_p)$ whose entries are the classical Teichmüller lifts of the entries of $A$. The map $A \mapsto [A]$ is obviously not a group homomorphism but only a map between sets. On the other hand, we use the notation $\hat{\mu}$ for the Teichmüller lift of a character $\mu$ of $T(\mathbb{F}_p)$.

We denote the standard lifts of simple reflections in $G(\mathbb{Q}_p)$ by

$$s_i = \begin{pmatrix} \text{Id}_{i-1} & 1 \\ 1 & \text{Id}_{n-i-1} \end{pmatrix}$$

for $1 \leq i \leq n - 1$. We also use the following notation

$$t_i = \begin{pmatrix} p\text{Id}_i \\ \text{Id}_{n-i} \end{pmatrix}$$

for $1 \leq i \leq n$. Let

$$\Xi_n := w^* t_1,$$

where $w^* := s_{n-1} \ast \cdots \ast s_1$. We recall the Iwahori subgroup $I$ and the pro-$p$ Iwahori subgroup $I(1)$ from the beginning of Section 3. Note that the operator $\Xi_n$ and the group $I$ actually generates the
normalizer of $I$ inside $G(Q_p)$. One easily sees that $\Xi_n$ is nothing else than the following matrix:

$$\Xi_n = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
p & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix} \in G(Q_p).$$

For each $1 \leq i \leq n-1$, we consider the maximal parabolic subgroup $P_i^-$ of $G$ containing lower-triangular Borel subgroup $B^-$ such that its Levi subgroup can be chosen to be $GL_i \times GL_{n-i}$ which embeds into $G$ in the standard way. We denote the unipotent radical of $P_i^-$ by $N_i^-$. Then we introduce

(4.7.2) $$U_n^i = \sum_{A \in N_i^-} t_i^{-1}[A]$$ for each $1 \leq i \leq n-1$.

Note that each $A \in N_i^-$ has the form

$$\begin{pmatrix}
\text{Id}_i & 0_{(n-i) \times i} \\
*_{i \times (n-i)} & \text{Id}_{n-i}
\end{pmatrix},$$

for each $1 \leq i \leq n-1$.

We recall the tuples $k_i$ and $k_i^{'*}$ from (4.2.3), and consider the characteristic 0 lift of Jacobi sums $S_n$ and $S_n^{'}$ as follows:

(4.7.3) $$\begin{cases}
\hat{S}_n = \sum_{A \in U(F_p)} \left( \prod_{i=1}^{n-1} [A^{k_i}_{i,i+1}] [A] \right) w_0; \\
\hat{S}_n^{'} = \sum_{A \in U(F_p)} \left( \prod_{i=1}^{n-1} [A^{k_i^{'*}}_{i,i+1}] [A] \right) w_0.
\end{cases}$$

The main result of this section is the following, which is a generalization of the case $n = 3$ in (3.2.1) of [HLM].

**Theorem 4.7.4.** Assume that the $n$-tuple of integers $(a_{n-1}, \cdots, a_0)$ is $n$-generic in the lowest alcove, and let

$$\Pi_p := \text{Ind}_B^{G(Q_p)}(\chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \cdots \otimes \chi_{n-2} \otimes \chi_{n-1} \otimes \chi_0)$$

be a tamely ramified principal series representation where the $\chi_i : Q_p^\times \to E^\times$ are smooth characters satisfying $\chi_i|_{Z_p} = \omega^{a_i}$ for $0 \leq i \leq n-1$.

On the 1-dimensional subspace $\Pi_p^{l(1),(a_1,a_2,\ldots,a_{n-1},a_0)}$ we have the identity:

$$\hat{S}_n \cdot (\Xi)^{n-2} = p^{n-2}\kappa_n \left( \prod_{k=1}^{n-2} \chi_k(p) \right) \hat{S}_n$$

for some $\kappa_n \in \mathcal{O}_E^\times$ such that

$$\kappa_n \equiv \varepsilon^* \mathcal{P}_n(a_{n-1}, \cdots, a_0) \mod \mathcal{W}_E$$

where $\varepsilon^* = \pm 1$ is a sign defined in (4.7.30) that depends only on $(a_{n-1}, \cdots, a_0)$ and $\mathcal{P}_n$ is an explicit rational function defined in (4.7.34).

Firstly, we need a lemma, which is a direct generalization of Lemma 3.2.5 in [HLM].

**Lemma 4.7.5.** Pick a non-zero element $\tilde{v} \in \Pi_p^{l(1),(a_1,a_2,\ldots,a_{n-1},a_0)}$. Then we have

$$U_n^{n-2} \tilde{v} = \left( \prod_{k=1}^{n-2} \chi_k(p) \right)^{-1} \tilde{v}$$
and moreover
\[(\Xi_n)^{n-2} \cdot U_n^{n-2} = \sum_{B \in (w^*)^{n-2}N_{n-2}^{-1}(F_p)} [B] \hat{v} \]

Note that $B \in (w^*)^{n-2}N_{n-2}^{-1}(F_p)$ is equivalent to that $B$ is running through the matrices in $G(F_p)$ of the form
\[
\begin{pmatrix}
*(n-2 \times 2)
\Id_{2}
\Id_{n-2}
0_{(2 \times n-2)}
\end{pmatrix}.
\]

**Proof.** The proof of this lemma is an immediate calculation which is parallel to that of [HLM], Lemma 3.2.5. \(\square\)

From now on, we fix a matrix $B$ of the form in (4.7.6), so that we may have
\[
w_0 \cdot B = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n-3} & \lambda_{1,n-2} & 0 & 1 \\
\lambda_{2,1} & \lambda_{2,2} & \cdots & \lambda_{2,n-3} & \lambda_{2,n-2} & 1 & 0
\end{pmatrix}.
\]

We now compute the Bruhat decomposition of the matrix $w_0B$. We apply the definition of $D_i, D'_i$ at the beginning of Section 4.3 as polynomials of entries of matrices to the matrix $w_0B$, namely we define
\[
D_i := \begin{cases}
\lambda_{2,1} & \text{if } i = 1; \\
\lambda_{2,i-1}\lambda_{1,i} - \lambda_{1,i-1}\lambda_{2,i} & \text{if } 2 \leq i \leq n - 2; \\
-\lambda_{1,n-2} & \text{if } i = n - 1.
\end{cases}
\]
and
\[
D'_i := \begin{cases}
\lambda_{1,1} & \text{if } i = 1; \\
-\lambda_{2,2} & \text{if } i = 2; \\
\lambda_{1,i-2}\lambda_{2,i} - \lambda_{2,i-2}\lambda_{1,i} & \text{if } 3 \leq i \leq n - 2; \\
\lambda_{1,n-3} & \text{if } i = n - 1.
\end{cases}
\]
Assume first that $D_i \neq 0$ for $1 \leq i \leq n - 1$, and let
\[
T_B = \text{diag} \left( \frac{D_1}{D_1}, \frac{D_2}{D_1}, \ldots, \frac{D_k}{D_{k-1}}, \ldots, \frac{1}{D_{n-1}} \right)
\]
and
\[
U_B = \begin{pmatrix}
1 & D'_{k+1} & \cdots & \ast & \ast & \cdots & \ast & \ast & \cdots & \cdots & \ast & \ast & \cdots & \ast & \ast & \cdots & \cdots & \ast
\end{pmatrix}.
\]

By a direct computation, we have
\[(U_Bw_0T_B)^{-1}w_0B \in U(F_p),\]
so that we may write

\[ w_0B = U_Bw_0T_BU_B' \]

for some matrix \( U_B' \) in \( U(F_p) \) (whose explicit form is not important for our purpose). We notice that

\[ w_0B \in B(F_p)w_0B(F_p) \] if and only if \( D_i \neq 0 \) for all \( 1 \leq i \leq n-1 \).

In general, if \( w_0B \in U_w(F_p)wB(F_p) \), we write

\[ w_0B = U_B^wT_B^wU_B^{w,j} \]

for \( U_B^w \in U_w(F_p) \), \( T_B^w \in T(F_p) \) and \( U_B^{w,j} \in U(F_p) \).

As a result, we deduce that

\[
\hat{S}_n \bullet (\Xi_n)^{n-2} \bullet U_n^{n-2} \hat{v} = \sum_{w \neq w_0} \hat{S}_w \hat{v} + \sum_{A \in U(F_p), B(w^*)^{n-2}N_{n-2}^-} \left( \prod_{i=1}^{n-1} [A_{i,i+1}]^{k_{i,i+1}} \right) \left[ AU_Bw_0T_B \right] \hat{v}
\]

where

\[
\hat{S}_w := \left( \sum_{A \in U(F_p)} \left( \prod_{i=1}^{n-1} [A_{i,i+1}]^{k_{i,i+1}} \right) \left[ A \right] \right) \cdot \left( \sum_{B \in B(w_0U_w(F_p)wB(F_p) \cap (w^*)^{n-2}N_{n-2}^-)} \left[ U_B^wT_B^wU_B^{w,j} \right] \right)
\]

Lemma 4.7.8. We have \( \hat{S}_w \hat{v} = 0 \) for each \( w \neq w_0 \).

The following proof is a direct generalization of Case 1 of Lemma 3.2.6 in [HLM].

Proof. We notice that

\[
\hat{S}_w \hat{v} = \sum_{A \in U(F_p)} \left( \prod_{i=1}^{n-1} [A_{i,i+1}]^{k_{i,i+1}} \right) \cdot \left( \sum_{B \in B(w_0U_w(F_p)wB(F_p) \cap (w^*)^{n-2}N_{n-2}^-)} \left[ AU_B^wT_B^w \right] \right) \hat{v}
\]

as \( \hat{v} \) is \( I(1) \)-invariant. After changing the variable \( A_w := AU_B^w \), we deduce

\[
\hat{S}_w \hat{v} = \sum_{A_w \in U(F_p)} \left( \prod_{i=1}^{n-1} [A_{i,i+1}]^{k_{i,i+1}} \right) \cdot \left( \sum_{B \in B(w_0U_w(F_p)wB(F_p) \cap (w^*)^{n-2}N_{n-2}^-)} \left[ A_wT_B^w \right] \right) \hat{v}
\]

where \( A_{i,i+1} \) is viewed as a rational function of \( A_{i,i+1}^w \) and the entries of \( B \).

For the given Weyl element \( w \neq w_0 \), we know that \( \Delta \cap (\Phi^+ \setminus \Phi_w^+) \neq \emptyset \). If \( \alpha_i \in \Delta \cap (\Phi^+ \setminus \Phi_w^+) \) for some \( 1 \leq i \leq n-1 \), then we have

\[ A_{i,i+1}^w = A_{i,i+1} \]

for the choice of \( i \) above.

By the definition of \( U_w \) we have the set theoretical decomposition

\[ U(F_p) = U_{w_0s_i}(F_p) \times U_{s_i}(F_p) \]

and thus we can write

\[ A_w = A_1^w \cdot A_2^w \]

for \( A_1^w \in U_{w_0s_i}(F_p) \) and \( A_2^w \in U_{s_i}(F_p) \) uniquely determined by \( A_w \).

As \( A_2^w \in wU(F_p) \), we deduce that

\[ \left[ A_2^wT_B^w \right] \hat{v} = \left[ A_1^wT_B^w \right] \hat{v} \]
and thus

\[
\hat{S}_n \hat{v} = \left( \sum_{\mathcal{U}_p \in U_p(F_p)} \mathcal{U}_p \right) \sum_{\mathcal{U}_p \in \mathcal{U}_p(F_p)} \left( \prod_{1 \leq j \leq n-1} [A_{j,j+1}]^{k_{j,j+1}} \right) \cdot \left( \sum_{B \in \mathcal{U}_p(F_p) \cap (w^*)^{n-2}N_{n-2}^{-1}} [A_1^w wT_B]^{\hat{v}} \right) = 0.
\]

Note that the sum \( \sum_{\mathcal{U}_p \in F_p} [A_{i,i+1}]^{k_{i,i+1}} \) is zero as the \( [A_{i,i+1}] \) are chosen to be Teichmüller lifts.

By Lemma 4.7.8 we may and do assume that \( D_i \neq 0 \) for \( 1 \leq i \leq n-1 \) from now on. As

\[
C := AU_B = \begin{pmatrix}
1 & A_{1,2} + \frac{D_{i-1}}{D_{i-1}} & \cdots & * & \cdots & * \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & * & \cdots \\
& & & 1 & A_{k,k+1} + \frac{D_k}{D_{k-1}} & \cdots & * \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \cdots & * \\
& & & & & & 1 & A_{n-1,n} + \frac{D_1}{D_1}
\end{pmatrix},
\]

we actually change the variable from \( A_{i,i+1} \) to \( C_{i,i+1} \) through \( C_{i,i+1} = A_{i,i+1} + \frac{D_{i-1}}{D_{i-1}} \) for \( 1 \leq i \leq n-1 \).

In other words, we have the equality

\[
(4.7.9) \quad \hat{S}_n^i \cdot (\Xi_n)^{n-2} \cdot U_n^{n-2} \hat{v} = \sum_{C \in U(F_p) \cap (w^*)^{n-2}N_{n-2}^{-1}(F_p)} \left( \prod_{i=1}^{n-1} C_{i,i+1} = \frac{D_i}{D_{i-1}} \right) [Cw_0 T_B]^{\hat{v}}.
\]

Note that we have

\[
(4.7.10) \quad [T_B]^{\hat{v}} = [D_1]^{a_{1}} [D_{n-1}]^{-a_{0}} \prod_{k=2}^{n} \frac{D_k}{D_{k-1}} \gamma_{a_k}^{a_k} = [D_{n-1}]^{a_{n-1} - a_{0}} \prod_{k=1}^{n-1} [D_k]^{a_k - a_{k+1}} \hat{v}.
\]

Combining (4.7.9) with (4.7.10), we obtain

\[
(4.7.11) \quad \hat{S}_n^i \cdot (\Xi_n)^{n-2} \cdot U_n^{n-2} \hat{v} = \sum_{C \in U(F_p) \cap (w^*)^{n-2}N_{n-2}^{-1}(F_p) \cap D_i \neq 0, \text{ for } 1 \leq i \leq n-1} \left[ X_0 \right]^{\hat{v}}.
\]

where

\[
X_0 := \left( \prod_{i=1}^{n-1} C_{i,i+1} = \frac{D_{i-1}}{D_{i-1}} \right) \left( [D_{n-1}]^{a_{n-1} - a_{0}} \prod_{k=1}^{n-2} [D_k]^{a_k - a_{k+1}} \right).
\]

Our main target in the rest of this section is to calculate (4.7.11) explicitly. The result (c.f. Theorem 4.7.3) is simple and clean. However, the intermediate step is a bit complicated. The sum (4.7.11) is essentially an exponential sum over \( F_p \)-points of an affine variety. In our case, it is possible for us to introduce an induction step to finally reduce the calculation of (4.7.11) to the special case \( n = 4 \). In other words, the induction step in Proposition 4.7.22 is trying to reduce the calculation of an exponential sum with many variables to another one with less variables. The main subtlety of the
induction is to carefully manipulate the affine varieties where the sums lie and to change the variables systematically.

Before we go into the calculation of (4.7.11), we start with recalling some standard facts about Jacobi sums and Gauss sums. We fix a primitive \( p \)-th root of unity \( \xi \in E \) and set \( \epsilon := \xi - 1 \). For each pair of integers \((a, b)\) with \( 0 \leq a, b \leq p - 1 \), we set

\[
J(a, b) := \sum_{\lambda \in F_p} [\lambda]^a [1 - \lambda]^b.
\]

We also set

\[
G(a) := \sum_{\lambda \in F_p} [\lambda]^a \xi^\lambda
\]

for each integer \( a \) with \( 0 \leq a \leq p - 1 \). For example, we have \( G(p - 1) = -1 \).

It is known by section 1.1, GS3 of [Lang] that if \( a + b \neq 0 \mod (p - 1) \), we have

\[
J(a, b) = G(a)G(b) = G(a + b).
\]

(4.7.13)

It is also obvious from the definition that if \( a, b, a + b \neq 0 \mod (p - 1) \) then

\[
J(b, a) = J(a, b) = (-1)^a J(b, [-a - b]) = (-1)^a J(a, [-a - b]) = (-1)^a J(a, [-a - b]).
\]

(4.7.14)

We introduce further notation. It is easy to see from (4.7.7) that there is an isomorphism of schemes

\[
(w^*)^{n-2} N_{n-2}^{-1} \cong M_{2,n-2}
\]

over \( Z \) where the right side is the \((2n - 4)\)-dimensional affine space, which can be viewed as the space of all \( 2 \times (n - 2) \)-matrices. As a result, we can replace the subscript \( B \in (w^*)^{n-2} N_{n-2}^{-1}(F_p) \) in (4.7.8) by \( \Delta \in M_{2,n-2}(F_p) \) where

\[
\Delta := \begin{pmatrix}
\lambda_{1,2} & \cdots & \lambda_{1,n-2} \\
\lambda_{2,2} & \cdots & \lambda_{2,n-2}
\end{pmatrix}.
\]

For each integer \( 2 \leq m \leq n - 2 \), we consider the space \( M_{2,m} \) of \( 2 \times m \)-matrices, and denote an arbitrary \( F_p \)-point of \( M_{2,m} \) by

\[
\Delta^m := \begin{pmatrix}
\lambda_{1,2} & \cdots & \lambda_{1,m} \\
\lambda_{2,2} & \cdots & \lambda_{2,m}
\end{pmatrix}.
\]

Hence, for each \( 3 \leq m \leq n - 2 \) we have a natural restriction map

\[
pr_{m,m-1} : M_{2,m}(F_p) \rightarrow M_{2,m-1}(F_p)
\]

by sending

\[
\begin{pmatrix}
\lambda_{1,2} & \cdots & \lambda_{1,m} \\
\lambda_{2,2} & \cdots & \lambda_{2,m}
\end{pmatrix} \mapsto \begin{pmatrix}
\lambda_{1,2} & \cdots & \lambda_{1,m-1} \\
\lambda_{2,2} & \cdots & \lambda_{2,m-1}
\end{pmatrix}.
\]

We define

\[
U_m := \{ \Delta^m \in M_{2,m}(F_p) \mid \lambda_{1,m} \neq 0 \land D_i \neq 0 \text{ for } 1 \leq i \leq m \}
\]

and thus \( U_m \) is the set of \( F_p \)-points of the open subscheme of \( M_{2,m} \) defined by the equations in (4.7.8).

For each subset \( U \subseteq U_m \), we also define

\[
L_m(U) := X_1 \cdot \left( \sum_{C \in U(F_p), \Delta^m \in U} Y_1 \cdot Z_1 \cdot [Cw_0]^v \right)
\]

(4.7.17)
where

\[ X_1 := (-1)^{(n-1-m)(a_0-a_{n-1})} \prod_{\ell=m+1}^{n-2} J([a_0-a_\ell_1, [a_\ell-a_\ell+1]_1 + n-2],J(a_{n-1}-a_\ell, [a_{\ell-1}-a_{n-1}]_1 + n-2) \]

\[ Y_1 := \left[ C_{n-1-m,n-m} + \frac{\lambda_{1,m-1}}{\lambda_{1,m}} \right]^{[a_m-a_{m+1}]_1+n-2} \left( \prod_{i=1}^{n-2-m} [C_{i,i+1}]^{[a_0-a_{n-1}]_1+n-2} \right) \]

and

\[ Z_1 := \left[ \lambda_{1,m} \right]^{a_{n-1}-a_0} \left[ D_m \right]^{a_{m}-a_{n-1}} \left( \prod_{k=1}^{m-1} [D_k]^{a_{k}-a_{k+1}} \right) \cdot \left( \prod_{i=n-m}^{n-1} \left[ C_{i,i+1} - \frac{D_{n-i}'}{D_{n-i}} \right]^{[a_{n-1}-a_{n-1}]_1+n-2} \right) . \]

It follows easily from this definition that, if \( U \) and \( U' \) are two subsets of \( U_m \) satisfying \( U \cap U' = \emptyset \), then for the disjoint union \( U \cup U' \subseteq U_m \) we have

\[ \mathcal{L}_m(U \cup U') = \mathcal{L}_m(U) + \mathcal{L}_m(U') . \]

**Proposition 4.7.19.** We have an equality

\[ \mathcal{L}_m(U) \cdot (\Xi_n)^{n-2} \cdot U_n^{n-2} \hat{v} = \mathcal{L}_m(U) \]

for each \( 2 \leq m \leq n-2 \).

We prove this proposition by a series of Lemmas.

**Lemma 4.7.21.** The equality \[ 4.7.20 \] is true for \( m = n - 2 \).

**Proof.** This is simply a reformulation of \[ 4.7.11 \]. \( \square \)

**Proposition 4.7.22.** If the equality \[ 4.7.20 \] is true for \( m \), then it is also true for \( m - 1 \).

Before we prove Proposition \[ 4.7.22 \] we need to define some further notation. We fix an integer \( 3 \leq m \leq n-2 \) until the end of the proof of Proposition \[ 4.7.22 \]. We define

\[ \mathcal{U}_m^{-1} := \text{pr}^{-1}_{m-1}(U_{m-1}) \subseteq U_m \quad \text{and} \quad \mathcal{U}_m^{-1,1} := U_m \setminus \mathcal{U}_m^{-1} \]

or more explicitly

\[ \mathcal{U}_m^{-1} = \{ \Lambda^m \in U_m \mid \lambda_{1,m-1} \neq 0 \} \quad \text{and} \quad \mathcal{U}_m^{-1,1} = \{ \Lambda^m \in U_m \mid \lambda_{1,m-1} = 0 \} . \]

We also define

\[ f_m(C,\Lambda^m) := \left[ \lambda_{1,m} \right]^{a_{n-1}-a_0} \left[ D_m \right]^{a_{m}-a_{n-1}} \left[ C_{n-1-m,n-m} + \frac{\lambda_{1,m-1}}{\lambda_{1,m}} \right]^{[a_m-a_{m+1}]_1+n-2} \]

\[ \cdot \left[ C_{n-m,n-m+1} - \frac{D_{n-i}'}{D_{n-i}} \right]^{[a_{n-1}-a_{n-1}]_1+n-2} . \]

Notice that \( f_m(C,\Lambda^m) \) is actually the Teichmüller lift of a rational function of \( C_{n-1-m,n-m}, C_{n-m,n-m+1}, \lambda_{1,m}, \lambda_{2,m}, \lambda_{1,m-1}, \lambda_{2,m-1}, \lambda_{1,m-2} \) and \( \lambda_{2,m-2} \).

Now we can rewrite \[ 4.7.17 \] as

\[ \mathcal{L}_m(U) = X_1 \cdot \left( \sum_{C \in U(\mathbb{F}_p) \mathcal{U}_m^{-1} \subseteq \text{pr}_{m-1}(U)} Y_2 \cdot Z_2 \cdot [Cw_0] \hat{v} \right) \]

for each subset \( U \subseteq U_m \), where

\[ Y_2 := \left( \prod_{k=1}^{m-1} [D_k]^{a_{k+1}-a_k} \right) \left( \prod_{i=1}^{n-2-m} [C_{i,i+1}]^{[a_0-a_{n-1}]_1+n-2} \right) \]

\[ \cdot \left( \prod_{i=n-m+1}^{n-1} \left[ C_{i,i+1} - \frac{D_{n-i}'}{D_{n-i}} \right]^{[a_{n-1}-a_{n-1}]_1+n-2} \right) . \]
and
\[ Z_2 := \sum_{\Delta^m \in \text{pr}_m^{-1}(\Lambda^m) \cap U} f_m(C, \Delta^m). \]

We emphasize that \( Y_2 \) only depends on \( C \) and \( \text{pr}_{m-1}(\Lambda^m) \). It is natural that the calculation of \( \mathcal{L}_m(U) \) for each \( U \subseteq U_m \) start with the calculation of \( Z_2 \).

**Lemma 4.7.24.** We have
\[ \mathcal{L}_m(U_m^{-1}) = \mathcal{L}_{m-1}(U_{m-1}). \]

**Proof.** For each \( \Delta^m \in U_m^{m-1} \), we have
\[
f_m(C, \Delta) = [\lambda_{1,m}]^{a_{m-1} - a_0} [\lambda_{2,m-1} \lambda_{1,m} - \lambda_{1,m-1} \lambda_{2,m}]^{a_m - a_{m-1}} \cdot \left[ C_{n-1-m,n-m} + \frac{\lambda_{1,m-1}}{\lambda_{1,m}} \right]^{[a_m - a_{m+1}]_1 + n - 2} \cdot \left[ C_{n-m,n-m+1} + \frac{\lambda_{1,m} - 2 \lambda_{2,m} - \lambda_{2,m-2} \lambda_{1,m}}{\lambda_{1,m} - 2 \lambda_{2,m} - \lambda_{2,m-1} \lambda_{1,m}} \right]^{[a_{m-1} - a_{m-1}]_1 + n - 2}.
\]

By a change of variable \( x = \lambda_{1,m} \in \mathbb{F}_p^x \) and \( y = \lambda_{2,m} \in \mathbb{F}_p \setminus \{ \lambda_{2,m-1} \} \), we deduce that
\[
f_m(C, \Delta) = [x]^{a_m - a_0} [\lambda_{2,m-1} - \lambda_{1,m-1} y]^{a_m - a_{m-1}} \cdot \left[ C_{n-1-m,n-m} + \frac{\lambda_{1,m-1}}{x} \right]^{[a_m - a_{m+1}]_1 + n - 2} \cdot \left[ C_{n-m,n-m+1} + \frac{\lambda_{1,m} - 2 y - \lambda_{2,m} - \lambda_{2,m-2}}{\lambda_{1,m} - 2 y - \lambda_{2,m}} \right]^{[a_{m-1} - a_{m-1}]_1 + n - 2}.
\]

If \( C_{n-1-m,n-m} = 0 \), then
\[
\sum_{x \in \mathbb{F}_p^x} f_m(C, \Delta) = \left( \sum_{x \in \mathbb{F}_p^x} [x]^{a_{m+1} - a_0 - n + 2} \right) \cdot (*) = 0
\]
where \(*\) is a certain term which is independent of \( x \). If \( C_{n-1-m,n-m} \neq 0 \), then we deduce that
\[
\sum_{x \in \mathbb{F}_p^x} f_m(C, \Delta) = \sum_{x \in \mathbb{F}_p^x} X_3 \cdot Y_3 \cdot Z_3
\]
where
\[
X_3 := [\lambda_{1,m-1}]^{a_m - a_0} [C_{n-1-m,n-m}]^{[a_0 - a_{m+1}]_1 + n - 2},
\]
\[
Y_3 := [\lambda_{2,m-1} - \lambda_{1,m-1} y]^{a_m - a_{m-1}} \left[ C_{n-m,n-m+1} + \frac{\lambda_{1,m} - 2 y - \lambda_{2,m-2}}{\lambda_{1,m-1} - 2 y - \lambda_{2,m}} \right]^{[a_{m-1} - a_{m-1}]_1 + n - 2},
\]
and
\[
Z_3 := \left[ \frac{C_{n-1-m,n-m} x}{\lambda_{1,m-1}} \right]^{a_m - a_0} \left[ 1 + \frac{\lambda_{1,m-1}}{C_{n-1-m,n-m} x} \right]^{[a_m - a_{m+1}]_1 + n - 2}.
\]
Therefore, by (4.7.12), we deduce that
\[
(4.7.25) \quad \sum_{x \in \mathbb{F}_p^x} f_m(C, \Delta) = (-1)^{a_0 - a_m} [J([a_0 - a_m]_1, [a_m - a_{m+1}]_1 + n - 2) \cdot X_3 \cdot Y_3).
\]

We emphasize that (4.7.25) still holds even if \( C_{n-1-m,n-m} = 0 \) as \( X_3 = 0 \) in that case.
One can rewrite $Y_3$ as follows:

$$Y_3 = \left[ \lambda_{2,m-1} - \lambda_{1,m-1}y \right]^{a_m-a_{n-1}} \cdot \left( \frac{C_{n,m,n-m+1} + \frac{\lambda_{1,m-2} \lambda_{1,m-1}}{\lambda_{1,m-1}} - \frac{\lambda_{1,m-2}}{\lambda_{1,m-1}}}{\frac{\lambda_{1,m-2}}{\lambda_{1,m-1}} - \lambda_{2,m-1}} \right)^{[a_m-a_{n-1}]+n-2}. $$

If $C_{n,m,n-m+1} + \frac{\lambda_{1,m-2}}{\lambda_{1,m-1}} = 0$, then we deduce that

$$\sum_{y \neq \frac{\lambda_{2,m-1}}{\lambda_{1,m-1}}} Y_3 = \left( \sum_{y \neq \frac{\lambda_{2,m-1}}{\lambda_{1,m-1}}} [\lambda_{1,m-1}y - \lambda_{2,m-1}]^{a_m-a_{m-1}+n+2} \right)(*) = 0$$

where $*$ is a certain term which is independent of $y$. If $C_{n,m,n-m+1} + \frac{\lambda_{1,m-2}}{\lambda_{1,m-1}} \neq 0$, then we deduce that

$$\sum_{y \neq \frac{\lambda_{2,m-1}}{\lambda_{1,m-1}}} Y_3 = \left( \sum_{y \neq \frac{\lambda_{2,m-1}}{\lambda_{1,m-1}}} [X_4]^{a_m-a_{m-1}}[1 - X_4]^{[a_m-a_{n-1}]+n-2} \right) \cdot Y_4$$

where

$$X_4 := \frac{\lambda_{1,m-2} \lambda_{2,m-1}}{\lambda_{1,m-1} \lambda_{2,m-1} - \lambda_{2,m-2}} (\frac{\lambda_{1,m-2}}{\lambda_{1,m-1}} - \lambda_{2,m-1}) (C_{n,m,n-m+1} + \frac{\lambda_{1,m-2}}{\lambda_{1,m-1}})$$

and

$$Y_4 := \left[ C_{n,m,n-m+1} + \frac{\lambda_{1,m-2} \lambda_{2,m-1}}{\lambda_{1,m-1}} \right]^{[a_m-a_{m-1}]+n-2} \left[ \frac{\lambda_{1,m-2} \lambda_{2,m-1}}{\lambda_{1,m-1}} - \lambda_{2,m-2} \right]^{a_m-a_{n-1}}.$$

By (4.7.12) we obtain that

$$\sum_{y \neq \frac{\lambda_{2,m-1}}{\lambda_{1,m-1}}} Y_3 = J(a_{n-1} - a_m, [a_{m-1} - a_{n-1}]_1 + n-2) Y_4.$$  

Combining (4.7.26) with (4.7.26), we deduce that

$$\sum_{x \in \mathcal{P}^* \setminus \frac{\lambda_{2,m-1}}{\lambda_{1,m-1}}} f_m(C_x) = (-1)^{a_0-a_m} J([a_0 - a_m]_1, [a_m - a_{m+1}]_1 + n-2) \cdot J(a_{n-1} - a_m, [a_{m-1} - a_{n-1}]_1 + n-2) \cdot X_3 \cdot Y_4.$$  

Now applying (4.7.27) back to (4.7.23), and then recalling the definition of $\mathcal{L}_m(U_m^{-1})$ from (4.7.17) by replacing $m$ by $m-1$, we conclude that $\mathcal{L}_m(U_m^{-1}) = \mathcal{L}_{m-1}(U_m^{-1})$, which completes the proof.

Lemma 4.7.28. We have

$$\mathcal{L}_m(U_m^{-1,t}) = 0.$$  

Proof. For each $\Delta^m \in U_m^{-1,t}$, we have

$$f_m(C_x) = \left[ \lambda_{1,m} \right]^{a_{m-1}-a_0} \left[ \lambda_{2,m-1} \lambda_{1,m} \right]^{a_m-a_{n-1}} \left[ C_{n-1,m,n-m} [a_{m-1} - a_{n-1}] + n-2 \right.$$  

$$\left. \left[ C_{n,m,n-m+1} + \frac{\lambda_{1,m-2} \lambda_{2,m-1} - \lambda_{2,m-2} \lambda_{1,m}}{-\lambda_{2,m-2} \lambda_{1,m}} \right]^{a_{m-1} - a_{n-1}] + n-2 \right).$$
By a change of variable $x = \lambda_{1,m} \in \mathbf{F}_p$ and $y = \frac{\lambda_{2,m}}{\lambda_{1,m}} \in \mathbf{F}_p$, we deduce that

$$f_m(C, \Delta) = \left[ x^m + a_{m+1} \cdot \frac{a_0 \cdot a_{m-1}}{\lambda_{2,m-1}} \cdot C_{n-1,m-n-1} \right] \cdot \left[ C_{n-m,n-m+1} + \frac{\lambda_{1,m} \cdot 2y - \lambda_{2,m-2}}{-\lambda_{2,m-1}} \right]^{a_{m-1} - a_{n-1}}.$$ 

Hence, we obtain

$$\sum_{y \in \mathbf{F}_p} f_m(C, \Delta) = \left( \sum_{y \in \mathbf{F}_p} \left[ C_{n-m,n-m+1} + \frac{\lambda_{1,m} \cdot 2y - \lambda_{2,m-2}}{-\lambda_{2,m-1}} \right]^{a_{m-1} - a_{n-1}} \right) (\ast) = 0$$

where $\ast$ is a certain term which is independent of $y$, and thus $L_m(U_{m-1}) = 0$. □

**Proof of Proposition 4.7.30** By Lemma 4.7.21 and Lemma 4.7.28 and the decomposition $U_m = U_{m-1} \sqcup U_{m-1}$, we deduce that

$$L_m(U_m) = L_m(U_{m-1}) + L_m(U_{m-1}) = L_{m-1}(U_{m-1})$$

which finishes the proof of Proposition 4.7.22. □

**Proof of Proposition 4.7.19** This follows directly from Proposition 4.7.22 and Lemma 4.7.21 by induction. □

We define

$$(4.7.29) \quad \kappa_n := (-1)^{(n-2)(a_0 \cdot a_{n-1})} p^{2-n} \prod_{m=1}^{n-2} \gamma_m$$

where

$$\gamma_m := J([a_0 - a_m]_1, [a_m - a_{m+1}]_1 + n - 2) \cdot J(a_{n-1} - a_m, [a_{m-1} - a_{n-1}]_1 + n - 2).$$

**Proposition 4.7.30.** We have

$$L_2(U_2) = p^{n-2} \kappa_n \mathcal{S}_n \bar{v}.$$ 

**Proof.** By the case $m = 2$ of (4.7.17), we have

$$L_2(U_2) := \sum_{C \in U(\mathbf{F}_p)} \sum_{\Delta^2 \in U_2} X_5 \cdot Y_5 \cdot Z_5 \cdot [Cw_0] \bar{v}$$

where

$$X_5 := \left( \prod_{m=3}^{n-2} \left. \varepsilon_m \right| \gamma_m \right) \cdot \left( \prod_{i=1}^{n-4} \left[ C_{i+1} \right]^{[a_0 - a_{n-1}]_1 + n - 2} \right),$$

$$Y_5 := \left[ \lambda_{2,1} \right]^{a_1 - a_2} \left[ \lambda_{1,2} \right]^{a_{n-1} - a_0} \left[ \lambda_{1,2} \lambda_{2,1} - \lambda_{1,1} \lambda_{2,2} \right]^{a_2 - a_{n-1}} \left[ C_{n-3,n-2} + \frac{\lambda_{1,1}}{\lambda_{1,2}} \right]^{[a_2 - a_3]_1 + n - 2},$$

and

$$Z_5 := \left[ C_{n-1,n} - \frac{\lambda_{1,1}}{\lambda_{2,1}} \right]^{[a_0 - a_{n-1}]_1 + n - 2} \left[ C_{n-2,n-1} - \frac{\lambda_{2,2}}{\lambda_{1,1} \lambda_{2,2} - \lambda_{1,2} \lambda_{2,1}} \right]^{[a_1 - a_{n-1}]_1 + n - 2}.$$

We define

$$\begin{align*}
U_2^1 & := \{ \Delta^2 \in U_2 \mid \lambda_{1,1} \neq 0 \}; \\
U_2^1' & := \{ \Delta^2 \in U_2 \mid \lambda_{1,1} = 0 \}. 
\end{align*}$$

It is obvious that $U_2 = U_2^1 \sqcup U_2^1'$ and so

$$L_2(U_2) = L_2(U_2^1) + L_2(U_2^1').$$
We start with the calculation of $L_2(U_2^{1/2})$. In this case we have

$$Y_5 \cdot Z_5 = [\lambda_{2,1}]^{a_1-a_2}[\lambda_{1,2}]^{a_{n-1}-a_0}[\lambda_{1,2} \lambda_{2,1}]^{a_2-a_{n-1}}$$

$$\cdot [C_{n-3,n-2}]^{a_2-a_3,1+n-2}[C_{n-1,n}]^{a_0-a_{n-1},1+n-2}[C_{n-2,n-1} + \frac{\lambda_{2,2}}{\lambda_{1,2} \lambda_{2,1}}]^{a_1-a_{n-1},1+n-2}$$

and thus

$$\sum_{\lambda_{2,2}\in F_p} Y_5 \cdot Z_5 = \left( \sum_{\lambda_{2,2}\in F_p} \left[ C_{n-2,n-1} + \frac{\lambda_{2,2}}{\lambda_{1,2} \lambda_{2,1}} \right]^{a_1-a_{n-1},1+n-2} \right) \cdot \left( \sum_{\lambda_{2,2}\in F_p} \left[ C_{n-2,n-1} - \frac{\lambda_{2,2}}{\lambda_{1,2} \lambda_{2,1}} \right]^{a_1-a_{n-1},1+n-2} \right) \equiv 0 \pmod{p}$$

where $*$ is a certain term which is independent of $\lambda_{2,2}$. Hence we conclude that

$$L_2(U_2^{1/2}) = 0.$$

We now compute $L_2(U_2^{1/2})$. By a change of variable $x = \lambda_{2,1} \in F_p^\times$ and $y = \frac{\lambda_{2,2}}{\lambda_{1,2}} \in F_p \setminus \{\lambda_{1,2}\}$ we can rewrite $Y_5 \cdot Z_5$ as

$$Y_5 \cdot Z_5 = \left[ x \right]^{a_1-a_{n-1}}[\lambda_{1,2}]^{a_{n-1}-a_0}[\lambda_{1,2} - \lambda_{1,1} y]^{a_2-a_{n-1}} \left[ C_{n-3,n-2} + \frac{\lambda_{1,1}}{\lambda_{1,2}} \right]^{a_2-a_3,1+n-2}$$

$$\cdot \left[ C_{n-1,n} - \frac{\lambda_{1,1}}{x} \right]^{a_0-a_{n-1},1+n-2} \left[ C_{n-2,n-1} - \frac{y}{\lambda_{1,1} y - \lambda_{1,2}} \right]^{a_1-a_{n-1},1+n-2}.$$}

If $C_{n-1,n} = 0$, then

$$\sum_{x \in F_p^\times} Y_5 \cdot Z_5 = \left( \sum_{x \in F_p^\times} \left[ x \right]^{a_1-a_0-n+2} \right) \equiv 0 \pmod{p}$$

where $*$ is a certain term that is independent of $x$. If $C_{n-1,n} \neq 0$, then we deduce from (4.7.12) that

$$\sum_{x \in F_p^\times} Y_5 \cdot Z_5 = \left[ \lambda_{1,2} - \lambda_{1,1} y \right]^{a_2-a_{n-1}} \left[ C_{n-2,n-1} - \frac{y}{\lambda_{1,1} y - \lambda_{1,2}} \right]^{a_1-a_{n-1},1+n-2}$$

$$\cdot \left[ C_{n-3,n-2} + \frac{\lambda_{1,1}}{\lambda_{1,2}} \right]^{a_2-a_3,1+n-2}$$

and

$$Y_6 := [\lambda_{1,1}]^{a_1-a_{n-1}}[\lambda_{1,2}]^{a_{n-1}-a_0} \left[ C_{n-3,n-2} + \frac{\lambda_{1,1}}{\lambda_{1,2}} \right]^{a_2-a_3,1+n-2}$$

as

$$\left[ x \right]^{a_1-a_{n-1}} \left[ C_{n-1,n} - \frac{\lambda_{1,1}}{x} \right]^{a_0-a_{n-1},1+n-2} \equiv \left[ C_{n-1,n} \right]^{[a_0-a_1,1+n-2][a_1-a_{n-1},1+n-2]} \left[ \frac{C_{n-1,n} x}{\lambda_{1,1}} \right]^{[a_1-a_{n-1},1+n-2]} \left[ 1 - \frac{\lambda_{1,1}}{C_{n-1,n} x} \right]^{[a_0-a_{n-1},1+n-2]}.$$
If $C_{n-2,n-1} - \frac{1}{\lambda_{1,1}} = 0$, then we deduce

$$X_6 = \left( \sum_{y \in F_p \backslash \{\frac{\lambda_{1,2}}{\lambda_{1,1}}\}} [\lambda_{1,1}y - \lambda_{1,2}]^{a_2 - a_1 - n + 2} \right) \left( \sum_{y \in F_p \backslash \{\frac{\lambda_{1,2}}{\lambda_{1,1}}\}} [\lambda_{1,1}y - \lambda_{1,2}]^{a_2 - a_1 - n + 2} \right) = 0$$

where $\ast$ is a certain term which is independent of $y$. If $C_{n-2,n-1} - \frac{1}{\lambda_{1,1}} \neq 0$, then we deduce

$$X_6 = \left[ C_{n-2,n-1} - \frac{1}{\lambda_{1,1}} \right]^{a_2 - a_1 - n + 2} \left[ \lambda_{1,2} \right]^{a_2 - a_1 - n + 2} [X_7]^{a_2 - a_1 - n + 2} [1 + X_7]^{a_2 - a_1 - n + 2}$$

where

$$X_7 := \frac{\lambda_{1,2}}{(\lambda_{1,2} - \lambda_{1,1}y) \left( C_{n-2,n-1} + \frac{1}{\lambda_{1,1}} \right)}.$$

Therefore, by (4.7.32) we obtain that

$$\sum_{y \in F_p \backslash \{\frac{\lambda_{1,2}}{\lambda_{1,1}}\}} X_6 = (-1)^{a_2 - a_1} J(a_{n-1} - a_2, [a_1 - a_{n-1}]_1 + n - 2) \left[ \lambda_{1,1} \right]^{a_2 - a_1 - n + 2} [\lambda_{1,2}]^{a_2 - a_1 - n + 2} [X_7]^{a_2 - a_1 - n + 2}$$

We emphasize that (4.7.32) still holds even if $C_{n-2,n-1} + \frac{1}{\lambda_{1,1}} = 0$.

Hence, we deduce that

$$\sum_{y \in F_p \backslash \{\frac{\lambda_{1,2}}{\lambda_{1,1}}\}} X_6 \cdot Y_6 = (-1)^{a_2 - a_1} J(a_{n-1} - a_2, [a_1 - a_{n-1}]_1 + n - 2) \left[ \lambda_{1,1} \right]^{a_2 - a_1 - n + 2} [\lambda_{1,2}]^{a_2 - a_1 - n + 2} [X_7]^{a_2 - a_1 - n + 2}$$

where

$$X_8 := [\lambda_{1,1}]^{a_2 - a_0} \left[ C_{n-2,n-1} - \frac{1}{\lambda_{1,1}} \right]^{a_2 - a_1 - n + 2}$$

and

$$Y_8 := \left[ \lambda_{1,2} \right]^{a_2 - a_0} \left[ C_{n-3,n-2} + \frac{\lambda_{1,1}}{\lambda_{1,2}} \right]^{a_2 - a_3 - n + 2}.$$

It is not difficult to see that

$$\sum_{\lambda_{1,1} \in F_p^x} X_8 = \left[ C_{n-2,n-1} \right]^{a_0 - a_2} [a_0 - a_1]_1 + n - 2 + J([a_0 - a_1]_1, [a_1 - a_2]_1 + n - 2);$$

$$\sum_{\lambda_{1,2} \in F_p^x} Y_8 = (-1)^{a_2 - a_0} \left[ C_{n-3,n-2} \right]^{a_0 - a_3} [a_0 - a_3]_1 + n - 2 J([a_0 - a_2]_1, [a_2 - a_3]_1 + n - 2).$$

Combining (4.7.31), (4.7.32) and (4.7.33), we finish the proof.

We define

$$P_n = \prod_{k=1}^{n-2} \prod_{j=1}^{n-2} \left[ \frac{a_k - a_{n-1}}{a_0 - a_k} \right]_1 + j = \prod_{k=1}^{n-3} a_k - a_{n-1} + j = \prod_{k=1}^{n-2} a_k - a_k + j \in \mathbb{Z}_p^x$$

and

$$\varepsilon^* := \prod_{m=1}^{n-2} (-1)^{a_0 - a_m}.$$
Lemma 4.7.36. We have
\[
\begin{aligned}
&\text{ord}_p(\kappa_n) = 0; \\
&\kappa_n \equiv \varepsilon^* P_n \quad \text{(mod } p) \\
\end{aligned}
\]

Proof. By (4.7.13), we deduce that
\[
\begin{aligned}
&\text{ord}_p(J([a_0 - a_m], [a_m - a_{m+1}]_1 + n - 2)) = 0; \\
&\text{ord}_p(J([a_{n-1} - a_m], [a_{m-1} - a_{n-1}]_1 + n - 2)) = 1,
\end{aligned}
\]

and thus \( \text{ord}_p(\kappa_n) = 0 \). On the other hand, still by (4.7.13), we obtain that
\[
\begin{aligned}
&J([a_0 - a_m], [a_m - a_{m+1}]_1 + n - 2) \equiv \frac{[a_0 - a_m]([a_m - a_{m+1}]_1 + n - 2)!}{([a_0 - a_m]_1 + n - 2)!} \quad \text{(mod } p); \\
p^{-1}J([a_{n-1} - a_m], [a_{m-1} - a_{n-1}]_1 + n - 2) \equiv \frac{[a_{n-1} - a_m]([a_{m-1} - a_{n-1}]_1 + n - 2)!}{([a_{m-1} - a_{n-1}]_1 + n - 2)!} \quad \text{(mod } p)
\end{aligned}
\]
for each \( 1 \leq m \leq n - 2 \), and thus
\[
\begin{aligned}
\kappa_n &\equiv (-1)^{(n-2)(a_0 - a_{n-1})} \prod_{m=1}^{n-2} [a_0 - a_m]^1([a_m - a_{m+1}]_1 + n - 2)! \\
&\quad \times \prod_{m=1}^{n-2} [a_{n-1} - a_m]^1([a_{m-1} - a_{n-1}]_1 + n - 2)! \\
&\equiv (-1)^{(n-2)(a_0 - a_{n-1})} \prod_{m=1}^{n-2} (-1)^{a_{n-1} - a_m} [a_0 - a_m]^1([a_m - a_{n-1}]_1 + n - 2)! \\
&\quad \times ([a_0 - a_m]_1 + n - 2)!([a_m - a_{n-1}]_1 + n - 2)! \\
&\equiv \varepsilon^* P_n \quad \text{(mod } p),
\end{aligned}
\]
which completes the proof. \( \square \)

Proof of Theorem 4.7.4 Theorem 4.7.4 follows from the combination of Lemma 4.7.5 Proposition 4.7.19 Proposition 4.7.30 and Lemma 4.7.36 \( \square \)

5. Mod \( p \) Local-Global Compatibility

In this section, we state and prove our main results on mod \( p \) local-global compatibility, which is a global application of our local results of Sections 3 and 4. In the first two sections, we recall some necessary known results on algebraic automorphic forms and Serre weights, for which we closely follow EGHK, HLM, and BLCC.

We first fix some notation for the whole section. Let \( P \supset B \) be an arbitrary standard parabolic subgroup and \( N \) its unipotent radical. We denote the opposite parabolic by \( P^- := w_0 P w_0 \) with corresponding unipotent radical \( N^- := w_0 N w_0 \). We fix a standard choice of Levi subgroup \( L := P \cap P^- \subseteq G \). We denote the positive roots of \( L \) defined by the pair \( (B \cap L, T) \) by \( \Phi_\pm^L \). We use
\[
X_L(T)_+ := \{ \lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Phi_\pm^L \}
\]
to denote the set of dominant weights with respect to the pair \( (B \cap L, T) \). We denote the Weyl group of \( L \) by \( W^L \) and identify it with a subgroup of \( W \). The longest Weyl element in \( W^L \) is denoted by \( w_0^L \).

We define the affine Weyl group \( \tilde{W}^L \) of \( L \) as the semi-direct product of \( W^L \) and \( X(T) \) with respect to the natural action of \( W^L \) on \( X(T) \). Therefore \( \tilde{W}^L \) has a natural embedding into \( \tilde{W} \). We define the subgroups \( \overline{P}, \overline{L}, \cdots \) of \( \overline{G} \) in the obvious similar fashion.

We also need to define several open compact subgroups of \( L(\mathbb{Q}_p) \). We define
\[
K^L := L(\mathbb{Z}_p),
\]
and via the mod \( p \) reduction map
\[
\text{red}^L : K^L = L(\mathbb{Z}_p) \to L(\mathbb{F}_p)
\]
we also define $K^L(1)$, $I^L(1)$, and $I^L$ as follows:

\[(5.0.2) \quad K^L(1) := (\text{red}^L)^{-1}(1) \subseteq I^L(1) := (\text{red}^L)^{-1}(U(F_p) \cap L(F_p)) \subseteq I^L := (\text{red}^L)^{-1}(B(F_p) \cap L(F_p)).\]

For any dominant weight $\lambda \in X(T)_+$, we let

\[H_0^L(\lambda) := \left( \text{Ind}_{T_0}^{\text{GL}_n} u_0^L \lambda \right)_{\text{alg}} \]

be the associated dual Weyl module of $L$. We also write $F^L(\lambda) := \text{soc}_F(H_0^L(\lambda))$ for its irreducible socle as an algebraic representation of $\overline{\text{T}}$. Through a similar argument presented at the beginning of Section 4.1, the notation $F^L(\lambda)$ is well defined as an irreducible representation of $L(F_p)$ if $\lambda \in T(F_p)$ is $p$-regular, namely lies in the image of $X^\text{reg}_n(T) \to X(T)/(p-1)X(T)$. We will sometimes abuse the notation $F^L(\lambda)$ for $F^L(\lambda) \otimes F_p$ or $F^L(\lambda)$ for $F^L(\lambda) \otimes F_p$ in the literature. We will emphasize the abuse of the notation $F^L(\lambda)$ each time we do so.

We introduce some specific standard parabolic subgroups of $G$. Fix integers $i_0$ and $j_0$ such that $0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1$, and let $i_1$ and $j_1$ be the integers determined by the equation

\[(5.0.3) \quad i_0 + i_1 = j_0 + j_1 = n - 1.\]

We let $P_{i_1,j_1} \supset B$ be the standard parabolic subgroup of $G = \text{GL}_n$ corresponding to the subset $\{\alpha_k \mid j_0 + 1 \leq k \leq i_0\}$ of $\Delta$. By specifying the notation for general $P$ to $P_{i_1,j_1}$, we can define $P^r_{i_1,j_1}$, $L_{i_1,j_1}$, $N_{i_1,j_1}$, and $N^-_{i_1,j_1}$. We can naturally embeds $\text{GL}_{j_1-i_1+1}$ into $G$ with its image denoted by $G_{i_1,j_1}$, such that $L_{i_1,j_1} = G_{i_1,j_1} T$:

\[(5.0.4) \quad \text{GL}_{j_1-i_1+1} \hookrightarrow G_{i_1,j_1} \hookrightarrow L_{i_1,j_1} \rightarrow P_{i_1,j_1} \hookrightarrow G.\]

We define $T_{i_1,j_1}$ to be the maximal tori of $G_{i_1,j_1}$, that is contained in $T$, and define $X(T_{i_1,j_1})$ to be the character group of $T_{i_1,j_1}$. If $i_1$ and $j_1$ are clear from the context (or equivalently $i_0$ and $j_0$ are clear) then we often write $P$, $P^r$, $L$, $N$, and $N^-$ for $P_{i_1,j_1}$, $P^r_{i_1,j_1}$, $L_{i_1,j_1}$, $N_{i_1,j_1}$, and $N^-_{i_1,j_1}$, respectively.

5.1. The space of algebraic automorphic forms. Let $F/Q$ be a CM field with maximal totally real subfield $F^+$. We write $c$ for the generator of $\text{Gal}(F/F^+)$, and let $S_p^+$ (resp. $S_p^-$) be the set of places of $F^+$ (resp. $F$) above $p$. For $v$ (resp. $w$) a finite place of $F^+$ (resp. $F$) we write $k_v$ (resp. $k_w$) for the residue field of $F_v^+$ (resp. $F_w$).

From now on, we assume that

\[\circ \quad F/F^+ \text{ is unramified at all finite places;}\]
\[\circ \quad p \text{ splits completely in } F.\]

Note that the first assumption above excludes $F^+ = Q$. We also note that the second assumption is not essential in this section, but it is harmless since we are only interested in $G_{Q_v}$-representations in this paper. Every place $v$ of $F^+$ above $p$ further decomposes and we often write $v = uv^c$ in $F$.

There exists a reductive group $G_{n/F^+}$ satisfying the following properties (cf. [BLGG], Section 2):

\[\circ \quad G_n \text{ is an outer form of } \text{GL}_n \text{ with } G_{n/F} \cong \text{GL}_n/F,\]
\[\circ \quad G_n \text{ is a quasi-split at any finite place of } F^+;\]
\[\circ \quad G_n(F_v^+) \cong U_n(R) \text{ for all } v|\infty.\]

By [CHT08], Section 3.3, $G_n$ admits an integral model $G_n$ over $O_{F^+}$ such that $G_n \times_{O_{F^+}} O_{F^+}$ is reductive if $v$ is a finite place of $F^+$ which splits in $F$. If $v$ is such a place and $w$ is a place of $F$ above $v$, then we have an isomorphism

\[(5.1.1) \quad \iota_v : G_n(O_{F_v^+}) \cong G_n(O_{F_w}) \cong \text{GL}_n(O_{F_w}).\]

We fix this isomorphism for each such place $v$ of $F^+$.

Define $F^+_p := F^+ \otimes Q_p$ and $O_{F^++p} := O_{F^+} \otimes \mathbb{Z}_p$. If $W$ is an $O_F$-module endowed with an action of $G_n(O_{F^+})$ and $U \subset G_n(A_{F^+}^\infty) \times G_n(O_{F^+})$ is a compact open subgroup, the space of algebraic
automorphic forms on $G_n$ of level $U$ and coefficients in $W$, which is also an $\mathcal{O}_E$-module, is defined as follows:

$$S(U, W) := \{ f : G_n(F^+) \setminus G_n(A_{F^+}^\infty) \to W \mid f(gu) = u_p^{-1}f(g) \forall g \in G_n(A_{F^+}^\infty), u \in U \}$$

with the usual notation $u = u^p u_\wp$ for the elements in $U$.

We say that the level $U$ is sufficiently small if

$$t^{-1}G_n(F^+)t \cap U$$

has finite order prime to $p$ for all $t \in G_n(A_{F^+}^\infty)$. We say that $U$ is unramified at a finite place $v$ of $F^+$ if it has a decomposition

$$U = G_n(\mathcal{O}_{F^+}^v)U^v$$

for some compact open $U^v \subset G_n(A_{F^+}^\infty)$. If $w$ is a finite place of $F$, then we say, by abuse of notation, that $w$ is an unramified place for $U$ or $U$ is unramified at $w$ if $U$ is unramified at $w|_{F^+}$.

For a compact open subgroup $U$ of $G_n(A_{F^+}^p) \times G_n(\mathcal{O}_{F^+,p})$, we let $\mathcal{P}_U$ denote the set consisting of finite places $v$ of $F$ such that

- $w|_{F^+}$ is split in $F$,
- $w \notin S_p$,
- $U$ is unramified at $w$.

For a subset $\mathcal{P} \subseteq \mathcal{P}_U$ of finite complement and closed with respect to complex conjugation we write $T^\mathcal{P} = \mathcal{O}_E[T_w^{(i)}], w \in \mathcal{P}, i \in \{0, 1, \ldots, n\}]$ for the universal Hecke algebra on $\mathcal{P}$, where the Hecke operator $T_w^{(i)}$ acts on $S(U, W)$ via the usual double coset operator

$$t^{-1}_w \left[ \text{GL}_n(\mathcal{O}_{F_w}) \left( \varpi_w \text{Id}_{i} \quad 0 \quad \text{Id}_{n-i} \right) \text{GL}_n(\mathcal{O}_{F_w}) \right]$$

where $\varpi_w$ is a uniformizer of $\mathcal{O}_{F_w}$ and $\text{Id}_i$ is the identity matrix of size $i$. The Hecke algebra $T^\mathcal{P}$ naturally acts on $S(U, W)$.

Recall that we assume that $p$ splits completely in $F$. Following [EGH15], Section 7.1 we consider the subset $(\mathbb{Z}^+_0)^{S_p}$ consisting of dominant weights $\underline{a} = (a_w)_w$ where $a_w = (a_{1,w}, a_{2,w}, \cdots, a_{n,w})$ satisfying

$$(5.1.2) \quad a_{i,w} + a_{n+1-i,w} = 0$$

for all $w \in S_p$ and $1 \leq i \leq n$. We let

$$W_{\underline{a}_w} := M_{\underline{a}_w}(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}} \mathcal{O}_E$$

where the $M_{\underline{a}_w}(\mathcal{O}_{F_w})$ is $\mathcal{O}_{F_w}$-specialization of the dual Weyl module associated to $\underline{a}_w$ (cf. [EGH15], Section 4.1.1); by condition (5.1.2), one deduces an isomorphism of $G_n(\mathcal{O}_{F^+})$-representations $W_{\underline{a}_w} \circ \iota_w \cong W_{\underline{a}_w} \circ \iota_w$. Therefore, by letting $W_{\underline{a}} := \bigotimes_{w \in S_p} W_{\underline{a}_w}$ for any place $w|v$, the $\mathcal{O}_E$-representation of $G_n(\mathcal{O}_{F^+,p})$

$$W_{\underline{a}} := \bigotimes_{v|p} W_{\underline{a}_w}$$

is well-defined.

For a weight $\underline{a} \in (\mathbb{Z}^+_0)^{S_p}$, let us write $S_\underline{a}(\mathbb{Q}_p)$ to denote the inductive limit of the spaces $S(U, W_{\underline{a}_w}) \otimes_{\mathcal{O}_E} \mathbb{Q}_p$ over the compact open subgroups $U \subset G_n(A_{F^+}^\infty) \times G_n(\mathcal{O}_{F^+,p})$. (Note that the transition maps are induced, in a natural way, from the inclusions between levels $U$.) Then $S_\underline{a}(\mathbb{Q}_p)$ has a natural left action of $G_n(A_{F^+}^\infty)$ induced by right translation of functions.

We briefly recall the relation between the space $\mathcal{A}$ of classical automorphic forms and the previous spaces of algebraic automorphic forms in the particular case which is relevant to us. Fix an isomorphism $\iota : \mathbb{Q}_p \to \mathbb{C}$ for the rest of the paper. As we did for the $\mathcal{O}_{F_w}$-specialization of the dual Weyl modules, we define a finite dimensional $G_n(F^+ \otimes \mathbb{Q} \mathbb{R})$-representation $\sigma_{\underline{a}} \cong \bigoplus_{v|\infty} \sigma_{\underline{a}_w}$ with $\mathbb{C}$-coefficients.

(We refer to [EGH15], Section 7.1.4 for the precise definition of $\sigma_{\underline{a}}$).
Lemma 5.1.3 ([EGH15], Lemma 7.1.6). The isomorphism \( \iota : \mathcal{O}_p \overset{\sim}{\to} \mathbb{C} \) induces an isomorphism of smooth \( G_n(\mathbb{A}_F^\infty) \)-representations
\[
S_{\mathfrak{a}}(\mathcal{O}_p) \otimes \mathcal{O}_{\mathfrak{p}^a}, \mathbb{C} \overset{\sim}{\to} \text{Hom}_{G_n(F^+ \otimes \mathcal{O})}(\sigma^\vee, A)
\]
for any \( \mathfrak{a} \in (\mathbb{Z}_p^n)_{\mathfrak{p}}^S \).

The following theorem guarantees the existence of Galois representations attached to automorphic forms on the unitary group \( G_n \). Let \( | \cdot | \) denote the unique square root of \( | \cdot |^{-1} \)
whose composite with \( \iota : \mathcal{O}_p \overset{\sim}{\to} \mathbb{C} \) takes positive values.

Theorem 5.1.4 ([EGH15]. Theorem 7.2.1). Let \( \Pi \) be an irreducible \( G_n(\mathbb{A}_F^\infty) \)-subrepresentation of \( S_{\mathfrak{a}}(\mathcal{O}_p) \).

Then there exists a continuous semisimple representation \( r_\Pi : G_F \to \text{GL}_n(\mathcal{O}_p) \) such that

1. \( r_\Pi \otimes \varepsilon^{n-1} \simeq r_\Pi^\vee \).
2. For each place \( w \) above \( p \), the representation \( r_\Pi|_{G_{F_w}} \) is de Rham with Hodge–Tate weights \( HT(r_\Pi|_{G_{F_w}}) = \{a_{1,w} + (n-1), a_{2,w} + (n-2), \ldots, a_{n,w}\} \).
3. If \( w \mid p \) is a place of \( F \) and \( v := w|_{F^+} \) splits in \( F \), then
   \[
   \text{WD}(r_\Pi|_{G_{F_w}})^{\text{F-ss}} \simeq \text{rec}_w((\Pi_v \circ \iota_w^{-1}) \otimes | \cdot |^{1/w}).
   \]

We note that the fact that (iii) holds without semi-simplification on the automorphic side is one of the main results of [Cara14]. We also note that property (iii) says that the restriction to \( G_{F_w} \) is compatible with the local Langlands correspondence at \( w \), which is denoted by \( \text{rec}_w \).

5.2. Serre weights and potentially crystalline lifts. In this section, we recall the relation of Serre weights and potentially crystalline lifts via (inertial) local Langlands correspondence.

Definition 5.2.1. A Serre weight for \( G_n \) is an isomorphism class of an absolutely irreducible smooth \( \mathbb{F}_p \)-representation \( V \) of \( G_n(\mathcal{O}_{F^+}) \). If \( v \) is a place of \( F^+ \) above \( p \), then a Serre weight at \( v \) is an isomorphism class of an absolutely irreducible \( \mathbb{F}_p \)-smooth representation \( V_v \) of \( G_n(\mathcal{O}_{F_v}) \). Finally, if \( w \) is a place of \( F \) above \( p \), a Serre weight at \( w \) is an isomorphism class of an absolutely irreducible \( \mathbb{F}_p \)-smooth representation \( V_w \) of \( \text{GL}_n(\mathcal{O}_{F_w}) \).

We will often say a Serre weight for a Serre weight for \( G_n \) if \( G_n \) is clear from the context. Note that if \( V_w \) is a Serre weight at \( w \), there is an associated Serre weight at \( w|v \) defined by \( V_v \circ \iota_w^{-1} \).

As explained in [EGH15], Section 7.3, a Serre weight \( V \) admits an explicit description in terms of \( \text{GL}_n(k_w) \)-representations. More precisely, let \( w \) be a place of \( F \) above \( p \) and write \( v := w|_{F^+} \). For any \( n \)-tuple of integers \( a_w := (a_{1,w}, a_{2,w}, \ldots, a_{n,w}) \in \mathbb{Z}_+^n \), that is restricted (i.e., \( 0 \leq a_{i,w} - a_{i+1,w} \leq p-1 \) for \( i = 1, 2, \ldots, n-1 \)), we consider the Serre weight \( F(a_w) := F(a_{1,w}, a_{2,w}, \ldots, a_{n,w}) \), as defined in [EGH15], Section 4.1.2. It is an irreducible \( \mathbb{F}_p \)-representation of \( \text{GL}_n(k_w) \) and of \( G_n(k_v) \) via the isomorphism \( \iota_w \). Note that \( F(a_{1,w}, a_{2,w}, \ldots, a_{n,w})^\vee \circ \iota_w \neq F(a_{1,w}, a_{2,w}, \ldots, a_{n,w}) \circ \iota_w \) as \( G_n(k_v) \)-representations, i.e., \( F(a_{w}) \circ \iota_w \neq F(a_{w}) \circ \iota_w \) if \( a_{i,w} + a_{n+1-i,w} = 0 \) for all \( 1 \leq i \leq n \). Hence, if \( \mathfrak{a} = (a_{w})_w \in (\mathbb{Z}_+^n)_{\mathfrak{p}}^S \) that is restricted, then we can set \( F_{\mathfrak{a}} := F(a_{w}) \circ \iota_w \) for \( w|v \). We also set
\[
F_{\mathfrak{a}} := \bigotimes_{v \mid p} F_{\mathfrak{a}}_v
\]
which is a Serre weight for \( G_n(\mathcal{O}_{F^+}) \). From [EGH15], Lemma 7.3.4 if \( V \) is a Serre weight for \( G_n \), there exists a restricted weight \( \mathfrak{a} = (a_{w})_w \in (\mathbb{Z}_+^n)_{\mathfrak{p}}^S \) such that \( V \) has a decomposition \( V \cong \bigotimes_{v \mid p} V_v \), where the \( V_v \) are Serre weights at \( v \) satisfying \( V_v \circ \iota_w^{-1} \cong F(a_{w}) \).

Recall that we write \( \mathbb{F} \) for the residue field of \( E \).
**Definition 5.2.2.** Let $\overline{\tau} : G_F \to \text{GL}_n(F)$ be an absolutely irreducible continuous Galois representation and let $V$ be a Serre weight for $G_n$. We say that $\overline{\tau}$ is automorphic of weight $V$ (or that $V$ is a Serre weight of $\overline{\tau}$) if there exists a compact open subgroup $U$ in $G_n(A_F^\infty \times O_{F_{p^+}})$ which is sufficiently small and unramified above $p$ and a cofinite subset $P \subseteq P_U$ such that $\overline{\tau}$ is unramified at each place of $P$ and

$$S(U, V)_{\overline{\tau}} \neq 0$$

where $m_{\tau}$ is the kernel of the system of Hecke eigenvalues $\overline{\tau} : T^P \to F$ associated to $\overline{\tau}$, i.e.

$$\det (1 - \overline{\gamma}(\text{Frob}_w)X) = \sum_{j=0}^{n} (-1)^j (N_{F/Q}(w))^{(j)} \overline{\alpha}(T_w^{(j)}) X^j$$

for all $w \in P$.

We write $W(\overline{\tau})$ for the set of automorphic Serre weights of $\overline{\tau}$. Let $w$ be a place of $F$ above $p$ and $v = w|_{F_{p^+}}$. We also write $W_w(\overline{\tau})$ for the set of Serre weights $F(\overline{\tau})$ such that

$$(F(\overline{\tau}) \circ t_w) \otimes \left( \bigotimes_{w' \in S^P \setminus \{v\}} V_{w'} \right) \in W(\overline{\tau})$$

where $V_{w'}$ are Serre weights of $G_n(O_{F_{p^+}})$ for all $v' \in S^P \setminus \{v\}$. We often write $W(\overline{\tau}|_{G_{F_w}})$ and $W_w(\overline{\tau}|_{G_{F_w}})$ for $W(\overline{\tau})$ and $W_w(\overline{\tau})$ respectively, when the given $\overline{\tau}$ is a restriction of an automorphic representation $\overline{\tau}$ to $G_{F_w}$.

Fix a place $w$ of $F$ above $p$ and let $v = w|_{F_{p^+}}$. We also fix a compact open subgroup $U$ of $G_n(A_F^\infty \times O_{F_{p^+}})$ which is sufficiently small and unramified above $p$. We may write $U = G_n(O_{F_{p^+}})^{\text{U}}$. If $W'$ is an $O_F$-module with an action of $\prod_{w' \in S^P \setminus \{v\}} G_n(O_{F_{p^+}})$, we define

$$S(U^v, W') := \lim_{\overline{v}} S(U^v \cdot U_v, W')$$

where the limit runs over all compact open subgroups $U_v$ of $G_n(O_{F_{p^+}})$, endowing $W'$ with a trivial $G_n(O_{F_{p^+}})$-action. Note that $S(U^v, W')$ has a smooth action of $G_n(F_v^{\infty})$ (given by right translation) and hence of $G_{n}(F_v)$ via $t_w$. We also note that $S(U^v, W')$ has an action of $T^P$ commuting with the smooth action of $G_n(F_v^{\infty})$, where $P$ is a cofinite subset of $P_U$.

**Lemma 5.2.3 ([EGL15], Lemma 7.4.3).** Let $U$ be a compact open subgroup of $G_n(A_F^\infty \times O_{F_{p^+}})$ which is sufficiently small and unramified above $p$, and $P$ a cofinite subset of $P_U$. Fix a place $w$ of $F$ above $p$ and let $v = w|_{F_{p^+}}$. Let $V \cong \bigotimes_{w' \in S^P} V_{w'}$ be a Serre weight for $G_n$. Then there is a natural isomorphism of $T^P$-modules

$$\text{Hom}_{G_n(O_{F_{p^+}})}(V^v, S(U^v, W')) \cong S(U, V)$$

where $V' := \bigotimes_{w' \in S^P \setminus \{v\}} V_{w'}$.

We now recall some formalism related to Deligne–Lusztig representations from Section 4.3. Let $w$ be a place of $F$ above $p$. For a positive integer $m$, let $k_{w,m}/k_w$ be an extension satisfying $[k_{w,m} : k_w] = m$, and let $T$ be a $F$-stable maximal torus in $\text{GL}_{n/k_w}$ where $F$ is the Frobenius morphism. We have an identification from [Her09], Lemma 4.7

$$T(k_w) \cong \prod_{j} k_{w,n_j}^\times$$

where $n \geq n_j > 0$ and $\sum_j n_j = n$; the isomorphism is unique up to $\prod_j \text{Gal}(k_{w,n_j}/k_w)$-conjugacy. In particular, any character $\theta : T(k_w) \to \overline{Q}_p$ can be written as $\theta = \otimes_j \theta_j$ where $\theta_j : k_{w,n_j}^\times \to \overline{Q}_p$. 
Given a $F$-stable maximal torus $T$ and a primitive character $\theta$, we consider the Deligne-Lusztig representation $R_T^\theta$ of $GL_n(k_w)$ over $\overline{Q}_p$ defined in Section 4.3. Recall from Section 4.3 that $\Theta(\theta_j)$ is cuspidal representation of $GL_{n_j}(k_w)$ associated to the primitive character $\theta_j$, we have

$$R_T^\theta \cong (-1)^{n-r} \cdot \text{Ind}_{P_w(k_w)}^{GL_n(k_w)}(\otimes_j \Theta(\theta_j))$$

where $P_w$ is the standard parabolic subgroup containing the Levi $\prod_j GL_{n_j}$ and $r$ denotes the number of its Levi factors.

Let $F_{w,m} := W(k_{w,m})[1/p]$ for a positive integer $m$. We consider $\theta_j$ as a character on $O_{F_{w,n_j}}^\times$ by inflation and we define the following Galois type $\text{rec}(\theta) : I_{F_w} \to GL_n(\overline{Q}_p)$ as follows:

$$\text{rec}(\theta) := \bigoplus_{j=1}^\tau \sigma(\theta_j \circ \text{Art}_{F_{w,n_j}}^{-1})$$

where $\theta_j$ is a primitive character on $k_w^{\times}$ of niveau $n_j$ for each $j = 1, \ldots, r$. Recall that $\text{Art}_{F_{w,n_j}} : F_{w,n_j}^\times \to W_{F_{w,n_j}}^\text{ab}$ is the isomorphism of local class field theory, normalized by sending the uniformizers to the geometric Frobenius.

We quickly review inertial local Langlands correspondence.

**Theorem 5.2.4** ([CEGGPS], Theorem 3.7 and [LLL], Proposition 2.3.4). Let $\tau : I_{Q_p} \to GL_n(\overline{Q}_p)$ be a Galois type. Then there exists a finite dimensional irreducible smooth $\overline{Q}_p$-representation $\sigma(\tau)$ of $GL_n(\mathbb{Z}_p)$ such that if $\pi$ is any irreducible $\overline{Q}_p$-representation of $GL_n(\mathbb{Q}_p)$ then $\pi|_{GL_n(\mathbb{Z}_p)}$ contains a unique copy of $\sigma(\tau)$ as a subrepresentation if and only if $\text{rec}_{Q_p}(\pi)|_{I_{Q_p}} \cong \tau$ and $N = 0$ on $\text{rec}_{Q_p}(\pi)$.

Moreover, if $\tau \cong \bigoplus_{j=1}^\tau \tau_j$ and the $\tau_j$ are pairwise distinct, then $\sigma(\tau) \cong R_T^\theta$ and $\tau \cong \text{rec}(\theta)$ for a maximal torus $T$ in $GL_n(F_w)$ and a primitive character $\theta : T(\mathbb{F}_p) \to \overline{Q}_p^\times$.

The following theorem provides a connection between Serre weights and potentially crystalline lifts, which will be useful for the main result, Theorem 5.2.6.

**Theorem 5.2.5** ([LLL], Proposition 4.2.5). Let $w$ be a place of $F$ above $p$, $T$ a maximal torus in $GL_{n/k_w}$, $\Theta = \bigotimes_{j=1}^\tau \Theta_j : T(k_w) \to \overline{Q}_p^\times$ a primitive character such that $\Theta_j$ are pairwise distinct, and $V_w$ a Serre weight at $w$ for a Galois representation $\tau : G_F \to GL_n(\mathbb{F}_p)$.

Assume that $V_w$ is a Jordan-Hölder constituent in the mod $p$ reduction of the Deligne–Lusztig representation $R_T^\theta$ of $GL_n(k_w)$. Then $\tau|_{G_{F_w}}$ has a potentially crystalline lift with Hodge–Tate weights $\{-n-1, -(n-2), \ldots, 0\}$ and Galois type $\text{rec}(\theta)$.

For a given automorphic Galois representation $\tau : G_F \to GL_n(\mathbb{F}_p)$, it is quite difficult to determine if a given Serre weight is a Serre weight of $\tau$. Thanks to the work of [BLGG], we have the following theorem, in which we refer to the reader to [BLGG] for the unfamiliar terminology.

**Theorem 5.2.6** ([BLGG], Theorem 4.1.9). Assume that if $n$ is even then so is $\frac{n(F^{+} \mathbb{Q}_p)}{2}$, that $\zeta_p \notin F$, and that $\tau : G_F \to GL_n(\mathbb{F}_p)$ is an absolutely irreducible representation with split ramification. Assume further that there is a RACSDC automorphic representation $\Pi$ of $GL_n(\mathbb{A}_F)$ such that

- $\tau \simeq \tau_{\Pi}$;
- For each place $w|p$ of $F$, $r_{\Pi}|_{G_{F_w}}$ is potentially diagonalizable;
- $\tau(G_{F(p)})$ is adequate.

If $\mathfrak{a} = (a_w)_{\text{w}} \in (\mathbb{Z}_p)^{S_F}$ and for each $w \in S_F$, $\tau|_{G_{F_w}}$ has a potentially diagonalizable crystalline lift with Hodge–Tate weights $\{a_{1,w} + (n+1), a_{2,w} + (n-2), \ldots, a_{n-1,w} + 1, a_{n,w}\}$, then a Jordan–Hölder factor of $W_{\mathfrak{a}} \otimes_{\mathbb{Z}_p} \mathbb{F}$ is a Serre weight of $\tau$. 


5.3. Weight elimination and automorphy of a Serre weight. In this section, we state our main Conjecture for weight elimination (Conjecture 5.3.1) which will be a crucial assumption in the proof of Theorem 5.7.6. This conjecture is now known by Bao V. Le Hung [LeH]. We also prove the automorphy of a certain obvious Serre weight under the assumptions of Taylor–Wiles type.

Throughout this section, we assume that \( \overline{\theta}_0 \) is always a restriction of an automorphic representation \( \tau: G_F \to \text{GL}_n(F) \) to \( G_{F_w} \) for a fixed place \( w \) above \( p \) and is generic (c.f. Definition 3.0.3). Recall that for \( 0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1 \) we have defined a tuple of integers \( (x_{i_0,j_0}, \cdots, x_{i_0,j_0}) \) in (5.7.7), which determines the Galois types as in (1.1.2). In many cases, we will consider the dual of our Serre weights, so that we define a pair of integers \((i_1, j_1)\) by the equation (5.0.3). We also let

\[
   b_k := -c_{n-1-k}
\]

for all \( 0 \leq k \leq n - 1 \). We will keep the notation \((i_1, j_1)\) and \( b_k \) for the rest of the paper.

For the rest of this section, we are mainly interested in the following characters of \( T(F_p) \): let

\[
   \mu^\square := (b_{n-1}, \cdots, b_0),
\]

and let

\[
   \mu^{i_1,j_1} := (x_{i_1-1}, x_{i_1-2}, \cdots, x_1, x_0),
\]

\[
   \mu^{i_1,j_1,\prime} := (x_{i_1-1}^\prime, x_{i_1-2}^\prime, \cdots, x_1^\prime, x_0^\prime),
\]

and

\[
   \mu^{\square, i_1,j_1} := (y_{i_1-1}, y_{i_1-2}, \cdots, y_1, y_0)
\]

where

\[
   x_j = \begin{cases} 
   b_j & \text{if } j > j_1 \text{ or } i_1 > j; \\
   b_{j_1+i_1-1-j} & \text{if } j_1 \geq j > i_1 + 1; \\
   b_{j_1} + j_1 - i_1 - 1 & \text{if } j = i_1 + 1; \\
   b_{i_1} - j_1 + i_1 + 1 & \text{if } j = i_1,
   \end{cases}
\]

\[
   x_j^\prime = \begin{cases} 
   b_j & \text{if } j > j_1 \text{ or } i_1 > j; \\
   b_{j_1+i_1-1-j} & \text{if } j_1 - 1 > j \geq i_1; \\
   b_{j_1} + j_1 - i_1 - 1 & \text{if } j = j_1;
   \end{cases}
\]

\[
   y_j = \begin{cases} 
   b_j & \text{if } j \notin \{j_1, i_1\}; \\
   b_{i_1} - j_1 + i_1 + 1 & \text{if } j = j_1; \\
   b_{j_1} + j_1 - i_1 - 1 & \text{if } j = i_1.
   \end{cases}
\]

As \( \overline{\theta}_0 \) is generic, each of the characters above is \( p \)-regular and thus uniquely determines a \( p \)-restricted weight up to a twist in \( (p-1)X_0(T) \), and, by abuse of notation, we write \( \mu^\square, \mu^{i_1,j_1}, \mu^{i_1,j_1,\prime}, \mu^{\square, i_1,j_1} \) for those corresponding \( p \)-restricted weights, respectively. We will clarify the twist in \( (p-1)X_0(T) \) whenever necessary. We also define two principal series representations

\[
   \pi^{i_1,j_1} := \text{Ind}_{B(F_p)}^{G(F_p)} \mu^{i_1,j_1} \quad \text{and} \quad \pi^{i_1,j_1,\prime} := \text{Ind}_{B(F_p)}^{G(F_p)} \mu^{i_1,j_1,\prime}.
\]

We now state necessary results of weight elimination to our proof of the main results, Theorem 5.7.6 in this paper.

**Conjecture 5.3.1.** Let \( \tau: G_F \to \text{GL}_n(F) \) be a continuous automorphic Galois representation with \( \overline{\tau}_{G_{F_w}} \cong \overline{\theta}_0 \) as in (5.0.1). Fix a pair of integers \((i_0, j_0)\) such that \( 0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1 \), and assume that \( \overline{\theta}_{i_0,j_0} \) is Fontaine–Laffaille generic and that \( \mu^{\square, i_1,j_1} \) is \( 2n \)-generic.

Then we have

\[
   W_w(\tau) \cap \text{JH}(\pi^{i_1,j_1}) \subseteq \{ F(\mu^\square)^\vee, F(\mu^{i_1,j_1})^\vee \}.
\]

Recently, we are informed that Bao V. Le Hung proved Conjecture 1.3.2 completely in his forthcoming paper [LeH]. Therefore, Conjecture 1.3.2 becomes a theorem based on the results in [LeH].

Finally, we prove the automorphy of the Serre weight \( F(\mu^\square)^\vee \).
Proposition 5.3.2. Keep the assumptions and notation of Conjecture 5.3.1. Assume further that if $n$ is even then so is $\frac{n[F^+ Q]}{2}$, that $\zeta_p \not\in F$, that $\tau : G_F \to \text{GL}_n(F)$ is an irreducible representation with split ramification, and that there is a RACSDC automorphic representation $\Pi$ of $\text{GL}_n(A_F)$ such that

- $\tau \simeq \tau_\Pi$;
- for each place $w' | p$ of $F$, $r_{\Pi} | G_{F_{w'}}$ is potentially diagonalizable;
- $\tau(G_{F_{(c_i)}})$ is adequate.

Then

$$\{ F(\mu^{\square})^\vee \} \subseteq W_w(\tau) \cap \text{JH}((\pi^{1, 1})^\vee).$$

Proof. We prove that $F(\mu^{\square})^\vee = F(c_{n-1}, c_{n-2}, \cdots, c_0) \in W_w(\tau)$ as well as $F(\mu^{\square})^\vee \in \text{JH}((\pi^{1, 1})^\vee)$.

Note that $(c_{n-1}, \cdots, c_0)$ is in the lowest alcove as $\overline{P}_0$ is generic, so that by Theorem 5.2.6 it is enough to show that $\overline{P}_0$ has a potentially diagonalizable crystalline lift with Hodge–Tate weights $\{c_{n-1} + (n - 1), \cdots, c_1 + 1, c_0\}$. Since $\overline{P}_0$ is generic, by [BLGGT], Lemma 1.4.3 it is enough to show that $\overline{P}_0$ has an ordinary crystalline lift with those Hodge–Tate weights. The existence of such a crystalline lift is immediate by [GHLSS], Proposition 2.1.10. On the other hand, we have $F(\mu^{\square})^\vee \in \text{JH}((\pi^{1, 1})^\vee)$ which is a direct corollary of Theorem 5.6.2. Therefore, we conclude that $F(\mu^{\square})^\vee \in W_w(\tau) \cap \text{JH}((\pi^{1, 1})^\vee)$. □

5.4. Some application of Morita theory. In this section, we will recall standard results from Morita theory to prove Proposition 5.4.3 and Corollary 5.4.5 which will be useful for the proof of Proposition 5.5.12 and Corollary 5.5.14 in the next section. We fix here an arbitrary finite group $H$ and a finite dimensional irreducible $E$-representation $V$ of $H$. By the Proposition 16.16 in [CR90], we know that for any $O_E$-lattice $V^\circ \subseteq V$, the set $\text{JH}_{F[H]}(V^\circ \otimes_{O_E} F)$ depends only on $V$ and is independent of the choice of $V^\circ$, and thus we will use the notation $\text{JH}_{F[H]}(V)$ from now on. Let $C$ be the category of all finitely generated $O_E$-module with a $H$-action which are isomorphic to subquotients of $O_E$-lattice in $V^\oplus_k$ for some $k \geq 1$. The irreducible objects of $C$ are $\sigma \in \text{JH}_{F[H]}(V)$. If $\sigma$ has multiplicity one in $V$, then we use $V^{\sigma}$ to denote a lattice (unique up to homothety by following the proof of Lemma 4.4.1 of [EGS15] as it actually requires only the multiplicity one of $\sigma$ in our notation) with cosocle $\sigma$.

By repeating the proof of Lemma 2.3.1, Lemma 2.3.2 and Proposition 2.3.3 in [Le15], we deduce the following.

Proposition 5.4.1. If $\sigma$ has multiplicity one in $V$, then the lattice $V^{\sigma}$ is a projective object in $C$.

We need to emphasize that the proof of Proposition 2.3.3 in [Le15] requires only the multiplicity one of $\sigma$, although it is necessary for all Jordan–Hölder $\sigma$ to have multiplicity one to have Proposition 2.3.4 in [Le15].

Corollary 5.4.2. Let $\Sigma$ be a subset of $\text{JH}_{F[H]}(V)$ such that each $\sigma \in \Sigma$ has multiplicity one in $V$. If $a O_E$-lattice $V^\circ \subseteq V$ satisfies

\[
(\text{cosoc}_H(V^\circ \otimes_{O_E} F) = \bigoplus_{\sigma \in \Sigma} \sigma
\]

then we have a surjection

\[
(\bigoplus_{\sigma \in \Sigma} V^{\sigma}) \to V^\circ.
\]

Proof. By (5.4.3) we have a surjection

\[
V^\circ \to \bigoplus_{\sigma \in \Sigma} \sigma.
\]

By Proposition 5.4.1 we know that $\bigoplus_{\sigma \in \Sigma} V^{\sigma}$ is a projective object in $C$. By the definition of $V^{\sigma}$ we know that there is a surjection

\[
\bigoplus_{\sigma \in \Sigma} V^{\sigma} \to \bigoplus_{\sigma \in \Sigma} \sigma
\]

which can be lifted by projectiveness to (5.4.4). □
Note in particular that (5.4.4) implies automatically the surjection
\[
\bigoplus_{\sigma \in \Sigma} V^\sigma \otimes \mathcal{O}_E \mathbf{F} \twoheadrightarrow V^\circ \otimes \mathcal{O}_E \mathbf{F}.
\]

**Proposition 5.4.6.** For a given \( \Sigma \) as in Corollary 5.4.2, there are a finite number of lattices (up to homothety) such that (5.4.3) holds. Moreover, if \( V^\circ \) is such a lattice, then we have
\[
\text{Hom}_{\mathcal{O}_E[\mathbb{H}]} \left( \bigoplus_{\sigma \in \Sigma} V^\sigma, V^\circ \right) \cong \bigoplus_{\sigma \in \Sigma} \left( \text{Hom}_{\mathcal{O}_E[\mathbb{H}]}(V^\sigma, V^\circ) \right) \cong \mathcal{O}_E^{[\Sigma]}.
\]

**Proof.** We fix an embedding \( V^\circ \hookrightarrow V \).

By (5.4.4) we have a surjection
\[
\bigoplus_{\sigma \in \Sigma} V^\sigma \twoheadrightarrow V^\circ,
\]
and thus we have the composition
\[
V^\sigma \hookrightarrow \bigoplus_{\sigma \in \Sigma} V^\sigma \twoheadrightarrow V^\circ \hookrightarrow V.
\]
We identify \( V^\sigma \) with its image in \( V \) via this composition, and hence we have
\[
V^\circ = \sum_{\sigma} V^\sigma \subseteq V.
\]
In particular, we have an inclusion
\[
V^\sigma \subseteq V^\circ
\]
for each \( \sigma \in \Sigma \).

If \( V^{\sigma_1} \subseteq V^{\sigma_2} \) for some \( \sigma_1 \neq \sigma_2 \in \Sigma \), then we have
\[
V^\circ = \sum_{\sigma \in \Sigma, \sigma \neq \sigma_1} V^\sigma,
\]
and thus
\[
\text{cosoc}_H(V^\circ \otimes \mathcal{O}_E \mathbf{F}) \hookrightarrow \bigoplus_{\sigma \in \Sigma, \sigma \neq \sigma_1} \sigma
\]
which is a contradiction to (5.4.3). As a result, we deduce that
\[
V^{\sigma_1} \not\subseteq V^{\sigma_2} \text{ for each } \sigma_1 \neq \sigma_2 \in \Sigma.
\]

We notice that for each \( \sigma_1 \neq \sigma_2 \in \Sigma \) and each \( V^{\sigma_1}, V^{\sigma_2} \), there exists an integer \( n \geq 1 \) such that
\[
\varpi_E^n V^{\sigma_1} \subseteq V^{\sigma_2} \subseteq \varpi_E^{-n} V^{\sigma_1}.
\]
We define the set
\[
\mathcal{E} := \{(V^\sigma)_{\sigma \in \Sigma}\} / \sim
\]
where \( V^\sigma \) runs through lattices in \( V \) with cosocle \( \sigma \), and \( \sim \) is the equivalence defined by
\[
(V^\sigma)_{\sigma \in \Sigma} \sim (V^{\sigma'})_{\sigma \in \Sigma} \iff V^{\sigma'} = \varpi_E^n V^\sigma \text{ for all } \sigma \in \Sigma \text{ and some } n \in \mathbb{Z}.
\]
Then we can define
\[
\mathcal{E}' := \{(V^\sigma)_{\sigma \in \Sigma} \in \mathcal{E} \text{ that satisfies (5.4.7)}\}
\]
as the condition (5.4.7) is preserved by the equivalence \( \sim \).

Now we can summarize that there exists a surjective map from the set \( \mathcal{E}' \) to the set of homothety class of lattices \( V^\circ \) satisfying (5.4.3). Therefore we only need to show that the set \( \mathcal{E}' \) is finite. By the equivalence \( \sim \), we only always fix a \( V^{\sigma_0} \) for a fixed element \( \sigma_0 \in \Sigma \) in advance. Then for each \( \sigma \in \Sigma \) such that \( \sigma \neq \sigma_0 \), we have only finite number of choices of \( V^\sigma \) by (5.4.8), and hence \( \mathcal{E}' \) is finite. \(\square\)

If \( \sigma \) has multiplicity one in \( V \), then we use \( V_{\sigma} \) to denote a lattice (unique up to homothety by following the proof of Lemma 4.4.1 of [EGS15]) with socle \( \sigma \).
Remark 5.4.11. Similar to the proof of Proposition 5.4.6, we can define the set
\[ \mathcal{E}'' := \{ (V_{\sigma})_{\sigma \in \Sigma} \text{ such that } V_{\sigma_1} \nsubseteq V_{\sigma_2} \text{ for each } \sigma_1 \neq \sigma_2 \in \Sigma \} / \sim \]
where \( \sim \) is the equivalence defined by simultaneous homothety as in the definition of \( \mathcal{E} \). Then there is a surjective map from the finite set \( \mathcal{E}'' \) to the set of homothety class of lattices \( V^\circ \) satisfying (5.4.10) by sending \( (V_{\sigma})_{\sigma \in \Sigma} \) to \( \bigcap_{\sigma \in \Sigma} V_{\sigma} \).

5.5. Complementary results on the local automorphic side. In this section, we establish further results on the local automorphic side that will be used in the proof of Proposition 5.6.13 and Theorem 5.7.6. One of the main result of this section is Corollary 5.5.14 (or rather Proposition 5.5.12), which will be used in the proof of Theorem 5.7.6 to deduce that certain lattice in a principal series comes from coinduction.

In this section, we will use the notation \( P \) (resp. \( N, L, P^- \cdots \)) for general standard parabolic subgroup (resp. unipotent radical, Levi, opposite parabolic subgroup, \( \cdots \)) as introduced at the beginning of Section 5.

We use our standard notation
\[ \pi = \text{Ind}_{B(F_p)}^{G(F_p)} \mu_\pi \]
for a principal series representation where
\[ \mu_\pi = (d_1, d_2, \cdots, d_n). \]

We will also consider the representation
\[ \pi^L := \text{Ind}_{B(F_p) \cap L(F_p)}^{L(F_p)} \mu_\pi. \]

We note that by the definition of these principal series we have natural surjection of \( L(F_p) \) representation
\[ \pi \mid_{L(F_p)} \twoheadrightarrow \pi \mid_{N(F_p)} \twoheadrightarrow \pi^L \]
where the left side is the \( N(F_p) \)-coinvariant of \( \pi \). We fix a non-zero vector \( v_\pi \in \pi^{U(F_p) \cdot \mu_\pi} \) and denote its image in \( \pi^{\mu_\pi} \) by \( v_\pi^L \).

Lemma 5.5.2. Fix an element \( w \in W^L \). The surjection (5.5.1) maps \( S_{L,F_p} v_\pi \) to \( S_{L,F_p} v_\pi^L \) and induces a bijection between the following two sets
\[ \{ S_{L,F_p} v_\pi \mid k = (k_\alpha)_{\alpha \in \Phi^+} \} \longleftrightarrow \{ S_{L,F_p} v_\pi^L \mid \bar{k} = (k_\alpha)_{\alpha \in \Phi^+_2} \}, \]
where \( S_{L,F_p} \) on the right side is interpreted as an element in \( F_p[L(F_p)] \) and \( S_{L,F_p} \) on the left side is interpreted as an element in \( F_p[G(F_p)] \) which is the image of the Jacobi sum on the right side via the natural embedding \( F_p[L(F_p)] \hookrightarrow F_p[G(F_p)] \).
Proof. We recall from (4.1.10) the decomposition
\[ \pi = \oplus_{w \in W} \pi_w. \]
Similarly, we also have
\[ \pi^L = \oplus_{w \in W^L} \pi^L_w. \]
We also recall from the proof of Proposition 4.1.17 the following decomposition
\[ \pi_w = \oplus_{A \in U_w(F_p)} \pi_{w,A}. \]
and similarly we have
\[ \pi^L_w = \oplus_{A \in U_w(F_p)} \pi^L_{w,A}, \]
where \( \pi^L_{w,A} \) is the subspace of \( \pi^L \) consisting of functions supported in \( B \cap L(F_p)w^{-1}A^{-1} \).
Notice that we have the following equality of set
\[ B(F_p)w^{-1}A^{-1} = B(F_p) \cap L(F_p) \cdot N(F_p)w^{-1}A^{-1} = B(F_p) \cap L(F_p)w^{-1}A^{-1}N(F_p) \]
as both \( w^{-1} \) and \( A^{-1} \) normalize \( N(F_p) \). Hence, by the definition of \( \pi_{w,A} \) and \( \pi^L_{w,A} \), we deduce that the morphism 5.5.1 maps \( \pi_{w,A} \) to \( \pi^L_{w,A} \). Then by the definition of Jacobi sum operators we conclude that 5.5.1 maps \( S_{L,w} \in F_p[L(F_p)] \mapsto F_p[G(F_p)] \)
and a bijection of basis stated in this lemma. \( \square \)

Lemma 5.5.3. For a representation \( V \) of \( G(F_p) \) and a representation \( W \) of \( L(F_p) \), we have the following form of Frobenius reciprocity
\[ \text{Hom}_{L(F_p)}(V_N(F_p), W) = \text{Hom}_{G(F_p)}(V, \text{coInd}^{G(F_p)}_P(F_p)W) \]
where
\[ \text{coInd}^{G(F_p)}_P(F_p)W := (\text{Ind}^{G(F_p)}_P(F_p)W)^\vee. \]
Here, \( (\cdot)^\vee \) is the dual.
Proof. It is easy to prove by chasing the definition:
\[ \text{Hom}_{L(F_p)}(V_N(F_p), W) = \text{Hom}_{L(F_p)}(W^\vee, (V_N(F_p))^\vee) \]
\[ = \text{Hom}_{G(F_p)}(\text{Ind}^{G(F_p)}_P(F_p)W^\vee, V^\vee) \]
\[ = \text{Hom}_{G(F_p)}(V, \text{coInd}^{G(F_p)}_P(F_p)W). \]
This completes the proof. \( \square \)

Remark 5.5.4. In fact, \( \text{coInd}^{G(F_p)}_P(F_p)W \) and \( \text{Ind}^{G(F_p)}_P(F_p)W \) has the same Jordan Holder factors with the same multiplicities. The relation between them is essentially that the graded pieces of the socle filtration of each of them is the graded pieces of the cosocle filtration of the other one. In fact, we also have the identification
\[ \text{coInd}^{G(F_p)}_P(F_p)(\cdot) \cong \text{Ind}^{G(F_p)}_P(F_p)(\cdot). \]
We use the notation \( \text{Inj}_{G(F_p)}(\cdot) \) (resp. \( \text{Inj}_{L(F_p)}(\cdot) \)) for the injective envelop in the category of finite dimensional \( F_p \)-representation of \( G(F_p) \) (resp. \( L(F_p) \)). We will abuse the shorten the notation \( F(\lambda) \otimes_{F_p} F_p \) (resp. \( F^L(\lambda) \otimes_{F_p} F_p \)) to \( F(\lambda) \) (resp. \( F^L(\lambda) \)) in the following Lemma 5.5.5 and Lemma 5.5.6.
**Lemma 5.5.5.** Fix a $\lambda \in X_{1}^{\text{reg}}(T)$. Then there are a surjection
\begin{equation}
\text{Inj}_{G(F_p)}^G F(\lambda) \twoheadrightarrow \text{Ind}_{P(F_p)}^G \left( \text{Inj}_{L(F_p)}^L F^L(\lambda) \right)
\end{equation}
and an injection
\begin{equation}
\text{coInd}_{P(F_p)}^G \left( \text{Inj}_{L(F_p)}^L F^L(\lambda) \right) \hookrightarrow \text{Inj}_{G(F_p)}^G F(\lambda).
\end{equation}

**Proof.** We notice that the injection (5.5.10) is just the dual of the surjection (5.5.9), so that we only need to prove the existence of the surjection (5.5.6).

As $\text{Inj}_{G(F_p)}^G F(\lambda)$ is indecomposable and also the projective with cosocle $F(\lambda)$, we deduce that the existence of a surjection
\[ \text{Inj}_{G(F_p)}^G F(\lambda) \twoheadrightarrow V \]
for a $\overline{F}_p$-representation $V$ of $G(F_p)$ is equivalent to the fact
\begin{equation}
\text{cosoc}_{G(F_p)} V = F(\lambda).
\end{equation}
Now we pick any $\mu \in X_1(T)$. By Frobenius reciprocity we have
\[ \text{Hom}_{G(F_p)}^G \left( \text{Inj}_{P(F_p)}^G \left( \text{Inj}_{L(F_p)}^L F^L(\lambda) \right), F(\mu) \right) = \text{Hom}_{L(F_p)}^L \left( \text{Inj}_{L(F_p)}^L F^L(\lambda), F(\mu)^{N(F_p)} \right). \]
As we know that
\[ \text{cosoc}_{L(F_p)} \left( \text{Inj}_{L(F_p)}^L F^L(\lambda) \right) = F^L(\lambda), \]
we deduce that
\[ \text{Hom}_{L(F_p)}^L \left( \text{Inj}_{L(F_p)}^L F^L(\lambda), F(\mu)^{N(F_p)} \right) \neq 0 \text{ if and only if } F^L(\lambda) \in \text{JH}_{L(F_p)}(F(\mu)^{N(F_p)}). \]

Then by Lemma 2.3 and 2.3 in [Her11], we can identify $F(\mu)^{N(F_p)}$ with $F^L(\mu)$, and hence
\[ \text{Hom}_{L(F_p)}^L \left( \text{Inj}_{L(F_p)}^L F^L(\lambda), F(\mu)^{N(F_p)} \right) \neq 0 \]
implies $\lambda = \mu$. In other word, we have shown that
\[ \text{cosoc}_{G(F_p)} \left( \text{Ind}_{P(F_p)}^G \left( \text{Inj}_{L(F_p)}^L F^L(\lambda) \right) \right) = F(\lambda), \]
and thus we finish the proof by (5.5.8). \hfill \square

**Lemma 5.5.9.** Fix a $\lambda \in X_{1}^{\text{reg}}(T)$. For any finite dimensional $\overline{F}_p$-representation $V$ of $L(F_p)$, if
\[ \text{soc}_{L(F_p)} V = F^L(\lambda), \]
then we have
\[ \text{soc}_{G(F_p)} \left( \text{coInd}_{P(F_p)}^G V \right) = F(\lambda). \]

**Proof.** By assumption we have an injection
\[ V \hookrightarrow \text{Inj}_{L(F_p)}^L F^L(\lambda). \]
By applying the exact functor $\text{coInd}_{P(F_p)}^G$ we deduce
\[ \text{coInd}_{P(F_p)}^G V \hookrightarrow \text{coInd}_{P(F_p)}^G \left( \text{Inj}_{L(F_p)}^L F^L(\lambda) \right). \]
We finish the proof by the second part of Lemma 5.5.5 and by observing $\text{soc}_{G(F_p)} \left( \text{Ind}_{P(F_p)}^G F(\lambda) \right) = F(\lambda).$ \hfill \square

**Remark 5.5.10.** Of course, we have a similar statement for the cosocle of an induction, which is just the dual statement of Lemma 5.5.9.
Remark 5.5.11. In the statement of Lemma 5.5.3 and Lemma 5.5.7 the coefficient of each representation is $\bar{F}_p$. In our future application, as long as the representation $V$ in Lemma 5.5.9 is given, we can fix a sufficiently large finite extension $F$ of $F_p$ such that the two equalities in Lemma 5.5.9 are defined over $F$.

We consider a principal series $\pi$ and together with the characteristic 0 principal series $\bar{\pi} := \text{Ind}_{B(F_p)}^{G(F_p)}(\bar{\mu}_\pi)$, where $\bar{\mu}_\pi$ is the Teichmüller lift of $\mu_\pi$. Here $\bar{\pi}$ is a $Q_p$-representation of $G(F_p)$ by definition. We use the notation $\bar{\pi}^\circ$ for a lattice in $\bar{\pi}$, which is a $Z_p$-subrepresentation of $\bar{\pi}$ such that

$$\bar{\pi}^\circ \otimes_{Z_p} Q_p = \bar{\pi}.$$ 

We also introduce similar notation $\bar{\pi}^L$ and $(\bar{\pi}^L)^\circ$ by replacing $\pi$ by $\bar{\pi}^L$.

**Proposition 5.5.12.** Let $\Sigma$ be a subset of $\text{JH}_{G(F_p)}(\pi)$. Assume that $F(\lambda)$ has multiplicity one in $\pi$ for each $F(\lambda) \in \Sigma$. Assume further that

$$F(\lambda) \in \text{JH}_{L(F_p)}(\pi^L)$$

for all $\lambda$ satisfying $F(\lambda) \in \Sigma$.

If a lattice $\bar{\pi}^\circ$ satisfies

$$\text{cosoc}_{G(F_p)}(\bar{\pi}^\circ \otimes_{Z_p} F_p) = \bigoplus_{F(\lambda) \in \Sigma} F(\lambda),$$

then there exists a lattice $(\bar{\pi}^L)^\circ$ of $\bar{\pi}^L$ such that

$$\bar{\pi}^\circ = \text{Ind}_{P(F_p)}^{G(F_p)}((\bar{\pi}^L)^\circ).$$

Moreover, we have

$$\text{cosoc}_{L(F_p)}((\bar{\pi}^L)^\circ \otimes_{Z_p} F_p) = \bigoplus_{F(\lambda) \in \Sigma} F(\lambda).$$

**Proof.** We will continue to use the notation in Proposition 5.4.6. Hence $\bar{\pi}^{F(\lambda)}$ is a lattice in $\bar{\pi}$ with cosocle $F(\lambda)$. We use the notation $\mathcal{E}'_\pi$ for the set $\mathcal{E}'$ defined in Proposition 5.4.6 if we replace $V$ by $\bar{\pi}$. By Proposition 5.4.10 we deduce the existence of an element $(\bar{\pi}^{F(\lambda)})|_{F(\lambda) \in \Sigma} \in \mathcal{E}'_\pi$ such that

$$\bar{\pi}^\circ = \sum_{F(\lambda) \in \Sigma} \bar{\pi}^{F(\lambda)} \subseteq \bar{\pi}.$$

On the other hand, as $F(\lambda)$ has multiplicity one in $\pi$, $F^L(\lambda)$ must have multiplicity one in $\pi^L$, and thus we have a unique (up to homothety) lattice $(\bar{\pi}^L)^{F^L(\lambda)}$ in $\bar{\pi}^L$ with cosocle $F^L(\lambda)$. Now we consider the lattice

$$\text{Ind}_{P(F_p)}^{G(F_p)}((\bar{\pi}^L)^{F^L(\lambda)}).$$

By applying Remark 5.5.10 to $\left(\text{Ind}_{P(F_p)}^{G(F_p)}((\bar{\pi}^L)^{F^L(\lambda)}) \otimes_{O_E} F\right)$ we deduce that

$$\text{cosoc}_{G(F_p)}\left(\text{Ind}_{P(F_p)}^{G(F_p)}((\bar{\pi}^L)^{F^L(\lambda)})\right) = F(\lambda).$$

Hence by the uniqueness of $\bar{\pi}^{F(\lambda)}$ up to homothety, we conclude the existence of $(\bar{\pi}^L)^{F^L(\lambda)}$ satisfying

$$(\bar{\pi}^L)^{F^L(\lambda)} = \text{Ind}_{P(F_p)}^{G(F_p)}((\bar{\pi}^L)^{F^L(\lambda)}).$$

for a given lattice $\bar{\pi}^{F^L(\lambda)}$.

Therefore for each element $(\bar{\pi}^{F(\lambda)})|_{F(\lambda) \in \Sigma} \in \mathcal{E}'_\pi$, there exists an element $((\bar{\pi}^L)^{F^L(\lambda)})|_{F(\lambda) \in \Sigma} \in \mathcal{E}'_{\bar{\pi}^L}$ such that (5.5.13) holds for all $F(\lambda) \in \Sigma$, where $\mathcal{E}'_{\bar{\pi}^L}$ is the finite set defined in Proposition 5.4.6 if we replace $V$ by $\bar{\pi}^L$. 

Finally, by the exactness of the functor $\text{Ind}_{P(F_p)}^{G(F_p)}$, we deduce the equality
\[
\tilde{\pi}^o = \sum_{F(\lambda) \in \Sigma} \tilde{\pi}^F(\lambda) = \sum_{F(\lambda) \in \Sigma} \left( \text{Ind}_{P(F_p)}^{G(F_p)}(\tilde{\pi}^L)^F(\lambda) \right) = \text{Ind}_{P(F_p)}^{G(F_p)} \left( \sum_{F(\lambda) \in \Sigma} (\tilde{\pi}^L)^F(\lambda) \right).
\]
Hence, letting
\[
(\tilde{\pi}^L)^o := \sum_{F(\lambda) \in \Sigma} (\tilde{\pi}^L)^F(\lambda)
\]
completes the proof. \qedhere

**Corollary 5.5.14.** Keep the notation and the assumption of Proposition 5.5.12.

If a lattice $\tilde{\pi}^o$ satisfies
\[
\text{soc}_{G(F_p)}(\tilde{\pi}^o \otimes_{\mathbb{Z}_p} F_p) = \bigoplus_{F(\lambda) \in \Sigma} F(\lambda),
\]
then there exists a lattice $(\tilde{\pi}^L)^o$ of $\tilde{\pi}^L$ such that
\[
\tilde{\pi}^o = \text{coInd}_{P(F_p)}^{G(F_p)}(\tilde{\pi}^L)^o.
\]
Moreover, we have
\[
\text{soc}_{L(F_p)}((\tilde{\pi}^L)^o \otimes_{\mathbb{Z}_p} F_p) = \bigoplus_{F(\lambda) \in \Sigma} F^L(\lambda).
\]

**Proof.** This is just the dual version of Proposition 5.5.12. \qedhere

**Lemma 5.5.15.** Let $H$ be an arbitrary finite group. The $p$-adic field $E$ is sufficiently large such that all irreducible representation of $H$ over $\mathbb{Q}_p$ are defined over $E$.

If we have an injection $V^o \hookrightarrow W^o$ of finite rank $\mathcal{O}_E$-representations of $H$, then the induced morphism
\[
(5.5.16)\quad V^o \otimes_{\mathcal{O}_E} F \rightarrow W^o \otimes_{\mathcal{O}_E} F
\]
is injective if and only if
\[
(5.5.17)\quad V^o = V^o \otimes_{\mathcal{O}_E} E \cap W^o \hookrightarrow W^o \otimes_{\mathcal{O}_E} E
\]
is.

Note that the injection $V^o \hookrightarrow W^o$ always induces a natural injection
\[
V^o \otimes_{\mathcal{O}_E} E \hookrightarrow W^o \otimes_{\mathcal{O}_E} E
\]
by the flatness of $E$ over $\mathcal{O}_E$.

**Proof.** We notice that (5.5.17) holds if and only if $W^o/V^o$ is $\varpi_E$-torsion free. On the other hand, by tensoring
\[
V^o \hookrightarrow W^o \rightarrow W^o/V^o
\]
with $E$, we deduce that the torsion free part of $W^o/V^o$ has rank the dimension of $E$-space $W^o \otimes_{\mathcal{O}_E} E/V^o \otimes_{\mathcal{O}_E} E$.

As we have
\[
\dim_F(\text{Ker}(V^o \otimes_{\mathcal{O}_E} F \rightarrow W^o \otimes_{\mathcal{O}_E} F))
\]
\[
= \dim_F(V^o \otimes_{\mathcal{O}_E} E) + \dim_F(\mathcal{O}_E) - \dim_F(W^o \otimes_{\mathcal{O}_E} E)
\]
\[
= \dim_E(\mathcal{O}_E) + \dim_E(\mathcal{O}_E) + \dim_E((W^o/V^o)^{\text{tor}} \otimes_{\mathcal{O}_E} E) - \dim_F(W^o \otimes_{\mathcal{O}_E} E)
\]
\[
= \dim_F((W^o/V^o)^{\text{tor}} \otimes_{\mathcal{O}_E} F)
\]
where $(W^o/V^o)^{\text{tor}}$ is the $\varpi_E$-torsion part of $W^o/V^o$, (5.5.16) is injective if and only if
\[
(W^o/V^o)^{\text{tor}} = 0
\]
or equivalently \( (5.5.14) \) holds.

5.6. Generalization of Section 4.2. In this section, we fix a pair of integers \((i_0, j_0)\) satisfying 
\[ 0 \leq j_0 < y_0 + 1 < i_0 \leq n - 1, \]
and determine \((i_1, j_1)\) by the equation \( (5.0.3) \). We will use the shorten notation \( P \) (resp. \( N, L, P^- \cdots \)) for \( P_{i_1, j_1} \) (resp. \( N_{i_1, j_1}, L_{i_1, j_1}, P^-_{i_1, j_1}, \cdots \)) as introduced at the beginning of Section 4. The main target of this section is to prove Proposition 5.6.13 using Corollary 4.4.9 and results in Section 5.5. Proposition 5.6.13 is crucial for the proof of Theorem 5.7.6.

Recall \( S_n, S_n'( \text{cf. } (4.2.4)\)) whose definitions are completely determined by fixing the data \( n \) and \((a_{n-1}, \cdots, a_0)\). We define \( S_{i_1, j_1}, S'_{i_1, j_1} \in F_p[GL_{i_1-1,j_0+1}(F_p)] \) by replacing \( n \) and \((a_{n-1}, \cdots, a_0)\) by
\[ j_1 - i_1 + 1 \quad \text{and} \quad (b_j + j_1 - i_1 - 1, b_{j_1 - 1}, \cdots, b_{i_1 - 1}, b_{i_1} - j_1 + i_1 + 1) \]
respectively with \( b_k \) as at the beginning of Section 5.3. Via the natural embedding \( (5.0.4) \) we also define \( S_{i_1, j_1}^{\prime} \) (resp. \( S'_{i_1, j_1}^{\prime} \)) to be the image of \( S_{i_1, j_1} \) (resp. of \( S'_{i_1, j_1} \)) in \( F_p[G(F_p)] \). More precisely, we have
\[ S_{i_1, j_1} := S_{\ell_{i_1, j_1}, W_{i_1, j_1}} \quad \text{and} \quad S_{i_1, j_1}^{\prime} := S_{\ell_{i_1, j_1}^{\prime}, W_{i_1, j_1}^{\prime}} \]
where \( \ell_{i_1, j_1} := (k_{i_1, j_1}^{i_1, j_1})_{i, j} \in \{0, \cdots, p-1\}_{\Phi_{W_{i_1, j_1}^{i_1, j_1}}} \) and \( \ell_{i_1, j_1}^{i_1, j_1} := (k_{i_1, j_1}^{i_1, j_1})_{i, j} \in \{0, \cdots, p-1\}_{\Phi_{W_{i_1, j_1}^{i_1, j_1}'}}, \)
defined by
\[ k_{i_1, j_1}^{i_1, j_1} := \begin{cases} [b_i - b_{i-1}] & \text{if } n - j_1 + 1 \leq i - j_1 \leq n - i_1 - 1; \\ i_1 - j_1 + 1 + [b_i - b_{j_1}] & \text{if } i = j - 1 = n - j_1; \\ 0 & \text{if } j \geq i + 2. \end{cases} \]
and
\[ k_{i_1, j_1}^{i_1, j_1} := \begin{cases} [b_{n-1-i} - b_{j_1}] & \text{if } n - j_1 \leq i = j - 1 \leq n - i_1 - 2; \\ j_1 - 1 + [b_i - b_{j_1}] & \text{if } i = j = n - i_1 - 1; \\ 0 & \text{if } j \geq i + 2. \end{cases} \]
We will also need the tuple \( \ell_{i_1, j_1}^{i_1, j_1, 0} = (k_{i_1, j_1}^{i_1, j_1, 0})_{i, j} \in \{0, \cdots, p-1\}_{\Phi_{W_{i_1, j_1}^{i_1, j_1} 0}} \),
defined by
\[ k_{i_1, j_1}^{i_1, j_1, 0} := \begin{cases} i_1 - j_1 + 1 + [b_i - b_{j_1}] & \text{if } n - j_1 \leq i = j - 1 \leq n - i_1 - 1; \\ 0 & \text{if } j \geq i + 2. \end{cases} \]
We state here a generalization of the Theorem 4.2.0

**Theorem 5.6.2.** Assume that \( \mu \subseteq i_1, j_1 \) is 2n-generic (cf. Definition 4.4.1). Then the constituent \( F(\mu^{\square}) \) has multiplicity one in \( \pi(i_1, j_1) \) (or equivalently in \( \pi(i_1, j_1') \)).

**Proof.** This is Corollary 4.4.9 if we replace \( \mu^{i_1, j_1} \) by \( \mu^{\square} \).
We define $V^{i_1,j_1}_1$ to be the subrepresentation of $\pi^{i_1,j_1}$ generated by $S^{i_1,j_1} \left( (\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) \right)$. Similarly, we define $V^{i_1,j_1}_0$ and $V^{i_1,j_1}_0$ to be the subrepresentations of $\pi^{i_1,j_1}$ and $\pi^{i_1,j_1}$ generated by $S^{i_1,j_1} \left( (\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) \right)$ and $S^{i_1,j_1} \left( (\pi^{i_1,j_1} U(F_p), (\mu^{i_1,j_1})^{w_0}) \right)$ respectively.

**Lemma 5.6.3.** Assume that $\mu^{i_1,j_1}$ is $\mu$-generic in the lowest alcove. Then we have

$$\dim_{F_p} ((\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}))(\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) = 1$$

and

$$S^{i_1,j_1} \left( (\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) \right) = S^{i_1,j_1} \left( (\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) \right) \neq 0.$$

**Proof.** By direct generalization of arguments in Lemma 4.3.10 and Proposition 4.5.11 we can deduce that

$$(5.6.4) \quad \mathcal{T}^{i_1,j_1}_{w^{i_1,j_1}}(S^{i_1,j_1} \left( (\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) \right)) = \mathcal{T}^{i_1,j_1}_{w^{i_1,j_1}}(S^{i_1,j_1} \left( (\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) \right)) = S^{\mu^{i_1,j_1}, w_0^{i_1,j_1}} \left( (\pi^{i_1,j_1} U(F_p), (\mu^{i_1,j_1})^{w_0}) \right) .$$

In other words, we have the surjections

$$(5.6.5) \quad V^{i_1,j_1}_1 \twoheadrightarrow V^{i_1,j_1}_0 \text{ and } V^{i_1,j_1}_1 \twoheadrightarrow V^{i_1,j_1}_0.$$

To lighten the notation, we pick a vector $v \in (\pi^{i_1,j_1} U(F_p), (\mu^{i_1,j_1})^{w_0})$. By Lemma 4.1.15 we can deduce that

$$S^{\mu^{i_1,j_1}, w_0^{i_1,j_1}} = X \cdot S^{\mu^{i_1,j_1}, w_0^{i_1,j_1}}$$

for some $X \in F_p[U(F_p)]$, and thus

$$(5.6.6) \quad S^{\mu^{i_1,j_1}, w_0^{i_1,j_1}} v \in V^{i_1,j_1}_0 .$$

On the other hand, by Lemma 4.1.19 we know that

$$F_p[S^{\mu^{i_1,j_1}, w_0^{i_1,j_1}} v] = (\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) ,$$

and thus by Frobenius reciprocity we have a non-zero morphism

$$\mathcal{T}^{i_1,j_1}_{w_0^{i_1,j_1}} : \pi^{i_1,j_1} \rightarrow \pi^{i_1,j_1}$$

for $\pi^{i_1,j_1} := \text{Ind}_{B(H)}^{G(F_p)} \mu^{i_1,j_1}$. such that

$$F_p[S^{\mu^{i_1,j_1}, w_0^{i_1,j_1}} v] = \mathcal{T}^{i_1,j_1}_{w_0^{i_1,j_1}} \left( (\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) \right) .$$

By (4.1.4), we know that

$$\mathcal{T}^{i_1,j_1}_{w_0^{i_1,j_1}} \cdot \mathcal{T}^{i_1,j_1}_{w_0^{i_1,j_1}} = c \mathcal{T}^{i_1,j_1}_{w_0^{i_1,j_1}}$$

for some $c \in F_p$ and thus

$$(5.6.7) \quad F_p \left[ \mathcal{T}^{i_1,j_1}_{w_0^{i_1,j_1}} \left( S^{\mu^{i_1,j_1}, w_0^{i_1,j_1}} v \right) \right] = (\pi^{i_1,j_1} U(F_p), \mu^{i_1,j_1})$$

by (4.1.3) applied to $\mathcal{T}^{i_1,j_1}_{w_0^{i_1,j_1}} v$. Combining (5.6.7) and (5.6.6) we deduce that

$$\mathcal{T}^{i_1,j_1}_{w_0^{i_1,j_1}} \left( V^{i_1,j_1}_1 \right) \neq 0 \text{ or equivalently } \mathcal{T}^{i_1,j_1}_{w_0^{i_1,j_1}} \left( S^{\mu^{i_1,j_1}, w_0^{i_1,j_1}} v \right) \neq 0.$$
We finish the proof by the following observation
\[ S^{i_1,j_1} \left( (\pi^{i_1,j_1}) U(F_p), \mu^{i_1,j_1} \right) = S^{i_1,j_1} \left( \mathbb{T}_{w_0, w_0} \left( (\pi^{i_1,j_1}) U(F_p), \mu^{i_1,j_1} \right) \right) \]
\[ = L \mathbb{T}_{w_0, w_0} \left( S^{i_1,j_1} \left( (\pi^{i_1,j_1}) U(F_p), \mu^{i_1,j_1} \right) \right) \]
\[ = L \left[ \mathbb{T}_{w_0, w_0} \left( S_{\mathbb{A}_{L}^{i_1,j_1},0, w_0} - \mu^{i_1,j_1} \right) \right] \]
and a similar observation for \( S^{i_1,j_1,j'} \left( (\pi^{i_1,j_1}) U(F_p), \mu^{i_1,j_1,j'} \right) \).

We recall the notation \( F^L(\lambda) \) from the beginning of Section 5.6. We define the representation
\[ \pi_{i_1,j_1,0} := \text{Ind}_{B(F_p)}^{L(F_p)} \left( \mu^{i_1,j_1} \right) \]
and also define \( \mathcal{V}^{i_1,j_1} \) (resp. \( \mathcal{V}^{i_1,j_1,j'} \)) to be the unique (up to isomorphism) quotient of \( \pi_{i_1,j_1} \) (resp. \( \pi_{i_1,j_1,j'} \)) with socle \( F(\mu) \), whose existence is ensured by Theorem 5.6.2.

Note that \( \mu^{i_1,j_1} \) is a permutation of both \( \mu^{i_1,j_1} \) and \( \mu^{i_1,j_1,j'} \) and thus \( F(\mu^{i_1,j_1}) \) has multiplicity one in both \( \pi_{i_1,j_1} \) and \( \pi_{i_1,j_1,j'} \). We define \( V^{i_1,j_1} \) (resp. \( V^{i_1,j_1,j'} \)) as the unique (up to isomorphism) quotient of \( \pi_{i_1,j_1} \) (resp. \( \pi_{i_1,j_1,j'} \)) with socle \( F(\mu^{i_1,j_1}) \).

**Lemma 5.6.8.** Assume that \( \mu \) is 3n-generic in the lowest alcove. Then we have
\[ 0 \neq S^{i_1-j,1} \left( (\mathcal{V}^{i_1,j_1}) U(F_p), \mu^{i_1,j_1} \right) \subseteq \mathcal{V}^{i_1,j_1} \]
and
\[ 0 \neq S^{i_1,j_1,j'} \left( (\mathcal{V}^{i_1,j_1,j'}) U(F_p), \mu^{i_1,j_1,j'} \right) \subseteq \mathcal{V}^{i_1,j_1,j'} \]
We also have
\[ F(\mu) \in \text{JH}(V^{i_1,j_1}) \cap \text{JH}(V^{i_1,j_1,j'}). \]

**Proof.** By the same argument as in the proof of Corollary 4.2.7, we only need to show the inclusion
\[ F(\mu) \in \text{JH}(V^{i_1,j_1}) \cap \text{JH}(V^{i_1,j_1,j'}). \]
By 5.6.5 we only need to show that
\[ F(\mu) \in \text{JH}(V_0^{i_1,j_1}). \]
To lighten the notation, we fix a vector \( v \in (\pi^{i_1,j_1}) U(F_p), \mu^{i_1,j_1} \) and denote its image under the composition
\[ \pi^{i_1,j_1} \to (\pi^{i_1,j_1}) N(F_p) \to \pi_{i_1,j_1}, \]
by \( v^L \). We recall the definition of the tuple \( L^{i_1,j_1,0} \) from 5.6.4. We define \( V^{i_1,j_1,0} \) to be the subrepresentation of \( \pi_{i_1,j_1,0} \) generated by \( S_{L^{i_1,j_1,0}, w_0} v^L \). By Lemma 5.6.2 we know that the vector \( S_{L^{i_1,j_1,0}, w_0} v^L \) is sent to \( S_{L^{i_1,j_1,0}, w_0} v^L \) under the composition 5.6.11, and thus we have natural surjections
\[ V^{i_1,j_1,0} \mid L(F_p) \to (V^{i_1,j_1,0}) N(F_p) \to V^{i_1,j_1,0}. \]
On the other hand, by replacing \( (a_{n-1}, \cdots, a_1, a_0) \) with \( (b_{j_1}, j_1), (b_{j_1-1}, \cdots, b_{j_1+1}, b_{j_1-j_1+i+1}) \) in Theorem 4.6.39 we have an inclusion
\[ F^L(\mu) \in \text{JH}_L(F_p)(V^{i_1,j_1,0}). \]
We use the notation \( V \) to denote the unique quotient of \( V^{i_1,j_1,0} \) with \( L(F_p) \)-socle \( F^L(\mu) \), and hence we have a surjection
\[ (V^{i_1,j_1,0}) N(F_p) \to V. \]
By Lemma 5.5.3 this gives a non-zero morphism
\[ V^{i_1,j_1} \to \text{coInd}_{L(F_p)}^{G(F_p)}(V). \]
Now by Lemma 5.6.9 we know that
\[ \text{soc}_{\text{G}(F_p)} \left( \text{coInd}_{\text{M}(F_p)}^{G(F_p)} V \right) = F(\mu^{\square}). \]

As the morphism 5.6.12 is non-zero, we thus deduce 5.6.10.

By definition of \( V^{i_{1},j_{1}}, V^{i_{1},j_{1},'} \) and \( V_{0}^{i_{1},j_{1}} \), we notice that we have inclusions
\[ V^{i_{1},j_{1}} \subseteq V^{i_{1},j_{1}} \text{ and } V_{0}^{i_{1},j_{1}} \subseteq V^{i_{1},j_{1}}, \]
and thus 5.6.9 also follows from 5.6.10. \( \square \)

**Proposition 5.6.13.** Let \( (\pi^{i_{1},j_{1}})^{\circ} \) be a lattice in \( \tilde{\pi}^{i_{1},j_{1}} \) satisfying
\[ \text{soc}_{\text{G}(F_p)} \left( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F \right) \hookrightarrow F(\mu^{\square}) \oplus F(\mu^{\square,i_{1},j_{1}}). \]

Then we have
\[ \dim_{F}(\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F)_{\mu^{i_{1},j_{1}}}^{U(F_p)} ; \quad \dim_{F}(\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F)_{\mu^{i_{1},j_{1},'} ;}^{U(F_p)} = 1 \]
and
\[ \mathcal{S}_{1}^{i_{1},j_{1}} \left( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F)_{U(F_p),\mu^{i_{1},j_{1}}}^{\circ,} \right) = \mathcal{S}_{1}^{i_{1},j_{1},'} \left( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F)_{U(F_p),\mu^{i_{1},j_{1},'} ;} \right) \neq 0. \]

**Proof.** By Bruhat decomposition 4.4.1, we have
\[ \dim_{E}(\pi^{i_{1},j_{1}})^{U(F_p),\mu^{i_{1},j_{1}}} = \dim_{E}(\pi^{i_{1},j_{1}})^{U(F_p),\mu^{i_{1},j_{1},'} ;} = 1 \]
Therefore by taking the intersection of \( (\pi^{i_{1},j_{1}})^{\circ} \) with the two one dimension \( E \)-spaces above and then taking reduction mod \( \mathfrak{m}, \) we have that
\[ \dim_{F}(\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F)_{\mu^{i_{1},j_{1}}} = 1 \quad \text{and} \quad \dim_{F}(\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F)_{\mu^{i_{1},j_{1},'} ;} = 1. \]

Then by Frobenius reciprocity and the fact that \( F(\mu^{i_{1},j_{1}}) \) and \( F(\mu^{i_{1},j_{1},'} ;) \) have multiplicity one in \( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F, \) we can deduce the equality (5.6.14).

If \( \text{soc}_{\text{G}(F_p)} \left( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F \right) \) is \( F(\mu^{\square,i_{1},j_{1}}), \) then (5.6.15) reduce to Lemma 5.6.9 Hence we may assume
\[ F(\mu^{\square}) \hookrightarrow \text{soc}_{\text{G}(F_p)} \left( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F \right) \hookrightarrow F(\mu^{\square}) \oplus F(\mu^{\square,i_{1},j_{1}}). \]

from now on.

As \( F(\mu^{i_{1},j_{1}}), F(\mu^{i_{1},j_{1},'} ;) \) and \( F(\mu^{i_{1},j_{1},'} ;) \) have multiplicity one in \( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F, \) we can define \( V^{\square,i_{1},j_{1}} \) (resp. \( V^{\square,i_{1},j_{1},'} ;) \) to be the unique subquotient of \( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F \) with cosocle \( F(\mu^{i_{1},j_{1}}) \) (resp. \( F(\mu^{i_{1},j_{1},'} ;) \)) and socle \( F(\mu^{i_{1},j_{1},'} ;) \) if either of them exists and zero otherwise. By Frobenius reciprocity, we have surjections
\[ \pi^{i_{1},j_{1}} \rightarrow V^{\square,i_{1},j_{1}} \quad \text{and} \quad \pi^{i_{1},j_{1},'} ; \rightarrow V^{\square,i_{1},j_{1},'} ;. \]

If \( V^{\square,i_{1},j_{1}} \) is non-zero, then by the definition of \( V^{i_{1},j_{1}} \) before Lemma 5.6.8 we deduce that
\[ V^{\square,i_{1},j_{1}} \cong V^{i_{1},j_{1}} \]
and thus by 5.6.9, we have
\[ F(\mu^{\square}) \in \text{JH}(V^{\square,i_{1},j_{1}}) \]
which contradicts the fact \( F(\mu^{\square}) \) has multiplicity one in \( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F \) and actually lies in the socle of \( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F. \) This contradiction means that we have
\[ V^{\square,i_{1},j_{1}} = 0. \]

Similarly, one can show that
\[ V^{\square,i_{1},j_{1},'} = 0. \]

As \( F(\mu^{i_{1},j_{1}}), F(\mu^{i_{1},j_{1},'} ;) \) and \( F(\mu^{\square}) \) have multiplicity one in \( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F, \) we can define \( V^{\square} \) (resp. \( V^{\square'} ;) \) to be the unique subquotient of \( (\pi^{i_{1},j_{1}})^{\circ} \otimes_{E} F \) with cosocle \( F(\mu^{i_{1},j_{1}}) \) (resp. \( F(\mu^{i_{1},j_{1},'} ;) \)) and socle \( F(\mu^{\square}) \). By (5.6.10) and the vanishing of \( V^{\square,i_{1},j_{1}} \) and \( V^{\square,i_{1},j_{1},'} ;, \) we deduce that
\[ V^{\square} \neq 0 \quad \text{and} \quad V^{\square'} ; \neq 0, \]
and that both $V^\square$ and $V^\square,\!\!j$ are actually subrepresentation of $(\bar{\pi}^i_{1, j_1})^0 \otimes O_E F$.

In fact, we obviously have the isomorphism

\[(5.6.17) \quad (V^\square)^U(F_p,\!\!j,\!\!i^{1, j_1}) \cong (\bar{\pi}^i_{1, j_1})^0 \otimes O_E F)^U(F_p,\!\!i^{1, j_1})\]

and

\[(5.6.18) \quad (V^\square,\!\!j)^U(F_p,\!\!i^{1, j_1, j_1'}) \cong (\bar{\pi}^i_{1, j_1})^0 \otimes O_E F)^U(F_p,\!\!i^{1, j_1, j_1'}).\]

By Frobenius reciprocity again, we have surjections

$\pi^i_{1, j_1} \rightarrow V^\square$ and $\pi^i_{1, j_1, j_1'} \rightarrow V^\square,\!\!j$

and thus we deduce the isomorphisms $V^\square \cong V^i_{1, j_1}$ and $V^\square,\!\!j \cong V^i_{1, j_1}$

by the definition of $V^i_{1, j_1}$ and $V^i_{1, j_1}$ before Lemma \[5.6.8\]. Therefore we can deduce

\[(5.6.19) \quad S^i_{1, j_1} \left( (V^\square)^U(F_p,\!\!i^{1, j_1}) \right) \neq 0 \quad \text{and} \quad S^i_{1, j_1, j_1'} \left( (V^\square,\!\!j)^U(F_p,\!\!i^{1, j_1, j_1'}) \right) \neq 0\]

from the first part of Lemma \[5.6.8\].

By Corollary \[5.4.9\] we deduce the existence of two lattices $\bar{\pi}^i_{1, j_1}$ and $\bar{\pi}^i_{1, j_1}$ in $\bar{\pi}^i_{1, j_1}$ such that

\[
\begin{align*}
\text{soc}_G(F_p) \left( \frac{\bar{\pi}^i_{1, j_1}}{\mu_{\square, i_{1, j_1}}} \right) \otimes O_E F & = F(\mu_{\square}); \\
\text{soc}_G(F_p) \left( \frac{\bar{\pi}^i_{1, j_1}}{\mu_{\square, i_{1, j_1}}} \right) \otimes O_E F & = F(\mu_{\square, i_{1, j_1}}); \\
(\bar{\pi}^i_{1, j_1})^0 & = \frac{\bar{\pi}^i_{1, j_1}}{F(\mu_{\square, i_{1, j_1}}) \cap \bar{\pi}^i_{1, j_1}} \subseteq \bar{\pi}^i_{1, j_1}.
\end{align*}
\]

Note in particular that we have an isomorphism

\[(5.6.20) \quad \bar{\pi}^i_{1, j_1} \otimes O_E F \cong \pi^i_{1, j_1}\]

by the uniqueness (up to homothety) of lattices with socle $F(\mu_{\square, i_{1, j_1}})$.

The inclusion $(\bar{\pi}^i_{1, j_1})^0 \hookrightarrow \bar{\pi}^i_{1, j_1}$ induce a non-zero morphism

\[(5.6.21) \quad (\bar{\pi}^i_{1, j_1})^0 \otimes O_E F \hookrightarrow \bar{\pi}^i_{1, j_1} \otimes O_E F\]

as \[(5.6.22)\] sends $F(\mu_{\square})$ in the left side into the socle of the right side by the proof of Corollary \[5.4.9\] and Proposition \[5.4.6\]. By the definition of $V^\square$ and $V^\square,\!\!j$, both of them are sent injectively into $\bar{\pi}^i_{1, j_1} \otimes O_E F$ by \[(5.6.21)\].

On the other hand, there exist a unique integer $k \geq 1$ such that

\[
\varpi^k_{E, F(\mu_{\square, i_{1, j_1}})} \subseteq \bar{\pi}^i_{1, j_1} \quad \text{and} \quad \varpi^k_{E, F(\mu_{\square, i_{1, j_1}})} \not\subseteq \varpi^k_{E, \bar{\pi}^i_{1, j_1}}.
\]

The inclusion $\varpi^k_{E, F(\mu_{\square, i_{1, j_1}})} \hookrightarrow \bar{\pi}^i_{1, j_1}$ induces a non-zero morphism

\[(5.6.22) \quad \varpi^k_{E, F(\mu_{\square, i_{1, j_1}})} \otimes O_E F \subseteq \bar{\pi}^i_{1, j_1} \otimes O_E F\]

as we have $\varpi^k_{E, F(\mu_{\square, i_{1, j_1}})} \not\subseteq \varpi^k_{E, \bar{\pi}^i_{1, j_1}}$.

By \[(5.6.23)\], the image of \[(5.6.22)\] can be identified with the unique quotient of $\pi^i_{1, j_1}$ with socle $F(\mu_{\square})$, which will be denoted by $V^i_{1, j_1}$. Then by \[5.6.22\] and the definition of $V^i_{1, j_1}$ and $V^i_{1, j_1, j_1'}$, we deduce that

\[
(5.6.23) \quad F(\mu_{\square, i_{1, j_1}}), F(\mu_{\square, i_{1, j_1, j_1'}}) \in \text{JH}(V^i_{1, j_1}),
\]

with multiplicity one, and thus we have the embeddings

\[(5.6.24) \quad V^\square \hookrightarrow V^i_{1, j_1} \quad \text{and} \quad V^\square,\!\!j \hookrightarrow V^i_{1, j_1}\]

by the definition of $V^\square$ and $V^\square,\!\!j$. 
As $V_{*}^{i_1,j_1}$ is a quotient of $\pi_{*}^{i_1,j_1}$, we deduce
\[ \dim_F(V_{*}^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) = \dim_F(V_{*}^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) = 1 \]
from
\[ \dim_F(\pi_{*}^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) = \dim_F(\pi_{*}^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) = 1 \]
and \eqref{5.6.24} together with Frobenius reciprocity.

Then the embeddings \eqref{5.6.24} induce the isomorphisms
\[ (V^\square)^* U(F_p), \mu^{i_1,j_1} \sim (V^\square)^* U(F_p), \mu^{i_1,j_1} \text{ and } (V^\square)^* U(F_p), \mu^{i_1,j_1} \sim (V^\square)^* U(F_p), \mu^{i_1,j_1} \].

Then by Lemma \ref{5.6.3} we can deduce that
\[ S^{i_1,j_1}(V_{*}^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) = S^{i_1,j_1}(V_{*}^{i_1,j_1} U(F_p), \mu^{i_1,j_1}) \leq (\pi_{i_1,j_1})^\circ \otimes_{\mathcal{O}_E} F \]
which finishes the proof of \eqref{5.6.15} by applying \eqref{5.6.19}, \eqref{5.6.17}, and \eqref{5.6.18}.

\[ \square \]

5.7. Main results. In this section, we state and prove our main results on mod $p$ local-global compatibility. Throughout this section, \mathcal{P}_0 is always assumed to be a restriction of a global representation $\mathfrak{T} : G_F \to \text{GL}_n(F)$ to $G_{F_p}$ for a fixed place $w$ of $F$ above $p$. Let $v := w|_p$, and assume further that $\mathfrak{T}$ is automorphic of a Serre weight $V = \bigotimes_v V_v$ with $V_v := V_v \circ w^{-1} \cong F(\mu_v)^\vee$. We may write $V_v \circ w^{-1} \cong F(\mathfrak{m}_w)^\vee$ for a dominant weight $\mathfrak{m}_w \in \mathcal{Z}_+$ where $w'$ is a place of $F$ above $w'$, and define
\[ V' := \bigotimes_{v \neq w} V_v \text{ and } \tilde{V}' := \bigotimes_{v \neq w} W_{\mathfrak{m}_w} . \]

From now on, we also assume that $\mathfrak{m}_w$ is in the lowest alcove for each place $w'$ of $F$ above $p$, so that $V' \cong \tilde{V}' \otimes_{\mathcal{O}_F} F$.

Let $U$ be a compact open subgroup of $G_n(A_f \otimes F) \times \mathcal{G}_n(\mathcal{O}_F, p)$, which is sufficiently small and unramified above $p$, such that $S(U, V)|\mathfrak{m}_w| \neq 0$ where $\mathfrak{m}_w$ is the maximal ideal of $T_F$ attached to $\mathfrak{T}$ for a cofinite subset $\mathcal{P}$ of $\mathcal{P}_U$.

We fix a pair of integers $(i_0, j_0)$ such that $0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1$, and determine a pair inter $(i_1, j_1)$ by the equation \eqref{5.6.27}. For brevity, we will use the general notation $P$ (resp. $N, L, P \cdots$) for the specific groups $P_{i_1,j_1}$ (resp. $N_{i_1,j_1}, L_{i_1,j_1}, P_{i_1,j_1}^- \cdots$) throughout this section, where $P_{i_1,j_1}, N_{i_1,j_1}, L_{i_1,j_1}, P_{i_1,j_1}^- \cdots$ are defined at the beginning of Section 5.

Recall $\hat{S}_n, \hat{S}_n^\prime$ (cf. \eqref{4.7.23}), $\kappa_n$ (cf. \eqref{4.7.29}), $\varepsilon^{\prime *}$ (cf. \eqref{4.7.35}), and $\mathcal{P}_n$ (cf. \eqref{4.7.34}), whose definitions are completely determined by fixing the data $n$ and $(a_{n-1}, \cdots, a_0)$. We define $S_{i_1,j_1}, \hat{S}_{i_1,j_1} \in \mathbf{Q}_p[\text{GL}_{j_1-i_1+1}(\mathbf{Q}_p)], \kappa_{i_1,j_1} \in \mathbb{Z}_p^\times, \varepsilon^{i_1,j_1} = \pm 1$, and $\mathcal{P}_{i_1,j_1}$ (resp. $b_{j_1-i_1+1}$) by replacing $n$ and $(a_{n-1}, \cdots, a_1, a_0)$ by $j_1-i_1+1$ and $(b_{j_1-i_1+1})$ respectively with $b_0$ as at the beginning of Section 5.3. Via the natural embedding \eqref{5.7.4} we also define $\hat{S}_{i_1,j_1}^\prime$ (resp. $\hat{S}_{i_1,j_1}^{\prime *}$) to be the image of $\hat{S}_{i_1,j_1}$ (resp. $\hat{S}_{i_1,j_1}^\prime$) in $\mathcal{Q}_p[\text{GL}_{j_1,i_1}(\mathbf{Q}_p)]$. Note that $S_{i_1,j_1}$ (resp. $\hat{S}_{i_1,j_1}$) is a Teichmüller lift of $S_{i_1,j_1}$ (resp. $\hat{S}_{i_1,j_1}$).

We recall the operator $\Xi_{j_1-i_1+1} \in \text{GL}_{j_1-i_1+1}(\mathbf{Q}_p)$ from \eqref{4.7.28} except that here we replace $n$ by $j_1-i_1+1$. Then we define
\[ \Xi_{i_1,j_1} := (\Xi_{j_1-i_1+1})^{j_1-i_1-1} \]
and denote the image of $\Xi_{i_1,j_1}$ via the embedding
\[ \text{GL}_{j_1-i_1+1}(\mathbf{Q}_p) \cong \text{GL}_{i_1,j_1}(\mathbf{Q}_p) \hookrightarrow L(\mathbf{Q}_p) \hookrightarrow \text{GL}_n(\mathbf{Q}_p) \]
by $\Xi_{i_1,j_1}$.
We define
\[
\begin{aligned}
M & := S(U^v, \hat{V}^v)_m \\
M^{i_1 j_1} & := S(U^v, \hat{V}^v)_{m_{i_1 j_1}} \\
M^{i_1 j_1, l} & := S(U^v, \hat{V}^v)_{m_{i_1 j_1, l}}
\end{aligned}
\]
then $M^{i_1 j_1}$ (resp. $M^{i_1 j_1, l}$) is a free $O_E$-module of finite rank as $M$ is a smooth admissible representation of $G(Q_p)$ which is $\omega_E$-torsion free. For any $O_E$-algebra $A$, we write $M_A^{i_1 j_1}$ for $M^{i_1 j_1} \otimes_{O_E} A$. We similarly define $M_A^{i_1 j_1, l}$ and $M_A$.

**Definition 5.7.3.** Two vectors $v^{i_1 j_1} \in M^{i_1 j_1} \lfloor_{m_\tau}$ and $v^{i_1 j_1, l} \in M^{i_1 j_1, l} \lfloor_{m_\tau}$ are said to be connected if there exists
\[
\tilde{v}^{i_1 j_1} \in M^{i_1 j_1} \quad \text{and} \quad \tilde{v}^{i_1 j_1, l} \in M^{i_1 j_1, l}
\]
that lifts $v^{i_1 j_1}$ and $v^{i_1 j_1, l}$ respectively such that $\tilde{v}^{i_1 j_1, l}$ and $\Xi^{i_1 j_1} \tilde{v}^{i_1 j_1}$ has the the same image in $(M_E)_{N-}(Q_p)$ via the coinvariant morphism
\[
M_E \twoheadrightarrow (M_E)_{N-}(Q_p).
\]
We also say that $v^{i_1 j_1, l}$ is a connected vector to $v^{i_1 j_1}$ if $v^{i_1 j_1}$ and $v^{i_1 j_1, l}$ are connected.

Let $T$ be the $O_E$-module that is the image of $T^p$ in $\text{End}_{O_E}(M^{i_1 j_1})$. Then $T$ is a local $O_E$-algebras with the maximal ideal $m_\tau$, where, by abuse of notation, we write $m_\tau \subseteq T$ for the image of $m_\tau$ of $T^p$. As the level $U$ is sufficiently small, by passing to a sufficiently large $E$ as in the proof of Theorem 4.5.2 of [HLM], we may assume that $T_E \cong E^r$ for some $r > 0$. For any $O_E$-algebra $A$ we write $T_A$ for $T \otimes_{O_E} A$. Similarly, we define $T'$ and $T_A'$ by replacing $M^{i_1 j_1}$ by $M^{i_1 j_1, l}$.

We have $M_E^{i_1 j_1} = \bigoplus_p M^{i_1 j_1} \lfloor_{pE}$, where the sum runs over the minimal primes $p$ of $T$ and $p_E := pT_E$. Note that for any such $p$ $T_E/p_E \cong E$. By abuse of notation, we also write $p$ (resp. $p_E$) for its inverse image in $T^p$ (resp. $T_E^p$) and for the corresponding minimal prime ideal of $T'$ (resp. $T'_E$). We also note that for any such $p$ we have a surjection $M[p] \twoheadrightarrow M[p] \lfloor_{m_\tau}$ as $m_\tau = p + \omega_E T^p$.

**Definition 5.7.4.** A non-zero vector $v^{i_1 j_1} \in M^{i_1 j_1} \lfloor_{pE}$ is said to be primitive if there exists a vector $\tilde{v}^{i_1 j_1} \in M^{i_1 j_1} \lfloor_{pE}$ that lifts $v^{i_1 j_1}$, for certain minimal prime $p$ of $T$.

Note that the $G(Q_p)$-subrepresentation of $M_E$ generated by $\tilde{v}^{i_1 j_1}$ is irreducible and actually lies in $M_E[pE]$.

We have to be careful that we only know that there is an inclusion
\[
\bigoplus_p M^{i_1 j_1} \lfloor_{pE} \subseteq M^{i_1 j_1},
\]
of $O_E$-modules, but we do not know if the equality holds. As we can always pick a minimal prime $p$ of $T$ and then pick an arbitrary vector $\tilde{v}^{i_1 j_1} \in M^{i_1 j_1} \lfloor_{pE}$ such that $\tilde{v}^{i_1 j_1} \not\in \omega_E M^{i_1 j_1} \lfloor_{pE}$, we can define $v^{i_1 j_1}$ as the image of $\tilde{v}^{i_1 j_1}$ and then deduce that $v^{i_1 j_1}$ is primitive. In other word, we have shown that a primitive vector in $M^{i_1 j_1}$ always exists, but as (5.7.5) might not be an equality, we do not know in general if all primitive vectors span the whole $F$-space $M^{i_1 j_1}$.

Now we can state our main results in this paper. Recall that by $\mathfrak{p}_0$ we always mean an $n$-dimensional ordinary representation of $G_{Q_p}$ as described in (4.0.4). We will shorten the notation $F(\lambda)_{\mathcal{F}}$ (resp. $F^L(\lambda)_{\mathcal{F}}$) to $F(\lambda)$ (resp. $F^L(\lambda)$) in the statement of the theorem and its proof.

**Theorem 5.7.6.** Fix a pair of integers $(i_0, j_0)$ satisfying $0 \leq j_0 < j_0 + 1 < i_0 \leq n - 1$, and let $(i_1, j_1)$ be a pair of integers such that $i_0 + i_1 = j_0 + j_1 = n - 1$. We also let $\mathfrak{p}_G : G_F \rightarrow GL_n(F)$ be an irreducible automorphic representation with $\mathfrak{p}_{\mathcal{G}_{F_n}} \cong \mathfrak{p}_0$. Assume that
\begin{itemize}
\item $\mu_{i_0, j_0}$ is $2n$-generic;
\item $\mathfrak{p}_{i_0, j_0}$ is Fontaine–Laffaille generic.
\end{itemize}
Assume further that
\[(5.7.7)\quad \{ F(\mu^{\square})^\vee \} \subseteq W_{\text{c}}(\mathfrak{P}) \cap \text{JH}(\mathcal{H}(\pi^{1,1}))^\vee \subseteq \{ F(\mu^{\square})^\vee, F(\mu^{\square,i,j_1})^\vee \}.
\]

Then there exists a primitive vector in $M_{\text{FL}}(\mathbb{N})^{(1), \mu^{1,1}_{i,j_1}}$. Moreover, for each primitive vector $v^{1,1}_{i,j_1} \in M_{\text{FL}}(\mathbb{N})^{(1), \mu^{1,1}_{i,j_1}}$ there exists a connected vector $v^{1,1}_{i,j_1} \in M_{\text{FL}}(\mathbb{N})^{(1), \mu^{1,1}_{i,j_1}}$ to $v^{1,1}_{i,j_1}$ such that $S^{1,1}_{i,j_1}v^{1,1} \neq 0$ and
\[
S^{1,1}_{i,j_1}v^{1,1} = \varepsilon^{1,1}_{i,j_1} \cdot p_{1,1,j_1}(b_{n-1}, \ldots, b_0) \cdot \text{FL}_{n}^{b_{i,j}}(\mathcal{P}_{G_F}) \cdot S^{1,1}_{i,j_1}v^{1,1}
\]
where
\[
\varepsilon^{1,1}_{i,j_1} = \prod_{k=i+1}^{j_1} (-1)^{b_{1,i}-b_{k,i}+i+1}
\]
and
\[
p_{1,1,j_1}(b_{n-1}, \ldots, b_0) = \prod_{k=i+1}^{j_1} \prod_{j=1}^{b_{1,i}-b_{k,i}+j-1} b_{1,i}-b_{k,i}-j \in \mathbb{Z}^\times.
\]

The right inclusion of \[(5.7.7)\] is just Conjecture \[5.3.1\] which is now a theorem of Bao V. Le Hung \[\text{[LHL]}\] (cf. Remark \[5.3.4\]). We also give an evidence for the left inclusion of \[(5.7.7)\] in Proposition \[5.3.2\] under some assumption of Taylor–Wiles type. As a result, the condition \[(6.1.7)\] can be removed under some standard Taylor–Wiles conditions.

**Proof.** We firstly point out that $M^{1,1}_{i,j_1} \neq 0$ (resp. $M^{1,1}_{i,j_1} \neq 0$), as $S(U, (F(\mu^{\square})^\vee \otimes \iota_{\mathfrak{P}}) \otimes V')_{\text{ss}} \neq 0$ and $F(\mu^{\square})$ is a factor of $\text{Ind}_{\mathfrak{P}}^\mu \mu^{1,1}_{i,j_1} = \text{Ind}_{\mathfrak{P}}^\mu \mu^{1,1}_{i,j_1}$ (resp. $\text{Ind}_{\mathfrak{P}}^\mu \mu^{1,1}_{i,j_1}$).

Picking an embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$, as well as an isomorphism $\iota : \overline{\mathbb{Q}}_p \overset{\sim}{\to} \mathbb{C}$, we see that
\[(5.7.8)\quad M^{1,1}_{i,j_1} \cong \bigoplus_{\Pi} m(\Pi) : \Pi^{1,1}_{i,j_1} \otimes (\Pi^{\infty,v})^{U,v},
\]
where the sum runs over irreducible representations $\Pi \cong \Pi^{\infty} \otimes \Pi_v \otimes \Pi^{\infty,v}$ of $G_{\text{an}}(\mathbb{A}_{\text{F}})$ over $\overline{\mathbb{Q}}_p$ such that $\Pi \otimes \iota, \mathbb{C}$ is a cuspidal automorphic representation of multiplicity $m(\Pi) \in \mathbb{Z}_{\geq 0}$ with $\Pi^{\infty} \otimes \iota, \mathbb{C}$ being determined by the algebraic representation $(\mathbb{V}')^\vee$ and with associated Galois representation $r_{\Pi}$ lifting $\mathfrak{P}'$ (cf. Lemma \[5.1.3\]).

We write $\delta$ for the modulus character of $B(\mathbb{Q}_p)$:
\[
\delta := | \cdot |^{-1} \otimes | \cdot |^{-2} \otimes \cdots \otimes | \cdot |^{1}
\]
where $| \cdot |$ is the (unramified) norm character sending $p$ to $p^{-1}$. For any $\Pi$ contributing to \[(5.7.8)\], we have

(i) $\Pi_v \cong \text{Ind}^G_{B(\mathbb{Q}_p)}(\psi \otimes \delta)$ for some smooth character
\[
\psi = \psi_{n-1} \otimes \psi_{n-2} \otimes \cdots \otimes \psi_1 \otimes \psi_0
\]
of $T(\mathbb{Q}_p)$ such that $\psi|_{T(\mathbb{Z}_p)} = \mu^{1,1}_{i,j_1}|_{T(\mathbb{Z}_p)}$, where $\psi_k$ are the smooth characters of $\mathbb{Q}_p$.

(ii) $r_{\Pi}^{\mathfrak{P}}(G_{\text{F}_{\mathfrak{P}}})$ is a potentially crystalline lift of $\mathfrak{P}$ with Hodge–Tate weights $\{- (n-1), -(n-2), \ldots, -1, 0\}$ and $\text{WD}(r_{\Pi}^{\mathfrak{P}}(G_{\text{F}_{\mathfrak{P}}}))_{\text{F-ss}} \cong \mathcal{O}_{\mathfrak{P}}^{n-1} \psi^{(-1)}$.

Here, part (i) follows from \[\text{[EGHI]}\], Propositions 2.4.1 and 7.4.4, and part (ii) follows from classical local-global compatibility (cf. Theorem \[5.1.4\]). Moreover, by Corollary \[3.7.3\] we have
\[(5.7.9)\quad \text{FL}_{n}^{i_{0,j_{0}},i_{j_{0}}}(p) = \frac{\prod_{k=j_{0}+1}^{i_{0}+1} \psi_{i_1-j_0+k}(p)}{p^{(i_{0}+j_{0}+j_{0}-j_{0}-1)}}.
\]

(Note that we may identify $\psi_{i_1-j_0+k}$ with $\Omega_k$ for $j_0 < k < i_0$, where $\Omega_k$ is defined in Corollary \[3.7.5\].) We use the shorten notation
\[
\tilde{C}(\chi) := \frac{\prod_{k=j_{0}+1}^{i_{0}-1} \psi_{i_1-j_0+k}}{p^{(i_{0}+j_{0}+j_{0}-j_{0}-1)}}
\]
for any smooth character $\chi := \psi \otimes \delta$, and we notice that
\[(5.7.10)\]  
\[\widetilde{C}(\chi) = \widetilde{C}(\chi') \text{ if } \chi |_{T_{1,j1}(Q_p)} = \chi' |_{T_{1,j1}(Q_p)} \]
for any two smooth characters $\chi, \chi' : T(Q_p) \to E$. Now we pick an arbitrary primitive vector $v_{1,j1} \in M_{1,j1}^{i_1,j_1}[m]$, with a lift $\tilde{v}_{1,j1} \in M_{1,j1}^{i_1,j_1}[p]$. We set
\[\tilde{\pi}_{1,j1} := (K \tilde{v}_{1,j1})E \subseteq M_E[p_E] \]
and thus $\tilde{\pi}_{1,j1}^0$ is an $O_E$-lattice in $\tilde{\pi}_{1,j1}$. Note that $M_{E}^{i_1,j_1}[p_E] \otimes E \overline{Q}_p$ is a direct summand of $(5.7.8)$ where $\Pi$ runs over a subset of $\text{automorphic representations}$ in $(5.7.8)$. The same argument as in the paragraph above (4.5.7) of [HLM] using Cebotarev density shows us that the local component $\Pi_{e}$ of each $\Pi$ occurring in this direct summand does not depend on $\Pi$.

By Lemma 5.5.15 and the definition of $(\tilde{\pi}_{1,j1})^0$, we obtain an injection
\[(5.7.11)\]  
\[(\pi_{1,j1})^0 \otimes O_E F \hookrightarrow (M[p]) \otimes O_E F = M_E[m] \]
as $p + \pi E T_p = m$. By the assumption $(5.7.11)$ (cf. Conjecture 5.3.1), we deduce that
\[\text{JH} \left(\text{soc}_{G(F_p)}(M_E[m])\right) \subseteq \{F(\mu^\square), F(\mu^\square_{i_1,j_1})\} \]
and therefore by $(5.7.11)$ we have
\[\text{JH} \left(\text{soc}_{G(F_p)}((\pi_{1,j1})^0 \otimes O_E F)\right) \subseteq \{F(\mu^\square), F(\mu^\square_{i_1,j_1})\}. \]

Pick an arbitrary vector
\[\tilde{w}_{1,j1} \in \left((\pi_{1,j1})^0 \otimes O_E F\right)U(F_p) \subseteq \pi_{1,j1} \cap M[p], \]
and denote its image in $((\pi_{1,j1})^0 \otimes O_E F)U(F_p) \subseteq \tilde{\pi}_{1,j1}^0$ by $v_{1,j1}$. Then, by Proposition 5.6.13 we obtain
\[\text{JH} \left(\text{soc}_{L(F_p)}((\pi_{1,j1})^0 \otimes O_E F)\right) \subseteq \{L(\mu^\square), L(\mu^\square_{i_1,j_1})\}. \]

We recall the open compact subgroups $K^L$, $K^L(1)$, $I^L$ and $I^L(1)$ of $L(Q_p)$ from (4.1.2). By Corollary 5.6.14 we know that there exists a $O_E$-lattice $\tilde{\pi}_{1,j1}^0(L)$ in
\[\tilde{\pi}_{1,j1}^0 := \text{Ind}_{K^L(1)}^{L(F_p)}(\tilde{\pi}_{1,j1}) \]
as a $K^L$-representation such that
\[(\pi_{1,j1})^0 = \text{coInd}_{P(F_p)}^{G(F_p)}(\tilde{\pi}_{1,j1}^0,L) \]
and
\[\text{JH} \left(\text{soc}_{L(F_p)}((\pi_{1,j1})^0 \otimes O_E F)\right) \subseteq \{L(\mu^\square), L(\mu^\square_{i_1,j_1})\}. \]
Since $\text{coInd}_{P(F_p)}^{G(F_p)}(\cdot) \cong \text{Ind}_{P^-(F_p)}^{G(F_p)}(\cdot)$ and $(\cdot)_{N^-(F_p)}$ are left and right adjoint functors of each other, we deduce the existence of surjections of $O_E$-representations of $L(F_p)$
\[(5.7.13)\]  
\[(\pi_{1,j1})^0 |_{L(F_p)} \twoheadrightarrow (\tilde{\pi}_{1,j1}^0)^0_{N^-(F_p)} \twoheadrightarrow (\tilde{\pi}_{1,j1}^0,L) \rightarrow (\pi_{1,j1}^L)^0. \]
We denote the composition $(5.7.13)$ by $\text{pr}$. If we write explicitly
\[\text{coInd}_{P(F_p)}^{G(F_p)}(\tilde{\pi}_{1,j1}^0,L) = \{f : G(F_p) \twoheadrightarrow (\pi_{1,j1}^L)^0 \mid f(p^- g) = p^- \cdot f(g) \forall p^- \in P^-(F_p)\}, \]
where $p^-$ acts on $(\pi_{1,j1}^L)^0$ through its image in $L(F_p)$, we can express $\text{pr}$ by
\[(5.7.14)\]  
\[\text{pr} : (\pi_{1,j1}^L)^0 |_{L(F_p)} \twoheadrightarrow (\tilde{\pi}_{1,j1}^0,L), \quad f \mapsto f(1). \]
By $(5.7.14)$ we obtain the following equalities
\[
\left\{ \begin{array}{c}
O_E \left[ \text{pr}(\tilde{\pi}_{1,j1}) \right] = \left((\pi_{1,j1}^L)^0 \otimes L(F_p)\right)_{\tilde{\pi}_{1,j1}^0}; \\
O_E \left[ \text{pr}(\tilde{\pi}_{1,j1}) \right] = \left((\pi_{1,j1}^L)^0 \otimes L(F_p)\right)_{\tilde{\pi}_{1,j1}^0}. 
\end{array} \right.
\]
By applying Proposition 5.6.13 to \((\tilde{\pi}^{i_1,j_1,L})^\circ \otimes_{G_E} F\) we deduce that
\[
(5.7.15) \quad 0 \neq F[S_{i_1,j_1} ((\text{pr} \otimes_{G_E} F) v^{i_1,j_1})] = F[S'_{i_1,j_1} ((\text{pr} \otimes_{G_E} F) v^{i_1,j_1,n})] \subseteq (\tilde{\pi}^{i_1,j_1,L})^\circ \otimes_{G_E} F.
\]

By Theorem 4.7.4 we have
\[
\text{Ind}_{B}^{G}(\tilde{\pi}^{i_1,j_1,L}) \cong \tilde{\pi}^{i_1,j_1,L},
\]
and thus together with (5.7.15) we deduce that
\[
(5.7.16) \quad \varpi_E(\tilde{\pi}^{i_1,j_1,L})^\circ \cong \mathcal{O}_E[S_{i_1,j_1} (\text{pr}(\tilde{\pi}^{i_1,j_1,L}))] = \mathcal{O}_E[S'_{i_1,j_1} (\text{pr}(\tilde{\pi}^{i_1,j_1,L}))] \subseteq \tilde{\pi}^{i_1,j_1,L}.
\]

We define
\[
\Pi^{i_1,j_1} := \langle G(Q)_{p} \tilde{\pi}^{i_1,j_1} \rangle_E.
\]
As \(\tilde{\pi}^{i_1,j_1}\) is primitive, by Definition 5.7.2 we deduce that \(\Pi^{i_1,j_1}\) is irreducible and there exists a smooth character \(\chi : T(Q) \rightarrow E^\times\) satisfying \(\chi|_{T(Z_\rho)} = \tilde{\pi}^{i_1,j_1}\) such that
\[
\Pi^{i_1,j_1} \cong \text{Ind}_{B(Q_p)}^{G(Q_p)} \chi.
\]

In particular, we notice that
\[
(5.7.17) \quad (\Pi^{i_1,j_1})^K(1) = \tilde{\pi}^{i_1,j_1}.
\]

We define
\[
B' := N^- \cdot (B \cap L),
\]
and thus \(B'\) is a Borel subgroup of \(G\) as it is conjugated to \(B\) via \(w_0w_1^L\).

By the intertwining between generic smooth principal series in characteristic zero in [Sha10], Chapter 4, we deduce the existence of a smooth character \(\chi' : T(Q) \rightarrow E^\times\) such that
\[
\text{Ind}_{B(Q_p)}^{G(Q_p)} \chi \cong \text{Ind}_{B'(Q_p)}^{G(Q_p)} \chi'.
\]
As \(T_{i_1,j_1}\) commutes with \(w_0w_1^L\), we observe from the above intertwining isomorphism that
\[
(5.7.18) \quad \chi'|_{T_{i_1,j_1}(Q_p)} = \chi|_{T_{i_1,j_1}(Q_p)}.
\]

Then we define
\[
\Pi^{i_1,j_1,L} := \text{Ind}_{B \cap L(Q_p)}^{L(Q_p)} \chi'
\]
and thus
\[
\Pi^{i_1,j_1} \cong \text{Ind}_{B(Q_p)}^{G(Q_p)} \chi \cong \text{Ind}_{B(Q_p)}^{G(Q_p)} \chi' = \text{Ind}_{P^-(Q_p)}^{G(Q_p)} \Pi^{i_1,j_1,L}.
\]

In particular, we also have
\[
(5.7.19) \quad (\Pi^{i_1,j_1,L})^{K_L}(1) = \tilde{\pi}^{i_1,j_1,L}
\]
As \(\text{Ind}_{P^-(Q_p)}^{G(Q_p)} (\cdot)\) and \((\cdot)_{N^-(Q_p)}\) are left and right adjoint functor of each other, we have surjections of \(L(Q_p)\)-representation
\[
(5.7.20) \quad \Pi^{i_1,j_1}|_{L(Q_p)} \twoheadrightarrow (\Pi^{i_1,j_1})_{N^-(Q_p)} \twoheadrightarrow \Pi^{i_1,j_1,L},
\]
and we denote the composition (5.7.20) by \(\text{Pr}\).

If we write explicitly
\[
\text{Ind}_{P^-(Q_p)}^{G(Q_p)} \Pi^{i_1,j_1,L} = \{ f : G(Q_p) \rightarrow \Pi^{i_1,j_1,L} \mid f(p^- g) = p^- f(g) \text{ for all } p^- \in P^-(Q_p) \}
\]
where \(p^-\) acts on \(\Pi^{i_1,j_1,L}\) through its image in \(L(Q_p)\), we can express \(\text{Pr}\) by
\[
(5.7.21) \quad \text{Pr} : \Pi^{i_1,j_1}|_{L(Q_p)} \rightarrow \Pi^{i_1,j_1,L}, \quad f \mapsto f(1).
\]
By (5.7.17), (5.7.19), (5.7.14) and (5.7.21), the morphism \( \text{pr} \) and \( \text{Pr} \) fit into the following commutative diagram:

\[
\begin{array}{ccc}
\Xi_{i,j} \circ \Omega_E F & \xrightarrow{\text{pr} \circ \Omega_E F} & \Xi_{i,j}(L) \circ \Omega_E F \\
\downarrow & & \downarrow \\
\Xi_{i,j} & \xrightarrow{\text{pr}} & \Xi_{i,j}(L)
\end{array}
\]

\[
\Xi_{i,j} = \left( \Pi_{i,j} \right)^{K(1)} \xrightarrow{\text{pr} \circ \Omega_E F} \Xi_{i,j}(L) = \left( \Pi_{i,j}(L) \right)^{K_L(1)}.
\]

It is clear from the commutative diagram (5.7.22) that we can use the notation \( \text{Pr}(v) \) instead of \( \text{pr}(v) \) for any \( v \in (\Xi_{i,j})^\circ \).

Since \( \Xi_{i,j} \in L(Q_p) \) lies in the normalizer of \( I^L(1) \) in \( L(Q_p) \), we deduce that

\[
\text{Pr}
\Xi_{i,j} (\Xi_{i,j} (\Xi_{i,j})) = \Xi_{i,j} (\text{Pr}(\Xi_{i,j} (\Xi_{i,j}))) \in \Xi_{i,j} (\Pi_{i,j}(L) I^L(1), \Pi_{i,j} (\Xi_{i,j})).
\]

Note that

\[
\Xi_{i,j} (\Pi_{i,j}(L) I^L(1), \Pi_{i,j} (\Xi_{i,j})) = \Xi_{i,j} (\Pi_{i,j}(L) I^L(1), \Pi_{i,j} (\Xi_{i,j})).
\]

As a result, we have

\[
(5.7.23)
E \left[ \text{Pr}
\Xi_{i,j} (\Xi_{i,j} (\Xi_{i,j})) \right] = E \left[ \text{Pr}(\Xi_{i,j} (\Xi_{i,j})) \right]
\]

By applying Theorem (4.7.4) to \( \Pi_{i,j}, L \) we deduce that

\[
(5.7.24)
\Xi_{i,j} (\Si_{i,j} \cdot \Xi_{i,j} (\text{Pr}(\Xi_{i,j}))) = \kappa_{i,j} \Si_{i,j} (\text{Pr}(\Xi_{i,j})),
\]

for some \( \kappa(\chi) \in \Omega_E \) and for \( \kappa_{i,j} \) satisfying

\[
(5.7.25)
\kappa_{i,j} \equiv \xi_{i,j} p^\kappa_{i,j} (b_{n-1}, \cdots, b_0) \quad (\mod \, \omega_E).
\]

Comparing (5.7.24) with (5.7.10) we deduce the existence of \( C_2 \in \Omega_E^\times \) such that

\[
(5.7.26)
\Xi_{i,j} (\text{Pr}(\Xi_{i,j})) = C_2 \Si_{i,j} (\text{Pr}(\Xi_{i,j})),
\]

and thus by (5.7.26) we obtain

\[
(5.7.27)
\Xi_{i,j} (\text{Pr}(\Xi_{i,j})) = C_2 \text{Pr}(\Xi_{i,j}) \in (\Xi_{i,j}(L) L(F_p), \Xi_{i,j} (\Xi_{i,j})).
\]

Now we let

\[
\Xi_{i,j, \lambda} := C_2 \text{Pr}(\Xi_{i,j} \Xi_{i,j} (\Xi_{i,j})) \in (\Xi_{i,j}(L) \circ \Omega_E F, \Xi_{i,j} (\Xi_{i,j})).
\]

and denote by \( \nu_{i,j, \lambda} \) the image of \( \Xi_{i,j, \lambda} \) in \((\Xi_{i,j}(L) \circ \Omega_E F, \Xi_{i,j} (\Xi_{i,j}))\). Then by Definition (5.7.3) we know that \( \nu_{i,j, \lambda} \) and \( \nu_{i,j, \lambda} \) are connected. Moreover, by definition of \( \Xi_{i,j, \lambda} \) we have

\[
(5.7.28)
\Xi_{i,j} (\text{Pr}(\Xi_{i,j})) = C_2 \text{Pr}(\Xi_{i,j} \Xi_{i,j} (\Xi_{i,j})),
\]

and we deduce from (5.7.28) the equality

\[
(5.7.29)
\Xi_{i,j} \nu_{i,j, \lambda} = \Xi_{i,j} \nu_{i,j, \lambda} = \Xi_{i,j} \nu_{i,j, \lambda} = \omega_E \nu \in (\Xi_{i,j} (\Xi_{i,j})).
\]

for some \( \Xi_{i,j} \in \Omega_E^\times \) that lifts \( C \) and for some \( \nu \in (\Xi_{i,j})^\circ \).

We consider the image of (5.7.28) under the morphism \( \text{Pr} \) (or rather \( \text{pr} \)):
or equivalently

\[ (5.7.29) \quad \tilde{S}_{i,j} \left( \Pr(\tilde{\rho}^{i,j}) \right) = \tilde{C} \tilde{S}_{i,j} \left( \Pr(\tilde{\rho}^{i,j}) \right) + \omega \Pr(\nu) \in (\tilde{\rho}^{i,j})^o. \]

Comparing \( (5.7.29) \) with \( (5.7.26) \), we deduce that

\[ \Pr(\nu) \in \Pr((\tilde{\rho}^{i,j})^o) \cap E \left[ \tilde{S}_{i,j} \left( \Pr(\tilde{\rho}^{i,j}) \right) \right]. \]

Note that

\[ \Pr((\tilde{\rho}^{i,j})^o) \cap E \left[ \tilde{S}_{i,j} \left( \Pr(\tilde{\rho}^{i,j}) \right) \right] = (\tilde{\rho}^{i,j,L})^o \cap E \left[ \tilde{S}_{i,j} \left( \Pr(\tilde{\rho}^{i,j}) \right) \right] = \mathcal{O}_E \left[ \tilde{S}_{i,j} \left( \Pr(\tilde{\rho}^{i,j}) \right) \right]. \]

By \( (5.7.24) \) we deduce

\[ \tilde{C} \equiv \kappa_{i,j} \tilde{C}(\chi') \mod \omega \mathcal{E}. \]

Therefore it follows from \( (5.7.13) \) and \( (5.7.10) \) as well as the congruences \( (5.7.9) \) and \( (5.7.25) \) that

\[ C = \kappa_{i,j} \tilde{C}(\chi') = \kappa_{i,j} \tilde{C}(\chi) = \nu^{i,j} P^{i,j}(b_n - 1, \ldots, b_0) \text{FL}_{n_j,2n}(\mathcal{P}_0), \]

which completes the proof. \( \square \)

**Remark 5.7.30.** In Theorem \( 5.7.6 \), we construct a vector \( v^{i,j} \) starting with a primitive vector \( v^{i,j} \) and \( v^{i,j} \) are connected. However, the definition of connected (c.f. Definition \( 5.7.3 \)) involves the lifts \( \tilde{\rho}^{i,j} \) (resp. \( \tilde{\rho}^{i,j} \)) of \( v^{i,j} \) (resp. \( v^{i,j} \)) in characteristic zero. We emphasize that our proof of Theorem \( 5.7.6 \) automatically implies that \( v^{i,j} \) is independent of the choice of \( \tilde{\rho}^{i,j} \) and the lift of the action of \( \tilde{\rho}^{i,j} \) on \( M_F \mathcal{P}_0 \) into characteristic zero, although we do not show how to construct \( v^{i,j} \) from \( v^{i,j} \) without lifting.

**Corollary 5.7.31.** Keep the notation of Theorem \( 5.7.6 \) and assume that each assumption in Theorem \( 5.7.6 \) holds for all \( (i_0, j_0) \) such that \( 0 \leq j_0 < j_0 + 1 < i_0 < n - 1 \).

Then the Galois representation \( \mathcal{P}_0 \) is determined by \( M_F \mathcal{P}_0 \) in the sense of Remark \( 5.7.30 \).

**Proof.** We follow the notation in Section 3.4 of [BH15]. As \( \mathcal{P}_0 \) is ordinary, we can view it as a morphism

\[ \mathcal{P}_0: \quad \mathcal{G}_q \to \tilde{B}(F) \subseteq \tilde{G}(F) \]

where \( \tilde{B} \) (resp. \( \tilde{G} \)) is the dual group of \( B \) (resp. \( G \)). The local class field theory gives us a bijection between smooth characters of \( \mathcal{G}_p \) and the smooth characters of the Weil group of \( \mathcal{Q}_p \) in characteristic zero. This bijection restricts to a bijection between smooth characters of \( \mathcal{G}_p \) and smooth characters of \( \text{Gal}(\mathcal{Q}_p/Q_p) \) both with values in \( \mathcal{O}_F^\times \). Taking \( \mod p \) reduction and then taking products we reach a bijection between smooth \( F \)-characters of \( T(\mathcal{Q}_p) \) and \( \text{Hom} \left( \text{Gal}(\mathcal{Q}_p/Q_p), \tilde{T}(F) \right) \).

We can therefore define \( \chi_{\mathcal{P}_0} \) as the character of \( T(\mathcal{Q}_p) \) corresponding to the composition

\[ \tilde{\chi}_{\mathcal{P}_0}: \quad \text{Gal}(\mathcal{Q}_p/Q_p) \to \tilde{B}(F) \to \tilde{T}(F). \]

In [BH15], a closed subgroup \( C_{\mathcal{P}_0} \subseteq \tilde{B} \) (at the beginning of section 3.2) and a subset \( W_{\mathcal{P}_0} \) \( ((2) \) before Lemma 2.3.6) of \( W \) is defined.

As we are assuming that \( \mathcal{P}_0 \) is maximally non-split, we observe that \( C_{\mathcal{P}_0} = B \) and \( W_{\mathcal{P}_0} = \{ 1 \} \) in our case. Therefore by the definition of \( \Pi^{\text{ord}}(\mathcal{P}_0) \) in [BH15] before Definition 3.4.3, we know that it is indecomposable with socle

\[ \text{Ind}_{\mu}^{\tilde{B}(F)}(\chi_{\mathcal{P}_0}) \cdot (\omega^{-1} \circ \theta) \]

where \( \theta \in X(T) \) is a twist character defined after Conjecture 3.1.2 in [BH15] which can be chosen to be \( \eta \) in our notation. Then as a Corollary of Theorem 4.4.7 in [BH15], we deduce that \( S(U^v, V^l)[\mathcal{P}_0] \) determines \( \chi_{\mathcal{P}_0} \) and hence \( \tilde{\chi}_{\mathcal{P}_0} \).

Now, we know that \( \mathcal{P}_0 \) is determined by the Fontaine–Laffaille parameters \( \{ \text{FL}_{n_j,2n_j}(\mathcal{P}_0) \in \mathbb{P}^1(F) \mid 0 \leq j_0 < i_0 + 1 < j_0 \leq n - 1 \} \) and \( \tilde{\chi}_{\mathcal{P}_0} \), up to isomorphism. Our conclusion thus follows from Theorem \( 5.7.6 \). \( \square \)
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