EFFECTIVE RATIONAL APPROXIMATION ON SPHERES

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Abstract. We prove an effective estimate for the counting function of Diophantine approximants on the sphere $S^n$. We use homogeneous dynamics on the space of orthogonal lattices, in particular effective equidistribution results and non-divergence estimates for the Siegel transform, developing on recent results of Alam-Ghosh and Kleinbock-Merrill.

Contents

1. Introduction and main result
2. Diophantine approximation on $S^n$ and dynamics on the space of lattices
3. Approximation of the counting function
   3.1. Siegel transform
   3.2. Non-divergence estimates
   3.3. Truncated Siegel transform
   3.4. Smooth approximation of the counting function
4. Effective equidistribution
   4.1. Effective double equidistribution of translated $K$-orbits
   4.2. Effective pointwise equidistribution along $A$-orbits
5. Effective estimate for the counting function
References
1. Introduction and main result

**Intrinsic Diophantine Approximation.** It is well-known in metric Diophantine approximation that for any \( c > 0 \) and Lebesgue-almost all \( \alpha \in \mathbb{R}^m \), there exist infinitely many solutions \((p, q) \in \mathbb{Z}^m \times \mathbb{N}\) to the inequality:\(^1\)

\[
\left\| \alpha - \frac{p}{q} \right\| < \frac{c}{q^{1+m}} .
\]

(1.1)

A first refinement of this problem is to count solutions up to a certain bound for the complexity \( q \) of the approximants, which leads to consider counting functions such as

\[
N_{T,c}(\alpha) := |\{(p, q) \in \mathbb{Z}^m \times \mathbb{N} : 1 \leq q < e^T \text{ and (1.1) holds}\}| .
\]

Accurate estimates of the counting function \( N_{T,c} \) have already been established. We mention in particular the effective estimate by W. Schmidt \[\text{Sch60}\], who proved for more general approximating functions that for Lebesgue-almost \( \alpha \in [0,1]^m \),

\[
N_{T,c}(\alpha) = C_{c,m} T + O_{\alpha,\epsilon} (T^{1/2+\epsilon}) ,
\]

(1.2)

for all \( \epsilon > 0 \), with a constant \( C_{c,m} \) depending only on \( c \) and \( m \).

Another refinement of this problem is the so-called *intrinsic* Diophantine approximation, where one considers a vector \( \alpha \) in a manifold \( X \subset \mathbb{R}^m \), for example level sets of a quadratic form, and is interested in counting rational approximants which also belong to \( X \). For rational approximation on spheres, Kleinbock and Merrill \[\text{KM13}\] proved an analog of Dirichlet’s theorem and a Kintchine-type dichotomy. For the divergence case in this last result, Alam and Ghosh \[\text{AG20}\] proved a quantitative estimate for the number of rational approximants on spheres with the critical Dirichlet exponent (Theorem 1.1). Our main result in this paper is to prove an estimate with error term analogous to (1.2) for intrinsic Diophantine approximation on the sphere \( S^n \).

Given \( T, c > 0 \) and \( \alpha \in S^n \), we consider the following inequality (with the critical Dirichlet exponent for intrinsic Diophantine approximation on \( S^n \))

\[
\left\| \alpha - \frac{p}{q} \right\| < \frac{c}{q} ,
\]

(1.3)

and the following counting function for intrinsic rational approximants

\[
N_{T,c}(\alpha) := |\{(p, q) \in \mathbb{Z}^{n+1} \times \mathbb{N} : \frac{p}{q} \in S^n, 1 \leq q < \cosh T \text{ and (1.3) holds}\}| .
\]

We recall the following quantitative result of Alam and Ghosh.

**Theorem 1.1 (AG20).** There exists a computable constant \( C_{c,n} > 0 \), depending only on \( c \) and \( n \), such that for almost every \( \alpha \in S^n \),

\[
\frac{N_{T,c}(\alpha)}{T} \sim C_{c,n} \quad \text{as } T \to \infty .
\]

(1.4)

In the following theorem we give an estimate with error term improving (1.4).

\(^1\)|\(\cdot\)| will denote the Euclidean norm.
Theorem 1.2. Let \( n \geq 2 \). There exist a computable constant \( C_{c,n} > 0 \), depending only on \( c \) and \( n \), and a constant \( \gamma < 1 \) depending only on \( n \), such that for almost every \( \alpha \in S^n \),

\[
N_{T,c}(\alpha) = C_{c,n} T + O(\alpha(T^\gamma)).
\]  

Outline of the paper. We develop further the approach of [KM13] and [AG20], starting with the embedding of \( S^n \) in the positive light cone \( C := \{ x \in \mathbb{R}^{n+1} \times \mathbb{R}_+ : Q(x) = 0 \} \) for a quadratic form \( Q \) of inertia \((n+1,1)\), and identifying good approximants \( p/q \in S^n \) with integer points \((p,q)\) in \( \mathbb{Z}^{n+2} \cap C \) whose images under certain rotations \( k \in K = \text{SO}(n+1) \) lie in a certain domain \( E_{T,c} \subset C \) (we recall more details about this correspondence in Section 2). The number of solutions \( N_{T,c} \) is then related to the number of lattice points in the domain \( E_{T,c} \), which can be approximated by a more convenient domain \( F_{T,c} \) and tessellated by the action of a hyperbolic subgroup \( \{a_t, t \in \mathbb{R}\} \subset \text{SO}(Q) \). Estimating \( N_{T,c} \) amounts then to analyzing ergodic averages of a counting function \( \hat{\chi} \), namely the Siegel transform of the characteristic function of a fixed domain \( F_{1,c} \), along \( K \)-orbits pushed by \( \{a_t\} \).

In [AG20], the authors use ergodicity of the \( \{a_t\} \)-action and Birkhoff’s Ergodic Theorem to obtain an asymptotic estimate for these ergodic averages. In order to obtain an estimate with error term, we use effective pointwise equidistribution along \( \{a_t\} \)-orbits, which we derive from effective double equidistribution of \( K \)-orbits pushed by \( \{a_t\} \) (Proposition 4.1), using a general method presented in [KSW17] to derive an “almost everywhere”-bound from an \( L^p \)-bound on ergodic averages (Proposition 4.2). However, using effective equidistribution requires to consider smooth and compactly supported test functions, whereas our counting function has typically none of these regularities. We address this issue in two steps.

We first introduce and study the integrability of the Siegel transform on the space of orthogonal lattices, then show (Proposition 3.6) that the Siegel transform \( \hat{f} \) of a compactly supported function \( f \) can be approximated by a truncated Siegel transform \( \hat{f}^{(L)} \) in a way to control the approximation on translated \( K \)-orbits, i.e. control \( |\hat{f} \circ a_t - \hat{f}^{(L)} \circ a_t| \) with respect to the probability measure on the orbits. In a second step (Proposition 3.7), we show that the characteristic function \( \chi \) of the elementary domain \( F_{1,c} \) can be approximated by a family of smooth compactly supported functions \( f_\varepsilon \), again in a way to keep control of the approximated Siegel transform \( \hat{f}_\varepsilon \) over translated \( K \)-orbits. In this process we also need non-divergence results for the Siegel transform with respect to the probability measure on \( K \)-orbits (Propositions 3.2 and 3.4). Thus, we can relate the counting function \( \hat{\chi} \) to a smooth and compactly supported function on the space of orthogonal lattices and then establish an effective almost-everywhere estimate for its ergodic averages (Theorem 4.2). We finally derive an effective estimate for the number of solutions \( N_{T,c} \) (Section 5).

2. Diophantine approximation on \( S^n \) and dynamics on the space of lattices

We recall the correspondence presented in [KM13] and [AG20] between Diophantine approximation on the sphere \( S^n \) and the dynamics of orthogonal lattices in \( \mathbb{R}^{n+2} \).

We consider the quadratic form \( Q : \mathbb{R}^{n+2} \rightarrow \mathbb{R} \) defined by

\[
Q(x) := \sum_{i=1}^{n+1} x_i^2 - x_{n+2}^2, \quad \text{for } x = (x_1, \ldots, x_{n+2}).
\]  

(2.1)
and the embedding of $S^n$ in the positive light cone

$$C := \{ x \in \mathbb{R}^{n+2} : Q(x) = 0, \ x_{n+2} > 0 \} ,$$

via $\alpha \mapsto (\alpha, 1)$, which yields a one-to-one correspondence between primitive integer points on the positive light cone, $(p, q) \in C \cap \mathbb{Z}_{\text{prim}}^{n+2}$, and rational points on the sphere, $\frac{p}{q} \in S^n$.

We denote by $G = \text{SO}(Q)^o \cong \text{SO}(n+1, 1)^o$ the connected component of the group of orientation-preserving linear transformations which preserve $Q$. We denote by $\Lambda_0 := C \cap \mathbb{Z}^{n+2}$ the set of integer points on the positive light cone. By a lattice $\Lambda$ in $C$ we mean a set of the form $g\Lambda_0$ for some $g \in G$. If we denote by $\Gamma$ the stabilizer of $\Lambda_0$ in $G$, then $\Gamma$ is a lattice in $G$ containing the subgroup $\text{SO}(Q, \mathbb{Z})$ of integer points in $G$, as a finite index subgroup. The space of lattices in $C$ can be identified with the homogeneous space $X := G/\Gamma$, endowed with the $G$-invariant probability measure $\mu_X$.

Let $K$ denote the subgroup of $G$ that preserves the last coordinate in $\mathbb{R}^{n+2}$, i.e.

$$K = \begin{pmatrix} \text{SO}(n+1) \\ 1 \end{pmatrix} \cong \text{SO}(n+1) ,$$

equipped with the Haar probability measure $\mu_K$.

The sphere $S^n$ can be realized as a quotient of $K$, endowed with a unique left $K$-invariant probability measure, giving a natural correspondence between full-measure sets in $K$ and those in $S^n$.

Let $\alpha \in S^n$. For $k \in K$ such that $k(\alpha, 1) = (0, \ldots, 0, 1, 1) \in C$, and $(p, q) \in \Lambda_0$, we write $k(p, q) = (x_1, x_2, \ldots, x_{n+2}) \in C$, with $x_{n+2} = q$, and observe the following correspondence ([AG20], Lemma 2.2.):

$$\frac{\| \alpha - \frac{p}{q} \|}{q} < \frac{c}{q} \Leftrightarrow \frac{\| q(\alpha, 1) - (p, q) \|}{q} < c ,$$

$$\Leftrightarrow \frac{\| qk(\alpha, 1) - k(p, q) \|}{q} < c ,$$

$$\Leftrightarrow \frac{\| (x_1, x_2, \ldots, x_n, x_{n+1} - x_{n+2}, 0) \|}{q} < c ,$$

$$\Leftrightarrow \frac{2x_{n+2}(x_{n+2} - x_{n+1})}{q^2} < c^2 \quad (\text{since } k(p, q) \in C) .$$

Hence, if we denote

$$E_{T,c} := \{ x \in C : 2x_{n+2}(x_{n+2} - x_{n+1}) < c^2, 1 \leq x_{n+2} < \cosh T \} ,$$

then, for $k \in K$ such that $k(\alpha, 1) = (0, \ldots, 0, 1, 1) \in C$, we have:

$$N_{T,c}(\alpha) = |E_{T,c} \cap k\Lambda_0| . \quad (\text{2.2})$$

We denote $Y := K\Lambda_0$, equipped with the Haar probability measure $\mu_Y$.

We also consider elements

$$a_t = \begin{pmatrix} I_n & \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix} \in G$$
and the corresponding one-parameter subgroup

\[ A = \{ a_t : t \in \mathbb{R} \} \]

endowed with the natural measure \( dt \).

We will denote by \( \chi_E \) the characteristic function of a given set \( E \) and use the notation \( a \asymp b \) (resp. \( a \ll b \)) when there exist positive constants \( C_1 \) and \( C_2 \) such that \( C_1 b \leq a \leq C_2 b \) (resp. \( a \leq C_2 b \)).

In order to use the dynamics of translates of \( Y \) for the Diophantine approximation problem (2.2), we first approximate \( E_{T,c} \) by a domain offering a convenient tessellation under the action of the subgroup \( A \). We recall briefly the approach as in [AG20].

**Approximation of \( E_{T,c} \).** We approximate \( E_{T,c} \) by the domain \( F_{T,c} \) defined by

\[ F_{T,c} := \{ x \in C : x_{n+2}^2 - x_{n+1}^2 < c^2, 1 \leq x_{n+2} + x_{n+1} < e^T \} \]

for large enough integers \( l \), with the sequence \( c_l := c \cdot \left( 1 - \frac{c^2}{l^2} \right)^{1/2} \), \( c_l \uparrow c \), a fixed constant \( r_0 > 0 \), and \( K \)-invariant sets

\[ C_0 := \{ x \in C : x_{n+2} \leq c^2 + 1 \}, \quad \text{and} \quad C_l := \{ x \in C : x_{n+2} \leq l \} \].

Then (2.2) and (2.3) imply, for all \( T > 0 \) and all \( k \in K \) as in (2.2),

\[ |(F_{T-r_0,c_l} \setminus C_l) \cap k \Lambda_0| \leq N_{T,c}(\alpha) + O(1) \leq |F_{T+r_0,c} \cap k \Lambda_0|. \quad (2.4) \]

We observe further that

\[ F_{1,c} \setminus F_{1,c_l} = \{ x \in C : c_l \leq (x_1^2 + \cdots + x_n^2)^{1/2} < c, 1 < x_{n+2} + x_{n+1} < e \} \]

hence, as \( l \to \infty \),

\[ \text{vol}(F_{1,c}) = \text{vol}(F_{1,c_l}) + O(c - c_l) = \text{vol}(F_{1,c_l}) + O(l^{-1}). \quad (2.5) \]

**Tessellation of \( F_{T,c} \).** We observe further that the domain \( F_{T,c} \) can be tessellated using translates of the set \( F_{1,c} \) under the action of \( \{a_t\} \). We have, for all \( N \geq 1 \),

\[ F_{N,c} = \bigcup_{j=0}^{N-1} a_{-j}(F_{1,c}). \quad (2.6) \]

We denote by \( \chi_{1,c} \) the characteristic function of \( F_{1,c} \), and \( \hat{\chi}_{1,c} \) its Siegel transform defined by

\[ \hat{\chi}_{1,c}(\Lambda) := \sum_{z \in \Lambda \setminus \{0\}} \chi_{1,c}(z), \quad \text{for all} \ \Lambda \in \mathcal{X}. \]

The tessellation (2.6) implies, for all \( T > 0 \) and all \( \Lambda \in \mathcal{X} \),

\[ \sum_{t=0}^{[T]-1} \hat{\chi}_{1,c}(a_t \Lambda) \leq |F_{T,c} \cap \Lambda| \leq \sum_{t=0}^{[T]} \hat{\chi}_{1,c}(a_t \Lambda). \quad (2.7) \]
We estimate the number of lattice points in \( C_t \) by \( O(t^n) \). It follows from \((2.4)\) and \((2.7)\), for all \( T > 0 \) and \( k \in K \) as in \((2.2)\),

\[
\sum_{t=0}^{[T-r_0]-1} \hat{\chi}_{1,c}(a_t k \Lambda_0) + O(t^n) \leq N_{T,c}(\alpha) + O(1) \leq \sum_{t=0}^{[T+r_0]} \hat{\chi}_{1,c}(a_t k \Lambda_0). \tag{2.8}
\]

Thus, estimating \( N_{T,c}(\alpha) \) amounts to analyzing ergodic sums of the form \( \sum_{t=0}^{N} \hat{\chi}_{1,c} \circ a_t \) on \( Y = K \Lambda_0 \). We will use for this purpose effective equidistribution results for unimodular lattices and specialize them to \( Y \). It will be important in our argument later that the error term in the effective equidistribution is explicit in terms of the \( C^l \)-norm, for some \( l \geq 1 \), of the test functions on \( X \). We introduce below the required notations.

Every \( Y \in \text{Lie}(G) \) defines a first order differential operator \( D_Y \) on \( C^\infty_c(X) \) by

\[
D_Y(\phi)(x) := \frac{d}{dt} \phi(\exp(tY)x)|_{t=0}.
\]

If \( \{Y_1, \ldots, Y_r\} \) is a basis of \( \text{Lie}(G) \), then every monomial \( Z = Y_1^{l_1} \cdots Y_r^{l_r} \) defines a differential operator by

\[
D_Z := D_{Y_1}^{l_1} \cdots D_{Y_r}^{l_r},
\tag{2.9}
\]

of degree \( \text{deg}(Z) = l_1 + \cdots + l_r \). For \( l \geq 1 \) and \( \phi \in C^\infty_c(X) \), we write

\[
||\phi||_l := ||\phi||_{C^l} = \sum_{\text{deg}(Z) \leq l} ||D_Z(\phi)||_\infty
\]

We recall in the following section some properties of the Siegel transform that we use later to analyze the ergodic sums \( \sum_{t=0}^{N} \hat{\chi}_{1,c} \circ a_t \).

3. APPROXIMATION OF THE COUNTING FUNCTION

3.1. Siegel transform. Given a bounded measurable function \( f : \mathbb{R}^{n+2} \to \mathbb{R} \) with compact support, its Siegel transform on the space \( \mathcal{L} \) of unimodular lattices in \( \mathbb{R}^{n+2} \) is defined by

\[
\hat{f}(\Lambda) := \sum_{z \in \Lambda \setminus \{0\}} f(z), \quad \text{for } \Lambda \in \mathcal{L}. \tag{3.1}
\]

The Siegel transform of a bounded function is typically unbounded, but its growth rate is controlled by an explicit function \( \alpha \) defined as follows.

Given a lattice \( \Lambda \in \mathcal{L} \), we say that a subspace \( V \) of \( \mathbb{R}^{n+2} \) is \( \Lambda \)-rational if the intersection \( V \cap \Lambda \) is a lattice in \( V \). If \( V \) is \( \Lambda \)-rational, we denote \( d_\Lambda(V) \) the covolume of \( V \cap \Lambda \) in \( V \). We define then

\[
\alpha(\Lambda) := \sup \left\{ d_\Lambda(V)^{-1} : V \text{ is a } \Lambda \text{-rational subspace of } \mathbb{R}^{n+2} \right\}.
\]

It follows from Mahler’s Compactness Criterion that \( \alpha \) is a proper map \( \mathcal{L} \to [1, +\infty) \). We recall below some important properties.

**Proposition 3.1** [Sch68]. If \( f : \mathbb{R}^{n+2} \to \mathbb{R} \) is a bounded function with compact support, then

\[
|\hat{f}(\Lambda)| \ll_{\text{supp}(f)} ||f||_\infty \alpha(\Lambda), \quad \text{for all } \Lambda \in \mathcal{L}.
\]

We restrict this function to the space \( X \) of lattices on the positive light cone and denote it also by \( \alpha \). Similarly to its \( L^p \)-integrability in \( \mathcal{L} \) (see [EMM98]), we verify in the following proposition that \( \alpha \) is also \( L^p \)-integrable in the space \( X \).
Proposition 3.2. The function $\alpha$ is in $L^p(\mathcal{X})$ for $1 \leq p < n$. In particular,

$$\mu_\mathcal{X}(\{\alpha \geq L\}) \ll_p L^{-p}, \quad \text{for all } p < n.$$ 

Proof. We show $L^p$-integrability of the function $\alpha$ using reduction theory and an explicit measure of integration on $G$. For this purpose we consider a more convenient coordinate system and write

$$Q(x) = x_1x_{n+2} + x_2^2 + \cdots + x_{n+1}^2, \quad \text{for } x = (x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2},$$

$$G = \text{SO}(Q)^0 \cong \text{SO}(n + 1, 1)^0, \quad \text{and } \Gamma \text{ the stabilizer of } \Lambda_0 \text{ in } G,$$

$$A = \{a_t : t \in \mathbb{R}\} \subset G \quad \text{with } \quad a_t := \text{diag}(e^t, 1, \ldots, 1, e^{-t}),$$

$$N = \{g \in G : a_1ga_t \to e \text{ as } t \to \infty\}, \quad \text{the expanding horospherical subgroup to } A,$$

$$K \cong \text{SO}(n + 1) \text{ a maximal compact in } G,$$

with their respective Haar measures $\mu_G, dt, \mu_N, \mu_K$, and consider the Iwasawa decomposition $G = KAN$.

We have $N = \exp(\mathcal{N})$ with the space of positive roots

$$\mathcal{N} := \bigoplus_{\rho(t) > 1} \{g \in \text{Lie}(G) : a_t ga_t = \rho(t)g\}.$$ 

One verifies easily that the unique positive root is $\rho_0(t) = e^t$ with multiplicity $n$, which yields a measure of integration in the coordinates $G = KAN$ given by

$$\mu_G = \prod_{\rho(t) > 1} \rho(t)d\mu_Kdtd\mu_N = e^{nt}d\mu_Kdt\mu_N.$$ 

For $\beta \in \mathbb{R}$, let $A_\beta := \{\text{diag}(e^t, 1, \ldots, 1, e^{-t}) : t < \beta\} \subset A$. By reduction theory on the space of lattices $\mathcal{X} \cong G/\Gamma$, we have that there exist $\beta > 0$ and a compact set $N_0 \subset N$ such that the union of finitely many translates of the Siegel set $S_\beta := KA_\beta N_0$ contains a fundamental domain for the action of $\Gamma$ on $G$. Thus, it is enough to verify the integrability of $\alpha$ on $S_\beta$. Moreover, since $N_0$ and $K$ are compact, $\{a_tna_{-t} : a_t \in A_\beta, n \in N_0\}$ uniformly bounded, and since there exists $C > 0$ such that $\alpha(tZ^n) \leq C\alpha(x)$ for all $x \in \mathcal{X}$ and uniformly for all $g$ in a compact set, we have for any $g = ka_n \in S_\beta$,

$$\alpha(gZ^{n+2}) = \alpha(ka_nna_{-t}Z^{n+2}) \ll \alpha(a_tZ^{n+2}).$$

By definition of $\alpha$, we have further $\alpha(a_tZ^{n+2}) = \max_{1 \leq j \leq n+1} \prod_{1 \leq i \leq j} a_{t, i, j} = e^{-t}$. Hence,

$$\int_{\mathcal{X}} \alpha(\Lambda)^p d\mu_\mathcal{X}(\Lambda) \ll \int_{S_\beta} (gZ^{n+2})^p d\mu_G(g)$$

$$\leq \int_{KA_\beta N_0} (ka_nZ^{n+2})^p e^{nt}d\mu_K(k)d\mu_N(n) \ll \int_{A_\beta} (a_tZ^{n+2})^{p}e^{nt}dt = \int_{-\infty}^{\beta} e^{-pt}e^{nt}dt < \infty,$$

for all $p < n$.

It follows in particular

$$\mu_\mathcal{X}(\{\alpha \geq L\}) \ll L^{-p}, \quad \text{for all } p < n.$$ 

□

Given a bounded measurable function $f : C \to \mathbb{R}$ with compact support, we define and denote similarly to (3.1) its Siegel transform on the space $\mathcal{X}$,

$$\hat{f}(\Lambda) := \sum_{z \in \Lambda \setminus \{0\}} f(z), \quad \text{for } \Lambda \in \mathcal{X}. \quad (3.2)$$
We recall the Siegel Mean Value Theorem in the space of unimodular lattices \( \mathcal{L} \) (see \cite{Sie45}) and verify in the following proposition an analogous result for the space \( \mathcal{X} \).

**Proposition 3.3.** If \( f : \mathcal{C} \to \mathbb{R} \) is a bounded Riemann integrable function with compact support, then

\[
\int_{\mathcal{X}} \hat{f}(\Lambda)d\mu_{\mathcal{X}}(\Lambda) = \int_{\mathcal{C}} f(x)dx
\]

for some \( G \)-invariant measure \( dx \) on \( \mathcal{C} \).

**Proof.** The map \( f \mapsto \int_{\mathcal{X}} \hat{f} \) defines a positive \( G \)-invariant linear functional on the space of continuous compactly supported functions on \( \mathcal{C} \), hence a \( G \)-invariant measure on \( \mathcal{C} \). Uniqueness up to multiplication by a scalar of the invariant measure yields the claim. \( \square \)

### 3.2. Non-divergence estimates.

In this subsection, we establish bounds for the Siegel transform \( \hat{f} \) on translated \( K \)-orbits by analyzing the escape of mass on submanifolds \( a_{t}Y \subset \mathcal{X} \).

Following the same argument as in \cite{BG18} (Proposition 4.5) and using effective equidistribution of translated \( K \)-orbits (which we establish later in Proposition 4.1) and \( L^{p} \)-integrability of the function \( \alpha \) established in Proposition 3.2 we verify an analogous non-escape of mass in \( a_{t}Y \).

**Proposition 3.4.** There exists \( \kappa > 0 \) such that for every \( L \geq 1 \) and \( t \geq \kappa \log L \),

\[
\mu_{Y}(\{y \in Y : \alpha(a_{t}y) \geq L\}) \ll_{p} L^{-p}, \quad \text{for all } p < n.
\]

**Proof.** Let \( \chi_{L} \) be the characteristic function of the set \( \{\alpha < L\} \subset \mathcal{X} \). By Mahler’s Compactness Criterion, \( \chi_{L} \) has compact support. Let \( \rho \in C_{c}^{\infty}(G) \) be a non-negative function with \( \int_{G} \rho = 1 \) and define

\[
\eta_{L}(x) := (\rho * \chi_{L})(x) = \int_{G} \rho(g)\chi_{L}(g^{-1}x)d\mu_{G}(g), \quad \text{for all } x \in \mathcal{X}.
\]

Since \( \mu_{\mathcal{X}} \) is \( G \)-invariant, we have

\[
\int_{\mathcal{X}} \eta_{L}d\mu_{\mathcal{X}} = \int_{\mathcal{X}} \chi_{L}d\mu_{\mathcal{X}} = \mu_{\mathcal{X}}(\{\alpha < L\}).
\]

Moreover, by invariance of \( \mu_{G} \), we have for any differential operator \( D_{Z} \) as in (2.9) that

\[
D_{Z} \eta_{L} = D_{Z}(\rho) * \chi_{L}, \quad \text{hence } \eta_{L} \in C_{c}^{\infty}(\mathcal{X}) \text{ with } ||\eta_{L}||_{C^{1}} \ll ||\rho||_{C^{1}}.
\]

Further, one can verify that there exists \( c > 1 \) such that \( \alpha(g^{-1}x) \geq c^{-1}\alpha(x) \) for every \( g \in \text{supp}(\rho) \) and \( x \in \mathcal{X} \), hence \( \{\alpha \circ g^{-1} < L\} \subset \{\alpha < cL\} \) and \( \eta_{L} \leq \chi_{cL} \). Thus,

\[
\mu_{Y}(\{y \in Y : \alpha(a_{t}y) < cL\}) = \int_{Y} \chi_{cL}(a_{t}y)d\mu_{Y}(y) \geq \int_{Y} \eta_{L}(a_{t}y)d\mu_{Y}(y).
\]

By exponential equidistribution of translated \( K \)-orbits, which we prove later in Proposition 4.1 there exist \( \gamma > 0 \) and \( l \geq 1 \) such that

\[
\int_{Y} \eta_{L}(a_{t}y)d\mu_{Y}(y) = \int_{\mathcal{X}} \eta_{L}d\mu_{\mathcal{X}} + O(e^{-\gamma t}||\eta_{L}||_{C^{1}})
\]

\[
= \mu_{\mathcal{X}}(\{\alpha < L\}) + O(e^{-\gamma t}),
\]

and by Proposition 3.2

\[
\mu_{\mathcal{X}}(\{\alpha \geq L\}) \ll_{p} L^{-p} \quad \text{for all } p < n.
\]
Altogether we obtain
\[ \mu_Y(\{y \in Y : \alpha(a_t y) < cL\}) \geq \mu_X(\{\alpha < L\}) + O(L^{-\gamma t}) = 1 + O_P(L^{-p} + e^{-\gamma t}), \]
thus
\[ \mu_Y(\{y \in Y : \alpha(a_t y) \geq cL\}) \ll_L L^{-p} + e^{-\gamma t}, \]
which yields the claim for \( s \geq \kappa \log L \) with \( \kappa = \frac{p}{\gamma}. \)
\[ \square \]

An important estimate in our argument later is the integrability of the Siegel transform \( \hat{f} \) on \( a_t Y \) uniformly in \( t \). We use the following integrability estimate for the function \( \alpha \).

**Proposition 3.5** ([EMM98]). If \( n \geq 2 \) and \( 0 < s < 2 \), then for any lattice \( \Lambda \) in \( \mathbb{R}^{n+2} \)
\[ \sup_{t>0} \int_K \alpha(a_t k \Lambda)^s d\mu_K(k) < \infty. \]

### 3.3. Truncated Siegel transform.

The Siegel transform of a smooth compactly supported function is typically not bounded. To be able to apply equidistribution results, we truncate the Siegel transform using a smooth cut-off function \( \eta_L \) built on the function \( \alpha \). We use the same construction as in [BG18, Lemma 4.9] which yields the following lemma.

**Lemma 3.1.** For every \( \xi > 1 \), there exists a family \( (\eta_L) \) in \( C_c^\infty (X) \) satisfying:
\[ 0 \leq \eta_L \leq 1, \quad \eta_L = 1 \text{ on } \{ \alpha \leq \xi^{-1} L \}, \quad \eta_L = 0 \text{ on } \{ \alpha > \xi L \}, \quad ||\eta_L||_{C^l} < 1. \]

For a bounded function \( f : C \to \mathbb{R} \) with compact support, we define the **truncated Siegel transform** of \( f \) by
\[ \hat{f}(L) := \hat{f} \cdot \eta_L. \]

We record in the following proposition some properties of the truncated Siegel transform \( \hat{f}(L) \) which we use later in our arguments.

**Proposition 3.6.** For a bounded measurable function \( f : C \to \mathbb{R} \) with compact support, the truncated Siegel transform \( \hat{f}(L) \) satisfies the following bounds:
\[ ||\hat{f}(L)||_\infty \ll_{\text{supp}(f)} L ||f||_\infty, \quad (3.3) \]
\[ \sup_{t \geq 0} ||\hat{f}(L) \circ a_t||_{L^p_Y} < \infty, \quad \text{for all } 1 \leq p < 2, \quad (3.4) \]
\[ ||\hat{f} - \hat{f}(L)||_{L^1_x} \ll_{\text{supp}(f), \tau} L^{-(\tau-1)} ||f||_{\infty}, \quad \text{for all } \tau < n, \quad (3.5) \]
\[ ||\hat{f} \circ a_t - \hat{f}(L) \circ a_t||_{L^p_Y} \ll_{\text{supp}(f), \tau} L^{-(\tau-1)} ||f||_{\infty}, \quad \text{for all } 1 \leq p < 2, \quad \tau < n \text{ and } t \geq \kappa \log L. \]
\[ (3.6) \]

Moreover, if \( f \in C_c^\infty (C) \) then \( \hat{f}(L) \in C_c^\infty (X) \) and satisfies
\[ ||\hat{f}(L)||_{C^l} \ll_{\text{supp}(f)} L ||f||_{C^l}, \quad \text{for all } l \geq 1. \]
\[ (3.7) \]

**Proof.** Since \( \text{supp}(\eta_L) \subset \{ \alpha \leq \xi L \} \), the first estimate follows from Proposition 3.1. Since \( 0 \leq \eta_L \leq 1 \), the second estimate follows immediately from Proposition 3.5. Further, since \( \eta_L = 1 \) on \( \{ \alpha \leq \xi^{-1} L \} \), it follows from Proposition 3.1 that
\[ \|\hat{f} - \hat{f}(L)\|_{L^1_x} = \int_X |\hat{f}| 1 - \eta_L |d\mu_x| \ll_{\text{supp}(f)} \int_{\{\alpha \geq \xi^{-1} L\}} \alpha d\mu_x \|f\|_\infty. \]
We apply Hölder’s Inequality with $1 \leq p < n$ and $q = (1 - 1/p)^{-1}$ and deduce
\[
\|\hat{f} - \hat{f}^{(L)}\|_{L^1} \ll \|\alpha\|_{L^1} \mu_X(\{\alpha \geq \xi^{-1} L\})^{1/q} \|f\|_{\infty}.
\]
Then Proposition 3.2 implies
\[
\|\hat{f} - \hat{f}^{(L)}\|_{L^1} \ll \text{supp}(f), p L^{-(p-1)} \|f\|_{\infty}.
\]
Similarly, Hölder’s Inequality with $1 \leq p < s < 2$ and $q = (1/p - 1/s)^{-1}$ gives
\[
\|\hat{f} \circ a_t - \hat{f}^{(L)} \circ a_t\|_{L^p} \ll \|\alpha \circ a_t\|_{L^p} \mu_Y(\{\alpha \circ a_t \geq \xi^{-1} L\})^{1/q} \|f\|_{\infty},
\]
then Propositions 3.4 and 3.5 imply, for all $t \geq \kappa \log L$ and $\varepsilon > 0$,
\[
\|\hat{f} \circ a_t - \hat{f}^{(L)} \circ a_t\|_{L^p} \ll \text{supp}(f), \ell, \varepsilon L^{-(n-\varepsilon)} \|f\|_{\infty}
\ll \text{supp}(f), \tau L^{-\tau(2-p)} \|f\|_{\infty},
\]
for all $\tau < n$.

For $f \in C^\infty_c(C)$ and any differential operator $D_Z$ as in (2.9), we observe that $D_Z(\hat{f}) = \widehat{D_Z(f)}$. Hence, Proposition 3.6 implies
\[
|D_Z(\hat{f})| \ll \text{supp}(f), \|f\|_{C^\ell}\alpha.
\]
Since $\text{supp}(\eta_{\varepsilon}) \subset \{\alpha \leq \xi L\}$ and $\|\eta_{\varepsilon}\|_{C^\ell} \ll 1$, it follows
\[
\|\hat{f}^{(L)}\|_{C^\ell} \ll \text{supp}(f), L\|f\|_{C^\ell}.
\]

3.4. Smooth approximation of the counting function. For simplicity we write $\chi := \chi_{F_{1,c}}$ for the characteristic function of the set $F_{1,c}$. We approximate $\chi$ by a family of non-negative functions $f_{\varepsilon} \in C^\infty_c(C)$ with support in an $\varepsilon$-neighborhood of $F_{1,c}$ such that
\[
\chi \leq f_{\varepsilon} \leq 1, \quad \|f_{\varepsilon} - \chi\|_{L^1} \ll \varepsilon, \quad \|f_{\varepsilon} - \chi\|_{L^2} \ll \varepsilon^{1/2}, \quad \|f_{\varepsilon}\|_{C^\ell} \ll \varepsilon^{-1}.
\]

(3.8)

The following proposition shows that this approximation of $\chi$ also yields a good approximation of its Siegel transform $\hat{\chi}$ on translated $K$-orbits in the following sense.

Proposition 3.7. There exists $\theta > 0$ such that for every $\varepsilon > 0$ and $t \geq -\frac{1}{\theta} \log \varepsilon$,
\[
\int |\hat{f}_{\varepsilon} \circ a_t - \hat{\chi} \circ a_t| d\mu_{\varepsilon} \ll \varepsilon
\]

Proof. We first observe that there exists $c_\varepsilon > c$ such that $c_\varepsilon = c + O(\varepsilon)$ and $f_{\varepsilon} \leq \chi_{\varepsilon}$, where $\chi_{\varepsilon}$ denotes the characteristic function of the set
\[
\{x \in C : 1 - \varepsilon \leq x_{n+2} + x_{n+1} \leq \varepsilon, \quad x_{n+2}^2 - x_{n+1}^2 < c_\varepsilon^2\}
\]
The difference $\chi_{\varepsilon} - \chi$ is bounded by the sum $\chi_{\varepsilon}^{(1)} + \chi_{\varepsilon}^{(2)} + \chi_{\varepsilon}^{(3)}$ of the characteristic functions of the sets
\[
\{x \in C : 1 - \varepsilon \leq x_{n+2} + x_{n+1} \leq 1, \quad x_{n+2}^2 - x_{n+1}^2 < c_\varepsilon^2\},
\]
\[
\{x \in C : \varepsilon \leq x_{n+2} + x_{n+1} \leq \varepsilon + \varepsilon, \quad x_{n+2}^2 - x_{n+1}^2 < c_\varepsilon^2\},
\]
\[
\{x \in C : 1 \leq x_{n+2} + x_{n+1} \leq \varepsilon, \quad c_\varepsilon^2 < x_{n+2}^2 - x_{n+1}^2 < c_\varepsilon^2\}.
\]
Since $0 \leq \chi \leq f_{\varepsilon} \leq \chi_{\varepsilon}$, it follows in particular
\[
\hat{f}_{\varepsilon}(a_t \Lambda) - \hat{\chi}(a_t \Lambda) \leq \hat{\chi}_{\varepsilon}^{(1)}(a_t \Lambda) + \hat{\chi}_{\varepsilon}^{(2)}(a_t \Lambda) + \hat{\chi}_{\varepsilon}^{(3)}(a_t \Lambda).
\]
We first consider $\chi^{(1)}_\varepsilon$. For $x$ in the corresponding set, we also have
\[ 0 \leq x_{n+2} - x_{n+1} < e^2/(1 - \varepsilon) \quad \text{and} \quad x_1^2 + \cdots + x_n^2 < e^2. \]

We write $I_{0,\varepsilon} := [0, c_\varepsilon], I_{1,\varepsilon} := [-c_\varepsilon/(1 - \varepsilon), 0], I_{2,\varepsilon} := [1 - \varepsilon, 1], k = (k_1, \ldots, k_{n+2})^T \in K$, and compute
\[
\int_y |\chi^{(1)}_\varepsilon \circ a_t| \, d\mu_y = \int_K \chi^{(1)}_\varepsilon(a_k \Lambda_0) \, d\mu_K(k) = \int_K \sum_{z \in \Lambda_0} \chi^{(1)}_\varepsilon(a_k z) \, d\mu_K(k)
\]
\[
= \sum_{z \in \Lambda_0} \int_K \chi^{(1)}_\varepsilon \left( \begin{array}{c}
\langle k_1, z \\
\vdots \\
\langle k_{n+1}, z \\
\langle k_{n+1}, z \rangle \cosh t - z_{n+2} \sinh t \\
\langle k_{n+1}, z \rangle ( - \sinh t) + z_{n+2} \cosh t 
\end{array} \right) \, d\mu_K(k)
\]
\[
= \sum_{z \in \Lambda_0} \int_K \chi_{I_{0,\varepsilon}} (||\langle k_1, z \rangle, \ldots, \langle k_{n+1}, z \rangle||) \chi_{I_{1,\varepsilon}} (e^{-t} (\langle k_{n+1}, z \rangle - z_{n+2})) \chi_{I_{2,\varepsilon}} (e^{-t} (\langle k_{n+1}, z \rangle + z_{n+2})) \, d\mu_K(k).
\]

We observe that the intersection $(e^{-t} I_{1,\varepsilon} + z_{n+2}) \cap (e^{-t} I_{2,\varepsilon} - z_{n+2})$ is non-empty only if $(1 - \varepsilon) e^t \leq 2z_{n+2} \leq e^t + \frac{e^2}{1 - \varepsilon} e^{-t}$, i.e. $z_{n+2} = e^t/2 + O(\varepsilon e^t + e^{-t})$. Moreover, writing each $z \in \Lambda_0$ as $z = z_{n+2} k_z v_0$ with some $k_z \in K$ and $v_0 = (0, \ldots, 0, 1, 1) \in \mathbb{C}$, and using invariance under $k_z$, we have
\[
\int_y |\chi^{(1)}_\varepsilon \circ a_t| \, d\mu_y
\]
\[
\leq \sum_{z \in \Lambda_0} \int_K \chi_{I_{0,\varepsilon}} (z_{n+2} ||\langle k_1, v_0 \rangle, \ldots, \langle k_{n+1}, v_0 \rangle||) \chi_{e^{-t} I_{1,\varepsilon}} (z_{n+2} (\langle k_{n+1}, v_0 \rangle - 1)) \cdot \chi_{e^{-t} I_{2,\varepsilon}} (z_{n+2} (\langle k_{n+1}, v_0 \rangle + 1)) \, d\mu_K(k)
\]
\[
\leq \sum_{z \in \Lambda_0} \int_K \chi_{e^{-t} I_{0,\varepsilon}} (||\langle k_1, v_0 \rangle, \ldots, \langle k_{n+1}, v_0 \rangle||) \chi_{e^{-2t} I_{1,\varepsilon}} (\langle k_{n+1}, v_0 \rangle - 1) \cdot \chi_{e^{-2t} I_{2,\varepsilon}} (\langle k_{n+1}, v_0 \rangle + 1) \, d\mu_K(k)
\]
\[
\leq \sum_{z \in \Lambda_0} \mu_K \left( \left\{ k \in K : |k_{i,n+1}| \ll e^{-t}, \quad i = 1, \ldots, n, \quad |k_{n+1,n+1} - 1| \ll \min(e^{-2t}, \varepsilon) \right\} \right)
\]
\[
\leq \sum_{z \in \Lambda_0} \mu_S^n \left( \left\{ v \in S^n : |v - v_0| \ll e^{-t} \right\} \right)
\]
\[
\ll \sum_{z \in \Lambda_0} e^{-nt}.
\]

\[
\ll \sum_{z \in \Lambda_0} e^{-nt}.
\]
We use further that there exist positive constants $C$ and $\theta$ such that, for all $n \geq 2$, we have
\[
|\{z \in C \cap \mathbb{Z}^{n+2} : 0 \leq z_{n+2} < T\}| = C T^n + O(T^{n-\theta}),
\]
hence
\[
|\{z \in C \cap \mathbb{Z}^{n+2} : (1 - \varepsilon)e^t \leq 2z_{n+2} < \varepsilon e^t + \frac{c^2}{1 - \varepsilon} e^{-t}\}| \leq \varepsilon e^{nt} + O(e^{(n-\theta)t}).
\]
It follows
\[
\int |\tilde{\chi}^{(1)}_\varepsilon \circ a_t| \, d\mu_y \ll \varepsilon + e^{-\theta t}.
\] (3.9)

We proceed similarly for $\chi^{(3)}_\varepsilon$. For $x$ in the corresponding set, we also have
\[
\frac{c^2}{\varepsilon} \leq x_{n+2} - x_{n+1} < \frac{c}{\varepsilon}^2 \quad \text{and} \quad c^2 < x_1^2 + \cdots + x_n^2 < \frac{c^2}{\varepsilon}.
\]
We write $I'_{0,\varepsilon} := [c, c\varepsilon]$, $I'_{1,\varepsilon} := [-c^2, -c^2/\varepsilon]$, $I_2' := [1, \varepsilon]$ and compute similarly
\[
\int |\tilde{\chi}^{(3)}_\varepsilon \circ a_t| \, d\mu_y = \int_K \sum_{z \in \Lambda_0} \chi^{(3)}_\varepsilon(a_kz) \, d\mu_K(k)
\]
\[
= \sum_{z \in \Lambda_0} \int_K \chi_{I'_{0,\varepsilon}}(\langle k_1, z \rangle, \ldots, \langle k_n, z \rangle) \chi_{I'_{1,\varepsilon}}(e^t(\langle k_{n+1}, z \rangle - z_{n+2})) \chi_{I_2'}(e^{-t}(\langle k_{n+1}, z \rangle + z_{n+2})) \, d\mu_K(k).
\]
We observe again that the intersection $(e^{-t}I'_{1,\varepsilon} + z_{n+2}) \cap (e^tI_2' - z_{n+2})$ is non-empty only if $C_1 e^t \leq z_{n+2} \leq C_2 e^t$ for some positive constants $C_1$ and $C_2$. Moreover, writing each $z \in \Lambda_0$ as $z = z_{n+2}k_z v_0$ with some $k_z \in K$ and $v_0 = (0, \ldots, 0, 1, 1) \in C$, and using invariance under $k_z$, we have
\[
\int |\tilde{\chi}^{(3)}_\varepsilon \circ a_t| \, d\mu_y
\]
\[
\leq \sum_{z \in \Lambda_0} \int_K \chi_{I'_{0,\varepsilon}}(z_{n+2}||\langle k_1, v_0 \rangle, \ldots, \langle k_n, v_0 \rangle||) \chi_{I'_{1,\varepsilon}}(z_{n+2}||\langle k_{n+1}, v_0 \rangle - 1||) \chi_{I_2'}(z_{n+2}||\langle k_{n+1}, v_0 \rangle + 1||) \, d\mu_K(k)
\]
\[
\leq \sum_{z \in \Lambda_0} \int_K \chi_{e^{-t}I_0'}(\langle k_1, v_0 \rangle, \ldots, \langle k_n, v_0 \rangle) \chi_{e^{-t}I_1'}(\langle k_{n+1}, v_0 \rangle - 1) \chi_{e^{-t}I_2'}(\langle k_{n+1}, v_0 \rangle + 1) \, d\mu_K(k)
\]
\[
\leq \sum_{z \in \Lambda_0} \mu_K(\{k \in K : ||kv_0 - v_0|| \ll \varepsilon e^{-t}\})
\]
\[
\ll \sum_{z \in \Lambda_0} \mu_{S^n}(\{v \in S^n : ||v - v_0|| \ll \varepsilon e^{-t}\})
\]
\[
\ll \sum_{z \in \Lambda_0} \varepsilon^n e^{-nt}.
\]
We use again the estimate
\[\{|z \in C \cap \mathbb{Z}^{n+2} : z_{n+2} \geq e^t\}| = O(e^t),\]
hence
\[\int_y |\hat{x}^{(3)}_e \circ a_t| \, d\mu_y \ll \varepsilon.\]
The bound for \(|\hat{x}^{(2)}_e \circ a_t||L_y^1|\) is obtained similarly as for \(\hat{x}^{(1)}_e\).
Altogether we obtain, for all \(t \geq -\frac{1}{9} \log \varepsilon\),
\[||f \circ a_t - \hat{x} \circ a_t||L_y^1 < \varepsilon.\]
\[\square\]

4. Effective equidistribution

4.1. Effective double equidistribution of translated \(K\)-orbits. In this section we prove an effective equidistribution of \(K\)-orbits by relating it to effective equidistribution of unstable horospherical orbits established in a more general setting in \([BG21]\). We recall the notations
\[G = SO(Q)^{\circ} \cong SO(n+1,1)^{\circ},\]
\[K = \left(\begin{array}{c}
SO(n+1) \\
1
\end{array}\right),\]
\[a_t = \left(\begin{array}{cc}
\cosh t & -\sinh t \\
-\sinh t & \cosh t
\end{array}\right) \in G, \quad \text{and} \quad A = \{a_t : t \in \mathbb{R}\}.
\]
We also consider the corresponding horospherical subgroups
\[U = \{g \in G : a_{-t}ga_t \to e \text{ as } t \to \infty\},\]
\[U^- = \{g \in G : a_tga_{-t} \to e \text{ as } t \to \infty\},\]
\[H = \{g \in G : a_tg = ga_t\},\]
and the probability measures \(d\mu_K, dt\) and \(d\mu_U\) on \(K, A\) and \(U\) respectively.

We denote \(B_r^K\) the ball of radius \(r > 0\) centered at the identity in \(K\).

Theorem 4.1 (specializes Theorem 1.1. in \([BG21]\)). There exists \(\delta > 0\) and \(l \geq 1\) such that, for every \(f \in C_c^\infty(U)\) and \(\varphi, \psi \in C_c^\infty(\mathcal{X})\) and every compact subset \(L \subset \mathcal{X}\), there exists \(C > 0\) such that for every \(\Lambda \in L\) and \(t_1, t_2 \geq 0\), one has
\[\left|\int_U f(u)\varphi(a_{t_1}u\Lambda)\psi(a_{t_2}u\Lambda)d\mu_U(u) - \int f \int \varphi \int \psi\right| \leq Ce^{-\delta \min(t_1, t_2, |t_1-t_2|)||f||_l||\varphi||_l||\psi||_l}.
\]
In the following proposition we prove an analogous effective double equidistribution for translated \(K\)-orbits.

Proposition 4.1. There exists \(\gamma > 0\) and \(l \geq 1\) such that, for every \(f \in C_c^\infty(K)\) and \(\varphi, \psi \in C_c^\infty(\mathcal{X})\) and every compact subset \(L \subset \mathcal{X}\), there exists \(C > 0\) such that for every \(\Lambda \in L\) and \(t_1, t_2 \geq 0\), we have
\[\left|I_{\Lambda,f,\varphi,\psi}(t_1, t_2) - \int_K f \int \varphi \int \psi\right| \leq Ce^{-\gamma \min(t_1, t_2, |t_1-t_2|)||f||_l||\varphi||_l||\psi||_l},\]
where \(I_{\Lambda,f,\varphi,\psi}(t_1, t_2) := \int_K f(k)\varphi(a_{t_1}k\Lambda)\psi(a_{t_2}k\Lambda)d\mu_K(k)\).
Proof. We consider the centralizer of $A$ in $K$, 
\[ M := \text{cent}_K(A) = K \cap H = \left( \text{SO}(n) \bigg/ I_2 \right) \cong \text{SO}(n), \]
and the submanifold $S \subset K$ defined via the exponential map by
\[ \text{Lie}(S) = \left\{ \begin{pmatrix} 0_n & s \\ -s^T & 0 \end{pmatrix} : s \in \mathbb{R}^n \right\}. \]
We have $\text{Lie}(K) = \text{Lie}(M) \oplus \text{Lie}(S)$ and the map $M \times S \to K$ is a diffeomorphism in a neighborhood of the identity, giving a unique decomposition $k = m(k)s(k)$ and also $d\mu_K = d\mu_M \times d\mu_S$, where we denote by $d\mu_M$ the Haar measure on $M$ and by $d\mu_S$ a smooth measure defined on a neighborhood of the identity in $S$.

Further, we consider the decomposition of $G$ as the product $U^- H U$ in a neighborhood of the identity, giving a unique decomposition $s = u^-(s)h(s)u(s)$. We verify that the coordinate map $S \to U$, $s \mapsto u(s)$ is a diffeomorphism in a neighborhood of the identity. We first observe that
\[ \dim(S) = \dim(K) - \dim(M) = \frac{(n+1)n}{2} - \frac{(n-1)n}{2} = n = \dim(U). \]
Moreover, for the product map $p : U^- \times H \times U \to G$, $(u^-, h, u) \mapsto u^- hu$, the derivative at the identity is given by $D(p)_e(x, y, z) = x + y + z$, for all $(x, y, z) \in \text{Lie}(U^-) \times \text{Lie}(H) \times \text{Lie}(U)$. Hence, for all $w \in \text{Lie}(G)$, the $U$-component of $D(p)_e^{-1}(w)$ is zero if and only if $w \in \text{Lie}(U^-) + \text{Lie}(H)$. Since $\text{Lie}(S) \cap (\text{Lie}(U^-) + \text{Lie}(H)) = 0$, the derivative of $s \mapsto u(s)$ is injective.

We localize the problem to a neighborhood of the identity by considering the partition of unity $1 = \sum_{i=1}^N \phi_i(kk^{-1})$ for some $k_i \in \text{supp}(f)$ and all $k \in \text{supp}(f)$, with non-negative functions $\phi_i \in C^\infty_{\text{loc}}(K)$ such that $\text{supp}(\phi_i) \subseteq B_{K}^r$, $\|\phi_i\|_1 \ll r^{-\nu}$ and $N \ll r^{-\lambda}$, for some $\nu$, $\lambda > 0$, and for $r > 0$ small enough to be fixed later.

We write for simplicity $k = m_k s_k = m_k u_{s_k}^a h_{s_k} u_{s_k}$, the unique decompositions of $k$ and $s$ in a neighborhood of the identity in $K$ and $S$. We also write $f_i(k) := f(kk_i)$ and $\Lambda_i := k_i \Lambda$. We compute
\[ I_{\Lambda, f, \phi, \psi}(t_1, t_2) = \sum_{i=1}^N \int_K \phi_i(k) f(kk_i) \psi(a_{t_1} kk_i \Lambda) \psi(a_{t_2} kk_i \Lambda) d\mu_K(k) \]
\[ = \sum_{i=1}^N \int_K \phi_i(k) f_i(k) \phi(m_k a_{t_1} u_{s_k}^a a_{-t_1} h_{s_k} a_{t_1} u_{s_k} \Lambda_i) \psi(m_k a_{t_2} u_{s_k}^a a_{-t_2} h_{s_k} a_{t_2} u_{s_k} \Lambda_i) d\mu_K(k). \]

By Lipschitz continuity of the coordinate maps $m_k$, $u_{s_k}^a$ and $h_{s_k}$ on $B_{r}^K$ with $r$ small enough, there exists a constant $C_1 > 0$ such that for all $k \in B_{r}^K$, we have
\[ a_{t} u_{s_k}^a a_{-t} \in B_{C_1 r^{-2t}}^K \] and $m_k, h_{s_k} \in B_{C_1 r}^K$.

By Lipschitz continuity of $\phi$ and $\psi$, it follows
\[ \left| I_{\Lambda, f, \phi, \psi}(t_1, t_2) - \sum_{i=1}^N \int_K \phi_i(k) f_i(k) \phi(a_{t_1} u_{s_k} \Lambda_i) \psi(a_{t_2} u_{s_k} \Lambda_i) d\mu_K(k) \right| \ll t \| f \|_{1} \| \phi \|_{1} \| \psi \|_{1}. \]
We use now the decomposition $d\mu_K = d\mu_M \times d\mu_S$ and apply the change of variable $u \mapsto s(u) = s_u$, with a density $\rho$ defined in a neighborhood of the identity in $U$ by

$$\int_S \Phi(s)d\mu_S(s) = \int_U \Phi(s(u))\rho(u)d\mu_U(u) \quad \text{for all } \Phi \in C_c(S) \text{ with supp}(\Phi) \subset B^S_r.$$  

We have

$$\sum_{i=1}^N \int_K \phi_i(k)f_i(k)\varphi(a_{t_1\cdot u_{s_k}\Lambda_i})\psi(a_{t_2\cdot u_{s_k}\Lambda_i})d\mu_K(k)$$

$$= \sum_{i=1}^N \int_{M \times S} \phi_i(ms)f_i(ms)\varphi(a_{t_1\cdot u_{s_k}\Lambda_i})\psi(a_{t_2\cdot u_{s_k}\Lambda_i})d\mu_S(s)d\mu_M(m)$$

$$= \int_M \left( \sum_{i=1}^N \int_U \phi_i(ms_u)f_i(ms_u)\varphi(a_{t_1\cdot u_{s_k}\Lambda_i})\psi(a_{t_2\cdot u_{s_k}\Lambda_i})\rho(u)d\mu_U(u) \right) d\mu_M(m). \quad (4.1)$$

Using Theorem [4.1] with the function $f_{m,i}(u) := \phi_i(ms_u)\rho(u)f_i(mu)$ and observing that $\|f_{m,i}\| \ll \|\phi_i\|\|\rho\|\|f_i\|\|l\|$ and that $\|\rho_B\psi\| \ll 1$, it follows that the integral $(4.1)$ is equal to

$$\int_M \sum_{i=1}^N \left( \int_U \phi_i(ms_u)f_i(ms_u)\rho(u)d\mu_U(u) \right) \int_X \varphi \int_X \psi$$

$$+ O \left( e^{-\delta \min(t_1,t_2,|t_1-t_2|)}\|\phi_i\|\|f\|\|\varphi\|\|\psi\|\|l\| \right) d\mu_M(m)$$

$$= \int_M \left( \sum_{i=1}^N \int_S \phi_i(ms)f_i(ms)d\mu_S(s) \right) d\mu_M(m) \int_X \varphi \int_X \psi$$

$$+ O \left( Ne^{-\delta \min(t_1,t_2,|t_1-t_2|)}\|\phi_i\|\|f\|\|\varphi\|\|\psi\|\|l\| \right)$$

$$= \int_K f \int_X \varphi \int_X \psi + O \left( r^{-\lambda}e^{-\delta \min(t_1,t_2,|t_1-t_2|)}r^{-\nu}\|f\|\|\varphi\|\|\psi\|\|l\| \right),$$

hence

$$I_{\Lambda,f,\varphi,\psi}(t_1,t_2) = \int_K f \int_X \varphi \int_X \psi + O \left( \left( r^{-\lambda-\nu}e^{-\delta \min(t_1,t_2,|t_1-t_2|)} + r \right) \|f\|\|\varphi\|\|\psi\|\|l\| \right).$$

We take $r = e^{-\gamma \min(t_1,t_2,|t_1-t_2|)}$ with $\gamma = \frac{\delta}{1+\lambda+\nu}$, which yields the claim. \hfill $\square$

### 4.2. Effective pointwise equidistribution along $A$-orbits.

In this subsection we use the quantitative double equidistribution of translated $K$-orbits established in Proposition [4.1] and previous estimates to derive a quantitative pointwise equidistribution along $\{a_t\}$-orbits of the averages $\frac{1}{N} \sum_{t=0}^{N-1} \chi \circ a_t$. The approach is presented in [KSW17] and works in a more general context to derive an almost-everywhere bound from an $L^p$-bound, $p > 1$, using Borel-Cantelli Lemma. We follow the same approach in Lemmas [4.1], [4.2] and Proposition [4.2] below.
**Proposition 4.2.** Let $(Y, \nu)$ be a probability space, and let $F : Y \times \mathbb{N} \to \mathbb{R}$ be a measurable function. Suppose there exist constants $p > 1$ and $C > 0$ such that for any integers $0 \leq a < b$,

$$ \int_Y \left( \sum_{t=a}^{b-1} F(y, t) \right)^p \, d\nu(y) \leq C(b - a). $$

Then, for every $\varepsilon > 0$ we have

$$ \sum_{t=0}^{N-1} F(y, t) = O \left( N^{\frac{1}{p}} \log^{1 + \frac{1}{p} + \varepsilon} N \right), $$

for $\nu$-almost every $y \in Y$.

In the following lemmas, the notations and assumptions are the same as in Proposition 4.2. For non-negative integers $m, l$ we write $[m..l] := [m, l] \cap \mathbb{N}$.

In order to satisfy in a later application the conditions $t \geq \kappa \log L$ and $t \geq -\frac{1}{2} \log L$ from Propositions 3.4 and 3.7, we need to use a different dyadic decomposition than the one used in the classical argument. For an integer $s \geq 2$ we consider the following set of dyadic subsets,

$$ L_s := \{[2^i..2^{i+1}) : 0 \leq i \leq s - 2 \} \cup \{[2^i..2^{i}(j+1)) : 2^i j \geq 2^{s-1}, 2^i(j+1) \leq 2^s \} \cup \{[0..1)\}, $$

where the sets of first type $[2^i..2^{i+1})$, $0 \leq i \leq s - 2$, together with $[0..1)$, are a decomposition of the set $[0..2^{s-1})$ from the classical Schmidt decomposition. For any integer $N \geq 2$ with $2^{s-1} \leq N - 1 < 2^s$, the set $[0..N)$ is the disjoint union of at most $2s - 1$ subsets in $L_s$ (namely $[0..1)$, the $s - 1$ subsets of the first type and at most $s - 1$ sets of the second type which can be constructed from the binary expansion of $N - 1$). We denote by $L(N)$ this set of subsets, i.e. $[0..N) = \bigsqcup_{I \in L(N)} I$.

**Lemma 4.1.** One has

$$ \sum_{I \in L_s} \int_Y \left| \sum_{t \in I} F(y, t) \right|^p \, d\nu(y) \leq Cs2^s. $$

**Proof.** Since $L_s$ is a subset of the set of all dyadic sets $[2^i..2^i(j+1))$ where $i, j$ are non-negative integers and $2^i(j+1) \leq 2^s$, we have

$$ \sum_{I \in L_s} \int_Y \left| \sum_{t \in I} F(y, t) \right|^p \, d\nu(y) \leq \sum_{i=0}^{s-1} \sum_{j=0}^{2^{s-1}-1} \int_Y \left| \sum_{t \in I} F(y, t) \right|^p \, d\nu(y) $$

$$ \leq \sum_{i=0}^{s-1} \sum_{j=0}^{2^{s-1}-1} C2^i $$

$$ \leq Cs2^s. $$

**Lemma 4.2.** For every $\varepsilon > 0$, there exists a sequence of measurable subsets $\{Y_s\}_{s \in \mathbb{N}}$ of $Y$ such that:

1. $\nu(Y_s) \leq Cs^{-(1 + p\varepsilon)}$. 


(2) For every integer \( N \geq 2 \) with \( 2^{s-1} \leq N - 1 < 2^s \) and every \( y \notin Y_s \) one has
\[
\left| \sum_{t=0}^{N-1} F(y, t) \right| \ll 2^{s} s^{1 + \frac{1}{p} + \varepsilon}.
\]

Proof. Consider
\[
Y_s = \left\{ y \in Y : \sum_{I \in L_s} \left| \sum_{t \in I} F(y, t) \right|^p > 2^{s} s^{2 + p \varepsilon} \right\}.
\]
The first assertion follows from Lemma 4.2 and Markov’s Inequality. Further, for \( 2^{s-1} \leq N - 1 < 2^s \) and \( y \notin Y_s \), and using that \( [0..N) = \bigcup_{I \in L(N)} I \) with \( L(N) \) of cardinality at most \( 2s - 1 \), we have
\[
\left| \sum_{t=0}^{N-1} F(y, t) \right|^p = \left| \sum_{I \in L(N)} \sum_{t \in I} F(y, t) \right|^p \\
\leq (2s - 1)^{p-1} \sum_{I \in L(N)} \left| \sum_{t \in I} F(y, t) \right|^p \quad \text{(by Hölder’s Inequality)} \\
\leq (2s - 1)^{p-1} \sum_{I \in L_s} \left| \sum_{t \in I} F(y, t) \right|^p \\
\ll_p s^{1 + p + p \varepsilon} 2^s \quad \text{(since } y \notin Y_s)\]
which yields the claim by raising to the power \( \frac{1}{p} \).

Proof of Proposition 4.2 (see [KSW17]). Let \( \varepsilon > 0 \) and choose a sequence of measurable subsets \( \{Y_s\}_{s \in \mathbb{N}} \) as in Lemma 4.2. Observe that
\[
\sum_{s=1}^{\infty} \nu(Y_s) \leq \sum_{s=1}^{\infty} C s^{-(1 + p \varepsilon)} < \infty.
\]
The Borel-Cantelli lemma implies that there exists a full-measure subset \( Y(\varepsilon) \subset Y \) such that for every \( y \in Y(\varepsilon) \) there exists \( s_y \in \mathbb{N} \) such that for all \( s \geq s_y \) we have \( y \notin Y_s \). Let \( N \geq 2 \) and \( s = 1 + \left\lfloor \log N - 1 \right\rfloor \), so that \( 2^{s-1} \leq N - 1 < 2^s \). Then, for \( N - 1 \geq 2^{s_y} \) we have \( s > s_y \) and \( y \notin Y_s \), thus
\[
\left| \sum_{t=0}^{N-1} F(y, t) \right| \ll 2^s s^{1 + \frac{1}{p} + \varepsilon} \\
\leq (2N)^{\frac{1}{p}} \log^{1 + \frac{1}{p} + \varepsilon} (2N).
\]
This implies the claim for \( y \in \cap_{N \in \mathbb{N}} Y(1/N) \).

We now apply Proposition 4.2 to the counting function \( \sum_t \hat{\chi} \circ a_t \). We denote by \( \text{vol}(F_{1,c}) \) the average of the Siegel transform from Proposition 3.3 for the function \( \chi = \chi_{F_1,c} \),
\[
\text{vol}(F_{1,c}) := \int_{C} \chi(x)dx = \int_{X} \hat{\chi}(\Lambda)d\mu_X(\Lambda).
\]
Theorem 4.2. There exists $\delta < 1$ depending only on the dimension $n$ such that for almost every $k \in K$ we have

$$
\sum_{t=0}^{N-1} \hat{\chi}(a_t k \Lambda_0) = N \text{vol}(F_{1,c}) + O_k(N^\delta).
$$

Proof. Using Proposition 4.2 it is enough to show that there exists $p > 1$ such that for every set $[a..b]$ in $L_a$ from lemmas 4.1 and 4.2, we have

$$
\left\| \sum_{t=a}^{b-1} \left( \hat{\chi} - \int_X \hat{\chi} \right) \circ a_t \right\|_{L^p(Y)}^p \ll (b - a).
$$

Using the estimates for the truncated Siegel transform from Proposition 3.6, we have for all $1 < p < 2$, $\frac{2p}{3p - 2} \leq \tau < n$ and $t \geq \kappa \log L$,

$$
\left\| (\hat{\chi} \circ a_t - \int_X \hat{\chi}) - (\hat{\chi}^{(L)} \circ a_t - \int_X \hat{\chi}^{(L)}) \right\|_{L^p_y} \leq \left\| \hat{\chi} \circ a_t - \hat{\chi}^{(L)} \circ a_t \right\|_{L^p_y} + \int_X |\hat{\chi} - \hat{\chi}^{(L)}|

\ll L \frac{r^{(2-p)}}{2p} + L^{-(\tau - 1)}

\ll L \frac{r^{(2-p)}}{2p}.
$$

Further, using Proposition 3.7 and the estimates from Proposition 3.6 and (3.8), we have for all $t \geq -\frac{1}{\delta} \log \varepsilon$,

$$
\left\| (\hat{\chi}^{(L)} \circ a_t - \int_X \hat{\chi}^{(L)}) - (\hat{f}_\varepsilon(L) \circ a_t - \int_X \hat{f}_\varepsilon(L)) \right\|_{L^p_y} \leq \left\| \hat{\chi}^{(L)} \circ a_t - \hat{f}_\varepsilon(L) \circ a_t \right\|_{L^p_y} + \int_X |\hat{\chi}^{(L)} - \hat{f}_\varepsilon(L)|

\ll L \frac{p-1}{p} \varepsilon^\frac{1}{p} + \varepsilon

\ll L \frac{p-1}{p} \varepsilon^\frac{1}{p}.
$$

Further, using the effective double equidistribution for smooth compactly supported functions from Proposition 4.1 and the estimates for the $C^\ell$-norm in (3.7) and (3.8), we have

$$
\left\| \sum_{t=a}^{b-1} (\hat{f}_\varepsilon(L) - \int_X \hat{f}_\varepsilon(L)) \circ a_t \right\|_{L^p_y} \leq \left\| \sum_{t=a}^{b-1} (\hat{f}_\varepsilon(L) - \int_X \hat{f}_\varepsilon(L)) \circ a_t \right\|_{L^p_y}

= \left( \sum_{t_1, t_2=a}^{b-1} \int_Y \hat{f}_\varepsilon(L) - \int_X \hat{f}_\varepsilon(L) \circ a_{t_1} \cdot (\hat{f}_\varepsilon(L) - \int_X \hat{f}_\varepsilon(L)) \circ a_{t_2} d\mu_Y \right)^{1/2}

\ll \left( \sum_{t_1, t_2=a}^{b-1} \|\hat{f}_\varepsilon(L)\|_l^2 e^{-\gamma \min(t_1, t_2, |t_1 - t_2|)} \right)^{1/2}

\ll \|\hat{f}_\varepsilon(L)\|_l (b - a)^{1/2} \ll L \varepsilon^{-1}(b - a)^{1/2},
$$

(4.6)
where we used the estimates
\[
\sum_{t_1, t_2 = a}^{b-1} e^{-\gamma \min(t_1, t_2, |t_1 - t_2|)} \leq \sum_{t_1, t_2 = a}^{b-1} e^{-\gamma t_1} \leq (b - a) \sum_{t_1 = a}^{b-1} e^{-\gamma t_1} \ll b - a,
\]
similarly
\[
\sum_{t_1, t_2 = a}^{b-1} e^{-\gamma \min(t_1, t_2, |t_1 - t_2|)} \ll b - a,
\]
and
\[
\sum_{t_1, t_2 = a}^{b-1} e^{-\gamma \min(t_1, t_2, |t_1 - t_2|)} \leq \sum_{l = a - b}^{b-1} \sum_{t_1 = a}^{t_1 \in [a + l, b - 1 + l]} e^{-\gamma |l|} \leq (b - a - |l|) e^{-\gamma |l|} \ll b - a.
\]
Combining (4.4), (4.5) and (4.6) we obtain
\[
\left\| \sum_{t = a}^{b-1} \left( \hat{\chi} - \int_X \hat{\chi} \right) \circ a_t \right\|_{L_y^p} \ll (b - a) L^{- \frac{2(p - p)}{2p}} + (b - a) L^{p-1} \varepsilon \frac{1}{p} + (b - a)^{1/2} L \varepsilon^{-l} \quad (4.7)
\]
Setting the summands in (4.7) to be equal, we have an optimal bound for
\[
\varepsilon = L^{1 - p - \frac{2(p - p)}{2p}} \quad \text{and} \quad L = (b - a)^{\frac{p}{2 + (1 + p)(r(2 - p) + 2(p - 1))}}.
\]
We verify that for all but finitely many sets \([a, b]\) in \(L_s\), we have that for all \(t \in [a, b]\) the conditions \(t \geq \kappa \log L\) and \(t \geq -\frac{1}{2} \log \varepsilon\) are satisfied. We write \(L = (b - a)^\beta\) with \(\beta > 0\). For sets \([a, b]\) of the first type, i.e. \([a, b] = [2^i, 2^i+1]\) with \(0 \leq i \leq s - 2\), we have
\[
\begin{align*}
\log L &= \log (2^i \beta) = i \log (2^i \\
&\leq \frac{1}{\kappa} 2^i \quad \text{for all} \quad C_1 \leq i \leq s - 2, \quad \text{for some fixed integer} \ C_1 \quad \text{and large enough} \ s, \\
&\leq \frac{1}{\kappa} t \quad \text{for all} \quad t \in [2^i, 2^i+1] \quad \text{for all} \quad C_1 \leq i \leq s - 2.
\end{align*}
\]
For sets \([a, b]\) of the second type, i.e. \([a, b] = [2^i j, 2^i (j+1)]\) with \(2^i j \geq 2^{s-1}\) and \(2^i (j+1) \leq 2^s\), the condition is then a fortiori satisfied, since those sets have all a larger lower bound and are all at most as large as the set of first type \([2^{s-2}, 2^{s-1}]\).
One verifies similarly that there exists a constant \(C_2 > 0\) such that
\[
\log \varepsilon \geq -\theta t \quad \text{for} \ t \ \text{in all sets} \ [a, b] \ \text{in} \ L_s \ \text{but those} \ [2^i, 2^{i+1}] \ \text{with} \ 0 \leq i < C_2.
\]
Further, since \(\sup_{t \geq 0} \| \hat{\chi} \circ a_t \|_{L_y^p} < \infty\), we have for all sets \([a, b] = [2^i, 2^{i+1}]\) with \(i < \max(C_1, C_2)\)
\[
\left\| \sum_{t = a}^{b-1} \left( \hat{\chi} - \int_X \hat{\chi} \right) \circ a_t \right\|_{L_y^p} \ll (b - a) \left\| \left( \hat{\chi} - \int_X \hat{\chi} \right) \circ a_t \right\|_{L_y^p} = O(1).
\]
Hence, (4.7) and (4.8) yield, for all sets \([a, b]\) in \(L_s\)
\[
\left\| \sum_{t = a}^{b-1} \left( \hat{\chi} - \int_X \hat{\chi} \right) \circ a_t \right\|_{L_y^p} \ll (b - a)^{\beta p}.
\]
with the exponent $\delta_p$ given by
\[
\delta_p = 1 - \frac{\tau(2 - p)}{4 + 2(pl + 1)(\tau(2 - p) + 2(p - 1))}.
\]
We observe that $\delta_p$ is an increasing function of $p$ for $1 < p < 2$, with $\lim_{p \to 1} \delta_p = 1 - \frac{\tau}{4 + 2\tau(l+1)} < 1$ and $\lim_{p \to 2} \delta_p = 1$ hence, by the intermediate value theorem, there exists $1 < p < 2$ with $\delta_p = \frac{1}{p} < 1$. We denote by $\delta$ this exponent and note that it depends only on the dimension $n$.

5. Effective estimate for the counting function

Proof of Theorem 1.2. Using Theorem 4.2 with the estimate (2.8) we have
\[
(|T - r_0| - 1) \text{vol}(F_{1,c}) + O(T^\delta) + O(t^n) \leq N_{T,c}(\alpha) + O(1) \leq (|T + r_0|) \text{vol}(F_{1,c}) + O(T^\delta).
\]
Using further (2.5) with $l = \left\lfloor T^{\frac{1}{n+1}} \right\rfloor$, we have
\[
T \text{vol}(F_{1,c}) + O(T^\delta) + O(t^n) = T \left( \text{vol}(F_{1,c}) + O(t^{-1}) \right) + O(T^\delta) + O(t^n)
\]
\[
= T \text{vol}(F_{1,c}) + O(T^\gamma), \quad \text{with } \gamma = \max \left( \frac{n}{n+1}, \delta \right),
\]
hence
\[
N_{T,c}(\alpha) = T \text{vol}(F_{1,c}) + O(T^\gamma).
\]
Since full-measure sets in $S^n$ correspond to full-measure sets in $K$, we conclude that this last estimate holds for almost every $\alpha \in S^n$.  \qed
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