Action of micro-differential operators on quantized contact transformations

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Abstract

Quantized contact transforms (QCT) have been constructed in [SKK]. We give here a complete proof of the fact that such QCT commute with the action of microdifferential operators. To our knowledge, such a proof did not exist in the literature. We apply this result to the microlocal Radon transform.

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1 Introduction

1.1 Overview of the results

For a manifold $M$, let us denote by $T^*M$ the cotangent bundle and $\hat{T}^*M$ the bundle $T^*M$ with the zero section removed. We will consider the following situation: let $X$ and $Y$ be two complex manifolds of the same dimension, a closed submanifold $Z$ of $X \times Y$, open subset $U \subset \hat{T}^*X$ and $V \subset \hat{T}^*Y$ and assume that the conormal bundle $\hat{T}^*_Z(X \times Y)$ induces a contact transformation $\hat{T}^*_Z(X \times Y) \cap (U \times V^o)$.

Let $F \in D^b(X)$ and let $\varphi_K(F)$ denote the contact transformation of $F$ with kernel $K \in D^b(X \times Y)$. We will prove an isomorphism between $\mu hom(F, \mathcal{O}_X)$ on $U \cap T^*_MX$ and $\mu hom(\varphi_K(F), \mathcal{O}_Y)$ on $V \cap T^*_NY$, which follows immediately from [KS90, Lem. 11.4.3]. Our main result will be the commutation of this isomorphism to the action of microdifferential operators. Although considered as well-known, the proof of this commutation does not appear clearly in the literature (see [SKK, p. 467]), and is far from being obvious. In fact, we will consider a more general setting, replacing sheaves of microfunctions with sheaves of the type $\mu hom(F, \mathcal{O}_X)$. Moreover, assume that $X$ and $Y$ are complexification of real analytic manifolds $M$ and $N$ respectively. Then, it is known that, under suitable hypotheses, one can quantize this contact transform and get an isomorphism between microfunctions on $U \cap T^*_MX$ and microfunctions on $V \cap T^*_NY$ [KKK86].

Then, we will specialize our results to the case of projective duality. We will study the microlocal Radon transform understood as a quantization of projective duality, both in the real and the complex case.

In the real case, denote by $P$ the real projective space (say of dimension $n$), by $P^*$ its dual and by $S$ the incidence relation:

$$S := \{(x, \xi) \in P \times P^*; \langle x, \xi \rangle = 0\}.$$  

In this setting, there is a well-known correspondence between distributions on $P$ and $P^*$ due to Gelfand, Gindikin, Graev [GGG82] and to Helgason [Hel80]. However, it is known since the 70s under the influence of the Sato’s school, that to well-understand what happens on real (analytic) manifolds, it may be worth to look at their complexification.

Hence, denote by $\mathbb{P}$ the complex projective space of dimension $n$, by $\mathbb{P}^*$ the dual projective space and by $S \subset \mathbb{P} \times \mathbb{P}^*$ the incidence relation. We have the correspondence

$$\hat{T}^*_S(\mathbb{P} \times \mathbb{P}^*) \sim \sim \hat{T}^*\mathbb{P} \sim \sim \hat{T}^*\mathbb{P}^*$$
This contact transformation induces an equivalence of categories between perverse sheaves modulo constant ones on the complex projective space and perverse sheaves modulo constant ones on its dual, as shown by Brylinski [Bry86], or between coherent $\mathcal{D}$-modules modulo flat connections, as shown by D’Agnolo-Schapira [DS94].

In continuation of the previous cited works, we shall consider the contact transform induced by (1.3)

\[
\begin{align*}
\hat{T}^*_S(\mathbb{P} \times \mathbb{P}^*) & \cap (\hat{T}^*_p\mathbb{P} \times \hat{T}^*_p\mathbb{P}^*) \\
\sim & \\
\hat{T}^*_p\mathbb{P} & \sim \hat{T}^*_p\mathbb{P}^*
\end{align*}
\]  

The above contact transformation leads to the well-known fact that the Radon transform establishes an isomorphism of sheaves of microfunctions on $\mathbb{P}$ and $\mathbb{P}^*$ (see [KKK86]). We will apply our main result to prove the commutation of this isomorphism to the action of microdifferential operators.

1.2 Main theorems

We will use the language of sheaves and $\mathcal{D}$-modules and we refer the reader to [KS90] and [Kas03] for a detailed development of these topics. We will denote by $k$ a commutative unital ring of finite global dimension.

Notations for integral transforms Let $X, Y$ be real manifolds and $S$ a closed submanifold $X \times Y$. Consider the diagrams $X \xleftarrow{f} S \xrightarrow{q} Y$, $X \xleftarrow{g} X \times Y \xrightarrow{q} Y$. Let $F \in D^b(k_Y)$ and $K \in D^b(k_{X \times Y})$. The integral transform of $F$ with respect to the kernel $K$ is defined to be $\Phi_K(F) := Rq_! (K \otimes q_2^{-1} F)$. We will denote $\Phi_S(F)$ the integral transform of $F$ with respect to the kernel $k_S[d_S - d_X]$.

Results on the functor $\mu_{hom}$ To establish our main results, we will need the following complement on the functor $\mu_{hom}$.

For $(M_i)_{i=1,2,3}$, three manifolds, we write $M_{ij} := M_i \times M_j$ ($1 \leq i, j \leq 3$). We consider the operation of composition of kernels:

\[
\begin{align*}
\circ_2 : D^b(k_{M_{12}}) \times D^b(k_{M_{23}}) & \to D^b(k_{M_{13}}) \\
(K_1, K_2) & \mapsto K_1 \circ_2 K_2 := Rq_{13!}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2) \\
& \simeq Rq_{13!}\delta_2^{-1}(K_1 \boxtimes K_2).
\end{align*}
\]  

We add a subscript $a$ to $p_j$ to denote by $p_j^a$ the composition of $p_j$ and the antipodal map on $T^*M_j$. We define the composition of kernels on cotangent bundles (see [KS90, Prop. 4.4.11])

\[
\begin{align*}
\circ_2^a : D^b(k_{T^*M_{12}}) \times D^b(k_{T^*M_{23}}) & \to D^b(k_{T^*M_{13}}) \\
(K_1, K_2) & \mapsto K_1 \circ_2^a K_2 := Rp_{13!}(p_{12}^{-1}K_1 \otimes p_{23}^{-1}K_2)
\end{align*}
\]
Let $F_i, G_i, H_i$ respectively in $D^b(k_{M_{12}}), D^b(k_{M_{23}}), D^b(k_{M_{34}})$, $i = 1, 2$. Let $U_i$ be an open subset of $T^*M_{ij}$ ($i = 1, 2, j = i + 1$) and set

$$U_3 = U_i \circ_2 U_j = p_{13}(p_{12}^{-1}(U_1) \cap p_{23}^{-1}(U_2))$$

In [KS90], a canonical morphism in $D^b(k_{T^*M_{13}})$ is constructed

$$\muhom(F_1, F_2)|_{U_1} \circ_2 \muhom(G_1, G_2)|_{U_2} \rightarrow \muhom(F_1 \circ_2 G_1, F_2 \circ_2 G_2)|_{U_3}. \tag{1.7}$$

We will see that the composition $\circ$ is associative and we will see also that the morphism (1.7) is compatible with associativity with respect to $\circ$.

**Complex contact transformations** Consider now two complex manifolds $X$ and $Y$ of the same dimension $n$, open $\mathbb{C}^\times$-conic subsets $U$ and $V$ of $\hat{T}X$ and $\hat{T}Y$, respectively, $\Lambda$ a smooth closed submanifold of $U \times V^a$ and assume that the projections $p_1|_{\Lambda}$ and $p_2^a|_{\Lambda}$ induce isomorphisms, hence a homogeneous symplectic isomorphism $\chi: U \sim \rightarrow V$:

\[\begin{array}{c}
\Lambda \subset U \times V^a \\
\hat{T}^*X \supset U \\
\sim \\
p_1 \\
\sim \\
\chi \\
p_2^a \\
V \subset \hat{T}^*Y
\end{array}\]

Let us consider a perverse sheaf $L$ on $X \times Y$ satisfying $(p_1^{-1}(U) \cup p_2^a^{-1}(V)) \cap SS(L) \subset \Lambda$ and a section $s$ of $\muhom(L, \Omega_{X \times Y/X})$ on $\Lambda$, where $\Omega_{X \times Y/X} := \mathcal{O}_{X \times Y} \otimes q_2^{-1}\mathcal{O}_Y \otimes q_1^{-1}\Omega_Y$. Recall that one denotes by $\mathcal{E}_X^\mathbb{R}$ sheaf of rings $\mathcal{E}_X := \muhom(\mathbb{C}_{\Delta_X}, \Omega_{X \times X/X})[d_X]$, and $\mathcal{E}_X$ the subsheaf of $\mathcal{E}_X^\mathbb{R}$ of finite order microdifferential operators. In the following theorem, the first statement (i) is well-known, see [SKK], (ii) is proved in [KS90], the fact that isomorphism (1.8) is compatible with the action of microdifferential operators was done at the germ level in [KS90], but from a global perspective, it was announced for microfunctions in various papers but no detailed proof exists to our knowledge. We will prove our main theorem:

**Theorem 1.1.** Let $G \in D^b(\mathbb{C}_Y)$ and assume to be given a section $s$ of $\muhom(L, \Omega_{X \times Y/X}),$ non-degenerate on $\Lambda$.

(i) For $W \subset U$, $P \in \mathcal{E}_X(W)$, there is a unique $Q \in \mathcal{E}_Y(\chi(W))$ satisfying $P \cdot s = s \cdot Q$ $(P, Q$ considered as sections of $\mathcal{E}_{X \times Y})$. The morphism induced by $s$

$$\chi^{-1}\mathcal{E}_Y|_V \rightarrow \mathcal{E}_X|_U$$

$$P \mapsto Q$$

is a ring isomorphism.
(ii) We have the following isomorphism in $D^b(C_U)$

\[(1.8) \quad \chi^{-1}\muhom(G, \mathcal{O}_Y)|_V \cong \muhom(\Phi_{L[n]}(G), \mathcal{O}_X)|_U\]

(iii) The isomorphism (1.8) is compatible with the action of $\mathcal{E}_Y$ and $\mathcal{E}_X$ on the left and right side of (1.8) respectively.

We will see that the action of microdifferential operators in Theorem 1.1 (iii) is derived from the morphism (1.7).

**Projective duality for microfunctions** For $M$ a real analytic manifold and $X$ its complexification, we might be led to identify $T^*_M X$ with $i \cdot T^* M$. We denote by $\mathcal{A}_M, \mathcal{B}_M, \mathcal{C}_M$ the sheaves of real analytic functions, hyperfunctions, microfunctions, respectively.

In this article, we will quantize the contact transform associated with the Lagrangian submanifold $\hat{T}^*_g(\mathbb{P} \times \mathbb{P}^*)$. We will construct and denote by $\chi$ the homogeneous symplectic isomorphism between $\hat{T}^* \mathbb{P}$ and $\hat{T}^* \mathbb{P}^*$.

For $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we denote by $\mathbb{C}_P(\varepsilon)$ the two locally constant sheaf of rank one on $P$ (see Section 4.1 for a precise definition).

Let an integer $p \in \mathbb{Z}$, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we will define the sheaves of real analytic functions $\mathcal{A}_P(\varepsilon, p)$, hyperfunctions $\mathcal{B}_P(\varepsilon, p)$, microfunctions $\mathcal{C}_P(\varepsilon, p)$, on $P$ resp. $P^*$ twisted by some power of the tautological line bundle.

For $X, Y$ either the manifold $\mathbb{P}$ or $\mathbb{P}^*$, for any two integers $p, q$, we note $\mathcal{O}_{X \times Y}(p, q)$ the line bundle on $X \times Y$ with homogeneity $p$ in the $X$ variable and $q$ in the $Y$ variable. Setting $\Omega_{X \times Y/X}(p, q) := \Omega_{X \times Y/X} \otimes_{\mathcal{O}_{X \times Y}} \mathcal{O}_{X \times Y}(p, q), \mathcal{E}_X^\varepsilon(p, q) := \muhom(\mathbb{C}_X, \Omega_{X \times Y/X}(p, q))[d_X]$ and we define accordingly $\mathcal{E}_X(\varepsilon, p, q)$. Let us notice that $\mathcal{E}_X^\varepsilon(-p, p)$ is a sheaf of rings.

Let $n$ be the dimension of $P$, (of course $n = d_\mathbb{P}$). For an integer $k$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we note $k^* := -n - 1 - k$, $\varepsilon^* := -n - 1 - \varepsilon \mod(2)$. We have:

**Theorem 1.2.** (i) Let $k$ be an integer such that $-n - 1 < k < 0$ and let $s$ be a global non-degenerate section on $\hat{T}^*_g(\mathbb{P} \times \mathbb{P}^*)$ of $H^1(\muhom(\mathbb{C}_S, \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}}(-k, k^*))$. For $P \in \mathcal{E}_\mathbb{P}(-k, k)$, there is a unique $Q \in \mathcal{E}_\mathbb{P}^*(-k^*, k^*)$ satisfying $P \cdot s = s \cdot Q$. The morphism induced by $s$

$$
\chi_* \mathcal{E}_\mathbb{P}(-k, k) \to \mathcal{E}_\mathbb{P}^*(-k^*, k^*)
$$

$$
P \mapsto Q
$$

is a ring isomorphism.

(ii) There exists such a non-degenerate section $s$.

In fact, we will see that the non-degenerate section of Theorem 1.2 is provided by the Leray section.

Now, from classical adjunction formulas for $\mathcal{E}$-modules, we get a correspondence between solutions of systems of microdifferential equations on the projective space and solutions of systems of microdifferential equations on its dual. We will prove the following theorem, which was proved in [DS96] for $\mathcal{D}$-modules,
Theorem 1.3. Let \( k \) be an integer such that \(-n-1 < k < 0\). Let \( \mathcal{N} \) be a coherent \( \mathcal{E}_\mathbb{P}(-k,k) \)-module and \( F \in \text{D}^b(\mathbb{P}) \). Then, we have an isomorphism in \( \text{D}^b(\mathbb{C}_+\mathbb{P}^n) \)

\[
\chi_* \text{RHom}_{\mathcal{E}_\mathbb{P}(-k,k)}(\mathcal{N}, \mu_{\text{hom}}(F, \mathcal{O}_\mathbb{P}(k))) \simeq \\
\text{RHom}_{\mathcal{E}_\mathbb{P}(-k^*,k^*)}(\Phi^\mathcal{E}_\mathbb{P}(\mathcal{N}), \mu_{\text{hom}}((\Phi^\mathcal{E}_\mathbb{P}[−1]F, \mathcal{O}_\mathbb{P}^*(k^*)))
\]

where \( \Phi^\mathcal{E}_\mathbb{P} \) it is the counterpart of \( \Phi^\mathcal{E} \) for \( \mathcal{E} \)-modules, and will be defined in Section 2.7.1.

Let us mention that, through a difficult result from [Kas+06], \( \mu_{\text{hom}}(F, \mathcal{O}_\mathbb{P}) \) is well-defined in the derived category of \( \mathcal{E} \)-modules.

Corollary 1.4. Let \( k \) be an integer such that \(-n-1 < k < 0\) and \( \varepsilon \in \mathbb{Z}/2\mathbb{Z} \). The section \( s \) of theorem 1.2 defines an isomorphism:

\[
\chi_* \mathcal{E}_\mathbb{P}(\varepsilon, k)|_{\mathbb{T}^*\mathbb{P}^n} \simeq \mathcal{E}_\mathbb{P}^*(\varepsilon^*, k^*)|_{\mathbb{T}^*\mathbb{P}^n}
\]

Moreover, this morphism is compatible with the respective action of \( \chi_* \mathcal{E}_\mathbb{P}(-k,k) \) and \( \mathcal{E}_\mathbb{P}^*(-k^*,k^*) \).

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2 Reminders on Algebraic Analysis and complements

In this section, we recall classical results of Algebraic Analysis, with the exception of section 2.8.

2.1 Notations for manifolds

(i) Let \( M_i (i = 1, 2, 3) \) be manifolds. For short, we write \( M_{ij} := M_i \times M_j \) (\( 1 \leq i, j \leq 3 \)), \( M_{123} = M_1 \times M_2 \times M_3 \), \( M_{1233} = M_1 \times M_2 \times M_2 \times M_3 \), etc.

(ii) \( \delta_{M_i} : M_i \to M_i \times M_i \) denote the diagonal embedding, and \( \Delta_{M_i} \) the diagonal set of \( M_i \times M_i \).

(iii) We will often write for short \( k_i \) instead of \( k_{M_i} \) and \( k_{\Delta_i} \) instead of \( k_{\Delta_{M_i}} \) and similarly with \( \omega_{M_i} \), etc., and with the index \( i \) replaced with several indices \( i\overline{j} \), etc.

(iv) We denote by \( \pi_i, \pi_{ij}, \) etc. the projection \( T^*M_i \to M_i, T^*M_{ij} \to M_{ij} \), etc.

(v) For a fiber bundle \( E \to M \), we denote by \( \hat{E} \to M \) the fiber bundle with the zero-section removed.

(vi) We denote by \( q_i \) the projection \( M_{ij} \to M_i \) or the projection \( M_{123} \to M_i \) and by \( q_{ij} \) the projection \( M_{123} \to M_{ij} \). Similarly, we denote by \( p_i \) the projection \( T^*M_{ij} \to T^*M_i \) or the projection \( T^*M_{123} \to T^*M_i \) and by \( p_{ij} \) the projection \( T^*M_{123} \to T^*M_{ij} \).
(vii) We also need to introduce the maps \( p_j \) or \( p_{ij} \), the composition of \( p_j \) or \( p_{ij} \) and the antipodal map \( a \) on \( T^*M_j \). For example,

\[
p_{12}(\langle x_1, x_2, x_3; \xi_1, \xi_2, \xi_3 \rangle) = (x_1, x_2; \xi_1, -\xi_2).
\]

(viii) We let \( \delta_2 : M_{123} \to M_{1233} \) be the natural diagonal embedding.

### 2.2 Sheaves

We follow the notations of [KS90].

Let \( X \) be a good topological space, i.e. separated, locally compact, countable at infinity, of finite global cohomological dimension and let \( k \) be a commutative unital ring of finite global dimension.

For a locally closed subset \( Z \) of \( X \), we denote by \( k_Z \), the sheaf, constant on \( Z \) with stalk \( k \), and 0 elsewhere.

We denote by \( D^b(k_X) \) the bounded derived category of the category of sheaves of \( k \)-modules on \( X \). If \( \mathcal{R} \) is a sheaf of rings, we denote by \( D^b(\mathcal{R}) \) the bounded derived category of the category of left \( \mathcal{R} \)-modules.

Let \( Y \) be a good topological space and \( f \) a morphism \( Y \to X \). We denote by \( Rf_*, f^{-1}, Rf_!, f^!, R\mathcal{H}om, \otimes \) the six Grothendieck operations. We denote by \( \boxtimes \) the exterior tensor product.

We denote by \( \omega_X \) the dualizing complex on \( X \), by \( \omega_X^{-1} \) the sheaf-inverse of \( \omega_X \) and by \( \omega_{Y/X} \) the relative dualizing complex.

In the following, we assume that \( X \) is a real manifold. Recall that \( \omega_X \simeq \omega_X \mid_{\dim X} \) where \( \omega_X \) is the orientation sheaf and \( \dim X \) is the dimension of \( X \). We denote by \( D_X(\bullet), D'_X(\bullet) \) the duality functor \( D_X(\bullet) = R\mathcal{H}om(\bullet, \omega_X), D'_X(\bullet) = R\mathcal{H}om(\bullet, k_X) \), respectively.

For \( F \in D^b(k_X) \), we denote by \( SS(F) \) its singular support, also called micro-support. For a a subset \( Z \subset T^*X \), we denote by \( D^b(k_X; Z) \) the localization of the category \( D^b(k_X) \) by the full subcategory of objects whose micro-support is contained in \( T^*X \setminus Z \).

For a closed submanifold \( M \) of \( X \), we denote by \( \nu_M, \mu_M, \mu_{hom} \), the functor of specialization, microlocalization along \( M \) and the functor of microlocalization of \( R\mathcal{H}om \) respectively.

Let \( M_i \) (\( i = 1, 2, 3 \)) be manifolds. We shall consider the operations of composition of kernels:

\[
\circ_2 : D^b(k_{M_{12}}) \times D^b(k_{M_{23}}) \to D^b(k_{M_{13}})
\]

\[
(K_1, K_2) \mapsto K_1 \circ_2 K_2 := Rq_{13!}(q_{12}^!K_1 \otimes q_{23}^!K_2)
\]

\[
\simeq Rq_{13!}\delta_2^{-1}(K_1 \boxtimes K_2)
\]

\[
\circ_{23} : D^b(k_{M_{12}}) \times D^b(k_{M_{23}}) \times D^b(k_{M_{34}}) \to D^b(k_{M_{14}})
\]

\[
(K_1, K_2, K_3) \mapsto K_1 \circ_{23} K_2 \circ_3 K_3 := Rq_{14!}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2 \otimes q_{34}^{-1}K_3)
\]

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Let us mention a variant of \(\circ\):

\[
* : D^b(k_{M_{12}}) \times D^b(k_{M_{23}}) \to D^b(k_{M_{13}})
\]

\[(K_1, K_2) \mapsto K_1 * K_2 := Rq_{13*}(q_2^{-1}\omega_2 \otimes \delta_2!(K_1 \boxtimes K_2))\]

There is a natural morphism \(K_1 \circ K_2 \to K_1 * K_2\).

We refer the reader to [KS90] for a detailed presentation of sheaves on manifolds.

### 2.3 \(O\)-modules and \(D\)-modules

We refer to [Kas03] for the notations and the main results of this section.

Let \((X, O_X)\) be a complex manifold. We denote by \(d_X\) its complex dimension and by \(D_X\) the sheaf of rings of finite order holomorphic differential operators on \(X\).

For an invertible \(O_X\)-module \(F\), we denote by \(F^{-1}\) the inverse of \(F\). Denote by \(\Omega_X\) the right \(D_X\)-module of holomorphic \(d_X\) forms.

Let \(D^b(D_X)\) be the bounded derived category of the category of left \(D_X\)-modules, \(D^b_{coh}(D_X)\) its full triangulated subcategory whose objects have coherent cohomology.

Let \(D^b_{good}(D_X)\) be the triangulated subcategory of \(D^b(D_X)\), whose objects have all cohomologies consisting in good \(D_X\)-modules (see [Kas03] for a classical reference).

We refer in the following to [Kas03]. Let \(f : Y \to X\) be a morphism of complex manifolds. We denote by \(D_Y \to X\) and \(D_X \leftarrow Y\) the transfer bimodules.

For \(M \in D^b(D_X)\), \(N \in D^b(D_Y)\), we denote by \(f^{-1}M, f_*N\), the pull-back and the direct image of \(D\)-modules respectively. We refer to [DS96] for functorial properties of inverse and direct image of \(D\)-modules.

### 2.4 \(\mathcal{E}\)-modules

We refer in the following to [SKK] (see also [Sch85] for an exposition). For a complex manifold \(X\), one denotes by \(\mathcal{E}_X\) the sheaf of filtered ring of finite order holomorphic microdifferential operators on \(T^*X\). We denote by \(D^b_{coh}(\mathcal{E}_X)\) the full triangulated subcategory of \(D^b(\mathcal{E}_X)\) whose objects have coherent cohomology.

For \(m \in \mathbb{Z}\), we denote by \(\mathcal{E}_X(m)\) the abelian subgroup of \(\mathcal{E}_X\) of microdifferential operators of order less or equal to \(m\). For a section \(P\) of \(\mathcal{E}_X\), we denote by \(\sigma(P)\) the principal symbol of \(P\).

Let \(\pi_X\) denote the natural projection \(T^*X \to X\). Let us recall that \(\mathcal{E}_X\) is flat over \(\pi^{-1}(D_X)\). To a \(D_X\)-module \(\mathcal{M}\), we associate an \(\mathcal{E}_X\)-module defined by

\[
\mathcal{E}_X \mathcal{M} := \mathcal{E}_X \otimes_{\pi_X^{-1}D_X} \pi_X^{-1} \mathcal{M}
\]

To a morphism of manifolds \(f : Y \to X\), we associate the diagram of natural morphisms:

\[
\begin{array}{ccc}
T^*Y & \xrightarrow{f^*} & T^*X \\
\pi_Y & \downarrow & \downarrow \pi_X \\
Y & \xrightarrow{f} & X
\end{array}
\]
where \( f_d \) is the transposed of the tangent map \( Tf: TY \to Y \times_X TX \).

For \( M, N \) objects of respectively \( D^b(\mathcal{E}_X) \) and \( D^b(\mathcal{E}_Y) \), we denote by \( f^{-1}M \) and \( f_*N \) the pull-back and the direct image of \( \mathcal{E} \)-modules respectively.

## 2.5 Hyperfunctions and microfunctions

Let \( M \) be a real analytic manifold and \( X \) a complexification of \( M \). We might be led to identify \( T^*_MX \) with \( i \cdot T^*M \). We denote by \( \mathcal{B}_M := \mathcal{O}_X|_M, \mathcal{C}_M := R\mathcal{H}\text{om}(\mathcal{D}'_X\mathcal{C}_M, \mathcal{O}_X), \mathcal{C}_M := \mu\text{hom}(\mathcal{D}'_X\mathcal{C}_M, \mathcal{O}_X) \), the sheaves of real analytic functions, hyperfunctions, microfunctions, respectively. Let us denote by \( sp \), the isomorphism

\[
sp: \mathcal{B}_M \xrightarrow{\sim} R\pi_M^*\mathcal{C}_M
\]

There is a natural action of the sheaf of microdifferential operators \( \mathcal{E}_X \) on \( \mathcal{C}_M \).

If \( Z \) is a closed complex submanifold of \( X \) of codimension \( d \), we note

\[
\mathcal{B}_Z|_X := H^d[Z](\mathcal{O}_X)
\]

the algebraic cohomology of \( \mathcal{O}_X \) with support in \( Z \).

## 2.6 Integral transforms for sheaves and \( D \)-modules

### 2.6.1 Integral transforms for sheaves

Let \( X \) and \( Y \) be complex manifolds of respective dimension \( d_X, d_Y \). Let \( S \) be a closed submanifold \( X \times Y \) of dimension \( d_S \). We set \( d_{S/X} := d_S - d_X \). Consider the diagram of complex manifolds

\[
\begin{array}{ccc}
S & \xrightarrow{f} & X \\
\downarrow{g} & & \downarrow{f} \\
Y & \xrightarrow{g} & X
\end{array}
\]

where the second diagram is obtained by interchanging \( X \) and \( Y \).

Let \( F \in D^b(\mathbb{C}_X), G \in D^b(\mathbb{C}_Y) \), we define

\[
\begin{align*}
\Phi_S(F) & := Rg_*f^{-1}F[d_{S/Y}], & \Phi_{\tilde{S}}(G) & := Rf_!g^{-1}G[d_{S/X}] \\
\Psi_S(F) & := Rg_*f^1F[d_{X/S}], & \Psi_{\tilde{S}}(G) & := Rf_*g^1G[d_{Y/S}]
\end{align*}
\]

For \( K \in D^b(\mathbb{C}_{X \times Y}) \), and given the diagram \( X \xrightarrow{q_1} X \times Y \xrightarrow{q_2} Y \), we define the integral transform of \( F \) with kernel \( K \)

\[
\Phi_K(F) := Rq_2!(K \otimes q_1^{-1}F)
\]
2.6.2 Integral transforms for $\mathcal{D}$-modules

Let $X, Y$ be complex manifolds of equal dimension $n > 0$, and $S$ a complex manifold. Consider again the situation (2.5).

We suppose

\begin{equation}
\begin{aligned}
(f, g) &\text{ are smooth and proper}, \\
S &\text{ is a complex submanifold of } X \times Y \text{ of codimension } c > 0.
\end{aligned}
\end{equation}

Let $\mathcal{M} \in \mathcal{D}^b(\mathcal{D}_X)$, $\mathcal{N} \in \mathcal{D}^b(\mathcal{D}_Y)$. Let us denote by $\tilde{S}$ the image of $S$ by the map $r : X \times Y \to Y \times X ; (x, y) \mapsto (y, x)$. One sets

$\Phi_S(\mathcal{M}) := g_* f^{-1} \mathcal{M}$, \hspace{1cm} $\Phi_{\tilde{S}}(\mathcal{N}) := f_* g^{-1} \mathcal{N}$

We refer to [DS94, Prop. 2.6.] for adjunction formulae related to these integral transforms.

Let us recall that we denote by $\Omega_X$ the sheaf of holomorphic $n$-forms and let

$\mathcal{B}^{(n,0)}_{S|X \times Y} := q_1^{-1} \Omega_X \otimes_{q_1^{-1} \mathcal{O}_X} \mathcal{B}_{S|X \times Y}$

This $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodule allows the computation of $\Phi_S$ because of the isomorphism, proven in [DS94, Prop 2.12]

$\mathcal{D}_Y \simeq S \otimes_{\mathcal{D}_S} \mathcal{D}_S \rightarrow X \sim \mathcal{B}^{(n,0)}_{S|X \times Y}$

leading to

$\Phi_S(\mathcal{M}) \simeq Rq_{2!}(\mathcal{B}^{(n,0)}_{S|X \times Y} \otimes_{q_1^{-1} \mathcal{D}_X} q_1^{-1} \mathcal{M})$

2.7 Microlocal integral transforms

2.7.1 Integral transforms for $\mathcal{E}$-modules

Let $X, Y$ be complex manifolds and $S$ is a closed submanifold of $X \times Y$. We consider again the diagram (2.5) under the hypothesis (2.6).

We define the functor

$\mathcal{D}^b(\mathcal{E}_X) \to \mathcal{D}^b(\mathcal{E}_Y)$, $\Phi^\mu_S(\mathcal{M}) := g_* f^{-1} \mathcal{M}$

We define the $\mathcal{E}_{X \times Y}$-module attached to $\mathcal{B}_{S|X \times Y}$,

$\mathcal{C}_{S|X \times Y} := \mathcal{E} \mathcal{B}_{S|X \times Y}$

and we consider the $(\mathcal{E}_Y, \mathcal{E}_X)$-bimodule

\begin{equation}
\mathcal{C}^{(n,0)}_{S|X \times Y} := \pi^{-1} q_1^{-1} \Omega_X \otimes_{\pi^{-1} q_1^{-1} \mathcal{O}_X} \pi^{-1} q_1^{-1} \mathcal{C}_{S|X \times Y}
\end{equation}
One can notice that
\[
E_Y \xleftarrow{\text{L}} S \otimes_{\mathcal{E}_S} \mathcal{E}_S \to X \xrightarrow{\text{L}} C_{(n,0)}^{(n,0)}
\]
and hence, we have

\[(2.8) \quad \Phi^\mu_S(M) \simeq R\pi_1^*(\mathcal{C}_{(n,0)}^{(n,0)} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y)
\]

Let \( M \in \text{D}^\mu_\text{good}(\mathcal{D}_X) \). The functors \( \Phi^\mu_S \) and \( \Phi_S \) are linked through the following isomorphism in \( \text{D}^\mu(\mathcal{C}_Y) \), (see [SS94])

\[(2.9) \quad \mathcal{E}(\Phi_S(M)) \simeq \mathcal{E}(\mathcal{M})
\]

### 2.7.2 Microlocal integral transform of the structure sheaf

Consider two open subsets \( U \) and \( V \) of \( T^*X \) and \( T^*Y \), respectively and \( \Lambda \) a closed complex Lagrangian submanifold of \( U \times V^a \)

\[(2.10) \quad T^*X \supset U \quad \xrightarrow{p_1} \quad U \times V^a \quad \xleftarrow{p_2} \quad V \subset T^*Y
\]

As detailed in Section 11.4 of [KS90], let \( K \in \text{D}^\mu(\mathcal{C}_{X \times Y}) \), \( SS(K) \) its micro-support and let us suppose that \( p_1|\Lambda, p_2|\Lambda \) are isomorphisms, \( K \) is cohomologically constructible simple with shift 0 along \( \Lambda \) and that \( \Lambda \) belongs to \( \text{D}^\mu(\mathcal{C}_{X \times Y}) \) and \( SS(K) \subset \Lambda \).

Let \( p \equiv (p_X, p_Y^\mu) \in \Lambda \) and let us consider some section \( s \in H^0(\mu\text{hom}(K, \Omega_{X \times Y}))/p \), where \( \Omega_{X \times Y} := \Omega_{X \times Y} \otimes_{q_1} \Omega_{X} q_1^{-1} \Omega_{X} \). The section \( s \) gives a morphism \( K \to \Omega_{X \times Y} \) in \( \text{D}^\mu(\mathcal{C}_{X \times Y}) \). Then, there is a natural morphism

\[(2.11) \quad \Phi_{K}[\mu_\Lambda](\mathcal{O}_X) \to \mathcal{O}_V
\]

We recall the result:

**Theorem 2.1** ([KS90, Th. 11.4.9]). There exists \( s \in H^0(\mu\text{hom}(K, \Omega_{X \times Y}))/p \) such that the associated morphism \( \Phi_{K}[\mu_\Lambda](\mathcal{O}_X) \to \mathcal{O}_V \) is an isomorphism in the category \( \text{D}^\mu(\mathcal{C}_{Y}) \). Moreover, this morphism is compatible with the action of microdifferential operators on \( \mathcal{O}_X \) in \( \text{D}^\mu(\mathcal{C}_{X}; \mu_\Lambda) \) and the action of microdifferential operators on \( \mathcal{O}_Y \) in \( \text{D}^\mu(\mathcal{C}_{Y}; \mu_\Lambda) \).

Also, we will make use of the following theorem proven in [KS90, Th. 7.2.1]:

**Theorem 2.2** ([KS90, Th. 7.2.1]). Let \( K \in \text{D}^\mu(X \times Y) \) and assume that

(i) \( K \) is cohomologically constructible

(ii) \( (p_1^{-1}(U) \cup (p_2^{-1}(V)) \cap SS(K) \subset \Lambda \)

(iii) the natural morphism \( \mu_\text{hom}(K, K)|_{\Lambda} \) is an isomorphism.

Then for any \( F_1, F_2 \in \text{D}^\mu(X; U) \), the natural morphism

\[
\chi_s\mu_\text{hom}(F_1, F_2) \to \mu_\text{hom}(\Phi_K(F_1), \Phi_K(F_2))
\]

is an isomorphism in \( \text{D}^\mu(Y; V) \).
2.8 Complements on the functor $\mu hom$

2.8.1 Associativity for the composition of kernels

The next result is well-known although no proof is written down in the literature, to our knowledge.

**Lemma 2.3.** Let $M_1, M_2, M_3$ be real manifolds, and $K, L, M$ be objects respectively of $D^b(k_{M_12})$, $D^b(k_{M_23})$, $D^b(k_{M_34})$, then the composition of kernels $\circ$ defined in 2.1 is associative. We have the following isomorphism

\[(K \circ L) \circ M \simeq K \circ (L \circ M)\]

such that for any $N \in D^b(k_{M_45})$, the diagram below commutes:

\[(2.12) \quad (K \circ L) \circ M \circ N \rightarrow K \circ (L \circ M) \circ N\]

\[\quad \rightarrow (K \circ (L \circ M)) \circ N \rightarrow K \circ (L \circ (M \circ N)).\]

**Proof.** Consider the following diagram

where the thick squares are cartesian, and where for clarity we enforced the notation: the projection $M_{ijk} \rightarrow M_{ij}$ by $q_{ij}^k$ (independently of order of appearance of the indices), and the projection $M_{ijkl} \rightarrow M_{ij}$ by $q_{ij}^{kl}$. We now have:

\[
Rq_{14!}^3(q_{13}^4 \cdot q_{13}^{-1} \cdot (Rq_{13!}^2(q_{12}^3 \cdot q_{23}^{-1} \cdot L)) \otimes q_{34}^{-1}M) \simeq Rq_{14!}^3(Rq_{134!}^2(q_{12}^3 \cdot q_{23}^{-1} \cdot L) \otimes q_{34}^{-1}M)
\]

\[
\simeq Rq_{14!}^3(q_{12}^{34} \cdot K \otimes q_{23}^{14} \cdot q_{23}^{-1}L \otimes q_{34}^{12}M)
\]

\[
:= K_1 \circ K_2 \circ K_3
\]
The same way, we get the isomorphism
\[ K_1 \circ K_2 \circ K_3 \simeq Rq_{14}^2(q_{12}^{-1}K \otimes (q_{24}^{-1}(Rq_{24}^3(q_{23}^{-1}L \otimes q_{34}^{-1}M)))) \]
which proves the isomorphism (2.12). And, it follows immediately that given \( N \in D^b(k_{M_{42}}) \), the diagram (2.13) commutes.

\[ \square \]

### 2.8.2 Associativity for the composition of \( \mu_{hom} \)

We define the composition of kernels on cotangent bundles (see [KS90, section 3.6, (3.6.2)]).

\[
\delta^2 : D^b(k_{T^*M_{12}}) \times D^b(k_{T^*M_{23}}) \to D^b(k_{T^*M_{13}}) \\
(2.14) \quad (K_1, K_2) \mapsto K_1 \circ K_2 := Rp_{13}(p_{12}^{-1}K_1 \otimes p_{23}^{-1}K_2) \\
\simeq Rp_{13}^*(p_{12}^{-1}K_1 \otimes p_{23}^{-1}K_2).
\]

There is a variant of the composition \( \circ \), constructed in [KS14]:

\[
\ast^2 : D^b(k_{M_{12}}) \times D^b(k_{M_{23}}) \to D^b(k_{M_{13}}) \\
(2.15) \quad (K_1, K_2) \mapsto K_1 \circ K_2 := Rp_{13}(q_{2}^{-1}\omega_2 \otimes \delta_2^2(K_1 \otimes K_2)).
\]

There is a natural morphism for \( K_1 \in D^b(k_{M_{12}}) \) and \( K_2 \in D^b(k_{M_{23}}) \), \( K_1 \circ K_2 \to K_1 \circ K_2 \).

Let us state a theorem proven in [KS90, Prop. 4.4.11] refined in [KS14].

**Theorem 2.4.** Let \( F_i, G_i, H_i \) respectively in \( D^b(k_{M_{12}}), D^b(k_{M_{23}}), D^b(k_{M_{34}}) \), \( i = 1, 2 \).

Let \( U_i \) be an open subset of \( T^*M_{ij} \) (\( i = 1, 2, j = i + 1 \)) and set \( U_3 = U_1 \circ U_2 \).

There exists a canonical morphism in \( D^b(k_{T^*M_{13}}) \), functorial in \( F_1 \) (resp. \( F_2 \)):

\[
\mu_{hom}(F_1, F_2)|_{U_1} \circ \mu_{hom}(G_1, G_2)|_{U_2} \to \mu_{hom}(F_1 \circ G_1, F_2 \circ G_2)|_{U_3},
\]

and hence

\[
\mu_{hom}(F_1, F_2)|_{U_1} \circ \mu_{hom}(G_1, G_2)|_{U_2} \to \mu_{hom}(F_1 \circ G_1, F_2 \circ G_2)|_{U_3}.
\]

We state the main theorem of this section.

**Theorem 2.5.** Let \( F_i, G_i, H_i \) respectively in \( D^b(k_{M_{12}}), D^b(k_{M_{23}}), D^b(k_{M_{34}}) \), \( i = 1, 2 \) then we have:

(a)

\[
\left( \mu_{hom}(F_1, F_2) \circ \mu_{hom}(G_1, G_2) \right) \circ \mu_{hom}(H_1, H_2) \simeq \mu_{hom}(F_1, F_2) \circ \left( \mu_{hom}(G_1, G_2) \circ \mu_{hom}(H_1, H_2) \right)
\]
The above isomorphism is compatible with the composition \( \circ \) in the sense that the following diagram commutes

\[
\begin{array}{ccc}
(\mu_{\text{hom}}(F_1, F_2) \circ_2 \mu_{\text{hom}}(G_1, G_2)) \circ_3 \mu_{\text{hom}}(H_1, H_2) & \sim & \mu_{\text{hom}}(F_1, F_2) \circ (\mu_{\text{hom}}(G_1, G_2) \circ_3 \mu_{\text{hom}}(H_1, H_2)) \\
\mu_{\text{hom}}(F_1 \circ_2 G_1, F_2 \circ_2 G_2) \circ_3 \mu_{\text{hom}}(H_1, H_2) & \sim & \mu_{\text{hom}}(F_1, F_2) \circ_2 \mu_{\text{hom}}(G_1 \circ_3 H_1, G_2 \circ_3 H_2) \\
\mu_{\text{hom}}((F_1 \circ_2 G_1) \circ_3 H_1, (F_2 \circ_2 G_2) \circ_3 H_2) & \sim & \mu_{\text{hom}}(F_1 \circ_2 (G_1 \circ_3 H_1), F_2 \circ_2 (G_2 \circ_3 H_2))
\end{array}
\]

**Proof.**
(a) This is a direct application of Lemma 2.3 with \( X, Y, Z \) taken to be respectively \( T^*M_{12}, T^*M_{13}, T^*M_{34} \).

(b) We shall skip the proof, which is tedious but straightforward. \( \square \)

## 3 Complex quantized contact transformations

### 3.1 Kernels on complex manifolds

Consider two complex manifolds \( X \) and \( Y \) of respective dimension \( d_X \) and \( d_Y \). We shall follow the notations of Section 2.1.

For \( K \in \text{D}^b(C_X \times Y) \), we recall that we defined the functor \( \Phi_K : \text{D}^b(C_Y) \to \text{D}^b(C_X) \), \( \Phi_K(G) = Rq_{1!}^*(K \otimes q_2^{-1}(G)) \), for \( G \in \text{D}^b(C_Y) \). With regards to the notation of Section 2.1, let us notice that \( \Phi_K(G) \) is \( K \circ G \). We refer also to Section 1.2 for a definition of \( \Omega_{X \times Y/X} \).

We recall the

**Lemma 3.1.** There is a natural morphism

\[
\Omega_{X \times Y/X} \circ \mathcal{O}_Y [d_Y] \to \mathcal{O}_X.
\]

**Proof.** We have

\[
\Omega_{X \times Y/X} \circ \mathcal{O}_Y [d_Y] = Rq_{1!}^*(\Omega_{X \times Y/X} \otimes q_2^{-1}(\mathcal{O}_Y[d_Y]))
\]

\[
\to Rq_{1!}^*(\Omega_{X \times Y/X}[d_Y]) \xrightarrow{f} \mathcal{O}_X,
\]

where the last arrow is the integration morphism on complex manifolds. \( \square \)

The following Lemma will be useful for the proof of Lemma 3.3. Let us first denote by \( M_i \) (\( i = 1, 2, 3, 4 \)) four complex manifolds, \( L_i \in \text{D}^b(C_{M_{i,i+1}}), 1 \leq i \leq 3 \). We set for short

\[
d_i = \dim_{\mathbb{C}} M_i, d_{ij} = d_i + d_j, \Omega_{ij/i} = \Omega_{M_{ij}/M_i} = \Omega^{(0,d_j)}_{M_{ij}}.
\]
Set for $1 \leq i \leq 3$,

\[
K_i = \mu_{\text{hom}}(L_i, \Omega_{i,j/i}[d_j]), \quad j = i + 1
\]

\[
L_{ij} = L_i \circ L_j \quad j = i + 1, \quad L_{123} = L_1 \circ L_2 \circ L_3,
\]

\[
\tilde{K}_{ij} = \mu_{\text{hom}}(L_{ij}, \Omega_{i,j/i}[d_j] \circ \Omega_{j,k/j}[d_k]) \quad j = i + 1, k = j + 1
\]

\[
\tilde{K}_{123} = \mu_{\text{hom}}(L_{123}, \Omega_{12/1}[d_2] \circ \Omega_{23/2}[d_3] \circ \Omega_{34/3}[d_4])
\]

\[
K_{ij} = \mu_{\text{hom}}(L_{ij}, \Omega_{i,k/i}[d_k]) \quad j = i + 1, k = j + 1,
\]

\[
K_{123} = \mu_{\text{hom}}(L_{123}, \Omega_{14/1}[d_4]).
\]

We recall that we have the sequence of natural morphisms:

\[
\Omega_{i,j/i} \circ \Omega_{j,k/j} = Rq_{i,k}((q_{i,j}^{-1}\Omega_{i,j/i} \otimes q_{j,k}^{-1}\Omega_{j,k/j}))
\]

\[
\rightarrow Rq_{i,k}((\Omega_{i,k/i}))
\]

\[
\rightarrow \Omega_{i,k/i}[−d_j]
\]

(3.1)

**Lemma 3.2.** The following diagram commutes:

\[
\begin{array}{ccc}
K_1 \circ K_2 \circ K_3 & \rightarrow & A \\
\tilde{K}_{12} \circ K_3 & \rightarrow & \tilde{K}_{123} & \longrightarrow & K_1 \circ \tilde{K}_{23} \\
K_1 \circ \tilde{K}_{23} & \longrightarrow & \tilde{K}_{123} & \rightarrow & K_1 \circ K_2 \circ K_3 \\
B & \downarrow & C & \downarrow & \\
K_1 \circ K_3 & \rightarrow & K_{123} & \rightarrow & K_1 \circ K_2 \circ K_3
\end{array}
\]

**Proof.** Diagram labelled A commutes by the associativity of the functor $\mu_{\text{hom}}$ (see Theorem 2.7.3). Let us prove that Diagram B and C commute. Of course, it is enough to consider Diagram B. To make the notations easier, we assume that $M_1 = M_4 = \text{pt}$. We are reduced to prove the commutativity of the diagram:

\[
\begin{array}{ccc}
\mu_{\text{hom}}(L_2, \Omega_2 \circ \Omega_{2,3/2}[d_{23}]) \circ \mu_{\text{hom}}(L_3, \mathcal{O}_3) & \longrightarrow & \mu_{\text{hom}}(L_{23}, \Omega_2 \circ \Omega_{2,3/2}[d_{23}] \circ \mathcal{O}_3) \\
\downarrow f_2 & & \downarrow f_2 \\
\mu_{\text{hom}}(L_2, \Omega_3[d_3]) \circ \mu_{\text{hom}}(L_3, \mathcal{O}_3) & \longrightarrow & \mu_{\text{hom}}(L_{23}, \Omega_3[d_3] \circ \mathcal{O}_3)
\end{array}
\]

For $F, F' \in \text{D}^b(k_{12})$, $G, G' \in \text{D}^b(k_{23})$, we saw in Theorem 2.5 (b) that the morphism $\mu_{\text{hom}}(F, F') \circ \mu_{\text{hom}}(G, G') \rightarrow \mu_{\text{hom}}(F \circ G, F' \circ G')$ is functorial in $F, F', G, G'$. This fact applied to the morphism

\[
\Omega_2 \circ \Omega_{2,3/2}[d_{23}] \rightarrow \Omega_3[d_3]
\]

gives that the above diagram commutes and so diagram $B$ commutes. \qed
Let $Z$ be a complex manifold and let $\Lambda \subset T^*(X \times Y)$ and $\Lambda' \subset T^*(Y \times Z)$ be two conic Lagrangian smooth locally closed complex submanifolds.

Let $L, L'$, be perverse sheaves on $X \times Y, Y \times Z$, with microsupport $SS(L) \subset \Lambda, SS(L') \subset \Lambda'$ respectively. We set

$$L'' := L[d_Y] \circ L'$$

Assume that

\begin{equation}
(3.2) \quad p_2^0|_\Lambda : \Lambda \to T^*Y \quad \text{and} \quad p_2|_{\Lambda'} : \Lambda' \to T^*Y
\end{equation}

are transversal

and that

\begin{equation}
(3.3) \quad \text{the map } \Lambda \times_{T^*Y} \Lambda' \to \Lambda \circ \Lambda' \text{ is an isomorphism.}
\end{equation}

Let us set

$$\mathcal{L} := \mu_{\mathrm{hom}}(L, \Omega_{X \times Y/X})$$

Note that $\mathcal{L} \in D^b(T^*(X \times Y))$ is concentrated in degree 0. Indeed, it is proven in [KS90, Th. 10.3.12] that perverse sheaves are the ones which are pure with shift zero at any point of the nonsingular locus of their microsupport. On the other hand, Theorem 9.5.2 of [KS85] together with Definition 9.5.1 of [KS85] show that the latter verify the property that, when being applied $\mu_{\mathrm{hom}}(\bullet, \Omega_{X \times Y/X})$, they are concentrated in degree 0. Moreover, $\mathcal{L}$ is a $(\mathcal{E}_X, \mathcal{E}_Y)$-bimodule. Indeed, such actions come from morphism (2.17) and the integration morphism (3.1). We define similarly $\mathcal{L}'$ and $\mathcal{L}''$.

Now consider two open subsets $U, V$ and $W$ of $\hat{T}^*X, \hat{T}^*Y, \hat{T}^*Z$, respectively.

Let $K'_{U \times V^a}$ be the constant sheaf on $(U \times V^a) \cap \Lambda$ with stalk $H^0\Gamma(U \times V^a; \mathcal{L})$, extended by 0 elsewhere.

$K'_{U \times V^a}$ is the constant sheaf on $(V \times W^a) \cap \Lambda'$ with stalk $H^0\Gamma(V \times W^a; \mathcal{L}')$, extended by 0 elsewhere.

$K''_{U \times V^a}$ is the constant sheaf on $(U \times W^a) \cap \Lambda \circ \Lambda'$ with stalk $H^0\Gamma(U \times W^a; \mathcal{L}'')$, extended by 0 elsewhere.

Let $s, s'$ be sections of $\Gamma(U \times V^a; \mathcal{L})$ and $\Gamma(V \times W^a; \mathcal{L}')$ respectively. We define the product $s \cdot s'$ to be the section of $\Gamma(U \times W^a; \mathcal{L}'')$, image of 1 by the following sequence of morphisms

$$\mathcal{C}_{\Lambda \circ \Lambda'} \leftarrow \mathcal{C}_\Lambda \circ \mathcal{C}_{\Lambda'} \leftarrow \mathcal{C}_\Lambda$$

\begin{align*}
&:= Rp_{13!}(p_{12}^{-1}\mathcal{C}_\Lambda \otimes p_{23}^{-1}\mathcal{C}_{\Lambda'}) \\
&\to Rp_{13!}(p_{12}^{-1}K_{U \times V^a} \otimes p_{23}^{-1}K_{V \times W^a}) \\
&\to Rp_{13!}(p_{12}^{-1}\mu_{\mathrm{hom}}(L, \Omega_{X \times Y/X}) \otimes p_{23}^{-1}\mu_{\mathrm{hom}}(L', \Omega_{Y \times Z/Y})) \\
&:= \mu_{\mathrm{hom}}(L, \Omega_{X \times Y/X}) \circ \mu_{\mathrm{hom}}(L', \Omega_{Y \times Z/Y}) \to \mathcal{L}''
\end{align*}

where the first isomorphism comes from the assumption 3.3.

**Lemma 3.3.** Assume that conditions 3.2 and 3.3 are satisfied. Let $s, s'$ be sections of $\Gamma(U \times V^a; \mathcal{L})$ and $\Gamma(V \times W^a; \mathcal{L}')$ respectively, and let $G \in D^b(\mathcal{C}_Y), H \in D^b(\mathcal{C}_Z)$. Then,
(i) $s$ defines a morphism

$$\alpha_G(s) : \mathbb{C}_A \circ \mu\text{hom}(G, \mathcal{O}_Y)|_V \to \mu\text{hom}(L[d_Y] \circ G, \mathcal{O}_X)|_U$$

(ii) Considering the morphism

$$\alpha_H(s \cdot s') : \mathbb{C}_{A \circ \Lambda} \circ \mu\text{hom}(H, \mathcal{O}_Z)|_W \to \mu\text{hom}(L[d_Y] \circ L'[d_Z] \circ H, \mathcal{O}_X)|_U$$

we have the isomorphism

$$\alpha_H(s \cdot s') \simeq \alpha_{L'[d_Z] \circ H}(s) \circ \Phi_{\mathbb{C}_A}(\alpha_H(s'))$$

Proof. (i) Given $s$ and two objects $G_1, G_2 \in \mathsf{D}^b(\mathbb{C}_Y)$, we have a morphism

$$\mathbb{C}_A \circ \mu\text{hom}(G_1, G_2)|_V \to \mu\text{hom}(L \circ G_1, \Omega_{X \times Y/X} \circ G_2)|_U$$

corresponding to the composition of morphisms:

$$\begin{align*}
R_{p_1!}(\mathbb{C}_A \otimes p_2^{-1} \mu\text{hom}(G_1, G_2)|_V) & \to R_{p_1!}(K_{U \times V^a} \otimes p_2^{-1} \mu\text{hom}(G_1, G_2)|_V) \\
\text{(3.4)} & \to R_{p_1!}(\mu\text{hom}(L, \Omega_{X \times Y/X} \circ G_2)|_V) \\
& \to \mu\text{hom}(L \circ G_1, \Omega_{X \times Y/X} \circ G_2)|_U
\end{align*}$$

where the second morphism comes from the natural morphism $K_{U \times V^a} \to \mu\text{hom}(L, \Omega_{X \times Y/X})$.

We conclude by choosing, $G_1 = G$, $G_2 = \mathcal{O}_Y$ and by using Lemma 3.1:

$$\mu\text{hom}(L \circ G_1, \Omega_{X \times Y/X} \circ \mathcal{O}_Y) \to \mu\text{hom}(L \circ G_1, \mathcal{O}_X[-d_Y]) \simeq \mu\text{hom}(L[d_Y] \circ G_1, \mathcal{O}_X)$$

(ii) Let $H \in \mathsf{D}^b(\mathbb{C}_Z)$. We denote by $\mathcal{H} := \mu\text{hom}(H, \mathcal{O}_Z)$. It suffices to prove that the following diagram commutes:
where we omitted the subscript $U \times V^a$ and $V \times W^a$, $H$, $L'[d_z] \circ H$ for $K_{U \times V^a}$, $K'_{V \times W^a}$, $\alpha_H$, $\alpha_{L'[d_z] \circ H}$, respectively.

We know from Theorem 2.4 that the operation $\circ$ is functorial, so that diagram $A$
and $B$ commute. For instance, diagram $A$ decomposes this way:

\[
\begin{array}{cccccc}
C_{\Lambda} \circ (C_{\Lambda'} \circ \mathcal{H}) \\
\downarrow \downarrow \downarrow \downarrow \\
C_{\Lambda} \circ (\mu_{\text{hom}}(L', \Omega_{Y \times Z/Y}) \circ \mathcal{H}) \\
\downarrow \downarrow \downarrow \downarrow \\
K \circ (K' \circ \mathcal{H}) \\
\end{array}
\]

Besides, diagram $C$ commutes by Lemma 3.2.

Finally, the bottom diagonal punctured line corresponds to $\alpha(s \cdot s')$, since the following diagram commutes

\[
\begin{array}{cccccc}
C_{\Lambda} \circ \mu_{\text{hom}}(G, \mathcal{O}_Y) \\
\downarrow \downarrow \downarrow \downarrow \\
C_{\Lambda} \circ \mu_{\text{hom}}(L'[d_Y] \circ H, \mathcal{O}_Y) \\
\end{array}
\]

Remark 3.4. In the following, unless necessary, we will omit the subscript for $\alpha$.

Theorem 3.5. Let $s \in \Gamma(U \times V^a; \mathcal{L})$, $G \in D^b(C_Y)$. Then,

(i) $s$ defines a morphism

\[
\alpha(s) : C_{\Lambda} \circ \mu_{\text{hom}}(G, \mathcal{O}_Y)|_U \to \mu_{\text{hom}}(L[d_Y] \circ G, \mathcal{O}_X)|_U.
\]

(ii) Moreover, if $P \in \Gamma(U; \mathcal{O}_X)$ and $Q \in \Gamma(V; \mathcal{O}_Y)$ satisfy $P \cdot s = s \cdot Q$, then the diagram below commutes

\[
\begin{array}{cccccc}
C_{\Lambda} \circ \mu_{\text{hom}}(G, \mathcal{O}_Y) \\
\downarrow \downarrow \downarrow \downarrow \\
C_{\Lambda} \circ \mu_{\text{hom}}(L[d_Y] \circ G, \mathcal{O}_X) \\
\end{array}
\]

Proof. (i) is already proven in Lemma 3.3.

(ii) With regards to the notation of Lemma 3.3, we consider the triplet of manifolds $X, X, Y$, $\Lambda = C_{\Delta_X}$, $\mathcal{L} := \mu_{\text{hom}}(C_{\Delta_X}[-n], \Omega_{X \times X/Y})$. Then, the assumption 3.2 is satisfied and noticing that $\Phi_{C_{\Delta_X}} \simeq Id_X$, we conclude by Lemma 3.3 that

\[
\alpha(P) \circ \alpha(s) \simeq \alpha(P \cdot s) \simeq \alpha(s \cdot Q) \simeq \alpha(s) \circ \Phi_{C_{\Lambda}}(\alpha(Q)).
\]
3.2 Main theorem

In this section, we will apply Theorem 3.5 when we are given a homogeneous symplectic isomorphism. Let us recall some useful results.

For $\mathcal{M}$ a left coherent $E_X$-module generated by a section $u \in \mathcal{M}$, we denote by $I_\mathcal{M}$ the annihilator left ideal of $E_X$ given by:

$$I_\mathcal{M} := \{ P \in E_X; Pu = 0 \}$$

and by $\overline{I}_\mathcal{M}$ the symbol ideal associated to $I_\mathcal{M}$:

$$\overline{I}_\mathcal{M} := \{ \sigma(P); P \in I_\mathcal{M} \}$$

Definition 3.6 ([Kas03]). Let $\mathcal{M}$ be a coherent $E_X$-module generated by an element $u \in \mathcal{M}$. We say that $(\mathcal{M}, u)$ is a simple $E_X$-module if $I_\mathcal{M}$ is reduced and $\overline{I}_\mathcal{M} = \{ \varphi \in O_{T^*X}; \varphi|_{\text{supp}(\mathcal{M})} = 0 \}$.

Consider two complex manifolds $X$ and $Y$, open subsets $U$ and $V$ of $T^*X$ and $T^*Y$, respectively, and denote by $p_1$ and $p_2$ the projections $U \xymatrix{\ar[r]^{p_1} & U \times V^a \ar[l]_{p_2}} V$. Let $\Lambda$ be a smooth closed submanifold Lagrangian of $U \times V^a$. We will make use of the following result from [SKK, Th. 4.3.1], [Kas03, Prop. 8.5]:

Theorem 3.7 ([SKK],[Kas03]). Let $(\mathcal{M}, u)$ be a simple $E_{X \times Y}$-module defined on $U \times V^a$ such that $\text{supp} \mathcal{M} = \Lambda$. Assume $\Lambda \to U$ is a diffeomorphism. Then, there is an isomorphism of $E_X$-modules:

$$E_X|_U \xymatrix{\ar[r]^{(p_1|_{U \times V^a})_* \mathcal{M}} & (p_2|_{U \times V^a})_* \mathcal{M}}$$

$$P \mapsto P \cdot u$$

Assume that the projections $p_1|_{\Lambda}$ and $p_2^a|_{\Lambda}$ induce isomorphisms. We denote by $\chi$ the homogeneous symplectic isomorphism $\chi := p_2|_{\Lambda} \circ p_1|_{\Lambda}^{-1}$.

(3.7)

$$\begin{array}{c}
\Lambda \subset U \times V^a \\
\xymatrix{U \ar@{~}[r]^{p_1|_{\Lambda}} \ar@{~}[d]^{\sim} & U \times V^a \ar@{~}[d]^{\sim} \\
\hat{T}^*X \ar[r]^{\chi} & V \subset \hat{T}^*Y}
\end{array}$$

Corollary 3.8. Let $(\mathcal{M}, u)$ be a simple $E_{X \times Y}$-module defined on $U \times V^a$. Assume $\text{supp} \mathcal{M} = \Lambda$. Then, in the situation of (3.7), we have an anti-isomorphism of algebras

$$\chi_* E_X|_U \simeq E_Y|_V$$

Consider two complex manifolds $X$ and $Y$ of the same dimension $n$, open subsets $U$ and $V$ of $\hat{T}^*X$ and $\hat{T}^*Y$, respectively, $\Lambda$ a smooth closed Lagrangian submanifold of $U \times V^a$ and assume that the projections $p_1|_{\Lambda}$ and $p_2^a|_{\Lambda}$ induce isomorphisms, hence a homogeneous symplectic isomorphism $\chi: U \xymatrix{\ar[r]^{\sim} & V}$. 

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We consider a perverse sheaf $L$ on $X \times Y$ satisfying

\[(3.9)\quad (p_1^{-1}(U) \cup p_2^{-1}(V)) \cap \text{SS}(L) = \Lambda.\]

and a section $s$ in $\Gamma(U \times V^a; \muhom(L, \Omega_{X \times Y/X}))$.

Let $G \in D^b(\mathbb{C}_Y)$. From Theorem 3.5 (i), the left composition by $s$ defines the morphism $\alpha(s)$ in $D^b(\mathbb{C}_U)$:

\[(3.10)\quad \mathbb{C}_\Lambda \circ \muhom(G, \mathcal{O}_Y)|_V \xrightarrow{\alpha(s)} \muhom(L[\eta] \circ G, \mathcal{O}_X)|_U\]

The condition (3.9) implies that $\text{supp}(\muhom(L, \Omega_{X \times Y/X})|_{p_2^{-1}(V)}) \subset \Lambda$. Since, $p_1$ is an isomorphism from $\Lambda$ to $U$ and that $\chi \circ p_1|_\Lambda = p_2^\alpha|_\Lambda$, we get a morphism in $D^b(\mathbb{C}_U)$

\[(3.11)\quad \chi^{-1}\muhom(G, \mathcal{O}_Y)|_V \xrightarrow{\alpha(s)} \muhom(\Phi_{L[\eta]}(G), \mathcal{O}_X)|_U\]

**Theorem 3.9.** Assume that the section $s$ is non-degenerate on $\Lambda$. Then, for $G \in D^b(\mathbb{C}_Y)$, we have the following isomorphism in $D^b(\mathbb{C}_U)$

\[(3.12)\quad \chi^{-1}\muhom(G, \mathcal{O}_Y)|_V \simeq \muhom(\Phi_{L[\eta]}(G), \mathcal{O}_X)|_U\]

Moreover, this isomorphism is compatible with the action of $\mathcal{O}_Y$ and $\mathcal{O}_X$ on the left and right side of (3.12) respectively.

**Proof.** Let us first prove the following lemma, whose proof is available at the level of germs in [KS90, Th. 11.4.9].

Let us prove that the morphism (3.11) is an isomorphism. Let $L^\ast$ be the perverse sheaf $r^{-1}R\mathcal{H}om(L, \omega_{X \times Y/Y})$ where $r$ is the map $X \times Y \to Y \times X, (x, y) \mapsto (y, x)$. Let $s'$ be a section of $\muhom(L^\ast, \Omega_{Y \times X/Y})$, non-degenerate on $r(\Lambda)$, then we apply the same precedent construction to get a natural morphism

$$\chi_\ast\muhom(\Phi_{L[\eta]}(G), \mathcal{O}_X)|_U \to \muhom(\Phi_{L^\ast[\eta]} \circ \Phi_{L[\eta]}G, \mathcal{O}_Y)|_V \simeq \muhom(\Phi_{L^\ast \circ L[\eta]}G, \mathcal{O}_Y)|_V$$

We know from [KS90, Th. 7.2.1] that $\mathbb{C}_\Delta \simeq L^\ast \circ L$, so that we get a morphism in $D^b(\mathbb{C}_Y)$

\[(3.13)\quad \chi_\ast\muhom(\Phi_{L[\eta]}(G), \mathcal{O}_X)|_U \xrightarrow{\alpha(s')} \muhom(G, \mathcal{O}_Y)|_V\]

We must prove that (3.11) and (3.13) are inverse to each other. By Lemma 3.3(ii), we get that the composition of these two morphisms is $\alpha(s' \cdot s)$, with $s' \cdot s \in \mathcal{O}_X$. 

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For any left $\mathcal{E}_X$-module $\mathcal{M}$, corresponds a right $\mathcal{E}_X$-module $\Omega_X \otimes_{\mathcal{E}_X} \mathcal{M}$. Fixing a non-degenerate form $t_X$ of $\Omega_X|_U$ (resp. $t_Y$ of $\Omega_Y|_V$), we apply now Theorem 3.7: $s$ and $s'$ are non-degenerate sections so that $(\mathcal{E}_{X\times Y}|_{U\times V'}, t_X \otimes s)$ and $(\mathcal{E}_{Y\times X}|_{V\times U'}, s' \otimes t_Y)$ are simple and so isomorphic to $(p_1^*\mathcal{E}_X|_U, 1) \simeq (p_2^*\mathcal{E}_Y|_V, 1)$. $\Omega_X$ resp. $\Omega_Y$ being invertible $\mathcal{E}_X$-module resp. $\mathcal{E}_Y$-module, we get as well for the left-right $(\mathcal{E}_X|_U, \mathcal{E}_Y|_V)$ bi-module, resp. left-right $(\mathcal{E}_Y|_V, \mathcal{E}_X|_U)$ bi-module generated by $s$ resp. $s'$, that they are both isomorphic to $p_1^*\mathcal{E}_X|_U \simeq p_2^*\mathcal{E}_Y|_V$.

Then, following the proof of [KS90, Th. 11.4.9], $s$ and $s'$, define ring isomorphisms associating to each $P \in \mathcal{E}_X(U)$, $P' \in \mathcal{E}_X(U)$, some $Q \in \mathcal{E}_Y(V)$, $Q' \in \mathcal{E}_Y(V)$, such that $P \cdot s = s \cdot Q$, $s' \cdot P' = Q' \cdot s'$, respectively. Hence, we get that $\alpha(s') \circ \alpha(s)$ is an automorphism $\mu_{\text{hom}}(G, \mathcal{E}_Y)|_V$, defined by the left action of $s' \cdot s \in \mathcal{E}_X$. Hence, we can choose $s'$ so that $\alpha(s') \circ \alpha(s)$ is the identity.

We are now in a position to prove Theorem 3.9: we constructed in the proof of the lemma, for each $P \in p_1^*\mathcal{E}_X|_U$, some $Q \in p_2^*\mathcal{E}_Y|_V$ such that $P \cdot s = s \cdot Q$ and we can apply Theorem 3.5 to conclude. 

\section{Radon transform for sheaves}

We are going to apply the results of the last chapter to the case of projective duality. Recall projective duality for $D$-modules were performed by D'Agnolo-Schapira [DS96]. We will extend their results in a microlocal setting.

\subsection{Notations}

In the following, we will quantize the contact transform associated with the Lagrangian submanifold $\hat{T}_p^*\mathbb{P} \times \mathbb{P}^*$, where $\mathbb{S}$ is the hypersurface of $\mathbb{P} \times \mathbb{P}^*$ defined by the incidence relation $\langle \xi, x \rangle = 0, (x, \xi) \in \mathbb{P} \times \mathbb{P}^*$.

We denote by $\hat{T}_p^*\mathbb{P}$, resp. $\hat{T}_p^*\mathbb{P}^*$, the conormal space to $P$ in $\hat{T}^*\mathbb{P}$, resp. to $P^*$ in $\hat{T}^*\mathbb{P}^*$, and we will construct and denote by $\chi$ the homogeneous symplectic isomorphism between $\hat{T}^*\mathbb{P}$ and $\hat{T}^*\mathbb{P}^*$.

For $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we denote by $\mathbb{C}_P(\varepsilon)$ the following sheaves: for $\varepsilon = 0$, we set

$$\mathbb{C}_P(0) := \mathbb{C}_P$$

for $\varepsilon = 1$, $\mathbb{C}_P(1)$ is the sheaf defined by the following exact sequence:

\begin{equation}
0 \to \mathbb{C}_P(1) \to q!\mathbb{C}_P \xrightarrow{tr} \mathbb{C}_P \to 0
\end{equation}

where $q$ is the $2 : 1$ map from the universal cover $\tilde{P}$ of $P$, to $P$ and $tr$ the integration morphism $tr : q!\mathbb{C}_P \simeq q!q^*\mathbb{C}_P \to \mathbb{C}_P$.

Let an integer $p \in \mathbb{Z}$, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we define the sheaves of real analytic functions, hyperfunctions on $P$ resp. $P^*$ twisted by some power of the tautological line bundle,

$$\mathcal{A}_P(\varepsilon, p) := \mathcal{A}_P \otimes_{\mathcal{O}_P} \mathcal{O}_P(p) \otimes_{\mathbb{C}} \mathbb{C}_P(\varepsilon)$$
We define the sheaves of microfunctions on $P$ resp. $P^*$ twisted by some power of the tautological bundle,

$$
\mathcal{C}_P(\varepsilon, p) := H^0(\mu hom(D_\varepsilon \mathcal{C}_P, \mathcal{O}_P(p))) \otimes \mathcal{C}_P(\varepsilon)
$$

and similarly with $P^*$ instead of $P$. We notice that for $n$ odd, $D_\varepsilon \mathcal{C}_P \simeq \mathcal{C}_P(0) = \mathcal{C}_P$, and for $n$ even $D_\varepsilon \mathcal{C}_P \simeq \mathcal{C}_P(1)$.

Let $n$ be the dimension of $P$, (of course $n = d_P$). For an integer $k$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we note

$$
k^* := -n - 1 - k$$

$$
\varepsilon^* := -n - 1 - \varepsilon \mod(2)
$$

### 4.2 Projective duality: geometry

#### 4.2.1 Notations

We refer to the notations of the sections 1.2. We recall that we denote by

$V$, $V$, an $(n+1)$-dimensional real and complex vector space, respectively,

$P$, $\mathbb{P}$, the $n$-dimensional real and complex projective space, respectively,

$S$, $S$, the real and complex incidence hypersurface in $P \times P^*$, $\mathbb{P} \times \mathbb{P}^*$, respectively.

When necessary, we will enforce the dimension by noting $\mathbb{P}_n$, resp. $\mathbb{P}_n^*$.

Let $X, Y$ be complex manifolds, we recall that we denote by $q_1$ and $q_2$ the respective projection of $X \times Y$ on each of its factor.

For $K \in D^b(\mathcal{C}_{X \times Y})$, we recall that we defined the functor:

$$
\Phi_K: D^b(\mathcal{C}_X) \to D^b(\mathcal{C}_Y)
$$

$$
F \mapsto Rq_2!(K \otimes q_1^{-1}F)
$$

For an integer $k$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we note $k^* = -n - 1 - k$ and $\varepsilon^* = -n - 1 - \varepsilon \mod(2)$.

We refer to Section 1.2 for the definition of the sheaves of twisted microfunctions $\mathcal{C}_P(\varepsilon, k)$, $\mathcal{C}_{P^*}(\varepsilon^*, k^*)$. 
4.2.2 Geometry of projective duality

For a manifold $X$, we denote by $P^*X$ the projectivization of the cotangent bundle of $X$. The following results are well-known. However, we will give a proof of Proposition 4.1 since it is more straightforward than the one usually found in the literature.

**Proposition 4.1.** There is an homogeneous complex symplectic isomorphism

\[
\hat{T}^*P \simeq \hat{T}^*P^*
\]

and a contact isomorphism

\[
P^*P \simeq S \simeq P^*P^*
\]

**Proof.** We have the natural morphism

\[
\mathbb{V} \setminus \{0\} \xrightarrow{\rho} P
\]

According to 2.3, this morphism, after removing the zero section, induces the following diagram

\[
\begin{array}{ccc}
T^*(\mathbb{V} \setminus \{0\}) & \xrightarrow{i^\rho'} & \mathbb{V} \setminus \{0\} \times_P T^*P \\
\downarrow & & \downarrow \\
\mathbb{V} \setminus \{0\} & \xrightarrow{i^\rho} & P
\end{array}
\]

We notice that $i^\rho'$ is an immersion. Let us denote by $\mathbb{H}, \mathbb{H}^*$, the incidence hypersurfaces:

\[
\mathbb{H} = \{(\xi, x) \in \mathbb{V}^* \times (\mathbb{V} \setminus \{0\}); \langle \xi, x \rangle = 0\}
\]
\[
\mathbb{H}^* = \{(x, \xi) \in \mathbb{V} \times (\mathbb{V}^* \setminus \{0\}); \langle x, \xi \rangle = 0\}
\]

Noticing that for $x \in \mathbb{V} \setminus \{0\}$, $\rho$ is constant along the fiber above $\rho(x)$, we see that $i^\rho'$ is an immersion into the incidence hypersurface $\mathbb{H}$. Besides, $i^\rho'$ is a morphism of fibered space and so, by a dimensional argument, we conclude that this immersion is also onto.

Removing the zero sections, we get the diagram

\[
\begin{array}{ccc}
T^*(\mathbb{V} \setminus \{0\}) & \xrightarrow{\sim} & \mathbb{H} \\
\downarrow & & \downarrow \\
T^*(\mathbb{V}^* \setminus \{0\}) & \xrightarrow{\sim} & \mathbb{H}^*
\end{array}
\]

where the isomorphism between $\mathbb{H}$ and $\mathbb{H}^*$ follows from the following symplectic isomorphism:

\[
\hat{T}^*(\mathbb{V} \setminus \{0\}) \simeq \hat{T}^*(\mathbb{V}^* \setminus \{0\})
\]

\[
(x, \xi) \mapsto (\xi, -x)
\]

\[24\]
Now, taking the quotient by the action of $\mathbb{C}^*$ on both sides of the isomorphism between $(\mathbb{V}\setminus\{0\}) \times_{\mathbb{P}} \hat{T}^*\mathbb{P}$ and $(\mathbb{V}^*\setminus\{0\}) \times_{\mathbb{P}^*} \hat{T}^*\mathbb{P}^*$, we get the isomorphism:

$$\hat{T}^*\mathbb{P} \simeq (\mathbb{V}\setminus\{0\}) \times_{\mathbb{P}} \mathbb{P}^* \simeq \hat{T}^*\mathbb{P}^*$$

This gives (4.2).

Besides, passing to the quotient by the action of $\mathbb{C}^* \times \mathbb{C}^*$ on the two central columns of diagram (4.4), we get (4.3).

**Proposition 4.2.** Consider the double fibrations

\[ (4.5) \]

\[
\begin{array}{ccc}
\hat{T}_S^*(\mathbb{P} \times \mathbb{P}^*) & \sim & (\hat{T}_p^*\mathbb{P} \times \hat{T}_p^*\mathbb{P}^*) \\
p_1 & \sim & p_2 \\
\hat{T}^*\mathbb{P} & \sim & \chi & \hat{T}^*\mathbb{P}^*
\end{array}
\]

Then, $p_1$ and $p_2^\prime$ are isomorphisms and $\chi = p_2^\prime \circ p_1^{-1}$ is a homogeneous symplectic isomorphism.

Now, we are going to prove the following

**Proposition 4.3.** The diagram (4.5) induces

\[ (4.5) \]

\[
\begin{array}{ccc}
\hat{T}_S^*(\mathbb{P} \times \mathbb{P}^*) \cap (\hat{T}_p^*\mathbb{P} \times \hat{T}_p^*\mathbb{P}^*) & \sim & \hat{T}_p^*\mathbb{P}^* \\
p_1 & \sim & p_2 \\
\hat{T}_p^*\mathbb{P} & \sim & \chi & \hat{T}_p^*\mathbb{P}^*
\end{array}
\]

### 4.3 Projective duality for microdifferential operators

Let $k, k'$ be integers and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. We follow the notations of the sections 1.2 and 2. We define similarly a twisted version of $B^{(n,0)}_{\mathbb{S}[\mathbb{P} \times \mathbb{P}^*]}$ and $C^{(n,0)}_{\mathbb{S}[\mathbb{P} \times \mathbb{P}^*]}$.

We set

$$B^{(n,0)}_{\mathbb{S}[\mathbb{P} \times \mathbb{P}^*]}(k, k') := q_2^{-1}\mathcal{O}_{\mathbb{P}^*}(k') \otimes q_1^{-1}\mathcal{O}_p \mathbb{B}_{\mathbb{S}[\mathbb{P} \times \mathbb{P}^*]} \otimes q_1^{-1}\mathcal{O}_p q_1^{-1}(\mathcal{O}_p(k) \otimes \mathcal{O}_p^* \mathcal{O}_p^*)$$

and the $(\mathcal{E}_p(-k, k), \mathcal{E}_{\mathbb{P}^*}(-k^*, k^*))$-module

$$\mathcal{E}^{(n,0)}_{\mathbb{S}[\mathbb{P} \times \mathbb{P}^*]}(k, k') := \mathcal{E}^{(n,0)}_{\mathbb{S}[\mathbb{P} \times \mathbb{P}^*]}(k, k')$$

We notice that $\mathcal{E}_p(-k, k)$ is nothing but $\mathcal{O}_p(-k) \mathcal{D} \otimes \pi_{\mathbb{P}^*}^{-1}\mathcal{D}_p \mathcal{E}_{\mathbb{P}^*} \otimes \pi_{\mathbb{P}^*}^{-1}\mathcal{D}_p \mathcal{D}\mathcal{O}_p(k)$. According to the diagram (4.5), we denoted by $\chi$ the homogeneous symplectic isomorphism

$$\chi := p_2^\prime|_{\hat{T}_S^*(\mathbb{P} \times \mathbb{P}^*)} \circ p_1^{-1}|_{\hat{T}_S^*(\mathbb{P} \times \mathbb{P}^*)}$$

We have:
Theorem 4.4 ([DS96, p. 469]). Assume \( -n - 1 < k < 0 \). There exists a section \( s \) of \( \mu \text{hom}(\mathcal{C}_S[-1], \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*)) \), non-degenerate on \( \hat{T}_S^*(\mathbb{P} \times \mathbb{P}^*) \).

Proof. From the exact sequence:

\begin{equation}
0 \rightarrow \mathcal{C}_{(\mathbb{P} \times \mathbb{P}^*) \setminus S} \rightarrow \mathcal{C}_{\mathbb{P} \times \mathbb{P}^*} \rightarrow \mathcal{C}_S \rightarrow 0
\end{equation}

we get the natural morphism

\begin{equation}
\Gamma((\mathbb{P} \times \mathbb{P}^*) \setminus S; \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*)) \rightarrow \Gamma(S; \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*))[1]
\end{equation}

\begin{equation}
\simeq \Gamma((\mathbb{P} \times \mathbb{P}^*) \setminus S; \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*))[1]
\end{equation}

\begin{equation}
\simeq \Gamma((\mathbb{P} \times \mathbb{P}^*) \setminus S; \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*))[1]
\end{equation}

\begin{equation}
\simeq \Gamma((\mathbb{P} \times \mathbb{P}^*) \setminus S; \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*))[1]
\end{equation}

\begin{equation}
\rightarrow \Gamma((\mathbb{P} \times \mathbb{P}^*) \setminus S; \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*))[1]
\end{equation}

\begin{equation}
\rightarrow \Gamma((\mathbb{P} \times \mathbb{P}^*) \setminus S; \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*))[1]
\end{equation}

Let \( z = (z_0, ..., z_n) \) be a system of homogeneous coordinates on \( \mathbb{P} \) and \( \zeta = (\zeta_0, ..., \zeta_n) \) the dual coordinates on \( \mathbb{P}^* \). As explained in [DS96], a non-degenerate section is provided by the Leray section, defined for \( (z, \xi) \in ((\mathbb{P} \times \mathbb{P}^*) \setminus S) \) by

\begin{equation}
s(z, \xi) = \frac{\omega'(z)}{\langle z, \xi \rangle^{n+1+k}}
\end{equation}

where \( \omega'(z) \) is the Leray form \( \omega'(z) = \sum_{k=0}^{n}(-1)^k z_k dz_0 \wedge ... \wedge dz_{k-1} \wedge dz_{k+1} \wedge ... \wedge dz_n \), Leray [Ler59].

Let \( s \) be a section of \( H^1(\mu \text{hom}(\mathcal{C}_S[-1], \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*))) \), non-degenerate on \( \hat{T}_S^*(\mathbb{P} \times \mathbb{P}^*) \).

Theorem 4.5. Assume \( -n - 1 < k < 0 \). Then, we have an isomorphism in \( D^b(\mathcal{C}_{\hat{T}^* \mathbb{P}}) \)

\[ \Phi^\mu_S(\mathcal{E}_\mathbb{P}(-k, k)|_{\hat{T}^* \mathbb{P}}) \simeq \mathcal{E}_\mathbb{P}(-k, k)|_{\hat{T}^* \mathbb{P}} \]

\[ \chi(\mathcal{E}_\mathbb{P}(-k, k)|_{\hat{T}^* \mathbb{P}}) \simeq \mathcal{E}_\mathbb{P}(-k, k)|_{\hat{T}^* \mathbb{P}} \]

Proof. Let \( \mathcal{F}, \mathcal{G} \) be line bundles on \( \mathbb{P} \), and \( \mathbb{P}^* \) respectively. We know from [SKK] that a global non-degenerate section \( s \in \Gamma((\hat{T}^* \mathbb{P} \times \hat{T}^* \mathbb{P}^*); \mathcal{E}_\mathbb{P}^{(n,0)} \otimes_{\mathcal{O}_{\mathbb{P}^*}} \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^*}} \mathcal{G}^{\otimes -1} \mathcal{E}) \) induces an isomorphism of \( \mathcal{E} \)-modules

\[ \Phi^\mu_S(\mathcal{E}_\mathbb{P}(k)|_{\hat{T}^* \mathbb{P}}) \simeq \mathcal{E}_\mathbb{P}(k)|_{\hat{T}^* \mathbb{P}} \]

Now, let us set \( \mathcal{F} = \mathcal{O}_\mathbb{P}(k), \mathcal{G} = \mathcal{O}_{\mathbb{P}^*}(k^*) \). Then, 4.4 provides such a non-degenerate section in \( \Gamma((\hat{T}^* \mathbb{P} \times \hat{T}^* \mathbb{P}^*); \mathcal{E}_\mathbb{P}^{(n,0)} \otimes_{\mathcal{O}_{\mathbb{P}^*}} \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^*}} \mathcal{O}_{\mathbb{P}^*}(k) \otimes_{\mathcal{O}_{\mathbb{P}^*}} \mathcal{G}^{\otimes -1} \mathcal{E}) \). So that, we have an isomorphism

\[ \Phi^\mu_S(\mathcal{E}_\mathbb{P}(-k, k)|_{\hat{T}^* \mathbb{P}}) \simeq \mathcal{E}_\mathbb{P}(-k, k)|_{\hat{T}^* \mathbb{P}} \]

On the other hand \( s \) is a non-degenerate section of \( \mathcal{E}_\mathbb{P}^{(n,0)} \), hence we can apply Theorem 3.7. Let us denote by

\[ \mathcal{E}_{\mathbb{P} \times \mathbb{P}^*}(k, k^*) := \mathcal{E}_\mathbb{P}(-k, k) \boxtimes_{\mathcal{O}_{\mathbb{P} \times \mathbb{P}^*}} \mathcal{E}_{\mathbb{P}^*}(-k^*, k^*) \]

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Theorem 3.7 gives the following isomorphisms

\[ \mathcal{E}_\mathbb{P}(-k, k)|_{\mathbb{T}^*}\mathbb{P} \simeq p_{1*}(\mathcal{E}_\mathbb{P}\times\mathbb{P}^* (k, k^*).s)|_{\mathbb{T}^*}\mathbb{P} \]
\[ p_{2*}(\mathcal{E}_\mathbb{P}\times\mathbb{P}^* (k, k^*).s)|_{\mathbb{T}^*}\mathbb{P} \simeq \mathcal{E}_{\mathbb{P}^*}(-k^*, k^*)|_{\mathbb{T}^*}\mathbb{P} \]

And so

\[ \chi_*\mathcal{E}_\mathbb{P}(-k, k)|_{\mathbb{T}^*}\mathbb{P} \simeq \mathcal{E}_{\mathbb{P}^*}(-k^*, k^*)|_{\mathbb{T}^*}\mathbb{P} \]

\[ \square \]

4.4 Projective duality for microfunctions

In the following, we will denote by \( K \) the object \( \mathbb{C}[n-1] \). In order to prove Proposition 4.7, we will need to compute \( \Phi_K(\mathbb{C}(1)) \), which is done in [DS96]:

Lemma 4.6 ([DS96]). We have

\[ \Phi_K(\mathbb{C}(1)) \simeq \begin{cases} \mathbb{C}_\mathbb{P}^*(1), & \text{for } n \text{ odd} \\ \mathbb{C}_{\mathbb{P}^*}\mathbb{P}^*[1], & \text{for } n \text{ even} \end{cases} \]

and

\[ H^j(\Phi_K(\mathbb{C}(0))) \simeq \begin{cases} \mathbb{C}_\mathbb{P}^*, & \text{for } j = n - 1 \\ \mathbb{C}_{\mathbb{P}^*}\mathbb{P}^*, & \text{for } j = -1 \text{ and } n \text{ odd} \\ \mathbb{C}_\mathbb{P}^*(1), & \text{for } j = 0 \text{ and } n \text{ even} \\ 0, & \text{in any other case} \end{cases} \]

We are in a proposition to prove:

Theorem 4.7. Assume \(-n-1 < k < 0\). Recall that any section \( s \in \Gamma(\mathbb{P} \times \mathbb{P}^*; \mathcal{E}_{\mathbb{S}}^{(n,0)}(-k, k^*)) \), defines a morphism in \( \mathbb{D}^b(\mathbb{C}_{\mathbb{T}^*}\mathbb{P}) \)

\[ (4.8) \quad \chi_*\mathcal{E}_\mathbb{P}(\varepsilon, k)|_{\mathbb{T}^*}\mathbb{P} \rightarrow \mathcal{E}_{\mathbb{P}^*}(-\varepsilon^*, k^*)|_{\mathbb{T}^*}\mathbb{P} \]

Assume \( s \) is non-degenerate on \( \mathbb{T}^*\mathbb{S}(\mathbb{P} \times \mathbb{P}^*) \). Then (4.8) is an isomorphism. Moreover, there exists such a non-degenerate section.

Remark 4.8. (i) This is a refinement of a general theorem of [SKK] and is a microlocal version of Theorem 5.17 in [DS96].

(ii) The classical Radon transform deals with the case where \( k = -n, k^* = -1 \).

Proof. We will deal with the case \( \varepsilon = 1 \) and \( n \) even, the complementary cases being proven the same way. Let us apply Theorem 2.2 in the following particular case

- \( U = \mathbb{T}^*\mathbb{P}, \ V = \mathbb{T}^*\mathbb{P}^*, \ \Lambda = \mathbb{T}^*\mathbb{S}(\mathbb{P} \times \mathbb{P}^*) \).
- \( K \) is \( \mathbb{C}[n-1] \).
- \( F_1 = \mathbb{C}(1) \) and \( F_2 = \mathcal{O}_\mathbb{P}(k) \)
- \( K \) verifies conditions (i),(ii),(iii) of Theorem 2.2
(i) is fulfilled as the constant sheaf on a closed submanifold of a manifold is cohomologically constructible.

(ii) is fulfilled since \( SS(\mathbb{C}_Z) \) is nothing but \( T^*_Z(\mathbb{P} \times \mathbb{P}^*) \).

(iii) \( C_{T^*_Z(\mathbb{P} \times \mathbb{P}^*)} \longrightarrow \mu \text{hom}(\mathbb{C}_Z, \mathbb{C}_S) \) is an isomorphism on \( T^*_Z(\mathbb{P} \times \mathbb{P}^*) \) (this follows from the fact that for a closed submanifold \( Z \) of a manifold \( X \), \( \mu_Z(\mathbb{C}_Z) \iso \mathcal{C}_{T^*_ZX} \), see [KS90, Prop. 4.4.3]).

By a fundamental result in [DS96, Th 5.17], we know that for \( -n - 1 < k < 0 \), a section \( s \in \Gamma(\mathbb{P} \times \mathbb{P}^*; \mathcal{B}_s^{n,0}(-k, k^*)) \), non-degenerate on \( \hat{T}^*_S(\mathbb{P} \times \mathbb{P}^*) \), induces an isomorphism

\[
\Phi_k(\mathcal{O}_\mathbb{P}(k)) \iso \mathcal{O}_\mathbb{P}(k^*)
\]

Formula (4.7) provides an example of such a non-degenerate section. Hence, applying Lemma 4.6, Theorem 2.2 gives:

\[
\chi \cdot \mu \text{hom}(\mathcal{C}_P(1), \mathcal{O}_\mathbb{P}(k)))|_{\hat{T}^*_\mathbb{P}} \iso \mu \text{hom}(\mathcal{C}_\mathbb{P} \setminus P^* \{1\}, \mathcal{O}_\mathbb{P}(k^*)))|_{\hat{T}^*_\mathbb{P}}.
\]

We have the exact sequence:

\[
0 \longrightarrow \mathbb{C}_\mathbb{P} \setminus P^* \longrightarrow \mathbb{C}_\mathbb{P} \longrightarrow \mathbb{C}_P \longrightarrow 0
\]

Now, for any \( F \in D^b(\mathbb{C}_\mathbb{P}) \), we have

\[
\text{supp}(\mu \text{hom}(\mathcal{C}_\mathbb{P}, F)|_{\hat{T}^*_\mathbb{P}}) \subset (SS(\mathbb{C}_\mathbb{P}) \cap \hat{T}^*\mathbb{P}^*) \cap SS(F) = \emptyset
\]

and hence,

\[
\mu \text{hom}(\mathcal{C}_\mathbb{P}, F)|_{\hat{T}^*_\mathbb{P}} \iso 0
\]

Applying the \( \mu \text{hom} \) functor to 4.9, we get

\[
\mu \text{hom}(\mathcal{C}_\mathbb{P} \setminus P^* \{1\}, \mathcal{O}_\mathbb{P}(k))|_{\hat{T}^*_\mathbb{P}} \iso \mu \text{hom}(\mathcal{C}_P, F)|_{\hat{T}^*_\mathbb{P}}
\]

Hence, we have proved in particular that

\[
\chi \cdot \mu \text{hom}(\mathcal{C}_P(1), \mathcal{O}_\mathbb{P}(k)))|_{\hat{T}^*_\mathbb{P}} \iso \mu \text{hom}(\mathcal{C}_P, \mathcal{O}_\mathbb{P}(k^*)))|_{\hat{T}^*_\mathbb{P}}.
\]

\[\square\]

4.5 Main results

We follow the notations of Section 1.2 and Section 2.

Let us consider the situation (4.5), where we denoted by \( \chi \) the homogeneous symplectic isomorphism between \( \hat{T}^*\mathbb{P} \) and \( \hat{T}^*\mathbb{P}^* \) through \( \hat{T}^*_S(\mathbb{P} \times \mathbb{P}^*) \). We set

\[
L := \mathcal{C}_S[-1]
\]

Then \( L \) is a perverse sheaf satisfying

\[
(p_1^{-1}(\hat{T}^*\mathbb{P}) \cup p_2^{-1}(\hat{T}^*\mathbb{P}^*)) \cap SS(L) = \hat{T}^*_S(\mathbb{P} \times \mathbb{P}^*)
\]

Recall Theorem 4.4, and let \( s \) be a section of \( \mu \text{hom}(\mathcal{C}_S[-1], \Omega_{\mathbb{P} \times \mathbb{P}^*/\mathbb{P}^*}(-k, k^*)) \), non-degenerate on \( \hat{T}^*_S(\mathbb{P} \times \mathbb{P}^*) \). We are in situation to apply Theorem 3.9.

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**Theorem 4.9.** Let $G \in D^b(C_P)$, $k$ an integer. Assume $-n - 1 < k < 0$. Then, we have an isomorphism in $D^b(C_{\hat{T}P})$:

\[
\chi^{-1} \muhom(G, \mathcal{O}_{\hat{T}P}(k^*)) \cong \muhom(\Phi_{C_S[n-1]}(G), \mathcal{O}_P(k))
\]

This isomorphism is compatible with the action of $\mathcal{O}_{\hat{T}P}(-k^*, k^*)$ and $\mathcal{O}_{\hat{T}P}(-k, k)$ on the left and right side of (4.11) respectively.

**Proof.** The isomorphism is directly provided by Theorem 3.9 in the situation where, using the notation in there, $U = \tilde{T}^*P$, $V = \hat{T}^*\mathbb{P}^*$ and $\Lambda = \hat{T}^*_S(\mathbb{P} \times \mathbb{P}^*)$ and where we twist by homogenous line bundles of $\mathbb{P}$, $\mathbb{P}^*$ as explained below.

Let us adapt (3.12) by taking into account the twist by homogeneous line bundles. We follow the exact same reasoning than sections of 3.1 and 3.2.

We have the natural morphism

\[
\Omega_{\hat{T}P}(p)(-k^*, k) \circ \mathcal{O}_{\hat{T}P}(k^*)[n] \to \mathcal{O}_P(k).
\]

Indeed, we have

\[
\Omega_{\hat{T}P}(p)(-k^*, k) \circ \mathcal{O}_{\hat{T}P}(k^*)[n] = Rq_{1!}(\mathcal{O}_{\hat{T}P}(p)(-k^*, k) \otimes q_{k^*}^{-1} \mathcal{O}_{\hat{T}P} \otimes q_{k^*}^{-1} \mathcal{O}_{\hat{T}P}(k^*)[n])
\]

\[
\to Rq_{1!}(\mathcal{O}_{\hat{T}P}(p)(k, 0) \otimes q_{k^*}^{-1} \mathcal{O}_{\hat{T}P} \otimes q_{k^*}^{-1} \mathcal{O}_{\hat{T}P})(k) \to \mathcal{O}_P(k)
\]

Given this morphism and considering $\mathcal{L} = \muhom(C_S[-1], \Omega_{\hat{T}P}(p)(-k^*, k))$, we mimic the proof of Theorem 3.5 so that for a section $s$ of $\mathcal{L}$ on $\hat{T}^*P \times \hat{T}^*\mathbb{P}$ and for $P \in \Gamma(\hat{T}^*P; \mathcal{O}_{\hat{T}P}(-k, k))$ and $Q \in \Gamma(\hat{T}^*\mathbb{P}; \mathcal{O}_{\hat{T}P}(-k^*, k^*))$ satisfying $P \cdot s = s \cdot Q$, the diagram below commutes:

\[
\begin{array}{ccc}
\mathcal{C}_S \circ \muhom(G, \mathcal{O}_{\hat{T}P}(k^*))|_{\hat{T}^*P} & \xrightarrow{\alpha(s)} & \muhom(C_S[n-1] \circ G, \mathcal{O}_{\hat{T}P}(k))|_{\hat{T}^*P} \\
\Phi_{C_S}(\alpha(Q)) & & \alpha(P)
\end{array}
\]

From there, given a non-degenerate section of $\mathcal{L}$ on $\hat{T}^*_S(\mathbb{P} \times \mathbb{P}^*)$, Theorem 3.9 gives the compatible action of micro-differential operators on each side of the isomorphism (4.11)

\[
\chi^{-1} \muhom(G, \mathcal{O}_{\hat{T}P}(k^*))|_{\hat{T}^*P} \cong \muhom(\Phi_{C_S[n-1]}(G), \mathcal{O}_P(k))|_{\hat{T}^*P}
\]

It remains to exhibit a non-degenerate section so that, for $P \in \Gamma(\hat{T}^*P; \mathcal{O}_{\hat{T}P}(-k, k))$, there is $Q \in \Gamma(\hat{T}^*\mathbb{P}; \mathcal{O}_{\hat{T}P}(-k^*, k^*))$ such that $P \cdot s = s \cdot Q$. Precisely, $s$ is given by Proposition 4.4.
Specializing the above proposition, we get

**Corollary 4.10.** Let $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. In the situation of Proposition 4.9, we have the isomorphism, compatible with the respective action of $p_1^{-1}E_P(-k, k)$ and $p_2^{-1}E_P^*(k^*, k^*)$

$$\chi_*\mathcal{C}_P(\varepsilon, k)|_{\mathcal{T}_p^*} \simeq \mathcal{C}_P^*(\varepsilon^*, k^*)|_{\mathcal{T}_p^*}$$

**Proof.** This is an immediate consequence of Proposition 4.9, where we consider the special case $G = C_P^*(\varepsilon^*)$. Indeed, we have, from Lemma 4.6, the isomorphism in $D^b(C_P^*; \mathcal{T}^*P^*)$

$$\mathbb{C}_S[n-1] \circ \mathbb{C}_P^*(\varepsilon^*) \simeq \mathbb{C}_P(\varepsilon)$$

We can state now

**Corollary 4.11.** Let $k$ be an integer. Let $\mathcal{N}$ be a coherent $\mathcal{E}_P(-k, k)$-module and $F \in D^b(\mathbb{P})$. Assume $-n - 1 < k < 0$. Then, we have an isomorphism in $D^b(C_T^*; \mathcal{T}^*\mathbb{P})$

$$\chi_*R\mathcal{H}om_{\mathcal{E}_P(-k, k)}(\mathcal{N}, \mu hom(F, \mathcal{O}_P(k))) \simeq R\mathcal{H}om_{\mathcal{E}_P^*(k^*, k^*)}(\Phi^\mathbb{C}_S[n-1]F, \mu hom((\Phi^\mathbb{C}_S[n-1]F, \mathcal{O}_P^*(k^*))))$$

**Proof.** It suffices to prove this statement for finite free $\mathcal{E}_P(-k, k)$-modules, which in turn can be reduced to the case where $\mathcal{N} = \mathcal{E}_P(-k, k)$. By Theorem 4.5, we have

$$\Phi^\mathbb{C}_S(\mathcal{E}_P(-k, k)|_{\mathcal{T}^*\mathbb{P}}) \simeq \mathcal{E}_P^*(k^*, k^*)|_{\mathcal{T}^*\mathbb{P}}$$

Then, by applying Proposition 4.9, we have

$$\chi_*\mu hom(F, \mathcal{O}_P(k))|_{\mathcal{T}^*\mathbb{P}} \simeq R\mathcal{H}om_{\mathcal{E}_P^*(k^*, k^*)}(\mathcal{E}_P^*(k^*, k^*), \mu hom((\Phi^\mathbb{C}_S[n-1]F, \mathcal{O}_P^*(k^*))|_{\mathcal{T}^*\mathbb{P}})$$

which proves the corollary.

**References**

[Bry86] Jean-Luc Brylinski. “Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques.” In: Astérisque 140.141 (1986), pp. 3–134.

[DS94] Andrea D’Agnolo and Pierre Schapira. “La transformée de Radon-Penrose des D-modules”. In: Comptes rendus de l’Académie des sciences. Série 1, Mathématique 319.5 (1994), pp. 461–466.

[DS96] Andrea D’Agnolo and Pierre Schapira. “Leray’s quantization of projective duality”. In: Duke Math. J. 84.2 (Aug. 1996), pp. 453–496.
[GGG82] I.M. Gelfand, S.G. Gindikin, and M.I. Graev. “Integral geometry in affine and projective spaces”. In: *Journal of Soviet Mathematics* 18.1 (1982), pp. 39–167.

[Hel80] Sigurdur Helgason. *The radon transform*. Vol. 2. Springer, 1980.

[Kas03] Masaki Kashiwara. “D-modules and microlocal calculus”. In: Translations of mathematical monographs (2003).

[KKK86] Masaki Kashiwara, Takahiro Kawai, and Tatsuo Kimura. *Foundations of Algebraic Analysis (PMS-37)*. Princeton University Press, 1986.

[KS85] Masaki Kashiwara and Pierre Schapira. *Microlocal study of sheaves*. Société mathématique de France, 1985.

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*. Vol. 292. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 1990, pp. x+512.

[KS14] Masaki Kashiwara and Pierre Schapira. “Microlocal Euler classes and Hochschild homology”. In: *J. Inst. Math. Jussieu* 13 (2014), pp. 487–516.

[Kas+06] Masaki Kashiwara et al. “Microlocalization of ind-sheaves”. In: (2006), pp. 171–221.

[Ler59] Jean Leray. “Le calcul différentiel et intégral sur une variété analytique complexe.(Problème de Cauchy. III.)” In: *Bulletin de la Société mathématique de France* 87 (1959), pp. 81–180.

[SKK] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara. “Microfunctions and pseudo-differential equations”. In: ().

[Sch85] Pierre Schapira. *Microdifferential systems in the complex domain*. Vol. 269. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 1985.

[SS94] Pierre Schapira and Jean-Pierre Schneiders. “Index theorem for elliptic pairs”. In: *Astérisque* 224 (1994), pp. 1–4.