Avoiding rainbow 2-connected subgraphs

Abstract: While defining the anti-Ramsey number Erdős, Simonovits and Sós mentioned that the extremal colorings may not be unique. In the paper we discuss the uniqueness of the colorings, generalize the idea of their construction and show how to use it to construct the colorings of the edges of complete split graphs avoiding rainbow 2-connected subgraphs. These colorings give the lower bounds for adequate anti-Ramsey numbers.

Keywords: Extremal edge-colorings, Anti-Ramsey number

MSC: 05C55, 05C35

1 Introduction

A subgraph of an edge-colored graph is called rainbow if all of its edges have different colors. In 1973 Erdős, Simonovits and Sós [1] introduced the anti-Ramsey numbers as follows. For a graph $H$ and a positive integer $n$, the anti-Ramsey number $\text{ar}(K_n, H)$ is the maximum number of colors in an edge-coloring of $K_n$ with no rainbow copy of $H$. The authors showed that these numbers are closely related to Turán numbers, especially for the cases when $H$ is neither bipartite nor become bipartite after deleting a single edge. Among others, they constructed colorings avoiding rainbow cycles. They also mentioned that these colorings may not be unique.

Since then, numerous results have been established for a variety of graphs $H$. The paper of Fujita, Magnant and Ozeki [2] presents the survey of results of that type. Apart from that, the definition of anti-Ramsey number was widen by substituting the complete graph, to be colored, by some other graph, namely, for graphs $G$ and $H$ anti-Ramsey number $\text{ar}(G, H)$ is the maximum number of colors in an edge-coloring of $G$ with no rainbow copy of $H$. For instance, bipartite graphs [3, 4] or hypercubes [5] were considered as host graphs.

In the paper we formalize the construction of extremal colorings given in [1] and discuss their uniqueness. We use the concept to construct the colorings of complete split graphs which avoid rainbow 2-connected subgraphs. These colorings establish the lower bound for anti-Ramsey number $\text{ar}(K_n + \overline{K}_s, H)$ for any 2-connected graph $H$.

Graphs considered below will always be simple, i.e. without loops and multiple edges. Throughout the paper we use the standard graph theory notation (see, e.g., [6]). $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of a graph $G$, respectively. $G \cup H$ stands for disjoint sum of graphs $G$ and $H$, $pG$ denotes a graph consisting of $p$ disjoint copies of a graph $G$ and $\overline{G}$ a complement of $G$. $K_n$, $C_n$ denote, respectively, the complete graph and the cycle on $n$ vertices. For a set $S$ by $|S|$ we denote the cardinality of $S$.

A join of graphs $G$ and $H$ is a graph $F = G + H$ such that $V(F) = V(G) \cup V(H)$ and the edge-set $E(F) = E(G) \cup E(H) \cup \{vw : v \in V(G), w \in V(H)\}$. A complete split graph $K_n + \overline{K}_s$ is a join of a complete graph $K_n$ and an empty graph $\overline{K}_s$.

In other words a graph is a complete split graph if it can be partitioned in a clique $K_n$ and an independent set $\overline{K}_s$ in such a way that every vertex in the independent set is adjacent to every vertex in the clique.
2 ESS coloring

As we mentioned in [1] Erdős, Simonovits and Sós presented the edge-coloring of $K_n$ avoiding rainbow cycles. This coloring is roughly described by the authors as follows. To avoid rainbow $C_k$ we divide the vertices of $K_n$ into $\frac{n}{k-1}$ of $k-1$ vertices. Then, we color all edges joining the vertices of the same group by different colors and edges joining the vertices from different groups by further $\frac{n}{k-1}$ colors in the following way: the vertices of the $i$-th group are joined by $i$-th extra color to the vertices of the $j$-th group if $j > i$.

Certainly, in a particular case, the appropriate floor and ceiling functions of $\frac{n}{k-1}$ should be used and one group with less than $k-1$ vertices can appear.

**Remark 2.1.** Note that the presented coloring also works if arbitrary rainbow 2-connected subgraph is forbidden.

The authors themselves mentioned that they do not focus on uniqueness of this coloring and conjectured that this construction uses maximum possible number of colors. The conjecture was proved by them for a triangle, for a 4-cycle by Alon [7], for $k = 5, 6$ Jiang and West [8] and finally completely confirmed by Montellano-Ballesteros and Neuman-Lara [9].

We try to generalize and formalize this very intuitive construction of Erdős, Simonovits and Sós.

**Definition 2.2.** Let $G$ be a connected graph and let $V(G) = \bigcup_{i=1}^{j} V_i$ be a partition of the vertex set of $G$, such that each $C^i = G[V_i]$ is connected and for each $i = 1, 2, \ldots, t - 1$ there exists at least one edge between $C^i$ and $C^j$, for certain $j \in \{i+1, \ldots, t\}$. We color the edges inside $C^i$ rainbowly and among the components in a following way. All edges $v_i v_j$ with $v_i \in V_i$, $v_j \in V_j$, $1 \leq i < j \leq t$ obtain $i$-th extra color. We say that we use ESS coloring to the partition into components ($C^1, C^2, \ldots, C^t$).

In case of components isomorphic to $C$ in the sequence we will write $(\ldots, pC, \ldots)$ instead of $(\ldots, C, C, \ldots, C, \ldots)$.

**Remark 2.3.** Note that in the original coloring of Erdős, Simonovits and Sós the components were cliques on $k-1$ vertices and maybe one clique with less vertices and all edges among components were present.

**Remark 2.4.** Note that we use $\bigcup_{i=1}^{j} |E(C^i)| + (t - 1)$ colors in ESS coloring.

**Remark 2.5.** Note that the ESS coloring avoids 2-connected rainbow subgraphs of order greater then $\max_{i} |V_i|$. Therefore, the idea of constructing these colorings is to choose components with equal number of vertices one less than the order of the forbidden graph $H$. To maximize the number of colors they should contain as many edges as possible, which means that they should be as close to complete graph as possible. One can imagine other reasons for avoiding rainbow $H$ than only the number of vertices. In this light coloring constructions presented by Axenovich et. al. in [4] avoiding rainbow even cycles in bipartite graphs can be seen as ESS colorings. The authors proposed the partition of $K_{m,n}$ into $(K_{k-1,n-(k-1)}, K_{m-(k-1),k-1})$ or $(m - k + 1)K_1, K_{k-1,n})$. In that way they avoid rainbow $C_{2k}$ and the reason is too few vertices in one of bipartition parts.

Further we discuss the uniqueness of the coloring. We do not give the formal definition of isomorphism of colorings. We simply understand that the colorings $c_1$ and $c_2$ are the same if there exists the permutation of colors $\pi$ such that $\pi(c_1) = c_2$.

**Remark 2.6.** Although Erdős et. al. do not focus on uniqueness of colorings, they give accidentally two different colorings avoiding rainbow triangles. One is the special case of their intuitive construction for $k = 3$. It means that the components in ESS coloring are independent edges and possible one single vertex. The coloring uses $\lceil \frac{n}{3} \rceil + \lceil \frac{n}{3} \rceil - 1 = n - 1$ colors. On the other hand, in the proof of the theorem for triangles the authors present another coloring where components are simply vertices of $K_n$. Actually all "between colorings" are suitable in that case.
Proposition 2.7. There are at least \( \left\lfloor \frac{n}{2} \right\rfloor \) different colorings of \( K_n \) with \( n - 1 \) colors avoiding the rainbow triangle.

Proof. Each partition \((pK_2, (n - 2p)K_1), p = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1\) together with ESS coloring is a coloring with \( n - 1 \) colors without rainbow triangle.

It is worth noticing that we end with parameter \( p = \left\lfloor \frac{n}{2} \right\rfloor - 1 \), because partitions \((\left\lfloor \frac{n}{2} \right\rfloor K_2, K_1)\) and \((\left\lfloor \frac{n}{2} \right\rfloor - 1)K_2, 3K_1\) generate the same colorings.

Apart from that, not all colorings are characterized by the above partition. For instance, for \( n = 5 \) coloring induced by \((2K_2, K_1)\) partition is different than that induced by \((K_1, 2K_2)\). The latter appears to be different from all of the type \((pK_2, (5 - 2p)K_1), p = 0, 1\). On the other hand, not all possible partitions give different colorings. For instance colorings induced by the partitions \((5K_1)\) and \((3K_1, K_2)\) are the same. In the Figure 1 colorings induced by all possible partitions of vertices of \( K_5 \) into components \( K_2 \) and \( K_1 \) are presented. In the first row there are colorings from Proposition 2.7 for \( p = 0, 1 \). In the remaining rows there are colorings induced by all other possible partitions. Colorings in the first and the second columns are the same. Second row contains all different colorings of \( K_5 \) without rainbow triangle.

Fig. 1. 4-colorings of edges of \( K_5 \) without rainbow \( K_3 \)

Observe that this diversity of colorings without rainbow triangles comes from the fact that the color of the only edge in the component \( K_2 \) can be sometimes exchanged into one color among two components, which arise by replacing this component by two components \( K_1 \). It means that in that case we can substitute "main" component \( K_1 = K_2 \) by two "reminder" components \( K_1 \). While avoiding \( C_4 \), for example, we cannot change the order of "main" components, which are \( K_3 \), in extremal colorings. Nevertheless, different colorings can be constructed by placing the remaining component \( K_r \), with \( r = n \mod 3, r \neq 0 \), in the different positions in the sequence of components. Apart from that, for \( r = 2 \) we can again sometimes substitute \( K_2 \) by \( 2K_1 \) to generate new coloring.

Therefore, it is a challenge to characterize all extremal colorings, but undoubtedly worth considering. It seems to be as complicated as characterization of Túran extremal graphs or Ramsey extremal graphs.

3 Application of ESS colorings to complete split graphs

In this section we show how to apply the ESS coloring to construct colorings of complete split graphs avoiding rainbow 2-connected subgraph \( H \). As a result we obtain the lower bounds for anti-Ramsey number \( ar(K_n + K_s, H) \) depending on \( n, s \) and \( k = |V(H)| \). As we always refer to ESS colorings in the proofs, we only indicate a procedure of creating a sequence of components. As we know (see Remark 2.1), the ESS coloring avoids 2-connected subgraph.

We start with the graphs which are large in comparison to a complete part of the split graph.
Theorem 3.1. Let \( n \geq 2, s \geq 1, k \geq 3 \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \geq n + 1 \) vertices. Then there exists an edge-coloring of \( K_n + K_s \) with \( \left( \begin{array}{c} n+3 \end{array} \right) + n(k-1-n) + s \) colors avoiding rainbow \( H \).

**Proof.** The sequence of components is as follows: firstly we take \( n - (k-1-n)K_1 \) from the empty part and the last component is the remaining complete split graph \( K_n + K_{k-1-n} \). Certainly we use \( \left( \begin{array}{c} n+3 \end{array} \right) + n(k-1-n) + s \) colors avoiding rainbow \( H \).

We obtain a straightforward corollary.

Corollary 3.2. Let \( n \geq 2, s \geq 1, k \geq 3 \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \geq n + 1 \) vertices. Then \( ar(K_n + K_s, H) \geq \left( \begin{array}{c} n+3 \end{array} \right) + n(k-1-n) + s \).

Now we focus on the cases when the order of the graph \( H \) is comparable with the order of a complete part of the split graph.

Theorem 3.3. Let \( n \geq 2, k \geq 3, s \leq \left\lfloor \frac{n}{k-2} \right\rfloor \), \( r = (n+s) - \left\lfloor \frac{n+s}{k-1} \right\rfloor (k-1) \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \leq n \) vertices. Then there exists an edge-coloring of \( K_n + K_s \) with \( \left\lfloor \frac{n+s}{k-1} \right\rfloor (k-1) + \left( \begin{array}{c} 2 \end{array} \right) + \left\lfloor \frac{n+s}{k-1} \right\rfloor - 1 \) colors avoiding rainbow \( H \).

**Proof.** The sequence of components is as follows. Firstly, we take \( nK_{k-2} \) contained in a complete part \( K_{n} \). To each clique \( K_{k-2} \) we add exactly one vertex from the empty part \( K_s \). In such a way we obtain \( s \) components \( K_{k-1} \). Then we divide the remaining vertices of a complete part into cliques \( K_{k-1} \) making next components being \( K_{k-1} \). We may obtain a certain remainder clique \( K_r \) as the last component. Note that this procedure is equivalent to division the whole vertex-set of the graph \( K_n + K_s \) into cliques of order \( k-1 \) in such a way that each clique contains at most one vertex from the empty part. It is possible since \( s \leq \left\lfloor \frac{n}{k-2} \right\rfloor \). Altogether we use \( \left\lfloor \frac{n+s}{k-1} \right\rfloor (k-1) + \left( \begin{array}{c} 2 \end{array} \right) + \left\lfloor \frac{n+s}{k-1} \right\rfloor - 1 \) colors.

Corollary 3.4. Let \( n \geq 2, k \geq 3, s \leq \left\lfloor \frac{n}{k-2} \right\rfloor \), \( r = (n+s) - \left\lfloor \frac{n+s}{k-1} \right\rfloor (k-1) \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \leq n \) vertices. Then \( ar(K_n + K_s, H) \geq \left( \begin{array}{c} n+3 \end{array} \right) + \left( \begin{array}{c} 2 \end{array} \right) + \left\lfloor \frac{n+s}{k-1} \right\rfloor - 1 \).

Theorem 3.5. Let \( n \geq 2, k \geq 3, \left\lfloor \frac{n}{k-2} \right\rfloor > \left\lfloor \frac{n}{k-1} \right\rfloor \), \( s > \left\lfloor \frac{n}{k-2} \right\rfloor \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \leq n \) vertices. Then there exists an edge-coloring of \( K_n + K_s \) with \( n(k-1-n) + s-1 \) colors avoiding rainbow \( H \).

**Proof.** The sequence of components is as follows. Firstly, we take \( s - \left\lfloor \frac{n}{k-2} \right\rfloor K_{k-2} \) contained in a complete part \( K_{n} \). To each clique \( K_{k-2} \) we add exactly one vertex from the empty part \( K_s \). In such a way we add \( nK_{k-2}K_{k-1} \) to the sequence of components. Altogether we use \( n(k-1-n) + s-1 \) colors.

Corollary 3.6. Let \( n \geq 2, k \geq 3, \left\lfloor \frac{n}{k-2} \right\rfloor > \left\lfloor \frac{n}{k-1} \right\rfloor \), \( s > \left\lfloor \frac{n}{k-2} \right\rfloor \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \leq n \) vertices. Then \( ar(K_n + K_s, H) \geq \left\lfloor \frac{n}{k-2} \right\rfloor (k-1) + s-1 \).

Theorem 3.7. Let \( n \geq 2, k \geq 3, \left\lfloor \frac{n}{k-2} \right\rfloor < \left\lfloor \frac{n}{k-2} \right\rfloor \), \( r = n - \left\lfloor \frac{n}{k-2} \right\rfloor (k-2) \), \( \left\lfloor \frac{n}{k-2} \right\rfloor \leq s \leq \left\lfloor \frac{n}{k-2} \right\rfloor + k - r - 1 \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \leq n \) vertices. Then there exists an edge-coloring of \( K_n + K_s \) with \( \left\lfloor \frac{n}{k-2} \right\rfloor (k-1-n) + \left( \begin{array}{c} 2 \end{array} \right) + r(s - \left\lfloor \frac{n}{k-2} \right\rfloor) + \left\lfloor \frac{n}{k-2} \right\rfloor - 1 \) colors avoiding rainbow \( H \).

**Proof.** The sequence of components is as follows. Firstly, we take \( \left\lfloor \frac{n}{k-2} \right\rfloor K_{k-2} \) contained in a complete part \( K_{n} \), to each clique \( K_{k-2} \) we add exactly one vertex from the empty part \( K_s \). In such a way we obtain \( \left\lfloor \frac{n}{k-2} \right\rfloor \) components \( K_{k-1} \). Then we form a complete split graph from remaining vertices as the last component. Altogether we use \( \left\lfloor \frac{n}{k-2} \right\rfloor (k-1-n) + \left( \begin{array}{c} 2 \end{array} \right) + r(s - \left\lfloor \frac{n}{k-2} \right\rfloor) + \left\lfloor \frac{n}{k-2} \right\rfloor - 1 \) colors.
Corollary 3.8. Let \( n \geq 2, k \geq 3, \left\lfloor \frac{n}{k-2} \right\rfloor < \left\lfloor \frac{n}{k-1} \right\rfloor, r = n - \left\lfloor \frac{n}{k-2} \right\rfloor (k-2), \left\lfloor \frac{n}{k-2} \right\rfloor \leq s \leq \left\lfloor \frac{n}{k-1} \right\rfloor + k - r - 1 \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \leq n \) vertices. Then \( ar(\bar{K}_n + \bar{K}_s, H) \geq \left\lfloor \frac{n}{k-2} \right\rfloor \left( \frac{k-1}{2} \right) + \left\lfloor \frac{n}{k-1} \right\rfloor (r/s) + (r-1)(k-1) + s \) colors avoiding rainbow \( H \).

Theorem 3.9. Let \( n \geq 2, k \geq 3, \left\lfloor \frac{n}{k-2} \right\rfloor < \left\lfloor \frac{n}{k-1} \right\rfloor, r = n - \left\lfloor \frac{n}{k-2} \right\rfloor (k-2), s > \left\lfloor \frac{n}{k-2} \right\rfloor + k - r - 1 \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \leq n \) vertices. Then there exists an edge-coloring of \( \bar{K}_n + \bar{K}_s \) with \( \left\lfloor \frac{n}{k-2} \right\rfloor \left( \frac{k-1}{2} \right) + \left\lfloor \frac{n}{k-1} \right\rfloor (r/s) + (r-1)(k-1) + s \) colors avoiding rainbow \( H \).

Proof. We color the edges of \( \bar{K}_n + \bar{K}_s \) as follows. Firstly, we take \((s - (\left\lfloor \frac{n}{k-2} \right\rfloor + k - r))K_1\) out of the empty part, then \( \left\lfloor \frac{n}{k-2} \right\rfloor \) components \( \bar{K}_{k-2} \) contained in a complete part \( \bar{K}_n \), to each clique \( \bar{K}_{k-2} \) we add exactly one vertex from the empty part \( \bar{K}_s \). In such a way we obtain next \( \left\lfloor \frac{n}{k-2} \right\rfloor \) components \( \bar{K}_{k-1} \) in the sequence. Finally, we take a complete split graph \( K_{r} + \bar{K}_{k-1-r} \) as the last component. Altogether we use \( \left\lfloor \frac{n}{k-2} \right\rfloor \left( \frac{k-1}{2} \right) + \left\lfloor \frac{n}{k-1} \right\rfloor (r/s) + s - (\left\lfloor \frac{n}{k-2} \right\rfloor + k - 1 - r) \) colors.

Corollary 3.10. Let \( n \geq 2, k \geq 3, \left\lfloor \frac{n}{k-2} \right\rfloor < \left\lfloor \frac{n}{k-1} \right\rfloor, r = n - \left\lfloor \frac{n}{k-2} \right\rfloor (k-2), s > \left\lfloor \frac{n}{k-2} \right\rfloor + k - r - 1 \) and \( n + s \geq k \). Let \( H \) be an arbitrary 2-connected graph on \( k \leq n \) vertices. Then \( ar(\bar{K}_n + \bar{K}_s, H) \geq \left\lfloor \frac{n}{k-2} \right\rfloor \left( \frac{k-1}{2} \right) + \left\lfloor \frac{n}{k-1} \right\rfloor (r/s) + (r-1)k - 1 - r + s \) colors.

3.1 Optimality and uniqueness

It is difficult to claim that the colorings presented in the previous section are always optimal (looking at the number of colors used). While constructing these colorings, we pay attention only to the order of the 2-connected graph \( H \) and the fact that it is bridge-free. So it is quite possible that for a particular graph \( H \) a better coloring can be constructed looking at its structure. Nevertheless, there are cases that the constructions are best possible in that sense.

Consider the simplest 2-connected graph \( H = K_3 \). For this graph we have \( k = 3 \) so by Theorems 3.3 and 3.5 we obtain the existence of the edge-coloring of \( \bar{K}_n + \bar{K}_s \) avoiding rainbow triangle with \( n + s - 1 \) colors. It is shown [10] that \( ar(\bar{K}_n + \bar{K}_s, K_3) = n + s - 1 \). It means that the coloring presented in Theorems 3.3 and 3.5 use maximal possible number of colors. Note that these colorings use maximum matchings and possible single vertices as components and ESS coloring. Similarly, as in the case of the complete graph considered as the host graph, changing the position of the component \( K_1 \) (if any), or replacing \( K_2 \) by \( 2K_1 \) we can obtain other colorings with \( n + s - 1 \) colors avoiding rainbow \( K_3 \). In other cases we can also create families of different colorings which avoid rainbow 2-connected subgraph. It is interesting if all of these colorings can be described using ESS colorings or maybe other constructions can appear. Another thing is to establish for which graphs presented colorings are optimal in the sense of number of colors used, i.e. which lower bounds for anti-Ramsey numbers are sharp.

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