Opportunistic Cooperation Strategies for Multiple Access Relay Channels with Compute-and-Forward

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Abstract—This paper studies the application of compute-and-forward to multiple access relay channels (MARC). Despite its promising advantage of improving network throughput, it is not straightforward to apply compute-and-forward to the MARC. This paper proposes two efficient cooperation strategies for the MARC with compute-and-forward. Both proposed strategies are opportunistic in the sense that the cooperation between the relay and the destination are performed only when it is needed for the sake of high transmission efficiency. In the first strategy, the relay helps the destination by sending its local optimal integer coefficient vector without taking into account that the resulting integer coefficient matrix at the destination is full rank. In contrast, in the second strategy, the relay forwards its “best” coefficient vector that always ensures a full rank coefficient matrix at the destination. Both of the proposed strategies achieve twice the network throughput of the existing strategies at high signal-to-noise power ratio (SNR). They also have lower outage probability, independent of relay placement. Furthermore, the first strategy nearly achieves diversity gain of order two, and the second one achieves exactly the diversity gain of order two, which cannot be achieved by the existing strategies.

Index Terms—Multiple access relay channel (MARC), cooperative networks, compute-and-forward, lattice codes, network coding.

I. INTRODUCTION

Network coding [3]–[6] has become an important networking strategy to improve the spectral efficiency of wireless communication networks. In contrast to simple forwarding, network coding allows intermediate nodes to “combine” the received messages before forwarding them to following nodes, to reduce the required number of transmissions. On the other hand, cooperative communication is an effective method to enlarge network coverage, increase transmission robustness, and improve power efficiency by exploiting spatial diversity without additional antennas [7]–[9]. However, the gains achieved by cooperative communications in practice come with a loss of spectral efficiency due to half-duplex operation [3]. Thus, it is beneficial to apply network coding to cooperative networks to achieve reliable communications with high spectral efficiency.

In this paper, we design network coding schemes for the multiple access relay channel (MARC), which is an important class of wireless cooperative networks. In the MARC, multiple sources want to deliver messages to one common destination with the assistance of one relay node [10]–[14]. The applications of such networks include sensor and ad-hoc networks and uplink for cellular networks with an intermediate node as a relay. The MARC also has found a use case in the LTE Advanced mobile communication standard [15], [16]. It has been shown that network coding significantly improves the spectral efficiency of the MARC. For example, in the conventional two-source MARC, four orthogonal transmissions are required, where the sources transmit their messages in turn and the relay forwards the messages one by one to the destination. With network coding, the number of transmissions is reduced to only three; the first and the second transmissions are used by the sources to transmit messages in turn, while the third transmission is used by the relay to forward the network coded version of the transmitted messages to the destination [11].

Recently, Nazer and Gastpar proposed a new network coding and relaying scheme, known as compute-and-forward [17]. It views interference as an advantage to exploit rather than a problem to avoid, and allows sources in a relay network to simultaneously transmit their messages via a non-orthogonal channel. Each relay directly computes an integer linear combination of the transmitted messages from the received superimposed signals without decoding each transmitted message separately, and then forwards it to the destination. Given a sufficient number of linear combinations, the destination can recover the transmitted messages so long as the coefficient matrix, that is the matrix composed of the coefficients of the linear combinations, is full rank.

Allowing sources to transmit their messages via one non-orthogonal channel is an appealing advantage of compute-and-forward. It is easy to see its potential for improving the spectral efficiency of the MARC. When compute-and-forward is applied to the MARC, the required number of transmissions can be reduced to only two; the first is used by the sources to broadcast their messages to the relay and destination at once and the second is used by the relay to forward its computed linear combination to the destination. Note that this advantage is applicable to any MARC with any number of sources, not limited to the MARC with two sources. Despite its promising advantage, however, it is not straightforward to efficiently apply compute-and-forward to MARC as the destination requires the resulting coefficient matrix to be full rank.

It is possible to naively apply the original compute-and-forward [17] to the MARC by allowing the destination and the relay to compute their local optimal integer linear combinations independently. However, it may result in a rank deficient coefficient matrix which causes a decoding failure. Therefore, cooperation between the destination and the relay in computing their integer linear combinations is necessary. In [18], Soussi et al. made an attempt to solve this issue. They proposed a global optimization technique such that the

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Our work related closely to this paper were presented in the 2017 IEEE International Conference on Communications (ICC) [1] and the 2018 International Symposium on Information Theory and Its Applications (ISITA) [2].
destination and the relay select the global optimal linearly independent combinations with respect to achievable rate. They showed that with this strategy, compute-and-forward achieves higher achievable symmetric-rate compared to other relaying strategies such as decode-and-forward and amplify-and-forward. The achievable rate improvement in their work, however, relies on the assumption that all channel state information (CSI) are known to all nodes. Insauti et al. [13] proposed another strategy for applying compute-and-forward to the MARC. Distinct from [13], the relay is allowed to select its local best linear combination; based on this the destination adjusts its linear combination ensuring a full rank coefficient matrix. This strategy is more efficient than the one proposed in [13], and also has a higher achievable rate. Both of the aforementioned works focused on the achievable rate performance without investigating outage probability performance. Because one of the main objectives of wireless cooperative networks is to increase transmission reliability, it is important to make sure that the outage probability performance is also improved when compute-and-forward is employed. In this work we design efficient cooperation strategies for the MARC employing compute-and-forward that improve outage probability performance as well.

The main contributions of this paper are the two efficient cooperation strategies proposed for applying compute-and-forward to the MARC. While improving the outage probability performance, the proposed strategies utilize transmission channels efficiently and allow the relay to help the destination only when necessary. In the first strategy, the relay helps the destination by forwarding its local best linear combination without taking into account the possibility that the resulting integer coefficient at the destination is not full rank. Contrarily, in the second strategy, the relay forwards its best linear combination that always ensures a full rank integer coefficient matrix at the destination. Semi-theoretical and numerical analyses are provided to show the performance of the proposed strategies in terms of outage probability, diversity gain, and network throughput. It is shown that independent of relay placement, the proposed strategies always have lower outage probability compared to the Soussi [13] and Insauti [13] strategies. It is also revealed that the first strategy nearly achieves diversity gain of order two, which is a significant improvement over the Soussi strategy [13]. On the other hand, the second strategy exactly achieves the diversity gain of order two, which is the full diversity gain of the MARC. Moreover, the results show that both of the proposed strategies yields twice the throughput of the existing strategies [13] and [15] at high signal-to-noise power ratio (SNR).

Notation: \( \mathbb{R} \), \( \mathbb{C} \), and \( \mathbb{Z} \) denote the real, complex, and integer numbers, respectively. \( \mathbb{F}_p \) represents the finite field of size \( p \), where \( p \) is a prime number. The Gaussian integers are denoted by \( \mathbb{Z}[i] \). Boldface lowercase letters denote vectors, e.g., \( \mathbf{a} \in \mathbb{Z}[i]^m \), while boldface uppercase letters denote matrices, e.g., \( \mathbf{A} \in \mathbb{Z}[i]^{m \times m} \). The identity matrix of size \( m \) is denoted as \( I_m \). For a vector \( \mathbf{a} \), we use \( a_i \) to denote the element with index \( i \).

\[^1\]In this paper, because all sources transmit independent messages, the full diversity gain of the MARC is of order two.

II. MULTIPLE ACCESS RELAY CHANNEL MODEL

We begin by describing the system model of the MARC considered in this paper. As illustrated in Fig. 1, our model consists of \( M \) sources denoted as \( s_m, m = 1, \ldots, M \), one destination \( d \) and one relay node \( r \). The sources want to transmit information messages to the destination. All wireless links are assumed to be Rayleigh block fading channels where the fading coefficients remain constant within a block of symbols, but change independently from one block to the other according to a circularly symmetric complex Gaussian distribution with zero mean and unit variance.

As shown in Fig. 1 there are three types of directed transmission links: sources-destination, sources-relay, and relay-destination links. All transmitters (sources and relay) have the same average transmit power \( P \). For \( i \in \{s_1, \ldots, s_M, r\} \), \( j \in \{r, d\} \), and \( i \neq j \), let \( \gamma_{ij}, \delta_{ij}, g_{ij}, \) and \( h_{ij} \) be the average SNR, distance, geometric gain, and channel gain of wireless link from node \( i \) to node \( j \), respectively. The geometric gain captures the effect of path loss which is a function of distance, i.e., \( g_{ij} = \delta_{ij}^{-\kappa} \), where \( \kappa \) is the path loss exponent [19], [20].

The channel gain \( h_{ij} \) and noise at every node is randomly distributed over \( CN(0, 1) \). Thus, the average SNR can be defined as \( \gamma_{ij} \triangleq P g_{ij} \). For simplicity, we assume that all sources have the same distance to the destination and also have the same distance to the relay. To be more precise, let \( \delta_{sd} \) and \( \delta_{sr} \) be positive real scalars. We assume that for all \( m \in \{1, \ldots, M\} \), \( \delta_{sm,d} = \delta_{sd} \) and \( \delta_{sm,r} = \delta_{sr} \). This assumption implies that the sources have one common geometric gain to the destination and another common geometric gain to the relay, i.e., \( \forall m \), \( g_{sm,d} = g_{sd} \) and \( g_{sm,r} = g_{sr} \) for positive scalars \( g_{sd} \) and \( g_{sr} \). Accordingly, the average SNR of each source to the destination...
can be defined as \( \gamma_{s_{n,d}} = \gamma_{sd} = Pg_{sd} \), and to the relay as \( \gamma_{s_{n,r}} = \gamma_{sr} = Pg_{sr} \).

III. MARC WITH COMPUTE-AND-FORWARD

In this section we provide an exposition of compute-and-forward and how it is adopted to the MARC. See [17] for an in-depth discussion of compute-and-forward.

A. Encoding Scheme

For simplicity, we consider a symmetric MARC where all sources transmit with the same rate \( R \). For a positive integer \( n \), consider nested lattices \( \Lambda_{c} \subseteq \Lambda_{l} \subseteq \mathbb{R}^{n} \) and let \( C \triangleq C(\Lambda_{c}/\Lambda_{l}) \) be a nested lattice code with rate \( R/2 \). \( \Lambda_{c} \) represents a fine lattice used for coding and \( \Lambda_{l} \) represents a coarse lattice used for shaping to ensure that the average power constraint is satisfied. The second moment of \( \Lambda_{c} \) is assumed to be \( P/2 \). For a prime number \( p \), let \( \mathcal{E} \) be a bijective mapping from \( \mathbb{F}_{p}^{R/2} \) to \( C \), i.e.,

\[
\mathcal{E} : \mathbb{F}_{p}^{R/2} \rightarrow C. \tag{1}
\]

The bijective mapping \( \mathcal{E} \) will be employed by the sources and the relay for encoding their messages to lattice codewords.

The encoding is performed as follows. Each source \( s_{m} \) randomly generates two information vectors \( \mathbf{w}_{m}^{Re}, \mathbf{w}_{m}^{Im} \in \mathbb{F}_{p}^{R/2} \). Together, these information vectors form \( \mathbf{w}_{m} = (\mathbf{w}_{m}^{Re}, \mathbf{w}_{m}^{Im}) \in \mathbb{F}_{p}^{R} \), which are then encoded to a complex-valued vector in the following way. \( \mathbf{w}_{m}^{Re} \) and \( \mathbf{w}_{m}^{Im} \) are respectively mapped to \( \mathbf{x}_{m}^{Re} \in C \) and \( \mathbf{x}_{m}^{Im} \in C \) using \( \mathcal{E} \), i.e.,

\[
\mathbf{x}_{m}^{Re} = \mathcal{E}(\mathbf{w}_{m}^{Re}), \tag{2}
\]
\[
\mathbf{x}_{m}^{Im} = \mathcal{E}(\mathbf{w}_{m}^{Im}). \tag{3}
\]

Subsequently, these real-valued vectors are used to form complex-valued vectors \( \mathbf{x}_{m} = \mathbf{x}_{m}^{Re} + i\mathbf{x}_{m}^{Im} \in \mathbb{C}^{n} \) which are broadcast to the destination and the relay. We assume that a dithering technique [24] is employed. However, we omit the description for ease exposition. Dithering is important for ensuring the resulting effective independent of the underlying lattice codewords. Furthermore, it ensures that each \( \mathbf{x}_{m} \) satisfies the average power constraint \( \mathbb{E}\{||\mathbf{x}_{m}||^2\} \leq nP \).

B. Transmission Rounds

The end-to-end information transmission is divided into two rounds. In the first round, the sources simultaneously broadcast \( \mathbf{x}_{m} \) to the destination and the relay. For \( j \in \{r, d\} \), let \( \mathbf{x}_{j} \) be a noise vector at node \( j \), and recall that \( \mathbf{h}_{ij} \) denotes the channel coefficient from node \( i \) to \( j \), \( i \in \{s_{1}, \ldots, s_{M}\} \). The destination and the relay respectively receive noisy superposition signals

\[
\mathbf{y}_{d}^{(1)} = \sum_{m=1}^{M} \sqrt{\gamma_{sd}} \mathbf{h}_{sd} \mathbf{x}_{m} + \mathbf{z}_{d}^{(1)}, \tag{4}
\]
\[
\mathbf{y}_{r}^{(1)} = \sum_{m=1}^{M} \sqrt{\gamma_{sr}} \mathbf{h}_{sr} \mathbf{x}_{m} + \mathbf{z}_{r}^{(1)}. \tag{5}
\]

At the end of the first round, the relay does not attempt to decode \( \mathbf{w}_{1}, \ldots, \mathbf{w}_{M} \) separately as usually done in conventional MARC schemes. Rather, it adopts the compute-and-forward technique to directly decode a linear combination of \( \mathbf{w}_{1}, \ldots, \mathbf{w}_{M} \). Let \( \mathbf{u}_{r} = f_{r}(\mathbf{w}_{1}, \ldots, \mathbf{w}_{M}) \) be the desired linear combination. In the second round, the relay encodes \( \mathbf{u}_{r} \) to a complex-valued vector \( \mathbf{x}_{r} \in \mathbb{C}^{n} \) using the same encoding scheme described in Subsection III-A and forwards it to the destination. We shall note that to increase the transmission efficiency, the second round is only used when certain conditions are met, which will be discussed further in Section IV. The received signal at the destination is given by

\[
\mathbf{y}_{d}^{(2)} = \sqrt{\gamma_{rd}} \mathbf{h}_{rd} \mathbf{x}_{r} + \mathbf{z}_{d}^{(2)}. \tag{6}
\]

C. Computing Linear Combinations

As mentioned above, by the end of the first transmission round, the relay decodes a combination of the transmitted information messages. In fact, it is not only the relay. Because at least \( M \) linear combinations are required to recover all the transmitted information messages, the destination also needs to decode at least \( M - 1 \) linear combinations. For simplicity, let us focus on how a receiver, which may represent either the destination or the relay, decodes some linear combinations.

Before going further, let us rewrite the received signals \( \{4\} \) or \( \{5\} \) in a simpler form, omitting the notations for the relay or the destination. Let \( \mathbf{h} = [h_{1}, \ldots, h_{M}] \in \mathbb{C}^{M} \) be the channel coefficient vector and \( g \) be the geometric gain from the sources to the receiver. The received signal is rewritten as

\[
\mathbf{y} = \sum_{m=1}^{M} \sqrt{\gamma} h_{m} \mathbf{x}_{m} + \mathbf{z}. \tag{7}
\]

Assume that the receiver expects to decode \( L \leq M \) linear combinations \( \mathbf{u}_{1}, \ldots, \mathbf{u}_{L} \in \mathbb{F}_{p}^{R} \). For \( l \in \{1, \ldots, L\} \), the receiver selects coefficients \( q_{lm}^{Re}, q_{lm}^{Im} \in \mathbb{F}_{p} \) and attempts to decode two equations

\[
\mathbf{u}_{l}^{Re} = \bigoplus_{m=1}^{M} q_{lm}^{Re} \mathbf{w}_{m}^{Re} \oplus (-d_{lm}) \mathbf{w}_{m}^{Im}, \tag{8}
\]
\[
\mathbf{u}_{l}^{Im} = \bigoplus_{m=1}^{M} q_{lm}^{Im} \mathbf{w}_{m}^{Re} \oplus (q_{lm}^{Re}) \mathbf{w}_{m}^{Im}. \tag{9}
\]

where \( (-q_{lm}) \) denotes the additive inverse of \( q_{lm} \). \( \bigoplus \) and \( \oplus \) denote the summation and addition in \( \mathbb{F}_{p} \), respectively. The linear combinations \( \mathbf{u}_{l} \) are then obtained as \( \mathbf{u}_{l} = (u_{l}^{Re}, u_{l}^{Im}) \).

Although the desired linear combinations are evaluated over the finite field \( \mathbb{F}_{p} \), the channels operate over the complex field \( \mathbb{C} \). This issue can be overcome by exploiting the real-valued decomposition of a complex-valued number. To be precise, the receiver selects an integer linear coefficient \( a_{l} = [a_{l1}, \ldots, a_{LM}] \in \mathbb{Z}[i]^{M} \) and decodes the corresponding lattice equation

\[
\mathbf{v}_{l} = \sum_{m=1}^{M} a_{lm} \mathbf{x}_{m} \mod \Lambda_{x}. \tag{10}
\]

2In-depth discussion about nested lattice codes can be found in [21]–[23].
Now, $v_l$ can be written as $v_l = v_l^{\text{Re}} + iv_l^{\text{Im}}$, where
\[ v_l^{\text{Re}} \triangleq \Re(v_l) = \left[ \sum_{m=1}^{M} \Re(a_{lm})x_m^{\text{Re}} - \Im(a_{lm})x_m^{\text{Im}} \right] \mod \Lambda_s \]
\[ v_l^{\text{Im}} \triangleq \Im(v_l) = \left[ \sum_{m=1}^{M} \Im(a_{lm})x_m^{\text{Re}} + \Re(a_{lm})x_m^{\text{Im}} \right] \mod \Lambda_s. \]
(11)
(12)
Once the destination obtains $v_l$, it can recover $u_l^{\text{Re}}$ and $u_l^{\text{Im}}$ using the inverse of $E$ as follows
\[ u_l^{\text{Re}} \triangleq E^{-1}(v_l^{\text{Re}}) = \sum_{m=1}^{M} q_{lm}^{\text{Re}}w_m^{\text{Re}} \oplus (-q_{lm}^{\text{Im}})w_m^{\text{Im}}, \]
\[ u_l^{\text{Im}} \triangleq E^{-1}(v_l^{\text{Im}}) = \sum_{m=1}^{M} q_{lm}^{\text{Im}}w_m^{\text{Re}} \oplus (q_{lm}^{\text{Re}})w_m^{\text{Im}}. \]
(13)
(14)
Given the choice of $a_l$, $q_{lm}^{\text{Re}}$ and $q_{lm}^{\text{Im}}$ are equivalent to $q_{lm}^{\text{Re}} = \Re(a_{lm}) \mod p$ and $u_l^{\text{Im}} = \Im(a_{lm}) \mod p$. This implies that the selection of integer coefficients in the Gaussian integer domain corresponds to the selection of coefficients in the finite field domain.

Now, in order to obtain the lattice equation $v_l$, the receiver first scales $y$ with a scaling factor $\alpha_l \in \mathbb{C}$ and computes
\[ \tilde{y}_l = [\alpha_ly] \mod \Lambda_s \]
\[ = \left[ \sum_{m=1}^{M} \alpha_l\sqrt{g_{lm}}x_m + \alpha_lz \right] \mod \Lambda_s \]
\[ = \left[ \sum_{m=1}^{M} a_{lm}x_m + \sum_{m=1}^{M} (\alpha_l\sqrt{g_{lm}}x_m - a_{lm}x_m) \right. \]
\[ + \alpha_lz \left. \right] \mod \Lambda_s \]
\[ = [v_l + z_{\text{eff}}(\alpha_l, a_l, h, g)] \mod \Lambda_s, \]
(15)
(16)
(17)
where
\[ z_{\text{eff}}(\alpha_l, a_l, h, g) = \sum_{m=1}^{M} (\alpha_l\sqrt{g_{lm}}x_m - a_{lm}x_m) + \alpha_lz \]
(18)
is the effective noise. Subsequently, it produces estimates of $v_l^{\text{Re}}$ and $v_l^{\text{Im}}$ by quantizing the real and imaginary components of $\tilde{y}_l$ with respect to $\Lambda_s$ and performs the modulo operation with respect to $\Lambda_s$, i.e.,
\[ \hat{v}_l^{\text{Re}} = Q_{\Lambda_s}(\Re(\tilde{y}_l)) \mod \Lambda_s \]
\[ \hat{v}_l^{\text{Im}} = Q_{\Lambda_s}(\Im(\tilde{y}_l)) \mod \Lambda_s. \]
(19)
(20)
Finally, the estimates of $u_l^{\text{Re}}$ and $u_l^{\text{Im}}$ are obtained using the inverse of $E$,
\[ \hat{u}_l^{\text{Re}} = E^{-1}(\hat{v}_l^{\text{Re}}), \]
\[ \hat{u}_l^{\text{Im}} = E^{-1}(\hat{v}_l^{\text{Im}}). \]
(21)
(22)
The estimate of $u_l$ is then recovered as $\hat{u}_l = (\hat{u}_l^{\text{Re}}, \hat{u}_l^{\text{Im}})$.

In order for the receiver to be able to decode $u_l$ with low error probability, the scaling factor $\alpha_l$ has to be chosen such that the variance of the effective noise $z_{\text{eff}}(\alpha_l, a_l, h, g)$ is minimized. Let $\sigma_{\text{eff}}^2(\alpha_l, a_l, h, g)$ be the variance of $z_{\text{eff}}(\alpha_l, a_l, h, g)$ defined as
\[ \sigma_{\text{eff}}^2(\alpha_l, a_l, h, g) \triangleq \frac{1}{n} \mathbb{E}\left( \|z_{\text{eff}}(\alpha_l, a_l, h, g)\|^2 \right) \]
\[ = \|\alpha_l\sqrt{\mathbb{E}}h - a_l\|^2 P + |\alpha_l|^2. \]
(23)
(24)
One can show that the optimal value for $\alpha_l$ is given by
\[ \alpha_l^{\text{opt}} = \arg\min_{\alpha_l} \sigma_{\text{eff}}^2(\alpha_l, a_l, h, g) \]
\[ = \frac{P\sqrt{\mathbb{E}}h^H a_l}{1 + Pg |h|^2}. \]
(25)
(26)
As a summary, the receiver decodes linear combination $u_l$ in the following way. It first selects an integer coefficient vector $a_l \in \mathbb{Z}[i]^M$, then computes $\alpha_l^{\text{opt}}$ and uses it as the scaling factor $\alpha_l$. Next, it scales the received signal and performs the modulo operation with respect to $\Lambda_s$. Finally, the desired linear combination is obtained by performing operations described in (19), (20), (21), and (22) sequentially.

Next, we discuss how to choose the integer coefficient vector $a_l$. In principle, it is possible for the receiver to choose any integer coefficient vector. However, the selection of the coefficient vector has a significant impact on the achievable computation rate and consequently on the outage probability. Therefore, $a_l$ has to be chosen carefully. The achievable computation rate of compute-and-forward in complex-valued channels is given in the following theorem.

**Theorem 1 (Computation Rate [17]):** Consider a complex-valued Gaussian network with $M$ transmitters that simultaneously transmit their messages with average power constraint $P$ to a receiver. Let $h = [h_1, \ldots, h_M]^T \in \mathbb{C}^M$ be the channel coefficients and $g$ be the geometric gain from the transmitters to the receiver. Given a coefficient vector $a = [a_1, \ldots, a_M] \in \mathbb{Z}[i]$, the receiver can decode the corresponding linear combination of transmitted messages with low error probability so long as the message rate is less than the computation rate given by
\[ R_{cp}(a, h, g) = \log^+ \left( \frac{\|a\|^2}{1 + Pg |h|^2} \right). \]
(27)

From the above theorem, it is clear that the selected integer coefficient $a$ is a component that determines the computation rate. Therefore, it should be chosen such that the computation rate is maximized.

For $l \in \{1, \ldots, L\}$, let $a_l$ be the integer coefficient vector of the desired linear combination $u_l$. Let $A = [a_1, \ldots, a_L]^T$ and $a_1, \ldots, a_L$ should be chosen to be linearly independent, and thus, rank$(A) = L$. The receiver selects $A$ such that
\[ A = \arg\max_{\text{rank}(A) = L} \min_{l=1, \ldots, L} R_{cp}(a_l, h, g). \]
(28)

It can be shown that the computation rate $R_{cp}(a_l, h, g)$ can be written as
\[ R_{cp}(a_l, h, g) = \log^+ \left( \frac{1}{\text{det}(\bar{M}a_l)} \right), \]
(29)
where

\[ M = I - \frac{Pg}{1 + Pg ||h||^2}hh^H. \] (30)

One can observe that \( M \) is a positive definite matrix, and thus, using Cholesky factorization, it can be decomposed into

\[ M = BB^H, \] (31)

where \( B \) is an upper triangular matrix.

The problem defined in (28) now can be transformed into

\[ A = \arg \max_{A=[a_1, \ldots, a_L] \in \mathbb{Z}[i]^{L \times M}, \text{rank}(A)=L} \min_{l=1, \ldots, L} \log^+ \left( \frac{1}{a_l^H M a_l} \right) \] (32)

\[ = \arg \min_{A=[a_1, \ldots, a_L] \in \mathbb{Z}[i]^{L \times M}, \text{rank}(A)=L} \max_{l=1, \ldots, L} a_l^H M a_l \] (33)

\[ = \arg \min_{A=[a_1, \ldots, a_L] \in \mathbb{Z}[i]^{L \times M}, \text{rank}(A)=L} \|B a_l\|^2. \] (34)

**Definition 1 (Successive Minima):** For an \( n \)-dimensional lattice \( \Lambda(G) \), the \( l \)-th successive minimum, \( 1 \leq l \leq n \), is defined as

\[ \lambda_l(G) = \min_{v_1, \ldots, v_l \in \Lambda(G)} \max\{\|v_1\|, \ldots, \|v_l\|\}, \] (35)

where the minimum is taken over all sets of \( l \) linearly independent vectors in \( \Lambda(G) \). In other words, \( \lambda_l(G) \) is the smallest real number \( r \) such that there exist \( l \) linearly independent vectors \( v_1, \ldots, v_l \in \Lambda(G) \) with \( \|v_1\|, \ldots, \|v_l\| \leq r \).

From (34) and Definition 1, it can be said that finding \( L \) "best" integer coefficient vectors \( a_1, \ldots, a_L \) with respect to the computation rate is equivalent to finding integer vectors providing \( L \) successive minima of the lattice with a generator matrix \( B \). Thus, to find \( A \), we can employ existing algorithms for finding the successive minima of a lattice such as the Fincke-Pohst algorithm [26], the Schnorr-Euchner algorithm [27], the LLL algorithm [28], and their variations [29]-[34].

**D. Recovering Information Messages**

We have discussed that upon receiving the noisy superposition signal in (4) and (5), the destination and the relay compute linear combinations of the transmitted messages. However, the ultimate goal of the destination is to recover the transmitted information messages \( w_1, \ldots, w_M \). This stage, the destination requires at least \( M \) linear combinations. Let us assume that the destination possesses linear combinations \( \hat{u}_1, \ldots, \hat{u}_M \). These linear combinations may be obtained either directly by the destination itself, or with the help of the relay. How these linear combinations are obtained will be addressed in the next section.

Let \( a_1, \ldots, a_M \in \mathbb{Z}[i]^M \) be the integer coefficient vectors corresponding to \( \hat{u}_1, \ldots, \hat{u}_M \) and let \( A = [a_1, \ldots, a_M]^T \); we refer to \( A \) as integer coefficient matrix or just coefficient matrix. The corresponding coefficient matrix in \( \mathbb{F}_p \) can be written as

\[ Q = \begin{bmatrix} \Re(A) & -\Im(A) \\ \Im(A) & \Re(A) \end{bmatrix} \mod p, \] (36)

where the modulo operation is element-wise. For all \( m \in \{1, \ldots, M\} \), let \( \hat{w}_m = (\hat{w}_{m, \Re}, \hat{w}_{m, \Im}) \) be the estimate of \( w_m \). The destination decodes the transmitted information messages by solving the following linear equation

\[ \begin{bmatrix} \hat{u}_{1, \Re} \\ \vdots \\ \hat{u}_{M, \Re} \end{bmatrix} = Q \begin{bmatrix} w_{1, \Re} \\ \vdots \\ w_{M, \Re} \end{bmatrix}, \] (37)

where all operations are performed over finite field \( \mathbb{F}_p \). It should be noted that the destination can solve the above linear equation system if and only if the matrix \( Q \) is full-rank.

In this section we propose two strategies for cooperation between the destination and the relay. In the first strategy, the relay helps the destination by providing its local "best" linear combination without taking into account whether the resulting coefficient matrix is full rank or not. In the second strategy, the relay assists the destination by forwarding the "best" linear combination that ensures the resulting coefficient matrix is full rank. Since the relay needs to know the linear combinations that the destination possesses, a sufficient amount of feedback is needed in the second strategy.

Before the cooperation stage begins, the destination and the relay find \( M \) linearly independent coefficient vectors. Note that at this pre-cooperation stage, the corresponding linear combinations are not yet decoded. Let \( A_d = \{a_{d,1}, \ldots, a_{d,M}\} \) and \( A_r = \{a_{r,1}, \ldots, a_{r,M}\} \) be the sets of coefficient vectors found by the destination and the relay, respectively. The elements of \( A_d \) and \( A_r \) are sorted based on the resulting computation rates. Let \( h_{d} = [h_{s,1}, \ldots, h_{s,M}] \) and \( h_{r} = [h_{s,1}, \ldots, h_{s,M}] \), and define

\[ R^{(m)}_{c_p, d} \triangleq R_{c_p}(a_{d,1}, h_{d}, g_{s,d}), \] (38)

\[ R^{(m)}_{c_p, r} \triangleq R_{c_p}(a_{r,1}, h_{r}, g_{s,r}). \] (39)

The coefficient vectors \( a_{d,1}, \ldots, a_{d,M} \) and \( a_{r,1}, \ldots, a_{r,M} \) are respectively sorted such that \( R^{(1)}_{c_p, d} \geq \cdots \geq R^{(M)}_{c_p, d} \) and \( R^{(1)}_{c_p, r} \geq \cdots \geq R^{(M)}_{c_p, r} \). Moreover, the elements of \( A_d \) and \( A_r \) are integer vectors that provide \( M \) successive minima corresponding to the matrix \( B \) described in (31). This implies that \( a_{d,i} \) and \( a_{r,i} \) are the local optimal coefficient vectors at the destination and the relay, respectively.

Using these \( A_d \) and \( A_r \), we propose two cooperation strategies as follows.
A. Limited Feedback Strategy

The first strategy is simple, yet it outperforms the existing cooperation strategies in the literature. In this strategy, there are two steps for decoding the transmitted messages. The first is, upon receiving $y_1^{(1)}$, the destination directly attempts to decode the transmitted information messages without the help of the relay. We refer to this step of decoding as direct decoding. Specifically, using the integer coefficient vectors in $A_d$, the destination decodes the corresponding $M$ linear combinations. Let $\hat{u}_d, \ldots, \hat{u}_{dM}$ be the decoded linear combinations and $A_d = [a_{d1}, \ldots, a_{dM}]^T$. Based on $\hat{u}_1, \ldots, \hat{u}_{dM}$ and $A_d$, the destination attempts to decode the transmitted messages by solving the resulting equation system similar to (37). If the destination successfully decodes the transmitted messages, the sources can transmit the next messages. Otherwise, the destination broadcasts feedback to the sources and the relay, asking the sources to wait and the relay to help the decoding. The feedback size is only 1-bit and is assumed to always be received correctly.

The second step of decoding, to which we refer as cooperative decoding, is carried out when the relay receives feedback from the destination. The relay chooses its local best coefficient vector $a_{r1}$, decodes the corresponding linear combinations and forwards it to the destination. Let $\hat{u}_r$ be the linear combination forwarded by the relay. The destination now has an additional linear combination $\hat{u}_r$ with coefficient vector $a_{r1}$. Let $A_{\text{cop}} = [a_{r1}, a_{d1}, \ldots, a_{dM}]^T$. Based on $\hat{u}_r, \hat{u}_1, \ldots, \hat{u}_{dM-1}$ and $A_{\text{cop}}$, the destination then again decodes the transmitted messages. Note that $A_{\text{cop}}$ may not be full-rank which will prevent the destination from decoding the information messages correctly.

B. Sufficient Feedback Strategy

The second strategy is similar to the first in the sense that it also uses two decoding steps. The first step is the same as the limited feedback strategy. The destination attempts to directly decode the information messages using its own linear combinations. Using $\hat{u}_d, \ldots, \hat{u}_{dM}$ and $A_d$ the destination recovers $w_1, \ldots, w_M$ by solving an equation system corresponding to (37). If direct decoding succeeds, the sources can transmit their next information messages. Otherwise, the destination sends feedback to the relay. The feedback must contain information about the $M-1$ best integer coefficient vectors of the destination, i.e., $a_{d1}, \ldots, a_{dM-1}$. Besides perfectly received, it is also assumed that the feedback size is negligible compared to the amount of information that can be transmitted within one coherence time. In practice, the feedback will require at least $(M-1)\log p$ bits.

The relay selects a coefficient vector that ensures that the resulting coefficient matrix is full rank while keeping the achievable rate as high as possible. Specifically, let $a_{r1}$ be the integer coefficient vector selected by the relay. Let $A_{r1} = [a_{r1}, a_{d1}, \ldots, a_{dM-1}]^T$. The relay must select $a_{r1}$ such that

$$a_{r1} = \arg \max_{a_r \in \mathbb{Z}} \frac{R_{(M)}^\text{cp,r}}{\text{rank}(A_r)} = M.$$  

Subsequently, the relay decodes the linear combination of the messages that corresponds to the selected coefficient vector $a_{r1}$, and then forwards it to the destination. Now, because the destination has enough linear combinations, it can re-decode the information messages by solving the resulting linear equation system according to (37).

V. PERFORMANCE ANALYSIS

A. Limited Feedback Strategy

We start from the performance analysis of the limited feedback (lim-FB) strategy. In this strategy, the destination has two possible ways of decoding the transmitted messages, direct decoding and cooperative decoding. In direct decoding, the destination attempts to decode the transmitted messages by itself. Specifically, it computes linear combinations with coefficients $a_{d1}, \ldots, a_{dM}$ and solves the corresponding linear equation system. Let $e_1$ be the outage event for direct decoding. $e_1$ is defined as

$$e_1 \triangleq \left\{ \min_{m \in \{1, \ldots, M\}} R_{(m)}^\text{cp,d} < R \right\} \quad \text{(41)}$$

$$= \left\{ R_{(M)}^\text{cp,d} < R \right\},$$  

where $R$ is the coding rate employed by the sources and the relay. Note that (42) is due to the fact $R_{(m)}^\text{cp,d} \geq \cdots \geq R_{(M)}^\text{cp,d}$, see Section [V]. Intuitively, we can think that the outage event for direct decoding is determined by the worst coefficient vector $a_{dM}$.

In cooperative decoding, the relay forwards its local best linear combination and the destination uses its $M-1$ best linear combinations and solves the resulting linear equation system to decode the transmitted messages. In order for cooperative decoding to succeed, all the linear combinations have to be correctly decoded and the resulting coefficient matrix has to be full rank. Let $e_2$ be the outage event for the cooperative decoding and $A$ be the resulting coefficient matrix. The outage event during cooperative decoding is defined as

$$e_2 \triangleq \left\{ \min_{m \in \{1, \ldots, M-1\}} R_{(m)}^\text{cp,d} < R \right\} \cup \left\{ R_{(1)}^\text{cp,d} < R \right\}$$

$$\cup \left\{ R_{(M-1)}^\text{cp,d} < R \right\} \cup \left\{ \text{rank}(A) < M \right\}$$

$$= \left\{ R_{(M-1)}^\text{cp,d} < R \right\} \cup \left\{ R_{(1)}^\text{cp,d} < R \right\} \cup \left\{ R_{ed} < R \right\}$$

$$\cup \left\{ \text{rank}(A) < M \right\},$$  

where $R_{ed} = \log(1 + |h_{rd}|^2 r_{rd})$ is the achievable rate of the point-to-point relay-destination link.

Let $P_{\text{def}} \triangleq \Pr(\text{rank}(A) < M)$. In the end, the destination fails to decode the transmitted messages if and only if both direct and the cooperative decodings fail. Therefore, the outage
probability is given by
\[
P_{\text{out}} \triangleq \Pr(e_1 \cap e_2)
\]
\[
= \Pr \left( \{ R_{\text{cp},d}^{(M)} < R \} \cap \{ R_{\text{cp},r}^{(M-1)} < R \} \cup \{ R_{\text{cp},r} < R \} \cup \{ R_{\text{rd}} < R \} \cup \{ \text{rank}(A) < M \} \right)
\]
\[
\leq \Pr(\{ R_{\text{cp},d}^{(M)} < R \}) + \Pr(\{ R_{\text{cp},d}^{(M)} < R \} \cap \{ R_{\text{cp},r}^{(M-1)} < R \}) + \Pr(\{ R_{\text{cp},d}^{(M)} < R \} \cap \{ \text{rank}(A) < M \})
\]
\[
+ \Pr(\{ R_{\text{cp},d}^{(M)} < R \} \cap \{ R_{\text{rd}} < R \} \cap \{ \text{rank}(A) < M \})
\]
\[
= \Pr(\{ R_{\text{cp},d}^{(M)} < R \}) + \Pr(\{ R_{\text{cp},d}^{(M)} < R \} \cap \{ R_{\text{cp},r}^{(M-1)} < R \})
\]
\[
+ \Pr(\{ R_{\text{cp},d}^{(M)} < R \} \cap \{ R_{\text{rd}} < R \}) + \min \{ \Pr(\{ R_{\text{cp},d}^{(M)} < R \}), P_{\text{det}} \}.
\]
\[
(44)
\]
where \((a)\) is due to the union bound and \((b)\) is because the channels of sources-destination, sources-relay, and relay-destination are independent. The last part of \((b)\) is due to the Frchet bound [35].

Besides outage probability, we are also interested in the diversity order achieved by the proposed strategies. Let us recall the definition of diversity order [36] achieved by a system.

**Definition 2** (Diversity order): For a system with outage probability \(P_{\text{out}}\), the diversity order of the system \(d\) is defined as
\[
d \triangleq - \lim_{\gamma \to \infty} \frac{\log P_{\text{out}}}{\log \gamma}.
\]
\[
(46)
\]
where \(\gamma\) is the average SNR of the channels.

For simplicity, we also use an alternative form \(P_{\text{out}} \triangleq \gamma^d\) to represent (46). The symbol \(\triangleq\) indicates the asymptotic equality for \(\gamma \to \infty\). The relation \(\leq\) indicates a similar meaning.

Equation (45) shows that the overall outage probability of the lim-FB strategy depends on the outage probability of the sources-destination, sources-relay, and relay-destination links, and the probability of rank deficient coefficient matrix during the cooperative decoding. Because the relay-destination is a point-to-point link, it has diversity order one, see [37] and [8]. For the sources-destination and sources-relay links, we have to evaluate the outage probability with respect to decoding linear combinations at the destination and the relay. In the following lemma, we show that the destination and the relay achieve full diversity order for decoding their local optimal linear combinations.

**Lemma 1**: Consider a compute-and-forward scheme with \(M\) transmitters, one receiver, and i.i.d. Rayleigh fading channels. The receiver wants to decode a linear combination of transmitted messages. Let \(\gamma\) and \(P_{\text{out}}\) be the average SNR and outage probability, respectively. The diversity order achieved by the receivers with respect to recovering a linear equation using its optimal integer coefficient vector is \(M\), i.e., \(P_{\text{out}} \leq \gamma^{-M}\).

**Proof.** See Appendix [A].

In addition to Lemma 1, which shows the achieved diversity order of a compute-and-forward scheme with the first-best (local optimal) integer coefficient vector, we also need to know the achieved diversity order when the \((M - 1)\)-th and \(M\)-th best integer coefficients are selected. To this end, we provide numerical results evaluating the outage probabilities of a compute-and-forward scheme with \(M\) best linear coefficient vectors in Figs. 2 and 3. By “best” here, we mean the coefficient vectors that provide successive minima of the resulting lattice \(B\) in (31). Fig. 2 shows outage probabilities...
written as
\[
P_{\text{out}} \leq \Pr (R_{\text{cp},d}^{(M-1)} < R) + \Pr (R_{\text{cp},d}^{(M)} \leq R) \Pr (R_{\text{cp},r}^{(1)} < R) + \Pr (R_{\text{cp},d}^{(M)} < R) \Pr (R_{\text{rd}} < R) + \min \{ \Pr (R_{\text{cp},d}^{(M)} < R), P_{\text{def}} \}.
\]
\[
\leq \frac{\xi_1}{\gamma_{sd}} + \frac{\xi_2}{\gamma_{sd} \gamma_{sd}^{M}} + \frac{\xi_2}{\gamma_{sd}} + \frac{\xi_2}{\gamma_{sd}^{3}} \tag{47}
\]
\[
\leq \frac{\xi}{\gamma_{sd}} \tag{48}
\]
where \(\xi, \xi_1, \xi_2, \xi_3, \) and \(\xi_4\) are positive constants. In (a), because \(k < 1\), in the high SNR regime, \(\min\{\Pr (R_{\text{cp},d}^{(M)} < R), P_{\text{def}}\} = \Pr (R_{\text{cp},d}^{(M)} < R)\). The above result indicates that the lim-FB cannot achieve the full-diversity gain of the MARC. However, the bound in (48) is loose because of the Frchet bound. As we will see in the next section, the lim-FB nearly achieves diversity gain of order two and its outage performance is significantly better compared to the existing strategies.

### B. Sufficient Feedback Strategy

The outage probability of the sufficient feedback (suf-FB) strategy is similar to that of the lim-FB strategy. In the suf-FB strategy, there are also two possible ways for the destination to decode the transmitted messages. The first one is direct decoding, which is exactly the same as that of the lim-FB. Therefore, the outage event for direct decoding in the suf-FB is also given by
\[
e_1 = \{ R_{\text{cp},d}^{(M)} < R \}. \tag{49}
\]

The second one is cooperative decoding, where the relay selects its best linear combination \(a_r\), that is linearly independent of the first \(M - 1\) linear combinations of the destination. Therefore, the resulting coefficient matrix is always full rank. As a result, the outage event for cooperative decoding depends only on the sources-destination, the sources-relay, and the relay-destination links. Let \(R_{\text{cp},r}^{(s)} \triangleq R_{\text{cp}}(a_r, h_{sr}, g_{sr})\). The outage probability of the suf-FB strategy is defined as
\[
e_2 = \{ \min_{m \in \{1, \ldots, M-1\}} R_{\text{cp},d}^{(m)} < R \} \cup \{ R_{\text{cp},r}^{(s)} < R \} \cup \{ R_{\text{rd}} < R \} \tag{50}
\]

Similar to the lim-FB strategy, the overall outage for the suf-FB strategy occurs if and only if both direct and cooperative decodings fail. Therefore, the outage probability is
\[
P_{\text{out}} \triangleq \Pr (e_1 \cap e_2) \tag{51}
\]
\[
= \Pr \left( \{ R_{\text{cp},d}^{(M)} < R \} \cap (\{ R_{\text{cp},d}^{(M-1)} < R \} \cup \{ R_{\text{cp},r}^{(s)} < R \}) \right.
\]
\[
\cup \{ R_{\text{rd}} < R \} \right)
\]
\[
\leq \Pr (R_{\text{cp},d}^{(M-1)} < R) + \Pr (R_{\text{cp},d}^{(M)} < R) \Pr (R_{\text{cp},r}^{(s)} < R) + \Pr (R_{\text{cp},d}^{(M)} < R) \Pr (R_{\text{rd}} < R) \tag{a}
\]
\[
\leq \Pr (R_{\text{cp},d}^{(M-1)} < R) + \Pr (R_{\text{cp},d}^{(M)} < R) \Pr (R_{\text{cp},r}^{(M)} < R) + \Pr (R_{\text{cp},d}^{(M)} < R) \Pr (R_{\text{rd}} < R) \tag{b}
\]
\[
+ \Pr (R_{\text{cp},d} < R) \Pr (R_{\text{rd}} < R) \tag{52}
\]
where \((a)\) is due to union bound and in \((b)\) we bound \(P_{\text{out}}\) by selecting the worst linear combinations at the relay.

Using the results shown in the previous subsection, we can see that the suf-FB strategy can achieve second-order diversity. Specifically, \(P_{\text{out}}\) can be written as

\[
P_{\text{out}} \leq \Pr \left( R_{\text{cp},d}^{(M-1)} < R \right) + \Pr \left( R_{\text{cp},d}^{(M)} < R \right) \Pr \left( R_{\text{cp},r}^{(M)} < R \right) + \Pr \left( R_{\text{cp},d}^{(M)} < R \right) \Pr \left( R_{\text{rd}} < R \right)
\]

\[
\leq \frac{\xi_1}{\gamma_{sd}} + \frac{\xi_2}{\gamma_{sd}} \frac{\xi_3}{\gamma_{sd}} + \frac{\xi_4}{\gamma_{sd}}
\]

\[
\leq \frac{\xi}{\gamma_{sd}}
\]

with other positive constants \(\xi, \xi_1, \xi_2, \xi_3,\) and \(\xi_4\).

VI. NUMERICAL EVALUATION

In this section, we provide results of computer simulations performed to evaluate the performance of the proposed cooperation strategies, and compare them with approaches found in the literature. Since we focus on the design of cooperation strategies for applying compute-and-forward to the MARC, we mainly compare our proposed strategies to the approaches proposed by Soussi et al. [18] and Insanusi et al. [15]. To the best of our knowledge, [18] and [15] are the only works available in the literature that addressed the problem of applying compute-and-forward to the MARC.

Before going into the details, let us first briefly describe the approaches proposed in [18] and [15]. In [18], Soussi et al. used a global optimization to choose linear combinations at the relay and the destination. Given channel state information (CSI) is known to all terminals, the relay and the destination select their optimal linear combinations maximizing the achievable rate. The relay then forwards its linear combination to the destination. Finally, having sufficient linear combinations, the destination attempts to recover the transmitted messages. It has been shown that this approach achieves better achievable rate compared to other relaying strategies such as amplify-and-forward and decode-and-forward. However, it has a drawback that it requires greater communication overhead due to the fact that all CSI is known to all nodes. Moreover, it can be proven that this approach does not achieve full-diversity gain as it has a bottleneck in the link between the relay and the destination.

In [15], Insanusi et al. proposed an approach where the relay is allowed to choose its best linear combination yielding optimal computation rate and to forward it to the destination. The destination then chooses linear combinations that are linearly independent of the one from the relay and decodes the transmitted messages. If the decoding fails, the destination computes one more linear combination from its received signal and again decodes the transmitted messages. This approach is quite similar to our lim-FB strategy. Indeed, the two strategies achieve the same outage probability performance as we will see later. They differ in the way they utilize the transmission rounds. In the Insanusi et al. approach, two transmission rounds are always used. While in our proposed strategy, only one transmission round is used when possible to increase transmission efficiency.

Now we describe conditions for the numerical evaluations. We assume all the sources have the same distance to the destination and also to the relay. The distance from the sources to the destination is denoted by \(\delta_{\text{sd}}\), and to the relay is denoted by \(\delta_{\text{sr}}\). The relay has distance \(\delta_{\text{rd}}\) to the destination. We normalize \(\delta_{\text{sd}} = 1\), and assume \(\delta_{\text{sr}} + \delta_{\text{rd}} = \delta_{\text{sd}}\). See the illustration in Fig. 1. The corresponding average SNRs are calculated using a path-loss exponent equal to 3.52 [19], [20]. In the simulations, we consider MARC with two sources and transmission rate \(R = 2\). The performance is evaluated in three scenarios as follows.

1) First scenario: The relay is closer to the sources than to the destination. Specifically, we set the distance from the sources to the relay \(\delta_{\text{sr}} = 0.25\), while from the relay to the destination is \(\delta_{\text{rd}} = 0.75\). This scenario is equivalent to the setting of average SNRs \(\gamma_{\text{sd}} = 21.19\text{ dB}\) and \(\gamma_{\text{rd}} = \gamma_{\text{sd}} + 4.39\text{ dB}\).

2) Second scenario: The distance from the sources to the relay is equal to the distance from the relay to the destination, i.e., \(\delta_{\text{sr}} = \delta_{\text{rd}} = 0.5\). In other words, the relay is half-way between the sources and the destination. With the same path-loss exponent, the resulting average SNRs are \(\gamma_{\text{sr}} = \gamma_{\text{rd}} = \gamma_{\text{sd}} + 10.59\text{ dB}\).

3) Third scenario: The relay is positioned closer to the destination than to the relay. In particular, we assume \(\delta_{\text{sr}} = 0.25\) and \(\delta_{\text{rd}} = 0.75\). As a result, the average \(\gamma_{\text{sr}} = \gamma_{\text{sd}} + 4.39\text{ dB}\) and \(\gamma_{\text{rd}} = \gamma_{\text{sd}} + 21.19\text{ dB}\).

The outage probability results for the first, second, and third scenarios are presented in Figs. 5-7, respectively. Additionally, we present the baseline outage probability for the case when the sources send their information to the destination without a relay so that we can see how much improvement is gained when the relay is employed. From here on we
refer to the strategy proposed by Soussi et al. [18] as Soussi strategy, and strategy proposed by Insausti et al. [15] as Insausti strategy.

Figs. 5 and 6 show that the Soussi strategy exhibits the highest outage probability. This is because even though the linear combinations selected by the relay and the destination are globally optimized, the destination can correctly recover the transmitted message if and only if it correctly decodes its own linear combination and the one from the relay. Therefore, if there is an outage in either the source-relay link, the source-destination link, or the relay-destination link, the final decoding at the destination will fail. This means that the relay does not act as a helper, and rather, its presence is mandatory. Also, point-to-point communication from the relay to the destination can only achieve first-order diversity gain, as can be confirmed in the numerical results, so the Soussi strategy suffers from a bottleneck performance at the relay-destination link. This fact can be seen from the three scenarios where the outage performance of the Soussi strategy gets better as the distance of the relay to the destination gets smaller, i.e., $\gamma_{rd}$ gets larger.

In Figs. 5 and 6, it is shown that a significant outage performance improvement over the Soussi strategy is achieved by the Insausti strategy in the first and the second scenarios. For the third scenario, even though at low SNR regime the Soussi strategy has lower outage probability, it can be predicted that eventually the Insausti strategy is better in high SNR regime as the slope of its outage probability curve is steeper. This improvement is a result of giving the destination two possible ways of decoding the transmitting messages. The first is with the help of the relay, and the second is by using linear combinations decoded by itself. Thus, it can be thought that the relay acts as a helper where its existence is not mandatory, i.e., it is possible for the destination to decode the transmitted messages without the relay. It can also be observed that our proposed lim-FB strategy achieves the same outage performance as the Insausti strategy. This is because they are quite similar in the sense that the destination has two possible ways for decoding the transmitted messages and treat the relay as a useful helper. If we carefully observe the slopes of the outage performance of the lim-FB and the Insausti strategies, they do not achieve second-order diversity gain. The main reason behind this is that the local best linear combination selected by the relay may not be linearly independent of the $M - 1$ best linearly combinations of the destination. We also observe that the performance of the Insausti and the lim-FB strategies degrades as the relay gets closer to the destination or as the average SNR from the sources to the relay gets smaller. This is related to the probability of the rank deficient coefficient matrix. As we have seen in Fig. 4, the smaller the difference between $\gamma_{sd}$ and $\gamma_{sr}$, the higher the probability of rank deficient coefficient matrix. Hence, for the lim-FB and the Insausti strategies, it is better to place the relay closer to the sources.

The best outage performance is achieved by the suf-FB strategy. Based on the slopes of the curves shown in Figs. 5 and 6, one can see that the suf-FB strategy achieves the second-order diversity gain. This agrees with our analysis in Subsection V-B. The main reason for this is that the destination has two possible ways in decoding the transmitted messages, the direct and the cooperative decoding. Moreover, unlike in the lim-FB, the resulting coefficient matrices in the suf-FB are guaranteed to always be full-rank.

Next, we evaluate network throughput performance which is defined as the ratio of the correctly received messages to number of transmission rounds utilized. For the proposed
this additional overhead is small enough compared to that of [18]. Moreover, the feedback is only sent when the destination fails to decode the transmitted messages by itself. Hence, in the higher SNR regime, only a small amount of feedback is required. As an alternative, one may choose the lim-FB strategy that is better than the existing strategy in terms of outage probability and network throughput with only one-bit feedback.

VII. CONCLUSIONS

In this paper, we have studied the application of compute-and-forward to multiple-access relay channels (MARC). We proposed two cooperation strategies between the relay and the destination. The proposed strategies are opportunistic in the sense that they use transmission rounds as few as possible to increase the transmission efficiency while improving outage probability performance of the MARC. We have shown that both of the proposed strategies improve network throughput remarkably, twice that of the existing strategies [15], [18]. It is also shown that both the proposed strategies always yield lower outage probability, independent of relay placement. Moreover, it is confirmed that the first strategy called the lim-FB strategy achieves diversity gain close to the second-order, which is a significant improvement over [18]. A better outage probability enhancement is achieved the second strategy, namely the suf-FB strategy, where the full-diversity gain of the MARC is achieved.

As future work, it is of interest to investigate the diversity multiplexing trade-off (DMT) of the MARC with compute-and-forward. Another direction is to allow the relay to help the destination several times, which can be regarded as an automatic repeat request (ARQ) scheme.

APPENDIX A

PROOF OF LEMMA 1

Let \( \mathbf{h} = [h_1, \ldots, h_M] \) be the channel coefficient vectors and \( R \) be the coding rate employed by the transmitters. Without loss of generality, assume the geometric gain from the transmitters to the receiver is a unit, i.e., \( g = 1 \) and the average SNR is \( \gamma \). Let \( \mathbf{a}_1 \) be the best (local optimal) integer coefficient vector selected by the receiver. The outage probability of decoding the linear combination corresponding to \( \mathbf{a}_1 \) is defined as

\[
P_{\text{out}} \triangleq \Pr(R_{\text{cp}}(\mathbf{a}_1, \mathbf{h}, g) < R)
\]

(58)

\[
= \Pr\left(\log^+ \left( \frac{1}{\mathbf{a}_1^T \mathbf{M} \mathbf{a}_1} \right) < R \right)
\]

(59)

\[
= \Pr\left(\log^+ \left( \frac{1}{\|\mathbf{B} \mathbf{a}_1\|^2} \right) < R \right)
\]

(60)

\[
= \Pr\left(\|\mathbf{B} \mathbf{a}_1\|^2 > 2^{-R} \right),
\]

(61)

where \( \mathbf{M} \) and \( \mathbf{B} \) are defined in (30) and (31).

As described in Section III-C, \( \mathbf{M} \) is a positive definite matrix and \( \mathbf{B} \) is its Cholesky factorization. And because \( \mathbf{a}_1 \) is the local optimal coefficient vector, \( \|\mathbf{B} \mathbf{a}_1\| \) is the first
successive minimum or the minimum distance of the lattice generated by $B$. The minimum distance of a lattice is related to the determinant of its generator matrix by the Hermite's constant, see [38]. Let $\Psi_M$ be the Hermite's constant of dimension $M$. With $\Psi_M$, now we have a relation

$$\Psi_M \geq \frac{\|Ba_1\|^2}{\det(B)^{M/2}}. \quad (62)$$

It can be shown that $\det(B) = 1/(1 + \gamma \|h\|^2)^{M/2}$. Thus,

$$\Psi_M \geq \frac{\|Ba_1\|^2}{1 + \gamma \|h\|^2}. \quad (63)$$

As a result, (61) can be written as

$$P_{out} = \Pr \left[ \|Ba_1\|^2 > 2^{-R} \right] \leq \Pr \left[ \frac{\Psi_M}{1 + \gamma \|h\|^2} > 2^{-R} \right] \leq \Pr \left[ \|h\| < \Psi_M^{2R - 1}/\gamma \right] \leq (a) \frac{\psi_M^{2R - 1}/\gamma}{\gamma^M}, \quad (66)$$

for a positive constant $\xi$, where $(a)$ is due to the generalization of [8] Fact 2 in Appendix 1]. We then can show that the full-diversity order is achieved for decoding an optimal linear combination of a compute-and-forward scheme as

$$-\lim_{\gamma \to \infty} \frac{\log P_{out}}{\log \gamma} \leq M, \quad (68)$$

or $P_{out} \leq \gamma^{-M}$.
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