Trajectory phase transitions in non-interacting systems: all-to-all dynamics and the random energy model

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We study the fluctuations of time-additive random observables in the stochastic dynamics of a system of $N$ non-interacting Ising spins. We mainly consider the case of all-to-all dynamics where transitions are possible between any two spin configurations with uniform rates. We show that the cumulant generating function of the time-integral of a normally distributed quenched random function of configurations, i.e. the energy function of the random energy model (REM), has a phase transition in the large $N$ limit for trajectories of any time extent. We prove this by determining the exact limit of the scaled cumulant generating function. This is accomplished by connecting the dynamical problem to a spectral analysis of the all-to-all quantum REM. We also discuss finite $N$ corrections as observed in numerical simulations.

1. Introduction

In statistical mechanics, we are used to studying static phase transitions from singularities in partition sums [1]: the value of a control parameter at which the
free energy becomes non-analytic (in the infinite-size limit) indicates that the equilibrium ensemble of configurations undergoes a phase change. The standard equilibrium ensemble method can be generalized straightforwardly to stochastic dynamics by replacing configurations with trajectories, static observables with (time-extensive) functions of trajectories and the partition sum with the corresponding moment generating function of the trajectory observable [2,3]. The ‘thermodynamics of trajectories’ approach [4–6] allows to study dynamical or ‘trajectory’ phase transitions, that is, singular changes in the nature of dynamical fluctuations that often are not reflected in (thermo)static properties or occur at different parameters of the model. The singularities of the relevant large deviation (LD) functions [7] reveal phase transitions in, for example, the dynamical activity of glassy systems [8–10], in time-integrated currents in exclusion processes [11–13] and in (active) work in active matter [14].

An interesting question is what occurs in a system of many degrees of freedom whose dynamics is non-interacting when one considers the fluctuations of a (quenched) random trajectory-observable that couples them. Our main object of interest will be a system of $N$ Ising spins which all flip independently from each other. For the case of non-random local observables and independent spins with single spin-flip dynamics recent results [15] show that in certain cases there is a phase transition in the LD function. While, naively, one might expect nothing interesting to occur due to the non-interacting nature of the dynamics, these results indicate that the optimal way to generate large fluctuations is by means of effectively highly correlated dynamics which is singularly different from the typical dynamics [16].

Here we start addressing the problem of random and long-ranged trajectory observables by considering the time integral of a function of configurations whose values are normally distributed with zero mean and variance $N$, that is, the energy function of the simplest mean-field spin glass: the random energy model (REM) [17,18]. For simplicity, we will consider dynamics which is all-to-all, that is, allowed configuration changes are those where any number of spins can flip simultaneously and independently. We also comment on the case of single-spin flips, which corresponds to the quantum random energy model (QREM).

The general problem we consider here has relevance in several areas. One is the minimization via trajectory sampling of (quasi-) random cost functions [19], which arises for example when training neural networks. A second one is in connection to measurement induced phase transitions in quantum systems [20,21], where the calculation of Renyi entropies reduces to computing the optimal dynamics of a random coupling function [22,23] in a system of classical replicas which evolve independently.

2. Unbiased dynamics

Any continuous-time Markov process with trajectories $\omega : [0, \infty) \to Q_N$ on the configuration space $Q_N := \{-1, 1\}^N$ of $N$ Ising spins is uniquely characterized in terms of the transition rates $w_{\sigma \to \tau}$ of spin configurations $\sigma = (\sigma_1, \ldots, \sigma_N) \in Q_N$ to any other configuration $\tau$, and the associated escape rates $r_\sigma := \sum_{\tau \neq \sigma} w_{\sigma \to \tau}$. The latter governs the law, $r_\sigma e^{-r_\sigma \Delta t}$, of the sojourn time $\Delta t$ until the next jump out of $\sigma$. In the following, we choose $w_{\sigma \to \tau} := N2^{-N}$ independent of the configuration. Since the connectivity of this jump process is then described by the complete graph on $2^N$ vertices (i.e. spin configurations), this dynamics is called the completely connected or all-to-all stochastic dynamics on Ising configurations. Using Dirac’s notation, in which $|\sigma\rangle |\sigma \in Q_N\rangle$ stands for the canonical orthonormal basis in the Hilbert space $\ell^2(Q_N) \equiv \bigotimes_{j=1}^N \mathbb{C}^2$, the generator of this Markov process is given by

$$W := \sum_{\sigma, \tau \in Q_N \atop \sigma \neq \tau} w_{\sigma \to \tau} |\sigma\rangle \langle \tau| - \sum_{\sigma \in Q_N} r_\sigma |\sigma\rangle \langle \sigma| = N (|\omega\rangle \langle \omega| - 1)$$

in terms of the orthogonal projection $|\omega\rangle \langle \omega|$ onto the ‘flat state’ defined by $|\sigma\rangle \langle \sigma| = 2^{-N/2}$. In its probabilistic interpretation, $W$ is considered an operator on $\ell^1(Q_N)$ and acts on probability
distributions \(|p_t|\), i.e. \(p_t(\sigma) \equiv (\sigma|p_t) \geq 0\) and \(\sum_{\sigma \in Q_N} p_t(\sigma) = 1\). The dynamics of any initial distribution is governed by the master equation

\[ \frac{\partial}{\partial t}|p_t\rangle = W|p_t\rangle. \]

The completely connected stochastic dynamics can be regarded as a further simplification of the dynamics of independent spin flips at infinite temperature. The latter is generated by \(\tilde{W} := \sum_{j=1}^{N}(X_j - 1)\), in terms of the Pauli-X matrices, which flip the \(j\)th spin, i.e. \(X_j|\sigma\rangle = |\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_N\rangle\). Both Markov processes are irreducible and share the equidistribution \(p_{\text{ss}}(\sigma) := 2^{-N}\) as its unique invariant measure. One difference is their spectral gap, which governs the rate of approach to the equidistribution. While the spectral gap is \(N\) in the case of \(W\), it is \(2\) in the case of \(\tilde{W}\). In this paper, we focus on the completely connected dynamics \(W\) and only comment on the single spin-flip dynamics \(\tilde{W}\).

The dynamics generated by \(W\) (and \(\tilde{W}\)) is ‘infinite temperature’ in the sense that transitions are completely independent of the initial and final states. The operator \(W\) is therefore bi-stochastic, \((-|W = 0, W|\rangle = 0\), with the first equality indicating conservation of probability, and the second that the stationary state is also the flat state (the stationary probability vector being \(2^{-N/2}|\rangle\)). Since the dynamics of all spins is independent, all correlation functions are unconnected.

3. Trajectory observable and random energy model

We study the statistics under the above defined all-to-all independent dynamics of a trajectory observable chosen to explore the energy landscape of the REM [17,18]. The REM, \(U : Q_N \rightarrow \mathbb{R}\), is a Gaussian random field (with randomness independent of the Markov process) in which the values \(U(\sigma)\) are distributed independently for all \(\sigma \in Q_N\) with identical normal law uniquely characterized by zero mean and covariance \(N\). The units are chosen so that the REM’s large deviations occur on order \(N\) which agrees with the norm of \(W\). In this context, we recall [17,24] that the REM’s minimum (and similarly for its maximum) satisfies the extremal value statistics:

\[ P(\min U \geq u_N(x)) = (1 - 2^{-N} e^{-x + o(1)})^{2N} \]

for any \(x\) with the scaling function \(u_N(x) := -\beta_c N + \left((\ln(N \ln 2) - \ln(4\pi))/2\beta_c\right) - (x/\beta_c)\), where \(\beta_c = \sqrt{2\ln 2}\) and \(P\) denotes the joint law of the REM. In particular, the minimal energy of the REM is roughly at \(-\beta_c N\).

The trajectory observable we consider is (up to a factor of \(t\)) the empirical average of the REM energy along a trajectory \(\omega\) of the Markov process

\[ U_t(\omega) := \int_0^t U(\omega(s)) \, ds, \quad t > 0. \]

We will be interested in the probability distribution of this quantity under the law \(P_t\) on trajectories associated with \(W\) up to time \(t\) with the initial spin configurations equally distributed. The main result of this short note is a proof of a large deviation principle for this distribution in the limit of large system size \(N\) (for trajectories of any time extent \(t\)). This large deviation principle is described in terms of the moment generating function

\[ Z(t, \lambda) := \int e^{-\lambda U_t(\omega)} P_t(d\omega) = \sum_{\sigma \in Q_N} 2^{-N} |\sigma| e^{t(W - \lambda U)} |\sigma\rangle = \langle -| e^{t(W - \lambda U)} |\rangle. \]

Here the second equality is due to the Feynman–Kac formula for the Markov process under consideration (cf. [25,26]). Crucially, this formula connects the question concerning the (a)typical behaviour of \(U_t\) to properties of the tilted generator

\[ W_\lambda := W - \lambda U, \]

which is a random matrix on \(\ell^2(Q_N)\). Note that by substituting \(W\) by \(\tilde{W}\), this random matrix coincides, up to a constant shift and change of sign, with the Hamiltonian of the QREM—one of the simplest quantum spin glass models [27–30]. In our case, the operator \(H_\lambda = -W_\lambda + N\).
instead corresponds to the Hamiltonian with an all-to-all kinetic energy studied in [31]. Owing to the symmetry of the REM’s distribution the parameter $\lambda$ can be taken non-negative without loss of generality, and the large deviation function also known as scaled cumulant generating function (SCGF) is then given by

$$
\theta(t, \lambda) := \lim_{N \to \infty} \frac{1}{N} \ln Z(t, \lambda).
$$

The SCGF plays the role of a free energy for trajectory ensembles.

It is important to emphasize that what we are considering here is very different from the study of classical thermal dynamics of the REM under Glauber or Metropolis schemes, as in e.g. [32–35]. In those cases, the dynamical Markov generator is interacting (as transitions depend on changes in $U$) and what is studied are the typical trajectories under that interacting dynamics. By contrast, we study rare trajectories under the non-interacting dynamics generated by $W$ with large fluctuations of $U$.  

### 4. Trajectory phase diagram

Our main result is the following:

**Theorem 4.1.** For any $t > 0$, $\lambda \geq 0$ and almost all realizations of the REM:

$$
\theta(t, \lambda) = \max \{0, t^{-1} p_0(\lambda) - 1\},
$$

with

$$
p_0(\beta) := \lim_{N \to \infty} \frac{1}{N} \ln \frac{1}{2^N} \sum_{\sigma} e^{-\beta U(\sigma)} = \begin{cases} 
\frac{\beta^2}{2}, & \beta \leq \beta_c := \sqrt{2 \ln 2} \\
\beta \beta_c - \ln 2, & \beta > \beta_c.
\end{cases}
$$

Before spelling out the short proof of theorem 4.1 in §7, let us put this result in some context and discuss some consequences. The quantity defined in (4.2) is the pressure corresponding to the REM’s static (normalized) partition function at inverse temperature $\beta$. The critical value $\beta_c = \sqrt{2 \ln 2}$ corresponds to the inverse of the REM’s freezing temperature into a spin glass phase with one-step replica symmetry breaking (cf. [17]). The phase diagram resulting from theorem 4.1 is thus composed of three regimes depicted in figure 1a:

(i) An *Active* dynamical phase in which the Markov generator $W$ dominates over the tilting, and which is characterized by $\theta(t, \lambda) = 0$ and the specific activity being unity (see below). It is separated from the remaining regimes by a first-order transition line. This regime persists for all $|\lambda| < \beta_c (2t)^{-1} + \beta_c^{-1}$ in case $t^{-1} < 2 \beta_c^{-2}$ and $|\lambda| < \sqrt{2t}^{-1}$ in case $t^{-1} \geq 2 \beta_c^{-2}$.

(ii) A regime of vanishing activity which occurs for $t^{-1} < 2 |\lambda| \beta_c^{-1} - 2 \beta_c^{-2}$ and which is dominated by the REM’s extreme values where the system localizes. This regime is related to the spin-glass phase of the REM. We call this the *Inactive-1* dynamical phase.

(iii) The remaining parameter regime corresponds to a second inactive regime which we term *Inactive-2* dynamical phase. It occurs only if $t^{-1} > 2 / \beta_c^2$ and is related to the classical paramagnetic phase of the REM.

In particular, in the long-time limit, $t \to \infty$, the value $\lambda = \beta_c^{-1}$ separates the Active and Inactive-1 phases, the latter dominating at large $\lambda$. Not surprisingly, this transition in the largest eigenvalue of the tilted generator $W_\lambda = W - \lambda U$ reflects the known location of its quantum-phase transition. As we will recall in §7, the eigenvector corresponding to the largest eigenvalue changes near $\lambda = \beta_c^{-1}$ from a delocalized state resembling $|-\rangle$ (indicating that trajectories visit all states equally giving rise to large activity) to a state localized at the REM’s maximizing spin configuration (corresponding to trajectories that are inactive as they do not move away from this configuration).

The classification above of the trajectory phases in terms of their activity is obtained as follows. The dynamical activity is the total number of configuration changes in a trajectory. It can be calculated through the same tilting method used above for the time-integrated REM energy. Specifically, if we define the doubly tilted partition sum $Z(t, \lambda, s) := \langle -e^{i W_{\lambda, s}} | - \rangle$ with $W_{\lambda, s} =$$
N e^{-\beta} \langle - \rangle - N(1 - 2^{-N}(1 - e^{-\beta}))I - \lambda U (where the additional tilting by e^{-\beta} of the off-diagonal part of W allows to count jumps in trajectories), we get the activity from $-\partial_s \log Z(t, \lambda, s)|_{s=0}$ Using the results above, it is easy to see that the average activity per unit space and time is unity in the active phase and zero in the two inactive phases.

Via the Gärtner–Ellis theorem [36], the rate function of the large deviation principle obeyed by $U_t$ is given by the Legendre–Fenchel transformation

$$\varphi(t, u) := \sup_{\lambda} (u \lambda - \theta(t, \lambda)) = \begin{cases} |u| \sqrt{\frac{2}{t}}, & |u| \leq \min \left\{ \sqrt{2t}, \beta_c \right\}, \\ 1 + \frac{u^2}{2t}, & \text{else,} \\ \infty, & |u| > \beta_c. \end{cases}$$

Note that, although in theorem 4.1 initially defined only for $\lambda \geq 0$, the function $\lambda \to \theta(t, \lambda)$ extends to all real values by symmetry. The rate function $u \to \varphi(t, u)$ is then symmetric as well. For times $t > \beta_c^2/2 = \ln 2$, the second case in the above equation is absent. As a corollary to theorem 4.1 and [36, Thm 2.3.6], we thus obtain the promised large deviation principle

$$- \inf_{u \in I} t \varphi(t, u) \leq \lim_{N \to \infty} \frac{1}{N} \ln \mathbb{P}_t((Nt)^{-1} U_t \in I) \leq \limsup_{N \to \infty} \frac{1}{N} \ln \mathbb{P}_t((Nt)^{-1} U_t \in I) = - \inf_{u \in I} t \varphi(t, u)$$

which holds for any Borel set $I \subset \mathbb{R}$ and any $t > 0$. The rate function is shown in figure 1b for two different times.

Clearly, under the a priori measure $\mathbb{P}_t$, which favours rapid changes of spin configurations at the rate $N(1 - 2^{-N})$, the typical value of the REM’s empirical energy density $N^{-1} U_t(\omega)$ along any trajectory $\omega$ is close to zero. The fluctuations about this typical behaviour are described by (4.3): close to $u = 0$, these fluctuations are linearly suppressed with a rate proportionally to $N \sqrt{2t}$. Tilting the a priori measure, one encounters one or two phase transitions depending on whether $t > \ln 2$ or not. If $t < \ln 2$, one enters a regime $\sqrt{2t} < |u| < \beta_c$ with Gaussian fluctuations. Beyond this, i.e. at energy densities of the order of the REM’s maximum or minimum (3.1), the energy
density effectively stops fluctuating. Trajectories freeze for long times in the REM’s extremal values.

5. Thermal phase diagram and generalizations

The dynamical partition sum of the stochastic system we are considering is reminiscent of a quantum (thermal and static) partition sum for the all-to-all version of the QREM. While the calculation of both is analogous, there are some important differences, which we address next. Specifically, if we consider the tilted generator \( W_{\lambda} \) as (minus) a Hamiltonian, the (specific) free energy of the associated quantum problem at temperature \( T \) is

\[
f(T, \lambda) := \lim_{N \to \infty} \frac{T}{N} \ln \frac{1}{2^N} \text{Tr} e^{(W_{\lambda}U)T}.
\]

As we will explain in §7, using results in [31] this quantity can easily be determined for a rather general Ising spin glass \( U \) in terms of its free energy.

**Theorem 5.1.** Let \( U : Q_N \to \mathbb{R} \) be such that \( P(U(\sigma) \in N \text{ dv}) = \varrho(v) \text{ dv} \) for all \( \sigma \in Q_N \) with an \( N \)-independent distribution \( \varrho \) which has up to four finite moments with \( \int v \varrho(v) \text{ dv} = 0 \) and \( \int v^2 \varrho(v) \text{ dv} = 1 \). If for almost all realizations the values \( U(\sigma) / N \) as \( N \to \infty \) fill a compact interval \([u_{\min}, u_{\max}]\) and the pressure \( p_0(\beta) = \lim_{N \to \infty} (1/N) \ln(1/2^N) \sum_{\sigma} e^{-\beta U(\sigma)} \) exists, then the free energy at temperature \( T \) of the tilted generator \( W_{\lambda} \) is given by

\[
f(T, \lambda) = \max \left\{ -T \ln 2, Tp_0 \left( \frac{\lambda}{T} \right) - 1 \right\}.
\]

Restricting attention to the REM to which the theorem is applicable, we note that (5.2) and the explicit expression for \( p_0 \) in (4.2) implies that the all-to-all QREM exhibits three different phases depending on coupling and temperature: a delocalized quantum paramagnetic phase (QPM), a localized spin-glass phase (SG) and a classical paramagnetic phase (PM) (figure 1c). These three static quantum phases are similar to the dynamical ones of the stochastic problem of theorem 4.1. But it is worth pointing out that at \( T > 0 \) the (thermo)static phase transitions described by \( f(T, \lambda) \) do not coincide with the dynamic phase transitions described by \( \theta(t, \lambda) \). These differences arise because the boundary vectors in the dynamical partition sum involve information about eigenvectors of \( W_{\lambda} \), also versus the trace in the static quantum one depends on the eigenvalues only. For a comparison of the phase diagrams, see figure 1a,c.

In contrast to the all-to-all simplification studied in theorem 5.1, the free energy of transversal field models with \( U \)’s having a correlation structure more complicated than the REM [27,29] or its generalized relatives [28,30] is not explicitly known. The most prominent example, to which theorem 5.1 also applies, is the all-to-all version of the Quantum Sherrington–Kirkpatrick (QSK) model. In contrast to its all-to-all simplification (5.2), the free energy and phase diagram of the QSK is predicted to consist of two phases only: a paramagnetic phase separated from a spin-glass phase at low temperatures and small transversal field (cf. [37,38] for numerical and [26,39] for recent analytical results). Clearly, the all-to-all version of the QSK will not exhibit an interesting entanglement structure predicted recently in [40].

In view of the general class of \( U \)’s covered by theorem 5.1, one might wonder whether theorem 4.1 is generalizable to the class of \( U \)’s covered above. In fact, from the proof in §7 it is easy to see that substituting in (4.1) the pressure of \( U \), the right-hand side remains a lower bound on \( \theta(t, \lambda) \), cf. (7.4). This lower bound is sharp only in case the eigenvectors corresponding to the bulk of eigenvalues remain localized (in the sense that \( \lim_{N \to \infty} N^{-1} \ln(-|1(e^{-\delta N}H_{\delta})(H_{\delta})|) = - \ln 2 \) for all sufficiently small \( \delta > 0 \) and \( E/N \in [u_{\min}, u_{\max}] \)). Motivated by results in [41], we conjecture this to hold for the SK model \( U \). However, this will not hold in case correlations of \( U \) cause its values to cluster.
6. Numerical illustration of finite-size corrections

The exact results above are for the limit \( N \to \infty \). At finite \( N \) there are of course finite-size corrections and sample-to-sample fluctuations between different realizations of the disorder \( U \). Using numerics, we now illustrate some of these finite-size effects. (A comprehensive numerical study of both the all-to-all and single spin-flip problem will be presented in a future publication.)

When the system size is not too large the dynamical partition sum (3.2) can be computed numerically using exact diagonalization (ED). We illustrate results for one disorder realization in a system of size \( N = 12 \) of the dynamical order parameter:

\[
u = \frac{1}{N \it Z(t, \lambda)} \int U_t[\omega] e^{-\lambda U_t[\omega]} \delta_{\it P}(d\omega) = - \frac{\partial}{\partial \lambda} \theta_N(t, \lambda) \tag{6.1}\]

Figure 2a shows the following: (i) for finite size the phase transitions turn into crossovers, as expected; (ii) for all \( t \) there is a crossover from \( u \approx 0 \) at \( \lambda = 0 \) to a large negative \( u \) for large \( \lambda \), eventually reaching the minimum of the potential (which changes from sample to sample); (iii) these crossovers are sharper the longer \( t \), also expected due to the preference of the boundary states in (3.2) for the delocalized state.

In figure 2b, we show representative trajectories for two values of \( \lambda \) for \( t = 10 \). We plot the instantaneous energy as a function of time in the trajectory. The top panel shows a typical trajectory of the dynamics corresponding to \( \lambda = 0 \), cf. the red square in figure 2a. This trajectory generated by \( W \) is sampled using standard (continuous-time) Monte Carlo [42]. Since the unbiased dynamics connects all configurations with equal rates the trajectory jumps between the energy values: it corresponds to the phase which has high activity and is delocalized. The bottom panel shows a characteristic trajectory for \( \lambda = 2 \), cf. the red circle in figure 2a. This is a rare event (exponentially suppressed in \( N \) and \( t \)) of the dynamics, and as such cannot be easily sampled from running Monte Carlo with \( W \) (since \( W_\lambda \) is not a stochastic operator). We obtain such rare trajectories instead by performing importance sampling in trajectory space using transition path sampling (TPS) [43], essentially a Monte Carlo method in trajectory space that aims to ‘equilibrate’ to a reweighted trajectory distribution \( Z(t, \lambda)^{-1} e^{-\lambda U_t[\omega]} \) (supplemented with bridge moves to improve acceptance; we will provide details of this method in a future publication). The inset to the lower panel shows the convergence of our TPS approach: it shows the evolution of the sampled trajectories with TPS iterations by showing their \( U_t[\omega] \) (per unit time). The \( U_t[\omega] \) in the
inset converges eventually to the value expected at $\lambda = 2$, showing that TPS converges to the tilted trajectory ensemble. The trajectory shown in the lower panel is the last trajectory from TPS. It is very different from the typical one in the upper panel: it has very low activity and is localized for most of the time in the minimum energy configuration, corresponding to the Inactive-1 dynamical phase. Note that while we only illustrate the numerics for the size $N = 12$, TPS can be used for larger system sizes in contrast to ED.

Figure 2 shows the location of the critical $\lambda$ for $t = \infty$, averaged over 20 realizations of the disorder, for different systems sizes. The transition point is inferred from the maximum of the dynamical susceptibility $\partial^2_0 \theta_N(\lambda)$, where $\theta_N(\lambda)$ is the largest eigenvalue of $W_\lambda$. This eigenvalue is calculated using (7.1), which allows to compute it for larger sizes than those accessible to ED. The figure suggests a convergence to the limiting value $\beta_c^{-1}$ for large $N$, as expected from the analytics above.

7. Proof of the large deviation results

The Feynman–Kac formula (3.2) reduces the large deviation problem to a spectral analysis of the random matrix $W_\lambda = W - \lambda U$. Up to a constant shift and rescaling, the spectrum of $W_\lambda$ for an REM $U$ has been analysed in [31] both on the macro and microscopic scale of the eigenvalue process. The main technical tool for studying $W_\lambda$, which applies to $U$ more generally than the REM, is rank-one perturbation theory. For non-degenerate values of $U$, the energy $E$ is an eigenvalue of $W_\lambda$ if and only if

$$1 = N\langle - | (E + N + \lambda U)^{-1} | - \rangle = \frac{1}{2N} \sum_\sigma \frac{N}{E + N + \lambda U(\sigma)}. \tag{7.1}$$

The corresponding eigenvectors $\psi_E$ satisfy for all $\sigma, \tau \in Q_N$:

$$\frac{\langle \sigma | \psi_E \rangle}{\langle \tau | \psi_E \rangle} = \frac{E + N + \lambda U(\tau)}{E + N + \lambda U(\sigma)}. \tag{7.2}$$

In the case of degeneracies of the values of $U$, the remaining eigenvalues remain unaffected by the rank-one perturbation. An immediate implication of this fact and (7.1) is the fact that all eigenvalues of $W_\lambda$ aside from the largest one are interlaced with the energies of $U$ and additionally shifted by $-N$ (cf. e.g. [44] and references therein for interlacing and finite-rank perturbation theory). In the case of the REM, all eigenvalues are in fact almost surely simple.

As we argue next, the largest solution of (7.1) is independent of the realization of $U$ up to small fluctuations. More precisely, we study solutions with $E/N \geq \lambda u_{\text{max}} - 1 + \epsilon$ for arbitrary $\epsilon > 0$ and all $N$ (with $u_{\text{max}} = \beta_c$ for the REM, cf. (3.1)). A strong law of large numbers implies that the right-hand side of (7.1) is then well approximated as $N \to \infty$ by the integral

$$\int_{-\infty}^\infty \frac{N}{E + N + \lambda \sqrt{N} v} \rho(v) \, dv = \frac{1}{1 + E/N} \left( 1 + \frac{\lambda^2}{N(1 + E/N)} + O(N^{-2}) \right).$$

This explains the following results on the largest eigenvalue $E_0 := \max_\sigma \sigma(W_\lambda)$, which are straightforward adaptations of results in [31]:

(i) In case $\lambda u_{\text{max}} < 1$, on an event with probability close to one, the largest eigenvalue is at $E_0 := \max_\sigma \sigma(W_\lambda) = \lambda^2 + O(N^{-1})$ and the corresponding eigenvector satisfies $\langle \sigma | \psi_{E_0} \rangle \propto (E_0 + N + \lambda U(\sigma))^{-1}$. Since $2^{-N} \sum_\sigma (E_0 + N + \lambda U(\sigma))^{-2}$ is of order one up to small fluctuations by the law of large numbers, this vector is hence still delocalized (as in the case $\lambda = 0$).

(ii) In case $\lambda u_{\text{max}} > 1$, the largest eigenvalue is at $E_0 = \max_\sigma \sigma(W_\lambda) = (\lambda u_{\text{max}} - 1)N + o(N)$. For the REM, one even has the fine asymptotics

$$E_0 = \max_\sigma \sigma(W_\lambda) = (\lambda \beta_c - 1)N - N2^{-N} \left( 1 - \frac{1}{\lambda \beta_c} \right)^{-1} + o(N2^{-N}) \tag{7.3}$$
and the corresponding eigenvector is mostly concentrated on the REM’s minimizing configuration $σ_0 : eqq arg min U$. This is specified through the ratios $(7.2)$. Note that the error term in the above equation only holds with a probability up to $1 − O(1/N)$ (cf. [31, App A]).

In particular, even in the case of degeneracies of $U$, the union of eigenvalues, $σ(W_λ)$, when divided by $N$, converges almost surely to the non-random set $[0] ∪ [λ u_{min} − 1, λ u_{max} − 1]$. Together with the interlacing property, one then easily arrives at (5.2) for the free energy of $W_λ$.

The proof of theorem 4.1 requires slightly more detailed knowledge, since $⟨−|e^{tW_λ}|−⟩$ involves properties of the eigenvectors, too. The rough picture established in [31] through a more detailed analysis of the characteristic equation (7.1) is the following:

(i) Delocalization of one eigenstate near energy 0 is shown to persist up to $λ < \sqrt{2}$. From that value on, $λ > \sqrt{2}$, this eigenstates ‘melts’ into a narrow band of semi-delocalized states near energy 0.

(ii) The eigenvalue process, when rescaled to order one at some fixed energy outside $−N$ and 0, is given by a Poisson process. Correspondingly, outside those special energies the normalized eigenvectors are localized.

We will need the following result, which is contained in [31, Proof of Thm. 6.3].

**Proposition 7.1.** For any $δ > 0$ and any $N$ there is some $a > 0$ and an event $Ω_N$ whose complement is summable, $\sum_N P(Ω_N^c) < \infty$, such that in the event $Ω_N$ any eigenvalue $E$ of $W_λ$ with $|E| > δN$ and $|E + N| > δN$ has a normalized eigenvector $ψ_E$, which satisfies $||−|ψ_E|2|^2 ≤ N^δ 2^{−N}$. Moreover, for any such $E$, there is some $σ_E ∈ \{−1, 1\}^N$ such that $|E + N − λ U(σ_E)| ≤ δN$.

**Proof of theorem 4.1.** The proof proceeds by establishing asymptotically coinciding upper and lower bounds. For the lower bound, we use Jensen’s inequality to conclude

$$\ln⟨−|e^{tW_λ}|−⟩ ≥ 1/(−|W_λ| −) = \frac{tλ}{2N} \sum_σ U(σ).$$

By the law of large numbers, this term converges to zero for almost all realizations of the REM. For another lower bound, which is sharper in case $t < p_0(tλ)$, we estimate

$$⟨−|e^{tW_λ}|−⟩ = \frac{1}{2N} \sum_σ t e^{tW_λ} |σ| ≥ \frac{1}{2N} \sum_σ |σ| e^{tW_λ} |σ| \geq \frac{1}{2N} \sum_σ \exp (tN(||σ|−|2 − 1) − tλ U(σ)), \quad (7.4)$$

where the last step is again by Jensen’s inequality. Using $||σ|−|2^2 = 2^{−N}$ and (4.2), the combination of the above estimates yields (4.1) as a lower bound.

A complementing upper bound is based on proposition 7.1. Expanding in eigenfunctions and splitting the sum over all eigenvalues in three parts corresponding to energies $E$ with $|E + N| ≤ δN$, $|E| ≤ δN$ and the rest, we write and estimate using proposition 7.1:

$$⟨−|e^{tW_λ}|−⟩ = \sum_E e^{tE}⟨−|ψ_E|⟩^2 ≤ e^{tN(δ−1)} + e^{tNδ} + \frac{N^δ}{2N} e^{tN(δ−1)} \sum_σ e^{−tλ U(σ)}. \quad (7.5)$$

In the event $Ω_N$ of proposition 7.1, we thus conclude

$$\limsup_{N→∞} \frac{1}{Nt} \ln⟨−|e^{tW_λ}|−⟩ ≤ max\{0, t^{−1} p_0(tλ) − 1\} + δ.$$

By a Borel–Cantelli argument, this establishes this almost-sure bound on the upper limit. Since $δ > 0$ is arbitrary, this concludes the proof.
8. Outlook: QREM

Let us conclude this note with some conjectures, partial results and comparison in case $W$ is replaced by the spin-flip dynamics generated by $\hat{W}$. In that case, the tilted generator $\hat{H}_\lambda := \hat{W} - \lambda U$ is the QREM. Its low-energy spectrum as well as the phase transitions in the free energy are well understood [29,45]. By the Feynman–Kac formula the dynamical phase transition is again described in terms of the asymptotic behaviour of $N^{-1} \ln (-|e^{\hat{H}_\lambda}|)$.

The phase transition in the largest eigenvalue $\hat{E}_0 := \max \sigma (\hat{H}_\lambda)$ occurs on order $N$ at the same location $\lambda = \beta_c^{-1}$ as for $H_\lambda$. However, the finite-volume corrections are different in the localization regime, i.e. for all realizations of the REM aside from a set of exponentially small probability (see [45] for details):

(i) if $\lambda > \beta_c^{-1}$ we have $\hat{E}_0 = -\lambda \min U + (\lambda \beta_c)^{-1} + \mathcal{O}(N^{-1/4})$,

(ii) if $\lambda < \beta_c^{-1}$ we have $\hat{E}_0 = \lambda^2 + \mathcal{O}(N^{-1/4})$.

Following the steps of the lower bound in the proof of theorem 4.1, it is easy to see that for almost all realizations of the REM one still has:

$$\lim \inf_{N \to \infty} \frac{1}{Nt} \ln (-|e^{\hat{H}_\lambda}|) \geq \max \{0, t^{-1} p_0(t\lambda) - 1\}.$$ (8.1)

We conjecture that this bound is sharp. In fact, using the spectral decomposing as in (7.5) and decomposing the sum into positive and negative energies we may again estimate

$$\langle -|e^{\hat{H}_\lambda}|\rangle = \sum_{E \in (\hat{H}_\lambda)} e^{12} |\langle -|\psi_E\rangle|^2 + 1.$$ $\langle -|e^{\hat{H}_\lambda}|\rangle \leq \sum_{E \in (\hat{H}_\lambda)} e^{12} |\langle -|\psi_E\rangle|^2 + 1.$

The first sum is estimated trivially by $e^{\hat{E}_0}$. In case $\lambda < \beta_c^{-1}$ this yields the upper bound $\limsup_{N \to \infty} (1/Nt) \ln (-|e^{\hat{H}_\lambda}|) \leq 0$, which coincides with the lower bound. In case $\lambda > \beta_c^{-1}$, we know from [45] that eigenvalues with energies $E > 0$ are in one-to-one correspondence with values $U(\sigma_E) = E + \mathcal{O}(1)$. We conjecture that the local density of states at these energies satisfies

$$\lim_{N \to \infty} N^{-1} \ln (-|1_{(E - \delta_N E, E + \delta_N)}(\hat{H}_\lambda)|) = -\ln 2$$

for all sufficiently small $\delta > 0$. This would prove that (8.1) is indeed sharp.

Data accessibility. Research data are available from the Nottingham Research Data Management Repository at http://doi.org/10.17639/nott.7196 [46].

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