Linearization for difference equations with infinite delay

Lokesh Singh

University of Rijeka, Croatia.
lokesh.singh@uniri.hr

Abstract. In this article, we construct a conjugacy map for a linear difference equation with infinite delay and corresponding nonlinear perturbation. We also prove that the conjugacy map is one one with some additional conditions. As an application of our result, we show that the cases of (uniform) exponential dichotomy follow from our result.

Keywords: Delay Difference equations, Hartman-Grobman theorem, Linearization.

1 Introduction

The classical Hartman and Grobman theorem \cite{5,6} is a fundamental result in the local theory of differential equations and dynamical systems. This celebrated theorem provides a topological conjugacy (near a hyperbolic equilibrium point $x^* = 0$) between the dynamics of a nonlinear differential equation $x' = Ax + f(x)$ (on a finite-dimensional space) and the dynamics corresponding to the linear equation $x' = Ax$. Later, this result was generalized by Pugh \cite{3} and Palis \cite{4} to the Banach space setting. Some other improvements of the theorem are due to Reinfelds \cite{7}, and Bates and Lu \cite{8}. It is well known that in general, the conjugacy in the Grobman-Hartman theorem is only locally Hölder continuous. On the other hand, there has been a significant amount of work concerned with formulating sufficient conditions under which the conjugacy exhibits higher regularity properties. Some of the early work in this direction is by Sternberg \cite{11} and Belitskii \cite{9,10}, while Rodrigues et. al. \cite{13}, ElBialy \cite{12} and Zhang along with Lu \cite{29,30,31} contributed recently.

Palmer \cite{14} proved the first version of the Grobman-Hartman theorem for nonautonomous differential equations by assuming that the linear part admits an exponential dichotomy (see Subsection (3.1) for definition). Later Shi and Xiong \cite{17} obtained an improvement on Palmer’s result and established Hölder continuity of conjugacies. The version of Palmer’s theorem in the case of discrete time was first established by Aulbach and Wanner \cite{15}. Castañeda and Robledo \cite{28} discussed the regularity of conjugacy map for nonautonomous differential equations. More recently, Barreira and Valls \cite{16} dealt with the case when the linear part admits a nonuniform exponential dichotomy. More recently,
Dragičević, Zhang and Zhang [18, 19] discussed the higher regularity of conjugacies in the nonautonomous setting. Some other important contributions to nonautonomous linearization are given in [20, 21, 27].

For delay differential equations, due to several complexities, there has not been much progress. As Sternberg mentioned in [26], in the case of delay differential equations, the solutions form semiflows (instead of flows) and sometimes solutions may not exist in backward time. Later, he obtained Hartman-Grobman theorem [25] for finite delay differential equations in the finite-dimensional setting under some restrictive conditions. Namely, he assumed the existence of a compact global attractor on which the semiflow is one-to-one. Benkhalti and Ezzinbi [24] obtained some improvement over Sternberg’s result. Farkas [22], proved the Hartman-Grobman theorem for autonomous delay differential equations admitting uniform exponential dichotomy in finite-dimensional settings. In recent work, Barreira and Valls [23] extended the result in continuous case of nonautonomous differential equations with finite delay in Banach space setting. They also assumed that the linear part admits a uniform exponential dichotomy.

In this work, our objective is to obtain a conjugacy map for nonautonomous discrete equations with infinite delay and corresponding nonlinear delay equations given in (2.1) and (2.6) respectively. To obtain the conjugacy map, we assume a sufficient condition in terms of a Green type function (given in (2.4)) associated with linear delay equation. With some additional assumptions, we also proved that the obtained conjugacy map is one-one. As an applications of our result, we showed that if the linear delay equation admits (uniform) exponential dichotomy, then the corresponding Green type function satisfies the assumptions of our Theorem 2 and the result follows. With respect to existing work, our work has novelty in two senses, we are considering infinite delay difference equation and we proved the result with more general conditions.

This article is organized as follows. We describe our setup in Section 2. In Section 3, we prove our main result and give an application of Theorem 2.

2 Preliminaries

Let $\mathbb{Z}, \mathbb{Z}^+$ and $\mathbb{Z}^-$ denote the set of all integers, set of nonnegative integers and set of nonpositive integers, respectively. Let $(X, \| \cdot \|)$ be an arbitrary Banach space. Given a sequence $x : \mathbb{Z} \to X$ and $m \in \mathbb{Z}$, we define $x_m : \mathbb{Z}^{-} \to X$ by

$$x_m(j) = x(m + j) \quad \text{for all } j \in \mathbb{Z}^-.$$ 

Next we consider a Banach space $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ of all sequences $\phi : \mathbb{Z}^- \to X$ satisfying the following assumption:

(A) There exists $J > 0$, and $K, M : \mathbb{Z}^+ \to [0, \infty)$, such that if $x : \mathbb{Z} \to X$ with $x_0 \in \mathcal{B}$, then for all $n \in \mathbb{Z}^+$,

$$x_n \in \mathcal{B} \quad \text{and} \quad J|x(n)| \leq \|x_n\|_{\mathcal{B}} \leq K(n) \sup_{0 \leq j \leq n} |x(j)| + M(n)\|x_0\|_{\mathcal{B}}.$$
As noted in [1], one such space satisfying the assumption (A) is Banach space \( (\mathcal{B}^\beta, \| \cdot \|_{\mathcal{B}^\beta}) \), defined by
\[
\mathcal{B}^\beta = \left\{ \phi : \mathbb{Z}^- \to X \left| \| \phi \|_{\mathcal{B}^\beta} < \infty \right. \right\}, \quad \| \phi \|_{\mathcal{B}^\beta} := \sup_{j \in \mathbb{Z}^-} |\phi(j)|e^{\beta j},
\]
where \( \beta \) is a real constant. Observe that the Banach Space \( (\mathcal{B}^\beta, \| \cdot \|_{\mathcal{B}^\beta}) \) satisfies (A) with \( J = 1 \), \( M(n) = e^{-\beta n} \) and \( K(n) = 1 \) if \( \beta \geq 0 \) and \( K(n) = e^{-\beta n} \) if \( \beta < 0 \).

Given a sequence of linear operators \( A_m : \mathcal{B} \to X, m \in \mathbb{Z}^+ \), we consider a linear difference equation given by,
\[
x(m + 1) = A_m x_m, \quad \text{for all } m \in \mathbb{Z}^+.
\]
(2.1)

Given \((n, \phi) \in \mathbb{Z}^+ \times \mathcal{B}\), there exists a unique sequence \( x : \mathbb{Z} \to X \) such that \( x_n = \phi \) and the sequence \( x(m) \) satisfies equation (2.1) for all \( m \geq n \geq 0 \). The sequence \( x \) is called a solution of equation (2.1) through \((n, \phi)\) and is denoted by \( x = x(\cdot; n, \phi) \).

Now we define a two parameter solution operator \( A(m, n) : \mathcal{B} \to \mathcal{B} \) of the linear equation (2.1) by,
\[
A(m, n) \phi = x_m(\cdot; n, \phi) \quad \text{for all } m \geq n \geq 0, \quad \phi \in \mathcal{B}.
\]
(2.2)

Clearly, \((A(m, n))_{m \geq n \geq 0}\) is a discrete evolution family corresponding to linear equation (2.1) and it satisfies
\[
A(n, n) = \text{Id}_B \quad \text{for all } n \geq 0,
\]
\[
A(m, k)A(k, n) = A(m, n) \quad \text{for all } m \geq k \geq n \geq 0.
\]

Here \( \text{Id}_B \) denotes the identity operator on \( \mathcal{B} \). Next we consider a sequence of projection maps \((P_n)_{n \in \mathbb{Z}^+}\) on \( \mathcal{B} \) such that
\[
P_m A(m, n) = A(m, n) P_n, \quad \text{for all } m \geq n \geq 0,
\]
(2.3)

and
\[
A(m, n) |_{\text{Ker} P_n} : \text{Ker} P_n \to \text{Ker} P_m \quad \text{is invertible for } m \geq n \geq 0.
\]

Also, for \( n \geq m \geq 0 \), we denote \( A(m, n) := (A(m, n) |_{\text{Ker} P_m})^{-1} : \text{Ker} P_n \to \text{Ker} P_m \) and \( Q_m := \text{Id}_B - P_m \) for each \( m \in \mathbb{Z}^+ \). Now, for \( m, n \in \mathbb{Z}^+ \), we define another operator
\[
G(m, n) := \begin{cases} 
A(m, n) P_n & \text{for } m \geq n; \\
-A(m, n) Q_n & \text{for } m < n.
\end{cases}
\]
(2.4)

This operator is usually called Green operator. With respect to projection maps \((P_n)_{n \in \mathbb{Z}^+}\), we have \( \mathcal{B} = E_n \oplus F_n \) for each \( n \in \mathbb{Z}^+ \). Here \( E_n \) and \( F_n \) are ranges of projections \( P_n \) and \( Q_n \) respectively. Now, let us consider a space \( \mathcal{M} \) consisting of continuous functions,
\[
\eta : \{(n, \phi) : n \in \mathbb{Z}^+, \phi \in F_n\} \to \mathcal{B}
\]
such that

$$\|\eta\|_\infty := \sup\{\|\eta(n, \phi)\|_B : n \in \mathbb{Z}^+ \text{ and } \phi \in F_n\} < \infty.$$ 

Clearly, \((M, \| \cdot \|_\infty)\) is a Banach space. We also write

$$\eta^n = \eta(n, \cdot) \quad \text{and} \quad h^n = Id_{F_n} + \eta^n.$$ 

Here \(Id_{F_n}\) is identity map on subspace \(F_n\). Furthermore, for each \(m \in \mathbb{Z}^+\), let \(f_m : B \to X\) be a sequence of maps such that \(f_m(0) = 0\) for all \(m \in \mathbb{Z}^+\) and there exist numbers \(c_m > 0\) satisfying

$$|f_m(\phi) - f_m(\psi)| \leq c_m \min (1, \|\phi - \psi\|_B), \quad (2.5)$$

for every \(m \in \mathbb{Z}^+\) and \(\phi, \psi \in B\). Finally, we consider a \emph{semilinear difference equation} given by

$$x(m + 1) = A_m x_m + f_m(x_m) \quad \text{for all } m \in \mathbb{Z}^+. \quad (2.6)$$

Given \((n, \phi) \in \mathbb{Z}^+ \times B\), there exists a unique sequence \(x : \mathbb{Z} \to X\) such that \(x_n = \phi\) and \(x\) satisfies the semilinear difference equation \((2.6)\). We write the solution of equation \((2.6)\) in terms of operator \(R(m, n)\) given by,

$$x_m = R(m, n) x_n \quad \text{for all } m \geq n \geq 0. \quad (2.7)$$

Now we recall \([1]\) and \([2]\) to give the variation of constants formula for a difference equation given by

$$x(m + 1) = A_m x_m + p_m, \quad (2.8)$$

where \((p_m)_{m \in \mathbb{Z}^+}\) is a sequence in \(X\). Define \(\Gamma : \mathbb{Z}^+ \to L(X)\) by

$$\Gamma(j) = \begin{cases} 
Id_X & \text{for } j = 0; \\
0_X & \text{for } j < 0.
\end{cases}$$

The symbol \(L(X)\) denotes the space of bounded linear operators in \(X\). For \(v \in X\), define \(\Gamma v : \mathbb{Z}^+ \to X\) by

$$\langle (\Gamma v)(j) \rangle := \Gamma(j) v = \begin{cases} 
v & \text{for } j = 0; \\
0 & \text{for } j < 0.
\end{cases}$$

If \(x : \mathbb{Z} \to X\) is defined by \(x(j) = 0\) for \(j \leq 0\) and \(x(j) = v\) for \(j > 0\), then \(x_0 = 0\) and \(x_1 = \Gamma v\). Since \(x_0 = 0 \in B\), by assumption (A), we have that \(\Gamma v = x_1 \in B\) and

$$\|\Gamma v\|_B \leq K(1)|v|. \quad (2.9)$$
Theorem 1. [1] Let $\phi \in B$. A sequence $x: \mathbb{Z} \to X$ is a solution of (2.8) with initial value $x_0 = \phi$ if and only if for $m \geq 0$, the segment $x_m$ satisfy the following relation in $B$,

$$x_m = A(m, 0)\phi + \sum_{k=0}^{m-1} A(m, k + 1)(\Gamma p_k), \quad m \geq 0. \quad (2.10)$$

As a corollary of above result, we have that $x: \mathbb{Z} \to X$ is a solution of (2.6) if and only if $\phi = x_0 \in B$ and

$$x_m = A(m, 0)\phi + \sum_{k=0}^{m-1} A(m, k + 1)(\Gamma f_k(x_k)), \quad m \geq 0.$$ 

Let $x = x(\cdot; 0, \phi)$ be a solution of equation (2.6) for $\phi \in B$, then using constants of variation formula from Theorem 1, for $m \geq n \geq 0$, we have

$$x_m = A(m, 0)\phi + \sum_{k=0}^{m-1} A(m, k + 1)(\Gamma f_k(x_k))$$

$$= A(m, 0)\phi + \sum_{k=0}^{n-1} A(m, k + 1)(\Gamma f_k(x_k)) + \sum_{k=n}^{m-1} A(m, k + 1)(\Gamma f_k(x_k))$$

$$= A(m, n)\{A(n, 0)\phi + \sum_{k=0}^{n-1} A(n, k + 1)(\Gamma f_k(x_k))\}$$

$$+ \sum_{k=n}^{m-1} A(m, k + 1)(\Gamma f_k(x_k))$$

$$= A(m, n)x_n + \sum_{k=n}^{m-1} A(m, k + 1)(\Gamma f_k(x_k)),$$

for all $m \geq n \geq 0$. Therefore, we obtain that $x: \mathbb{Z} \to X$ is a solution of (2.6) if and only if $x_0 \in B$ and for $m \geq n \geq 0$

$$x_m = A(m, n)x_n + \sum_{k=n}^{m-1} A(m, k + 1)(\Gamma f_k(x_k)). \quad (2.11)$$

3 Main Result

The following theorem is our main result.

Theorem 2. Assume that the semilinear equation (2.6) admits

$$q := \sup_{m \in \mathbb{Z}^+} \left( \sum_{n \in \mathbb{Z}^+} c_n\|G(m, n + 1)\| \right), \quad \text{with } K(1)q < 1. \quad (3.1)$$
Then there exists a function $\eta \in M$, such that for every $m \geq n \geq 0$

$$h^m \circ A(m, n) = R(m, n) \circ h^n, \quad \text{on } F_n.$$  \hfill (3.2)

Moreover, each map $h^n$ is one-to-one provided that the condition (3.1) holds with a constant sequence of Lipschitz constants $(c_n)_{n \in \mathbb{Z}^+}$.

First we prove a lemma which will be used to establish that $h^n$ is one-one for $n \in \mathbb{Z}^+$. Set

$$a_{m,n} := \|G(m, n)\|. \quad \hfill (3.3)$$

**Lemma 1.** Assume that the sequence of Lipschitz constants $(c_n)_{n \in \mathbb{Z}^+}$ in (2.5) is constant and that (3.1) holds. Then, for each fixed $m \in \mathbb{Z}^+$, we have that

$$\lim_{n \to \infty} a_{m,n} = 0.$$  \hfill (3.4)

**Proof.** Let $C > 0$ be such that $c_n = C$ for all $n \in \mathbb{Z}^+$. By (3.1), it follows that

$$\sup_{m \in \mathbb{Z}^+} \left( \sum_{n \in \mathbb{Z}^+} c_n \|G(n, n + 1)\| \right) = C \sup_{m \in \mathbb{Z}^+} \left( \sum_{n \in \mathbb{Z}^+} a_{m,n+1} \right) < \infty.$$  

Therefore, for each fixed $m \in \mathbb{Z}^+$,

$$\sum_{n \in \mathbb{Z}^+} a_{m,n+1} < \infty.$$  

Hence, $\lim_{n \to \infty} a_{m,n+1} = 0$ for each fixed $m \in \mathbb{Z}^+$. Equivalently,

$$\lim_{n \to \infty} a_{m,n} = 0 \quad \text{for each fixed } m \in \mathbb{Z}^+. \quad \hfill (3.4)$$

Now we give our proof of Theorem 2.

**Proof.** Let us consider a map $F : M \to M$ given by

$$F(\eta)(n, \phi) = \sum_{m \in \mathbb{Z}^+} G(n, m + 1) f_m(A(m, n) \phi + \eta^m(A(m, n) \phi)), \quad \hfill (3.5)$$

where $\eta \in M$ and $(n, \phi) \in \mathbb{Z}^+ \times F_n$. Furthermore, since $f_m(0) = 0$ for each $m \in \mathbb{Z}^+$ and using (2.5), we have,

$$|f_m(\phi)| \leq c_m \min(1, \|\phi\|_B) \leq c_m, \quad \text{for all } \phi \in B.$$  

Therefore, using above estimate,

$$\|F(\eta)\|_\infty \leq \sup_{n \in \mathbb{Z}^+} \left( K(1) \sum_{m \in \mathbb{Z}^+} c_m \|G(n, m + 1)\| \right) = K(1)q < \infty.$$
Therefore, the operator $F$ is well defined. We now claim that $F$ is a contraction map. For each $\eta, \xi \in M$ we have
\[
|F(\eta)(n, \phi) - F(\xi)(n, \phi)| \\
\leq K(1) \sum_{m \in \mathbb{Z}^+} \|\mathcal{G}(n, m + 1)\| \left| f_m(A(m, n)\phi + \eta^m(A(m, n)\phi)) - f_m(A(m, n)\phi + \xi^m(A(m, n)\phi)) \right| \\
\leq K(1) \sum_{m \in \mathbb{Z}^+} \|\mathcal{G}(n, m + 1)\| c_m |\eta^m(A(m, n)\phi) - \xi^m(A(m, n)\phi)| \\
\leq K(1) \sum_{m \in \mathbb{Z}^+} c_m \|\mathcal{G}(n, m + 1)\| \|\eta - \xi\|_{\infty}.
\]
Therefore,
\[
\|F(\eta) - F(\xi)\|_{\infty} \leq K(1)q \|\eta - \xi\|_{\infty}.
\]
As $K(1)q < 1$, the operator $F$ is a contraction map. Therefore it has a unique fixed point function, say $\eta$. i.e. $F(\eta) = \eta$. Now, for $\phi \in F_n$, we have
\[
\eta^n(\phi) = \sum_{m \in \mathbb{Z}^+} \mathcal{G}(n, m + 1) \Gamma f_m(A(m, n)\phi + \eta^m(A(m, n)\phi)) \\
= \sum_{m \in \mathbb{Z}^+} \mathcal{G}(n, m + 1) \Gamma f_m(h^m(A(m, n)\phi)). \quad (3.6)
\]
Note that,
\[
Q_n \eta^n(\phi) = - \sum_{m \geq n} A(n, m + 1) Q_{m+1} \Gamma f_m(h^m(A(m, n)\phi)). \quad (3.7)
\]
Take $0 \leq p \leq n$. By (3.7), for $F_p \ni \psi = A(p, n)\phi$ we have that
\[
Q_p \eta^p(\psi) = - \sum_{m \geq p} A(p, m + 1) Q_{m+1} \Gamma f_m(h^m(A(m, p)\psi)). \quad (3.8)
\]
Also from (3.7),
\[
A(p, n)Q_n \eta^n(\phi) = - \sum_{m \geq n} A(p, m + 1) Q_{m+1} \Gamma f_m(h^m(A(m, n)\phi)).
\]
Using (3.8) and the above relation, we obtain that
\[
A(p, n)Q_n \eta^n(\phi) = Q_p \eta^p(A(p, n)\phi) \\
+ \sum_{m=p}^{n-1} A(p, m + 1) Q_{m+1} \Gamma f_m(h^m(A(m, n)\phi)).
\]
Equivalently,
\[ Q_n \eta^n(\phi) = A(n, p)Q_p \eta^p(A(p, n)\phi) \]
\[ + \sum_{m=p}^{n-1} A(n, m + 1)Q_{m+1} \Gamma f_m \left(h^m(A(m, n)\phi)\right). \] (3.9)

Similarly, from (3.6), it follows that
\[ P_n \eta^n(\phi) = \sum_{m<n} A(n, m + 1)P_{m+1} \Gamma f_m \left(h^m(A(m, n)\phi)\right). \] (3.10)

For \( F_p \ni \psi = A(p, n)\phi \), we have that
\[ P_p \eta^p(\psi) = \sum_{m<p} A(p, m + 1)P_{m+1} \Gamma f_m \left(h^m(A(m, p)\psi)\right). \] (3.11)

Hence,
\[ P_n \eta^n(\phi) = A(n, p)P_p \eta^p(A(p, n)\phi) \]
\[ + \sum_{m=p}^{n-1} A(n, m + 1)P_{m+1} \Gamma f_m \left(h^m(A(m, n)\phi)\right). \] (3.12)

By adding (3.9) and (3.12), we obtain that
\[ \eta^n(\phi) = A(n, p)\eta^p(A(p, n)\phi) + \sum_{m=p}^{n-1} A(n, m + 1)\Gamma f_m \left(h^m(A(m, n)\phi)\right). \]

Since \( h^n = Id_{F_n} + \eta^n \), we conclude that
\[ h^n(\phi) = A(n, p)(h^p(A(p, n)\phi)) + \sum_{m=p}^{n-1} A(n, m + 1)\Gamma f_m \left(h^m(A(m, n)\phi)\right). \]

Moreover,
\[ h^n(A(n, p)\psi) = A(n, p)h^p(\psi) + \sum_{m=p}^{n-1} A(n, m + 1)\Gamma f_m \left(h^m(A(m, p)\psi)\right) \] (3.13)

On the other hand, by the variation of constants formula in (2.11) and the definition of \( R(m, n) \) in (2.7), for all \( n \geq p \geq 0 \), we have,
\[ R(n, p)h^p(\psi) = A(n, p)h^p(\psi) + \sum_{m=p}^{n-1} A(n, m + 1)\Gamma f_m \left(R(m, p)h^p(\psi)\right). \]

Comparing above relation with (3.13), and using strong induction on \( n \), we obtain,
\[ R(n, p)h^p(\psi) = h^n(A(n, p)\psi), \quad \text{for all } n \geq p \geq 0 \text{ and } \psi \in F_p. \] (3.14)
Hence, we proved that (3.2) holds.

Now we show that the map \( h^p \) is one-to-one for \( p \in \mathbb{Z}^+ \). Observe that
\[
\| G(p,n) \| = \| A(p, n)Q_n \| = a_{p,n} \quad \text{for all } n > p \geq 0.
\]

Now let \( n > p \geq 0 \) and \( \phi_1, \phi_2 \in F_p \) such that
\[
h^p(\phi_1) = h^p(\phi_2).
\]

Using above relation and (3.14), we have,
\[
h^n(A(n,p)\phi_1) = h^n(A(n,p)\phi_2).
\]

Thus,
\[
A(n,p)(\phi_1 - \phi_2) = -(\eta^n(A(n,p)\phi_1) - \eta^n(A(n,p)\phi_2)). \tag{3.15}
\]

Note that, as \( \phi_1, \phi_2 \in F_p \),
\[
\phi_1 - \phi_2 = A(p, n)A(n, p)Q_p(\phi_1 - \phi_2) = A(p, n)Q_nA(n, p)(\phi_1 - \phi_2).
\]

Also, \( A(p, n)Q_n \) is a nonzero map in \( B \), as \( A(p, n)|_{F_n} \) is invertible. Therefore, we have,
\[
\| A(n, p)(\phi_1 - \phi_2)\|_B \geq \frac{\| \phi_1 - \phi_2\|_B}{\| A(p, n)Q_n\|} \geq \frac{\| \phi_1 - \phi_2\|_B}{a_{p,n}}.
\]

If \( \phi_1 \neq \phi_2 \), then it follows from (3.4) that the function \( n \to A(n,p)(\phi_1 - \phi_2) \) is unbounded. However, the right hand side of (3.15), is bounded by \( 2\| \eta \|_{\infty} \), which leads to a contradiction. Therefore \( \phi_1 = \phi_2 \) and hence \( h^p \) is one-one for all \( p \in \mathbb{Z}^+ \).

This completes the proof of our Theorem 2.

3.1 Uniform Exponential Dichotomy Case:

Let us assume that (2.1) admits uniform exponential dichotomy, i.e. for each \( n \in \mathbb{Z}^+ \), there exists projection \( P_n \) such that:

1. \( A(m,n)P_n = P_mA(m,n) \) for all \( m \geq n \geq 0 \);
2. for \( m \geq n \), \( A(m,n)\big|_{\text{Ker} \ P_n} : \text{Ker} \ P_n \to \text{Ker} \ P_m \) is an invertible map and we denote the inverse map \( (A(m,n)\big|_{\text{Ker} \ P_n})^{-1} \) by \( A(n,m) \);
3. there exists \( D, \lambda > 0 \), such that
\[
\| A(m,n)P_n \| \leq De^{-\lambda(m-n)} \quad \text{for all } m \geq n;
\]

and
\[
\| A(m,n)(Id_B - P_n) \| \leq De^{-\lambda(n-m)} \quad \text{for all } m < n.
\]
In particular, we have that,
\[
\| G(m, n) \| \leq D e^{-\lambda|m-n|} \quad \text{for all } m, n \in \mathbb{Z}^+.
\]

By comparing the above expression with (3.3), we have that \( a_{m,n} \leq D e^{-\lambda|m-n|} \).

Finally, let \( C > 0 \) be a constant such that \( c_m = C \) for all \( m \in \mathbb{Z}^+ \). Using above estimates, we have

\[
C \sup_{m \in \mathbb{Z}^+} \left( \sum_{n \in \mathbb{Z}^+} a_{m,n+1} \right) \leq C \sup_{m \in \mathbb{Z}^+} \left( \sum_{n \in \mathbb{Z}^+} D e^{-\lambda|m-n-1|} \right)
= CD \sup_{m \in \mathbb{Z}^+} \left( \frac{e^{-\lambda (1-e^{-\lambda(m-1)})}}{1-e^{-\lambda}} + \frac{1}{1-e^{-\lambda}} \right) < \infty.
\]

Using the constant \( C \), we can make sure that equation (3.1) is satisfied. Therefore, if the linear delay difference equation (2.1) satisfies uniform exponential dichotomy then our Theorem 2 is applicable.

Acknowledgements Author would like to thanks Prof. D. Dragičević for his valuable suggestions throughout the process of solving and writing this article. The Author is supported by Croatian Science Foundation under the project IP-2019-04-1239.

References

1. Murakami, S.: Representation of solutions of linear functional difference equations in phase space. Nonlinear Anal. 30, pp. 1153-1164 (1997).
2. Dragičević, D., and Pituk, M.: Shadowing for nonautonomous difference equations with infinite delay. Applied Mathematics Letters, 120, 107284 (2021).
3. Pugh, C.: On a Theorem of P. Hartman. American Journal of Mathematics, 91(2), pp. 363–367 (1969).
4. Palis J.,: On the local structure of hyperbolic points in Banach spaces. An. Acad. Brasil. Cienc. 40, pp. 263–266 (1968).
5. Hartman, P.: On the local linearization of differential equations. Proc. Amer. Math. Soc., 14, pp. 568-573 (1963).
6. Grobman, D.: Homeomorphism of systems of differential equations. Dokl. Akad. Nauk SSSR, 128, pp. 880-881 (1959).
7. Reinifelds, A.: A generalized theorem of Groban and Hartman. Latv. Mat. Ezhegodnik, 29, pp. 84-88 (1985).
8. Bates, W., Lu, K.: A Hartman-Grobman theorem for the Cahn-Hilliard and phase-field equations. J. Dynam. Differential Equations, 6(1), pp. 101-145 (1994).
9. Belitskii, G.R.: Functional equations and the conjugacy of diffeomorphism of finite smoothness class. Funct. Anal. Appl. 7, pp. 268–277 (1973).
10. Belitskii, G.R.: Equivalence and normal forms of germs of smooth mappings. Russian Math. Surv. 33, pp. 107–177 (1978).
11. Sternbger, S.: Local contractions and a theorem of Poincaré. Am. J. Math. 79, pp. 809–824 (1957).
12. ElBialy, M.S.: Smooth conjugacy and linearization near resonant fixed points in Hilbert spaces. Houston J. Math. 40, pp. 467–509 (2014).
13. Rodrigues, H.M., Solà-Morales, J.: Invertible contractions and asymptotically stable ODEs that are not C1-linearizable. J. Dyn. Differ. Equ. 18, pp. 961–974 (2006).
14. Palmer, K.: A generalization of Hartman’s linearization theorem. J. Math. Anal. Appl. 41, pp. 753–758 (1973).
15. Aulbach, B., Wanner, T.: Topological simplification of nonautonomous difference equations. J. Differ. Equ. Appl. 12, pp. 283–296 (2006).
16. Barreira L., Valls C.: A Grobman–Hartman theorem for nonuniformly hyperbolic dynamics. J. Math. Anal. Appl. 228, pp. 285–310 (2006).
17. Shi, J.L., Xiong, K.Q.: On Hartman’s linearization theorem and Palmer’s linearization theorem. J. Math. Anal. Appl. 192, pp. 813–832 (1995).
18. Dragićević, D., Zhang, W., Zhang, W.: Smooth linearization of nonautonomous difference equations with a nonuniform dichotomy. Math. Z. 292, pp. 1175–1193 (2019).
19. Dragićević, D., Zhang, W., Zhang, W.: Smooth linearization of nonautonomous differential equations with a nonuniform dichotomy. Proc. Lond. Math. Soc. 121, pp. 32–50 (2020).
20. Backes, L., Dragićević, D.: A generalized Groban–Hartman theorem for nonautonomous dynamics. Collect. Math. (2021).
21. Barreira, L., Dragićević, D., Valls C.: Existence of conjugacies and stable manifold theorem via suspensions. Elec. J. of Diff. Equa. Vol. 2017, No. 172, pp. 1–11 (2017).
22. Farkas G.: A Hartman-Grobman result for retarded functional differential equations with an application to the numerics around hyperbolic equilibria, Z. Angew. Math. Phys. 52, pp. 421-432 (2001).
23. Barreira, L., Valls, C.: Perturbations of delay equations. J. Differ. Equ. 269, pp. 7015–7041 (2020).
24. Benkhalti, R., Ezzinbi, K.: A Hartman and Grobman theorem for some partial functional differential equations, Int. J. of Bif. and Cha., Vol. 10, No. 5, pp. 1165–1169 (2000).
25. Sternberg, N.: A Hartman-Grobman Theorem for a Class of Retarded Functional Differential Equations, J. Math. Anal. Appl. 176, pp. 156-165 (1993).
26. Sternberg, N.: A Hartman-Grobman theorem for maps, in: Ordinary and Delay Differential Equations, Edinburg, TX, 1991, in: Pitman Res. Notes Math. Ser., vol. 272, Longman Sci. Tech., Harlow, pp. 223-227 (1992).
27. Dragićević, D.: Global smooth linearization of nonautonomous contractions on Banach spaces, Electron. J. Qual. Theory Differ. Equ., Paper No. 12, pp. 1-19 (2022).
28. A. Castañeda, G. Robledo.: Differentiability of Palmer’s linearization theorem and converse result for density function, J. Differential Equations 259, pp. 4634–4650 (2015).
29. W. M. Zhang, W. N. Zhang.: C1 linearization for planar contractions, J. Funct. Anal. 260, No. 7, 2043–2063 (2011).
30. W. M. Zhang, W. N. Zhang, Sharpness for C1 linearization of planar hyperbolic diffeomorphisms, J. Differential Equations 257, pp. 4470–4502 (2014).
31. W. M. Zhang, K. Lu, W. N. Zhang.: Differentiability of the conjugacy in the Hartman-Grobman Theorem, Trans. Amer. Math. Soc. 369, 4995-5030 (2017).