A new Kim’s type Bernoulli and Euler Numbers and related identities and zeta and \(L\)-functions

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Abstract. In this paper, by using \(q\)-deformed bosonic \(p\)-adic integral, we give \(\lambda\)-Bernoulli numbers and polynomials, we prove Witt’s type formula of \(\lambda\)-Bernoulli polynomials and Gauss multiplicative formula for \(\lambda\)-Bernoulli polynomials. By using derivative operator to the generating functions of \(\lambda\)-Bernoulli polynomials and generalized \(\lambda\)-Bernoulli numbers, we give Hurwitz type \(\lambda\)-zeta functions and Dirichlet’s type \(\lambda\)-\(L\)-functions; which are interpolated \(\lambda\)-Bernoulli polynomials and generalized \(\lambda\)-Bernoulli numbers, respectively. We give generating function of \(\lambda\)-Bernoulli numbers with order \(r\). By using Mellin transforms to their function, we prove relations between multiply zeta function and \(\lambda\)-Bernoulli polynomials and ordinary Bernoulli numbers of order \(r\) and \(\lambda\)-Bernoulli numbers, respectively. We also study on \(\lambda\)-Bernoulli numbers and polynomials in the space of locally constant. Moreover, we define \(\lambda\)-partial zeta function and interpolation function.

§0. Introduction, definitions and notations

Throughout this paper, \(\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p\) and \(\mathbb{C}_p\) will be denoted by the ring of rational integers, the ring of \(p\)-adic integers, the field of \(p\)-adic rational numbers and the completion of the algebraic closure of \(\mathbb{Q}_p\), respectively. Let \(\nu_p\) be the normalized exponential valuation of \(\mathbb{C}_p\) with \(|p|_p = p^{-\nu_p(p)} = \frac{1}{p}\), cf. [2,3,4,5,6,7,8,9,16,17, 21,27].

When one talks of \(q\)-extension, \(q\) considered in many ways such as an indeterminate, a complex number \(q \in \mathbb{C}\), as \(p\)-adic number \(q \in \mathbb{C}_p\). If \(q \in \mathbb{C}\) one normally assumes that \(|q| < 1\). If \(q \in \mathbb{C}_p\), we normally assume that \(|q - 1|_p < p^{-\nu(p)}\) so that \(q^x = \exp(x \log q)\) for \(|x|_p \leq 1\). We use the following notations:

\[
[x] = [x : q] = \frac{1 - q^x}{1 - q}, \quad \text{cf [3, 4, 5, 6, 8, 9, 24, 25, 27]}. 
\]

Observe that when \(\lim_{q \to 1}[x] = x\), for any \(x\) with \(|x|_p \leq 1\) in the present \(p\)-adic case

\[
[x : a] = \frac{1 - a^x}{1 - a}. 
\]

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Let \( d \) be a fixed integer and let \( p \) be a fixed prime number. For any positive integer \( N \), we set

\[
X = \lim_{N \to \infty} \left( \mathbb{Z}/dp^N \mathbb{Z} \right),
\]

\[
X^* = \bigcup_{0 < a < dp} \left\{ (a + dp \mathbb{Z}) \right\}
\]

\[
a + dp^N \mathbb{Z}_p = \left\{ x \in X | x \equiv a \pmod{dp^n} \right\},
\]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \). We assume that \( u \in \mathbb{C}_p \) with \( |1 - u|_p \geq 1 \). cf. [3,4,5,6,7,8,24, 27].

For \( x \in \mathbb{Z}_p \), we say that \( g \) is a uniformly differentiable function at point \( a \in \mathbb{Z}_p \), and write \( g \in \text{UD} (\mathbb{Z}_p) \), the set of uniformly differentiable functions, if the difference quotients,

\[
F_g(x,y) = \frac{g(y) - g(x)}{y - x},
\]

have a limit \( l = g'(a) \) as \((x,y) \to (a,a)\). For \( f \in \text{UD}(\mathbb{Z}_p) \), the \( q \)-deformed bosonic \( p \)-adic integral was defined as

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x)
\]

\[
= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \mu_q(x + p^N \mathbb{Z}_p) \quad (A)
\]

\[
= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \frac{q^x}{[p^N]}, \; [4,5,9].
\]

By Eq-(A), we have

\[
\lim_{q \to -q} I_q(f) = I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x).
\]

This integral, \( I_{-q}(f) \), give the \( q \)-deformed integral expression of fermionic. The classical Euler numbers were defined by means of the following generating function:

\[
\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}, \; |t| < 2\pi. \; [6, 7, 19, 21].
\]

Let \( u \) be algebraic in complex number field. Then Frobenius-Euler polynomials \([6,7,19,21]\) were defined by

\[
\frac{1 - u}{e^t - u} e^{xt} = e^{H(u,x)t} = \sum_{m=0}^{\infty} H_m(u, x) \frac{t^m}{m!}, \quad (A1)
\]
where we use technical method’s notation by replacing $H^m(u, x)$ by $H_m(u, x)$ symbolically. In case $x = 0$, $H_m(u, 0) = H_m(u)$, which is called Frobenius-Euler number. The Frobenius-Euler polynomials of order $r$, denoted by $H_n^{(r)}(u, x)$, were defined by

$$
\left( \frac{1 - u}{e^t - u} \right)^r e^{tx} = \sum_{n=0}^{\infty} H_n^{(r)}(u, x) \frac{t^n}{n!}
$$

cf. [7, 10, 26, 27].

The values at $x = 0$ are called Frobenius-Euler numbers of order $r$. When $r = 1$, these numbers and polynomials are reduced to ordinary Frobenius-Euler numbers and polynomials. In the usual notation, the $n$-th Bernoulli polynomial were defined by means of the following generating function:

$$
\left( \frac{t}{e^t - 1} \right) e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
$$

For $x = 0$, $B_n(0) = B_n$ are said to be the $n$-th Bernoulli numbers. The Bernoulli polynomials of order $r$ were defined by

$$
\left( \frac{t}{e^t - 1} \right)^r e^{tx} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}
$$

and $B_n^{(r)}(0) = B_n^{(r)}$ are called the Bernoulli numbers or order $r$. Let $x, w_1, w_2, \ldots, w_r$ be complex numbers with positive real parts. When the generalized Bernoulli numbers and polynomials were defined by means of the following generating function:

$$
\frac{w_1 w_2 \cdots w_r t^r e^{xt}}{(e^{w_1 t} - 1)(e^{w_2 t} - 1) \cdots (e^{w_r t} - 1)} = \sum_{n=0}^{\infty} B_n^{(r)}(x \mid w_1, w_2, \ldots, w_r) \frac{t^n}{n!}
$$

and $B_n^{(r)}(0 \mid w_1, w_2, \ldots, w_r) = B_n^{(r)}(w_1, w_2, \ldots, w_r)$, cf. [13,15].

The Hurwitz zeta function is defined by

$$
\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(x + n)^s},
$$

$\zeta(s, 1) = \zeta(s)$, which is Riemann zeta function. The multiple zeta functions [12,27] were defined by

$$
\zeta_r(s) = \sum_{0 < n_1 < n_2 < \cdots < n_r} \frac{1}{(n_1 + \cdots + n_r)^s}.
$$

(C)

We summarize our paper as follows:
In section 1, by using \( q \)-deformed bosonic \( p \)-adic integral, generating function of \( \lambda \)-Bernoulli numbers and polynomials are given. We obtain many new identities related to these numbers and polynomials. We proved Gauss multiplicative formula for \( \lambda \)-Bernoulli numbers. Witt's type formula of \( \lambda \)-Bernoulli polynomials is given.

In section 2, by using \( \left( \frac{d}{dt} \right)^{k} \bigg|_{t=0} \) derivative operator to the generating function of the \( \lambda \)-Bernoulli numbers, we define new relations and Hurwitz type \( \lambda \)-function, which interpolates \( \lambda \)-Bernoulli polynomials at negative integers.

In section 3, by using same method of section 2, we give Dirichlet type \( \lambda \)-L-function which interpolates generalized \( \lambda \)-Bernoulli numbers.

In section 4, generating function of \( \lambda \)-Bernoulli numbers of order \( r \) is defined, by using Cauchy residue theorem and Mellin transforms to this function, we proved relation between multiple zeta function and \( \lambda \)-Bernoulli numbers of order \( r \).

In section 5, we give some important identities related to generalized \( \lambda \)-Bernoulli numbers of order \( r \).

In section 6, we study on \( \lambda \)-Bernoulli numbers and polynomials in the space of locally constant. In this section, we also define \( \lambda \)-partial zeta function which interpolates \( \lambda \)-Bernoulli numbers at negative integers.

In section 7, we give \( p \)-adic interpolation functions.

### §1. \( \lambda \)-Bernoulli numbers

In this section, by using Eq-(A), we give integral equation of bosonic \( p \)-adic integral. By using this integral equation we define generating function of \( \lambda \)-Bernoulli polynomials. We give fundamental properties of the \( \lambda \)-Bernoulli numbers and polynomials. We also give some new identities related to \( \lambda \)-Bernoulli numbers and polynomials. We prove Gauss multiplicative formula for \( \lambda \)-Bernoulli numbers as well. Witt’s type formula of \( \lambda \)-Bernoulli polynomials is given.

To give the expression of bosonic \( p \)-adic integral in Eq-(A), we consider the limit

\[
I_{1}(f) = \lim_{q \to 1} I_{q}(f) = \int_{Z_{p}} f(x) d\mu_{1}(x). \tag{0}
\]

Bosonic \( p \)-adic integral on \( Z_{p} (= p \)-adic invariant integral on \( Z_{p} \))

\[
I_{1}(f_{1}) = I_{1}(f) + f'(0), \tag{1}
\]

where \( f_{1}(x) = f(x+1) \), integral equation for bosonic \( p \)-adic integral. Let \( C_{p^{n}} \) be the space of primitive \( p^{n} \)-th root of unity,

\[
C_{p^{n}} = \{ \zeta \mid \zeta^{p^{n}} = 1 \}.
\]
Then, we denote 
\[ T_p = \lim_{n \to \infty} C_{p^n} = \bigcup_{n \geq 0} C_{p^n}. \]

For \( \lambda \in \mathbb{Z}_p \), we take \( f(x) = \lambda^x e^{tx} \), and \( f_1(x) = e^t \lambda f(x) \). Thus we have
\[ f_1(x) - f(x) = (\lambda e^t - 1) f(x). \] (2)

By substituting (2) into (1), we get
\[ (\lambda e^t - 1) I_1(f) = f'(0). \] (2a)

Consequently, we have
\[ \log \lambda + t \frac{\lambda e^t - 1}{\lambda e^t - 1} := \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}. \] (3)

By using Eq-(3), we obtain
\[ \lambda(B(\lambda) + 1)^n - B_n(\lambda) = \begin{cases} 
\log \lambda, & \text{if } n = 0 \\
1, & \text{if } n = 1 \\
0, & \text{if } n > 1,
\end{cases} \]

with the usual convention of replacing \( B_n(\lambda) \) by \( B^n(\lambda) \). We give some \( B_n(\lambda) \) numbers as follows:
\[ B_0(\lambda) = \frac{\log \lambda}{\lambda - 1}, \quad B_1(\lambda) = \frac{\lambda - 1 - \lambda \log \lambda}{(\lambda - 1)^2}, \ldots. \]

We note that, if \( \lambda \in T_p \), for some \( n \in \mathbb{N} \), then Eq-(2a) is reduced to the following generating function:
\[ \frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}. \] (3a)

If \( \lambda = e^{2\pi i / f} \), \( f \in \mathbb{N} \) and \( \lambda \in \mathbb{C} \), then Eq-(3) is reduced to (3a). Eq-(3a) is obtained by Kim [3]. Let \( u \in \mathbb{C} \), then by substituting \( x = 0 \) into Eq-(A1), we set
\[ \frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}. \] (3b)

\( H_n(u) \) is denoted Frobenius-Euler numbers. Relation between \( H_n(u) \) and \( B_n(\lambda) \) is given by the following theorem:
Theorem 1. Let $\lambda \in \mathbb{Z}_p$. Then

$$B_n(\lambda) = \frac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \frac{n H_{n-1}(\lambda^{-1})}{\lambda - 1},$$

$$B_0(\lambda) = \frac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1}).$$

Proof. By using Eq-(3), we have

$$\sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!} = \frac{\log \lambda + t}{\lambda e^t - 1} = \frac{\log \lambda}{\lambda e^t - 1} + \frac{t}{\lambda e^t - 1} = \frac{1 - \lambda^{-1}}{(1 - \lambda^{-1}) \lambda} \left( \frac{\log \lambda}{e^t - \lambda^{-1}} \right) - \frac{(1 - \lambda^{-1})}{(e^t - \lambda^{-1})} \cdot t \quad \lambda(1 - \lambda^{-1})$$

$$= \frac{\log \lambda}{\lambda - 1} \sum_{n=0}^{\infty} H_n(\lambda^{-1}) \frac{t^n}{n!} + \frac{t}{\lambda - 1} \sum_{n=0}^{\infty} H_n(\lambda^{-1}) \frac{t^n}{n!},$$

the next to the last step being a consequence of Eq-(3b). After some elementary calculations, we have

$$\sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!} = \frac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1}) + \sum_{n=1}^{\infty} \left( \frac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \frac{n H_{n-1}(\lambda^{-1})}{\lambda - 1} \right) \frac{t^n}{n!}.$$

By comparing coefficient $\frac{t^n}{n!}$ in the above, then we obtain the desired result. □

Observe that, if $\lambda \in T_p$ in Eq-(4), then we have, $B_0(\lambda) = 0$ and $B_n(\lambda) = \frac{n H_{n-1}(\lambda^{-1})}{\lambda - 1}$, $n \geq 1$.

By Eq-(3) and Eq-(4), we obtain the following formula:

For $n \geq 0$, $\lambda \in \mathbb{Z}_p$

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = \begin{cases} \frac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1}), & n = 0 \\ \frac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \frac{n H_{n-1}(\lambda^{-1})}{\lambda - 1}, & n > 0 \end{cases}$$

and

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = B_n(\lambda), \quad n \geq 0.$$
Now, we define $\lambda$–Bernoulli polynomials, we use these polynomials to give the sums powers of consecutive. The $\lambda$-Bernoulli polynomials are defined by means of the following generating function:

$$\frac{\log \lambda + t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(\lambda; x) \frac{t^n}{n!}. \quad (5)$$

By Eq-(3) and Eq-(5), we have

$$B_n(\lambda; x) = \sum_{k=0}^{n} \binom{n}{k} B_k(\lambda) x^{n-k}.$$  

The Witt’s formula for $B_n(\lambda; x)$ is given by the following theorem:

**Theorem 2.** For $k \in \mathbb{N}$ and $\lambda \in \mathbb{Z}_p$, we have

$$B_n(\lambda; x) = \int_{\mathbb{Z}_p} (x + y)^n \lambda^y d\mu_1(y). \quad (6)$$

**Proof.** By substituting $f(y) = e^{t(x+y)} \lambda^y$ into Eq-(1), we have

$$\int_{\mathbb{Z}_p} e^{t(x+y)} \lambda^y d\mu_1(y) = \sum_{n=0}^{\infty} B_n(\lambda; x) \frac{t^n}{n!} = \frac{(\log \lambda + t)e^{tx}}{\lambda e^t - 1}.$$  

By using Taylor expansion of $e^{tx}$ in the left side of the above equation, after some elementary calculations, we obtain the desired result. \qed

We now give the distribution of the $\lambda$–Bernoulli polynomials.

**Theorem 3.** Let $n \geq 0$, and let $d \in \mathbb{Z}^{+}$. Then we have

$$B_n(\lambda; x) = d^{n-1} \sum_{a=0}^{d-1} \lambda^a B_n \left( \lambda^d, \frac{x + a}{d} \right). \quad (7)$$
Proof. By using Eq-(6),

\[ B_n(x; \lambda) = \int_{\mathbb{Z}_p} (x + y)^n \lambda^y d\mu(y) \]

\[ = \lim_{N \to \infty} \frac{1}{dp^N} \sum_{y=0}^{dN-1} (x + y)^n \lambda^y \]

\[ = \lim_{N \to \infty} \frac{1}{dp^N} \sum_{a=0}^{d-1} \sum_{y=0}^{pN-1} (a + dy + x)^n \lambda^a dy \]

\[ = d^{n-1} \lim_{N \to \infty} \frac{1}{pN} \sum_{a=0}^{d-1} \sum_{y=0}^{pN-1} \left( \frac{a + x}{d} + y \right)^n (\lambda^{d^y}) \]

Thus, we have the desired result. □

By substituting \( x = 0 \) into Eq-(7), we have the following corollary:

**Corollary 1.** For \( m, n \in \mathbb{N} \), we have

\[ mB_n(\lambda) = \sum_{j=0}^{n} \binom{n}{j} B_j(\lambda^m) m^j \sum_{a=0}^{m-1} \lambda^a a^{n-j}. \]  

( Gauss multiplicative formula for \( \lambda \)-Bernoulli numbers).

By Eq-(8), we have

**Theorem 4.** For \( m, n \in \mathbb{N} \) and \( \lambda \in \mathbb{Z}_p \), we have

\[ mB_n(\lambda) - m^n[m] \lambda B_n(\lambda^m) = \sum_{j=0}^{n-1} \binom{n}{j} B_j(\lambda^m) m^j \sum_{k=1}^{m-1} \lambda^k k^{n-j}. \]  

**Theorem 5.** Let \( k \in \mathbb{Z} \), with \( k > 1 \). Then we have

\[ B_l(\lambda; k) - \lambda^{-k} B_l(\lambda) = \lambda^{-k} \sum_{n=0}^{k-1} \lambda^n n^{l-1} + (\lambda^{-k} \log \lambda) \sum_{n=0}^{k-1} n^l \lambda^n. \]  

(10)
Proof. We set

\[ - \sum_{n=0}^{\infty} e^{(n+k)t} \lambda^n + \sum_{n=0}^{\infty} e^{nt} \lambda^{n-k} = \sum_{n=0}^{\infty} e^{nt} \lambda^{n-k} \quad (10a) \]

\[ = \sum_{l=0}^{\infty} \left( \lambda^{-k} \sum_{n=0}^{k-1} n^l \lambda^n \right) \frac{t^l}{l!} \]

\[ = \sum_{l=1}^{\infty} \left( \lambda^{-k} l \sum_{n=0}^{k-1} n^l \lambda^n \right) \frac{t^{l-1}}{l!}. \]

Multiplying \((t + \log \lambda)\) both side of Eq-(10a), then by using Eq-(3) and Eq-(5), after some elementary calculations, we have

\[ \sum_{l=0}^{\infty} \left( B_l(\lambda; k) - \lambda^{-k} B_l(\lambda) \right) \frac{t^l}{l!} \]

\[ = \sum_{l=0}^{\infty} \left( \lambda^{-k} l \sum_{n=0}^{k-1} n^l \lambda^n \right) + \lambda^{-k} \log \lambda \sum_{n=0}^{k-1} n^l \lambda^n \frac{t^l}{l!} \quad (10b) \]

By comparing coefficient \(\frac{t^l}{l!}\) in both sides of Eq-(10b). Thus we arrive at the Eq-(10). Thus we complete the proof of theorem. □

Observe that \(\lim_{\lambda \to 1} B_l(\lambda) = B_l\). For \(\lambda \to 1\), then Eq-(10) reduces the following:

\[ B_l(k) - B_l = l \sum_{n=0}^{k-1} n^{l-1}. \]

If \(\lambda \in T_p\), then Eq-(10) reduces to the following formula:

\[ B_l(\lambda; k) - \lambda^{-k} B_l(\lambda) = \lambda^{-k} l \sum_{n=0}^{k-1} n^l \lambda^{n-1}. \]

Remark. Garrett and Hummel [1B] proved combinatorial proof of \(q\)-analogue of

\[ \sum_{k=1}^{n} k^3 = \binom{n+1}{2} \]

as follows:

\[ \sum_{k=1}^{n} q^{k-1} [k]_q^2 \left( \binom{k-1}{2} q^2 + \binom{k+1}{2} q^2 \right) = \binom{n+1}{2}, \]

where \([k]_q\) denotes the \(q\)-analogue of \(k\), and \(\binom{n}{k}_q\) denotes the \(q\)-analogue of \(\binom{n}{k}\).
where \( [n]_q = \prod_{j=1}^{k} \frac{[n+1-j]_q}{[j]_q} \), \( q \)-binomial coefficients. Garrett and Hummel, in their paper, asked for a simpler \( q \)-analogue of the sums of cubes. As a response to Garrett and Hummel’s question, in [11], Kim constructed the following formula

\[
S_{n,q}^h(k) = \sum_{l=0}^{k-1} q^l h[l]^n = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1-j}{j} q_j k_j [k]^{n+1-j} - \frac{(1 - q^{n+1}) \beta_{n+1,q}}{n+1},
\]

where \( \beta_{j,q} \) are the \( q \)-Bernoulli numbers which were defined by

\[
e^{\frac{t}{1 - q t}} \log q - t \sum_{n=0}^{\infty} q^n t^n e^{[n+x]t} = \sum_{n=0}^{\infty} \frac{\beta_{n,q}(x)}{n!} t^n, \quad |q| < 1, |t| < 1,
\]

\( \beta_{n,q}(0) = \beta_{n,q} \), cf. [11]

Schlosser [20] gave for \( n = 1, 2, 3, 4, 5 \) the value of \( S_{n,q}^h[k] \). In [28], the authors also gave another proof of \( S_{n,q}(k) \) formula.

\[\text{§2. Hurwitz's type } \lambda \text{-zeta function}\]

In this section, by using generating function of \( \lambda \)-Bernoulli polynomials, we construct Hurwitz’s type \( \lambda \)-zeta function, which is interpolate \( \lambda \)-Bernoulli polynomials at negative integers. By Eq-(5), we get

\[
F_\lambda(t; x) = \frac{\lambda + t}{\lambda e^t - 1} e^{xt} = -(\log \lambda + t) \sum_{n=0}^{\infty} \lambda^n e^{(n+x)t} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!},
\]

By using \( \frac{d^k}{dt^k} \) derivative operator to the above, we have

\[
B_k(\lambda; x) = \frac{d^k}{dt^k} F_\lambda(t; x) \bigg|_{t=0},
\]

\[
B_k(\lambda; x) = -\log \lambda \sum_{n=0}^{\infty} \lambda^n (n+x)^k - k \sum_{n=0}^{\infty} (n+x)^{k-1} \lambda^n.
\]

Thus we arrive at the following theorem:
Theorem 6. For \( k \geq 0 \), we have
\[
-\frac{1}{k} B_k(\lambda; x) = \frac{\log \lambda}{k} \sum_{n=0}^{\infty} \lambda^n (n + x)^k + \sum_{n=0}^{\infty} \lambda^n (n + x)^{k-1}.
\]

Consequently, we define Hurwitz type zeta function as follows:

**Definition 1.** Let \( s \in \mathbb{C} \). Then we define
\[
\zeta_{\lambda}(s, x) = \frac{\log \lambda}{1 - s} \sum_{n=0}^{\infty} \frac{\lambda^n}{(n + x)^s} + \sum_{n=0}^{\infty} \frac{\lambda^n}{(n + x)^s}.
\]

Note that \( \zeta_{\lambda}(s, x) \) is analytic continuation, except for \( s = 1 \), in whole complex plane. By Definition 1 and Theorem 6, we have the following:

**Theorem 7.** Let \( s = 1 - k \), \( k \in \mathbb{N} \),
\[
\zeta_{\lambda}(1-k, x) = -\frac{B_k(\lambda, x)}{k}.
\]

§3. Generalized \( \lambda \)-Bernoulli numbers associated with Dirichlet type \( \lambda \)-\( L \)-functions

By using Eq-(0), we define
\[
I_1(f_d) = I_1(f) + \sum_{j=0}^{d-1} f'(j),
\]
where \( f_d(x) = f(x + d) \), \( \int_X f(x) d\mu(x) = I_1(f) \).

Let \( \chi \) be Dirichlet character with conductor \( d \in \mathbb{N}^+ \), \( \lambda \in \mathbb{Z}_p \).

By substituting \( f(x) = \lambda^x \chi(x)e^{tx} \) into Eq-(12), then we have
\[
\int_X \chi(x)\lambda^x e^{tx} d\mu_1(x) = \sum_{j=0}^{d-1} \frac{\chi(j)\lambda^j e^{tj} (\log \lambda + t)}{\lambda^d e^{dt} - 1}
= \sum_{n=0}^{\infty} B_{n,\chi}(\lambda) \frac{t^n}{n!}.
\]

By Eq-(12a), we easily see that
\[
B_{n,\chi}(\lambda) = \int_X \chi(x)x^n \lambda^x d\mu_1(x).
\]
From Eq-(12a), we define generating function of generalized Bernoulli number by
\[ F_{\lambda,\chi}(t) = \sum_{j=0}^{d-1} \frac{\chi(j) \lambda^j e^{jt} (\log \lambda + t)}{\lambda^d e^{dt} - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}. \tag{12c} \]
Observe that if \( \lambda \in T_p \), then the above formula reduces to
\[ F_{\lambda,\chi}(t) = \sum_{j=0}^{d-1} \chi(j) \lambda^j e^{jt} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!} \quad \text{(for detail see cf. [3,16,18,22,23,24]).} \]
From the above, we easily see that
\[ F_{\lambda,\chi}(t) = -(\log \lambda + t) \sum_{m=1}^{\infty} \chi(m) \lambda^m e^{tm} = \sum_{n=0}^{\infty} B_{n,\chi}(\lambda) \frac{t^n}{n!}. \]
By applying \( \frac{d^k}{dt^k} \) derivative operator both sides of the above equation, we arrive at the following theorem:

**Theorem 8.** Let \( k \in \mathbb{Z}^+, \lambda \in \mathbb{Z}_p \) and let \( \chi \) be Derichlet character with conductor \( d \). Then we have
\[ \sum_{m=1}^{\infty} \chi(m) \lambda^m m^{k-1} + \frac{\log \lambda}{k} \sum_{m=1}^{\infty} \lambda^m \chi(m) m^k = -\frac{B_{k,\chi}(\lambda)}{k}. \tag{13} \]

**Definition 2 (Dirichlet type \( \lambda \)-L function).** For \( \lambda, s \in \mathbb{C} \), we define
\[ L_\lambda(s, \chi) = \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^s} - \log \lambda \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^s - 1}. \tag{14} \]
Relation between \( L_\lambda(s, \chi) \) and \( \zeta_\lambda(s, y) \) is given by the following theorem:

**Theorem 9.** Let \( s \in \mathbb{C} \) and \( d \in \mathbb{Z}^+ \). Then we have
\[ L_\lambda(s, \chi) = d^{-s} \sum_{a=1}^{d} \lambda^a \chi(a) \zeta_{\lambda^d} \left( s, \frac{a}{d} \right). \]

**Proof.** By substituting \( m = a + dk, \ a = 1, 2, \cdots, d, \ k = 0, 1, \cdots, \infty, \) into Eq-(14), we have
\[ L_\lambda(s, \chi) = \sum_{a=1}^{d} \sum_{k=0}^{\infty} \frac{\lambda^{a+dk} \chi(a + dk)}{(a + dk)^s} - \frac{\log \lambda}{s-1} \sum_{a=1}^{d} \sum_{k=0}^{\infty} \frac{\lambda^{a+dk} \chi(a + dk)}{(a + dk)^{s-1}} \]
\[ = d^{-s} \sum_{a=1}^{d} \left( \sum_{k=0}^{\infty} (\lambda^d)^k \frac{(a + k)^s - (a + k + \frac{a}{d})^s}{(k + \frac{a}{d})^s} \right). \]
By using Eq-(11) in the above we obtain the desired result. \( \square \)
**Theorem 10.** For $k \in \mathbb{Z}^+$, we have

$$L_{\lambda}(1-k, \chi) = -\frac{1}{k} B_{k, \chi}(\lambda), \ k > 0.$$  

**Proof.** By substituting $s = 1 - k$ in Definition 2 and using Eq-(13), we easily obtain the desired result. □

**Remark.** If $\lambda \in T_p$, then from Definition 2, we have

$$L_{\lambda}(s, \chi) = \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^s}.$$  

In [18], Koblitz studied on this function. He gave the name of this function “twisted $L$-function” for $\lambda$ is $r$-th root of 1. In [22,23,24], Simsek studied of this functions. He gave fundamental properties of this function as well.

In [16], Kim et al. gave $\lambda-(h, q)$ zeta function and $\lambda-(h, q)$ $L$-function. These functions interpolate $\lambda-(h, q)$-Bernoulli numbers. Observe that, if we take $s = 1 - k$ in Theorem 9, and then using Eq-(12) in Theorem 7, we get another proof of Theorem 10.

§4. $\lambda$-Bernoulli numbers of order $r$ associated with multiple zeta function

In this section, we define generating function of $\lambda$-Bernoulli numbers of order $r$. By using Mellin transforms and Cauchy residue theorem, we obtain multiple zeta function which is given in Eq-(C). We also gave relations between $\lambda$-Bernoulli polynomials of order $r$ and multiple zeta function at negative integers. This relation is important and very interesting. Let $r \in \mathbb{Z}^+$. Generating function of $\lambda$-Bernoulli numbers of order $r$ is defined by

$$F^{(r)}_{\lambda}(t) = \left( \frac{\log \lambda + t}{\lambda e^t - 1} \right)^r = \sum_{n=0}^{\infty} B^{(r)}_{n}(\lambda) \frac{t^n}{n!}. \quad (15)$$

Generating function of $\lambda$-Bernoulli polynomials of order $r$ is defined by

$$F^{(r)}_{\lambda}(t, x) = F^{(r)}_{\lambda}(t)e^{tx} = \sum_{n=0}^{\infty} B^{(r)}_{n}(\lambda) \frac{t^n}{n!}.$$

Observe that when $r = 1$, Eq-(15) reduces to Eq-(3). By applying Mellin transforms to the Eq-(15) we get

$$\frac{1}{\Gamma(s)} \int_{0}^{\infty} \lambda^r e^{-tr} F^{(r)}_{\lambda}(-t)(t - \log \lambda)^{s-r-1} dt = \sum_{n_1, \ldots, n_r=0}^{\infty} \frac{1}{(n_1 + n_2 + \cdots + n_r + r)^s}.$$  

Thus, we get, by (C)

$$\zeta_r(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \lambda^r e^{-tr} F^{(r)}_{\lambda}(t)(t - \log \lambda)^{s-r-1} dt.$$  

By using the above relation, we obtain the following theorem:
Theorem 11. Let \( r, m \in \mathbb{Z}^+ \). Then we have
\[
\zeta_r(-m) = (-\lambda)^r m! \sum_{j=0}^{\infty} \binom{-m - r - 1}{j} (\log \lambda)^j \frac{B_{m+r+j}(\lambda; r)}{(m + r + j)!}.
\] (D1)

Remark. If \( \lambda \to 1 \), the above theorem reduces to
\[
\zeta_r(-m) = (-1)^r m! \frac{B_{m+r+1}(1; r)}{(m + r)!}
\] (D2)
which is given Theorem 6 in [12].

By (D1) and (D2), we obtain relation between \( \lambda \)-Bernoulli polynomials of order \( r \) and ordinary Bernoulli polynomials of order \( r \) as follows:
\[
B_{m+r}(r) = \lambda^r \sum_{j=0}^{\infty} \binom{-m - r - 1}{j} (\log \lambda)^j \frac{B_{m+r+j}(\lambda; r)}{(m + r + j)!} (m + r)!
\]
where \( m, r \in \mathbb{Z}^+ \).

We now give relations between \( B_n^{(r)}(\lambda) \) and \( H_n^{(r)}(\lambda^{-1}) \) as follows:

If \( \lambda \in T_p \), then Eq-(15) reduces to the following equation
\[
\frac{t^r}{(\lambda e^t - 1)^r} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!}.
\]

Thus by the above equation, we easily see that
\[
t^r = (\lambda e^t - 1)^r e^{B^{(r)}(\lambda) t}
\]
\[
= \sum_{l=0}^{r} \lambda^l (-1)^{r-l} e^{(B^{(r)}(\lambda) + l) t}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{r} \lambda^l (-1)^{r-l} (B^{(r)}(\lambda) + l)^n \right) \frac{t^n}{n!}.
\]

Consequently we have
\[
\sum_{l=0}^{r} \lambda^l (-1)^{r-l} (B^{(r)}(\lambda) + l)^n = \begin{cases} 
0 & \text{if } n \neq r \\
1 & \text{if } n = r
\end{cases}.
\]

By Eq-(15) we obtain
\[
\sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!} = \frac{t^r}{(\lambda - 1)^r} \sum_{n=0}^{\infty} H_n^{(r)}(\lambda^{-1}) \frac{t^n}{n!}.
\]
By comparing coefficient \( \frac{t^n}{n!} \) in the both sides of the above equation, we have

\[
B_n^{(r)}(\lambda) = \frac{\Gamma(n + r + 1)}{\Gamma(n + 1)} \frac{1}{(\lambda - 1)^r} H_n^{(r)}(\lambda^{-1}).
\]

Observe that, if we take \( r = 1 \), then the above identity reduce to Eq-(4), that is

\[
B_{n+1}(\lambda) = \frac{(n + 1)}{\lambda - 1} H_n(\lambda^{-1}).
\]

§5. \( \lambda \)-Bernoulli numbers and polynomials associated with multivariate \( p \)-adic invariant integral

In this section, we give generalized \( \lambda \)-Bernoulli numbers of order \( r \). Consider the multivariate \( p \)-adic invariant integral on \( \mathbb{Z}_p \) to define \( \lambda \)-Bernoulli numbers and polynomials.

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \lambda^{w_1 x_1 + \cdots + w_r x_r} e^{(w_1 x_1 + \cdots + w_r x_r) t} d\mu_1(x_1) \cdots d\mu_1(x_r)
\]

\[
= \frac{(w_1 \log \lambda + w_1 t) \cdots (w_r \log \lambda + w_r t)}{(\lambda^{w_1 e^{w_1 t}} - 1) \cdots (\lambda^{w_r e^{w_r t}} - 1)}
\]

\[
= \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; w_1, w_2, \cdots, w_r) \frac{t^n}{n!},
\]

where we called \( B_n^{(r)}(\lambda; w_1, w_2, \cdots, w_r) \) \( \lambda \)-extension of Bernoulli numbers. Substituting \( \lambda = 1 \) into Eq-(16), \( \lambda \)-extension of Bernoulli numbers reduce to Barnes Bernoulli numbers as follows:

\[
\frac{(w_1 t) \cdots (w_r t)}{(e^{w_1 t} - 1) \cdots (e^{w_r t} - 1)} = \sum_{n=0}^{\infty} B_n^{(r)}(w_1, \cdots, w_r) \frac{t^n}{n!},
\]

where \( B_n^{(r)}(w_1, \cdots, w_r) \) are denoted Barnes Bernoulli umbers and \( w_1, \cdots, w_r \) complex numbers with positive real parts [1A,7,27]. Observe that when \( w_1 = w_2 = \cdots = w_r = 1 \) in Eq-(16), we obtain the \( \lambda \)-Bernoulli numbers of higher order as follows:

\[
\left( \frac{\log \lambda + t}{\lambda e^t - 1} \right)^r = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!}.
\]

We note that \( B_n^{(r)}(\lambda; 1, 1, \cdots, 1) = B_n^{(r)}(\lambda) \).
Consider
\[
\left( \frac{\log \lambda + t}{\lambda e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x) \frac{t^n}{n!}.
\]

Observe that
\[
\sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x) \frac{t^n}{n!} = \left( \frac{\log \lambda + t}{\lambda e^t - 1} \right)^r e^{(\log \lambda+t)x} \lambda^{-x}
\]
\[
= \frac{1}{\lambda^x} \sum_{m=0}^{\infty} B_m^{(r)}(\lambda; x) \frac{(t + \log \lambda)^m}{m!}
\]
\[
= \frac{1}{\lambda^x} \sum_{m=0}^{\infty} B_m^{(r)}(\lambda; x) \sum_{l=0}^{m} \binom{m}{l} (\log \lambda)^l m^{m-l}
\]
\[
= \sum_{m=0}^{\infty} \left( \frac{1}{\lambda^x} \sum_{l=0}^{\infty} B_{n+l}^{(r)}(\lambda; x) \frac{(\log \lambda)^l}{l!} \right) \frac{t^n}{n!}.
\]

Now, comparing coefficient \( \frac{t^n}{n!} \) both sides of the above equation, we easily arrive at the following theorem:

**Theorem 12.** For \( n, r \in \mathbb{N} \) and \( \lambda \in \mathbb{Z}_p \), we have
\[
B_n^{(r)}(\lambda; x) = \frac{1}{\lambda^x} \sum_{l=0}^{\infty} B_{n+l}^{(r)}(\lambda; x) \frac{(\log \lambda)^l}{l!},
\]
where \( 0^l = \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{if } l \neq 0 \end{cases} \).

**Remark.** In Theorem 12, we see that
\[
\lim_{\lambda \to 1} B_n^{(r)}(\lambda; x) = \begin{cases} B_n^{(r)}(x) & \text{if } l = 0, \\ 0 & \text{if } l \neq 0. \end{cases}
\]

§6. \( \lambda \)-Bernoulli numbers and polynomials in the space of locally constant

In this section, we construct partial \( \lambda \)-zeta functions, we need this function in the following section. We need this function in the following section. By Eq-(3b), Frobenius-Euler polynomials are defined by means of the following generating function:
\[
\left( \frac{1-u}{e^t-u} \right) e^{xt} = \sum_{n=0}^{\infty} H_n(u, x) \frac{t^n}{n!}.
\]
As well known, we note that the Frobenius-Euler polynomials of order \( r \) were defined by
\[
\left( \frac{1 - u}{e^t - u} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(u, x) \frac{t^n}{n!}.
\]
The case \( x = 0 \), \( H_n^{(r)}(u, 0) = H_n^{(r)}(u) \), which are called Frobenius-Euler numbers of order \( r \).

If \( \lambda \in T_p \), then \( \lambda \)-Bernoulli polynomials of order \( r \) are given by
\[
\frac{t^r}{(\lambda e^t - 1)^r} e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x) \frac{t^n}{n!}.
\]
Hurwitz type \( \lambda \)-zeta function is given by
\[
\zeta_{\lambda}(s, x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(n + x)^s}, \quad \lambda \in T_p.
\]
(17)

Thus, from Theorem 7, we have
\[
\zeta_{\lambda}(1 - k, x) = -\frac{1}{k} B(\lambda; x), \quad k \in \mathbb{Z}^+.
\]
(17a)

We now define \( \lambda \)-partial zeta function as follows
\[
H_{\lambda}(s, a|F) = \sum_{m \equiv a \pmod{F}} \frac{\lambda^m}{m^s}.
\]
(17b)

From (17), we have
\[
H_{\lambda}(s, a|F) = \frac{\lambda^a}{F^s} \zeta_{\lambda^F}(s, a|F),
\]
(17c)

where \( \zeta_{\lambda^F}(s, a|F) \) is given by Eq-(17). By Eq-(17a) we have
\[
H_{\lambda}(1 - n, a|F) = -\frac{F^{n-1} \lambda^a B_n(\lambda^F; a|F)}{n}, \quad n \in \mathbb{Z}^+.
\]
(18)

If \( \lambda \in T_p \), then by Eq-(14), we have
\[
L_{\lambda}(s, \chi) = \sum_{n=1}^{\infty} \frac{\lambda^n \chi(n)}{n^s},
\]
where $s \in \mathbb{C}$, $\chi$ be the primitive Dirichlet character with conductor $f \in \mathbb{Z}^+$. By Theorem 9, Eq-(17c) and Eq-(18) we easily see that

$$L_\lambda(s, \chi) = \sum_{a=1}^{F} \chi(a) H_\lambda \left( s, \frac{a}{F} \right),$$

and

$$L_\lambda(1 - k, \chi) = -\frac{B_{k, \chi}(\lambda)}{k}, \quad k \in \mathbb{Z}^+,$$

where $B_{k, \chi}(\lambda)$ is defined by

$$\sum_{a=0}^{F-1} \frac{t^a \chi(a) e^{at}}{\lambda^F e^{Ft} - 1} = \sum_{a=0}^{\infty} B_{n, \chi}(\lambda) \frac{t^n}{n!}, \quad \lambda \in T_p$$

and $F$ is multiple of $f$.

**Remark.**

$$\frac{B_m(\lambda)}{m} = \frac{1}{\lambda - 1} H_{n-1}(\lambda^{-1}), \quad \lambda \in T_p.$$

§7. $p$-adic interpolation function

In this section we give $p$-adic $\lambda$-$L$ function. Let $w$ be the Teichimuller character and let \( x = \frac{x}{w(x)} \).

When $F$ is multiple of $p$ and $f$ and $(a, p) = 1$, we define

$$H_{\lambda, s}(a|F) = \frac{1}{s-1} \lambda^a < a >^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} \left( \frac{F}{a} \right)^j B_j(\lambda^F).$$

From this we note that

$$H_{\lambda, 1-n}(a|F) = -\frac{1}{n} \lambda^a \sum_{j=0}^{n} \binom{n}{j} \left( \frac{F}{a} \right)^j B_j(\lambda^F)$$

$$= -\frac{1}{n} F^{n-1} \lambda^a w^{-n}(a) B_n(\lambda^F; \frac{a}{F})$$

$$= w^{-n}(a) H_\lambda(1 - n; \frac{a}{F}),$$

since by Theorem 3 for $\lambda \in T_p$, Eq-(18).
By using this formula, we can consider $p$-adic $\lambda$-$L$-function for $\lambda$-Bernoulli numbers as follows:

$$L_{p,\lambda}(s, \chi) = \sum_{a=1}^{F} \chi(a)H_{p,\lambda}\left(s, \frac{a}{F}\right).$$

By using the above definition, we have

$$L_{p,\lambda}(1-n, \chi) = \sum_{a=1}^{F} \chi(a)H_{p,\lambda}\left(1-n, \frac{a}{F}\right)$$

$$= -\frac{1}{n}\left(B_{n,\chi w^{-n}}(\lambda) - p^{n-1}\chi w^{-n}(p)B_{n,\chi w^{-n}}(\lambda^p)\right).$$

Thus, we define the formula

$$L_{p,\lambda}(s, \chi) = \frac{1}{F} \frac{1}{s-1} \sum_{a=1}^{F} \chi(a)\lambda^a < a > \sum_{j=0}^{\infty} \frac{1-s}{j} B_j(\lambda^F)$$

for $s \in \mathbb{Z}_p$.

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