A NOTE ON THE HOLOMORPHIC INVARIANTS OF TIAN-ZHU

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In this short note, we compute the holomorphic invariants defined by Tian and Zhu on smooth hypersurfaces of $\mathbb{C}P^n$. The holomorphic invariants, which generalize the famous Futaki invariants, are obstructions towards the existence of Kähler-Ricci solitons.

For a Kähler manifold with the first positive Chern class, the existence of the Kähler-Ricci soliton can be reduced to the existence of the solution of a non-linear equation of Monge-Ampère type. In general, solving such an equation is highly non-trivial. Similar to the Futaki invariants, the Tian-Zhu invariants gives the obstruction before one need to solve the equation. It is thus very important to compute it concretely. In this paper, in the case of hypersurfaces, we give an explicit formula.

Let $M \subset \mathbb{C}P^n$ be a smooth hypersurface defined by a homogeneous polynomial $F = 0$ of degree $d$. Let $v$ and $X$ be two holomorphic vector fields on $\mathbb{C}P^n$. For the sake of simplicity, we assume that $v = \sum_{i=0}^{n} v_i Z_i \frac{\partial}{\partial Z_i}$, and $X = \sum_{i=0}^{n} X_i Z_i \frac{\partial}{\partial Z_i}$, where $[Z_0, \cdots, Z_n]$ is the homogeneous coordinate of $\mathbb{C}P^n$, $(v_0, \cdots, v^n) \in \mathbb{C}^{n+1}$, $(X_0, \cdots, X^n) \in \mathbb{C}^{n+1}$. We further assume that

\begin{equation}
\sum_{i=0}^{n} v_i = 0, \quad \sum_{i=0}^{n} X_i = 0.
\end{equation}

If $v$ and $X$ are tangent vector fields of $M$, then there are complex numbers $\lambda$ and $\kappa$ such that

\begin{equation}
vF = \kappa F, \quad XF = \lambda F.
\end{equation}

Let $\omega$ be the Kähler form of the Fubini-Study metric of $\mathbb{C}P^n$. Then $(n-d+1)\omega$ restricts to a representative of the first Chern class $c_1(M)$ of $M$. Thus there is a smooth function $\xi$ on $M$ such that

\[
\text{Ric}((n-d+1)\omega|_M) - (n-d+1)\omega|_M = \bar{\partial}\bar{\partial}\xi.
\]

Date: May 24, 2001.

1991 Mathematics Subject Classification. Primary: 53A30; Secondary: 32C16.

Key words and phrases. Kähler-Ricci soliton, Futaki invariants, and Kähler-Einstein metric.

Research supported by NSF grant DMS 0196086.
For fixed holomorphic vectors $X$ and $v$, the holomorphic invariant defined by Tian-Zhu [4], in our context, is

$$ F_X(v) = (n - d + 1)^{n-1} \int_M v(\xi - (n - d + 1)\theta_X) e^{(n-d+1)\theta_X} \omega^{n-1}, $$

where $\theta_X$ is defined as

$$ i(X)\omega = \frac{\sqrt{-1}}{2\pi} \partial \theta_X, $$

$$ \int_M e^{(n-d+1)\theta_X} \omega^{n-1} = d. $$

The main property of the Tian-Zhu invariants is the following (cf. [4]):

**Theorem 1.** Let $F_X(v)$ be the Tian-Zhu invariant. Then we have

1. If the Kähler-Ricci soliton exists, that is, we have
   $$ \text{Ric}(\omega) - \omega = L_X \omega $$
   for some Kähler metric $\omega$. Then $F_X(v) \equiv 0$.

2. $F_X(v)$ is independent of the choice of the Kähler metric $\omega$ within the first Chern class.

In this note, we give a “computable” expression of $F_X(v)$. Our main result is as follows:

**Theorem 2.** Using the notations as above, defined the function

$$ \varphi(X) = \sum_{k=0}^{\infty} \frac{n!(n-d+1)^k}{(n+k)!} \sum_{\alpha_0 + \cdots + \alpha_n = k} X_0^{\alpha_0} \cdots X_n^{\alpha_n}, $$

where $\alpha_0, \cdots, \alpha_n \in \mathbb{Z}^{n+1}$ are nonnegative integers. Let

$$ \sigma(X) = \left(-\frac{\lambda(n-d+1)}{n} + d\right) \varphi(X) + \frac{d}{n} \sum_{i=0}^{n} X_i \frac{\partial \varphi(X)}{\partial X_i}. $$

Then the invariants defined by Tian-Zhu can be explicitly expressed as

$$ F_X(v) = -(n - d + 1)^{n-1} d \left( \kappa + \sum_{i=0}^{n} v_i \frac{\partial \log \sigma(X)}{\partial X_i} \right). $$

**Corollary 1.** The Futaki invariant for the hypersurface $M$ is

$$ F(v) = -(n - d + 1)^{n-1} \frac{(n+1)(d-1)}{n} \kappa. $$

The rest of this note is devoted to the proof Theorem 2. We define

$$ \bar{\theta}_X = \lambda_0 |Z_0|^2 + \cdots + \lambda_n |Z_n|^2. $$

Then we have

$$ i(X)\omega = \bar{\partial} \bar{\theta}_X. $$
By comparing the above equation with (11), we have
\begin{equation}
\theta_X = \tilde{\theta}_X + c_X
\end{equation}
for a constant \( c_X \). First, we have the following lemma

**Lemma 1.**
\[ \int_{\mathbb{P}^n} e^{(n-d+1)\tilde{\theta}_X} \omega^n = \varphi(X), \]
where \( \varphi(X) \) is defined in (5).

**Proof.** This follows from the expansion
\[ e^{(n-d+1)\tilde{\theta}_X} = \sum_{k=0}^{\infty} \frac{(n-d+1)^k}{k!} \tilde{\theta}_X^k, \]
and the elementary Calculus. \( \square \)

**Lemma 2.** Using the same notation as above, we have
\[ F_X(v) = (n-d+1)^{n-1} \left( -\kappa d - \int_M (n-d+1)\theta_v e^{(n-d+1)\theta_X} \omega^{n-1} \right). \]

**Proof.** By [3, Theorem 4.1], we have
\[ \text{div} \, v + v(\xi) + (n-d+1)\theta_v = -\kappa, \]
where \( \theta_v \) is the function on \( \mathbb{P}^n \) defined by
\[ \theta_v = \frac{v_0|Z_0|^2 + \cdots + v_n|Z_n|^2}{|Z_0|^2 + \cdots + |Z_n|^2}, \]
and \( \kappa \) is defined in (2). Then (3) becomes
\begin{equation}
F_X(v) = (n-d+1)^{n-1} \left( -\kappa d - \int_M (n-d+1)\theta_v e^{(n-d+1)\theta_X} \omega^{n-1} \right). \end{equation}
We also have
\begin{equation}
\text{div} \, (e^{(n-d+1)\theta_X} v) = e^{(n-d+1)\theta_X} (\text{div} \, v + (n-d+1)v(\theta_X)). \end{equation}
The lemma follows from (4), (11), (12) and the divergence theorem. \( \square \)

The following key lemma transfers the integration on \( M \) to the integrations on \( \mathbb{P}^n \).

**Lemma 3.**
\begin{equation}
(n-d+1) \int_M \theta_v e^{(n-d+1)\theta_X} \omega^{n-1} = \sum_{i=0}^{d} v_i \frac{\partial \log \sigma}{\partial X_i}, \end{equation}
where \( \sigma(X) \) is defined in (6).
Proof. Let

\[ \eta = \log \frac{|F|^2}{(|Z_0|^2 + \cdots + |Z_n|^2)^d}. \]

Then \( \eta \) is a smooth function on \( CP^n \) outside \( M \). We have the following identity:

\[
\overline{\partial}(e^{(n-d+1)\theta_X} \partial_X \eta \wedge \omega^{n-1}) = -e^{(n-d+1)\theta_X} \overline{\partial} \eta \wedge \omega^{n-1} - \frac{n-d+1}{n} e^{(n-d+1)\theta_X} \lambda (\lambda - d \tilde{\theta}_X) \omega^n.
\]

Since on \( CP^n \), there are no \((2n+1)\) forms, the left hand side of the above equation is the divergence of some vector field. Integrate the equation on both side and use the divergence theorem, we have

\[
\int_{CP^n} e^{(n-d+1)\theta_X} \partial_X \eta \wedge \omega^{n-1} = -\frac{n-d+1}{n} \int_{CP^n} (\lambda - d \tilde{\theta}_X) e^{(n-d+1)\theta_X} \omega^n.
\]

By [2, page 388], in the sense of currents, we have

\[
\partial \overline{\partial} \eta = [M] - d\omega.
\]

Thus from (16),

\[
\int_M e^{(n-d+1)\theta_X} \omega^{n-1} = \left( -\frac{\lambda(n-d+1)}{n} + d \right) \int_{CP^n} e^{(n-d+1)\theta_X} \omega^n + \frac{d(n-d+1)}{n} \int_{CP^n} \tilde{\theta}_X e^{(n-d+1)\theta_X} \omega^n.
\]

From Lemma 1, we have

\[
\sum_{i=0}^n X^i \frac{\partial \varphi(X)}{\partial X^i} = (n-d+1) \int_{CP^n} \tilde{\theta}_X e^{(n-d+1)\theta_X} \omega^n.
\]

By (10), (18) and (19)

\[
\int_M e^{(n-d+1)\theta_X} \omega^{n-1} = \sigma(X) e^{c_X}.
\]

From the above equation, we have

\[
(n-d+1) \int_M \theta_v e^{(n-d+1)\theta_X} \omega^{n-1} = \sum_{i=0}^n v^i \frac{\partial \sigma(X)}{\partial X^i} e^{c_X}.
\]

On the other hand, from (20), we have

\[
d = \sigma(X) e^{c_X},
\]

by (4). Lemma 3 follows from (21) and (22).

Theorem 2 follows from Lemma 2 and Lemma 3. \qed
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