Gauge Fields and Space-Time

A.M. Polyakov
Joseph Henry Laboratories
Princeton University
Princeton, New Jersey 08544

Abstract

In this article I attempt to collect some ideas, opinions and formulae which may be useful in solving the problem of gauge/string/space-time correspondence. This includes the validity of D-brane representation, counting of gauge-invariant words, relations between the null states and the Yang-Mills equations and the discussion of the strong coupling limit of the string sigma model. The article is based on the talk given at the "Odyssey 2001" conference.

October 2001
1 Introduction

As time goes by, it seems that in fundamental physics we are circling around a particular set of themes and ideas. One of the important examples of such "limit cycles" is the relations between gauge fields and strings. The logic of these connections is as following. We begin with an observation (by K. Wilson [1]) that in the strong coupling limit of a lattice gauge theory the elementary excitations are represented by closed strings formed by the color-electric fluxes. In the presence of quarks these strings open up and end on the quarks, thus guaranteeing quark confinement. Moreover, in the SU(N) gauge theory the strings interaction is weak at large N. This fact makes it reasonable to expect that also in the physically interesting continuous limit (not accessible by the strong coupling approximation) the best description of the theory should involve the flux lines (strings) and not fields, thus returning us from Maxwell to Faraday. In other words it is natural to expect an exact duality between gauge fields and strings. The challenge is to build a precise theory on the string side of this duality.

It is also possible to turn this problem around. We know that there is a class of string theories, the superstrings, which contain quantum gravity and (as the enthusiasts say) everything else. It might be possible to visualize the superstrings as flux lines of some unknown gauge theory. That would give us a completely novel view of what we call a space-time. The view is that in a deep sense it does not exist, being only a quasiclassical limit of some abstract gauge theory, residing nowhere. Instead, our observables must be sets of gauge invariant operators, formed out of the products of some elementary ones, just as words are formed from the alphabet. The theory should provide us with the expectation values for the different combination of letters.

In recent years a serious progress has been achieved in realization of these ideas (for a review see [2] and [3]). In this lecture I will discuss some of my latest attempts to find the dynamical correspondence between strings and words. The main idea of the discussion below is quite simple. String theories possess infinite number of gauge symmetries in the target space, generated by the sequence of the zero norm states on the world sheet. The lowest of these symmetries is general covariance. It is well known that the general covariance on the string side leads to the conservation of the energy momentum tensor of the gauge theory. We will see that the higher gauge symmetries generate
relations between the words, satisfied as a consequence of the equations of motion of the gauge theory.

2 The general picture

The origin of the gauge fields in general / strings duality (and AdS/CFT in particular) can be traced to the relation between open and closed strings, known from the very beginning of string theory. This relation in its simplest form tells us the amplitude of an annulus represents either a partition function of the open string or a propagation of the closed one. In the d+1-dimensional space the amplitude the annulus with the external particles attached to its outer circle and the inner circle placed at the distance $R$, has a representation

$$Z = \int_0^\infty \frac{d\tau}{\tau^2} f(\tau) \exp\left(-\frac{R^2}{2\tau}\right) \int_0^{2\pi} \prod d\alpha_k \exp\left(-\sum p_ip_j \Delta(\alpha_i - \alpha_j | \tau)\right)$$

where we assumed the Dirichlet boundary condition in the d+1 direction, Neuman condition in all the others and take $p_i$ to be d-dimensional momenta; $f(\tau)$ is the corresponding determinant and $\Delta$ is a propagator on a cylinder

$$\Delta(\alpha) = \sum_{n \neq 0} \frac{e^{ina}}{n \tanh(n\tau)}$$

We see that for the small $\tau$ (a thin cylinder) this propagator is that of a particle on a circle, while as $\tau \to \infty$ the inner circle of the annulus can be replaced by an insertion of a local operator. In the leading order in the bosonic string this operator is the closed string tachyon. For a general value of $\tau$ it is still possible to replace the inner circle by a combination of local operators $\{O_n\}$. These operators are the Ishibashi states, annihilated by the boundary energy momentum tensor. They are related to the primary operators $V_n$ by the well known formulas

$$O_n = (1 + \frac{1}{2\Delta} L_{-1}L_{-1} + ... )V_n$$

where $\Delta$ is the dimension of $V$. The contribution of an annulus is equal to the contribution of a disc with an operator $V_H$, describing the hole, inserted
at the middle. This operator has an expansion

\[ V_H = \sum_n c_n(\tau)O_n \]  

(4)

with the coefficient functions behaving as

\[ c_n \sim e^{-n\tau} \]  

(5)

at large \( \tau \). When we have several holes in the world sheet they all can be replaced by the hole operators \( V_H \).

In order to get a general idea of gauge/strings correspondence, imagine for a moment that this procedure converges (which is not true in a simple bosonic string). We know that we can obtain the Yang-Mills amplitudes from the string by taking a limit of zero slope. In this limit the integral in (1) is dominated by the small \( \tau \) and represents a loop of a vector particle. When we have a disk with several holes, it will shrink to a planar Feynman diagram of the Yang-Mills theory. But if there is a convergence in (1), the same set of diagrams can be viewed as a string disk diagram in some general background generated by the \( V_H \) insertions. We would conclude that the amplitudes of gauge theory in \( d \) dimensions are equal to the amplitudes of string theory in some non-trivial background field.

However there is a possible divergence of the above procedure which can easily invalidate it. Namely, if the theory possesses an open string tachyon, it will manifest itself in the divergence at small \( \tau \). Hence we should be looking for theories without such tachyons. As was suggested in [2] from a different point of view, we must consider a non-critical fermionic NSR string with the non-chiral GSO projection. The gravitational background (induced by the hole operators) for these strings must have the form

\[ ds^2 = d\varphi^2 + a^2(\varphi)dx^2 \]  

(6)

where the \( \varphi \) direction comes from the Liouville field of the non-critical string theory. The GSO projection comes from the summation on the spin structures of the NSR fermions. This summation is indispensable for having modular invariant amplitudes. From the open string point of view it eliminates the open string tachyon. From the point of view of the closed string background, allowing the world sheet fermions to change sign as we go around
the hole, means that there must be the Ramond-Ramond fields in the background, coupled to the spin operators on the world sheet. Thus we have the connection: modular invariance - absence of the open tachyons - RR background.

If the tachyon is absent for other reasons this is just as well. In this case the RR background is not needed. As an example consider $c \leq 1$ string theories. In this case we can consider a disc with the holes and set the boundary condition $e^{\phi} \to \infty$, recently considered in [4]. According to this work the open strings with this boundary conditions do not have excited states, at least for the simplest bootstrap solution. To obtain the sum of planar diagrams we have to supply these strings with the Chan-Paton factors. Hence, from the open string point of view all degrees of freedom are described by the matrix model which can be shown to coincide with the Kontsevich model. From the closed string point of view we have a Liouville theory in a certain background. Along these lines it is possible to resolve a long standing question - why the matrix models describe the Liouville theory.

The picture we are describing is just the standard D-brane picture of gauge fields (see e.g [3]). What we are trying to explain is why the stack of the D-branes in flat space can be replaced by the closed string backgrounds. The full quantitative derivation of this is still lacking, but we will see how the above arguments work in the different situations. It is important to realize though, that D-branes, being a beautiful and useful tool, are neither necessary nor sufficient to derive even AdS/CFT correspondence, to say nothing of the general case. A more powerful method to attack these problems is based on the loop equations [5] But it also is not developed enough to provide a complete understanding.

3 The entropy of gauge invariant words.

In this section we discuss the structure and combinatorics of the gauge invariant words in preparation to the comparison with the string theory. A simplest candidates for the gauge invariant words are expressions of the type of $Tr(\nabla^{k_1}F...\nabla^{k_n}F)$ where $F$ stands for the Yang-Mills field strength and $\nabla$ for a covariant derivative. These words are related to the closed string states. But this particular alphabet, consisting of the letters $\nabla^k F$ is not convenient for our purposes because the letters are not independent. There
are kinematical relations coming from the Bianchi identities and from the relation $F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$. There are also the dynamical relations coming from the Yang-Mills equations. A better alphabet, which will allow us to count the words, is obtained by taking a radial gauge

$$x_\mu A_\mu = 0$$

(7)

By differentiating this relation and setting $x = 0$ we find a set of letters

$$\Omega_{\alpha_1...\alpha_n;\beta} = \partial_{\alpha_1}...\partial_{\alpha_n} A_{\beta}$$

(8)

with a single relation

$$\Omega_{\alpha_1...\alpha_n;\beta} + \text{permutations} = 0$$

(9)

Therefore $\Omega$ transforms as $(n,1)$ Young tableau. Let us consider first the gauge theory without matter in the weak coupling limit and count the number of letters of given dimension

$$\#(\Omega) = \sum_{\{n_j\}} (d\delta_{n_j \sum n_j} - \delta_{n+1, \sum n_j})$$

(10)

Here $d$ is the number of dimensions, $n_j$ is the number of indices equal to $j$ and the second term is the number of the constraints (9) which reduce the number of $\Omega$-s. To find the generating function for $\Omega$ we simply notice that in the weak coupling limit it has a dimension $\Delta = n + 1$ and thus contributes $x^{n+1}$. Therefore the partition function $F(x)$ for the letters has the form

$$F(x) = \frac{dx}{(1-x)^d} - \frac{1}{(1-x)^d} + 1$$

(11)

If the matter fields are present, they can be easily included in this counting. Now we have to find the partition function for the words. A slight complication here is that the cyclic symmetry of the words must be properly accounted. For example, a word $Tr(AABB)$ has four cyclic reincarnations and thus must have a weight $\frac{1}{4}$. At the same time a similar word $Tr(ABAB)$ has only two cyclic versions and thus the weight is $\frac{1}{2}$. This combinatorial problem can be handled by the Polya theory. It is based on the observation that each
transformation of the symmetry group splits the set of words onto cycles of different length and introduces the cyclic index

$$\Phi(z_1 \ldots z_n) = \frac{1}{[G]} \sum_{g \in G} z_1^{b_1} \ldots z_n^{b_n} \ldots$$  \hspace{1cm} (12)$$

where $G$ is the symmetry group (the cyclic permutations in our case), $[G]$ is the number of its elements and $b_k$ is the number of the cycles of the length $k$. According to Polya, the partition function is expressed in terms of the cyclic index by the formula

$$Z = \Phi(\sum x_k, \sum x_k^2, \ldots \sum x_k^n \ldots)$$  \hspace{1cm} (13)$$

Incidentally, the meaning of these formulas can be easily understood by the use of path integrals with the sums over permutations, as was done by Feynman to derive the Bose statistics.

Using these methods we obtain the following answer for the partition function of the words

$$Z = -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log(1 - F(x^n)) = -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log\left[\frac{1 - dx^n}{(1 - x^n)^d}\right]$$  \hspace{1cm} (14)$$

Here $\varphi(n)$ is the Euler function equal to the number of integers relatively prime to $n$. It takes care of the subtlety with the weights described above.

In the case of the $N=4$ Yang Mills theory we have to add the scalar fields $\Phi_i$ to our alphabet the letters $\partial_{\alpha_1} \ldots \partial_{\alpha_n} \Phi_i$ where $i = 1 \ldots 10 - d$. An obvious modification of the above counting gives in this case

$$Z = -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log\left[\frac{1 - 10 x^n}{(1 - x^n)^d}\right]$$  \hspace{1cm} (15)$$

We do not include letters containing fermions, although it is also easy to do. These formulas count the off-shell words not constrained by the equation of motion, which generate some further relations between the words. To find the on-shell entropy we notice that from the linearized Yang-Mills equations

$$\partial^2 A_\alpha - \partial_\alpha \partial_\lambda A_\lambda = 0$$  \hspace{1cm} (16)$$

it follows after multiple differentiation that

$$\Omega_{\beta_1 \ldots \beta_{n-2} \lambda \lambda; \alpha} - \Omega_{\beta_1 \ldots \beta_{n-2} \lambda \alpha; \lambda} = 0$$  \hspace{1cm} (17)$$
The number of extra constraints is now easy to count. The tensor in (17) has zero trace when $\alpha$ and $\beta$ are contracted and otherwise all its components are independent. By the same method as before we obtain the partition function for the letters on shell, $\tilde{F}(x)$

$$\tilde{F}(x) = F(x) - \frac{dx^3}{(1-x)^d} + \frac{x^4}{(1-x)^d} = 1 - \frac{(1-x^2)(1-dx+x^2)}{(1-x)^d}$$ (18)

It is interesting to notice that for $d=4$ the on-shell $\Omega$-s transform according to $\left(\frac{n+1}{2}, \frac{n-1}{2}\right) \oplus \left(\frac{n-1}{2}, \frac{n+1}{2}\right)$ representation of SO(4). Correspondingly $\tilde{F}=2\sum n(n+2)x^{n+1}$.

Using these formulas we can evaluate the number of words $\mathcal{N}$ of the given dimension $\Delta$ by the formula

$$\mathcal{N} = \oint dx x^{\Delta+1} Z(x)$$ (19)

When $\Delta$ is large the asymptotic is determined by the singularities of $Z$ and we get (on shell)

$$\mathcal{N} \sim e^{b\Delta}$$ (20)

$$b = \log\left(\frac{d}{2} - \sqrt{\frac{d^2}{4} - 1}\right)$$ (21)

An exponential behavior in a similar problem has been numerically observed in [6]. This formula gives the number of states in the limit of zero gauge coupling. In the opposite limit of the large gauge coupling $\lambda$ the states are described by the almost free string theory. According to [7,8] in this case $\Delta \sim \lambda^{1/4} M$, where $M$ is the mass of the string state. Since the number of states in the closed string grows exponentially with the mass, it is safe to conjecture that $\mathcal{N} \sim e^{b(\lambda)\Delta}$ with $b(\lambda) \rightarrow const$ at small $\lambda$ and $b(\lambda) \sim \lambda^{-\frac{1}{4}}$ at large $\lambda$.

### 4 Words and vertices

The essence of the gauge fields - strings correspondence is the statement that there is an isomorphism between the gauge invariant operators and the vertex operators of a certain closed string theory in the background (6 ).
This statement is presumed to be true for a general gauge theory, although it has been thoroughly checked only for the Yang-Mills theory with maximal supersymmetry and in a few related cases. Even in these cases the derivation from the first principles is still lacking. To set the stage let us review this isomorphism, taking as an example conformal gauge theories, for which the background metric is known \[9\]. Namely the conformal group on the gauge side must be present as a group of motion for the metric on the string side. That leads us to the space of constant negative curvature, the AdS. It is convenient in this case to rewrite (6) in the form

\[
ds^2 = \sqrt{\lambda} dy^2 + \frac{d\bar{x}^2}{y^2}
\]

where the Liouville field \(\varphi \sim \log y\) and \(\lambda\) measure the curvature and will be related to the Yang-Mills coupling. For the case of the four-dimensional gauge theories the symmetry group of this metric is O(5,1). The string theory in question is described by the two-dimensional \(\sigma\)-model on the world sheet the basic field of which is \(n(\xi)\), the field of the Lorentzian unit vector, \(n_0^2 - \bar{n}^2 = 1\). The six dimensional unit vector \(n\) is related to \(x\) and \(y\) in (22) by the standard parametrization

\[
\begin{align*}
n_- &= n_0 - n_1 = y^{-1} \\
n_+ &= n_0 + n_1 = y^{-1}(y^2 + \bar{x}^2) \\
n_\mu &= y^{-1} x_\mu
\end{align*}
\]

The Lagrangian of the \(\sigma\)-model consists of several pieces. For the case of the N=4 Y.-M. theory it contains, apart from the \(n\) field describing AdS\(_5\) another field \(N\), describing six scalars and having the \(R\) symmetry O(6). This is a field of \(S_5\) \([9]\), making the background ten dimensional, AdS\(_5\) \(\times\) S\(_5\). In the pure gauge theory this field would be absent. However strings in the critical dimension (ten) are simpler than in the non-critical one and this is precisely why presently we know more about N=4 Yang-Mills theory then about pure gauge theory (but it is the latter that is our final goal). We have the following structure

\[
L = \frac{\sqrt{\Lambda}}{2} [(\partial_\alpha n)^2 + (\partial_\alpha N)^2] + L_F(n) + L_F(N) + e^{-\phi/2} \sum(n) \sum(N) + L_{ghost}
\]

where \(L_F\) is the standard Lagrangian for the NSR fermions which are orthogonal to \(n\) and \(N\) correspondingly. The \(L_{ghost}\) describes the bosonic ghosts of
the NSR string and the field $e^{-\frac{\chi}{2}}$ is the spin operator for these ghosts, while the operators $\Sigma$ are the spin operators for the NSR fermions. Notice that the fields $n$ and $N$ interact only via the spin term (which creates antiperiodic boundary conditions for all NSR fermions simultaneously). Without this interaction the sigma model would not be conformally invariant on the world sheet. Instead the coupling for the $N$ field would be asymptotically free, while the coupling for $n$ would flow in the opposite direction. The spin-spin interaction locks them together.

The vertex operators are simply various primary operators of this CFT with dimension (1,1). Let us discuss their structure, concentrating on the most non-trivial ones formed out of the non-compact $n$-field with the group $O(5,1)$. A convenient trick here is to start with the compact case of $O(6)$ and then perform an analytic continuation. The simplest operators are those which do not contain derivatives (we will call them "level zero" operators). They have the form

$$V = \Psi_{i_1...i_l}n_{i_1}...n_{i_l} + ...$$  \hspace{1cm} (27)

here $\Psi$ is a totally symmetric traceless tensor and we dropped the fermionic terms which can mix (in higher orders) with the written one. The anomalous world sheet dimension $\delta$ of this term in the one loop approximation is given by the well known formula

$$\delta = \frac{1}{\sqrt{\lambda}} l(l + 4)$$  \hspace{1cm} (28)

where the first factor is simply the sigma model coupling constant, while the second is the Casimir operator for our representation. To perform the analytic continuation let us consider the highest weight component in $V$

$$V_+ = n_+^l = y^{-l}(y^2 + \bar{x}^2)^l$$  \hspace{1cm} (29)

and notice that after going to $O(5,1)$ the formula for the anomalous dimension (28) changes sign, because we change the sign of the curvature; so now

$$\delta = -\frac{1}{\sqrt{\lambda}} l(l + 4)$$  \hspace{1cm} (30)

Another consequence of the non-compactness is that the angular momentum $l$ doesn’t have to be an integer anymore. Moreover from (29) we see its new
meaning - it defines the space-time scaling of \(V\) \((V_+ \sim x^l)\). However so far we found only the highest weigh component of the representation. To find all other operators is very easy. We have to exploit the fact that translations in \(x\) (because they are the part of \(\text{SO}(5.1)\)) leave us inside the representation, and hence \(V_+ (x + a)\), where \(a\) is an arbitrary vector, transform under the same representation as \(V_+ (x)\). In fact these objects with different \(a\) form a complete basis for the representation. It is convenient to make a linear transformation of this basis by taking the Fourier transformation in \(a\) and deal with the operators

\[
V_{p\Delta} = \int d^4 an_+^\Delta (x+a, y)e^{ipa} = y^\Delta e^{ipx} \int d^4 a e^{ipa} \frac{1}{(a^2 + y^2)^{\Delta}} = N(\Delta)p^{\Delta-2}y^2K_{\Delta-2}(py)e^{ipx}
\]

where we introduced the space-time scaling dimension \(\Delta = -l\), \(N(\Delta)\) is an irrelevant normalization factor, \(K\) is the modified Bessel function; we wrote these formulas for the four dimensional case, while in the dimension \(d\) we would get \(K_{\Delta-d}\) in them. Under the \(\text{O}(5,1)\) transformations the operators with different \(\Delta\) do not mix, while \(p\) plays the role of the magnetic quantum number. We see from (30) that the space-time and the world sheet dimensions are related as

\[
\delta = \frac{1}{\sqrt{\lambda}} \Delta(4 - \Delta)
\]

So far we treated the operators without world sheet derivatives (level zero). Let us consider next level two operators. They have a general form

\[
V = \Psi_{ab,i_1...i_l}n_{i_1}...n_{i_l}\partial_{z}n_a\partial_{\bar{z}}n_b
\]

These operators are reducible from the point of view of \(\text{O}(6)\), and transform as a product of \((1,0,0)\times(1,0,0)\times(l,0,0)\), where \((f_1f_2f_3)\)is the representation with the lengths of the rows in the Young tableau equal to these numbers (recall that \(\text{O}(6)\) has rank 3). A convenient way to parametrize these numbers is to consider first the Lorentz properties of \(V\). Since \(\text{O}(4) \approx \text{O}(3) \times \text{O}(3)\), its representations are labeled by the two integer angular momenta \((j_1,j_2)\). In order to characterize a representation of \(\text{O}(5,1)\) we simply add \(\Delta\) to this list, getting a triple \((j_1,j_2,\Delta)\). The world sheet dimensions are in general different for different representations, although in the limits of strong and
weak couplings certain degeneracies occur. For the operators (33) we have the formula
\[
\delta = 1 + \frac{1}{\sqrt{\lambda}}[\Delta(4 - \Delta) + \varepsilon(j_1, j_2)] + o\left(\frac{1}{\sqrt{\lambda}}\right)
\] (34)

In this formula the first term is just the naive dimension of (33), the second term, as in (30), represents the center of mass motion of the string on $\text{AdS}_5$ and the correction $\varepsilon(j_1, j_2)$ (which is not hard to calculate but it has not been done) is the interaction between the center of mass motion of the string and its oscillations. Notice that this last effect is not present in the flat space where the Hamiltonian is the sum of the oscillator part and the center of mass part. In the curved space such a separation does not exist and $\varepsilon$-like corrections are different for different levels. We will further discuss it in section 5.

By the same method as before we can express our operator in terms of $\vec{x}$ and $y$. Let notice that in the level zero case we can describe the operator $V_{p\Delta}(\vec{x}, y)$ as an eigenfunction of the Laplace operator with the asymptotic behaviour $V_{p\Delta} \to y^{-\Delta}$ as $y \to 0$. The level two operator has the structure
\[
V \sim y^{-\Delta-2}\Psi_{mn}(py)\partial_{x^m}\partial_{x^n}e^{ipx}
\] (35)

where $m, n = 0, ..., 4$ and we define $x^0 = y$; we also adjusted the power of $y$ so that the space-time dimension of $V$ is equal to $\Delta$. As we explained above, for a fixed $\Delta$ the operator must have definite Lorentz properties. For a tensor operator $V^\mu_{\nu\rho\sigma}$ we must choose the function $\Psi_{mn}^{\mu\nu}$ in such a way that it is an eigenfunction of the tensor Laplacian in AdS and has the asymptotic behaviour at $y \to 0$
\[
\Psi^{\mu\nu}_{\lambda\gamma} \to \delta^\mu_\lambda \delta^\nu_\gamma
\] (36)

Notice that (36) specifies only spatial components of $\Psi$. The gauge conditions satisfied by $V$ will determine the ”time” components which can’t be fixed independently. As a result we obtain a uniquely defined tensor vertex operator. In general we should expect that the traceless part of $V^\mu_{\nu}$ and its trace have different dimensions. However in the special case of $\mathbb{N}=4$ supersymmetry they coincide, being related to the chiral operators.

Continuing in this way we obtain all possible operators of the above sigma model. The vertex operator form a subset defined by the conditions that they are conformal primaries on the world sheet and have dimensions (1,1).
The last condition gives us an equation for determining their space-time dimensions
\[ \delta(j_1, j_2, \Delta) = 1 \]  
(37)
This is the on-shell condition. It is important to realize that the 4d momentum \( p \) is not constrained by this condition. Sometimes it is more convenient to use the coordinate representation for the integrated vertex operator which (omitting the Lorentz indices) has the structure
\[ \Omega(x') = \int d^2 \xi V(x' + \xi(x), y) \]  
(38)

The gauge - strings duality is the isomorphism between the set of operators \( \Omega \), obtained from the various (1,1) primaries of the above sigma model and the gauge- invariant words. Let us explain the above conjectures, using (to shorten the notations) the ”holomorphic halves” of the vertex operators. On the level one we consider an operator
\[ \Omega_1 = \Psi_m(x) \partial_z x^m \]  
(39)
For small curvatures we can use the standard technic \[ 10 \] to determine the change \( \Omega \) of this operator when we apply an external Liouville field \( \sigma \). A simple one loop computation gives
\[ \hat{\Omega} = \sigma \nabla^2 \Psi_m \partial_z x^m + \partial_z \sigma \nabla^m \Psi_m \]  
(40)
The first term comes from the action of \( L_0 \) Virasoro generator while the second - from \( L_1 \). If we require that the operator should be \( \sigma \) - independent, that is conformal primary, we get the equations
\[ \nabla^2 \Psi_m = 0 \]  
(41)
\[ \nabla^m \Psi_m = 0 \]  
(42)

We see that the second equation determines the \( y \) -component of \( \Psi \), while the solution of the first one is determined by the asymptotic condition
\[ \Psi_\mu \sim y^{\Delta - 1} \psi_\mu(x) \]  
(43)
The exponent \( \Delta - 1 \) (the value of which is determined by the on-shell condition (37)) is defined so that \( \psi_\mu(x) \) scales like \( x^{-\Delta} \). This follows from the
The fact that $\Psi_m$, being a vector in AdS, scales as $x^{-1}$ under its isometries. Let us also notice that if we require the $\sigma$-independence only of the integrated $\Omega$ we get a gauge unfixed version of the equations (41, 42).

The full vertex operator is defined by the function in the 4d space $\psi_\mu(x)$ or by its Fourier transform $\psi_\mu(p)$. A corresponding object in the standard string theory would be a photon polarization. It is important to realize that while any $\psi_\mu$ generates a physical vertex operator, some of these operators can have zero norm. In the above simple example this happens in the following way. The above vertex has the form

$$V_\mu(p) = \int dz\Psi_{\mu m}(p, y)\partial_z x^m e^{ipx}$$

where the function $\Psi_{\mu m}(p, y)$ is a solution of (41) with the asymptotic conditions

$$\Psi_{\lambda \mu}(p, y) \to \delta_{\lambda \mu} y^{\Delta - 1}$$

From this we conclude that

$$p_\mu \Psi_{\mu m}(p, y) = \nabla_m \chi(p, y)$$

if $\chi$ satisfies the Laplace equation with the conditions $\chi \to y^{\Delta - 1}$. After plugging (46) into (44) we see that the vertex operator satisfies the conservation equation

$$p_\mu V_\mu = 0$$

This is just a rephrasing of the statement that in the critical string theory we have a zero norm state at the level one, generated by the Virasoro operator $L_{-1}$. To apply it to our closed string case we have to do the usual doubling of the vertex $V_\mu$ and obtain the tensor vertex (33). Then the vertex is identified with the energy momentum tensor of the gauge theory and the existence of the null state on the string side implies a conservation law on the gauge theory side. As we proceed to the level two, we expect that another null state, generated by the $L_{-2} + \frac{3}{2} L_{-1}^2$ will play the similar role, providing some relations between gauge-invariant words. The (holomorphic half) at the level two has the form

$$\Omega = \psi_m \nabla_z \partial_z x^m + \chi_{mn} \partial_z x^m \partial_z x^n$$
In the one loop approximation (small AdS curvature) it is easy to find the physical state conditions. In the expansion similar to (40) we find that the change of $\Omega$ with respect to the external Liouville field has the form

$$\dot{\Omega} = \sigma L_0 \Omega + \partial_z \sigma L_1 \Omega + \partial^2 \sigma L_2 \Omega + (\partial_z \sigma)^2 L^2 \Omega \quad (49)$$

This formula can be easily derived by passing to the light-cone gauge on the world sheet (I didn’t find it in the literature). To ensure that the state is physical we have to calculate (using the method of [10]) the corresponding terms and set them to zero. That gives

$$\left( \frac{1}{\sqrt{\lambda}} \nabla^2 - 1 \right) \psi_m = 0 \quad (50)$$

$$\nabla^m \psi_m + \frac{1}{2} \chi_k^k = 0 \quad (51)$$

$$\frac{1}{\sqrt{\lambda}} \nabla^m \chi_{mn} + \psi_m = 0 \quad (52)$$

Once again, the solution of these equations is specified by fixing the asymptotic behaviour of the 4d components of these tensor fields, $\psi_\mu(\vec{x}^\mu)$ and $\chi_{\mu\nu}(\vec{x}^\mu)$. Similarly to the scalar case (38) we have the vertex operators $V_\mu(\vec{x}^\mu)$ and $V_{\mu\nu}(\vec{x}^\mu)$ given by the formulas

$$V_\mu(\vec{x}^\mu) = \int dz (g_{\mu m}(\vec{x}^\mu + \vec{x}(z), y(z)) \partial_z x^m + h_{\mu mn}(\vec{x}^\mu + \vec{x}(z), y(z)) \partial_z x^m \partial_z x^n) \quad (53)$$

and analogously for $V_{\mu\nu}$. Here $g$ and $h$ are the corresponding propagators for the equations (50 - 52).

The existence of the null vectors at the levels one and two implies that certain linear combinations of these vertex operators have zero matrix elements. It is easy to see that while the above vertices from the point of view of SO(4) transform as $\left( \frac{1}{2}, \frac{1}{2} \right) \oplus (1, 1) \oplus (0, 0)$, the elimination of the null states leaves only $(1, 1)$ representation. This is because the corresponding gauge parameters are scalar (at the level two) and vector (at the level one).

Our main conjecture is that any physical (that is satisfying the Virasoro constraints) vertex operator corresponds to a certain combination of the gauge-invariant words. Setting the null states to zero generates certain linear combinations between the vertex operators. These relations must be
equivalent to the Yang-Mills equations of motion. In other words the condition

\[ < V_{\text{null}} \ldots > = 0 \]  \hspace{1cm} (54)

can be considered as a variational principle for the D-branes describing the Yang-Mills theory.

To clarify this statement let us consider as an example the null state generated by \( L_{-1} \) and assume that we have a D-brane located at \( Y^i = 0 \), where \( Y \) is a transverse coordinate. It is well known that the above null state corresponds (in the closed strings) to an infinitesimal diffeomorphisms \( Y^i \Rightarrow Y^i + \varepsilon^i(x) \) in the sense that its insertion in the middle of the world sheet disc describing the D-brane generates this transformation at the boundary. Let \( S_D[Y] \) be the Born-Infeld action describing the D-brane. Then the condition that the null state decouples from the D-brane is that the action remains unchanged under this diffeomorphism, or

\[ \delta S_D \delta Y^i = 0. \]

The check of our conjecture requires a better understanding of the correspondence between the words and the string states then we presently have. The problem is to relate the words in the weak Yang-Mills coupling limit, described by the formulae (8) and (17), to the string states, which are easy to describe in the opposite limit. Only after that we will be able to translate explicitly the condition (54) to the gauge theory language.

While the full correspondence is unknown, we can formulate a principle which governs it. Consider the (space-time) anomalous dimensions \( \Delta_1(\lambda) \) and \( \Delta_2(\lambda) \) of two operators \( \Omega_1 \) and \( \Omega_2 \), where once again \( \lambda \) is the gauge coupling. As we change \( \lambda \) from zero to infinity these dimensions flow from the fixed integer values, defined by the free fields to the arbitrary large values following from the formula (37) (we consider unprotected operators). We shall use the following "non-intersection principle" : the trajectories of the operators with the same symmetry do not intersect. It is easy, by the use of the conformal perturbation theory, to reduce this statement to the standard secular equation argument of quantum mechanics. As an example of an application of this principle, consider (modulo fermions and gauge fields) an operator \( Tr(F_{\alpha\beta}F_{\gamma\delta}) \) from the point of view of SO(4) its content is given by the representations \( 2(0,0) \oplus (1,1) \oplus (2,0) \oplus (0,2) \). Scalar and tensor representations correspond to the graviton, dilaton and axion and are protected by the supersymmetry. The unprotected operator is thus \( (2,0) \oplus (0,2) \). Now as the coupling increases it must go to the first available level of the weakly coupled
string theory. This is because otherwise this level will be taken by an operator with the same SO(4) quantum numbers but with the higher naive dimension. In these circumstances the intersection of the trajectories $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ will be unavoidable. As we saw, at the level two we have a state $(1, 1)$ in the holomorphic part of the vertex operator. When we take a product of the holomorphic and antiholomorphic parts we get among other representations the desired $(2, 0) \oplus (0, 2)$. We conclude that it must be occupied by our gauge operator.

However, in order to check quantitatively that at each level decoupling of the null states corresponds to passing from the off-shell gauge invariant words to the on-shell ones, one needs a much finer combinatorial analyses. Namely, it is necessary to find the number of the gauge operators of given dimension and given SO(4) quantum numbers and compare them with the corresponding decomposition on the string side. This work is now in progress.

5 Weak and strong coupling

The gauge / string correspondence is a duality in the sense that the strong coupling limit on the gauge side corresponds to the weak coupling in string sigma model (that is a small curvature of the target space) and vice versa. The main difficulty of the subject stems from our temporary inability to solve exactly the above sigma model and to analyze gauge theory in terms of its physical states. In this section we will introduce some further conjectures concerning these states.

Let us notice first of all, that the strong coupling limit must be simple. Indeed, the spectrum of the space-time anomalous dimension in this case must coincide with that of the free gauge fields and thus all the dimensions must be integer. At the same time, formulae (32) and (37) show that at large $\lambda$ we have a behaviour $\Delta \sim \lambda^{4/5}$ for a very large $\lambda$ [7,8] corresponding to the small curvature limit (the space-time curvature $R \sim \frac{1}{\sqrt{\lambda}}$). If we try to extrapolate blindly this formula to the small $\lambda$ we would get a completely wrong behaviour - namely all space-time dimensions will become degenerate. This happens because in this extrapolation the world-sheet dimensions blow up in this limit.

Another strange feature of the above result is the following. Consider large but finite $\lambda$ in gauge theory. It seems that typically in perturbation
theory for the operators of very high dimension and twist the normal (integer) part of dimension is larger then the anomalous contribution. Hence it is natural to expect that at any $\lambda$ we have highly excited states retaining (asymptotically) their normal dimensions.

We conjecture here that the resolution of these puzzles lies in the structure of the spectrum of anomalous dimensions in the sigma model. Namely, we assume that while the dimension of a generic operator blows up in the strong coupling limit, there is a special subset of operators with finite dimensions. These operators have integer space-time dimensions and correspond to the free gauge fields. The same operators dominate at finite $\lambda$ but at very high levels.

Let us discuss a possible origin of these special operators. As a first example consider an operator of the form

$$V = n^{-\Delta}(\partial_z n \bar{\partial}_{\bar{z}} n)^s (\partial_z n)^{2p} (\bar{\partial}_{\bar{z}} n)^{2p} + ...$$

where the dots refer to the similar terms with smaller $s$ but the same overall dimension. In the one loop approximation dimensions of these operators have a peculiar feature first noticed in the context of localization theory in [11] and in O(d) sigma models in [12]. Adapting the results of these papers to our case we get the following formula for anomalous dimensions

$$\delta = s + 2p + \frac{1}{\sqrt{\lambda}} \left[ \Delta(4 - \Delta) + 2s(s - 1) - 16p \right] + o(\frac{1}{\sqrt{\lambda}})$$

We see that at large $s$ we have from $\delta = 1$ condition (if we trust the one loop approximation) $\Delta \approx \sqrt{2}s$ independently of $\lambda$. We also see that due to the peculiar sign of the $s$-dependent term, which tends to decrease the dimension in the compact case, the world sheet dimension $\delta$ doesn’t blow up as $\lambda \to 0$. There are other operators with the same properties and different Lorentz structure.

In the one loop approximation world sheet supersymmetry and RR backgrounds don’t change these results. However we can consider them only as a hint of the true structure, before we have higher approximation under control. This problem is not solved yet. In what follows I shall describe a possible approach to its solution.

If we set $\lambda$ to zero in the eq. (26) we are left with the fermionic $\lambda$-independent Lagrangian. Hopefully it will describe the above subset of
operators with finite dimensions. The structure of the $L_F$ term is as following

$$L_F = \overline{\psi} \gamma_\mu (\partial_\mu + \omega_\mu + A_\mu) \psi + \sqrt{\lambda} A_\mu^2$$

(57)

where $\omega_\mu$ is a spin-connection on the hyperboloid in question and $A_\mu$ is an auxiliary field needed to reproduce the four fermion term in the supersymmetric sigma model. We see that as $\lambda \to 0$ the last term disappears and we can shift $A_\mu + \omega_\mu \Rightarrow A_\mu$. As a result we end up with the fermions with "zero current" condition, the system which is easy to analyze. This system, which is equivalent to the Thirring model at infinite coupling, in which only gauge invariant operators have finite dimensions and they can be classified. Unfortunately this doesn’t solve our problem. We have to treat the spin-spin (or RR) interaction in (26). The zero current systems with the RR-interaction has not been studied so far, although they are much simpler then the original sigma model. The spin operators should play a very important role in this analyses, since their dimensions, being related to the central charge, do not blow up. Perhaps they form the building blocks for the free gauge fields.

Finally, let us stress that although in this section we dealt with the theories which are conformally invariant in space-time and seemingly excluded the most interesting cases in which this invariance is broken, this is not a serious limitation. Because of the asymptotic freedom all theories are almost invariant in the large curvature limit. More technically one can consider the $4 + \varepsilon$ dimensional space in which the above symmetry is strict.

6 The Outlook

The picture which slowly arises from the above considerations is that of the space-time gradually disappearing in the regions of large curvature. The natural description in this case is provided by a gauge theory in which the basic objects are the texts formed from the gauge-invariant words. The theory provides us with the expectation values assigned to the various texts, words and sentences. These expectation values can be calculated either from the gauge theory or from the strongly coupled 2d sigma model. The coupling in this model is proportional to the target space curvature. This target space can be interpreted as a usual continuous space-time only when the curvature is small. As we increase the coupling, this interpretation becomes more
and more fuzzy and finally completely meaningless. Since the theory is not complete, we can’t give an explicit demonstration of this mechanism.

Apart from the cases considered above it is perhaps worthwhile to examine an old example of the similar phenomenon given in [13]. In this work I have looked at the sigma model with the Lagrangian

$$ L = (\partial \varphi)^2 + a^2(\varphi)(\partial N)^2 + ... $$

(58)

This theory has two interpretations. First, in the weak coupling domain it describes the Friedman universe with positive curvature. The one loop approximation for the function $a$ gives the Einstein-like equations for this metric and not surprisingly the solution has a Big Bang singularity at some $\varphi = \varphi_0$, $a(\varphi_0) = 0$. In the second interpretation this is an $O(3)$ sigma model coupled to 2d gravity with $a^{-2}$ being a running coupling constant. The one loop renormalization group once again gives a singularity due to the asymptotic freedom. But this singularity is a fiction! As well known, the $\bar{N}$ field develops a mass gap and the theory is completely non-singular. However this target space can not be interpreted as a 4d space-time. Instead it is characterized by a set of amplitudes derived from the above sigma model, which has the space-time interpretation only in the quasi-classical domain. In the very early universe space-time is not getting singular - it simply doesn’t exist. It would be very interesting to find the gauge theory corresponding to the above model. The running of $a(\varphi)$is related to the running of the gauge coupling.

Another related problem is to give the gauge theory interpretation to the conjectured [13] infrared screening of the cosmological constant. This screening must also be related to the gauge coupling, since this quantity measures the curvature of space-time. Unfortunately we are still far from finding the ultimate gauge theory which describes the Universe.

I am grateful to B. Altshuler for drawing my attention to the papers [11,12]. This work was supported by the NSF grant PHY9802484.
REFERENCES
[1] K. G. Wilson Phys. Rev. D10, 2445 (1974)
[2] A. M. Polyakov Int. Journ. of Mod. Phys. A 14 (1999) 645 [ hep-th/9809057]
[3] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri, Y. Oz Phys. Rept. 323, 183 (2000)
[4] A. Zamolodchikov, Al. Zamolodchikov hep-th/0101152
[5] A. M. Polyakov, V. Rychkov Nucl. Phys. B581 (2000) 116 [ hep-th/0002106]
[6] D. Gross, Ig. Klebanov, A. Matytsin, A. Smilga Nucl. Phys. B461 (1996)109 [ hep-th/9511104]
[7] S. Gubser, Ig. Klebanov, A.M. Polyakov Phys.Lett B428 (1998) 105 [ hep-th /9802109]
[8] E. Witten Adv. Theor. Phys. 2 (1998) 253 [ hep-th/ 9802150]
[9] J. Maldacena Adv. Theor. Phys. 2 (1998) 231 [hep-th / 9711200]
[10] C. Callan, Z. Gan Nucl. Phys. B 272 (1986) 647
[11] V. Kravtsov, I. Lerner, V. Yudson Phys. Lett. A134 (1989) 245
[12] F. Wegner Z. Phys. B78 (1990) 33
[13] A. M. Polyakov Proceedings of Les Houches (1992) [hep-th /9304146]