Weakly tame systems, their characterizations and application

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Abstract

We explore the notion of discrete spectrum and its various characterizations for ergodic measure preserving actions of an amenable group on a compact metric space. We introduce a notion of ‘weak-tameness’, which is a measure theoretic version of a notion of ‘tameness’ introduced by E. Glasner, based on the work of A. Köhler\footnote{A. Köhler introduced this notion and call such systems “regular”.}, and characterize such topological dynamical systems as systems for which every invariant measure has a discrete spectrum. Using the work of M. Talagrand, we also characterize weakly tame as well as tame systems in terms of the notion of ‘witness of irregularity’ which is based on ‘up-crossings’. Then we establish that ‘strong Veech systems’ are tame. In particular, for any countable amenable group $T$, the flow on the orbit closure of the translates of a ‘Veech function’ $f \in \mathcal{K}(T)$ is tame. Thus Sarnak’s Möbius orthogonality conjecture holds for this flow and as a consequence, we obtain an improvement of Motohashi-Ramachandra 1976’s theorem on the Mertens function in short interval. We further improve Motohashi-Ramachandra’s bound to $1/2$ under Chowla conjecture.

Keywords Topological dynamics, discrete spectrum, enveloping semigroup, $\mu$-tame systems, $\mu$-mean-equicontinuity, Sarnak’s Möbius disjointness conjecture, Mertens function, amenable group.

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1 Preliminaries and notation

This note results from trying to understand whether the notion of ‘discrete spectrum’ of a compact, metric, ergodic dynamical system $(X, T, \mu)$ can be captured in terms of the regularity properties of the elements its enveloping semigroup. It turns out that even though in general this type of characterization of systems with discrete spectrum is not possible, our study allows us to obtain other characterizations for more general acting groups $T$. In the second section we shall introduce the notion of ‘$\mu$-tameness’, which is a weakening of the notion of ‘tameness’ introduced by E. Glasner [19] based on the work of A. Köhler [32] (such systems in [32] are called “regular”). This notion also exists in a dormant form in the work of Bourgain [6]. The third section is devoted to the study of $\mu$-mean equicontinuity. In the fourth section we shall also characterize ‘weakly-tame’ and tame systems using the work of M. Talagrand on Glivenko-Cantelli families and the notion of ‘witness of irregularity’. This section can be viewed as our
efforts to give a simple and aseptic exposition, in the language of dynamical systems, to the results generated by the 1976 paper of J. Bourgain and the 1987 paper of M. Talagrand. These two papers are 'difficult to digest' and have generated a huge amount of literature.

In the fifth section we study – what we shall call 'Veech systems’. Professor W. Veech introduced an interesting class $\mathcal{K}(\mathbb{Z})$ of functions on integers which properly contains the class of all weakly almost periodic functions. Translation flow on the orbit closure of such a function is an example of a Veech system. The final section is where we apply our results to translations flow of a special Veech function will allow us to improve ‘Motohashi-Ramachandra estimates’ of the Mertens function in short interval. Thus, our method yields a ‘dynamical approach’ to these type of number theoretic estimates. As another application, we also establish, by a simple observation of Rauzy [46], Besicovich almost periodicity of a number theoretic function arising from the $B$-free integers integers.

We begin by introducing the notation and basic definitions. By a topological dynamical system $(X,T)$ we mean a compact, Hausdorff space $X$ on which a topological group $T$ acts (on the right), with a jointly continuous action $(x,t) \to \pi(x,t) \equiv \pi_t(x) \equiv xt$, $x \in X$ and $t \in T$. In what follows topology of $T$ will not play any part and so one may as well assume $T$ to be discrete. The set $O(x) = \{xt \mid t \in T\}$ is the orbit of $x \in X$. A subset $M \subset X$ is invariant if $O(x) \subset M$ for all $x \in M$. System $(X,T)$ is point transitive if it has a dense orbit and is minimal if all orbits are dense, (equivalently there are no proper closed invariant sets).

Following Prof. R. Ellis’s algebraic approach to dynamics, we try to capture the asymptotic properties of the system in terms of the algebraic properties of a suitable compactification of the acting group $T$. We begin by introducing three important compactifications we need, (1) the Stone-Čech compactification $(\beta T, T)$, (2) the enveloping or Ellis semigroup $E(X,T)$ and (3) the ‘ergodic analog’ of $E(X,T)$-namely $(\Omega_\mu, T)$. As the notation indicates, all of these compactifications are themselves going to be topological dynamical systems where the underlying compactification will be a compact Hausdorff space with a semigroup structure which has the common additional property of being an $\mathcal{E}$-semigroup. Before describing the compactifications, we recall the definition of an $\mathcal{E}$-semigroup.

**Definition 1.1** A set $E$ is an $\mathcal{E}$ semigroup if (i) it is a semigroup, (ii) it has a compact, Hausdorff topology and (iii) in this topology the left multiplication map $L_p : E \to E$, $L_p(q) = pq$, $p,q \in E$ is continuous.

1. **Stone-Čech Compactification** $\beta T$: Recall that element of $\beta T$ are ultrafilters on $T$. In fact $\beta T$ is an $\mathcal{E}$-semigroup with multiplication of ultrafilters $p,q \in \beta T$ given by

$$A \in pq \text{ if and only if } A * p = q,$$

where $A * p = \{t \in T \mid At^{-1} \in q\}$.

This compactification also has the universality property that any continuous map from $T$ to any compact, Hausdorff space has a unique continuous extension to $\beta T$, (here $T$ has the discrete topology). This universal property allows one to extend the $T$ action on a compact, Hausdorff space $X$ to an action of the semigroup $\beta T$. In particular, if a net $\{t_\alpha\}$ in $T$ converges to $p \in \beta T$, then $xp = \lim_{\alpha} xt_\alpha$, for $x \in X$.

Furthermore, this also implies that the dynamical system $(\beta T, T)$ is a universal point-transitive system...

2. **Enveloping Semigroup** $E(X,T)$: Let $E(X,T) = \{\pi_t \mid t \in T\}$, where the closure is in the topology of pointwise convergence on all maps from $X$ to $X$. Then $E(X,T)$ itself is an $\mathcal{E}$-semigroup and $(E(X), T)$ is a point transitive dynamical system.

3. **Measure theoretic enveloping semigroup** $(\Omega_\mu, T)$: Let $(X,T)$ be a compact, metric dynamical system with a $T$ invariant Borel probability measure $\mu$ on $X$. Let $H = L^2(X, \mu)$ and let $U_t[f] = [ft]$
where for a measurable function \( f : X \to \mathbb{C} \), \([f]\) denotes its equivalence class \((\text{mod } \mu)\) and \( f_t(x) = f(xt) \). Then \( t \to U_t \) unitary representation of \( T \) on \( H \). It is important that we distinguish between a measurable function \( f \) and its equivalence class \([f]\). Let \( \Omega_\mu = \{U_t \mid t \in T\} \), where the closure is in the weak operator topology. Then \( \Omega_\mu \) is a \( \mathcal{E} \)-semigroup and \((\Omega_\mu, T)\) is itself a point transitive dynamical system. The dynamical system \((\Omega_\mu, T)\) is weakly almost periodic (see [12], [13]). We shall list its special properties shortly.

Next we recall a few general facts about \( \mathcal{E} \)-semigroups, (see also [4], [11]). Let \( E \) be an \( \mathcal{E} \)-semigroup. A subset \( M \subset E \) is a right ideal if it is closed and \( m \in M, e \in E \) implies \( me \in M \). The following lemma summarizes the structure of minimal right ideals.

**Proposition 1.2** Let \( E \) be a \( \mathcal{E} \) semigroup and \( M \subset E \) be a minimal right ideal of \( E \). Then

1. The set \( J_M = \{v \in M \mid v^2 = v\} \) of idempotents in \( M \) is non-empty.
2. For each \( v \in J_M \), the set \( Mv \) is a subgroup of \( M \) with identity \( v \).
3. \( vp = p \), for each \( v \in J_M \) and \( p \in M \), i.e. each \( v \in J_M \) is a left identity in \( M \).
4. Any two minimal right ideals of \( E = \beta T \) are isomorphic.

As mentioned above, all of the previous three examples are \( \mathcal{E} \)-semigroups but \( \Omega_\mu \) has many additional features which we list now, (see ([12], [13] for proofs).

**Proposition 1.3** (1) The flow \((\Omega_\mu, T)\) is weakly almost periodic, in particular,

2. The right multiplication \( R_p(q) = qp, p, q \in \Omega_\mu \) is also continuous.
3. There is only one minimal right ideal in \( \Omega_\mu \), which we denote by \( I_\mu \).
4. The ideal \( I_\mu \) has a unique idempotent, which we denote by \( P_\mu \) and which commutes with all elements of \( I_\mu \).
5. The ideal \( I_\mu \) is closed under \( \ast \)-the operator adjoint,
6. In fact \( I_\mu \) is a compact topological group of operators and the weak and strong operator topologies on \( I_\mu \) coincide.

**Remark 1.4** Let \( \nu \) be the normalized Haar measure on the compact topological group \( I_\mu \) and let \( C_\mu = \int_{I_\mu} g d\nu \). Then the operator \( C_\mu \) is the projection on \( T \) invariant functions. Thus

1. \((X, T, \mu)\) is ergodic iff \( C_\mu = C \)-the the projection on constants,
2. \((X, T, \mu)\) has discrete spectrum if and only if \( P_\mu = I \)-the identity operator and in this case \( \Omega_\mu = I_\mu \).
3. \((X, T, \mu)\) is weakly mixing if and only if \( P_\mu = C \), (see [13] for details).
The projection maps \( p \to \rho_p : \beta T \to E(X,T) \) and \( p \to U_p : \beta T \to \Omega_\mu \).

(a) Since \((\beta T,T)\) is a universal point transitive flow and \(E(X,T)\) is point transitive, there is a canonical factor map \( p \to \rho_p : \beta T \to E(X,T) \) such that \( \rho_e = i_X \), i.e. this maps the identity \( e \) of \( T \) to the identity map \( i_X \) on \( X \). Equivalently, given a point transitive flow \((X,T,x_0)\), there is a unique continuous extension to \( \beta T \) of the map \( t \to x_0t : T \to X \). This defines a \( \beta T \) action on \( X \) given by

\[
x \cdot p = \rho_p(x), \quad (x \in X, \ p \in \beta T).
\]

(b) Again, since \((\beta T,T)\) is a universal point transitive flow, the map

\[
p \to U_p : \beta T \to \Omega_\mu,
\]

is the unique continuous extension of the semigroup homomorphism \( t \to U_t : T \to \Omega_\mu \) that takes the identity of \( T \) to \( I \)- the identity operator. It is also a semigroup homomorphism. Thus, if \( \{t_\alpha\} \) is a net in \( \beta T \) such that \( t_\alpha \to p \) in \( \beta T \), then

\[
U_p = \lim_{t_\alpha \to p} U_{t_\alpha}.
\]

Now fix a minimal (right) ideal \( M \subset \beta T \), (which ideal hardly matters because they are all isomorphic). Then \( M \) is a closed, \( T \)-invariant set of \((\beta T,T)\). Hence it is also a minimal set of the dynamical system \((\beta T,T)\). Since \((\beta T,T)\) is a universal point transitive flow, it follows that \((M,T)\) is a universal minimal flow. Thus, the restriction of the above map gives a canonical projection

\[
p \to U_p : M \to I_\mu \subset \Omega_\mu.
\]

Note that since \( \Omega_\mu \) has a unique minimal set \( I_\mu \), all minimal ideals will project onto \( I_\mu \) and since \( I_\mu \) is a group, all idempotents in any minimal ideal will be mapped into the projection operator \( P_\mu \) which is the identity of \( I_\mu \). Thus, given a net \( t_\alpha \to p \) in \( M \),

\[
U_p = \lim_{t_\alpha \to p} P_\mu U_{t_\alpha} = \lim_{t_\alpha \to p} U_{t_\alpha} P_\mu .
\]

Note that \( U_v = P_\mu \) for all idempotents \( v \in M \).

Now, given a measurable map \( f : X \to \mathbb{C} \), let \([f]\) denote its equivalence class determined by the relation defined by equality modulo a set of \( \mu \) measure zero. Then for \([f] \in L^2(X,\mu)\) and \( p \in \beta T \), we set

\[
[f]_p = U_p[f].
\]

Remark 1.5

1. Even though \([f]_t = [f]_1\) for \( t \in T \), we cannot replace \( t \in T \subset \beta T \) by a general \( p \in \beta T \) in this equation. To begin with, in general \( f_p \) may not be even measurable, so \([f]_p\) makes no sense. Even in the special case when \( \rho_p = i_X \), obviously \( f(xp) = f(x) \) but even in this case we cannot say \( U_p[f] = [f] \), as the following example will show.

2. Note that for the transformation \( T(x,y) = (x+\alpha, x+y) \) on the 2-torus \( \mathbb{T}^2 \), (where \( \alpha \notin \mathbb{Q} \)), \( \rho_v = i_X \) for all idempotents \( v \in \beta T \), (since \( T \) is distal), and \( U_v = P_\mu \neq I \), for all minimal idempotents \( v \in M \), (where \( \mu \) is the usual Lebesgue measure on \( \mathbb{T}^2 \)).
For systems with discrete spectrum, if \( \nu = \nu^2 \in M \), then \( U_\nu = I \). However \( U_\nu = I \) may not imply \( \rho_\nu = \iota_X \) as the following simple example shows. Let \( (X,T) \) be a minimal Sturmian shift which is always uniquely ergodic and has discrete spectrum with respect to the unique invariant measure \( \mu \). Then \( U_u = I \), for any minimal idempotent in \( M \) but \( \rho_u \neq \iota_X \) for some \( u = u^2 \in M \), as there are non-trivial proximal pairs in the system.

2 \( \mu \)-compact vectors and \( \mu \)-tame vectors.

**Definition 2.1** Let \((X,T,\mu)\) be a compact metric, ergodic dynamical system. A function \( f \in L^2(X,\mu) \) is a compact vector if the orbit \( \{U_t f \mid t \in T\} \) of \( f \) has compact closure in the norm topology on \( L^2(X,\mu) \).

With this definition, the following is a corollary to Proposition 1.3.

**Proposition 2.2** Let \((X,T,\mu)\) be a compact metric system and let \( f \in L^2(X,\mu) \). Then the following statements are equivalent,

1. \( f \) is a compact vector,
2. \( P_\mu(f) = f \),
3. the weak and the strong topologies on the set \( \overline{O(f)} \) of orbit closure of \( f \) coincide,
4. the system \( (\overline{O(f)},T) \) is minimal,
5. for some \( m \in M \), (where \( M \) is some minimal right ideal of \( \beta T \)), there is a sequence \( \{t_n\} \) in \( T \) such that \( U_{t_n}[f] \equiv [f]_{t_n} \rightarrow U_m[f] \) in \( L^2(X,\mu) \).

Next, we introduce the notion of a \( \mu \)-tame function.

**Definition 2.3** (a) Let \( f \in L^2(X,\mu) \). Then \( f \) is said to be \( \mu \)-tame if there exists a \( q \in \beta T \), a Borel set \( N \subset X \) with \( \mu(N) = 0 \) and a sequence \( \{t_n\} \) in \( T \) such that

1. \( f_{t_n} \rightarrow f_q \) pointwise on \( X \setminus N \). Thus, in particular the map \( f_q \equiv f \circ \rho_q : X \setminus N \rightarrow \mathbb{C} \) is a Borel map and
2. \( U_q \in I_\mu \). This will imply that there exists some \( m \in M \) such that \( U_m[f] = [1_{X \setminus N} f_q] = U_q[f] \),

where \( 1_{X \setminus N} \) is the indicator function of the set \( X \setminus N \).

(b) System \((X,T,\mu)\) will be called \( \mu \)-tame if each \( f \in L^2(X,T) \) is a \( \mu \)-tame vector.

**Remark 2.4** We recall the notion of a tame dynamical system \((X,T)\) introduced by Glasner-Khöler, (see [19], [32]).

**Definition 2.5** A compact, Hausdorff topological dynamical system \((X,T)\) is tame if each element of \( E(X,T) \) is a Baire-1 class function.
It follows that if $(X,T)$ is a tame system then $(X,T,\mu)$ is a $\mu$-tame system for any invariant Borel probability measure $\mu$ on $X$.

**Proposition 2.6** Let $f \in L^2(X,\mu)$. Then $f$ is $\mu$-compact if and only if it is $\mu$-tame.

**Proof.** Suppose $f$ is $\mu$-compact. Pick any $m \in M$, where $M$ is any minimal right ideal in $\beta T$. Then select a sequence $\{t_n\}$ in $T$ such that $U_{t_n}[f] \to U_m[f]$ in $L^2(X,\mu)$. This implies that by passing to a subsequence, (which we again denote by $\{t_n\}$), we can assume that $f_{t_n}$ converges pointwise on a set $X \setminus N$ for some Borel set $N$ with $\mu(N) = 0$. Now viewing the sequence $\{t_n\}$ as a net in $\beta T$, we can find a convergent subnet (which may not be a subsequence), converging to some $q \in \beta T$. It follows that the pointwise limit of $f_{t_n}$ on $X \setminus N$ is a Borel function on $X \setminus N$ and equals $f_q$. Note that $U_q[f] = U_m[f] \in I_\mu$.

Conversely, the hypothesis implies that for some sequence $\{t_n\}$ in $T$, $U_{t_n}[f] \to U_m[f]$ for some $m \in M$. Hence by (5) of Proposition (2.2), $f$ is $\mu$-compact. $lacksquare$

**Remark 2.7** It is immediate that if $f$ is tame then it is $\mu$-tame for any invariant Borel probability measure $\mu$ on $(X,T)$. In particular, with respect to any invariant measure, a tame system $(X,T)$ has discrete spectrum. The following example shows that system may not be tame even if all invariant ergodic measures have discrete spectrum.

**Example 2.8** Consider the system on the 2-torus $X = \mathbb{T}^2$ given by $T(x,y) = (x, x + y)$. Note that any ergodic measure for this system is of the form $\delta_x \times \nu$, where $\nu$ is either the Lebesgue measure $\lambda$ on the unit circle $\mathbb{T}^1$ if $x$ is irrational, or the uniform probability on the finite orbit $y + nx \mod 1$ if $x$ is rational. It follows from Namioka’s work, (see ([43])) that for this system $E(X,T) = \{ id \times f \mid f : \mathbb{T} \to \mathbb{T} \text{ is any homomorphism} \}$. Thus, a ‘large number’ of elements of $E(X,T)$ are not even measurable. Thus this system is not tame. But it is easy to check that it is $\mu$-tame for any invariant ergodic measure.

This simple example also illustrates that $\mu$-tame with respect to all invariant ergodic measures does not imply $\mu$-tame for any invariant measure. Since $\lambda \times \lambda$ is a non-ergodic invariant measure, with respect to which $T$ does not have discrete spectrum, this example also shows that systems can have discrete spectrum with respect to all invariant ergodic measure but may fail to have discrete spectrum with respect to all invariant measures.

**Remark 2.9** E. Glasner proved that if a distal system is tame, then it is equicontinuous. The above example being distal, shows that the analogue of Glasner’s result is false if ‘tame’ is replaced by ‘$\mu$-tame’. However some analogue of this might be true. For example, we would like to know if $(X,T,\mu)$ is minimal, distal and $\mu$-tame, then is it equicontinuous? and for such non-minimal systems we ask whether the system is equicontinuous on the support of $\mu$.

### 3 $\mu$-mean equicontinuous vectors.

We first recall a few necessary things about amenable groups. Let $T$ be a countable (discrete) group.

**Definition 3.1**
(1) Given finite sets \( F, K \subset T \), \( F \) is \((K, \epsilon)\)-invariant if \( |KF \Delta F| < \epsilon |F| \).

(2) A sequence \( \{F_n\} \) of finite subsets of \( T \) is a Følner sequence if given any \( \epsilon > 0 \) and a finite set \( K \subset T \), there exists a \( n_0 \in \mathbb{N} \) such that \( F_n \) is \((K, \epsilon)\)-invariant for all \( n > n_0 \). This is equivalent to saying that
\[
\lim_{n \to \infty} \frac{|tF_n \Delta F_n|}{|F_n|} = 0, \quad \text{for each } t \in T.
\]

(3) A Følner sequence \( \{F_n\} \) is tempered if there exists a constant \( C > 0 \) such that
\[
| \bigcup_{k \leq n} F_k^{-1} F_{n+1} | \leq C |F_{n+1}|, \quad \text{for all } n \in \mathbb{N}.
\]

Remark 3.2 Every Følner sequence has a tempered subsequence, (see [34]).

Definition 3.3 Fix a Følner sequence \( \mathcal{F} = \{F_n\} \). Let \( A \subset T \).

(1) The (asymptotic) density \( \bar{d}_F \) of \( A \) with respect to \( \mathcal{F} \) is given by
\[
\bar{d}_F = \limsup_{n \to \infty} \frac{|A \cap F_n|}{|F_n|}.
\]

(2) For \( A \subset T \) and a finite subset \( F \subset T \), set
\[
D^*_F(A) = \sup_{t \in T} \frac{|A \cap Ft|}{|F|},
\]
\[
D^*(A) = \inf \left\{ D^*_F(A) \mid F \subset T, \ |F| < \infty \right\}.
\]

Then \( D^*(A) \) is called the upper Banach density of \( A \).

Lemma 3.4

(1) Let \( \{F_n\} \) be a Følner sequence in \( T \) and \( A \subset T \). Then
\[
D^*(A) = \lim_{n \to \infty} D^*_{F_n}(A).
\]
In particular the above limit exist and is independent of the choice of Følner sequence.

(2) Furthermore
\[
D^*(A) = \sup_{\mathcal{F}} \left( \limsup_{n \to \infty} D^*_{F_n}(A) \right),
\]
where \( \mathcal{F} = \{F_n\} \) varies over all Følner sequences in \( T \).

We shall use the following pointwise ergodic theorem for \( L^1 \) functions, (see [34]).
Theorem 3.5 Let \((X,T,\mu)\) be an ergodic, probability preserving system with \(T\) amenable. Let \(\mathcal{F} = \{F_n\}\) be a tempered Følner sequence and \(f \in L^1(X,\mu)\). Then
\[
\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{t \in F_n} f(\tau^t) = \int_X f(x)\,d\mu(x), \quad \text{a.e. } x.
\]

Definition 3.6 (Besicovitch seminorm and Besicovitch functions on \(\mathbb{T}\)) Fix a tempered Følner sequence \(\mathcal{F} = \{F_n\}\).

1. On the space of complex valued maps on \(\mathbb{T}\), define the Besicovitch seminorm \(\|\cdot\|_{B_1}\) by setting,
\[
\|f\|_{B_1} = \limsup_{n \to \infty} \frac{1}{|F_n|} \sum_{t \in F_n} |f(t)|.
\]

2. A map \(f : \mathbb{T} \to \mathbb{C}\) is Besicovitch if \(\|f\|_{B_1} < \infty\).

Remark 3.7 Next, we want to define the notion of ‘Besicovitch almost periodic function’. When \(T\) is abelian, the classical definition says \(T \to \mathbb{C}\) is Besicovitch almost periodic if given any \(\varepsilon > 0\), there exists a trigonometric polynomial \(P\) such that \(\|f - P\|_{B_1} < \varepsilon\). By a trigonometric polynomial we mean a finite linear combination of characters of \(T\). For non-abelian groups ‘trigonometric polynomials’ will have to be replaced by the matrix coefficient functions of finite dimensional irreducible, unitary representations of \(T\). For non-abelian \(T\) these irreducible, unitary representations are not necessarily one dimensional and hence one cannot just add the matrix coefficients functions and demand \(f\) be approximated by them. A proper way to do this is to consider the given \(f\) as an element of the Hilbert space \(L^2(T)\), decompose the left regular representation of \(T\), assume that it decomposes in to an orthogonal direct sum of finite dimensional irreducible unitary representations and then demand that the projection of \(f\) on each irreducible subspace be approximable by a vector valued map on \(\mathbb{T}\) with coefficients given by the matrix coefficients of the underlying ‘piece of the unitary representation’ from the decomposition.

Now given a compact, metric ergodic dynamical system \((X,T,\mu)\) with amenable \(T\) and \(f \in L^2(X,\mu)\), a.e \(x \in X\) we get a complex valued function \(\psi_{x,f}(t) = f(\tau^t)\). Since we shall be interested in maps on \(\mathbb{T}\) arising this way, we may make the above notion precise by considering the unitary representation \(t \to U_t\) on \(L^2(X,\mu)\) instead of the left regular representation and try to see when \(\psi_{x,f}\) is ‘Besicovitch almost periodic’ for almost all \(x \in X\). As one can guess, this is exactly the case when \(f\) is \(\mu\)-compact. The next lemma puts all of this discussion on a more formal footing.

Lemma 3.8 Let \((X,T,\mu)\) be ergodic and \(f \in L^2(X,\mu)\) be \(\mu\)-compact. Consider the closed subspace \(H = H_f\) of \(L^2(X,\mu)\) generated by the span of \(\{U_t[f] \mid t \in \mathbb{T}\}\). Then \(H\) can be written as an orthogonal direct sum \(H = \bigoplus V_k\), where each \(V_k\) is a finite dimensional, \(U_t\) invariant subspace. The representation \(U_t\) restricted to each \(V_i\) is irreducible. Let \(P_t f\) denote the orthogonal projection of \(f\) onto \(V_i\), (\(i \in \mathbb{N}\)). Then \(f = \sum_{i \in \mathbb{N}} P_t f\) and each \(P_t f\) is of the form
\[
P_t f(x) = \sum_{j=1}^{d_i} (f^j_t(x), w^j) w^j,
\]
where \(d_i = \text{dim}(V_i)\), \(\{w^j_i \mid 1 \leq j \leq d_i\}\) is a fixed basis of \(V_i\) and \(f^j_t : X \to V_j \equiv \mathbb{C}^{d_j}\) are measurable maps.
Proof. This is just an application of the Peter-Weyl theorem to the compact topological group $I_\mu$, the unique minimal ideal of $\Omega_\mu$. Note that, since $f$ is $\mu$-compact, $P_\mu f = f$ and $H$ is the closed linear space of $\{Uf \mid U \in I_\mu\}$. Thus the compact topological group of unitary operators $I_\mu$ has a natural unitary representation on $H$. By Peter-Weyl theorem $H = \bigoplus V_k$ where each $V_k$ is a finite dimensional, $U_t$ invariant subspace. The representation $U_t$ restricted to each $V_i$ is irreducible. The rest of the lemma is a trivial consequence of linear algebra. □

Remark 3.9

(1) The above representations of $f$ and $P_i f$ are to be understood as expressions in $L^2(X, \mu)$.

(2) Note that if $f$ is $\mu$-compact and $f$ has the above representation, then

$$U_t f(x) = \sum_{i \in \mathbb{N}} U_t P_i f = \sum_{i \in \mathbb{N}} \sum_{j = 1}^{d_i} \langle U_t f_i^j(x), w_i^j \rangle w_i^j.$$ Again, this representation is to be understood as an expression in $L^2(X, \mu)$.

Definition 3.10

(1) A function which is a finite sum of functions of the form $t \to \langle U_t v, w \rangle$ will be called ‘generalized trigonometric polynomials on $T$, where $t \to U_t$ is a unitary representation of $T$ on a finite dimensional vector space $V$ and $v, w \in V$.

(2) A function $f \in L^2(T)$ that can be approximated in the $\| \cdot \|_{B_1}$ norm by a generalized trigonometric polynomial will be called a Besicovitch almost periodic function.

The following theorem is a generalization to ergodic amenable group actions of a known characterization of discrete spectrum for abelian group actions.

Theorem 3.11 Let $(X, T, \mu)$ be an ergodic system, with $T$ amenable. Then the following statements are equivalent.

(1) A vector $f \in L^2(X, \mu)$ is a $\mu$-compact vector.

(2) For $\mu$-almost all $x \in X$, the map $\psi_{x,f}(t) = f(xt)$ is a Besicovitch almost periodic function, in the sense that given $\varepsilon > 0$, there exists a measurable map $P : X \to \mathbb{C}$ such that (i) for almost all $x$ the map $t \to P(xt) \equiv U_t P$ is a generalized trigonometric polynomial and (ii) $\| \psi_{x,f} - \psi_{x,P} \|_{B_1} < \varepsilon$.

Proof. (1) implies (2): Since $f$ is $\mu$-compact, we have the representation

$$\psi_{x,f} = U_t f(x) = \sum_{i \in \mathbb{N}} U_t P_i f = \sum_{i \in \mathbb{N}} \sum_{j = 1}^{d_i} \langle U_t f_i^j(x), w_i^j \rangle w_i^j.$$ Given $\varepsilon > 0$, select $k \in \mathbb{N}$ such that $\|f - P\|_2 < \varepsilon$, (and hence $\|f - P\|_1 < \varepsilon$), where $P(x) = \sum_{i=1}^{k} P_i f(x) = \sum_{i=1}^{k} \sum_{j=1}^{d_i} \langle f_i^j(x), w_i^j \rangle w_i^j$. But, by the ergodic theorem, for almost all $x \in X$ we have
\[ \|\psi_{x,f} - \psi_{x,P}\|_{B_1} = \|f - P\|_1 < \varepsilon. \] This proves that for almost all \( x \in X \), the map \( t \to \psi_{x,f}(t) = f(xt) \) is Besicovitch almost periodic.

(2) implies (1): It follows from our assumption that given \( \varepsilon > 0 \), \( \|\psi_{x,f} - \psi_{x,P}\|_{B_1} < \varepsilon \) for almost all \( x \), where \( t \to U_t P \) is a generalized trigonometric polynomial. Again by the ergodic theorem \( \|f - P\|_1 = \|\psi_{x,f} - \psi_{x,P}\|_{B_1} < \varepsilon \), (for suitable \( x \)'s). since \( t \to U_t P \) is a generalized trigonometric polynomial, \( P \)-is a \( \mu \)-compact vector. Thus, we have shown that there is a sequence \( P_n \) of \( \mu \)-compact vectors that converge to \( f \) in the \( L^1(X) \) norm. But then there is a subsequence of \( \{P_n\} \) that converges pointwise almost everywhere and hence in the \( L^2(X) \) norm to \( f \). Whence, \( f \) is \( \mu \)-compact.

**Corollary 3.12** Let \( (X,T,\mu) \) be uniquely ergodic with discrete spectrum and \( f \in C(X) \). Then \( t \to f(xt) \) is Besicovitch almost periodic for every \( x \in X \).

**Proof.** This follows from the argument used in (1) implies (2) of the above theorem, since each \( x \in X \) is \((f,\mu)\) generic.

Next, we generalize to amenable group actions, another characterization of \( \mu \)-compact vectors in terms of \( \mu \)-mean equicontinuous vectors. This result is originally due to B. Scarpellini, (see [50]) and more recently to García-Ramos, see [24], see also [28]. We begin by defining the notion of \( \mu \)-mean equicontinuity.

**Definition 3.13 (\( \mu \)-Mean Equicontinuity)** Let \( (X,T,\mu) \) be a compact, metric dynamical system. Let \( T \) be amenable with a given Følner sequence \( \mathcal{F} = \{F_n\} \).

(1) Let \( K \subset X \). A vector \( f \in L^2(X,\mu) \) is called a \( \mu \)-mean equicontinuous vector on \( K \) if given any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( x, y \in K \) and \( d(x, y) < \delta \) then \( \|\psi_{x,f} - \psi_{y,f}\|_{B_1} < \varepsilon \).

(2) A vector \( f \in L^2(X,\mu) \) is called a \( \mu \)-mean equicontinuous if given any \( \eta > 0 \), there exists a compact set \( K \subset X \) such that \( \mu(K) > 1 - \eta \) and \( f \) is \( \mu \)-mean equicontinuous on \( K \).

(3) A dynamical system \( (X,T,\mu) \) is \( \mu \)-mean equicontinuous if each \( f \in L^2(X,\mu) \) is a \( \mu \)-mean equicontinuous vector.

**Proposition 3.14** Let \( (X,T,\mu) \) be a compact metric, ergodic dynamical system with \( T \) amenable. Let \( f \in L^2(X,\mu) \) be a compact vector. Then \( f \) is a \( \mu \)-mean equicontinuous vector.

**Proof.** Since \( f \) is \( \mu \)-compact, it has a representation \( f(x) = \sum_{i \in \mathbb{N}} P_i f \), where \( P_i \) is the orthogonal projection operator on to subspace \( V_i \), (we shall use the previous notation in this proof). Thus given \( \varepsilon > 0 \) we can select \( k \in \mathbb{N} \) such that \( \|f - P\|_2 < \varepsilon/3 \), where \( P = \sum_{i=1}^k P_i f \). Let \( M_1 \subset X \) be the of set points at which the ergodic average of \( f - P \) converges to \( \|f - P\|_1 \). Thus \( \mu(M_1) = 1 \) and if \( x \in M_1 \),

\[ \frac{\varepsilon}{3} \geq \|f - P\|_2 \geq \|f - P\|_1 = \|\psi_{x,f} - \psi_{x,P}\|_{B_1}. \]

Thus if \( x, y \in M_1 \), then

\[ \|\psi_{x,f} - \psi_{y,f}\|_{B_1} \leq \|\psi_{x,f} - P(x)\|_{B_1} + \|P(x) - P(y)\|_{B_1} + \|P(y) - \psi_{y,f}\|_{B_1} \leq \frac{2\varepsilon}{3} + \|P(x) - P(y)\|_{B_1}. \]
We show that \( \|P(x) - P(y)\|_{B_1} < \frac{\varepsilon}{3} \), if \( x \) and \( y \) are close enough. Recall that \( P \) has the form

\[
P(x) = \sum_{i=1}^{k} \sum_{j=1}^{d_i} \langle f_j^i(x), w_j^i \rangle w_j^i,
\]

where \( w_j^i \in V_i \subset L^2(X, \mu) \) and \( f_j^i : X \to \mathbb{C}^{d_i} \) are measurable. Given \( \varepsilon > 0 \), by Egorov's theorem pick a compact set \( M_2 \subset X \) such that \( f_j^i \) and \( w_j^i \) are continuous on \( M_2 \). Let \( K \subset M_1 \cap M_2 \), be compact such that \( \mu(K) > 1 - \varepsilon \). Select \( \delta > 0 \) such that

\[
\text{if } d(x, y) < \delta, \ x, y \in K, \text{ then } \sum_{i=1}^{k} d_i \| f_j^i(x) - f_j^i(y) \| \| w_j^i \| < \frac{\varepsilon}{3}.
\]

Now for \( x, y \in K \), with \( d(x, y) < \delta \), the following pointwise representation for \( U_t P \) gives

\[
\left| (U_t P)(x) - (U_t P)(y) \right| = \left| \sum_{i=1}^{k} \sum_{j=1}^{d_i} \langle (U_t f_j^i)(x) - U_t f_j^i(y), w_j^i \rangle w_j^i \right|,
\]

\[
\leq \left| \sum_{i=1}^{k} \sum_{j=1}^{d_i} \| U_t f_j^i(x) - U_t f_j^i(y) \| \| w_j^i \| \right|
\]

\[
\leq \sum_{i=1}^{k} d_i \| f_j^i(x) - f_j^i(y) \| \| w_j^i \| < \frac{\varepsilon}{3}.
\]

Thus,

\[
\|P(x) - P(y)\|_{B_1} = \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{t \in F_n} \left| (U_t P)(x) - (U_t P)(y) \right| < \frac{\varepsilon}{3}.
\]

Whence, if \( x, y \in K \) and \( d(x, y) < \delta \), then \( \| f(x) - f(y) \|_{B_1} < \varepsilon \) and the proof is complete.

For abelian acting groups \( T \), the converse of the above theorem is true and there are several proofs, using different arguments, (see [24], [28]). We shall present a result with yet another argument and weaker assumptions, which in particular will yield the converse. In the following theorem we weaken the ‘condition of continuity’ in the notion of mean equicontinuity to obtain a sufficiency condition for discrete spectrum. All we need is just one point having three key properties. As for the converse of the above theorem for non-abelian acting groups none of these proofs will generalize in a straightforward way.

**Theorem 3.15** Let \( (X, T, \mu) \) be a compact, metric ergodic dynamical system with \( T \) abelian. Let \( f \in L^2(X, \mu) \). Suppose there exists a point \( x_0 \in X \) satisfying the following conditions:

1. \( x_0 \) is \((f, \mu)\) generic, i.e. the ergodic average of \( f \) converges at \( x_0 \) to \( \int_X f \, d\mu \),

2. \( x_0 \) is a point of continuity of the map \( x \to \psi_{x,f} \), i.e. given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( d(x, y) < \delta \), then \( \| \psi_{x,f} - \psi_{y,f} \|_{B_1} < \varepsilon \).

3. For \( \delta > 0 \), let \( R_{x_0}(\delta) = \{ t \in T \mid d(x_0, x_0 t) < \delta \} \). Suppose for any \( \delta > 0 \) there exists a minimal ideal \( M \equiv M_\delta \subset \beta T \) such that \( R_{x_0}(\delta)R_{x_0}(\delta)^{-1} \cap M \neq \emptyset \), here the closure is in the topology on \( \beta T \).
Then \( f \) is a \( \mu \)-compact vector.

**Proof.** First, we claim that hypothesis (3) above, implies that there exists a unitary operator \( V \in I_\mu \subset \Omega_\mu \) such that given any \( n \in \mathbb{N} \) and \( g, h \in L^2(X, \mu) \), there exists \( t, s \in R_{x_0}(\frac{1}{n}) \) such that

\[
|\langle U_{ts^{-1}}g, h \rangle - \langle Vg, h \rangle| < \frac{1}{n}.
\]

To prove the claim consider, \( F_n = \{U_{ts^{-1}} \mid t, s \in R_{x_0}(\frac{1}{n})\} \cap I_\mu \), where the closure is in the topology on \( \Omega_\mu \), i.e. in the weak operator topology. Since the family of non-empty closed sets \( \{F_n\} \) has the finite intersection property and \( I_\mu \) is compact, \( \bigcap_{n \in \mathbb{N}} F_n \neq \emptyset \). Pick any \( V \in \bigcap_{n \in \mathbb{N}} F_n \). The claim follows from this.

Next, let \( \varepsilon > 0 \) and \( h \in L^\infty(X, \mu) \subset L^2(X, \mu) \) with \( ||h||_\infty \leq 1 \) be given. Using hypothesis (2) select \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \frac{\varepsilon}{3} \) and

\[
\text{if } d(x_0, y) < \frac{1}{n}, \text{ then } ||\psi_{x_0, f} - \psi_{y, f}||_{B_1} < \frac{\varepsilon}{3}.
\]

For this \( n \) and taking \( g = f \) in the above claim, select \( t, s \in R_{x_0}(\frac{1}{n}) \) such that

\[
|\langle f_{ts^{-1}}, h \rangle - \langle Vf, h \rangle| < \frac{1}{n} < \frac{\varepsilon}{3}.
\]

Now, since \( T \) is abelian, \( \psi_{xt, s} = f(xts) = f(xst) = f_t(xs) = \psi_{x_0, f_t}(s) \), for any \( x \in X, t, s \in T \). Thus, if \( t \in R_{x_0}(\frac{1}{n}) \). Then, by our choice of \( n \), we have

\[
\frac{\varepsilon}{3} \geq ||\psi_{x_0, f} - \psi_{x_0, f_t}||_{B_1} = ||\psi_{x_0, f} - \psi_{x_0, f_t}||_{B_1} = ||\psi_{x_0, (f-f_t)}||_{B_1} = ||f - f_t||_1.
\]

The last equality comes from hypothesis (1) and the fact that if \( (x_0, f) \) is \( \mu \) generic then so is \( (x_0t, f) \). Similarly \( ||f - f_s||_1 < \frac{\varepsilon}{3} \), if \( s \in R_{x_0}(\frac{1}{n}) \). Thus \( ||f - f_{ts^{-1}}||_1 \leq ||f - f_t||_1 + ||f - f_s||_1 < \frac{2\varepsilon}{3} \), if \( t, s \in R_{x_0}(\frac{1}{n}) \).

Now, note that

\[
|\langle f, h \rangle - \langle Vf, h \rangle| \leq |\langle f, h \rangle - \langle f_{ts^{-1}}, h \rangle| + |\langle f_{ts^{-1}}, h \rangle - \langle Vf, h \rangle| \\
\leq ||f - f_{ts^{-1}}||_1 ||h||_\infty + |\langle f_{ts^{-1}}, h \rangle - \langle Vf, h \rangle| < \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, \( \langle f - Vf, h \rangle = 0 \) and since \( h, (||h||_\infty \leq 1) \), is arbitrary, \( f = Vf \). Since \( V \in I_\mu \), \( f \) is \( \mu \)-compact.

As a consequence of the above proof, for abelian \( T \), we shall give yet another proof of the converse of Proposition 3.14. But, first, we state a result attributed to V. Bergelson, (see [44]), that we shall need here, as well as later.

**Lemma 3.16** Let \((X, T)\) be a dynamical system, \( T \) is countable and \( \mu \) be an ergodic Borel probability measure on \( X \). Let \( A \subset X \) be a Borel set with \( \mu(A) > 0 \). Let \( R = R(x, A) = \{t \in T \mid xt \in A\} \). Then \( RR^{-1} \) is a \( \Delta^* \) set for any \( x \in \text{Supp}(\mu) \). In particular, \( RR^{-1} \) is syndetic and hence its closure in \( \beta T \) intersects every minimal right ideal of \( \beta T \).

We recall that a set \( A \subset T \) is \( \Delta^* \) if and only if it intersects every difference set, i.e. given any infinite sequence \( \{t_n\} \) in \( T \), \( A \cap \{t_n t_m^{-1} \mid n > m\} \neq \emptyset \).
Proposition 3.17 Let $(X,T,\mu)$ be a compact metric ergodic dynamical system with $T$ abelian and let $f \in L^2(X,\mu)$ be $\mu$-mean equicontinuous. Then $f$ is $\mu$-compact.

Proof. Let $\eta > 0$ be given. Let $M_c \subset X$ be a Borel set such that $\mu(M_c) > 1 - \eta$ and the restriction to $M_c$ of the map $x \rightarrow \psi_{x,f}$ is continuous. Consider sets,

$$M_e = \{x \in X \mid \text{ergodic average of } f \text{ converges at } x\},$$

$$M_s = \text{Supp}(\mu),$$

By the ergodic theorem $\mu(M_e) = 1$ and $\mu(M_s) = 1$ always holds. Pick $x_0 \in M_c \cap M_e \cap M_s$. Since $x_0 \in M_e$, it satisfies hypothesis (1) of Theorem 3.15. The proof is exactly as the proof of Theorem 3.15, except that the set $R \equiv R_{x_0}(\delta)$ will be replaced by the set $R_{x_0}(\delta) \cap M_c$. Note that since $x_0 \in \text{Supp}(\mu)$ and $\mu(R) > 0$, above lemma can be applied to conclude that the closure of the set $RR^{-1}$ in $\beta T$ intersects every minimal right ideal of $\beta T$. Thus hypothesis (3) and ‘modified hypothesis’ (2) of Theorem 3.15 holds. So the proof follows exactly as before by selecting $t,s \in R_{x_0}(\frac{1}{n}) \cap M_c$. \[\blacksquare\]

Remark 3.18

(1) Hypothesis (3) of Theorem 3.15 demands that the return time set $R \equiv R_{x_0}(\delta)$ satisfy that $RR^{-1}$ is a ‘large’ set. In fact $R$ may be of density zero. An example of such a case is a set that is piecewise syndetic. In act, this hypothesis is much weaker, we only need $RR^{-1}$ to be piecewise syndetic. Thus our theorem will yield stronger corollaries than the one above but we leave this to the reader.

(2) Recall that in the definition of $\mu$-mean equicontinuity of a vector $f$, given $\varepsilon > 0$, we have a compact set $K$ with large measure and a set $S$ with large density so that if $x$ and $y$ in $K$ are sufficiently close, then $f(xs)$ and $f(ys)$ are within $\varepsilon$ for $s \in S$. This is much weaker than demanding equicontinuity of the family $\{f_s \mid s \in S\}$ on $K$. On the other hand if we demand equicontinuity of this family, where the set $S$ may even have zero density, we can still prove $\mu$-compactness of $f$ if we assume that $S$ is ‘large’ but not in the sense of density. The following proposition can be proved by similar arguments to those in Theorem 3.15, however, here $T$ can be any (infinite) group. We shall leave the proof to the reader.

Theorem 3.19 Let $(X,T,\mu)$ be a compact, metric ergodic dynamical system. Let $f \in L^2(X,\mu)$. Suppose for any $\varepsilon > 0$ there exists (i) a compact $K \subset X$ such that $\mu(K) > 1 - \varepsilon$, (ii) a minimal ideal $M \in \beta T \setminus T$ and (iii) a set $S \subset T$ such that the following holds.

1. $\{f_s \cdot \chi_K \mid s \in S\}$ is an equicontinuous family and

2. for any infinite sequence $\{s_n\}$ in $S$, the difference set $D(S) \equiv \overline{\{s_n s_m^{-1} \mid n > m\}} \cap M \neq \emptyset$, (where the closure is in $\beta T$).

Then $f$ is a $\mu$-compact vector.
4 A Characterization of $\mu$-compact vectors using the notion of ‘witness of irregularity’

Following ideas of M. Talagrand and others, we shall see another characterization of $\mu$-compact vectors using the notion of a ‘Glivenko-Cantelli family’, (see Theorem (4.10)). In probability theory one studies ‘Glivenko-Cantelli family’ of random variables to investigate conditions on the family under which the law of large numbers has a ‘uniform version’. In the later half of this section we compare ‘weakly tame’ with tame systems, (Theorem (4.12)). It is our hope that this exposition and proofs will make contributions in references [16], [21],[23], [51],[52] more accessible from the dynamics point of view. First, we begin by recalling the notion of a Glivenko-Cantelli family.

Definition 4.1 Let $(X, B, \mu)$ be a complete probability space. Let $L^1 \equiv L^1(X, \mu)$ denote the space of measurable functions $f$ such that $E(f) \overset{\text{def}}{=} \int_X f \, d\mu < \infty$. We shall not identify functions in $L^1$ with their classes in $L^1(X, \mu)$.

(1) A subset $Z \subset L^1$ is ordered bounded if there exists a $u \in L^1$, $u \geq 0$ such that for each $f \in Z$, we have $|f(x)| \leq u(x), \forall x \in X$.

(2) We also recall the following notion of uniform ergodicity à la Glivenko-Cantelli. A subset or a family $Z \subset L^1$ is $\mu$-Glivenko-Cantelli if

$$\lambda-a.e. \lim_{N \to \infty} \sup_{f \in F} \left| \frac{1}{N} \sum_{k=1}^{N} f(\omega_k) - \int f \, d\mu \right| = 0.$$ 

Here $\omega = (\omega_0, \omega_1, \omega_2, \cdots)$ is an i.i.d. process with common distribution $\mu$;

First let $T = Z$. We shall now show that when our underlying probability space is a compact metric dynamical systems $(X, T, \mu)$, for $f \in L^2(X, \mu)$, the $\mu$-compactness of $f$ is equivalent to the family $Z \equiv F = \{f \circ T^n \mid n \in \mathbb{Z} \}$ being $\mu$-Glivenko-Cantelli. We begin with one of the implication.

Proposition 4.2 Suppose $(X,T,\mu)$ be a compact, metric ergodic dynamical system. Let $f \in L^2(X, \mu)$ be a $\mu$ compact vector. Then the family $\{f \circ T^n \mid n \in \mathbb{Z} \}$ is a $\mu$-Glivenko-Cantelli family.

Proof. First we verify this when $f$ is an $L^2$ eigenfunction, i.e. $f(Tx) = e^{2\pi i \lambda} f(x)$, a.e $x$. By ergodicity, we can assume that $f$ is bounded almost surely. We need to verify that the family $\{f \circ T^n \mid n \in \mathbb{Z} \}$ is $\mu$-Glivenko-Cantelli. Note that

$$\sup_{m} \left| \frac{1}{N} \sum_{k=1}^{N} f \circ T^m(\omega_k) \right| = \sup_{m} \left| \lambda^m \frac{1}{N} \sum_{k=1}^{N} f(\omega_k) \right|$$

$$= \frac{1}{N} \left| \sum_{k=1}^{N} f(\omega_k) \right|. \quad (4.1)$$

The last term converge to zero by the law of large numbers. Now, let $f \in L^2(X, \mu)$ be $\mu$-compact. Then we can write $f = \sum_{j=0}^{+\infty} \alpha_j f_j$, where each $f_j$ is an eigenfunction and $\sum_{j=0}^{+\infty} |\alpha_j|^2 < \infty$ and the rest is clear.

Now, we prove the converse.
Proposition 4.3 Suppose \((X,T,\mu)\) be a compact, metric ergodic dynamical system. Suppose the family \(\{f \circ T^n \mid n \in \mathbb{Z}\}\) is a \(\mu\)-Glivenko-Cantelli family. Then \(f \in L^2(X,\mu)\) is a \(\mu\) compact vector.

Proof. First write the ‘compact-weak-mixing decomposition’ of \(f, f = f_c + f_{wm}\), where \(f_c\) is \(\mu\)-compact and \(f_{wm}\) is a non-constant \(\mu\)-weak mixing vectors in \(L^2(X,\mu)\). From the above proposition we know that \(\{f_c \circ T^n \mid n \in \mathbb{Z}\}\) is \(\mu\)- Glivenko-Cantelli. Thus, the hypothesis implies that \(\{(f - f_c) \circ T^n \mid n \in \mathbb{Z}\}\) is \(\mu\)-Glivenko-Cantelli, i.e. the family \(\{f_{wm} \circ T^n \mid n \in \mathbb{Z}\}\) is \(\mu\)-Glivenko-Cantelli. Since \(f_{wm}\) is weakly-mixing, along a subsequence \((n_j)\) of density 1, we have,

\[
\lim_{j \to \infty} \left\langle f_{wm} \circ T^{n_j}, g \right\rangle = \int f_{wm} d\mu \int g d\mu, \quad \text{for any } g \in L^\infty(X,\mu).
\]

Since the family \(\{f_{wm} \circ T^n, n \in \mathbb{Z}\}\) is \(\mu\)-Glivenko-Cantelli, by a lemma of M. Talagrand, ([52, Proof of Proposition 2.5, p. 379], or [16, Chap. 46, pp. 59-60]), there exist a finite sub-algebra \(\mathcal{P}\) of the Borel sigma algebra of \(X\) such that, for any \(j \in \mathbb{N}\), one has

\[
\left\| f_{wm} \circ T^{n_j} - E(f_{wm} \circ T^{n_j} \mid \mathcal{P}) \right\|_1 < \varepsilon, \quad (4.2)
\]

where \(E(\cdot \mid \mathcal{P})\) is the projection operator of the conditional expectation with respect to \(\mathcal{P}\). Moreover, by the property of this projection operator we have,

\[
E(f_{wm} \circ T^{n_j} \mid \mathcal{P}) = \sum_{P \in \mathcal{P}} \frac{1}{\mu(P)} \left( \int f_{wm} \circ T^{n_j} d\mu \right) \chi_P(x). \quad (4.3)
\]

Letting \(j \to \infty\), we get,

\[
\lim_{j \to \infty} E(f_{wm} \circ T^{n_j} \mid \mathcal{P}) = \int f_{wm} d\mu \sum_{P \in \mathcal{P}} \chi_P(x), \quad (4.4)
\]

for almost all \(x \in X\). This combined with \(4.2\) yields

\[
\limsup_{j \to \infty} \left\| f_{wm} \circ T^{n_j} - \sum_{P \in \mathcal{P}} \chi_P(x) \int f_{wm} d\mu \right\|_1 \leq \varepsilon. \quad (4.5)
\]

Whence

\[
\left\| f_{wm} - \int f_{wm} d\mu \right\|_1 \leq \varepsilon, \quad (4.6)
\]

which is impossible since, \(f_{wm}\) is a non-constant weak-mixing vector. The proof of the proposition is complete.

\[\blacksquare\]

Remark 4.4 Now we try to capture \(\mu\) compactness of the vector \(f\) in terms of ‘up-crossings’. The notion of ‘witness of irregularity’ captures the case when the ‘up-crossings of the family \(F = \{f_t \mid t \in T\}\) are ‘wild’ and \(\mu\)-compactness of \(f\) can be characterized when this does not happen. Following M. Talagrand, two non-negative numbers are introduced which are in some sense another notions of ‘entropy’ for the above family \(F\). Again, \(\mu\)-compactness of \(f\) can be characterized by vanishing of these two numbers and in this sense they ‘better’ serve the purpose of characterizing \(\mu\)-compactness of \(f\) than the usual measure theoretic or topological entropy. These notions will also allow us to generalize above two proposition to the setting of ergodic actions of countable, amenable groups. For the following concepts we refer to [52].

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Definition 4.5  We shall first recall our more general set up, where \((X, \mathcal{B}, \mu)\) be a complete probability space. Let \(Z\) be a uniformly bounded subset of real valued measurable maps on \(X\). First we introduce quantities \(N^1(Z,n,\varepsilon)\) and \(N^\infty(Z,n,\varepsilon)\) whose logarithmic growth rate, (as \(n \to \infty\)), will capture a version of the ‘notion of entropy’ for the family \(Z\).

(1) For a fixed \(n \in \mathbb{N}\), consider the following norms on \(\mathbb{R}^n\),
\[
\|x\|_1 = \frac{1}{n} \sum_{k=1}^{n} x_k, \quad \text{and} \quad \|x\|_\infty = \max\{|x_k| \mid 1 \leq k \leq n\}.
\]

Let \(X_k : \Omega \to X, (k \in \mathbb{N})\) be an \(X\) valued i.i.d. whose common distribution is \(\mu\). Consider the random set
\[
\Sigma_Z(\omega, n) \equiv \{(f \circ X_1(\omega), \cdots, f \circ X_n(\omega)) \mid f \in Z\} \subset \mathbb{R}^n.
\]

Let \(N^1(Z,n,\varepsilon)(\omega)\) and \(N^\infty(Z,n,\varepsilon)(\omega)\) denote the minimum number of balls of radius \(\varepsilon\), (in the respective metric on \(\mathbb{R}^n\)), required to cover the set \(\Sigma_Z(\omega, n)\).

(2) Next, we recall the notion of ‘shattering’. As before \(Z\) is a family of measurable, real valued maps on \(X\). Let \(\alpha < \beta\) be two real numbers. A finite subset \(F \subset X\) is said to be \((\alpha, \beta)\)-shattered by \(Z\) if for each fixed \(\omega < \beta\), let \(\omega\) be a uniform bounded countable family \(Z\) of \(X^n\) given by
\[
S_n^{(\alpha, \beta)} = \{(x_1, \cdots, x_n) \in X^n \mid \text{the set } F = \{x_1, \cdots, x_n\} \text{ has } n \text{ elements and is } (\alpha, \beta) \text{-shattered by } Z\}
\]
is a measurable subset, since \(Z\) is countable. The largest \(n \in \mathbb{N}\) for which \(\mu^n(S_n^{(\alpha, \beta)}) > 0\) will be called the \((\alpha, \beta, \mu)\)-shattering dimension of the family \(Z\).

(4) A measurable set \(A \subset X\) is a \((\alpha, \beta)\)-witness of irregularity, (of the family \(Z\)), if (i) the restriction of \(\mu\) to \(A\) is non-atomic and (ii) for each \(n \in \mathbb{N}\), almost all \((x_1, \cdots, x_n) \in A^n\) belong to \(S_n^{(\alpha, \beta)}\).

(5) A countable family \(Z\) is said to have no witness of irregularity, (with respect to \(\mu\)), if any measurable set \(A \subset X\) that is a \((\alpha, \beta)\) witness of irregularity for some \(\alpha < \beta\), then \(\mu(A) = 0\).

Then the following is a summary of results proved in [52].

Theorem 4.6 Let \((X, \mathcal{B}, \mu)\) be a complete probability space. Let \(Z\) be a uniformly bounded countable subset of real valued measurable maps on \(X\). Then the following statements are equivalent.

(1) The family \(Z\) is a Glivenko-Cantelli family,

(2) \(\lim_{n \to \infty} \frac{1}{n} \mathbb{E}(\log(N^1(Z,n,\varepsilon))) = 0\), for each \(\varepsilon > 0\),

(3) \(\lim_{n \to \infty} \frac{1}{n} \mathbb{E}(\log(N^\infty(Z,n,\varepsilon))) = 0\), for each \(\varepsilon > 0\),

(4) family \(Z\) has no witness of irregularity with respect to \(\mu\).

(5) for any \(\alpha < \beta\), \(\lim_{n \to \infty} \mu^n(S_n^{(\alpha, \beta)})^{1/n} = 0\).
Here $\mathcal{E}$ denotes the expectation of the random variables on $X^n$.

Given a compact metric topological dynamical system $(X,T,\mu)$ with $T$ countable, amenable and $f \in C(X)$, we can apply above theorem to the family $Z = \{f_t \mid t \in T\}$ to get the following characterization of a $\mu$-compact vector $f$.

**Proposition 4.7** Let $(X,T,\mu)$ be a compact, metric ergodic dynamical system with $T$ countable, amenable. Let $f \in C(X)$, then the following statements are equivalent.

(1) $f$ is a $\mu$-compact vector,

(2) the family $\mathcal{F} \equiv \{f_t \mid t \in T\}$ is Glivenko-Cantelli, where $f_t(x) = f(xt), x \in X, t \in T$.

(3) $\lim_{n \to \infty} \frac{1}{n} \mathcal{E}(\log(N^i(\mathcal{F},n,\varepsilon))) = 0$, for each $\varepsilon > 0$, where $i = 1$ or $\infty$.

(4) $\lim_{n \to \infty} \mu^n(\mathcal{S}_{n}^{(\alpha,\beta)})^{1/n} = 0$, for any $\alpha < \beta$.

(5) The family $\mathcal{F}$ has no witness of irregularity.

**Proof.** We shall prove that (1) implies (3) and (2) implies (1), the equivalence of (2), (3), (4) and (5) is established by the previous theorem.

(1) implies (3): Consider the set $\Sigma_{\omega,n} \equiv \Sigma_{\omega}(x_1,\ldots,x_n) \in \mathbb{R}^n$ defined above, (where $Z = \{f_t \mid t \in T\}$ and $\omega = (x_1,\ldots,x_n) \in X^n$). We shall show that for any $\varepsilon > 0$, $N^\infty(Z,n,\varepsilon)(\omega)$ is bounded independent of $n$, for all $\omega = (x_1,\ldots,x_n) \in X^n$ outside a set of arbitrarily small $\mu^n$ measure. This will prove (3).

Recall from Proposition 3.1, on a set $K$ of measure arbitrarily close to 1, $f(xt)$ can be written as $U_tP(x)$, where $P(x) = \sum_{i=1}^{k} \sum_{j=1}^{d_i} \langle f_j^i(x),w_j^i \rangle w_j^i$, (here we are using the notation of Proposition 3.1). Here each $f_j^i : K \to \mathbb{C}^d$ is continuous. Now we notice that it is enough to show that $N^\infty(Z_j,n,\varepsilon)(\omega)$ is bounded independent of $n$ for $\omega \in K^n$ for each $i$ and $j$, where $H(Z_j^i) = \{((f_j^i)_t \mid t \in T\}$. Of course this family is a family of $\mathbb{R}^d$ valued maps on $X$ instead of being real valued, but that does not change anything. Now note that $f_j^i(xt) = U_t f_j^i(x)$, where $U_t$ is a finite dimensional unitary operator acting on a vector $f_j^i(x) \in \mathbb{C}^d$. Given $\varepsilon > 0$ select a compact neighbourhood $V$ of the identity in the Unitary group $U(d_i)$ so that $||v - U_1v|| < \varepsilon$ for all $v \in \mathbb{C}^d$. Let $L$ be the number of translates of $V$ that cover the whole unitary group $D(d_i)$. Then it follows that the set $\{(U_t f_j^i(x_1),\ldots,U_t f_j^i(x_n)) \mid t \in T\}$ is covered by at most $L$ balls of radius $\varepsilon$, for $(x_1,\ldots,x_n) \in K^n$. Note that (i) $L$ is independent of $n$ and depends only on $\varepsilon$ and (ii) for $\omega$’s outside $K^n$ - a set of arbitrarily small $\mu^n$ measure - the number $N^\infty(Z_j,n,\varepsilon)(\omega)$ is bounded by $\left(\frac{M}{\varepsilon^2}\right)^n$, where $M$ is the maximum of $f_j^i$. Thus, given $n$ and $\varepsilon$, choosing set $K$ to be a compact set with measure very close to 1, we can make the expectation of the map $\omega \to \log(N^\infty(Z_j^i,n,\varepsilon)(\omega))$ bounded independent of $n$. This proves (3).

(2) implies (1): This proof is exactly as in the case of $T = \mathbb{Z}$. Because $T$ is countable amenable, if we have a weak-mixing vector $f \in L^2(X,\mu)$, then $f_n \to 0$ weakly along a set $\{t_n \mid n \in \mathbb{N}\}$ of density one, (density is with respect to some fixed Følner sequence). Then the rest of the proof is as in the case of integer group.

**Remark 4.8** It is easy to see that if for any $\alpha < \beta$, the $(\alpha,\beta)$-shattering dimension is finite, then $f$ is $\mu$-compact. The converse may not hold, but the above condition on the limit in (4) guarantees the converse.
Now we introduce the notion of ‘weak-tameness’.

**Definition 4.9** A compact, metric dynamical system \((X,T)\) is weakly tame if each \(f \in C(X)\) is \(\mu\)-tame for and every invariant Borel probability measure \(\mu\) on \(X\).

The following characterization already follows from the results proved earlier.

**Theorem 4.10** Let \((X,T)\) be a compact, metric system where \(T\) is countable and amenable. Then the following statements are equivalent.

1. \((X,T)\) is weakly tame,
2. \((X,T,\mu)\) has discrete spectrum with respect to every invariant Borel probability \(\mu\) on \(X\),
3. The family \(\{f_t \mid t \in T\}\) is \(\mu\)-Glivenko-Cantelli for every \(f \in C(X)\) and every invariant Borel probability measure \(\mu\) on \(X\).
4. For every \(f \in C(X)\), the family \(\{t_t \mid t \in T\}\) has no witness of irregularity with respect to \(\mu\) for every invariant Borel measure \(\mu\) on \(X\).

**Remark 4.11** We would like to pose a question: Can we give a characterization of a subclass of weakly tame dynamical systems for which for any \(\alpha < \beta\), the \((\alpha,\beta)\)-shattering dimension is finite and bounded as a function of the invariant measure \(\mu\)?

In the following we shall compare ‘tameness’ with ‘weak tameness’. The above characterization of weak-tameness is based on the up-crossing behavior of the family \(F \equiv \{f_t \mid t \in T\}\). In topological dynamics one tries to capture dynamical features in terms of the regularity properties of functions in the family \(\{f_p \mid p \in \beta T\}\), which is the closure of the family \(F\) in the topology of point-wise convergence. Recall that \((X,T)\) is tame if and only if each \(f_p\) is a Baire-1 function, for each \(f \in C(X)\) and \(p \in \beta T\). Thus, in some sense tameness can be studied by a ‘topological version’ of up-crossings. A more formal way to study this is by introducing the notion of ‘independence’ for a sequence of a pair disjoint subsets of \(X\). But before introducing this notion, first we shall state a characterization of tameness and ask the reader to compare it with the previous theorem.

**Theorem 4.12** Let \((X,T,\mu)\) be a compact metric topological dynamical system with \(T\) countable. Then the following statements are equivalent;

1. \((X,T)\) is tame,
2. The family \(\{f_t \mid t \in T\}\) is \(\mu\)-Glivenko-Cantelli for every \(f \in C(X)\) and every Borel probability measure \(\mu\) on \(X\).
3. For every \(f \in C(X)\), the family \(\{f_t \mid t \in T\}\) has no witness of regularity with respect to \(\mu\) for every Borel probability measure \(\mu\) on \(X\).

Before getting into the technicalities of this proof we make some observations. We shall mention two theorems whose proofs share a certain ‘common part’ with the proof of the above theorem. The first one is due to D. Kerr and H. Li (see Proposition 6.4 of [31]).
Theorem 4.13  [Kerr-Li Theorem] A system \((X, T)\) is not tame if and only if there is a non-trivial ‘IT pair’, (this notion will be defined shortly).

The second one is the famous dichotomy theorem of J. Bourgain, D. Fremlin and M. Talagrand. The following is its ‘dynamical formulation’ due to E. Glasner.

Theorem 4.14  [BFT-dichotomy Theorem] A compact, metric dynamical system \((X, T)\), (with \(T\) countable), is not tame if and only if the enveloping semi-group \(E(X, T)\) contains a topological copy of \(\beta T\).

As mentioned before, the argument in proofs of the last three theorems have a certain ‘common part’. This is the part that involves ‘combinatorics’, namely a ‘dichotomy argument’ using the ‘Nash-Williams Theorem’. In the following we shall show how the arguments in the above theorems can be sketched in a self contained way using only section 5 of S. Todorcevic’s book \[54\] and a lemma from \[25\]. We begin by recalling the notion of ‘independence’.

Definition 4.15  (1) A sequence \(\{(A^n_0, A^n_1)\}_{n \in \mathbb{N}}\) of disjoint pairs of subsets of a set \(X\) is independent if for every finite subset \(F \subset \mathbb{N}\), and \(G \subset F\), we have
\[
\bigcap_{j \in G} A^n_0 \bigcap \bigcap_{j \notin F \setminus G} A^n_1 \neq \emptyset.
\]

(1) The same sequence is \(\sigma\)-independent if for every subset \(F \subset \mathbb{N}\) we have,
\[
\bigcap_{j \in F} A^n_0 \bigcap \bigcap_{j \notin F} A^n_1 \neq \emptyset.
\]

(2) Let \(F = \{f_n \mid n \in \mathbb{N}\}\) be a countable family of real valued functions on a set \(X\). This family is said to be independent (or \(\sigma\)-independent) at level \((\alpha, \beta)\), (where \(\alpha < \beta\) are reals), if the sequence of sets \(A^n_0 = f^{-1}_n(-\infty, \alpha]\) and \(A^n_1 = f^{-1}_n[\beta, \infty)\).

(3) Let \((X, T)\) be a compact, metric dynamical system. A pair \((x, y) \in X \times X\) is said to be an IT-pair if any product neighborhood \(U \times V\) of \((x, y)\) has an infinite independent set, i.e. there is an infinite countable set \(B \subset \mathbb{T}\) such that the sequence \(\{(U_t, V_t) \mid t \in B\}\) is an independent sequence, (without loss of generality, we are assuming that \(U \cap V = \emptyset\)).

Proof of Theorem 4.12

Proof.  First we recall the Rosenthal dichotomy theorem and its proof, (Chapter 5, \[54\]), (which is a weaker form of the BFT theorem). A strong version is needed to prove the BFT theorem, see \[6\], Theorem 11.

Theorem 4.16 (Rosenthal Dichotomy Theorem) Let \(X\) be a compact metric space and \(\{f_n\}\) be a point-wise bounded sequence in \(C_p(X)\)- the space of continuous real valued maps on \(X\) with the topology of pointwise convergence. Then either \(\{f_n\}\) contains a convergent sequence, (i.e. as a sequence in \(C_p(X)\)) or it contains a subsequence whose closure is homeomorphic to \(\beta \mathbb{N}\).

If we look at the proof of this theorem, (page 22-23 of \[54\]), it uses the Nash-Williams theorem to conclude that if the point-wise closure of \(\{f_n \mid n \in \mathbb{N}\}\) does not contain a (point-wise) convergent subsequence then for some \(\alpha < \beta\), there is a sub-sequence \(\{f_{n_i} \mid n_i \in \mathbb{N}\}\) such that this sub-sequence is
actually-σ independent at level \((\alpha, \beta)\). This fact is what we called ‘the ‘common part’ to the proofs of all of the three theorems mentioned above. In the following we very briefly describe the arguments in the proofs of these theorems.

(i) After this ‘common part’, to prove the Rosenthal dichotomy theorem (or the BFT theorem) one only needs some facts about the topology of a zero dimensional space, (see [54], Chapter 5 and 13).

(ii) In the proof of Kerr-Li Theorem, to get a non-trivial IT pair from an independent sub-sequence one has to use ‘more combinatorics’ and a compactness argument, (see Proposition 3.9 and 6.4 of [31]). A ‘Ramsey type’ combinatorial property one needs is the following : If \(U \times V\) has an infinite independent set and \(U = U_1 \cup U_2\) then either \(U_1 \times V\) or \(U_2 \times V\) has an infinite independent set.

Now we go to the proof of Theorem 4.12. As mentioned in the above remark, either the system \((X, T)\) is tame or for some \(f \in C(X)\), the family \(F_B = \{f_t \mid t \in B \subset T\}\) is a σ-independent family, where \(B\) is a (countable) infinite set. Now we use Theorem 1.3 of [25] which says that non existence of a σ-independent infinite sub-sequence in the family \(F = \{f_t \mid t \in T\}\) is equivalent to saying that the family \(F\) is ‘universally Glivenko-Cantelli’ i.e. it is \(\mu\)-Glivenko-Cantelli for every Borel probability measure \(\mu\) on \(X\).

So the dichotomy argument in the proof of Rosenthal’s theorem followed by the above theorem of van Handel gives a proof of Theorem 4.12. It is worth noting that existence of a σ-independent sub-sequence \(F_B\) allows one to construct a Borel probability measure \(\nu\) on \(X\) such that the family \(F_B\) is not \(\nu\)-Glivenko compact. This construction is Theorem A.1 in the appendix of [25] and should be of independent interest despite work summarized in Theorem 4.6.

Remark 4.17 For \(T = \mathbb{Z}\), the statement of Theorem (4.12) is motivated by Theorem 8.20 and Theorem 8.16 of [21]. However, the authors have neither given a proof or a hint to Theorem 8.20, nor do they give any reference to Theorem 8.16., (they attribute this theorem to M. Talagrand). In this paper the authors define notion of ‘topological stability’ of the family \(F\) by requiring that such family do not have any ‘topologically critical’ set. The authors define a closed set \(A \subset X\) ‘topologically critical’ if for some \(\alpha < \beta\) the set \(S_{n}^{(\alpha, \beta)}\) is dense in \(A^n\) for every \(n \in \mathbb{N}\). A more natural definition of ‘topologically critical’ would have been given by demanding that \(\mu^n\) almost every point of \(A^n\) belongs to \(S_{n}^{(\alpha, \beta)}\) for every \(n\) and every probability measure \(\mu\). These two notions may not be the same. In any case these notions are trying to capture the topological analogue of \((\alpha, \beta)\)-witness of irregularity.

Finally we state a couple of consequences of various things discussed above.

Corollary 4.18 A dynamical system is weakly tame if and only if has a bounded measure theoretic complexity with respect to any invariant measure.

Proof. By Proposition 4.1 from [27], if \(\mu\) is an invariant measure and the dynamical system \((X, \mathcal{A}, \mu, T)\) has a discrete spectrum, then its measure complexity is bounded. Conversely, by the main result of [28] ([Theorem 4.3]) or [57, Theorem 3.2], if the measure complexity is bounded then the system is \(\mu\)-mean equicontinuous, hence, its spectrum is discrete. Therefore the dynamical system has a bounded complexity for any invariant measure if and only if for each invariant measure its spectrum is discrete. The proof of the corollary is complete.

Now let \(T = \mathbb{Z}\). We recall the notion of a null system.
Definition 4.19 Let \((X,T)\) be a compact metric dynamical system. Given a sequence \(S = \{s_i\} \subset \mathbb{Z}\) and a finite open cover \(\mathcal{O}\) of \(X\), we define the topological entropy of \((X,T)\) with respect to \(S\) and \(\mathcal{O}\) by

\[
h_{\text{top}}(S,\mathcal{O}) = \lim_{n \to +\infty} \frac{\log \left( N(\bigvee_{i=1}^{n} T^{s_i}(\mathcal{O})) \right)}{n},
\]

where \(N(.)\) is the minimal cardinality of a subcover. The sequential topological entropy of \(T\) along \(S\) is given by

\[
h_{\text{top}}(S) = \sup \left\{ h_{\text{top}}(S,\mathcal{U}), \mathcal{U} \text{ is an open cover of } X \right\}.
\]

System \((X,T)\) is said to be null if its sequential topological entropy is zero for any subsequence.

Corollary 4.20 Suppose \((X,T)\) is null, then \((X,T)\) is weakly tame.

Proof. By Kushnirenko [33], for null systems any invariant measure has discrete spectrum and hence by our result the system is weakly tame. □

Remark 4.21 (1) D. Kerr and H. Li have characterized null systems as those that do not admit non-trivial IN-pairs, (see [31] for details).
(2) Very recently, Fuhrmann and Kwietniak proved that there is a tame dynamical system which is non-null [18] (Tame in the sense of Glasner-Köhler).

5 The Veech systems and \(\mathbb{K}(T)\)

In [55] Professor W. Veech introduced a structure which he called ‘a bi-topological flow’. The following is a slight modification of his original definition.

Definition 5.1 Let \((X,T)\) be a topological dynamical system, where \(\tau_1\) denotes the topology on \(X\). Let \(T\) be countable. The system \((X,T)\) is said to be a Veech system if \(X\) has another topology \(\tau_2\) such that the following properties hold.

1. Topology \(\tau_2\) is a metric topology generated by a metric \(D : X \times X \to [0, \infty)\),
2. \(\tau_1 \leq \tau_2\),
3. any \(\tau_2\) open set is \(\tau_1\)-Borel, (i.e. is in the sigma algebra generated by the \(\tau_1\) open sets).
4. The \(T\) action preserves metric \(D\), i.e. \(D(xt,yt) = D(x,y)\), for all \(x,y \in X\) and \(t \in T\).
5. The space \((X,\tau_2)\) is separable.
Remark 5.2 (1) In fact W. Veech introduced this structure in two of his papers, first in [55] and much later in [56]. In the first paper instead of condition (3) above, he requires a much stronger condition of \( \tau_1 \)-continuity of the map \( y \mapsto D(x,y) \) for a generic set of \( x \)'s. In his later paper he weakened it by demanding that \( y \mapsto D(x,y) \) be lower semi-continuous. In the second paper, his main interest was in studying the special case of the translation flow on orbit closure of functions of class \( K(\mathbb{Z}) \) (which we shall recall below). For this system the condition in his first paper does not hold but the one in the second paper holds. In his study, he posed the question: Whether the Sarnak conjecture\(^2\) holds for the translation flow on the orbit closure of functions of class \( K(\mathbb{Z}) \). In a recent paper [27] the authors claim to have proved this, [27, Theorem 5.1]. However this proof has a gap. We shall discuss this and present a correct proof of this conjecture.

(2) W. Veech introduces this structure for uncountable acting groups \( T \) as well. In general, for such groups technicalities arise due to non-separability of \( l^\infty(T) \) and hence even the definition of \( K(T) \) becomes cumbersome. So we restrict ourselves to countable \( T \)'s.

As mentioned above, a prime example of Veech-system is the translation flow on the orbit closure of a function of class \( K(T) \). The precise definition follows.

Definition 5.3 Consider \( l^\infty(T) \)-the space of bounded, complex valued function on \( T \) with the weak* topology as a dual of \( l^1(T) \). Let \( f \in l^\infty(T) \) and \( X_f \) be the closure of the orbit \( \{ f_t \mid t \in T \} \) with respect to the weak* topology, where \( f_t(s) = f(st) \). A function \( f \in l^\infty(T) \) belongs to the class \( K(T) \) if \( X_f \) is separable with respect to the topology induced by the restriction of the \( l^\infty(T) \) norm to \( X \). It is not difficult to verify that the translation flow \( (X_f, T) \) is a Veech system. Here \( \tau_1 \) is the weak* topology and \( \tau_2 \) is the \( l^\infty(T) \) norm topology on \( X_f \).

Remark 5.4 One can show that if \( f \in K(T) \) then \( X_f \subset K(T) \) and \( K(T) \) is a subalgebra containing the subalgebra \( WAP(T) \) of weakly almost periodic functions on \( T \). The following is a concrete example when \( T = \mathbb{Z} \), that shows this containment is proper.

Example 5.5 Here \( T = \mathbb{Z} \). Let \( S \equiv \{ n_k \} \) be a sequence in \( \mathbb{N} \) such that \( n_{k+1} - n_k \) increases to \( \infty \). Let \( \varepsilon \equiv \{ \varepsilon_k \} \in \{ -1, 1 \} \). Corresponding to \( (S, \varepsilon) \), define a map \( f \equiv f(S, \varepsilon) : \mathbb{Z} \to \{ -1, 0, 1 \} \) by setting

\[
\begin{align*}
f(n) &= 0 \text{ if } n \leq 0, \\
&= \varepsilon_k \text{, if } n_k \leq n < n_{k+1}, k \in \mathbb{N}.
\end{align*}
\]

Consider \( f \) to be a point in \( \{ -1, 0, 1 \}^\mathbb{Z} \) and let \( X_f \) be the orbit closure of \( f \) under the left shift map.

Lemma 5.6 Consider the above example, then its enveloping semigroup \( E(X_f, \mathbb{Z}) \) is given by

\[
E(X_f, \mathbb{Z}) = \mathbb{Z} \cup \{ \hat{p}, \hat{q}, \hat{z} \}
\]

where the elements \( \hat{p}, \hat{q} \) and \( \hat{z} \) of \( E(X_f, \mathbb{Z}) \) will be described in the proof. In particular, \( f \in K(\mathbb{Z}) \) and \( (X_f, \mathbb{Z}) \) is a Veech system. Furthermore, every element of \( E(X_f, \mathbb{Z}) \) is a Baire-1 function and hence \( (X_f, \mathbb{Z}) \) is tame. Finally \( f \notin WAP(\mathbb{Z}) \) i.e \( (X_f, \mathbb{Z}) \) is not weakly almost periodic.

\(^2\)See sections 6. for more details.
Proof. Recall that \( f \equiv f^{(S, \varepsilon)} \) is given. To avoid confusion, we shall denote the point \( f \) of \( X_f \) by \( x^* \). Denote by \([a_k, b_k]\) the ‘middle third’ of the interval \([n_k, n_{k+1}]\). Let

\[
P = \bigcup \left\{ n \in [a_k, b_k] \mid f(a_k) = 1 \right\}, \quad Q = \left\{ n \in [a_k, b_k] \mid f(a_k) = -1 \right\} \quad \text{and} \quad Z = \mathbb{Z} \setminus (P \cup Q).
\]

The partition \( \{P, Q, Z\} \) of the set \( \mathbb{Z} \) induces a partition \( \{\bar{P}, \bar{Q}, \bar{Z}\} \) of \( \beta \mathbb{Z} \), given by their closures in \( \beta \mathbb{Z} \). Let \( p \in \bar{P} \), we describe the map \( \hat{p} : X_f \to X_f \) giving its action on \( X_f \). Let \( U \in p \). Note that for arbitrarily large \( k \in \mathbb{N}, U \cap [a_k, b_k] \neq \emptyset \), where \([a_k, b_k] \subset P \). Thus given any \( m \in \mathbb{N} \), select \( k_m \in \mathbb{Z} \) and \( t(U, k_m) \in U \cap [a_k, b_k] \), where \( k_m \) is ‘slightly less’ than one third \( n_{k+1} - n_{k} \). Note that the net \( \{t(U, k_m)\} \) converges to \( p \)

\[
x^* p(t) = \lim_{t(U, k_m)} x^* t(U, k_m)(t) = \lim_{t(U, k_m)} x^* (t(U, k_m) + t) = 1. \quad \text{for all } |t| < m.
\]

Since \( n_{j+1} - n_j \to \infty \), it follows that \( x^* p(t) = 1 \) for all \( t \in \mathbb{Z} \). Denoting the constant sequences \( 1, -1 \) and \( 0 \) respectively, we have shown that \( x^* \cdot p \equiv f_p = 1 \), if \( p \in \bar{P} \). Actually the same argument is valid for any translate of \( f \) as well. Thus, \( (x^*)p = 1 \) for all \( t \in \mathbb{Z} \). Similarly, we can see that \( (x^*)r = -1 \) if \( r \in Q \) and \( (x^*)r = 0 \), if \( r \in Z \), for all \( t \in \mathbb{Z} \). The last fact can be proved similarly, by considering a net, (or a sequence), \( k_t \to -\infty \) and observing that \( k_t \to r \in Z \) and arguing as above. This shows that the only elements in the orbit closure of \( x^* \), under the action of \( E(X_f, \mathbb{Z}) \) are \( 1, -1 \) and \( 0 \), i.e. \( X_f = \{x^* t \mid t \in \mathbb{Z} \} \cup \{1, -1, 0\} \). Now it is easy to verify that each element \( r \in \beta \mathbb{Z} \) fixes these three elements. Thus, we have a complete description of the elements of \( E(X_f, \mathbb{Z}) \setminus \mathbb{Z} \), they are the maps \( \hat{p}, \hat{q} \) and \( \hat{z} \) given by,

\[
(x^*)\hat{p} = 1, \quad \hat{p} \text{ fixes } 1, -1, 0,
(x^*)\hat{q} = -1, \quad \hat{q} \text{ fixes } 1, -1, 0,
(x^*)\hat{z} = 0, \quad \hat{z} \text{ fixes } 1, -1, 0,
\]

(5.1)

where \( t \in \mathbb{Z} \).

Finally, to show that \( (X_f, \mathbb{Z}) \) is not weakly almost periodic, pick a sequence \( k_t \to -\infty \) and \( p \in \bar{P} \). Then \( (x^* k_t)p = 1 \), for each \( k_t \). But \( \lim_{k_t \to -\infty} x^* k_t \cdot p = 0 \cdot p = 0 \). This shows that the map \( p \), is discontinuous at \( 0 \). Thus \( (X_f, \mathbb{Z}) \) cannot be weakly almost periodic, (since for such systems all elements of the enveloping semigroup are continuous (see [13])).

In fact, the following more general observation proves that countable, compact dynamical systems are tame.

**Lemma 5.7** Let \((X, T)\) be a compact countable dynamical system. Then \((X, T)\) is tame.

**Proof.** Let \( f : X \to \mathbb{R} \) be any map. We want to show that \( f \) is of Baire class one. This will show that any element of the enveloping semigroup \( E(X, T) \) is a Baire class one function and hence \((X, T)\) is tame. We need to show that the set \( S \equiv \{x \in X \mid f \text{ is not continuous at } x\} \) is a set of first category. If \( S \) is finite, this is obvious. So suppose \( S \) is countable, say \( S = \{y_j \mid j \in \mathbb{N}\} \). Note that for any \( j \), \( \overline{\{y_j\}} = \{y_j\}^0 = \emptyset \). If not, then \( y_j \) is an isolated point and hence is not a point of discontinuity of \( f \). Thus, \( S \) is of first category.

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Remark 5.8 Of course, the proof of Lemma 5.7 can be obtained directly by applying Bourgain-Fremelin-Talagrand dichotomy theorem, since the cardinality of $\beta T$ is at most $2^{80}$. But, here, our arguments are much simpler.

The next result describes the nature of minimal sets and the support of an invariant ergodic measure on a Veech system. The proof presented by W. Veech in [56] is primarily for the special case $(X_f, T)$, where $f \in \mathbb{K}(T)$. To prove analogous result for general Veech systems we need to modify arguments and use the enveloping semigroup machinery.

Theorem 5.9 Let $(X, T)$ be a Veech system. Let $\mu$ be any invariant ergodic, Borel probability measure on $X$ with support $C(\mu)$. Then

1. $C(\mu)$ is a $\tau_1$-minimal set.
2. In addition $T$ be amenable. Then every minimal subset of $X$ is almost automorphic, (in particular point distal) and is an ‘isometric extension’ (in the sense of [10]), i.e. is a measure theoretic isometric extension of its maximal equicontinuous factor.
3. With $T$ amenable, every ergodic invariant measure on $X$ has discrete spectrum.

Proof. (1): Let $\{x_m \mid m \in \mathbb{N}\}$ be a countable $\tau_2$-dense subset of $X$. Fix any $\varepsilon > 0$. Then $\{B_\varepsilon(x_m) \mid m \in \mathbb{N}\}$ is a cover of $X$, (recall that $B_\varepsilon(x_m)$ is the $\varepsilon$ ball centered at $x_m$ in metric $D$). Let $\Sigma(\varepsilon) \subset \mathbb{N}$ be a countable set such that $m \in \Sigma(\varepsilon)$ if and only if $\mu(C_m(\varepsilon)) = B_\varepsilon(x_m) \cap C(\mu)$. Then $C(\varepsilon) = \bigcup_{m \in \Sigma(\varepsilon)} C_m(\varepsilon)$ is a $\tau_1$-Borel subset of $C(\mu)$ of measure 1 for every $\varepsilon > 0$.

Since $\mu$ is ergodic, by Lemma 3.16, there exists a $y_m \in C_m$ and a syndetic set $S_m \subset T$ such that $t \in S_m \equiv S_m(\varepsilon)$ implies $y_m t \in C_m$.

Claim: If $y \in C_m(\varepsilon)$ and $t \in S_m$, then $D(y, yt) < 3\varepsilon$. This follows from the $T$ invariance of metric $D$ and following triangle inequality

$$D(y, yt) \leq D(y, y_m) + D(y_m, y_m \cdot t) + D(y_m t, yt) < 3\varepsilon.$$ 

Let $C_1 = \bigcap_{n \in \mathbb{N}} C_m(\frac{\varepsilon}{2^n})$. Then $\mu(C_1) = 1$ and if $y \in C_1(\mu)$, we have shown that the orbit of $y$ returns to its $3\varepsilon$ neighbourhood, (in $D$ metric), in a syndetic set, for every $\varepsilon > 0$. In particular, since $\tau_1 \leq \tau_2$, it returns to its given $\tau_1$-neighbourhood in a syndetic set. Since $(X, \tau_1)$ is compact, this means that $y$ is a $\tau_1$-almost periodic point, i.e. its $\tau_1$-orbit closure is a $\tau_1$-minimal set. Note that we cannot say this with respect to the $\tau_2$ topology. To conclude the $\tau_2$-compactness of the $\tau_2$-orbit closure one would need some additional special properties, such as local compactness of the $\tau_2$-topology, which in general we do not have.

(2): In fact, we can improve the previous claim to: If $y \in C_m(\varepsilon)$ and $t, s \in S_m$, then $D(y, y(ts^{-1})) < 4\varepsilon$. This follows from the inequality,

$$D(y, y(ts^{-1})) \leq D(y, y_m) + D(y_m, y_m(ts^{-1})) + D(y_m(ts^{-1}), y(ts^{-1})) 
\leq D(y, y_m) + D(y_m s, y_m t) + D(y_m, y) 
\leq D(y, y_m) + D(y_m s, y) + D(y, y_m t) + D(y_m, y) < 4\varepsilon.$$
This shows that every point in $C_1$ returns to its $4\varepsilon$ neighbourhood in a set of times which is a $\Delta^*$ set, for every $\varepsilon > 0$. As before, since $\tau_1 \leq \tau_2$, the return time set of any point $y \in C_1$ to any of its $\tau_1$-neighbourhood is a $\Delta^*$ set. Hence such a $y$ is $\tau_1$-almost automorphic.

Now, let

$$C_2 = \left\{ y \in C(\mu) \mid \tau_1\text{-orbit closure of } y \text{ equals } C(\mu) \right\}.$$ 

Since $\mu$ is ergodic, $\mu(C_2) = 1$. Let $C = C_1 \cap C_2 \subset C(\mu)$. Then $\mu(C) = 1$ and if $y \in C$, then $\tau_1$-orbit closure of $y$ is $C(\mu)$ and $y$ is almost automorphic. Even though $\mu(C(\mu) \setminus C) = 0$, unfortunately, for general Veech systems we are unable to show that $C(\mu) \setminus C$ is the empty set. This would prove that $C(\mu)$ is actually minimal equicontinuous. We shall later proved this for the special case of $(X_f, T)$, with $f \in \mathbb{K}(T)$.

We have shown that $C(\mu)$ is an almost automorphic minimal set. Now, we observe that it is a ‘regular’ almost automorphic set and hence $(C(\mu), T, \mu)$ is an measure theoretical ‘isometric extension’ of its maximal equicontinuous factor, (see [17] and [10] for these notions). First, note that since $T$ is amenable, by a well known theorem of D. McMahon the regional proximality relation $Q(C(\mu))$ on $C(\mu)$ is an ‘icer’, i.e. invariant, closed equivalence relation. Next, we recall the ‘Veech relation’ $V(Y, T)$ on any dynamical system $(Y, T)$, (see [5]),

$$V(Y, T) = \left\{ (y_1, y_2) \in Y \times Y \mid \text{there exists a net } t_\alpha \in T \text{ and } z \in Y \text{ such that } y_1t_\alpha \to z \text{ and } zt_\alpha^{-1} \to y_2 \right\}.$$ 

Since each $x \in C \subset C(\mu)$ is almost automorphic, $V[x] \overset{\text{def}}{=} \{ y \in X \mid (x, y) \in V(X, T) \} = \{ x \}$. By Theorem 13 of [5], the cell $V[x]$ is dense in the cell $Q[x]$ of the regional proximality relation. Thus, if $\pi : C(\mu) \to C(\mu)/Q(C(\mu))$ is the canonical factor map from $C(\mu)$ onto its maximal equicontinuous factor, then $\pi^{-1}(\pi(x)) = \{ x \}$ for all $x \in C$. Thus $\pi$ is one to one on set $C$, a set of full measure. Thus, $C(\mu)$ is an isometric extension of its maximal equicontinuous factor.

Finally, since $T$ is amenable, every minimal set $M$ is the support $C(\mu)$ of some ergodic invariant measure $\mu$. It follows that $(M, T)$ is minimal almost automorphic and is an isometric extension of its maximal equicontinuous factor.

\((3)\): This immediately follows from (2), since $(C(\mu), T, \mu)$ is measure theoretically isomorphic to a minimal equicontinuous system, namely, its maximal equicontinuous factor.

**Remark 5.10** The above theorem describes the structure of minimal sets in a general Veech system. However, for such systems (i) we cannot say much about the regularity properties of the elements of its enveloping semigroup and (ii) in general the discrete nature of the spectrum cannot be easily extended to non-ergodic measures. Now we shall show that for the special case of $(X_f, T)$, $(f \in \mathbb{K}(T))$, more can be said regarding these two issues.

**Theorem 5.11** Let $T$ be a countable group and $f \in \mathbb{K}(T)$. Then every element of $E(X_f, T)$ is Borel.

**Proof.** We start with a general Veech system $(X, T)$. Recall that $D$ is the metric on $X$ generating the $\tau_2$-topology. Let $p \in E(X, T)$ and let $\rho_p : X \to X$ be $\rho_p(x) = xp$. We show that $\rho_p$ is a $\tau_1$-Borel map. It is enough to show that if $U \subset X$ is $\tau_1$-open, then $\rho_p^{-1}(U)$ is $\tau_1$-Borel.
For each \( y \in U \) let \( \varepsilon \equiv \varepsilon(y) > 0 \) be such that \( B_\varepsilon(y) \subset U \), (recall that \( B_\varepsilon(y) = \{ y \in X \mid D(x, y) < \varepsilon \} \)) and this choice is possible since \( \tau_1 \leq \tau_2 \). Since \((X, D)\) is separable, there exists a countable set \( \{ y_n \mid n \in \mathbb{N} \} \subset U \) such that \( U = \bigcup_{y \in U} B_\varepsilon(y) = \bigcup_{n \in \mathbb{N}} B_\varepsilon(y_n) \). Thus,

\[
\rho_p^{-1}(U) = \rho_p^{-1}\left( \bigcup_{n=1}^{\infty} B_\varepsilon(y_n) \right) = \bigcup_{n=1}^{\infty} \rho_p^{-1}(B_\varepsilon(y_n)) .
\]

Therefore, it is enough to show that \( \rho_p^{-1}(B_\varepsilon(y_n)) \) is \( \tau_1 \)-Borel for each \( y_n \). Next, let

\[
\Sigma_n = \rho_p^{-1}(y_n) = \{ z \in X \mid zp = y_n \} .
\]

So far \((X, T)\) was a general Veech system. The following lemma is where we restrict to the case \( X = X_f, f \in \mathbb{K}(T) \).

**Lemma 5.12** With the notation as above, \( \rho_p^{-1}(B_\varepsilon(y_n)) = \bigcup_{z \in \Sigma_n} B_\varepsilon(z) \).

Assuming this lemma and again using separability of \((X, D)\), we can write \( \bigcup_{z \in \Sigma_n} B_\varepsilon(z) \) as a countable union of such balls. Since each ball in the \( D \) metric is a \( \tau_1 \)-Borel set, \( \rho_p^{-1}(B_\varepsilon(y_n)) \) is Borel for each \( n \in \mathbb{N} \) and the proof is complete.

**Proof of Lemma 5.12:**

In this proof \( \langle x, \xi \rangle \) will denote the ‘pairing’ of vectors \( x \in \ell^\infty(T) \) and \( \xi \in \ell^1(T) \) as vectors in dual space. First we claim that

\[
\|xp - yp\|_\infty \leq \|x - y\|_\infty \text{ for any } x, y \in X_f \subset \ell^\infty(T) .
\]

Let \( t_\alpha \to p \) in \( \beta T \), where \( \{ t_\alpha \} \) is a net in \( T \). Consider,

\[
\|x - y\|_\infty = \|xt_\alpha - yt_\alpha\|_\infty , \quad \text{(since } T \text{ action preserves the } \ell^\infty \text{ metric)}
\]

\[
\geq |\langle xt_\alpha - yt_\alpha, z \rangle| \geq |\langle xp - yp, z \rangle| ,
\]

where \( z \in \ell^1(T) \) is any vector with \( \|z\|_1 \leq 1 \). Now we pick \( z \) such that \( |\langle xp - yp, z \rangle| = \|xp - yp\|_\infty \) and the claim is proved. This claim implies \( B_\varepsilon(z) \subset \rho_p^{-1}(B_\varepsilon(y_n)) \) for each \( z \in \Sigma_n \).

To prove the reverse inclusion, we need to show that \( \bigcap_{z \in \Sigma_n} B_\varepsilon(z)^c \subset \left( \rho_p^{-1}(B_\varepsilon(y_n)) \right)^c \), where \( A^c \) denotes the complement of set \( A \). Let \( x \in X \cap \bigcap_{z \in \Sigma_n} B_\varepsilon(z)^c \). Thus \( \|x - z\|_\infty \geq \varepsilon \) for any \( z \in \Sigma_n \). Now, for any \( z \in \Sigma_n \) and \( \xi \in \ell^1(T) \) with \( \|\xi\|_1 \leq 1 \), we have

\[
|\langle (x - z)p, \xi \rangle| = \lim_{\alpha} |\langle xt_\alpha - zt_\alpha, \xi \rangle| .
\]

Whence, for each \( \alpha \) we can choose \( \xi_\alpha \) with \( \|\xi_\alpha\|_\infty \leq 1 \) such that \( \|xt_\alpha - zt_\alpha\|_\infty = |\langle (x - z)t_\alpha, \xi_\alpha \rangle| \). Thus,

\[
|\langle (x - z)p, \xi_\alpha \rangle| = |\langle (x - z)t_\alpha \rangle| = \|x - z\|_\infty \geq \varepsilon , \quad \text{(by the hypothesis)} .
\]

Now, select \( \xi^* \in \ell^1(T) \) with \( \|\xi^*\| \leq 1 \) such that

\[
\|xp - zp\|_\infty = |\langle (x - z)p, \xi^* \rangle| = \sup \{ |\langle (x - z)p, \xi \rangle| \mid \|\xi\|_1 \leq 1 \} ,
\]

Thus \( \|xp - zp\|_\infty \geq \|x - z\|_\infty \geq \varepsilon \). This shows that \( x \notin \rho_p^{-1}(B_\varepsilon(y_n)) \) and the proof is complete.

Using the previous theorem we can now prove the following.
Proposition 5.13 Let $T$ be amenable. Then, any minimal set of $(X_f, T)$ is equicontinuous, where $f \in \mathbb{K}(T)$.

Proof. We recall the notation used in the proof of Theorem 5.9. Any minimal set can be taken to be of the form $C(\mu)$ for some ergodic invariant measure $\mu$. We need to show that $C(\mu) = C$. Suppose this is not true. Then pick $x \in C$ and $y \notin C$. Since $(C(\mu), T)$ is minimal, there exists $p \in \beta T$ such that $\rho_p(x) = xp = y$. Since the map $\rho_p : X_f \to X_f$ contracts the $\ell^\infty$ metric on $X_f$, (which is the metric $D$ in the notation of Theorem (5.9)), for any $\varepsilon > 0$, $B(\varepsilon(x)) \subset B(\varepsilon(y)) = B(\varepsilon(y))$, (recall that $B_\varepsilon(x)$ denotes the $\varepsilon$ ball in metric $D$ centered at $x$). Note that since $y \notin C$, $\mu(B_\varepsilon(y)) = 0$ for all small enough positive $\varepsilon$'s and since $x \in C$, $\mu(B_\varepsilon(x)) > 0$ for all positive $\varepsilon$'s. Now we show that since $\rho_p$ is Borel, it preserves $\mu$ and this will lead to a contradiction.

Consider the map $\eta : \beta T \to \Omega_\mu(C(\mu)) : p \mapsto U_p$, where $U_p[f] = [f]p$, ($f \in L^2(C(\mu), \mu)$). Since $\rho_p : X \to X$ is Borel, $U_p[f] = [f]p = [f \circ \rho_p] = [f_p]$. Given $\delta > 0$ and any Borel set $A \subset X_f$, consider the open neighborhood $W_{A,\delta}$ of $p$ in $\beta T$ defined by

$$W_{A,\delta} = \{ q \in \beta T \mid |\langle U_q \chi_A, 1 \rangle - \langle U_p \chi_A, 1 \rangle| < \delta \}.$$

Pick $t \in W_{A,\delta}$ and note that

$$\langle U_t \chi_A, 1 \rangle = \int_{C(\mu)} \chi_A(\omega t) d\mu(\omega) = \mu(At^{-1}) = \mu(A), \text{ and}$$

$$\langle U_p \chi_A, 1 \rangle = \int_{C(\mu)} \chi_A \circ \rho_p(\omega) d\mu = \mu(\rho_p^{-1}(A)). \quad (5.3)$$

Thus, $|\mu(\rho_p^{-1}(A)) - \mu(A)| < \delta$. Since $\delta$ is arbitrary, $\mu(\rho_p^{-1}(A)) = \mu(A)$. Now if $\varepsilon > 0$ is small enough, using the fact that $B_\varepsilon(x) \subset \rho_p^{-1}(B_\varepsilon(y))$, we have

$$0 = \mu(B_\varepsilon(y)) = \mu(\rho_p^{-1}(B_\varepsilon(y)) \geq \mu(B_\varepsilon(x)) > 0,$$

a contradiction. Thus $C(\mu) = C$ and each point of $C(\mu)$ is almost automorphic, $C(\mu)$ being minimal, it follows that $(C(\mu), T)$ is equicontinuous, (see [5] Corollary 8).

Remark 5.14 Next we study the spectral feature of an invariant measure on $(X_f, T)$, $(f \in \mathbb{K}(T))$. Professor W. Veech had posed the question : ‘Is the Sarnak conjecture valid for the flow $(X_f, T)$’? This question is answered affirmatively if one shows that every invariant measure has discrete spectrum. In a recent paper, (see [27]) the authors attempt to give a proof of this for $T = \mathbb{Z}$. But to us, the proof appears to be incomplete! We shall discuss the underlying issues with their proof and shall present a different proof. Thus proving Sarnak conjecture for $\mathbb{K}(T)$, for any countable amenable $T$.

Theorem 5.15 Any invariant measure on $(X_f, T)$, $f \in \mathbb{K}(T)$ has discrete spectrum.

A discussion on the proof.

Consider a general Veech system $(X, T)$ and let $\{ x_m \mid m \in \mathbb{N} \}$ be a $\tau_2$-dense subset of $X$. Using $\tau_1$-compactness of $X$, given any sequence $\{ t_n \} \in T$, by the ‘diagonal argument’ we can pick a subsequence $\{ t_{n_k} \}$ such that the sequence $\{ x_m t_{n_k} \}$ is $\tau_1$-convergent for each $m \in \mathbb{N}$. The key issue is to show that the sequence $\{ x t_{n_k} \}$ is $\tau_1$-convergent for each $x \in X$.
To do this one needs to use the special structure given by the $T$-invariant metric $D$ generating the $\tau_2$-topology. Note that by viewing $\{t_n\}$ as a net in $\beta T$ there is a subnet, (which may not be a subsequence), that converges to some $q \in \beta T$. Since $\{x_m t_{n_k}\}_{k \in \mathbb{N}}$ converges for each $m$, it follows that it must $\tau_1$-converge to $x_m q$. Now we make a note of the following points:

1. We know that for each $x \in X$, there is a subnet of $\{x t_{n_k}\}$ that $\tau_1$-converges to $x q$ and this subnet will depend on $x$. The crucial point is to show that the sequence $\{x t_{n_k}\}$ itself $\tau_1$-converges to $x q$ for each $x$.

2. To do this, one may think of using the following triangle inequality,

$$D(x t_{n_k}, x q) \leq D(x t_{n_k}, x_m t_{n_k}) + D(x_m t_{n_k}, x_m q) + D(x_m q, x q),$$

and try to show that each terms on the right hand side gets small as $n_k \to \infty$. Convergence in $D$ metric will yield $\tau_1$-convergence.

($2_a$) One has to be careful about ‘interchanging the limits’. That is, suppose $x_m \to x$ in the $\tau_1$ topology, in general $\lim_{m \to \infty} \lim_{k \to \infty} x_m t_{n_k}$ may not exist and even if it does, may not be equal to $\lim_{k \to \infty} \lim_{m \to \infty} x_m t_{n_k}$. Of course, the second limit exists and it is equal to $x q$. One could do this if $(X, T)$ is weakly almost periodic, (à la ‘Grothendieck’, see [12]), but not for a general Veech system. Thus, for a general Veech system making the third term $D(x_m q, x q)$ small is a problem. We have proved, (see Lemma 5.12), that for $(X_f, T)$, $(f \in \mathbb{K}(T))$, the map $\rho_q$ is not only Borel but it is in fact $D$ contracting. This will enable us to make the third term small as $n_k \to \infty$.

($2_b$) Making the second term small is even more problematic, because $x_m t_{n_k} \to x_m q$ only in $\tau_1$-topology. This is due to $\tau_1$-compactness of $X$. The $\tau_2$-topology given by the metric $D$ is not compact. This is a real hurdle in directly proving that $x t_{n_k} \to x q$. A way out is to work with continuous functions on $X$ rather than $X$ itself. We shall follow this approach, as in [27].

($2_c$) The first term in the above triangle inequality is exactly where one uses the $T$ invariance of metric $D$. However, just making these terms small in $D$ metric will not be enough, we need to do this in the $\tau_1$-topology, to use the $\tau_1$-compactness of $X$. We also need a ‘certain uniformity’ to get rid of the dependence on sequence $\{t_{n_k}\}$.

Thus, summarizing, to get $x t_{n_k} \tau_1$-converge to $x q$, we need (a) a certain ‘uniform mechanism’ that will give us ‘$\tau_1$-closeness’ from ‘$\tau_2$-closeness’. This will be used after making the first and the third term small in $D$ metric. (b) To make the second term small, we have to abandon the above triangle inequality and consider its analogue ‘for a continuous function’.

We again point out that the authors of [27] tacitly move pass the above issues by claiming ‘it is not hard’, (see [27, p.849]), without giving any indication of how to resolve these issues. This makes their proof of Theorem (5.1) incomplete. We shall prove why $\{x t_{n_k}\}$ converges for each $x \in X$ for the system $(X_f, T)$, $f \in \mathbb{K}(T)$ and for general Veech systems provided they satisfy an additional ‘uniformity condition’. Now we introduce this additional condition that the topologies $\tau_1$ and $\tau_2$ have to satisfy in order to carry out the above line of argument and this will lead to showing that any invariant measure on such systems has discrete spectrum.

**Definition 5.16** A Veech system $(X, T)$ is said to be a strongly Veech if in addition to the five properties in the definition of Veech systems, we also have the following sixth property:

6. Given a $\tau_1$-open set $V \subset X \times X$ containing the diagonal $\Delta_X$, there exists a $\delta > 0$ such that $B_{\delta}(x) \times B_{\delta}(x) \subset V$ for all $x \in X$. 

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Lemma 5.17 The Veech system \((X_f, T)\), \(f^* \in \mathcal{K}(T)\) is strongly Veech.

Proof. First observe that, given a \(f \in X_f, \tau\)-open neighbourhood of \(f\) is given by \(V_{g, \eta}(f)\), where \(g \in l^1(T)\) and \(\eta > 0\) and
\[
V_{g, \eta}(f) = \{ h \in X_f, \tau \in l^\infty(T) \mid \langle h - f, g \rangle < \eta \},
\]
where \(\langle \cdot, \cdot \rangle\) is the canonical pairing between vectors in \(l^\infty(T)\) and \(l^1(T)\).

Let \(V \subset X_{f, \tau} \times X_{f, \tau}\) be a \(\tau\)-open set containing the diagonal. Pick a \(\tau\)-open set \(V_1\) such that
\[
\Delta_{X_f, \tau} \subset V_1 \subset V \quad \text{and}
\]
and \(\{V_{g, \eta}(f_i) \mid 1 \leq i \leq \ell\}\) is a cover of \(X_{f, \tau}\). Compactness of \(X_{f, \tau}\) makes this possible.

Now we claim that, given \(f \in X_{f, \tau}\), \(B_\delta(f) \subset V_{g, \eta}(f_i)\) for some \(i \in \{1, \cdots, \ell\}\), where \(0 < \delta < \min\{\frac{\eta}{2||g||_\infty} \mid 1 \leq i \leq \ell\}\). To see this, first pick an \(i\) such that \(f \in V_{g, \eta}(f_i)\), let \(h \in B_\delta(f)\) and observe that
\[
\|\langle h - f_i, g_i \rangle\| \leq \|\langle (h - f) + (f - f_i), g_i \rangle\|
\leq \|\langle h - f, g_i \rangle\| + \|\langle f - f_i, g_i \rangle\|
\leq \|h - f\|_\infty \|g_i\|_1 + \frac{\eta_i}{2}, \quad \text{since } f \in V_{g_i, \eta}(f_i)
\leq \delta \|g_i\|_1 + \frac{\eta_i}{2} = \frac{\eta_i}{2} = \eta_i.
\]
Hence \(h \in V_{g_i, \eta}(f_i)\). Thus, \(B_\delta(f) \times B_\delta(f) \subset V_1 \subset V\) .

Proof of Theorem 5.15:
It is enough to show that each \(g \in C(X_f) \subset L^2(X_f, \mu)\) is \(\mu\)-compact vector. To do this we show that given any sequence \(\{t_n\}\) in \(T\), it has a subsequence \(\{t_{n_k}\}\) such that \(g_{t_{n_k}}\) converges pointwise on \(X_f\), (and hence by the dominated convergence theorem, in the \(L^2\) norm on \((X_f, \mu)\)). This will prove \(\mu\)-compactness of \(g\).

So, as discussed before, by the ‘diagonal procedure’ select a subsequence \(\{t_{n_k}\}\) such that the sequence \(x_{m,t_{n_k}}\) converges, (as \(k \to \infty\)), for each \(m \in \mathbb{N}\). Now a subnet of \(\{t_{n_k}\}\) converges to some \(q \in \beta T\), (in the topology on \(\beta T\)). Since \(\{x_{m,t_{n_k}}\}\) converges, it will converge to \(x_{m,q}\), \((m \in \mathbb{N})\).

Now we show that the sequence \(g(x_{n_k})\) converges for each \(x \in X_f\). So fix any \(x \in X_f\) and let \(\varepsilon > 0\) be given. Since \((x, y) \to g(x) - g(y)\) is continuous and \(X\) is \(\tau\)-compact, we can find a \(\tau\)-open neighbourhood \(V\) of the diagonal \(X_f \times X_f\) such that if \((x, y) \in V\) then \(|g(x) - g(y)| < \frac{\varepsilon}{3}\). For this \(V\), pick \(\delta > 0\) as in ‘Property (6)’, (see the definition of strong Veech systems). Pick \(m \in \mathbb{N}\) such that \(D(x, x_m) < \delta\). Note that (i) \(D(x_{m,q}, x_q) \leq D(x_m, x)\) by (5.2) and (ii) \(D(x_{n_k}, x_{m,t_{n_k}}) = D(x, x_m) < \delta\).

Thus, \((x_{n_k}, x_{m,t_{n_k}}) \in V\) and \((x_{m,q}, x_q) \in V\). Now consider the inequality,
\[
|g(x_{t_{n_k}}) - g(x_q)| \leq |g(x_{t_{n_k}}) - g(x_{m,t_{n_k}})| + |g(x_{m,t_{n_k}}) - g(x_{m,q})| + |g(x_{m,q}) - g(x_q)|
\leq \frac{\varepsilon}{3} + |g(x_{m,t_{n_k}}) - g(x_{m,q})| + \frac{\varepsilon}{3}.
\]
Thus, there exists \(k_0\) such that if \(k > k_0\), then \(|g(x_{t_{n_k}}) - g(x_q)| < \varepsilon\). This proves pointwise convergence of \(g(x_{t_{n_k}})\).
**Remark 5.18** Actually a tiny modification of the arguments in above proof yields the same conclusion for any strongly Veech system.

**Theorem 5.19** Let $T$ be amenable, then any invariant measure on a strongly Veech system $(X, T)$ has discrete spectrum.

**Proof.** With the notation as in the previous theorem, we need to show that the sequence $g(x t_{n_k})$ converges for each $x \in X_f$. We can show that it is a Cauchy sequence by considering the inequality

$$|g(x t_{n_k}) - g(x t_{n_l})| \leq |g(x t_{n_k}) - g(x m t_{n_k})| + |g(x m t_{n_k}) - g(x m t_{n_l})| + |g(x m t_{n_l}) - g(x t_{n_l})|.$$  

$$\leq \varepsilon/3 + |g(x m t_{n_k}) - g(x m t_{n_l})| + \varepsilon/3.$$  

The rest of the argument is as before. 

Finally, one would like to know whether $(X_f, T) , (f \in \mathbb{K}(T))$ is tame, or more generally any strongly Veech system is tame? We answer this question below.

**Theorem 5.20** Let $(X, T)$ be a strongly Veech system with $T$ amenable.

1. If $(X, \tau_1)$ is metrizable, then $(X, T)$ is tame.
2. In particular $(X_f, T)$ is tame, where $f \in \mathbb{K}(T)$, (recall that $T$ is countable, amenable).
3. As a consequence, metrizable, strongly Veech systems have zero topological entropy and
4. the Sarnak conjecture holds for such systems.\(^3\)

**Proof.** (1): Our assumption implies that $C(X)$ the space of continuous real valued functions on $X$ with the sup-topology is separable. Fix a countable dense set $g_n \in C(X)$. In the above theorem we have already showed that given any $\alpha \in \beta T \setminus T$, and $g \in C(X)$, there exists a sequence $\{t_k\}$ in $T$ such that the sequence $g(x t_k)$ converges to $g(x \alpha)$ for all $x \in X$. Again, by the arguments in the previous theorem, given any $\alpha \in \beta T \setminus T$, we can find a sequence $\{t_k\}$ such that $g_n(x t_k)$ converges to $g_n(x \alpha)$ as $k \to \infty$, for each $x \in X$ and $n \in \mathbb{N}$. Since $\{g_n \mid n \in \mathbb{N}\}$ is dense in $C(X)$, this implies $x t_k \to x \alpha$, (in $\tau_1$ topology), for each $x \in X$. This shows that $\rho_{\alpha} \in E(X, T)$ is of Baire class 1, for every $\alpha \in \beta T \setminus T$. Thus $(X, T)$ is tame.

(2): We only need to observe that $(X_f, \tau_1)$ is metrizable. Note that since $T$ is countable and $f$ is bounded, with out loss of generality $|f(t)| \leq 1$, $t \in T$. Let $\psi : X_f \to [0, 1]^T$ be the map $(\psi(x))_i = xt$. Then $\psi$ is an injective map onto its image and it intertwines the $T$ action on $X_f$ with the shift action on the Hilbert cube. Observe that $\psi$ is a homeomorphism where its domain has the weak* topology and the range has the (restriction of) the product topology on $[0, 1]^T$. The later topology being metric, it follows that $(X, \tau_1)$ is metrizable.

(3) and (4): Now these results follow from the fact that every invariant measure on $X$ has discrete spectrum and $X$ is metrizable. 

**Remark 5.21** After the first version of this paper was posted on Arxiv, M. Megrelishvili informed us that one can prove (2) of Theorem 5.20 by applying [22, Theorem 8.2.4] combined with [22, Theorem 9.12] and [23, Theorem 6.1]. However, our proof for amenable acting groups is direct and self contained.

\(^3\)See Section 6. for more details.
6 Applications to number theory

Möbius disjointness. In this section, we are interested in the applications of our results on Veech systems to Number theory. Precisely, our applications are related to the so called Möbius randomness law as formulated by P. Sarnak in his striking paper [47]. This law is about the dynamical behavior of the Möbius and Liouville functions.

We recall that the integer is square-free if its prime decomposition does not contain any square. The Liouville function $\lambda$ is defined as 1 if the number of the prime factor of the integer is even and $-1$ if not, and the Möbius function $\mu$ coincide with the Liouville function on its support which is the subset of square-free integers.

The Möbius randomness law à la Sarnak state that the statistical average or Cesàro average of the values of a continuous map along a orbit of any point $x$ with respect to any transformation with zero topological entropy, averaged with weights given by the Möbius function, converge to zero. Formally, this law can be stated as follows:

Sarnak’s Möbius disjointness Conjecture. Let $(X, T, \mu)$ be a compact metric, topological dynamical system with topological entropy zero, then, for any $x \in X$, for any continuous function $f : X \to \mathbb{R}$, the following should hold.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) f(T^n x) = 0.$$ 

This law is also known as Sarnak’s conjecture or Möbius disjointness conjecture. We proved that the Sarnak conjecture holds for the system $(X_f, \mathbb{Z})$, where $f \in K(\mathbb{Z})$. A bit later we shall see a number theoretic consequence of this. But first we recall that for the simplest zero entropy dynamical system—the irrational rotation of the circle, Sarnak’s conjecture is a consequence of the following Davenport estimate, (see [9]),

$$\max_{\theta \in \mathbb{T}} \left| \sum_{k \leq x} \mu(k) e^{ik\theta} \right| \leq \frac{x}{\log(x)^A}, \quad \text{where } A > 0.$$ 

We view this as ‘Möbius disjointness’ for the almost periodic map $k \to e^{ik\theta} : \mathbb{Z} \to \mathbb{R}$. Now we can extend this ‘Möbius disjointness’ to Besicovitch almost periodic functions on $\mathbb{Z}$ by the following simple argument. Let $f : \mathbb{Z} \to \mathbb{R}$ be a Besicovitch almost periodic map. Thus, there is a sequence $\{g_j\}$ of (Bohr) almost periodic maps from $\mathbb{Z}$ to $\mathbb{R}$ such that given $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that $\|f - g_k\|_{B_1} < \varepsilon$. Now for any $N \geq 1$, we have

$$\left| \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) f(n) - \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) g_k(n) \right| \leq \sup_{j} \left( \frac{1}{N} \sum_{n=1}^{N} \left| f(n) - g_k(n) \right| \right) = \|f - g_k\|_{B_1} < \varepsilon.$$

This extension of ‘Möbius disjointness’ from Bohr almost periodic to Besicovitch almost periodic functions immediately yields the following.

**Theorem 6.1** Let $(X, T, \mu)$ be uniquely ergodic system with discrete spectrum. Then, the Möbius disjointness holds.
Proof. This follows immediately from Corollary 3.12.

As an immediate consequence, we have the following.

Corollary 6.2 The Möbius disjointness holds for any weakly almost periodic system.

Proof. Weakly almost periodic systems are uniquely ergodic with discrete spectrum, (see [13]).

Remark 6.3 With our notion of weak tameness, Theorem 1.2 of [27] says that the Sarnak conjecture holds for weakly tame systems. Here, we have used this theorem to prove the validity of Sarnak conjecture for Veech systems. Unfortunately, this theorem does not say anything about the validity of Möbius disjointness for the simplier example 2.8. Furthermore, even if that theorem is improved to establish Möbius disjointness for systems with only countably many ergodic measures with discrete spectrum, it still does not say anything about example 2.8. In addition, one also observes that the results of a recent paper [15] do not apply to our example to validate ‘logarithmic Möbius disjointness’. On the other hand it is easy to check that this example satisfy Möbius disjointness conjecture. Notice further that the results of a recent paper [15] do not apply to the graph maps and dendrites maps.

In the forthcoming paper [2], the authors proved that Sarnak’s Möbius disjointness holds if each invariant measure has a singular spectrum. Therefore, it suffices to establish that the conjecture holds only for the system for which invariant measures that have a Lebesgue component. We further establish that the spectral measure of the Möbius function is absolutely continuous with respect to the Lebesgue measure. We would like also to point out that therein the authors present a ‘dissection of Möbius flow à la Veech’ and use the result of Rokhlin-Sinai which assert that for any dynamical system with positive entropy has the relatively Kolmogorov property with respect to Pinsker algebra. This was also observed and popularized by Jean-Paul Thouvenot [53]. Accordingly, now it is obvious to deduce that the dynamical system \((x, y) \in \mathbb{T} \mapsto (x, x + y)\) can not be a factor of the Möbius flow.

Improving Motohashi-Ramachandra estimate.

Here, we will gives a simple argument which yields a slight improvement of an old result of Motohashi-Ramachandra [42], [45] on the behavior of Mertens function \(M(x) := \sum_{n \leq x} \mu(n)\) on the short interval. We start by recalling Motohashi-Ramachandra’s result.

Lemma 6.4 (Motohashi-Ramachandra’s theorem [42], [45]) The Mertens function satisfy,

\[
|M(x + h) - M(x)| = o(h),
\]

uniformly in \(h\) satisfying \(x^\tau \leq h \leq x\), whenever \(\tau > \frac{7}{12}\).

However, let us mention that in the same year, using the so-called Hooley-Huxley contour, K. Ramachandra obtain the following estimations.

Lemma 6.5 (Ramachandra’s theorem [45]) The Mertens function satisfy, for any \(A > 0\),

\[
\sum_{x \leq n \leq x + h} \mu(n) = O_{\varepsilon, A}\left(\frac{h}{\log(x)^A} + x^{\frac{1}{A} + \varepsilon}\right) \quad \text{and, as } x \to +\infty
\]

(6.1)

\[
\frac{1}{X} \int_X^{2X} \left| \sum_{x \leq n \leq x + h} \mu(n) \right|^2 \, dx = O_{\varepsilon, A}\left(\frac{h}{\log(X)^A} + X^{\frac{1}{A} + \varepsilon}\right), \quad \text{as } X \to +\infty.
\]

(6.2)
In his 2016’s paper [56], Professor W. Veech observes that no progress was made on the behavior of Mertens function in the short interval since Motohashi and Ramachandra original papers. It turns out that in the same year, Matomaki-Radzwill in [37] improved (6.2) by establishing that for any \( \varepsilon > 0 \) and \( h \leq X^\varepsilon \), we have

\[
\frac{1}{X} \int_X^{2X} \left| \sum_{x \leq n \leq x+h} \mu(n) \right|^2 \, dx = o(Xh^2).
\]  

(6.3)

Notice that it is easy to obtain the following corollary from Motohashi-Ramachandra’s theorem.

**Corollary 6.6** Let \((x_n)\) a sequence of positive real numbers and \( \tau > \frac{7}{12} \). Suppose that \( x_n + (x_{n+1} - x_n)^\tau \leq x_{n+1} \leq 2x_n \), for a large \( n \). Then,

\[
\sum_{k=1}^{n} \left| M(x_{k+1}) - M(x_k) \right| = o(x_{n+1}).
\]

One can state similar corollary by applying (6.3).

Here, our Theorem (5.20) showing that Sarnak conjecture holds for \((X_f, \mathbb{Z})\), for \( f \in K(\mathbb{Z}) \), will allow us to obtain a stronger result, namely the following.

**Theorem 6.7** Let \((x_n)\) a sequence of positive real numbers such that \( x_{n+1} - x_n \to +\infty \) as \( n \to +\infty \). Then,

\[
\sum_{k=1}^{n} \left| M(x_{k+1}) - M(x_k) \right| = o(x_{n+1}).
\]

**Proof.** Let \( k \in \mathbb{N} \) and put

\[
\varepsilon_k = \begin{cases} 
\text{Sg}(M(x_{k+1}) - M(x_k)) & \text{if } M(x_{k+1}) - M(x_k) \neq 0, \\
1, & \text{if not},
\end{cases}
\]

where \( \text{Sg}(x) = \frac{x}{|x|} \), for \( x \neq 0 \). Now, define a sequence \( f = f(\varepsilon_k) \) by

\[
f(n) = \sum_{k \geq 1} \varepsilon_k 1_{[x_k, x_{k+1}]}(n).
\]

Clearly \( f \) is in \( \ell^\infty(\mathbb{Z}) \) and as shown before, \( f \in K(\mathbb{Z}) \). Since Sarnak’s conjecture holds for \((X_f, \mathbb{Z})\),

\[
\sum_{k=x_1}^{x_{n+1}} \mu(k) f(k) = o(x_{n+1}).
\]

(6.4)
But
\[
\sum_{k=x_1}^{x_{n+1}} \mu(k)f(k) = \sum_{j=1}^{n} \sum_{k=x_j}^{x_{j+1}} \mu(k)f(k) \\
= \sum_{j=1}^{n} \sum_{k=x_j}^{x_{j+1}} \mu(k) \\
= \sum_{j=1}^{n} Sg(M(x_{j+1}) - M(x_j)) \sum_{k=x_j}^{x_{j+1}} \mu(k) \\
= \sum_{j=1}^{n} \left| M(x_{j+1}) - M(x_j) \right|. \\
\tag{6.5}
\]

The last inequalities follows from the definition of \((\epsilon_k)\) and \(M\). Combining (6.4) and (6.5), we obtain the desired estimation, and the proof of the proposition is complete.\(^4\)

\begin{remark}
Note that once we show that translation flow on Veech function is tame, validity of Sarnak conjecture for this flow follows from to Theorem 2.1 from [27]. It is a common misunderstanding that one uses the work of Motomaki-Radzwill\[^{37}\] for this purpose. It is a result of Matomaki-Radzwill-Tao on the validity of averaged form of Chowla of order two, (see [36]) was used in [27] for the proof of Sarnak conjecture for systems for which every invariant measure has discrete spectrum. Let us notice further that this improvement can be obtained also as a consequence of Matomaki-Radzwill’s result [37]. However, our proof avoid the heavy analytic Number Theory machinery. We point out that the Matomaki-Radzwill-Tao result on the validity of averaged form of Chowla of order two, (in [36]), does not need a more elaborate machinery of analytic number theory like the result of Matomaki-Radzwillin [37] as it is shown in the appendix. Indeed, the only ingredient needed for the proof is Davenport estimate. Thus, even though Theorem 6.7 can be derived also from Matomaki-Radzwill result of [37], our approach considerably reduces the input from Number Theory.

Having said this, we also point out that our proof of Sarnak conjecture for systems with singular spectrum in reference [2] bypasses even Matomaki-Radzwill-Tao and makes our approach to Mertens’s growth far more dynamical/ergodic-theoretic with only a minimal number theory input, (namely, uses only Davenport estimate).

The previous result can be improved by assuming Chowla conjecture which asserts that for any distinct integers \(s_1, s_2, \cdots, s_k, k \geq 1\), we have
\[
\frac{1}{N} \sum_{n=1}^{N} \lambda(n + s_1) \cdots \lambda(n + s_k) \xrightarrow{N \to +\infty} 0.
\]

It follows that for any distinct integers \(a_1, a_2, \cdots, a_k, k \geq 1\), we have
\[
\frac{1}{N} \sum_{n=1}^{N} \lambda^{a_1}(n + s_1) \cdots \lambda^{a_k}(n + s_k) \xrightarrow{N \to +\infty} 0.
\]
\[^{4}\]This result can be obtained also as a consequence of Matomaki-Radzwill’s result [37]. However, our proof avoid the heavy analytic Number Theory machinery.
We thus get that $\lambda$ is normal, that is, generic for the Bernoulli measure $dB(1/2) = \otimes_{k \in \mathbb{N}}^{1/2}(1/2 + 1/2 \delta_{-1})$.

Let $X = \{-1, 1\}^\mathbb{N}$ and $X_\lambda$ be the orbit generated by $\lambda$ under the shift map $S : \omega \mapsto S(\omega) = (\omega(n+1))$.

For any $\omega \in X_\lambda$, we define the random Mertens function by

\[ M_\omega(x) = \sum_{n \leq x} \omega(n). \]

**Definition 6.9** Let $f$ be an arithmetic function ($f : \mathbb{N} \rightarrow \mathbb{C}$) and $\tau_0 \geq 0$. $f$ is said to satisfy Motohashi-Ramachandra property of order $\tau_0$ if for any $\tau > \tau_0$, we have

\[ \sum_{x \leq n \leq x+h} f(n) = o(h), \]

uniformly in $h \geq x^\tau$, that is, there is a sequence $(\delta_x)$ such that $(\delta_x) \xrightarrow{x \rightarrow +\infty} 0$, and there exist a constant $C_\tau > 0$ for which we have

\[ \sup_{x^\tau \leq h \leq x} \left| \frac{1}{h} \sum_{n \leq x} f(n) \right| \leq C_\tau \delta_x. \]

At this point, let us point out that Motohashi [42] and independently Ramachandra [45] proved that the Möbius function satisfy their property for $\tau_0 = \frac{7}{12}$ (Lemma 6.4). Later, T. Zhan extended Motohashi-Ramachandra’s result by proving that the estimation of Davenport holds on the short interval [58], that is,

\[ \sup_{x^\tau \leq h \leq x} \left| \frac{1}{h} \sum_{x \leq n \leq x+h} \mu(n)e^{in\theta} \right| \leq \frac{C_{\tau, \epsilon}}{\log(x)^{\epsilon}}, \]

for any $\tau > \frac{5}{8}$ and $\epsilon > 0$, uniformly in $\theta$. For the quadratic case, J. Liu and T. Zhan improved Hua’s result on the sum of two primes and a prime square in [26] by establishing that the bound can be $\frac{11}{16}$ and $\frac{2}{3}$ under GRH [35]. Besides, D. Hajela and J. Smith conjectured, (see [29]), that for any $\tau > \frac{1}{2},$

\[ \sup_{\theta} \left| \frac{1}{x^{\tau}} \sum_{n \leq x} \mu(n)e^{in\theta} \right| \leq \delta_{\tau, x}, \]

with $\delta_{\tau, x} \xrightarrow{x \rightarrow +\infty} 0.$

When this paper was in final preparation, Igor Shparlinski informed us that Matomäki- Teräväinen improved the bounded to $11/20$ [38]. Here, in the spirit of the dichotomy à la Sarnak [48], [49] we establish the following:

**Theorem 6.10** Under Chowla conjecture, we have for almost all $\omega \in X_\lambda$, for any $\tau > \frac{1}{7},$

\[ \sup_{x^\tau \leq h \leq x} \left| \frac{1}{h} \left( M_\omega(x+h) - M_\omega(x) \right) \right| \xrightarrow{x \rightarrow +\infty} 0. \]
Proof. We start by noticing that under Chowla conjecture, the system \((X_\lambda, B, S, dB(\frac{1}{2}))\) is a Bernouilli system. Therefore, the sequence of random Merstens function \((M_\omega(x))\) is a martingale. We thus get, by Doob-Kolmogorov inequality
\[
\sup_{x^\tau \leq h \leq x} \left\| \frac{1}{h} \left( M_\omega(x + h) - M_\omega(x) \right) \right\|_2 \leq \sup_{x^\tau \leq h \leq x} \left\| \frac{1}{h} M_\omega(x + h) \right\|_2 + \frac{1}{x^\tau} \left\| M_\omega(x) \right\|_2 \\
\leq \frac{1}{x^\tau} \left\| M_\omega(2x) \right\|_2 + \frac{1}{x^\tau} \left\| M_\omega(x) \right\|_2 \leq (\sqrt{2} + 1)x^{\frac{1}{2} - \tau}.
\]
(6.6)

(6.7)
Now, we apply Etemadi’s trick. Take \(\tau > \frac{1}{2}\) and \(x = [\rho^y], \rho > 1\), to see that
\[
\sum_x \sup_{x^\tau \leq h \leq x} \left\| \frac{1}{h} \left( M_\omega(x + h) - M_\omega(x) \right) \right\|_2 < +\infty.
\]
This gives that, for almost all \(\omega \in X_\lambda\),
\[
\sup_{x^\tau \leq h \leq x} \left\| \frac{1}{h} \left( M_\omega(x + h) - M_\omega(x) \right) \right\|_2 \xrightarrow{x \to +\infty} 0.
\]
We finish the proof by letting \(\rho \to 1\).

Conjecture. We conjecture that for any \(\tau > \frac{1}{2}\),
\[
\left| M(x + h) - M(x) \right| = o(h),
\]
uniformly in \(h\) provided \(x^\tau \leq h \leq x\).

Besicovitch almost periodicity of certain number theoretic functions.

Now, we would like to mention that G. Rauzy pointed out that the square of the M"obius function is a Besicovitch almost periodic sequence (i.e. a Besicovitch almost periodic function), (see [46, p.99]). Here, let us notice that this fact can be extended to the analogous number theoretic map in the more general setting of \(B\)-free integers. We recall this notion of P. Erd"os [14].

Definition 6.11 Let \(B = \{b_k \mid k \in \mathbb{N}\} \subset \{n \in \mathbb{N} \mid n \geq 2\}\) be a subset of natural numbers which have the following properties:
\[
\text{for all } 1 \leq k < k', b_k \text{ and } b_{k'} \text{ are relatively prime and } \sum_{k \geq 1} \frac{1}{b_k} < \infty. \tag{6.8}
\]
Integers with no factors in \(B\) are called \(B\)-free integers and the set of \(B\)-free integers will be denoted by the set \(B\).

Let \(\chi_B\) denote the indicator function of the set \(B\). The set of square-free integers is a special case when \(B\) is the set of all squares primes. L. Mirsky had studied, (see [39], [40], [41]), the distribution of patterns in the characteristic function of \(r\)-free numbers, that is, the numbers which are not divisible by the \(r\)-th power of any prime \((r \geq 2)\).

To establish that the indicator function of \(B\)-free numbers is a Besicovitch sequence, it suffices to prove that the indicator function \(\chi_{m_B}\) of the subset \(m_B \overset{\text{def}}{=} \{x \mid x \equiv 0 \mod b_k \text{ for some } k \geq 1\}\) is a
Besicovitch sequence. For that let \( K \geq 1 \) and \( \chi_{m_{B_{K}}} \) the indicator function of the subset \( m_{B_{K}} \stackrel{\text{def}}{=} \{ x \mid x \equiv 0 \mod b_{K} \text{ for some } k \in \{1, \cdots , K\} \} \). It follows that

\[
\lim \sup \frac{1}{N} \sum_{n=1}^{N} |\chi_{m_{B}}(n) - \chi_{m_{B_{K}}}(n)| \leq \sum_{k>K} \frac{1}{b_{k}} \to 0, \text{ as } K \to \infty. \tag{6.9}
\]

Furthermore, \( \chi_{m_{B_{K}}} \) is a periodic function. Taking into account that Mirsky’s theorem can be extended to \( B \)-free integers ([4]), (that is, the indicator function of \( B \)-free integers is a ‘generic point’ for the Mirsky measure), our Theorem 3.11 shows that the subshift generated by \( \chi_{B} \) its Mirsky measure has discrete spectrum. This gives a new and simple proof of Cellarosi-Sinai theorem [8] and el Abdalaoui-Lemańczyk-de-la-Rue extension of it [3].

We need to point out here that the principal tool in the proof of Mirsky theorem is based on the notion of admissibility. This notion is crucial in the studies of the dynamical behavior of \( B \)-free systems. It is also fundamental in the structure of Möbius flow and the well-know Chowla conjecture. For more details, we refer to [1].

We recall that the subset \( A \) of positive integers is \( B \)-admissible if for any \( k \geq 1 \), the image of \( A \) under the maps \( x \in \mathbb{N}^{*} \mapsto \pi \in \mathbb{Z}/b_{k}\mathbb{Z} \) is proper, that is,

\[ \{|y \in \mathbb{Z}/b_{k}\mathbb{Z} : \exists n \in A, n = y [b_{k}]\} | < b_{k} . \]

An infinite sequence \( x = (x_{n})_{n \in \mathbb{N}^{*}} \in \{0,1\}^{\mathbb{N}} \) is said to be \( B \)-admissible if its support \( \{ n \in \mathbb{N}^{*} : x_{n} = 1 \} \) is \( B \)-admissible. In the same way, a finite block \( x_{1} \cdots x_{N} \in \{0,1\}^{N} \) is \( B \)-admissible if \( \{ n \in \{1, \cdots , N\} : x_{n} = 1 \} \) is \( B \)-admissible.

Let us notice that the approximation of \( \chi_{B} \) by the periodic function \( \chi_{m_{B_{K}}} \) can not be uniform in the following sense

\[ \lim \sup \sup _{N} \left( \frac{1}{N} \sum_{n=1}^{N} |\chi_{B}(n+k) - \chi_{m_{B_{K}}}(n+k)| \right) = 0 , \]

since the flow generated by the indicator function of \( B \)-free numbers has a positive topological entropy. We can also see this directly. Indeed, for any \( x > 0 \), the sequence \( 00 \cdots 0 \) is an admissible sequence.

Therefore, for any fixed \( x \) there is a positive density of \( k \)'s for which \( \chi_{B}(n+k) = 0 \), for \( n = 1, \cdots , [x] \). Moreover, if \( k \) is a multiple of the period \( c \) of \( \chi_{m_{B_{K}}} \), then we have \( \chi_{m_{B_{K}}}(n+k) = \chi_{m_{B_{K}}}(n) \). Thus, we get

\[ \lim \sup \sup _{N} \left( \frac{1}{N} \sum_{n=1}^{N} |\chi_{B}(n+k) - \chi_{m_{B_{K}}}(n+k)| \right) \]

\[ \geq \lim \sup \left( \frac{1}{N} \sum_{n=1}^{N} |\chi_{m_{B}}(n+c) - \chi_{m_{B_{K}}}(n+k.c)| \right) \]

\[ \geq \lim \sup \left( \frac{1}{N} \sum_{n=1}^{N} \chi_{m_{B_{K}}}(n) \right) \]

\[ = \prod_{k=1}^{K} \left( 1 - \frac{1}{b_{k}} \right) > 0 . \]
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Appendix. On the average Chowla of order two.

In this short note, by applying a Bourgain’s observation, we present a simple proof of Matomäki-Radziwiłł-Tao theorem on the average Chowla of order two \[^{36}\] based on Davenport theorem. In their inequality \(H\) is allowed to grow very slowly with respect to \(X\). Here, for \(H = X\) we obtain a bound for the speed of convergence. Notice that this is the only ingredient needed for the proof of the validity of Sarnak conjecture for systems with discrete spectrum in Huang-Wang-Ye’s result.

**Theorem 12** Let \(\nu\) be a Möbius or Liouville function. Then, for any \(N \geq 2\),

\[
\frac{1}{N} \sum_{m=1}^{N} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n + m) \right| \leq \frac{C}{\log(N)^\kappa},
\]

where \(C\) is some positive constant.

**Proof.** By Cauchy-Schwarz inequality, we have

\[
\frac{1}{N} \sum_{m=1}^{N} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n + m) \right| \leq \left( \frac{1}{N} \sum_{m=1}^{N} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n + m) \right|^2 \right)^{\frac{1}{2}},
\]

and by Bourgain’s observation \[^{7}\], equations (2.5) and (2.7)], we have

\[
\sum_{m=1}^{N} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n + m) \lambda^{n+m} \right|^2 = \sum_{m=1}^{N} \left| \int_{T} \left( \frac{1}{N} \sum_{n=1}^{N} \nu(n)z^{-n} \right) \left( \sum_{p=1}^{2N} \nu(p)\lambda^p z^p \right)^2 \right| dz \leq \int_{T} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)z^{-n} \right|^2 \left| \sum_{p=1}^{2N} \nu(p)\lambda^p \right|^2 dz \leq \sup_{z \in T} \left( \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)z^{-n} \right|^2 \right) \int_{T} \left| \sum_{p=1}^{2N} \nu(p)\lambda^p \right|^2 dz.
\]
The inequality (1.3) is due to Parseval inequality. Indeed, by putting

\[ \Phi_N(z) = \left( \frac{1}{N} \sum_{n=1}^{N} \nu(n) z^{-n} \right) \left( \sum_{p=1}^{2N} \nu(p) (\lambda z)^p \right). \]

We see that for any \( m \in \mathbb{Z} \),

\[ \hat{\Phi}_N(m) = \int_{T} \left( \frac{1}{N} \sum_{n=1}^{N} \nu(n) z^{-n} \right) \left( \sum_{p=1}^{2N} \nu(p) (\lambda z)^p \right) z^{-m} dz. \]

and

\[
\begin{align*}
\sum_{m=1}^{N} \left| \int_{T} \left( \frac{1}{N} \sum_{n=1}^{N} \nu(n) z^{-n} \right) \left( \sum_{p=1}^{2N} \nu(p) (\lambda z)^p \right) z^{-m} dz \right|^2 \\
\leq \sum_{m=1}^{N} \left| \hat{\Phi}_N(m) \right|^2 \\
\leq \int_{T} |\Phi_N(z)|^2 dz.
\end{align*}
\]

Now, by appealing to Davenport Theorem, we get

\[
\begin{align*}
\sum_{m=1}^{N} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) \nu(n+m) \lambda^{n+m} \right|^2 \\
\leq \frac{C_\epsilon}{\log(N)^\alpha} \quad (14)
\end{align*}
\]

From this, we obtain the desired inequality and the proof is complete.

\[ \blacksquare \]

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