Some Exceptional Beauville Structures

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Abstract

We first show that every quasisimple sporadic group, including the Tits group $2F_4(2)'$, possesses an unmixed strongly real Beauville structure aside from the Mathieu groups $M_{11}$ and $M_{23}$ (and possibly $2B$ and $M$). We go on to show that no almost simple sporadic group possesses a mixed Beauville structure. We go on to use the exceptional nature of the alternating group $A_6$ to give a strongly real Beauville structure for this group explicitly correcting an earlier error of Fuertes and González-Diez. In doing so we complete the classification of alternating groups that possess strongly real Beauville structures. We conclude by discussing mixed Beauville structures of the groups $A_6 : 2^{(2)}$.

Keywords: Beauville structure, Beauville surface, sporadic simple group

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1. Introduction

1.1. Beauville Surfaces, Structures and Groups

Complex surfaces lie in the intersection of algebraic geometry, differential geometry and complex variable theory and as such enjoy applications as far afield as number theory, topology and even superstring theory. Finding examples of such surfaces that are easy to work with is thus more important than ever. One approach that has proved particularly fruitful over the past ten years or so is the concept of a Beauville surface: a class of 2-dimensional complex algebraic varieties that are rigid, in the sense of admitting no non-trivial deformations, whose study was recently initiated by Bauer, Catanese and Grunewald in [2, 3, 7]. These surfaces are defined over the field $\mathbb{Q}$ of algebraic numbers, providing a geometric action of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By generalizing Beauville’s original example [4, p.159], they can...
be constructed from finite groups acting on suitable pairs of algebraic curves, and here we give some new examples of surfaces of this kind.

**Definition 1.** A Beauville surface of unmixed type is a compact complex surface $S$ such that

(a) $S$ is isogenous to a higher product, that is, $S \cong (\mathcal{C}_1 \times \mathcal{C}_2)/G$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are algebraic curves of genus at least 2 and $G$ is a finite group acting by the diagonal action freely on $\mathcal{C}_1 \times \mathcal{C}_2$ by holomorphic transformations;

(b) If $G_0 < G$ denotes the subgroup consisting of the elements which preserve each of the factors, then $G_0$ acts effectively on each curve $\mathcal{C}_i$ so that $\mathcal{C}_i/G_0 \cong \mathbb{P}^1$ and $\mathcal{C}_i \to \mathcal{C}_i/G_0$ ramifies over three points.

Condition (b) is equivalent to each curve $\mathcal{C}_i$ admitting a regular dessin in the sense of the theory of dessin d’enfants due to Grothendieck [8, 17, 29], or equivalently an orientably regular hypermap [19], with $G$ acting as the orientation-preserving automorphism group.

One particularly attractive feature of this class of curves is the fact that the above definition can be translated into more finitely combinatorial terms that ‘internalize’ the structure of the surface into the group $G$ in the following way.

**Definition 2.** Let $G$ be a group. An unmixed Beauville structure of $G$ is a pair of generating sets $\{x_i, y_i, z_i\} \subset G$ with $l_i := o(x_i)$, $m_i := o(y_i)$ and $n_i := o(z_i)$ for $i = 1, 2$ such that the following holds for $i = 1, 2$.

1. $x_i y_i z_i = 1$;
2. $l_i^{-1} + m_i^{-1} + n_i^{-1} < 1$ and
3. Defining

$$\Sigma(x_i, y_i, z_i) := \bigcup_{g \in G} \bigcup_{j=0}^{\infty} \{gx_i^j g^{-1}, gy_i^j g^{-1}, gz_i^j g^{-1}\}$$

we have

$$\Sigma(x_1, y_1, z_1) \cap \Sigma(x_2, y_2, z_2) = \{e\}.$$

We say that the Beauville structure has type $((l_1, m_1, n_1), (l_2, m_2, n_2))$. We call a group possessing an unmixed Beauville structure a Beauville group.
It was conjectured by Bauer, Catanese and Grunewald that every non-abelian finite simple group is a Beauville group, with the sole exception of the alternating group $A_5$ [3, Conjecture 1]. Several authors settled special cases of this conjecture [3, 12, 13, 16]. Finally the full conjecture was recently verified by the author, Magaard and Parker in [11, theorem 1.3] where we prove the following more general theorem.

**Theorem 3.** Aside from the groups $SL_2(5)$ and $PSL_2(5) (\cong A_5)$ every non-abelian finite quasisimple group possesses an unmixed Beauville structure.

Similar results were obtained at around the same time by Guralnick and Malle in [18] and by Garion, Larsen and Lubotzky in [15].

1.2. Real Surfaces and Mixed Structures

Now that we know that almost every quasisimple group is a Beauville group, we are in a position to address the more general issue of what these Beauville structures and surfaces actually look like. One more specific instance of this somewhat vague question is to ask when a complex surface $S$ is ‘real’ (ie there is a biholomorphic map $\sigma : S \rightarrow \overline{S}$ such that $\sigma^2$ is the identity). As is the ‘zeitgeist’ of Beauville constructions, this topological property can be translated into finitery combinatorial terms inside the corresponding group.

**Definition 4.** We say that a Beauville surface is real and its corresponding Beauville structure $\{x_i, y_i, z_i|i = 1, 2\}$ and group $G$ are strongly real if there exist automorphisms $\phi_i \in \text{Aut}(G)$ for $i = 1, 2$ that differ only in an inner automorphism of $G$ such that $x_i^{\phi_i} = x_i^{-1}$ and $y_i^{\phi_i} = y_i^{-1}$ for $i = 1, 2$. If such an automorphism exists we say that $G$ is strongly $(l_i, m_i, n_i)$ generated.

It has been conjectured by Bauer, Catanese and Grunewald [2, Conjecture 3] that all but finitely many of the finite simple groups are strongly real Beauville groups. Given the progress made on the wider class of quasisimple groups in theorem [3] it seems natural to make following stronger conjecture.

**Conjecture 5.** All but finitely many finite quasisimple groups possess strongly real unmixed Beauville structures.

Clearly settling the status as strongly real Beauville groups of the sporadic groups makes no impact whatsoever on this conjecture, since there are only finitely many sporadic groups. (Note that throughout we shall use the term
sporadic group to refer to the 26 ‘traditional’ sporadic groups as well as the
Tits group $^{2}F_{4}(2)'$. Nonetheless it is the opinion of the author that settling
these questions for the sporadic groups is no less important than, for example,
determining their symmetric genii or determining which of them are Hurwitz
groups [9, 27]. It is, however, arguably more useful to settle this matter for
the sporadic groups since curves and surfaces associated with them are likely
to be very exceptional in nature and much of the original motivation for the
study of Beauville surfaces was for their use as counterexamples [2].

**Theorem 6.** (a) The Mathieu groups $M_{11}$ and $M_{23}$ are not strongly real
Beauville groups.

(b) Every other quasisimple sporadic group (except possibly the groups $2\bar{B}$
and $M$) is a strongly real Beauville group.

Our computationally intensive methods are unable to handle the groups
$2\bar{B}$ and $M$ (though we are able to show that the simple group $B$ is a strongly
real Beauville group). We make no apologies for this: to resolve the similar
problem of settling the Monster’s status as a Hurwitz group took almost ten
years of computing time [27, p.370]! Nonetheless, the vast majority of the
conjugacy classes in both of these groups are strongly real (see [24]) and since
the only problem the groups $M_{11}$ and $M_{23}$ encounter is a lack of strongly real
classes (see Section 4) we make the following conjecture.

**Conjecture 7.** Both of the groups $2\bar{B}$ and $M$ are strongly real Beauville
groups.

In the case of $M$ we make several specific conjectural remarks concerning
how a strongly real Beauville structure for $M$ might be obtained in Section
5 - the problem is not a lack of theoretical ideas or knowledge about the
monster, but simply a lack of computational power!

A mixed Beauville structure is a Beauville structure in which the action
of our group interchanges the two curves defining our surface. This can also
be ‘internalized’ into $G$, though we postpone this definition until Section 6
where we prove the following theorem.

**Theorem 8.** Let $G$ be an almost simple group such that the derived subgroup
$[G, G]$ is sporadic. Then $G$ does not possess an mixed Beauville structure.

Finally, our attention turns to the question of which alternating groups
possess a strongly real Beauville structure and in doing so we prove the
following theorem.
Theorem 9. The alternating group $A_6$ has a strongly real Beauville structure of type $((4,4,4),(5,5,5))$.

When combined with the structures explicitly constructed in the proof of [12, theorem 2] we have the following corollary.

Corollary 10. The alternating group $A_n$ is a strongly real unmixed Beauville group if and only if $n \geq 6$.

(Note that in [12, theorem 2] Fuertes and González-Diez claim to prove the above result with the bound $n \geq 6$ replaced with $n \geq 7$ - an error that the above theorem explicitly corrects. Interestingly, this correction requires the use of the exceptional nature of $Aut(A_6)$, so is clearly very different to the $n \geq 7$ cases. We further note that in [13, theorem 3.1] strongly real Beauville structures for the groups $PSL_2(q)$ are obtained. This appears to also correct the above error since $PSL_2(9) \cong A_6$, but it is only by delegating this case to the reader that they achieve this. Given that this strongly real Beauville structure can only be constructed by using an automorphism that is exceptional, regardless of whether this group is viewed as $PSL_2(9)$ or $A_6$, explicitly resolving this case is clearly desirable.) We conclude with a brief discussion of mixed Beauville structures of groups of the form $A_6 : 2^{(2)}$.

2. Strongly Real Sporadic Beauville groups

2.1. Our Construction

Roughly speaking, our method of showing that a group is a strongly real Beauville group, which in principal may be applied to any perfect group of even order that possesses a strongly real Beauville structure, is as follows.

We recall the following from [6]. Let $t, g \in G$ be such that $t$ is an involution.

- If $o(tt^g) = 2r$ for some integer $r$ then $(tt^g)^r \in C_G(t)$.
- If $o(tt^g) = 2r + 1$ for some integer $r$ then $g(tt^g)^r \in C_G(t)$.

Let $x_i := tt^{g_i}$ for some elements $g_i \in G$ for $i = 1, 2$. The above observations makes it easy to find some element $u \in G$ that commutes with $t$ and does not normalize the subgroup $\langle x_1 \rangle$. We can then define the elements $y_i := (x_i^{j(i)})^u$ for $i = 1, 2$, the value of the integer $j(i)$ being chosen to make the product $x_i y_i$ ‘nice’ (ie we ensure that the conditions of definition 2 are satisfied and...
when necessary hopefully makes it easier to see from the subgroup structure of $G$ that $\langle x_i, y_i \rangle = G$. This gives a Beauville structure for $G$. Since $u$ and $t$ commute we also have that $x_i^t = x_i^{-1}$ and $y_i^t = y_i^{-1}$ thus the Beauville structure just constructed must be strongly real.

2.2. The Structures

In this section we describe and tabulate the Beauville structures that we construct here.

Standard generators for the sporadic groups are given on the online atlas [28] and are named $a$ and $b$. In each case we have that $o(a) = 2$, so where possible we use this element to define the automorphisms needed when constructing strongly real Beauville structures. For background information on standard generators more generally see the original article by Wilson [25].

The types of the Beauville structures we construct here are given in Table 1. The words used to define our Beauville structures are given in Table 2.

We remark that whilst it is common to use lower case letters for the standard generators of a simple group and upper case letters for their covering groups. For the sake of aesthetics we use lower case letters in both cases, it being clear which are the non-simple cases.

In some cases it is either necessary or desirable to use an involution other than $a$ that we call $c$. The words in the standard generators used to define these elements $c$ are given in Table 3. In each case the fact that the given elements generate may be verified using either permutation or matrix representations of these groups available on [28], either directly or by the observations made in the next section.

2.3. Homomorphic images

So far we have only shown that the full covering group of each of the groups of part (b) of theorem 6 are strongly real Beauville groups. In this subsection we consider the cases of the quasisimple sporadic groups with non-trivial centers and their homomorphic images.

Given a group $G$, it is tempting to look for a Beauville structure in the quotient $G/N$ by some normal subgroup $N \triangleleft G$, and to try to lift this back to $G$. However, a triple that generates $G/N$ need not lift back to a triple generating $G$, and even if it does, the condition (2) of definition 2 may not be satisfied. In this situation, the following two lemmata are of great use (whilst the proofs of these results may seem trivial to the group theorist we
Table 1: The types of the Beauville structures defined by the words given in Tables 2 and 3. See definition 2.

| G     | Type                  | G     | Type                  |
|-------|-----------------------|-------|-----------------------|
| J1    | ((19,19,11),(15,15,7)) | 3'O'N | ((28,28,12),(19,19,19)) |
| 2'M12 | ((5,5,3),(11,11,11))  | Co3   | ((7,7,23),(5,5,24))   |
| 12'M22| ((5,5,5),(12,12,6))   | Co2   | ((16,16,8),(11,11,7)) |
| 2'J2  | ((7,7,7),(12,12,8))   | 6'Fi22| ((7,7,5),(13,13,13)) |
| 2'F4(2)'| ((5,5,5),(4,4,4))    | HN    | ((5,5,5),(6,6,6))    |
| 2'HS  | ((15,15,5),(8,8,7))   | Ly    | ((67,67,40),(37,37,21)) |
| 3'J3  | ((17,17,19),(9,9,9))  | Th    | ((19,19,19),(13,13,13)) |
| M24   | ((5,5,5),(6,6,11))    | Fi23  | ((5,5,5),(6,6,4))    |
| 3'McL | ((5,5,5),(6,6,6))     | 2'Co1 | ((5,5,5),(6,6,6))    |
| He    | ((3,3,6),(17,17,17))  | J4    | ((43,43,11),(29,29,6)) |
| 2'Ru  | ((4,4,10),(13,13,7))  | 3'Fi24| ((9,9,9),(11,11,26)) |
| 6'Suz | ((13,13,13),(12,12,10)) | ℬ    | ((13,13,19),(12,12,20)) |

Lemma 11. If G is a perfect group, N is a central subgroup of G, and S is a subset of G such that the image of S in G/N generates G/N, then S generates G.

Proof. See [13, lemma 4.1].

Lemma 12. Let G have generating triples \((x_i, y_i, z_i)\) with \(x_i y_i z_i = 1\) for \(i = 1, 2\), and a normal subgroup N such that at least one of these triples is faithfully represented in G/N. If the images of these triples correspond to a Beauville structure for G/N, then these triples correspond to a Beauville structure for G.

Proof. See [13, lemma 4.2].

From the types of the Beauville structures obtained in the previous section and from the orders of the centers of the relevant groups it is clear that the above lemmata may be applied to the Beauville structures obtained in the previous section.
for $G$ given in Table 3. In some cases it was necessary/desirable to use the standard generators element labeled $c$ in each case the elements structure for the full covering group of each of the sporadic simple groups considered here.

Table 2: Words in terms of the standard generators defining a strongly real Beauville structure for the full covering group of each of the sporadic simple groups considered here. In each case the elements $a$ and $b$ are the standard generators. In cases where the use of an element labeled $c$ is required, words in the standard generators defining these elements are given in Table 3. In some cases it was necessary/desirable to use the standard generators for $G:2$ rather than $G$. These cases we write in bold font.
| $G$  | $c$                          | $G$  | $c$                          |
|------|------------------------------|------|------------------------------|
| 2- HS | $(bab^2ab^3a)^5$             | 6- Fi$_{22}$ | $(abab^{10})^6$             |
| He   | $(ab^3)^4$                   | 2- Co$_1$    | $(ab)^{20}$                  |
| 2- Ru | $b^2$                        | J$_4$        | $(abab^2)^5$                |
| 6- Suz | $(bab^2(ab^2)^{28}$       | 3- Fi$\ '_{24}$ | $((ab)^4b)^{18}$         |
| Co$_2$ | $(ab(ba)^2b((bab)^2(ab^2)^2)^2)^3$ |              |                             |

Table 3: Words in terms of the standard generators $a$ and $b$ defining an involution $c$ in cases the involution $a$ is unable to define a strongly real Beauville structure via the construction described in Section 2.1.

3. Large Strongly Real Beauville Groups

In this section we prove that the Beauville structures defined in Section 2 do indeed generate the groups claimed in the cases where the representations of the groups in question are too cumbersome for this to be verified directly. In doing so we complete the proof of part (b) of theorem 6. In each case it is taken for granted that the elements refered to from the previous section do indeed have the stated orders and we focus only on the question of generation in each case. Any direct calculation refered to in the below proofs may easily be performed in Magma [5] or GAP [14].

Lemma 13. The Harada-Norton group $HN$ possesses a strongly real Beauville structure of type $((5,5),(6,6,6))$.

Proof. From the list of maximal subgroups of $HN$, as listed in the ATLAS [10, p.166], we see that no proper subgroup contains elements of order 22 and order 25. Direct computation shows that $o(x_1y_1x_2^2y_1^4) = o(x_2y_2^3x_2y_2^4) = 22$ and $o(x_1y_1x_2^2y_1^4) = o(x_2y_2x_2x_2^2) = 25$, hence $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$. □

Lemma 14. The Lyons group $Ly$ possesses a strongly real Beauville structure of type $((67,67,40),(37,37,21))$.

Proof. From the list of maximal subgroups of $Ly$, as listed in the ATLAS [10, p.174], we see that an element of order 67 is contained in only one maximal subgroup, a copy of the Frobenious group 67:22. This clearly contains no elements of order 40. Similarly we see that an element of order 37 is contained in only one maximal subgroup, a copy of the Frobenious group 37:18. Since this clearly contains no elements of order 21 we must have $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$. □
Lemma 15. The Thompson group $Th$ possesses a strongly real Beauville structure of type $((19,19,19),(13,13,13))$.

Proof. From the list of maximal subgroups of $Th$, as listed in [20, 21] (note that list given in the Atlas [10, p.177] is incomplete), we see that the only maximal subgroups containing elements of order 31 are isomorphic to either $2^5L_5(2)$ or the Frobenious group 31:15. These subgroups clearly contain no elements of order 19 or 13. Direct computation shows that $o(x_1y_1x_1^7y_1^5) = o(x_2y_2x_2^2y_2^{11}) = 31$, and so $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$.  

Lemma 16. The Janko group $J_4$ possesses a strongly real Beauville structure of type $((43,43,11),(29,29,6))$.

Proof. From the list of maximal subgroups of $J_4$, as listed in the Atlas [10, p.190], we see that an element of order 43 is contained in only one maximal subgroup, a copy of the Frobenious group 43:14. This clearly contains no elements of order 11. Similarly we see that an element of order 29 is contained in only one maximal subgroup, a copy of the Frobenious group 29:28. Since this clearly contains no elements of order 6 we must have $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$.  

Lemma 17. The Baby Monster $B$ possesses a strongly real Beauville structure of type $((13,13,19),(12,12,20))$.

Proof. From the list of maximal subgroups of $B$, as listed in [26] (note that list given in the Atlas [10, p.217] is incomplete), we see that the only maximal subgroup containing elements of order 47 is isomorphic the Frobenious group 47:23. This subgroup clearly contains no elements of order 13 or 12. Direct computation shows that $o(x_1y_1x_1^3y_1^5) = o(x_2y_2x_2^3y_2^7) = 47$, and so $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$.  

4. Non-Strongly Real Beauville Groups

In this short section we prove that the sporadic groups $M_{11}$ and $M_{23}$ are not strongly real Beauville groups. In doing so we complete the proof of theorem 6. Note that the strongly real classes of the sporadic simple group were classified by Suleiman in [24].

Lemma 18. The groups $M_{11}$ and $M_{23}$ are not strongly real Beauville groups.
| Element | Order | Element | Order |
|---------|-------|---------|-------|
| $p$     | 42    | $s$     | 39    |
| $qr$    | 19    | $q^2r^2$| 57    |
| $qsr$   | 35    | $qs^2r$ | 105   |

Table 4: Some elements of $\mathbb{M}$ inverted by conjugation by $g$ and their orders.

| $(x,y,xy)$ | $(o(x), o(y), o(xy))$ | $(x,y,xy)$ | $(o(x), o(y), o(xy))$ |
|-----------|------------------------|-----------|------------------------|
| $(p, s, ps)$ | (42,39,19)             | $(qr, s, qrs)$ | (19,39,22) |
| $(p^2, s, p^2s)$ | (21,39,39)             | $(qr, s^2, qrs^2)$ | (19,39,66) |
| $(p^3, s, p^3s)$ | (14,39,56)             | $(p^2, qr, p^2qr)$ | (21,19,60) |
| $(p, qr, pqr)$ | (42,19,42)             | $(p^2, s^2, p^2s^2)$ | (21,39,55) |
| $(p, qsr, pqsr)$ | (42,35,57)             | $(qsr, s, qsrs)$ | (35,39,105) |

Table 5: Some sets of elements of $\mathbb{M}$ that could potentially strongly $(a, b, c)$-generate the group.

Proof. In both cases the only strongly real classes are classes of elements of order at most 6. In each group there is only one class of elements of order 2, one of order 3 and one of order 5.

Computer calculations show that in each case, the group is only strongly $(5, 5, m)$ generated if the integer $m$ is 4 or 6 and that neither group is strongly $(3, 3, m)$ generated for any integer $m$. \hfill \Box

We remark that in [2, p.35] Bauer, Catanese and Grunewald state that they were unable to find a strongly real Beauville structure for $M_{11}$ (among other groups). The above lemma explains why.

5. The Monster

We give a brief discussion as to how a strongly real Beauville structure of the monster group $\mathbb{M}$ might be obtained.

In [27] Wilson proves that $\mathbb{M}$ can be generated by a pair of elements $g$ and $h$ such that $g$ is in class 2B, $h$ is in class 3B and $gh$ is in class 7B. In the process of proving this Wilson defines the following four elements

$$ p = ghgh^2, \quad q = ghghgh^2, \quad r = ghgh^2gh^2, \quad s = ghghgh^2gh^2. $$
Firstly, to apply our construction of Section 1.3 we need an involution of $M$ - naturally we take the element $g$.

The orders of several short words in the elements above are given in [27, Table 1]. In particular we have that $o(p) = 42$. Now, for our Bray-type element, $u$, observe that $p^g = p^{-1}$ and so $g \neq p^{21} \in Z\langle p, g \rangle$. Whilst other short words in the above elements, such as those appearing in Table 4, are inverted by conjugation by $g$, these words often have odd order and so there is no guarantee that the involution produced will be distinct from $g$.

For our elements $x_i$, $i = 1, 2$ (which immediately give us the elements $y_i := x_i^u$) we note that several of the words given in [27, Table 1] are inverted by conjugation by $g$, such as those given in Table 4 and their powers, and any one of these provide candidates for our $x_i$s.

A slightly different approach is as following. In several cases, the products of elements found in Table 4 also have their orders listed in [27, Table 1]. We can thus define at least one of our (potential) generating pairs by taking these elements themselves. We list a few of these possibilities in Table 5.

We remark that proving that a proposed generating set $M$ does in fact generate is easier than it first appears. Whilst the maximal subgroups of $M$ have yet to be classified, a substantial amount of information is known. In particular, a complete classification of the maximal subgroups that contain elements of class 2A is known - see [23]. An immediate corollary of this classification is that the only maximal subgroups of $M$ containing elements of order 94 are copies of $2^\infty B$. Finding a word in our set of proposed generators of order 94 forces the set to be contained in some copies of $2^\infty B$ and another word in our set of proposed generators that cannot lie in such a subgroup proves that the set generates. This is precisely how Wilson showed in [27] that the above $g$ and $h$ generate $M$ - it turns out that $o(ppqsrpqsqrq) = 94$ and $o(ppqsrqprq) = 41$.

Whilst multiplying elements of $M$ together is extremely difficult, computing the order of such a word is somewhat easier - the method described in [22], computing orders by analyzing orbits of specially chosen vectors in the natural 196882 dimensional $\mathbb{F}_2$ module, being the method used to calculate the orders given above.

6. Mixed Beauville structures

In this short section we consider the mixed case and prove theorem 8. Recall that a Beauville structure is mixed if the action of the group interchanges
the two curves being used to define the surface. As with unmixed Beauville structures, the concept of a mixed Beauville structure can be ‘internalised’ to the group.

**Definition 19.** Let $G$ be a group. A mixed Beauville structure of $G$ is a set \{$x, y, z$\} $\subset G$ that generates an index 2 subgroup $G_0 < G$ such that for every $g \in G \setminus G_0$ we have

1. $xyz = 1$;
2. $\Sigma(x, y, z) \cap \Sigma(x^g, y^g, z^g) = \{1\}$ and
3. $g^2 \notin \Sigma(x, y, z)$ for $i = 1, 2$.

Clearly no simple group can possess an mixed Beauville structure, however this doesn’t rule out the possibility of an almost simple group possessing one. Few examples of mixed Beauille structures are known \cite{1} so finding more is of great interest.

The following easy lemma is extremely useful.

**Lemma 20.** Let $(C \times C)/G$ be a Beauville surface of mixed type and $G_0$ the subgroup of $G$ consisting of the elements which preserve each of the factors, then the order of any element in $G \setminus G_0$ is divisible by 4.

**Proof.** See \cite{12, lemma 5}.

Of the 27 sporadic groups thirteen of them (namely $M_{12}$, $M_{22}$, $J_2$, $^2F_4(2)'$, HS, $J_3$, McL, He, Suz, O’N, $F_{i_{22}}$, HN and $F_{i'_{24}}$) posses outer automorphisms. From their character tables, which can be reconstructed from the data given in \cite{10}, we see that, apart from the Tits group $^2F_4(2)'$, all of the almost simple groups whose derived subgroup is in the above list have involutions lying outside $G_0$ and so by the above lemma none of these groups can possess a mixed Beauville structure.

The case of the almost simple Tits group $^2F_4(2)$ is more delicate. In this case we see from the character table \cite{10, p.75} that every element that is outside the simple group has order divisible by 4 and so lemma 20 cannot be used to block the existence of a mixed Beauville structure. We can, however, also see the following. Condition 3 of definition 19 forces the orders $x$, $y$ and $z$ to be odd since every involution of $G$ has the property that there is an element of order 4 in $G \setminus G_0$ that squares to it. The only elements of odd order have order 3, 5 or 13 and in each case there is only one class of cyclic subgroups of that order making it impossible to satisfy condition 2 of definition 19. The group $^2F_4(2)$ thus has no mixed Beauville structure, proving theorem 8.
7. The Alternating Groups

In this final section we prove theorem 9 and corollary 10. To do this we first recall some standard facts about automorphisms of alternating groups.

If \( n \neq 2, 3 \) or 6 then \( \text{Aut}(A_n) \cong S_n \), the full symmetric group. (If \( n = 2, 3 \) then \( \text{Aut}(A_n) \cong S_{n-1} \).) If \( n = 6 \) then we have that \( S_6 \) is an index 2 subgroup of \( \text{Aut}(A_6) \) which has structure \( A_6 : 2 \). An immediate consequence of this fact is the result that \( \text{Aut}(A_6) \) has three index 2 subgroups, each of structure \( A_6 : 2 \). One is isomorphic to the linear group \( \text{PGL}_2(9) \) (the exceptional isomorphism \( A_6 \cong \text{PSL}_2(9) \) gives us the fact that \( \text{Aut}(A_6) \cong \text{PGL}_2(9) \)); another to \( S_6 \); and the final one to the Mathieu group \( M_{10} \).

**Proof. of theorem 9.** Consider the permutations

\[
x_1 := (2, 9, 5, 6)(3, 4, 7, 8), \quad y_1 := (1, 3, 8, 5)(2, 6, 10, 4),
\]

\[
x_2 := (1, 9, 4, 6, 2)(3, 5, 7, 10, 8), \quad y_2 := (1, 3, 2, 5, 7)(4, 8, 6, 10, 9),
\]

and

\[
g := (1, 10)(2, 8)(3, 6)(4, 5)(7, 9).
\]

Easy calculation gives \( o(x_1) = o(y_1) = o(x_1y_1) = 4 \) and \( o(x_2) = o(y_2) = o(x_2y_2) = 5 \). Easy computations further show that \( \langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = A_6 \). From their orders it is clear that these elements also satisfy conditions 2 and 3 of definition 2 and so these permutations define a Beauville structure for \( A_6 \) of type \( ((4,4,4),(5,5,5)) \). We claim that this Beauville structure is strongly real.

Easy computations show that \( \langle x_1, y_1, g \rangle = \langle x_2, y_2, g \rangle = \text{PGL}_2(9) \), one of the groups of the form \( A_6 : 2 \) not isomorphic to the symmetric group \( S_6 \) (or the Mathieu group \( M_{10} \)). Further direct calculation reveals that \( x_i^g = x_i^{-1} \) and \( y_i^g = y_i^{-1} \) for \( i = 1, 2 \) and so this (outer) automorphism of \( A_6 \) shows that this Beauville structure is strongly real.

**Proof. of corollary 10.** For \( n \geq 7 \) these are explicitly constructed in the proof of [12, theorem 2]. If \( n = 6 \) this is the above theorem. If \( n \leq 5 \) then it is easily verified that \( A_n \) does not even possess a Beauville structure let alone a strongly real one.

In [12] Fuertes and González-Diez use lemma 20 to show that \( S_6 \) does not possess a mixed Beauville structure. In the case of \( \text{PGL}_2(9) \) there are involutions lying outside the derived subgroup and so this same lemma ensures
that $PGL_2(9)$ also does not possess a mixed Beauville structure. Remarkably, in the case of the group $M_{10}$ the only elements lying outside the derived subgroup all have order 4 or 8, so lemma 20 is of no use here.

In this case, however, we can say the following. Since $M_{10}$ has only one class of involutions it must be the case that, as in the case of $^2F_4(2)$, the elements defining a mixed Beauville structure must have odd order (ie order 3 or 5) by condition 3 of definition 19. Again, there is only one class of cyclic subgroups of each order, so Condition 2 of definition 19 cannot be satisfied, so there is no mixed Beauville structure in this case. The group $PGL_2(9)$ also cannot have an mixed Beauville structure since for each of the index 2 subgroups there is a class of involutions lying outside the subgroup blocking the existence of a mixed Beauville structure by lemma 20. It follows that no group of the form $A_6 : 2^{(2)}$ possesses a mixed Beauville structure.

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References

[1] N. Barker, N. Boston, N. Peyerimhoff and A. Vdovina “New examples of Beauville surfaces”, to appear Monatsh. Math.

[2] I.C. Bauer, F. Catanese and F. Grunewald “Beauville surfaces without real structures I’ in Geometric Methods in Algebra and Number Theory, Progr. Math. 235, Birkhäuser Boston, Boston, 2005, pp. 1-42

[3] I.C. Bauer, F. Catanese and F. Grunewald “Chebycheff and Belyi polynomials, dessins denfants, Beauville surfaces and group theory”, Mediterr. J. Math. 3 (2006), 121-146

[4] A. Beauville “Surfaces algébriques complexes”, Astérisque 54, Soc. Math. France, Paris, 1978

[5] W. Bosima, J. Cannon and C. Playoust “The MAGMA algebra system I. The user language”, J. Symbolic Comput. 24 (3-4): 235–265, 1997
[6] J.N. Bray “An improved method for generating the centralizer of an involution”, Arch. Math. 74 (2000) 241–245

[7] F. Catanese “Fibred surfaces, varieties isogenous to a product and related moduli spaces”, Am. J. Math. 122 (2000), 1-44

[8] P.B. Cohen, C. Itzykson and J. Wolfart “Fuchsian triangle groups and Grothendieck dessins: Variations on a theme of Belyi”, Comm. Math. Phys. 163 (1994), 605-627

[9] M.D.E. Conder, RA Wilson and AJ Woldar “The symmetric genus of sporadic”, Proc. Amer. Math. Soc., 116 (1992), no. 3, 653–663

[10] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson “An ATLAS of finite groups” (Oxford University Press, 1985)

[11] B.T. Fairbairn, K. Magaard and C.W. Parker “Quasisimple groups, triangle groups and an application to Beauville structures”, preprint 2010

[12] Y. Fuertes and G. González-Diez “On Beauville structures on the groups $S_n$ and $A_n$”, Math. Z. 264, No. 4, 959-968 (2010)

[13] Y. Fuertes and G. Jones “Beauville surfaces and finite groups”, preprint 2009, available at the time of writing from http://arxiv.org/abs/0910.5489

[14] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.4.12; 2008 (http://www.gap-system.org)

[15] S. Garion, M. Larsen and A. Lubotzky “Beauville surfaces and finite simple groups”, preprint 2010, http://arxiv.org/abs/1005.2316

[16] S. Garion and M. Penegini “New Beauville surfaces, moduli spaces and finite groups”, preprint 2009, available at the time of writing from http://arxiv.org/abs/0910.5402

[17] A. Grothendieck “Esquisse dun Programme”, pp. 5-84 in Geometric Galois Actions 1. Around Grothendiecks Esquisse dun Programme, ed. P. Lochak, L. Schneps, London Math. Soc. Lecture Note Ser. 242, Cambridge University Press, 1997
[18] R. Guralnick and G. Malle “Simple groups admit Beauville structures”, preprint 2010, http://arxiv.org/abs/1009.6183

[19] G.A. Jones and D. Singerman “Belyi functions, hypermaps and Galois groups”, Bull. London Math. Soc. 28 (1996), 561-590

[20] S.A. Linton “The maximal subgroups of the Thompson group”, J. London Math. Soc. (2) 39 (1989), no. 1, 79–88

[21] S.A. Linton “Corrections to: The maximal subgroups of the Thompson group”, J. London Math. Soc. (2) 43 (1991), no. 2, 253–254

[22] S.A. Linton, R.A. Parker, P.G. Walsh and R.A. Wilson “Computer construction of the Monster”, J. Group Theory 1 (1998), 307–337

[23] S.P. Norton and R.A. Wilson “Anatomy of the Monster: II”, Proc. London Math. Soc. (3) 84 (2002) 581–598

[24] I. Suleiman “Strongly real elements in sporadic groups and alternating group” Jordan Journal of Mathematics and Statistics 1(2) (2008), 97–103

[25] R.A. Wilson “Standard generators for the sporadic simple groups” J. Algebra 184, (1996) 505–515

[26] R.A. Wilson “The Maximal Subgroups of the Baby Monster, I”, J. Algebra 211, (1999) 1–14

[27] R.A. Wilson “The Monster is a Hurwitz group” J. Group Theory 4 (2001) 367–374

[28] R.A. Wilson et al “Atlas of Finite Group Representations - Version 3”, available at the time of writing from http://brauer.maths.qmul.ac.uk/Atlas/v3

[29] J. Wolfart “ABC for polynomials, dessin denfants, and uniformization a survey”, pp. 313-345 in Elementare und Analytische Zahlentheorie (Tagungsband), Proceedings ELAZConference May 2428, 2004 (ed. W. Schwarz, J. Steuding), Steiner Verlag Stuttgart 2006 (available at the time of writing from http://www.math.uni-frankfurt.de/~wolfart/).