Symmetry is a fascinating notion in science, primarily perhaps due to the fact that the symmetry of geometric objects evokes the aesthetic and elegant perception of every human being. More fundamentally, the symmetries of spacetime are closely related to conservation laws in physics. In this note we show how one can use a symmetry of a two-dimensional surface to prove a monotonicity property of a one-dimensional function.

A couple of years ago Jan Ubøe gave an unconventional proof of the fact that a certain mapping \( f(x) \) which depends on a positive parameter \( a \) is increasing in \( x \) if \( a > 1 \) and decreasing if \( a < 1 \), see [1]. In fact, \( f \) is given by

\[
f(x) = \frac{\phi(ax) - \phi(x)}{x(\Phi(ax) + \Phi(x))}
\]

where \( \phi(x) = e^{-x^2} \) is the derivative of \( \Phi(x) = \int_0^x \phi(u) \, du \). This behaviour of \( f \) is used to describe certain effects in a so-called newsvendor model in operations research [2]. The author of [1] reckons that Euler would have liked his proof (perhaps because of transferring the problem to cumbersome tasks involving infinite sums), but also that a very short proof is lurking around somewhere. We believe that our alternative proof offers simpler geometric arguments and is more tractable and appealing.

The proof that we present makes repeated use of the facts that:

(a) \( \phi \) can be extended as an even function on \( \mathbb{R} \);
(b) the derivative satisfies a recurrent formula, namely \( \phi'(x) = -2x\phi(x) \).

First we observe that the sign of \( f'(x) \) is determined by the sign of

\[
h(x) = (ax\phi'(ax) - x\phi'(x))(\Phi(ax) + \Phi(x))
- (\phi(ax) - \phi(x))(\Phi(ax) + \Phi(x))
- (\phi(ax) - \phi(x))(ax\phi(ax) + x\phi(x)).
\]

Then we rewrite each of the sums of two terms as integrals with help of properties (a) and (b) above and arrive at a double-integral representation

\[
h(x) = \int_{-x}^{ax} \int_{-x}^{ax} \phi(u)\phi(v) \, \Gamma(u, v) \, dudv
\]

with

\[
\Gamma(u, v) = 2(u + v)(u - v)^2.
\]
Notice that the integrand in the last double integral is odd with respect to the diagonal \( v = -u \) and positive above it. It is due to the obvious symmetry of \( \phi \) stated in (a) above together with a hidden symmetry encoded in the convolution kernel

\[
\Gamma(u, v) = -\Gamma(-u, -v)
\]

valid for all \( u, v \in \mathbb{R} \). Consequently, since we are integrating over the rectangle \((-x, ax) \times (-x, ax)\), the odd symmetry immediately implies that (cf. Figure 1):

- for \( a = 1 \) the domains of positivity and negativity of \( \Gamma \) are equilibrated, so \( h(x) \) equals zero (and \( f \) is constant, in fact it is identically equal to zero);
- for \( a > 1 \) the domain of positivity of \( \Gamma \) outweighs the one of negativity, so \( h(x) \) is positive (and \( f \) is strictly increasing);
- for \( a < 1 \) the domain of negativity of \( \Gamma \) outweighs the one of positivity, so \( h(x) \) is negative (and \( f \) is strictly decreasing).

In summary, we have obtained the desired monotonicity of the one-dimensional function \( f \) as a consequence of the symmetry properties of the two-dimensional surface \( \Gamma \).

![Graph of the surface Γ and its contour plot.](image)

Figure 1: Graph of the surface \( \Gamma \) and its contour plot. Integration over the green square (corresponding to \( x = 1 \) and \( a = 1 \) in the picture) gives zero because of the odd symmetry of \( \Gamma \) with respect to the line \( u = -v \), while the integral over the rectangle expanded by the lighter green colour into the positive (brownish) region of \( \Gamma \) (corresponding to \( a = 1.4 \) in the picture) gives a positive number.

In the rest of this note we explain the passage from (1) to (2). For the first term appearing in (1) we have

\[
ax\phi'(ax) - x\phi'(x) = -2 \left( ax^2 \phi(ax) - x^2 \phi(x) \right) = -2 \int_x^{ax} (u^2 \phi(u))' \, du
\]

\[
= -4 \int_x^{ax} u\phi(u) \, du + 4 \int_x^{ax} u^3 \phi(u) \, du
\]

\[
= -4 \int_{-x}^{x} u\phi(u) \, du + 4 \int_{-x}^{ax} u^3 \phi(u) \, du.
\]
In the last equality we have used the fact that $u\phi(u)$ and $u^3\phi(u)$ are odd functions of $u$. Similarly we write
\[
\Phi(ax) + \Phi(x) = \int_{-x}^{ax} \phi(u) \, du,
\]
\[
\phi(ax) - \phi(x) = \int_{-x}^{ax} \phi'(u) \, du = -2\int_{-x}^{ax} u\phi(u) \, du,
\]
\[
ax\phi(ax) + x\phi(x) = \int_{-x}^{ax} (u\phi(u))' \, du = \int_{-x}^{ax} \phi(u) \, du - 2\int_{-x}^{ax} u^2\phi(u) \, du.
\]
Putting these formulae together one can rewrite $h$ as follows:
\[
h(x) = 4 \left( \int_{-x}^{ax} u^3\phi(u) \, du \right) \left( \int_{-x}^{ax} \phi(u) \, du \right) - 4 \left( \int_{-x}^{ax} u\phi(u) \, du \right) \left( \int_{-x}^{ax} u^2\phi(u) \, du \right).
\]
Now the main idea is to rewrite the product of integrals $(\int g(u) \, du)(\int j(v) \, dv)$ as a double integral $\iint g(u)j(v) \, dudv$ and realise that
\[
h(x) = \int_{-x}^{ax} \int_{-x}^{ax} \phi(u)\phi(v) \, \tilde{\Gamma}(u, v) \, dudv = \int_{-x}^{ax} \int_{-x}^{ax} \phi(u)\phi(v) \, \tilde{\Gamma}(v, u) \, dudv,
\]
where $\tilde{\Gamma}(u, v) = 4(u^3 - uv^2)$. Here, the second equality is due to the fact that the role of variables $u$ and $v$ can be interchanged. Therefore, $\tilde{\Gamma}(u, v)$ can be replaced by $\tilde{\Gamma}(v, u)$ and also by the arithmetic mean
\[
\Gamma(u, v) = \frac{\tilde{\Gamma}(u, v) + \tilde{\Gamma}(v, u)}{2} = 2(u + v)(u - v)^2,
\]
so that (2) holds.

Acknowledgment

D.K. was partially supported by the GACR grant No. 18-08835S and by FCT (Portugal) through project PTDC/MAT-CAL/4334/2014.

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