Unconditional security from noisy quantum storage

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Abstract

We consider the implementation of two-party cryptographic primitives based on the sole assumption that no large-scale reliable quantum storage is available to the cheating party. We construct novel protocols for oblivious transfer and bit commitment, and prove that realistic noise levels provide security even against the most general attack. Such unconditional results were previously only known in the so-called bounded-storage model which is a special case of our setting. Our protocols can be implemented with present-day hardware used for quantum key distribution. In particular, no quantum storage is required for the honest parties.

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1 The noisy-storage model: definition and results

1.1 Motivation: security from physical assumptions

The security of most cryptographic systems currently in use is based on the premise that a certain computational problem is hard to solve for the adversary. Concretely, this relies on the assumption that the adversary’s computational resources are limited, and the underlying problem is hard in some precise complexity-theoretic sense. While the former assumption may be justified in practice, the latter statement is usually an unproven mathematical conjecture. In contrast, quantum cryptographic schemes are designed in such a way that they provide security based solely on the validity of quantum physics. No assumptions on the adversary’s computational power nor the validity of some complexity-theoretic statements are needed.

Unfortunately, not even the laws of quantum physics allow us to realize all desirable cryptographic functionalities without further assumptions [32, 38, 34, 33, 39]. An example of such a functionality is (fully randomized) oblivious transfer, where Alice receives two random strings $S_0, S_1$ while Bob receives one of the strings $S_C$ together with the index $C$. Security for this primitive means that neither Alice nor Bob can obtain any information beyond this specification. A protocol which securely implements oblivious transfer is desirable because any two-party computation, such as secure identification, can be based on this building block [24, 19].

In light of this state of affairs, it is natural to consider other physical assumptions: Motivated by similar classical models [36, 37], the authors of [14, 12] and [53, 55, 48] propose to assume that the adversary’s quantum storage is bounded and noisy, respectively. The assumption of bounded quantum storage deals with the noiseless case (but assumes a small amount of storage), whereas the noisy-storage model deals with the case of noise (but possibly a large amount of storage). Here, we take a more general point of view which incorporates both the amount of storage and noise. We refer to this simply as the noisy-storage model. The previously considered settings are special cases, as we will explain below.

Compared to the classical world, the assumption of limited and noisy quantum storage is particularly realistic in view of the present state of the art, and the considerable challenges faced when trying to build scalable quantum memories. Further motivation for considering noise as a resource for security comes from the fact that the transfer of the state of a (photonic) qubit used during the execution of the protocol onto a different carrier (such as an atomic ensemble) used as a quantum memory is typically already noisy.

1.2 The noisy-storage model

Let us first describe more formally what we mean by a noisy quantum memory. We think of a device whose input states are in some Hilbert space $\mathcal{H}_m$. A state $\rho$ stored in the device decoheres over time. That is,
the content of the memory after some time \( t \) is a state \( \mathcal{F}_t(\rho) \), where \( \mathcal{F}_t : \mathcal{B}(\mathcal{H}_{in}) \rightarrow \mathcal{B}(\mathcal{H}_{out}) \) is a completely positive trace-preserving map corresponding to the noise in the memory. The family of maps \( \{\mathcal{F}_t\}_{t \geq 0} \) completely describes the behavior of the storage. We assume that the noise is Markovian, that is, the family \( \{\mathcal{F}_t\}_{t \geq 0} \) is a continuous one-parameter semigroup

\[
\mathcal{F}_0 = I \quad \text{and} \quad \mathcal{F}_{t_1 + t_2} = \mathcal{F}_{t_1} \circ \mathcal{F}_{t_2} .
\]

Assumption (1) is implicit in \[55\] and is essential to ensure that the adversary cannot gain any information by delaying the readout. More concretely, we will consider protocols which have certain time delays \( \Delta t \) which force any adversary to use his storage device for a time at least \( \Delta t \). Property (1) now tells us that the best he can do is to read out the information from the device immediately after time \( \Delta t \), as any further delay will only degrade his information further. We can thus focus on the channel \( \mathcal{F} = \mathcal{F}_{\Delta t} \) instead of the family \( \{\mathcal{F}_t\}_{t \geq 0} \).

We also assume that all actions, including computation, communication, measurement and state preparation, are instantaneous (compared to \( \Delta t \)). In our protocol, such an assumption is well justified as the honest parties merely need to prepare and measure BB84-encoded qubits and do not require any quantum storage. The assumption implies that a malicious player only needs to store quantum information when a time delay \( \Delta t \) is specified in the protocol.

Summarizing, our security model assumes that

- The adversary has unlimited classical storage, and (quantum) computational resources.
- Whenever the protocol requires the adversary to wait for a time \( \Delta t \), he has to measure/discard all his quantum information except what he can encode (arbitrarily) into \( \mathcal{H}_{in} \). This information then undergoes noise described by \( \mathcal{F} \).

To see how previously analyzed cases fit into our model, note that the bounded-storage model corresponds to the case where \( \mathcal{H}_{in} \) is of limited input dimension, and \( \mathcal{F} \) is the identity on \( \mathcal{H}_{in} \). Concretely, \[13\] considers protocols with \( n \) qubits of communication and \( \mathcal{H}_{in} \cong (\mathbb{C}^2)^{\otimes n} \) for some parameter \( \nu > 0 \) which we call the storage rate. Security of certain protocols was established for \( \nu < 1/4 \). Furthermore, the protocol proposed by Crépeau \[3\] for oblivious transfer is secure if the adversary cannot store any quantum information at all, corresponding to a storage rate of \( \nu = 0 \). Previous work on the noisy-storage model \[55\] analyzed protocols with \( n \) qubits of communication, where the noise \( \mathcal{F} \equiv \mathcal{N}^{\otimes n} \) is an \( n \)-fold tensor product of a noisy single-qubit channel \( \mathcal{N} : \mathcal{B}(\mathbb{C}^2) \rightarrow \mathcal{B}(\mathbb{C}^2) \) (i.e., \( \mathcal{H}_{in} \cong (\mathbb{C}^2)^{\otimes n} \) and \( \nu = 1 \)). Note, however, that in \[55\] the adversary was further restricted to performing product measurements on the qubits received in the protocol (albeit otherwise fully arbitrary).

1.3 Main result

In this paper, we establish security in the noisy-storage model against fully general attacks for arbitrary channels \( \mathcal{F} : \mathcal{B}(\mathcal{H}_{in}) \rightarrow \mathcal{B}(\mathcal{H}_{out}) \). We now provide an overview of our results, and state precise definitions of cryptographic primitives and their security later on. First, we show that a sufficient condition for security is that the number of classical bits that can be sent through the noisy storage-channel \( \mathcal{F} \) is limited. We thus relate the security of our protocols to the problem of sending information through the channel. More formally, let

\[
P_{\text{succ}}(n) := \max_{\{D_x\}_{x \in \{0,1\}^n}} \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \text{tr}(D_x \mathcal{F}(\rho_x))
\]

be the maximal success probability of correctly decoding a randomly chosen \( n \)-bit string \( x \in \{0,1\}^n \). Here, the maximum is over families of code states \( \{\rho_x\}_{x \in \{0,1\}^n} \) on \( \mathcal{H}_{in} \) and decoding POVMs \( \{D_x\}_{x \in \{0,1\}^n} \) on \( \mathcal{H}_{out} \). We show that security can be obtained for channels with the property that the decoding probability decays exponentially above a certain threshold.
Theorem 1.1 (Informal statement). Suppose that the decoding probability of a channel $F$ satisfies
\[
\lim_{n \to \infty} -\frac{1}{n} \log P_{\text{succ}}^F(nR) > 0
\]
for some $R > 0$ with
\[
R < \frac{1}{2}.
\]
Then oblivious transfer and bit commitment can be implemented using $O(n)$ qubits of communication against an adversary whose noisy storage is described by $F$. Moreover, the security is exponential in $n$.

Remarkably, the statement of Theorem 1.1 does not require any knowledge of the channel beyond its relation to the coding problem. In particular, we do not need to assume that $F$ is of tensor product form. We discuss possible extensions and limitations of our approach in Section 6.

We provide explicit security parameters for channels of the form $F = N^\otimes n$, where $n$ is the number of qubits sent in the protocol, and $\nu \geq 0$ is the storage rate. Our proof connects the security of protocols in the noisy-storage model for such channels to the classical capacity $C_N$ of $N$. This is the main conceptual novelty, and requires somewhat different protocols and analysis. It provides a quantitative expression of our intuition that noisy channels which are of little use for classical information transmission give rise to security in the noisy-storage model. To see what Theorem 1.1 implies, observe first that (3) clearly requires that the classical capacity $C_N$ of the channel is smaller than $1/2$. This, however, is not sufficient, since $R > C_N$ is not generally known to imply (3) for $F = N^\otimes n$. We are therefore interested in channels $N$ which satisfy the following strong-converse property: The success probability (2) decays exponentially for rates $R$ above the capacity, i.e., it takes the form
\[
P_{\text{succ}}^{N^\otimes n}(nR) \leq 2^{-n\gamma^N(R)} \quad \text{where} \quad \gamma^N(R) > 0 \quad \text{for all} \quad R > C_N.
\]
(4)

In [31], property (4) was shown to hold for a large class of channels, including the depolarizing channel (see (5) below).

Combining Theorem 1.1 with (4), we obtain the following statement:

Corollary 1.2 (Informal statement). Let $\nu \geq 0$, and suppose that $N$ satisfies the strong-converse property (4). If
\[
C_N \cdot \nu < \frac{1}{2},
\]
then oblivious transfer and bit commitment can be implemented with polynomial resources (in $n$) and exponential security against an adversary with noisy storage $F = N^\otimes n$.

An important example for which we obtain security is the $d$-dimensional depolarizing channel $N_r : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ defined for $d \geq 2$ as
\[
N_r(\rho) := r\rho + (1-r)\frac{I}{d}
\]
(5)
for some fixed $0 \leq r \leq 1$, which replaces the input state $\rho$ with the completely mixed state with probability $1-r$. For $d = 2$, this means that the adversary can store $\nu n$ qubits, which are affected by independent and identically distributed noise. It has been shown that the depolarizing channel exhibits the strong-converse property [31]. To see for which values of $r$ we can obtain security, we need to consider the classical capacity of the depolarizing channel as evaluated by King [26]. For $d = 2$, i.e., qubits, it is given by
\[
C_{N_r} = 1 + \frac{1+r}{2} \log \frac{1+r}{2} + \frac{1-r}{2} \log \frac{1-r}{2}.
\]
Figure 1: Our results applied to depolarizing noise $\mathcal{F} = \mathcal{N}_r^{\otimes \nu n}$: The vertical axis represents the noise parameter $r$, while the horizontal axis represents the storage rate $\nu$. Our protocols are secure when the pair $(r, \nu)$ is in the lower region bounded by the solid blue curve. Security is still possible in the region labeled with '?' , but cannot be obtained from our analysis.

Figure 1 shows the region in the $(r, \nu)$-plane corresponding to the noise channel $\mathcal{F} = \mathcal{N}_r^{\otimes \nu n}$, where we allow $n$ qubits of communication in the protocol. This is obtained from Corollary 1.2 (The depolarizing channel $\mathcal{N}_r$ satisfies the corresponding conditions).

**Comparison to the bounded-storage model: depolarizing noise**

It was previously observed [47] that the case of depolarizing storage noise (i.e., $r < 1$) can be dealt with using results obtained in the bounded-storage model (i.e., $r = 1$) when the noise is sufficiently strong. More precisely, the results of [13] can be extended to give non-trivial statements if the “effective” dimension of the storage system to be less than $n/4$, where $n$ is the number of qubits communicated in the protocol $^2$. We sketch such a simple dimensional analysis to illustrate that our model offers significant improvements over the bounded-storage analysis: we obtain security even at lower noise levels and higher storage rates.

Concretely, consider the noise channel $\mathcal{F} = \mathcal{N}_r^{\otimes \nu n} : B((\mathbb{C}^2)^{\otimes \nu n}) \to B((\mathbb{C}^2)^{\otimes \nu n})$ (cf. 13 for $d = 2$). Applying depolarizing noise to any of the $\nu n$ systems $\mathbb{C}^2$ means that the state on this system is replaced by the completely mixed state with probability $1 - r$. We can think of an indicator random variable $E_{\nu n} = (E_1, \ldots, E_{\nu n}) \in \{0, 1\}^{\nu n}$, where $E_i$ is 1 if and only if the $i$-th qubit is replaced by the completely mixed state. These “erasure” variables are independent and identically distributed Bernoulli variables with parameter $r = P[E_i = 0]$. In particular, the number of erasures

$$|E_{\nu n}| = \sum_{i=1}^{\nu n} E_i$$

is distributed according to the binomial distribution with $\nu n$ trials, each of which succeeds with probability $1 - r$.

We now assume that the adversary is given the location of the erasures $E_{\nu n}$ in addition to the output of the channel. Note that this can only make the adversary more powerful. Conditioned on the locations $E_{\nu n}$, the “effective dimension” of his channel is equal to $2^{\nu n - |E_{\nu n}|}$. Hence, we may think of an “effective” storage

$^1$More generally, the result of 31 applies to channels with certain covariance properties and additive minimum output $\alpha$-Rényi entropy. Examples are all unital qubit channels, the Werner-Holevo channel and the depolarizing channel.

$^2$We compare the randomized oblivious transfer protocol of 13 to our protocol based on weak string erasure.
Figure 2: Security for depolarizing noise parameters $(1, \nu)$ with $\nu < 1/4$ was established in the bounded-storage model (BSM). The naïve argument presented in this section extends this security to the region $(r, \nu)$ bounded by the dashed red curve. The more refined analysis given in this paper extends this region significantly (solid blue curve). This curve shows that our analysis also improves on results previously obtained in the bounded-storage model, which corresponds to the noise-free case ($r = 1$).

rate $\nu_{\text{eff}}$ given by the random variable

$$\nu_{\text{eff}} = \nu - \frac{|E^{\nu n}|}{n}.$$

We know from the bounded storage model analysis that for $\nu_{\text{eff}} < \frac{1}{4}$, the previously studied protocols provide security. Overall, we therefore conclude that security can be obtained from the noisy channel $\mathcal{F}$ if $\Pr[\nu_{\text{eff}} > \frac{1}{4}]$ is exponentially small. Note that by Chernoff’s inequality

$$\Pr[\nu_{\text{eff}} > \frac{1}{4}] = \Pr[|E^{\nu n}| < (1 - \delta)\mu] < e^{-\mu\delta^2/2} \quad \text{if} \quad \delta = \frac{1 - 4\nu r}{4\nu(1 - r)} > 0,$$

where $\mu = n\nu(1 - r)$.

In particular, we conclude that we obtain security for

$$\nu r < \frac{1}{4}. \quad (6)$$

Figure 2 compares the curve of this equation to the results we will derive below. We see that for the noiseless case ($r = 1$), our analysis provides security for storage rates $\nu < 1/2$, extending previous results (i.e., $\nu < 1/4$ in [13]) in the bounded-storage model. This improvement stems from the fact that (for oblivious transfer) our protocol uses a different classical post-processing based on interactive hashing instead of the min-entropy splitting tool of [13]. Note that this requires additional rounds of classical communication.

We stress that the naïve analysis outlined here does not apply to other channels (such as the two-Pauli channel considered below), while our more refined analysis gives results even in such cases.

### 1.4 Techniques: weak string erasure

Before describing our protocols and proving Theorem 1.1, we give a short overview of the techniques involved.

First, we introduce a primitive called weak string erasure, which may be of independent interest. Our protocols for oblivious transfer and bit commitment are then based on this primitive. Weak string erasure provides Alice with a random bit-string $X^n \in \{0, 1\}^n$, while Bob receives a randomly chosen substring $X_I = \ldots$
weak string erasure roughly means that Bob will remain ignorant about a significant amount of information about $X^n$, while security against Alice means that she does not learn anything about $I$ (for a precise definition, we defer the reader to Section 3).

We provide a protocol for weak string erasure in the noisy-storage model. This protocol can be implemented with present-day hardware used for quantum key distribution. In particular, it does not require the honest parties to have any form of quantum memory. We prove security of this protocol for channels $F$ as stated in Theorem 1.1. Security against (even an all-powerful) Alice follows from the fact that the protocol only involves one-way communication from Alice to Bob. The security analysis in the presence of a malicious Bob limited by storage noise $F$ is more involved. Our proof combines an entropic uncertainty relation involving post-measurement information \[1,13\] with a reformulation of the problem as a coding scheme: Essentially, the uncertainty relation implies that with high probability (over measurement outcomes), Bob’s classical information about $X^n$ before using his storage is limited. We then show that this implies that any successful attacker Bob needs to encode classical information at a high rate into his storage device. However, the assumed noisiness of $F$ precludes this.

Having built a protocol for weak string erasure, we proceed to present protocols for bit commitment and oblivious transfer. The case of bit commitment is particularly appealing: It is essentially only based on weak string erasure and a classical code, and requires little additional analysis. Our approach to realizing oblivious transfer is somewhat more involved: Here weak string erasure is combined with a technique called interactive hashing \[16\]. The output of interactive hashing is a pair of substrings of classical $X$, respectively, and a number of tools, namely privacy amplification \[44\] is then used to extract completely random bits. The security analysis of this protocol requires the use of entropy sampling with respect to a quantum adversary \[27\].

As a side remark, note that Kilian \[24\] showed that oblivious transfer is universal for secure two-party computation. In particular, bit commitment could be built from oblivious transfer, but this reduction is generally inefficient.

## 2 Tools

We briefly introduce all necessary notation as well as several important concepts we will need throughout the paper. For weak string erasure we require the notion of min-entropy (Section 2.2.1), uncertainty relations (Section 2.2.2), as well as an understanding of how storage noise leads to information loss for the cheating party (Section 2.3). In our protocols for bit-commitment and oblivious transfer from weak string erasure, we additionally require the concepts of smooth min-entropy (Section 2.2.3), as well as an understanding of how storage noise leads to information loss for the cheating party (Section 2.3). In our protocols for bit-commitment and oblivious transfer from weak string erasure, we additionally require the concepts of smooth min-entropy (Section 2.2.3), as well as an understanding of how storage noise leads to information loss for the cheating party (Section 2.3). In our protocols for bit-commitment and oblivious transfer from weak string erasure, we additionally require the concepts of smooth min-entropy (Section 2.2.3), as well as an understanding of how storage noise leads to information loss for the cheating party (Section 2.3).

### 2.1 Notation

For an integer $n$, let $[n] := \{1, \ldots, n\}$. We use $2^{[n]} := \{S \mid S \subseteq [n]\}$ to refer to the set of all possible subsets of $[n]$, including the empty set $\emptyset$. For an $n$-tuple $x^n = (x_1, \ldots, x_n) \in \mathcal{X}^n$ over a set $\mathcal{X}$ and a (non-empty) set $I = \{i_1, \ldots, i_t\} \in 2^{[n]}$, we write $x_I$ for the subtuple $x_I = (x_{i_1}, \ldots, x_{i_t}) \in \mathcal{X}^t$.

We use upper case letters to denote a random variable $X$ distributed according to a distribution $P_X$ over a set $\mathcal{X}$, and use lower case letters $x$ for elements $x \in \mathcal{X}$. Joint distributions of e.g., three random variables $(X,Y,Z)$ on $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ are denoted by $P_{XYZ}$. Given a function $f : \mathcal{X} \rightarrow \mathcal{Y}$, any distribution $P_X$ of a random variable $X$ gives rise to another jointly distributed random variable $Y = f(X)$: The joint distribution $P_{XY} \equiv P_{Xf(X)}$ is given by

$$P_{Xf(X)}(x,y) = P_X(x)\delta_{f(x),y}, \quad \text{(7)}$$

where $\delta_{i,j}$ is the Kronecker symbol. An important example is the case where $X^n \in \{0,1\}^n$ is a random bitstring and $I \in 2^{[n]}$ is a random subset of $[n]$, where $X^n$ and $I$ have joint distribution $P_{X^nI}$. In this case,
the joint distribution \( P_{X^nIIZ} = P_{X^nIXZ} \) describes e.g., a situation where some bits \( Z = X_Z \) of a string \( X^n \) are given, together with a specification \( I \) of where these bits are located in the original string.

We use \( B(\mathcal{H}) \) to denote the set of bounded operators on a Hilbert space \( \mathcal{H} \). A (quantum) state is a Hermitian operator \( \rho \in B(\mathcal{H}) \) satisfying \( \text{tr}(\rho) = 1 \) and \( \rho \geq 0 \). Quantum states can be used to encode classical probability distributions: for a (finite) set \( \mathcal{X} \), we fix a Hilbert space \( \mathcal{H}_X \cong (\mathbb{C}^{|\mathcal{X}|}) \) and an orthonormal basis \( \{|x\rangle \mid x \in \mathcal{X}\} \) of \( \mathcal{H}_X \). This will be referred to as the computational basis. A probability distribution \( P_X \) on \( \mathcal{X} \) can then be encoded into the classical state \( (\text{e}-state) \)

\[
\rho_X = \sum_{x \in \mathcal{X}} P_X(x)|x\rangle \langle x| .
\]

Of particular interest is the uniform distribution over \( \mathcal{X} \), which gives rise to the completely mixed state on \( \mathcal{H}_X \) denoted by the shorthand

\[
\tau_X := \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} |x\rangle \langle x| .
\]

States describing classical information (random variables) and truly quantum information simultaneously are termed classical-quantum or cq-states. They are described by bipartite systems, where the classical part of the state is diagonal with respect to the computational basis. Concretely, let \( \mathcal{H}_Q \) be an additional Hilbert space. A state \( \rho_{XQ} \) on \( \mathcal{H}_X \otimes \mathcal{H}_Q \) is a cq-state if it has the form

\[
\rho_{XQ} = \sum_{x \in \mathcal{X}} P_X(x)|x\rangle \langle x| \otimes \rho_x \otimes \rho_{Q} . \quad (8)
\]

In other words, such a state \( \rho_{XQ} \) encodes an ensemble of states \( \{P_X(x), \rho_x\}_{x \in \mathcal{X}} \) on \( \mathcal{H}_Q \), where \( \rho_x \) is the conditional state on \( Q \) given \( X = x \). The notion of cq-states directly generalizes to multipartite systems, where classical parts are diagonal with respect to the computational basis. We often fix an ordering of the multipartite parts, and indicate by \( c \) or \( q \) whether a part is classical or quantum. We can also apply functions to classical parts as before. For a function \( f : \mathcal{X} \rightarrow \mathcal{Y} \),

\[
\rho_{Xf(X)Q} = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{Xf(X)}(x,y)|x\rangle \langle y| \otimes |y\rangle \langle y| \otimes \rho_x \otimes \rho_{Q} . \quad (9)
\]

is the ccq-state encoding the pair \( (X, f(X)) \) of classical random variables \( (cc) \) distributed according to \( (7) \) as well as the quantum information \( Q \) \( (q) \) (which depends only on \( X \) in this case). Note that in \( (9) \), the systems on the rhs. are uniquely determined by the expression on the lhs. We will therefore omit the braces below. Given a state \( \rho_{Q_1Q_2} \) on systems \( Q_1 \) and \( Q_2 \), we also use \( \rho_{Q_2} = \text{tr}_{Q_1}(\rho_{Q_1Q_2}) \) to denote the state obtained by tracing out \( Q_2 \).

The Hadamard transform is the unitary described by the matrix

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

in the computational basis \( \{|0\rangle, |1\rangle\} \) of the qubit Hilbert space \( \mathbb{C}^2 \). For the \( n \)-qubit Hilbert space, we let

\[
H^{\theta^n} |x^n\rangle := H^{\theta_1} |x_1\rangle \otimes \ldots \otimes H^{\theta_n} |x_n\rangle \quad \text{for } x^n = (x_1, \ldots, x_n), \theta^n = (\theta_1, \ldots, \theta_n) \in \{0,1\}^n .
\]

We also call states of this form BB84-states.

Finally, we need a distance measure for quantum states on a Hilbert space \( \mathcal{H} \). We use the distance determined by the trace norm \( \|A\|_1 := \text{tr} \sqrt{A^*A} \) for bounded operators \( A \in B(\mathcal{H}) \). We will say that two states \( \rho, \sigma \in B(\mathcal{H}) \) are \( \varepsilon \)-close if \( \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \), which we also write as

\[
\rho \approx_\varepsilon \sigma .
\]
2.2 Quantifying adversarial information

2.2.1 Min-entropy and measurements

One of the main properties of the weak string erasure-primitive is that the adversary’s (quantum) information \( Q \) about the generated bit-string \( X \) is limited. To make this statement precise, we first need to introduce an appropriate measure of information. Throughout, we are interested in the case where the adversary holds some (possibly quantum) information \( Q \) about a classical random variable \( X \). This situation is described by a cq-state \( \rho_{XQ} \) as in \( \mathbb{S} \). A natural measure for the amount of information \( Q \) gives about \( X \) is the maximal average success probability that a party holding \( Q \) has in guessing the value of \( X \). For a given cq-state \( \rho_{XQ} \), this guessing probability can be written as

\[
P_{\text{guess}}(X|Q) := \max_{\{D_x\}_x} \sum_x P_X(x) \text{tr}(D_x \rho_x) ,
\]

where the maximization is over all POVMs \( \{D_x\}_{x \in X} \) on \( \mathcal{H}_Q \). It will be convenient to turn (10) into an conditional entropy-like quantity, called the min-entropy, which is given by

\[
H_\infty(X|Q) := -\log P_{\text{guess}}(X|Q) .
\]

Note that the min-entropy was originally defined \( [43] \) for arbitrary bipartite states \( \rho_{AB} \), as we will discuss in more detail below.

As an illustrative, yet important, example consider the following ccq-state on \( \mathcal{H}_X \otimes \mathcal{H}_\Theta \otimes \mathcal{H}_Q \cong (\mathbb{C}^2)^{\otimes 3} \)

\[
\rho_{X\Theta Q} = \frac{1}{4} \sum_{x, \theta \in \{0,1\}} |x\rangle \langle x| \otimes |\theta\rangle \langle \theta| \otimes H^\theta|x\rangle \langle x| H^\theta .
\]

This state arises when encoding a uniformly random bit \( X \) using either the computational basis (\( \Theta = 0 \)) or the Hadamard basis (\( \Theta = 1 \)) chosen uniformly at random. Clearly, we have

\[
H_\infty(X) = 1 , \quad H_\infty(X|\Theta) = 1 \quad \text{and} \quad H_\infty(X|Q,\Theta = \theta) = 0 ,
\]

where the last identity is a consequence of the fact that given \( \Theta = \theta \), the operation \( H^\theta \) can be undone, such that a subsequent measurement in the computational basis provides \( X \) with certainty. Note that this is a special case of the identity

\[
H_\infty(X|Q,\Theta = \theta) = -\log E_{\theta \leftarrow \mathcal{P}_\Theta} \left[ 2^{-H_\infty(X|Q,\Theta = \theta)} \right] ,
\]

for a general cq-state \( \rho_{XQ\Theta} \) with classical part \( \Theta \), where \( E_{\theta \leftarrow \mathcal{P}_\Theta} \) denotes the expectation value over the choice of \( \Theta \), and \( H_\infty(X|Q,\Theta = \theta) \) is the min-entropy of the conditional state

\[
\rho_{X|Q,\Theta = \theta} = \sum_{x \in \{0,1\}} P_{X|\Theta = \theta}(x) |x\rangle \langle x| \otimes H^\theta|x\rangle \langle x| H^\theta .
\]

Returning to the state (12), it can also be shown \( [1] \) that

\[
H_\infty(X|Q) = -\log \left( \frac{1}{2} + \frac{1}{2\sqrt{2}} \right) .
\]

\[^3\text{All logarithms are taken to base 2.}\]
2.2.2 Smooth min-entropy

When building oblivious transfer from weak string erasure, we will need to employ a more general definition of the min-entropy given in [43]. For arbitrary (not necessarily unit-trace, or cq) bipartite density operators $\rho_{AB}$ this quantity is defined as

$$H_\infty(A|B)_\rho = -\log \inf \{ \text{tr}(\sigma_B) \mid \sigma_B \geq 0 \text{ and } \rho_{AB} \leq I_A \otimes \sigma_B \}$$  \hspace{1cm} (14)

where we use the subscript $\rho$ to indicate what state the quantity refers to. In [29], it was shown via semidefinite programming duality that for a cq-state $\rho_{XQ}$, definition (14) of $H_\infty(X|Q)$ coincides with definition (11) in terms of the guessing-probability $P_{\text{guess}}(X|Q)$. The advantage of (14) is that it allows us to maximize over neighborhoods of $\rho_{XQ}$. This leads to the definition of smooth entropy [43], which is defined as

$$H^\varepsilon_\infty(X|Q)_\rho := \sup_{\bar{\rho}_{XQ} \geq 0, \|\bar{\rho}_{XQ} - \rho_{XQ}\|_1 \leq \text{tr}(\rho_{XQ}) \epsilon} H^\varepsilon_\infty(X|Q)_{\bar{\rho}}.$$  \hspace{1cm} (15)

We will also use the fact that if $\rho_{XQ}$ is a cq-state, the supremum can be restricted to density operators $\bar{\rho}_{XQ}$ where $X$ is classical and has the same range as the original $X$. Definition (15) will be convenient for our proof: Roughly, we will construct some state that has high min-entropy. We then show that the state created during a real execution of the protocol is $\epsilon$-close to this state. By the above definition, the actual state generated in the protocol has high smooth min-entropy.

A useful property of the smooth min-entropy is that it obeys a chain rule [43, Theorem 3.2.12], which is a cq-state (i.e., an encoded joint distribution $\bar{\rho}_{XYQ}$). By the above definition, the actual state generated in the protocol has high smooth min-entropy.

$$H^\varepsilon_\infty(X|YQ)_\rho \geq H^\varepsilon_\infty(X|Q)_\rho - \log |\mathcal{Y}|,$$  \hspace{1cm} (16)

where $|\mathcal{Y}|$ is the size of the support of $Y$.

2.2.3 Uncertainty relations for post-measurement information

When showing the security of weak string erasure, we need to consider a setting where an adversary can first extract some classical information $K$ given access to a quantum system $Q$ and later obtains some additional information $\Theta$. His objective is to guess the value of a random variable $X$. Suppose he applies a measurement described by a POVM $\{E_k\}_k$ to $Q$, and retains only the measurement result $k$. We can think of this as a completely positive trace-preserving map (CPTPM) $\mathcal{K} : \mathcal{B}(\mathcal{H}_Q) \to \mathcal{B}(\mathcal{H}_K)$. When he performs this measurement on the $Q$-part of a cq-state $\rho_{XQ}$, we get

$$\rho_{XK(Q)} := (I_X \otimes \mathcal{K})(\rho_{XQ}) = \sum_k \text{tr}_Q ((I_X \otimes E_k)\rho_{XQ}) \otimes |k\rangle\langle k|,$$

which is a cc-state (i.e., an encoded joint distribution $P_{XK}$) if $X$ is classical. Due to its definition, the min-entropy $H_\infty(X|Q)$ is intimately connected with such measurements, and in fact it is easy to see that

$$H_\infty(X|Q) = \min_K H_\infty(X|K(Q)).$$  \hspace{1cm} (17)

This important identity relates min-entropies given quantum information $Q$ to min-entropies given classical information $K = \mathcal{K}(Q)$.

Returning to the example given in [12], let us consider what happens if the adversary learns the basis information $\Theta$ after the measurement $K$. In [1, Theorem 4.7] it was shown that the minimal post-measurement min-entropy optimized over all measurements $\mathcal{K}$ obeys

$$\min_K H_\infty(X|\mathcal{K}(Q)|\Theta) = -\log \left( \frac{1}{2} + \frac{1}{2\sqrt{2}} \right),$$

Unlike in [13], we require that half the 1-norm is bounded. This ensures that $H^\varepsilon_\infty(X|Q)_\rho \geq H_\infty(X|Q)_\rho$ if $\rho_{XQ} \approx_{\epsilon} \sigma_{XQ}$.
which in the case of our example matches the min-entropy $H_\infty(X|Q)$ without post-measurement information $\Theta$. In our security proof, we will need to consider $n$ repetitions of the state (12), that is,

$$\rho_{X^n\Theta^n} = \rho_{X^{\otimes n}}\otimes \rho_{\Theta^{\otimes n}} ,$$

where $X^n = (X_1, \ldots, X_n)$ and $\Theta^n = (\Theta_1, \ldots, \Theta_n)$ are $n$-bit strings, and $H_Q \cong (\mathbb{C}^2)^{\otimes n}$. It follows from [54, Lemma 2] and [4] that

$$\min_H H_\infty(X^n|K(Q)\Theta^n) = -n \log \left( \frac{1}{2} + \frac{1}{2\sqrt{2}} \right).$$

A generalization of this relation to smooth min-entropy is

$$\min_H H_\varepsilon(X^n|K(Q)\Theta^n) \geq n \left( \frac{1}{2} - 2\delta \right) \text{ where } \delta \in [0, \frac{1}{2}] \text{ and}$$

$$\varepsilon = \exp \left( -\frac{\delta^2 n}{32(2 + \log \frac{1}{\delta})^2} \right).$$

This relation follows from [13, Corollary 3.4] using the standard purification trick (cf. [54, Lemma 2.3]). Our construction of a protocol for weak string erasure will make essential use of (18) and (19).

### 2.2.4 Secure keys and what it means to be ignorant

We will often informally say that an adversary “does not know anything” or “does not learn anything” or “is ignorant” about some random variable $X$, even when he holds some (quantum) information $Q$. In terms of the cq-state $\rho_{XQ}$ this means that $X$ is uniformly distributed on $X$, and independent of $Q$, that is,

$$\rho_{XQ} = \tau_X \otimes \rho_Q .$$

Clearly, for such a state, the uncertainty about $X$ given $Q$ is maximal, which in terms of the min-entropy means that $H_\infty(X|Q) = \log |X|$. For $\rho_{XQ}$ as in (21), $X$ is also referred to as an ideal key with respect to $Q$.

In practice, we are generally forced to work with approximately ideal keys, where $X$ is called an $\varepsilon$-secure key with respect to $Q$ if $\rho_{XQ}$ is $\varepsilon$-close to the ideal state $\tau_X \otimes \rho_Q$, that is,

$$\rho_{XQ} \approx_\varepsilon \tau_X \otimes \rho_Q .$$

This notion of a secure key behaves nicely under composition [3, 14, 28].

### 2.3 Processes that increase uncertainty

To show the security of weak string erasure, we need to capture the amount of “uncertainty” that an adversary has as a result of his noisy storage $F$. First, let us consider general processes which increase uncertainty. Note that from the definitions, it is immediate [43, Theorem 3.1.12] that the min-entropy satisfies the following monotonicity property: for every CPTPM $F : B(H_Q) \rightarrow B(H_{Q'})$, we have

$$H_\infty(X|F(Q)) \geq H_\infty(X|Q) .$$

An important case is where $H_Q = H_{Q_1Q_2}$ is bipartite, and $F = \text{tr}_{Q_2}$ is the partial trace over the second system $Q_2$. We then get

$$H_\infty(X|Q_1) \geq H_\infty(X|Q_1Q_2) ,$$

reflecting the fact that “forgetting” information makes it harder to guess $X$. 

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Inequality (23) is insufficient for our purposes, and we will need a more quantitative estimate on the increase of entropy due to a channel $F$ representing the adversary’s memory. Clearly, such an estimate will depend on properties of $F$. Here we express the bound in terms of the function $P^F_{\text{succ}}(n)$ introduced in (2). Intuitively, the following lemma shows that the uncertainty about $X$ after application of $F$ to $Q$ is related to the problem of transmitting classical bits through the channel $F$, where the number of bits is given by the min-entropy of $X$.

**Lemma 2.1.** Consider an arbitrary cq-state $\rho_{XQ}$ and a CPTPM $F : \mathcal{B}(\mathcal{H}_Q) \to \mathcal{B}(\mathcal{H}_\text{out})$. Then $H_\infty(X|F(Q)) \geq - \log P^F_{\text{succ}}(\lfloor H_\infty(X) \rfloor)$.

*Proof.* Let $k := \lfloor H_\infty(X) \rfloor$. It is well-known (see e.g., [43]) that probability distributions $P_X$ with min-entropy at least $k$ are convex combinations of “flat” distributions, i.e., uniform distributions over subsets of $X$ of size $2^k$. In other words, there is a joint distribution $P_{X\mathcal{T}Q}$, where $T$ is distributed over subsets of size $2^k$, such that

$$P_X(x) = \sum_{t} P_T(t) P_{X|T=t}(x),$$

and $P_{X|T=t}$ is uniform on $t \subset X$.

The distribution $P_{X\mathcal{T}}$ together with $\rho_{XQ}$ gives rise to a state $\rho_{XQT}$ whose partial trace is equal to $\rho_{XQ}$. Again using (24), we get

$$H_\infty(X|F(Q)) \geq H_\infty(X|F(Q)T).$$

By property (13) of the min-entropy when conditioning on classical information, we have

$$H_\infty(X|F(Q)T) = - \log \mathbb{E}_{t \sim P_T} \left[ 2^{-H_\infty(X|F(Q), T=t)} \right],$$

(25)

where $\mathbb{E}_{t \sim P_T}$ denotes the expectation value, and $H_\infty(X|F(Q), T=t)$ is the min-entropy of the conditional state

$$\rho_{X,F(Q)|T=t} = \sum_{x} P_{X|T=t}(x) |x\rangle \langle x| \otimes F(\rho_x).$$

Now we use the fact that $P_{X|T=t}$ is uniform over a set of size $2^k$, and the definition of $P^F_{\text{succ}}(n)$. This leads to

$$H_\infty(X|F(Q), T=t) \geq - \log P^F_{\text{succ}}(k) \quad \text{for all } t \text{ in the support of } P_T. \quad (26)$$

Combining (25) with (26) gives the claim. \hfill $\Box$

We now give a straightforward but important generalization of this result.

**Lemma 2.2.** Consider an arbitrary ccq-state $\rho_{X\mathcal{T}Q}$, and let $\varepsilon, \varepsilon' \geq 0$ be arbitrary. Let $F : \mathcal{B}(\mathcal{H}_Q) \to \mathcal{B}(\mathcal{H}_\text{out})$ be an arbitrary CPTPM. Then

$$H_{\infty}^{\varepsilon\varepsilon'}(X|TF(Q)) \geq - \log P^F_{\text{succ}} \left( [H_\infty(X|T) - \log \frac{1}{\varepsilon'}] \right).$$

*Proof.* Clearly, the statement for $\varepsilon = 0$ implies the statement for any $\varepsilon > 0$ because a CPTPM cannot increase distance. To prove the statement for $\varepsilon = 0$, we consider the quantities $2^{-H_\infty(X|T=t)}$ of the conditional states $\rho_{X|T=t}$, together with the distribution $P_T$ over $T$ defined by the state $\rho_{X\mathcal{T}Q}$. Applying Markov’s inequality $Pr[Z \geq c] \leq \frac{\mathbb{E}[Z]}{c}$ for any real-valued random variable $Z$ and constant $c > 0$, we obtain

$$Pr_{t \sim P_T} \left[ 2^{-H_\infty(X|T=t)} \geq 2^{-H_\infty(X|T)+\log \frac{1}{\varepsilon'}} \right] \leq \varepsilon' \left( 2^{-H_\infty(X|T)} \right)^{-1} \mathbb{E}_{t \sim P_T} \left[ 2^{-H_\infty(X|T=t)} \right] = \varepsilon'.$$
This implies that the distribution \( P_T \) has weight at least \( 1 - \epsilon' \) on the set

\[
\mathcal{G}_{\text{Good}} = \left\{ t \in T \mid H_\infty(X|T = t) \geq \left[H_\infty(X|T) - \log \frac{1}{\epsilon'}\right] \right\}.
\] (27)

Accordingly, we can rewrite \( \rho_{XTQ} \) as a convex combination

\[
\rho_{XTQ} = (1 - p) \cdot \rho_{XTQ|T \notin \mathcal{G}_{\text{Good}}} + p \cdot \rho_{XTQ|T \in \mathcal{G}_{\text{Good}}}
\]

where \( p = P_T(\mathcal{G}_{\text{Good}}) \geq 1 - \epsilon' \). (28)

Set \( \sigma_{XTQ} := \rho_{XTQ|T \in \mathcal{G}_{\text{Good}}} \). From (28), we conclude that \( \frac{1}{2} \| \rho_{XTQ} - \sigma_{XTQ} \|_1 \leq \epsilon' \). By the monotonicity of the distance under CPTP, it therefore suffices to show that

\[
H_\infty(X|T|\mathcal{F}(Q))_{\sigma} \geq -\log P_{\text{succ}}^E \left( \left| H_\infty(X|T)_{\rho} - \log \frac{1}{\epsilon'} \right| \right).
\] (29)

For this purpose, note that \( \sigma_{XT|\mathcal{F}(Q)} \) is given by the expression

\[
\sigma_{XT|\mathcal{F}(Q)} = \sum_{t \in \mathcal{G}_{\text{Good}}} P_{T|T \in \mathcal{G}_{\text{Good}}}(t) |t\rangle \langle t| \otimes \rho_{X|T = t}.
\]

In particular, by using (13) again, we have

\[
2^{-H_\infty(X|T|\mathcal{F}(Q))_{\sigma}} = \mathbb{E}_{t \sim P_{T|T \in \mathcal{G}_{\text{Good}}}} \left[ 2^{-H_\infty(X|T = t)_{\rho}} \right].
\] (30)

Using Lemma 2.1 (applied to the conditional state \( \rho_{X|T = t} \)), we conclude that

\[
H_\infty(X|T = t)_{\rho} \geq -\log P_{\text{succ}}^E \left( \left| H_\infty(X|T)_{\rho} - \log \frac{1}{\epsilon'} \right| \right)
\]

for all \( t \in \mathcal{G}_{\text{Good}} \). (31)

The claim (29) immediately follows from (31) and (30).

\[\square\]

2.4 Defeating a quantum adversary: essential building blocks

In order to build oblivious transfer and bit commitment from weak string erasure, we will employ three additional tools: first, we require privacy amplification against a quantum adversary [43, 44] as explained in Section 2.4.1. For oblivious transfer, we also need the notion of min-entropy sampling outlined in Section 2.4.2. In particular, we discuss how min-entropy about classical information is approximately preserved when considering randomly chosen subsystems. We then show in Section 2.4.3 how random subsets can be chosen in a cryptographically secure manner with a protocol called interactive hashing.

2.4.1 Privacy amplification

Intuitively, privacy amplification allows us to turn a long string \( X \), about which the adversary holds some quantum information \( Q \), into a shorter string \( Z = \text{Ext}(X, R) \) about which he is almost entirely ignorant. The maximal length of this new string is directly related to the min-entropy \( H_\infty(X|Q) \) from Section 2.2. In order to obtain this new string, we will need a 2-universal hash function: Formally, a function \( \text{Ext} : \{0,1\}^n \otimes \mathcal{R} \rightarrow \{0,1\}^\ell \) is called 2-universal if for all \( x \neq x' \in \{0,1\}^n \) and uniformly chosen \( r \in_R \mathcal{R} \), we have

\[
\Pr[\text{Ext}(x, r) = \text{Ext}(x', r)] \leq 2^{-\ell}.
\]

**Theorem 2.3** (Privacy amplification [43, 44]). Consider a set of 2-universal hash functions \( \text{Ext} : \{0,1\}^n \otimes \mathcal{R} \rightarrow \{0,1\}^\ell \), and a cq-state \( \rho_{X^n \otimes Q} \), where \( X^n \) is an \( n \)-bit string. Define \( \rho_{X^n \otimes RQ} = \rho_{X^n \otimes \tau_\mathcal{R}} \), i.e., \( R \) is a random variable uniformly distributed on \( \mathcal{R} \), and independent of \( X^n Q \). Then

\[
\rho_{\text{Ext}(X^n, R)Q} \approx e^{\epsilon'} \cdot \tau_{\{0,1\}^\ell} \otimes \rho_{RQ}
\]

for \( \epsilon' := 2^{-\frac{1}{2}(H_\infty(X^n|Q) - \ell)} - 1 + 2\epsilon \) for all \( \epsilon > 0 \).
It is important to stress that the extracted key $\text{Ext}(X^n, R)$ is secure even if the adversary is given $R$ in addition to $Q$. Theorem 2.3 immediately gives rise to a procedure allowing parties sharing some random variable $X^n$ to extract a key secure against an adversary holding $Q$. Indeed, one party can simply use independent randomness to pick $r \in R$ uniformly at random, and distribute (publicly) the value of $r$. Because 2-universal hash functions can be efficiently constructed (e.g., using linear functions [8]), this privacy amplification protocol is efficient [22, 23].

2.4.2 Min-entropy sampling with adversarially chosen partitions

For oblivious transfer, we will make use of the sampling property of min-entropy which was first established by Vadhan [52] in the classical case, and in [27] for the classical-quantum case. Consider a cq-state $\rho_{X^n, Q}$, where $X^n = (X_1, \ldots, X_n)$ is an $n$-bit string. An important property of smooth min-entropy is that the min-entropy rate

$$\frac{H_{\infty}(X^n|Q)}{n}$$

is approximately preserved when considering a randomly chosen substring $X_S$ of $X^n$. In some sense, we can therefore think of [52] as the (average) min-entropy of an individual bit $X_i$ given $Q$.

The corresponding technical statement is slightly more involved. In essence, it requires to pick a subset $S$ from a distribution $P_S$ over subsets of $[n]$ with certain properties ($P_S$ needs to be a so-called averaging sampler, see e.g., [18]). For concreteness, we consider the special case where $P_S$ is uniformly distributed over subsets of size $s = |S|$. Vadhan’s result for the classical case [52] then shows that, for sufficiently large $s$, we have

$$\frac{H_{\infty}(X_S|C)}{s} \geq \frac{H_{\infty}(X^n|C)}{n} - \delta,$$

with high probability over the choice of $S$, for some small $\varepsilon > 0$ and $\delta > 0$. An analogous statement for the cq-case is given in [27]. A major difference is that the result of [27] for the quantum setting requires $X_i$ to be a block, i.e., a $\beta$-bit string instead of a single bit.

Since our work is mainly a proof of principle, we do not yet care about optimality or efficiency. We therefore choose $S$ to be uniform over all subsets of a fixed size $s$. Furthermore, it is sufficient for our purposes to ensure that the min-entropy rate decreases by at most a factor of 2. This leads to the following statement, which we derive by specializing the results of [27] (see appendix A.1 for details).

**Lemma 2.4** (Min-entropy sampling [27]). Let $\rho_{Z^n}$ be a cq-state, where $Z = (Z_{i,\alpha})_{(i,\alpha) \in [m] \times [\beta]} \in \mathbb{M}_{m \times \beta}(\{0, 1\})$ is an $m \times \beta$-matrix with entries in $\{0, 1\}$. Let $Z_i := (Z_{i,1}, \ldots, Z_{i,\beta}) \in \{0, 1\}^\beta$ be the $i$-th row of $Z$, such that $Z^n = (Z_1, \ldots, Z_m) = Z$. Let

$$\frac{H_{\infty}(Z|Q)}{m \beta} \geq \lambda$$

be a lower bound on the smooth min-entropy rate of $Z$ given $Q$. Assume $s, \beta \in \mathbb{N}$ are such that

$$s \geq m/4 \quad \text{and} \quad \beta \geq \max \left\{ 67, \frac{256}{\lambda^2} \right\},$$

and let $P_S$ be the uniform distributions over subsets of $[m]$ of size $s$. Then

$$\Pr_{S} \left[ \frac{H_{\infty}(Z_S|Q)}{s \beta} \geq \frac{\lambda}{2} \right] \geq 1 - \delta^2 \quad \text{where} \quad \delta = 2^{-m \lambda^2/800}.$$

In Lemma 2.4, we have implicitly partitioned $\beta m$ bits into $m$ blocks $Z_1, \ldots, Z_m$ of $\beta$ bits each, by arranging the bits in the matrix $Z$. However, in our protocol we need to extend this result to the case where
this partition is chosen arbitrarily, even in an adversarial manner. To formally state the corresponding generalization of Lemma 2.4, first observe that any partition of $3m$ bits into $m$ blocks is described by a permutation $\pi : [m] \times [\beta] \rightarrow [m] \times [\beta]$ where $\pi \in S_{m, \beta}$. Given a matrix $Z$ and a permutation $\pi$, let $\pi(Z)_i = (Z_{\pi(i,1)}, \ldots, Z_{\pi(i,\beta)})$ denote the $i$-th row of the matrix $\pi(Z)$ defined by permuting the entries of $Z$ using $\pi$, and let $\pi(Z)^m = (\pi(Z)_1, \ldots, \pi(Z)_m)$ be the $m$-tuple of these $\beta$-bit rows. We are interested in the min-entropy of the $s\beta$-bit substring $\pi(Z)_{S} = (\pi(Z)_{i_1}, \ldots, \pi(Z)_{i_s})$, where $S = \{i_1, \ldots, i_s\} \subset [m]$. Note that the identity permutation $\pi(i,\alpha) = (i,\alpha)$ corresponds to the case of Lemma 2.4.

It will be important that the permutation $\Pi = \pi$ is a random variable which may depend on the adversary’s quantum information $Q$. More precisely, we will assume that $\Pi$ is the result of a CPTPM applied to $Q$. Such a CPTPM takes the form $\mathcal{E} : \mathcal{B}(\mathcal{H}_Q) \rightarrow \mathcal{B}(\mathcal{H}_{Q'} \otimes \mathcal{H}_\Pi)$, and has the property that $\Pi$ is classical and a permutation in $S_{m, \beta}$ for any input state. We now generalize Lemma 2.4 to deal with arbitrarily chosen partitions. The generalized version essentially follows from the easily verified fact that the min-entropy is invariant under reordering, i.e.,

\[ H_\infty^\pi(Z|Q) = H_\infty^\pi(\pi(Z)|Q) \quad \text{for all permutations} \quad \pi \in S_{m, \beta}. \quad (34) \]

Again, we refer to appendix A.1 for the proof of the following statement.

**Lemma 2.5.** Let $\rho_{ZQ}$ be a cq-state, where $Z = (Z_{i,\alpha})_{(i,\alpha) \in [m] \times [\beta]} \in \mathbb{M}_{m \times \beta}(\{0,1\})$ is a $m \times \beta$-matrix with entries in $\{0,1\}$. Assume that

\[ \frac{H_\infty^\pi(Z|Q)}{m\beta} \geq \lambda, \]

and that $\lambda$ and $s, \beta \in \mathbb{N}$ satisfy condition (33) of Lemma 2.4. Let $\mathcal{E} : \mathcal{B}(\mathcal{H}_Q) \rightarrow \mathcal{B}(\mathcal{H}_{Q'} \otimes \mathcal{H}_\Pi)$ be a permutation-computing CPTPM, as explained above, and let $\rho_{ZQ'\Pi} = (I_Z \otimes \mathcal{E})\rho_{ZQ}.$

Finally, let $P_S$ be the uniform distribution over subsets of $[m]$ of size $s$. Then

\[ \Pr[ \frac{H_\infty^\pi(\Pi(Z)|Q|Q'\Pi)}{s\beta} \geq \frac{\lambda}{2} ] \geq 1 - \delta^2 \quad \text{where} \quad \delta = 2^{-m\lambda^2/800}. \]

### 2.4.3 Interactive hashing

A final tool we need is interactive hashing [16, 46] first introduced in [41]. This is a two-party primitive where Bob inputs some string $W^t$, and Alice has no input. The primitive then generates two strings $W^t_0$, $W^t_1$, with the property that one of the two equals $W^t$. For a protocol implementing this primitive, security is intuitively specified by the following conditions: Alice does not learn which of the two strings is indeed equal to $W^t$. Conversely, Bob should have very little control over the other string created by the protocol. Figure 8 depicts the idealized version of this primitive.

More formally, the following is achieved in [16, Theorem 5.6], where we refer to [46] for the exact parameters used in the security condition for Alice.

**Lemma 2.6** (Interactive Hashing [10, 46]). There exists a protocol called interactive hashing (IH) between two players, Alice and Bob, where Alice has no input, Bob has input $W^t \in \{0,1\}^t$ and both players output $(W^t_0, W^t_1) \in \{0,1\}^t \times \{0,1\}^t$, satisfying the following:

**Correctness:** If both players are honest, then $W^t_0 \neq W^t_1$ and there exists a $D \in \{0,1\}$ such that $W^t_D = W^t$. Furthermore, the distribution of $W^t_{1-D}$ is uniform on $\{0,1\}^t \setminus \{W^t\}$.

**Security for Bob:** If Bob is honest, then $W^t_0 \neq W^t_1$ and there exists a $D \in \{0,1\}$ such that $W^t_D = W^t$. If Bob chooses $W^t$ uniformly at random, then $D$ is uniform and independent of Alice’s view.

**Security for Alice:** If Alice is honest, then for every subset $S \subseteq \{0,1\}^t$,

\[ \Pr[W^t_0 \in S \text{ and } W^t_1 \in S] \leq 16 \cdot \frac{|S|}{2^t}. \]
3 Weak string erasure in the noisy-storage model

We are now ready to introduce our main primitive. After giving a precise security definition in Section 3.1, we present a protocol for realizing this primitive in the noisy-storage model. We will subsequently show that the protocol satisfies the given security definition.

3.1 Definition

“Strong” versus weak string erasure

In an ideal world, string erasure would realize the ideal functionality depicted in Figure 4. It takes no inputs, but provides Alice with a uniformly distributed string $n$-bit string $X^n = (X_1, \ldots, X_n) \in \{0, 1\}^n$, while Bob receives a random subset $\mathcal{I} = \{i_1, \ldots, i_{|\mathcal{I}|}\} \subset 2^{[n]}$ and the substring $X_\mathcal{I}$. The set of indices $\mathcal{I}$ would thereby be randomly distributed over all the set $2^{[n]}$ of all subsets of $[n]$. Intuitively, we think of the complement of $\mathcal{I}$ as the locations of the “erased” bits.

Figure 4: The ideal functionality of string erasure. The actual weak string erasure is somewhat weaker, however, a dishonest party cannot gain significantly more information from the protocol than provided by the “box” depicted above.

Ideally, we would like to realize the functionality in Figure 4 in such a way that even a dishonest party cannot learn anything at all beyond what is provided by the box. Unfortunately, this definition is too stringent to be achieved by our protocol. We therefore relax our functionality to weak string erasure, where the players may gain a small amount of additional information. More precisely, we allow a dishonest Bob to learn some information about $X^n$ possibly different from $(\mathcal{I}, X_\mathcal{I})$. However, we demand that his total information about $X^n$ is limited: given a dishonest Bob’s system $B'$, he still has some residual amount of uncertainty about $X^n$. For a dishonest Alice, we essentially retain the strong security property that she does not learn anything about the set of indices $\mathcal{I}$ that Bob receives. In order to obtain bit commitment and oblivious transfer later on, we also demand one additional property that may seem superfluous from a classical perspective, namely that Alice is “committed” to a choice of $X^n$ at the end of the protocol. This difficulty arises since unlike in a classical setting, a dishonest Alice may for example store some quantum
information and perform measurements only at a later time. This may allow her to determine parts of $X^n$ after the protocol is completed. Security against such attacks is subtle to define in a quantum setting. To address this problem, we define security in terms of an “ideal” state $\sigma_{A'X^nIX^I}$ that could have been obtained by an honest Alice by preparing some state on $A'$ using $X^n$ (i.e., by post-processing). Our security definition then demands that the actual state $\rho_{A'B}$ shared by dishonest Alice and honest Bob after the execution of the protocol has the same form as the partial trace of the ideal state, that is, $\rho_{A'B} = \sigma_{A'IX^I}$.

**Formal definition**

In the following definition of weak string erasure, we write $\rho_{AB}$ for the resulting state at the end of the protocol if both parties are honest, $\rho_{A'B}$ is Alice is dishonest and $\rho_{AB}'$ if Bob is dishonest. Our definition is phrased in terms of ideal states denoted by $\sigma$ that exhibit all the desired properties of weak string erasure.

**Definition 3.1.** An $(n, \lambda, \varepsilon)$-weak string erasure (WSE) scheme is a protocol between Alice and Bob satisfying the following properties:

**Correctness:** If both parties are honest, then the ideal state $\sigma_{X^nIX^I}$ is defined such that

1. The joint distribution of the $n$-bit string $X^n$ and subset $I$ is uniform:
   \[ \sigma_{X^nI} = \tau_{\{0,1\}^n} \otimes \tau_{2^n}, \]  

2. The joint state $\rho_{AB}$ created by the real protocol is equal to the ideal state:
   \[ \rho_{AB} = \sigma_{X^nIX^I}, \]  

where we identify $(A, B)$ with $(X^n, IX^I)$.

**Security for Alice:** If Alice is honest, then there exists an ideal state $\sigma_{X^nB'}$ such that

1. The amount of information $B'$ gives Bob about $X^n$ is limited:
   \[ \frac{1}{n} H_\infty(X^n|B'), \sigma \geq \lambda \]

2. The joint state $\rho_{AB'}$ created by the real protocol is $\varepsilon$-close to the ideal state:
   \[ \sigma_{X^nB'} \approx_\varepsilon \rho_{AB'}, \]

where we identify $(X^n, B')$ with $(A, B')$.

**Security for Bob:** If Bob is honest, then there exists an ideal state $\sigma_{A'\hat{X}^nI}$, where $\hat{X}^n \in \{0,1\}^n$ and $I \subseteq [n]$ such that

1. The random variable $I$ is independent of $A'\hat{X}^n$ and uniformly distributed over $2^n$:
   \[ \sigma_{A'\hat{X}^nI} = \sigma_{A'\hat{X}^n} \otimes \tau_{2^n}, \]

2. The joint state $\rho_{A'B}$ created by the real protocol is equal to the ideal state:
   \[ \rho_{A'B} = \sigma_{A'(I\hat{X}^I)} \]

where we identify $(A', B)$ with $(A', I\hat{X}^I)$.

Note that we do not require $X^n$ to be uniform when Bob is dishonest. To show security of bit commitment and oblivious transfer we will only require that $X^n$ has high min-entropy. The condition that the real state is close to an ideal state having high min-entropy means that the real state has smooth min-entropy as outlined in Section 2. We allow the protocol to abort. However, we demand that the above conditions are satisfied if the protocol is not aborted.
Figure 5: The protocol as a circuit. Alice chooses a random string $x^n = (x_1, \ldots, x_n) \in \{0,1\}^n$. She then encodes the bits in random bases specified by $\theta^n = (\theta_1, \ldots, \theta_n) \in \{0,1\}^n$ and sends the corresponding quantum states to Bob. Bob measures in random bases specified by $\tilde{\theta}^n = (\tilde{\theta}_1, \ldots, \tilde{\theta}_n) \in \{0,1\}^n$ obtaining measurement outcomes $\tilde{x}^n = (\tilde{x}_1, \ldots, \tilde{x}_n)$. Upon reception of the basis string $\theta^n$, Bob determines the locations where he measured in the same basis by computing the bit-wise xor $\theta^n \oplus \tilde{\theta}^n = (\theta_1 \oplus \tilde{\theta}_1, \ldots, \theta_n \oplus \tilde{\theta}_n)$. He subsequently discards the bits he measured in the wrong bases (indicated by $\perp$: this replaces the classical input symbol by an erasure symbol).

3.2 Protocol

We now consider a simple protocol achieving weak string erasure in the noisy-storage model using BB84-states. Other encodings are certainly possible, and we will discuss some of the implications of this choice of encoding in Section 6. This protocol is essentially identical to the first step of known protocols for quantum key distribution [58, 4]. However, as explained in the last section, our security requirements differ greatly as we are dealing with two mutually distrustful parties.

**Protocol 1: Weak String Erasure (WSE)**

**Outputs:** $x^n \in \{0,1\}^n$ to Alice, $(I, z^{[I]}) \in 2^{[n]} \times \{0,1\}^{|I|}$ to Bob.

1. Alice: Chooses a string $x^n \in_R \{0,1\}^n$ and bases-specifying string $\theta^n \in_R \{0,1\}^n$ uniformly at random. She encodes each bit $x_i$ in the basis given by $\theta_i$ (i.e., as $H^{\theta_i} |x_i\rangle$) and sends it to Bob.

2. Bob: Chooses a basis string $\tilde{\theta}^n \in_R \{0,1\}^n$ uniformly at random. When receiving the $i$-th qubit, Bob measures it in the basis given by $\tilde{\theta}_i$ to obtain outcome $\tilde{x}_i$.

Both parties wait time $\Delta t$.

3. Alice: Sends the basis information $\theta^n$ to Bob, and outputs $x^n$.

4. Bob: Computes $I := \{i \in [n] \mid \theta_i = \tilde{\theta}_i\}$, and outputs $(I, z^{|I|}) := (I, \tilde{x}_I)$.

If Alice does not adhere to the designated waiting time $\Delta t$, but instead delays her subsequent message considerably, Bob aborts the protocol. Note that in any protocol, a dishonest party may decide to stop communicating altogether. Our main claim is the following:

**Theorem 3.2 (Weak string erasure).** *(i) Let $\delta \in [0, \frac{1}{2}]$ and let Bob’s storage be given by $F : B(\mathcal{H}_{\text{in}}) \rightarrow B(\mathcal{H}_{\text{out}})$. Then Protocol 1 is an $(n, \lambda(\delta), \varepsilon(\delta))$-weak string erasure protocol with min-entropy rate

$$\lambda(\delta) = -\lim_{n \to \infty} \frac{1}{n} P_{\text{succ}}^{F} \left( \left( \frac{1}{2} - \delta \right) \cdot n \right),$$

and error

$$\varepsilon(\delta) = 2 \exp \left( -\frac{\delta^2}{512(4 + \log \frac{1}{\delta})^2} \cdot n \right).$$

(37)*
(ii) Suppose $\mathcal{F} = \mathcal{N}^{\otimes n}$ for a storage rate $\nu > 0$, $\mathcal{N}$ satisfying the strong-converse property \(^4\) and having capacity $C_\mathcal{N}$ bounded by

$$C_\mathcal{N} \cdot \nu < \frac{1}{2}.$$ 

Let $\delta \in [0, \frac{1}{2} - C_\mathcal{N} \cdot \nu]$. Then Protocol I is an $(n, \lambda(\delta), \varepsilon(\delta))$-weak string erasure protocol for sufficiently large $n$, where

$$\lambda(\delta) = \nu \cdot \gamma^{C_\mathcal{N} \cdot \nu} \left(\frac{1/2 - \delta}{\nu}\right).$$

It is easy to see that the protocol is correct if both parties are honest: if Alice is honest, her string $X^n = x^n$ is chosen uniformly at random from $\{0, 1\}^n$ as desired, and if Bob is honest, he will clearly obtain $\tilde{x}_i = x_i$ whenever $i \in \mathcal{I}$ for a random subset $\mathcal{I} \subseteq [n]$. The remainder of Section 3 is thus devoted to proving security if one of the parties is dishonest: In Section 3.3 we use the properties of the channel $\mathcal{F}$ to show that the protocol is secure against a dishonest Bob. In Section 3.4 we argue that the protocol satisfies Definition 3.1 when Alice is dishonest.

### 3.3 Security for honest Alice

We now show that for any cheating strategy of a dishonest Bob, his min-entropy about the string $X^n = (X_1, \ldots, X_n)$ is large. Before turning to the proof, we first explain in Figure 6 how our model restricts the actions of Bob in our protocol. At time $t$, Bob receives an encoding of a classical string $x^n = (x_1, \ldots, x_n)$ which he would like to reconstruct as accurately as possible. To this end, he can apply any CPTPM $\mathcal{E} : \mathcal{B}(\mathbb{C}^2)^{\otimes n} \rightarrow \mathcal{B}(\mathcal{H}_{in} \otimes \mathcal{H}_K)$ with the following property: For any input state $\rho$ on $(\mathbb{C}^2)^{\otimes n}$, he obtains an output state $\rho_{Q_{in}, K} = \mathcal{E}(\rho)$, where $Q_{in}$ is the quantum information he will put into his quantum storage, and $K$ is any additional classical information he retains. Note that we allow an arbitrary amount of classical storage, that is, $\mathcal{H}_K$ may be arbitrarily large\(^5\). We call the map $\mathcal{E}$ Bob’s encoding attack.

We can think of the encoding attack $\mathcal{E}$ as being composed of two steps, $\mathcal{E} = (\mathcal{E}_{Q_{in}} \otimes \mathcal{M}) \circ \mathcal{E}_1$, where Bob first applies an arbitrary CPTPM $\mathcal{E}_1 : \mathcal{B}(\mathbb{C}^2)^{\otimes n} \rightarrow \mathcal{B}(\mathcal{H}_{in} \otimes \mathcal{H}_Q)$, and subsequently performs a measurement $\mathcal{K} : \mathcal{B}(\mathcal{H}_Q) \rightarrow \mathcal{B}(\mathcal{H}_K)$ on $\mathcal{H}_Q$. The outcome of this measurement forms his classical information $K = \mathcal{K}(\tilde{Q})$. For example, Bob can measure some of the incoming qubits, or encode some information using an error-correcting code. The joint state before his storage noise is applied is hence given by

$$\rho_{X^n = x^n K_{in}} = \frac{1}{(2^n)^2} \sum_{x^n, \theta^n \in \{0, 1\}^n} P_{K|X^n = x^n, \Theta^n = \theta^n}(k) |x^n⟩⟨x^n| \otimes |\theta^n⟩⟨\theta^n| \otimes |k⟩⟨k| \otimes ζ_{x^n = \theta^n k} ,$$

(38)

where $\zeta_{x^n = \theta^n k}$ is the conditional state on $\mathcal{H}_{in}$ conditioned on the string $X^n = x^n$, the basis choice $\Theta^n = \theta^n$ and Bob’s classical measurement outcome $K = k$. The state (38) is completely determined by Bob’s encoding attack $\mathcal{E}$ at time $t$.

Bob’s storage $Q_{in}$ then undergoes noise described by $\mathcal{F} : \mathcal{B}(\mathcal{H}_{in}) \rightarrow \mathcal{B}(\mathcal{H}_{out})$, and the state evolves to $\rho_{X^n = x^n K_{F}(Q_{in})}$. At time $t + \Delta t$, Bob additionally receives the basis information $\Theta^n = \theta^n$. The joint state is now given by

$$\rho_{X^n = x^n K_{F}(Q_{in})} = \frac{1}{(2^n)^2} \sum_{x^n, \theta^n \in \{0, 1\}^n} P_{K|X^n = x^n, \Theta^n = \theta^n}(k) |x^n⟩⟨x^n| \otimes |\theta^n⟩⟨\theta^n| \otimes |k⟩⟨k| \otimes \mathcal{F}(\zeta_{x^n = \theta^n k}) ,$$

(39)

where Bob holds $B’ = \Theta^n K_{F}(Q_{in})$. We now show that Bob’s information $B’$ about $X^n$ is limited for large $n$.

---

\(^4\) It is sufficient for any adversary to store $2^n$ bits, one for each possible basis string $\Theta^n$.\(^5\)
Figure 6: The most general structure of a cheating Bob. Bob’s action at time $t$ consists of a CPTPM $\mathcal{E}_1$, followed by a (partial) measurement $K$, where he may use an arbitrary ancilla $\rho_{aux}$. At time $t+\Delta t$, Bob can try to reconstruct $x^n = (x_1, \ldots, x_n)$ given the content $\mathcal{F}(Q_{in})$ of the storage device, the classical measurement result $K = K(\tilde{Q})$, and the basis information $\theta^n = (\theta_1, \ldots, \theta_n)$.

**Theorem 3.3** (Security for Alice). Fix $\delta \in ]0, \frac{1}{2}[$ and let

$$\varepsilon = 2 \exp \left( -\frac{\delta^2/4}{32(2 + \log \frac{1}{\delta})^2} \cdot n \right).$$

Then for any attack of a dishonest Bob with storage $\mathcal{F} : \mathcal{B}(\mathcal{H}_{in}) \to \mathcal{B}(\mathcal{H}_{out})$, there exists a cq-state $\sigma_{X^nB'}$ such that

1. $\sigma_{X^nB'} \approx_\varepsilon \rho_{X^nB'}$;
2. $\frac{1}{n} H_\infty(X^n|B')_\sigma \geq -\frac{1}{n} \log P_{\text{succ}}^\mathcal{F}(\Theta - \delta)n$,

where $\rho_{X^nB'}$ is given by (39). In particular, if, for some $R < \frac{1}{2}$, we have $\lim_{n \to \infty} -\frac{1}{n} \log P_{\text{succ}}^\mathcal{F}(nR) > 0$, then $\rho_{X^nB'}$ is exponentially close (in $n$) to a state $\sigma_{X^nB'}$ with constant min-entropy rate $\frac{1}{n} H_\infty(X^n|B')_\sigma$.

**Proof.** We use the notation introduced in (39). By definition (16) of the smooth min-entropy, statements (1) and (2) follow if we show that the smooth min-entropy rate $\frac{1}{n} H_\infty(X^n|B')_\rho$ is lower bounded by the expression on the rhs. in (2). By the uncertainty relation (19), we have

$$H_{\infty}^{\varepsilon/2}(X^n|\Theta nK)_\rho \geq \frac{n}{2} - \frac{n\delta}{2}.$$

Using Lemma 27 applied to $T = (\Theta^n, K)$, we conclude that for $Q_{out} = \mathcal{F}(Q_{in})$ after the noise $\mathcal{F}$

$$H_\infty^\varepsilon(X^n|\Theta^nKQ_{out})_\rho \geq -\log P_{\text{succ}}^\mathcal{F}\left( \frac{n}{2} - \frac{n\delta}{2} - \log \frac{2}{\varepsilon} \right) \geq -\log P_{\text{succ}}^\mathcal{F}\left( \frac{n}{2} - \frac{n\delta}{2} - \frac{n\delta}{2} \right),$$

where the final inequality follows from the monotonicity property of the success probability $P_{\text{succ}}^\mathcal{F}(m)$ for $m \geq m'$ and the fact that $\log \frac{2}{\varepsilon} \leq \frac{2}{2n}$ because $(\delta/4)^2/(32(2 - \log \delta/4)^2) \leq \delta/2$ for any $0 < \delta < 1/2$.

Let us specialize Theorem 3.3 to the case where $\mathcal{F}$ is a tensor product channel.

**Corollary 3.4.** Let Bob’s storage be described by $\mathcal{F} = \mathcal{N}^{\otimes n}$ with $\nu > 0$, where $\mathcal{N}$ satisfies the strong-converse property [14], and

$$C_\mathcal{N} \cdot \nu < \frac{1}{2}.$$

Fix $\delta \in ]0, \frac{1}{2} - C_\mathcal{N} \cdot \nu[$, and let $\varepsilon = \varepsilon(\delta)$ be defined by (37). Then for any attack of a dishonest Bob, there exists a cq-state $\sigma_{X^nB'}$ such that
Figure 7: This circuit shows the interaction between a dishonest party Alice and an honest Bob: Alice sends some $n$-qubit register $Q_A$ and $n$ classical bits $\theta^n$ to Bob, and also retains some possibly quantum register $T$. Honest Bob computes $I$ and $\tilde{X}_n$ as before. This generates an overall state $\rho_{\Theta^n TI\tilde{X}_n}$, where Alice’s information $A'$ after execution consists of the classical string $\Theta^n$ and $T$.

1. $\sigma_{X^n B'} \approx_{\varepsilon} \rho_{X^n B'}$,
2. $\frac{1}{n} H_{\infty}(X^n | B') \geq \nu \cdot \gamma^N \left( \frac{1/2 - \delta}{\nu} \right) > 0$,

where $\rho_{X^n B'}$ is given by (39).

Proof. Substituting $n$ by $\nu n$ and $R$ by $R/\nu$, the strong-converse property (4) turns into

$$\frac{1}{n} \log P_{\text{succ}}^{N_{\text{equiv}}}(nR) \geq \nu \cdot \gamma^N(R/\nu) .$$

The claim then follows from Theorem 3.3 by setting $R := \frac{1}{2} - \delta$.

Theorem 3.3 and Corollary 3.4 establish the first part of Theorem 3.2. It remains to analyze the security against a dishonest Alice.

3.4 Security for honest Bob

When Alice is dishonest, it is intuitively obvious that she is unable to gain any information about the index set $I$, since she never receives any information from Bob during our protocol. Yet, in order to obtain bit commitment and oblivious transfer from weak string erasure we require a more careful security analysis. Figure 7 depicts the form of any interaction between a cheating Alice and an honest Bob. Since Alice takes no input in the protocol, her actions are completely specified by the state $\rho_{Q_A \Theta^n T}$ she outputs, where $H_{Q_A} \sim (C^2)^{\otimes n}$ is an $n$-qubit register that she sends to Bob (in the case where Alice is honest, this encodes the string $X^n$), $\Theta^n$ is some classical $n$-bit string (in the case where Alice is honest, this encodes the bases), and $H_T$ is an auxiliary register of Alice corresponding to the (quantum) information she holds after execution of the protocol. In the actual protocol, an honest Bob proceeds as shown in Figure 5 that is,

1. Upon receipt of $Q_A$ at time $t$, an honest Bob measures in randomly chosen bases specified by the string $\tilde{\Theta}^n = (\tilde{\Theta}_1, \ldots, \tilde{\Theta}_n) \in \{0, 1\}^n$, obtaining measurement outcomes $\tilde{X}_n = (\tilde{X}_1, \ldots, \tilde{X}_n)$.
2. After receiving $\Theta^n = (\Theta_1, \ldots, \Theta_n)$ at time $t + \Delta t$, he computes the intersecting set $I$ defined by $\tilde{\Theta}^n$ and $\Theta^n$, and the corresponding substring $\tilde{X}_I$.

The protocol thus creates some state $\rho_{A' IX_T}$, where $A' = (\Theta^n T)$ is Alice’s information, and $B = (I \tilde{X}_I)$ is the information obtained by Bob. Note that this state can be obtained from $\rho_{A' X_T \tilde{X}_T}$, because $I$ is a function of $\Theta^n$ and $\tilde{\Theta}^n$, and $\tilde{X}_I$ is a function of $\tilde{X}_I$ and $I$.

Theorem 3.5 (Security for Bob). Protocol 1 satisfies security for honest Bob.
ideal state on the registers held by Alice and Bob. Figure 8 summarizes the actions of the simulator:

Proof. We now construct a state $\sigma_{A'(\hat{X}_t)}$ with the required properties. For simplicity, we give an algorithmic description of this state. It is obtained by letting Alice and Bob interact with a simulator which has perfect quantum memory. Note that this simulator is purely imaginary and is merely used to specify the desired ideal state $\sigma_{A'(\hat{X}_t)}$. However, we will later show that the real state created during the protocol equals this ideal state on the registers held by Alice and Bob. Figure 8 summarizes the actions of the simulator:

1. First, the simulator measures the $n$-qubits $Q_A$ in the bases specified by the bits $\Theta^n = (\Theta_1, \ldots, \Theta_n)$, obtaining measurement outcomes $\hat{X}_n = (\hat{X}_1, \ldots, \hat{X}_n)$.

2. Second, the simulator re-encodes the measurement outcomes $\hat{X}_n$ using randomly chosen bases specified by $\hat{\Theta}_n = (\hat{\Theta}_1, \ldots, \hat{\Theta}_n) \in R \{0, 1\}^n$. He then sends the corresponding qubits to Bob (i.e., the states $\hat{\rho}_n = \hat{x}_i$). We call this quantum register $\hat{Q}_A$.

3. Finally, the simulator provides Bob with the basis string $\hat{\Theta}_n = (\hat{\Theta}_1, \ldots, \hat{\Theta}_n)$.

An honest Bob proceeds as before, but with $\Theta^n$ replaced by the simulator’s string $\hat{\Theta}_n$, and $Q_A$ replaced by the simulator’s quantum message $\hat{Q}_A$. As before, Alice’s information $A' = (T \Theta^n)$ consists of the string $\Theta^n$ and her (quantum) system $T$. The state $\sigma_{A'(\hat{X}_t)}$ held by Alice and Bob can be obtained from $\sigma_{A'(\hat{X}_t)} = \sigma_{A'(\hat{X}_t)} \otimes \tau_{2^{|I|}}$, noting that $\hat{X}_t = \hat{X}_t$.

Let us argue that $\sigma_{A'(\hat{X}_t)}$ has the properties required by Definition 3.1. First, observe that

$$\sigma_{A'(\hat{X}_t)} = \sigma_{A'(\hat{X}_t)} \otimes \tau_{\{0,1\}^n},$$

since both $\hat{\Theta}_n$ and $\hat{\Theta}_n$ are chosen uniformly and independently at random by the simulator and Bob, respectively. Since the set $I$ consists of those indices where $\hat{\Theta}_n$ and $\hat{\Theta}_n$ agree, we conclude that $I$ is uniform on the set of subsets of $[n]$, and independent of $A' \Theta^n$. That is, the previous identity implies

$$\sigma_{A'(\hat{X}_t)} = \sigma_{A'(\hat{X}_t)} \otimes \tau_{2^{|I|}},$$

as desired.

It remains to prove that the state created during the real protocol equals this ideal state, that is,

$$\rho_{A'B} = \sigma_{\Theta^n T I \hat{X}_t}.$$  \hspace{1cm} (41)

Observe that, since an honest Bob measures in randomly chosen bases, the random variable $I$ in $\rho_{\Theta^n I}$ is uniform over subsets of $[n]$ and independent of $\Theta^n$. From (41) and the fact that $\sigma_{\Theta^n} = \rho_{\Theta^n}$ by construction of $\sigma_{\Theta^n T I \hat{X}_t}$, we conclude that

$$\rho_{\Theta^n I} = \sigma_{\Theta^n I} = \rho_{\Theta^n} \otimes \tau_{2^{|I|}}.$$
have the same distribution. To establish (41), it therefore suffices to prove that the conditional states agree, that is,

$$\rho_{T\hat{X}|\theta^n=\theta, I=\{i_1,\ldots,i_r\}} = \sigma_{T\hat{X}|\theta^n=\theta, I=\{i_1,\ldots,i_r\}}$$

for every subset $I=\{i_1,\ldots,i_r\} \subseteq [n]$ and every string $\theta^n \in \{0,1\}^n$. Consider first the lhs. of this identity: By definition, we have

$$\rho_{T\hat{X}|\theta^n=\theta, I=\{i_1,\ldots,i_r\}} = \rho_{T\hat{X}_{i_1} \cdots \hat{X}_{i_r}|\theta^n=\theta, I=\{i_1,\ldots,i_r\}} = \rho_{T\hat{X}_{i_1} \cdots \hat{X}_{i_r}|\theta^n=\theta} ,$$

where we used the fact that $I$ is independent of $T\hat{X}^n\Theta^n$. A similar reasoning gives

$$\sigma_{T\hat{X}|\theta^n=\theta, I=\{i_1,\ldots,i_r\}} = \sigma_{T\hat{X}_{i_1} \cdots \hat{X}_{i_r}|\theta^n=\theta} .$$

Inspecting the honest behavior of Bob in Figure 5 and comparing with Figure 8 shows that $\rho_{T\hat{X}_{i_1} \cdots \hat{X}_{i_r}|\theta^n=\theta}$ and $\sigma_{T\hat{X}_{i_1} \cdots \hat{X}_{i_r}|\theta^n=\theta}$ both correspond to measuring the subsets of qubits $\{i_1,\ldots,i_n\}$ in the bases specified by $(\theta_{i_1},\ldots,\theta_{i_r})$, for the conditional state $\rho_{T\hat{X}|\theta^n=\theta}$. We conclude that

$$\rho_{TY|\theta^n=\theta, I=\{i_1,\ldots,i_r\}} = \sigma_{T\hat{X}|\theta^n=\theta, I=\{i_1,\ldots,i_r\}} ,$$

from which the claim follows.

**3.5 Application to concrete tensor product channels**

We examine the security parameters we can obtain for several well-known channels. A simple example is the $d$-dimensional depolarizing channel defined in [35], which replaces the input state $\rho$ with the completely mixed state with probability $1-r$. Another simple example is the one-qubit two-Pauli channel [25]

$$\mathcal{N}_{\text{Pauli}}(\rho) := r\rho + \frac{1-r}{2} X\rho X + \frac{1-r}{2} Z\rho Z .$$

Both these channels obey the strong-converse property (4) (see [31]), allowing us to obtain security of weak string erasure by Corollary 3.4.

For simplicity, we first consider the case where the storage rate is $\nu = 1$, that is, Bob’s storage system is $(\mathbb{C}^d)^\otimes n$, i.e., $n$ copies of a $d$-dimensional system, and his noise channel is $\mathcal{F} = \mathcal{N}^\otimes n$. We first determine the values of $r$ that allow for a secure implementation of weak string erasure. By Corollary 3.4, the capacity of the channel $\mathcal{N}$ must be bounded by $C_N < \frac{1}{2}$. The table given in Figure 9 summarizes the relevant parameters.

| Channel            | Capacity $C_N$                                                                 | Reference | Threshold |
|--------------------|--------------------------------------------------------------------------------|-----------|-----------|
| Qubit depolarizing | $1 + \frac{1-r}{2} \log \frac{1-r}{2} + \frac{1-r}{2} \log \frac{1-r}{2}$     | [26]      | $r \leq 0.77$ |
| Qutrit depolarizing| $\log 3 + (r + \frac{1-r}{3}) \log (r + \frac{1-r}{3}) + 2\frac{1-r}{3} \log \frac{1-r}{3}$ | [26]      | $r \leq 0.61$ |
| Two-Pauli          | $1 - h \left(\frac{1 + \max(r,2r-1)}{2}\right)$                              | [25]      | $r \leq 0.77$ |

Figure 9: A sufficient condition for achieving security (for storage rate $\nu = 1$) is that the noise parameter $r$ lies below the threshold given above. This is equivalent to $C_N < \frac{1}{2}$.

When allowing storage rates other than $\nu = 1$, we may again consider the regime where our proof provides security. Figure 10 examines this setting for the qudit depolarizing channel and the two-Pauli channel, respectively.
To determine the exact security of the protocol, we need to compute the min-entropy rate
\[ \lambda(\delta) = \nu \cdot \gamma^N \left( \frac{1/2 - \delta}{\nu} \right), \]
as stated in Corollary 3.4. For the class of channels \( N : B(\mathbb{C}^d) \to B(\mathbb{C}^d) \) considered in [34], the strong converse property [4] was shown to be satisfied with the function \( \gamma^N \) given by
\[ \gamma^N(R) = \max_{\alpha \geq 1} \frac{\alpha - 1}{\alpha} \left( R - \log d + S_{\alpha}^{\min}(N) \right), \]
where \( S_{\alpha}^{\min}(N) \) is the minimum output \( \alpha \)-Rényi-entropy of the channel. For the \( d \)-dimensional depolarizing channel (see [5]) we may rewrite this expression [26] as
\[ \gamma^N(R) = \max_{\alpha \geq 1} \frac{\alpha - 1}{\alpha} \left( R - \log d + \frac{1}{1 - \alpha} \log \left( \left( r + \frac{1 - r}{d} \right)^{\alpha} + (d - 1) \left( \frac{1 - r}{d} \right)^{\alpha} \right) \right). \]

Figure 11 shows how the min-entropy rate \( \lambda(\delta) \) relates to the noise parameter \( r \) for the qubit and qutrit depolarizing channels for a storage rate of \( \nu = 1 \) and error \( \delta = 0.01 \). The figure shows that the min-entropy rate we can achieve in our protocol is directly related to the amount of noise in the storage.

4 Bit commitment from weak string erasure

4.1 Definition
Informally, a standard commitment scheme consists of a Commit and an Open primitive between two parties Alice and Bob. First, Alice and Bob execute the Commit primitive, where Alice has input \( Y^\ell \in \{0, 1\}^\ell \), and Bob has no input. As output, Bob receives a notification that Alice has chosen an input \( Y^\ell \). Afterwards, they may execute the Open protocol, during which Bob either accepts or rejects. If both parties are honest, Bob always accepts and receives the value \( Y^\ell \). If Alice is dishonest, however, we still demand that Bob either outputs the correct value of \( Y^\ell \) or rejects (binding). If Bob is dishonest, he should not be able to gain any information about \( Y^\ell \) before the Open protocol is executed (hiding).
Figure 11: The value of the min-entropy rate $\lambda$ for the qubit depolarizing channel (dashed red line) and the qutrit depolarizing channel (solid blue line) as a function of the noise parameter $r$, for $\nu = 1$ and $\delta = 0.01$. Using qutrits means that the dimension of the overall storage system is higher, and we expect the resulting higher capacity to lead to a smaller min-entropy rate $\lambda$. Our analysis confirms this intuition.

Here, we make use of a randomized version of a commitment as depicted in Figure 12. This simplifies both our definition, as well as the protocol. Instead of inputting her own string $Y^\ell$, Alice now receives a random string $C^\ell$ from the Commit protocol. Note that if Alice wants to commit to a value $Y^\ell$ of her choice, she may simply send the xor of her value with the random commitment $Y^\ell \oplus C^\ell$ to Bob at the end of the Commit protocol.

![Randomized string commitment](image)

Figure 12: Randomized string commitment: Alice receives a random $C^\ell \in \mathbb{R}\{0,1\}^\ell$ from Commit. During the Open phase, Bob outputs $\tilde{C}^\ell$ and $F$. If both parties are honest, then $\tilde{C}^\ell = C^\ell$ and $F = accept$. If Alice is dishonest, Bob outputs $F \in \{accept, reject\}$, but $\tilde{C}^\ell = C^\ell$ if $F = accept$. To obtain a standard commitment, Alice can send the extra message indicated by the dashed line.

To give a more formal definition, note that we may write the Commit and the Open protocol as CPTPMs $C_{AB}$ and $O_{AB}$ respectively, consisting of the local actions of honest Alice and Bob, together with any operations they may perform on messages that are exchanged. When both parties are honest, the output of the Commit protocol will be a state $C_{AB}(\rho_m) = \rho_{C^\ell U V}$ for some fixed input state $\rho_m$, where $C^\ell \in \{0,1\}^\ell$ is the classical output of Alice, and $U$ and $V$ are the internal states of Alice and Bob respectively. Clearly, if Alice is dishonest, she may not follow the protocol, and we use $C_{A'B}$ to denote the resulting map. Note that $C_{A'B}$ may not have output $C^\ell$, and we hence simply write $\rho_{A' U V}$ for the resulting output state, where $A'$
denotes the register of a dishonest Alice. Similarly, we use $C_{AB'}$ to denote the CPTPM corresponding to the case where Bob is dishonest, and write $\rho_{C^tU'B'}$ for the resulting output state, where $B'$ denotes the register of a dishonest Bob.

The Open protocol can be described similarly. If both parties are honest, the map $O_{AB} : \mathcal{B}(H_U) \rightarrow \mathcal{B}(H_{C^tF})$ creates the state $\eta_{C^t\hat{C}^tF} := (I_{C^t} \otimes O_{AB})(\rho_{C^tU'V}),$ where $C^t \in \{0,1\}^t$ and $F \in \{\text{accept, reject}\}$ is the classical output of Bob. Again, if Alice is dishonest, we write $O_{A'B}$ to denote the resulting CPTPM with output $\eta_{A'\hat{C}^tF}$, and if Bob is dishonest, we write $O_{AB'}$ for the resulting CPTPM with output $\eta_{C^tB'}$. The following definition is similar to the one given in [13], but slightly more general.

**Definition 4.1.** An $(\ell, \varepsilon)$-randomized string commitment scheme is a protocol between Alice and Bob satisfying the following properties:

**Correctness:** If both parties are honest, then the ideal state $\sigma_{C^t\hat{C}^tF}$ is defined such that

1. The distribution of $C^t$ is uniform, and Bob accepts the commitment:
   \[ \sigma_{C^tF} = \tau_{\{0,1\}^t} \otimes |\text{accept}\rangle\langle\text{accept}|. \]

2. The joint state $\eta_{C^t\hat{C}^tF}$ created by the real protocol is $\varepsilon$-close to the ideal state:
   \[ \eta_{C^t\hat{C}^tF} \approx_\varepsilon \sigma_{C^t\hat{C}^tF}, \]
   where we identify $(A, B)$ with $(C^t, \hat{C}^tF)$.

**Security for Alice ($\varepsilon$-hiding):** If Alice is honest, then for any joint state $\rho_{C^tB'}$ created by the Commit protocol, Bob does not learn $C^t$:

\[ \rho_{C^tB'} \approx_\varepsilon \tau_{\{0,1\}^t} \otimes \rho_{B'} . \]

**Security for Bob ($\varepsilon$-binding):** If Bob is honest, then there exists an ideal cqq-state $\sigma_{C^tA'V}$ such that for all $O_{A'B}$:

1. Bob almost never accepts $\hat{C}^t \neq C^t$:
   
   \[ \text{For } \psi_{C^tA'\hat{C}^tF} = (I_{C^t} \otimes O_{A'B})(\sigma_{C^tA'V}), \text{ we have } \Pr[C^t \neq \hat{C}^t \text{ and } F = \text{accept}] \leq \varepsilon . \]

2. The joint state $\rho_{A'V}$ created by the real protocol is $\varepsilon$-close to the ideal state:
   \[ \rho_{A'V} \approx_\varepsilon \sigma_{A'V} . \]

### 4.2 Protocol

To construct our protocol based on weak string erasure, we will need a binary $(n, k, d)$-linear code $C \subseteq \{0,1\}^n$, i.e., a linear code with $2^k$ elements and minimal distance $d := 2 \log 1/\varepsilon$. Let $\text{Syn} : \{0,1\}^n \rightarrow \{0,1\}^{n-k}$ be a function that outputs a parity-check syndrome for the code $C$. Let $\text{Ext} : \{0,1\}^n \times \mathcal{R} \rightarrow \{0,1\}^t$ be a 2-universal hash function as defined in Section 2.4.4.

**Protocol 2a: Commit**

Inputs: none. Outputs: $c^t \in \{0,1\}^t$ to Alice.

1. **Alice and Bob:** Execute $(n, \lambda, \varepsilon)$-WSE. Alice gets $x^n \in \{0,1\}^n$, and Bob gets $I \subset [n]$ and $s = x_I$.

2. **Alice:** Chooses $r \in_R \mathcal{R}$ and sends $r$ and $w := \text{Syn}(x^n)$ to Bob.

3. **Alice:** Outputs $c^t := \text{Ext}(x^n, r)$ and stores $x^n$. Bob stores $(r, w, I, s)$.
Our main claim of this section is the following.

**Theorem 4.2 (String commitment).** The pair \((2a, 2b)\) of protocols \((Commit, Open)\) is an \((\lambda n - (n - k) - 2\log 1/\varepsilon, 3\varepsilon)\)-randomized string commitment scheme based on one instance of \((n, \lambda, \varepsilon)\)-WSE.

The length \(\ell := \lambda n - (n - k) - 2\log 1/\varepsilon\) of the commitment depends on our choice of code \(C\). Since we require that information \(C^\ell\) is negligible. The intuition behind this proof is that weak string erasure ensures that Bob’s information about the string \(X^n\) is limited. Via privacy amplification we then obtain that his information about \(C^\ell\), which is the output of a 2-universal hash function applied to \(X^n\), is negligible.

**Lemma 4.3 (Security for Alice).** The pair of protocols \((Commit, Open)\) is \(3\varepsilon\)-hiding.

**Proof.** Let \(\rho_{X^nB'}\) the cq-state created by the execution of WSE. From the properties of WSE it follows that there exists a state \(\sigma_{X^nB'}\) such that \(H_\infty(X^n|B')_{\sigma} \geq \lambda n\) and \(\rho_{X^nB'} \approx_\varepsilon \sigma_{X^nB'}\). This implies that

\[
H_\infty^c(X^n|B')_{\rho} \geq \lambda n.
\]

By the chain rule (see [16]), we get

\[
H_\infty^c(X^n|B'Syn(X^n))_{\rho} \geq \lambda n - (n - k) = \ell + 2\log 1/\varepsilon.
\]

Using privacy amplification (Theorem [2.3]), we then get that

\[
\frac{1}{2} \| \rho_{C'\cup B'} - \tau_{\{0,1\}^\ell} \otimes \rho_{B'} \|_1 \leq 2\varepsilon + 2^{-\frac{1}{2}2\log 1/\varepsilon - 1} \leq 3\varepsilon,
\]

as promised.

To show security for honest Bob, we need the following property of linear codes. Note that the function \(Syn\) is linear, i.e., for all codewords \(x^n\) and \(\bar{x}^n\), we have \(Syn(x^n \oplus \bar{x}^n) = Syn(x^n) \oplus Syn(\bar{x}^n)\). Therefore, for any \(x^n\) and \(\bar{x}^n\) with \(x^n \neq \bar{x}^n\) and \(Syn(x^n) = Syn(\bar{x}^n)\), we have that the string \(Syn(x^n \oplus \bar{x}^n)\in \{0,1\}^{n-k}\) is the all zero string \(0^{n-k}\). From this it follows that \(x^n \oplus \bar{x}^n\) is a codeword different from \(0^n\). Since all codewords except \(0^n\) have weight at least \(d\), it follows that \(x^n\) and \(\bar{x}^n\) have distance at least \(d\).

The intuition behind the following proof is the observation that weak string erasures ensures that Bob knows the substring \(X_\ell\) of a string \(X\). The properties of the error-correcting code limit the set of strings \(X^n\) consistent with this substring and the given syndrome \(W\); this implies that Alice will be detected with high probability if she attempts to cheat.
Lemma 4.4 (Security for Bob). The pair of protocols \((\text{Commit}, \text{Open})\) is \(\varepsilon\)-binding.

Proof. Let \(\rho_{A,B}\) be the state shared by Alice and Bob after the execution of WSE. From the properties of WSE it follows that there exists a state \(\sigma_{A',X^n} = \sigma_{A,X^n} \otimes \tau_{2^n} \) such that \(\rho_{A,B} = \sigma_{A'(LX^n)}\), where \(B = (LX^n)\). Let \(\hat{X}^n\) be the closest string to \(X^n\) that satisfies \(\text{Syn}(\hat{X}^n) = W\), and let \(C^\ell := \text{Ext}(X^n, R)\). We will now show that the state \(\sigma_{C^\ell}(\text{RW}_{\text{WSE}})\) created during the Commit protocol satisfies the binding condition.

First of all, note that if Alice sends \(X^n = \hat{X}^n\), then Bob outputs \(\hat{C}^\ell = C^\ell\). It thus remains to analyze the case of \(X^n \neq \hat{X}^n\). Note that we may write

\[
\Pr[C^\ell \neq \hat{C}^\ell \text{ and } F = \text{accept}] = \Pr_{R,X^n,\hat{X}^n} \left[ \begin{array}{c}
\text{Ext}(\hat{X}^n, R) \neq \text{Ext}(X^n, R) \text{ and } F = \text{accept} \\
\text{Syn}(X^n) \neq \text{Syn}(\hat{X}^n)
\end{array} \right] + \Pr_{R,X^n,\hat{X}^n} \left[ \begin{array}{c}
\text{Ext}(\hat{X}^n, R) \neq \text{Ext}(X^n, R) \text{ and } F = \text{accept} \\
\text{Syn}(X^n) = \text{Syn}(\hat{X}^n)
\end{array} \right]
\]

where the last equality follows from the fact that Bob always rejects if \(\text{Syn}(X^n) \neq \text{Syn}(\hat{X}^n)\).

We now show that the remaining term is small. Note that if \(\text{Syn}(X^n) = \text{Syn}(\hat{X}^n)\), and \(X^n \neq \hat{X}^n\), the distance between \(X^n\) and \(\hat{X}^n\) is at least \(d\). We also know that for our choice of \(X^n\), the distance between \(X^n\) and \(\hat{X}^n\) is at most \(d/2\). Hence, \(X^n\) has distance at least \(d/2\) to \(\hat{X}^n\). Since Alice does not know \(I\) and every \(i \in [n]\) is in \(I\) with probability \(\frac{1}{2}\), Bob accepts with probability at most \(\varepsilon = 2^{-d/2}\). Hence, we obtain

\[
\Pr[C^\ell \neq \hat{C}^\ell \text{ and } F = \text{accept}] \leq \varepsilon ,
\]

as promised.

It remains to show that the protocol is correct. This follows essentially from the properties of weak string erasure. However, we still need to demonstrate that the state we obtain from weak string erasure has \(C^\ell\) close to uniform.

Lemma 4.5 (Correctness). The pair of protocols \((\text{Commit}, \text{Open})\) satisfies correctness with an error of at most \(3\varepsilon\).

Proof. Let \(\eta_{C^\ell,\hat{C}^\ell}\) be the state at the end of the protocol. It follows directly from the properties of WSE that \(\eta_{C^\ell,\hat{C}^\ell} = \eta_{C^\ell,C^\ell}\). It remains to show that this state is close to the ideal state \(\sigma_{C^\ell,\hat{C}^\ell}\). By the same arguments as in Lemma 4.3 it follows that \(\frac{1}{2}\|\eta_{C^\ell,\hat{C}^\ell} - \sigma_{C^\ell,\hat{C}^\ell}\|_1 \leq 3\varepsilon\). Hence, we also have \(\frac{1}{2}\|\eta_{C^\ell,\hat{C}^\ell} - \sigma_{C^\ell,C^\ell}\|_1 \leq 3\varepsilon\).

5 1-2 oblivious transfer from weak string erasure

5.1 Definition

We now show how to obtain 1-2 oblivious transfer given access to weak string erasure. Usually, one considers a non-randomized version of 1-2 oblivious transfer, in which Alice has two inputs \(Y^\ell_0, Y^\ell_1 \in \{0,1\}^\ell\), and Bob has as input a choice bit \(D \in \{0,1\}\). At the end of the protocol Bob receives \(Y^\ell_D\), and Alice receives no output. The protocol is considered secure if the parties do not gain any information beyond this specification, that is, Alice does not learn \(D\) and there exists some input \(Y^\ell_{1-D}\) about which Bob remains ignorant.

Here, we again make use of fully randomized oblivious transfer. Fully randomized oblivious transfer takes no inputs, and outputs two strings \(S^\ell_0, S^\ell_1 \in \{0,1\}^\ell\) to Alice, and a choice bit \(C \in \{0,1\}\) and \(S^\ell_C\) to Bob. Security means that if Alice is dishonest, she should not learn anything about \(C\). Similar to weak string
erasure, we also demand that two strings $S_0^\ell$ and $S_1^\ell$ are created by the protocol. Intuitively, this ensures that just like in a classical protocol, we can again think of the protocol as being completed once Alice and Bob have exchanged their final message. If Bob is dishonest, we demand that there exists some random variable $C$ such that Bob is entirely ignorant about $S_1^\ell - C$. That is, he may learn at most one of the two strings which are generated.

Fully randomized oblivious transfer can easily be converted into “standard” oblivious transfer as depicted in Figure 13 using the protocol presented in [6] (see also [2]). To obtain non-randomized 1-2 oblivious transfer, Bob sends Alice a message indicating whether $C = D$. Note that since Alice does not know $C$, she also does not know anything about $D$. If $C = D$, Alice sends Bob $Y_0^\ell \oplus S_0^\ell$, and $Y_1^\ell \oplus S_1^\ell$, otherwise she sends $Y_0^\ell \oplus S_1^\ell$ and $Y_1^\ell \oplus S_0^\ell$. Clearly, if Bob does not learn anything about $S_1^\ell - C$, he can learn at most one of $Y_0^\ell$ and $Y_1^\ell$.

![Figure 13: Fully randomized 1-2-oblivious transfer when Alice and Bob are honest. Intuitively, if one of the parties is dishonest, he/she should not be able to obtain more information from the primitive as depicted above. The dashed messages are exchanged to obtain non-randomized oblivious transfer from FROT.](image)

We now provide a more formal definition, which is very similar to the definitions in [13, 17].

**Definition 5.1.** An $(\ell, \varepsilon)$-fully randomized oblivious transfer (FROT) scheme is a protocol between Alice and Bob satisfying the following:

**Correctness:** If both parties are honest, then the ideal state $\sigma_{S_0^\ell S_1^\ell C S_C^\ell}$ is defined such that

1. The distribution over $S_0^\ell$, $S_1^\ell$ and $C$ is uniform:
   \[
   \sigma_{S_0^\ell S_1^\ell C} = \tau_{\{0,1\}^\ell} \otimes \tau_{\{0,1\}^\ell} \otimes \tau_{\{0,1\}}.
   \]

2. The real state $\rho_{S_0^\ell S_1^\ell CY}$ created during the protocol is $\varepsilon$-close to the ideal state:
   \[
   \rho_{S_0^\ell S_1^\ell CY} \approx_{\varepsilon} \sigma_{S_0^\ell S_1^\ell C S_C^\ell},
   \]
   where we identify $A = (S_0^\ell, S_1^\ell)$ and $B = (C, Y^\ell)$.

**Security for Alice:** If Alice is honest, then there exists an ideal state $\sigma_{S_0^\ell S_1^\ell B C}$, where $C$ is a random variable on $\{0,1\}$, such that

1. Bob is ignorant about $S_1^\ell - C$:
   \[
   \sigma_{S_1^\ell - C S_C^\ell B C} \approx_{\varepsilon} \tau_{\{0,1\}^\ell} \otimes \sigma_{S_C^\ell B C}.
   \]

2. The real state $\rho_{S_0^\ell S_1^\ell B}$ created during the protocol is $\varepsilon$-close to the ideal state:
   \[
   \rho_{S_0^\ell S_1^\ell B} \approx_{\varepsilon} \sigma_{S_0^\ell S_1^\ell B}.
   \]
Security for Bob: If Bob is honest, then there exists an ideal state \( \sigma_{A' \Sigma_1 S_1 C} \) such that

1. Alice is ignorant about \( C \):
   \[
   \sigma_{A' \Sigma_1 S_1 C} = \sigma_{A' S_1} \otimes \tau_{\{0,1\}} .
   \]
2. The real state \( \rho_{A'C} \) created during the protocol is \( \varepsilon \)-close to the ideal state:
   \[
   \rho_{A'C} \approx_{\varepsilon} \sigma_{A'CS_1} ,
   \]
   where we identify \( B = (C, Y^\ell) \).

Again, we allow the protocol implementing this primitive to abort, but demand that the security conditions are satisfied if the protocol does not abort.

5.2 Protocol

We now show how to obtain a fully randomized oblivious transfer given access to weak string erasure. To obtain some intuition for the actual protocol, consider the following naïve protocol, which we only state informally. It makes use of a 2-universal hash function \( \text{Ext} : \{0,1\}^{n/4} \times \mathcal{R} \rightarrow \{0,1\}^\ell \).

**Protocol 3’**: Naïve Protocol (informal)

Outputs: \( (s_0^l, s_1^l) \in \{0,1\}^{\ell} \times \{0,1\}^\ell \) to Alice, and \( (c, y^\ell) \in \{0,1\} \times \{0,1\}^\ell \) to Bob

1. **Alice and Bob**: Execute WSE. Alice gets a string \( x^n \in \{0,1\}^{n} \), Bob a set \( \mathcal{I} \subset [n] \) and a string \( s = x_\mathcal{I} \). If \( |\mathcal{I}| < n/4 \), Bob aborts. Otherwise, he randomly truncates \( \mathcal{I} \) to size \( n/4 \), and deletes the corresponding values in \( s \).

2. **Alice and Bob**: Execute interactive hashing with Bob’s input \( w \) equal to a description of \( \mathcal{I} = \text{Enc}(w) \). Interpret the outputs \( w_0 \) and \( w_1 \) as descriptions of subsets \( \mathcal{I}_0 \) and \( \mathcal{I}_1 \) of \( [n] \).

3. **Alice**: Chooses \( r_0, r_1 \in R \mathcal{R} \) and sends them to Bob.

4. **Alice**: Outputs \( (s_0^l, s_1^l) := (\text{Ext}(x_{\mathcal{I}_0}, r_0), \text{Ext}(x_{\mathcal{I}_1}, r_1)) \).

5. **Bob**: Computes \( c \in \{0,1\} \) with \( \mathcal{I} = \mathcal{I}_c \), and \( x_\mathcal{I} \) from \( s \). He outputs \( (c, y^\ell) := (c, \text{Ext}(s, r_c)) \).

For now, let us neglect the fact that the outputs of interactive hashing are strings, and assume that the subset \( \mathcal{I}_{1-c} \) generated by the interactive hashing protocol is uniformly distributed over subsets of size \( n/4 \) not equal to \( \mathcal{I} \). The string \( x_{\mathcal{I}_{1-c}} \) is then obtained by sampling from the string \( x^n \), which by the definition of weak string erasure has high min-entropy. We therefore expect the value \( s_{1-c}^l \) to be uniform and independent of Bob’s view. This should imply security for Alice, whereas security for Bob immediately follows from the properties of interactive hashing.

In this intuitive argument, we have ignored the fact that the sampling result only applies to blocks, and not individual bits. To make use of the sampling results we hence need to make slight modification to the simple protocol given above. We partition \( x^n \) (where \( n = \beta m \)) into \( m \) blocks of \( \beta \) bits each. It will be convenient to arrange the bits of \( x^n \) into a matrix \( z \in \mathbb{M}_{m \times \beta} \{0,1\} \), where \( z_{j,\alpha} := x_{(j-1) \beta + \alpha} \). Note, however, that we cannot simply sample from the rows of \( z \), since the subset \( \mathcal{I} \) of bits known to Bob does not correspond to the union of certain rows. We therefore allow Bob to permute the entries of \( z \) by picking a permutation \( \pi : [m] \times [\beta] \rightarrow [m] \times [\beta] \) such that he knows a subset \( \mathcal{J} \subset [m] \) of \( |\mathcal{J}| = m/4 \) rows of \( \pi(z) \). More formally,

\[
\exists \mathcal{J} \subset [m], |\mathcal{J}| = m/4 : \text{if } j' \in \mathcal{J}, \alpha' \in [m] \text{ then } (j-1) \cdot \beta + \alpha \in \mathcal{I} \text{ with } (j, \alpha) = \pi^{-1}(j', \alpha') . \tag{43}
\]
Bob announces the permutation $\pi$ to Alice, and both parties continue to use $\pi(z)$ instead of $z$. It turns out that picking $\pi$ at random subject to (43) ensures that Bob’s input $J$ to the interactive hashing protocol does not reveal any significant information about $c$ to Alice. This will be shown below.

To use interactive hashing in conjunction with subsets, the actual protocol needs an encoding of subsets $\text{Enc}: \{0,1\}^t \rightarrow T$, where $T$ is the set of all subsets of $[m]$ of size $m/4$ (we assume without loss of generality that $m$ is a multiple of 4). Here we choose $t$ such that $2^t \leq \binom{m}{m/4} \leq 2 \cdot 2^t$, and an injective encoding $\text{Enc}: \{0,1\}^t \rightarrow T$, i.e., no two strings are mapped to the same subset. Note that this means that not all possible subsets are encoded, but at least half of them. We refer to [16, 46] for details on how to obtain such an encoding.

Protocol 3: WSE-to-FROT

Parameters: Integers $n, \beta$ such that $m := n/\beta$ is a multiple of 4. Set $t := m/2$. Outputs: $(s^0_r, s^1_r) \in \{0,1\}^t \times \{0,1\}^t$ to Alice, and $(c,y^t) \in \{0,1\} \times \{0,1\}^t$ to Bob.

1: Alice and Bob: Execute $(n, \lambda, \epsilon)$-WSE. Alice gets a string $x^n \in \{0,1\}^n$, Bob a set $I \subset [n]$ and a string $s = x_I$. If $|I| < n/4$, Bob aborts. Otherwise, he randomly truncates $I$ to the size $n/4$, and deletes the corresponding values in $s$.

We arrange $x^n$ into a matrix $x \in M_{m \times \beta}([0,1])$, by $z_{j,\alpha} := x_{(j-1) \cdot \beta + \alpha}$ for $(j, \alpha) \in [m] \times [\beta]$.

2: Bob:

1. Randomly chooses a string $w^t \in \mathcal{R} \{0,1\}^t$ corresponding to an encoding of a subset $\text{Enc}(w^t)$ of $[m]$ with $m/4$ elements.
2. Randomly partitions the $n$ bits of $x^n$ into $m$ blocks of $\beta$ bits each: He randomly chooses a permutation $\pi: [m] \times [\beta] \rightarrow [m] \times [\beta]$ of the entries of $z$ as in Lemma 2.3, such that he knows $\pi(z_{\text{Enc}(w^t)})$ (that is, these bits are permutation of the bits of $s$). Formally, $\pi$ is uniform over all permutations satisfying the following condition: for all $(j, \alpha) \in [m] \times [\beta]$ and $(j', \alpha') := \pi(j, \alpha)$, we have $(j - 1) \cdot \beta + \alpha \in I \iff j' \in \text{Enc}(w^t)$.

3. Bob sends $\pi$ to Alice.

3: Alice and Bob: Execute interactive hashing with Bob’s input equal to $w^t$. They obtain $w^t_0, w^t_1 \in \{0,1\}^t$ with $w^t \in \{w^t_0, w^t_1\}$.

4: Alice: Chooses $r_0, r_1 \in \mathcal{R} \mathcal{R}$ and sends them to Bob.

5: Alice: Outputs $(s^0_r, s^1_r) := (\text{Ext}(\pi(z)_{\text{Enc}(w^t_0)}, r_0), \text{Ext}(\pi(z)_{\text{Enc}(w^t_1)}, r_1))$.

6: Bob: Computes $c$, where $w^t = w^t_c$, and $\pi(z)_{\text{Enc}(w^t)}$ from $s$. He outputs $(c, y^t) := (c, \text{Ext}(\pi(z)_{\text{Enc}(w^t)}, r_c))$.

Theorem 5.2 (Oblivious transfer). For any $\beta \geq \max\{67, 256/\lambda^2\}$, the protocol WSE-to-FROT implements an $(\ell, 41 \cdot 2^{-\min\{2\beta-\lambda^2\} n + 2\epsilon})$-FROT from one instance of of $(n, \lambda, \epsilon)$-WSE, where $\ell := \left\lceil (\frac{\lambda}{16} - \frac{\lambda^2}{8000}) n - \frac{1}{2} \right\rceil$.

5.3 Security proof

We first show that the protocol is secure against a cheating Alice. Intuitively, the properties of weak string erasure ensure that Alice does not know which bits $x_I$ of $x^n$ are known to Bob, that is, she is ignorant about the index set $I$. This implies that essentially any partition of the bits is consistent with Alice’s view. In particular, she does not gain much information from the particular partition chosen by Bob. Finally, the properties of interactive hashing ensure that she cannot gain much information about which of the two final strings is known to Bob.
Lemma 5.3 (Security for Bob). Protocol WSE-to-FROT satisfies security for Bob.

Proof. Let \( \hat{\rho}_{A''\cdot CY'} \) denote the joint state at the end of the protocol, where \( A'' \) is the quantum output of a malicious Alice and \( (C, Y^t) \) is the classical output of an honest Bob. We construct an ideal state \( \tilde{\sigma}_{A''\cdot W'\cdot CY'} = \tilde{\sigma}_{A''\cdot W'\cdot CY'}^\otimes \sigma_{\tau(0, 1)} \) that satisfies \( \hat{\rho}_{A''\cdot CY'} = \tilde{\sigma}_{A''\cdot W'\cdot CY'}^\otimes \).

First, we divide a malicious Alice into two parts. The first part interacts with Bob in the WSE protocol, after which the state shared by Alice and Bob is \( \rho_{A'X'\cdot I} \). From the properties of WSE it follows that there exists an ideal state \( \tilde{\sigma}_{A'\cdot X'\cdot I} \) such that the reduced state satisfies \( \rho_{A'X'\cdot I} = \tilde{\sigma}_{A'\cdot X'\cdot I} \).

The second part of Alice takes \( A' \) as input and interacts with Bob in the rest of the protocol. To analyze the resulting joint output state \( \hat{\rho}_{A''\cdot CY'} \), we can use the properties of weak string erasure, and let the second part of Alice interact with honest Bob starting from the state \( \tilde{\sigma}_{A'\cdot X'\cdot I} \). The protocol outputs a state \( \tilde{\sigma}_{A''\cdot X'\cdot CY'\cdot M} \), where \( M \) denotes all classical communication during the protocol. Note that the values \( II, W_0^t, W_1^t, R_0 \) and \( R_1 \) can be computed from \( M \). Let \( S_i^t := \text{Ext}(\Pi(\tilde{Z}_i), R_i) \) for \( i \in \{0, 1\} \), where for \( (j, a) \in [m] \times [\beta] \) we have \( \tilde{Z}_{j, a} := \tilde{X}_{(j-1)\cdot \beta + a} \). We obtain the state \( \tilde{\sigma}_{A''\cdot S_i^t\cdot CY'} \) by taking the partial trace of \( \tilde{\sigma}_{A''\cdot S_i^t\cdot CY'} \cdot M \). From the construction of this state and the fact that \( \rho_{A'X'\cdot I} = \sigma_{A'\cdot X'\cdot I} \) it follows directly that \( \hat{\rho}_{A''\cdot CY'} = \tilde{\sigma}_{A''\cdot CY'} \) and \( \tilde{\sigma}_{A''\cdot S_i^t\cdot CY'} = \tilde{\sigma}_{A''\cdot S_i^t\cdot CY'}^\otimes \). Hence
\[
\hat{\rho}_{A''\cdot CY'} = \tilde{\sigma}_{A''\cdot S_i^t\cdot CY'}^\otimes \cdot
\]

It remains to be shown that Alice does not learn anything about \( C \), that is, \( \tilde{\sigma}_{A''\cdot S_i^t\cdot CY'}^\otimes \cdot \tilde{\sigma}_{A''\cdot S_i^t\cdot CY'}^\otimes \cdot \tau_{\{0, 1\}} \). From the properties of WSE it follows that \( \sigma_{A''\cdot X'\cdot I} = \sigma_{A''\cdot X'\cdot I}^\otimes \). Since Bob randomly truncates \( I \) such that \( |I| = n/4 \), also the truncated set \( I \) is uniformly distributed over all subsets of size \( n/4 \) and independent of \( A' \). Hence, conditioned on any fixed \( W^t = w^t \), the permutation \( II \) is uniform and independent of \( A' \). It follows that the string \( W^t \) is also uniform and independent of \( A' \) and \( II \). From the properties of interactive hashing we are guaranteed that \( C \) is uniform and independent of Alice’s view afterwards, and hence,
\[
\tilde{\sigma}_{A''\cdot S_i^t\cdot CY'}^\otimes \cdot \tau_{\{0, 1\}} = \tilde{\sigma}_{A''\cdot S_i^t\cdot CY'}^\otimes \cdot \tau_{\{0, 1\}}.
\]

Second, we show that the protocol is secure against a cheating Bob. We again first give an intuitive argument. We have from weak string erasure that Bob gains only a limited amount of information about the string \( X^n \). The properties of interactive hashing ensure that Bob has very little control over one of the subsets of blocks chosen by the interactive hashing. Therefore, by the results on min-entropy sampling, Bob has only limited information about the bits of \( X^n \) in these blocks. Privacy amplification can then be used to turn this into almost complete ingnorance.

Lemma 5.4 (Security for Alice). Protocol WSE-to-FROT satisfies security for Alice with an error of
\[
41 \cdot 2^{-\frac{\lambda^2}{2\ln n}} + 2\varepsilon.
\]

Proof. Let \( \rho_{X^n \cdot B'} \) be the cq-state created by the execution of WSE. From the properties of WSE it follows that there exists a state \( \sigma_{X^n \cdot B'} \) such that \( H_{\infty}(X^n|B')_\rho \geq \lambda n \) and \( \rho_{X^n \cdot B'} \approx_{\varepsilon} \sigma_{X^n \cdot B'} \), which implies that
\[
H_{\infty}(Z|B')_\rho = H_{\infty}(X^n|B')_\rho \geq \lambda n.
\]

Recall that our goal is to show that Bob has high min-entropy about the string \( X^n \) restricted to one of the subsets of blocks generated by the interactive hashing protocol. Our first step is to count the subsets of blocks which are bad for Alice in the sense that Bob has a lot of information about \( X^n \) in such blocks. We then show that the probability that both sets chosen via the interactive hashing primitive lie in the bad set of blocks is exponentially small in \( n \).

32
First of all, note that the permutation \( \Pi \) is generated by Bob. In particular, if Bob is dishonest, \( \Pi \) may depend on his quantum information \( B' \). This corresponds to the situation studied in Lemma 2.5. With Lemma 2.5 we therefore conclude that for the uniform\(^1\) distribution over subsets \( S \subseteq [m] \) of size \( m/4 = |S| \)

\[
\Pr_{\mathcal{S}} \left[ H_{\infty}^{+45}(\Pi(\mathcal{Z})_S|\mathcal{S}B'') < \frac{\lambda n}{8} \right] \leq \delta^2 , \tag{44}
\]

where \( \delta = 2^{-m\lambda^2/800} \) and \( B'' \) is Bob’s part of the shared quantum state after he has sent \( \Pi \) to Alice. Let \( \text{Bad} \) be the set of all subsets of size \( m/4 \) that result in small min-entropy, i.e.,

\[
\text{Bad} := \left\{ S \subseteq [m] \mid |S| = \frac{m}{4} \text{ and } H_{\infty}^{+45}(\Pi(\mathcal{Z})_S|\mathcal{S}B'') < \frac{\lambda n}{8} \right\} .
\]

Since we have considered the uniform distribution over all subsets of size \( m/4 \), we conclude from (44) that

\[
|\{w^t \in \{0,1\}^t \mid \text{Enc}(w^t) \in \text{Bad}\}| \leq |\text{Bad}| \leq \left( \frac{m}{m/4} \right) \delta^2 \leq 2 \cdot 2^{t} \delta^2 . \tag{45}
\]

In the first inequality, we have used the fact that \( \text{Enc} \) is injective, i.e., every element in the image has exactly one preimage. In the last inequality, we used the fact that \( \left( \frac{m}{m/4} \right) \leq 2 \cdot 2^{t} \). By the third property of the interactive hashing, we conclude that

\[
\Pr \left[ \text{Enc}(W^t_0) \in \text{Bad} \text{ and } \text{Enc}(W^t_1) \in \text{Bad} \right] \leq 16 \cdot \frac{2^t \delta^2}{2^t} \leq 32 \delta^2 . \tag{46}
\]

Let \( \tilde{\rho}_{ZW'_0W'_1\Pi B''} \) be the shared quantum state after the interactive hashing, where \( B'' \) is Bob’s part of that state. From (16) it follows that there exists a \( C \subseteq \{0,1\} \), or more precisely, there exists an ideal state \( \tilde{\sigma}_{ZW'_0W'_1\Pi B''} = \tilde{\sigma}_{ZW'_0W'_1\Pi B''}, \) such that

\[
\Pr_{\tilde{\sigma}} \left[ H_{\infty}^{+45}(\Pi(\mathcal{Z})_{\text{Enc}(W^t_1 \cdot C)}|W^t_0W^t_1\Pi B'')_{\tilde{\sigma}} \geq \lambda n \right] \geq 1 - 32 \delta^2 . \tag{47}
\]

Note that Bob may use his quantum state during the interactive hashing, but he cannot increase the probability of (46) this way. Furthermore, any processing may only increase his uncertainty. Let \( \mathcal{A} \) be the event that the inequality in the argument on the l.h.s. of (47) holds. Let

\[
\tilde{\sigma}_{ZW'_0W'_1\Pi B''}^{CR_0R_1} := \tilde{\sigma}_{ZW'_0W'_1\Pi B''} \otimes \tau_R \otimes \tau_R
\]

and let \( S^t_0 \) and \( S^t_1 \) be calculated as stated in the protocol. Using the chain rule (see (16)) and the fact that \( (R_0, R_1) \) are independent, we get

\[
H_{\infty}^{+45}(\Pi(\mathcal{Z})_{\text{Enc}(W^t_1 \cdot C)}|S^t_0C^t_0CR_0R_1W^t_0W^t_1\Pi B'')_{\tilde{\sigma}} \geq \frac{\lambda n}{8} - \ell - 1 .
\]

Using privacy amplification (Theorem 2.3), we then have conditioned on the event \( \mathcal{A} \) that

\[
\frac{1}{2} \left\| \tilde{\sigma}_{S^t_1 \cdot C, S^t_0C^t_0CR_0R_1W^t_0W^t_1\Pi B''} - \tau_{\{0,1\}}^t \otimes \tilde{\sigma}_{S^t_0C^t_0CR_0R_1W^t_0W^t_1\Pi B''} \right\|_1 \leq \delta + 2\varepsilon + 8\delta , \tag{48}
\]

since

\[
\frac{n\lambda}{8} - 2\ell - 1 \geq 2 \log 1/\delta = 2 \cdot \frac{\lambda^2 m}{800} ,
\]

\(^6\)Note that, in the protocol, we do not actually sample from the uniform distribution over subsets; the bound (44) is merely used in a counting argument here to establish that the number of “bad” subsets is limited, cf. (15) below.
which follows from

\[ \ell \leq \frac{\lambda n}{16} - \frac{\lambda^2 m}{800} - \frac{1}{2}. \]

Let \( B^* := (R_0 R_1 B''') \) be Bob's part in the output state. Since \( \Pr[A] \geq 1 - 32\delta^2 \), we get

\[ \tilde{\sigma}_{S_1 C S_0 B^*} C \approx 32\delta^2 + 95 + 2\epsilon \tau_{(0,1)} \otimes \tilde{\sigma}_{S_0} B^* C \]

and

\[ \tilde{\sigma}_{S_0 S_1 B^*} = \tilde{\rho}_{S_0 S_1 B^*}. \]

Since \( \delta^2 \leq \delta \), this implies the security condition for Alice, with a total error of at most \( 41\delta + 2\epsilon \).

Finally, we show that the protocol is correct when Alice and Bob are both honest.

**Lemma 5.5 (Correctness).** Protocol WSE-to-FROT satisfies correctness with an error of

\[ 3 \cdot 2^{-\frac{3\epsilon n}{\nu}} \leq 3 \cdot 2^{-\frac{3\lambda^2 m}{800}}. \]

**Proof.** Let \( \xi := 2^{-n/16} \). We have to show that the state \( \tilde{\rho}_{S_0 S_1 C Y} \) at the end of the protocol is \( 3\xi \)-close to the given ideal state \( \tilde{\sigma}_{S_0 S_1 C Y} \). Using the Hoeffding bound [51], the probability that a random subset of \([n]\) has less than \( n/4 \) elements is at most \( \exp(-n/8) \leq \xi \). Hence, the probability that the protocol aborts when both parties are honest is at most \( \xi \). Let \( A \) be the event that the protocol does not abort. It remains to show that the state \( \tilde{\rho}_{S_0 S_1 C Y} |A \) is \( 2\xi \)-close to the given ideal state \( \sigma_{S_0 S_1 C Y} |A \). Note that the correctness condition of WSE ensures that the state created by WSE is equal to \( \rho_{X^m I X^I} = \sigma_{X^m I X^I} \), where \( \sigma_{X^m I} = \tau_{(0,1)} \otimes \tau_{2^n} \).

Since \( I_0 \) and \( I_1 \) are chosen independently of \( X^n \), \( X_{I_0} \), and \( X_{I_1} \) are independent and have a min-entropy of \( n/4 \) each. Since \( \ell \leq n/16 \leq n/4 - 2 \log 1/\xi \), it follows from Theorem 2.3 that \( S_0^\ell \) and \( S_1^\ell \) are independent and each of them is \( \xi \)-close to uniform. Furthermore, by the same arguments as in Lemma 5.3 we have that \( C \) is uniform and independent of \( S_0^\ell \) and \( S_1^\ell \). Hence,

\[ \rho_{S_0^\ell S_1^\ell |A} \approx 2\xi \sigma_{S_0 S_1 |C}. \]

Since the extra condition on the permutation \( \Pi \) implies that Bob can indeed calculate \( \Pi(Z)_{\text{Enc}(W)} \) from \( X_I \), we have that \( Y^\ell = S_0^\ell \). Using \( \Pr[A] \geq 1 - \xi \), we get

\[ \rho_{S_0^\ell S_1^\ell |C} \approx 3\xi \sigma_{S_0 S_1 |C S_0^\ell}. \]

Finally, \( \lambda \leq 1 \) and \( \beta > 1 \) give us \( 1/16 > \lambda^2/(800\beta) \) from which the claim follows.

6 Conclusions and open problems

We have shown that secure bit commitment and oblivious transfer can be obtained with unconditional security in the noisy-storage model. We have connected the security of our protocols to the information-carrying capacity of the noisy channel describing the malicious party’s storage. We found a natural tradeoff between the (classical) capacity of the storage channel and the rates at which oblivious transfer and bit commitment can be performed: higher noise levels lead to stronger security.

While the connection between capacities of channels and security turns out to be directly applicable to a number of settings of practical interest, our work raises several immediate questions concerning the exact requirements for security in the noisy-storage model.

**Extending security:** Clearly, it is desirable to extend the security guarantee to a wider range of noisy channels. The limiting factor in obtaining security from a noisy storage described by \( \mathcal{F} = N^{\otimes m/n} \) was the fact that we require the sufficiency condition \( C_N \cdot \nu < 1/2 \) to hold (see Corollary 1.2), where \( \nu \) is the storage rate and \( C_N \) is the classical capacity of \( N \). The constant 1/2 is a result of using BB84-states, and stems
from a corresponding uncertainty relation using post-measurement information [35]. It is a natural question whether other encodings offer us an advantage. For example, if we were to use 3 mutually unbiased bases instead of only the computational and Hadamard basis, the corresponding uncertainty relation gives a value of $2/3$ [45, 56]. Of course, using more than 2 encodings will also require some modification of our subsequent protocols, and a careful analysis of the resulting security regime (see Figure 1) is needed.

For channels with small classical capacity, our work reduces security to proving a strong converse for coding. Of considerable practical interest are continuous-variable channels: our results are also applicable in this case, given a suitable bound on the information-carrying capacity.

A more challenging question is to extend security to entirely different classes of channels than considered here. Our results are currently restricted to channels without memory. Possibly the most important class of channels to which our results do not apply are those with high classical capacity. This includes for example the dephasing channel whose classical capacity is 1. Security tradeoffs for such a channel are known [23] for the case of individual storage attacks [57]. For the fully general case considered here, it is not a priori clear whether small classical capacity is a necessary condition for security: Our security proof overestimates the capabilities of the malicious party by expressing his power purely by his ability to preserve classical information. Completely different techniques may be required to address this question.

Another way to extend our security analysis is to combine our protocols with computationally secure protocols to achieve security if the adversary either has noisy quantum storage or is computationally bounded. This can be achieved by using combiners (see [21, 20, 40]). For oblivious transfer, the same can be achieved using the techniques of [6, 59, 10, 12], which only requires the use of a computationally secure bit commitment scheme.

Limits for security: We have found sufficient conditions for security in the noisy-storage model. For concrete channels, these conditions give regions in the plane parametrized by the storage rate and the noise level (cf. Figure 1) where security is achievable. Establishing outer bounds on the achievability region is an interesting open problem. Corresponding necessary conditions could become practically relevant as technology advances.

Note that when the adversarial player is restricted to individual storage attacks, the optimal attacks are known [48]. It is an open problem whether the fully general coherent attacks considered here actually reduce the achievability region. In contrast, both kinds of attacks are known to be equivalent in QKD [43].

Our work is merely a proof of principle. For practical realizations of our protocols, the following issues need to be addressed:

Efficiency: One can reduce the amount of classical computation and communication needed to execute our protocols by using techniques from derandomization. In particular, we could use the constant-round interactive hashing protocol and the efficient encoding of subsets from [16], randomness-efficient samplers (see e.g., [18]), and extractors (see e.g. [49, 30, 50]) instead of two-universal hash functions.

In practice, both the security parameter $\varepsilon$ and the number $\ell$ of bits in the commitment or oblivious transfer are fixed constants. Savings in communication may then be obtained by using alternative uncertainty relations (i.e., generalizations of [19], which is tight for $\varepsilon = 0$).

Composability: We have shown security of oblivious transfer and bit commitment with respect to security definitions that are motivated by composability considerations: This should ensure that the protocols remain secure even when executed many times e.g., sequentially. It is, however, an open problem to show formal composability in our model as has been done in the setting of bounded-storage [57, 17]. To this end, a nice composability framework for our setting needs to be established.

Robustness: We have considered an idealized setting where the operations of the honest parties are error-free. In particular, the communication channel connecting Alice and Bob was assumed to be noiseless. In real applications, both the BB84-state preparation by (honest) Alice, the communication, and the measurement of (honest) Bob will be affected by noise. To guarantee security even in such a setting, we can apply the error-correction techniques of [48]. However, it remains to determine the exact tradeoff between the amount of tolerable noise of the communication channel (parametrized e.g., by the bit error rate) and the amount of noise in the malicious player’s storage device [14].
We conclude with a few speculative remarks on potential applications of our work. Note that, in contrast to key distribution, general two-party computation is also interesting at short (physical) distances. An example is the problem of secure identification [15], where Alice wants to identify herself to Bob (possibly an ATM machine) without ever giving her password away. Our approach could be extended to realize this primitive in a similar way as in [48]. It would be interesting to find a new and more efficient protocol based directly on weak string erasure. The setting of secure identification is especially suitable for our model, since the short distance between Alice and Bob implies that their communication channel is essentially error-free. At such short range, we could also use visible light for which much better detectors exist than are presently used in quantum key distribution. Note that Alice only needs to carry a device capable of generating BB84-states and allowing her to enter her password on a keypad. This device does not need to store any information about Alice herself and hence each user could carry an identical device which is completely exchangeable among different (trusted) users at any time. In particular, this means means that Alice’s password is not compromised even if the device is lost. Finally, note that Alice’s technological requirements are minimal: She only needs a device capable of generating BB84-states. This could potentially be small enough to be carried on a key chain.

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A Proofs for min-entropy sampling

A.1 The parameters for sampling – proof of Lemma 2.4

For the proof of Lemma 2.4, we first recall the definition of a sampler:

**Definition A.1.** An \((m, \xi, \gamma)\)-averaging sampler is a probability distribution over subsets \(S \subset [m]\) with the property that for all \((\mu_1, \ldots, \mu_m) \in [0, 1]^m\) we have

\[
\Pr_S \left[ \frac{1}{|S|} \sum_{i \in S} \mu_i \leq \frac{1}{m} \sum_{i=1}^m \mu_i - \xi \right] \leq \gamma.
\]

Choosing subsets of a fixed size at random is a prime example of a sampler; this is the sampler we will use. The parameters of this sampler are as follows

**Lemma A.2.** Let \(s < m\) and let \(P_S\) be the uniform distribution over subsets \(S \subset [m]\) of size \(|S| = s\). Then \(P_S\) is an \((m, \xi, 2^{-s\xi^2/2})\)-sampler for every \(s > 0\) and \(\xi \in [0, 1]\).

**Proof.** Fix \(s > 0\) and \(\xi \in [0, 1]\). In [27, Lemma 2.2], \(P_S\) was shown to be a \((m, \xi, e^{-s\xi^2/2})\)-sampler. The claim then follows from the fact that \(e^{-s\xi^2/2} \leq 2^{-s\xi^2/2}\). \(\square\)

Replacing \(h_{\min}\) by \(H_\infty\), the following lemma follows directly from [27, Lemma 6.15 and Lemma 6.20].

**Lemma A.3.** Let \(\rho_{Z^m Q}\) be a cq-state, where \(Z^m = (Z_1, \ldots, Z_m)\) with \(Z_i \in \{0, 1\}^\beta\), and let \(P_S\) be an \((m, \xi, \gamma)\)-averaging sampler supported on subsets \(S\) of size \(s = |S|\). Then

\[
\Pr \left[ \frac{H_{\infty}^{1/4} (Z_S | SQ)}{s^\beta} \geq \frac{H_\infty (Z^m | Q)}{m^\beta} - c \right] \geq 1 - \sqrt{\gamma}
\]

where

\[
c = \frac{\log 1/\gamma}{2m^\beta} + \xi + 2\kappa \log 1/\kappa
\]

and \(\kappa = \frac{m}{s^\beta} \leq 0.15\).

We are now ready to give the

**Proof of Lemma 2.4**. Because of the definition of smooth min-entropy and the fact that partial traces do not increase distance, it suffices to establish the claim for \(\varepsilon = 0\). By Lemma A.2 and Lemma A.3, we have

\[
\Pr \left[ \frac{H_{\infty}^{1-2s^{-2\xi^2/2}} (Z_S | SQ)}{s^\beta} \geq \frac{H_\infty (Z^m | Q)}{m^\beta} - c \right] \geq 1 - 2^{-s\xi^2/4}
\]

where

\[
c = \frac{s^2}{4m^\beta} + \xi + 2\sqrt{\kappa} \leq \frac{5}{4} \xi + 2\sqrt{\kappa}
\]

if \(\kappa = \frac{m}{s^\beta} \leq 0.06\). Here we used the inequalities

\[
\kappa \log 1/\kappa \leq \sqrt{\kappa} \quad \text{for } \kappa \leq 0.06 ,
\]

\[
\beta \geq 1 , \quad s \leq m \quad \text{and} \quad \xi \leq 1 .
\]

Note that the condition \(\kappa \leq 0.06\) is satisfied if \(s \geq m/4\) and \(\beta \geq 67\). Setting \(\xi := \lambda/5\) and using \(s \geq m/4\) again, we get

\[
c \leq \frac{\lambda}{4} + 2\sqrt{\frac{4}{\beta}} \leq \frac{\lambda}{2}
\]
if $256/\lambda^2 \leq \beta$. In particular, this implies that

$$\frac{H_\infty(Z^m|Q)}{m\beta} - c \geq \frac{\lambda}{2}. \tag{50}$$

Combining (49), (50) with $s\xi^2 = s\lambda^2/25 \geq m\lambda^2/100$ and $\delta = 2^{-m\lambda^2/800}$ gives the claim. \hfill \Box

### A.2 Generalized min-entropy sampling – proof of Lemma 2.5

**Proof.** By definition of the smooth min-entropy, there exists a (possibly subnormalized) cq-state $\bar{\rho}_{ZQ}$ with $Z \in M_{m\times\beta}(\{0,1\})$ such that

$$\frac{1}{2}\|\bar{\rho}_{ZQ} - \rho_{ZQ}\|_1 \leq \varepsilon$$

and

$$H_\infty(Z|Q)_{\bar{\rho}} \geq \lambda. \tag{51}$$

By the monotonicity of the trace distance under CPTPMs, we therefore have

$$\frac{1}{2}\|\bar{\rho}_{ZQ'|\Pi} - \rho_{ZQ'|\Pi}\|_1 \leq \varepsilon \tag{52}$$

for the state $\bar{\rho}_{ZQ'|\Pi} := (I_Z \otimes \mathcal{E})(\bar{\rho}_{ZQ})$, and by the fact that CPTPMs can only increase min-entropy (cf. (23)),

$$H_\infty(Z|Q'\Pi)_{\bar{\rho}} \geq H_\infty(Z|Q)_{\bar{\rho}}. \tag{53}$$

With (33) (applied to the conditional states $\bar{\rho}_{ZQ'|\Pi=\pi}$), it is easy to see that

$$H_\infty(\Pi(Z)|Q'\Pi)_{\bar{\rho}} = H_\infty(Z|Q'\Pi)_{\bar{\rho}}. \tag{54}$$

Combining (51), (53) and (54) gives

$$H_\infty(\Pi(Z)|Q'\Pi)_{\bar{\rho}} \geq \lambda.$$

Applying Lemma 2.4 to $\bar{\Pi}(Z|Q'\Pi)$ (with $Q = Q'\Pi$, $\Pi(Z)$ instead of $Z$ and $\varepsilon = 0$) therefore leads to

$$\Pr_{\bar{S}}\left[\frac{H_\infty(\Pi(Z)_{\bar{S}}|Q'\Pi)_{\bar{\rho}}}{s\beta} \geq \frac{\lambda}{2}\right] \geq 1 - \delta^2 \quad \text{where } \delta = 2^{-m\lambda^2/800}.
$$

The claim follows from the monotonicity of the trace distance under CPTPMs and (52), since these imply

$$\frac{1}{2}\|\bar{\Pi}(Z)_{\bar{S}Q'\Pi} - \rho_{\Pi(Z)_{\bar{S}Q'\Pi}}\|_1 \leq \varepsilon$$

for every subset $S \subset [m]$. \hfill \Box