THE CAUCHY PROBLEM
FOR PROPERLY HYPERBOLIC EQUATIONS
IN ONE SPACE VARIABLE

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Dedicated to our friend Enrico Jannelli

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1. Introduction
In this paper we deal with the non characteristic Cauchy Problem
\begin{align}
\mathcal{L} u &= f(t,x), \quad t \in \mathbb{R}^+, \ x \in \mathbb{R}, \\
\partial_t^j u(0,x) &= \varphi_j(x), \quad 0 \leq j \leq m-1, \quad x \in \mathbb{R},
\end{align}
where \( \mathcal{L} \) is an operator of order \( m \geq 2 \) in \( \mathbb{R}_t \times \mathbb{R}_x \), with smooth coefficients, of the form
\begin{equation}
\mathcal{L} u \equiv \mathcal{L}(t,x; \partial_t, \partial_x) u = P(x; \partial_t, \partial_x) u - \sum_{d=0}^{m-1} R_d(t,x; \partial_t, \partial_x) u.
\end{equation}

Hypothesis A. The principal symbol
\begin{equation}
P(x; \tau, \xi) \equiv \tau^m + a_1(x) \tau^{m-1} \xi + \cdots + a_m(x) \xi^m
\end{equation}
depends only on the spatial variable \( x \), and is weakly hyperbolic, i.e.,
\begin{equation}
P(x; \tau, \xi) = \prod_{j=1}^{m} (\tau - \tau_j(x) \xi)
\end{equation}
where the characteristic roots \( \tau_j(x) \) are real functions, not necessarily distinct.

The lower order terms
\begin{equation}
R_d(t,x; \tau, \xi) \equiv \sum_{k=0}^{d} r_{d,k}(t,x) \tau^{d-k} \xi^k, \quad d = 0, 1, \ldots, m-1,
\end{equation}
are homogeneous polynomials in \( (\tau, \xi) \) of degree \( d \), depending on both variables.

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Notations. Throughout this paper we shall write shortly:

\begin{equation}
(1.7) \quad P(x; \tau) \equiv P(x; \tau, 1), \quad R_d(t, x; \tau) \equiv R_d(t, x; \tau, 1)
\end{equation}

Let $S_0$ be the initial line $\{ t = 0 \}$. We say that the Cauchy Problem \((1.1)-(1.2)\) is \textit{locally well-posed} (in $C^\infty$) at a point $y_0 \in S_0$ if, for every neighborhood $\mathcal{I}$ of $y_0$ in $\mathbb{R}^+ \times \mathbb{R}$, there exists a neighborhood $\mathcal{I}' \subseteq \mathcal{I}$, with $\mathcal{I}' \cap S_0 = \mathcal{I} \cap S_0$, such that for all $\varphi_j \in C^\infty(\mathcal{I} \cap S_0)$ and $f \in C^\infty(\mathcal{I})$ there is a unique solution $u \in C^\infty(\mathcal{I}')$. When, for each strip $\mathcal{I} = [0, T] \times \mathbb{R}$, we can take $\mathcal{I}' = \mathcal{I}$, we say that the Problem is \textit{globally well-posed}.

In the \textit{strictly hyperbolic} case, i.e., when the characteristic roots are distinct, the Cauchy Problem \((1.1)-(1.2)\) is well-posed with no further assumption on the lower order terms. On the other side, in case of multiple characteristic roots some additional conditions are needed, both on the principal symbol and on the lower order terms.

In \[ST1\] we considered the case of an \textit{homogeneous operator}, that is when $R_d \equiv 0$ for all $d$, and we proved the well-posedness for \((1.1)-(1.2)\) under the following condition:

**Hypothesis B.** There exists a constant $M > 0$ such that

\begin{equation}
(1.8) \quad \tau_2^2(x) + \tau_k^2(x) \leq M \left( \tau_j(x) - \tau_k(x) \right)^2, \quad 1 \leq j < k \leq m.
\end{equation}

Note that, by Newton's Theorem on symmetric polynomials, \((1.8)\) can be explicitly expressed in terms of the coefficients $a_j(x)$ of the principal symbol (see \[ST1\] Remark 1.2 for some examples).

In order to treat the case of non homogeneous operators, some conditions on the lower order terms are needed, even if the coefficients are constant. Indeed, in the constant coefficients case the following condition on the full symbols $L(\tau, \xi)$ is \textit{necessary and sufficient} for the Problem \((1.1)-(1.2)\) to be well-posed in $C^\infty$ (see \[G\] \[H\] \[Sv\]):

\begin{equation}
(1.9) \quad \text{there exists } C > 0 \text{ such that } L(\tau, \xi) \neq 0 \text{ if } \xi \in \mathbb{R} \text{ and } |\text{Im } \tau| > C.
\end{equation}

This is the classical Gårding condition. Various conditions equivalent to \((1.9)\) have been found. We consider here the formulation given by Peyser \[P1\]. In order to explain this condition, we give the following definition:

**Definition 1.** If $P(x; \tau) \equiv (\tau - \tau_1(x)) \cdots (\tau - \tau_m(x))$ is a hyperbolic polynomial, we say that a polynomial $R(x; \tau)$ of degree $\leq m - 1$ has a \textit{proper decomposition} w.r.t. $P(x; \tau)$ if there exist some functions $\ell_k \in L^\infty(\mathbb{R})$ such that

\begin{equation}
(1.10) \quad R(x; \tau) = \sum_{k=1}^{m} \ell_k(x) P_k(x; \tau), \quad \text{for all } x \in \mathbb{R},
\end{equation}

where

\begin{equation}
(1.11) \quad P_k(x; \tau) \equiv \prod_{j=1, \ldots, m}^{k} (\tau - \tau_j(x)), \quad k = 1, \ldots, m.
\end{equation}

The polynomials $P_k$ are called the \textit{reduced} (or \textit{incomplete}) polynomials of $P$.

**Example 1.** The polynomials $\partial_x P(x; \tau)$ and $\partial_x P(x; \tau)$ have a proper decomposition w.r.t. $P(x; \tau)$. The first assertion is obvious, indeed

\begin{equation}
(1.12) \quad \partial_x P(x; \tau) = \sum_{h=1}^{m} P_h(x; \tau).
\end{equation}
On the other side, recalling the Lemma of Bronšteĭn on Lipschitz continuity of the characteristic roots of an hyperbolic polynomial ([B]; cf. [W2] [M] [Tar]), we find

\[
\partial_x P(x; \tau) = -\sum_{h=1}^{m} \tau'_h(x) P_h(x; \tau),
\]

so that \( \partial_x P \) has a proper decomposition w.r.t. \( P \).

**Remark 1.1.** If, for some fixed \( x \in \mathbb{R} \), the polynomial \( P(x; \tau) \) has simple (real) roots, the \( \ell_k \)s in (1.10) are uniquely determined by Newton-Lagrange formula:

\[
\ell_j(x) = \frac{R(x; \tau_j(x))}{P_k(x; \tau_j(x))}.
\]

In this paper, recalling (1.7), we assume:

**Hypothesis C.** For each \( d = 1, \ldots, m-1 \), the polynomials \( R_d(t, x; \tau) \) have a proper decomposition w.r.t. \( \partial_{\tau}^{m-1-d} P(x; \tau) \).

This means that \( R_d(t, x; \tau) \) is a linear combination of the reduced polynomials of the polynomial \( \partial_{\tau}^{m-1-d} P(x; \tau) \), with coefficients uniformly bounded with respect to \( t, x \).

Following Peyser [P1] we call *properly hyperbolic* the operators verifying Hypothesis C.

In the constant coefficients case, Peyser [P1] derived suitable energy estimates ensuring the \( C^\infty \) well-posedness for the operators that verify Hypothesis C. Dunn [D] extended the result of Peyser proving that Hypothesis C is sufficient to ensure the well-posedness also if the lower order terms have variable coefficients. On the other side Wakabayashi [W1] proved that the proper hyperbolicity is also necessary.

All of the previously mentioned results concern the case of operators with constant coefficient principal part. In this paper we consider operators with variable coefficients principal part satisfying Hypothesis B and we prove:

**Theorem 1.** Let \( a_k \in C^\infty(\mathbb{R}) \) for \( 1 \leq k \leq m \), \( r_{d,k} \in L^\infty([0, T] \times \mathbb{R}) \) for \( 0 \leq k \leq d \leq m-1 \). Then, if Hypothesis [A] [B] and [C] are satisfied, the Cauchy Problem (1.1) - (1.2) is locally well-posed. If in addition the \( a_k(x) \)'s are bounded on \( \mathbb{R} \), we have the global well-posedness.

**Remark 1.2.** An analogous result holds true for operators with principal symbol depending only on the time variable. Condition (1.8) was considered for the first time in [CO], where the \( C^\infty \) well-posedness was proved for homogeneous equations with analytic-type coefficients depending only on time. The result was then extended in [KS] to operators with non-analytic coefficients and in [Tag] and [GR] to non-homogeneous operators.

**Remark 1.3.** Our result is restricted to operators in one space dimension, since Hypothesis [B] is no longer sufficient for \( C^\infty \) well-posedness in the higher-dimensional case. Indeed, in [BB] they prove that the Cauchy problem for the equation

\[
\partial_t^2 u + a_1(x, y, \partial_x, \partial_y) \partial_t u + a_2(x, y, \partial_x, \partial_y) u = 0,
\]

where

\[
a_1(x, y; \xi, \eta) = -2x\eta, \quad a_2(x, y; \xi, \eta) = -\xi^2 + x^3\eta^2,
\]
is not locally well-posed at the origin, although this equation fulfills the analogue of Hypothesis $[3]$ near $(x,y) = (0,0)$, viz.

$$a_t^2(x,y; \xi, \eta) \leq C \left\{ (a_t^2(x,y; \xi, \eta) - 4a_2(x,y; \xi, \eta) \right\}, \quad \text{for all } (x,y; \xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2.$$  

**Sketch of the proof of Theorem 1**

Firstly, we transform equation (1.1) into a first order system. Then, resorting to Jannelli’s symmetrizer $[J1, J2]$ and to the Hypotheses B and C, we get an apriori energy estimate. Finally, using a form of the Lemma of Nuij $[N]$, we approximate a sequence of strictly hyperbolic equations, thus obtaining a sequence of approximating solutions which converge to a solution of (1.1).

To make the proof more understandable, in the next section we shall briefly recall the homogeneous case, i.e., that of equation (1.3) with $R_d \equiv 0$ for all $d$. This case was already treated in $[ST1]$. Then, in §§3 and §4, we shall treat the general case.

**Remark 1.4.** The (standard) symmetrizer is a useful tool to study homogeneous operators $[ST1, JT1]$ or operators in which the behaviour is essentially determined by the principal symbol $[N2, NP1, NP2]$. A more powerful tool to treat operators with non-smooth coefficients $[JT2]$, Levi conditions $[Tag, GR]$, or non-linear problems $[Sp]$, $[ST2]$, is given by the so-called quasi-symmetrizer, introduced in $[DAS]$ and extensively studied in $[J2]$ (see also $[Tag]$ and $[N1]$).

The quasi-symmetrizer is a perturbation of the standard symmetrizer:

$$Q_\varepsilon = Q_0 + \varepsilon^2 Q_1 + \cdots + \varepsilon^{2(m-1)} Q_{m-1}$$

where $Q_0$ is the standard symmetrizer of $P$, $Q_1$ is the sum of the symmetrizers of the reduced polynomials $P$ (cf. (1.11)), $Q_2$ is the sum of the symmetrizers of the bi-reduced polynomials $P$ (see (5.7) below) and so on.

In this paper we use a different approach to modify the symmetrizer: we construct a symmetrizer

$$Q = Q_0 \oplus Q^{(1)} \oplus \cdots \oplus Q^{(m-1)}.$$ 

where $Q_0$ is still the standard symmetrizer of $P$, whereas $Q^{(1)}$ is the standard symmetrizer of $\partial_t P$, $Q^{(2)}$ is the standard symmetrizer of $\partial_x^2 P$, and so on.

**2. THE HOMOGENEOUS CASE AND THE JANNELLI’S SYMMETRIZER**

In this section we assume $R_d \equiv 0$ for all $d = 0, \ldots, m-1$. Thus, equation (1.1) becomes

$$(2.1) \quad P(x; \partial_t, \partial_x) u \equiv \partial_t^m u + a_1(x) \partial_t^{m-1} \partial_x u + \cdots + a_m(x) \partial_x^m u = f(t,x).$$

Now we introduce the $m$-vectors:

$$(2.2) \quad u(t,x) := \begin{pmatrix} \partial_t^{m-1} u \\ \partial_t \partial_x^{m-2} u \\ \vdots \\ \partial_x^{m-1} u \end{pmatrix}, \quad f(t,x) := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f \end{pmatrix},$$

and the $m \times m$ *Sylvester matrix* associated to the polynomial $P(x, \tau)$:

$$(2.3) \quad A(x) \equiv \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \\ -a_m(x) & -a_{m-1}(x) & \cdots & -a_1(x) \end{pmatrix}$$
Clearly, the equation (2.1) is equivalent to the system
\[ \mathbf{u}_t = A(x) \mathbf{u}_x + f(t, x). \]

To get an apriori estimate for the solutions to this system, we resort to the energy function based on the Jannelli's symmetrizer [J1] [J2]. We recall that a symmetrizer of a $m \times m$ matrix $A$ is a $m \times m$ matrix $Q$ such that
\[ Q^T = Q \geq 0, \quad (QA)^T = QA. \]

Correspondingly to a given symmetrizer, we define the energy of a solution of (2.1) as:
\[ E(t, \mathbf{u}) := \frac{1}{2} \int_{I_t} (Q(x) \mathbf{u}(t, x), \mathbf{u}(t, x)) \, dx, \]
where $I_t$ is a suitable real interval, and we look for an apriori estimate.

In the case that the matrix $A$ (hence also $Q$) is constant, and $I_t = \mathbb{R}$, by differentiating (2.1) in $t$ we find
\[ E'(t, \mathbf{u}) = \frac{1}{2} \int_{\mathbb{R}} \{ (QA u_x, u) + (Qu, Au_x) \} \, dx = \int_{\mathbb{R}} \partial_x (QA u, u) \, dx = 0 \]
for any solution $\mathbf{u}(t, \cdot)$ with compact support in $\mathbb{R}$. Thus, $E(t, \mathbf{u})$ keeps constant.

In the general case, in order to get an estimate of $E'(t, \mathbf{u})$ in term of $E(t, \mathbf{u})$, we have to estimate $QA$ and $\partial_x (QA)$ in terms of $Q$. Moreover, in order to derive an estimate of $\mathbf{u}$, we need some coercivity of the symmetrizer $Q(x)$. The Jannelli’s symmetrizer, under Hypothesis 3 enjoys of all these properties (Proposition 2.2 below).

We now recall the construction of such a symmetrizer (cf. [J1], [J2]): First of all we introduce a notation.

**Notations.** Given a polynomial $p(\tau) = c_0 + c_1 \tau + \cdots + c_d \tau^d$ of degree $d$, we define the row vectors:
\[ \text{vec}(p) := (c_d, \ldots, c_1, c_0) \in \mathbb{R}^{d+1} \]
\[ \mathbf{w}_k(x) := \text{vec} \left( \mathbf{P}_k(x; \cdot) \right) \in \mathbb{R}^m, \quad k = 1, \ldots, m, \]
where the $\mathbf{P}_k$ are the reduced polynomials of $\mathbf{P}$, and we introduce the $m \times m$ matrix
\[ W(x) := \begin{pmatrix} w_1(x) \\ \vdots \\ w_m(x) \end{pmatrix}. \]

If $\{\tau_1(x) \leq \cdots \leq \tau_m(x)\}$ are the eigenvalues of the matrix $A(x)$, that is the roots of $\mathbf{P}(x, \tau) = 0$, from the above definition we easily derive (see [ST1]) that, up to dilatations, the row vector $\mathbf{w}_k(x)$ is the unique left eigenvector of $A(x)$ corresponding to the eigenvalue $\tau_k(x)$, i.e.,
\[ \mathbf{w}_k(x) A(x) = \tau_k(x) \mathbf{w}_k(x). \]

**Definition 2.** We call Jannelli’s (or standard) symmetrizer of $A(x)$ the $m \times m$ matrix
\[ Q(x) := W^T(x) W(x). \]

Note the identity:
\[ (Q(x) \mathbf{v}, \mathbf{v}) = \sum_{k=1}^m (\mathbf{w}_k(x), \mathbf{v})^2, \quad \text{for all } \mathbf{v} \in \mathbb{R}^m, \ x \in \mathbb{R}. \]
Later on, we shall use the following characterisation of the proper decomposition:

**Lemma 2.1.** A polynomial \( R(x; \tau) \) of degree \( \leq m - 1 \) has a proper decomposition w.r.t. \( P(x; \tau) \) (see (1.10)-(1.11)) if and only if there exists some constant \( C_0 \) for which

\[
(\text{vec}(R(x; \cdot)), v)^2 \leq C_0 \langle Q(x)v, v \rangle, \quad \text{for all } v \in \mathbb{R}^m, \ x \in \mathbb{R}.
\]

**Proof.** If we define the vector

\[
r(x) := \text{vec}(R(x; \cdot)),
\]

and recall (1.10)-(2.7), we see that \( R \) has a proper decomposition w.r.t. \( P \) if and only if there are some \( \ell_k \in L^\infty(\mathbb{R}) \) for which

\[
r(x) = \sum_{k=1}^m \ell_k(x) w_k(x), \quad \text{for all } x \in \mathbb{R}.
\]

Hence, if \( R \) has a proper decomposition w.r.t. \( P \), by (2.12) and (2.10) we derive:

\[
(\ell_k(x), v)^2 \leq \sum_{k=1}^m \ell_k^2(x) \cdot \sum_{k=1}^m \langle w_k(x), v \rangle^2 \leq C \langle Q(x)v, v \rangle,
\]

with \( C = \sum \|\ell_k\|_{L^\infty}^2 \).

Conversely, let us assume (2.11). If we define, for each fixed \( x \in \mathbb{R} \), the linear space

\[
Z(x) := \text{Span}\{w_1(x), \ldots, w_m(x)\} \subset \mathbb{R}^m
\]

and note that \( \langle Q(x)v, v \rangle = 0 \) if \( v \in Z(x)^\perp \) (see (2.10)), we get

\[
(\ell_k(x), v) = 0 \quad \text{for all } v \in Z(x)^\perp.
\]

Consequently we see that \( r(x) \in Z(x) \). Thus, there are \( \ell_1(x), \ldots, \ell_m(x) \) such that

\[
r(x) = \sum_{k=1}^m \ell_k(x) w_k(x),
\]

where, after a slight modification, we can assume that \( \ell_h = \ell_k \) whenever \( w_h = w_k \).

We prove that

\[
|\ell_k(x)| \leq \sqrt{C_0}, \quad \text{for each fixed } k \text{ and } x.
\]

Indeed, if the eigenvalue \( \tau_k(x) \) has multiplicity \( \nu \) \( (1 \leq \nu \leq m) \), let \( h_1, \ldots, h_\nu \) be the indices \( h \) for which \( \tau_h(x) = \tau_k(x) \), and consequently \( w_h(x) = w_k(x) \) and \( \ell_h(x) = \ell_k(x) \). Therefore taking a vector \( \vec{v} \in \mathbb{R}^m \) which satisfies \( \langle w_k(x), \vec{v} \rangle = 1 \), and \( \langle w_h(x), \vec{v} \rangle = 0 \) if \( h \notin \{h_1, \ldots, h_\nu\} \), and recalling (2.13), we find:

\[
(\ell_k(x), \vec{v}) = \sum_{h=1}^m \ell_h(x) \langle w_h(x), \vec{v} \rangle = \nu \ell_k(x).
\]

On the other side, by (2.10) we know that

\[
\langle Q(x)\vec{v}, \vec{v} \rangle = \sum_{h=1}^m \langle w_h(x), \vec{v} \rangle = \nu,
\]

thus, inserting (2.15) and (2.16) in (2.11), we get

\[
|\ell_k(x)| \leq \sqrt{\frac{C_0}{\nu}}.
\]

Hence (2.14) holds true. \( \square \)
Next, putting
\[ \tau_{\max} := \max_{x \in \mathbb{R}} \max_{1 \leq j \leq m} |\tau_j(x)|, \quad \psi_k := \sum_{1 \leq j_1 < \cdots < j_k \leq m} \tau_{j_1}^2 \cdots \tau_{j_k}^2, \]
we define the diagonal matrices:
\[
(2.17) \quad \Lambda(x) := \begin{pmatrix} \tau_1 & 0 & \cdots & 0 \\ 0 & \tau_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \tau_{m-1} & 0 \\ 0 & \cdots & 0 & \tau_m \end{pmatrix}, \quad \Psi(x) := \begin{pmatrix} \psi_{m-1} & 0 & \cdots & 0 \\ 0 & \psi_{m-2} & 0 & \vdots \\ \vdots & \ddots & \ddots & \psi_1 \\ 0 & \cdots & \psi_1 & 1 \end{pmatrix}.
\]

By (2.8) we have
\[ W(x)A(x) = \Lambda(x)W(x), \]
whence
\[ QA = W^T \Lambda W. \]
Thus, the matrices \( Q \) and \( QA \) are both symmetric, so \( Q \) is a symmetrizer of \( A \).

In the following Proposition we resume the properties of the Jannelli’s symmetrizer (2.9). We write for brevity \( Q \equiv Q(x), A \equiv A(x), \Psi \equiv \Psi(x), \tau_j \equiv \tau_j(x) \).

**Proposition 2.2.** The entries of the symmetrizer \( Q(x) \) in (2.9) are polynomials in the coefficients of \( P(x; \tau) \), in particular are smooth functions of \( x \).

It holds:
\[ \det Q = \prod_{1 \leq j < k \leq m} (\tau_j - \tau_k)^2. \]

Moreover, if the entries of \( A(x) \) verify
\[ \|a_j^{(k)}\|_{L^\infty(\mathbb{R})} \leq L < +\infty, \quad \text{for } j, k = 0, \ldots, m, \]
then there exists a positive constant \( \Gamma_0 \equiv \Gamma_0(m, L) \) such that, for any \( v \in \mathbb{R}^m \), it holds:
\[ \|QAV, v\| \leq \tau_{\max} |v|^2, \]
\[ \|(QA)'v, v\| \leq \Gamma_0(v, v), \]
\[ \|(QA'v, A'v)\| \equiv |A'v|^2 \leq \Gamma_0(v, v). \]

Finally, if Hypothesis [H] is satisfied, the symmetrizer \( Q(x) \) is a nearly diagonal matrix, in the sense that there exist two positive constants \( \Gamma_j \equiv \Gamma_j(m, M), j = 1, 2 \), for which
\[ \Gamma_1(\Psi v, v) \leq (Qv, v) \leq \Gamma_2(\Psi v, v). \]

**Proof.** See [ST1] Propositions 1.10 and 1.11].

Always in [ST1] we define the energy
\[ E(t, u) := \frac{1}{2} \int_{I_t} (Q(x)u(t, x)u(t, x)) \, dx, \]
where
\[ I_t := [x_0 - \rho(t), x_0 + \rho(t)], \quad \rho(t) := \rho_0 - \tau_{\max} t, \]
and, using the properties (2.20)–(2.22) of \( Q(x) \), we prove the apriori estimate
\[ E(t, u) \leq C(t) \left\{ E(0, u) + \int_0^t \int_{I_s} |f(s, x)| \, dx \, ds \right\}, \quad 0 \leq t \leq \rho_0/\tau_{\max}. \]
From (2.26) we cannot derive an estimate on \( u(t, \cdot) \): indeed to his end we need an estimate like \( \| u(t, \cdot) \| \leq C E(t, u) \), which holds only when \( Q(x) \) is coercive (see (2.24)).

Now, by (2.18) it follows that \( Q(x) \) is coercive if and only if \( P(x; \tau) \) is strictly hyperbolic, whereas it degenerates at those points \( x \) where the characteristic roots are multiple.

However, thanks to Hypothesis \( \mathbb{H} \), a weak coercivity estimate holds. Indeed, taking into account that the last element of the matrix \( \Psi(x) \) is 1, by (2.23) we derive:

\[
(2.27) \quad (Q(x) \nu, \nu) \geq \Gamma_1 |v_m|^2, \quad \text{for all } \nu = (v_1, \ldots, v_m)^T \in \mathbb{R}^m.
\]

Hence we get:

\[
\| \partial_t^{m-1} u(t, \cdot) \|_{L^2(I_t)} \leq \frac{2}{\Gamma_1} E(t, u),
\]

which yields, by integration in \( t \), an apriori estimate of \( u(t, \cdot) \) in \( L^2(I_t) \).

Differentiating repeatedly (2.27) in \( x \), and using a suitable estimate of the derivatives \( A^{(h)}(x) \) in terms of the \( \psi_j(x) \)'s, we get an estimate of \( u(t, \cdot) \) in \( H^k(I_t) \).

### 3. Reduction to a first order system

Let us go back to an equation of the general type:

\[
(3.1) \quad L(x; \partial_t, \partial_x) u = P(x; \partial_t, \partial_x) u - \sum_{d=0}^{m-1} R_d(t, x; \partial_t, \partial_x) u = 0,
\]

with principal symbol

\[
(3.2) \quad P(x; \tau, \xi) = \tau^m + a_1(x) \tau^{m-1} \xi + \ldots + a_m(x) \xi^m.
\]

Proceeding as in the homogeneous case, we transform such an equation into an equivalent first order system. Before to proceed, we fix some notation.

**Notations.**

- \( P(x; \tau) := P(x; \tau, 1) \), \( R_d(x; \tau) := R_d(x; \tau, 1) \).
- For each \( d = 1, \ldots, m \), the monic \( \tau \)-derivative of order \( m - d \) of \( P(x; \tau) \) is:

\[
(3.3) \quad P^{(d)}(x; \tau) := \frac{d!}{m!} \delta_{\tau}^{m-d} P(x; \tau) = \sum_{k=0}^{d} a_{d,k}(x) \tau^{d-k} = \prod_{\ell=1}^{d} (\tau - \tau^{(d)}(x)).
\]

Note that the functions \( a_{d,k}(x) \) are, up to constants, the coefficients \( a_j(x) \) of the principal symbol (3.2).

- For each \( d = 1, \ldots, m \), \( A_d(x) \) denotes the Sylvester matrix associated to \( P^{(d)}(x; \tau) \):

\[
(3.4) \quad A_d(x) := \begin{pmatrix}
0 & 1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 \\
-a_{d,d}(x) & -a_{d,d-1}(x) & \ldots & -a_{d,1}(x)
\end{pmatrix}.
\]

If \( d = m \), \( A_m(x) \) coincides with the matrix \( A(x) \) considered in §2 (see (2.3)).

Let

\[
p(\tau) = c_0 \tau^{d} + \ldots + c_j \tau^{d-j} + \ldots + c_d
\]

be a generic polynomial of degree \( d \). Therefore:

- \( \text{vec}(p) \) is the row vector in \( \mathbb{R}^{d+1} \) formed by the coefficients of \( p \):

\[
\text{vec}(p) := (c_d, c_{d-1}, \ldots, c_1, c_0) \in \mathbb{R}^{d+1}.
\]
• Mat\(_j(p)\) is the \(j \times (d + 1)\) matrix obtained from the null matrix by replacing the entries of the last line with the coefficients of \(p(\tau)\):

\[
\text{Mat}_j(p) := \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
c_d & c_{d-1} & \cdots & c_1 & c_0
\end{pmatrix}.
\]

• Finally, we define the \((d + 1) \times (d + 1)\)-matrices

\[
\alpha_d(x) \equiv \text{Mat}_d(P^{(d)}(x; \cdot)) := \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
a_{d,d}(x) & a_{d,d-1}(x) & \cdots & a_{d,1}(x) & 0
\end{pmatrix},
\]

\[
\rho_d(t, x) \equiv \text{Mat}_m(R_d(t, x; \cdot)) := \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
r_{d,d}(t, x) & r_{d,d-1}(t, x) & \cdots & r_{d,0}(t, x)
\end{pmatrix}.
\]

Now, for \(u = u(t, x)\) and \(0 \leq d \leq m - 1\), we consider the vector formed by the derivatives of order \(d\) of \(u\):

\[
(3.5) \quad u_d := (u_{d,0}, \ldots, u_{d,d})^T \in \mathbb{R}^{d+1}, \quad u_{d,j} := \partial_t^j \partial_x^{d-j} u, \quad 0 \leq j \leq d.
\]

i.e.,

\[
(3.6) \quad u_0 := u, \quad u_1 := \left(\frac{\partial_x u}{\partial_t u}\right), \quad \ldots, \quad u_{m-1} := \left(\frac{\partial_x^{m-1} u}{\partial_t^{m-2} u}, \ldots, \frac{\partial_x u}{\partial_t u}, \frac{\partial_x^2 u}{\partial_t^2 u}, \ldots, \frac{\partial_x^{m-1} u}{\partial_t^{m-1} u}\right).
\]

By a simple calculation we see that, if \(u(t, x)\) is a solution of (3.1), then \(u_d\) satisfies the system

\[
(3.7) \quad \partial_t u_d = A_{d+1}(x) \partial_x u_d + \alpha_{d+1}(x) u_{d+1}, \quad \text{if} \quad d = 0, \ldots, m - 2,
\]

\[
(3.8) \quad \partial_t u_{m-1} = A_m(x) \partial_x u_{m-1} + \sum_{j=0}^{m-1} \rho_j(t, x) u_j + f(t, x),
\]

where \(f = (0, \ldots, 0, f)^T \in \mathbb{R}^{d+1}\). Next, setting

\[
(3.9) \quad \mu := \frac{m(m - 1)}{2},
\]

we define the \(\mu\)-vectors

\[
(3.10) \quad U := \begin{pmatrix}
u_0 \\
u_1 \\
\vdots \\
u_{m-1}
\end{pmatrix}, \quad F := \begin{pmatrix}0 \\
0 \\
\vdots \\
f
\end{pmatrix}.
\]
Combining (3.7) and (3.8), we conclude that the equation (3.1) is equivalent to the system

\[ U_t = \mathcal{A}(x) U_x + \mathcal{B}(t, x) U + F(t, x), \]

where \( \mathcal{A} \) and \( \mathcal{B} \) are \( \mu \times \mu \) matrix having a block structure with blocks of different size:

\[
\mathcal{A}(x) := \begin{pmatrix}
A_1(x) & 0 & \cdots & 0 \\
0 & A_2(x) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & A_m(x)
\end{pmatrix},
\]

\[
\mathcal{B}(t, x) := \begin{pmatrix}
0 & \alpha_1(x) & 0 & \cdots & 0 \\
0 & \alpha_2(x) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{m-1}(x)
\end{pmatrix} + \begin{pmatrix}
\rho_0(t, x) & \rho_1(t, x) & \cdots & \rho_{m-1}(t, x)
\end{pmatrix}.
\]

**Example 2.** If \( m = 3 \), we have:

\[
\partial_t \begin{pmatrix} u \\ u_x \\ u_{tx} \\ u_{tt} \end{pmatrix} = \begin{pmatrix}
-a_{2,1} & 0 & 0 & 0 \\
0 & -a_{1,2} & -a_{1,1} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -a_{0,3} -a_{0,2} -a_{0,1}
\end{pmatrix} \partial_x \begin{pmatrix} u \\ u_x \\ u_{tx} \\ u_{tt} \end{pmatrix} + \begin{pmatrix}
0 & a_{2,1} & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
r_0,0 & r_{0,1} & r_{1,0} & r_{1,1} & r_{2,0}
\end{pmatrix} \begin{pmatrix} u \\ u_x \\ u_{tx} \\ u_{tt} \end{pmatrix}.
\]

In order to construct a good symmetrizer for the matrix \( \mathcal{A}(x) \), we prove that each block \( A_d(x) \) is the Sylvester matrix coming from a hyperbolic polynomial which satisfies (1.8), and we take the corresponding Jannelli symmetrizer. Now, \( A_d(x) \) is the Sylvester matrix of the polynomial \( P^{(d)}(x; \tau) \) (cf. (3.3)), which is proportional to the derivative \( \partial_{\tau}^{m-d}P(x; \tau) \); thus, we have only to prove the following result:

**Proposition 3.1.** Let \( P(x; \tau) \) be a hyperbolic polynomial with roots satisfying (1.8) for some constant \( M \). Then \( \partial_{\tau}P(x; \tau) \) is a hyperbolic polynomial and its roots satisfy (1.8) with a constant \( \tilde{M} = \bar{M}(m, M) \) independent of \( x \). Consequently, the polynomials \( P^{(d)}(x; \tau) \) enjoy the same property.

**Remark 3.2.** Before proving Proposition 3.1 we note that, if \( p(\tau) \) is a hyperbolic polynomial, then \( p'(\tau) \) is also hyperbolic, and the roots of the two polynomials are interlaced, that is, denoting by \( \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \) the roots of \( p(\tau) \), and by \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{m-1} \) those of \( p'(\tau) \), one has

\[
\tau_j \leq \lambda_j \leq \tau_{j+1}, \quad j = 1, \ldots, m - 1.
\]

Indeed, (3.14) follows from the classical Rolle’s theorem, taking into account that any root of \( p(\tau) \) with multiplicity \( m > 1 \) is a root of \( p'(\tau) \) with multiplicity \( m - 1 \).
Let us now recall that, in [2], Peyser refined the estimate (3.14) by proving that
\begin{equation}
\tau_j + \frac{\tau_{j+1} - \tau_j}{m-j+1} \leq \lambda_j \leq \tau_{j+1} - \frac{\tau_{j+1} - \tau_j}{j+1}, \quad j = 1, \ldots, m-1.
\end{equation}
Thus, Proposition 3.1 is a direct consequence of the following Lemma.

**Lemma 3.3.** Let \( p(\tau) \) and \( r(\tau) \) be hyperbolic polynomials, of degree \( m \) and \( m-1 \) respectively, whose roots, \( \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \) and \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{m-1} \), satisfy
\begin{equation}
\tau_j \leq \lambda_j \leq \tau_{j+1} - \eta (\tau_{j+1} - \tau_j), \quad 1 \leq j \leq m-1,
\end{equation}
for some constant \( \eta \in [0,1] \). Therefore, if
\begin{equation}
\tau_j^2 + \tau_k^2 \leq M (\tau_j - \tau_k)^2, \quad 1 \leq j < k \leq m,
\end{equation}
for some constant \( M \), then
\begin{equation}
\lambda_j^2 + \lambda_k^2 \leq \tilde{M} (\lambda_j - \lambda_k)^2, \quad 1 \leq j < k \leq m,
\end{equation}
with
\begin{equation}
\tilde{M} := \frac{4M}{\eta^2} + 2.
\end{equation}

**Proof.** It is sufficient to prove (3.18) for \( k = j + 1 \); indeed, once proved (3.18) for \( k = j + 1 \), the general case can be reached as follows:
\begin{align*}
\lambda_j^2 + \lambda_k^2 &\leq \sum_{h=j}^{k-1} (\lambda_h^2 + \lambda_{h+1}^2) \leq \sum_{h=j}^{k-1} \tilde{M} (\lambda_h - \lambda_{h+1})^2 \\
&\leq \tilde{M} \left( \sum_{h=j}^{k-1} (\lambda_h - \lambda_{h+1}) \right)^2 = \tilde{M} (\lambda_j - \lambda_k)^2.
\end{align*}
From the second inequality in (3.16) it follows that
\begin{equation}
\tau_{j+1} - \tau_j \leq \frac{1}{\eta} (\tau_{j+1} - \lambda_j),
\end{equation}
hence
\begin{equation}
\tau_{j+1} - \tau_j \leq \frac{1}{\eta} (\lambda_{j+1} - \lambda_j).
\end{equation}
Then, noting that \( \tau_j \leq \lambda_j \leq \tau_{j+1} \), and using (3.17), we get:
\begin{equation}
\lambda_j^2 \leq \tau_j^2 + \tau_{j+1}^2 \leq M (\tau_{j+1} - \tau_j)^2 \leq \frac{M}{\eta^2} (\lambda_{j+1} - \lambda_j)^2.
\end{equation}
Analogously, noting that \( \tau_{j+1} \leq \lambda_{j+1} \leq \tau_{j+1} + (\lambda_{j+1} - \lambda_j) \), we get
\begin{align*}
\lambda_{j+1}^2 &\leq \tau_{j+1}^2 + \left( \tau_{j+1} + (\lambda_{j+1} - \lambda_j) \right)^2 \leq 3 \tau_{j+1}^2 + 2 (\lambda_{j+1} - \lambda_j)^2 \\
&\leq 3 M (\tau_{j+1} - \tau_j)^2 + 2 (\lambda_{j+1} - \lambda_j)^2 \leq \frac{3M}{\eta^2} + 2 (\lambda_{j+1} - \lambda_j)^2.
\end{align*}
Summing up, we obtain (3.18) with \( \tilde{M} \) given by (3.19). This concludes the proof of Lemma 3.3 and hence of Proposition 3.1.

**Remark 3.4.** If, for some \( \eta \in [0,1] \), we replace (3.16) by
\begin{equation}
\tau_j + \eta (\tau_{j+1} - \tau_j) \leq \lambda_j \leq \tau_{j+1}, \quad 1 \leq j \leq m-1,
\end{equation}
we get the same conclusion.
Remark 3.5. From (3.14) it follows that
\[ |τ^{(d)}_\ell(x)| \leq τ_{\text{max}}, \quad \text{for } 1 \leq \ell \leq d \leq m. \]

We are now in the position to construct the wished symmetrizer for \(\mathcal{A}(x)\). Indeed, the matrix \(A_d(x)\) has the same properties of the matrix \(A(x)\) in §2, thus it admits a good symmetrizer \(Q_d(x)\) with the same properties of \(Q(x)\) listed in Proposition 2.2.

Consequently, the \(\mu \times \mu\) matrix

\[ \mathcal{Q}(x) := \begin{pmatrix} Q_1(x) & 0 & \cdots & 0 \\ \vdots & Q_2(x) & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & Q_m(x) \end{pmatrix} \]

is a symmetrizer of \(\mathcal{A}(x)\). Note that, if \(U\) is given by (3.10), then:

\[ (\mathcal{Q}(x) U, U) = \sum_{d=0}^{m-1} (Q_{d+1}(x) u_d, u_d). \]

Next we define the block diagonal matrix (with blocks of different size)

\[ \Xi(x) := \begin{pmatrix} \Psi_1(x) & 0 & \cdots & 0 \\ \vdots & \Psi_2(x) & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Psi_m(x) \end{pmatrix}, \]

where \(\Psi_d(x)\) is the \(d \times d\) matrix analogous to \(\Psi(x)\) in (2.17) related to the polynomial \(P^{(d)}(x; \tau)\) (see (3.3)). Then, writing \(\mathcal{A} \equiv \mathcal{A}(x), \mathcal{Q} \equiv \mathcal{Q}(x), \ldots\), we prove:

**Proposition 3.6.** The matrix \(\mathcal{Q}(x)\) in (3.20) is a symmetrizer of \(\mathcal{A}(x)\). If the coefficients \(a_j(x)\) of the principal symbol (3.2) verify (2.19), there exists a constant \(\Gamma_0 = \Gamma_0(m, L)\) such that, for all \(V \in \mathbb{R}^\mu\), it holds

\[
\begin{align*}
(\mathcal{Q} V, V) &\leq \Gamma_0 |V|^2, \\
(\mathcal{Q} \mathcal{A} V, V) &\leq \tau_{\text{max}} (\mathcal{Q} V, V), \\
((\mathcal{Q} \mathcal{A})_x V, V) &\leq \Gamma_0 (\Xi V, V), \\
((\mathcal{Q} \mathcal{A})_x V, \mathcal{A}_x V) &\equiv |\mathcal{A}_x V|^2 \leq \Gamma_0 (\Xi V, V).
\end{align*}
\]

Moreover, if the operator \(L\) in (3.1) verifies Hypothesis \(\text{H}\), there are two positive constants \(\Gamma_j, j = 1, 2\), such that:

\[ \Gamma_1 (\Xi V, V) \leq (\mathcal{Q} V, V) \leq \Gamma_2 (\Xi V, V). \]

Finally, if \(L\) verifies also Hypothesis \(\text{C}\), there exists a constant \(\Gamma_3\) such that:

\[ |(\mathcal{Q}(x) \mathcal{B}(t, x) V, V)| \leq \Gamma_3 (\mathcal{Q}(x)V, V), \quad \text{for all } V \in \mathbb{R}^\mu. \]

**Proof.** The first part of proof is an immediate consequence of Propositions 2.2 and 3.1 so we omit it, and we prove only (3.28):
By Schwarz inequality and (3.23), we can find a constant $C_1$ such that
\begin{equation}
(3.29) \quad |(\mathcal{H} V, V)| \leq (\mathcal{H} V, V)^{1/2} (\mathcal{L} V, V)^{1/2} \leq C_1 |\mathcal{B} V| (\mathcal{L} V, V)^{1/2},
\end{equation}
for all $V \equiv (v_0, \ldots, v_m)^T \in \mathbb{R}^d$, where $v_d \in \mathbb{R}^{d+1}$. Therefore, by (3.13), we have:
\begin{equation}
V := \begin{pmatrix}
v_0 \\
v_1 \\
\vdots \\
v_{m-2} \\
v_{m-1}
\end{pmatrix}, \quad \mathcal{B} V = \begin{pmatrix}
\alpha_1 v_1 \\
\vdots \\
\alpha_{m-1} v_{m-1} \\
\sum_{d=0}^{m-1} \rho_d v_d
\end{pmatrix},
\end{equation}
with
\begin{equation}
\alpha_d v_d = \begin{pmatrix}
0 \\
\vdots \\
0 \\
\text{vec}(P^{(d)}) \cdot v_d
\end{pmatrix}, \quad \rho_d v_d = \begin{pmatrix}
0 \\
\vdots \\
0 \\
\text{vec}(R_d) \cdot v_d
\end{pmatrix},
\end{equation}
and hence
\begin{equation}
|\mathcal{B} V| \leq \sum_{d=0}^{m-1} \left| \text{vec}(P^{(d)}) \cdot v_d \right| + \sum_{d=0}^{m-1} \left| \text{vec}(R_d) \cdot v_d \right|.
\end{equation}
Now, according to Example 1, $P^{(d)}$ has a proper decomposition w.r.t. $P^{(d+1)}$, whereas, by Hypothesis C, $R_d$ has a proper decomposition w.r.t. $P^{(d+1)}$. Hence, thanks to Lemma 2.1 and (3.21), we have
\begin{equation}
(3.30) \quad |\mathcal{B} V| \leq C_2 \sum_{d=0}^{m-1} (Q_{d+1} v_d, v_d)^{1/2} \leq C_2 (\mathcal{L} V, V)^{1/2},
\end{equation}
for some constant $C_2$, and inserting the inequality (3.30) in (3.29) we obtain (3.28). \hfill \Box

4. THE ENERGY ESTIMATE

Thanks to the (Propositions 3.1 and 3.6), we are in the position to prove an energy estimate for the solutions $U \equiv U(t, x)$ to the system
\begin{equation}
(4.1) \quad U_t = \mathcal{A}(x) U_x + \mathcal{B}(t, x) U + F(t, x).
\end{equation}
Indeed, defining the energy
\begin{equation}
\mathcal{E}(t, U) := \frac{1}{2} \int_I \left( \mathcal{L}(x) U(t, x), U(t, x) \right) dx,
\end{equation}
where $\mathcal{L}(x)$ is the symmetrizer of $\mathcal{A}(x)$ in (3.20), $I$ the real interval (2.25), we prove:

**Proposition 4.1.** Assume that the operator $\mathcal{L}$ in (3.1) satisfies the Hypothesis A, B, and C, and the coefficients $a_j(x)$ belong to $C^\infty \cap L^\infty$. Then, for each solution $U$ to (4.1), it holds the estimate:
\begin{equation}
(4.2) \quad \mathcal{E}(t, U) \leq C \left\{ \mathcal{E}(0, U) + \int_0^t \| F(s, \cdot) \|^2_{L^2(I_x)} ds \right\}, \quad 0 \leq t \leq T_0,
\end{equation}
with $T_0 := \min \{ \rho_0 / \tau_{\text{max}}, T \}$ and $C$ independent of $t$ and $x$. 

\textbf{Proof.} The proof is similar to that of Lemma 6.1 in [ST1]. Differentiating $\mathcal{E}(t)$, we find:
\[ \mathcal{E}'(t, U) = \int_{t_i} (\partial(x)U(t, U) \, dx + I_4(t) = I_1(t) + I_2(t) + I_3(t) + I_4(t), \]
where
\[ I_1(t) := \int_{t_i} (\partial \mathcal{A} U_{x,t}, U) \, dx, \quad I_2(t) := \int_{t_i} (\partial \mathcal{B} U, U) \, dx, \quad I_3(t) := \int_{t_i} (\partial F, U) \, dx, \]
and
\[ I_4(t) := -\frac{\tau_{\text{max}}}{2} \left\{ (\mathcal{D}U, U) \big|_{x=x_0+\rho(t)} + (\mathcal{D}U, U) \big|_{x=x_0-\rho(t)} \right\}. \]
Since the matrix $\partial(x)\mathcal{A}(x)$ is symmetric, the identity
\[ ((\partial \mathcal{A})U, U)_{x} = ((\partial \mathcal{A})_{x}U, U) + 2((\partial \mathcal{A})_{x}U_{x}, U) \]
holds true; hence, recalling (3.24), (3.25) and (3.27), we derive
\[ I_1(t) = \frac{1}{2} \left\{ (\partial \mathcal{A}U, U) \big|_{x=x_0+\rho(t)} - (\partial \mathcal{A}U, U) \big|_{x=x_0-\rho(t)} \right\} - \frac{1}{2} \int_{t_i} (\partial \mathcal{A})_{x}U, U) \, dx \]
\[ \leq \frac{\tau_{\text{max}}}{2} \left\{ (\mathcal{D}U, U) \big|_{x=x_0+\rho(t)} + (\mathcal{D}U, U) \big|_{x=x_0-\rho(t)} \right\} + \frac{\Gamma_0}{\Gamma_1} \mathcal{E}(t). \]
Consequently:
\[ I_1(t) + I_4(t) \leq \frac{\Gamma_0}{\Gamma_1} \mathcal{E}(t). \]
On the other hand, by Schwarz’ inequality and (3.23), we have
\[ I_2(t) \leq \Gamma_3 \mathcal{E}(t), \]
\[ I_3(t) \leq \left\{ \int_{t_i} (\partial F, F) \, dx \right\}^{1/2} \sqrt{2 \mathcal{E}(t)} \leq \frac{1}{2} \int_{t_i} (\partial F, F) \, dx + \mathcal{E}(t) \]
\[ \leq \frac{\Gamma_0}{2} \|F(t, \cdot)\|_{L^2(t_i)}^2 + \mathcal{E}(t). \]
Summing up, we find a constant $C = C(\Gamma_j, d_0) = C(m, L, M, d_0)$ for which
\[ \mathcal{E}'(t, U) \leq C \left\{ \mathcal{E}(0, U) + \|F(t, \cdot)\|_{L^2(t_i)}^2 \right\}, \quad 0 \leq t \leq T_0, \]
so, by Grönwall Lemma, we get (4.2). \qed

The next goal will be to derive from (4.2) an estimate for the Sobolev norms of the solution $U(t, x)$. This is not easy because the symmetrizer $\partial(x)$ is not coercive. First of all we observe that
\[ \mathcal{E}(t, U) \geq \frac{\Gamma_1}{2} \sum_{d=0}^{m-1} \int_{t_i} (\Psi_d(t) u_d, u_d) \, dx, \]
and hence, since each $\Psi_d$ is a diagonal matrix with $(d,d)$-entry 1,
\[ \mathcal{E}(t, U) \geq \frac{\Gamma_1}{2} \sum_{d=0}^{m-1} \int_{t_i} |u_{d,d}|^2 \, dx = \frac{\Gamma_1}{2} \sum_{d=0}^{m-1} \int_{t_i} |\partial^d u|^2 \, dx. \]
Thus (4.2) provides an estimate for the time derivatives of the solution $u$ to (1.1)–(1.2):
\[ \sum_{d=0}^{m-1} \|\partial^d u(t, \cdot)\|_{L^2(t_i)}^2 \leq C \left\{ \sum_{j=0}^{m-1} \|\partial^j u(t, 0)\|_{L^2(t_0)}^2 + \int_0^t \|f(s, \cdot)\|_{L^2(t_s)}^2 \, ds \right\}, \quad 0 \leq t \leq T_0. \]
In §5 we derive a similar estimate for the space derivatives, by proving that $\partial_x^k u$ is a solution of an equation, like (1.1), which satisfies the Hypothesis A, B and C.

5. ESTIMATE OF THE SPACE-DERIVATIVES

We have

$$\partial_x \{ P(x; \partial_t, \partial_x) u \} = P(x; \partial_t, \partial_x) u_x + P'(x; \partial_t, \partial_x) u_x,$$

where $P'(x; \partial_t, \partial_x)$ is the differential operator of order $m - 1$ with symbol

$$P'(x; \tau, \xi) := \frac{1}{\xi} \partial_x P(x; \tau, \xi) \equiv \sum_{j=1}^{m} a_j'(x) \tau^{m-j} \xi^{j-1}.$$  

Similarly, recalling (1.6), we find:

$$\partial_x \{ R_d(t, x; \partial_t, \partial_x) u \} = R_d(t, x; \partial_t, \partial_x) u_x + \partial_x r_{d,0}(t, x) \partial_t^d u + R'_d(t, x; \partial_t, \partial_x) u_x,$$

where $R'_d(t, x; \partial_t, \partial_x)$ is the differential operator of order $d - 1$ with symbol

$$R'_d(t, x; \tau, \xi) := \frac{1}{\xi} \partial_x \left\{ R_d(t, x; \tau, \xi) - r_{d,0}(t, x) \tau^d \right\} \equiv \sum_{j=1}^{d} \partial_x r_{d,j}(t, x) \tau^{d-j} \xi^{j-1}.$$  

Differentiating (1.1) w.r.t. $x$, we see that $u_x$ satisfies the equation

$$L^{(1)}(t, x; \partial_t, \partial_x) u_x = f^{(1)}(t, x),$$

where

$$L^{(1)}(t, x; \partial_t, \partial_x) := P(x; \partial_t, \partial_x) - \sum_{d=0}^{m-1} R_d^{(1)}(t, x; \partial_t, \partial_x)$$

and

$$R_d^{(1)}(t, x; \tau, \xi) := \begin{cases} R_{m-1}(t, x; \tau, \xi) - P'(x; \tau, \xi) & \text{if } d = m - 1, \\ R_d(t, x; \tau, \xi) + R'_{d+1}(t, x; \tau, \xi) & \text{if } d = 1, \ldots, m - 2, \end{cases}$$

$$f^{(1)}(t, x) := \partial_x f(t, x) - \sum_{d=0}^{m-1} \partial_x r_{d,0}(t, x) \partial_t^d u.$$  

Since the principal symbol of (5.3) is the same of that of (1.1), Hypothesis A and B for the operator $L^{(1)}(t, x; \partial_t, \partial_x)$ are fulfilled. Thus, we check that $L^{(1)}$ satisfies Hypothesis C, i.e., the polynomial $R_d^{(1)}(t, x; \tau, \xi)$ has a proper decomposition w.r.t. $\partial_x^d P(x; \tau, \xi)$.

Now, according to Example 1 the polynomial $P'(x; \tau)$ in (5.1) has a proper decomposition w.r.t. $P'(x; \tau)$, while $R_d(t, x; \partial_t, \partial_x)$ has a proper decomposition w.r.t. $P^{(d)}(x; \tau)$, hence it remains only to prove that

$$R_{d+1}^{(1)}(t, x; \tau) \text{ has a proper decomposition w.r.t. } P^{(d)}(x; \tau), \quad d = 1, \ldots, m - 1.$$  

More precisely, we prove:

**Proposition 5.1.** Let $P(x; \tau, \xi)$ be a hyperbolic polynomial in $(\tau, \xi)$ of degree $m$ with characteristic roots verifying (1.8), and $R(t, x; \tau, \xi)$ an homogeneous polynomial in $(\tau, \xi)$ of degree $m - 1$ having a proper decomposition w.r.t. $P$. Then, the polynomial

$$R'(t, x; \tau, \xi) := \frac{1}{\xi} \partial_x \left\{ R(t, x; \tau, \xi) - r_0(t, x) \tau^{m-1} \right\},$$

where $r_0$ is the leading coefficient of $R$, has a proper decomposition w.r.t. $\partial_x P$. Consequently (5.5) holds true and the operator $L^{(1)}$ in (5.1) satisfies also Hypothesis C.
Consequently, we can apply Proposition 4.1 to the equation (5.3) to estimate the \( x \)-derivative of the solutions \( u \) of (3.1). Summing up we obtain:

**Corollary 5.2.** If the operator \( L(t, x; \partial_t, \partial_x) \) in equation (5.1) verifies the Hypothesis \( A \) \( B \) and \( C \), then also \( L^{(1)}(t, x; \partial_t, \partial_x) \) in (5.3) verifies these Hypothesis. As a consequence every smooth solution \( u(t, x) \) of (5.1) satisfies an apriori estimate like (4.3) with the \( H^1 \)-norm in place of the \( L^2 \)-norm. Iterating, we derive an apriori estimate for any \( H^k \)-norm of \( u \).

The proof of Proposition 5.1 is not very direct and needs some preliminary Lemmas. We start by fixing notation and terminology. To simplify the presentation, throughout this section we omit the \( t \) and \( \xi \) variables, and we consider the polynomials:

\[
P(x; \tau) = \sum_{h=0}^{m} a_h(x) \tau^{m-h} \equiv \prod_{j=1}^{m} (\tau - \tau_j(x)),
\]

where \( \{\tau_1(x) \leq \cdots \leq \tau_m(x)\} \) are real valued functions, and

\[
R(x; \tau) = \sum_{h=0}^{m-1} r_h(x) \tau^{m-1-h} \equiv \prod_{j=1}^{m-1} (\tau - \lambda_j(x)).
\]

The \( \lambda_j \)'s are not necessarily real valued functions.

We recall that \( R \) has a proper decomposition (of first order) w.r.t. \( P \) if

\[
R(x; \tau) = \sum_{h=1}^{m} \ell_h(x) P_{\eta}(x; \tau)
\]

for some \( \ell_h \in L^\infty(\mathbb{R}) \), where the \( P_{\eta} \) are the reduced polynomials of \( P \) defined in (1.11). Analogously, we say that a polynomial \( R \) of degree \( \leq m-2 \) has a proper decomposition of second order w.r.t. \( P \) if

\[
R(x; \tau) = \sum_{1 \leq h < k \leq m} \ell_{h,k}(x) P_{\eta,\xi}(x; \tau)
\]

for some \( \ell_{h,k} \in L^\infty(\mathbb{R}) \), where the \( P_{\eta,\xi} \) are the bi-reduced polynomials of \( P \), that is:

\[
P_{\eta,\xi}(x; \tau) := \prod_{\ell=1, \ldots, m} (\tau - \tau_{\ell}(x)).
\]

If also the roots \( \lambda_j \) of \( R \) are real, we say that the polynomials \( P \) and \( R \) are interlaced if

\[
\tau_j(x) \leq \lambda_j(x) \leq \tau_{j+1}(x), \quad \text{for all } x \in \mathbb{R}, \ j = 1, \ldots, m-1.
\]

Now we give a series of Lemmas that will be used to prove Proposition 5.1.

**Lemma 5.3.** If \( P(x; \tau) \) verifies Hypothesis \( B \) then \( \tau^{m-1} \) has a proper decomposition w.r.t. \( P(x; \tau) \).

**Proof.** As \( \text{vec}(\tau^{m-1}) = (0, \ldots, 0, 1) \), we have

\[
(\text{vec}(\tau^{m-1}), v) = v_m, \quad \text{for any } v = (v_1, \ldots, v_m)^T \in \mathbb{R}^m.
\]

Thus the proof follows from (2.27) and Lemma 2.1.

In the next Lemma we consider the relation between proper decomposition and interlacing property.
Lemma 5.4.

(1) Let the leading term of $R$ be positive. Then $P$ and $R$ are interlaced if and only if $R$ has a proper decomposition w.r.t. $P$ with coefficients $\ell_h(x)$ in (5.6) positive.

(2) Let $R$ have a proper decomposition w.r.t. $P$. Then $R$ is the sum of two hyperbolic polynomials of degree $m-1$ both interlaced with $P$. More precisely, there exists a constant $\zeta > 0$ and a monic hyperbolic polynomial $\tilde{P}_\zeta(x;\tau)$ interlaced with $P(x;\tau)$, such that

$$R(x;\tau) = \zeta \partial_\tau P(x;\tau) + (r_0(x) - m\zeta) \tilde{P}_\zeta(x;\tau).$$

Proof. The first part of the Lemma is already known: see Lemma 1.20 in [F]. We prove the second part. First recall that the polynomial $P_\tau$ is interlaced with $P$ (Remark 3.2). On the other side, according to (5.6), and (1.12),

$$\zeta \partial_\tau P(x;\tau) - R(x;\tau) = \sum_{h=1}^{m} (\zeta - \ell_h(x)) P_\tau(x;\tau).$$

Thus, choosing $\zeta$ large enough, the coefficients $\zeta - \ell_h(x)$ are all positive, and hence, by the part (1), the polynomial

$$\tilde{P}_\zeta(x;\tau) := \frac{1}{m\zeta - r_0(x)} \left\{ \zeta \partial_\tau P(x;\tau) - R(x;\tau) \right\}$$

is monic, hyperbolic and interlaced with $P(x;\tau)$. \hfill \Box

Here we give some condition in order to have a proper decomposition of second order.

Lemma 5.5. Let $S(x;\tau)$ be a polynomial of degree $\leq m-2$, and assume that one of the following conditions is fulfilled:

(1) $S$ has a proper decomposition w.r.t. $P$;

(2) there exists a polynomial $R$ of degree $\leq m-1$ interlaced with $P$ such that $S$ has a proper decomposition w.r.t. $R$.

Then $S(x;\tau)$ has a proper decomposition of second order w.r.t. $P$.

Proof. (1) Since $S(x;\tau)$ is of degree $\leq m-2$, by considering the terms of degree $m-1$ in (5.6), we see that

$$\sum_{h=1}^{m} \ell_h(x) = 0,$$

thus:

$$S(x;\tau) = \sum_{h=1}^{m} \ell_h(x) P_\tau(x;\tau) - \left\{ \sum_{h=1}^{m} \ell_h(x) \right\} P_\tau(x;\tau) = \sum_{h=2}^{m} \ell_h(x) \left\{ P_\tau(x;\tau) - P_\tau(x;\tau) \right\}.$$

To derive the wished conclusion, we have only to observe that, for any $h \neq k$, it holds

$$P_h(x;\tau) - P_k(x;\tau) = (\tau - \tau_k) P_{h\tau}(x;\tau) - (\tau - \tau_h) P_{k\tau}(x;\tau) = (\tau_h - \tau_k) P_{h\tau}(x;\tau).$$

(2) By linearity, we can assume, with no loss of generality, that $S(x;\tau) = R_j(x;\tau)$, for some $j$. As $R_j$ is interlaced with $P_j$, thanks to Lemma 5.4(1), it has a proper decomposition w.r.t. $P_j$, hence it has a proper decomposition of second order w.r.t. $P$. \hfill \Box

The converse of Lemma 5.5 is false in general.
Example 3. Let

\[ P(x; \tau) = (\tau^2 - x^2) (\tau^2 - 1) \]
\[ S(x; \tau) = (\tau^2 - 1). \]

\( S(x; \tau) \) has a proper decomposition of second order w.r.t. \( P(x; \tau) \), but is not properly decomposable w.r.t. \( P(x; \tau) \). Indeed, there’s a unique way in which we can write \( S(x; \tau) \) as a linear combination of the reduced polynomials of \( P(x; \tau) \) (cf. Remark 1.1), i.e.,

\[ \tau^2 - 1 = \frac{1}{2\tau} (\tau + x) (\tau^2 - 1) - \frac{1}{2\tau} (\tau - x) (\tau^2 - 1), \]

and the coefficients are not bounded. On the other side, if

\[ R(x; \tau) = \tau (\tau^2 - 4x^2), \]

\( P \) and \( R \) are interlaced polynomials, but \( S \) is not properly decomposable w.r.t. \( R \), since any linear combination of the reduced polynomials of \( R \) vanishes for \( x = 0 \) and \( \tau = 0 \).

This example shows that, in order to reverse Lemma 5.5 (2), if \( \tau_j(x) < \lambda_j(x) < \tau_{j+1}(x) \) and \( \lambda_j(x) \) approaches \( \tau_j(x) \), then \( \tau_{j+1}(x) \) should approach them too. Roughly speaking, \( \lambda_j(x) \) has to remain in some “central part” of the interval \([\tau_j(x), \tau_{j+1}(x)]\).

Motivated by Peyser’s inequality (5.15), we now consider the estimate

\[ \tau_j(x) + \eta (\tau_{j+1}(x) - \tau_j(x)) \leq \lambda_j(x) \leq \tau_{j+1}(x) - \eta (\tau_{j+1}(x) - \tau_j(x)), \]

for some \( 0 < \eta < 1/2 \), and all \( j = 1, \ldots, m-1 \).

Note that from the interlacing property (5.8) it follows that the ratios

\[ \frac{\tau_{j+1}(x) - \lambda_j(x)}{\lambda_{j+1}(x) - \lambda_j(x)} \quad \frac{\lambda_{j+1}(x) - \tau_{j+1}(x)}{\lambda_{j+1}(x) - \lambda_j(x)}, \]

are bounded, whereas from (5.10) it follows

\[ \eta (\tau_{j+2}(x) - \tau_j(x)) \leq \lambda_{j+1}(x) - \lambda_j(x) \leq (1 - \eta) (\tau_{j+2}(x) - \tau_j(x)), \]

hence also the ratios

\[ \frac{\lambda_j(x) - \tau_j(x)}{\lambda_{j+1}(x) - \lambda_j(x)} \quad \frac{\lambda_{j+1}(x) - \tau_j(x)}{\lambda_{j+1}(x) - \lambda_j(x)} \]

\[ \frac{\tau_{j+2}(x) - \lambda_j(x)}{\lambda_{j+1}(x) - \lambda_j(x)} \quad \frac{\tau_{j+2}(x) - \lambda_{j+1}(x)}{\lambda_{j+1}(x) - \lambda_j(x)}, \]

are bounded.

Lemma 5.6. Assume that \( P \) and \( R \) are interlaced and their roots verify (5.10). Then any polynomial \( S \) having a proper decomposition of second order w.r.t. \( P \) has a proper decomposition w.r.t. \( R \).

Proof. We give the proof in the case \( m = 3 \). The proof in the general case requires only additional technical effort. 

If \( m = 3 \) we have

\[ \tau_1(x) \leq \lambda_1(x) \leq \tau_2(x) \leq \lambda_2(x) \leq \tau_3(x), \]

and we have to prove that

\[ \tau - \tau_1(x), \quad \tau - \tau_2(x), \quad \tau - \tau_3(x) \]
are linear combination with bounded coefficients of
\[ \tau - \lambda_1(x), \quad \tau - \lambda_2(x). \]

Now, by Newton Lagrange formula, we have
\[
\begin{align*}
\tau - \tau_1(x) &= \frac{\lambda_2(x) - \tau_1(x)}{\lambda_2(x) - \lambda_1(x)} (\tau - \lambda_1(x)) + \frac{\lambda_1(x) - \tau_1(x)}{\lambda_1(x) - \lambda_2(x)} (\tau - \lambda_2(x)) \\
\tau - \tau_2(x) &= \frac{\lambda_2(x) - \tau_2(x)}{\lambda_2(x) - \lambda_1(x)} (\tau - \lambda_1(x)) + \frac{\lambda_1(x) - \tau_2(x)}{\lambda_1(x) - \lambda_2(x)} (\tau - \lambda_2(x)) \\
\tau - \tau_3(x) &= \frac{\lambda_2(x) - \tau_3(x)}{\lambda_2(x) - \lambda_1(x)} (\tau - \lambda_1(x)) + \frac{\lambda_1(x) - \tau_3(x)}{\lambda_1(x) - \lambda_2(x)} (\tau - \lambda_2(x)).
\end{align*}
\]

Thanks to (5.11), (5.12) and (5.13) all the coefficients are bounded. \(\square\)

Finally, from Peyser’s inequality (cf. (3.15)) we derive the following result:

**Lemma 5.7.** Any polynomial \(S(x; \tau)\) having a proper decomposition of second order w.r.t. \(P(x; \tau)\) has a proper decomposition w.r.t. \(\partial_\tau P(x; \tau)\).

Now we have all the tools necessary to prove Proposition 5.1

**Proof of Proposition 5.1.** By (5.9) we derive the identity:
\[
\begin{align*}
\partial_\tau \left\{ R(x; \tau) - r_0(x) \tau^{m-1} \right\} &= \zeta \partial_{x\tau}^2 P(x; \tau) + (r_0(x) - m\zeta) \partial_x \tilde{P}_\zeta(x; \tau) + r_0'(x)(\tilde{P}_\zeta(x; \tau) - \tau^{m-1}),
\end{align*}
\]

and we prove that each of the terms on the left has a proper decomposition w.r.t. \(\partial_\tau P\).

Since \(\partial_\tau P(x; \tau)\) is a hyperbolic polynomial, using (1.13) with \(\partial_\tau P\) in place of \(P\), we see that \(\partial_\tau^2 P\) has a proper decomposition w.r.t. \(\partial_\tau P\).

For the second term, as \(\tilde{P}_\zeta\) is a monic hyperbolic polynomial, using (1.13) with \(\tilde{P}_\zeta\) in place of \(P\) we see that \(\partial_\tau \tilde{P}_\zeta\) has a proper decomposition w.r.t. \(\tilde{P}_\zeta\). As the roots of \(\tilde{P}_\zeta\) are interlaced with those of \(P\), by Lemma 5.3 (2), \(\partial_\tau \tilde{P}_\zeta\) has a proper decomposition of second order w.r.t. \(P\), hence by Lemma 5.7 it has a proper decomposition w.r.t. \(\partial_\tau P\).

For the third term, as \(\tilde{P}_\zeta\) is interlaced with \(P\), by Lemma 5.4 (1), it has a proper decomposition w.r.t. \(P\). Moreover, by Lemma 5.3 \(\tau^{m-1}\) has a proper decomposition w.r.t. \(P\) too. Thus \(\tilde{P}_\zeta(x; \tau) - \tau^{m-1}\) is a polynomial of order \(\leq m-2\) which has a proper decomposition w.r.t. \(P\), hence also a proper decomposition of second order w.r.t. \(P\) and by Lemma 5.7 a proper decomposition w.r.t. \(\partial_\tau P\). \(\square\)

6. **Null’s approximation**

We have proved an apriori estimate for each space-time derivative of the solutions to \(Lu = f\), where \(L \equiv P - \sum R_{m-1} + \cdots + R_0\) is the operator (1.3).

To construct a solution for the corresponding Cauchy Problem, we proceed as in [ST1]: we approximate the operator \(L\) by a sequence \(\{L_\varepsilon\}\) of strictly hyperbolic operators, to get a sequence \(\{u_\varepsilon\}\) of approximating solutions converging to some solution of (1.1).

More precisely we approximate the principal symbol of \(L\), i.e., the polynomial \(P(x; \tau, \xi)\), by the polynomials
\[
P_\varepsilon := \mathcal{N}_\varepsilon^{m-1}(P),
\]

where \(\mathcal{N}_\varepsilon^{m-1}(P)\) is a polynomial of order \(m-1\) which is a polynomial of order \(m-1\).
where $\mathcal{N}_ε$ is the Nuij’s map:

\[
\mathcal{N}_ε: \ P \mapsto P - ε \partial_x P.
\]

In [N], Nuij proved that, if $P(τ)$ is a hyperbolic polynomial with maximum multiplicity $r$, then $P - ε \partial_x P$ is a hyperbolic polynomial with maximum multiplicity $r - 1$. Thus the polynomial $\mathcal{N}_ε^{m-1}P$ is strictly hyperbolic. Then we define

\[
L_ε := P_ε - \{R_{m-1} + \cdots + R_0\}.
\]

Now, let $u_ε$ be the solution of the Cauchy Problem (1.1)-(1.2) with $L_ε$ in place of $L$. Before applying our apriori estimate to the functions $u_ε$, we check that the polynomials $L_ε(t, x; τ, ξ)$ verify the Hypothesis [A] and [C] uniformly w.r.t. $ε$. Now $L_ε(t, x; τ, ξ)$ verify Hypothesis [A] by Nuij’s Lemma, and Hypothesis [C] by Lemma 1.9 in [ST]. To check Hypothesis [C] we use the following result of S. Waka Bayashi (see [W2]) which precises the Nuij’s approximation:

**Lemma 6.1.** Let $P(τ)$ an hyperbolic polynomial with degree $m$, and $P_ε = \mathcal{N}_ε(P)$. Denote by $\{τ_j\}$ and $\{τ_{j,ε}\}$ the roots of $P$ and $P_ε$. Therefore there are two constants, depending only on $P$, such that, for all $j, k = 1, \ldots, m$ with $j \neq k$, it results:

1. $|τ_{j,ε}(x) - τ_j(x)| ≤ C_1 ε$,
2. $|τ_{k,ε}(x) - τ_{j,ε}(x)| ≥ C_2 ε$.

Then we prove:

**Lemma 6.2.** Let

\[
P_ε(x; τ, ξ) = \prod_{j=1}^{m} (τ - τ_{j,ε}(x) ξ)
\]

be a sequence of strictly hyperbolic polynomial satisfying the properties (1) and (2) of Lemma 6.1 for some constant $C_1$ and $C_2$ independent of $ε$ and $x$.

Therefore, if $R(t, x; τ, ξ)$ has a proper decomposition w.r.t. $P$, it has also a proper decomposition w.r.t. $P_ε$, with coefficients bounded uniformly w.r.t. $ε$.

**Proof.** We know that

\[
R(t, x; τ, ξ) = \sum_{k=1}^{m} \ell_k(x)P_k(τ)
\]

where the $P_k$ are the reduced polynomial of $P$. Hence, by linearity, it is sufficient to prove that each $P_k$ has a proper decomposition w.r.t. $P_ε$. With no loss of generality we can assume that $k = 1$. By the Lagrange interpolation formula we have

\[
P_1(τ) = \sum_{k=1}^{m} \frac{P_1(τ_{j,ε})}{P_{k,ε}(τ_{j,ε})} P_k(τ).
\]

If $k = 1$, we have

\[
\left| \frac{P_1(τ_{j,ε})}{P_{1,ε}(τ_{j,ε})} \right| = \prod_{h=2}^{m} \frac{τ_{1,ε} - τ_h}{τ_h - τ_{j,ε}} = \prod_{h=2}^{m} \left| 1 + \frac{τ_h - τ_1}{τ_{j,ε} - τ_h} \right| ≤ \left(1 + \frac{C_1}{C_2}\right)^{m-1}.
\]

If $k \neq 1$, we have

\[
\left| \frac{P_1(τ_{j,ε})}{P_{1,ε}(τ_{j,ε})} \right| = \frac{τ_{k,ε} - τ_1}{τ_{k,ε} - τ_1} \prod_{h=2, m \neq k} \frac{τ_{k,ε} - τ_h}{τ_h - τ_{j,ε}} ≤ \frac{C_1}{C_2} (1 + \frac{C_1}{C_2})^{m-2}.
\]
Thus we get the development

\[ R(t,x;\tau,\xi) = \sum_{k=1}^{m} \ell_{k,\varepsilon}(x)P_{k,\varepsilon}(\tau), \]

with some \( |\ell_{k,\varepsilon}(x)| \leq C. \)

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