FINDING ALL MONOMIALS IN A POLYNOMIAL IDEAL

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Abstract. Given a $d \times n$ integer matrix $A$, the main result is an elementary, simple-to-state algorithm that finds the largest $A$-graded ideal contained in any ideal $I$ in a polynomial ring $k[x]$ in $n$ variables. The special case where $A$ is an identity matrix yields that $(t.I) \cap k[x]$ is the largest monomial ideal in $I$, where the generators of $t.I$ are those of $I$ but with each variable $x_i$ replaced by $t_i x_i$ for an invertible variable $t_i$.

It is easy to tell whether an ideal $I$ in a polynomial ring $k[x] = k[x_1, \ldots, x_n]$ contains at least one monomial: it does so if and only if the saturation $(I : (x_1 \cdots x_n)^\infty)$ is the unit ideal. Being more precise about the monomials in $I$ makes the problem a little harder. Here are three equivalent ways to formulate it.

**Question 1.** Fix an ideal $I \subseteq k[x]$.

1. What is the set of monomials in $I$?
2. What is the largest $\mathbb{N}^n$-graded ideal contained in $I$?
3. What is the smallest $(k^*)^n$-scheme containing the zero scheme of $I$?

**Answer 2.** Let $t = t_1, \ldots, t_n$ be a new set of variables. Inside of the Laurent polynomial ring $k[x][t^{\pm 1}]$, let $t.I$ denote the ideal whose generators are those of $I$ where each variable $x_i$ is replaced by $t_i x_i$. The biggest monomial ideal contained in $I$ is $(t.I) \cap k[x]$.

This answer appears, with non-invertible $t$-variables, as Algorithm 4.4.2 in [SST00]. An elementary proof is given there. It is obvious, for instance, that every monomial in $I$ lies in $(t.I) \cap k[x]$, since the $t$ variables are units; and intuitively, there is no way to clear all of the $t$ variables simultaneously from all of the monomials in a given polynomial with more than one term. That said, viewing Question 1 as a special case of a more general problem from multigraded algebra lends insight. For notation, if $A \in \mathbb{Z}^{d \times n}$ is a $d \times n$ matrix of integers, to say that the polynomial ring $k[x]$ is $A$-graded means that each monomial $x^b \in k[x]$ is assigned the $A$-degree $\deg(x^b) = Ab$, the linear combination of the $n$ columns of the matrix $A$ with coefficients $b = b_1, \ldots, b_n$. An ideal $I$ is $A$-graded if it is generated by polynomials whose terms all have the same $A$-degree.

**Theorem 3.** Fix an ideal $I \subseteq k[x]$ and a $d \times n$ matrix $A$ with column vectors $a_1, \ldots, a_n$. Let $t = t_1, \ldots, t_n$ be a new set of variables. Denote by $t.I \subseteq k[x][t^{\pm 1}]$ the ideal whose generators are those of $I$ with each variable $x_i$ replaced by $t^a_i x_i$. The largest $A$-graded ideal contained in $I$ equals the intersection $(t.I) \cap k[x]$.

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After the first version of this note was posted, the authors of [KR05] pointed out that the statement of Theorem 3 is essentially Tutorial 50(a) in their book, an exercise with a suggested proof that is different from the one here.

Remark 4. Details on $A$-graded algebra in general can be found in [MS05, Chapter 8]. The $A$-grading on $k[x]$ corresponds uniquely to the action of a torus $T \cong (k^*)^d$ on $k^n$. (References for this are hard to locate. An exposition appears in Appendix A.1 of the first arXiv version of [KM05], at http://arxiv.org/abs/math/0110058v1.) Under this correspondence, $A$-graded ideals correspond to subschemes of $k^n$ that carry $T$-actions. Therefore the zero scheme of the ideal $(t.I) \cap k[x]$ in Theorem 3 is the smallest $T$-scheme containing the zero scheme $Z(I)$.

Proof of Theorem 3. Let $X = T \times k^n$. Create a subbundle $Y \subseteq X$ over $T$ whose fiber over $\tau \in T$ is the translate $\tau^{-1}.Z(I)$ of the zero-scheme $Z(I)$ by $\tau^{-1}$. The image of the projection of $Y$ to $k^n$ is the minimal $T$-stable scheme containing $Z(I)$ by construction: it is the union of all $T$-translates of $Z(I)$. Therefore the vanishing ideal of the image of the projection is the maximal $A$-graded subideal of $I$. The scheme $Y$ is expressed, in coordinates, as the zero scheme of $t.I$, and the image of its projection to $k^n$ is the zero scheme of $(t.I) \cap k[x]$. □

Remark 5. In contrast to the monomial situation, the binomial analogue of Question 1.1, which begins with, “Is there a binomial in $I$?”, appears to be much harder than the monomial question, as observed by Jensen, Kahle, and Katthän [JKK16]. They note, for example, that for each $d$ there is an ideal in $k[x, y]$ that contains no binomials of degree less than $d$ but nonetheless has a quadratic Gröbner basis and contains a binomial of degree $d$.

References

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