Steklov-type 1D inequalities (a survey)

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Abstract

We give a survey of classical and recent results on sharp constants and symmetry/asymmetry of extremal functions in 1-dimensional functional inequalities.

Keywords: One-dimensional functional inequalities, symmetry, symmetry breaking, sharp constants.

1 Introduction

The first version of this survey was an extended variant of the talk given at the International Conference “Qualitative Theory of Differential Equations” in December 2020. In comparison with it, the present version is supplemented with both historical references and new results. The text has partial intersection with the survey [42] where an extensive bibliography on multidimensional functional inequalities is given.

Sharp constants in one-dimensional functional inequalities are important in various fields of mathematics such as the theory of functions (see [45], [68]), mathematical physics (see, e.g., [21], [18]), mathematical statistics (see [54, §6.2], [44]) etc.

The problem of sharp constants is closely related with the problem of symmetry and symmetry breaking of corresponding extremal functions. The first author was introduced to this topic more than 25 years ago by Yakov Yu. Nikitin and Vladimir A. Kondratiev.

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The pioneer results of this type were sharp constants in the inequalities
\[
\int_0^\ell u^2(x) \, dx \leq \left( \frac{\ell}{\pi} \right)^2 \int_0^\ell [u'(x)]^2 \, dx, \quad \int_0^\ell u(x) \, dx = 0; \quad (1) \quad \text{eq:Steklov1}
\]
\[
\int_0^\ell u^2(x) \, dx \leq \left( \frac{\ell}{\pi} \right)^2 \int_0^\ell [u'(x)]^2 \, dx, \quad u(0) = u(\ell) = 0; \quad (2) \quad \text{eq:Steklov2}
\]
\[
\int_0^\ell u^2(x) \, dx \leq \left( \frac{\ell}{2\pi} \right)^2 \int_0^\ell [u'(x)]^2 \, dx, \quad u(0) = u(\ell), \quad \int_0^\ell u(x) \, dx = 0. \quad (3) \quad \text{eq:Steklov3}
\]

These constants were found by Steklov [63] and [64], respectively (see also [65], and Almansi [2] (see also [41]). See [47, Ch. II], [12] for a comprehensive history of inequalities (1)–(3).

Notice that the sharp constants in (1) and (2) are attained by functions \( \cos(\pi \frac{x}{\ell}) \) and \( \sin(\pi \frac{x}{\ell}) \), respectively (up to a multiplicative constant). So, the extremal functions are symmetric (respectively, odd and even) about the middle of the interval; see Fig. 1. The extremal functions in (3) are given by any linear combination of \( \cos(\pi \frac{x}{2\ell}) \) and \( \sin(\pi \frac{x}{2\ell}) \).

\[\text{Fig1} \]

Figure 1: The graphs of extremal functions for the inequalities (1) (thin line) and (2) (bold line).

## 2 A simplest extension of the inequality (2)

We begin with the inequality
\[
\|u\|_{L^q(0,\ell)} \leq \lambda_1^{\frac{4}{2q} + \frac{1}{p}} \|u'\|_{L^p(0,\ell)}, \quad u(0) = u(\ell) = 0. \quad (4) \quad \text{eq:Friedrichs}
\]

\(^1\)In the literature, inequalities (1)–(3) are often called Poincaré inequalities or Wirtinger inequalities.
(here and below $1 \leq p, q \leq \infty$, and $p'$ stands for the Hölder conjugate exponent to $p$). By dilation, it is easy to see that $\lambda_1$ depends only on $p$ and $q$. So, it is sufficient to consider $\ell = 1$.

Inequality (4) is equivalent to several ones. We list three examples:

\begin{align*}
\|u\|_{L^q(0,1)} &\leq 2\lambda_1\|u'\|_{L^p(0,1)}, \quad u(0) = 0; \\
\|u\|_{L^q(0,1)} &\leq \lambda_1\|u'\|_{L^p(0,1)}, \quad u(0) + u(1) = 0; \\
\|u\|_{L^q(0,1)} &\leq \frac{1}{2}\lambda_1\|u'\|_{L^p(0,1)}, \quad u(0) = u(1), \quad \min u + \max u = 0.
\end{align*}

The following statement holds:

\begin{tfont}{Theorem 1.}
The sharp constant in (4) is given by

$$\lambda_1(p, q) = \frac{\mathfrak{F}_{\frac{1}{p'}} + \frac{1}{q}}{2\mathfrak{F}_{\frac{1}{p'}} \mathfrak{F}_{\frac{1}{q}}},$$

where $\mathfrak{F}(s) = \frac{\Gamma(s+1)}{s^s}$. The corresponding extremal function $U_{p,q}$ can be expressed in quadratures, does not change sign and is even with respect to $x = \frac{1}{2}$, see Fig. 2.

Figure 2: The graphs of extremal functions in (4) for $p = 2$: $q = 1$ (thin line) and $q = \infty$ (bold line).

**Remark 1.** For $p > 1$, the natural space for $u$ in (4) is the Sobolev space $W^{1}_p(0,1)$. For $p = 1$ the sharp constant in (4) is not achieved in the Sobolev space. However, in this case one can consider $u \in BV(0,1)$ and understand the right-hand side of (4) in the sense of measures. In this case, the statement of Theorem 1 is true (except for the case $p = 1, q = \infty$, where there are both symmetric and asymmetric extremals).
The history of Theorem 1 is given in the Table 1.

Remark 2. Notice that the result of Theorem 1 and its particular cases were later rediscovered several times even in the 21st century; see, for instance, [6, p. 50–51], [66, p.220], [9, p.377], [67, p.357], [17, Theorem 5.1], [4], [19].

3 An extension of the inequality (1)

Now we consider the following inequality:

\[
\|u\|_{L^q(0,\ell)} \leq \lambda_2 \ell^{\frac{1}{q} + \frac{1}{p}} \|u'\|_{L^p(0,\ell)}, \quad \int_0^\ell |u(x)|^r u(x) \, dx = 0 \quad (7)
\]

(here \(1 \leq p, q, r \leq \infty\); for \(r = \infty\) the last relation is understood in the limit sense). As in (1), \(\lambda_2 = \lambda_2(p, q, r)\) does not depend on \(\ell\), and we put \(\ell = 1\).

Under additional restriction that \(u\) is 1-periodic, inequality (7) holds with the sharp constant \(\frac{1}{2} \lambda_2\). In the case \(r = 2\), (7) is equivalent to several other inequalities. We again list three examples:

\[
\begin{align*}
\|u'\|_{L^1(0,1)} &\leq \frac{1}{2} \lambda_2 \|u''\|_{L^1(0,1)}, & u(0) = u(1) = u'(0) = u'(1) = 0; \\
\|u - \overline{u}\|_{L^1(0,1)} &\leq \frac{1}{2} \lambda_2 \|u'\|_{L^1(0,1)}, & u(0) = u(1) = 0; \\
\|u^{(k)}\|_{L^1(0,1)} &\leq \frac{1}{2} \lambda_2 \|u^{(k+1)}\|_{L^1(0,1)}, & u \text{ is 1-periodic, } k \in \mathbb{N}
\end{align*}
\]

(\(\overline{u}\) stands for the mean value of \(u\)).

Notice that if the extremal function in (7) is odd w.r.t. \(x = \frac{1}{2}\) then the integral restriction is fulfilled for any \(r\). Therefore, in this case, \(\lambda_2\) does not depend on \(r\). However, the general picture is more complicated.

\(^2\)In fact, Hardy–Littlewood–Pólya and Levin dealt with the first inequality in (5) whereas Schmidt considered the third inequality in (5).
Theorem 2. If \( q \leq (2r - 1)p \) then the following equality holds:

\[
\lambda_2(p, q, r) = \lambda_1(p, q),
\]

see (6). The corresponding extremal function \( V_{p,q} \) is (up to a multiplicative constant) given by formula

\[
V_{p,q}(x) = \begin{cases} 
U_{p,q}(x + \frac{1}{2}) & \text{if } x \leq \frac{1}{2}, \\
-U_{p,q}(x - \frac{1}{2}) & \text{if } x \geq \frac{1}{2},
\end{cases}
\]

where \( U_{p,q} \) is introduced in Theorem 1. In particular, \( V_{p,q} \) is odd w.r.t. \( x = \frac{1}{2} \).

In contrast, if \( q > (2r - 1)p \) then \( \lambda_2(p, q, r) > \lambda_1(p, q) \), and the extremal function \( V_{p,q} \) has no symmetry; see Fig. 3.

Figure 3: The graphs of extremal functions in (7) for \( p = 2, r = 2 \): \( q = 1 \) (thin line) and \( q = \infty \) (bold line).

Remark 3. Similarly to Theorem 1, for \( p = 1 \) the statement of Theorem 2 is true if one understands the right-hand side of (7) in the sense of measures.

If \( r = 1 \) then the statement is true for \( q \leq p \) whereas for \( q > p \) the sharp constant in (7) is not achieved.\(^3\)

For \( r = \infty \) the last relation in (7) is understood in the limit sense as \( \min u + \max u = 0 \). The statement of Theorem 2 is true.

The history of Theorem 2 is given in Table 2.\(^4\)

\(^3\)However, for \( q > p \) any normalized minimizing sequence converges to a non-symmetric function. See [28], §2; cf. also [11].

\(^4\)In fact, Bohr, Dacorogna–Gangbo–Subia, Belloni–Kawohl and Croce–Dacorogna dealt with (7) for periodic functions, while Egorov and Buslaev–Kondratiev–Nazarov considered the first inequality in (8).
| Year | Authors | Symmetry | Asymmetry |
|------|---------|----------|-----------|
| 1896 | Steklov [63] | $r = 2, q = p = 2$ |          |
| 1935 | Bohr [7] | $r = 2, q = p = \infty$ |          |
| 1992 | Dacorogna et al [15] | $r = q$ |          |
| 1992 | [15] | $r = 2, q \leq 2p$ | $r = 2, q >> 1$ |
| 1997 | Egorov [20] |          | $r = 2, q > 4p - 1$ |
| 1998 | Buslaev et al [10] | $r = 2, q \leq 2p + \varepsilon$ |          |
| 1999 | Belloni, Kawohl [3] | $r = 2, q \leq 2p + 1$ |          |
| 2002 | Nazarov [50] | $r = 2, q \leq 3p$ |          |
| 2002 | Abessolo [1] | $q \leq rp + \varepsilon$ | $q > r^2p - (r - 1)^2$ |
| 2003 | Croce, Dacorogna [14] | $q \leq rp + r - 1$ | $q > (2r - 1)p$ |
| 2011 | Gerasimov, Nazarov [28] | $q \leq (2r - 1)p$ |          |
| 2018 | Ghisi et al [29] | $q \leq (2r - 1)p$ |          |

Table 2: The history of inequality (7)

Remark 4. In [15, Sect. 1.1.19], the equality $\lambda_3(p, p, 2) = \lambda_1(p, p)$ is attributed to A. Stanoyevitch [61]. However, the proof in [61] turned out to be incorrect. A part of Theorem 2 proved in [15] was also rediscovered later. See [22] for the case $p = q = r$ and [5] for $r = 2, q \leq 2p$. On the other hand, it is erroneously stated in [17, Theorem 5.2] that $\lambda_2(p, q, 2) = \lambda_1(p, q)$ for any $q$.

Up to our knowledge, a unique explicit expression for $\lambda_2$ in the asymmetry region of parameters is as follows.

Proposition 1 ([10, §4; see also [62], 7]). $\lambda_2(p, \infty, 2) = (p' + 1)^{-\frac{1}{p'}}$.

4 A higher-order extension of inequality (2)

Further, we consider the following inequality:

$$
\|u^{(k)}\|_{L^q(0, \ell)} \leq \lambda_3 \ell^{\alpha - k + \frac{1}{p'} - \frac{1}{p}}\|u^{(n)}\|_{L^p(0, \ell)}, \quad u \in W_p^n(0, \ell) \tag{9} \text{ eq:high}
$$

(here $n, k \in \mathbb{Z}_+, n > k$).

Remark 5. As earlier, $\lambda_3 = \lambda_3(n, k, p, q)$ does not depend on $\ell$, and we can put $\ell = 1$. For $p = 1$ the right-hand side of (9) is understood in the sense of measures. Evidently, $\lambda_3(1, 0, p, q) = \lambda_1(p, q)$. Moreover, by (8) we have $\lambda_3(2, 1, p, q) = \frac{1}{2}\lambda_2(p, q, 2)$.

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5A technical gap in the proof was fixed in [39].

6In [28], a computer-assisted proof was given, while Ghisi–Gobbino–Rovellini succeeded in a pure analytical proof.

7Stechkin considered (7) and some higher-order inequalities for periodic functions.
Remark 6. Recall that for $u \in \dot{W}_p^m(0, \ell)$, all derivatives of $u$ up to $(n - 1)$-th order vanish at the endpoints of the interval. In this connection we mention the paper [8], where the inequality

$$
\|u\|_{L^p(0, \ell)} \leq \hat{\lambda}_3 \ell^{2+\frac{1}{q}} \|u''\|_{L^p(0, \ell)}, \quad u(0) = u(\ell) = 0,
$$

was investigated. In particular, it was proved that the extremal function in this inequality is even w.r.t. the middle of the interval. Also (somewhat unexpectedly) in the case $q = p'$ the sharp constant in this inequality was calculated explicitly for general $p$. It turns out that

$$
\hat{\lambda}_3(p, p') = \lambda_3(2, p'),
$$

and the extremal function is just $U_{2, p'}$, see Theorem 1.

The results listed earlier, as well as some calculations, lead to the following conjecture; see [48].

**Conjecture 1.** If $k$ is even, then the extremal function in the problem (9) is even w.r.t. $x = \frac{1}{2}$ for all admissible $n, p, q$ (except for the case $p = 1, q = \infty, n = k + 1$). If $k$ is odd, then for all admissible $n$ and $p$ there exists $\hat{q}(n, k, p) > p$ such that the extremal is even w.r.t. $x = \frac{1}{2}$ for $q \leq \hat{q}$ and is non-symmetric for $q > \hat{q}$.

The known values of $\lambda_3$, besides $n = 1, k = 0$, and $n = 2, k = 1$, concern the cases $p = 2 (q = 1, 2)$ or $q = \infty$.

The following statement was proved in [35] for $k = n - 1$ and in [56] in the general case.

**Theorem 3.** $\omega = \lambda_3^{\frac{1}{n-k}}(n, k, 2, 2)$ is the least positive root of the function

$$
\Phi^{(n,k)}(\omega) = \det \left[D^{(n,k)}(\omega)\right],
$$

where $D^{(n,k)}(\omega)$ is the $(n-k) \times (n-k)$-matrix with entries

$$
D_{jm}^{(n,k)}(\omega) = (\omega z^m)^{2k+2j-1} \frac{J_{2k+2j-1}(\omega z^m)}{2}, \quad j, m = 0, \ldots, n-k-1
$$

(here $z = e^{\pi i k}$, whereas $J_{\nu}$ is the Bessel function of the first kind). The corresponding extremal function is even w.r.t. $x = \frac{1}{2}$, see Fig. 4.

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8Evidently, $\hat{\lambda}_3(p, q) > \lambda_3(2, 0, p, q)$ for all $p, q$.

9In this case extremal function can be as symmetric as asymmetric; cf. [27] Theorem 3], where it was shown that $\lambda_3(n, n - 1, 1, \infty) = \frac{1}{2}$.

10In [38], two-sided estimates of $\lambda_3(n, 0, 2, q)$ were obtained for general $q$.

11The case $n = 2$ was considered in [33]. The announcement without proof was given in [34]. Later, the result of [35] was rediscovered in [52].

12See also [13] and [36] (without proof), and [60] for $k = n - 2$. 

7
Figure 4: The graphs of extremal functions in (9) for $p = 2$, $q = 2$: $n = 4$, $k = 2$ (thin line) and $n = 3$, $k = 1$ (bold line).

For $q = \infty$ and general $p$, the answers are known only for $n = 2$, $k = 0$ \cite{53,70} and $n = 3$, $k = 0$ \cite{70}.

**Theorem 4.** The following equalities hold:

$$\lambda_3(2, 0, p, \infty) = \frac{1}{8} (p' + 1)^{-\frac{1}{p'}}.$$

$$\lambda_3(3, 0, p, \infty) = \frac{1}{16} \cdot \min_{\alpha \in (0, 1)} \left( \int_0^1 x'^r |x - \alpha|^{p'} dx \right)^{\frac{1}{p'}}.$$

The corresponding extremal function is even w.r.t. $x = \frac{1}{2}$.

It is convenient to introduce the function

$$A_{n,k,p}(a) = \max\{|u^{(k)}(a)| : u \in W^n_p(0, 1), \|u^{(n)}\|_{L^p(0,1)} \leq 1\}, \quad a \in (0, 1).$$

**Remark 7.** It is evident that $\lambda_3(n, k, p, \infty) = \max_{a \in (0, 1)} A_{n,k,p}(a)$. Moreover, it is easy to see that the extremal function in (9) with $1 < p < \infty$ and $q = \infty$ is even w.r.t. $x = \frac{1}{2}$ if and only if $\max_{a \in (0, 1)} A_{n,k,p}(a) = A_{n,k,p}(\frac{1}{2})$.

**Remark 8.** The function $A_{n,k,2}(a)$ was introduced in \cite{37}. In particular, it was shown in \cite{37} that $A_{n,k,2}^2(a)$ is a degree $2n - 1$ polynomial of the variable $t = a - a^2 \in (0, \frac{1}{4})$.

The following statement holds:

\cite{13}Recently Garmanova and Sheipak \cite{27} derived a general relation between $\lambda_3(n, k, p, \infty)$ and the best approximation of a special spline by polynomials in $L^p(0,1)$. However, at the moment the explicit result is obtained only in some particular cases, see below.
Theorem 5. Let $k$ be even. Then

$$\lambda_3(n, k, 2, \infty) = A_{n,k,2}(\frac{1}{2}) = \frac{(k-1)!!}{2^{n-\frac{3k}{2}-1}(n-\frac{k}{2}-1)!\sqrt{2n-2k-1}}.$$ 

The corresponding extremal function is even w.r.t. $x = \frac{1}{2}$.

In contrast, if $k$ is odd, then $\lambda_3(n, k, 2, \infty) > A_{n,k,2}(\frac{1}{2})$, and the extremal function has no symmetry; see Fig. 5.

![Fig5](image)

Figure 5: The graphs of extremal functions in (9) for $p = 2, q = \infty$: $n = 5, k = 2$ (left) and $n = 5, k = 3$ (right).

If $k$ is odd then explicit expressions for $\lambda_3(n, k, 2, \infty)$ are known only for $k = 1$ [37], $k = 3, 5$ [59].

Theorem 6. The following equalities hold:

$$\lambda_3(n, 1, 2, \infty) = A_{n,1,2}(a_{1,2}) = \frac{1}{(n-1)!} \left( \frac{n-1}{2(n-1)} \right)^{n-1} \sqrt{\frac{2n-1}{2n-3}},$$

where $a_{1,2} = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2n-1}} \right);$ 

$$\lambda_3(n, 3, 2, \infty) = A_{n,3,2}(a_{3,4}) = \left( \frac{(n-2)(2n-3) + \sqrt{(n-2)(2n-3)}}{2(2n-1)(2n-3)} \right)^{n-\frac{7}{2}} \times \sqrt{\frac{3(n-2)(2n-5) - (2n-7)}{(n-2)(2n-1)}} \sqrt{\frac{3(2n-3)}{2(2n-7)}},$$

where

$$a_{3,4} = \frac{1}{2} \left( 1 + \sqrt{1 - 2 \left( \frac{(n-2)(2n-3) + \sqrt{3(n-2)(2n-3)}}{2(2n-1)(2n-3)} \right)} \right);$$

[14] In [25], two-sided estimates of $\lambda_3(n, k, 2, \infty)$ were obtained for odd $k.$
\[ \lambda_3(n, 5, 2, \infty) = A_{n,5,2}(a_{5,6}) \equiv A_{n,5,2}(\frac{1}{2}(1 + \sqrt{1 - 4t})) , \]

where

\[
t = \frac{n-3}{2(2n-1)} + \frac{\sqrt{5(n-3)}}{(2n-1)\sqrt{2n-3}} \cos \left( \frac{1}{3} \arccos \left( -2n-11 \sqrt{\frac{2n-3}{5(n-3)}} \right) \right). \]

The explicit expression for \( \lambda_3(n, 5, 2, \infty) \) is quite complicated, and we omit it.

Figure 6 shows examples of functions \( A_{n,k,2}(a) \) for even and odd \( k \).

For \( p = q = \infty \), the sharp constant is known in the cases \( k = n - 1 \) [27] and \( k = 0 \) [40].

**Theorem 7.**

1. Let \( k = n - 1 \) be even. Then

\[
\lambda_3(n, n-1, \infty, \infty) = A_{n,n-1,\infty}(\frac{1}{2}) = \tan \left( \frac{\pi}{2(n+1)} \right) .
\]

The corresponding extremal function is even w.r.t. \( x = \frac{1}{2} \).

2. Let \( k = n - 1 \) be odd. Then

\[
\lambda_3(n, n-1, \infty, \infty) = A_{n,n-1,\infty}(a_{\frac{n}{2}}) = A_{n,n-1,\infty}(a_{\frac{n}{2}+1}) = \tan \left( \frac{\pi}{2(n+1)} \right) \sin \left( \frac{\pi n}{2(n+1)} \right),
\]

where \( a_j = \sin^2 \frac{\pi j}{4(n+1)} \), \( j = 1, \ldots, n \).

The corresponding extremal function is even w.r.t. \( x = \frac{1}{2} \), see Fig. 7.
Remark 9. Notice that for odd \( k = n - 1 \) the extremal function is even, despite the fact that \( \max_{a \in (0,1)} A_{n,n-1,\infty}(a) > A_{n,n-1,\infty}(\frac{1}{2}) \). This distinguishes the case \( p = \infty \) from the case \( p < \infty \), cf. Remark 7.

The symmetry of the extremal function was not discussed in [27]. We prove it in the Appendix.

Figure 8 shows examples of functions \( A_{n,n-1,\infty}(a) \) for odd and even \( n \). The algorithm for constructing these graphs was given in the paper [16].

\[ \int_0^1 (1 - x^2)^n \frac{1}{1 + (1 - x^2)^n} \, dx. \]

This integral can be expressed in terms of hypergeometric functions, see [40].

The corresponding extremal function is even w.r.t. \( x = \frac{1}{2} \).
Now we turn to the case $q = 1$. The first result here was obtained quite recently [31].

**Theorem 9.** The following equality holds:

$$\lambda_3(n, 0, 2, 1) = \frac{n!}{(2n)!\sqrt{2n + 1}}.$$  

The corresponding extremal function equals $x^n(1-x)^n$ (up to a multiplicative constant). Evidently, it is even w.r.t. $x = \frac{1}{2}$.

**Conjecture 2.** For any $n \geq 2$ and $1 \leq p \leq \infty$, the following equality holds:

$$\lambda_3(n, 1, p, 1) = \lambda_3(n, 0, p, \infty).$$

Corresponding extremal functions coincide.

The history of Theorems 3–9 is given in Table 3.

| Year  | Authors             | $n$ | $k$ | $p$ | Symm. | Asymm.   |
|-------|---------------------|-----|-----|-----|-------|----------|
| 1940  | Schmidt [58]        | 1   | 0   | $\forall$ | $\forall$ | $q > 3p$ |
| 1998  | Buslaev et al [10] | 2   | 1   | $\forall$ | $q \leq 3p$ |
| 2002  | Nazarov [50]        | 2   | 1   | $\forall$ | $q = 2$   |
| 1931  | Janet [55]          |    | $n-1$ | 2   | $q = 2$ |
| 2017  | Yu. Petrova [56]    |    | $\forall$ | 2   | $q = \infty$ |
| 2008  | Oshime [55, 17]     | 2   | 0   | $\forall$ | $q = \infty$ |
| 2009  | Watanabe et al [70] | 3   | 0   | $\forall$ | $q = \infty$ |
| 2010  | Kalyabin [37, 18]   | 1   | 0, 2 | 2   | $q = \infty$ |
| 2010  |                     | 1   | 0, 1 | 2   | $q = \infty$ |
| 2014  | Mukoseeva, Nazarov [48] | $\forall$ | 4, 6 | 2   | $q = \infty$ |
| 2014  |                     | 1   | even | 2   | $q = \infty$ |
| 2021  | Garmanova, Sheipak [26] | $\forall$ | even | 2   | $q = \infty$ |
| 2024  | Garmanova, Sheipak [27, 19] | $\forall$ | $n-1$ | $\infty$ | $q = \infty$ |
| 2024  | Kazimirov, Sheipak [40] | $\forall$ | 0   | $\infty$ | $q = \infty$ |
| 2024  | Hindov et al [31]   | $\forall$ | 0, 2 | 2   | $q = 1$ |

Table 3: The history of inequality (9)

15 See also [52].
16 See also [60] for the case $n = k + 2$, and [46].
17 See also [70].
18 See also [71] for the case $k = 0$.
19 See also Appendix.
5 A non-homogeneous inequality for periodic functions

Finally, we consider an estimate related to the inequality (3):

\[ \|u\|_{L^p(0,\ell)}^p \leq \mu \|u\|_{W^{1,p}_0(0,\ell)}^p = \mu \int_0^\ell (|u'(x)|^p + |u(x)|^p) \, dx, \quad u(0) = u(\ell). \]  

(10)  

By dilation, we can reduce (10) to the case \( \ell = 1 \):

\[ \|u\|_{L^p(0,1)}^p \leq \tilde{\mu} \int_0^1 (|u'(x)|^p + m|u(x)|^p) \, dx, \quad u(0) = u(1). \]  

(11)  

Here \( m > 0 \), and \( \tilde{\mu} \) depends on \( p, q, \) and \( m \).

For \( q \leq p \), using the Hölder inequality we conclude that the extremal function in (11) is constant and thus \( \tilde{\mu}(p, q, m) \equiv m^{-1} \).

For \( q > p \), the problem of sharp constant in (11) is more delicate and depends on \( p \). In [49] it was shown that for \( p > 2 \) and any \( q > p \) the extremal function in (11) is non-constant and therefore \( \tilde{\mu}(p, q, m) > m^{-1} \). In contrast, if \( p < 2 \) then the constant function is a local extremal in (11) for arbitrary \( q \). However, given \( q > p \), for sufficiently small \( m \) the extremal function is constant whereas for sufficiently large \( m \) the extremal function is non-constant [20].

The most interesting case is \( p = 2 \). In this case we consider a more general inequality\(^{21}\) with “magnetic term”

\[ \|u\|_{L^q(0,2\pi)}^2 \leq \lambda_4 \int_0^{2\pi} \left( |(e^{ix}u(x))'|^2 + m|u(x)|^2 \right) \, dx, \quad u(0) = u(2\pi) \]  

(12)  

\( \text{eq:magnetic} \)

(here \( \lambda_4 = \lambda_4(\alpha, m, q) \)).

It is easy to see that both sides of (12) are invariant w.r.t. replacement \( \alpha \mapsto \alpha + k \) and \( u(x) \mapsto u(x)e^{-ikx}, \, k \in \mathbb{Z} \). Therefore, without loss of generality, we may assume that \( |\alpha| \leq \frac{1}{2} \). Then the necessary and sufficient condition of the validity of inequality (12) is \( m + \alpha^2 > 0 \).

Under these assumptions, the following statement holds:

\(^{20}\)This problem was considered earlier in [39], but the conclusion in this paper is not correct.

\(^{21}\)For convenience we choose \( \ell = 2\pi \).
Theorem 10. Let \( \alpha^2(q + 2) + m(q - 2) \leq 1 \). Then
\[
\lambda_4(\alpha, m, q) = \frac{(2\pi)^{\frac{q-1}{2}}}{m + \alpha^2}.
\]
The corresponding extremal function is constant.

In contrast, if \( \alpha^2(q + 2) + m(q - 2) > 1 \) then \( \lambda_4(\alpha, m, q) > \frac{(2\pi)^{\frac{q-1}{2}}}{m + \alpha^2} \), and the extremal function is non-constant.

Remark 10. In the borderline case \( \alpha^2(q + 2) + m(q - 2) = 1 \) we conclude from \( |\alpha| \leq \frac{1}{2} \)
\[
q = \frac{1 + 2(m - \alpha^2)}{m + \alpha^2} \geq 2.
\]
In particular, this means that for \( q \leq 2 \) the extremal function in (12) is always constant.

The history of Theorem 10 is given in Table 4.

| Year | Authors               | Parameters |
|------|-----------------------|------------|
| 1999 | Nazarov \[49]\[^{22}\] | \( \alpha = 0 \) |
| 2004 | Nazarov \[51]\[^{23}\] | \( \alpha = 0 \) |
| 2018 | Nazarov, Shcheglova \[53\] | \( m = 0 \) |
| 2018 | Dolbeaut et al \[18\] | \( \forall \alpha, m \) |

Table 4: The history of inequality (12)

Up to our knowledge, a unique explicit expression for \( \lambda_4 \) in the non-constancy region of parameters is as follows.

Theorem 11 (\[23\], §4, and \[24\]\[^{24}\]). Let \( m + \alpha^2 > 0 \). Then
\[
\lambda_4(m, \alpha, \infty) = \begin{cases} 
\sin(2\pi\sqrt{m}) \cos(2\pi\alpha), & m > 0; \\
\frac{1}{2\sqrt{m}} \cosh(2\pi\sqrt{m}) \cos(2\pi\alpha), & m = 0; \\
\frac{1}{2\sqrt{-m}} \sin(2\pi\sqrt{-m}) \cos(2\pi\alpha), & m < 0.
\end{cases}
\]

\[^{22}\] for some values of \( m \).
\[^{23}\] In \[51\], inequalities of arbitrary order were considered. See also \[69\] for a related result.
\[^{24}\] Galunov and Oleinik considered also some higher-order inequalities. See also \[32\] for a related result.
Here we prove that the extremal function in the inequality \((9)\) for \(p = q = \infty\) and \(k = n - 1\) is even w.r.t. \(x = \frac{1}{2}\).

Theorem 2 in [27] implies that
\[
A_{n,n-1,\infty}(a) = \min_P \|\chi_{[0,a]} - P\|_{L^1(0,1)},
\]
where the minimum is taken over the set of polynomials of degree no greater than \(n - 1\).

Let \(P_*\) be a minimizing polynomial in \((13)\). Then the necessary condition of minimum reads as follows (see Statement 2 in [27] or [57, Theorem 2]):
\[
\int_0^1 x^{k-1} \text{sign}(\chi_{[0,a]} - P_*) \, dx = 0, \quad k = 1, \ldots, n,
\]
and thus the difference \(\chi_{[0,a]} - P_*\) should change sign at \(N \geq n\) points (denote them \(a_1 < a_2 < \cdots < a_N\)). On the other hand, since the values of \(P_*\) at all points \(a_j\), except maybe for \(a_j = a\), are prescribed, we have \(N \leq n + 1\).

Let \(P\) be another minimizing polynomial. Then Proposition 2.4 in [57] shows that
\[
(\chi_{[0,a]} - P_*)(\chi_{[0,a]} - P) \geq 0 \quad \text{a.e. on } [0, 1].
\]
This means that the difference \(\chi_{[0,a]} - P\) changes sign at the same points.

Statement 3 in [27] implies that the function \(w_{n,a}\) providing the value \(A_{n,n-1,\infty}(a)\) is the \(n\)-th primitive of the function \(\text{sign}(\chi_{[0,a]} - P_*)\). So, if we choose \(a\) as the maximum point of \(A_{n,n-1,\infty}\) given in Theorem 7, then \(w_{n,a}^{(n-1)}\) reaches its maximum at \(a\), and therefore \(a\) belongs to the set \(\{a_j\}_{j=1}^N\).

The condition \((14)\) can be rewritten as follows:
\[
a_1^k - a_2^k + \cdots + (-1)^{N-1}a_N^k + \frac{(-1)^N}{2} = 0, \quad k = 1, \ldots, n.
\]
It is easy to prove (cf. [12, Corollary 13]) that
\[
a_j = a_j = \sin^2 \frac{\pi j}{4(n+1)}, \quad j = 1, \ldots, n,
\]
satisfy the equations \((15)\) with \(N = n\). Notice that the maximum point of \(A_{n,n-1,\infty}\) belongs to the set \(\{a_j\}_{j=1}^n\), as required.

It remains to observe that equalities \((16)\) imply symmetry of the function \(\text{sign}(\chi_{[0,a]} - P_*)\): it is even or odd w.r.t. \(x = \frac{1}{2}\) if \(n\) is even or odd, respectively. Since the extremal function in the original inequality is its \(n\)-th primitive, it is even w.r.t. \(x = \frac{1}{2}\) in both cases, and we are done.

Notice that \(L_1(0,1)\) is not a strictly convex space, so in general we have no uniqueness of the best approximation function.
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