Cosmological parameter estimation: impact of CMB aberration

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Abstract. The peculiar motion of an observer with respect to the CMB rest frame induces an apparent deflection of the observed CMB photons, i.e. aberration, and a shift in their frequency, i.e. Doppler effect. Both effects distort the temperature multipoles $a_{\ell m}$'s via a mixing matrix at any $\ell$. The common lore when performing a CMB based cosmological parameter estimation is to consider that Doppler affects only the $l = 1$ multipole, and neglect any other corrections. In this paper we reconsider the validity of this assumption, showing that it is actually not robust when sky cuts are included to model CMB foreground contaminations. Assuming a simple fiducial cosmological model with five parameters, we simulated CMB temperature maps of the sky in a WMAP-like and in a Planck-like experiment and added aberration and Doppler effects to the maps. We then analyzed with a MCMC in a Bayesian framework the maps with and without aberration and Doppler effects in order to assess the ability of reconstructing the parameters of the fiducial model. We find that, depending on the specific realization of the simulated data, the parameters can be biased up to one standard deviation for WMAP and almost two standard deviations for Planck. Therefore we conclude that in general it is not a solid assumption to neglect aberration in a CMB based cosmological parameter estimation.

Keywords: cosmological parameters from CMBR, CMBR theory

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1 Introduction

The Cosmic Microwave Background is used as a fundamental tool to test Cosmological Models and to quantitatively extract the parameters of such models. This is usually done by extracting the Temperature and the Polarization power spectra from the maps and by fitting them with a ΛCDM background model with an almost scale-invariant gaussian spectrum of density perturbations. However, if we observe the CMB with a large velocity \( \beta \equiv v/c \) relative to such background, the image undergoes distortions due to the Doppler effect and due to the aberration effect and it is natural to wonder whether there is any sizable bias on such spectra and so also on cosmological parameter estimation due to the fact that we are not at rest with the CMB.

It is always commonly assumed that the only sizable effect of an observer’s velocity on the CMB map is the generation of a large dipole and any other possible effect is commonly ignored in all analyses. In fact, if we consider a perfectly homogeneous map, there is a Doppler effect which induces a dipole of order \( \beta \), while higher \( n \)-th multipoles are of order \( \beta^n \). Since we observe a dipole of order \( 10^{-3} \), while other multipoles are \( O(10^{-5}) \), the standard lore is to interpret this large dipole as due to our velocity (which is then inferred to be \( \beta = (1.23 \pm 0.003) \times 10^{-3} \) [1]), and otherwise to ignore completely any other effect of our velocity.

However this would be true only on a perfectly homogenous sky. In reality we have a nontrivial map with nonzero coefficients \( a_{\ell m} \), which are measured to be of order \( 10^{-5} \) for any \( \ell > 1 \). Applying a Doppler effect induces a mixing between multipoles [2–4]. Precisely there is a mixing at order \( \beta \) between neighboring multipoles \( \ell \) and \( \ell \pm 1 \) and a mixing at order \( \beta^n \) among \( \ell \) and \( \ell \pm n \). Moreover there is another important distortion, that is aberration, which
changes the direction of observation of incoming photons, similarly to what lensing does. Aberration [2–4] can be computed for small $\beta$ and induces an additional coupling between neighboring multipoles of order $(\beta \ell)^n$. Such a distortion is therefore even more important because it becomes large at high $\ell$. Note that this is in fact the most important secondary effect on the CMB and it is not clear a priori if it induces a negligible bias on the power spectra and on the cosmological parameters: an $O(\beta)$ or $O(\beta \ell)$ effect on all multipoles could a priori lead to a large effect, when summed over all $\ell$ and $m$’s. Moreover for aberration this effect grows as $\beta \ell$ and becomes dominant at large $\ell$: using the value $\beta = 1.23 \times 10^{-3}$ the distortion is order 1 on a single coefficient already at $\ell \gtrsim 800$! What happens in fact is that the aberration is deflecting the arrival direction of the photons by an angle of order $\beta$ and so if we are looking at angles smaller than $\beta$ there is an order 1 effect on the CMB.

However, a negative result came from [2] where it was found that, defining as usual $C_\ell \equiv 1/(2\ell + 1) \sum_m a^*_{\ell m} a_{\ell m}$, the bias on the ensemble average $\langle C_\ell \rangle$ undergoes some cancellations leading to a very small deviation which does not grow with $\ell$, precisely $\delta \langle C_\ell \rangle / \langle C_\ell \rangle = O(\beta^2)$ at leading order in $\beta$.

More recently instead it has been realized that such conclusion is not the final word for several reasons. First of all such calculations are based on a perturbative approach in $\beta$ and so, as we mentioned before, they apply only up to $\ell \lesssim 800$, while we do not know if a clear bias could show up at higher $\ell$’s [4, 5]. Second, such an estimate is based on a full-sky definition for $C_\ell$: in fact, in full-sky, because of the sum over $m$ there are cancellations between the corrections on the individual $a_{\ell m}$’s. However we never observe a full-sky in an experiment and it can be easily seen that as long as one part of the sky is removed such cancellations do not hold and we can get again effects of order $\beta \ell$, which are larger, on the pseudo-$C_\ell$’s which have to be used in a partial sky estimate. This was noticed in [6], where the bias on the pseudo-$C'_\ell$s has been analyzed for small $\beta \ell$ and for very low $\ell$, less than about 20. This estimate becomes very expensive computationally at higher $\ell$ and anyway not feasible within perturbation theory at large $\ell \gtrsim 1/\beta$. For all these reasons the question whether the effect of velocity is negligible on the power spectrum is still lacking an answer.

The scope of the present paper is to resolve this issue by analyzing the bias on cosmological parameter estimation due to a nonzero $\beta$ at any $\ell$ and for both full-sky and cut-sky. As we have stressed this could be done perturbatively analytically only for low-$\ell$, while another possibility, valid at any $\ell$, would be to write down the exact mixing matrix between any multipole without perturbative expansions. This would amount to numerically compute integrals of highly oscillating Legendre polynomials, which becomes very slow at high-$\ell$, and becomes extremely slow for a cut-sky, since it involves also a window function. This approach has been addressed by recursive formulae in [7] and by approximate fitting functions in [5].

The way we proceed here is a more direct and straightforward one: we simulate maps of the CMB sky and we directly apply on the maps the boost trasformation before extracting the $a_{\ell m}$’s, therefore bypassing the need of computing the mixing coefficients. Then we extract the $a_{\ell m}$’s and the $C_\ell$’s and finally we run a Markov Chain Monte Carlo (MCMC) to estimate cosmological parameters and evaluate the bias between the moving observer and the rest frame observer. This approach also allows us to easily implement sky-cuts and to check whether the effect is larger on a realistic cut-sky experiment. We apply all this procedure to Temperature maps only and both for WMAP-like resolution and for Planck-like resolution.

The paper is organized as follows: in section 2 we review the impact of aberration and Doppler on CMB observables. Section 3 is instead devoted to a detailed description of the procedure followed to simulate CMB temperature maps properly including Doppler and...
aberration effects. These maps are then analyzed in section 4. In section 5 we check that using fitting functions from [5] we can reconstruct the aberrated $C_\ell$’s analytically. Finally section 6 summarizes our results.

2 CMB aberration and Doppler

For an observer in motion with respect to a light source, the aberration phenomenon consists in the apparent deflection of the observed light bundles due to the relative motion between the observer and the source. In the case of the cosmic microwave background (CMB), as a consequence of aberration, an observer moving with respect to the CMB rest frame assigns to a photon emitted along the direction $\hat{n}$ at the last scattering surface the arrival direction $\hat{n}'$, where $\hat{n}$ and $\hat{n}'$ are unit vectors. The aberration angle $\alpha \equiv \hat{n}' - \hat{n}$ can be calculated from the velocity transformation relating the CMB and the observer reference frames: assuming that the observer moves along the direction of the z-axis (identified by the unit vector $\hat{z}$) with speed $\beta$ compared to the CMB reference frame, one finds

$$\alpha \cdot \hat{z} = \frac{\beta \sin^2 \theta}{1 + \beta \cos \theta}$$

(2.1)

where $\cos \theta = \hat{n} \cdot \hat{z}$. We refer to the relative motion between the observer and the CMB rest frame as the peculiar motion of the observer. Besides an aberration of the CMB radiation, a peculiar motion of the observer induces also a change in the frequency of the observed photons. According to Special Relativity the frequency $\nu'$ in the moving frame is related to the frequency $\nu$ measured at rest by the usual Doppler effect:

$$\nu' = \nu \gamma (1 + \beta \hat{n} \cdot \hat{z})$$

(2.2)

where $\gamma = 1/\sqrt{1 - \beta^2}$. Doppler and aberration effects unavoidably distort the CMB temperature and polarization maps currently measured by experiments such as for instance the WMAP or Planck satellites. This can be seen as follows: CMB experiments directly measure the brightness of the observed radiation. The brightness $I(\nu)$ is then conventionally translated into an equivalent value for the thermodynamic temperature $T$ through the black body law

$$T = \frac{I(\nu)}{\nu^2} \frac{e^{\nu/T} - 1}{\nu/T}.$$  

(2.3)

By solving eq. (2.3) one finds

$$T = \frac{\nu}{\log \left(1 + \frac{\nu^3}{I(\nu)}\right)}$$

(2.4)

which is therefore the temperature indirectly measured by CMB experiments. Analogously, in the case of the CMB polarization one translates the quantities actually measured into the value of an effective temperature obtained replacing in eq. (2.4) the brightness $I(\nu)$ with the relevant Stokes parameters. Since the ratio $\nu^3/I(\nu)$ is invariant with respect to frame transformations [2], one can conclude from eq. (2.4) that under a Lorentz boost the temperature has to transform as follows

$$T'(\hat{n}') = \gamma (1 + \beta \hat{n} \cdot \hat{z}) T(\hat{n})$$

(2.5)

where $T'$ is the temperature in the moving frame and $T$ is the temperature measured by an observer at rest (with respect to the CMB). Eq. (2.5) together with eq. (2.1) thus describes
how a temperature map is distorted consequently to aberration and Doppler effects: the temperature associated with the direction $\hat{n}'$ by an observer in motion with peculiar velocity $\beta \hat{z}$ is equal to the temperature that an observer at rest would assign to the direction $\hat{n}$, multiplied by a corrective Doppler factor which depends on $\beta$. Expanding in spherical harmonics both $T'(\hat{n}') = \sum_{\ell m} a'_{\ell m} Y_{\ell m}(\hat{n}')$ and $T(\hat{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n})$, eq. (2.5) can be rewritten as follows\(^1\)

$$a'_{\ell m} = \sum_{\ell} \int d\hat{n} a_{\ell m'} [\gamma (1 + \beta \hat{n} \cdot \hat{z})]^{-2} Y_{\ell m'}^*(\hat{n}') Y_{\ell m}(\hat{n}).$$

(2.6)

This expression can be then used to compute the desired correlation functions and compare these results with observations, properly accounting for aberration and Doppler effects. Eq. (2.6) is exact, though it relies on an integral computationally very expensive for high multipoles.

In the following subsections we will review two approximations recently discussed in the literature to estimate the coefficients $a'_{\ell m}$ of eq. (2.6). In sections 3 and 4, instead, we will show how starting from eq. (2.5) one can actually include aberration and Doppler effects exactly in a CMB based cosmological parameter estimation, by transforming directly simulated maps of the sky in the boosted frame.

### 2.1 Perturbative expansions

Since our peculiar velocity is small in natural units, i.e. $\beta \simeq 1.23 \times 10^{-3}$, eq. (2.6) can be evaluated expanding the integrand around $\beta$ equal to zero. The leading corrections in $\beta$ necessary to evaluate the $C_\ell$'s are given by

$$a'_{\ell m} \simeq c_{\ell m} a_{\ell-1 m} + c'_{\ell m} a_{\ell+1 m} + a_{\ell m} (1 + d_{\ell m})$$

(2.7)

where

$$c_{\ell m}^+ = \beta (\ell + 1) G_{\ell+1, m},$$

$$c_{\ell m}^- = -\beta \ell G_{\ell, m},$$

$$d_{\ell m} = \frac{\beta^2}{2} \left[ (\ell + 1)(\ell + 2) G_{\ell+1, m}^2 + \ell(\ell - 1) G_{\ell, m}^2 - \ell(\ell + 1) + m^2 - 1 \right]$$

(2.8)

and $G_{\ell, m} \equiv \sqrt{\ell^2 - m^2}/2\ell + 1$. Similar expressions can be derived for the analogous coefficients corresponding to the Stokes parameters describing the CMB polarization. We refer the reader to refs. [2–5] for an explicit derivation of such expressions; note that the correct expressions for Temperature fluctuations for the $c_{\ell m}^\pm$ have been derived first in [3] and for $d_{\ell m}$ in [5] and they slightly differ from [2], which would be valid for Intensity instead of Temperature. We note here that the corrections increase as powers of $\beta \ell$, and so become large for high $\ell$ and so for $\ell > 1/\beta$ the perturbative approach breaks down. This happens because the aberration effect is a distortion on the angle of order $1/\beta$, and this is not a small correction if we are looking at angular scales smaller than $1/\beta$. Starting from eq. (2.7) one can easily compute the desired correlation functions. In the case of the angular power spectrum $C_\ell' \equiv \langle a'_{\ell m} a'_{\ell m} \rangle$,
for instance, one obtains

\[
C'_\ell = \sum_{\ell'} C_{\ell'} \left\{ \delta_{\ell\ell'} \left( 1 - \frac{1}{3} \beta^2 (\ell^2 + \ell + 1) \right) + \delta_{\ell(\ell+1)} \beta^2 \frac{\ell^3}{3(2\ell + 1)} \right\}.
\]

(2.9)

Thus, for \( \ell < 1/\beta \) the leading corrections to the primordial angular power spectrum \( C_\ell \equiv \langle a_{\ell m} a_{\ell m}^* \rangle \) are small, i.e. \( O(\beta^2 \ell^2) \), and moreover they are subject to a further suppression since they would give zero in the limit of flat angular power spectrum when \( C_\ell \sim C_{\ell+1} \). This is the reason why in the literature the aberration has been not considered so far in the cosmological parameter probabilistic inference: according to eq. (2.9) the aberration only mildly affects the observed \( C'_\ell \) and can be therefore safely neglected studying with standard MCMC methods the Bayesian credible regions and frequentist confidence levels of the cosmological parameters. However as already stressed in the Introduction, this is not completely safe for two reasons. First of all, when going at \( \ell \gtrsim 1/\beta \) the corrections due to aberration and Doppler are not under perturbative control and could turn out to be large also for the \( C'_\ell \)'s. Second, eq. (2.9) and (2.8) are correct only in the full sky approximation, which does not account for any foreground contamination of the observed maps. In a more realistic computation, in fact, one has to consider that indeed only a fraction of the sky is experimentally accessible. This can be done by multiplying the original full sky map by an opportune window function to mask the background dominated portions of the sky and this leads to a larger bias in the \( C'_\ell \)'s: in fact the small result obtained in (2.9) can be understood as a cancellation of a larger effect when summing over the sphere, but such cancellation does not hold when we look only at a portion of the sky. More precisely: (1) an effect of \( O(\beta \ell) \) shows up [6] for asymmetric maps (2) even for symmetric maps the above mentioned suppression which acts when \( C_\ell \sim C_{\ell+1} \) becomes less effective, because the sum involves a larger number of \( C_\ell \)'s and not only the neighbours. In the present paper we analyze only symmetric maps and we leave to further work the analysis with asymmetric maps.

The procedure of applying a window function leads to a fit of the data based on the concept of pseudo-\( C_\ell \). We will review this approach in the next section emphasizing that the CMB aberration appreciably modifies the pseudo-\( C_\ell \). This in turn will have a non negligible impact on any CMB based cosmological parameter estimation. We conclude this subsection mentioning that, because of aberration and Doppler effects, not only the primordial \( C_\ell \) are distorted, but also the off-diagonal terms of the covariance matrix (i.e. \( \langle a_{\ell_1m} a_{\ell_2m}^* \rangle \) with \( \ell_1 \neq \ell_2 \)). In a series of recent work it has been shown that this property of the off-diagonal terms can be used to effectively measure the magnitude and direction of our peculiar motion. In ref. [4], for instance, it has been found perturbatively that an experiment such as Planck will be able to measure \( \beta \) with an accuracy of about 30%, and the direction of this motion with an error of about 20 degree, while ref. [5] shows non-perturbatively that such accuracy would be of about 20% for Planck and down to 10% or even 5% for future planned experiments. Perturbative corrections to the covariance matrix of the \( a_{\ell m} \) were also calculated beyond the first order approximation. A computation of \( \langle a_{\ell_1m} a_{\ell_2m}^* \rangle \) at second order in \( \beta \) can be found in refs. [2, 5], while higher order corrections have been calculated in some cases by [5] and by using recurrence formulas in ref. [7].

\footnote{Note that such correction goes to zero only if computed with the correct coefficients eq. (2.7), derived in [5] instead of going to \( 4\beta^2 \), as previously obtained by [2].}
2.2 Fitting formulas

Eq. (2.6) can be rewritten in matrix form as follows

\[ a'_{\ell m} = \sum_{\ell'} K_{\ell' \ell m} a_{\ell m} \]  
(2.10)

where the kernel \( K_{\ell' \ell m} \) is equal to

\[ K_{\ell' \ell m} = \int_{-1}^{1} \frac{dx}{\gamma (1 - \beta x)} \tilde{P}_{\ell'}^m (x) \tilde{P}_\ell^m \left( \frac{x - \beta}{1 - \beta x} \right) \]  
(2.11)

The \( \tilde{P}_\ell^m (x) \) are the associated Legendre polynomials. The \( K_{\ell' \ell m} \) (the so-called aberration kernel), which relates the aberrated coefficients \( a'_{\ell m} \) to the primordial ones \( a_{\ell m} \), are of course given at leading order in \( \beta \ell \ll 1 \) by expressions such as eq. (2.8), but it is more problematic to compute them at large \( \ell \) beyond a perturbative approach. Such problem has been recently addressed in [5] proposing an approximate solution to the oscillating integrals of eq. (2.6).

In this paper simple fitting formulas were constructed which, approximating the oscillatory behavior of the relevant integrals by appropriate Bessel functions, reproduce the exact result (verified numerically in a few representative cases) for the elements of the aberration kernel with an accuracy of about 0.1 per cent. For example, for the temperature aberration kernel an accurate fitting formula is

\[ K_{\ell - 1 \ell m} \simeq J_1 \left( -2 e^-_{\ell m} \right) \]  
(2.13)

\[ K_{\ell + 1 \ell m} \simeq J_1 \left( 2 e^+_{\ell m} \right) \]  
(2.14)

where \( J_1 \) is a Bessel function of the first kind. Similar expressions also exist for the polarization aberration kernels. We refer the reader to [5] for a complete list of fitting formulas and a detailed discussion of the derivation and regime of validity. In the next section, we will go beyond the approximation schemes discussed in these subsections showing how to include exactly aberration and Doppler effects in a MCMC scan of the cosmological parameter space.

3 Simulated maps including aberration and Doppler

The ultimate goal of this analysis is to determine the impact of aberration and Doppler on a CMB based cosmological parameter estimation. This requires a method to simulate CMB maps including aberration and Doppler, in particular when windows functions are considered in the analysis to model the CMB maps foreground contamination. Mock data accounting for these phenomena can be generated and analyzed by using a straightforward generalization of the pseudo-\( C_\ell \) approach. In the following, after introducing the basic formulas to compute the pseudo-\( C_\ell \) including aberration and Doppler, we will discuss their implementation in a modified version of the HEALPix\textsuperscript{3} code [8]. This numerical tool will allow us to generate mock data measured by an observer moving with respect to the CMB rest frame. The next sections will be then devoted to a MCMC analysis of these simulated data.

\textsuperscript{3}http://healpix.jpl.nasa.gov
3.1 The boosted pseudo-$C_\ell$

To simplify the expression of the aberration kernel (see eq. (2.11)), we assumed so far that the direction of the observer peculiar motion coincides with $\hat{z}$, namely the direction of the $z$-axis. However, when dealing with window functions, it is convenient to work using Galactic coordinates and assume therefore that the $z$-axis identifies a direction perpendicular to the Galactic plane. We know however that our peculiar motion does not occur perpendicularly to the Galactic plane, but it is instead associated with the direction of the CMB dipole, measured along the direction identified by the longitude $l \simeq 249$ and the latitude $b \simeq 48$. This implies that to express the coefficients computed in eq. (2.6) consistently with our observed peculiar motion, we need to rotate them with a Wigner matrix $D^\ell_{m'm'}(\phi, \theta, \gamma)$, where $\phi, \theta, \gamma$ are the Euler angles required to align the peculiar velocity, initially along $\hat{z}$, with the CMB dipole.

This Wigner rotation reads as follows

$$\hat{a}_\ell^m = \sum_{-\ell \leq m' \leq \ell} D^\ell_{m'm'}(\phi, \theta, \gamma) a^m_{\ell m'}$$  

(3.1)

where the coefficients $a^m_{\ell m'}$ were calculated in eq. (2.6). The coefficients $\hat{a}_\ell^m$ are now consistently expressed in the frame of an observer moving with velocity $\beta$ in the direction of the CMB dipole. These equations are correct in a full sky harmonic decomposition. However, current CMB experiments observe only a portion of the whole CMB sky. From a theoretical point of view, this limitation can be modeled through opportune windows functions which set to zero the contributions to the computed quantities coming from regions of the sky background dominated (e.g. the Galactic plane). A window function $W(\hat{n})$ acts multiplicatively on the original full sky map. Therefore, if we denote by $\hat{T}(\hat{n})$ a full sky temperature map, its cut sky counterpart $\tilde{T}(\hat{n})$ reads as follows

$$\tilde{T}(\hat{n}) = W(\hat{n})\hat{T}(\hat{n})$$

$$= \sum_{\ell} \sum_{-\ell \leq m' \leq \ell} w_\ell^m Y_{\ell m}(\hat{n})\hat{T}(\hat{n})$$  

(3.2)

where in the second line we expanded the window function in spherical harmonics and denoted by $w_\ell^m$ the coefficients of this decomposition. A further expansion of the full sky map $\hat{T}(\hat{n}) = \sum_\ell a_\ell^m Y_{\ell m}(\hat{n})$ and of the cut sky map $\tilde{T}(\hat{n}) = \sum_\ell \tilde{a}_\ell^m Y_{\ell m}(\hat{n})$ finally leads to

$$\tilde{a}_{\ell_1 m_1} = \sum_{\ell_2} \sum_{-\ell_2 \leq m_2 \leq \ell_2} F_{\ell_1 m_1 \ell_2 m_2} \hat{a}_{\ell_2 m_2}$$  

(3.3)

where the kernel $F_{\ell_1 m_1 \ell_2 m_2}$ relating the full sky coefficients $\hat{a}_{\ell_2 m_2}$ to the cut sky coefficients $\tilde{a}_{\ell_1 m_1}$ admits the following representation in terms of the Wigner $3-j$ symbols

$$F_{\ell_1 m_1 \ell_2 m_2} = \sum_{\ell_3} \sum_{-\ell_3 \leq m_3 \leq \ell_3} w_{\ell_3 m_3} (-1)^{-m_2} \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi}}$$

$$\times \frac{\ell_1 \ell_2 \ell_3}{0 0 0} \frac{\ell_1 \ell_2 \ell_3}{m_1 - m_2 m_3}.$$  

(4.3)

In the present analysis the $\hat{a}_{\ell m}$ coefficients are the ones given in eq. (3.1). The coefficients $\tilde{a}_{\ell m}$ are then used to construct the pseudo-$C_\ell$, which are denoted by $\tilde{C}_\ell$ and defined as follows

$$\tilde{C}_\ell \equiv \frac{1}{2\ell + 1} \sum_{-\ell \leq m \leq \ell} |\tilde{a}_{\ell m}|^2.$$  

(3.5)
Combining now eqs. (3.3) and (3.5), we can finally obtain the desired expression for the pseudo-$C_{\ell}$ which includes also aberration and Doppler. For the ensemble average of the $C_{\ell}$ one finds

$$\langle \tilde{C}_{\ell} \rangle = \frac{1}{2 \ell_1 + 1} \sum_{-\ell_1 \leq m_1 \leq \ell_1} \sum_{\ell_2 m_2} \sum_{\ell_3 m_3} \mathcal{F}_{\ell_1 m_1 \ell_2 m_2} \mathcal{F}_{\ell_3 m_3} \langle \hat{a}_{\ell_2 m_2} \hat{a}_{\ell_3 m_3}^* \rangle. \quad (3.6)$$

An analytic expression for $\langle \tilde{C}_{\ell} \rangle$ at first order in $\beta$ has been obtained in ref. [6]. In the limit $\beta = 0$ and assuming statistical isotropy eq. (3.6) reduces to

$$\langle \tilde{C}_{\ell} \rangle = \sum_{\ell_2} \mathcal{M}_{\ell_1 \ell_2} C_{\ell_2} \quad (3.7)$$

where the mode-coupling matrix $\mathcal{M}_{\ell_1 \ell_2}$ reads as follows

$$\mathcal{M}_{\ell_1 \ell_2} = \frac{2 \ell_2 + 1}{4 \pi} \sum_{\ell_3} (2 \ell_3 + 1) \mathcal{W}_{\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 \quad (3.8)$$

and $\mathcal{W}_{\ell_3} = 1/(2 \ell_3 + 1) \sum_{m_3} |w_{\ell_3 m_3}|^2$ is the window function power spectrum. The pseudo-$C_{\ell}$ formalism is very convenient to construct estimators of the full sky $C_{\ell}$. The best estimate of the ensemble average $\langle \tilde{C}_{\ell} \rangle$ is given by $C_{\ell}^{\text{exp}}$, the value of $\tilde{C}_{\ell}$ actually observed or extracted from simulations. Then, by inverting the mode-coupling matrix $\mathcal{M}_{\ell_1 \ell_2}$, one can construct from eq. (3.7) an unbiased estimator, denoted by $C_{\ell}^\dagger$, of the full sky $C_{\ell}$

$$C_{\ell}^\dagger = \sum_{\ell_2} \mathcal{M}^{-1}_{\ell_1 \ell_2} C_{\ell_2}^{\text{exp}}. \quad (3.9)$$

Now, to extract an estimator of the full sky $C_{\ell}$ from eq. (3.6) is instead extremely more complicated than in eq. (3.7), since the boosted $\tilde{C}_{\ell}$ involve the full covariance matrix instead of only its diagonal part. We will come back to this important observation in section 4 discussing its role in the cosmological parameter estimation.

In the usual case of ignoring $\beta$, the goodness of estimators like eq. (3.9) has been tested in various works [12]. In the next sections, when referring to $C_{\ell}^\dagger$, we will use for simplicity an estimator obtained replacing in eq. (3.9) the mode-coupling matrix with the $f_{\text{sky}}$ parameter (defined as the portion of the sky covered divided by $4 \pi$). This choice has been shown to be fairly robust in the context of weak lensing analyses [13] and it is accurate enough for the present investigation, which does not aim at proposing a new estimator to account for aberration but rather at determining the error induced by neglecting the peculiar motion of the observer when performing analyses based on eq. (3.9) (with $\mathcal{M}^{-1}_{\ell_1 \ell_2} \rightarrow f_{\text{sky}}^{-1}$).

### 3.2 Numerical implementation

The equations introduced in the previous sections involve oscillatory angular integrals (see eq. (2.6)) and rotations in harmonic space (e.g. eqs. (3.3) and (3.1)). To handle these complications we will make use of a modified version of the HEALPix code [8], a numerical package to sample functions defined on the sky through a hierarchical, equal area, iso-latitude pixelisation of the sphere. The aim is to simulate CMB data including aberration and Doppler. This requires a fiducial cosmological model for which the angular power spectrum $C_{\ell}^{(f)}$ is assumed to be known. Our choice will be specified at end of this section. The data simulation is then articulated in five steps which for clarity and completeness we list here in the following together with the HEALPix routines used in the generation of the mock data.
• **synfast**: the synfast program generates full sky maps sampling the corresponding harmonic coefficients $a_{\ell m}^J$ from a Gaussian distribution with zero mean and variance equal to the $C_{\ell}^{(J)}$. We modified this code including the possibility of applying the transformation law given in eq. (2.5) directly on the pixels. In this way the aberration and Doppler effects are included directly in position space rather than in harmonic space. As a consequence, one can avoid the evaluation of the complicated oscillatory integral in eq. (2.6). We set the map resolution parameter $N_{\text{side}}$ to the value 2048.

• **anafast**: we analyze the temperature map obtained in the previous step with the standard HEALPix version of the anafast code, expanding it in spherical harmonics and determining the corresponding multipole coefficients. These are the exact (i.e. non perturbative) $a'_{\ell m}$ coefficients of eq. (2.6).

• **alteralm**: we also modified the alteralm program in order to rotate these $a'_{\ell m}$ in a new reference frame where the peculiar velocity inducing the aberration has the direction of the CMB dipole (this transformation is not implemented in the the standard version of alteralm). This corresponds to evaluate eq. (3.1).

• **synfast**: then we performed a second call to the synfast program to generate the temperature map $\hat{T}(\hat{n})$ associated with the rotated $a_{\ell m}$ coefficients.

• **anafast**: A final call to the anafast program allows to introduce the desired window function and obtain the observed $\hat{C}_\ell^{\exp}(\beta)$, which are in general a function of the peculiar velocity $\beta$.

This procedure provides the CMB datasets analyzed in the next section. The fiducial model used in this paper is similar to the one studied in ref. [9]. It corresponds to a toy cosmological model with five free parameters, namely the baryon density in units of the critical density times the square of the present value of the Hubble rate $\Omega_b h^2$, the analogous quantity for the dark matter component $\Omega_{dm} h^2$, the spectral index $n_s$, the amplitude of the primordial power spectrum $A_s$ and, finally, 100 times the ratio of the sound horizon to the angular diameter distance at recombination $\theta$. The fiducial model is characterized by the following parameter values: $\Omega_b h^2 = 0.022$, $\Omega_b h^2 = 0.12$, $n_s = 1$, $A_s = 2.3 \times 10^{-9}$ and $\theta$ has been set to a value corresponding to $h = 0.7$. The other cosmological parameters have been fixed as in the file test_params.ini which can be downloaded at the web page [14]. This simplified choice of the cosmological parameters allowed us to study the impact of aberration within a model for which the Likelihood approximation implemented in the next section has been carefully tested in ref. [9].

### 3.3 Datasets

Starting from 30 different HEALPix random seeds we simulated 30 pairs of temperature maps with Planck resolution ($l_{\text{max}} = 2100$) and a sky cut of 20 degrees around the Galactic plane: within a pair of temperature maps, one map was generated in the CMB rest frame, while the second in a frame moving with $\beta = 1.23 \times 10^{-3}$, including Doppler and aberration following the procedure described in the previous subsection. Then for all these maps we calculated the angular power spectrum, i.e. $\hat{C}_\ell^{\exp}(\beta)$ in the moving frame and $\hat{C}_\ell^{\exp}(0)$ in the CMB rest frame, obtaining the datasets analyzed with the CosmoMC program in the next sections. For one pair of temperature maps, associated with a simulation labelled by $S$, 

• **synfast**: the synfast program generates full sky maps sampling the corresponding harmonic coefficients $a_{\ell m}$ from a Gaussian distribution with zero mean and variance equal to the $C_{\ell}^{(J)}$. We modified this code including the possibility of applying the transformation law given in eq. (2.5) directly on the pixels. In this way the aberration and Doppler effects are included directly in position space rather than in harmonic space. As a consequence, one can avoid the evaluation of the complicated oscillatory integral in eq. (2.6). We set the map resolution parameter $N_{\text{side}}$ to the value 2048.
Figure 1. Left panel: relative difference $\Delta C_\ell / C_\ell \equiv 1 - \tilde{C}_\ell^{\exp}(\beta) / \tilde{C}_\ell^{\exp}(0)$ as a function of the multipole $\ell$ for the simulation $S$, in a full sky configuration (cyan) and in a cut sky configuration (black). The red dashed line corresponds instead to the binned version of the black curve with a bin size equal to 50 multipoles. Right panel: as in the left panel but with the cosmic variance in the denominator.

We also evaluated these angular power spectra in a full sky configuration, therefore without applying the mentioned azimuthal cut.

In the left panel of figure 1 we show the relative difference $\Delta C_\ell / C_\ell \equiv 1 - \tilde{C}_\ell^{\exp}(\beta) / \tilde{C}_\ell^{\exp}(0)$ as a function of the multipole $\ell$ for the simulation $S$, in a full sky configuration (cyan) and in a cut sky configuration (black). From the full sky curve, which is more directly comparable with theoretical expectations, we can appreciate that at large $\ell$ perturbation theory breaks down and the curve oscillates between $-0.1$ and $0.1$. Regarding the cut sky configuration, instead, we notice that $\Delta C_\ell / C_\ell$ is considerably larger than in the full sky case, at least up to $\ell \sim 1000$. For larger $\ell$, also in this case the curve oscillates between $-0.1$ and $0.1$. In the right panel of figure 1, we finally compare $\Delta C_\ell \equiv \tilde{C}_\ell^{\exp}(0) - \tilde{C}_\ell^{\exp}(\beta)$ with the cosmic variance, which is given by $\sigma_\ell = \sqrt{2/(2\ell + 1) f_{\text{sky}} \tilde{C}_\ell^{\exp}(0)}$, again for a full sky configuration (cyan) and a cut sky configuration (black). In this figure one can see that for a given $\ell$ the effect is very large (a few times the cosmic variance!). However the difference between the power spectrum estimators oscillates around zero without exhibiting clear trends. This is also confirmed, in both panels, by the behavior of the red dashed curve, which has been obtained binning the black curve (cut sky case) with a bin size of 50 multipoles. The exact shape of this curve obviously depends from the bin size. This dependence has been studied in figure 2, where varying the bin size from 10 to 50, we checked that a bin size equal to 50 provides a good compromise between a sufficiently large number of points in each bin and a large enough number of bins.

4 MCMC analysis

Let us assume that a CMB experiment has measured a certain temperature map, say $T^{\exp}(\hat{n})$, with associated angular power spectrum $C_\ell^{\exp}$. If we wanted to use this map to infer the underlying cosmological model, one possibility would be to use directly eq. (3.9) to derive
Figure 2. Binned version of the black curve shown in the left panel of figure 1. The three curves shown here assume different bin sizes, varying from 10 to 50. This figure shows that a bin size equal to 50, giving rise to a curve close to the one corresponding to $\Delta \ell = 25$, provides a good compromise between a sufficiently large number of points in each bin and a large enough number of bins.

from $C_{\ell}^{\exp}$ the corresponding estimator of the full sky angular power spectrum, $C_{\ell}^d$, and implement this quantity in a Likelihood based analysis. This approach would be correct if the experiment were comoving with the CMB, but it would instead introduce a “bias” in the final parameter reconstruction if the observer were moving with a peculiar velocity different from zero, as one can deduce by comparing eqs. (3.6) and (3.7). The common lore in this case is to assume that such a bias is negligible. In the following we will reconsider this assumption performing 30+30 independent cosmological parameter estimations, one for every dataset considered in section 3.3. We will also perform additional 1+1 parameter estimations associated with the full sky power spectra related to the simulation $S$. In these analyses we will always use the estimator $C_{\ell}^d$, defined in eq. (3.9) (with $\mathcal{M}^{-1}_{\ell_1,\ell_2} \rightarrow f_{\text{sky}}^{-1}$), as an experimental input for our Likelihood based reconstruction procedure (described in detail in the next sections). In this way, depending from which dataset we are considering, the resulting parameter reconstruction will be biased or not (in the sense explained above). Comparing then biased and unbiased reconstructions we can estimate the impact of a peculiar motion on the cosmological parameter estimation. Summarizing, there are two possible cases:

- The dataset has been generated including aberration and Doppler. In this case an appropriate estimator of the full sky $C_{\ell}$ could be found solving eq. (3.7) for the covariance matrix of the primordial $a_{\ell m}$. Using instead eq. (3.9) in the Likelihood introduces an error in the analysis which consists in interpreting the data with a wrong theoretical assumption, namely assigning a null peculiar velocity to the observer. These considerations apply to the datasets $\tilde{C}_{\ell}^{\exp}(\beta)$.

- The dataset has been generated without introducing aberration and Doppler according to eq. (2.5). In this case an analysis based on eq. (3.9) is perfectly correct. This applies to the datasets $\tilde{C}_{\ell}^{\exp}(0)$.
Figure 3. Results obtained from the simulation $S$ using the associated full sky power spectra. The top left panel refers to $\Omega_b h^2$, the top right panel to $\Omega_{dm} h^2$, the bottom left to $n_s$ and, finally, the bottom right to $A_s$. In every panel, the black solid line and the black dot-dashed line correspond respectively to the marginal posterior pdf and the mean likelihood derived from the dataset $\tilde{C}^{\exp}_\ell(0)$. The red dashed and dotted lines correspond instead to respectively the posterior and mean likelihood obtained from the dataset $\tilde{C}^{\exp}_\ell(\beta)$. In all panels the fiducial model is represented by a green dot. In this full sky analysis the statistical outputs obtained from these two datasets (with and without aberration and Doppler) exhibit only minor differences. We implemented in the analysis an experimental noise corresponding to a Planck-like experiment.

Comparing therefore the results obtained using eq. (3.9) with $\tilde{C}^{\exp}_\ell(\beta)$ and the same equation but with $\tilde{C}^{\exp}_\ell(0)$ will enable us to determine, in one full sky and in 30 cut sky simulations, whether the use of the estimator (3.9), that is, neglecting the peculiar motion of the observer, is indeed a completely solid assumption or not. As we will see in this section, this is not true a priori in a cut sky analysis.

4.1 Likelihood

We will tackle the problem of reconstructing the parameters of our fiducial cosmological model from the data simulated in section 3.3 within a Bayesian framework. Bayesian methods combined with MCMC scanning techniques allow an effective comparison between observations and theoretical predictions. The outputs of this type of analyses are marginal posterior probability density functions (pdf) and credible intervals for the underlying model parameters (for a review on these subjects, see for instance ref. [10]). In a Bayesian analysis the experimental input are encoded in the Likelihood function, which in our case will be a function of the model parameters and of the simulated data. We use here the Likelihood approximation
introduced in ref. [9], according to which the Likelihood $L$ can be written as follows

$$-2 \log L \simeq \sum_{\ell \ell'} g(C^\dagger_\ell / C_\ell) C^{(f)}_{\ell'} \left[ M^{(f)} \right]^{-1}_{\ell \ell'} C^{(f)}_{\ell'} g(C^\dagger_{\ell'} / C_{\ell'})$$ \hspace{1cm} (4.1)

where the function $g$ is explicitly given by

$$g(x) = \text{sgn}(x-1) \sqrt{2(x-\ln(x)-1)},$$ \hspace{1cm} (4.2)

$[M^{(f)}]_{\ell \ell'}$ is the covariance matrix of the estimator (3.9) evaluated at the fiducial model, $C^{(f)}_\ell$ is the angular power spectrum of the fiducial model and $C_\ell$ is the predicted angular power spectrum (e.g. computed with CAMB). For the fiducial model considered in this analysis, this matrix has been already estimated in ref. [9] and can be downloaded in [14]. In the presence of isotropic noise with known angular power spectrum $N_\ell$, it is straightforward to include its contribution to eq. (4.1). It is in fact enough to sum to all the power spectra in eq. (4.1) a noise contribution $N_\ell$. Concerning $N_\ell$, we use two noise maps: one which reproduces the sensitivity of the WMAP experiment and another for the Planck experiment. Regarding the Planck noise we refer to ref. [9], implementing in our analysis the same noise map used in that analysis. The WMAP noise has been instead taken from [15]. Given a signal estimator $C^\dagger_\ell$ (see section 3.3), a covariance matrix for it $[M^{(f)}]_{\ell \ell'}$ and a noise power spectrum $N_\ell$, the model parameter reconstruction can be performed with the CosmoMC program [11], using these quantities. We restrict our analysis to multipoles $\ell > 30$ to avoid complication related to the choice of the Likelihoods at small $\ell$. Note that, even if those are few datapoints, in this way we might actually underestimate the bias we are looking for: in fact, as we will see, the power spectrum is more biased due to $\beta$ in a cut sky at small $\ell$ rather than at large $\ell$; however our conclusion that it is not safe to neglect aberration and Doppler effect would possibly be only strengthened in the case in which also the low-$\ell$ contribute significantly to the bias. Finally, the window function used in this work and applied in the last call to the anafast program in the data simulation process (see section 3.2) consists in an azimuthal cut of 20 degrees around the Galactic plane. This corresponds to a value of the $f_{\text{sky}}$ parameter equal to 0.826.

### 4.2 Full sky results

We now compare the results of two cosmological parameter estimations performed using the two full sky power spectra associated with the simulation $S$: one evaluated in the CMB rest frame and the other one obtained including aberration and Doppler. As expected, in this full sky case the bias induced in the parameter reconstruction by neglecting Doppler and aberration is very small; we therefore limited our analysis to this full sky simulation only. In the four panels of figure 3 we plot the marginal posterior pdf’s and mean likelihoods referring to four parameters of our toy cosmological model, namely $\Omega_b h^2$, $\Omega_{dm} h^2$, $n_s$ and $A_s$. For every parameter we show the results obtained for the simulation $S$ using $C^{\text{exp}}_\ell(0)$ (black lines) and $C^{\text{exp}}_\ell(\beta)$ (red lines), in order to estimate the impact of aberration and Doppler on this parameter reconstruction. More specifically, in every panel, the black solid line and the black dot-dashed line correspond respectively to the marginal posterior pdf and the mean likelihood derived from $C^{\text{exp}}_\ell(0)$. The red dashed and dotted lines, instead, correspond to respectively the posterior and mean likelihood obtained from the dataset $C^{\text{exp}}_\ell(\beta)$. From this single map analysis we can conclude that in the case of a full sky parameter reconstruction to neglect aberration and Doppler introduces only a minor modification to the relevant statistical
results from the simulation $S$ using the associated cut sky power spectra. In this figure the means of the parameters obtained from $\hat{C}_\ell^{\text{exp}}(0)$ (no aberration and Doppler) and from $\hat{C}_\ell^{\text{exp}}(\beta)$ (with aberration and Doppler) differ up to almost 2σ (see table 1). The experimental noise corresponds here to a Planck-like experiment.

outputs. More precisely, defining quantities $\Delta \equiv \langle p \rangle - \langle p' \rangle$, where $\langle p \rangle$ and $\langle p' \rangle$ are the mean values in the two frames for a parameter $p$ and $\sigma_p$ is the error on the parameter we find that $\Delta$ is typically of order a few times $10^{-2}$. This is still much larger than a naive $O(\beta^2)$ result, but practically not relevant. Given this result we only show the results for Planck-like sensitivity (in the cut sky case, instead, we will also consider a WMAP-like experiment). As we will see, however, this conclusion is not a priori correct in a cut sky analysis.

4.3 Cut sky results

We now present the outcome of our 30 cut sky simulations. Every simulation produced a pair of temperature maps: one in the CMB rest frame and one including Doppler and aberration. We analyzed these maps as explained above by means of the program CosmoMC. As in the case of a full sky analysis, in a single figure we report here, for one of these CosmoMC runs, the relevant statistical outputs for the parameters $\Omega_b h^2$, $\Omega_{\text{dm}} h^2$, $n_s$ and $A_s$. More precisely, figure 4 shows the results obtained using the cut sky power spectra $\hat{C}_\ell^{\text{exp}}(\beta)$ and $\hat{C}_\ell^{\text{exp}}(0)$ associated with the simulation $S$. We are using therefore the same HEALPix random seed previously used in the full sky analysis. Comparing both marginal posterior pdf’s and mean likelihoods in this figure, we find an interesting result: the statistical outputs of this analysis are biased in a non-negligible way. This is the consequence of having analyzed the dataset $\hat{C}_\ell^{\text{exp}}(\beta)$ assuming that the observer is comoving with the CMB, i.e. assuming
therefore $\beta = 0$. For this specific simulation, we also show in table 1 for each parameter the means of the pdf’s found using $\tilde{C}_\ell^\text{exp}(\beta)$ and $\tilde{C}_\ell^\text{exp}(0)$. For every parameter we also report the corresponding value of $\Delta$, namely the difference between the two means in units of the standard deviation associated with the analysis based on the dataset $\tilde{C}_\ell^\text{exp}(0)$. For this specific Planck-like simulated experiment the means of the cosmological parameters could deviate up to one, or almost two standard deviations. In particular the most affected parameters in this simulation seem to be $\Omega_m h^2$ and $\theta$. We also find a similar effect, though less sizable, when assuming a WMAP sensitivity. We show this for the same simulation $S$ in figure 5; the shift in the parameters in this case can go up to one sigma, as shown in table 2. The most affected parameter in this case are given by $\Omega_m h^2$ and the amplitude of the perturbations $A_s$, which are both shifted by approximately one sigma.

To check the solidity of this conclusion, derived from a single simulation, we evaluated for the other 29 simulations at disposal and for every model parameter the bias previously defined $\Delta$. The results of this study are shown in figure 6. We find that varying the simulation number (the simulation $S$ is the simulation number six) the bias oscillates around zero. But on a single simulation, or when averaged on a subset of different simulations, it is always significantly different from zero. The standard deviation for $\Delta$ depends from the specific parameter considered, being equal to 0.57, 0.72, 0.68, 0.59, and 0.65 for respectively $\Omega_b h^2$, $\Omega_{dm} h^2$, $\theta$, $n_s$ and $\log(10^{10} A_s)$. A second interesting result is that $A_s$ and $\Omega_{dm} h^2$ are clearly correlated and the same is true for the other three parameters. The two subsets of parameters
Figure 6. The bias $\Delta$ as a function of the simulation number. The standard deviation for $\Delta$ depends from the specific parameter considered, being equal to 0.57, 0.72, 0.68, 0.59, and 0.65 for respectively $\Omega_b h^2$, $\Omega_{dm} h^2$, $\theta$, $n_s$ and $\log(10^{10} A_s)$. $A_s$ and $\Omega_{dm} h^2$ are clearly correlated and the same is true for the other three parameters. The two subsets of parameters are however anticorrelated.

Table 1. Means and standard deviations for the cosmological model parameters derived by analyzing the datasets $\tilde{C}^{\exp}_\ell(0)$ and $\tilde{C}^{\exp}_\ell(\beta)$ using a Planck-like noise. For every parameter we also list in the last column the absolute value of $\Delta$, namely the difference of the two means (second and fourth column) divided by the standard deviation given in the third column. $\Delta$ provides a quantitative estimation of the error associated with neglecting aberration and Doppler effects.

Although a more careful analysis performed eventually using also $\ell \lesssim 30$, including polarization and considering a more realistic cosmological model, is necessary in order to achieve a more robust determination of the impact of aberration and Doppler on the cosmological parameter estimation, from the simulations discussed in this paper, we can already draw the conclusion that the common lore of neglecting the transformation law (2.5) in a CMB based parameter analysis is not in general a solid assumption.
Table 2. As in table 1 but for a WMAP-like experiment.

5 Analytical estimates

In this section we present how to perform analytical checks and estimates of the results that we have found numerically. First, in [5] fitting formulas given in terms of Bessel functions have been found to reproduce to high accuracy (0.2%) the exact coefficients of eq. (2.11) at least up to $\ell = 700$. In particular for the Temperature coefficients they read:

$$K_{\ell - n \ell m}^X \simeq J_n \left(-2 \beta \prod_{k=0}^{n-1} \left[ (\ell - k) G_{\ell-km} \right]^{1/n} \right),$$

$$K_{\ell + n \ell m}^X \simeq J_n \left(2 \beta \prod_{k=1}^{n} \left[ (\ell + k) G_{\ell+km} \right]^{1/n} \right),$$

(5.1)

where $J_n$ is the Bessel function of the $n$-th kind and where

$$G_{\ell m} \equiv \sqrt{\frac{\ell^2 - m^2}{4\ell^2 - 1}},$$

(5.2)

while for $n = 0$:

$$K_{\ell \ell m}^X \simeq J_0 \left(\beta \sqrt{2} \left[-(\ell + 1) (\ell + 2) (s G_{\ell+1m})^2 - \ell (\ell - 1) (s G_{\ell m})^2 + \ell (\ell + 1) - m^2 + 1 \right]^{1/2} \right).$$

(5.3)

Then we have checked that we can reproduce to high accuracy the same $C_\ell$’s of our numerical program, using such fitting formulas to transform a set of randomly generated $a_{\ell m}$’s, as long as we include mixing with a sufficiently large number of neighbours in eq. (2.10). For instance, including 2 neighbours we obtain agreement between the two methods of the order $(C_\ell^\text{exp} - C_\ell^\text{fit})/C_\ell^\text{exp} \approx 10^{-6}$ at $\ell = 100$, where $C_\ell^\text{exp}$ are the numerical results and $C_\ell^\text{fit}$ are the ones obtained with fitting functions. The precision goes down to $10^{-7}$ when including 3 or more neighbours. At higher $\ell$ (between 1000 and 2000) the error rapidly grows with $\ell$ if we consider only two neighbours. When considering 4 neighbours the error still grows with $\ell$ but it is always smaller than $5 \times 10^{-4}$, up to $\ell = 2000$, while with 5 neighbours it goes down to $10^{-4}$. As we will see later such a precision is sufficient to have the same result on the bias on a cosmological parameter than the numerical simulation.

We have therefore good control over the full-sky $C_\ell$’s at least up to $\ell = 700$. Note that the fact that we get a quite sizable effect on the cosmological parameters is true already for
the WMAP simulation which goes up to roughly $\ell = 700$, where the coefficients are under control.

Given the $C_\ell$’s it is possible to have an estimate on the size of the effect on a cosmological parameter, by considering an idealized case in which the CMB depends multiplicatively on a single parameter, which we call $A$, so that the $\chi^2$ is given by:

$$\chi^2(A) = \sum_\ell \left( \frac{C^\text{exp}_\ell - A \hat{C}^\text{th}_\ell}{\sigma^2_\ell} \right)^2$$

where $\hat{C}^\text{th}_\ell$ is the theoretical spectrum when $A = 1$. The best fit value $A_{bf}$ for $A$ is obtained when $\partial(\chi^2)/\partial A = 0$ which gives:

$$A_{bf} = \sum_\ell \frac{C^\text{exp}_\ell \hat{C}^\text{th}_\ell}{\sigma^2_\ell} \frac{1}{N}$$

where

$$N \equiv \sum_\ell \frac{(\hat{C}^\text{th}_\ell)^2}{\sigma^2_\ell} = \frac{1}{A_{bf}} \sum_\ell \frac{2\ell + 1}{2}$$

The difference $A'_{bf} - A_{bf}$, between the best-fit values in two frames with observed values $C^\text{exp}_\ell$ and $C'^\text{exp}_\ell$ is therefore given by:

$$\frac{\delta A_{bf}}{A_{bf}} = \frac{A'_{bf} - A_{bf}}{A_{bf}} = \frac{1}{A_{bf} \cdot N} \sum_\ell \frac{\delta C_\ell \cdot \hat{C}^\text{th}_\ell}{\sigma^2_\ell} = \left( \sum_\ell \frac{\delta C_\ell}{C^\text{th}_\ell} (2\ell + 1) \right) / \left( \sum_\ell 2\ell + 1 \right)$$

The typical values we obtain for $\delta A_{bf}/A_{bf}$ are of order $10^{-5}$ for the full-sky case and $10^{-3}$ for the cut-sky case, which is much larger than a naive $O(\beta^2)$ effect. As a consequence for instance defining a parameter such as $f \equiv \log(10^{10} A)$, we find $\Delta \equiv \frac{\delta f}{f}$ equal to roughly $5 \times 10^{-2}$ for the full-sky case and about 0.5 for any of the 30 simulations in the cut-sky case, in agreement with the full numerical results.

Finally note that through eq. (5.4) we can also check which is the precision needed for the $C_\ell$’s to obtain the same result on the bias $\Delta$ either using the $C^\text{exp}_\ell$ or the $C'^\text{exp}_\ell$. It is straightforward to estimate $\delta A_{bf}/A_{bf}$ for the 30 simulations, using for instance for $C^\text{th}_\ell$ the fiducial model used to generate them and summing up to $\ell = 2100$, for Planck-like resolution. The typical values we obtain for $\delta A_{bf}/A_{bf}$ are of order $10^{-5}$ for the full-sky case and $10^{-3}$ for the cut-sky case, which is much larger than a naive $O(\beta^2)$ effect. As a consequence for instance defining a parameter such as $f \equiv \log(10^{10} A)$, we find $\Delta \equiv \frac{\delta f}{f}$ equal to roughly $5 \times 10^{-2}$ for the full-sky case and about 0.5 for any of the 30 simulations in the cut-sky case, in agreement with the full numerical results.

We have further checked the goodness of our fitting function approximation also by considering again one simulation of the CMB sky with Planck-like resolution, namely the simulation $S$ of section 3.3. From this original full-sky map we have then obtained a new set
of $a_{\ell m}$’s in the boosted frame, applying the 5-neighbours fitting function transformation to the $a_{\ell m}$’s of the original map. Then we have extracted the pseudo-$C_\ell$’s applying the same sky-cut that we use throughout the paper with the standard HEALPix package exactly as in section 3. Finally we have performed MCMC parameter estimation runs on this dataset as in section 4. We define the relative bias between this dataset and the numerically obtained map as $\Delta_{rel} \equiv \langle p_{\text{num}} \rangle - \langle p_{\text{fitting}} \rangle \sigma_p$, where $\langle p_{\text{num}} \rangle$ and $\langle p_{\text{fitting}} \rangle$ are the mean values for a parameter $p$ respectively using the numerical data or the fitting functions, while $\sigma_p$ as usual is the error on the parameter. We performed 6 times the MCMC simulation on the same dataset to control for possible sources of error due to different seeds of the MCMC. We find that $\Delta_{rel}$ is always less than 0.1 for all MCMC runs and for all parameters. We also performed an average of the $\langle p_{\text{fitting}} \rangle$ values over these 6 MCMC runs and in this case the relative bias goes down even to less than $4 \times 10^{-2}$ for all parameters.

So, the fact that two completely independent methods, namely our numerical procedure and the Bessel fitting functions, give results which coincide with high precision provides a further argument in favor of the correctness of our approach.

6 Conclusions

In this paper we have considered the effect of our peculiar velocity on the CMB maps and thus on cosmological parameter estimation. We have assumed a peculiar velocity of $\beta = 1.23 \times 10^{-3}c$ in the direction of the CMB dipole and we have applied a boost transformation (eqs. (2.1) and (2.5)) to this frame directly in pixel space on simulated maps, rather than on the $a_{\ell m}$’s, which allows us to overcome the challenging problem of evaluating efficiently a large number of integrals of highly oscillating Legendre polynomials. For this purpose we have simulated 30 temperature maps with $\ell < 2100$ and with a noise which reproduces the Planck satellite resolution as well as one map with WMAP resolution, with $\ell < 1000$, as a random realization of a fiducial five-parameters cosmological model, for which a simple Likelihood functions has been tested in [9]. For simplicity we have also discarded the first 30 multipoles, which are usually treated with a different likelihood. Finally we have performed, both for full sky and for cut sky configurations, an analysis of the maps by running a MCMC on each map with or without the aberration plus Doppler effect.

We have also checked that using fitting functions for the highly oscillating integrals as in [5] we can reconstruct the aberrated $C_\ell$’s analytically with high precision and we have shown that such precision is enough to lead to the same results for the bias on Cosmological parameters.

As a conclusion we find that neglecting aberration and Doppler would induce some bias on cosmological parameters which could result in a shift of the mean values of the parameters which could go, depending on the specific random realization of the map, up to one or almost two standard deviations for Planck and up to one standard deviation for WMAP. While our analysis is not exhaustive because of many simplifications, we have shown that it is not harmless to ignore aberration and Doppler in order to do cosmological parameter estimations for high resolution experiments such as Planck and even for WMAP, and that therefore it would be appropriate before analyzing the experimental maps to apply a Lorentz transformation which transforms them back to the CMB rest frame. In order to do this it is crucial however to rely on the assumption that our velocity is approximately given by the CMB dipole, unless it can be measured to some extent in other ways, as proposed in [4, 5].
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