The spinorial $R$-matrix

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Abstract
The $R$-matrix acting in the tensor product of two spinor representation spaces of Lie algebra $so(d)$ is considered thoroughly. The corresponding Yang–Baxter relation is proved and the underlying local Yang–Baxter equation is established.

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1. Introduction

In this paper, we will prove certain relations concerning the $so(N)$-invariant $L$-operator, which was constructed in [1]. Further, we will consider thoroughly the corresponding numerical $R$-matrix defining Yangian for $so(N) \simeq \text{spin}(N)$.

Let $\mathcal{A}$ be a Lie algebra and $T_a$ ($a = 0, 1, 2, \ldots$) be representations of $\mathcal{A}$ in spaces $V_a$. Consider operators $R_{ab}(u) \in \text{End}(V_a \otimes V_b)$, where $u$ denotes a spectral parameter. We say that $\mathcal{A}$ is the symmetry algebra of the operator $R_{ab}(u)$, if $\forall g \in \mathcal{A}$, we have

$$(T_a(g) \otimes I_b + I_a \otimes T_b(g))R_{ab}(u) = R_{ab}(u)(T_a(g) \otimes I_b + I_a \otimes T_b(g)),$$

where $I_a$ and $I_b$ are unit operators in $V_a$ and $V_b$, respectively. Consider a set of Yang–Baxter RRR-equations for the operators $R_{ab}(u)$:

$$R_{ab}(u - v) R_{bc}(u) R_{ab}(v) = R_{bc}(v) R_{ab}(u) R_{bc}(u - v) \in \text{End}(V_a \otimes V_b \otimes V_c)$$  \hspace{1cm} (1.1)

where the representation spaces $V_a$, $V_b$, $V_c$ are different in a general situation. There is an efficient procedure, which enables us to construct nontrivial solutions of cubic Yang–Baxter equations (1.1) starting with the known one. The procedure can be illustrated by the following sequence of specializations in the Yang–Baxter relations (1.1):

$$V_0 \otimes V_0 \otimes V_0 \rightarrow V_0 \otimes V_0 \otimes V_0 \rightarrow V_0 \otimes V_0 \otimes V_0 \rightarrow V_0 \otimes V_0 \otimes V_0 \rightarrow V_0 \otimes V_0 \otimes V_0 \rightarrow V_0 \otimes V_0 \otimes V_0 \rightarrow \cdots$$

and the corresponding sequence of solutions
Indeed, one starts with the simplest known solution \(R_{0,0}\) of the Yang–Baxter equation (1.1) defined in the space \(V_0 \otimes V_0 \otimes V_0\), where \(V_0\) is the space of the simplest faithful representation, e.g. the defining representation for the matrix Lie algebra \(A\). Further, one introduces another representation \(T_a\), which acts in the space \(V_a\) (finite-dimensional or infinite-dimensional) and solves the Yang–Baxter equation (1.1) restricted to \(V_a \otimes V_0 \otimes V_0\). It happens to be a quadratic equation on the operator \(R_{0,0}\), which in special cases represents the Yangian of the corresponding matrix Lie algebra \(A\) (\(V_0\) is the space of the defining representation and \(V_a\) is the space of the representation of the Yangian). In the next step, one solves the Yang–Baxter relation (1.1) restricted to the space \(V_a \otimes V_a \otimes V_0\) and obtains \(R_{a,a}\). There is a well known argumentation (based on the associativity relations) why \(R_{a,a}\) respects the Yang–Baxter equation (1.1) defined in the space \(V_a \otimes V_a \otimes V_a\). Nevertheless, it can be proved directly. Thus, the solution \(R_{a,a}\) of the cubic Yang–Baxter equation is constructed in several steps starting with the simplest one \(R_{0,0};\) in each step, linear or quadratic relations have to be solved.

To be more concrete, let us recall [1, 2] how it works for the algebra \(so(p, q)\) as well where \(p + q = d\). The corresponding fundamental \(R\)-matrix \(R^0\) (defined in the tensor product \(V_0 \otimes V_0\) of two fundamental (defining) \(d\)-dimensional representations of \(so(d)\) can be represented as

\[
(R^0)_{j,i}^{k,l}(u) = u \delta_{j}^{[i} \delta_{l]}^{k} + \delta_{j}^{[i} \delta_{l]}^{k} - \frac{u}{u + \frac{d}{2}} \delta_{j}^{[i} \delta_{l]}^{k},
\]  

(1.2)

and depicted as follows

\[
R^0(u) = u \bigotimes + \bigotimes \bigotimes - \frac{u}{u + \frac{d}{2}} \bigotimes \bigotimes.
\]  

(1.3)

\(R^0\) respects the Yang–Baxter equation

\[
R^0_{12}(u - v) R^0_{13}(u) R^0_{12}(v) = R^0_{12}(v) R^0_{13}(u - v) R^0_{12}(v) \in \text{End}(V_0 \otimes V_0 \otimes V_0)
\]  

(1.4)

and can be considered as the simplest solution in the hierarchy of solutions of the universal Yang–Baxter equation (1.1) related to \(so(d)\) Lie algebra. It was found in 1978 by Zamolodchikov and Zamolodchikov [3].

In the next step, we introduce a spinor representation of \(so(d)\) acting in the space \(V\) with dimension \(2^d\). Let \(\gamma_a\) \((a = 1, \ldots, d)\) be \(2^d\)-dimensional gamma-matrices in \(\mathbb{R}^d\), which act in \(V\) as linear operators. Operators \(\gamma_a\) represent generators of the Clifford algebra

\[
\gamma_a \gamma_b + \gamma_b \gamma_a = 2 \delta_{ab} \cdot 1.
\]  

(1.5)

As a vector space, the Clifford algebra has dimension \(2^d\). The standard basis in this space is formed by antisymmetrized products of the \(\gamma\)-matrices

\[
\gamma_{a_1 \cdots a_k} = \frac{1}{k!} \sum_s (-1)^{p(s)} \gamma_{s(a_1)} \cdots \gamma_{s(a_k)} \equiv \gamma_{A_k} \quad (\forall k \leq d), \quad \gamma_{A_k} = 0 \quad (\forall k > d),
\]  

(1.6)

where the summation is taken over all permutations \(s\) of \(k\) indices \([a_1, \ldots, a_k] \rightarrow [s(a_1), \ldots, s(a_k)]\) and \(p(s)\) denote the parity of the permutation \(s, A_k\) is a multi-index \(a_1 \ldots a_k\).

Then, according to the procedure outlined above, we look for the operator \(L^0(u)\) defined in the space \(V_0 \otimes V\) which respects quadratic relation

\[
R^0_{12}(u - v) L^0_{12}(u) L^0_{13}(v) = L^0_{12}(v) L^0_{13}(u) R^0_{12}(u - v) \in \text{End}(V \otimes V_0 \otimes V_0),
\]  

(1.6b)
where $R^0_{23}(u)$ is fundamental $R$-matrix (1.2). The solution of the above equation has been found in [2] (see also [4–7, 16]). It has the form

$$L^0(u) = u \mathbf{1} \otimes I_n - \frac{1}{4} [\gamma^a, \gamma^b] \otimes e_{ab}$$  \hspace{1cm} (1.7)$$

where $e_{ab}$ are matrix units, $\mathbf{1}$ and $I_n$ are identity operators in spinor and defining representation spaces respectively and summation over repeated indices is implied.

Further, from the universal Yang–Baxter equation (1.1), we obtain a linear equation for the $R$-matrix $R_{12}(u)$ acting in the tensor product $V \otimes V$ of two spinor representations

$$R_{12}(u-v) L_{12}^0(u) L_{12}^0(v) = L_{12}^0(v) L_{12}^0(u) R_{12}(u-v) \in \text{End}(V \otimes V \otimes V_0).$$  \hspace{1cm} (1.8)$$

In [2] (see also [4–7]) spinorial $R$-matrix has been sought for in $\text{SO}(d)$-invariant form

$$R(u) = \sum_{k=0}^{\infty} \frac{R_k(u)}{k!} \gamma_{\alpha_1...\alpha_k} \otimes \gamma^{\alpha_1...\alpha_k} \in \text{End}(V \otimes V).$$  \hspace{1cm} (1.9)$$

For convenience, in the rhs of (1.9), the summation over $k$ runs up to infinity. However, we note that this summation is automatically truncated due to condition $k \leq d$ (see (1.6)). It has been claimed in [2] that the $R$-matrix (1.9) satisfies the RLL-relation (1.8) if coefficient functions $R_k(u)$ obey the recurrent relation

$$R_{k+2}(u) = \frac{u+k}{k-(u+d-2)} R_k(u).$$  \hspace{1cm} (1.10)$$

As far as we know, it has not been checked directly so far that $R(u)$ satisfies the Yang–Baxter relation defined in the space $V \otimes V \otimes V$

$$R_{12}(u) R_{23}(u+v) R_{12}(v) = R_{23}(v) R_{12}(u+v) R_{23}(u) \in \text{End}(V \otimes V \otimes V)$$  \hspace{1cm} (1.11)$$

owing to the complicated gamma-matrix structure one has to deal with. One of the aims of this paper is to carry out a corresponding calculation. In order to avoid multiple summation over repeated indices, we apply the generating functions technique and rewrite the sum in (1.9) as an integral over the auxiliary parameter. It enables us to perform the calculation in a concise manner. We undertake this calculation in section 3. The derivation of the recurrent relations (1.10) is given in the appendix.

Now we proceed to explore thoroughly the spinorial $R$-matrix (1.9). At first, we are going to deduce its basic properties, which can be obtained on a rather general argumentation and do not need an intricate calculation technique. Further, we formulate the other properties, which we prove in subsequent sections.

It is well known that spinor representation $T$ for generators $M_{ab}$ of the Lie algebra $\text{so}(d)$ can be constructed out of gamma-matrices $T(M_{ab}) = \frac{1}{2} \gamma_{ab}$ (1.6). Then one can easily check that $R$-matrix (1.9) is invariant under $\text{spin}(d)$ action, i.e.

$$(\gamma_{ab} \otimes \mathbf{1} + \mathbf{1} \otimes \gamma_{ab}) R(u) = R(u) (\gamma_{ab} \otimes \mathbf{1} + \mathbf{1} \otimes \gamma_{ab}), \quad \forall a, b,$$

moreover it satisfies the commutation relation

$$(\gamma_{d+1} \otimes \gamma_{d+1}) R(u) = R(u) (\gamma_{d+1} \otimes \gamma_{d+1}),$$  \hspace{1cm} (1.12)$$

which demonstrates an additional $u(1)$ symmetry of $R(u)$. In the (1.12) matrix, $\gamma_{d+1}$ is defined as follows

$$\gamma_{d+1} = a \gamma_{1-d}, \quad a^2 = (-)^{\frac{d}{2}}; \quad \gamma_{d+1}^2 = \mathbf{1}; \quad \{\gamma_{d+1}, \gamma_a\} = 0 \quad \text{at} \quad a = 1, \ldots, d$$  \hspace{1cm} (1.13)$$

and in the appropriate representation of gamma-matrices, it takes the form $\gamma_{d+1} = \text{diag}(I, -I)$, where $I$ is a $2^{\frac{d}{2}-1}$-dimensional identity matrix.
Let us note that recurrence equations (1.10) for the series of even coefficient functions \( R_{2k}(u) \) and odd ones \( R_{2k+1}(u) \) are independent. The general solutions of these equations are

\[
R_{2k}(u) = A(u) \left( -1 \right)^k \frac{\Gamma\left( k + \frac{1}{2} \right)}{\Gamma\left( \frac{k}{2} \right)},
\]

\[
R_{2k+1}(u) = B(u) \left( -1 \right)^k \frac{\Gamma\left( k + \frac{1}{2} \right)}{2 \Gamma\left( \frac{k+1}{2} \right)} \Gamma\left( \frac{k}{2} \right),
\]

(1.14)

where \( A \) and \( B \) are arbitrary functions of spectral parameter \( u \). For example, if \( A \) and \( B \) are polynomials of the spectral parameter, then the coefficient functions in (1.14) are normalized to be polynomials as well. Thus it is convenient to decompose the spinorial \( R \)-matrix (1.9) in the sum \( R(u) = R^+(u) + R^-(u) \) where (1.6)

\[
R^+(u) = \sum_{k=0}^{\infty} \frac{R_{2k}(u)}{(2k)!} \gamma_{\lambda_2} \otimes \gamma^{\lambda_2}, \quad R^-(u) = \sum_{k=0}^{\infty} \frac{R_{2k+1}(u)}{(2k+1)!} \gamma_{\lambda_{2k+1}} \otimes \gamma^{\lambda_{2k+1}}.
\]

(1.15)

We refer to \( R^+(u) \) and \( R^-(u) \) as even and odd parts of spinorial \( R \)-matrix, respectively.

Consider the decomposition of the spinorial \( R \)-matrix in the sum

\[
R(u) = P^+ R(u) + P^- R(u),
\]

(1.16)

where \( P^\pm \) are projectors

\[
P^\pm = \frac{1}{2} \left( 1 \otimes 1 \pm \gamma_{d+1} \otimes \gamma_{d+1} \right); \quad P^+ P^- = P^- P^+ = 0, \quad (P^\pm)^2 = P^\pm.
\]

(1.17)

**Proposition 1.** Even and odd parts (1.15) of the spinorial \( R \)-matrix can be singled out by projectors \( P^\pm \), i.e. we have

\[
R^+(u) = P^+ R(u), \quad R^-(u) = P^- R(u),
\]

(1.18)

\[
P^\pm R^\mp(u) = R^\mp(u), \quad P^\pm R^T(u) = 0, \quad R^\mp(u) R^T(v) = 0.
\]

(1.19)

**Proof.** Due to (1.14) we see that coefficient functions in (1.15) satisfy the reciprocal conditions

\[
R_{2k}(u) = (-1)^k R_{d-2k}(u), \quad R_{2k+1}(u) = (-1)^k R_{d-2k+1}(u).
\]

(1.20)

Taking into account (1.6), we deduce

\[
\left( \gamma_{d+1} \otimes \gamma_{d+1} \right) \frac{1}{k!} \gamma_{\lambda_k} \otimes \gamma^{\lambda_k} = \frac{(-1)^{d-k}}{d-k!} \gamma_{\lambda_{d-k}} \otimes \gamma^{\lambda_{d-k}},
\]

(1.21)

where \( \lambda_{d-k} \) is the multi-index, such that \( \lambda_{d-k} \cap \lambda_k = \emptyset \) and \( \lambda_{d-k} \cup \lambda_k = [1, 2, \ldots, d] \). Then, using (1.20) and (1.21), we immediately obtain (1.18). Equations (1.19) follow from (1.17) and (1.18).

**Proposition 2.** The Yang–Baxter equation (1.11) is equivalent to the following relations for \( R^+ \) and \( R^- \) (1.15):

\[
\begin{align*}
R_{23} R_{12} R_{23} &= R_{12} R_{23} R_{12}, \quad R_{23} R_{12} R_{23} = R_{12} R_{23} R_{12}, \\
R_{23} R_{12} R_{23} &= R_{12} R_{23} R_{12}, \quad R_{23} R_{12} R_{23} = R_{12} R_{23} R_{12}, \\
R_{23} R_{12} R_{23} &= 0, \quad R_{23} R_{12} R_{23} = 0, \quad R_{23} R_{12} R_{23} = 0, \quad R_{23} R_{12} R_{23} = 0, \\
R_{12} R_{23} R_{12} &= 0, \quad R_{12} R_{23} R_{12} = 0, \quad R_{12} R_{23} R_{12} = 0, \quad R_{12} R_{23} R_{12} = 0,
\end{align*}
\]

(1.22)

where the dependence on the spectral parameters is the same as in (1.11).
The Yang–Baxter equation \( R_{12}^+ R_{12}^- R_{23}^- = R_{12}^- R_{12}^+ R_{23}^- \) is deduced from (1.11) if we act on it by projectors \( P_{12}^+ \) and \( P_{23}^+ \) from the left and right and use commutation relations
\[
P_{12}^+ R_{23}^- = R_{23}^- P_{12}^+, \quad P_{23}^+ R_{12}^- = R_{12}^- P_{23}^+.
\]
The relation \( R_{23}^+ R_{12}^- R_{23}^- = 0 \) is obtained as follows:
\[
R_{23}^+ R_{12}^- R_{23}^- = R_{23}^- R_{12}^- P_{23}^+ R_{23}^- = R_{23}^- P_{23}^- R_{12}^- R_{23}^- = 0,
\]
where we use \( P_{12}^+ P_{23}^- = P_{23}^- R_{12}^- \), etc.

We stress that in view of (1.22) the Yang–Baxter equation (1.11) is satisfied for any linear combination \( R(u) = \alpha(u) R^+ (u) + \beta(u) R^- (u) \) with arbitrary coefficient functions \( \alpha(u) \) and \( \beta(u) \). It means that \( A(u) \) and \( B(u) \) in (1.14) are not fixed by the equation (1.11). Moreover, one can check by using (1.22) that the Yang–Baxter equation (1.11) is satisfied if we transform the solution \( R \) as following (we write this transformation in terms of even and odd parts \( R^+ (u), R^- (u) \)):
\[
R^+ \rightarrow R^+, \quad R^- \rightarrow \pm R^- (\gamma_{d+1} \otimes 1) R^+,
\]
\[
R^+ \rightarrow R^+, \quad R^- \rightarrow \pm R^- (1 \otimes \gamma_{d+1}) R^-.
\]

**Proposition 3.** Even and odd parts (1.15) of the spinorial \( R \)-matrix satisfy projected unitarity relations
\[
R^+ (u) R^+ (-u) = h^+ (u) P^+, \quad R^- (u) R^- (-u) = h^- (u) P^-.
\]
where functions \( h^+ (u), h^- (u) \) are constructed out of coefficients \( R_k (u) \) (1.14)
\[
h^+ (u) = 2 \sum_{k=0}^{d/2} \left( \frac{d}{2k} \right) R_{2k} (u) R_{2k} (-u) = A(u) A(-u) \prod_{k=0}^{\frac{d}{2}-1} (k^2 - u^2),
\]
\[
h^- (u) = 2 \sum_{k=0}^{d/2-1} \left( \frac{d}{2k + 1} \right) R_{2k+1} (u) R_{2k+1} (-u) = B(u) B(-u) \prod_{k=1}^{\frac{d}{2}-1} (k^2 - u^2).
\]
Let us emphasize that in the rhs of the relations, (1.24) projectors \( P^\pm \) (1.17) appear.

**Proof.** At first, in view of (1.14), one obtains that at a special value of the spectral parameter, the spinorial \( R \)-matrix reduces to projector (1.17): \( R^+ (\epsilon) = \epsilon \Gamma \left( \frac{d}{2} \right) P^+ + O(\epsilon^2) \) at \( \epsilon \to 0 \). Then the first Yang–Baxter relation in (1.22) at \( v = -u + \epsilon \) and \( \epsilon \to 0 \) leads to \( R_{23}^+ (u) P_{12}^+ R_{23}^- (-u) = R_{23}^- (-u) P_{12}^- R_{23}^+ (u) \). The latter relation is equivalent to \( P_{12}^+ R_{23}^+ (u) R_{23}^- (-u) = R_{23}^- (-u) R_{23}^+ (u) P_{12}^+ \), that implies \( R_{12}^+ (u) R_{12}^- (-u) \sim P_{12}^+ \). Coefficient functions \( h^+ (u), h^- (u) \) (1.24) are calculated in subsection 3.3 using the generating function technique.

The spinorial \( R \)-matrix satisfies another significant relation, namely the cross-unitarity relation (see [15] equation (3.10))
\[
\text{tr}_2 (R_{12} (u) R_{23} (2 - d - u)) = h(u) P_{13}
\]
where \( P_{13} \) denotes the permutation operator and \( h(u) \) is a polynomial of the spectral parameter. This relation is a consequence of the unitarity and the crossing symmetry identities for factorizable \( S \)-matrices (see, e.g., [3] and [15]). In subsection 3.4, we prove that if coefficient functions \( A \) and \( B \) in (1.14) respect relation
\[
A(u) A(2 - d - u) \prod_{k=0}^{\frac{d}{2}-1} (u + 2k)^2 = -B(u) B(2 - d - u) \prod_{k=1}^{\frac{d}{2}-1} (u + 2k - 1)^2
\]
then the $R$-matrix does satisfy the cross-unitarity relation (1.25) with $h(u) = A(u)A(2 - d - u)(-2)^{-\frac{d}{2}}\prod_{k=0}^{d-1}(u + 2k)^2$.

Our considerations are aimed at checking the Yang–Baxter equation (1.11) for the $R$-matrices (1.9), (1.14) and the verification of their properties. For this, we need to perform a rather complicated computation with Clifford algebra of gamma-matrices. To succeed, we appeal to the technique of the generating functions developed in [8]. We briefly describe this technique in the next section.

2. Clifford algebra

2.1. Fermionic interpretation of Clifford algebra

Let $\Gamma_a$, $a = 1, \ldots, d$, be a set of $d$ generators of the Clifford algebra satisfying the standard relations (see (1.5))

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2 \delta_{ab} 1.$$  \hspace{1cm} (2.1)

Note that the Clifford algebra generators can be represented as [12]

$$\Gamma_a = \theta_a + \partial \theta_a$$  \hspace{1cm} (2.2)

where $\partial \theta_a = \frac{\partial}{\partial \theta_a}$ and $\theta_a$ ($a = 1, \cdots, d$) are fermionic variables (generators of the Grassmann algebra)

$$[\theta_a, \theta_b] = 0, \quad [\partial \theta_a, \partial \theta_b] = 0, \quad [\partial \theta_a, \theta_b] = \delta_{ab}.$$  \hspace{1cm} (2.3)

Below, the fermionic interpretation of the operators $\Gamma_a$ will be important for us and to distinguish them from the matrices $\gamma_a$, we use different notation $\gamma_a \rightarrow \Gamma_a$.

The Clifford algebra is a vector space with dimension $2^d$. The standard basis in this space is formed by anti-symmetrized products of $\Gamma_a$. For example, (see (1.6))

$$\Gamma_{A_0} = 1, \quad \Gamma_{A_1} = \Gamma_a, \quad \Gamma_{A_2} = \Gamma_{a_1a_2} = \frac{1}{2!} [\Gamma_{a_1} \Gamma_{a_2} - \Gamma_{a_2} \Gamma_{a_1}], \ldots$$  \hspace{1cm} (2.4)

$$\Gamma_{A_k} = \Gamma_{a_1\cdots a_k} = \text{As}(\Gamma_{a_1} \cdots \Gamma_{a_k}) = \frac{1}{k!} \sum_s (-1)^{\mu(s)} \Gamma_{s(a_1)} \cdots \Gamma_{s(a_k)}.$$  \hspace{1cm} (2.5)

Here we use the notion of an antisymmetric product $\text{As}$ of $\Gamma_a$-operators. Inside the $\text{As}$-product, the operators $\Gamma_a$ behave like anti-commuting variables.

Now we introduce the generating function for the basis elements $\Gamma_{A_k}$ (2.3)

$$\sum_{k=0}^{\infty} \frac{1}{k!} u^a_1 \cdots u^a_k \text{As}(\Gamma_{a_1} \cdots \Gamma_{a_k}) = \sum_{k=0}^{\infty} \frac{1}{k!} (u^a \Gamma_a)^k = \exp(u \cdot \Gamma) = \text{As}[\exp(u \cdot \Gamma)].$$  \hspace{1cm} (2.6)

Here $u \cdot \Gamma = u^a \Gamma_a$, $u^a$ are anti-commuting auxiliary variables: $u^a u^b = -u^b u^a$ and we also adopt that $u^a \Gamma_b = -\delta_{ab} \Gamma_a$ to have natural commutation relations $u^a (u \cdot \Gamma) = (u \cdot \Gamma) u^a$.

Formula (2.6) implies that the basis elements $\Gamma_{A_k}$ (2.3) can be obtained from $\exp(u \cdot \Gamma)$ as

$$\Gamma^{a_1 \cdots a_k} = \partial \theta_{a_1} \cdots \partial \theta_{a_k} \exp(u \cdot \Gamma)|_{u=0}.$$  \hspace{1cm} (2.7)

Further, we indicate two basic relations that will be used extensively in our calculations with Clifford algebra.

Proposition 4. The product of generating functions (2.4) is evaluated as

$$e^{u_1 \cdot \Gamma} \cdots e^{u_k \cdot \Gamma} = e^{-\sum_{i<j} u_i u_j} e^\left(\sum_i u_i \Gamma_i\right).$$  \hspace{1cm} (2.8)
Let $u^a$, $v^a$, $\alpha^a$, $\beta^a$ be anticommuting variables, $x$ and $y$ are commuting variables. Then we have the following identity

$$
\exp(x\partial_u \cdot \partial_v)\exp(u\cdot \alpha + v \cdot \beta + y u \cdot v)|_{y=u=0} = (1 - xy)^d \exp\left(\frac{x}{1 - xy} \alpha \cdot \beta\right),
$$

where we have used shorthand notation $\partial_u \cdot \partial_v \equiv \frac{\partial}{\partial u^a} \frac{\partial}{\partial v^a}$.

Proof. The formula (2.6) is a consequence of the Baker–Hausdorff formula $e^A e^B = e^{A + B + \frac{1}{2}[A, B]}$, where we take $A = u \cdot \Gamma$, $B = v \cdot \Gamma$ and $[A, B] = -u^a v^b (\Gamma_a \Gamma_b + \Gamma_b \Gamma_a) = -2 u \cdot v$.

Identity (2.7) can be easily deduced by taking into account the standard representation (9–12) of the operator $\exp(x \partial_u \cdot \partial_v)$ as a Gaussian integral over $2d$ anticommuting variables $\theta_a$ and $\bar{\theta}_a$. Indeed

$$
\exp(x\partial_u \cdot \partial_v) = x^d \int \prod_{a=1}^{d} d\theta_a d\bar{\theta}_a \exp(x^{-1} \bar{\theta} \cdot \theta + \bar{\theta} \cdot \partial_u + \partial_v \cdot \theta),
$$

so that all operations of differentiations lead to the simple shifts $u \rightarrow u + \bar{\theta}$, $v \rightarrow v - \theta$ and then the lhs of (2.7) takes the form of the Gaussian integral again

$$
x^d \int \prod_{a=1}^{d} d\theta_a d\bar{\theta}_a \exp((x^{-1} - y) \bar{\theta} \cdot \theta + \bar{\theta} \cdot \alpha + \beta \cdot \theta) = x^d (x^{-1} - y)^d \exp\left(\frac{x}{1 - xy} \alpha \cdot \beta\right).
$$

In fact, all subsequent calculations are based on (2.6) and (2.7). \( \square \)

Note that the topic of this section has an evident interpretation in the language of quantum field theory. The formula (2.6) is one of the variants of Wick’s theorem and expresses the result of reduction to the normal form. The topic of this section can be considered as an application of the general field-theoretical functional technique [10] to a very special example; exactly this point of view was elaborated in the paper [8]. It is possible to use the language of symbols of fermionic operators [11] as well. For simplicity, we have derived all needed formulae in a very naive and straightforward way.

### 2.2. Fermionic realization of R-matrix

Dealing with the Yang–Baxter equation (1.11) as well as with the RLL-relation (1.8), we have to handle the tensor product of several spinor representation spaces. In fact we need gamma-matrices $\gamma_a$ acting in the tensor product of two spaces. Since instead of gamma-matrices $\gamma_a$ we consider the generators of Clifford algebra, we need here two types of generators $(\Gamma_1)_a$, $(\Gamma_2)_a$, which anticommute to each other

$$
(\Gamma_1)_a (\Gamma_2)_b = -(\Gamma_2)_b (\Gamma_1)_a.
$$

(2.8)

It is rather natural due to the emphasized above fermionic nature of representation (2.2).

Moreover, the convention (2.8) makes the formulae much simpler.

$SO(d)$-invariant fermionic R-matrix (1.9) is constructed out of tensor products $(\Gamma_1)_a (\Gamma_2)^b_a$. Let us rewrite this gamma-matrix structure in a more appropriate form

$$(\Gamma_1)_a (\Gamma_2)^b_a = A_s[\Gamma_{i_1n_1} \cdots \Gamma_{i_nn_n} [\Gamma_2^{a_1} \cdots \Gamma_2^{a_2}]] = A_s[\Gamma_{i_1n_1} \cdots \Gamma_{i_nn_n} [\Gamma_2^{a_1} \cdots \Gamma_2^{a_2}]] = A_s(1)[\Gamma_{i_1n_1} \cdots \Gamma_{i_nn_n} \Gamma_2^a] = A_s(1)[(\Gamma_1 \cdot \Gamma_2)^4]$$

(2.9)

where $s_k \equiv (-1)^{\frac{k(k+1)}{2}}$ and we denote by $A_s(1)$ the operation $A_s$ applied only for the product of $(\Gamma_1)_a$. Below we will omit index (1) in the notation $A_s(1)$ since for the expressions of the type (2.9) we have $A_s(1) = A_s(2)$. At the first step in (2.9), taking into account definition (1.6), one
can forget about one of the symbols $A$ due to the convolution of two antisymmetric tensors. Next, it is possible to accomplish rearrangements taking into account that $\Gamma_1 \cdot \Gamma_2 = -\Gamma_2 \cdot \Gamma_1$. The last equality in (2.9) implies that $A_s[\varepsilon^a \Gamma_1 \Gamma_2]$ is a generating function for the set of tensor products $(\Gamma_1)_{A_k} (\Gamma_2)^{A_k}$

$$A_s[\varepsilon^a \Gamma_1 \Gamma_2] = \sum_k \frac{s_k}{k!} \varepsilon^a \Gamma_1 \Gamma_2_{A_k} (\Gamma_2)^{A_k}. \quad (2.10)$$

Thus we have succeeded in rewriting the multiple summation over repeated indices in a compact form.

**Proposition.**

$$A_s[\varepsilon^a \Gamma_1 \Gamma_2] = e^{x_k A_k} e^{x_0 \Gamma_1 + v^0 \Gamma_2} |_{u = v = 0}, \quad (2.11)$$

**Proof.** Using (2.5) we obtain

$$s_k(\Gamma_1)_{A_k} (\Gamma_2)^{A_k} = s_k \partial_{x_k_1} \cdots \partial_{x_k_N} e^{x_k \Gamma_1} \partial_v \cdots \partial_v e^{x_0 \Gamma_1} e^{x_0 \Gamma_2} |_{u = v = 0} = (\partial_v \cdots \partial_v)^k e^{x_k \Gamma_1} e^{x_0 \Gamma_2} |_{u = v = 0}. \quad (2.12)$$

Substituting (2.12) into (2.10) gives (2.11). □

Consider a fermionic analogue of the operator (1.9) where coefficient functions are assumed to be arbitrary. Using generating function (2.10) we represent this operator in several equivalent forms

$$R(u) = \sum_{k=0}^\infty \frac{R_k(u)}{k!} (\Gamma_1)_{A_k} (\Gamma_2)^{A_k} = \sum_{k=0}^\infty \frac{R_k(u)}{k!} a_k^A A_s[\varepsilon^a \Gamma_1 \Gamma_2]_{|_{u = v = 0}} = R(u|x) \ast A_s[\varepsilon^a \Gamma_1 \Gamma_2], \quad (2.13)$$

where we have used shorthand notation $R(x) \ast F(x) \equiv R(\partial_v) F(x)|_{x = 0}$. Note that all the information about coefficient functions of the operator $R$ in (2.13) is encoded in just one function $R(u|x)$

$$R(u|x) = \sum_{k=0}^\infty \frac{R_k(u)}{k!} a_k^A \Gamma_{A_k}. \quad (2.14)$$

At the end of this subsection, we show how to represent fermionic operators (2.13) in the matrix form. Consider two matrix representations $\rho'$ and $\rho''$ for the fermionic Clifford algebra with generators $\Gamma_{\rho_1}$ and $\Gamma_{\rho_2}$ and defining relations (2.1), (2.8):

$$\rho'(\Gamma_{\rho_1}) = \gamma_\rho \otimes 1, \quad \rho'(\Gamma_{\rho_2}) = \gamma_{\rho+1} \otimes \gamma_\rho, \quad (2.15)$$

where $\gamma_\rho$, $\gamma_{\rho+1}$ are standard $\gamma$-matrices defined in (1.5) and (1.13). For the even and odd parts of (2.13)

$$R^+ = \sum_{k=0}^\infty \frac{R_{2k}(u)}{(2k)!} (\Gamma_1)_{A_{2k}} (\Gamma_2)^{A_{2k}}, \quad R^- = \sum_{k=0}^\infty \frac{R_{2k+1}(u)}{(2k+1)!} (\Gamma_1)_{A_{2k+1}} (\Gamma_2)^{A_{2k+1}},$$

we obtain, using (2.15), the following representations

$$\rho'(R^+) = \sum_{k=0}^\infty \frac{R_{2k}(u)}{(2k)!} \gamma_{\rho_{2k}} \otimes \gamma^{A_{2k}}, \quad \rho'(R^-) = \left( \sum_{k=0}^\infty \frac{R_{2k+1}(u)}{(2k+1)!} \gamma_{A_{2k+1}} \otimes \gamma^{A_{2k+1}} \right) \left( \gamma_{\rho+1} \otimes 1 \right), \quad (2.16)$$
\[ \rho^\nu(R^+)(\rho^\nu(R^-)) = \sum_{k=0}^\infty \frac{R_{2k}}{(2k)!} \gamma_{A_k} \otimes \gamma^A_{2k}, \quad \rho^\nu(R^-) = -\left( \sum_{k=0}^\infty \frac{R_{2k+1}}{(2k+1)!} \gamma_{A_{2k+1}} \otimes \gamma^A_{2k+1} \right) (1 \otimes \gamma_d^{d+1}). \]

(2.17)

Taking into account the fact that the solutions of the Yang–Baxter equation (1.11) admit transformations (1.23), we can use the following convention to construct the matrix representation \( \rho \) of the Yang–Baxter solutions (2.13):

\[ \rho \left( \sum_{k=0}^\infty \frac{R_k}{k!} (\gamma_1 \gamma_2)^A_k \right) = \sum_{k=0}^\infty \frac{R_k}{k!} \gamma_{A_k} \otimes \gamma^A_{k}. \]

(2.18)

Let us note that at even \( d \) the corresponding calculation is implemented in detail in the appendix.

Here we omit arbitrary constant multipliers. To be more concrete, let us rewrite the previous expression for the operator \( P \) in the following form

\[ P = \sum_{k=0}^\infty \frac{R_k}{k!} \left( (\gamma_1 \gamma_2)^A_k \right) = \sum_{k=0}^\infty \left( \sum_{a_1, \ldots, a_n < x_{a_1}, \ldots, x_{a_n}} \gamma_{a_1 \gamma_{a_2} \cdots \gamma_{a_n}} \gamma_{a_1 \gamma_{a_2} \cdots \gamma_{a_n}} \cdots \gamma_{a_1 \gamma_{a_2} \cdots \gamma_{a_n}} \right). \]

The proof presented below will serve as a simple example to demonstrate typical calculations with generating functions. Firstly we prove identities

\[ e^{\nu \Gamma_1} \text{As}(e^{\nu \Gamma_1} \Gamma_2) = \text{As}(e^{\nu \Gamma_1} \Gamma_2) \text{As}(e^{\nu \Gamma_1} \Gamma_2) ; \quad P = e^{-\nu} \text{As}(e^{\nu \Gamma_1} \Gamma_2) = \text{As}(e^{-\nu} \Gamma_1 \Gamma_2). \]

(2.21)

Here we omit arbitrary constant multipliers. To be more concrete, let us rewrite the previous expression for the operator \( P \) in the following form

\[ P = \sum_{k=0}^\infty \frac{R_k}{k!} (\gamma_1 \gamma_2)^A_k \]

(2.20)

\[ (\gamma_1)^n (\gamma_2)^m = \sum_{\kappa_1, \kappa_2, \cdots, \kappa_n} \sum_{\kappa_1, \kappa_2, \cdots, \kappa_m} \gamma_{\kappa_1} \gamma_{\kappa_2} \cdots \gamma_{\kappa_n} \Gamma_{\kappa_1} \Gamma_{\kappa_2} \cdots \Gamma_{\kappa_m} \]

The proof presented below will serve as a simple example to demonstrate typical calculations with generating functions. Firstly we prove identities

\[ e^{\nu \Gamma_1} \text{As}(e^{\nu \Gamma_1} \Gamma_2) = \text{As}(e^{\nu \Gamma_1} \Gamma_2) \text{As}(e^{\nu \Gamma_1} \Gamma_2) ; \quad P = e^{-\nu} \text{As}(e^{\nu \Gamma_1} \Gamma_2) = \text{As}(e^{-\nu} \Gamma_1 \Gamma_2). \]

(2.22)

Here we apply successively (2.11) and (2.6). Thus (2.22) is proven. In fact, this calculation set the pattern for subsequent manipulations with generating functions.

We rewrite equation (2.20) with the help of generating functions (2.5), (2.10)

\[ P(x) = \delta \binom{\nu \Gamma_1 \Gamma_2}{0} \left[ (\gamma_1)^x (\gamma_2)^y \text{As}(e^{\nu \Gamma_1} \Gamma_2) \right]|_{x, y = 0} = P(x) = \delta \text{As}(e^{\nu \Gamma_1} \Gamma_2) \]

(2.23)

Substituting (2.22) in (2.23) and calculating the derivatives with respect to \( s \) and \( t \), we obtain

\[ P(x) = \text{As} \left[ (\Gamma_{2a} + x \Gamma_{2a}) e^{\nu \Gamma_1} \Gamma_2 \right] = P(x) = \text{As} \left[ (\Gamma_{2a} + x \Gamma_{2a}) e^{\nu \Gamma_1} \Gamma_2 \right], \]

or equivalently

\[ [P(x) - \delta \binom{P(x)}{0} \left[ (\gamma_1)^x (\gamma_2)^y \text{As}(e^{\nu \Gamma_1} \Gamma_2) \right]|_{x, y = 0} = [P(x) - \delta \binom{P(x)}{0} \left[ (\gamma_1)^x (\gamma_2)^y \text{As}(e^{\nu \Gamma_1} \Gamma_2) \right]. \]

(2.24)

In (2.24), we obtain the differential equation on the function \( P(x) \)

\[ \delta_x P(x) = P(x) \implies P(x) = \text{const} \cdot e^x, \]

which finishes the proof of (2.21).

In (2.22), we have found the product of the generating functions (2.4) and (2.10). It is exactly what we need to check RLL-relations (1.8), (A.1) giving rise to the condition (1.10).

The corresponding calculation is implemented in detail in the appendix.
2.4. Generating function for Yang–Baxter, unitarity and cross-unitarity relations

In this subsection, we examine thoroughly the gamma-matrix structure of the Yang–Baxter (1.11), unitarity (1.24) and cross-unitarity (1.25) relations. In order to apply the technique outlined above and in view of matrix representation \( \rho \) (2.18), we consider instead their fermionic analogues, i.e. the relations for fermionic operators (2.13).

We start with a fermionic version of the Yang–Baxter relation (1.11) whose rhs is a sum of the operator tensor products

\[
(\Gamma_2)_{A_1} (\Gamma_3)^{B_k} (\Gamma_1)_{B_k} (\Gamma_2)^{B_k} (\Gamma_2)_{C_k} (\Gamma_3)^{C_k}
\]

(2.25)
multiplied by appropriate coefficient functions of spectral parameters. According to our approach, instead of simplifying the products of fermionic generators of Clifford algebra in (2.25), we multiply the corresponding generating functions (2.10) depending on parameters \( x, y \) and \( z \)

\[
\text{As}(e^x t_2; \Gamma_1)\text{As}(e^y t_1; \Gamma_2)\text{As}(e^z t_3; \Gamma_3) = (1 - xy)^2 \text{As}(e^{\frac{2 e^{xy}}{1-xy} \gamma_1} \gamma_2 + \frac{2e^{xy}}{1-xy} \gamma_1 + \frac{2e^{xy}}{1-xy} \gamma_1 \gamma_3).
\]

(2.26)

Expanding the latter formula into a series over \( x, y, z \) and picking out the appropriate term, one obtains (2.25). Let us outline the derivation of (2.26). Using (2.11), one can rewrite the product of the three generating functions in (2.26) as follows

\[
e^{x^2 + h} e^{y^2, h} e^{z^2 + v} e^{x y z} e^{x^2 + v} e^{x y z} e^{x^2 + v} e^{x y z} e^{x y z} e^{x y z} e^{x y z} e^{x y z}.
\]

Then due to (2.6)

\[
e^x \gamma_1 + v^2 \gamma_2, e^y \gamma_1 + v^2 \gamma_2, e^z \gamma_1 + q^2 \gamma_2, e^x \gamma_1 + q^2 \gamma_2, e^y \gamma_1 + q^2 \gamma_2, e^z \gamma_1 + q^2 \gamma_2, e^x \gamma_1 + q^2 \gamma_2, e^y \gamma_1 + q^2 \gamma_2, e^z \gamma_1 + q^2 \gamma_2,
\]

and applying several times (2.7) one obtains the desired result (2.26). In much the same way, the generating function of the tensor product structure in the lhs of the fermionic Yang–Baxter relation (1.11) has the form

\[
\text{As}(e^x t_1; \Gamma_1)\text{As}(e^y t_2; \Gamma_1)\text{As}(e^z t_3; \Gamma_1) = (1 - xy)^2 \text{As}(e^{\frac{2 e^{xy}}{1-xy} \gamma_1} \gamma_2 + \frac{2e^{xy}}{1-xy} \gamma_1 + \frac{2e^{xy}}{1-xy} \gamma_1 \gamma_3).
\]

(2.27)

Let us mention that expressions (2.26) and (2.27) are almost identical.

Dealing with the unitarity relation (1.24) for the spinorial \( R \)-matrix, we calculate

\( R_{12}(u)R_{12}(-u) \) which forces us to consider the tensor products of fermionic generators (\( \Gamma_1 \) \( \Gamma_2 \) \( \Gamma_3 \)) \( \text{As}(e^x t_1; \Gamma_1)\text{As}(e^y t_2; \Gamma_1)\text{As}(e^z t_3; \Gamma_1) \)

(2.28)

and obtaining the desired result in (2.29).

Finally, the cross-unitarity relation (1.25) leads to examining the product \( R_{12} R_{13} \) whose generating function of tensor product structure can be obtained from (2.26) at \( x = 0 \). Indeed, tracing it over the second space, one has

\[
\text{tr}_2 \text{As}(e^x t_1; \Gamma_1)\text{As}(e^y t_2; \Gamma_1) = \text{tr}_2 \text{As}(e^x t_1; \Gamma_2 + y, \Gamma_1, \gamma_1, \gamma_1, \gamma_1) = 2^d \text{As}(e^x t_1; \Gamma_1),
\]

(2.29)

where relations \( \text{tr}_2 \gamma_2 = 2 \gamma_2 \) and \( \text{tr}_2 \gamma_1 = 0 \) at \( k > 0 \) are taken into account.

Thus, we have indicated the generating functions for the tensor product structure of the relevant fermionic relations for the spinorial \( R \)-matrix (2.13). We will prove these relations in the subsequent sections using the obtained results.

Remark. Equations (2.26), (2.27), (2.28) give the identities for the exchange operators (2.21):

\[
P_{12}P_{23}P_{12} = P_{23}P_{12}P_{23}, \quad P_{12}^P P_{23} = P_{23}^P P_{12}^P P_{23},
\]

\[
PP = \frac{e^{xy}}{d!} \text{As}(\Gamma_1, \Gamma_2)^d, \quad PP' = \frac{(-2)^d}{d!} \text{As}(\Gamma_1, \Gamma_2)^d, \quad PP'' = PP = 2^d I.
\]
2.5. The local Yang–Baxter relation

Let us note that due to (2.26) and (2.27), the following local Yang–Baxter relation takes place

\[(1 - xy)^{-d} \text{As}(e^{i\Gamma_1}, e^{i\Gamma_2}) \text{As}(e^{i\Gamma_2}, e^{i\Gamma_3}) \text{As}(e^{i\Gamma_3}, e^{i\Gamma_1}) = (1 - x'y')^{-d} \text{As}(e^{i\Gamma_1'}, e^{i\Gamma_2'}) \text{As}(e^{i\Gamma_2'}, e^{i\Gamma_3'}) \text{As}(e^{i\Gamma_3'}, e^{i\Gamma_1'}),\]

(2.30)

where parameters \(x, y, z\) and \(x', y', z'\) are related by equations

\[
\begin{align*}
    x + y &= z(1 + x'y'), \\
    x(1 + xy) &= x'y' + y, \\
    z(1 + xy) &= x'y' + y', \\
    z(x - y) &= z'(x' - y').
\end{align*}
\]

(2.31)

Further, we consider the case of real parameters \(x, y, z\) and \(x', y', z'\). The last relation in (2.31) and the product of the first two relations in (2.31) show that the functions

\[
\lambda_1 = \frac{z(x - y)}{(1 - xy)}, \quad \lambda_2 = \frac{z(x + y)(1 + xy)}{(1 - xy)^2},
\]

are invariant under the transformation \(x, y, z \rightarrow x', y', z'\). Thus, the points \((x, y, z)\) and \((x', y', z')\) lie on the curve \(C_{a,b} \subset \mathbb{R}^3\) defined by the equations

\[
\begin{align*}
    z(x - y) &= b(1 - xy), \\
    (x + y)(1 + xy) &= a(x - y)(1 - xy)
\end{align*}
\]

(2.32)

where \(b = \lambda_1\) and \(a = \frac{1}{y^2}\) are parameters that fix the curve. In the generic case, the geometrical picture is the following. The second equation in (2.32) defines the family of curves parameterized by \(a\), which are projections of \(C_{a,b}\) onto the plane \((x, y)\). Thus, it is possible to introduce new coordinates \((x, y) \rightarrow (a, t)\) in the plane, where \(t\) is a coordinate on the curve specified by \(a\). The variable \(t\) is a coordinate on \(C_{a,b}\) as well. Then due to the first equation in (2.32), the coordinate \(z\) is determined by \(b\) and \((x, y)\) or equivalently by \(b\) and \((a, t)\). The transformation \((x, y, z) \rightarrow (x', y', z')\) is equivalent to the change of coordinates \(t \rightarrow t'\) on the curve \(C_{a,b}\). Now we specify the coordinate \(t\) on the curve and choose, according to (2.32), new variables \((a, b, t)\) instead of \((x, y, z)\):

\[
a = \frac{1 + xy x + y}{1 - xy x - y}, \quad b = \frac{x - y}{1 - xy}, \quad t = \frac{x - y}{1 + xy}.
\]

(2.33)

In terms of these new variables, the transformation \(x, y, z \rightarrow x', y', z'\) looks very simple

\[
a \rightarrow a' = a, \quad b \rightarrow b' = b, \quad t \rightarrow t' = \frac{b}{at}.
\]

The \(t \rightarrow t'\) transformation follows from the second relation in (2.31), which can be written as \(b/t = a't'\).

At the end of this section, we note that the local Yang–Baxter equations were introduced in [13] and applied to the investigations of 3d integrable systems in many papers (see, e.g., [14]).

3. Yang–Baxter relation, unitarity and crossing

In order to prove crucial properties of the spinorial \(R\)-matrix (1.9), we need to transform it to a more appropriate form. In subsection 3.1, we rewrite the spinorial \(R\)-matrix, which is presented in fermionic realization (2.13) as an integral over auxiliary parameter

\[
R(u) = \int_0^\infty \frac{dx x^{a-1}}{(1 + x^2)^{a/2}} [a(u) \text{As}(e^{i\Gamma_1}, e^{i\Gamma_2}) + b(u) \text{As}(e^{-i\Gamma_1}, e^{-i\Gamma_2})] \]

(3.1)

where \(a(u)\) and \(b(u)\) are arbitrary functions, related to \(A(u)\) and \(B(u)\) which appeared in (1.14). Representation (3.1) happens to be very helpful since it enables one to avoid multiple
sums over repeated indices in (1.9). Moreover, the finite summation over $k$ in (1.9) is
substituted by an integral over the auxiliary parameter. Thus the Yang–Baxter equation (1.11),
which would assert equality of the two cumbersome multiple sums if we use representation
(1.9), turns into an equality of two integrals. Using the representation (3.1), we check directly
that the Yang–Baxter equation (1.11) is satisfied. More concretely, we show that the equation
(1.11) is equivalent to the symmetry of a certain integral taken over the space of auxiliary
parameters.

3.1. Spinorial $R$-matrix

We previously showed that the gamma-matrix structure of the spinorial $R$-matrix (1.9) can
be simplified considerably using fermionic realization (2.13). Now we are going to make one
more step rewriting the function $R(u|x)$ in (2.13) that contains all the information about the
coefficient functions $R_k(u)$. Let us remind the reader that coefficient functions respect the
recurrence relations (1.10). Above, we have already found their solutions (1.14) containing
two arbitrary functions of the spectral parameter. Using this freedom, coefficient functions can
be expressed in terms of the Euler beta function

$$R_{2k}(u) = (-)^k A(u) \frac{\Gamma(k + \frac{d}{2}) \Gamma\left(\frac{u + d}{2} - k\right)}{\Gamma(u + \frac{d}{2})},$$

$$R_{2k+1}(u) = (-)^k B(u) \frac{\Gamma(k + \frac{u+1}{2}) \Gamma\left(\frac{u+d-1}{2} - k\right)}{\Gamma(u + \frac{d}{2})}$$

where $A(u)$ and $B(u)$ are arbitrary functions of spectral parameter. Then we separate even and
odd terms in (2.14)

$$R(u|y) = \sum_{k=0}^{\infty} \frac{R_k(u) s_k}{k!} y^k = \sum_{k=0}^{\infty} \frac{R_{2k}(u) s_{2k} y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{R_{2k+1}(u) s_{2k+1} y^{2k+1}}{(2k+1)!},$$

take into account $s_{2k} = s_{2k+1} = (-)^k$, resort to an integral representation of the B-function

$$\frac{\Gamma(k + \frac{d}{2}) \Gamma\left(\frac{u + d}{2} - k\right)}{\Gamma(u + \frac{d}{2})} = 2 \int_0^{\infty} \frac{dx \: x^{u-1} \: x^{2k}}{(1 + x^2)^{u+d/2}}$$

$$\frac{\Gamma(k + \frac{u+1}{2}) \Gamma\left(\frac{u+d-1}{2} - k\right)}{\Gamma(u + \frac{d}{2})} = 2 \int_0^{\infty} \frac{dx \: x^{u-1} \: x^{2k+1}}{(1 + x^2)^{u+d/2}}$$

and sum up the series obtaining

$$R(u|y) = \int_0^{\infty} \frac{dx \: |x|^{u-1}}{(1 + x^2)^{u+d/2}} [(A + B) e^{xy} + (A - B) e^{-xy}].$$

Thus, we have managed to substitute the finite set of coefficient functions appearing in (1.9)
by the integral over the auxiliary parameter. Finally, applying (2.13), we deduce the desired form
(3.1) of the spinorial $R$-matrix claimed above. In (1.16), we indicated a natural decomposition
of the spinorial $R$-matrix in the sum of even $R^+$ and odd $R^-$ parts (1.15). The formulae (3.1)
and (3.3) imply the second natural decomposition

$$R(u) = A(u) R^+(u) + B(u) R^-(u) = a(u) \mathcal{R}^+(u) + b(u) \mathcal{R}^-(u)$$

where

$$\mathcal{R}^+(u) = \int_0^{\infty} \frac{dx \: |x|^{u-1}}{(1 + x^2)^{u+d/2}} \mathrm{As}(e^{i\gamma_1, r_2}) \quad \mathcal{R}^-(u) = \int_0^{\infty} \frac{dx \: |x|^{u-1}}{(1 + x^2)^{u+d/2}} \mathrm{As}(e^{-i\gamma_1, r_2}),$$

and $a = A + B, b = A - B.$
3.2. Integral identity

Now we are ready to establish the Yang–Baxter relation (1.11). More exactly, we will first prove the Yang–Baxter relation for the spinorial $R$-matrix in fermionic realization. Its tensor product structure has already been discussed in subsection 2.4. To be more precise, the corresponding generating functions for its rhs (2.26) and lhs (2.27) have been indicated. Then, in the previous subsection, we found out that the coefficient functions of the spinorial $R$-matrix can be arranged in a sole function (3.3). Further, let us note that the Yang–Baxter relation (1.11) in fermionic realization is equivalent to the set of eight three–term relations for $R^+, R^-$ (3.5)

$$R^i_{12}(u) R^k_{23}(u+v) R^j_{12}(v) = R^j_{23}(v) R^k_{12}(u+v) R^i_{12}(u)$$

(3.6)

where $i, j, k = +, -$ since $a(u)$ and $b(u)$ in the expression of spinorial $R$-matrix (3.4) are arbitrary functions. At first, the Yang–Baxter relation will be proven for $R^+(u)$.

Taking into account (2.26), (2.27) and (3.3) one can easily see that the Yang–Baxter relation (3.6) at $i = j = k = +$ is equivalent to

$$\text{As}[I^a, \Gamma_1 \cdot \Gamma_2, \Gamma_1 \cdot \Gamma_3, \Gamma_2 \cdot \Gamma_3] = \text{As}[I^a, \Gamma_2 \cdot \Gamma_3, \Gamma_1 \cdot \Gamma_3, \Gamma_1 \cdot \Gamma_2]$$

(3.7)

where

$$I^{a\cdot}(A, B, C) = \int_D dx \, dy \, dz \, |x|^{a-1} |y|^{v-1} |z|^{w-v-1} \frac{(1-xy)^d}{(1+x^2)^{w+y+1}(1+y^2)^{w+x+1}(1+z)^{w+v+1}}$$

(3.8)

the integration domain $D = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$. Instead of verifying (3.7), we are going to check the more general relation

$$I^{a\cdot}(A, B, C) = I^{b\cdot}(C, B, A)$$

(3.9)

where the left and right hand sides are to be understood as the formal power series in $A, B, C$, which are unspecified operators. The discrete symmetry (3.9) of the integral (3.8) will be established by means of the integration variable change $(x, y, z) \rightarrow (x', y', z')$ defined by the local Yang–Baxter equation (2.30), which leads to the system of relations (2.31). One can easily see that, under this transformation of variables, the external parameters $A$ and $C$ are interchanged in the exponential factor in (3.8). However, it is rather nontrivial that the other factors in the integrand transform in the right way, such that (3.9) is satisfied.

To see it, we appeal to geometric interpretations of the transformation (2.31), which have been discussed in section 2.5, where we proposed to change the variables $(x, y, z) \rightarrow (a, b, t)$, according to (2.33). In this case, the integration domain $D$ can be represented as $\bigcup_{a, b} C_{a,b}$, where $C_{a,b}$ is a curve parameterized by $a$ and $b$. After all, we make in (3.8) the natural change of the integration variables $(x, y, z) \rightarrow (a, b, t)$, presented in (2.33), for which the Jacobian determinant has a rather simple form

$$\det \left( \frac{\partial(t, a, b)}{\partial(x, y, z)} \right) = 2 \frac{(1+x^2)(1+y^2)}{(1+xy)(1-xy)}.$$  

The formulae (2.33) map domain $D$ onto disconnected domain $G$

$$G = \{(a, b, t) : a \geq 1, b \geq 0, -\infty < t < \infty\} \cup \{(a, b, t) : a \leq -1, b \leq 0, -\infty < t < \infty\},$$

(3.10)

which is illustrated in figures 1 and 2. After a simple calculation, one obtains

$$I^{a\cdot}(A, B, C) = \frac{1}{2^{a+b+1}} \int_G da \, db \, dt \, |t|^{-1} |t|^{-1} \exp \left(Aat - Bb + Cht^{-1}\right).$$

(3.11)

Since the integral (3.8) is rewritten in the form (3.11), it is straightforward to prove the symmetry (3.9), applying the integration variable change $(a, b, t) \rightarrow (a, b, t')$
Figure 1. Projection of the domain $D$ onto the plane $(x, y)$. It is separated by the curves $x = y$, $xy = 1$ into four parts marked by different colors, each mapped on the corresponding domain in figure 2.

where $t' = \frac{b}{at}$. It corresponds to the transposition $at \rightleftharpoons bt^{-1}$ in (3.11). Indeed the integrand in (3.11) transforms correctly and the integration domain $G$ is mapped onto itself.
Thus the Yang–Baxter relation (3.6) at $i = j = k = +$ is established. In a similar way, the other seven three-term relations (3.6) can be checked. To realize it, we note that expressions (3.5) for $R^+$ and $R^-$ are almost identical. They can be obtained from each other reflecting the integration variable $x \rightarrow -x$. In other words $R^+$ and $R^-$ differ solely in integration contour. In the first case, one integrates over the positive semiaxis and in the second case over the negative one. Consequently, to check one of the three-term relations (3.6), we have to consider the integral (3.8) taken over the appropriate reflected domain $D$. For example, at $i = j = -$, $k = +$ we integrate over $x \leq 0$, $y \leq 0$, $z \geq 0$ in (3.8). When the variable change (2.33) is performed, it leads to the integral (3.11) with a certain integration domain that can be found in figure 3. The symmetry (3.9) is established as before by means of the variable change $at \rightarrow bt^{-1}$ in (3.11), which preserves the integration domain, as one can easily see. Let us stress that algebraic manipulations needed to prove that the three-term relations (3.6) are the same in all eight cases. The only difference is in the integration domains in (3.8) or (3.11).

Finally, we have checked eight three-term relations (3.6) and hence we have proved the Yang–Baxter relation (1.11) for the spinorial $R$-matrix in fermionic realization. Using the decomposition (3.4) of the $R$-matrix in the sum of even and odd parts, we obtain eight three-term

$$R^i_{12}(u) R^j_{23}(u + v) R^k_{13}(v) = R^i_{23}(v) R^j_{12}(u + v) R^k_{13}(u) \quad (3.12)$$

where $i, j, k = +, -$ (compare with (1.22)). However, let us emphasize that we have always used above the fermionic representation for the $R$-matrix.

Figure 3. Projection of the domain $D$ and of its three reflections onto the plane $(x, y)$. Images of the subregions in the space $(a, b, t)$ are indicated (see (2.33)). It is assumed $z > 0$. If otherwise $z < 0$ then $b$ to be substituted on the figure by $-b$. 

At the end of subsection 2.2, we have shown how to represent fermionic operators (2.13) in the matrix form (2.18). It can be easily checked that three-term relations (3.12) remain valid.
in both matrix representations \( \rho' \) and \( \rho'' \) (2.15). Thus, the Yang–Baxter relation (1.11) for the spinorial \( R \)-matrix (1.9) is checked.

3.3. Unitarity relation

In the Introduction, we have formulated unitarity relations (1.24) and proven them up to an explicit calculation of coefficient functions \( h_+ (u) \), \( h_- (u) \). Now we are going to fill this gap. We use the fermionic realization of the \( R \)-matrix. In view of (2.13) and (2.28), one has

\[
R^+ (u) R^+ (-u) = R^+ (u|x) R^+ (-u|y) \ast (1 - xy)^d \text{As}(e^{-i \pi \rho}; \Gamma_1, \Gamma_2).
\]

(3.13)

Since we know that \( R^+ (u) R^+ (-u) \) is proportional to projector \( P^+ \), formula (3.13) contains only fermionic structures \( I \) and \( \text{As} (\Gamma_1 \cdot \Gamma_2)^d \). Coefficients at the other structures are equal to zero. Thus it will be sufficient for us to calculate numerical coefficient for \( I \) in (3.13) that is equal to

\[
R^+ (u|x) R^+ (-u|y) \ast (1 - xy)^d = \sum_{k=0}^{d/2} \binom{d}{2k} R_{2k} (u) R_{2k} (-u).
\]

Similarly, calculating the numerical coefficient for \( \text{As} (\Gamma_1 \cdot \Gamma_2)^d \) in (3.13), one obtains

\[
R^+ (u|x) R^+ (-u|y) \ast (x + y)^d = \sum_{k=0}^{d/2} (-1)^k \binom{d}{2k} R_{2k} (u) R_{d-2k} (-u).
\]

Further, using matrix representations \( \rho' \) (2.16) or \( \rho'' \) (2.17), one obtains (2.18)

\[
\rho' (R^+ (u)) \rho' (R^+ (-u)) = \rho'' (R^+ (u)) \rho'' (R^+ (-u)) = \rho (R^+ (u)) \rho (R^+ (-u))
\]

that leads finally to the first unitarity relation (1.24) in view of (2.19).

The previous arguments are valid also for \( R^- (u) R^- (-u) \). The coefficient function for \( I \) is equal to

\[
R^- (u|x) R^- (-u|y) \ast (1 - xy)^d = - \sum_{k=0}^{d/2-1} \binom{d}{2k+1} R_{2k+1} (u) R_{2k+1} (-u).
\]

The matrix realization of the second unitarity relation (1.24) is provided by

\[
\rho' (R^- (u)) \rho' (R^- (-u)) = \rho'' (R^- (u)) \rho'' (R^- (-u)) = - \rho (R^- (u)) \rho (R^- (-u)).
\]

Remark. Unitarity relations (1.24) can be established as well by means of an integral representation for the \( R \)-matrix (3.1) using

\[
\int_0^\infty \int_0^\infty dx \, dy \, u^{d-1} (1 + x^2)^{-u} (1 + y^2)^{-u} = \begin{cases} \frac{2\pi}{u \sin \pi u}, & \text{at } k = 0 \\ - \frac{2\pi}{u \cot \pi u}, & \text{at } k = d \\ 0, & \text{at } k = 1, \ldots, d - 1. \end{cases}
\]

3.4. Cross-unitarity relation

We proceed to the cross-unitarity relation (1.25) where the \( R \)-matrix is taken in fermionic realization and \( P \) is the exchange operator (2.21). The generating function of its tensor product
structure is indicated in (2.29). Then, using an integral representation for coefficient functions (3.3), one obtains that the lhs of the crossing relation (1.25) is equal to

\[
2^2 \int_0^\infty \int_0^\infty \frac{dx \, dy \, x^{d-1}y^{d-1}}{(1 + x^2)(1 + y^2)(1 - x^2y^2)} \text{As}[(a(u)a(2 - u - d) + b(u)b(2 - u - d)) e^{\sqrt{\Gamma_1} \cdot \Gamma_3} + (b(u)a(2 - u - d) + a(u)b(2 - u - d)) e^{\sqrt{\Gamma_1} \cdot \Gamma_3}]
\]

\[\equiv \sum_{k=0}^{\infty} \frac{c_k}{k!} \text{As}(\Gamma_1 \cdot \Gamma_3)^k.\]  

(3.14)

Calculating integrals in (3.14) and taking into account \( a = A + B, \ b = A - B \) one obtains

\[
c_{2k} = \frac{2^{d+1}\pi}{1 - \frac{d}{2} - u} \cot\left(\frac{\pi u}{2}\right) A(u) A(2 - u - d),
\]

\[
c_{2k+1} = \frac{2^{d+1}\pi}{1 - \frac{d}{2} - u} \tan\left(\frac{\pi u}{2}\right) B(u) B(2 - u - d).
\]

(3.15)

Let us note that coefficient functions \( c_{2k} \) and \( c_{2k+1} \) happen to be independent of index \( k \). Comparing (3.14) with exchange operator \( P_{13} = \text{As}(e^{\gamma_1 \cdot \gamma_3}) \) (2.21) one concludes \( c_{2k} = c_{2k+1} \) that gives the restriction on \( A, B \): \( \tan\left(\frac{\pi u}{2}\right) = -\frac{A(u)A(2 - u - d)}{B(u)B(2 - u - d)} \). Finally, using matrix representation \( R \rightarrow \rho(R) \) (2.13), it is easy to see that the crossing relation remains literally the same.

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**Appendix**

In [1] we introduced the representation space \( V' \), which we assumed to be infinitely dimensional in general and that it restricted the universal Yang–Baxter equation (1.1) to the space \( V \otimes V \otimes V' \)

\[ R_{12}(u - v) L_{13}(v) L_{23}(u) R_{12}(u - v) = L_{13}(u) L_{23}(v) R_{12}(u - v) \subseteq \text{End}(V \otimes V \otimes V'). \]  

(A.1)

The operator \( L(u) \), which is defined in the tensor product \( V \otimes V' \) of spinor and arbitrary representation \( T' \) spaces, has been sought for in the form

\[ L(u) = u + \frac{i}{4} \gamma_{ab} \otimes T'(M^{ab}). \]  

(A.2)

Here notation (2.3) is used and \( M_{ab} \) \( (a, b = 1, \ldots, d) \) are generators of \( so(d) \) subjected to relations

\[ [M_{ab}, M_{cd}] = i(\delta_{bd}M_{ac} + \delta_{ad}M_{bc} - \delta_{ac}M_{bd}) \]  

(A.3)

In [1], we claimed that RLL-relations (A.1) with spinorial \( R \)-matrix (1.9) is satisfied if representation \( T' \) is such that

\[ T'([M_{ab}, M_{cd}]) = 0, \]  

(A.4)
where \( [A,B] = AB + BA \) is the anticommutator and square brackets denote antisymmetrization. We will undertake the corresponding calculation in the first part of this appendix using the generating function technique. We are going to show that the RLL-relation (A.1) with the \( L \)-operator (A.2) and the \( R \)-matrix of the form (1.9) leads to the recurrence relation (1.10) for coefficient functions \( R_k(u) \) and sets up the restriction (A.4) on the representation \( T' \) in the quantum space.

Let us multiply (A.4) by the product of four gamma-matrices \( \gamma^a \gamma^b \gamma^c \gamma^d \) that leads to a concise form of the restriction on the examined representations

\[
M^2 - 2i(d-2)M + 2C_2 = 0, \tag{A.5}
\]

where \( M \equiv T'(M_{ab}) \gamma^b \) and \( C_2 \equiv T'(M_{ab} M^{ab}) \) is the quadratic Casimir operator. Using a characteristic identity (A.5), one can immediately obtain the unitarity relations for the \( L \)-operators (A.2):

\[
L(u) \cdot L(d/2 - 1 - u) = (d/2 - 1 - u) u + \frac{1}{8} C_2. \tag{A.6}
\]

In the paper [16] the set of \( \mathfrak{so}(d) \)-type \( L \)-operators (A.2) was investigated for several irreducible finite-dimensional representations \( T' \) of \( \mathfrak{so}(d) \). In these cases, the conditions (A.5), (A.6) were indicated in [16] in the form when the Casimir operator \( C_2 \) is fixed by its numerical values related to the representations \( T' \).

In order to avoid misunderstandings, let us note that in a special case where \( d = 6 \), we have the isomorphism \( \mathfrak{so}(6, \mathbb{C}) = \mathfrak{sl}(4, \mathbb{C}) \), where the corresponding eight-dimensional \( L \)-operator (A.2) is a direct sum of two four-dimensional \( L \)-operators of \( \mathfrak{sl}(4) \) algebra, the spinorial \( R \)-matrix (1.9) reduces to the Yang-R matrix under Weyl projections and condition (A.4) on representation \( T' \) happens to be superfluous. We demonstrate it in the second part of this appendix.

### A.1. RLL-relation

Further by abuse of notation we denote \( T'(M_{ab}) \rightarrow M_{ab} \). The following calculation is very similar to the one presented in subsection 2.3; it uses the generating function technique. We are going to prove the fermionic version of the RLL-relation (A.1). Then, taking the matrix representation \( \rho' \) or \( \rho'' \) (2.15), one obtains immediately (A.1) for the spinorial \( R \)-matrix (1.9) and the \( L \)-operator (A.2).

The substitution of spinorial \( R \)-matrix (2.13) in the fermionic realization and the fermionic analogue of \( L \)-operator (A.2) with the unspecified representation \( T' \) in the quantum space in the RLL-relation (A.1) gives

\[
\sum_{k=0}^{\infty} \frac{R_k(u-v)}{k!} (\Gamma_1)_a (\Gamma_2)_b \left( u + \frac{i}{4} (\Gamma_1)_{ab} M^{ab} \right) \left( v + \frac{i}{4} (\Gamma_2)_{cd} M^{cd} \right) \tag{A.7}
\]

This relation contains terms linear and quadratic in generators \( M_{ab} \). The product of two generators can be transformed by means of Lie algebra commutation relations (A.3)

\[
M_{ab} M_{cd} = \frac{1}{2} [M_{ab}, M_{cd}] + \frac{1}{2} [M_{ab}, M_{cd}]
\]

\[
= \frac{i}{2} [g_{bc} M_{ad} - g_{ac} M_{bd} - g_{ab} M_{cd} + g_{bd} M_{ca}] + \frac{1}{2} [M_{ab}, M_{cd}]
\]

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so that

\[(\Gamma_1)_{ab} (\Gamma_2)_{bc} M^{ab}_{cd} M^{cd} = -2i (\Gamma_1)_{a^c} (\Gamma_2)_{b^c} M^{ab} + \frac{1}{2} (\Gamma_1)_{ab} (\Gamma_2)_{cd} \left\{ M^{ab}, M^{cd} \right\} .\]

All terms in (A.7) linear on spectral parameters are combined in a single one \(\sim (u-v)\) due to relation

\[(\Gamma_1) A_k (\Gamma_2)_{ab} = (\Gamma_1)_{ab} (\Gamma_1) A_k (\Gamma_2)_{ab} = (\Gamma_1) A_k (\Gamma_2)_{ab} (\Gamma_2)_{ab} \]

which is a consequence of the so(d) invariance

\[\{ (\Gamma_1)_{ab} + (\Gamma_2)_{ab} , (\Gamma_1) A_k (\Gamma_2)_{ab} \} = 0 .\]

After all intertwining relation (A.7) is reduced to the form

\[\sum_{k=0}^{\infty} R^k (u) \frac{k!}{k!} M^{ab} (\Gamma_1) A_k (\Gamma_2)_{ab} - (\Gamma_1)_{ab} (\Gamma_1) A_k (\Gamma_2)_{ab} = 0 .\]

Using the reference formulae for products of generating functions (see (2.22))

\[\text{As}(e^{\Gamma_1+\Gamma_2}) = \text{As}(e^{\Gamma_1+\Gamma_2+\Gamma_1+\Gamma_2}) ; \quad e^{\Gamma_1} \text{As}(e^{\Gamma_1+\Gamma_2}) = \text{As}(e^{\Gamma_1+\Gamma_2+\Gamma_1+\Gamma_2}) ,\]

it is easy to derive a compact expression for the first term in (A.8)

\[\sum_{k=0}^{\infty} R^k (u) \frac{k!}{k!} M^{ab} (\Gamma_1) A_k (\Gamma_2)_{ab} - (\Gamma_1)_{ab} (\Gamma_1) A_k (\Gamma_2)_{ab} = 0 .\]

In a similar way, using

\[\text{As}(e^{\Gamma_1+\Gamma_2}) e^{\Gamma_1+\Gamma_2} = \text{As}(e^{(\Gamma_1+\Gamma_2)+(\Gamma_1+\Gamma_2)}) ,\]

\[e^{\Gamma_1+\Gamma_2} \text{As}(e^{\Gamma_1+\Gamma_2}) = \text{As}(e^{(\Gamma_1+\Gamma_2)+(\Gamma_1+\Gamma_2)}) ,\]

the second term in (A.8) can be rearranged as follows

\[\sum_{k=0}^{\infty} R^k (u) \frac{k!}{k!} M^{ab} (\Gamma_1) A_k (\Gamma_2)_{ab} - (\Gamma_1)_{ab} (\Gamma_1) A_k (\Gamma_2)_{ab} = 0 .\]

and the last term in (A.8) takes the form

\[\sum_{k=0}^{\infty} R^k (u) \frac{k!}{k!} M^{ab} (\Gamma_1) A_k (\Gamma_2)_{ab} - (\Gamma_1)_{ab} (\Gamma_1) A_k (\Gamma_2)_{ab} = 0 .\]
\[ \sum_{k=0}^{\infty} \frac{R_k}{k!} \left( (\Gamma_1)_{ab}(\Gamma_1)_{cd} - (\Gamma_1)_{ad}(\Gamma_1)_{bc} \right) [M^{ab}, M^{cd}] \]

\[ = 4R(x) \times (x^3 - x) [M^{ab}, M^{cd}] \text{As e}^{i\Gamma_1 \Gamma_3} \left[ \Gamma_{1\alpha} \Gamma_{1\beta} \Gamma_{1\gamma} \Gamma_{1\delta} - \Gamma_{2\alpha} \Gamma_{2\beta} \Gamma_{2\gamma} \Gamma_{2\delta} \right] \]

\[ = 4\left[ \partial_x^3 R(x) - \partial_x R(x) \right] \times [M^{ab}, M^{cd}] \text{As e}^{i\Gamma_1 \Gamma_3} \left[ \Gamma_{1\alpha} \Gamma_{1\beta} \Gamma_{1\gamma} \Gamma_{1\delta} - \Gamma_{2\alpha} \Gamma_{2\beta} \Gamma_{2\gamma} \Gamma_{2\delta} \right] \]

Thus finally we obtain that (A.8) is equivalent to the relation

\[ \left[ x\partial_x^3 R(x) + x\partial_x R(x) - (d - 2)\partial_x^2 R(x) - u(\partial_x^2 R(x) - R(x)) \right] \times [M^{ab}, M^{cd}] \text{As e}^{i\Gamma_1 \Gamma_3} \left[ \Gamma_{1\alpha} \Gamma_{1\beta} \Gamma_{1\gamma} \Gamma_{1\delta} - \Gamma_{2\alpha} \Gamma_{2\beta} \Gamma_{2\gamma} \Gamma_{2\delta} \right] \]

\[ - \frac{i}{2} [\partial_x^2 R(x) - \partial_x R(x)] \times [M^{ab}, M^{cd}] \text{As e}^{i\Gamma_1 \Gamma_3} \left[ \Gamma_{1\alpha} \Gamma_{1\beta} \Gamma_{1\gamma} \Gamma_{1\delta} - \Gamma_{2\alpha} \Gamma_{2\beta} \Gamma_{2\gamma} \Gamma_{2\delta} \right] \]

\[ = 0. \tag{A.9} \]

There are two independent gamma-matrix structures in the latter formula, so that the differential equation for the coefficient function \( R(x) \)

\[ x\left[ \partial_x^3 R(x) + \partial_x R(x) - (d - 2)\partial_x^2 R(x) - u(\partial_x^2 R(x) - R(x)) \right] \times [M^{ab}, M^{cd}] \text{As e}^{i\Gamma_1 \Gamma_3} \left[ \Gamma_{1\alpha} \Gamma_{1\beta} \Gamma_{1\gamma} \Gamma_{1\delta} - \Gamma_{2\alpha} \Gamma_{2\beta} \Gamma_{2\gamma} \Gamma_{2\delta} \right] \]

and requirement \([M_{ab}, M_{cd}] = 0\) (see (A.4)) arise. The differential equation produces the recurrence relation (see (1.10)) for the coefficients \( R_k(u) \):

\[ R(x) = \sum_{k=0}^{\infty} \frac{R_k(u)}{k!} x^k \rightarrow R_{k+2}(u) = -\frac{u + k}{u + d - 2 - k} R_k(u). \]

A.2. R-matrix in the special case \( d = 6 \)

Now we proceed to the special case \( d = 6 \). The recurrent relations (1.10) for odd and even coefficients are independent, which enables us to fix \( R_0(u) = (u + 4)/8 \) and \( R_1(u) = 0 \). Hence \( R \)-matrix (1.9) takes the form

\[ R(u) = R_0(u) \mathbf{1} \otimes 1 + \frac{R_2(u)}{2!} \gamma_{a1a2} \otimes \gamma^{a1a2} + \frac{R_4(u)}{4!} \gamma_{a1...a8} \otimes \gamma^{a1...a8} + \frac{R_6(u)}{6!} \gamma_{a1...a16} \otimes \gamma^{a1...a16} \]

\[ \tag{A.10} \]

where

\[ R_0(u) = (u + 4)/8, \quad R_2(u) = -u/8, \quad R_4(u) = u/8, \quad R_6(u) = -(u + 4)/8 \]

and the last term in (A.9), which is responsible for the condition (A.4), reduces to

\[ \frac{2}{3!} [R_2(u) + R_4(u)] [M^{ab}, M^{cd}] \gamma_{ab} \gamma_{cd} \otimes \gamma^{ab} \gamma^{cd}. \]

\[ \tag{A.11} \]

All the other terms vanish because of the special form of coefficients \( R_k(u) \) and owing to the finiteness of the Clifford algebra of gamma-matrices. Next we note that owing to \( \alpha \gamma_{ab} \gamma_{cd} = \epsilon_{ab} \gamma_{ce} \gamma_{fd} \gamma_{gf} = 1 \) (1.13), the gamma-matrix structure in (A.11) can be transformed as follows

\[ \gamma_{ab} \gamma_{cd} \otimes \gamma^{ef} \gamma^{gh} = \gamma^a \otimes \gamma^b \gamma_{ce} \gamma_{fg} \gamma_{de} \gamma_{gh} = 120 \gamma^a \otimes \gamma^b \gamma^c \gamma^d \gamma^e \gamma^f \gamma^g \gamma^h \gamma^i \gamma^j \gamma^k \gamma^l \gamma^m \gamma^n \gamma^o \gamma^p \gamma^q \gamma^r \gamma^s \gamma^t \gamma^u \gamma^v \gamma^w \gamma^x \gamma^y \gamma^z \]

Consequently (A.11) which is proportional to

\[ [M^{ab}, M^{cd}] \left[ \delta_{ac} \gamma_{bd} - \delta_{bc} \gamma_{ad} + \delta_{de} \gamma_{ab} \right] = 2[M^{ab}, M^{cd}] \gamma_{bc} = 0 \]

turns to zero. In the last expression, the parentheses (...) denote symmetrization. Therefore, the RLL-equation (A.7) is valid for arbitrary representation of generators \([M_{ab}]\) of the algebra \( so(6) \).

Let us rewrite the expression for the \( R \)-matrix (A.10) in a more transparent form. All gamma-matrix structures in (A.10) have a block-diagonal form in Weyl representation for
gamma-matrices. Therefore, it is reasonable to consider projections of (A.10) on corresponding irreducible subspaces. We introduce subspaces $V_+$ and $V_-$ obtained by Weyl projections: $V_+ = \frac{1 + \Gamma_3}{2} V$ and $V_- = \frac{1 - \Gamma_3}{2} V$. At first we note that relations

$$\mathbf{1} \otimes \left[ 1 - \frac{1}{6!} \gamma_{A_1} \otimes \gamma^{A_1} \right]_{V_+ \otimes V_-} = \left[ \frac{1}{2} \gamma_{A_1} \otimes \gamma^{A_1} - \frac{1}{4!} \gamma_{A_2} \otimes \gamma^{A_2} \right]_{V_+ \otimes V_-} = 0$$

lead to $R(u)|_{V_+ \otimes V_-} = R(u)|_{V_- \otimes V_+} = 0$. Further a pair of relations

$$\mathbf{1} \otimes \left[ 1 + \frac{1}{6!} \gamma_{A_1} \otimes \gamma^{A_1} \right]_{V_+ \otimes V_-} = \left[ \frac{1}{2} \gamma_{A_1} \otimes \gamma^{A_1} + \frac{1}{4!} \gamma_{A_2} \otimes \gamma^{A_2} \right]_{V_+ \otimes V_-} = 0$$

leads to the Yang–R matrix

$$R(u)|_{V_+ \otimes V_-} = [2 R_0(u) \mathbf{1} \otimes \mathbf{1} + R_2(u) \gamma_{ab} \otimes \gamma^{ab}]_{V_+ \otimes V_-} = \mathbf{1} \otimes \mathbf{1} + u P$$

where $P$ is a permutation operator and we take into account $-\frac{1}{8} \gamma_{ab} \otimes \gamma^{ab}|_{V_+ \otimes V_-} = P - \frac{1}{4} \mathbf{1} \otimes \mathbf{1}$. Analogously, one concludes that $R(u)|_{V_- \otimes V_+} = 1 \otimes 1 + u P$.

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