On convergency properties of meromorphic functions and mappings

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0. Introduction.

This note has two purposes. First is to discuss the possible notions of convergency of meromorphic functions and more generally meromorphic mappings with values in general complex spaces. Concerning the second, it was observed already by Cartan and Thullen in [C-T] that the study of the domains of existence of holomorphic functions leads to the results on the sets of their convergency (Konvergenzbereiche). In this paper this connection (for the case of meromorphic mappings) works in both directions.

We also give an application to the study of Fatou sets of meromorphic self-maps of compact complex surfaces.

When one runs quickly on literature around questions of convergency of meromorphic functions and more recently of mappings (usually with values in $\mathbb{CP}^n$) one founds a number of unequivalent definitions of the notion of convergency (even in the case of functions). The reason is that different authors study a different questions and adapt their definitions to their problems. Let us give a simple example.

When one wishes to study the development of, say elementary functions, into the entire series one founds the following definition to be convenient, see for example [R-1] p.255.

Definition 1. The series $\sum_{n=1}^{\infty} \lambda_n$ converge compactly on $\Omega$ if for every compact $K \subset\subset \Omega$ there is a number $m = m(K)$ such that $\lambda_n$ has no poles on $K$ for $n \geq m$ and $\sum_{n=0}^{m-1} \lambda_n |_K$ converge uniformly on $K$.

For the sequence $\{f_n := \Sigma_{k=1}^{\infty} \lambda_n\}$ of partial sums this notion of convergency means the following. For any compact $K \subset\subset \Omega$ represent each $f_n$ as a some of its principal part on $K$ and holomorphic function, i.e. $f_n = P_n + h_n$, where $P_n(z) = \Sigma_{j=1}^{l_n} \frac{c_{n,j}}{(z-a_{n,j})^{n_j}}$. Here $a_{n,j}$ run over the poles of $f_n$ contained in $K$, and the constants $c_j$ are uniquely determinated by $f_n$.

Definition 1’. One says that $\{f_n\}$ compactly converges on $\Omega$ if for any compact $K \subset\subset \Omega$ the principal parts for $K$ of $f_n$ stabylise for $n$ sufficiently big, and $\{h_n\}$ converge uniformly on $K$.

From the point of view of this definition the sequence $\{\frac{1}{z-1/n}\}$ doesn’t converge in any neighborhood of zero. However it should converge just by common sence. So the following definition looks also natural.

Definition 2. A sequence $\{f_n\}$ of meromorphic functions in $\Omega$ converge uniformly on compacts in $\Omega$ if for any point $z_0 \in \Omega$ there is a closed disk $\Delta(z_0, \varepsilon)$ and a natural $m = \ldots$
m(z₀, ε) such that either all \( f_n, n \geq m \) are holomorphic in \( \bar{\Delta}(z₀, ε) \) and uniformly converge there, or all \( 1/f_n, n \geq m \) are holomorphic on \( \bar{\Delta}(z₀, ε) \) and uniformly converge there.

The second definition means, in other words, that \( f_n \) converges uniformly on compacts in \( \Omega \) as a holomorphic mappings from \( \Omega \) to the Riemann sphere \( \mathbb{C}P^1 \).

In fact meromorphic functions on plain domains are just a holomorphic mappings into \( \mathbb{C}P^1 \), and at least from the geometrical point of view they are not really meromorphic objects.

If \( \Omega \geq 2 \) the meromorphic functions could have the points of indeterminancy. No specific value can be prescribed to a meromorphic function at such point, and thus from analytic point of view indeterminancies should be excluded from the domains of convergency, see [FS-1], (where this point of view comes naturally from dynamical study of holomorphic selfmaps of \( \mathbb{C}P^2 \)). However, if one looks on meromorphic function (or mapping) as on the analytic set - its graph, one is forced to study the indeterminancies as the most interesting points, carrying an essential information about the behavior of the converging sequence.

Consider for example a Cremona transformation of \( \mathbb{C}P^2 \): \( f : [z₀ : z₁ : z₂] \to [\frac{1}{z₀} : \frac{1}{z₁} : \frac{1}{z₂}] \). Then the sequence of its iterates \( \{f^n\} \) consists of \( f \) and identity. So the maximal open set where the family \( \{f^n\} \) is relatively compact (i.e. Fatou set of \( f \)) should be the whole \( \mathbb{C}P^2 \). However if we exclude the indeterminancies (and a fortiori their preimages) the Fatou set will be \( \mathbb{C}P^2 \) minus three lines.

This note consists from two parts. In Part I we give several possible definitions of convergency of meromorphic mappings. We discuss them giving pro and contra in each case and give some statements without proofs (not to interrupt the exposition). In Part II we give the main results, such as Rouche principle and apriori estimate of the volume, and prove the statements from Part I.

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Part I. Possible notions of convergency of meromorphic mappings.

Let \( \Omega \) and \( X \) be complex spaces. All complex spaces, which we consider in this paper are supposed to be reduced and normal.

We say that a sequence \( f \) holomorphic mappings from a complex space \( \Omega \) to a complex space \( X \) converge uniformly on compacts in \( \Omega \) to a holomorphic mapping \( f : \Omega \to X \) if for any compact \( K \subset \subset \Omega \) there is a compact \( P \subset \subset X \) such that \( f(K), f_n(K) \subset P \) for all \( n \)
and \( \lim_{n \to \infty} \sup_{x \in K} d(f_n(x), f(x)) = 0 \). Here \( X \) is equipped with some Hermitian metric. Of course this notion of convergency doesn’t depend on the choice of this metric.

1.1. Strong convergency.

Recall that a meromorphic mapping from \( \Omega \) to \( X \) is defined by a holomorphic map \( f : \Omega \setminus F \to X \) where \( F \) is analytic subset of \( \Omega \) of codimension at least two, such that the closure \( \overline{\Gamma_f} \) of its graph is an analytic subset of the product \( \Omega \times X \). This subset, which from now on will be noted as \( \Gamma_f \) (withought bar) is called the graph of the meromorphomic map (again denoted as \( f \)) and clearly possesess the following properties:

(i) \( \Gamma_f \) is an irreducible analytic subset of \( \Omega \times X \).

(ii) The restriction \( \pi |_{\Gamma_f} \) extends holomorphically on the whole \( C \).

This notion of meromorphicity is due to Remmert, see [Re-2], and is based on the (important for us) observation that meromorphic functions on \( \Omega \) are precisely the meromorphic mappings, i.e. mappings into \( \mathbb{P}^1 \) that \( \Gamma_f \) is biholomorphic to \( \mathbb{C}^n \). This following couple of examples.

Example 1. (Meromorphic functions) A meromorphic function \( f \) on the complex space \( \Omega \) is given by the open covering \( \{ \Omega_j \} \) and a pairs \( (h_j, g_j) \in \mathcal{O}(\Omega_j), g \neq 0 \), such that on \( \Omega_i \cap \Omega_j \) one has \( h_j g_j = h_j g_i \). One can allways suppose that \( h_j \) and \( g_j \) have relatively prime germs at all their common zeroes. Let \( [z_0 : z_1] \) denote the homogeneous coordinates on \( \mathbb{P}^1 \). Observe now that the analytic set \( \Gamma_f := \{ (x, [z_0 : z_1]) \in \Omega_j \times \mathbb{P}^1 : h_j(x)z_0 - g_j(x)z_1 \} \) is correctly defined and irreducible. This will be the graph of the meromorphic map from \( \Omega \) to \( \mathbb{P}^1 \).

Take for example \( \Omega = \mathbb{C}^2 \) with coordinates \( x = (z_0, z_1) \) and \( f(x) = \frac{z_0}{z_1} \). This \( f \) is holomorphic on \( \mathbb{C}^2 \setminus \{0\} \), but \( f(0) = \mathbb{P}^1 \).

Example 2. (Modification) If \( \Gamma_f = \{ (z_1, z_2; [w_0 : w_1]) \in \mathbb{C}^2 \times \mathbb{C}^1 : z_0 w_1 = z_1 w_0 \} \) denotes the graph of the mapping \( f(z) = \frac{z_1}{z_0} \), then it is easy to check that \( \Gamma_f \) is a manifold and that \( \Gamma_f \setminus \{ 0 \} \) is biholomorphic to \( \mathbb{C}^2 \setminus \{ 0 \} \) under the projection onto \( \mathbb{C}^2 \). One notes this \( \Gamma_f \) usually as \( \hat{\mathbb{C}}_0^2 \) - blown-up \( \mathbb{C}^2 \) at origin.

Example 3. (Nonextendability) Meromorphic mappings, say from \( \mathbb{C}^2 \setminus \{0\} \) with values in general complex manifold are not allways extendable to zero (Hartogs theorem in the case of holomorphic functions, i.e. mappings into \( \mathbb{C} \), and E. Levi theorem in the case of meromorphic functions, i.e. mappings into \( \mathbb{C}^4 \)). Take, for example a Hopf surface \( X = \mathbb{C}^2 \setminus \{0\}/(z \sim 2z) \). Then the natural projection \( \pi : \mathbb{C}^2 \setminus \{0\} \to X \) cannot be extended meromorphically to zero because \( \lim_{z \to 0} \pi(z) = X \).
Let us introduce our first notion of convergency. Let \( \{f_n\} \) be some sequence of meromorphic maps from \( \Omega \) into \( X \).

**Definition 1.1.1.** We shall say that \( \{f_n\} \) strongly converges (s-converges) on compacts in \( \Omega \) to a meromorphic map \( f: \Omega \to X \) if for any compact \( K \subset \Omega \)

\[
\mathcal{H} \lim_{n \to \infty} \Gamma f_n \cap (K \times X) = \Gamma f \cap (K \times X)
\]

Here by \( \mathcal{H} \)–lim we denote the limit in the Hausdorff metric, supposing that both \( \Omega \) and \( X \) are equipped with some Hermitian metrics. Remark that this notion of convergency doesn’t depend on a choice of metrics on \( \Omega \) and \( X \).

This definition is well related with the usual notion of convergency of holomorphic mappings. Namely, in Part II we shall prove the following

**Theorem 1 (Rouche principle).** Let a sequence of meromorphic mappings \( \{f_n\} \) between normal complex spaces \( \Omega \) and \( X \) strongly converge on compacts in \( \Omega \) to a meromorphic map \( f: \Omega \to X \). Then:

(a) If \( f \) is holomorphic then for any relatively compact open subset \( D_1 \subset D \) all restrictions \( f_n \mid_{D_1} \) are holomorphic for \( n \) big enough, and \( f_n \to f \) on compacts in \( D \) in the usual sense.

(b) If \( \{f_n\} \) are holomorphic then \( f \) is also holomorphic and \( f_n \to f \) on compacts in \( D \) in the usual sense.

We shall write \( f_n \to f \) in \( D \) to denote that \( \{f_n\} \) strongly converges on compacts in \( D \) to a meromorphic map \( f \). The notion of strong convergency is well related with an extension properties of meromorphic mappings. In one direction we have the following:

**Proposition 1.1.1.** Let \( D \) be a domain in a normal Stein space and let \( \hat{D} \) be its envelope of holomorphy. Further let \( \{f_n\} \) be a sequence of meromorphic maps of \( \hat{D} \) into a complex space \( X \). Suppose that \( f_n \to f \) in \( D \) and that \( f \) meromorphically extends onto \( \hat{D} \). Then there exists an analytic subset \( A \) of \( \hat{D} \) of codimension at least two, such that \( f_n \to f \) on \( \hat{D} \setminus A \). Moreover \( A \subset I(f) \).

The proof is given in 2.2.

**Definition 1.1.2.** Recall, that the complex space \( X \) possesses a meromorphic (holomorphic) extension property in dimension \( n \) if for any domain \( D \subset \mathbb{C}^n \) any meromorphic (holomorphic) mapping \( f: D \to X \) extends meromorphically (holomorphically) onto the unit polydisk \( \Delta^n \).

Denote by

\[
H^n(r) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n: \|z\| < r, |z_n| < 1, r^2 < \|z\| < 1, 1-r < |z_n| < 1\}
\]  

(1.1.1)
a \( n \)-dimensional Hartogs figure. Here \( z' = (z_1,\ldots,z_{n-1}) \) and \( 0 < r < 1 \). Recall, see [Sh],[Iv-3], that complex space \( X \) possesses a meromorphic (holomorphic) extension property in dimension \( n \) iff any meromorphic (holomorphic) mapping \( f: H^n(r) \to X \) extends meromorphically (holomorphically) onto the unit polydisk \( \Delta^n \). \( \|\cdot\| \) stands here for the polydisk norm in \( \mathbb{C}^n \).

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If \( X \) possesses a hol.ext.prop. in some dimension \( n \geq 2 \) then \( X \) possesses this property in all dimensions, see [Sh]. We shall just say that the space \( X \) possesses the hol.ext.prop. However in [Iv-5] a 3-dimensional compact complex manifold was constructed, which possesses a mer.ext.prop. in dimension two but there exists a meromorphic map from punctured 3-ball into this manifold which doesn’t extend to origin.

The Proposition 1.1.1 implies immediately the following

**Corollary 1.1.2.** If the space \( X \) possesses the hol.ext.prop. then the maximal open subset \( D \subset \Omega \) where \( \{f_n\} \) converge is Levi-pseudoconvex. If \( X \) possesses a mer.ext.prop. in dimension \( n = \dim \Omega \) then the maximal open set \( D \subset \Omega \) where \( \{f_n\} \) s-converge is equal to a Levi-pseudoconvex set minus variety.

Vice versa, if a meromorphic mapping from the domain \( D \) in Stein space is a strong limit of meromorphic mappings, which are defined od the envelope \( \hat{D} \) of \( D \), then \( f \) itself extends onto \( \hat{D} \), provided the image space \( X \) posesses a pluriclosed Hermitian metric form, see Proposition 2.1 in Part II. This is always the case when \( X \) is a compact complex surface.

Already in the Corollary we see that the notion of s-convergency is not perfect, when one wishes to study the maximal sets where the sequence converge. The difference becomes even more striking when one looks for the maximal set where a given family is relatively compact. Let \( F = \{f_\lambda : \lambda \in \Lambda\} \) be an arbitrary family of meromorphic mappings of space \( \Omega \) into a space \( X \).

**Definition 1.1.3.** One says that \( F \) is (strongly) relatively compact on the open subset \( D \) of \( \Omega \) if for any sequence \( \{f_n\} \subset F \) there exists a subsequence \( \{f_{n_k}\} \) which (strongly) converges on compacts in \( D \) to some meromorphic map \( f : D \rightarrow X \). The maximal open subset \( N_s \) of \( \Omega \) such \( F \) is (strongly) relatively compact on \( N_s \) we shall call the set of (strong) convergency of the family \( F \).

In general nothing good can be said about sets of strong convergency of families of meromorphic (and also holomorphic) mappings into a non-Stein spaces. Namely one has the following

**Example 4.** Let \( X \) be \( \mathbb{CP}^3 \) blown up in one point. Then for every open subset \( D \) of \( C^2 \) one can find a sequence of holomorphic mappings of \( C^2 \) into \( X \) with \( D \) as its set of strong convergency.

To see this, let \((z_1, z_2, z_3)\) be coordinates of an affine part of \( \mathbb{CP}^3 \). We suppose that the blown-up point is zero in this coordinates. For \( a = (a_1, a_2) \) and \( n \in \mathbb{N} \) define a mapping \( f_{n,a} : \mathbb{C}^2 \rightarrow X \) as follows: \( f_{n,a}(z_1, z_2) = (z_1 - a_1, z_2 - a_2, 1/n) \). If one takes \( A \) to be the set of all points in \( \mathbb{C}^2 \setminus \hat{D} \) with rational ordinates, then \( F = \{f_{n,a} : n \in \mathbb{N}, a \in A\} \) will be the family with \( N_s = D \).

1.2. Weak convergency.

The Proposition 1.1.1 in fact shows to us how to modify the notion of convergency to obtain the better picture for the sets of convergency.

**Definition 1.2.1.** We shall say that the sequence of meromorphic mappings from \( \Omega \) to \( X \) weakly converge (w-converge) in open subset \( D \) to a meromorphic map \( f : D \rightarrow X \) if there exists an analytic subset \( A \) of \( D \) of codimention at least two such that \( f_n \rightarrow f \) in \( D \setminus A \).
We denote this fact as $f_n \rightharpoonup f$. Changing the word strong to weak in the Definition 1.1.3 we obtain the notion of the set of weak convergence of the family of meromorphic mappings $F$. We shall denote it by $N_w$.

The set of weak convergence $N_w$ of the family $F$ of meromorphic mappings from $\Omega$ into $X$ is now the maximal open subset of $\Omega$ such that $F$ is weakly relatively compact on $N_w$.

The following corollary from Proposition 1.1.1, shows the advantage of the second definition. Recall that open subset $D$ of complex space $\Omega$ is called Levi-pseudoconvex if for any point $p \in \partial D$ one can found an open neighborhood $U$ of $p$ in $\Omega$ such that $U \cap D$ is Stein. We say that a complex space possesses a meromorphic extension property if for any domain $D$ in Stein space any meromorphic mapping $f : D \rightarrow X$ extends to a meromorphic map $\hat{f} : \hat{D} \rightarrow X$ of the envelope of holomorphy $\hat{D}$ of the domain $D$ into $X$. For example any compact Kähler manifold possesses a meromorphic extension property, see [Iv-3].

Corollary 1.2.1. (a) Set of weak convergence always exist.

(b) If space $X$ possesses a mer. ext. prop. in dimension $n = \dim \Omega$, then the set of weak convergence is Levi-pseudoconvex for any family of meromorphic maps of $\Omega$ into $X$.

In fact the situation is essentially better then it is described in this statement. Namely, from our characterisation of obstructions for the extendibility of meromorphic mappings from domains in $\mathbb{C}^n$ into the spaces with pluriclosed metrics, we shall derive that if a meromorphic map $f : D \rightarrow X$ is a limit on $D$ (weak or strong, doesn’t matter) of meromorphic maps $f_n : D \rightarrow X$, then $f$ has extra extension properties. In fact $f$ extends onto the envelope $\hat{D}$, and thus $f_n \rightharpoonup f$ on $\hat{D}$! See 2.3.

Remarks 1. Let us give one more reason, why the second definition of convergence for meromorphic mappings is natural. Take $\mathbb{CP}^N$ as $X$ now. Then one can show that any meromorphic map $f : \Omega \rightarrow \mathbb{CP}^N$ can be locally presented as $f = [f_0 : ... : f_N]$ in homogeneous coordinates, where $f_j$ are holomorphic functions. One can show that the following is true

Proposition 1.2.1. $f_n \rightharpoonup f$ iff there can be find such local presentations $f_n = [f_n^1 : ... : f_n^N]$ that $f_n^j \rightharpoonup f^j$ as holomorphic functions for all $j = 1,...,N$.

Such type of convergence of meromorphic mapping with values in projective manifolds was considered by Fujimoto, see [Fj].

2. Observe that for the family $F$ constructed in Example 4 $N_w = \mathbb{C}^2$.

3. However, we should remark that notion of weak convergence is not as well related with convergence of holomorphic mappings as strong does. In particular the Rouche principle is not longer valid in this case. To see this consider in Example 4 the sequence $f_n(z_1, z_2) = (z_1, z_2, \frac{1}{n})$. $\{f_n\}$ are holomorphic, and they weakly converge to mapping $f$ from $\mathbb{C}^2$ to $X$, which has indeterminacy at zero.

1.3. $\Gamma$-convergency.

Fix some Hermitian metric forms $w_X$ and $w_\Omega$ on $X$ and $\Omega$ respectively. By $p_1 : \Omega \times X \rightarrow \Omega$ and $p_2 : \Omega \times X \rightarrow X$ we denote the projections onto the first and second factors. On the product $\Omega \times X$ we consider the metric form $w = p_1^* w_\Omega + p_2^* w_X$. It will be convenient sometimes for us to consider instead of mappings $f : \Omega \rightarrow X$ their graphs $\Gamma_f$. 


By $\hat{f} = (z, f(z))$ we shall denote the mapping into the graph $\Gamma_f \subset \Omega \times X$. The volume of the graph $\Gamma_f$ of the mapping $f$ is given by

$$\text{vol}(\Gamma_f) = \int_{\Gamma_f} w^q = \int_\Omega (f^*w_X + w_\Omega)^q$$  \hspace{1cm} (1.3.1)

Here by $f^*w_X$ we denote the preimage of $w_X$ under $f$, i.e. $\phi^*w_X = (p_1)_*p_2^*w_X$.

Recall that the Hausdorff distance between two subsets $A$ and $B$ of the metric space $(Y, \rho)$ is a number $\rho(A, B) = \inf\{\varepsilon : A^\varepsilon \supset B, B^\varepsilon \supset A\}$. Here by $A^\varepsilon$ we denote the $\varepsilon$-neighborhood of the set $A$, i.e. $A^\varepsilon = \{y \in Y : \rho(y, A) < \varepsilon\}$.

Note that if the family $\{F \subset \text{Mer}(\Omega, X)\}$ is $s$-normal on $\Omega$ then currents $\{(w_\Omega + f_\lambda^*w_X)^j : f_\lambda \in F\}$ have uniformly bounded masses on compacts in $\Omega$ for $j = 1, \ldots, q = \dim X$.

This means, in other words that the volumes of the graphs $\Gamma_f$ are uniformly bounded over the compacts in $\Omega$. So, one naturally has one more notion of convergency of the sequence $\{f_n\}$ of meromorphic mappings of the complex space $\Omega$ into the complex space $X$.

**Definition 1.3.1.** We shall say that $\{f_n\}$ $\Gamma$- converge on the compacts in $\Omega$ if for every relatively compact open $D_\varepsilon \subset \Omega$ the sequence graphs $\Gamma_{f_n} \cap (D \times X)$ converge in the Hausdorff metric on $D \times X$.

One has the following  

**Proposition 1.3.1.** Let $\{f_n\}$ be a sequence of meromorphic mappings into a complex space $X$. Suppose that there exists a compact $K \subset X$ and a constant $C < \infty$ such that:

1) the sequence $\{\Gamma_{f_n}\}$ converges in the Hausdorff metric to the analytic subset $\Gamma$ of $\Omega \times X$ of pure dimension $q$;

2) $\Gamma = \Gamma_\phi \cup \hat{\Gamma}$, where $\Gamma_\phi$ is the graph of some meromorphic mapping $f : \Omega \rightarrow X$, and $\hat{\Gamma}$ is a pure $q$-dimensional analytic subset of $\Omega \times X$, mapped by the projection $p_1$ onto $A$;

3) $f_{n_j} \rightarrow f$ on compacts in $\Omega \setminus A$;

4) one has

$$\lim_{j \rightarrow \infty} \text{vol}(\Gamma_{f_{n_j}}) \geq \text{vol}(\Gamma_f) + \text{vol}(\hat{\Gamma})$$  \hspace{1cm} (1.3.2)

5) For every $1 \leq p \leq \dim X - 1$ there exists a positive constant $\nu_p = \nu_p(K, h)$ such that the volume of every pure $p$-dimensional compact analytic subset of $X$ which is contained in $K$ is not less then $\nu_p$.

6) Put $\hat{\Gamma} = \bigcup_{p=0}^{q-1} \Gamma_p$, where $\Gamma_p$ is a union of all irreducible components of $\hat{\Gamma}$ such that $\dim[p_1(\Gamma_p)] = p$. Then

$$\text{vol}_{2q}(\hat{\Gamma}) \geq \sum_{p=0}^{q-1} \text{vol}_{2p}(A_p) \cdot \nu_{q-p}$$  \hspace{1cm} (1.3.3)
where \( A_p = p_1(\Gamma_p) \).

The proof uses the Harvey-Shiffman generalisation of Bishop’s convergency theorem for analytic sets and can be found in [Iv-5].

The notion of \( \Gamma \)-convergency seems to be to week to reflect the fact the it is the mappings are converging and not just an analytic sets. However it might be interesting from the measure theoretic point of vew. Indeed the set \( N_\Gamma \) of \( \Gamma \)-convergency of the family \( \mathcal{F} \) is exactly the maximal open set where the currents \( \{(w_\Omega + f^*w_X)^n : f \in \mathcal{F}\} \) have uniformly bounded masses. We shall discuss this in 2.4.

1.4. Other types of convergency.

One can give also other definitions of convergency of sequences of meromorphic functions and mappings. One is in the spirit of Definition 1 from the Introduction. Let a sequence of meromorphic maps \( f_n : \Omega \to X \) is given.

**Definition 1.4.1.** One says that \( \{f_n\} \) converges on compacts in \( \Omega \) if for any relatively compact open \( D \subset \Omega \) there are

1. a proper modification \( \pi_D : \hat{D} \to D \); 
2. \( N \) depending on \( D \);

such that the pullbacks \( f_n \circ \pi_D : \hat{D} \to X \) are holomorphic on \( \hat{D} \) for \( n \geq N \) and converge there as the sequence of holomorphic maps.

Of course this is a very restrictive notion, which means that indeterminancies of the sequence stabylise. So, it is a direct analog of the first definition from the Introduction, and is reasonable in the context of studying of developping of entire (for example) functions into the series of their principal parts.

Such convergency implies the strong one, but not vice versa. Really, consider a sequence of meromorphic functions \( f_n(z_1,z_2) = \frac{z_1 - \frac{1}{z_2}}{z_2} \). This sequence converge in the strong sence but indeterminancies do not stabylise.

In the dynamical study of holomorphic and meromorphic selfmaps the sequence \( \{f^n\} \) of itertates of mapping \( f : X \to X \) is said to be normal on \( D \subset X \) if it is equicontinuous there, [FS-1]. In particular this exludes from \( D \) the indeterminancies of \( f \) and their preimages. See example of Cremona transformation from the Introduciton in this regard.

**Part II. Proofs of the statements.**

2.1. Rouche principle.

**Proof.**

Let \( \{f_n : \Omega \to X\} \) our sequence, which strongly converge on compacts in \( \Omega \) to a meromorphic map \( f : \Omega \to X \).

(a) Suppose that \( f \) is holomorphic. Take a point \( a \in \Omega \) and let \( V \ni a \) and \( W \ni f(a) \) are Stein neighborhoods such that \( \overline{f(V)} \subset W \). Then \( \Gamma_{f|V} \cap \partial(V \times W) \subset \partial V \times W \). So the natural projection \( p_1 |_{\Gamma_{f(V)} \cap (V \times W)} : \Gamma_f \cap (V \times W) \to V \) is proper and in fact bijective.

From the strong convergency of our sequence we have for \( n >> 0 \) \( \Gamma_{f_n|V} \cap \partial(V \times W) \subset \partial V \times W \) and thus \( p_1 |_{\Gamma_{f_n(V)} \cap (V \times W)} : \Gamma_{f_n} \cap (V \times W) \to V \) is proper and surjective. Now \( V \times W \) is Stein, so doesn’t contain a compact subvarieties of positive dimension. So \( \Gamma_{f_n} \cap (V \times W) \)
is a graph of (may be multivalued) holomorphic correspondence from $V$ to $W$. But over a dense set $V \setminus I(f_n)$ this correspondence is one-valued. Thus $\Gamma_{f_n} \cap (V \times W)$ is a graph of a holomorphic mapping.

(b) Suppose now that $f$ is not holomorphic. We must prove that $f_n$ are also not holomorphic for $n$ big enough. Take a point $a$ on $I(f)$ such that $k := \dim f[a] \geq 1$. Consider two cases.

Case 1. $k < \dim X$.

Put $p = \dim \Omega$. Fix two distinct points $b_1, b_2 \in f[a]$. Take a nonintersecting neighborhoods $V_1 \ni b_1$ and $V_2 \ni b_2$ of those points in $X$ and proper holomorphic embeddings $\phi_j : V_j \to \Delta^k \times \Delta^{m_j}$ which extend to the neighborhood of $V_j$ and such that $\phi_j(V_j \cap f[a]) \cap (\Delta^k \times \partial \Delta^{m_j}) = \emptyset$. Take a neighborhood $W \ni a$ in $\Omega$ and consider an embeddings $\Phi_j : W \times V_j \to W \times (\Delta^k \times \Delta^{m_j})$ given by $\Phi_j(w, v) = (w, \phi_j(v))$. Consider a projections $\pi_j : W \times (\Delta^k \times \Delta^{m_j}) \to W \times \Delta^k$. If $W$ was choosen sufficiently small then still $\Phi_j(\Gamma_j \cap (W \times V_j)) \cap (W \times \Delta \partial \Delta^{m_j}) = \emptyset$. Thus the restrictions $\pi_j |_{\Phi_j(\Gamma_j \cap (W \times V_j))} : \Phi_j(\Gamma_j \cap (W \times V_j)) \to W \times \Delta^k$ will remain proper. Denote by $A_j = \pi_j(\Phi_j(\Gamma_j \cap (W \times V_j)))$ the $p$-dimensional analytic subsets of $W \times \Delta^k$. Remark that $A_1 \cap A_2 \ni a$ and that $\dim A_1 + \dim A_2 = 2p > p + k = \dim (W \times \Delta^k)$, while $k = \dim f[a] \leq p - 1$. Choosing the coordinate morphisms $\phi_j$ generically, we can suppose that $A_1 \neq A_2$.

For $n$ big enough we have also $\Phi_j(\Gamma_{f_n} \cap (W \times V_j)) \cap (W \times \Delta^k) \times \partial \Delta^{m_j} = \emptyset$. Thus $A^n_j := \pi_j(\Phi_j(\Gamma_{f_n} \cap (W \times V_j)))$ will be an analytic subsets of dimension $p$ in $W \times \Delta^k$. We have by the definition of strong convergency, that $\Gamma_{f_n} \cap (W \times V_j) \to \Gamma_f \cap (W \times V_j)$ in Haussdorff metric. This impyes that $A^n_j \to A_j$ in Haussdorff metric in $W \times \Delta^k$.

The usual Rouche theorem for holomorphic functions imply now that $A^n_1 \cap A^n_2 \neq \emptyset$ for $n$ big enough. This gives us a point $a_n \in W$ for which $f_n[a_n]$ consists of more that one point. I.e. $a_n$ is a point of indeterminacy of $f_n$.

Case 2. $k = \dim X$, i.e. $f[a] = X$.

In this case take two germs of hypersurfaces $F_j \ni b_j$ in $X$. Put $A_j := \pi((F_j \times W) \cap \Gamma_f)$, where $\pi : \Omega \times X \to \Omega$ is a natural projection. For $b_1, b_2$ general enough $A_1$ and $A_2$ will be different. Now repeat the reasonings of Case 1 to get the same conclusion.

q.e.d.

2.2. Propagation of convergency by extension.

In this paragraph we shall prove the Proposition 1.1.1 and Corollary 1.2.1 from Part I, which establish the main properties of the sets of convergency of the sequences of meromorphic mappings.

**Proof of Proposition 1.1.1.** Let $D'$ be a maximal open subset of $\hat{D}$ where $f$ is holomorphic and $\{f_n\}$ converge to $f$ on compacts as a sequence of holomorphic maps. Suppose that there exists a point $p \in \partial D' \setminus I(f)$ such that $D'$ is not pseudoconvex at $p$. Take a Stein neighborhood $V \ni p$ such that the envelope of holomorphy of $(D' \cup V)$ contains $p$ and $V \cap I(f) = \emptyset$.

In the product $\hat{D} \times X$ take a Stein neighborhood $W$ of $\Gamma_{f|\hat{v}}$. Now we have a sequence of holomorphic maps $f_n : V \to W$ converging to $f$, which holomorphically extends onto $\hat{V}$.
$W$ is Stein, so from the maximum principle it follows that $f_n$ converge to $f$ on $\hat{V}$. This contradicts the maximality of $D'$.

This implies that $D' \supset D \setminus I(f)$.

q.e.d.

Proof of Corollary 1.2.1. (a) We need to prove that if for open $N_1, N_2 \subset \Omega$ the family $F$ is weakly relatively compact on $N_i, i = 1, 2$, then $F$ is weakly relatively compact on $N_1 \cup N_2$. For this take some sequence $\{f_n\} \subset F$. We find an analytic subset $A_1 \subset N_1$ of codimension at least two and substruct from from $\{f_n\}$ a subsequence (still denoted as $\{f_n\}$), which converge on compacts in $N_1 \setminus A_1$ to some meromorphic map $f_1 : N_1 \rightarrow X$. From this subsequence we can substruct a subsequence once more, which will converge on compacts in $N_2 \setminus A_2$ for some analytic of codimension at least two set $A_2$ in $N_2$ to a meromorphic map $f_2 : N_2 \rightarrow X$.

$f_1$ and $f_2$ will clearly coinside on $(N_1 \cap N_2)$ and our subsequence will strongly converge to $f := f_1 = f_2$ on $(N_1 \cup N_2) \setminus (A_1 \cup A_2)$. From Proposition 1.1.1 we have that $I(f) \subset (A_1 \cup A_2)$. If there would exists some $a \in (A_1 \cup A_2) \setminus I(f)$, then again by Proposition 1.1.1 our subsequence would converge to $f$ in the neighborhood of $a$. Thus $f_{n_k} \rightarrow f$ on compacts in $(N_1 \cup N_2) \setminus I(f)$.

(b) This is again immediate corollary of Proposition 1.1.1.

q.e.d.

2.3. Propagation of extension by convergency.

In fact the picture from the previous paragrah can be in some cases reversed. Namely, from our characterisation of obstructions for the extendib ility of meromorphic mappings from domains in Stein manifolds into the spaces with pluriclosed metrics, we shall here derive that if a meromorphic map $f : D \rightarrow X$ is a limit on $D$ (weak or strong, doesn’t matter) of meromorphic maps $f_n : \hat{D} \rightarrow X$, then $f$ has extra extension properties. In fact $f$ extends onto the envelope $\hat{D}$, and thus $f_n \rightarrow f$ on $\hat{D}$.

We call a Hermitian metric form $w$ on complex space $X$ pluriclosed if $dd^c w = 0$. One can show that the following duality holds:

either compact complex manifold $X$ admits a pluriclosed metric form, or $X$ carries
a bidimension $(2,2)$ current $T$ such that $dd^c T$ is also positive.

We shall restrict ourselves with the following class of spaces:

Definition 2.3.1. Call the complex space $X$ disk-convex if for any compact $K \subset X$ there is a compact $\hat{K} \subset X$ such that for every analytic disk $\phi : \Delta \rightarrow X$ with $\phi(\partial \Delta) \subset K$ one has $\phi(\Delta) \subset \hat{K}$.

In [Iv-3] the following theorem is proved:

Theorem 2.3.1. Let $f : D \rightarrow X$ be a meromorphic mapping from a domain in Stein manifold $\Omega$ into a disk-convex complex space $X$ which possesses a pluriclosed Hermitian metric form $w$. Then $f$ extends onto $\hat{D} \setminus A$ where $A$ is an analytic subset of $D$ of pure codimension two (may be empty). Moreover, if $A$ is nonempty then for any $S^3 \subset \hat{D} \setminus A$, such that $S^3$ is not homologous to zero in $\hat{D} \setminus A$ the image $f(S^3)$ is also not homologous to zero in $X$. 

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Recall, see [Iv-2], that a spherical shell of dimension two in complex space $X$ is an image $\Sigma$ of the standard sphere $S^3 \subset \mathbb{C}^2$ under the holomorphic map of some neighborhood of $S^3$ into $X$, such that $\Sigma$ is not homologous to zero in $X$. This notion is close to the notion of the global spherical shell, introduced by Kato, see [Ka]. Thus we obtain the following

**Corollary 2.3.2.** Let $X$ be a disk-convex complex space which possess a pluriclosed Hermitian metric form. Then the following is equivalent:

(a) $X$ possesses a meromorphic extension property in all dimensions.

(b) $X$ contains no spherical shells.

We turn now to the sets of convergency of families of meromorphic mappings.

**Corollary 2.3.3.** If $X$ admits a pluriclosed Hermitian metric, then for any family of meromorphic mappings from $\Omega$ to $X$ the set of weak convergency is Levi-pseudoconvex.

*Proof.* Let $D = N_w$ be the set of weak convergency of our family $F$. Suppose that $p \in \partial D$ is not pseudoconvex boundary point. Let $U \ni p$ be a neighborhood such that for some component of $U \cap D$, say $V$, the projection of the envelope of holomorphy of $V$ into $U$ contains a neighborhood $W$ of $p$.

Take some sequence $\{f_n\} \subset F$. Then some subsequence $\{f_k\}$ weakly converge on $D$ to a meromorphic map $f : D \rightarrow X$. This means by definition that there is an analytic set $A$ of codimension at least two in $D$ such that $f_{n_k} \rightarrow f$ on compacts in $D \setminus A$. By Theorem 2.3.1 $f$ extends meromorphically to $W$ minus analytic set of codimension two. By Proposition 1.1.1 $f_{n_k} \rightarrow f$ on $W \setminus (A \cup B)$.

All that left to prove is that $B \subset A$, i.e. $f$ is meromorphic on the whole $W$. Would $B \not\subset A$ then there by Theorem 2.3.1 would be a three-sphere $S^3 \subset W \setminus B$ such that $f(S^3) \not\sim 0$ in $X$. But $f_{n_k} \rightarrow f$ on $S^3$ and $f_{n_k}(S^3) \sim 0$ in $X$ because $f_{n_k}$ are meromorphic on the whole $W$. This is a contradiction.

q.e.d.

### 2.4. Apriori estimate of the volume.

Our aim in this paragraph is the following

**Theorem 2.** Let $\{f_\lambda\}$ be a family of meromorphic mappings from $\Delta^2$ to a disk-convex Kähler space $(X, w)$ such that $f_\lambda$ are holomorphic on $A^2(1/2, 1) := \Delta^2(1) \setminus \Delta^2(1/2)$ and equicontinuous there. Then $\{\text{vol}(\Gamma_{f_\lambda})\}$ are uniformly bounded.

*Proof.* Consider a family of currents $T_\lambda := f^* w$ on $\Delta^2$. Write $T_\lambda = i/2 t_\lambda^{\alpha\beta} dz_\alpha \wedge dz_\beta$, where $t_\lambda^{\alpha\beta}$ are distributions on $\Delta^2$, smooth on $\Delta^2 \setminus S_\lambda$. Here $S_\lambda$ is a finite set for each $\lambda$.

Consider functions

$$
\mu_\lambda(z_1) = i/2 \int_{\Delta_{z_1}} t_\lambda^{2\bar{2}} dz_2 \bar{dz}_2.
$$

(2.4.1)

$\{\mu_\lambda\}$ are uniformly bounded on $A(1/2, 1)$. Moreover

$$
\frac{\partial}{\partial z_1} \mu_\lambda = i/2 \int_{\Delta_{z_1}} \frac{\partial}{\partial z_1} t_\lambda^{2\bar{2}} dz_2 \bar{dz}_2 = i/2 \int_{\Delta_{z_1}} \frac{\partial}{\partial z_2} t_\lambda^{1\bar{2}} dz_2 \bar{dz}_2 =
$$

$$
i/2 \int_{\partial \Delta_{z_1}} t_\lambda^{1\bar{2}} \bar{dz}_2.
$$

(2.4.2)
This gives us the boundedness of the differentials of \( \{ \mu_\lambda \} \) on \( \Delta \) and thus the boundedness in \( L^1(\Delta^2) \) of \( \{ t_{\lambda}^{22} \} \).

In the same way we get a boundedness in \( L^1(\Delta^2) \) of \( \{ t_{\lambda}^{11} \} \). From positivity of \( T_\lambda \) we get

\[
\int_{\Delta^2} |t_{\lambda}^{11}|^2 \leq \int_{\Delta^2} \sqrt{t_{\lambda}^{11}} \cdot \sqrt{t_{\lambda}^{22}} \leq \sqrt{\int_{\Delta^2} t_{\lambda}^{11} \cdot \int_{\Delta^2} t_{\lambda}^{22}}.
\]

(2.4.3)

This gives us the boundedness of \( \{ T_\lambda \} \) in \( L^1(\Delta^2) \).

To estimate \( \text{vol} \Gamma_f = \int_{\Delta^2} (f^*w + dd^c||z||^2)^2 \) we need to estimate also \( \int_{\Delta^2} (f^*w)^2 \).

Take a smooth function \( 1 \geq \eta \geq 0 \) in \( \Delta^2 \), \( \eta \mid_{\Delta^2(3/4)} \equiv 1 \), \( \eta \mid_{A^2(7/8,1)} \equiv 0 \) and consider the following potentials:

\[
U_\lambda(z) = -\int_{C^2} \eta(x) \frac{T_\lambda(x) \wedge dd^c||x||^2}{||x - z||^2} = (\eta(x) \sum_{j=1}^2 t_{\lambda}^{ij}(x)) \ast K(z)
\]

(2.4.4)

where \( K(z) := \frac{1}{||z||} \). \( U_\lambda \) are bounded in \( L^1(\Delta^2) \) because \( T_\lambda \) are so. From [Sk], p.376, we have that

\[
\frac{\partial^2 U_\lambda(z)}{\partial z_\alpha \partial \bar{z}_\beta} = \eta(z) \cdot t_{\lambda}^{\alpha \bar{\beta}}(z) + \sum_{j=1}^2 \frac{\partial K}{\partial z_\alpha} \ast (\frac{\partial \eta}{\partial x_\beta} t_{\lambda}^{j\bar{\beta}} - \frac{\partial \eta}{\partial x_j} \bar{t}_{\lambda}^{\alpha \bar{\beta}}) + \sum_{j=1}^2 \frac{\partial K}{\partial \bar{z}_j} \ast (\frac{\partial \eta}{\partial x_\alpha} \bar{t}_{\lambda}^{\bar{j} \beta} - \frac{\partial \eta}{\partial x_j} \bar{t}_{\lambda}^{\alpha \bar{\beta}}).
\]

(2.4.5)

Using the fact that \( \text{supp} \nabla \eta \subset A^2(3/4,7/8) \) and usual properties of convolutions, we see that the family \( \{ dd^c U_\lambda \} \) is uniformly \( C^\infty \)-bounded on \( A^2(1/2,3/4) \). So \( \{ U_\lambda \} \) are \( C^\infty \)-bounded on \( A^2(1/2,3/4) \). Denote by \( t_{\lambda \varepsilon}^{\alpha \bar{\beta}} \) the smoothing of \( t_{\lambda}^{\alpha \bar{\beta}} \) by convolution and by \( S_\lambda^\delta \) the \( \delta \)-neighborhood of \( S_\lambda \). We have that

\[
\int_{\Delta^{2(1/2)} \setminus S_\lambda^\delta} (f^*w)^2 = \lim_{\varepsilon \searrow 0} \int_{\Delta^{2(1/2)} \setminus S_\lambda^\delta} T_{\lambda \varepsilon} \wedge T_{\lambda \varepsilon} \leq \lim_{\varepsilon \searrow 0} \int_{\Delta^{2(1/2)}} T_{\lambda \varepsilon} \wedge T_{\lambda \varepsilon} =
\]

\[
= \lim_{\varepsilon \searrow 0} \int_{\Delta^{2(1/2)}} \varepsilon_{\alpha \beta \delta \gamma} t_{\lambda \varepsilon}^{\alpha \gamma} \cdot t_{\lambda \varepsilon}^{\delta \beta} dV = \lim_{\varepsilon \searrow 0} \int_{\Delta^{2(1/2)}} \varepsilon_{\alpha \beta \delta \gamma} \frac{\partial^2 U_{\lambda \varepsilon}}{\partial z_\alpha \partial \bar{z}_\gamma} - \frac{\partial K}{\partial z_\alpha} \ast (\frac{\partial \eta}{\partial x_\beta} t_{\lambda \varepsilon}^{j\bar{\beta}} - \frac{\partial \eta}{\partial x_j} \bar{t}_{\lambda \varepsilon}^{\alpha \bar{\beta}}) - \sum_{j=1}^2 \frac{\partial K}{\partial \bar{z}_j} \ast (\frac{\partial \eta}{\partial x_\alpha} \bar{t}_{\lambda \varepsilon}^{\bar{j} \beta} - \frac{\partial \eta}{\partial x_j} \bar{t}_{\lambda \varepsilon}^{\alpha \bar{\beta}})
\]

\[
= \frac{\partial^2 U_{\lambda \varepsilon}}{\partial z_\delta \partial \bar{z}_\beta} - \sum_{j=1}^2 \frac{\partial K}{\partial z_\delta} \ast (\frac{\partial \eta}{\partial x_\beta} t_{\lambda \varepsilon}^{j\bar{\beta}} - \frac{\partial \eta}{\partial x_j} \bar{t}_{\lambda \varepsilon}^{\alpha \bar{\beta}}) - \sum_{j=1}^2 \frac{\partial K}{\partial \bar{z}_j} \ast (\frac{\partial \eta}{\partial x_\alpha} \bar{t}_{\lambda \varepsilon}^{\bar{j} \beta} - \frac{\partial \eta}{\partial x_j} \bar{t}_{\lambda \varepsilon}^{\alpha \bar{\beta}})
\]

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where by $J_{\lambda \varepsilon}$ we denote the appropriate expression, which remains after integrating. This $J_{\alpha \varepsilon}$ is clearly $C^\infty$-bounded on $A^2(1/2,3/4)$ uniformly on $\lambda$ and $\varepsilon$.

q.e.d.

One has the following obvious

**Corollary 2.4.1.** Let $\mathcal{F}$ be a family of meromorphic mappings from $\Delta^2$ to a disk-convex Kähler space $X$, which is equicontinuous on the Hartogs domain $H^2(r) \subset \Delta^2$. Then for any $\rho < 1$ there is a constant $C = C_{\rho,\mathcal{F}}$ such that $\text{vol}(\Gamma_{f_\lambda}) \leq C$ on $\Delta^2_\rho$, for all $f_\lambda \in \mathcal{F}$.

Remark that this statement doesn’t follows from the Oka-type estimates for the volumes of analytic sets of masses of currents, compare [FS-2]. We shall need this estimate in the proof of Theorem 3 below.

In this regard we whant also to propose the following

**Conjecture.** Let $\mathcal{F}$ be a family of meromorphic mappings from $\Delta^n$ to a disk-convex Kähler space $X$, which is equicontinuous on the Hartogs domain $H^n(r) \subset \Delta^n$. Then for any $\rho < 1$ there is a constant $C = C_{\rho,\mathcal{F}}$ such that $\text{vol}(\Gamma_{f_\lambda}) \leq C$ on $\Delta^n_\rho$, for all $f_\lambda \in \mathcal{F}$.

Equicontinuity condition here, as well as in Corollary 2.4.1, means in particular, that $f_\lambda$ are holomorphic on $H^n(r)$. Example of Shiffman and Taylor, see [Si-2], shows that one should essentially use the fact that currents $T_\lambda$ are preimages of the Kähler form by the mappings $f_\lambda$!

**2.5. Fatou sets of meromorphic self-maps of compact complex surfaces.**

We shall consider here a special case, when our family $\mathcal{F}$ is the family $\{f^n\}$ of iterates of some meromorphic self-map $f : X \rightarrow X$ of an algebraic surface $X$, then strong Fatou set (i.e. set of convergency of the family of iterates) coincides with weak Fatou set unless a quite special case occurs. Namely we have the following

**Theorem 3.** Let $f$ be a meromorphic self-map of a compact Kähler surface $X$. Denote by $\Phi_s$ the (strong) Fatou set of $f$ and by $\Phi_w$ the weak Fatou set of $f$. Then:

(i) $\Phi_w$ is a Levi-pseudoconvex open subset of $X$;

(ii) If $\Phi_s$ is not equal to $\Phi_w$, then:

(a) $\Phi_w \supset X \setminus C$, where $C$ is a rational curve in $X$;

(b) any weakly converging subsequence $\{f^{n_k}\}$ converge strongly on $X \setminus (C \cup \{ \text{finite set}\})$, and its weak limit $f_\infty$ is a degenerate mapping of $X$ onto $C$.

First let us start with a simple Lemma. Let a meromorphic map $f : \Omega \rightarrow X$ of complex spaces is given. By an image of a point $a$ (or more generally a set $A$) one understands $f[A] := \{x \in X : \exists a \in A \text{ s.t. } (a,x) \in \Gamma_f\}$. If $A$ is an analytic subset, then by $f|_A (A)$ we understand the image of $A$ under the restriction of $f$ onto $A$. An analytic set $D(f) := p_1 (\{(z,x) \in \Omega \times X : \dim_{(z,x)} p_2^{-1}(x) \geq 1\})$ is called a locus of degeneration of $f$. $f : \Omega \rightarrow X$ is degenerate if $D(f) = \Omega$.

**Lemma 2.5.1.** Let $z \in \Phi_s$ (corr. in $\Phi_w$) and take some $l \geq 1$. Then $f^l | z \setminus f^l |_{D(f^l)} (D(f^l)) \subset \Phi - s$ (corr. $\Phi_w$).
Proof. Let \( v \in f^l[z] \backslash f^l[D(f^l)](D(f^l)) \), then there are neighborhoods \( U \ni z \) and \( V \ni v \) such that \( f^{-l} : V \to U \) is a multivalued holomorphic map. Take some sequence \( \{ f^{n_k} \} \subset \{ f^n \} \). From sequence \( \{ f^{n_k+1} \} \) by assumption one can subtract a converging (corr. weakly converging) subsequence \( \{ f^{n_{k_i}+1} \} \) on \( U \). So \( f^{n_{k_i}} = f^{n_{k_i}+1} \circ f^{-l} \) will converge (corr. weakly converge) on \( V \).

q.e.d.

Proof of Theorem 4. (i) This is consequence of Corollary 1.2.1 and the Hartogs-type extension theorem for meromorphic mappings into Kähler manifolds, see [Iv-2].

(ii) If \( \Phi_s \neq \Phi_w \) then there exists a point \( p \in \Phi_w \), a ball \( \mathbb{B} \) centered at \( p \), a subsequence of iterates \( \{ f^{n_k} \} \), which s-converge on \( \mathbb{B} \setminus \{ p \} \) to a meromorphic map \( f_{\infty} : \mathbb{B} \to X \), but not converging at any neighborhood of \( p \). In particular this means that \( p \in I(f_{\infty}) \) by Rouche principle. Taking \( \mathbb{B} \) small enough, we can suppose that \( p \) is the only fundamental point of \( f_{\infty} \) in \( \mathbb{B} \).

By Theorem 2 \( \text{vol}(\Gamma_{f_{\infty}}) \) are uniformly bounded on \( \mathbb{B} \). So \( \{ \Gamma_{f_{\infty}} \} \) are converging (after going to subsequence) in Hausdorff metric to \( \Gamma_{f_{\infty}} \cup \{ p \} \times X \). Put \( C = f_{\infty}[p] \). \( C \) is a finite union \( \bigcup_{i=1}^{N} C_i \) of rational curves. Take a point \( q \in X \setminus C \). Then for \( k \gg 1 \) \( q \in f^{n_k}(\mathbb{B} \setminus \{ p \}) \). If moreover \( q \notin f^{n_k}(D(f^{n_k})) \) then \( q \in \Phi_w \) by Lemma 2.5.1. But \( \bigcup_k f^{n_k}(D(f^{n_k})) \) is at most countable set of points and \( \Phi_w \) is Levi-pseudoconvex. So \( \Phi_w \supset X \setminus C \).

Take now a point \( x \in C \). Suppose that \( \Gamma_{f_{\infty}} \cap (X \times \{ x \}) \) has \( (p,x) \) as isolated point. Then we can find neighborhoods \( W \ni p \) and \( V \ni x \) with \( (\partial W \times \bar{V}) \cap \Gamma_{f_{\infty}} = \emptyset \). Thus \( \partial W \times \bar{V} \cap \Gamma_{f_{\infty}} = \emptyset \) for \( k \) big enough. So by \( f^{n_k}(W) \supset V \). Thus \( \Phi_w \supset C \).

Let us distinguish two cases.

Case 1. \( C \) has such a point \( x \).

By pseudoconvexity \( \Phi_w \) contains a component of \( C \), to which \( x \) belongs. By connectivity of \( C \) \( \Phi_w \supset C \) and thus \( \Phi_w = X \). In this case our sequence \( \{ f^{n_k} \} \) strongly converges on \( X \setminus \{ p_1,...,p_N \} \). From Theorem 2 we see that \( \text{vol}(\Gamma_{f_{\infty}}) \) are uniformly bounded but not less then \( N \cdot \text{vol}X \). This is possible only if \( f \) has degree one, \( N = 1 \) and \( f_{\infty} \) is degenerate mapping onto \( C \). \( C \) in this case should consist only from one component.

Case 2. For all points \( x \in C \) \( \text{dim}(p,x)\Gamma_{f_{\infty}} \cap X \times \{ x \} > 0 \).

Then \( f_{\infty} \) is a degenerate mapping of \( X \setminus C \) onto \( C \). By the same reasoning as in the previous case \( p \) is a single point in \( \Phi_w \setminus \Phi_s \).

q.e.d

The following example shows that the situation described in part (ii) of this theorem can really happen. Let \( X = \mathbb{CP}^2 \) and \( f : [z_0 : z_1 : z_2] \to [z_0 : 2z_1 : 2z_2] \). Then for this \( f \) we have the phenomena described above with \( p = [1 : 0 : 0] \) and \( C = \{ z_0 = 0 \} \).

Remark. 1. The statement (i) of this Theorem is valid for meromorphic self-maps of compact Kähler manifolds of any dimension for the same reason. It is also valid for all compact complex surfaces. This follows from Corollary 2.3.3 and the fact that every compact Kähler surface carries a pluriclosed metric form.

2. In the first Case, see the proof, \( f \) is a bimeromorphic automorphism of \( X \). Probably one should expect that if \( \Phi_s \neq \Phi_w \) then \( f \) is nessessarily a bimeromorphic automorphism. The dynamics of birational automorphisms of \( \mathbb{CP}^2 \) where recently studied in [D].
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