The little Grothendieck theorem and Khintchine inequalities for symmetric spaces of measurable operators

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Abstract

We prove the little Grothendieck theorem for any 2-convex noncommutative symmetric space. Let $M$ be a von Neumann algebra equipped with a normal faithful semifinite trace $\tau$, and let $E$ be an r.i. space on $(0, \infty)$. Let $E(M)$ be the associated symmetric space of measurable operators. Then to any bounded linear map $T$ from $E(M)$ into a Hilbert space $H$ corresponds a positive norm one functional $f \in E(2)^{\ast}(M)$ such that

$$\forall x \in E(M) \quad \|T(x)\|^2 \leq K^2 \|T\|^2 f(x^* x + xx^*),$$

where $E(2)$ denotes the 2-concavification of $E$ and $K$ is a universal constant. As a consequence we obtain the noncommutative Khintchine inequalities for $E(M)$ when $E$ is either 2-concave or 2-convex and $q$-concave for some $q < \infty$. We apply these results to the study of Schur multipliers from a 2-convex unitary ideal into a 2-concave one.

1 Introduction

Let $C(\Omega)$ denote the space of continuous functions on a compact topological space $\Omega$, equipped with the uniform norm. The classical little Grothendieck theorem asserts that for any bounded linear map $T$ from $C(\Omega)$ into a Hilbert space $\mathcal{H}$ there exists a probability measure $\mu$ on $\Omega$ such that

$$\forall x \in C(\Omega) \quad \|T(x)\|^2 \leq K^2 \|T\|^2 \int_{\Omega} |x|^2 d\mu,$$

where $K$ is an absolute positive constant. This result was extended by Maurey [9] to maps defined on any 2-convex Banach lattice $\Lambda$. Namely, if $T : \Lambda \to \mathcal{H}$ is bounded, then there exists a positive norm one functional $f \in (\Lambda(2))^{\ast}$ such that

$$\forall x \in \Lambda \quad \|T(x)\|^2 \leq K^2 \|T\|^2 f(|x|^2).$$

Here $\Lambda(2)$ denotes the 2-concavification of $\Lambda$. The reader is referred to [8] for all notions on Banach lattices used in this paper.

On the other hand, the noncommutative analogue of the little Grothendieck theorem was obtained by Pisier [12] (see also [14]). More precisely, let $A$ be a C*-algebra, and let $T : A \to \mathcal{H}$ be a bounded linear map. Then there exists a state $f$ on $A$ such that

$$\forall x \in A \quad \|T(x)\|^2 \leq K^2 \|T\|^2 f(x^* x + xx^*).$$
In the spirit of Pisier’s theorem, the first named author of the present paper extended in \cite{5} Maurey’s inequality to unitary ideals of operators on a Hilbert space, and more generally, to symmetric spaces of measurable operators, provided that the underlying r.i. spaces are 2-convex and satisfy an additional condition (see the discussion following Theorem \ref{1.1} below for more details). It was conjectured in \cite{5} that this additional condition should be irrelevant.

The main objective of this paper is to remove the additional condition mentioned above from the main result of \cite{5}, so we obtain the full noncommutative analogue of Maurey’s inequality. On the other hand, the arguments of \cite{5} are rather lengthy, and unfortunately, contain some obscure points about polar decomposition (see \cite{5} Lemma IV.5). Our proof of Theorem \ref{1.1} below is simpler and more readable. To state our main result we need to introduce symmetric spaces of measurable operators.

Let $M$ be a von Neumann algebra, equipped with a semifinite normal faithful trace $\tau$, and let $L_0(M, \tau)$, or, simply $L_0(M)$ denote the topological *-algebra of all operators which are measurable with respect to $(M, \tau)$. The topology of $L_0(M)$ is determined by convergence in measure. For $x \in L_0(M)$ and $t > 0$, $\mu_t(x)$ denotes the $t$-th generalized singular number of $x$. The function $t \mapsto \mu_t(x)$ is called the generalized singular number function and is denoted by $\mu(x)$. Recall that $\mu(x)$ is nonincreasing and $\mu(x) = \mu(x^r) = \mu(|x|)$, where $|x| = (x^*x)^{1/2}$ is the absolute value of $x$. The reader is referred to \cite{2} for more details on generalized singular numbers.

Let $E$ be an r.i. space on $(0, \infty)$ in the sense of \cite{3}. The symmetric space $E(M, \tau)$ of measurable operators associated with $M$ and $E$ is defined as the space of all measurable operators $x \in L_0(M)$ such that $\mu(x) \in E$. $E(M, \tau)$ is a Banach space equipped with the norm $\|x\|_{E(M, \tau)} = \|\mu(x)\|_E$. $E(M, \tau)$ is often denoted simply by $E(M)$. The spaces $E(M)$ are the so-called noncommutative symmetric spaces, studied in detail for the first time by Ovchinnikov \cite{10}. Note that if $M = B(\ell_2)$ and $\tau$ is the usual trace on $B(\ell_2)$, $E(M)$ is a unitary ideal of operators on $\ell_2$. On the other hand, if $\tau$ is finite, $E$ can be taken to be an r.i. space on $[0, \tau(1)]$. Recall that if $E = L_p(0, \infty)$, $E(M) = L_p(M, \mu)$, the noncommutative $L_p$-space associated with $(M, \tau)$.

For $r > 1$, $E^{(r)}$ and $E_{(r)}$ denote the $r$-convexification and $r$-concavification of $E$, respectively. Recall that if $E$ is a $p$-convex and $q$-concave r.i. space, $E^{(r)}$ is a $pr$-convex and $qr$-concave r.i. space. If in addition $p \geq r$ and the $p$-convexity constant of $E$ is equal to 1, then $E^{(r)}$ is a $p/r$-convex and $q/r$-concave r.i. space. In particular, if $E$ is 2-convex with constant 1, $E_{(2)}$ is an r.i. space. Recall that if $E$ is $p$-convex and $q$-concave, $E$ can be renormed into an r.i. space which is $p$-convex and $q$-concave with constant 1.

The following is our main result. In the remainder of the paper, unless explicitly stated otherwise, $M$ will denote a von Neumann algebra equipped with a normal faithful semifinite trace $\tau$, and $E$ will be an r.i. space on $(0, \infty)$. $K$ will denote a universal positive constant, which may change from line to line.

**Theorem 1.1** Assume that $E$ is 2-convex with constant 1. Let $\mathcal{H}$ be a Hilbert space. Then for any bounded linear map $T : E(M) \to \mathcal{H}$ there exists a positive norm one functional $f \in E_{(2)}(M)^*$ such that

$$\forall \ x \in E(M) \quad \|T(x)\|^2 \leq K^2 \|T\|^2 f(x^*x + xx^*).$$

This theorem is stated in \cite{3} with the stronger assumption that $E$ is $p$-convex with $p > 2$. For unitary ideals (i.e. when $M = B(\ell_2)$ equipped with the usual trace), the $p$-convexity assumption is weakened to 2-convexity plus an additional mild condition.

We should also emphasize the universality of the constant $K$ in Theorem 1.1, which is of independent interest. In some special cases, it is much easier to prove the little Grothendieck inequality with a constant depending on the space $E$ in consideration. This is, for instance, the case for $E = L_p(0, \infty)$ with $2 \leq p < \infty$ (see \cite{13} Theorem 6.6). The proof of Theorem 1.1 will be given in the next section. It depends on two other equivalent statements. One of them is the (difficult) lower estimate in the noncommutative Khintchine
inequalities for the dual space \( E(M)^* \) of \( E(M) \), which is important for its own right. To state this equivalence it is more convenient to work with the noncommutative symmetric space \( E'(M) \) instead of \( E(M)^* \), where \( E' \) denotes the Köthe dual of \( E \), which is the subspace of \( E^* \) consisting of all integrals. Let us recall the well-known relations between \( E^* \) and \( E' \). \( E' \) is a norming subspace of \( E^* \). If \( E \) is order continuous, \( E^* = E' \). On the other hand, if \( E \) is maximal (i.e. \( E = E'' \)) and is \( p \)-convex with \( p > 1 \), then \( E = E'^* \). Indeed, the \( p \)-convexity of \( E \) implies that the restriction of \( E \) to \([0, 1]\) is not order isomorphic to \( L_1[0, 1] \). Thus by Proposition 2.a.3 and the remark following it in [5], \( E = F^* \), where \( F \) is the closure of simple functions in \( E' \). However, since \( E' \) is \( p' \)-concave (\( p' \) denoting the conjugate index of \( p \)), \( E' \) is order continuous. It follows that \( E' = F \). Also observe that if \( E \) is an r.i. space on \([0, 1]\) and is not order continuous, then \( E \) is maximal. Indeed, if \( E \) is minimal and non separable, \( E \) is order isomorphic to \( L_\infty(0,1) \), so maximal. Consequently, every r.i. space on \([0, 1]\) either is order continuous or has the Fatou property.

Now if \( E \) is order continuous, \( E(M)^* = E'^*(M) \). This is [23] Lemma 1 if \( \tau(1) < \infty \) and is stated in [1] p. 745] for the general case. Let us also note that the latter case easily follows from the former by a standard approximation argument using the semifiniteness of \( \tau \).

We are ready to state the equivalence theorem of [23]. \((\varepsilon_k)\) denotes a Rademacher sequence on a probability space, and \( E \) is the corresponding expectation.

**Theorem 1.2** Assume that \( E \) is 2-convex with constant 1. Then the following statements are equivalent

i) There exists a positive constant \( C_1 \) such that to any bounded map \( T \) from \( E(M) \) into a Hilbert space \( H \) corresponds a positive norm one functional \( f \in E_{(2)}(M)^* \) satisfying

\[
\forall x \in E(M) \quad \|T(x)\| \leq C_1 \|T\| (|f(x^*x + xx^*)|)^{1/2}.
\]

ii) There exists a positive constant \( C_1 \) such that for any bounded map \( T : E(M) \to H \) and any finite sequence \( (x_k) \subset E(M) \)

\[
\left( \sum_k \|T(x_k)\|^2 \right)^{1/2} \leq C_1 \|T\| \left( \sum_k x_k^* x_k + x_k x_k^* \right)^{1/2}.
\]

iii) There exists a positive constant \( C_2 \) such that for any finite sequence \( (x_k) \subset E'(M) \)

\[
\inf \left\{ \left( \sum_k a_k^* a_k \right)^{1/2} + \left( \sum_k b_k^* b_k \right)^{1/2} \right\} \leq C_2 \|E\| \left( \sum_k \varepsilon_k x_k^2 \right)^{1/2},
\]

where the infimum runs over all decompositions \( x_k = a_k + b_k \) in \( E'(M) \).

Moreover, the constants \( C_1 \) and \( C_2 \) above satisfy the relations: \( C_1 \leq C_2 \leq KC_1 \).

By standard arguments we obtain the following general noncommutative Khintchine inequalities for symmetric spaces of measurable operators. They generalize the Khintchine inequalities for noncommutative \( L_p \)-spaces in [1] and [23].

**Theorem 1.3**  
i) If \( E \) is 2-concave with constant 1, then for every finite sequence \( (x_k) \subset E(M) \)

\[
(E\| \sum_k \varepsilon_k x_k \|^2)^{1/2} \leq \inf \left\{ \left( \sum_k a_k^* a_k \right)^{1/2} + \left( \sum_k b_k^* b_k \right)^{1/2} \right\} \leq K(E\| \sum_k \varepsilon_k x_k \|^2)^{1/2},
\]

where the infimum runs over all decompositions \( x_k = a_k + b_k \) in \( E(M) \).

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ii) If $E$ is 2-convex and $q$-concave with constant 1 for some $q < \infty$, then for every finite sequence $(x_k) \subset E(M)$

$$K_q^{-1} \{ \mathbb{E} \left\| \sum_k \varepsilon_k x_k \right\|^2 \}^{1/2} \leq \max \left\{ \left\| \left( \sum_k x_k \varepsilon_k \right)^{1/2} \right\|, \left\| \left( \sum_k x_k x_k^* \right)^{1/2} \right\| \right\} \leq \left( \mathbb{E} \left\| \sum_k \varepsilon_k x_k \right\| \right)^{1/2},$$

where $K_q$ depends only on $q$. Moreover, $K_q \leq K_q$.

**Proof.** The second inequality of i) follows from Theorem 1.11 and Theorem 1.12. The first one is obtained by using the 2-convexity of $E$ as in [7]. In the same way, the second inequality of ii) is a consequence of the 2-convexity of $E$. Thus it remains to prove the first one of ii). This is done via duality by using the second inequality of i). To this end, we need the K-convexity of $E(M)$ (cf. e.g. [15] for the definition of K-convexity). Under the assumption of ii), by [21], $E(M)$ is of type 2 with a constant $T_q$ depending only on $q$, so $E(M)$ is K-convex. Alternately, we can also use [15] Theorem 7.11. Indeed, by [13], there exists an r.i. space $E_0$ such that $E = (E_0, L_2(0, \infty))_g$, where $\theta = 2/q$. Then $E(M) = (E_0(M), L_2(M))_g$. Thus it follows that $E(M)$ is K-convex with constant majorized by $K'q$ for some universal constant $K'$. Therefore, using the second inequality of ii) and duality, we deduce the first inequality of ii) with $K_q \leq K'q$.

Note that the $q$-concavity condition in Theorem 1.13 ii) is necessary. Indeed, under the 2-convexity assumption of $E$, the first inequality of ii) implies that $E$ is of type 2, and so is of finite concavity. On the other hand, if $E = L_q(0, \infty)$ with $2 \leq q < \infty$, the optimal order of the constant $K_q$ above is $O(\sqrt{q})$. We do not know whether this is true in the general case.

We end this section with some open problems. The first one concerns the noncommutative Khintchine inequalities. Theorem 1.13 gives a deterministic characterization of the expression $\mathbb{E} \left\| \sum_k \varepsilon_k x_k \right\|$ only when $E$ satisfies one of the two conditions there. Recall that if $E$ is a $q$-concave Banach lattice for some $q < \infty$, then for any finite sequence $(x_k) \subset E$

$$\mathbb{E} \left\| \sum_k \varepsilon_k x_k \right\| \approx \left( \sum_k |x_k|^2 \right)^{1/2}$$

with relevant constants depending only on $q$ and the $q$-concavity constant of $E$. At the time of this writing, we do not know how to characterize deterministically $\mathbb{E} \left\| \sum_k \varepsilon_k x_k \right\|_{E(M)}$ for a general $E$.

**Problem 1.4** Let $E$ be a $q$-concave r.i. space with $q < \infty$. Find a deterministic characterization of $\mathbb{E} \left\| \sum_k \varepsilon_k x_k \right\|$ for any finite sequence $(x_k) \subset E(M)$.

The second problem is on the big Grothendieck theorem in the setting of this paper.

**Problem 1.5** Let $E$ and $F$ be two 2-convex r.i. spaces with constant 1. Let $u : E(M) \times F(M) \to \mathbb{C}$ be a bounded bilinear form. Do there exist two positive norm one functionals $f \in E(2)(M)^{*}$ and $g \in F(2)(M)^{*}$ such that

$$\forall x \in E(M), \forall y \in F(M) \quad |u(x, y)| \leq K \left\| \left[ f(x^*x + xx^*) \right]^{1/2} \left[ g(y^*y + yy^*) \right]^{1/2} \right\|^2 ?$$

We can state the following more general problem.

**Problem 1.6** Let $E$ be a 2-convex r.i. space with constant 1 and $Y$ a Banach space of cotype 2. Let $T : E(M) \to Y$ be a bounded linear map. Do there exist a positive norm one functional $f \in E(2)(M)^{*}$ and a positive constant $C$ (depending only on the cotype 2 constant of $Y$) such that

$$\forall x \in E(M) \quad \|T(x)\|^2 \leq C \|T\|^2 \left\| f(x^*x + xx^*) \right\|^2 ?$$

This can be reformulated as follows. Does there exist a positive constant $C$ (depending only on the cotype 2 constant of $Y$) such that

$$\forall x_k \in E(M) \quad \left\| \sum_k T(x_k) \right\|^2 \leq C \|T\| \left\| \left( \sum_k x_k^*x_k + x_kx_k^* \right)^{1/2} \right\|^2 ?$$
In the case of $E = \ell_\infty(0, \infty)$ (i.e. $E(M) = M$) the previous problem has a positive solution. In this case $M$ can be replaced by any C*-algebra (see [13]).

## 2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We require two auxiliary lemmas.

**Lemma 2.1** Assume that $\tau(1) = 1$ and $E = F^*$ for an order continuous r.i. space $F$ on $[0, 1]$.

i) $E(M) = F(M)^*$.

ii) For any $T : E(M) \to H$ with $\|T\| \leq 1$ there exists a net $(T_i)$ of $w^*$-continuous finite rank maps from $E(M)$ into $H$ such that $\|T_i\| \leq 1$ and $T_i \to T$ strongly (i.e. in the point-norm topology).

**Proof.** i) is [23] Lemma 1. To show ii) we use standard duality for Banach space tensor product. We have

$$B(E(M), H) = (E(M) \hat{\otimes} H^*)^*,$$

where $\hat{\otimes}$ denotes the projective tensor product for Banach spaces, and where the duality is determined as follows. For $T \in B(E(M), H)$ and $x \otimes \xi \in E(M) \otimes H^*$

$$\langle T, x \otimes \xi \rangle = (\xi, T(x)).$$

On the other hand,

$$E(M) \hat{\otimes} H^* = (F(M) \hat{\vee} H)^*,$$

where $\hat{\vee}$ denotes the injective tensor product for Banach spaces. It follows that $B(E(M), H)$ is the bidual of $F(M) \hat{\vee} H$. Therefore, the unit ball of $B(E(M), H)$ is the weak* closure of that of $F(M) \hat{\vee} H$.

Recall that $F(M) \hat{\vee} H$ can be identified as the norm closure in $B(E(M), H)$ of $w^*$-continuous finite rank maps (i.e. those associated with vectors in the algebraic tensor product $F(M) \otimes H$). Now let $T : E(M) \to H$ with $\|T\| \leq 1$. Then we deduce a net $(T_i)$ such that each $T_i$ is a $w^*$-continuous finite rank map from $E(M)$ to $H$, $\|T_i\| \leq 1$ and $T_i \to T$ in the $w^*$-topology. Thus $T_i(x) \to T(x)$ weakly in $H$ for any $x \in E(M)$. Therefore, an appropriate net of convex combinations of the $T_i$'s converges to $T$ strongly.

**Lemma 2.2** Assume $\tau(1) = 1$ and $E$ is an order continuous r.i. space on $[0, 1]$. Let $a \in E(M)$ be a positive operator and $e = s(a)$ the support projection of $a$. Then $\{ha + ah + e^\frac{1}{2}he^\frac{1}{2} : h \in M_h\}$ is dense in $E(M)_h$, where $e^\frac{1}{2} = 1 - e$ and $E(M)_h$ denotes the selfadjoint part of $E(M)$, i.e. $E(M)_h = \{x \in E(M) : x^* = x\}$.

**Proof.** The order continuity of $E$ implies that $E^* = E'$ is again an r.i. space on $[0, 1]$. Thus by Lemma 2.1, $E(M)^* = E^*(M) = E'(M)$. Let $y \in E'(M)$ be such that

$$\forall h \in M_h \quad \tau(y(ha + ah + e^\frac{1}{2}he^\frac{1}{2})) = 0.$$

Then

$$\forall x \in M \quad \tau((ya + ay)x + e^\frac{1}{2}ye^\frac{1}{2}x)) = 0.$$

This implies in particular

$$\forall x \in M \quad \tau(e^\frac{1}{2}ye^\frac{1}{2}x) = 0;$$

whence

$$e^\frac{1}{2}ye^\frac{1}{2} = 0.$$
Thus by (2.3) we deduce that $ya = -ay$, so $ya^2 = a^2y$. Therefore, $y$ commutes with all polynomials in $a^2$, thus by functional calculus, with $(a^2)^{1/2} = a$ too. It follows that $ay = ya = 0$; whence $ey = ye = 0$. Combining this with (2.2), we get $y = 0$, which implies the desired density. □

Proof of Theorem 1.1 We will prove one of the three equivalent statements in Theorem 1.2 according to different cases. We start the proof by reducing $\tau$ to a finite trace. To this end we consider the noncommutative Khintchine inequality in Theorem 1.2. Note that $E'$ is 2-concave for $E$ is 2-convex. Thus $E'$ is order continuous. On the other hand, by the seminilteness of $\tau$, we have an increasing net $(e_i)$ of projections in $M$ such that $\tau(e_i) < \infty$ for each $i$ and $e_i \to 1$ strongly. Then the order continuity of $E'$ implies that $e_ixe_i \to x$ in $E'(M)$ for every $x \in E'(M)$ (see [22, Lemma 4.5]). Therefore, we need only to prove inequality (K) for all $x_k = e_ix_1e_i$ and for every fixed $i$. Namely, it suffices to show (K) for $E'(e_iMe_i)$. Now the restriction of $\tau$ to $e_iMe_i$ is finite. Thus we are reduced to the finite trace case.

In the sequel, $\tau$ is a normal faithful finite trace on $M$, so by normalization, we can further assume $\tau(1) = 1$. Accordingly, $E$ can be taken to be a 2-convex r.i. space on $[0,1]$. The remainder of the proof is divided into several cases.

Case 1: $E$ is $p$-convex and $q$-concave with constant 1 for some $p > 2$ and $q < \infty$. This is the main part of the whole proof. We will prove the little Grothendieck theorem for $E(M)$. The pattern of the following argument is modelled on Haagerup’s proof of the little Grothendieck theorem for $C^*$-algebras (see [3]). It is clear that it suffices to do this for every finite dimensional Hilbert space $\mathcal{H}$. So in this part $\mathcal{H}$ is assumed finite dimensional. Fix a map $T : E(M) \to \mathcal{H}$ such that $\|T\| = 1$. The $p$-convexity and $q$-concavity of $E$ implies that $E(M)$ is uniformly convex and uniformly smooth by virtue of [21]. In particular, $E(M)$ is reflexive. Then $T$ is weakly continuous, so the weak compactness of the unit ball of $E(M)$ implies that $T$ attains its norm (recalling that $\dim \mathcal{H} < \infty$). Thus there exists $a \in E(M)$ such that $\|a\| = 1$ and $\|T(a)\| = 1$. We consider two subcases according to $a \geq 0$ or not.

Subcase 1: $a \geq 0$. Let $h \in M$ be a selfadjoint operator. Then $e^{ith}$ is unitary for any $t \in \mathbb{R}$. Consequently,

$$e^{ith}a e^{ith} \in E(M) \quad \text{and} \quad \|e^{ith}a e^{ith}\| = 1.$$

Writing

$$e^{ith}a e^{ith} = a - t^2b + it(ha + ah) + o(t^2),$$

where $b = (h^2a + ah^2)/2 + hah$, we have

$$\|a - t^2b + it(ha + ah)\| \leq 1 + o(t^2).$$

By the selfadjointness of $h$,

$$\|a - t^2b + it(ha + ah)\| \leq \|a - t^2b - it(ha + ah)\|.$$

Thus

$$\mathbb{E}\|a - t^2b + it\varepsilon(ha + ah)\|^2 = \|a - t^2b + it(ha + ah)\|^2,$$

where $\varepsilon$ is a Rademacher function and $\mathbb{E}$ the corresponding expectation. Then we deduce that

$$\|T(a - t^2b)\|^2 + t^2 \|T(ha + ah)\|^2 = \mathbb{E}\|T(a - t^2b) + it\varepsilon T(ha + ah)\|^2 \leq \mathbb{E}\|a - t^2b + it\varepsilon(ha + ah)\|^2 \leq 1 + o(t^2).$$

Therefore,

$$t^2 \|T(ha + ah)\|^2 \leq 2t^2 \text{Re}(T(a), T(b)) + o(t^2);$$

whence

$$(2.3) \quad \|T(ha + ah)\|^2 \leq 2 \text{Re}(T(a), T(b)).$$
Let \( \varphi = T^*T(a) \). (More rigorously, \( \varphi = \overline{T^*T(a)} \) with \( \overline{T^*}: H = \overline{H} \to E(M)^* \), where \( \overline{X} \) denotes the complex conjugate of a Banach space \( X \).) Then \( \varphi \in E(M)^* = E^*(M) \) and \( \|\varphi\| \leq 1 \). On the other hand, \( \varphi(a) = 1 \). Consequently, \( \|\varphi\| = 1 \) and \( \varphi \) is a supporting functional of \( a \), which is unique by virtue of the smoothness of \( E(M) \). \( \varphi \) must be positive and \( s(\varphi) \leq e \), where \( e = s(a) \) is the support projection of \( a \). Indeed, it is easy to see that the absolute value of \( \varphi \) is also a supporting functional of \( a \), which must coincide with \( \varphi \) by uniqueness. In the same way, \( e\varphi e = \varphi \) for \( e\varphi e \) is again a supporting functional for \( a \). (In fact, one can easily show that \( \varphi \) is affiliated with the von Neumann subalgebra generated by the spectral projections of \( a \).)

Next, let \( E(2) \) be the 2-convicification of \( E \). \( E(2) \) is \( p/2 \)-convex and \( q/2 \)-concave (and so \( E(2)(M) \) is also uniformly smooth). Consider the one dimensional subspace \( \mathbb{C}a^2 \subset E(2)(M) \) generated by \( a^2 \), and the functional \( f_0 : \mathbb{C}a^2 \to \mathbb{C} \) defined by \( f_0(\lambda a^2) = \lambda \). Then \( \|f_0\| = 1 \) and \( f_0(a^2) = 1 \). By the Hahn-Banach theorem, \( f_0 \) extends to a norm one functional \( f \) on \( E(2)(M) \). Then \( f \) is the unique supporting functional of \( a^2 \), and the preceding argument shows that \( efe = f \geq 0 \). Let \( \psi = af \). We claim that \( \psi \) is a norm one functional on \( E(M) \) and supports \( a \). Indeed, for any \( x \in E(M) \), by the Cauchy-Schwarz inequality

\[
|\psi(x)| = |\tau(xaf)| = |\tau(f^{1/2}xaf^{1/2})| \\
\leq \||f^{1/2}x\|_2 \||af^{1/2}\|_2 = (f(|x|^2))^{1/2} (f(a^2))^{1/2} \\
\leq \||x|^2\|^{1/2}_{E(2)(M)} \|x\|_{E(M)}.
\]

Thus \( \|\psi\| \leq 1 \). However, \( \varphi(a) = f(a^2) = 1 \). Then our claim follows. Therefore, by uniqueness, \( \varphi = \psi \), i.e. \( \varphi = af \). Passing to adjoints, we also have \( \varphi = fa \).

Now since \( fa = af \), inequality (2.3) becomes

\[
||T(ha + ah)||^2 \leq 2\varphi(h^2a + hah) = 2f(h^2a^2 + haha).
\]

On the other hand (recalling that \( f \geq 0 \)),

\[
f((ha + ah)^2) = 2f(haha) + f(h^2a^2) + f(hah^2) \geq f(h^2a^2 + haha).
\]

Therefore,

\[
(2.4) \quad ||T(ha + ah)||^2 \leq 2f((ha + ah)^2).
\]

We will apply Lemma (2.2). To this end we need to deal with operators supported by \( e^\perp \). We claim that \( T(x) = 0 \) for every \( x \in E(M) \) such that \( e^\perp xe^\perp = x \). It suffices to consider the case where \( x \) is selfadjoint. Then

\[
||T(a)||^2 + t^2||T(x)||^2 = E||T(a) + t\varepsilon T(x)||^2 \leq E||a + t\varepsilon x||^2.
\]

Since \( a \) and \( x \) are of disjoint support, by considering the commutative von Neumann subalgebra generated by \( a \) and \( x \), we can assume that \( a \) and \( x \) are functions of disjoint support. Thus the \( p \)-convexity of \( E \) implies that

\[
||a + t\varepsilon x|| = ||(|a|^p + t^p|x|^p)^{1/p}|| \leq (||a||^p + t^p||x||^p)^{1/p}.
\]

Combining the preceding inequalities (recalling that \( ||a|| = ||T(a)|| = 1 \)), we get

\[
t^2||T(x)||^2 \leq O(t^p);
\]

whence the claim as \( t \to 0 \) by the assumption that \( p > 2 \).

Now let \( h \in M_h \) and \( x = ha + ah + e^\perp he^\perp \). Using the previous claim, (2.4) and the fact that \( f \) is supported by \( e \), we have

\[
||T(x)||^2 = ||T(ha + ah)||^2 \leq 2f((ha + ah)^2) = 2f(x^2).
\]
By the density of \( \{ ha + ah + e^{\pm}he^{\pm} : h \in M_h \} \) in \( E(M)_h \) given by Lemma 2, we deduce that \( \| T(x) \|^2 \leq 2f(x^2) \) for any selfadjoint \( x \in E(M) \). It then follows that
\[
\forall \, x \in E(M) \quad \| T(x) \|^2 \leq 2f(x^*x + xx^*) ;
\]
Namely, (G) holds in this subcase with \( K = \sqrt{2} \).

Subcase 2: \( a \geq 0 \). Let \( a = w|a| \) be the polar decomposition of \( a \). Let \( e_1 = u^*u \) and \( e_2 = uu^* \). Then \( e_1 \) and \( e_2 \) are two equivalent projections of \( M \). Since \( M \) is finite, their complementary projections \( e_1^⊥ \) and \( e_2^⊥ \) are also equivalent (see [12 Proposition V.1.38]). Therefore, there exists a partial isometry \( v \in M \) such that \( v^*v = e_1^⊥ \) and \( vv^* = e_2^⊥ \). Set \( w = u + v \). Then \( w \) is a unitary and \( a = w|a| \).

Now consider a new map \( S : E(M) \to \mathcal{H} \) defined by \( S(y) = T(wy) \). Then \( S \) has norm one and attains its norm at \( |a| \). Therefore, by Subcase 1.1, there exists a norm one positive functional \( g \in E_2(M)^* \) such that
\[
\forall \, y \in E(M) \quad \| S(y) \| \leq 2g(y^*y + yy^*) ;
\]
whence (by writing \( y = w^*x \))
\[
\forall \, x \in E(M) \quad \| T(x) \| \leq 2g(x^*x + w^*xx^*w) \leq 4f(x^*x + xx^*) ,
\]
where \( f = (g + wgw^*)/2 \). Therefore, we still have the Grothendieck factorization for \( E(M) \) with \( K = 2 \). Thus the proof of Case 1 is complete.

Case 2: \( E \) is \( p \)-convex with constant 1 for some \( p > 2 \). We will show the noncommutative Khintchine inequality for \( F(M) \), where \( F = F^p \). To this end note that \( F \) is \( p' \)-concave with constant 1, where \( p' \) denotes the conjugate index of \( p \). In particular, \( F \) is order continuous. Consequently, \( M \) is dense in \( F(M) \) (see [22, Lemma 4.5]). Thus in order to prove inequality (K) we need only to consider finite sequences \( (x_k) \) in \( M \).

Now let \( r > 1 \) and consider the \( r \)-convexification \( F^{(r)} \) of \( F \). The order continuity of \( F^{(r)} \) implies that for any \( x \in M \)
\[
\lim_{r \to 1} \| x \|_{F^{(r)}(M)} = \| x \|_{F(M)} .
\]
Thus we are reduced to show inequality (K) with \( F^{(r)}(M) \) in place of \( F(M) \) for all \( r \) close to 1. However, by Theorem 1.3, this is equivalent to the validity of the little Grothendieck theorem for \( G(M) \), where \( G \) is the dual space of \( F^{(r)} \). Since \( F^{(r)} \) is \( r \)-convex and \( rp'^{-} \)-concave with constant 1, \( G \) is \( r' \)-concave and \( s \)-convex with constant 1, where \( s = rp/(1 + (r - 1)p) \). For \( r > 1 \) sufficiently close to 1 we still have \( s > 2 \). Thus \( G \) verifies the condition of Case 1. Consequently, the little Grothendieck theorem holds for \( G(M) \), so Case 2 is done.

Case 3: The general case. Recall that \( E \) is a 2-convex r.i. space on \([0,1]\). We will show the 2-concavity inequality (C). To this end fix a map \( T : E(M) \to \mathcal{H} \) with \( \| T \| \leq 1 \). Let \( r > 1 \) and consider the \( r \)-convexification \( E^{(r)} \) of \( E \). By the Hölder inequality, \( E^{(r)} \subset E \) and the inclusion has norm 1; so \( E^{(r)}(M) \subset E(M) \) is also a norm one inclusion. Let \( \tilde{T} = T \circ \iota \), where \( \iota \) is the natural inclusion from \( E^{(r)}(M) \) into \( E(M) \). Thus \( \tilde{T} : E^{(r)}(M) \to \mathcal{H} \) is a contraction. Now \( E^{(r)} \) is \( 2r \)-convex with \( 2r > 2 \). Therefore, applying Case 2 to \( E^{(r)}(M) \), we obtain that for any finite sequence \( (x_k) \subset M \)
\[
(\sum_k \| \tilde{T}(x_k) \|^2)^{1/2} \leq K \| (\sum_k x_k^*x_k + x_kx_k^*)^{1/2} \|_{E^{(r)}(M)} .
\]
Namely,
\[
(\sum_k \| T(x_k) \|^2)^{1/2} \leq K \| (\sum_k x_k^*x_k + x_kx_k^*)^{r/2} \|_{E^{(r)}(M)}^{1/r} .
\]
As in Case 2 we also have
\[
\forall \, x \in M, \ x > 0 \quad \lim_{r \to 1} \| x^r \|_{E(M)} = \| x \|_{E(M)} .
\]
This follows from the order continuity or the Fatou property of $E$. Therefore,

$$
(\sum_k \|T(x_k)\|^2)^{1/2} \leq K \| (\sum_k x_k^* x_k + x_k x_k^*)^{1/2} \|_{E(M)}.
$$

That is, inequality (C) holds for all finite sequences $(x_k) \subset E$. To pass from $M$ to $E(M)$ we use approximation as usual in such a situation. Indeed, if $E$ is order continuous, $M$ is dense in $E(M)$, so we are done in this case. Otherwise, $E = F^*$ with $F = F'$. By Lemma 2.1 there exists a net $(T_i)$ of $w^*$-continuous finite rank maps in the unit ball of $B(E(M), H)$ such that $T_i \to T$ strongly. Since inequality (C) is stable under strong limit, we are reduced to prove (C) for each $T_i$. Replacing $T$ by $T_i$ if necessary, we can assume that $T$ itself is $w^*$-continuous and of finite rank. Now fix a finite sequence $(x_k) \subset E(M)$. For each $n \in \mathbb{N}$ let $x_{k,n} = x_k \mathbb{I}_{[0,n]}(|x_k|)$, where $\mathbb{I}_{[0,n]}(x)$ denotes the spectral projection of a positive operator $x$ corresponding to the interval $[0, n]$. Then $x_{k,n} \in M$, so by the preceding inequality

$$
(\sum_k \|T(x_{k,n})\|^2)^{1/2} \leq K \| (\sum_k x_{k,n}^* x_{k,n} + x_{k,n} x_{k,n}^*)^{1/2} \|_{E(M)}.
$$

However, for each $k$, $x_{k,n} \to x_k$ in $E(M)$ relative to the $w^*$-topology as $n \to \infty$. It follows that $T(x_{k,n}) \to T(x_k)$ in $H$ by virtue of the $w^*$-continuity of $T$. On the other hand,

$$
\sum_k x_{k,n}^* x_{k,n} + x_{k,n} x_{k,n}^* \leq \sum_k x_k^* x_k + x_k x_k^*.
$$

Therefore, we deduce

$$
(\sum_k \|T(x_k)\|^2)^{1/2} \leq K \| (\sum_k x_k^* x_k + x_k x_k^*)^{1/2} \|_{E(M)},
$$

as desired. Thus the proof of Theorem \ref{thm:main} is complete. \hfill \Box

### 3 Applications to Schur multipliers

In this section we present some applications of our little Grothendieck theorem to Schur multipliers. We characterize the Schur multipliers from a 2-convex unitary ideal into a 2-concave one. Now the von Neumann algebra $M$ is $B(\ell_2)$ and the trace $\tau$ is the usual trace $\text{Tr}$. Accordingly, instead of r.i. spaces on $(0, \infty)$, we consider r.i. spaces on $\mathbb{N}$, i.e. symmetric sequence spaces. Given a symmetric sequence space $E$, we denote the associated unitary ideal by $S_E$. Namely, $S_E = E(B(\ell_2), \text{Tr})$ in the previous notation. Note that if $E = \ell_p$, $S_E$ becomes the usual Schatten class $S_p$. In particular, $S_1$ is the trace class, $S_\infty = B(\ell_2)$, and $S_2 = \ell_2(\mathbb{N}^2)$ is the Hilbert-Schmidt class. As usual, the operators in $S_E$ are represented by infinite matrices. Let $e_{ij}$ be the canonical matrix units. Then every $x \in S_E$ is given by an infinite matrix $x = (x_{ij})_{i,j \geq 0}$, i.e.

$$
x = \sum_{i,j \geq 0} x_{ij} e_{ij}.
$$

Equally, $x$ can be also viewed as a function on $\mathbb{N}^2$. In the sequel we will not distinguish an infinite matrix and the corresponding function on $\mathbb{N}^2$.

Let $E$ and $F$ be two symmetric sequence spaces. Let $\varphi = (\varphi_{ij})$ be an infinite matrix. We call $\varphi$ a Schur multiplier from $S_E$ to $S_F$ if the map $M_\varphi : x \mapsto (\varphi_{ij} x_{ij})_{i,j \geq 0}$ defines a bounded map from $S_E$ into $S_F$. More generally, if $X$ and $Y$ are two Banach spaces of complex functions on $\mathbb{N}^2$, a Schur multiplier from $X$ into $Y$ is a function $\varphi$ on $\mathbb{N}^2$ such that $M_\varphi$ induces a bounded map from $X$ into $Y$.  

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Let $1 \leq p \leq \infty$. Recall that $E(\ell_p)$ is the space of complex matrices $\varphi = (\varphi_{ij})$ such that the sequence $i \mapsto \|\varphi_i\|_{\ell_p} = (\sum_j |\varphi_{ij}|^p)^{1/p}$ belongs to $E$ (with the usual convention for $p = \infty$). The norm of $E(\ell_p)$ is given by

$$\|\varphi\|_{E(\ell_p)} = \left(\|\varphi_i\|_{\ell_p}\right)_{i \geq 0} \|\varphi\|_E.$$  

Let $^tE(\ell_p) = \{\varphi : ^t \varphi \in E(\ell_p)\}$, equipped with the natural norm, where $^t \varphi$ is the transpose of $\varphi$, i.e. $^t \varphi_{ij} = \varphi_{ji}$. Note that $E(\ell_p)$ and $^tE(\ell_p)$ are Köthe function spaces on $\mathbb{N}^2$. If $X$ and $Y$ are two Banach spaces of functions on $\mathbb{N}^2$, $X + Y$ and $X \cap Y$ denote their sum and intersection, respectively. Recall that the norm of $X + Y$ and $X \cap Y$ are given by

$$\|z\|_{X + Y} = \inf \{\|x\|_X + \|y\|_Y : z = x + y, x \in X, y \in Y\}$$

and

$$\|z\|_{X \cap Y} = \max (\|z\|_X, \|z\|_Y).$$

**Lemma 3.1** Let $E$ be a 2-convex symmetric sequence space with constant 1, and let $\varphi$ be a function on $\mathbb{N}^2$. Then the following assertions are equivalent

i) $\varphi$ is a Schur multiplier from $S_E$ to $S_2$;

ii) $\varphi$ is a Schur multiplier from $E(\ell_2) \cap ^tE(\ell_2)$ to $\ell_2(\mathbb{N}^2)$;

iii) $\varphi \in G(\ell_\infty) + ^tG(\ell_\infty)$, where $G = ((E(\mathbb{Z}))')^{(2)}$.

Moreover,

$$\|M_\varphi : S_E \to S_2\| \approx \|M_\varphi : E(\ell_2) \cap ^tE(\ell_2) \to \ell_2(\mathbb{N}^2)\| = \|\varphi\|_{G(\ell_\infty) + ^tG(\ell_\infty)},$$

where the equivalence constants are controlled by a universal constant.

**Proof.** i) $\Rightarrow$ ii). Let $\varphi$ be a Schur multiplier from $S_E$ to $S_2$. Let $x$ be a finite matrix. Then by Theorem 1.1 and inequality (C) in Theorem 1.2.

$$\|M_\varphi(x)\|_{S_2} = \left(\sum_{i,j} |\varphi_{ij} x_{ij}|^2\right)^{1/2}$$

$$\leq K \|M_\varphi\| \left[\left(\sum_{i,j} |x_{ij}|^2(e_{ij}^* e_{ij}^* + e_{ij} e_{ij}^*)\right)^{1/2}\right]_{S_E}$$

$$= K \|M_\varphi\| \left[\left(\sum_{i,j} |x_{ij}|^2(e_{jj} + e_{ii})\right)^{1/2}\right]_{S_E}$$

$$\leq 2K \|M_\varphi\| \|x\|_{E(\ell_2) \cap ^tE(\ell_2)}.$$  

Therefore, $\varphi$ is a Schur multiplier from $E(\ell_2) \cap ^tE(\ell_2)$ to $\ell_2(\mathbb{N}^2)$.

ii) $\Rightarrow$ i). First observe that $S_E$ embeds contractively into $E(\ell_2) \cap ^tE(\ell_2)$. Indeed, let $x \in S_E$, and let $a_i = \sum_j x_{ij} e_{ij}$. Then by Theorem 1.2 ii)

$$\|x\|_{E(\ell_2)} = \|\left(\sum_i a_i a_i^*\right)^{1/2}\|_{S_E} \leq (E \|\|\sum_i e_i a_i\|^2_{S_E}) = \|x\|_{S_E};$$

whence the observation. It then follows that

$$\|M_\varphi : S_E \to S_2\| \leq \|M_\varphi : E(\ell_2) \cap ^tE(\ell_2) \to \ell_2(\mathbb{N}^2)\|.$$  

ii) $\Leftrightarrow$ iii). Let $X$ be a 2-convex Köthe function space on $\mathbb{N}^2$. Then it is clear that $\varphi$ is a Schur multiplier from $X$ to $\ell_2(\mathbb{N}^2)$ iff $\varphi \in ((X(\mathbb{Z})))^{(2)}$. Therefore, the equivalence ii) $\Leftrightarrow$ iii) follows.  

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Remark 3.2 It is clear that the symmetric sequence space \( G \) in Lemma 3.1 (iii) is equal to the space of multipliers from \( E \) to \( \ell_2 \).

Let \( G \) and \( H \) be two 2-convex symmetric sequence spaces with constant 1. Define \( GH \) by

\[
GH = \{ xy : x \in G, y \in H \} \quad \text{and} \quad \| z \|_{GH} = \inf \{ \| x \|_G \| y \|_H : x \in G, y \in H \}.
\]

It is easy to see that \( GH \) is again a symmetric sequence space.

Theorem 3.3 Let \( E \) and \( F \) be two symmetric sequence spaces. Assume that \( E \) and \( F \) are respectively 2-convex and 2-concave with constant 1. Then a function \( \varphi \) on \( \mathbb{N}^2 \) is a Schur multiplier from \( S_E \) to \( S_F \) iff \( \varphi \in L(\ell_\infty) + \ell_L(\ell_\infty) \), where \( L = GH \) with \( G = ((E(2)')')^{(2)} \) and \( H = (((F'(2))')')^{(2)} \).

Moreover,

\[
\| M_\varphi : S_E \to S_F \| \approx \| \varphi \|_{L(\ell_\infty) + \ell_L(\ell_\infty)}
\]

with universal constants.

Proof. Let \( \varphi = \psi \omega \) with \( \psi \in G(\ell_\infty) \) and \( \omega \in H(\ell_\infty) \). By Lemma 3.1 \( \psi \) is a Schur multiplier from \( S_E \) to \( S_2 \) and \( \omega \) a Schur multiplier from \( S_F \) to \( S_2 \). Passing to adjoint, we see that \( \omega \) is also a Schur multiplier from \( S_2 \) to \( S_F \). It follows that \( \varphi \) is a Schur multiplier from \( S_E \) to \( S_F \). Consequently, every function in \( L(\ell_\infty) + \ell_L(\ell_\infty) \) is a Schur multiplier from \( S_E \) to \( S_F \).

Conversely, let \( \varphi \) be a Schur multiplier from \( S_E \) to \( S_F \). To prove that \( \varphi \in L(\ell_\infty) + \ell_L(\ell_\infty) \), by the Fatou property of \( L(\ell_\infty) + \ell_L(\ell_\infty) \), we can clearly assume that \( \varphi \) is a finite matrix, say an \( n \times n \) matrix. Thus in the remainder of the proof all matrices are assumed to be of order \( n \).

Since \( F \) and \( E' \) are 2-concave with constant 1, by [20], \( S_F \) and \( S_E^* \) are of cotype 2 with universal constants. Therefore, by [14] Theorem 4.1, \( M_\varphi : S_E \to S_F \) factors through a Hilbert space \( \mathcal{H} \) as \( M_\varphi = VU \) with \( \| V \| \| U \| \leq K \| M_\varphi \| \), where \( K \) is a universal constant. The point now is to show that we may take \( \mathcal{H} = S_2 \) and assume that \( U \) and \( V \) are Schur multipliers. The following argument is well-known and standard.

By Theorem [14] there exists a norm one positive functional \( f \in (S_{E(2)})^* \) such that

\[
\forall x \in S_E \quad \| U(x) \|^2 \leq K^2 f(x^* x + xx^*).
\]

Hence

\[
\forall x \in S_E \quad \| M_\varphi(x) \|^2 \leq K^2 f(x^* x + xx^*).
\]

Now let \( \varepsilon = (\varepsilon_i) \) be a Rademacher sequence. Let \( D_\varepsilon \) be the diagonal matrix whose diagonal entries are the \( \varepsilon_i \)'s. Let \( \varepsilon' \) be an independent copy of \( \varepsilon \) and \( D_{\varepsilon'} \) the associated diagonal matrix. Recall that the norm of \( S_F \) is unitary invariant (in fact, what is needed here is the invariance of the norm by left and right multiplications by unitary diagonal matrices). Thus by the previous inequality, for any \( x \in S_E \) we have

\[
\| M_\varphi(x) \|^2 = \| D_{\varepsilon} M_\varphi(x) D_{\varepsilon'} \|^2 = \| M_\varphi(D_{\varepsilon} x D_{\varepsilon'}) \|^2 \leq K^2 f(D_{\varepsilon'} x^* x D_{\varepsilon'} + D_{\varepsilon} x x^* D_{\varepsilon}) = K^2 [D_{\varepsilon'} f D_{\varepsilon'}(x^* x) + D_{\varepsilon} f D_{\varepsilon}(x x^*)].
\]

Taking expectation, we deduce that

\[
\| M_\varphi(x) \|^2 \leq K^2 g(x^* x + xx^*),
\]

where \( g = \mathbb{E}(D_{\varepsilon} f D_{\varepsilon}) \). Note that \( g \in (S_{E(2)})^* \) is a positive diagonal matrix, so its diagonal sequence belongs to \( (E(2))' \). The preceding inequality can be rewritten as

\[
\| M_\varphi(x) \|^2 \leq K^2 (g^{1/2} x^* x^{1/2})^2 + g^{1/2} x^* x^{1/2},
\]

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It follows that there exist two bounded maps $v$ and $v'$ from $S_2$ to $S_F$ such that

$$M_\omega = vu + v'u',$$

where $u$ and $u'$ are respectively the left and right multiplications by $g^{1/2}$. Note that $u = M_\omega$ and $u' = M_{\omega'}$ with $\psi_{ij} = \alpha_i$ and $\psi'_{ij} = \alpha_j$, where $\alpha_i = g_i^{1/2}$. Using an average argument as above, we can further assume that $v$ and $v'$ are also given by Schur multipliers $M_\omega$ and $M_{\omega'}$. Therefore, $\omega$ and $\omega'$ are Schur multipliers from $S_2$ to $S_F$, and hence also from $S_F$ to $S_2$. Thus by Lemma 3.4, $\omega, \omega' \in F(\ell_\infty) + \iota F(\ell_\infty)$.

Now it is easy to show that $\varphi = \psi\omega + \psi'\omega'$ belongs to $L(\ell_\infty) + \iota L(\ell_\infty)$. Indeed, let $\omega = \delta + \gamma$ with $\delta \in F(\ell_\infty)$ and $\gamma \in \iota F(\ell_\infty)$. It is clear that $\psi\delta \in L(\ell_\infty)$. We next show that $\psi\gamma \in L(\ell_\infty) + \iota L(\ell_\infty)$. To this end, by permutations of rows and columns if necessary, we may assume that the sequence $(\alpha_i)$ and $(\beta_j)$ are nonincreasing, where $\beta_j = \sup_k |\gamma_{ij}|$. Define $\gamma'_{ij} = \gamma_{ij}$ if $i \leq j$ and $\gamma'_{ij} = 0$ if $i > j$. Set $\gamma'' = \gamma - \gamma'$. Then $\sup_i |\gamma''|_{ij} \leq \beta_i \alpha_i$ and $\sup_i |\gamma''|_{ij} \leq \beta_j \alpha_j$. It follows that $\psi\gamma' \in L(\ell_\infty)$ and $\psi\gamma'' \in \iota L(\ell_\infty)$, so $\psi\gamma \in L(\ell_\infty) + \iota L(\ell_\infty)$. Consequently, $\psi\omega \in L(\ell_\infty) + \iota L(\ell_\infty)$. Similarly, $\psi'\omega' \in L(\ell_\infty) + \iota L(\ell_\infty)$. Therefore, $\varphi \in L(\ell_\infty) + \iota L(\ell_\infty)$. Thus the proof of the theorem is complete.

The preceding theorem extends the characterization of Schur multipliers from $S_q$ to $S_p$ for $1 \leq p \leq 2 \leq q \leq \infty$ in [24] (see also [17] for the case of $p = 1$ and $q = \infty$). If one of $E$ and $F$ is an $\ell_p$, the space $L$ in Theorem 3.4 is easy to be determined. For instance, let us consider the case where $F = \ell_p$ with $1 \leq p \leq 2$ (and $E$ is still 2-convex with constant 1). By Remark 6.2, $L = \mathcal{GH}$ coincides with the space of multipliers from $E$ to $F$. Thus if $F = \ell_p$, this latter space is equal to $((E(p))')^{(p)}$. Thus we get the following

**Corollary 3.4** Let $E$ be a 2-convex symmetric sequence space with constant 1 and $1 \leq p \leq 2$. Then a function $\varphi$ on $\mathbb{N}^2$ is a Schur multiplier from $S_E$ to $S_p$ iff $\varphi \in G_1(\ell_\infty) + \iota G_1(\ell_\infty)$, where $G_1 = ((E(p))')^{(p)}$.

The previous arguments apply equally to the case where one of the unitary ideals $S_E$ and $S_F$ is replaced by a Kőthe function space on $\mathbb{N}^2$. By symmetry, it suffices to consider the case where the second ideal $S_F$ is replaced by a 2-concave Kőthe function space on $\mathbb{N}^2$.

**Theorem 3.5** Let $E$ be a 2-convex symmetric sequence space with constant 1, and let $X$ be a 2-concave Kőthe function space on $\mathbb{N}^2$ with constant 1. Then a function $\varphi$ on $\mathbb{N}^2$ is a Schur multiplier from $S_E$ to $X$ iff $\varphi \in [G(\ell_\infty) + \iota G(\ell_\infty)]Y$, where $G = ((E(2))')^{(2)}$ and $Y = (((X(2)))')^{(2)}$. Moreover, the relevant constants are controlled by a universal constant.

In particular, if $X = \ell_p(\mathbb{N}^2)$ with $1 \leq p \leq 2$, then $\varphi$ is a Schur multiplier from $S_E$ to $\ell_p(\mathbb{N}^2)$ iff $\varphi \in G_1(\ell_q) + \iota G_1(\ell_q)$, where $G_1 = ((E(p))')^{(p)}$ and $q = 2p/(2 - p)$.

**Proof.** This proof is almost the same as that of Theorem 3.3. The only difference is that the space of Schur multipliers from $S_2$ to $X$ coincides with the space $Y$, that makes simpler the present proof. We leave the details to the reader.

The theorem above in the case of $S_E = B(\ell_2)$ and $X = \ell_1(\mathbb{N}^2)$ goes back to [9] Example b](see also [11] Theorem 4.1).

**References**

[1] P. Dodds, and T. Dodds and B. de Pagter. Noncommutative Kőthe duality. *Trans. Amer. Math. Soc.*, 339:717–750, 1993.

[2] Th. Fack and H. Kosaki. Generalized $s$-numbers of $\tau$-measurable operators. *Pacific J. Math.*, 123:269–300, 1986.
[3] U. Haagerup. The Grothendieck inequality for bilinear forms on $C^*$-algebras. *Adv. Math.*, 56:93–116, 1985.

[4] F. Lust-Piquard. Inégalités de Khintchine dans $C_p$ ($1 < p < \infty$). *C. R. Acad. Sci. Paris*, 303:289–292, 1986.

[5] F. Lust-Piquard. A Grothendieck factorization theorem on 2-convex Schatten spaces. *Israel J. Math.*, 79:331–365, 1992.

[6] F. Lust-Piquard. On the coefficient problem: a version of the Kahane-Katznelson-De Leeuw theorem for spaces of matrices. *J. Funct. Anal.*, 149:352–376, 1997.

[7] F. Lust-Piquard and G. Pisier. Noncommutative Khintchine and Paley inequalities. *Ark. Mat.*, 29:241–260, 1991.

[8] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. II*. Springer-Verlag, Berlin, 1979.

[9] B. Maurey. *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces $L^p$*. Astérisque, No. 11, 1974.

[10] V. I. Ovčinnikov. Symmetric spaces of measurable operators. *Dokl. Akad. Nauk SSSR*, 191:769–771, 1970.

[11] A. Pelczynski and F. Sukochev. Some remarks on Toeplitz multipliers and Hankel matrices. *Studia Math.*, to appear.

[12] G. Pisier. Grothendieck’s theorem for noncommutative $C^*$-algebras, with an appendix on Grothendieck’s constants. *J. Funct. Anal.*, 29:397–415, 1978.

[13] G. Pisier. Factorization of operators through $L_{p\infty}$ or $L_{p1}$ and noncommutative generalizations. *Math. Ann.*, 276:105–136, 1986.

[14] G. Pisier. *Factorization of linear operators and geometry of Banach spaces*. CBMS Regional Conference Series in Mathematics, vol. 60, Washington, DC, 1986.

[15] G. Pisier. *The volume of convex bodies and Banach space geometry*. Cambridge Tracts in Mathematics, vol. 94, Camb. Univ. Press, 1989.

[16] G. Pisier. Some applications of the complex interpolation method to Banach lattices. *J. Analyse Math.* 35:264–281, 1979.

[17] G. Pisier and D. Shlyakhtenko. Grothendieck’s theorem for operator spaces. *Invent. Math.*, 150:185–217, 2002.

[18] G. Pisier and Q. Xu. Non-commutative $L^p$-spaces. In *Handbook of the geometry of Banach spaces*, Vol. 2, pages 1459–1517. North-Holland, Amsterdam, 2003.

[19] M. Takesaki. *Theory of operator algebras. I*. Springer-Verlag, New York, 1979.

[20] N. Tomczak-Jaegermann. Uniform convexity of unitary ideals. *Israel J. Math.* 48:249–254, 1984.

[21] Q. Xu. Convexité uniforme des espaces symétriques d’opérateurs mesurables. *C. R. Acad. Sci. Paris*, 309:251–254, 1989.

[22] Q. Xu. Analytic functions with values in lattices and symmetric spaces of measurable operators. *Math. Proc. Cambridge Philos. Soc.*, 109:541–563, 1991.

[23] Q. Xu. Radon-Nikodým property in symmetric spaces of measurable operators. *Proc. Amer. Math. Soc.*, 115:329–335, 1992.
[24] Q. Xu. Operator space Grothendieck inequalities for noncommutative $L_p$-spaces. *Duke Math. J.*, 131:525–574, 2005.