We discuss the non perturbative approach to the problem of high–energy hadron–hadron (dipole–dipole) scattering at low momentum transfer by means of numerical simulations in Lattice Gauge Theory.

1. Introduction

The prediction from first principles of total cross sections at high energy is one of the oldest open problems of hadronic physics. Present–day experimental data are well described by a universal pomeron–like power–law behaviour (see, for example, Ref. [1] and references therein),

\[ \sigma_{\text{tot}}^{(\text{hh})}(s) \sim \frac{s}{s_0}^{\epsilon_P} \]

where the so–called soft pomeron intercept is \( \epsilon_P \approx 0.08 \), but this is forbidden as a true asymptotic behaviour by the well–known Froissart–Lukaszuk–Martin theorem [2]. As we believe QCD to be the fundamental theory of strong interactions, it should predict the correct asymptotic behaviour; nevertheless, a satisfactory explanation is still lacking.

The problem of total cross sections is part of the more general problem of high–energy scattering at low transferred momentum, the so–called soft high–energy scattering. As soft high–energy processes possess two different energy scales, the total center–of–mass energy squared \( s \) and the transferred momentum squared \( t \), smaller than the typical energy scale of strong interactions (\( |t| \lesssim 1 \text{ GeV}^2 \ll s \)), we cannot fully rely on perturbation theory. A genuine non perturbative approach in the framework of QCD has been proposed by Nachtmann in [3], and further developed in [4,5,6,7,8]: using a functional integral approach, high–energy hadron–hadron elastic scattering amplitudes are shown to be governed by the correlation function of certain Wilson loops defined in Minkowski space. Moreover, as it has been shown in [9,10], such a correlation function can be reconstructed by analytic continuation from its Euclidean counterpart, i.e., the correlation function of two Euclidean Wilson loops, that can be calculated using the non perturbative methods of Euclidean Field Theory.

In [11] we have investigated this problem by means of numerical simulations in Lattice Gauge Theory (LGT). Although we cannot obtain an analytic expression in this way, nevertheless this is a first–principle approach that provides (inside the errors) the true QCD expectation for the relevant correlation function. In this contribution, after a quick survey of the non perturbative approach to soft high–energy scattering in the case of meson–meson elastic scattering, we will present our numerical approach based on LGT, and we will show how the numerical results can be compared to the existing analytic models.

2. Meson–meson elastic scattering amplitudes and Wilson–loop correlators

We sketch here the non perturbative approach to soft high–energy scattering; see [11] for a more detailed presentation. As it has been shown in [4,5,6], the elastic meson–meson scattering amplitude can be reconstructed from the scattering amplitude of two \( q \bar{q} \) colour dipoles, after averaging over the transverse sizes and the longitudinal momentum fractions of the dipoles. The central quantity in this approach is a certain (properly normalised) correlation function (in the sense of the QCD functional integral) of two Wilson loops in the fundamental representation, defined in Minkowski space–time, running along the paths made up of the quark and antiquark straight–line
classical trajectories and closed at proper times $\pm T$ by straight–line paths in the transverse plane.

In [910] (see also [1213]) it has been shown that, under certain analyticity hypotheses, this correlation function can be reconstructed from the Euclidean correlation function of two Euclidean Wilson loops, $\mathcal{W}_1$ and $\mathcal{W}_2$,

$$G_E(\theta; T; \vec{z}_1; 1, 2) \equiv \frac{\langle \mathcal{W}_1(T) \mathcal{W}_2(T) \rangle}{\langle \mathcal{W}_1(T) \rangle \langle \mathcal{W}_2(T) \rangle} - 1$$

(1)

(here “1” and “2” stand respectively for $\vec{R}_{1\perp}, f_1$ and $\vec{R}_{2\perp}, f_2$); the two loops run along two rectangular paths $\vec{C}_1$ and $\vec{C}_2$, made up of the “Euclidean trajectories” of the partons,

$$\vec{C}_1 : X_1^{a[q]}(\tau) = z + \frac{p_1}{m} e^{at} + f_1^{a[q]} R_{1E},$$

$$\vec{C}_2 : X_2^{a[q]}(\tau) = \frac{p_2}{m} e^{at} + f_2^{a[q]} R_{2E}$$

(2)

where $p_{1/2}|E| = m \left( - [\sin \frac{\theta}{2}, 0, \cos \frac{\theta}{2} \right)$, $R_{iE} = (0, \vec{R}_{i\perp}, 0)$, $f_i^a = 1 - f_i$, $f_i^\perp = - f_i$ ($i = 1, 2$), $z_E = (0, \vec{z}_1, 0)$, with $\vec{R}_{i\perp}$ and $f_i$ the transverse sizes and longitudinal momentum fractions of the two dipoles, and $\vec{z}_1$ the impact–parameter distance between the two loops in the transverse plane. The paths are closed at proper times $\pm T$ by straight–line paths in the transverse plane; here $T$ acts as an IR cutoff which has to be removed in the end. As the elastic scattering amplitude of two meson states is expected to be an IR–finite physical quantity [14], we expect the limit $T \to \infty$ to be finite, so that we can define $C_E(\theta; \vec{z}_1; 1, 2) \equiv G_E(\theta; T \to \infty; \vec{z}_1; 1, 2)$. Note that $G_E$ is a real function, as can be shown making use of the charge–conjugation invariance of the functional integral.

Finally, the meson–meson scattering amplitudes can be written as

$$M_{(hh)}(s, t; 1, 2) = -i \cdot 2s \int d^2 \vec{z}_1 e^{i\vec{q}_1 \cdot \vec{z}_1}$$

$$\times \int d\vec{d}_1 |\psi_1(1)|^2 \int d\vec{d}_2 |\psi_2(2)|^2$$

$$\times \mathcal{C}_M(\theta \to - i \log (s/m^2); \vec{z}_1; 1, 2);(3)$$

here $\vec{q}_1$ is the (transverse) transferred momentum ($t = -[\vec{q}_1]^2$), and $\psi_1$ and $\psi_2$ are the wave functions which describe the two interacting mesons.

Total cross sections are then recovered via the optical theorem.

In the following we will set for simplicity $f_1 = f_2 = 1/2$, which is known to be a good approximation for hadron–hadron interactions [17].

3. Wilson–loop correlators on the lattice

The gauge–invariant Wilson–loop correlation function $G_E$ is a natural candidate for a lattice computation, but some care has to be taken due to the explicit breaking of $O(4)$ invariance on a lattice. As straight lines on a lattice can be either parallel or orthogonal, we are forced to use off–axis Wilson loops to cover a significantly large set of angles. To stay as close as possible to the continuum case, the loop sides are evaluated on the lattice paths that minimise the distance from the true, continuum paths: this can be easily accomplished making use of the well–known Bresenham algorithm [15] to find the required “minimal distance paths” corresponding to the sides of the loops. The relevant Wilson loops $\tilde{W}_L(\vec{l}_i; \vec{r}_L; n)$ are then characterised by the position $n$ of their center and by two two–dimensional vectors $\vec{l}_i$ and $\vec{r}_i$, corresponding respectively to the longitudinal and transverse sides of the loop.

On the lattice we then define the correlator

$$G_L(\vec{l}_1; \vec{l}_2; \vec{d}_i; \vec{r}_{1\perp}, \vec{r}_{2\perp})$$

$$\equiv \frac{\langle \tilde{W}_L(\vec{l}_1); \vec{r}_{1\perp}; d \rangle \tilde{W}_L(\vec{l}_2); \vec{r}_{2\perp}; 0)\rangle}{\langle \tilde{W}_L(\vec{l}_1); \vec{r}_{1\perp}; d \rangle \langle \tilde{W}_L(\vec{l}_2); \vec{r}_{2\perp}; 0\rangle)$$

(4)

where $d = (0, \vec{d}_i; 0)$, $\vec{d}_i = (d_2, d_3)$, and moreover

$$C_L(\vec{l}_1; \vec{l}_2; \vec{d}_i; \vec{r}_{1\perp}, \vec{r}_{2\perp})$$

$$\equiv \lim_{L_1, L_2 \to \infty} G_L(\vec{l}_1; \vec{l}_2; \vec{d}_i; \vec{r}_{1\perp}, \vec{r}_{2\perp})$$

(5)

where $L_i \equiv |\vec{l}_i|$ are defined to be the lengths of the longitudinal sides of the loops in lattice units, and $\vec{L}_i \equiv L_i \vec{l}_i$. In the continuum limit, where $O(4)$ invariance is restored, we expect

$$C_L(\vec{l}_1; \vec{l}_2; \vec{d}_i; \vec{r}_{1\perp}, \vec{r}_{2\perp})$$

$$\approx C_E(\theta; a\vec{d}_i; a\vec{r}_{1\perp}, 1/2, a\vec{r}_{2\perp}, 1/2)$$

(6)

where $\vec{l}_1 \cdot \vec{l}_2 \equiv \cos \theta$ defines the relative angle $\theta$, and $a$ is the lattice spacing.
Figure 1. Lattice data plotted against $\theta$ for various lengths of the loops.

4. Numerical results

As already pointed out in the Introduction, numerical simulations cannot provide the analytic expression for the relevant correlation function, but nevertheless, as these simulations are first-principles calculations, they provide the “correct” (inside the errors) prediction of QCD. Approximate analytic calculations have then to be compared with the lattice data, in order to test the goodness of the approximations involved. In particular, we are interested in the dependence on the relative angle $\theta$, as it encodes the energy dependence of the scattering amplitudes, which is recovered after the proper analytic continuation.

In Fig. 1 we show, as an example, the lattice data for $G_L$ in the case of parallel transverse sides with $|\vec{r}_1| = |\vec{r}_2| = 1$ at $d = 0$, plotted against the angle $\theta$ for various lengths of the loops. These data are obtained using Wilson action for $SU(3)$ pure–gauge (quenched) theory, on a $16^4$ lattice at $\beta = 6.0$. The data are quite stable against variations of the lengths, so that we can take the largest–length data as a reasonable approximation of $C_L$.

In Figs. 2 and 3 we compare $C_L$ with the prediction of various models (the loop configuration is the same as in Fig. 1). While the Stochastic Vacuum Model (SVM) [16] provides a fully quantitative prediction, that can be directly compared with the data, the Instanton Liquid Model (ILM) [17] and the AdS/CFT correspondence [15] give only the qualitative dependence on the angle $\theta$, so that a comparison can be made by trying to fit the data with the given functional form. In Fig. 2 we show the SVM prediction, together with a best–fit with the SVM functional form; in Fig. 3 we show the best–fits with the expressions obtained in perturbation theory to lowest order [19,16,10], in the ILM and using the AdS/CFT correspondence. A detailed discussion of the results is given in [11]; here we simply note that the agreement of the numerical data with the various models is not fully satisfactory, and further investigations have to be made, both on the numerical and on the analytical side. We want also to remark that while perturbative effects...
seem to be dominant at short distances between the loops, non perturbative effects are already relevant at distances of about 0.2 fm; however, as the correlation function is rapidly decreasing with the distance \(d = |\vec{d}_\perp|\) between the centers of the loops (see Fig. 4), a detailed study at large distances is difficult, and requires the use of noise reduction techniques.

Lattice data show also the presence of odderon contributions to dipole–dipole scattering. Indeed, as explained in [12], making use of the crossing–symmetry relations for loops one can show that the crossing–odd component of the dipole–dipole scattering amplitudes is related via the usual analytic continuation to the antisymmetric (with respect to \(\pi/2\)) component \(C_{E}^-\) of \(C_E\): this quantity is shown in Fig. 5 together with the coresponding SVM prediction (the loop configuration is the same as in Fig. 4).

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