Anomalous electron states: sonoluminescence, neutrons, and energy

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Unexpected electron states in atom are proposed. The states are bound to the electrostatic field of atomic nucleus cut off on its size. For the stationary nucleus these states are singular and thus non-physical. In an acoustically driven liquid the macroscopic velocity, acquired by the atom in a collapsing gas bubble, cuts off the singularity. The state becomes physical with the binding energy in the MeV range. Electron transitions to this anomalous state provide the different source of sonoluminescence. In a solid under mechanical stress atoms jump with a finite velocity to neighbor positions due to defect motion, microcracks, etc. Electron transitions to the resulting anomalous state can directly activate nuclear deformation modes. The matrix element is enhanced because of the smeared singularity. This results in neutron emission under mechanical stress. Occupation of the anomalous state, associated with the lead nucleus, releases the energy of 32.5 MeV.

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I. INTRODUCTION

Properties of electron in the electrostatic field of atomic nucleus are described in textbooks [1, 2]. Solutions of the Dirac equation in harmonic potential are investigated in [3–6]. It seems unlikely to add something unusual to these fields.

The singular solution, which is \( \psi \sim 1/r \) at small \( r \), of the Schrödinger equation \((-\hbar^2\nabla^2/2m + U - E)\psi = 0\) does not exist even formally since it requires the non-existing source \( \delta(\vec{r}) \) in the right-hand side. Analogously the singular Coulomb potential should be supported by a point charge.

A quite different situation may be in relativistic quantum mechanics. The Dirac equation in the electrostatic nuclear field, cut off by the nuclear radius, can have singular solutions resulting in the divergent electron density \( 1/r^4 \) at small distance \( r \) from the nucleus center. These solutions of the Dirac equations really exist since the singularity is of algebraic origin. It does not come from a differential equation requiring an artificial source like \( \delta(\vec{r}) \). The states correspond to an electron binding energy in the MeV region due to smallness of nuclear radius. In frameworks of the Dirac quantum mechanics those solutions are non-physical because of the singularity.

But the situation becomes different, when the atom is a lattice site of a solid. Due to quantum fluctuations of this site the singularity takes various positions in space resulting in smoothness. The state thus becomes physical. This anomalous state is additional to the usual atomic ones and does not exist in its bare form. The state is assisted by a heavy cloud of phonons and spontaneous creation of this multi-phonon object has negligible probability in random processes in a solid.

However, there exists a real way to create anomalous states, when atoms in condensed matter move fast under the action of some macroscopic perturbations. In this case the non-physical singularity is also cut off. When the macroscopic motion dominates the natural background, the phonon cloud is a correction and thus the matrix element of transition to the anomalous state is negligible.

In a liquid a fast macroscopic motion of atoms can be due to the acoustically driven implosion of a gas bubble [7–9]. This can result in formation of anomalous states. As argued in the paper, the anomalous states provide the different source of sonoluminescence. The electromagnetic radiation, emitted under the electron transitions to anomalous levels, is converted into sonoluminescence.

In a solid a strong mechanical stress results in defect motion, microcracks, etc., when an atom jumps to a neighbor position with a large speed [10, 11]. The electron transitions to the associated anomalous level may directly activate collective nuclear modes. The matrix element is enhanced because of the smeared singularity. As a result, under macroscopic mechanical perturbation the neutron emission is possible. This looks paradoxical. However, the concept of anomalous states links those different issues. The neutron emission, caused by the mechanical stress, was reported in [12, 13].

Under occupation of the anomalous state in the lead nucleus the hidden energy of 32.5 MeV is released as shown in this paper. At sufficiently large atomic velocities nuclei of the sample are expected to undergo to anomalous state. This provides a different way of energy release. Besides the matter heating to overcome an energy barrier in fusion processes, one can apply a mechanical perturbation to get fast atoms.

II. DIRAC QUANTUM MECHANICS

One starts with the Dirac equation for electron in the standard representation, when the total bispinor consists of two spinors \( \Phi(\vec{r}, t) \) and \( \Theta(\vec{r}, t) \) [2]. The central potential well \( U(r) \) is supposed to satisfy the condition of harmonic oscillator \( U(r) \sim U(0) + U''(0)r^2/2 \) at \( r \rightarrow 0 \).
An atomic electron is acted by the nucleus electrostatic field produced by the electric charge $Ze$. The nuclear charge density is supposed to be spherically symmetric and homogeneously distributed within the sphere of the radius $r_N$ \[4\]. In this case

$$U(r) = \begin{cases} 
-Ze^2/r, & r \ll r_N \\
-3Ze^2/2r_N + \lambda r^2, & r \ll r_N,
\end{cases} \quad (1)$$

where $\lambda = Ze^2/2r_N^3$. The radiative correction to the Coulomb field (due to vacuum polarization) \((2e^2/3\pi\hbar)c\ln(0.24h/mc)\) is negligible at $r \sim r_N$. As shown below, short distances are mainly significant, whereas an influence of other atomic electrons is minor.

For the usual carbon isotope $^{12}\text{C}$ \((Z = 6)\) the nuclear radius is $r_N \approx 2.47 \times 10^{-13}\text{cm}$ and $U(0) \approx -3Ze^2/2r_N \approx -5.2\text{ MeV}$. For oxygen $^{16}\text{O}$ \((Z = 8)\) the nuclear radius is $r_N \approx 2.7 \times 10^{-13}\text{cm}$ and $U(0) \approx -6.3\text{ MeV}$. For iron $^{56}\text{Fe}$ \((Z = 26)\) the nuclear radius is $r_N \approx 3.73 \times 10^{-13}\text{cm}$ and $U(0) \approx -14.9\text{ MeV}$. For lead $^{207}\text{Pb}$ \((Z = 82)\) the nuclear radius is $r_N \approx 3.49 \times 10^{-13}\text{cm}$ and $U(0) \approx -32\text{ MeV}$. For thorium $^{228}\text{Th}$ \((Z = 90)\) the nuclear radius is $r_N \approx 5.75 \times 10^{-13}\text{cm}$ and $U(0) \approx -33\text{ MeV}$. This nucleus emits $\alpha$-particle with a half-life of 1.92 years.

The Dirac equations for two spinors have the form \[2\]

$$\begin{align*}
\left[ i\frac{\partial}{\partial t} - U(r) - m \right] \Phi_1(\vec{r}, t) &= -i\vec{\sigma} \nabla \Theta_1 \\
\left[ i\frac{\partial}{\partial t} - U(r) + m \right] \Theta_1(\vec{r}, t) &= -i\vec{\sigma} \nabla \Phi_1.
\end{align*} \quad (2)$$

Here $\vec{\sigma}$ is the Pauli matrix and $h = c = 1$. For the eigenfunction $\Phi_1(\vec{r}, t) = \Phi_2(\vec{r})\exp(-i\varepsilon t)$ and analogously $\Theta_1(\vec{r}, t)$

$$\begin{align*}
\varepsilon - U(r) \Phi_2 + i\vec{\sigma} \nabla \Theta_2 &= m \Phi_2 \\
\varepsilon - U(r) \Theta_2 + i\vec{\sigma} \nabla \Phi_2 &= -m \Theta_2.
\end{align*} \quad (4)$$

The spinor $\Theta_2(\vec{r})$ from \[5\] is expressed through $\Phi_2(\vec{r})$

$$\Theta_2 = \frac{-i\vec{\sigma} \nabla \Phi_2}{\varepsilon - U(r) + m} \quad (6)$$

and substituted into \[4\]. The result is

$$\begin{align*}
-\nabla^2 \Phi_2 - \frac{\nabla U}{\varepsilon - U(r)} \nabla \Phi_2 - i\vec{\sigma} \times \nabla \Phi_2 \\
&+ m^2 \Phi_2 = (\varepsilon - U(r))^2 \Phi_2.
\end{align*} \quad (7)$$

The form \[7\] can be conveniently used for obtaining non-relativistic limit when the energies $E = \varepsilon - m$ and $U(r)$ are small compared to $m$. In this case the term with $\nabla U$ is small \((\sim 1/c^2\) in the physical units) and Eq. \[7\] turns into the conventional Schrödinger equation for the spinor function $\Phi_2$ \[1\]

$$-\frac{1}{2m} \nabla^2 \Phi_2 + U(r) \Phi_2 = E \Phi_2. \quad (8)$$

## A. Singularity

### 1. Basic statements

There is another way to reduce Eqs. \[4\] and \[5\] to an equation for one spinor. One should express $\Phi$ from \[1\] and substitute into Eq. \[5\]. This way an unusual feature of the solution is revealed. It follows that

$$\Phi_2(\vec{r}) = -\frac{i\vec{\sigma} \nabla \Theta_2(\vec{r})}{\varepsilon - U(r) - m} \quad (9)$$

and the equation for the spinor $\Theta_2$, if to introduce the function $q(r) = \varepsilon - U(r) - m$, is

$$-\nabla^2 \Theta_2 + \frac{\nabla q}{q} \left( \nabla \Theta_2 - i\vec{\sigma} \times \nabla \Theta_2 \right) + m^2 \Theta_2 = (\varepsilon - U)^2 \Theta_2. \quad (10)$$

The spinor $\Theta_2$ is supposed to be isotropic. Since $U(r)$ is also isotropic, there is no term $\vec{\sigma} \times \nabla \Theta$ in \[10\] and this equation takes the form

$$-\frac{q}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Theta_2}{\partial r} \right) + m^2 \Theta_2 = (\varepsilon - U)^2 \Theta_2. \quad (11)$$

At $\varepsilon = \varepsilon_b$, where $\varepsilon_b = U(0) + m$, at small $r$ the function $q(r) \approx -\lambda r^2$. On these distances it can be two solutions of Eq. \[11\], $\Theta_{\pm} \sim 1$ and $\Theta_{\mp} \sim r$. On large distances these solutions turn into two waves $\exp(\pm ir\sqrt{\varepsilon_b - m^2})$. That is the solution $\Theta_{\pm}(r)$ is not singular at any $r$. Contrary, $\Theta_{\mp} \sim 1/r^2$ is singular as follows from \[9\]. Thus there exists a formal singular solution of the Dirac equation. This is an exact result. Note that the singular solution $1/r$ of the equation $\nabla^2 1/r = 0$ does not exist even formally since it requires the non-existing source $4\pi\delta(\vec{r})$ in the right-hand side. In contrast, in our case the singularity is of algebraic origin.

### 2. Detailed solutions

Above statements are detailed below. At $r \ll r_N$ one can use the approximation $q(r) \simeq -U''(0)(r^2 - r_0^2)/2$, where the classical turning point $r_0$ is determined by $r_0^2 = 2(\varepsilon - \varepsilon_b)/U''(0)$. The energy $\varepsilon$ is well below $-m$.

Suppose the energy $\varepsilon$ to be very close to $\varepsilon_b$ so that $r_0$ is much less than $r_N$. In Eq. \[11\] there is the singularity at $r = r_0$. At $(r - r_0) \ll r_0$ it should be $\partial \Theta_2/\partial r \sim (r - r_0)$ to compensate this singularity. As follows from \[9\] and the estimate of terms in \[11\] at $r \ll r_N$,

$$\Phi_2 = \frac{2im}{3} (\vec{\sigma} \vec{r}) c_0, \quad (12)$$

$$\Theta_2 = \left[ 1 + \frac{mU''(0)}{3} \int_0^r (r_1^2 - r_0^2) dr_1 \right] c_0. \quad (13)$$

where $c_0$ is a constant spinor. At large distance there are two waves $\exp(\pm ir\sqrt{\varepsilon_b - m^2}/r)$. The form, with the
asymptotic (12) and (13), is a part of the solution corresponding to the usual continuous spectrum at \( \varepsilon < -m \).

Eq. (11) has another solution besides (13). At small distances it can be found from the condition \((r^2/q)\partial \Theta_\varepsilon / \partial r = \text{const.}\) When \( r_0 \ll r_N \), it reads

\[
\frac{\partial \Theta_\varepsilon}{\partial r} = \frac{r^2 - r_0^2}{r} c_b, \tag{14}
\]

where \( c_b \) is a constant spinor. If \( \varepsilon \neq \varepsilon_b \) (that is \( r_0 \neq 0 \)), the part \( r^2/r^2_0 \) does not exist even formally. As mentioned above, it requires the non-existing \( \delta \)-source as in the equation \( \nabla^2 1/r = 4\pi \delta(\vec{r}) \). This is similar to electrodynamics, when a point charge supports the singular Coulomb potential.

Therefore, besides the usual continuous spectrum at \( \varepsilon < -m \), there exists the separate state with the energy \( \varepsilon_b \). This state is not physical due to the singularity \( \Phi_{\varepsilon_b} \sim 1/r^2 \) but nevertheless it is a formal solution of the Dirac equation. This is true since that singularity is of algebraic origin (denominator in (9)) but not due to a direct solution of a differential equation.

Two Dirac spinors have the form

\[
\Phi(\vec{r}, t) = -\frac{\text{i}(\vec{\sigma}\vec{r})w(r)}{r(U(0) - U(r))} \exp(-\text{i}t\varepsilon_b), \tag{15}
\]

\[
\Theta(\vec{r}, t) = w(r) \exp(-\text{i}t\varepsilon_b), \tag{16}
\]

where \( w(r) = \Theta_{\varepsilon_b}(r) \). The differential equation for \( w(r) \) follows from (11)

\[-\frac{\partial}{\partial r} \left( \frac{r^2}{U(0) - U(r)} \frac{\partial w}{\partial r} \right) = r^2 \left[ 2m + U(0) - U(r) \right] w. \tag{17}\]

One can show from (17), after a little algebra, that on short distances the total solution of (17) consists of two independent parts expanded in even and odd powers of \( r \)

\[
w(r) = \left[ 1 + \frac{mU''(0)}{12} r^4 + \ldots \right] c_0 + r \left( 1 - \frac{r^2}{4r_N^2} + \ldots \right) c_b. \tag{18}\]

On large distances there are two solutions \( \sin(r\sqrt{\varepsilon_b^2 - m^2}/r) \) and \( \cos(r\sqrt{\varepsilon_b^2 - m^2}/r) \). Here the Coulomb phases [2] proportional, in physical units, to

\[\frac{Ze^2}{\hbar c} \int_0^r \frac{dr_1}{\sqrt{r_1^2 + r_N^2}}\]  

are omitted.

The part with \( c_0 \) is the usual state of the continuous spectrum with the energy \( \varepsilon_b \). This part coincides with (13) at \( r_0 = 0 \). The part with the spinor \( c_b \),

\[
c_b = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{20}\]

corresponds to (14) at \( r_0 = 0 \).

Eq. (17) have two solutions which are symmetric and antisymmetric with respect to the formal transformation \( r \rightarrow -r \). At small \( r \) they are two parts in (18). That symmetry property also takes place at large \( r \). According to this, it follows from (15) and (10) for the state \( b \)

\[
\Phi_{\varepsilon_b} = c_b \frac{i\vec{\sigma}\vec{r} r_3^3}{r^3 Ze^2} \left\{ 2, \begin{array}{l} r \ll r_N \\ r_N \ll r \end{array} \right\} \tag{21}
\]

\[
\Theta_{\varepsilon_b} = c_b \begin{pmatrix} r, \\ \tilde{\alpha}(r_N^2/r) \cos rp_b, \end{pmatrix} \begin{array}{l} r \ll r_N \\ r_N \ll r \end{array}, \tag{22}\]

where \( \varepsilon_b = U(0) + m, \ p_b = \sqrt{U(0)[2m + U(0)]}, \) and \( c_b \) is a constant spinor to be chosen from a normalization condition. Note that \( r \rightarrow -r \) symmetry does not hold for usual wave functions for the Coulomb potential [2].

The numerical parameter \( \tilde{\alpha} \sim 1 \) is determined by exact solution of (17) to match two asymptotics. The level \( \varepsilon_b = U(0) + m \) corresponds to the singular function \( \Phi_b \). This is electron-like level since it joins the set of levels around \( \varepsilon = m \) under an adiabatic reduction of the potential \( U(r) \).

There is an interpretation of the solution of Eq. (10). At small \( r \), \( \Theta_{\varepsilon_b} = c_b r \) and that equation is of the Poisson type

\[\nabla^2 \Theta_{\varepsilon_b} = \frac{2c_b}{r}\]  

with the “electric charge” density proportional to \( 1/r \). The solution of (23) for “electrostatic potential” is self consistent, \( \Theta_{\varepsilon_b} = c_b r \), and no artificial source is required.

Electrostatic analogy of the expression (21) at short distance is the dipole potential \( \varphi = (\vec{\sigma}\vec{r})/r^3 \), where \( \vec{\sigma} \) is a usual vector of dipole moment. The projection of the electric field on the direction \( \vec{\sigma} \), proportional to

\[\vec{\sigma}\nabla\varphi = \frac{\sigma^2}{r^3} - \frac{3(\vec{\sigma}\vec{r})(\vec{\sigma}\vec{r})}{r^5}, \tag{24}\]

is not zero. But that expression is zero if \( \vec{\sigma} \) is the Pauli matrix [1]. The form (24) is analogous to the term \( i\vec{\sigma}\nabla\Phi_\varepsilon \) in Eq. (5).

Eqs. (6) and (9) are of the same type differing by signs of mass. One can say the same about the pair (14) and (10). Thus one can apply the above formalism to Eqs. (9) and (7). This allows to conclude about the identical state but with the energy \( \varepsilon_a = U(0) - m \) and the wave vector \( p_a = \sqrt{U(0)[-2m + U(0)]} \). For the state \( a \)

\[
\Phi_{\varepsilon_a} = c_a \begin{pmatrix} r, \\ \tilde{\alpha}(r_N^2/r) \cos rp_a, \end{pmatrix} \begin{array}{l} r \ll r_N \\ r_N \ll r \end{array}, \tag{25}\]

\[
\Theta_{\varepsilon_a} = c_a \frac{i\vec{\sigma}\vec{r} r_3^3}{r^3 Ze^2} \left\{ 2, \begin{array}{l} r \ll r_N \\ r_N \ll r \end{array} \right\} \tag{26}\]

The level \( \varepsilon_a = U(0) - m \) corresponds to the singular function \( \Theta_a \). The levels \( b \) and \( a \), additional to the conventional continuous spectrum, are shown in Fig. [1].
The obtained forms exist as formal mathematical solutions of the Dirac equation. Due to the singularities they are non-physical and should be disregarded if we remain in frameworks of Dirac quantum mechanics. 

Particle density is given by the expression \( n = \psi \ast \psi = \Phi \ast \Phi + \Theta \ast \Theta \). \( (27) \)

In Dirac quantum mechanics \( \Phi \) particle current is

\[ j = \psi \ast \gamma^0 \overline{\psi} = \Phi \ast \partial \Theta + \Theta \ast \partial \Phi = 2 \text{Re}(\Theta \ast \partial \Phi). \] \( (28) \)

One can rotate the space to get one component of the spinor \( \Phi \) to be zero. Each state, \( a \) or \( b \), is degenerated double with \( j = 1/2, m = 1/2 \) \( (c_2 = 0) \) and \( j = 1/2, m = -1/2 \) \( (c_1 = 0) \). Here \( j \) is a quantum number of the total angular momentum.

### B. Different types of nuclear potential

The condition of isotropic potential \( U(r) \) is not a crucial aspect. When \( U(r) - U(0) \propto \alpha x^2 + \beta y^2 + z^2 \) close to the minimum of \( U(r) \), the spinor

\[ \Theta_{\pm \epsilon} = r \left[ a(\theta, \phi) + ib(\theta, \phi)\sigma \right] + \ldots \] \( (29) \)

is also expanded in odd powers of \( r \) as in \( \Theta \). Forms of the functions \( a(\theta, \phi) \) \( b(\theta, \phi) \) follow from \( \Theta \). As in the isotropic case, the spinor \( \Theta_{\pm \epsilon} \) is smooth but \( \Theta_{\pm \epsilon} \sim 1/[U(0) - U(r)] \) is also proportional to \( 1/r^2 \). The energy \( \epsilon_\pm \) has the same form as above. In the isotropic case \( (\alpha = \beta = 1) \) \( a = 1 \) and \( b = 0 \) as in Eq. \( \Theta \).

For a model of the Dirac harmonic oscillator \( U(r) = m\Omega^2 r^2/2 \) \( \Theta \) the results of Sec. \( \text{IIA} \) are also valid. In this case \( \epsilon_{b,a} = \pm m \). The singular state \( \epsilon_b \), at the bottom of the well, is additional to the usual discrete levels of the harmonic oscillator.

When the nucleus is proton, the nuclear charge density \( \rho(r) \) is linear at small \( r \) \( \Theta \) and hence the nuclear electrostatic potential satisfies the condition \( |U(r) - U(0)| \sim r^3 \) at small \( r \). Eqs. \( \text{[15]} - \text{[17]} \) are valid for this situation. Analogously to \( \text{[18]} \), at \( r \ll r_N \) two solutions of \( \text{[17]} \) are

\[ w(r) = \left[ 1 + \frac{mU^{(0)}(r)}{45} r^5 + \ldots \right] c_4 + r^2 (1 + \ldots) c_5. \] \( (30) \)

The term with \( c_5 \) leads to \( \Phi \sim 1/\epsilon \) \( \text{[15]} \) as before. At \( r_N \ll r \) the solution is \( \Theta \) but with a different phase.

One can conclude that the singular solution, proportional to \( 1/r^2 \), of the Dirac equation exists in a nucleus with a real distribution of charge density.

### III. FULL SET OF SINGULAR STATES

For the central potential \( U(r) \) one can easily reformulate the problem \( \text{[11]} \) and \( \text{[5]} \) in terms of spherical spinors \( \Phi \). In this method

\[ \Phi_\epsilon = f(r)\Omega_{jlm}, \quad \Theta_\epsilon = (-1)^{(1+l+1')/2}g(r)\Omega_{j'l'm}, \] \( (31) \)

where \( l = j \pm 1/2 \) and \( l' = 2j - l \). The spherical spinors are expressed through spherical harmonics \( Y_{lm}(\theta, \phi) \) \( \text{[1]} \)

\[ \Omega_{l+1/2,l,m} = \frac{1}{\sqrt{2j+1}} \left\{ \sqrt{j+m+1} Y_{l,m-1/2} - \sqrt{j-m+1} Y_{l,m+1/2} \right\} \] \( (32) \)

\[ \Omega_{l-1/2,l,m} = \frac{1}{\sqrt{2j+1}} \left\{ -\sqrt{j-m+1} Y_{l,m-1/2} + \sqrt{j+m+1} Y_{l,m+1/2} \right\} \] \( (33) \)

The functions in Eqs. \( \text{[31]} \) satisfy the equations \( \text{[2]} \)

\[ [\epsilon - U(r) - m] f + g' + \frac{1 - \kappa}{r} g = 0 \] \( (34) \)

\[ [\epsilon - U(r) + m] g - f' - \frac{1 + \kappa}{r} f = 0 \] \( (35) \)

where

\[ \kappa = \begin{cases} -(l + 1), & j = l + 1/2 \\ l, & j = l - 1/2 \end{cases} \] \( (36) \)

Let us consider states with the energy \( \epsilon_b \) (Sec. \( \text{IIA} \)). In this case Eqs. \( \text{[34]} \) and \( \text{[55]} \) take the forms

\[ g(r)f + g' + \frac{1 - \kappa}{r} g = 0, \] \( (37) \)

\[ 2m + [g(r)] f - f' - \frac{1 + \kappa}{r} f = 0, \] \( (38) \)

where the function \( g(r) \) \( \sim -\lambda r^2 \) at small \( r \) (Sec. \( \text{IIA} \)). In the states with \( \epsilon_b \) the function \( g(r) \) is less singular at \( r \to 0 \) compared to \( f(r) \). Thus the last two singular terms in \( \text{[38]} \) have to compensate each other. This is possible, when at \( r \to 0 \)

\[ f(r) = \frac{1}{r^{3/2+j}}, \quad g(r) = \frac{\lambda}{2 - 2j} r^3 f. \] \( (39) \)
This singular function $f(r)$ corresponds to the case $\kappa = l = j + 1/2$ in (39). That is for a given $j$ one should use in Eqs. (31) $l = j + 1/2$ and $l' = j - 1/2$. This set $b$ relates to the energy $\varepsilon = \varepsilon_b$. Analogously one can consider the set $a$ with $\varepsilon = \varepsilon_a$. In this case $\kappa = -l - 1$, $l = j - 1/2$, and $l' = j + 1/2$ result in $f \sim r^3 g$ and $g \sim r^{-3/2-j}$.

At large $r$ the states with $j > 1/2$ behave similar to asymptotics (21) - (22) and (25) - (26).

The states, studied in Sec. II A, relate to $j = 1/2$. One can easily show from Eqs. (31) - (33) that, for the state $b$, at small $r$

$$\Psi_b(r) = \frac{i(\vec{\sigma} \vec{r})}{r} \Theta_b(r) \lambda^{1/2}, \quad \Theta_b(r) = r c_b(m). \quad (40)$$

This is equivalent to (13) and (16) and the expansion (18). The spinor $c_b(m)$ has the form

$$c_b(1/2) = -\frac{\lambda}{\sqrt{4\pi}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c_b(-1/2) = -\frac{\lambda}{\sqrt{4\pi}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (41)$$

Thus the formalism of Sec. II A and Sec. III (for $j = 1/2$) lead to the same results.

Eqs. (31) - (33) define the full set, consisting of the sets $a$ ($2j + 1$ states) and $b$ ($2j + 1$ states), of singular solutions of the Dirac equations. The orbital quantum number could be $j = 1/2, 3/2, \ldots$. As in the particular case of $j = 1/2$ in Sec. II A the singular solutions at $1/2 < j$ also formally exist and are non-physical due to the singularity.

**IV. ANOMALOUS STATES**

The states in the Coulomb field $U(r)$ of a nucleus, obtained in Sec. II A, are singular and thus non-physical. But the situation can be different, when the nucleus is a lattice site in a solid. In this case the electron is acted by the potential $U(|\vec{r} - \vec{a}|)$, where $\vec{a}$ is a displacement of the lattice site from the equilibrium position. The quantum fluctuation of $\vec{a}$ occur with the mean squared displacement $\langle a^2 \rangle \frac{2 \hbar}{c m} \lambda^{10/3}$. As the result, the electron wave function becomes a continuous superposition of singularities at various positions in space. The related state is not singular and therefore physical. This state can be qualified as anomalous.

The positional fluctuations of a lattice site in a solid occur with the typical frequency $\omega_D \sim 10^{13} s^{-1}$. They are very adiabatic compared to the electron motion, $\hbar \omega_D / \varepsilon_b \sim 10^{-8}$. For this reason, one can consider the Born-Oppenheimer rule, when an electron distribution is mainly determined by an instant value of a nucleus position. See also [16, 17]. In the totally adiabatic approximation ($\vec{a} = 0$) the electron density would be written in the form

$$n(\vec{r}) = \int d^3 u \left[ |\Phi(\vec{r} - \vec{u})|^2 + |\Theta(\vec{r} - \vec{u})|^2 \right] K(u), \quad (42)$$

where $\varepsilon = \varepsilon_{a,b}$. The certain function $K(u) \sim \exp(-3u^2/2u^2)$ has the property $\int d^3 u K(u) = 1$. It is accounted here that $\langle u_i^2 \rangle = (u^2)/3 (i = 1, 2, 3)$.

The first defect of the adiabatic approximation (42) is divergence of the integration at $\vec{u} = \vec{R}$ (compare with (39)). The second defect of (42) is exhibited at large distance $r \gg \sqrt{\langle u^2 \rangle}$. In this case, as follows from (12), (21) and (22),

$$n(\vec{r}) \sim |\Phi(\vec{r})|^2 + |\Theta(\vec{r})|^2 \sim \frac{1}{r^2}. \quad (43)$$

This density cannot be normalized. To eliminate those defects one should include non-adiabatic effects.

In quantum electrodynamics (QED) the electron propagator can be obtained from one for a macroscopic electromagnetic field. One has to average this propagator on the macroscopic field with the certain weight function $\frac{S}{i\varepsilon}$. In non-relativistic quantum mechanics, according to Feynman [19], this weight function is exp $(iS/\hbar t)$, where $S$ is the classical action for the macroscopic field.

In our case of lattice quantum vibrations also one can perform calculations for a macroscopic displacement $\vec{u}(t)$ and then average on it. Thus one should study first the system with the macroscopic displacement $\vec{u}(t)$.

Since now in the Dirac equations (42) and (43) the potential is of the form $U(|\vec{r} - \vec{a}(t)|)$ it is convenient to present the wave function as $\Phi(\vec{r} - \vec{a}(t), t)\exp(-i\varepsilon \vec{u} \cdot \vec{R})$ and analogously the function $\Theta(\vec{r} - \vec{a}(t), t)\exp(-i\varepsilon \vec{u} \cdot \vec{R})$ (the state $b$ is considered). The equations, analogous to (2) and (3) are

$$i\frac{\partial}{\partial t} + \varepsilon_b - U(R) - m - \vec{i} \vec{u} \vec{\nabla} \Phi(\vec{R}, t) = -i\vec{\sigma} \vec{\nabla} \Theta(\vec{R}, t) \quad (44)$$

$$i\frac{\partial}{\partial t} + \varepsilon_b - U(R) + m + \vec{i} \vec{u} \vec{\nabla} \Theta(\vec{R}, t) = -i\vec{\sigma} \vec{\nabla} \Phi(\vec{R}, t) \quad (45)$$

where $\vec{\nabla} = \partial / \partial \vec{R}$. In study of electron-acoustic interaction in solids the new coordinate $\vec{R} = \vec{r} - \vec{u}(t)$ is analogous to a transition to the co-system, where a crystal lattice is undisturbed [20].

**A. Spatial distribution of singularities**

In the adiabatic limit ($\vec{u} = 0$) Eqs. (44) and (45) lead to the same singular wave functions as in Sec. II A but depending on $\vec{R}$ instead of $\vec{r}$. A non-zero $\vec{u}$ cannot be accounted for by perturbation theory since in the term $\vec{u} \vec{\nabla} \Phi$ the initial singularity $\Phi \sim 1/R^2$ is enhanced. It follows that one should exactly account for the non-adiabatic term $\vec{u}(t)$, when all orders of perturbation theory work.

That is the number of participating phonons $N$ is large and the typical frequency $N\omega_D$ is much larger than the characteristic frequency $\omega_D$ of $\vec{u}(t)$. For this reason, in Eqs. (44) and (45) one can treat $\vec{u}(t)$ as an “instant”
parameter at each $t$. With the new functions, introduced by the relations

$$\Phi(\vec{R}, t) \simeq F(\vec{R}) \exp \left[ -i \int^t dt_1 \Delta \varepsilon(t_1) \right]$$

$$\Theta(\vec{R}, t) \simeq G(\vec{R}) \exp \left[ -i \int^t dt_1 \Delta \varepsilon(t_1) \right],$$

(Eqs. (44) and (45) are reduced to

$$\left[ \Delta \varepsilon(t) + q(\vec{R}) - i \vec{u}(t) \vec{\nabla} \right] F = -i \vec{\sigma} \nabla G$$

(47)

$$\left[ \Delta \varepsilon(t) + 2m + q(\vec{R}) - i \vec{u}(t) \vec{\nabla} \right] G = -i \vec{\sigma} \nabla F$$

(48)

where $t$ is just a parameter and $q(\vec{R}) = -\lambda R^2$ is determined in Sec. II A. Below the quantum number $j = 1/2$ is considered. In this case the right-hand side of (47) is not singular. When $\vec{u}(t) = 0$ and therefore $\Delta \varepsilon = 0$, the part of (47) in square brackets is proportional to $R^2$ recovering the singularity $\Phi \sim 1/R^2$ of Sec. II A. Thus one has to pay attention to the left-hand side of (47).

In terms of macroscopic displacement $\vec{u}(t)$ the left-hand side in (47) turns to zero at different complex $R$, when time is different. In the language of average on quantum fluctuations of the displacement this means that the singularity is smeared out within the certain spatial domain. The size of this domain $R \sim \delta$ and the energy uncertainty $\Delta \varepsilon$ can be estimated from the condition, when three terms in the left-hand side of (47) are on the same order of magnitude, $\Delta \varepsilon \sim \lambda R^2 \sim v/R$. Here the velocity $v$ is given by $\langle A, 10 \rangle$. It follows for the radius of the singularity cut off

$$\delta \sim r_N \left( \frac{\hbar v}{Ze^2} \right)^{1/3} \sim 10^{-2} \frac{r_N}{Z^{1/3}}$$

(49)

for and the uncertainty of the electron energy

$$\Delta \varepsilon \sim |\varepsilon_b| \left( \frac{\hbar v}{Ze^2} \right)^{2/3} \sim 100 Z^{1/3} eV.$$  

(50)

The results (49) and (50) are non-perturbative with respect to velocity $\vec{u} \sim v$. This occurs since the electron-phonon coupling, $\vec{u}/R$ in (47), is large close to the singularity. The above study, conducted for the state $b$, also relates to the state $a$. The mechanism of singularity cut off works also for states with quantum numbers $j > 1/2$, when the singularity is stronger as follows from $\langle A, 90 \rangle$.

The value of $\vec{u}(t)$ is weakly time-dependent ($t \sim 1/\omega_D \sim 10^{-13}s$) and therefore the electron system acquires low frequency (phonon) harmonics. This results in slow energy leakage, with the typical time $1/\omega_D$, to the phonon system. The successive quanta absorption excites the lattice ground state (the limit of zero temperature is considered) up to the phonon energy, which is on the order of $\Delta \varepsilon$.

The anomalous electron, bound to the hosting atom, becomes accompanied by a cloud consisting of $N \sim \Delta \varepsilon/\hbar \omega_D \sim 10^4$ virtual phonons. That object can be referred to as a non-conventional polaron. It corresponds to strong electron-phonon coupling and can exist even in metals. The associated phonon cloud, containing the big number of phonons ($\sim 10^4$), remains virtual that is with no macroscopic displacement. This contrast to the conventional strong-coupling polaron $\langle 10, 11, 21, 28 \rangle$.

The exact energy $\varepsilon^R_b$ of the anomalous state is determined by electron interaction with the lattice ground state and the excited phonons, so that $(\varepsilon^R_b - \varepsilon_b) \sim \Delta \varepsilon \sim 100 eV$. The positions of the exact energy levels $\varepsilon^R_{a,b}$ hardly differ from ones in Fig. 1. The anomalous levels $a$ and $b$ are additional to the usual set of electron levels and they are not mixed with usual ones of the same energy. The proper matrix element is negligible because of the multi-phonon component of the anomalous states.

As one can see, accounting for non-adiabatic effects (connected to $\vec{u}$) eliminates the first defect of Eq. (42) related to the divergence at $\vec{r} = \vec{u}$. Now at small $(\vec{r} - \vec{u})$ one has to do the substitution in the expression (42)

$$|\Phi_{\varepsilon_b}(\vec{r} - \vec{u})|^2 \rightarrow \frac{|C_b|^2}{p_b(\vec{r} - \vec{u})^2 + \delta^2}.$$  

(51)

This formula accounts for the cut off radius $\delta$ in Eq. (21). The dimensionless constant spinor $C_b$ is to be determined.

### B. Long distance from the nucleus

On large distance from the nucleus in Eqs. (44) and (45) one can put $U(\vec{R}) = 0$. In those equations $i \partial/\partial t \sim \Delta \varepsilon$ is much larger than $\vec{u} \nabla \sim v p_b \sim |\varepsilon_b| v/c$ Thus one can drop the terms $i \vec{u} \nabla$ in (44) and (45). The electron becomes free, when

$$\Phi(\vec{R}, t) = \int \frac{d^3 \vec{P}}{(2\pi)^3} \varphi(\vec{p}) \exp \left( i p R + it \varepsilon_b - it \sqrt{p^2 + m^2} \right).$$

(52)

Here $\varepsilon_b = \sqrt{p_b^2 + m^2}$. The weight function $\varphi(\vec{p})$ provides the momentum distribution according to the restriction

$$\left( \sqrt{p^2 + m^2} - \varepsilon_b \right) \sim \Delta \varepsilon.$$  

This energy uncertainty follows from the condition at small $R$ (50). Due to the momentum uncertainty $\Delta p \sim \Delta \varepsilon$ the superposition of the functions $\sin \left( \vec{R} p_b + \vec{\nabla} \right) / R$ exponentially decays at $R > l \sim 1/\Delta p$, where (in physical units)

$$l = \frac{\hbar c}{\Delta \varepsilon} \sim r_N \left( \frac{c}{v} \right)^{2/3} \left( \frac{\hbar c}{Ze^2} \right)^{1/3} \sim 10^{-9} \left( \frac{\hbar c}{Ze^2} \right)^{1/3} \text{cm}.$$  

(53)

It is accounted that the nuclear radius is $r_N \sim 10^{-13} \text{cm}$ and $v/c \sim 10^{-6}$. It follows from here and (21)

$$\Phi_{\varepsilon_b} \sim \frac{\delta R^3}{\Delta \varepsilon} C_b p_b^{3/2} \left\{ \begin{array}{ll} 1/p_b^2 (R^2 + \delta^2), & R < 1/p_b \\ \sin R p_b / R p_b, & 1/p_b < R < l \end{array} \right.$$  

(54)
In the second limiting case the $\Delta p$-uncertainty of momentum is not essential. Ignoring the small energy uncertainty $\Delta \varepsilon$, the index $\varepsilon_b$ is ascribed to the function $\Phi_{\varepsilon_b}$ indicating that it belongs to the state $b$.

Since the argument $(\vec{r} - \vec{u})$ in (42) is not larger than $l$, the particle density can be normalized as it decays on the distance $l$. In this way, non-adiabatic effects, related to $\vec{u}$, eliminate the second defect of the expression (42) resulting in the tail (43). The lengths $l$ (53) and $\sqrt{\langle u^2 \rangle}$ (4.7) do not significantly differ and thus the particle density (42) is localized on that scale. The property

$$\int d^3 r \left[ |\Phi_{\varepsilon_b}(\vec{r} - \vec{u})|^2 + |\Theta_{\varepsilon_b}(\vec{r} - \vec{u})|^2 \right] = 1$$

holds due to the contribution of the length $R \sim l$. It follows from here that the spinor

$$C_b \sim \left( \frac{v}{\gamma} \right)^{1/3} \left( \frac{\hbar c}{Ze^2} \right)^{1/3}.$$  (56)

For the state $a$ the coefficient $C_a$ has the same evaluation.

### C. Anomalous states

The anomalous states, discussed above, are associated with strong coupling of the electron and phonons. This could be classified as non-conventional polaron. The anomalous state is smooth (cut off on the radius $\delta$) and thus physical. This state does not exist in its bare form. It is an electron bound to the nucleus and heavily dressed by the phonon cloud. The anomalous state unifies the singular quantum mechanical states and the polaron concept.

As one can see, for generation of the anomalous state one should create the polaronic cloud of virtual phonons dressing the electron. The background of thermal phonons in a solid does not destroy anomalous states. The anomalous state at zero temperature has the binding energy in the $MeV$ range. Finite temperature cannot violate such deep level.

The multi-phonon process, with the phonon number $N \sim \Delta \varepsilon/\hbar \omega_D \sim 10^4$, is the reason of the exponentially small probability of spontaneous creation of the anomalous state. Indeed, if the probability to create one phonon is $w$, then the probability of production of the sequence of $N \sim \Delta \varepsilon/\hbar \omega_D \sim 10^4$ phonons is estimated as $w^N = \exp(-B)$, where $B = N \ln(1/w) \sim 10^4$ (see [29] and references therein). The number exp$(−10^4)$ does not exist in nature. That is spontaneous creation of those electron-phonon states is impossible. However, there exists a way to artificially create anomalous states.

### V. MACROSCOPIC MOTION OF ATOMS

Suppose a solid or a liquid to be influenced by an acoustic or mechanical pulse. Under its action a particular atom moves with the macroscopic displacement $\vec{u}(t)$. The total displacement of that atom is $\vec{U}(t) + \vec{u}(t)$, where the fluctuating part $u(t) \sim 10^{-9} cm$ ($(\vec{u}) = 0$) is a matter of average on the phonon field as in Sec. [IV]. See also Appendix. The total displacement has to be put in Eqs. (17) and (48) instead of pure fluctuating part $\vec{u}$.

When the macroscopic velocity $\vec{U}$ of the nucleus is zero or smaller than the natural background $\sqrt{\langle \vec{u}^2 \rangle} \sim 10^5 cm/s$, the anomalous states exist but they are accompanied by the phonon cloud (Sec. [IV.C]). In this case the probability of population of those states is negligible. In the opposite limit of large $\xi$ the natural background plays secondary role. It slightly renormalizes the $\xi$-dominated anomalous state and formation of anomalous states is expected at every nucleus of the matter. In the intermediate case ($\xi \sim 10^5 cm/s$) only a fraction of all atomic nuclei participates.

When the external perturbation is strong enough, the quantum mechanical approach is applicable with time dependent wave functions. The adiabatic shift $\Phi(\vec{r} - \vec{u})$ is substituted by $\Phi(\vec{r} - \vec{\xi})$. The macroscopic displacement $\vec{\xi}(t)$ is also adiabatic compared to electron degrees of freedom.

The radius of the singularity cut off and energy shift of the anomalous state become time-dependent. They are given by the same expressions (19) and (14), where one has make the substitution $v \rightarrow \vec{\xi}(t)$

$$\delta(t) \sim r_N \left[ \frac{\hbar \vec{\xi}(t)}{Ze^2} \right]^{1/3}, \quad \Delta \varepsilon(t) \sim |\varepsilon_b| \left[ \frac{\hbar \vec{\xi}(t)}{Ze^2} \right]^{2/3}. \quad (57)$$

At every moment of time the state is a superposition of ones with energies in the interval $\Delta \varepsilon(t)$. The parameters (57) adiabatically depend on $t$ and make sense at those moments of time, when $\vec{\xi}(t)$ exceeds $\sqrt{\langle \vec{u}^2 \rangle} \sim 10^5 cm/s$. Otherwise the anomalous state turns into phonon-mediated one studied in Sec. [IV]

Acoustic or mechanical pulses, acting on solid or liquid, can be shock waves, artificial stress, etc. A strong mechanical stress in a solid results in a defect motion or microcracks, when jumping atoms acquire the velocity of $\sim 10^5 cm/s$. Also under collapse of bubbles in a cavitating liquid atoms velocity is of the same scale. In both cases formation of anomalous states is expected.

### VI. MACROSCOPIC ELECTROMAGNETIC PULSE

Suppose that under external conditions the macroscopic velocity $\vec{\xi}(t)$ of an atom exceeds the typical fluctuation velocity $\vec{u} \sim 10^4 cm/s$. In this case one can consider solely macroscopic displacement $\vec{\xi}(t)$ of the nucleus. The function $\Phi_{\varepsilon_b}(\vec{R})$ is given by (54) and $\Theta_{\varepsilon_b}(\vec{R})$ differs by substitution of sine by cosine. At small $R$ the function $\Theta_{\varepsilon_b}(\vec{R})$ is small.

As follows from Fig. 1 transitions to the levels $a$ or $b$ with photon emission are possible from higher levels
of the continuous spectrum. The wave functions of this conventional spectrum turn to plane waves \(^2\)

\[
\left(\Phi_{\rho}(\vec{R})\right)_{\theta_{\rho}(\vec{R})} = \frac{1}{2\varepsilon_{\rho}} \left(\sqrt{\varepsilon_{\rho} - m} (\bar{q}\bar{\sigma}) c_{\rho}/q\right) \exp(-i\bar{q}\bar{R})
\]

(58)
on large distance, \(R \gg r_N\), from the nucleus. Now \(\bar{R} = \bar{r} - \bar{\xi}(t)\). Two components of the spinor \(c_{\rho}\) satisfies the condition \(|c_{\rho 1}|^2 + |c_{\rho 2}|^2 = 1\). The proper energy eigenvalue is \(\varepsilon_{\rho} = \sqrt{q^2 + m^2}\). Thus these functions are marked by the wave vector \(\bar{q}\).

Non-adiabatic effects, as in Sec. [V B] also result in exponential decay of plane wave \([58]\) on large distances \(R > l_q\). The length \(l_q\) can be estimated, analogously to Sec. [V B] from the relation \(\Delta \varepsilon_q \sim \hbar c/l_q \sim \bar{\xi} q\). Since \(q \sim k_b\), the length \(l_q \sim r_N (c/\xi)(\hbar c/Z e^2) \sim 10^{-7}(\hbar c/Z e^2) cm\). This length is larger than \(l\) and thus one can use the plane wave approximation on large distances.

The transition probability per unit time to the level \(b\) form higher levels of the continuous spectrum, with the emission of photon energy \(k\), is \([18]\)

\[
\frac{1}{\tau_{ph}} = \frac{e^2}{4\pi} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3k}{k} \langle q|\gamma^\mu \exp(i\bar{k}\bar{R})|b\rangle \\
\times \langle b|\gamma^\mu \exp(-i\bar{k}\bar{R})|q\rangle \delta(\varepsilon_q - \varepsilon_b - k),
\]

(59)

where the small difference between \(\varepsilon^b_q\) and \(\varepsilon_b = \sqrt{p^2 + m^2}\) is ignored. The Dirac matrices \(\gamma^\mu = (\gamma^1, \gamma^0)\) are

\[
\gamma = \begin{pmatrix} 0 & \bar{\sigma} \\ -\bar{\sigma} & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(60)

Using the definitions of gamma-matrices, one can easily show that

\[
\langle q|\gamma^\mu \exp(i\bar{k}\bar{R})|b\rangle \langle b|\gamma^\mu \exp(-i\bar{k}\bar{R})|q\rangle
\]

\[
= \int d^3 R \left(\Phi^*_{\rho} \Theta_{\rho 1} + \Theta^*_{\rho 1} \Phi_{\rho}\right) \exp(i\bar{k}\bar{R}) \\
\times \int d^3 R_t \left(\Phi^*_{\rho 1} \bar{\sigma} \Theta_{\rho 1} + \Theta^*_{\rho 1} \bar{\sigma} \Phi_{\rho}\right) \exp(-i\bar{k}\bar{R}_t)
\]

(61)

and analogously for the part with \(\gamma^0\). All functions in \([41]\) correspond to the same atom and thus they depend on equally shifted by \(\bar{u}\) argument.

The function \([22]\) changes its asymptotics at \(R \sim 1/p_b\). Thus the exact function has a singularity at \(R \sim i/p_b\) in the complex plane. According to this, the expression \([41]\), containing the matrix elements, is estimated as

\[
\frac{|C_b|^2}{p_b^6} \exp\left(-\text{const} \left|\frac{q - \bar{k}}{p_b}\right|\right).
\]

(62)

For the anomalous states with \(j = 1/2\) (Sec. [II A]) the region \(R \sim \delta\) hardly affects the matrix elements. The main contribution to \([59]\) comes from the largest \(q\) and \(k\) which are on the order of \(p_b\). This gives the estimate

\[
1/\tau_{ph} \sim e^2 p_b |C_b|^2. \quad \text{With the estimates } p_b \sim Ze^2/r_N \text{ and } \([60]\), in the physical units

\[
\frac{\hbar}{\tau_{ph}} \sim \frac{e^2}{r_N} \left(\frac{\xi}{c}\right)^{2/3} \left(\frac{Ze^2}{\hbar c}\right)^{1/3}.
\]

(63)

For the oxygen nucleus \(\hbar/\tau_{ph} \sim 10 eV\) and the life-time \(\tau_{ph} \sim 10^{-16}s\).

The anomalous states with \(j > 1/2\) (Sec. [III]) are more singular at \(r = 0\) (before smearing) and thus the region \(r \sim \delta\) can mainly contribute to the matrix element. This case will be studied elsewhere.

Besides those transitions from continuous positron-like levels in Fig. 1 a transition from a discrete levels, shown in Fig. 1 to anomalous a and b ones are also possible. For these processes in the probability \([59]\) one should formally put \(q = 0\) and substitute \(d^3q\) by \(1/a_B^2\), where the Bohr radius \(a_B\) characterizes the atomic discrete states. The proper probability is \(1/p_b^6 a_B^6 \sim 10^{-9}\) fraction of the value \([63]\).

A. Macroscopic electromagnetic pulse

Transitions to the anomalous states occur from any positron-like level of the continuous spectrum. Thus the spectrum of the emitted pulse is also continuous with the photon wave vectors in the interval \(0 < k < (\sqrt{p_b^2 + m^2} - m) \approx p_b\). For the transitions to the anomalous level \(a\) wave vectors are in the interval \(0 < k < p_a\). It follows from \([63]\) that the emission of the pulse elapses \(10^{-16}s\). The emitted electromagnetic pulse moves from the nucleus as an expanded spherical layer of the width \(\delta R = c\tau_{ph} \sim 10^{-6}cm\) shown in Fig. 2.

The electromagnetic action for the pulse, proportional to \(c \tau_{ph} \sim 10^5 h\), is large compared to the Planck constant. Thus the emitted multi-photon pulse is macroscopic. The density of the electromagnetic energy in this
macroscopic pulse in Fig. 2 is

\[ \frac{E^2(R)}{4\pi} = \frac{\varepsilon_a}{4\pi R^2 \delta R}, \]  

where \( E \) is the electric field of the pulse. It is expressed in the form

\[ \frac{E^2(R)}{E_{at}^2} = \frac{Z}{\delta R} \left( \frac{\alpha B}{\tau_N} \right)^3 \left( \frac{\alpha B}{\tau_N} \right) \left( \frac{\delta R}{R} \right) \sim \left( \frac{\delta R}{R} \right)^2, \]  

where \( E_{at} = e/a_B^2 \) is the atomic electric field and \( a_B \sim 10^{-8} \text{cm} \) is the Bohr radius. The electric field is on the order of magnitude of the Rydberg potential.

The pulse acts on atomic electrons of surrounding matter as a macroscopic perturbation of the energy of the anomalous electron, forming a macroscopic pulse. The probability of transition of electrons to higher energy is

\[ w \sim \left( \frac{2Ea_B}{\hbar} \right)^2 \sim \frac{E^2(R)}{E_{at}^2} \left( \frac{\tau_{ph} Ry}{\hbar} \right)^2 \sim \frac{E^2(R)}{E_{at}^2}, \]  

with the Rydberg energy of \( \sim 10 \text{eV} \). This probability becomes close to unity at \( R \lesssim \delta R \). Thus within the sphere of radius \( \delta R \) around the nucleus, atoms become excited up to \( h/\tau_{ph} \sim 10 \text{eV} \). In a matter the number of atoms within this sphere is \( N \sim (\delta R/10^{-8} \text{cm})^3 \sim 10^9 \). That is with \( \varepsilon_a/N \sim 10 \text{eV} \) extra energy per atom. In other words, the energy \( \varepsilon_a \) of the anomalous electron state is mainly consumed for the excitation of surrounding atoms by the macroscopic pulse.

As follows from (65), the electric field of the macroscopic pulse is of the atomic scale within the radius \( \delta R \). This results in the non-stationary uncertainty of \( 10 \text{eV} \) of atomic levels. That is the energy, obtained by the surrounding atoms, is emitted in the continuous electromagnetic spectrum of \( \sim 10 \text{eV} \) width. There is no emission of high-energy quanta of the \( 10^{-8} \text{MeV} \) scale since the duration of the anomalous pulse, \( 10^{-16} \text{s} \), is larger than \( h/(1 \text{MeV}) \).

There is the chain process around the nucleus: (1) electron transitions to the anomalous level, (2) resulting emission of the macroscopic electromagnetic pulse during \( 10^{-16} \text{s} \), (3) absorption of the pulse energy \( (\sim 10 \text{MeV}) \) by surrounding atoms, \( \sim 10 \text{eV} \) per atom.

The excitation of the atoms in the process (3) occurs simultaneously with emission of energy and re-absorption of it. The radiation becomes isotropic on the mean free path of the macroscopic pulse \( \delta R \sim 10^{-6} \text{cm} \). Due to multiple pulses, coming from other nuclei, there is pumping of the electromagnetic energy.

VII. ELECTROMAGNETIC INTERACTION

Energy levels of hydrogen atom are slightly shifted under electromagnetic interaction. This Lamb shift is calculated on the basis of QED [2]. There is a different approach, when an electron “vibrates”, with the displacement \( \vec{u}_{em} \), under electromagnetic fluctuations [33]. The mean squared displacement, for the conventional atomic states, is evaluated as

\[ \langle u_{em}^2 \rangle = \frac{4\pi^2}{\pi} \frac{c^2}{\hbar c^2} \ln \frac{\hbar c}{e^2} \simeq (0.82 \times 10^{-11} \text{cm})^2. \]  

In this approach the electron moves in the averaged potential

\[ \langle U(\vec{r} - \vec{u}_{em}) \rangle \simeq U(r) + \frac{\langle u_{em}^2 \rangle}{6} \nabla^2 U(r). \]  

The quantum mechanical expectation value of the last term in (65) is the Lamb shift [30–32]. The mean squared displacement is formed by photons of energies between zero and approximately \( mc^2 \). For usual atomic states, \( \sqrt{\langle u_{em}^2 \rangle} \) is much smaller than the electron distribution on the Bohr radius resulting in perturbation theory for this reason.

For anomalous states that approach does not work because the mean electromagnetic displacement \( \sqrt{\langle u_{em}^2 \rangle} \) is much larger than the width \( \delta \) of the electron state. In this case one has to directly use the expression for electromagnetic shift of the anomalous level [18]. That expression is obtained from (59) by the substitution \( \pi \delta(q - \varepsilon_b - k) \rightarrow 1/(\xi - \varepsilon_b - k) \) with the principal value of the integral. The electromagnetic shift of the anomalous level

\[ \Delta \varepsilon_{em} \sim 10 \left( \frac{Z e^2}{\hbar c} \right)^{1/3} \text{eV}. \]

This expression is of the type of (63) and the region \( R \sim \delta \) hardly affects the matrix elements. The shift (69) is smaller than the preexisting shift (59) or (57). Thus in (47) and (48) \( \Delta \varepsilon \) is hardly renormalized by QED effects.

Therefore the electromagnetic effects play secondary role in formation of anomalous states characterized by the parameters (67).

VIII. NEUTRON EMISSION

Before the nucleus was treated as a rigid object interacting via Coulomb forces with electrons. According to the liquid drop model of nucleus, collective oscillations of the nuclear matter are possible with frequencies in a wide range on the order of \( 10 \text{MeV} \) (nuclear giant resonance [34–36]). An external \( \gamma \)-radiation, absorbed by those collective modes, can lead to nuclear deformations, generic with nuclear fission, resulting in neutron emission [38]. There is another mechanism of neutron emission, when it is caused by incident high-energy electrons. The direct interaction of incident electrons with nucleus is weaker compared to \( \gamma \)-radiation. However, those high-energy electrons can convert their kinetic energy into photons and also lead to neutron emission [34].

In our case the electrons, occupying the positron-like states and falling to the anomalous levels in Fig. 1 emit photons. But the resulting electromagnetic pulse in
Fig. 2 with the harmonics of $1/\tau_{ph} \sim 10^{16} s^{-1}$ frequency, cannot excite nuclear degrees of freedom. Nevertheless, the electron transitions to the anomalous level can directly activate collective nuclear modes. This occurs because of the Coulomb mode-electron interaction. In this process the electrons give up the energy $-\varepsilon_a$ to nuclear collective modes. A subsequent nuclear deformation can lead to neutron emission analogously to [38].

The absorption of the anomalous electron by the iron nucleus corresponds to the process

$$^{56}_{26}\text{Fe} + e_A \to ^{55}_{25}\text{Mn} + n + \nu_e + \gamma,$$  \hspace{1cm} (70)

where $\nu_e$ is the electron neutrino and the symbol $e_A$ stays for electron in the anomalous state. The mass of the iron nucleus is $M_{\text{Fe}} \simeq 55.93493$ u (1 u $\simeq 931.49$ $MeV$). Analogously $M_{\text{Mn}} \simeq 54.93804$ u and $M_n \simeq 1.00866$ u. According to these estimates, the threshold of the process (70) corresponds to excitation (by the electron $e_A$) of the iron nucleus up to the energy of $10.96$ $MeV$. In our case the excitation energy $-\varepsilon_a = 15.4$ $MeV$ exceeds that threshold and thus the reaction (70) is energetically possible. Note that the minimal excitation energy of copper or lead nucleus, to emit neutrons, is around of $10$ $MeV$ [35].

### A. Electron interaction with collective nuclear modes

To estimate the mean time of occupation of the anomalous level, by excitation of nucleus modes, general arguments are used instead of a detailed theory. As follows from Fig. 1, transitions to the levels $\varepsilon_a$ are possible from higher levels $\varepsilon_q$ of the continuous spectrum described by usual wave functions $\psi_q(\vec{R})$. These functions turn to plane waves at large $\vec{R}$ and one can approximately evaluate $\psi_q(\vec{R}_N) \sim 1$.

In those transitions the electrons can emit not only photons, as in Sec. VI but also quanta of collective nuclear modes. To estimate the strength of this process one can use a quantum mechanical approach, when the nuclear modes are initially considered as macroscopic co-ordinates. In this approach the probability of transition to the anomalous level $a$ [1]

$$w(t) = \int \frac{d^3 q}{(2\pi)^3} \left| \int d \xi \psi_a(\vec{R}) V(\vec{R}, t) \psi_q(\vec{R}) \right|^2,$$  \hspace{1cm} (71)

where the matrix element

$$V(t) = \int d^3 R \psi_a(\vec{R}) V(\vec{R}, t) \psi_q(\vec{R}).$$  \hspace{1cm} (72)

Here the potential $V(\vec{R}, t)$ describes the Coulomb (nuclear mode)-(electron) interaction. This potential depends on time since the nuclear modes are initially treated as macroscopic variables. Nuclear collective modes correspond, for example, to elliptical deformation $\delta r_N$ of the spherical nucleus. The quantum average on macroscopic variables is

$$\langle V(t_1)V(t_2) \rangle = \left( \frac{Z e^2}{r_N^3} \right)^2 D(t_1 - t_2).$$  \hspace{1cm} (73)

The correlator (73) is similar to the photon propagator in QED formalism of Sec. VI. $D(t)$ decays on $t \sim \hbar/\varepsilon_a$ and one can evaluate (in physical units)

$$D \sim \left\langle (\delta r_N)^2 \right\rangle \sim \frac{1}{r_N^2} \left( \frac{\hbar^2}{\varepsilon_a M_{\text{Fe}}} \right) \sim 10^{-1}.$$  \hspace{1cm} (74)

The parameter in the parenthesis is the mean-squared displacement of the oscillator with the frequency $\varepsilon_a/\hbar$. At large $t$ the time-dependent part of (71) can be substituted as

$$\int_{-\infty}^{t_i} dt_1 \int_{t_i}^{t} dt_2 D(t_1 - t_2) \exp[i(t_1 - t_2)(\varepsilon_q - \varepsilon_a)]$$

$$\to t \int_{-\infty}^{t} d\theta D(\theta) \exp[i \theta(\varepsilon_q - \varepsilon_a)] \sim t \frac{10^{-1}}{\varepsilon_a}.$$  \hspace{1cm} (75)

According to [54], one can make the substitution in Eq. (71) $\psi_a(\vec{R}) \to C_a / (R^2 \sqrt{p_0})$. Also $d^3 q \sim p_0^3$. The probability $w(t)$ becomes on the order of unity at $t \sim \tau_{\text{nuc}}$, where the time $\tau_{\text{nuc}}$ is estimated from (71) - (76)

$$\frac{\hbar}{\tau_{\text{nuc}}} \sim 0.1 \frac{Z e^2}{r_N^3} \left\langle C_a^2 \right\rangle \left( \frac{Z e^2}{\hbar c} \right)^2$$

$$\sim 0.1 \frac{Z e^2}{r_N^3} \left( \frac{Z e^2}{\hbar c} \right)^{4/3} \left( \frac{\dot{\xi}}{c} \right)^{2/3}.$$  \hspace{1cm} (76)

Here the relation [56] is taken into account. Since $\dot{\xi}/c \sim 10^{-6}$, for the iron nucleus $\tau_{\text{nuc}} \sim 10^{-16}$ s.

Summarizing, during the time $\tau_{\text{nuc}} \sim 10^{-16}$ s the collective modes in the iron nucleus are excited up to the energy of $15.4$ $MeV$. This excitation energy is sufficient to initiate the process (70) getting the emitted neutron with the energy up to $4.4$ $MeV$.

As in the case of photon emission (Sec. VI), the superposition of excited collective nuclear modes is in the continuous spectrum since transitions occur from the continuous positron-like states. Analogously, the $\gamma$-emission in the process (70) is also expected in the continuous spectrum. The emitted macroscopic pulse is of the duration $\tau_{\text{nuc}} \sim 10^{-16}$ s. For this reason, registration of the $MeV$-energy quanta is impossible. This situation is similar to Sec. VIA.

### B. General scenario

Switching on an external mechanical perturbation results in the atomic (nuclear) velocity in the $10^9 cm/s$ range. This makes real the possibility of anomalous states.
Electron transitions, within the time $\tau_{ph} \sim 10^{-16}s$, to that anomalous level from the positron-like states result in emission of the light with the continuous spectrum in the $10\ eV$ range. The total released energy, in the $MeV$ range, cannot be observed as a separate high-energy quantum. The luminescence of the continuous spectrum is to be registered instead.

In parallel with that photon emission, the electron transitions to the anomalous level excite, within the time $\tau_{nuc} \sim 10^{-16}s$, collective modes of the nucleus.

Thus the anomalous states, initiated by the macroscopic perturbation, result in the electromagnetic and neutron emissions. A $\gamma$-emission cannot be registered.

IX. EXPERIMENTS

The phenomena, proposed in this paper, are expected to occur in any experiment, where accelerated atoms (nuclei) reach the velocity of approximately $10^5 cm/s$.

A. Sonoluminescence

This is a phenomenon, when ultrasonically driven gas bubbles in liquids emit light. That phenomenon was of challenge. In the multiplicity of theories a diverse array of physical effects was proposed as a condition to finally produce sonoluminescence. The sonoluminescence energy was supposed to be mechanically transferred from the moving bubble surface. That energy was evaluated by modeling parameters.

The different source of sonoluminescence is revealed in this paper. When the collapsing bubble reaches the certain radius, the velocity $\xi$ of its surface becomes close to $10^5 cm/s$. This is a condition of anomalous states. Sonoluminescence originates from the electromagnetic energy of the emitted macroscopic pulses with the continuous spectrum in the $10\ eV$ range.

The macroscopic pulses are emitted from various atoms on the bubble surface and accumulated inside the bubble. The bubble surface reflects electromagnetic waves assisting the accumulation. At the certain moment the radiation out the bubble is triggered off. This radiation is sonoluminescence.

B. Neutron registration

In the experiments the application of ultrasound was made to a solution containing stable isotopes of iron. Instead of sonoluminescence, neutron measurements were performed. The authors of the paper claimed the evidence of neutron emission. The general feature of the collapsing gas bubble is the velocity ($\sim 10^5 cm/s$) of its surface.

In the experiments the strong ultrasound was applied to the metallic bar of iron. More general mechanical conditions are described in. See also. The emission of neutrons in the $MeV$ range was reported. Under motion of defects in a solid, microcracks, etc, atoms jump to a neighbor position having the approximately sound velocity $\sim 10^5 cm/s$.

That is the anomalous state condition in is fulfilled. This allows to conclude that the observed neutron emission is likely due to the mechanism of anomalous states. The absence of the $MeV$ scale $\gamma$-radiation corresponds to that mechanism.

Two drastically different phenomena, macroscopic mechanical stress and nuclear reactions, are hardly expected to be connected. However, the concept of anomalous states is likely the mechanism linking these worlds.

C. Energy release

Under occupation of the anomalous state $a$, related to the lead nucleus, the hidden energy of $32.5\ MeV$ is released in the form of light and products of nuclear processes. This total yield, excepting neutrinos, is absorbed by the surrounding matter.

The creation of anomalous states is initiated by the artificial velocity $\xi$ acquired by the nucleus in experiments. It has to exceed $\sim 10^4 cm/s$. Starting with this range, the fraction of anomalous nuclei in the matter increases (Sec. VLA). At sufficiently large $\xi$ all nuclei $N_0$ of the surrounding matter are expected to undergo to anomalous state. This would imply the total energy release of $32.5N_0 MeV$.

One gram of lead (atomic mass of $3.44 \times 10^{-22}g$) contains $N_0 \approx 2.9 \times 10^{21}$ nuclei resulting in the energy release of $0.94 \times 10^{23} MeV \approx 1.5 \times 10^{10}J$. Three grams of lead can release the energy produced by explosion of 10 tons of trotyl.

X. DISCUSSIONS

Properties of electron in the electrostatic field of atomic nucleus are described in textbooks. It seems unlikely to add something unusual to this field. In quantum mechanics the solution of the equation $-(1/2m)\nabla^2\psi - E\psi = 0$ can have a tendency to form the singularity $\psi \sim 1/r$ at small $r$. But this singular solution is false since, analogously to electrostatics, it requires the artificial $\delta(r)$ term in the right-hand side of that equation.

A quite different situation may occur in the Dirac quantum mechanics in the electrostatic field of an atomic nucleus cut off on its size. In this case the Dirac spinor can be singular, $\Phi \sim 1/r^2$, at small $r$. This is a real solution of the Dirac equation because the singularity is of algebraic origin. It does not come from a differential equation requiring an artificial source like $\delta(r)$.

That non-physical singular solution had low chances to be regarded due to invisible link between it and a
category of physical states. The link exists as shown in this paper.

The singularity, obtained in frameworks of relativistic quantum mechanics, becomes smeared out under electron-phonon interaction. This happens because of nucleus quantum vibrations in a solid, when the singularity takes various positions in space. Therefore such additional states become physical. They are classified as anomalous and are added to the conventional atomic spectrum of electrons. The binding energy of these states is in the range of tens of $\text{MeV}$.

That anomalous electron state does not exist in its bare form. It is assisted by a polaronic cloud consisting of a large, $\sim 10^4$, number of virtual phonons. This non-conventional polaron is the electron-phonon state with the energy uncertainty in the phonon cloud $\sim 100\,\text{eV}$. This is the true ground state of an atom in condensed matter but spontaneous creation of such multi-phonon state has negligible probability in random processes in a solid.

However, there exists a real way to create anomalous states. Atoms (nuclei) in condensed matter can move fast under some macroscopic perturbation. In a liquid it can be acoustically driven implosion of a gas bubble. The velocity of its wall can reach $10^3\,\text{cm/s}$. In a solid a strong mechanical stress can result in defect motion or microcracks, when an atom jumps to a neighbor position having the velocity of $\sim 10^5\,\text{cm/s}$.

In those cases the velocity of a nucleus in the matter is a pulse as a function of time. This velocity can dominate the quantum mechanical background produced by positional quantum fluctuations of an atom in the matter. Then the macroscopic velocity itself cuts off the singularity making the state physical. This anomalous state is macroscopically mediated and does not require a heavy phonon cloud. Thus transitions to the anomalous level, with emission of the electromagnetic pulse, are possible.

As argued in the paper, the anomalous states provide the different source of sonoluminescence. It originates from the electromagnetic energy of the emitted macroscopic pulses with the continuous spectrum in the $10\,\text{eV}$ range. In this mechanism the luminescence is not mechanically converted from moving bubble surface.

That is not the full story. The electron transitions to the anomalous level can directly activate collective nuclear modes. This occurs because of the Coulomb mode-electron interaction. The proper matrix element is enhanced by the smeared singularity. As a result, under macroscopic mechanical perturbation of a solid the neutron emission is possible. This looks paradoxical. However, the concept of anomalous states likely links those different worlds.

Simultaneously with the neutron emission, the electromagnetic energy in the $\text{MeV}$ range is emitted as the macroscopic pulse. It cannot be registered as a $\gamma$-quantum of that energy.

That is again not a full story. Under occupation of the anomalous state in the lead nucleus the hidden energy of $32.5\,\text{MeV}$ is released in the form of light and products of nuclear processes. This total yield, excepting neutrinos, is absorbed by the surrounding matter. Three grams of lead, under a strong mechanical perturbation, can release the energy produced by explosion of 10 tons of tritroyl. This is a different concept of energy release. Besides the matter heating to overcome an energy barrier in fusion processes, one can apply a mechanical perturbation to get atoms moved faster than $10^5\,\text{cm/s}$.

There is another aspect of the phenomenon proposed. Anomaly in quantum field theory corresponds to any phenomenon that arises, when a quantity that becomes zero, according to quantum mechanics, acquires a finite value, when quantum field theory is used. A non-trivial example is chiral anomaly in QED [41, 42] (see also [43–45]). In the Dirac massless quantum mechanics the chiral current $\psi\gamma^5\gamma^\mu\psi$ conserves. When moving from the quantum mechanics to QED, this conservation violates. In our case the quantum mechanical state of the zero size becomes physical with a finite width by application of quantum fields. Therefore this phenomenon can be treated as anomaly.

**XI. CONCLUSIONS**

Anomalous electron states

- are the different source of sonoluminescence,
- likely link mechanical stress and neutron emission,
- provide the different mechanism of energy release.

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**Appendix: LATTICE FLUCTUATIONS**

Suppose that in a crystal the displacement from an equilibrium position at the lattice site $\vec{l}$ is $\vec{v}_l$ [10, 11]. It can be Fourier transformed

$$
\vec{v}_l = \sqrt{\frac{a^3}{V}} \sum_k \vec{a}_k \exp(i\vec{k}\cdot\vec{l}),
$$

where $V$ is the volume and the summation is restricted by the Brillouin zone. Here $a^3$ is the volume of the unit cell. The classical kinetic energy of the system is

$$
E_{\text{kin}} = \frac{M}{2} \sum_l \vec{v}_l^2 = \frac{M}{2} \sum_k \vec{v}_k \cdot \vec{v}_{-k},
$$
where $M$ is the mass of the site. Analogously, the elastic energy is

$$E_{el} = \frac{M}{2} \sum_k \omega_k^2 \tilde{v}_k \tilde{v}_{-k}, \quad (A.3)$$

where $\omega_k$ is the phonon spectrum. In this approach, there is no difference between longitudinal and transverse modes. For each independent degree of freedom $\tilde{v}_k$ the Schrödinger equation follows from the classical energies $\langle A.2 \rangle$ and $\langle A.3 \rangle$

$$-\frac{1}{2M} \frac{\partial^2 \psi (\tilde{v}_k)}{\partial \tilde{v}_k^2} + \frac{M \omega_k^2}{2} \tilde{v}_k^2 \psi = \omega_k \left[ \frac{3}{2} + n_1(\tilde{k}) + n_2(\tilde{k}) + n_3(\tilde{k}) \right] \psi. \quad (A.4)$$

Here $\tilde{v}_k$ and $\tilde{v}_{-k}$ are not distinguished since this does not influence the final result. One can also use the momentum representation, when $n_1 + n_2 + n_3 = 2n_r + l$ and $n_r = 0, 1, 2, \ldots$. According to probability distribution in oscillator $\langle 46 \rangle$, for the ground state of the lattice the mean squared value is

$$\langle \tilde{v}_k^2 \rangle = \frac{3}{2M\omega_k}. \quad (A.5)$$

The mean squared displacement $\langle \tilde{u}^2 \rangle = \langle \tilde{v}_l^2 \rangle$ of a lattice site does not depend on site number $l$ and

$$\langle \tilde{u}^2 \rangle = \sum_k a^3 \langle \tilde{v}_k^2 \rangle V = \int_{BZ} \frac{a^3 d^3k}{(2\pi)^3} \frac{3}{2M\omega_k}. \quad (A.6)$$

where the integration is restricted by the Brillouin zone. The volume of the Brillouin zone is on the order of $1/a^3$. It follows from $\langle A.6 \rangle$ that the mean squared displacement corresponds to a particle of the mass $M$ in a harmonic potential of the Debye frequency $\omega_D$

$$\langle u^2 \rangle \sim \frac{1}{M\omega_D} \sim (10^{-9} \text{cm})^2. \quad (A.7)$$

The mean squared velocity distribution $\langle \tilde{u}^2 \rangle$ can be found noting that $M \tilde{v}_k \rightarrow \tilde{p}_k = -i\partial / \partial \tilde{v}_k$, where $\tilde{p}_k$ is momentum. The Schrödinger equation, analogous to $\langle A.4 \rangle$, has the form for the ground state $\langle 1 \rangle$

$$-\frac{M \omega_k^2}{2} \frac{\partial^2 \phi (\tilde{p}_k)}{\partial \tilde{p}_k^2} + \frac{\tilde{p}_k^2}{2M} \phi = \frac{3\omega_k}{2} \phi. \quad (A.8)$$

Analogously to $\langle A.5 \rangle$ and $\langle A.6 \rangle$, it follows at zero temperature

$$\langle \tilde{u}_k^2 \rangle = \frac{3\omega_k}{2M} \quad (A.9)$$

and

$$\langle \tilde{u}^2 \rangle = v^2 = \int_{BZ} \frac{a^3 d^3k}{(2\pi)^3} \frac{3\omega_k}{2M} \sim \omega_D^2 \langle u^2 \rangle \sim (10^4 \text{cm/s})^2. \quad (A.10)$$

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