Quantum Gravity: A Mathematical Physics Perspective

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1. Introduction

The problem of quantum gravity is an old one and over the course of time several distinct lines of thought have evolved. However, for several decades, there was very little communication between the two main communities in this area: particle physicists and gravitation theorists. Indeed, there was a lack of agreement on even what the key problems are. By and large, particle physics approaches focused on perturbative techniques. The space-time metric was split into two parts: \( g_{\mu\nu} = \eta_{\mu\nu} + G h_{\mu\nu} \), \( \eta_{\mu\nu} \) being regarded as a flat kinematic piece, \( h_{\mu\nu} \) being assigned the role of the dynamical variable and Newton’s constant \( G \) playing the role of the coupling constant. The field \( h_{\mu\nu} \) was then quantized on the \( \eta_{\mu\nu} \)-background and perturbative techniques that had been so successful in quantum electrodynamics were applied to the Einstein-Hilbert action. The key problems then were those of handling the infinities. The gravity community, on the other hand, felt that a central lesson of general relativity is that the space-time metric plays a dual role: it is important that one and the same mathematical object determine geometry and encode the physical gravitational field. From this perspective, an ad-hoc split of the metric goes against the very spirit of the theory and must be avoided. If one does not carry out the split, however, a theory of quantum gravity would be simultaneously a theory of quantum geometry and the notion of quantum geometry raises a variety of conceptual difficulties. If there is no background space-time geometry— but only a probability amplitude for various possibilities—how does one do physics? What does causality mean? What is time? What does dynamics mean? Gravity theorists focused on such conceptual issues. To simplify mathematics, they often truncated the theory by imposing various symmetry conditions and thus avoided the field theoretic difficulties. Technically, the emphasis was on geometry rather than functional analysis. It is not that each community was completely unaware of the work of the other (although, by and large, neither had fully absorbed what the other side was saying). Rather, each side had its list of central problems and believed that once these issues were resolved, the remaining ones could be handled without much difficulty. To high energy theorists, the conceptual problems of relativists were perhaps analogous to the issues in foundations of quantum mechanics which they considered to be “unimportant for real physical predictions.” To relativists, the field theoretic difficulties of high energy physicists were technicalities which could be sorted out after the conceptual issues had been resolved. This is of course a simplified picture. My aim is only to provide an impression of the general state of affairs.

Over the last decade, however, there has been a certain rapprochement of ideas on quantum gravity. Each side has become increasingly aware of the difficulties that were...
emphasized by the other. By and large, high energy theorists now agree that the non-perturbative techniques are critical. They see that the underlying diffeomorphism invariance should be respected. Even if one introduces a background structure for, e.g., regularization of operators, the final result should not make reference to such structures (unless of course they are physically important). Relativists have come to recognize that the field theoretic divergences have to be faced squarely. There is now a general agreement that although the truncated theories are interesting and provide insights into the conceptual (and certain mathematical) problems faced by the full theory, they are essentially toy models whose value, in the final analysis, is quite limited since they have only a finite number of degrees of freedom. Thus, the sets of goals of the two communities have moved closer.

These recognitions do not imply, however, that there is a general consensus on how all these problems are to be resolved. Thus, there are again many approaches. But this diversity in the lines of attack is very healthy. In a problem like quantum gravity, where directly relevant experimental data is scarce, it would be an error if everyone followed the same path. As Feynman (1965) put it:

\textit{It is very important that we do not all follow the same fashion. ... It’s necessary to increase the amount of variety ... and the only way to do is to implore you few guys to take a risk with your lives that you will not be heard of again, and go off in the wild blue yonder to see if you can figure it out.}

And quite a few groups have taken the spirit of this advice seriously and “gone off in the yonder to figure it out.” What is striking is that, in spite of the diversity of their methods, some their results are qualitatively similar. One message that seems to keep coming back is that not only can one not assume a flat Minkowskian geometry at the Planck scale but in fact even the more general notions from Riemannian geometry would fail. The continuum picture itself is likely to break down. The lesson comes from certain computer simulations of 4-dimensional Euclidean gravity, (see e.g. Agishtein & Migdal (1992)) from string theory (see e.g. Gross & Mende (1988), Amati et al (1990), Aspinwall (1993)) and from canonical quantization of 4-dimensional general relativity (Ashtekar et al (1992)). The detailed pictures of the micro-structure of space-time that arises in these approaches is quite different at least at first sight and it is not clear that these pictures can be reconciled with one another in detail. Nonetheless, there are certain similarities in the results and most of them are obtained by using genuinely non-perturbative techniques.

The purpose of this article is to give a flavor of these ideas and techniques to mathematical physicists. I should emphasize that this is not a systematic survey. In particular, I will concentrate just on one approach –non-perturbative, canonical treatment of 4-dimensional, Lorentzian general relativity. Even within this approach, there are over 350 papers and I can not do justice to them in this limited space. (For detailed reviews, see, e.g., Ashtekar (1991,1992) and Pullin (1993).) Rather, I will just present a few results that may be of interest to this audience, indicating, wherever possible, the degree of precision and rigor of the underlying calculations.

However, since the goal of the conference is to look to future, “towards the 21st century,” and since the organizers asked us to try to “inspire rather than merely inform,” I will begin in section 2 with a few remarks that may perhaps seem provocative to some mathe-
matical physicists. In doing so, however, I am following the lead of other speakers and my hope is the same as theirs: these remarks may lead to stimulating exchanges of ideas and perhaps a re-examination of some of the basic premises. In section 3, the main difficulties of quantum gravity are outlined from the perspective of mathematical physics. Section 4 summarizes some recent developments. While general relativity is normally regarded as a theory of metrics, it can be recast as a dynamical theory of connections (Ashtekar, 1987). This shift of emphasis has two important consequences. First, it brings general relativity closer to theories of other interactions and one can draw on the numerous techniques that have been developed to quantize these theories. Second, the shift simplifies the basic equations considerably making them low order polynomials in the basic variables. (They are non-polynomial in the metric variables.) Recently, a number of rigorous results have been obtained to analyze theories of connections, particularly the ones such as general relativity in which there is an underlying diffeomorphism invariance. The main idea here is to use algebraic methods to develop an integration theory on the space $A/G$ of connections modulo gauge transformations and to explicitly construct a measure which is diffeomorphism invariant. These results now provide the foundation for a non-perturbative approach to quantum gravity based on Hamiltonian methods. They may also have other applications in mathematical physics. In section 5, this framework is used to show the existence of states in the full, non-perturbative quantum gravity which approximate a classical metric when coarse-grained at scales much larger than the Planck length and which exhibit a specific discrete structure at the Planck scale. If such a state is used in place of Minkowski space-time, the ultra-violet difficulties of Minkowskian field theories may disappear altogether. This is a concrete illustration of the results in non-perturbative quantum gravity that may have an impact on quantum field theory.

2. The ultraviolet catastrophe: A matter of gravity?

As we all know, for over 40 years, quantum field theory has been in a somewhat peculiar situation as far as realistic models are considered. On the one hand, perturbative treatments are available for, say, the electro-weak interaction and the results are in excellent agreement with experiments. It is clear therefore that there is something “essentially right” about these theories. On the other hand, their mathematical status has continued to be dubious and it has not been possible to say precisely what is right with them. And this is not because of lack of effort. Already by early sixties, quantum field theory had become an intellectually coherent subject (recall that PCT spin and statistics and all that was published in 1964). A considerable amount of imaginative—and often heroic—effort has gone in to the field since then. And yet none of the physically realistic quantum field theories has reached a sound status in 4 space-time dimensions.

I think it is fair to say that the general attitude in the mathematical physics community is that the difficulties are of a mathematical nature. Realistic 4-dimensional theories are extremely involved; as Professor Wightman put it at the conference, “handling them with the present techniques is hellishly complicated.” So, the overall feeling seems to be that there is no obstacle of principle to construct a non-Abelian gauge theory such as QCD in 4-dimensions, but new mathematical tools are needed to make the task practicable.
One might worry about the fact that in all these theories, one uses Minkowski space as the underlying space-time, thereby ignoring all the Planck scale effects. Could this be a source of the difficulty? The general belief in the community seems to be that this is not the case; no new physical input from the Planck scale is needed to rigorously construct the quantum theory that underlies the standard model of particle physics. (This view was, for example, expressed in Professor Buchholtz’s talk.) Sure, the theory would be an approximation in that one would not be able to trust its predictions below, say $10^{-17}$ cm. But it would be internally consistent and agree with Nature as far as laboratory physics is concerned. Sure, we are idealizing the space-time geometry. But one does this all the time and such idealizations always work: our general experience in physics tells us that, somehow, the phenomena at different scales decouple approximately. Indeed, very little progress could have occurred in absence of this decoupling. After all, when engineers build bridges, they don’t have to worry about the fact that it is quantum mechanics that governs the atoms of the bridge; they just use classical, Newtonian physics. They succeed because there is a factor of $10^{12}$ between the scale of the bridge and the atomic scale. There is a factor of $10^{16}$ between (say) the weak-interaction scale and the Planck length! Surely, one says, the Planck-size effects are unimportant to the problem of constructing a consistent (and therefore in particular, finite) quantum theory underlying the standard model.

I would like to argue that this is not necessarily the case. Recall, first, that the key difficulties of quantum field theory are the ultra-violet divergences. These arise precisely because we allow virtual processes involving arbitrary number of loops, each carrying an integral over arbitrarily large momenta. And it is not our choice to allow or disallow them: we are forced into it by the general principles of quantum field theory which include Poincaré invariance. Surely, as the energy involved becomes bigger, the approximation that one can ignore the gravitational effects becomes worse. If the microstructure of space-time is qualitatively different from that given by the continuum picture, the whole procedure is flawed; we shouldn’t— and couldn’t—integrate to arbitrarily high momenta since that is equivalent to integrating to arbitrarily small distances. Let us consider an analogy with atomic physics where we can successfully use non-relativistic quantum mechanics. Suppose for a moment that there was a general requirement that arose from the quantum principles which forced us, in the calculation of, say, the ground state energy of the hydrogen atom, to consider electrons with velocities arbitrarily close to the speed of light. Then, had we ignored special relativity altogether, we would probably have got an inconsistent theory: The hypothetical quantum principle would have forced us to bring special relativity into the treatment. The standard treatment of atomic physics in non-relativistic quantum mechanics is internally consistent because the calculation scheme does not involve steps which violate the basic premise and limitation of non-relativistic quantum mechanics. (It agrees well with experiments because, in addition, these approximations are met in Nature.) Let us return to quantum field theory. In spite of the fact that we are interested here in processes in which physical energies (and masses of particles) are small compared to the Planck mass, we are forced to allow virtual processes involving arbitrarily high energies and these do probe the Planck-scale structure of the space-time geometry. But we insist on ignoring this structure altogether. That, it may well be, is the physical source of the ultraviolet catastrophe; it may be a matter of gravity.
Indeed, there are other instances in physics where mathematical difficulties signalled the need for changing the basic physical premise. Consider for instance the action at a distance models in classical, relativistic physics. The system of integro-differential equations one obtains is hard to manage mathematically and, if taken seriously, raises questions of predictability in relativistic physics. The “origin” of these difficulties lies in the physical inadequacy of the basic assumptions. Once we bring in the field degrees of freedom, these mathematical problems go away. Now, classical physics is described by a system of hyperbolic differential equations and causality is manifest. Another example is the birth of quantum mechanics itself. The mathematical description of the black-body radiation broke down in the framework of classical physics and pointed to the necessity of a radical revision of that framework. There are a number of other examples. Indeed, most radical changes of the conceptual framework are inspired, at least in part, by the fact that the older framework ran into serious mathematical difficulties. This does not of course imply that the same must happen with the ultra-violet divergences. These are only analogies. And there is no irrefutable evidence that more sophisticated mathematical techniques will not suffice to construct a consistent quantum field theory for the standard model. However, the examples suggest that we should be open to such a possibility.

I will conclude with a related but somewhat different point. Let me again use an analogy. Suppose, for a moment that special relativity had not been discovered. One might have learnt painfully that the predictions of Newtonian mechanics are not quite correct and have to be supplemented by powers of \(v/c\) where \(v\) is the velocity of the object under consideration. If we had been sufficiently clever, we would have discovered that to compute any physical effect, one can use a perturbation series in the powers of \(v/c\). One could get sophisticated and worry about whether such series actually converge. This is perhaps similar to the present situation with the perturbation theory for the electro-weak interaction. In the hypothetical case of “special relativity”, proving convergence and other mathematical properties of the resulting series would be instructive. Like Lorentz and Poincaré, one could have even discovered the Lorentz transformations and found all sorts of equations which are “true.” However, without the physical shift of scenario that was provided to us by Einstein –that there is no absolute simultaneity– we would still be missing key insights. In a real sense, we would not really “understand” what the equations were telling us. The situation could be similar with the standard model. Suppose we do succeed in giving a mathematical meaning to the perturbative results. We will have a nice, consistent theory with convergence proofs. But it is possible that we may still be missing some key insights because we ignored the Planck scale physics. Indeed, it is not obvious that all effects of the quantum nature of geometry will be confined to the Planck scale. Let us take special relativity. It is true that most effects are corrections in powers of \(v/c\) to the predictions of non-relativistic physics. But now and again, there are also qualitative predictions that have nothing to do with how large the velocity \(v\) of the particles involved is. Conversion of mass into enormous amount of energy happens in nuclear processes where all velocities are small. There is a prediction that associated with every particle there is an anti-particle. There is the CPT theorem. These are all qualitative effects completely unrelated to how fast the particles in question are moving.

\[ \text{† This example was suggested to me by Jose Mourão} \]
They come about because special relativity shifts the very paradigm within which one operates. The entire mathematical framework of quantum physics changes abruptly. The problems change. The tools change. The concepts change. There is a possibility that quantum field theory may undergo a similar radical change once the concepts from quantum gravity are brought in. Indeed, we will see in the subsequent sections that quantum gravity does make strong demands on how one should formulate and analyze problems. It insists on diffeomorphism invariance whence there is no background metric or causal structure; it trivializes the Hamiltonian and puts the burden of dynamics on the constraints of the theory; it introduces an essential non-locality through physical observables. The ground rules therefore seem quite different from those we are used to in Minkowskian quantum field theories.

Mathematical physicists can raise an immediate objection against all these possibilities. After all, there do exist well-defined, consistent quantum field theories in 2 and 3 space-time dimensions. Why don’t the Planck scale problems raise their annoying head there?† It turns out that there is a dramatic change in the properties of the gravitational field starting precisely at 4 dimensions! It acquires its local degrees of freedom only in 4 and higher dimensions. In dimension 3 or lower, there are no gravity waves and no gravitons. Therefore, the ultraviolet problems of field theories are, in a sense, decoupled from the quantum gravitational field. Indeed, one can turn the argument around and ask if it is a pure coincidence that the number of dimensions for which the ultra-violet problems of field theories seem so difficult to handle happens to be precisely the one at which gravity comes on its own. Or, is there a lesson lurking here that we have ignored?

I want to emphasize again that I do not regard the arguments given in this section as conclusive. It is a viable, logical possibility that a consistent quantum field theory incorporating the standard model will exist in 4-dimensions and will contain all the physics that is relevant to the scale of these interactions. Quantum gravity may have no effect whatsoever on this physics. However, I feel that this “mainstream” viewpoint in mathematical physics is also not watertight. There are differences between the current situation in field theory and previous examples in the history of physics where a clean decoupling occurred between physics at one scale and that at another. And, in the interest of variety, I believe it is important not to ignore altogether the possibility that the ultra-violet catastrophe may, in the end, be a matter of gravity.

3. Difficulties of quantum gravity

The importance of the problem of unification of general relativity and quantum theory was recognized as early as 1930’s and a great deal of effort has been devoted to it in the last three decades. Yet, we seem to be far from seeing the light at the end of the tunnel. Why is the problem so hard? What are the principal difficulties? Why can we not apply the quantization techniques that have been successful in theories of other interactions? In this section, I will address these questions from the perspective of mathematical physics.

The main difficulty, of course, is the lack of experimental data with a direct bearing on quantum gravity. One can argue that this need not be an unsurmountable obstacle. After

† Indeed, Professor Chayes did ask this question during my talk!
all, one hardly had any experimental data with a direct bearing on general relativity when
the theory was invented. Furthermore, the main motivation came from the incompatibility
of Newtonian gravity with special relativity. We face a similar situation; we too are driven
by what appears to be a fundamental tension between general relativity and quantum
theory. However, it is also clear that the situation with discovery of general relativity is an
anomaly rather than a rule. Most new physical theories—including quantum mechanics—
arose and were continually guided and shaped by experimental input. In quantum gravity,
we are trying to make a jump by some twenty orders of magnitude—from a fermi to a
Planck length. The hope that there is no dramatically new physics in the intermediate
range is probably just that—a hope.

The experimental status, however, makes the situation even more puzzling. If there
is hardly any experimental data, theorists should have a ball; without these “external,
bothersome constraints,” they should be able to churn out a theory a week. Why then
do we not have a single theory in spite of all this work? The brief answer, I think, is
that it is very difficult to do quantum physics in absence of a background space-time. We
have very little experience in constructing physically realistic, diffeomorphism invariant
field theories. Indeed, until recently, there were just a handful of examples, obtained by
truncating general relativity in various ways. It is only in the last three years or so that a
significant number of diffeomorphism invariant models with an infinite number of degrees
of freedom has become available, still, however, in low space-time dimensions.

As mentioned in the Introduction, one way out of this quandary—tried by the high
energy physicists—was to simply break the diffeomorphism invariance to start with and
introduce a flat background metric. As is well-known, however, the resulting perturbative
quantum field theory is non-renormalizable. In the high energy community, this was con-
sidered a fatal flaw. At first, it was thought that the problem is with the starting point
—general relativity. Therefore, attempts were made to modify Einstein’s theory. Perhaps
the most notable of these modifications were the higher derivative theories and supergrav-
ity. However, these attempts at defining a local quantum field theory for gravity (with
matter) which is consistent order by order in the perturbation expansion failed. Finally,
these developments led, in the mid-eighties, to string theory. Since there were several talks
on this subject in the conference, I will restrict myself just to a one line summary here:
The perturbation series in string theory is believed to be finite order by order (in the string
tension) but the series is believed not to be even Borel-summable. As a result, a great deal
of effort is being devoted to constructing the theory non-perturbatively.

Returning to quantum general relativity, the failure of perturbation theory would,
presumably, not upset the mathematical physicists a great deal. After all, they know
that in spite of perturbative non-renormalizability, a quantum field theory can exist non-
perturbatively. This point was discussed in some detail in Professor Klauder’s talk and
Professor Wightman commented on it in the context of the Gross-Neveu model in 3-
dimensions. Indeed, in the case of (GN)3, there appears to be no fundamental difference
from renormalizable models. In particular, we learnt in this conference that the conjec-
ture that such models should be based on distributions which are worse behaved than
tempered distributions has been shown to be false. So, at a basic mathematical level,
non-renormalizability seems not be a fundamental consideration. Returning to gravity,
as I will indicate below, there is some evidence from numerical simulations that quantum
general relativity itself may exist non-perturbatively. One might therefore wonder: why
have the standard methods developed by mathematical physicists not been applied to the
problem of quantum gravity? What are the obstacles? Let me therefore consider some
natural strategies that one may be tempted to try and indicate the type of difficulties one
encounters.

First, one might imagine defining the goal properly by writing down a set of axioms. In
Minkowskian field theories the Wightman and the Haag-Kastler axioms serve this purpose.
Can we write down an analogous system for quantum gravity, thereby spelling out the goals
in a clean fashion? Problems arise right away because both systems of axioms are rooted
in the geometry of Minkowski space and in the associated Poincaré group. Let me consider
the Wightman system (Streater & Wightman 1964) for concreteness. The zeroth axiom
asks that the Hilbert space of states carry an unitary representation of the Poincaré group
and that the 4-momentum operator have a spectrum in the future cone; the second axiom
states how the field operators should transform; the third axiom introduces micro-causality,
i.e., the condition that field operators should commute at space-like separations; and, the
fourth and the last axiom requires asymptotic completeness, i.e., that the Hilbert spaces
\( \mathcal{H}^\pm \) of asymptotic states be isomorphic with the total Hilbert space. Thus, four of the five
axioms derive their meaning directly from Minkowskian structure. It seems quite difficult
to extend the zeroth and the fourth axioms already to quantum field theory in topologically
non-trivial space-times, leave alone to the context in which there is no background metric
what so ever. And if we just drop these axioms, we are of course left with a framework
that is too loose to be useful.

The situation is similar with the Osterwalder-Schrader system. One might imagine
foregoing the use of specific axioms and just using techniques from Euclidean quantum field
theory to construct a suitable mathematical framework. This is the view recently adapted
by some groups using computer simulations. These methods have had a great deal of
success in certain exactly soluble 2-dimensional models. The techniques involve dynamical
triangulations and have been extended to the Einstein theory in 4 dimensions (see, e.g.,
Agishtein & Migdal (1992)). Furthermore, there is some numerical evidence that there is a
critical point in the 2-dimensional parameter space spanned by Newton’s constant and the
cosmological constant, suggesting that the continuum limit of the theory may well exist.
This is an exciting development and interesting results have now been obtained by several
groups. Let us be optimistic and suppose that a well-defined Euclidean quantum theory of
gravity can actually be constructed. This would be a major achievement. Unfortunately, it
wouldn’t quite solve the problem at hand. The main obstacle is that, as of now, there is no
obvious way to pass from the Euclidean to the Lorentzian regime! The standard strategy
of performing a Wick-rotation simply does not work. First, we don’t know which time
coordinate to Wick-rotate. Second, even if we just choose one and perform the rotation,
generically, the resulting metric will not be Lorentzian but complex. The overall situation
is the following. Given an analytic Lorentzian metric, one can complexify the manifold
and extend the metric analytically. However, the resulting complex manifold need not admit any Euclidean
metric and the resulting complex space-time need not have any Lorentzian section.) This
is not just an esoteric, technical problem. Even the Lorentzian Kerr metric, which is stationary, does not admit an Euclidean section. Thus, even if one did manage to solve the highly non-trivial problem of actually constructing an Euclidean theory using the hints provided by the computer work, without a brand new idea we would still not be able to answer physical questions that refer to the Lorentzian world. More generally –i.e., going beyond computational physics– one can hope to use the Euclidean techniques for some specific calculations tailored to suitable approximations. However, for the construction of a consistent quantum field theory, as of now, the Euclidean techniques seem to be of little help.

Why not then try canonical quantization? This method lacks manifest covariance. Nonetheless, as we will see, one can construct a Hamiltonian framework without having to introduce any background fields, thereby respecting the diffeomorphism invariance of the theory. However, from the perspective of mathematical physicists, the structure of the resulting framework has an unusual feature which one has never seen in any of the familiar field theories: it is a dynamically constrained system. That is, most of the non-trivial content of the theory lies in its constraints. To see how this comes about, let us first consider Yang-Mills theory in Minkowski space. As is well-known, the Hamiltonian description of this theory has a constraint –the Gauss law. It tells us that, only those points \((A^i_a(x), E^a_i(x))\) of the phase space represent physical states of the classical theory for which the electric field \(E^a_i\) has zero covariant divergence, i.e., \((A^i_a, E^a_i)\) satisfies the constraint \(D_a E^a_i = 0\). The canonical transformations generated by this functional corresponds precisely to gauge transformations on \((A^i_a, E^a_i)\). In quantum theory, the corresponding operator equation is imposed on wave functionals \(\Psi(A)\) to select the physical states: A state is physical if and only if \(\hat{D}_a \hat{E}^a \circ \Psi(A) = 0\). This operator equation tells us simply that the physical states are gauge invariant: \(\Psi(A) = \Psi(A^g)\), where \(A^g\) is the transform of \(A\) under a local gauge transformation \(g(x)\). The dynamics on these states is generated by the Hamiltonian operator which, being gauge invariant, maps physical states to physical states. Let us return to general relativity. Now, the group of space-time diffeomorphisms is the “gauge group” of the theory. Hence, in the Hamiltonian framework, there are four constraints, three corresponding to “spatial” diffeomorphism on the 3-manifold fixed in the construction of the phase space and one corresponding to diffeomorphisms in “time-like directions” transverse to this surface. The canonical transformation generated by this last constraint functional defines the dynamics of the theory; it is therefore called the Hamiltonian constraint. Thus, we have a peculiar situation: on physical classical states, the generator of dynamics –the Hamiltonian– vanishes identically! Suddenly, then, mathematical physicists find themselves in an unfamiliar territory and all the experience gained from canonical quantization of field theories in 2 or 3 dimensions begins to look not so relevant. Normally, the key problem is that of finding a suitable representation of the CCRs –or, an appropriate measure on the space of states– which lets the Hamiltonian be

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† For simplicity I am assuming that the spatial slices are compact. In the asymptotically flat case, the Hamiltonian generating dynamics does not vanish identically; it equals a surface term. The total energy of the theory is thus analogous to the charge integral in QED. This situation is generic to diffeomorphism invariant theories; it is not restricted to the Einstein-Hilbert action.
self-adjoint. Now, the Hamiltonian seems trivial but all the difficulty seems concentrated in the quantum constraints. Furthermore, while the representation of the CCRs is to be chosen prior to the imposition of constraints, the scalar product on the space of states has physical meaning only after the constraints are solved. One thus needs to develop new strategies and modify the familiar quantization program appropriately.

Finally, one may imagine using techniques from geometrical quantization (see, e.g., Woodhouse 1980). However, since the key difficulty lies in the constraints of the theory, a “correct” polarization would be the one which is preserved, in an appropriate sense, by the Hamiltonian flows generated by the constraint functions on the phase space (Ashtekar & Stillermann, 1986). The problem of finding such a polarization is closely related to that of obtaining a general solution to Einstein’s equation and therefore seems hopelessly difficult in 4 dimensions. (Incidentally, in 3 dimensions, the strategy does work but only because there are no local degrees of freedom; every solution to the field equations is flat.) One may imagine using instead the Dirac (1964) approach to quantization of constrained systems: as in Yang-Mills theory, use the operator constraints to select physical states. This is in fact the procedure one uses in familiar examples such as a free relativistic particle: the classical constraint, $P^\alpha P_\alpha + \mu^2 = 0$ provides us, via the Dirac strategy, the Klein-Gordon equation $\eta^{\alpha\beta} \partial_\alpha \partial_\beta \Phi - \mu^2 \Phi = 0$ which incorporates quantum dynamics. Indeed, in the next two sections, we will use this strategy. We will see however that the representation best suited for solving the quantum constraints in this framework does not arise from any polarization what so ever on the phase space. Thus, unfortunately, geometric quantization techniques do not seem to be well suited to this problem. Finally, one might imagine group theoretic method of quantization. An appropriate canonical group was in fact found (Isham & Kakas, 1984a,b) and it does “interact” well with the constraints of the theory which generate spatial diffeomorphisms (and triad rotations). However, the problem of incorporating the Hamiltonian constraint which generates dynamics again seems hopelessly difficult.

As the discussion suggests, one needs a new quantization program that can handle the peculiarities of general relativity. Such a program does exist. The basic ideas were introduced by Dirac already in the sixties and have been refined over the years by many authors. The approach to quantum general relativity that I will discuss in the next two sections is based on the framework developed in Ashtekar (1991) (chapter 10; see also Ashtekar & Tate 1993).

4. A mathematical framework for quantum general relativity

In this section, I will present a number of recent results that provide a mathematical framework for non-perturbative, canonical quantization of general relativity in 3 and 4 space-time dimensions. This is not an exhaustive treatment of canonical quantum gravity. I will focus only on a few topics and omit most of the details. I will, however, provide references where these can be found. My aim is to illustrate the type of results that have been obtained and provide a feel of the current status of the field from the perspective of mathematical physics.

This section is divided into three parts. In the first, I will collect some basic results that relate general relativity to theories of connections. In the second—and the main part—
I will report on some recent work on calculus on the space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations. In the third part, I will indicate how these results are now being used in quantum gravity.

4.1 Preliminaries

As I mentioned in the Introduction, it is possible to regard general relativity as a dynamical theory of connections. In 3 space-time dimensions, the connection in question is simply the spin-connection which enables one to transport the $SU(1,1)$ spinors along curves. In 4 dimensions, the appropriate connection turns out to be chiral; it enables one to transport (say) left-handed spinors along curves. In both cases, one can construct a Hamiltonian formulation in which all the equations are low order polynomials in the connection and its “canonically conjugate variable” —the analog of the electric field of Yang-Mills theory. However, unlike in the Yang-Mills theory, this field, $E^a_i$, has a dual interpretation: it can be regarded as a square root of the spatial metric. (In the 4-dimensional theory, this is just a spatial triad.) This dual interpretation enables one to pass back and forth between Yang-Mills theory and general relativity. In particular, the effect of Yang-Mills gauge transformations on the electric field can be re-interpreted as a triad rotation, which leaves the metric invariant—it is thus a gauge transformation also from the perspective of general relativity. Consequently, the field $E^a_i$ is constrained to be divergence-free also in general relativity. However, as noted in section 3, general relativity has additional constraints, a vectorial constraint that implies that physical states should be invariant under spatial diffeomorphisms and a scalar (or Hamiltonian) constraint which encodes dynamics of the theory.

Thus, in this connection-dynamics formulation of general relativity, the phase space, to begin with, is the same as in the Yang-Mills theory. Furthermore, the constraint sub-manifold of the phase space in general relativity is embedded in the constraint sub-manifold of the Yang-Mills theory. The difference lies, of course, in dynamics. In the Yang-Mills theory, it is generated by the Hamiltonian,

$$H(A, E) = \int d^3 x (E^a_i E^i_a + B^a_i B^i_a), \quad (4.1)$$

which is (gauge invariant but otherwise) unrelated to the constraint (i.e., the Gauss law). In general relativity, there are additional constraints and time-evolution is coded in one of them. Hence, if one can solve the quantum constraints, one has essentially tackled the issue of dynamics. We will not need the explicit form of the constraints in what follows.

Let us conclude the preliminaries by noting that a key simplification arises in 3 dimensions: the general relativity constraints imply that the connection must be flat. Thus, in this case, the essence of the entire theory is contained in just two constraints: the first is the Gauss law which ensures gauge invariance and the second is flatness which implies that the $SU(1,1)$ connection—which serves as the configuration variable—must be flat. In the classical theory, the two constraints tell us that all dynamical trajectories correspond to flat metrics; there are no local degrees of freedom, no gravity waves. In the quantum theory, the imposition of these constraints implies that the physical states $\Psi(A)$ are gauge invariant functions of flat connections —i.e., are functions on the moduli space of flat connections on the given spatial, 2-manifolds. Since these spaces are finite dimensional, we
have quantum mechanics rather than quantum field theory. In 4 dimensions, of course, no such simplification occurs. We again have the general relativity constraints in addition to the Gauss law. However, these do not imply that the connections must be flat. In the classical theory, there are local degrees of freedom, gravity waves and rich, albeit complicated, dynamics. In the quantum theory, we have a genuine field theory with infinite number of degrees of freedom.

For details, see, e.g., Ashtekar (1987, 1991) and Romano (1993).

4.2 Integration on the space $A/G$

As we saw in sections 1 and 3, any attempt at non-perturbative quantization of gravity faces a host of conceptual problems. 3-dimensional general relativity is an excellent toy model to see how these problems can be faced since it has been solved exactly (see, e.g., Witten (1988), Ashtekar et al (1989), Nelson & Regge (1989).) However, one needs a new mathematical framework and a variety of new techniques. Indeed, even the finished theory contains a number of unfamiliar notions. The basic observables are essentially non-local; the familiar operator-valued distributions are notably absent. To begin with, there is no Hamiltonian; indeed, no time to evolve anything in. There is no underlying space-time and hence no microcausality. Yet, the Hilbert space is well-defined and the observables are represented by self-adjoint observables. One can, if one wishes, regard a suitable dynamical variable as time and see how states “evolve” relative to this “internal clock.” One can show that this evolution is unitary. There are physical predictions. The theory is in a form that is unfamiliar from, say, constructive quantum field theory but has as much physical content as one can hope for. Therefore, in my talk I discussed this case in some detail. In this report, however, I will forego that discussion since there are a number of reviews on the subject (see, e.g., Carlip (1990, 1993) and Ashtekar (1991), chapter 17) Instead, I will focus here on recent developments which encompass the 4-dimensional theory.

A key mathematical problem is to develop integration theory on the space $A/G$ of connections modulo gauge transformations since, heuristically, this is the domain space of quantum states. In the 3-dimensional case, this problem is easy to solve because the (appropriate components of the) moduli space of flat $SU(1, 1)$ connections can be naturally given the structure of a finite dimensional symplectic manifold (see e.g. Ashtekar(1991), chapter 17); one can simply use the Liouville volume element to perform integration. Thus, although the domain space of quantum states is non-linear, the integration theory is simple because the space is finite dimensional. In 4 space-time dimensions, the situation is again simple for the case of linearized gravity –the theory of free gravitons in Minkowski space. This theory can be cast in the language of connections (see, e.g., Ashtekar (1991), chapter 11). Integration theory is again well-developed; the domain space is now linear and one can simply use the Gaussian measure as in free field theories in Minkowski space. Thus, in this case, in spite of the presence of an infinite number of degrees of freedom, the integration theory is straightforward because of the underlying linearity. In the case of full, non-linear general relativity –and also for Yang-Mills theory– in 4-dimensions the problem is significantly more difficult because $A/G$ is both non-linear and infinite dimensional. Fortunately, a rigorous approach has now become available to tackle this problem. In particular, a diffeomorphism invariant measure has been found. I will now outline these developments and indicate in the next sub-section how they can be used in quantum gravity.
Fix an analytic 3-manifold $\Sigma$ which will represent a Cauchy surface in space-times to be considered. We will consider $SU(2)$ connections on $\Sigma$. Since any $SU(2)$ bundle over a 3-manifold is trivial, we can represent any connection by a Lie-algebra valued 1-form $A^a_i$ on $\Sigma$, where $a$ is the spatial index and $i$, the internal. Denote by $\mathcal{A}$ the space of smooth (say $C^2$) $SU(2)$-connections equipped with one of the standard (Sobolev) topologies (see, e.g. Mitter & Viallet (1981)). $\mathcal{A}$ has the structure of an affine space. However, what is of direct interest to us is the space $\mathcal{A}/G$ obtained by taking the quotient of $\mathcal{A}$ by $(C^3)$ local gauge transformations. In this projection, the affine structure is lost; $\mathcal{A}/G$ is a genuinely non-linear space with complicated topology. To define the integration theory, we will begin by constructing a sub-algebra of the Abelian $C^\star$-algebra of bounded functions on $\mathcal{A}/G$ and the desired measures will arise from positive linear functionals on this $C^\star$-algebra. Given any closed loop $\alpha$ on the 3-manifold, we can define the Wilson-loop functional $T_\alpha$ on $\mathcal{A}/G$:

$$T_\alpha(A) := \frac{1}{2} \text{Tr} \mathcal{P} \exp G \oint_\alpha A.dl, \quad (4.2)$$

where the trace is taken in the fundamental representation of $SU(2)$ and the Newton’s constant $G$ appears because, in general relativity, it is $G A^a_i$ that has the dimensions of a connection; in gauge theories, of course, this factor would be absent. For technical reasons, we will have to restrict ourselves to piecewise analytic loops $\alpha$. (This is why we needed $\Sigma$ to be analytic. Note that the loops need not be smooth; they can have kinks and intersections but only at a finite number of points.) It turns out that, due to $SU(2)$ trace identities, product of any two Wilson-loop functionals can be expressed as a sum of other Wilson loop functionals. Therefore, the vector space generated by finite complex-linear combinations of these functions has the structure of a $\star$-algebra. The functionals $T_\alpha$ are all bounded (between $-1$ and 1). Hence, the sup-norm (over $\mathcal{A}/G$) is well-defined and we can take the completion to obtain a $C^\star$-algebra. We will call it the holonomy $C^\star$-algebra and denote it by $\mathcal{H}A$. Elements of $\mathcal{H}A$ are to be thought of as the configuration variables of the theory.

Since $\mathcal{H}A$ is an Abelian $C^\star$-algebra with identity, we can apply the Gel’fand theory and conclude that $\mathcal{H}A$ is isomorphic with the $C^\star$-algebra of all continuous functions on a compact, Hausdorff space $\text{sp}(\mathcal{H}A)$, the spectrum of the given $C^\star$-algebra $\mathcal{H}A$. Furthermore, since elements of $\mathcal{H}A$ suffice to separate points of $\mathcal{A}/G$, it follows that $\mathcal{A}/G$ is densely embedded in $\text{sp}(\mathcal{H}A)$. To emphasize this point, from now on, we will denote the spectrum by $\overline{\mathcal{A}/G}$ and regard it as a completion of $\mathcal{A}/G$ (in the Gel’fand topology). Integration theory will be defined on $\overline{\mathcal{A}/G}$. This is in accordance with the common occurrence in quantum

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† Results reported in this section for which explicit references are not provided are all taken from Ashtekar and Isham (1992) and Ashtekar and Lewandowski (1993a,b). The last two papers and those by Baez (1993a,b) contain significant generalizations which include allowing more general gauge groups, allowing the manifold $\Sigma$ to be of arbitrary dimension and allowing the connections to live in non-trivial bundles.

‡ The letter $T$ stands for trace. We will later define other $T$-variables which have the information about the “electric field” $E^a_i$ as well. Although defined on $\mathcal{A}$, being gauge invariant, functions $T_\alpha$ project down to $\overline{\mathcal{A}/G}$ unambiguously.
field theory: while the classical configuration (or phase) space may contain only smooth fields (typically taken to belong to be the Schwartz space), the domain space of quantum states is a completion of this space in an appropriate topology (the space of distributions).

A key difficulty with the use of the Gel'fand theory is that one generally has relatively little control on the structure of the spectrum. In the present case, however, we are more fortunate: a simple and complete characterization of the spectrum is available. To present it, I first need to introduce a key definition. Fix a base point \( x_o \) in the 3-manifold \( \Sigma \) and regard two (piecewise analytic) closed loops \( \alpha \) and \( \alpha' \) to be equivalent if the holonomy of any connection in \( \mathcal{A} \), evaluated at \( x_o \), around \( \alpha \) is the same as that around \( \alpha' \). We will call each equivalence class (a holonomically equivalent loop or ) a \( \text{hoop} \) and denote the hoop to which a loop \( \alpha \) belongs by \( \tilde{\alpha} \). For example, \( \alpha \) and \( \alpha' \) define the same hoop if they differ by a reparametrization or by a line segment which is immediately re-traced\(^\dagger\). The set of hoops has, naturally, the structure of a group. We will call it the \( \text{hoop group} \) and denote it by \( HG \). In terms of this group, we can now present a simple characterization of the Gel'fand spectrum \( \mathcal{A}/G \):

Every homomorphism \( \hat{H} \) from the hoop group \( HG \) to the gauge group \( SU(2) \) defines an element \( \tilde{A} \) of the spectrum \( \mathcal{A}/G \) and every \( \tilde{A} \) in the spectrum defines a homomorphism \( \hat{H} \) such that \( \hat{A}(\tilde{\alpha}) = \frac{1}{2} \text{Tr} \hat{H}(\tilde{\alpha}) \). This is a 1-1 correspondence modulo the trivial ambiguity that homomorphisms \( \hat{H} \) and \( g^{-1} \cdot \hat{H} \cdot g \) define the same element \( \tilde{A} \) of the spectrum.

Clearly, every regular connection \( A \) in \( \mathcal{A} \) defines the desired homomorphism simply through the holonomy operation: \( \hat{H}(\tilde{\alpha}) := \mathcal{P} \exp G \oint_{\alpha} A.dl \), where \( \alpha \) is any loop in the hoop \( \tilde{\alpha} \). However, there are many homomorphisms which do not arise from smooth connections. This leads to “generalized connections” –i.e. elements in \( \mathcal{A}/G - \mathcal{A}/G \). In particular, there exist \( \tilde{A} \) in \( \mathcal{A}/G \) which have support at a single point and are thus “distributional. Note that this characterization of the spectrum \( \mathcal{A}/G \) is \textit{completely algebraic}; there is no continuity assumption on the homomorphisms. This property makes the characterization very useful in practice.

From the general representation theory of \( C^*\)-algebras, it follows that positive linear functionals on \( HA \) are in 1-1 correspondence with regular measures on (the compact Hausdorff space) \( \mathcal{A}/G \). It turns out that the positive linear functions, in turn, are determined completely by certain \"generating functionals\" \( \Gamma(\alpha) \) on the space \( \mathcal{L}_{x_o} \) of loops based at \( x_o \):

There is a 1-1 correspondence between positive linear functionals on \( HA \) (and hence regular measures on \( \mathcal{A}/G \)) and functional \( \Gamma(\alpha) \) on \( \mathcal{L}_{x_o} \) satisfying:

1. \( \sum_i a_i T_{\alpha_i} = 0 \Rightarrow \sum_i a_i \Gamma(\alpha_i) = 0 \); and,
2. \( \sum_{i,j} a_i a_j (\Gamma(\alpha_i \circ \alpha_j) + \Gamma(\alpha_i \circ \alpha_j^{-1}) \geq 0 \),

for all loops \( \alpha_i \) and complex numbers \( a_i \).

The first condition implies that the functional \( \Gamma \) is well-defined on hoops. Hence we could have taken it to be a functional on \( HG \) from the beginning. Thus, we see that there is a

\(^\dagger\) For piecewise analytic loops and \( SU(n) \) connections, these two are the most general operations; two loops define the same hoop if and only if they are related by a combination of reparametrizations and retracings.
nice “non-linear duality” between the spectrum $\overline{A/G}$ and the hoop group $\mathcal{H}G$: Elements of $\overline{A/G}$ are homomorphisms from $\mathcal{H}G$ to $SU(2)$ and regular measures on $\overline{A/G}$ correspond to certain functionals on $\mathcal{H}G$. Finally, if one is interested in measures on $\overline{A/G}$ which are invariant under the (induced) action of diffeomorphisms on $\Sigma$, one is led to seek functionals $\Gamma(\alpha)$ which depend not on the individual loops $\alpha$ but rather on the (generalized) knot class to which $\alpha$ belongs. (The qualification “generalized” refers to the fact that here we are considering loops which can have kinks, overlaps and self-intersections. Until recently, knot theorists considered only smoothly embedded loops.) Thus, there is an interesting –and potentially powerful– interplay between knot theory and representations of the holonomy algebra $\mathcal{H}A$ in which the diffeomorphism group of $\Sigma$ is unitarily implemented.

Finally, we can make the integration theory more explicit. Consider a subgroup $S_n$ of the hoop group $\mathcal{H}G$ which is generated by $n$ (independent) hoops. We can introduce the following equivalence relation on $\overline{A/G}$: $\bar{A} \equiv \bar{A}'$ if and only if their action on all elements of $S_n$ coincides, i.e., if and only if $\bar{A}(\bar{\alpha}) = g^{-1} \cdot \bar{A}'(\bar{\alpha}) \cdot g$ for all $\bar{\alpha} \in S_n$ and some (hoop independent) $g \in SU(2)$. It turns out that the quotient space is isomorphic to $[SU(2)]^n/Ad$. Therefore, we can introduce a notion of cylindrical functions on $\overline{A/G}$: A function $f$ on $\overline{A/G}$ will be said to be cylindrical if it is the pull-back to $\overline{A/G}$ of a smooth function $\tilde{f}$ on $[SU(2)]^n/Ad$ for some sub-group $S_n$ of the hoop group. Finally, we can define integrals of these functions $f$ on $\overline{A/G}$ through their integrals on $[SU(2)]^n/Ad$, provided of course we equip $[SU(2)]^n/Ad$ with suitable measures $d\mu_n$ for each $n$. We can then define a positive linear functional $\Gamma'$ on the space of cylindrical functions $f$ via:

$$\Gamma'(f) := \int_{[SU(2)]^n/Ad} \tilde{f} d\mu_n .$$

(4.3)

For the functional to be well-defined, of course, the family of measures $d\mu_n$ on $[SU(2)]^n/Ad$ must satisfy certain consistency conditions. It turns out that these requirements can be met and, furthermore, the resulting functionals $\Gamma'(f)$ define regular measures on $\overline{A/G}$. A particularly natural choice (and, not surprisingly, the first to be discovered) is to let $d\mu_n$ be simply induced on $[SU(2)]^n/Ad$ by the Haar-measure on $SU(2)$. We then have the following results:

i) The consistency conditions are satisfied; the left side of (4.3) is well-defined for all cylindrical functions $f$ on $\overline{A/G}$;

ii) The generalized holonomies $T_{\bar{\alpha}}$ are cylindrical functionals on $\overline{A/G}$ and $\Gamma(\alpha) := \Gamma'(T_{\bar{\alpha}})$ defined via (4.3) serves as a generating functional for a faithful, cyclic representation of the homonomy $C^*$-algebra $\mathcal{H}A$ which ensures that $d\mu$ is a regular, strictly positive measure on $\overline{A/G}$;

iii) The measure $d\mu$ is invariant under the induced action of the diffeomorphism group on $\Sigma$.

(The knot invariant defined by $d\mu$ is a genuinely generalized one; roughly, it counts the number of self-overlaps in any given loop.)

This measure is in some ways analogous to the Gaussian measure on linear vector spaces. Both can be obtained by a “cylindrical construction.” The Gaussian measure uses the natural metric on $\mathbb{R}^n$ while the above measure uses the natural (induced) Haar
measure on $[SU(2)]^n/Ad$. They are both regular and strictly positive. This leads us to ask if other properties of the Gaussian measure are shared. For instance, we know that the Gaussian measure is concentrated on distributions; although the smooth fields are dense in the space of distributions in an appropriate topology, they are contained in a set whose total measure is zero. Is the situation similar here? The answer turns out to be affirmative.

The classical configuration space $A/G$ with which we began is dense in the domain space $\overline{A/G}$ of quantum states in the Gel’fand topology. However, $A/G$ is contained in a set whose total measure is zero. The measure is again concentrated on “generalized” connections in $\overline{A/G}$ (Marolf & Mourão (1993)). In a certain sense, just as the Gaussian measures on linear spaces originate in the harmonic oscillator, the new measure on $\overline{A/G}$ originates in a (generalized) rotor (whose configuration space is the $SU(2)$ group-manifold). However, the measure is, so to say, “genuinely” tailored to the underlying non-linearity. It is not obtained by “perturbing” the Gaussian measure.

With the measure $d\mu$ at hand, we can consider the Hilbert space $L^2(\overline{A/G}, d\mu)$ and introduce operators on it. This is not the Hilbert space of physical states of quantum gravity since we have not imposed constraints. It is a fiducial, kinematical space which enables us to regularize various operators (in particular, the quantum constraint operators). The configuration operators are associated with the generalized Wilson loop functionals: $\hat{T}_{\tilde{\alpha}} \circ \Psi(\tilde{A}) = \tilde{A}(\tilde{\alpha})\Psi(\tilde{A})$. One can show that there are bounded, self-adjoint operators on the Hilbert space. There are also “momentum operators” – associated with closed, 2-dimensional ribbons or strips in the 3-manifold $\Sigma$ – which are gauge invariant and linear in the electric field. One can show that these are also self-adjoint (but unbounded). Finally, since $d\mu$ is invariant under the induced action of the diffeomorphism group of $\Sigma$, this group acts unitarily.

The next task is to represent quantum constraints as well-defined operators on the Hilbert space and then solve them, i.e., find their kernels. The first step has been completed for the three vector constraints of general relativity. As for the kernel, typically, the constraint operators are self-adjoint on the kinematical Hilbert space and zero is in the continuous part of their spectrum. The physical states –elements of the kernel– are thus not normalizable; they do not belong to the Hilbert space. Rather, they belong to the (appropriately constructed) rigged Hilbert spaces (Hajiček, 1993). Consider for example, the simple case of a free relativistic particle, where the classical constraint is $P^\alpha P_\alpha + \mu^2 = 0$. In this case, the kinematical Hilbert space can be taken to be $L^2(\mathbb{R}^4)$. This space is needed to translate the classical constraint function to a well-defined operator (whose kernel can then be found). The operator, of course, is $\eta^{\alpha\beta} \partial_\alpha \partial_\beta - \mu^2$. No (non-zero) element in its kernel is normalizable in $L^2(\mathbb{R}^4)$. These elements belong to the rigged Hilbert space; in the momentum space, they are distributions with support on the mass shell. One wishes to carry out a similar construction in the gravitational case. For this, one needs to introduce the appropriate rigged Hilbert spaces. This is an open problem where input from the mathematical physics community would be most useful. To summarize, the present status is that the operators representing the vector constraints of general relativity are well-defined and self-adjoint on $L^2(\overline{A/G}, d\mu)$. As we will see in the next sub-section, it is intuitively clear what their kernel is. What is needed is a precise, rigorous result. For the Hamiltonian constraint, we are yet to show that the operator is well-defined and all
considerations are heuristic at this stage.

Thus, the integration theory based on the measure $d\mu$ is being used as the mathematical basis in the quantization of general relativity in the connection-dynamics approach. However, these techniques may be used also in other theories of connections which are diffeomorphism invariant and perhaps even in Yang-Mills theory which is not diffeomorphism invariant. We saw that the full domain space of quantum theory, $\overline{A/G}$, can be thought of as the space of homomorphisms from the full hoop group $\mathcal{H}$ to the gauge group $SU(2)$. Given a finitely generated sub-group $S_n$ of the hoop group, we can consider the space of homomorphisms from it to $SU(2)$. This provides the space $[SU(2)]^n/Ad$ which is precisely the domain space of quantum states of a lattice gauge theory where the lattice is not rectangular but tailored to the given subgroup $S_n$ of the gauge group. Thus, what we have is a set of “floating lattices,” each associated with a finitely generated subgroup of the hoop group. The space $\overline{A/G}$ can be rigorously recovered as a projective limit of the configuration spaces of lattice theories (Marolf & Mourão 1993). This construction is potentially quite powerful; it may enable one to take continuum limits of operators of lattice theories in a completely new fashion. The limit is obtained not by taking the lattice separation to zero but by enlarging lattices to probe the continuum connections better and better, i.e., by considering larger and larger subgroups of the hoop group.

4.3 Loop representation of quantum gravity

In non-perturbative quantum gravity, to date, most progress has been made in the so-called “loop representation” in which quantum states arise as functionals of closed loops on the 3-manifold. This representation can also be used in, e.g., Yang-Mills theories (Loll 1992) and has led to concrete results in the lattice formulations (Brügmann 1991). However, the representation is particularly well-suited in the gravitational case since it seems to be best suited for solving the quantum constraints. In particular, the solutions to the vector (or diffeomorphism) constraints, referred to in the last sub-section, are explicit in this representation. In this sub-section, I will outline various results that have been obtained so far in this framework. From the perspective of mathematical physics, some of the results I will report here are still heuristic. However, there does not seem to be any difficulty of principle in making them rigorous. This may well be a fertile area for young researchers in the field.

The loop representation was introduced as a heuristic device by Rovelli and Smolin (1990) in the context of quantum gravity (and somewhat earlier, but in a somewhat different fashion, by Gambini and Trias (1986) in the context of Yang-Mills theory.) These ideas can be now made rigorous using the framework outlined in the last sub-section. Consider, as before, the Hilbert space $L^2(\overline{A/G}, d\mu)$ the elements of which are functionals $\Psi(\overline{A})$ of generalized connections $\overline{A}$. Since the generalized Wilson loop functionals $T_\alpha(\overline{A}) := \text{Tr} \overline{A}(\alpha)$ belong to this space, we can define the following transform:

$$\psi(\alpha) := \int_{\overline{A/G}} T_\alpha(\overline{A})\Psi(\overline{A}) \ d\mu , \quad (4.4)$$

to pass from functionals $\Psi(\overline{A})$ of generalized connections to functionals $\psi(\alpha)$ of hoops†.

† Since the left side is a function of hoops, we should, strictly, use the terms hoop repre-
This has some similarities with the Fourier transform

\[ \psi(k) := \int_{-\infty}^{\infty} e^{ik \cdot x} \Psi(x) \, dx , \]  

(4.5)

that enables one to pass from the position to the momentum representation. The role of the integral kernel, \( \exp ik \cdot x \) is now played by the generalized Wilson loop functional. The Rovelli-Smolin transform is faithful. However, a nice characterization of the space of functionals \( \psi(\tilde{\alpha}) \) obtained through this transform is not available. In particular, the integration theory on the hoop group \( \mathcal{HG} \) has not yet been developed and therefore, at present, there is no analog of the Plancharel theorem which makes the Fourier transform so powerful. This is one of the open problems looking longingly to mathematical physicists for help.

The transform as it is written above is however well-defined. Therefore one can take operators from the connection side and write them on the hoop side. Typically, these involve simple geometric operations on the loop arguments – composing, breaking, and rerouting loops. For example, the generalized Wilson-loop operators which act on \( L^2(\mathcal{A}/\mathcal{G}, d\mu) \) simply by multiplication, \( \hat{T}_\beta \circ \Psi(A) = A(\beta)\Psi(A) \), can be transported to the loop states and their action is given by:

\[ (\hat{T}_\beta \circ \psi)(\tilde{\alpha}) = \frac{1}{2} (\psi(\tilde{\alpha} \cdot \tilde{\beta}) + \psi(\tilde{\alpha} \cdot \tilde{\beta}^{-1})) , \]  

(4.6)

where \( \tilde{\alpha} \cdot \tilde{\beta} \) is the hoop obtained by composing \( \tilde{\alpha} \) and \( \tilde{\beta} \) in the hoop group. Similarly, one can transport other operators. Of particular interest are the vector or the diffeomorphism constraint operators. There is one such operator \( \hat{C}(V) \) associated with every vector field \( V^a \) on \( \Sigma \) and their action is given simply by:

\[ \hat{C}(V) \circ \psi(\alpha) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (\psi(\alpha^V_\epsilon) - \psi(\alpha)) , \]  

(4.7)

where \( \alpha^V_\epsilon \) is the loop obtained by displacing \( \alpha \) along the integral curves of \( V^a \) an affine parameter distance \( \epsilon \). Thus, the regularized operator corresponding to the diffeomorphism constraint does what one intuitively expects it to do: it drags the loop in the argument of the wave function along the diffeomorphism. Therefore, it is intuitively clear that the kernel of this constraint consists of functionals of loops which remain unchanged if the loop is replaced by a diffeomorphic one. That is, the elements of the kernel are functions of generalized knot classes on the 3-manifold. It is remarkable that in the loop representation, one can write down the general solution to the three of the four constraints of general relativity in a simple, geometrical way and the solutions relate quantum gravity with knot theory. As I remarked earlier, one would like to make these results rigorous by specifying the regularity conditions on the permitted knot invariants using rigged Hilbert spaces.

sentation and the hoop transform. However, in various calculations, it is often convenient to lift these functionals from the hoop group \( \mathcal{HG} \) to the space of loops \( \mathcal{L}_{x_0} \). Therefore, as in most of the literature on the subject, we will not keep a careful distinction between loops and hoops in what follows.
The remaining (Hamiltonian) constraints are more difficult. They have been transcribed in the loop representation and various apparently distinct methods of doing so have led to equivalent results (Brügmann & Pullin (1993)). Thus, there is a general feeling that one is on the “right track.” However, these results are not rigorous. A promising new direction is being pursued by Gambini and his collaborators where the hoops group is replaced by a group of “smoothened out” hoops (Dì Bortolo et al, 1993). This group has the structure of a Lie group and offers a new approach to the problem of regularization of various operators including the Hamiltonian constraints. This approach has already led to some interesting, new solutions to the Hamiltonian constraints which are related to some well-known knot invariants (Brügmann et al, 1992a,b). This general area of research related to the Hamiltonian constraint is one of the centers of current activities although one seems quite far from finding the generic solution to these constraints even at a heuristic level.

5. Waving a classical geometry with quantum threads

I will now discuss two striking results that have emerged –already at a kinematical level, prior to the imposition of quantum constraints– from the loop representation (Ashtekar et al 1992). The first is that certain operators representing geometrical observables can be regulated in a way that respects the diffeomorphism invariance of the underlying theory. What is more, these regulated operators are finite without any renormalization. Using these operators, one can ask if there exist loop states which approximate smooth geometry at large scales. One normally takes for granted that the answer to such questions would be obviously “yes.” However, in genuinely non-perturbative treatments, this is by no means clear a priori; one may be working in a sector of a theory which does not admit the correct or unambiguous classical limit. For example, the sector may correspond to a confined phase which has no classical analog or the limit may yield a wrong number even for the macroscopic dimensions of space-time! The second main result of this subsection is that not only is the answer to the question raised above in the affirmative but, furthermore, these states exhibit a discrete structure of a definite type at the Planck scale. (For further details, see, e.g., Rovelli & Smolin (1990) and Ashtekar (1992), Smolin (1993.).)

Let us begin with the issue of regularization. As noted in section 4, in the present framework, the spatial metric is constructed from products of “electric fields” $E^a_i$. It is a “composite” field given by $q^{ab}(x) = E^{ai}(x)E^b_i(x)$. In the quantum theory, therefore, this operator must be regulated. The obvious possibility is point splitting. One might set $q^{ab}(x) = \lim_{y \to x} E^{ai}(x)E^b_i(y)$. However, the procedure violates gauge invariance since the internal indices at two different points have been contracted. There is, however, a suitable modification that will ensure gauge invariance. Consider the field $T^{a\alpha}[\alpha](y', y)$, labelled

\footnote{More precisely, the situation is as follows. Since there is no background metric, the “momenta” $E^a_i$ are actually vector densities of weight one, whence the composite field $q^{ab}$ is of density weight two; it is the determinant of the covariant metric multiplied by the contravariant metric. In what follows, these density weights are important for the details of the arguments. However, for brevity, I will not dwell on this point any further.}
by a closed loop $\alpha$ and points $y$ and $y'$ thereon, defined in the classical theory by:

$$T^{aa'}[\alpha](y, y') := \frac{1}{2} \text{Tr} \left[ (P \exp G \int_{y}^{y'} A_b dl^b) E^a(y') (P \exp G \int_{y}^{y'} A_c dl^c) E^{a'}(y) \right]. \quad (5.1)$$

In the limit $\alpha$ shrinks to zero, $T^{aa'}[\alpha](y, y')$ tends to $-4q^{aa'}$.

Now, in quantum theory, one can define the action of the operator $\hat{T}^{aa'}[\alpha](y, y')$ directly on the loop states $\psi(\beta)$. The explicit form will not be needed here. We only note that using the bra-ket notation, $\psi(\beta) = \langle \beta | \psi \rangle$ the action can be specified easily. Indeed, $\langle \beta | \circ \hat{T}^{aa'}[\alpha](y, y')$ is rather simple: if a loop $\beta$ does not intersect $\alpha$ at $y$ or $y'$, the operator simply annihilates the bra $\langle \beta |$ while if an intersection does occur, it breaks and re-routes the loop $\beta$, each routing being assigned a specific weight. One may therefore try to define a quantum operator $\hat{q}^{aa'}$ as a limit of $\hat{T}^{aa'}[\alpha]$ as $\alpha$ shrinks to zero. The resulting operator does exist after suitable regularization and renormalization. However (because of the density weights involved) the operator necessarily carries a memory of the background metric used in regularization. Thus, the idea of defining the metric operator again fails. In fact one can give general qualitative arguments to say that there are no local, operators which carry the metric information and which are independent of background fields (used in the regularization). Thus, in quantum theory, the absence of background fields introduces new difficulties. That such difficulties would arise was recognized quite early by Chris Isham and John Klauder.

There do exist, however, non-local operators which can be regulated in a way that respects diffeomorphism invariance.

As the first example, consider the function $Q(\omega)$ representing the smeared 3-metric on the classical phase space, defined by

$$Q(\omega) := \int d^3x \ (q^{ab}\omega_a\omega_b)^{\frac{1}{2}}, \quad (5.2)$$

where $\omega_a$ is any smooth 1-form of compact support. It is important to emphasize that, in spite of the notation, $Q(\omega)$ is not obtained by smearing a distribution with a test field; because of the square-root, $Q(\omega)$ is not linear in $\omega$. We can, nonetheless define the corresponding quantum operator as follows. Let us choose on $\Sigma$ test fields $f_\epsilon(x, y)$ (which are densities of weight one in $x$ and) which satisfy:

$$\lim_{\epsilon \to 0} \int_{\Sigma} d^3x f_\epsilon(x, y) g(x) = g(y) \quad (5.3)$$

for all smooth functions of compact support $g(x)$. If $\Sigma$ is topologically $\mathbb{R}^3$, for example, we can construct these test fields as follows:

$$f_\epsilon(x, y) = \sqrt{\frac{h(x)}{\pi^2\epsilon^3}} \exp -|x - y|^2, \quad (5.4)$$

† Note that the integral is well-defined without the need of a background volume element because $q^{ab}$ is a density of weight two.

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where $\vec{x}$ are the cartesian coordinates labeling the point $x$ and $h(x)$ is a “background” scalar density of weight 2. Next, let us define

$$q_\epsilon^{a'a'}(x) = -\frac{1}{4} \int_\Sigma d^3y \int_\Sigma d^3y' f_\epsilon(x,y)f_\epsilon(x,y')T^{aa'}(y,y'). \tag{5.5}$$

As $\epsilon$ tends to zero, the right side tends to $q^{ab}$ because the test fields force both the points $y$ and $y'$ to approach $x$, and hence the loop passing through $y,y'$, used in the definition of $T^{aa'}(y,y')$, to zero. It is now tempting to try to define a local metric operator $\hat{q}^{aa'}$ corresponding to $q^{aa'}$ by replacing $T^{aa'}(y,y')$ in (5.5) by its quantum analog and then taking the limit. One finds that the limit does exist provided we first renormalize $\hat{q}^{aa'}$ by an appropriate power of $\epsilon$. However, as before, the answer depends on the background structure (such as the density $h(x)$) used to construct the test fields $f_\epsilon(x,y)$. If, however, one tries to construct the quantum analog of the non-local classical problem, we can set

$$\frac{1}{4} \int_\Sigma d^3x(q_\epsilon^{a'a'}\omega_a\omega_{a'})^{\frac{1}{2}}, \tag{5.6}$$

The required quantum operator $\hat{Q}(\omega)$ on the loop states can now be obtained by replacing $T^{aa'}(y,y')$ by the operator $\hat{T}^{aa'}(y,y')$. A careful calculation shows that: i) the resulting operator is well-defined on loop states; ii) no renormalization is necessary, i.e., the limit is automatically finite; and, iii) the final answer carries no imprint of the background structure (such as the density $h(x)$ or, more generally, the specific choice of the test fields $f_\epsilon(x,y)$) used in regularization. To write out its explicit expression, let me restrict myself to smooth loops $\alpha$ without any self-intersection. Then, the action is given simply by:

$$\langle \alpha \mid \circ \hat{Q}(\omega) = l_P^2 \int_\alpha ds |\dot{\alpha}^a\omega_a| \cdot |\alpha|, \tag{5.7}$$

where $l_P = \sqrt{G\hbar}$ is the Planck length, $s$, a parameter along the loop and $\dot{\alpha}^a$ the tangent vector to the loop. In this calculation, the operation of taking the square-root is straightforward because the relevant operators are diagonal in the loop representation. This is analogous to the fact that, in the position representation of non-relativistic quantum mechanics, we can set $<x|\circ (\hat{X}^2)^{\frac{1}{2}} =< x| \cdot |x|$ without recourse to the detailed spectral theory. The $G$ in $l_P$ of (5.7) comes from the fact that $GA^i_a$ has the usual dimensions of a connection while $\hbar$ comes from the fact that $\hat{E}_i^a$ is $\hbar$ times a functional derivative. The final result is that, on non-intersecting loops, the operator acts simply by multiplication: the loop representation is well-suited to find states in which the 3-geometry—rather than its time evolution—is sharp.

The second class of operators corresponds to the area of 2-surfaces. Note first that, given a smooth 2-surface $S$ in $\Sigma$, its area $A_S$ is a function on the classical phase space. We first express it using the classical loop variables. Let us divide the surface $S$ into a large number $N$ of area elements $S_I, I = 1, 2...N$, and set $A_I^{\text{appr}}$ to be

$$A_I^{\text{appr}} = -\frac{1}{4} \left[ \int_{S_I} d^2x \eta_{abc} \int_{S_I} d^2x' \eta_{a'b'c'} T^{aa'}(x,x') \right]^\frac{1}{2}, \tag{5.8}$$

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where \( \eta_{abc} \) is the (metric independent) Levi-Civita density of weight \(-1\). It is easy to show that \( A_{I}^{\text{appr}} \) approximates the area function (on the phase space) defined by the surface elements \( S_I \), the approximation becoming better as \( S_I \) and hence loops with points \( x \) and \( x' \) used in the definition of \( T^{aa'} \) shrink. Therefore, the total area \( A_S \) associated with \( S \) is given by

\[
A_S = \lim_{N \to \infty} \sum_{I=1}^{N} A_{I}^{\text{appr}}. \tag{5.9}
\]

To obtain the quantum operator \( \hat{A}_S \), we simply replace \( T^{aa'} \) in (5.8) by the quantum loop operator \( \hat{T}^{aa'} \). This somewhat indirect procedure is necessary because, as indicated above, there is no well-defined operator-valued distribution that represents the metric or its area element at a point. Again, the operator \( \hat{A}_S \) turns out to be finite. Its action, evaluated on a nonintersecting loop \( \alpha \) (for simplicity), is given by:

\[
\langle \alpha | \circ \hat{A}_S = \frac{l_P^2}{2} I(S, \alpha) \cdot \langle \alpha |, \tag{5.10}
\]

where \( I(S, \alpha) \) is simply the unoriented intersection number between the 2-surface \( S \) and the loop \( \alpha \). (One obtains the unoriented intersection number here and the absolute sign in the integrand of (5.7) because of the square-root operation involved in the definition of these operators.) Thus, in essence, “a loop \( \alpha \) contributes half a Planck unit of area to any surface it intersects.”

The fact that the area operator also acts simply by multiplication on non-intersecting loops lends further support to the idea that the loop representation is well-suited to “diagonalize” operators describing the 3-geometry. Indeed, we can immediately construct a large set of simultaneous eigenbras of the smeared metric and the area operators. There is one, \( \langle \alpha | \), associated to every nonintersecting loop \( \alpha \). Note that the corresponding eigenvalues of area are quantized in integral multiples of \( l_P^2/2 \). There are also eigenstates associated with intersecting loops which, however, I will not go into to since the discussion quickly becomes rather involved technically.

With these operators on hand, we can now turn to the construction of quantum loop states that approximate the classical metric \( h_{ab} \) on \( \Sigma \) on a scale large compared to the Planck length. The basic idea is to weave the classical metric out of quantum loops by spacing them so that, on an average, precisely one line crosses any surface element whose area, as measured by the given \( h_{ab} \) is one Planck unit. Such loop states will be called weaves. Note that these states are not uniquely picked out since our requirement is rather weak. Indeed, given a weave approximating a given classical metric, one can obtain others, approximating the same classical metric.

Let us begin with a concrete example of such a state which will approximate a flat metric \( h_{ab} \). To construct this state, we proceed as follows. Using this metric, let us introduce a random distribution of points on \( \Sigma = \mathbb{R}^3 \) with density \( n \) (so that in any given volume \( V \) there are \( nV(1 + O(1/\sqrt{nV})) \) points). Center a circle of radius \( a = (1/n)^{\frac{1}{3}} \) at each of these points, with a random orientation. We assume that \( a << L \), so that there is a large number of (non-intersecting but, generically, linked) loops in a macroscopic volume \( L^3 \). Denote the collection of these circles by \( \Delta_a \). As noted in section 4, due to trace
identities, products of Wilson loop functionals $T_\alpha$ can be expressed as linear combinations of Wilson loop functionals. As a consequence, it turns out that the bras defined by multiloops are equivalent to linear combinations of single loop bras. Therefore, for each choice of the parameter $a$, there is a well-defined bra $\langle \Delta_a \mid$. This is our candidate weave state.

Let us consider the observable $\hat{Q}[\omega]$. To see if $\langle \Delta_a \mid$ reproduces the geometry determined by the classical metric $h_{ab}$ on a scale $L >> l_p$, let us introduce a 1-form $\omega_a$ which is slowly varying on the scale $L$ and compare the value $Q[\omega](h)$ of the classical $Q[\omega]$ evaluated at the metric $h_{ab}$, with the action of the quantum operator $\hat{Q}[\omega]$ on $\langle \Delta_a \mid$. A detailed calculation yields:

$$\langle \Delta_a \mid \circ \hat{Q}[\omega] = \left[ \frac{\pi}{2} \left( \frac{l_p}{a} \right)^2 Q[\omega](h) + O\left( \frac{a}{L} \right) \right] \cdot \langle \Delta_a \mid. \quad (5.11)$$

Thus, $\langle \Delta_a \mid$ is an eigenstate of $\hat{Q}[\omega]$ and the corresponding eigenvalue is closely related to $Q[\omega](h)$. However, even to the leading order, the two are unequal unless the parameter $a$ — the average distance between the centers of loops — equals $\sqrt{\pi/2} l_p$. More precisely, (5.11) can be interpreted as follows. Let us write the leading coefficient on the right side of this equation as $(1/4)(2\pi a/l_p)(n!^3)$. Since this has to be unity for the weave to reproduce the classical value (to leading order), we see that $\Delta_a$ should contain, on an average, one fourth Planck length of curve per Planck volume, where lengths and volumes are measured using $h_{ab}$.

The situation is the same for the area operators $\hat{A}_S$. Let $S$ be a 2-surface whose extrinsic curvature varies slowly on a scale $L >> l_p$. One can evaluate the action of the area operator on $\langle \Delta_a \mid$ and compare the eigenvalue obtained with the value of the area assigned to $S$ by the given flat metric $h_{ab}$. Again, the eigenvalue can be re-expressed as a sum of two terms, the leading term which has the desired form, except for an overall coefficient which depends on the mean separation $a$ of loops constituting $\Delta_a$, and a correction term which is of the order of $O\left( \frac{a}{L} \right)$. We require that the coefficient be so adjusted that the leading term agrees with the classical result. This occurs, again, precisely when $a = \sqrt{\pi/2} l_p$. It is interesting to note that the details of the calculations which enable one to express the eigenvalues in terms of the mean separation are rather different for the two observables. In spite of this, the final constraint on the mean separation is precisely the same.

Let us explore the meaning and implications of these results.

1) It is generally accepted that, to obtain classical behavior from quantum theory, one needs two things: i) an appropriate coarse graining, and, ii) special states. In our procedure, the slowly varying test fields $\omega_a$ and surfaces $S$ with slowly varying extrinsic curvature enable us to perform the appropriate coarse graining while weaves — with the precisely tuned mean separation $a$ — are the special states. There is, however, something rather startling: The restriction on the mean separation $a$ — i.e., on the short distance behavior of the multi-loop $\Delta_a$ — came from the requirement that $\langle \Delta_a \mid$ should approximate the classical metric $h_{ab}$ on large scales $L$!

2) In the limit $a \to \infty$, the eigenvalues of the two operators on $\langle \Delta_a \mid$ go to zero. This is not too surprising. Roughly, in a state represented by any loop $\alpha$, one expects the quantum geometry to be excited just at the points of the loops. If the loops are very far away from each other as measured by the fiducial $h_{ab}$, there would be macroscopic
regions devoid of excitations where the quantum geometry would seem to correspond to a zero metric.

3) The result of the opposite limit, however, is surprising. One might have naively expected that the best approximation to the classical metric would occur in the continuum limit in which the separation $a$ between loops goes to zero. However, the explicit calculation outlined above shows that this is not the case: as $a$ tends to zero, the leading terms in the eigenvalues of $\hat{Q}[\omega]$ and $A_S$ actually diverge! (One’s first impulse from lattice gauge theories may be to say that the limit is divergent simply because we are not rescaling, i.e., renormalizing the operator appropriately. Note, however, that, in contrast to the calculations one performs in lattice theories, here, we already have a well defined operator in the continuum. We are only probing the properties of its eigenvectors and eigenvalues, whence there is nothing to renormalize.) It is, however, easy to see the reason underlying this behavior. Intuitively, the factors of the Planck length in (5.7) and (5.10) force each loop in the weave to contribute a Planck unit to the eigenvalue of the two geometrical observables. In the limit $a \to 0$, the number of loops in any fixed volume (relative to the fiducial $h_{ab}$) grows unboundedly and the eigenvalue diverges.

4) It is important to note the structure of the argument. In non-perturbative quantum gravity, there is no background space-time. Hence, terms such as “slowly varying” or “microscopic” or “macroscopic” have, a priori, no physical meaning. One must do some extra work, introduce some extra structure to make them meaningful. The required structure should come from the very questions one wants to ask. Here, the questions had to do with approximating a classical geometry. Therefore, we could begin with classical metric $h_{ab}$. We used it repeatedly in the construction: to introduce the length scale $L$, to speak of “slowly varying” fields $\omega_a$ and surfaces $S$, and, to construct the weave itself. The final result is then a consistency argument: If we construct the weave according to the given prescription, then we find that it approximates $h_{ab}$ on macroscopic scales $L$ provided we choose the mean separation $a$ to be $\sqrt{\pi/2l_p}$, where all lengths are measured relative to the same $h_{ab}$.

5) Note that there is a considerable non-uniqueness in the construction. As we noted already, a given 3-geometry can lead to distinct weave states; our construction only serves to make the existence of such states explicit. For example, there is no reason to fix the radius $r$ of the individual loops to be $a$. For the calculation to work, we only need to ensure that the loops are large enough so that they are generically linked and small enough so that the values of the slowly varying fields on each loop can be regarded as constants plus error terms which we can afford to keep in the final expression. Thus, it is easy to obtain a 2-parameter family of weave states, parametrized by $r$ and $a$. The condition that the leading order terms reproduce the classical values determined by $h_{ab}$ then gives a relation between $r$, $a$ and $l_p$ which again implies discreteness. Clearly, one can further enlarge this freedom considerably: There are a lot of eigenbras of the smeared-metric and the area operators whose eigenvalues approximate the classical values determined by $h_{ab}$ up to terms of the order $O(l_p/\sqrt{L})$ since this approximation ignores Planck scale quantum fluctuations.

6) Finally, I would like to emphasize that, at a conceptual level, the important point is
that the eigenvalues of $\hat{Q}[\omega]$ and $\mathcal{A}[S]$ can be discrete.

Let me conclude the discussion on weaves with two remarks. First, it is not difficult to extend the above construction to obtain weave states for curved metrics $g_{ab}$ which are slowly varying with respect to a flat metric $h_{ab}$. Given such a metric, one can find a slowly varying tensor field $t^a_b$, such that the metric $g_{ab}$ can be expressed as $t^c_a t^d_b h_{cd}$. Then, given a weave of the type $\langle \Delta \rvert \omega \rangle$ considered above approximating $h_{ab}$, we can “deform” each circle in the multi-loop $\Delta$ using $t^a_b$ to obtain a new weave $\langle \Delta \rvert_t \omega \rangle$ which approximates $g_{ab}$ in the same sense as $\langle \Delta \rvert \omega \rangle$ approximates $h_{ab}$. (See, also, Zegwaard (1992) for the weave corresponding to the Schwarzschild black-hole.) The second remark is that since the weaves are eigenbras of the operators that capture the 3-geometry, they do not approximate 4-geometries. To obtain a state that can approximate Minkowski space-time, for example, one has to consider a loop state that resembles a “coherent state” peaked at the weave $\Delta_a$. In that state, neither the 3-geometry nor the time-derivative thereof would be sharp; but they would have minimum spreads allowed by the uncertainty principle. This issue has been examined in detail by Iwasaki and Rovelli (1993).

Since these results are both unexpected and interesting, it is important to probe their origin. We see no analogous results in familiar theories. For example, the eigenvalues of the fluxes of electric or magnetic fields are not quantized in QED nor do the linearized analogs of our geometric operators admit discrete eigenvalues in spin-2 gravity. Why then did we find qualitatively different results? The technical answer is simply that the familiar results refer to the Fock representation for photons and gravitons while we are using a completely different representation here. Thus, the results are tied to our specific choice of the representation. Why do we not use Fock or Fock-like states? It is not because we insist on working with loops rather than space-time fields such as connections. Indeed, one can translate the Fock representation of gravitons and photons to the loop picture. (See, e.g., Ashtekar et al (1991) and Ashtekar & Rovelli, (1992).) And then, as in the Fock space, the discrete structures of the type we found in this section simply disappear. However, to construct these loop representations, one must use a flat background metric and essentially every step in the construction violates diffeomorphism invariance. Indeed, there is simply no way to construct “familiar, Fock-like” representations without spoiling the diffeomorphism invariance. Thus, the results we found are, in a sense, a direct consequence of our desire to carry out a genuinely non-perturbative quantization without introducing any background structure. However, we do not have a uniqueness theorem singling out the measure $d\mu$ which was used to define the loop transform and hence to construct the loop representation used here. There do exist other diffeomorphism invariant measures which will lead to other loop representations. The measure we have used is the most natural and the simplest among the known ones. Whether the results presented here depend sensitively on the choice of the measure is not known. Therefore, it would be highly desirable to have a uniqueness theorem which tells us that, on physical grounds, we should restrict ourselves to a specific (class of) measure(s).

My overall viewpoint is that one should simultaneously proceed along two lines: i) one should take these results as an indication that we are on the right track and push heuristic calculations within this general framework; and, ii) one should try to put the available heuristic results on rigorous mathematical footing to avoid the danger of “wandering off”
in unsound directions.

6. Conclusion

In this article I have reported on some recent developments in non-perturbative quantum general relativity in 4 dimensions. Due to the space limitation, I could not go into details, and, to avoid the discussion of certain technical points in the Hamiltonian framework of general relativity, I have omitted a couple of important issues. However, I hope I have given the general flavor of the type of problems we face and the strategies that have been devised. Many of the problems may indeed be unfamiliar in the world of rigorous quantum field theory since they are peculiar to general relativity. There is no background space-time, no obvious notion of microcausality, no reference to the asymptotic states of scattering theory. And yet, significant progress could be made. In 3 space-time dimensions, in particular, a complete, rigorous, mathematical treatment is available.

In 4 space-time dimensions, the program is far from being complete. But as I have tried to argue in the last two sections, a number of interesting results have emerged. The loop representation, in particular, furnishes brand new tools to tackle problems arising from the diffeomorphism invariance. Furthermore, the general program has matured at least to the extent that there is a well-defined mathematical framework in the background. Indeed, a number of open problems are precisely of the type that mathematical physicists can now make major contributions to the field. Finally, the results obtained so far may themselves have interesting implications for quantum field theory in general, quite outside the realm of quantum gravity.

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