On the global structure of some natural fibrations of Joyce manifolds

Chien-Hao Liu

Department of Mathematics
University of Texas at Austin
Austin, Texas 78712-1082

Abstract

The study of fibrations of the target manifolds of string/M/F-theories has provided many insights to the dualities among these theories or even as a tool to build up dualities since the work of Strominger, Yau, and Zaslow on the Calabi-Yau case. For M-theory compactified on a Joyce manifold $M^7$, the fact that $M^7$ is constructed via a generalized Kummer construction on a 7-torus $T^7$ with a torsion-free $G_2$-structure $\varphi$ suggests that there are natural fibrations of $M^7$ by $T^3$, $T^4$, and K3 surfaces in a way governed by $\varphi$. The local picture of some of these fibrations and their roles in dualities between string/M-theory have been studied intensively in the work of Acharya. In this present work, we explain how one can understand their global and topological details in terms of bundles over orbifolds. After the essential background is provided in Sec. 1, we give general discussions in Sec. 2 about these fibrations, their generic and exceptional fibers, their monodromy, and the base orbifolds. Based on these, one obtains a 5-step-routine to understand the fibrations, which we illustrate by examples in Sec. 3. In Sec. 4, we turn to another kind of fibrations for Joyce manifolds, namely the fibrations by the Calabi-Yau threefolds constructed by Borcea and Voisin. All these fibrations arise freely and naturally from the work of Joyce. Understanding how the global structure of these fibrations may play roles in string/M-theory duality is one of the major issues for further pursuit.

Key words: M-theory, duality, Joyce manifold, asymptotically associative/coassociative submanifold, torus/K3-fibration, orbifold, Borcea-Voisin threefold.

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1E-mail: chienliu@math.utexas.edu
Fibrations of Joyce Manifolds

0. Introduction and outline.

Introduction.

The study of fibrations of the target manifolds of string/M/F-theories has provided many insights to the dualities among these theories or even as a tool to build up dualities since the work of Strominger, Yau, and Zaslow on the Calabi-Yau case. For M-theory compactified on a Joyce manifold $M^7$, the fact that $M^7$ is constructed via a generalized Kummer construction on a 7-torus $T^7$ with a torsion-free $G_2$-structure $\varphi$ suggests that there are natural fibrations of $M^7$ by $T^3$, $T^4$, and K3 surfaces in a way governed by $\varphi$. The local picture of these fibrations and their roles in dualities between string/M-theory have been studied intensively in the work of Acharya. In this present work, based on the work of Joyce, we set up a 5-step-routine which provides the global and topological details to these fibrations in the terms of flat bundles over orbifold. After illuminating this routine by several examples, we turn to a discussion on some natural fibrations of Joyce manifolds of the first kind by the Calabi-Yau threefolds constructed by Borcea and Voisin. This renders Joyce manifolds of the first kind to be similar to the 7-spaces obtained by rolling Calabi-Yau threefolds, as discussed in [Li].

Since all these fibrations arise freely and naturally from the work of Joyce, one is surely very curious about how such fibrations, particularly their global structures, should enter string/M-theory dualities. This will have to await a future pursuit.

Convention. Since both real and complex manifolds are involved in this article, to avoid confusion, a real $n$-dimensional manifold will be called an $n$-manifold while a complex $n$-dimensional manifold an $n$-fold. Also, a real $n$-dimensional orbifold will be called an $n$-orbifold.

Outline.

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3. Examples in the 5-step-routine.
4. Fibrations of Joyce manifolds by Borcea-Voisin threefolds.
5. Remarks on further questions/works.
1 Essential mathematical background for physicists.

To set up some notations and to provide some essential background for physicists, three key ingredients of the work: orbifolds, Joyce manifolds, and Borcea-Voisin threefolds, are concisely explained in this section. Readers are referred to the listed references for more details.

**Orbifolds** ([Th1], also [B-S], [F-M], [Mo] and [Sc]). (Cf. Remark 2.1.4.)

An $n$-dimensional orbifold $Q$ is a topological space locally modelled by a quotient $U = \tilde{U}/\Gamma_U$, where $\tilde{U}$ is a connected open set in $\mathbb{R}^n$ and $\Gamma_U$ is a finite group that acts on $\tilde{U}$ effectively. Associated to each point $p$ in $Q$ is a group $\Gamma_p$ that is isomorphic to the stabilizer of any preimage point of $p$ in a local model $U = \tilde{U}/\Gamma_U$ around $p$. The set $\Sigma_Q = \{ p \in Q | \Gamma_p \neq \{1\} \}$ is called the singular locus of $Q$. In general $\Sigma_Q$ is stratified by manifolds of various dimensions (cf. Figures 2-1, 3-1-1, 3-6-1). In particular, when a discrete group $\Gamma$ acts on a manifold $X$ effectively and properly discontinuously but not necessarily freely, then the quotient $X/\Gamma$ is an orbifold with $\Sigma_{X/\Gamma}$ descending from the set of fixed points in $X$ of some element in $\Gamma$. The group $\Gamma_p$ associated to each point $p$ in $X/\Gamma$ is the stabilizer $\text{Stab}(\tilde{p})$ of any preimage point $\tilde{p}$ of $p$ in $X$.

A covering orbifold of an orbifold $Q$ is an orbifold $\tilde{Q}$ with a projection $\phi : X_{\tilde{Q}} \to X_Q$ between the underlying spaces such that each $x \in X_Q$ has a neighborhood, modelled by $U = \tilde{U}/\Gamma_U$, for which each component $V_i$ of $\phi^{-1}(U)$ is isomorphic to $\tilde{U}/\Gamma_i$, where $\Gamma_i$ is a subgroup in $\Gamma$ and the isomorphisms are compatible with $\phi$.

**Fact 1.1 [universal covering orbifold].** ([Th1]: Proposition 13.2.4). Any orbifold $Q$ has a universal covering orbifold $\tilde{Q}$. In other words, if $* \in X_Q - \Sigma_Q$ is a base point for $Q$, then $\phi : \tilde{Q} \to Q$ is a connected covering orbifold with base point $\tilde{*} \in \phi^{-1}(*)$ such that, for any covering orbifold $\phi' : \tilde{Q}' \to Q$ with base point $\tilde{*}' \in \phi'^{-1}(*)$, there is an orbifold covering map $\chi : \tilde{Q} \to \tilde{Q}'$ such that $\chi(\tilde{*}) = \tilde{*}'$ and $\phi = \phi' \circ \chi$.

By definition, such $\tilde{Q}$ is unique. The fundamental group $\pi_1^{\text{orb}}(Q)$ of an orbifold $Q$ is then defined to be the group of deck transformations of its universal covering orbifold $\tilde{Q}$.

The following example will be important to us:

**Example 1.2 [toroidal orbifold].** Let $\mathbb{T}^n = \mathbb{R}^n/\Lambda$ be an Euclidean $n$-torus determined by a lattice $\Lambda$ in $\mathbb{R}^n$ and $Q$ be an $n$-orbifold obtained by the quotient of $\mathbb{T}^n$ by a discrete group $\Gamma$ of isometries. Then the universal covering orbifold of $Q$ is the same as the universal covering space of $\mathbb{T}^n$, which is $\mathbb{R}^n$. Let $\bar{\Gamma}$ be a lifting of the $\Gamma$-action on $\mathbb{T}^n$ to $\mathbb{R}^n$. Then $\pi_1^{\text{orb}}(Q) = \langle \bar{\Gamma}, \Lambda \rangle$, the group of isometries on $\mathbb{R}^n$ generated by $\bar{\Gamma}$ and the translation group determined by $\Lambda$.

In the above example, let $\phi$ be the universal covering map, which is the composition $\mathbb{R}^n \to \mathbb{T}^n \to Q$. Then $\langle \bar{\Gamma}, \Lambda \rangle$ acts on $\phi^{-1}(Q - \Sigma_Q)$ freely. Hence, if $* \in Q - \Sigma_Q$ is a
base point for \( Q \) and \( \bar{y} \in \phi^{-1}(\ast) \) is a base point for \( \mathbb{R}^n \), then, for each \( y \) in the orbit of \( \ast \), there is a unique deck transformation \( g \in (\hat{\Gamma}, \Lambda) \) that sends \( \ast \) to \( y \). Let \( \hat{\gamma} \) be a path from \( \bar{y} \) to \( y \), then its projection \( \gamma = \phi(\hat{\gamma}) \) is a loop at \( \ast \) in \( Q \) that represents \( g \in \pi_1^\text{orb}(Q) \).

In terms of homotopy classes of loops at \( \ast \), one has a tautological group homomorphism \( \tau : \pi_1(Q - \Sigma Q) \to \pi_1^\text{orb}(Q) \). (Cf. Figure 2-1.)

**Joyce manifolds ([Jo1]).**

Consider the standard oriented Euclidean space \( \mathbb{R}^7 \) and the 3-form

\[
\varphi_0 = e^1 \wedge e^2 \wedge e^7 + e^1 \wedge e^3 \wedge e^6 + e^1 \wedge e^4 \wedge e^5 + e^2 \wedge e^3 \wedge e^5 - e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^4 \wedge e^7 + e^5 \wedge e^6 \wedge e^7.
\]

The group of orientation-preserving linear isomorphisms that preserves \( \varphi_0 \) is the exceptional group \( G_2 \) and lies in \( SO(7) \). The same holds also for its Hodge dual

\[
\ast \varphi_0 = e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^2 \wedge e^5 \wedge e^6 - e^1 \wedge e^3 \wedge e^5 \wedge e^7 + e^2 \wedge e^3 \wedge e^6 \wedge e^7 + e^2 \wedge e^4 \wedge e^5 \wedge e^7 + e^3 \wedge e^4 \wedge e^5 \wedge e^6.
\]

Thus a 3-form \( \varphi \) on an oriented 7-manifold \( M^7 \) that is pointwise oriented-isomorphic to \( \varphi_0 \) determines a \( G_2 \)-structure and, via which, a Riemannian metric \( g_\varphi \) on \( M^7 \). When \( \varphi \) is torsion-free, the holonomy of \( g_\varphi \) is contained in \( G_2 \), which implies that \( g_\varphi \) is Ricci-flat. This feature singles out the role in string theory of closed 7-manifolds with torsion-free \( G_2 \)-structures.

In [Jo1], Joyce has constructed numerous such examples from desingularizations of the following two kinds of toroidal orbifolds from the standard \( (\mathbb{R}^7, \varphi_0) \) above:

**Class (1):** The quotient \( T^7/\Gamma \), where \( T^7 = \mathbb{R}^7/\mathbb{Z}^7 \) with coordinates \((x_1, \ldots, x_7)\), where \( x_i \in \mathbb{R}/\mathbb{Z} \), and \( \Gamma \) is the group \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), generated by

\[
\begin{align*}
\alpha(x_1, \ldots, x_7) &= (-x_1, -x_2, -x_3, -x_4, -x_5, x_6, x_7), \\
\beta(x_1, \ldots, x_7) &= (b_1 - x_1, b_2 - x_2, x_3, x_4, -x_5, -x_6, x_7), \\
\gamma(x_1, \ldots, x_7) &= (c_1 - x_1, -x_2, c_3 - x_3, x_4, c_5 - x_5, x_6, -x_7),
\end{align*}
\]

where \( b_1, b_2, c_1, c_3, \) and \( c_5 \) are some constants in \( \{0, \frac{1}{7}\} \).

**Class (2):** The quotient \( (\mathbb{C}^3 \times \mathbb{R})/\langle \Gamma, \Lambda \rangle \), where \( \mathbb{C}^3 \times \mathbb{R} \) is parametrized by \((z_1, z_2, z_3, x)\), \( \Gamma \) is the dihedral group \( D_a \) of \( 2a \) elements, generated by

\[
\begin{align*}
\alpha(z_1, z_2, z_3, x) &= (u z_1, v z_2, \overline{u v} z_3, x + \frac{1}{a}), \\
\beta(z_1, z_2, z_3, x) &= (-\overline{z_1}, -\overline{z_2}, -\overline{z_3}, -x)
\end{align*}
\]

with \( u, v \) being unit complex numbers and \( a \) being the smallest positive integer such that \( u^a = v^a = 1 \), and \( \Lambda \) is a lattice in \( \mathbb{C}^3 \times \mathbb{R} \) invariant under \( \Gamma \).
Let $W^7$ be a such quotient. Then its singular set $S$ arises from the collection of all the fixed 3-tori $T^3$ of some element in $\Gamma$. We require that the constants that appear in the definition of $\Gamma$ are chosen so that the tubular neighborhood $\nu(S_0)$ of a component $S_0$ of $S$ is modelled by either (a) or (b) below:

**Model (a):** $T^3 \times (\mathbb{C}^2/\langle -1 \rangle)$, where $-1$ stands for negation of both coordinates.

**Model (b):** $\{T^3 \times (\mathbb{C}^2/\langle -1 \rangle)\}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $T^3 \times (\mathbb{C}^2/\langle -1 \rangle)$ in Model (a) freely.

We shall called these *transverse A$_1$-singularities* along $T^3$ or $T^3/\mathbb{Z}_2$.

Both Model (a) and Model (b) appear for admissible $(u, v, \Lambda)$, Model (a) is the only kind of singularity that appears. They can be resolved as follows:

- For Model (a), $S_0$ is a $T^3$ and $\nu(S_0)$ can be resolved by the transverse blowup $Id \times \psi$ that resolves the $A_1$-singularity in each $\mathbb{C}^2/\langle -1 \rangle$ along $S_0$:

$$Id \times \psi : T^3 \times T^*\mathbb{C}^1 \to T^3 \times (\mathbb{C}^2/\langle -1 \rangle),$$

where $Id$ is the identity map on $T^3$ and $\psi : T^*\mathbb{C}^1 \to \mathbb{C}^2/\langle -1 \rangle$ is the resolution of the isolated $A_1$-singularity of $\mathbb{C}^2/\langle -1 \rangle$. Note that, since the exceptional locus for $\psi$ is the 0-section of $T^*\mathbb{C}^1$, which is a $\mathbb{C}^1$, the exceptional locus of $Id \times \psi$ is $T^3 \times \mathbb{C}^1$.

- For Model (b), $S_0$ is a free quotient $T^3/\mathbb{Z}_2$. Depending on the ways the fiber $\mathbb{R}^4/\langle -1 \rangle$ of $\nu(S_0)$ is identified with $\mathbb{C}^2/\langle -1 \rangle$, there are two inequivalent free $\mathbb{Z}_2$-actions on $T^3 \times (\mathbb{C}^2/\langle -1 \rangle)$: one generated by a holomorphic map of the form $(*, (z_1, z_2)) \mapsto (*, (z_1, -z_2))$ and the other generated by an antiholomorphic map of the form $(*, (z_1, z_2)) \mapsto (*, (\overline{z_1}, \overline{z_2}))$. Each has a unique natural lifting to a free $\mathbb{Z}_2$-action on $T^3 \times T^*\mathbb{C}^1$ equivariant with the $\mathbb{Z}_2$-action on $T^3 \times (\mathbb{C}^2/\langle -1 \rangle)$ with respect to $Id \times \psi$ in Case (a). Each quotient $(T^3 \times T^*\mathbb{C}^1)/\mathbb{Z}_2$ gives then a different resolution for $\nu(S_0)$. The exceptional locus is $(T^3 \times \mathbb{C}^1)/\mathbb{Z}_2$.

From a combination of three independent 1-forms on $T^3$ and three 2-forms on $T^*\mathbb{C}^1$ from its hyper Kähler structure, any above resolution $\nu(S_0)$ of $\nu(S_0)$ admits a family of torsion-free $G_2$-structures depending on a real parameter $t$. The singular $\nu(S_0)$ is recovered as $t \to 0$.

Hence, after resolving the singular set $S$ of $W^7$ as above, the resulting smooth closed 7-manifold $M^7$ admits a $G_2$-structure $\varphi_t$ by sewing the $G_2$-structure of $W^7$ inherited from $(\mathbb{R}^7, \varphi_0)$ and that of $\nu(S)$ via a partition of unity. By construction, $\varphi_t$ is torsion-free outside a ring domain $A$ around the boundary $\partial \nu(S)$ but has torsion in $A$. However, Joyce shows that, for $t$ small enough, one can deform $\varphi_t$ by exact 3-forms $d\eta_t$ so that the new $G_2$-structure $\tilde{\varphi} = \varphi_t + d\eta_t$ is torsion-free all over $M^7$. In this way, he constructed families.
of closed Riemannian 7-manifolds with holonomy $G_2$. They are called Joyce manifolds.
For convenience, we shall call those from quotients in Class (1) Joyce manifolds of the first kind, with notation $J(b_1, b_2, c_1, c_3, c_5)$, and those from quotients in Class (2) Joyce manifolds of the second kind, with notation $J(u, v, \Lambda)$.

**Borcea-Voisin threefolds** ([Bo], [G-W], [Ni], [Vo]).

Let $X$ be a K3 surface with an involution $\iota$ that acts by $(-1)$ on the holomorphic 2-form of $X$. Let $\Sigma$ be the set of fixed points of $\iota$. Then ([Ni]) $\Sigma$ is a disjoint union of smooth curves in $X$ as classified below:

1. $\Sigma$ is empty;
2. $\Sigma = C_1 \cup C'_1$, where $C_1$ and $C'_1$ are both elliptic curves;
3. $\Sigma = C_g + E_1 + \cdots + E_k$, where $C_g$ is a curve of genus $g$ and $E_i$ are rational curves.

In all cases, the quotient $X/\iota$ is a smooth surface.

Let $E$ be an elliptic curve, and $j$ be the involution on $E$ acting by negation. The set $S_0$ of fixed-points of $(\iota, j)$ on $X \times E$ consists of four disjoint copies of $\Sigma$, and the quotient $(X \times E)/(\iota, j)$ has $A_1$-singularities along $S_0$. Simultaneous blowup along the transverse directions to $S_0$ resolves these singularities and the result is a smooth Calabi-Yau threefold $Y$. Such Calabi-Yau manifolds and their mirror symmetry are considered first in the work of Borcea [Bo] and Voisin [Vo] and we shall call them Borcea-Voisin threefolds. Let $n$ be the number of components of $\Sigma$ and $n'$ be the sum of the genus of all these components. Then the Hodge numbers of a Borcea-Voisin threefold $Y$ are determined by ([Bo] and [Vo])

$$h^{1,1}(Y) = 11 + 5n - n' \quad \text{and} \quad h^{2,1}(Y) = 11 + 5n' - n.$$ 

Such Calabi-Yau threefolds could admit K3- and special Lagrangian $T^3$-fibrations:

- **K3-fibration**: By construction, one has the natural K3-fibration $\pi$ from $(X \times E)/(\iota, j)$ onto $E/j$, which is an $S^2(2222)$-orbifold (cf. Remark 2.1.4 in Sec. 2.1). The fixed-point set $S_0$ is contained in the exceptional fibers of $\pi$ with one $\Sigma$ in each such fiber. Hence, when resloving the singularities of $E/j$ along $S_0$ to obtain $Y$, the K3-fibration $\pi$ is lifted to a K3-fibration $\tilde{\pi}$ from $Y$ onto the same base $E/j$. The exceptional fiber over an orbifold point now becomes $X/\iota \cup V$, where $V = \Sigma \times \mathbb{CP}^1$ intersects $X/\iota$ transversely at the $\Sigma$ from $S_0$.

- **$T^3$-fibration** ([G-W]): If $X$ has an elliptic fibration invariant under $\iota$ with generic fiber a special Lagrangian submanifold with respect to the Kähler form on $X$, then one can choose a $T^1$-fibration of $E$ and combine these into a special Lagrangian $T^3$-fibration for $Y$. The base of this fibration is a 3-orbifold $Q$ whose underlying topology is $S^3$ with the singular locus $\Sigma_Q$, which corresponds to the set of critical values of the fibration, as indicated in Figure 1-1.

With these preparations, let us now turn to the themes of this paper.
2 Fibrations of Joyce manifolds from decompositions of $\mathbb{R}^7$ governed by the $G_2$-structure $\varphi_0$.

We discuss in Sec. 2.1 how some natural fibrations for a Joyce manifold can arise from $(\mathbb{R}^7, \varphi_0)$ and discuss how they can be understood in terms of a bundle $\pi : T^7/\Gamma \to Q$ over an orbifold $Q$. In Sec. 2.2 the effect to $\pi$ after resolving the singular set $S$ of $T^7/\Gamma$ is studied and, hence, one completes the picture.

2.1 Fibrations in terms of bundles over orbifolds.

The following definitions from [H-L] (also [Ha], [Jo1], and [McL]) is the starting point of all the fibrations to be discussed in this paper.

Definition 2.1.1 [calibration, (co)associative submanifold]. A calibration on a Riemannian manifold $M$ is a closed $k$-form $\omega$ whose restriction to any tangent $k$-plane to $M$ is less than or equal to the volume of the $k$-plane. A calibrated submanifold of $(M, \omega)$ is a $k$-dimensional submanifold $N$ in $M$ such that $\omega|_N$ is equal to the volume-form of the induced metric on $N$. Let $M^7$ be a 7-manifold with torsion-free $G_2$-structure $\varphi$. Then both $\varphi$ and $*\varphi$ are calibrations on $M^7$. An associative submanifold of $M^7$ is a calibrated submanifold of $(M^7, \varphi)$ and a coassociative submanifold of $M^7$ is a calibrated submanifold of $(M^7, *\varphi)$.

These definitions generalize automatically to orbifolds and their sub-orbifolds by considering the complement of their singular loci.
Joyce manifolds of the second kind, as the total space of the trivial bundle \(T\). Regard the singularity of \((C, T)\) to be small, then the fibration of associated to an admissible decomposition of \(R\). Consider How the fibrations arise.

such fibration as asymptotically associative/coassociative fibration. From the explicit expression of \(\varphi\) by Joyce in [Jo1], the lattice \(\Lambda\) is decomposable into \(\Lambda_{abc}\), \(\Lambda_{ijkl}\), \(\Lambda_{3456}\), \(\Lambda_{1256}\), and \(\Lambda_{1234}\) are preserved by \(\Gamma\), each of which descends to a pair of associative-coassociative foliations for \((C^3 \times R)/\langle\Lambda, \Gamma\rangle\). In general these may not be realizable as a nice fibration. However, for most of the concrete examples given by Joyce in [Jo1], the lattice \(\Lambda\) is decomposable into \(\Lambda_{abc} \oplus \Lambda_{ijkl}\), where \(\Lambda_{abc}\) is a lattice in \(R_{abc}\), \(\Lambda_{ijkl}\) is a lattice in \(R_{ijkl}\), and \(\Lambda_{abc} \oplus \Lambda_{ijkl}\) is one of the admissible associative-coassociative product decomposition above. In this good case, an above decomposition descends to an \(\Gamma\)-invariant associative-coassociative product decomposition \(T_{abc} \times T_{ijkl}\) for \(T^7 = R^7/\Lambda\). This then leads to an a.a./a.c. fibration for the Joyce manifold after resolving the singularity of \((C^3 \times R)/\langle\Lambda, \Gamma\rangle\) as in the case for Joyce manifolds of the first kind.

The associated bundles over orbifolds before resolving \(S\).

Regard \(T^7\), either from \(R^7/\mathbb{Z}^7\) for Joyce manifolds of the first kind or \((C^3 \times R)/\Lambda\) for Joyce manifolds of the second kind, as the total space of the trivial bundle \(Y = Z \times X\) associated to an admissible decomposition of \(R^7\) listed above, with \(Z\) being the base and
$X$ the associative/coassociative fiber. Then $\Gamma$ acts effectively on $Y$ as a finite group of bundle automorphisms. This induces a $\Gamma$-action on the base $Z$. Let $\Gamma_0$ be the normal subgroup in $\Gamma$ that consists of all the elements in $\Gamma$ which acts trivially on $Z$. Then $\Gamma_0$ is contained in the stabilizer of every fiber of $Y$, after taking the quotient by $\Gamma$, one obtains a bundle $\pi$ and the commutative diagram:

$$
\begin{array}{ccc}
Y & \rightarrow & Y/\Gamma \\
\downarrow & & \downarrow \pi \\
Z & \xrightarrow{pr} & Q = Z/\Gamma .
\end{array}
$$

The generic fiber of $\pi$ is $X/\Gamma_0$. The following lemma, which can be justified by tracing definitions, gives the basic relations among fibers in $Y/\Gamma$, $\Gamma_p$ for $p$ in $Q$, and the stabilizer subgroup of fibers in $Y$:

**Lemma 2.1.2.** Given a point $p$ in $Q$, let $\bar{p}$ be a point in the preimage of $p$ in $Z$, $F_{\bar{p}}$ be the fiber in $Y$ over $\bar{p}$, and $\text{Stab}_0(F_{\bar{p}})$ be the smallest subgroup in $\text{Stab}(F_{\bar{p}})$ that contains $\Gamma_0$ and all the elements in $\Gamma$ that act trivially on $F_{\bar{p}}$. Then the group $\Gamma_p$ associated to $p$ is the quotient $\text{Stab}(F_{\bar{p}})/\Gamma_0$. The fiber $F_p$ in $Y/\Gamma$ over $p$ is the quotient $F_{\bar{p}}/\text{Stab}(F_{\bar{p}})$ with multiplicity equal to the cardinality of $\text{Stab}(F_{\bar{p}})/\text{Stab}_0(F_{\bar{p}})$.

For Joyce manifolds of the first kind $J(b_1, b_2, c_1, c_3, c_5)$, $Q$ can be immediately read off by projecting the $\Gamma$-action on $\mathbb{T}^7$ to some appropriate $\mathbb{T}_{abc}$ or $\mathbb{T}^4_{ijkl}$ factor, as listed below:

- **For a.a. fibrations:** Let $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$, with coordinates $(x, y, z, w)$, $x, y, z, w \in \mathbb{R}/\mathbb{Z}$. Then $Q$ is the quotient of $\mathbb{T}^4$ by one of the following abelian groups:

  \begin{align*}
  \Gamma_{3456} &= \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z, w) = (-x, -y, z, w), \\
  \beta(x, y, z, w) &= (x, y, -z, -w), \gamma(x, y, z, w) = (c_3 - x, y, c_5 - z, w); \\
  \Gamma_{2457} &= \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z, w) = (-x, -y, z, w), \\
  \beta(x, y, z, w) &= (b_2 - x, y, -z, -w), \gamma(x, y, z, w) = (x, y, c_5 - z, -w); \\
  \Gamma_{2367} &= \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z, w) = (-x, -y, z, w), \\
  \beta(x, y, z, w) &= (b_2 - x, y, -z, -w), \gamma(x, y, z, w) = (x, c_3 - y, z, -w); \\
  \Gamma_{1467} &= \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z, w) = (-x, -y, z, w), \\
  \beta(x, y, z, w) &= (b_1 - x, y, -z, -w), \gamma(x, y, z, w) = (c_1 - x, y, z, -w); \\
  \Gamma_{1357} &= \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z, w) = (-x, -y, z, w), \\
  \beta(x, y, z, w) &= (b_1 - x, y, -z, -w), \gamma(x, y, z, w) = (c_1 - x, c_3 - y, c_5 - z, -w); \\
  \Gamma_{1256} &= \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z, w) = (-x, -y, z, w), \\
  \beta(x, y, z, w) &= (b_1 - x, b_2 - y, -z, -w), \gamma(x, y, z, w) = (c_1 - x, y, c_5 - z, w); \\
  \Gamma_{1234} &= \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z, w) = (-x, -y, -z, -w), \\
  \beta(x, y, z, w) &= (b_1 - x, b_2 - y, z, w), \gamma(x, y, z, w) = (c_1 - x, y, c_3 - z, w).
  \end{align*}
Such 4-orbifold $Q^4$ can be described in terms of a flat $T^2$, $A^2$, or $S^2(2222)$-fibrations over a toroidal 2-orbifolds with the exceptional fibers and the monodromy understood in exactly the same way we try to understand the more complicated bundle $\pi$. We shall come back to this at the end of this section. (Cf. Remark 2.1.4; also Examples 3.4 and 3.6 in Sec. 3.)

- For a.c. fibrations: Let $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with coordinates $(x, y, z)$, where $x, y, z$ are in $\mathbb{R}/\mathbb{Z}$. Then $Q$ is the quotient of $T^3$ by one of the following abelian groups:

  \[
  \Gamma_{127} = \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z) = (-x, -y, z), \\
  \beta(x, y, z) = (b_1 - x, b_2 - y, z), \gamma(x, y, z) = (c_1 - x, y, -z);
  \]

  \[
  \Gamma_{136} = \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z) = (-x, -y, z), \\
  \beta(x, y, z) = (b_1 - x, y, -z), \gamma(x, y, z) = (c_1 - x, c_3 - y, z);
  \]

  \[
  \Gamma_{145} = \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z) = (-x, -y, z), \\
  \beta(x, y, z) = (b_1 - x, y, -z), \gamma(x, y, z) = (c_1 - x, y, c_5 - z);
  \]

  \[
  \Gamma_{235} = \langle \alpha, \beta, \gamma \rangle, \text{ where } \alpha(x, y, z) = (-x, -y, z), \\
  \beta(x, y, z) = (b_2 - x, -z), \gamma(x, y, z) = (x, c_3 - y, c_5 - z);
  \]

  \[
  \Gamma_{246} = \langle \alpha, \beta \rangle, \text{ where } \alpha(x, y, z) = (-x, -y, z), \beta(x, y, z) = (b_2 - x, y, -z);
  \]

  \[
  \Gamma_{347} = \langle \alpha, \gamma \rangle, \text{ where } \alpha(x, y, z) = (-x, -y, z), \gamma(x, y, z) = (c_3 - x, y, -z);
  \]

  \[
  \Gamma_{567} = \langle \beta, \gamma \rangle, \text{ where } \beta(x, y, z) = (-x, -y, z), \gamma(x, y, z) = (c_5 - x, y, -z).
  \]

Such 3-orbifold $Q^3$ can be understood by choosing a fundamental domain $\Omega$ of the $\Gamma_{abc}$-action on $T^3$ and examining how $\Gamma_{abc}$ acts on the boundary $\partial \Omega$. (Cf. Examples 3.1 and 3.2 in Sec. 3.)

For Joyce manifolds of the second kind $J(u, v, \Lambda)$, since the choice of $\Lambda$ varies with $(u, v)$, one no longer has general forms as above. However, once $(u, v, \Lambda)$ is explicitly written down, the base orbifold $Q$ of $\pi$ can still be understood in the same way as in the case $J(b_1, b_2, c_1, c_3, c_5)$. (Cf. Examples 3.5 and 3.6 in Sec. 3.)

The monodromy associated to $\pi$.

The flat connection on the trivial bundle $Y = Z \times X$ over $Z$ induced from the product structure is $\Gamma$-invariant; hence it descends to a flat connection on the fibration $\pi$ of $Y/\Gamma$ over $Q$. Associated to this is a monodromy homomorphism $\rho: \pi^b_{\text{orb}}(Q) \to \text{Aut}(X/\Gamma_0)$. For the Joyce manifolds under consideration, from the explicit expression of the $\Gamma$-action on $Y$ following Sec. 1, any $g$ in $\Gamma$ acting on $Y$ can be splitted into $(\bar{g}, g^f)$, where $\bar{g}$ acts on $Z$ and $g^f$ acts on $X$; in other words, the $\Gamma$-action on $Y$ is the diagonal action of $(\bar{T}, \Gamma^f)$, where $\bar{T}$ is the projection of the $\Gamma$-action on $Y$ to $Z$ and $\Gamma^f$ is the projection of the $\Gamma$-action on $Y$ to $X$. Now recall from Sec. 1 that $\pi^b_{\text{orb}}(Q) = (\bar{T}, \bar{\Lambda})$, one thus has

\[
\rho(\bar{g}) = \bar{g}^f \quad \text{and} \quad \rho(\bar{\Lambda}) = \{\text{Id}\},
\]
Remark 2.1.3. For physicists who feel less familiar with orbifold fundamental group \( \pi_1^{orb}(Q) \), one can also consider directly the monodromy \( \rho' \) on the usual \( \pi_1(Q - \Sigma_Q) \) as follows. Let \( \text{pr} : Z \to Q \) be the quotient map. Then \( \text{pr} : Z - \text{pr}^{-1}(\Sigma_Q) \to Q - \Sigma_Q \) is a regular covering map ([Sp]) since any two liftings of a closed loop in \( Q - \Sigma_Q \) differ by a transformation via an element in \( \Gamma \) and hence must be either both closed or both open. This implies that \( \text{pr}_*(\pi_1(Z - \text{pr}^{-1}(\Sigma_Q))) \) is a normal subgroup in \( \pi_1(Q - \Sigma_Q) \) and the quotient \( \pi_1(Q - \Sigma_Q)/\text{pr}_*(\pi_1(Z - \text{pr}^{-1}(\Sigma_Q))) \) is the group of deck transformations of \( \text{pr} \), which is \( \Gamma/\Gamma_0 \) by construction. Let \( \rho_0 : \pi_1(Z - \text{pr}^{-1}(\Sigma_Q)) \to \text{Aut}(X/\Gamma_0) \) be the monodromy homomorphism associated to the flat connection on \( Y \) and \( \rho' : \pi_1(Q - \Sigma_Q) \to \text{Aut}(X/\Gamma_0) \) be the monodromy homomorphism associated to the flat connection on \( Y/\Gamma \). Then in general we only know that \( \rho' \) is an extension of \( \rho_0 \) in the sense that \( \rho' \circ \text{pr} = \rho_0 \). In our case, \( \rho_0 \) is trivial since \( Y/\Gamma_0 \) is a trivial bundle. From the explicit expression of elements in \( \Gamma \), one has a well-defined chain of group homomorphisms:

\[
\pi_1(Q - \Sigma_Q) \to \pi_1(Q - \Sigma_Q)/\text{pr}_*(\pi_1(Z - \text{pr}^{-1}(\Sigma_Q))) = \Gamma/\Gamma_0 \to \text{Aut}(X/\Gamma_0),
\]

where \( j \) is the projection of the \( \Gamma/\Gamma_0 \)-action on \( Y/\Gamma_0 = Z \times (X/\Gamma_0) \) to the \( X/\Gamma_0 \) factor and \( \text{Aut}(X/\Gamma_0) \) is the group of automorphisms of \( X/\Gamma_0 \). By construction, the homomorphism from \( \pi_1(Q - \Sigma_Q) \) to \( \text{Aut}(X/\Gamma_0) \) obtained by the composition of these homomorphisms coincides with the monodromy homomorphism \( \rho' \). Recall from Sec. 1 the tautological homomorphism \( \tau : \pi_1(Q - \Sigma_Q) \to \pi_1^{orb}(Q) \), then indeed \( \rho' = \rho \circ \tau \).

For Joyce manifolds of the first kind, \( \Gamma \) is abelian; hence \( \rho' \) descends to and is determined by a homomorphism

\[
\overline{\rho'} : H_1(Q - \Sigma_Q; \mathbb{Z}) \to j(\Gamma/\Gamma_0).
\]

When the underlying topology of \( Q \) is \( S^3 \), as in many examples, \( H_1(Q - \Sigma_Q; \mathbb{Z}) \) is generated by the set \( \mathcal{C} \) of meridians associated to real co-dimension 2 strata of \( \Sigma_Q \) and hence \( \overline{\rho'}(H_1(Q - \Sigma_Q; \mathbb{Z})) = j(\Gamma/\Gamma_0) \) is generated by \( \overline{\rho'}(\mathcal{C}) \), which can be read off also directly from \( \Gamma_0 \) associated to points \( p \) that lie in these strata of \( \Sigma_Q \).

\[\square\]

Remark 2.1.4 [Conway’s notation, toroidal 2-orbifolds and the base of a.a. fibrations]. There are several ways to understand the base 4-orbifold \( Q^4 \) of an a.a. fibration \( \pi \) above. A most economic one is to borrow the above discussion for \( \pi \) and to regard \( Q \) itself as the total space of a flat bundle over a toroidal 2-orbifold \( Q^2 \). For this and future reference, let us introduce the Conway’s notation to label a 2-orbifold in terms of its underlying topology and the combinatorial structure of its singular locus.
Conway's notation: (Cf. [Th1].)

- The default topology for the underlying space of a 2-orbifold is $S^2$.
- Each $n_1n_2\cdots$ not preceded by a * indicates cone-points of order $n_1, n_2, \cdots$, that lie in the interior of the underlying topology.
- Each $^*n_1n_2\cdots$ indicates a disk is removed from $S^2$ to form a silvered boundary with a string of corner-reflectors of order $n_1, n_2, \cdots$ sitting along.
- Additional topology is indicated by the symbol °. Before a $|$, ° denotes a handle, while after it it denotes a cross-cap.

(Cf. Figure 2-1.) Though redundant, whenever available we shall combine the above notation with the symbol for the underlying topology to denote a 2-orbifold. For example, $S^2(22)$ is the 2-orbifold (22) since its underlying topology is $S^2$ and $D^2(*)$ is the silvered 2-disk (*), etc..

There are 17 toroidal 2-orbifolds (e.g. [Mo] and [Th1]):

| $T^2(c)$ | (torus) |
|----------|---------|
| $S^2(2222)$ | $S^2(236)$ | $S^2(244)$ | $S^2(333)$ | $A^2(\ast\ast)$ | (annulus) |
| $D^2(\ast2222)$ | $D^2(\ast236)$ | $D^2(\ast244)$ | $D^2(\ast333)$ | $M^2(\ast\ast\ast)$ | (Möbius strip) |
| $D^2(22\ast)$ | $D^2(4\ast2)$ | $D^2(3\ast3)$ | $K^2(\ast\ast\ast)$ | (Klein bottle) |
| $D^2(\ast22)$ | $P^2(22\ast)$ | (projective plane) |

Some 2-orbifolds $Q^2$ in the list together with a set of generators for $\pi_1^{\text{orb}}(Q^2)$ represented by based-loops in $Q^2$ are illustrated in Figure 2-1.

![Figure 2-1. Some toroidal 2-orbifolds $Q^2$ with their Conway notation. The boundary that is modelled on $\mathbb{C}/\langle z \mapsto z \rangle$ is indicated by a double-line. Such boundary is said to be silvered. A generating set of $\pi_1^{\text{orb}}(Q^2)$ is represented by a collection of based loops.]

With these facilities, the base 4-orbifold $Q^4$ can be easily visualized. Let us illustrate this by an example.
Example. $Q^1_{3456} = T^4_{3456}/\Gamma_{3456}$ with $(c_3, c_5) = (0, 0)$ (cf. [Jo1]: II, Example 4). Explicitly, $T^4_{3456}$ is parametrized by $(x_3, x_4, x_5, x_6)$ and $\Gamma_{3456} = \left\langle \alpha, \beta, \gamma \right\rangle$, where

$$
\alpha(x_3, x_4, x_5, x_6) = (-x_3, -x_4, x_5, x_6), \quad \beta(x_3, x_4, x_5, x_6) = (x_3, x_4, -x_5, -x_6),
$$
$$
\gamma(x_3, x_4, x_5, x_6) = (-x_3, x_4, -x_5, x_6).
$$

Regard $T^4_{3456}$ as the trivial bundle $T^2_{34} \times T^2_{56}$, with base $T^2_{34}$ and fiber $T^2_{56}$, and $\Gamma_{3456}$ as a group of bundle automorphisms. Then all previous discussions for $\pi$ apply and one has a fibration $\pi_{3456} : Q^4_{3456} \to Q^2_{34}$. By inspection, $Q^2_{34}$ is a $D^2(\ast 2222)$ and the generic fiber an $S^2(2222)$. $\Sigma_p$ for $p \in \Sigma Q^2_{34}$ is $\langle \beta, \gamma \rangle = D_2$ and the exceptional fiber thereover is a $D^2(\ast 2222)$ of multiplicity 2. The global structure of $\pi_{3456}$ is coded in the associated monodromy $\rho_{3456}$, which can be read off directly from the above explicit expression of $\alpha$, $\beta$, and $\gamma$. With some abuse of notations, let $t_3(x_3, x_4) = (x_3+1, x_4)$ and $t_4(x_3, x_4+1)$ be translations on $\mathbb{R}^2_{34}$; then $\pi_{3456}$ is generated by $\alpha, \gamma, t_3, \text{ and } t_4$. And $\rho_{3456}$ is determined by

$$
\rho_{3456}(\alpha) = \rho_{3456}(t_3) = \rho_{3456}(t_4) = \text{Id}.
$$

(Figure 2-2.)

![Figure 2-2](image)

**Figure 2-2.** The monodromy along the loop representing $\gamma$. Notice that the letter “R” on the front face of the $S^2(2222)$-fiber over $\ast$ is sent to the back face after the round trip along $\gamma$.

The same method can also be applied to understand an exceptional fiber of an a.c. fibration. This concludes the remark.

\[\square\]
2.2 Adjustment to the fibration and the monodromy after resolving $S$.

So far the discussion has been centered around the toroidal orbifold $\mathbb{T}^7/\Gamma$. To relate it to the Joyce manifold $M^7$ obtained by resolving the singular locus $S$ of $\mathbb{T}^7/\Gamma$, one needs to understand the effect to $\pi$ when resolving $S$. This can be accomplished by understanding

1. how components of $S$ sit in $\mathbb{T}^7/\Gamma$ over $Q$, which gives the information of how components in $S$ intersect the fibers of $\pi$, and

2. the restriction of the exceptional locus to a fiber when resolving $S$, which gives the information of the new fiber after resolving $S$.

Adjustments to the base $Q$ and the monodromy $\rho$ will then follow.

Remark 2.2.1. Recall that a component of $S$ comes from a fixed $\mathbb{T}^3$ of some element in $\Gamma$. For Joyce manifolds of the first kind, any such fixed $\mathbb{T}^3$ is a coordinate 3-torus $\mathbb{T}^3_{a'b'c'}$. Much of the answer to Item (1) above follows from the label $(a'b'c')$. This suggests a computer code to assist the work. (Cf. Sec. 5, Issue (1).)

Let us now consider the fibrations case by case.

a.a. fibrations for a Joyce manifold of the first kind.

For such fibrations, $Y = Z \times X$ with $Z = \mathbb{T}^4$, $X = \mathbb{T}^3$, and $\Gamma_0$ is always trivial. Hence the generic fiber of $\pi$ is always $\mathbb{T}^3$ and the singular set $S$ has to lie over the singular locus $\Sigma_Q$ of $Q$. Consider now what happens to the fibers of $\pi$ when resolving $S$. Since a generic fiber does not intersect $S$, resolving $S$ does not affect it. Let $F_p$ now be a fiber over a point $p$ in $\Sigma_Q$ and $S_0$ be a component of $S$. Let $\widetilde{F}_p$, $\widetilde{S}_0$ be a connected component of their respective preimage in $Y$. Then $\widetilde{F}_p$ is one of $\mathbb{T}_{137}^3$, $\mathbb{T}_{136}^3$, $\mathbb{T}_{145}^3$, $\mathbb{T}_{235}^3$, $\mathbb{T}_{236}^3$, $\mathbb{T}_{347}^3$, and $\mathbb{T}_{356}^7$. On the other hand, since $\widetilde{S}_0$ is a fixed $\mathbb{T}^3$ for some element of $\Gamma$, it can only be one of $\mathbb{T}_{567}^3$, $\mathbb{T}_{347}^3$, $\mathbb{T}_{246}^3$, $\mathbb{T}_{127}^3$, $\mathbb{T}_{136}^3$, $\mathbb{T}_{145}^3$, and $\mathbb{T}_{235}^3$. Comparing the two lists, one has that $\widetilde{F}_p \cap \widetilde{S}_0$ can only be empty, $\mathbb{T}_{1}^3$, or $\mathbb{T}_{a'b'c'}^3$, in the last case $\widetilde{F}_p = \widetilde{S}_0$. Together with the local models (a) and (b) of $\nu(S_0)$ described in Sec. 1, one concludes that the possible $F_p \cap S_0$ and the effect to $F_p$ when resolving $S_0$ can only be one of the following:

(a) When $\nu(S_0) = \mathbb{T}^3 \times (\mathbb{C}^2/\langle -1 \rangle)$:

- $F_p \cap S_0$ empty: No effect on $F_p$ when resolving $S_0$.
- $F_p \cap S_0 = \text{some disjoint } n \text{ copies of } \mathbb{T}^1$: After resolving $S_0$, the new fiber $F'_p$ at $p$ becomes $F_p \cup n \mathbb{T}^1 \times \mathbb{CP}^1$ with normal crossing singularities along $F_p \cap S_0$.
- $F_p = S_0$: Then $p$ is an isolated $A_1$-singularity of $Q$ and resolving $S_0$ changes the neighborhood $\nu(p) \cong \mathbb{C}^2/\langle -1 \rangle$ of $p$ in $Q$ to $\nu(p) \cong T^* \mathbb{CP}^1$. Recall that a fiber of $T^* \mathbb{CP}^1$ here is regarded as a subcone $\mathbb{C}/\langle -1 \rangle$ in $\mathbb{C}^2/\langle -1 \rangle$ and the union of their $\mathbb{Z}_2$-cone points form the exceptional locus $E$ of the resolution, which is the 0-section.
of $T^*\mathbb{CP}^1$. Thus, after resolving $S_0$, $Q$ is changed to a new orbifold $Q'$ with singular locus $\Sigma_{Q'} = (\Sigma_Q - \{p\}) \cup E$, where $\Gamma_p = \mathbb{Z}_2$ for $p \in E$. The associated new fibration inherited from $\pi$ is the product $\nu(p) \times T^3$ over $\nu(p)$.

(b) When $\nu(S_0) = \{T^3 \times (\mathbb{C}^2/(-1))\} / \mathbb{Z}_2$:

- $F_p \cap S_0$ empty: No effect on $F_p$ when resolving $S_0$.

- $F_p \cap S_0$ = some disjoint $n$ copies of $T^1$: Since each component of $F_p \cap S_0$ descends from the intersection $T^1$ of two 3-tori in $Y$, $F_p \cap S_0$ is liftable to $T^3 \times (\mathbb{C}^2/(-1))$ and the induced $\mathbb{Z}_2$-action on the set of lifted $T^1$ in $T^3 \times (\mathbb{C}^2/(-1))$ can only be either free or trivial. In the former case, the new fiber $F'_p$ at $p$ after resolving $S_0$ becomes $F_p \cup n T^1 \times \mathbb{CP}^1$ with normal crossing singularities along $F_p \cap S_0$, while in the latter case it becomes $F_p \cup n (T^1 \times \mathbb{CP}^1)/\mathbb{Z}_2$ with normal crossing singularities along $F_p \cap S_0$.

- $F_p = S_0$: In this case $F_p$ is the fiber over $(0,0)$ in the natural $T^3$-fibration of $\{T^3 \times (\mathbb{C}^2/(-1))\} / \mathbb{Z}_2$ over $\nu(p) = \mathbb{C}^2/((1) \oplus \mathbb{Z}_2)$, where the $\mathbb{Z}_2$ acts on $\mathbb{C}^2$ either holomorphically, as generated by $(z_1, z_2) \mapsto (z_1, -z_2)$, or antiholomorphically, as generated by $(z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2})$, following the holomorphicity of the $\mathbb{Z}_2$-action on $T^3 \times (\mathbb{C}^2/(-1))$. Let us consider these two cases separately:

  - $\mathbb{Z}_2$-action holomorphic: Then $\Sigma_Q \cap \nu(p)$ consists of the image of $\{z_1z_2 = 0\}$ in $\mathbb{C}^2/((-1) \oplus \mathbb{Z}_2)$, which is isomorphic to the neighborhood of a double point in a complex curve. $\Gamma_p = \mathbb{Z}_2$ for $p$ regular point of $\Sigma_Q \cap \nu(p)$ and $= (-1) \oplus \mathbb{Z}_2$ for $p$ the double point. The fiber over $\Sigma_Q \cap \nu(p)$ is $\mathbb{C}^2/\mathbb{Z}_2$. When $S_0$ is resolved, the base $Q$ is changed to a new $Q'$ with $\nu(p)$ resolved to $\nu(p) = T^*\mathbb{CP}^1/\mathbb{Z}_2$. The exceptional locus $E$ of the resolution is $\mathbb{CP}^1/\mathbb{Z}_2$, where $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is the 0-section of $T^*\mathbb{CP}^1$ parametrized by $z$ and the $\mathbb{Z}_2$-action is generated by $z \mapsto -z$. Since the locus $\{z_1z_2 = 0\}$ selects two complex lines in $T_{(0,0)}\mathbb{C}^2$ that are inequivalent under the $(-1) \oplus \mathbb{Z}_2$-action, the new singular locus $\Sigma_{Q'}$ is the union of the resolution of $\Sigma_Q$ at the double point by normalization and the new component $E$ as in Figure 2-3 (i). $\Gamma_p = (-1)$ for $p \in E - \{p_0, p_{\infty}\}$ and $= (-1) \oplus \mathbb{Z}_2$ for $p = p_0$ or $p_{\infty}$. (Intrinsically $E$ is an $S^2(22)$-orbifold.)

  - $\mathbb{Z}_2$-action antiholomorphic: Then $\Sigma_Q \cap \nu(p)$ consists of the image of $C = \{(z_1, z_2) | z_1, z_2$ both real or both purely imaginary $\}$ in $\mathbb{C}^2/((-1) \oplus \mathbb{Z}_2)$, whose topology is still the same as a neighborhood of a double point in a complex curve. $\Gamma_p = \mathbb{Z}_2$ for $p$ regular point of $\Sigma_Q \cap \nu(p)$ and $= (-1) \oplus \mathbb{Z}_2$ for $p$ the double point. The fiber over $\Sigma_Q \cap \nu(p)$ is $\mathbb{C}^2/\mathbb{Z}_2$. When $S_0$ is resolved, the base $Q$ is changed to $Q'$ with $\nu(p)$ resolved to $\nu(p) = T^*\mathbb{CP}^1/\mathbb{Z}_2$. The exceptional locus $E$ of the resolution is $\mathbb{CP}^1/\mathbb{Z}_2$, where $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is the 0-section of $T^*\mathbb{CP}^1$ parametrized by $z$ and the $\mathbb{Z}_2$-action is generated
by \( z \mapsto \overline{z} \). The locus \( C \) now selects an \( S_1 \)-family of complex lines in \( T_{(0,0)}\mathbb{C}^2 \) that are inequivalent under the \( (-1) \oplus \mathbb{Z}_2 \)-action; hence, the new singular locus \( \Sigma_Q' \) is the union of the resolution of \( \Sigma_Q \) at the double point by tubing and the new component \( E \) in Figure 2-3 (ii). \( \Gamma_p = \mathbb{Z}_2 \) for \( p \) in the interior of \( E \) and \( = (-1) \oplus \mathbb{Z}_2 \) for \( p \) in its boundary. (Intrinsically \( E \) is a silvered 2-disk \( D^2(*) \).

\[
\begin{array}{c}
\gamma
\end{array}
\]

\( Q \rightarrow Q' \)

\( E \) is a silvered 2-disk.

Figure 2-3 The two local models for \( \nu(S_0) \) in Case (b) lead to different resolutions \( \tilde{\nu}(p) \) of \( \nu(p) \). The shaded part is the exceptional locus \( E \), which becomes an additional component in \( \Sigma_{Q'} \). In (i), \( E \) is an \( S^2(22) \)-orbifold, while in (ii) \( E \) is a silvered 2-disk. The singular locus \( \Sigma_Q \cap \nu(p) \) and what it is resolved to are also indicated.

**a.c. fibrations for a Joyce manifold of the first kind.**

For such fibrations, \( Y = Z \times X \) with \( Z = \mathbb{T}^3 \) and \( X = \mathbb{T}^4 \). One can check that there are two cases:

1. \( \Gamma_0 = 0 \): In this case \( Y/\Gamma \) is a \( \mathbb{T}^4 \)-fibration over \( Q \) and \( S \) lies over \( \Sigma_Q \).
2. \( \Gamma_0 = \langle g \rangle = \mathbb{Z}_2 \) for some \( g \) in \( \Gamma \): In this case \( g \) acts on \( X = \mathbb{T}^4 \) by negation and the generic fiber of the fibration \( \pi : Y/\Gamma \to Q = Z/\Gamma \) is the singular K3 surface \( X/\langle g \rangle \). The components of \( S \) that are associated to the fixed 3-tori of \( g \) now appear as multi-sections of \( \pi \) while all other components of \( S \) must sit over \( \Sigma_Q \).

Let us now consider what happens to a fiber of \( \pi \) when resolving \( S \).

Consider first the special components of \( S \) that come from the fixed 3-tori of \( g \) in Case (2) above. Since their preimages in \( Y \) are transverse to \( X \), they are transverse to all fibers in \( Y/\Gamma \). Conversely, if a component \( \mathbb{T}^3 \) of \( S \) is transverse to some fiber, then its preimage in \( Y \) must be transverse to \( X \) and hence must come from a fixed \( \mathbb{T}^3 \) of \( g \). Thus they accounts for all the isolated singularities in a fiber - generic or exceptional alike - of \( \pi \).
Resolving such components of $S$ will resolve simultaneously all the isolated singularities in fibers and, hence, leads to a $K3$-fibration for the Joyce manifold $M^7$.

For all other components of $S$, let $F_p$ be a fiber of $\pi$ over $p$ and $S_0$ be a component of $S$ that lies over $\Sigma_Q$. Let $\tilde{F}_p$, $\tilde{S}_0$ be a connected component of their respective preimage in $Y$. A similar consideration as in the discussion for a.a. fibrations implies that $\tilde{F}_p$ is a coordinate $T^4$, $\tilde{S}_0$ is a coordinate $T^3$, and their intersection can only be either empty or a $T^2$-fibration. Together with the local model (a) and (b) of $\nu(S_0)$ described in Sec. 1, one concludes that the possible connected component of $F_p \cap S_0$ and the effect to $F_p$ when resolving $S_0$ can only be one of the following:

(a) When $\nu(S_0) = T^3 \times (\mathbb{C}^2/\langle -1 \rangle)$:

- $F_p \cap S_0$ empty: No effect on $F_p$ when resolving $S_0$.
- $F_p \cap S_0 = \text{some disjoint } n \text{ copies of } T^2$: After resolving $S_0$, the new fiber $F'_p$ at $p$ becomes $F_p \cup n T^2 \times \mathbb{CP}^1$ with normal crossing singularities along $F_p \cap S_0$.

(b) When $\nu(S_0) = \{T^3 \times (\mathbb{C}^2/\langle -1 \rangle)\}/\mathbb{Z}_2$:

- $F_p \cap S_0$ empty: No effect on $F_p$ when resolving $S_0$.
- $F_p \cap S_0 = \text{some disjoint } n \text{ copies of } T^2$: Again $F'_p \cap S_0$ is liftable to $T^3 \times (\mathbb{C}^2/\langle -1 \rangle)$ and the induced $\mathbb{Z}_2$-action on the set of lifted $T^2$ in $T^3 \times (\mathbb{C}^2/\langle -1 \rangle)$ is either free or trivial. In the former case, the new fiber $F'_p$ at $p$ after resolving $S_0$ becomes $F_p \cup n T^2 \times \mathbb{CP}^1$ with normal crossing singularities along $F_p \cap S_0$, while in the latter case it becomes $F_p \cup n (T^2 \times \mathbb{CP}^1)/\mathbb{Z}_2$ with normal crossing singularities along $F_p \cap S_0$.

### a.a./a.c. fibrations for Joyce manifolds of the second kind.

For such fibrations, $\Gamma_0$ is always trivial. The singular set $S$ of $Y/\Gamma$ sits over $\Sigma_Q$. From Sec. 1, the tubular neighborhood $\nu(S_0)$ of a component $S_0$ of $S$ is always homeomorphic to $T^3 \times (\mathbb{C}^2/\langle -1 \rangle)$; thus, if a fiber $F_p$ happens to be some $S_0$ in an a.a. fibration $\pi$, resolving $S_0$ changes $Q$ to $Q'$ obtained by blowing up $Q$ at $p$. Other possible intersections of $F_p$ and $S_0$ and the effect to $F_p$ when resolving $S_0$ are similar to the discussion for Joyce manifolds of the first kind. For such Joyce manifold, an a.a. fibration is a $T^3$-fibration and an a.c. fibration is a $T^4$-fibration.

### Adjustment to the monodromy.

Let $\bar{\pi} : M^7 \rightarrow \bar{Q}$ be the induced fibration from $\pi : Y/\Gamma \rightarrow Q$ after resolving $S$. Then the flat connection on $\pi$ can be lifted to a flat connection on $\bar{\pi}$. (For $\bar{\pi}$ a $K3$-fibration, see more details below.) Let $\bar{\rho}$ be the associated monodromy of $\bar{\pi}$.

(i) When $\pi$ is an a.a. fibration:
Both \( \pi \) and \( \tilde{\pi} \) are \( \mathbb{T}^3 \)-fibrations. When \( \pi \) contains no component of \( S \) as a fiber, then \( \tilde{Q} = Q \) and the restriction of \( \tilde{\pi} \) to \( \tilde{Q} - \Sigma_{\tilde{Q}} \) coincides with the restriction of \( \pi \) to \( Q - \Sigma_Q \). Thus \( \tilde{\rho} = \rho \).

On the other hand, if \( \pi \) contains some components of \( S \) as fibers, then one has a resolution \( r : \tilde{Q} \to Q \) as discussed earlier. Since the restriction of \( \tilde{\pi} \) to \( \tilde{Q} - \Sigma_{\tilde{Q}} \) still coincides with the restriction of \( \pi \) to \( Q - \Sigma_Q \), \( \tilde{\rho} = \rho \circ r_* \), where \( r_* : \pi_{1}^{\text{orb}}(\tilde{Q}) \to \pi_{1}^{\text{orb}}(Q) \) is the induced homomorphism of \( r \).

**Remark 2.2.1.** From the local model of \( \nu(S) \), the monodromy along a meridian \( C \) associated to an exceptional locus \( E \) in \( \Sigma_{\tilde{Q}} \) of the resolution \( r \) above is always trivial since \( C \) is homologous to the unit circle in a fiber of \( T^*\mathbb{C}P^1 \) and the fibration is trivial when restricted thereover.

(ii) When \( \pi \) is an a.c. fibration:

For such fibration, \( \tilde{Q} \) and \( Q \) are always the same. If \( \Gamma_0 = \{0\} \), then both \( \pi \) and \( \tilde{\pi} \) are \( \mathbb{T}^4 \)-fibrations and the restriction of \( \tilde{\pi} \) to \( \tilde{Q} - \Sigma_{\tilde{Q}} \) coincides with the restriction of \( \pi \) to \( Q - \Sigma_Q \). Thus \( \tilde{\rho} = \rho \).

On the other hand, if \( \Gamma_0 = \mathbb{Z}_2 \), then resolving \( S \) - though does not change the base \( Q \) - alters the generic fiber from a toroidal 4-orbifold to a smooth K3 surface via the resolution of \( A_1 \)-singularities \( f : \tilde{X}/(\sim -1) \to X/(\sim -1) \). However, from Sec. 1, as bundle automorphisms of \( Y \) over \( Z \), elements in \( \Gamma \) restricted to fibers are either holomorphic or anti-holomorphic. Thus \( \Gamma \) can be lifted to a group of bundle automorphisms of \( Y' = Z \times \tilde{X} \) over \( Z \), where \( \tilde{X} \) is the blow up of \( X \) at the fixed points of \( \Gamma_0 \). The flat connection on \( Y' \) induces then a flat connection on \( \tilde{\pi} \) which is compatible with the flat connection on \( Y/\Gamma \) under the resolution \( M^7 \to Y/\Gamma \) that resolves \( S \); hence \( \tilde{\rho} \) is a lifting of \( \rho \). Indeed, if one recalls Remark 2.1.3, then since any map in \( \gamma(\Gamma/\Gamma_0) \) is either holomorphic or anti-holomorphic and the resolution \( \tilde{X}/(\sim -1) \to X/(\sim -1) \) is through blowups, \( \gamma(\Gamma/\Gamma_0) \) is liftable in a unique way into \( \text{Aut}(X/(\sim -1)) \), which gives the monodromy \( \tilde{\rho} \) of \( \tilde{\pi} \).

This concludes our general discussions for the a.a./a.c. fibrations of a Joyce manifold.

### 3 Examples in the 5-step-routine.

The discussion in Sec. 2 suggests a 5-step-routine to understand the a.a./a.c. fibrations of any given Joyce manifold. In this section, we choose \( J(0, \frac{1}{2}, \frac{1}{2}, 0) \) in [Jo1] (I) and \( J(e^{\pi i/3}, e^{2\pi i/3}, \Lambda) \) in [Jo1] (II) to illuminate this procedure.

**Example 3.1** \([J(0, \frac{1}{2}, \frac{1}{2}, 0) : \text{a.c. K3-fibration}] \). (Cf. [Jo1]: I; also II, Example 3.)\( \Gamma \) is generated by

\[
\alpha(x_1, \cdots, x_7) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7),
\]

\[
\beta(x_1, \cdots, x_7) = (-x_1, \frac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7),
\]

\[
\gamma(x_1, \cdots, x_7) = \left(\frac{1}{2} - x_1, x_2, \frac{1}{2} - x_3, x_4, -x_5, x_6, -x_7\right).
\]
From [Jo1] the quotient $\mathbb{T}^7/\Gamma$ has a singular set $S$ of $A_1$-singularities that consists of 12 disjoint $\mathbb{T}^3$ arising from the fixed $\mathbb{T}^3$ of $\alpha$, $\beta$, and $\gamma$. The tubular neighborhood of each component of $S$ is modelled on $\mathbb{T}^3 \times (\mathbb{C}^2/\langle -1 \rangle)$. After resolving $S$, one obtains a Joyce manifold.

Consider the associative-coassociative decomposition $Y = \mathbb{T}^7 = \mathbb{T}^3_{567} \times \mathbb{T}^4_{1234}$ with $Z = \mathbb{T}^3_{567}$ and $X = \mathbb{T}^4_{1234}$. One then has the fibration $\pi_{567} : \mathbb{T}^7/\Gamma \to Q^3_{567}$, where $Q^3_{567}$ is the orbifold $\mathbb{T}^3_{567}/\langle \beta, \gamma \rangle$ with $\beta(x_5, x_6, x_7) = (-x_5, -x_6, x_7)$ and $\gamma(x_5, x_6, x_7) = (-x_5, x_6, -x_7)$.

(i) The base orbifold $Q^3_{567}$: By choosing a fundamental domain $\Omega$ of the $\langle \beta, \gamma \rangle$-action on $\mathbb{T}^3_{567}$ to be, e.g., $[0, 1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and examining how $\langle \beta, \gamma \rangle$ acts on the boundary $\partial \Omega$, one concludes that $Q^3_{567}$ is a Euclidean 3-orbifold whose underlying topology is $S^3$ and whose singular locus $\Sigma_Q$ is the 1-skeleton of a 3-cube. The group $\Gamma_p$ associated to an interior point $p$ of each edge of $\Sigma_Q$ is $\mathbb{Z}_2$ - generated by $\beta$, $\gamma$, or $\beta$, depending on whether the edge is parallel to $x_5$, $x_6$, or $x_7$-axis respectively - and to each vertex $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle \beta, \gamma \rangle$.

(Figure 3-1-1.)

(ii) The fibers: $\Gamma_0 = \mathbb{Z}_2 = \langle \alpha \rangle$ in this example. The fiber $F_p$ of the fibration $\pi_{567}$ over a point $p$ in $Q^3_{567}$ is listed below:

- For $p \in Q^3_{567} - \Sigma_Q$,

  $$F_p = X_0 = \mathbb{T}^4_{1234}/\langle \alpha \rangle = \mathbb{T}^4_{1234}/(\langle x_1, x_2, x_3, x_4 \rangle \mapsto (-x_1, -x_2, -x_3, -x_4)),$$

  which is a singular K3 surface with 16 $A_1$-singularities.

- For $p$ an interior point of an edge of $\Sigma_Q$, $\Gamma_p$ is $\langle \alpha, \beta \rangle$, $\langle \alpha, \gamma \rangle$, or $\langle \alpha, \beta \rangle$, depending on whether the edge is parallel to $x_5$, $x_6$, or $x_7$-axis respectively. Thus $F_p = X_e$.
is the (isomorphic) \( \mathbb{Z}_2 \)-quotient \( X_0/\langle \beta \gamma \rangle, X_0/\langle \gamma \rangle \), or \( X_0/\langle \beta \rangle \) accordingly, where it is understood that, e.g.

\[
X_0/\langle \beta \gamma \rangle = T^7_{1234}/\langle \alpha, \beta \gamma \rangle = T^4_{1234}/\langle (x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, -x_3, -x_4), (x_1, x_2, x_3, x_4) \mapsto (\frac{1}{2} + x_1, \frac{1}{2} - x_2, -x_3, -x_4) \rangle.
\]

Such \( F_p \) has multiplicity 2.

• For \( p \) a vertex of \( \Sigma_Q \), \( \Gamma_p = \Gamma \). Thus \( F_p = X_v = X_0/\langle \beta, \gamma \rangle \), which has multiplicity 4.

These exceptional fibers have various realizations as a \( \mathbb{T}^2 \)-bundle over toroidal 2-orbifolds as demonstrated by the example in Remark 2.1.4.

(iii) How \( S \) sits over \( Q_{567}^3 \): By inspection, how the 12 components of \( S \) sit in \( T^7/\Gamma \) with respect to \( \pi_{567} \) is listed below:

• Since \( \Gamma_0 = \langle \alpha \rangle \), the 4 \( T^3 \) of \( S \) that descend from the 16 fixed \( T^3 \) of \( \alpha \) become 4 disjoint multi-sections \( \sigma_i, i = 1, \ldots, 4 \), of \( \pi_{567} \). The union \( \bigcup_i \sigma_i \) contains exactly the isolated singularities of fibers \( F_p \).

• The 4 \( T^3 \) from the fixed 3-tori of \( \beta \) are mapped under \( \pi_{567} \) onto the 4 edges of \( \Sigma_Q \) that are parallel to the \( x_7 \)-axis with one \( T^3 \) to one edge; and similarly for the other 4 \( T^3 \) from the fixed 3-tori of \( \gamma \) to the edges of \( \Sigma_Q \) that are parallel to \( x_6 \)-axis. Up to permutations of coordinates, the restriction of \( \pi_{567} \) to each of these \( T^3 \) is of the form

\[
T^3 \rightarrow [0, \frac{1}{2}], \quad (x, y, z) \mapsto \overline{z}, \quad \text{where } x, y, z \in \mathbb{R}/\mathbb{Z} \text{ and } \overline{z} = \begin{cases} 
    z & \text{if } z \in [0, \frac{1}{2}] \\
    1 - z & \text{if } z \in [\frac{1}{2}, 1] 
\end{cases},
\]

(i.e. folding of \( T^1 \))

\[\text{Figure 3-1-2. How the components of } S \text{ sit over } \Sigma_Q \text{ is indicated.}\]
(iv) Adjustment after resolving $S$: Following Sec. 2.2, after resolving $S$, one has a fibration $\tilde{\pi}_{567}$ of $M^7$ over $Q^3_{567}$ with generic fiber $\tilde{X}_0$, which is a smooth K3 surface, and exceptional fibers:

- $\tilde{X}_e$ over the interior of the edges of $\Sigma_Q$ parallel to $x_5$, which is a smooth $\mathbb{Z}_2$-quotient of $\tilde{X}_0$;
- $\tilde{X}_e \cup 2 \mathbb{T}^2 \times \mathbb{C}P^1$ over the interior of other edges of $\Sigma_Q$;
- $\tilde{X}_v \cup 2 \mathbb{T}^2 \times \mathbb{C}P^1$ over a vertex of $\Sigma_Q$.

(v) Monodromy: From Sec. 2, the monodromy of $\tilde{\pi}_{567}$ is determined by a representation $\varphi$ from $H_1(Q^3_{567} - \Sigma_Q; \mathbb{Z})$ to $\text{Aut}(X_0)$. Since $Q^3_{567} - \Sigma_Q$ is homeomorphic to the complement of the bouquet $\vee_5 S^1$ of five circles in $S^3$, $H_1(Q^3_{567} - \Sigma_Q; \mathbb{Z}) = \mathbb{Z}^5$, whose generators may be chosen to be the meridians $C_{5,1}, C_{5,2}, C_{6,1}, C_{6,2},$ and $C_{6,3}$ as indicated in Figure 3-1-3. One then has $\varphi(C_{5,i}) = j(\gamma)$ for $i = 1, 2, 3$, and $\varphi(C_{6,i}) = j(\gamma)$ for $i = 1, 2, 3$.

![Figure 3-1-3](image)

Figure 3-1-3. The set of generating cycles $\{C_{5,1}, C_{5,2}, C_{6,1}, C_{6,2}, C_{6,3}\}$ for $H_1(Q^3_{567} - \Sigma_Q; \mathbb{Z})$ are indicated by thick loops.

This concludes the example.

Example 3.2 $[J(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0): \text{a.c. K3-fibration}]$. In Example 3.1, consider instead the decomposition $Y = T^7 = T^3_{347} \times T^4_{1256}$ with $Z = T^3_{347}$ and $X = T^4_{1256}$. One then has the fibration $\pi_{347}: T^7/\Gamma \to Q^3_{347}$, where $Q^3_{347}$ is the orbifold $T^3_{347}/\langle \varpi, \tau \rangle$ with $\varpi(x_3, x_4, x_7) = (-x_3, -x_4, x_7)$ and $\tau(x_3, x_4, x_7) = (\frac{1}{2} - x_3, x_4, -x_7)$.

(i) The base orbifold $Q^3_{347}$: By choosing a fundamental domain $\Omega$ in $T^3_{347}$ to be $[0, 1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and examining how $\langle \varpi, \tau \rangle$ acts on $\partial \Omega$, one concludes that $Q^3_{347}$ is an Euclidean 3-orbifold whose underlying topology is $S^3$ and whose singular locus $\Sigma_Q$ is a doubled Hopf...
link, which has 4 components: $S_{4,1}^1$ and $S_{4,2}^1$ parallel to the $x_4$-axis, and $S_{7,1}^1$ and $S_{7,2}^1$ parallel to the $x_7$-axis. For $p$ in $S_{4,i}^1$, $\Sigma_p = \langle \gamma \rangle$; and, for $p$ in $S_{7,i}^1$, $\Sigma_p = \langle \alpha \rangle$. (FIGURE 3-2-1.)

(ii) The fibers: In this example, $\Gamma_0 = \mathbb{Z}_2 = \langle \beta \rangle$. The fiber $F_p$ of the fibration $\pi_{347}$ over a point $p$ in $Q_{347}^3 - \Sigma_Q$ is listed below:

- For $p$ in $Q_{347}^3 - \Sigma_Q$,
  
  $F_p = X_0 = \mathbb{T}^7_{1256}/\langle \beta \rangle = \mathbb{T}^4_{1256}/\langle (x_1, x_2, x_5, x_6) \mapsto (-x_1, \frac{1}{2} - x_2, -x_5, -x_6) \rangle \,$
  
  which is a singular K3 surface with 16 $A_1$-singularity.

- For $p$ in $S_{4,i}^1$ (resp. $S_{7,i}^1$), $\Gamma_p$ is $\langle \beta, \gamma \rangle$ (resp. $\langle \alpha, \beta \rangle$). Thus $F_p = X_e$ is the (isomorphic) $\mathbb{Z}_2$-quotient $X_0/\langle \gamma \rangle$ or $X_0/\langle \alpha \rangle$ accordingly.

(iii) How $S$ sits over $Q_{347}^3$:

- The 4 $\mathbb{T}^3$ of $S$ that descend from the 16 fixed $\mathbb{T}^3$ of $\beta$ become 4 disjoint multi-sections $\sigma_i$, $i = 1, \ldots, 4$, of $\pi_{347}$. The union $\cup_i \sigma_i$ contains exactly the isolated singularities of fibers $F_p$.

- Two of the 4 $\mathbb{T}^3$ in $S$ from the fixed 3-tori of $\alpha$ are mapped onto $S_{4,1}^1$ under $\pi_{347}$ and the other two onto $S_{4,2}^1$. Similarly for the 4 $\mathbb{T}^3$ in $S$ from the fixed 3-tori of $\gamma$: two onto $S_{7,1}^1$ and two onto $S_{7,2}^1$. The restriction of $\pi_{347}$ to each of these $\mathbb{T}^3$ is simply a standard projection of $\mathbb{T}^3$ to $\mathbb{T}^1$.

(iv) Adjustment after resolving $S$: After resolving $S$, one obtains a different K3-fibration for the same $M^7$ in Example 3.1: $\tilde{\pi}_{347} : M^7 \rightarrow Q_{347}^3$. Its set of critical values is $\Sigma_Q$ and its degenerate fiber is $\tilde{X}_e \cup 2 \mathbb{T}^2 \times \mathbb{C} \mathbb{P}^1$. 

Figure 3-2-1. After the identification of $\partial \Omega$ governed by $\langle \alpha, \gamma \rangle$, one obtains the Euclidean orbifold $Q_{347}^3$ with underlying topology $S^3$ and singular locus $\Sigma_Q$ a doubled Hopf link.
$(v)$ Monodromy: For monodromy $\overline{\rho}: H_1(Q^3_{347} - \Sigma_Q; \mathbb{Z}) \to Aut(X_0)$, $H_1(Q^3_{347} - \Sigma_Q; \mathbb{Z}) = \mathbb{Z}^4$ is generated by the meridians $C_{4,1}$, $C_{4,2}$, $C_{7,1}$, and $C_{7,2}$ associated to $S^1_{3,1}$, $S^1_{3,2}$, $S^1_{7,1}$, and $S^1_{7,2}$ respectively, as indicated in Figure 3-2-2. One thus has $\overline{\rho}(C_{4,i}) = f(\gamma)$ and $\overline{\rho}(C_{7,i}) = f(\alpha)$, $i = 1, 2$.

This concludes the example. 

\[\square\]

Remark 3.2.1. Examples 3.1 and 3.2 together show that a Joyce manifold can have inequivalent a.a. K3-fibrations.

Example 3.3 $[J(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0): =$ a.c. $\mathbb{T}^4$-fibration$]$. In Example 3.1, consider instead the decomposition $Y = \mathbb{T}^7 = \mathbb{T}^3_{127} \times \mathbb{T}_{3456}$ with $Z = \mathbb{T}^3_{127}$ and $X = \mathbb{T}^4_{3456}$. One then has the fibration $\pi_{127}: \mathbb{T}^7/\Gamma \to Q^3_{127}$, where $Q^3_{127}$ is the orbifold $\mathbb{T}^3_{127}/\langle \overline{\gamma}, \overline{\beta}, \overline{\alpha}\rangle$ with $\overline{\gamma}(x_1, x_2, x_7) = (-x_1, -x_2, x_7)$, $\overline{\beta}(x_1, x_2, x_7) = (-x_1, \frac{1}{2} - x_2, x_7)$, and $\overline{\alpha}(x_1, x_2, x_7) = (\frac{1}{2} - x_1, x_2, -x_7)$.

(i) The base orbifold $Q^3_{127}$: By choosing a fundamental domain $\Omega$ in $\mathbb{T}^3_{127}$ to be $[0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and examining how $\langle \overline{\pi}, \overline{\beta}, \overline{\alpha}\rangle$ acts on $\partial \Omega$, one concludes that $Q^3_{127}$ is an Euclidean 3-orbifold whose underlying topology is $S^3$ and whose singular locus $\Sigma_Q$ is a doubled Hopf link, which has 4 components: $S^1_{2,1}$ and $S^1_{2,2}$ parallel to the $x_2$-axis, and $S^1_{7,\alpha}$ and $S^1_{7,\beta}$ parallel to the $x_7$-axis. For $p$ in $S^1_{2,i}$, $\Sigma_p = \langle \overline{\gamma}\rangle$; for $p$ in $S^1_{7,\alpha}$, $\Sigma_p = \langle \overline{\alpha}\rangle$; and, for $p$ in $S^1_{7,\beta}$, $\Sigma_p = \langle \overline{\beta}\rangle$. (Figure 3-3-1.)

(ii) The fibers: In this example, $\Gamma_0 = \{0\}$. The fiber $F_p$ of the fibration $\pi_{127}$ over a point $p$ in $Q^3_{127}$ is listed below:

- For $p$ in $Q^3_{347} - \Sigma_Q$, $F_p = X = \mathbb{T}^4_{3456}$.
- For $p$ in $S^1_{2,i}$ (resp. $S^1_{7,\alpha}$, $S^1_{7,\beta}$), $\Gamma_p$ is $\langle \overline{\gamma}\rangle$ (resp. $\langle \overline{\alpha}\rangle$, $\langle \overline{\beta}\rangle$). Thus $F_p$ is the $\mathbb{Z}_2$-quotient $X/\langle \gamma\rangle$ (resp. $X/\langle \alpha\rangle$, $X/\langle \beta\rangle$). They are all isomorphic to $X_e = S^2(2222) \times \mathbb{T}^2$. 

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For monodromy $(v)$ Monodromy $T^3$ is generated by the meridians $S_i$.

(iii) How $S$ sits over $\Sigma_Q$: The 4 $T^3$ in $S$ from the fixed 3-tori $T^3_{1234}$ of $\alpha$ are mapped onto $S_{2,\alpha}^1$ by $\pi_{347}$ as a standard projection. Similarly for the 4 $T^3$ in $S$ from the fixed 3-tori $T^3_{246}$ of $\gamma$ are mapped onto $S_{2,\beta}^1$ by $\pi_{347}$ as a composition of a standard projection and a double covering map from $S^1$ to $S^1$. Similarly for the remaining 2 $T^3$ of $S$ onto $S_{2,\gamma}^1$.

(iv) Adjustment after resolving $S$: After resolving $S$, one obtains $T^4$-fibration for the same $M^7$ in Example 3.1: $\tilde{\pi}_{127} : M^7 \to Q^4_{127}$. Its set of critical values is $\Sigma_Q$ and its degenerate fiber is $X_\epsilon \cup 4T^2 \times \mathbb{C}P^1$.

(v) Monodromy: For monodromy $\rho : H_1(Q^4_{347} - \Sigma_Q; \mathbb{Z}) \to Aut(X_0)$, $H_1(Q^4_{347} - \Sigma_Q; \mathbb{Z}) = \mathbb{Z}^4$ is generated by the meridians $C_{2,1}, C_{2,2}, C_{7,\alpha}$, and $C_{7,\beta}$ associated to $S_{2,1}^1, S_{2,2}^1, S_{2,\alpha}^1$, and $S_{2,\beta}^1$ respectively (cf. Figures 3-2-2 and 3-3-1). One thus has $\rho(C_{2,i}) = j(\gamma), i = 1, 2, \rho(C_{7,\alpha}) = j(\alpha)$, and $\rho(C_{7,\beta}) = j(\beta)$.

This concludes the example.

Example 3.4 [J(0, 1/2, 1/2, 1/2, 0): a.a. $T^3$-fibration]. In Example 3.1, consider instead the decomposition $Y = \tilde{T}^7 = T^4_{1234} \times \tilde{T}^3_{567}$ with $Z = T^4_{1234}$ and $X = T^3_{567}$. One then has the fiber $\pi_{1234} : T^7/\Gamma \to Q^4_{1234}$, where $Q^4_{1234}$ is the orbifold $\tilde{T}^4_{1234}/(\pi, \beta, \gamma)$ with

\[\begin{align*}
\overline{\alpha}(x_1, x_2, x_3, x_4) & = (-x_1, -x_2, -x_3, -x_4), \\
\overline{\beta}(x_1, x_2, x_3, x_4) & = (-x_1, \frac{1}{2} - x_2, x_3, x_4), \\
\overline{\gamma}(x_1, x_2, x_3, x_4) & = (\frac{1}{2} - x_1, x_2, \frac{1}{2} - x_3, x_4).
\end{align*}\]

(i) The base orbifold $Q^4_{1234}$: Similar to the discussions in [Jo1], one can check that the only fixed-points of the $\Gamma$-action on $T^4_{1234}$ are the 16 fixed-points of $\overline{\alpha}$, the 4 fixed $T^3_{34}$ of
In terms of \(β\), and the 4 fixed \(\mathbb{T}_3^2\) of \(γ\). These fixed points or 2-tori are all disjoint and the action of \(⟨β, γ⟩\) on 16 fixed-points of \(π\), \(⟨π, β⟩\) on the set of 4 fixed \(\mathbb{T}_3^2\) of \(β\), and \(⟨π, β⟩\) on the set of 4 fixed \(\mathbb{T}_3^2\) of \(γ\) are all free. Thus the singular set \(Σ_Q\) of the orbifold \(Q_{1234}^4\) consists of 4 isolated points \(\{q_1, q_2, q_3, q_4\}\) and two disjoint copies \(\mathbb{T}^2\), denoted by \(\mathbb{T}_3^2,β\) and \(\mathbb{T}_3^2,γ\) respectively.

The decomposition \(T_{1234}^4 = T_{12}^2 \times T_{34}^2\) induces a \(T_{34}^2\)-fibration \(π_{12}^{1234} : Q_{1234}^4 \to Q_{12}^2 = D^2(22^∗)\). By inspection, one may write \(Σ_Q^{12} = S^1_γ \setminus \{p_α, p_β\}\) with \(Γ_{p_α} = ⟨π⟩\), \(Γ_{p_β} = ⟨β⟩\), and \(Γ_p = ⟨γ⟩\) for \(p ∈ S^1_γ\). The exceptional fiber is an \(S^2(2222)\) over \(p_α\) and an \(A^2(∗∗)\) over \(p ∈ S^1_γ\). (The fiber over \(p_β\) is a regular \(T_3^2\).) The monodromy \(ρ_{12}^{1234} : π_{1 orb}^1(Q_{12}^4) \to Aut(T_{34}^2)\) is determined by

\[
\begin{align*}
ρ_{12}^{1234}(π) &= \overline{α}f = (x_3, x_4) ↦ (-x_3, x_4), \\
ρ_{12}^{1234}(β) &= \overline{γ}f = (x_3, x_4) ↦ (\frac{1}{2} - x_3, x_4), \quad \text{and} \\
ρ_{12}^{1234}(β) &= ρ_{12}^{1234}(t_1) = ρ_{12}^{1234}(t_2) = Id.
\end{align*}
\]

In terms of \(π_{12}^{1234}\), the \(Σ_Q\) sits in \(Q_{1234}^4\) as illustrated in Figure 3-4-1.

For \(p ∈ Q_{1234}^4 - Σ_Q\), \(F_p = X = T_{567}^3\).

For \(p ∈ \{q_1, q_2, q_3, q_4\}\), \(Γ_p = ⟨π⟩\); thus \(F_p = T_{567}^3/⟨α⟩ = T_{567}^3\), a regular fiber.

For \(p ∈ T_{34, β}^2\), \(Γ_p = ⟨β⟩\); thus \(F_p = T_{567}^3/⟨β⟩\). Likewise, for \(p ∈ T_{24, γ}^2\), \(Γ_p = ⟨γ⟩\); thus \(F_p = T_{567}^3/⟨γ⟩\). They are all isomorphic to \(X_e = S^2(2222) × S^1\).

(ii) The fibers:

- For \(p \in Q_{1234}^4 - Σ_Q\), \(F_p = X = T_{567}^3\).
- For \(p \in \{q_1, q_2, q_3, q_4\}\), \(Γ_p = ⟨π⟩\); thus \(F_p = T_{567}^3/⟨α⟩ = T_{567}^3\), a regular fiber.
- For \(p \in T_{34, β}^2\), \(Γ_p = ⟨β⟩\); thus \(F_p = T_{567}^3/⟨β⟩\). Likewise, for \(p \in T_{24, γ}^2\), \(Γ_p = ⟨γ⟩\); thus \(F_p = T_{567}^3/⟨γ⟩\). They are all isomorphic to \(X_e = S^2(2222) × S^1\).
(iii) How $S$ sits: The $4 \mathbb{T}^3$ in $S$ from the fixed 3-tori $\mathbb{T}^3_{567}$ of $\alpha$ are mapped onto $\{q_1, q_2, q_3, q_4\}$ by $\pi_{1234}$. Each of the $4 \mathbb{T}^3$ in $S$ from the fixed 3-tori $\mathbb{T}^3_{347}$ of $\beta$ onto $\mathbb{T}^2_{34,\beta}$ as standard projection. Similarly, each of the $4 \mathbb{T}^3$ in $S$ from the fixed 3-tori $\mathbb{T}^3_{246}$ of $\gamma$ are mapped onto $\mathbb{T}^2_{24,\gamma}$ as standard projection.

(iv) Adjustment after resolving $S$: After resolving $S$, one obtains a $\mathbb{T}^3$-fibration for the same $M^7$ in Example 3.1: $\tilde{\pi}_{1234}: M^7 \to \tilde{Q}^4_{1234}$. where $r: \tilde{Q}^4_{1234}$ is the resolution of $Q^4_{1234}$. $\Sigma_Q$ is isomorphic to $\Sigma_Q \cap 4 \mathbb{CP}^1$. The set of critical values of $\tilde{\pi}_{1234}$ is the $\Sigma_Q$-part of $\Sigma_Q$ and the exceptional fiber is $X_e \cup 4 \mathbb{T}^1 \times \mathbb{CP}^1$ with normal crossing singularities.

(v) Monodromy: Let $t_i$ be the translation of $x_i$ in $\mathbb{R}$ by 1; then $\pi_1^{orb}(Q^4_{1234}) = \langle \overline{\alpha}, \overline{\beta}, \overline{\gamma}; t_1, t_2, t_3, t_4 \rangle$ and $\overline{\rho} = \rho \circ r_\ast: \pi_1^{orb}(\tilde{Q}^4_{1234}) \to Aut(\mathbb{T}^3_{567})$ is determined by

$$
\begin{align*}
\rho(\overline{\alpha}) &= \beta f = \langle (x_5, x_6, x_7) \mapsto (-x_5, -x_6, x_7) \rangle, \\
\rho(\overline{\beta}) &= \gamma f = \langle (x_5, x_6, x_7) \mapsto (-x_5, x_6, -x_7) \rangle, \quad \text{and} \\
\rho(\overline{\gamma}) &= \rho(t_i) = Id \, , \, i = 1, 2, 3, 4.
\end{align*}
$$

This concludes the example.

Example 3.5 $[J(e^{\pi i/3}, e^{2 \pi i/3}, \Lambda): a.c. \mathbb{T}^4$-fibration]. (Cf. [Jo1]: II, Example 11.)

$\Gamma$ is generated by

$$
\begin{align*}
\alpha(z_1, z_2, z_3, x) &= (e^{\pi i/3}z_1, e^{2\pi i/3}z_2, -z_3, x + \frac{i}{3}) \\
\beta(z_1, z_2, z_3, x) &= (-\overline{z_1}, -\overline{z_2}, -\overline{z_3}, -x),
\end{align*}
$$

where $z_1 = x + ix_2$, $z_2 = x_3 + ix_4$, $z_3 = x_5 + ix_6$, $x = x_7$; and

$$
\Lambda = (\mathbb{Z} + e^{2\pi i/3}\mathbb{Z}) \oplus (\mathbb{Z} + e^{2\pi i/3}\mathbb{Z}) \oplus (\mathbb{Z} + i\mathbb{Z}) \oplus \mathbb{Z}.
$$

From [Jo1] (II), the quotient $\mathbb{T}^7/\Gamma$ has a singular set $S$ of $A_1$-singularities that consists of 4 disjoint copies $\mathbb{T}^3$. 2 copies arising from the fixed $\mathbb{T}^3$ of any element in $\Gamma$ conjugate to $\beta$ and another 2 copies arising from the fixed $\mathbb{T}^3$ of any element in $\Gamma$ conjugate to $\beta \alpha$. The tubular neighborhood of each component of $S$ is modelled on $\mathbb{T}^3 \times (\mathbb{C}^2/\langle -1 \rangle)$. After resolving $S$, one obtains a Joyce manifold $M^7$.

Consider the associative-coassociative decomposition $Y = \mathbb{T}^7 = \mathbb{T}^3_{127} \times \mathbb{T}^4_{3456}$, where $Z = \mathbb{T}^3_{127} = \{\mathbb{C}/(\mathbb{Z} \oplus e^{2\pi i/3}\mathbb{Z})\} \times \mathbb{R}/\mathbb{Z}$ is parametrized by $(z_1, x)$ and

$$
X = \mathbb{T}^4_{3456} = \{\mathbb{C}/(\mathbb{Z} \oplus e^{2\pi i/3}\mathbb{Z})\} \times \{\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})\}
$$

is parametrized by $(z_2, z_3)$. One then has the fibration $\pi_{127}: J = Q^3_{127}$, where $Q^3_{127}$ is the orbifold $\mathbb{T}^3_{127}/\langle \overline{\alpha}, \overline{\beta} \rangle$ with $\overline{\alpha}(z_1, x) = (e^{\pi i/3}z_1, x + \frac{i}{3})$ and $\overline{\beta}(z_1, x) = (-\overline{z_1}, -x)$. 

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(i) The base orbifold $Q^3_{127}$: There are two ways to understand $Q^3_{127}$. Consider first the general method by studying a fundamental domain $\Omega$ of the $\Gamma$-action on $T^3_{127}$. One can choose $\Omega$ to be $T^3_{12} \times [0, \frac{1}{12}]$, then $\partial \Omega = T^3_{12} \times \{0, \frac{1}{12}\}$. Since $\text{Stab}(T^3_{12} \times \{0\}) = \langle \beta \rangle$ and $\text{Stab}(T^3_{12} \times \{\frac{1}{12}\}) = \langle \alpha \beta \rangle$, $Q^3_{127}$ is obtained from $\Omega$ by identifying $p$ with $\beta(p)$ for $p \in T^3_{12} \times \{0\}$ and $p'$ with $\alpha \beta(p')$ for $p' \in T^3_{12} \times \{\frac{1}{12}\}$. These orientation-reversing maps on $\partial \Omega$ and their fixed locus are indicated in Figure 3-5-1. The quotient of $\partial \Omega$ by them are two Möbius strips. From this, one concludes that the underlying topology of $Q^3_{127}$ is a closed orientable 3-manifold and the singular locus $\Sigma_Q$ consists of two copies of $S^3$; one from the fixed locus of $\beta$ in $T^3_{12} \times \{0\}$ and the other from the fixed locus of $\alpha \beta$ in $T^3_{12} \times \{\frac{1}{12}\}$. We shall denote the former by $S^3_{\beta}$ and the latter by $S^3_{\alpha \beta}$.

Denote the underlying 3-manifold of $Q^3_{127}$ by $M^3$. Then one can show that $\pi_1(M^3)$ is trivial as follows: Regard $M^3$ as pasted from the two halves, $M^-_3$ from $T^3_{12} \times [0, \frac{1}{12}]$ and $M^+_3$ from $T^3_{12} \times \{\frac{24}{12}, \frac{1}{12}\}$, along the shared boundary $T^3_{12} \times \{\frac{12}{12}\}$. Then $\pi_1(M^3) = Z$ is generated by the core loop $\gamma$ of the Möbius strip $(T^3_{12} \times \{0\})/\langle \beta \rangle$ since $M^3$ is strong deformation retractable to $\gamma$. Similarly $\pi_1(M^3) = Z$ is generated by the core loop $\gamma$ of the Möbius strip $(T^3_{12} \times \{\frac{12}{12}\})/\langle \alpha \beta \rangle$. By choosing a generating set for $\pi_1(T^3_{12} \times \{\frac{12}{12}\})$ to be $\{u_1, u_2\}$, where $u_1$ corresponds to the path in $C$ from 0 to 1 and $u_2$ from 0 to $e^{2\pi i/3}$, and using the Van Kampen’s theorem [Sp], one obtains that $\pi_1(M^3)$ has a presentation $\langle [\gamma_-], [\gamma_+] | [\gamma_+] = 1, [\gamma_-] = [\gamma_+]^2 \rangle$, which is $\{1\}$. Thus, if one assumes that the Poincare conjecture is correct, then $M^3$ is an $S^3$ and $\Sigma_Q$ is a two-component link in $S^3$. Though one may try to push on to gain more information on $Q^3_{127}$, we shall now turn to the second method.

The second method is due to the observation that, for the current example, if one identifies $(z_1, x)$ in $T^3_{127}$ with $(z_1, 2\pi x)$ in the unit tangent bundle $T^3_{12} \times T^3_{12}$, where $(z_1, \theta = 0)$ corresponds to a global unit tangent vector field $\xi$ on $T^3_{12}$ that is parallel to the fixed direction of $\beta$ on $T^3_{12}$, then the $\Gamma$-action on $T^3_{12}$ is indeed the induced action of the $\Gamma$-action on $T^3_{12}$, defined by $a_\beta(z_1) = e^{\pi i/3} z_1$ and $\beta(z_1) = -\frac{z_1}{x}$. Consequently, $Q^3_{127}$ is the
unit tangent bundle of the 2-orbifold $T_1^2/\langle \pi, \overline{\beta}\rangle$, which is the $D^2(*236)$-orbifold. Recall now the following fact from [Th1]:

**Fact 3.5.1.** ([Th1]: Sec. 13.4.) Let $Q^2$ be a 2-orbifold whose combinatorial type is a polygon. Then its unit tangent bundle $T_1Q^2$ is a 3-orbifold whose underlying topology is an $S^3$ and the singular locus is a link, constructed as indicated in Figure 3-5-2.

![Figure 3-5-2](image)

**Figure 3-5-2.** The unit tangent bundle $Q^3 = T_1Q^2$ of a 2-orbifold $Q^2 = D^2(m_1 \cdots m_l \ast n_1 \cdots n_k)$ and its singular locus $\Sigma_Q$. $\Sigma_p = \mathbb{Z}_2$ for $p \in \Sigma_Q$. Notice that, though the cone-points of $Q^2$ influence the bundle structure of $Q^3$, they play no role in determining $\Sigma_Q$ as a link in $S^3$.

Thus $Q^3_{127} = T_1D^2(*236)$ has the underlying topology $S^3$ and $\Sigma_Q$ as in Figure 3-5-3, which is a two-component link. To see which component is $S^1_{\beta}$ and which component is $S^1_{\alpha\beta}$, recall from Sec. 1 that, up to an overall conjugation by $\Gamma$, one has subgroups $\Gamma_{(2)}$, $\Gamma_{(3)}$, and $\Gamma_{(6)}$ in $\Gamma$ associated to the corner reflectors labelled in Figure 3-5-3 by (2), (3), and (6) respectively and subgroups $\Gamma_{(23)}$, $\Sigma_{(26)}$, and $\Sigma_{(36)}$ in $\Gamma$ associated to points in the silvered edge of $D^2(*236)$ that connects corner reflectors $\{(2),(3)\}$, $\{(3),(6)\}$, and $\{(2),(6)\}$ respectively. One can check directly that $\Gamma_{(23)}$ and $\Gamma_{(36)}$ conjugate in $\Gamma_{(3)}$ and both are conjugate to $\langle \overline{\beta} \rangle$ in $\Gamma$ while $\Gamma_{(26)}$ is conjugate to $\langle \overline{\alpha\beta} \rangle$. Consequently, in Figure 3-5-3, $K_1$ corresponds to $S^1_{\beta}$ while $K_2$ corresponds to $S^1_{\alpha\beta}$.

(ii) The fibers:

- For $p$ in $Q^3_{127} - \Sigma_Q$, $F_p = X = \mathbb{T}^4_{3456}$.
- For $p$ in $S^1_{\beta}$, $\Sigma_p = \langle \overline{\beta} \rangle$; hence, $F_p = \mathbb{T}^4_{3456}/\langle \beta \rangle$. For $p$ in $S^1_{\alpha\beta}$, $\Sigma_p = \langle \overline{\alpha\beta} \rangle$; hence, $F_p = \mathbb{T}^4_{3456}/\langle \alpha\beta \rangle$. These fibers are isomorphic and will be denoted by $F_e$. They have multiplicity 2 and can be realized as a $\mathbb{T}^2$-bundle over a toroidal 2-orbifold.

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(iii) How $S$ sits: Under $\pi_{127}$, the 2 copies of $T^3$ in $S$ associated to $\beta$ are mapped onto $S^1_\beta$, while the other 2 copies of $T^3$ in $S$ are mapped onto $S^1_{\alpha\beta}$. All these maps are standard projections.

(iv) Adjustment after resolving $S$: After resolving $S$, one obtains a $T^4$-fibration: $\tilde{\pi}_{127} : M^7 \to \tilde{Q}_{127}^3$. Its set of critical values is $\Sigma_Q$ and its exceptional fiber is $X_e \cup 2T^2 \times \mathbb{C}P^1$.

(v) Monodromy: Let $u_1(z_1, x) = (z_1 + 1, x)$, $u_2(z_1, x) = (z_1 + e^{2\pi i/3}, x)$, and $t_7(z_1, x) = (z_1, x + 1)$; then

$$\pi_1^{orb}(\tilde{Q}_{127}^3) = \pi_1^{orb}(Q_{127}^3) = \langle \alpha, \beta; u_1, u_2, t_7 \rangle$$

and $\tilde{\rho} = \rho : \pi_1^{orb}(Q_{127}^3) \to \text{Aut}(T^4_{3456})$ is determined by

$$\rho(\alpha) = \alpha f = (z_2, z_3) \mapsto (e^{2\pi i/3}z_2, -z_3),$$

$$\rho(\beta) = \beta f = (z_2, z_3) \mapsto (-\overline{z_2}, -\overline{z_3}),$$

and

$$\rho(u_1) = \rho(u_2) = \rho(t_7) = Id.$$ 

This concludes the example.

Example 3.6 [$J(e^{\pi i/3}, e^{2\pi i/3}, \Lambda)$: a.a. $T^3$-fibration]. In Example 3.5, consider the same $Y = T^7 = T^3_{127} \times T^4_{3456}$ but now with $Z = T^4_{3456}$ and $X = T^3_{127}$. One then has the fibration $\pi_{3456} : T^7/\Gamma \to Q^3_{3456}$, where $Q^3_{3456}$ is the orbifold $T^3_{3456}/\langle \alpha, \beta \rangle$ with $\overline{\alpha}(z_2, z_3) = (e^{2\pi i/3}z_2, -z_3)$ and $\overline{\beta}(z_2, z_3) = (-\overline{z_2}, -\overline{z_3})$.

(i) The base orbifold $Q^3_{3456}$: Denote $T^4_{3456} = \{ \mathbb{C}/(\mathbb{Z} \oplus e^{2\pi i/3}\mathbb{Z}) \} \times \{ \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \}$ by $T^4_{34}(e^{2\pi i/3}) \times T^4_{36}(i)$ and regard it as a trivial bundle with base $T^3_{34}(e^{2\pi i/3})$, fiber $T^2_{36}(i)$, and $\Gamma$ as a group of bundle automorphisms. Since this action has the subgroup $\langle \alpha \rangle$ that
acts trivially on the base, while acting as negation on the fiber, the induced fibration \( \pi_{3456}^3 : T_{3456}^3/\Gamma \to Q_{34}^2 = T_{34}^2(e^{2\pi i/3})/\Gamma \) is an \( S^2(2222) \)-fibration over a \( D^2(*333) \). All the exceptional fibers are isomorphic to \( D^2(*2222) \) and have multiplicity 2.

Let us now locate the singular locus \( \Sigma_Q \) of \( Q_{34}^4 \) in \( \pi_{3456}^3 \). As bundle automorphisms, \( \pi^4 \) has three invariant \( T_{56}^2 \)-fibers (at \( z_2 = 0, \frac{2\pi i}{3}, \frac{4\pi i}{3} \)), on each of which there are 4 fixed points; they correspond to the 4 corner-reflectors of the exceptional \( D^2(*2222) \)-fiber over the 3 corner-reflectors of \( Q_{34}^2 \). \( \pi^4 \) has the above three \( D^2(*2222) \)-fibers as the fixed-point set. The 4 fixed \( T_{34}^2 \) of \( \pi^3 \) correspond to the 4 sections of \( \pi_{3456}^3 \) whose union consists of the 4 cone-points of a generic fiber \( S^2(2222) \) and the corner-reflectors of an exceptional fiber \( D^2(*2222) \). This takes care of the rotation part of \( \Gamma \). For the reflection part, since they form two conjugacy classes, \( \{ \overline{\beta}, \pi^3 \overline{\beta}, \pi^3 \overline{\beta} \} \) and \( \{ \overline{\beta}, \pi^3 \overline{\beta}, \pi^3 \overline{\beta} \} \), one only needs to consider the fixed locus of \( \overline{\beta} \) and the fixed locus of \( \pi^3 \overline{\beta} \) on \( T_{3456}^4 \). By inspection, both have 2 copies of \( T^2 \) as fixed locus. Each descends to a silvered annulus \( A^2(**) \) in \( Q_{34}^4 \) mapped onto \( \Sigma_{Q_{34}^2} \). When appropriately parametrized, the map is modeled by \( (y_{34}, y_{56}) \to (y_{34}, -y_{56}) \), where \( y_{34}, y_{56} \in S^1 \). Together these 4 annuli form a 2-torus \( \Sigma_{Q_{34}^2} \times \Sigma_{D^2(*2222)} \). Thus one concludes the following decomposition

\[
\Sigma_Q = 2A^2_{\overline{\beta}}(**) \cup 2A^2_{\overline{\beta} \beta}(**) \cup 3D^2_{\alpha \beta}(*2222) \cup 4D^2_{\alpha \beta}(333).
\]

From the discussion, \( \Gamma_p \) for \( p \in Q_{3456}^3 \) can also be obtained. (Figure 3-6-1.)

\[Q_{34}^2 = D^2(*333)\]

**Figure 3-6-1.** The decomposition of \( \Sigma_Q \). Topologically, \( \Sigma_Q \) is a 2-complex obtained by a \( T^2 \) (i.e. \( 2A^2_{\overline{\beta}} \cup 2A^2_{\overline{\beta} \beta} \)) attached by 4 2-disks (i.e. \( D^2_{\alpha \beta} \)) along disjoint \((1, 0)\)-loops and 3 2-disks (i.e. \( D^2_{\alpha \beta} \)) along disjoint \((0, 1)\)-loops. (The \( T^2 \) sits above \( \partial Q_{34}^2 \) and is transparent.)

(ii) **The fibers:** The generic fiber of \( \pi_{3456} \) is \( T^3_{127} \). Its exceptional fibers follow from **Figure**
3-6-1: (Note that in the following table $p|Q^2$ means $p$ is a regular point of $Q^2$ and $m$ is the multiplicity of the fiber $F_p$ over $p$.)

| $p$ | $D_{αβ}^2(*333)$ | $D_{αβ}^2(*2222)$ | $A_{β}^2(∗∗)$ | $A_{αβ}^2(∗∗)$ | $D_{αβ}^2 ∩ A_{αβ}^2$ |
|-----|-----------------|-----------------|---------------|----------------|-----------------|
| $F_p$ | $T^3_{127}/(α^3)$ | $T^3_{127}/(α^2)$ | $T^3_{127}/(β)$ | $T^3_{127}/(αβ)$ | $T^3_{127}/(α, β)$ |
| $m$ | 2                | 3               | 2             | 2              | 12              |

| $D_{αβ}^2 ∩ A_{αβ}^2$ | $D_{αβ}^2 ∩ A_{β}^2$ | $D_{αβ}^2 ∩ D_{αβ}^2 ∩ A_{αβ}^2$ |
|----------------------|----------------------|----------------------------------|
| $T^3_{127}/(α, β)$  | $T^3_{127}/(α, β)$  | $T^3_{127}/(α, β)$              |
| 6                    | 6                    | 12                               |

All these exceptional fibers are realizable as Seifert 3-orbifolds ([B-S]).

(iii) How $S$ sits: Recall the 4 $T^3$-components of $S$ from Example 3.5. The standard projection $T^7 → T^4_{3456}$ takes a fixed $T^3$ of $β$ (resp. $αβ$) on $T^7$ to a fixed $T^2$ of $β$ (resp. $πβ$) on $T^4_{3456}$, modelled by $(y_{12}, y_{34}, y_{56}) → (y_{34}, y_{56})$, where $y_{12}, y_{34}, y_{56} ∈ S^1$. Composed further by the quotient map $T^4_{3456} → Q^4_{3456}$, one concludes that $S$ sits over $2 A_β^2(∗∗) ∪ 2 A_β^2(αβ)(∗∗)$, with one $T^3$ to one $A^2$. The restriction of $π_{3456}$ to a such $T^3 → A^2$ is modelled by the folding $(y_{12}, y_{34}, y_{56}) → (y_{34}, y_{56})$, where $y_{12}, y_{34}, y_{56} ∈ ℝ/ℤ$ and $y_{56} = y_56$ if $y_{56} ∈ [0, 1/2]$, $= 1 − y_{56}$ if $z ∈ [1/2, 1]$. From Step (iii) above, only the fibers of $π_{3456}$ that sit over $2 A_β^2 ∪ 2 A_β^2$ is modified, by $F_p → F_p ∪ 2 T^1 × ℂP^1$; all other fibers of $π_{3456}$ go to the fibers of $π_{3456}$ unchanged.

(iv) Adjustment after resolving $S$: After resolving $S$, one obtains a $T^3$-fibration $π_{3456}$ of $M^7$ over $Q^4_{3456}$. From Step (iii) above, only the fibers of $π_{3456}$ that sit over $2 A_β^2 ∪ 2 A_β^2$ is modified, by $F_p → F_p ∪ 2 T^1 × ℂP^1$; all other fibers of $π_{3456}$ go to the fibers of $π_{3456}$ unchanged.

(v) Monodromy: Let $u_3(z_2, z_3) = (z_2 + 1, z_3)$, $u_4(z_2, z_3) = (z_2 + e^{2πi/3}, z_3)$, $u_5(z_2, z_3) = (z_2, z_3 + 1)$, and $u_6(z_2, z_3) = (z_2, z_3 + i)$; then

$π^1_{orb}(Q^4_{3456}) = π^1_{orb}(Q^4_{3456}) = (α, β; u_3, u_4, u_5, u_6)$

and the monodromy $ρ = ρ: π^1_{orb}(Q^4_{3456}) → Aut(T^3_{127})$ is determined by

$ρ(α) = α^f = ((z_1, x) ↦ (e^{πi/3}z_1, x + 1/6)),$

$ρ(β) = β^f = ((z_1, x) ↦ (−z_1, −x)),$

and $ρ(u_3) = ρ(u_4) = ρ(u_5) = ρ(u_6) = Id$.

This concludes the example.

We hope these examples are enough to demonstrate the method (cf. Sec. 5, Issue (1)).
4 Fibrations of Joyce manifolds by Borcea-Voisin threefolds.

In addition to the a.a/a.c. fibrations discussed so far, a Joyce manifold of the first kind admits also natural fibrations by Borcea-Voisin threefolds, as we shall now discuss.

Fibrations by Borcea-Voisin threefolds.

Recall from Sec. 1 the group $\Gamma = \langle \alpha, \beta, \gamma \rangle$ that acts on $(T^7, \varphi_0)$ by

$$\alpha(x_1, \ldots, x_7) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7),$$

$$\beta(x_1, \ldots, x_7) = (b_1 - x_1, b_2 - x_2, x_3, x_4, -x_5, -x_6, x_7),$$

$$\gamma(x_1, \ldots, x_7) = (c_1 - x_1, x_2, c_3 - x_3, x_4, c_5 - x_5, x_6, -x_7)$$

with $b_1, b_2, c_1, c_3, c_5$ some appropriate constants in $\{0, \frac{1}{2}\}$ and how this leads to Joyce manifolds $M^7$ in $J(b_1, b_2, c_1, c_3, c_5)$.

The 7-orbifold $T^7/\Gamma$ can be realized as the quotient of $(T^7/\langle \alpha \rangle)$ by $\langle \beta, \gamma \rangle$. If one chooses the complex coordinates on $T^7_{123456}$ by $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, $z_3 = x_5 + ix_6$, then $\beta$ acts on $T^7/\langle \alpha \rangle$ holomorphically while $\gamma$ acts on $T^7/\langle \alpha \rangle$ antiholomorphically. Both lift to an automorphism of the resolution $K3 \times T^2_{56} \times T^1_7$ of $T^7/\langle \alpha \rangle$ with $\beta$ holomorphic and $\gamma$ antiholomorphic on the $(K3 \times T^2_{56})$-component. Furthermore, when the lifted $\beta$ restricts to the $(K3 \times T^2_{56})$-component, it serves as the automorphism $(i, j)$ in the construction of Borcea-Voisin threefolds since the $\beta$-action on the $K3$-component of $K3 \times T^2_{56}$ is a holomorphic automorphism that acts by $(-1)$ on its holomorphic 2-form lifted from $dz_1 \wedge dz_2$ on $T^1_{1234}$ and on the $T^2_{56}$ by negation. The set of fixed-points of the $i$ in this case is the union of two copies of $T^2_{34}$. Let $Y$ be the Borcea-Voisin threefold obtained by resolving the singularities of $(K3 \times T^2_{56})/\langle \beta \rangle$; then the corresponding resolution $W^7_{\alpha, \beta}$ of $T^7/\langle \alpha, \beta \rangle$ is a trivial bundle $T^1_7 \times Y$ over $T^4_7$. Now $\gamma$ acts on $W^7_{\alpha, \beta}$ as a bundle automorphism with two invariant fibers: one over $x_7 = 0$ and the other over $x_7 = \frac{1}{2}$. Hence one obtains a fibration $\pi_7 : W^7_{\alpha, \beta}/\langle \gamma \rangle \rightarrow [0, \frac{1}{2}] = T^1_7/(−1)$.

Let $S_\gamma$ be the singular set of $W^7_{\alpha, \beta}/\langle \gamma \rangle$; then it consists of the 3-tori or the free $\mathbb{Z}_2$-quotient of 3-tori descending from the fixed $T^3$'s of $\gamma$, $\alpha \gamma$, $\beta \gamma$, or $\alpha \beta \gamma$ on $T^7$ and is contained in the two exceptional fibers $F_e = Y/\langle \gamma \rangle$ of $\pi_7$. On the other hand, since all the singularities involved arise from the fixed $T^3$'s in $\Gamma$ and the resolution is by transverse blowups, $W^7_{\alpha, \beta}/\langle \gamma \rangle$ is the same orbifold as obtained from $T^7/\Gamma$ by resolving the $T^3$-components in the singular set $S$ that arise from the fixed tori associated to $\langle \alpha, \beta \rangle$ on $T^7$. Thus, after resolving $S_\gamma$, one recovers the Joyce manifold $M^7$, which now fibers over $[0, \frac{1}{2}]$ with generic fiber the Borcea-Voisin threefold $Y$ and exceptional fibers over $x_7 = 0$ and $x_7 = \frac{1}{2}$. These two exceptional fibers are isomorphic since the construction is invariant under the map $(x_1, \ldots, x_6, x_7) \rightarrow (x_1, \ldots, x_6, \frac{1}{2} - x_7)$, which flips $x_7 = 0$ and $x_7 = \frac{1}{2}$ to each other. Both fibers have multiplicity 2.

Remark 4.1 [topology of $Y$]. From Sec. 1, the Borcea-Voisin threefold $Y$ has Hodge numbers $h^{1,1}(Y) = h^{2,1}(Y) = 19$. 

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Remark 4.2 [exceptional fiber]. The exceptional fiber $\widetilde{F}_e$ in $M^7$ is a connected 6-space that is a 6-manifold-with-boundary except at some singularities on the boundary. To see this, first notice that, over $x_7 \in \{0, \frac{1}{2}\}$ the fixed-point set of the $\gamma$-action on $Y$ coincides with $S_\gamma$; hence, $F_e = Y/\langle \gamma \rangle$ is a manifold except at $S_\gamma$. Now let $S_0$ be a component of $S_\gamma$ and consider the effect of resolving $S_0$ to $F_e$ case by case, following the local models in Sec. 1 for the tubular neighborhood $\nu(S_0)$ of $S_0$:

- **Case (a) $\nu(S_0) = T^3_{246} \times (\mathbb{R}^4_{1357}/(-1))$:** Identify $\mathbb{R}^4_{1357}$ with $\mathbb{C}^2$, say, via the decomposition $\mathbb{R}^3_{13} \times \mathbb{R}^2_{7}$ and recall from Sec. 1 the resolution $\psi : T^*\mathbb{C}P^1 \to \mathbb{C}^2/(-1)$, whose exceptional locus $E$ is the 0-section of $T^*\mathbb{C}P^1$. The transverse directions to $S_0$ in $F_e$ is given by $\mathbb{R}^3_{135}/(-1) = (\mathbb{C} \times \mathbb{R}^5)/(-1)$ in $\mathbb{C}^2/(-1)$. Thus resolving $S_0$ changes $\nu(S_0)$ to $T^3_{246} \times \{\psi^{-1}((\mathbb{C} \times \mathbb{R}^5)/(-1))\}$. To understand $\psi^{-1}((\mathbb{C} \times \mathbb{R}^5)/(-1))$, recall the foliation $\mathcal{F}$ of $(C - \{(0,0)\})/(-1)$ by the complex cone-lines $C_1/(-1)$ = $C \cdot (1, \lambda)/(-1)$, where $\lambda = \mathbb{C} - \{0\}$ and $\lambda \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, and the fact that $\psi$ is obtained by blowing up the singularity $(0,0)$ in $\mathbb{C}^2/(-1)$ to the leaf space $\mathbb{C}P^2$ of $\mathcal{F}$, which is identified with $E$. $\mathcal{F}$ now induces an equivalence relation $\sim$ on $(\mathbb{C} \times \mathbb{R}^5 - \{(0,0)\})/(-1)$ by setting $(z, x_5) \sim (z', x_5')$ if both lie in some $C_1/(-1)$. Since every $C_1/(-1)$ intersects $(\mathbb{C} \times \mathbb{R}^5 - \{(0,0)\})/(-1)$, $E$ is also the exceptional locus for the restriction of $\psi$ on $\psi^{-1}((\mathbb{C} \times \mathbb{R}^5)/(-1))$. When restricted to the unit projective space $\mathbb{RP}^2 = S^2/(-1)$ in $(\mathbb{C} \times \mathbb{R}^5)/(-1)$, each $\sim$-equivalence class is simply a point in $\mathbb{RP}^2$ except the class associated to $C^*/(-1)$, which is a generating circle $C$ for $\pi_1(\mathbb{RP}^2)$. Since $\nu(C)$ in $\mathbb{RP}^2$ is a Möbius strip, one has that $\psi^{-1}((\mathbb{C} \times \mathbb{R}^5)/(-1))$ is a manifold with boundary $E = S^2 = \mathbb{RP}^2/\sim$, on which there is an isolated singularity $\ast$ whose link $\partial \nu(\ast)$ is a Möbius strip instead of a 2-disk for all other points on the boundary. This implies that resolving such $S_0$ contributes to $\widetilde{F}_e$ a boundary component $T^3 \times S^2$ with a singular locus $T^3 \times \{\ast\}$.

- **Case (b) $\nu(S_0) = \{T^3 \times (\mathbb{C}^2/(-1))\}/\mathbb{Z}_2$:** Since the free $\mathbb{Z}_2$-action on $T^3 \times (\mathbb{C}^2/(-1))$ arises from a subgroup in $\langle \alpha, \beta \rangle$ and its restriction to the $\mathbb{C}^2/(-1)$-component is either holomorphic or anti-holomorphic, it leaves $(\mathbb{C} \times \mathbb{R})/(-1)$ in Case (a) invariant and is compatible with the equivalence relation $\sim$. Together with (a), this implies that resolving such $S_0$ contributes to $\widetilde{F}_e$ a boundary component, which is the free quotient $(T^3 \times S^2)/\mathbb{Z}_2$ with a singular locus $(T^3/\mathbb{Z}_2) \times \{\ast\}$.

This concludes the remark.

Remark 4.3 [non-uniqueness]. In general, the fibration of a Joyce manifold of the first kind by Borcea-Voisin threefolds is not unique. One can identify the complex-real coordinates $(z_1, z_2, z_3, x)$ with $(x_1, \cdots, x_7)$ in several ways. Each may give different factorization $K3 \times T^2 \times T^1$ in the intermediate step, which then leads to different fibrations of $M^7$ by Borcea-Voisin threefolds $Y$. However the Borcea-Voisin manifolds $Y$ that appear in these different fibrations are homeomorphic with Hodge numbers as in Remark 4.1.

Remark 4.4 [7-space from rolling Calabi-Yau]. The existence of fibrations of a Joyce manifold $M^7$ of the first kind by Borcea-Voisin threefolds renders such $M^7$ similar to the

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7-spaces obtained by rolling Calabi-Yau threefolds, as discussed in [Li].

The harmony with a.a./a.c. fibrations.

The Borcea-Voisin threefold $Y$ constructed above is naturally fibred over the 2-orbifold $Q^2(2222)$ with generic fiber the K3 surface $X$ obtained by resolving $\mathbb{R}^4_{1234}/\langle -1 \rangle$. This in turn induces a fibration of $M^7$ by $X$ over a 3-orbifold. On the other hand, since the fibration of $Y$ by $X$ can be regarded as descending from the decomposition $\mathbb{T}_{123456}^6 = T_{1234}^1 \times T_{246}^2$, which is compatible with the associative-coassociative product decomposition $\mathbb{T}^7 = T_{567}^3 \times T_{1234}^4$, the induced fibration of $M^7$ is simply an a.c. K3-fibration discussed in Sec. 2.

Likewise, with respect to the complex structure on $X$ induced from the above identification of $\mathbb{R}^4_{1234}$ with $\mathbb{C}^2$, the special Lagrangian $\mathbb{T}^3$-fibrations for $Y$ as constructed in [G-W] come from either of the decompositions $\mathbb{T}_{123456}^6 = T_{135}^3 \times T_{246}^3$ and $\mathbb{T}_{123456}^6 = T_{136}^3 \times T_{245}^3$, which are compatible with the associative-coassociative product decompositions $\mathbb{T}^7 = T_{246}^3 \times T_{1357}^4$ and $\mathbb{T}^7 = T_{136}^3 \times T_{2457}^4$ respectively. Hence the induced $\mathbb{T}^3$-fibrations for $M^7$ are a.a. $\mathbb{T}^3$-fibrations discussed in Sec. 2. In addition to these, one also has $\mathbb{T}^3$-fibrations of $Y$ induced from the decompositions $\mathbb{T}_{145}^3 \times T_{236}^3$ (from $\mathbb{T}_{145}^3 \times T_{236}^3$) and $\mathbb{T}_{235}^3 \times T_{146}^4$ (from $\mathbb{T}_{235}^3 \times T_{146}^4$) of $\mathbb{T}_{123456}^6$. These induce also a.a. $\mathbb{T}^3$-fibrations for $M^7$ discussed in Sec. 2.

**Example 4.5** $[J(0, \frac{1}{2}, \frac{1}{4}, 0, 0)]$. ([Jo1]: II, Example 4.) Following [Jo1] and previous discussions, the fixed 3-tori $T_{246}^3$ of $\gamma$ on $\mathbb{T}^7$ descends to $S_{\gamma}$, which consists of 8 copies of $T_{\alpha, \beta}^3/\mathbb{Z}_2$ in $W_{\alpha, \beta}/\langle \gamma \rangle$ with 4 copies over $x_0 = 0$ and 4 copies over $x_7 = \frac{1}{2}$. After resolving $S_{\gamma}$, $M^7$ has a fibration over $[0, \frac{1}{2}]$ with generic fiber $Y$. From Remark 4.2, the exceptional fiber $\tilde{F}_e$ is a connected 6-space that is a manifold with 4 boundary components $\{T^3 \times S^2\}/\mathbb{Z}_2$ except for a singular locus $T^3/\mathbb{Z}_2$ on each boundary component. The K3-fibration of $Y$ induces the a.c. K3-fibration of $M^7$. The standard projection from $T_{567}^3$ to $T_7^2$ induces a fibration of $Q_{567}^3(2222)$ by $Q_{567}^3(2222)$. Likewise, the decomposition $\mathbb{T}_7^6 = T_{135}^3 \times T_{246}^3$ induces a $\mathbb{T}^3$-fibration of $Y$ over a 3-orbifold $Q_{135}^3$, which then induces an a.a. $\mathbb{T}^3$-fibration of $M^7$. The projection of $T_{135}^3$ to $\mathbb{R}^7$ induces a fibration of $Q_{135}^3$ by a 3-orbifold $Q_{135}^4$. Diagrammatically, one has

![Diagram](image)

where “→” stands for “the generic fiber of” and “↓” stands for “fibred over”. The discussion for all other cases are similar.

\[\square\]
This concludes our discussions on the fibrations of Joyce manifolds.

5 Remarks on further questions/works.

As readers may have noticed, the various fibrations of Joyce manifolds discussed and illustrated by examples in this paper come freely and naturally from the work of Joyce; we strongly feel that there must be some place for them in string/M-theory. Indeed this has been pursued vigorously by Acharya ([Ar1] and [Ar2]). On the other hand, the issue of dualities in string/M-theory requires us to go beyond this present work. Among many further questions, let us list three issues\(^2\) with some comments for future pursuit/cooperations:

- (1) **Complete table**: Give a complete table of the a.a./a.c. fibrations of all the Joyce manifolds of the first and the second kind, up to fibration isomorphism (cf. Table 2 in [Jo1]:II). Design some elegant notations to code the combinatorial data of the fibrations. Give more informations of the exceptional fibers, e.g. their various numerical topological invariants. This work seems endurable, though experience told us that it could be very tedious or even brain-racking. Since many steps are straightforward, likely a computer code should be able to realize quite a part of the work.

- (2) **How far from associative/coassociative fibrations**: The associative/coassociative fibrations for a 7-manifold of holonomy $G_2$ is the analogue of the fibrations of a Calabi-Yau threefold by supersymmetric cycles (i.e. special Lagrangian submanifolds). Thus, one likes to design some quantity of measuring how far the fibrations given in the table in Issue (1) are from being a true associative/coassociative fibration and to understand the physical meaning of this quantity.

- (3) **The role of global structure**: Before a comprehensive table in Issue (1) is completed, following the 5-step-routine in Sec. 3, one can still understand the global structure of an a.a./a.c. fibrations of a Joyce manifold. For examples of Joyce manifolds that has already appeared in physics literature (e.g. [Ar1] and [Ar2]), one of the the next major questions is then:

  **Q.** *How could its global structure play roles in string/M-theory dualities?*

  In other words, in a string/M-theory duality involving fibrations of Joyce manifolds, where is the global structure of the fibration hidden?

  With these open question/work/project in mind, let us conclude this paper.

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