Sharpened Information-Theoretic Uncertainty Relations and the Histories Approach to Quantum Mechanics

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Abstract

In this paper alternative formulations of the conventional uncertainty relation are studied in the context of decoherent histories. The results are given in terms of Shannon information. A variety of methods are developed to evaluate the upper bound for the probability of two or more projection histories. The methods employed give improved limits for the maximal achievable probability and an improved lower bound for the Shannon information. The results are then applied to a number of physically relevant situations.

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I. INTRODUCTION

In this paper, we will explore some mathematical aspects of the formulation of quantum mechanics based on the idea of a quantum-mechanical history, defined to be a time-ordered sequence of projection operators acting on an initial state. The projection operators at each moment of time satisfy

\[ P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha, \]  
(I.1)

and

\[ \sum_\alpha P_\alpha = 1. \]  
(I.2)

The probability of a n-projection history with an initial state represented by a density matrix \( \rho \) is given by

\[ p(\alpha_1, \cdots, \alpha_n) = Tr[P_{\alpha_n}^n(t_n) \cdots P_{\alpha_1}^1(t_1) \rho P_{\alpha_1}^1(t_1) \cdots P_{\alpha_n}^n(t_n)] \]  
(I.3)

with the time evolution operator \( e^{-iHt/\hbar} \) allowing us to write for the m’th-projection

\[ P_{\alpha_m}^m(t_m) = e^{iH(t_m-t_0)/\hbar} P_{\alpha_m}^m(t_0) e^{-iH(t_m-t_0)/\hbar}. \]  
(I.4)

A general form of this probability functional plays a central role in the decoherent histories approach. For more details about the decoherent histories approach see [9] and references therein. An axiomatic foundation of the decoherent histories approach is given in [11], [12]. In standard quantum mechanics, the uncertainty principle takes the form

\[ \Delta p \Delta q \geq \frac{\hbar}{2}. \]  
(I.5)

It relies very much on the notion of the state at a fixed moment of time. In this paper we are concerned with the question of how the uncertainty principle arises in a formulation based on the expression for the probability of histories, (I.3). This question has been addressed by Halliwell [10], who argued that the uncertainty principle may arise as a lower bound on
the Shannon information of the probability \([\text{I}3]\). He derived approximate although explicit expressions for the lower bound in certain situations.

The aim of the present paper is to obtain more accurate and more general versions of the known information-theoretic inequalities by introducing more sophisticated mathematical techniques which are likely to be of use in a wide variety of related problems. The other main aim is to understand how information can be used to measure uncertainty in standard and generalized versions of quantum mechanics \([12]\). It means that the role of information theory has to be reconsidered in the common understanding of quantum mechanics.

In Section II we introduce Shannon information and briefly describe some of its properties. The next few sections are taken up with the development of methods for calculating the maximum probabilities of quantum-mechanical histories. These results are then used in section VII to obtain upper and lower bounds for the maximum probabilities in a variety of situations. Section VIII is used to calculate the upper bounds for the probabilities of N-projection histories. In section IX the results of the previous sections are applied to gain a lower bound for Shannon information based on \(I \geq -\log(p_{\text{max}})\).

There are two more points of interest: This paper is motivated by the decoherent histories approach, but all the calculations are totally independent of it and can be carried out in standard quantum mechanics.

Secondly we are mainly concerned with physically realizable projections and quantum-mechanically (experimentally) interesting situations. We will later make a statement about the physical relevance of approximate (Gaussian) projections. In this regime \(\sigma_x\sigma_yM/2\hbar\), \(\sigma_x\sigma_k/2\hbar\) or similar constants are small. The constants used are the mass \(M\), the time separation between the projections \(T\) and the size of the exact position and momentum projections \(\sigma_x, \sigma_y\) and \(\sigma_k\).
II. INFORMATION THEORY

We give a short introduction to certain aspects of Shannon information theory [19], [3] which are useful in finding alternative formulations of the standard quantum-mechanical uncertainty principle. The information theoretic approach has been used extensively in [1] and [10].

In the discrete case we have a probability function \( p(n) \) defined on the set \( n = 1, \ldots, N \), satisfying \( 0 \leq p(n) \leq 1 \) and

\[
\sum_{n=0}^{N} p(n) = 1. \tag{II.1}
\]

In the continuous case we have

\[
\int dx p_c(x) = 1. \tag{II.2}
\]

The corresponding information in the discrete case is defined as

\[
I_d = -\sum_n p(n) \log p(n)
\]

and the continuous case it is

\[
I_c = -\int dx p_c(x) \log p_c(x).
\]

The relationship between discrete and continuous information can be understood best by studying an example. In particular for a Gaussian position projection the two kinds of information are related by

\[
I_d(\bar{X}) \equiv -\sum_n p(n) \log p(n) \\
\geq -\int dx \langle x|\rho|x \rangle \log \langle x|\rho|x \rangle - \log \sigma \alpha \\
\equiv I_c(X) - \log(\sigma \alpha)
\]

with

\[
p(n) = \text{Tr}[P^n_x \rho]. \tag{II.3}
\]
and $\sigma_\alpha$ the width of the Gaussian sampling function. This case is discussed and the notation explained in detail in [10].

In the discrete case an upper and lower bound for $I_d$ can easily be given:

$$0 \leq I_d \leq \log N.$$  \hspace{1cm} (II.4)

The upper bound is reached for $p(n)$ equal to $1/N$ for all $n$.

### III. TWO-TIME HISTORIES

For most of this paper we will concentrate on the case of histories characterized by position and/or momentum projections at two moments of time. The probability for a two-time history can be written as

$$p(\alpha, \beta, T) = Tr(P_\alpha e^{iHT} P_\beta e^{-iHT} P_\alpha \rho).$$ \hspace{1cm} (III.1)

Before we seek to maximize the value of the probability over all alternatives $\alpha$, $\beta$, and over all initial states $\rho$, we will demonstrate with the help of Weyl-Heisenberg coherent states that we can choose $\alpha$ and $\beta$ arbitrarily without changing the maximum probability. This means the maximum probability of any two-projection history is translation invariant, depending only on the size of the projections, but not on their positions in space. This will give us a lower bound for the Shannon information. This result has been derived previously by Halliwell [10] using the Wigner transform. Our method is not only simpler, but can also be generalized to a large variety of physical situations.

In the case of $\rho = |\psi\rangle \langle \psi|$ there is a set of wavefunctions $\psi_{\text{max}}$ which maximize the probability. We can restrict $\rho$ to be a pure state without changing the upper bound for the probability. If the wave function $\psi_{\text{max}}$, achieving maximal probability for one particular configuration, is known, then $U(\bar{x}, \bar{k})\psi_{\text{max}}$ is the corresponding wave-function in the new translated situation.
A. Two-Position Histories

We study the probability of a history consisting of position samplings at two moments of time. In this case $P_{\alpha_1}$ and $P_{\alpha_2}$ are two position projections and the probability-functional has the form

$$p(\alpha_1, \alpha_2, T) = Tr \left[ P_{\alpha_1}^x e^{-iHT} P_{\alpha_2}^x \rho P_{\alpha_2}^x e^{iHT} \right]$$

(III.2)

where the projection has the general form

$$P_{\alpha}^x = \int dx \Upsilon(x - \bar{x}_{\alpha})|x\rangle\langle x|$$

(III.3)

with the sampling function satisfying

$$\int dx \Upsilon(x - \bar{x}_{\alpha}) = \sigma_x$$

(III.4)

$$\sum_{\alpha} \Upsilon(x - \bar{x}_{\alpha}) = 1$$

(III.5)

and

$$\bar{x}_{\alpha} = \alpha \sigma_x .$$

(III.6)

For exact projections the sampling function $\Upsilon(x - \bar{x}_{\alpha})$ is

$$\Theta\left(\frac{x - \bar{x}_{\alpha} + \frac{1}{2} \sigma_x}{\sigma_x}\right) \Theta\left(\frac{-x + \bar{x}_{\alpha} + \frac{1}{2} \sigma_x}{\sigma_x}\right)$$

(III.7)

where $\sigma_x$ is the width, $\bar{x}_{\alpha}$ is the center of the exact projection and $\alpha$ is an integer. Now we will show that $p(\alpha_1, \alpha_2, T)$ is equal to $Tr(U(\bar{k}, \bar{x})\Omega_x U(\bar{k}, \bar{x})\rho)$ where

$$\Omega_x = \frac{1}{2\pi \hbar} \int dx dy dy' \Upsilon(x) \Upsilon(y) \Upsilon(y') e^{i \frac{m}{\hbar} \left( (x-y)^2 - (x-y')^2 \right)} |y\rangle\langle y'| .$$

(III.8)

This can be done with the help of Weyl-Heisenberg coherent-states. We define:

$$U(p, q) = e^{\frac{i}{\hbar}(p\hat{Q} - q\hat{P})}$$

(III.9)

$$U(p, q)|x\rangle = |x + q\rangle e^{\frac{i}{\hbar}p(x+q)}$$

(III.10)

$$U(p, q)|k\rangle = |k + p\rangle e^{\frac{i}{\hbar}q(k-p)}$$

(III.11)
The commutation relation is
\[ e^{\hat{Q} + \hat{P}} = e^{\hat{Q}} e^{\hat{P}} e^{-[\hat{Q}, \hat{P}] / 2} \]  
(III.12)
and the free particle-evolution is given by
\[ \langle x, t | y, 0 \rangle = \sqrt{\frac{m}{2\pi \hbar}} \exp \left( \frac{im}{2\hbar} (x - y)^2 \right) \]  
(III.13)
which we assume to be a good approximation in general and exactly true for the cases we analyse in the next two sections\(^\dagger\). In this situation our history is \( P_x e^{iHt} P_y \) and we set
\[ \bar{k} = \frac{(\bar{x} - \bar{y})m}{\hbar} . \]  
(III.14)
By writing the equation for the probability
\[ \text{Tr}(U(\bar{k}, \bar{x}) \Omega_x U^\dagger(\bar{k}, \bar{x}) \rho) \]  
(III.15)
explicitly and doing a straightforward substitution, the translational invariance of the maximum probability is derived.

**B. Momentum-Position History**

For a position-momentum history the probability \( p(\sigma_x, \sigma_k, T) \) is given by the integral
\[ \frac{1}{2\pi \hbar} \int dk \int dy \int dx \Upsilon(x - \bar{x}) \Upsilon(y - \bar{y}) \Gamma(k - \bar{k}) \langle x | e^{iHT} | k \rangle \langle k | e^{-iHT} | y \rangle \psi(x) \psi^*(y) \]
where \( \Gamma(k - \bar{k}) \) is the sampling function for the momentum. In the short time limit the terms containing \( H \) cancel out. The goal again is to reformulate the equation (9) containing \( P_\alpha \) and \( P_\beta \) into the form \( U(\bar{k}, \bar{x}) \Omega_k U(\bar{k}, \bar{x}) \) with
\[ \Omega_k = \frac{1}{2\pi \hbar} \int dx \int dy \int dk \Upsilon(x) \Upsilon(y) \Gamma(k) e^{ik(x-y)/\hbar} |x \rangle \langle y | \]  
(III.16)
\(^\dagger\) The proof for cases analysed in later sections follows trivially from the one given above.
using $\langle x|k \rangle = e^{ikx/\hbar}$. Applying $U$-operators explicitly and using substitution gives the expected result

$$Tr[P^x_\alpha e^{-iHT} P^k_\beta e^{-iHT} P^x_\alpha \rho] = Tr[U(\bar{k}, \bar{x}) \Omega_k U^\dagger(\bar{k}, \bar{x}) \rho] .$$

(III.17)

IV. CALCULATION FOR 2 PROJECTION HISTORIES IN THE FREE-PARTICLE CASE

Now we do the explicit calculation for the free-particle case and exact projections. The probability is

$$p(\sigma_x, \sigma_y, T) = \int_{-\frac{A'}{2}}^{\frac{A'}{2}} dx \int_0^{\sigma y} dy K(x, T; y, 0)\psi(y, 0)^2$$

$$= \frac{A'}{\pi} \int_{-\frac{A'}{2}}^{\frac{A'}{2}} dx \int_0^{\sigma y} dy e^{iA'(x-y)^2} \psi(y, 0)^2$$

$$= \int_0^{\sigma y} dy \int_0^{\sigma y} dy' e^{iA'y^2} e^{-iA'y'^2} \sin(A'(y-y')) \psi(y, 0)\psi^*(y', 0)$$

where $A' = \sigma_x m/2\hbar T$ and the free particle propagator is

$$K(x, T; y, 0) = \sqrt{\frac{m}{2\hbar T}} \exp\left(\frac{im}{2\hbar T}(x - y)^2\right).$$

(IV.1)

The change of notation from $p(\alpha_1, \alpha_2, T)$ to $p(\sigma_x, \sigma_y, T)$ for the probability is beneficial, because we concentrate in the next sections on calculating an upper bound for the probability of a two-position history with arbitrary $\alpha_1$ and $\alpha_2$, but with fixed $\sigma_x, \sigma_y, m$ and $T$.

To simplify the problem a redefinition of $\psi$ to $\psi(y)e^{iA'y^2}$ is useful. Next we approximate $\psi$ by piecewise constant functions. This can be justified using the basic ideas of multiresolution analysis [4], [3]. We start with a sequence of successive approximation spaces $V_N$. These subspaces of $L^2(\mathbb{R})$ satisfy the following criteria:

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$$

(IV.2)

with
\[ \bigcup_{N \in \mathbb{Z}} V_N = L^2(\mathbb{R}) \]  

(IV.3)

and

\[ \bigcap_{N \in \mathbb{Z}} V_N = 0 \cdot \]  

(IV.4)

We also impose the additional scaling requirement that \( \psi \in V_N \) implies, and is implied by, \( \psi(2^N \cdot) \in V_0 \) and the invariance under integer translations so that \( \psi \in V_0 \) implies, and is implied by, \( \psi(\cdot - k) \in V_0 \) for all \( k \in \mathbb{Z} \).

The orthogonal projection \( P_N \) onto \( V_N \) allows us to write

\[ \lim_{N \to \infty} P_N \psi = \psi \]  

(IV.5)

for all \( \psi \in L^2(\mathbb{R}) \). The simplest example for this ladder of spaces is given by

\[ V_N = \{ \psi \in L^2(\mathbb{R}) ; \psi \text{ piecewise constant on the half open interval } [2^{-N}n, 2^{-N}(n+1)], n \in \mathbb{Z} \} \]

The orthonormal basis of \( V_N \) is

\[ \{ \psi_{n,N} ; n \in \mathbb{Z} \} \]  

(IV.6)

where \( \psi_{n,N} \) is defined as

\[ \psi_{n,N}(x) = \begin{cases} 1 & n2^{-N} \leq x \leq (n+1)2^{-N} \\ 0 & \text{otherwise} \end{cases} \]

If \( ||\psi|| = 1 \) then

\[ \lim_{N \to \infty} \sum_{n \in \mathbb{Z}} |a_{n,N}|^2 2^{-N} = 1 \]  

(IV.7)

In our case we are interested only in the values of the wave-function in the interval \([0, \sigma_y] \). This allows us to restrict the range of \( n \) from the integers to the natural numbers between 0 and \( 2^N \), if we redefine \( \psi_{n,N} \) to be

\[ \psi_{n,N}(x) = \begin{cases} 1 & n\sigma_y 2^{-N} \leq x \leq (n+1)\sigma_y 2^{-N} \\ 0 & \text{otherwise} \end{cases} \]
The integral can now be rewritten as

\[ p(\sigma_x, \sigma_y, T) = \lim_{N \to \infty} \sum_{n=0}^{2N} \sum_{m=0}^{2N} \frac{1}{\pi} \int_{0}^{\sigma_y} dy \int_{0}^{\sigma_y} dy' \sin(A'(y-y')) \psi_{N,n}(y) a_{N,m}^{*} \psi_{N,m}(y') \]

\[ = \lim_{N \to \infty} \sum_{n=0}^{2N} \sum_{m=0}^{2N} \frac{1}{\pi} \int_{n2^{-N}\sigma_y}^{(n+1)2^{-N}\sigma_y} dy \int_{m2^{-N}\sigma_y}^{(m+1)2^{-N}\sigma_y} dy' \frac{\sin(A'(y-y'))}{\pi(y-y')} a_{N,n} a_{N,m}^{*} \]  

(IV.8)

with

\[ \psi(y, 0) = \lim_{N \to \infty} \sum_{n=0}^{2N} a_{n,N} \psi_{n,N}(x) \]  

(IV.9)

and

\[ a_{n,N} = \int_{n2^{-N}\sigma_y}^{(n+1)2^{-N}\sigma_y} dy \psi(y, 0). \]  

(IV.10)

The convergence of (IV.9) is given in the $L^2$-norm, for more details see Meyer [14].

Now we can evaluate the integral on each interval with $a_{n}a_{m}^{*} = 2^{N} \sigma_{y}$:

\[ c_{\delta} = \frac{A2^{-N}}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{0}^{1} dy' \int_{\delta}^{\delta+1} dy \ e^{iA(y-y')x2^{-N}} \]

\[ = \frac{1}{\pi} \int_{0}^{1} dy' \int_{\delta}^{\delta+1} dy \ \sin(A(y-y')2^{-N}) \]

\[ = \frac{1}{\pi} \int_{\delta}^{\delta+1} dy \ Si(Ay2^{-N}) - Si(A(y-1)2^{-N}) \]

\[ = \frac{1}{\pi} \left\{ 2^{N} \cos(A(\delta - 1)2^{-N}) - 2^{N+1} \cos(A\delta2^{-N}) + 2^{N} \cos(A(\delta + 1)2^{-N}) \right\} \]

\[ + (\delta - 1)Si(A(\delta - 1)2^{-N}) - 2\delta Si(A\delta2^{-N}) + (\delta + 1)Si(A(\delta + 1)2^{-N}) \]

with $\delta = n - m$, $c_{nm} = c_{n-m}$ and $A := A'\sigma_{y}$. This integral is solved most easily by substituting new variables for $y + \delta y'$ and $y - \delta y'$. We also define

\[ Si(k) := \int_{0}^{k} dx \ \frac{\sin(x)}{x} \]  

(IV.11)

which can be approximated by
\[
\sum_{n=0}^{N} (-1)^n \frac{k^{2n+1}}{(2n+1)! (2n+1)}.
\] (IV.12)

This means our integral-operator has been transformed into matrix form. Instead of
\[\int dy \int dx \psi(x)f(x - y)\psi^*(y) = \lambda,\]
we now work with its discretized version: \[\sum_{n,m} b_n C_{nm} b_m^* = \lambda\] with \[C_{nm} = c_{n-m}\] with \[\sum_{n=0}^{2N} |b_n|^2 = 1.\] The bilinear form is maximal if \[b_m\] is an eigenvector of \[C_{nm}.\]

This allows us to use two simple approximations for the eigenvalues of the matrix in the
case when \[A < \frac{\pi}{2}\] because the matrix coefficients are all positive.

We know that for any matrix \[C_{nm}\] all eigenvalues are smaller than
\[\max_j \sum_{i=0}^{2N} |C_{ij}|\] (IV.13)

In our case this gives as an upper bound for the eigenvalue for \[N \to \infty\] of \[\frac{2}{\pi} Si\left(\frac{A}{2}\right).\] This is
gained by letting \[N \to \infty\] in the sum
\[\sum_{n=-2N-1}^{2N-1} c_n.\] (IV.14)

The upper bound for the probability is
\[p(\sigma_x, \sigma_y, T) \leq \frac{2}{\pi} Si\left(\frac{A}{2}\right).\] (IV.15)

**A. Töplitz-Method**

The next step is to study the eigenvalues by exploiting the fact that \[c_{nm}\] is a Töplitz
matrix [17], [13], [8]. The characteristic property of a Töplitz matrix \[c_{nm}\] is the equality of
the coefficients on the diagonals.

\[K = \begin{pmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-1} \\
c_{-1} & c_0 & c_1 & \cdots & c_{n-2} \\
c_{-2} & c_{-1} & c_0 & \cdots & c_{n-3} \\
& \ddots & \ddots & \ddots & \ddots \\
c_{-n+1} & c_{-n+2} & c_{-n+3} & \cdots & c_0
\end{pmatrix}\] (IV.16)
It is also normally assumed $c_n = c_{-n}$, but in our case $c_n = c_{-n} = c^*_n$. This allows us to view the $c_n$ as Fourier-coefficients of a real function:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\Theta \ f(\Theta) e^{-i n \Theta} \quad n = 0, \pm 1, \pm 2, \ldots \quad (IV.17)$$

$$f_N(\Theta) = \sum_{n=-2^N}^{2^N} c_{n,N} e^{i n \Theta} \quad (IV.18)$$

and

$$f_N(\Theta) = c_0 + \sum_{n=1}^{2^N} 2 c_{n,N} \cos(n\Theta) \quad (IV.19)$$

or, for fixed $N$,

$$f_N(\Theta) = \lim_{L \to \infty} \sum_{n=-L}^{L} c_{n,N} e^{i n \Theta} \quad . \quad (IV.20)$$

For any fixed $N$ we assume

$$m_N \leq f_N(\Theta) \leq M_N \quad (IV.21)$$

for all $\Theta \in [-\pi, \pi]$. Then we know that the eigenvalues $\lambda$ of the matrix $C_{nm}$ for a fixed $N$ satisfy

$$m_N \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_L \leq M_N \quad . \quad (IV.22)$$

Now to complete the process we study $A > \frac{\pi}{2}$. In this case some of the elements of the Töplitz matrix are negative and the function $f_N$ does not automatically reach its maximum at $\Theta = 0$. One has to study $f'_N(\Theta)$. In the case of $N$ very large we approximate $c_n$ by $\sin\left(\frac{nA}{2^n}\right)/n\pi$ and get

$$f'(\Theta) = \sum_{n=1}^{2^N} -n 2 c_n \sin(n\Theta) \approx \sum_{n=1}^{2^N} -2 \sin\left(\frac{nA}{2^N}\right) \frac{\sin(n\Theta)}{n\pi} \quad . \quad (IV.23)$$

As mentioned before, this case is of secondary importance and will not be studied in any detail.
B. Perron-Frobenius Theorem

In this sub-section the Perron-Frobenius theorem for non-negative matrices is used to get a lower limit for the maximum eigenvalue of the matrix $C_{nm}$:

**Theorem:** Suppose $C$ is an $n \times n$ non-negative primitive matrix. Then there exists an eigenvalue $\lambda_m$ such that:

(a) $\lambda_m > 0$;

(b) $\lambda_m > \lambda$ for any eigenvalue $\lambda_m \neq \lambda$;

(c) the eigenvectors associated with $\lambda_m$ are unique up to constant multiples.

A proof can be found in [18].

The maximum eigenvalue can be calculated in the following way:

$$\lambda_{\text{max}} = \sup_{||x||=1} \min_i \frac{\sum_j C_{ij}x_j}{x_i}$$  \hspace{1cm} (IV.24)

This supremum is attained for some sequence $x$ with $l^2$-norm equal to one and every element unequal to zero.

For example, setting all $a_n$ equal produces

$$\lambda_{\text{max}} \geq \frac{S_i(A)}{\pi}$$  \hspace{1cm} (IV.25)

for $N \to \infty$.

As a result we now have an upper and lower bound for the maximum probability in the physically most relevant case. Expanding our solutions also gives us information about $p_{\text{max}}$:

$$p_{\text{max}} = \frac{A}{\pi} - O(A^3) \quad \text{for} \quad A < \frac{\pi}{2}.$$  \hspace{1cm} (IV.26)

\footnote{A square non-negative matrix $T$ is said to be primitive if there exist a positive integer $k$ such that $T^k > 0$.}

\footnote{The extension from the finite to the countable case is possible and necessary in our problem, but will not be given explicitly for reasons of simplicity.}
A further improvement of the bounds, omitted for reasons of brevity, can be gained by using variational methods. Improvement of the bounds can also be gained by exploiting the following theorem:

If \( r \) is the Perron-Frobenius eigenvalue of an irreducible\(^4\) matrix \( \{c_{ij}\} \), then for any vector \( x \in P \), where \( P = \{x; x > 0\} \):

\[
\min_i \sum_j c_{ij} x_j / x_i \leq r \leq \max_i \sum_j c_{ij} x_j / x_i
\]

This was proven by Collatz \[4\] in 1942.

C. Young’s-Inequality

The inequality

\[
\int dx \int dy f(x) g(x - y) h(y) < D_{rst} ||f||_r ||g||_s ||h||_t
\]

proven independently by Beckner \[3\] and by Brascamp-Lieb \[3\] is based on Young’s inequality and gives another way to calculate an upper bound for \( p(\sigma_x, \sigma_y, T) \).

The relevant theorem is from the paper of Brascamp-Lieb.

**Theorem:** For \( f \in L^r, g \in L^s, h \in L^t \), \( r, s, t \geq 1 \) and \( 1/r + 1/s + 1/t = 2 \), then

\[
|\int dx \int dy f(x) g(x - y) h(y)| < D_{rst} ||f||_r ||g||_s ||h||_t
\]

with \( D_{rst} = r^{1/r'} s^{1/s'} t^{1/t'} \) and \( r' = (1 - 1/r)^{-1} \).

The inequality is only sharp if \( f, g, h \) are certain types of Gaussian. In our case \( r, t = 2, s = 1 \) and

\textsuperscript{4}An \( n \times n \) non-negative matrix \( c_{ij} \) is irreducible if for every pair \( i, j \) of its index set, there exists a positive integer \( m \equiv m(i, j) \) such that \( c_{ij}^{(m)} > 0 \)
\[ f(x) = \psi(x)\Theta(x + \frac{1}{2})\Theta(\frac{1}{2} - x) \]
\[ g(z) = \frac{\sin(Az)}{\pi z}\Theta(z + 1)\Theta(1 - z) \]
\[ h(y) = \psi^*(y)\Theta(y + \frac{1}{2})\Theta(\frac{1}{2} - y) . \]

This leads to

\[ p(\sigma_x, \sigma_y, T) \leq \frac{2}{\pi} Si(A) \quad \text{(IV.29)} \]

if \( A < \pi \).

V. POSITION AND MOMENTUM SAMPLINGS

Now we use methods developed in section III to calculate the maximum probability in the case of a position-momentum history. First we have to rewrite the probability-equation in an usable form:

\[
p(\sigma_k, \sigma_x, T) = \frac{1}{2\pi \hbar} \int_{-rac{\sigma_k}{\hbar}}^{\frac{\sigma_k}{\hbar}} dk \int_{-rac{\sigma_x}{2}}^{\frac{\sigma_x}{2}} dx \int_{-rac{\sigma_x}{2}}^{\frac{\sigma_x}{2}} dx' \psi(x)e^{iHT}k\langle k|e^{-iHT}|x'\rangle \psi(x')\psi^*(x')
\]

\[
= \frac{1}{\pi} \int_{-rac{\sigma_x}{2}}^{\frac{\sigma_x}{2}} dx \int_{-rac{\sigma_x}{2}}^{\frac{\sigma_x}{2}} dx' \int_{-rac{1}{2}}^{\frac{1}{2}} dk \ e^{iA(x-x')}k\psi(x)\psi^*(x')
\]

\[
= \frac{1}{\pi} \int_{0}^{\sigma_x} dx \int_{0}^{\sigma_x} dx' \sin(A(x-x'))\psi(x)\psi^*(x') \quad \text{(V.1)}
\]

with \( A = \sigma_k\sigma_x/2\pi \hbar \). This time no redefinition of the wave-function is necessary and we can discretize \( \psi \) directly. The rest of the calculation is now identical to section II. The result is an upper bound for the probability:

\[
p(\sigma_k, \sigma_x, T) \leq \frac{2Si(A)}{\pi} . \quad \text{(V.2)}
\]

A. Local Uncertainty Relation

An upper bound can also be calculated using the local uncertainty relation developed by Price and Faris. First a general inequality proven by Price [16]:
Theorem: Suppose $E \subseteq \mathbb{R}^d$ is measurable and $\alpha > \frac{d}{2}$. Then

$$\int_E |\hat{f}(k)| dk < K_1 m(E) \|f\|_2^{2-\frac{d}{\alpha}}$$  \hfill (V.3)

and

$$\int_E dk |\hat{f}|^2 \leq \text{const} \ m(E) \ |f|_2^{2-\frac{d}{\alpha}} \ |x|^\alpha \ f \ |l|_{2} \ |x|^\alpha \ f \ |l|_{2}$$  \hfill (V.4)

for all $f \in L^2(R)$, where

$$K_1 = \frac{\theta(d)}{2\alpha} \Gamma \left( \frac{d}{2\alpha} \right) \Gamma \left( 1 - \frac{d}{2\alpha} \right) \left( \frac{d}{\alpha} - 1 \right)^{\frac{d}{\alpha}} \left( 1 - \frac{d}{2\alpha} \right)^{-1}.$$  \hfill (V.5)

In our special case we have

$$\int_E |\hat{f}(k)|^2 dk \leq 2\pi m(E) \ |f|_2 \ |xf|_2.$$

We assume $f = \psi$ and $||x\psi||_2 \leq \frac{\sigma_x}{\sqrt{2}} ||\psi||_2$ to get

$$p(\sigma_x, \sigma_k, T) \leq \frac{\sigma_x \sigma_k}{2\pi \hbar}.$$  \hfill (V.6)

We do not have to study the momentum-position-case independently because of time reversibility in quantum mechanics. In the inequality this just means that the Fourier transform of $\psi$ replaces the original wave-function.

VI. TWO-MOMENTUM HISTORIES

Now we calculate the probability of a two-momentum history. In the case of free-particle propagation the result is trivial, because momentum is a conserved quantity. The simplest case worth studying is the harmonic oscillator. First we have to rewrite the probability-equation for the harmonic oscillator in a simpler form:

$$p(\sigma_{k'}, \sigma_k, T) = \int_{\frac{-\pi}{\sigma_{k'}}}^{\frac{\pi}{\sigma_{k'}}} dk' \int_{\frac{-\pi}{\sigma_k}}^{\frac{\pi}{\sigma_k}} dl \int_{\frac{-\pi}{\sigma_{k'}}}^{\frac{\pi}{\sigma_{k'}}} dl' \hat{K}(l, T; k', 0) \hat{K}^*(l', T; k', 0) \hat{\psi}(l) \hat{\psi}^*(l')$$

$$= \int_{\frac{-\pi}{\sigma_{k'}}}^{\frac{\pi}{\sigma_{k'}}} dk' \int_{\frac{-\pi}{\sigma_k}}^{\frac{\pi}{\sigma_k}} dl \int_{\frac{-\pi}{\sigma_{k'}}}^{\frac{\pi}{\sigma_{k'}}} dl' \frac{-\sin(\omega T)}{\pi m \omega h} \exp \left\{ -i \frac{\sin(\omega T)}{\hbar \omega m} \left[ q'^2 - \frac{2k'l'}{\cos(\omega T)} \right] \right\}$$

$$\exp \left\{ i \frac{\sin(\omega T)}{\hbar \omega m} \left[ q'^2 - \frac{2k'l'}{\cos(\omega T)} \right] \right\} \hat{\psi}(l) \hat{\psi}^*(l')$$
This time a simple redefinition of the wave-function is necessary before we can use the methods developed previously directly. The upper bound for the probability is given by

\[ p(\sigma_k', \sigma_k, T) \leq \frac{2}{\pi} Si\left( \frac{\sigma_k \sigma_k' \sin(\omega T)}{2m\hbar} \right) \]

### VII. TWO-PROJECTION HISTORIES FOR OTHER LAGRANGIANS

In the next few subsections the methods developed above are applied to other Lagrangians. In all the cases we analyse, the maximum probability remains translation invariant. This can be seen directly by just redefining our wave-function appropriately.

#### A. Harmonic oscillator

For the harmonic oscillator the Lagrangian has the form

\[ L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \omega^2 m^2 x^2 \]  \hspace{1cm} (VII.1)

The corresponding Green’s function (propagator) is

\[ K(x, T; y, 0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega T)}} \exp\left\{ \frac{im\omega}{2\hbar \sin(\omega T)}[(x^2 + y^2) \cos(\omega T) - 2xy]\right\} \]  \hspace{1cm} (VII.2)

We will show that the probability for a history in this potential is

\[ p(\sigma_x, \sigma_y, T) \leq \frac{2}{\pi} Si\left( \frac{m\omega \sigma_x \sigma_y}{4\hbar \sin(\omega T)} \right) \]  \hspace{1cm} (VII.3)

The derivation is sketched in the next few lines and is again based on methods developed in the previous sections. The probability is

\[ p(\sigma_x, \sigma_y, T) = \int_{-\frac{\sigma_x}{2}}^{\frac{\sigma_x}{2}} dx \int_{-\frac{\sigma_y}{2}}^{\frac{\sigma_y}{2}} dy e^{iA(x^2 + y^2) \cos(\omega T) - 2xy} \psi(y, 0) \psi^*_{\text{new}}(y, 0) \]  \hspace{1cm} (VII.4)

There are two points of interest: To avoid having to evaluate complicated integrals, it is again necessary to redefine the wave-function. Secondly, the similarity between the maximal probability \( \psi \) in the free-particle case and the case just described allows us to generalize our method to all linear cases.
\textbf{B. Constant Electric-field}

In this case of a constant electric-field the Lagrangian has the form:

\begin{equation}
L = \frac{1}{2}m \dot{x}^2 - e x .
\end{equation}

(VII.5)

And the relevant propagator is

\begin{equation}
K(x, T; y, 0) = \sqrt{\frac{m}{2\pi i\hbar T}} \exp \left\{ \frac{i}{\hbar} \frac{m(x - y)^2}{2T} - \frac{1}{2} eT(x + y) - \frac{e^2 T^3}{24 m} \right\} .
\end{equation}

(VII.6)

The probability for history in this potential is accordingly

\begin{equation}
p(\sigma_x, \sigma_y, T) \leq \frac{2}{\pi} S_i \left( \frac{m \sigma_x \sigma_y}{4 \hbar T} \right) .
\end{equation}

(VII.7)

\textbf{C. Time-dependent Electric-field and Harmonic Oscillator}

As a generalization of this case we can also calculate the maximum probability for the Lagrangian

\begin{equation}
L = \frac{1}{2}m \dot{x}^2 - e(t) x - \frac{m\omega^2}{2} x^2
\end{equation}

(VII.8)

where the propagator is

\begin{equation}
K(x, T; y, 0) = \sqrt{\frac{m \omega}{2\pi i\hbar \sin(\omega T)}} \exp \left\{ \left( \frac{im \omega}{2 \hbar \sin(\omega T)} \right) \left[ \cos(\omega T)(x^2 + y^2) - 2xy 
\right.ight.

\left. - 2 \frac{y}{m \omega} \int_0^T dt e(t) \sin(\omega(t - T)) \right.

\left. - 2 \frac{x}{m \omega} \int_0^T dt e(t) \sin(\omega(T - t)) \right.

\left. - 2 \frac{1}{m^2 \omega^2} \int_0^T dt \int_0^T dt' e(t)e(t') \sin(\omega(t - T)) \sin(\omega(t' - T)) \right\} .
\end{equation}

The calculation is identical to the previous sub-section, except that we have to redefine $\psi$ to $\psi e^{iAy^2} e^{\frac{2m \omega}{\hbar} \int_0^T dt e(t) \sin(\omega(t - T))}$.

The resulting probability is

\begin{equation}
p(\sigma_x, \sigma_y, T) \leq \frac{2}{\pi} S_i \left( \frac{m \sigma_x \sigma_y \omega}{4 \hbar \sin(\omega t)} \right)
\end{equation}

(VII.9)
for $m\omega x_y/h\sin(\omega t) > \frac{\pi}{2}$. As in all the previous cases a lower bound for the maximal probability can be computed which is

$$p_{\text{max}}(\sigma_x, \sigma_y, T) \geq \frac{1}{\pi} Si\left(\frac{m\sigma_x \sigma_y \omega}{2h\sin(\omega t)}\right).$$

(VII.10)

This means that any influence by an electrical field can be counteracted by choosing appropriate initial conditions.

**D. General Lagragians**

In the case of arbitrary linear systems the propagator has the form

$$\langle x'', t''|x', t'\rangle = \Delta(t'', t') \exp\left\{\frac{i}{\hbar}S(x'', t''|x', t')\right\}.$$  

(VII.11)

A corresponding upper bound for the probability is then

$$p(\sigma_x, \sigma_y, T) \leq \sigma_x \sigma_y |\Delta(t'', t')|^2$$

(VII.12)

which follows directly from the Hölder inequality\[.\] For reasons of brevity no further examples are given. It should be clear by now that a very large class of physical problems can be studied by the methods developed in the previous sections.

**VIII. N-PROJECTIONS**

We can use the same generalizations to give a simple improvement to the N-projection probability given in \[.\] At first we assume that the N-projection history is made up out of $N-1$ two-projection histories. The new upper bound for the probability is then:

$$p(\sigma_{x,1}, ..., \sigma_{x,n}) \leq \left(\frac{2}{\pi}\right)^{N-1} \prod_{n=1}^{N-1} Si\left(\frac{\sigma_{x,n}\sigma_{x,n+1}m}{4\pi\hbar T_{ij}}\right)$$

(VIII.1)

where $T_{ij}$ is the time difference between the successive projections. Analogous upper bounds for the probability can be given for all the other cases studied before.

---

\[5\] If $f \in L^p$ and $g \in L^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$ then: $\int f(x)g(x)dx \leq ||f||_p ||g||_{p'}$
IX. INFORMATION THEORY

An lower bound for the information is given by

\[ I \geq -\log(p_{\text{max}}) \]  

(IX.1)

because of \( \sum p = 1 \).

In our case this gives an lower bound for the information of

\[ I \geq -\log \left( \frac{2Si(A)}{\pi} \right) \]  

(IX.2)

This is true in all the examples described above. The constant \( A \) is dependent on the physical situation and has been given in previous sections.

Next we show how one can calculate an \( I_d \) for different conventionally used wavefunctions. At first we look at two-position histories. We assume \( \psi \) to be divided again in piecewise constant functions with

\[ \psi_n(x) = \begin{cases} \sqrt{\sigma_y} & n\sigma_y \leq x \leq (n + 1)\sigma_y \\ 0 & \text{otherwise} \end{cases} \]

and

\[ \sum_n |a_n|^2 = 1 \]  

(IX.3)

The transition probability is

\[ p(n, m) = A \frac{1}{\pi} \int_{\sigma_x n}^{\sigma_x (n+1)} dx \int_{\sigma_y m}^{\sigma_y (m+1)} dy \int_{\sigma_y m}^{\sigma_y (m+1)} dy' \exp \left\{ i \frac{2Ax(y - y')}{\sigma_x \sigma_y} \right\} \psi(y) \psi^*(y') \]  

(IX.4)

The fact that \( \psi_n \) is constant in each \( m \)-interval, leads to

\[ p(n, m) = p'(n)p(m) \]  

(IX.5)

\[ p(m) = |a_m|^2 \]  

(IX.6)

and
\[
p'(n) = \frac{A}{\pi} \int_n^{n+1} dx \int_0^1 dy \int_0^1 dy' e^{i2Ax(y-y')}
\]
\[
= \frac{A}{\pi} \int_0^1 dx \int_0^1 dy \int_0^1 dy' e^{i2A(x+n)(y-y')}
\]
\[
= \frac{1 - (n + 1) \cos(2An) + n \cos(2A(n + 1))}{2\pi n(n + 1)A} + \frac{Si(2A(n+1)) - Si(2nA)}{\pi}
\]
for \( n > 0 \). The corresponding information is
\[
I_d = -\sum p(n,m) \log p(n,m)
\]
\[
= -\sum p(n)p'(m) \log(p(n)p'(m))
\]
\[
= -\sum n p(n) \log p(n) + p'(n) \log p'(n)
\]
(IX.7)

For any \( \sigma_y \) and \( \sigma_x \) and given \( a_n \) we can now calculate the information explicitly. This sum has its minimum when \( p(n) = \delta_{nm} \).

Another interesting case involves setting \( \sigma_y \) and \( \sigma_x \) to be much less than one, thereby approximating the Shannon information required for the continuous case. The information is
\[
I_{\text{min},d} = -\sum n p'(n) \log p'(n).
\]
(IX.8)

For other two-projection histories the process is similar.

X. DISCUSSION AND OUTLOOK

In this paper several mathematical methods are developed to calculate the maximum probability and minimum Shannon information of histories under varying conditions. Not only upper but also lower bounds for the maximum probability are calculated. Special importance is placed on the similarities between the histories for different Lagrangians. Next a lower bound for the corresponding information is calculated.

These results can be used to do explicit calculations in the framework of decoherent histories and to construct experiments to verify quantum mechanics, by comparing the maximal experimentally achievable probability with our theoretical bounds.
This paper also can be used to support the view that Gaussian slits are not as unphysical as normally assumed, but can have experimental relevance. This can be shown by comparing the maximal probabilities and lower bounds for the Shannon information for Gaussian, as described in [10], and exact projections in a variety of situations.

There are three major areas where the formalism should be extended:

• There is the need to gain an exact bound for the maximal probability and Shannon information for a general N-projection history.

• One needs to generalize the results from one to three spacial dimensions.

• One needs to study projection smeared in time.

This will be done in a forthcoming paper.

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REFERENCES

[1] A. Anderson and J.J. Halliwell, Phys. Rev. D 48, 1580 (1992).

[2] W. Beckner, Ann. of Math., 102, 159-82 (1975).

[3] H. J. Brascamp and E. H. Lieb, Advances in Math. 20, 151-173 (1976).

[4] L. Collatz, Mathematische Zeitschrift, 48, 221-6 (1942).

[5] T. M. Cover and J. A. Thomas, Elements of Information Theory, (Wiley, New York, 1991).

[6] I. Daubechies, Ten Lectures on Wavelets, (SIAM, Philadelpbia, PA, 1992).

[7] W. G. Faris, J. Math. Phy. 19, 461-466 (1978).

[8] U. Grenader and G. Szegö, Töplitz Forms and their applications, (Univ. California Press, Berkley and L.A., 1958).

[9] R. Griffiths, J. Stat. Phy. 36, 219, (1984); M. Gell-Mann and J.B. Hartle, in Complexity, Entropy and the Physics of Information, edited by W. Zurek, SFI Studies in the Science of Complexity Vol. VIII (Addison-Wesley, Reading, MA, 1990); R. Omnes, Rev. Mod. Phys. 64, 339 (1992).

[10] J.J. Halliwell, Phys. Rev. D 48, 2739 (1993).

[11] J. B. Hartle, Spacetime quantum mechanics and the quantum mechanics of spacetime, (Proceedings on the 1992 Les Houches School, Gravitation and Quantisation, 1993).

[12] C. J. Isham, IMPERIAL/TP/92-93/39, appearing in J. Math. Phy..

[13] M. Kac, W.L. Murdock & G. Szegö, Journal of Rational Mechanics and Analysis, Vol. 2, No. 4, Oct. 1953, 767-799.

[14] Y. Meyer, Wavelets and Operators, (Cambridge University Press, 1992).
[15] M. H. Partovi, Phys.Rev.Lett. 50, 1883(1983).

[16] J. F. Price, Studia Mathematica 85, 26-45 (1987).

[17] I. Schur, Über einen Satz von Caratheodory, Sitzungsbericht der Kgl. Preussischen Akademie der Wissenschaften, 1912, 4-15.

[18] E. Seneta, *Non-Negative Matrices*, (George Allen & Unwin, London, 1973).

[19] C.E. Shannon and W.W. Weaver, The Mathematical Theory of Communication, (University of Illinois Press, Urbana, IL., 1949).

[20] G.Segö, Mathematische Zeitschrift 6, 167-202 (1921).