Dialogue-based Activities and Manipulatives to Engage Liberal Arts Majors in Mathematics

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Abstract: This article presents four inquiry-based learning activities developed for a liberal arts math course. The activities cover four topics: the Pythagorean theorem, interest theory, optimization, and the Monty Hall problem. Each activity consists of a dialogue, with a theme and characters related to the topic, and a manipulative, that allow students to physically interact with the mathematics they are doing. The overall goal is to create a new way for liberal arts students to engage in mathematics, while simultaneously cultivating an appreciation of the subject.

Keywords: Inquiry-based learning, liberal arts mathematics, number systems, Pythagorean theorem, financial mathematics, interest theory, problem solving, modeling, optimization, probability, simulation, Monty Hall problem.

1. INTRODUCTION

Presented here are four inquiry-based learning activities\(^1\) that were developed for college mathematics and based partly on the pedagogical recommendations of CRAFTY (MAA’s commitee on Curriculum Renewal Across the First Two Years, [1]), and Process-Oriented Guided Inquiry Learning (POGIL, [3]). Each activity is 50 minutes long and designed to be done in groups of four to five students. They are composed of two parts: a dialogue\(^2\) and manipulative.

The dialogues require students to fill-in blanks, draw and label diagrams, reason geometrically and algebraically, construct tables and models, and run simulations. They include a theme and characters, along with references to various historical events and figures, literary and scientific works, and popular

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\(^1\)Video descriptions of the first three activities can be found at [4].

\(^2\)The dialogues can be found in the online appendices.
culture. These are designed to enhance the students overall experience and serve as discussion points.

The manipulatives are used in tandem with the dialogues to explore the main mathematical topics of each activity, which are listed below:

1. The Pythagorean theorem– discovery, consequences, and proof.
2. Interest theory– simple, compound, and continuous.
3. Optimization– problem solving and modeling.
4. The Monty Hall problem– probability and simulation.

They are simple to make, and constructed out of common office supplies and household goods. Most importantly, they enable students to physically interact with the mathematics they are doing; essentially, bringing the mathematics to life.

2. CLASS PERIOD STRUCTURE

At the start of class, students gather into groups of four or five, after which, the dialogue and manipulative are handed out and introduced. The introduction frequently consists of a YouTube video [5], followed by a short introduction by the teacher, highlighting the main points, themes, characters, and inquiry-based learning approach of the activity. The students are then left to choose their parts and start the dialogue.

During the activity, the teacher walks through the classroom to facilitate learning. It is important that the teacher does not answer a group’s question directly; instead, a question is redirected back to the group in the form of another question. If several groups become stuck, then the activity is paused, and a mini-lecture is given.

A review and discussion session is held at the end of each activity. It begins with a review of the dialogue and manipulative, covering the main points. The session concludes by using the activity as a platform to discuss the philosophy and applications of mathematics, and their impact on our world.

3. ACTIVITY 1: THE PYTHAGOREAN THEOREM

The activity, *The Pythagorean Theorem*, answers the question, “Why is $a^2 + b^2 = c^2$?” The dialogue is a historical narrative on the discovery, consequences, and proof of the Pythagorean theorem. It is given in the form of a three-act play inspired by [2] and Shakespeare’s *Macbeth*. The characters include the Narrator, Hint Master, Pythagoras, Egyptian Rope-Stretcher, and Hippasus. Four 3-4-5 right triangles made out of index cards, serve as the manipulative, which is used to give a proof of the Pythagorean theorem.
In Act I, Discovering the Pythagorean Theorem, Pythagoras stumbles upon an Egyptian roper-stretcher, constructing a right triangle with just a rope. The group answers a series of questions about triangles and is asked to replicate this construction with a piece of string and three tacks on a cork board (see Figure 1(a)). From this model, the group follows Pythagoras’s possible line of thought through a set of questions, leading them to conjecture that $a^2 + b^2 = c^2$.

In the next act, Discovering a New Number and Murder, the Pythagorean cult and one of its more prominent members, Hippasus, are introduced. It begins with a conversation between Pythagoras and Hippasus on the discovery of the irrational number $\sqrt{2}$, as a consequence of the Pythagorean theorem. Eventually, a set of problems, and fill-in the blanks lead to an indirect proof that $\sqrt{2}$ is irrational. A portion of this part of the activity is given below.

**Pythagoras:** Surely, this number is rational?

**Hippasus:** Sadly, it is not. There is no way to create it by dividing two integers. Here is my lamentable proof.

Assume $\sqrt{2}$ is rational. Then,

$$\sqrt{2} = \frac{a}{b}$$

where $a$ and $b$ are natural numbers with nothing in common. By multiplying both sides by $b$, we have

$$\sqrt{2}b = \text{______},$$

which implies that

![Figure 1. Manipulatives. (a) right triangle and (b) four right triangles.](image-url)
Since the left-hand side of the equation is even, the right-hand side must also be even. Thus $a$ is an even number, for the square of an odd number is always odd. Hence,

$$2b^2 = (\text{even})^2 = (\text{even})(\text{even}),$$

which implies $b$ must also be even. If $a$ and $b$ are both even, then they must have what number in common\? (Write this number below.)

But $a$ and $b$ have nothing in common, so they cannot have this number in common. This is called a contradiction, so our assumption that $\sqrt{2}$ was rational is wrong. Therefore, $\sqrt{2}$ is not a rational number.

Q.E.D

**Pythagoras:** Startling and perplexing! Tell no one of your discovery or it will be the end of us.

At this point, the activity is paused by the teacher to discuss “proof by contradiction.” The act ends with the mysterious death of Hippasus.

Act III, *Discovering a Proof of the Pythagorean Theorem*, begins with Pythagoras trying to prove his theorem. He grapples with the question, “Why $a^2 + b^2 = c^2$ and not $a + b = c$?" His efforts to focus are thwarted by the ghosts of the Egyptian rope-stretcher and Hippasus. The group is asked to help Pythagoras prove his theorem by using the four 3-4-5 right triangles, and some algebra.

A series of hints are given by the teacher to help them do this. The first hint, “try to construct a square out of the four right triangles,” is used to construct the square in Figure 1(b). The next two hints, “the sum of the area of the parts = area of the whole,” and “area four triangles + area little square = area big square,” are used in conjunction with Figure 1(b) to get the expression, $2ab + c^2 = (a + b)^2$. The dialogue ends when the Pythagorean relation, $a^2 + b^2 = c^2$, is found by using the distributive property to multiply out the right-hand side of the equation and subtracting $2ab$ from both sides.

4. ACTIVITY 2: DISCOVERING INTEREST FORMULAS

*Discovering Interest Formulas* helps students compare and contrast the theories of simple, compound, and continuous interest. The dialogue, based partly on Galileo’s work, *Dialogue Concerning the Two Chief World Systems*, has three
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parts: Discovering the Simple Interest Formula, Discovering the Compound Interest Formula, and Discovering the Continuous Interest Formula. The characters include the Narrator, Equanimity, Malfeance, Simplicio, and Banker. The main manipulative is fake money, used in conjunction with the interest diagrams in Figure 2, to simulate each type of interest.

The dialogue begins as Malfeance asks Simplicio for a $1000 dollar loan at a simple interest rate of 5%. Simplicio, who is suspicious of Malfeance, requires a table and formula for this loan. The group is asked to create this repayment table for the first 5 years, and work through an analysis to derive the simple interest formula. Based on this information Simplicio decides to accept this offer, but before he can do so, Equanimity steps in to say, “I see a great inequality here.”

The second part of the activity explores the similarities and differences between and among simple and compound interest. This is done by using fake money, in conjunction, with the interest diagrams in Figure 2, to model each type of interest. Simple interest is simulated in the following manner:

Equanimity: Banker, please take out a $1000 and place it on the table. Simplicio, this represents your loan to Malfeance. Every year it generates $50 in interest. Banker would you please put $50 dollars next to the $1000 dollar pile (Figure 2(a)). So after one year the $1000-pile generates $50. The next year it will generate another $50 (Banker put another fifty on the $50-pile); then another fifty (another fifty banker), and so on. Notice that the $1000-pile is working hard for you, but the $50-pile is do nothing. It is doing no work at all. In order to get the best deal, you need to make interest off your interest. This is called compounding interest, and it means that the $50-pile needs go to work.

![Interest diagrams. (a) simple interest and (b) compound interest.](image)
This is then compared to a loan compounded annually at 5%, which is modeled in a similar way using Figure 2(b):

**Equanimity:** Banker, clear all the piles and put down $1000. This is the amount borrowed. After a year, it generates $50 in interest. Banker, put down $50 in a new pile. After another year, the $1000-pile generates another $50 and the $50-pile generates $50 \times 0.05 = $2.50. Banker, add another $50 to the $50-pile and put down $2.50 in a new pile.

Finally, each group compounds interest one more time on their own, which allows them to fill-in a repayment table for the first 3 years of the loan. An analysis of this work allows each group to derive the yearly compound interest formula. The activity is paused at this point, so the teacher can compare and contrast these two types of interest with the simulations.

Once the activity resumes, the groups discover that Simplicio can make more money by compounding semi-annually. They follow a similar, prior procedure, using the fake money, together with an interest diagram, for interest compounded semi-annually, at a nominal interest rate of 5%. The compound interest formula is motivated from this, and used by the groups to fill-in a 1 year repayment table dependent on interest compounded yearly, semi-annually, monthly, and daily. From this, the following conclusion is made that leads naturally to the theory of continuous interest, given in the next section:

**Simplicio:** Amazing! The more I compound interest, the more I get from Malfeasance. It is sort of like throwing logs onto a fire. The more logs I throw on the fire (the more times I compound), the hotter the fire becomes (the more money I make). I could make a fortune off of Malfeasance. I could just keep on compounding and get more and more money from him.

**Equanimity:** Not quite. Just like there is a limit to how hot a fire can get, there is also a limit to the amount you can squeeze out of poor Malfeasance.

The final part of the activity illustrates how the continuous interest formula is connected to the compound interest formula. The groups begin by working through a simple thought experiment designed to explain the notion of a limit. This is followed by the construction of a repayment table, in which interest is compounded daily, hourly, and every minute. Consequently, the group discovers the actual limit that Simplicio can get back from Malfeasance. In addition, they learn how the continuous interest formula arises naturally, from the compound interest formula by compounding the loan more often,

$$
\lim_{n \to \infty} PV \left(1 + \frac{i}{n}\right)^{nt} = PVe^{it}.
$$
The dialogue ends with Simplicio offering Malfeasance a $1000 loan at a continuous interest rate of 5%. Malfeasance replies with a witty, popular culture reference, “Darn you Equanimity and your meddling interest.”

5. ACTIVITY 3: THE MUSIC OF OPTIMIZATION

This activity illustrates that mathematics and technology can be used to model and create something real. This is done by solving the following optimization problem and constructing it out a piece of paper with the help of a ruler and four sticky dots:

A rectangular tube with a square base is made by taking a flat rectangular piece of paper and folding it in equal parts three times (Figure 3). If the perimeter of the flat piece of paper is 24 inches, then determine the largest possible volume of the tube.

The dialogue mimics the rehearsal of a symphony and is a parody of the 1980’s pop hit, Rock Me Amadeus, by Falco. There are five parts, and the characters include Conductor, Singer, Ludwig, Amadeus, and Johann.

The first part of the activity introduces the problem to the group and has them work through a series of examples and fill-in the blanks leading them to the following conclusion:

Ludwig: Exactly! In fact, there are an infinite number of possible volumes, because there are an infinite number of pieces of paper that have a perimeter of 24 inches. But there will be just one piece of paper that gives us the largest volume as a rectangular tube.

Amadeus: Amazing! It is like my music. There are an infinite number of sequences of notes, but there is just one, perfect sequence for each piece I compose. Let us find this optimal piece of paper!

In the second part, each group works with the picture in Figure 3 and attempts to come up with a labeling of the parts in the following way:

Figure 3. Optimization problem.
Amadeus: Ah, you are right! I need to label things first. What should we use as labels?

Singer: Use x. Use x. Use x and y!

Ludwig: But there are three sides to label on our tube. How can we use two letters to label three things?

Conductor: Johann!

Johann: There are a number of ways to solve this problem. Instead of labeling the tube, we could label the rectangle on the left. The length would be labeled _____, while the width would be labeled y (label this on the picture). This means that the length of the tube would be labeled _____, the width of the tube is also labeled _____ since the base is a _____, and the height is labeled _____ (label this on the picture).

The activity is paused at this point by the teacher, in order to explain why the length of the square tube is \( x/4 \) when the length of the paper is \( x \). Often, the example of a triangular tube is used to lead each group to this conclusion.

The third part of the dialogue, involves a series of fill-in the blanks, like the ones given below, that lead to the volume formula for the rectangular tube in terms of \( x \).

Amadeus: The 24 inches! This is the perimeter of the rectangular piece of paper. The perimeter of the rectangle is given by

\[
\text{Perimeter} = 2 \times \text{length} + 2 \times \text{width}.
\]

In our case, this can be expressed as \( _____ = 2 _____ + 2y \).

Solving this for \( y \), we get \( y = _____ \).

Therefore,

\[
V = \frac{x}{4} \times \frac{x}{4} \times (_____).
\]

This formula is then graphed, using a graphing calculator, in the forth part, and the “maximum” function is used to determine the dimensions of the paper with maximum volume. Finally, each student constructs the model out of a piece of paper, using a ruler to measure the dimensions and four sticky dots to secure the edges. Once the construction is complete, the students write-up their work on the final page of the dialogue titled, Musical Score: The Music of Optimization.

6. ACTIVITY 4: THE MONTY HALL PROBLEM

The last activity is based on a problem posed in 1975 by Steve Selvin [6] and popularized in Marylin vos Savant’s column, “Ask Marylin,” in Parade magazine [7]:

Suppose you’re on a game show, and you’re given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what’s behind each door, opens another door, say No. 3, which has a goat. He then says to you, “Do you want to pick door No. 2?” Is it to your advantage to switch your choice?

The dialogue is in the form of a game show, based on *Let’s Make a Deal*, hosted by Monty Hall. The characters are Monty Hall, Fermat, Pascal, The Goat, and The Audience.

The activity begins with an introduction to the game, in which Fermat, Pascal, and The Goat, are working together to determine which door to choose. To do this, the groups create a tree diagram and list all the possible outcomes in a table. This is then used to determine the sample space and probability that the car will be behind door 1. This leads the groups to prove that each door has the same likelihood of having the car.

In the next part, the deal is made, in which Monty Hall tells the contestants the car is not behind door 3, and gives them the option of switching doors. A discussion ensues among the characters as to what they should do. During this discussion, a new sample space is formed by the group with the condition that the car is not behind door 3, and the probability of the car being behind door 1 is recalculated. The conclusion being it does not matter if they switch; however, Fermat disagrees, and presents the table in Figure 4, to argue that they are more likely to win if they do switch.

At this point, the groups are not sure what to do; the first analysis led to the conclusion that it does matter if they switch, where as, the second analysis led to the opposite conclusion. The final part of this activity is a simulation that ultimately decides the switch/not switch question. This is done with three playing cards: one red and two black. The red card represents the door with the car behind it, whereas the two black cards represent the animals. The cards are mixed up, laid face down in a row, and flipped over. The results are recorded in a table, like the one given in Figure 5 with mock data, and repeated 12 times by each group. The number of times a win occurs from switching versus not switching are gathered from the groups and recorded by the teacher on the board (see Figure 6). The end result are 72 or more simulations that lead to the decision to switch doors.

| Door1 | Door2 | Door3 |
|-------|-------|-------|
| Car   | Animal| Animal|
| Animal| Car   | Animal|
| Animal| Animal| Car   |

*Figure 4. Fermat’s table.*
| Trial | Door1 | Door 2 | Door3 | Switch | Not Switch |
|-------|-------|--------|-------|--------|------------|
| 1     | Red   | Black  | Black | Lose   | Win        |
| 2     | Black | Red    | Black | Win    | Lose       |
| ...   | ...   | ...    | ...   | ...    | ...        |
| 12    | Black | Black  | Red   | Win    | Lose       |

Figure 5. Simulation table.

| Decision | Switch | Not Switch |
|----------|--------|------------|
| Win      | 48     | 24         |
| Lose     | 24     | 48         |
| Probability of Winning | 67% | 33% |

Figure 6. Simulation analysis.

7. CONCLUSION

These activities were created in the Fall of 2011. Their intent is to offer a new way for liberal arts students to experience mathematics, leading to a deeper understanding and appreciation. They have run successfully every semester since then, and the students have responded positively to them.

There are two key points to running this type of activity successfully: a good introduction and well managed group work. A good introduction piques curiosity, while covering the main points, themes, and characters of the activity. In addition, it is vital to point out that these are inquiry-based learning activities, in which no question is answered directly; instead, a question is met by another question directed back to group. Without this declaration, students often become agitated when their question is not answered directly, and the activity can quickly turn sour.

As for well managed group work, it important to help each group keep up with the pace of the activity, and stay on track. From experience, the most common reason for a group to fall behind is that they are not seated in a way conducive to conducting group work. To avoid this, students should be told to sit, or arranged by the teacher, in a way they can directly see each other; and the closer they are arranged together, the better they will be able to communicate. It is also important for the teacher to stop and review the activity, at least once every 20 minutes. This will ensure that the groups stay on track and do not lose focus.

Overall, students have enjoyed these activities, and commented positively on them. A poll was taken during the first semester these activities were offered, as whether to continue with the dialogue-based activities. They unanimously voted for the activities to continue. At times, they can be a bit silly; however,
they are designed this way to elicit positive reactions, such as laughter, in an often infamous subject for liberal arts majors.

8. APPENDIX: ACTIVITIES

I. Activity 1: The Pythagorean Theorem
   Manipulative I: Piece of string, ruler, three tacks, and a piece of cork board.
   Manipulative II: Four, 3-4-5 right triangles made out of index cards.

II. Activity 2: Discovering Interest Formulas
   Manipulative: Fake money created from a template found on the internet.

III. Activity 3: The Music of Optimization
   Manipulative: A piece of paper, ruler, and four sticky dots.

IV. Activity 4: The Monty Hall Problem
   Manipulative: Three playing cards: one red and two black.

SUPPLEMENTAL MATERIAL

Supplemental data for this article can be accessed on the publisher’s website.

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BIOGRAPHICAL SKETCH

James C. Price is an Assistant Professor of Mathematics at the University of Arkansas at Fort Smith. He earned his Ph.D. from Purdue University, where he received numerous teaching awards. Beyond liberal arts mathematics, his research interests include algebraic geometry, history of mathematics, and actuarial science. He is passionate about changing the world, through the way people think about mathematics and education.