DIRECT PROOFS OF THE FEIGIN–FUCHS CHARACTER FORMULA
FOR UNITARY REPRESENTATIONS OF THE VIRASORO ALGEBRA

Antony Wassermann, CNRS, Institut des Mathématiques de Luminy

ABSTRACT. Previously we gave a proof of the Feigin–Fuchs character formula for the irreducible unitary discrete series of the Virasoro algebra with $0 < c < 1$. The proof showed directly that the multiplicity space arising in the coset construction of Goddard, Kent and Olive was irreducible, using the elementary part of the unitarity criterion of Friedan, Qiu and Shenker, giving restrictions on $h$ for $c = 1 - 6/m(m+1)$ with $m \geq 3$. In this paper we consider the same problem in the limiting case of the coset construction for $c = 1$. Using primary fields, we directly establish that the Virasoro algebra acts irreducibly on the multiplicity spaces of irreducible representations of the corresponding affine Kac–Moody algebra. This gives a direct proof that the only singular vectors in these representations are those given by Goldstone’s formulas, which also play an important part in the proof. For this proof, the theory is developed from scratch in a self-contained semi-expository way. Using the Jantzen filtration and the Kac determinant formula, we give an additional independent proof for the case $c = 1$ which generalises to the case $0 < c < 1$, where it provides an alternative approach to that of Astashkevich.

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References.

1. Introduction. The Witt algebra $\mathfrak{witt}$ is the Lie algebra of complex vector fields $f(\theta) \cdot d/d\theta$ on the unit circle with $f(\theta)$ a trigonometric polynomial. It has a basis

$$\ell_n = i e^{in\theta} \frac{d}{d\theta}$$

for $n \in \mathbb{Z}$ and commutation relations

$$[\ell_m, \ell_n] = (m - n) \ell_{m+n}.$$ 

This Lie algebra has a central extension by $\mathbb{C}$, called the Virasoro algebra $\mathfrak{vir}$, with basis $L_n$ ($n \in \mathbb{Z}$), $C$ and the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C.$$ 

In this article we will be interested in positive energy unitary representations of the Virasoro algebra. By this we mean an inner product space $\mathcal{H}$ on which the Lie algebra $\mathfrak{vir}$ acts by operators $\pi(L_n)$ satisfying $\pi(L_n)^* = \pi(L_{-n})$, $\pi(C) = \pi(C)^*$, with $\pi(C)$ acting as a scalar $cI$ and such that $\mathcal{H}$ has a decomposition as an algebraic orthogonal direct sum of eigenspaces of $\pi(L_0)$

$$\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}(n)$$
with $\mathcal{H}(n)$ finite–dimensional and $\pi(L_0)\xi = (h + n)\xi$ for $\xi \in \mathcal{H}(n)$. Vectors in $\mathcal{H}(n)$ are said to have energy $h + n$. Necessarily $h$ and $c$ are real; and it is easy to see that $c \geq 0$ and $h \geq 0$. Every positive energy unitary representation can be written as a direct sum of irreducible representations which are uniquely determined by the lowest eigenvalue $h$ of $\pi(L_0)$. In this case $\mathcal{H}(0)$ is one–dimensional and generated by a unit vector $\xi_h$. The character of a positive energy representation on $\mathcal{H}$ is defined as the formal sum

$$\chi_{\mathcal{H}}(q) = q^h \sum_{n \geq 0} d(n) q^n,$$

where $d(n) = \dim \mathcal{H}(n)$. We shall often write $L_n$ in place of $\pi(L_n)$, regarding $\mathcal{H}$ as a module over the Virasoro algebra. The above character is then formally

$$\sum_{n \geq 0} \text{Tr}_{\mathcal{H}(n)} q^{L_0} = \text{Tr}_\mathcal{H} q^{L_0}.$$  

The work of Friedan–Qiu–Shenker \[12\] shows that the irreducible positive energy unitary representations are given precisely by a continuous series $c \geq 1$ and $h \geq 0$ together with a discrete series

$$c = 1 - \frac{6}{m(m+1)}, \quad h = h_{p,q}(m) \equiv \frac{(p(m+1) - qm)^2 - 1}{4m(m+1)}, \quad m \geq 2, \quad 1 \leq q \leq p \leq m - 1.$$ 

The case $m = 2$ corresponds to $c = 0$ and had already been analysed by Gomes: only the trivial one–dimensional representation can occur ($h = 0$). Otherwise the representations are all on infinite–dimensional inner product spaces $L(c,h)$. The problem of computing the character $\chi_{c,h}(q) = \chi_{\mathcal{H}}(q)$ when $\mathcal{H}$ is the irreducible representation with parameters $(c,h)$ was first solved by Feigin and Fuchs \[8\]. We outline their proof and some of its subsequent simplifications below. It relies on a detailed knowledge of singular vectors and non–unitary representations.

The vector $\xi = \xi_h$ satisfies $L_n \xi = 0$ for $n > 0$ and $L_0 \xi = h \xi$. The space $\mathcal{H}$ is therefore spanned by all monomials $L_{-k}^{n_k} \cdots L_{-1}^{n_1} \xi$ with $n_i \geq 0$. On the other hand there is a universal module for the Virasoro algebra, the Verma module $M(c,h)$, which has as basis such vectors. Formally it can be defined as the module induced from the one–dimensional representation $L_n = 0$ ($n > 0$), $L_0 = h$ and $C = c$ of the Lie subalgebra spanned by $L_n$ ($n \geq 0$) and $C$. Its existence is standard and follows from the Poincaré–Birkhoff–Witt theorem applied to the universal enveloping algebra of the Virasoro algebra. (By that result the universal enveloping algebra has a basis consisting of monomials of the above form post–multiplied by monomials $C^k L_0^m L_1^{n_1} \cdots$ with $k \geq 0$ and $n_1 \geq 0$.) By definition $M = M(c,h)$ has positive energy and $d(n) = \dim M(n)$ is given by the number of partitions $P(n)$ of $n$. Thus

$$\chi_{M(c,h)}(q) = q^h \prod_{n \geq 1} (1 - q^n)^{-1}. \quad (1)$$

By the definition of $M(h,c)$, the irreducible representation $L(c,h)$ is the quotient of $M(c,h)$ by its unique maximal submodule.

The Verma module is in general not an inner product space, but for $c, h$ real, it does carry a unique Hermitian form $(\xi, \eta)$ such that $(\xi_h, \xi_h) = 1$ and $(L_n \xi, \eta) = (\xi, L_{-n} \eta)$, the so–called Shapovalov form. It is easy to see that the maximal submodule of $M(c,h)$ coincides with the kernel of the Shapovalov form. Thus for all real values of $c$ and $h$, the Shapovalov form defines a non–degenerate Hermitian form on $\mathcal{H} = L(c,h)$. The representation is unitary precisely when this Hermitian form is positive–definite on $\mathcal{H}$, or equivalently, since the subspaces $\mathcal{H}(n)$ are orthogonal, on each $\mathcal{H}(n)$.

In \[36\] we gave a direct method of determining the characters of the discrete series $0 < c < 1$ involving only unitary representations, which we summarise below. When $c > 1$ the character formula is particularly simple because the Verma module is itself irreducible so the character formula is given by (1). This remains true in the limiting case $c = 1$ provided $h \neq m^2/4$ with $m \in \mathbb{Z}$. When $h = m^2/4$ or equivalently $j^2$ with $j$ a non–negative half–integer, the Verma module is no longer irreducible and the character formula takes the form

$$\chi_{L(1,j^2)}(q) = (q^j - q^{(j+1)^2}) \prod_{n \geq 1} (1 - q^n)^{-1}. \quad (2)$$
The main result of this article is a direct proof of this formula using only unitary representations; it is the counterpart of our previous proof for the discrete series and relies only on a detailed knowledge of the boson–fermion correspondence in conformal field theory.

Before explaining the proof for $c = 1$, it will be helpful to recall the method used for the discrete series $0 < c < 1$ which relies on the coset construction of Goddard–Kent–Olive [15]. The discrete series at $c = 1$ can be regarded as a limiting case of this construction. The complexification $\mathfrak{g}$ of the Lie algebra of $SU(2)$ can be identified with the $2 \times 2$ complex matrices of trace zero. It is closed under taking adjoints and the Lie algebra of $SU(2)$ can be identified with the skew–adjoint matrices of trace zero. The affine Lie algebra $L\mathfrak{g}$ is defined as the space of trigonometric polynomial maps into $\mathfrak{g}$ with the pointwise bracket. It has generators $X_n = e^{i\theta X}$ $(X \in \mathfrak{g})$ with commutation relations $[X_m, Y_n] = [X, Y]_{m+n}$. The Witt algebra acts naturally by differentiation, so that $[\ell_n, X_k] = -kX_{k+n}$. In particular $\ell_0$ defines a derivation $[d, X_n] = -nX_n$. The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is defined as a central extension of $L\mathfrak{g} \rtimes \mathbb{C}$ by $\mathbb{C}$. It has generators $X(n), D$ and $C$ (central) satisfying the commutation relations

$$[X(m), Y(n)] = [X, Y](m + n) + m\delta_{n+m,0}\text{Tr}(XY) \cdot C, \quad [D, X(n)] = -nX(n).$$

The Lie subalgebra generated by the $X(n)$ and $C$ is denoted $L\mathfrak{g}$: it is a central extension of $L\mathfrak{g}$ by $\mathbb{C}$. Again we can define positive energy unitary representations on an inner product space $\mathcal{H}$ by requiring that $C^* = C$ acts as a scalar $\ell I$, $D^* = D$. $X(n)^* = X^*(-n)$ and that $\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}(n)$, where the $\mathcal{H}(n)$'s are finite–dimensional eigenspaces of $D$ corresponding to the eigenvalue $n$. Any such representation decomposes as a direct sum of reducible positive energy representations of the same form (possibly by adjusting $D$ by subtracting a positive scalar). For infinite–dimensional irreducible representations, $\ell$ must be a positive integer, called the level, and $\mathcal{H}(0)$ an irreducible representation of the Lie subalgebra $\mathfrak{g}(0) = \mathfrak{g}$. Each such $V_j$ is specified by a non–negative half–integer spin $j$ with dim $V_j = 2j + 1$. At level $\ell$, the only spins that occur are those satisfying $0 \leq j \leq \ell/2$. The Segal–Sugawara construction shows that in each irreducible positive energy representation operators $L_n$ can be defined with

$$L_0 = D + h \cdot I, \quad h = \frac{j^2 + j}{2(\ell + 2)}$$

such that $[L_n, X(m)] = -mX(n + m), L_n^* = L_{-n}$ and

$$[L_m, L_n] = (m - n)L_{n+m} + \frac{m^3 - m}{12}c_\ell \cdot I$$

where $c_\ell = 3\ell/$(\ell + 2$). The character of the representation is defined as the formal sum

$$\chi_{\ell,j}(q, g) = q^h \sum_{n \geq 0} q^n \text{Tr}_{\mathcal{H}(n)}(g),$$

where $g \in SU(2)$. Since this only depends on the conjugacy class of $g$, as usual we may take

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

with $|\zeta| = 1$. Note that formally

$$\chi_{\ell,j}(q, g) = \text{Tr}_{\mathcal{H}(g) L_0}.$$

These characters are given by the Weyl–Kac character formula (see [23], [31])). In [36] this formula was proved purely in terms of unitary representations together using the supersymmetric operators of Kazama–Suzuki; see also [26].

Let $\mathcal{H}_{\ell,j}$ denote the irreducible representation of level $\ell$ and spin $j$. The algebraic tensor product $\mathcal{H} = \mathcal{H}_{\ell,r} \otimes \mathcal{H}_{m,s}$ can be expanded as a direct sum

$$\mathcal{H} = \mathcal{H}_{\ell,r} \otimes \mathcal{H}_{m,s} = \bigoplus \mathcal{H}_{m+\ell,t} \otimes M_t,$$
where $t - r - s$ is an integer and $M_t$ is a multiplicity space that can be identified with $\text{Hom}_{\mathbb{Z}}^0(\mathcal{H}_{m+t}, \mathcal{H})$. Now if $T \in \text{Hom}_{\mathbb{Z}}^0(\mathcal{H}_{m+t}, \mathcal{H})$ so too is $L_nT - TL_n$ where the action on the right is by the Segal–Sugawara construction on $\mathcal{H}_{m+t}$ and the action on the left on $\mathcal{H} = \mathcal{H}_{t,r} \otimes \mathcal{H}_{m,s}$ is given the tensor product of the Segal–Sugawara constructions on $\mathcal{H}_{t,r}$ and $\mathcal{H}_{m,s}$. Hence the operator defined on the tensor product as $\pi_t(L_n) \otimes I + I \otimes \pi_m(L_n) - \pi_{t+m}(L_n)$ acts on the multiplicity space $M_t$. This is the Goddard–Kent–Olive construction. It gives a representation of the Virasoro algebra with central charge $c = c_t + c_m - c_{t+m}$. In particular when $m = 1$, we have a decomposition

$$\mathcal{H} = \mathcal{H}_{t,r} \otimes \mathcal{H}_{1,s} = \mathcal{H}_{t+1,t} \otimes M_t,$$

with $s = 0$ or $1/2$, $r - t \in s + \mathbb{Z}$ and

$$c = \frac{3\ell}{\ell + 2} + 1 - \frac{3(\ell + 1)\ell}{\ell + 3} = 1 - \frac{6}{(\ell + 2)(\ell + 3)}.$$

The character of the multiplicity space is determined directly by the Weyl–Kac characters at level $1$, $\ell$ and $\ell + 1$ and this gives the lowest eigenvalue of $L_0$ in $M_t$, which turns out to be $h = h_{p,q}(m)$ with $p = 2r + 1$ and $q = 2t + 1$. The representation $L(c, h)$ therefore occurs as a summand of $M_t$. This shows that for this value of $c$, all the $h$ appearing in the list of Friedan–Qiu–Shenker are indeed unitary. To complete the proof we just have to show that $M_t$ is irreducible.

This is accomplished by invoking the easy part of the Friedan–Qiu–Shenker theorem. Indeed as observed in [36], their proof of the classification theorem splits naturally into two parts. The first part shows that for $c$ fixed at $1 - 6/m(m + 1)$, the values of $h$ have to be one of those in their list. The proof requires the Kac determinant formula for $M = M(c, h)$ (see [22]). This computes the determinant of $(v_i, v_j)$ where the $v_i$’s run across the canonical monomial basis of $M(n)$. If $c$ is fixed, this is a constant times a polynomial in $h$. On the other hand, since $\chi_{M_t}$ gives an upper bound for $\chi_{L(c, h)}$, it follows that $M = M(c, h)$ has a singular vector in $M(pq)$. The Kac determinant for $M(n)$ therefore must vanish when $n \geq pq$. This implies that $h - h_{p,q}(c)$ divides the Kac determinant if $m$ is given by $c = 1 - 6/m(m + 1)$. By the structure of the Verma modules, it occurs with multiplicity at least $P(n - pq)$ where $P(n)$ is the partition function. Thus the Kac determinant is divisible by

$$\prod_{p,q:1 \leq pq \leq n} (h - h_{p,q}(c))^{P(n-pq)}.$$

Since it can be checked that this product has the same degree as the Kac determinant and they agree up to a constant in the highest power, the product above is proportional to the Kac determinant. We will use the same strategy of proof to establish a similar product formula, due to Feigin and Fuchs, for a polynomial arising elsewhere in the representation theory of the Virasoro algebra.

Finally to prove that $M_t$ is irreducible, on the one hand the Kac determinant formula gives a lower bound for the character of $\mathcal{H} = L(c, h)$. It follows that the $M_t(n)$ agrees with $\mathcal{H}(n)$ for $n < m(m + 1) - (m + 1)p + mq = M$. On the other hand if $M_t$ is reducible it is a sum of representation $L(c, h')$ from the list of Friedan–Qiu–Shenker. But it is elementary to verify that $h' < M + h$ for all such $h'$. It follows that $M_t$ must be irreducible.

The proof of the character formula for $c = 1$ and $h = j^2$ proceeds similarly by indentifying a multiplicity space on which the Virasoro algebra acts and proving that it is irreducible. Graeme Segal [32] gave a direct proof of the first case when $h = 0$. Our proof is different in this case. It avoids the Segal–Sugawara construction and the Kac determinant formula, relying instead on various well known aspects of the fermion–boson correspondence as tools, including elementary parts of the theory of vertex algebras (see [6], [12], [14], [24]).

Indeed in this case a single complex fermion field is given by a set of operators $(e_n)$, $d$ acting on an inner product space $\mathcal{H}$ subject to the anticommutation relations

$$e_m e_n + e_n e_m = 0, \quad e_m e_n + e_n e_m = \delta_{m+n,0} \cdot I, \quad d = d^*, \quad [d, e_n] = -(n + \frac{1}{2}) e_n.$$

Slightly relaxing the positive energy condition, we require that $\mathcal{H}$ has a decomposition $\mathcal{H} = \oplus \mathcal{H}(r)$ where $r$ runs over non–negative half integers with $\mathcal{H}(r) = \{ \xi : d\xi = r\xi \}$ finite–dimensional. There is essentially only
one such irreducible representation on an inner product space $F_f$ generated by a vacuum vector $\Omega \in F_f(0)$. $F_f$ is called fermionic Fock space and can be identified with an exterior algebra.

A single boson field is formed by operators $(a_n)$, $d$ acting on an inner product space $K$ and satisfying the commutation relations

$$a_m a_n - a_n a_m = m \delta m + n, 0 \cdot I, \quad a_n^* = a_{-n}, \quad d = d^*, \quad [d, a_n] = -na_n.$$  

Again there is an essentially unique irreducible positive energy representation, this time with non-negative integer eigenvalues for $d$. It is generated by $\Omega \in K(0)$ satisfying $a_0 \Omega = \rho \Omega$. A unitary charge operator $U$ can be added to this bosonic system satisfying the additional relations

$$Ua_n U^* = a_n + \delta_{n,0} I, \quad UdU^* = d + a_0 + \frac{1}{2} I.$$  

The “discrete” system $U$, $a_0$ and $d$ has an essentially unique positive energy representation with $d$ having non-negative half integer eigenvalues. From this it follows that the charged boson system $(a_n, U, d)$ has an essentially unique positive energy representation on an inner product space $F_b$, obtained as a tensor product. $F_b$ is called bosonic Fock space and can be identified with a symmetric algebra tensored with the algebraic group algebra of $\mathbb{Z}$.

Boson–fermion duality is the statement that there is a natural unitary isomorphism between $F_f$ and $F_b$ compatible with the operators, carrying $\Omega_f$ onto $\Omega_b$. Identifying the spaces, the compatibility conditions state that $d$ should be preserved and that

$$[a_n, e_m] = e_{n+m}, \quad U e_n U^* = e_{n+1}.$$  

On the other hand it is straightforward to see that there are essentially unique operators on $F_f$ satisfying these relations and that they act irreducibly. In fact $U$ arises as an explicit shift operator on $F_f$ and the $a_n$’s can be written as linear combination of operators of the form $e_i e_j^*$. These operators act irreducibly on $F_f$, so in this sense charged bosons can be constructed from fermions. To proceed in the other direction, i.e. to construct fermions from charged bosons, requires the introduction of vertex operators. For $m \in \mathbb{Z}$, these are defined as formal power series in $z$ and $z^{-1}$ by the formula

$$\Phi_m(z) = U^m z^{-m_0} \exp \left( \sum_{n<0} \frac{m z^n a_n}{n} \right) \exp \left( \sum_{n>0} \frac{m z^n a_n}{n} \right).$$  

The fundamental identities are then

$$\Phi_1(z) = \sum e_n z^{-n-1}, \quad \Phi_{-1}(z) = \sum e_{-n}^* z^{-n}.$$  

Identifying $F_f$ and $F_b$, which we write simply as $F$, there are Virasoro operators $L_n$ with $c = 1$ satisfying the covariance relations

$$[L_n, e_m] = -(m + \frac{1}{2}(m + 1))e_{m+n}, \quad [L_n, a_m] = -ma_{m+n}. \quad (3)$$  

The operators $L_n$ can be written either as linear combination of operators of the form $e_i e_j^*$ or of the form $a_i a_j$. The identity (3) is a special case of the more general Fubini–Veneziano relation

$$[L_n, \Phi_m(z)] = z^{n+1} \Phi'_m(z) + \frac{m^2}{2} (n+1) z^n \Phi_m(z). \quad (4)$$  

Note that the operators $L_m$ commute with $a_0$ and so leave invariant its eigenspaces. The action on the non–zero eigenspaces turns out to be irreducible. The action on the zero eigenspace, however, is not irreducible. We will see that it is a direct sum of representations $L(1, m^2)$ with $m$ a non–negative integer. Similarly the Virasoro algebra acts on the irreducible representation of $(a_n)$, $d$ with $a_0 = \frac{1}{2} I$, which it will turn out decomposes as a direct sum of representations $L(1, (m + \frac{1}{2})^2)$. 

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A natural way to understand and study these decompositions is by passing to $\mathcal{F}^{\otimes 2} = \mathcal{F} \otimes \mathcal{F}$. The description of $\mathcal{F}$ as an exterior algebra shows that there is a natural action on $\mathcal{F}^{\otimes 2}$ of the group $SU(2)$ and thus its Lie algebra. This action commutes with the operator $d \otimes I + I \otimes d$. Conjugating the action of $a_n \otimes I$ by elements of $SU(2)$ then leads to operators $E_{ij}(n)$ with $E_{11}(n) = a_n$ satisfying $E_{ij}(n)^* = E_{ji}(-n)$ and

$$[X(m), Y(n)] = [X, Y](n + m) + m\delta_{m+n,0} \text{Tr} XY,$$

if $X = \sum x_{ij} E_{ij}$ and $Y = \sum y_{ij} E_{ij}$. In particular taking $X \in \mathfrak{sl}_2$, the matrices with zero trace, we get a positive energy representation of the Kac–Moody Lie algebra at level 1. These operators commute with the operators $A_n = \frac{1}{2}(a_n \otimes I + I \otimes a_n)$. In particular the $A_n$’s commute with the operators $B_m = H(m), E(m)$ and $F(m)$ where $H = \frac{1}{4}(E_{11} - E_{22}), E = E_{12}$ and $F = E_{21}$ is the standard basis of $\mathfrak{sl}_2$. These operators are related to the vertex operators by the formulas

$$\Phi_1(z) \otimes \Phi_{-1}(z) = \sum E(n)z^{-n-1}, \quad \Phi_{-1}(z) \otimes \Phi_1(z) = \sum F(n)z^{-n-1},$$

identities originally due to Frenkel–Kac [10] and Segal [32].

The space $\mathcal{F}^{\otimes 2}$ decomposes as a sum of two components

$$\mathcal{F}^{\otimes 2} = \mathcal{H}_0 \otimes M_1 \oplus M_1 \otimes \mathcal{H}_1/2 \otimes M_{1/2},$$

with the affine Kac–Moody algebra acting irreducibly on $\mathcal{H}_j$ and the bosonic operator irreducibly on the multiplicity spaces $M_j$. The action of the Virasoro algebra on $\mathcal{F}^{\otimes 2}$ preserves this decomposition. Its actions on the tensor factors coincides with the natural action associated with $(B_n)$ and $(A_n)$. The action on $H_j$ commutes with the action of $SU(2)$ and satisfies $[L_n, X(m)] = -mX(n+m)$. Each space $\mathcal{H}_j$ may be further decomposed according to the action of $SU(2)$

$$\mathcal{H}_j = \bigoplus_{i \geq 0, i-j \in \mathbb{Z}} V_i \otimes \mathcal{K}_{ij}.$$

The operators $L_n$ commute with $SU(2)$, so preserve the multiplicity spaces $\mathcal{K}_{ij}$, each of which contains a copy of $L(1, i^2)$ generated by a singular vector in the lowest energy space. This action on the multiplicity spaces can be regarded as a limiting case of the coset construction of Goddard–Kent–Olive. The singular vectors were described explicitly by Goldstone [21] and we shall refer to them as Goldstone vectors. Goldstone’s formulas were shown by Segal [32] to be a direct consequence of the vertex operator formulas in (5). Although the formulas can be developed without any advanced theory of symmetric functions [27, 29], the simplest way to describe them involves the combinatorics of the Weyl character formula for $U(n)$ [40]. In fact setting $X_n = a_{-n}$ and regarding these as the symmetric functions with signature $(n, 0, 0, 0, \ldots)$, the Goldstone vectors correspond to the symmetric functions $X_q$ with signature $(k + m, k + m, \ldots, k + m, 0, 0, \ldots)$, where $k + m$ appears $k$ times applied to a singular vector for the $a_n$ with $n > 0$. (If we took $X'_n = \sum z^n$, then $X'_q$ would just be $z^{f_+ n-i}/z^{f_+ n-i}$, the character of the irreducible representation $V_q$ of $U(n)$ with signature $f$.) Goldstone conjectured that these vectors were the only singular vectors in $\mathcal{H}_j$ so that consequently the Virasoro algebra acted irreducibly on the multiplicity spaces. Since $\text{ch} \mathcal{K}_{ij} = (q^{i^2} - q^{(i+1)^2}) \cdot \varphi(q)$, this establishes the character formulas for $L(1, i^2)$.

To prove this directly, it suffices to show that in $\mathcal{H}_j$ the Goldstone vectors are the only singular vectors in a fixed eigenspace $\mathcal{K}$ of $H(0)$. It is easy to check that $\mathcal{K}$ is an irreducible representation of the $H(n)$’s and that, up to scalar multiples, there is at most one singular vector at any fixed energy level in $\mathcal{K}$. Moreover as module over $\text{vir}$, $\mathcal{K}$ is a direct sum of irreducible representations, with one component for each singular vector. If there were a component $\mathcal{K}'$ isomorphic to $L(1, p)$ with $p$ not of the form $(j + k^2)$, then by irreducibility, for some component $\mathcal{K}''$ generated by a Goldstone vector, $H(a)\mathcal{K}''$ would have to have a non–zero projection on $\mathcal{K}'$. But if $P'$ and $P''$ denotes the orthogonal projections onto $\mathcal{K}'$ and $\mathcal{K}''$, then

$$\Psi(z) = P'H(z)P''$$

would give a formal power series of operators from $\mathcal{K}''$ to $\mathcal{K}'$ satisfying

$$[L_k, \Psi(z)] = z^{k+1} d\Psi'(z).$$

(6)
Such family of operators is a special case of what is called a primary field. Writing \( \Psi(z) = \sum \psi(n)z^{-n} \), (6) can be rewritten

\[
[L_k, \psi(n)] = -n\psi(n + k).
\]

This prompts the introduction of the density modules \( V_{\lambda, \mu} \) giving the natural representations of the Witt algebra on expressions of the form \( f(\theta)e^{i\mu\theta}(d\theta)^\lambda \), with \( f \) a trigonometric polynomial. Multiplication and integration over the circle gives a natural pairing with \( V_{1 - \lambda, -\mu} \). A primary field, for appropriate \( \lambda \) and \( \mu \), then defined to be a linear map \( \mathcal{K}' \otimes V_{\lambda, \mu} \rightarrow \mathcal{K}' \) commuting with the action of the Virasoro algebra. By duality it is the same as an equivariant map \( \mathcal{K}' \rightarrow \mathcal{K}' \otimes V_{\lambda', \mu'} \), where \( \lambda' = 1 - \lambda \) and \( \mu' = -\mu \). The formula (6) is then replaced by the more general relation

\[
[L_k, \Psi(z)] = z^{k+1}d\Psi'(z) + \lambda z^k\Psi(z).
\]

Returning to (6) and (7), the action there is simply on functions, so that \( \lambda = 0 = \mu \). But then \( V_{0,0} = V_{1,0} \), the space of differentials \( f(\theta)d\theta \). On the other hand if \( \mathcal{K}' \) is isomorphic to \( L(1, i^2) \) then it is the quotient of a Verma module \( M(1, i^2) \) by a submodule containing at least one singular vector \( w \) of energy \((i + 1)^2\). If \( v \) is the lowest energy vector of \( M(1, i^2) \), then \( w \) has the form \( \Lambda v \) for some non–commuting polynomial in \( L_{-k} \). Taking the component of the lowest energy vector of \( K_1 \), it is then easy to see that a necessary condition for the existence of a primary field \( \Psi(z) \) is that the action of \( P \) on one of the standard basis vectors of \( V_{\lambda', \mu'} \) must be zero. But the action of the Witt algebra is given in this basis as shift operators weighted by polynomials, so the condition is equivalent to the vanishing of a polynomial. A formula for this polynomial was given without proof by Feigin–Fuchs [8]. In this particular case where \( c = 1 \), sufficiently many factors of the polynomial can be produced by explicitly exhibiting enough primary fields to determine completely the polynomial. (These are constructed as operators between different \( L(1, i^2) \) by compressing \( \Phi_{\lambda}(z) \otimes \Phi_{\mu}(z) \).) Irreducibility then follows by noting that this polynomial does not vanish for the hypothetical field \( \Psi(z) \) constructed above.

The proof that there are sufficiently many primary fields is essentially the first step in showing that under fusion the representations \( L(1, i^2) \) for \( i \) a non–negative half integer behave exactly like the irreducible representations \( V_i \) of \( SU(2) \) under tensor product (see [38]). Now if \( \Psi(z) \) is a primary field and \( L_{-1}w = 0 \), we see that \( F(z) = \Psi(z)w \) satisfies the so–called “equation of motion”

\[
\frac{dF}{dz} = L_{-1}F,
\]

so that

\[
F(z) = e^{zL_{-1}}\Psi(0)\Omega.
\]

This identity allows the existence of sufficiently many primary fields to be reduced to checking that \( L_1^{|f|}X_Fv \neq 0 \) where \( v \) is a singular vector of the \( H(n) \)'s for \( n > 0 \) with \( H(0)v = pv \) with \( p \) a positive integer. But it is easy to see that \( (|f|!)^{-1}L_1^{|f|}X_Fv = av \) for some constant \( a \). Like \( X_F \), the value of \( a \) is given by a determinant, but this time of binomial coefficients. It can be calculated as an explicit product using either the Weyl dimension formula or an elementary matrix computation. In particular, in the case of interest, it does not vanish: indeed remarkably \( a \) equals the dimension of the irreducible representation \( V_f \) of \( U(p) \) when this makes sense and vanishes otherwise. This can be summarised by saying that, if the Goldstone vectors are given formally by formulas relating to the Weyl character formula, then their non–degeneracy properties follow from the corresponding Weyl dimension formula.

Sections 2 to 8 give a self–contained step–by–step account of this method of establishing the character formula for \( L(1, j^2) \). Appendix A contains a alternative direct verification of the Fubini–Veneziano relations for vertex operators. In Appendices B and C, we give an alternative approach to the product formula of Feigin–Fuchs using an elegant algorithm of Bauer, Di Francesco, Itzykson and Zuber ([3], [7]) for determining the polynomial \( P \), related to an earlier formula of Benoit and St Aubin [4]: we give a simplification of their treatment in Appendix A. The most general Feigin–Fuchs formula can be proved using the coset construction for primary fields for the unitary discrete series with \( 0 < c < 1 \). Indeed the primary fields for the affine Kac–Moody algebra \( \mathfrak{sl}_2 \) can be constructed explicitly and then used to construct primary fields for the Virasoro algebra [28], [39]. In this case the irreducible representations for \( c = 1 - 6/m(m + 1) \) with \( m \) large are
indexed by non-negative half integers \((i, j)\) with \(i - j\) and integer. The corresponding representation has \(h = h_{p,q}(m)\) with \(p = 2i + 1\) and \(q = 2j + 1\). The singular vector at level \(pq\) acts on the density module as a multiple of \(\prod (h - h_{r,s}(m))\) where \(V_i \otimes V_j = \bigoplus V_{l_r}\) and \(V_j \otimes V_i = \bigoplus V_{l_j}\). Other methods of proving the Feigin–Fuchs formula are given in [7] and [13].

In Section 9 we describe a simple method for determining the character formula for \(L(1, j^2)\) using only the Kac determinant formula and the Jantzen filtration [20]. It is generalised in Section 10 to the case \(0 < c < 1\) and gives a different method of proof to that of Astashkevich [1]. Let \(A(x) = \sum_{i \geq 0} A_i x^i\) be an analytic family of non-negative self-adjoint matrices defined for \(x\) real and small on \(V = \mathbb{R}^n\) or \(\mathbb{C}^n\). We define a filtration \(V(i) \ (i \geq 0)\) by \(V(0) = V\) and for \(m \geq 1\)

\[
V(m) = \bigcap_{i=0}^{m-1} \ker A_i.
\]

In the examples \(A(x)\) will be invertible for \(x \neq 0\) in a neighbourhood of 0. It follows that \(V(m) = (0)\) for \(m\) sufficiently large. The fundamental identity, which is easy to prove, is that the order of 0 as a root of \(\det A(x)\) equals

\[
\sum_{i \geq 1} \dim V(i).
\]

Taking the standard inner product on \(V\), we can define a Hermitian form on \(V\) by \((v_1, v_2)_x = (A(x)v_1, v_2)\). This filtration has an important functorial property. Suppose we are given two such space \((V, A(x))\) and \((W, B(x))\) and in addition analytic maps \(X(x) : V \to W, Y(x) : W \to V\) such that

\[
(X(x)v, w)_x = (v, Y(x)w)_x.
\]

It then follows easily that \(X(0)\) and \(Y(0)\) induce maps between \(V(i)\) and \(W(i)\) preserving the canonical Hermitian forms induced by \(A_i\) and \(B_i\).

Now let \(M = M(1, 1, j^2)\). Then \(V = M(n)\) is independent of \(x\) with a canonical basis, so can be identified with \(\mathbb{C}^P(n)\). We consider the Jantzen filtration for \(M = M(1, j^2)\) associated with the parameter \(x\) in \(c = 1 + x\). The last property above implies that each \(M(i) \equiv \oplus M(n)\) is a submodule for the Virasoro algebra and comes with an invariant Hermitian form. By the Kac determinant formula we can explicitly compute \(\sum_{i \geq 1} \dim M(i)\). Now the determinant formula or the coset construction above show that \(M(1, j^2)\) has a filtration by Verma modules \(M(1, (j + k)^2)\) for \(k \geq 1\). Let \(M(n_k)\) be the last term in the Jantzen filtration containing \(M(1, (j + k)^2)\); it turns out that \((n_k)\) is strictly increasing (we set \(n_0 = 0\)). But a lower bound for the character sum above is given by \(\sum_{k \geq 1} \chi M(1, (j + k)^2)\) and hence \(\sum_{k \geq 1} \chi M(1, (j + k)^2)\). However this sum actually equals the character sum. Hence \(M(k) = M(1, (j + k)^2)\) and hence \(L(1, j^2) = \varphi(q) \cdot (q^2 - q^{(j+1)^2}).\)

The proof for the discrete series is similar but slightly but more intricate. Let \(c = 1 - 6/m(m+1)\) and \(M = M(c(m), h_{p,q}(m))\) with \(1 \leq q \leq p \leq m - 1\). Again the Kac determinant formula produces a sequence of singular vectors in \(M\) which generate a series of Verma submodules \(A_i, B_i\) for \(i \geq 1\) such that \(A_i\) and \(B_j\) both contain \(A_k\) and \(B_k\) for \(k > i\). Thus \(A_k \cap B_k \supseteq A_{k+1} + B_{k+1}\). Again, this time using \(x = h - h_{p,q}(m)\) as a parameter, we can compute the character sum \(\sum_{i \geq 1} \chi M^{(i)}\) as well as that for \(A_1\) corresponding to the singular vector with energy \(h + pq\). Analogous inequalities to those above prove that \(A_k \cap B_k = A_{k+1} + B_{k+1}\) for \(k \geq 1\) and that \(M(k) = A_k + B_k\) for \(k \geq 1\). Using the isomorphisms \(A_k \oplus B_k/(A_{k+1} + B_{k+1}) = A_k + B_k\), the character formula for \(L(c(m), h_{p,q}(m))\) follows.

Although we have not checked this, it seems likely that variants of the two proofs above can be used to simplify substantially the proof of Feigin and Fuchs. Such a simplification had been given by Astashkevich [1] for all cases except \(\text{III}^0\). The method of Feigin–Fuchs in this case was considerably more complicated than what is required for the discrete series \(c < 1\) (the case \(\text{III}^-\)) and invokes the Riemann–Roch theorem for holomorphic vector bundles on the sphere.

We end this overview by briefly describing the Feigin and Fuchs’ original proof of the character formula for \(L(1, j^2)\) in [8]—their case \(\text{III}^0\). In addition to a knowledge of the discrete series characters and their
Jantzen filtrations, their proof requires the fact that the formula for the singular vector is a polynomial in $t$ of degree $2j + 1$, with constant term $L_{2j+1}^{2j+1}$ and leading coefficient $[(2j)!!2L_{-2j-1}$-1]. This was proved indirectly by Astashkevich and Fuchs [2] for $M(1,j^2)$. In this case we show in Appendix A that it is an easy consequence of the formula of Bauer, Di Francesco, Itzykson and Zuber [3] for the singular vector of $M(b_{2j+1,1}(t), 13 - 6t - 6t^{-1})$.

Instead of the Riemann–Roch theorem, all that is needed is the classical theory of holomorphic vector bundles on the Riemann sphere. Recall that according to the Grothendieck–Birkhoff theorem, any such bundle can be written uniquely as a sum of line bundles, each classified by their degree (the sum with multiplicities of the degrees of poles and zeros of any meromorphic section). Grothendieck’s original proof [19] was algebraic–geometric using the language of divisors, although it was soon realized that the theorem followed from the Birkhoff factorization theorem (see [16], [31], [38]). Indeed it is known [9] that a rank $n$ holomorphic vector bundle is trivial when restricted to either of the open discs $z < R$ and $|z| > r$ with $r < 1 < R$ (see Appendix D for a short analytic proof). But then it is specified by a holomorphic map $f$ from the annulus $r < |z| < R$ into $GL_n(\mathbb{C})$. By Birkhoff’s factorization theorem the restriction of $f$ to the circle $|z| = 1$ can be written $f(z) = f_-(z)D(z)f_+(z)$ where $D(z)$ is a matrix with entries $z^n$ on the diagonal, $f_\pm(z)$ is a function holomorphic on $D_\pm(z) = \{z : |z|^{\pm 1} < 1\}$ and smooth on its closure. The matrix $D(z)$ is uniquely determined up to the order of its entries. Since $f$ extends to $r < |z| < R$ it follows immediately from the Schwarz reflection principle that $f_+(z)$ and $f_-(z)$ extend to holomorphic function on $|z| < R$ and $|z| > r$. But then the clutching function on the annulus can be replaced by $D(z)$ which corresponds to a direct sum of holomorphic line bundles. (Each of these can in turn be understood in terms of holomorphic sections of a homogeneous line bundle for the group $G = SU(2)$, identifying the Riemann sphere $S$ with $G/T$, where $T$ is the subgroup of diagonal matrices.)

Given a representation such as $L(1,j^2)$, the method of Feigin and Fuchs proceeds as follows. Let $r = 2j + 1$. At each point of the curve $C$ given by $\varphi_{r,1}(c,h) = 0$, there is a singular vector of energy $2j + 1$, in the Verma module $M = M(c, h)$. The action of the operators $L_n$ for $n > 0$ is independent of $c$ and $h$ so the Verma module can be identified along the curve. The curve on the other hand can be parametrised by $c = 13 - 6t - 6t^{-1}$ and $h = (j^2 + j)t - j$ with $t \in \mathbb{C}^*$. The singular vector $w(t) \in M(2j + 1)$ depends polynomially on $t$ (see Appendix B) and thus defines a line bundle on the Riemann sphere: its degree $d$ is determined by the degree of the polynomial, since the constant term is non–vanishing. Let $W_t$ be the Verma submodule of $M$ generated by $w(t)$. At any fixed energy level $k$ it is a sum of line bundles of degree $d$. So $M(k)/W_t(k)$ defines a holomorphic vector bundle $E$ over the Riemann sphere, which is a sum of line bundles with the sum of the degrees determined. But if the rank of $E$ is $m$, $F = \lambda^m E$ is a line bundle of known degree. Now the Kac determinant formula implies that generically $M/W_t$ is irreducible on the curve $\varphi_{r,1}(c,h) = 0$. The Kac determinant in the quotient produces a meromorphic section of $F^*$, the line bundle dual to $F$. The degree of $F^*$ is known. It can also be computed in terms of the order of poles and zeros of the meromorphic section. The contributions from $0$ and $\infty$ can be computed directly, since there are two explicit bases independent of $t$ in $V$ complementary to $W_t$ for $t \neq 0, \infty$. The computation reduces to that of a principal minor where the diagonal terms dominate. Only real points given by the intersection of the curve with other curves $\varphi_{p,q}(c,h) = 0$ with $pq \leq N$ give other contributions. For $c \neq 1$, these correspond to Verma modules of type III. Having the same structure of singular vectors as the discrete series with $0 < c < 1$. These will have a singular vector of type $(r, 1)$ amongst the other singular vectors (possibly of lower energy). Since the structure of these modules is known explicitly, the Jantzen filtration on $M/W_t$ corresponding to the (real part of the) curve $C$ can be computed and hence the contribution from these points. Adding up these contributions, it then follows that the contribution from $(1,j^2)$ is 0, as required.

2. **Constructions of positive energy representations.** We define the bosonic algebra $\mathfrak{a}$ to be the Lie algebra with basis $a_n$ ($n \in \mathbb{Z}$), $c$ and $d$, with non–zero brackets

$$[a_m, a_n] = m\delta_{m+n,0} \cdot c, \quad [d, a_n] = -na_n.$$ (Thus $c$ is central.) A unitary positive energy representation of $\mathfrak{a}$ consists of an inner product space $\mathcal{H}$ which can be written as the orthogonal algebraic direct sum of finite–dimensional spaces $\mathcal{H}(n)$ ($n \geq 0$)

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}(n)$$
on which \( \mathfrak{a} \) acts such that \( d\xi = (n + h)\xi \) for \( \xi \in \mathcal{H}(n) \), with \( h \in \mathbb{R} \), \( c = I \) and \( a^* = a_{-n} \). \( d \) is called the energy operator and an element of \( \mathcal{H}(n) \) is said to have energy \( h + n \). Note that replacing \( d \) by \( d + tI \) gives another essentially equivalent representation.

Note that if an operator \( a \) satisfies the canonical commutation relations \( a^*a - aa^* = \mu I \) and \( a^*v = 0 \), then an easy induction argument shows that

\[
(a^m v, a^n v) = \delta_{m,n} m! \mu^m. \tag{1}
\]

Now let \( V = \mathbb{C}[x_1, x_2, \ldots] \) and for \( n > 0 \) define operators \( a_{-n} p = x_n p \), \( a_n p = n\partial_{x_n} p \) and \( a_0 p = \lambda p \). We define \( \text{deg} x_n = n \) and let \( D \) be the operator which multiplies a monomial by its degree, so that

\[
D = \sum_{n \geq 1} n \cdot x_n \partial_{x_n}. \tag{2}
\]

This gives a representation of the Lie algebra \( \mathfrak{a} \). Since \( p(x_1, x_2, \ldots) = p(a_1, a_2, \ldots)1 \), repeated applications of (1) show that it is unitary for the inner product making monomials orthogonal with

\[
||x_1^m x_2^{m_2} \cdots||^2 = \prod m_n ! n^{m_n}.
\]

It is irreducible because any non-trivial \( \mathfrak{a} \)-submodule must contain a non-zero vector \( p \) of lowest energy. Thus \( a_n p = 0 \) for all \( n > 0 \), i.e. \( \partial_{x_n} p = 0 \) for all \( n > 0 \). But then \( p \) is a constant and the submodule must be the whole of \( V \). Conversely if \( W \) is any positive energy unitary representation, then by (1) any lowest energy vector unit \( w \) will satisfy \( (p(a_{-1}, a_{-2}, \ldots)w, q(a_{-1}, a_{-2}, \ldots)w) = (p, q) \), the inner product on \( V \) constructed above. Let \( W_1 \) be the \( \mathfrak{a} \)-module generated by \( w \). Adjusting \( a_0 \) and \( d \) by constants if necessary, it follows from (1) that \( W_1 \) is unitarily equivalent to \( V \) by the map \( Up = p(a)w \). The orthogonal complement of \( W_1 \) is also a positive energy unitary representation so likewise contains an irreducible submodule \( W_2 \) of the same form. Continuing in this way, the positive energy condition shows that we can write \( W = \bigoplus_{m \geq 1} W_m \), an orthogonal direct sum where each \( W_m \) is isomorphic to \( V \) but with \( a_0 \) acting as \( \lambda_m I \) and \( d \) as \( D + h_m I \). Thus \( \lambda_m \) and \( h_m \) are the eigenvalues of \( a_0 \) and \( d \) on the lowest energy vector \( w_m \): clearly this determines the representation uniquely. Representations of the Virasoro algebra can be define on these spaces by the generalising the formula (1) as follows:

\[
L_0 = \frac{1}{2}a_0^2 + \sum a_{-n}^a a_n, \quad L_n = \frac{1}{2} \sum_{r+s=n} a_r a_s (n \neq 0).
\]

Note that in the second formula the operators \( a_r \) and \( a_s \) commute. By definition \( L_n \) changes energy by \( -n \). It is easy to check that \( [L_n, a_m] = -ma_{m+n} \) and hence that \( A_{m,n} = [L_m, L_n] - (m - n)L_{m+n} \) commutes with all the \( a_r^s \). It leaves each \( W_k \) invariant. By uniqueness of the lowest energy vector in \( W_k \), it must act as a scalar \( \nu_{m,n} \) and therefore preserve energy. Thus \( A_{n,m} = 0 \) unless \( n = -m \). Otherwise for \( n > 0 \)

\[
([L_n, L_{-n}] - 2nL_0)w_k, w_k) = -n(a_0^2 w_k, w_k) + \frac{1}{2} (L_n \sum_{r+s=-n} a_r a_s w_k, w_k)
\]

\[
= -n(a_0^2 w_k, w_k) + \frac{1}{2} \sum_{r+s=-n} -r(a_{n+r} a_s w_k, w_k) - s(a_r a_{n+s} w_k, w_k)
\]

\[
= \frac{1}{2} \sum_{t=1}^n t(n - t)
\]

\[
= (n^3 - n)/6.
\]

Thus

\[
[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} I.
\]

We now make similar constructions using the canonical anticommutation relations. In this case the notion of positive energy representation has to be slightly generalised to allow the eigenvalues of the energy
operator to have the form \( n + h \) with \( n \in \frac{1}{2}\mathbb{Z} \). We define \( V \) to be the inner product space spanned by the mutually orthogonal unit vectors

\[
e_i \land e_{i_2} \land e_{i_3} \land \cdots,
\]

with \( i_k \in \mathbb{Z} \), \( i_1 < i_2 < i_3 < \cdots \) and \( i_{k+1} = i_k + 1 \) for \( k \) sufficient large. We define the vacuum vector by

\[
\Omega = e_0 \land e_1 \land e_2 \land \cdots
\]

and make the \( e_i \)'s act by exterior multiplication. It is easily checked that the \( e_j^* \)'s act by interior multiplication and that

\[
e_m e_n^* + e_n e_m^* = \delta_m n I, \quad e_m e_n + e_n e_m = 0.
\]

The space \( V \) can be canonically identified with the exterior algebra in the anticommuting variables \( e_{-n} \) for \( n > 0 \) and \( e_n^* \) for \( n \geq 0 \). We can define the energy and charge of a basis vector in (3) as follows. Firstly it can be regarded as a monomial in the exterior algebra in the \( e_{-n} \)'s \((n > 0)\) and the \( e_n^* \)'s \((n \geq 0)\). A term \( e_{-n} \) or \( e_n^* \) contributes \( n - 1/2 \) to the energy and \(+1\) and \(-1\) to the charge. The vector

\[
\Omega = \Omega_0 = e_0 \land e_1 \land e_2 \land \cdots
\]

has energy 0 and charge 0. Let \( d \) and \( a \) be the operators multiplying the basis vectors by their energy and charge respectively. Thus \( a^* = a \) and \( d^* = d \). Moreover \( [d, e_i] = -(i + 1/2) e_i \) and \( [a, e_i] = e_i \). Taking adjoints we get \( [d, e_i^*] = (i + 1/2) e_i^* \) and \( [a, e_i^*] = -e_i^* \). Alternatively a basis vector can be written

\[
v = e_{m_1} \land \cdots \land e_{m_j} \land e_{j+k} \land e_{j+k+1} \land \cdots,
\]

where \( m_1 < m_2 < \cdots < m_j < j + k \). Thus \( m_i < k + i \). The vector will have charge \(-k\) and energy \( \sum_{i=1}^{j} (k + 1/2 - i - m_i) \). Indeed the vector

\[
\Omega_k = e_k \land e_{k+1} \land \cdots
\]

has charge \(-k\) and energy \( k^2/2 \). The operators \( e_n e_n^* \) commute with \( a \) so preserve charge. Since \( v \) is obtained from \( \Omega_k \) by successively applying the operators \( e_m e_{k-i}^* \), which raises the energy by \( k - i - m_i \), it follows that \( v \) has energy \( \sum_{i=1}^{j} (k - i - m_i) \). We define

\[
a_k = \sum_{n-m=k} e_n e_m^*, \quad a_0 = \sum_{n \geq 0} e_{-n} e_n^* - \sum_{n > 0} e_n^* e_n
\]

and

\[
L_k' = \sum_{n-m=k} -(m + 1/2 + k/2) e_n e_m^*, \quad L_0' = \sum_{n > 0} (n - 1/2) e_{-n} e_n^* + \sum_{n \geq 0} (n + 1/2) e_n^* e_n.
\]

It is straightforward to check that \( a_n^* = a_{-n} \), \( (L_n')^* = L_{-n}' \) and

\[
[L_n', e_m] = -(m + n/2 + 1/2)e_{m+n}, \quad [L_n', e_m^*] = (m - n/2 - 1/2)e_{m-n}, \quad [a_n, e_m] = e_{n+m}, \quad [a_n, e_m^*] = -e_{n-m}^*.
\]

The operators \( d - L_0' \) and \( a - a_0 \) commute with the \( e_n \)'s and \( e_n^* \)'s and annihilate \( \Omega \). By cyclicity of \( \Omega \), it follows that \( d = L_0' \) and \( a = a_0 \). The operator \([a_n, a_m]\) commute with the \( e_k \)'s and \( e_k^* \)'s and raise energy by \(-n - m \). If \( n + m < 0 \) it must annihilate \( \Omega \) and hence vanish. Since \([a_n, a_m]^* = [a_{-m}, a_{-n}]. \) it follows that

\[
[a_n, a_m] = 0 \text{ if } n + m \neq 0.
\]

Now up to a scalar multiple \( \Omega \) is the unique vector of energy 0, so that if \( n > 0 \),

\[
[a_n, a_{-n}] \Omega = \lambda_n \Omega, \quad [a_n, a_{-n}] = \lambda_n \delta_{m+n,0} I.
\]

To evaluate \( \lambda_n \) we have

\[
\lambda_n = [a_n, a_{-n} \Omega] = -\|a_{-n} \Omega\|^2 / n.
\]

Thus

\[
[a_m, a_n] = m \delta_{n+m,0} I.
\]
Similarly $[L_n', L_n'] - (m-n)L_{m+n}'$ commutes with the $e_k$'s and $e_k^*$'s and raise energy by $-n-m$. Using adjoints as before, it must be 0 if $n+m \neq 0$ and a scalar $\mu_m$ if $m+n = 0$. Clearly $\lambda_{-m} = -\lambda_m$ using adjoint and if $n > 0$

$$\lambda_n = ([L_n', L_n']|\Omega, \Omega)$$

$$= ||L_n'\Omega||^2$$

$$= \sum_{m=1}^{n} (m-n/2-1/2)^2$$

$$= n(n+1)(2n+1)/6 + n(n+1)^2/4 - (n+1)^2n/2$$

$$= \frac{1}{12}n(n+1)[4n + 2 + 3(n+1) - 6(n+1)] = \frac{1}{12}(n^3 - n).$$

Hence we have

$$[L_n', L_n'] = (m-n)L_{m+n}' + \delta_{m+n,0}m^3 - mI.$$

**Lemma (Gomes).** The only irreducible positive energy unitary representation of the Virasoro algebra with $c = 0$ is the trivial representation.

**Proof.** Let $\Omega$ be a lowest energy unit vector with $L_0\Omega = h\Omega$ and $L_n\Omega = 0$ for $n > 0$. Let $\xi_1 = L_{-2n}\Omega$ and $\xi_2 = L_{-n}\Omega$. Then $||\xi_1||^2 = 4nh$, $||\xi_2||^2 = 4n^2h(2h+n)$ and $(\xi_1, \xi_2) = 6n^2h$. Hence

$$\det(\xi_1, \xi_2) = 16n^3h^2(2h+n) - 36n^4h^2 = 4n^3h^2(8h-5n).$$

To be non-negative for all $n$ we must have $h = 0$. But then unitarity implies that $L_{-n}\Omega = 0$ for all $n > 0$.

We have constructed the positive energy representation of the fermionic operators $e_n$ and $e_n^*$ on $F$ with vacuum vectors $\Omega$. We have constructed operators, bilinear in fermions, $a_n$ and $L_n'$ satisfying $[a_m, a_n] = m\delta_{m+n,0}I$, $[L_n', e_m] = -me_{n+m}$ and $[L_n', a_m] = -ma_{m+n}$. The $L_n'$'s define a representation of the Virasoro algebra with $c = 1$. On the other hand we may also define operators $L_n$, bilinear in the bosons $a_n$, which give another representation of the Virasoro algebra with $c = 1$ satisfying $[L_n, a_m] = -ma_{m+n}$. The coset operators $L''_n = L_n' - L_n$ define a positive energy representation of the Virasoro algebra with $c = 0$ on the multiplicity spaces $M_i = \{\xi | a_0\xi = i\xi, a_n\xi = 0 \ (n > 0)\}$. So $L_n = L_n'$ on $M_i$ and hence everywhere.

**Corollary.** $L_n' = L_n$.

We now generalize this contruction to produce representations of the Virasoro algebra with $c = 1$ and $h = j^2$ with $j$ a non-negative half-integer. The space $F$ is canonically $\mathbb{Z}_2$-graded and the grading is compatible with its identification with an exterior algebra. The operators $e_i$ and $e_i^*$ act as odd operators. On the $\mathbb{Z}_2$-graded tensor product $F_2 = F \otimes F$ there are operators $e_n^{(1)} = e_i \otimes I$ and $e_n^{(2)} = I \otimes e_i$. The space $F_2$ can be identified with the exterior algebra on the anticommuting variables $e_n^{(j)} (n > 0)$ and $(e_n^{(j)})^*$ (n ≥ 0). If for $v = (\alpha, \beta) \in \mathbb{C}^2$ we set $v_i = \alpha e_n^{(1)} + \beta e_n^{(2)}$ and $v^*_i = \pi(e_n^{(1)})^* + \pi(e_n^{(2)})^*$, then the natural actions of $G = U(2)$ on $V$ and $V^*$ (the conjugate space) extend to a unitary action $\pi$ on the exterior algebra and hence $F_2$. Evidently

$$v_nw_m + w_mv_n = 0, \ v_nw_m + w_mv_n = \delta_{n,m}(v, w)I, \ \pi(g)v_i\pi(g)^* = (gv)_i.$$

For $i, j = 1, 2$ we define

$$E_{ij}(k) = \sum_{n=m+k} e_n^{(i)}(e_n^{(j)})^*, \ E_{ij}(0) = \sum_{n \geq 0} e_n^{(i)}(e_n^{(j)})^* - \sum_{n > 0} (e_n^{(j)})^* e_n^{(i)}.$$

Identifying $E_{ij}$ with the canonical basis of End $V = M_2(\mathbb{C})$, we can extend this definition by linearity to define operators $X(n)$ for $X \in \text{End } V$. Note that these operators are even. We define $D = L_0 \otimes I + I \otimes L_0$, so that $[D, v_i] = -iv_i$ and $[D, X(n)] = -nX(n)$. More generally set $L_n = L_n \otimes I + I \otimes L_n$. Thus

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}m^3 - mI, \ [L_m, v_n] = -nv_{n+m}.$$
Lemma. (a) \( \pi(g)X(n)\pi(g)^* = (gXg^*)(n) \) for \( g \in U(2) \).
(b) \( E_{11}(n) = a_n \otimes I \), \( E_{22}(n) = I \otimes a_n \).
(c) \( X(n)^* = X^*(-n) \).
(d) \( [X(n), v_i] = (Xv)_{i+n} \).
(e) \( [L_m, X(n)] = -nX(n + m) \).
(f) \( [X(m), Y(n)] = [X, Y](m + n) + m\delta_{m+n, 0} \cdot \mathrm{Tr}(XY)I \).

Proof. The first three identities are immediate. By complex linearity it suffices to check the remaining relations for \( X, Y \) skew–adjoint, i.e. in the Lie algebra of \( U(2) \). Relation (d) is evident for diagonal operators using (b) and the relation \( a_m, c_n = c_{n+m} \). Using (a) and the relation \( \pi(g)v_i\pi(g)^* = (gv)_i \), it follows for skew–adjoint \( X \), since we can choose \( g \in U(2) \) so that \( gXg^* \) is diagonal. Similarly (e) is evident for diagonal \( X \) and follows for skew–adjoint \( X \) from the fact that the operators \( \pi(g) \) and \( L_n \) commute, because \( L_n = L'_n \).

From (d), it follows that \( [X(m), Y(n)] - [X, Y](m + n) \) commutes with the \( v_i \)’s. Taking adjoints, it also commutes with the \( v_i^* \)’s. It changes energy by \(-n - m\), so vanishes if \( n + m \neq 0 \). Otherwise, if \( n = -m \), it preserves energy and therefore takes the vacuum vector \( \Omega \otimes \Omega \) to a scalar multiple of itself. Since the vacuum vector is cyclic, it acts as this scalar everywhere, so that

\[
[X(m), Y(n)] - [X, Y](m + n) = \delta_{m+n, 0} \cdot \lambda_m(X, Y).
\]

Evidently \( \lambda_m(X, Y) \) is bilinear in \( X \) and \( Y \) and, conjugating the left hand side by \( \pi(g) \), satisfies

\[
\lambda_m(gXg^*, gYg^*) = \lambda_m(X, Y).
\]

Thus \( \lambda_m(X, Y) = \mathrm{Tr}((X \otimes Y)Z) \) where \( Z \in \mathrm{End}V \otimes V \) commutes with \( U(2) \), or equivalently, since \( Z \) automatically commutes scalars, with \( SU(2) \). But as a representation of \( SU(2) \), \( V = V_{1/2} \) and \( V_{1/2} \otimes V_{1/2} = V_0 \oplus V_1 \). Thus by Schur’s lemma, the space of invariant bilinear forms is two–dimensional, with an easily identified basis:

\[
\lambda_m(X, Y) = \alpha_m \cdot \mathrm{Tr}(X) \cdot \mathrm{Tr}(Y) + \beta_m \cdot \mathrm{Tr}(XY).
\]

Since \( E_{11}(m) = a_m \otimes I \) and \( E_{22}(n) = I \otimes a_n \) commute and \( [a_m \otimes I, a_n \otimes I] = m\delta_{m+n, 0}I \), it follows that \( \alpha_m = 0 \) and \( \beta_m = m \), as claimed.

We have thus constructed an action of \( \mathfrak{u}_2 \) on \( F_2 \) by operators \( X(n) \). Let \( L_n^{(1)} = L_n \otimes I \) and \( L_n^{(2)} = I \otimes L_n \). From the above, if \( H_i = E_{ii} \), then \( L_n^{(i)} \) is the Virasoro algebra associated to the bosonic system \( H_i(k) \) and \( \mathcal{L}_n = L_n^{(1)} + L_n^{(2)} \) satisfies \( [L_n, X(m)] = -mX(m + n) \). Set \( H = \frac{1}{2}(H_1 - H_2), \ K = \frac{1}{2}(H_1 + H_2), \ E = E_{12} \) and \( F = E_{21} \). Then the operators \( K(n) \) commutes with the operators \( E(m), F(m) \) and \( H(m) \). Let \( L_n^{SU(2)} \) and \( \mathcal{L}_n^{(1)} \) be the Virasoro operators constructed using the two oscillator algebras \((\sqrt{2}H(n))\) and \((\sqrt{2}K(n))\).

Since \( H_1^2 + H_2^2 = 2(H^2 + K^2) \), we have \( L_n = L_n^{SU(2)} + \mathcal{L}_n^{(1)} \). Explicitly we have

\[
L_0^{SU(2)} = \mathrm{H}(0)^2 + 2 \sum_{n > 0} \mathrm{H}(-n)\mathrm{H}(n), \quad L_0^{(1)} = K(0)^2 + 2 \sum_{n > 0} K(-n)K(n).
\]

Since \( K(n) \) commutes with \( X(m) \) for \( X \in \mathrm{Lie}SU(2) \), it follows that \( [L_n^{SU(2)}, X(m)] = -mX(m + n) \). By construction, since the operators \( \mathrm{H}(m) \) leave any \( \mathfrak{sl}_2 \)–submodule invariant, so do the Virasoro operators \( L_n^{SU(2)} \); moreover \( \mathcal{L}_n^{(1)} \) leaves invariant the multiplicity spaces of the distinct level 1 representations of \( \mathfrak{sl}_2 \).

There are at most two such representations, classified by their lowest energy spaces \( \mathcal{H}(0) \), isomorphic to either \( V_0 \) or \( V_{1/2} \). In fact let \( V \) be an irreducible \( SU(2) \)–submodule of \( \mathcal{H}(0) \) and let \( \mathcal{K} \) be the \( \mathfrak{sl}_2 \)–submodule generated by \( V \). By the commutation relations any monomial in the \( (X(n))'s \) can be written as a sum of monomials of the form \( RDL \) with \( R \) a monomial in energy raising operators \( X(-n) \) \( (n > 0) \), \( D \) a monomial in constant energy operators \( X(0) \), and \( L \) a monomial in energy lowering operators \( X(n) \) \( (n > 0) \). Hence \( \mathcal{K} \) is spanned by products \( Rv \). But then clearly \( \mathcal{K}(0) = V \). By irreducibility \( \mathcal{H} = \mathcal{K} \) and hence \( \mathcal{H}(0) = V \). We claim that if \( V = V_j \), then \( j = 0 \) or \( 1/2 \). Indeed let \( e = F(1), f = E(-1) \) and \( 2h = [e, f] = (E_{22}(0) - E_{11}(0)) + I = -2H + I \). Thus \( h^2 = h, e^* = f, [h, e] = e \) and \( [h, f] = -f \). Suppose that \( \mathcal{H}(0) \cong V_j \) and that \( v \in V_j \) satisfies \( Hv = jv \). So \( hv = (1 - 2j)v \) and \( ev = 0 \). By standard \( \mathfrak{sl}_2 \) theory,
it follows that \( 1 - 2j \geq 0 \), i.e. \( j = 0 \) or \( j = 1/2 \). Note that if \( \mathcal{H}(0) \cong V_j \) with \( j = 0 \) or \( j = 1/2 \), then each \( \mathcal{H}(n) \) decomposes as a sum of integer \( (j = 0) \) or half–integer spin \( (j = 1/2) \). Since the decomposition of \( \mathcal{F}_2 \) contains both integer and half–integer spin representations of \( SU(2) \), it follows that there are irreducible positive energy representations with \( \mathcal{H}(0) = V_j \) \( (j = 0, 1/2) \).

To prove uniqueness, note that any monomial \( A \) in operators from \( \mathcal{F}_2 \) is a sum of monomials \( RDL \) with \( R \) a monomial in energy raising operators \( X(-n) \) \( (n > 0) \), \( D \) a monomial in constant energy operators \( X(0) \), and \( L \) a monomial in energy lowering operators \( X(n) \) \( (n > 0) \). If \( v, w \in \mathcal{H}(0) \), then the inner products \( (A_I v, A_I w) \) are uniquely determined by \( v, w \) and the monomials \( A_I \). Indeed \( A_2^2 A_1 \) is a sum of terms \( RDL \) and \( (RDL v, w) = (DL v, R^* w) \), with \( R^* \) an energy lowering operator. Hence if \( \mathcal{H}' \) is another irreducible positive energy representation with \( \mathcal{H}(0) \cong \mathcal{H}'(0) \) by a unitary isomorphism \( v \mapsto v' \), then \( U(A v) = A' v \) defines a unitary map of \( \mathcal{H} \) onto \( \mathcal{H}' \) intertwining the action of \( \mathcal{F}_2 \).

**3. Character formulas for affine Lie algebras.** The character of the spin \( j \) representation \( V_j \) of \( SU(2) \), and hence \( sl_2 \), is given by

\[
\chi_j(\zeta) = \frac{\zeta^{2j+1} - \zeta^{-2j+1}}{\zeta - \zeta^{-1}},
\]

writing \( \zeta \) for the element \( \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \) of \( SU(2) \). The character of the level 1 spin \( j \) representation \( \mathcal{H}_j \) \( (j = 0, 1/2) \) is given by

\[
X_j(\zeta, q) = ch \mathcal{H}_j = \sum_{k=0}^{\infty} \chi_j(\zeta)(q^{(j+k)^2} - q^{(j+k+1)^2}) \phi(q) = \sum_{n \in \mathbb{Z}^+} \zeta^{2n} q^n \phi(q),
\]

where

\[
\phi(q) = \prod_{n \geq 1} (1 - q^n)^{-1}.
\]

This formula can be deduced directly from first principles or by decomposing the fermionic representation on \( \mathcal{F}^{0,2} \). In both cases we use the fact that there are just two level one representations of \( \mathcal{L}sl_2 \), \( \mathcal{H}_j \) with \( j = 0, 1/2 \), \( \mathcal{H}_0 \) involving only integer spin representations of \( SU(2) \) and \( \mathcal{H}_{1/2} \) representations only non–integer spins. We use the fact that the operators \( L_n^{SU(2)} \) are given by the construction of the Virasoro algebra associated with the bosonic system \( (H(k)) \).

**Method I. Direct treatment.** We consider first the \( \mathcal{H} \) is generated by a lowest energy vector \( \Omega \) with \( L_0 \Omega = 0 \). Let \( \mathcal{H}[i] = \{ \xi \in \mathcal{H}[0] : H(0)\xi = i \zeta \} \). Since \( \Omega \) is cyclic, these eigenspaces are invariant under the Virasoro algebra and are non–zero only if \( i \) is an integer. We know that \( H(0) \) is non–zero. We claim that \( \mathcal{H}[i] \) is non-zero for every \( i \in \mathbb{Z} \). Indeed \( E(-n), F(n) \) and \( X = \frac{1}{2}[E(-n), F(n)] \) give a copy of \( sl_2 \) with \( X\Omega = -n\Omega \). Hence \( \xi_n = E(-n)^n\Omega \neq 0 \), \( \mathcal{H}[n] \). Similarly using \( E(n), F(-n) \) and \( Y = \frac{1}{2}[E(n), F(-n)] \) as the copy of \( sl_2 \), it follows that \( \xi_{-n} = F(-n)^n\Omega \neq 0 \) and \( \mathcal{H}[-n] \). Now \( \mathcal{H}[0] \) has a unique lowest energy vector and is an irreducible oscillator module. For any singular vector \( \xi_\nu \) would satisfy \( L_n^{SU(2)} \xi_\nu = 0 \) and hence be a multiple of the vacuum vector.

We claim that the \( \xi_\nu \) is a lowest energy vector in \( \mathcal{H}[n] \) and generates \( \mathcal{H}[n] \) as an oscillator module. Clearly \( [L_0, E(-n)^n] = n^2E(-n)^n \) and \( [L_0, F(-n)^n] = n^2F(-n)^n \) so that \( L_0\xi_n = n^2 \). But with the present normalisation \( L_0 = H(0)^2 + 2 \sum_{m>0} H(-m)H(m) \), so that \( H(m)\xi_n = 0 \). As before an \( sl_2 \) argument shows that, up to scalar multiples, these are the only singular vectors in \( \mathcal{H}[0] \). They have energy \( (n+1) + 1/4 = (n+1/2)^2 \). As before an \( sl_2 \) argument shows that, up to scalar multiples, these are the only singular vectors in \( \mathcal{H}[(\pm(n+1/2)] \). Hence the character of the spin \( 1/2 \) representation is \( \sum_{n \in \mathbb{Z}^+} q^n z^n \phi(q) \).

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Method II. Treatment using fermions. Using the direct sum decomposition from the previous section, as a representation of $U(1)$, $\mathcal{F}$, as a representation of $U(1)$, has character

$$\theta(z, q) = \sum_{n \in \mathbb{Z}} z^n q^{n^2/2} \varphi(q),$$

for $z \in U(1)$. We now use the identity:

$$\theta(z, q) \theta(z^{-1}, q) = \sum_{m, n \in \mathbb{Z}} (z^2)^m (z)^n q^{(m^2 + n^2)/2} \varphi(q)^2$$

$$= \sum_{j=0}^{1/2} \sum_{a, b \in \mathbb{Z}} \sum_{m} z^{2n} q^{a^2 + b^2} \varphi(q)^2$$

$$= \sum_{j=0}^{1/2} X_j(z, q) \Psi_j(z, q),$$

where

$$\Psi_j(z, q) = \sum_{m \in \mathbb{Z}} z^{2m} q^{m^2} \varphi(q).$$

We have already exhibited explicit singular vectors $\Omega_p \in \mathcal{F}$ with $p \in \mathbb{Z}$. By boson–fermion duality these exhaust the singular vectors. This yields singular vectors $\Omega_p \otimes \Omega_q$ in $\mathcal{F} \otimes \mathcal{F}$. If we split up the character of $\mathcal{F} \otimes \mathcal{F}$ as the sum of characters of integer and non–integer spin $\Theta_j(z, \zeta, q)$, then $\Theta_j(z, \zeta, q) = X_j(z, \zeta, q) \cdot \Psi_j(z, q)$, with $X_j(z, \zeta, q)$ the character of $\mathcal{H}_j$ and $\Psi_j(z, q)$ the character of the multiplicity space of $\mathcal{H}_j$. If we fix a spin $j = 0, 1/2$, then there are certainly singular vectors of charge $k + j$ for each $k \in \mathbb{Z}$ for the $H(n)$'s. Thus

$$X_j(z, \zeta, q) \geq X_j(z, \zeta, q) = \sum_{n \in \mathbb{Z}} \zeta^{2n} q^{n^2} \varphi(q), \quad (1)$$

where the inequality is to be taken coefficient by coefficient. Similarly, for the character of the multiplicity space,

$$\Psi_j(z, q) \geq \Psi(z, q) = \sum_{n \in \mathbb{Z}} z^{2n} q^{n^2} \varphi(q). \quad (2)$$

On the other hand the identity above implies that

$$\sum_j X_j(z, \zeta, q) \Psi_j(z, q) = \sum X_j(z, \zeta, q) \Psi_j(z, q),$$

so that equality must hold in (1) and (2).

Remark. In Section 8 we will introduce shift operators which together with the $H(n)$'s and the $K(n)$'s act irreducibly on the $H_j$'s and their multiplicity spaces. This gives a more conceptual operator–theoretic explanation of the second proof.

4. Existence of singular vectors in the oscillator representations. According to the character formula $\mathcal{H}_j$ has a series of vectors $\xi_m$ $(m \in j + \mathbb{Z})$ with $H(0) \xi_m = m \xi_m$ and $H(n) \xi_m = 0$ for $m > 0$. Moreover $\mathcal{H}_j$ is the direct sum of the cyclic oscillator modules $\mathcal{K}_m$ generated by these vectors; note that, if $H = H(0)$, then $\mathcal{K}_m$ is the $m$–eigenspace of $H$. Following Graeme Segal, we construct non-zero singular vectors in $\mathcal{K}_j \subset \mathcal{H}_j$. In fact the vector $\xi_m$ satisfies $H \xi_m = m \xi_m$. If $m \geq 0$ then $\eta = E \xi_m$ lies in $\mathcal{K}_{m+1}$ and satisfies $L_0 \eta = \frac{1}{2} m^2 \eta$. Since the lowest energy level in $\mathcal{K}_{m+1}$ is $\frac{1}{2} (m+1)^2$ it follows that $\eta = 0$, i.e. $E \xi_m = 0$. Similarly if $m \leq 0$, $F \xi_m = 0$. Thus $\xi_m$ generates an irreducible $\mathfrak{s}l_2$–representation of dimension $2|m| + 1$. If $m > 0$, then $F^{[m]} \xi_m$ is a non–zero singular vector in $\mathcal{K}_j$ of energy $m^2/2$. If $m < 0$ then $E^{[m]} + 1 \xi_m$ is a non–zero singular vector in $\mathcal{K}_j$ of energy $m^2/2$. This proves:

Proposition. The oscillator representation with $H(0) = jI$ has non–singular singular vectors for the Virasoro algebra with energy $(j + n)^2/2$ for $n \geq 0$. The non–zero singular vector constructed above in $\mathcal{K}_j$ of energy $m^2/2$ $(m \geq 0)$ generates a copy of $V_m$ as an $\mathfrak{s}l_2$–module.
We shall call these singular vectors \textit{Goldstone vectors}. In Section 8, following Segal, we will use vertex operators to deduce the explicit form of Goldstone for these vectors.

5. Uniqueness of singular vectors in the oscillator representations. Let \((a_m)\) satisfy \([a_m, a_n] = m \delta_{m+n, 0} I\). These act as operators on \(V = \mathbb{C}[x_1, x_2, \ldots]\) via \(a_0 = \sqrt{2} \mu\), \(a_n = n \partial_x x_n\) and \(a_{-n} = x_n\) for \(n > 0\). Set \(L_0 = a_0^2/2 + \sum_{n>0} a_n a_n\) and \(L_k = \frac{1}{2} \sum_{\mu+q=k} a_\mu a_q\) for \(k \neq 0\). Thus \([L_k, a_n] = -na_{n+k}\) and 
\[
[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n, 0}(m^3 - m)/12.
\]

**Proposition.** For \(n \geq 0\) let \(V(n) = \{ \xi : L_0 \xi = (\mu^2 + n)\xi \} \) with \(\mu \in \mathbb{R}\). Then \(\dim V(n) = \mathcal{P}(n)\) the partition function and the subspace of singular vectors, i.e. \(\text{solutions of } L_0 \xi = 0 = L_2 \xi\), is at most one-dimensional.

**Proof.** If we define the degree of \(x_i\) to be \(i\), then \(V(\mu^2 + n)\) is spanned by all monomials of total degree \(n\), so that \(\dim V(n) = \mathcal{P}(n)\). We shall treat a more general case which evidently implies the proposition:

**Lemma.** Let \(W = \mathbb{C}[x_1, x_2, \ldots]\) with \(\deg x_i = i\) and let \(W(n)\) be the space spanned by monomials of total degree \(n\), so that \(\dim W(n) = \mathcal{P}(n)\). For \(a, b \in \mathbb{C}\) let 
\[
A = a \partial_{x_1} + \sum_{n>0} (n+1) x_n \partial_{x_{n+1}}, \quad B = \frac{1}{2} \partial_{x_1}^2 + b \partial_{x_2} + \sum_{n \geq 1} (n+2) x_n \partial_{x_{n+2}}.
\]

Then the space of solutions of \(W_0(n) = \{p \in W(n) : Ap = 0, \quad Bp = 0\}\) is at most one-dimensional.

**Proof.** For \(i \geq 0\), let \(W(n, i) = \{x_1^{n-i} q(x_2, \ldots) : \deg q = i\}\). Evidently 
\[
W(n) = \bigoplus_{i \geq 0} W(n, i)
\]
so that any element \(p \in W(n)\) has a decomposition \(p = \sum p_i\) with \(p_i \in W(n, i)\). Thus \(p_i = x_1^{n-i} q_i(x_2, x_3, \ldots)\) with \(\deg q_i = i\). We assume that the leading coefficient \(p_0\) (a multiple of \(x_1^n\)) is zero and prove by induction on \(i\) that \(q_i = 0\). Each \(q_i\) is a sum of linear combination of monomials \(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\) with \(\sum_{k \geq 2} k \alpha_k = i\). Suppose that \(\alpha_2 > 0\). Then 
\[
Ax_1^{n-i} x_1^{\alpha_1} = a(n-i)x_1^{n-i-1}x_1^{\alpha_1} + 2x_1^{n-i+1}x_2^{\alpha_2 - 1}x_3^{\alpha_3} \cdots + 3x_1^{n-i+2}x_2^{\alpha_2}x_3^{\alpha_3 - 1}x_4^{\alpha_4} + \cdots.
\]

Because of the inductive hypothesis, this is the only monomial which produces a term \(x_1^{n-i+1} x_2^{\alpha_2 - 1} x_3^{\alpha_3} \cdots\) when \(A\) is applied. It follows that its coefficient must be zero. So only the coefficients of monomials 
\(x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots\). The image under \(B\) of such a monomial is 
\[
Bx_1^{n-i} x_3^{\alpha_3} x_4^{\alpha_4} \cdots = x_1^{n-i-2} x_3^{\alpha_3} x_4^{\alpha_4} + 3x_1^{n-i+1} x_3^{\alpha_3 - 1} x_4^{\alpha_4} + \cdots.
\]

Because of the inductive hypothesis this is the only monomial which produces a term \(x_1^{n-i-2} x_3^{\alpha_3} x_4^{\alpha_4} \cdots\) when \(B\) is applied. It follows that its coefficient must be zero. Thus \(p_i = 0\) and hence \(p = 0\), as required.

6. Density modules, primary fields and the Feigin–Fuchs product formula. For \(\lambda, \mu \in \mathbb{C}\), we define the density module \(V_{\lambda, \mu}\) to be the vector space with basis \(v_n\) \((n \in \mathbb{Z})\) and define operators \(\ell_k\) on \(V_{\lambda, \mu}\) by 
\[
\ell_k v_n = -(n + \lambda k + \mu) v_{n+k}.
\]

It is easy to check that \([\ell_m, \ell_n] = (m-n) \ell_{m+n}\), so this defines a representation of the Virasoro algebra with \(c = 0\). Clearly the change of basis by the shift operator \(v_n \mapsto v_{n+k}\) gives a natural isomorphism between \(V_{\lambda, \mu}\) and \(V_{\lambda, \mu+k}\).

Recall that if \(\pi : g \rightarrow \text{End } W\) is a representation of a Lie algebra \(g\), then the \textit{dual representation} on the dual space \(W'\) is given by \((\pi'(X)\xi, v) = -\langle \xi, X v \rangle\) for \(v \in W\). Let \(w_n\) be the canonical basis of \(V_{1-\lambda, -\mu}\). We identify \(U\) with a subspace of the dual \(V_{1, \mu}\) via the pairing \(\langle v_n, w_m \rangle = \delta_{n-m,0}\). It is immediately verified that the dual representation restricts to the natural representation on \(V_{1-\lambda, -\mu}\). With an obvious abuse of notation, we shall write \(V_{1-\lambda, -\mu} \cong V_{\lambda, \mu}\). These representations are the infinitesimal version of the action of
Hence \( (\Phi(n)) \), or more properly a certain central extension, on the densities \( e^{i\mu \theta} f(\theta) (d\theta)^\lambda \). The duality between \( V_{\lambda,\mu} \) and \( V_{1-\lambda,-\mu} \) is given by the pairing into 1-forms

\[
(f(\theta)e^{i\mu \theta} (d\theta)^\lambda, g(\theta)e^{-i\mu \theta} (d\theta)^{1-\lambda}) \mapsto fg d\theta
\]

followed by integration over the circle, i.e. \( \frac{1}{2\pi} \int_0^{2\pi} fg d\theta \). Identifying \( v_n \) with \( e^{i\mu \theta} e^{i\mu \theta} (d\theta)^\lambda \), we see that

\[
\ell_k(f e^{i\mu \theta} (d\theta)^\lambda)) = -e^{ik\theta}(i \frac{d}{d\theta} + k\lambda)(f e^{i\mu \theta}) (d\theta)^\lambda.
\]

Thus it is the action on \( \mathbb{C}[e^{i\theta}, e^{-i\theta}] e^{i\mu \theta} \) given by the operators

\[
\ell_k = -e^{ik\theta}(i \frac{d}{d\theta} + k\lambda).
\]

Changing variables to \( z = e^{i\theta} \), regarded as a formal variable, and introducing an extra factor \( z^\lambda \), it may also be identified with the action on \( \mathbb{C}[z, z^{-1}] z^{\mu + \lambda} \) given by

\[
\ell_k = -z^{k+1} \frac{d}{dz} - (k + 1)\lambda z^k.
\]

For applications here \( \lambda \) and \( \mu \) will be in \( \mathbb{Z} \) (for applications to fusion we require \( \lambda, \mu \in \frac{1}{2}\mathbb{Z} \)).

For \( i = 1, 2 \), let \( \mathcal{H}_i = L(1, h_i) \) with lowest energy vectors \( \xi_i = \xi_{h_i} \). We define a primary field of type \( (\lambda, \mu) \) between \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) to be a map \( \Phi : \mathcal{H}_1 \otimes V_{\lambda,\mu} \rightarrow \mathcal{H}_2 \) commuting with the action of the Virasoro algebra. Let \( \Phi(n)\xi = \phi(\xi \otimes v_n) \). Then

\[
[L_k, \Phi(n)] = -(n + k\lambda + \mu)\Phi(n + k).
\]

We have the following uniqueness result:

**Lemma A.** If \( (\Phi(n)\xi_1, \xi_2) = 0 \) then \( \Phi(n) = 0 \) for all \( n \in \mathbb{Z} \). If \( \Phi \) is non-zero, then \( m = h_1 - h_2 - \mu \) is an integer and \( \Phi(m)\xi_1 \) is a non-zero multiple of \( \xi_2 \).

**Proof.** Suppose that \( (\Phi(n)\xi_1, \xi_2) = 0 \). From the commutation relations it follows that for \( n_i > 0 \)

\[
(\Phi(n)L_{-n_k} \cdots L_{-n_1} \xi_1, \xi_2) = 0.
\]

Hence \( (\Phi(n)\xi, \xi_2) = 0 \) for all \( \xi \). From the commutation relations is then follows that for \( n_1 > 0 \)

\[
(\Phi(n)\xi, L_{-n_k} \cdots L_{-n_1} \xi_2) = 0,
\]

so that \( (\Phi(n)\xi, \eta) = 0 \) for all \( \xi, \eta \) and hence \( \Phi(n) = 0 \) for all \( n \in \mathbb{Z} \). Finally if \( (\Phi(m)\xi_1, \xi_2) \neq 0 \), then \( \Phi(m)\xi_1 \) is a non-zero multiple of \( \xi_2 \). Hence \( h_2 = h_1 - m - \mu \).

**Remark.** This result allows us to normalise a primary field by shifting numbering of the modes \( \Phi(n) \) to \( \Phi(n+k) \). Thus, if a primary field exist, we may assume that \( \mu = h_1 - h_2 \) and that \( \Phi(0)\xi_1 = \alpha\xi_2 \) with \( \alpha \neq 0 \).

In the language of vertex operators and vertex algebras, primary fields are usually described in terms of generating functions in a formal variable \( z \). We set

\[
\Phi(z) = \sum_{n \in \mathbb{Z}} \Phi(n) z^{-n-\delta}.
\]

Then

\[
[L_k, \Phi(z)] = z^{k+1} \frac{d\Phi(z)}{dz} + (k + 1)\Delta z^k \Phi(z).
\]
Taking coefficients of \( z^{-n-\delta} \), we get

\[
[L_k, \Phi(n)] = -(n + k + \delta - (k + 1)\Delta)\Phi(n + k).
\]

Thus \( \lambda = -\Delta + 1 \) and \( \mu = \delta - \Delta \). Hence \( \Delta = 1 - \lambda \) and \( \delta = \mu + \lambda - 1 \).

Now the Verma module representation \( M(1,j^2) \) \((j = 0, 1/2, 1, 3/2, \ldots)\) certainly has one singular vector at energy \((j + 1)^2 = j^2 + d\) with \( d = 2j + 1 \), since the representation \( L(1,j^2) \) is a subrepresentation of the multiplicity space. This corresponds to a element of the universal enveloping algebra \( \mathcal{U}_k \) generated by \( L_{-k} \) \((k \geq 1)\). Let \( \mathcal{U}_k \) be the universal enveloping algebra generated by \( L_{-i} \) with \( i \geq k \). It is the universal enveloping algebra of the Lie algebra generated by \( L_i \) for \( i \geq k \). This is a Lie ideal in the Lie algebra generated by \( L_{-j} \) with \( j \geq 0 \). As a consequence every element of the algebra \( \mathcal{U}_k \) can be written uniquely in the form \( P = \sum_{k \geq 0} q_k L_{-1}^k \) and \( \text{ad} L_{-1} \) defines a derivation of \( \mathcal{U}_2 \). In particular the singular vector has the form \( P_{d^2} \) where

\[
P = P_d = \sum_{k=0}^{d} q_k L_{-1}^k.
\]

Each term is a \( q_k \) is a sum of monomials \( L_{-i_1} \cdots L_{-i_r} \) with \( i_1 + \cdots + i_r = d - k \). In particular \( q_d \) is a scalar. The following uniqueness result was stated by Fuchs.

**Lemma B.** The coefficient of \( L_{-1}^d \) in \( P_d \) is non-zero; in particular if a singular vector exists, it is unique up to a scalar multiple.

**Proof.** Suppose that \( q_d = 0 \). We prove that \( P_d = 0 \). Each \( q_d \) can be written as a linear combination of monomials \( L_{-s}^{m_s} \cdots L_{-2}^{m_2} \) with \( n_p \geq 0 \). Amongst all monomials with non-zero coefficients, we can find one \( p \geq 2 \) minimal with \( n_p > 0 \). We can also choose this monomial so that \( n = n_p \) is also minimal. Suppose that it occurs in \( q_c \) with \( 0 \leq c < d \). We may in addition assume that \( c \) is chosen to be maximal. Let \( w = P_d \) with \( v \) the cyclic lowest energy vector in Verma module. Then \( L_{-1} w = 0 \). This can be written as

\[
\sum_{k=0}^{c} L_{p-1} q_k L_{-1}^k v = \sum_{k=0}^{c} [L_{p-1}, q_k] L_{-1}^k v + q_k L_{p-1} L_{-1}^k v.
\]

We look for terms ending with \( L_{-1}^{n-1} L_{-1}^{c+1} \) in this expression. We first note that it follows by induction on \( k \) that if \( j \geq 1 \) then \( L_j L_{-1}^k v \) lies in \( \text{lin} \{ AL_{-1}^i : i < k, \ A \in \mathcal{U}_2 \} \). Indeed

\[
L_j L_{-1}^{k-1} v = (j + 1) L_{j-1} L_{-1}^{k-1} v + L_{-1} L_j L_{-1}^{k-1} v,
\]

which has the same form since \( L_{-1} \mathcal{U}_2 = \mathcal{U}_2 \) and \( L_{-1}^{k-1} v \) is an eigenvector of \( L_0 \). If \( k \neq c \), then the minimality of \( p \) forces \( q_k \) to be in \( \mathcal{U}_p \). So either \( k > c \), in which case any monomial \( q_k \) could have a non-zero contribution from a monomial \( L_{-p}^{m_p} \cdots L_{-2}^{m_2} \) with \( m_p > n_p \). Clearly taking the Lie bracket with \( L_{-p} \) can diminish the exponent of \( L_{-p} \) by at most one and at the same time must increase the exponent of \( L_{-1} \) by at least one. So no monomials ending with \( L_{-1}^{n-1} L_{-1}^{c+1} \) can appear with non-zero coefficient. If \( k < c \), there is no way to increase the power of \( L_{-1}^k \) to \( L_{c+1}^{c+1} \) by taking the Lie bracket with \( L_{p-1} \). For \( k = c \) and the terms \( [L_{p-1}, AL_{-1}^n] \) with \( A \) a monomial in \( \mathcal{U}_{p+1} \), we have

\[
[L_{p-1}, AL_{-1}^n] = [L_{p-1}, A] L_{-1}^n + A[L_{p-1}, L_{-1}^n].
\]

The first term lies in \( \mathcal{U}_2 \) while for the second

\[
[L_{p-1}, L_{-1}^n] v = (2p - 1) \sum_{a+b=n-1} L_{-1}^a L_{-1}^b v = (2p - 1) n L_{-1}^{n-1} L_{-1}^1 + B,
\]

where \( B \in \mathcal{U}_2 \). There could be several terms in \( q_c \) with \( n_p = n \), but on bracketing with \( L_{p-1} \) and taking the term ending with \( L_{-p}^{n-1} L_{-1}^{c+1} \) is equal to \( AL_{-1}^{n-1} L_{-1}^{c+1} \). Thus all these terms are distinct. But then there can be no cancellation and the coefficient of \( L_{-1}^c \cdots L_{-p}^{n-1} L_{-1}^{c+1} v \) must therefore be non-zero, a contradiction.
Remark. Fuchs used a slightly different ordering on monomials, which can also be used to give a prove the uniqueness theorem. For a monomial $L_{n_1}^{\lambda_1} \cdots L_{n_k}^{\lambda_k}$, we set $n_1 = d - k$ and use the lexicographic ordering determined by $(n_1, n_2, \ldots)$. The proof is almost word-for-word that used in the second proof of Fuchs' algorithm in Section 10.

Let $P_d \in \mathcal{U}$ be the element giving the singular vector normalized so that the coefficient of $L^d_{-1}$ equals 1. In the representation $V_{\lambda, \mu}$, we evidently have

$$P_d v_0 = a_d(\lambda, \mu) v_{-d}$$

where $a_d(\lambda, \mu)$ is a inhomogenous polynomial of degree $d$ in $\lambda$ and $\mu$. Indeed since the $L^d_{-1}$ appears with coefficient 1, the formula for the action shows that the coefficient of $\mu^d$ is $(-1)^d$. Using the coset construction in Section 8 and an explicit formula for $P_d$ in Appendix C, we give two methods to establish the following product formula of Feigin–Fuchs, which we state first in the two simplest cases:

**Feigin–Fuchs product formula.** Let $j = 0, 1/2, 1, 3/2, \ldots$ and set $S = \{-j, -j + 1, -j + 2, \ldots\}$.

(a) $a_d(0, \mu) = (-1)^d \prod_{k \in S}(\mu + j^2 - k^2)$.
(b) $a_d(1, \mu) = (-1)^d \prod_{k \in S}(\mu + j^2 - (k + 1)^2)$

(c) $a_d(p^2, \mu) = (-1)^d \prod_{k \in S}(\mu + j^2 - (k + p)^2)$ for $p \geq 0$.

(d) $a_d(\lambda, \mu)^2 = \prod_{k \in S}(\lambda - \mu - j^2 + k^2)^2 - 4\lambda k^2$.

We have the following necessary condition for the existence of a primary field:

**Lemma C.** If there is a non-zero primary field from $L(1, j^2)$ to $L(1, h)$ of type $(\lambda, \mu)$, then

$$a_d(1 - \lambda, h - j^2) = 0.$$

**Proof.** Let $\Phi$ be a non-zero primary field from $L(1, j^2)$ to $L(1, h)$. Then we may assume that $\Phi$ is normalised so that $\mu = j^2 - h$ and $\Phi(0)\xi, \eta = \alpha \xi_h$ for some constant $\alpha \neq 0$. Let $\xi = \xi_\lambda$, $\eta = \xi_\mu$. Then $(\Phi(0)\xi, \eta) \neq 0$. On the other hand $P_d \xi = 0$. Let $P_d = \sum a_m L_{-1}^m L_{-2}^m \cdots$. For an operator we set $\pi(L_j) A = [L_j, A]$. This defines an action of the Virasoro algebra with $c = 0$. Now

$$0 = (\Phi(d) P_d \xi, \eta) = \sum a_m (\pi(-L-n)^m \cdots \pi(-L-2)^m \pi(-L-1)^m) \cdot \Phi(d) \xi, \eta).$$

On the other hand

$$\sum a_m \pi(-L-n)^m \cdots \pi(-L-2)^m \pi(-L-1)^m \cdot \Phi(d) = b \Phi(0),$$

where for the basis $(v_i)$ of $V_{\lambda, \mu}$

$$\sum a_m (-L-n)^m \cdots (-L-2)^m \cdot (-L-1)^m v_d = b v_0.$$ 

Thus we require that $b$ should vanish. To calculate $b$, we take $(w_j)$ to be a dual basis of $V_{1-\lambda,-\mu}$. Then

$$b = \sum a_m (-L-n)^m \cdots (-L-2)^m (-L-1)^m v_d, w_0) = (v_d, P_d w_0).$$

Thus $P_d w_0 = bw_{-d}$ in $V_{1-\lambda,-\mu}$ and hence

$$b = a_d(1 - \lambda, -\mu) = a_d(1 - \lambda, h - j^2).$$

**Remark.** This result can be seen a little more transparently if the dual of $V_{\lambda, \mu}$ is used. Indeed, passing to a suitable completion of the tensor product, a primary field of type $(\lambda, \mu)$ from $L(c, h_1)$ to $L(c, h_2)$ can be regarded as a map $\mathcal{H}_1 \to \mathcal{H}_2 \otimes V_{\lambda', \mu'}$, where $\lambda' = 1 - \lambda$ and $\mu' = -\mu$. Explicitly the map becomes

$$\xi \mapsto \Phi(z) \xi = \sum \varphi(n) \xi z^{-n},$$
Hence \( \Phi(\eta) \) here is identified with a basis element of \( V_{\lambda',\mu'} \). If \( v_{h_2} = \phi(-k)v_{h_1} \), then, picking out the coefficient of \( v_{h_2} \), the condition \( P(L_{-1}, L_{-2}, \ldots) v_{h_1} = 0 \) evidently implies that \( P(\ell_{-1}, \ell_{-2}, \ldots) z^k = 0 \), i.e. the condition stated in Lemma C. It is therefore natural to refer to the primary field as having dual type \((\lambda',\mu')\). Thus if the primary field is normalised so that \( \Phi(0)\xi_1 = \alpha\xi_2 \) with \( \alpha \neq 0 \), then \( \mu' = h - j^2 \) and Lemma C requires that \( (a_d(\lambda',\mu') = 0). \)

7. Multiplicity one theorem. We wish to prove the following theorem:

**Theorem A.** The action of the Virasoro algebra on the multiplicity space \( M_m \) of \( V_m \) (\( m \geq 0, m \in j + \mathbb{Z} \)) in \( H_j \) is irreducible for \( j = 0, 1/2 \).

Let us start by showing that it suffices to prove:

**Theorem B.** The Goldstone vectors are the only singular vectors in \( K_m = \{ \xi \in H_j : H\xi = m\xi \} \) for \( m \in j + \mathbb{Z} \).

**Proof that Theorem B implies Theorem A.** The multiplicity space \( M_m \) can be identified with

\[
M_m = \{ \xi \in H_j : H\xi = m\xi, E\xi = 0 \} \subset K_m = \{ \xi : H\xi = m\xi \}.
\]

If the action on \( M_m \) is not irreducible, then \( M_m \) would have to contain a singular vector of higher energy. But by Theorem B any such vector would have to be a Goldstone vector in \( K_m \). But, by the proposition in Section 4, \( \xi_m \) is the only Goldstone vector in \( K_m \) annihilated by \( E \).

Thus we have only to prove Theorem B. By the uniqueness theorem it suffices to show that there are non singular vectors in \( K_j \) with energy not of the form \( m^2/2 \) with \( m \in j + \mathbb{Z} \). For this we require the notion of a particularly simple kind of primary field. Let \( V_{0,0} = \mathbb{C}[z,z^{-1}] \) with \( L_n \) acting as the operator \( \ell_n = -z^{n+1}d/dz \). A \((0,0)\)–primary field is an operator

\[
\Phi : L(1,h_1) \otimes V_{0,0} \to L(1,h_2)
\]

which intertwines the action of the Virasoro algebra.

**Lemma A.** Let \( \Phi(n)\xi = \Phi(\xi \otimes z^n) \) for \( \xi \in L(1,h_1) \). Then \( [L_m, \Phi(n)] = -n\Phi(n + m) \). Moreover any such assignment corresponds to a \((0,0)\)–primary field via the above formula.

**Proof.** Equivariance implies that

\[
L_m\Phi(n)\xi = L_m(\Phi(\xi \otimes z^n)) = \Phi(L_m\xi \otimes z^n) + \Phi(\xi \otimes \ell_m z^n) = \Phi(n) L_m \xi - m\Phi(x,m + n).
\]

**Lemma B.** If \( (\Phi(n)) \) is a \((0,0)\)–primary field so is \( \Psi(n) = \Phi(-n)^* \).

**Proof.** We have

\[
[L_m, \Psi(n)] = [L_m, \Phi(-n)^*] = -[L_{-m}, \Phi(-n)]^* = -m\Phi(-n - m)^* = -m\Psi(n + m).
\]

**Lemma C.** If \( (\Phi(n)\xi_1, \xi_2) = 0 \) with \( \xi_i \) the lowest energy vector in \( L(1,h_i) \), then \( \Phi(n) = 0 \) for all \( n \in \mathbb{Z} \).

**Proof.** Suppose that \( (\Phi(n)\xi_1, \xi_2) = 0 \). From the commutation relations it follows that for \( n_i > 0 \)

\[
(\Phi(n) L_{-n_i} \cdots L_{-n_1}\xi_1, \xi_2) = 0.
\]

Hence \( (\Phi(n)\xi, \xi_2) = 0 \) for all \( \xi \). From the commutation relations is then follows that for \( n_i > 0 \)

\[
(\Phi(n)\xi, L_{-n_i} \cdots L_{-n_1}\xi_2) = 0,
\]

so that \( (\Phi(n)\xi, \eta) = 0 \) for all \( \xi, \eta \) and hence \( \Phi(n) = 0 \) for all \( n \in \mathbb{Z} \).

**Remark.** Note that if \( L_0\xi_i = h_i\xi_i \), then such a \( \Phi \) can exist only if \( h_1 - h_2 \) is an integer.
Proposition. If \( j \) is a non-negative half-integer and \( k \neq (j + m)^2 \) with \( m \in \mathbb{Z} \) there are no non-zero \((0,0)\)-primary fields from \( L(1,j^2) \) to \( L(1,k) \) or from \( L(1,k) \) to \( L(1,j^2) \).

Proof. Using adjoints it suffices to prove the first assertion. If a non–zero primary field exists, then by Lemma C of Section 6, necessarily \( a_d(j^2, j^2 - k) \) would have to vanish. On the other hand by the Feigin–Fuchs product formula, if \( S = \{ -j, -j + 1, \ldots, j - 1, j \} \), we know that

\[
a_d(j^2, j^2 - k) = \prod_{t \in S} ((t - 1)^2 - k) \neq 0.
\]

Proof of Theorem B. We have to prove that the Goldstone vectors are up to scalar multiples the only singular vectors in \( K_m \). Let \( P \) be the projection onto the submodule generated by the Goldstone vectors in \( K_m \), and \( P_0 \) the projection onto the submodule generated by a Goldstone vector \( \xi \). If there is another singular vector \( \eta \) not proportional to a Goldstone vector then its energy does not have the form \((j + k)^2\) with \( k \) an integer, by the uniqueness theorem. Let \( Q \) be the projection on submodule generated by \( \eta \). Then \( \Phi(n) = QH(n)P_0 \) and \( P_0\Phi(n)Q \) are \((0,0)\)-primary fields since \( L_m \) commutes with \( P_0 \) and \( Q \) and \( [L_m, H(n)] = -nH(n + m) \). By the proposition, we must have \( QH(n)P_0 = 0 \) and \( P_0H(n)Q = 0 \). Hence it follows that \( QH(n)P = 0 = PH(n)Q \). But then \( H(n) \) leaves the subspace \( PK_m \) invariant. Since the \( H(n) \)'s act irreducibly on \( K_m \) it follows that \( P = I \), a contradiction. So there are no singular vectors other than the Goldstone vectors.

Corollary 1. The character of \( L(1,m^2) \) for \( m \) a non–negative half-integer is \((q^{m^2} - q^{(m+1)^2})\varphi(q)\).

Corollary 2. Up to a scalar multiple \( P_m \xi_m \) is the unique singular vector of energy \((m+1)^2\) in \( M(1,m^2) \) and the quotient module is irreducible. The other singular vectors in \( M(1,m^2) \) are given by \( P_{m+k-1} \cdots P_{m+1}P_m\xi_m \) (of energy \((m + k)^2\)).

Proof. By the character formula, there is only one singular vector \( \eta \) of energy \((m+1)^2\) in \( M = M(1,m^2) \) and no singular vectors of lower energy except the generating vector of energy \( m^2 \). The submodule \( N \) generated by \( \eta \) is isomorphic to \( M(1,(m+1)^2) \). By the character formula \( M(1,m^2)/N \) is irreducible. Any singular vector in \( M \) not in \( N \) its image would give a non–zero singular vector in \( L(1,m^2) \) and would have to be a multiple of the generating vector. So it follows that all singular vectors of energy greater than or equal to \((m + 1)^2 \) lie in \( N \), whence the result.

8. Proof of the Feigin–Fuchs product formula using primary fields. We shall prove below that if \( j, j_1, j_2 \) are integers then there is a primary field of dual type \((j_1^2, j_2^2 - j^2)\) from \( L(1,j_1^2) \) to \( L(1,j_2^2) \) provided that \( V_{j_2^2} \) occurs as a component in \( V_j \)). Thus \( a_d(j_1^2, j_2^2 - j^2) = 0 \), by Lemma C in Section 6. This is enough information to compute \( a_d(\lambda, \mu) \) completely.

We fix \( j \), with \( d = 2j + 1 \), and freeze \( j_2 \), and hence \( \mu = j_2^2 - j^2 \), with \( j_2 \geq j \). Then the possible values of \( j_1 \) are \( j_2 + k \) with \( k \in S = \{ -j, -j + 1, \ldots, j - 1, j \} \). But \( a_d(\lambda, j_2^2 - j^2) \) has degree at most \( d \) in \( \lambda \) and \( d = |S| \). Hence

\[
a_d = a_d(\lambda, j_2^2 - j^2) = C \prod_{k \in S} (\lambda - (j_2 + k)^2),
\]

where the constant \( C \) might depend on \( \mu \). But then, using the fact that \( -S = S \), we can write

\[
a_d(\lambda, j_2^2 - j^2)^2 = C^2 \prod_{k \in S} ((\lambda - (j_2^2 - k^2 - 2j_2k)(\lambda - j^2 - k^2 + 2j_2k))
\]

\[
= C^2 \prod_{k \in S} ((\lambda - j_2^2 - k^2)^2 - 4j_2^2k^2)
\]

\[
= C^2 \prod_{k \in S} ((\lambda - \mu - j_2^2 - k^2)^2 - 4(j_2^2 + \mu)k^2).
\]

Since \( \mu \) can take infinitely many values, it follows that \( C^2 = 1 \) and

\[
a_d(\lambda, \mu)^2 = \prod_{k \in S} ((\lambda - \mu - j_2^2 - k^2)^2 - 4(j_2^2 + \mu)k^2).
\]

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On the other hand
\[(\lambda - \mu - j^2 - k^2)^2 - 4(j^2 + \mu)k^2 = (\lambda + \mu)^2 - 2(\lambda - \mu)(j^2 + k^2) + (j^2 + k^2)^2 - 4j^2k^2 - 4\mu k^2
\]
\[= \lambda^2 - 2\lambda \mu + \mu^2 - 2\lambda(j^2 + k^2) + 2\mu(j^2 - k^2) + (j^2 - k^2)^2
\]
\[= (-\mu - j^2 + k^2)^2 + 2\lambda(-\mu - j^2 + k^2) + \lambda^2 - 4k^2\lambda
\]
\[= (\lambda - \mu - j^2 + k^2)^2 - 4k^2\lambda.
\]
Thus we have
\[a_d(\lambda, \mu)^2 = \prod_{k \in S} [(\lambda - \mu - j^2 + k^2)^2 - 4k^2\lambda].
\]
When \(k = p^2\), this can be rewritten
\[a_d(p^2, \mu)^2 = \prod_{k \in S} (-\mu - j^2 + (k + p)^2)^2,
\]
again using the fact that \(S = -S\). Since the coefficient of \(\mu^d\) in \(a_d(\lambda, \mu)\) is \((-1)^d\), we deduce that
\[a_d(p^2, \mu) = (-1)^d \prod_{k \in S} (\mu + j^2 - (k + p)^2),
\]
as required.

We now explicitly construct all these primary fields. First we recall a version of the vertex operator formulation of the boson–fermion correspondence. We define the shift operator \(U\) on fermionic Fock space \(\mathcal{F}\) by
\[U(e_1 \wedge e_2 \wedge e_3 \wedge \cdots) = e_{i+1} \wedge e_{i+2} \wedge e_{i+3} \wedge \cdots.
\]
By definition \(U\) is unitary and \(UE_i U^* = e_{i+1}\). Thus uniquely specifies \(U\) up to a scalar multiple, because if \(U'\) is another such unitary, then \(V = U'U^*\) commutes with the \(e_i\)'s and their adjoints. Since the vacuum vector is characterised up to a scalar multiple by the conditions \(e_i \Omega = 0\) for \(i \geq 0\) and \(e_i^* \Omega = 0\) for \(i < 0\), \(V \Omega = \zeta \Omega\) for some \(\zeta \in \mathbb{T}\). Since \(\Omega\) is cyclic for the \(e_i\)'s and their adjoints, it follows that \(V = \zeta I\). The shift operator has the following properties, which show in particular that it defines by conjugation an automorphism of \(\mathfrak{a}\):

**Lemma A.**

(a) \(UE_i U^* = e_{i+1}\)

(b) \(UL_k U^* = L_k + a_k + \frac{1}{2} \delta_{k,0} I\)

(c) \(Ua_n U^* = a_n + \delta_{n,0} I\)

(d) The system \(\mathfrak{a}\), \(U\) with the relations \(Ua_n U^* = a_n + \delta_{n,0} c, UdU^* = d + a_0 + \frac{1}{2} c\) and \(d = a_0^2/2 + \sum_{n>0} a_n a_n\)

has a unique irreducible positive energy representation with \(c\) acting as the scalar \(I\) and \(a_0\) having integer eigenvalues.

(e) If \(m\) is a positive integer, the system \(\mathfrak{a}\), \(V\) with the relations \(Va_n V^* = a_n + ma_0 + \frac{m^2}{2} c\) and \(d = a_0^2/2 + \sum_{n>0} a_n a_n\)

has \(m\) inequivalent irreducible positive energy representations with \(c\) acting as the scalar \(I\) and \(a_0\) having integer eigenvalues. These representations can be characterised as having a lowest energy vector of charge \(i\) and energy \(i^2/2\) with \(i = 0, \ldots, m - 1\). For each \(i\), the other eigenvalues of \(a_0\) are congruent to \(i\) modulo \(m\).

**Proof.** (a) has already been proven. For (b), note that
\[
[L_k - UL_k U^*, e_i] = -(i + (k + 1)/2)e_{i+k} + (i + (k - 1)/2)e_{i+k} = -e_{i+k} = [-a_k, e_i].
\]

Hence \(UL_k U^* - L_k - a_k\) commutes with the \(e_i\)'s; the same relation for \(-k\) in place of \(k\), shows that it also commutes with their adjoints. Thus it carries \(\Omega\) onto a scalar multiple of itself and by cyclicity is equal to multiplication by this scalar. The scalar need only be computed for \(k \geq 0\), by hermiticity. When \(k > 0\) we get 0 while for \(k = 0\), we get \(UL_0 U^* = L_0 + a_0 + 1/2\). For (c), note that \(Ua_n U^* - a_n\) commutes with the \(e_i\)'s and their adjoints so is a scalar operator, which only need be calculated for \(n \geq 0\) by looking at how it acts on \(\Omega\). For \(n > 0\) we get 0, while for \(n = 0\), we get \(Ua_0 U^* - a_0 = I\). Finally to prove (d), in
the inner product space $\mathcal{H}$ take a vector $\xi_0$ of lowest energy which is an eigenvector of $a_0$. Thus $d\xi = h\xi$ and $a_0\xi = \mu\xi$. Then the vectors $\xi_n = U^{-n}\xi_0$ satisfy $a_0\xi_n = (\mu + n)\xi_n$ and $d\xi_n = \frac{1}{2}n^2\xi_n$. Each $\xi_n$ generates an irreducible $\mathfrak{a}$-module, necessarily mutually orthogonal since the eigenvalues of $a_0$ are different on each. By irreducibility their direct sum gives the whole inner product space. But this decomposition shows that, after subtracting $hI$ from $d$ and $\mu I$ from $a_0$, $\mathcal{H}$ can be written as a tensor product $K \otimes E$ where $K$ is the standard irreducible representation of $(a_n) (n \neq 0)$ and $d$ and $E$ is the standard inner product space with orthonormal basis $e_n$ on which $Ue_n = e_{n+1}$, $de_n = \frac{1}{2}n^2e_n$ and $a_0e_n = ne_n$. To prove (e), note that taking $V = U^m$, the space $E = \mathbb{C}[\mathbb{Z}]$ splits up into $m$ cosets for the subgroup $m\mathbb{Z}$ corresponding to the subgroup generated by $V$. Moreover each coset gives an irreducible summand with the vectors $\Omega_i (i = 0, \ldots, m - 1)$ giving the corresponding lowest energy vectors. The representations are inequivalent because $i$ is specified as the smallest non-negative solution $\lambda$ of $a_0v = \lambda v$, $L_0v = \frac{1}{2}\lambda^2v$.

**Corollary.** In the setting of (e), the Virasoro operators associated with the $a_n$’s satisfy $VL_kV^* = L_k + ma_k + \frac{m^2}{2}k_0 I$.

**Proof.** It suffices to note that this holds for $V = U^m$.

Now for $m \in \mathbb{Z}$ we define the formal power series $z$ and $z^{-1}$

$$
\Phi_m(z) = U^{-m}z^{ma_0} \exp(\sum_{n>0} \frac{m \cdot z^n a_{-n}}{n}) \exp(\sum_{n<0} \frac{m \cdot z^n a_{-n}}{n}) = \sum \Phi_m(n) z^{-n}.
$$

If we define

$$
E^m_\pm(z) = \exp(\sum_{\pm n>0} \frac{-m \cdot z^{-n} a_n}{n}),
$$

then

$$
\Phi_m(z) = U^{-m}z^{ma_0} E^m_\pm(z) E'^m_\pm(z).
$$

To see that this is well defined, note that, when applied to a finite energy vector $\xi$, this becomes a formal power series in $z$ multiplied by a power of $z^{-1}$, since $E^m_\pm(z)\xi$ is a polynomial in $z^{-1}$ with vector coefficients. We shall simply write $E^m_\pm(z)$ for $E'^m_\pm(z)$. The notation has been chosen consistently so that, for $m > 0$,

$$
E^m_\pm(z) = E_\pm(z)^m.
$$

**Fubini–Veneziano relations.** $[a_i, \Phi_m(z)] = mz^i \Phi_m(z)$ and $[L_k, \Phi_m(z)] = z^{k+1} \Phi'_m(z) + \frac{m^2}{2}(k+1)z^k \Phi_m(z)$.

This result can be proved in a number of ways, the most conceptual using operator product expansions, which is now usually formulated within the language of vertex algebras. The original proof was along those lines and is described in [14]. Here we give a very elementary proof relying on the boson–fermion duality and no detailed computations. In Appendix A we give two other proofs of the relations: the first is a simple indirect proof based on a uniqueness result for primary fields associated with the system $(a_n), U, L_0$; the second is a direct verification by brute force along the lines sketched by Frenkel and Kac [10].

**Lemma B.** Let $A$ be a formal power series in $z$ (or $z^{-1}$) with operator coefficients and let $D$ be an operator such that $[D, A]$ commutes with $A$. Then $[D, e^A] = [D, A]e^A$.

**Proof.** We have $[D, A^N] = \sum_{p+q=N-1} A^p[D, A]A^q = N[D, A]A^{N-1}$.

**Corollary.** (a) $[a_n, E_+(z)] = 0 = [a_{-n}, E_-(z)]$ if $n \geq 0$.
(b) $[a_n, E_-(z)] = z^n E_-(z)$, $[a_{-n}, E_+(z)] = z^{-n} E_+(z)$ if $n > 0$.
(c) $[L_0, E_+(z)] = (\sum_{n>0} a_nz^{-n}) E_+(z)$, $[L_0, E_-(z)] = (\sum_{n>0} a_{-n}z^{-n}) E_-(z)$.

**Proof.** These formulas are straightforward consequences of the lemma, setting $D = a_n$ or $L_0$ and

$$
A = \sum_{\pm n>0} \frac{m}{n} a_n z^{-n}.
$$
Proof of the Fubuni–Veneziano relations. If \( i \neq 0 \), then \( a_i \) commutes with \( U \) and \( A_0 \), while
\[
[a_i, E_-(z)E_+(z)] = mz^iE_-(z)E_+(z).
\]
Hence \([a_i, \Phi_m(z)] = mz^i\Phi_m(z)\). For \( i = 0 \), we have \( U^m a_0 U^{-m} = a_0 + mI \) and hence \([a_0, U^{-m}] = mU^{-m}\), so that \([a_0, \Phi_m(z)] = m\Phi_m(z)\). It follows that \( \Phi_m(z) \) carries \( F[k] \) into \( F[k + m] \).

If \( k = 0 \), \( UL_0U^* = L_0 + a_0 + \frac{1}{2} I \), so that it follows by induction that
\[
U^m L_0 U^{-m} = L_0 + ma_0 + \frac{m^2}{2} I.
\]
Premultiplying by \( U^{-m} \), we get
\[
[L_0, U^{-m}] = U^{-m}(ma_0 + \frac{m^2}{2} I).
\]
Hence
\[
[L_0, \Phi_m(z)] = U^{-m}z^{mav}(ma_0 + \frac{m^2}{2})E_-(z)E_+(z) + U^{-m}z^{mav}(B_-E_-E_+ + E_-B_+E_+),
\]
where \( B_{\pm}(z) = \sum_{\pm n > 0} ma_n z^{-n} \). On the other hand
\[
\frac{dz\Phi_m(z)}{dz} = U^{-m}(ma_0)z^{ma_0}E_-E_+ + U^{-m}z^{-ma_0}(B_-E_-E_+ + E_-B_+E_+).
\]
Thus
\[
[L_0, \Phi_m(z)] = z\Phi_m'(z) + \frac{m^2}{2} \Phi_m(z).
\]
We also have the relations
\[
U\Phi_m(z)U^* = z^m\Phi_m(z), \quad z^{-km}\Phi_m(z)|_{z=0} = \Omega_{k+m}.
\]
The first is an immediate consequence of the relation \( Ua_0U^* = a_0 + I \). The second follows from the first and the relation \( \Phi_m(z)\Omega|_{z=0} = \Omega_m \).

We shall say that \( \phi(n) : F \to F \) is a primary field for the system of operators \((a_n)\), \( U \) and \( L_0 \) if
\[
[a_n, \phi(k)] = m\phi(k + n), \quad [L_0, \phi(n)] = -(n + \mu)\phi(n), \quad U\phi(n)U^* = \phi(n + m),
\]
for some \( m \in \mathbb{Z} \) and \( \mu \in \mathbb{R} \). In particular \( \phi(n) = \Phi_m(n) \) satisfies these conditions with \( \mu = -m^2/2 \). Note that if \( \phi(n) \) is a primary field, so too is the shifted field \( \psi(n) = \phi(n + k) \) for any \( k \in \mathbb{Z} \). We now show that after a suitable shift \( \phi(n) \) is proportional to the modes of the field \( \Phi_m(z) \). Since \( U, a_n \) act cyclically on \( \Omega \), if \( \phi(n)\Omega = 0 \), then \( \phi \) is identically zero. Take \( n \) minimal subject to \( \phi(n)\Omega \neq 0 \): this is possible because \( F \) is a positive energy representation. Shifting \( \phi \) if necessary we may assume \( n = 0 \). Hence \( a_i\xi = 0 \) for all \( i > 0 \). Moreover \( a_0\xi = m\xi \). It follows that \( \xi \) is proportional to \( \Omega_m \). Scaling if necessary, we may assume that \( \xi = \Omega_m \). Taking \( k = n \) in \([L_0, \phi(k)] = -(k + \mu)\phi(k)\), we get \( \mu = m - m^2/2 \). We have the following uniqueness result for primary fields:

**Lemma C.** (a) Let \( K_i \) be irreducible representations of \( a \) with lowest energy vectors \( \xi_i \) with \( a_0\xi_i = \lambda_i\xi_i \). Let \( \phi(n) : K_1 \to K_2 \) be operators such that \([d, \phi(n)] = -(n + \mu)\phi(n) \) and \([a_k, \phi(n)] = \nu\phi(n + k) \) for \( n \in \mathbb{R} \) fixed. If \( \phi(n)\xi_1 = 0 \) for all \( n \), then \( \phi(n) \equiv 0 \).

(b) Let \((a_n, d, U)\) be the bosonic system of operators acting on \( F \) and suppose that \( \phi(n) : F \to F \) satisfies
\[
[d, \phi(n)] = -(n + \mu)\phi(n), \quad [a_k, \phi(n)] = \nu\phi(n + k) \quad \text{and} \quad U\phi(n)U^* = \phi(n + a) \quad \text{for some} \ \mu, \nu, a \in \mathbb{Z}.
\]
Then if \( \phi(n)\Omega = 0 \), we have \( \phi(n) \equiv 0 \). In particular if \( \phi(n)\Omega, \Omega_0 \equiv 0 \), then \( \phi(n) \equiv 0 \).

(c) Let \((a_n, d, U)\) be the bosonic system of operators acting on \( F \) and define \( a_n^{(1)} = a_n \otimes I, a_n^{(2)} = I \otimes a_n, U_1 = U \otimes I, U_2 = I \otimes U \) on (the \( \mathbb{Z}_2 \)-graded tensor product) \( F \otimes \mathbb{Z}_2 \). Suppose that \( \phi(n) : F \otimes \mathbb{Z}_2 \to F \otimes \mathbb{Z}_2 \) satisfies
\[
[d, \phi(n)] = -(n + \mu)\phi(n), \quad [a_k^{(i)}, \phi(n)] = \nu_i\phi(n + k) \quad \text{and} \quad U_i\phi(n)U_i^* = \phi(n + a) \quad \text{for some} \ \mu, \nu, a \in \mathbb{Z}.
\]
Then if \( \phi(n)\Omega \otimes \Omega = 0 \), we have \( \phi(n) \equiv 0 \).
Proof. (a) From the commutation conditions it follows that $(\phi(n)p(a_{-1}, a_{-2}, \ldots))\xi_1, \xi_2) = 0$ for any polynomial $p$. Thus $(\phi(n)\xi, \xi_2) = 0$ for all $\xi$. But then by the commutation relations $(\phi(n)\xi, p(a_{-1}, a_{-2}, \ldots)\xi_2) = 0$ for any polynomial $p$. But then $(\phi(n)\xi, \eta) = 0$ for all $\xi, \eta$ and hence $\phi(n) \equiv 0$.

(b) Note that by (a) it suffices to show that $(\phi(n)\Omega_r, \Omega_s) = 0$ for all $r, s$. But then

$$(\phi(n)U^r\Omega, U^s\Omega) = (\phi(n-ar)\Omega, U^{s-r}\Omega) = (\phi(n-ar)\Omega, \Omega_{r-s}) = 0.$$  

Finally $\phi(n) : \mathcal{F}_0 \to \mathcal{F}_\nu$, so by (b), the second condition implies that $\Phi(n)\Omega = 0$.

(c) Let $\Omega'$ be an eigenvector of $d$ in $\mathcal{F}$ and define $\psi(n)$ by

$$(\psi(n)\xi, \eta) = (\phi(n)\Omega \otimes \xi, \Omega' \otimes \eta).$$

Then by $\psi(n)$ satisfies the conditions in (b), so $\psi(n) = 0$. Since $\Omega'$ was arbitrary, it follows that $\phi(n)\Omega \otimes \xi = 0$ for all $\xi$. Fixing $\xi_2, \eta_2$ and defining $\psi(n)$ by

$$(\psi'(n)\xi, \eta) = (\phi(n)\xi \otimes \xi_2, \eta \otimes \eta_2),$$

we deduce that $\psi'(n) = 0$. Hence $\phi(n) \equiv 0$.

This uniqueness result allows us to prove the boson–fermion correspondence in its vertex operator formulation.

Example 1. $\Phi_1(z) = \sum e_n z^{-n-1}$ and $\Phi_{-1}(z) = \sum e^*_n z^{-n}$. Indeed both sides of these identities satisfy the same covariance relations with $\Phi_1(-1)\Omega = \Omega_1 = e_{-1}\Omega$ and $\Phi_{-1}(0)\Omega = \Omega_{-1} = e_0^*\Omega$.

As a consequence we get

$$\frac{[L_n, \Phi_{-1}(z)]}{z^{n+1}\Phi_{-1}(z)} = \frac{1}{2}\frac{n+1}{n+1} z^n\Phi_{-1}(z).$$

We claim that this implies the Fubini–Veneziano relations for $\Phi_{\pm m}(z)$. Indeed $(\ast)$ immediately implies the relations for the $m^2$-fold $\mathbb{Z}_2$-graded tensor product

$$\Theta_{\pm}(z) = \Phi_{\pm 1}(z) \otimes \Phi_{\pm 1}(z) \otimes \cdots \otimes \Phi_{\pm 1}(z)$$

on $\mathcal{F}^\otimes m^2$. But this is just the vertex operator constructed with the operators

$$A_n = a_n \otimes I \otimes I \otimes \cdots \otimes I \otimes \cdots \otimes I \otimes a_n$$

and

$$V = U \otimes U \otimes \cdots \otimes U$$

on $\mathcal{F}^\otimes m^2$. Now

$$[A_p, A_q] = m^2 p\delta_{p+q,0} I, \quad VA_0V^* = A_0 + m^2 \cdot I.$$

Thus letting $a'_n = m^{-1}A_n$, we have

$$[a'_p, a'_q] = p\delta_{p+q,0} I, \quad VA_0V^* = a'_0 + m \cdot I.$$

The operators $L_k$ on $\mathcal{F}^\otimes m^2$ satisfy

$$[L_k, a'_n] = -na'_{n+k}, \quad VL_kV^* = L_k + ma'_k + \frac{m^2}{2}\delta_{k,0} I, \quad [L_k, \Theta_{\pm}(z)] = z^{k+1}\frac{d}{dz}\Theta_{\pm}(z) + \frac{m^2}{2}(k+1)z^k\Theta_{\pm}(z).$$

The first two relations are also true for the Virasoro operators $L'_k$ constructed from the $a'_n$’s, by the corollary to Lemma A. Thus $L_k - L'_k$ commutes with $a'_n$ and $V$, and therefore the third identity is valid with $L'_k$ replacing $L_k$. This holds on any irreducible submodule for the $a'_n$’s and $V$, in particular those on which $a'_0$ has integer eigenvalues. It is easy to see that the eigenvalues of $A_0$ run through all possible integers, so that
those of $a_0^* = m^{-1}a_0$ in particular include all integers. Thus the Fubini–Veneziano relations are satisfied
on any irreducible representation of the type discussed in part (e) of Lemma A. But these are exactly the irreducible
representations for $U^m$ and a that occur in $\mathcal{F}$, so the Fubini–Veneziano relations hold on $\mathcal{F}$.

We now pass to $\mathcal{F}_2 = \mathcal{F} \otimes \mathcal{F}$. The unitary operators $V = U \otimes U^*$ and $W = U \otimes U$ act on this space. Now that $V$ commutes with the action of $d_n = a_n \otimes I + I \otimes a_n$ and therefore leaves its multiplicity space invariant, since these can be indentified with its space of singular vectors as explained above. More explicitly invariant under $b_n = a_n \otimes I - I \otimes a_n$ is invariant under $V$. Similarly any module invariant under $b_n$ is invariant under $W$.

As a consequence the two level one representations of $\mathcal{H}_2$ are invariant under the vertex operators

$$\Psi_m(z) = \Phi_m(z) \otimes \Phi_{-m}(z) = V^{-m} z^{mb_0} \exp\left(\sum_{n>0} \frac{z^m b_{-n}}{n}\right) \exp\left(\sum_{n<0} \frac{m z^n b_{-n}}{n}\right) = \sum \Psi_m(n) z^{-n}.$$ 

The two systems $(b_n)$, $V$ and $(d_n)$, $W$ are examples of systems $(A_n)$, $U$

$$[A_m, A_n] = 2m \delta_{m+n,0} \cdot I, \quad U A_m U^* = A_m + 2 \delta_{m,0} I.$$ 

Thus if $L_0 = \frac{1}{2} A_0^2 + \sum_{n>0} A_{-n} A_n$, then we have

$$U L_0 U^* - L_0 = \frac{1}{2} [(A_0 + 2I)^2 - A_0^2] = A_0 + 2I.$$ 

Such a system has two inequivalent positive energy representations with $A_0$ acting with integer eigenvalues, either all odd or all even. Thus both sets of operators $H(n), V$ and $K(n), w$ have only two irreducible representations with $H(0)$ and $K(0)$ having half integer eigenvalues. In one the eigenvalues run through the whole of $\mathbb{Z}$, and in the other through $\frac{1}{2} + \mathbb{Z}$. It follows that operators $H(n), V$ act irreducibly on each $\mathcal{H}_j$ and that the $K(n), W$ act irreducibly on the associated multiplicity spaces. The operator $V$ also has a compatibility with the operators $E(n)$ and $F(n)$ on $\mathcal{F}^\otimes 2$:

$$V E(n) V^* = E(n+2), \quad V F(n) V^* = F(n-2).$$

In fact more generally

$$U_1 E(n) U_1^* = E(n+1), \quad U_2 E(n) U_2^* = E(n-1), \quad U_1 F(n) U_1^* = F(n+1), \quad U_2 F(n) U_2^* = F(n+1).$$

Indeed it suffices to show the result for $E(n)$ since $E(n)^* = F(-n)$. The differences $X = U_1 E(n) U_1^* - E(n+1)$ and $Y = U_2 E(n) U_2^* - E(n-1)$ commute with the fermionic operators $v_i$ and $v_i^*$, so that $X(\Omega \otimes \Omega)$ and $Y(\Omega \otimes \Omega)$ are proportional to $\Omega \otimes \Omega$. Hence $X$ and $Y$ are scalar operators. Since $[H, X] = X$ and $[H, Y] = Y$, we must have $X = 0 = Y$.

**Example 2 (Frenkel–Kac–Segal).** $E(z) = \Psi_1(z)$ and $F(z) = \Psi_{-1}(z)$ on $\mathcal{F}^\otimes 2$ and hence on the representations $H_0$ and $H_{1/2}$. Indeed the result follows from part (c) of Lemma C since, as we have just checked, both sides of these identities satisfy the same covariance relations and

$$\Psi_1(0)(\Omega \otimes \Omega) = \Omega_1 \otimes \Omega_{-1} = E(-1)(\Omega \otimes \Omega), \quad \Psi_{-1}(0)(\Omega \otimes \Omega) = \Omega_{-1} \otimes \Omega_1 = F(-1)(\Omega \otimes \Omega).$$

Graeme Segal found explicit formulas for the Goldstone vectors in the vacuum representation $H_0$. The same method can be applied in representation $H_{1/2}$. Let $x_i = b_{-i}$ and define operators $c_n (n > 0)$ by the formal power series expansion

$$\Psi_{-}(z) = \exp \sum_{n>0} \frac{b_{-n}}{n} z^n = \sum_{n \geq 0} c_n z^n.$$ 

Thus $c_0 = 1$. For $n < 0$, we define $c_n = 0$. Similarly we can define

$$\Psi_+(z) = \exp \sum_{n<0} \frac{b_{-n}}{n} z^n,$$

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so that
\[ \Psi_m(z) = V^{-m} z^m \psi_0 \Psi_m(z)^m . \]
Evidently
\[ \Psi_+(z) = E_+(z) \otimes E_+(z)^{-1}, \quad \Psi_-(z) = E_-(z) \otimes E_-(z)^{-1}. \]
For any signature \( f_1 \geq f_2 \geq f_3 \geq 0 \), we define
\[ X_f = \det c_{f_1-i+j}. \]

**Remark.** The proofs below can be understood in terms of the combinatorics of the Weyl character formula for \( U(n) \). The characters of the irreducible representations \( \pi_f \) of \( U(n) \) are indexed by signatures \( f_1 \geq f_2 \geq \cdots \geq f_n \) with the character given by \( \chi_f(g) = \text{Tr} \pi_f(g) = \det z_i^{f_i+n-j} / \det z_i^{n-j} \) if \( z_1, z_2, \ldots, z_n \) are the eigenvalues of \( g \). The denominator is the Vandermonde determinant given by \( \Delta(z) = \prod_{i<k} (z_i - z_k) \). The group \( U(n) \) acts irreducibly on \( S^k C^n \): it is the representation with signature \((k,0,\ldots,0)\). The group \( U(m) \times U(n) \) acts on \( S^k (\otimes C^m \otimes C^n) \) and the character if this representation on an element \((g,h)\) is given by \( \sum f_i \geq 0, |f| = k \chi_f(g) \chi_f(h) \), where \(|f| = f_1 + \cdots + f_n\). From this it follows that \( \chi_f(g) = \det C_{k,\cdot-i+j} \) (the Jacobi-Trudy identity) if \( C_k = \chi_{(k,0,\ldots,0)}(g) \). Moreover the following generating function identity holds:
\[
\prod_{i=1}^n f(z_i) = \sum_{j \geq 0} \chi_f(z) \cdot \det c_{j-i+j},
\]
where \( f(z) = 1 + c_1 z + c_2 z^2 + \cdots \) (with \( c_0 = 1 \) and \( c_i = 0 \) for \( i < 0 \)). Note that the generating function for the \( X_j \)'s giving the characters has \( X_j = \sum z_i^j \).

**Lemma D.** If \( A \) and \( B \) are formal power series of operators in \( z \) and \( w^{-1} \) such that \( C = [A,B] \) is a multiple of the identity operator, then \( e^A e^B = e^{C} e^B e^A \).

**Proof.** By Lemma B, \([e^A,B] = Ce^A \) so that \( e^A B e^{-A} = B + C \). Hence
\[ e^A e^B e^{-A} = \exp e^A B e^{-A} = e^{B+C} = e^C e^B. \]

**Corollary.**
(a) \( E_+(z) E_-(w) = (1 - w/z)^{-1} E_-(w) E_+(z) \).
(b) \( E_+(z)^{-1} E_-(w)^{-1} = (1 - w/z)^{-1} E_-(w)^{-1} E_+(z)^{-1} \).

**Proof.** To prove (a), let \( A = \sum_{n>0} a_n z^{-n} / n \) and \( B = \sum_{n<0} a_n w^{-n} / n \). Then
\[ C = [A,B] = \sum_{n>0} z^{-n} w^{-n} / n = - \log(1 - w/z), \]
so that
\[ e^C = \exp - \log(1 - w/z) = (1 - w/z)^{-1}. \]

(b) can be proved similarly or by taking formal inverses in (a).

We now establish the formulas for Goldstone's vectors. As in Section 4, where the Goldstone vectors are defined, let \( \xi_j \) be a lowest energy unit vector in \( H_j \) with \( H \xi_j = j \xi_j \). Then for \( m \in \mathbb{Z} \) we may take \( \xi_{m+j} = V^{-m} \xi_j \).

**Lemma E (Goldstone’s formulas).** If \( k \geq 0 \) is a half integer and \( m \geq 0 \) is an integer, the Goldstone vector in \( H_{j[-k]} \) of energy \((k+m)^2\) is proportional to \( \xi_f \) where \( f_1 = 2k + m, \ldots, f_m = 2k + m \) and \( f_i = 0 \) for \( i > m \) and the Goldstone vector in \( H_{j[k]} \) of energy \((k+m)^2\) is proportional to \( \xi_f \) where \( f_1^t = m, \ldots, f_{2k+m} = m \) and \( f_i^t = 0 \) for \( i > 2k + m \) (the transposed signature).

**Remark.** In [32], Segal only treats the case \( j = 0 \) and his statement is not quite correct since it omits the transpose. Note that the passage from \(-k\) to \( k \) is equivalent to replacing the variables \( X_n \) by \(-X_n\).
(and hence inverting the generating series $\sum c_n z^n$). It corresponds to the action of the Weyl group element $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which establishes an isomorphism between $\mathcal{H}_j[-k]$ and $\mathcal{H}_j[k]$. Indeed the automorphism $b_n \mapsto -b_n$ fixes the Virasoro algebra generators and changes the sign of the eigenvalue of $b_0$.

**Proof.** By the corollary to Lemma D,

$$\Psi_+(z)\Psi_-(w) = (1 - \frac{w}{z})^{-2}\Psi_-(w)\Psi_+(z).$$

On the other hand the Goldstone vector $\xi_+ \in \mathcal{H}_j[k]$ is proportional to $E(0)^{m+2k}\xi_{-m-k}$ and the Goldstone vector $\xi_- \in \mathcal{H}_j[-k]$ is proportional to $E(0)^{m}\xi_{-m-k}$. Let $p = p_\pm = m + k \pm k$. Thus $\xi_\pm$ is proportional to $E(0)^{p}\xi_{-m-k}$. Now $\Psi(z) \equiv \Psi^1(z) = \sum E(n)z^{-n-1}$. Hence $\xi_\pm$ is proportional to the coefficient of $z_i^{-1}\cdots z_j^{-1}$ in $\Psi(z_1)\cdots\Psi(z_p)\xi_{-k-m}$. On the other hand if $C(z) = \sum c_n z^n$, then

$$\Psi(z_1)\cdots\Psi(z_p)\xi_{-k-m} = \prod_{i<j} (1 - \frac{z_i}{z_j})^2 z_1^{2(p-1)}z_2^{2(p-2)}\cdots z_p^{2(p-2)} \cdot \Psi(z_1)\cdots\Psi(z_p)\xi_{\pm k}$$

$$= \prod_{i<j} (z_1z_2\cdots z_p)^{-2k-2m+p}C(z_1)\cdots C(z_p) \prod_{i<j} (z_i - z_j)(z_i^{-1} - z_j^{-1})\xi_{\pm k}.$$ 

Now we already noted that

$$C(z_1)\cdots C(z_p) = \sum_{g_{p+1}=0} \chi_g(z) \cdot \det c_{g_{i-i+j}}.$$ 

Thus we are looking for the constant coefficient in

$$(z_1z_2\cdots z_p)^{-2k-2m+p}C(z_1)\cdots C(z_p) \prod_{i<j} (z_i - z_j)(z_i^{-1} - z_j^{-1})\xi_{\pm k}.$$ 

By the combinatorics of the Weyl integration formula, the constant coefficient in

$$\chi_f(z^{-1})\chi_g(z) \prod_{i<j} (z_i - z_j)(z_i^{-1} - z_j^{-1})$$

equals $p!\delta_{f,g}$ (the usual orthogonality relations). Thus $\xi_-$ is proportional to $X_f\xi_{-k}$ with $f_1 = 2k + m, \ldots, f_m = 2k + m$ and $f_i = 0$ for $i > m$ and $\xi_+$ is proportional to $X_f\xi_{k}$ with $f'_1 = m, \ldots, f'_{2k+m} = m$ and $f'_i = 0$ for $i > 2k + m$. Thus $f'$ is the transposed Young diagram.

Following Segal, we now check this directly without using any improved combinatorial results on symmetric functions. We have

$$\Delta(z^{-1}) = \det z_i^{-(p-j)}$$

and similarly

$$\Delta(z) = \sum_{\sigma \in S_p} \varepsilon(\sigma) z_1^{p-1}\cdot \sigma(2)\cdots z_{\sigma(p-1)}.$$ 

We are looking for the constant term in

$$C(z_1)\cdots C(z_p)(z_1 \cdots z_p)^{-2k-2m+p}\Delta(z^{-1}) \sum_{\sigma \in S_p} \varepsilon(\sigma) z_1^{p-1}\cdots z_{p-1}.$$ 

Since $\Delta(z^{-1})$ satisfies $\sigma\Delta(z^{-1}) = \varepsilon(\sigma)\Delta(z^{-1})$ and the other terms are invariant under permutation, this is the same as the constant term in

$$p!C(z_1)\cdots C(z_p)(z_1 \cdots z_p)^{-2k-2m+p} \det(z_i^{-(p-j)}) z_1^{p-1}\cdots z_{p-1}$$

$$= p!C(z_1)\cdots C(z_p) \det z_i^{-(2k-2m+p-i)}.$$
But, if the determinant is written as an alternating sum over the symmetric group $S_p$, this is clearly proportional to $\det c_{2k+2m-p+i-j}$, as required.

**Remark.** The proof above is formally similar to the proof of the Weyl character formula for $U(N)$ (or $SU(N)$). Wallach made this link explicit by noticing that the vertex operator formalism allows a map to be defined from central functions on $U(N)$ into the oscillator representations. This map intertwines the action of Virasoro operators with the natural action of the operators defined from central functions on $\mathcal{U}$.

Hence by the Leibniz rule, we have

$$L_1 = \frac{1}{4} \sum_{p+q=1} b_p b_q$$

where $b_0 = 2\mu I$, so that $H(0) = \mu I$. This can be identified with the action on $C[X]$ with $b_{-n} = \sqrt{2}X_n$ and $b_n = \sqrt{2n\mu} \partial X_n$ for $n > 0$. The operator $L_1$ is given by the formula

$$L_1 = \frac{1}{2} b_0 b_1 + \frac{1}{2} \sum_{p \geq 1} b_{-p} b_{p+1}$$

This operator can thus be described in terms of two derivations $D_1$ and $D_2$ on the ring $C[X]$, namely $D_1X_n = nX_{n-1}$ $(n \geq 2)$, $D_1X_1 = 0$ and $D_2 = \partial X_1$. Now if $f(z) = \exp \sum_{n \geq 0} X_n t^n/n = \sum c_n t^n$, then $D_1 f = (\sum_{n \geq 0} X_n t^{n+1}) f = t^2 f'(t)$. It follows that $D_1 c_n = (n-1)c_{n-1}$. Similarly $D_2 f = tf$, so that $D_2 c_n = c_{n-1}$. Let $D = D_1 + \mu D_2$, also a derivation. Since $X_f$ has degree $|f|$ and $D$ decreases the degree by 1, it follows that $D^{|f|} X_f$ is a scalar.

**Lemma F.** $(|f|!)^{-1} D^{|f|} \det c_{f_i-i+j} = \det (\mu^{f_{i-1+j} - f_i-i+j})$.

**Proof.** Consider a term in the determinant $c_{m_1} \cdots c_{m_n}$. The term $c_{m_i}$ has degree $m_i$ so the product has degree $\sum m_i = |f|$. Since $Dc_k = (k-1 + \mu)c_{k-1}$, we have

$$D^k c_k = \mu(\mu+1) \cdots (\mu + k - 1) = k! \cdot \left(\frac{\mu - 1 + k}{k}\right).$$

Hence by the Leibniz rule,

$$D^{|f|} c_{m_1} \cdots c_{m_n} = \frac{|f|!}{m_1! \cdots m_n!} D^{m_1}(c_{m_1}) D^{m_2}(c_{m_2}) \cdots D^{m_n}(c_{m_n})$$

$$= \frac{|f|!}{m_1! \cdots m_n!} \prod_{i=1}^n \left(\frac{\mu - 1 + m_i}{m_i}\right).$$

So applying $(|f|!)^{-1} D^{f_i}$ to the determinant has the effect of replacing each term $c_k$ by $(\mu^{-1} + k)$, as claimed.

**Corollary.** If $p$ is a half-integer, then $(|f|!)^{-1} L_1^{f_i}(X_f) = \alpha_p(f) \xi_p$ where $\alpha_p(f) = \det (2^{p_{f_i-i+j} - f_{i-1+j}})$.

We now calculate the above determinant for the signature $f_i = N$ $(i = 1, m)$, $f_i = 0$ $(i > m)$.
Lemma G. If $N \geq m$ then,

$$\det \left( \frac{\lambda + N - 1 + j - i}{N + j - i} \right)_{i,j=1,m} = \frac{\prod_{j=1}^{N} \prod_{i=1}^{m} (\lambda - i + j)}{\prod_{j=1}^{N} \prod_{i=1}^{m} (N + i - j)}. $$

First proof (Weyl’s character and dimension formulas). Recall that, if $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$ is a signature, then the character of the corresponding irreducible representation $\pi_f$ of the unitary group $U(n)$ is given by

$$\chi_f(g) = \text{Tr} \pi_f(g) = \frac{\det z_i^{f_i + n - i}}{\det z_i^{n - i}},$$

where $z_1, \ldots, z_n$ are the eigenvalues of $g$. The signature defines a Young diagram; the hook length of any node in the Young diagram is defined to be one plus the sum of the number of nodes in the same column or row after that node. If we place $n - i + j$ in at the $j$th node of the $i$th row, then the dimension $d_f$ of $\pi_f$ is the product of these numbers divided by the product of the hooklengths. It is equivalently given by the usual Weyl dimension formula

$$d_f = \frac{\prod_{j<i} (f_i - f_j - i + j)}{(n - 1)!!},$$

although this formula is not manifestly written as a polynomial in $n$. If $f_1 = r$ and $f_1 = 0$ for $i > n$, then $\chi_r$ is the character of the symmetric power $S^r C^n$, which has dimension $\left(\frac{n+r-1}{r}\right)$.

On the other hand $\chi_f$ can be expressed as a determinant in the $\chi_i$’s:

$$\chi_f = \det \chi_{f_i + j - i},$$

where we set $\chi_r = 0$ for $r < 0$. In particular evaluating the character at 1, we get a formula expressing a determinant of binomial coefficients as a product of linear factors in $n$. Taking a Young diagram with $m$ rows and $K + m$ boxes in each row, we have $f_i = K + m$ ($1 \leq i \leq m$) and $f_i = 0$ for $i > m$. Hence

$$\det \left( \frac{n - 1 + K + m + j - i}{K + m + j - i} \right)_{i,j=1,...,m} = \frac{\prod_{j=1}^{K+m} \prod_{i=1}^{m} (n - i + j)}{\prod_{j=1}^{K+m} \prod_{i=1}^{m} (K + m + i - j)}. $$

The stated determinantal formula follows by setting $\lambda = n$ and $N = K + m$.

Remark. Although we shall not need this result, the dimension formula implies more generally that $\det \left( \frac{\mu + f_i - j + 1}{f_i - i + 1} \right)$ is the product of factors $\mu + j - i$, when there is a box in the $i$th row and $j$th column of the Young diagram, divided by the product of hooklengths of the diagram. In other words if $\mu = -p$, with $p \geq 1$ an integer, then $((f!!)^{-1} D^{|J|} X_f)$ equals the dimension of the representation $\pi_f$ of $U(p)$. It is therefore strictly positive when $f_{p+1} = 0$ and vanishes if $f_{p+1} \neq 0$. Since $L_{-1} \xi_0 = 0$, all these expressions vanish for $p = 0$.

Second proof. We have to show that

$$\det \left( \frac{a + N + i - j}{N + i - j} \right)_{i,j=1,m} = \prod_{p=1}^{N} \prod_{q=1}^{m} \frac{a + 1 + p - q}{N - p + q}.$$

We subtract a multiple of the second column from the first column to make the last entry of the first column 0; then we subtract a multiple of the third column from the second column to make the last entry of the second column 0, and so on. After these modification the $(i,j)$ entry for $i,j < n$ is modified from $\left( \frac{N+a+i-j}{a} \right)$ to $\left( \frac{N-1+a+i-j}{a} \right)(m-i)/(N + m - j)$. Thus writing $F_m(a,N) = \det \left( \frac{a + N + i - j}{N + i - j} \right)_{i,j=1,m}$, we have

$$F_m(a,N) = \left( \frac{a+N}{N+m} \right) \cdot F_{m-1}(a,1,N-1).$$
This may be rewritten as
\[ F_m(a, N) = \frac{(a + 1) \cdots (a + N)}{m(m + 1) \cdots (m + N - 1)} \cdot F_{m-1}(a - 1, N). \]

Thus
\[ F_m(a, N) = (m - 1)! \prod_{i=1}^{m} \prod_{j=1}^{N} a + 1 + j - i \prod_{i=1}^{m} \prod_{j=1}^{N} (N + i - j). \]

**Corollary.** If \( f_i = N \) for \( 1 \leq i \leq m \) and \( f_i = 0 \) for \( i > m \), then \( (|f|!)^{-1} L_1^{|f|} (X_f \xi_p) = \alpha_f(p) \xi_p \) where
\[ \alpha_p(f) = \prod_{j=1}^{N} \prod_{i=1}^{m} (p - i + j) \prod_{j=1}^{N} \prod_{i=1}^{m} (N + i - j). \]

**Lemma H.** For \( k \) an integer, \( \Psi_k(z) \xi_0 = e^{zL_{-1}} \xi_k \).

**Proof.** Let \( f(z) = \Psi_k(z) \xi_0 \). We have \([L_{-1}, \Psi_k(z)] = d\Psi_k/dz\) and \( L_{-1} \xi_0 = 0 \). Hence \( df(z)/dz = L_{-1} f(z) \).

Since \( f(0) = \xi_k \), the result follows.

**End of proof.** We shall just use the fact that if \( f \) is a signature as above with \( f_{p+1} = 0 \) with \( p \geq 1 \), then
\[ (L_1^{|f|} X_f \xi_p, \xi_p) \neq 0. \]

We fix an integer \( m \geq 1 \) and a half integer \( k \geq 0 \). Since we can take adjoints, it will suffice to show that if \( i \) is a non-negative half integer with \( i \geq k \) an integer and \( |m - k| \leq i \leq m + k \), then there are Goldstone vectors \( \xi_1, \xi_2 \) in \( \mathcal{H}_i \) with \( \xi_1 \) having energy \( k^2 \) or \( i^2 \) and \( \xi_2 \) energy \( i^2 \) or \( k^2 \) such that \( (\Psi_m(z) \xi_1, \xi_2) \neq 0 \). In this case if \( P_1 \) is the projection onto the \( \text{vir} \)-module generated by \( \xi_i \), then \( P_2 \Psi_m(z) P_1 \) or its adjoint is the required primary field of type \( m^2 \) between \( L(1, k^2) \) and \( L(1, i^2) \).

Assume first that \( k \) is an integer and set \( s = |k - m| \). Take \( \xi_k \) in \( H_0 \) or \( H_{1/2} \) and suppose \( i \) satisfies \( |k - m| \leq i \leq k + m \) and \( i - k \in \mathbb{Z} \). We assume further that \( m \geq k \) and \( i \geq m \) and set\[ A(z) = (\Psi_m(z) \xi_{-i}, X_f \xi_{i+m}). \]

where \( X_f \xi_{i+m} \) is a Goldstone vector of energy \( k^2 \). Thus \( f_r = k + i - m \) for \( r = 1, \ldots, k + m - i \) and \( f_r = 0 \) for \( r > k + m - i \). We must show that \( A(z) \neq 0 \).

But
\[
A(z) = (\Psi_m(z) V^r \xi_0, X_f V^{i-m} \xi_0) = z^{-2mi} (\Psi_m(z) \xi_0, X_f \xi_m) = z^{-2mi} (e^{zL_{-1}} \xi_m, X_f \xi_m) = z^{-2mi} (\xi_m, e^{zL_{-1}} X_f \xi_m).
\]

This is non-zero because \( f_{m+1} = 0 \) since \( m + 1 > k + m - i \).

Now suppose that \( m \geq k \) and \( i \leq m \). In this case we consider
\[ B(z) = (\Psi_m(z) \xi_{-k}, X_f \xi_{m-k}), \]

where \( X_f \xi_{m-k} \) is a Goldstone vector of energy \( i^2 \). Thus \( f_r = m - k + i \) for \( r = 1, \ldots, i + k - m \) and \( f_r = 0 \) for \( r > i + k - m \). Then
\[
B(z) = (\Psi_m(z) V^k \xi_0, X_f V^{-m+k} \xi_0) = z^{-2km} (\Psi_m(z) \xi_0, X_f \xi_m) = z^{-2km} (e^{zL_{-1}} \xi_m, X_f \xi_m) = z^{-2km} (\xi_m, e^{zL_{-1}} X_f \xi_m).
\]
which again is non–vanishing since \( m \geq i + k - m \) and hence \( f_{m+1} = 0 \).

Thus if \( k \leq m \) we have shown that there is a non–zero primary field of type \( m^2 \) from \( L(1, k^2) \) to \( L(1, i^2) \) if \( V_i \leq V_m \otimes V_k \), taking adjoints it follows that there is a non–zero primary field of type \( m^2 \) from \( L(1, i^2) \) to \( L(1, k^2) \) if \( i \geq m \) and \( k \leq m \) and \( V_k \leq V_m \otimes V_i \). So now assume that \( k \geq m \) and \( i \geq m \) with \( V_i \leq V_m \otimes V_k \). Since may take adjoints, we many assume that \( i \geq k \). But then the computation with \( A(z) \) above shows that the corresponding primary field exists.

Now suppose that \( k \in 1/2 + \mathbb{Z} \). Let \( W \) be a shift operator such that \( W\xi_0 = f_{1/2} \). Let \( k = p + \frac{1}{2} \) and \( i = q + \frac{1}{2} \). If \( k < m \) and \( i > m \) or \( k > m \) and \( i > m \), then as before we take

\[
A(z) = (\Psi_m(z)\xi_{-i}, X_j\xi_{-i+m}),
\]

with the same choice of \( f \) as before. Then we have

\[
A(z) = (\Psi_m(z)V^qW\xi_0, X_jV^{q-m}W\xi_0) = z^{-2im}(\Psi_m(z)\xi_0, X_j\xi_m) = z^{-2im}(\varepsilon^{-L-1}\xi_m, X_j\xi_m) = z^{-2im}(\xi_m, \varepsilon^{L+1}X_j\xi_m),
\]

which as before is non–zero. Thus the primary fields exist in this case. If \( i < m \) and \( k < m \) we take

\[
B(z) = (\Psi_m(z)\xi_{-i}, X_j\xi_{-i-m}),
\]

with the same choice of \( f \) as before. In this case

\[
B(z) = (\Psi_m(z)V^qW\xi_0, X_jV^{q-m}W\xi_0) = z^{-2km}(\Psi_m(z)\xi_0, X_j\xi_m) = z^{-2km}(\varepsilon^{-L-1}\xi_m, X_j\xi_m) = z^{-2km}(\xi_m, \varepsilon^{L+1}X_j\xi_m),
\]

which as before is non–zero. It follows that in all cases there is a non–zero primary field of type \( m^2 \) between \( L(1, k^2) \) and \( L(1, i^2) \) if \( V_i \leq V_m \otimes V_k \).

9. **Proof of the character formula for \( L(1, j^2) \) using the Jantzen filtration.** Let \( A(x) = \sum_{i \geq 0} A_i x^i \) be an analytic family of non–negative self–adjoint matrices defined for \( x \) real and small on \( V = \mathbb{R}^n \) or \( \mathbb{C}^n \). We define a filtration \( V^{(i)} (i \geq 0) \) by \( V^{(0)} = V \) and for \( m \geq 1 \)

\[
V^{(m)} = \bigcap_{i=0}^{m-1} \ker A_i.
\]

In the examples \( A(x) \) will be invertible for \( x \neq 0 \) in a neighbourhood of 0. It follows that \( V^{(m)} = (0) \) for \( m \) sufficiently large. We call \( (V^{(m)}) \) the Jantzen filtration associated with \( A(x) \).

**Lemma.** The order of 0 as a root of \( \det A(x) \) equals \( k = \sum_{i \geq 1} \dim V^{(i)} \).

**Proof.** We set \( U_0 = V^{(0)} = V \), \( U_1 = V^{(n_1)} \) the first \( k > 0 \) with \( V^{(k)} \neq V^{(0)} \), \( U_2 = V^{(n_2)} \) the first \( k > n_1 \) with \( V^{(k)} \neq V^{(n_1)} \), and so on. We set \( n_0 = 0 \). Let \( v_1, \ldots, v_{n_1} \) be an orthonormal basis of \( U_0 \oplus U_1 \), \( v_{n_1+1}, \ldots, v_{n_2} \) an orthonormal basis of \( U_1 \oplus U_2 \) and so on. Thus \( (A(x) v_i, v_j) = x^{n_s} (A_{n_s} v_i, v_j) \) with higher powers of \( x \) for \( i > m_s \). Moreover the first term vanishes if \( j > m_{s+1} \). On the other hand the matrix \( (A_{n_s} v_i, v_j) \) is invertible. Thus we can divide rows \( m_s + 1, \ldots, m_{s+1} \) by \( x^{n_s} \) and then set \( x = 0 \). The resulting matrix is block triangular with invertible blocks on the diagonal, so it is itself invertible. Hence the order of 0 as a root of \( \det A(x) \) equals

\[
k = \sum_{s=1}^{n_s} (\dim U_{s-1} - \dim U_s) = \sum_{i \geq 1} \dim V_i,
\]
as claimed.

Note that multiplying a bilinear form by (or matrix $A(x)$) by a factor $c_0 + c_1 x + c_2 x^2 + \cdots$ with $c_0 \neq 0$ does not change the Jantzen filtration. In this case we will say that the forms are equivalent.

We shall need an additional simple functoriality property of the Jantzen filtration:

**Lemma.** Let $V$, $W$ be finite-dimensional inner product spaces and for $x \in (-\varepsilon, \varepsilon)$ let $A(x)$, $B(x)$ be self-adjoint non-negative matrices depending analytically on $x$. Let $X(x) \in \text{Hom}(V, W)$ and $Y(x) \in \text{Hom}(W, V)$ be analytic functions of $x$ such that $(X(x)A(x)v, w) = (B(x)Y(x)v, w)$ for all $v \in V$, $w \in W$. Then $X(0)$ and $Y(0)^*$ induce maps of the corresponding Jantzen filtrations which are dual to each other on $V^{(i)}/V^{(i+1)}$ and $W^{(i)}/W^{(i+1)}$.

**Proof.** The condition implies that $X(x)A(x) = B(x)Y(x)$. If $A(x) = \sum A_n x^n$, $B(x) = \sum B_n x^n$, $X(x) = \sum X_n x^n$ and $Y(x) = \sum Y_n x^n$, then

$$\sum_{i+j=n} X_i B_j = \sum_{i+j=n} A_i Y_j.$$  

Thus $X_0 A_0 = B_0 Y_0$ on $V$. The equation (*) implies that $X_0 A_1 = B_1 Y_0$ on $V^{(1)} = \ker A_0 \cap \cdots \cap \ker A_{i-1}$. But then $X_0$ and $Y_0^*$ induce dual maps between $V^{(i)}/V^{(i+1)}$ and $W^{(i)}/W^{(i+1)}$, as claimed.

Now consider a representation $M(1, j^2)$. The point $(c, h) = (1, j^2)$ lies on the curve $\varphi_{p, 1}(c, h) = 0$ where $p = 2j + 1$. The curve is parametrized by a variable $t$ via $c(t) = 13 - 6t - 6t^{-1}$ and $h(t) = (j^2 + j)t - j$, the point $(1, j^2)$ corresponding to the value $t = 1$. If $t \neq 0, \infty$, the Verma module $M_t = M(c(t), h(t))$ is defined. It is a direct sum of finite energy spaces $M_t(n)$ on which $L_0$ acts as multiplication by $n + h(t)$. The space $M_t$ has a canonical invariant bilinear form $B_t(v, w)$ defined on it. Invariance implies that the subspaces $M_t(n)$ are orthogonal. If we take as basis of $M_t(n)$ elements $v_i = L_{-i_1} \cdots L_{-i_s} v_s$ with $i_1 \leq i_2 \leq \cdots$ and $\sum s_i = n$, then all these spaces can be identified with the same space $V = \mathbb{R}^{P(n)}$. Thus $A_t^{(n)}(t) = B_t(v_i, v_j)$ is a symmetric matrix, the entries of which are polynomials in $t$ and $t^{-1}$ with real coefficients. Since the Shapovalov form is invariant under the Virasoro algebra, it follows from the second lemma above that of $M = M(1, j^2)$ then, for $t = 1 + x$ with $x$ small, the Jantzen filtration defines a filtration of $M$ by irreducible submodules $M^{(i)}$. In particular we can calculate $\sum_{i \geq 1} \dim M^{(i)}(n)$. It is exactly the degree of $t = 1$ as a root of the Kac determinant at level $n$ which is known explicitly.

We now use the Jantzen filtration to compute the character of $M(1, j^2)$. From the coset construction, we know that $M(1, j^2)$ contains a Verma submodule $M(1, (j + 1)^2)$; this is also immediate from the Kac determinant formula for $c = 1$:

$$\det_N(1, h) = \prod_{p,q \leq N} (h - \frac{(p-q)^2}{4})^{p(N-pq)}.$$  

Continuing in this way, it has a decreasing chain of Verma submodules $M(1, (j + k)^2)$ with $k \geq 1$. If we fix $h = j^2$ and set $c = 1 + x$, then as a function of $x$ the determinant is proportional to

$$\Psi(x) = \prod_{1 \leq r \leq N} (j^2 + \frac{r^2 - 1}{24} \cdot x) \cdot \prod_{1 \leq s \leq r \leq N} [ (j^2 - \frac{(r-s)^2}{4})^2 
+ ((r^2 + s^2 - 2)j^2 + \frac{1}{2}(rs + 1)(r-s)^2)] \cdot \frac{x}{24} + (r^2 - 1)(s^2 - 1)(\frac{x}{24})^2].$$  

We take $x$ as the parameter of the Jantzen filtration. Since the operators $L_n$ depend polynomially on the parameters $t$ and $t^{-1}$, and hence analytically on $x$, it is evident that when $h = j^2$, they preserve the filtration $M^{(i)}$ of $M = M(1, j^2)$. If $j \geq 0$, then from (1) we get

$$X_{j^2}(q) = \sum_{i \geq 1} \text{ch } M^{(i)} = \sum_{N \geq 0} a(N)q^{N+j^2}.$$  

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where
\[ a(N) = \sum_{1 \leq r \leq N} P(N - r(r + 2)). \]

Thus we obtain
\[ X_{j^2}(q) = \sum_{i \geq 1} \text{ch } M^{(i)} = \varphi(q) \cdot \left( \sum_{r \geq 1} q^{(r + j)^2} \right). \]

(2)

Let \( v_r \) be the singular vector in \( M(1, j^2) \) of energy \((j + r)^2\). Let \( n_r > 0 \) be maximal subject to \( v_r \) \( \in M^{(n_r)} \).

Thus the Verma module generated by \( v_r \) lies in \( M^{(n_r)} \) and \( \beta(w_1, w_2) = (w_1, w_2)_{x} |_{x = 0} \) restricts to a multiple of the Shapovalov form. In particular \( n_r \) is the order of the zero of \( (v_r, v_r)_x \). (Note that for \( x > 0 \) the Shapovalov form is positive definite, so that \( (v_r, v_r)_x > 0 \) for \( x > 0 \).) Since the Shapovalov form vanishes on any submodule, we have \( \beta(v_{r+1}, v_{r+1}) = 0 \) and hence \( n_r < n_{r+1} \). Moreover \( M(1, (j + r)^2) \subseteq M^{(i)} \) for \( n_{r-1} < i \leq n_r \). Hence
\[ X_{j^2}(q) \geq \sum_{N \geq 1} \text{ch } M^{(i)} \geq \sum_{r \geq 1} (n_r - n_{r+1}) \text{ch } M(1, (j + r)^2) = \varphi(q) \cdot \sum_{r \geq 1} (n_r - n_{r+1}) q^{(j+r)^2}, \]

(3)

where the inequality is between coefficients of \( q^k \) and we set \( n_0 = 0 \). Comparing (3) with (1) and (2), it follows that \( n_r = r \) for \( r \geq 1 \) and that equality holds in (3). Hence \( M^{(1)} = M(1, (j + 1)^2) \) and
\[ \text{ch } L(1, j^2) = (q^{j^2} - q^{(j+1)^2}) \varphi(q), \]
as required.

10. Proof of the character formula for the discrete series \( 0 < c < 1 \) using the Jantzen filtration. The proof of the character formula for the discrete series
\[ c = 1 - \frac{6}{m(m+1)}, \quad h = h_{r,s} = \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)}, \quad (1 \leq s \leq r \leq m - 1) \]
is proved similarly using \( x = h - h_{r,s} \) as a parameter. Let
\[ \varphi_{r,s}(c, h) = h + \frac{1}{24} (r^2 - 1)(c - 1) \]
and for \( r, s \geq 1 \) with \( r \neq s \),
\[ \varphi_{r,s}(c, h) = \left( h - \frac{(r - s)^2}{4} \right)^2 + \frac{h}{24} (r^2 + s^2 - 2)(c - 1) + \frac{1}{24^2} (r^2 - 1)(s^2 - 1)(c - 1)^2 + \frac{1}{48} (c - 1)(r - s)^2(rs + 1). \]

We shall need the following result, implicit in [8]. We give two proofs, the second of which is an expanded and slightly corrected version of a result from [13].

Proposition (Fuchs algorithm). If \( \varphi_{r,s}(c, h) = 0 \), then \( M(c, h) \) has a non-zero singular vector at energy level \( h + rs \).

Proof 1. A singular vector at energy level \( h + rs \) has the form \( Pv_h \) where
\[ P_d = \sum_{k=0}^{rs} q_k L_{-1}^k. \]

Each term \( q_k \) is a sum of monomials \( L_{-1}^{n_1} \cdots L_{-1}^{n_2} \) where \( 2n_2 + 3n_3 + \cdots = d - k \). By the uniqueness result of Fuchs, we may assume \( q_d = 1 \), where \( d = rs \). Set \( n_1 = d - k \) and order these monomials lexicographically on \( (n_2, n_3, \ldots) \). Thus the first term is \( L_{-1}^{d-1} \).
Now we have $L_nL_{-1}v$ and look for terms ending with $L_nL_{-1}v$ in the singular vector $w$. Let $k = d - n_1$. Then since for a singular vector we would require $L_p v = 0$, in this case we just require that the coefficient of
\[
\cdots L_{-p}^{n+1} L_{-p}^{n_p} f^{d-n_1} v
\]
in $L_{-p} w$ must be zero. We claim that this coefficient equals $n_p(2p-1)a$ plus a sum of lower order coefficients times polynomials in $c$ and $h$. In particular $a$ is again a polynomial in $c$ and $h$.

If $A$ is an element of the enveloping algebra of the Virasoro algebra we shall write $Av \sim 0$ if $Av$ is a combination of terms lower than $Bv$ in the lexicographic order, with coefficients polynomials in $c$ and $h$. Now we have
\[
L_{p-1} w = \sum_{j \geq 0} L_{p-1} q_j L_{-1}^j v = \sum_{k \geq 0} [L_{p-1}, q_j] L_{-1}^j v + q_j L_{p-1} L_{-1}^j v.
\]

We look for terms ending with $L_{-p}^{n-1} L_{-1}^{k+1}$ in this expression. We first note that it follows by induction on $k \geq 1$ that if $j \geq 1$ then $L_j L_{-1}^j v$ lies in $\text{lin} \{ AL_{-1}^i : i < k, A \in \mathcal{U}_2 \}$. Indeed
\[
L_j L_{-1} L_{-1}^{k-1} v = (j + 1) L_{j-1} L_{-1}^{k-1} v + L_{-1} L_{j} L_{-1}^{k-1} v,
\]
which has the same form since $L_{-1} \mathcal{U}_2 = \mathcal{U}_2 L_1$ and $L_{-1}^{k-1} v$ is an eigenvector of $L_0$. If $k \neq c$, then the induction hypothesis forces $q_c \sim D$ with $D \in \mathcal{U}_p$. So either $k > c$, in which case any monomial $q_k$ could have a non–zero contribution from a monomial $L_{-s}^{m_s} \cdots L_{-p}^{m_p}$ with $m_p > n_p$. Clearly taking the Lie bracket with $L_{p-1}$ can increase the exponent of $L_{-1}$ by at most one while diminishing the exponent of $L_{-p}$ by at most one. So monomials ending with $L_{-p}^{n-1} L_{-1}^{c+1}$ have coefficients that are polynomials in $c$ and $h$. If $k < c$, there is no way to increase the power of $L_{-1}^k$ to $L_{-1}^{c+1}$ by taking the Lie bracket with $L_{p-1}$. For $k = c$ and the terms $[L_{p-1}, AL_{-p}^n]$ with $A$ a monomial in $\mathcal{U}_{p+1}$, we have
\[
[L_{p-1}, AL_{-p}^n] = [L_{p-1}, A] L_{-p}^n + A[L_{p-1}, L_{-p}^n].
\]
The first term lies in $\mathcal{U}_2$ while for the second
\[
[L_{p-1}, L_{-p}^n] v = (2p - 1) \sum_{a+b=n-1} L_{-p}^a L_{-1} L_{-p}^b v = (2p - 1)n L_{-p}^{n-1} L_{-1} + B,
\]
where $B \in \mathcal{U}_2$. There could be several terms in $q_k$ with $n_p = n$; but in this particular case on bracketing with $L_{p-1}$ and taking the term ending with $L_{-p}^{n-1} L_{-1}^{c+1}$ we obtain $AL_{p-1}^{n-1} L_{-1}^{c+1}$. It follows that all these terms are linearly independent. But then there can be no cancellation and the coefficient of $L_{-p}^{n-1} \cdots L_{-p}^{c+1} v$ must be $n_p(2p-1)a$ plus a sum of lower order coefficients times polynomials in $c$ and $h$, as claimed.

Now given this vector $w = \sum a_\alpha(c, h) L_\alpha L_{-1}^{d-n_1}$ where the coefficients $a_\alpha(c, h)$ are polynomials in $c$ and $h$, the condition $L_1 w = 0 = L_{2w}$ gives a series of polynomials $b_\beta(c, h)$ which must vanish for $w$ to be a singular vector. By the coset construction, we know that $L(c_m, h_{r,s}(q))$ has a singular vector at energy $h_{r,s}(m) + rs$ if $1 \leq s \leq r \leq m - 1$. Thus $b_\beta(c_m, h_{r,s}(m)) = 0$. This $b_\beta(c, h)$ vanishes at a point of accumulation on the real curve $\varphi_{r,s}(c, h) = 0$. It follows that $b_\beta(c, h) = 0$ whenever $\varphi_{r,s}(c, h) = 0$. Thus $w$ defines a non–zero singular vector if $\varphi_{r,s}(c, h) = 0$, as required.

**Proof 2.** A singular vector at energy level $h + rs$ has the form $P v_h$ where
\[
P_d = \sum_{k=0}^{rs} q_k L_{-1}^k.
\]

Each term $q_k$ is a sum of monomials $L_{-k}^{n_1} \cdots L_{-1}^{n_1}$ where $2n_2 + 3n_3 + \cdots = d - k$. By the uniqueness result of Fuchs, we may assume $q_d = 1$, where $d = rs$. Set $n_1 = d - k$ and order these monomials lexicographically on $(n_1, n_2, n_3, \ldots)$. Thus the first term is $L_{-1}^{n_1}$. 35
Consider a term \( \cdots L_{p-1}^{n+1} L_{p}^{n} f^{d-n} v \) with \( n_p > 0 \) and suppose that we have already determined the coefficients of all previous monomials in the lexicographic order to be polynomials in \( c \) and \( h \). Let \( a \) be the coefficient of \( B v = \cdots L_{p-1}^{n+1} L_{p}^{n} f^{d-n} v \) in the singular vector \( w \). Let \( k = d - n_1 \). Then since for a singular vector we would require \( \hat{L}_{p-1} w = 0 \), in this case we just require that the coefficient of \( \cdots L_{p-1}^{n+1} L_{p}^{n} L_{-1}^{k+1} v \)

in \( \hat{L}_{p-1} w \) must be zero. We claim that this coefficient equals \( n_p (2p - 1) a \) plus a sum of lower order coefficients times polynomials in \( c \) and \( h \). In particular \( a \) is again a polynomial in \( c \) and \( h \).

If \( A \) is an element of the enveloping algebra of the Virasoro algebra we shall write \( A v \sim 0 \) if \( A v \) is a combination of terms lower than \( B v \) in the lexicographic order, with coefficients polynomials in \( c \) and \( h \).

We have\n
\[
L_{p-1} w = \sum_{k=0}^{e} L_{p-1} q_k L_{-1}^k v = \sum_{k=0}^{e} \{ L_{p-1}, q_k \} L_{-1}^k v + q_k L_{p-1} L_{-1}^k v.
\]

We look for terms ending with \( L_{-1}^{k-1} L_{-1}^{e+1} \) in this expression. We first note that it follows by induction on \( k \) that if \( j \geq 1 \) then \( L_j L_{-1}^k v \) lies in \( \text{lin} \{ AL_{-1}^i v : i < k, A \in \mathcal{U}_2 \} \). Indeed

\[
L_j L_{-1}^k v = (j + 1) L_{j-1} L_{-1}^{k-1} v + L_{-1} L_j L_{-1}^{k-1} v,
\]

which has the same form since \( L_{-1} \mathcal{U}_2 = \mathcal{U}_2 L_1 \) and \( L_{-1}^{k-1} v \) is an eigenvector of \( L_0 \).

By the inductive hypothesis, all terms with \( k > e \) are have uniquely determined polynomial coefficients. A monomial with \( k \leq e \) can be written \( u = B \cdot L_{-1}^n A \cdot L_{-1}^{e+1} v \) with \( B \) a monomial in \( L_{-j} \)’s for \( j > p \) and \( A \) a monomial in the \( L_{-j} \)’s with \( 2 \leq j \leq p \). Then

\[
L_{p-1} u = [L_{p-1}, B] L_{-1}^n A L_{-1}^{e+1} v + B \cdot (L_{p-1}, L_{-1}^n) A L_{-1}^{e+1} v + B \cdot (L_{p-1}, L_{-1}^n) \cdot A L_{-1}^{e+1} v + B L_{-1}^n L_{p-1} A L_{-1}^{e+1} v.
\]

Since \( [L_{p-1}, B] \) lies in \( \mathcal{U}_2 \), the first term in this expression has no terms ending in \( L_{-1}^{e+1} \). For the second note that

\[
[L_{p-1}, L_{-1}^e] = (2p - 1) \sum_{a+b=n-1} L_{-p}^a L_{-1}^b = (2p - 1) n L_{-p}^{n-1} L_{-1} + D,
\]

where \( D \in \mathcal{U}_2 \). Clearly, if \( k < e \), then neither \( B D L_{-1}^{e+1} v \) nor \( B L_{-1} A L_{-1}^{e+1} v \) contain terms ending in \( L_{-1}^{e+1} v \). Finally for the last term, note that if \( X \) is a monomial in \( L_{-j} \) for \( 1 \leq j \leq t \), then for \( 1 \leq s \leq t \) we have \( L_s X v = \sum a_i X_i v \), where \( X_i \) has the same form as \( X \) with the exponent of \( L_{-1} \) increased by at most one. Thus no terms ending in \( L_{-1}^{n_p-1} L_{-1}^{e+1} \) arise this way if \( k < e \).

When \( k = e \), then by the inductive hypothesis, we only need to consider terms of the form \( u = B \cdot L_{-1}^n L_{-1}^{e+1} v \) with either \( n \geq n_p > 0 \) or zero. If \( n = 0 \), then

\[
L_{p-1} u = L_{p-1} B L_{-1}^{e+1} v = [L_{p-1}, B] L_{-1}^n v + B L_{p-1} L_{-1}^{e+1} v.
\]

The first term contains no terms ending in \( L_{-1}^{e+1} v \). Nor does the second, since if \( kj \), \( L_j L_{-1}^{e+1} v \) lies in \( \text{lin} \{ L_{-1}^i v : 0 < i < e \} \). If \( n > n_p \), the none of the terms in (1) give rise to a monomial ending in \( L_{-1}^{n_p-1} L_{-1}^{e+1} v \).

Finally there could be several terms \( B L_{p-1} L_{-1}^e \) in \( u \), with \( n = n_q \). However taking the term ending with \( L_{-1}^{n_p-1} L_{-1}^{e+1} v \) in (1), we get \( B L_{p-1} L_{-1}^{e+1} v \). Thus all these terms are distinct. But then there can be no cancellation and thus the coefficient of \( L_{-1}^{n_p-1} \cdots L_{-1}^{n_1-1} L_{-1}^{e+1} v \) must be \( n_p (2p - 1) a \) plus a sum of lower order coefficients times polynomials in \( c \) and \( h \).

Now given this vector \( w = \sum a_i(c, h) L_{-1}^{d-|c|} \) where the coefficients \( a_i(c, h) \) are polynomials in \( c \) and \( h \), the condition \( L_1 w = 0 = L_{-2} v \) gives a series of polynomials \( b_i(c, h) \) which must vanish for \( w \) to be a singular vector. By the coset construction, we know that \( L(c_m, h, r, s(q)) \) has a singular vector at energy \( h_{-1}^r s(m) + pq \) if \( 1 \leq s \leq r \leq m - 1 \). Thus \( b_i(c_m, h, r, s(q)) = 0 \). Thus \( b_i(c, h) \) vanishes at a point of accumulation on the real curve \( c_{r,s}(c, h) = 0 \). It follows that \( b_i(c, h) = 0 \) whenever \( c_{r,s}(c, h) = 0 \). Thus \( w \) defines a non-zero singular vector if \( c_{r,s}(c, h) = 0 \), as required.
**Corollary.** If \( r, s \geq 1 \) and \( c = 1 - 6/m(m+1) \), then \( M(c, h_{r,s}) \) has a singular vector of energy \( h = h_{r,s} + rs \).

**Proposition.** Let \( c = 1 - 6/m(m+1) \), \( 1 \leq s \leq r \leq m - 1 \), \( r' = m - r \), \( s' = m + 1 - q \). Then \( M(c, h_{r,s}) \) has singular vectors \( a_1, b_1, a_2, b_2, \cdots \) where for \( k \geq 1 \), \( a_k \) has energy \( \alpha_k = h_{r',k(m+1)+s'} = h_{r,s-k(m+1)} \) and \( b_k \) has energy \( \beta_k = h_{r,s+k(m+1)} = h_{r',s'-k(m+1)} \). The energies of the vectors are strictly increasing in the above order. Moreover if \( A_k \) and \( B_k \) are the Verma modules generated by \( a_k \) and \( b_k \), then \( A_{k+1} + B_{k+1} \subseteq A_k \cap B_k \).

**Proof.** Since
\[
\alpha_{k+1} = \alpha_k + r'(k(m+1) + s'), \quad \beta_{k+1} = \alpha_k + r(s - k(m+1))
\]
and
\[
\alpha_{k+1} = \beta_k + r(s + k(m+1)), \quad \beta_{k+1} = \beta_k + r'(s' - k(m+1)),
\]
existence follows by successive applications of the previous corollary. The uniqueness of singular vectors as a particular energy level in a Verma module implies the result on the set theoretic inclusion.

For fixed \( m \) and \( 1 \leq s \leq r \leq m - 1 \), take \( x = h - h_{r,s}(m) \). As in the previous section let \( m_k > 0 \) be the order of the zero of \((a_k, a_k)_x\) and \( n_k \) that of \((b_k, b_k)_x\). For this to make sense, we have to check that, if \( \xi = a_k \) or \( b_k \), then \((\xi, \xi)_x\) does not vanish identically. But \( m(\xi, \xi)_x > 0 \) for \( x \) sufficiently large, because the Kac determinant formula and the unitarity of Verma modules for \( c \geq 1 \) together imply that, at a fixed energy level, the Shapovalov form is positive definite for \( c > 0 \) and \( h \) sufficiently large. The same argument as used in the Section 9 shows that \( m_k < m_{k+1}, n_{k+1} \) and \( n_k < m_{k+1}, n_{k+1} \) since \( A_{k+1} + B_{k+1} \subseteq A_k \cap B_k \).

**Proposition.** \( A_1 + B_1 \) is the maximal submodule of \( M = M(c, h_{r,s}) \) and \( A_{k+1} + B_{k+1} = A_k \cap B_k \) for \( k \geq 1 \).

Moreover \( m_k = n_k = 1 \) and
\[
\sum_{i \geq 1} \text{ch} M(i) = \text{ch}(A_1 + B_1) + \sum \text{ch} A_{2k} + \text{ch} B_{2k} = \sum_{k \geq 0} \text{ch} A_{2k+1} + \text{ch} B_{2k+1}.
\]

**Proof.** From the Kac determinant formula \( h_{r,s} = h_{r_1,s_1} \) for \((r, s) = (r_1, s_1) + a(m, m+1) \) with \( a \geq 0 \) or \((r, s) = -(r_1, s_1) + b(m, m+1) \) with \( b \geq 1 \). Hence
\[
\sum \text{ch} M(i) = \varphi(q) \cdot \sum_{a \in \mathbb{Z}_k} q^{r(a+m)(s+a(m+1))}.
\]

On the other hand if \( M_k = \max(m_k, n_k) \) and \( N_k = \min(m_k, n_k) \), then \( N_k \leq M_k < N_{k+1} \leq M_{k+1} \). On the other hand
\[
\sum \text{ch} M(i) \geq \sum_{k \geq 1} (N_{k+1} - M_k) \text{ch}(A_k + B_k) \geq \sum_{k \geq 1} \text{ch}(A_k + B_k).
\]

Thus
\[
\sum \text{ch} M(i) \geq \sum_{k \geq 1} \text{ch}(A_k + B_k).
\]

We wish to prove the same inequality with \( A_k \cap B_k \) replacing \( A_{k+1} + B_{k+1} \) for all \( k \). Indeed suppose that \( R_k \) is chosen maximal such that \( A_k \cap B_k \subseteq M(R_k) \). Then \( M_k + 1 \leq R_k < N_k \). Hence \( R_k < R_{k+1} \) and so
\[
\sum \text{ch} M(i) \geq \text{ch}(A_1 + B_1) + \sum_{k \geq 1} (R_k - R_{k-1}) \text{ch} A_k \cap B_k \geq \text{ch}(A_1 + B_1) + \sum_{k \geq 1} \text{ch} A_k \cap B_k.
\]

Now we have a short exact sequence
\[
0 \rightarrow A_k \cap B_k \rightarrow A_k \oplus B_k \rightarrow A_k + B_k \rightarrow 0.
\]

Hence
\[
\text{ch}(A_1 + B_1) + \text{ch} A_1 \cap B_1 = \text{ch} A_1 + \text{ch} B_1,
\]
and for \( k \geq 1 \)
\[
\text{ch} A_{2k} \cap B_{2k} + \text{ch} A_{2k+1} \cap B_{2k+1} \geq \text{ch}(A_{2k+1} + B_{2k+1}) + \text{ch} A_{2k+1} \cap B_{2k+1} = \text{ch} A_{2k+1} + \text{ch} B_{2k+1}.
\]
Hence
\[ \sum \text{ch } M^{(i)} \geq \text{ch } (A_1 + B_1) + \sum_{k \geq 1} \text{ch } A_k \cap B_k \geq \sum_{k \geq 0} \text{ch } A_{2k+1} + \text{ch } B_{2k+1}. \] (3)

On the other hand the right hand side equals
\[ \varphi(q) \cdot \sum_{a \in \mathbb{Z}} q^{(r+am)(s+a(m+1))}, \]
so it coincides with the left hand side. So we have equality in (3), so that
\[ A_{2k} \cap B_{2k} = A_{2k+1} + B_{2k+1} \]
for all \( k \). Now instead of \( M \) we take \( M' = A_1 \) and set \( A'_{k+1} = B_{k+1}, B'_{k+1} = A_{k+1} \) for \( k \geq 1 \). Repeating the argument above we get
\[ \sum_{i \geq 1} \text{ch } (M')^{(i)} \geq \sum_{k \geq 1} \text{ch } A_k \cap B_k \geq \sum_{k \geq 1} \text{ch } A_{2k} + \text{ch } B_{2k}. \] (4)

By (1), the left hand side of (4) equals
\[ \varphi(q) \cdot \sum_{a \in \mathbb{Z}} q^{(r_1+am)(s_1+a(m+1))}, \]
where \( r_1 = r \) and \( s_1 = -s \). But this equals the last term on the right hand side. Hence equality holds in (4), so that
\[ A_{2k-1} \cap B_{2k-1} = A_{2k} + B_{2k} \]
for \( k \geq 1 \). Returning to equation (2), we have equality between the furthest terms and hence \( N_{k+1} - M_k = 1 \) for all \( k \). Evidently \( m_1 < m_2 < \cdots \) and \( n_1 < n_2 < \cdots \). Now we have \( M^{(1)} \supseteq A_1 + B_1 \), so \( m_1 = n_1 \). Hence \( M_1 = 1 \). So \( N_2 = \min(m_2, n_2) = 2 \). Note that if \( N_k < M_k \) then we would have an extra contribution of \( \text{ch } A_k \) or \( \text{ch } B_k \) on the right hand side of (2). Since equality holds, it follows that \( N_k = M_k \), so that \( n_k = m_k \). Since \( m_1 = n_1 = 1 \), we finally get \( m_k = k = n_k \).

**Corollary.** \( \text{ch } L(c, h_{r,s}) = \varphi(q) \cdot \sum_{k \in \mathbb{Z}} (-1)^k q^{b_{r,s+k(m+1)}}. \)

**Proof.** We have
\[
\text{ch } L(c, h_{pq}) = \frac{\text{ch } M(c, h_{r,s})}{(A_1 + B_1)} = \frac{\text{ch } M(c, h_{r,s}) - \text{ch } (A_1 + B_1)}{A_1 + B_1} = \frac{\text{ch } M(c, h_{r,s}) - \sum_{k \geq 0} \text{ch } A_{2k+1} + \text{ch } B_{2k+1} - \sum_{k \geq 0} (\text{ch } A_{2k+1} + \text{ch } B_{2k+1})}{A_1 + B_1} = \varphi(q) \cdot \sum_{k \in \mathbb{Z}} (-1)^k q^{b_{r,s+k(m+1)}}.
\]

**APPENDIX A:** Alternative proofs of Fubini–Veneziano relations.

**First indirect proof.** Recall that \( \phi(n) : \mathcal{F} \to \mathcal{F} \) is a primary field for the system of operators \( (a_n), U \) and \( L_0 \) if
\[ [a_n, \phi(k)] = m \phi(k+n), \quad [L_0, \phi(n)] = -(n + \mu) \phi(n), \quad U \phi(n) U^* = \phi(n + m), \]
for some \( m \in \mathbb{Z} \) and \( \mu \in \mathbb{R} \). We now fix \( k \) and consider
\[ \psi_k(n) = [L_k, \phi(n-k)] + (n-k) \phi(n). \]
It is easy to check that $\psi$ satisfies the same conditions as $\phi$ with the same choice of $\mu$. So by uniqueness, $\psi_k(n) = c(k)\phi(n)$ for some constant $c(k)$:

$$[L_k, \phi(n - k)] + (n - k)\phi(n) = c(k)\phi(n). \tag{1}$$

The Jacobi relation, the relation $[L_a, [L_b, T]] - [L_b, [L_a, T]] = (a - b)[L_{a+b}, T]$ and the non-vanishing of $\phi(0)$ imply that $$\left( a - b \right) c(a + b) = ac(a) - bc(b).$$

But this functional equation implies that $c$ is an affine function $c(a) = \alpha a + \beta$. Indeed the $c$ satisfying this functional equation form a vector space. So subtracting an affine function from $c$, we may assume that $c(0) = 0 = c(1)$ and must then show that $c \equiv 0$. Taking $a = -b$, we see that $c(a) = -c(-a)$ for all $a$. But for $a > 1$, $(a - 1)c(a + 1) = ac(a)$. Hence $c(a) = 0$ for all $a > 0$ and hence for all $a$. So from (1) we have

$$[L_k, \phi(n)] = (-n - \alpha k - \beta)\phi(n + k). \tag{2}$$

Now on the one hand we have $\Phi_m(z) = \sum \phi(n)z^{-n-\delta}$ while on the other, regardless of the labelling of the $\phi(n)$’s, we have

$$\Phi_m(z)\Omega|_{z=0} = \Omega_m.$$\text{It follows that $\delta = 0$ and $\phi(n)\Omega = 0$ for $n > 0$, with $\phi(0)\Omega = \Omega_m$. Since $L_0\Omega_m = \frac{m^2}{2}\Omega_m$, we must have $\beta = -m^2/2$. From equation (2) we have that}

$$[L_k, \Phi_m(z)] = z^{k+1}\Phi_m'(z) + \Delta^k(k+1)\Phi_m(z),$$

where $\Delta = 1 - \alpha$ and $\delta = \alpha - \beta - 1$. But $\delta = 0$. Hence $\alpha = 1 - m^2/2$ and $\Delta = m^2/2$. This proves the Fubini–Veneziano relations.

**Second proof by direct verification.** We start by noting the adjoint relation

$$(\Phi_m(z^{-1}))^* = z^{-m^2}\Phi_{-m}(z)$$

follows immediately because

$$\Phi_m(z^{-1})^* = \exp\left(\sum_{n<0} -\frac{z^n m a_n}{n}\right)\exp\left(\sum_{n>0} \frac{-m z^n a_n}{n}\right)U^m$$

$$= z^{-ma_0}U^m \exp\left(\sum_{n<0} -\frac{z^n m a_n}{n}\right)\exp\left(\sum_{n>0} -m z^n a_n\right)$$

$$= z^{-m^2}\Phi_{-m}(z).$$

We shall need the following generalisation of Lemma B in Section 8.

**Lemma.** Let $A$ be a formal power series in $z$ (or $z^{-1}$) with operator coefficients and let $D$ be an operator such that $C = [A, B]$ is a multiple of the identity operator, where $B = [D, A]$. Then $[D, e^A] = (B + C/2)e^A$.

**Proof.** We have

$$[D, A^N] = \sum_{p+q=N-1} A^pBA^q$$

$$= \sum_{p+q=N-1} BA^{p+q} + pCA^{p+q-1}$$

$$= NBA^{N-1} + \frac{N(N-1)}{2}CA^{N-2}.$$\text{Hence}

$$[D, e^A] = (B + C/2)e^A.$$

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Corollary. (a) $[L_k, E_+(z)] = (- \sum_{n>0} a_{n+k} z^{-n}) E_+(z)$,  
and $[L_{-k}, E_-(z)] = (- \sum_{n>0} a_{-n-k} z^{-n}) E_-(z)$ if $k \geq 0$. 
(b) $[L_{-k}, E_+(z)] = (- \sum_{n>0} a_{n-k} z^{-n} + \frac{m^2}{2} (k-1) z^k) E_+(z)$  
and $[L_k, E_-(z)] = (- \sum_{n>0} a_{-n-k} z^{-n} - \frac{m^2}{2} (k+1) z^{-k}) E_-(z)$ for $k > 0$. 

Proof. These formulas are straightforward consequences of the lemma, setting $D = L_k$ and  
$$A = \sum_{n \geq 0} \frac{m}{n} a_n z^{-n}.$$  
For example to prove the first formula in (b), we have  
$$B = [D, A] = - \sum_{n>0} ma_n z^{-n} = - \sum_{i>-k} ma_i z^{-i+k}$$  
so that  
$$[A, B] = - \left[ \sum_{n>0} \frac{m}{n} a_n z^{-n}, \sum_{i>-k} ma_i z^{-i+k} \right]$$  
$$= - \left\{ \begin{array}{ll} \sum_{n=1}^{k-1} & a_n, a_{-n} \frac{m^2}{n} z^k \\ \sum_{n=1}^{k-1} & a_n, a_{-n} \frac{m^2}{n} z^k \\ \end{array} \right.$$  
$$= -m^2 (k-1) z^k.$$  

Proof of the Fubini–Veneziano relations. We first check the commutation relations with $L_k$ for $k \neq 0$.  
For $k > 0$, we have $UL_{-k} U^* = L_{-k} + a_{-k}$, so that $U^m L_{-k} U^{-m} = L_{-k} + ma_{-k}$ and hence  
$$[L_{-k}, U^m] = -m U^m a_{-k}.$$  
Thus  
$$[L_{-k}, \Phi_m(z)] = U^m z^{-ma_0} (B_- E_- E_+ + E_- B_+ E_+),$$  
where  
$$B_- = -ma_{-k} - \sum_{n>0} ma_{-n-k} z^{-n}, B_+ = - \sum_{n>0} ma_{n-k} z^{-n} - \frac{m}{2} (z^k - 1)/(z-1).$$  
On the other hand  
$$z^{-k+1} \Phi'_m(z) = U^m z^{-ma_0} z^{-k} (-ma_0) E_- E_+ + U^m z^{-ma_0} (C_- E_- E_+ + E_- C_+ E_+),$$  
where  
$$C_- = -\sum_{n>0} ma_{-n} z^{-n-k}, C_+ = -\sum_{n>0} ma_n z^{-n-k}.$$  
Thus  
$$B_- = C_- - A$$  
and  
$$B_+ = C_+ + A - \frac{m^2}{2} (k-1) z^{-k} I$$  
where  
$$A = \sum_{i=0}^{k-1} ma_i z^{-i-k}.$$  
On the other hand  
$$[A, E_-(z)] = \sum_{i=0}^{k-1} m[a_i, E_-(z)] z^{i-k}$$  
$$= \sum_{i=1}^{k-1} m^2 z^{-k} E_-(z)$$  
$$= m^2 z^{-k} (-1) k E_-(z).$$
Hence
\[
[L_{-k}, \Phi_m(z)] - z^{-k+1}\Phi_m'(z) = \left[ \frac{m^2}{2}(k-1) + m^2(k-1) \right] \Phi_m(z) - z^{-k} \Phi_m(z),
\]
as required. The relation \([L_k, \Phi_m(z)]\) can be proved similarly or follows from this one by taking adjoints.

**APPENDIX B: Explicit construction of singular vectors and asymptotic formulas of Feigin–Fuchs.** In this appendix we give a slightly simplified account of the approach of [3], [7] to singular vectors in \(M(c, h, r, t)\), where \(c(t) = 13 - 6t - 6t^{-1} \), \(r = 2j + 1\) for \(j\) a non–negative half integer and \(h, r, t\) a non–negative half integer. The case of interest in the text has \(t = 1\), so that \(c = 1\) and \(h = j^2\). The treatment of [8], however, uses the more general case, when the singular vector is a polynomial in \(t\) and a knowledge of the constant term and leading coefficient is required. A proof of their formulas in a more general setting was given in [2], who commented that in the particular case the method of [3] for determining the singular vector could not be used to derive the result. We show on the contrary that the formulas are a trivial consequence of the algorithm of [3], which provided an alternative approach to explicit formulas for the singular vectors of Benoit and St–Aubin [4]. Let \(E, F, H, F\) be a canonical basis of \(sl_2\) satisfying \([E, F] = 2H, [H, E] = F\) and \([H, F] = -F\) and let \(C = H^2 + (EF + FE)/2\), the Casimir element. Let \(V = V_j\) be the irreducible representation of \(sl_2\) of spin \(j\) with \((\text{non–orthonormal})\) basis \(v_{-j}, v_{-j+1}, \cdots, v_j\) satisfying \(Hv_k = kv_k, F^k v_j = v_{j-k}\) for \(k > 0\) and \(F^{j+1}v_j = 0\). Let \(W = \bigoplus m \in \mathbb{Z} \) be a representation of the Virasoro algebra with \(L_0w = (h + m)w\) for \(w \in W(m)\) and \(\dim W(m) < \infty\) for all \(m\). We make operators \(A\) on \(V\) and \(B\) on \(W\) act on \(V \otimes W\) as \(A \otimes I\) and \(I \otimes B\) respectively. With this convention, we define
\[
N = -F + \sum_{m \geq 0} (-tE)^m L_{-m-1}, \quad M = L_0 = H - tC, \quad L = L_1 + E(t(H-1) + 1), \quad K = L_2 - tE^2(t(H - \frac{3}{2}) + \frac{7}{4}).
\]

**Remark.** More generally, if we define \(L_{-1} = N, \ L_0 = M\) and
\[
L_k = L_k - (-E)^k H - (t(H - \frac{k+1}{2}) + \frac{3k+1}{4})
\]
for \(k \geq 1\) (so that \(L_1 = L\) and \(L_2 = K\)), then for \(p \geq 0\)
\[
[L_p, L_{-1}] = \sum_{q \geq 0} (p+1+q)t^q E^q L_{p+q}
\]
and for \(p, q \geq 0\)
\[
[L_p, L_q] = (p-q)L_{p+q}.
\]

**Proof.** This is a straightforward verification using the commutation relations.

**Lemma B.** Let \(\xi \in W\) satisfy \(L_0\xi = h(t)\xi\). Then there is a unique vector \(w = \sum_{k=0}^{2j+1} \xi_{-j+k} \otimes \xi_{j+k}, \xi_{-j} = \xi\) with \(L_0\xi_k = (h(t) + k)\xi_k\) and \(Nw = v_j \otimes \eta\). In this case \(\eta = P_j \xi_0\) with \(P_j\) a uniquely determined homogeneous polynomial of total degree \(2j + 1\) in the universal enveloping algebra \(U\) of the Lie algebra \(\mathfrak{vir}_-\) with basis \(L_{-k}\) \((k > 0)\) with coefficients polynomials in \(t\) of degree \(\leq 2j\). The constant coefficient is \(L_1^r\) and the coefficient of \(t^{2j}\) is \((2j)!^2 L_{-r}\). More generally, the coefficient of \(L_{-1}^r\) is 1.

**Remark.** We will check below that \(P_j\) is non–zero by calculating its action on a specific module. In fact it can also be seen directly on a 1–dimensional module for \(\mathfrak{vir}_-\). For \(z \in \mathbb{C}\) define that algebra homomorphism \(\psi_z : \mathfrak{vir}_- \rightarrow \mathbb{C}\) by \(\psi_z(L_{-1}) = z\) and \(\psi_z(L_k) = 0\) for \(k > 1\). Then \(\psi_z(P_j) = z^{2j+1}\). Indeed we have to solve \(Nw = av_j\) with \(w = \sum a_k v_k\), \(a_{-j} = 1\) and \(N^r = -F + zI\). The solution is \(w = v_{-j} + zv_{-j+1} + z^2v_{-j+2} + \cdots\) since we can check that \((zI - F)w = z^{2j+1}v_j\). Hence \(\psi_z(P_j) = z^{2j+1}\). (Alternatively this can be proved using the Lie algebra endomorphism of \(\mathfrak{vir}_-\) defined by \(\theta(L_{-1}) = L_{-1}\) and \(\theta(L_k) = 0\) for \(k \leq -2\).)
\textbf{Proof.} The vectors $\xi_k$ are defined inductively by $\xi_{-j} = \xi$ and for $k = -j, \ldots, j$

\[ \xi_{k+1} \otimes v_k = L_{-1} \xi_k \otimes v_k - L_{-2} \xi_{k-1} \otimes tE v_{k-1} + L_{-3} t^2 E^2 v_{k-2} + \cdots \]

This proves uniqueness. By induction $\xi_{-j+k}$ has the form $Q_k \xi$, with $Q_k$ a polynomial in $U_-$ of total degree $k$. Performing this process for the Verma module $M(1, j)^2 \cong U_-$, gives a uniquely determined polynomial $P_j \in U_-$. Since there is a natural homomorphism of $M(c, h) \to W$ which is compatible with the equations defined by $N$, it is clear that $\eta = P_j \xi$.

Since the Lie algebra with basis $L_{-k}$ ($k \geq 0$) is the semidirect product of the algebra with basis $L_{-k}$ ($k \geq 2$) and $\mathbb{C}L_{-1}$, it makes sense to talk about the coefficients of powers of $L_{-1}$ in $U_-$. Taking the homomorphism $\pi \to \eta = P_j \xi$.

Thus the highest power of $t$ in $\eta$ is $t^{2j}$ and the coefficient is $cL_{-j} \xi$, where $cv_j = E^{2j} v_{-j}$. On the other hand $C = H^2 - H + EF$ so that

\[ Ev_k = EF v_{k+1} = (j^2 + j - (k + 1)^2 + k + 1)v_{k+1} = (j^2 - j - k^2 - k)v_{k+1} = (j - k)(j + k + 1)v_{k+1}. \]

Thus $E^{2j} v_{-j} = ((2j)!)^2 v_j$ and hence $c = ((2j)!)^2$.

\textbf{Lemma C.} In the Verma module $M(c(t), h(t))$ generated by the vector $\xi$, there is a non-zero homogeneous polynomial $P_j$ of total degree $(2j + 1)$ in the universal enveloping algebra $U_-$ of the Lie algebra with basis $L_{-k}$ ($k > 0$) such that $P_j \xi$ is a non-zero singular vector.

\textbf{Proof.} Since the map $A \to A \xi$ gives an isomorphism between $U_-$ and the Verma module, $P_j \neq 0$ forces $P_j \xi \neq 0$. We check that

\[ L_0 \eta = (j + 1)^2 \eta, \quad L_1 \eta = 0 = L_2 \eta \]

Since $L_1$ and $L_2$ generate the Lie algebra with basis $L_{-k}$ ($k > 0$), the second identities imply that $L_k \xi_0 = 0$ for all $k > 0$, i.e. that $\eta$ is a singular vector.

Let $w$ be the vector in Lemma B with $\xi_{-j} = \xi$. $Nw = v_j \otimes \eta$ and let $P = EN$. Thus $Pw = 0$ and any solution $w' \in V \otimes M_j^2$ of $Pw' = 0$ must satisfy $Nw' = v_j \otimes \eta'$ for some $\eta' \in M_2$. On the other hand the uniqueness statement in Lemma B, if $w' = \sum v_i \otimes \xi'_i$ satisfies $\xi_{-j} = 0$ and $Pw' = 0$ then $w' = 0$.

We now show that $w' = Aw$ with $A = M, L, K$ satisfies these hypotheses. First we check that the coefficient of $v_{-j}$ in $Aw'$ is zero. For $A = M$, we have

\[ M(v_{-j} \otimes \xi_{-j}) = (L_0 - H - tC)(v_{-j} \otimes \xi_{-j}) \]

\[ = (L_0 - H - t(j^2 + j))(v_{-j} \otimes \xi_{-j}) \]

\[ = ((j^2 + j)t - j + j - (j^2 + j)t)(v_{-j} \otimes \xi_{-j}) \]

\[ = 0. \]

(In fact a similar computation shows directly that $M(v_i \otimes \xi_i) = 0$.) The result for $A = L$ and $A = K$ holds because $L_1 \xi_0 = L_2$ and $Ev_1$ and $E^2 v_2$ are multiples of $v_{i+1}$ and $v_{i+2}$ respectively.

Now we check that $PAw = 0$ for the three choices of $A$. We first observe that, since $EH = (H - I)E$, we have $EA = A'E$ with

\[ M' = L_0 - H - tC + I, \quad L' = L_1 - E(t(H - 2) + 1), \quad K' = L_2 - tE^2(t(H - \frac{5}{2}) + \frac{7}{4}). \]

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Since \([N, M] = aN\), we have \(NM = MN + aN\) so that

\[
ENM = EMN + aEN = M'EN + aEN.
\]

Hence

\[
PM = M'P + aP.
\]

So that \(PMw = 0\), since \(Pw = 0\). By uniqueness, \(Mw = 0\). Similarly \(NL = LN + bEN + cM\) so that

\[
ENL = ELN + bE^2N + cEM = L'EN + bE^2N + cEM.
\]

Hence \(PLw = 0\), since \(Pw = 0\) and \(Mw = 0\). By uniqueness, \(Lw = 0\). Finally \(NK = KN + dE^2N + eEM + fL\) so that

\[
ENK = K'EN + dE^2N + eE^2M + fEL.
\]

Hence \(PKw = 0\), since \(Nw = 0\), \(Mw = 0\) and \(Lw = 0\). By uniqueness, \(Kw = 0\).

**APPENDIX C: Proof of Feigin–Fuchs product formula using explicit formula for singular vectors.**

**Lemma A.** Let \(A\) be an upper triangular \(n \times n\) matrix, \(B\) the lower triangular matrix with 1’s below the diagonal and 0’s elsewhere and suppose \(A - B\) is invertible. Then the solution \(v = \sum v_ie_i\) of \((A - B)v = e_1\) has \(v_n = \det(A - B)^{-1}\).

**Proof.** Let \(X = A - B\) and \(Y = X^{-1}\). Thus \(v_n = y_{n1} = (-1)^{n-1} \det D / \det(X)\) where \(D\) is the \((1, n)\) minor obtained by deleting the first row and last column of \(X\). This matrix is upper triangular with entries \(-1\) on the diagonal, so that \(\det D = (-1)^{n-1}\). Hence \(v_n = (\det X)^{-1}\) as required.

**Lemma B.** There is a unique vector \(w = \sum w_ie_i\) with \(w_n = 1\) such that \((A - B)w = ae_1\). For this solution \(a = \det(A - B)\).

**Proof.** The solution satisfies the recurrence relations \(w_n = 1\) and for \(1 \leq k \leq n\)

\[
w_k = \sum_{i=k}^{n} a_{ki}w_i
\]

and is therefore uniquely determined. If \(A - B\) is invertible the result follows immediately from Lemma A. If not then \((A - B)u = 0\) has a non–zero solution \(u = \sum u_ie_i\). If \(u_n \neq 0\), then, rescaling if necessary, we may assume that \(u_n = 1\). But then by uniqueness \(a = 0 = \det(A - B)\), as required. If \(u_n = 0\), then \(w' = w + u\) would satisfy \(w'_n = 1\) and \((A - B)w' = e_1\), so that by uniqueness \(u = 0\), a contradiction. The result follows.

**Lemma C.** Let \(P_j\) be the operator giving the singular vector in the Verma module \(M(1, j^2)\) with \(j\) a non–negative half integer. Let \(p \geq 0\) be an integer. Then for the natural action \(\ell_n = -z^{n+1}d/dz - p^2(n+1)z^n\) on \(\mathbb{C}[z, z^{-1}]z^\mu\) with \(c = 0\), we have \(P_jz^{\mu+p^2} = az^{-d+\mu+p^2}\) where \(a = (-1)^d \det(-F + (I + E)^{-1}(-\mu I - p^2I + (2p + 1)H + jI))\).

**Remark.** Only the case \(p = 1\) is needed in our applications.

**Proof.** Let \(\lambda = p^2\) and \(\nu = \mu + \lambda\). We set

\[
u(z) = \sum_{i=0}^{2j} a_iz^{-i+\nu}v_{-j+i}, \tag{1}
\]

with \(a_0 = 1\). The coefficient \(a\) is determined as the unique solution of

\[
av_jz^{\nu - 2j - 2} = -Fu(z) - \sum_{m \geq 0} (-E)^m z^{-m}u'(z) - \sum_{m \geq 0} \lambda m(-E)^m z^{-m+1}u(z).
\]

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Thus

\[ av_j z^{-2j-2} = -Fu(z) + \sum_{m \geq 0} (-E^m)z^{-m}(\mu z^{v-1}v_{j-} + \sum_{i=1}^{2j}(-i+\nu)z^{-i-1}v_{j+i}) + \lambda E(I+Ez^{-1})^{-2}u(z). \]  

Let \( w(z) = z^{-\nu}u(z) \). Thus multiplying by \( z^{-\nu} \), we get

\[ av_j z^{-2j-2} = -Fw(z) + \sum_{m \geq 0} (-E)^m z^{-m}(\nu z^{-1}v_{j-} + \sum_{i=1}^{2j}(-i+\nu)z^{-i-1}v_{j+i}) + \lambda E(I+Ez^{-1})^{-2}w(z). \]

Let \( w = w(1) = v_{j-} + a_1v_{j+1} + a_2v_{j+2} + \cdots \) and set \( z = 1 \) in (3). Setting \( a_0 = 1 \), this yields

\[ av_j = -Fw + \sum_{m \geq 0} (-E)^m \sum_{i=0}^{2j}(-i+\mu)a_iv_{j+i} + \lambda E(I+E)^{-2}w. \]

Now

\[ Hv_{j+i} = (-j+i)v_{j-i}, \]

so that

\[ (-i-\mu)v_{j+i} = (-H-j-\mu)v_{j+i}. \]

Hence we get

\[ av_j = -Fw + \sum_{m \geq 0} (-E)^m (H+j)w + \lambda E(I+E)^{-2}w, \]

\[ = [-F + (I+E)^{-1}((-\mu I - H - jI))w + \lambda E(I+E)^{-2}w. \]

It follows from Lemma B that

\[ a = \det(-F + (I+E)^{-1}((-\mu I - H - jI)) + \lambda E(I+E)^{-2}). \]

When \( \lambda = 0 = p \), this immediately gives the result for \( p = 0 \). When \( \lambda = 1 = p \), we have \( HE = EH + E \), so that \( H(I+E) = (E+I)H + E \) and hence

\[ (I+E)^{-1}H(I+E) = H + (I+E)^{-1}E. \]

Similarly \( FE = EF - 2H \), so that \( F(I+E) = (I+E)F - 2H \) and hence

\[ (I+E)^{-1}F(I+E) = F - 2(I+E)^{-1}H. \]

So in this case

\[ -F + (I+E)^{-1}(-\nu I + H + jI)) + E(I+E)^{-2} = (I+E)^{-1}(-F + (I+E)^{-1}(-\nu I + 3H + jI)(I+E), \]

and hence

\[ a = \det(-F + (I+E)^{-1}(-\nu I + 3H + jI). \]

It follows by induction from (5) and (6) that for \( p \geq 1 \)

\[ (I+E)^{-p}H(I+E)^p = H + p(I+E)^{-1}E, \]

\[ (I+E)^{-p}F(I+E)^p = F - 2p(I+E)^{-1}H - (p^2 - p)(I+E)^{-2}E. \]

But then

\[ -F + (I+E)^{-1}(H + \alpha I) + p^2(I+E)^{-2}E = (I+E)^{-p}(-F + (I+E)^{-1}(H + \alpha I))(I+E)^p. \]
On the other hand, 

$$a = \det(-F + (I + E)^{-1}(H + jI - \nu I)),$$

as required.

**Lemma D.** Let $P_j$ be the operator giving the singular vector in the Verma module $M(1, j^2)$ with $j$ a non-negative half integer. Let $p \geq 0$. Then for the natural action $\ell_n = -z^{n+1}d/dz - p^2(n+1)z^n$ on $\mathbb{C}[z, z^{-1}]z^{n+p^2}$ with $c = 0$, we have $P_{2n+p^2} = az^{-2j-1+p^2}$ where

$$a = (-1)^d \prod_{k \in S}(\mu - j^2 + k^2),$$

and $S = \{-j, -j+1, \cdots, j-1, j\}$ is the set of eigenvalues of $H$ on $V_j$.

**Proof.** By Lemmas B and C, we have

$$a = \det(-F + (I + E)^{-1}(-\mu - p^2 + j + (2p+1)H)).$$

Since $\det(I + E) = 1$, it follows that

$$a = \det(-F - EF - \mu + j + H).$$

On the other hand

$$C = \frac{1}{2}(EF + FE + 2H^2) = EF - H + H^2$$

is a central operator acting on $v_{-j}$ as $j^2 + j$. Hence

$$-EF = -j - j^2 - H + H^2,$$

so that

$$a = \det(-F - \mu - p^2 - j - H - j^2 + H^2 + j + (2p+1)H)$$

$$= \det(j^2 - 2pH - \mu - p^2 + H^2)$$

$$= \det(-\mu - (H + p)^2 + j^2)$$

$$= (-1)^d \prod_{k \in S}(\mu - j^2 + (k + p)^2),$$

as required.

**APPENDIX D:** Holomorphic vector bundles and flat connections. Given a holomorphic vector bundle on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, we can restrict it to the covering by two discs $\{z : |z| < 2R\}$ and $\{z : |z| > r/2\}$ where $R > 1 > r$ which can be refined to smaller discs $\{z : |z| < R\}$ and $\{z : |z| > r\}$. We may identify each disc with a disc $D' = \{z : |z| < 1 + \delta\}$ and the smaller disc with the unit disc $D = \{z : |z| < 1\}$. Because it arises by restriction, there is a finite covering of $D'$ by opens $U_i$ and holomorphic maps $g_{ij} : U_i \cap U_j \to GL_n(\mathbb{C})$ with $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. Taking a connection on the corresponding complex vector bundle, parallel transport along lines radiating from the origin trivialises the vector bundle. Thus there are smooth maps $h_i : U_i \to GL_n(\mathbb{C})$ such that $g_{ij} = h_i h_j^{-1}$. Thus $g_{ij} \cdot \partial_{\overline{z}}h_j = \partial_{\overline{z}}h_i$ and hence

$$h_i^{-1}\partial_{\overline{z}}h_i = h_j^{-1}\partial_{\overline{z}}h_j$$

on $U_i \cap U_j$. It follows that there is a $C^\infty$ map $A : D' \to M_n(\mathbb{C})$ such that $A = h_i^{-1}\partial_{\overline{z}}h_i$ on $U_i$. We claim that there is a smooth map $f : D' \to GL_n(\mathbb{C})$ with

$$\partial_{\overline{z}}f = -Af$$

(*)

For then

$$\partial_{\overline{z}}(h_i f) = \partial_{\overline{z}}(h_i) f + h_i \partial_{\overline{z}}(f) = h_i (Af + \partial_{\overline{z}}(f)) = 0.$$
Thus $k_i = h_i f$ is holomorphic on $U_i$ with $g_{ij} = k_i k_j^{-1}$, as required. We need the following generalisation of Dolbeault’s lemma.

**Lemma A.** Let $A \in C^\infty(D_1, M_s)$ and $h \in C^\infty(D_1, \mathbb{C}^s)$. Then we can find a solution $f \in C^\infty(D, \mathbb{C}^s)$ of the homogeneous equation $\partial_z^* f = -A f$ on $D$. The inhomogenous equation $\partial_z^* f = -A f + h$ on $D_r$ is solvable if the dual equation has the analytic continuation property: with $R$ fixed in $(1,1+\delta)$, any solution of $\partial_z f = A^* f$ in $D_R = \{ z : |z| < R \}$ vanishing on some open $U$ in $D_R$ is identically zero in $D_R$.

**Remark.** In the example above the solutions of $\partial_z^* f = -A f$ in $U \subset D$ correspond to holomorphic sections of the holomorphic vector bundle over $U$. Indeed $\xi_i(z) = h_i(z) f(z)$ satisfies $g_{ij} \xi_j = \xi_i$, so it is a section, and $\partial_z \xi_i = 0$. So it is holomorphic. Conversely, if $\xi_i$ is a holomorphic section, then $h_i^{-1} \xi_i$ fit together to give a function $f(z)$ which evidently satisfies $\partial_z^* f = -A f$ in $U \subset D$. Similarly if we define dual bundle by $g^*_{ij}(z) = (g_{ij}(z)^*)^{-1}$ then $h^*_i(z) = (h_i(z))^{-1}$ and

$$A'(z) = h^*_i(\partial_z h_i^{-1})^* = -A(z)^*.$$  

Similarly the corresponding quantities for the conjugate antiholomorphic bundle $g^*_{ij}(z) = (g_{ij}(z)^*)^{-1}$ are $h_i^*(z) = (h_i(z)^*)^{-1}$ and $A^*(z) = -A(z)^*$. Thus any solution of $\partial_z f = A^* f$ in $U$ yields an antiholomorphic section of the antiholomorphic vector bundle over $U$.

**Proof.** Let $\psi \in C^\infty_c(D)$ be a bump function equal to 1 on $D$, and 0 off $D$, for some $r < \rho < 1$. Replacing $A$ by $\psi A$ and $h$ by $\psi h$, we may assume that $X$ lies in $C^\infty_c(D, M_s)$ and $h$ lies in $C^\infty_c(D, \mathbb{C}^s)$ and both therefore extend to the whole of $\mathbb{C}$. To solve the equation $\partial_z^* f = -A f + h$ on $D$, we include the disc $D$ in a large square $F = [-R, R] \times [-R, R]$. By identifying opposite sides, doubly periodic functions on $F$ can be identified with functions on a torus $T = \mathbb{T}^2$. The operator $\partial z^* = \partial_x + i \partial_y$. Let $H_k(T)$ be the $L^2$ Sobolev spaces for $T$ constructed using the Laplacian operator $\Delta = \partial_x^2 + \partial_y^2$ (see [5], [18], [35] or [38]). The operator $D$ defines a Fredholm operator of index 0 from $H_k(T)$ to $H_{k-1}(T)$, since it is diagonalised in the natural basis. Its kernel consists of the constant functions and its image is the orthogonal complement of the constant functions. The operator $D + A$ is therefore also Fredholm of index zero from $H_k(T) \otimes M_s$ to $H_{k-1}(T) \otimes M_s$. Thus the equation

$$(D + A) f = h + g$$

is soluble provided $(g + h, w_i)_{(k-1)} = 0$ for finitely many vectors $w_1, \ldots, w_p \in H_{k-1}(T) \otimes M_s$. But if $U$ is an open set $[-R, R] \times [-R, R]$ with $\overline{U} \cap \overline{D} = \emptyset$, then $C^\infty_c(U)$ is embedded in $H_{k-1}(T)$. We need to justify why there will be a vector $g$ in $C^\infty_c(U)$ satisfying $(g + h, w_i)_{(k-1)} = 0$ ($i = 1, \ldots, p$). In the homogeneous case, when $h = 0$, this is clear because the image of $C^\infty_c(U)$ contains a subspace of dimension $p + 1$.

In the inhomogenous case, the $w_i$ will be the smooth functions in the finite-dimensional kernel of $D^* + A^* = -\partial_z + A(z)^*$. By assumption their restrictions to some $U$ with $\overline{U} \cap \overline{D} = \emptyset$ are linearly dependent, i.e. regarded as elements of $C^0 \subset \mathbb{C}^s$. Since $C^\infty_c(U, \mathbb{C}^s)$ is dense in $L^2(U, \mathbb{C}^s)$, we can find a solution $g \in C^\infty_c(U, \mathbb{C}^s)$ of $(g + h, w_i) = 0$ ($i = 1, \ldots, p$). Hence the conditions are satisfied with $k = 1$.

**Corollary.** The inhomogenous equation is always soluble in the example coming from a holomorphic vector bundle.

**Proof.** Linearly independent antiholomorphic sections of the conjugate bundle stay linearly independent on any open $U$. Indeed in this case if $w$ is in the kernel of $D^* + A^* = -\partial_z + A(z)^*$ and vanishes outside $\overline{D}$ then it vanishes everywhere (by antiholomorphicity inside $D_r$ for $r < R < 1$).

**Lemma B.** If $A \in C^\infty_c(D, M_n)$ and $D_r = \{ z : |z| < r \}$ for $0 < r < 1$, then we can find $B \in C^\infty_c(D, M_n)$ such that $\partial_z B + A$ and $\partial_z + B$ commute on $D_r$.

**Proof.** The commutativity condition on $D_r$ is equivalent to

$$\partial_z B = \partial_z A - \text{ad}(A(z)) \cdot B,$$

which can be solved by Lemma A with $s = n^2$, $f = B$, $h = \partial_z A$ and the operator $A(z)$ given by $-\text{ad}(A(z))$.

**Lemma C.** If $A \in C^\infty_c(D, M_n)$ and $D_r = \{ z : |z| < r \}$ for $0 < r < 1$, then we can find $f \in C^\infty_c(D, M_n)$ such that $\partial_z f = -A f$ on $D_r$ with $f(z)$ invertible on $D_r$. 

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Proof. Taking $B(z)$ as in Lemma B, we get operators $\partial_z + A$ and $\partial_z + B$ commuting on $D_r$. These define a flat connection so that parallel transport along any path from 0 results in a smooth map $f$ of $D_r$ into invertible matrices such that $f(0) = I$ and

$$\partial_z f = -Af, \quad \partial_z f = -Bf$$

Remarks. 1. Standard approximation techniques by polynomials (see [9]) can be applied to deduce that any holomorphic vector bundle on the unit disc is trivial.

2. The same techniques can be used to prove that a holomorphic vector bundle on an open ball $B$ in $\mathbb{C}^n$ is trivial. In that case the we have commuting operators $\partial_z + A_i$ and must find commuting operators $\partial_z + B_i$. This can be accomplished by successively using operators on $L^2$ Sobolev spaces of $T^{2n}$ corresponding to the de Rham complex in the first $2k$ real variables in $T^{2n}$ and the Dolbeault complex in the last $2n - 2k$ real variables (regarded as $n - k$ complex variables).

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