EXPERIMENTAL STABILITY FOR A TRANSMISSION PROBLEM OF A NONLINEAR VISCOELASTIC WAVE EQUATION

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Abstract. We are concerned with the transmission problem of nonlinear viscoelastic waves in a heterogeneous medium, establishing the well-posedness of solutions and the exponential stability of the related energy functional. We introduce an auxiliary problem to prove the exponential stability and the proof combines an observability inequality and microlocal analysis tools.

1. Introduction. In the present paper we establish the well-posedness of solutions for the transmission problem of nonlinear viscoelastic waves with hereditary memory, as well as the exponential stability result. The transmission problem is posed in a heterogeneous medium composed by two different materials with a common boundary where the coupling term is placed.

Consider $\Omega$ and $\Omega_2$ open, bounded and connected sets of $\mathbb{R}^N$, $N \geq 2$, with smooth boundaries $\Gamma = \partial \Omega$ and $\Gamma_2 = \partial \Omega_2$, respectively, such that $\Omega_2 \subset \Omega$. Denote $\Omega_1 := \Omega \setminus \Omega_2$ and $\partial \Omega_1 := \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \Gamma$. This scenario is represented in Figure 1. We are

\begin{figure}[h]
\centering
\includegraphics[width=0.25\textwidth]{domain}
\caption{Domain}
\end{figure}

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interested in studying the problem

\[
\begin{aligned}
    &\left\{
    \begin{array}{l}
    \rho_1(x)u_{tt} - \text{div}(K(x)\nabla u) + \int_{-\infty}^{t} g(t-s) \text{div}(K(x)\nabla u(s)) \, ds + f(u) \\
    = 0 \text{ in } \Omega_1 \times (0, \infty), \\
    \rho_2(x)v_{tt} - \text{div}(Q(x)\nabla v) + h(v) = 0 \text{ in } \Omega_2 \times (0, \infty), \\
    u = 0 \text{ on } \Gamma_1 \times \mathbb{R}, \\
    u = v \text{ on } \Gamma_2 \times (0, \infty), \\
    \frac{\partial v}{\partial \nu^{\mathcal{K}}} = \frac{\partial u}{\partial \nu^{\mathcal{K}}} - \int_{-\infty}^{t} g(t-s) \frac{\partial u}{\partial \nu^{\mathcal{K}}}(s) \, ds \text{ on } \Gamma_2 \times (0, \infty), \\
    u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega_1, \\
    v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega_2, \\
    (u(x, t) = u^0(x, t), \quad u_t(x, t) = \partial_t u^0(x, t), \quad x \in \Omega_1, \quad t \leq 0,
    \end{array}
    \right.
\end{aligned}
\]

where \( u^0 : \Omega_1 \times (-\infty, 0] \rightarrow \mathbb{R} \) is the prescribed past history of \( u \), \( \frac{\partial u}{\partial \nu^{\mathcal{K}}} \) denotes the co-normal derivative of \( u \) along \( \Gamma_2 \), which is defined by

\[
\frac{\partial u}{\partial \nu^{\mathcal{K}}}(x) = \sum_{i,j=1}^{N} k_{ij}(x) \frac{\partial u}{\partial x_i}(x) \nu_j(x), \quad x \in \Gamma_2
\]

wherein \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \) is the outward unit normal vector to \( \partial \Omega_1 = \Gamma_1 \cup \Gamma_2 \) and \( k_{ij} \) are the functions that constitute the inputs of the \( N \times N \) matrix \( K = K(x) \).

Equivalently, \( \frac{\partial v}{\partial \nu^{\mathcal{K}}} \) is the co-normal derivative of \( v \) along \( \Gamma_2 \), that is,

\[
\frac{\partial v}{\partial \nu^{\mathcal{K}}}(x) = \sum_{i,j=1}^{N} q_{ij}(x) \frac{\partial v}{\partial x_i}(x) \nu_j(x), \quad x \in \Gamma_2
\]

where \( q_{ij} \) are the inputs of the \( N \times N \) matrix \( Q = Q(x) \).

We observe that problem (1.1) is a non-autonomous problem and using the past history framework, introduced by Dafermos in his pioneering paper [14], we consider the following change of variables

\[
\eta^t(x, s) = \eta(x, t, s) = u(x, t) - u(x, t - s), \quad x \in \Omega_1, \quad t \geq 0, \quad s \in (0, \infty),
\]

and rewrite (1.1) as

\[
\begin{aligned}
    &\left\{
    \begin{array}{l}
    \rho_1(x)u_{tt} - \hat{g}_0 \text{div}(K(x)\nabla u) - \int_{0}^{\infty} g(s) \text{div}(K(x)\nabla \eta(s)) \, ds + f(u) \\
    = 0 \text{ in } \Omega_1 \times (0, \infty), \\
    \rho_2(x)v_{tt} - \text{div}(Q(x)\nabla v) + h(v) = 0 \text{ in } \Omega_2 \times (0, \infty), \\
    \eta_t + \eta_s = u_t \text{ in } \Omega_1 \times (0, \infty) \times (0, \infty), \\
    u = 0 \text{ on } \Gamma_1 \times (0, \infty), \quad \eta^t = 0 \text{ on } \Gamma_1 \times (0, \infty) \times (0, \infty), \\
    u = v \text{ on } \Gamma_2 \times (0, \infty), \\
    \frac{\partial v}{\partial \nu^{\mathcal{K}}}(x, t) = \hat{g}_0 \frac{\partial u}{\partial \nu^{\mathcal{K}}}(x, t) + \int_{0}^{\infty} g(s) \frac{\partial \eta^t}{\partial \nu^{\mathcal{K}}}(s) \, ds \text{ on } \Gamma_2 \times (0, \infty), \\
    u(x, 0) = u_0(x) := u^0(x, 0); \quad u_t(x, 0) = u_1(x) := \partial_t u^0(x, t) \big|_{t=0}; \quad x \in \Omega_1, \\
    v(x, 0) = v_0(x); \quad v_t(x, 0) = v_1(x); \quad x \in \Omega_2, \\
    \eta^t(x, 0) = 0; \quad \eta^0(x, s) = \eta^0(x, s) = u^0(x, 0) - u^0(x, -s); \quad x \in \Omega_1, \quad s \in (0, \infty),
    \end{array}
    \right.
\end{aligned}
\]

where \( \hat{g}_0 = 1 - k_0, \quad g_0 := \int_{0}^{\infty} g(s) \, ds < 1. \)
1.1. Previous literature. The problem of studying materials with memory has attracted the attention of many mathematicians during the years since they model some important physical phenomena, such as: viscoelasticity, population dynamics, heat flow in real conductors. The correct modelling of this kind of materials has always been a challenging task and, thanks to the works of Boltzmann and Volterra [4, 5, 39, 40], the connective bridge between the notion of memory and elastic materials was built.

Besides, transmission problems have been studied by several authors or, more generally, interface problems. The question of boundary controllability of transmission problems was considered, for example, in the work of Lions [27] and was generalized in Lagnese [26] and complemented in the work of Liu [28]. We also refer the works of Nicaise [31, 32], where the question of exact boundary controllability for the wave equation on two-dimensional networks in the euclidian n-dimensional setting was considered and the recent work of Gagnon [19]. Regarding the asymptotic stability of transmission problems, we quote the work of Liu and Williams [30], where they employed the classical energy method and the multiplier technique to prove that the system was exponentially stable. In [37], Muñoz Rivera and Oquendo considered the wave equation in a one-dimensional heterogeneous domain and they proved that the viscoelastic effect was responsible for stabilizing the system.

In addition, transmission problems in the context of thermoelasticity were considered by Muñoz Rivera and Naso [36] and Fernández Sare and Muñoz Rivera [18]. Concerning transmission problems with viscoelasticity of Kelvin-Voigt type, we quote the works of Alves et al. [1, 2]. We finish, recounting the works of Cardoso and Vodev [7] and Ignatova et al. [25], and the references therein.

There has been extensive work in the existing literature dealing with viscoelastic equations with hereditary memory. We recall the pioneering work of Dafermos [14] and the work of Liu and Liu [29], where in the latter the authors considered both, the Kelvin-Voigt model and the Boltzmann model posed on a one-dimensional domain. We briefly recount other works regarding viscoelastic problems in the past history framework, such as: Appleby et al. [3], Pata [34, 8, 9, 11, 12, 13, 15, 17, 21, 22, 23, 24] and others. Cavalcanti et al. established in [10] the exponential stability for a transmission problem of viscoelastic wave model with hereditary memory posed in a heterogeneous media. To stabilize the system, the authors needed to supplement the model with a frictional damping on the common boundary of the two parts of the domain, but they did not assume any relation between the wave propagation speed in each media.

1.2. Our contribution. In the present paper we improve substantially the result proved in [10] in what concerns the dissipative effects we need to driven the system to the exponential stability. In [10] the localized frictional damping played an essential role to control the flow of the rays of the geometric optics in the transmission boundary. So, taking advantage of this feature, instead of proving the exponential stability result for problem (1.5) directly, we introduce an auxiliary problem and the auxiliary problem is the key point to obtain the exponential stability. To be more precise, to establish that problem (1.5) is exponential stable we prove the fundamental observability inequality, that is, we prove that for $T > T_0$ and $R > 0$ there exists $C = C(T, R) > 0$ such that

$$E_U(0) \leq C \int_0^T \int_0^{\infty} (-g'(s)) \int_{\Omega_1} K(x)|\nabla \eta(t,s)|^2 dxdsdt$$

(1.6)
whenever $E_U(0) \leq R$, taking advantage of the auxiliary problem. Since its solutions satisfy the corresponding observability inequality: Consider $T > T_0$. There exists $k_0 \geq 1$ such that, for every $k \geq k_0$, the corresponding solution $U_k$ to the auxiliary problem (4.2) satisfies

$$E_{U_k}(0) \leq C \left( \int_0^T \int_0^\infty -g'(s) \int_{\Omega_1} K(x) |\nabla \eta^k_t(x,s)|^2 dx ds dt \right.$$  
$$+ \frac{1}{k} \int_0^T \int_{\Omega_1} b(x)|\partial_t u_k(t)|^2 dx dt + \frac{1}{k} \int_0^T \int_{\Omega_2} b(x)|\partial_t v_k(t)|^2 dx dt),$$

passing to the limit when $k \to \infty$, inequality (1.6) is obtained.

Summarizing, the novelty of our results can be seen from the following aspects:

- The viscoelastic effect given by the memory term with past history is enough to stabilize the system, unlike the previous work [10]. Our results give a positive answer to question (Q.2) and generalizes substantially the previous result. To control the flow in terms of the rays of the geometric optics is the main difficulty in removing the frictional damping acting in the transmission boundary. We succeed introducing an auxiliary problem, inspired by an idea in Özsari et al. [33], we obtain suitable estimates on the approximate solutions and passing to a subsequence, it converges to the solution of the original problem and we obtain the desired exponential stability.
- We also did not make any imposition regarding the wave propagation speed in each part of the domain, we just assume that they are different which is natural considering the heterogeneous media.

The present paper is organized as follows. The next section is devoted to the definition of the functional spaces which are going to be used throughout the paper and to introduce the main assumptions. In section 3, we establish the well-posedness result, while in section 4 we prove the well-posedness of solutions to the auxiliary problem and prove that the limit of solutions to this problem is the solution to the original one. In section 5, we obtain the exponential stability result taking advantage of the properties of the auxiliary problem.

2. Assumptions and functional spaces.

**Assumption 2.1.** $\rho_i \in C^\infty(\Omega_i)$ and there exist positive constants $c_i, d_i$; verifying

$$c_i \leq \rho_i(x) \leq d_i, \quad \forall x \in \Omega_i, \quad i = 1, 2. \quad (2.1)$$

**Assumption 2.2.** Consider $k_{ij} \in C^\infty(\Omega_1)$ and $q_{ij} \in C^\infty(\Omega_2)$ such that

$$K : \Omega_1 \to M_N(\mathbb{R})$$
$$x \mapsto K(x) = (k_{ij}(x))_{N \times N}$$

and

$$Q : \Omega_2 \to M_N(\mathbb{R})$$
$$x \mapsto Q(x) = (q_{ij}(x))_{N \times N}$$

are symmetric matrices satisfying

$$\alpha_1|\xi|^2 \leq \sum_{1 \leq i,j \leq N} k_{ij}(x)\xi_i \xi_j \leq \beta_1|\xi|^2, \quad \forall x \in \Omega_1 \text{ and } \xi \in \mathbb{R}^N \quad (2.2)$$
and
\[ \alpha_2 |\xi|^2 \leq \sum_{1 \leq i, j \leq N} q_{ij}(x) \xi_i \xi_j \leq \beta_2 |\xi|^2, \quad \forall x \in \Omega_2 \text{ and } \xi \in \mathbb{R}^N \]  \tag{2.3}
where \( \alpha_i \) and \( \beta_i \) are positive constants for \( i = 1, 2 \).

**Assumption 2.3.** \( g \in C^1([0, \infty)) \cap W^{1,1}([0, \infty)) \) is a positive, non-increasing function verifying
\[ g_0 := \int_0^\infty g(s) \, ds < 1. \]

**Assumption 2.4.** \( f \) and \( h \) satisfies
\[ f, h \in C^1(\mathbb{R}) \] \tag{2.4}
\[ f(0) = 0 = h(0) \] \tag{2.5}
\[ |f'(s)| \leq \gamma (1 + |s|)^{p-1}, \quad \forall s \in \mathbb{R} \] \tag{2.6}
\[ |h'(s)| \leq \gamma (1 + |s|)^{p-1}, \quad \forall s \in \mathbb{R}, \] \tag{2.7}
where \( \gamma \) is a positive constant, \( p \geq 1 \) for \( N = 2 \) and \( 1 \leq p < \frac{N}{N-2} \) for \( N \geq 3 \).

Now we are going to define the Hilbert spaces which are going to be used throughout the paper.
\[ L^2 = L^2(\Omega_1) \times L^2(\Omega_2), \]
\[ H^1 = H^1(\Omega_1) \times H^1(\Omega_2), \]
\[ H^1_{\Gamma_1} = \{ (u, v) \in H^1 : u|_{\Gamma_1} = 0, u = v \text{ on } \Gamma_2 \}, \]
where \( L^2 \) and \( H^1_{\Gamma_1} \) are endowed, respectively, with the following inner products
\[ ((\varphi_1, \psi_1), (\varphi_2, \psi_2))_{L^2} = \int_{\Omega_1} \rho_1(x) \varphi_1 \varphi_2 \, dx + \int_{\Omega_2} \rho_2(x) \psi_1 \psi_2 \, dx. \]
and
\[ ((\varphi_1, \psi_1), (\varphi_2, \psi_2))_{H^1_{\Gamma_1}} = \tilde{g}_0 \int_{\Omega_1} \nabla \varphi_1 \nabla \varphi_2 \, dx + \int_{\Omega_2} Q(x) \nabla \psi_1 \nabla \psi_2 \, dx. \]

Let’s define
\[ V = \{ w \in L^2(\Omega_1) ; \nabla w \in L^2(\Omega_1) \text{ and } w = 0 \text{ on } \Gamma_1 \}, \]
endowed with the topology
\[ (w, z)_V = \int_{\Omega_1} K(x) \nabla w \nabla z \, dx \]
and the Hilbert space
\[ M := L^2_g(\mathbb{R}^+ ; V) = \left\{ \eta : \mathbb{R}^+ \to V ; \int_0^\infty g(s) \| \eta(s) \|^2_V \, ds < +\infty \right\}, \]
endowed with the inner product
\[ (\eta, \xi)_M = \int_0^\infty g(s) (\eta(s), \xi(s))_V \, ds, \quad \forall \eta, \xi \in M. \]
We observe that Assumption 2.2 is fundamental in obtaining that \( V \) is a Hilbert space when the above inner product is considered.

In addition, we shall consider the closed operator
\[ T : \quad D(T) \subset M \to M \quad \eta \quad \mapsto \quad T(\eta) = -\eta, \]
where

\[ D(T) = \{ \eta \in \mathcal{M}; \; \eta_s \in \mathcal{M}, \; \eta(0) = 0 \}. \]

We are going to consider the following additional assumption:

**Assumption 2.5.**

\[
F(s) = \int_0^s f(t) dt \quad \text{and} \quad H(s) = \int_0^s h(t) dt, \tag{2.8}
\]

satisfy

\[
- \frac{\beta_1}{2} |s|^2 \leq F(s) \leq f(s)s + \frac{\beta_1}{2} |s|^2, \quad \forall s \in \mathbb{R} \tag{2.9}
\]

and

\[
- \frac{\beta_2}{2} |s|^2 \leq H(s) \leq h(s)s + \frac{\beta_2}{2} |s|^2, \quad \forall s \in \mathbb{R}, \tag{2.10}
\]

where \( \beta_1, \beta_2 \in [0, \lambda) \) and \( \lambda \) is the first eigenvalue of the linear problem

\[ \mathbf{A} \mathbf{u} = \lambda \mathbf{u} \]

where

\[ \mathbf{A} : D(\mathbf{A}) \subset \mathbb{H}_{\Gamma_1}^1 \to \mathbb{L}^2 \]

is the operator defined by the triple \( \{ \mathbb{H}_{\Gamma_1}^1, \mathbb{L}^2, a(\cdot, \cdot) \} \), wherein

\[ a : \mathbb{H}_{\Gamma_1}^1 \times \mathbb{H}_{\Gamma_1}^1 \to \mathbb{R} \]

\[ (u, v) \mapsto a((u_1, v_1), (u_2, v_2)) \]

and

\[ a((u_1, v_1), (u_2, v_2)) = \tilde{g}_0 \int_{\Omega_1} K(x) \nabla u_1 \nabla u_2 dx + \int_{\Omega_2} Q(x) \nabla v_1 \nabla v_2 dx. \]

In this case

\[ D(\mathbf{A}) = \left\{ (u, v) \in \mathbb{H}_{\Gamma_1}^1; \begin{array}{c}
(u_2, v_2) \in \mathbb{H}_{\Gamma_1}^1, \\
\tilde{g}_0 \frac{\partial u_2}{\partial \nu K} = \frac{\partial v_2}{\partial \nu Q} \text{ on } \Gamma_2,
\end{array} \begin{array}{c}
\mathrm{div}(K(x) \nabla u), \mathrm{div}(Q(x) \nabla v) \in \mathbb{L}^2,
\end{array} \right\}. \]

We define the phase space

\[ \mathcal{H} = \mathbb{H}_{\Gamma_1}^1 \times \mathbb{L}^2 \times \mathcal{M} \]

and the operator

\[ \mathbf{L} : D(\mathbf{L}) \subset \mathcal{H} \to \mathcal{H} \]

where

\[ D(\mathbf{L}) \]

\[ \begin{cases}
U \in \mathcal{H}; \\
\left\{ \begin{array}{c}
\tilde{g}_0 \mathrm{div} \left\{ K(x) \left[ \nabla u_1 + \frac{1}{\tilde{g}_0} \int_0^\infty g(s) \nabla \eta(s) ds \right] \right\} \begin{array}{c}
(u_2, v_2) \in \mathbb{H}_{\Gamma_1}^1, \\
\mathrm{div}(Q(x) \nabla v_1) \in \mathbb{L}^2,
\end{array} \\
\eta \in D(T), \\
\frac{\partial v_1}{\partial \nu K} = \tilde{g}_0 \frac{\partial}{\partial \nu K} \left[ u_1 + \frac{1}{\tilde{g}_0} \int_0^\infty g(s) \eta(s) ds \right] \text{ on } \Gamma_2.
\end{array} \right. \end{cases} \]
for $U = ((u_1, v_1), (u_2, v_2), \eta)$, and

$$LU = \begin{bmatrix}
\frac{\tilde{g}_0}{\rho_1(x)} \text{div}(K(x) \nabla u_1) + \frac{1}{\rho_1(x)} \int_0^\infty g(s) \text{div}(K(x) \nabla \eta ds)
\vspace{0.5em}
\frac{1}{\rho_2(x)} [\text{div}(Q(x) \nabla v_1)]
\vspace{0.5em}
u_2 - \eta_s
\end{bmatrix}$$

(2.11)

for $U = ((u_1, v_1), (u_2, v_2), \eta) \in D(L)$.

Problem (1.5) is equivalent to

$$\begin{cases}
U(t) - LU(t) = FU(t), & \forall t > 0 \\
U(0) = U_0
\end{cases}$$

(2.12)

with initial condition $U_0 = ((u_0, v_0), (u_1, v_1), \eta_0)$, where $F: \mathcal{H} \to \mathcal{H}$ is defined by

$$FU = \begin{bmatrix}
0 \\
- \frac{1}{\rho_1(x)} f(u_1) \\
- \frac{1}{\rho_2(x)} h(v_1) \\
0
\end{bmatrix}$$

(2.13)

for every $U = ((u_1, v_1), (u_2, v_2), \eta) \in \mathcal{H}$.

3. The Well-posedness result. The main result of this section reads as follows.

**Theorem 3.1** (Well-posedness). Assume the validity of Assumptions 2.1 - 2.5. If $U_0 = ((u_0, v_0), (u_1, v_1), \eta_0) \in H^1 \times L^2 \times \mathcal{M}$, there exists a unique weak solution $U(t) = (u(t), v(t), \eta)$ of (1.5) satisfying

$$(u, v) \in C^0([0, \infty); H^1) \cap C^1([0, \infty); L^2), \quad \eta \in C^0(0, \infty; \mathcal{M}).$$

(3.1)

In addition, if $U_0 = ((u_0, v_0), (u_1, v_1), \eta_0) \in D(L)$, the solution is regular.

Proof. We are going to prove that the operator $L$ is a m-dissipative operator. Indeed, let us consider $U = ((u_1, v_1), (u_2, v_2), \eta) \in D(L)$, then

$$(LU, U)_\mathcal{H} = \tilde{g}_0 \int_{\Omega_1} K(x) \nabla u_2 \nabla u_1 dx + \int_{\Omega_2} Q(x) \nabla v_2 \nabla v_1 dx$$

$$+ \int_0^\infty g(s) \int_{\Omega_1} K(x) \nabla (u_2 - \eta_1(s)) \nabla \eta ds dx ds$$

$$+ \tilde{g}_0 \int_{\Omega_1} \text{div}(K(x) \nabla \left[u_1 + \frac{1}{\tilde{g}_0} \int_0^\infty g(s) \eta ds \right]) u_2 dx$$

$$+ \int_{\Omega_2} \text{div}(Q(x) \nabla v_1)v_2 dx.$$
In fact, proceeding formally, we obtain from (3.3) that is,

\[ \eta(s) = u_2(1 - e^{-s}) + \int_0^s e^{\tau-s} \xi(\tau) d\tau, \]

which combined with (3.3)1 and (3.3)3 yields

\[ \rho_1(x) u_2 - \tilde{g}_1 \text{div}(K(x)\nabla u_2) \]

\[ = \rho_1(x) f_2 + \tilde{g}_0 \text{div}(K(x)\nabla f_1) + \int_0^\infty \int_0^s g(s)e^{\tau-s} \text{div}(K(x)\nabla \xi(\tau)) d\tau ds \in H^{-1}(\Omega_1), \]

where \( \tilde{g}_1 = \tilde{g}_0 + \int_0^\infty g(s)(1 - e^{-s}) ds \).

In light of identity (3.5) we are motivated to define the bilinear form \( B \), which is continuous and coercive,

\[ B : H^1_{\Gamma_1} \times H^1_{\Gamma_1} \rightarrow \mathbb{R} \]

\[ (w, z) \mapsto B(w, z) \]

given by

\[ B(w, z) = \int_{\Omega_1} \rho_1(x) w_1 z_1 dx + \int_{\Omega_2} \rho_2(x) w_2 z_2 dx + \tilde{g}_1 \int_{\Omega_1} K(x) \nabla w_1 \nabla z_1 dx + \int_{\Omega_2} Q(x) \nabla w_2 \nabla z_2 dx; \]

where \( w = (w_1, w_2), z = (z_1, z_2) \in H^1_{\Gamma_1} \).

From (3.3)2 and (3.3)4, we have

\[ \rho_2(x) v_2 - \text{div}(Q(x)\nabla v_2) = \rho_2(x) g_2 + \text{div}(Q(x)\nabla g_1) \in H^{-1}(\Omega_2). \]
and we define the linear and continuous functional \( \Lambda \)
\[
\Lambda : H^1_1 \to \mathbb{R}
\]
\[
z \mapsto \Lambda(z)
\]

where
\[
\Lambda(z) = \int_{\Omega_1} \rho_1(x) f_2 z_1 dx + \int_{\Omega_2} \rho_2(x) g_2 z_2 dx - \tilde{g}_0 \int_{\Omega_1} K(x) \nabla f_1 \nabla z_1 dx - \int_{\Omega_2} Q(x) \nabla g_1 \nabla z_2 dx
\]
\[
- \int_{\Omega_2} Q(x) \nabla g_1 \nabla z_2 dx - \int_{\Omega_1} \int_{0}^{s} g(s)e^{\tau - s} \int_{\Omega_1} K(x) \nabla \xi(s) \nabla z_1 dx d\tau ds;
\]
\[
z = (z_1, z_2) \in H^1_1.
\]

Consequently, the Lax-Milgram Theorem yields the existence of a unique \((u^*, v^*)\) \(\in H^1_1\), verifying
\[
\mathbf{B}((u^*, v^*), (z_1, z_2)) = \Lambda(z_1, z_2), \quad \forall (z_1, z_2) \in H^1_1,
\]
that is,
\[
\int_{\Omega_1} \rho_1(x) u^* z_1 dx + \int_{\Omega_2} \rho_2(x) v^* z_2 dx + \tilde{g}_1 \int_{\Omega_1} K(x) \nabla u^* \nabla z_1 dx + \int_{\Omega_2} Q(x) \nabla v^* \nabla z_2 dx
\]
\[
= \int_{\Omega_1} \rho_1(x) f_2 z_1 dx + \int_{\Omega_2} \rho_2(x) g_2 z_2 dx - \tilde{g}_0 \int_{\Omega_1} K(x) \nabla f_1 \nabla z_1 dx - \int_{\Omega_2} Q(x) \nabla g_1 \nabla z_2 dx
\]
\[
- \int_{\Omega} \int_{0}^{s} g(s)e^{\tau - s} \int_{\Omega_1} K(x) \nabla \xi(s) \nabla z_1 dx d\tau ds,
\]
for every \(z = (z_1, z_2) \in H^1_1\).

Consider \(z_1 \in D(\Omega_1)\) and define \(z = (z_1, 0) \in H^1_1\). We derive from (3.7) that
\[
- \operatorname{div} \left( K(x) \nabla [\tilde{g}_1 u^* + \tilde{g}_0 f_1 + \int_{0}^{\infty} \int_{0}^{s} g(s)e^{\tau - s} \xi(s) d\tau ds] \right) = \rho_1(x) f_2 - \rho_1(x) u^* \text{ in } D'(\Omega_1).
\]
Since \(\rho_1(x) f_2 - \rho_1(x) u^* \in L^2(\Omega_1)\), we conclude that
\[
\rho_1(x) u^* = - \operatorname{div} \left( K(x) \nabla \left\{ \tilde{g}_1 u^* + \tilde{g}_0 f_1 + \int_{0}^{\infty} \int_{0}^{s} g(s)e^{\tau - s} \xi(s) d\tau ds \right\} \right)
\]
\[
= \rho_1(x) f_2 \text{ a.e. in } \Omega_1. \quad (3.8)
\]

Similarly, taking \(z_2 \in D(\Omega_2)\) and defining \(z = (0, z_2) \in H^1_1\), it follows from (3.7) that
\[
\rho_2(x) v^* = - \operatorname{div} (Q(x) \nabla [v^* + g_1]) = \rho_2(x) g_2 \text{ a.e. in } \Omega_2. \quad (3.9)
\]
Replacing identities (3.8) and (3.9) in (3.7) and using Green’s Formula, we obtain for every \(z = (z_1, z_2) \in H^1_1\)
\[
- \left[ \frac{\partial}{\partial \nu_{\mathcal{K}}} \left\{ \tilde{g}_1 u^* + \tilde{g}_0 f_1 + \int_{0}^{\infty} \int_{0}^{s} g(s)e^{\tau - s} \xi(s) d\tau ds \right\}, z_1 \right]_{H^\frac{1}{2}(\Gamma_2) \times H^\frac{1}{2}(\Gamma_2)}
\]
\[
+ \left[ \frac{\partial}{\partial \nu_{\mathcal{K}}} (v^* + g_1), z_2 \right]_{H^\frac{1}{2}(\Gamma_3) \times H^\frac{1}{2}(\Gamma_2)} = 0. \quad (3.10)
\]

Defining
\[
u_2 = u^*, u_1 = u^* + f_1
\]
\[
u_2 = v^*, v_1 = v^* + g_1
\]
and considering \(\eta\) given by expression (3.4), we obtain that \(U = (u_1, v_1), (u_2, v_2), \eta) \in D(L)\) which finishes the proof of (3.2) and, as a consequence, we obtain that
Performing similar computations and observing that $U, V$ for $\mathcal{L}$ is a m-dissipative operator. The Lumer-Phillips Theorem yields that $\mathcal{L}$ is the infinitesimal generator of a $C_0$ semigroup of contractions on $\mathcal{H}$ and $D(\mathcal{L})$ is dense in $\mathcal{H}$.

Besides, from Assumption 2.4, we derive

$$|f(s_1) - f(s_2)| \leq c_1(1 + |s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|$$  \hspace{1cm} (3.11)

and

$$|h(s_1) - h(s_2)| \leq c_2(1 + |s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|.$$  \hspace{1cm} (3.12)

In light of the above inequalities, we are going to conclude that $\mathcal{F}$ is locally Lipschitz. Indeed, let $\mathcal{B}$ be a bounded subset of $\mathcal{H}$ and $U, V \in \mathcal{B}$, where $U = ((u_1, v_1), (u_2, v_2), \eta)$ and $V = ((w_1, z_1), (w_2, z_2), \xi)$.

From (3.11) and (3.12) we obtain

$$\|\mathcal{F}(U) - \mathcal{F}(V)\|_\mathcal{H}^2 = \int_{\Omega_1} |f(w_1) - f(u_1)|^2 dx + \int_{\Omega_2} |h(z_1) - h(v_1)|^2 dx$$

$$\leq C_f \int_{\Omega_1} (1 + |u_1|^{2(p-1)} + |w_1|^{2(p-1)})|u_1 - w_1|^2 dx$$

$$+ C_h \int_{\Omega_1} (1 + |v_1|^{2(p-1)} + |z_1|^{2(p-1)})|v_1 - z_1|^2 dx,$$

where $C_f$ and $C_h$ are positive constants depending on $f$ and $h$, respectively.

Taking advantage of Hölder’s inequality, with $p$ and $q = \frac{p}{p-1}$, we have

$$\int_{\Omega_1} |w_1|^{2(p-1)}|u_1 - w_1|^2 dx \leq \|w_1\|_{L^{2p}(\Omega_1)}^2 \|u_1 - w_1\|_{L^{2p}(\Omega_1)}^2.$$  \hspace{1cm} (3.13)

Performing similar computations and observing that $U, V \in \mathcal{B}$, we conclude

$$\|\mathcal{F}(U) - \mathcal{F}(V)\|_\mathcal{H}^2 \leq C\|u_1 - w_1\|_{L^{2p}(\Omega_1)}^2 + \|v_1 - z_1\|_{L^{2p}(\Omega_2)}^2 \leq C\|U - V\|_\mathcal{H}^2,$$  \hspace{1cm} (3.14)

for $U, V \in \mathcal{B}$; where $C$ represents different positive constants which depend on $\mathcal{B}$.

According to Theorem 1.4 of Chapter 6 and the Remark of session 6.1 in [35], we obtain the existence of mild and regular solutions to problem (1.5) on the interval $[0, T_{\text{max}})$, and that these solutions are unique.

In addition,

$$c\|U(t)\|_{\mathcal{H}}^2 \leq E(t) \leq E(0), \hspace{0.5cm} \forall t \in [0, T_{\text{max}}),$$

where

$$E(t) = \frac{1}{2} \left[ \int_{\Omega_1} \rho_1(x)|u_1(t)|^2 dx + \int_{\Omega_2} \rho_2(x)|v_1(t)|^2 dx + \bar{g}_0 \int_{\Omega_1} K(x)|\nabla u(t)|^2 dx$$

$$+ \int_{\Omega_2} Q(x)|\nabla v(t)|^2 dx + \int_0^t g(s) \int_{\Omega_1} K(x)|\nabla \eta| dx ds \right]$$

$$+ \int_{\Omega_1} F(u(t)) dx + \int_{\Omega_2} H(v(t)) dx$$  \hspace{1cm} (3.15)

is the energy functional related to problem (1.5). Then, $T_{\text{max}} = +\infty$.

The well-posedness Theorem 3.1 is proved. \hfill \Box
4. **Auxiliary problem.** Observe that given \((u_0, v_0), (u_1, v_1), \eta_0) \in \mathcal{H}\), consider a sequence \[((u_{0,k}, v_{0,k}), (u_{1,k}, v_{1,k}), \eta_{0,k}))\) for \(k \in \mathbb{N}\) in \(D(L)\), where

\[
D(L) = \left\{ U \in \mathcal{H}; \begin{cases}
\tilde{g}_0 \text{div} \left\{ \int_0^{\infty} g(s) \nabla \eta(s) ds \right\}, \text{div}(Q(x)\nabla v_1) \in \mathbb{L}^2, \\
(u_2, v_2) \in \mathbb{H}^1_{0,1}, \\
\eta \in D(T), \\
\frac{\partial v_1}{\partial \nu} = \tilde{g}_0 \frac{\partial}{\partial \nu} \left[ u_1 + \frac{1}{\tilde{g}_0} \int_0^{\infty} g(s) \eta(s) ds \right] \text{ on } \Gamma_2
\end{cases} \right\}
\]

for \(U = (u_1, v_1), (u_2, v_2), \eta) \in \mathcal{H}\), and

\[
((u_{0,k}, v_{0,k}), (u_{1,k}, v_{1,k}), \eta_{0,k}) \to ((u_0, v_0), (u_1, v_1), \eta_0) \text{ in } \mathcal{H}. \quad (4.1)
\]

In this section, instead of studying problem (1.5) we are going to study the well-posedness of the following auxiliary problem, for each \(k \in \mathbb{N}\):

\[
\begin{cases}
\rho_1(x)\partial_t^2 u_k - \tilde{g}_0 \text{div}(K(x)\nabla u_k) - \int_0^{\infty} g(s) \text{div}(K(x)\nabla \eta_k(s)) ds \\
+ \frac{1}{k} b(x) \partial_t u_k + f_k(u_k) = 0 \text{ in } \Omega_1 \times (0, \infty), \\
\rho_2(x)\partial_t^2 v_k - \text{div}(Q(x)\nabla v_k) + \frac{1}{k} b(x) \partial_t v_k + h_k(v_k) = 0 \text{ in } \Omega_2 \times (0, \infty), \\
\partial_t \eta_k + \partial_x \eta_k = \partial_t u_k \text{ in } \Omega_1 \times (0, \infty) \times (0, \infty), \\
u_k = 0 \text{ on } \Gamma_1 \times (0, \infty), \quad \eta_k^0 = 0 \text{ on } \Gamma_1 \times (0, \infty) \times (0, \infty), \\
u_k(x, 0) = u_{0,k}(x), \partial_t u_k(x, 0) = u_{1,k}(x), \quad x \in \Omega_1, \\
v_k(x, 0) = v_{0,k}(x), \partial_t v_k(x, 0) = v_{1,k}(x), \quad x \in \Omega_2, \\
\eta_k^0(x, 0) = 0; u^0_k(x, s) = u_{0,k}(x, s) - u^0_k(x, -s); \quad x \in \Omega_1, s \in (0, \infty), \quad (4.2)
\end{cases}
\]

where \(u^0_k(x, t) : \Omega_1 \times (-\infty, 0) \to \mathbb{R}\) is the prescribed past history of \(u_k\).

In addition, the function \(b\) is a non-negative real function such that \(b = b(x) \in L^\infty(\Omega)\) and \(f_k, h_k : \mathbb{R} \to \mathbb{R}\) are defined by

\[
f_k(s) := \begin{cases}
f(s), |s| \leq k, \\
f(k), s > k, \\
f(-k), s < -k;
\end{cases} \quad (4.3)
\]

and

\[
h_k(s) := \begin{cases}
h(s), |s| \leq k, \\
h(k), s > k, \\
h(-k), s < -k.
\end{cases} \quad (4.4)
\]

Our goal is to prove that the sequence \(\{(u_k, v_k, \eta_k)\}_{k \in \mathbb{N}}\) of solutions to problem (4.2) converges to the unique solution to the problem (1.5). Indeed, we begin observing that the distributional derivatives \(f'_k, h'_k\) are essentially bounded and given by

\[
f'_k(s) := \begin{cases}
f'(s), |s| < k, \\
0, s > k, \\
0, s < -k;
\end{cases} \quad (4.5)
\]
and
\[ h'_k(s) := \begin{cases} h'(s), & |s| < k, \\ 0, & s > k, \\ 0, & s < -k. \end{cases} \tag{4.6} \]

Besides, for each \( k \in \mathbb{N} \), there exists positive constants \( C_k, D_k \) verifying
\[ |f_k(r) - f_k(s)| \leq C_k |r - s| \text{ for every } r, s \in \mathbb{R}, \tag{4.7} \]
and
\[ |h_k(r) - h_k(s)| \leq D_k |r - s| \text{ for every } r, s \in \mathbb{R}. \tag{4.8} \]

Thus, for each \( k \in \mathbb{N} \), \( f_k, h_k \) are globally Lipschitz functions.

From standard Semigroup Theory, similarly to Theorem 3.1, we derive that problem (4.2) possesses a unique regular solution \((u_k, v_k, \eta_k)\) that is,
\[ (u_k, v_k, \eta_k) \in C^0([0, \infty); D(L)) \cap C^1([0, \infty); \mathcal{H}). \]

The energy functional associated to the auxiliary problem (4.2) is given by
\[ E_k(t) = \frac{1}{2} \int_{\Omega_1} \rho_1(x)|\partial_t u_k(t)|^2 dx + \int_{\Omega_2} \rho_2(x)|\partial_t v_k(t)|^2 dx + \bar{g}_0 \int_{\Omega_1} K(x)|\nabla u_k(t)|^2 dx \\
+ \int_{\Omega_2} Q(x)|\nabla v_k(t)|^2 dx + \int_{0}^{\infty} g(s) \int_{\Omega_1} K(x)|\nabla \eta_k|^2 dx ds \]
\[ + \int_{\Omega_1} F_k(u_k(t)) dx + \int_{\Omega_2} H_k(v_k(t)) dx, \tag{4.9} \]
where
\[ F_k(t) = \int_{0}^{t} f_k(s) ds \quad \text{and} \quad H_k(t) = \int_{0}^{t} h_k(s) ds. \tag{4.10} \]

Consequently, for \( 0 < \tau < \infty \),
\[ E_k(\tau) - \frac{1}{2} \int_{0}^{\tau} \int_{0}^{\infty} g'(s) \int_{\Omega_1} K(x)|\nabla \eta_k^t(x, s)|^2 dx ds dt \]
\[ + \frac{1}{k} \int_{0}^{\tau} \int_{\Omega_1} b(x)|\partial_t u_k(t)|^2 dx dt + \frac{1}{k} \int_{0}^{\tau} \int_{\Omega_2} b(x)|\partial_t v_k(t)|^2 dx dt = E_k(0). \tag{4.11} \]

We also observe that
\[ F_k(s) := \begin{cases} \int_{0}^{s} f(\xi) d\xi, & |s| \leq k, \\ \int_{0}^{k} f(\xi) d\xi + f(k)|s - k|, & s > k, \\ f(-k)|s + k| + \int_{0}^{-k} f(\xi) d\xi, & s < -k. \end{cases} \tag{4.12} \]
and
\[ H_k(s) := \begin{cases} \int_{0}^{s} h(\xi) d\xi, & |s| \leq k, \\ \int_{0}^{k} h(\xi) d\xi + h(k)|s - k|, & s > k, \\ h(-k)|s + k| + \int_{0}^{-k} h(\xi) d\xi, & s < -k. \end{cases} \tag{4.13} \]
Assumptions (2.4) and (2.5), together with (3.11) and (3.12) imply that
\[ |f(s)| \leq c|s| + |s|^p \quad \text{and} \quad F(s) \leq f(s)s + \frac{\beta_1}{2}|s|^2, \quad \forall s \in \mathbb{R} \]
and
\[ |h(s)| \leq c|s| + |s|^p \quad \text{and} \quad H(s) \leq h(s)s + \frac{\beta_2}{2}|s|^2, \quad \forall s \in \mathbb{R}. \]
Then, we infer that
\[ |F_k(s)|, |H_k(s)| \leq c|s|^2 + |s|^{p+1}, \quad \text{for all } s \in \mathbb{R} \text{ and } k \in \mathbb{N}, \tag{4.14} \]
that is,
\[ \int_{\Omega_1} |F_k(u_{0,k})| \, dx \leq c \int_{\Omega} [u_{0,k}|^2 + |u_{0,k}|^{p+1}] \, dx \tag{4.15} \]
and
\[ \int_{\Omega_1} |H_k(v_{0,k})| \, dx \leq c \int_{\Omega} [v_{0,k}|^2 + |v_{0,k}|^{p+1}] \, dx. \tag{4.16} \]
So, convergence (4.11), the energy identity (4.11) and inequalities (4.15), (4.16), yield a subsequence of \{(u_k, v_k, \eta_k)\}, reindexed by \{(u_k, v_k, \eta_k)\}, verifying
\begin{align*}
(u_k, v_k) &\to (u, v) \text{ weakly star in } L^\infty(0,T; H^1_{\text{loc}}) , \tag{4.17} \\
(\partial_t u_k, \partial_t v_k) &\to (u_t, v_t) \text{ weakly star in } L^\infty(0,T; L^2) , \tag{4.18} \\
\frac{1}{\sqrt{k}} \sqrt{b(x)} \partial_t u_k &\to 0 \text{ strongly in } L^2_{\text{loc}}(0,\infty; L^2(\Omega_1)) , \tag{4.19} \\
\frac{1}{\sqrt{k}} \sqrt{b(x)} \partial_t v_k &\to 0 \text{ strongly in } L^2_{\text{loc}}(0,\infty; L^2(\Omega_2)) , \tag{4.20} \\
\eta_k &\to \eta \text{ weakly star in } L^\infty(0,T; \mathcal{M}). \tag{4.21} 
\end{align*}

Thanks to the standard compactness result (see Simon [38]) we also deduce that
\[ (u_k, v_k) \to (u, v) \text{ strongly in } L^\infty(0,T; L^{2^*-\zeta}(\Omega_1)) \times L^{2^*-\zeta}(\Omega_2)); \quad \forall T > 0, \tag{4.22} \]
where \(2^* := \frac{2N}{N-2}\) and \(\zeta > 0\) is small enough. In addition, from (4.22), we obtain for every \(T > 0\),
\[ u_k \to u \text{ a.e. in } \Omega_1 \times (0,T) \quad \text{and} \quad v_k \to v \text{ a.e. in } \Omega_2 \times (0,T). \tag{4.23} \]
On the other hand, from (4.23) and the continuity of the functions \(f, h\) we get, for \(T > 0\),
\[ f_k(u_k) \to f(u) \text{ a.e. in } \Omega_1 \times (0,T) \quad \text{and} \quad h_k(v_k) \to h(v) \text{ a.e. in } \Omega_2 \times (0,T). \tag{4.24} \]
Indeed, the convergence (4.23) guarantees the existence of a positive constant \(L = L(x,t) > 0\) verifying \(|u_k(x,t)|, |v_k(x,t)| < L\) for every \(k \in \mathbb{N}\). Then, using the definition of \(f_k, h_k\), we obtain that
\[ \text{if } |u_k(x,t)| < L, \forall k \text{ then } f_k(u_k(x,t)) = f(u_k(x,t)) \quad \text{and} \quad h_k(u_k(x,t)) = h(u_k(x,t)); \quad k \geq L. \tag{4.25} \]
The continuity of \(f, h\) yields the convergences (4.24).

In addition, thanks to Sobolev imbeddings and the boundedness of the sequences \(\{u_k\} \text{ and } \{v_k\}\), we derive for all \(k \in \mathbb{N}\) the following inequalities, for \(T > 0\):
\[ \int_0^T \int_{\Omega_1} |f_k(u_k(x,t))|^{\frac{p+1}{p}} \, dx dt \leq C \quad \text{and} \quad \int_0^T \int_{\Omega_2} |h_k(v_k(x,t))|^{\frac{p+1}{p}} \, dx dt \leq C. \tag{4.26} \]
In light of (4.24), (4.26) and Lions’ Lemma, we deduce that
\[ f(u) \in L^{\frac{n+1}{2}}(\Omega_1 \times (0, T)), \quad h(u) \in L^{\frac{n+1}{2}}(\Omega_2 \times (0, T)), \]
and
\[ f_k(u_k) \rightharpoonup f(u) \quad \text{weakly in } L^{\frac{n+1}{2}}(\Omega_1 \times (0, T)) \]  
(4.27)
and
\[ h_k(v_k) \rightharpoonup h(v) \quad \text{weakly in } L^{\frac{n+1}{2}}(\Omega_2 \times (0, T)). \]  
(4.28)
Going back to problem (4.2), multiplying by \( \varphi \theta \), where \( \varphi \in C_0^\infty(\Omega), \theta \in C_0^\infty(0, T) \), performing integration by parts, considering convergences (4.17) - (4.21), (4.27), (4.28) and passing to the limit, we obtain
\[
\begin{align*}
\rho_1(x)\partial_t^2 u - \tilde{g}_0 \text{div}(K(x)\nabla u) \\
- \int_0^\infty g(s) \text{div}(K(x)\nabla \eta(s)) \, ds + f(u) = 0 \quad \text{in } D'(\Omega_1 \times (0, T))
\end{align*}
\]
and
\[
\begin{align*}
\rho_2(x)\partial_t^2 v - \text{div}(Q(x)\nabla v) + h(v) = 0 \quad \text{in } D'(\Omega_2 \times (0, T)).
\end{align*}
\]
Consequently,
\[
\begin{align*}
\rho_1(x)\partial_t^2 u - \tilde{g}_0 \text{div}(K(x)\nabla u) \\
- \int_0^\infty g(s) \text{div}(K(x)\nabla \eta(s)) \, ds + f(u) = 0 \quad \text{in } L^\infty(0, T; L^{\frac{n+1}{2}}(\Omega_1) + H^{-1}(\Omega_1))
\end{align*}
\]
and
\[
\begin{align*}
\rho_2(x)\partial_t^2 v - \text{div}(Q(x)\nabla v) + h(v) = 0 \quad \text{in } L^\infty(0, T; L^{\frac{n+1}{2}}(\Omega_2) + H^{-1}(\Omega_2)).
\end{align*}
\]
Now, we are going to prove that the limit of solutions to the auxiliary problem (4.2), which solves problem (1.5), has the required regularity, that is,
\[
(u, v) \in C^0([0, T]; H^1_{\Gamma_1}), \quad (\partial_t u, \partial_t v) \in C^0([0, T]; \mathbb{L}^2), \quad \eta \in C^0([0, T]; \mathcal{M})
\]
and
\[
(\partial_t^2 u, \partial_t^2 v) \in C^0 \left([0, T]; \left(H^{-1}(\Omega_1) + L^{\frac{n+1}{2}}(\Omega_1)\right) \times \left(H^{-1}(\Omega_2) + L^{\frac{n+1}{2}}(\Omega_2)\right)\right).
\]
In addition, that
\[
(u_k, \partial_t u_k) \to (u, \partial_t u) \quad \text{in } C^0([0, T]; V) \times C^0([0, T]; L^2(\Omega_1)).
\]
When \( N = 2 \) the regularity is trivially obtained. So, in what follows, we are going to consider \( N \geq 3 \). In fact, first of all, we shall prove that
\[
f_k(u_k) \to f(u) \quad \text{strongly in } L^2(\Omega_1 \times (0, T))
\]
and
\[
h_k(v_k) \to h(v) \quad \text{strongly in } L^2(\Omega_2 \times (0, T)).
\]
Indeed, we observe that
\[
\int_0^T \int_{\Omega_1} |f_k(u_k) - f(u)|^2 \, dx \, dt \lesssim \int_0^T \int_{\Omega_1} |f_k(u_k) - f(u_k)|^2 \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega_1} |f(u_k) - f(u)|^2 \, dx \, dt
\]  
(4.35)
and 
\[
\int_{\Omega_1} |f(u_k) - f(u)|^2 \, dx \lesssim \int_{\Omega_1} |u_k - u|^2 \, dx + \int_{\Omega_1} |u_k^{2(p-1)}|u_k - u|^2 \, dx \\
+ \int_{\Omega_1} |u|^{2(p-1)}|u_k - u|^2 \, dx.
\]

Observe that from (4.22) we derive that \(I_{1,k} \to 0\) as \(k \to +\infty\). Besides, Hölder inequality yields 
\[
I_{2,k} \lesssim \left( \int_{\Omega_1} |u_k|^{2p} \right)^{\frac{p-1}{p}} \left( \int_{\Omega_1} |u_k - u|^{2p} \right)^{\frac{1}{p}} 
\]
and, since \(p < \frac{N}{2}\), from (4.17) and (4.22) we deduce that \(I_{2,k} \to 0\) as \(k \to +\infty\). Similarly, we obtain that \(I_{3,k} \to 0\) as \(k \to +\infty\).

Thus, 
\[
\int_0^T \int_{\Omega_1} |f(u_k) - f(u)|^2 \, dx \, dt \to 0 \text{ as } k \to \infty. \tag{4.36}
\]

On the other hand, define, for each fixed \(t \in [0, T]\), the set 
\[
\Omega_k^t := \{x \in \Omega_1; |u_k(x,t)| > k\}.
\]

Since, 
\[
f_k(u_k) - f(u_k) = 0, \text{ if } |u_k(x,t)| \leq k,
\]
we have 
\[
\int_{\Omega_1} |f_k(u_k) - f(u_k)|^2 \, dx \\
= \int_{\Omega_k^t} |f_k(u_k) - f(u_k)|^2 \, dx \lesssim \left[ \int_{\Omega_k^t} |f(u_k)|^2 \, dx + \int_{\Omega_k^t} |f(-k)|^2 \, dx + \int_{\Omega_k^t} |f(k)|^2 \, dx \right] \\
\lesssim \left[ \int_{\Omega_k^t} |u_k|^2 + |u_k|^{2p} \, dx + \int_{\Omega_k^t} |k|^2 + |k|^{2p} \, dx \right] \lesssim \int_{\Omega_k^t} |u_k|^{2p} \, dx \, dx. \tag{4.37}
\]

Considering that, for all \(t \in [0, T]\), 
\[
\left( \int_{\Omega_k^t} k^{\frac{2N-2}{p-2}} \, dx \right) \lesssim \left( \int_{\Omega_k^t} |u_k|^{\frac{2N-2}{p-2}} \, dx \right) \lesssim [E_{u_k}(0)]^{\frac{2N-2}{p-2}} \leq C, \tag{4.38}
\]
where \(C\) is a positive constant which does not depend on \(k\) and \(t\), we conclude 
\[
\text{meas}(\Omega_k^t) \lesssim k^{-\frac{2N+4}{N-2}}, \forall t \in [0, T]. \tag{4.39}
\]

Defining \(\beta := \frac{2N}{2p(N-2)}\), since \(p < \frac{N}{N-2}\), we get that \(\beta > 1\). Then, considering \(\alpha := \frac{2N}{2N-2p(N-2)}\), we have \(\frac{1}{\alpha} + \frac{1}{\beta} = 1\) and from Hölder inequality and (4.39) we derive 
\[
\int_0^T \int_{\Omega_k^t} |u_k|^{2p} \, dx \lesssim k^{-\frac{2N+4}{N-2}} \left( \frac{2N-2p(N-2)}{2N} \right) \int_0^T \|u_k(t)\|^{2p} \, L^{\frac{2N}{N-2}}(\Omega) \, dt \\
\lesssim k^{(-\frac{2N+4}{N-2})} \left( \frac{2N-2p(N-2)}{2N} \right) [E_{u_k}(0)]^p. \tag{4.40}
\]
Observing that \( E_k(0) \leq C \) for all \( k \in \mathbb{N} \) and \( \left( \frac{-2N+1}{N-2} \right) \left( \frac{2N-(2p)(N-2)}{2N} \right) < 0 \), the above inequality yields
\[
\int_0^T \int_{\Omega_k} |u_k|^2 \, dx \to 0, \quad \text{as } k \to +\infty. \quad (4.41)
\]

From inequality (4.37) and convergence (4.41), we conclude the proof of (4.33). Analogously, we prove the convergence (4.34).

Defining the sequences \( z_{\mu, \sigma} = u_\mu - u_\sigma \) and \( w_{\mu, \sigma} = v_\mu - v_\sigma ; \mu, \sigma \in \mathbb{N} \), we derive from problem (4.2), after integrating over \((0, t) \times \Omega_1 \) and \((0, t) \times \Omega_2 \), that
\[
\frac{1}{2} \left\{ \left\| (\partial_t z_{\mu, \sigma}(t), \partial_t w_{\mu, \sigma}(t)) \right\|^2_{L^2} + \left\| (z_{\mu, \sigma}(t), w_{\mu, \sigma}(t)) \right\|^2_{L^2} \right\} + \left\| \eta_\mu(t) - \eta_\sigma(t) \right\|^2_M \\
\leq \frac{1}{2} \left\{ \left\| (u_{1, \mu} - u_{1, \sigma}, v_{1, \mu} - v_{1, \sigma}) \right\|^2_{L^2} + \left\| (u_{0, \mu} - u_{0, \sigma}, v_{0, \mu} - v_{0, \sigma}) \right\|^2_{L^2} \right\} \\
+ \left\| \eta_{0, \mu} - \eta_{0, \sigma} \right\|^2_M + \left[ \frac{1}{\mu} + \frac{1}{\sigma} \right] \int_0^t \int_{\Omega_1} b(x)(|\partial_t u_\mu|^2 + |\partial_t v_\sigma|^2) \, dx \, ds \\\n+ \int_0^t \int_{\Omega_2} (f_\mu(u_\mu) - f_\sigma(u_\sigma)) (\partial_t u_\mu - \partial_t v_\sigma) \, dx \, ds \\\n+ \int_0^t \int_{\Omega_2} (h_\mu(v_\mu) - h_\sigma(v_\sigma)) (\partial_t v_\mu - \partial_t v_\sigma) \, dx \, ds. \quad (4.42)
\]

The convergences (4.1), (4.18), (4.33) and (4.34) imply that the terms on the RHS of (4.42) converge to zero as \( \mu, \sigma \to +\infty \) and, consequently,
\[
(u_\mu, v_\mu) \to (u, v) \text{ in } C^0([0, T]; \mathbb{H}^1) \cap C^1([0, T]; L^2) \\\n\eta_\mu \to \eta \text{ in } C^0([0, T]; \mathcal{M}) \quad (4.43)
\]
for all \( T > 0 \).

To finish this section, we shall prove that \((u, v, \eta)\), obtained in (4.43), satisfies the energy functional associated to problem (1.5) as a limit of solutions. First of all, we need to perform some computations regarding \( F_k(u_k) \) and \( H_k(v_k) \). We are going to explicit the computations concerning the function \( F_k \) since the ideas regarding \( H_k \) are absolutely the same.

Indeed, inequality (4.14) gives
\[
|F_k(s)| \leq c(|s|^2 + |s|^{p+1}),
\]
for all \( s \in \mathbb{R} \) and \( k \in \mathbb{N} \).

Since \( 1 < p < \frac{N}{N-2}, \, N \geq 3, \) we obtain \( 2 \leq p + 1 < \frac{2N}{N-2} = 2^* \) and there exists \( \zeta > 0 \) such that \( p + 1 + \zeta = 2^* \). Then, \( H^1(\Omega_1) \hookrightarrow L^{p+1+\zeta}(\Omega_1) \) and
\[
\int_{\Omega_1} |F_k(u_k)|^{\frac{p+1+\zeta}{p+1}} \, dx \leq c \int_{\Omega_1} \left[ |u_k|^{\frac{2(p+1+\zeta)}{p+1}} + |u_k|^{p+1+\zeta} \right] \, dx \leq CE_k(0), \quad (4.44)
\]
for all \( t \in [0, T] \). The boundedness of \( E_k(0) \) implies that there exists \( \chi \in L^{\frac{2^*}{p+1+\zeta}}(\Omega_1 \times (0, T)) \) verifying the following convergence:
\[
F_k(u_k) \rightharpoonup \chi \text{ weakly in } L^{\frac{2^*}{p+1+\zeta}}(\Omega_1 \times (0, T)), \quad \text{as } k \to +\infty. \quad (4.45)
\]
On the other hand, observe that $H^1(\Omega_1) \hookrightarrow L^{\frac{n+1+c/2}{r-1}}(\Omega_1)$ and from the Aubin-Lions-Simon’s Theorem we derive $u_k \rightarrow u$ strongly in $L^\infty(0,T;L^{\frac{n+1+c/2}{r-1}}(\Omega_1))$. Then,  
$$u_k(x,t) \rightarrow u(x,t) \text{ a.e. in } \Omega_1 \times (0,T).$$  
(4.46)  
Note that,  
$$|F_k(u_k(x,t)) - F(u(x,t))|$$  
$$\leq |F_k(u_k(x,t)) - F(u_k(x,t))| + |F(u_k(x,t)) - F(u(x,t))|. \quad (4.47)$$  
In light of convergence (4.46) and the continuity of $F$, we trivially obtain  
$$F(u_k(x,t)) \rightarrow F(u(x,t)) \text{ a.e. in } \Omega_1 \times (0,T). \quad (4.48)$$  
In addition, for each $(x,t)$, there exists a positive constant $M=M(x,t)>0$ such that  
$$\left|F_k(u_k(x,t)) - F(u_k(x,t))\right| = \left|\int_0^{u_k(x,t)} f_k(s)ds - \int_0^{u_k(x,t)} f(s)ds\right|$$  
$$\leq \int_{-M}^M |f_k(x,t) - f(s)|ds = 0, \quad \text{if } k \geq M. \quad (4.49)$$  
From (4.47)-(4.49), convergence (4.45) and Lions Lemma we derive  
$$F_k(u_k) \rightarrow F(u) \text{ weakly in } L^{\frac{2^*}{r}}(\Omega_1 \times (0,T)), \quad \text{as } k \rightarrow +\infty, \quad (4.50)$$  
proving that $\chi = F(u)$.

Besides, employing Strauss’s Lemma:  
Let $\mathcal{O}$ be an open and bounded subset of $\mathbb{R}^N$, $N \geq 1$, $1 < q < +\infty$, and $\{u_n\}_{n \in \mathbb{N}}$ a sequence which is bounded in $L^q(\mathcal{O})$. If $u_n \rightarrow u$ a.e. in $\mathcal{O}$, then $u \in L^q(\mathcal{O})$ and $u_n \rightarrow u$ weakly in $L^q(\mathcal{O})$. In addition, if $1 \leq r < q$ we also have $u_n \rightarrow u$ strongly in $L^r(\mathcal{O})$, we also deduce that  
$$F_k(u_k) \rightarrow F(u) \text{ strongly in } L^r(\Omega_1 \times (0,T)), \quad \text{as } k \rightarrow +\infty, \quad (4.51)$$  
for all $1 \leq r < \frac{2^*}{1}$.

Then, the energy is obtained as a limit of solutions, that is,  
$$E(t) = \frac{1}{2} \left[ \int_{\Omega_1} \rho_1(x)|u_t(x)|^2dx + \int_{\Omega_2} \rho_2(x)|\eta_t(x)|^2dx + \int_{\Omega_1} k(x)|\nabla u(x)|^2dx \right.$$  
$$+ \int_{\Omega_2} Q(x)|\nabla v(x)|^2dx + \int_0^\infty g(s) \int_{\Omega_1} k(x)|\nabla \eta(x)|^2dxds \right]$$  
$$+ \int_{\Omega_1} F(u(x))dx + \int_{\Omega_2} H(v(x))dx. \quad (4.52)$$  

5. **Exponential stability.** Consider the following additional assumptions on $g$, $b$, $f$ and $h$.

**Assumption 5.1.** The function $g : [0, \infty) \rightarrow \mathbb{R}^+$ also verifies, for a positive constant $c > 0$,  
$$g'(s) \leq -cg(s), \quad \forall s \geq 0. \quad (5.1)$$  

**Assumption 5.2.** The nonnegative function $b = b(x) \in L^\infty(\Omega)$ satisfies  
$$b(x) \geq b_0 > 0, \quad \forall x \in \omega, \quad (5.2)$$  
where $\omega$ is a neighbourhood of $\Gamma_2 = \partial \Omega_2$, $\omega_1 := \omega \cap \Omega_1$, $\omega_2 := \omega \cap \Omega_2$ and $\omega_2$ geometrically controls $\Omega_2$, that is, there exists $T_0 > 0$ such that every geodesic of $\Omega_2$, travelling with speed 1 and issued at $t = 0$, enters the closure of $\omega$ in a time $t < T_0$. 


Assumption 5.3. In addition to Assumption 2.4, $f$ and $h$ satisfy the following hypotheses:

$$f, h \in C^2(\mathbb{R}), \quad (5.3)$$

$$|f''(s)| \leq \gamma (1 + |s|)^{p-2}, \quad \forall s \in \mathbb{R}, \quad (5.4)$$

$$|h''(s)| \leq \gamma (1 + |s|)^{p-2}, \quad \forall s \in \mathbb{R}. \quad (5.5)$$

Besides, assume that

Assumption 5.4. For every $T > 0$, the only solution $u \in C(0, T; L^2(\Omega)) \cap C(0, T; H^{-1}(\Omega))$ for

$$\begin{align*}
\rho(x)u_{tt} - \text{div}(Q(x)\nabla u) + \Psi(x, t)u &= 0 \quad \text{in } \Omega \times (0, T) \\
u &= 0 \quad \text{on } \partial \Omega \times (0, T) \\
u &= 0 \quad \text{in } \omega \times (0, T)
\end{align*}$$

where $\Psi = \Psi(x, t) \in L^\infty(0, T; L^p(\Omega))$ for some $p \in [N, +\infty]$, is the trivial one, that is, $u \equiv 0$.

Remark 1. We observe that in case that $\Psi \equiv 0$, Assumption 5.4 is satisfied in light of the results due to Burq and Gerard [6]. When a potential is in place we use the results of Duyckaerts et al. [16] to obtain the desired Unique Continuation Principle.

The goal of this section is to prove that the energy functional associated to problem (1.5)

$$E_U(t) = \frac{1}{2} \left[ \int_{\Omega_1} \rho_1(x)|u_1(t)|^2 \, dx + \int_{\Omega_2} \rho_2(x)|u_2(t)|^2 \, dx + \tilde{g}_0 \int_{\Omega_1} K(x)|\nabla u(t)|^2 \, dx \\
+ \int_{\Omega_2} Q(x)|\nabla v(t)|^2 \, dx + \int_0^\infty g(s) \int_{\Omega_2} K(x)|\nabla \eta(t)|^2 \, dx \, ds \\
+ \int_{\Omega_1} F(u(t)) \, dx + \int_{\Omega_2} H(v(t)) \, dx \right], \quad (5.6)$$

where $F$ and $H$ are given in (2.8) and $U(t) = (u(t), v(t), \eta(t))$ is the unique solution of (1.5) established in Theorem 3.1; decays exponentially to zero when $t$ tends to $+\infty$.

Observe that the first derivative of $E_U$ satisfies the identity

$$\frac{dE_U}{dt}(t) = \frac{1}{2} \int_0^\infty g'(s) \int_{\Omega_1} K(x)|\nabla \eta^t(x, s)|^2 \, dx \, ds, \quad (5.7)$$

which, in light of Assumption 5.1, proves that $E_U$ is a non-increasing functional.

The exponential stability result is given below.

Theorem 5.1 (Exponential Stability). Assume that the assumptions of Theorem 3.1 are in place and that Assumptions 5.1 and 5.4 holds. Let $R > 0$ be a positive constant such that $E_U(0) \leq R$. Then, there exist $T_0 > 0$ and constants $C, \hat{\gamma} > 0$, depending on $R > 0$, satisfying

$$E_U(t) \leq C(R)e^{-\hat{\gamma}t}E_U(0), \quad \forall t > T_0, \quad (5.8)$$

for every weak solution $U = (u, v, \eta)$ of (1.5).
Our first step is to prove the observability inequality to the auxiliary problem\ equation (4.2), whose energy functional, its derivative and the energy identity are, respectively, given by

\[
E_{U_k}(t) = \frac{1}{2} \left[ \int_{\Omega_1} \rho_1(x) |\partial_t u_k(t)|^2 dx + \int_{\Omega_2} \rho_2(x) |\partial_t v_k(t)|^2 dx + g_0 \int_{\Omega_1} K(x) |\nabla u_k(t)|^2 dx \\
+ \int_{\Omega_2} Q(x) |\nabla v_k(t)|^2 dx + \int_0^\infty g(s) \int_{\Omega_1} K(x) |\nabla \eta_k|^2 dx ds \\
+ \int_{\Omega_1} F_k(u_k(t)) dx + \int_{\Omega_2} H_k(v_k(t)) dx, \frac{dE_{U_k}}{dt}(t) \\
= \int_0^\infty g'(s) \int_{\Omega_1} K(x) |\nabla \eta_k(x,s)|^2 dx ds - \frac{1}{k} \int_{\Omega_1} b(x) |\partial_t u_k(t)|^2 dx \\
- \frac{1}{k} \int_{\Omega_2} b(x) |\partial_t v_k(t)|^2 dx,
\]

and

\[
E_{U_k}(t_2) - E_{U_k}(t_1) = \frac{1}{2} \int_0^{t_2} \int_{\Omega_1} g'(s) \int_{\Omega_1} K(x) |\nabla \eta_k(x,s)|^2 dx ds dt - \frac{1}{k} \int_0^{t_2} \int_{\Omega_1} b(x) |\partial_t u_k(t)|^2 dx dt \\
- \frac{1}{k} \int_0^{t_2} \int_{\Omega_2} b(x) |\partial_t v_k(t)|^2 dx dt,
\]

for all 0 \leq t_1 \leq t_2 < +\infty and \( U_k(t) = (u_k(t), v_k(t), \eta_k) \) denote the unique solution of (4.2).

Let \( T_0 > 0 \) be associated to the geometric control condition, that is, every ray of the geometric optics enters \( \omega \) in a time \( T^* < T_0 \), where \( \omega \) is defined in Assumption 5.2. Our first step is to prove the observability inequality to the auxiliary problem (4.2), which reads as follows:

**Consider** \( T > T_0 \). There exists \( k_0 \geq 1 \) such that, for every \( k \geq k_0 \), the corresponding solution \( U_k \) to the auxiliary problem (4.2) satisfies the inequality

\[
E_{U_k}(0) \leq C \left( \int_0^T \int_0^\infty g'(s) \int_{\Omega_1} K(x) |\nabla \eta_k(x,s)|^2 dx ds dt \\
+ \frac{1}{k} \int_0^T \int_{\Omega_1} b(x) |\partial_t u_k(t)|^2 dx dt + \frac{1}{k} \int_0^T \int_{\Omega_2} b(x) |\partial_t v_k(t)|^2 dx dt \right), \tag{5.12}
\]

for some constant \( C > 0 \) depending on \( R > 0 \).

**Remark 2.** Before proving the above inequality it is important to point out that the constant \( C \) furnished by (5.12) is uniform with respect to the parameter \( k \), and the answer lies in its proof. Looking at the proof of (5.12), if \( U_0 = \)
verifying (5.13), whose corresponding solution
we can take

$$U_k$$

every

$k$

and

$U$

verifying (4.1), for

$k$

and

$u$


\[ \text{Recall that the energy functional is nondecreasing due to (5.10). Then, we get} \]

$$U_0 = 0$$

and

$U_0 \neq 0$. If

$U_0 = 0$, we consider the null sequence

\[ \{(u_{0,k}, v_{0,k}, u_{1,k}, v_{1,k}, \eta_{0,k})\}_{k \in \mathbb{N}} = 0 \in D(L), \]

verifying (4.1), for

$k \geq 1$, and the corresponding unique solution to the auxiliary problem (4.2) will be

$$(u_k, v_k, \eta_k) \equiv 0$$. As a result, (5.12) is verified.

When

$U_0 \neq 0$, there exists a positive number

$\Upsilon > 0$

such that

$$0 < \|(u_0, v_0), (u_1, v_1), \eta_0\|_H < \Upsilon$$

and

$k_0 \geq k_0$.

We are going to prove that under condition (5.13) on the initial datum, the corresponding solution

$U_k$

to (4.2) satisfies (5.12). We proceed by contradiction. For every

$k \geq 1$ and

$n \in \mathbb{N}$, there exists an initial datum

$$((u^n_{0,k}, v^n_{0,k}, u^n_{1,k}, v^n_{1,k}, \eta^n_{0,k})$$

verifying (5.13), whose corresponding solution

$U^n_k = (u^n_k, v^n_k, \eta^n_k)$

violates (5.12).

The contradiction arguments guarantee that

$$E_{U_k}(0)$$

is bounded and

\[
\lim_{n \to \infty} \left[ \int_0^T \int_\Omega (-g'(s)) \nabla \eta^n_k(s)^2 \, dx \, ds \, dt \right. \\
+ \left. \frac{1}{k} \int_0^T \int_\Omega b(x)|\partial_t u^n_k(x,t)|^2 \, dx \, dt + \frac{1}{k} \int_0^T \int_\Omega b(x)|\partial_t v^n_k(x,t)|^2 \, dx \, dt \right] = 0. \tag{5.14}
\]

We derive from Assumption 5.1 that

\[
\lim_{n \to \infty} \left[ \int_0^T \int_\Omega g(s) \nabla \eta^n_k(s)^2 \, dx \, ds \, dt \right. \\
+ \left. \frac{1}{k} \int_0^T \int_\Omega b(x)|\partial_t u^n_k(x,t)|^2 \, dx \, dt + \frac{1}{k} \int_0^T \int_\Omega b(x)|\partial_t v^n_k(x,t)|^2 \, dx \, dt \right] = 0. \tag{5.15}
\]

Recall that the energy functional is nondecreasing due to (5.10). Then, we get

$$E_{U_k}(t) \leq E_{U_k}(0)$$

for all

$t > 0$

and, consequently, there exists a subsequence of

$$\{(u^n_k, v^n_k, \eta^n_k)\}_{n \in \mathbb{N}}$$

still denoted by

$$\{(u^n_k, v^n_k, \eta^n_k)\}_{n \in \mathbb{N}}$$

such that (5.15) is verified and passing to the limit as

$n \to \infty$, we obtain

$$u^n_k \rightharpoonup (u_k, v_k) \text{ weakly star in } L^\infty(0,T;H^1_1)$$

and

$$(\partial_t u^n_k, \partial_t v^n_k) \rightharpoonup (\partial_t u_k, \partial_t v_k) \text{ weakly star in } L^\infty(0,T;L^2).$$

In addition, Aubin-Lions-Simon Theorem yields for all

$q \in [2, \frac{2n}{n-2})$

\[
(u^n_k, v^n_k) \to (u_k, v_k) \text{ strongly in } L^\infty(0,T;L^q(\Omega_1 \times \Omega_2)). \tag{5.18}
\]
Defining $\mathcal{F}_k : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathcal{F}_k((t, s)) = (f_k(t), h_k(s))$, it follows that
\[ \mathcal{F}_k((u^n_k, v^n_k)) \to \mathcal{F}_k((u_k, v_k)) \text{ weakly in } L^2(0, T; L^2). \]  

We are going to consider two cases: $(u_k, v_k) \neq 0$ and $(u_k, v_k) = 0$.

**Case 1**: $(u_k, v_k) \neq 0$. For each $n, k \in \mathbb{N}$, consider $(u^n_k, v^n_k, \eta^n_k)$ the regular solution to the problem
\[
\begin{aligned}
\rho_1(x)\partial^2_t u^n_k - \tilde{g}_0 \text{div}(K(x)\nabla u^n_k) - \int_0^\infty g(s) \text{div}(K(x)\nabla \eta^n_k(s)) \, ds \\
+ \frac{k}{k} b(x)\partial_t u^n_k + f_k(u^n_k) = 0 \text{ in } \Omega_1 \times (0, T), \\
\rho_2(x)\partial^2_t v^n_k - \text{div}(Q(x)\nabla v^n_k) + \frac{k}{k} b(x)\partial_t v^n_k + h_k(v^n_k) = 0 \text{ in } \Omega_2 \times (0, T), \\
\eta^n_{k,t} + \eta^n_{k,s} = \partial_t u^n_k \text{ in } \Omega_1 \times (0, T) \times (0, \infty), \\
u^n_k = 0 \text{ on } \Gamma_1 \times (0, T), \quad \eta^n_k = 0 \text{ on } \Gamma_1 \times (0, T) \times (0, \infty), \\
u^n_k = v^n_k \text{ on } \Gamma_2 \times (0, T), \\
\frac{\partial v^n_k}{\partial n} = \tilde{g}_0 \frac{\partial \eta^n_k}{\partial n}(t) + \int_0^\infty g(s) \frac{\partial \eta^n_k}{\partial n}(s) \, ds \text{ on } \Gamma_2 \times (0, T), \\
(u^n_k(0), v^n_k(0)) = (u^n_{1,k}, v^n_{1,k}), (\partial_t u^n_k(0), \partial_t v^n_k(0)) = (u^n_{1,k}, v^n_{1,k}), \eta^n_k(0) = \eta^n_{0,k}
\end{aligned}
\]  

Take $\theta \in \mathcal{D}(0, T)$ and $\varphi \in \mathcal{D}(\Omega)$, multiply equations (5.20) by $\theta \varphi$ and integrate over $(0, T) \times \Omega_1$ and $(0, T) \times \Omega_2$, respectively. Applying Green’s formula and considering the transmission conditions, we obtain
\[
- \int_0^T \int_{\Omega_1} \rho_1(x)u^n_k \partial_t \varphi_1 \, dxdt + \tilde{g}_0 \int_0^T \int_{\Omega_1} K(x)\nabla u^n_k \nabla (\theta \varphi_1) \, dxdt \\
+ \frac{1}{k} \int_0^T \int_{\Omega_1} b(x)u^n_k \partial_t \varphi_1 \, dxdt + \int_0^T \int_{\Omega_1} g(s) \int_{\Omega_1} K(x)\nabla \eta^n_k(s) \nabla (\theta \varphi_1) \, dxds \\
+ \int_0^T \int_{\Omega_2} f_k(u^n_k) \varphi_1 \partial_t \varphi_2 \, dxdt - \int_0^T \int_{\Omega_2} \rho_2(x)v^n_k \theta \varphi_2 \partial_t \varphi_2 \, dxdt \\
+ \int_0^T \int_{\Omega_2} Q(x)\nabla v^n_k \nabla (\theta \varphi_2) \, dxdt + \frac{1}{k} \int_0^T \int_{\Omega_2} b(x)v^n_k \theta \varphi_2 \partial_t \varphi_2 \, dxdt \\
+ \int_0^T \int_{\Omega_1} h_k(v^n_k) \theta \varphi_2 \partial_t \varphi_2 \, dxdt = 0,
\]  

where $\varphi_1 = \varphi|_{\Omega_1}$ and $\varphi_2 = \varphi|_{\Omega_2}$.

Convergences (5.15), (5.19) yield the following identity, for all $\theta \in \mathcal{D}(0, T)$ and $(\varphi_1, \varphi_2) = \varphi \in \mathcal{D}(\Omega)$
\[
- \int_0^T \int_{\Omega_1} \rho_1(x)u_k \partial_t \varphi_1 \, dxdt + \tilde{g}_0 \int_0^T \int_{\Omega_1} K(x)\nabla u_k \nabla (\theta \varphi_1) \, dxdt \\
+ \int_0^T \int_{\Omega_1} f_k(u_k) \theta \varphi_1 \partial_t \varphi_2 \, dxdt - \int_0^T \int_{\Omega_2} \rho_2(x)v_k \theta \varphi_2 \partial_t \varphi_2 \, dxdt \\
+ \int_0^T \int_{\Omega_2} Q(x)\nabla v_k \nabla (\theta \varphi_2) \, dxdt + \int_0^T \int_{\Omega_2} h_k(v_k) \theta \varphi_2 \partial_t \varphi_2 \, dxdt = 0.
\]  

In particular, considering $\theta \in \mathcal{D}(0, T)$ and $\varphi = (\varphi_1, 0)$, where $\varphi_1 \in \mathcal{D}(\Omega_1)$, we derive from (5.22)
\[
\rho_1(x)\partial^2_t u_k - \tilde{g}_0 \text{div}(K(x)\nabla u_k) + f_k(u_k) = 0 \text{ in } L^2(0, T; H^{-1}(\Omega_1)) \tag{5.23}
\]

and considering $\theta \in \mathcal{D}(0, T)$ and $\varphi = (0, \varphi_2)$ where $\varphi_2 \in \mathcal{D}(\Omega_2)$, we obtain
\[
\rho_2(x)\partial^2_t v_k - \text{div}(Q(x)\nabla v_k) + h_k(v_k) = 0 \text{ in } L^2(0, T; H^{-1}(\Omega_2)). \tag{5.24}
\]
On the other hand, define the auxiliary function
\[ w_n^u(t) = \tilde{g}_0 u_n^u(t) + \int_0^\infty g(s)\eta_n^u(s)ds \in L^2(\Omega_1). \] (5.25)

Taking the derivative with respect to \( t \), we obtain
\[ w_n^u(t) = u_n^u(t) + \int_0^\infty g'(s)\eta_n^u(s)ds. \]

Convergences (5.14) and (5.17) imply that
\[ w_n^u(t) \rightarrow u_{k,t} \text{ weakly in } L^2(0, T; L^2(\Omega_1)). \] (5.26)

In addition, convergences (5.15) and (5.16) yield
\[ u_n^u \rightarrow \tilde{g}_0 u_k \text{ strongly in } L^2(0, T; L^2(\Omega_1)). \] (5.27)

Hence, from (5.26) and (5.27) we conclude
\[ u_{k,t} = \tilde{g}_0 u_{k,t} \text{ in } L^2(0, T; L^2(\Omega_1)), \]
and since \( \tilde{g}_0 \neq 1 \),
\[ u_{k,t} \equiv 0 \text{ a.e. in } \Omega_1 \times (0, T), \] (5.28)
for every \( k \in \mathbb{N} \).

From identity (5.23) and (5.28), we obtain, for almost every \( t \in (0, \infty) \),
\[ -\tilde{g}_0 \div (K(x)\nabla u_k(t)) + f_k(u_k(t)) = 0 \text{ in } L^2(\Omega_1). \] (5.29)

Going back to (5.24), taking the derivative with respect to the variable \( t \) and denoting \( z_k = v_{k,t} \), we establish the following problem
\[
\begin{aligned}
\rho_2(x)z_{k,t} - \div(Q(x)\nabla z_k) + \Psi(x,t)z_k &= 0 \quad \text{in } \Omega_2 \times (0, T) \\
z_k &= 0 \quad \text{in } \Gamma_2 \times (0, T) \\
z_k &= 0 \quad \text{in } \omega_2 \times (0, T)
\end{aligned}
\] (5.30)

where \( \Psi(x,t) = h'_k(v_k(x,t)) \) in \( \Omega_2 \times (0, T) \). We observe that \( v_{k,t} = u_{k,t} = 0 \) on \( \Gamma_2 \) and since \( b(x) \geq b_0 > 0 \) in \( \omega \), we obtain from converge (5.14) that
\[ v_n^u \rightarrow 0 \text{ strongly in } L^2(0, T; L^2(\omega_2)), \] (5.31)

where \( \omega_2 = \Omega_2 \cap \omega \).

The hypothesis on \( h_k \) implies that \( \Psi \in L^\infty(0, T; L^N(\Omega_2)) \) and from Assumption 5.4 we conclude that \( z_k \) is zero everywhere, that is,
\[ z_k = v_{k,t} \equiv 0 \text{ a.e. in } \Omega_2 \times (0, T). \] (5.32)

Then, we obtain from (5.24) that, for almost every \( t \in (0, \infty) \),
\[ \div(Q(x)\nabla v_k(t)) = h_k(v_k(t)) \text{ in } L^2(\Omega_2), \] (5.33)

Multiplying (5.29) by \( u_k \) and (5.33) by \( v_k \) and integrating over \( \Omega_1 \) and \( \Omega_2 \), respectively, we obtain
\[ \tilde{g}_0 \int_{\Omega_1} K(x)\nabla u_k \nabla u_k dx + \int_{\Omega_2} Q(x)\nabla v_k \nabla v_k dx = - \int_{\Omega_1} f_k(u_k)u_k dx - \int_{\Omega_2} h_k(v_k) v_k dx. \]

In light of (2.9) and (2.10), we conclude
\[ \tilde{g}_0 \int_{\Omega_1} K(x) |\nabla u_k|^2 dx + \int_{\Omega_2} Q(x) |\nabla v_k|^2 dx \leq \beta_1 \int_{\Omega_1} |u_k|^2 dx + \beta_2 \int_{\Omega_2} |v_k|^2 dx < \lambda \left( \int_{\Omega_1} |u_k|^2 dx + \int_{\Omega_2} |v_k|^2 dx \right), \]
that is,
\[ \| (u_k, v_k) \|_{H^1_0}^2 < \lambda \| (u_k, v_k) \|_{L^2}^2. \]

Since \((u_k, v_k) \neq 0\), we have
\[ \frac{\| (u_k, v_k) \|_{H^1_0}^2}{\| (u_k, v_k) \|_{L^2}^2} < \lambda \]
and
\[ \lambda = \inf_{(w, z) \in H^1_0 \setminus \{0\}} \frac{\| (w, z) \|_{H^1_0}^2}{\| (w, z) \|_{L^2}^2} \leq \frac{\| (u_k, v_k) \|_{H^1_0}^2}{\| (u_k, v_k) \|_{L^2}^2} < \lambda, \]
which yields the desired contradiction.

**Case II:** \((u_k, v_k) = 0\). Let us define for each \(k, n \in \mathbb{N}\),
\[ \alpha_k^n = (E_k^n(0))^{\frac{1}{2}}, \quad (w_k^n, z_k^n) = \frac{1}{\alpha_k^n}(u_k^n, v_k^n) \quad \text{and} \quad \xi_k^n = \frac{1}{\alpha_k^n}\eta_k^n. \]
As a consequence, \(W_k^n = (w_k^n, z_k^n, \xi_k^n)\) is the solution to the following problem:

\[
\begin{align*}
\rho_1(x)w_{k,t,t}^n - \tilde{g}_0 \dive(K(x)\nabla w_k^n) - \int_0^\infty g(s) \dive(K(x)\nabla \xi_k^n(s)) \, ds \\
+ \frac{1}{k} b(x) w_k^n,t + \frac{1}{\alpha_k^n} f_k(\alpha_k^n w_k^n) = 0 \quad &\text{in } \Omega_1 \times (0, T), \\
\rho_2(x)z_{k,t,t}^n - \dive(Q(x)\nabla z_k^n) + \frac{1}{k} b(x) z_k^n,t + \frac{1}{\alpha_k^n} h_k(\alpha_k^n z_k^n) = 0 \quad &\text{in } \Omega_2 \times (0, T), \\
\xi_{k,t}^n + \xi_{k,s} = w_k^n &\text{ in } \Omega_1 \times (0, T) \times (0, \infty), \\
w_k^n = 0 &\text{ on } \Gamma_1 \times (0, T), \quad \xi_k^n = 0 \quad \text{on } \Gamma_1 \times (0, T) \times (0, \infty), \\
w_k^n = z_k^n &\text{ on } \Gamma_2 \times (0, T), \\
\frac{\partial z_k^n}{\partial x_q} = \tilde{g}_0 \frac{\partial w_k^n}{\partial x_q}(t) + \int_0^\infty g(s) \frac{\partial \xi_k^n}{\partial x_q}(s) \, ds &\text{on } \Gamma_2 \times (0, T), \\
(w_k^n(0), z_k^n(0)) = \frac{1}{\alpha_k^n}(u_{0,k}^n, v_{0,k}^n), (w_k^n(0), z_k^n(0)) = \frac{1}{\alpha_k^n}(u_{0,k}^n, v_{0,k}^n), \\
\xi_k^n(0) = \frac{1}{\alpha_k^n}\eta_k^n. &
\end{align*}
\tag{5.34}
\]

Note that the energy functional related to problem (5.34) is given by
\[
E_{W_k^n}(t) = \frac{1}{2} \left[ \int_{\Omega_1} \rho_1(x)|w_{k,t}^n(t)|^2 \, dx + \int_{\Omega_2} \rho_2(x)|z_{k,t}^n(t)|^2 \, dx \\
+ \tilde{g}_0 \int_{\Omega_1} K(x)|\nabla w_k^n(t)|^2 \, dx + \int_{\Omega_2} Q(x)|\nabla z_k^n(t)|^2 \, dx \\
+ \int_0^\infty g(s) \int_{\Omega_1} K(x)|\nabla \xi_k^n(s)|^2 \, dx \, ds \right] \\
+ \frac{1}{\alpha_k^n} \int_{\Omega_1} F_k(\alpha_k^n w_k^n(t)) \, dx + \frac{1}{\alpha_k^n} \int_{\Omega_2} H_k(\alpha_k^n z_k^n(t)) \, dx 
\tag{5.35}
\]
and its derivative verifies, for all \(t > 0\), the identity
\[
\frac{dE_{W_k^n}}{dt}(t) = \frac{1}{2} \int_0^\infty g'(s) \int_{\Omega_1} K(x)|\nabla \xi_k^n(s, x)|^2 \, dx \, ds \\
- \frac{1}{k} \int_{\Omega_1} b(x)|w_{k,t}^n(x, t)|^2 \, dx - \frac{1}{k} \int_{\Omega_2} b(x)|z_{k,t}^n(x, t)|^2 \, dx, \tag{5.36}
\]
which, from Assumption 5.1, derives that $E_{W_n}$ is a non-increasing functional verifying

$$E_{W_n}(0) = E_{W_n}(T) - \frac{1}{2} \int_0^T \int_{\Omega_1} g'(s) \int K(x) |\nabla \xi_k^n(s)|^2 \, dx \, ds \, dt + \frac{1}{k} \int_0^T \int_{\Omega_2} b(x) |w^{n,k}_{t,i}(t)|^2 \, dx \, dt + \frac{1}{k} \int_0^T \int_{\Omega_2} b(x) |z^{n,k}_{t,i}(t)|^2 \, dx \, dt. \quad (5.37)$$

In addition,

$$E_{W_n}(0) = \frac{1}{(\alpha_k^n)^2} E_{k}^{n}(0) = 1, \quad \forall \ n, k \in \mathbb{N}. \quad (5.38)$$

Observe that the contradiction argument provides

$$\lim_{n \to \infty} \left[ \int_0^T \int_{\Omega_2} (-g'(s)) \int K(x) |\nabla \xi_k^n(s)|^2 \, dx \, ds \, dt + \frac{1}{k} \int_0^T \int_{\Omega_2} b(x) |w^{n,k}_{t,i}(x, t)|^2 \, dx \, dt + \int_0^T \int_{\Omega_2} b(x) |z^{n,k}_{t,i}(x, t)|^2 \, dx \, dt \right] = 0, \quad (5.39)$$

Passing to the limit in (5.37) and observing convergences (5.38) and (5.39), we obtain

$$1 = \lim_{n \to \infty} E_{W_n}(0) = \lim_{n \to \infty} E_{W_n}(T).$$

If we prove that

$$\lim_{n \to \infty} E_{W_n}(T) = 0, \quad (5.40)$$

the desired contradiction is obtained.

In fact, gathering together (5.38) and the fact that the energy functional is non-decreasing due to (5.36), passing to the limit as $n \to +\infty$, we conclude

$$(w^{n}_{k,t}, z^{n}_{k,t}) \rightharpoonup (w_{k,t}, z_{k,t}) \text{ weakly star in } L^\infty(0, T; L^2) \quad (5.41)$$

and

$$(w^{n}_{k}, z^{n}_{k}) \rightharpoonup (w_{k}, z_{k}) \text{ weakly star in } L^\infty(0, T; \tilde{H}^1_{\Gamma_1}) \quad (5.42)$$

and using Aubin-Lions Theorem,

$$(w^{n}_{k}, z^{n}_{k}) \to (w_{k}, z_{k}) \text{ strongly in } L^2(0, T; L^2). \quad (5.43)$$

In addition, convergences (5.39) and Assumption 5.2 provide

$$\xi_k^n \to 0 \text{ strongly in } L^2(0, T; M), \quad (5.44)$$

$$w^{n}_{k,t} \to 0 \text{ strongly in } L^2(0, T; L^2(\omega)), \quad (5.45)$$

$$z^{n}_{k,t} \to 0 \text{ strongly in } L^2(0, T; L^2(\omega_2)). \quad (5.46)$$

as $n \to +\infty$.

Extracting a subsequence if it is necessary, $\alpha_k^n \to \alpha_k \in [0, \infty)$, as $n \to +\infty$. If $\alpha_k > 0$, then

$$(w^{n}_{k}, z^{n}_{k}, \xi_k^n) = \frac{1}{\alpha_k^n} (w^{n}_{k}, z^{n}_{k}, \xi_k^n) = \frac{1}{\alpha_k^n} (w^{n}_{k}, v^{n}_{k}, \eta_k^n) \to 0 \text{ strongly in } L^2(0, T; L^2 \times M)$$

and $$(w_k, z_k) = (0, 0).$$

If $\alpha_k = 0$, first, we are going to prove that

$$\frac{1}{\alpha_k} f_k(\alpha_k^n) w_k \to f'(0) w_k \text{ weakly in } L^2(0, T; L^2(\Omega_1)) \text{ as } n \to \infty. \quad (5.47)$$
Since
\[
\frac{1}{\alpha_k^2} f_k(\alpha_k^n w_k^n) - f'(0) w_k = \frac{1}{\alpha_k^2} f_k(\alpha_k^n w_k^n) - \frac{1}{\alpha_k^2} f(\alpha_k^n w_k^n) + \frac{1}{\alpha_k^2} f(\alpha_k^n w_k^n) - f'(0) w_k,
\]
to prove (5.47), it is sufficient to prove that
\[
\frac{1}{\alpha_k^2} f_k(\alpha_k^n w_k^n) - \frac{1}{\alpha_k^2} f(\alpha_k^n w_k^n) \to 0 \text{ strongly in } L^2(0,T;L^2(\Omega_1))
\]
(5.48)
and
\[
\frac{1}{\alpha_k^2} f(\alpha_k^n w_k^n) - f'(0) w_k \to 0 \text{ weakly in } L^2(0,T;L^2(\Omega_1)),
\]
(5.49)
as \(n \to \infty\).

In fact, define \(\Omega_{n,k}^t = \{x \in \Omega_1; |u_k^n(x,t)| > k\}\) and observe that
\[
f_k(\alpha_k^n w_k^n) = f(\alpha_k^n w_k^n) \text{ in } \Omega \setminus \Omega_{n,k}^t.
\]

We will consider \(p > 1\) since the case \(p = 1\) is trivially valid from standard Semigroup Theory. As such, given that \(p > 1, k \geq 1\) and \(w_k^n = \alpha_k^n w_k^n\), we conclude
\[
\left\| \frac{1}{\alpha_k^2} f_k(\alpha_k^n w_k^n) - \frac{1}{\alpha_k^2} f(\alpha_k^n w_k^n) \right\|^2_{L^2(0,T;L^2(\Omega_1))}
\]
\[
= \frac{1}{(\alpha_k^2)^2} \int_0^T \int_{\Omega_{n,k}^t} |f_k(\alpha_k^n w_k^n) - f(\alpha_k^n w_k^n)|^2 \, dx dt
\]
\[
\leq \int_0^T \int_{\Omega_{n,k}^t} |f_k|^2 - |f(-k)|^2 \, dx dt + \frac{1}{(\alpha_k^2)^2} \int_0^T \int_{\Omega_{n,k}^t} |\alpha_k^n w_k^n|^2 + |\alpha_k^n w_k^n|^{2p} \, dx dt
\]
\[
\leq \frac{1}{(\alpha_k^2)^2} \int_0^T \int_{\Omega_{n,k}^t} |u_k^n|^2 + |u_k^n|^{2p} \, dx dt + \frac{1}{(\alpha_k^2)^2} \int_0^T \int_{\Omega_{n,k}^t} |\alpha_k^n w_k^n|^2 + |\alpha_k^n w_k^n|^{2p} \, dx dt
\]
\[
\leq \frac{1}{(\alpha_k^2)^2} \int_0^T \int_{\Omega_{n,k}^t} |\alpha_k^n w_k^n|^{2p} \, dx dt \leq (\alpha_k^n)^{2(p-1)} \|w_k^n \|_{L^{2p}(0,T;L^2(\Omega_1))}^2 \to 0, \text{ as } n \to \infty,
\]
which proves (5.48).

Besides, from Assumption 5.3 and applying Taylor’s Formula, we obtain
\[
f(s) = f(0) + f'(0)s + \frac{f''(\zeta_0)}{2} s^2 = f'(0)s + \frac{f''(\zeta_0)}{2} s^2,
\]
(5.50)
for some \(\zeta_0 \in (0,s)\).

Also, there exists \(\zeta_1 \in (0,s)\) such that
\[
f(s) = f(s) - f(0) = f'(\zeta_1)(s - 0) = f'(\zeta_1)s,
\]
thus (2.6) yields
\[
|f(s)| = |f'(\zeta_1)| |s| \leq C_p (1 + |s|^{p-1}) |s|,
\]
therefore
\[
|f(s)| \leq C_1 (|s| + |s|^p).
\]
(5.51)

Define \(R_f(s) = \frac{f''(\zeta_0)}{2} s^2\). From (5.4) we derive the following:
\[
|f''(\zeta_0)| \leq C(1 + |\zeta_0|^{p-2}) = C \frac{(1 + |\zeta_0|)^{p-1}}{1 + |\zeta_0|} \leq C_p (1 + |\zeta_0|^{p-1}),
\]
(5.52)
since \(1 + |\zeta_0| \geq 1\) and, consequently, \(\frac{1}{1 + |\zeta_0|} \leq 1\). Then,
\[
|R_f(s)| \leq C_2 (|s|^2 + |s|^{p+1}),
\]
(5.53)
for $s > 0$ such that $\zeta_0 \in (0, s)$.

We are going to prove that $\frac{R_f(\alpha_k^n w_k^n)}{\alpha_k^n} \to 0$ strongly in $L^2(0, T; L^2(\Omega_1))$, $n \to +\infty$.

Indeed, observe that $\mathcal{R}_f(s) = f(s) - f'(0)s$, then

$$
\left\| \frac{R_f(\alpha_k^n w_k^n)}{\alpha_k^n} \right\|_{L^2(0, T; L^2(\Omega_1))}^2 = \int_0^T \left\| \frac{R_f(\alpha_k^n w_k^n(t))}{\alpha_k^n} \right\|_{L^2(\Omega_1)}^2 dt
$$

$$
= c_p \int_0^T \int_{\Omega_1} |f(\alpha_k^n w_k^n(x, t))|^2 \alpha_k^n dx dt + c_p \int_0^T \int_{\Omega_1} |f'(0)w_k^n(x, t)|^2 \alpha_k^n dx dt
$$

$$
\leq c_p \frac{1}{(\alpha_k^n)^2} \int_0^T \int_{\Omega_1} (|\alpha_k^n w_k^n(x, t)| + |\alpha_k^n w_k^n|^p)^2 \alpha_k^n dx dt + c \int_0^T \int_{\Omega_1} |w_k^n(x, t)|^2 \alpha_k^n dx dt
$$

$$
= c \int_0^T \int_{\Omega_1} |w_k^n(x, t)|^2 \alpha_k^n dx dt + C(\alpha_k^n)^{2p-2} \int_0^T \int_{\Omega_1} |\alpha_k^n w_k^n|^p \alpha_k^n dx dt
$$

$$
\leq c \|w_k^n\|_{L^2(0, T; L^2(\Omega_1))}^2 + C(\alpha_k^n)^{2p-1} \|w_k^n\|_{L^p(0, T; L^{2p}(\Omega_1))}^p.
$$

Using Sobolev’s embeddings, the boundedness of the sequence $\{w_k^n\}$ and that $p \geq 1$, we obtain that there exists $C > 0$ verifying

$$
\left\| \frac{\mathcal{R}_f(\alpha_k^n w_k^n)}{\alpha_k^n} \right\|_{L^2(0, T; L^2(\Omega_1))} < C,
$$

and, consequently, there is $r \in L^2(0, T; L^2(\Omega_1))$ such that

$$
\frac{\mathcal{R}_f(\alpha_k^n w_k^n)}{\alpha_k^n} \rightharpoonup r \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega_1)).
$$

On the other hand, using (5.53) we have

$$
\left\| \frac{R_f(\alpha_k^n w_k^n)}{\alpha_k^n} \right\|_{L^1(0, T; L^1(\Omega_1))}
$$

$$
= \int_0^T \int_{\Omega_1} \left| R(\alpha_k^n w_k^n(x, t)) \right| \alpha_k^n dx dt \leq \int_0^T \int_{\Omega_1} \frac{1}{\alpha_k^n} \left[ |\alpha_k^n w_k^n(x, t)|^2 + |\alpha_k^n w_k^n(x, t)|^{p+1} \right] \alpha_k^n dx dt
$$

$$
= \alpha_k^n \int_0^T \int_{\Omega_1} |w_k^n(x, t)|^2 dx dt + (\alpha_k^n)^p \int_0^T \int_{\Omega_1} |w_k^n(t)|^{p+1} dx dt
$$

$$
= \alpha_k^n \|w_k^n\|_{L^2(0, T; L^2(\Omega_1))}^2 + (\alpha_k^n)^p \|w_k^n\|_{L^{p+1}(0, T; L^{p+1}(\Omega_1))}^{p+1}.
$$

Since $\alpha_k^n \rightarrow 0$, $n \rightarrow +\infty$, from the above inequality and convergence (5.55) we derive that

$$
\frac{\mathcal{R}_f(\alpha_k^n w_k^n)}{\alpha_k^n} \rightharpoonup 0 \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega_1)).
$$

Then, from (5.50), we conclude

$$
\frac{1}{\alpha_k^n} f(\alpha_k^n w_k^n) \rightarrow f'(0)w_k \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega_1))
$$

(5.56)

and we finish the proof of convergence (5.47).

Analogously, we prove that

$$
\frac{1}{\alpha_k^n} h_k(\alpha_k^n z_k^n) \rightarrow h'(0)z_k \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega_2)).
$$

(5.57)
Now, considering \( \varphi \in D(\Omega) \), \( \theta \in D(0, T) \), multiplying (5.34) and (5.34) by \( \theta \varphi \), integrating over \((0, T) \times \Omega_1 \), \((0, T) \times \Omega_2 \), respectively, observing the transmission conditions and convergences (5.39) - (5.42), (5.44) and (5.58) we derive

\[
\frac{1}{\alpha_k^2}(f_k(\alpha_k^n w_k^n), h_k(\alpha_k^n z_k^n)) \rightarrow (f'(0)w_k, h'(0)z_k) \quad \text{weakly in} \quad L^2(0, T; L^2). \tag{5.58}
\]

Then, taking the derivative of (5.61) with respect to \( t \) we obtain that

\[
\rho_1(x)w_k,t \theta \varphi_1 dx dt + \tilde{g}_0 \int_0^T \int_{\Omega_1} K(x) \nabla w_k \nabla (\theta \varphi_1) dx dt
\]

\[
+ \int_0^T \int_{\Omega_2} f'(0)w_k \theta \varphi_1 dx dt - \int_0^T \int_{\Omega_2} \rho_2(x) z_k,t \theta \varphi_2 dx dt
\]

\[
+ \int_0^T \int_{\Omega_2} Q(x) \nabla z_k \nabla (\theta \varphi_2) dx dt + \int_0^T \int_{\Omega_2} h'(0) z_k \theta \varphi_2 dx dt = 0, \tag{5.59}
\]

for \( \theta \in D(0, T), \varphi \in D(\Omega) \).

In particular, when \( \varphi \in D(\Omega) \) verifies \( \text{supp}(\varphi) \subset \Omega_1 \), equality (5.59) becomes

\[
\rho_1(x)w_k,t \theta \varphi_1 dx dt + \tilde{g}_0 \int_0^T \int_{\Omega_1} K(x) \nabla w_k \nabla (\theta \varphi_1) dx dt
\]

\[
+ \int_0^T \int_{\Omega_1} f'(0)w_k \theta \varphi_1 dx dt = 0.
\]

Then,

\[
\rho_1(x)w_k,t - \tilde{g}_0 \text{div}(K(x) \nabla w_k) + f'(0)w_k = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_1 \times (0, T)). \tag{5.60}
\]

Analogously, considering \( \varphi \in D(\Omega) \) verifying \( \text{supp}(\varphi) \subset \Omega_2 \), we obtain from identity (5.59) that

\[
\rho_2(x)z_k,t - \text{div}(Q(x) \nabla z - k) + h'(0)z_k = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_2 \times (0, T)). \tag{5.61}
\]

Similarly to (5.25)- (5.28), defining, for \( t \in (0, T) \),

\[
\chi_k^n(t) = \tilde{g}_0 w_k^n(t) + \int_0^\infty g(s) \xi_k^n(s) ds \quad \text{in} \quad L^2(\Omega_1),
\]

we conclude

\[
w_{k,t} = 0 \text{ a.e. in } \Omega_1 \times (0, T). \tag{5.62}
\]

On the other hand, taking the derivative of (5.61) with respect to \( t \) we obtain that \( \gamma_k = z_{k,t} \) satisfies

\[
\begin{align*}
\rho_2(x) \gamma_{k,t} - \text{div}(Q(x) \nabla \gamma_k) + h'(0) \gamma_k &= 0 \text{ in } \Omega_2 \times (0, T) \\
\gamma_k &= 0 \text{ on } \Gamma_2 \times (0, T) \\
\gamma_k &= 0 \text{ in } \omega_2 \times (0, T) \tag{5.63}
\end{align*}
\]

and, following the same ideas we have used to obtain (5.32), we get \( \gamma_k \equiv 0 \), that is,

\[
z_{k,t} \equiv 0 \text{ a.e. in } \Omega_2 \times (0, T). \tag{5.64}
\]

We aim to prove that \( (w_k, z_k) = 0 \) in \( L^2(0, T; L^2) \). In fact, if \( (w_k, z_k) \neq 0 \) and gathering together (5.60)- (5.62) and (5.64) we conclude

\[
- \tilde{g}_0 \text{div}(K(x) \nabla w_k) + f'(0)w_k = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_1 \times (0, T)) \tag{5.65}
\]

and

\[
- \text{div}(Q(x) \nabla z_k) + h'(0)z_k = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_2 \times (0, T)). \tag{5.66}
\]
Once \((w_k, z_k) \in L^\infty(0, T; \mathbb{H}^1_1)\), multiplying (5.65) by \(w_k\) and (5.66) by \(z_k\), it yields
\begin{align*}
\bar{g}_0 \int_{\Omega_1} K(x) |\nabla w_k|^2 \, dx + \int_{\Omega_2} Q(x) |\nabla z_k|^2 \, dx + \int_{\Omega_1} f'(0) |w_k|^2 \, dx \\
+ \int_{\Omega_2} h'(0) |z_k|^2 \, dx = 0,
\end{align*}
which equals to
\[\|(w_k, z_k)\|_{L^2(\Omega_1)}^2 + f'(0) \|w_k\|_{L^2(\Omega_1)}^2 + h'(0) \|z_k\|_{L^2(\Omega_2)}^2 = 0.\]
Observing hypothesis (2.9), we conclude for \(s \neq 0\) that \(-\beta_1 \leq \frac{L(s)}{s}\), that is, \(-\beta_1 \leq f'(0)\). An analogous inequality is obtained to \(h\). Then,
\[f'(0) + \beta_1 \geq 0 \quad \text{and} \quad h'(0) + \beta_2 \geq 0.\]
In addition,
\[\lambda = \inf_{(w, z) \in \mathbb{H}^1_1} \frac{\|(w, z)\|_{L^2(\Omega_1)}^2 + f'(0) \|w_k\|_{L^2(\Omega_1)}^2 + h'(0) \|z_k\|_{L^2(\Omega_2)}^2}{\|(w, z)\|_{L^2(\Omega_1)}^2 + f'(0) \|w_k\|_{L^2(\Omega_1)}^2 + h'(0) \|z_k\|_{L^2(\Omega_2)}^2} \leq 0,
\]
that is,
\[\lambda \|(w, z)\|_{L^2(\Omega_1)}^2 \leq \|(w, z)\|_{L^2(\Omega_1)}^2.
\]
Thereby, from (5.68) we derive that
\[\lambda \|(w_k, z_k)\|_{L^2(\Omega_1)}^2 + f'(0) \|w_k\|_{L^2(\Omega_1)}^2 + h'(0) \|z_k\|_{L^2(\Omega_2)}^2 = 0.
\]
Defining \(c = \min\{f'(0) + \lambda, h'(0) + \lambda\} > 0\),
\[c \|(w_k, z_k)\|_{L^2(\Omega_1)}^2 \leq f'(0) \|w_k\|_{L^2(\Omega_1)}^2 + \lambda \|w_k\|_{L^2(\Omega_1)}^2 + h'(0) \|z_k\|_{L^2(\Omega_2)}^2 + \lambda \|z_k\|_{L^2(\Omega_2)}^2 \leq 0,
\]
therefore \((w_k, z_k) = 0\) in \(\mathbb{H}^2\).
So, in both cases, \((w_k, z_k) = 0\) and it follows from (5.41)-(5.43) that
\begin{align*}
(w_k^n, z_k^n) &\to (0, 0) \text{ weakly star in } L^\infty(0, T; \mathbb{H}^1_1), \\
(w_k^{n,t}, z_k^{n,t}) &\to (0, 0) \text{ weakly star in } L^\infty(0, T; L^2), \\
(w_k^k, z_k^k) &\to (0, 0) \text{ strongly in } L^2(0, T; L^2).
\end{align*}
From (5.54) and convergences (5.71) - (5.73) we conclude that
\[\mathcal{R}_f (\alpha_k^n w_k^n) \to 0 \quad \text{strongly in } L^2(0, T; L^2(\Omega_1)).
\]
The above convergence and the analogous steps for function \(h\), imply that
\[\frac{1}{\alpha_k^n} (f_k(\alpha_k^n w_k^n), h_k(\alpha_k^n z_k^n)) \to (0, 0) \quad \text{strongly in } L^2(0, T; \mathbb{L}^2).
\]
Define \(\phi_k^n(x, t) := \int_0^\infty g(s) \xi_k^n(x, s) ds\) then, \(\phi_k^n \in L^2(0, T; V)\). In addition, consider \(\theta \in D(0, T)\) verifying
\[0 \leq \theta(t) \leq 1, \quad \theta(t) = 1 \text{ in } (\varepsilon, T - \varepsilon) \quad \text{and} \quad \text{supp}(\theta) \subset (0, T),
\]
and \(\varphi \in C^\infty(\Omega_1)\) is such that
\[0 \leq \varphi(x) \leq 1, \quad \varphi(x) = 1 \text{ in } \Omega_1 \setminus \omega_1 \quad \text{and} \quad \varphi(x) = 0, \text{ in } V_{\omega_1}.
\]
where \( \omega_1 \) is given in Assumption 5.2 and \( V_{\omega_1} \) a neighbourhood of \( \Gamma_2 \) in \( \Omega_1 \) verifying \( V_{\omega_1} \subset \omega_1 \).

Multiplying (5.34) by \( \theta \phi_k^n \), integrating over \( (0,T) \times \Omega_1 \) and integrating by parts, we obtain

\[
- \int_0^T \int_{\Omega_1} \rho_1(x)w^n_{k,t} \theta \phi_k^n \phi dx dt - \int_0^T \int_{\Omega_1} \rho_1(x)w^n_{k,t} \theta_t \phi_k^n \phi dx dt
\]

\[
+ \tilde{g}_0 \int_0^T \int_{\Omega_1} K(x) \nabla w^n_k \nabla \phi_k^n \theta dx dt + \tilde{g}_0 \int_0^T \int_{\Omega_1} K(x) \nabla w^n_k \nabla \phi_k^n \phi dx dt
\]

\[
+ \int_0^T \int_0^{\infty} g(s) \int_{\Omega_1} K(x) \nabla \xi_k^n \nabla \phi_k^n \theta dx ds dt
\]

\[
+ \int_0^T \int_0^{\infty} g(s) \int_{\Omega_1} K(x) \nabla \xi_k^n \nabla \phi_k^n \phi dx ds dt + \frac{1}{k} \int_0^T \int_{\Omega_1} b(x)w^n_k \theta \phi_k^n dx dt
\]

\[
+ \frac{1}{\alpha_k^n} \int_0^T \int_{\Omega_1} f(\alpha_k^n w^n_k) \phi_k^n \phi dx dt = 0.
\]

From (5.44), (5.71) and (5.72) we derive

\[
J_{2n}, J_{3n}, J_{4n}, J_{7n} \to 0 \quad \text{as} \quad n \to \infty.
\]

We also observe that

\[
|J_{5n}| \leq \frac{k_0 N \gamma_{K,\varphi}}{2} \int_0^T \int_0^{\infty} g(s) \int_{\Omega_1} \frac{N \gamma_{K,\varphi}}{2} \left[ \|\nabla \xi_k^n(s)\|^2_{L^2(\Omega_1)} \right] \theta |d\tau| ds dt
\]

\[
= \frac{k_0 N \gamma_{K,\varphi}}{2} \int_0^T \int_0^{\infty} g(s) \|\nabla \xi_k^n(s)\|^2_{L^2(\Omega_1)} \theta |d\tau| ds dt + N k_0 \gamma_{K,\varphi} \int_0^T \int_0^{\infty} g(\tau) \|\xi_k^n(\tau)\|^2_{L^2(\Omega_1)} \theta |d\tau| dt \to 0
\]

Analogously, we have

\[
|J_{6n}| \leq N k_0 \gamma_{K,\varphi} \int_0^T \int_0^{\infty} g(s) \int_{\Omega_1} \|\nabla \xi_k^n(s)\|^2_{L^2(\Omega_1)} \theta |d\tau| ds dt \to 0.
\]

Finally, convergence (5.44) combined with the boundedness of \( \alpha_k^n, k \in \mathbb{N} \), imply that

\[
|J_{8n}| = \frac{1}{\alpha_k^n} \int_0^T \int_0^{\infty} g(s) \int_{\Omega_1} f(\alpha_k^n w^n_k) \xi_k^n \phi \theta dx ds dt
\]

\[
\leq \frac{c_1}{\alpha_k^n} \int_0^T \int_0^{\infty} g(s) \int_{\Omega_1} \left[ (1 + |\alpha_k^n w^n_k|^{p-1}) |\alpha_k^n w^n_k| \|\xi_k^n \phi \theta\| dx ds dt
\]

\[
= c_1 \int_0^T \int_0^{\infty} g(s) \int_{\Omega_1} |w^n_k \xi_k^n \phi \theta| dx ds dt
\]
\[ + c_1 (\alpha_k^n)^{p-1} \int_0^T \int_0^\infty g(s) \int_{\Omega_1} |w_{k,t}^n|^p |\xi_{k,s}^n \theta| dx ds dt \rightarrow 0. \]

Gathering together the whole convergences, we obtain, for \( i = 2, \ldots, 8 \), that
\[ J_{in} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

Then,
\[ \lim_{n \rightarrow \infty} J_{1n} = 0. \quad (5.78) \]

Recalling that \( \xi_{k,t}^n + \xi_{k,s}^n = w_{k,t}^n \), we obtain
\[ J_{1n} = \int_0^T \int_0^\infty g(s) \int_{\Omega_1} \rho_1(x) w_{k,t}^n \theta \xi_{k,t}^n \varphi dx ds dt \]
\[ = \int_0^T \int_0^\infty g(s) \int_{\Omega_1} \rho_1(x) |w_{k,t}^n|^2 \theta \varphi dx ds dt - \int_0^T \int_0^\infty g(s) \int_{\Omega_1} \rho_1(x) w_{k,s}^n \theta \xi_{k,s}^n \varphi dx ds dt \]
\[ = g_0 \int_0^T \int_{\Omega_1} \rho_1(x) |w_{k,t}^n|^2 \theta \varphi dx ds dt - \int_0^T \int_0^\infty g(s) \int_{\Omega_1} \rho_1(x) w_{k,s}^n \theta \xi_{k,s}^n \varphi dx ds dt. \]

From convergences (5.72) and \( \int_0^\infty g(s) \xi_{k,s}^n ds \rightarrow 0 \) strongly in \( L^2(0, T; L^2(\Omega_1)) \), we get
\[ \lim_{n \rightarrow \infty} \int_0^T \int_0^\infty g(s) \int_{\Omega_1} \rho_1(x) w_{k,s}^n \theta \xi_{k,s}^n \varphi dx ds dt = 0 \]
and, consequently, in light of (5.78) it yields
\[ \lim_{n \rightarrow \infty} k_0 \int_0^T \int_{\Omega_1} \rho_1(x) |w_{k,t}^n|^2 \theta \varphi dx dt = 0. \]

Due to (5.76), (5.77) and the arbitrariness of \( \varepsilon > 0 \), we obtain
\[ \lim_{n \rightarrow \infty} \int_0^T \int_{\Omega_1 \setminus \omega_1} |w_{k,t}^n|^2 dx dt = 0. \quad (5.79) \]

On the other hand, thanks to (5.2) and (5.39) we get
\[ \lim_{n \rightarrow \infty} \int_0^T \int_{\omega_1} |w_{k,t}^n|^2 dx dt = 0. \quad (5.80) \]

Hence, from (5.79) and (5.80), we obtain
\[ \lim_{n \rightarrow \infty} \int_0^T \int_{\Omega_1} |w_{k,t}^n|^2 dx dt = 0. \quad (5.81) \]

To prove (5.40), it remains to prove that
\[ z_{k,t}^n \rightarrow 0 \quad \text{strongly in} \quad L^2(\Omega_2 \times (0, T)). \quad (5.82) \]

To obtain the above convergence, we are going to use microlocal analysis arguments, see Gérard [20].

In fact, let \( \mu_2 \) be the microlocal defect measure associated with \( \{ z_{k,t}^n \}_{n \in \mathbb{N}} \) in \( L^2(\Omega_2 \times (0, T)) \). Note that from convergence (5.39) we get
\[ z_{k,t}^n \rightarrow 0 \quad \text{strongly in} \quad L^2(\omega_2 \times (0, T)). \quad (5.83) \]

In what follows we are going to use the results of Burq-Gérard [6].

The strong convergence (5.83) implies that \( \mu_2 = 0 \) in \( \omega_2 \times (0, T) \), that is,
\[ \text{supp } \mu_2 \subset [\Omega_2 \setminus \omega_2] \times (0, T). \]
On the other hand, by (5.34) and (5.39) we have
\[ \rho_2(x)z_{k,t}^n - \text{div}(Q(x)\nabla z_k^n) \]
\[ = - b(x)z_{k,t}^n - \frac{1}{\alpha_n} h(\alpha_k^n z_k^n) \to 0 \text{ strongly in } L^2((0,T) \times \Omega_2), \quad (5.84) \]
consequently,
\[ \partial_t \left[ \rho_2(x)z_{k,t}^n - \text{div}(Q(x)\nabla z_k^n) \right] \to 0 \text{ strongly in } H^{-1}_{loc}((0,T) \times \Omega_2). \quad (5.85) \]
The convergence (5.85) is equivalent to
\[ \text{supp}(\mu_2) \subset \left\{ (t,x,\tau,\xi) : \tau^2 = \frac{Q(x)}{\rho_2(x)}||\xi||^2 \right\} \]
and supp(\(\mu_2\)) is the union of curves which are the bicharacteristics of the principal symbol
\[ p(\tau,\xi) = \tau^2 - \frac{Q(x)}{\rho_2(x)}\xi \cdot \xi \]
associated with the following operator:
\[ Pu = \rho_2(x)\partial_t u - \text{div}(Q(x)\nabla u). \]
Then, we conclude that \(\mu_2 \equiv 0\) in \(\Omega_2 \times (0,T)\) and
\[ z_{k,t}^n \to 0 \text{ strongly in } L^2((0,T) \times \Omega_2). \quad (5.86) \]
Coming back to problem (5.34), multiplying equations (5.34) by \(\theta w_k^n\) and \(\theta z_k^n\), respectively, where \(\theta \in D(0,T)\) verifies (5.76), we obtain
\[ - \int_0^T \theta(t) \int_{\Omega_1} \rho_1(x)w_{k,t}^n w_k^n dx dt - \int_0^T \theta(t) \int_{\Omega_1} \rho_1(x)|w_{k,t}^n|^2 dx dt \\
+ \int_0^T \theta(t) \int_0^\infty g(s) \int_{\Omega_2} K(x)\nabla \xi_k^n(s)\nabla w_k^n dx ds dt \\
+ \frac{1}{k} \int_0^T \theta(t) \int_{\Omega_1} b(x)w_{k,t}^n w_k^n dx dt + \int_0^T \theta(t) \int_{\Omega_1} \frac{1}{\alpha_k} f_k(\alpha_k^n w_k^n) dx dt \\
- \int_0^T \theta(t) \int_{\Omega_2} \rho_2(x)z_{k,t}^n z_k^n dx dt - \int_0^T \theta(t) \int_{\Omega_2} \rho_2(x)|z_{k,t}^n|^2 dx dt \\
+ \frac{1}{k} \int_0^T \theta(t) \int_{\Omega_2} b(x)z_{k,t}^n z_k^n dx dt + \int_0^T \theta(t) \int_{\Omega_2} \frac{1}{\alpha_k} h_k(\alpha_k^n z_k^n) dx dt \\
+ \tilde{g}_0 \int_0^T \theta(t) \int_{\Omega_1} K(x)\nabla w_k^n^2 dx dt + \int_0^T \theta(t) \int_{\Omega_2} Q(x)\nabla z_k^n^2 dx dt \]
\[ := M_{1,n} + \cdots + M_{11,n} = 0. \]
Thus, from (5.73)-(5.72), (5.81) and (5.86), we derive that
\[ \lim_{n \to \infty} M_{j,n} = 0, \quad j \in \{1,\ldots,9\} \]
therefore,
\[ \lim_{n \to \infty} (M_{10,n} + M_{11,n}) = 0. \]
Hence,
\[ \lim_{n \to \infty} \left[ \int_0^T \int_{\Omega_1} |\nabla w_k^n|^2 dx dt + \int_0^T \int_{\Omega_2} |\nabla z_k^n|^2 dx dt \right] = 0. \]
Due to the arbitrariness of $\varepsilon > 0$ it yields
\[
\lim_{n \to \infty} \int_0^T \int_{\Omega_1} |\nabla w_n^k|^2 \, dx \, dt = \lim_{n \to \infty} \int_0^T \int_{\Omega_2} |\nabla z_n^k|^2 \, dx \, dt = 0. \tag{5.87}
\]
Integrating $E_{W_n^k}$ over $(0, T)$ and from convergences (5.39), (5.82), (5.81), (5.87), we derive
\[
\lim_{n \to \infty} \int_0^T E_{W_n^k}(t) \, dt = 0.
\]
Since the energy associated to (5.34) is a non-increasing functional, we have
\[
0 = \lim_{n \to \infty} \int_0^T E_{W_n^k}(t) \, dt \geq \lim_{n \to \infty} T E_{W_n^k}(T) \geq 0,
\]
and
\[
\lim_{n \to \infty} E_{W_n^k}(T) = 0,
\]
which proves (5.40) and the inequality (5.12).

In the next steps we will conclude the exponential stability to the problem (1.5). Passing to the limit, as $k \to +\infty$, in the inequality (5.12) and observing convergences (4.19), (4.20), (4.21), (4.43), and (4.51), we obtain the observability inequality associated to the original problem (1.5), that is,
\[
E_U(0) \leq C \int_0^T \int_0^\infty -g'(s) \int_{\Omega_1} K(x) |\nabla \eta^t(x,s)|^2 \, dx \, ds \, dt, \quad \forall T \geq T_0, \tag{5.88}
\]
where $C > 0$ is a constant which depends on $R$. Also, passing to the limit as $k \to +\infty$ and considering the same convergences, identity (5.11) yields the energy identity associated to the problem (1.5)
\[
E_U(t_2) - E_U(t_1) = \frac{1}{2} \int_{t_1}^{t_2} \int_0^\infty g'(s) \int_{\Omega_1} K(x) |\nabla \eta^t(x,s)|^2 \, dx \, ds \, dt, \tag{5.89}
\]
for all $0 \leq t_1 < t_2 < +\infty$.

The inequality (5.88) and the identity (5.89) are the main ingredients to prove the exponential decay to problem (1.5), which proves Theorem 5.1. \hfill \Box

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