Finite Element Error Estimates under Geometric Uncertainty

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Abstract

We develop error estimates for the finite element approximation of elliptic partial differential equations under geometric uncertainty, i.e. when the computational domain does not match the real geometry. The result shows that the uncertainty related to the domain can be a dominating factor in the finite element discretization error. The main result consists of $H^1$- and $L^2$-error estimates for the Laplace problem. Theoretical considerations are validated by a computational examples.

1 Introduction

The main aim of this work is to develop finite element (FE) error estimates in the case when there is uncertainty with respect to the computational domain. We consider the question of how a domain related error coincides with the finite element discretization error. We use the conforming finite element method (FEM) which is well established in the scientific computing community and allows for a rigorous analysis of the approximation error \cite{7}.

Our motivation is as follows. The steps to obtain a mesh for FE computations often come with some uncertainty, for example related to empirical measurements or image processing techniques (e.g. medical image segmentation). Therefore we often perform computations on a domain which is an approximation of the real geometry, i.e., the computational domain is close to but does not match the real domain. In this work we do not specify the source of the error, but we take the error into account by explicitly using the error laden reconstructed domains.
This theoretical result is of great importance for scientific computations. Vast numbers of engineering branches rely on the results of computational fluid dynamics simulations, where there is often uncertainty connected to the computational domain.

A prime example of this is computational based medical diagnostics, where shapes are reconstructed from inverse problems, like e.g. tomography. The assessment of error attributed to the limited spatial resolution of magnetic resonance technique has been discussed in [14,15]. For a survey on computational vascular fluid dynamics, where modelling and reconstruction related issues are discussed, we refer to [17]. Error analysis of computational models is a key factor for assessing the reliability for virtual predictions.

Uncertainties in the computational domain have been studied from the numerical perspective. Rigorous bounds for elliptic problems on random domains have been derived, for approximated problems defined on a sequence of domains that is supposed to converge in the set sense to a limit domain, for both Dirichlet [2] and Neumann [1] boundary conditions.

When measurement data is available the accuracy of numerical predictions can be improved by data assimilation techniques. Applications of variational data assimilation in computational hemodynamics have been revised in [6]. For recent developments we refer to [9] and [16].

On the other hand, the treatment of boundary uncertainty can be cast into a probabilistic framework. The domain mapping method is based entirely on stochastic mappings to transform the original deterministic/stochastic problem in a random domain into a stochastic problem in a deterministic domain, see [10,20,21]. The perturbation method starts with a prescribed perturbation field at the boundary of a reference configuration and uses a shape Taylor expansion with respect to this perturbation field to represent the solution [11]. Moreover, the fictitious domain approach and a polynomial chaos expansion have been applied in [5]. We note, that the probabilistic approach is beyond the scope of this work and the introduction of the boundary uncertainty as random variable increases the complexity of the problem.

This paper is organized as follows. After a brief introduction, in Section 2 we introduce the mathematical setting and some required auxiliary results. In Section 3 we describe the finite element discretization and prove the main results of this work. We finally illustrate our result with a computational example in Section 4.
2 Mathematical setting and auxiliary result

We consider the Laplace equation on a domain $\Omega \subset \mathbb{R}^d$ with dimension $d \in \{2, 3\}$, a right hand side $f \in L^2(\Omega)$ and homogeneous Dirichlet boundary conditions,

$$- \Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega. \tag{1}$$

In variational formulation, this problem is given as

$$u \in H^1_0(\Omega) : (\nabla u, \nabla \phi)_\Omega = (f, \phi)_\Omega \quad \forall \phi \in H^1_0(\Omega), \tag{2}$$

where $H^1_0(\Omega)$ is the Sobolev space of $L^2(\Omega)$ functions with first weak derivative in $L^2(\Omega)^d$ and trace zero on the boundary, $(\cdot, \cdot)_\Omega$ the $L^2$-scalar product.

The boundary $\partial \Omega$ is supposed to have a parametrization in $C^{m+2}$, where $m \in \mathbb{N}$. Given the additional regularity $f \in H^m(\Omega)$, with the notation $H^0(\Omega) := L^2(\Omega)$, there exists a unique solution satisfying the bound (see [8])

$$\|u\|_{H^{m+2}(\Omega)} \leq c\|f\|_{H^m(\Omega)}. \tag{3}$$

In the following we assume that the real domain $\Omega$ is not exactly known but only given up to an uncertainty. We hence define a second domain, the reconstructed domain $\Omega_r$ that for $\Upsilon \in \mathbb{R}$, with $\Upsilon > 0$, satisfies

$$\text{dist}(\partial \Omega, \partial \Omega_r) := \sup_{x \in \partial \Omega} \inf_{y \in \partial \Omega_r} |x - y| < \Upsilon.$$

This distance $\Upsilon$ is not necessarily small. When it comes to spatial discretization we will be interested in both cases, $h \ll \Upsilon$ as well as $\Upsilon \ll h$, where $h > 0$ is the mesh size. The two domains do not match and either domain can protrude from the other, see Figure 1.

On $\Omega_r$ we define the solution $u_r \in H^1_0(\Omega_r)$ to the disturbed Laplace problem

$$(\nabla u_r, \nabla \phi_r)_{\Omega_r} = (f_r, \phi_r)_{\Omega_r} \quad \forall \phi_r \in H^1_0(\Omega_r), \tag{4}$$

where $f_r$ denotes an extension of $f$ from $\Omega$ to $\Omega_r$. Given $f_r \in H^m(\Omega_r)$ and given that the boundary $\partial \Omega_r$ has a $C^{m+2}$ parametrization, the unique solution to (4) satisfies the bound (see [8])

$$\|u_r\|_{H^{m+2}(\Omega_r)} \leq c\|f_r\|_{H^m(\Omega_r)}. \tag{5}$$

**Remark 1** (Extension of the right hand side). Given $f \in H^m(\Omega)$ we assume that there exists an extension $f_r$ of $f$ from $\Omega$ to $\Omega_r$ such that

$$\|f_r - f\|_{\Omega \cap \Omega_r} = 0, \quad \|f_r\|_{H^m(\Omega_r)} \leq c\|f\|_{H^m(\Omega)}. \tag{6}$$
The difficulty to meet this assertion is strongly problem dependent. If the right hand side is a simple volume force like the gravity, the extension is straightforward. If the right hand side however strongly depends on the domain $\Omega$, e.g. if it models normal stresses on $\partial \Omega$, an extension must be constructed case by case and the bound (6) must be shown separately.

Remark 2 (Extension of the solutions). A technical difficulty for deriving error estimates is found in the different domains of definition for the solutions $u$ on $\Omega$ and $u_r$ on $\Omega_r$. Since the domains do not match, $u$ must not be defined on all of $\Omega$ and vice versa. To give the expression $u - u_r$ a meaning on all domains we extend both solutions by zero outside their defining domains, i.e. $u := 0$ on $\Omega_r \setminus \Omega$ and $u_r := 0$ on $\Omega \setminus \Omega_r$. Globally, both functions still have the regularity $u, u_r \in H^1(\Omega \cup \Omega_r)$. We will use the same notion for discrete functions $u_h \in V_h$ defined on a mesh $\Omega_h$ and extend them by zero to $\mathbb{R}^d$.

As a preliminary result we collect two standard estimates that can be considered as variants of the trace inequality and of Poincaré’s estimate, respectively.

Lemma 3. Let $\gamma \in \mathbb{R}_{\geq 0}$, $V \subset \mathbb{R}^d$ and $V_r \subset \mathbb{R}^d$ for $d \in \{2, 3\}$ be two domains with boundaries $\partial V$ and $\partial V_r$ that allow for $C^2$ parametrizations or that are piecewise polygonal with distance
\[
dist(\partial V, \partial V_r) < \gamma.
\]
Further, for each boundary point \( x \in \partial V \) the complete line to the closest point \( y \in \partial V_r \) is in \( V \cup V_r \). For \( \psi \in C^1(\Omega \cap \Omega_r) \cup C(\bar{\Omega}) \) it holds
\[
\| \psi \|_{\partial V_r \cap V} \leq c\left(\| \psi \|_{\partial V} + \gamma^{\frac{3}{2}} \| \nabla \psi \|_{\partial \Omega} \right), \quad \| \psi \|_{V \setminus V_r} \leq c\gamma^{\frac{1}{2}} \left(\| \psi \|_{\partial V} + \frac{\gamma}{2} \| \nabla \psi \|_{\partial \Omega} \right),
\]
and, in the case \( \psi = 0 \) on \( \partial V \) it holds
\[
\| \psi \|_{\partial V_r \cap V} \leq c\gamma^{\frac{1}{2}} \| \nabla \psi \|_{V \setminus V_r}, \quad \| \psi \|_{V \setminus V_r} \leq c\gamma \| \nabla \psi \|_{\partial \Omega}.
\]
(7)

Proof. For the proof we refer to Figure 2. Let \( x_{\partial V_r} \in \partial V_r \cap V \) and \( x_{\partial V} \in \partial V \) with \( |x_{\partial V_r} - x_{\partial V}| \leq \gamma \) and such that the connecting line segment is entirely within \( V \cap V_r \). It holds
\[
|\psi(x_{\partial V_r})|^2 \leq 2|\psi(x_{\partial V})|^2 + 2\gamma \int_{x_{\partial V_r}}^{x_{\partial V}} |\nabla \psi(s)|^2 \, ds.
\]
Integration over the boundary segment \( \partial V_r \cap V \) gives
\[
\| \psi \|_{\partial V_r \cap V} \leq c\gamma^{\frac{1}{2}} \| \nabla \psi \|_{V \setminus V_r}.
\]
Next, we consider a point \( x \in V \setminus V_r \) and \( x_{\partial V} \in \partial V \). By the same arguments it holds
\[
|\psi(x)|^2 \leq 2|\psi(x_{\partial V})|^2 + 2\gamma \int_{x_{\partial V}}^{x_{\partial V_r}} |\nabla \psi(s)|^2 \, ds,
\]
and integration over \( V \setminus V_r \) shows
\[
\| \psi(x) \|_{V \setminus V_r} \leq c\gamma^{\frac{1}{2}} \| \psi \|_{\partial V} + c\gamma \| \nabla \psi \|_{\partial \Omega}.
\]
(8)

We start by estimating the difference between the solutions of the Laplace equations on \( \Omega \) and on \( \Omega_r \).

Lemma 4. Let \( \Omega \) and \( \Omega_r \) have \( C^2 \) boundaries satisfying \( \text{dist}(\Omega, \Omega_r) < \Upsilon \) and let \( f \in L^2(\Omega) \) be an extension of \( f_r \in L^2(\Omega_r) \) satisfying \( f_r = f \) on \( \Omega \cap \Omega_r \) and \( \| f_r \|_{\Omega_r} \leq c\| f \|_{\Omega} \). For the solutions \( u \in H^1_0(\Omega) \cap H^2(\Omega) \) and \( u_r \in H^1_0(\Omega_r) \cap H^2(\Omega_r) \) to (3) and (4), respectively, it holds
\[
\| u - u_r \|_{\Omega} + \Upsilon^{\frac{3}{2}} \| \nabla(u - u_r) \|_{\Omega} \leq c\Upsilon \| f \|_{\Omega}.
\]
Figure 2: Figure for the illustration of the proof to Lemma 3. Two domains \(V\) and \(V_r\) have the distance \(\gamma\). The shaded area on the right is the remainder \(V \setminus V_r\).

Proof. (i) By means of Remark 2 we extend \(u\) and \(u_r\) beyond their defining domain by zero, such that \(u - u_r \in H^1(\Omega \cup \Omega_r)\) is well defined. It holds

\[
\|\nabla (u - u_r)\|^2 = \left(\nabla (u - u_r), \nabla (u - u_r)\right)_\Omega
\]

\[
= -\left(\Delta (u - u_r), u - u_r\right)_\Omega + \langle \partial_n(u - u_r), u - u_r \rangle_{\partial \Omega} + \langle [\partial_n(u - u_r)], u - u_r \rangle_{\partial \Omega \cap \Omega},
\]

(9)

where we denote by \(\langle \cdot, \cdot \rangle_\Gamma\) the \(L^2\) scalar product on the \(d - 1\) dimensional manifold \(\Gamma\), e.g. \(\Gamma = \partial \Omega\), and \([\partial_n \psi]\) is the jump of the normal derivative of \(\psi\), i.e. for \(x \in \Gamma\) with normal \(\vec{n}\)

\[
[\partial_n \psi](x) := \lim_{h \to 0} \partial_n \psi(x + h\vec{n}) - \lim_{h \to 0} \partial_n \psi(x - h\vec{n}).
\]

(10)

In \(\Omega \cap \Omega_r\) it holds \(f = f_r\) and hence (weakly) \(-\Delta (u - u_r) = 0\), such that

\[
-\left(\Delta (u - u_r), u - u_r\right)_\Omega = -\left(\Delta (u - u_r), u - u_r\right)_{\Omega \cap \Omega_r} - \left(\Delta u, u\right)_{\Omega \setminus \Omega_r} = (f, u)_{\Omega \setminus \Omega_r}.
\]

(11)

On \(\partial \Omega\) it holds \(u = 0\) and on \(\partial \Omega_r \cap \Omega\) it holds \(u_r = 0\). Further, since \(u \in H^2(\Omega)\) it holds \([\partial_n u] = 0\) on \(\partial \Omega_r \cap \Omega\). Finally, \(u_r = 0\) on \(\Omega \setminus \Omega_r\), such that the boundary terms reduce to

\[
\langle \partial_n(u - u_r), u - u_r \rangle_{\partial \Omega} + \langle [\partial_n(u - u_r)], u - u_r \rangle_{\partial \Omega \cap \Omega} = -\langle \partial_n(u - u_r), u_r \rangle_{\partial \Omega \cap \Omega} - \langle \partial_n u_r, u \rangle_{\partial \Omega \cap \Omega}.
\]

(12)

Combining (9)-(12) we estimate

\[
\|\nabla (u - u_r)\|^2 \leq \|f\|_{\Omega \setminus \Omega_r} \|u\|_{\Omega \setminus \Omega_r} + \|\partial_n(u - u_r)\|_{\partial \Omega \cap \Omega} \|u_r\|_{\partial \Omega \cap \Omega} + \|\partial_n u\|_{\partial \Omega \cap \Omega} \|u\|_{\partial \Omega \cap \Omega}.
\]

(13)
Since \( u, u_r \in H^2(\Omega \cap \Omega_r) \), the trace inequality gives
\[
\| \nabla (u - u_r) \|^2_{\Omega} \leq \| f \|^2_{\Omega_r \cap \Omega} \| u \|_{\Omega_r \cap \Omega} + c \left( \| u \|_{H^2(\Omega)} + \| u_r \|_{H^2(\Omega_r)} \right) \left( \| u \|_{\partial \Omega \cap \Omega} + \| u_r \|_{\partial \Omega \cap \Omega_r} \right).
\]
(14)

We use Lemma 3 twice, applied to \( \psi = u \) and to \( \psi = \nabla u \) (same for \( u_r \)), to bound
\[
\| u \|_{\partial \Omega \cap \Omega} \leq c\gamma^\frac{1}{2} \| \nabla u \|_{\Omega_r \cap \Omega} \leq c\gamma \left( \| u \|_{H^1(\partial \Omega)} + \gamma \frac{1}{2} \| u \|_{H^2(\Omega)} \right),
\]
\[
\| u_r \|_{\partial \Omega \cap \Omega_r} \leq c\gamma \frac{1}{2} \| \nabla u_r \|_{\Omega_r \cap \Omega} \leq c\gamma \left( \| u_r \|_{H^1(\partial \Omega_r)} + \gamma \frac{1}{2} \| u_r \|_{H^2(\Omega_r)} \right).
\]

With the trace inequality and the a priori estimates \( \| u \|_{H^2(\Omega)} \leq c \| f \|_{\Omega} \) and \( \| u_r \|_{H^2(\Omega_r)} \leq c \| f_r \|_{\Omega_r} \leq c \| u_r \|_{\Omega} \) we obtain the bounds
\[
\| u \|_{\partial \Omega \cap \Omega} \leq c\gamma \| f \|_{\Omega}, \quad \| u_r \|_{\partial \Omega \cap \Omega_r} \leq c\gamma \| f \|_{\Omega}.
\]
(15)

Lemma 3 more precisely (8) followed by (7) and the trace inequality is used to bound
\[
\| u \|_{\Omega_r \cap \Omega} \leq c\gamma \| \nabla u \|_{\Omega_r \cap \Omega} \leq c\gamma \left( \| u \|_{H^2(\Omega)} + \gamma \frac{1}{2} \| u \|_{H^2(\Omega)} \right) \leq c\gamma \frac{3}{2} \| f \|_{\Omega},
\]
(16)

where we used the a priori bound \( \| u \|_{H^2(\Omega)} \leq c \| f \| \). Further, we estimate \( \| f \|_{\Omega_r \cap \Omega} \leq \| f \|_{\Omega} \) by extending to the complete domain, combine (14) with (15) and (16) to estimate
\[
\| \nabla (u - u_r) \|^2_{\Omega} \leq c\left( \gamma \frac{3}{2} + \gamma \right) \| f \|^2_{\Omega},
\]
which shows the \( H^1 \)-norm estimate.

(ii) For the \( L^2 \)-estimate we introduce the adjoint problem
\[
z \in H^1_0(\Omega) : \quad -\Delta z = \frac{u - u_r}{\| u - u_r \|_{\Omega}} \quad \text{in} \ \Omega,
\]
which allows for a unique solution satisfying \( \| z \|_{H^2(\Omega)} \leq c_s \). Testing with \( u - u_r \) and integrating by part twice gives
\[
\| u - u_r \|_{\Omega} = -\langle z, \Delta (u - u_r) \rangle_{\Omega} + \langle z, \partial_n(u - u_r) \rangle_{\partial \Omega} + \langle z, [\partial_n(u - u_r)] \rangle_{\partial \Omega \cap \Omega} - \langle \partial_n z, u - u_r \rangle_{\partial \Omega}.
\]
It holds \( z = 0 \) and \( u = 0 \) on \( \partial \Omega \), \( \partial_n u = 0 \) in \( \Omega \) and \( -\Delta (u - u_r) = 0 \) in \( \Omega \cap \Omega_r \) such that we get
\[
\| u - u_r \|_{\Omega} \leq \langle z, f \rangle_{\Omega \cap \Omega_r} + \langle z, \partial_n u_r \rangle_{\partial \Omega \cap \Omega} + \langle z, \partial_n z \rangle_{\partial \Omega} + \| u_r \|_{\partial \Omega}.
\]
The boundary terms $\|z\|_{\partial \Omega \cap \Omega}$ and $\|u_r\|_{\partial \Omega}$ are estimated with lemma 3, namely (8), the normal derivatives by the trace inequality and the terms on $\Omega \setminus \Omega_r$ again by (8)

$$\|u-u_r\|_{\Omega} \leq c\Upsilon^{\frac{3}{2}}\|z\|_{H^2(\Omega)} + c\Upsilon\|u_r\|_{H^2(\Omega)} + c\Upsilon\|z\|_{H^2(\Omega)}\|u_r\|_{H^2(\Omega)}.$$ 

The $L^2$-norm estimate follows by using the bounds $\|u\|_{H^2(\Omega)} \leq c\|f\|_{\Omega}$, $\|u_r\|_{H^2(\Omega_r)} \leq c\|f\|_{\Omega}$ and $\|z\|_{H^2(\Omega)} \leq c$.

**Remark 5.** The estimate $\|f\|_{\Omega \cap \Omega_r} \leq c\|f\|_{\Omega}$ is not optimal. Further powers of $\Upsilon$ are easily generated. Also, the estimate $\|\partial_n(u-u_r)\| \leq c\|u\|_{H^2(\Omega)} + \|u_r\|_{H^2(\Omega_r)}$ by Cauchy Schwarz and the trace inequality could be enhanced to produce powers of $\Upsilon$. The limiting term in (9) however is the boundary integral $|\langle \partial_n u_r, u_r \rangle_{\partial \Omega \cap \Omega}| = O(\Upsilon^{\frac{1}{2}})$ which is optimal in the $H^1$-estimate. At the end of Section 3 in Remark 8 and Corollary 9 we present an estimate on the intersected domain $\Omega \cap \Omega_r$ that allows us to improve the order to $O(\Upsilon)$ in the $H^1$-case by avoiding exactly this boundary integral.

## 3 Discretization

Starting point of a finite element simulation is the discretization $\Omega_h$ of the domain $\Omega$. In our setting we do not discretize $\Omega$ directly, because the domain $\Omega$ is not exactly known. Instead, we consider that $\Omega_h$ is a triangulation of the reconstruction $\Omega_r$.

We partition $\Omega_r$ into a parametric triangulation (or mesh) $\Omega_h$, consisting of open elements $T \subset \mathbb{R}^d$. Each element $T \in \Omega_h$ stems from a unique reference element $\hat{T}$ which is a simple geometric structure like a triangle, quadrilateral or tetrahedron (the numerical examples in section 4 are based on quadrilateral meshes). The map $T_T : \hat{T} \rightarrow T$ is a polynomial of degree $r \in \mathbb{N}$. We will consider iso-parametric finite element spaces, that are based on polynomials of the same degree $r$. We assume structural and shape regularity of the mesh such that standard interpolation estimates will hold. See [19, section 4.2.2] for a detailed description.

On the reference element $\hat{T}$ let $\hat{P}$ be a polynomial space of degree $r$, e.g.

$$\hat{P} \cong Q^r := \text{span}\{x_1^{\alpha_1} \cdots x_d^{\alpha_d} : 0 \leq \alpha_1, \ldots, \alpha_d \leq r\}$$

on quadrilateral and hexahedral meshes. Then, the finite element space $V_h^r$ on the mesh $\Omega_h$ is defined as

$$V_h = \{ \phi_h \in C(\hat{\Omega}_h) : \phi_h \circ T_T \in \hat{P} \text{ on every } T \in \Omega_h \}.$$
This parametric finite element space does not exactly match the domain \( \Omega_r \). Given an iso-parametric mapping of degree \( r \) it holds \( \text{dist}(\partial \Omega_r, \partial \Omega_h) = \mathcal{O}(h^{r+1}) \) and finite element approximation error and geometry approximation error are balanced. From [19, theorem 4.37] we cite the following approximation result for the iso-parametric approximation of the Laplace equation on curved domains.

**Theorem 6.** Let \( m \in \mathbb{R}_{\geq 0} \) and \( \Omega_r \) a domain with a boundary that allows for a parametrization of degree \( m + 2 \). Let \( f_r \in H^m(\Omega_r) \) and \( u_h \in V_h^r \cap H^1_0(\Omega_h) \) be the iso-parametric finite element discretization of degree \( 1 \leq r \leq m + 1 \)

\[
(\nabla u_h, \nabla \phi_h)_{\Omega_h} = (f_r, \phi_h)_{\Omega_h} \quad \forall \phi_h \in V_h^r.
\]

It holds

\[
\| u - u_h \|_{H^1(\Omega_r)} \leq c h^r \| f_r \|_{H^{r-1}(\Omega_r)},
\]

\[
\| u - u_h \|_{L^2(\Omega_r)} \leq c h^{r+1} \| f_r \|_{H^{r-1}(\Omega_r)}.
\]

We formulated the error estimate on the domain \( \Omega_r \) although the finite element functions are given on \( \Omega_h \) only. To give theorem 6 sense, we consider all functions extended by zero by means of remark 2. With these preparations we formulate the main result.

**Theorem 7.** Let \( m \in \mathbb{N}_{\geq 0} \), \( \Omega \) and \( \Omega_r \) be domains with \( C^{m+2} \) boundary. Let \( \Omega_h \) be the iso-parametric mesh of \( \Omega_r \) with degree \( 1 \leq r \leq m + 1 \). For the finite element error between the fully discrete solution \( u_h \in V_h^r \)

\[
(\nabla u_h, \nabla \phi_h)_{\Omega_h} = (f_r, \nabla \phi_h)_{\Omega_h} \quad \forall \phi_h \in V_h^r.
\]

and the real solution \( u \in H^1_0(\Omega) \cap H^{m+2}(\Omega) \) it holds

\[
\| u - u_h \|_{H^1(\Omega)} \leq c \left( \frac{1}{2} + h^r \right) \| f \|_{H^{r-1}(\Omega)},
\]

as well as

\[
\| u - u_h \|_{L^2(\Omega)} \leq c (1 + h^{r+1}) \| f \|_{H^{r-1}(\Omega)}.
\]

**Proof.** (i) To obtain the \( H^1 \)-norm estimate we split the error by introducing the solution on the reconstructed domain \( \pm u_r \in H^1_0(\Omega_r) \)

\[
\| \nabla (u - u_h) \|_\Omega^2 = (\nabla u, \nabla (u - u_h))_{\Omega} - (\nabla u_h, \nabla (u - u_h))_{\Omega \cap \Omega_h}
\]

\[
= (\nabla (u - u_r), \nabla (u - u_h))_{\Omega} + (\nabla u_r, \nabla (u - u_h))_{\Omega \cap \Omega_r} - (\nabla u_h, \nabla (u - u_h))_{\Omega \cap \Omega_h}
\]

9
where we used that \( u_h = 0 \) on \( \Omega \setminus \Omega_h \) and \( u_r = 0 \) on \( \Omega \setminus \Omega_r \). Noting the small discrepancy between \( \Omega \cap \Omega_r \) and \( \Omega \cap \Omega_h \) we get

\[
\| \nabla (u - u_h) \|_{L^2(\Omega)}^2 \leq \| \nabla (u - u_r) \|_{L^2(\Omega)}^2 \| \nabla (u - u_h) \|_{L^2(\Omega)}^2 + \| \nabla (u_r - u_h) \|_{L^2(\Omega \cap \Omega_r)} \| \nabla (u - u_h) \|_{L^2(\Omega \cap \Omega_r)} + \| \nabla u_h \|_{L^2((\Omega \cap \Omega_h) \setminus \Omega_r)} \| \nabla (u - u_h) \|_{L^2((\Omega \cap \Omega_h) \setminus \Omega_r)} \tag{17}
\]

The terms on the remainder \( S_h := (\Omega \cap \Omega_h) \setminus \Omega_r \) are estimated by lemma 3, which – in the discrete setting – is also given in [19] lemma 4.34. Lemma 3 is applied element per element on each \( T \cap S_h \) for \( T \in \Omega_h \). As \( \partial \Omega_h \) is an approximation of \( \partial \Omega_r \) of degree \( r \), these remainders are very small, with \( \gamma = \mathcal{O}(h^{r+1}) \) in the context of the lemma. In this spirit we get – on each \( T \cap S_h \) – the estimate

\[
\| \nabla u_h \|_{L^2(T \cap S_h)}^2 \leq c h^{r+1} \| \nabla u_h \|_{L^2(\partial \Omega_r \cap T)}^2 + c h^{2r+2} \| \nabla^2 u_h \|_{L^2(T)}^2. \tag{18}
\]

Using the trace inequality and the inverse estimate it holds

\[
\| \nabla u_h \|_{L^2(T \cap S_h)}^2 \leq c h^r \left( \| \nabla u_h \|_{L^2(T)} + h^2 \| \nabla^2 u_h \|_{L^2(T)} \right) + c h^{2r+2} \| \nabla^2 u_h \|_{L^2(T)}^2 \leq c h^r \| \nabla u_h \|_{L^2(T)}^2,
\]

such that summation over all element \( T \cap S_h \) gives

\[
\| \nabla u_h \|_{S_h} \leq \| \nabla u_h \|_{\Omega_h}. \tag{19}
\]

Next, we apply (18) locally on every \( T \cap S_h \) to the error \( \psi := \nabla (u - u_h) \in H^2(T \cap S_h) \)

\[
\| \nabla (u - u_h) \|_{L^2(T \cap S_h)}^2 \leq c h^{r+1} \| \nabla (u - u_h) \|_{L^2(\partial \Omega_r \cap T)}^2 + c h^{2r+2} \| \nabla^2 (u - u_h) \|_{L^2(T)}^2.
\]

We insert an interpolation \( \pm I_h u \in V_h \) to both terms and use the trace inequality (locally on each element \( T \)) as well as the inverse estimate

\[
\| \nabla (u - u_h) \|_{L^2(S_h \cap T)}^2 \leq c h^{r+1} \| \nabla (u - I_h u) \|_{L^2(\partial \Omega_r \cap T)}^2 + c h^r \| \nabla (I_h u - u_h) \|_{L^2(T)}^2 + c h^{2r+2} \| \nabla^2 (u - I_h u) \|_{L^2(S_h \cap T)}^2 + c h^{2r} \| \nabla (I_h u - u_h) \|_{L^2(S_h \cap T)}^2.
\]

By inserting \( \pm u \) and using interpolation estimates, summing over all elements of \( S_h \) we obtain the estimate

\[
\| \nabla (u - u_h) \|_{S_h} \leq \| h^2 \|_{L^2(S_h)} u \|_{H^{r-1}(\Omega)} + \| h^2 \|_{L^2(S_h)} \nabla (u - u_h) \|_{\Omega}. \tag{20}
\]

We combine this estimate with (19) and (17) to get

\[
\| \nabla (u - u_h) \|_{L^2(\Omega)}^2 \leq \| \nabla (u - u_r) \|_{L^2(\Omega)}^2 \| \nabla (u - u_h) \|_{L^2(\Omega)}^2 + \| \nabla (u_r - u_h) \|_{L^2(\Omega \cap \Omega_r)} \| \nabla (u - u_h) \|_{L^2(\Omega \cap \Omega_r)} + c h^{2r} \| u \|_{H^{r+1}(\Omega)} \| \nabla u_h \|_{\Omega_h} + \| h^2 \|_{L^2(S_h)} \nabla (u - u_h) \|_{\Omega}.
\]

10
Using Young’s inequality we hide the $\|\nabla (u - u_h)\|_{\Omega_h}$-terms on the left hand side and use the bounds $\|\nabla u_h\|_{\Omega_h} \leq c\|f_r\|_{\Omega_r} \leq c\|f\|_{\Omega}$ and $\|u\|_{H^{r+1}(\Omega)} \leq c\|f\|_{H^{r-1}(\Omega)}$.

$$\|\nabla (u - u_h)\|_{\Omega} \leq c\left(\|\nabla (u - u_r)\|_{\Omega}^2 + \|\nabla (u_r - u_h)\|_{\Omega_0 \cap \Omega_r}^2 + ch^2r\|f\|_{H^{r-1}(\Omega)}^2\right).$$

The $H^1$-error estimate follows by combining lemma 4 and theorem 6.

(ii) For estimating the $L^2$-error we start by introducing the reconstruction and transferring the finite element error from $\Omega$ to $\Omega_r$.

$$\|u - u_h\|_{\Omega}^2 = (u, u - u_h)_\Omega - (u_h, u - u_h)_{\Omega_0} = (u - u_r, u - u_h)_\Omega + (u_r - u_h, u - u_h)_{\Omega_0 \cap \Omega_r} - (u_h, u - u_h)_{(\Omega_0 \cap \Omega) \setminus \Omega_r}.$$ 

Estimation with Cauchy Schwarz and Young’s inequality gives

$$\|u - u_h\|_{\Omega}^2 \leq c\left(\|u - u_r\|_{\Omega}^2 + \|u_r - u_h\|_{\Omega}^2 + \|u_h\|_{(\Omega_0 \cap \Omega) \setminus \Omega_r}^2\right),$$

where bounds for the first and second term are given in lemma 4 and theorem 6.

The product on the remainder $S_h := (\Omega_h \cap \Omega) \setminus \Omega_r$ must be treated similar to the $H^1$ error case. Similarly to (18) we apply Lemma 3 to $\psi = u_h$ with $u_h = 0$ on $\partial \Omega_h$ such that together with (19) it holds

$$\|u_h\|_{S_h}^2 \leq ch^{2r+2}\|\nabla u_h\|_{S_h}^2 \leq ch^{3r+2}\|\nabla u_h\|_{\Omega}^2.$$ 

which finishes the proof.

**Remark 8** (Optimality of the estimates). Two ingredients form the error estimates.

1. First, a geometrical error of order $O(\Upsilon^\frac{1}{2})$ and $O(\Upsilon)$, respectively, that describes the discrepancy between $\Omega$ and $\Omega_r$. This term is optimal which is easily understood by considering a simple example illustrated in Figure 3, namely $-\Delta u = 4$ on the unit disc $\Omega = B_1(0)$ and $-\Delta u_r = 4$ on the shifted domain $\Omega_r = B_1(\Upsilon)$. The errors in $H^1$ norm and $L^2$ norms expressed on the complete domain $\Omega$ are estimated by

$$\|u - u_r\|_{\Omega} = \sqrt{\pi} \Upsilon + O(\Upsilon^3), \quad \|\nabla (u - u_r)\|_{\Omega} = \sqrt{8\Upsilon} + O(\Upsilon).$$
On $\Omega = B_1(0)$ and $\Omega_r = B_1(\Upsilon)$ consider $-\Delta u = 4$ and $-\Delta u_r = 4$, respectively with homogeneous Dirichlet conditions and the solutions

$$u(x, y) = 1 - x^2 - y^2, \quad u_r(x, y) = 1 - (x - \Upsilon)^2 - y^2$$

and the errors

$$\|\nabla (u - u_r)\|_{\Omega} = \sqrt{8\Upsilon^2} + O(\Upsilon), \quad \|u - u_r\|_{\Omega} = \sqrt{\pi}\Upsilon^2 + O(\Upsilon^3).$$

Figure 3: Illustration concerning Remark 8. The error estimates for $u - u_h$ are optimal, if the error is evaluated on $\Omega$. The lowest order terms $O(\Upsilon^{1/2})$ appear in the shaded area $\Omega \setminus \Omega_r$ where $u_r$ and (most of) $u_h$ are zero.

A closer analysis shows that the main error – in the $H^1$-case – occurs on the small shaded stripe $\Omega \setminus \Omega_r$ such that

$$\|\nabla (u - u_r)\|_{\Omega \setminus \Omega_r} = O(\Upsilon^{1/2}), \quad \|\nabla (u - u_r)\|_{\Omega \cap \Omega_r} = O(\Upsilon),$$

while the $L^2$-error in $\Omega \cap \Omega_r$ is optimal

$$\|u - u_r\|_{\Omega \cap \Omega_r} = O(\Upsilon^{3/2}), \quad \|u - u_r\|_{\Omega \setminus \Omega_r} = O(\Upsilon).$$

2. Second, the usual Galerkin error $\|u_r - u_h\|_{\Omega_r} + h\|\nabla (u - u_r)\|_{\Omega_r} = O(h^{r+1})$ of iso-parametric finite element approximations contributes to the overall error. For $\Omega = \Omega_r$, i.e. $\Upsilon = 0$ this would be the complete error. This estimate is optimal, as it shows the same order as usual finite element bounds on meshes that resolve the geometry.

In Section 4 we discuss that the optimality of the error estimates is difficult to verify which is mainly due to the technical problems in evaluating norms on the domain remainders $\Omega \setminus \Omega_r$, where no finite element mesh is given. These remainders contribute the lowest order parts $\Upsilon^{1/2}$ of the overall error. The following corollary is closer to the setting of the numerical examples and it holds approximation of order $\Upsilon$ in the $H^1$-norm error.

**Corollary 9.** In addition to the assumptions of Theorem 7 let there be a $C^1$-diffeomorphism

$$T_r : \Omega \rightarrow \Omega_r$$

satisfying

$$\|I - \det(\nabla T_r)\nabla T_r^{-1}\nabla T_r^{-T}\|_{L^\infty(\Omega)} = O(\Upsilon). \quad (22)$$
Further let the following additional regularity of problem data and solution be given
\[ \|f\|_{W^{1,\infty}(\Omega)} + \|u\|_{W^{2,\infty}(\Omega)} + \|u_r\|_{W^{2,\infty}(\Omega_r)} \leq c. \] (23)

Then, it holds
\[ \|\nabla(u - u_h)\|_{\Omega \cap \Omega_r} \leq c(1 + h^r). \]

**Proof.** We start by splitting the error into domain approximation and finite element approximation
\[ \|\nabla(u - u_h)\|_{\Omega \cap \Omega_r} \leq \|\nabla(u - u_r)\|_{\Omega \cap \Omega_r} + \|\nabla(u_r - u_h)\|_{\Omega \cap \Omega_r}. \] (24)

An optimal order estimate of the finite element error
\[ \|\nabla(u_r - u_h)\|_{\Omega \cap \Omega_r} \leq \|\nabla(u_r - u_h)\|_{\Omega_r} = \mathcal{O}(h^r) \] (25)
is given in Theorem 6. To estimate the first term in (24) we introduce the function
\[ \hat{u}_r(x) := u_r(T(x)), \]
which satisfies \( \hat{u}_r \in H^1_0(\Omega) \) and solves the problem
\[ (J_rF_r^{-1}F^{-T}_r\nabla \hat{u}_r, \nabla \hat{\phi}_r)_{\Omega} = (\hat{f}_r, \hat{\phi}_r) \quad \forall \hat{\phi}_r \in H^1_0(\Omega), \]
where \( \hat{f}_r(x) := f_r(T_r(x)) \) and where \( F_r := \nabla T_r \) and \( J_r := \text{det}(F_r) \). See [19, Section 2.1.2] for this transformation of the variational formulation. For estimating the domain approximation error in (24) we introduce \( \pm \hat{u}_r \) to obtain
\[ \|\nabla(u - u_r)\|_{\Omega \cap \Omega_r} \leq \|\nabla(u - \hat{u}_r)\|_{\Omega \cap \Omega_r} + \|\nabla(\hat{u}_r - u_r)\|_{\Omega \cap \Omega_r}. \] (26)

We introduce the notation \( e_r := u - \hat{u}_r \), extend the first term from \( \Omega \cap \Omega_r \) to \( \Omega \) and insert \( \pm J_rF_r^{-1}F^{-T}_r\nabla \hat{u}_r \) to obtain
\[ \|\nabla(u - u_r)\|_{\Omega \cap \Omega_r} \leq \|\nabla(u - \hat{u}_r)\|_{\Omega \cap \Omega_r} \]
\[ = (\nabla u, \nabla e_r)_{\Omega} - (J_rF_r^{-1}F^{-T}_r\nabla \hat{u}_r, \nabla e_r)_{\Omega} + (J_rF_r^{-1}F^{-T}_r\nabla \hat{u}_r, \nabla e_r)_{\Omega} - (\nabla u_r, \nabla e_r)_{\Omega} \]
\[ = (f - \hat{f}_r, e_r)_{\Omega} + ([J_rF_r^{-1}F^{-T}_r - I]\nabla \hat{u}_r, \nabla e_r)_{\Omega} \]
\[ \leq \|f - \hat{f}_r\|_{L^\infty(\Omega)}\|\nabla e_r\|_{\Omega} + \|J_rF_r^{-1}F^{-T}_r - I\|_{L^\infty(\Omega)}\|\nabla e_r\|_{\Omega}, \] (27)
where we also used Poincaré’s estimate. For the estimation of \( f - \hat{f}_r \) we consider a point \( x \in \Omega \cap \Omega_r \) and introduce \( \pm f_r(x) \)
\[ |f(x) - \hat{f}_r(x)| \leq |f(x) - f_r(T_r(x))| = |f(x) - f_r(x)| + |f_r(x) - f_r(T_r(x))|. \]
Given Remark 1 it holds $f = f_r$ in $\Omega$ such that the first term vanishes. The second term is estimated by a Taylor expansion

$$|f_r(x) - f_r(T_r(x))| = |\nabla f_r(\xi) \cdot (T_r(x) - x)| \leq \Upsilon |\nabla f_r(\xi)|,$$

where $\xi \in \Omega$ is some point on the line from $x$ to $T_r(x)$. We take the square and integrate over $\Omega$ to get the estimate

$$\|f - \hat{f}_r\|_{W^{1,\infty}} \leq c\Upsilon \|f_r\|_{W^{1,\infty}}. \quad (28)$$

This argument is also applicable to the second term of (26) such that it holds

$$\|\nabla (\hat{u}_r - u_r)\|_M \leq c\Upsilon \|u_r\|_{W^{2,\infty}(\Omega \cap \Omega_r)} \leq c\Upsilon.$$

This, together with (24), (25), (26) and (27) finishes the proof. \qed

The application of this corollary must be discussed case by case and it will depend on the existence of a suitable map $T_r : \Omega \to \Omega_r$. Here a construction is possible in correspondence to the ALE map, common in fluid-structure interactions, see [19, section 2.5.2] which can be constructed by means of a domain deformation $\hat{d} : \Omega \to \mathbb{R}^2$

$$T_r(x) = x + \hat{d}(x), \quad F_r(x) = I + \nabla \hat{d}(x)$$

Given that $|\hat{d}|, |\nabla \hat{d}| = \mathcal{O}(\Upsilon)$ it holds

$$\|J_r\|_{L^\infty(\Omega)} = 1 + \mathcal{O}(\Upsilon), \quad \|I - J_r F_r^{-1} F_r^{-T}\|_{L^\infty(\Omega)} = \mathcal{O}(\Upsilon).$$

While the assumption $|\hat{d}| = \mathcal{O}(\Upsilon)$ is easy to satisfy since $\text{dist}(\partial \Omega, \partial \Omega_r) \leq \Upsilon$, the condition $|\nabla \hat{d}| = \mathcal{O}(\Upsilon)$ will strongly depend on the shape and regularity of the boundary.

We conclude by discussing a simple application of this corollary. Figure 4 illustrates the setting. Let $\Omega$ be the unit sphere, $\Omega_r$ be an ellipse

$$\Omega = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}, \quad \Omega_r = \{x \in \mathbb{R}^2 : (1+\Upsilon)^2 x_1^2 + (1+\Upsilon)^{-2} x_2^2 < 1\}.$$

It holds $\text{dist}(\partial \Omega, \partial \Omega_r) \leq \Upsilon$ and we define the map $T_r : \Omega \to \Omega_r$ by

$$T_r(x) = \begin{pmatrix} (1 + \Upsilon)^{-1} x_1 \\ (1 + \Upsilon) x_2 \end{pmatrix}, \quad F_r = \nabla T_r = \begin{pmatrix} (1 + \Upsilon)^{-1} & 0 \\ 0 & (1 + \Upsilon) \end{pmatrix}, \quad J_r = 1.$$

This map satisfies the assumptions of the corollary

$$I - J_r F_r^{-1} F_r^{-T} = \Upsilon (\Upsilon + 2) \begin{pmatrix} -1 & 0 \\ 0 & (1 + \Upsilon)^{-2} \end{pmatrix}, \quad \|I - J_r F_r^{-1} F_r^{-T}\|_\infty = 2\Upsilon + \Upsilon^2.$$

14
4 Numerical illustration

In this section we aim to illustrate the theoretical considerations from the previous section. We compute the Laplace problem on a series of domains. Moreover, we numerically extend the analytical predictions and show that a similar behaviour holds for the Stokes system.

We consider $\Omega$ to be a unit ball in two and three dimensions and define a family of perturbed domains $\Omega_\Upsilon$, with the amplitude of the perturbation being dependent on the coefficient $\Upsilon$, cf. Figure 5. The description of the boundary of the domain $\Omega_\Upsilon$ for

$$\Upsilon \in \{0, 0.0125, 0.025, 0.05, 0.1\}$$

in the polar coordinates $(\rho, \varphi)$ (two dimensions) reads

$$\partial \Omega_\Upsilon = \{(1 - \Upsilon/5 + \Upsilon \sin(8\varphi), \varphi) \text{ for } \varphi \in [0, 2\pi)\},$$

and in the spherical coordinates $(\rho, \theta, \varphi)$ (three dimensions)

$$\partial \Omega_\Upsilon = \{(1 - \Upsilon/5 + \Upsilon \sin(3\varphi) \sin(3\theta), \theta, \varphi) \text{ for } \theta \in [0, \pi), \varphi \in [0, 2\pi)\}.$$ 

All computations are performed on a series of refined meshes. The dependence between the mesh size $h$ and the refinement level $L$ reads $h = 2^{-L}$. We denote the mesh approximating $\Omega_\Upsilon$, with a mesh size $h$, by $\Omega_{h_\Upsilon}$.

The numerical implementation is realized in the software library Gascoigne 3D [4] and using equal-order iso-parametric finite elements of degree 1 and 2. A detailed description of the underlying numerical methods is given in [19].
4.1 Laplace equation in two and three dimensions

We consider the following problem

$$- \Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,$$  \hspace{1cm} (29)

where $\Omega$ is the unit ball in 2 dimensions and the unit sphere in 3 dimensions.

To compute errors we choose a rotationally symmetric analytical solution to (29) (in 2d and 3d) as

$$u(r) = - \cos \left( \frac{\pi}{2} r \right)$$

with $r = \sqrt{x^2 + y^2}$ in two and $r = \sqrt{x^2 + y^2 + z^2}$ in three dimensions, respectively, which results in the right hand sides

$$f_{2d}(r) = \frac{\pi}{2r} \sin \left( \frac{\pi}{2} r \right) + \frac{\pi^2}{4} \cos \left( \frac{\pi}{2} r \right), \quad f_{3d}(r) = \frac{\pi}{r} \sin \left( \frac{\pi}{2} r \right) + \frac{\pi^2}{4} \cos \left( \frac{\pi}{2} r \right).$$
Figure 6: $L^2$- and $H^1$-errors w.r.t. mesh-size $h_{\text{max}}$ for varying parameter $\Upsilon$ computed for the Laplace problem in three-dimensions with linear finite elements.

Figure 7: $L^2$- and $H^1$-errors w.r.t. mesh-size $h_{\text{max}}$ for varying parameter $\Upsilon$ computed for the Laplace problem in two-dimensions with FE. Top: linear finite elements. Bottom: quadratic finite elements.
For the evaluation of $H^1$- and $L^2$-norms we use truncated domains

\begin{align*}
\Omega_{2d}^2 &= \{(\varphi, \rho) \text{ for } \varphi \in [0, 2\pi) \text{ and } \rho \in (0, 0.88)\}, \\
\Omega_{2d}^3 &= \{(\varphi, \theta, \rho) \text{ for } \theta \in [0, \pi), \varphi \in [0, 2\pi) \text{ and } \rho \in (0, 0.88)\},
\end{align*}

see also remark 8 and the following discussion. We hence do not compute the errors $\|\nabla (u - u_h)\|$ and $\|u - u_h\|$ on the remainders $\Omega \setminus \Omega_r$. Therefore we expect optimal order convergence in the spirit of corollary 9. The restriction of the domain to an area within $\Omega_h$ is also by technical reasons, as the evaluation of integrals outside the finite element mesh is usually not possible in finite element implementations such as Gascoigne 3D [1].

![Graph showing $L^2$ and $H^1$-errors w.r.t. parameter $\Upsilon$ computed for the Laplace problem in two and three-dimensions with linear and quadratic finite elements.](image)

**Figure 8:** $L^2$- and $H^1$-errors w.r.t. parameter $\Upsilon$ computed for the Laplace problem in two and three-dimensions with linear and quadratic finite elements.

In Figures 6 and 7, we see the resulting $L^2$- and $H^1$-errors. We observe that for finer meshes, $\Upsilon$ becomes the dominating factor of the error. In particular, the use of quadratic finite elements shows a strong disbalance between FE error and geometric error, which quickly dominates as seen in the lower part of Fig. 7. The result is consistent with Corollary 9. As soon as the FE error is smaller than the geometry perturbation $\Upsilon$, we do not observe any further improvement of the error. In Fig. 8, we show the convergence in both norms in terms of the geometry parameter $\Upsilon$. Linear convergence is clearly observed. The apparent decay of convergence rate in case of the $L^2$-error in three dimensions is due to the still dominating FE error in this case.
4.2 Stokes system in two dimensions

To go beyond the Laplace problem, we investigate the behaviour of the solution to the Stokes system with respect to the domain variation in two spatial dimensions. Velocity $u$ pressure $p$ obey

$$\text{div } u = 0, \quad -\Delta u + \nabla p = f,$$

with homogenous Dirichlet condition $u = 0$ on the boundary $\partial \Omega$ and a right hand side vector $f$. System (30) is solved with equal-order iso-parametric finite elements using pressure stabilisation by local projections, see [3].

We prescribe an analytical solution for comparison with the finite element approximation

$$u(x, y) = \cos\left(\frac{\pi}{2} (x^2 + y^2)\right) \begin{pmatrix} y \\ -x \end{pmatrix},$$

where the corresponding forcing term reads

$$f(x, y) = \pi \cos\left(\frac{\pi}{2} (x^2 + y^2)\right) \begin{pmatrix} yr^2\pi + 4(y - x) \tan\left(\frac{\pi}{2} (x^2 + y^2)\right) \\ -xr^2\pi - 4(x + y) \tan\left(\frac{\pi}{2} (x^2 + y^2)\right) \end{pmatrix}.$$

In Figure 9 we see the resulting $L^2$- and $H^1$-errors. Again we observe that $\Upsilon$ becomes the dominant factor for finer meshes. This result is not covered by the theoretical findings, shows however that geometric uncertainty should be taken into account for the simulations of flow models.

![Figure 9: $L^2$- and $H^1$-errors w.r.t. mesh-size $h_{\text{max}}$ for varying parameter $\Upsilon$ computed for the Stokes problem in two-dimensions with linear finite elements.](image)

19
5 Conclusions

We have demonstrated that small boundary variations have crucial impact on the result of the finite element simulations. The developed error estimates are linear with respect to the maximal distance $\Upsilon$ between the real and the approximated domains, cf. Theorem 7. We have illustrated the sharp nature of this bound in the computations performed in Section 4.

In particular, in the case of first and second order approximation we observe how the relation between the mesh size $h$ and aforementioned $\Upsilon$ impact the resulting $L^{2}$- and $H^{1}$-errors. The same behaviour has been demonstrated for the Stokes system.

In practice we do not have control on the accuracy of the domain reconstruction. This has shown that it is worth to take into account the geometric uncertainty when deciding on the mesh-size in order to avoid unnecessary computational effort.

In this work we have focused on the Laplace problem (2). Additionally, the Stokes system has been treated numerically and it exhibits similar features. In future work we will extend this consideration to flow models, in particular the Navier-Stokes equations [13]. Among the additional challenges in extending the present work to the Navier-Stokes system are the consideration of the typical saddle-point structure of incompressible flow models introducing a pressure variable [18] and the difficulty of nonlinearities introduced by the convective term, and thus the non-uniqueness of solutions [12].

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