On the Stability for the Cauchy Problem of Timoshenko thermoelastic Systems with Past History: Cattaneo and Fourier Law

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Abstract
In this paper, we investigate the decay properties of thermoelastic Timoshenko systems with past history in the whole space where the thermal effects are modeled by Cattaneo and Fourier laws. We establish rates of decay of order $\left(1 + t\right)^{-\frac{1}{8}}$ for both systems, Timoshenko-Fourier and Timoshenko-Cattaneo, satisfying the regularity-loss type property. Moreover, for the Cattaneo case, we show that the decay rates depend on a new condition $\chi_{0,\tau}$ which has been recently introduced to study the asymptotic behavior of Timoshenko systems in bounded domains. We found that this number also plays an important role in unbounded situations, affecting the decay rate of the solution.

Keywords: Decay rate, Heat conduction, Timoshenko system, Thermoelasticity, Regularity loss phenomenon

1. Introduction
In the literature concerning Timoshenko systems, the stability nature of solutions have a relationship with the wave speeds of propagation, essentially, when these speeds are equal or different. In this context, denoting by $\chi_0$ the difference of propagation’s speed (see equation (6)), we investigate how the decay rates of solutions of thermoelastic Timoshenko systems depend of $\chi_0$, in particular when the thermal effects are given by Cattaneo and Fourier law, both with additional history terms. As we can see from the references, the constant $\chi_0$ will play an important role in the characterization of asymptotic behavior of solutions.

Recalling that Cattaneo’s law is a hyperbolic heat model implying that the temperature has a finite speed of propagation, we can observe the impact of this heat model on the stability of Timoshenko systems. Being more specific, recently in [6, 30], the authors proved that Cattaneo’s law modifies the stability number $\chi_0$ when the model is formulated in bounded domains. Since this hyperbolic model generates dissipative thermal effects weaker than the parabolic Fourier model, the authors introduce a new stability number $\chi_{0,\tau}$ (see equation (3)), which generalizes the previous one $\chi_0$ in the sense that, when $\tau = 0$, Cattaneo’s law turns into the Fourier law and the conditions over the new number $\chi_{0,\tau}$ are equivalent to the old stability number $\chi_0$.

In this line of research, our goal in this paper is to investigate the relation between $\chi_{0,\tau}$ and $\chi_0$ with the decay rates of the solution of Timoshenko system posed in the whole real line. In fact, we consider the Cauchy problem of the Timoshenko system with the heat conduction described by the Cattaneo and Fourier

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For this system, in bounded domains $(0, L)$, polynomially with rate $\tau$, and the heat flow, respectively. The integral term represents a history term with kernel $g$ satisfying the following hypotheses:

\begin{itemize}
    \item[(H1)] $g(\cdot)$ is a non negative function.
    \item[(H2)] There exist positive constants $k_1$ and $k_2$, such that, $-k_1g(s) \leq g'(s) \leq -k_2g(s)$.
    \item[(H3)] $a := b - b_0 > 0$, where $b_0 = \int_0^\infty g(s)ds$.
\end{itemize}

For this system, in bounded domains $(0, L)$, see for instance [3]; the associated stability number is given by

$$\chi_{0,\tau} = \left( \tau - \frac{\rho_1}{\rho_3k} \right) \left( \rho_2 - \frac{b\rho_1}{k} \right) - \frac{\tau\rho_1\delta^2}{\rho_3k}.$$  \hfill (3)

As mentioned in several references, condition (3) is more mathematical than physical, because is not realistic to assume that the propagation speeds associated to system (1) will satisfy condition (3). Note that, when $\tau = 0$, the system (1) has a thermal effect given by the Fourier law ($g = -\beta\theta_x$). Indeed, formally the Timoshenko-Cattaneo system (1) is reduced to the following Timoshenko-Fourier system

\begin{itemize}
    \item[(H2)] $\theta(0, \cdot) = \theta_0(\cdot), \quad \psi(0, \cdot) = \psi_0(\cdot),\quad \psi(0, \cdot) = \psi_0(\cdot),$ \quad $\forall s \in (-\infty, 0]$
\end{itemize}

with initial data

$$\varphi(0, \cdot) = \varphi_0(\cdot), \quad \psi(0, \cdot) = \psi_0(\cdot), \quad \theta(0, \cdot) = \theta_0(\cdot), \quad \forall s \in (-\infty, 0],$$

and stability number given by

$$\chi_0 = \rho_2 - \frac{b\rho_1}{k}.$$  \hfill (6)

As we said previously, the main purpose of this article is to investigate the relationship between damping terms, the stability numbers $\chi_{0,\tau}, \chi_0$ and their influence on the decay rate of solutions of systems (1)-(2) and (4)-(5), respectively.

Let us start by giving some references on Timoshenko systems. The original Timoshenko system was first introduced by Timoshenko [34, 35] and describes the vibration of a beam taking into account the transversal displacement and the rotational angle of the beam filaments. An initial boundary value problem associated to (1) and (2) under hypotheses $(H_1), (H_2), (H_3)$ was considered by Fernández Sare and Racke in [6]. They prove that the energy of the solution for the Timoshenko-Cattaneo model with history does not decay exponentially as $t \to \infty$ if $\chi_0 = 0$, while for the Timoshenko-Fourier system the energy decays exponentially if and only if $\chi_0 = 0$. This result has been recently improved by Fatori et al in [3] where, for the Cattaneo’s case, the exponential stability is obtained if and only if a new condition on the wave speeds of propagation is satisfied, i.e, the energy of solution of a IBVP Timoshenko-Cattaneo decay exponentially if and only if $\chi_{0,\tau} = 0$, where $\chi_{0,\tau}$ is given by (3). Furthermore, if $\chi_{0,\tau} \neq 0$, they prove that the energy decays polynomially with rate $t^{-\frac{1}{2}}$. 

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There are many other references on Timoshenko systems in bounded domains with interesting results. In particular, the problem of stability for Timoshenko-type systems in bounded domains has received much attention in the last years, and quite a number of results concerning uniform and asymptotic decay of energy have been established, see for instance [1, 2, 3, 4, 13, 16, 17, 18, 20, 21, 22, 23, 31] and references therein. As a matter of fact, in bounded domains the proofs of stability results for Timoshenko systems are based on Poincaré inequalities and boundary conditions of the systems.

In this paper we are specially interested in the unbounded situation: when the system is formulated in the whole space $\mathbb{R}$. This kind of problem has been considered in recent papers because it exhibits the regularity-loss phenomenon that usually appears in the pure Cauchy problems; see for instance [10, 11, 24, 25, 36] and references therein. Roughly speaking, a decay rate of solution is of regularity-loss type when it is obtained only by assuming some additional regularity on the initial conditions. In this direction, we can mention some recent results on stabilization of Cauchy Timoshenko systems. For instance, in Ide-Haramoto-Kawashima [11], Ide-Kawashima [12] and Racke-Houari [24, 25], the authors consider Timoshenko systems with normalized coefficients proving that the assumptions $b = 1$ or $b \neq 1$ play decisive roles in showing whether or not the decay estimates of solutions are of regularity-loss type.

For Cauchy problems associated to Timoshenko systems in thermoelasticity, as far as we know, the decay rate of solutions has been first studied by Said-Houari and Kasimov in [28, 29]. In particular, the authors considered the following Timoshenko systems with normalized coefficients proving that the assumptions $b = 1$ or $b \neq 1$ play decisive roles in showing whether or not the decay estimates of solutions are of regularity-loss type.

The main goal of this paper is to investigate the decay rate of the Cauchy problems (1) and (4). We show that the respective solutions of Timoshenko-Cattaneo and Timoshenko-Fourier with history term, are of regularity-loss type and decay slowly with the rate $(1 + t)^{-\frac{1}{\beta}}$ in the $L^2$-norm. Our proofs are based on some estimates for the Fourier image of the solution, Plancherel Theorem, as well as on a suitable linear combination of series associated to energy estimates. Here, the decay rate $(1 + t)^{-\frac{1}{\beta}}$ will be obtained by taking regular initial data $u_0 \in H^s(\mathbb{R})$, for some $s \in \mathbb{R}$. This regularity loss comes from the analysis of the Fourier image, $\hat{U}(\xi, t)$, of the solution $U(\xi, t)$. In fact, we will obtain the estimate

$$\left| \hat{U}(\xi, t) \right|^2 \leq Ce^{-\beta \rho(\xi)t} \left| \hat{U}(\xi, 0) \right|^2,$$

where $C, \beta$ are positive constants and

$$\rho(\xi) = \begin{cases} \frac{\xi^4}{(1 + \xi^2)^2}, & \text{if } \chi_{0, \tau} = 0 \text{ (resp, } \chi_0 = 0), \\ \frac{\xi^4}{(1 + \xi^2)}, & \text{if } \chi_{0, \tau} \neq 0 \text{ (resp, } \chi_0 \neq 0). \end{cases}$$
As we will see, the decay estimates for Timoshenko-Cattaneo and Timoshenko-Fourier, depend on the properties of the function $\rho(\xi)$. In fact, this function $\rho(\xi)$ behaves like $\xi^4$ in the low frequency region ($|\xi| \leq 1$) and like $\xi^{-2}$ near infinity, whenever $\chi_{0,r} = 0$ (resp, $\chi_0 = 0$). Otherwise, if $\chi_{0,r} \neq 0$ (resp, $\chi_0 \neq 0$), the function $\rho(\xi)$ behaves also like $\xi^4$ in the low frequency region but like $\xi^{-4}$ near infinity, which means that the dissipation in the high frequency region is very weak and produces the regularity loss phenomenon. It is known that this regularity loss causes some difficulties in the nonlinear cases, see for example [11, 12] for more details.

This paper is organized as follows. Section 2 is dedicated to state the problems. In section 3 we will present the energy method in the Fourier space and the construction of the Lyapunov functionals. The main results, Theorems 4.1 and 4.2 are formulated in Section 4.

2. Setting of the Problem

In order to establish the decay rates of the Timoshenko systems (11) and (13), we have to transform the original problems to a first-order (in variable $t$) systems, defining new variables. Then, we apply the energy method in the Fourier space to prove some point wise estimates which will help in the proof of the decay estimates.

2.1. The Cattaneo Model

We consider the Timoshenko system with history and Cattaneo law. Using the change of variable, introduced in 3,

$$
\eta(t, s, x) := \psi(t, x) - \psi(t - s, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \quad s \geq 0,
$$

the system (11), can be rewritten as

$$
\begin{align*}
\rho_1 \phi_t - k (\phi_x - \psi)_x &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\
\rho_2 \psi_t - a \psi_{xx} - m \int_0^\infty g(s) \eta_{xx}(s) ds - k (\phi_x - \psi) + \delta \theta_x &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\
\rho_3 \theta_t + q_x + \delta \psi_{xt} &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\
\eta_x + \eta_s - \psi_t &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\
\eta(\cdot, 0, \cdot) &= 0
\end{align*}
$$

(9)

where $a = a(b, g)$ is a positive constant given by $(H_3)$ and operator $T\eta = -\eta_s$ is the usual operator defined in problems with history terms, see for instance [22, 6] and references therein. Here, the last two equations of system (9) are obtained differentiating equation (8). We define also the initial data

$$
\begin{align*}
\phi(0, \cdot) &= \phi_0(\cdot), \quad \psi(0, \cdot) = \psi_0(\cdot), \quad \theta(0, \cdot) = \theta_0(\cdot), \\
\phi_t(0, \cdot) &= \phi_1(\cdot), \quad \psi_t(0, \cdot) = \psi_1(\cdot), \quad q(0, \cdot) = q_0(\cdot), \\
\eta(0, s, \cdot) &= \psi(0, \cdot) - \psi(0, -s, \cdot).
\end{align*}
$$

Furthermore, we can rewrite the system (9) by considering the following change of variables

$$
\begin{align*}
u &= \phi_t, \quad z = \psi_x, \quad y = \psi_t, \quad v &= \phi_x - \psi.
\end{align*}
$$

Then, (9) takes the form

$$
\begin{align*}
\nu_t - \nu_x + y &= 0, \\
\rho_1 \nu_t - kv_x &= 0, \\
z_t - y_x &= 0, \\
\rho_2 \psi_t - az_x - m \int_0^\infty g(s) \eta_{xx}(s) ds - kv + \delta \theta_x &= 0, \\
\rho_3 \theta_t + q_x + \delta y_x &= 0, \\
tq_t + \beta q + \theta_x &= 0, \\
\eta_t + \eta_s - y &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\
\eta(\cdot, 0, \cdot) &= 0
\end{align*}
$$

(10)
Now, we define the solution of (10) by the vector $U$, which is given by

$$U(t, x) = (v, u, z, y, \theta, q, \eta)^T.$$  

The initial condition can be written as

$$U_0(x) = U(0, x) = (v_0, u_0, z_0, y_0, \theta_0, q_0, \eta_0)^T,$$

where $u_0 = \varphi_1$, $z_0 = \psi_{0,x}$, $y_0 = \psi_1$, $v_0 = \varphi_{0,x} - \psi_0$ and $\eta_0 = \eta(0, s, \cdot)$ which is defined, as usual, in the history space $L^2_g(\mathbb{R}^+, H^1(\mathbb{R}))$, endowed with the norm

$$||\eta||^2 := \int_{\mathbb{R}} \int_0^\infty g(s)|\eta_x(s)|^2 ds dx.$$

### 2.2. The Fourier Model

Similarly to Section 2.1, we consider the Timoshenko system (1) with history and the Fourier law, i.e., when $\tau = 0$. Indeed, we can eliminate $q$ easily and obtain the following differential equation for $\theta$:

$$\rho_2 \theta_t - \tilde{\beta} \theta_{xx} + \delta \psi_{xt} = 0,$$

where $\tilde{\beta} = \beta^{-1} > 0$. Then, introducing $\eta$ as in the previous subsection, we have the differential equations

$$
\begin{cases}
\rho_1 \varphi_{tt} - k (\varphi_x - \psi)_x = 0 \\
\rho_2 \psi_{tt} - a \psi_{xx} - m \int_0^\infty g(s) \eta_{xx}(s) ds - k (\varphi - \psi) + \delta \theta_x = 0 \\
\rho_2 \theta_t - \tilde{\beta} \theta_{xx} + \delta \psi_{xt} = 0 \\
\eta_t + \eta_s - \psi_1 = 0 \\
\eta(\cdot, 0, \cdot) = 0
\end{cases}
$$

with initial data

$$\varphi(0, \cdot) = \varphi_0(\cdot), \quad \psi(0, \cdot) = \psi_0(\cdot), \quad \theta(0, \cdot) = \theta_0(\cdot), \quad \varphi_t(0, \cdot) = \varphi_1(\cdot), \quad \psi_t(0, \cdot) = \psi_1(\cdot), \quad \eta(0, s, \cdot) = \psi(0, \cdot) - \psi(-s, \cdot).$$

As in the previous section, we can rewrite the system as a first-order system, by defining the following variables

$$u = \varphi_t, \quad z = \psi_x, \quad y = \psi_1, \quad v = \varphi_x - \psi.$$

Then, (12) takes the form,

$$
\begin{cases}
v_t - u_x + y = 0, \\
\rho_1 u_t - k v_x = 0, \\
z_t - y_x = 0, \\
\rho_2 y_t - a z_x - m \int_0^\infty g(s) \eta_{xx}(s) ds - k v + \delta \theta_x = 0, \\
\rho_2 \theta_t - \tilde{\beta} \theta_{xx} + \delta \psi_{xt} = 0, \\
\eta_t + \eta_s - y = 0.
\end{cases}
$$

We define the vector solution $V$ of the system (13), as

$$V(t, x) = (v, u, z, y, \theta, \eta)^T.$$

Thus, the initial condition can be written

$$V_0(x) = V(x, 0) = (v_0, u_0, z_0, y_0, \theta_0, \eta_0)^T,$$

where $u_0 = \varphi_1$, $z_0 = \psi_{0,x}$, $y_0 = \psi_1$, $v_0 = \varphi_{0,x} - \psi_0$ and $\eta_0 = \eta(0, s, \cdot)$ defined in the history space given in the Cattaneo’s version.
3. The energy method in the frequency space

This section is devoted to showing the relationship between the rate of decay of solutions and the new condition (see [4])

\[ \chi_{0, \tau} = \left( \tau - \frac{\rho_1}{\rho_3 k} \right) \left( \rho_2 - \frac{b\rho_1}{k} \right) - \frac{\tau \rho_1 \delta^2}{\rho_3 k}. \]

For this reason, we will discuss two cases: the case where \( \chi_{0, \tau} = 0 \) and the case where \( \chi_{0, \tau} \neq 0 \). Moreover, for the Timoshenko-Fourier model, i.e., when \( \rho = 0 \), we consider the usual wave speeds propagation given by

\[ \chi_0 = \rho_2 - \frac{b\rho_1}{k}. \]

In each case, we use a delicate energy method to build appropriate Lyapunov functionals in the Fourier space.

3.1. The Timoshenko-Cattaneo Law

We consider the Fourier image of the Timoshenko-Cattaneo model with history and we show that the heat damping induced by Cattaneo law and the past history are strong enough to stabilize the whole system. Thus, taking Fourier Transform in (10), we obtain the following integro-differential system:

\begin{align*}
\hat{v}_t - i\xi \hat{u} + \hat{y} &= 0, \quad (15) \\
\rho_1 \hat{u}_t - ik\xi \hat{v} &= 0, \quad (16) \\
\hat{z}_t - i\xi \hat{y} &= 0, \quad (17) \\
\rho_2 \hat{y}_t - i\alpha \hat{\xi} + m\xi^2 \int_0^{\infty} g(s) \hat{\eta}(s) ds - k\hat{v} + i\delta \hat{\xi} \hat{\theta} &= 0, \quad (18) \\
\rho_3 \hat{\theta}_t + i\xi \hat{\varphi} + i\delta \xi \hat{\varphi} &= 0, \quad (19) \\
\tau\hat{\varphi} + \beta \hat{\varphi} + i\xi \hat{\varphi} &= 0, \quad (20) \\
\hat{\eta}_t + \hat{\eta}_s - \hat{y} &= 0. \quad (21)
\end{align*}

Here, the solution vector and initial data are given by \( \hat{U}(\xi, t) = (\hat{v}, \hat{u}, \hat{z}, \hat{\varphi}, \hat{\theta}, \hat{\varphi})^T \) and \( \hat{U}(\xi, 0) = \hat{U}_0(\xi) \), respectively. The energy functional associated to the above system is defined as:

\[ E(\xi, t) = \rho_1 |\hat{u}|^2 + \rho_2 |\hat{\varphi}|^2 + \rho_3 |\hat{\theta}|^2 + k |\hat{\varphi}|^2 + a|\hat{\xi}|^2 + \tau |\hat{\varphi}|^2 + m\xi^2 \int_0^{\infty} g(s)|\hat{\eta}(s)|^2 ds. \quad (22) \]

**Lemma 3.1.** The energy (22) satisfies the following estimate:

\[ \frac{d}{dt} E(\xi, t) \leq -2\beta |\hat{\varphi}|^2 - k_1 m\xi^2 \int_0^{\infty} g(s)|\hat{\varphi}(s)|^2 ds, \quad (23) \]

where the constant \( k_1 > 0 \) is given by (H2).

**Proof.** Multiplying (15) by \( k\xi \), (16) by \( \hat{u} \), (17) by \( a\xi \), (18) by \( \hat{\varphi} \), (19) by \( \hat{\theta} \) and (20) by \( \hat{\varphi} \), adding and taking real part, it follows that

\[ \frac{1}{2} \frac{d}{dt} \left( \rho_1 |\hat{u}|^2 + \rho_2 |\hat{\varphi}|^2 + \rho_3 |\hat{\theta}|^2 + k |\hat{\varphi}|^2 + a|\hat{\xi}|^2 + \tau |\hat{\varphi}|^2 \right) = -\beta |\hat{\varphi}|^2 - \text{Re} \left( m\xi^2 \int_0^{\infty} g(s)\hat{\eta}(s)\overline{\hat{\varphi}} ds \right). \quad (24) \]

On the other hand, taking the conjugate of equation (21), multiplying the resulting equation by \( g(s)\hat{\eta}(t, s, x) \) and integrating with respect to \( s \), we obtain

\[ \text{Re} \left( \int_0^{\infty} g(s)\overline{\hat{\eta}(s)} \hat{\varphi}(s) ds \right) = \frac{1}{2} \frac{d}{dt} \int_0^{\infty} g(s)|\hat{\varphi}(s)|^2 ds + \frac{1}{2} \int_0^{\infty} g(s) \frac{d}{ds} |\hat{\eta}(s)|^2 ds. \quad (25) \]
Integrating by parts the last term in the left hand side of (25), we have

\[ \text{Re} \left( \int_0^\infty g(s)\dot{\eta}(s)\overline{\theta}\,ds \right) = \frac{1}{2} \frac{d}{dt} \int_0^\infty g(s)|\dot{\eta}(s)|^2\,ds - \frac{1}{2} \int_0^\infty g'(s)|\dot{\eta}(s)|^2\,ds. \]

Plugging the above equation in (24), it follows that

\[ \frac{d}{dt} \dot{E}(\xi, t) = -2\beta|\dot{q}|^2 + m\xi^2 \int_0^\infty g'(s)|\dot{\eta}(s)|^2\,ds. \]

Using \((H_2)\), we obtain (23). \(\square\)

With this energy dissipation in hands, the following questions arise:

*Does \(\dot{E}(t) \to 0\) as \(t \to \infty\)? If it is the case, can we find the decay rate of \(\dot{E}(t)\)?*

The following Theorem provides a positive answer establishing the exponential decay of the integro-differential system (15)-(21). This result is a fundamental ingredient in the proof of our main results.

**Theorem 3.2.** Let

\[ \chi_{0, \tau} = \left( \tau - \frac{\rho_1}{\rho_3k} \right) \left( \rho_2 - \frac{b\rho_1}{k} \right) - \frac{\tau\rho_1\delta^2}{\rho_3k}. \]

Then, for any \(t \geq 0\) and \(\xi \in \mathbb{R}\), the energy of system (15)-(21) satisfies

\[ \dot{E}(\xi, t) \leq Ce^{-\lambda\tau(\xi)} \dot{E}(0, \xi), \]

where \(C, \lambda\) are positive constants and the function \(\rho(\cdot)\) is given by

\[ \rho(\xi) = \begin{cases} \frac{\xi^4}{(1 + \xi^2)^4} & \text{if } \chi_{0, \tau} = 0, \\ \frac{\xi^4}{(1 + \xi^2)^2} & \text{if } \chi_{0, \tau} \neq 0. \end{cases} \]

Following the ideas contain in \([29]\), we construct some functionals to capture the dissipation of all the components of the vector solution. These functionals allow us to build an appropriate Lyapunov functional equivalent to the energy. The proof of Theorem 3.2 is based on the following lemmas:

**Lemma 3.3.** Consider the functional

\[ J_1(\xi, t) = -\tau\rho_2 \text{Re}(\bar{v}\overline{\theta}u) - \frac{\alpha\tau\rho_1}{k} \text{Re}(\bar{v}\overline{\theta}u) + \delta\rho_1 \left( \tau + \frac{1}{\delta^2} \left( \rho_2 - \frac{b\rho_1}{k} + \tau\rho_0\rho_3 \right) \right) \text{Re}(\bar{v}\overline{\theta}u) \]

\[ - \frac{\tau}{\delta} \left( \rho_2 - \frac{b\rho_1}{k} + \tau\rho_0\rho_3 \right) \text{Re}(\bar{v}\overline{\theta}u) \]

Then, for any \(\varepsilon > 0\), \(J_1\) satisfies

\[ \frac{d}{dt} J_1(\xi, t) + \tau k(1 - \varepsilon)|\bar{v}|^2 \leq \tau\rho_2|\dot{q}|^2 + C(\varepsilon)\xi^4 \int_0^\infty g(s)|\dot{\eta}(s)|^2\,ds \]

\[ + \chi_{0, \tau} \text{Re}(i\xi\overline{\theta}u) + \frac{1}{\delta} \left( \chi_{0, \tau} + \tau\rho_0\rho_3 \left( \tau - \frac{\rho_1}{\rho_3k} \right) \right) \text{Re}(i\xi\overline{\theta}u) \]

\[ + \frac{\tau}{\delta} \left( \rho_2 - \frac{b\rho_1}{k} + \tau\rho_0\rho_3 \right) \text{Re}(g\overline{\theta}) + C(\varepsilon)|\dot{\eta}|^2. \]

where \(C(\varepsilon)\) is a positive constant and \(\chi_{0, \tau}\) is given by (29).
Proof. Multiplying (15) by $-\rho_2 \hat{y}$ and taking real part, we obtain
\[-\rho_2 \text{Re} (\hat{v} \hat{y}) + \rho_2 \text{Re} (i \xi \hat{u} \hat{y}) - \rho_2 |\hat{y}|^2 = 0.\]

Multiplying (15) by $-\hat{u}$ and taking real part, it follows that
\[-\rho_2 \text{Re} (\hat{q} \hat{v}) + a \text{Re} (i \xi \hat{z} \hat{v}) + |\hat{v}|^2 - \text{Re} \left( m \xi^2 \hat{v} \int_0^\infty g(s) \hat{\eta}(s) ds \right) - \delta \text{Re} (i \xi \hat{\theta} \hat{v}) = 0.\]

Adding the above identities,
\[-\rho_2 \frac{d}{dt} \text{Re} (\hat{v} \hat{y}) + |\hat{v}|^2 - \rho_2 |\hat{y}|^2 - a \text{Re} (i \xi \hat{z} \hat{v}) + \text{Re} \left( m \xi^2 \hat{v} \int_0^\infty g(s) \hat{\eta}(s) ds \right) - \rho_2 \text{Re} (i \xi \hat{u} \hat{y}) + \delta \text{Re} (i \xi \hat{\theta} \hat{v}). \]

On the other hand, multiplying (16) by $-\hat{u}$, (17) by $-\frac{a \rho_1 \hat{y}}{k}$, adding the results and taking real part, it follows that
\[-\frac{a \rho_1}{k} \frac{d}{dt} \text{Re} (\hat{z} \hat{u}) = -\frac{a \rho_1}{k} \text{Re} (i \xi \hat{y} \hat{u}) - a \text{Re} (i \xi \hat{v} \hat{z}). \]

Moreover, multiplying (16) by $\frac{\delta \rho_1}{k}$ and taking real part,
\[
\frac{\delta \rho_1}{k} \text{Re} (\hat{u} \hat{\theta}) - \delta \text{Re} (i \xi \hat{v} \hat{\theta}) = 0.
\]

Next, multiplying (19) by $\frac{\delta \rho_1}{\rho_3 k}$ and taking real part,
\[
\frac{\delta \rho_1}{k} \text{Re} (\hat{u} \hat{\theta}) + \frac{\delta \rho_1}{\rho_3 k} \text{Re} (i \xi \hat{q} \hat{u}) + \frac{\delta^2 \rho_1}{\rho_3 k} \text{Re} (i \xi \hat{y} \hat{u}) = 0.
\]

Adding the above identities, we obtain
\[
\frac{\delta \rho_1}{k} \frac{d}{dt} \text{Re} (\hat{u} \hat{\theta}) = -\frac{\delta \rho_1}{\rho_3 k} \text{Re} (i \xi \hat{q} \hat{u}) - \frac{\delta^2 \rho_1}{\rho_3 k} \text{Re} (i \xi \hat{y} \hat{u}) + \delta \text{Re} (i \xi \hat{v} \hat{\theta}). \]

Furthermore, multiplying (19) by $-\hat{v}$ and taking real part,
\[-\tau \text{Re} (\hat{q} \hat{v}) - \beta \text{Re} (i \xi \hat{q} \hat{v}) - \text{Re} (i \xi \hat{\theta} \hat{v}) = 0.\]

Multiplying (15) by $-\hat{q}$ and taking real part,
\[-\tau \text{Re} (\hat{v} \hat{q}) + \tau \text{Re} (i \xi \hat{u} \hat{q}) - \tau \text{Re} (\hat{y} \hat{q}) = 0.\]

Adding the above identities, it follows that
\[-\tau \frac{d}{dt} \text{Re} (\hat{y} \hat{q}) = -\tau \text{Re} (i \xi \hat{u} \hat{q}) + \tau \text{Re} (\hat{y} \hat{q}) + \beta \text{Re} (i \xi \hat{q} \hat{v}) + \text{Re} (i \xi \hat{\theta} \hat{v}).\]

Now, computing (30) + (31) + (32), we have
\[
\frac{d}{dt} \left\{-\rho_2 \text{Re} (\hat{v} \hat{y}) - \frac{a \rho_1}{k} \text{Re} (\hat{z} \hat{u}) + \frac{\delta \rho_1}{k} \text{Re} (\hat{u} \hat{\theta}) \right\} + k|\hat{v}|^2 = \rho_2 |\hat{y}|^2 + \text{Re} \left( m \xi^2 \hat{v} \int_0^\infty g(s) \hat{\eta}(s) ds \right) + \left( \rho_2 - \frac{a \rho_1}{k} - \frac{\delta^2 \rho_1}{\rho_3 k} \right) \text{Re} (i \xi \hat{u} \hat{y}) - \frac{\delta \rho_1}{\rho_3 k} \text{Re} (i \xi \hat{y} \hat{u}) \right\}. \tag{33}
\]

Computing $\tau (30) + \tau (31) + \left( \tau + \frac{1}{\delta^2} \left( \rho_2 - \frac{b \rho_1}{k} + \tau b_0 \rho_3 \right) \right)(32)$, we find that
\[
\frac{d}{dt} \left\{-\tau \rho_2 \text{Re} (\hat{v} \hat{y}) - \frac{a \tau \rho_1}{k} \text{Re} (\hat{z} \hat{u}) + \frac{\Gamma \delta \rho_1}{k} \text{Re} (\hat{u} \hat{\theta}) \right\} + \tau k|\hat{v}|^2 + \tau m \text{Re} \left( \xi^2 \hat{v} \int_0^\infty g(s) \hat{\eta}(s) ds \right) + \left( \tau \left( \rho_2 - \frac{a \rho_1}{k} \right) - \frac{\delta \rho_1}{\rho_3 k} \Gamma \right) \text{Re} (i \xi \hat{u} \hat{y}) + \tau \delta \text{Re} (i \xi \hat{v} \hat{\theta}) - \frac{\delta \rho_1}{\rho_3 k} \Gamma \text{Re} (i \xi \hat{y} \hat{u}) + \Gamma \delta \text{Re} (i \xi \hat{\theta} \hat{v}). \tag{34}
\]
Hence,

\[
\frac{d}{dt} \left\{ -\tau \rho_2 \Re (\hat{v} \hat{y}) - \frac{\alpha \rho_1}{k} \Re (\hat{z} \hat{u}) + \frac{\Gamma \delta \rho_1}{k} \Re (\hat{b} \hat{u}) + \tau k |\hat{v}|^2 = \tau \rho_2 |\hat{y}|^2 + \tau m \Re \left( \chi_0 \hat{y} \right) + \frac{k \rho_1}{k} - \frac{\delta^2 \rho_1}{\rho_3 k} \Re (\hat{\xi} \hat{y}) + \delta (\tau - \Gamma) \Re (\hat{\xi} \hat{y}) \right\} + \tau k |\hat{v}|^2
\]

Multiplying (33) by \( \Gamma \) and adding the result to (35), it follows that

\[
\frac{d}{dt} \left\{ -\tau \rho_2 \Re (\hat{v} \hat{y}) - \frac{\alpha \rho_1}{k} \Re (\hat{z} \hat{u}) + \frac{\Gamma \delta \rho_1}{k} \Re (\hat{b} \hat{u}) \right\} + \tau k |\hat{v}|^2
\]

Adding the above identities, we obtain

\[
\frac{d}{dt} J_1(\xi, t) + \tau k |\hat{v}|^2 = \tau \rho_2 |\hat{y}|^2 + \tau m \Re \left( \chi_0 \hat{y} \right) + \left( -\delta \tau (\tau - \Gamma) - \frac{\delta \rho_1 \Gamma}{\rho_3 k} \right) \Re (\hat{\xi} \hat{y})
\]

Applying Young inequality, (29) follows.

**Lemma 3.4.** Consider the functional

\[
J_2(\xi, t) = \rho_1 \Re (\hat{\xi} \hat{v} \hat{u}) + \rho_2 \Re (\hat{\xi} \hat{y} \hat{z}) + \delta \tau \Re (\hat{\xi} \hat{y} \hat{q})
\]

For any \( \varepsilon > 0 \), the estimate

\[
\frac{d}{dt} J_2(\xi, t) + \rho_1 (1 - \varepsilon) \xi^2 |\hat{u}|^2 + a(1 - \varepsilon) \xi^2 |\hat{z}|^2 \leq C(\varepsilon)(1 + \xi^2) |\hat{v}|^2 + C(\varepsilon)(1 + \xi^2) |\hat{y}|^2
\]

\[
+ C(\varepsilon)(1 + \xi^2) |\hat{q}|^2 + C(\varepsilon) \xi^4 \int_0^\infty g(s) \hat{\eta}(s) ds \right|^2
\]

is satisfied.

**Proof.** Multiplying (15) by \( i \rho_1 \hat{u} \) and taking real part,

\[
\rho_1 \Re (\hat{\xi} \hat{v} \hat{u}) + \rho_1 \xi^2 |\hat{u}|^2 + \rho_1 \Re (\hat{\xi} \hat{y} \hat{u}) = 0.
\]

Multiplying (16) by \( -i \hat{v} \) and taking real part,

\[
-\rho_1 \Re (\hat{\xi} \hat{u} \hat{v}) - k \xi^2 |\hat{v}|^2 = 0.
\]

Adding the above identities, we obtain

\[
\rho_1 \frac{d}{dt} \Re (\hat{\xi} \hat{v} \hat{u}) + \rho_1 \xi^2 |\hat{u}|^2 = k \xi^2 |\hat{v}|^2 - \rho_1 \Re (\hat{\xi} \hat{y} \hat{u})
\]

Moreover, multiplying (17) by \( -i \rho_2 \hat{y} \) and taking real part,

\[
-\rho_2 \Re (\hat{\xi} \hat{z} \hat{y}) - \rho_2 \xi^2 |\hat{y}|^2 = 0.
\]
Multiplying (18) by $i\xi \overline{z}$ and taking real part,
\[
\rho_2 \text{Re} \left( i\xi \hat{y} \overline{z} \right) + a\xi^2 |\hat{z}|^2 - k \text{Re} \left( i\xi \hat{v} \overline{z} \right) + m \text{Re} \left( i\xi^3 \overline{z} \int_0^\infty g(s)\hat{\eta}(s) ds \right) - \delta \text{Re} \left( \xi^2 \overline{\theta} \right) = 0.
\]
Adding the above identities,
\[
\rho_2 \frac{d}{dt} \text{Re} \left( i\xi \hat{y} \overline{z} \right) + a\xi^2 |\hat{z}|^2 = \rho_2 \xi^2 |\hat{y}|^2 + k \text{Re} \left( i\xi \hat{v} \overline{z} \right) - m \text{Re} \left( i\xi^3 \overline{z} \int_0^\infty g(s)\hat{\eta}(s) ds \right) + \delta \text{Re} \left( \xi^2 \overline{\theta} \right).
\]
(39)

Multiplying (20) by $-i\xi \overline{z}$ and taking real part,
\[-\delta \text{Re}(i\xi \hat{q} \overline{z}) - \beta \delta \text{Re}(i\xi \hat{q} \overline{z}) + \delta \text{Re}(\xi^2 \overline{\theta}) = 0.
\]

Multiplying (17) by $i\delta \tau \hat{q}$ and taking real part,
\[
\delta \tau \text{Re}(i\xi \hat{z} \hat{q}) + \delta \tau \text{Re}(\xi^2 \hat{q} \hat{q}) = 0.
\]

Adding the above identities, we obtain
\[
\delta \tau \frac{d}{dt} \text{Re}(i\xi \hat{q} \overline{z}) = -\delta \tau \text{Re}(\xi^2 \hat{q} \hat{q}) + \beta \delta \text{Re}(i\xi \hat{q} \overline{z}) - \delta \text{Re}(\xi^2 \overline{\theta}).
\]
(40)

Computing (38) + (39) + (40), we find that
\[
\frac{d}{dt} \left\{ \rho_1 \text{Re} \left( i\xi \hat{v} \overline{u} \right) + \rho_2 \text{Re} \left( i\xi \hat{y} \overline{z} \right) + \delta \tau \text{Re}(i\xi \hat{q} \overline{z}) \right\} + a\xi^2 |\hat{z}|^2 + \rho_1 \xi^2 |\hat{u}|^2 = \rho_2 \xi^2 |\hat{y}|^2 + k \text{Re} \left( i\xi \hat{v} \overline{z} \right) - m \text{Re} \left( i\xi^3 \overline{z} \int_0^\infty g(s)\hat{\eta}(s) ds \right) - \delta \tau \text{Re}(\xi^2 \hat{q} \hat{q}) + \beta \delta \text{Re}(i\xi \hat{q} \overline{z}) + \delta \tau \text{Re}(\xi^2 \overline{\theta}) - \rho_1 \text{Re} \left( i\xi \hat{v} \overline{u} \right).
\]

Hence,
\[
\frac{d}{dt} J_2(\xi, t) + a\xi^2 |\hat{z}|^2 + \rho_1 \xi^2 |\hat{u}|^2 \leq \rho_2 \xi^2 |\hat{y}|^2 + k \|\xi\| |\hat{v}| |\hat{z}|
\]
\[
+ m |\xi|^3 |\hat{z}| \int_0^\infty g(s)\hat{\eta}(s) ds + \delta \tau \xi^2 |\hat{y}| |\hat{q}| + \beta \delta |\xi| |\hat{q}| |\hat{z}| + k \xi^2 |\hat{v}|^2 + \rho_1 |\xi| |\hat{y}| |\hat{u}|
\]
applying Young’s inequality, we obtain (37).

**Lemma 3.5.** Consider the functional
\[
J_3(\xi, t) = -\rho_2 \text{Re} \left( \xi^2 \overline{y} \int_0^\infty g(s)\hat{\eta}(s) ds \right).
\]
Then, for any $\varepsilon > 0$, the following estimate
\[
\frac{d}{dt} J_3(\xi, t) + \rho_2 b_0(1 - \varepsilon)\xi^2 |\hat{y}|^2 \leq C(\varepsilon)\xi^2 \int_0^\infty g(s) |\hat{\eta}(s)|^2 ds + a |\xi|^3 |\hat{z}| \int_0^\infty g(s)\hat{\eta}(s) ds
\]
\[
+ m |\xi|^4 \int_0^\infty g(s)\hat{\eta}(s) ds + k \xi^2 |\hat{v}| \int_0^\infty g(s)\hat{\eta}(s) ds + \delta |\xi|^3 |\hat{\theta}| \int_0^\infty g(s)\hat{\eta}(s) ds
\]
(41)
holds.

**Proof.** Multiplying (18) by $-\xi^2 g(s) \overline{\eta}$ and taking the integration with respect to $s$ for the real parts, it follows that
\[
-\rho_2 \text{Re} \left( \xi^2 \hat{y} \int_0^\infty g(s)\overline{\eta}(s) ds \right) + a \text{Re} \left( i\xi^3 \hat{z} \int_0^\infty g(s)\overline{\eta}(s) ds \right)
\]
\[
- m \text{Re} \left( \xi^4 \int_0^\infty g(s)\hat{\eta}(s) ds \int_0^\infty g(s)\overline{\eta}(s) ds \right) + k \text{Re} \left( \xi^2 \hat{v} \int_0^\infty g(s)\overline{\eta}(s) ds \right)
\]
\[
- \delta \text{Re} \left( i\xi^3 \hat{\theta} \int_0^\infty g(s)\overline{\eta}(s) ds \right) = 0.
\]
Multiplying (21) by $-\rho_2 \xi^2 g(s)\overline{y}$ and taking the integration with respect to $s$ for the real parts, it follows that

$$-\rho_2 \text{Re} \left( \xi^2 \overline{y} \int_0^\infty g(s)\overline{\eta}_s(s) \, ds \right) - \rho_2 \text{Re} \left( \xi^2 \overline{y} \int_0^\infty g(s)\overline{\eta}_s(s) \, ds \right) + \rho_2 \xi^2 \int_0^\infty g(s) \, ds |\overline{y}|^2 = 0.$$  

Adding the above identities, we obtain that

$$\frac{d}{dt} K_3(\xi, t) + \rho_2 \xi^2 b_0 |\overline{y}|^2 = \rho_2 \text{Re} \left( \xi^2 \overline{y} \int_0^\infty g(s)\overline{\eta}_s(s) \, ds \right) - a \text{Re} \left( i\xi^3 \dot{z} \int_0^\infty g(s)\overline{\eta}(s) \, ds \right) + \text{Re} \left( i\xi^3 \dot{\theta} \int_0^\infty g(s)\overline{\eta}(s) \, ds \right).$$

Now, integrating by parts the first term on the right-hand side of the above equality, we get

$$\frac{d}{dt} K_3(\xi, t) + \rho_2 \xi^2 b_0 |\overline{y}|^2 = -\rho_2 \text{Re} \left( \xi^2 \overline{y} \int_0^\infty g'(s)\overline{\eta}_s(s) \, ds \right) - a \text{Re} \left( i\xi^3 \dot{z} \int_0^\infty g(s)\overline{\eta}(s) \, ds \right) + \text{Re} \left( i\xi^3 \dot{\theta} \int_0^\infty g(s)\overline{\eta}(s) \, ds \right).$$

Hence,

$$\frac{d}{dt} K_3(\xi, t) + \rho_2 \xi^2 b_0 |\overline{y}|^2 \leq \rho_2 \xi^2 |\overline{y}| \left| \int_0^\infty g'(s)\overline{\eta}_s(s) \, ds \right| + a |\xi^3| \left| \int_0^\infty g(s)\overline{\eta}(s) \, ds \right|$$

$$+ m \xi^4 \left| \int_0^\infty g(s)\overline{\eta}_s(s) \, ds \right|^2 + k \xi^2 |\nabla| \left| \int_0^\infty g(s)\overline{\eta}(s) \, ds \right| + \delta |\xi|^3 \left| \int_0^\infty g(s)\overline{\eta}(s) \, ds \right|.$$  

Young inequality yields

$$\frac{d}{dt} K_3(\xi, t) + \rho_2 \xi^2 b_0 (1 - \varepsilon) |\overline{y}|^2 \leq C(\varepsilon) \xi^2 \left| g(\xi) |\overline{\eta}(s)|^2 ds + a |\xi^3| \left| \int_0^\infty g(s)\overline{\eta}(s) \, ds \right|$$

$$+ m \xi^4 \left| \int_0^\infty g(s)\overline{\eta}_s(s) \, ds \right|^2 + k \xi^2 |\nabla| \left| \int_0^\infty g(s)\overline{\eta}(s) \, ds \right| + \delta |\xi|^3 \left| \int_0^\infty g(s)\overline{\eta}(s) \, ds \right|.$$  

Using the hypothesis on $g'$, the result follows.  

**Lemma 3.6.** Consider the functional

$$J_4(\xi, t) = \tau \rho_3 \text{Re} \left( i\xi \dot{\theta} \overline{\eta} \right).$$  

(42)

For any $\varepsilon > 0$, the following estimate

$$\frac{d}{dt} J_4(\xi, t) + \rho_3 (1 - \varepsilon) \xi^2 |\dot{\theta}|^2 \leq \tau \delta \xi^2 |\overline{y}| |\dot{\eta}| + C(\varepsilon)(1 + \xi^2)|\dot{\eta}|$$

(43)

holds.

**Proof.** Multiplying (33) by $i\tau \xi \overline{\eta}$ and taking real part,

$$\tau \rho_3 \text{Re} \left( i\xi \dot{\theta} \overline{\eta} \right) - \tau \xi^2 |\dot{\eta}|^2 - \tau \delta \text{Re} \left( \xi^2 \overline{y} \right) = 0.$$  

Multiplying (34) by $-i\rho_3 \dot{\theta}$ and taking real part,

$$-\tau \rho_3 \text{Re} \left( i\xi \dot{\theta} \overline{\eta} \right) - \beta \rho_3 \text{Re} \left( i\xi \dot{y} \overline{\eta} \right) + \nu \xi^2 |\dot{\theta}|^2 = 0.$$  

Adding up the above identities, it follows that

$$\frac{d}{dt} J_4(\xi, t) + \rho_3 \xi^2 |\dot{\theta}|^2 \leq \tau \xi^2 |\overline{y}|^2 + \nu \delta \xi^2 |\dot{y}| |\dot{\eta}| + \beta \rho_3 |\xi||\dot{\eta}||\dot{\theta}|.$$  

Applying Young’s inequality, the result follows.  

Proof of Theorem 3.2: In order to make the proof clear, we will consider several cases:

I. Case $\chi_0, \tau = 0$: Consider the expressions

$$\lambda_1 \xi^2 J_1(\xi, t), \quad \lambda_2 \frac{\xi^2}{1 + \xi^2} J_2(\xi, t), \quad \lambda_3 J_3(\xi, t)$$

where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are positive constants to be fixed later. Thus, Lemmas 3.3 and 3.4 imply that

$$\frac{d}{dt} \left\{ \lambda_1 \xi^2 J_1(\xi, t) + \lambda_2 \frac{\xi^2}{1 + \xi^2} J_2(\xi, t) \right\} + k [\lambda_1 \tau (1 - \varepsilon) - C(\varepsilon) \lambda_2] \xi^2 |\hat{u}|^2 + \rho_1 (1 - 2\varepsilon) \lambda_2 \frac{\xi^4}{1 + \xi^2} |\hat{\varepsilon}|^2$$

$$+ a(1 - \varepsilon) \lambda_2 \frac{\xi^4}{1 + \xi^2} |\hat{\varepsilon}|^2$$

$$\leq C(\varepsilon, \lambda_1, \lambda_2) \xi^2 |\hat{u}|^2 + C(\varepsilon, \lambda_1, \lambda_2) \xi^6 \int_0^\infty g(s) \eta(s) ds^2 + C(\varepsilon, \lambda_1, \lambda_2) \xi^2 (1 + \xi^2) |\hat{\eta}|^2. \quad (44)$$

Furthermore, applying Young’s inequality in equation (44) of Lemma 3.5, it follows that

$$\frac{d}{dt} \lambda_3 J_3(\xi, t) + \rho_2 \lambda_3 b_0 (1 - \varepsilon) \xi^2 |\hat{u}|^2 \leq C(\varepsilon, \lambda_3) \xi^2 \int_0^\infty g(s) |\hat{\eta}(s)| ds^2 + a \lambda_2 \varepsilon \frac{\xi^4}{1 + \xi^2} |\hat{\varepsilon}|^2$$

$$+ C(\varepsilon, \lambda_1, \lambda_2) \xi^2 (1 + \xi^2) \left| \int_0^\infty g(s) \eta(s) ds \right|^2 + \lambda_3 m \xi^4 \left| \int_0^\infty g(s) \eta(s) ds \right| + k \lambda_1 \tau \xi^2 |\hat{\varepsilon}|^2$$

$$+ C(\varepsilon, \lambda_1, \lambda_3) \xi^2 \left| \int_0^\infty g(s) \eta(s) ds \right|^2 + \lambda_3 \delta |\hat{\eta}|^2 \left| \int_0^\infty g(s) \eta(s) ds \right|.$$

Thus,

$$\frac{d}{dt} \lambda_3 J_3(\xi, t) + \rho_2 \lambda_3 b_0 (1 - \varepsilon) \xi^2 |\hat{u}|^2 \leq C(\varepsilon, \lambda_3) \xi^2 \int_0^\infty g(s) |\hat{\eta}(s)| ds^2 + a \lambda_2 \varepsilon \frac{\xi^4}{1 + \xi^2} |\hat{\varepsilon}|^2 + k \lambda_1 \tau \xi^2 |\hat{\varepsilon}|^2$$

$$+ C(\varepsilon, \lambda_1, \lambda_2, \lambda_3) \xi^2 (1 + \xi^2) \left| \int_0^\infty g(s) \eta(s) ds \right|^2 + \lambda_3 \delta |\hat{\eta}|^2 \left| \int_0^\infty g(s) \eta(s) ds \right|. \quad (45)$$

Computing (44) + (45), we obtain

$$\frac{d}{dt} \left\{ \lambda_1 \xi^2 J_1(\xi, t) + \lambda_2 \frac{\xi^2}{1 + \xi^2} J_2(\xi, t) + \lambda_3 J_3(\xi, t) \right\} + k [\lambda_1 \tau (1 - 2\varepsilon) - C(\varepsilon) \lambda_2] \xi^2 |\hat{u}|^2 + \rho_1 (1 - 2\varepsilon) \lambda_2 \frac{\xi^4}{1 + \xi^2} |\hat{\varepsilon}|^2$$

$$+ a(1 - 2\varepsilon) \lambda_2 \frac{\xi^4}{1 + \xi^2} |\hat{\varepsilon}|^2 + \rho_2 \lambda_3 b_0 (1 - \varepsilon) - C(\varepsilon, \lambda_1, \lambda_2) \xi^2 |\hat{\eta}|^2$$

$$\leq C(\varepsilon, \lambda_3) \xi^2 \int_0^\infty g(s) |\hat{\eta}(s)| ds^2 + C(\varepsilon, \lambda_1, \lambda_2, \lambda_3) \xi^2 (1 + \xi^2) \left| \int_0^\infty g(s) \eta(s) ds \right|^2$$

$$+ C(\varepsilon, \lambda_1, \lambda_2, \lambda_3) \xi^2 (1 + \xi^2) |\hat{\eta}|^2 + \lambda_3 \delta |\hat{\eta}|^2 \left| \int_0^\infty g(s) \eta(s) ds \right|.$$

Now, consider the following functional

$$\mathcal{L}_1(\xi, t) = \lambda_1 \xi^2 J_1(\xi, t) + \lambda_2 \frac{\xi^2}{1 + \xi^2} J_2(\xi, t) + \lambda_3 J_3(\xi, t) + J_4(\xi, t).$$
Finally, we define the following Lyapunov functional:

$$
\frac{d}{dt} \mathcal{L}_1(\xi, t) + k [\lambda_1 \tau(1 - 2\varepsilon) - C(\varepsilon) \lambda_2] \xi^2 |\dot{\xi}|^2 + \rho_1 (1 - 2\varepsilon) \lambda_2 \frac{\xi^4}{1 + \xi^2} |\dot{\xi}|^2 \\
+ a(1 - 2\varepsilon) \lambda_2 \frac{\xi^4}{1 + \xi^2} |\dot{\xi}|^2 + \rho_2 [3 \lambda_0 (1 - 2\varepsilon) - C(\varepsilon, \lambda_1, \lambda_2)] \xi^2 |\dot{\theta}|^2 + \rho_3 (1 - 2\varepsilon) \xi^2 |\dot{\theta}|^2
\leq C(\varepsilon, \lambda_3) \xi^2 \int_0^\infty g(s) |\dot{\eta}(s)|^2 ds + C(\varepsilon, \lambda_1, \lambda_2, \lambda_3) (1 + \xi^2)^2 |\dot{\eta}|^2.
$$

Now, using the following inequality:

$$
\left| \int_0^\infty g(s) \dot{\eta}(s) ds \right|^2 = \left| \int_0^\infty \dot{g}(s) g \dot{\eta}(s) ds \right|^2 \\
\leq \left( \int_0^\infty g(s) ds \right)^{\frac{1}{2}} \left( \int_0^\infty \dot{g}(s) |\dot{\eta}(s)|^2 ds \right)^{\frac{1}{2}} \\
= b_0 \int_0^\infty g(s) |\dot{\eta}(s)|^2 ds,
$$

we obtain that

$$
\frac{d}{dt} \mathcal{L}_1(\xi, t) + k [\lambda_1 \tau(1 - 2\varepsilon) - C(\varepsilon) \lambda_2] \xi^2 |\dot{\xi}|^2 + \rho_1 (1 - 2\varepsilon) \lambda_2 \frac{\xi^4}{1 + \xi^2} |\dot{\xi}|^2 \\
+ a(1 - 2\varepsilon) \lambda_2 \frac{\xi^4}{1 + \xi^2} |\dot{\xi}|^2 + \rho_2 [3 \lambda_0 (1 - 2\varepsilon) - C(\varepsilon, \lambda_1, \lambda_2)] \xi^2 |\dot{\theta}|^2 + \rho_3 (1 - 2\varepsilon) \xi^2 |\dot{\theta}|^2
\leq C_1 (1 + b_0) \xi^2 (1 + \xi^2)^2 \int_0^\infty g(s) |\dot{\eta}(s)|^2 ds + C_1 (1 + \xi^2)^2 |\dot{\eta}|^2,
$$

where $C_1$ is a positive constant that depends on $\varepsilon$ and $\lambda_j$ for $j = 1, 2, 3$. Then, we can choose the constants to make all the coefficients in the right side in (47) positive. First, let us fix $\varepsilon$ such that $\varepsilon < \frac{1}{2}$. Thus, we can take first choose $\lambda_2 > 0$ and

$$
\lambda_1 > \frac{C(\varepsilon) \lambda_2}{\tau(1 - 2\varepsilon)}, \quad \lambda_3 > \frac{C(\varepsilon, \lambda_1, \lambda_2)}{b_0 (1 - 2\varepsilon)}.
$$

Then, from (48) and some trivial inequalities such as

$$
\frac{\xi^2}{1 + \xi^2} \leq 1 \quad \text{and} \quad \frac{1}{1 + \xi^2} \leq 1,
$$

we can deduce the existence of a positive constant $M_1$ such that

$$
\frac{d}{dt} \mathcal{L}_1(\xi, t) \leq -M_1 \frac{\xi^4}{1 + \xi^2} \left\{ k |\dot{\xi}|^2 + \rho_1 |\dot{\xi}|^2 + a |\dot{\xi}|^2 + \rho_3 |\dot{\theta}|^2 + \rho_2 |\dot{\theta}|^2 \right\} \\
+ C_1 (1 + b_0) \xi^2 (1 + \xi^2)^2 \int_0^\infty g(s) |\dot{\eta}(s)|^2 ds + C_1 (1 + \xi^2)^2 |\dot{\eta}|^2.
$$

Finally, we define the following Lyapunov functional:

$$
\mathcal{L}(\xi, t) = \mathcal{L}_1(\xi, t) + N (1 + \xi^2)^2 \dot{\mathcal{E}}(\xi, t),
$$

where $N$ is a positive constant to be fixed later. Note that the definition of $\mathcal{L}_1$ together with inequality (40), imply that

$$
|\mathcal{L}_1(\xi, t)| \leq M_2 \{ |J_1(\xi, t)| + |J_2(\xi, t)| + |J_3(\xi, t)| + |J_4(\xi, t)| \} \\
\leq M_2 (1 + \xi^2)^2 \mathcal{E}(\xi, t).
$$
Hence, we obtain
\[(N - M_2)(1 + \xi^2)\hat{E}(\xi, t) \leq \mathcal{L}(\xi, t) \leq (N + M_2)(1 + \xi^2)\hat{E}(\xi, t).\] (52)

On the other hand, taking the derivative of \( \mathcal{L} \) with respect to \( t \) and using the estimates (50) and Lemma 3.1, it follows that
\[
\frac{d}{dt} \mathcal{L}(\xi, t) \leq -M_1 \frac{\xi^4}{1 + \xi^2} \left\{ k|\dot{v}|^2 + \rho_1 |\dot{u}|^2 + a|\dot{z}|^2 + \rho_3 |\dot{\theta}|^2 + \rho_2 |\dot{y}|^2 \right\} 
- (2N\beta - C_1) (1 + \xi^2)|\dot{y}|^2 
- (k_1 N m - C_1 (1 + b_0)) (1 + \xi^2) \xi^2 \int_0^\infty g(s)|\dot{y}(s)|^2 ds.
\]

Now, choosing \( N \) such that \( N \geq \max \left\{ M_2, C_1 \frac{C_1 (1 + b_0)}{k_1 m} \right\} \) and using the inequality \((1 + \xi^2)^2 \geq \frac{\xi^4}{1 + \xi^2}\), there exists a positive constant \( M_3 \) such that
\[
\frac{d}{dt} \mathcal{L}(\xi, t) \leq -M_3 \frac{\xi^4}{1 + \xi^2} \hat{E}(\xi, t).
\]

Estimate (52) implies that
\[
\frac{d}{dt} \mathcal{L}(\xi, t) \leq -\Gamma \frac{\xi^4}{(1 + \xi^2)^3} \mathcal{L}(\xi, t)
\]
where \( \Gamma = \frac{M_3}{N + M_2} \). By Gronwall’s inequality, it follows that
\[
\mathcal{L}(\xi, t) \leq e^{-\Gamma \rho(\xi)t} \mathcal{L}(\xi, 0), \quad \rho(\xi) = \frac{\xi^4}{(1 + \xi^2)^3}.
\]

Again by using (52), we have that
\[
\dot{E}(\xi, t) \leq C e^{-\Gamma \rho(\xi)t} \hat{E}(\xi, 0), \quad \text{where} \quad C = \frac{N + M_2}{N - M_2} > 0.
\]

**II. Case \( \chi_{0, \tau} \neq 0 \):** Similar to previous case, we introduce positive constants \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \) that will be fixed later. Next, we estimate the following terms by applying Young’s inequality,
\[
\left| \chi_{0, \tau} \text{Re}(i\xi \dot{\bar{u}}) \right| \leq \frac{\rho_1 \gamma_2 \varepsilon}{2\gamma_1} \frac{\xi^2}{1 + \xi^2}|\dot{u}|^2 + C(\varepsilon, \gamma_1, \gamma_2)(1 + \xi^2)|\dot{y}|^2,
\]
\[
\frac{1}{\delta} \left| \left( \chi_{0, \tau} + \tau b_0 \rho_3 \left( \tau - \frac{\rho_1}{\rho_3 k} \right) \right) \text{Re}(i\xi \dot{\bar{u}}) \right| \leq \frac{\rho_1 \gamma_2 \varepsilon}{2\gamma_1} \frac{\xi^2}{1 + \xi^2}|\dot{u}|^2 + C(\varepsilon, \gamma_1, \gamma_2)(1 + \xi^2)|\dot{y}|^2.
\]

Hence, from Lemma 3.3 we can rewrite (29) as
\[
\frac{d}{dt} J_1(\xi, t) + k\tau(1 - \varepsilon)|\dot{v}|^2 \leq \frac{\rho_1 \gamma_2 \varepsilon}{\gamma_1} \frac{\xi^2}{1 + \xi^2}|\dot{u}|^2 + C(\varepsilon, \gamma_1, \gamma_2)(1 + \xi^2)|\dot{y}|^2
+ C(\varepsilon, \gamma_1, \gamma_2)(1 + \xi^2)|\dot{y}|^2 + C(\varepsilon) \xi^4 \int_0^\infty g(s)|\dot{y}(s)|ds \right|^2.
\] (53)
Thus, the Lemma 3.4 and (53) imply that
\[
\frac{d}{dt} \left\{ \frac{\gamma_1 \xi^2}{1 + \xi^2} J_1(\xi, t) + \frac{\gamma_2 \xi^2}{(1 + \xi^2)^2} J_2(\xi, t) \right\} + k \left[ \gamma_1 (1 - \varepsilon) - C(\varepsilon) \gamma_2 \right] \frac{\xi^2}{1 + \xi^2} |\dot{\theta}|^2 + \rho_1 (1 - 2\varepsilon) \gamma_2 \frac{\xi^4}{(1 + \xi^2)^2} |\dot{\xi}|^2 \\
+ a(1 - \varepsilon) \gamma_2 \frac{\xi^4}{(1 + \xi^2)^2} |\dot{\xi}|^2 \\
\leq C(\varepsilon, \gamma_1, \gamma_2) \xi^2 |\dot{\gamma}|^2 + C(\varepsilon, \gamma_1, \gamma_2) \xi^2 |\dot{\gamma}|^2 + C(\varepsilon, \gamma_1) \frac{\xi^6}{1 + \xi^2} \left| \int_0^\infty g(s) \hat{\eta}(s) ds \right|^2 \\
+ C(\varepsilon, \gamma_2) \frac{\xi^2}{1 + \xi^2} |\dot{\gamma}|^2 + C(\varepsilon, \gamma_2) \frac{\xi^2}{1 + \xi^2} |\dot{\gamma}|^2 + C(\varepsilon, \gamma_1, \gamma_2) (1 + \xi^2)^2 \left| \int_0^\infty g(s) \hat{\eta}(s) ds \right|^2.
\] (54)

Furthermore, applying Young’s inequality in the equation (41) of the Lemma 3.5, it follows that
\[
\frac{d}{dt} \gamma_3 J_3(\xi, t) + \rho_2 \gamma_3 b_0 (1 - \varepsilon) \xi^2 |\dot{\gamma}|^2 \leq C(\varepsilon, \gamma_3) \xi^2 \left[ \int_0^\infty g(s) |\hat{\theta}(s)|^2 ds + a \gamma_2 \varepsilon \frac{\xi^4}{(1 + \xi^2)^2} |\dot{\xi}|^2 \\
+ C(\varepsilon, \gamma_2, \gamma_3) \xi^2 (1 + \xi^2)^2 \left[ \int_0^\infty g(s) \hat{\eta}(s) ds \right]^2 + \gamma_3 \delta |\xi|^3 |\dot{\theta}| \left| \int_0^\infty g(s) \hat{\eta}(s) ds \right|^2.
\] (55)

Thus,
\[
\frac{d}{dt} \gamma_3 J_3(\xi, t) + \rho_2 \gamma_3 b_0 (1 - \varepsilon) \xi^2 |\dot{\gamma}|^2 \leq C(\varepsilon, \gamma_3) \xi^2 \left[ \int_0^\infty g(s) |\hat{\theta}(s)|^2 ds + a \gamma_2 \varepsilon \frac{\xi^4}{(1 + \xi^2)^2} |\dot{\xi}|^2 + k \gamma_1 \tau \varepsilon \frac{\xi^2}{1 + \xi^2} |\dot{\xi}|^2 \\
+ C(\varepsilon, \gamma_1, \gamma_2, \gamma_3) \xi^2 (1 + \xi^2)^2 \left[ \int_0^\infty g(s) \hat{\eta}(s) ds \right]^2 + \gamma_3 \delta |\xi|^3 |\dot{\theta}| \left| \int_0^\infty g(s) \hat{\eta}(s) ds \right|^2.
\] (56)

Adding (54) and (55), we obtain
\[
\frac{d}{dt} \left\{ \frac{\gamma_1 \xi^2}{1 + \xi^2} J_1(\xi, t) + \frac{\gamma_2 \xi^2}{(1 + \xi^2)^2} J_2(\xi, t) + \gamma_3 J_3(\xi, t) \right\} + k \left[ \gamma_1 (1 - 2\varepsilon) - C(\varepsilon) \gamma_2 \right] \frac{\xi^2}{1 + \xi^2} |\dot{\xi}|^2 \\
+ \rho_1 (1 - 2\varepsilon) \gamma_2 \frac{\xi^4}{(1 + \xi^2)^2} |\dot{\xi}|^2 + a(1 - 2\varepsilon) \gamma_2 \frac{\xi^4}{(1 + \xi^2)^2} |\dot{\xi}|^2 + \rho_2 \left[ \gamma_3 b_0 (1 - \varepsilon) - C(\varepsilon, \gamma_1, \gamma_2) \right] \xi^2 |\dot{\xi}|^2 \\
\leq C(\varepsilon, \gamma_3) \xi^2 \left[ \int_0^\infty g(s) |\hat{\theta}(s)|^2 ds + C(\varepsilon, \gamma_1, \gamma_2, \gamma_3) \xi^2 (1 + \xi^2)^2 \left[ \int_0^\infty g(s) \hat{\eta}(s) ds \right]^2 \\
+ \gamma_3 \delta |\xi|^3 |\dot{\theta}| \left| \int_0^\infty g(s) \hat{\eta}(s) ds \right|^2 + C(\varepsilon, \gamma_1, \gamma_2, \gamma_3) \xi^2 |\dot{\xi}|^2.
\] (56)

Now, consider the following functional
\[
\mathcal{L}_2(\xi, t) = \gamma_1 \xi^2 J_1(\xi, t) + \gamma_2 \frac{\xi^2}{1 + \xi^2} J_2(\xi, t) + \gamma_3 J_3(\xi, t) + J_4(\xi, t).
\]

From Lemma 3.6, applying Young inequality to (56) and (53), together with (40), we have
\[
\frac{d}{dt} \left\{ \frac{\gamma_1 \xi^2}{1 + \xi^2} J_1(\xi, t) + \frac{\gamma_2 \xi^2}{(1 + \xi^2)^2} J_2(\xi, t) + \gamma_3 J_3(\xi, t) \right\} + k \left[ \gamma_1 (1 - 2\varepsilon) - C(\varepsilon) \gamma_2 \right] \frac{\xi^2}{1 + \xi^2} |\dot{\xi}|^2 \\
+ \rho_1 (1 - 2\varepsilon) \gamma_2 \frac{\xi^4}{(1 + \xi^2)^2} |\dot{\xi}|^2 + a(1 - 2\varepsilon) \gamma_2 \frac{\xi^4}{(1 + \xi^2)^2} |\dot{\xi}|^2 + \rho_2 \left[ \gamma_3 b_0 (1 - \varepsilon) - C(\varepsilon, \gamma_1, \gamma_2) \right] \xi^2 |\dot{\xi}|^2 \\
+ \rho_3 (1 - 2\varepsilon) \xi^2 |\dot{\theta}|^2 \leq C_1 (1 + b_0) \xi^2 (1 + \xi^2)^2 \left[ \int_0^\infty g(s) |\hat{\theta}(s)|^2 ds + C_1 (1 + \xi^2) |\dot{\xi}|^2,
\] (57)
where $C_1$ is a positive constant that depends on $\varepsilon$ and $\gamma_j$ for $j = 1, 2, 3$. In order to make all the coefficients in the right side in (57) positive, we have to choose appropriate constant $\gamma_i$. First, let us fix $\varepsilon$ such that $\varepsilon < \frac{1}{2}$. Thus, we can take any $\gamma_2 > 0$ and

$$
\gamma_1 > \frac{C(\varepsilon)\gamma_2}{\tau(1 - 2\varepsilon)}, \quad \gamma_3 > \frac{C(\varepsilon, \gamma_1, \gamma_2)}{b_0(1 - 2\varepsilon)}.
$$

Then, from (59) and (58), we can deduce the existence of a positive constant $M_1$ such that

$$
\frac{d}{dt}L_2(\xi, t) \leq -M_1 \frac{\xi^4}{(1 + \xi^2)^2} \left\{ k|\dot{\xi}|^2 + \rho_1|\dot{\eta}|^2 + a|\dot{\xi}|^2 + \rho_2|\dot{\eta}|^2 + \rho_3|\dot{\theta}|^2 \right\} + C_1(1 + \xi^2)^2 \xi^2 \int_0^\infty g(s)|\dot{\eta}(s)|^2 ds + C_1(1 + \xi^2)^2 |\dot{\theta}|^2.
$$

Finally, we define the following Lyapunov functional:

$$
L(\xi, t) = L_2(\xi, t, t) + N(1 + \xi^2)^2 \hat{E}(\xi, t),
$$

where $N$ is a positive constant to be fixed later. Note that the definition of $L_2$ together with inequality (46) imply that

$$
|L_2(\xi, t)| \leq M_2 \{ |J_1(\xi, t)| + |J_2(\xi, t)| + |J_3(\xi, t)| + |J_4(\xi, t)| \}
$$

$$
\leq M_2(1 + \xi^2)^2 \hat{E}(\xi, t).
$$

Hence, we obtain

$$
(N - M_2)(1 + \xi^2)^2 \hat{E}(\xi, t) \leq L(\xi, t) \leq (N + M_2)(1 + \xi^2)^2 \hat{E}(\xi, t).
$$

On the other hand, taking the derivative of $L$ with respect to $t$ and using the estimates (59) and Lemma 3.1, it follows that

$$
\frac{d}{dt}L(\xi, t) \leq -M_1 \frac{\xi^4}{(1 + \xi^2)^2} \left\{ k|\dot{\xi}|^2 + \rho_1|\dot{\eta}|^2 + a|\dot{\xi}|^2 + \rho_2|\dot{\eta}|^2 + \rho_3|\dot{\theta}|^2 \right\} - (2N\beta - C_1)(1 + \xi^2)^2 |\dot{\eta}|^2 - (Nk_1 - C_1(1 + b_0))(1 + \xi^2)^2 \xi^2 \int_0^\infty g(s)|\dot{\eta}(s)|^2 ds.
$$

Now, choosing $N$ such that $N \geq \max \left\{ M_2, \frac{C_1}{2\beta}, \frac{C_1(1 + b_0)}{k_1m} \right\}$ and using the inequality $(1 + \xi^2)^2 \geq \frac{\xi^4}{(1 + \xi^2)^2}$, there exists a positive constant $M_3$ such that

$$
\frac{d}{dt}L(\xi, t) \leq -M_3 \frac{\xi^4}{(1 + \xi^2)^2} \hat{E}(\xi, t).
$$

(60) implies that

$$
\frac{d}{dt}L(\xi, t) \leq -\Gamma \frac{\xi^4}{(1 + \xi^2)^2} L(\xi, t)
$$

where $\Gamma = \frac{M_3}{N + M_2}$. By Gronwall’s inequality, it follows that

$$
L(\xi, t) \leq L(\xi, 0)e^{-\Gamma\rho(\xi)t}, \quad \rho(\xi) = \frac{\xi^4}{(1 + \xi^2)^4},
$$

again by using (60), we have that

$$
\hat{E}(\xi, t) \leq C \hat{E}(\xi, 0)e^{-\Gamma\rho(\xi)t}, \quad C = \frac{N + M_2}{N - M_2} > 0.
$$

\[\square\]
3.2. The Timoshenko-Fourier Law

Similarly to the previous case, taking Fourier transform in (61), we obtain the following integro differential system

\begin{align*}
\dot{v}_t - i\xi \dot{u} + \dot{y} &= 0, \\
\rho_1 \ddot{u}_t - i k \xi \ddot{v} &= 0, \\
\ddot{z}_t - i\xi \ddot{y} &= 0, \\
\rho_2 \ddot{y}_t - i a \xi \ddot{z} + m \xi^2 \int_0^\infty g(s)\dot{y}(s)ds - k \ddot{v} + i \delta \ddot{\theta} &= 0, \\
\rho_3 \dddot{\theta} + i \delta \dddot{\xi} + i \delta \dddot{\xi} &= 0, \\
\ddot{\eta}_t + \dot{\eta}_s - \ddot{y} &= 0,
\end{align*}

where the solution vector and initial data are given by \( \hat{V}(\xi, t) = (\hat{v}, \hat{u}, \hat{z}, \hat{\theta}, \hat{\eta})^T \) and \( \hat{V}(\xi, 0) = \hat{V}_0(\xi) \), respectively. Furthermore, the energy functional associated to the above system is defined as

\[ \dot{E}(t, \xi) = \rho_1 |\ddot{u}|^2 + \rho_2 |\ddot{y}|^2 + \rho_3 |\dddot{\theta}|^2 + k|\ddot{v}|^2 + a |\dddot{z}|^2 + m \xi^2 \int_0^\infty g(s)|\dot{y}(s)|^2ds. \]

**Lemma 3.7.** The energy of the system (61)-(66), satisfies

\[ \frac{d}{dt} \dot{E}(\xi, t) \leq -2 \beta \xi^2 |\ddot{\theta}|^2 - k m \xi^2 \int_0^\infty g(s)|\dot{y}(s)|^2ds, \]

where the constant \( k_1 > 0 \) is given by (H2).

**Proof.** Multiplying (61) by \( k \ddot{\theta} \), (62) by \( \ddot{u} \), (63) by \( \ddot{z} \), (64) by \( \ddot{y} \) and (65) by \( \dddot{\theta} \), adding and taking real part, it follows that

\[ \frac{1}{2} \frac{d}{dt} \left\{ \rho_1 |\ddot{u}|^2 + \rho_2 |\ddot{y}|^2 + \rho_3 |\dddot{\theta}|^2 + k|\ddot{v}|^2 + a |\dddot{z}|^2 \right\} = -2 \beta \xi^2 |\ddot{\theta}|^2 - Re \left( m \xi^2 \int_0^\infty g(s)|\dot{y}(s)|\ddot{\theta} ds \right). \]

On the other hand, taking the conjugate of equation (65), multiplying the resulting equation by \( g(s)\dot{y}(t, s, x) \) and integrating with respect to \( s \), we obtain

\[ \int_0^\infty g(s)\dot{y}(s)\ddot{\theta} ds = \frac{1}{2} \frac{d}{dt} \int_0^\infty g(s)|\dot{y}(s)|^2ds + \frac{1}{2} \int_0^\infty g(s) \frac{d}{ds} \dot{y}(s)|\dot{y}(s)|^2ds. \]

Integrating by parts the last term in the left side of (70) and substituting in (69), it follows that

\[ \frac{d}{dt} \dot{E}(\xi, t) = -2 \beta \xi^2 |\ddot{\theta}|^2 - k m \xi^2 \int_0^\infty g'(s)|\dot{y}(s)|^2ds. \]

By using (H2), we obtain (68).

Since the energy of the system (61)-(66) is dissipative, we expect the exponential decay like in the previous subsection. The principal result of this subsection reads as follows:

**Theorem 3.8.** Let

\[ \chi_0 = \left( \rho_2 - \frac{bp_1}{k} \right). \]

Then, for any \( t \geq 0 \) and \( \xi \in \mathbb{R} \), we obtain the following decay rates for the energy of the system (61)-(66):

\[ \dot{E}(\xi, t) \leq Ce^{-\lambda \chi_0(\xi)} \dot{E}(0, \xi), \]

where \( C, \lambda \) are positive constants and the function \( \rho(\cdot) \) is given by

\[ \rho(\xi) = \begin{cases} 
\frac{\xi^4}{(1 + \xi^2)^4} & \text{if } \chi_0 = 0, \\
\frac{\xi^4}{(1 + \xi^2)^4} & \text{if } \chi_0 \neq 0.
\end{cases} \]
Similarly to for Cattaneo’s law, we need establish some preliminary results.

**Lemma 3.9.** Consider the functional

\[ K_1(\xi, t) = -\rho_2 \text{Re} (\bar{v}y) - \frac{\alpha_1}{k} \text{Re} (\bar{z}u) + \frac{b_0 \beta_1 \rho_3}{k \delta} \text{Re} (\bar{\theta}u). \]

Then, for any \( \varepsilon > 0 \), \( K_1 \) satisfies

\[
\frac{d}{dt} K_1(\xi, t) + k(1 - \varepsilon)|\dot{\varepsilon}|^2 \leq \rho_2 |\dot{y}|^2 + \chi_0 \text{Re} (i\xi \bar{u}y) + C(\varepsilon)\varepsilon^2 |\dot{\varepsilon}|^2 + C(\varepsilon) \varepsilon^4 \left| \int_0^\infty g(s) \bar{\eta}(s) ds \right|^2 + \frac{b_0 \beta_1 \rho_3}{k \delta} \varepsilon^2 |\dot{\theta}| \bar{u} |, \tag{72}
\]

for \( t \geq 0 \), where \( C(\varepsilon) \) is a positive constant and \( \chi_0 = \left( \rho_2 - \frac{b_1}{k} \right) \).

**Proof.** Multiplying (61) by \( -\rho_2 \bar{y} \) and taking real part, we obtain

\[-\rho_2 \text{Re} (\dot{v} \bar{y}) + \rho_2 \text{Re} (i\xi \dot{u} \bar{y}) - \rho_2 |\dot{y}|^2 = 0.\]

Multiplying (64) by \(-\bar{v}\) and taking real part, it follows that

\[-\rho_2 \text{Re} (\bar{y} \dot{v}) + a \text{Re} (i\xi \bar{z} \dot{v}) + k |\dot{v}|^2 - m \text{Re} \left( \xi^2 \bar{v} \right) \int_0^\infty g(s) \bar{\eta}(s) ds - \delta \text{Re} (i\xi \bar{\theta} \bar{v}) = 0.\]

Adding the above identities,

\[-\rho_2 \frac{d}{dt} \text{Re} (\dot{v} \bar{y}) + k |\dot{v}|^2 = \rho_2 |\dot{y}|^2 - a \text{Re} (i\xi \bar{z} \dot{v}) + m \text{Re} \left( \xi^2 \bar{v} \right) \int_0^\infty g(s) \bar{\eta}(s) ds - \rho_2 \text{Re} (i\xi \dot{u} \bar{y}) + \delta \text{Re} (i\xi \bar{\theta} \bar{v}). \tag{73}\]

On the other hand, multiplying (62) by \(-\bar{\xi} \bar{v}\), (63) by \(-\frac{\alpha_1}{k} \bar{u}\), adding the results and taking real part, it follows that

\[-\frac{\alpha_1}{k} \frac{d}{dt} \text{Re} (\dot{z} \bar{u}) = -\frac{\alpha_1}{k} \text{Re} (i\xi \dot{g} \bar{u}) - a \text{Re} (i\xi \dot{v} \bar{z}). \tag{74}\]

Adding (73) and (74), we have

\[
\frac{d}{dt} \left\{ -\rho_2 \text{Re} (\dot{v} \bar{y}) - \frac{\alpha_1}{k} \text{Re} (\dot{z} \bar{u}) \right\} + k |\dot{v}|^2 = \rho_2 |\dot{y}|^2 + m \text{Re} \left( \xi^2 \bar{v} \right) \int_0^\infty g(s) \bar{\eta}(s) ds
+ \left( \rho_2 - \frac{\alpha_1}{k} \right) \text{Re} (i\xi \dot{g} \bar{u}) + \delta \text{Re} (i\xi \bar{\theta} \bar{v}). \tag{75}\]

Moreover, multiplying (62) by \( \bar{\theta} \) and taking real part,

\[
\frac{\delta \rho_1}{k} \text{Re} (\dot{u} \bar{\theta}) - \delta \text{Re} (i\xi \dot{v} \bar{\theta}) = 0.
\]

Next, multiplying (65) by \( \frac{\delta \rho_1}{\rho_3 k} \bar{u} \) and taking real part,

\[
\frac{\delta \rho_1}{k} \text{Re} (\dot{u} \bar{\theta}) + \frac{\delta \beta \rho_1}{\rho_3 k} \text{Re} (\xi^2 \bar{u} \bar{\theta}) + \frac{\delta^2 \rho_1}{\rho_3 k} \text{Re} (i\xi \dot{g} \bar{u}) = 0.
\]

Adding the above identities, we obtain

\[
\frac{\delta \rho_1}{k} \frac{d}{dt} \text{Re} (\dot{u} \bar{\theta}) = -\frac{\delta \beta \rho_1}{\rho_3 k} \text{Re} (\xi^2 \bar{u} \bar{\theta}) - \frac{\delta^2 \rho_1}{\rho_3 k} \text{Re} (i\xi \dot{g} \bar{u}) + \delta \text{Re} (i\xi \dot{v} \bar{\theta}). \tag{76}\]
Computing \( \frac{d}{dt} \left\{ -\rho_2 \text{Re} (\bar{v} \tilde{y}) - \frac{a \rho_1}{k} \text{Re} (\bar{z} \tilde{y}) + \frac{b_0 \rho_1}{k \delta} \text{Re} (\tilde{\theta} \tilde{u}) \right\} + k|\tilde{v}|^2 = \rho_2 |\tilde{y}|^2 + m \text{Re} \left( \xi^2 \tilde{v} \int_0^\infty g(s) \tilde{\eta}(s) ds \right) \\
+ \left( \rho_2 - \frac{a \rho_1}{k} - \frac{b_0 \rho_1}{k} \right) \text{Re} (i \xi \tilde{w} \tilde{y}) + \left( \delta - \frac{b_0 \rho_3}{\delta} \right) \text{Re} (i \xi \tilde{\theta} \tilde{v}) - \frac{b_0}{k \delta} \text{Re} (\xi^2 \tilde{\theta} \tilde{u}). \)

Then,

\[
\frac{d}{dt} K_1(\xi, t) + k|\tilde{v}|^2 \leq \rho_2 |\tilde{y}|^2 + m |\xi|^2 |\tilde{v}|^2 \leq C(\varepsilon) (1 + \varepsilon^2)|\tilde{v}|^2 + C(\varepsilon) (1 + \varepsilon^2)|\tilde{y}|^2 \\
+ C(\varepsilon)|\xi|^2 |\tilde{\theta}|^2 + C(\varepsilon) k^4 \left| \int_0^\infty g(s) \tilde{\eta}(s) ds \right|^2 \tag{78}
\]

is satisfied.

**Proof.** Multiplying \(61\) by \(i \rho_1 \bar{\tilde{u}}\) and taking real part,

\[ \rho_1 \text{Re} (i \xi \tilde{v} \bar{\tilde{u}}) + \rho_1 |\xi|^2 |\tilde{u}|^2 + \rho_1 \text{Re} (i \xi \tilde{y} \bar{\tilde{u}}) = 0. \]

Multiplying \(62\) by \(-i \bar{\tilde{u}}\) and taking real part,

\[ -\rho_1 \text{Re} (i \xi \tilde{u} \bar{\tilde{v}}) - k |\xi|^2 |\tilde{u}|^2 = 0. \]

Adding the above identities, we obtain

\[ \rho_1 \frac{d}{dt} \text{Re} (i \xi \tilde{v} \bar{\tilde{u}}) + \rho_1 |\xi|^2 |\tilde{u}|^2 = k |\xi|^2 |\tilde{u}|^2 - \rho_1 \text{Re} (i \xi \tilde{y} \bar{\tilde{u}}). \tag{79} \]

Moreover, multiplying \(63\) by \(-i \rho_2 \bar{\tilde{y}}\) and taking real part,

\[ -\rho_2 \text{Re} (i \xi \tilde{z} \bar{\tilde{y}}) - \rho_2 |\xi|^2 |\tilde{y}|^2 = 0. \]

Multiplying \(64\) by \(i \xi \bar{\tilde{z}}\) and taking real part,

\[ \rho_2 \text{Re} (i \xi \tilde{y} \bar{\tilde{z}}) + a |\xi|^2 |\tilde{z}|^2 - k \text{Re} (i \xi \tilde{v} \bar{\tilde{z}}) + m \text{Re} \left( i \xi^2 \tilde{z} \int_0^\infty g(s) \tilde{\eta}(s) ds \right) - \delta \text{Re} \left( \xi^2 \tilde{\theta} \bar{\tilde{z}} \right) = 0. \]

Adding the above identities,

\[ \rho_2 \frac{d}{dt} \text{Re} (i \xi \tilde{y} \bar{\tilde{z}}) + a |\xi|^2 |\tilde{z}|^2 = \rho_2 |\xi|^2 |\tilde{y}|^2 + k \text{Re} (i \xi \tilde{v} \bar{\tilde{z}}) - m \text{Re} \left( i \xi^2 \tilde{z} \int_0^\infty g(s) \tilde{\eta}(s) ds \right) + \delta \text{Re} \left( \xi^2 \tilde{\theta} \bar{\tilde{z}} \right). \tag{80} \]

Therefore, computing \(79\) + \(80\), it follows that

\[
\frac{d}{dt} K_2(\xi, t) + \rho_1 |\xi|^2 |\tilde{u}|^2 + a |\xi|^2 |\tilde{z}|^2 \leq \rho_2 |\xi|^2 |\tilde{y}|^2 + k |\xi|^2 |\tilde{v}|^2 + k |\xi||\tilde{v}| |\tilde{z}| \\
- m |\xi|^2 |\tilde{z}| \left| \int_0^\infty g(s) \tilde{\eta}(s) ds \right| + \delta |\xi|^2 |\tilde{\theta}| |\tilde{z}| + \rho_1 |\xi||\tilde{y}||\tilde{u}|. 
\]

Applying Young’s inequality, \(78\) holds. \qed
Lemma 3.11. Consider the functional

$$K_3(\xi, t) = -\rho_2 \text{Re} \left( \xi^2 \overline{\eta} \int_0^\infty g(s) \eta(s) ds \right).$$

Then, for any \( \varepsilon > 0 \), the following estimate

$$\frac{d}{dt} K_3(\xi, t) + \rho_2 b(t_0 - \epsilon) \xi^2 |\hat{y}|^2 \leq C(\varepsilon) \xi^2 \int_0^\infty g(s) |\eta(s)|^2 ds + a |\xi|^3 |\hat{z}| \int_0^\infty g(s) \eta(s) ds + m \xi^4 \int_0^\infty g(s) \eta(s) ds + k \xi^2 |\hat{v}| \int_0^\infty g(s) \eta(s) ds + \delta |\xi|^3 |\hat{\theta}| \int_0^\infty g(s) \eta(s) ds.$$

holds.

Proof. Proceeding as proof of Lemma 3.9, we obtain (81). Indeed, we have to multiply (60) by \(-\xi^2 g(s) \overline{\eta}\) and (66) by \(-\rho_2 \xi^2 g(s) \overline{\eta}\), next we take the integration with respect to \(s\) for the real parts. We omit the details. \(\square\)

Proof of Theorem 3.8 As Theorem 3.2 we will consider several cases:

I. Case \( \chi_0 = 0 \): Consider the expressions

$$\zeta_1 \xi^2 K_1(\xi, t), \quad \frac{\xi^2}{1 + \xi^2} K_2(\xi, t), \quad \zeta_3 K_3(\xi, t),$$

where \( \zeta_1, \zeta_2 \) and \( \zeta_3 \) are positive constants to be fixed later. Lemmas 3.9 and 3.10 imply that

$$\frac{d}{dt} \left( \zeta_1 \xi^2 K_1(\xi, t) + \zeta_2 \frac{\xi^2}{1 + \xi^2} K_2(\xi, t) \right) + k [\zeta_1 (1 - \varepsilon) - C(\varepsilon) \zeta_2] \xi^2 |\hat{v}|^2 + \rho_1 (1 - \varepsilon) \zeta_2 \xi^4 \frac{1}{1 + \xi^2} |\hat{u}|^2$$

$$+ a (1 - \varepsilon) \zeta_2 \xi^4 \frac{1}{1 + \xi^2} |\hat{z}|^2 \leq C(\varepsilon, \zeta_1, \zeta_2) \xi^2 |\hat{y}|^2 + C(\varepsilon, \zeta_1, \zeta_2) \xi^4 |\hat{\theta}|^2$$

By Young’s inequality,

$$\frac{d}{dt} \left( \zeta_1 \xi^2 K_1(\xi, t) + \zeta_2 \frac{\xi^2}{1 + \xi^2} K_2(\xi, t) \right) + k [\zeta_1 (1 - \varepsilon) - C(\varepsilon) \zeta_2] \xi^2 |\hat{v}|^2 + \rho_1 (1 - 2 \varepsilon) \zeta_2 \xi^4 \frac{1}{1 + \xi^2} |\hat{u}|^2$$

$$+ a (1 - \varepsilon) \zeta_2 \xi^4 \frac{1}{1 + \xi^2} |\hat{z}|^2 \leq C(\varepsilon, \zeta_1, \zeta_2) \xi^2 |\hat{y}|^2 + C(\varepsilon, \zeta_1, \zeta_2) \xi^4 (1 + \xi^2) |\hat{\theta}|^2$$

Further, applying Young’s inequality in (81) in the Lemma 3.11 it follows that

$$\frac{d}{dt} \zeta_3 K_3(\xi, t) + \rho_2 \xi^2 b(t_0 - \epsilon) \xi^2 |\hat{y}|^2 \leq C(\varepsilon, \zeta_1, \xi^2) \xi^2 \int_0^\infty g(s) |\hat{s}(s)|^2 ds + a \zeta_2 \xi^4 \frac{1}{1 + \xi^2} |\hat{z}|^2$$

$$+ C(\varepsilon, \zeta_2, \zeta_3) \xi^2 (1 + \xi^2) \int_0^\infty g(s) \eta(s) ds \left( 2 + \zeta_m \xi^2 \int_0^\infty g(s) \eta(s) ds \right) + k \zeta_1 \varepsilon \xi^2 |\hat{v}|^2$$

$$+ C(\varepsilon, \zeta_1, \xi_3) \xi^2 (1 + \xi^2) \int_0^\infty g(s) \eta(s) ds \left( 2 + \zeta_3 \xi^2 |\hat{s}(s)|^2 \right) + k \zeta_1 \varepsilon \xi^2 |\hat{v}|^2$$

$$+ C(\varepsilon, \zeta_1, \xi_3) \xi^2 (1 + \xi^2) \int_0^\infty g(s) \eta(s) ds \left( 2 + \zeta_3 \xi^2 |\hat{s}(s)|^2 \right) + k \zeta_1 \varepsilon \xi^2 |\hat{v}|^2.$$
Thus,
\[
\frac{d}{dt} C_3 K_3(\xi, t) + \rho_2 \zeta_3 b_0 (1 - \varepsilon) \varepsilon^2 \hat{y} |\hat{y}|^2 \leq C(\varepsilon, \zeta_3) \xi^2 \int_0^\infty g(s) |\hat{y}(s)|^2 ds + a \xi_2 \varepsilon \xi \int_0^\infty \xi |\hat{y}|^2 + k \zeta_1 \varepsilon \xi^2 |\hat{y}|^2
\]
\[
+ C(\varepsilon, \zeta_1, \zeta_2, \zeta_3) \xi^2 (1 + \xi^2) \int_0^\infty g(s) |\hat{y}(s)|^2 ds + C(\varepsilon) \xi^4 |\hat{y}|^2. \quad (83)
\]

Computing (82) + (83), we obtain
\[
\frac{d}{dt} \left\{ \zeta_1 \xi^2 K_1(\xi, t) + \zeta_2 \frac{\xi^2}{1 + \xi^2} K_2(\xi, t) + \zeta_3 K_3(\xi, t) \right\} + k \left[ \zeta_1 (1 - 2 \varepsilon) - C(\varepsilon, \zeta_2) \right] \xi^2 |\hat{y}|^2
\]
\[
+ \rho_1 (1 - 2 \varepsilon) \zeta_2 \frac{\xi^4}{1 + \xi^2} |\hat{y}|^2 + a (1 - 2 \varepsilon) \zeta_2 \frac{\xi^4}{1 + \xi^2} |\hat{y}|^2 + \rho_2 \left[ \zeta_3 b_0 (1 - \varepsilon) - C(\varepsilon, \zeta_1, \zeta_2) \right] \xi^2 |\hat{y}|^2
\]
\[
\leq C(\varepsilon, \zeta_3) \xi^2 \int_0^\infty g(s) |\hat{y}(s)|^2 ds + C(\varepsilon, \zeta_1, \zeta_2, \zeta_3) \xi^2 (1 + \xi^2) \int_0^\infty g(s) |\hat{y}(s)|^2 ds + C(\varepsilon, \zeta_1, \zeta_2, \zeta_3) \xi^2 (1 + \xi^2) |\hat{y}|^2. \quad (84)
\]

From inequality (84), we conclude that
\[
\frac{d}{dt} \left\{ \zeta_1 \xi^2 K_1(\xi, t) + \zeta_2 \frac{\xi^2}{1 + \xi^2} K_2(\xi, t) + \zeta_3 K_3(\xi, t) \right\} + k \left[ \zeta_1 (1 - 2 \varepsilon) - C(\varepsilon, \zeta_2) \right] \xi^2 |\hat{y}|^2
\]
\[
+ \rho_1 (1 - 2 \varepsilon) \zeta_2 \frac{\xi^4}{1 + \xi^2} |\hat{y}|^2 + a (1 - 2 \varepsilon) \zeta_2 \frac{\xi^4}{1 + \xi^2} |\hat{y}|^2 + \rho_2 \left[ \zeta_3 b_0 (1 - \varepsilon) - C(\varepsilon, \zeta_1, \zeta_2) \right] \xi^2 |\hat{y}|^2
\]
\[
\leq C(\varepsilon, \zeta_1, \zeta_2, \zeta_3) (1 + b_0) \xi^2 (1 + \xi^2) \int_0^\infty g(s) |\hat{y}(s)|^2 ds + C(\varepsilon, \zeta_1, \zeta_2, \zeta_3) \xi^2 (1 + \xi^2) |\hat{y}|^2. \quad (85)
\]

In order to make all coefficients in the right-hand side in (84) positive, we have to choose appropriate constant $\zeta_1$. First, let us fix $\varepsilon$, such that $\varepsilon < \frac{1}{2}$. Thus, we can take any $\zeta_2 > 0$ and
\[
\zeta_1 > \frac{C(\varepsilon) \zeta_2}{1 - 2 \varepsilon}, \quad \zeta_3 > \frac{C(\varepsilon, \zeta_1, \zeta_2)}{b_0 (1 - \varepsilon)}. \quad (85)
\]

Then, from (85) and the estimate $\frac{\xi^2}{1 + \xi^2} \leq 1$, we can deduce the existence of a positive constant $M_1$ such that
\[
\frac{d}{dt} Q_1(\xi, t) \leq -M_1 \frac{\xi^4}{1 + \xi^2} \left\{ k |\hat{u}|^2 + \rho_2 |\hat{u}|^2 + a |\hat{z}|^2 + \rho_2 |\hat{y}|^2 \right\}
\]
\[
+ C_1 (1 + b_0) (1 + \xi^2) \xi^2 \int_0^\infty g(s) |\hat{y}(s)|^2 ds + C_1 (1 + \xi^2) \xi^2 |\hat{y}|^2, \quad (86)
\]
where
\[
Q_1(\xi, t) = \zeta_1 \xi^2 K_1(\xi, t) + \zeta_2 \frac{\xi^2}{1 + \xi^2} K_2(\xi, t) + \zeta_3 K_3(\xi, t)
\]
and $C_1$ is a positive constant that depends of $\varepsilon$ and $\zeta_j$ for $j = 1, 2, 3$. Finally, we define the following Lyapunov functional:
\[
Q(\xi, t) = Q_1(\xi, t, t) + N (1 + \xi^2)^2 \hat{K}(\xi, t),
\]
where $N$ is a positive constant to be fixed later. Note that the definition of $Q_1$ together with (85) imply that
\[
|Q_1(\xi, t)| \leq M_2 \left\{ K_1(\xi, t) + K_2(\xi, t) + K_3(\xi, t) \right\}
\]
\[
\leq M_2 (1 + \xi^2) \hat{K}(\xi, t).
\]

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Hence, we obtain
\[(N - M_2)(1 + \xi^2)\hat{\hat{\psi}}(\xi, t) ≤ Q(\xi, t) ≤ (N + M_2)(1 + \xi^2)\hat{\hat{\psi}}(\xi, t).\]  
(87)

On the other hand, taking the derivative of \(Q\) with respect to \(t\) and using the estimates (86) and Lemma 3.7, it follows that
\[
\frac{d}{dt} Q(\xi, t) ≤ -M_1 \xi^4 \{k|\dot{v}|^2 + \rho_1 |\dot{u}|^2 + a|\ddot{z}|^2 + \rho_2 |\dot{y}|^2\} - (2N\beta - C_1) (1 + \xi^2) \xi^2 |\dot{\theta}|^2 - (Nk_1m - C_1(1 + b_0)) (1 + \xi^2)^2 \int_0^\infty g(s)|\dot{\theta}(s)|^2 ds.
\]

Now, choosing \(N\) such that \(N ≥ \max \left\{ M_2, \frac{C_1}{2\beta}, \frac{C_1(1 + b_0)}{k_1 m}\right\}\) and using the inequalities \((1 + \xi^2)^2 ≥ \frac{\xi^4}{1 + \xi^2}\) and \((1 + \xi^2)^2 ≥ \frac{\xi^2}{1 + \xi^2}\), there exists a positive constant \(M_3\) such that
\[
\frac{d}{dt} Q(\xi, t) ≤ -M_3 \xi^4 (1 + \xi^2) \hat{\hat{\psi}}(\xi, t).
\]

Estimate (3.7) implies that
\[
\frac{d}{dt} Q(\xi, t) ≤ -\Gamma \frac{\xi^4}{(1 + \xi^2)^3} Q(\xi, t)
\]
where \(\Gamma = \frac{M_3}{(N + M_2)}\). By Gronwall’s inequality, it follows that
\[
Q(\xi, t) ≤ e^{-\Gamma t(\xi)^4} Q(\xi, 0), \quad \rho(\xi) = \frac{\xi^4}{(1 + \xi^2)^3}.
\]

Again by using (3.7), we have that
\[
\hat{\hat{\psi}}(\xi, t) ≤ C e^{-\Gamma t(\xi)^4} \hat{\hat{\psi}}(\xi, 0), \quad \text{where } C = \frac{N + M_2}{N - M_2} > 0.
\]

II. Case \(\xi_0 ≠ 0\): Similar to previous case, we introduce positive constants \(\kappa_1, \kappa_2\) and \(\kappa_3\) that will be fixed later. Next, we estimate the following term by applying Young’s inequality,
\[
\left|\xi_0 Re(i\xi \dot{u} \dot{y})\right| ≤ \frac{\rho_1 \kappa_2 \xi}{2\kappa_1} \frac{\xi^2}{1 + \xi^2} |\dot{u}|^2 + C(\xi, \kappa_1, \kappa_2)(1 + \xi^2) |\dot{y}|^2,
\]
\[
\frac{b_0 \beta \kappa_1}{k \delta} \xi^2 |\dot{\theta}| |\dot{u}| ≤ \frac{\rho_1 \kappa_2 \xi}{2\kappa_1} \frac{\xi^2}{1 + \xi^2} |\dot{u}|^2 + C(\xi, \kappa_1, \kappa_2)(1 + \xi^2) \xi^2 |\dot{\theta}|^2.
\]

Hence, (72) can be written as
\[
\frac{d}{dt} K_1(\xi, t) + k(1 - \varepsilon) |\dot{u}|^2 ≤ \frac{\rho_1 \kappa_2 \xi}{\kappa_1} \frac{\xi^2}{1 + \xi^2} |\dot{u}|^2 + C(\xi, \kappa_1, \kappa_2)(1 + \xi^2) |\dot{y}|^2 + C(\xi, \kappa_1, \kappa_2)(1 + \xi^2) \xi^2 |\dot{\theta}|^2 + C(\xi, \kappa_1, \kappa_2)(1 + \xi^2) \xi^4 \int_0^\infty g(s)|\dot{\theta}(s)|^2 ds.
\]
(88)

Thus, inequalities (73) and (88) imply that
\[
\frac{d}{dt} \left\{ \kappa_1 \frac{\xi^2}{1 + \xi^2} K_1(\xi, t) + \kappa_2 \frac{\xi^2}{1 + \xi^2} K_2(\xi, t) \right\} + k [\kappa_1 (1 - \varepsilon) - C(\xi) \kappa_2] \frac{\xi^2}{1 + \xi^2} |\dot{u}|^2 + \rho_1 (1 - 2\varepsilon) \kappa_2 \frac{\xi^4}{(1 + \xi^2)^2} |\dot{u}|^2 + C(\xi, \kappa_1, \kappa_2) \xi^2 |\dot{y}|^2 + C(\xi, \kappa_1, \kappa_2) \xi^4 |\dot{\theta}|^2 + C(\xi, \kappa_1, \kappa_2) \xi^4 \int_0^\infty g(s)|\dot{\theta}(s)|^2 ds \leq C(\xi, \kappa_1, \kappa_2) \xi^2 |\dot{y}|^2 + C(\xi, \kappa_1, \kappa_2) \xi^4 |\dot{\theta}|^2 + C(\xi, \kappa_1, \kappa_2) \xi^4 \int_0^\infty g(s)|\dot{\theta}(s)|^2 ds.
\]
(89)
Furthermore, applying Young’s inequality in (81) in the Lemma 3.11 it follows that

\[
\frac{d}{dt} \kappa_3 K_3(\xi, t) + \rho_2 \kappa_3 b_0 (1 - \varepsilon) \xi_2^2 \dot{y}^2 \leq C(\varepsilon, \kappa_3) \xi_2^2 \int_0^\infty g(s) |\dot{y}(s)|^2 ds + a \kappa_2 \varepsilon \frac{\xi_4^4}{(1 + \xi_2^2)^2} |\dot{z}|^2 + C(\varepsilon, \kappa_2, \kappa_3) \xi_2^2 (1 + \xi_2^2) \left( \int_0^\infty g(s) |\dot{y}(s)|^2 ds \right)^2 + \kappa_3 \varepsilon \xi_2^2 t |\dot{\theta}|^2 + C(\varepsilon, \kappa_1, \kappa_3) \xi_2^2 (1 + \xi_2^2) \left( \int_0^\infty g(s) |\dot{y}(s)|^2 ds \right)^2 + C(\varepsilon) \xi_4^4 |\dot{\theta}|^2.
\]

Thus,

\[
\frac{d}{dt} \kappa_3 K_3(\xi, t) + \rho_2 \kappa_3 b_0 (1 - \varepsilon) \xi_2^2 \dot{y}^2 \leq C(\varepsilon, \kappa_3) \xi_2^2 \int_0^\infty g(s) |\dot{y}(s)|^2 ds + a \kappa_2 \varepsilon \frac{\xi_4^4}{(1 + \xi_2^2)^2} |\dot{z}|^2 + \kappa_1 \varepsilon \xi_2^2 |\dot{\theta}|^2 + C(\varepsilon, \kappa_1, \kappa_2, \kappa_3) \xi_2^2 (1 + \xi_2^2) \left( \int_0^\infty g(s) |\dot{y}(s)|^2 ds \right)^2.
\]

Computing (93) + (94), we obtain

\[
\frac{d}{dt} \left\{ \kappa_1 \frac{\xi}{1 + \xi^2} K_1(\xi, t) + \kappa_2 \frac{\xi^2}{1 + \xi^2} K_2(\xi, t) + \kappa_3 K_3(\xi, t) \right\} + k \left[ \kappa_1 (1 - 2 \varepsilon) - C(\varepsilon) \kappa_2 \right] \frac{\xi^2}{1 + \xi^2} |\dot{y}|^2 + \rho_1 (1 - 2 \varepsilon) \kappa_2 \frac{\xi^4}{1 + \xi^2} |\ddot{u}|^2 + a (1 - 2 \varepsilon) \kappa_2 \frac{\xi^4}{1 + \xi^2} |\dot{z}|^2 + \rho_2 (1 - \varepsilon - C(\varepsilon, \kappa_2) \xi^2 |\dot{y}|^2 \leq C(\varepsilon, \kappa, \kappa_2, \kappa_3) (1 + \xi^2) \int_0^\infty g(s) |\dot{y}(s)|^2 ds + C(\varepsilon, \kappa_1, \kappa_2, \kappa_3) \xi^2 (1 + \xi^2) |\dot{\theta}|^2.
\]

From inequality (95), we conclude that

\[
\frac{d}{dt} \left\{ \kappa_1 \frac{\xi^2}{1 + \xi^2} K_1(\xi, t) + \kappa_2 \frac{\xi^2}{1 + \xi^2} K_2(\xi, t) + \kappa_3 K_3(\xi, t) \right\} + k \left[ \kappa_1 (1 - 2 \varepsilon) - C(\varepsilon) \kappa_2 \right] \frac{\xi^2}{1 + \xi^2} |\dot{y}|^2 + \rho_1 (1 - 2 \varepsilon) \kappa_2 \frac{\xi^4}{1 + \xi^2} |\ddot{u}|^2 + a (1 - 2 \varepsilon) \kappa_2 \frac{\xi^4}{1 + \xi^2} |\dot{z}|^2 + \rho_2 (1 - \varepsilon - C(\varepsilon, \kappa_2) \xi^2 |\dot{y}|^2 \leq C(\varepsilon, \kappa_1, \kappa_2, \kappa_3) (1 + \xi^2) \int_0^\infty g(s) |\dot{y}(s)|^2 ds + C(\varepsilon, \kappa_1, \kappa_2, \kappa_3) \xi^2 (1 + \xi^2) |\dot{\theta}|^2.
\]

In order to make all coefficients in the right-hand side in (97) positive, we have to choose appropriate constant \(\kappa_1\). First, let us fix \(\varepsilon\), such that \(\varepsilon < \frac{1}{2}\). Thus, we can take \(\kappa_2 > 0\) and

\[
\kappa_1 > \frac{C(\varepsilon) \kappa_2}{1 - 2 \varepsilon}, \quad \kappa_3 > \frac{C(\varepsilon, \kappa_1, \kappa_2)}{b_0 (1 - \varepsilon)}.
\]

Then, from (92) and the estimate \(\frac{\xi^2}{1 + \xi^2} \leq 1\), we can deduce the existence of a positive constant \(M_1\) such that

\[
\frac{d}{dt} Q_1(\xi, t) \leq -M_1 \frac{\xi^4}{(1 + \xi^2)^2} \left\{ k |\dot{v}|^2 + \rho_1 |\ddot{u}|^2 + a |\dot{z}|^2 + \rho_2 |\dot{y}|^2 \right\} + C_1 (1 + b_0) (1 + \xi^2) \xi^2 \xi^2 \int_0^\infty g(s) |\dot{y}(s)|^2 ds + C_1 (1 + \xi^2) \xi^2 |\dot{\theta}|^2,
\]

where

\[
Q_1(\xi, t) = \kappa_1 \frac{\xi^2}{1 + \xi^2} K_1(\xi, t) + \kappa_2 \frac{\xi^2}{1 + \xi^2} K_2(\xi, t) + \kappa_3 K_3(\xi, t)
\]

and \(C_1\) is a positive constant that depends of \(\varepsilon\) and \(\kappa_j\) for \(j = 1, 2, 3\). Finally, we define the following Lyapunov functional:

\[
Q(\xi, t) = Q_1(\xi, t) + N(1 + \xi^2)^2 \dot{E}(\xi, t),
\]

where

\[
\dot{E}(\xi, t) = \frac{\xi^2}{1 + \xi^2} E_1(\xi, t) + \frac{\xi^2}{1 + \xi^2} E_2(\xi, t) + E_3(\xi, t)
\]

and

\[
\frac{d}{dt} E_1(\xi, t) = -M_1 \frac{\xi^4}{(1 + \xi^2)^2} \left\{ k |\dot{v}|^2 + \rho_1 |\ddot{u}|^2 + a |\dot{z}|^2 + \rho_2 |\dot{y}|^2 \right\} + C_1 (1 + b_0) (1 + \xi^2) \xi^2 \xi^2 \int_0^\infty g(s) |\dot{y}(s)|^2 ds + C_1 (1 + \xi^2) \xi^2 |\dot{\theta}|^2,
\]

and

\[
\frac{d}{dt} E_2(\xi, t) = -M_1 \frac{\xi^4}{(1 + \xi^2)^2} \left\{ k |\dot{v}|^2 + \rho_1 |\ddot{u}|^2 + a |\dot{z}|^2 + \rho_2 |\dot{y}|^2 \right\} + C_1 (1 + b_0) (1 + \xi^2) \xi^2 \xi^2 \int_0^\infty g(s) |\dot{y}(s)|^2 ds + C_1 (1 + \xi^2) \xi^2 |\dot{\theta}|^2,
\]

and

\[
\frac{d}{dt} E_3(\xi, t) = -M_1 \frac{\xi^4}{(1 + \xi^2)^2} \left\{ k |\dot{v}|^2 + \rho_1 |\ddot{u}|^2 + a |\dot{z}|^2 + \rho_2 |\dot{y}|^2 \right\} + C_1 (1 + b_0) (1 + \xi^2) \xi^2 \xi^2 \int_0^\infty g(s) |\dot{y}(s)|^2 ds + C_1 (1 + \xi^2) \xi^2 |\dot{\theta}|^2.
\]
where $N$ is a positive constant to be fixed later. Note that the definition of $Q_1$ together with the inequality (46) imply that

$$|Q_1(\xi,t)| \leq M_2(1 + \xi^2)^2 \hat{\mathcal{E}}(\xi,t)$$

for some positive constant $M_2$. Hence, we obtain

$$(N - M_2)(1 + \xi^2)^2 \hat{\mathcal{E}}(\xi,t) \leq Q(\xi,t) \leq (N + M_2)(1 + \xi^2)^2 \hat{\mathcal{E}}(\xi,t).$$

(94)

On the other hand, taking the derivative of $Q$ with respect to $t$ and using the estimates (93) and Lemma 3.7, it follows that

$$\frac{d}{dt} Q(\xi,t) \leq -M_1 \frac{\xi^4}{(1 + \xi^2)^2} \left\{ k|\dot{v}|^2 + \rho_1|\ddot{u}|^2 + a|\ddot{\xi}|^2 + \rho_2|\dot{\gamma}|^2 \right\} - \left(2N\beta - C_1\right)(1 + \xi^2)^2 \xi^2 |\dot{\theta}|^2 - (Nk_1m - C_1(1 + b_0))(1 + \xi^2)^2 \xi^2 \int_0^\infty g(s)|\dot{\eta}(s)|^2 ds.$$

Now, choosing $N$ such that $N \geq \max\left\{ M_2, \frac{C_1}{2 \beta}, \frac{C_1(1 + b_0)}{k_1 m} \right\}$ and noting that $(1 + \xi^2)^2 \geq \frac{\xi^4}{(1 + \xi^2)^2}$, there exists a positive constant $M_3$ such that

$$\frac{d}{dt} Q(\xi,t) \leq -M_3 \frac{\xi^4}{(1 + \xi^2)^2} \hat{\mathcal{E}}(\xi,t).$$

Estimate (91) implies that

$$\frac{d}{dt} Q(\xi,t) \leq -\Gamma \frac{\xi^4}{(1 + \xi^2)^2} Q(\xi,t)$$

where $\Gamma = \frac{M_3}{N + M_2}$. By Gronwall’s inequality, it follows that

$$Q(\xi,t) \leq e^{-\Gamma \rho(\xi)t} Q(\xi,0), \quad \rho(\xi) = \frac{\xi^4}{(1 + \xi^2)^2},$$

again by using (52), we have that

$$\hat{\mathcal{E}}(\xi,t) \leq C e^{-\Gamma \rho(\xi)t} \hat{\mathcal{E}}(\xi,0), \quad \text{where} \quad C = \frac{N + M_2}{N - M_2} > 0. \quad \square$$

4. The decay estimates

In this section, we establish the decay rates of solutions $U(x,t)$, $V(x,t)$ of systems (10)-(11) and (13)-(14), respectively. By using the energy inequalities in the Fourier space, we show that the decay rates depend of condition $\chi_{0,\tau} = 0$ or $\chi_{0,\tau} \neq 0$ (resp. $\chi_0 = 0$ or $\chi_0 \neq 0$). In any case, the regularity loss phenomenon is present. This first main result reads as follows:

**Theorem 4.1.** Let $s$ be a nonnegative integer and

$$\chi_{0,\tau} = \left( \tau - \frac{\rho_1}{\rho_3 k} \right) \left( \rho_2 - \frac{b\rho_1}{k} \right) - \frac{\tau \rho_1 \delta^2}{\rho_3 k}. \quad (95)$$

Suppose that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, where

$$H^s(\mathbb{R}) := [H^s(\mathbb{R})]^6 \times L^2_0(\mathbb{R}; H^{1+s}(\mathbb{R})) \quad \text{and} \quad L^1(\mathbb{R}) := [L^1(\mathbb{R})]^6 \times L^2_0(\mathbb{R}; W^{1,1}(\mathbb{R})).$$

Then, the solution $U$ of the system (10), satisfies the following decay estimates,
(i) If $\chi_{0, \tau} = 0$, then
\[ \| \partial_x^k U(t) \|_2 \leq C(1 + t)^{-\frac{1}{2} - \frac{k}{4}} \| U_0 \|_1 + C(1 + t)^{-\frac{1}{2}} \| \partial_x^{k+l} U_0 \|_2, \quad t \geq 0. \] (96)

(ii) If $\chi_{0, \tau} \neq 0$, then
\[ \| \partial_x^k U(t) \|_2 \leq C(1 + t)^{-\frac{1}{2} - \frac{k}{4}} \| U_0 \|_1 + C(1 + t)^{-\frac{1}{2}} \| \partial_x^{k+l} U_0 \|_2, \quad t \geq 0. \] (97)

where $k + l \leq s$, $C$ and $c$ are two positive constants.

**Proof.** Applying the Plancherel’s identity, we have
\[ \| \partial_x^k U(t) \|_2^2 = \| (i\xi)^k \hat{U}(t) \|_2^2 = \int |\xi|^{2k} \left| \hat{U}(\xi, t) \right|^2 d\xi. \]

It is easy to see that
\[ c_1 \left| \hat{U}(\xi, t) \right|^2 \leq \hat{E}(\xi, t) \leq c_2 \left| \hat{U}(\xi, t) \right|^2, \] (98)

for some positive constant $c_1$ and $c_2$. Thus, it follows that
\[ \| \partial_x^k U(t) \|_2^2 \leq \frac{1}{c_1} \int |\xi|^{2k} \hat{E}(\xi, t) d\xi. \]

From Theorems 3.2 and (98), there exist a postive constant $M > 0$, such that
\[
\begin{align*}
\| \partial_x^k U(t) \|_2^2 &\leq M \int |\xi|^{2k} e^{-\lambda \rho(\xi) t} \left| \hat{U}(0, \xi) \right|^2 d\xi \\
&\leq M \int_{|\xi| \leq 1} |\xi|^{2k} e^{-\lambda \rho(\xi) t} \left| \hat{U}_0(\xi) \right|^2 d\xi + M \int_{|\xi| \geq 1} |\xi|^{2k} e^{-\lambda \rho(\xi) t} \left| \hat{U}_0(\xi) \right|^2 d\xi.
\end{align*}
\]

**Case $\chi_{0, \tau} = 0$:** It is not difficult to see that the function $\rho(\cdot)$ satisfies
\[
\begin{cases}
\rho(\xi) \geq \frac{1}{2} \xi^4 & \text{if } |\xi| \leq 1, \\
\rho(\xi) \geq \frac{1}{8} \xi^{-2} & \text{if } |\xi| \geq 1.
\end{cases}
\] (99)

Thus, we estimate $I_1$ as follows,
\[
I_1 \leq M \| \hat{U}_0 \|_{L^\infty}^2 \int_{|\xi| \leq 1} |\xi|^{2k} e^{-\frac{4}{5} \xi^4 t} d\xi \leq C_1 \| \hat{U}_0 \|_{L^\infty}^2 (1 + t)^{-\frac{3}{5}(1 + 2k)} \leq C_1 (1 + t)^{-\frac{3}{5}(1 + 2k)} \| U_0 \|_{L^1}^2.
\]

On the other hand, by using the second inequality in (99), we obtain
\[
I_2 \leq M \int_{|\xi| \geq 1} |\xi|^{2k} e^{-\frac{2}{5} \xi^{-2} t} \left| \hat{U}_0(\xi) \right|^2 d\xi \leq M \sup_{|\xi| \geq 1} \{ |\xi|^{-2} e^{-\frac{2}{5} \xi^{-2} t} \} \int_{|\xi| \geq 1} |\xi|^{2(k+t)} \left| \hat{U}_0^2(\xi) \right|^2 d\xi \leq C_2 (1 + t)^{-\frac{3}{5} k} \| \partial_x^{k+l} U_0 \|_2^2.
\]

Combining the estimates of $I_1$ and $I_2$, we obtain (99).

**Case $\chi_{0, \tau} \neq 0$:** In this case, the function $\rho(\cdot)$ satisfies
\[
\begin{cases}
\rho(\xi) \geq \frac{1}{16} \xi^4 & \text{if } |\xi| \leq 1, \\
\rho(\xi) \geq \frac{1}{16} \xi^{-4} & \text{if } |\xi| \geq 1
\end{cases}
\] (100)
Thus, we estimate $I_1$ as following,

$$I_1 \leq M \| \hat{U}_0 \|_{L^2}^2 \int_{|\xi| \leq 1} |\xi|^{2k} e^{-\frac{s}{4} \xi^4} \ d\xi \leq C_1 \| \hat{V}_0 \|_{L^\infty}^2 (1 + t)^{-\frac{s}{4}(1+2k)} \leq C_1 (1 + t)^{-\frac{s}{8}(1+2k)} \| U_0 \|_{L^1}^2.$$  

Moreover, by using the second inequality in \cite{100}, it follows that

$$I_2 \leq M \int_{|\xi| \geq 1} |\xi|^{2k} e^{-\frac{s}{4} \xi^4} |\hat{U}_0(\xi)|^2 \ d\xi \leq M \sup_{|\xi| \geq 1} \{|\xi|^{2t} e^{-\frac{s}{4} \xi^4} \} \int_{\mathbb{R}} |\xi|^{2(k+l)} |\hat{U}_0^2(\xi)|^2 \ d\xi$$

$$\leq C_2 (1 + t)^{-\frac{s}{8}} \| \partial_x^{k+l} U_0 \|_{L^2}^2.$$  

Combining the estimates of $I_1$ and $I_2$, we obtain \cite{37}.

Similar to the proof of Theorem \cite{11}, we establish decay estimates of the solution $V(x,t)$ of Timoshenko-Fourier system \cite{13-14}. The proof of next theorem is carried out by the same technique as that of Theorem \cite{11}. Therefore, we omit it.

**Theorem 4.2.** Let $s$ be a nonnegative integer and

$$\chi_0 = \left( \rho_2 - \frac{b_{p_1}}{k} \right).$$  

\par Suppose that $V_0 \in \mathbb{H}^s(\mathbb{R}) \cap L^1(\mathbb{R})$, where

$$\mathbb{H}^s(\mathbb{R}) := [H^s(\mathbb{R})]^5 \times L^2_0(\mathbb{R}; H^{1+s}(\mathbb{R})) \quad \text{and} \quad L^1(\mathbb{R}) := \left[ L^1(\mathbb{R}) \right]^5 \times L^2_0(\mathbb{R}; W^{1,1}(\mathbb{R})).$$

Then, the solution $V$ of the system \cite{13}, satisfies the following decay estimates,

(i) If $\chi_0 = 0$, then

$$\| \partial_x^k V(t) \|_2 \leq C_1(1 + t)^{-\frac{s}{8}} \| V_0 \|_1 + C_2(1 + t)^{-\frac{s}{8}} \| \partial_x^{k+l} V_0 \|_2, \quad t \geq 0.$$  

(ii) If $\chi_0 \neq 0$, then

$$\| \partial_x^k V(t) \|_2 \leq C_1(1 + t)^{-\frac{s}{8}} \| V_0 \|_1 + C_2(1 + t)^{-\frac{s}{8}} \| \partial_x^{k+l} V_0 \|_2, \quad t \geq 0.$$  

where $k + l \leq s$, $C_1$, $C_2$ are two positive constants.

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