Multiple $\mathbb{S}^1$-orbits for the Schrödinger-Newton system

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Abstract

We prove existence and multiplicity of symmetric solutions for the Schrödinger-Newton system in three dimensional space using equivariant Morse theory.

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1 Introduction

The Schrödinger-Newton system in three dimensional space has a long standing history. It was firstly proposed in 1954 by Pekar for describing the quantum mechanics of a polaron. Successively it was derived by Choquard for describing an electron trapped in its own hole and by Penrose \cite{27, 28, 29} in his discussions on the selfgravitating matter.

For a single particle of mass $m$ the system is obtained by coupling together the linear Schrödinger equation of quantum mechanics with the Poisson equation from Newtonian mechanics. It has the form

\begin{equation}
\begin{aligned}
-\frac{\hbar^2}{2m} \Delta \psi + V(x) \psi + U \psi &= 0, \\
-\Delta U + 4\pi \kappa |\psi|^2 &= 0,
\end{aligned}
\end{equation}

where $\psi$ is the complex wave function, $U$ is the gravitational potential energy, $V$ is a given potential, $\hbar$ is Planck’s constant, and $\kappa := Gm^2$, $G$ being Newton’s constant.

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Rescaling \( \psi(x) = \frac{1}{\sqrt{\varepsilon}} \psi(x) \), \( V(x) = \frac{1}{2m} \tilde{V}(x) \), \( U(x) = \frac{1}{2m} \tilde{U}(x) \), system (1) becomes equivalent to the single nonlocal equation
\[
-\hbar^2 \Delta \tilde{\psi} + \tilde{V}(x) \tilde{\psi} = \frac{1}{\hbar^2} \left( \frac{1}{|x|} \ast |\tilde{\psi}|^2 \right) \tilde{\psi}.
\] (2)

The existence of one solution can be traced back to Lions’ paper [19]. Successively equation (2) and related equations have been investigated by many authors, see e.g. [2, 12, 16, 13, 15, 20, 21, 24, 22, 25, 30, 31, 8, 23] and the references therein. Semiclassical analysis for equation (2) has been studied in [33] and in [10] for a more general convolution potential, not necessarily radially symmetric.

In this work we shall consider the nonlocal equation (2) in presence of a magnetic potential \( A \) and an electric potential \( V \) which satisfy specific symmetry. Precisely, we consider \( G \) a closed subgroup of the group \( O(3) \) of linear isometries of \( \mathbb{R}^3 \) and assume that \( A: \mathbb{R}^3 \to \mathbb{R}^3 \) is a \( C^1 \)-function, and \( V: \mathbb{R}^3 \to \mathbb{R} \) is a bounded continuous function with \( \inf_{\mathbb{R}^3} V > 0 \), which satisfy
\[
A(gx) = gA(x) \quad \text{and} \quad V(gx) = V(x) \quad \text{for all} \quad g \in G, \ x \in \mathbb{R}^3. \] (3)

Given a continuous homomorphism of groups \( \tau: G \to S^1 \) into the group \( S^1 \) of unit complex numbers. A physically relevant example is a constant magnetic field \( B = \text{curl} A = (0,0,2) \) and the group \( G_m = \{e^{2\pi ik/m} | k = 1, \ldots, m \} \) for \( m \in \mathbb{N}, \ m \geq 1 \); see Subsection 5.1 for more details.

We are interested in semiclassical states, i.e. solutions as \( \varepsilon \to 0 \) to the problem
\[
\left\{ \begin{array}{l}
(\varepsilon^2 \nabla + A)^2 u + V(x)u = \frac{1}{\varepsilon^2} \left( \frac{1}{|x|} \ast |u|^2 \right) u, \\
u \in L^2(\mathbb{R}^3, \mathbb{C}), \\
\varepsilon \nabla u + iAu \in L^2(\mathbb{R}^3, \mathbb{C}^3),
\end{array} \right. \] (4)

which satisfy
\[
u(gx) = \tau(g) \nu(x) \quad \text{for all} \quad g \in G, \ x \in \mathbb{R}^3. \] (5)

This implies that the absolute value \( |u| \) of \( u \) is \( G \)-invariant and the phase of \( u(gx) \) is that of \( u(x) \) multiplied by \( \tau(g) \).

Recently in [2] the authors have showed that there is a combined effect of the symmetries and the electric potential \( V \) on the number of semiclassical \( \tau \)-intertwining solutions to (4). More precisely, we showed that the Lusternik-Schnirelmann category of the \( G \)-orbit space of a suitable set \( M_\tau \), depending on \( V \) and \( \tau \), furnishes a lower bound on the number of solutions of this type. In this work we shall apply equivariant Morse theory for better multiplicity results than those given by Lusternik-Schnirelmann category. Moreover equivariant Morse theory provides information on the local behavior of a functional around a critical orbit. The main result is established in Theorem 5.3. For the local case, similar results are obtained in [7]. For other results about magnetic Schrödinger equations, we refer to [4, 5].

Finally, concerning magnetic Pekar functional, we mention the recent results in [14].

2 The variational problem

Set \( \nabla_{\varepsilon,A} u = \varepsilon \nabla u + iAu \) and consider the real Hilbert space
\[
H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C}) := \{u \in L^2(\mathbb{R}^3, \mathbb{C}) \mid \nabla_{\varepsilon,A} u \in L^2(\mathbb{R}^3, \mathbb{C}^3) \}
\]
with the scalar product
\[
\langle u, v \rangle_{\varepsilon,A,V} = \text{Re} \int_{\mathbb{R}^3} \left( \nabla_{\varepsilon,A} u \cdot \overline{\nabla_{\varepsilon,A} v} + V(x)u \overline{v} \right). \] (6)
We write
\[ \|u\|_{\varepsilon,A,V} = \left( \int_{\mathbb{R}^3} \left( |\nabla_{\varepsilon,A} u|^2 + V(x) |u|^2 \right) \right)^{1/2} \]
for the corresponding norm.

If \( u \in H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C}) \), then \( |u| \in H^1(\mathbb{R}^3, \mathbb{R}) \) and the diamagnetic inequality \([13]\) holds
\[ \varepsilon \left| \nabla |u(x)| \right| \leq |\varepsilon \nabla u(x) + iA(x)u(x)| \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{7} \]

Set
\[ D(u) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} \, dx \, dy. \]

We need some basic inequalities about convolutions. A proof can be found in \([18, \text{Theorem 4.3}]\) and in \([17]\).

**Theorem 2.1.** If \( p, q \in (1, +\infty) \) satisfy \( 1/p + 1/3 = 1 + 1/q \) and \( f \in L^p(\mathbb{R}^3) \) then
\[ \| |x| * f \|_{L^q(\mathbb{R}^3)} \leq N_p \|f\|_{L^p(\mathbb{R}^3)} \tag{8} \]
for a constant \( N_p > 0 \) that depends on \( p \) but not on \( f \). More generally, if \( p, t \in (1, +\infty) \) satisfy \( 1/p + 1/t + 1/3 = 2 \) and \( f \in L^p(\mathbb{R}^3) \), \( g \in L^t(\mathbb{R}^3) \), then
\[ \left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{f(x)g(y)}{|x-y|} \, dx \, dy \right| \leq N_p \|f\|_{L^p(\mathbb{R}^3)} \|g\|_{L^t(\mathbb{R}^3)} \tag{9} \]
for some constant \( N_p > 0 \) that does not depend on \( f \) and \( g \).

Theorem 2.1 yields that
\[ D(u) \leq C \|u\|_{L^{12/5}(\mathbb{R}^3)} \tag{10} \]
for every \( u \in H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C}) \).

The energy functional \( J_{\varepsilon,A,V} : H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C}) \to \mathbb{R} \) associated to problem (3), defined by
\[ J_{\varepsilon,A,V}(u) = \frac{1}{2} \|u\|^2_{\varepsilon,A,V} - \frac{1}{4\varepsilon^2} D(u), \]
is of class \( C^1 \), and its first derivative is given by
\[ J'_{\varepsilon,A,V}(u)[v] = \langle u, v \rangle_{\varepsilon,A,V} - \frac{1}{\varepsilon^2} \text{Re} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |u|^2 \right) v \overline{u}. \]

Moreover we can write the second derivative
\[ J''_{\varepsilon,A,V}(u)[v,w] = \langle w, v \rangle_{\varepsilon,A,V} - \frac{1}{\varepsilon^2} \text{Re} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |u|^2 \right) w \overline{v} - \frac{2}{\varepsilon^2} \text{Re} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast (u \overline{v}) \right) w. \]

By (9) it is easy to recognize that
\[ |J''_{\varepsilon,A,V}(u)[v,w]| \leq \|v\|_{\varepsilon,A,V} \|w\|_{\varepsilon,A,V} + C \|u\|^2_{L^{12/5}(\mathbb{R}^3)} \|v\|_{L^{12/5}(\mathbb{R}^3)} \|w\|_{L^{12/5}(\mathbb{R}^3)} \leq K \|v\|_{\varepsilon,A,V} \|w\|_{\varepsilon,A,V}. \]

We postpone the proof that \( J_{\varepsilon,A,V} \) is of class \( C^2 \) to the Appendix.
The solutions to problem (4) are the critical points of $J_{E,A,V}$. The action of $G$ on $H^1_{E,A}(\mathbb{R}^3, \mathbb{C})$ defined by $(g,u) \mapsto u_g$, where

$$(u_g)(x) = \tau(g) u(g^{-1}x),$$

satisfies

$$\langle u_g, v \rangle_{E,A,V} = \langle u, v \rangle_{E,A,V} \quad \text{and} \quad \mathbb{D}(u_g) = \mathbb{D}(u)$$

for all $g \in G, u, v \in H^1_{E,A}(\mathbb{R}^3, \mathbb{C})$. Hence, $J_{E,A,V}$ is $G$-invariant. By the principle of symmetric criticality [26, 34], the critical points of the restriction of $J_{E,A,V}$ to the fixed point space of this $G$-action, denoted by

$$H^1_{E,A}(\mathbb{R}^3, \mathbb{C})^{\tau} = \{ u \in H^1_{E,A}(\mathbb{R}^3, \mathbb{C}) \mid u_g = u \} = \{ u \in H^1_{E,A}(\mathbb{R}^3, \mathbb{C}) \mid u(gx) = \tau(g)u(x) \quad \forall x \in \mathbb{R}^3, g \in G \},$$

are the solutions to problem (4) which satisfy (5).

Let us define the Nehari manifold

$$\mathcal{N}_{E,A,V}^{\tau} = \left\{ u \in H^1_{E,A}(\mathbb{R}^3, \mathbb{C})^{\tau} \mid u \neq 0 \text{ and } \varepsilon^2 \| u \|^2_{E,A,V} = \mathbb{D}(u) \right\},$$

which is a $C^2$-manifold radially diffeomorphic to the unit sphere in $H^1_{E,A}(\mathbb{R}^3, \mathbb{C})^{\tau}$. The critical points of the restriction of $J_{E,A,V}$ to $\mathcal{N}_{E,A,V}^{\tau}$ are precisely the nontrivial solutions to (4) which satisfy (5).

Since $S^1$ acts on $H^1_{E,A}(\mathbb{R}^3, \mathbb{C})^{\tau}$ by scalar multiplication: $(e^{i\theta}, u) \mapsto e^{i\theta} u$, the Nehari manifold $\mathcal{N}_{E,A,V}^{\tau}$ and the functional $J_{E,A,V}$ are invariant under this action. Therefore, if $u$ is a critical point of $J_{E,A,V}$ on $\mathcal{N}_{E,A,V}^{\tau}$ then so is $\gamma u$ for every $\gamma \in S^1$. The set $S^1 u = \{ \gamma u \mid \gamma \in S^1 \}$ is then called a $\tau$-intertwining critical $S^1$-orbit of $J_{E,A,V}$. Two solutions of (4) are said to be geometrically different if their $S^1$-orbits are different.

Recall that $J_{E,A,V} : \mathcal{N}_{E,A,V}^{\tau} \to \mathbb{R}$ is said to satisfy the Palais-Smale condition $(PS)_{c}$ at the level $c$ if every sequence $(u_n)$ such that

$$u_n \in \mathcal{N}_{E,A,V}^{\tau}, \quad J_{E,A,V}(u_n) \to c, \quad \nabla_{\mathcal{N}_{E,A,V}^{\tau}} J_{E,A,V}(u_n) \to 0$$

contains a convergent subsequence. Here $\nabla_{\mathcal{N}_{E,A,V}^{\tau}} J_{E,A,V}(u)$ denotes the orthogonal projection of $\nabla J_{E,A,V}(u)$ onto the tangent space to $\mathcal{N}_{E,A,V}^{\tau}$ at $u$.

In Lemma 3.4 of [8] the following result was proved for $\varepsilon = 1$.

**Proposition 2.2.** For every $\varepsilon > 0$, the functional $J_{E,A,V} : \mathcal{N}_{E,A,V}^{\tau} \to \mathbb{R}$ satisfies $(PS)_{c}$ at each level

$$c < \varepsilon^3 \min_{x \in \mathbb{R}^3 \setminus \{0\}} \#G x V_{\infty}^{3/2} E_1,$$

where $V_{\infty} = \liminf_{|x| \to \infty} V(x)$.

### 3 The limit problem

For any positive real number $\lambda$ we consider the problem

$$
\begin{align*}
-\Delta u + \lambda u &= \left( \frac{1}{|x|^2} * u^2 \right) u, \\
u \in H^1(\mathbb{R}^3, \mathbb{R}).
\end{align*}
$$

(11)
Its associated energy functional \( J_\lambda : H^1(\mathbb{R}^3, \mathbb{R}) \to \mathbb{R} \) is given by
\[
J_\lambda (u) = \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{4} \mathbb{D}(u), \quad \text{with} \quad \|u\|_\lambda^2 = \int_{\mathbb{R}^3} \left( |\nabla u|^2 + \lambda u^2 \right) .
\]
Its Nehari manifold will be denoted by
\[
\mathcal{M}_\lambda = \left\{ u \in H^1(\mathbb{R}^3, \mathbb{R}) \mid u \neq 0, \ \|u\|_\lambda^2 = \mathbb{D}(u) \right\} .
\]
We set
\[
E_\lambda = \inf_{u \in \mathcal{M}_\lambda} J_\lambda (u).
\]
The critical points of \( J_\lambda \) on \( \mathcal{M}_\lambda \) are the nontrivial solutions to \( (\text{i}) \). Note that \( u \) solves the real-valued problem
\[
\begin{align*}
-\Delta u + u &= \left( \frac{1}{|x|^2} u^2 \right) u, \\
 u &\in H^1(\mathbb{R}^3, \mathbb{R})
\end{align*}
\]
if and only if \( u_\lambda (x) = \sqrt{\lambda} u(\sqrt{\lambda} x) \) solves \( (\text{i}) \). Therefore,
\[
E_\lambda = \lambda^{3/2} E_1,
\]
where \( E_1 \) is the least energy of a nontrivial solution to \( (\text{ii}) \). Minimizers of \( J_\lambda \) on \( \mathcal{M}_\lambda \) are called ground states. The existence and uniqueness of ground states up to sign and translations was established by Lieb in \( [16] \). We denote by \( \omega_\lambda \) the positive solution to problem \( (\text{ii}) \) which is radially symmetric with respect to the origin.

Fix a radial function \( \rho \in C^\infty(\mathbb{R}^3, \mathbb{R}) \) such that \( \rho (x) = 1 \) if \( |x| \leq \frac{1}{2} \) and \( \rho (x) = 0 \) if \( |x| \geq 1 \). For \( \varepsilon > 0 \) set \( \rho_\varepsilon (x) = \rho (\sqrt{\varepsilon} x) \), \( \omega_{\lambda, \varepsilon} = \rho_\varepsilon \omega_\lambda \) and
\[
v_{\lambda, \varepsilon} = \frac{\|\omega_{\lambda, \varepsilon}\|_1}{\sqrt{\mathbb{D}(\omega_{\lambda, \varepsilon})}} \omega_{\lambda, \varepsilon} .
\]
Note that \( \text{supp}(v_{\lambda, \varepsilon}) \subset B(0, 1/\sqrt{\varepsilon}) = \{ x \in \mathbb{R}^3 \mid |x| \leq 1/\sqrt{\varepsilon} \} \) and \( v_{\lambda, \varepsilon} \in \mathcal{M}_\lambda \). An easy computation shows that
\[
\lim_{\varepsilon \to 0} J_\lambda (v_{\lambda, \varepsilon}) = \lambda^{3/2} E_1 .
\]
Now we define
\[
\ell_{G, V} = \inf_{x \in \mathbb{R}^3} (\#Gx) V^{3/2}(x)
\]
and consider the set
\[
M_\tau = \{ x \in \mathbb{R}^3 \mid (\#Gx) V^{3/2}(x) = \ell_{G, V} , \ G_x \subset \ker \tau \}.
\]
Here \( Gx = \{ gx \mid g \in G \} \) is the \( G \)-orbit of the point \( x \in \mathbb{R}^3 \), \( \#Gx \) is its cardinality, and \( G_x = \{ g \in G \mid gx = x \} \) is its isotropy subgroup. Observe that the points in \( M_\tau \) are not necessarily local minima of \( V \).

In what follows we will assume that there exists \( \alpha > 0 \) such that the set
\[
\left\{ y \in \mathbb{R}^3 \mid (\#Gy) V^{3/2}(y) \leq \ell_{G, V} + \alpha \right\}
\]
is compact. Then
\[
M_{G, V} = \left\{ y \in \mathbb{R}^3 \mid (\#Gy) V^{3/2}(y) = \ell_{G, V} \right\}
\]
is a compact $G$-invariant set and all $G$-orbits in $M_{G,V}$ are finite. We split $M_{G,V}$ according to the orbit type of its elements, choosing subgroups $G_1, \ldots, G_m$ of $G$ such that the isotropy subgroup $G_x$ of every point $x \in M_{G,V}$ is conjugate to precisely one of the $G_i$’s, and we set

$$M_i = \{ y \in M_{G,V} \mid G_y = gG_x g^{-1} \text{ for some } g \in G \}.$$  

Since isotropy subgroups satisfy $G_{gx} = gG_x g^{-1}$, the sets $M_i$ are $G$-invariant and, since $V$ is continuous, they are closed and pairwise disjoint, and

$$M_{G,V} = M_1 \cup \cdots \cup M_m.$$  

Moreover, since $|G/G_i| = 3/2(\#Gy)$ for all $y \in M_i$, the potential $V$ is constant on each $M_i$. Here $|G/G_i|$ denotes the index of $G_i$ in $G$. We denote by $V_i$ the value of $V$ on $M_i$.

It is well known that the map $G/G_\xi \to G\xi$ given by $gG_\xi \mapsto g\xi$ is a homeomorphism, see e.g. [11].

Let $u_{l,e} = v_{l,e}$ be defined as in [13] with $\lambda = V_i$. Set

$$\psi_{\xi,e}(x) = \sum_{g\xi \in G_\xi} \tau(g) u_{l,e} \left( \frac{x - g\xi}{e} \right) e^{-\lambda \xi (\frac{x - g\xi}{e})}.$$  

Let $\pi_{e,\xi} : H^1_{l,A}(\mathbb{R}^3, \mathbb{C}) \setminus \{0\} \to \mathcal{M}^\tau_{l,A,V}$ be the radial projection given by

$$\pi_{e,A,V}(u) = \frac{\epsilon ||u||_{A,V}}{D(u)} u.$$  

We can derive the following results, arguing as in Lemmas 2 in [6] (see also Lemma 4.2 in [9]).

**Lemma 3.1.** Assume that $G_i \subset \ker \tau$. Then, the following hold:

(a) For every $\xi \in M_i$ and $\epsilon > 0$, one has that

$$\psi_{e,\xi}(gx) = \tau(g) \psi_{e,\xi}(x) \quad \forall g \in G, \; x \in \mathbb{R}^3.$$  

(b) For every $\xi \in M_i$ and $\epsilon > 0$, one has that

$$\tau(g) \psi_{e,\xi}(x) = \psi_{e,\xi}(x) \quad \forall g \in G, \; x \in \mathbb{R}^3.$$  

(c) One has that

$$\lim_{\epsilon \to 0} \epsilon^{-3} J_{e,A,V} [\pi_{e,A,V}(\psi_{e,\xi})] = \ell_{G,V} E_1.$$  

uniformly in $\xi \in M_i$.  

Let
\[ M_\tau = \{ y \in M_{G,V} \mid G_y \subset \ker \tau \} = \bigcup_{G \subset \ker \tau} M_I. \]

As immediate consequence of Lemma 3.1, we derive the following result.

**Proposition 3.2.** The map \( \hat{\iota}_\varepsilon : M_\tau \to M_{\varepsilon A,V} \) given by
\[ \hat{\iota}_\varepsilon (\xi) = \pi_{\varepsilon A,V} (\psi_{\xi, \varepsilon}) \]
is well defined and continuous, and satisfies
\[ \tau(g) \hat{\iota}_\varepsilon (g \xi) = \hat{\iota}_\varepsilon (\xi) \quad \forall \xi \in M_\tau, g \in G. \]

Moreover, given \( d > \ell_{G,V} E_1 \), there exists \( \varepsilon_d > 0 \) such that
\[ \varepsilon^{-3} J_{\varepsilon A,V} (\hat{\iota}_\varepsilon (\xi)) \leq d \quad \forall \xi \in M_\tau, \xi \in (0, \varepsilon_d). \]

**4 The baryorbit map**

Let us consider the real-valued problem
\[
\begin{cases}
-\varepsilon^2 \Delta v + V(x)v = \frac{1}{\varepsilon^2} \left( \frac{1}{|v|^4} + u^2 \right) u, \\
v \in H^1(\mathbb{R}^3, \mathbb{R}), \\
v(gx) = v(x) \quad \forall x \in \mathbb{R}^3, \ g \in G.
\end{cases}
\]  

(17)

Set
\[ H^1(\mathbb{R}^3, \mathbb{R})^G = \{ v \in H^1(\mathbb{R}^3, \mathbb{R}) \mid v(gx) = v(x) \ \forall x \in \mathbb{R}^3, \ g \in G \} \]
and write
\[ \|v\|^2_V = \int_{\mathbb{R}^3} \left( \varepsilon^2 |\nabla v|^2 + V(x)v^2 \right). \]

The nontrivial solutions of (17) are the critical points of the energy functional
\[ J_{\varepsilon,V}(v) = \frac{1}{2} \|v\|^2_{\varepsilon,V} - \frac{1}{4\varepsilon^2} D(v) \]
on the Nehari manifold
\[ \mathcal{M}^G_{\varepsilon,V} = \left\{ v \in H^1(\mathbb{R}^3, \mathbb{R})^G \mid v \neq 0, \|v\|^2_{\varepsilon,V} = \varepsilon^{-2} D(v) \right\}. \]

Set
\[ c_{\varepsilon,V}^G = \inf_{\mathcal{M}^G_{\varepsilon,V}} J_{\varepsilon,V} = \inf_{v \in H^1(\mathbb{R}^3, \mathbb{R})^G, v \neq 0} \frac{\varepsilon^2 \|v\|^4_{\varepsilon,V}}{4D(v)}. \]  

(18)

As proved in Lemma 5.1 in [9] we have

**Lemma 4.1.** There results
\[ 0 < (\inf_{\mathbb{R}^3} V)^{3/2} E_1 \leq \varepsilon^{-3} c_{\varepsilon,V}^G \quad \text{for every } \varepsilon > 0, \]
and
\[ \limsup_{\varepsilon \to 0} \varepsilon^{-3} c_{\varepsilon,V}^G \leq \ell_{G,V} E_1, \]
We fix $\hat{\rho} > 0$ such that

$$
\begin{cases}
   |y - gy| > 2\hat{\rho} & \text{if } g y \neq y \in M_{G,W}, \\
   \text{dist}(M_i, M_j) > 2\hat{\rho} & \text{if } i \neq j,
\end{cases}
$$

(19)

where $G_i$, $M_i$, $V_i$ are the groups, the sets and the values defined as in Section 3.

For $\rho \in (0, \hat{\rho})$, let

$$
M_{\rho}^0 = \{ y \in \mathbb{R}^3 : \text{dist}(y, M_i) \leq \rho, \ G_y = g G_i g^{-1} \text{ for some } g \in G \},
$$

and for each $\xi \in M_{\rho}^0$ and $\varepsilon > 0$, define

$$
\Theta_{\varepsilon, \xi}(x) = \sum_{g \xi \in G_\xi} \omega_i \left( \frac{x - g \xi}{\varepsilon} \right),
$$

where $\omega_i$ is unique positive ground state of problem (11) with $\lambda = V_i$ which is radially symmetric with respect to the origin. Set

$$
\Theta_{\rho, \varepsilon} = \{ \Theta_{\varepsilon, \xi} : \xi \in M_{\rho}^0 \cup \cdots \cup M_m^0 \}.
$$

Arguing as in Proposition 5 in [8], we can derive the following result.

**Proposition 4.2.** Given $\rho \in (0, \hat{\rho})$ there exist $d_\rho > \ell_{G,V} E_1$ and $\varepsilon_\rho > 0$ with the following property:

For every $\varepsilon \in (0, \varepsilon_\rho)$ and every $v \in J_{E,V}$ with $J_{E,V}(v) \leq \varepsilon^3 d_\rho$ there exists precisely one $G$-orbit $G_{\xi_{\varepsilon,v}}$ with $\xi_{\varepsilon,v} \in M_{\rho}^0 \cup \cdots \cup M_m^0$ such that

$$
\varepsilon^{-3} \left\| |v| - \Theta_{\varepsilon, \xi_{\varepsilon,v}} \right\|_{E,V}^2 = \min_{\theta \in \Theta_{\rho, \varepsilon}} \left\| |v| - \theta \right\|_{E,V}^2.
$$

For every $c \in \mathbb{R}$ we set

$$
J_{E,V}^c = \{ v \in J_{E,V} : J_{E,V}(v) \leq c \}.
$$

Proposition 4.2 allows us to define, for each $\rho \in (0, \hat{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$, a local baryorbit map

$$
\hat{\beta}_{\rho, \varepsilon, 0} : J_{E,V}^{\varepsilon d_\rho} \longrightarrow (M_{\rho}^0 \cup \cdots \cup M_m^0) / G
$$

by taking

$$
\hat{\beta}_{\rho, \varepsilon, 0}(v) = G_{\xi_{\varepsilon,v}},
$$

where $G_{\xi_{\varepsilon,v}}$ is the unique $G$-orbit given by the previous proposition.

Coming back to our original problem, for every $c \in \mathbb{R}$ set

$$
J_{E,A,V}^c = \{ u \in J_{E,A,V} : J_{E,A,V}(u) \leq c \}.
$$

The following holds.

**Corollary 4.3.** For each $\rho \in (0, \hat{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$, the local baryorbit map

$$
\hat{\beta}_{\rho, \varepsilon} : J_{E,A,V}^{\varepsilon d_\rho} \longrightarrow (M_{\rho}^0 \cup \cdots \cup M_m^0) / G,
$$

given by

$$
\hat{\beta}_{\rho, \varepsilon}(u) = \hat{\beta}_{\rho, \varepsilon, 0}(\hat{\kappa}_\varepsilon(|u|)),
$$
where \( \hat{\nu}_\epsilon : H^1(\mathbb{R}^3, \mathbb{R})^G \setminus \{0\} \rightarrow \mathcal{M}^G_\epsilon \) is the radial projection, is well defined and continuous. It satisfies
\[
\hat{\nu}_\epsilon(\gamma u) = \hat{\nu}_\epsilon(u) \quad \forall \gamma \in S^1,
\]
\[
\hat{\nu}_\epsilon(\xi) = \xi \quad \forall \xi \in M_\tau \text{ with } J_{E,A,V}(t_\tau(\xi)) \leq \epsilon^3 d_\rho,
\]
where \( \hat{\tau}_\epsilon \) is the map defined in Proposition 3.2.

**Proof.** If \( u \in \mathcal{N}_{E,A,V} \) then \( \hat{\nu}_\epsilon(|u|) \in \mathcal{M}^G_\epsilon \). The diamagnetic inequality yields
\[
J_{E,V}(\hat{\nu}_\epsilon(|u|)) \leq J_{E,A,V}(u).
\]
Inequality (20) yields \( \lim_{\epsilon \to \infty} \epsilon^{-3} c^\tau_{E,A,V} = \ell_{G,V} E_1 \).

**Corollary 4.4.** If there exists \( \xi \in \mathbb{R}^3 \) such that \( (\#G_\xi)V^{3/2}(\xi) = \ell_{G,V} \) and \( G_\xi \subset \ker \tau \), then
\[
\lim_{\epsilon \to \infty} \epsilon^{-3} c^\tau_{E,A,V} = \ell_{G,V} E_1,
\]
where \( c^\tau_{E,A,V} = \inf_{E \in \mathcal{N}_{E,A,V}} J_{E,A,V} \).

**Proof.** Inequality (20) yields \( c^\tau_{E,V} = \inf_{E \in \mathcal{N}_{E,A,V}} J_{E,V} \leq \inf_{E \in \mathcal{N}_{E,A,V}} J_{E,A,V} = c^\tau_{E,A,V} \). By statement (c) of Lemma 3.1
\[
\ell_{G,V} E_1 = \lim_{\epsilon \to \infty} \epsilon^{-3} c^\tau_{E,V} \leq \liminf_{\epsilon \to 0} \epsilon^{-3} c^\tau_{E,A,V} \leq \limsup_{\epsilon \to \infty} \epsilon^{-3} c^\tau_{E,A,V} \leq \ell_{G,V} E_1.
\]

### 5 Multiplicity results via Equivariant Morse theory

We start by reviewing some well known facts on equivariant Morse theory. We refer the reader to [3][32] for further details.

**Definition 5.1.** Let \( \Gamma \) be a compact Lie group and \( X \) be a \( \Gamma \)-space.

- The \( \Gamma \)-orbit of a point \( x \in X \) is the set \( \Gamma x := \{ \gamma x \mid \gamma \in \Gamma \} \).
- A subset \( A \) of \( X \) is said to be \( \Gamma \)-invariant if \( \Gamma x \subset A \) for every \( x \in A \). The \( \Gamma \)-orbit space of \( A \) is the set \( A/\Gamma := \{ \Gamma x : x \in A \} \) with the quotient space topology.
- \( X \) is called a free \( \Gamma \)-space if \( \gamma x \neq x \) for every \( \gamma \in \Gamma, x \in X \).
- A map \( f : X \to Y \) between \( \Gamma \)-spaces is called \( \Gamma \)-invariant if \( f \) is constant on each \( \Gamma \)-orbit of \( X \), and it is called \( \Gamma \)-equivariant if \( f(\gamma x) = \gamma f(x) \) for every \( \gamma \in \Gamma, x \in X \).

We fix a field \( \mathbb{K} \) and denote by \( \mathcal{H}^*(X,A) \) the Alexander-Spanier cohomology of the pair \( (X,A) \) with coefficients in \( \mathbb{K} \). If \( X \) is a \( \Gamma \)-pair, i.e. if \( X \) is a \( \Gamma \)-space and \( A \) is a \( \Gamma \)-invariant subset of \( X \), we write
\[
\mathcal{H}^*(\Gamma \times \Gamma X, \Gamma \times \Gamma A) := \mathcal{H}^*(E\Gamma \times \Gamma X, E\Gamma \times \Gamma A)
\]
for the Borel-cohomology that pair. \( E\Gamma \) is the total space of the classifying \( \Gamma \)-bundle and \( E\Gamma \times \Gamma X \) is the orbit space \( (E\Gamma \times X)/\Gamma \) (see e.g. [11 Chapter III]). If \( X \) is a free \( \Gamma \)-space, as will be the case in our application, then the projection \( E\Gamma \times \Gamma X \to X/\Gamma \) is a homotopy equivalence and it induces an isomorphism
\[
\mathcal{H}^*(\Gamma \times \Gamma X, \Gamma \times \Gamma A) \cong \mathcal{H}^*(X/\Gamma, A/\Gamma).
\]
In our setting, $\Gamma = S^1$; if $A \subset X$ are $S^1$-invariant subsets of $\mathcal{M}^r_{e,A,V}$ we denote by $X/S^1$ and $A/S^1$ their $S^1$-orbit spaces and by (21) it is legitimate to write

$$\mathcal{H}^r_{S^1}(X,A) \simeq \mathcal{H}^r(X/S^1,A/S^1).$$

If $S^1u$ is an isolated critical $S^1$-orbit of $J_{e,A,V}$ its $k$-th critical group is defined as

$$C^k_{S^1}(J_{e,A,V}, S^1u) = \mathcal{H}^k(J^r_{e,A,V} \cap U, (J^r_{e,A,V} \setminus S^1u) \cap U),$$

where $U$ is an $S^1$-invariant neighborhood of $S^1u$ in $\mathcal{M}^r_{e,A,V}$, $c = J_{e,A,V}(u)$. Its total dimension

$$\mu(J_{e,S^1u}) = \sum_{k=0}^{\infty} \dim C^k_{S^1}(J_{e,A,V}, S^1u)$$

is called the multiplicity of $S^1u$. If $S^1u$ is nondegenerate and $J_{e,A,V}$ satisfies the Palais-Smale condition in some neighborhood of $c$, then

$$\dim C^k_{S^1}(J_{e,A,V}, S^1u) = 1$$

if $k$ is the Morse index of $J_{e,A,V}$ at the critical submanifold $S^1u$ of $\mathcal{M}^r_{e,A,V}$ and it is 0 otherwise.

Moreover, for $\rho > 0$ we set

$$B_\rho M_\tau = \{x \in \mathbb{R}^3 \mid \text{dist}(x, M_\tau) \leq \rho\}$$

and write $i_\rho : M_\tau/G \hookrightarrow B_\rho M_\tau/G$ for the embedding of the $G$-orbit space of $M_\tau$ in that of $B_\rho M_\tau$. We will show that this embedding has an effect on the number of solutions of (4) for $\epsilon$ small enough.

**Lemma 5.2.** For every $\rho \in (0, \bar{\rho})$ and $d \in (\ell_{G,V} E_1, d_0)$, with $d_0$ as in Proposition 3.2 there exists $\epsilon_{\rho,d} > 0$ such that

$$\dim \mathcal{H}^k(J^r_{e,A,V}/S^1) \geq \text{rank} \left( i_\rho^* : \mathcal{H}^k(B_\rho M_\tau/G) \rightarrow \mathcal{H}^k(M_\tau/G) \right)$$

for every $\epsilon \in (0, \epsilon_{\rho,d})$ and $k \geq 0$, where $i_\rho : M_\tau/G \hookrightarrow B_\rho M_\tau/G$ is the inclusion map.

**Proof.** Let $\epsilon_{\rho,d} = \min\{\epsilon_d, \epsilon_\rho\}$ where $\epsilon_\rho$ is as in Proposition 4.2 and $\epsilon_d$ is as in Proposition 3.2. Fix $\epsilon \in (0, \epsilon_{\rho,d})$. Then,

$$J_{e,A,V}(i_\epsilon(\xi)) \leq \epsilon^3 d \quad \text{and} \quad \beta_{\rho,x}(i_\epsilon(\xi)) = \xi \quad \forall \xi \in M_\tau.$$ 

By Proposition 3.2 and Corollary 4.3 the maps

$$M_\tau/G \xrightarrow{\iota_{\epsilon}} J^r_{e,A,V}/S^1 \xrightarrow{\beta_{\rho,x}} B_\rho M/G$$

given by $i_\epsilon(G_\xi) = i_\epsilon(\xi)$ and $\beta_{\rho,x}(S^1u) = \beta_{\rho,x}(u)$ are well defined and satisfy $\beta_{\rho,x}(i_\epsilon(G_\xi)) = G_\xi$ for all $\xi \in M_\tau$. Note that $M_\tau = \bigcup \{M_i \mid G_i \subset \ker \tau\}$ is the union of some connected components of $M$. Moreover, our choice of $\bar{\rho}$ implies that $B_\rho M_\tau \cap B_\rho (M \setminus M_\tau) = \emptyset$. Therefore the inclusion $i_{\tau,\rho} : B_\rho M_\tau/G \hookrightarrow B_\rho M/G$ induces an epimorphism in cohomology. Since $\beta_{\rho,x} \circ i_\epsilon = i_{\tau,\rho} \circ i_\rho$ we conclude that

$$\text{rank}(\beta_{\rho,x} \circ i_\epsilon) = \text{rank}(\beta_{\rho,x} \circ i_{\tau,\rho} \circ i_\rho) = \text{rank}(i_\rho^* : \mathcal{H}^k(B_\rho M/G) \rightarrow \mathcal{H}^k(M_\tau/G))$$

as claimed. \qed
We are ready to prove our main theorem.

**Theorem 5.3.** Assume there exists $\alpha > 0$ such that the set
\[
\{ x \in \mathbb{R}^3 \mid (\# Gx) V^{3/2}(x) \leq \ell_{G,V} + \alpha \}.
\] (22)
is compact. Then, given $\rho > 0$ and $\delta \in (0, \alpha)$, there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ one of the following two assertions holds:

(a) $J_{E,A,V}$ has a nonisolated $\tau$-intertwining critical $S^1$-orbit in the set $J_{E,A,V}^{-1}[\varepsilon^3(\ell_{G,V}E_1 - \delta), \varepsilon^3(\ell_{G,V}E_1 + \delta)]$.

(b) $J_{E,A,V}$ has finitely many $\tau$-intertwining critical $S^1$-orbits $S^1 u_1, S^1 u_2, \ldots, S^1 u_m$ in $J_{E,A,V}^{-1}[\varepsilon^3(\ell_{G,V}E_1 - \delta), \varepsilon^3(\ell_{G,V}E_1 + \delta)]$. They satisfy

\[
\sum_{j=1}^m \dim C_{S^1}(J_{E,A,V}(S^1 u_j)) \geq \text{rank}(i_{\rho}^* : \mathcal{H}^k(B_{\rho}M_\tau/G) \to \mathcal{H}^k(M_\tau/G))
\]
for every $k \geq 0$.

In particular, if every $\tau$-intertwining critical $S^1$-orbit of $J_{E,A,V}$ in the set $J_{E,A,V}^{-1}[\varepsilon^3(\ell_{G,V}E_1 - \delta), \varepsilon^3(\ell_{G,V}E_1 + \delta)]$ is nondegenerate then, for every $k \geq 0$, there are at least

\[
\text{rank}(i_{\rho}^* : \mathcal{H}^k(B_{\rho}M_\tau/G) \to \mathcal{H}^k(M_\tau/G))
\]
of them having Morse index $k$ for every $k \geq 0$.

**Proof.** Assume $M_\tau \neq \emptyset$ and let $\rho > 0$ and $\delta \in (0, \alpha E_1)$ be given. Without loss of generality we may assume that $\rho \in (0, \bar{\rho})$. Assumption (22) implies that

\[
\ell_{G,V} + \alpha \leq \min_{x \in \mathbb{R}^3 \setminus \{0\}} (\# Gx)V^{3/2}
\]
where $V_\infty = \limsup_{|x| \to \infty} V(x)$. By Proposition 2.2 the functional

\[
J_{E,A,V} : N_{E,A,V}^\tau \to \mathbb{R}
\]
satisfies $(PS)_c$ at each level $c \leq \varepsilon^3(\ell_{G,V}E_1 + \delta)$ for every $\varepsilon > 0$. By Corollary 4.4 there exists $\varepsilon_0 > 0$ such that

\[
\ell_{G,V}E_1 - \delta < \varepsilon^{-3} \inf_{u \in J_{E,A,V}} \forall \varepsilon \in (0, \varepsilon_0).
\]

Let $d \in (\ell_{G,V}E_1, \min\{d_\rho, \ell_{G,V}E_1 + \delta\})$ with $d_\rho$ as in Proposition 4.2 and $\bar{\varepsilon} = \min\{\varepsilon_0, \varepsilon_{\rho,d}\}$ with $\varepsilon_{\rho,d}$ as in Lemma 5.2. Fix $\varepsilon \in (0, \bar{\varepsilon})$ and for $u \in N_{E,A,V}$ with $J_{E,A,V}(u) = c$ set

\[
C_{S^1}(J_{E,A,V}(S^1 u)) = \mathcal{H}^k((J_{E,A,V}^{-1}(U)/S^1, ((J_{E,A,V}^{-1}(S^1 u) \cup U)/S^1).
\]

Assume that every critical $S^1$-orbit of $J_{E,A,V}$ lying in $J_{E,A,V}^{-1}[\varepsilon^3(\ell_{G,V}E_1 - \delta), \varepsilon^3(\ell_{G,V}E_1 + \delta)]$ is isolated. Since $J_{E,A,V} : N_{E,A,V}^\tau \to \mathbb{R}$ satisfies $(PS)_c$ at each $c \leq \varepsilon^3(\ell_{G,V}E_1 + \delta)$ there are only finitely many of them. Let $S^1 u_1, \ldots, S^1 u_m$ be those critical $S^1$-orbits of $J_{E,A,V}$ in $N_{E,A,V}^\tau$ which satisfy $J_{E,A,V}(u_i) < \varepsilon^3 d$. 

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Applying Theorem 7.6 in \cite{3} to $J_{\varepsilon,A,V}: \mathcal{M}_{\varepsilon,A,V} \to \mathbb{R}$ with $a = \varepsilon^3(\ell_{G,V} E_1 - \delta)$ and $b = \varepsilon^3 d$ and Lemma 5.2 we obtain that

$$\sum_{j=1}^{m} \dim \mathcal{C}^j_{\mathbb{S}^1}(J_{\varepsilon,A,V}, \mathbb{S}^1 u_i) \geq \dim \mathcal{H}^k(J_{\varepsilon,A,V}/\mathbb{S}^1)$$

$$\geq \text{rank} \left( \mathcal{H}^k(B_p M_{\varepsilon}/G) \to \mathcal{H}^k(M_{\varepsilon}/G) \right)$$

for every $k \geq 0$, as claimed. The last assertion of Theorem 5.3 is an immediate consequence of Theorem 7.6 in \cite{3}. \hfill \Box

If the inclusion $i_\rho : M_{\varepsilon}/G \hookrightarrow B_p M_{\varepsilon}/G$ is a homotopy equivalence then

$$\text{rank} \left( \mathcal{H}^k(B_p M_{\varepsilon}/G) \to \mathcal{H}^k(M_{\varepsilon}/G) \right) = \dim \mathcal{H}^k(M_{\varepsilon}/G).$$

An immediate consequence of Theorem 5.3 is the following.

**Corollary 5.4.** If assumption \cite{22} holds then, given $\rho > 0$ and $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon > 0$ problem \cite{4} has at least

$$\sum_{k=0}^{\infty} \text{rank}(i_\rho^* : \mathcal{H}^k(B_p M_{\varepsilon}/G) \to \mathcal{H}^k(M_{\varepsilon}/G))$$

geometrically different solutions in $J_{\varepsilon,A,V}^{-1}[\{\varepsilon^3(\ell_{G,V} E_1 - \delta), \varepsilon^3(\ell_{G,V} E_1 + \delta)\}]$, counted with their multiplicity.

### 5.1 Examples

As a typical application of our existence result, we consider the constant magnetic field $B(x_1, x_2, x_3) = (0, 0, 2)$ in $\mathbb{R}^3$. We can consider its vector potential $A(x_1, x_2, x_3) = (-x_2, x_1, 0)$, and identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$. With this in mind, we write $A(z, t) = (iz, 0)$, with $z = x_1 + ix_2$. We remark that $A(e^{i\theta} z, t) = e^{i\theta} A(z, t)$ for every $\theta \in \mathbb{R}$.

Given $m \in \mathbb{N}$, $m \geq 1$ and $n \in \mathbb{Z}$, we look for solutions $u$ to problem \cite{4} which satisfy the symmetry property

$$u \left( e^{2\pi i k/m}z, t \right) = e^{2\pi i n/k} u(z, t)$$

for every $k = 1, \ldots, m$ and $(z, t) \in \mathbb{C} \times \mathbb{R}$. We assume that $V$ satisfies

(a) $V \in C^2(\mathbb{R}^3)$ is bounded and $\inf_{\mathbb{R}^3} V > 0$; moreover

$$\inf_{x \in \mathbb{R}^3} V^{3/2}(x) < \liminf_{|x| \to +\infty} V^{3/2}(x).$$

(b) There exists $m_0 \in \mathbb{N}$ such that

$$m_0 \inf_{x \in \mathbb{R}^3} V^{3/2}(x) < \inf_{t \in \mathbb{R}} V^{3/2}(0, t)$$

$$V \left( e^{2\pi i k/m_0}z, t \right) = V(z, t)$$

for every $k = 1, \ldots, m_0$ and $(z, t) \in \mathbb{C} \times \mathbb{R}$.
For each $m$ that divides $m_0$ (in symbols: $m|m_0$), we consider the group

$$G_m = \left\{ e^{2\pi i k/m} \mid k = 1, \ldots, m \right\}$$

acting by multiplication on the $z$-coordinate of each point $(z, t) \in \mathbb{C} \times \mathbb{R}$. It is easy to check that $A$ and $V$ match all the assumptions of Theorem 5.3 for each $G = G_m$: the compactness condition (22) follows from the two inequalities in (a) and (b). If $\tau: G_m \to S^1$ is any homeomorphism, we have that

$$M_\tau = \left\{ x \in \mathbb{R}^3 \mid V(x) = \inf_{y \in \mathbb{R}^3} V(y) \right\}.$$

Given $n \in \mathbb{Z}$, we consider the homeomorphism $\tau(e^{2\pi i k/m}) = e^{2\pi i nk/m}$. In particular, given $\rho, \delta > 0$, for $\varepsilon$ small enough we have

$$\sum_{m|m_0} \sum_{k=0}^\infty m \text{rank} \left( i^*_\rho : \mathcal{H}^k(B_\rho M/G_m) \to \mathcal{H}^k(M/G_m) \right).$$

geometrically distinct solutions, counted with multiplicity.

**Remark 1.** Our multiplicity result cannot be obtained, in general, via standard category arguments. For a concrete example, consider $M = \bigcup_{n \geq 1} S_n$, where

$$S_n = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \left( x_1 - \frac{1}{n} \right)^2 + x_2^2 + x_3^2 = \frac{1}{n^2} \right\}.$$

The category of $M$ is then 2, whereas

$$\lim_{\rho \to 0} \text{rank} \left( i^*_\rho : \mathcal{H}^2(B_\rho M) \to \mathcal{H}^2(M) \right) = +\infty.$$

For a short proof, we refer to [7, Example 1, pag. 1280]

### 6 Appendix

**Proposition 6.1.** The second derivative $J''_{\varepsilon, A, V}$ is continuous.

**Proof.** We first prove that $J''_{\varepsilon, A, V}$ is continuous at zero. Let $\{u_n\}_n$ be a sequence in $H^1_{\varepsilon, A}(\mathbb{R}^3, \mathbb{C})$ converging to zero. By Sobolev’s embedding theorem, $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for $r \in [2, 6)$. From (9) it follows that

$$\left| \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|u_n(y)|^2}{|x-y|} \, dy \right) w(x)\overline{v(x)} \, dx \right| 
\leq C \|u_n\|_{L^{12/5}(\mathbb{R}^3)} \|v\|_{L^{12/5}(\mathbb{R}^3)} \|w\|_{L^{12/5}(\mathbb{R}^3)} 
\leq o(1) \|v\|_{\varepsilon, A, V} \|w\|_{\varepsilon, A, V}$$

(23)

This implies that

$$\lim_{n \to +\infty} \left| \Re \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|u_n(y)|^2}{|x-y|} \, dy \right) w(x)\overline{v(x)} \, dx \right| = 0$$

(24)

whenever $u_n \to 0$ strongly in $H^1_{\varepsilon, A, V}(\mathbb{R}^3, \mathbb{C})$. 

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Similarly, we use (9) to prove that
\[
\left| \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast (u_n \mathbf{v}) \right) u_n \mathbf{w} \, dx \right| = \left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u_n(x)\overline{w(x)}u_n(y)\overline{v(y)}}{|x-y|} \, dx \, dy \right| \leq C \|u_n \mathbf{w}\|_{L^{6/5}(\mathbb{R}^3)} \|u_n \mathbf{v}\|_{L^{6/5}(\mathbb{R}^3)} \leq C \|u_n\|^2_{L^{12/5}(\mathbb{R}^3)} \|\mathbf{v}\|_{L^{12/5}(\mathbb{R}^3)} \|\mathbf{w}\|_{L^{12/5}(\mathbb{R}^3)}
\]
which implies that
\[
\lim_{n \to +\infty} \left| \text{Re} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast (u_n \mathbf{v}) \right) u_n \mathbf{w} \, dx \right| = 0 \tag{25}
\]
whenever \(u_n \to 0\) strongly in \(H^1_{e,AV}(\mathbb{R}^3, \mathbb{C})\). It is now easy to conclude that \(J''_{e,AV}(u_n) \to J''_{e,AV}(0)\).

If \(u_n \to u\) in \(H^1_{e,AV}(\mathbb{R}^3, \mathbb{C})\), we replace \(|u_n|^2\) in (23) with \(u_n^2 = |u_n|^2 - |u_n - u|^2 - |u|^2\) and find
\[
\left| \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|u_n(y)|^2}{|x-y|} \, dy \right) w(x) \overline{v(x)} \, dx \right| \leq C \|u_n^2\|_{L^{6/5}(\mathbb{R}^3)} \|w\|_{e,AV} \|v\|_{e,AV} \leq o(1) \|w\|_{e,AV} \|v\|_{e,AV}.
\]

Analogously
\[
\left| \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|u_n(y) - u(y)|^2}{|x-y|} \, dy \right) w(x) \overline{v(x)} \, dx \right| \leq C \|u_n - u\|^2_{L^{12/5}(\mathbb{R}^3)} \|w\|_{e,AV} \|v\|_{e,AV} \leq o(1) \|w\|_{e,AV} \|v\|_{e,AV},
\]
we conclude that
\[
\left| \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|u_n(y)|^2 - |u(y)|^2}{|x-y|} \, dy \right) w(x) \overline{v(x)} \, dx \right| \leq o(1) \|w\|_{e,AV} \|v\|_{e,AV}. \tag{26}
\]

Switching to the second term of \(J''_{e,AV}(u_n) - J''_{e,AV}(u)\), we notice that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u_n(x)\overline{w(x)}u_n(y)\overline{v(y)}}{|x-y|} \, dx \, dy - \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(x)\overline{w(x)}u(y)\overline{v(y)}}{|x-y|} \, dx \, dy = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{[(u_n(y) - u(y))u_n(x) + (u_n(x) - u(x))u(y)] v(y)w(x)}{|x-y|} \, dx \, dy,
\]
so that
\[
\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u_n(x)\overline{w(x)}u_n(y)\overline{v(y)}}{|x-y|} \, dx \, dy - \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(x)\overline{w(x)}u(y)\overline{v(y)}}{|x-y|} \, dx \, dy \right| \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|(u_n(y) - u(y))u_n(x)| v(y)w(x)}{|x-y|} \, dx \, dy + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|(u_n(x) - u(x))u(y)| v(y)w(x)}{|x-y|} \, dx \, dy \leq o(1) \|v\|_{e,AV} \|w\|_{e,AV} \tag{27}
\]
because \(u_n \to u\). Recalling that \(|\text{Re} z| \leq |z|\) for every \(z \in \mathbb{C}\) and putting together (26) and (27), we conclude that \(J''_{e,AV}(u_n) \to J''_{e,AV}(u)\).
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