Characterization of parameters with a mixed bias property

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Abstract

In this article we characterize a class of parameters in large non-parametric models that admit rate doubly robust estimators. An estimator of a parameter of interest which relies on non-parametric estimators of two nuisance functions is rate doubly robust if it is consistent and asymptotically normal when one succeeds in estimating both nuisance functions at sufficiently fast rates, with the possibility of trading off slower rates of convergence for estimators of one of the nuisance functions with faster rates of convergence of the estimator of the other nuisance function. Our class is defined by the property that the bias of the one step estimator of the parameter of interest is the mean of the product of the estimation errors of the two nuisance functions. We call this property the mixed bias property. We show that our class strictly includes two recently studied classes of parameters that satisfy the mixed biased property and which include many important parameters of interest in causal inference. For parameters in our class we characterize their form and the form of their influence functions. Furthermore, we derive two functional moment equations, each being solved at one of the two nuisance functions. In addition, we derive two functional loss functions, each having expectation that is minimized at one of the two nuisance functions. Both the moment equations and the loss functions are important because they can be used to derive loss based penalized estimators of the nuisance functions.

1 Introduction

Suppose that we are given a sample \( \mathcal{D}_n \) of \( n \) i.i.d. copies of a random vector \( O \) with law \( P \) which is known to belong to a collection of laws \( \mathcal{M} = \{ P_\eta : \eta \in \eta \} \) where \( \eta \) is a large, non-Euclidean, parameter

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space. Our goal is to estimate the value taken by a scalar parameter \( \chi(\eta) \) at \( P \). Suppose \( O \) includes a vector \( Z \) with sample space \( Z \subset \mathbb{R}^d \). We are interested in parameters \( \chi(\eta) \) which depend on \( \eta \) through some unknown function of the covariates \( Z \), such as a conditional mean given \( Z \) or a density of \( Z \).

Given an initial estimator \( \hat{\eta} \), the plug-in estimator \( \chi(\hat{\eta}) \) is a natural choice for estimating \( \chi(\eta) \). However, except for special estimators \( \hat{\eta} \) targeted to specific parameters \( \chi(\eta) \), \( \chi(\hat{\eta}) \) is not \( \sqrt{n} \)-consistent.

A strategy for reducing the bias of \( \chi(\hat{\eta}) \) is to subtract from it an estimate \(-\mathbb{P}_n\chi^1_\eta \) of its first order bias \cite{Robins2017}, where for each \( \eta \), \( \chi^1_\eta \equiv \chi^1_\eta(O) \) is an adequately chosen random variable and \( \mathbb{P}_n h \) is the empirical mean operator \( n^{-1} \sum_{i=1}^n h(O_i) \). This strategy yields the one step estimator \( \hat{\chi} \equiv \chi(\hat{\eta}) + \mathbb{P}_n\chi^1_\eta \). It is well known from Semiparametric theory that a good choice for \( \chi(\eta) \) is of order \( O\left(\sqrt{n}\right) \) times the sample average of mean zero random variables, so it converges to a normal distribution.

Many parameters \( \chi(\eta) \) of interest in Causal Inference and Econometrics have influence functions. Thus we pick \( \chi(\eta) \) of interest in Causal Inference and Econometrics have influence functions.

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\begin{align}
\chi(\hat{\eta}) - \chi(\eta) & = \mathbb{P}_n \left\{ \chi^1_\eta \right\} + \mathbb{G}_n \left\{ \chi^1_\eta - \chi_\eta \right\} + \mathbb{G}_n \left\{ \chi_\eta \right\}.
\end{align}

where \( \mathbb{G}_n \left\{ \chi^1_\eta \right\} \equiv \left\{ \chi^1_\eta(o) dP_n(o) \right\} \) and \( \mathbb{G}_n \left\{ \chi^1_\eta \right\} \equiv \sqrt{n} \mathbb{P}_n \left\{ \chi^1_\eta - \chi_\eta \right\} \).

The term \( \mathbb{G}_n \left\{ \chi^1_\eta \right\} \) is a \( \sqrt{n} \) times the sample average of mean zero random variables, so it converges to a normal distribution. On the other hand, if model \( M \) is not too big, then for estimators \( \hat{\eta} \) converging to \( \eta \) one may expect \( \mathbb{G}_n \left\{ \chi^1_\eta - \chi^1_\eta \right\} \) to be \( o_p \left(1\right) \). One can make this term be \( o_p \left(1\right) \) without making model size requirements by splitting the sample \( D_n \) into two samples, and computing \( \hat{\eta} \) from one subsample and the one step estimator from the other subsample. The efficiency lost due to sample splitting can be recovered by computing a second one step estimator switching the roles of the two subsamples and then computing the final estimator \( \hat{\chi} \) of \( \chi(\eta) \) as the average of the two one step estimators. This process is known as cross-fitting. \cite{Schick1986, Van der Vaart2000, Ayyagari2010} Ph.D. thesis (subsequently published as \cite{Robins2013} and \cite{Zheng2011}).

If cross-fitting is employed, then \( \sqrt{n} \left\{ \hat{\chi} - \chi(\eta) \right\} \) converges to a normal distribution if

\begin{align}
\chi(\hat{\eta}) - \chi(\eta) - E_\eta \left( \chi^1_\eta \right)
\end{align}

is \( o_p \left(n^{-1/2}\right) \). For estimators \( \hat{\eta} \) such that \( n^{1/4} ||\hat{\eta} - \eta|| = o_p \left(1\right) \) for some norm \( ||\cdot|| \) , this is achieved if \( \hat{\eta} \) is of order \( O\left(||\hat{\eta} - \eta||^2\right) \). This suggests that we pick \( \chi^1_\eta \) to be an influence function of \( \chi(\eta) \) since for such choice, \( E_\eta \left( \chi^1_\eta \right) \) acts like minus the functional derivative of \( \chi(\eta) \) in the direction \( \hat{\eta} - \eta \), so that \( \hat{\eta} \) acts like the residual from a first order Taylor’s expansion of \( \chi(\eta) \), and hence is of order \( O\left(||\hat{\eta} - \eta||^2\right) \).

See \cite{Chapter 25, Van der Vaart2000} for the definition of influence function. Parameters that admit influence functions are called regular parameters. Such parameters have a unique influence function if the model \( M \) is non-parametric. By non-parametric we mean that the closed linear span of the scores for all parametric submodels at \( P \) of model \( M \) is equal to model \( L_2(P) \). Throughout we will assume that \( \chi(\eta) \) is regular and that \( M \) is non-parametric.

Many parameters \( \chi(\eta) \) of interest in Causal Inference and Econometrics have influence functions which satisfy the following property.

\begin{definition}[Mixed bias property]
For each \( \eta \) there exist functions \( a(Z) \equiv a(Z; \eta) \) and \( b(Z) \equiv b(Z; \eta) \) such that for any \( \eta' \):

\begin{align}
\chi(\eta') - \chi(\eta) + E_\eta \left( \chi^1_\eta \right) = E_\eta \left[ S_{ab} \{ a'(Z) - a(Z) \} \{ b'(Z) - b(Z) \} \right]
\end{align}

\end{definition}
where \( a'(Z) \equiv a(Z; \eta') \), \( b'(Z) \equiv b(Z; \eta') \) and \( S_{ab} \equiv s_{ab}(O) \) and \( a \to s_{ab}(o) \) is a known function, i.e. it that does not depend on \( \eta \).

As we will see in the next section, the mixed bias property implies that \( \chi(\eta) + \chi^1_\eta \) depends on \( \eta \) only through \( a \) and \( b \) and, consequently, the one step estimator depends on \( \hat{\eta} \) only through estimators \( \hat{a} \) and \( \hat{b} \). The property implies that for estimators \( \hat{a} \) and \( \hat{b} \) satisfying \( \int \{ \hat{a}(z) - a(z) \}^2 dP(z) = O_p(\gamma_{a,n}) \) and \( \int \{ \hat{b}(z) - b(z) \}^2 dP(z) = O_p(\gamma_{b,n}) \), equation (1) is of order \( O_p(\gamma_{a,n}\gamma_{b,n}) \). This in turn implies that, when sample-splitting and cross-fitting is employed, \( \sqrt{n} \{ \hat{\chi} - \chi(\eta) \} \) converges to a mean zero Normal distribution if \( \gamma_{a,n} = o(1) \), \( \gamma_{b,n} = o(1) \) and \( \gamma_{a,n}\gamma_{b,n} = o\left(n^{-1/2}\right) \). Because the rates of convergence \( \gamma_{a,n} \) and \( \gamma_{b,n} \) of estimators \( \hat{a} \) and \( \hat{b} \) depend on the complexity of \( a \) and \( b \), the mixed bias property essentially implies that \( \hat{\chi} \) has the doubly-robust property of being \( \sqrt{n} \)– consistent and asymptotically normal even if one of the functions \( a \) or \( b \) is very complex so long as the other is simple enough.

Recent articles in the Biostatistical and Econometrics literature have identified two distinct classes of parameters \( \chi(\eta) \) with the mixed bias property. Specifically, Robins et al. (2008) have shown that parameters with influence function of the form

\[
\chi^1_\eta = S_{a}a(Z)b(Z) + S_{a}a(Z) + S_{b}b(Z) + S_{0} - \chi(\eta)
\]

where \( S_{a} \) and \( S_{b} \) are statistics, satisfy the mixed bias property. These authors gave a long list of examples of important parameters of interest in causal inference that satisfy (3). These include the examples given in Section 4. In particular, the popular average treatment effect under the assumption of unconfoundedness conditional on \( Z \) is the difference of two parameters with influence functions of the form (3). Likewise, the treatment effect on the treated contrast is the difference of the conditional mean of \( Y \) among the treated minus a parameter with influence function of the form (3).

In the Econometrics literature, Chernozhukov et al. (2018) have shown that parameters of the form \( \chi(\eta) = E_{\eta}[d(O,a)] \) where \( a(Z) \equiv E_{\eta}(Y|Z) \) and \( d(O,a) \) is such that the map \( h \in L_2(P_{y,Z}) \to E_{\eta}[d(O,h)] \) is a continuous and affine linear, also satisfy the mixed bias property. These authors also provide many interesting examples, several of which agree with the examples illustrating Robins et al. (2008) class. They also exhibit some examples which, as our Examples 1a) and b) below, do not fall in the Robins et al. (2008) class.

In this paper we will show that neither the class Robins et al. (2008) nor that of Chernozhukov et al. (2018) is contained in the other. More importantly, we characterize the influence functions, under a non-parametric model \( M \), of parameters in the class of those that satisfy the mixed bias property and show that this class strictly includes the union of the classes of parameters of Robins et al. and of Chernozhukov et al. We will show that parameters that satisfy the mixed bias property are necessarily of the form

\[
\chi(\eta) = E_{\eta}[m_1(O,a)] + E_{\eta}[S_0] = E_{\eta}[m_2(O,b)] + E_{\eta}[S_0]
\]

for some statistic \( S_0 \), and some \( m_1 \) and \( m_2 \) such that the maps \( h \in A \to m_1(O,h) \) and \( h \in B \to m_2(O,h) \) are linear, where \( A \equiv \{a(Z;\eta) : \eta \in \eta\} \) and \( B \equiv \{b(Z;\eta) : \eta \in \eta\} \).

In addition, we will prove a number
of results about the structure of \( a \) and/or \( b \) in special cases. In particular, we will show that, under mild regularity conditions, when \( a \) does not depend on the marginal distribution of \( Z \), then, up to regularity conditions, a necessary and sufficient condition for \( \chi(\eta) \) to have the mixed bias property is that
\[
\chi(\eta) = E_\eta [m_1(O,a)] + E_\eta [S_0] \text{ for a statistic } S_0, \text{ a linear map } h \in A \rightarrow m_1(O,h) \text{ and } a(Z) \text{ a ratio of two conditional means given } Z. \]
We will also show that for parameters \( \chi(\eta) \) that satisfy the mixed bias property the influence function naturally yields two loss functions whose expectations are minimized at \( a \) and \( b \) respectively. These loss functions can then be used to construct loss-based machine-learning estimators of \( a \) and \( b \) such as support vector machine estimators (Christmann and Steinwart, 2008).

2 Characterization of the influence functions with the mixed bias property

Our first result establishes the fact anticipated in the introduction that for parameters \( \chi(\eta) \) that satisfy the mixed bias property, \( \chi(\eta) + \chi^1_\eta \) depends on \( \eta \) only through \( a \) and \( b \). We will establish this result under the following regularity condition.

\textbf{Condition 1 (Condition R.1)} There exists a dense set \( H_a \) of \( L_2(P_\eta,Z) \) such that for each \( \eta \) and for each \( h \in H_a, \) there exists \( \varepsilon(\eta,h) > 0 \) such that \( a+th \in A \) if \( |t| < \varepsilon(\eta,h) \) where \( a(Z) \equiv a(Z;\eta) \). Furthermore, \( H_a \cap A \neq \emptyset \). The same holds replacing \( a \) with \( b \) and \( A \) with \( B \). Furthermore \( E_\eta [|S_{ab}(Z) h(Z)|] < \infty \) for \( h \in H_a \) and \( E_\eta [|S_{ab}(Z) h(Z)|] < \infty \) for \( h \in H_b \).

\textbf{Proposition 1} In a non-parametric model \( \mathcal{M} \) if a parameter \( \chi(\eta) \) satisfies the mixed bias property and condition R.1 holds, then \( \chi(\eta) + \chi^1_\eta \) depends on \( \eta \) only through \( a \) and \( b \).

The next Theorem characterizes the influence functions of parameters with the mixed bias property.

\textbf{Theorem 1} In a non-parametric model \( \mathcal{M} \), if \( \chi(\eta) \) satisfies the mixed bias property and Condition R.1 holds, then there exist a statistic \( S_0 \) and maps \( h \in A \rightarrow m_1(O,h) \) and \( h \in B \rightarrow m_2(O,h) \) such that the maps \( h \in A \rightarrow E_\eta [m_1(O,h)] \) and \( h \in B \rightarrow E_\eta [m_2(O,h)] \) are linear and such that (4) and

\[
\chi^1_\eta = S_{ab}b + m_1(O,a) + m_2(O,b) + S_0 - \chi(\eta). \tag{5}
\]

hold. Furthermore, for all \( h \in A, E_\eta [S_{ab}b + m_1(O,h)] = 0 \) and for all \( h \in B, E_\eta [S_{ab}a + m_2(O,h)] = 0 \). In addition, if for all \( h \in L_2(P_\eta) \) it holds that \( m_1(O,h) \in L_2(P_\eta) \) then the map \( h \in A \rightarrow m_1(O,h) \) is linear a.s.\( (P_\eta) \). Likewise, if for all \( h \in L_2(P_\eta) \) it holds that \( m_2(O,h) \in L_2(P_\eta) \) then the map \( h \in B \rightarrow m_2(O,h) \) is linear a.s.\( (P_\eta) \).

Part (i) of the next result establishes that under a slightly stronger requirement on \( m_1 \) and \( m_2 \) and some regularity conditions, the reverse of Theorem 1 also holds. The theorem also establishes several additional results that we will comment after its statement.
Theorem 2 In a non-parametric model $M$, suppose that for each $\eta$ there exist functions $a(Z) \equiv a(Z; \eta)$ and $b(Z) \equiv b(Z; \eta)$ such that Condition R.1 holds and such that the influence function of $\chi(\eta)$ is of the form (1) for $m_1$ and $m_2$ that satisfy that for each $\eta$, the maps

$$ h \in L_2(P_{\eta,Z}) \to E_{\eta}[m_1(O,h)] \quad \text{and} \quad h \in L_2(P_{\eta,Z}) \to E_{\eta}[m_2(O,h)] $$

are continuous and linear with Riesz representers $R_1(Z)$ and $R_2(Z)$ respectively. Moreover, $E_{\eta}[m_1(O,a)]$ and $E_{\eta}[m_2(O,b)]$ exist. Furthermore, suppose that for each $\eta$, $E_{\eta}(S_{ab}|Z)a(Z)$ and $E_{\eta}(S_{ab}|Z)b(Z)$ are in $L_2(P_{\eta,Z})$. Then,

(i) for each $\eta'$ such that $a'(Z) \equiv a(Z;\eta'), b'(Z) \equiv b(Z;\eta')$ satisfy that $a' - a \in L_2(P_{\eta,Z})$ and $b' - b \in L_2(P_{\eta,Z})$ it holds that (2) holds.

(ii) for all $h \in L_2(P_{\eta,Z})$ it holds that

$$ E_{\eta}[S_{ab}ha + m_2(O,h)] = 0 \quad \text{and} \quad E_{\eta}[S_{ab}hb + m_1(O,h)] = 0 $$

(iii) if $E_{\eta}(S_{ab}|Z) \neq 0 \text{ a.s.}(P_{\eta,Z})$, then $a(Z) = -R_2(Z)/E_{\eta}(S_{ab}|Z)$. Likewise, if $E_{\eta}(S_{ab}|Z) \neq 0 \text{ a.s.}(P_{\eta,Z})$, then $b(Z) = -R_1(Z)/E_{\eta}(S_{ab}|Z)$.

(iv)

(iv.a) if $a \in L_2(P_{\eta,Z})$ or if $\exists \varepsilon > 0$ such that $(1+t)a \in A$ for $0 < t < \varepsilon$ or for $-\varepsilon < t < 0$ then $\chi(\eta) = E_{\eta}[m_2(O,b)] + E_{\eta}[S_0]$.

(iv.b) Likewise, if $b \in L_2(P_{\eta,Z})$ or if $\exists \varepsilon > 0$ such that $(1+t)b \in B$ for $0 < t < \varepsilon$ or for $-\varepsilon < t < 0$ then $\chi(\eta) = E_{\eta}[m_1(O,a)] + E_{\eta}[S_0]$.

(iv.c) if the conditions of parts (iv.a) and (iv.b) hold then $\chi(\eta) = -E_{\eta}[S_{ab}ab] + E_{\eta}[S_0]$.

(v) if $a \in L_2(P_{\eta,Z}), b \in L_2(P_{\eta,Z})$ and $E_{\eta}(S_{ab}|Z) > 0 \text{ a.s.}(P_{\eta,Z})$, then for the loss functions

$$ T_1(O,h) \equiv S_{ab}\frac{h^2}{2} + m_2(O,h) \quad \text{and} \quad T_2(O,h) \equiv S_{ab}\frac{h^2}{2} + m_1(O,h) $$

it holds that

$$ a = \arg \min_{h \in L_2(P_{\eta,Z})} E_{\eta}[T_1(O,h)] \quad \text{and} \quad b = \arg \min_{h \in L_2(P_{\eta,Z})} E_{\eta}[T_2(O,h)] $$

Note that part (ii) of Theorem2 provides unbiased moment equations for $a$ and $b$ respectively without requiring that $a$ or $b$ be in $L_2(P_{\eta,Z})$. In Chernozhukov et al. (2018), the moment equation for $b$ is used to derive $\ell_1$ regularized estimators of $b$ under an approximately sparse linear model for it. Likewise, Smucler et al. (2019) exploit the moment equations for $a$ and $b$ under approximately sparse generalized linear models for both nuisance functions. Part (iii) of the theorem provides the formulae for $a$ and $b$ in terms of the Riesz representers of the maps. Part (iv) shows that under a strengthening on the requirements on $a$ and $b$, the representation in (2) holds. Note that the
requirement that \((1 + t)b \in \mathcal{B}\) for \(0 < t < \varepsilon\) is rather mild. For instance, for \(b(Z) = 1/P(D = 1|Z)\), as in example 1, the requirement is satisfied since the only restriction the elements \(b'(z)\) of \(\mathcal{B}\) satisfy is that for each \(z\), \(b'(z)\) must be greater than or equal 1. The loss functions derived in part (v) of the Theorem could in principle be used to derive other machine learning, loss-based estimators of these parameters, such as support vector machines (Christmann and Steinwart, 2008).

### 3 Characterization of the nuisance functions

An interesting question is what can be said about the restrictions that the nuisance functions \(a\) and \(b\) of parameters with the mixed bias property must satisfy. In this section we explore this question in the special case in which \(a\) does not depend on the marginal law of \(Z\). We will show that such \(a\) must be a ratio of two conditional expectations given \(Z\). We will need the following regularity condition:

**Condition 2** \(\chi(\eta)\) satisfies the mixed bias property and there exists \(b' \in \mathcal{B}\) such that for all \(\eta\), (i) \(b'(Z) \neq 0\) a.s.(\(P_{\eta,Z}\)), and (ii) for the map \(m_2\) defined in the proof of Theorem 1, \(E_{\eta}\{m_2(O,b)|Z\} + E_{\eta}(S_{ab}|Z)b(Z)a(Z)\) is in \(L_2(P_{\eta,Z})\) and \(m_2(O,b) - m_2(O,b)/b'(Z)\) is in \(L_2(P_{\eta})\).

**Proposition 2** Suppose that in a non-parametric model \(\mathcal{M}\), the parameter \(\chi(\eta)\) satisfies the mixed bias property, conditions (1) and (2) hold and \(E_{\eta}[S_{ab}|Z] \neq 0\) a.s. (\(P_{\eta,Z}\)). If \(a\) depends on \(\eta\) only through \(\eta_2\) (i.e. through the law of \(O|Z\)), then there exists a statistic \(q(O)\) such that \(a(Z) = -E_{\eta}\{q(O)|Z\}/E_{\eta}(S_{ab}|Z)\). Furthermore, the influence function of \(\chi(\eta)\) satisfies (6)

\[
\chi_{\eta}^1 = S_{ab}ab + m_1(O,a) + q(O)b + S_0 - \chi(\eta).
\]

for some linear map \(h \in \mathcal{A} \rightarrow m_1(O,h)\).

**Proposition 3** Suppose that \(a(Z) = -E_{\eta}\{q(O)|Z\}/E_{\eta}(S_{ab}|Z)\) is in \(L_2(P_{\eta,Z})\). Suppose also that the map \(h \in L_2(P_{\eta,Z}) \rightarrow E_{\eta}\{m_1(O,a)\}\) is linear and continuous with Riesz representer \(\mathcal{R}_1(Z)\). Then, \(\chi(\eta) = E_{\eta}\{m_1(O,a)\} + E_{\eta}(S_0)\) has influence function that satisfies (6) for \(b(Z) = -\mathcal{R}_1(Z)/E_{\eta}(S_{ab}|Z)\).

In addition, if \(E_{\eta}\{q(O)|Z\} \in L_2(P_{\eta,Z})\), then \(\chi(\eta)\) has the mixed bias property.

For a given parameter \(\chi(\eta)\) there can exist more than one function \(a(Z) \equiv a(Z,\eta)\) independent of the law of \(Z\) such that the mixed bias property holds for some \(b(Z) \equiv b(Z,\eta)\). An instance is the parameter in Example 5 below, since \(a(Z)\) can be either \(E_{\eta}(Y|Z)\) or \(E_{\eta}(D|Z)\). However, in that example, if \(a = E_{\eta}(Y|Z)\) then \(b = E_{\eta}(D|Z)\) and vice versa, if \(a = E_{\eta}(D|Z)\) then \(b = E_{\eta}(Y|Z)\). An open question is whether or not there exist two distinct triplets \((S_{ab},a,b)\) and \((S_{ab}^*,a^*,b^*)\) with \((a,b) \neq (a^*,b^*)\) such that the parameter \(\chi(\eta)\) satisfies the mixed bias property for both triplets. This is important because if it such distinct triplets existed, then there would exist two different pairs of nuisance functions of the same covariate \(Z\) that one could choose to estimate in order to construct doubly robust estimators of \(\chi(\eta)\).
In the preceding propositions we have assumed a given partition of the data $O$ into a given ‘covariate’ vector $Z$ and the remaining variables in $O$. Interestingly, there exist parameters $\chi(\eta)$ that satisfy the mixed bias property for two different partitions of $O$, one with ‘covariate’ vector $Z$ and another with a different ‘covariate’ vector $Z^*$. Specifically, in Example 1 we show that for $\chi(\eta)$ equal to the mean of an outcome missing at random, there exist two possible partitions of $O$, into two different ‘covariate’ vectors $Z$ and $Z^*$, such that for all $\eta$ and $\eta'$

$$S_{ab}\{a(Z, \eta) - a(Z, \eta')\} \{b(Z, \eta) - b(Z, \eta')\} = S_{ab}^{*}\{a^{*}(Z^{*}, \eta) - a^{*}(Z^{*}, \eta')\} \{b^{*}(Z^{*}, \eta) - b^{*}(Z^{*}, \eta')\},$$

where $a^{*}(Z^{*}) \equiv a^{*}(Z^{*}, \eta)$ and $b^{*}(Z^{*}) \equiv b^{*}(Z^{*}, \eta)$ are different from $a(Z)$ and $b(Z)$, $a(Z)$ depends on the law of $O$ given $Z$, $a^{*}(Z^{*})$ depends on the law of $O$ given $Z^*$ and the statistic $S_{ab}^{*}$ is different from $S_{ab}$. Consequently, the parameter $\chi(\eta)$ satisfies the mixed bias property for the functions $a$ and $b$, but also for the functions $a^{*}$ and $b^{*}$. In this example, $S_{ab}$ is not a constant but $S_{ab}^{*}$ is a constant, so in view of part (i) of Proposition 2, $a(Z)$ is a ratio of two conditional expectations given $Z$, whereas $a^{*}$ is a conditional expectation of a specific statistic $q(O)$ given $Z^*$. This example raises the following interesting question: suppose that $\chi(\eta)$ satisfies the mixed bias property for a function $a(Z)$ that is a strict ratio of two conditional expectations given $Z$, is it always possible to find a different covariate vector $Z^*$ such that $\chi(\eta)$ satisfies the mixed bias property for a function $a^{*}(Z^{*})$ that is a conditional mean of a statistic given $Z^*$? The answer to this question is negative, as our Example 6 illustrates. This example proves that the class of parameters that satisfy the mixed bias property strictly includes the class of parameters considered in Chernozhukov et al. (2018).

We conclude our analysis answering to the next natural question of whether a characterization exist of nuisance functions that depend also on the marginal law of $Z$. The answer to this question is negative. This can be understood from Proposition 3 because when the map $h \in L_2(P_{\eta,Z}) \to E_{\eta}[m_1(O,h)]$ is linear and continuous, $b(Z) = -R_1(Z)/E_{\eta}(S_{ab}|Z)$ where $R_1(Z)$ is the Riesz representer of the map. The representer $R_1(Z)$ can be many different functionals of the marginal law of $Z$, depending on the map it represents. The examples in the next section illustrate this point.

4 Examples

In this section we present several examples of parameters satisfying the mixed bias property. These examples demonstrate that the parameter classes of Chernozhukov et al. (2018) and of Robins et al. (2008) intersect, but neither is included in the other. Furthermore, they demonstrate that the class of parameters with the mixed bias property strictly includes the union of both classes.

4.1 Parameters in the Robins et al. class and also in the Chernozhukov et al. class

Example 1 (Mean of an outcome that is missing at random) Suppose that $O = (DY, D, Z)$ where $D$ is binary, $Y$ is an outcome which is observed if and only if $D = 1$ and $Z$ is a vector of always observed
covariates. If we make the untestable assumption that the density \( p(y|D = 0, Z) \) is equal to the density \( p(y|D = 1, Z) \), i.e. that the outcome \( Y \) is missing at random then, for \( P = P_\eta \), the mean of \( Y \) is equal to

\[
\chi(\eta) = E_\eta[a(Z)]
\]

where \( a(Z) \equiv E_\eta(DY|Z)/E_\eta(D|Z) \). If \( a(Z) \in L_2(P_{\eta,Z}) \) and \( E_\eta(D|Z) > 0 \), then the parameter \( \chi(\eta) \) satisfies the conditions of Proposition 3 with \( m_1(O, h) \equiv h \) and \( S_0 = 0 \). The map \( h \in L_2(P_{\eta,Z}) \rightarrow E_\eta[m_1(O, h)] \) is continuous with Riesz representer \( R_1(Z) = 1 \), and \( a(Z) = -E_\eta(q(O)|Z)/E_\eta(S_{ab}|Z) \) for \( S_{ab} = -D \) and \( q(O) = DY \). Consequently, \( \chi(\eta) \) has the mixed bias property for \( a(Z) \) as defined and \( b(Z) = 1/E_\eta(D|Z) \). Since \( m_1(O, a) = a \), equation (6) implies that \( \chi(\eta) \) is in the class of parameters considered by Robins et al. (2008). Interestingly, as shown in Chernozhukov et al. (2018) and anticipated in the previous section, the parameter \( \chi(\eta) \) is also in the class of Chernozhukov et al. (2018), but for a different ‘covariate’ \( Z^* \). Specifically, let \( Z^* \equiv (D, Z) \) and \( a^*(Z^*) \equiv E_\eta(DY|Z^*) \). Then, we can re-express \( \chi(\eta) \) as \( \chi(\eta) = E_\eta[m_1^*(O, a^*)] \) where for any \( h^*(D, Z) \), \( m_1^*(O, h^*) \equiv h^*(D = 1, Z) \). The map \( h^* \in L_2(P_{\eta,(D,Z)} \rightarrow E_\eta[m_1^*(O, h^*)] \) is linear and it is continuous when \( E_\eta[P_\eta(D = 1|Z)^{-1}] < \infty \) and has Riesz representer \( R_1^*(Z^*) = D/E_\eta(D|Z^*) \). Thus, under the latter condition, the parameter falls in the class of Chernozhukov et al. (2018). Because \( a^*(Z^*) \) is a conditional expectation given \( Z^* \), Proposition 3 implies that \( \chi(\eta) \) has the mixed bias property for \( a^*(Z^*) \) as defined, \( S^*_{ab} = 1 \) and \( b^*(Z^*) = D/E_\eta(D|Z) \).

**Example 2 (Mean of outcome missing at random in the non-respondents)** With the notation and assumptions of Example 1, \( E_\eta[(1 - D)a(Z)]/E_\eta[1 - D) \) where again, \( a(Z) \equiv E_\eta(DY|Z)/E_\eta(D|Z) \), is equal to the mean of \( Y \) among the non-respondents, i.e. in the population with \( D = 0 \). If \( a(Z) \in L_2(P_{\eta,Z}) \) and \( E_\eta(D|Z) > 0 \), then the parameter \( \chi(\eta) \equiv E_\eta[(1 - D)a(Z)] \) satisfies the conditions of Proposition 3 with \( m_1(O, h) \equiv (1 - D)h \) and \( S_0 = 0 \). The map \( h^* \in L_2(P_{\eta,Z}) \rightarrow E_\eta[m_1^*(O, h^*)] \) is linear and it is continuous when \( E_\eta[P_\eta(D = 1|Z)^{-1}] < \infty \) and has Riesz representer \( R_1^*(Z^*) = D/E_\eta(D|Z^*) \). Thus, under the latter condition, the parameter falls in the class of Chernozhukov et al. (2018). Because \( a^*(Z^*) \) is a conditional expectation given \( Z^* \), Proposition 3 implies that \( \chi(\eta) \) has the mixed bias property for \( a^*(Z^*) \) as defined, \( S^*_{ab} = 1 \) and \( b^*(Z^*) = D/E_\eta(D|Z) \).

**Example 3 (Population average treatment effect)** Suppose that \( O = (Y, D, Z) \) where \( D \) is a binary treatment indicator, \( Y \) is an outcome and \( Z \) is a baseline covariate vector. Under the assumption of unconfoundedness given \( Z \), the population average treatment effect contrast is \( ATE(\eta) \equiv \chi_1(\eta) - \chi_2(\eta) \) where \( \chi_1(\eta) \equiv E_\eta[a_1(Z)] \) and \( \chi_2(\eta) \equiv E_\eta[a_2(Z)] \) with \( a_1(Z) \equiv E_\eta(DY|Z)/E_\eta(D|Z) \) and \( a_2(Z) \equiv E_\eta(DY|Z)/E_\eta(D|Z) \).
$E \eta \{ (1-D) Y | Z \} / E \eta \{ (1-D) | Z \}$. Regarding $1-D$ as another missing data indicator, example 1 implies that $ATE(\eta)$ is a difference of two parameters, $\chi_1(\eta)$ and $\chi_2(\eta)$, each in the class of Robins et al. (2008) and of Chernozhukov et al. (2018).

Example 4 (Treatment effect on the treated) With the notation and assumptions of Example 3, the parameter $ATT(\eta) \equiv E(Y|D=1) - \chi(\eta)/E(\eta)$ where $\chi(\eta) \equiv E_\eta[Da(Z)]$ and $a(Z)$ defined as $E_\eta \{ (1-D) Y | Z \} / E_\eta \{ (1-D) | Z \}$ the parameter $ATT(\eta)$ is the average treatment effect on the treated. Once again, regarding $1-D$ as another missing data indicator, Example 2 implies that $ATT(\eta)$ is a continuous function of a parameter $\chi(\eta)$ in the class of Robins et al. (2008) and of Chernozhukov et al. (2018), and other parameters $E(Y|D=1)$ and $E_\eta(D)$ whose estimation does not require the estimation of high dimensional nuisance parameters.

Example 5 (Expected conditional covariance) Let $O = (Y, D, Z)$, where $Y$ and $D$ are real valued. Let $\chi(\eta) \equiv E_\eta[\text{cov}_\eta(D, Y|Z)]$ be the expected conditional covariance between $D$ and $Y$. When $D$ is a binary treatment, $\chi(\eta)$ is an important component of the variance weighted average treatment effect Robins et al. (2008). We can re-write $\chi(\eta) = E_\eta(DY) + E_\eta[m_1(O, a)]$ where $m_1(O, h) \equiv -Dh$, $a(Z) \equiv E_\eta(Y|Z) = -E_\eta(q(O)|Z)/E(Sab|Z)$ with $q(O) = Y$ and $Sab = -1$. Then $\chi(\eta)$ has the mixed bias property with $a(Z)$ as defined and $b(Z) = R_1(Z) = -E_\eta(D|L)$ the Riesz representer of the map $h \in L_2(P_{\eta,Z}) \rightarrow E_\eta[m_1(O, h)]$. Thus, $\chi(\eta)$ is in Chernozhukov et al. (2018) and in the Robins et al. (2008) classes with $S_a = -D$, $S_b = Y$ and $S_0 = DY$.

4.2 A parameter in the Robins et al. class but not in the Chernozhukov et al. class

Example 6 (Mean of an outcome that is missing not at random) Suppose that $O = (DY, D, Z)$ where $D$ is binary, $Y$ is an outcome which is observed if and only if $D = 1$ and $Z$ is a vector of always observed covariates. If we make the untestable assumption that the density $p(y|D = 0, Z)$ is a known exponential tilt of the density $p(y|D = 1, Z)$, i.e.

$$p(y|D = 0, Z) = p(y|D = 1, Z) \exp(\delta y) / E[\exp(\delta Y)|D = 1, Z]$$

where $\delta$ is a given constant, then under $P = P_\eta$ the mean of $Y$ is $\chi(\eta) = E_\eta[DY + (1-D)a(Z)]$ assuming $a(Z) \equiv E_\eta[DY \exp(\delta Y)|Z]/E_\eta[D\exp(\delta Y)|Z]$ exists. Estimation of $\chi(\eta)$ under different fixed values of $\delta$ has been proposed in the literature as a way of conducting sensitivity analysis to departures from the missing at random assumption Scharfstein et al. 1996. Under the sole restriction that the law $P$ of the observed data $O$ is unrestricted, and hence the model for $P$ is non-parametric. If $a(Z) \in L_2(P_{\eta,Z})$ and $E_\eta[D\exp(\delta Y)|Z] > 0$, then the parameter $\psi(\eta) \equiv E_\eta[m_1(O, a)]$ with $m_1(O, h) \equiv (1-D)h$ has the mixed bias property because it satisfies the conditions of Proposition 3 with $q(O) = DY \exp(\delta Y)$, $S_{ab} = -D \exp(\delta Y)$ and Riesz representer $R_1(Z) = E_\eta(1 - D|Z)$ and $b(Z) \equiv -E_\eta((1-D)|Z)/E_\eta[D\exp(\delta Y)|Z]$. Thus, $\chi(\eta)$ also satisfies the mixed bias property with $S_{ab}$, $a$ and $b$ as defined. The influence function of $\chi(\eta)$ was derived in Robins and Rotnitzky 2004.
and was shown to be in the \textcite{Robins2008} class in that paper. In the Appendix we show that when $\delta \neq 0$, unlike the case of missing at random in example \textcite{1}, We also show that there exists no linear and continuous map $h^* \in L_2(P_{\eta}(Z)) \rightarrow E_\eta[m_1^*(O, h^*)]$, such that $\psi(\eta) = E_\eta[m_1^*(O, a^*)]$ for $a^*(Z) = E_\eta\{q(O)\mid Z\}$ and $q(O)$ some statistic. This shows that $\psi(\eta)$, and consequently $\chi(\eta)$, is not in the class studied in \textcite{Chernozhukov2018}.

4.3 Parameters in the Chernozhukov et al. class but not in the Robins et al. class

\textbf{Example 7} In the next two examples, $O = (Y, D, L)$, $Z = (D, L)$, $Y$ and $D$ are real valued, $D$ is a treatment variable taking any value in $[0, 1]$ and $L$ is a covariate vector. Furthermore, $Y_d$ denotes the counterfactual outcome under treatment $D = d$ and in the following we assume that $E_\eta(Y_d|L) = E_\eta(Y|D = d, L)$.

\begin{enumerate}
\item \textbf{Causal effect of a treatment taking values on a continuum} The parameter $\chi(\eta) \equiv E_\eta[m_1(O, a)]$ with $(D, L) \equiv E_\eta(Y|D, L)$, $m_1(O, a) \equiv \int_0^1 a(u, L)w(u)\,du$ where $w(\cdot)$ is a given scalar function satisfying $\int_0^1 w(u)\,du = 0$ agrees with the treatment effect contrast $\int_0^1 E_\eta(Y_u)w(u)\,du$. The map $h \in L_2(P_{\eta}(D, Z)) \rightarrow E_\eta[m_1(O, h)]$ where $m_1(O, h) \equiv \int_0^1 h(u, L)w(u)\,du$ is continuous if $E_\eta\left[\left\{w(D)/f(D|L)\right\}^2\right] < \infty$ with Riesz representer $R_1(Z) = w(D)/f(D|L)$. In such case, the parameter $\chi(\eta)$ is in the class studied in \textcite{Chernozhukov2018}. Thus, by Proposition \textcite{8} it has the mixed bias property with $S_{ab} = -1$, $a$ as defined, and $b(Z) = R_1(Z) = w(D)/f(D|L)$. However, in the Appendix we show that $\chi(\eta)$ is not in the class of \textcite{Robins2008}.

\item \textbf{Average policy effect of a counterfactual change of covariate values} Let $\chi(\eta) \equiv \psi(\eta) - E_\eta(Y)$ where $\psi(\eta) = E_\eta[a(t(D), L)]$ with $(D, L) \equiv E_\eta(Y|D, L)$ is the average policy effect of a counterfactual change $d \rightarrow t(d)$ of treatment values as in \textcite{Stock1989}. Note that $\psi(\eta) = E_\eta[m_1(O, a)]$ where for any $h(D, L)$ $m_1(O, h) = h(t(D), L)$. The functional $h \in L_2(P_{\eta}(D, Z)) \rightarrow E_\eta[m_1(O, h)]$ is continuous if $E_\eta\left[\left\{f_t(D|L)/f(D|L)\right\}^2\right] < \infty$ where $f_t(D|L)$ is the density of $t(D)$ given $L$. The Riesz representer is in the class studied in \textcite{Chernozhukov2018}, and thus $\chi(\eta)$ has the mixed bias property, with $S_{ab} = -1$, $a(Z)$ as defined, and $b(Z) = R_1(Z) = f_t(D|L)/f(D|L)$. However, it can be shown that $\chi(\eta)$ is not in the class of \textcite{Robins2008}.
\end{enumerate}

4.4 A parameter that is in neither the class of Chernozhukov. et al class nor in the class of Robins et al,

\textbf{Example 8} The following toy example illustrates that there exist parameters $\chi(\eta)$ that have the mixed bias property but that are in neither the class of \textcite{Chernozhukov2018} nor in the class of \textcite{Robins2008}. Let $O = (Y_1, Y_2, Z)$ for $Y_1$ and $Y_2$ continuous random variables, $Y_2 > 0$ and $Z$ a
scalar vector taking any values in \([0, 1]\). The parameter \(\chi(\eta) \equiv \int_0^1 a(z) \, dz\) where \(a(Z) \equiv E_\eta(Y_1|Z)/E_\eta(Y_2|Z)\) can be written as \(\chi(\eta) = E_\eta[m_1(O,a)]\) where for any \(h(z), m_1(O,h) \equiv \int_0^1 h(z) \, dz\) does not depend on \(O\). The map \(h \in L_2(P_\eta,Z) \to E_\eta[m_1(O,h)]\) is linear. It is continuous if \(E_\eta[f(Z)^{-2}] < \infty\) and has Riesz representer \(\mathcal{R}_1(Z) = f(Z)^{-1}\). In such case, by proposition 3, \(\chi(\eta)\) satisfies the mixed bias property with \(S_{ab} = -Y_2\), \(a\) as defined and \(b(Z) = \{f(Z) E_\eta(Y_2|Z)\}^{-1}\). However, it can be shown that the parameter is in neither the class studied in Chernozhukov et al. (2018) nor in the class proposed in Robins et al. (2008).

5 Final remarks

In this article we have characterized the parameters that satisfy the mixed bias property as defined in Section 1. Parameters with this property are of interest because they enjoy the ‘rate double robustness’ property that they can be estimated at rate \(\sqrt{n}\) so long as one can estimate at a sufficiently fast rate one of two nuisance functions even if the second nuisance function can only be estimated at slow rates. While it is true that parameters with the mixed bias property have the rate double robustness property the opposite is not necessarily true. For instance, consider \(\psi(\eta) = g(\chi(\eta))\) for a non-linear continuously differentiable function \(g\). If \(\chi(\eta)\) has the mixed bias property, by directly computing the influence function of \(g(\chi(\eta))\) one can immediately verify that it is not of the form (5) and consequently if cannot have the mixed bias property. However, the delta method and the fact that \(\chi(\eta)\) has the rate double robustness property, imply that \(\psi(\eta)\) also has the rate double robustness property.

In a separate article Smucler et al. (2019) we report a unified strategy for the construction of one-step, sample-split and cross-fitted estimators of parameters that have the mixed bias property with the nuisance functions estimated via \(\ell_1\) regularization.
6 Appendix

Proof: [Proof of Proposition 1] Let \( \eta' \) be such that \( a' = a \) and \( b' = b \). Without loss of generality consider a local variation independent parameterization \( \eta = (a, b, \tau) \) and a regular parametric submodel \( t \to \eta_t = (a_t, b_t, \tau_t) \). Then,

\[
\frac{d}{dt} \chi(\eta_t) \bigg|_{t=0} = \frac{d}{dt} \chi(a_t, b, \tau) \bigg|_{t=0} + \frac{d}{dt} \chi(a, b_t, \tau) \bigg|_{t=0} + \frac{d}{dt} \chi(a, b, \tau_t) \bigg|_{t=0}
\]

By (2),

\[
\chi(a_t, b, \tau) = E_{(a_t,b,\tau)} \left[ \chi(\eta') + \chi_{\eta'} \right]
\]

and

\[
\chi(a, b_t, \tau) = E_{(a,b_t,\tau)} \left[ \chi(\eta') + \chi_{\eta'} \right]
\]

Then,

\[
\frac{d}{dt} \chi(\eta_t) \bigg|_{t=0} = \frac{d}{dt} E_{(a_t,b,\tau)} \left[ \chi(\eta') + \chi_{\eta'} \right] \bigg|_{t=0} + \frac{d}{dt} E_{(a,b_t,\tau)} \left[ \chi(\eta') + \chi_{\eta'} \right] \bigg|_{t=0} + \frac{d}{dt} E_{(a,b,\tau_t)} \left[ \chi(\eta') + \chi_{\eta'} \right] \bigg|_{t=0}
\]

\[
= \frac{d}{dt} E_{\eta} \left[ \chi(\eta') + \chi_{\eta'} \right] \bigg|_{t=0} = E_{\eta} \left[ \{ \chi(\eta') + \chi_{\eta'} \} g \right]
\]

Consequently,

\[
\chi_{\eta}^{1} = \chi(\eta') + \chi_{\eta'} - E_{\eta} \left[ \chi(\eta') + \chi_{\eta'} \right]
\]

\[
= \chi(\eta') + \chi_{\eta'} - \chi(\eta)
\]

Thus, \( \chi_{\eta}^{1} + \chi(\eta) = \chi(\eta') + \chi_{\eta'}^{1} \) which proves the proposition. \( \square \)

Proof: [Proof of Theorem 1] Fix \( a^* \) and \( b^* \) and define

\[
S_0^* \equiv (\chi + \chi^1)_{(a^*,b^*)} - S_{ab} a^* b^*
\]

\[
m_1^* (O,a) \equiv \left[ (\chi + \chi^1)_{(a,b^*)} - S_{ab} a b^* \right] - S_0^*
\]

\[
m_2^* (O,b) \equiv \left[ (\chi + \chi^1)_{(a^*,b)} - S_{ab} a^* b \right] - S_0^*
\]
For any \( h \in A \) we have

\[
E_\eta [m_1^*(O, h)] = E_\eta \left[ (\chi + \chi^1)_{(h,b^*)} - S_{ab}hb^* - S_0^* \right]
\]

\[
= E_\eta \left[ (\chi + \chi^1)_{(h,b^*)} - S_{ab}hb^* \right] - E_\eta \left[ (\chi + \chi^1)_{(a^*, b^*)} - S_{ab}a^*b^* \right]
\]

\[
= E_\eta \left[ (\chi + \chi^1)_{(h,b^*)} - \chi (\eta) - S_{ab}hb^* \right] - E_\eta \left[ (\chi + \chi^1)_{(a^*, b^*)} - \chi (\eta) - S_{ab}a^*b^* \right]
\]

\[
= E_\eta [-S_{ab}hb^*] + E_\eta [S_{ab} (a - h) (b - b^*)] - E_\eta [S_{ab} (a - a^*) (b - b^*)] + E_\eta [S_{ab}a^*b^*]
\]

\[
= -E_\eta [S_{ab}hb] + E_\eta [S_{ab}a^*b]
\]

(8)

Likewise, by symmetry we have established that

\[
E_\eta [m_2^*(O, h)] = -E_\eta [S_{ab}ha] + E_\eta [S_{ab}ab^*]
\]

We will next show that

\[
\chi_\eta = S_{ab}ab + m_1^*(O, a) + m_2^*(O, b) + S_0^* - \chi (\eta)
\]

To do so, it suffices to show that

(I) \( E_\eta [S_{ab}ab + m_1^*(O, a) + m_2^*(O, b) + S_0^*] = \chi (\eta) \) and

(II) \( \frac{d}{dt} E_\eta [S_{ab}a_t b_t + m_1^*(O, a_t) + m_2^*(O, b_t) + S_0^*]_{t=0} = 0 \) for any regular submodel \( t \to P_{\eta t} \)

To show (I) we write

\[
E_\eta [m_1^*(O, a) + m_2^*(O, b) + S_0^*]
\]

\[
= E_\eta \left[ (\chi + \chi^1)_{(a,b^*)} - S_{ab}a^*b - S_0 \right]
\]

\[
= E_\eta \left[ (\chi + \chi^1)_{(a,b^*)} - \chi (\eta) - S_{ab}ab^* + (\chi + \chi^1)_{(a^*, b^*)} - \chi (\eta) - S_{ab}a^*b - [S_0 - \chi (\eta)] \right] + \chi (\eta)
\]

\[
= E_\eta [-S_{ab}ab^* - S_{ab}a^*b] - E_\eta [S_0 - \chi (\eta)] + \chi (\eta)
\]

\[
= E_\eta [-S_{ab}ab^* - S_{ab}a^*b] - E_\eta \left[ (\chi + \chi^1)_{(a^*, b^*)} - \chi (\eta) - S_{ab}a^*b^* \right] + \chi (\eta)
\]

\[
= E_\eta [-S_{ab}ab^* - S_{ab}a^*b] - E_\eta [S_{ab} (a - a^*) (b - b^*)] + E_\eta [S_{ab}a^*b^*] + \chi (\eta)
\]

\[
= -E_\eta [S_{ab}ab] + \chi (\eta)
\]

which shows (I)

To show (II) we note that by \( \mathbb{S} \)

\[
E_\eta [S_{ab}a_t b + m_1^*(O, a_t)] = E_\eta [S_{ab}a^*b]
\]

and the right hand side does not depend on \( a_t \). Likewise,

\[
E_\eta [S_{ab}ab_t + m_2^*(O, b_t)] = E_\eta [S_{ab}ab^*]
\]
Thus,

\[
\frac{d}{dt} E_\eta \left[ S_{ab} a_t b_t + m^*_1 (O,a_t) + m^*_2 (O,b_t) + S_0^* \right]_{t=0} = 0
\]

This shows part (II) and thus concludes the proof of (9).

Next, take \(a^\dagger \in H_a \cap A\) and \(b^\dagger \in H_b \cap B\) which we know exist by Condition R.1. Also, by Condition R.1 we know that \(a^{**} \equiv a^* + \varepsilon a^\dagger \in A\) and \(b^{**} \equiv b^* + \varepsilon b^\dagger \in B\) for an \(\varepsilon > 0\) sufficiently small. Now, define \(S_0^{**}, m_1^{**} (O,a)\) and \(m_2^{**} (O,b)\) like \(S_0^*, m_1^* (O,a)\) and \(m_2^* (O,b)\) but using \(a^{**}\) and \(b^{**}\) instead of \(a^*\) and \(b^*\).

Then,

\[
\chi^1_\eta = S_{ab} a + m_1^{**} (O,a) + m_2^{**} (O,b) + S_0^{**} - \chi (\eta).
\]

So, combining this equality with (9) we conclude that

\[
m_1^{**} (O,a) + m_2^{**} (O,b) + S_0^{**} = m_1^* (O,a) + m_2^* (O,b) + S_0^*.
\]

Thus,

\[
m_1^{**} (O,a) - m_1^* (O,a) = m_2^* (O,b) - m_2^{**} (O,b) + S_0^* - S_0^{**}.
\]

The right hand side depends on \(b\) and the data \(O\), but the left hand side depends on \(a\) and the data \(O\). Thus, we conclude that \(m_1^{**} (O,a) - m_1^* (O,a)\) is a statistic \(Q_1^1\) that does not depend on \(\eta\).

Now, by (8),

\[
E_\eta \left( Q_1^1 \right) = E_\eta \left[ m_1^{**} (O,a) \right] - E_\eta \left[ m_1^* (O,a) \right]
\]

Next, let \(S_0^\dagger, m_1^\dagger (O,a)\) and \(m_2^\dagger (O,b)\) be defined like \(S_0^*, m_1^* (O,a)\) and \(m_2^* (O,b)\) but using \(a^\dagger\) and \(b^\dagger\) instead of \(a^*\) and \(b^*\). By (8) applied to \(m_1^\dagger (O,a)\) and \(a^\dagger\) instead of \(m_1^* (O,a)\) and \(a^*\) we have that

\[
E_\eta \left[ m_1^\dagger (O,h) \right] = -E_\eta \left[ S_{ab} h b \right] + E_\eta \left[ S_{ab} a \right]
\]

Consequently,

\[
m_1 (O,h) \equiv m_1^\dagger (O,h) - Q_1^1 / \varepsilon
\]

satisfies

\[
E_\eta \left[ m_1 (O,h) \right] = -E_\eta \left[ S_{ab} h b \right] \text{ for all } h \in A.
\]
and therefore the map \( h \in \mathcal{A} \to E_\eta \left[ m_1 (O, h) \right] \) is linear. In fact, for any \( h_1, h_2 \in \mathcal{A} \) and constants \( \alpha_1 \) and \( \alpha_2 \) such that \( \alpha_1 h_1 + \alpha_2 h_2 \in \mathcal{A} \), we know that

\[
E_\eta [m_1 (O, \alpha_1 h_1 + \alpha_2 h_2)] = \alpha_1 E_\eta [m_1 (O, h_1)] + \alpha_2 E_\eta [m_1 (O, h_2)]
\]

is true for all \( \eta \). Then, for all \( \eta' \)

\[
E_{\eta'} [m_1 (O, \alpha_1 h_1 + \alpha_2 h_2) - \{\alpha_1 m_1 (O, h_1) + \alpha_2 m_1 (O, h_2)\}] = 0
\]

By assumption the random variable \( r (O) \equiv m_1 (O, \alpha_1 h_1 + \alpha_2 h_2) - \{\alpha_1 m_1 (O, h_1) + \alpha_2 m_1 (O, h_2)\} \) is in \( L_2 \left( P_\eta \right) \). The linearity a.s.\( (P_\eta) \) of the map \( h \in \mathcal{A} \to m_1 (O, h) \) follows from Lemma A.1 below which implies that \( r (O) = 0 \) a.s.\( (P_\eta) \).

Likewise, we can show that there exists \( Q_1^2 \) and \( m_2 (O, h) \equiv m_2^1 (O, h) - Q_2^1 / \varepsilon \) such that \( h \in \mathcal{B} \to E_\eta [m_2 (O, h)] = -E_\eta [S_{ab} a h] \) and the map \( h \in \mathcal{B} \to m_2 (O, h) \) is linear. Finally, define \( S_0 = S_0^1 + Q_1^1 / \varepsilon + Q_2^1 / \varepsilon \) and conclude from \( \chi_1^2 = S_{ab} a b + m_1^1 (O, a) + m_2^1 (O, b) + S_0 - \chi (\eta) \) that

\[
\chi_1^2 = S_{ab} a b + m_2 (O, a) + m_2 (O, b) + S_0 - \chi (\eta).
\]

In addition, from (10) and its analogous for \( b \), we have that

\[
E_\eta [S_{ab} a b + m_1 (O, a)] = 0
\]

and

\[
E_\eta [S_{ab} a b + m_2 (O, b)] = 0.
\]

Consequently,

\[
\chi (\eta) = E_\eta [m_1 (O, a)] + E_\eta [S_0] = E_\eta [m_2 (O, b)] + E_\eta [S_0]
\]

thus showing (4) holds. This concludes the proof of the Theorem. \( \square \)

**Lemma 1 (Lemma A.1)** Suppose that \( r (O) \) is in \( L_2 \left( P_\eta \right) \) and that for all \( \eta' \), \( E_{\eta'} \{ r (O) \} = 0 \). Then \( r (O) = 0 \) a.s.\( (P_\eta) \).

**Proof:** [Proof of Lemma A.1.] Suppose first that \( r (O) \) is bounded. Then consider the submodel \( t \to p_t (O) = p_\eta (O) (1 + t r (O)) \). Note that for \( t \) sufficiently small, \( p_t > 0 \) (by the boundedness of \( r (O) \)) and \( p_t \) integrates to 1 because \( E_\eta \{ r (O) \} = 0 \). Then score of the submodel is \( r (O) \). Then, since by assumption the mean \( E_t (r (O)) \) of \( r (O) \) under \( p_t \) satisfies \( E_{p_t} (r (O)) = 0 \) for all \( t \), we have

\[
0 = \left. \frac{d}{dt} E_t (r (O)) \right|_{t=0} = E_\eta \{ r (O)^2 \}
\]

Consequently \( r (O) = 0 \) a.s.\( (P_\eta) \). Next, given an arbitrary \( r (O) \) in \( L_2 \left( P_\eta \right) \) such that \( E_{\eta'} \{ r (O) \} = 0 \) for all \( \eta' \), consider define \( r_n (O) \equiv r (O) I_{(-n,n)} (r (O)) - E_\eta \{ r (O) I_{(-n,n)} (r (O)) \} \). Then \( r_n (O) \) satisfies \( E_\eta [r_n (O)] = 0 \) and is bounded. So, \( r_n (O) = 0 \) a.s.\( (P_\eta) \). However, \( r_n (O) \) converges in \( L_2 \left( P_\eta \right) \) to \( r (O) \) so \( r (O) = 0 \) a.s.\( (P_\eta) \). \( \square \)
Proof: [Proof of Theorem 2] For any fixed $h \in H_a$, and a given $\eta = (a, b, \tau)$ consider a parametric submodel $t \to P_{\eta}$ where $\eta_t = (a_t, b, \tau)$ with $a_t = a + th$ and $|t| < \varepsilon(\eta, h)$ as in Condition R.1. Then, since $\chi^1_\eta$ is an influence function of the form \(\text{[5]}\) we have

\[
0 = \frac{d}{dt} E_\eta [S_{ab}a_t b + m_1(O,a_t) + m_2(O,b) + S_0]_{t=0} = \frac{d}{dt} E_\eta [S_{ab}(a + th) b + m_1(O,a) + tm_1(O,h)]_{t=0} = E_\eta [S_{ab}hb + m_1(O,h)].
\]

The continuity of the maps $h \in L_2(P_{\eta,Z}) \to E_\eta[m_1(O,h)]$ and $h \in L_2(P_{\eta,Z}) \to E_\eta[S_{ab}h(Z)b(Z)]$ implies that the map $h \in L_2(P_{\eta,Z}) \to E_\eta[S_{ab}hb + m_1(O,h)]$ is continuous. Then, since we have just shown that this map evaluates to 0 at a dense set of $L_2(P_{\eta,Z})$, it must equal to 0 for all $h \in L_2(P_{\eta,Z}).$ Reasoning analogously, we arrive at the conclusion that

\[
E_\eta[S_{ab}ha + m_2(O,h)] = 0 \quad (11)
\]

for all $h \in L_2(P_{\eta,Z})$, thus showing part (ii) of the Theorem. Next, suppose that $a' - a \in L_2(P_{\eta,Z})$ and $b' - b \in L_2(P_{\eta,Z}).$ Then, applying part (ii) we have that $E_\eta[S_{ab}(a' - a) b + m_1(O,(a' - a))] = 0$ and $E_\eta[S_{ab}(b' - b) a + m_2(O,(b' - b))] = 0.$ Consequently,

\[
E_\eta[\chi(\eta') + \chi^1_\eta] - \chi(\eta) = E_\eta[\chi(\eta') + \chi^1_\eta] - E_\eta[\chi(\eta) + \chi^1_\eta] = E_\eta[S_{ab}a'b' + m_1(O,a') + m_2(O,b')] - E_\eta[S_{ab}ab + m_1(O,a) + m_2(O,b)]
= E_\eta[S_{ab}a'b' + m_1(O,a') + m_2(O,b')] - E_\eta[S_{ab}ab + m_1(O,a) + m_2(O,b)]
= E_\eta[S_{ab}(a' - a) b + m_1(O,(a' - a))] - E_\eta[S_{ab}(b' - b) a + m_2(O,(b' - b))]
= E_\eta[S_{ab} (a - a')(b - b)]
\]

thus showing part (i) of the Theorem.

Turn now to the proof of part (iii). Equation \(\text{[11]}\) implies that for all $h \in L_2(P_{\eta,Z}),$

\[
0 = E_\eta(S_{ab}ha + R_2h).
\]

Thus, if $h^*(Z) \equiv E_\eta(S_{ab}|Z)a(Z) + R_2(Z)$ is in $L_2(P_{\eta,Z}),$ then specializing at $h = h^*$ the preceding identity we conclude that a.s.$(P_{\eta,Z})$

\[
E_\eta(S_{ab}a + R_2|Z) = 0
\]

or equivalently $a(Z) = -R_2(Z)/E_\eta(S_{ab}|Z)$ if $E_\eta(S_{ab}|Z) \neq 0$ a.s.$(P_{\eta,Z}).$ The assertion for $b(Z)$ is proved analogously.
Next, we prove part (iv). If \( b \) is in \( L_2(P_{\eta,Z}) \), then specializing (11) at \( h = b \) we obtain
\[
\chi(\eta) = E_\eta [S_{ab}ab + m_1(O,a) + m_2(O,b) + S_0]
\]
(12)

On the other hand, if \( b \not\in L_2(P_{\eta,Z}) \) but \((1 + t)b \in B\) for \( 0 < t < \varepsilon \) then, given \( \eta = (a,b,\tau) \) consider the parametric submodel \( t \to P_{\eta_t} \) where \( \eta_t = (a,b_t,\tau) \) with \( b_t = b + tb \) and \( 0 < t < \varepsilon \). Then, by \( \chi_\eta^1 \) of the form (5) being an influence function and with \( \frac{d}{dt} \) denoting the right derivative, we have
\[
0 = \frac{d}{dt^+} E_\eta [S_{ab}ab_t + m_1(O,a) + m_2(O,b_t) + S_0] \bigg|_{t=0}
\]
\[
= \frac{d}{dt^+} E_\eta [S_{ab}ab(b + tb) + m_2(O,b) + tm_2(O,b)] \bigg|_{t=0}
\]
\[
= E_\eta [S_{ab}ab + m_2(O,b)].
\]
So, applying again (12) we arrive at \( \chi(\eta) = E_\eta [m_1(O,a) + S_0] \). The same reasoning, but now taking left derivatives, yields to the same conclusion if \((1 + t)b \in B\) for \(-\varepsilon < t < 0\). This shows (iv.b). Part (iv.a) is proved analogously. Finally, part (iv.c) follows from
\[
\chi(\eta) = E_\eta [S_{ab}ab + m_1(O,a) + m_2(O,b) + S_0]
\]
\[
= E_\eta [S_{ab}ab + m_1(O,a) + m_2(O,b) + S_{ab}ab + S_0 - S_{ab}ab]
\]
\[
= E_\eta [S_0 - S_{ab}ab].
\]

Turn now to the proof of part (v). By part (iii) we have that a.s.(\( P_{\eta,Z} \))
\[
a(Z) = -\frac{\mathcal{R}_2(Z)}{E_\eta(S_{ab}|Z)} \text{ and } b(Z) = -\frac{\mathcal{R}_1(Z)}{E_\eta(S_{ab}|Z)}
\]

Next, for any \( h \in L_2(P_{\eta,Z}) \), write
\[
E_\eta \{ T_2(O,h) \}
\]
\[
= E_\eta \left\{ \frac{h^2}{2} + m_1(O,h) \right\}
\]
\[
= E_\eta \left\{ E_\eta(S_{ab}|Z) \frac{h(Z)^2}{2} + \mathcal{R}_1(Z) h(Z) \right\}
\]
\[
= E_\eta \left\{ \frac{E_\eta(S_{ab}|Z)}{2} \left[ h(Z)^2 + 2 \frac{\mathcal{R}_1(Z)}{E_\eta(S_{ab}|Z)} h(Z) + \left( \frac{\mathcal{R}_1(Z)}{E_\eta(S_{ab}|Z)} \right)^2 \right] \right\} - E_\eta \left\{ \frac{E_\eta(S_{ab}|Z)}{2} \left( \frac{\mathcal{R}_1(Z)}{E_\eta(S_{ab}|Z)} \right)^2 \right\}
\]
\[
= E_\eta \left\{ \frac{E_\eta(S_{ab}|Z)}{2} \left[ h(Z) + \frac{\mathcal{R}_1(Z)}{E_\eta(S_{ab}|Z)} \right]^2 \right\} - E_\eta \left\{ \frac{E_\eta(S_{ab}|Z)}{2} \left( \frac{\mathcal{R}_1(Z)}{E_\eta(S_{ab}|Z)} \right)^2 \right\}.
\]
So

\[
\begin{align*}
\arg \min_{h \in L_2(P_{\eta,Z})} E_\eta[T_2(O,h)] &= \arg \min_{h \in L_2(P_{\eta,Z})} E_\eta \left\{ \frac{E_\eta (S_{ab}|Z)}{2} \left[ h(Z) + \frac{R_1(Z)}{E_\eta (S_{ab}|Z)} \right]^2 \right\} \\
&= -\frac{R_1(Z)}{E_\eta (S_{ab}|Z)} = b(Z).
\end{align*}
\]

The assertion for \(T_1\) is proved analogously, interchanging the roles of \(a\) and \(b\), and of \(T_2\) and \(T_1\). \(\square\)

**Proof:** [Proof of Proposition 2] With the definition of \(m_2\) given in the proof of Theorem 1 we have that \(E_\eta [m_2 (O,h)] = -E_\eta [S_{ab}h]\) for all \(h \in B\). Fix \(b' \in B\) such that \(b' (Z) \neq 0\) a.s.\((P_\eta,Z)\). Then,

\[
E_{\eta_1} [\{E_{\eta_2} [m_2 (O,b') | Z] + E_{\eta_2} (S_{ab}|Z) b'(Z) a(Z)\}] = 0 \text{ for all } \eta_1.
\]

Since by assumption \(a\) does not depend on \(\eta_1\), then \(s_{\eta_2} (Z) \equiv E_{\eta_2} [m_2 (O,b') | Z] + E_{\eta_2} (S_{ab}|Z) b'(Z) a(Z)\) is a fixed function of \(Z\) (i.e. independent of \(\eta_1\)) with mean zero under any marginal law of \(Z\). Hence, since by condition 2, \(s_{\eta_2} (Z)\) is in \(L_2(P_{\eta,Z})\), then by Lemma A.1, \(s_{\eta_2} (Z) = 0\) a.s.\((P_\eta,Z)\), from where we conclude that \(a(Z) = -E_{\eta_2} [q(O) | Z] / E_{\eta_2} [S_{ab}|Z] \) for \(q(O) \equiv m_2 (O,b') / b'(Z)\).

Next, write for any \(\eta\)

\[
0 = E_\eta [S_{ab}a + m_2 (O,b)] = E_\eta \left[ S_{ab} \left( -\frac{E_\eta [q(O) | Z]}{E_\eta [S_{ab}|Z]} \right) b + m_2 (O,b) \right] = E_\eta [\frac{E_\eta [q(O) | Z]}{E_\eta [S_{ab}|Z]} b] \]

and since in the last display \(-q(O)b + m_2(O,b)\) is a statistic independent of \(\eta\), which, by condition 2, is in \(L_2(P_{\eta,Z})\) and the display holds for all \(\eta\) then \(-q(O)b + m_2(O,b) = 0\) a.s.\((P_\eta)\) for all \(\eta\). This shows that \(m_2(O,h) = q(O)h\). \(\square\)

**Proof:** [Proof of Proposition 3]

For a regular parametric submodel \(t \to \eta_t\) with score \(g\) at \(t = 0\) (with \(\eta_{t=0} = \eta\),

\[
\frac{d}{dt} \chi(\eta_t) \bigg|_{t=0} = \frac{d}{dt} E_{\eta_t} [m_1 (O,a_t)] + E_{\eta_t} [S_0] \bigg|_{t=0} = E_{\eta} [m_1 (O,a) g] + \frac{d}{dt} E_{\eta} [m_1 (O,a)] \bigg|_{t=0} + E_{\eta} [S_0 g].
\]

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But,
\[
\frac{d}{dt} E_\eta [m_1 (O, a_t)]_{t=0} = \frac{d}{dt} E_\eta [\mathcal{R}_1 (Z) a_t]_{t=0} = -E_\eta \left[ \mathcal{R}_1 (Z) \frac{d}{dt} \left( \frac{E_\eta [q (O) | Z]}{E_\eta [S_{ab} | Z]} \right) \right]_{t=0} = -E_\eta \left[ \mathcal{R}_1 (Z) \left( \frac{E_\eta \{q (O) - E_\eta [q (O) | Z]\} g | Z| - E_\eta [q (O) | Z] \frac{E_\eta \{S_{ab} - E_\eta [S_{ab} | Z]\} g | Z|}{E_\eta [S_{ab} | Z]^2} \right) \right] = E_\eta \left[ -\frac{\mathcal{R}_1 (Z)}{E_\eta [S_{ab} | Z]} \left( q (O) - \frac{E_\eta [q (O) | Z]}{E_\eta [S_{ab} | Z]} S_{ab} \right) g \right] = E_\eta [b (Z) (q (O) + a (Z) S_{ab}) g] \]

where
\[b (Z) \equiv -\frac{\mathcal{R}_1 (Z)}{E_\eta [S_{ab} | Z]} \]

Thus,
\[\chi_1 = m_1 (O, a) + b (Z) (q (O) + a (Z) S_{ab}) + S_0 = -E_\eta [m_1 (O, a) + b (Z) (q (O) + a (Z) S_{ab}) + S_0] = S_{ab} a b + m_1 (O, a) + q (O) b + S_0 - \chi (\eta) - E_\eta [b (Z) (q (O) + a (Z) S_{ab})] \]

But,
\[E_\eta [b (Z) (q (O) + a (Z) S_{ab})] = E_\eta [b (Z) (E_\eta [q (O) | Z] + a (Z) E_\eta [S_{ab} | Z])] = 0 \]

where the last identity follows by definition of a (Z). The last assertion of the Theorem follows by Theorem 2.

Proof: [Proof that the parameter ψ (η) in Example 6 is not in the class studied in Chernozhukov et al. (2018)] Let \(O = (DY, D, Z)\). Notice that given \(D = 0\), \(O\) depends only on \(Z\). Let
\[\psi (\eta) \equiv E_\eta \left[ (1 - D) \frac{E_\eta [DY \exp (\delta Y) | Z]}{E_\eta [D \exp (\delta Y) | Z]} \right] = E_\eta [(1 - D) a(Z)] \]
where \( a(Z) \equiv E_\eta [D \exp (\delta Y) | Z] / E_\eta [D \exp (\delta Y) | Z] \) for \( \delta \neq 0 \). Suppose that there exists \( m_*^1 (O, \cdot) \) such that for each \( \eta \), the map \( h \in L_2 (P_\eta (D,Z)) \rightarrow E_\eta [m_*^1 (O, h)] \) is continuous and linear and such that

\[
\sigma (\eta) = E_\eta [m_*^1 (O, a^*)] + E_\eta [S_0^*]
\]

where \( a^* (D, Z) = E_\eta [q (O) | D, Z] \) for some statistic \( q (O) \).

Without loss of generality we can assume that \( q (O) = D q^* (Y, Z) \). To see this write \( q (O) = D q^* (Y, Z) + (1 - D) q^{**} (Z) \). Then,

\[
a^* (D, Z) = E_\eta [q (O) | D, Z] = a^*_1 (D, Z) + a^*_0 (D, Z)
\]

where \( a^*_1 (D, Z) \equiv E_\eta [D q^* (Y, Z) | D, Z] \) and \( a^*_0 (D, Z) \equiv (1 - D) q^{**} (Z) \). Then, by the assumed linearity of the map \( h \in L_2 (P_\eta (D,Z)) \rightarrow E_\eta [m_*^1 (O, h)] \) we can now write

\[
E_\eta [m_*^1 (O, a^*)] + E_\eta [S_0^*] = E_\eta [m_*^1 (O, a^*_1)] + E_\eta [m_*^1 (O, a^*_0)] + E_\eta [S_0^*]
\]

where \( S_0^{**} = m_*^1 (O, a^*_0) + S_0^* \) is a statistic because \( a^*_0 (D, Z) \) does not depend on \( \eta \).

So, from now on we will assume \( a^* (D, Z) \equiv E_\eta [q (O) | D, Z] \) where \( q (O) = D q^* (Y, Z) \) for some \( q^* \). Note that \( q (Y, Z) \) depends on \( Y \), for otherwise \( a^* (D, Z) \) would not depend on \( \eta \).

Because \( \sigma (\eta) \) is the same functional as \( \psi (\eta) \) then their unique influence functions \( \sigma^1_\eta \) and \( \psi^1_\eta \) must agree. We shall compute next the influence function \( \psi^1_\eta \) of \( \psi (\eta) \). For any path \( t \rightarrow \eta_t \) through \( \eta_{t=0} = \eta \) with score \( g \) we have

\[
\frac{d}{dt} E_{\eta_t} \left[ (1 - D) \frac{E_{\eta_t} [D \exp (\delta Y) | Z]}{E_{\eta_t} [D \exp (\delta Y) | Z]} \right]_{t=0} = E_{\eta_t} \left\{ (1 - D) \frac{E_{\eta_t} [D \exp (\delta Y) | Z] - \psi (\eta)}{E_{\eta_t} [D \exp (\delta Y) | Z]} g \right\} + E_{\eta_t} \left[ E_{\eta_t} [1 - D | Z] \frac{d}{dt} E_{\eta_t} [D \exp (\delta Y) | Z] \right]_{t=0} \right]_{t=0} + E_{\eta_t} \left[ E_{\eta_t} [1 - D | Z] \frac{D \exp (\delta Y) - E_{\eta_t} [D \exp (\delta Y) | Z]}{E_{\eta_t} [D \exp (\delta Y) | Z]} \right] g
\]

\[
+ E_{\eta_t} \left[ E_{\eta_t} [1 - D | Z] \frac{D \exp (\delta Y) - E_{\eta_t} [D \exp (\delta Y) | Z]}{E_{\eta_t} [D \exp (\delta Y) | Z]^2} \right] g.
\]
So, we conclude that

\[
\psi^1_\eta = (1 - D) \frac{E_\eta [DY \exp(\delta Y) | Z] - \psi(\eta)}{E_\eta [D \exp(\delta Y) | Z]} + \psi(\eta) + E_\eta [1 - D | Z] \frac{D \exp(\delta Y) - E_\eta [DY \exp(\delta Y) | Z]}{E_\eta [D \exp(\delta Y) | Z]}
\]

\[
- E_\eta [1 - D | Z] E_\eta [DY \exp(\delta Y) | Z] \frac{D \exp(\delta Y) - E_\eta [D \exp(\delta Y) | Z]}{E_\eta [D \exp(\delta Y) | Z]^2}
\]

\[
= (1 - D) \frac{E_\eta [DY \exp(\delta Y) | Z]}{E_\eta [D \exp(\delta Y) | Z]} + DY \exp(\delta Y) \frac{E_\eta [1 - D | Z]}{E_\eta [D \exp(\delta Y) | Z]}
\]

\[
- D \exp(\delta Y) \frac{E_\eta [1 - D | Z] E_\eta [DY \exp(\delta Y) | Z]}{E_\eta [D \exp(\delta Y) | Z]^2} - \psi(\eta).
\]

On the other hand, letting \( R^*_\eta (D, Z) \) be the Riesz resolvent of the map \( \eta \in L_2 (P_{\eta, (D, Z)}) \rightarrow E_\eta [m_1^* (O, h)] \), we have

\[
\frac{d}{dt} E_\eta [m_1^* (O, a^*_\eta)] + E_\eta [S_0^*]
\]

\[
eq E_\eta [\{m_1^* (O, a^*) - E_\eta [m_1^* (O, a^*)]\} g] + \frac{d}{dt} E_\eta [R^*_\eta (D, Z) E_\eta [q (O) | D, Z]] |_{t=0}
\]

\[
eq E_\eta [m_1^* (O, a^*) g] + E_\eta [R^*_\eta (D, Z) \{q (O) - E_\eta [q (O) | D, Z]\} g] + E_\eta (S_0^*) - \sigma(\eta)
\]

from where we conclude that

\[
\sigma^*_\eta = m_1^* (O, a^*) + R^*_\eta (D, Z) \{q (O) - E_\eta [q (O) | D, Z]\} + S_0^* - \sigma(\eta).
\]

The uniqueness of influence functions \( \sigma^*_\eta \) and \( \psi^1_\eta \) implies that

\[
(1 - D) \frac{E_\eta [DY \exp(\delta Y) | Z]}{E_\eta [D \exp(\delta Y) | Z]} + D \exp(\delta Y) \frac{E_\eta [1 - D | Z]}{E_\eta [D \exp(\delta Y) | Z]}
\]

\[
- D \exp(\delta Y) \frac{E_\eta [1 - D | Z] E_\eta [DY \exp(\delta Y) | Z]}{E_\eta [D \exp(\delta Y) | Z]^2}
\]

\[
= m_1^* (O, a^*) + R^*_\eta (D, Z) \{q (O) - E_\eta [q (O) | D, Z]\} + S_0^*
\]

(13)

(14)

Now, taking \( \eta \) and \( \eta' \) that agree on the law of \( Y, D | Z \), but disagree on the marginal of \( Z \), the left hand side agree on these two laws as well as \( a^* \) so, subtracting one from the other we obtain

\[
0 = \{R^*_\eta (D, Z) - R^*_{\eta'} (D, Z)\} \{q (O) - E_\eta [q (O) | D, Z]\}
\]

\[
= \{R^*_\eta (D = 1, Z) - R^*_{\eta'} (D = 1, Z)\} D \{q^*(Y, Z) - E_\eta [q^*(Y, Z) | D = 1, Z]\}
\]

Since \( q^*(Y, Z) \) depends on \( Y \) then \( R^*_\eta (D = 1, Z) - R^*_{\eta'} (D = 1, Z) \) must be equal to 0. So, we conclude that \( R^*_\eta (D = 1, Z) \) depends on \( \eta \) only through the law of \( Y, D | Z \).
Next, for any $h(D, Z) = Du(Z)$, we have

$$E_\eta \left[ E_\eta \{ m_1^* (O, h) | Z \} \right] = E_\eta \left[ E_\eta \{ \mathcal{R}_\eta^* (D = 1, Z) Du(Z) | Z \} \right] \text{ for all } \eta.$$ 

So, since $\mathcal{R}_\eta^* (D = 1, Z)$ does not depend on the marginal law of $Z$, we conclude that

$$E_\eta \{ m_1^* (O, h) | Z \} = E_\eta \{ \mathcal{R}_\eta^* (D = 1, Z) Du(Z) | Z \}$$

or equivalently

$$E_\eta \{ m_1^* (O, h) | D = 0, Z \} E_\eta (1 - D | Z) + E_\eta \{ m_1^* (O, h) | D = 1, Z \} E_\eta (D | Z) = \mathcal{R}_\eta^* (D = 1, Z) u(Z) E_\eta (D | Z).$$

Suppose that $z^*$ is such that $u(z^*) = 0$. Then,

$$m_1^* ((0, 0, z^*), h) + \left[ E_\eta \{ m_1^* (O, h) | D = 1, Z = z^* \} - m_1^* ((0, 0, z^*), h) \right] E_\eta (D | Z = z^*) = 0$$

where to arrive at the left hand side we have used the fact that $E_\eta \{ m_1^* (O, h) | D = 0, Z \} = m_1^* ((0, 0, Z), h)$. Now, since $E_\eta \{ m_1^* (O, h) | D = 1, Z = z^* \}$ does not depend on the law of $D | Z$, then letting $E_\eta (D | Z = z^*) \to 0$ we conclude that $m_1^* ((0, 0, z^*), h) = 0$ and consequently also $E_\eta \{ m_1^* (O, h) | D = 1, Z = z^* \} = 0$.

Next, for any $Z = z$ such that $u(z) \neq 0$ we write

$$\frac{1}{E_\eta (D | Z = z)} m_1^* ((0, 0, z), h) + \frac{E_\eta \{ m_1^* (O, h) | D = 1, Z = z \} - m_1^* ((0, 0, z), h)}{u(z)}$$

and since $\mathcal{R}_\eta^* (D = 1, z)$ does not depend on $u$, then taking any other $u^*$ with $u^*(z) \neq 0$, we have

$$0 = \frac{1}{E_\eta (D | Z = z)} \left[ \frac{m_1^* ((0, 0, z), h)}{u(z)} - \frac{m_1^* ((0, 0, z), h^*)}{u^*(z)} \right]$$

$$+ \frac{\left[ E_\eta \{ m_1^* (O, h) | D = 1, Z = z \} - m_1^* ((0, 0, z), h) \right]}{u(z)}$$

$$- \frac{\left[ E_\eta \{ m_1^* (O, h^*) | D = 1, Z = z \} - m_1^* ((0, 0, z), h^*) \right]}{u^*(z)}.$$

Once again, since none of the terms in squared brackets depend on the law of $D | Z$, the right hand side is a linear function of $\alpha \equiv \frac{1}{E_\eta (D | Z = z)}$ which can take any value in $(1, \infty)$, but the left hand side is identically equal to 0. Therefore,

$$\frac{m_1^* ((0, 0, z), h)}{u(z)} - \frac{m_1^* ((0, 0, z), h^*)}{u^*(z)} = 0.$$

So, we conclude that there exists a function $c(z)$ independent of $\eta$ such that for all $h(D, Z) = Du(Z)$

$$m_1^* ((0, 0, z), h) = c(z) u(z) \quad (15)$$

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Next, return to the equations (13) and (14) and evaluate them at \( D = 0 \), to obtain

\[
\frac{E_\eta [Y \exp (\delta Y) | D = 1, Z]}{E_\eta [\exp (\delta Y) | D = 1, Z]} = m_1^* (0, 0, Z, a^*) + t (Z)
\]

where \( t (Z) \) is equal to \( S_0^* (0, 0, Z) \), i.e. to \( S_0^* \) evaluated at \( D = 0 \). Next, recalling that \( a^* (D, Z) = DE [q^* (Y, Z) | D = 1, Z] \) and invoking (15) we conclude that

\[
\frac{E_\eta [Y \exp (\delta Y) | D = 1, Z]}{E_\eta [\exp (\delta Y) | D = 1, Z]} = c (Z) E_\eta [q^* (Y, Z) | D = 1, Z] + t (Z)
\]

where \( c (z) \) and \( t (z) \) are functions of \( z \) that do not depend on \( \eta \). We will now show that the last equality cannot hold for all \( \eta \) if \( \delta \neq 0 \). To do so, we re-write the last identity as

\[
\frac{E_\eta [DY \exp (\delta Y) | Z]}{E_\eta [D \exp (\delta Y) | Z]} = c (Z) \frac{E_\eta [Dq^* (Y, Z) | Z]}{E_\eta [D | Z]} + t (Z).
\]

(16)

If this identity holds for all \( \eta \), then taking expectations on both sides we have that for all \( \eta \)

\[
E_\eta \left\{ E_\eta \left[ \frac{DY \exp (\delta Y) | Z}{E_\eta [D \exp (\delta Y) | Z]} \right] \right\} = E_\eta \left\{ c (Z) \frac{E_\eta [Dq^* (Y, Z) | Z]}{E_\eta [D | Z]} \right\} + E_\eta \{ t (Z) \}.
\]

(17)

or equivalently, for all \( \eta \)

\[
E_\eta \left\{ \frac{DY \exp (\delta Y)}{E_\eta [D \exp (\delta Y) | Z]} \right\} = E_\eta \left\{ c (Z) \frac{Dq^* (Y, Z)}{E_\eta [D | Z]} \right\} + E_\eta \{ t (Z) \}.
\]

Since the functionals on the left and right hand-sides are identical, their influence functions must agree. Then, taking an arbitrary submodel \( t \to \eta \) with score \( g \) at \( \eta_{t=0} = \eta \) we have

\[
\frac{d}{dt} E_\eta \left\{ \frac{DY \exp (\delta Y)}{E_\eta [D \exp (\delta Y) | Z]} \right\} \bigg|_{t=0} = \frac{d}{dt} E_\eta \left\{ c (Z) \frac{Dq^* (Y, Z)}{E_\eta [D | Z]} \right\} \bigg|_{t=0} + \frac{d}{dt} E_\eta \{ t (Z) \} \bigg|_{t=0}
\]

from where we conclude that

\[
E_\eta \left[ \left[ \frac{DY \exp (\delta Y)}{E_\eta [D \exp (\delta Y) | Z]} - E_\eta \left\{ \frac{DY \exp (\delta Y)}{E_\eta [D \exp (\delta Y) | Z]} \right\} g \right] \right] = \left\{ E_\eta \left[ \frac{Dq^* (Y, Z)}{E_\eta [D | Z]} \right] \bigg( D - E_\eta [D | Z] \bigg) g \right\} + E_\eta \{ t (Z) \} - E_\eta \{ t (Z) \} g
\]

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Consequently,

\[
\begin{align*}
&\frac{DY \exp(\delta Y)}{E_\eta[D \exp(\delta Y) | Z]} - E_\eta \left\{ \frac{DY \exp(\delta Y)}{E_\eta[D \exp(\delta Y) | Z]} \right\} \\
&= \frac{E_\eta[D \exp(\delta Y) | Z]^2}{E_\eta[D \exp(\delta Y) | Z]} \{ D \exp(\delta Y) - E_\eta[D \exp(\delta Y) | Z] \} + t(Z) - E_\eta \{ t(Z) \}.
\end{align*}
\]

Invoking the equalities (16) and (17), the last identity is the same as

\[
\frac{1}{E_\eta[D \exp(\delta Y) | Z]} DY \exp(\delta Y) - \frac{E_\eta[D Y \exp(\delta Y) | Z]}{E_\eta[D \exp(\delta Y) | Z]^2} D \exp(\delta Y)
\]

or equivalently

\[
\frac{1}{E_\eta[D \exp(\delta Y) | Z]} Dq^* (Y, Z) - Dc(Z) \frac{E_\eta[D q^*(Y, Z) | Z]}{E_\eta[D | Z]^2}.
\]

The last equality is equivalent to

\[
\frac{D \exp(\delta Y)}{E_\eta[\exp(\delta Y) | D = 1, Z]} \left\{ Y - \frac{E_\eta[Y \exp(\delta Y) | D = 1, Z]}{E_\eta[\exp(\delta Y) | D = 1, Z]} \right\} = Dc(Z) \{ q^*(Y, Z) - E_\eta[q^*(Y, Z) | D = 1, Z] \}
\]

The last equation cannot hold for all \( \eta \). To see this, evaluate the left and right and sides at \( y \) and \( y^* \) with \( y \neq y^* \), and subtract one from the other, to obtain

\[
\frac{D \exp(\delta y)}{E_\eta[\exp(\delta Y) | D = 1, Z]} \left\{ y - \frac{E_\eta[Y \exp(\delta Y) | D = 1, Z]}{E_\eta[\exp(\delta Y) | D = 1, Z]} \right\} = Dc(Z) \{ q^*(y, Z) - q^*(y^*, Z) \}
\]

The left hand side depends on \( \eta \) whereas the right hand side does not. We will show that this cannot occur when \( \delta \neq 0 \). To do so, let \( \eta \) and \( \eta' \) correspond to two arbitrary distinct laws. Then evaluating the
left hand side at \( \eta \) and at \( \eta' \) and subtracting one from the other, we obtain

\[
D \left\{ \frac{1}{E_\eta \left[ \exp (\delta Y) \right] |D = 1, Z|} - \frac{1}{E_{\eta'} \left[ \exp (\delta Y) \right] |D = 1, Z|} \right\} \exp (\delta y) y - \\
D \exp (\delta y) \left\{ \frac{E_\eta \left[ Y \exp (\delta Y) |D = 1, Z| \right] - E_{\eta'} \left[ Y \exp (\delta Y) |D = 1, Z| \right]}{E_\eta \left[ \exp (\delta Y) \right] |D = 1, Z|^2} \right\} - \\
D \left\{ \frac{1}{E_\eta \left[ \exp (\delta Y) \right] |D = 1, Z|} - \frac{1}{E_{\eta'} \left[ \exp (\delta Y) \right] |D = 1, Z|} \right\} \exp (\delta y^*) y^* - \\
D \exp (\delta y^*) \left\{ \frac{E_\eta \left[ Y \exp (\delta Y) |D = 1, Z| \right] - E_{\eta'} \left[ Y \exp (\delta Y) |D = 1, Z| \right]}{E_\eta \left[ \exp (\delta Y) \right] |D = 1, Z|^2} \right\} = 0
\]

This holds for all \( y \) and \( y^* \). Then, regarding \( y^*, \eta \) and \( \eta' \) as fixed and \( y \) as a free variable the preceding display is of the form

\[
k_1 (D, z) \exp (\delta y) y - k_2 (D, z) \exp (\delta y) + k_3 (D, z) = 0.
\]

Next, since \( \exp (\delta y) y \) and \( \exp (\delta y) \) are not the same function of \( y \), then the preceding identity can only hold if \( k_j (D, z) = 0 \) for \( j = 1, 2, 3 \). In particular, the equality \( k_j (D, z) = 0 \) implies that

\[
E_\eta \left[ \exp (\delta Y) \right] |D = 1, Z| = E_{\eta'} \left[ \exp (\delta Y) \right] |D = 1, Z|.
\]

But since \( \eta \) and \( \eta' \) are arbitrary, this implies that \( E_\eta \left[ \exp (\delta Y) \right] |D = 1, Z| \) does not depend on \( \eta \). This is a contradiction when \( \delta \neq 0 \) because it would imply that \( \exp (\delta Y) = \mathcal{C} (Z) \) for some function \( \mathcal{C} \). This shows that \( \sigma (\eta) \) cannot be equal to \( \psi (\eta) \).

Next, we will show that \( \psi (\eta) \) cannot be equal to any parameter of the form

\[
\kappa (\eta) \equiv E_\eta \left[ m_1^* (O, a^*) \right] + E_\eta \left[ S_0^* \right]
\]

where \( a^* (Z) = E_\eta \left[ q (O) | Z \right] \) for some statistic \( q (O) \), the map \( h \in L_2 (P_{\eta,Z}) \rightarrow E_\eta \left[ m_1^* (O, h) \right] \) is continuous and linear for each \( \eta \) and \( S_0^* \) is a statistic. To proceed, just as before, we start by arguing that if the parameter \( \kappa (\eta) \) is the same as \( \psi (\eta) \) then their unique influence functions must agree. We have already computed the influence function \( \psi_\eta^1 \) of \( \psi (\eta) \).

On the other hand, it is easy to see that the influence function \( \kappa_\eta^1 \) of \( \kappa (\eta) \) is

\[
\kappa_\eta^1 = m_1^* (O, a^*) + \mathcal{R}_\eta^* (Z) \left\{ q (O) - E_\eta \left[ q (O) | Z \right] \right\} + S_0^* - \kappa (\eta)
\]

where \( \mathcal{R}_\eta^* (Z) \) is the Riesz representer of the map \( h \in L_2 (P_{\eta,Z}) \rightarrow E_\eta \left[ m_1^* (O, h) \right] \). Consequently, equating \( \kappa_\eta^1 \) with \( \psi_\eta^1 \) we obtain

\[
(1 - D) \frac{E_\eta \left[ DY \exp (\delta Y) | Z \right]}{E_\eta \left[ D \exp (\delta Y) | Z \right]} + D \exp (\delta Y) \frac{E_\eta \left[ Y \exp (\delta Y) | Z \right] - E_\eta \left[ \exp (\delta Y) | Z \right]}{E_\eta \left[ D \exp (\delta Y) | Z \right]} - (20)
\]

\[
D \exp (\delta Y) \frac{E_\eta \left[ Y \exp (\delta Y) | Z \right]}{E_\eta \left[ D \exp (\delta Y) | Z \right] - E_\eta \left[ \exp (\delta Y) | Z \right]^2} = m_1^* (O, a^*) + \mathcal{R}_\eta^* (Z) \left\{ q (O) - E_\eta \left[ q (O) | Z \right] \right\} + S_0^*.
\]

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Now, taking \( \eta \) and \( \eta' \) that agree on the law of \( Y,D|Z \), but disagree on the marginal of \( Z \), the left hand side agree on these two laws as well as \( a^* \) so, subtracting one from the other we obtain

\[
0 = \left\{ R^*_\eta(Z) - R^*_{\eta'}(Z) \right\} \{ q(O) - E_\eta [q(O)|Z] \}.
\]

Since \( q(O) \) depends on \( (D,Y) \) then \( \{ R^*_\eta(Z) - R^*_{\eta'}(Z) \} \) must be equal to 0. So, we conclude that \( R^*_\eta(Z) \) depends on \( \eta \) only through the law of \( Y,D|Z \).

Next, for any \( h(Z) \), we have

\[
E_\eta [E_\eta \{ m^*_1(O,h) | Z \}] = E_\eta \left[ R^*_\eta(Z) h(Z) \right] \text{ for all } \eta.
\]

So, since \( R^*_\eta(Z) \) does not depend on the marginal law of \( Z \), we conclude that

\[
E_\eta \{ m^*_1(O,h) | Z \} = R^*_\eta(Z) h(Z).
\]

The last equality is the same as

\[
m^*_1(0,0,Z,h) E_\eta (1 - D|Z) + E_\eta \{ m^*_1(O,h) | D = 1, Z \} E_\eta (D|Z) = R^*_\eta(Z) h(Z)
\]

or equivalently

\[
m^*_1(0,0,Z,h) + [E_\eta \{ m^*_1(O,h) | D = 1, Z \} - m^*_1(0,0,Z,h)] E_\eta (D|Z) = R^*_\eta(Z) h(Z)
\]

If \( z^* \) is such that \( h(z^*) = 0 \), then

\[
m^*_1(0,0,z^*,h) + [E_\eta \{ m^*_1(O,h) | D = 1, Z = z^* \} - m^*_1(0,0,z^*,h)] E_\eta (D|Z = z^*) = 0
\]

and since \( E_\eta \{ m^*_1(O,h) | D = 1, Z = z^* \} \) does not depend on the law of \( D|Z \), then since \( E_\eta (D|Z = z^*) \) can take any value in \((0,1)\), we conclude that \( m^*_1(0,0,z^*,h) = 0 \).

On the other hand, for \( z \) such that \( h(z) \neq 0 \), we have

\[
\frac{m^*_1(0,0,z,h)}{h(z)} + \left[ E_\eta \{ m^*_1(O,h) | D = 1, Z = z \} - m^*_1(0,0,z,h) \right] \frac{E_\eta (D|Z = z)}{h(z)} = R^*_\eta(Z = z)
\]

Consequently, for any other \( h^* \) such that \( h(z^*) \neq 0 \), we have

\[
0 = \left\{ \frac{m^*_1(0,0,z,h)}{h(z)} - \frac{m^*_1(0,0,z,h^*)}{h^*(z)} \right\} + \left[ E_\eta \{ m^*_1(O,h) | D = 1, Z = z \} - m^*_1(0,0,z,h) \right] \frac{E_\eta (D|Z = z)}{h(z)} - \left[ E_\eta \{ m^*_1(O,h^*) | D = 1, Z = z \} - m^*_1(0,0,z,h^*) \right] \frac{E_\eta (D|Z = z)}{h^*(z)}.
\]

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Once again, since $E_{\eta} \{ m^*_1(O, h^*) \mid D = 1, Z = z \}$ and $E_{\eta} \{ m^*_1(O, h) \mid D = 1, Z = z \}$ do not depend on the law of $D \mid Z$, and since $E_{\eta} (D \mid Z = z)$ can take any value in $(0, 1)$ we conclude that

$$\frac{m^*_1(0, 0, Z, h)}{h(Z)} - \frac{m^*_1(0, 0, Z, h^*)}{h^*(Z)} = 0$$

Consequently, there exists a function $c(Z)$ such that for all $h$

$$m^*_1(0, 0, Z, h) = c(Z) h(Z). \tag{22}$$

Now, evaluating (20) and (21) at $D = 0$

$$\frac{E_{\eta} [DY \exp(\delta Y) \mid Z]}{E_{\eta} [D \exp(\delta Y) \mid Z]} = m^*_1(0, 0, Z, a^*) + R^*_\eta(Z) \{ q(0, 0, Z) - E_{\eta} [q(0, 0, Z) \mid Z] \} + S^*_0(0, 0, Z).$$

So, with $t(Z) \equiv S^*_0(0, 0, Z)$ and with $a^*(Z) = E_{\eta} [q(O) \mid Z]$ substituted for $h$ in (22) we arrive at the conclusion that the following equality must hold for all $\eta$

$$\frac{E_{\eta} [DY \exp(\delta Y) \mid Z]}{E_{\eta} [D \exp(\delta Y) \mid Z]} = c(Z) E_{\eta} [q(O) \mid Z] + t(Z).$$

Therefore,

$$E_{\eta} \left\{ \frac{DY \exp(\delta Y)}{E_{\eta} [D \exp(\delta Y) \mid Z]} \right\} = E_{\eta} [c(Z) q(O) + t(Z)].$$

Again equating the influence functions of the functionals on the right and left hand sides we conclude that

$$\frac{DY \exp(\delta Y)}{E_{\eta} [D \exp(\delta Y) \mid Z]} - \frac{E_{\eta} [DY \exp(\delta Y) \mid Z]}{E_{\eta} [D \exp(\delta Y) \mid Z]^2} \{ D \exp(\delta Y) - E_{\eta} [D \exp(\delta Y) \mid Z] \} = c(Z) q(O) + t(Z)$$

which leads to a contradiction when $\delta \neq 0$. To arrive at the contradiction we would reason just as we did to show that the equality (19) leads to a contradiction. \hfill \Box

**Proof:** [Proof that the parameter in Example 7 (a), is in the class of Chernozhukov et al. (2018) but not in the class of Robins et al. (2008)] Let

$$\chi(\eta) \equiv E_{\eta} \left[ \int_0^1 E_{\eta} (Y \mid D = u, L) w(u) \, du \right].$$

Its influence function is

$$\chi^1_{\eta} = \int_0^1 E_{\eta} (Y \mid D = u, L) w(u) \, du + \frac{w(D)}{\int_{\eta} (D|L)} \{ Y - E_{\eta} (Y \mid D, L) \} - \chi(\eta).$$

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Suppose the parameter $\chi(\eta)$ is in the class of [Robins et al. (2008)] for some functions $a^*(D, L)$ and $b^*(D, L)$ which are non-constant in $D$ and in $L$. Then, there would exist statistics $S_{ab}, S_a, S_b$ and $S_0$ such that

$$a^*(D, L) = -\frac{E_\eta(S_b|D, L)}{E_\eta(S_{ab}|D, L)}, b^*(D, L) = -\frac{E_\eta(S_a|D, L)}{E_\eta(S_{ab}|D, L)}$$

and such that

$$\chi^1_\eta = S_{ab}a^*(D, L)b^*(D, L) + S_a a^*(D, L) + S_b b^*(D, L) + S_0 - \chi(\eta).$$

Then, equating the influence functions we arrive at

$$\int_0^1 E_\eta(Y|D = u, L) w(u) du + \frac{w(D)}{f_\eta(D|L)} \{Y - E_\eta(Y|D, L)\} = S_{ab}a^*(D, L)b^*(D, L) + S_a a^*(D, L) + S_b b^*(D, L) + S_0$$

The right hand side does not depend on $f_\eta(D|L)$. On the other hand, in the left hand side

$$\int_0^1 E_\eta(Y|D = u, L) w(u) du$$

does not depend on $f_\eta(D|L)$ but

$$\frac{w(D)}{f_\eta(D|L)} \{Y - E_\eta(Y|D, L)\}$$

depends on $f_\eta(D|L)$. This is a contradiction.

Now, suppose that we re-define $Z = L$, so that we now partition $O$ into $(Y, D)$ and $Z = L$. If the parameter $\chi(\eta)$ was in the class of [Robins et al. (2008)] for some functions $a^*(L)$ and $b^*(L)$ which are non-constant in $L$, then, there would exist statistics $S_{ab}, S_a, S_b$ and $S_0$ such that

$$a^*(L) = -\frac{E_\eta(S_b|L)}{E_\eta(S_{ab}|L)}, b^*(L) = -\frac{E_\eta(S_a|L)}{E_\eta(S_{ab}|L)}$$

and such that

$$\chi^1_\eta = S_{ab}a^*(L)b^*(L) + S_a a^*(L) + S_b b^*(L) + S_0 - \chi(\eta).$$

Then equating the influence functions we would arrive at

$$\int_0^1 E_\eta(Y|D = u, L) w(u) du + \frac{w(D)}{f_\eta(D|L)} \{Y - E_\eta(Y|D, L)\} = S_{ab}a^*(L)b^*(L) + S_a a^*(L) + S_b b^*(L) + S_0.$$
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