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SUPERSYMMETRIES AND COHOMOLOGY OF GRAPH COMPLEXES.

SERGUEI BARANNIKOV

1. INTRODUCTION.

I study the noncommutative generalisation of the Chern-Simons type construction of cohomology classes of graph complexes from [K2]. I propose the extension of the initial framework beyond the paradigm of homological algebra. My main initial ingredient is the h-invertible supersymmetry generator $I$ acting on finite-dimensional algebra, whose square is nonzero in general, $I^2 \neq 0$. The second principal result is the construction of boundaries on the stable ribbon graph complex associated to a pair of certain operators. It is used to construct the Virasoro algebra action on the stable ribbon graph complexes in [B3].

Notations. For an element $a$ from super vector space $A$ I denote by $\pi a \in \Pi A$ the same element considred with inversed parity. $\text{char}(k) = 0$. The parity of $a$ is denoted by $\varepsilon(a)$.

Let $A$ be finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra with odd invariant inner product

$$g(\cdot, \cdot) : S^2 A \rightarrow \Pi k$$

$$g(ab, c) = g(a, bc)$$

$$g(a, b) = (-1)^{\varepsilon(a)\varepsilon(b)} g(b, a)$$

If the algebra has an identity then such an inner product is the same as an odd trace

$$l : A/[A, A] \rightarrow \Pi k; g(a, b) = l(ab)$$

In general the linear functional $l$ is well defined via $l(ab) = g(a, b)$ on the image of the multiplication map $A^{\otimes 2} \rightarrow A$.

Examples: $q(n)$, $H^*_{DR}$ on odd-dimensional manifold, $Cl_1$.

2. SUPERSYMMETRY.

Let $I$ be an odd derivation acting on $A$, preserving the inner product:

$$g(Ix, y) = (-1)^{\varepsilon(x)\varepsilon(I)} g(x, Iy)$$

or, if the algebra has an identity, equivalently,

$$l_{|\text{im}(I)} = 0$$

Examples: $[\Lambda, \cdot]$ with $\Lambda = 1$, $i_v$ with $\varepsilon = 0$, ...

Notice that the square of the derivation $I$ is an even derivation of $A$, which is nonzero in the general case.

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Condition 1. Let the kernel of the derivation $I$ is isotropic with respect to the pairing:

$$g(\cdot, \cdot)|_{\ker(I)} = 0$$

or, if the algebra has an identity, equivalently,

$$I|_{\ker(I)} = 0$$

Proposition 1. Any such derivation is invertible in the following homotopic sense. There exists an odd self-adjoint operator $\tilde{I}$, acting on $A$, such that:

$$(2.1) \ [I, \tilde{I}] = Id$$

Proof. If $\ker(I) = 0$ then $I$ is a linear isomorphism. Then I take simply the one-half of the inverse to $I$

$$\tilde{I} = \frac{1}{2} I^{-1}$$

In the general case, the condition (1) implies $\ker(I)$ belongs to the orthogonal complement of itself, or since $I$ is anti-selfadjoint,

$$\ker(I) \subset \text{Im}(I)$$

Then the restriction of $I$ on $\ker(I)$ induces an odd linear isomorphism

$$(\ker(I^2))/\ker(I) \cong \ker(I)$$

One can choose subspaces $L_i$, so that $L_i \oplus \ker(I)_i = \ker(I^2)_i$, $i = 0, 1$, and that $L_0$ is orthogonal to $L_1$. Define $\tilde{I}$ on $\ker(I)$ as the inverse to the odd isomorphism $L \cong \ker(I)$, where $L = L_0 \oplus L_1$. Extend $\tilde{I}$ to $\ker(I^2)$ by $\tilde{I}|_{L} = 0$. Then $[I, \tilde{I}]|_{\ker(I^2)} = Id$. Let $L'$ be the orthogonal complement to $\ker(I^2)$ in $A$. The inner product is non-degenerate on $\ker(I^2)$ and we have the direct sum decomposition

$$(2.2) \ L' \oplus \ker(I^2) = A,$$

preserved by $I$. Then restriction of $I$ on $L'$ is a linear isomorphism and I extend $\tilde{I}$ to $A$ by

$$\tilde{I}|_{L'} = \frac{1}{2} I^{-1}|_{L'}.$$ 

It is easy to check that $\tilde{I}$ is self-adjoint. 

Definition 1. I call such a derivation $I$ satisfying (2.1) $h$-invertible.

This notion extends the notion of contractible differential to the case when $I^2 \neq 0$.

3. The ribbon graph complex.

Recall the definition of the (even) ribbon graph complex $(C_*, d)$, see ([K2],[GK]).

Definition 2. A ribbon graph $G$ is a triple $(\text{Flag}(G), \sigma, \eta)$, where $\text{Flag}(G)$ is a finite set, whose elements are called flags, $\sigma$ is a permutation from $\text{Aut}(\text{Flag}(G))$ with orbits of length greater than two, and $\eta$ is a fixed-point free involution acting on $\text{Flag}(G)$. 

The vertices of the graph correspond to the cycles of the permutation $\sigma$. The set of vertices is denoted by $\text{Vert}(G)$. The subset of $\text{Flag}(G)$ corresponding to vertex $v$ is denoted by $\text{Flag}(v)$. The set $\text{Flag}(v)$ has natural cyclic order since it is a cycle of $\sigma$. The cardinality of $\text{Flag}(v)$ is called the valence of $v$ and is denoted $n(v)$. It is assumed that $n(v) \geq 3$. The edges of the graph are the pairs of flags forming a two-cycle of the involution $\eta$. The set of edges is denoted $\text{Edge}(G)$. The subset of edges which are loops is denoted by $\text{Loop}(G)$. The edges which are not loops are called "regular" and I denote the corresponding subset of $\text{Edge}(G)$ by $\text{Edge}_r(G)$. The two element subset of $\text{Flag}(G)$ corresponding to an edge $e \in \text{Edge}(G)$ is denoted $\text{flag}(e)$. I denote by $S_G$ the Riemann surface associated with the ribbon graph $G$ (see [K1], [B1] and references therein). I denote by $|G|$ the one-dimensional CW-complex which is the geometric realisation of the underlying graph $G$.

**Definition 3.** The even ribbon graph complex is the vector space generated by equivalence classes of pairs $(G, \text{or}(G))$, where $G$ is a connected ribbon graph, $\text{or}(G)$ is an orientation on the vector space

$$
\bigotimes_{v \in \text{Vert}(G)} (k^{\text{Flag}(v)} \oplus k)
$$

and the relation $(G, -\text{or}(G)) = -(G, \text{or}(G))$ is imposed. The differential is

$$
d(G, \text{or}(G)) = \sum_{e \in \text{Edge}_r(G)} (G/\{e\}, \text{induced orientation})
$$

The identification of my definition of the orientation with that one from ([K2]) follows from ([GK], 4.14). A choice of orientation can be fixed by a choice of a flag from $\text{Flag}(v)$ for all vertices $v$ having even number of flags and a choice of an order on the set of such vertices. Two different choices are related by a set of some cyclic permutations on $\text{Flag}(v)$ for vertices $v$ having even number of flags and a permutation on the set of such vertices. The two corresponding orientations differ by the sign equal to the product of signs of all these permutations. The induced orientation on $G/\{e\}$ can be easily worked out. In the case when the edge $e$ connects two vertices $v_1$ and $v_2$ with even number of flags, assume that the order on the set of vertices with even number of flags is such that $v_1 < v_2$ and no other such vertices are between them, and that the flags $f_i \in \text{Flag}(v_i)$, $i = 1, 2$, fixing the orientation, form the edge $e$. Then the induced orientation for $G/\{e\}$ is defined by placing the new vertex $v_0$, obtained from $v_1$ and $v_2$ after shrinking of $e$, to the place of $v_1$ and by choosing the element corresponding to $\sigma_G(f_1)$ in $\text{Flag}(v_0)$. In the case when the edge $e$ connects two vertices $v_1$ and $v_2$ with odd number of flags, the induced orientation for $G/\{e\}$ is defined by placing the new vertex $v_0$, to the lowest level in the hierarchy of vertices with even number of flags, and by choosing the flag corresponding to $\sigma_G(f)$ in $\text{Flag}(v_0)$, where $f$ is either of the two flags forming $e$.

I denote via $(C^*, \delta)$ the complex dual to the (even) ribbon graph complex. We identify the generators of both complexes corresponding to oriented ribbon graphs when it does not seem to lead to a confusion.

**4. The partition function.**

Since $g$ is odd, the symmetric dual inner product $g^{-1}$ is defined on $\Pi \text{Hom}(A, k)$, the dual space with parity inversed. If $P$ denotes the isomorphism $A \to \Pi \text{Hom}(A, k)$
induced by \( g \), then
\[
g^{-1}(\varphi, \psi) = g(P^{-1} \varphi, P^{-1} \psi).
\]

Denote by \( \tilde{I}^* \) the dual to \( \tilde{I} \), odd symmetric operator, acting on \( \Pi \text{Hom}(A, k) \):
\[
\tilde{I}^* \varphi(a) = (-1)^{\text{parity}} \varphi(\tilde{I}a),
\]

one can check that
\[
\tilde{I}^* = P \tilde{I} P^{-1},
\]

since \( \tilde{I} \) is symmetric. The odd symmetric operator \( \tilde{I}^* \) defines the even symmetric pairing \( g^{-1}_{\tilde{I}} \) on \( \Pi \text{Hom}(A, k) \):
\[
g^{-1}_{\tilde{I}}(\varphi, \psi) = g^{-1}(\tilde{I}^* \varphi, \psi), \quad g^{-1}_{\tilde{I}} \in (\Pi A)^{\otimes 2}
\]

**Definition 4.** Given a ribbon graph \( G \), the tensor product of the even symmetric tensors \((g^{-1}_{\tilde{I}})_{e}\), associated with every edge \( e \) of \( G \), \((g^{-1}_{\tilde{I}})_{e} \in (\Pi A)^{\otimes \text{flag}(e)}\), defines the canonical element
\[
g^{-1}_{\tilde{I},G} = \otimes_{e \in \text{Edge}(G)} (g^{-1}_{\tilde{I}})_{e}, \quad g^{-1}_{\tilde{I},G} \in (\Pi A)^{\otimes \text{Flag}(G)}.
\]

**Definition 5.** Define the \((-1)^{n+1}\)-cyclically symmetric tensors \( \alpha_{n} \in \text{Hom}((\Pi A)^{\otimes n}, k) \), \( n \geq 2 \)
\[
\alpha_{n}(\pi a_{1}, \ldots, \pi a_{n}) = (-1)^{\sum_{i=1}^{n-1} j i} \pi_{1} I(a_{1} \cdot a_{2} \cdot \ldots \cdot a_{n-1} \cdot a_{n})
\]

(the tensor from \( V^{\otimes n} \) is \((-1)^{n+1}\)-cyclically symmetric if under the elementary cyclic shift it is multiplied by the Koszul sign resulting from parities of elements of \( V \) plus the sign of the cyclic permutation, which is \((-1)^{n+1}\))

Notice that the space \( \text{Flag}(v) \) is canonically oriented for a vertex \( v \in \text{Vert}(G) \) of odd valency, \( n(v) \equiv 1 \mod 2 \), thanks to the cyclic order. For a vertex \( v \in \text{Vert}(G) \) of valency \( n(v) \equiv 0 \mod 2 \), let \( o(v) \) denotes an orientation on the vector space \( \text{Flag}(v) \). Because of cyclic order on \( \text{Flag}(v) \), \( o(v) \) is fixed canonically by a choice of an element \( f \in \text{Flag}(v) \).

**Definition 6.** The tensors \( \alpha_{2n+1}, n \in \mathbb{N} \) are cyclically symmetric and define canonical elements
\[
\alpha_{v} \in \text{Hom}((\Pi A)^{\otimes \text{Flag}(v)}, k)
\]
for all vertices of odd valence. The tensors \( \alpha_{2n}, n \in \mathbb{N} \), are cyclically antisymmetric, they define canonical elements \( \alpha_{v} \in \text{Hom}((\Pi A)^{\otimes \text{Flag}(v)}, k) \) for all vertices of even valence equipped with a choice of orientation \( o(v) \).

**Proposition 2.** Given an oriented ribbon graph \( G \) with orientation \( o(G) \), the product over all vertices \( v \) of the \((-1)^{n(v)+1}\)-cyclically symmetric tensors \( \alpha_{v} \) defines the canonical element
\[
\alpha_{G,o(G)} = \otimes_{v \in \text{Vert}(G)} \alpha_{v}, \quad \alpha_{G,o(G)} \in \text{Hom}((\Pi A)^{\otimes \text{Flag}(G)}, k).
\]

**Proof.** The orientation \( o(G) \) is a choice of an element of \( \text{Flag}(v) \) for all vertices \( v \) of even valence, and a choice of an order on the total set of such vertices of even valencies. The first part gives a choice of orientation for all vertices of even valence. Notice that the parity of \( \alpha_{n} \) is even for odd \( n \), and odd for even \( n \). Therefore the choice of order on the set of vertices of even valency the product \( \otimes_{v \in \text{Vert}(G)} \alpha_{v} \) gives a well-defined element from \( \text{Hom}((\Pi A)^{\otimes \text{Flag}(G)}, k) \). In more details, the orientation \( o(G) \) fixes the signs in the definition of \( \alpha_{G,o(G)} \in \text{Hom}((\Pi A)^{\otimes \text{Flag}(G)}, k) \) by dictating for every vertex of even valence \( n(v) \) which element \( a_{i} \) is to be placed the first inside \( \alpha_{n(v)} \), and then in which order the tensors \( \alpha_{n(v)} \) are to be multiplied,
Definition 7. The partition function $Z_i^{G, \text{or}(G)}$ of an oriented ribbon graph $G$ is the contraction of $\otimes_{e \in \text{Vert}(G)} \alpha_e$ with $\otimes_{e \in \text{Edge}(G)} (g_i^{-1})_e$. The sum over all equivalence classes of connected ribbon graphs defines the cochain $Z_I$

\begin{equation}
Z_I = \sum_{[G]} Z_i^{G, \text{or}(G)}(G, \text{or}(G)), \; Z_I \in (C^*, \delta)
\end{equation}

Notice that the element $Z_i^{G, \text{or}(G)}(G, \text{or}(G))$ of $(C^*, \delta)$ does not depend on the choice of $\text{or}(G)$, since a change of orientation changes the signs of both $Z_i^{G, \text{or}(G)}$ and $(G, \text{or}(G))$.

5. Action of $I$ on the tensors $g_i^{-1}$ and $\alpha_n$.

I denote by $I^*$ the dual to $I$ odd antisymmetric operator, acting on $\text{Hom}(\Pi A, k)$:

$$I^* \varphi(a) = (-1)^\tau \varphi(I a)$$

The action of any endomorphism of the super vector space $\text{Hom}(\Pi A, k)$ is naturally extended to its tensor algebra $\bigoplus_{n} (\text{Hom}(\Pi A, k))^\otimes n$ as a derivation, by the Leibnitz rule. The dual action on $\bigoplus_{n} (\Pi A)^\otimes n$ corresponds to the similar extension of the action of $I$ on $\Pi A$.

Proposition 3. The result of action of the operator $I$ on $g_i^{-1}$ is

$$I(g_i^{-1}) = g_i^{-1}$$

Proof.

$$I(g_i^{-1}) = g_i^{-1}(I^* \varphi, \psi) + (-1)^\tau g_i^{-1}(\varphi, I^* \psi) = g_i^{-1}([I^*, I^*] \varphi, \psi) = g_i^{-1}(\varphi, \psi) \quad \square$$

Consider the partition functions of an oriented ribbon graph $G$ and the graph $G' = G/\{e'\}$ obtained from $G$ by contracting the edge $e' \in \text{Edge}_c(G)$, with its induced orientation. The previous proposition shows that acting by $I$ on the two-tensor associated with the edge $e'$ gives

$$I(g_i^{-1})_{e'} = g_i^{-1} \cdot e'$$

where $g_i^{-1} \in (\Pi A)^\otimes (f, f')$, $e' = (f f')$, denotes the two-tensor inverse to $g$, associated with the edge $e'$. I claim that inserting $g_i^{-1}$ instead of $(g_i^{-1})_{e'}$ for the edge $e'$ which is not a loop gives $Z_i^{G/\{e'\}, \text{or}(G/\{e'\})}$.

Proposition 4. The partition function $Z_i^{G/\{e'\}, \text{or}(G/\{e'\})}$, for a regular edge $e'$, is equal to the contraction of $\otimes_{e \in \text{Vert}(G)} \alpha_e$ with $\otimes_{e \in \text{Edge}(G)} h_e$ where $h_e = (g_i^{-1})_e$ for $e \neq e'$ and $h_{e'} = g_i^{-1}$.

Proof. The part of the contraction involving $g_i^{-1}$ is

$$\sum_{\mu, \nu} (-1)^\tau \alpha_{k+1}(v_1 \ldots v_k u_\mu) g_i^{-1}(w^\mu, w^\nu) \alpha_{n-k+1}(u_\nu v_{k+1} \ldots v_n)$$

the other choices are irrelevant because the sign of cyclic permutation is always positive for odd number of elements, and because $\alpha_n$ defines an even tensor for odd $n$. \square
where \( \{ u_\mu \} \) is a basis in \((\Pi A)\) and \( \{ u^\mu \} \) is the dual basis, \( u_\mu \) and \( u^\nu \) represent the two flags of the edge \( e' \), and

\[
v_1 \otimes \ldots \otimes v_k \otimes v_{k+1} \ldots \otimes v_n \in (\Pi A)^{\otimes \text{Flag}(v)}
\]

represents the flags corresponding to the new vertex of \( G/\{ e' \} \) to which the edge \( e' \) shrinks. Using the linear algebra identity

\[
b = \sum_{\mu} u_\mu u^\mu(b) = \sum_{\mu \nu} (-1)^{\nu} u_\mu g^{-1}(u^\mu, u^\nu) g(u_\nu, b).
\]

with \( b = v_{k+1} \cdot \ldots \cdot v_n \) I get

\[
\alpha_n(v_1 \ldots v_n) = \sum_{\mu, \nu} (-1)^{\nu} \alpha_{k+1}(v_1 \ldots v_k u_\mu) g^{-1}(u^\mu, u^\nu) \alpha_{n-k+1}(u_\nu v_{k+1} \ldots v_n)
\]

i.e. the tensor \( \alpha_\nu \) associated with the new vertex. \( \square \)

Next proposition shows that acting by \( I^* \) on \( \alpha_n \) gives zero because \( I \) is a derivation of \( A \), preserving \( g \).

**Proposition 5.**

\[
I^*(\alpha_n) = 0
\]

**Proof.** It follows from

\[
(5.2) \quad \sum_{i=1}^n (-1)^{i-1} l(a_1 \cdot \ldots \cdot (I a_i) \cdot \ldots \cdot a_n) = l(I(a_1 \cdot \ldots \cdot a_n)) = 0
\]

where \( \varepsilon_i = \sum_{j=1}^{i-1} \eta_j \), \( n \geq 2 \). \( \square \)

6. **The boundary of the cochain \( Z_I \).**

Combining the three propositions we get the following theorem.

**Theorem 1.** The boundary of the cochain \( Z_I \) (4.1) is given by the sum over graph with loops, each loop contributing the term similar to the one described in proposition 4 for regular edges:

\[
\delta Z_I = \sum_{[G], \text{Loop}(G) \neq \emptyset} (G, \text{or}(G)) Z_I^{\text{loop}, (G, \text{or}(G))}
\]

where

\[
Z_I^{\text{loop}, (G, \text{or}(G))} = - \left( \bigotimes_{e \in \text{Edge}_r(G)} (g_{I e}^{-1})_{e} \right) I \left( \bigotimes_{l \in \text{Loop}(G)} (g_{I l}^{-1})_{l} \right) \left( \bigotimes_{v \in \text{Vert}(G)} \alpha_v \right).
\]

**Proof.** The boundary of \( Z_I \)

\[
\delta Z_I = \sum_{[G']} Z_I^{(G', \text{or}(G'))} \delta(G', \text{or}(G')) = \sum_{[G]} (G, \text{or}(G)) \sum_{e \in \text{Edge}_r(G)} Z_I^{(G/\{ e \}, \text{or}(G/\{ e \}))}
\]

For any ribbon graph \( G \) we have
0 \overset{\text{Prop. 5}}{=} \left( \bigotimes_{e \in \text{Edge}(G)} (g^{-1}_I)_e \right) I^* \left( \bigotimes_{v \in \text{Vert}(G)} \alpha_v \right) = \\
= I \left( \bigotimes_{e \in \text{Edge}(G)} (g^{-1}_I)_e \right) \left( \bigotimes_{v \in \text{Vert}(G)} \alpha_v \right) \overset{\text{Prop. 3,4}}{=} \left( \sum_{e \in \text{Edge}(G)} Z^I_{G / \{e\}, \text{or}(G / \{e\})} \right) + Z^\text{loop}_{G, \text{or}(G)} \quad \square

7. **Compactification and the stable ribbon graph complex (even scalar product case).**

To remedy the problem with contribution of loops, since they are not contracted in the ribbon graph complex, one has to consider the complex in which all the edges, including the loops, are allowed to contract. This leads to the complex of stable ribbon graphs and the combinatorial compactification of the moduli space of Riemann surfaces, which was introduced by Kontsevich in [K1], two years before [K2], from completely different point of view in connection with the proof of Witten conjecture. As far as I know it was not until the paper ([B1]) that this compactification reappeared in a new context more than ten years later, in my work devoted to study of noncommutative Batalin-Vilkovisky equation. In loc.cit I have related the complex of stable ribbon graphs with some modular variant of cyclic associative algebras. This leads, in particular the dual to the map from loc.cit., proposition 11.3, and the same propagator \((g^{-1}_I)\), to the construction of weights on the stable ribbon graphs, completing the cochain \(Z^I\) to a coboundary free cochain.

**Definition 8.** A stable ribbon graph \(\hat{G}\) is a data \((\text{Flag}(\hat{G}), \sigma, \lambda, \gamma_v)\), where \(\text{Flag}(\hat{G})\) is a finite set, whose elements are called flags, \(\lambda\) is a partition on the set \(\text{Flag}(\hat{G})\), \(\sigma\) is a permutation from \(\text{Aut}(\text{Flag}(\hat{G}))\), stabilizing \(\lambda\), \(\eta\) is a fixed-point free involution acting on \(\text{Flag}(\hat{G})\), and \(\gamma_v \in \mathbb{Z}_{\geq 0}\) is a nonnegative integer attached to every cluster \(v\) of \(\lambda\). The clusters of the partition \(\lambda\) are the vertices of stable ribbon graph \(\hat{G}\). The edges are the orbits of the involution \(\eta\). This data satisfy the stability condition ...

In particular to any vertex \(v\) corresponds the integer \(\gamma_v\) and a permutation \(\sigma_v\) on the subset of flags \(\text{Flag}(v)\) from the cluster \(v\), so that \(\sigma = \Pi_v \sigma_v\). I denote by \(\text{Cycle}(v)\) the set of cycles of the permutation \(\sigma_v\), attached to a vertex \(v \in \text{Vert}(\hat{G})\), and by \(\text{Cycle}(\hat{G})\) the union of such sets which is the set of cycles of \(\sigma\). For \(c \in \text{Cycle}(\hat{G})\) I denote by \(\text{Flag}(c)\) the corresponding subset of \(\text{Flag}(\hat{G})\). Denote \(b_v = |\text{Cycle}(v)|\)

7.1. **Contraction of edges on the stable ribbon graphs.** In order to describe the action of the differential on the stable ribbon graph \(\hat{G}\) one needs to describe the result of the contraction of an arbitrary edge \(\hat{G} / \{e\}\). The contraction of edges in the stable ribbon graphs is described combinatorially via compositions and contractions on permutations, see ([B1]) and references therein, representing the corresponding geometric operations on \(S_{\hat{G}}\). For the reader convenience I rephrase it here. For an edge ending at vertices with \(\gamma_v = 0\), \(b_v = 1\) this reproduces the standard contraction on ribbon graphs.
I identify the permutation $\sigma^\hat{G}$ with multicyclic order on $Flag(\hat{G})$, i.e. the collection of cyclic orders on the orbits of $\sigma$. Then the result of the contraction of an edge $e = (f, f')$, $f, f' \in Flag(\hat{G})$ is the stable ribbon graph $\hat{G}/\{e\}$ such that

$$Flag(\hat{G}/\{e\}) = Flag(\hat{G})\setminus\{f, f'\},$$

$$\sigma^\hat{G}/\{e\} = \left((f, f') \circ \sigma^\hat{G}\right) |_{Flag(\hat{G})\setminus\{f, f'\}},$$

as a multicyclic order, i.e. $\sigma^\hat{G}/\{e\}$ is the multi-cyclic order induced on $Flag(\hat{G})\setminus\{f, f'\}$ by the multi-cyclic order on $Flag(\hat{G})$ represented by the product $(f, f') \circ \sigma^\hat{G}$, naturally

$$\eta^\hat{G}/\{e\} = \eta^\hat{G}|_{Flag(\hat{G})\setminus\{f, f'\}}$$

as a map, i.e. the involution $\eta^\hat{G}/\{e\}$ is the restriction of the involution $\eta^\hat{G}$ on the subset $Flag(\hat{G})\setminus\{f, f'\}$. If an edge $e = (f, f')$, $f, f' \in Flag(\hat{G})$ is not a loop, i.e. $f, f'$ are from two distinguished clusters $v$ and $v'$ of $\lambda$, then the clusters $v$ and $v'$ collide to the new cluster having $\gamma = \gamma_v + \gamma_{v'}$. If an edge $e$ is a loop then no vertices must collide and the partition $\lambda^\hat{G}/\{e\}$ on $Flag(\hat{G})\setminus\{f, f'\}$ is induced from $\lambda^\hat{G}$, if the flags $f, f'$ are from the same cycle of $\sigma^\hat{G}_v$ then $\gamma^\hat{G}_v(e) = \gamma^\hat{G}_v$ and if they are from different cycles of $\sigma^\hat{G}_v$ then $\gamma^\hat{G}_v(e) = \gamma^\hat{G}_v$. Lastly if $f, f'$ are neighbors in a cycle of $\sigma^\hat{G}_v$, i.e. say $\sigma^\hat{G}_v(f) = f'$, then, by definition, $\hat{G}/\{e\} = \emptyset$, so that such loop do not contribute to the boundary operator. This exception is dictated by the relation of the stable ribbon graphs with the combinatorial compactification of the moduli spaces from ([K1]), and leads to the interesting condition on the algebra $A$.

8. The differential on the stable ribbon graph complex.

**Definition 9.** The (even) stable ribbon graph complex is the vector space generated by equivalence classes of pairs $(\hat{G}, or(\hat{G}))$, where $\hat{G}$ is a stable ribbon graph, $or(\hat{G})$ is an orientation on the vector space

$$\otimes_{v \in Vert(\hat{G})}(k^{Flag(v)} \oplus k^{Cycle(v)})$$

and the relation $(\hat{G}, -or(\hat{G})) = -(\hat{G}, or(\hat{G}))$ is imposed. The differential is

$$d(\hat{G}, or(\hat{G})) = \sum_{e \in Edge(\hat{G})}(\hat{G}/\{e\}, induced~orientation)$$

Using the multi-cyclic order on $Flag(\hat{G})$ as in section *** above, a choice of orientation is fixed by a choice of a flag from every cycle of $\sigma$ of even length and a choice of order on the total set of such cycles. The induced orientation is also analogous to the case of usual ribbon graphs, see ([B1]) for details.

9. Weights on stable ribbon graphs.

I start by constructing the tensor $\alpha_{\sigma_v, \gamma_v} \in Hom((\bigotimes Flag_v)^\otimes Flag(v), k)$ for any given vertex $v \in Vert(\hat{G})$ with permutation $\sigma_v \in Aut(Flag(v))$ and integer $\gamma_v$ attached, together with a choice of orientation $or(v)$ from $k^{Flag(v)} \oplus k^{Cycle(v)}$. Let $\sigma_v = (\rho_1 \ldots \rho_r) \ldots (\tau_1 \ldots \tau_t)$ be a representation of $\sigma_v$ compatible with $or(v)$, in the
sense that the order \( \rho_1 < \ldots < \tau_t \) on flags together with the order \( (\rho_1 \ldots \rho_r) < \ldots (\tau_1 \ldots \tau_t) \) on cycles gives \( or(v) \), put

\[
(9.1) \quad \alpha_{\sigma, \gamma_v}(a_{\rho_1} \otimes \ldots \otimes a_{\tau_t}) = \\
l \left( (-1)^{l} \left( \sum_{\rho_1 = \ldots = \rho_{b_{v-1}}} e_{\rho_1} \ldots e_{\rho_{b_{v-1}}} a_{\tau_1} \ldots a_{\tau_t} \right) \prod_{i=1}^{\gamma_v} \left( \sum_{\xi_i, \zeta_i} e_{\xi_i} e_{\zeta_i} e_{\xi_i} e_{\zeta_i} \right) \right)
\]

where \( \{e_{\mu}\}, \{e_{\nu}\} \) is a pair of dual bases in \( A \), \( g(e_{\mu}, e_{\nu}) = \delta_{\mu}^{\nu} \), \( \epsilon \) is the Koszul sign (taking into account the passing of all \( \pi \)'s to the left and then putting \( a_i \) inside \( (-1)^{\delta_{\mu}^{\nu}} e_{\mu} a_{\mu} \ldots e_{\mu_{b_{v-1}}} a_{\mu_{b_{v-1}}} \)).

**Proposition 6.** For any \( a, b \in A \)

\[
(9.2) \quad \sum_{\mu} (-1)^{\epsilon(\pi+\overline{\pi}+1)} e_{\mu} a_{\mu} b = (-1)^{\pi} \sum_{\nu} (-1)^{\epsilon(\pi+\overline{\pi}+1)} e_{\nu} b a_{\nu}.
\]

For any \( a, b \in A \),

\[
(9.3) \quad \sum_{\mu} (-1)^{\epsilon(\pi+1)} e_{\mu} a_{\mu} b = (-1)^{\pi} \sum_{\nu} (-1)^{\epsilon(\pi+1)} e_{\nu} b a_{\nu}.
\]

**Proof.** If

\[
b_{\mu} = \sum_{\nu} b_{\mu} e_{\nu}
\]

then

\[
\beta_{\mu}^{\nu} = g(e_{\nu}, b_{\mu}) = g(e_{\nu} b, e_{\mu})
\]

so

\[
e_{\nu} b = \sum_{\mu} \beta_{\mu}^{\nu} e_{\mu}.
\]

Therefore,

\[
\sum_{\mu} (-1)^{\epsilon(\pi+1)} e_{\mu} a_{\mu} b = \sum_{\mu} (-1)^{\epsilon(\pi+1)} e_{\mu} a \sum_{\nu} \beta_{\mu}^{\nu} e_{\nu} = \\
= (-1)^{\pi+1} \sum_{\mu, \nu} (-1)^{\epsilon(\pi+1)} \beta_{\mu}^{\nu} e_{\mu} a_{\nu} = (-1)^{\pi} \sum_{\nu} (-1)^{\epsilon(\pi+1)} e_{\nu} b a_{\nu}
\]

The proof of (9.3) is analogous. \( \square \)

**Remark 1.** It follows that \( \sum_{\mu} (-1)^{\epsilon(\pi)} e_{\mu} a_{\mu} b \) and \( \sum_{\xi, \zeta} (-1)^{\epsilon(\xi+\zeta)} e_{\xi} e_{\zeta} e_{\xi} e_{\zeta} \) are in the center of \( A \).

It follows from (9.2) and (9.3) that for every cycle \( (\rho_1 \ldots \rho_r) \) of \( \sigma_v \), the expression (9.1) is \( (-1)^{r+1} \)-cyclically symmetric with respect to cyclic permutations of \( \pi a_{\rho_1} \ldots \pi a_{\rho_r} \). And that the expression (9.1) is invariant under the changing the order of cycles in the representation of \( \sigma \), up to the sign taking into account that the total parity of \( e_{\mu_1} a_{\rho_1} \ldots e_{\mu_1} a_{\rho_1} \) differs from the parity of \( \pi a_{\rho_1} \ldots \pi a_{\rho_r} \) by the factor \( (-1)^{r+1} \). Therefore, for a fixed choice of \( or(v) \), the expression (9.1) gives a well-defined element \( \alpha_{\pi, \gamma_v} \in Hom((\mathbb{IA})^{\otimes Flag(v)}, k) \). Notice that for a vertex with \( \gamma_v = 0 \), \( b_v = 1 \) this gives the tensor \( \alpha_v \), defined above.

A choice of orientation \( or(\hat{G}) \) on stable ribbon graph can be identified with a choice of orientation \( or(v) \) on every vertex plus a choice of order on the set of vertices with odd number of cycles of even length. These are precisely the vertices
for which the tensor $\alpha_{\sigma_v,\gamma_v}$ is odd. Therefore for a choice of $\text{or}(\hat{G})$ the product
\[ \otimes_{v \in \text{Vert}(\hat{G})} \alpha_{\sigma_v,\gamma_v} \] giving a well-defined element from $\text{Hom}((\text{IIA})^{\otimes \text{Flag}(\hat{G})}, k)$.

**Definition 10.** A choice of orientation $\text{or}(\hat{G})$ on a stable ribbon graph $\hat{G}$ allows to define the linear functional $\alpha_{\hat{G},\text{or}(\hat{G})}$ on $(\text{IIA})^{\otimes \text{Flag}(\hat{G})}$:
\[ \alpha_{\hat{G},\text{or}(\hat{G})} = \otimes_{v \in \text{Vert}(\hat{G})} \alpha_{\sigma_v,\gamma_v} \]

For the propagator I use the same bilinear form associated with $h$-inverse of the odd supersymmetry $I$.

**Definition 11.** Given a ribbon graph $G$, the tensor product of the even symmetric tensors $(g_I^{-1})_e$, associated with every edge $e$ of $G$, $(g_I^{-1})_e \in (\text{IIA})^{\otimes \text{Flag}(e)}$, defines the canonical element
\[ g_I^{-1} = \otimes_{e \in \text{Edge}(G)} (g_I^{-1})_e, \quad g_I^{-1} \in (\text{IIA})^{\otimes \text{Flag}(\hat{G})}. \]

**Definition 12.** The partition function $\hat{Z}_I^{G,\text{or}(\hat{G})}$ of an oriented stable ribbon graph $\hat{G}$ is the contraction of $\otimes_{v \in \text{Vert}(\hat{G})} \alpha_{\sigma_v,\gamma_v}$ with $\otimes_{e \in \text{Edge}(\hat{G})} (g_I^{-1})_e$. As above, the element $\hat{Z}_I^{G,\text{or}(\hat{G})}(\hat{G},\text{or}(\hat{G}))$ of $(\hat{G}^*,\delta)$ does not depend on the choice of $\text{or}(\hat{G})$. The sum over all equivalence classes of connected stable ribbon graphs defines the cochain $\hat{Z}_I$
\[ \hat{Z}_I = \sum_{[G]} \hat{Z}_I^{G,\text{or}(\hat{G})}(\hat{G},\text{or}(\hat{G})), \quad \hat{Z}_I \in (\hat{G}^*,\delta) \]

10. THE CANCELLATION OF ANOMALY.

In this section I discuss the condition imposed on $A$ in order that the weights introduced below behave correctly with respect to the contraction of a loop encircling a boundary component.

**Condition 2.** For any $a \in A$ the super trace of the operator of multiplication by $a$ is zero:
\[ \sum_{\mu} (-1)^{\overline{\nu}} g(u^\mu, a \cdot u_\mu) = 0 \]

Below I describe different sufficient conditions for (10.1) to hold. All of them trivially hold in our basic examples.

Assume that $I(\sum_i [p_i, q_i]) = 0$, $p_i, q_i$- symplectic basis in $L_0$ for $g(I, \cdot)$ in particular, for example if $\text{ker}(I)_1 = 0$.

**Proposition 7.** Assume that $I(\sum_i [p_i, q_i]) = 0$, where $p_i, q_i$- symplectic basis in $L_0$ for $g(I, \cdot)$ then (10.1) holds

**Proposition 8.** Assume that $A_0$ is semisimple, then (10.1) holds

**Proof.** The statement is trivial for $a \in A_1$. Let $\{x_\mu | x_\mu \in A_0\}$ be a linear basis in $A_0$ and $\{\xi_\mu | \xi_\mu \in A_1\}$ be the dual basis in $A_1$, so that
\[ g(x_\mu, \xi_\nu) = \delta_{\mu\nu} \]

Let us denote by $K_a$ the operator of left multiplication by $a$ acting on $A$ and by $R_a$ the operator of right multiplication. From
\[ g(x_\mu, a \xi_\mu) = g(x_\mu a, \xi_\mu) \]
it follows that
\[ \text{Tr} K_a|_{A_1} = \text{Tr} R_a|_{A_0} \]
Let us show that \( \text{Tr} R_a|_{A_0} = \text{Tr} K_a|_{A_0} \). If \( A_0 \) is semisimple then \( t(a, b) = Tr K_{ab}|_{A_0} \) is a nondegenerate invariant even inner product on \( A_0 \) and
\[ Tr K_a|_{A_0} = \sum_\mu t(\bar{c}_\mu a, a \bar{c}_\mu) = \text{Tr} R_a|_{A_0} \]
where \( \bar{c}_\mu \) is the orthonormal basis for \( t \) in \( A_0 \otimes_k \bar{k} \).

Proposition 9. Assume that the factor algebra by the nil radical \( A_0/\text{nil}(A_0) \) is simple, then (10.1) holds.

Proposition 10. Contracting any loop whose flags are neighbors in the sense that \( \sigma(f) = f' \), so that the loop encircles some boundary component of the surface \( S(\hat{G}) \), gives zero.
\[ \sum_\mu \nu (-1)^{\nu} \alpha_{k+2}(v_1 \ldots v_k u_\mu u_\nu) g^{-1}(u^\mu, u^\nu) = 0 \]

Proof. Taking \( a = v_1 \cdot \ldots \cdot v_k \) the statement is reduced to (10.1).

11. Action of \( I \) on the tensors \( \alpha_{\sigma v, \gamma_v} \) and \( g_{\hat{I}, \hat{G}}^{-1} \).

11.1. Action of \( I \) on the tensors \( \alpha_{\sigma v, \gamma_v} \).

Proposition 11.

(11.1) \[ I^*(\alpha_{\sigma v, \gamma_v}) = 0 \]

Proof. It follows from the fact that \( I \) is a derivation of the multiplication and of the bilinear form. It follows from
\[ g(I e^\nu, e_\mu) + (-1)^{\nu} g(e^\nu, I e_\mu) = 0 \]
that if \[ I e_\mu = \sum_\nu I^\nu_\mu e_\nu \]
then
\[ I e^\nu = (-1)^{\nu+1} \sum_\mu I^\nu_\mu e^\mu. \]
Therefore
\[ \sum_\mu (-1)^{\nu+1} I(e^\mu) a e_\mu + \sum_\nu (-1)^{\nu+1} e^\nu a I(e_\nu) = 0 \]
for any \( a \in A \). Now (11.1) follows from (5.2).

The immediate consequence is the invariance of \( \alpha_{\hat{G}, \text{or}(\hat{G})} \) under the action of derivation \( I \).

Proposition 12. \[ I^* \alpha_{\hat{G}, \text{or}(\hat{G})} = 0 \]
11.2. **Action of $I$ on $g_{I,G}^{-1}$ and contractions of edges.** As it was shown in the proposition 3, acting by $I$ on the two-tensor associated with an edge $e'$ gives

$$I(g_{I}^{-1})e' = g_{e'}^{-1},$$

where $g_{e'}^{-1} \in (IIA)^{\otimes \{f,f'\}}$, $e' = (ff')$, denotes the two-tensor inverse to $g$. The next proposition shows that, inserting $g_{e'}^{-1}$ instead of $(g_{I}^{-1})_{e'}$ for arbitrary edge $e'$ and contracting with $\alpha_{G,\text{or}(\hat{G})}$ gives $\tilde{Z}_{I}(G/(e'),\text{or}(G/(e'))) \equiv \text{partition function of the oriented stable ribbon graph } \hat{G}/\{e'\}.$

**Proposition 13.** For arbitrary edge $e'$ the partition function $\tilde{Z}_{I}(G/(e'),\text{or}(G/(e'))) \equiv \text{partition function of the oriented stable ribbon graph } \hat{G}/\{e'\}.$

**Proof.** Consider first the case when $e'$ is not a loop. Then the term of the contraction of $\otimes_{e \in \text{Vert}(\hat{G})} \alpha_{\nu}$ with $\otimes_{e \in \text{Edge}(\hat{G})} h_{e}$, involving $g_{e'}^{-1}$ is

$$\sum_{\mu,\nu} (-1)^{\gamma_{\mu,\gamma_{\nu}}} g^{-1}(U_{\mu},U_{\nu}) \alpha_{\mu,\nu}(U_{\nu},a_{k+1},...,a_{n})$$

where $\{U_{\mu}\}$ is a basis in (IIA) and $\{U_{\nu}\}$ is the dual basis in $\text{Hom}(\text{IIA},k)$, $U_{\mu}$ and $U_{\nu}$ represent the two flags of the edge $e'$. Using the identities (9.2) and (9.3) the tensor $\alpha_{\mu,\nu}$ can be represented in the form:

$$l \left( (-1)^{f} \sum_{\mu_{1},...,\mu_{n-1}} U_{\mu_{1}}a_{\tau_{1}}...a_{\tau_{f}}e_{\mu_{1}}a_{\rho_{1}}...a_{\rho_{f}},e_{\mu_{f}}e_{\mu_{f+1}}...e_{\mu_{n-1}} \prod_{i=1}^{n} e_{\xi_{i}}e_{\xi_{i}}e_{\xi_{i}} \right)$$

Using as above the linear algebra identity

$$b = \sum_{\mu} U_{\mu} g^{-1}(U_{\mu},U_{\nu}) g(U_{\nu},b).$$

with

$$b = (-1)^{f} \sum_{\mu_{1},...,\mu_{n-1}} a_{\tau_{1}}...a_{\tau_{f}}e_{\mu_{1}}a_{\rho_{1}}...a_{\rho_{f}},e_{\mu_{f}}e_{\mu_{f+1}}...e_{\mu_{n-1}} \prod_{i=1}^{n} e_{\xi_{i}}e_{\xi_{i}}e_{\xi_{i}}$$

I get the tensor $\alpha_{\mu,\nu}$ with $b$ substituted instead of $U_{\mu}$. Using again (9.2) and (9.3) to bring it to the standard form (9.1), I get precisely the tensor associated with the new vertex in $\hat{G}/\{e'\}$, i.e. with the two permutations $\sigma_{e'}$ and $\sigma_{e'}$ merged at the flags of $e'$, with $\gamma_{\mu_{\mu_{\nu}} = \gamma_{e} + \gamma_{e'}}$ etc. Let’s now $e' = (ff')$ is a loop, $f, f' \in \text{flag}(v)$. Then if $f$ and $f'$ are in the same cycle of $\sigma_{e}$, assuming that flags $f, f'$ are not neighbors, $\sigma(f) \neq f', \sigma(f') \neq f$:

$$\sum_{\mu,\nu,\kappa} (-1)^{f} g^{-1}(U_{\mu},U_{\nu}) e_{\kappa} a_{\rho_{1}}...a_{\rho_{f}}U_{\mu}a_{\rho_{f+1}}...a_{\rho_{n-1}}U_{\nu}a_{\rho_{n}}...a_{\rho_{f}},e_{\kappa} =$$

$$= \sum_{\mu,\kappa} e_{\kappa} a_{\rho_{1}}...a_{\rho_{f}} e_{\mu} a_{\rho_{f+1}}...a_{\rho_{n-1}} e_{\mu} a_{\rho_{n}}...a_{\rho_{f}}, e_{\kappa}$$

Using (9.3) it is transformed to

$$\sum_{\mu,\kappa} e_{\kappa} a_{\rho_{1}}...a_{\rho_{n-1}} e_{\mu} a_{\rho_{n}}...a_{\rho_{f}}, e_{\kappa}$$
And the tensor $\alpha_{\sigma_v, \gamma_v}$ is transformed to $\alpha_{\sigma_{v}^{new}, \gamma_v}$ corresponding to the permutation $\sigma_{v}^{new}$ obtained from $\sigma_v$ by dissecting one cycle into two at flags $f$ and $f'$. Similarly if $f$ and $f'$ are in the two different cycles then the result is the tensor $\alpha_{\sigma^{new} v, \gamma_{v}^{new}}$ corresponding to the permutation $\sigma^{new} v$ obtained from $\sigma v$ by merging the two cycles at flags $f$ and $f'$. Similarly if $f$ and $f'$ are in the two different cycles then the result is the tensor $\alpha_{\sigma^{new} v, \gamma_{new} v}$ corresponding to the permutation $\sigma^{new} v$ obtained from $\sigma v$ by merging the two cycles at flags $f$ and $f'$ and with $\gamma_{new} = \gamma_v + 1$. In both cases I get precisely the tensor corresponding to the transformed vertex $v$ in $\hat{G}/\{e'\}$. Finally if $f$, $f'$ are neighbors, say $\sigma(f) \neq f'$, then the insertion of $g^{-1}_e$ instead of $(g^{-1}_e)_{e'}$ gives zero because of the property (10.1) imposed on $A$. □

12. The boundary of the extended cochain $\tilde{Z}_I$.

Combining the results from the previous sections I get the following theorem.

Theorem 2. The boundary of the cochain $\tilde{Z}_I$ (9.4) is zero

$$\delta \tilde{Z}_I = 0$$

Proof. The boundary of $\tilde{Z}_I$

$$\delta \tilde{Z}_I = \sum_{[\hat{G}']} \tilde{Z}_I^{(G', \text{or}(G'))} \delta(\hat{G'}, \text{or}(G')) = \sum_{[\hat{G}]} \tilde{Z}_I^{(\hat{G}, \text{or}(\hat{G}))} \sum_{e \in \text{Edge}(\hat{G})} \tilde{Z}_I^{(\hat{G}/\{e\}, \text{or}(\hat{G}/\{e\}))}$$

For any stable oriented ribbon graph $\hat{G}$ I have

$$\left( \sum_{e \in \text{Edge}(\hat{G})} \tilde{Z}_I^{(\hat{G}/\{e\}, \text{or}(\hat{G}/\{e\}))} \right) \quad \text{(Prop. 13)}$$

$$= I \left( \bigotimes_{e \in \text{Edge}(\hat{G})} (g^{-1}_I)_e \right) \left( \bigotimes_{v \in \text{Vert}(\hat{G})} \alpha_{\sigma_v, \gamma_v} \right) =$$

$$= \left( \bigotimes_{e \in \text{Edge}(\hat{G})} (g^{-1}_I)_e \right) I^* \left( \bigotimes_{v \in \text{Vert}(\hat{G})} \alpha_{\sigma_v, \gamma_v} \right) \quad \text{(Prop. 12)}$$

$$= 0$$

□

13.

14. Gauge transformations and coboundaries.

Same arguments as in the previous section show that if one modifies $\tilde{I}$ via

$$\tilde{I} \to \tilde{I} + [I, B]$$

then the cohomology cochain is modified by a coboundary .

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