Every finitely generated two-sided ideal of a Leavitt path algebra is a principal ideal

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Abstract
Let $E$ be an arbitrary graph and $K$ be any field. For every non-graded ideal $I$ of the Leavitt path algebra $L_K(E)$, we give an explicit description of the generators of $I$. Using this, we show that every finitely generated ideal of $L_K(E)$ must be principal. In particular, if $E$ is a finite graph, then every ideal of $L_K(E)$ must be principal ideal.

1 Introduction
The notion of Leavitt path algebras of a graph $E$ was introduced and initially studied in [1], [3] as algebraic analogues of $C^*$-algebras and the analysis of the structure of their two-sided ideals has received much attention in recent years. For instance, Tomforde [8] described all the graded ideals in a Leavitt path algebra in terms of their generators. In [6] and [2] generating sets for arbitrary ideals of a Leavitt path algebra were established while in [5] and [7] the prime ideal structure of a Leavitt path algebra was described. In this note, complementing Tomforde’s theorem on graded ideals, we first give an explicit description of a set of generators for non-graded ideals in the Leavitt path algebra $L_K(E)$ of an arbitrary graph $E$ over a field $K$. Using this we prove that every finitely generated ideal in $L_K(E)$ must be a principal ideal. As a corollary, we show that if $E$ is a finite graph, then every ideal of $L_K(E)$ must be a principal ideal. The method involves a judicious selection of finitely many mutually orthogonal generators to replace a given finite set of generators of the ideal $I$. The sum of these orthogonal generators will then be the desired single generator for $I$. 

1
2 Preliminaries

All the graphs $E$ that we consider here are arbitrary in the sense that no restriction is placed either on the number of vertices in $E$ (such as being a countable graph) or on the number of edges emitted by any vertex (such as being a row-finite graph). We shall follow \[2, 7\] for the general notation, terminology and results. For the sake of completeness, we shall outline some of the concepts and results that we will be using.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets $E^0$ and $E^1$ together with maps $r, s : E^1 \rightarrow E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ edges. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called row-finite.

If a vertex $v$ emits no edges, that is, if $s^{-1}(v)$ is empty, then $v$ is called a sink. A vertex $v$ is called an infinite emitter if $s^{-1}(v)$ is an infinite set, and $v$ is called a regular vertex if $s^{-1}(v)$ is a finite non-empty set. A path $\mu$ in a graph $E$ is a finite sequence of edges $\mu = e_1 \ldots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n - 1$. In this case, $n$ is the length of $\mu$; we view the elements of $E^0$ as paths of length 0. We denote by $\mu^0$ the set of vertices of the path $\mu$, i.e., the set $\{s(e_1), r(e_1), \ldots, r(e_n)\}$.

A path $\mu = e_1 \ldots e_n$ is closed if $r(e_n) = s(e_1)$, in which case $\mu$ is said to be based at the vertex $s(e_1)$. A closed path $\mu$ as above is called simple provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \ldots, n$. The closed path $\mu$ is called a cycle if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$. An exit for a path $\mu = e_1 \ldots e_n$ is an edge $e$ such that $s(e) = s(e_i)$ for some $i$ and $e \neq e_i$. We say that $E$ satisfies Condition (L) if every simple closed path in $E$ has an exit, or, equivalently, every cycle in $E$ has an exit. A graph $E$ is said to satisfy Condition (K) provided no vertex $v \in E^0$ is the base of precisely one simple closed path, i.e., either no simple closed path is based at $v$, or at least two are based at $v$.

We define a relation $\geq$ on $E^0$ by setting $v \geq w$ if there exists a path in $E$ from $v$ to $w$. A subset $H$ of $E^0$ is called hereditary if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is saturated if, for any regular vertex $v$, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

For each $e \in E^1$, we call $e^*$ a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$.

Given an arbitrary graph $E$ and a field $K$, the Leavitt path $K$-algebra $L_K(E)$ is defined to be the $K$-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

1. $s(e)e = e = er(e)$ for all $e \in E^1$.
2. $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
3. (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$. 

2
(4) (The "CK-2 relations") For every regular vertex \( v \in E^0 \),

\[
v = \sum_{e \in E^1, s(e) = v} ee^*.
\]

If \( \mu = e_1 \ldots e_n \) is a path in \( E \), we denote by \( \mu^* \) the element \( e_1^* \ldots e_n^* \) of \( L_K(E) \).

A useful observation is that every element \( a \) of \( L_K(E) \) can be written as \( a = \sum_{i=1}^{n} k_i \alpha_i \beta_i^* \), where \( k_i \in K \), \( \alpha_i, \beta_i \) are paths in \( E \) and \( n \) is a suitable integer (see \([8]\)).

The following concepts and results from \([8]\) will be used in the sequel. A vertex \( w \) is called a breaking vertex of a hereditary saturated subset \( H \) if \( w \in E^0 \setminus H \) is an infinite emitter with the property that \( 1 \leq |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty \). The set of all breaking vertices of \( H \) is denoted by \( B_H \). For any \( v \in B_H \), \( v^H \) denotes the element \( v - \sum_{s(e) = v, r(e) \notin H} ee^* \). Given a hereditary saturated subset \( H \) and a subset \( S \subseteq B_H \), \((H, S)\) is called an admissible pair and \( I_{(H, S)} \) denotes the ideal generated by \( H \cup \{v^H : v \in S\} \). It was shown in \([8]\) that the graded ideals of \( L_K(E) \) are precisely the ideals of the form \( I_{(H, S)} \) for some admissible pair \((H, S)\). Moreover, it was shown that \( I_{(H, S)} \cap E^0 = H \) and \( \{v \in B_H : v^H \in I_{(H, S)}\} = S \).

Given an admissible pair \((H, S)\), the corresponding quotient graph \( E \setminus (H, S) \) is defined as follows:

\[
(E \setminus (H, S))^0 = (E^0 \setminus H) \cup \{v' : v \in B_H \setminus S\};
\]

\[
(E \setminus (H, S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \setminus S\}.
\]

Further, \( r \) and \( s \) are extended to \( (E \setminus (H, S))^0 \) by setting \( s(e') = s(e) \) and \( r(e') = r(e') \). Note that, in the graph \( E \setminus (H, S) \), the vertices \( e' \) are all sinks.

Theorem 5.7 of \([8]\) states that there is an epimorphism \( \phi : L_K(E) \to L_K(E \setminus (H, S)) \) with \( \ker \phi = I_{(H, S)} \) and that \( \phi(v^H) = v' \) for \( v \in B_H \setminus S \). Thus \( L_K(E) / I_{(H, S)} \cong L_K(E \setminus (H, S)) \). This theorem has been established in \([8]\) under the hypothesis that \( E \) is a graph with at most countably many vertices and edges; however, an examination of the proof reveals that the countability condition on \( E \) is not utilized. So the Theorem 5.7 of \([8]\) holds for arbitrary graphs \( E \).

### 3 Generators of non-graded ideals of \( L_K(E) \)

As noted earlier, Tomforde \([8]\) described a generating set for the graded ideals of a Leavitt path algebra \( L_K(E) \). In this section, as a complement to Tomforde’s theorem, we give an explicit description of a set of generators for the non-graded ideals in \( L_K(E) \). These generators are then used in proving the main theorem of the next section.

We begin with the following useful result from \([2]\).
Theorem 1. Let $E$ be an arbitrary graph and $K$ be any field. Then any non-zero ideal of the $L_K(E)$ is generated by elements of the form

$$(u + \sum_{i=1}^{k} k_i g^r_i)(u - \sum_{e \in X} ee^*)$$

where $u \in E^0$, $k_i \in K$, $r_i$ are positive integers, $X$ is a finite (possibly empty) proper subset of $s^{-1}(u)$ and, whenever $k_i \neq 0$ for some $i$, then $g$ is a unique cycle based at $u$.

The next Lemma is an extension of Lemma 3.3 in [7] showing that ideals of $L_K(E)$ containing no vertices are generated by a set of mutually orthogonal polynomials over cycles.

Lemma 2. Suppose $E$ is an arbitrary graph and $K$ is any field. If $N$ is a non-zero ideal of $L_K(E)$ which does not contain any vertices of $E$, then $N$ is a non-graded ideal and possesses a generating set of mutually orthogonal generators of the form $y_j = (v_j + \sum_{i=1}^{n_j} k_{ji} g_j^r)$ where (i) $g_j$ is a (unique) cycle without exits based at the vertex $v_j$, (ii) $k_{ji} \in K$ with at least one $k_{ji} \neq 0$ and $v_r \neq v_s$ (so $y_r y_s = 0$) if $r \neq s$.

Proof. Since $N$ is non-zero and since $H = N \cap E^0$ is the empty set, $N$ must be a non-graded ideal, because if $N$ was a graded ideal, then $N$ must be $\{0\}$ since, by Tomforde [S], $N$ is generated by $H \cup \{v^H \in B_H \cap N\}$ and $H$, $B_H$ are both empty sets. From Theorem [H] we know that $N$ is generated by elements of the form $y = (u + \sum_{i=1}^{n_j} k_{ji} g_j^r)(u - \sum_{e \in X} ee^*) \neq 0$ where $g$ is a unique cycle in $E$ based at the vertex $u$ and where $X$ is a finite proper subset of $s^{-1}(u)$.

We wish to show that, for each such generator $y = (u + \sum_{i=1}^{n_j} k_{ji} g_j^r)(u - \sum_{e \in X} ee^*)$, the corresponding cycle $g$ has no exits in $E$ and that $X$ must be an empty set, so that $y = (u + \sum_{i=1}^{n_j} k_{ji} g_j^r)$. By hypothesis, there is an $f \in s^{-1}(u) \backslash X$. Let $r(f) = w$. This must be the initial edge of $g$. Because otherwise $f^* g = 0$ and $(\sum_{e \in X} ee^*) f = 0$, and we obtain $f^* y f = f^* (u + \sum_{i=1}^{n_j} k_{ji} g_j^r) f = f^* u f = r(f) = w \in N$, a contradiction since $N$ contains no vertices. So we can write $g = f \alpha$ and let $h$ denote the cycle $\alpha f$ (based at $w$). Note that, in this case, $f^* y f = f^* (u + \sum_{i=1}^{n_j} k_{ji} g_j^r)(u - \sum_{e \in X} ee^*) f = f f^* + f^* \sum_{i=1}^{n_j} k_{ji} g_j^r f = w + \sum_{i=1}^{n_j} k_{ji} h_j^r \in N$.

Then $\alpha^* (w + \sum_{i=1}^{n_j} k_{ji} h_j^r) \alpha = u + \sum_{i=1}^{n_j} k_{ji} g_j^r \in N$. Suppose, by way of contradiction, there is an exit $e$ at a vertex $u'$ on $g$. Let $\beta$ be the part of $g$ connecting $u$ to $u'$ (where we take $\beta = u$ if $u' = u$) and $\gamma$ be the part of $g$ from $u'$ to $u$ (so that $g = \beta \gamma$). Then, denoting the cycle $\gamma \beta$ (based at $u'$) by $d$, we get $e^* \beta^* (u + \sum_{i=1}^{n_j} k_{ji} g_j^r) \beta e = e^* (u' + \sum_{i=1}^{n_j} k_{ji} d_j^r) e = e^* e = r(e) \in N$, a contradiction.
Thus the cycle $g$ has no exits. In particular, $|s^{-1}(u)| = 1$ and this implies that $X$ must be an empty set, as $X$ is a proper subset of $s^{-1}(u)$.

Thus the generators of $N$ are of the form $y = (u + \sum_{i=1}^{n} k_i g^{r_i})$. If there is another generator of $N$ of the form $y' = u + \sum_{i=1}^{n'} k'_i (g')^{r_i}$ with the same vertex $u$, then, by the uniqueness of $g$, $g' = g$. Using the convention that $g^0 = u$, we can write $y = p(g)$ and $y' = q(g)$ where $p(x) = 1 + \sum_{i=1}^{n} k_i x^{r_i}$ and $p'(x) = 1 + \sum_{i=1}^{n'} k'_i x^{r_i}$ both belonging to $K[x]$. If $d(x)$ is the gcd of $p(x)$ and $q(x)$ in $K[x]$, then we can assume, without loss of generality, that $d(0) = 1$. Moreover, we can write $d(x) = a(x)p(x) + b(x)q(x)$ for suitable $a(x), b(x) \in K[x]$. Clearly $d(g) = a(g)p(g) + b(g)q(g) \in I$ and we can then replace both $y = p(g)$ and $y' = q(g)$ by $d(g)$. Iteration of this process guarantees that different generators $y_j$ and $y_k$ involve different vertices $v_j$ and $v_k$ and so $y_jy_k = 0 = y_ky_j$ for $j \neq k$, resulting in a mutually orthogonal set of generators for the ideal $I$. ■

Since Condition (L) on a graph demands that cycles have exits, an immediate consequence of Lemma 2 is the following well-known result.

Corollary 3 [4] Let $E$ be an arbitrary graph. If $E$ satisfies Condition (L), then every non-zero two-sided ideal of $L_K(E)$ contains a vertex.

The next theorem gives an explicit description of the generators of the non-graded ideals of a Leavitt path algebra.

**Theorem 4** Let $I$ be a non-zero ideal of $L_K(E)$ with $I \cap E^0 = H$ and $S = \{v \in B_H : v^H \in I\}$. Then $I$ is generated by $H \cup \{v^H : v \in S\} \cup Y$, where $Y$ is a set of mutually orthogonal elements of the form $(u + \sum_{i=1}^{n} k_i g^{r_i})$ in which (i) $g$ is a (unique) cycle without exits in $E^0 \setminus H$ based at a vertex $u$ in $E^0 \setminus H$ and (ii) $k_i \in K$ with at least one $k_i \neq 0$. Moreover, $I$ is non-graded if and only if $Y$ is non-empty.

**Proof.** Let $J = I_{(H,S)}$ be the ideal of $L_K(E)$ generated by $H \cup \{v^H : v \in S\}$. We may assume that $J \subseteq I$ since there is nothing to prove if $I = J$. By Tomforde [8], $L_K(E)/J \cong L_K(E \setminus (H,S))$. Identifying $L_K(E)/J$ with $L_K(E \setminus (H,S))$ via this isomorphism, we note that the non-zero ideal $I/J$ contains no vertices of $E \setminus (H,S)$ and so by Lemma 2 $I/J$ is generated by elements of the form $(u + \sum_{i=1}^{n} k_i g^{r_i})$ where $g$ is a (unique) cycle without exits in $E \setminus (H,S)$ based at a vertex $u \in (E \setminus (H,S))^0 = E^0 \setminus H \cup \{v' : v \in B_H \setminus S\}$ and $k_i \in K$ with at least one $k_i \neq 0$. It is then clear that the ideal $I$ is generated by $H \cup \{v^H : v \in S\} \cup Y$, where $Y$ is the set of mutually orthogonal elements of the form $y = (u + \sum_{i=1}^{n} k_i g^{r_i})$ where $g$ is a (unique) cycle without exits in $E \setminus (H,S)$ based
at a vertex \( u \in (E \setminus (H, S))^0 \) and \( k_i \in K \) with at least one \( k_i \neq 0 \). Observe that since the \( v' \in (E \setminus (H, S))^0 \) are all sinks, both \( u \) and the vertices on \( g \) all belong to \( E^0 \setminus H \). ■

Since Condition (K) on the graph \( E \) implies that the set \( Y \) in above theorem must be empty, the following well-known result (see, for eg. [8]) can be derived immediately from Theorem 4.

**Corollary 5** Let \( E \) be an arbitrary graph. Then \( E \) satisfies Condition (K) if and only if every ideal of \( L_K(E) \) is graded.

## 4 Finitely generated ideals of \( L_K(E) \)

Here we show that any finitely generated two-sided ideal \( I \) in a Leavitt path algebra must be a principal ideal. The main idea of the proof is to start with a generating set of the ideal \( I \) as given Theorem 4 and to replace any finite subset of these generators by an appropriate finite set of mutually orthogonal generators. The sum of these orthogonal generators will be a desired single generator. As a consequence, we derive that if \( E \) is a finite graph, then the Leavitt path algebra \( L_K(E) \) will be a two-sided principal ideal ring, that is, every ideal of \( L_K(E) \) will be a principal ideal.

**Theorem 6** Let \( E \) be an arbitrary graph. Then every finitely generated ideal of \( L_K(E) \) is a principal ideal.

**Proof.** Suppose \( E \) is an arbitrary graph and \( I \) is an ideal of \( L_K(E) \) generated by a finite set of elements \( a_1, \ldots, a_m \) in \( L_K(E) \). By Theorem 4 \( I \) also has a generating set \( H \cup \{ v^H : v \in S \} \cup Y \) where \( H = I \cap E^0, S = \{ v \in B_H : v^H \in I \} \) and \( Y \) is a set of elements of the form \( y = (u + \sum_{i=1}^{n} k_ig^r_i) \) where \( g \) is a (unique) cycle without exits in \( E \setminus (H, S) \) based at a vertex \( u \) in \( (E \setminus (H, S))^0 = E^0 \setminus H \) and \( k_i \in K \) with at least one \( k_i \neq 0 \). Since each \( a_i \) can be written as a finite sum of elements of the form \( \sum_{i=1}^{n} \beta^r_i x^r_i y^r_i z^r_i \) where \( x \in H \cup \{ v^H : v \in S \} \cup Y \), we may assume without loss of generality that the ideal \( I \) is generated by a finite set of elements \( x_1, \ldots, x_n \) where \( x_i \in H \cup \{ v^H : v \in S \} \cup Y \). We wish to re-choose the generators \( x_i \) such that for \( i \neq j \), \( x_ix_j = 0 \). This property clearly holds if \( x_i, x_j \) are different elements in either \( H \cup \{ v^H : v \in S \} \) or \( H \cup Y \).

So we need only to consider the case when \( x_i \in \{ v^H : v \in S \} \) and \( x_j \in Y \) with \( x_ix_j \neq 0 \) so that \( x_i = v - \sum_{e \in s^{-1}(v), r(e) \notin H} ee^* \) and \( x_j = v + \sum_{i=1}^{n} k_ig^r_i \) where \( v \in B_H \) and \( g \) is a cycle without exits based at \( v \) in \( E^0 \setminus H \). Since \( g \) has no exits in \( E \setminus (H, S) \), \( x_i = v - ee^* \) with \( e \) the initial edge of \( g \). Then \( x_ix_j = (v - ee^*)(v + \sum_{i=1}^{n} k_ig^r_i) = v - ee^* + \sum_{i=1}^{n} k_ig^r_i - \sum_{i=1}^{n} k_ig^{r_i} = v - ee^* = x_i \) and so we remove \( x_i \) from the list of generators of \( I \). Repeating this process a finite number of times, we obtain a finite set of generators \( y_1, \ldots, y_t \) of the ideal \( I \).
where \( y_i = v_i y_i v_i \) for all \( i \) and \( v_1, ..., v_t \) are distinct vertices in \( E \). An arbitrary element \( z \) of \( I \) will then be of the form

\[
z = \sum_{i=1}^{r_1} k_{1i} \alpha_{1i} \beta_{1i}^* y_1 \gamma_{1i} \delta_{1i}^* + \cdots + \sum_{i=1}^{r_t} k_{ti} \alpha_{ti} \beta_{ti}^* y_t \gamma_{ti} \delta_{ti}^*
\]

where, for \( s = 1, ..., t \), \( k_{si} \in K \), \( \alpha_{si}, \beta_{si}, \gamma_{si}, \delta_{si} \) are all paths in \( E \) for various \( i \).

Then

\[
z = \sum_{i=1}^{r_1} k_{1i} \alpha_{1i} \beta_{1i}^* a \gamma_{1i} \delta_{1i}^* + \cdots + \sum_{i=1}^{r_t} k_{ti} \alpha_{ti} \beta_{ti}^* a \gamma_{ti} \delta_{ti}^*
\]

where \( a = y_1 + \cdots + y_t \in I \). This shows that \( I \) is the principal ideal generated by the element \( a \). \( \blacksquare \)

It was shown in [6] that if \( E \) is a finite graph, then every ideal of \( L_K(E) \) is finitely generated. From Theorem 6 we then obtain the following stronger conclusion.

**Corollary 7** Let \( E \) be a finite graph. Then every ideal of \( L_K(E) \) is a principal ideal.

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