FORWARD OMEGA LIMIT SETS OF NONAUTONOMOUS DYNAMICAL SYSTEMS

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Dedicated to Jürgen Scheurle on his 65th birthday.

ABSTRACT. The forward $\omega$-limit set $\omega_B$ of a nonautonomous dynamical system $\varphi$ with a positively invariant absorbing family $B = \{B(t), t \in \mathbb{R}\}$ of closed and bounded subsets of a Banach space $X$ which is asymptotically compact is shown to be asymptotically positive invariant in general and asymptotic negative invariant if $\varphi$ is also strongly asymptotically compact and eventually continuous in its initial value uniformly on bounded time sets independently of the initial time. In addition, a necessary and sufficient condition for a $\varphi$-invariant family $A = \{A(t), t \in \mathbb{R}\}$ in $B$ of nonempty compact subsets of $X$ to be a forward attractor is generalised to this context.

1. Introduction. Omega limit sets play a fundamental role in characterising the behaviour of dynamical systems. Indeed, the attractor of an autonomous dynamical system is an omega limit set. In such systems the omega limit sets are invariant.

The situation is more complicated in nonautonomous dynamical systems because the behaviour now depends on the actual time and not on the time that has elapsed since starting. This leads to new concepts of nonautonomous attractors that consist of families of compact subsets. Two kinds have been distinguished, pullback attractors which depend on information from the distant past and forward attractors which depend on the future behaviour of the system. The constituent subsets depend on time and are mapped into each other by the system, i.e., are invariant in a new nonautonomous sense.

Vishik [15] also proposed another kind of nonautonomous attractor, which he called a uniform attractor. It is essentially a forward omega limit set which attracts bounded sets uniformly in initial times. No claim about its invariance was made by Vishik, but see [1, 2, 4, 7] for a later discussion for skew product semiflows. In addition, the convergence uniformly in all initial times is far too strong, because forward attractor is concerned only about what happens in the distant future and not in the past.

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Recently, Kloeden & Yang [12] showed that the forward omega limit set $\omega_B$ of a nonautonomous dynamical system $\varphi$ is asymptotic positive invariant for nonautonomous difference equations with a positively invariant compact absorbing set $B$. This concept has been long known in the differential equations literature [8, 13]. Also in [9] it was shown that $\omega_B$ is also asymptotic negative invariant provided $\varphi$ is eventually continuous in its initial value inside the compact absorbing set $B$ uniformly on bounded time sets independently of the initial time. These results were used to show the upper semi continuous dependence of $\omega_B$ on parameters.

These results are generalised here to a nonautonomous dynamical system $\varphi$ with a positively invariant absorbing family $B = \{B(t), t \in \mathbb{R}\}$ of closed and bounded subsets of $X$. This requires a successive strengthening of assumptions on compactness, such as asymptotic compactness and strong asymptotic compactness, and some eventual uniform properties.

In addition, a necessary and sufficient condition in [10] for a $\varphi$-invariant family $A = \{A(t), t \in \mathbb{R}\}$ of nonempty compact subsets of $X$ to be a forward attractor is shown to hold when, amongst other things, the absorbing family is strongly asymptotically compact and the process is eventually continuous in its initial value uniformly on bounded time sets independently of the initial time.

2. Dissipative nonautonomous systems. Let $\text{dist}_X(\cdot, \cdot)$ be the Hausdorff semi-distance between nonempty closed and bounded subsets of a Banach space $X$ and define

$$\mathbb{R}^2_\geq := \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0\}.$$ 

Since forward asymptotic behaviour depends on the future of the system and not on its past, it can be restricted to initial times $t_0 \geq T^*$ for some $T^*$. In fact, the system need only be defined for such times, i.e., the time set $\mathbb{R}^2_{t_0, \geq} := \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0 \geq T^*\}$.

Consider a nonautonomous dynamical system $\varphi$ on $X$ defined as a process or two-parameter semi-group, i.e.,

**Definition 2.1.** A process on the state space $X$ is a mapping $\varphi : \mathbb{R}^2_\geq \times X \rightarrow X$, which satisfies the initial value, 2-parameter semigroup and continuity properties:

(i) $\varphi(t_0, t_0, x_0) = x_0$ for all $t_0 \in \mathbb{R}$ and $x_0 \in X$;
(ii) $\varphi(t_2, t_0, x_0) = \varphi(t_2, t_1, \varphi(t_1, t_0, x_0))$ for all $t_0 \leq t_1 \leq t_2$ in $\mathbb{R}$ and $x_0 \in X$;
(iii) the mapping $(t, t_0, x_0) \mapsto \varphi(t, t_0, x_0)$ of $\mathbb{R}^2 \times X$ into $X$ is continuous.

We assume that the nonautonomous system $\varphi$ is dissipative. In particular, we assume that $\varphi$ has an absorbing family $B = \{B(t), t \in \mathbb{R}\}$ of closed and bounded subsets of $X$ which is $\varphi$-positively invariant, i.e.,

$$\varphi(t, t_0, B(t_0)) \subset B(t), \quad \forall t \geq t_0.$$ 

**Assumption 1.** There exists a $\varphi$-positive invariant family $B = \{B(t), t \in \mathbb{R}\}$ of closed and bounded subsets of $X$ such that for any bounded subset $D$ of $X$ and every $t_0 \geq T^*$ there exists a $T_{D, t_0} \geq 0$ for which

$$\varphi(t, t_0, x_0) \in B(t) \quad \text{for all } t \geq t_0 + T_{D, t_0}, x_0 \in D.$$

In addition, we assume that $\varphi$ is asymptotically compact on $B$. 

Assumption 2. The process $\varphi$ is asymptotically compact for the $\varphi$-positive invariant family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ of closed and bounded subsets of $X$ if there is a compact subset $K$ of $X$ such that
\[ \lim_{t \to \infty} \text{dist}_X (\varphi(t, t_0, B(t_0)), K) = 0. \]

2.1. Omega limit sets. Let $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ be $\varphi$-positive invariant family of closed and bounded subsets of $X$. The omega limit set starting in $B(t_0)$ at time $t_0 \geq T^*$ is defined as
\[ \omega_{B(t_0)}(t_0) := \bigcap_{t \geq t_0} \bigcup_{s \geq t} \varphi(s, t_0, B(t_0)). \]
Clearly, $y \in \omega_{B(t_0)}(t_0)$ if and only if there exist sequences $t_n \to \infty$ and $x_n \in B(t_0)$ such that $\phi(t_n, t_0, x_n) \to y$. Note that $\omega_{B(t_0)}(t_0) \subset \omega_{B(t_0)}(t_0')$ if $t_0 \leq t_0'$.

Lemma 2.2. Assume that $\varphi$ is asymptotically compact for the family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$, i.e., satisfies Assumption 2. Then the omega limit set $\omega_{B(t_0)}(t_0)$ starting in $B(t_0)$ at time $t_0$ is nonempty and compact with $\omega_{B(t_0)}(t_0) \subset K$ for each $t_0 \in \mathbb{R}$. Moreover,
\[ \lim_{t \to \infty} \text{dist}_X (\varphi(t, t_0, B(t_0)), \omega_{B(t_0)}(t_0)) = 0. \quad (2.1) \]
Proof. Consider a sequence $y_n = \varphi(t_n, t_0, b_n) \in \varphi(t_n, t_0, B(t_0))$, where $t_n \to \infty$. Then
\[ \text{dist}_X (y_n, K) \leq \text{dist}_X (\varphi(t_n, t_0, B(t_0)), K) \to 0 \text{ as } n \to \infty. \]
Since $K$ is compact there is a $k_n \in K$ such that
\[ \|y_n - k_n\| = \text{dist}_X (y_n, K) \to 0 \text{ as } n \to \infty. \]
Moreover, there is a convergent subsequence $k_{n_j} \to k \in K$. Hence
\[ \|y_{n_j} - k\| \leq \text{dist}_X (y_{n_j}, K) \to 0 \text{ as } n_j \to \infty. \]
Clearly, $k \in \omega_{B(t_0)}(t_0)$. Thus $\omega_{B(t_0)}(t_0)$ is nonempty.

Now let $\omega_n$ be a sequence in $\omega_{B(t_0)}(t_0)$. Then there exist sequences $\varphi(t_n, t_0, b_n)$ with $b_n \in B(t_0)$ and $t_n \to \infty$ such that
\[ \|\varphi(t_n, t_0, b_n) - \omega_n\| < \frac{1}{n}, \quad n \in \mathbb{N}. \]
Hence,
\[ \text{dist}_X (\omega_n, K) \leq \|\varphi(t_n, t_0, b_n) - \omega_n\| + \text{dist}_X (\varphi(t_n, t_0, B(t_0)), K) \]
\[ \leq \frac{1}{n} + \text{dist}_X (\varphi(t_n, t_0, B(t_0)), K) \to 0 \text{ as } n \to \infty, \]
so $\text{dist}_X (\omega_n, K) \to 0$. Arguing as above, there is a subsequence $\omega_{n_j}$ and a $\bar{\omega} \in K$ such that $\omega_{n_j} \to \bar{\omega}$ as $n_j \to \infty$. Moreover, $\bar{\omega} \in \omega_{B(t_0)}(t_0)$, since $\omega_{B(t_0)}(t_0)$ is closed. Hence, $\omega_{B(t_0)}(t_0)$ is compact.

Similarly, let $\omega \in \omega_{B(t_0)}(t_0)$. Then there exist sequences $\varphi(t_n, t_0, b_n)$ with $b_n \in B(t_0)$ and $t_n \to \infty$ such that $\|\varphi(t_n, t_0, b_n) - \omega\| \to 0$ as $n \to \infty$. Moreover,
\[ \text{dist}_X (\omega, K) \leq \|\omega - \varphi(t_n, t_0, b_n)\| + \text{dist}_X (\varphi(t_n, t_0, b_n), K) \]
\[ \leq \|\omega - \varphi(t_n, t_0, b_n)\| + \text{dist}_X (\varphi(t_n, t_0, B(t_0)), K) \to 0 \text{ as } n \to \infty, \]
so $\text{dist}_X (\omega, K) = 0$. Hence, $\omega \in K$. Since the choice of $\omega \in \omega_{B(t_0)}(t_0)$ was arbitrary, this implies that $\omega_{B(t_0)}(t_0) \subset K$ for each $t_0 \in \mathbb{R}$. 
Finally, suppose that the forward convergence (2.1) does not hold. Then there exist an \( \varepsilon_0 > 0 \) and a sequence \( t_n \to \infty \) as \( n \to \infty \) such that
\[
\text{dist}_X (\varphi(t_n, t_0, B(t_0)), \omega_{B(t_0)}(t_0)) \geq 2\varepsilon_0, \quad n \in \mathbb{N}.
\]
The set \( \varphi(t_n, t_0, B(t_0)) \) may not be compact, so the supremum defining the distance
\[
\text{dist}_X (\varphi(t_n, t_0, B(t_0)), \omega_{B(t_0)}(t_0)) = \sup_{y \in \varphi(t_n, t_0, B(t_0))} \text{dist}_X (y, \omega_{B(t_0)}(t_0))
\]
need not be attained. However, for each \( n \in \mathbb{N} \) there is a point \( y_n \in \varphi(t_n, t_0, B(t_0)) \) such that
\[
\text{dist}_X (\varphi(t_n, t_0, B(t_0)), \omega_{B(t_0)}(t_0)) \geq \varepsilon_0, \quad n \in \mathbb{N}.
\]
Since \( y_n \in \varphi(t_n, t_0, B(t_0)) \) and
\[
\text{dist}_X (y_n, K) \leq \text{dist}_X (\varphi(t_n, t_0, B(t_0)), K) \to 0 \quad \text{as} \quad n \to \infty,
\]
there exists a convergent subsequence \( y_{n_k} \to \bar{y} \in K \). Moreover, \( \bar{y} \in \omega_{B(t_0)}(t_0) \) by definition.

However, \( \text{dist}_X (y_{n_k}, \omega_{B(t_0)}(t_0)) \geq \varepsilon_0 \), so \( \text{dist}_X (\bar{y}, \omega_{B(t_0)}(t_0)) \geq \varepsilon_0 \), which is a contradiction. Hence the forward convergence (2.1) must hold.

3. Asymptotic invariance. Let \( B = \{B(t), t \in \mathbb{R}\} \) be the \( \varphi \)-positive invariant family of closed and bounded subsets of \( X \) in Assumption 1. Then Assumption 2 through Lemma 2.2 ensures that the omega limit set
\[
\omega_B := \bigcup_{t_0 \geq T_*} \omega_{B(t_0)}(t_0)
\]
of \( B \) is nonempty, compact and included in \( K \). Since the sets \( \omega_{B(t_0)}(t_0) \) are increasing with \( t_0 \) this union is in fact a limit.

The simple example \( \dot{x} = -x + e^{-t} \) with \( \omega_B = \{0\} \) shows that the set \( \omega_B \) need not be invariant or even positive invariant.

3.1. Asymptotic positive invariance.

**Definition 3.1.** A set \( A \) is said to be *asymptotically positive invariant* if for any monotonic decreasing sequence \( \varepsilon_p \to 0 \) as \( p \to \infty \) there exists a monotonic increasing sequence \( T_p \to \infty \) as \( p \to \infty \) such that
\[
\varphi(t, t_0, A) \subset B_{\varepsilon_p}(A), \quad t \geq t_0,
\]
for each \( t_0 \geq T_p \), where \( B_{\varepsilon_p}(A) := \{x \in X : \text{dist}_X (x, A) < \varepsilon_p\} \).

The following result was proved in [12] for difference equations and in ordinary differential equations in [9]. The proof is similar, but with critical difference due to the fact that the absorbing family is not just one set or compact. It requires an additional assumption.

Assumption 1 implies that the compact set \( K \) in Assumption 2 is absorbed into \( B = \{B(t), t \in \mathbb{R}\} \). Specifically, \( \varphi(\tau, t_0, K) \subset B(\tau) \) for \( \tau \geq T_{K,t_0} \). By the 2-parameter semigroup property,
\[
\varphi(t, t_0, K) = \varphi(t, \tau, \varphi(\tau, t_0, K)) \subset \varphi(t, \tau, B(\tau)), \quad t \geq \tau \geq T_{K,t_0} + t_0,
\]
Assumption 3. The compact subset $K$ of $X$ in Assumption 2 is such that, for every sequences $t_{0,j} \leq t_j$ with $t_{0,j} \to \infty$,  

$$\lim_{j \to \infty} \text{dist}_X (\varphi(t_j, t_{0,j}, K), K) = 0. \quad (3.2)$$

The following stronger assumption is needed to ensure limiting dynamics remains asymptotically compact. It essentially says that compact subset $K$ is asymptotically positive invariant.

**Assumption 3.** The compact subset $K$ of $X$ in Assumption 2 is such that, for every sequences $t_{0,j} \leq t_j$ with $t_{0,j} \to \infty$,  

$$\lim_{j \to \infty} \text{dist}_X (\varphi(t_j, t_{0,j}, K), K) = 0. \quad (3.2)$$

Theorem 3.2. Let Assumptions 1, 2 and 3 hold. Then $\omega_B$ is asymptotically positive invariant.

**Proof.** For $\varepsilon_1 > 0$ fixed, we prove by contradiction that there exists $t_1 = t_1(\varepsilon_1)$ large enough such that  

$$\text{dist}_X (\varphi(t, t_0, \omega_B), \omega_B) < \varepsilon_1 \quad \text{for} \quad t \geq t_0 \geq t_1(\varepsilon_1). \quad (3.3)$$

If it is not the case, then there are sequences $t_{0,j} \leq t_j \leq t_{0,j} + T_0(t_{0,j}, \varepsilon_1)$ with $t_{0,j} \to \infty$ as $j \to \infty$ such that  

$$\text{dist}_X (\varphi(t_j, t_{0,j}, \omega_B), \omega_B) \geq \varepsilon_1, \quad j \in \mathbb{N}. \quad (3.3)$$

Since $\omega_B$ is compact and $x_0 \mapsto \varphi(t_j, t_{0,j}, x_0)$ is continuous for each fixed $j$, the set $\varphi(t_j, t_{0,j}, \omega_B)$ is compact, so there exists an $\omega_j \in \omega_B \subseteq K$ such that  

$$\text{dist}_X (\varphi(t_j, t_{0,j}, \omega_j), \omega_B) = \text{dist}_X (\varphi(t_j, t_{0,j}, \omega_B), \omega_B) \geq \varepsilon_1, \quad j \in \mathbb{N}. \quad (3.3)$$

Define $y_j := \varphi(t_j, t_{0,j}, \omega_j)$. Then $y_j \in \varphi(t_j, t_{0,j}, K)$, so  

$$\text{dist}_X (y_j, K) \leq \text{dist}_X (\varphi(t_j, t_{0,j}, K), K).$$

Hence, by Assumption 3,  

$$\text{dist}_X (y_j, K) \to 0 \quad \text{as} \quad j \to \infty.$$ 

As in the proof of Lemma 2.2 it follows that there exists a convergent subsequence $y_{j_k} \to \tilde{y} \in K$. Moreover, $\tilde{y} \in \omega_B$ by the definition. However, $\text{dist}_X (y_{j_k}, \omega_B) \geq \varepsilon_1$, so $\text{dist}_X (\tilde{y}, \omega_B) \geq \varepsilon_1$, which is a contradiction. Thus for this $\varepsilon_1 > 0$ there exists $t_1 = t_1(\varepsilon_1)$ large enough such that (3.3) holds.

Repeating inductively with $\varepsilon_{p+1} < \varepsilon_p$ and $t_{p+1}(\varepsilon_{p+1}) > t_p(\varepsilon_p)$, it follows that $\omega_B$ is asymptotically positively invariant. 

3.2. **Asymptotic negative invariance.**

**Definition 3.3.** A set $A$ is said to be **asymptotic negatively invariant** if for every $a \in A$, $\varepsilon > 0$ and $T > 0$, there exist $t_\varepsilon \geq T + T^*$ and $a_\varepsilon \in A$ such that  

$$\|\varphi(t_\varepsilon, t_\varepsilon - T, a_\varepsilon) - a\| < \varepsilon.$$ 

To show that this property holds requires some further assumptions on the future uniform behaviour of the process. Let $B = \{B(t), t \in \mathbb{R}\}$ be the $\varphi$-positive invariant family of closed and bounded subsets of $X$ in Assumption 1. First, the asymptotic compactness in Assumption 2 needs to be strengthened to strongly asymptotically compact.
Assumption 4. The nonautonomous system $\varphi$ is strongly asymptotically compact for the $\varphi$-positive invariant family $B = \{B(t), t \in \mathbb{R}\}$ of closed and bounded subsets of $X$ in the sense that there is a compact subset $K$ of $X$ such that for every sequences $t_{0,j} < t_j$ with $t_{0,j} \to \infty$ nd $t_j - t_{0,j} \to \infty$,

$$\lim_{j \to \infty} \text{dist}_X (\varphi(t_j, t_{0,j}, B(t_{0,j})), K) = 0.$$ 

An alternative version of Assumption 4 would be to assume that the sets $B(t)$ are compact rather than just closed and bounded. However, in practice, it is much harder to show that a set is compact. The current version holds when the system has a Vishik uniform attractor and the sets $B(t)$ are uniformly bounded.

Note that Assumption 4 implies Assumption 2. Indeed, if $\text{dist}_X(\varphi(t_j, t_0, B(t_0)), K)$ does not converge to 0 for some sequence $t_j \to \infty$, a contradiction will occur as by positive invariance and Assumption 4 we have

$$\text{dist}_X (\varphi(t_j, t_0, B(t_0)), K) = \text{dist}_X (\varphi(t_j, t_j - 1, \varphi(t_j - 1, t_0, B(t_0))), K) \leq \text{dist}_X (\varphi(t_j, t_j - 1, B(t_j - 1)), K) \to 0, \quad \text{as } j \to \infty.$$

We also need the process to be continuous in its initial state on time intervals of bounded length uniformly in the starting time.

Assumption 5. The mapping $x_0 \to \varphi(t, t_0, x_0)$ is continuous in $x_0 \in B(t_0) \cup K$ uniformly on any time interval of finite length, i.e. for every $\varepsilon > 0$ and $T > 0$ there exists $\delta = \delta(\varepsilon, T)$ such that

$$\|\varphi(t, t_0, x_0) - \varphi(t, t_0, y_0)\| < \varepsilon \quad \text{for} \quad \|x_0 - y_0\| < \delta, \ x_0, y_0 \in B(t_0) \cup K$$

(3.4)

for all $t \in [t_0, t_0 + T]$ and $t_0 \geq T^*$. 

Theorem 3.4. Let Assumptions 1, 3, 4 and 5 hold. Then $\omega_B$ is asymptotic negatively invariant.

Proof. To show this let $\omega \in \omega_B$, $\varepsilon > 0$ and $T > 0$ be given. Then there exist sequences $b_n \in B(\tau_n)$ and $\tau_n < t_n$ with $\tau_n \to \infty$, $t_n - \tau_n \to \infty$, and an integer $N(\varepsilon)$ such that

$$\|\varphi(t_n, \tau_n, b_n) - \omega\| < \frac{1}{2} \varepsilon, \quad n \geq N(\varepsilon).$$

Define $a_n := \varphi(t_n - T, \tau_n, b_n)$. Then by Assumption 4, there exists a convergent subsequence $a_{n_j} := \varphi(t_{n_j} - T, \tau_{n_j}, b_{n_j}) \to \omega_{\varepsilon} \in K$ as $n_j \to \infty$. By definition, $\omega_{\varepsilon} \in \omega_B$.

From Assumption 5 the process $\varphi$ is continuous in initial conditions uniformly on finite time intervals of the same length, i.e., (3.4). Hence, for $\hat{N}(\varepsilon, T)$ large enough,

$$\|\varphi(t_{n_j}, t_{n_j} - T, a_{n_j}) - \varphi(t_{n_j}, t_{n_j} - T, \omega_{\varepsilon})\| < \frac{1}{2} \varepsilon, \quad n_j \geq \hat{N}(\varepsilon, T).$$

(3.5)

By the 2-parameter semi-group property

$$\varphi(t_{n_j}, t_{n_j} - T, a_{n_j}) = \varphi(t_{n_j}, t_{n_j} - T, \varphi(t_{n_j} - T, \tau_{n_j}, b_{n_j})) = \varphi(t_{n_j}, \tau_{n_j}, b_{n_j}).$$

$$\text{dist}_X (\varphi(t_{n_j}, t_{n_j} - T, a_{n_j}), K) = \text{dist}_X (\varphi(t_{n_j}, t_{n_j} - T, \varphi(t_{n_j} - T, \tau_{n_j}, b_{n_j})), K) \leq \text{dist}_X (\varphi(t_{n_j}, t_{n_j} - T, \omega_{\varepsilon}), K) \to 0, \quad \text{as } n_j \to \infty.$$
Then
\[ \|\omega - \varphi(t_{n_j}, t_{n_j} - T, \omega)\| \leq \|\omega - \varphi(t_{n_j}, t_{n_j} - T, a_{n_j})\| \\
+ \|\varphi(t_{n_j}, t_{n_j} - T, a_{n_j}) - \varphi(t_{n_j}, t_{n_j} - T, \omega)\| \\
= \|\omega - \varphi(t_{n_j}, \tau_{n_j}, b_{n_j})\| \\
+ \|\varphi(t_{n_j}, t_{n_j} - T, a_{n_j}) - \varphi(t_{n_j}, t_{n_j} - T, \omega)\| \\
< \frac{1}{2} + \frac{1}{2} = \epsilon. \]

This is the desired result. \hspace{1cm} \Box

**Remark 3.5.** Assumptions 1-5 can be satisfied, for example, for non-autonomous dynamical system \( \varphi \) which is asymptotically autonomous satisfying, for every sequences \( t_{0,j} \leq t_j \) with \( t_{0,j} \to \infty \) and bounded set \( E \),
\[
\sup_{x \in E} \text{dist}_X (\varphi(t_j, t_{0,j}, x), \phi(t_j - t_{0,j}, x)) \to 0, \quad j \to \infty, \quad (3.6)
\]
where \( \phi \) is the limit semigroup, with global attractor \( A \). In this case, the compact set \( K \) can be taken as the global attractor \( A \), provided that the \( \varphi \)-positive invariant family \( B \) of closed and bounded subsets is uniformly bounded, i.e., \( \cup_{t \geq T} B(t) \) is bounded. For example, the reaction-diffusion equation \( 4u_t - \Delta u + u + u^3 = g(x, t) \) defined on \( \mathbb{R} \) satisfies \((3.6)\), if \( g \in L^2_{t,loc}(T^*; L^2(\mathbb{R})) \) is asymptotically autonomous such that for some \( g_0 \in L^2(\mathbb{R}) \)
\[
\int_{t_{0,j}}^{t_j} e^{(t_j - s)} \|g(s) - g_0\|^2 ds \to 0, \quad j \to \infty,
\]
for all \( t_{0,j} \leq t_j \) with \( t_{0,j} \to \infty \). This condition is quite strong, but others are possible. A simple example is for the switching term \( g(x, t) = -g_0(x) \) for \( t < 0 \) and \( g(x, t) = g_0(x) \) for \( t \geq 0 \).

4. **Upper semi continuity in a parameter.** Now consider a parameterised family of processes \( \varphi^\nu(t, t_0, x_0) \), where \( \nu \in [0, \nu^*] \), on \( X \).

Assumptions 1, 3 and 4 need to be strengthened so the parameterised family of processes is absorbing in a family \( B = \{B(t), t \in \mathbb{R}\} \) of closed and bounded subsets of \( X \) uniformly in \( \nu \in [0, \nu^*] \) and is asymptotically compact for \( B \) with a common compact set \( K \) for all \( \nu \in [0, \nu^*] \).

**Assumption 6.** There exists a family \( B = \{B(t), t \in \mathbb{R}\} \) of closed and bounded subsets of \( X \) which is \( \varphi^\nu \)-positive invariant for each \( \nu \in [0, \nu^*] \) such that for any bounded subset \( D \) of \( X \) and every \( t_0 \geq T^* \) there exists a \( T_{D,t_0} \geq 0 \) (independent of \( \nu \)) for which
\[
\varphi^\nu(t, t_0, x_0) \in B(t), \quad \forall t \geq t_0 + T_{D,t_0}, x_0 \in D, \nu \in [0, \nu^*].
\]

**Assumption 7.** Each system \( \varphi^\nu, \nu \in [0, \nu^*] \), satisfies Assumptions 3 and 4 for the same compact set \( K \).

In addition, the following uniform continuous convergence of the processes in the parameter is needed.
Assumption 8. For every $\varepsilon > 0$ and $T > 0$ there exists a $\delta(\varepsilon, T) > 0$ such that
\[
\|\varphi' (t, t_0, b) - \varphi^0 (t, t_0, b)\| < \varepsilon, \quad t_0 \leq t \leq t_0 + T, b \in B(t_0) \cup K,
\]
for $|\nu| < \delta(\varepsilon, T)$ and $t_0 \geq T^*$.

Finally, the uniform attraction of the set $\omega_B^0$ for the system $\varphi^0$ is also required.

Assumption 9. $\omega_B^0$ uniformly attracts $K$, i.e., for every $\varepsilon > 0$ there exists a $T(\varepsilon)$, which is independent of $t_0 \geq T^*$, such that
\[
\text{dist}_{X} (\varphi^0 (t, t_0, K), \omega_B^0) < \varepsilon, \quad t \geq t_0 + T(\varepsilon), t_0 \geq T^*.
\]

Under the above assumptions the upper semi continuous convergence of the omega limit sets in the parameter holds.

Theorem 4.1. Suppose that Assumptions 6, 7, 8 and 9 hold. Then
\[
\text{dist}_{X} (\omega^\nu_B, \omega_B^0) \to 0, \quad \text{as } \nu \to 0.
\]

Proof. A proof by contradiction will be used. Suppose for some sequence of parameters $\nu_j \to 0$ that the above limit is not true, i.e., there exists an $\varepsilon_0 > 0$ such that
\[
\text{dist}_{X} (\omega^\nu_B, \omega_B^0) \geq 4\varepsilon_0, \quad j \in \mathbb{N}.
\]
Since $\omega^\nu_B$ is compact, there exists $\omega_j \in \omega^\nu_B$ such that
\[
\text{dist}_{X} (\omega_j, \omega_B^0) = \text{dist}_{X} (\omega^\nu_B, \omega_B^0) \geq 4\varepsilon_0, \quad j \in \mathbb{N}. \tag{4.1}
\]
By Assumption 9 there is a $T = T(\varepsilon_0)$ such that for any $t_0 \geq T^*$
\[
\text{dist}_{X} (\varphi^0 (t_0 + T, t_0, K), \omega_B^0) < \varepsilon_0.
\]
Then use Assumption 8 with this $T$ to pick a $\nu_j < \delta(\varepsilon_0, T)$ to ensure that
\[
\|\varphi^\nu_j (t_0, t_0 - T, b) - \varphi^0 (t_0, t_0 - T, b)\| < \varepsilon_0, \quad b \in B(t_0 - T) \cup K, t_0 \geq T^* + T.
\]
Fix such a $\nu_j$ and use the asymptotical negative invariance of $\omega^\nu_B$ to obtain the existence of an $\omega_j, \omega \in \omega^\nu_B \subset K$ and a $t^*_j \geq T^* + T$ so that
\[
\|\varphi^\nu_j (t^*_j, t^*_j - T, \omega_j, T) - \omega_j\| < \varepsilon_0.
\]
Then, with $t_0$ taken as $t^*_j$ above,
\[
\text{dist}_{X} (\omega_j, \omega_B^0) \leq \|\omega_j - \varphi^\nu_j (t^*_j, t^*_j - T, \omega_j, T)\| + \|\varphi^\nu_j (t^*_j, t^*_j - T, \omega_j, T) - \varphi^0 (t^*_j, t^*_j - T, \omega_j, T)\| + \text{dist}_{X} (\varphi^0 (t^*_j, t^*_j - T, \omega_j, T), \omega_B^0)
\]
\[
< \varepsilon_0 + \varepsilon_0 + \varepsilon_0 = 3\varepsilon_0,
\]
which contradicts the assumption (4.1).
5. **Nonautonomous forward attractors.** A \( \varphi \)-invariant family \( A = \{A(t), t \in \mathbb{R}\} \) of nonempty compact subsets of \( X \) is called a **forward attractor** if it forward attracts all families \( D = \{D(t), t \in \mathbb{R}\} \) of nonempty bounded subsets of \( X \), i.e.,

\[
\lim_{t \to \infty} \operatorname{dist}_X (\varphi(t, t_0, D(t_0)), A(t)) = 0, \quad (\text{fixed } t_0).
\] (5.1)

A forward attractor \( A = \{A(t), t \in \mathbb{R}\} \) is Lyapunov asymptotic stable, i.e., both forward attracting (5.1) and Lyapunov stable. Simple examples show that a forward attractor need not be unique, even when uniformly bounded \([10, 12]\), but can be asymptotically unique \([5]\).

It is possible to construct the component subsets of candidates for forward attractors in the same way as for a pullback attractor \([10]\). This is based on the observation that a \( \varphi \)-positively invariant family of nonempty compact subsets contains a maximal \( \varphi \)-invariant family of nonempty compact subsets. Essentially, it is formed by all the entire trajectories in the \( \varphi \)-positively invariant family.

When the component subsets of the absorbing family \( B = \{B(t), t \in \mathbb{R}\} \) are only closed and bounded but not compact, an additional assumption such as pullback asymptotic compactness of \( \varphi \) is needed to ensure that the intersection of nested subsets in (5.2) are nonempty, i.e. the sequences \( \{\varphi(t, t_0, b_{0,n})\}_{n \in \mathbb{N}} \) are precompact for all \( b_{0,n} \in B(t_0,n) \) and \( t_0,n \to -\infty \). Then the following theorem adapted from \([10]\) holds.

**Theorem 5.1.** Suppose that a process \( \varphi \) on \( X \) has a \( \varphi \)-positively invariant family \( B = \{B(t), t \in \mathbb{R}\} \) of nonempty closed and bounded subsets of \( X \) and is pullback asymptotically compact on \( B \). Then \( \varphi \) has a maximal \( \varphi \)-invariant family \( A = \{A(t), t \in \mathbb{R}\} \) in \( B \) of nonempty compact subsets of \( X \) determined by

\[
A(t) = \bigcap_{t_0 \leq t} \varphi(t, t_0, B(t_0)) \quad \text{for each } t \in \mathbb{R}.
\] (5.2)

This pullback construction is used only inside the \( \varphi \)-positively invariant family \( B \). It is equivalent to

\[
\lim_{t_0 \to -\infty} \operatorname{dist}_X (\varphi(t, t_0, B(t_0)), A(t)) = 0, \quad (\text{fixed } t).
\] (5.3)

Theorem 5.1 does not, however, imply that the subsets given by (5.2) form a pullback attractor, since nothing has been assumed about what is happening outside of the sections \( B(t) \) of \( B \). With the additional assumption that \( B \) is pullback absorbing, then the family \( A \) is a pullback attractor.

To ensure that the family \( A \) constructed by the pullback method in (5.2) is forward attracting the set of omega limit points of solutions starting in \( A \) needs to coincide with the set of omega limit points of solutions starting in the family \( B \), rather than to be just a proper subset. In addition, the family \( B \) should be forward absorbing.

### 5.1. Conditions ensuring forward convergence

Let \( B = \{B(t), t \in \mathbb{R}\} \) be \( \varphi \)-positively invariant family of nonempty closed and bounded subsets of \( X \) and recall that for each \( t_0 \in \mathbb{R} \), the forward omega limit set starting in \( B(t_0) \) at times \( t_0 \) is defined by

\[
\omega_{B(t_0)}(t_0) := \bigcap_{t \geq t_0} \bigcup_{s \geq t} \varphi(s, t_0, B(t_0)).
\]
In addition,
\[ \omega_B := \bigcup_{t_0 \geq T^*} \omega_{B(t_0)}(t_0). \]

Suppose that Assumptions 1 and 2 hold. Then the above sets are nonempty compact subsets of \( K \) and, by Lemma 2.2,
\[ \lim_{t \to \infty} \text{dist}_X (\varphi(t, t_0, B(t_0)), \omega_{B(t_0)}(t_0)) = 0, \quad (\text{fixed } t_0). \quad (5.4) \]
Since \( A(t_0) \subset B(t_0) \) and \( A(t) = \varphi(t, t_0, A(t_0)) \subset \varphi(t, t_0, B(t_0)) \), it follows that
\[ \lim_{t \to \infty} \text{dist}_X (A(t), \omega_{B(t_0)}(t_0)) = 0, \quad (\text{fixed } t_0). \quad (5.5) \]

The set of omega limit points for dynamics starting inside the family of sets \( A = \{ A(t), t \in \mathbb{R} \} \) is defined by
\[ \tilde{\omega}_A := \bigcap_{t_0 \in \mathbb{R}} \bigcup_{t \geq t_0} \varphi(t, t_0, A(t_0)) = \bigcap_{t_0 \in \mathbb{R}} \bigcup_{t \geq t_0} A(t), \]
which is nonempty and compact as a family of nested compact sets.

Obviously, \( \tilde{\omega}_A \subset \omega_B \subset K \). The inclusions here may be strict. The existence of omega limit points in \( \omega_B \) that are not in \( \tilde{\omega}_A \) means that \( A \) cannot be forward attracting from within \( B \). The converse also holds. The following theorem is adapted from [10].

**Theorem 5.2.** Suppose that Assumptions 1, 3, 4 and 5 hold and that the assumptions of Theorem 5.1 hold. Then \( A \) is forward attracting from within \( B \), i.e., the forward convergence (5.1) holds, if and only if \( \tilde{\omega}_A = \omega_B \).

**Proof.** (Sufficiency) Suppose that the forward convergence (5.1) does not hold. Then there is an \( \varepsilon_0 > 0 \) and a sequence \( t_n \to \infty \) as \( n \to \infty \) such that
\[ \text{dist}_X (\varphi(t_n, t_0, B(t_0)), A(t_n)) \geq 2\varepsilon_0 \quad \text{for all } n \in \mathbb{N}. \]
The set \( \varphi(t_n, t_0, B(t_0)) \) is not compact, so the supremum defining the distance
\[ \text{dist}_X (\varphi(t_n, t_0, B(t_0)), A(t_n)) = \sup_{y \in \varphi(t_n, t_0, B(t_0))} \text{dist}_X (y, A(t_n)) \]
need not be attained. However, arguing as in the proof of Lemma 2.2, for each \( n \in \mathbb{N} \) there is point \( y_n \in \varphi(t_n, t_0, B(t_0)) \) such that
\[ \text{dist}_X (\varphi(t_n, t_0, B(t_0)), A(t_n)) - \varepsilon_0 \leq \text{dist}_X (y_n, A(t_n)) \leq \text{dist}_X (\varphi(t_n, t_0, B(t_0)), A(t_n)), \]
which means that
\[ \text{dist}_X (y_n, A(t_n)) \geq \varepsilon_0, \quad n \in \mathbb{N}. \]

Now
\[ \|y_n - a_n\| \geq \text{dist}_X (y_n, A(t_n)) \geq \varepsilon_0, \quad n \in \mathbb{N}, \]
for any points \( a_n \in A(t_n) \). Since the \( a_n, y_n \in \varphi(t_n, t_0, B(t_0)) \) and
\[ \text{dist}_X (a_n, K) \vee \text{dist}_X (y_n, K) \leq \text{dist}_X (\varphi(t_n, t_0, B(t_0)), K) \to 0 \text{ as } n \to \infty, \]
there exist convergent subsequences \( y_{n_j} \to \tilde{y} \in K \) and \( a_{n_j} \to \tilde{a} \in K \). It follows that
\[ \|\tilde{y} - \tilde{a}\| \geq \varepsilon_0. \]
From the definitions \( \tilde{y} \in \omega_{B(t_0)}(t_0) \subset \omega_B \) and \( \tilde{a} \in \tilde{\omega}_A \). Since the \( a_n \) and hence \( \tilde{a} \) were otherwise arbitrary, it follows that
\[ \text{dist}_X (\omega_B, \tilde{\omega}_A) \geq \text{dist}_X (\tilde{y}, \tilde{\omega}_A) \geq \varepsilon_0. \]
Hence $\hat{\omega}_A \neq \omega_B$.

(Necessity) This will also be proved by contradiction. Suppose $\hat{\omega}_A \neq \omega_B$. Since $\hat{\omega}_A \subset \omega_B$, there is a point $x \notin \omega_B \setminus \hat{\omega}_A$. Without loss of generality, assume that $x \in \omega_B(t_0) \setminus \hat{\omega}_A$ for some $t_0 \geq T^*$ such that $\operatorname{dist}_X(x, \hat{\omega}_A) \geq 4\delta_0$ for some $\delta_0 > 0$.

By the forward attraction,
$$\operatorname{dist}_X(\phi(t, t_0, B(t_0)), A(t)) < \delta_0$$
for all $t$ large enough. Since $x \in \omega_B(t_0)$ indicates that there are sequences $x_n \in B(t_0)$ and $t_n \to \infty$ such that $\phi(t_n, t_0, x_n) \to x$ as $n \to \infty$, we have
$$\operatorname{dist}_X(x, A(t_n)) \leq \operatorname{dist}_X(x, \phi(t_n, t_0, x_n)) + \operatorname{dist}_X(\phi(t_n, t_0, B(t_0)), A(t_n)) < 2\delta_0$$
for $t_n$ large enough. This implies
$$\operatorname{dist}_X\left(x, \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A(t_m) \right) \leq 2\delta_0.$$
Hence,
$$\operatorname{dist}_X(x, \hat{\omega}_A) \leq \operatorname{dist}_X\left(x, \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A(t_m) \right) \leq 2\delta_0,$$
which contradicts the assumption that $\operatorname{dist}_X(x, \hat{\omega}_A) \geq 4\delta_0$. \hfill \qed

**Remark 5.3.** Generally, the existence of forward attractors is fairly hard to verify in general applications. In [6], several conditions are given for asymptotically autonomous dynamical systems to ensure the tails of the pullback attractor converge forwards and backwards to the global attractor of the limit semigroup. In this case, i.e., if $\lim_{m \to \infty} \operatorname{dist}_H(A(t), A) = 0$ and $\lim_{m \to -\infty} \operatorname{dist}_H(A(t), A) = 0$, where $\operatorname{dist}_H$ is the full Hausdorff metric, $A = \{A(t)\}_{t \in \mathbb{R}}$ is the pullback attractor and $A$ is the global attractor of the limit semigroup, then the pullback attractor $A$ is the forward attractor by Theorem 5.2 since $\hat{\omega}_A = \omega_B = A$. Several particular ODE examples of forward attractors are also given in [5].

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