Gravity and Discrete Symmetry

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Abstract

Using the differential calculus on discrete group, we study the general relativity in the space-time which is the product of a four dimensional manifold by a two-point space. We generalize the usual concept of frame and connection in our space-time, and from the generalized torsion free condition we obtain an action of a scalar field coupled to Einstein gravity, which may be related to the Jordan-Brans-Dicke theory.

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1 Introduction

In order to overcome some essential problems in elementary particle models, the concept of non-commutative geometry has been introduced in physics in recent years. It was Connes who first put forward a picture in which his non-commutative geometry was used to construct particle physical model \[1\],\[2\]. Then, many others expounded his thoughts in other methods which are not so abstruse in mathematics. There are so many works on this problem that we can only give out uncompleted references, for example see \[3, 4, 5, 8, 9, 10\]. Some improvements have been made in this direction. Especially, taking the Higgs field as the gauge field on the internal discrete group, we can reconstruct the standard model for electroweak-strong interactions in an elegant and consistent way \[5, 8, 9\]. Although we still don’t know how to quantize these reconstructed models, we have a possible explanation of geometrical origin of the Higgs field and the symmetry breaking mechanisms, which had puzzled physicists for many years.

Another interesting use of non-commutative geometry is in gravity. Due to our shortage of knowledge on how to describe the space-time at tiny distance, how to describe gravity at distance of order Plank-length is a problem for us. It seems that we should modify our classical geometrical concepts. So some persons have studied the gravity using the non-commutative geometrical method. First, Chameseddine et al. \[6\] study the general relativity in the framework of Connes’ non-commutative differential geometry. Based on a space-time which is the product of four dimensional manifold by a two-point space, they obtain a model of a scalar field coupled to Einstein gravity. And they interpret this scalar field as describing the distance between the two points in the internal space. Very recently, some other attempts have been made in this area \[7, 11\].

In this paper, by extending the formalism of \[8\], we investigate the relation between the discrete symmetry and gravity. We also take the space-time as the product of four dimensional manifold by a two-point space. In this space-time, we generalize the usual concepts of frame and connection. Beyond the ordinary frame and connection
components on four dimensional space-time, we introduce an extra component which
associates with discrete symmetry. After imposing our generalized torsion free condi-
tion, we get some relations among connections. We then calculate the curvature tensor,
and obtain a gravity action which describes a scalar field coupled to Einstein gravity.
This scalar field is closely related to the discrete symmetry, and we would like to take it
as the Brans-Dicke field [12]. We believe our formalism is more mathematically concise
and elegant than others.

This paper is scheduled as follows: In §2, we review the differential calculus on
discrete groups briefly. In §3, we give out our generalization of frame, connection, and
the consequence of generalized torsion free condition. In §4, we calculate the curvature
tensor and give out our gravity action. The final section is devoted to some conclusions
and remarks.

2 Introduction to differential calculus on $M \otimes Z_2$

The details of differential calculus on $M \otimes Z_2$ can be found in [1], [8]. In this section we
give out some basic and necessary information associated with our following discussion.

Let $\mathcal{A}$ be the algebra of complex valued function on $M \otimes Z_2$ and let $d_M, d_{Z_2}$ be
the external derivative on the differential algebra on $M, Z_2$ respectively. The external
derivative $d_{Z_2}$ acts on $\mathcal{A}$ as follows:

$$d_{Z_2}f = \partial_{Z_2}f \chi = (f - R(f))\chi, \quad \forall f \in \mathcal{A}.$$  

(1)

Here $\chi$ is a given one-form of $Z_2$ and $R$ is the right action on $\mathcal{A}$:

$$(Rf)(x, e) = f(x, r), \quad (Rf)(x, r) = f(x, e), \quad e, r \in Z_2.$$  

(2)

We denote the one form space as $\Omega^1$ and its basis as $(dx^m, \chi)$. Then we can give out
the external derivative on $\mathcal{A}$:

$$df = \partial_\mu f \cdot dx^\mu + \partial_{Z_2}f \cdot \chi.$$  

(3)
In order to satisfy the properties of the ordinary external derivative operator, we should add two conditions on \( d_{Z_2} \):

\[
\chi f = (Rf)\chi, \tag{4}
\]
\[
d_{Z_2}\chi = -2\chi \otimes \chi, \tag{5}
\]

and the nilpotency of \( d \) requires that

\[
dx^\mu \otimes dx^\nu = -dx^\nu \otimes dx^\mu, \tag{6}
\]
\[
dx^\mu \otimes \chi = -\chi \otimes dx^\mu. \tag{7}
\]

Let us define the metric \( g \) as a form on the left module of one-forms valued in the algebra \( A \) and bilinear over algebra \( A \),

\[
g : \Omega^1 \otimes \Omega^1 \rightarrow A. \tag{8}
\]

Imposing the middle-linearity and hermity conditions on metric, we obtain the following results:

\[
< dx^\mu, dx^\nu > = g^{\mu\nu}, \quad < dx^\mu, \chi > = < \chi, dx^\mu > = 0, \quad < \chi, \chi > = \eta, \tag{9}
\]

where \( \eta \) is a real parameter. Finally we introduce the Haar integral as a complex valued linear functional on \( A \) that remains invariant under the action of \( R \)

\[
\int_{Z_2} f = \frac{1}{2}(f(x,e) + f(x,r)). \tag{10}
\]

3 Generalized Connection and Torsion free condition

Einstein’s general relativity is based on the four-dimensional space-time which is not suited to describe the gravity at small distance. Now we take our space-time to be the product of four-dimensional manifold by a two-point space, i.e. \( M^4 \otimes Z_2 \). In this
space-time, the usual concept of frame and connection should be generalized. Our generalization are as follow:

Frame

\[ E^A = \begin{cases} E^a = e^a_m dx^m & a, m = 1, 2, 3, 4 \\ E^5 = \lambda \chi \end{cases} \]

(11)

Connection

\[ \Omega^{AB} = \begin{cases} \Gamma^A_mB^m & \Gamma^A_mB^m = -\Gamma^B_mA^m \\ \Gamma^A_A\chi \end{cases} \]

(12)

Here \( A = a, 5 \) and \( B = b, 5 \). \( \lambda \) is a real scalar field. We must note that our notation is a little different from the usual one, here is the correspondence between two notations

\[ \Gamma \rightarrow \Omega, \quad \gamma \rightarrow \Gamma \]

(13)

i.e. \( \Gamma \) to be spin connection and \( \gamma \) to be affine connection.

We assume that the torsion free condition should be maintain, i.e. \( T^A = 0 \). Using the generalized frame and connection, we calculate the torsion as:

\[
T^A = dE^A + \Gamma^A_mB^m E^B + \Gamma^A_A\chi E^B,
\]

\[
T^a = dE^a + \Gamma^a_m b^m \otimes E^b + \Gamma^a_5 d^m \otimes E^5 + \Gamma^a_5 \chi \otimes E^b + \Gamma^a_5 \chi \otimes E^5
\]

\[
= (d_x + d_z)E^a + \Gamma^a_m b^m \wedge e^b_n dx^n
\]

\[
+ \Gamma^a_5 d^m \otimes \chi + \Gamma^a_5 \chi \otimes e^b_n dx^n + \Gamma^a_5 \chi \otimes \chi
\]

\[
= \partial_m e^a_n dx^n \wedge dx^n + [e^a_n - R(e^a_n) - \chi \otimes dx^n + \Gamma^a_m b^m d^m \wedge dx^n
\]

\[
+ \Gamma^a_5 \chi \otimes dx^n + \Gamma^a_5 R(\chi) \chi \otimes \chi.
\]

(14)

From the torsion free condition, we get the usual torsion free condition

\[
(\partial_m e^a_n + \Gamma^a_m b^m)dx^n \wedge dx^n = 0
\]

(16)

and

\[
[e^a_n - R(e^a_n) - \Gamma^a_5 \lambda + \Gamma^a_5 R(e^a_n) - \chi \otimes dx^n = 0.
\]

(17)

For simplicity, we consider that the frame and connection on two sheets be the same, i.e.

\[
e^a_n = R(e^a_n), \quad \Gamma^A_BC = R(\Gamma^A_BC), \quad \lambda = R(\lambda),
\]

(18)
then we find
\[ \Gamma^a_5 \lambda = \Gamma^a_{cb} e^b_c. \] (19)

The other result from the torsion free condition is
\[ \Gamma^a_5 R(\lambda) \chi \otimes \chi = 0 \implies \Gamma^a_5 = 0. \] (20)

Another component of torsion is
\[ T^5 = dE^5 + \Gamma^5_B dx^m E^B + \Gamma^5_B \chi \otimes E^B. \]
\[ = d(\lambda \chi) + \Gamma^5_b dx^m \otimes E^b + \Gamma^5_m dx^m \otimes \lambda \chi + \Gamma^5_5 \chi \otimes E^b + \Gamma^5_5 \chi \otimes \lambda \chi \]
\[ = \partial_m \lambda dx^m \otimes \chi + (\lambda - R(\lambda)) \chi \otimes \chi - 2 \lambda \chi \otimes \chi + \Gamma^5_b dx^m \wedge e^b_n dx^n \]
\[ + \Gamma^5_5 dx^m \otimes \lambda \chi + \Gamma^5_5 \chi \otimes e^b_n dx^n + \Gamma^5_5 \chi \otimes \lambda \chi \]
\[ = \partial_m \lambda dx^m \otimes \chi + (\lambda - R(\lambda)) \chi \otimes \chi + \Gamma^5_5 b dx^m \wedge dx^n \]
\[ + \Gamma^5_5 \lambda dx^m \otimes \chi + \Gamma^5_5 R(e^b_n) \chi \otimes dx^n + \Gamma^5_5 R(\lambda) \chi \otimes \chi. \] (21)

Also from the torsion free condition, we have
\[ (1) \quad \Gamma^5_m e^b_n dx^m \wedge dx^n = 0 \implies \Gamma^5_m = 0 \]
\[ (2) \quad [\partial_m \lambda + \Gamma^5_5 \lambda - \Gamma^5_5 R(e^b_m) - dx^m \otimes \chi = 0 \]
\[ \implies \partial_m \lambda = \Gamma^5_5 e^b_m \]
\[ (3) \quad [-\lambda - R(\lambda) + \Gamma^5_5 R(\lambda) - \chi \otimes \chi = 0 \]
\[ if \quad \lambda = R(\lambda) \implies \Gamma^5_5 = 2, \]
\[ (if \quad \lambda = -R(\lambda) \implies \Gamma^5_5 = 0). \] (22)

In summary, from the torsion free condition we can get
\[ \Gamma^a_5 = 0, \] (23)
\[ \Gamma^5_m = 0 \quad Lra \Gamma^5_5 = 0, \] (24)
\[ \partial_m \lambda = \Gamma^5_5 e^b_m, \] (25)
\[ \Gamma^5_5 = 2, \] (26)

The above results are very important and can be used to calculate the curvature tensor and gravity action in the next section.
4 Curvature tensor and Gravity action

In §3 we generalized the frame and connection in $M^4 \otimes Z_2$ and from the generalized torsion free condition we get some important relations. Further in this section we use the relations to calculate the curvature tensor in our space-time. Finally we obtain the gravity action which describes a scalar field coupled to the Einstein gravity.

From the frame and connection given above, we reach the curvature tensor

$$ R^{AB} = d\Omega^{AB} + \Omega^AC \otimes \Omega^{CB} $$

$$ = d(\Gamma^{AB}_m dx^m + \Gamma^{AB}_n \chi) + (\Gamma^AC dx^m + \Gamma^AC \chi) \otimes (\Gamma^{CB}_n dx^n + \Gamma^{CB}_\chi) $$

$$ = \partial_n \Gamma^{AB}_m dx^n \wedge dx^m + \Gamma^{AC} \Gamma^{CB}_n dx^m \wedge dx^n $$

$$ + d_z \Gamma^{AB}_m dx^m + d_\chi \Gamma^{AB}_n \chi + \Gamma^{AC} \chi \otimes \Gamma^{CB}_m dx^n + \Gamma^{AC} dx^m \otimes \Gamma^{CB}_\chi $$

$$ + d_z \Gamma^{AB}_n \chi + \Gamma^{AC} \chi \otimes \Gamma^{CB}_m $$

$$ = (\partial_n \Gamma^{AB}_m + \Gamma^{AC} \Gamma^{CB}_n) dx^m \wedge dx^n + \Gamma^{AB}_m - R(\Gamma^{AB}_m) + \Gamma^{AC} R(\Gamma^{CB}_m) - \chi \otimes dx^m $$

$$ + (\partial_m \Gamma^{AB} + \Gamma^{AC} \Gamma^{CB}) dx^m \otimes \chi + [- \Gamma^{AB} + \Gamma^{AB} - R(\Gamma^{AB}) + \Gamma^{AC} R(\Gamma^{CB}) - \chi \otimes \chi] $$

The gravity action density can be calculated in usual way:

$$ I = < R^{AB}, E^A \otimes E^B > $$

$$ = < R^{ab}_{mn} dx^m \wedge dx^n, E^a \otimes E^b > + < (R^{5b}_{m} - R^{5b}_{m}) \chi \otimes dx^m, E^5 \otimes E^b > $$

$$ + < (R^{a5}_{m} - R^{a5}_{m}) dx^m \otimes \chi, E^a \otimes E^5 > + < R^{55}_{5} \chi \otimes \chi, E^5 \otimes E^5 >, \tag{28} $$

where from (27) we find

$$ R^{ab}_{mn} = \partial_m \Gamma^{ab}_n + \Gamma^{ac}_{mn} \Gamma^{cb}_n \tag{29} $$

$$ R^{5b}_{m} = \Gamma^{5c} R(\Gamma^{cb}_m) = \Gamma^{5c} \Gamma^{cb}_m \tag{30} $$

$$ R^{5b}_{m} = \partial_m \Gamma^{5b} + \Gamma^{5c} \Gamma^{cb}_n = \partial_m \Gamma^{5b} \tag{31} $$

$$ R^{a5}_{m} = \partial_m \Gamma^{a5} + \Gamma^{ac} \Gamma^{c5}_m = 0 \tag{32} $$

$$ R^{a5}_{m} = \Gamma^{ac} R(\Gamma^{c5}_m) = 0 \tag{33} $$

$$ R^{55}_{5} = - \Gamma^{55} - R(\Gamma^{55}) + \Gamma^{5c} \Gamma^{c5} $$

$$ = (-2 + \Gamma^{55}) \Gamma^{55} \tag{34} $$
Finally, we obtain

\[ I = (\partial_m \Gamma^{ab}_n + \Gamma^{ac}_m \Gamma^{cb}_n - \delta_m \Gamma^{ab}_n) < dx^m \wedge dx^n, dx^p \wedge dx^q > \epsilon_p^a \epsilon_q^b \]

\[ + (\Gamma^c_m \Gamma^{eb}_m - \partial_m \Gamma^{eb}_m) < \chi \otimes dx^m, \lambda \chi \otimes \epsilon_p^b dx^p > \]

\[ + (-2 + \Gamma^{55}_m)\Gamma^{55}_m < \chi \otimes \chi, \lambda \chi \otimes \lambda \chi >. \quad (35) \]

The first term in (35) is nothing but the usual curvature. But the second term is not so obvious and needs some skills. From the equation for frame

\[ \nabla_m e^b_p = \partial_m e^b_p + \Gamma^{bd}_m e^d_p - \gamma^n_{mp} e^b_n = 0, \quad (36) \]

we get

\[ \Gamma^5_b \Gamma^{bd}_m e^d_p - \partial_m \Gamma^5_b e^b_p \]

\[ = \Gamma^5_b \Gamma^{bd}_m e^d_p - \partial_m (\Gamma^5_b e^b_p) + \Gamma^5_b \partial_m e^b_p \]

\[ = \Gamma^5_b (\partial_m e^b_p + \Gamma^{bd}_m e^d_p) - \partial_m (\partial_p \lambda) \]

\[ = \partial_n \lambda \gamma^n_{mp} e^b_n - \partial_m (\partial_p \lambda) \]

\[ = \partial_n \lambda \gamma^n_{mp} - \partial_m (\partial_p \lambda), \quad (37) \]

where (25) has been used. The last equation is just the covariant derivative

\[ \nabla_m V_p = \partial_m V_p - \gamma^n_{mp} V_n, \quad V_p = \partial_p \lambda, \quad (38) \]

so the second term in (35) becomes

\[ - \nabla_m (\partial_p \lambda) \lambda g^{mp} \eta. \quad (39) \]

The third term in (35) is more simple if we use (20), we can easily find that it vanishes. Since the frame and connection on two sheets are the same, and we take \( \lambda \) to be a real scalar field, the Haar integral of \( I \) makes no difference. At last we obtain the gravity action

\[ I \sim R - \nabla_m (\partial_p \lambda) \lambda g^{mp} \eta \]

\[ \sim R + \partial_p \lambda \partial_m \lambda g^{mp} \eta. \quad (40) \]
5 Conclusions and Remarks

In this paper, we consider the gravity action in $M^4 \otimes Z_2$. Our formalism is based on differential calculus on discrete groups. Analogue to usual construction in ordinary four-dimensional space-time, we generalize the frame and connection in our space-time. Have considering the generalized torsion free condition, we calculate the curvature tensor and then obtain our final results. Our gravity action is very like the one in [6], but you can easily find that our formalism is more concise and elegant than others [6], [7], [11], so it is easier to accept.

Finally, we want to give a few comments on the meaning of the scalar field in our action. It is obvious that it must be related to the discrete symmetry. In [8], it was interpreted as describing the distance between the two points in the internal space. That is true. In our opinion we would like to take it as a Brans-Dicke field. So, we obtain a Jordan-Brans-Dicke theory. This is a very interesting consequence.

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