Modelling solar and stellar differential rotation

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Abstract. We present a model of stellar differential rotation based on the mean-field theory of fluid dynamics. DR is driven by Reynolds stress and anisotropic heat transport caused by the Coriolis force. Our model reproduces the rotation pattern in the solar convection zone and allows predictions for other stars with outer convection zones. We present results for a range of spectral types, including the limiting case of very shallow convection zones, and discuss the dependence of DR on the rotation rate and spectral type for main sequence stars.

1. Introduction

Differential rotation (DR) is a powerful generator of magnetic fields and therefore a key ingredient in stellar dynamo models. The surface DR of the sun has been known for a long time from the tracking of sunspots. The rotation period at the solar equator is about 30 percent shorter than that at the poles. Helioseismology has revealed that this pattern persists throughout the entire convection zone while the radiative core rotates rigidly. Between the core and the convection zone there is a transition layer with strong radial shear [1] which is often called the tachocline. Surface DR is also found for other main-sequence stars. Photometry of spotted stars shows a variation of the rotation period with the stellar activity cycle [2]. Such stellar butterfly diagrams (in analogy to the solar butterfly diagram) give a lower estimate of the surface DR but can not distinguish between solar-type and anti-solar rotation without additional information about the star’s activity cycle [3]. The Mt. Wilson Ca II HK project which monitors stellar Ca II activity found butterfly diagrams similar to those from photometry [4]. Spectroscopic measurement of DR with the Fourier transform method can distinguish between solar-type and anti-solar rotation but requires fast rotation and high luminosity. Consequently, it has been carried out for stars of spectral types A and F only [5]. Doppler imaging derives the DR from the motions of surface spots [6]. Like the Fourier transform method, it needs fast rotation. Space-based photometry using the MOST satellite has determined the DR of the stars ε Eri and κ1 Ceti [7, 8].

So far no MS star has been found showing anti-solar DR, i.e. a polar cap rotating with a shorter period than the stellar equator. Stellar DR is usually characterized by the surface shear,

\[ \delta \Omega = \Omega_{\text{eq}} - \Omega_{\text{pole}}, \]

where \( \Omega_{\text{eq}} \) and \( \Omega_{\text{pole}} \) are the rotation rates at the equator and the poles, respectively. The surface shear is related to the lapping time, \( t_{\text{lap}} = \frac{2\pi}{\delta \Omega} \). For stars, a surface rotation law of the form

\[ \Omega = \Omega_{\text{eq}}(1 + k \cos^2 \theta) \]
is usually assumed, where $\theta$ is the colatitude. With that type of rotation law we have

$$\delta \Omega = \frac{2\pi |k|}{P_{\text{rot}}},$$

(3)

where $P_{\text{rot}}$ is the rotation period at the equator. While rotation laws of the form (2) are widely used for fitting observation data it should be noted that in the rotation law derived from Doppler shifts for the solar surface [9],

$$\Omega(\theta) = (14.050 - 1.492 \cos^2 \theta - 2.606 \cos^4 \theta) \text{deg/day},$$

(4)

the $\cos^4 \theta$ term exceeds the $\cos^2 \theta$ term. Rotation laws derived from the observation of sunspots have $k = -0.2$ [10, 11], which is somewhat smaller than the equator-pole difference from the Doppler shifts and corresponds to $\delta \Omega = 0.05$ rad/day and a lapping time of 135 days. From the rotation law (4) we find $\delta \Omega = 0.07$ and $t_{\text{lap}} = 90$ days.

While observations so far show no systematic dependence of $\delta \Omega$ on the stellar rotation period, a temperature dependence of the form

$$\delta \Omega \propto T^{8.92 \pm 0.31}$$

(5)

has been found [12]. Theory thus not only has to explain the rotation pattern found in the solar convection zone but also the variation of stellar DR along the main sequence.

2. Model

DR can be explained with angular momentum transport by the convective gas motions. The mean-field approach of magnetohydrodynamics treats the very complex gas motion in a stellar convection zone by applying an average and solving the equation of motion for the mean gas motion only. In that equation the small-scale motions (i.e. the convection pattern and all motions on scales smaller than that) only appear through a correlation tensor, which acts as an additional stress. The latter is called the Reynolds stress and depends on certain statistical properties of the gas motion only [13].

Our model combines DR, meridional flow, and convective heat transport. The equation of motion for the mean gas flow in a stellar convection zone reads

$$\rho \left[ \frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} \right] = \nabla (\pi + R) - \nabla P + \rho g,$$

(6)

where $\bar{u}$ is the mean velocity field, $P$ the gas pressure, $\rho$ the gas density, and $g$ the local acceleration due to gravity. In first term on the RHS, $\pi$ and $R$ are the molecular and Reynolds stress tensors, respectively.

The convective heat transport is described by the equation:

$$\rho T \frac{\partial \delta s}{\partial t} = -\nabla \cdot (F_{\text{conv}} + F_{\text{rad}} + \rho T \bar{u} \delta s),$$

(7)

with $\delta s$ defined as

$$\delta s = \bar{s} - s_0, \quad \delta s \ll \bar{s}, s_0.$$  

(8)

In Eq. 8, $\bar{s}$ is the mean entropy and the constant value $s_0$ describes perfectly adiabatic stratification of the reference state [14]. The radiative heat flux is given by

$$F_{\text{rad}}^i = - \frac{16 \sigma T^3}{3 \kappa \rho} \nabla_i T$$

(9)
with the Stefan-Boltzmann constant $\sigma$ and the opacity $\kappa$. The convective heat transport is described by the flux vector

$$ F_{i}^{\text{conv}} = \rho c_p \langle u'_i T' \rangle $$

(10)

where $u'$ and $T'$ denote the fluctuations of velocity and temperature, respectively. The correlation $\langle u'_i T' \rangle$ can be rewritten in terms of mean quantities:

$$ \langle u'_i T' \rangle = \chi_t \Phi_{ij} \beta_j $$

(11)

where $\chi_t$ is a scalar diffusion coefficient and $\Phi_{ij}$ a dimensionless tensor [15]. The diffusion coefficient is determined by the stratification of the convection zone:

$$ \chi_t = \tau_c g \alpha^2 H_p^2 \langle \beta_r \rangle / 12 T $$

(12)

where $\tau_c$ is the turbulent eddy correlation time, $\alpha$ the classical mixing-length parameter, $H_p$ the pressure scale height, $\langle \beta_r \rangle$ the horizontally-averaged radial component of the super-adiabatic gradient,

$$ \beta = g/c_p - \nabla T, $$

(13)

and $c_p$ the specific heat capacity at constant pressure. The tensor $\Phi_{ij}$ depends on the Coriolis number,

$$ \Omega^* = 2\pi \tau_c \Omega, $$

(14)

which is closely related to the Rossby number,

$$ \text{Ro} = P_{\text{rot}} / \tau_c = 4\pi \Omega^*/\Omega^*. $$

(15)

For slow rotation, $\Omega^* \ll 1$, $\Phi_{ij}$ is reduced to the Kronecker $\delta$ and $\chi_t$ is the same as in standard mixing length theory. For fast rotation, $\Omega^* \gg 1$, both the magnitude of the diffusion coefficient and the structure of the tensor $\Phi_{ij}$. The convective heat flux is then no longer aligned with the temperature gradient. Instead, it is tilted towards the rotation axis so that there is a horizontal heat flux from the equator to the poles.

The stress tensor, $R_{ij} = -\rho Q_{ij}$, is determined by the gas density and the one-point correlation tensor of the velocity fluctuations: $Q_{ij} = \langle u'_i u'_j \rangle$. It can be expressed in terms of the mean gas motion, in which case it takes the form:

$$ Q_{ij} = -N_{ijkl} \frac{\partial \Omega_k}{\partial x_l} + \Lambda_{ijkl} \Omega_k. $$

(16)

The first term is zero for rigid rotation. In the equation of motion it acts purely diffusive and thus represents a viscosity. The second term, the $\Lambda$-effect, exists even in case of rigid rotation and therefore can cause DR.

Like the convective heat transport, we write the viscosity tensor as a product of a scalar function and a dimensionless tensor:

$$ N_{ijkl} = \nu_t \Psi_{ijkl} $$

(17)

where $\nu_t = \tau_c g \alpha^2 H_p^2 \langle \beta_r \rangle / 15 T$ and $\Psi_{ijkl}(\Omega^*)$ dimensionless functions of the Coriolis number [15]. In spherical polar coordinates, the $\Lambda$-effect only appears in two components of the correlation tensor:

$$ Q_{r\phi}^\Lambda = \nu_t V \sin \theta \Omega, \quad V = V^{(0)} + V^{(1)} \sin^2 \theta $$

(18)

$$ Q_{\theta\phi}^\Lambda = \nu_t H \cos \theta \Omega, \quad H = V^{(1)} \sin^2 \theta. $$

(19)
with the dimensionless scalar functions $V^{(0)}(\Omega^*)$, and $V^{(1)}(\Omega^*)$ [16]. For isotropic, homogeneous fluctuations the $\Lambda$-effect vanishes. The Reynolds stress is then reduced to usual turbulence viscosity. A stellar convection zone, however, is stratified and rotates. The gas motions are therefore neither isotropic nor homogeneous and the $\Lambda$-effect appears in the stress tensor. For slow (but non-zero) rotation angular momentum is mainly transported in the radial direction while for fast rotation the horizontal effect dominates, especially in thin convection zones.

Figure 1. The convective turnover time as a function of depth for an M dwarf, the sun, and a $1.4 M_\odot$ main sequence star (from left to right). The rotation periods are 5 d, 27 d, and 1 d, respectively.

3. Results
The first application of our model is the solar rotation. Figure 2 shows the resulting rotation and meridional flow patterns. As the model contains the outer convection zone only, the tachocline is missing. Nevertheless the rotation shows the main characteristics of the solar DR, namely a fast equator, a slow pole, and little variation with radius in the bulk of the convection zone. Both the surface flow and the horizontal shear are remarkably close to the observed values. The meridional flow shows one flow cell per hemisphere with the surface flow directed towards the poles and the bottom flow towards the equator. The amplitude is about 16 m/s at the top and 8 m/s at the bottom. Note that the return flow is confined to a shallow layer at the bottom of the convection zone. The difference between equatorial and polar rotation rates is about 25 percent. A least-squares fit to a rotation law of the type (2) gives $k = -0.21$. The corresponding values of $\delta\Omega$ are 0.05 rad/d and 0.06 rad/s, respectively, the lapping times are 119 and 100 days.

Next we apply our model to a star on the lower main sequence. Figure 3 shows the meridional
Figure 2. Meridional flow and DR for the sun with a rotation period of 27d. Left: stream lines of meridional flow. Blue indicates counter-clockwise flow, red clockwise. Center: color contour plot of the normalized rotation rate. Right: the meridional flow speed at the top (blue) and bottom (red) of the convection zone.

Figure 3. Meridional flow and DR for a fully-convective low-mass star. Left: stream lines of meridional flow. Blue indicates counter-clockwise flow, red clockwise. Center: color contour plot of the (normalized) rotation rate. Right: meridional flow speed at the top (blue) and bottom (red) of the convection zone.
flow and rotation of an M dwarf rotating with a period of 5 d. As in the solar convection zone, there is one flow cell per hemisphere with the surface flow directed towards the poles. The flow amplitude at the surface is 6 m/s. A small core with a radius of 5 percent of the stellar radius is present for numerical reasons. The color contour plot in Fig. 3 shows little variation expect close to the rotation axis. Variation inside the cylinder surrounding the core is an artifact caused by the boundary condition. For the equator-pole difference of the rotation period, we find a value of 2.8 percent of the equatorial rotation rate or a total of 0.035 rad/d. A fit to a cos²θ law of type (2) yields $k = -0.028$ and 0.035 rad/d. The lapping time is 180 d. Despite the fact that the rotation profile appears much more rigid in the color contour plot, the total shear $\delta \Omega$ is more than half the solar value. The rotation pattern is much closer to the Taylor-Proudman state, though, with $\Omega$ increasing with radius in the bulk of the convection zone.

At the other end of the lower main sequence we have the F-dwarf with 1.4 solar masses. The depth of the convection zone in this star is only five percent of the stellar radius. Figure 4 shows the meridional flow and rotation patterns for this star at a rotation period of 1 d. In the left and center diagrams, the depth of the convection zone has been artificially increased by a factor of five for clarity. The meridional flow is very fast in comparison with the sun. There is one flow cell per hemisphere with the surface flow directed towards the poles. The maximum flow speed is 140 m/s at the surface and 100 m/s at the bottom. The rotation pattern is similar to the solar rotation, with an equator-pole difference of 17 percent of the equatorial rotation rate. For the total shear we find $\delta \Omega = 1.04 \text{rad/d}$, corresponding to a lapping time of only 6 d. From the cos²θ fit we get $k = -0.2$, $\delta \Omega = 1.25 \text{rad/d}$, and a lapping time of 5 d.

Figure 5 summarizes observations and models for a number of stars. In the left diagram, the lines denote the total surface shear $\delta \Omega$ as a function of the rotation period for four types of main sequence star from F8 to M2. The triangles show observed surface DR from Doppler imaging for four rapidly rotating young stars. The diamonds show the sun and the stars ε Eri and κ¹ Ceti. DR of the latter two was derived from their light curves as recorded by the MOST satellite. The model shows a rather modest variation of the surface shear both with the rotation period and with spectral type. Hotter stars have more DR than cool ones and fast rotators.
Figure 5. Left: differential rotation vs. rotation period for MS stars. The solid lines show the surface shear for model stars of spectral types F8, G2, K5, and M2. The symbols indicate observations. The triangles denote results from Doppler imaging for the stars Speedy Mic, AB Dor, and PZ Tel. The diamonds in the left half of the diagram indicate surface DR derived from photometry for \( \epsilon \) Eri and \( \kappa^1 \) Ceti. The diamond in the right part marks the sun. Right: differential rotation vs. temperature. The line indicates the \( T^{8.92} \) law from [12]. The diamonds denote the results of model calculations. In some cases the rotation period has been varied for the same star, leading to vertical scatter.

have more shear than slow ones but both trends are rather weak. So far we have not been able to compute models for rotation periods as short as 0.5 days. Extrapolating from the available models to the rotation periods where Doppler imaging is possible, we find very good agreement between theory and observations.

The right diagram of Fig. 5 shows the surface shear as a function of the effective temperature. The line denotes the \( T^{8.92} \) law of [12], the diagrams indicate results from our model for various types of star and rotation periods. In the right half of the diagram, the diamonds scatter around the curve, in the left they lie above it. This indicates a weaker temperature dependence than found by [12]. In the right half, on the other hand, we find some quite large values for \( \delta \Omega \), especially for the F-dwarf discussed above.

4. Discussion
Based on the mean-field theory of hydrodynamics, the second order correlation approximation, and the mixing length theory of stellar convection, we have constructed models for the rotation and meridional flow of main sequence stars with outer convection zones. This model successfully reproduces the rotation pattern of the solar convection zone and the meridional flow. It also allows predictions for other stars. For a given type of star, the dependence of the surface shear on the rotation period is weak. Variation with spectral type is moderate at the lower end of the MS, but steep for effective temperatures above \( \approx 6500 \) K. For masses above 1.4 \( M_{\odot} \), the convection zone is very shallow and the convective time scale is short. These objects must therefore rotate fast for convection to be significantly affected by rotation. On the other hand the Fourier transform method can only detect differential rotation if \( k > 0.1 \). Together with the short rotation periods required, this results in large values of \( \delta \Omega \) for stars with observed surface DR. Stars with weaker DR or slow rotators would not be detected as differential rotators. In our model, the large values of \( \delta \Omega \) for stars with masses above 1.4 solar masses are the result of the
short convective time scales in the shallow convection zones of these stars. They require very fast rotation in order to reach $\Omega^* \approx 1$. In a star with 1.4 solar masses rotating with an average period of 27 d, the Coriolis number would be as small as 0.16 at the bottom of the CZ and the impact of rotation on the convection pattern weak. The resulting surface shear would be small and not be detected by the Fourier transform method.

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