Entropy functions for accelerating black holes

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We introduce an entropy function for supersymmetric accelerating black holes in $AdS_4$, that uplift on general Sasaki-Einstein manifolds $X_7$ to solutions of M-theory. This allows one to compute the black hole entropy without knowing the explicit solutions. A dual holographic microstate counting would follow from computing certain supersymmetric partition functions of Chern-Simons-matter theories compactified on a spindle. We make a general prediction for a class of such partition functions in terms of “blocks”, with each block being constructed from the partition function on a three-sphere.

INTRODUCTION

Obtaining a precise microstate counting interpretation of black hole entropy is one of the major achievements of string theory. This was first studied in the context of supersymmetric and asymptotically flat black holes in $\text{Illum}$ and this led to an enormous literature of further work. More recently, starting with $\text{[2,3]}$, there has been a growing body of similar work in the context of supersymmetric and asymptotically $AdS$ black holes. In contrast to the earlier work, where it is the Cardy formula that underlies the microstate counting, instead the results for black holes in $AdS$ use holography and exact localization results for supersymmetric partition functions.

The present paper builds on $\text{[3,7]}$, which introduced an entropy function for a large class of supersymmetric $AdS_4$ black holes in M-theory. The seven-dimensional internal space $X_7$ is taken to be an arbitrary Sasaki-Einstein manifold, with the three-dimensional holographic duals being Chern-Simons-matter theories. The entropy function, similar in spirit to that of Sen $\text{[8]}$, allows one to compute the entropy without knowing the explicit supergravity solutions: the inputs are only $X_7$, the topology of the black hole horizon $\Sigma$, taken to be a Riemann surface, and the magnetic charges. This entropy was shown to match a dual computation using the localization results of $\text{[9,10]}$, for infinite families of black holes.

In this note we extend $\text{[4,6]}$ to accelerating black holes in $AdS_4$. This leads to a number of novel features $\text{[11]}$: the black holes have different horizon topology, with conical deficit angles; supersymmetry is preserved in a novel way; and when the deficit angles are appropriately quantized, so that the horizon $\Sigma$ is an orbifold known as a spindle, remarkably the uplifted $D = 11$ solutions are completely smooth on and outside the horizon.

We will explain how to compute the entropy function for a general class of accelerating $AdS_4$ black holes, which takes a simple “gravitational block” form, vastly extending $\text{[12,13]}$. This leads to a general prediction for the partition functions of supersymmetric Chern-Simons-matter theories compactified on a spindle, with magnetic fluxes switched on for flavour and baryonic global symmetries, in the large $N$ limit $\text{[15]}$. We also point out a striking relation of our entropy function to the on-shell action of the black holes in various examples, and comment on including angular momentum and electric charges in this formalism.

SUPERSYMMETRIC AdS_2 SOLUTIONS

Our starting point is the following general class of supersymmetric $AdS_2$ solutions to $D = 11$ supergravity introduced in $\text{[10]}$ and clarified in $\text{[17]}$:

\[
\begin{align*}
    ds_{11}^2 &= e^{-2B/3} (ds_{AdS_2}^2 + ds_Y^2) , \\
    G &= \text{vol}_{AdS_2} \wedge [J - d(e^{-B} \eta)] .
\end{align*}
\]

Here $ds_{AdS_2}^2$ is a unit radius metric on $AdS_2$, with volume form $\text{vol}_{AdS_2}$ and $G$ is the $D = 11$ four-form. The GK space $Y_9$ has a canonically defined Killing vector field, $\xi$, called the $R$-symmetry vector, and it plays a central role. The metric on $Y_9$ takes the form

\[
    ds_Y^2 = \eta^2 + e^B ds_T^2 ,
\]

where the one-form $\eta$ is dual to $\xi$, $ds_T^2$ is a Kähler metric transverse to $\eta$, with associated Kähler two-form $J$ and Ricci two-form $d\eta = \rho$. The function $e^B = \frac{1}{2} R$, where $R > 0$ is the scalar curvature of the Kähler metric.

The metric and four-form in $\text{(1)}$ give supersymmetric solutions to $D = 11$ supergravity provided also

\[
    \Box R = \frac{1}{2} R^2 - R_{ab} R^{ab} ,
\]

where $R_{ab}$ denotes the Ricci tensor for the Kähler metric, and $\Box$ is the Laplacian operator. However, to define our entropy function we wish to go off-shell $\text{[1]}$, and in particular we will not directly impose $\text{(3)}$ in what follows.
NEAR HORIZON LIMITS OF BLACK HOLES

For the solutions of interest the $D = 11$ vacuum solution, without the black hole, is $AdS_4 \times X_7$, where $X_7$ is a Sasaki-Einstein manifold. A putative black hole may then carry conserved charges associated to various massless $U(1)$ gauge fields in $AdS_4$. The latter arise from Kaluza-Klein reduction on $X_7$, either from isometries of $X_7$ (“flavour symmetries”) or from homology cycles (“baryonic symmetries”).

Sasaki-Einstein manifolds $X_7$ have a $U(1)^s$ isometry, where necessarily $s \geq 1$, and we may choose an associated normalized basis of Killing vector fields $\partial_{\phi_i}$, $i = 1, \ldots, s$. $X_7$ is equipped with a Killing spinor, and without loss of generality we choose the basis so that this spinor has charge $\frac{1}{2}$ under $\partial_{\phi_1}$, and is uncharged under the remaining vector fields. Via the Kaluza-Klein mechanism, massless $U(1)$ gauge fields $A_i$ in $AdS_4$ are obtained by gauging $d\phi_i \rightarrow d\phi_i + A_i$, in the metric on $X_7$, together with adding a corresponding term to the $D = 11$ six-form potential $C_6$, given in (4) below, where $dC_6 = *_7G$. On the other hand, if $\Sigma_I \subset X_7$ form a basis of five-cycles, $I = 1, \ldots, b_5(X_7) = \dim H_5(X_7, \mathbb{R})$, then reducing $C_6$ on each five-cycle $\Sigma_I$ also leads to massless $U(1)$ gauge fields $A_I$ in $AdS_4$. Altogether we have the linear perturbation

$$\delta C_6 = \sum_{i=1}^s A_i \wedge \omega_i + \sum_{I=1}^{b_5(X_7)} A_I \wedge \omega_I.$$  

(4)

Here both $\omega_i$ and $\omega_I$ are co-closed five-forms on $X_7$, but $\omega_I$ is closed while $d\omega_i = \partial_{\phi_i} \text{vol}_{X_7}$, for a suitably normalized volume form on $X_7$. Notice that in (4) we are free to shift $\omega_i \rightarrow \omega_i + \sum_I c_I^I \omega_I$, for arbitrary constants $c_I^I$, which is precisely the freedom to mix baryonic symmetries into flavour symmetries in field theory. This correspondingly shifts $A_I \rightarrow A_I - \sum_I c_I^I A_I$, and hence the notion of baryonic fluxes in the reduced theory on $AdS_4$.

Consider introducing a supersymmetric extremal black hole into this $AdS_4$ vacuum. The near horizon limit should be $AdS_2 \times \Sigma$, where the two-dimensional surface $\Sigma$ is the black hole horizon \[19\]. For an accelerating black hole we take $\Sigma$ to be a spindle \[11\]. This is topologically a two-sphere, but with conical deficit angles $2\pi(1 - 1/m_{\pm})$ at the poles, specified by two coprime positive integers $m_{\pm}$. The non-accelerating case is recovered simply by setting $m_{\pm} = 1$, so $\Sigma = S^2$.

Now consider turning on flavour magnetic charges, for the gauge fields originating from isometries of $X_7$, with

$$\frac{1}{2\pi} \int_{\Sigma} dA_i = \frac{p_i}{m_- m_+},$$

(5)

the magnetic flux through the horizon. This precisely fibres $X_7$ over $\Sigma$ to give a GK geometry of the form

$$X_7 \hookrightarrow Y_9 \rightarrow \Sigma.$$  

(6)

The fibration is well-defined \[20\] when the flavour magnetic charges $p_i$ are integers \[21\]. Imposing supersymmetry requires \[21\] that

$$p_i = -\sigma m_+ - m_-,$$  

(7)

where recall that the first copy of $U(1)$ is singled out by the Killing spinor being charged under this symmetry. Here $\sigma = \pm 1$ are called twist and anti-twist, respectively.

On the other hand, the $D = 11$ seven-form flux satisfies the Dirac quantization condition

$$\frac{1}{(2\pi \ell_p)^6} \int_Y dC_7 = N_7 \in \mathbb{Z},$$  

(8)

where $\ell_p$ is the $D = 11$ Planck length, and $Y \subset Y_9$ is any seven-cycle. When $Y_9$ takes the fibred form \[9\] there is a distinguished such cycle, namely a copy $Y = X_7$ of the fibre, and we identify $N = N_{X_7}$ with the number of M2-branes generating the original $AdS_4 \times X_7$ vacuum \[22\]. If we pick representatives of the five-cycles $\Sigma_I \subset X_7$ that are invariant under the $U(1)^s$ action, then via \[9\] these will fibre over the spindle $\Sigma$ to give associated seven-cycles $\Upsilon_I \subset Y_9$. We denote the corresponding flux numbers in (8) as $N_I = N_{\Sigma_I}$, and analogously to \[9\] define flux magnetic charges $P_I = N_I/N$ \[23\]. Notice that via \[9\] these fluxes will in general include contributions from the flavour magnetic charges $p_i$ in (4), and also baryonic magnetic charges $\frac{1}{2\pi} \int_{\Sigma} dA_I$. However, as explained above, defining the latter in general requires (arbitrary) choices, and so we instead parametrize the baryonic magnetic charges of the black hole via the $P_I$. This accounts for all quantized fluxes on $Y_9$.

ENTROPY FUNCTION

We have seen how fixing the magnetic charges $p_i$, $P_I$ of the black hole encodes the twisting of the fibration \[9\], and also quantized flux numbers \[8\] for the corresponding near horizon $AdS_2 \times Y_9$ solution. This can be related to geometric data on $Y_9$ as follows \[14\]. First, evaluating the left hand side of (8) on the background \[14\] gives

$$\frac{1}{(2\pi \ell_p)^6} \int_Y \eta \wedge \rho \wedge \frac{1}{2} J^2 = N_7,$$  

(9)

while imposing that the integral of \[3\] over $Y_9$ holds gives

$$\int_{Y_9} \eta \wedge \rho^2 \wedge J^2 = 0.$$  

(10)

The integrals \[9\], \[10\] are functions of Kähler class parameters $[J]$ and the R-symmetry vector, which we may write as

$$\xi = \sum_{\mu=0}^s b_\mu \partial_{\phi_\mu}.$$  

(11)
Here $\partial_{x_0}$ is a Killing vector field rotating the spindle $\Sigma$, fixing its poles. The Kähler class parameters lie in the basic cohomology $[J] \in H^2_B(F_\xi)$ associated to the foliation $F_\xi$ defined by $\xi$. One can show that the total number of such parameters is $\dim H_5(X_7, \mathbb{R}) = 2$. On the other hand, fixing the flux magnetic charges $P_1$, together with $N$ and imposing (10), imposes the same number of constraints. Although we do not have a general argument, in all examples fixing $p_1$, which determines the topology of $Y_0$, together with $P_1$ and $N$ fixes all the Kähler class parameters. This leaves the R-symmetry vector $\vec{b}$ still unspecified, apart from the constraint $b_1 = 1$ which corresponds to the Killing spinor necessarily having charge $1$.

The main result of [3] is that solutions to the PDE (3) extremize the entropy function

$$\mathcal{S} \equiv \frac{4\pi}{(2\pi)^7 \rho p_0} \int_{Y_0} \eta \wedge \rho \wedge \frac{1}{3!} J^3.$$  

(12)

For fixed $X_7$, spindle data $m_\pm$, magnetic charges $p_1$, $P_1$ and $N$, we have $\mathcal{S} = \mathcal{S}(\vec{b}_i)$ is a function only of the R-symmetry vector. The near horizon $AdS_2$ solution necessarily extremizes this, as a function of $(b_0, b_1 = 1, b_2, \ldots, b_s)$, with the black hole entropy $S_{BH} = \mathcal{S}(\vec{b}_i^\ast)$ being the entropy function evaluated at the critical point.

**GRAVITATIONAL BLOCKS**

Using Stokes’ theorem one can show that the entropy function (12) can be written in the “block” form

$$\mathcal{S} = \frac{4\pi}{(2\pi)^7 \rho p_0} \left[ \text{Vol}(X_7^+ \Sigma) - \text{Vol}(X_7^- \Sigma) \right],$$  

(13)

(see [24] for details). Here $X_7^\pm$ are the copies of $X_7$ over the two poles of the spindle [25], and $\text{Vol}(X_7^\pm) = \int_{X_7^\pm} \eta \wedge \frac{1}{3!} J^3$ is the volume induced by the choice of Kähler class. For toric $X_7$, which by definition have $s = 4$ and so at least $U(1)^4$ isometry, this was called the “master volume” in [5] and references [5,24] describe in detail how to compute this master volume in terms of toric data.

In practice [9], [10] are quadratic in the Kähler class parameters, and for more than one such parameter it is typically difficult to solve for $[J]$ in closed form, and thus obtain the entropy function $\mathcal{S}$ as described after equation (12). In the remainder of this paper we will hence focus on a restricted, but still very rich, class of examples that we refer to as **flavour twists**. This generalizes a similar class studied in [6] where by definition we impose that $[J] |_{X_7^\pm} \propto [\rho] |_{X_7^\pm}$. It can be shown that (13) leads to the result

$$\mathcal{S} \equiv \frac{8\pi^3 N^{3/2}}{3\sqrt{6}b_0} \left( \frac{1}{\sqrt{\text{Vol}_S(X_7)}} |_{\vec{b}_i^+} - \frac{\sigma}{\sqrt{\text{Vol}_S(X_7)}} |_{\vec{b}_i^-} \right),$$  

(14)

Here $\sigma = \pm 1$ as in (7), and $\text{Vol}_S(X_7)$ is the Sasakian volume of $X_7$, introduced in [27]. This is a function only of the R-symmetry vector $\vec{b} = (b_1 = 1, b_2, \ldots, b_s)$ (i.e. excluding the $b_0$ spindle direction), with

$$\vec{b}_i^+ \equiv \vec{b} - \frac{b_0}{m_+} (a_+, -a_+, \ldots, -a_+, a_+),$$  

$$\vec{b}_i^- \equiv \vec{b} + \frac{b_0}{m_-} (a_-, a_-, \ldots, -a_-),$$  

(15)

where $a_\pm$ are integers satisfying $a_- a_+ + a_- a_+ = 1$. Such $a_\pm$ exist by Bezout’s lemma, as $m_\pm$ are coprime. They are not unique, but different choices amount to a different choice of basis for the $U(1)^{s+1}$ action on $Y_0$, with generators $\partial_{x_i}$, $\mu = 0, 1, \ldots, s$, and the black hole entropy which extremizes (14) is independent of this choice.

For fixed $X_7$, the entropy function (14) is manifestly a function of only $m_\pm$, $N$ the flavour magnetic charges $p_1$, and the R-symmetry vector $(b_0, b_1 = 1, b_2, \ldots, b_s)$. The flavour charges $p_2, \ldots, p_s$ are here arbitrary, but in this class of examples the flux charges $P_1$ are determined by the remaining data. Specifically, one can show [21]

$$P_1 = \frac{\pi}{3b_0} \left( \frac{\text{Vol}_S(\Sigma_i)}{\text{Vol}_S(X_7)|_{\vec{b}_i^+}} - \frac{\text{Vol}_S(\Sigma_i)}{\text{Vol}_S(X_7)|_{\vec{b}_i^-}} \right),$$  

(16)

where these are again Sasakian volumes. Various methods for computing these volumes, for different classes of $X_7$, were given in [27], including using toric geometry, a fixed point theorem and a limit of an equivariant index.

The entropy function (14) may thus be written down for infinite families of accelerating $AdS_4$ black holes in M-theory, with general flavour magnetic charges $p_1$, and extremized over the R-symmetry vector to obtain the entropy. We present some examples in the next section.

The $AdS_4/CFT_3$ correspondence relates the free energy of the dual field theory (here typically Chern-Simons-matter theories) on the three-sphere $S^3$ [28,30] to a gravitational quantity via $\mathcal{F}_{S^3}(\vec{b}) = 2^{1/2} 3^{-3/2} \pi^2 / \sqrt{\text{Vol}_S(X_7)} |_{\vec{b}}$. This has been shown in many examples, although we are not aware of a general proof. We may then write (14) as

$$\mathcal{S} = \frac{4}{b_0} \left[ \mathcal{F}_{S^3}(\vec{b}_i^+) - \sigma \mathcal{F}_{S^3}(\vec{b}_i^-) \right].$$  

(17)

Here the free energy blocks are functions of the shifted R-symmetry vectors $\vec{b}_i^\pm$, which in field theory correspond to certain shifted trial R-symmetry assignments for the fields.

Finally, consider setting $m_\pm = 1$, so that the horizon $\Sigma = S^2$ and there is no acceleration, and also taking the limit $b_0 \rightarrow 0$ so that the R-symmetry vector is purely tangent to $X_7$. From (14) (or (17)) we then obtain

$$\mathcal{S} = 4 \sum_{i=1}^s p_1 \frac{\partial}{\partial b_i} \sqrt{\frac{2\pi^6}{27 \text{Vol}_S(X_7)}} |_{\vec{b}} N^{3/2},$$  

(18)
and here we should take \( b_1 = 1 \) after taking the derivative. This recovers the results of [2, 3], where the derivative operator precisely acts on \( F_{S^3}(\vec{b}) \).

**EXAMPLES**

The are two particularly interesting classes of examples of the flavour twist construction described above, where in particular cases we may also make contact with various explicit solutions. The first is when there are no baryonic symmetries, i.e. \( H_5(X_7, \mathbb{R}) = 0 \). In this case \( J \) is necessarily exact on \( X_7 \), and the condition \( (16) \) is vacuous. A simple example is \( X_7 = S^7 \), for which \( s = 4 \) and in a natural choice of basis for \( U(1)^4 \) we have

\[
\text{Vol}_{S}(S^7)\big|_g = \frac{\pi^4}{3b_2b_4(b_1 - b_2 - b_3 - b_4)}.
\]

One can check that the entropy function \( (14) \) agrees with the conjectured entropy function in [14] (after a simple linear change of variable), and moreover extremizing to obtain the entropy the result agrees with the explicit near horizon supergravity solutions in [31]. Instead the non-accelerating result \( (18) \) was in this case already known [22] to reproduce the entropy of the family of STU supergravity black hole solutions in [3]. Another example in this class, treated in [22], is \( X_7 = V_{5,2} \), for which no explicit supergravity solutions are known.

The second class of examples are referred to as the universal anti-twist. These correspond to the explicit accelerating black hole solutions constructed in [11]. They are universal in the sense that the solutions exist for arbitrary choice of Sasaki-Einstein \( X_7 \) with rational R-symmetry vector (see \( (20) \) below). Moreover the solutions exist only in the anti-twist case, with \( \sigma = -1 \), as we shall see momentarily [23]. The universal anti-twist may be characterized geometrically by saying that the flavour twisting is only along the R-symmetry direction of the Sasaki-Einstein metric. This is equivalent to imposing

\[
\vec{p} = \frac{p_1}{b_1} \vec{b}^+ + \frac{p_1}{b_1} \vec{b}^-.
\]

Using a homogeneity property of the Sasakian volume, one can show [24] that \( (14) \) leads to the simple result

\[
\mathcal{S} = \frac{1}{4b_0} \left[ (\vec{b}_1^+)^2 - \sigma (\vec{b}_1^-)^2 \right] F_{S^3}.
\]

Here the free energy \( F_{S^3} = F_{S^3}(\vec{b}^+) \) is computed using the (extremal) Sasaki-Einstein metric. In [21] one should set \( b_1 = 1 \) and extremize over \( b_0 \) to obtain the entropy. If \( \sigma = -1 \) there are no extrema, forcing \( \sigma = -1 \). One can check that the (positive) extremal value \( S_{\text{BH}} = \mathcal{S}^* \) is given by

\[
S_{\text{BH}} = \frac{(2m_+^2 + 2m_-^2)^{1/2} - m_- - m_+}{2m_- m_+} F_{S^3},
\]

which precisely agrees with the entropy of the explicit family of supersymmetric accelerating black holes in [11].

**ON-SHELL ACTION**

One might wonder how the entropy function we have introduced is related to other approaches to computing black hole entropy, and the associated thermodynamics. An immediate issue for extremal black holes in AdS\(_4\) is that the infinite AdS\(_4\) throat leads to a (IR) divergence in the holographically renormalized on-shell action \( I \), which is thus ill-defined without some form of regularization.

In [31] a complex locus of supersymmetric but non-extremal accelerating black holes was considered, that in addition have non-zero rotation and electric charge. This complex locus has well-defined but complex action:

\[
I = \pm \frac{1}{i\pi} \left[ \varphi^2 + \frac{\left( \frac{m_- - m_+}{4m_- m_+} \right)^2 \omega^2}{\mathcal{F}_{S^3}} \right]. \tag{23}
\]

Here the two signs correspond to two different complex branches, and \( \varphi, \omega \) are chemical potentials associated to electric charge and rotation. These satisfy the constraint \( \varphi = \frac{1}{4} \omega \pm i \pi \), where \( \chi = (m_+ - m_-)/m_- m_+ \) is the orbifold Euler characteristic of the spindle horizon. The entropy is obtained in a standard way from this, as minus the Legendre transform of \( I \), passing from grand canonical to microcanonical ensemble. Remarkably \( (21) \) and \( (23) \) satisfy \( I = -\mathcal{S} \), via the change of variable

\[
\omega = \mp 2\pi i b_0. \tag{24}
\]

The Legendre transform of \( I \) thus extremizes \( \mathcal{S} \), and since \( \omega \) is a chemical potential for rotation of the horizon, and \( b_0 \) is the component of the R-symmetry vector rotating the horizon, \( (21) \) is a natural identification.

On the other hand, \( I \) is an on-shell quantity for the AdS\(_4\) black holes, while \( \mathcal{S} \) is off-shell quantity for the associated near horizon AdS\(_2\) solutions. It is therefore hard to see how these might be related physically, although by construction both are “entropy functions”, in the sense that extremizing both gives the (same) black hole entropy. There is a similar relation between the entropy function \( (18) \) in the case of \( X_7 = S^7 \) and the on-shell action of the STU black holes computed in [22] (see also [14, 36]), suggesting this relation is not accidental.

**ANGULAR MOMENTUM AND ELECTRIC CHARGE**

It should be possible to generalize the analysis of this paper and also turn on both angular momentum \( J \) and electric charges \( q_i, Q_i \) for the AdS\(_4\) black holes. The fact that these are zero here is simply due to [4]: adding rotation and electric charge modifies this ansatz [37].
In [24] some additional observables in gravity are also introduced, namely the geometric R-charges [38]

\[
R_a^+ = \frac{4 \pi m_a^+}{(2\pi\ell_p)^6 N} \int_{S_a^+} \eta^\wedge \frac{J_2}{2!},
\]

\[
R_a^- = \frac{4 \pi m_a^-}{(2\pi\ell_p)^6 N} \int_{S_a^-} \eta^\wedge \frac{J_2}{2!}.
\] (25)

Here \(S_a^\pm\) are a set of \(U(1)^a\)-invariant supersymmetric five-submanifolds of the fibres \(X_7^\pm\), and note that these exist even when \(\dim H_5(X_7,\mathbb{R}) = 0\). These geometric R-charges are dual to R-charges of baryonic operators associated with M5-branes wrapping the submanifolds. When \(X_7\) is toric, the cones over these are precisely the toric divisors in the Calabi-Yau cone \(C(X_7)\), labelled by \(a = 1, \ldots, d\). In this toric case we have the identity [24]

\[
\frac{1}{2} \sum_{a=1}^d (R_a^+ + R_a^-) = 2 - \frac{m_- - \sigma m_+}{m_+ m_-} b_0.
\] (26)

In the universal anti-twist case, with the identification \(\varphi = \pm \frac{\pi}{2} \sum_{a=1}^d (R_a^+ + R_a^-)\).

For the special case of \(X_7 = S^7\), where the index \(a\) may be identified with the flavour index \(i\), we can define the master entropy function

\[
S = \mathcal{S} - i \left[ 4 b_0 J - \frac{1}{4} \sum_{a=1}^d (R_a^+ + R_a^-) \right] \mathcal{F}_{S^3}.
\] (27)

Here \(\mathcal{S}\) is the entropy function already introduced, depending on spindle data, magnetic charges and R-symmetry vector \(b_0, b_i, i = 1, \ldots, 4\). In [27] we have further introduced angular momentum \(J\), conjugate to \(b_0\), and electric charges \(q_a\), conjugate to the R-charges [25]. This is a natural generalization of the entropy function conjectured for the non-accelerating STU black holes in [39], and moreover we have checked that extremizing \(S\) and imposing that \(S\) and the conserved charges are real, precisely leads to the entropy of the family of near horizon solutions constructed in [30]. These were conjectured to be the near horizon limits of general dyonically charged, rotating and accelerating black holes in \(AdS_4\) in STU gauged supergravity, which uplift on \(X_7 = S^7\) to solutions of M-theory. In the case with only a single dyonic pair of charges turned on for the graviphoton, these are precisely the black hole solutions in [34].

More generally the flavour and baryonic charges are naturally combined in toric geometry via the index \(a = 1, \ldots, d\), and it is natural to conjecture that [27] is the correct entropy function with general charges, not just for \(X_7 = S^7\) but for more general classes of \(X_7\).

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[1] A. Strominger and C. Vafa, Phys. Lett. B 379, 99 (1996), arXiv:hep-th/9601029
[2] F. Benini, K. Hristov, and A. Zaffaroni, JHEP 05, 054 (2016), arXiv:1511.04085 [hep-th]
[3] F. Benini, K. Hristov, and A. Zaffaroni, Phys. Lett. B771, 462 (2017), arXiv:1608.07294 [hep-th]
[4] C. Couzens, J. P. Gauntlett, D. Martelli, and J. Sparks, JHEP 01, 212 (2019), arXiv:1810.11026 [hep-th]
[5] J. P. Gauntlett, D. Martelli, and J. Sparks, JHEP 06, 140 (2019), arXiv:1904.04282 [hep-th]
[6] S. M. Hosseini and A. Zaffaroni, JHEP 07, 174 (2019), arXiv:1904.04269 [hep-th]
[7] H. Kim and N. Kim, JHEP 11, 050 (2019), arXiv:1904.05344 [hep-th]
[8] A. Sen, JHEP 09, 038 (2005) arXiv:hep-th/0506177
[9] S. M. Hosseini and A. Zaffaroni, JHEP 08, 064 (2016), arXiv:1604.03122 [hep-th]
[10] S. M. Hosseini and N. Mekareeya, JHEP 08, 089 (2016), arXiv:1604.03397 [hep-th]
[11] P. Ferrero, J. P. Gauntlett, J. M. P. Ipiña, D. Martelli, and J. Sparks, Phys. Rev. D 104, 046007 (2021), arXiv:2012.08530 [hep-th]
[12] S. M. Hosseini, K. Hristov, and A. Zaffaroni, JHEP 12, 168 (2019), arXiv:1909.10550 [hep-th]
[13] S. M. Hosseini, K. Hristov, and A. Zaffaroni, JHEP 07, 182 (2021), arXiv:2104.11249 [hep-th]
[14] F. Faedo and D. Martelli, JHEP 02, 101 (2022), arXiv:2111.13660 [hep-th]
[15] It would be of much interest to generalise the entropy functions of this paper to incorporate higher-derivative corrections and hence go beyond the large \(N\) limit.
[16] N. Kim and J.-D. Park, JHEP 09, 041 (2006), arXiv:hep-th/0607093
[17] J. P. Gauntlett and N. Kim, Commun. Math. Phys. 284, 897 (2008) arXiv:0710.2590 [hep-th]
[18] S. Benvenuti, L. A. Pando Zayas, and Y. Tachikawa, Adv. Theor. Math. Phys. 10, 395 (2006), arXiv:hep-th/0601054
[19] While this is generically expected to be true, it is important to understand the necessary and sufficient conditions.
[20] \(V_0\) is free of orbifold singularities provided the \(p_i\) are co-prime to both of \(m_+\).
[21] P. Ferrero, J. P. Gauntlett, and J. Sparks, JHEP 01, 102 (2022), arXiv:2112.01543 [hep-th]
[22] We must have \(N = m_+ N^X_+ = m_- N^X_-\) with \(N^X_\pm \in \mathbb{Z}\) and hence \(N = m_+ m_- N_0\) with \(N_0 \in \mathbb{Z}\) [23].
[23] The normalization factor of \(N\) is due to the fact that baryonic operators, dual to M5-branes wrapped on \(\Sigma_I\), arise as \(N \times N\) determinants in Chern-Simons-matter duals, and \(P_I\) is then the charge of the associated field.
[24] A. Boido, J. P. Gauntlett, D. Martelli, and J. Sparks, (2022), arXiv:2211.02662 [hep-th]
[25] More precisely these fibres are \(X_7^+ / \mathbb{Z}_{m_+}\), so \(X_7^+\) are really covering spaces of the fibres. The orientations of \(X_7^+\) are discussed in more detail in [24].
[26] Tools for analysing toric examples are developed in [24].
[27] D. Martelli, J. Sparks, and S.-T. Yau, Commun. Math. Phys. 280, 611 (2008) [arXiv:hep-th/0603021 [hep-th]].
[28] D. Martelli and J. Sparks, Phys. Rev. D84, 046008 (2011) [arXiv:1102.5289 [hep-th]].
[29] S. Cheon, H. Kim, and N. Kim, JHEP 05, 134 (2011) [arXiv:1102.5565 [hep-th]].
[30] D. L. Jafferis, I. R. Klebanov, S. S. Pufu, and B. R. Safdi, JHEP 06, 102 (2011) [arXiv:1103.1181 [hep-th]].
[31] C. Couzens, JHEP 03, 078 (2022) [arXiv:2112.04462 [hep-th]].
[32] S. M. Hosseini and A. Zaffaroni, JHEP 03, 108 (2019) [arXiv:1901.05977 [hep-th]].

Switching off rotation and electric charge gives solutions where the spindle degenerates at the AdS$_4$ boundary \[11\].

[33] D. Cassani, J. P. Gauntlett, D. Martelli, and J. Sparks, Phys. Rev. D 104, 086005 (2021) [arXiv:2106.05571 [hep-th]].
[34] D. Cassani and L. Papini, JHEP 09, 079 (2019) [arXiv:1906.10148 [hep-th]].
[35] P. Ferrero, M. Inglese, D. Martelli, and J. Sparks, Phys. Rev. D 105, 126001 (2022) [arXiv:2109.14625 [hep-th]].
[36] C. Couzens, E. Marcus, K. Stemerdink, and D. van de Heisteeg, JHEP 05, 194 (2021) [arXiv:2011.07071 [hep-th]].

[38] \[24\] also shows that associated with $U(1)^4$ invariant non-trivial five-cycles $\Sigma_I$ on $X_7$, (16) can be recast in the form $P_I = \frac{1}{2\pi_0} ( R_I^+ - R_I^- )$. 
