On Banach spaces of vector-valued random variables and their duals motivated by risk measures

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Abstract
We introduce Banach spaces of vector-valued random variables motivated from mathematical finance. So-called risk functionals are defined in a natural way on these Banach spaces and it is shown that these functionals are Lipschitz continuous. The risk functionals cannot be defined on strictly larger spaces of random variables which creates a particular interest for the spaces presented. We elaborate key properties of these Banach spaces and give representations of their dual spaces in terms of vector measures with values in the dual space of the state space.

Keywords: Vector-valued random variables, Banach spaces of random variables, rearrangement invariant spaces, dual representation, risk measures, stochastic dominance

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1 Introduction
This paper introduces Banach spaces for vector-valued random variables in a first part. These spaces extend rearrangement spaces for functions in two ways. First, random variables are considered on a probability space and second, we extend them to vector-valued (i.e., $\mathbb{R}^d$, or more general Banach space-valued) random variables.

It is natural to address differences/similarities between $L^1$ and $L^p$ spaces and we elaborate on extensions in the second part of the paper. We fully describe the duals of the new spaces. The duality theory for these spaces differs essentially from $L^p$ spaces. The new spaces are larger than $L^\infty$, but not an $L^p$ space in general and further, their dual is not even similar to $L^p$ spaces. However, they are reflexive. The duality theory is particularly nice in case that the dual of the state space enjoys the Radon–Nikodým property.

An important motivation for considering these spaces derives from recent developments in mathematical finance. Vector-valued functions or portfolio vectors are naturally present in many real life situations. An example is given by considering a portfolio with investments in $d$, say, different currencies. The random outcome is in $\mathbb{R}^d$ in this motivating example, the related random variable is said to be vector valued. Here, we consider more generally Banach space-valued random variables. The spaces can be associated with risk functionals and we demonstrate that the spaces introduced are as large as possible such that the associated risk functionals remain continuous.

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Rüschendorf [26] introduces and considers vector valued risk functionals first. Svindland [27], Filipović and Svindland [13], Kupper and Svindland [17] and many further authors consider and discuss different domain spaces for risk measures on portfolio vectors, for example Orlicz spaces (as done in Cheridito and Li [5] and Bellini and Rosazza Gianin [3]). Ekeland and Schachermayer [11] consider the domain space $L^\infty$ for these risk measures. Ekeland et al. [12] provide the first multivariate generalization of a Kusuoka representation for risk measures on vector-valued random variables on $L^2$. In contrast, the present paper extends these spaces and presents the largest possible Banach spaces for which those functionals remain continuous. The resulting spaces are neither Orlicz nor Lebesgue spaces, as considered in the earlier literature.

The spaces, which we consider, are in a way related to function spaces (rearrangement spaces) introduced by Lorentz [19, 20], following earlier results obtained by Halperin [15]. For unexplained notions from the theory of vector measures we would like to refer the reader to the book Diestel and Uhl [10].

**Outline of the paper.** The following section (Section 2) provides the mathematical setting including the relation to mathematical finance. The Banach spaces $L^p_\sigma(P,X)$ of $X$-valued random variables, introduced in Section 3, constitute the natural domains of risk functionals. We demonstrate that risk functionals are continuous with respect to the norm of the space introduced. In Section 4 we give a representation of the dual spaces of these Banach spaces in the scalar-valued case. This representation is used in Section 5 to derive representations of the duals in the general vector-valued case.

## 2 Mathematical setting and motivation

We consider a probability space $(\Omega, \mathcal{F}, P)$ and denote the distribution function (cdf) of a $\mathbb{R}$-valued random variable $Y$ by

$$F_Y(q) := P(Y \leq q) = P(\{\omega : Y(\omega) \leq q\}).$$

The generalized inverse is the nondecreasing and left-continuous function

$$F_Y^{-1}(\alpha) := \inf \{q : P(Y \leq q) \geq \alpha\},$$

also called the quantile or Value-at-Risk.

With $X = (X, \| \cdot \|)$ we denote a Banach space and by $X^*$ its continuous dual space. We use the notation $\langle \varphi, x \rangle$ for $\varphi \in X^*$ and $x \in X$. As usual we denote for $p \in [1, \infty)$ by $L^p(P,X)$ the Bochner-Lebesgue space of $p$-Bochner integrable $X$-valued random variables $Y$ on $(\Omega, \mathcal{F}, P)$ whose norm we denote by $\| \cdot \|_p$. Recall that for $Y \in L^p(P,X)$

$$\|Y\|_p = \left( \int_0^1 F_Y^{-1}(u)^p du \right)^{1/p} = \left( \int_0^\infty p t^{p-1} (1 - F_Y(t)) \, dt \right)^{1/p}. \tag{1}$$

In this paper Banach spaces of vector-valued, strongly measurable random variables are introduced by weighting the quantiles in a different way than (1). The present results extend and generalize characterizations obtained in Pichler [25], where only real valued random variables and $p = 1$ are considered (and elaborated in a context of insurance).
Remark 1. We shall assume throughout the paper that the probability space $(\Omega, \mathcal{F}, P)$ is rich enough to carry a $[0,1]$-valued, uniform distribution.\footnote{If $P(U \leq u) = u$ for all $u \in [0,1]$.} If this is not the case, then one may replace $\Omega$ by $\tilde{\Omega} := \Omega \times [0,1]$ with the product measure $\tilde{P}(A \times B) := P(A) \cdot \text{Lebesgue measure}(B)$. Every random variable $Y$ on $\tilde{\Omega}$ extends to $\Omega$ by $\tilde{Y}(\omega, u) := Y(\omega)$ and $U(\omega, u) := u$ is a uniform random variable, as $\tilde{P}(U \leq u) = \tilde{P}(\Omega \times [0,u]) = u$. We denote the set of $[0,1]$-valued uniform random variables on $(\Omega, \mathcal{F}, P)$ by $\mathcal{U}(0,1)$.

With an $\mathbb{R}$-valued random variable $Y$ one may further associate its generalized quantile transform

$$F(y,u) := (1-u) \cdot \lim_{y' \uparrow y} F_Y(y') + u \cdot F_Y(y).$$

The random variable $F(Y,U)$ is uniformly distributed again and $F(Y,U)$ is coupled in a comonotone way with $Y$, i.e., the inequality $(F(Y,U)(\omega) - F(Y,U)(\omega'))(Y(\omega) - Y(\omega')) \geq 0$ holds $P \otimes P$ almost everywhere (see, e.g., Pflug and Römisch [24, Proposition 1.3]).

### Relation to mathematical finance: risk measures and their continuity properties.

Risk measures on $\mathbb{R}$-valued random variables have been introduced in the pioneering paper Artzner et al. [2]. An $\mathbb{R}$-valued random variable is typically associated with the total, or accumulated return of a portfolio in mathematical finance. (The prevalent interpretation in insurance is the size of a claim, which happens with a probability specified by the probability measure $P$.)

The aggregated portfolio is composed of individual components, as stocks. From the perspective of comprehensive risk management it is desirable to understand not only the risk of the accumulated portfolio, but also its components. These more general risk measures on $\mathbb{R}^d$-valued random variables have been considered first in Burgert and Rüschendorf [4] and further progress was made, for example, by Rüschendorf [26], Ekeland et al. [12] and Ekeland and Schachermayer [11].

Ekeland and Schachermayer [11, Theorem 1.7] obtain a Kusuoka representation (cf. Kusuoka [18]) for risk measures based on $\mathbb{R}^d$-valued random variables. The risk functional identified there in the “regular case” for the homogeneous risk functional on random vectors is

$$\rho_Z(Y) := \sup \{ \mathbb{E} \langle Z, Y' \rangle : Y' \sim Y \}, \tag{2}$$

where $Y \sim Y'$ indicates that $Y$ and $Y'$ enjoy the same law in $\mathbb{R}^d$.\footnote{That is, $P(Y_1 \leq y_1, \ldots, Y_d \leq y_d) = P(Y'_1 \leq y_1, \ldots, Y'_d \leq y_d)$ for all $(y_1, \ldots, y_d) \in \mathbb{R}^d$.} $\rho_Z$ is called the maximal correlation risk measure in direction $Z$.

The rearrangement inequality (see e.g. McNeil et al. [21, Theorem 5.25(2)], also known as Chebyshev’s sum inequality, cf. Hardy et al. [16, Section 2.17]) provides an upper bound for the natural linear form in (2) by

$$\|\mathbb{E} \langle Z, Y \rangle\| \leq \mathbb{E} \| Z \|_* \cdot \| Y \| \leq \mathbb{E} K \cdot \| Z \|_{l_1^d} \cdot \| Y \| \leq K \cdot \int_0^1 F_{\| z \|_{l_1^d}}^{-1}(u) \cdot F_{\| Y \|}^{-1}(u) du, \tag{3}$$

where the norms $\| \cdot \|$ and $\| \cdot \|_*$ are dual to each other on $\mathbb{R}^d$ (here, $K > 0$ is the constant linking the norms by $\| \cdot \|_* \leq K \cdot \| \cdot \|_{l_1^d}$ on (the dual of) $\mathbb{R}^d$).
The maximal correlation risk measure \((2)\) employs the linear form \(\mathbb{E} \langle Z, Y \rangle\), which satisfies the bounds \((3)\). This motivates fixing the function

\[
\sigma(\cdot) := F^{-1}_{\|Z\|_Y}(\cdot)
\]

and to consider an appropriate vector space of random variables endowed with

\[
\|Y\|_\sigma := \int_0^1 \sigma(u) \cdot F^{-1}_{\|Y\|}(u)du.
\]

It turns out that \(\|\cdot\|_\sigma\) is a norm (theorem \(4\) below) on this vector space of random variables and that the maximal correlation risk measure is continuous with respect to the norm (proposition \(7\)).

### 3 The vector-valued Banach spaces \(L^p_\sigma(P, X)\)

Motivated by the observations made in the previous section we introduce the following notions.

**Definition 2.** A nondecreasing, nonnegative function \(\sigma: [0, 1) \to [0, \infty)\), which is continuous from the left and normalized by \(\int_0^1 \sigma(u)du = 1\), is called a distortion function (in the literature occasionally also spectrum function, cf. Acerbi \([1]\)).

**Definition 3.** For a distortion function \(\sigma\), a Banach space \((X, \|\cdot\|)\) and a probability space \((\Omega, \mathcal{F}, P)\) we define for \(p \in [1, \infty)\) and a strongly measurable \(X\)-valued random variable \(Y\) on \((\Omega, \mathcal{F}, P)\)

\[
\|Y\|_{\sigma, p} := \sup_{U \text{ uniform}} \mathbb{E}\sigma(U)\|Y\|^p = \sup_{U \text{ uniform}} \int_\Omega \sigma(U(\omega))\|Y(\omega)\|^p dP(\omega),
\]

where the supremum is taken over all \(U \in \mathcal{U}(0, 1)\), i.e., over all \([0, 1]\)-valued, uniformly distributed random variables \(U\) on \((\Omega, \mathcal{F}, P)\). Moreover, we set

\[
L^p_\sigma(P, X) := \{Y : \Omega \to X \text{ strongly measurable and } \|Y\|_{\sigma, p} < \infty\},
\]

where as usual we identify \(X\)-valued random variables which coincide \(P\)-almost everywhere.

Obviously, for \(\sigma = 1\) one obtains the classical Bochner-Lebesgue spaces \(L^p(P, X)\) which are well-known to be Banach spaces.

**Theorem 4.** \(L^p_\sigma(P, X)\) is a vector space and \(\|\cdot\|_{\sigma, p}\) is a norm on \(L^p_\sigma(P, X)\) turning it into a Banach space which embeds contractively into \(L^p(P, X)\).

Moreover, for each \(X\)-valued, strongly measurable \(Y\) on \((\Omega, \mathcal{F}, P)\) and every \(U \in \mathcal{U}(0, 1)\) which is coupled in comonotone way with \(\|Y\|\) it follows that

\[
\|Y\|_{\sigma, p} = \mathbb{E}(\sigma(U)\|Y\|^p) = \left(\int_0^1 \sigma(u)F^{-1}_{\|Y\|}(u)^p du\right)^{1/p}.
\]

**Proof.** We denote the probability measure on \((\Omega, \mathcal{F})\) with \(P\)-density \(\sigma \circ U\) for some \(U \in \mathcal{U}(0, 1)\) by \(\sigma(U)P\) and the expectation of a non-negative random variable \(Z\) on \((\Omega, \mathcal{F}, \sigma(U)P)\) by \(\mathbb{E}_U(Z)\). We obviously have

\[
\|Y\|_{\sigma, p} = \sup_{U \in \mathcal{U}(0, 1)} \mathbb{E}_U\|Y\|^p,
\]
which implies that \( L_\sigma^p(P, X) \) is a subspace of the intersection of Banach spaces \( \bigcap_{U \in \mathcal{U}} L^p(\sigma(U)P, X) \) and that \( \| \cdot \|_{\sigma, p} \) is a seminorm on \( L_\sigma^p(P, X) \).

By the rearrangement inequality (see, e.g., McNeil et al. [21, Theorem 5.25(2)]), the well-known fact that \( F_{\sigma(U)}^{-1} = \sigma \) and \( (F_{\| \cdot \|_Y}^{-1})^p = F_{\| \cdot \|_Y}^{-1}^p \) it follows for every \( U \in \mathcal{U} \) and each \( X \)-valued, strongly measurable \( Y \) on \( (\Omega, \mathcal{F}, P) \), that

\[
\mathbb{E} (\sigma(U)\|Y\|^p) \leq \int_0^1 \sigma(u) F_{\| \cdot \|_Y}^{-1}(u)^p du
\]

so that

\[
\|Y\|_{\sigma, p}^p \leq \int_0^1 \sigma(u) F_{\| \cdot \|_Y}^{-1}(u)^p du. \tag{6}
\]

Moreover, if we fix for a \( X \)-valued, strongly measurable \( Y \) on \( (\Omega, \mathcal{F}, P) \) some \( U \in \mathcal{U} \) such that \( U \) and \( \| \cdot \|_Y \) are coupled in a comonotone way (such \( U \) exists due to our general assumption on \( (\Omega, \mathcal{F}, P) \) made in remark 1) then (Kusuoka [18])

\[
\mathbb{E} (\sigma(U)\|Y\|^p) = \int_0^1 \sigma(u) F_{\| \cdot \|_Y}^{-1}(u)^p du.
\]

Together with (6) we obtain for each \( X \)-valued, strongly measurable \( Y \) on \( (\Omega, \mathcal{F}, P) \) that there is \( U \in \mathcal{U} \) such that

\[
\|Y\|_{\sigma, p}^p = \mathbb{E} (\sigma(U)\|Y\|^p) = \int_0^1 \sigma(u) F_{\| \cdot \|_Y}^{-1}(u)^p du,
\]

proving (5).

In order to see that the seminorm \( \| \cdot \|_{\sigma, p} \) on \( L_\sigma^p(P, X) \) is in fact a norm we apply the continuous version of Chebychev’s inequality (see, e.g., Gradshteyn and Ryzhik [14, Eq. 12.314]) to the nonnegative, nondecreasing functions \( \sigma \) and \( (F_{\| \cdot \|_Y}^{-1})^p \) on \( [0, 1] \) to obtain

\[
\int_0^1 \sigma(u) F_{\| \cdot \|_Y}^{-1}(u)^p du \geq \int_0^1 \sigma(u) du \cdot \int_0^1 F_{\| \cdot \|_Y}^{-1}(u)^p du = \int_0^1 F_{\| \cdot \|_Y}^{-1}(u)^p du = \mathbb{E} (\|Y\|^p),
\]

where the last equality follows from \( (F_{\| \cdot \|_Y}^{-1})^p = F_{\| \cdot \|_Y}^{-1}^p \). In particular, together with (5) we obtain for every \( X \)-valued, strongly measurable \( Y \)

\[
\mathbb{E} (\|Y\|^p) \leq \|Y\|_{\sigma, p}^p,
\]

which proves that \( L_\sigma^p(P, X) \) embeds contractively into \( L^p(P, X) \) and that \( \|Y\|_{\sigma, p} = 0 \) implies \( Y = 0 \) so that \( \| \cdot \|_{\sigma, p} \) is indeed a norm.

Finally, in order to prove that \( L_\sigma^p(P, X) \) is a Banach space when equipped with the norm \( \| \cdot \|_{\sigma, p} \), we first note that a Cauchy sequence \( (Y_n)_{n \in \mathbb{N}} \) in \( L_\sigma^p(P, X) \) is also a Cauchy sequence in \( L^p(P, X) \) so that there is \( Y \in L^p(P, X) \) with \( Y = \lim_{n \to \infty} Y_n \) in \( L^p(P, X) \). From this we conclude that \( Y = \lim_{n \to \infty} Y_n \) \( P \)-almost everywhere on \( \Omega \) for some subsequence \( (Y_{n_k})_{k \in \mathbb{N}} \) of \( (Y_n)_{n \in \mathbb{N}} \). Since for each \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) such that for all \( U \in \mathcal{U} \)

\[
\varepsilon^p > \mathbb{E} (\sigma(U)\|Y_n - Y_m\|^p)
\]
whenever \( n, m \geq N \) it follows with Fatou’s Lemma that for every \( U \in \mathcal{B}(0, 1) \) and each \( n \geq N \) we have

\[
\mathbb{E}(\sigma(U)\|Y - Y_n\|^p) = \mathbb{E}(\lim_{k \to \infty} \sigma(U)\|Y_{n_k} - Y_n\|^p) \leq \liminf_{k \to \infty} \mathbb{E}(\sigma(U)\|Y_{n_k} - Y_n\|^p) \leq \varepsilon^p;
\]

i.e., \( \|Y - Y_n\|_{\sigma, p} \leq \varepsilon^p \) for every \( n \geq N \). Thus, we conclude that

\[
Y = (Y - Y_N) + Y_N \in L^p_\sigma(P, X)
\]

and that \( (Y_n)_{n \in \mathbb{N}} \) converges to \( Y \) in \( L^p_\sigma(P, X) \).

**Remark 5.** By (5) \( L^p_\sigma(P, X) \)-membership of \( Y \) only depends on the quantile function \( F^{-1}_{\|Y\|} \) so that \( L^p_\sigma(P, X) \) is invariant with respect to rearrangements. From the definition of \( \| \cdot \|_{\sigma, p} \) it follows immediately that \( L^p_\sigma(P, X) \) is a \( L^\infty(P) \)-module and that \( \|\alpha Y\|_{\sigma, p} \leq \|\alpha\|_\infty \|Y\|_{\sigma, p} \) for all \( \alpha \in L^\infty(P) \) and each \( Y \in L^p_\sigma(P, X) \).

We next show that the \( L^p_\sigma(P, X) \)-spaces behave like the classical Bochner-Lebesgue spaces \( L^p(P, X) \) when one varies the exponent \( p \in [1, \infty) \).

**Proposition 6.** Let \( p, p' \in [1, \infty) \) be such that \( p < p' \).

i) \( L^p_\sigma(P, X) \subseteq L^p(P, X) \) and \( \|Y\|_{\sigma, p} \leq \|Y\|_{\sigma, p'} \) for every \( Y \in L^p_\sigma(P, X) \).

ii) If with \( r := p'/(p' - p) \) the distortion function \( \sigma \) satisfies \( \int_0^1 \sigma'(u) du < \infty \) then even \( L^p_\sigma(P, X) \subseteq L^p(P, X) \) and \( \|Y\|_{\sigma, p} \leq \|Y\|_{p'} \) for every \( Y \in L^p_\sigma(P, X) \).

**Proof.** Setting \( r := p'/(p' - p) \) it follows from (5), \( 1/r + 1/(p'/p) = 1 \) and Hölder’s inequality that for each \( X \)-valued, strongly measurable \( Y \) on \((\Omega, \mathcal{F}, P)\)

\[
\|Y\|_{\sigma, p}^p = \int_0^1 \sigma^r(u)\sigma^{p/r}(u) F^{-1}_{\|Y\|}(u)^p du \leq \left( \int_0^1 \sigma(u) du \right)^{1/r} \left( \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p du \right)^{p/r} = \|Y\|_{\sigma, p'}^p,
\]

which proves i) while ii) follows from (5), \( 1/r + 1/(p'/p) = 1 \), and Hölder’s inequality since

\[
\|Y\|_{\sigma, p}^p = \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p du \leq \left( \int_0^1 \sigma'(u) du \right)^{1/r} \left( \int_0^1 F^{-1}_{\|Y\|}(u)^{p'/r} du \right)^{p/r} = \left( \int_0^1 \sigma'(u) du \right)^{1/r} \left( \mathbb{E}\|Y\|_{p'} \right)^{p/r}
\]

holds for each \( X \)-valued, strongly measurable \( Y \) on \((\Omega, \mathcal{F}, P)\).

For a Banach space \( X \) with (continuous) dual space \( X^* \) we write as usual \( \langle x^*, x \rangle := x^*(x) \), \( x \in X, x^* \in X^* \). The dual norm on \( X^* \) will also be denoted by \( \| \cdot \| \). If \( Z \) is a \( X^* \)-valued, Bochner integrable random variable on \((\Omega, \mathcal{F}, P)\) such that \( \mathbb{E}\|Z\| = 1 \) then \( \sigma_Z := F^{-1}_{\|Z\|} \) is a distortion function. For two \( X \)-valued, strongly measurable \( Y_1, Y_2 \) on \((\Omega, \mathcal{F}, P)\) we write \( Y_1 \sim Y_2 \) if they have the same law, i.e., if \( P^{Y_1} = P^{Y_2} \).
**Proposition 7.** Let $X$ be a real Banach space and let $Z$ be a $X^*$-valued, Bochner integrable random variable on $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}\|Z\| = 1$. Then, for every $p \in [1, \infty)$

$$\rho_Z : L^p_{\sigma_Z}(P, X) \to \mathbb{R}, \ Y \mapsto \sup\{\mathbb{E}\langle Z, Y' \rangle : Y \sim Y'\}$$

is a well-defined subadditive, convex functional. Moreover, for $Y_1, Y_2 \in L^p_{\sigma_Z}(P, X)$ we have

$$|\rho_Z(Y_1) - \rho_Z(Y_2)| \leq \|Y_1 - Y_2\|_{\sigma,p}.$$

**Proof.** It follows from $Y \sim Y'$ that $F^{-1}_{\|Y\|} = F^{-1}_{\|Y'\|}$. Hence, $Y' \in L^p_{\sigma_Z}(P, X)$ whenever $Y \in L^p_{\sigma_Z}(P, X)$ by (5) in theorem 4. From the strong measurability of $Z$ and $Y \in L^p_{\sigma_Z}(P, X)$ it follows immediately that $\omega \mapsto \langle Z(\omega), Y(\omega) \rangle$ is a $\mathbb{R}$-valued random variable on $(\Omega, \mathcal{F}, P)$. The rearrangement inequality, the definition of $\sigma_Z$, (5) in theorem 4 and proposition 6 imply that for $Y' \sim Y \in L^p_{\sigma_Z}(P, X)$

$$|\mathbb{E}(Z, Y')| \leq \mathbb{E}(\|Z\|, \|Y'\|) \leq \int_0^1 \sigma_Z(u)F^{-1}_{\|Y\|}(u)du = \|Y\|_{\sigma,1} \leq \|Y\|_{\sigma,p},$$

which proves that $\rho_Z$ is well-defined and that

$$|\rho_Z(Y)| \leq \|Y\|_{\sigma,p}. \quad (7)$$

Obviously, $\rho_Z(\lambda Y) = \lambda \rho_Z(Y)$ for all $\lambda > 0$. Moreover, from the definition of $\rho_Z$ and strong measurability it follows immediately that $\rho_Z$ is subadditive. Therefore,

$$\rho_Z(Y_1) = \rho(Y_2 + Y_1 - Y_2) \leq \rho_Z(Y_2) + \rho_Z(Y_1 - Y_2).$$

Interchanging the roles of $Y_1, Y_2$ in the above inequality gives

$$|\rho_Z(Y_1) - \rho_Z(Y_2)| \leq \rho_Z(Y_1 - Y_2),$$

which together with (7) proves $|\rho_Z(Y_1) - \rho_Z(Y_2)| \leq \|Y_1 - Y_2\|_{\sigma,p}$. \qed

In the remainder of this section we have a closer look at the Banach spaces $L^p_{\sigma}(P, X)$.

**Proposition 8.** Let $X \neq \{0\}$. Then the following are equivalent.

i) For all $p \in [1, \infty)$ the spaces $L^p_{\sigma}(P, X)$ and $L^p(P, X)$ are isomorphic as Banach spaces.

ii) There is $p \in [1, \infty)$ such that $L^p_{\sigma}(P, X) = L^p(P, X)$ as sets.

iii) $\sigma$ is bounded.

**Proof.** Obviously, i) implies ii). By theorem 4, $L^p_{\sigma}(P, X)$ embeds contractively into $L^p(P, X)$. Thus, if ii) holds, this embedding is onto so that by Banach’s Isomorphism Theorem there is $C > 0$ such that

$$\forall Y \in L^p(P, X) : \sup_{U \in \Psi(0,1)} \int_{\Omega} \sigma(U(\omega))\|Y(\omega)\|^p dP(\omega) \leq C \int_{\Omega} \|Y(\omega)\|^p dP(\omega),$$

7
where \( \mathcal{U}(0, 1) \) is defined as before. Choose \( f \in L^1(P, \mathbb{R}) \) and \( x \in X \) with \( \|x\| = 1 \). Then \( Y(\omega) := |f(\omega)|^{1/p} x \) defines an element of \( L^p(P, X) \) so that for any \( U \in \mathcal{U}(0, 1) \) we have

\[
\left| \int_{\Omega} \sigma(U(\omega)) f(\omega) dP(\omega) \right| \leq C \int_{\Omega} \|Y(\omega)\|^p dP(\omega)
\]

where \( C \) is a constant. By the strong measurability of \( Y \), we have

\[
\|Y\|_{L^p} \leq \|Y\|_{L^\infty} \leq \|Y\|_{L^p(\omega)} = C \int_{\Omega} |f(\omega)| dP(\omega) < \infty.
\]

Since \( f \in L^1(P, \mathbb{R}) \) was chosen arbitrarily it follows that \( \sigma \circ U \in L^\infty(P, \mathbb{R}) \) which by \( U \in \mathcal{U}(0, 1) \) and by the fact that \( \sigma \) is nondecreasing implies boundedness of \( \sigma \). Thus, iii) follows from ii).

Finally, iii) and the fact that \( L^p_{\sigma}(P, X) \) embeds contractively into \( L^p(P, X) \) for any \( p \in [1, \infty) \) by theorem 4 implies i).

**Proposition 9.** We have the following:

i) For every \( p \in [1, \infty) \), \( L^\infty(P, X) \) embeds contractively into \( L^p_{\sigma}(P, X) \).

ii) Simple functions are dense in \( L^p_{\sigma}(P, X) \) for every \( p \in [1, \infty) \).

**Proof.** It follows from the definition of quantile function that \( 0 \leq F^{-1}_{\|Y\|} \leq \|Y\| \leq \|Y\|_{L^\infty} \) for every \( X \)-valued, strongly measurable \( Y \) on \( (\Omega, \mathcal{F}, P) \) which implies by (5) in theorem 4

\[
\|Y\|_{L^p} = \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p du \leq \|Y\|_{L^\infty},
\]

proving i).

In order to prove ii) let \( Y \in L^p_{\sigma}(P, X) \) and fix \( \varepsilon \in (0, 1) \). We choose \( u_\varepsilon \in (0, 1) \) such that \( \int_{u_\varepsilon}^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p du < \varepsilon^p \). By the strong measurability of \( Y \) there are \( N \in \mathcal{F} \) with \( P(N) = 0 \) and a separable, closed subspace \( X_1 \) of \( X \) such that \( \mathbb{1}_{N^c} = \mathbb{1}_{X_1} \)-valued. Let \( \{x_j; j \in \mathbb{N}\} \) be a dense subset of \( X_1 \). Denoting the open ball about \( x_j \) with radius \( \varepsilon \) in \( X \) by \( B_\varepsilon(x_j) \) we choose Borel subsets \( E_j \subseteq B_\varepsilon(x_j) \) such that \( X_1 \subseteq \bigcup_{j \in \mathbb{N}} E_j \) and such that the \( E_j \) are pairwise disjoint. Then \( \{(\mathbb{1}_{N^c} - 1)(E_j)\}_{j \in \mathbb{N}} \) is a pairwise disjoint sequence in \( \mathcal{F} \) such that \( P(\bigcup_{j \in \mathbb{N}} (\mathbb{1}_{N^c} - 1)(E_j)) = 1 \). Let \( n \in \mathbb{N} \) be such that

\[
\sum_{j=1}^n P((\mathbb{1}_{N^c} - 1)(E_j)) > u_\varepsilon \quad (8)
\]

and set \( E := \bigcup_{j=1}^n (\mathbb{1}_{N^c} - 1)(E_j) \).

Obviously, for \( t \geq 0 \) we have \( \{\|Y\| \leq t\} \supseteq \{\|Y\| \leq t\} \) so that \( F_{1_{E^c}}(\|Y\|)(t) \geq F_{\|Y\|}(t) \). Therefore,

\[
\forall u \in [0, 1]; \{t \geq 0; F_{1_{E^c}}(\|Y\|)(t) \geq u\} \supseteq \{t \geq 0; F_{\|Y\|}(t) \geq u\},
\]

which implies \( F^{-1}_{1_{E^c}}(\|Y\|)(u) \leq F^{-1}_{\|Y\|}(u) \). Furthermore,

\[
F_{1_{E^c}}(\|Y\|)(0) = P(1_{E^c} \|Y\| = 0) \geq P(E)
\]
so that \( F_{\parallel Y}^{-1}(u) = 0 \) for every \( u \in [0, P(E)] \), which together with \( F_{\parallel Y}^{-1} = (F_{\parallel Y}^{-1})^p \) and (8) yields for all \( u \in [0, 1] \)
\[
F_{\parallel Y}^{-1}(u) \leq \mathbb{1}_{(P(E), 1]}(u) F_{\parallel Y}^{-1}(u)^p \leq \mathbb{1}_{(u, 1]}(u) F_{\parallel Y}^{-1}(u)^p.
\]

Defining \( Y_\varepsilon := \sum_{j=1}^n \mathbb{1}_{(\varepsilon N, Y)}^{-1}(E_j) x_j \) it follows from the definition of \( E \) that \( \| \mathbb{1}_{N} Y - Y_\varepsilon \| \leq \varepsilon \) on \( E \) while \( Y_\varepsilon = 0 \) on \( E^c \). For every \( U \in \mathcal{Y}(0, 1) \) we obtain
\[
\int_{\Omega} \sigma(U) \| \mathbb{1}_{N} Y - Y_\varepsilon \| dP = \int_{\Omega} \sigma(U) \| \mathbb{1}_{N} Y - Y_\varepsilon \| dP + \int_{\Omega} \sigma(U) \| \mathbb{1}_{E_c} Y \| dP
\leq \varepsilon \int_{\Omega} \sigma(U) dP + \int_{0}^{1} \sigma(u) F_{\mathbb{1}_{E_c} Y}^{-1} d\varepsilon
\leq \varepsilon \int_{\Omega} \sigma(u) F_{\mathbb{1}_{E_c} Y}^{-1} d\varepsilon < 2\varepsilon^p,
\]
where we used the rearrangement inequality (see McNeil et al. [21, Theorem 5.25(2)]) in the first inequality and (9) in the second one while the last inequality follows form the choice of \( u_\varepsilon \). Thus, \( \| Y - Y_\varepsilon \| < \sqrt{2\varepsilon} \), proving ii).

**Theorem 10.** For \( X \neq \{0\} \) the following are equivalent.

i) \( L_{\sigma}^p(P, X) \) is a Hilbert space.

ii) \( X \) is a Hilbert space, \( p = 2 \), and \( \sigma = 1 \) on \( (0, 1) \).

**Proof.** Obviously, ii) implies i). Assume that i) holds. Since
\[
X \to L_{\sigma}^p(P, X), \ x \mapsto (\omega \mapsto x)
\]
is an isometry it follows that \( X \) is a Hilbert space. By our assumption on the existence of \([0, 1]\)-valued, uniformly distributed random variables on \((\Omega, \mathcal{F}, P)\) for every \( \alpha \in [0, 1] \) there is \( X_\alpha \in \mathcal{F} \) with \( P(E_\alpha) = \alpha \). For \( \alpha \in (0, 1) \) and \( x \in X \) a straightforward calculation gives for \( Y = \mathbb{1}_{E_c} x \) that
\[
F_{\parallel Y}^{-1}(u) = \| x \| \mathbb{1}_{[1-\alpha, 1]}(u) \cdot \mathbb{1}_{(\alpha, 1]}(u) F_{\parallel Y}^{-1}(u)^p.
\]
Moreover, for \( x_1, x_2 \in X \) with \( \| x_1 \| = \| x_2 \| = 1 \) and \( Y_1 := \mathbb{1}_{E_c} x_1, Y_2 := \mathbb{1}_{E_c} x_2 \) we have \( \| Y_1 \pm Y_2 \|_{\sigma, p} = 1 \). Thus, by the parallelogram identity and (5) we obtain for arbitrary \( \alpha \in [0, 1] \)
\[
1 = \frac{1}{2} (\| Y_1 + Y_2 \|_{\sigma, p}^2 + \| Y_1 - Y_2 \|_{\sigma, p}^2) = \| Y_1 \|_{\sigma, p}^2 + \| Y_2 \|_{\sigma, p}^2
\leq \left( \int_{1-\alpha}^{1} \sigma(u) du \right)^{2/p} + \left( \int_{\alpha}^{1} \sigma(u) du \right)^{2/p}.
\]
Since \( \sigma \) is continuous from the left \( \alpha \mapsto \int_{1-\alpha}^{1} \sigma(u) du \) is differentiable from the left on \( (0, 1) \) with left derivative \( -\sigma(u) \) and \( \alpha \mapsto \int_{1-\alpha}^{1} \sigma(u) du \) is differentiable from the right on \( (0, 1) \) with right derivative \( \sigma(1 - \alpha) \). Because \( x \mapsto x^{2/p} \) is increasing on \( [0, \infty) \) it follows together with (10) that both \( \alpha \mapsto (\int_{\alpha}^{1} \sigma(u) du)^{2/p} \) and \( \alpha \mapsto (\int_{1-\alpha}^{1} \sigma(u) du)^{2/p} \) are differentiable on \( (0, 1) \) and that for each \( \alpha \in (0, 1) \)
\[
\left( \int_{\alpha}^{1} \sigma(u) du \right)^{2/p - 1} \cdot \sigma(\alpha) = \left( \int_{1-\alpha}^{1} \sigma(u) du \right)^{2/p - 1} \cdot \sigma(1 - \alpha).
\]

\[9\]
We now assume $p > 2$ so that $\frac{2}{p} - 1 < 0$. Thus, $x \mapsto x^{2/p-1}$ is decreasing on $(0, \infty)$ and it follows that both $\alpha \mapsto \left(\int_{1-\alpha}^{1} \sigma(u)du\right)^{2/p-1}$ and $\alpha \mapsto \sigma(\alpha)$ are nondecreasing (and nonnegative) on $[0, 1)$ while $\alpha \mapsto \left(\int_{1-\alpha}^{1} \sigma(u)du\right)^{2/p-1}$ and $\alpha \mapsto \sigma(1-\alpha)$ are both nonincreasing (and nonnegative) on $[0, 1)$. Thus, the left hand side in (11) is nondecreasing in $\alpha$ while the right hand side is nonincreasing. We conclude that there is $c \in [0, \infty)$ such that
\[
c = \left(\int_{1-\alpha}^{1} \sigma(u)du\right)^{2/p-1} \sigma(1-\alpha)
\]
for all $\alpha \in (0, 1)$. This implies
\[
0 = \lim_{\alpha \downarrow 0} \left(\int_{1-\alpha}^{1} \sigma(u)du\right)^{1-2/p} = \lim_{\alpha \downarrow 0} \sigma(1-\alpha) = \sup_{\beta \in [0,1)} \sigma(\beta),
\]
where we used that $\sigma$ is nondecreasing in the last step. Using again that $\sigma$ is nondecreasing and that $\sigma \geq 0$ we conclude $\sigma = 0$ contradicting $\int_{0}^{1} \sigma(u)du = 1$.

Therefore we have $p \in [1, 2]$ so that $2/p \in [1, \infty)$. Because $\int_{\beta}^{1} \sigma(u)du \leq 1$ for all $\beta \in (0, 1)$ it follows $(\int_{\beta}^{1} \sigma(u)du)^{2/p} \leq \int_{\beta}^{1} \sigma(u)du$ so that by (10) we obtain
\[
1 \leq \int_{1-\alpha}^{1} \sigma(u)du + \int_{\alpha}^{1} \sigma(u)du
\]
for all $\alpha \in (0, 1)$. Because $\sigma$ is nondecreasing we conclude
\[
\int_{0}^{\alpha} \sigma(u)du = 1 - \int_{\alpha}^{1} \sigma(u)du \leq \int_{1-\alpha}^{1} \sigma(u)du \leq \int_{0}^{1} \sigma(1-\alpha)du = \alpha \sigma(1-\alpha)
\]
that is
\[
\frac{1}{\alpha} \int_{0}^{\alpha} \sigma(u)du \leq \sigma(1-\alpha)
\]
for all $\alpha \in (0, 1)$. Therefore,
\[
1 = \lim_{\alpha \uparrow 1} \frac{1}{\alpha} \int_{0}^{\alpha} \sigma(u)du \leq \lim_{\alpha \uparrow 1} \inf_{\alpha \downarrow 1} \sigma(1-\alpha) = \inf_{\beta \in (0,1)} \sigma(\beta).
\]
Since $\sigma$ is nondecreasing and $\int_{0}^{1} \sigma(u)du = 1$ it follows that $\sigma = 1$ on $(0, 1)$. Combining this with (11) yields for all $\alpha \in (0, 1)$
\[
(1-\alpha)^{2/p-1} = \alpha^{2/p-1} \Leftrightarrow 0 = (2/p - 1) \ln \frac{\alpha}{1-\alpha},
\]
which finally proves $p = 2$. \hfill \Box

4 The dual space in the scalar valued case

In this section we are going to determine the dual space of $L^p_k := L^p_k(P) := L^p_k(P, K), K \in \{\mathbb{R}, \mathbb{C}\}$. For $\varphi \in L^p_k(P)^*$ we denote the dual norm of $\varphi$ by $\|\varphi\|_{L^p_k}$. Some of the results presented in this section are inspired by Lorentz [20].
Definition 11. As usual we denote by $L^0(P)$ the set of $\mathbb{K}$-valued random variables on $(\Omega, \mathcal{F}, P)$, where random variables which coincide $P$-almost surely are identified. We define the Köthe dual of $L^p_\sigma(P)$ as

$$L^p_\sigma(P)^\times := \{Z \in L^0(P); \ \forall Y \in L^p_\sigma(P): ZY \in L^1(P)\}.$$ 

Since $L^\infty(P) \subseteq L^p_\sigma(P)$ for all $p \in [1, \infty)$ it follows from taking $Y = \mathbb{1}_{\{Z \neq 0\}} \frac{Z}{|Z|}$ that $Z \in L^1(P)$ whenever $Z \in L^p_\sigma(P)^\times$.

Proposition 12. For every $Z \in L^p_\sigma(P)^\times$

$$\sup\{||E(ZY)||; \ |Y|_{\sigma,p} \leq 1\} < \infty.$$ 

Moreover, 

$$\varphi_Z : L^p_\sigma(P) \to \mathbb{K}, \ \varphi_Z(Y) = E(ZY)$$ 

belongs to $L^p_\sigma(P)^\times$ and 

$$\Phi : L^p_\sigma(P)^\times \to L^p_\sigma(P)^*$$ 

is a linear isomorphism with 

$$\forall Z \in L^p_\sigma(P)^\times: ||\Phi(Z)||_{\sigma,p} = \sup\{||E(ZY)||; \ |Y|_{\sigma,p} \leq 1\}. \quad (12)$$ 

Proof. Obviously, $\varphi_Z$ is a well-defined, linear functional on $L^p_\sigma(P)$ for every $Z \in L^p_\sigma(P)^\times$. The assumption 

$$\infty = \sup\{||E(ZY)||; \ |Y|_{\sigma,p} \leq 1\}$$ 

implies the existence of a sequence $(Y_k)_{k \in \mathbb{N}}$ in the unit ball of $L^p_\sigma(P)$ such that 

$$\forall k \in \mathbb{N}: k^2 \leq |E(ZY_k)| = E(|ZY_k|).$$ 

Because $\tilde{Y}_k := \mathbb{1}_{\{Y_k \neq 0\}} \frac{Z}{|Y_k|} Y_k$ belongs to the unit ball of $L^p_\sigma(P)$, $k \in \mathbb{N}$ the completeness of $L^p_\sigma(P)$ implies that $(\sum_{k=1}^n \frac{1}{k^2} \tilde{Y}_k)_{n \in \mathbb{N}}$ converges in $L^p_\sigma(P)$ to some $Y$. As $Z \in L^p_\sigma(P)^\times$ it follows that $ZY \in L^1(P)$.

But on the other hand, since $L^p_\sigma(P)$ embeds contractively into $L^p(P)$ by theorem 4 it follows that some subsequence $(\sum_{k=1}^{n_l} \frac{1}{k^2} \tilde{Y}_k)_{l \in \mathbb{N}}$ also converges $P$-almost surely to $Y$. Therefore, $P$-almost surely we have

$$ZY = Z \lim_{l \to \infty} (\sum_{k=1}^{n_l} \frac{1}{k^2} \tilde{Y}_k) = \lim_{l \to \infty} \sum_{k=1}^{n_l} \frac{1}{k^2} Z \tilde{Y}_k = \lim_{l \to \infty} \sum_{k=1}^{n_l} \frac{1}{k^2} |Z \tilde{Y}_k| \quad (13)$$

and by an application of the Monotone Convergence Theorem we conclude

$$E(ZY) = E(\lim_{l \to \infty} \sum_{k=1}^{n_l} \frac{1}{k^2} |Z \tilde{Y}_k|) = \lim_{l \to \infty} \sum_{k=1}^{n_l} \frac{1}{k^2} E(|Z \tilde{Y}_k|) \geq \lim_{l \to \infty} \sum_{k=1}^{n_l} 1$$

which contradicts $ZY \in L^1(P)$. Hence,

$$\infty > \sup\{||E(ZY)||; \ |Y|_{\sigma,p} \leq 1\}$$

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that \( \varphi_Z \in L_0^p(P)^* \) with
\[
\| \varphi_Z \|_{\sigma,p} = \sup \{ \| E Y \| : Y \|_{\sigma,p} \leq 1 \}. \tag{14}
\]

This implies that \( \Phi \) is a well-defined linear mapping which satisfies (12). In order to show that \( \Phi \) is injective choose \( Z \in L_0^p(P)^\times \) with \( \Phi(Z) = 0 \). We set \( Y := \mathbb{1}_{\{ \{Z \neq 0 \} \}} \). Since simple functions belong to \( L_0^p(P) \) it follows easily that \( Y \in L_0^p(P) \). It follows
\[
0 = \Phi(Z)(Y) = \mathbb{E}(\| Z \|),
\]
so that \( Z = 0 \).

In order to prove surjectivity of \( \Phi \) let \( \varphi \in L_0^p(P) \). For \( E \in \mathcal{F} \) and \( Y = \mathbb{1}_E \) we have \( F_{|Y|}^{-1} = \mathbb{1}_{(1-P(E),1]} \) so that by (5)
\[
|\varphi(\mathbb{1}_E)| \leq \| \varphi \|_{\sigma,p} \| \mathbb{1}_E \|_{\sigma,p} = \| \varphi \|_{\sigma,p} (\int_{1-P(E)}^1 \sigma(u) du)^{1/p}.
\]

Using this inequality, it is straightforward to show that
\[
\mu : \mathcal{F} \to \mathbb{K}, \; \mu(E) := \varphi(\mathbb{1}_E)
\]
is a complex measure which is \( P \)-continuous, i.e., \( \mu(E) = 0 \) whenever \( P(E) = 0 \). An application of the Radon-Nikodým Theorem yields some \( Z \in L^1(P) \) such that \( \mu(E) = \int_O \mathbb{1}_E Z dP = \mathbb{E}(Z \mathbb{1}_E) \) for all \( E \in \mathcal{F} \). For simple functions \( Y \) it follows \( \varphi(Y) = \mathbb{E}(ZY) \). As soon as we have shown that \( Z \in L_0^p(P)^\times \) it follows from the above and theorem 9 that \( \varphi = \Phi(Z) \).

In order to show \( Z \in L_0^p(P)^\times \) we first observe that \( \alpha Y \in L_0^p(P) \) and \( \| \alpha Y \|_{\sigma,p} \leq \| \alpha \|_\infty \| Y \|_{\sigma,p} \)
for every \( Y \in L_0^p(P) \) and each \( \alpha \in L^\infty(P) \). Therefore, by setting \( E_n := \{ |Z| \leq n \}, \) \( n \in \mathbb{N} \), we have \( \| \mathbb{1}_{E_n} Y \|_{\sigma,p} \leq \| Y \|_{\sigma,p} \) for each \( Y \in L_0^p(P) \) which implies \( \varphi_n \in L_0^p(P)^* \) and \( \| \varphi_n \|_{\sigma,p} \leq \| \varphi \|_{\sigma,p} \)
where \( \varphi_n(Y) := \varphi(\mathbb{1}_{E_n} Y) \). For simple functions \( Y \) we have \( \varphi_n(Y) = \mathbb{E}(Z \mathbb{1}_{E_n} Y) \). Additionally, by Hölder’s inequality and theorem 4 we obtain for arbitrary \( Y \in L_0^p(P) \)
\[
\mathbb{E}(\| Z \mathbb{1}_{E_n} Y \|) \leq n \mathbb{E}(\| Y \|) \leq n \| Y \|_{\sigma,p},
\]
so that \( Z \mathbb{1}_{E_n} \in L_0^p(P)^\times \). Because simple functions are dense in \( L_0^p(P) \) by theorem 9 we conclude from the above \( \Phi(Z \mathbb{1}_{E_n}) = \varphi_n \). Finally, since
\[
\mathbb{E}(\| Z \mathbb{1}_{E_n} Y \|) = |\varphi_n(\mathbb{1}_{\{ Z \neq 0 \}} \frac{Z Y}{\| Z \|} Y)| \leq \| \varphi_n \|_{\sigma,p} \| \mathbb{1}_{\{ Z \neq 0 \}} \frac{Z Y}{\| Z \|} Y \|_{\sigma,p}
\]
\[
\leq \| \varphi \|_{\sigma,p} \| Y \|_{\sigma,p},
\]
it follows with the aid of the Monotone Convergence Theorem that
\[
\mathbb{E}(\| Z Y \|) = \lim_{n \to \infty} \mathbb{E}(\| Z \mathbb{1}_{E_n} Y \|) \leq \| \varphi \|_{\sigma,p} \| Y \|_{\sigma,p}
\]
for each \( Y \in L_0^p(P) \) so that \( Z \in L_0^p(P)^\times \). \qed

Remark 13. With the aid of the fact that for \( \alpha \in L^\infty(P) \) the linear mapping \( Y \mapsto \alpha Y \) is well defined and continuous from \( L_0^p(P) \) into itself it is straightforward to see that \( |Z| \in L_0^p(P)^\times \)
whenever \( Z \in L_0^p(P)^\times \) and that in this case \( \| \varphi_Z \|_{\sigma,p} = \| \varphi_{|Z|} \|_{\sigma,p} \).
We next aim at giving a representation of $L^p_0(P)^\times$ and thus of the dual space of $L^p_\alpha(P)$. For this purpose we introduce the following notion.

**Definition 14.** For a distortion function $\sigma$ we define

$$S_\sigma := S : [0, 1] \rightarrow \mathbb{R}, \quad S_\sigma(\alpha) = \int_{\alpha}^{1} \sigma(u) du.$$  

**Remark 15.** Obviously, $S$ is a continuous, nonincreasing function with $S(0) = 1, S(1) = 0$. If we set $u_0 := \inf\{ u > 0; \sigma(u) > 0 \}$ we have $u_0 < 1$, $S_{|[0,u_0]} = 1$ and $S_{|[u_0,1]}$ is an increasing bijection from $[u_0, 1]$ to $[0, 1]$. By abuse of notation we denote the inverse of $S_{|[u_0,1]}$ by $S^{-1}$.

For $\alpha_1, \alpha_2 \in [0, 1], \alpha_1 < \alpha_2$ and $\lambda \in (0, 1)$ it follows from the fact that $\sigma$ is nondecreasing that

$$S(\lambda_1 + (1-\lambda)\alpha_2) - S(\alpha_1) = -\int_{\alpha_1}^{\lambda_1+\lambda(\alpha_2-\alpha_1)} \sigma(u) du$$

and

$$S(\alpha_2) - S(\lambda_1 + (1-\lambda)\alpha_2) = -\int_{\alpha_1}^{\lambda_1+\lambda(\alpha_2-\alpha_1)} \sigma(u) du$$

so that

$$\frac{S(\lambda_1 + (1-\lambda)\alpha_2) - S(\alpha_1)}{(1-\lambda)(\alpha_2 - \alpha_1)} \geq -\sigma(\lambda_1 + (1-\lambda)\alpha_2)$$

which implies $S(\lambda_1 + (1-\lambda)\alpha_2) \geq \lambda S(\alpha_1) + (1 - \lambda) S_{|[0,1]}$. In particular, $S$ is differentiable from the left and from the right on $(0, 1]$, $(0, 1)$, resp.) and since $\sigma$ is continuous from the left it is straightforward to show that for the left derivative we have $S'_{|[0,1]}(\alpha) = -\sigma(u), u \in (0, 1]$.

Recall that for a non-negative random variable $Z$ the average value-at-risk of level $\alpha \in [0, 1)$ is defined as $AV@R_\alpha(Z) = \frac{1}{1-\alpha} \int_{\alpha}^{1} F_{Z}^{-1}(u) du$.

**Definition 16.** For a distortion function $\sigma$, $Z \in L^0(P)$, and $\alpha \in [0, 1)$ we define

$$|Z|_{\sigma, \infty} := \sup_{\alpha \in [0,1)} \frac{AV@R_\alpha(|Z|)}{1-\alpha} S_{\sigma}(\alpha) = \sup_{\alpha \in (0,1)} \frac{\int_{\alpha}^{1} F^{-1}_{|Z|}(u) du}{\int_{\alpha}^{1} \sigma(u) du}.$$  

(15)

Moreover, we say that $Z' \in L^0(P)$ $\sigma$-dominates $Z$ (in symbols $Z' \succ \sigma Z$) if there is a uniform random variable $U \in \mathcal{U}(0, 1)$ such that

$$AV@R_\alpha(\sigma(U)|Z'|) \geq AV@R_\alpha(|Z|) \text{ for all } \alpha < 1.$$  

(16)
Further we define, for \( p \in (1, \infty) \),
\[
|Z|^{*}_{\sigma,q} := \inf \left\{ \|Z'\|_{\sigma,q} : Z' \succ \succ Z \right\}
\]  
(17)
where \( q \in (1, \infty) \) is the conjugate exponent to \( p \), i.e., \( 1/p + 1/q = 1 \) and where as usual inf \( \emptyset := \infty \).

Finally, for \( p \in [1, \infty) \) with conjugate exponent \( q \), i.e. \( 1/p + 1/q = 1 \), we set \( L^{*}_{\sigma,q}(P) := \{ Z \in L^{0}(P) : |Z|^{*}_{\sigma,q} < \infty \} \) (and we identify random variables which coincide \( P \)-almost everywhere).

From the definition of quantile functions it follows for \( Z_{1}, Z_{2} \in L^{0}(P) \) with \( |Z_{1}| \leq |Z_{2}| \) that \( F^{-1}_{|Z_{1}|} \leq F^{-1}_{|Z_{2}|} \) which implies \( |Z_{1}|^{*}_{\sigma,q} \leq |Z_{2}|^{*}_{\sigma,q} \). Since also \( F^{-1}_{|\alpha Z_{1}|} = |\alpha| F^{-1}_{|Z_{1}|} \) for \( \alpha \in \mathbb{K} \) it follows also \( |\alpha Z_{1}|^{*}_{\sigma,q} = |\alpha| |Z_{1}|^{*}_{\sigma,q} \). Since \( \text{AV@R}_{\alpha} \) is subadditive (cf. Pflug and Römisch [24]) it follows easily that \( L^{*}_{\sigma,q}(P) \) is a subspace of \( L^{0}(P) \).

**Remark 17** (Stochastic dominance of second order). The definition of \( |\cdot|^{*}_{\sigma,\infty} \) reflects the duality of risk functionals. Indeed, the supremum (15) can be restated as
\[
|Z|^{*}_{\sigma,\infty} = \inf \left\{ \eta \geq 0 : \text{AV@R}_{\alpha}(|Z|) \leq \frac{\eta}{1-\alpha} \cdot \int_{\alpha}^{1} \sigma(u)du \text{ for all } \alpha < 1 \right\}.
\]
By the rearrangement inequality (cf. McNeil et al. [21, Theorem 5.25(2)]) this equivalent formulation involves the statement
\[
\text{AV@R}_{\alpha}(|Z|) \leq \text{AV@R}_{\alpha}(\eta \sigma(U)),
\]
(18)
where \( U \in \mathcal{U}(0,1) \). Choosing \( U \) to be coupled in a comonotone way with \( |Z| \) it follows
\[
|Z|^{*}_{\sigma,\infty} = \inf \left\{ \eta \geq 0, U \in \mathcal{U}(0,1) : \text{AV@R}_{\alpha}(|Z|) \leq \text{AV@R}_{\alpha}(\eta \sigma(U)) \text{ for all } \alpha < 1 \right\}
\]
Following Ogryczak and Ruszczyński [22], (18) is equivalent to saying that \( |Z| \) is dominated by \( |Z|^{*}_{\sigma,\infty} \cdot \sigma(U) \) in second stochastic order.\footnote{Cf. Dentcheva and Ruszczyński [6, 7, 8] for stochastic dominance of second order.}

**Remark 18.**

i) By the choice \( \alpha = 0 \) in (15) it follows that
\[
|Z|^{*}_{\sigma,\infty} \geq \text{AV@R}_{0}(|Z|) = \int_{0}^{1} F^{-1}_{|Z|}(u)du = \mathbb{E}[|Z|] = \|Z\|_{1},
\]
(19)
so that \( L^{*}_{\sigma,\infty}(P) \subseteq L^{1}(P) \).

ii) Since for \( Z, Z' \in L^{0}(P) \) with \( Z' \succ \succ Z \) we have for \( p \in [1, \infty) \) with proposition 6
\[
\|Z\|_{1} = \int_{0}^{1} F^{-1}_{|Z|}(u)du \leq \int_{0}^{1} \sigma(u)F^{-1}_{|Z|}(u)du = \|Z'\|_{\sigma,1} \leq \|Z'\|_{\sigma,p}
\]
it also follows that \( L^{*}_{\sigma,q}(P) \subseteq L^{1}(P) \).

**Proposition 19.** For \( Z \in L^{0}(P) \) we have
\[
|Z|^{*}_{\sigma,\infty} = \inf \{ \eta \geq 0; \forall F : [0,1) \to [0,\infty) \text{ nondecreasing:} \}
\]
\[
\int_{0}^{1} F^{-1}_{|Z|}(u)F(u)du \leq \eta \int_{0}^{1} \sigma(u)F(u)du,
\]

\footnote{Cf. Dentcheva and Ruszczyński [6, 7, 8] for stochastic dominance of second order.}
and for \( p \in (1, \infty) \)
\[
|Z|_{\sigma,q}^* = \inf\{\|Z\|_{\sigma,q} : Z' \in L^p(\mathcal{P}) \text{ such that } \forall F : [0, 1) \to [0, \infty) \text{ nondecreasing:}
\]
\[
\int_0^1 F_{|Z|}^{-1}(u)F(u)du \leq \int_0^1 \sigma(u)F_{|Z|}^{-1}(u)F(u)du,
\]
where as usual \( 1/p + 1/q = 1 \).

**Proof.** Since \( \mathbb{1}_{[\alpha, 1]} \) is a nondecreasing, non-negative function for every \( \alpha \in [0, 1) \) it follows
\[
\{ \eta \geq 0; \forall F : [0, 1) \to [0, \infty) \text{ nondecreasing: } \int_0^1 F_{|Z|}^{-1}(u)F(u)du \leq \eta \int_0^1 \sigma(u)F(u)du \}
\]
\[
\subseteq \{ \eta \geq 0; \forall \alpha \in [0, 1) : \int_0^1 F_{|Z|}^{-1}(u)du \leq \eta \int_1^1 \sigma(u)du \}.
\]

On the other hand, if for some \( \eta \geq 0 \) we have
\[
\forall \alpha \in [0, 1) : \int_0^1 F_{|Z|}^{-1}(u)du \leq \eta \int_0^1 \sigma(u)du
\]
it follows for all \( \gamma_1, \ldots, \gamma_n \in [0, \infty) \) and every choice of \( \alpha_1 < \ldots < \alpha_n \in [0, 1) \) that
\[
\int_0^1 F_{|Z|}^{-1}(u) \sum_{j=1}^n \gamma_j \mathbb{1}_{[\alpha_j, 1]}(u)du \leq \eta \int_0^1 \sigma(u) \sum_{j=1}^n \gamma_j \mathbb{1}_{[\alpha_j, 1]}(u)du.
\]
Since every non-negative, nondecreasing function \( F : [0, 1) \to [0, \infty) \) is the pointwise limit of a non-decreasing sequence of such step functions \( \sum_{j=1}^n \gamma_j \mathbb{1}_{[\alpha_j, 1]} \) it follows from the Monotone Convergence Theorem that
\[
\int_0^1 F_{|Z|}^{-1}(u)F(u)du \leq \eta \int_0^1 \sigma(u)F(u)du
\]
for all such \( F \). Hence it also holds
\[
\{ \eta \geq 0; \forall F : [0, 1) \to [0, \infty) \text{ nondecreasing: } \int_0^1 F_{|Z|}^{-1}(u)F(u)du \leq \eta \int_0^1 \sigma(u)F(u)du \}
\]
\[
\sup \{ |\mathbb{E}(ZY)| : Y \in L^p_\sigma(\mathcal{P}), \|Y\|_{\sigma,p} \leq 1 \} \leq |Z|_{\sigma,q}^*
\]
(20)
for every \( Z \in L^*_\sigma,q(\mathcal{P}) \), where \( q \) is the conjugate exponent to \( p \).

**Proposition 20.** For a distortion function \( \sigma \) and \( p \in [1, \infty) \) we have \( L^*_\sigma,q(\mathcal{P}) \subseteq L^p_\sigma(\mathcal{P}) \)× and
\[
\sup \{ |\mathbb{E}(ZY)| : Y \in L^p_\sigma(\mathcal{P}), \|Y\|_{\sigma,p} \leq 1 \} \leq |Z|_{\sigma,q}^*
\]
(20)
for every \( Z \in L^*_\sigma,q(\mathcal{P}) \), where \( q \) is the conjugate exponent to \( p \).

**Proof.** Let \( p = 1 \). For \( Z \in L^*_\sigma,\infty(\mathcal{P}) \) it follows for arbitrary \( Y \in L^1_\sigma(\mathcal{P}) \) from the rearrangement inequality combined with proposition 19
\[
\mathbb{E}(|ZY|) \leq \int_0^1 F_{|Z|}^{-1}(u)F_{|Y|}^{-1}(u)du \leq |Z|_{\sigma,\infty}^* \int_0^1 \sigma(u)F_{|Y|}^{-1}(u)du
\]
\[
= |Z|_{\sigma,\infty}^* \|Y\|_{\sigma,1}.
\]

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Hence, \( Z \in L^1_\sigma(P)^* \) and the above inequality also implies that \( |Z|_{\sigma,\infty} \) is an upper bound for 
\[ \sup\{\|E(ZY)\|; \ Y \in L^1_\sigma(P), \|Y\|_{\sigma,1} \leq 1\}. \]

Next let \( p \in (1, \infty) \) and \( q \) the corresponding conjugate exponent. For \( Z \in L_{\sigma,q}^*(P) \) let \( Z' \in L^p_\sigma(P) \) with \( Z'_{\sigma} \succ Z \). For arbitrary \( Y \in L^1_\sigma(P) \) it follows from the rearrangement inequality combined with proposition 19 and Hölder’s inequality

\[
\mathbb{E}(|ZY|) \leq \int_0^1 F^{-1}_{|Z|}(u)F^{-1}_Y(u)du \leq \int_0^1 \sigma(u)F^{-1}_{|Z|}(u)F^{-1}_{|Y|}(u)du
\]

\[
\leq \|Z'\|_{\sigma,q}\|Y\|_{\sigma,p}.
\]

Thus, \( Z \in L^p_\sigma(P)^* \) and because \( Z' \in L^p_\sigma(P) \) with \( Z'_{\sigma} \succ Z \) was chosen arbitrarily, it follows that 
\[ \sup\{\|E(ZY)\|; \ Y \in L^1_\sigma(P), \|Y\|_{\sigma,p} \leq 1\} \] is bounded by \( |Z|_{\sigma,q}^* \).

In order to show that in fact \( L_{\sigma,q}^*(P) = L^p_\sigma(P)^* \) holds as well as equality in inequality (20) we have to distinguish the cases \( p = 1 \) and \( p \in (1, \infty) \). We begin with the case \( p = 1 \).

**Proposition 21.** For a \( \mathbb{K} \)-valued random variable \( Z \) on \( (\Omega, \mathcal{F}, P) \) and \( \alpha \in [0, 1) \) there is \( E_\alpha \in \mathcal{F} \) such that \( P(E_\alpha) = 1 - \alpha \) and

\[
\mathcal{A}\mathcal{V}@\mathcal{R}_\alpha(|Z|) = \frac{1}{1 - \alpha} \mathbb{E}(|Z|\mathbb{1}_{E_\alpha}).
\]

**Proof.** Let \( E \in \mathcal{F} \) with

\[
\{Z > F^{-1}_{|Z|}(\alpha)\} \subseteq E \subseteq \{Z \geq F_{|Z|}(\alpha)\}
\]

be arbitrary. Denoting the positive part of a \( \mathbb{R} \)-valued function \( f \) as usual by \( f_+ \) it follows

\[
\mathcal{A}\mathcal{V}@\mathcal{R}_\alpha(|Z|) = \frac{1}{1 - \alpha} \int_0^1 F^{-1}_{|Z|}(u)du = F^{-1}_{|Z|}(\alpha) + \frac{1}{1 - \alpha} \int_0^1 (F^{-1}_{|Z|}(u) - F^{-1}_{|Z|}(\alpha))_+ du
\]

\[
= F^{-1}_{|Z|}(\alpha) + \frac{1}{1 - \alpha} \mathbb{E}(\langle |Z| - F^{-1}_{|Z|}(\alpha)\rangle \mathbb{1}_E)
\]

\[
= F^{-1}_{|Z|}(\alpha) + \frac{1}{1 - \alpha} \mathbb{E}(\|Z\|_E) - \frac{1}{1 - \alpha} F^{-1}_{|Z|}(\alpha)P(E)
\]

\[
= \frac{1}{1 - \alpha} \mathbb{E}(\|Z\|_E) + (1 - \frac{1}{1 - \alpha} P(E))F^{-1}_{|Z|}(\alpha).
\]

From the definition of \( F^{-1}_{|Z|} \) it follows immediately that

\[
P(|Z| < F^{-1}_{|Z|}(\alpha)) \leq \alpha \leq P(|Z| \leq F^{-1}_{|Z|}(\alpha))
\]

so that

\[
P(|Z| > F^{-1}_{|Z|}(\alpha)) \leq 1 - \alpha \quad \text{and} \quad P(|Z| \geq F^{-1}_{|Z|}(\alpha)) \geq 1 - \alpha.
\]

Let \( U \) be a \([0, 1]\)-valued, uniformly distributed random variable on \((\Omega, \mathcal{F}, P)\). We define

\[
E_\beta := \{Z > F^{-1}_{|Z|}(\alpha)\} \cup \left(\{Z = F^{-1}_{|Z|}(\alpha)\} \cap \{U \in [0, \beta]\}\right)
\]

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for $\beta \in [0, 1]$ and set 
\[ f : [0, 1] \to [0, 1], f(\beta) := P(E_\beta). \]
From the properties of a probability measure it follows easily that $f$ is continuous as well as 
\[ f(0) = P(|Z| > F_{|Z|}^{-1}(\alpha)) \leq 1 - \alpha \text{ and } f(1) = P(|Z| \geq F_{|Z|}^{-1}(\alpha)) \geq 1 - \alpha. \]
Hence, there is $\beta_0 \in [0, 1]$ such that for $E_{\beta_0}$ we have $P(E_{\beta_0}) = 1 - \alpha$ and it follows from (21) that $E_{\beta_0}$ has the desired property.

For the case $p = 1$ we can now give the desired intrinsic description of $L_{\sigma}^s(P)^\times$.

**Proposition 22.** For a distortion function $\sigma$ it holds $L_{\sigma}^s(P)^\times = L_{\sigma,\infty}^s(P)$ and for every $Z \in L_{\sigma}^1(P)^\times$ we have 
\[ |Z|_{\sigma,\infty}^s = \sup\{|E(ZY)|; Y \in L_{\sigma}^1(P), \|Y\|_{\sigma,1} \leq 1\}. \]

**Proof.** Let $Z \in L_{\sigma}^1(P)^\times$. By proposition 21, for any $\alpha \in [0,1)$ there is $E_\alpha \in \mathcal{F}$ such that $AV@R_\alpha(|Z|) = \frac{1}{1 - \alpha}E(|Z|1_{E_\alpha})$ and $P(E_\alpha) = 1 - \alpha$. Employing the notation from proposition 12 we obtain 
\[ AV@R_\alpha(|Z|) = |\phi Z(1 - \alpha)(Z \neq 0)\frac{Z}{Z}1_{E_\alpha}| \leq \|\phi Z\|_{\sigma,1}^s \frac{1}{1 - \alpha}1_{E_\alpha}\|Z\|_{\sigma,1} \]
\[ = \|\phi Z\|_{\sigma,1}^s \frac{1}{1 - \alpha} \int_0^1 \sigma(u)F_{1-P(\sigma)}^{-1}(u)du \]
\[ = \|\phi Z\|_{\sigma,1}^s \frac{1}{1 - \alpha} \int_{1-P(E_\alpha)}^1 \sigma(u)du = \|\phi Z\|_{\sigma,1}^s \frac{1}{1 - \alpha}S_\alpha(\alpha), \]
so that $|Z|_{\sigma,\infty}^s$ is finite and bounded above by $\|\phi Z\|_{\sigma,1}^s = \sup\{|E(ZY)|; Y \in L_{\sigma}^1(P), \|Y\|_{\sigma,1} \leq 1\}$. Proposition 20 now yields the rest of the claim.

Combining propositions 12 and 22 we immediately derive the next result.

**Theorem 23.** Let $\sigma$ be a distortion function. Then $|\cdot|_{\sigma,\infty}^s$ is a norm on $L_{\sigma,\infty}^s(P)$ turning it into a Banach space. Moreover,
\[ \Phi : (L_{\sigma,\infty}^s(P), |\cdot|_{\sigma,\infty}^s) \to \left(L_{\sigma}^1(P), \|\cdot\|_{\sigma,1}^s\right), \quad Z \mapsto (Y \mapsto \Phi(Z)(Y) := E(ZY)) \]
is an isometric isomorphism.

In order to derive an analogous representation for the case $p \in (1, \infty)$ we need an equivalent result to 22 for this case. This requires some preparation. We begin by recalling a notion from [20].

**Definition 24.** Let $\sigma$ be a distortion function. A function $H : [0, 1] \to \mathbb{R}$ is called $S_\sigma$-concave if whenever $y, b \in \mathbb{R}$ are such that $yS_\sigma(\alpha_1) + b = H(\alpha_1)$ and $yS_\sigma(\alpha_2) + b = H(\alpha_2)$ for some $\alpha_1 < \alpha_2 \in [0, 1]$ then $H(\alpha) \geq yS(\alpha) + b$ for each $\alpha \in [\alpha_1, \alpha_2]$.

The next proposition is essentially contained in [20, Proof of Theorem 3.6.2]. Nevertheless, we include its proof for the reader’s convenience.
Proposition 25. Let $\sigma$ be a distortion function and let $u_0 := \inf\{u > 0; \sigma(u) > 0\}$. Moreover, let $\mathcal{H}$ be a set of $S_\sigma$-concave functions such that $H_{|[0,u_0]}$ is constant for every $H \in \mathcal{H}$. Assume that
\[\forall \alpha \in [0,1]: F(\alpha) := \inf\{H(\alpha); H \in \mathcal{H}\} > -\infty.\]
Then, $F$ is $S_\sigma$-concave.

Proof. Let $y, b \in \mathbb{R}$ and let $\alpha_1 < \alpha_2 \in [0,1]$ such that $yS_\sigma(\alpha_j) + b = F(\alpha_j), j = 1, 2$. Let $H \in \mathcal{H}$ be arbitrary. Since $H_{|[0,u_0]}$ is constant there are $\tilde{y}, \tilde{b} \in \mathbb{R}$ such that $\tilde{y}S_\sigma(\alpha_j) + \tilde{b} = H(\alpha_j), j = 1, 2$.

In case of $y - \tilde{y} \geq 0$ it follows that $\tilde{y}S_\sigma + \tilde{b} - (yS_\sigma + b)$ is nonincreasing while $\tilde{y}S_\sigma + \tilde{b} - (yS_\sigma + b)$ is nondecreasing in case of $y - \tilde{y} \leq 0$. Therefore, $S_\sigma$-concavity of $H$ together with $yS_\sigma(\alpha_j) + b \leq \tilde{y}S_\sigma(\alpha_j) + \tilde{b}, j = 1, 2$ implies
\[\forall \alpha \in [\alpha_1, \alpha_2]: yS_\sigma(\alpha) + b \leq \tilde{y}S_\sigma(\alpha) + \tilde{b} \leq H(\alpha).\]
Since $H \in \mathcal{H}$ was arbitrary, we conclude that $F \geq yS_\sigma + b$ on $[\alpha_1, \alpha_2]$.

Proposition 26. Let $\sigma$ be a distortion function, $u_0 := \inf\{u > 0; \sigma(u) > 0\}$, and let $y_1, y_2, b_1, b_2 \in \mathbb{R}$. Moreover, let $H$ be $S_\sigma$-concave, continuous from the right in $u_0$ such that $H_{|[0,u_0]}$ is constant.

If for $y, b \in \mathbb{R}$ and $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 < \alpha_2$ and $u_0 < \alpha_2$ we have $H(\alpha_j) = yS_\sigma(\alpha_j) + b, j = 1, 2$, then it follows that $yS_\sigma + b \geq H$ on $[0,1] \setminus (\{\alpha_1, \alpha_2\}$.

Proof. It is straightforward to show that if for $\alpha, \beta \in [0,1], \alpha < \beta$, it holds $y_1S_\sigma(\alpha) + b_1 = y_2S_\sigma(\alpha) + b_2$ and $y_1S_\sigma(\beta) + b_1 > y_2S_\sigma(\beta) + b_2$ then $\beta > u_0$ and
\[\forall \gamma \in (\max\{\alpha, u_0\}, \beta]: y_1S_\sigma(\gamma) + b_1 > y_2S_\sigma(\gamma) + b_2 \quad (22)\]
while for $\alpha, \beta \in [u_0,1], \alpha < \beta$, the conditions $y_1S_\sigma(\alpha) + b_1 > y_2S_\sigma(\alpha) + b_2$ and $y_1S_\sigma(\beta) + b_1 = y_2S_\sigma(\beta) + b_2$ imply
\[\forall \gamma \in [\alpha, \beta]: y_1S_\sigma(\gamma) + b_1 > y_2S_\sigma(\gamma) + b_2. \quad (23)\]

In case of $\alpha_1 \leq u_0$ it follows from the hypothesis that $H_{|[0,u_0]}$ is constant that trivially $yS + b \geq H$ on $[0, \alpha_1]$. Now let $u_0 < \alpha_1$. We assume that $yS(\alpha) + b < H(\alpha)$ for some $\alpha \in (u_0, \alpha_1)$. Because $S_\sigma$ is strictly decreasing in $[u_0,1]$ there are $\tilde{y}, \tilde{b} \in \mathbb{R}$ such that
\[\tilde{y}S_\sigma(\alpha) + \tilde{b} = H(\alpha) \text{ and } \tilde{y}S_\sigma(\alpha_2) + \tilde{b} = H(\alpha_2).\]
The $S_\sigma$-concavity of $H$ hence implies $H \geq \tilde{y}S_\sigma + \tilde{b}$ on $[\alpha, \alpha_2]$. In particular
\[yS_\sigma(\alpha_1) + b = H(\alpha_1) \geq \tilde{y}S_\sigma(\alpha_2) + \tilde{b}. \quad (24)\]
On the other hand
\[\tilde{y}S_\sigma(\alpha) + \tilde{b} = H(\alpha) > yS_\sigma(\alpha) + b \text{ and } \tilde{y}S_\sigma(\alpha_2) + \tilde{b} = H(\alpha_2) = yS_\sigma(\alpha_2) + b\]
so that by (23) we obtain $\tilde{y}S_\sigma + \tilde{b} > yS_\sigma + b$ on $[\alpha, \alpha_2)$ which contradicts (24). Therefore $yS + b \geq H$ on $(u_0, \alpha_1)$. Since $S_\sigma$ is continuous and $H$ is continuous from the right in $u_0$ the same inequality holds on $[u_0, \alpha_1]$. Because $S_\sigma$ and $H$ are constant on $[0,u_0]$ we obtain $yS + b \geq H$ on $[0, \alpha_1]$.

It remains to show that $yS + b \geq H$ on $[\alpha_2,1]$ as well. Assume there is $\alpha \in [\alpha_2,1]$ with $yS_\sigma(\alpha) + b < H(\alpha)$. Since $\alpha_2 > u_0$ there are again $\tilde{y}, \tilde{b} \in \mathbb{R}$ such that
\[\tilde{y}S_\sigma(\alpha_1) + \tilde{b} = H(\alpha_1) \text{ and } yS(\alpha) + b = H(\alpha).\]
Because $H$ is $S_\sigma$-concave this implies
\[ \forall \beta \in [\alpha_1, \alpha] : H(\beta) \geq \tilde{y} S_\sigma(\beta) + \tilde{b}. \tag{25} \]

On the other hand
\[ \tilde{y} S_\sigma(\alpha_1) + \tilde{b} = H(\alpha_1) = y S_\sigma(\alpha_1) + b \text{ and } \tilde{y} S_\sigma(\alpha) + \tilde{b} = H(\alpha) > y S_\sigma(\alpha) + b. \]

By (22) it follows $\tilde{y} S_\sigma + \tilde{b} > y S_\sigma + b$ on $(\max\{\alpha_1, u_0\}, \alpha]$. In particular
\[ y S_\sigma(\alpha_2) + \tilde{b} > y S_\sigma(\alpha_2) + b = H(\alpha_2) \]
which contradicts (25).

\[ \square \]

**Definition 27.** Let $\sigma$ be a distortion function. For a continuous function $G : [0, 1] \to \mathbb{R}$ we define
\[ G_\sigma^* : [0, \infty) \to \mathbb{R}, G_\sigma^*(y) := \inf_{\alpha \in [0, 1]} y S_\sigma(\alpha) - G(\alpha). \]

Then $G(\alpha) + G_\sigma^*(y) \leq y S_\sigma(\alpha)$ for all $\alpha \in [0, 1], y \geq 0$ so that
\[ G_\sigma : [0, 1] \to \mathbb{R}, G_\sigma(\alpha) := \inf_{y \geq 0} y S_\sigma(\alpha) - G_\sigma^*(y) \]
is well-defined and satisfies $G_\sigma \geq G$.

**Remark 28.**

i) Setting as before $u_0 := \inf\{u > 0; \sigma(u) > 0\}$ we have that $S_\sigma[u_0, 1]$ is a bijection from $[u_0, 1]$ onto $[0, 1]$. Denoting by abuse of notation its inverse with $S^{-1}$ it follows
\[ \forall \alpha \in [0, 1] : G_\sigma(S^{-1}(\alpha)) = \inf_{y \geq 0} y \alpha - G_\sigma^*(y) \]
so that
\[ \tilde{G}_\sigma : [0, 1] \to \mathbb{R}, \tilde{G}_\sigma(\alpha) := \inf_{y \geq 0} y \alpha - G_\sigma^*(y) \]
is well-defined. Being the infimum of nondecreasing and concave functions $\tilde{G}_\sigma$ is nondecreasing and concave, too. Therefore, $\tilde{G}_\sigma$ is differentiable from the right on $[0, 1)$ with non-negative and nonincreasing right derivative. We obviously have $G_\sigma = \tilde{G}_\sigma \circ S_\sigma$ so that the concavity of $S_\sigma$ implies that $G_\sigma$ is concave, too. Denoting left and right derivatives by $\frac{d}{d\alpha}$ and $\frac{d}{d\alpha}$ respectively, an appropriate adaption of your favorite proof of the chain rules yields
\[ \forall \alpha \in (0, 1] : \frac{d}{d\alpha} G_\sigma(\alpha) = \frac{d}{d\alpha} \tilde{G}_\sigma(S_\sigma(\alpha)) \frac{d}{d\alpha} S_\sigma(\alpha) = - \frac{d}{d\alpha} \tilde{G}_\sigma(S_\sigma(\alpha)) \sigma(\alpha). \]

Combined with $G_\sigma(1) - G_\sigma(\alpha) = \int_{\alpha}^{1} \frac{d}{d\alpha} G_\sigma(u) du$ we obtain
\[ \forall \alpha \in [0, 1] : G_\sigma(\alpha) = G_\sigma(1) + \int_{\alpha}^{1} H(u) \sigma(u) du \tag{26} \]
for a non-negative, nondecreasing function $H$ on $[0, 1]$ which is continuous from the left.
ii) For \( y \geq 0 \) the function \( yS_\sigma - G^*_\sigma(y) \) is obviously \( S_\sigma \)-concave. It therefore follows from proposition 25 that \( G_\sigma \) is \( S_\sigma \)-concave. Moreover, being the infimum of nonincreasing functions \( G_\sigma \) is nonincreasing.

iii) Because \( S_\sigma\{0, u_0\} = 1 \) it follows that \( G_\sigma \) is constant on \( [0, u_0] \). Hence, for all \( \alpha_1, \alpha_2 \in [0, 1] \), \( \alpha_1 < \alpha_2 \), there are \( y \geq 0, b \geq G^*_\sigma(y) \) with \( G_\sigma(\alpha_j) = yS_\sigma(\alpha_j) - b \) for \( j = 1, 2 \).

Indeed, if \( \alpha_2 > u_0 \) we choose \( y = \frac{G_\sigma(\alpha_1) - G_\sigma(\alpha_2)}{S_\sigma(\alpha_2) - S_\sigma(\alpha_1)} \) which is well-defined and non-negative because \( S_\sigma \) is strictly decreasing on \( [u_0, 1] \) and \( G_\sigma \) is nonincreasing. Then

\[
b = yS_\sigma(\alpha_2) - G_\sigma(\alpha_2) \geq \inf_{\alpha \in [0, 1]} yS_\sigma(\alpha) - G_\sigma(\alpha)
\]

\[
= \inf_{\alpha \in [0, 1]} yS_\sigma(\alpha) - (\inf_{y \geq 0} G^*_\sigma(y)) \geq G^*_\sigma(y).
\]

In case of \( \alpha_2 \leq u_0 \) we may choose \( y = 0 \) so that

\[
b = -G_\sigma(\alpha_2) = -G_\sigma(0) = -\inf_{y \geq 0} y - G^*_\sigma(y) \geq G^*_\sigma(0).
\]

If additionally \( G \) is nonincreasing it holds

\[
G_\sigma(0) = \inf_{y \geq 0} y - G^*_\sigma(y) \leq G^*_\sigma(0) = -\inf_{\alpha \in [0, 1]} (-G(\alpha)) = \sup_{\alpha \in [0, 1]} G(\alpha) = G(0)
\]

so that because \( G \leq G_\sigma \) we conclude

\[
b = -G_\sigma(0) = -G(0) = G^*_\sigma(0).
\]

**Proposition 29.** Let \( G : [0, 1] \rightarrow \mathbb{R} \) be continuous and nonincreasing. If \( \alpha \in (0, 1) \) is such that \( G_\sigma(\alpha) > G(\alpha) \) then there are \( 0 \leq \alpha_1 < \alpha < \alpha_2 \leq 1 \) and \( y \geq 0 \) such that

\[
\forall \beta \in (\alpha_1, \alpha_2) : G_\sigma(\beta) = yS_\sigma(\beta) - G^*_\sigma(y).
\]

**Proof.** By continuity of \( S_\sigma \) and \( G_\sigma \) there are \( 0 \leq \alpha_1 < \alpha < \alpha_2 \leq 1 \) such that \( G_\sigma(\alpha_2) > G(\alpha_1) \). From remark 28 iii) we conclude the existence of \( y \geq 0 \) and \( b \geq G^*_\sigma(y) \) such that \( G_\sigma(\alpha_j) = yS_\sigma(\alpha_j) - b, j = 1, 2 \) and such that \( y = 0 \) in case of \( \alpha_2 \leq u_0 \).

Because \( S_\sigma \) and \( G_\sigma \) are nonincreasing and \( y \geq 0 \) it follows from

\[
\inf_{\beta \in (\alpha_1, \alpha_2)} G_\sigma(\beta) = G_\sigma(\alpha_2) > G(\alpha_1) = \sup_{\beta \in (\alpha_1, \alpha_2)} G(\beta)
\]

that \( yS_\sigma - b \geq G \) on \( (\alpha_1, \alpha_2) \).

If \( \alpha_2 \leq u_0 \) we use have seen in remark iii) that without loss of generality we may assume \( y = 0 \) and \( b = -G(0) \). Since \( G \) is nonincreasing it thus follows \( yS_\sigma - b = G(0) \geq G \) on \( [0, 1] \).

If \( \alpha_2 > u_0 \) we apply proposition 26 to \( G_\sigma \) to conclude \( yS_\sigma - b \geq G_\sigma \) on \( [0, 1] \). Since \( G_\sigma \geq G \) and \( yS_\sigma - b \geq G \) on \( (\alpha_1, \alpha_2) \) we obtain also in this case \( yS_\sigma - b \geq G \) on \( [0, 1] \).

So in both cases \( yS_\sigma - b \geq G \) or equivalently \( yS_\sigma - G \geq b \) on \( [0, 1] \) so that \( G^*_\sigma(y) = \inf_{\alpha} yS_\sigma(\alpha) - G(\alpha) \geq b \). Since also \( b \geq G^*_\sigma(y) \) it follows \( b = G^*_\sigma(y) \).

Finally, since \( G_\sigma \) is \( S_\sigma \)-concave and \( G_\sigma(\alpha_j) = yS_\sigma(\alpha_j) - G^*_\sigma(\alpha_j) \) holds for \( j = 1, 2 \) it follows for \( \beta \in (\alpha_1, \alpha_2) \)

\[
yS_\sigma(\beta) - G^*_\sigma(y) \leq G_\sigma(\beta) = \inf_{\bar{y} \geq 0} yS_\sigma(\beta) - G^*_\sigma(\bar{y}) \leq yS_\sigma(\beta) - G^*_\sigma(y)
\]

which proves the claim. \( \square \)
Proposition 30. Assume that $G : [0, 1] \to \mathbb{R}$ is continuous, nonincreasing, and that $G(1) = 0$. Then $G_\sigma(0) = G(0)$ and $G_\sigma(1) = 0$.

Proof. We already observed in remark 28 iii) that $G_\sigma(0) = G(0)$. Using the compactness of $[0, 1]$, $G(1) = 0$, and that $S_\alpha(\alpha) = 0$ implies $\alpha = 1$ it follows

$$\forall n \in \mathbb{N} \exists k_n \in \mathbb{N} \forall \alpha \in [0, 1] : G(\alpha) > k_n S_\sigma(\alpha) + \frac{1}{n}$$

which implies that for every $n \in \mathbb{N}$ there is $k_n \in \mathbb{N}$ with $-1/n \leq G_\sigma(k_n)$. Because $S_\alpha(1) = 0$ we derive

$$G_\sigma(1) = \inf_{y \geq 0} (-G_\sigma(y)) \leq \inf_{n \in \mathbb{N}} (-G_\sigma(k_n)) \leq 0 = G(1) \leq G_\sigma(1)$$

which gives $G_\sigma(1) = 0$. \qed

Combining propositions 29 and 30 we immediately obtain the next result.

Proposition 31. Let $G : [0, 1] \to \mathbb{R}$ be continuous and nonincreasing such that $G(1) = 0$. If $\alpha \in (0, 1)$ satisfies $G_\sigma(\alpha) > G(\alpha)$ there are $0 \leq \alpha_1 < \alpha < \alpha_2 \leq 1$ and $y \geq 0$ such that $G_\sigma = yS_\sigma - G_\sigma(y)$ on $(\alpha_1, \alpha_2)$ and $G_\sigma(\alpha_j) = G(\alpha_j)$ for $j = 1, 2$.

We have now everything at hand to derive the analogue of proposition 22.

Lemma 32. Let $\sigma$ be a distortion function and $p \in (1, \infty)$ with conjugate exponent $q$. Then $L^p_\sigma(P)^\times = L^*_\sigma,q(P)$, for every $Z \in L^2_\sigma(P)^\times$ it holds $|Z|^{\star,q}_{\sigma,p} = \|\varphi_Z\|^{\star}_{\sigma,p}$, and there is $Y \in L^p_\sigma(P)$ with $\|Y\|_{\sigma,p} = 1$ such that $\varphi_Z(\varphi_Y) = \|\varphi_Z\|^{\star}_{\sigma,p}$.

Proof. By proposition 20 we only have to show $L^p_\sigma(P)^\times \subseteq L^*_\sigma,q(P)$ and that $\sup\{E(ZY); Y \in L^p_\sigma(P), \|Y\|_{\sigma,p} \leq 1\}$ is an upper bound for $|Z|^{\star,q}_{\sigma,p}$ for any $Z \in L^p_\sigma(P)^\times$.

So we fix $Z \in L^p_\sigma(P)^\times$. By remark 13 we also have $|Z| \in L^p_\sigma(P)^\times$. We define

$$G : [0, 1] \to [0, \infty), G(\alpha) := \int_0^1 F^{-1}_{|Z|}(u) du$$

and observe that $G$ is well-defined by $|Z| \in L^1(P)$. $G$ is obviously continuous, differentiable from the left, nonincreasing with $G(1) = 0$. By proposition 31 and remark 28 i) there is a nonnegative, nondecreasing function $H$ on $[0, 1]$ which is continuous from the left such that $G_\sigma(\alpha) = \int_0^\alpha H(u) du$.

If there is $\sigma \in (0, 1)$ with $G_\sigma(\alpha) > G(\alpha)$ it follows immediately from proposition 31 that there are $0 \leq \alpha_1 < \alpha < \alpha_2 \leq 1$ and $y \geq 0$ such that $H(u) = y$ for $u \in (\alpha_1, \alpha_2)$ and $G_\sigma(\alpha_j) = G(\alpha_j), j = 1, 2$ so that

$$\int_{\alpha_1}^{\alpha_2} H(u)^q \sigma(u) du = y^{q-1} \int_{\alpha_1}^{\alpha_2} H(u) \sigma(u) du = y^{q-1}(G_\sigma(\alpha_1) - G_\sigma(\alpha_2))$$

$$= y^{q-1}(G(\alpha_1) - G(\alpha_2)) = \int_{\alpha_1}^{\alpha_2} H(u)^{q-1} F^{-1}_{|Z|}(u) du.$$
Combining these arguments gives
\[
\forall \alpha_1, \alpha_2 \in [0, 1]: \left( G_\sigma(\alpha_1) = G(\alpha_1), G_\sigma(\alpha_2) = G(\alpha_2) \right) \Rightarrow \int_{\alpha_1}^{\alpha_2} H(u)^q \sigma(u) \, du = \int_{\alpha_1}^{\alpha_2} H(u)^{q-1} F^{-1}_\sigma(u) \, du.
\]  

In order to proceed, we distinguish two cases. First we assume that there is a strictly increasing sequence \((\alpha_n)_{n \in \mathbb{N}}\) in \((0, 1)\) converging to 1 such that \(G(\alpha_n) = G_\sigma(\alpha_n)\). We define
\[
Y_n := (\mathbb{I}_{[0, \alpha_n]} H^{q-1}) \circ U,
\]
where \(U \in \mathcal{U}(0, 1)\) is coupled in a comonotone way with \(|Z|\). From \(1/p + 1/q = 1\) it follows that \(|Y_n^p| = (\mathbb{I}_{[0, \alpha_n]} H^q) \circ U\) which implies
\[
\|Y_n\|_{\sigma, p}^p = \int_0^1 F_{|Z|}^{-1}(u)^p \sigma(u) \, du = \int_0^{\alpha_n} H^q(u) \sigma(u) \, du < \infty
\]
for all \(n \in \mathbb{N}\). Using that \(H\) in nondecreasing and \(\int_0^1 \sigma(u) \, du = 1\) so that \(Y_n = |Y_n| \in L^p_\sigma(P)\). Using the notation from proposition 12, because \(|Z|\) and \(U\) are coupled in a comonotone way we have by (28) and (27) applied to \(\alpha_1 = 0\) and \(\alpha_2 = \alpha_n\)
\[
\int_0^{\alpha_n} H(u)^q \sigma(u) \, du = \int_0^{\alpha_n} H^{q-1}(u) F_{|Z|}^{-1}(u) \, du = \mathbb{E}(|Y_n| | |Z|)
\]
\[
\leq \|\varphi|Z\|_{\sigma, p} \|Y_n\|_{\sigma, p} = \|\varphi|Z\|_{\sigma, p} \left( \int_0^{\alpha_n} H(u)^q \sigma(u) \, du \right)^{1/p}
\]
which gives
\[
\left( \int_0^1 \mathbb{I}_{[0, \alpha_n]} H(u)^q \sigma(u) \, du \right)^{1/q} \leq \|\varphi|Z\|_{\sigma, p}^*
\]
for all \(n \in \mathbb{N}\). Using that \(\lim_{n \to \infty} \alpha_n = 1\) an application of the Monotone Convergence Theorem yields
\[
\left( \int_0^1 H(u)^q \sigma(u) \, du \right)^{1/q} \leq \|\varphi|Z\|_{\sigma, p}^*
\]
so that \(Z' := H \circ U\) belongs to \(L^p_\sigma(P)\). Because \(Z' = |Z'|\) and \(F_{|Z'|}^{-1} = H\) it follows from
\[
\forall \alpha \in [0, 1]: \int_0^1 H(u)^q \sigma(u) \, du = G_\sigma(\alpha) \geq G(\alpha) = \int_0^1 F_{|Z'|}^{-1}(u) \, du
\]
that \(Z' \sigma \geq Z\) which combined with (29) yields \(Z \in L^*_{\sigma, q}\) and \(|Z|_{\sigma, q}^* \leq \|\varphi|Z\|_{\sigma, p}^* = \|\varphi|Z\|_{\sigma, p}^*\) where we have used remark 13 in the last equality. Since also \(\|Z\|_{\sigma, p}^* \leq |Z|_{\sigma, q}^*\) we obtain from (29)
\[
\|\varphi Z\|_{\sigma, p}^* = \left( \int_0^1 H(u)^q \sigma(u) \, du \right)^{1/q} = \inf\{\|Z'|_{\sigma, q}; Z' \sigma \geq Z\}\]

Now we define
\[
Y := \mathbb{I}_{\{Z \neq 0\}} \frac{Z}{|Z|} H^{q-1}(U).
\]
Then the same arguments used in deriving (28) combined with (29) show that $Y \in L^p_\sigma(P)$ and

$$\|Y\|_{\sigma,p} = \left( \int_0^1 H^q(u)\sigma(u)du \right)^{1/p}.$$  

Moreover, using that $|Z|$ and $U$ are coupled in a comonotone way, (27) applied to $\alpha_1 = 0$ and $\alpha_2 = 1$, and (30) give

$$\varphi_Z(Y) = \mathbb{E}(ZY) = \mathbb{E}(|ZY|) = \int_0^1 H(u)^{q-1}F^{-1}_{|Z|}(u)du$$

$$= \int_0^1 H(u)^q\sigma(u)du = \left( \int_0^1 H(u)^q\sigma(u)du \right)^{1/q} \left( \int_0^1 H(u)^q\sigma(u)du \right)^{1/p}$$

$$= \|\varphi_Z\|_{\sigma,p}^* Y_{\sigma,p}.$$ 

Next, if there is no strictly increasing sequence $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ in $(0,1)$ converging to 1 such that $G(\alpha_n) = G_\sigma(\alpha_n)$ there is $\beta \in (0,1)$ such that $G(u) < G_\sigma(u)$ for all $\alpha \in (\beta,1)$ and such that $G(\beta) = G_\sigma(\beta)$. It therefore follows from proposition 31 that there is $y \geq 0$ such that $H = y$ on $(\beta,1)$. Because $H$ is nondecreasing this implies that $H$ is bounded so that trivially

$$\int_0^1 H(u)^q\sigma(u) < \infty.$$ 

By repeating the arguments from the first part of the proof it follows for $U \in \mathcal{U}(0,1)$ coupled in a comonotone way with $|Z|$ that $Z' := H \circ U$ satisfies $Z' \in L^p_\sigma(P)$ and $Z'_{\sigma,p} \succeq Z$ which gives $Z \in L^p_{\sigma,q}(P)^*$ and $|Z|_{\sigma,q}^* = \|\varphi_Z\|_{\sigma,p}^* = \|Z\|_{\sigma,q}$. Defining $Y$ as in (31) finally gives again $\varphi_Z(Y) = \|\varphi_Z\|_{\sigma,p}^* Y_{\sigma,p}$ which proves the claim. 

Combining proposition 12 and lemma 32 we immediately derive the next result.

**Theorem 33.** Let $\sigma$ be a distortion function and $p \in (1,\infty)$ with conjugate exponent $q$. Then $|\cdot|_{\sigma,q}^*$ is a norm on $L^\infty_{\sigma,q}(P)$ turning it into a Banach space. Moreover,

$$\Phi : (L^\infty_{\sigma,q}(P), |\cdot|_{\sigma,q}^*) \to (L^p_\sigma(P)^*, \|\cdot\|_{\sigma,p}), Z \mapsto (Y \mapsto \Phi(Z)(Y) := \mathbb{E}(ZY))$$

is an isometric isomorphism. Moreover, for every $\varphi \in L^\infty_\sigma(P)^*$ there is $Y \in L^p_\sigma(P)$ with $\|Y\|_{\sigma,p} = 1$ such that $\varphi(Y) = \|\varphi\|_{\sigma,p}^*$. 

**Corollary 34.** For a distortion function $\sigma$ and $p \in (1,\infty)$ the Banach space $L^p_\sigma(P)$ is reflexive.

**Proof.** This is an immediate consequence of James’ Theorem (see e.g. [9, Theorem I.3]) and theorem 33. 

**Proposition 35.** Simple functions (and thus $L^\infty$) are dense in $L^p_\sigma$, whenever $q < \infty$.

**Proof.** Let $\mathfrak{F}$ contain all finite sigma algebras $\mathcal{F}$ for which the measure $P$ is defined. Note that $(\mathfrak{F}, \subseteq)$ is a filter, and the proof of proposition 9 actually demonstrates that

$$\|\mathbb{E}(Y|\mathcal{F}) - Y\|_{\sigma,p} \to 0$$

whenever $\mathcal{F} \in \mathfrak{F}$ increases.
Recall first that $\text{AV}\circ\mathcal{R}_\alpha(\mathbb{E}(Y|\mathcal{F})) \leq \text{AV}\circ\mathcal{R}_\alpha(Y)$. Indeed, it follows from the conditional Jensen inequality (cf. Williams [28, Section 34]) that $(\mathbb{E}(Y|\mathcal{F}) - q)_+ \leq \mathbb{E}((Y - q)_+|\mathcal{F})$, and hence, using Pflug [23],

$$\text{AV}\circ\mathcal{R}_\alpha(\mathbb{E}(Y|\mathcal{F})) = \min_{q \in \mathbb{R}} q + \frac{1}{1 - \alpha} \mathbb{E}(\mathbb{E}(Y|\mathcal{F}) - q)_+ \leq \min_{q \in \mathbb{R}} q + \frac{1}{1 - \alpha} \mathbb{E}((Y - q)_+|\mathcal{F})$$

$$= \min_{q \in \mathbb{R}} q + \frac{1}{1 - \alpha} \mathbb{E}((Y - q)_+|\mathcal{F}) = \text{AV}\circ\mathcal{R}_\alpha(Y).$$

Suppose that $Z' \succ_{\sigma} Z$. It follows that

$$\int_\alpha^1 \sigma(u)F'_{Z}(u)du \geq \int_\alpha^1 F'_{Z}(u)du \geq \int_\alpha^1 F'_{Z}(u)du$$

for every $\alpha \leq 1$, that is $Z' \succ_{\sigma} Z(\mathcal{F})$ and thus $\|\mathbb{E}(Z(\mathcal{F}))\|_{\sigma,q}^* < \|Z\|_{\sigma,q}^*$. The assertion follows as $\{\mathbb{E}(Z(\mathcal{F}) : \mathcal{F} \in \mathcal{F}\}$ is arbitrarily close to $Z$ in the norm $\|\cdot\|_{\sigma,q}$ by Proposition 9.

We close this section by having a closer look at $L^1_\alpha(P)$ and its dual space.

**Theorem 36.** The dual space of $L^1_\alpha(P)$ is not separable.

**Proof.** It is enough to assume that $\sigma$ is unbounded, as for bounded $\sigma$ we have that $L^1_\alpha(P)$ is isomorphic to $L^1(P)$ by proposition 8 and its dual $L^\infty(P)$ is not separable.

For $\beta \in [0,1]$ consider the random variables

$$Z_\beta := \begin{cases} \sigma(U) & \text{if } U \leq \beta, \\ \sigma(1 + \beta - U) & \text{if } U > \beta \end{cases}$$

for a (fixed) uniform random variable $U \in \mathcal{U}(0,1)$. Notice, that $\|Z_\beta\|_{\sigma,1}^* = 1$, since $Z_\beta$ is a rearrangements of $\sigma(U)$. Assume that $\beta < \gamma$ and observe that

$$Z_\gamma - Z_\beta = \sigma(1 + \gamma - U) - \sigma(1 + \beta - U) \geq \sigma(1 + \gamma - U) - \sigma(1 + \beta - \gamma)$$

whenever $U > \gamma$. Then it holds that

$$\|Z_\gamma - Z_\beta\|_{\sigma,1}^* \geq \limsup_{\alpha \to 1} \frac{1}{1 - \alpha} \int_\alpha^1 \sigma(u)du \geq \limsup_{\alpha \to 1} \frac{\text{AV}\circ\mathcal{R}_\alpha(\sigma(1 + \beta - U) - \sigma(1 + \beta - \gamma))}{\frac{1}{1 - \alpha} \int_\alpha^1 \sigma(u)du}.$$ 

Now, as $\sigma$ is unbounded, the denominator is unbounded as well (indeed, $\frac{1}{1 - \alpha} \int_\alpha^1 \sigma(u)du \geq \sigma(\alpha)$) and hence

$$\|Z_\gamma - Z_\beta\|_{\sigma,1}^* \geq \limsup_{\alpha \to 1} \frac{\text{AV}\circ\mathcal{R}_\alpha(\sigma(1 + \beta - U) - \sigma(1 + \beta - \gamma))}{\frac{1}{1 - \alpha} \int_\alpha^1 \sigma(u)du} \geq \lim_{\alpha \to 1} \frac{1}{1 - \alpha} \int_\alpha^1 \sigma(u)du - \sigma(1 + \beta - \gamma) = 1. \quad (32)$$

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Suppose finally that there is a dense sequence \((D_k)_{k \in \mathbb{N}} \subset L_P^*(P)\). For \(\beta \in [0,1]\) fixed there is \(k \in \mathbb{N}\) such that \(\|Z_\beta - D_k\|_{\sigma,1}^* < \frac{1}{2}\). But \(1 \leq \|Z_\beta - Z_\gamma\|_{\sigma,1}^* \leq \|Z_\beta - D_k\|_{\sigma,1}^* + \|D_k - Z_\gamma\|_{\sigma,1}^*\), from which follows that \(\|D_k - Z_\gamma\|_{\sigma,1}^* > \frac{1}{2}\) whenever \(\gamma \neq \beta\). Hence only countably many \(Z_\beta\) can be approximated by the sequence \((D_k)_{k \in \mathbb{N}}\) with a distance \(\|Z_\beta - D_k\|_{\sigma}^* < \frac{1}{2}\) and \((D_k)_{k \in \mathbb{N}}\) thus is not dense giving the desired contradiction. 

\[\square\]

## 5 The dual space in the vector-valued case

In this section we determine the dual space of \(L_P^p(P,X)\) for arbitrary Banach spaces \(X\) over \(K \in \{\mathbb{R}, \mathbb{C}\}\). We denote the space of \(X\)-valued simple functions on \((\Omega, F, P)\) by \(S(X)\), i.e.,

\[S(X) = \{Y : \Omega \to X; Y(\Omega) \text{ is finite and } \forall x \in X : Y^{-1}\{x\} \in F\}.\]

Then it is straightforward to see and well-known that \(\{\varphi : S(X) \to K; \varphi \text{ linear}\}\) and \(\{\mu : F \to X^*; \mu \text{ vector measure}\}\) are isomorphic via the linear mapping

\[
\Phi : \varphi \mapsto (\mu_\varphi(E)(x) := \varphi(\mathbb{1}_E x), x \in X, E \in F).
\]

(33)

For a vector measure \(\mu\) we denote by \(|\mu|\) its variation.

**Lemma 37.** For a linear mapping \(\varphi : S(X) \to K\) we have

\[
\sup\{\|\varphi(\sum_{j=1}^n \mathbb{1}_{E_j} x_j)\|; E_j \in F \text{ partition of } \Omega, x_j \in X, \|\sum_{j=1}^n \mathbb{1}_{E_j} x_j\|_{\sigma,p} \leq 1\}
\]

\[
= \sup\{\|\varphi\|_{\mu_\varphi(E_j)}||; E_j \in F \text{ partition of } \Omega, \alpha_j \in K, \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\}
\]

\[
= \sup\{\int_\Omega \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\| d|\mu_\varphi||; E_j \in F \text{ partition of } \Omega, \alpha_j \in K, \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\},
\]

where \(\mu_\varphi\) is defined as in (33).

**Proof.** For a partition \(E_1, \ldots, E_n \in F\) of \(\Omega\), \(\alpha_1, \ldots, \alpha_n \in K\), and \(z_1, \ldots, z_n \in X\) with \(\|z_j\| = 1\) we have

\[
\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j} z_j\|_{p,p}^p = \sup_{U \in \mathcal{U}(0,1)} \int_\Omega \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j} z_j\|_{p}^p dP = \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j} z_j\|_{p,p}^p,
\]

where the norm on the left hand side is the one on \(L_P^p(P,X)\) while the norm on the right hand side.
denotes the one on \( L^2_\sigma(P) \). Therefore we conclude

\[
\sup\{\|\varphi(\sum_{j=1}^n \mathbb{1}_{E_j} x_j)\|; \ E_j \in \mathcal{F} \ \text{partition of} \ \Omega, \ x_j \in X, \ \|\sum_{j=1}^n \mathbb{1}_{E_j} x_j\|_{\sigma,p} \leq 1\}
\]

\[
= \sup\{\|\varphi(\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j} z_j)\|; \ E_j \in \mathcal{F} \ \text{partition of} \ \Omega, \ \alpha_j \geq 0, \ z_j \in X, \ \|z_j\| = 1, \ \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\}
\]

\[
= \sup\{\|\varphi(\sum_{j=1}^n \|\mu_\varphi(E_j)\|)\|; \ E_j \in \mathcal{F} \ \text{partition of} \ \Omega, \ \alpha_j \in \mathbb{K}, \ \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\},
\]

which gives the first equality. Using the definition of \( |\mu_\varphi| \) we continue

\[
\sup\{\sum_{j=1}^n \|\alpha_j\| |\mu_\varphi(E_j)|\|; \ E_j \in \mathcal{F} \ \text{partition of} \ \Omega, \ \alpha_j \in \mathbb{K}, \ \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\}
\]

\[
= \sup\{\sum_{\alpha_j \text{pairwise different}} |\alpha| \|\mu_\varphi(E_j)|\|; \ E_j \in \mathcal{F} \ \text{partition of} \ \Omega, \ \alpha_j \in \mathbb{K}, \ \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\}
\]

\[
= \sup\{\sum_{\alpha_j \text{pairwise different}} |\alpha| |\mu_\varphi|\left(\bigcup_{j: \alpha_j = \alpha_j} E_j\right)\|; \ E_j \in \mathcal{F} \ \text{partition of} \ \Omega, \ \alpha_j \in \mathbb{K}, \ \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\}
\]

\[
= \sup\{\int_\Omega |\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}| d|\mu_\varphi|\|; \ E_j \in \mathcal{F} \ \text{partition of} \ \Omega, \ \alpha_j \in \mathbb{K}, \ \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\}
\]

which proves the second equality.

\[\square\]

**Definition 38.** For a distortion function \( \sigma, p \in [1, \infty) \), and a Banach space \( X \) we define

\[
\mathcal{L}_{\sigma,p}(S(X)) := \{ \varphi : S(X) \to \mathbb{K}; \ \varphi \text{ linear and continuous with respect to } \|\cdot\|_{\sigma,p} \}.
\]

**Lemma 39.** Let \( \Phi \) be the natural isomorphism from (33). Then \( \Phi(\mathcal{L}_{\sigma,p}(S(X))) \) coincides with the set

\[
\{ \mu : \mathcal{F} \to X^*; \ \mu \text{ is a } \sigma \text{-additive vector measure of bounded variation such that } |\mu| \ll P \text{ and } \frac{d|\mu|}{dP} \in L^*_\sigma(P) \},
\]

and

\[
\forall \varphi \in \mathcal{L}_{\sigma,p}(S(X)); \ \|\varphi\| = \left| \frac{d\varphi}{dP} \right|_{\sigma,q},
\]

where \( q \) is the conjugate exponent to \( p \).
Proof. For $\varphi \in L_\sigma(p(S(X)))$ it follows from the density of $S(X)$ in $L_\sigma^p(P,X)$ that $\varphi$ extends to a unique element of $L_\sigma^p(P,X)^{\times}$ which we still denote by $\varphi$. For a pairwise disjoint sequence $(E_j)_{j \in \mathbb{N}}$ in $F$ and its union $E$ it follows for arbitrary $x \in X$

$$|\mu_\varphi(E)(x) - \mu_\varphi\left(\bigcup_{j=1}^m E_j\right)(x)| = |\mu_\varphi\left(\bigcup_{j=m+1}^\infty E_j\right)(x)| = |\varphi(\mathbb{1}_{\bigcup_{j=m+1}^\infty E_j}x)|$$

$$\leq \|\varphi\|_{\sigma,p}^* \|x\| \|\mathbb{1}_{\bigcup_{j=m+1}^\infty E_j}\|_{\sigma,p}$$

$$= \|\varphi\|_{\sigma,p}^* \|x\| \left(\int_{1-\sum_{j=m+1}^\infty P(E_j)}^1 \sigma(u)du\right)^{1/p}.$$ 

With the aid of Lebesgue’s Dominated Convergence Theorem it follows

$$\|\mu_\varphi(E) - \sum_{j=1}^m \mu_\varphi(E_j)\| \leq \|\varphi\|_{\sigma,p}^* \left(\int_{1-\sum_{j=m+1}^\infty P(E_j)}^1 \sigma(u)du\right)^{1/p} \to m \to \infty 0.$$

Thus $\Phi(\varphi) = \mu_\varphi$ is a $\sigma$-additive vector measure. Moreover, for every finite partition $E_1, \ldots, E_n \in F$ of $\Omega$ and $x_1, \ldots, x_n \in X$ with $\|x_j\| \leq 1$ we have

$$\sum_{j=1}^n |\mu_\varphi(E_j)(x_j)| = \sum_{j=1}^n |\varphi(\mathbb{1}_{E_j}x_j)| = \sum_{j=1}^n \text{sign} \left(\varphi(\mathbb{1}_{E_j}x_j)\right) \varphi(\mathbb{1}_{E_j}x_j)$$

$$= |\varphi\left(\sum_{j=1}^n \text{sign} \left(\varphi(\mathbb{1}_{E_j}x_j)\right) \mathbb{1}_{E_j}x_j\right)|$$

$$\leq \|\varphi\|_{\sigma,p}^* \|\sum_{j=1}^n \text{sign} \left(\varphi(\mathbb{1}_{E_j}x_j)\right) \mathbb{1}_{E_j}x_j\|_{\sigma,p}$$

$$\leq \|\varphi\|_{\sigma,p}^* \left(\sup_{U \in \mathcal{U}(0,1)} \int \sum_{j=1}^n \text{sign} \left(\varphi(\mathbb{1}_{E_j}x_j)\right)^p \mathbb{1}_{E_j}x_j\|x_j\|^p \sigma(U)dP\right)^{1/p}$$

$$\leq \|\varphi\|_{\sigma,p}^* \left(\sup_{U \in \mathcal{U}(0,1)} \int \sigma(U)dP\right)^{1/p} = \|\varphi\|_{\sigma,p}^*,$$

where for a complex number $\alpha$ as usual sign $(\alpha) = \frac{\pi}{|\alpha|}$ in case $\alpha \neq 0$, resp. sign $(0) = 0$. Thus, for arbitrary $\varepsilon > 0$ it follows for suitable choices $x_j^\varepsilon \in X$ from the above inequality that

$$\sum_{j=1}^n \|\mu_\varphi(E_j)\| \leq \sum_{j=1}^n \left(\|\mu_\varphi(E_j)(x_j^\varepsilon)\| + \frac{\varepsilon}{n}\right) \leq \|\varphi\|_{\sigma,p}^* + \varepsilon,$$

i.e., $\sum_{j=1}^n \|\mu_\varphi(E_j)\| \leq \|\varphi\|_{\sigma,p}^*$ which in turn implies $|\mu_\varphi(\Omega)| \leq \|\varphi\|_{\sigma,p}^*$. Hence $\Phi(\varphi) = \mu_\varphi$ is of bounded variation.

Since $\mu_\varphi$ is $\sigma$-additive the same holds for $|\mu_\varphi|$ (see [10, Proposition I.1.9]), i.e., $|\mu_\varphi|$ is a (finite) measure on $F$. If $E \in F$ satisfies $P(E) = 0$ it follows for $x \in X$

$$\|\mathbb{1}_E x\|_{\sigma,p} = \|x\| \left(\sup_{U \in \mathcal{U}(0,1)} \int_E \sigma(U)dP\right)^{1/p} = 0$$

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and therefore \(|\mu_\varphi(E)| = 0\). If \(E_1, \ldots, E_n \in \mathcal{F}\) is a partition of \(E\) it follows \(P(|E|) = 0\) and thus \(\sum_{j=1}^n |\mu_\varphi(E_j)| = 0\) which implies \(|\mu_\varphi(E)| = 0\). By an application of the Radon-Nikodým Theorem we obtain \(g_\varphi \in L^1(P)\), \(g_\varphi \geq 0\) such that
\[
\forall E \in \mathcal{F} : \int_E g_\varphi \, dP = |\mu_\varphi|(E).
\]
From the fact that \(\mathcal{S}(\mathbb{K})\) is dense in \(L^p_F(P)\) and \(\mathcal{S}(\mathbb{K})\) is dense in \(L^p_F(P)\) it follows with lemma 39
\[
\|\varphi\|_{\sigma,p}^* = \sup\{|\varphi(\sum_{j=1}^n \mathbb{1}_{E_j} x_j)| ; E_j \in \mathcal{F} \text{ partition of } \Omega, x_j \in X, \|\sum_{j=1}^n \mathbb{1}_{E_j} x_j\|_{\sigma,p} \leq 1\} = \sup\{\int_\Omega |\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}| \, d|\mu_\varphi| ; E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K}, \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\} = \sup\{\int_\Omega |\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}| \, g_\varphi \, dP ; E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K}, \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\} = \sup\{\int_\Omega |f| \, |g_\varphi| \, dP ; f \in L^p_F(P), \|f\|_{\sigma,p} \leq 1\},
\]
so that in particular \(g_\varphi \in L^p_F(P)\) and \(|\varphi\|_{\sigma,p}^* = |g_\varphi|_{\sigma,q}^*\). Since \(\varphi \in \mathcal{L}_{\sigma,p}(\mathcal{S}(\mathbb{K}))\) was chosen arbitrarily this finally shows that \(\Phi(\mathcal{L}_{\sigma,p}(\mathcal{S}(\mathbb{K})))\) is contained in the set of \(X^*\)-valued, \(\sigma\)-additive vector measures of bounded variation such that their bounded variation measure admits a \(P\)-density in \(L^*_{\sigma,q}(P)\).

Next let \(\mu\) be such a measure and set \(\varphi := \Phi^{-1}(\mu)\). We have to show that \(\varphi\) belongs to \(\mathcal{L}_{\sigma,p}(\mathcal{S}(\mathbb{K}))\). But from the density of \(\mathcal{S}(\mathbb{K})\) in \(L^p_F(P)\) it follows immediately together with lemma 37 that
\[
\sup\{|\varphi(\sum_{j=1}^n \mathbb{1}_{E_j} x_j)| ; E_j \in \mathcal{F} \text{ partition of } \Omega, x_j \in X, \|\sum_{j=1}^n \mathbb{1}_{E_j} x_j\|_{\sigma,p} \leq 1\} = \sup\{\int_\Omega |\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}| \, d|\mu| \, dP ; E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K}, \|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\|_{\sigma,p} \leq 1\} = |\frac{d|\mu|}{dP}|\|_{\sigma,p}^* < \infty,
\]
which shows \(\varphi \in \mathcal{L}_{\sigma,p}(\mathcal{S}(\mathbb{K}))\).

\textbf{Definition 40.} Let \(X\) be a Banach space, \(\sigma\) a distortion function, and \(p \in [1, \infty)\) with conjugate exponent \(q\). Then we define
\[
L^*_{\sigma,q}(P, X^*) := \{\mu : \mathcal{F} \to X^* ; \mu\text{ is a }\sigma\text{-additive vector measure of bounded variation such that }|\mu| \ll P \text{ and } \frac{d|\mu|}{dP} \in L^*_{\sigma,q}(P)\},
\]
which is obviously a subspace of the space of all \(X^*\)-valued vector measures on \(\mathcal{F}\). Moreover, for \(\mu \in L^*_{\sigma,q}(P, X^*)\) we set \(|\mu|_{\sigma,q} := |\frac{d|\mu|}{dP}|_{\sigma,q}^*\). Then, \(|\cdot|_{\sigma,q}^*\) is obviously a norm on \(L^*_{\sigma,q}(P, X^*)\).
Remark 41. For $\mu \in L_{p,q}^*(P,X^*)$ it follows from lemma 39 and the density of $\mathcal{S}(X)$ in $L_p^p(P,X)$ that $\Phi^{-1}(\mu)$ can be extended in a unique way to a continuous linear functional on $L_p^p(P,X)$ which we again denote by $\Phi^{-1}(\mu)$. For $Y \in L_p^p(P,X)$ we also write for obvious reasons

$$\int_\Omega Y d\mu := \Phi^{-1}(\mu)(Y).$$

With this notation the following theorem is an immediate consequence of lemma 39, proposition 22, and lemma 32.

Theorem 42. Let $X$ be a Banach space, $\sigma$ a distortion function, and $p \in [1, \infty)$ with conjugate exponent $q$. Then $(L_{p,q}^*(P,X^*), |\cdot|_{p,q}^*)$ is a Banach space and the mapping

$$\Psi : (L_{p,q}^*(P,X^*), |\cdot|_{p,q}^*) \to (L_p^p(P,X)^*, \|\cdot\|_{p,p}^*), \mu \mapsto (Y \mapsto \int_\Omega Y d\mu)$$

is an isometric isomorphism.

Definition 43. For a Banach space $X$, $p \in [1, \infty)$ with conjugate exponent $q$ we define

$$L_{p,q}^*(P,X) := \{Z : \Omega \to X; Z \text{ strongly measurable, } \|Z\| \in L_{p,q}^*(P)\}$$

and for $Z \in L_{p,q}^*(P,X)$ we set $|Z|_{p,q}^* := \|Z\|_{p,q}^*$, where as usual we identify random variables which coincide $P$-almost everywhere. It follows easily that $L_{p,q}^*(P,X)$ is a vector space and $|\cdot|_{p,q}^*$ a norm.

Remark 44. For $Z \in L_{p,q}^*(P,X^*)$ it follows from $\|Z\| \in L_{p,q}^*(P) \subseteq L^1(P)$ that

$$\mu_Z : \mathcal{F} \to X^*, \mu_Z(E) := \int_E Z dP$$

is a well-defined, $\sigma$-additive vector measure of bounded variation with $|\mu_Z|(E) = \int_E \|Z\| dP$ (see, e.g., [10, Theorem II.2.4]). A straightforward calculation gives for $Y \in \mathcal{S}(X)$

$$\int_\Omega Y d\mu_Z = \int_\Omega \langle Z(\omega), Y(\omega) \rangle dP(\omega).$$

Moreover, for $Z \in L_{p,q}^*(P,X^*)$ and $Y \in L_p^p(P,X)$ it follows from $\|Z\| \in L_{p,q}^*(P)$ and $\|Y\| \in L_p^p(P)$

$$\int_\Omega |\langle Z(\omega), Y(\omega) \rangle| dP(\omega) \leq \int_\Omega \|Z(\omega)\| \|Y(\omega)\| dP(\omega) \leq \|Z\|_{p,q}^* \|Y\|_{p,p}^*$$

which implies that $\psi_Z : L_p^p(P,X) \to \mathbb{K}, Y \mapsto \mathbb{E}(|\langle Z, Y \rangle|)$ is a well-defined continuous linear functional which coincides on the dense subspace $\mathcal{S}(X)$ with $\Psi(\mu_Z)$. Together with theorem 42 this shows that

$$\nu : (L_{p,q}^*(P,X^*), |\cdot|_{p,q}^*) \to L_p^p(P,X^*), Z \mapsto \left( Y \mapsto \mathbb{E}(|\langle Z, Y \rangle|) \right)$$

is an isometry.

As in the case of Bochner-Lebesgue spaces we have the following result.
Theorem 45. For a Banach space $X$, a distortion function $\sigma$, and $p \in [1, \infty)$ with conjugate exponent $q$ the isometry

$$\iota : (L^q_\sigma(P,X^*), |\cdot|_{q,\sigma}^*) \to L^p_\sigma(P,X)^*, \ Z \mapsto \left(Y \mapsto \mathbb{E}(\langle Z,Y \rangle)\right)$$

is an isomorphism if and only if $X^*$ has the Radon-Nikodým property with respect to $(\Omega, \mathcal{F}, P)$.

Proof. Assume first, that $X^*$ has the Radon-Nikodým property with respect to $(\Omega, \mathcal{F}, P)$. By remark 44 we only have to show surjectivity of $\iota$. For an arbitrary $\varphi \in L^q_\sigma(P,X)^*$ there is by theorem 42 a $\sigma$-additive $X^*$-valued vector measure of bounded variation such that $|\mu| \ll P$ and $\frac{d|\mu|}{dP} \in L^q_{\sigma,q}(P)$ with $\|\varphi\| = |\frac{d|\mu|}{dP}|_{\sigma,q}$. By the Radon-Nikodým property of $X^*$ it follows that there is $Z \in L^1(P,X^*)$ such that $\mu(E) = \int_E Z \ dP$ for all $E \in \mathcal{F}$. Since $|\mu|(E) = \int_E |Z| \ dP$ (see e.g. [10, Theorem II.2.4]) it follows that $Z \in L^q_\sigma(P,X^*)$ and $\iota(Z) = \mu$ showing the surjectivity of $\iota$.

Now, let $\iota$ be an isometric isomorphism. The proof that $X^*$ has the Radon-Nikodým property is along the same lines as the proof of the corresponding implication of [10, Theorem IV.1.1]. However, we include the proof for the reader’s convenience. So, let $\mu : \mathcal{F} \to X^*$ be a $P$-continuous vector measure of bounded variation and fix $E_0 \in \mathcal{F}$ such that $P(E_0) > 0$. By the Hahn Decomposition Theorem applied to the signed measure $kP - |\mu|$ for large enough $k > 0$ gives the existence of $B \in \mathcal{F}, B \subseteq E_0, P(B) > 0$ such that $|\mu|(E) \leq kP(E)$ for all $E \in \mathcal{F}, E \subseteq B$. For $Y \in \mathcal{S}(X), Y = \sum_{j=1}^n 1_{E_j} x_j$ with pairwise disjoint $E_j \in \mathcal{F}$ and $x_j \in X$ we define

$$\varphi(Y) = \sum_{j=1}^n \mu(E_j \cap B)(x_j).$$

Denoting the norm in $L^1(P,X)$ as usual by $\|\cdot\|_1$ theorem 4 then gives

$$|\varphi(Y)| \leq \sum_{j=1}^n k\|\mu(E_j \cap B)(x_j)\| \leq k\|Y\|_1 \leq k\|Y\|_{\sigma,p}$$

so that the obviously linear mapping $\varphi$ on $\mathcal{S}(X)$ is continuous with respect to $\|\cdot\|_{\sigma,p}$. By proposition 9 $\varphi$ extends (in a unique way) to an element of $L^q_\sigma(P,X^*)$ which we still denote by $\varphi$. Since $\iota$ is supposed to be surjective there is $Z \in L^q_\sigma(P,X^*) \subseteq L^1(P,X^*)$ such that

$$\forall Y \in L^q_\sigma(P,X) : \varphi(Y) = \mathbb{E}(\langle Z,Y \rangle).$$

Since $\mu(E \cap B)(x) = \varphi(1_{E \cap B} x) = \int_{E \cap B} Z(\omega) \ dP(\omega) = \langle f, Z \rangle dP(\omega), x \rangle$ for all $E \in \mathcal{F}, x \in X$ it follows that $\mu(E \cap B) = \int_E Z \ dP$.

Because $E_0 \in \mathcal{F}$ with $P(E_0) > 0$ was chosen arbitrarily, it follows from [10, Corollary III.2.5] that there is $Z \in L^1(P,X^*)$ such that $\mu(E) = \int_E Z \ dP$ for all $E \in \mathcal{F}$ which proves the Radon-Nikodým property of $X^*$ with respect to $(\Omega, \mathcal{F}, P)$.

6 Summary

This paper introduces Banach spaces, which naturally carry risk measures for vector-valued returns. Risk measures are continuous on these spaces, and the spaces are as large as possible. The spaces are built based on duality, and in this sense are natural for risk measures involving vector-valued
returns. We provide a complete characterization of the topological dual, which essentially simplifies if the dual of the state space enjoys the Radon–Nikodym property.

It is a key property of these spaces that the corresponding risk functional is continuous (in fact, Lipschitz continuous) with respect to any of the associated norms introduced, such that they all qualify as a domain space for the risk measure.

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