Λ–buildings and base change functors

Petra N. Schwer (née Hitzelberger)∗ Koen Struyve†

October 8, 2009

Abstract

We prove an analog of the base change functor of Λ–trees in the setting of generalized affine buildings. The proof is mainly based on local and global combinatorics of the associated spherical buildings. As an application we obtain that the class of generalized affine building is closed under taking ultracones and asymptotic cones. Other applications involve a complex of groups decompositions and fixed point theorems for certain classes of generalized affine buildings.

1 Introduction

The so-called Λ–trees have been studied by Alperin and Bass [AB87], Morgan and Shalen [MSS83] and others and have proven to be a useful tool in understanding properties of groups acting nicely on such spaces. The Λ–trees are a natural generalization of R–trees. Here Λ is an arbitrary ordered abelian group replacing the copies of the real line in the concept of an R–tree or the geometric realizations of simplicial trees.

Since simplicial trees are precisely the one-dimensional examples of affine buildings and real trees the one-dimensional R–buildings, it was natural to ask whether there is a higher dimensional object generalizing Λ–trees and affine buildings at the same time.

These objects, the so called Λ–affine buildings or generalized affine buildings, were introduced by Curtis Bennett [Ben94] and studied by the first author in [Hit09a] and [Hit09b]. A recent application of them is a short proof of the Margulis conjecture by Kramer and Tent [KT04].

In the present paper, we address a generalization of an important geometric property of Λ–trees: the existence of a base change functor. Easy to prove in the tree case, compare for example [Chi01], the generalization to Λ–affine buildings turns out to be much harder.

We will prove that a morphism $e : \Lambda \rightarrow \Gamma$ of ordered abelian groups gives rise to a base change functor $\phi$ mapping a generalized affine building $X$ defined over $\Lambda$ to another building $X'$ which is defined over $\Gamma$. In case $e$ is an epimorphism, we will see, that the pre-image $X''$ under $\phi$ of a point in $X'$ is again an affine building, defined over the kernel of $e$.

After having established our main results, we will present several applications. One of the consequences of our base change theorem is the proof of the fact that the class of generalized affine buildings is closed under taking asymptotic cones and ultracones.

∗The first author is supported by the “SFB 478 Geometrische Strukturen in der Mathematik” at the Institute of Mathematics, University of Münster
†The second author is supported by the Fund for Scientific Research – Flanders (FWO – Vlaanderen)
1.1 Our main results

A set $X$ together with a collection $\mathcal{A}$ of charts $f : A \to X$ from a model space $A$ into $X$ is a generalized affine building if certain compatibility and richness conditions, as stated in Definition 2.1 are satisfied. The model space $A$ is defined with respect to a spherical root system $R$ and a totally ordered abelian group $\Lambda$. Therefore it is sometimes denoted by $A(R, \Lambda)$. As a set it is isomorphic to the space of formal sums $\{\sum_{\alpha \in B} \lambda_{\alpha} : \lambda_{\alpha} \in \Lambda\}$, where $B$ is a basis of $R$.

The model space carries an action of an affine Weyl group $W$ and the transition maps of charts are given by elements of $W$. One of the conditions $(X, \mathcal{A})$ has to satisfy is, that every pair of points $x$ and $y$ is contained in a common apartment $f(A)$ with $f \in \mathcal{A}$.

Given a morphism $e : \Lambda \to \Lambda'$ of ordered abelian $\mathbb{Q}[[\alpha' \beta]]_{\alpha, \beta \in R}$ modules $\Lambda$ and $\Lambda'$ and let $(X, \mathcal{A})$ be an affine building with model space $A(R, \Lambda)$ (or shortly $A$) and distance function $d$ which is induced by the standard distance on the model space. Then we have the following:

**Theorem 1.1.** There exists a $\Lambda'$–building $(X', \mathcal{A}')$ and (functorial) map $\phi : X \to X'$ such that $\phi$ maps $A$ to $A'$ and such that

$$d'(\phi(x), \phi(y)) = e(d(x, y))$$

for $x, y \in X$ and $d'$ the distance function defined on $(X', \mathcal{A}')$. Furthermore, the spherical buildings $\partial_A X$ and $\partial_{A'} X'$ at infinity are isomorphic.

The map $\phi$ will be referred to as the base change functor associated to $e$.

Every morphism of abelian groups can be written as the composition of an epimorphism followed by a monomorphism. The kernel of an epimorphism of ordered abelian groups is convex in the sense that given some $x > 0$ in this kernel then $x \geq y \geq 0$ implies that $y$ is also contained in the kernel. Therefore the ordering of an abelian group $\Lambda$ induces an order on the quotient of $\Lambda$ by the kernel of an epimorphism. Hence morphisms of ordered abelian groups can also be decomposed into an epimorphism followed by a monomorphism.

We will prove the two cases, of an epimorphism and a monomorphism, separately. Theorem 1.1 is a direct consequence of the first and third Main Result.

In case $e : \Lambda \to \Lambda'$ is an epimorphism one defines the base change functor $\phi$ and the building $X'$ is as follows. Let two points $x, y \in X$ be equivalent, denoted by $x \sim y$, when $d(x, y) \in \ker(e)$ and let $X'$ be the quotient of $X$ defined by this equivalence relation. Defining a metric $d'$ on $X'$ by $d'(\phi(x), \phi(y)) := e(d(x, y))$ the quotient map $\phi : X \to X'$ turns out to satisfy the properties needed for the first main result.

It is possible to define a set of charts $\mathcal{A}'$ from $A' = A(R, \Lambda')$ into $X'$ such that the following theorem holds. For details compare Section 3

**Main Result 1.** Let $e : \Lambda \to \Lambda'$ be an epimorphism of ordered abelian groups and $(X, \mathcal{A})$ a $\Lambda$–building. Then the following hold:

1. There exists a $\Lambda'$–building $(X', \mathcal{A}')$ and map $\phi : X \to X'$, called the base change functor associated to $e$, such that the apartment system $\mathcal{A}$ is mapped onto $\mathcal{A}'$ by $\phi$ and such that $\phi$ satisfies

$$d'(\phi(x), \phi(y)) = e(d(x, y))$$

(1)
for all \( x, y \in X \), where \( d, d' \) are the distance functions defined respectively on \((X, A)\) and \((X', A')\). Moreover, the spherical buildings \( \partial_A X \) and \( \partial_A X' \) at infinity are isomorphic.

2. If \((Z, d'')\) is another metric space and \( \psi : X \to Z \) is a map satisfying (1) with respect to the metric \( d'' \) on \( Z \) in place of \( d' \), then there exists an isometry \( \mu : X' \to Z \) with \( \mu \circ \phi = \psi \). The space \( Z \) can be endowed with an induced atlas such that \( X' \) and \( Z \) are isomorphic as affine buildings.

3. These base change functors act as a functor on the category of \( \Lambda \)-buildings to the one of the \( \Lambda' \)-buildings. In particular, let \( G \) be a group acting on \( X \) by isometries, then \( G \) acts on \( X' \) by isometries and the map \( \phi \) is \( G \)-equivariant.

As mentioned above, the kernel of an ordered abelian group is again such. It turns out that one can prove that the “kernel” of the base change functor is again an affine building. Details of the proof can be found in Section 4.

**Main Result 2.** Let \( \phi \) be a base change functor associated to an ordered abelian group epimorphism \( e : \Lambda \to \Lambda' \), acting on a \( \Lambda \)-building \((X, A)\). For all elements \( x \) of \( X \) the following is true

1. The set \( X'' = \phi^{-1}(\phi(x)) \) admits a set of charts \( A'' \) making it into a \( \ker(e) \)-building with as distance function \( d'' \) the distance function inherited from \((X, A)\).

2. There is a natural isomorphism between \( \partial_{A''} X'' \) and \( \Delta_{\phi(x)} X' \), where \((X', A')\) is as in the first main result.

One can prove a result similar to the first main one for monomorphisms of ordered abelian groups. The construction of the building \( X' \) is again of explicit nature. The basic idea is to take the product of the old charts with the new, enlarged, model space and consider equivalence classes of these products. Since this construction is more involved than the one in the first main result, let us postpone details to Section 5. There we will prove

**Main Result 3.** Let \( e : \Lambda \to \Lambda' \) be a monomorphism of ordered abelian groups and \((X, A)\) a \( \Lambda \)-building. Then the following assertions hold

1. There exists a \( \Lambda' \)-building \((X', A')\) and map \( \phi : X \to X' \) satisfying

\[
d'(\phi(x), \phi(y)) = e(d(x, y))
\]

for all \( x, y \in X \), where \( d, d' \) are the distance functions defined respectively on \((X, A)\) and \((X', A')\). Further \( \phi \) maps the apartment system \( A \) to \( A' \) and the spherical buildings \( \partial_A X \) and \( \partial_A X' \) at infinity are isomorphic.

2. These base change functors act as a functor on the category of \( \Lambda \)-buildings to the one of the \( \Lambda' \)-buildings, but only for isometries mapping apartments to apartments. In particular, let \( G \) be a group acting on \( X \) by isometries stabilizing the system of apartments, then \( G \) acts on \( X' \) by isometries stabilizing the system of apartments and the map \( \phi \) is \( G \)-equivariant.

The proofs of the main results 1 to 3 can be found in Sections 3 to 5. Before defining generalized affine buildings in Section 2 we will use the last subsection of the introduction to state our main applications.
1.2 Applications

Let us quickly summarize the applications proved in Section 6.

Asymptotic cones

Asymptotic cones of metric spaces capture the ‘large scale structure’ of the underlying space. The main idea goes back to Gromov \cite{Gro93} and was later generalized by van den Dries and Wilkie \cite{vdDW84}. Asymptotic cones provide interesting examples of metric spaces and have proven useful in the context of geometric group theory.

In Section 6.1 we will prove, using the base change functor, that the class of generalized affine buildings is closed under ultraproducts, asymptotic cones and ultracones. The main results read as follows:

**Theorems 6.3 and 6.6**

- The ultraproduct of a sequence \((X_i, A_i)_{i \in I}\) of \(\Lambda_i\)-affine buildings defined over the same root system \(R\) is again a generalized affine building over \(R\).
- Asymptotic cones and ultracones of generalized affine buildings are again such.
- Furthermore, if \((X, A)\) is a building modeled on \(A(R, \Lambda)\), then its asymptotic cone \(\text{Cone}(X)\) has model space \(A(R, \text{Cone}(\Lambda))\) and its ultracone \(\text{UCon}(X)\) is modeled on \(A(R, \text{UCon}(\Lambda))\).

Fixed point theorems

The base change functors can be used to reduce problems of generalized affine buildings to the (easier) case of \(R\)-buildings. In Section 6.2 we illustrate this with a fixed point theorem for a certain class of \(\Lambda\)-buildings (we postpone the description of this class to the aforementioned section). The result then reads:

**Theorem 1.2.** A finite group of isometries of a generalized affine building \((X, A)\) of this class admits a fixed point.

Complex of groups decompositions

One can use the first main result to conclude that groups acting nicely on certain affine buildings do admit a complex of groups decomposition. We will not carry out the details of the proof, but let us make the statement a bit more precise.

Assume that \((X, A)\) is modeled over an abelian group \(\Lambda := \mathbb{R} \times \Lambda'\), where the two components are ordered lexicographically, and assume further that the image of the base change functor associated to the projection \(e: \Lambda \to \mathbb{R}\) is a simplicial affine building.

Then, if \(G\) is a subgroup of the automorphism group of \(X\) such that the induced action on \((X', A')\) is simplicial, the group \(G\) has a complex of groups decomposition where each vertex group acts on a \(\Lambda'\)-building. In addition, if the action of \(G\) on \(X\) is free, then each vertex group acts freely on a \(\Lambda'\)-building.
2 Preliminaries

For a detailed study of Λ–affine buildings and proof of the results in this introductory section we refer to [Ben94] and [Hit09a].

2.1 Definition of apartments and buildings

We will first define the model space for apartments in Λ–buildings and examine its metric structure. We conclude this subsection with the definition of a Λ–building.

For a (not necessarily crystallographic) spherical root system $R$ let $F$ be a subfield of the reals containing the set $\{\langle \beta, \alpha^\vee \rangle : \alpha, \beta \in R \}$ of all evaluations of co-roots on roots. Assume that Λ is a (non-trivial) totally ordered abelian group admitting an $F$–module structure and define the model space of a generalized affine building of type $R$ to be the set

$$A(R, \Lambda) = \text{span}_F(R) \otimes_F \Lambda.$$ 

Notice that $F$ can always be chosen to be the quotient field of $Q[\{\langle \beta, \alpha^\vee \rangle : \alpha, \beta \in R \}]$. If $R$ is crystallographic this is $F = Q$. We will sometimes abbreviate $A(R, \Lambda)$ by $A$.

A fixed basis $B$ of the root system $R$ provides natural coordinates for the model space $A$. The vector space of formal sums

$$\left\{ \sum_{\alpha \in B} \lambda_\alpha \alpha : \lambda_\alpha \in \Lambda \right\}$$

is canonically isomorphic to $A$. The evaluation of co-roots on roots $\langle \cdot, \cdot \rangle$ is linearly extended to elements of $A$. Let $o$ be the point of $A$ corresponding to the zero vector.

By $B$ a set of positive roots $R^+ \subset R$ is defined which determines the fundamental Weyl chamber

$$C_f := \{ x \in A : \langle x, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+ \}$$

with respect to $B$. By replacing some (which might be all or none) of the inequalities in the definition of $C_f$ by equalities we obtain faces of the fundamental Weyl chamber.

The spherical Weyl group $W$ of $R$ acts by reflections $r_\alpha, \alpha \in R$, on the model space $A$. The fixed point sets of the $r_\alpha$ are called hyperplanes and are denoted by $H_\alpha$. One has $H_\alpha = \{ x \in A : r_\alpha(x) = x \} = \{ x \in A : \langle x, \alpha^\vee \rangle = 0 \}$.

An affine Weyl group is the semidirect product of a group of translations $T$ of $A$ by $W$. If $T$ equals $A$, then $W := W \rtimes T$ is called the full affine Weyl group. The actions of $W$ and $T$ on $A$ induce an action of $W$ on $A$. An (affine) reflection is an element of $W$ which is conjugate in $W$ to a reflection $r_\alpha, \alpha \in R$. A hyperplane $H_r$ in $A$ is the fixed point set of an affine reflection $r$. It determines two half-spaces of $A$ called half-apartments.

We define a Weyl chamber in $A$ to be an image of a fundamental Weyl chamber under the affine Weyl group $W$. The image of the faces of the fundamental Weyl chamber then define the faces of this Weyl chamber. A face of a Weyl chamber will also be called a Weyl simplex. Note that a Weyl simplex $S$ contains exactly one point $x$ which is the intersection of all bounding hyperplanes of $S$. We call it the base point of $S$ and say $S$ is based at $x$.

Let $\Lambda$ be a totally ordered abelian group and let $X$ be a set. A metric on $X$ with values in $\Lambda$, short a $\Lambda$–valued metric, is a map $d : X \times X \mapsto \Lambda$ satisfying the usual axioms of a
metric. That is positivity, symmetry \(d(x, y) = d(y, x)\), equality \(d(x, y) = 0\) if and only if \(x = y\) and the triangle inequality for arbitrary triples of points. The pair \((X, d)\) is called \(\Lambda\)-metric space.

A particular \(W\)-invariant \(\Lambda\)-valued metric on the model space \(\mathbb{A}\) is defined by

\[
d(x, y) = \sum_{\alpha \in \mathbb{R}^+} |\langle y - x, \alpha \rangle|.
\]

A subset \(Y\) of \(\mathbb{A}\) is called convex if it is the intersection of finitely many half-apartments. The convex hull of a subset \(Y \subset X\) is the intersection of all half-apartments containing \(Y\).

Note that Weyl simplices and hyperplanes, as well as finite intersections of convex sets are convex.

**Definition 2.1.** Let \(X\) be a set and \(\mathcal{A}\) a collection of injective maps \(f : \mathbb{A} \hookrightarrow X\), called charts. The images \(f(\mathbb{A})\) of charts \(f \in \mathcal{A}\) are called apartments of \(X\). Define Weyl chambers, hyperplanes, half-apartments, ... of \(X\) to be images of such in \(\mathbb{A}\) under any \(f \in \mathcal{A}\). The set \(X\) is a (generalized) affine building with atlas (or apartment system) \(\mathcal{A}\) if the following conditions are satisfied

(A1) Given \(f \in \mathcal{A}\) and \(w \in W\) then \(f \circ w \in \mathcal{A}\).

(A2) Given two charts \(f, g \in \mathcal{A}\) with \(f(\mathbb{A}) \cap g(\mathbb{A}) \neq \emptyset\). Then \(f^{-1}(g(\mathbb{A}))\) is a convex subset of \(\mathbb{A}\). There exists \(w \in W\) with \(f|_{f^{-1}(g(\mathbb{A}))} = (g \circ w)|_{f^{-1}(g(\mathbb{A}))}\).

(A3) For any two points in \(X\) there is an apartment containing both.

(A4) Given Weyl chambers \(S_1\) and \(S_2\) in \(X\) there exist sub-Weyl chambers \(S'_1, S'_2\) in \(X\) and \(f \in \mathcal{A}\) such that \(S'_1 \cup S'_2 \subset f(\mathbb{A})\).

(A5) For any apartment \(A\) and all \(x \in A\) there exists a retraction \(r_{A,x} : X \to A\) such that \(r_{A,x}\) does not increase distances and \(r_{A,x}^{-1}(x) = \{x\}\).

(A6) If \(f, g\) and \(h\) are charts such that the associated apartments intersect pairwise in half-apartments then \(f(\mathbb{A}) \cap g(\mathbb{A}) \cap h(\mathbb{A}) \neq \emptyset\).

The dimension of the building \(X\) is \(n = \text{rank}(\mathbb{R})\), where \(\mathbb{A} \cong \Lambda^n\).

Condition (A1)-(A3) imply the existence of a \(\Lambda\)-distance on \(X\), that is a function \(d : X \times X \mapsto \Lambda\) satisfying all conditions of the definition of a \(\Lambda\)-metric but the triangle inequality. Given \(x, y\) in \(X\) fix an apartment containing \(x\) and \(y\) with chart \(f \in \mathcal{A}\) and let \(x', y'\) in \(\mathbb{A}\) be defined by \(f(x') = x, f(y') = y\). The distance \(d(x, y)\) between \(x\) and \(y\) in \(X\) is given by \(d(x', y')\). By Condition (A2) this is a well defined function on \(X\). Therefore it makes sense to talk about a distance non-increasing function in (A5). Note further that, by (A5), the defined distance function \(d\) satisfies the triangle inequality. Hence \(d\) is a metric on \(X\).

Recently the first author showed that Condition (A5) follows from the other conditions, see [SnH09] for details. However, in the proofs in the current paper either Condition (A5) will be either nearly trivial to prove, or we will use another (less involved) result of the same paper [SnH09] proving (A5).
2.2 Local and global structure of \( \Lambda \)–affine buildings

There are two types of spherical buildings associated to an affine \( \Lambda \)–building \((X, \mathcal{A})\) of type \( \Lambda \langle R, \Lambda \rangle \): the spherical building \( \partial_{\mathcal{A}}X \) at infinity and at each point \( x \in X \) a so-called residue \( \Delta_x \).

Two subsets \( \Omega_1, \Omega_2 \) of a \( \Lambda \)–metric space are at parallel if there exists \( N \in \Lambda \) such that for all \( x \in \Omega_i \) there exists an \( y \in \Omega_j \) such that \( d(x, y) \leq N \) for \( \{i, j\} = \{1, 2\} \). Note that parallelism is an equivalence relation. One can prove

**Proposition 2.2.** [Ben94, Section 2.4] Let \( \Lambda \langle R, \Lambda \rangle \) be the model space equipped with the full affine Weyl group \( W \).

1. Two hyperplanes, Weyl simplices are parallel if and only if they are translates of each other by elements of \( W \).
2. For any two parallel Weyl chambers \( S \) and \( S' \) there exists a Weyl chamber \( S'' \) contained in \( S \cap S' \) and parallel to both.

A simplex in the spherical building at infinity is a parallel class \( \partial S \) of a Weyl simplex \( S \) in \( X \). Hence as a set of simplices

\[
\partial_{\mathcal{A}}X = \{ \partial S : S \text{ is a Weyl simplex of } X \}.
\]

One simplex \( \partial S_1 \) is contained in a simplex \( \partial S_2 \) if there exist representatives \( S'_1, S'_2 \) which are contained in a common apartment with chart in \( \mathcal{A} \), having the same base point and such that \( S'_1 \) is contained in \( S'_2 \).

**Proposition 2.3.** The set \( \partial_{\mathcal{A}}X \) defined above is a spherical building of type \( R \) with apartments in one to one correspondence with apartments of \( X \).

**Proof.** See [Ben90, 3.6] or [Hit09a, 5.7].

To define a second type of equivalence relation on Weyl simplices we say that two of them, \( S \) and \( S' \), share the same germ if both are based at the same point and if \( S \cap S' \) is a neighborhood of \( x \) in \( S \) and in \( S' \). It is easy to see that this is an equivalence relation on the set of Weyl simplices based at a given point. The equivalence class of \( S \), based at \( x \), is denoted by \( \Delta_x S \) and is called germ of \( S \) at \( x \). The germs of Weyl simplices based at a point \( x \) are partially ordered by inclusion: \( \Delta_x S_1 \subset \Delta_x S_2 \) if there exist representatives \( S'_1, S'_2 \) contained in a common apartment such that \( S'_1 \) is a face of \( S'_2 \). Let \( \Delta_x X \) be the set of all germs of Weyl simplices based at \( x \).

**Proposition 2.4.** [Hit09a, 5.17] For all \( x \in X \) the set \( \Delta_x X \) is a spherical building of type \( R \) which is independent of \( \mathcal{A} \).

Let \( \mu \) be a germ of a Weyl simplex \( S \) based at \( x \). We say that \( \mu \) is contained in a set \( K \) if there exists an \( \varepsilon > 0 \) in \( \Lambda \) such that \( B_\varepsilon(x) \cap S \) is contained in \( K \).

The following properties will be of use in subsequent proofs of the present paper.

**Proposition 2.5.** Let \((X, \mathcal{A})\) be an affine building of type \( \Lambda \langle R, \Lambda \rangle \). Then:

1. Given two Weyl chambers \( S, T \) based at the same point \( x \). If their germs are opposite in \( \Delta_x X \) then there exists a unique apartment containing \( S \) and \( T \).
2. For any germ $\mu \in X$ the affine building $X$ is, as a set, the union of all apartments containing $\mu$.

Proof. Compare 5.23 and 5.13 of [Hit09a] for a proof. \hfill \square

The proof of the following proposition is the same as of Proposition 1.8 in [Par00].

Proposition 2.6. Let $(X,A)$ be an affine building and $c$ a chamber in $\partial A X$. For a Weyl chamber $S$ based at a point $x \in X$ there exists an apartment $A$ with chart in $A$ containing a germ of $S$ at $x$ and such that $c$ is contained in the boundary $\partial A$.

Given a germ of a Weyl chamber in a fixed apartment $A$ one can define a retraction of the building onto $A$ as follows.

Definition 2.7. Fix a germ $\mu$ of a Weyl chamber in $X$. Given a point $x$ in $X$ let $g$ be a chart in $A$ such that $x$ and $\mu$ are contained in $g(A)$. Define

$$r_{A,\mu}(x) := (f \circ w \circ g^{-1})(x)$$

where $w \in W$ is such that $g|_{g^{-1}(f(A))} = (f \circ w)|_{g^{-1}(f(A))}$. The map $r_{A,\mu}$ is called the retraction onto $A$ centered at $\mu$.

By item 2 of Proposition 2.5 this retraction is well defined. Furthermore, as proved in Appendix C of [Hit09a], it is distance non-increasing. Furthermore, the restriction of $r_{A,\mu}$ to an apartment containing $\mu$ is an isomorphism onto $A$.

We end these preliminaries by pointing out that by our main results (in particular Main Result 2, part 2) allow for more spherical buildings to be defined from a $\Lambda$-building than the two constructions mentioned in this section. In fact, one can associate a spherical building to each set of points with distance in a convex subgroup $\Lambda'$ of $\Lambda$ from a certain point of $X$. The spherical building at infinity and the residues correspond to the choices $\Lambda' = \Lambda$ and $\Lambda' = \{0\}$.

3 Proof of the first main result

Given an epimorphism $e : \Lambda \to \Lambda'$ of ordered abelian $\mathbb{Q}[\{\alpha \vee (\beta)\}_{\alpha, \beta \in \mathbb{R}}]$-modules $\Lambda$ and $\Lambda'$, we define the base change functor as follows. Let $(X,A)$ be an affine building with model space $\mathbb{A}(\mathbb{R},\Lambda)$ (or shortly $\mathbb{A}$) and distance function $d$ which is induced by the standard distance on the model space.

The relation “$\sim$” with $x \sim y$ when $d(x,y) \in \ker(e)$ is an equivalence relation (due to the triangle inequality). Let $X'$ be the quotient of $X$ defined by this equivalence relation. The associated quotient map $\phi : X \to X'$ is surjective by definition. One can define a metric $d'$ on $X'$ by putting $d'(\phi(x), \phi(y)) := e(d(x,y))$. This metric is well-defined due to the triangle inequality, one also easily checks it is indeed a metric. Let $\mathbb{A}'$ be the model space $\mathbb{A}(\mathbb{R},\Lambda')$ and $W'$ the associated affine Weyl group. In the same way as for $X$ one can define a map $\phi_{\mathbb{A}}$ from the model space $\mathbb{A}$ to $\mathbb{A}'$. Because the preimages of points under $\phi$ and $\phi_{\mathbb{A}}$ are the same, one can define for each chart $f \in A$, an injective map $f'$ such that $\phi \circ f$ equals $f' \circ \phi_{\mathbb{A}}$. 8
This way we have defined a set of charts \( \mathcal{A}' \) from \( \mathcal{A}' \) into \( X' \). Automatically we also have defined (half-)apartments, hyperplanes, Weyl chambers, \ldots in \( X' \). By construction these objects are the images under \( \phi \) of similar objects in \( X \).

Conditions (A1) and (A3)-(A6) for \((X',\mathcal{A}')\) are easy consequences of the fact that these conditions are already satisfied by \((X,\mathcal{A})\). The only non-trivial condition to check is Condition (A2).

### 3.1 Auxiliary lemmas

**Lemma 3.1.** Let \( x \) and \( x' \) be two points of the model space \( \mathbb{A} \) lying in two Weyl simplices \( S \) and \( S' \) both based at some point \( y \). Suppose that \( d(x,x') \in \ker e \). Then if \( S \) and \( S' \) do not have Weyl simplices in common, other than the base point, one has that \( d(x,y),d(x',y) \in \ker e \).

**Proof.** The images of the two Weyl simplices \( S \) and \( S' \) under \( \phi_\mathbb{A} \) are again two Weyl simplices \( \phi_\mathbb{A}(S) \) and \( \phi_\mathbb{A}(S') \) having no common Weyl simplices. So the intersection of \( \phi_\mathbb{A}(S) \) and \( \phi_\mathbb{A}(S') \) is the singleton \( \{ \phi_\mathbb{A}(y) \} \). As the point \( \phi_\mathbb{A}(x) = \phi_\mathbb{A}(x') \) lies in this intersection, one has that \( \phi_\mathbb{A}(x) = \phi_\mathbb{A}(x') = \phi_\mathbb{A}(y) \). By the definition of \( \phi_\mathbb{A} \), we conclude that \( d(x,y),d(x',y) \in \ker e \). \( \Box \)

**Lemma 3.2.** Suppose that some subset \( K \) of the model space \( \mathbb{A} \) is closed under taking convex hulls of pairs of points of \( K \). Then a germ based at a point \( k \in K \) lies in \( K \), if and only if, there is a point \( x \in K \) contained in the Weyl simplex \( S \) in \( \mathbb{A} \) corresponding to that germ, and \( S \) is the minimal Weyl simplex containing \( x \).

**Proof.** Let \( k \) be a point of \( K \) and \( \mu \) a germ of a Weyl simplex \( S \) based at \( k \). We have to prove that there exists an \( \varepsilon > 0 \) such that \( B_\varepsilon(k) \cap S \subset K \) if and only if there is a point \( x \in K \) lying in the Weyl simplex \( S \) corresponding to that germ, such that \( S \) is the minimal Weyl simplex containing \( x \).

First assume that there is such a point \( x \) in \( K \). Consider the minimal Weyl simplex \( T \) based at \( x \) containing \( k \). One has that \( \partial T \) is opposite \( \partial S \) in \( \partial \mathbb{A} \). By assumption, the convex hull of \( x \) and \( k \) is contained in \( K \). But since this convex hull is the intersection of \( S \) and \( T \) we have that there is an \( \varepsilon \leq d(x,k) \) such that \( S \cap B_\varepsilon(k) \) is contained in \( K \).

Conversely assume that \( \mu \) is a germ of a Weyl simplex \( S \) based at \( k \) contained in \( K \). So there exists an \( \varepsilon > 0 \) such that \( S \cap B_\varepsilon(k) \) is contained in \( K \). If we can prove that there exists a point \( x \) in \( S \) but not on a non-maximal face of this Weyl simplex, with distance less than \( \varepsilon \) to \( k \), we are done.

Let \( R, B \) and \( F \) be as in Section 2.1. Consider the submodule \( M \) of \( \Lambda \) spanned by \( \varepsilon \), this submodule is isomorphic to \( F \). Consider only the linear combinations \( \sum_{\alpha \in B} v_\alpha \alpha \), with \( v_\alpha \in M \), as points (see Section 2.1). This way, the problem is reduced to the case where \( \Lambda \) is isomorphic to \( F \), a subfield of the reals. Assume we are in this case, let \( y \) be a point of \( S \), not on a non-maximal face of the Weyl simplex. Due to the field nature of \( \Lambda \) one can now find an \( f \in F \) such that the product of \( f \) with distance \( d(o,y) \) is \( \varepsilon/2 \). Taking the scalar product of \( f \) with the vector corresponding to \( y \), one obtains a vector corresponding to a point \( x \) with the same properties as \( y \) but at distance \( \varepsilon/2 \) from \( o \). This concludes the proof. \( \Box \)
Lemma 3.3. Let $K$ be a convex subset of the model space $\mathbb{A}$, and $x$ a point of $\mathbb{A} \setminus K$. Then there exists a point $y \in K$ and Weyl simplex $S$ based at $y$, containing $x$, such that the intersection $S \cap K$ is exactly $y$.

Proof. Let $S$ be a Weyl simplex based at $x$ of minimal dimension while having a non-empty intersection with $K$. By the minimality no face of $S$ contains points of $K$ (except $S$ itself). Consider for each point $k$ in the intersection $S \cap K$ the Weyl simplex $S'_k$ based at $k$ and in the opposite direction of $S$. It follows that each such $S'_k$ contains $x$.

Let $y$ be a point of $S \cap K$ such that the face of the germ of $S'_y$ in $S \cap K$ is minimal. Suppose that $S'_y \cap S \cap K$ contains more than just the point $y$. If this is the case we can find some one-dimensional face $R$ of the Weyl simplex $S'_y$ such that $R \cap S \cap K$ contains more than just $y$. It is impossible that $R$ lies completely in $K$ as no (non-maximal) face of $S$ contains points of $K$ (while $R$ will contain a point of a face of $S$). So $R \cap K$ is a line segment bounded two endpoints, one is $y$, call the other $y'$ (the reason that there are two endpoints is convexity).

Consider the Weyl simplex $S'_{y'}$. By the previous lemma if a face of the germ of $S'_{y'}$ lies in $S \cap K$, then the corresponding face of the germ $S'_y$ based at $y$ obtained by translation will also be contained in $S \cap K$. Moreover the face of the germ of $S'_y$ in $S \cap K$ is strictly larger than the one of $S'_{y'}$, because the germ of $R$ is also contained in this last face. This violates minimality. It follows that $S'_y \cap S \cap K = \{y\}$.

As no (non-maximal) face of $S$ contains points of $K$, it follows from the convexity of $K$ that $S \cap K = \pi \cap K$ with $\pi$ the subspace (of the apartment) spanned by $S$. As $S'_y$ lies in $\pi$, this implies that $\{y\} = S'_y \cap S \cap K = S'_y \cap \pi \cap K = S'_y \cap K$. We conclude that the point $y$ and Weyl simplex $S'_y$ have the desired properties.

Lemma 3.4. Let $K$ and $K'$ be two convex subsets of respectively apartments $A$ and $A'$ of the affine building $(X, A)$, such that their intersection $K \cap K'$ is non-empty. If $x \in K$ and $x' \in K'$ are two points with $d(x, x') \in \ker e$, then there exists a point $z \in K \cap K'$ with $d(x, z), d(x', z) \in \ker e$.

Proof. Note that if we find a point $z \in K \cap K'$ such that $d(x, z) \in \ker e$, then also $d(x', z) \in \ker e$ holds by the triangle inequality. In order to exclude trivialities, we assume that $x, x' \notin K \cap K'$.

Let $K \cap K'$ and $x$ be the convex subset and point on which we apply Lemma 3.3. This lemma yields us a point $a$ of $K \cap K'$. Let $S$ and $S'$ be the minimal Weyl simplices in respectively $A$ and $A'$, both based at $a$ and containing respectively $x$ and $x'$. The germs of $S$ and $S'$ (which lie in respectively $K$ and $K'$ due to minimality and convexity) have no simplices in common, because otherwise the intersection of the Weyl simplex $S$ with $K$ and $K'$ would contain more than just $a$, contradicting the construction of $a$.

Let $R$ be a Weyl chamber in $A$ based at $a$ and containing $S$. Consider the retraction $r_{A,R}$ (see Definition 2.7). Let $R'$ be the image of $S'$ under this retraction. As the germs of $S$ and $S'$ have no simplices in common, the Weyl simplices $S$ and $R'$ do not have Weyl simplices in common either. Let $y$ be the image of $x'$ under the retraction, as the retraction is distance non-increasing, we have that $d(x, y) \in \ker e$. By Lemma 3.3 and the previous discussion, we obtain that $d(x, a) \in \ker e$. Put $z := a$ - this ends the proof.

Corollary 3.5. Let $S$ and $S'$ be two Weyl simplices of the $\Lambda$–building $(X, A)$ having non-empty intersection. If $x \in S$ and $x' \in S'$ such that $d(x, x') \in \ker e$, then there exists a point $z \in S \cap S'$ such that $d(x, z), d(x', z) \in \ker e$. 

10
Proof. Directly from the above lemma.

\textbf{Corollary 3.6.} Let $S$ and $S'$ be two parallel Weyl chambers of the $\Lambda$-building $(X, A)$, based at respectively $y$ and $y'$ with $d(y, y') \in \ker e$. Then there exists an isometry $\tau$ from $S$ to $S'$, such that if $x \in S$, then $d(x, \tau(x)) \in \ker e$.

Proof. As the intersection of $S$ and $S'$ is not empty, the above corollary shows the existence of a point $z$ in $S \cap S'$ such that $d(y, z), d(y', z) \in \ker e$. Consider the Weyl chamber $S''$ based at $z$ parallel to $S$ and $S'$. It is a sub-Weyl chamber of both $S$ and $S'$.

Let $A$ be an apartment containing $S$. There exists a translation of $A$ mapping $y$ to $z$ (and so also $S$ to $S''$). Each point of $A$ is mapped to another point of $A$ with the distance between both in $\ker e$. In the same way as we have defined an isometry from $S$ to $S''$ here, one can define an isometry from $S''$ to $S'$. Combining these two isometries forms an isometry $\tau$ from $S$ to $S'$ such that for each point $x$ in $S$ one has that $d(x, \tau(x)) \in \ker e$.

\textbf{Corollary 3.7.} If two apartments $A$ and $B$ of $(X, A)$ already intersect before applying $\phi$, then $(A2)$ will be satisfied for each pair of charts of the apartments $\phi(A)$ and $\phi(B)$ of $(X', A')$.

Proof. It is clear that $\phi(A \cap B)$ is a convex subset of $\phi(A)$, as the images of the finite number of half-apartments of $A$ with $A \cap B$ as intersection, will be a finite number of half-apartments of $\phi(A)$ with $\phi(A \cap B)$ as intersection.

Let $x \in A$ be a point such that $\phi(x) \in \phi(B)$. So there exists a point $x' \in B$ such that $\phi(x) = \phi(x')$, or equivalently $d(x, x') \in \ker e$. Lemma 3.4 implies that there is a point $z \in A \cap B$ with $d(x, z), d(x', z) \in \ker e$. So $\phi(x) = \phi(x') = \phi(z)$. We can conclude that $\phi(A) \cap \phi(B) = \phi(A \cap B)$, which is a convex set.

The second part of $(A2)$, this being the existence of a $w' \in W'$ with certain properties, follows directly from Condition $(A2)$ for the original building $(X, A)$.

\section*{3.2 Germs in $(X', A')$}

Although we did not prove Condition $(A2)$ yet, one can define germs in $(X', A')$ in the same way as in Section 2.2. These set of germs based at some point $\phi(x) \in X'$ forms again a simplicial complex $\Delta_{\phi(x)}X'$. The goal of this section that $\Delta_{\phi(x)}X'$ is again a spherical building. Denote by $X''$ all the points in $X$ such that the distance to the point $x$ lies in $\ker e$. The germs of Weyl chambers at $\phi(x)$ are the chambers of $\Delta_{\phi(x)}X'$.

\textbf{Lemma 3.8.} Two germs at $\phi(x)$ lie in a common apartment of $(X', A')$.

Proof. This follows directly from a proposition with the same statement and proof of Proposition 2.6.

\textbf{Lemma 3.9.} Two Weyl chambers $S_1$ and $S_2$ in $(X, A)$ based at respectively $x_1$ and $x_2$, such that $\phi(x_1) = \phi(x_2) = \phi(x)$, give rise to the same germ $\Delta_{\phi(x)}\phi(S_1) = \Delta_{\phi(x)}\phi(S_2)$ in $X'$, if and only if, there exists a point $x' \in X'' \cap S_1 \cap S_2$ such that the sub-Weyl chambers of $S_1$ and $S_2$ based at $x'$ are identical restricted to $X''$. 

11
Proof. Assume that $S_1$ and $S_2$, with properties as stated in the lemma, give rise to the same germ $\Delta_{\phi(x)}\phi(S_1) = \Delta_{\phi(x)}\phi(S_2)$. Corollary 3.5 implies the existence of a point $z$ in $S_1 \cap S_2$ such that $d(z, x_i) \in \ker(e)$ for $i = 1, 2$. Hence $\phi(z) = \phi(x_i) = \phi(x)$ for $i = 1, 2$. Let $S'_1$ denote the sub Weyl chamber of $S_1$ based at $z$. It remains to prove that $S'_1 \cap X'' = S'_2 \cap X''$.

Apply Corollary 3.6 to the pairs $S_i$, $S'_i$ to obtain that $\Delta_{\phi(x)}\phi(S'_1) = \Delta_{\phi(x)}\phi(S'_2)$, so $\Delta_{\phi(x)}\phi(S'_1) = \Delta_{\phi(x)}\phi(S'_2)$. This makes it possible to find points $z_1$ of $S'_1$ and $z_2$ of $S'_2$ such that $\phi(z_1) = \phi(z_2)$, and such that this image does not lie on the (non-maximal) faces of the Weyl chambers $\phi(S'_1)$ and $\phi(S'_2)$. Apply Lemma 3.10 to $S'_1$ and $S'_2$ and obtain a point $z' \in S'_1 \cap S'_2$ such that $\phi(z) = \phi(z_1) = \phi(z_2)$.

The convex hull of $z$ and $z'$ lies in both $S'_1$ and $S'_2$, and contains both $S'_1 \cap X''$ and $S'_2 \cap X''$, this proves that $S'_1 \cap X'' = S'_2 \cap X''$, and thus the first implication follows.

We now prove the other direction. Let $F_1$ and $F_2$ be two one-dimensional Weyl simplices of the same type, with source $x'$, and assume that they are faces of the Weyl chambers parallel to respectively $S_1$ and $S_2$ with source $x'$. The intersection $F_1 \cup F_2$ is convex in $F_1$, thus either $F_1 = F_2$, or there exists a point $a \in F_1$ such that $F_1 \cap F_2$ equals the convex hull of $a'$ and $a$. As $F_1 \cap X'' \subset F_1 \cap F_2$, the distance between $x'$ and $a$ does not lie in $\ker(e)$. This implies that $\Delta_{\phi(x)}\phi(F_1) = \Delta_{\phi(x)}\phi(F_2)$. Repeating this argument proves eventually that $\Delta_{\phi(x)}\phi(S_1) = \Delta_{\phi(x)}\phi(S_2)$. \hfill $\square$

Corollary 3.10. If the intersection $\phi(A) \cap \phi(B)$ of the images of two apartments $A$ and $B$ contains a germ of a Weyl chamber, then the intersection $A \cap B$ is non-empty.

Proof. By applying the above lemma to two Weyl chambers in $A$ and $B$ for which the base points have an identical image under $\phi$, and such that the images of the Weyl chambers have the same germs. \hfill $\square$

Note that this implies that if two apartments are identical after the base change functor, then they are also identical before it. Once we proved that $(X', A')$ is indeed a $\Lambda'$–building, it follows that the buildings at infinity of $(X, A)$ and $(X', A')$ are isomorphic.

Lemma 3.11. The set of germs $\Delta_{\phi(x)}X'$ forms a spherical building.

Proof. The germs in $\Delta_{\phi(x)}X'$ form a simplicial complex. Each two simplices lie in an apartment due to Lemma 3.8. The second property to check is if one has two apartments $\phi(A)$ and $\phi(B)$ containing $\phi(x)$, that when the corresponding apartments $\Delta_{\phi(x)}\phi(A)$ and $\Delta_{\phi(x)}\phi(B)$ share a chamber $C$ and simplex $D$, that there exist an isomorphism from $\Delta_{\phi(x)}A$ to $\Delta_{\phi(x)}B$, preserving $C$ and $D$ pointwise. The existence of this isomorphism now follows from Corollary 3.7 and the fact that $A$ and $B$ intersect due to the above corollary. \hfill $\square$

3.3 Proof of Condition (A2)

To prove the first main result the only non-trivial condition to check is (A2). In the following let $A$ and $B$ be two apartments of $(X, A)$ such that the intersection $K := \phi(A) \cap \phi(B)$ is non-empty.

Lemma 3.12. Suppose $x \in A$ is a point such that $\phi(x) \in K$. Let $S$ be a Weyl chamber of $A$ based at $x$. Then there is a convex set $K_{S,x} \subset S$, containing $x$ and such that $\phi(K_{S,x}) = K \cap \phi(S)$. 

12
Proof. As \( \phi(x) \in K \) there exists a point \( x' \in B \) such that \( \phi(x) = \phi(x') \). Let \( S' \) be the Weyl chamber parallel to \( S \) and based at \( x' \). Corollary \[3.6\] defines an isometry \( \tau \) from \( S' \) to \( S \). The intersection \( K' \) of \( S' \) with \( B \) will be convex, also its image under \( \tau \) will be convex. Denote this image by \( K_{S,x} \). This is a closed and convex subset of \( S \) and hence also of \( A \).

If \( y \) is an element of \( K_{S,x} \), then its preimage \( y' \) under \( \tau \) will by construction be in \( B \). Corollary \[3.3\] tells us that \( d(y,y') = d(\tau(y'),y') \in \ker e \), so \( \phi(y) = \phi(y') \in K \cap \phi(S) \). This implies that \( \phi(K_{S,x}) \) is contained in \( K \cap \phi(S) \).

Now let \( z \) be a point of \( S \) such that \( \phi(z) \in K \). This implies that there exists a \( z' \in B \) with \( \phi(z) = \phi(z') \). Let \( z'' \in S' \) be the preimage of \( z \) under \( \tau \). Note that the triangle inequality implies that \( d(z',z'') \in \ker e \). As both \( B \) and \( S' \) are convex subsets of apartments, we can apply Corollary \[3.3\] to find a point \( a \) in \( K' = B \cap S' \) having distances in \( \ker e \) to both \( z' \) and \( z'' \). So we conclude that \( \phi(\tau(a)) = \phi(z') = \phi(z) \), and thus \( \phi(K_{S,x}) \supset K \cap \phi(S) \), or by combining both inequalities \( \phi(K_{S,x}) = K \cap \phi(S) \).

\( \square \)

Lemma 3.13. The set \( K \) is convex in \( \phi(A) \).

Proof. Let \( x \) be a point of \( A \) such that \( \phi(x) \in K \). For each Weyl chamber \( S_i \) based at \( x \) (with \( i \in \{1, \ldots, n\} \), \( n \) being the number of Weyl chambers in \( A \) based at \( x \)), we can define a convex subset \( K_i := K_{S_i,x} \) of \( S_i \) using the previous lemma. We have that \( \phi(\bigcup_i K_i) = K \).

Label the roots in \( R \) with values in \( \{1,2,\ldots,m\} \), with \( m = |R| \). So we have roots \( r_1,\ldots,r_m \). For the convex subset \( K_i \), let \( K_i^j \) be the minimal positive half-apartment defined by the root \( r_j \) containing \( K_i \). That is \( K \) is a set \( H_{i,k_i}^+ := \{ x : \langle x, r_j \rangle \geq k_i \} \) with \( k_i \in \Lambda \cup \{-\infty\} \) as small as possible. For the purpose of the proof, we interpret \( H_{i,\infty}^+ \), which is the whole apartment, also as a half-apartment. As each \( K_i \) is convex, it follows easily that \( K_i = \bigcap_j K_i^j \).

For \( j \in \{1,\ldots,m\} \), define \( K_j \) to be the maximal half-apartment corresponding to the root \( r_j \) in the set \( \{K_1^j, K_2^j,\ldots,K_n^j\} \).

Let \( K' \) be the intersection \( \bigcap_j \phi(K_j^j) \). It is clear that \( K \subset K' \) as each \( \phi(K_i) \) will be a subset of \( K' \). Also, \( K' \) will by construction be the intersection of all half-apartments of \( \phi(A) \) containing \( K \), and so \( K' \) and \( \phi(K_j) \) will be independent of the choice of \( x \). If we show that \( K = K' \), then the lemma is proven. Suppose that \( k' \) is a point of \( K' \) but not of \( K \).

Let \( F \) be the minimal Weyl simplex based at \( x \), containing \( k' \). Then applying Lemma \[3.3\] to the set \( F \cap K \) (which is convex by the previous lemma), one obtains a point \( k \) and a minimal Weyl simplex \( S \) in \( \phi(A) \), containing \( k' \) and based at \( k \), such that \( S \cap F \cap K = \{k\} \). Note that again by the previous lemma the set \( S \cap K \) is a convex subset of the apartment \( \phi(A) \), which certainly contains \( k \), if it would contain more than this contradicts Lemma \[3.2\]. One can conclude that we have constructed a point \( k \) and Weyl simplex \( S \) in \( \phi(A) \), containing \( k' \), based at \( k \), such that \( S \cap K = \{k\} \).

Lemma \[3.11\] shows that \( \Delta_k X' \) is a spherical building in which \( \phi(A) \) and \( \phi(B) \) define two apartments \( \Delta_k \phi(A) \) and \( \Delta_k \phi(B) \). The intersection of both is a convex subset of \( \Delta_k \phi(A) \). By [AB08] Prop. 3.137 convex subsets of apartments are finite intersections of half-apartments. So there exists a half-apartment \( \Delta_k H \) of \( \Delta_k \phi(A) \) in the spherical building \( \Delta_k X' \), which does not contain \( \Delta_k \phi(S) \).

Let \( H \) be the half-apartment of \( A \) corresponding to \( \Delta_k H \). Using Lemma \[3.2\] it follows that this half-apartment contains \( K \) completely, but intersects \( S \) in exactly \( \{k\} \). So \( k' \) does not lie in this half-apartment. We have obtained a contradiction and proved convexity. \( \square \)
Lemma 3.14. Given two charts \( f, g \in A' \) with \( f(A') \cap g(A') \neq \emptyset \). Then there exists a \( w \in W \) with \( f|_{f^{-1}(g(A'))} = (g \circ w)|_{f^{-1}(g(A'))} \).

Proof. Denote the apartment defined by the chart \( f \) by \( A \), the one defined by the chart \( g \) by \( B \). First note that if \( A \) and \( B \) do not have points in common, there is nothing to prove. So suppose that there exists a point \( k \in A \cap B \). Using Condition (A1) we can assume without loss of generality that \( f(o) = g(o) = k \).

Consider the spherical building \( \Delta_k X' \) (see Lemma 3.11), of which \( \Delta_k A \) and \( \Delta_k B \) are apartments. The fundamental Weyl chamber of \( \Delta \) defines a certain Weyl chamber based at \( k \) in the apartment \( A \), and hence a chamber in \( \Delta_k A \), denote this chamber by \( C_A \). Similarly define \( C_B \) in \( \Delta_k B \). A well known property of spherical buildings tells us that there is an isomorphism \( \sigma \) from the apartment \( \Delta_k A \) to the apartment \( \Delta_k B \) preserving the intersection of both.

Let \( \bar{w} \in W \) be the unique element in the spherical Weyl group acting on \( \Delta_k B \) which maps \( C_B \) to \( \sigma(C_A) \). Interpreting this element in the larger affine Weyl group one sees that \( h := g \circ \bar{w} \circ f^{-1} \) is an isometry from \( A \) to \( B \) mapping the germ corresponding to \( C_A \) to \( \sigma(C_A) \).

In order to show that \( f|_{f^{-1}(g(A'))} = (g \circ w)|_{f^{-1}(g(A'))} \), we need to prove that if \( k' \in A \cap B \) then \( h(k') = k' \). If we take a look at the action of \( h \) the germs in \( \Delta_k A \cap \Delta_k B \), we see that these are fixed. This follows from the fact that if \( h \) maps \( C_A \) to \( \sigma(C_A) \) that each germ \( F_A \) in \( \Delta_k A \) will be mapped to \( \sigma(F_A) \) in \( \Delta_k B \), so those in the intersection are fixed.

Now let \( k' \) be an arbitrary point in \( A \cap B \) different from \( k \). Let \( S_A \) be the minimal Weyl simplex in \( A \) based at \( k \) and containing \( k' \). The germ of this simplex is fixed, as it is in the intersection of \( A \) and \( B \) (see Lemma 3.2). It now easily follows from \( h \) being an isometry that \( k' \) is fixed.

This completes the proof of 1 of the first main result. The remaining parts are done in the subsequent section.

3.4 Remaining parts of the proof

It remains to prove assertions 2 and 3 of the first main result.

Proposition 3.15. Let \((X,d), e, \phi \) and \( \Lambda \) and \( \Lambda' \) be as in the first main result and let \((Z,d')\) be a \( \Lambda' \)-metric space. If \( \psi : X \to Z \) is a surjective map such that for all \( x,y \in X \) we have that

\[ e(d(x,y)) = d'(\psi(x),\psi(y)), \]

then there exists an isometry \( \mu : X' \to Z \) with \( \mu \circ \phi = \psi \). The space \( Z \) can be endowed with an induced atlas such that \( X' \) and \( Z \) are isomorphic as affine buildings.

Proof. First we will prove that there exist an isometry (i.e. surjective isometric embedding) \( \mu : X' \to Z \) such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X' \\
\downarrow{\psi} & & \downarrow{\mu} \\
\ & Z & \\
\end{array}
\]
We define \( \mu : X' \to Z \) by putting \( \mu(x') \) to be \( \psi(x) \) for some \( x \in \phi^{-1}(x') \). It is easy to see that \( \phi(x) = \phi(y) \) if and only if \( \phi(x) = \phi(y) \) and hence if and only if \( d(x, y) \in \ker(e) \). Hence \( \mu \) is well defined.

Given \( z \in Z \) we can pick a point \( z' \in X \) with \( \psi(z') = z \). It is easy to verify that \( \mu(\phi(z)) = z \) and hence \( \mu \) is onto.

To prove that \( \mu \) is an isometric embedding let \( x' \) and \( y' \) be points in \( X' \). Choose preimages \( x, y \in X \) of \( x', y' \), respectively, under \( \phi \). We can calculate

\[
d'(\mu(x'), \mu(y')) = d'(\psi(x), \psi(y)) = e(d(x, y)) = d'(\phi(x), \phi(y)) = d'(x', y').
\]

We can define \( \nu : Z \to X' \) by \( \nu(z') = \phi(z) \), where \( z \) is chosen such that \( \psi(z) = z' \). With this we obtain, similar as above, an isometry from \( Z \) to \( X' \).

It is clear that we can take the isometric images of apartments in \( X' \) to be the apartments of \( Z \) and thus obtain an atlas for \( Z \) making \( Z \) into a building which is, by construction, isomorphic to \( X' \).

If one has two \( \Lambda \)–buildings \((X_1, A_1)\) and \((X_2, A_2)\) with an isometric embedding \( \psi \) from \( X_1 \) to \( X_2 \), then it is easy to find a map \( \phi' \) between the resulting \( \Lambda' \)–buildings \((X_1', A_1')\) and \((X_2', A_2')\) by the base change functors \( \phi_1 \) and \( \phi_2 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\psi} & X_2 \\
\downarrow{\phi_1} & & \downarrow{\phi_2} \\
X_1' & \xrightarrow{\psi'} & X_2'
\end{array}
\]

**Lemma 3.16.** With assumptions and notation as in the first main result let \( G \) be a group acting on \( X \) by isometries, then \( G \) acts on \( X' \) by isometries and the map \( \phi \) is \( G \)–equivariant, i.e. assertion \( \heartsuit \) of the first main result holds.

**Proof.** By functoriality. \( \square \)

This completes the proof of the first main result.

### 4 Proof of the second main result

Let \((X, A)\) be an affine building with model space \( \mathbb{A} := \mathbb{A}(R, \Lambda) \). As always assume that \( \Lambda \) admits an \( F \)–module structure and let \( e : \Lambda \to \Lambda' \) be an epimorphism of ordered abelian \( F \)–modules. The kernel of \( e \) is again an ordered abelian group admitting a natural \( F \)–module structure.

With notation as above denote by \( \phi \) the base change functor associated to \( e \), as defined at the beginning of Section 3. Then for arbitrary \( x \in \mathbb{A} \) the set \( \phi^{-1}(\phi(x)) \) is canonically isomorphic to \( \mathbb{A}'' := \mathbb{A}(R, \ker(e)) \). According to the first main result, the image of \( \phi \) carries the structure of a \( \Lambda' \)–building \((X', A')\) with model space \( \mathbb{A}' = \mathbb{A}(R, \Lambda') = \phi(\mathbb{A}) \).

Fix some point \( x \in X \) for the remainder of the section. Define \( X'' \) to be the set \( \phi^{-1}(\phi(x)) \). This will be the set of points of the \( \ker(e) \)–building we want to construct. Denote with \( o \) the point in \( \mathbb{A} \) corresponding to the zero vector. Let \( \mathbb{A}'' \) be the points \( y \) in \( \mathbb{A} \) such that \( d(o, y) \in \ker e \). Using the coordinate description of \( \mathbb{A} \) it follows directly that \( \mathbb{A}'' \) is
canonically isomorphic to $\mathcal{A}'' = \mathcal{A}(\mathbb{R}, \ker(e))$. The Weyl group $W'' = \mathcal{W}T''$ of $\mathcal{A}''$, with $T'' = T \cap \mathcal{A}''$, can canonically be interpreted as a subgroup of $W$. Note that the elements of $W''$ are those elements $w$ of $W$ such that $w.o \in \mathcal{A}''$.

Let $\bar{A}$ be the set of charts $f$ in $\mathcal{A}$ such that $\phi(x) \in \phi(f(\mathcal{A}''))$. As the charts in $\mathcal{A}$ are isometries, it follows from the triangle inequality that for each $f \in \bar{A}$ one has $f(\mathcal{A}'') = f(\mathcal{A}) \cap X''$. From this we can define injections $\mathcal{A}'' = \{f|_{\mathcal{A}''} : f \in \bar{A}\}$ from $\mathcal{A}''$ into $X''$. We now claim that $(X'', \mathcal{A}'')$ is a $\ker(e)$-building.

A first observation is that apartments in $X''$ will be intersections of $X''$ with apartments of $X$ containing a point of $X''$ (because $\forall f \in \bar{A}: f(\mathcal{A}'') = f(\mathcal{A}) \cap X''$).

**Lemma 4.1.** The pair $(X'', \mathcal{A}'')$ satisfies Conditions (A1)-(A3) and (A5).

**Proof.** To prove (A1) let $f''$ be an element of $\mathcal{A}''$ and $w \in W''$. By definition $f''$ is the restriction of a chart $f \in \mathcal{A}$. Condition (A1) applied to $(X, \mathcal{A})$ implies that $f \circ w \in \mathcal{A}$. As $\phi(x) \in \phi(f(\mathcal{A}'')) = \phi((f \circ w)(\mathcal{A}''))$, one has that $f \circ w \in \bar{A}$. So $f \circ w|_{\mathcal{A}''} \in \mathcal{A}''$. This chart is essentially $f'' \circ w$, so we have proven (A1).

The first part of (A2) is easily seen to be true because the intersection of a convex set in $\mathcal{A}$ with $\mathcal{A}'$ is a convex set of $\mathcal{A}''$ (as the intersection of a finite number of half-apartments of $\mathcal{A}$ with $\mathcal{A}''$ is the intersection of a finite number of half-apartments of $\mathcal{A}''$). The second part follows if we can show that if one has $f \circ w(y) = g(y)$ with $f, g \in \bar{A}$, $w \in W$ and $y \in \mathcal{A}'$, then $w \in W''$. If this was not the case then $w.y \notin \mathcal{A}''$ and $(f \circ w)(y) \notin X''$ as $f(\mathcal{A}'') = f(\mathcal{A}) \cap X''$. But $g(y) \in X''$, so we can conclude that $w \in W''$ and that (A2) holds.

Condition (A3) follows from the fact that apartments in $X''$ are the intersections of $X''$ with apartments of $X$ containing a point of $X''$.

In order to prove (A5) observe that apartments of $X''$ are, as sets, intersections of apartments of $X$ with $X''$. Hence given a point $x$ and apartment $A''$ of $(X'', \mathcal{A}'')$ containing $x$ the restriction to $X''$ of a retraction based at $x$ onto an apartment $A$ of $X$ such $A \cap X'' = A''$ satisfies the desired conditions, since retractions are distance non-increasing. $\blacksquare$

To finish the proof of the second main result we have to verify that (A4) and (A6) hold as well. This is done in the subsequent propositions.

**Proposition 4.2.** The pair $(X'', \mathcal{A}'')$ satisfies (A4).

**Proof.** Let $S''$ and $T''$ be two Weyl chambers of $(X'', \mathcal{A}'')$. These Weyl chambers are restrictions of Weyl chambers respectively $S$ and $T$ of $(X, \mathcal{A})$ (not necessarily unique) to $X''$. These two Weyl chambers give rise to chambers $\Delta_{\phi(x)}\phi(S)$ and $\Delta_{\phi(x)}\phi(T)$ of the spherical building $\Delta_{\phi(x)}X'$. Lemma 3.8 implies that there exists an apartment $A'$ of $(X', \mathcal{A}')$ containing both germs. By construction of $(X', \mathcal{A}')$ there also exists an apartment $A$ of $(X, \mathcal{A})$ such that $\phi(A) = A'$. Let $S'$ and $T'$ be Weyl chambers of $A$, both based at some point $y \in X''$, for which the images correspond respectively to the germs $\Delta_{\phi(x)}\phi(S)$ and $\Delta_{\phi(x)}\phi(T)$.

The (non-empty) intersection $X'' \cap A$ is an apartment $A''$ of $(X'', \mathcal{A}'')$. The Weyl chambers $S$ and $S'$ of $(X, \mathcal{A})$ both give rise to the same germ $\Delta_{\phi(x)}\phi(S)$ of $(X', \mathcal{A}')$, so Lemma 3.9 implies that the apartment $A''$ of $(X'', \mathcal{A}'')$ contains a sub-Weyl chamber of $S''$. A similar argument asserts that it also contains a sub-Weyl chamber of $T''$. This proves (A4) for $(X'', \mathcal{A}'')$. $\blacksquare$
With Condition (A1)-(A5) we have enough information to construct a spherical building \( \partial_{x'}X'' \) at infinity of \( (X'', A'') \). The next lemma shows that this spherical building is essentially \( \Delta_{\phi(x)}X' \).

**Lemma 4.3.** There is a canonical isomorphism between \( \partial_{A'}X'' \) and \( \Delta_{\phi(x)}X' \).

**Proof.** Lemma 3.9 implies a bijection between the chambers of these two spherical buildings. It is easily seen that this bijection preserves adjacency, from which follows that this bijection defines an isomorphism between the two buildings. \( \square \)

**Proposition 4.4.** The pair \( (X'', A'') \) satisfies (A6).

**Proof.** Let \( A_1, A_2 \) and \( A_3 \) be three apartments of \( X'' \) which pairwise intersect in half-apartments. The boundaries \( \partial A_i \) do as well intersect in half-apartments and correspond to apartments \( a_i \) in the residue of \( \phi(x) \) in \( X' = \phi(X) \) by Lemma 4.3. There are two cases: either a) there exists a half-apartment \( \alpha \) in \( \Delta_{\phi(x)}X' \) such that \( a_i \cap a_j = \alpha \) for all \( i \neq j \) with \( \{i, j\} \subset \{1, 2, 3\} \), or b) the intersections \( a_i \cap a_j \) are distinct for all three pairs of elements of \( \{1, 2, 3\} \), that is \( a_1 \cup a_2 \cup a_3 = (a_1 \cap a_2) \cup (a_1 \cap a_3) \cup (a_2 \cap a_3) \).

In case b) choose an apartment \( B'_i \) in \( X' \) such that \( \Delta_{\phi(x)}B'_i = a_1 \). Let \( c \) be a chamber in \( (a_2 \cap a_3) \) sharing a panel with \( a_1 \). Let \( T \) be the unique Weyl chamber \( T \) based at \( \phi(x) \) with germ \( c \) such that \( \partial T \) shares a panel with \( \partial B_1 \). Then there exist precisely two Weyl chambers \( S_2, S_3 \) in \( B_1 \) such that \( \partial S_i \) is opposite \( \partial T \) in \( \partial_{A'}X' \) for \( i = 2, 3 \). By construction \( \Delta_{\phi(x)}S_i \) is opposite \( c \) in \( \Delta_{\phi(x)}X' \). Hence there exist, by [1] of [2.5] unique apartments \( B_2 \) and \( B_3 \) containing \( T \) and \( S_2 \), respectively \( S_3 \). It is easily seen that the apartments \( B_i \) pairwise intersect in half-apartments. By construction \( \phi(x) \) is contained in their intersection which is, using (A6) of \( X' \) nonempty. From the definition of apartments in \( (X', A') \) it follows that there are three apartments \( B_i \) such that \( \phi(B_i) = B'_i \) for all \( i = 1, 2, 3 \). The intersection \( B_1 \cap B_2 \cap B_3 \) is nonempty by (A6) for \( X \). Lemma 3.9 implies that the boundary of \( B_i \cap X'' \) in \( X'' \) is the same as the boundary of \( A_i \), since their images under \( \phi \) induce the same germs in \( \Delta_{\phi(x)}X' \). Therefore \( B_i \cap X'' = A_i \) for all \( i \). The fact that \( \phi(x) \) is contained in \( B_1 \cap B_2 \cap B_3 \) implies that \( A_1 \cap A_2 \cap A_3 = B_1 \cap B_2 \cap B_3 \cap X'' \neq \emptyset \), and that (A6) holds for \( X'' \) in case b).

Assume that we are in case a). Let \( c \) be a chamber of the half-apartment \( \alpha \). With this chamber there corresponds a parallel class of Weyl chambers of \( (X'', A'') \). By applying Condition (A4) each two of these Weyl chambers have (at least) a common sub-Weyl chamber. If one takes three Weyl chambers \( S_1, S_2 \) and \( S_3 \) in this parallel class respectively in apartments \( A_1, A_2 \) and \( A_3 \), then it follows that these three Weyl chambers have a common intersection, so also the apartments have a common intersection. This completes the proof of the proposition. \( \square \)

Combining the previous propositions the second Main Result follows.

## 5 Proof of the third main result

Let \( R \) be a root system and let \( F \) be a subfield of \( \mathbb{R} \) containing the set of evaluations of co-roots on roots. Let \( e : \Lambda \rightarrow \Lambda' \) be a monomorphism of ordered abelian groups \( \Lambda \) and \( \Lambda' \) both admitting an \( F \)-module structure.
Using this monomorphism one can define a natural embedding of the model space \( \mathbb{A} = \mathbb{A}(R, \Lambda) \) into \( \mathbb{A}' = \mathbb{A}(R, \Lambda' ) \) as follows. Choose a basis \( B \) of \( R \), then each element \( x \) of \( \mathbb{A} \) has a presentation as \( x = \sum_{\alpha \in B} \lambda_{\alpha} \alpha \). Define \( \iota : \mathbb{A} \rightarrow \mathbb{A}' \) by

\[
x = \sum_{\alpha \in B} \lambda_{\alpha} \alpha \mapsto \iota(x) = \sum_{\alpha \in B} e(\lambda_{\alpha}) \alpha.
\]

Since \( e \) is a monomorphism \( \iota \) is injective. Given points \( x = \sum_{\alpha \in B} \lambda_{\alpha} \alpha \) and \( y = \sum_{\alpha \in B} \mu_{\alpha} \alpha \) in \( \mathbb{A} \) we can calculate the distance \( d' \) in \( \mathbb{A}' \) of the images under \( \iota \) of these two points of \( \mathbb{A} \):

\[
d'(\iota(x), \iota(y)) = \sum_{\beta \in R^+} |(\iota(y) - \iota(x), \beta^\vee)|
= \sum_{\beta \in R^+} |(\sum_{\alpha \in B} (e(\mu_{\alpha}) - e(\lambda_{\alpha})) \alpha, \beta^\vee)|
= e(\sum_{\beta \in R^+} |(\sum_{\alpha \in B} (\mu_{\alpha} - \lambda_{\alpha}) \alpha, \beta^\vee)|)
= e(d(x,y)),
\]

hence \( \iota \) is an embedding of \( \mathbb{A} \) into \( \mathbb{A}' \). Similarly one can embed the affine Weyl group \( W \) of \( \mathbb{A} \) into the affine Weyl group \( W' \) of \( \mathbb{A}' \). Each element \( w \) of \( W \) is a product of a translation \( t \) and an element \( \overline{w} \in \overline{W} \) of the spherical Weyl group, Since elements of the translation subgroup \( T \) of \( W \) (respectively \( T' \) of \( W' \)) can canonically be identified with points in \( \mathbb{A} \), (respectively points in \( \mathbb{A}' \)), we can define a group monomorphism \( \hat{i} : W \rightarrow W' \) by putting \( w = \overline{w}t \mapsto \hat{i}(w) := \overline{w}t \). Hence \( W \) can be naturally identified with a subgroup of \( W' \).

Let \( H \) be a half-apartment of \( \mathbb{A} \). If \( \iota \) is not an isomorphism then the image of \( H \) under it is not a half-apartment of \( \mathbb{A}' \), but one do can easily find a half-apartment \( H' \) of \( \mathbb{A}' \) such that \( \forall a \in A : a \in H \) if and only if \( \iota(a) \in H' \). This allows us to define a map \( \hat{i} \) from the half-apartments of \( \mathbb{A} \) to half-apartments of \( \mathbb{A}' \). As each convex subset of \( \mathbb{A} \) is the intersection of a finite number of half-apartments of \( \mathbb{A} \), one can extend the map \( \hat{i} \) to send convex subsets of \( \mathbb{A} \) to convex subsets of \( \mathbb{A}' \), such that for each point \( a \) and convex subset \( K \) of \( \mathbb{A} : a \in K \Leftrightarrow \iota(a) \in \hat{i}(K) \). It also easily follows for a convex subset \( K \) of \( \mathbb{A} \) and group element \( w \in W \) that \( \hat{i}(w.K) = \hat{i}(w).\hat{i}(K) \).

Let \((X, A)\) be an affine building with model space \( \mathbb{A} = \mathbb{A}(\Lambda, R) \) and \( \iota \) an embedding of \( \mathbb{A} \) into \( \mathbb{A}' \) induced by a monomorphism \( e \) as above. We now define a space \( X' \) and a set of charts \( \mathcal{A}' \) which will turn out to be the generalized affine building whose existence is claimed in the third main result.

We denote by \( \tilde{X} \) the set of pairs \( A \times A' \). Let \((f, a)\) and \((g, b)\) be two pairs of \( \tilde{X} \). By Condition (A2) we know that the inverse image \( Z := f^{-1}(g(\tilde{A})) \) is a convex subset of \( \mathbb{A} \) and that there exists a \( w \in W \) such that \( f|Z = g \circ w|Z \). We now define \((f, a)\) and \((g, b)\) to be equivalent (denoted by \((f, a) \sim (g, b)\)) if and only if \( a \in \hat{i}(Z) \) and \( \iota(w).a = b \). This is independent of the choice of \( w \in W \), as an other choice \( w' \) would necessarily have the same action on \( Z \), and so also the action of \( \iota(w) \) and \( \iota(w') \) on \( \hat{i}(Z) \) would be the same.

**Lemma 5.1.** Let \((f, a)\) and \((g, b)\) be two pairs of \( \tilde{X} \) and \( v \in W \), then \((f, a) \sim (g, b) \Leftrightarrow (f, a) \sim (g \circ v^{-1}, \hat{i}(v).b) \).
Proof. First of all notice that \( Z = f^{-1}(g(\mathbb{A})) = f^{-1}((g \circ v^{-1})(\mathbb{A})) \), and that if \( w \) is an element of \( W \) such that \( f|_Z = g \circ w|_Z \), then also \( f|_Z = (g \circ v^{-1}) \circ (v \circ w)|_Z \).

The condition for \((f, a) \sim (g, b)\) to hold is that \( a \in \iota(Z) \) and \( \iota(w).a = b \) with \( w \) as above. On the other hand the condition for \((f, a) \sim (g \circ v^{-1}, \iota(v).b)\) is \( a \in \iota(Z) \) and \( \iota(v \circ w).a = \iota(v).b \). One easily sees that these conditions are the same. \( \square \)

Corollary 5.2. Let \((f, a)\) and \((g, b)\) be two pairs of \( \tilde{X} \) and \( w \in W \) such that \( f|_Z = g \circ w|_Z \) with \( Z = f^{-1}(g(\mathbb{A})) \), then \((f, a) \sim (g, b) \iff (f, a) \sim (g \circ w, a)\) and \( \iota(w).a = b \).

Proof. Directly from the previous lemma. \( \square \)

Lemma 5.3. One has that \((f, a) \sim (f, b)\) if and only if \( a = b \).

Proof. Directly from the definition of the relation, and the fact that for the element in \( W \) mentioned in this definition one can take the identity. \( \square \)

Lemma 5.4. The relation \( \sim \) on \( \tilde{X} \) is an equivalence relation.

Proof. Reflexivity is clear from the previous lemma. We now proof symmetry. Given \((f, a) \sim (g, b)\) there exists \( w \in W \) such that \( f|_Z = g \circ w|_Z \) where \( Z := f^{-1}(g(\mathbb{A})) \).

Further, \( a \) is contained in the set \( \iota(Z) \) and \( \iota(w).a = b \), so \( b = \iota(w)^{-1}.a \). It remains to prove that \( b \) is contained in the set \( \iota(Y) \) with \( Y := g^{-1}(f(\mathbb{A})) \). By assumption \( w.Z = Y \) and so \( \iota(w).\iota(Z) = \iota(Y) \). This implies that \( b = \iota(w)a \) is contained in \( \iota(Y) \) and that the relation is symmetric.

The last property to check is that the equivalence is transitivity. Note that Lemma 5.1 already shows a weak version of transitivity. Using this weak version and Corollary 5.2 we can assume that we have three pairs \((f, a), (g, a)\) and \((h, a)\), with \((f, a) \sim (g, a)\) and \((f, a) \sim (h, a)\) such that \( f|_Z = g|_Z \) and \( f|_{Z'} = h|_{Z'} \) with \( Z := f^{-1}(g(\mathbb{A})) \) and \( Z' := f^{-1}(h(\mathbb{A})) \).

Consider the convex set \( K := Z \cap Z' \), by taking restrictions to both \( Z \) and \( Z' \) it follows that \( g|_K = h|_K \). In particular one observes that \( K \subset Z'' := g^{-1}(h(\mathbb{A})) \), and that \( a \in \iota(K) \subset \iota(Z''). \) By Condition (A2) we know that there exists a \( w \in W \) such that \( g|_{Z''} = h \circ w|_{Z''} \), and so also \( g|_K = h \circ w|_K \). Because \( h \) and \( g \) are injections it follows that \( w \) leaves \( K \) invariant. This also implies that \( \iota(w) \) leaves \( \iota(K) \) invariant and that \( \iota(w).a = a \). We can conclude that \((g, a) \sim (h, a)\) and that \( \sim \) is a transitive relation. \( \square \)

Definition 5.5. Let \( X' = (\mathcal{A} \times \mathbb{A}')/\sim \) and define for each chart \( f \in \mathcal{A} \) a map \( \psi(f) \) from \( \mathbb{A}' \) to \( X' \) as follows:

\[
\psi(f) = [a \mapsto (f, a)/\sim].
\]

We write \( \mathcal{A}' \) for the set of maps \( \psi(f) \circ w' \), with \( f \in \mathcal{A} \) and where \( w' \) ranges over all \( w' \in W' \).

We will prove that \((X', \mathcal{A}')\) satisfies the assertion of the third main result. Note that Condition (A1) follows already from the definition and part two of the following lemma.

Lemma 5.6. The elements of \( \mathcal{A}' \) satisfy:

1. each map \( f' \in \mathcal{A}' \) is injective,
2. if \( w \in W \) and \( f \in \mathcal{A} \), then \( \psi(f \circ w) = \psi(f) \circ \iota(w) \).
Proof. The first property is a consequence of Lemma 5.3. To prove the second observe that
\[ \psi(f \circ w) = [a \mapsto (f \circ w, a)/\sim] \]
and that
\[ \psi(f) \circ i(w) = [a \mapsto (f, i(w).a)/\sim]. \]
It therefore remains to prove that \((f, i(w).a)\) and \((f \circ w, a)\) are equivalent, which follows from Lemma 5.1.

Let \(x\) be a point of \(X\) lying in some apartment \(\Sigma\). Let \(f \in \mathcal{A}\) be a chart defining this apartment, and \(a \in \mathcal{A}\) such that \(f(a) = x\). We now define \(\phi(x)\) to be the equivalence class of the pair \((f, \iota(a))\).

**Lemma 5.7.** The map \(\phi\) is well defined.

**Proof.** Let \(x\) be a point of \(X\), and \(f, g\) two charts in \(\mathcal{A}\) such that for the two points \(a\) and \(b\) in \(\mathcal{A}\) it holds that \(f(a) = g(b) = x\). Observe that \(a \in Z := f^{-1}(g(\mathcal{A}))\) and that if \(f|_{Z} = g \circ w|_{Z}\) for some \(w \in W\), then \(w.a = b\). This implies that \((f, \iota(a)) \sim (g, \iota(b))\) and that the map is well defined. \(\Box\)

**Lemma 5.8.** The map \(\phi\) is an injection.

**Proof.** Let \(x\) and \(y\) be two points of \(X\). By Condition (A3) there exists a chart \(f \in \mathcal{A}\) and two points \(a, b\) of the model space \(\mathcal{A}\) such that \(x = f(a)\) and \(y = f(b)\). One has that \(\phi(x) = \phi(y)\) if and only if \((f, a) \sim (f, b)\). Injectivity of \(\phi\) now follows from Lemma 5.3. \(\Box\)

We now check Condition (A2). We may assume without loss of generality that the two charts in \(\mathcal{A}'\) are of the form \(\psi(f)\) and \(\psi(g)\) with \(f\) and \(g\) two charts in \(\mathcal{A}\). Let \(Z\) be the set \(f^{-1}(g(\mathcal{A}))\) and \(Z'\) the set \(\psi(f)^{-1}(\psi(g)(\mathcal{A}'))\). A point \(a\) lies in \(Z'\) if there exists a point \(b \in \mathcal{A}'\) such that \((f, a) \sim (g, b)\). The necessary and sufficient condition for this equivalence to happen is that \(a\) is an element of \(\iota(Z)\). So we can conclude that \(Z' = \iota(Z)\) and that \(Z'\) is convex. The second part of Condition (A2) follows easily from Corollary 5.2.

Using part two of Lemma 5.6 one sees that an apartment of \((X, \mathcal{A})\) defines in a bijective way an apartment (i.e. an image of a chart in \(\mathcal{A}'\)) of \((X', \mathcal{A}')\). We can think of the map \(\phi\) as embedding \(X\) into a larger set \(X'\) which adds points to each apartment. The above discussion for Condition (A2) implies that the intersection of two apartments before and after the embedding stays convex and also of the “same shape” (there is a map between both induced by \(\iota\)). So if two apartments share a Weyl chamber, then they also do after “adding the extra points”.

We now prove (A6), as it is needed to prove certain results involving germs, which in turn are needed to prove (A3).

**Lemma 5.9.** Let \(A', B'\) and \(C'\) be three apartments of \((X', \mathcal{A}')\) intersecting pairwise in half-apartments. Then the intersection of all three is non empty.

**Proof.** Let \(A, B\) and \(C\) be the three apartments of \((X, \mathcal{A})\) which define respectively \(A'\), \(B'\) and \(C'\). By the construction it follows that \(A, B\) and \(C\) also intersect pairwise in half-apartments, so they will have some point \(x\) in common because of Condition (A6) for \((X, \mathcal{A})\). The point \(\phi(x)\) lies in both \(A', B'\) and \(C'\), so the intersection of these three apartments is not empty. \(\Box\)
Before we prove Condition (A3), we need to investigate Weyl chambers and their germs in \((X', A')\) and show that the residues are spherical buildings. It is easily seen we can define Weyl chambers in the apartments of \((X', A')\). Let \(S\) be some Weyl chamber in some apartment \(A'\) of \((X', A')\). Let \(A\) be the corresponding apartment in \((X, A)\). One can consider parallel classes of Weyl chambers in this apartment \(A\). Using the correspondence again, there corresponds a parallel class of Weyl chambers in \(A'\) to each class in \(A\). This way we can correspond to the Weyl chamber \(S\) in \(A'\) a chamber \(c\) in \(\partial A \subset \partial A X\), we say that: “\(S\) has the chamber \(c\) at infinity”. From the already proven Condition (A2) it then follows that this correspondence to chambers in \(\partial A X\) is independent of the choice of apartment containing \(S\).

**Lemma 5.10.** If two Weyl chambers \(S_1\) and \(S_2\) of \((X', A')\) have the same chamber at infinity, then they have a sub-Weyl chamber in common.

**Proof.** So suppose one has two Weyl chambers \(S_1\) and \(S_2\) (respectively in apartments \(A'_1\) and \(A'_2\)), both having the chamber \(c \in \partial A X\) at infinity. Let \(A_1\) and \(A_2\) be the apartments of \((X, A)\) corresponding to respectively the apartments \(A'_1\) and \(A'_2\) of \((X', A')\). The apartments at infinity \(\partial A_1\) and \(\partial A_2\) have the chamber \(c\) in common, so \(A_1\) and \(A_2\) both contain a Weyl chamber \(T\) (with some base point \(x \in A_1 \cap A_2\) having \(c\) at infinity. We go back to \((X', A')\) and obtain a Weyl chamber \(T'\) based at \(\phi(x)\) in both \(A'_1\) and \(A'_2\) having \(c\) at infinity. This Weyl chamber \(T'\) has with both Weyl chambers \(S_1\) and \(S_2\) sub Weyl chambers in common, which on its turn implies that \(S_1\) and \(S_2\) have a sub Weyl chamber in common. \(\square\)

We use the above lemma to show Condition (A4). Let \(S_1\) and \(S_2\) be two Weyl chambers in \((X', A')\), let \(c_1\) and \(c_2\) be the two corresponding chambers at infinity in \(\partial A X\). As \(\partial A X\) is a spherical building, there exists an apartment \(A\) in \((X, A)\) such that \(\partial A\) contains both chambers \(c_1\) and \(c_2\). Let \(A'\) be the apartment in \((X', A')\) corresponding with \(A\). This apartment \(A'\) contains Weyl chambers \(S'_1\) and \(S'_2\) having respectively chambers \(c_1\) and \(c_2\) at infinity. Lemma 5.10 proves that \(A'\) contains sub Weyl chambers of both \(S_1\) and \(S_2\), and so Condition (A4) holds.

One can define germs in \(X'\) as well, and so also residues \(\Delta_{x'} X'\) for points \(x' \in X'\). We now list some lemmas, already listed in the introduction, which stay true in this case and are also proved in the same way.

**Lemma 5.11.** Let \(c\) be a chamber of \(\partial A X\). For a Weyl chamber \(S\) based at a point \(x \in X'\) there exists an apartment \(A\) with chart in \(A'\) containing a germ of \(S\) at \(x\) and such that \(c\) is contained in the boundary \(\partial A\).

**Proof.** See Proposition 2.6 \(\square\)

**Lemma 5.12.** For all \(x \in X'\) one has that the residue \(\Delta_x X'\) is a spherical building of type \(R\).

**Proof.** See Proposition 2.4 \(\square\)

**Lemma 5.13.** Given two Weyl chambers \(S, T\) based at the same point \(x \in X'\). If their germs are opposite in \(\Delta_x X'\) then there exists a unique apartment of \((X', A')\) containing \(S\) and \(T\).

**Proof.** See Proposition 2.5 item 1 \(\square\)

21
Proposition 5.14. Let $A_1$ and $A_2$ be two apartments of $(X', A')$. Let $\mathcal{C} := \mathcal{C}(A_1, A_2)$ be the set of apartments containing at infinity two chambers $c_i \in \partial A_i$, $i = 1, 2$, which are opposite in $\partial A X$. Then $\mathcal{C}$ is a finite set of apartments such that each pair of points $(x, y) \in A_1 \times A_2$ is contained in one of these apartments in $\mathcal{C}$.

Proof. We first prove that if two points, one in each of the apartments $A_1$ and $A_2$, are contained in a common apartment that they also lie in an apartment in $\mathcal{C}$. So suppose $B$ is an apartment containing points $x_i \in A_i$, $i = 1, 2$. We choose an $x_1$–based Weyl chamber $S_1$ in $B$ containing $x_2$. Lemma 5.12 implies the existence of an $x_1$–based Weyl chamber $T_1$ contained in $A_1$ whose germ is opposite the germ of $S_1$ at $x_1$. By Lemma 5.13 the set $T_1 \cup S_1$ will be contained in some apartment $B'$.

Let $T_2$ be the $x_2$–based Weyl chamber parallel to $T_1$, i.e. $\partial T_2 = \partial T_1$. It is easy to see that $T_2$ contains $x_1$. We denote by $T'_2$ the $x_2$–based Weyl chamber in $A_2$ whose germ at $x_2$ is opposite $\Delta x_2 T_2$. Again by Lemma 5.13 we obtain a unique apartment $B''$ containing $T_2$ and $T'_2$. This apartment $B''$ contains $x_1$ and $x_2$. Since apartments of $(X', A')$ are in one to one correspondence with apartments in $\partial A X$ the apartment $B''$ is uniquely determined by the chambers $\partial T_2$ and $\partial T'_2$ in its boundary. By construction the apartment $B''$ is contained in $\mathcal{C}$. So we conclude that if two points $x_i \in A_i$, $i = 1, 2$ lie in a common apartment, that they also lie in a common apartment in $\mathcal{C}$.

Now suppose that two points $x_i \in A_i$ do not lie in one apartment. Let $K$ be the set of points in $A_1$ which do lie in an apartment with $x_2$. Note that due to the above discussion and the fact that (A2) is already proven for $(X', A')$ we have that $K$ is a finite union of convex sets. Let $\partial S$ be a chamber at infinity of the apartment $A_1$. Lemma 5.11 implies that there exists an apartment $A$ in $(X', A')$ containing $x_2$ and $\partial S$ at infinity. By applying Lemma 5.10 one obtains that $A$ and $A_1$ in $(X', A')$ share at least a Weyl chamber. So $K$ is not empty, but also not the entirety of $A_1$ as $x_1 \notin K$. Because $K$ is a finite union of convex sets, one can find a point $y$ in $K$ so that not all the germs based at $y$ lie in $K$ (a point at the “border” of $K$). Let $\Delta_y R$ be such a germ, and $c$ a chamber at infinity of a Weyl chamber $U$ based at $y$ containing $x_2$ (possible because there exists an apartment containing both). Lemma 5.11 yields that there exists an apartment $A'$ containing the germ $\Delta_y R$ and $c$ at infinity. Apartment $A'$ also contains the Weyl chamber $U$ (so also $x_2$) due to (A2). The germ $\Delta_y R$ lies in $K$, contradicting the way we have chosen $\Delta_y R$. So we obtain that $K$ contains all points of $A_1$. The proposition is hereby proven. □

Corollary 5.15. Each two points of $X'$ lie in an apartment.

Proof. Directly from the above proposition. □

So we have proven Condition (A3). Combining this with Lemma 5.11 we obtain that each germ and point of $(X', A')$ lies in an apartment (also called Condition (A3')). Also the above proposition shows that if one has an apartment $A$ and point $x$ of $(X', A')$ then a finite number of apartments containing $x$ cover $A$ (this is known as Condition (FC')).

One can prove Condition (A5) using (A1), (A2), (A3') and (FC'), as carried out by the first author [SnH09, Section 7]. So $(X', A')$ is indeed a $A'$–building. The next lemma states a link between the distance functions of both generalized affine buildings.

Lemma 5.16. Let $x$ and $y$ be two points of $X$, then $d'(\phi(x), \phi(y)) = e(d(x, y))$, where $d$, $d'$ are the distance functions defined respectively on $(X, A)$ and $(X', A')$. 

22
Proof. Let $f \in \mathcal{A}$, $a, b \in \mathcal{A}$ such that $x = f(a)$ and $y = f(b)$ (possible by Condition (A3)). Then $d(x, y)$ equals $d(a, b)$, and $d'(\phi(x), \phi(y))$ equals $d'(\iota(a), \iota(b))$. The equality we need to prove follows from the discussion in the beginning of the section.

This concludes the proof of the first assertion of the third Main Result.

For two $\Lambda$–buildings $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ with an isometric embedding $\psi$ from $X_1$ to $X_2$ mapping apartments to apartments it is easy to find a map $\phi'$ between the resulting $\Lambda'$–buildings $(X'_1, \mathcal{A}'_1)$ and $(X'_2, \mathcal{A}'_2)$ (also mapping apartments to apartments) by the base change functors $\phi_1$ and $\phi_2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\psi} & X_2 \\
\phi_1 \downarrow & & \phi_2 \\
X'_1 & \xrightarrow{\psi'} & X'_2
\end{array}
$$

This assures functoriality.

6 Applications

6.1 Asymptotic cones of $\Lambda$–buildings

We will use the base change functor to prove that asymptotic cones as well as ultracones of generalized affine buildings are again generalized affine buildings.

An ultrafilter $\mu$ on an infinite set $I$ is a finite additive measure $\mu : 2^I \to \{0, 1\}$ such that $\mu(I) = 1$. We say that $\mu$ is principal if it is a Dirac measure, i.e., concentrated in one element of $I$. Sometimes we identify $\mu$ with the subset $F_\mu$ of the powerset $2^I$ consisting of the sets having measure one. Therefore another way of stating that $\mu(A) = 1$ is to say that $A$ is contained in $F_\mu$. To simplify (and slightly abuse) notation we write $A \in \mu$ instead of $A \in F_\mu$.

**Definition 6.1.** Given an ultrafilter $\mu$ on an infinite set $I$ we may define the ultraproduct of a sequence of set $(X_i)_{i \in I}$ as follows: Let $\tilde{X} := \Pi_{i \in I} X_i$, be the cartesian product of the $X_i$. We define a relation $\sim$ on $\tilde{X}$ by

$$
x \sim y \iff \mu(\{i \in I : x_i = y_i\}) = 1.
$$

It is easy to see that $\sim$ is an equivalence relation on $\tilde{X}$. The ultraproduct $^*X$ of the sequence $(X_i)_{i \in I}$ is the space of equivalence classes $^*x$ of sequences $x = (x_i)_{i \in I}$ with respect to this equivalence relation. In case that $X_i = X$ for all $i \in I$ we call $^*X$ the ultrapower of $X$.

Loś’s Theorem, compare 2.1 on page 90 in [BS69], tells us that anything which may be stated in a first order language on the level of the components $X_i$ is still true for the ultraproduct of the sequence.

One easily observes that, defining multiplication componentwise, the ultraproduct of a sequence of abelian groups $\Lambda_i$ is again an abelian group. If each $\Lambda_i$ is totally ordered, the ultraproduct carries a natural ordering defined as follows. An element $^*(a_n)_{n \in \mathbb{N}}$ of $^*\Lambda$ is smaller or equal to $^*(b_n)_{n \in \mathbb{N}} \in ^*\Lambda$ if the set $\{n \in I : a_n \leq b_n\}$ has measure one.
The ultraproduct $^\ast X$ of a sequence $(X_i, d_i)_{i \in I}$ of $\Lambda_i$-metric spaces carries a natural $^\ast \Lambda$-valued metric. The distance function $^\ast d$ on $^\ast X$ is defined by

$$^\ast d(x, y) := (^\ast (d_i(x_i, y_i)))_{i \in I} \in ^\ast \Lambda,$$

which is simply the equivalence class of the sequence of distances of the components $x_i$ and $y_i$ in the corresponding metric space $X_i$.

**Remark 6.2.** If $\mu$ is a principal ultrafilter, then there exists $i$ such that $\mu(\{i\}) = 1$ and hence $^\ast X = X_i$. So the interesting cases are the ones with $\mu$ non-principal.

Fix a spherical root system $R$ and denote by $W$ its spherical Weyl group. We will consider a sequence $(X_i, A_i)_{i \in I}$ of generalized affine buildings, where $(X_i, A_i)$ is modeled on $\Lambda_i = \Lambda(R, A_i)$, with $\Lambda_i$ a totally ordered abelian group. Here it is important that the underlying spherical root system is the same for all factors. Further assume that each $X_i$ carries a $\Lambda_i$-valued metric $d_i$.

We will prove that the ultraproduct $^\ast X$ of $(X_i, A_i)_{i \in I}$ is again a generalized affine building. In order to do so, we need to take the product of all structural features, such as charts or apartments, the Weyl groups, the distance function, . . . simultaneously. This process might be formalized in terms of first order languages and formulas on a certain set which involves all these structures. Taking the product of such a “universal setting” will allow us to talk about the same structures in the ultraproduct which we already had in the components themselves. However, we will not carry out the details here. We refer the interested reader to chapter 5 of Bell and Slomson’s book [BS69] on models and ultraproducts.

**Proposition 6.3.** The ultraproduct $(^\ast X, ^\ast A)$ is a generalized affine building modeled on $\Lambda(R, ^\ast \Lambda)$ and carries a natural $^\ast \Lambda$-metric $^\ast d$ induced by $d$.

**Proof.** The points and charts of $(^\ast X, ^\ast A)$ are equivalence classes of sequences $(x_i)_{i \in I}$ (resp. $(f_i)_{i \in I}$), where $x_i$ is a point in $X_i$ and $f_i$ a chart in $A_i$.

First we will prove that the ultraproduct $^\ast \Lambda$ of the model spaces $\Lambda_i$ equals $\Lambda(R, ^\ast \Lambda)$. Identifying points of $\Lambda_i$ with elements of the full affine Weyl group $W_i := W \ltimes \Lambda_i^n$, with $n = \text{rank}(R)$, this question can be answered by proving that $^\ast W := (^\ast (W_i))_{i \in I} = W \ltimes (^\ast \Lambda)^n$.

Abbreviate $\Lambda_i^n$ by $T_i$. Since the spherical Weyl group is finite, every sequence $(w_i)_{i \in I}$, with $w_i \in W$, is equivalent to a constant sequence $(u)_{n \in I}$. Hence given representatives $(u_i, s_i)_{i \in I}$ and $(v_i, t_i)_{i \in I}$ of elements $^\ast u, ^\ast v$ of $^\ast W$, we observe

$$(u_i, s_i)_{i \in I} \cdot (v_i, t_i)_{i \in I} = (u_i v_i, s_i + r_{u_i}(t_i))_{i \in I} \sim (u v, s_i + r_u(t_i))_{i \in I}$$

The equivalence class w.r.t. $\mu$ of the sequence $(u v, s_i + r_u(t_i))_{i \in I}$ is obviously an element of $^\ast W \ltimes ^\ast T$.

Interpreting Conditions (A1) to (A6) componentwise on sequences and using the fact that each factor $(X_i, A_i)$ satisfies these Conditions it is easy to deduce from Loš’s Theorem, that ultraproduct satisfies all the Conditions as well. Hence the claim.

From now on we will restrict ourselves to ultrapowers. Fix once and for all an ordered abelian group $\Lambda$. The subset of finite elements $^\ast \Lambda_{fin}$ of the ultrapower of an ordered abelian group $\Lambda$ is defined by

$$^\ast \Lambda_{fin} = \{ \alpha \in ^\ast \Lambda : -c < \alpha < c \text{ for some } c \in \Lambda_+ \}.$$ 

An easy computation shows that $^\ast \Lambda_{fin}$ is a convex subgroup of $^\ast \Lambda$. 

24
An element \( \lambda \in \Lambda \) is the standard part of an element \(*\lambda \) in \(*\Lambda_{fin} \) if the absolute value of the difference satisfies \(|\lambda - \lambda| < n \) for all \( n \in \Lambda_+ \).

**Definition 6.4.** We define an equivalence relation \( \sim_{fin} \) on \(*X \) by

\[ *x \sim_{fin} *y \iff \text{std}(*d(*x,*y)) \leq n \ \text{for some} \ n \in \Lambda. \]

The ultracone \( \text{UCon}(X) = \*X/_{\sim_{fin}} \) of the metric space \( X \) is again a metric space whose metric \( *d_{fin} \) is the \(*\Lambda/\Lambda_{fin} \)-quotient-metric induced by \(*d\).

**Definition 6.5.** Let \( \mu \) be a non-principal ultrafilter, \( \alpha = (\alpha_i)_{i \in I} \) a non-finite element of \(*\Lambda \) and assume that \((X,d)\) is a \( \Lambda \)-metric space with chosen base point \( o \). Clearly \(*X \) is a \( \Lambda \)-metric space with the constant sequence \(*o = (o)_{i \in I} \) as its basepoint. Set

\[ X_{<\alpha>} := \{ *x \in \*X : *d(*x,*o) \cdot \alpha^{-1} \in \*\Lambda_{fin} \}. \]

We define \(*x \sim_{\alpha} *y \) on \( X_{<\alpha>} \) if \(*d(*x,*y) \cdot \alpha^{-1} \) has standard part \( o \). The set

\[ \text{Cone}(X) := X_{<\alpha>}/_{\sim_{\alpha}} \]

is called asymptotic cone of \( X \). It carries a natural \( \Lambda \)-metric defined by

\[ d(x,y) = \text{std}(*d(x,y) \cdot \alpha^{-1}), \]

where \( \text{std} \) denotes the standard part of the element \(*d(x,y) \cdot \alpha^{-1} \).

Note that ultracones and asymptotic cones depend on the choices of \( \alpha \) and \( \mu \).

**Theorem 6.6.** The class of generalized affine buildings is closed under taking asymptotic cones and ultracones. Furthermore, if \((X,A)\) is modeled on \( \mathbb{A}(\mathbb{R},\Lambda) \), then \( \text{Cone}(X) \) has model space \( \mathbb{A}(\mathbb{R},\text{Cone}(\Lambda)) \) and \( \text{UCon}(X) \) is modeled on \( \mathbb{A}(\mathbb{R},\text{UCon}(\Lambda)) \).

**Proof.** Let \((X,A)\) be a generalized affine building modeled on \( \mathbb{A}(\mathbb{R},\Lambda) \) and let \( o \) be a base point in \( X \) and \( d : X \times X \to \Lambda \) a metric.

First simply view \( X \) as a metric space and consider its ultracone \( \text{UCon}(X) \). There is then an obvious projection \( \pi \) from \( \*X \) onto \( \text{UCon}(X) \) such that

\[ *d_{fin}(\pi(*x),\pi(*y)) = e(*d(*x,*y)), \]

where \( e \) is the projection from \(*\Lambda \) to \(*\Lambda_{fin} \). By 1 of our Main Result \( \square \) there exists a \(*\Lambda_{fin} \)-affine building \((X',A')\), which is the image of the base change functor \( \phi \) associated to \( e : \*\Lambda \to \*\Lambda_{fin} \). Using the observations made above we may deduce from 2 of our first Main Result that \( \text{UCon}(X) \) is isomorphic to \( X' \) and carries the structure of a \(*\Lambda_{fin} \)-affine building.

We now apply similar arguments in the case of asymptotic cones. By our second Main Result \( \square \) the set \( X'' := \phi^{-1}(\phi_e(0)) \) is a \( \text{ker}(e) \)-affine building. It is easy to see that \( X'' \) coincides with \( X_{<\alpha>} \), which was defined to be the set of sequences whose distance to the constant sequence \( o \) is an element of \(*\Lambda_{fin} \), that is a finite element of \(*\Lambda \).

The asymptotic cone of \( X \) is the quotient space of \( X_{<\alpha>} \) by the relation \( \sim_{\alpha} \), that is identifying sequences having infinitesimal distance to \( o \). This corresponds to the image of the base change functor \( \phi' \) associated to the projection \( e' : \*\Lambda_{fin} \to \text{Cone}(\Lambda) \). Hence by Main Result \( \square \) the asymptotic cone \( \text{Cone}(X) \) is isomorphic to the \( \text{Cone}(\Lambda) \)-affine building \( \phi'(X_{<\alpha>} \).

The asymptotic cone of the reals is again \( \mathbb{R} \), hence we have the following corollary, which was proven earlier by Kleiner and Leeb in [KL97].

**Corollary 6.7.** The asymptotic cone of an affine \( \mathbb{R} \)-building is again an affine \( \mathbb{R} \)-building.
6.2 Reducing to the $\mathbb{R}$-building case: a fixed point theorem

This last application is an example how our base change functors can reduce problems of $\Lambda$-buildings to the more known and familiar case of $\Lambda = \mathbb{R}$. We will demonstrate this by proving a fixed point theorem for certain $\Lambda$-buildings. The main tool herein is an embedding theorem by Hahn. First we need to define the Hahn product. Given an ordered set $I$ and collection $(\Lambda_i)_{i \in I}$ of ordered abelian groups of order, then the Hahn product is the subgroup of $\prod_{i \in I} \Lambda_i$ where the set $I' \subset I$ of indices with non-zero entries of element is always well-ordered w.r.t. the reverse ordering of $I$ (this means that each non-empty subset of $I'$ has a maximal element). This subgroup carries a natural lexicographical ordering.

**Theorem 6.8** (Hahn’s embedding theorem, [Hah07]). Given an ordered abelian group $\Lambda$, then there exists a ordered set $I$ such that $\Lambda$ is isomorphic as ordered abelian group to a subgroup of the Hahn product of copies of the real numbers $\mathbb{R}$ over an ordered set $I$.

Let $(X, A)$ be a $\Lambda$–building, by the third main result and the above theorem one can embed this building in a $\Lambda'$–building where $\Lambda'$ is the Hahn product of copies of the real numbers $\mathbb{R}$ over an ordered set $I$. So assume $\Lambda$ is of this form. Additionally assume that $I$ is well-ordered (so every non-empty subset of $I$ has a least element). Our fixed point theorem is now as follows.

**Theorem 6.9.** A finite group of isometries of a generalized affine building $(X, A)$ with above properties admits a fixed point.

**Proof.** For an arbitrary non-zero element $h$ of $\Lambda$ let $i_h \in I$ be the maximal element of the set of indices in $i$ with non-zero entries in the representation of $h$ as a product. If $h$ is zero, then we set $i_h$ to be $-\infty$. For a non-zero element $g$ of $\Lambda$, one can now define the following two convex subgroups of $\Lambda$:

$$M_g := \{ \lambda \in \Lambda : i_\lambda \leq i_g \},$$

$$N_g := \{ \lambda \in \Lambda : i_\lambda < i_g \}.$$

Note that $M_g / N_g$ is isomorphic to $\mathbb{R}$.

Choose some point $x_0$ of $X_0 := X$, the orbit of $x_0$ is finite, hence it is bounded by some element $g_0 \in \Lambda$. If $g_0$ is zero then we have found a fixed point, so suppose this is not the case. The points of $X$ at distance in $M_{g_0}$ from $x_0$ form a $M_g$–building (by applying the second main result on the canonical epimorphism $\Lambda \to \Lambda / M_{g_0}$). Note that $G$ stabilizes this $M_g$–building. Consider the canonical epimorphism $M_{g_0} \to M_{g_0} / N_{g_0}$, so using the first main result we obtain an $\mathbb{R}$–building on which $G$ acts, still as a finite group of isometries (we mark this step by (*) for further reference). By a result of the second author, this action has a fixed point $y_0$ (see [Str09]). In the original $\Lambda$–building $(X, A)$ this point $y_0$ corresponds to a set $X_1$ of points with distance in $N_{g_0}$ from a certain point $x_1$. By the second main result we can consider the set $X_1$ as a $N_{g_0}$–building stabilized by $G$. So we can repeat this algorithm. Because $I$ is well-ordered, the algorithm has to stop at some point where $g_i = 0$ is the trivial group. When this happens we have obtained a fixed point of $G$ in $(X, A)$. □

This result should not be interpreted as a full investigation into fixed point theorems of $\Lambda$–buildings, but as a quick example of how our results combined with Hahn’s embedding theorem can reduce problems to the $\mathbb{R}$–building case. Even when the theorems one obtains...
in this way do not hold in full generality (like this fixed point theorem), they might help to understand generalized affine buildings and point out where possible difficulties might occur.

We end with an example of this last thing. In [Str09] there is proved that a finitely generated bounded group of isometries of an $\mathbb{R}$-building has a fixed point, so what happens if we only ask this weaker condition to be fulfilled in the above theorem? An analogous proof would fail in step (*), one cannot show that this new action is bounded. Indeed, consider the lexicographically ordered abelian group $G := \mathbb{R} \times \mathbb{R}$ as a generalized affine building with only one apartment, and define for each $k \in \mathbb{Z}$ the isometry $G \to G : (x, y) \mapsto (x, y + kx)$. All these isometries form a finitely generated bounded group (as each orbit is bounded by $(1, 0)$), but it has no fixed point. So this exercise shows that there lies a difficulty in the notion of boundedness.

Bibliography

[AB87] R. Alperin and H. Bass. Length functions of group actions on $\Lambda$-trees. In *Combinatorial group theory and topology (Alta, Utah, 1984)*, volume 111 of *Ann. of Math. Stud.*, pages 265–378. Princeton Univ. Press, Princeton, NJ, 1987.

[AB08] P. Abramenko and K. S. Brown. *Buildings, Theory and applications*, volume 248 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[Ben90] C. D. Bennett. Affine $\Lambda$-buildings. *Dissertation, Chicago Illinois*, pages 1–106, 1990.

[Ben94] C. D. Bennett. Affine $\Lambda$-buildings. I. *Proc. London Math. Soc. (3)*, 68(3):541–576, 1994.

[BS69] J. L. Bell and A. B. Slomson. *Models and ultraproducts: An introduction*. North-Holland Publishing Co., Amsterdam, 1969.

[Chi01] I. Chiswell. *Introduction to $\Lambda$-trees*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Hah07] H. Hahn. Über die nichtarchimedischen grösensysteme. *Sitz. K. Akad. Wiss.*, 116:601–655, 1907.

[Hit09a] P. Hitzelberger. Generalized affine buildings: Automorphisms, affine Suzuki-Ree-buildings and Convexity. *arXiv:0902.1107v1*, 2009.

[Hit09b] P. Hitzelberger. Nondiscrete affine buildings and convexity. *arXiv:0906.4925v1*, 2009.

[KL97] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, (86):115–197, 1997.
[KT04] L. Kramer and K. Tent. Asymptotic cones and ultrapowers of Lie groups. *Bull. Symbolic Logic*, 10(2):175–185, 2004.

[MS84] J. W. Morgan and P. B. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.

[Par00] A. Parreau. Immeubles affines: construction par les normes et étude des isométries. In *Crystallographic groups and their generalizations (Kortrijk, 1999)*, volume 262 of *Contemp. Math.*, pages 263–302. Amer. Math. Soc., Providence, RI, 2000.

[SnH09] P. N. Schwer (née Hitzelberger). Axioms of affine buildings. *arXiv:0909.2967v1*, 2009.

[Str09] K. Struyve. (non-)completeness of $\mathbb{R}$-buildings and fixed point theorems. *arXiv:0909.3202v1*, 2009.

[vdDW84] L. van den Dries and A. J. Wilkie. Gromov’s theorem on groups of polynomial growth and elementary logic. *J. Algebra*, 89(2):349–374, 1984.