Energy Complexity of Distance Computation in Multi-hop Networks*

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Abstract

Energy efficiency is a critical issue for wireless devices operated under stringent power constraint (e.g., battery). Following prior works, we measure the energy cost of a device by its transceiver usage, and define the energy complexity of an algorithm as the maximum number of time slots a device transmits or listens, over all devices. In a recent paper of Chang et al. (PODC 2018), it was shown that broadcasting in a multi-hop network of unknown topology can be done in $\text{poly log } n$ energy. In this paper, we continue this line of research, and investigate the energy complexity of other fundamental graph problems in multi-hop networks. Our results are summarized as follows.

Breadth-first search. To avoid spending $\Omega(D)$ energy, the broadcasting protocols of Chang et al. (PODC 2018) do not send the message along a BFS tree, and it is open whether BFS could be computed in $o(D)$ energy, for sufficiently large $D$. In this paper we devise an algorithm that uses $\tilde{O}(\sqrt{n})$ energy.

Diameter. We show that the framework of the $\tilde{\Omega}(n)$ round lower bound proof for computing diameter in CONGEST of Abboud et al. (DISC 2017) can be adapted to give an $\tilde{\Omega}(n)$ energy lower bound in the wireless network model (with no message size and runtime constraint), and this lower bound applies to $O(\log n)$-arboricity graphs. From the upper bound side, we show that the energy complexity of $\tilde{O}(\sqrt{n})$ can be attained for bounded-genus graphs (which includes planar graphs).

Minimum Cut. Our upper bound for computing diameter can be extended to other graph problems. We show that exact global minimum cut or approximate $s-t$ minimum cut can be computed in $\tilde{O}(\sqrt{n})$ energy for bounded-genus graphs. In contrast, there are $\Omega(n)$ energy lower bounds for computing the exact value of (i) a $s-t$ minimum cut (for planar graphs), and (ii) a global minimum cut (for unit disc graphs).

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1The notation $\tilde{O}(\cdot)$ suppresses any poly log $n$ factor.
1 Introduction

In many wireless networks (e.g., sensor networks), devices are operated under a limited power constraint (e.g., capacity of the battery). To maximize the lifetime of a network, it is important that the devices operate in an energy efficient manner. For small devices, the majority of energy is often spent on radio transceiver usage (sending and receiving packets), and not on computation. Rather than assigning a different energy cost per each mode of operation, we simply assume that a device spends one unit of energy when it sends a message or listens to the channel. This is a common assumption; see, e.g., [8, 19, 20, 22, 21].

Wireless Network Models. A network is modeled as a connected undirected graph $G = (V, E)$, where each vertex is a device, and each edge represents a communication link. We assume that the local clock of each device is synchronized to a global time (e.g., they can use GPS technology). Communication proceeds in synchronized rounds, and all devices agree on the same starting time. In each round, each device can listen to the channel, send a message to its neighbors, or be idle. We do not allow a device to simultaneously send and listen (i.e., we are in the half-duplex model). We assume that there is no message size constraint. The energy cost of a device is measured by the total number of channel accesses (i.e., the number of rounds that the device sends or listens). The energy cost of an algorithm is the maximum energy cost of a device, among all devices.

In one round, if a device $v$ listens to the channel, and exactly one device $u$ in $N(v)$ transmits a message, then $v$ receives the message sent by $u$. If the number of transmitting devices in $N(v)$ is not 1, then $v$ receives a channel feedback which depends on the underlying model.

Without Collision Detection: No-CD. If the number of transmitting devices in $N(v)$ is not 1, then $v$ receives “silence”. With Collision Detection: CD. If the number of transmitting devices in $N(v)$ is greater than 1, then $v$ receives “noise”. If the number of transmitting devices in $N(v)$ is 0, then $v$ receives “silence”.

Each transmitting or idle device does not receive any feedback from the communication channel. In particular, a transmitting device does not know whether its message is successfully received by its neighbors.

Throughout the paper, unless otherwise stated, we are always in the randomized No-CD model. We assume that each device has access to an unlimited local random source, but there is no shared randomness between devices. We do not consider deterministic algorithms in this paper. The maximum allowed failure probability is, by default, $f = 1/poly(n)$, and the term “with high probability” refers to probability of $1 - 1/poly(n)$. Note that randomized algorithms can generate private random bits to break symmetry, e.g., they can assign themselves $O(\log n)$-bit IDs, which are distinct with high probability. Therefore, we may assume that each device already has a distinct ID of length $O(\log n)$.

The graph topology of the underlying network $G$ is initially unknown to all devices; but we assume that the two parameters $n = |V|$ and $\Delta = \max_v \deg(v)$ are initially known to everyone.

Whether the assumption that transmitting and listening both cost one unit of energy is valid depends on the underlying wireless network technology. Note that, in general, there is also a difference between the energy consumption of receiving packets and that of idle listening (i.e., the transceiver circuits are active, but no message is received). There are examples of networks where the difference between these costs is small; see e.g., [21 Table 1].

In real world, a wireless device might be allowed to adjust the power of transmission, which affects the range that its message can reach. In this paper, we assume that all devices send with a fixed power. This is not an uncommon assumption; see e.g., [5].

Message-Passing Models. The LOCAL model considers the setting where there is no collision at all, and each message sent along each edge is successfully transmitted [27, 28]. The CONGEST model is a variant of the LOCAL model that requires each message to have length at most $O(\log n)$.

1.1 Related Works

Energy saving is a critical issue for wireless networks, and there have been a lot of efforts in designing energy efficient algorithms in networking and systems research. In what follows, we briefly summarize prior works
in theory research that considered channel accesses as a complexity measure.

**Single-hop Networks.** First, we consider an important special case where all devices are within a single hop (i.e., $G$ is a clique), but the number of devices $n$ is unknown. The computation model for single-hop networks is simpler than that of general mult-hop networks. Variants of the models that are often considered include: (i) whether or not a device can simultaneously send and listen (i.e., full duplex and half duplex), (ii) the ability to detect collision (i.e., CD and No-CD), (iii) whether to allow randomness.

There has been a line of research on the energy complexity of fundamental problems such as leader election (all vertices in the network agree on a leader), and approximate counting (all vertices estimate the number of vertices within a constant factor) [19, 21, 22, 23, 4, 8]. In the randomized half-duplex model, both problems have energy complexity $\Theta(\log^2 n)$, in CD and No-CD, respectively, for poly($n$)-time algorithms that succeed with probability $1 - 1/\text{poly}(n)$.

**Multi-hop Networks.** Chang et al. [7] extended the above single-hop network model to the multi-hop setting, and they showed that broadcasting can be done in poly log $n$ energy. They presented randomized algorithms for CD and No-CD using energy $O((\log n \log \log \Delta) + \log \Delta \log^2 n)$, respectively, but it takes super-linear time to attain these energy complexities. There is a constant $k_0 > 0$ such that for any $k > k_0$, if one is allowed to spend $O(\log^k n)$ energy, then broadcasting can be solved in $D^{1+O(1/k)} \cdot O(\log^k n)$ time.

They also showed three lower bounds: (i) an $\Omega(\log(D))$ lower bound, which applies to even the LOCAL model on path graphs, (ii) an $\Omega(\log \Delta \log n)$ lower bound for randomized No-CD, and (iii) an $\Omega(\log n)$ lower bound for randomized CD.

Klonowski and Pajak [25] investigated a variant of the model where only transmitting costs energy, and they showed that in No-CD, for any $1 \leq \varphi \leq O(\log n / \log \log n)$, broadcasting can be solved in $O((D + \varphi)n^{1/\varphi})$ time using $O(\varphi)$ transmission per vertex.

Other than [7, 25], there are a number of works that studied energy efficiency of multi-hop networks from different perspectives. In wireless networks, the power of a signal received is proportional to $O(1/d^\alpha)$, where $d$ is the distance to the sender, and $\alpha$ is a constant related to environmental factors. Kirousis et al. [24] studied the optimization problem of assigning transmission ranges of devices subject to some connectivity and diameter constraints, and the goal is to minimize the total power consumption; see also [22, 9, 2] for related works.

A major cost of energy loss is due to collision of multiple signals. A number of papers studied the problem of minimizing the number of rounds or transmissions to realize certain communication task [11, 30, 5]. In the setting of known network topology, Gsieniec et al. [15] designed a randomized protocol for broadcasting in $O(D + kn^{1/(k-2)} \log^2 n)$ rounds such that each device transmits at most $k$ times.

### 1.2 Our Contribution

As observed in [7], for graphs of bounded degree $\Delta = O(1)$, all graph problems can be solved in $O(\log n)$ energy (we will see this in Section 2). Thus, the main challenge for achieving energy efficiency is to deal with “vertex congestion” at high degree vertices. If the energy budget is $x$, then only (at most) $x$ messages can be transmitted across each vertex $v$. For comparison, the well-studied distributed CONGEST model only captures the issue of “edge congestion”. For simpler problems like broadcasting, vertex congestion might not be a critical issue. For more complicated tasks like computing diameter, it might be necessary to have a large amount of messages sent across a vertex $v$ to solve the problem. Our results are briefly summarized as follows.

**Breadth-first Search.** In the BFS problem, we are given a specific source vertex $s$, and the goal is to let each vertex $v$ learn $\text{dist}(v, s)$. Note that the broadcasting protocols of Chang et al. [7] do not send the message along a BFS tree so as to avoid spending $\Omega(D)$ energy, and it is open whether BFS can be computed in $o(D)$ energy, for sufficiently large $D$. In this paper, we show that there is a randomized algorithm for BFS in $O(n^{1.5})$ time and $\tilde{O}(\sqrt{n})$ energy. For large $D$, this improves upon the trivial/standard approach (which constructs the BFS tree along the BFS tree itself) that takes $\tilde{O}(D)$ time and energy.

Our BFS algorithm for Theorem 3 is a result of a combination of some communication building blocks and a known approach in distributed and parallel shortest path algorithms [14, 15, 23, 10]. The algorithm
samples $\tilde{O}(\sqrt{n})$ distinguished vertices, each of them builds a local BFS tree of depth $\tilde{O}(\sqrt{n})$, and the final BFS tree rooted at $s$ can be constructed by combining the local BFS trees.

**Diameter.** We show that the framework of the CONGEST $\tilde{O}(n)$ time lower bound for computing diameter of Abboud et al. \cite{ABB15} can be adapted to give an $\tilde{O}(n)$ energy lower bound in the wireless network model, even allowing unbounded message size and runtime. This lower bound applies to even $\tilde{O}(\log n)$-arboricity graphs. This adaptation is very specific to this lower bound proof and does not apply to other CONGEST lower bounds, in general. For instance, the $\Omega(\sqrt{n})$ lower bounds in \cite{CR12} do not seem to extend to the energy complexity in wireless networks (with no message size constraint), and we are not aware of any non-trivial $\tilde{O}(\sqrt{n})$ energy lower bound.

Due to the $\Omega(n)$ energy lower bound for $O(\log n)$-arboricity graphs, in order to design energy efficient algorithms for computing diameter, we have to consider graph classes that are “more specialized” than low arboricity graphs. Note that arboricity is a graph parameter that measures the density of the graph. A graph has arboricity $k$ if its edge set can be partitioned into $k$ forests.

We show that the energy complexity of $\tilde{O}(\sqrt{n})$ can be attained for the class of bounded-genus graphs, which includes planar graphs. Our algorithm is based on a partition of vertices into a high degree part and a low degree part, and a classification of small degree components into three types. We show that for some small degree components, we only need to extract a small amount of information in order to compute the diameter.

**Minimum Cut.** Our approach for computing diameter in bounded-genus graphs is sufficiently generic that it can be applied to other graph problems as well. In particular, we show that both exact global minimum cut and approximate $s$–$t$ minimum cut can be computed in $\tilde{O}(\sqrt{n})$ energy for bounded-genus graphs. In contrast, there are $\tilde{Ω}(n)$ energy lower bounds for computing the exact value of (i) an $s$–$t$ minimum cut (for planar graphs), and (ii) a global minimum cut (for unit disc graphs).

**Remark.** Our results about bounded-genus graphs also fit into a recent line of research on distributed computing for planar or near-planar graph classes. Ghaffari and Haeupler \cite{GH17} showed that minimum spanning tree can be computed in $\tilde{O}(D)$ time for planar graphs via low-congestion short cuts. This approach has been extended to any graph class that can be characterized by an exclusion of a set of minors (which includes bounded-genus graphs) by Haeupler, Li, and Zuzic \cite{HLZ18}.

## 2 Basic Building Blocks and a Simple BFS Algorithm

In this section we present the basic tools that we use in our algorithms. Some of them are extensions of the techniques developed in \cite{ABB15}. Based on these building blocks, we devise a simple BFS algorithm that achieves $\tilde{O}(\sqrt{n})$ energy complexity.

### 2.1 SR-communication

Let $S$ and $R$ be two not necessarily disjoint vertex sets. The task $\text{SR-comm}$ \cite{ABB15} is defined as follows. Each vertex $u \in S$ holds a message $m_u$ that it wishes to transmit, and each vertex $v \in R$ wants to receive a message from vertices in $N^+(v) \cap S$, where $N^+(v) = N(v) \cup \{v\}$ is the inclusive neighborhood of $v$. An algorithm for $\text{SR-comm}$ guarantees that for each $v \in R$ with $N^+(v) \cap S \neq \emptyset$, the vertex $v$ receives a message $m_u$ from at least one vertex $u \in N^+(v) \cap S$, with high probability. Several variants of $\text{SR-comm}$ are defined as follows.

**Finding Minimum:** $\text{SR-comm}^{\text{min}}$. The message $m_u$ sent from each vertex $u \in S$ contains a key $k_u$ from the key space $[K] = \{1, 2, \ldots, K\}$, and we require that w.h.p., every $v \in R$ with $N^+(v) \cap S \neq \emptyset$ receives a message $m_u$ such that $k_u = \min_{u' \in N^+(v) \cap S} k_{u'}$. We define $\text{SR-comm}^{\text{max}}$ analogously.

**Multiple Messages:** $\text{SR-comm}^{\text{multi}}$. Each vertex $u \in S$ holds a set of messages $M_u$. For each message $m$, all vertices holding $m$ have access to shared random bits (associated with $m$). We assume that for each $v \in R$, the number of distinct messages in $\bigcup_{u \in N^+(v) \cap S} M_u$ is upper bounded by a number $M$. We define $\text{SR-comm}^{\text{multi}}$ analogously.
which is known to all vertices. We require that each vertex \( v \in R \) receives all distinct messages in \( \bigcup_{u \in N^+(v) \cap S} M_u \) w.h.p.

**All Messages:** SR-comm\(^{\text{all}}\). Every \( v \in R \) needs to receive the message \( m_u \), for each \( u \in N^+(v) \cap S \) w.h.p. We let \( \Delta' \) be an upper bound on \(|S \cap N(v)|\), for each \( v \in R \). If \( \Delta' \) is unknown, then we can set \( \Delta' = \Delta \).

**Approximate Sum:** SR-comm\(^{\text{apx}}\). The message \( m_u \) sent from each vertex \( u \in S \) is an integer within the range \([W]\). Every \( v \in R \) needs to approximately learn the summation \( \sum_{u \in N^+(v) \cap S} m_u \) within an \((1 + \epsilon)-factor\) w.h.p.

The following table summarizes the time and energy complexity of our algorithms for these tasks; see Appendix A for proofs. These algorithms serve as basic communication primitives in subsequent discussion. Note that we do not optimize some of the complexities.

| Task         | Time           | Energy                  |
|--------------|----------------|-------------------------|
| SR-comm      | \(O(\log \Delta \log n)\) | \(O(\log \Delta \log n)\) |
| SR-comm\(^{\text{all}}\) | \(O(\Delta' \log n)\) | \(O(\Delta' \log n)\) |
| SR-comm\(^{\text{min}}\) | \(O(M \log \Delta \log^2 n)\) | \(O(M \log \Delta \log^2 n)\) |
| SR-comm\(^{\text{min}}\) | \(O(K \log \Delta \log n)\) | \(O(K \log \Delta \log n)\) |
| SR-comm\(^{\text{apx}}\) | \(O((1/e^\epsilon) \log W \log \Delta \log n)\) | \(O((1/e^\epsilon) \log W \log \Delta \log n)\) |

For the special case of \( S \cap R = \emptyset \) and \(|R \cap N(v)| \leq 1\) for each \( v \in S \), the runtime of SR-comm\(^{\text{min}}\) can be improved to \(O(\log K \log \Delta \log n)\).

### 2.2 Vertex Labeling and Broadcasting

A good labeling is a vertex labeling \( L: V(G) \mapsto \{0, \ldots, n - 1\} \) such that each vertex \( v \in L(v) > 0 \) has a neighbor \( u \) with \( L(u) = L(v) - 1 \) \( [7] \). A vertex \( v \) is called a layer-\( i \) vertex if \( L(v) = i \). Note that if there is a unique layer-0 vertex \( r \) (which is also called the root), then \( L \) represents a tree \( T \) rooted at \( r \). However, since a vertex could have multiple choices of its parent, the tree \( T \) might not be unique. The following lemma was proved in \([7]\).

**Lemma 1.** A good labeling \( L \) with a unique layer-0 vertex \( r \) can be constructed in \(O(n \log \Delta \log^2 n)\) time and \(O(\log \Delta \log^2 n)\) energy. We are allowed to choose whether or not to designate a specific vertex \( r \).

A good labeling allows us to broadcast messages in an energy-efficient manner. In particular, we have the following lemma.

**Lemma 2.** Suppose that we are given a good labeling \( L \) with a unique layer-0 vertex \( r \). Then we can achieve the following.

1. It takes \(O(n \Delta \log n)\) time and \(O(\Delta \log n)\) energy for every vertex to broadcast a message to the entire network.

2. It takes \(O(nx \Delta \log n)\) time and \(O(x \log \Delta \log n)\) energy for \( x \) vertices to broadcast messages to the entire network.

**Proof.** For the first task, consider the following algorithm. We relay the message of each vertex to the root \( r \) using the following converge cast algorithm. For \( i = n - 1 \) down to 1, do SR-comm\(^{\text{all}}\) with \( S \) being the set of all layer-\( i \) vertices, and \( R \) being the set of all layer-\( (i-1) \) vertices. For each execution of SR-comm\(^{\text{all}}\), each vertex in \( S \) transmits not only its message but also all other messages that it has received so far. Although we perform SR-comm\(^{\text{all}}\) \( n - 1 \) times, each vertex only participates at most twice. Thus, the cost is \(O(n \Delta \log n)\) time and \(O(\Delta \log n)\) energy.

After that, the root \( r \) has gathered all messages. The root \( r \) then broadcasts this information to all vertices via the diverge cast algorithm, as follows. For \( i = 0 \) to \( n - 2 \), do SR-comm with \( S \) being the set of all layer-\( i \) vertices, and \( R \) being the set of all layer-\( (i+1) \) vertices. Similarly, although we perform SR-comm for \( n - 1 \) times, each vertex only participates at most twice. Thus, the cost is \(O(n \log \Delta \log n)\) time and \(O(\log \Delta \log n)\) energy.
The second task can be solved similarly. Let $X$ be a size-$x$ set of vertices that attempt to broadcast a message. We present two different algorithms solving this task. Both of them take $O(nx \log \Delta \log n)$ time and $O(x \log \Delta \log n)$ energy.

We can solve this task by first doing a converge cast (using $\text{SR-comm}^\text{multi}$ with $M = x$) to gather all $x$ messages to the root, and then do a diverge cast (using $\text{SR-comm}$) to broadcast these messages from root to everyone. Note that in order to use $\text{SR-comm}^\text{multi}$, the initial holder of each message $m$ needs to first generate a sufficient amount of random bits and attach them to the message; these random bits serve as the shared randomness associated with the message $m$ (which is needed in $\text{SR-comm}^\text{multi}$). The cost of $\text{SR-comm}^\text{multi}$ with $M = x$ is $O(nx \log \Delta \log n)$ time and $O(x \log \Delta \log n)$ energy.

Alternatively, the task can be solved using only $\text{SR-comm}$. Consider the following procedure which broadcasts at least one message $m$ among all $x$ messages. Do a converge cast using $\text{SR-comm}$, and then it is guaranteed that at least one message $m$ is reached to the root $r$. Next, we do a diverge cast using $\text{SR-comm}$ to broadcast the message $m$ to everyone. This takes $O(n \log \Delta \log n)$ time and $O(\log \Delta \log n)$ energy. To solve the task we repeat this procedure for $x$ times, and so the total cost is $O(nx \log \Delta \log n)$ time and $O(x \log \Delta \log n)$ energy.

**A Note about Energy Efficiency for Small Degree Graphs.** For the $\text{LOCAL}$ model, it is possible to construct a good labeling $\mathcal{L}$ with a unique layer-0 vertex $r$ in $O(\log n)$ energy [7]. Note that in $\text{LOCAL}$ each vertex is able to gather the list of IDs of its neighbors in one round. Given $\mathcal{L}$, the root $r$ is able to gather the entire graph topology in $O(1)$ energy in $\text{LOCAL}$ (using a converge cast to relay the information to the root layer-by-layer). Thus, all graph problems in the $\text{LOCAL}$ model can be solved using merely $O(\log n)$ energy.

In [7], they also showed that we can simulate any $\text{LOCAL}$ algorithm in the $\text{CD}$ or $\text{No-CD}$ models on bounded degree graphs (i.e., $\Delta = O(1)$) with only $O(1)$-factor overhead on both time and energy costs. Thus, all graph problems in wireless network models on bounded degree graphs can be solved using $O(\log n)$ energy.

We can establish this result formally via Lemma 2(1). Note that we can let each vertex $v$ learn the ID list of $\mathcal{L}$ with probability 1. With probability $1 - \frac{1}{\sqrt{n}}$, each vertex $v$ learns the ID list of $\mathcal{L}$ in one round. Given $\mathcal{L}$, the root $r$ with probability $1 - \frac{1}{\sqrt{n}}$ joins $\mathcal{L}$, where the message of each vertex is its ID; this takes $O(\Delta \log n)$ time and energy. After each vertex knows the ID list of $\mathcal{L}$, we apply Lemma 2(1) to let every vertex learn the entire network topology; it takes $O(n \Delta \log n)$ time and $O(\Delta \log n)$.

In view of the above, in this work we only focus on unbounded degree graphs in wireless network models.

### 2.3 A Simple Breadth-first Search Algorithm

In the BFS problem, we are given a specific source vertex $s$, and the goal is to let each vertex $v$ learn $\text{dist}(v, s)$. It is straightforward to see that a BFS labeling can be constructed in $O(D)$ time and energy (using $\text{SR-comm}$ to construct the BFS tree layer-by-layer). In this section, we show that BFS can be solved in $O(\sqrt{n})$ energy, for a general graph $G = (V, E)$. This is more energy efficient than the trivial/standard method for large diameter graphs.

**Theorem 3.** There is a randomized algorithm for BFS taking $\tilde{O}(n^{1.5})$ time and $\tilde{O}(\sqrt{n})$ energy.

The BFS algorithm for Theorem 3 is based on a combination of the building blocks and a known approach in distributed and parallel shortest path algorithms [14, 18, 33, 10]. More specifically, we sample a set $U$ of distinguished vertices such that each vertex $v \neq s$ joins $U$ independently with probability $\log n / \sqrt{n}$ (and $s \in U$ with probability 1). Note that $|U| < 2\sqrt{n} \log n$ with high probability (by a Chernoff bound). We have the following lemma, which first appeared in [33].

**Lemma 4.** Let $s$ be a specified source vertex. Let $U$ be a subset of vertices such that each vertex $v \neq s$ joins $U$ with probability $\log n / \sqrt{n}$, and $s$ joins $U$ with probability 1. With probability $1 - n^{-\Omega(C)}$, the following is true. For each vertex $t \neq s$, there is an $s$-$t$ shortest path $P_{s,t}$ that contains no $C\sqrt{n}$-vertex subpath $P'$ such that $P' \cap U = \emptyset$.

**Proof.** Given any set $S$ of $C\sqrt{n}$ vertices, the probability that $|S \cap U| = 0$ is at most $(1 - \log n / \sqrt{n})^{C\sqrt{n}} = n^{-\Omega(C)}$. The lemma follows by a union bound over $n - 1$ choices of $t$, and a union bound over at most $O(n)$ choices of $C\sqrt{n}$-vertex subpaths of a given $s$-$t$ shortest path $P_{s,t}$. \(\square\)
In view of Lemma 3 if each vertex \( t \) knows (i) \( \text{dist}(t,v) \) for each \( v \) within distance \( C\sqrt{n} \) to \( t \), and (ii) all-pair distances of vertices in \( U \), then \( t \) has enough information to calculate \( \text{dist}(s,t) \). The reason is as follows. Let \( P_{s,t} \) be an \( s-t \) shortest path. Then we can decompose \( P \) into subpaths \((s, v_1)\), \((v_1, v_2)\), \((v_2, v_3)\), ..., \((v_k, t)\) such that \( s, v_1, ..., v_k \in U \), and each subpath has length at most \( C\sqrt{n} \). Since \( t \) knows the length of all these subpaths, \( t \) can calculate its distance to \( s \). In what follows, we show how to let each vertex \( t \) acquire this information.

**Step 1: Local BFS Searches.** At a high level, we let each \( u \in U \) initiate a distance-\( C\sqrt{n} \) local BFS-tree rooted at \( u \), in parallel; we assign each local BFS search with a random starting time so as to minimize the amount of local contention. That is, we show that during the algorithm, for each time slot, and for each vertex \( v \), the number of local BFS searches hitting \( v \) is small. The idea of using random starting time to schedule multiple algorithms is from [27], and this technique is later extended in [12].

Our goal in Step 1 is to let each vertex \( v \in V \) know \( \text{dist}(v,u) \) for each \( u \in U \) such that \( \text{dist}(v,u) \leq C\sqrt{n} \). We aim at devising an algorithm that takes \( O(\sqrt{n}) \) time and \( O(\sqrt{n}) \) energy. The algorithm consists of \( h = 2\sqrt{n}\log n + C\sqrt{n} \) epochs. Each epoch consists of \( \log n \) time slots, which is enough for executing \( \text{SR-comm}^{\text{multi}} \) with \( M = O(\log n/\log \log n) \).

Each vertex \( u \in U \) selects a number \( \tau_u \) (indicating the starting epoch) uniformly at random from the numbers \([2\sqrt{n}\log n]\); the following lemma is straightforward.

**Lemma 5.** Suppose \( |U| < 2\sqrt{n}\log n \). For each vertex \( v \), and for each integer \( d = 1, \ldots, h \), with high probability, the number of vertices \( u \in U \) such that \( \tau_u + \text{dist}(u,v) = d \) is at most \( O(\log n/\log \log n) \).

*Proof.* Given two vertices \( u \in U \) and \( v \in V \), and a fixed number \( d \), the probability that \( \tau_u + \text{dist}(u,v) = d \) is at most \( 1/(2\sqrt{n}\log n) \). Let \( X \) be the total number of \( u \in U \) such that \( \tau_u + \text{dist}(u,v) = d \). The expected value of \( X \) is at most \( \mu = |U|/(2\sqrt{n}\log n) < 1 \), since \( |U| < 2\sqrt{n}\log n \). By a Chernoff bound, for any \( \delta > 0 \), we have \( \Pr[X \geq (1 + \delta) \cdot 1] < \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{1+\delta} \). As long as \( \delta = \Omega(\log n/\log \log n) \), we have \( \Pr[X \geq (1 + \delta) \cdot 1] = 1/\text{poly}(n) \). \( \square \)

Each vertex \( u \in U \) starts its local BFS search at epoch \( \tau_u \). For each \( i = 0, \ldots, C\sqrt{n} - 1 \), at the beginning of epoch \( \tau_u + i \), (by the inductive hypothesis) all vertices that are within distance \( i \) to \( u \) already know their distance to \( u \); by the end of epoch \( \tau_u + i \), we require that all vertices that are within distance \( i + 1 \) to \( u \) know their distance to \( u \). This can be achieved via an application of \( \text{SR-comm}^{\text{multi}} \) with \( M = O(\log n/\log \log n) \). Recall (Lemma 5) that in one epoch each vertex \( v \) only needs to learn at most \( O(\log n/\log \log n) \) distances (each corresponds to a local BFS search initiated at a distinct vertex \( u \in U \)), and so the number of distinct messages \( v \) needs to receive is at most \( O(\log n/\log \log n) \).

In what follows, we describe the details of \( \text{SR-comm}^{\text{multi}} \) in an epoch \( e \). We set \( R = V \). For each \( u \in U \) such that \( x = \tau_u + i \) for some \( i \in \{0, \ldots, C\sqrt{n} - 1\} \), the layer-\( i \) vertices \( \{v \in V \mid \text{dist}(u,v) = i - 1\} \) in the local BFS tree rooted at \( u \) are included in the set \( S \), and these vertices transmit the same message containing the following three components: (i) the number \( i \), (ii) the ID of \( u \), and (iii) the (shared) random bits generated by \( u \). Recall that \( \text{SR-comm}^{\text{multi}} \) requires that all holders of the same message have shared randomness. That is, we need the vertices participating in the local BFS search initiated by \( u \in U \) to agree on the same choices of random bits. We let \( u \) generate all random bits needed at the beginning of the algorithm, and these random bits are sent to other vertices through \( \text{SR-comm}^{\text{multi}} \) along the local BFS tree rooted at \( u \).

To summarize, we perform \( h = 2\sqrt{n}\log n + C\sqrt{n} = O(\sqrt{n}) \) number of \( \text{SR-comm}^{\text{multi}} \) with \( M = O(\log n/\log \log n) \), and the runtime for each \( \text{SR-comm}^{\text{multi}} \) is \( O(1) \), and so the time complexity of Step 1 is \( O(\sqrt{n}) \).

**Step 2: Gathering Distance Information.** Recall that by the end of the previous step, each vertex \( v \in V \) knows \( \text{dist}(v,u) \) for each \( u \in U \) such that \( \text{dist}(v,u) \leq C\sqrt{n} \). If we let each vertex \( v \) gather all information stored in all vertices in \( U \), then \( v \) is able to locally calculate the all-pair distances among vertices in \( U \). The task of letting all vertices in \( U \) broadcast a message can be done via Lemma 2(2) with \( x = 2\sqrt{n}\log n > |U| \). The cost is \( O(n^{1.5}) \) time and \( O(\sqrt{n}) \) energy.
3 Diameter

In this section, we show that the energy complexity of computing diameter is \(\Omega(n/\log^2 n)\) in both CD and No-CD, even for \(O(\log n)\)-arboricity graphs. For the upper bound side, we show that for bounded-genus graphs, diameter can be computed in \(\tilde{O}(\sqrt{n})\) energy. The genus of a graph \(G\) is the minimum number \(g\) such that \(G\) can be drawn on an oriented surface of \(g\) handles without crossing. A graph with genus \(g = O(1)\) is called bounded-genus. For example, planar graphs are genus-0 graphs; graphs that can be drawn on a torus without crossing have genus at most 1. Bounded-genus graphs is a much wider graph class than planar graphs. For instance, the 5-vertex complete graph \(K_5\) is not planar, but the genus of \(K_x\) is \([(x-3)(x-4)/12]\); see [17] p. 118.

**Theorem 6.** There is a randomized algorithm for computing diameter in \(\tilde{O}(n^{1.5})\) time and \(\tilde{O}(\sqrt{n})\) energy for bounded-genus graphs.

**Theorem 7.** The energy complexity of computing diameter is \(\Omega(n/\log^2 n)\) in both CD and No-CD. This is true even for graphs of arboricity \(O(\log n)\).

Note that the lower bound proof naturally extends to the setting where the maximum degree is \(\Delta\). In this setting, computing diameter takes \(\Omega(\Delta/\log^2 \Delta)\) energy.

3.1 A Partition of the Vertex Set

Let \(G = (V, E)\) be a bounded-genus graph. Let \(V_H\) be the set of vertices that have degree at least \(\sqrt{n}\); let \(V_L = V \setminus V_H\). We assume \(|V_H| \geq 1\); since otherwise we can already solve all problems in \(\tilde{O}(\sqrt{n})\) energy, in view of the discussion in Section 2.2.

We divide the connected components induced by vertices in \(V_L\) into three types. For each component \(S\), denote \(G[S]\) as the subgraph induced by all edges that have at least one endpoint in \(S\)

**Type 1.** A connected component \(S\) induced by vertices in \(V_L\) is of type-1 if \(|S| \leq \sqrt{n}\) and \(\bigcup_{w \in S} N(w) \cap V_H = \{u\}\). For each vertex \(u \in V_H\), we write \(C(u)\) to denote the set of type-1 components \(S\) such that \(\bigcup_{w \in S} N(w) \cap V_H = \{u\}\).

**Type 2.** A connected component \(S\) induced by vertices in \(V_L\) is of type-2 if \(|S| \leq \sqrt{n}\) and \(\bigcup_{w \in S} N(w) \cap V_H = \{u, v\}\). For each pair of two distinct vertices \(\{u, v\} \subseteq V_H\), we write \(C(u, v)\) to denote the set of type-2 components \(S\) such that \(\bigcup_{w \in S} N(w) \cap V_H = \{u, v\}\).

**Type 3.** A connected component \(S\) induced by vertices in \(V_L\) is of type-3 if it is neither of type-1 nor of type-2.

Since \(G\) is of bounded-genus, we have \(|E(G)| = O(n)|\), and so \(|V_H| = O(\sqrt{n})\). In what follows, we show that the number of type-3 components is also \(O(\sqrt{n})\).

**Lemma 8.** Let \(G\) be a bipartite graph with bipartition \(V = X \cup Y\) and genus at most \(g\). If \(\deg(v) \geq 3\) for each \(v \in X\), then \(|X| \leq 2|Y| + 4(g - 1)\).

**Proof.** Let \(E, V,\) and \(F\) be the edge set, vertex set, and face set of \(G\). Note that in a bipartite graph, each face has at least 4 edges, and each edge appears in at most 2 faces, and so \(|E| \geq 2|F|\). Combining this inequality with Euler’s polyhedral formula \(|V| - |E| + |F| \geq 2 - 2g\) (note that the genus of \(G\) is at most \(g\)), we obtain that \(2|V| - |E| \geq 4(1 - g)\).

Since \(\deg(v) \geq 3\) for each \(v \in X\), we have \(|E| \geq 3|X|\). Note that we also have \(|V| = |X| + |Y|\), and so \(2|V| - |E| \leq 2(|X| + |Y|) - 3|X| = 2|Y| - |X|\). Therefore, \(2|Y| - |X| \geq 4(1 - g)\), as desired.

Therefore, as long as \(G\) is of bounded-genus, we have \(|X| \leq 2|Y| + O(1)\) by Lemma 8. The reader might wonder whether we can replace the bounded-genus requirement to the weaker requirement of being bounded-minor-free. It is not hard to see that being bounded-minor-free is in general not enough to guarantee \(|X| = O(|Y|)|\), as the complete bipartite graph \(K_{3, x}\) does not contain \(K_5\) as a minor, for any \(x\).

\[\text{In most of the literature, } G[S]\text{ is used to denote the subgraph induced by } S; \text{ but in this paper we write } G[S]\text{ to denote the subgraph induced by } \bigcup_{v \in S} N(v)\.]
Lemma 9. Given that $G$ is of bounded-genus, the number of type-3 components is at most $O(\sqrt{n})$.

Proof. By its definition, if a type-3 component $S$ satisfies $|\bigcup_{w \in S} N(w) \cap V_H| \leq 2$, then $|S| > \sqrt{n}$. Thus, the number of type-3 components $S$ such that $|\bigcup_{w \in S} N(w) \cap V_H| \leq 2$ is at most $\sqrt{n}$. Let $X$ be the set of all type-3 components $S$ such that $|\bigcup_{w \in S} N(w) \cap V_H| > 2$.

Consider a bipartite graph $G^*$ with the bipartition $X \cup V_H$, where each $S \in X$ is adjacent to all $v \in \bigcup_{w \in S} N(w) \cap V_H$. Note that $\deg(S) \geq 3$ for each $S \in X$. The property that the genus is at most $k$ is closed under edge contraction and vertex removal. Note that $G^*$ can be obtained from $G$ via a sequence of edge contractions and vertex removals, and so $G^*$ is of bounded-genus. By Lemma 8, we have $|X| \leq 2|V_H| + O(1) = O(\sqrt{n})$. □

We let $G_H$ be the graph defined by the vertex set $V_H$ and the edge set $\{(u, v) : |C(u, v)| > 0\}$. The following lemma is useful in subsequent discussion.

Lemma 10. Given that $G$ is of bounded-genus, the number of edges in $G_H$ is at most $O(\sqrt{n})$. Furthermore, there is an edge orientation of $G_H$ such that each vertex has out-degree $O(1)$.

Proof. The graph $G_H$ can be obtained from $G$ via a sequence of edge contractions and vertex removals, and so $G_H$ is of bounded-genus. Note that bounded-genus graphs have arboricity $O(1)$, and so the number of edges in $G_H$ is at most linear in the number of vertices in $G_H$, which is $O(\sqrt{n})$, and we can orient the edges of $G_H$ in such a way that each vertex has out-degree of $O(1)$. □

Note that the number of type-3 components is still $O(\sqrt{n})$ even if the graph genus is as high as $O(\sqrt{n})$, and so in a sense Lemma 10 is the bottleneck of our approach.

3.2 Computing Diameter via Graph Partition

Before we proceed, we briefly discuss our proof idea. First of all, learning the entire graph topology of the subgraph induced by $V_H$ and all type-3 components is doable using $O(\sqrt{n})$ energy via Lemma 2; this is based on the following facts: (i) $|V_H| = O(\sqrt{n})$, (ii) $\deg(v) = O(\sqrt{n})$ for each $v$ in a type-3 component, and (iii) the number of type-3 components is $O(\sqrt{n})$.

More specifically, we can use SR-comm$^{\text{approx}}$ to let each vertex $v \in V$ approximately learn its degree. Then, we use SR-comm$^{\text{all}}$ to let all vertices in $V_L$ learn the set of all its neighbors. Since $|V_H| = O(\sqrt{n})$, we can do another SR-comm$^{\text{all}}$ to let each $v \in V_H$ learn $N(v) \cap V_H$. For each component $S$ of $V_L$, we use Lemma 1 to let each vertex $v \in S$ broadcast the IDs of vertices in $N(v)$ to all vertices in $S$; after that, each vertex $v \in S$ knows the topology of $G[S]$. Recall that (i) $|V_H| = O(\sqrt{n})$ and (ii) the number of type-3 components is $O(\sqrt{n})$, and so we can use Lemma 2 with $x = O(\sqrt{n})$ to do the following.

- Each $v \in V_H$ broadcasts the IDs of vertices in $N(v) \cap V_H$.
- A representative of each type-3 component $S$ broadcasts the topology of $G[S]$.

At this point, all vertices know the topology of the subgraph induced by $V_H$ and all type-3 components. See Section 3.3 for details.

It is much more difficult to extract information from type-1 and type-2 components. For example, a vertex $u \in V_H$ could be connected to $\Theta(n)$ type-1 components (i.e., $|C(u)| = \Theta(n)$). Since the energy budget for $u$ is $O(\sqrt{n})$, throughout the entire algorithm $u$ can only receive information from at most $O(\sqrt{n})$ components in $C(u)$. The challenge is to show that the diameter can still be calculated with limited amount of carefully chosen information about type-1 and type-2 components.

The goal of this section is to define a set of parameters of type-1 and type-2 components, and to show that with these parameters, the diameter can be calculated. In subsequent discussion, denote eccentricity$(u, S)$ as $\max_{v \in S} \text{dist}(u, v)$. By default, all distances are with respect to the underlying network $G$; we use subscript to indicate distances within a specific vertex set, edge set, or subgraph.
Parameters for Type-1 Components. We first consider type-1 components in \( C(u) \), for a vertex \( u \in V_H \).

\( A_i[u] \), \( a_i[u] \): Denote \( A_i[u] \) as a component \( S \in C(u) \) that maximizes eccentricity(\( u, S \)), and denote \( A_2[u] \) as a component \( S \in C(u) \) \( \setminus \{ A_1[u] \} \) that maximizes eccentricity(\( u, S \)). For \( i \in \{1, 2\} \), denote \( a_i[u] = \text{eccentricity}(u, A_i[u]) \).

\( B[u] \), \( b[u] \): Denote \( B[u] \) as a component \( S \in C(u) \) that maximizes \( \max_{s,t \in S \cup \{ u \}} \text{dist}(s, t) \), and denote \( b[u] \) as \( \max_{s,t \in B[u] \cup \{ u \}} \text{dist}(s, t) \).

Some of the above definitions are undefined when \( |C(u)| \) is too small. For example, if \( |C(u)| = 1 \), then \( A_2[u] \) and \( a_2[u] \) are undefined. In such a case, we set these parameters to their default values: zero for numerical parameter (e.g., \( a_2[u] = 0 \)), and empty set for vertex set parameter (\( A_2[u] = \emptyset \)). It is also possible that there are multiple choices of a parameter, and we may break ties arbitrarily.

Observe that any path connecting a vertex in \( \bigcup_{S \subseteq C(u)} S \) to the rest of the graph must passes the vertex \( u \in V_H \). We only need to extract the following information from \( C(u) \) for the calculation of diameter:

- The longest distance between two vertices within \( \bigcup_{S \subseteq C(u)} S \cup \{ u \} \), and this is \( \max\{ b[u], a_1[u] + a_2[u] \} \).
- The longest distance between \( u \) and a vertex in \( \bigcup_{S \subseteq C(u)} S \), and this is \( a_1[u] \).

Thus, regardless of the size of \( C(u) \), we only need to learn \( a_1[u] \), \( a_2[u] \), and \( b[u] \) from the components of \( C(u) \). These parameters can be learned via SR-comm\(^{\max} \), as follows. For each \( S \subseteq C(u) \), there will be a representative vertex \( r_{S,u} \in S \) responsible of communicating with \( u \), and the vertex \( r_{S,u} \) already knows the topology of \( G[S] \). Then, \( u \) can learn one of the above parameters by doing one SR-comm\(^{\max} \) with \( S = \{ r_{S,u} \mid S \subseteq C(u) \} \) and \( R = \{ u \} \), and this takes only \( \tilde{O}(1) \) time. See Section 5.3 for details.

Parameters for Type-2 Components. The issue about type-2 components is more complicated, as we need to extract \( O(\sqrt{n}) \) parameters from each \( C(u, v) \). We first describe the parameters, and later explain their purposes. Consider two distinct vertices \( u, v \in V_H \).

\( R[u, v] \), \( r[u, v] \): Let \( R[u, v] \) be a component \( S \subseteq C(u, v) \) that minimizes \( \text{dist}_{G[S]}(u, v) \), and we write \( r[u, v] = \text{dist}_{G[R[u, v]]}(u, v) \). Intuitively, \( R[u, v] \) corresponds a component that offers the shortest route between \( u \) and \( v \), among all components in \( C(u, v) \).

\( A^k_i[u, v] \), \( a^k_i[u, v] \): For a component \( S \subseteq C(u, v) \), denote \( S^{u,k} \) as the set of vertices \( \{ w \in S \mid \text{dist}_{G[S]}(w, u) \leq k \} \). Intuitively, \( S^{u,k} \) is the set of all vertices in \( S \) whose distance to \( u \) (in the subgraph \( G[S] \)) is shorter than that to \( v \) by at least \( k \) units.

Denote \( A^k_i[u, v] \) as a component \( S \subseteq C(u, v) \) that maximizes eccentricity\(_{G[S]}(u, S^{u,k}) \), and denote \( A^k_{ij}[u, v] \) as a component \( S \subseteq C(u, v) \) \( \setminus \{ A^k_i[u, v] \} \) that maximizes eccentricity\(_{G[S]}(u, S^{u,k}) \). We write \( a^k_i[u, v] = \text{eccentricity}_{G[A^k_i[u, v]]}(u, A^k_i[u, v]) \). We only consider \( k \in \{-\sqrt{n}, \ldots, \sqrt{n} \} \).

\( B^l[u, v] \), \( b^l[u, v] \): For a component \( S \subseteq C(u, v) \), denote \( G[S] \) as the graph resulting from adding to \( G[S] \) a length-\( l \) path connecting \( u \) and \( v \); and denote \( \phi^l(S) \) as the maximum value of \( \text{dist}_{G[S]}(s, t) \) over all pairs of vertices \( s, t \in S \cup \{ u, v \} \). A useful observation is that if \( \text{dist}_{V \setminus S}(u, v) = l \), then \( \phi^l(S) \) equals the maximum value of \( \text{dist}_{G}(s, t) \) over all pairs of vertices \( s, t \in S \cup \{ u, v \} \).

Denote \( B^l[u, v] \) as a component \( S \subseteq C(u, v) \) \( \setminus \{ R[u, v] \} \) that maximizes \( \phi^l(S) \), and write \( b^l(u, v) = \phi^l(B^l[u, v]) \). We only consider \( l \in \{1, \ldots, \sqrt{n} \} \).

Similar to parameters of type-1 components, all above parameters are set to their default values if undefined. Note that the definition of \( a^k_i[u, v] \) and \( A^k_i[u, v] \) are asymmetric in the sense that we might have \( a^k_i[u, v] \neq a^k_i[v, u] \) and \( A^k_i[u, v] \neq A^k_i[v, u] \). All remaining parameters for type-2 components are symmetric.

We explain the relevance of the above parameters to the calculation of diameter. Let \( P = (s, \ldots, t) \) be an \( s-t \) shortest path in \( G \) whose length equals the diameter. Consider three possible ways that \( P \) involves vertices in \( \bigcup_{S \subseteq C(u, v)} S \).

- The two endpoints \( s \) and \( t \) are within \( G[S] \), for a component \( S \subseteq C(u, v) \). In this case, if \( \text{dist}_{V \setminus S}(u, v) = l \), then the length of \( P \) equals \( \phi^l(S) = b^l(u, v) \).
• There is a subpath \( P' = (u, \ldots, v) \) of \( P \) whose intermediate vertices are all in \( \bigcup_{S \in C(u, v)} S \); in this case, the length of \( P' \) equals \( r[u, v] \).

• Suppose \( s \in S \) for a component \( S \in C(u, v) \), but \( t \) is not in \( G[S] \). Suppose \( P' = (s, \ldots, u) \) is a subpath of \( P \) such that all vertices in \( P' \setminus \{u\} \) are all in \( S \). If \( \dist(t, v) - \dist(t, u) = k \), then we must have \( s \in S^{u,k} \), and so the length of \( P' \) equals \( a^k_1[u, v] \) (for the case \( t \notin A^k_1[u, v] \)) or \( a^k_2[u, v] \) (for the case \( t \in A^k_1[u, v] \)).

The above discussion also explains why we need to consider different values of \( l \) and \( k \). For example, the choice of the vertex \( t \) and \( \dist(t, v) - \dist(t, u) \) are unknown to vertices in a component \( S \in C(u, v) \), and so there is a need to go over all possibilities to ensure that all candidate shortest paths are considered.

Even though each \( C(u, v) \) is associated with \( O(\sqrt{n}) \) parameters, all these parameters can still be learned using \( O(\sqrt{n}) \) energy per vertex. By Lemma 10 there is an assignment \( F : E(G_H) \rightarrow V_H \) mapping each pair \( \{u, v\} \in E(G_H) \) to one vertex in \( \{u, v\} \) such that each \( w \in V_H \) is mapped at most \( O(1) \) times, and so we can let each \( w \in V_H \) be responsible of learning at most \( O(\sqrt{n}) \) parameters. Learning one parameter can be done using \( \text{SR-comm}^{\text{max}} \) or \( \text{SR-comm}^{\text{min}} \) in \( O(1) \) time. See Section 3.3 for details.

The Graph \( G^* \). To show that one can compute the diameter from a set of parameters extracted from type-1 and type-2 components (and the topology of the subgraph induced by \( V_H \) and all type-3 components), a plausible proof strategy is to classify all shortest paths into classes, and to show how to compute the longest path length in each class. This approach would probably end up being a long case analysis, and so we adopt a different approach. Since we have no message size constraint, instead of only learning the aforementioned distance parameters associated with type-1 and type-2 components, we also learn the topology of \( G[S] \) for each component \( S \) associated with these distance parameters.

Define \( G^* \) as the subgraph induced by (i) \( V_H \), (ii) all type-3 components, (iii) \( A_1[u], A_2[u], \) and \( B[u] \), for each \( u \in V_H \), and (iv) \( A^k_1[u, v], A^k_2[u, v], B^k[u, v], \) and \( R[u, v] \), for each pair of distinct vertices \( \{u, v\} \subseteq V_H \), \( i \in \{1, 2\}, k \in \{-\sqrt{n}, \ldots, \sqrt{n}\} \), and \( l \in \{1, \ldots, \sqrt{n}\} \).

In the rest of this section, we prove that the diameter of \( G \) equals the diameter of \( G^* \), and so the task of computing the diameter of \( G \) is reduced to learning the topology of \( G^* \).

**Lemma 11.** The diameter of \( G \) equals the diameter of \( G^* \).

**Proof.** The proof strategy is to show that the following two statements are correct.

(S1) For each pair of vertices \( \{s, t\} \) in the graph \( G^* \), we have \( \dist_G(s, t) = \dist_{G^*}(s, t) \); this is proved in Lemma 12.

(S2) For each pair of vertices \( \{s, t\} \) in the graph \( G \), there exists a pair of vertices \( \{s', t'\} \) in the graph \( G^* \) satisfying \( \dist_G(s, t) \leq \dist_{G^*}(s', t') \); this is proved in Lemma 14.

The two statements together imply that \( G \) and \( G^* \) have the same diameter. \( \square \)

**Lemma 12.** For any two vertices \( s \) and \( t \) in \( G^* \), we have \( \dist_G(s, t) = \dist_{G^*}(s, t) \).

**Proof.** We choose \( P \) to be an \( s \)-\( t \) path in \( G \) whose length is \( \dist_G(s, t) \) that uses the minimum number of vertices not in \( G^* \). If \( P \) is entirely in \( G^* \), then we are done. In what follows, suppose that \( P \) is not entirely in \( G^* \). Then \( P \) contains a subpath \( P' = (u, \ldots, v) \) whose intermediate vertices are all within a type-2 component \( S \in C(u, v) \) that is not included to \( G^* \). By the definition of \( R[u, v] \), the length of \( P' \) is not longer than the shortest path between \( u \) and \( v \) via \( R[u, v] \), which has length \( r[u, v] \). Therefore, we can replace \( P' \) with a path via vertices in \( R[u, v] \) (which is within \( G^* \)) without increasing the path length. This contradicts our choice of \( P \). \( \square \)

**Lemma 13.** Let \( P \) be a shortest path between two vertices \( s \) and \( t \) in \( G \) such that \( s \in S \) for some type-1 or type-2 component \( S \) not included in \( G^* \), and \( t \) does not reside in \( G[S] \). Then there exists a vertex \( s' \) in \( G^* \) such that \( \dist_G(s', t) \geq \dist_G(s, t) \).
Proof. Suppose that \( S \in C(u) \) is of type-1. Let \( i \in \{1, 2\} \) be an index such that \( t \) is not in \( A_i[u] \). Consider the subpath \( P = (s, \ldots, u) \) of \( P \). The length of \( P \) must not be longer than \( a_i[u] \), and there exists a vertex \( s' \in A_i[u] \) such that the length of the shortest path between \( s' \) and \( u \) is \( a_i[u] \). Thus, we have

\[
\text{dist}_G(s', t) = \text{dist}_G(s', u) + \text{dist}_G(u, t) \geq \text{dist}_G(s, u) + \text{dist}_G(u, t) = \text{dist}_G(s, t).
\]

Next, consider the case that \( S \in C(u, v) \) is of type-2. The path \( P \) must contain at least one of \( u \) and \( v \). Without loss of generality, assume that \( u \) is in \( P \), and there is a subpath \( \tilde{P} = (s, \ldots, u) \) of \( P \) such that \( v \) is not in \( \tilde{P} \). Note that all vertices in \( \tilde{P} \) other than \( u \) are in \( S \), and so the length of \( \tilde{P} \) equals \( \text{dist}_{G[S]}(s, u) + \text{dist}_G(u, t) \).

Let \( k = \text{dist}_{G[S]}(s, v) - \text{dist}_{G[S]}(s, u) \). Note that at least one of \( A^k_1[u, v] \) and \( A^k_2[u, v] \) does not contain \( t \). We choose \( S' = A_i^k[u, v] \) as any one of them that does not contain \( t \). We choose \( s' \in S' \) as a vertex such that \( \text{dist}_{G[S']}((s', u) = a^k_i[u, v] \) and \( \text{dist}_{G[S']}((s', v) - \text{dist}_{G[S']}((s', t) \geq k \). The existence of such \( s' \) is guaranteed by the definition of \( A^k_i[u, v] \).

Our plan is to show that (i) \( a^k_i[u, v] + \text{dist}_G(u, t) \geq \text{dist}_G(s, t) \) and (ii) \( \text{dist}_G(s', t) = a^k_i[u, v] + \text{dist}_G(u, t) \).

Combining these two inequalities give us the desired result \( \text{dist}_G(s', t) \geq \text{dist}_G(s, t) \).

Proof of (i). By the definition of \( A^k_i[u, v] \), we must have \( \text{dist}_{G[S]}((s', u) = a^k_i[u, v] \geq \text{dist}_{G[S]}((s, u) \), and so we have \( a^k_i[u, v] + \text{dist}_G(u, t) \geq \text{dist}_{G[S]}((s, u) + \text{dist}_G(u, t) = \text{dist}_G(s, t) \).

Proof of (ii). Suppose that (ii) is not true. Then any shortest path between \( s' \) and \( t \) must contain a subpath \( P'' = (s', \ldots, v) \) such that \( u \) is not in \( P'' \), and so we have:

\[
\text{dist}_G(s', t) = \text{dist}_{G[S]}((s', v) + \text{dist}_G(v, t) < \text{dist}_{G[S]}((s', u) + \text{dist}_G(u, t).
\]

Combining this inequality with the known fact \( \text{dist}_{G[S]}((s', v) - \text{dist}_{G[S]}((s', u) \geq k \), we have:

\[
\text{dist}_G(u, t) - \text{dist}_G(v, t) > \text{dist}_{G[S]}((s', v) - \text{dist}_{G[S]}((s', u) \geq k,
\]

which implies that \( \text{dist}_G(v, t) < \text{dist}_G(u, t) - k \). We calculate an upper bound of \( \text{dist}_G(s, t) \).

\[
\text{dist}_G(s, t) \leq \text{dist}_{G[S]}((s, v) + \text{dist}_G(v, t)
= (k + \text{dist}_{G[S]}((s, u)) + \text{dist}_G(v, t) 
= (k + \text{dist}_{G[S]}((s, u)) + (\text{dist}_G(u, t) - k)
= \text{dist}_{G[S]}((s, u)) + \text{dist}_G(u, t).
\]

This contradicts the fact that \( P \) is a shortest path between two vertices \( s \) and \( t \) in \( G \), as the length of \( P \) equals \( \text{dist}_{G[S]}((s, u)) + \text{dist}_G(u, t) \).

\[
\qed
\]

Lemma 14. For any two vertices \( s \) and \( t \) in the graph \( G \), there exist two vertices \( s' \) and \( t' \) in the graph \( G^* \) such that \( \text{dist}_{G}(s, t) \leq \text{dist}_{G^*}(s', t') \).

Proof. If both \( s \) and \( t \) are already in \( G^* \), then we are done by setting \( s' = s \) and \( t' = t \). In what follows, assume that at least one of \( s \) and \( t \) is not in \( G^* \).

Case 1: Consider the case where \( s \) and \( t \) are within \( G[S] \), for a type-1 or a type-2 component \( S \) that is not included in \( G^* \). If \( S \in C(u) \) for some \( u \in V_H \), then in the component \( B[u] \subset C(u) \) there reside two vertices \( s' \) and \( t' \) such that \( \text{dist}_{G}(s', t') = b[u] \geq \text{dist}_{G}(s, t) \) according to the definition of \( B[u] \).

Next, consider the situation \( S \in C(u, v) \) for some \( u, v \in V_H \). Let \( l = \text{dist}_{G \setminus S}(u, v) \). Note that \( l \leq r[u, v] \), since \( S \not\subseteq R[u, v] \) and the existence of \( S \) guarantees that \( R[u, v] \neq \emptyset \). Consider the component \( B_i[u, v] \subset C(u, v) \). We also have \( l = \text{dist}_{G \setminus B_i[u, v]}(u, v) \), since the shortest \( u-v \) path via \( R[u, v] \) is not longer than any \( u-v \) path via \( S \) or \( B_i[u, v] \), according to our choice of \( R[u, v] \); also note that our definition of \( B_i[u, v] \) prevents \( B_i[u, v] = R[u, v] \). Since \( l = \text{dist}_{G \setminus B_i[u, v]}(u, v) \), by definition of \( B_i[u, v] \), there exist two vertices \( s' \) and \( t' \) in \( G[B_i[u, v]] \) such that \( \text{dist}_{G}(s', t') \geq \text{dist}_{G}(s, t) \), since otherwise we would have selected \( B_i[u, v] = S \).
Case 2: Next, consider the case where there are $s$ and $t$ do not completely reside in the same subgraph $G[S]$, for any type-1 or type-2 component $S$. If $s$ is not in $G^*$, we apply Lemma 13 to find a vertex $s'$ in $G^*$ such that $\text{dist}(s, t) \leq \text{dist}(s', t)$; otherwise we simply set $s' = s$. If $t$ is not in $G^*$, we apply Lemma 13 again to find a vertex $t'$ in $G^*$ such that $\text{dist}(s', t) \leq \text{dist}(s', t')$; otherwise we simply set $t' = t$. Our choice of $s'$ and $t'$ satisfies $\text{dist}(s, t) \leq \text{dist}(s', t')$.  

3.3 Learning the Graph Topology of $G^*$

In this section we show how to let all vertices learn the graph topology of $G^*$. Throughout this section, we implicitly assume that the communication network $G$ is of bounded-genus. Recall that $G_H$ is the graph defined by the vertex set $V_H$ and the edge set $\{\{u, v\} : |C(u, v)| > 0\}$, and we know that $E(G_H) = O(\sqrt{n})$ from Lemma 10. By Lemma 10 there exists an assignment $F : E(G_H) \to V_H$ mapping each pair $\{u, v\} \in E(G_H)$ to one vertex in $\{u, v\}$ such that each $w \in V_H$ is mapped at most $O(1)$ times.

Let $A'$ be any deterministic algorithm that finds such an assignment $F$, and we fix $F^*$ to be the outcome of $A'$ on the input $G_H$. Note that if each vertex $v \in V$ already knows the graph $G_H$, then $v$ can locally calculate $F^*$.

To learn $G^*$, we first let each vertex $u \in V$ learn the following information. Note that $I_1(u)$ and $I_2(u)$ contain nothing if $u \in V_L$.

Basic Information $I_0(u)$: For each vertex $u \in V$, $I_0(u)$ contains the following information: (i) whether $u \in V_H$ or $u \in V_L$; (ii) the list of vertices in $N(u) \cap V_H$; and (iii) the set of all pairs $\{u', v'\} \in E(G_H)$.

Information about Type-1 Components $I_1(u)$: For each $u \in V_H$, $I_1(u)$ contains the graph topology of $G[S]$, for each $S' = A_1[u], A_2[u], B[u]$.

Information about Type-2 Components $I_2(u)$: For each $u \in V_H$, $I_2(u)$ contains the graph topology of $G[S']$, for each $S' = A_1^k[u], A_2^k[u], B^l[u, v], R[u, v]$, for each $i \in \{1, 2\}, k \in \{-\sqrt{n}, \ldots , \sqrt{n}\}$, and $l \in \{1, \ldots , \sqrt{n}\}$.

Note that the information $I_0(u)$ allows each vertex $u$ to calculate $F^*$ locally. The following lemma shows that letting each vertex $u \in V$ learn $I_0(u), I_1(u)$, and $I_2(u)$ is all we need.

Lemma 15. Given that each $u \in V$ already knows $I_0(u), I_1(u)$, and $I_2(u)$, in $\tilde{O}(n^{1.5})$ time and $\tilde{O}(\sqrt{n})$ energy, we can let each vertex in $G$ learn the graph topology of $G^*$.

Proof. To learn $G^*$, it suffices to know (i) $I_1(u)$ and $I_2(u)$ for each $u \in V_H$, (ii) the graph topology of $G[S]$ for each type-3 component $S$, and (iii) the graph topology of the subgraph induced by $V_H$. For each type-3 component $S$, let $r_S$ be the smallest ID vertex in $S$. In view of the above, to let each vertex learn the topology of $G^*$, it suffices to let the following vertices broadcast the following information:

- For each $u \in V_H$, $u$ sends $I_1(u), I_2(u)$, and the list of vertices $N(u) \cap V_H$ (which is contained in $I_0(u)$).
- For each $u \in V_L$ such that $u = r_S$ for a type-3 component $S$, $u$ sends the graph topology of $G[S]$. Note that each vertex $u \in V_L$ can decide locally using information in $I_0(u)$ whether or not $u$ itself is $r_S$ for a type-3 component $S$.

Since $|V_H| = O(\sqrt{n})$ and the number of type-3 components is also $O(\sqrt{n})$ by Lemma 9, the number of vertices that need to broadcast is $O(\sqrt{n})$. Thus, we can use Lemma 2 with $x = O(\sqrt{n})$ to broadcast all the above information. This is done in time $\tilde{O}(n^{1.5})$, and energy $\tilde{O}(\sqrt{n})$.  

Next, we consider the task of learning the basic information $I_0(u)$.

Lemma 16. In $\tilde{O}(\sqrt{n})$ time and energy, each vertex $v$ is able to detect whether $v \in V_H$ or $v \in V_L$. Moreover, if $v \in V_H$, $v$ learns the list of vertices in $N(v) \cap V_H$; if $v \in V_L$, $v$ learns the two lists of vertices $N(v) \cap V_L$ and $N(v) \cap V_H$.  

12
Proof. First, use $\text{SR-comm}^{\text{apx}}$ (with $W = 1$, $\epsilon = 1$, $S = R = V$, and $m_u = 1$ for each $v \in S$) to let each $v \in V$ estimate $\deg(v)$ up to a factor of 2. This step takes $O(1)$ time.

Next, do $\text{SR-comm}^{\text{all}}$ with $S = V$ and $R$ being the set of all vertices $v$ whose estimate of $\deg(v)$ is at most $2\sqrt{n}$. The message $m_v$ for each vertex $v$ is its ID, and we use the bound $\Delta' = 4\sqrt{n}$ for $\text{SR-comm}^{\text{all}}$. Recall that $V_L$ is the set of vertices of degree less than $\sqrt{n}$, and so we must have $V_L \subseteq R$. After this step, each vertex $v \in V$ has enough information to decide whether $v \in V_H$ or $v \in V_L$. Furthermore, if $v \in V_L$, then $v$ knows the list of all vertices $N(v)$. This step takes $O(\sqrt{n})$ time.

Lastly, for each vertex to learn all the required vertex lists, do $\text{SR-comm}^{\text{all}}$ again with a different setting. Let $S = V_H$, and $R = V$. The message $m_v$ for each vertex $v \in S$ is its ID. We use the bound $\Delta' = \sqrt{n} \geq |V_H|$. After $\text{SR-comm}^{\text{all}}$, each vertex $v \in V$ knows the list of vertices in $N(v) \cap V_H$. For each $v \in V_L$, since $v$ already knows the list of all vertices $N(v)$, it can locally calculate the list $N(v) \cap V_L$. This step takes $O(\sqrt{n})$ time. \qed

Lemma 17. In $O(n^{1.5})$ time and $O(\sqrt{n})$ energy, each vertex in each component $S$ of $V_L$ is able to learn (i) the list of vertices in $S$, and (ii) the topology of the subgraph $G[S]$.

Proof. First, apply Lemma 10 to let each vertex $v \in S$ learn the two lists $N(v) \cap V_L$ and $N(v) \cap V_H$. To let each vertex learn the required information, it suffices to let each $v \in S$ broadcast the two lists $N(v) \cap V_L$ and $N(v) \cap V_H$ to all vertices in $S$.

We use Lemma 1 to let each component $S$ of $V_L$ compute a good labeling with one root vertex $r_S$. Then, we invoke Lemma 2(1) to let each vertex $v \in S$ broadcast the two lists $N(v) \cap V_L$ and $N(v) \cap V_H$ to all vertices in $S$. Recall that the degree of any vertex in $V_L$ is less than $\sqrt{n}$, and so the cost of applying Lemma 2(1) is $O(n^{1.5})$ time and $O(\sqrt{n})$ energy. \qed

After executing the algorithm of Lemma 17 each vertex $w \in S$ is able to determine the type of $S$. If $S$ is of type-1, $w$ knows the vertex $u \in V_H$ such that $S \subseteq \{u\}$; if $S$ is of type-2, $w$ knows the two vertices $u, v \in V_H$ such that $S \subseteq \{u, v\}$.

Recall that $G_H$ is the graph defined by the vertex set $V_H$ and the edge set $\{(u, v) : |C(u, v)| > 0\}$, and we know that $E(G_H) = O(\sqrt{n})$ from Lemma 10.

Lemma 18. Suppose that each vertex in every type-2 component $S$ already knows (i) the list of vertices in $S$, and (ii) the topology of the subgraph $G[S]$. Then, in $O(n^{1.5})$ time and $O(\sqrt{n})$ energy, all vertices in the graph can learn the set of all pairs $\{(u, v) \in E(G_H)\}$.

Proof. First of all, we let all vertices in $V_H$ agree on a fixed ordering $V_H = \{v_1, \ldots, v_{|V_H|}\}$ as follows. We use Lemma 1 to compute a good labeling (on the entire graph) with one root vertex $r$. Then, we invoke Lemma 2(2) (with $x = \sqrt{n}$) to let each vertex $v \in V_H$ broadcast ID($v$). Then we order $V_H = \{v_1, \ldots, v_{|V_H|}\}$ by increasing ordering of ID. This step takes $O(n^{1.5})$ time and $O(\sqrt{n})$ energy.

Next, we show how to let each $u \in V_H$ learn the list of all $v \in V_H$ such that $C(u, v) \neq \emptyset$ by $|V_H|$ invocations of $\text{SR-comm}$. Given a type-2 component $S \subseteq C(u, v)$, define $z_{u,S}$ as the smallest ID vertex in $N(v) \cap S$. The vertex $z_{u,S}$ will be responsible for letting $v$ to know that $C(u, v) \neq \emptyset$. For $i = 1$ to $|V_H|$, we do an $\text{SR-comm}$ with $S$ being the set of all vertices that are $z_{u,S}$ for some type-2 component $S$ (with $v_i \in G[S]$), and $R$ being the set of all vertices in $V_H$. Note that a vertex $u \in V_H$ receives a message during the $i$th iteration if and only if $C(u, v_i) \neq \emptyset$, i.e., $\{u, v_i\} \in E(G_H)$. This step takes $|V_H| \cdot O(1) = O(\sqrt{n})$ time.

At this moment, each $u \in V_H$ knows the list of all $v \in V_H$ such that $C(u, v) \neq \emptyset$. Lastly, we apply Lemma 2(2) (with $x = \sqrt{n}$) to let each vertex $u \in V_H$ broadcast this information to all vertices. This step takes $O(n^{1.5})$ time and $O(\sqrt{n})$ energy. \qed

Lemma 19. In $O(n^{1.5})$ time and $O(\sqrt{n})$ energy, we can let each $u \in V$ learn $I_0(u)$.

Proof. This follows from Lemma 17 and Lemma 18. \qed

Next, we consider the task of learning $I_1(u)$ and $I_2(u)$. Each $u \in V_H$ needs to learn 3 parameters in $I_1(u)$ and $O(\sqrt{n})$ parameters in $I_2(u)$. Recall that for each $u \in V_H$, the number of pairs $\{u, v\}$ such that $F^*(\{u, v\}) = u$ is at most $O(1)$.

For each $u \in V_H$, and for each type-1 component $S \subseteq C(u)$ or type-2 component $S \subseteq C(u, v)$ (for some $v$ such that $F^*(\{u, v\}) = u$), we let $r_{S,u}$ be the smallest ID vertex in the set $S \cap N(u)$. Intuitively, $r_{S,u}$ will be the one responsible of communicating with $u$ about information associated with $S$. \qed

13
A Generic Approach for Learning a Parameter in $I_1(u)$ and $I_2(u)$. We present a generic approach that lets a vertex $u \in V_H$ learn one parameter in $I_1(u)$ and $I_2(u)$. The cost of learning one parameter is $\tilde{O}(1)$ time and energy for all vertices that are involved. For example, if we apply this approach to let $u$ learn $R[u,v]$, then the only vertices that are involved are $u$ itself and $r_{S,u}$ for all $S \in C(u,v)$. We only describe how to let each $u \in V_H$ learn $A_1[u]$ and $A_2[u]$; the rest are similar.

Learning $A_1[u]$. Recall that $A_1[u]$ is a component $S' \in C(u)$ that maximizes eccentricity($u, S'$). To learn $A_1[u]$, we use SR-comm$^{\text{max}}$ with $S = \{r_{S,u} \mid S \in C(u)\}$ and $R = \{u\}$. The message of the vertex $v = r_{S,u}$ is $m_v = \text{topology of } G[S]$, and the key is $k_v = \text{eccentricity}(u, S)$. Since each type-1 and type-2 component satisfies $|S| \leq \sqrt{n}$, the maximum possible value of eccentricity($u, S$) is $\sqrt{n}$, and so the size of the key space for SR-comm$^{\text{max}}$ is $K = \sqrt{n}$.

If $|C(u)| > 0$, the message that $u$ receives from SR-comm$^{\text{max}}$ is the topology of $G[S']$, for a component $S' \in C(u)$ that attains the maximum value of eccentricity($u, S'$) among all components in $C(u)$, and so $u$ sets $A_1[u] = S'$. If $|C(u)| = 0$, the vertex $u$ receives nothing from SR-comm$^{\text{max}}$, and so $u$ sets $A_1[u] = \emptyset$. The cost of SR-comm$^{\text{max}}$ is $O(\log K \log \Delta \log n) = \tilde{O}(1)$.

Learning $A_2[u]$. The procedure for learning $A_2[u]$ is almost exactly the same as that for $A_1[u]$, with only one difference. Recall that $A_2[u]$ is a component $S' \in C(u) \setminus \{A_1[u]\}$ that maximizes eccentricity($u, S'$), and so we need to exclude the component $A_1[u]$ from participating. To do so, before we apply SR-comm$^{\text{max}}$, we use one round to let $u$ send ID($r_{A_1[u],u}$) to all vertices $\{r_{S,u} \mid S \in C(u)\}$. This allows each $r_{S,u}$ to know whether or not $S = A_1[u]$.

Lemma 20. Suppose that each $v \in V$ already knows $I_0(v)$. In $\tilde{O}(n^{1.5})$ time and $\tilde{O}(\sqrt{n})$ energy we can let each vertex $u \in V_H$ learn $I_1(u)$ and $I_2(u)$.

Proof. The total number of parameters needed to be learned in $I_1(u)$ and $I_2(u)$ over all $u \in V_H$ is at most $|E(G_H)| \cdot \tilde{O}(\sqrt{n}) = O(n)$. We fix any ordering of these parameters, and learn each of these parameters one by one using the above generic approach. The time and energy cost of learning one parameter is $\tilde{O}(1)$. Since there are $O(n)$ parameters to learn, the total time complexity is $O(n) \cdot \tilde{O}(1) = O(n)$. Each vertex is involved in learning at most $O(\sqrt{n})$ parameters, and so the total energy complexity is $O(\sqrt{n}) \cdot \tilde{O}(1) = O(\sqrt{n})$.

Combining Lemma 11, Lemma 15, Lemma 14 and Lemma 20 we conclude the proof of Theorem 6.

3.4 Lower Bound

In this section, we prove Theorem 7. The proof is based on a reduction from the set-disjointness problem of communication complexity, which is defined as follows. Consider two players $A$ and $B$, each of them holds a subset of $\{0, \ldots, n\}$, and their task is to decide whether their subsets are disjoint. If the maximum allowed failure probability is $f < 1/2$, then they need to communicate $\Omega(n)$ bits [8]. This is true even if the two players have access to an infinite amount of public random bits.

At a high level, our approach is similar to that of [1], which shows that computing diameter takes $\Omega(n/\log^2 n)$ time in the CONGEST model, or more generally $\Omega\left(\frac{n}{B \log n}\right)$ time in the message-passing model with $B$-bit message size constraint.

Note that a time lower bound in CONGEST does not in general transform to an energy lower bound in the wireless network model, if we make no message size constraint. The main challenge for proving Theorem 7 is that we allow messages of unbounded length.

Lower Bound Graph Construction. Let $S_A = \{a_1, \ldots, a_\ell\}$ and $S_B = \{b_1, \ldots, b_\ell\}$ be two subsets of $\{0, \ldots, k\}$ corresponding to an instance of set-disjointness problem. We assume that $k = 2^\ell$, for some positive integer $\ell$, and so each element $s \in S_A \cup S_B$ is represented as a binary string of length $\ell = \log k$. We write $\text{Ones}(s) \subseteq [\ell] = \{1, \ldots, \ell\}$ to denote the set of indices $i$ in $[\ell]$ such that $s[i] = 1$ (i.e., the $i$th bit of $s$ is 1); similarly, $\text{Zeros}(s) = [\ell] \setminus \text{Ones}(s)$ is the set of indices $i$ in $[\ell]$ such that $s[i] = 0$. For example, if the binary representation of $s$ is $10110010$ ($\ell = 8$), then $\text{Ones}(s) = \{1, 3, 4, 7\}$ and $\text{Zeros}(s) = \{2, 5, 6, 8\}$.

Define the graph $G = (V, E)$ as follows. The graph $G$ has at most $n = 2(k+1) + 2\log k + 2$ vertices, and the arboricity of $G$ is $O(\log k) = O(\log n)$.
Each vertex following assumptions about algorithms in this modified model. Next, we discuss the consequences of these extra powers. In particular, we show that we can make the diameter of $G$ is 2; otherwise (a no-instance for the set-disjointness problem) the diameter of $G$ is 3. This can be seen as follows. First, of all, observe that we must have $\text{dist}(s, t) \leq 2$, unless $s \in V_A$ and $t \in V_B$ (or $s \in V_B$ and $t \in V_A$). Now suppose $s = u_i \in V_A$ and $t = v_j \in V_B$.

- Consider the case $a_i \neq b_j$. We show that $\text{dist}(s, t) = 2$. Note that there is an index $l \in [\ell]$ such that $a_i$ and $b_j$ differ at the $l$th bit. If the $l$th bit of $a_i$ is 0 and the $l$th bit of $b_j$ is 1, then $(u_i, x_l, v_j)$ is a length-2 path between $s$ and $t$. If the $l$th bit of $a_i$ is 1 and the $l$th bit of $b_j$ is 0, then $(u_i, w_l, v_j)$ is a length-2 path between $s$ and $t$.

- Consider the case $a_i = b_j$. We show that $\text{dist}(s, t) = 3$. Note that there is no index $l \in [\ell]$ such that $a_i$ and $b_j$ differ at the $l$th bit. Thus, each $w_l \in V_C$ and $x_l \in V_D$ is adjacent to exactly one of $\{u_i, v_j\}$.

Hence there is no length-2 path between $s$ and $t$.

Therefore, if $S_A \cap S_B = \emptyset$, then $\text{dist}(s, t) = 2$ for all pairs $\{s, t\}$, and so the diameter is 2; otherwise, there exist $s = u_i \in V_A$ and $t = v_j \in V_B$ such that $\text{dist}(s, t) = 3$, and so the diameter is 3.

**Reduction.** Suppose that there is a randomized distributed algorithm $A$ that is able to compute the diameter with $O(n/\log^2 n)$ energy in CD or No-CD models, with failure probability $f = 1/poly(n)$. We show that the algorithm $A$ can be transformed into a randomized communication protocol that solves the set-disjointness problem with $o(n)$ bits of communication, and with the same failure probability $f = 1/poly(n)$.

The main challenge in the reduction is that we do not impose any message size constraint. To deal with this issue, our strategy is to consider a modified computation model $M'$. We will endow the vertices in the modified computation model $M'$ with strictly more capabilities than the original wireless network model. Then, we argue that in the setting of $M'$, we can assume that each message has size $O(\log k)$.

**Modified Computation Model $M'$.** We add the following extra powers to the vertices:

**P1** All random bits used in all vertices are known to everyone at the beginning of the algorithm. That is, they have shared randomness, and they know the list of the IDs of all vertices. We assume that $\text{ID}(w_i) = i$ for each $w_i \in V_C$; $\text{ID}(x_i) = \ell + i$ for each $x_i \in V_D$; $\text{ID}(u^*) = 2\ell + 1$; $\text{ID}(v^*) = 2\ell + 2$. Thus, for each $v \in V_C \cup V_D \cup \{u^*, v^*\}$, its role can be inferred from $\text{ID}(v)$.

**P2** All messages successfully received by vertices in $V_C \cup V_D \cup \{u^*, v^*\}$ are also delivered to all vertices. For example, if $v \in V_C \cup V_D \cup \{u^*, v^*\}$ receives a message $m$ from a vertex $u \in V$ at time $t$, then by the end of time $t$ all vertices in $V$ know that “$v$ receives $m$ from $u$ at time $t$”.

**P3** Each vertex $v \in V_A \cup V_B$ knows the list of the IDs of its neighbors initially.

Next, we discuss the consequences of these extra powers. In particular, we show that we can make the following assumptions about algorithms in this modified model $M'$. 

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15
Vertices in $V_U \cup V_D \cup \{u^*, v^*\}$ never transmit. The above (P1) and (P2) together imply that each vertex in the graph is able to locally simulate the actions of all vertices in $V_U \cup V_D \cup \{u^*, v^*\}$ in each round. Intuitively, this means that all vertices in $V_U \cup V_D \cup \{u^*, v^*\}$ do not need to transmit at all throughout the algorithm.

Note that each vertex $v \in V$ already knows the list of $N(v) \cap (V_U \cup V_D \cup \{u^*, v^*\})$. If $v \in V_A \cup V_B$, then $v$ knows this information via (P3). If $v \in V_U \cup V_D \cup \{u^*, v^*\}$, then $v$ knows this information via (P1); the role of each vertex in $V_U \cup V_D \cup \{u^*, v^*\}$ can be inferred from its ID, which is a public to everyone.

Thus, right before the beginning of each time $t$, each vertex $v \in V$ already knows exactly which vertices in $N(v) \cap (V_U \cup V_D \cup \{u^*, v^*\})$ will transmit at time $t$ and their messages. Thus, in the modified model $\mathcal{M}'$, we can simulate the execution of an algorithm which allows the vertices in $V_U \cup V_D \cup \{u^*, v^*\}$ to transmit by another algorithm that forbid them to do so.

Messages Sent by Vertices in $V_A \cup V_B$ Have Length $O(\log k)$. Next, we argue that we can assume that each message $m$ sent by a vertex $v' \in V_A \cup V_B$ can be replaced by another message $m'$ which contains only the list of all neighbors of $v'$, and this can be encoded as an $O(\log k)$-bit message, as follows. Recall that $N(v')$ is a subset of $V_U \cup V_D \cup \{u^*, v^*\}$, and so we can encode $N(v')$ as a binary string of length $|V_U \cup V_D \cup \{u^*, v^*\}| = 2\ell + 2 = O(\log k)$.

The message $m$ is a function of all information that $v'$ has. Since no vertex in $V_U \cup V_D \cup \{u^*, v^*\}$ transmits any message, $v'$ never receive a message, and so the information that $v'$ has consists of only the following components.

- The shared randomness and the ID list of all vertices (due to (P1)).
- The history of vertices in $V_U \cup V_D \cup \{u^*, v^*\}$ (due to (P2)).
- The list of neighbors of $v'$ (due to (P3)).

The only private information that $v'$ has is its list of neighbors. If a vertex $u' \in V$ knows the list of neighbors of $v'$, then $u'$ is able to calculate $m$ locally, and so $v'$ can just send its list of neighbors, and not send $m$.

Algorithm $\mathcal{A}'$. To sum up, given the algorithm $\mathcal{A}$ (in a wireless network model, CD or No-CD), we can transform it into another algorithm $\mathcal{A}'$ in the modified computation model $\mathcal{M}'$ that uses only $O(\log k)$-bit messages, and $\mathcal{A}'$ achieves what $\mathcal{A}$ does. Note that the energy cost of $\mathcal{A}'$ is at most the energy cost $\mathcal{A}$.

Solving Set-Disjointness. Now we show how to transform $\mathcal{A}'$ into a protocol for the set-disjointness problem using only $o(k)$ bits of communication. The protocol for the set-disjointness problem is simply a simulation of $\mathcal{A}'$. The shared random bits used in $\mathcal{A}'$ are based on the public random bits available to the two players $A$ and $B$.

Each player $X \in \{A, B\}$ is responsible for simulating vertices in $V_X$. In view of above, vertices in $V_A$ and $V_B$ never receive messages, and so all we need to do is to let both players $A$ and $B$ know the messages sent to $V_U \cup V_D \cup \{u^*, v^*\}$ (in view of (P2)).

We show how to simulate one round $\tau$ of $\mathcal{A}'$. We write $Z(\tau)$ to denote the subset of vertices in $V_U \cup V_D \cup \{u^*, v^*\}$ that listens at time $\tau$. Recall that everyone can predict the action of every vertex in $V_U \cup V_D \cup \{u^*, v^*\}$. Consider a vertex $u' \in Z(\tau)$ that listens at time $\tau$. Let $Q_A$ be the number of vertices in $N(u') \cap V_A$ transmitting at time $\tau$. We define $m_{u', \tau, A}$ as follows.

$$m_{u', \tau, A} = \begin{cases} 
0 & \text{if } Q_A = 0, \\
\log(Z(\tau)) & \text{if } Q_A \geq 2, \\
(v', m') & \text{if } Q_A = 1, \text{ and } v' \text{ is the vertex in } N(u') \cap V_A \text{ sending } m' \text{ at time } \tau.
\end{cases}$$

We define $m_{u', \tau, B}$ analogously. Note that the size of $m'$ must be $O(\log k)$.

The protocol for simulating round $\tau$ is simply that $A$ sends $m_{u', \tau, A}$ (for each $u' \in Z(\tau)$) to $B$, and $B$ sends $m_{u', \tau, B}$ (for each $u' \in Z(\tau)$) to $A$. This offers enough information for both player to know the channel feedback (noise, silence, or a message $m$) received by each vertex in $Z(\tau)$. Note that the number of bits exchanged by $A$ and $B$ due to the simulation of round $\tau$ is $O(|Z(\tau)| \log k)$. 16
Recall that the energy cost of each vertex in an execution of $\mathcal{A}'$ is $o(k/\log^2 k)$, and we have $|V_C \cup V_D \cup \{u^*, v^*\}| = O(\log k)$. Thus, the total number of bits exchanged by the two players $A$ and $B$ is
\[
\sum_{\tau} O(|\mathcal{Z}(\tau)| \log^2 k) = |V_C \cup V_D \cup \{u^*, v^*\}| \cdot o(k/\log^2 k) \cdot O(\log k) = o(k).
\]

4 Minimum Cut

The generic framework introduced in Section 3 can be applied to other graph problems as well. In this section, we show how to find (i) exact global minimum cut and (ii) approximate $s$-$t$ minimum cut of a bounded-genus graph in $\tilde{O}(\sqrt{n})$ energy. These results are complemented with two simple hardness proofs: It takes $\Omega(n)$ energy for any algorithm to find an exact solution of either (i) an $s$-$t$ minimum cut (for planar graphs), or (ii) a global minimum cut (for unit disc graphs).

4.1 Upper Bounds

In this section we prove the upper bounds. Both upper bounds follow the same framework introduced in Section 3 via applications of $\text{SR-comm}$ and its variants.

For a cut $C = (X, V \setminus X)$, the two vertex sets $X \neq \emptyset$ and $V \setminus X \neq \emptyset$ are called the two parts of $C$; the cut edges of $C$ are defined as $\{(u, v) | u \in X, v \in V \setminus X\}$. The value of $C$, which we denote as $|C|$, is the number of cut edges. A minimum cut of a graph is a cut $C$ that minimizes $|C|$ over all possible cuts. An $s$-$t$ minimum cut of a graph is a cut $C$ that minimizes $|C|$ over all possible cuts subject to the constraint that $s$ and $t$ belong to different parts. We define the following notations.

- $c(S)$: For a type-1 component $S$, let $c(S)$ be the minimum cut value of $G[S]$.
- $c'(S)$: For a type-2 component $S \in C(u, v)$, let $c'(S)$ be the $u$-$v$ minimum cut value of $G[S]$.
- $c''(S)$: For a type-2 component $S \subseteq C(u, v)$, let $c''(S)$ be the minimum cut value of $G[S]$ among all cuts such that both $u$ and $v$ are within one part.

**Lemma 21.** Let $C = (X, V \setminus X)$ be any minimum cut of $G$. Then the following is true.

- For a vertex $u \in V_H$, either (i) one part of the cut contains all vertices in $\bigcup_{S \subseteq C(u)} S \cup \{u\}$, or (ii) the value of the cut is $\min_{S \subseteq C(u)} c(S)$.
- For two distinct vertices $u, v \in V_H$, either (i) one part of the cut contains all vertices in $\bigcup_{S \subseteq C(u, v)} S \cup \{u, v\}$, or (ii) the value of the cut is $\min_{S \subseteq C(u, v)} c''(S)$, or (iii) $u$ and $v$ are within different parts of the cut, and the number of cut edges that have at least one endpoint in $\bigcup_{S \subseteq C(u, v)} S'$ is $\sum_{S \subseteq C(u, v)} c'(S)$.

**Proof.** Consider the first statement. Suppose that (i) is not met. Then there exists a component $S \subseteq C(u)$ such that $S \cup \{u\}$ is not entirely within one part of the cut. Then $C' = (X \cap (S \cup \{u\}), (V \setminus X) \cap (S \cup \{u\}))$ is a cut of $G[S]$, and we must have $c(S) \leq |C'| \leq |C|$. Since $C$ is a minimum cut of $G$, we have $\min_{S \subseteq C[u]} c(S) = c(S) = |C|$.

Consider the second statement. Suppose that (i) is not met. We first consider the case where $u$ and $v$ are in one part of the cut. Then there exists a component $S \subseteq C(u, v)$ such that $S \cup \{u, v\}$ is not entirely contained in one part of the cut. Then $C' = (X \cap (S \cup \{u, v\}), (V \setminus X) \cap (S \cup \{u, v\}))$ is a cut of $G[S]$ where $u$ and $v$ are within one part. We must have $c''(S) \leq |C'| \leq |C|$. Since $C$ is a minimum cut of $G$, we have $\min_{S \subseteq C(u, v)} c''(S) = c''(S) = |C|$.

Next, consider the case where $u$ and $v$ are in different parts of the cut. For each $S \subseteq C(u, v)$, write $Z_S$ to denote the number of cut edges of $C$ that have at least one endpoint in $S$. Then we must have
we remove all type-1 components. Then, for each pair \( \{u, v\} \) of distinct vertices in \( V_H \) with \( |C(u, v)| > 0 \), replace \( C(u, v) \) by an edge with weight \( \min \{k^*, \sum_{S \in C(u, v)} c'(S)\} \). In view of Lemma 21 the minimum cut value of \( G \) is the minimum over the following numbers:

- The minimum over all \( \min_{S \in C(u)} c(S) \) (for all \( u \in V_H \) such that \( |C(u)| > 0 \)).
- The minimum over all \( \min_{S \in C(u, v)} c''(S) \) (for all \( u, v \in V_H \) such that \( |C(u, v)| > 0 \)).
- The minimum cut value of \( G^* \).

For each \( u \in V_H \), let \( I_1(u) \) contain the parameter \( \min_{S \in C(u)} c(S) \), and let \( I_2(u) \) contain the two parameters \( \min_{S \in C(u, v)} c'(S) \) and \( \min_{S \in C(u, v)} c''(S) \) for each pair \( \{u, v\} \in E(G_H) \) with \( F^*\{u, v\} = u \). The definition of \( I_0(u) \) is the same as that in Section 3.

We use the algorithms for Lemma 13 and Lemma 20 to let each vertex \( u \in V \) learn the information \( I_0(u) \), \( I_1(u) \), and \( I_2(u) \). Note that the exact value of the parameter \( \min \{k^*, \sum_{S \in C(u, v)} c'(S)\} \) can be learned by applying the generic approach of learning parameters described in Section 3 using \( \text{SR-comm} \) with \( \epsilon = 1/(k^* + 1) \) and \( W = k^* \). More specifically, during \( \text{SR-comm} \), we let the message of the representative \( S \) be \( \min \{k^*, c'(S)\} \).

The minimum cut value of \( G \) can be calculated from the following information: (i) \( I_1(u) \) and \( I_2(u) \) for each \( u \in V_H \), (ii) the topology of \( G[S] \) for each type-3 component \( S \), and (iii) the topology of the subgraph induced by \( V_H \). Using the algorithm of Lemma 13 we can let everyone learn this information in \( \tilde{O}(n^{1.5}) \) time and \( \tilde{O}(\sqrt{n}) \) energy.

**Theorem 23.** There is a randomized algorithm for computing an \((1 \pm \epsilon)\)-approximate \( s-t \) minimum cut value in \( \tilde{O}(n^{1.5}) \cdot \text{poly}(1/\epsilon) \) time and \( \tilde{O}(\sqrt{n}) \cdot \text{poly}(1/\epsilon) \) energy for bounded-genus graphs.

**Proof.** The proof is similar to that of Lemma 22. The main differences are the following. If \( s \) or \( t \) happens to be within a type-1 or a type-2 component \( S \), then we additionally need to learn the topology of \( G[S] \). Any type-1 component that does not contain \( s \) or \( t \) is irrelevant to the \( s-t \) minimum cut value.

For each \( x \in \{s, t\} \), let \( S_x \) be the type-1 or type-2 component containing \( x \): if \( x \) is not contained in any type-1 or type-2 component, then let \( S_x = \emptyset \). Define \( G^* \) as the result of the following operations. Remove all type-1 components except \( S_s \) and \( S_t \). For each pair \( \{u, v\} \) of distinct vertices in \( V_H \) with \( |C(u, v) \setminus \{S_s, S_t\}| > 0 \), replace all components in \( C(u, v) \setminus \{S_s, S_t\} \) by an edge with weight \( \sum_{S \in C(u, v) \setminus \{S_s, S_t\}} c'(S) \).

It is straightforward to see that the minimum \( s-t \) cut value in \( G^* \) is the same as the minimum \( s-t \) cut value of \( G \) (via a proof similar to that of Lemma 21 detail is omitted). If the edge weights of \( G^* \) are off by a factor of at most \( 1 \pm \epsilon \), then the minimum \( s-t \) cut value in \( G^* \) is an \((1 \pm \epsilon)\)-approximation of the minimum \( s-t \) cut value of \( G \).

In view of the above, for each \( u \in V_H \), let \( I_1(u) = 0 \), and let \( I_2(u) \) contain the parameter \( \sum_{S \in C(u, v)} c'(S) \) for each pair \( \{u, v\} \in E(G_H) \) with \( F^*(\{u, v\}) = u \). To calculate the minimum \( s-t \) cut value of \( G \), it suffices to learn the parameters in \( I_2(u) \) within an approximation factor of at most \( 1 \pm \epsilon \), and this can be done by applying the generic approach of learning parameters described in Section 3 with \( \text{SR-comm} \). The rest of the proof is the same as that of Lemma 22.

**4.2 Lower Bounds**

We first show that the randomized energy complexity of computing \( s-t \) minimum cut is \( \Omega(n) \) in both \( \text{CD} \) and \( \text{No-CD} \), even for planar graphs. The lower bound graph is a complete bipartite graph \( K_{2, \Delta} \), with the vertex partition \( \{s, t\} \) and \( \{v_1, \ldots, v_\Delta\} \). It is clear that the exact \( s-t \) minimum cut is \( \Delta \). We show that it takes \( \Omega(n) \) energy for the graph to learn the exact value of \( \Delta \).
Theorem 24. The randomized energy complexity of computing exact s–t minimum cut value is $\Omega(n)$ in both CD and No-CD. This is true even for complete bipartite graphs $K_{2,\Delta}$.

Proof. Let $G$ be a complete bipartite graph with the vertex partition $\{s,t\}$ and $\{v_1,\ldots,v_\Delta\}$. Let $G'$ be $G - v_\Delta$. It is clear that the $s$–$t$ minimum cut of $G$ is $\Delta$, and the $s$–$t$ minimum cut of $G'$ is $\Delta - 1$. In such graphs, the value of $s$–$t$ minimum cut is identical to $\text{deg}(s)$.

Let $E = \Delta/5$. Suppose that there is a randomized algorithm $A$ that is able to let the vertex $s$ compute the $s$–$t$ minimum cut with (at most) $E$ energy, then $A$ is also able to let $s$ distinguish between $G$ and $G'$ with $E$ energy.

Consider an execution of $A$ on $G$. Let $S$ be the subset of $\{v_1,\ldots,v_\Delta\}$ such that $v_i \in S$ if there is a time slot $\tau$ where (i) $v_i$ transmits, (ii) the number of vertices in $\{v_1,\ldots,v_\Delta\}$ that transmit is at most 2, and (iii) at least one of $s$ and $t$ listens.

We claim that $|S| \leq 4E = 4\Delta/5$. Let $T$ be the set of all time slots $\tau$ such that the above (i), (ii), and (iii) hold for at least one $v_i \in \{v_1,\ldots,v_\Delta\}$. Then we must have $|T| \geq |S|/2$ (in view of (ii)). Note that if $\tau \in T$, then at least one of $s$ and $t$ must listen at time $\tau$. Thus, the energy cost of one of $s$ and $t$ must be at least $|T|/2 \geq |S|/4$, and so we have the inequality $E \geq |S|/4$.

Let $E$ be the event that $v_\Delta \notin S$ if we execute $A$ on $G$. Note that whether or not $E$ occurs depends only on the random bits associated with the vertices $\{s,t\}$ and $\{v_1,\ldots,v_\Delta\}$. Since $|S| \leq 4\Delta/5$, we have $\Pr[E] \geq (\Delta/5)/\Delta = 1/5$.

Suppose that all vertices in $\{s,t\}$ and $\{v_1,\ldots,v_\Delta\}$ have decided their random bits in advance; and then with probability 1/2 we run $A$ on $G$; with probability 1/2 we run $A$ on $G'$. Note that if $E$ occurs, then the execution of $A$ on both $G$ and $G'$ gives identical results for all vertices (other than $v_\Delta$). Therefore, given that the event $E$ occurs, the probability that the vertex $s$ correctly decide whether the underlying graph is $G$ or $G'$ is at most 1/2 (i.e., $s$ can only guess randomly).

Since $\Pr[E] \geq 1/5$, the probability that the vertex $s$ fails to correctly decide whether the underlying graph is $G$ or $G'$ is at least $(1/2) \cdot (1/5) = 1/10$, and so $s$ fails to correctly calculate the $s$–$t$ minimum cut with probability at least 1/10. This contradicts the assumption that $A$ is able to compute the $s$–$t$ minimum cut with high probability.

For graphs of maximum degree $\Delta$, the above proof gives us an $\Omega(\Delta)$ energy lower bound. It is important that we consider bipartite graphs. If the middle $\Delta$ vertices form a clique, then the lower bound fails, as there exist an energy efficient algorithm for every vertices in a clique to broadcast a message. For example, consider an $n$-vertex single-hop network, where each vertex has a unique ID within $[N]$; we assume $N$ is known to everyone, but $n \leq N$ is unknown. There is a simple folklore $O(\log N)$-energy deterministic algorithm [3 Section 4.2] that allows each vertex to broadcast a message to everyone.

Next, we consider the task of finding an exact global minimum cut. The lower bound graphs in the proof are $K_n$ and $K_n - e$. Note that these graphs are unit disc graphs.

Theorem 25. The randomized energy complexity of computing an exact global minimum cut value is $\Omega(n)$ in both CD and No-CD. This is true even if the underlying graph is either (i) an $n$-vertex complete graph $K_n$, or (ii) an $n$-vertex complete graph minus one edge $K_n - e$.

Proof. Throughout the proof, we consider the scenario where the underlying graph is $K_n$ with probability 1/2, and is $K_n - e$ with probability 1/2. The edge $e$ is chosen uniformly at random. Let $A$ be any randomized algorithm that solves the minimum cut problem using at most $E = (n-1)/8$ energy. Note that the minimum cut of $K_n$ is $n-1$; and the minimum cut of $K_n - e$ is $n-2$, and so $A$ is able to distinguish between $K_n$ and $K_n - e$. We make the following assumptions which only increase the capability of the vertices.

- Each vertex has a distinct ID from $[n]$.
- All vertices have access to shared random bits.
- By the end of each time slot $t$, each vertex knows the following information: (i) the IDs of the vertices transmitting at time $t$, (ii) the IDs of the vertices listening at time $t$, and (iii) the channel feedback (i.e., noise, silence, or a message $m$) for each listening vertex.
With the above extra capabilities, all vertices share the same history. Since the actions of the vertices at time $t + 1$ depend only on the shared history of all vertices and their shared random bits, by the end of time $t$ all vertices are able to predict the actions (i.e., transmit a message $m$, listen, or idle) of all vertices at time $t + 1$.

We say that time $t$ is good for a pair $\{u, v\}$ if the following conditions are met. Intuitively, if $t$ is not good for $\{u, v\}$, then what happens at time $t$ does not reveal any information as to whether $\{u, v\}$ is an edge.

- The number of transmitting vertices at time $t$ is either 1 or 2,
- One of the two vertices $\{u, v\}$ listens at time $t$, and the other one transmits at time $t$.

Given the shared random bits for all vertices, define $X_{\text{bad}}$ as the set of pairs $\{u, v\}$ such that there is no time $t$ that is good for $\{u, v\}$ in an execution of $A$ on $K_n$; and define $X_{\text{good}}$ as the set of all remaining pairs. Note that $X_{\text{bad}}$ and $X_{\text{good}}$ are determined solely by the shared random bits.

We claim that $\Pr[\{u, v\} \in X_{\text{bad}}] \geq 1/2$, for each pair $\{u, v\}$. Recall that if a time $t$ is good for some pair, then the number of transmitting vertices is at most 2. Thus, if $t$ is good for $x$ pairs, then at least $x/2$ vertices listen at time $t$, and so the summation of energy spent in all vertices in all time is at least $|X_{\text{good}}|/2$, and so $nE = n(n - 1)/8 \geq |X_{\text{good}}|/2$, which implies $|X_{\text{bad}}| \geq n(n - 1)/4$. Hence $\Pr[\{u, v\} \in X_{\text{bad}}] \geq 1/2$.

Let $\mathcal{E}$ be the event $e \in X_{\text{bad}}$. Note that $\mathcal{E}$ depends only on (i) the random bits for choosing $e$, and (ii) the shared random bits for all vertices. Given that $\mathcal{E}$ occurs, the execution of $A$ is identical on both $K_n$ and $K_n - e$, and so the success probability of $A$ is at most $1/2$. Thus, $A$ fails with probability at least $(1/2) \Pr[\mathcal{E}] \geq 1/4$. This contradicts the assumption that $A$ is able to compute the minimum cut with high probability.

5 Conclusion

We have shown that for a variety of graph problems, the energy complexity $\tilde{O}(\sqrt{n})$ can be achieved. A good future work topic is to further investigate the landscape of energy complexity of fundamental graph problems. In particular, can we prove a lower bound for BFS that is higher than the known lower bounds for broadcasting? Another potential future work direction is to study graph classes that are more realistic, such as unit disc graphs. It is worth noted that several known energy lower bounds techniques rely on the bipartite graph structure, such as Theorem 7, Theorem 24, and some lower bounds for broadcasting in [7]; all these lower bounds do not apply to unit disc graphs.

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A Algorithms for SR-communication

In this section we present our algorithms for SR-comm and its variants.

Lemma 26. SR-comm can be solved in time $O(\log \Delta \log n)$ and energy $O(\log \Delta \log n)$.

Proof. This is a well-known result from [3]. Note that each $v \in S \cap R$ is not required to receive any message from other vertices, since we already have $v \in N^+(v) \cap S$. Thus, in what follows, we assume that $S \cap R = \emptyset$.

The algorithm is as follows. Repeat the following routine for $\Theta(\log n)$ times. For $i = 1$ to $\log \Delta$, let each vertex $u \in S$ sends with probability $2^{-i}$. Each $v \in R$ is always listening throughout the procedure.

Consider a vertex $v \in R$ such that $N(v) \cap S \neq \emptyset$. Let $i^*$ be the largest integer $i$ such that $2^i \leq 2|N(v) \cap S|$. Consider a time slot $t$ where each vertex $u \in S$ sends with probability $2^{-i^*}$. For notational simplicity, we write $n' = |N(v) \cap S|$ and $p' = 2^{-i^*}$. Note that $1/n' \geq p' \geq 1/(2n')$. The probability that exactly one vertex in the set $N(v) \cap S$ sends is $n'(1 - p')^{n' - 1} \geq 1/(2e)$. If this occurs, then $v$ successfully receives a message $m_u$ from a vertex $u \in N(v) \cap S$. The probability that $v$ does not receive any message from vertices in $N(v) \cap S$ is at most $(1 - 1/(2e))^{\Theta(\log n)} = 1/poly(n)$.

This algorithm is known as decay [3], and it has been used as a basic communication building block in various algorithms in wireless networks; see also [7]. \qed
Lemma 27. SR-comm$^\text{all}$ can be solved in time $O(\Delta' \log n)$ and energy $O(\Delta' \log n)$.

Proof. Recall that $\Delta'$ is an upper bound on $|S \cap N(v)|$, for each $v \in \mathcal{R}$. The goal of SR-comm$^\text{all}$ is to let each vertex in $\mathcal{S} \cap N(v)$ deliver a message to $v \in \mathcal{R}$, for each $v \in \mathcal{R}$.

Consider the algorithm which repeats the following routine for $\Theta(\Delta' \log n)$ rounds. In each round, each vertex $u \in \mathcal{S}$ sends $m_u$ with probability $1/\Delta'$; for each $u \in \mathcal{R}$, if $u$ does not send in this round, then $u$ listens.

Let $e = \{u, v\}$ be any edge with $u \in \mathcal{S}$ and $v \in \mathcal{R}$. In one round of the above algorithm, $u$ successfully sends a message to $v$ if (i) all vertices in $\{v\} \cup (\mathcal{S} \cap N(v)) \setminus \{u\}$ do not send, and (ii) $u$ sends. Thus occurs with probability at least $(1 - 1/\Delta')|\mathcal{S} \cap N(v)| \cdot (1/\Delta') \geq (1 - 1/\Delta')^2 \cdot (1/\Delta') \geq 1/(2e\Delta')$.

Therefore, the probability that $u$ does not successfully send a message to $v$ throughout the $\Theta(\Delta' \log n)$ rounds is at most $(1 - 1/(2e\Delta'))^{\Omega(\Delta' \log n)} = 1/poly(n)$. \hfill \square

Lemma 28. SR-comm$^\text{multi}$ can be solved in time $O(M \log \Delta \log^2 n)$ and energy $O(M \log \Delta \log^2 n)$.

Proof. We repeat the following procedure for $\Theta(M \log n)$ times. Do SR-comm with the sets $(\mathcal{S}', \mathcal{R}')$ chosen as follows. Let $\mathcal{R}'$ be a random subset of $\mathcal{R}$ such that each $v \in \mathcal{R}$ joins $\mathcal{R}'$ with probability $1/2$. Let $\mathcal{S}'$ be a random subset of $\mathcal{S} \setminus \mathcal{R}'$ defined as follows. For each message $m$, all vertices in $\mathcal{S} \setminus \mathcal{R}'$ that hold $m$ join $\mathcal{S}'$ with probability $1/M$ (using the shared randomness for the message $m$).

Before we proceed to the analysis, we clarify some possible ambiguities of the above algorithm. Note that a vertex $u \in \mathcal{S} \setminus \mathcal{R}'$ might have $k \geq 1$ distinct messages, and in this case, $u$ joins $\mathcal{S}'$ with probability $Pr[Binomial(k, 1/M) \geq 1]$. That is, if the shared randomness associated with at least one message $m \in \mathcal{M}_u$ lets $u$ to join $\mathcal{S}'$, then $u$ joins $\mathcal{S}'$. Due to the shared randomness, if a vertex $u$ joins $\mathcal{S}'$ because of the shared randomness for message $m$, then all vertices $w \in \mathcal{S} \setminus \mathcal{R}'$ holding the message $m$ also join $\mathcal{S}'$.

In what follows, we analyze the algorithm. Consider a vertex $v \in \mathcal{R}$, and consider one iteration of the above algorithm. Let $\mathcal{M}$ be the set of distinct messages in $\bigcup_{u \in \mathcal{N}(v) \cap \mathcal{S}} \mathcal{M}_u \setminus \mathcal{M}_v$. Consider a message $m \in \mathcal{M}$. Observe that $v$ receives $m$ if the following two events $\mathcal{E}_1$ and $\mathcal{E}_2$ occur: $\mathcal{E}_1$ is the event that all vertices $w \in \mathcal{N}(v) \cap \mathcal{S}'$ hold the message $m$, and $\mathcal{E}_2$ is the event that $v \in \mathcal{R}'$.

The probability that $\mathcal{E}_1$ occurs is at least $(1/2) \cdot (1/M) \cdot (1 - 1/M)^{M - 1}$. The probability that $\mathcal{E}_2$ occurs is $1/2$. Thus, the probability that $v$ receives $m$ in one iteration is at least $(1/4) \cdot (1/M) \cdot (1 - 1/M)^{M - 1} \geq 1/(4eM)$, and so the probability that $v$ does not receive $m$ in all iterations is at most $(1 - 1/(4eM))^{\Omega(M \log n)} = 1/poly(n)$. \hfill \square

Lemma 29. Each of SR-comm$^\text{min}$ and SR-comm$^\text{max}$ can be solved in time $O(K \log \Delta \log n)$ and energy $O(\log K \log \Delta \log n)$. For the special case of $\mathcal{S} \cap \mathcal{R} = \emptyset$ and $|\mathcal{R} \cap \mathcal{N}(v)| \leq 1$ for each $v \in \mathcal{S}$, the runtime is reduced to $O(\log K \log \Delta \log n)$.

Proof. We only consider SR-comm$^\text{min}$, as the proof for SR-comm$^\text{max}$ is the same. The proof is analogous to the analysis of a deterministic version of SR-comm in [2]. Note that we can do SR-comm once to let each $v \in \mathcal{R}$ test whether or not $N^+(v) \cap \mathcal{S} \neq \emptyset$. If a vertex $v \in \mathcal{R}$ knows that $N^+(v) \cap \mathcal{S} = \emptyset$, then $v$ can simply removes itself from $\mathcal{R}$. Thus, in subsequent discussion, we assume $N^+(v) \cap \mathcal{S} \neq \emptyset$ for each $v \in \mathcal{R}$.

Let $v \in \mathcal{R}$, and we define $f_v = \min_{u \in N^+(v) \cap \mathcal{S}} k_u$. The high level idea of our algorithm is to conduct a binary search to determine all log $K$ bits of binary representation of $f_v$.

General Case. Suppose at some moment each vertex $v \in \mathcal{R}$ already knows the first $x$ bits of $f_v$. The following procedure allows each $v \in \mathcal{R}$ to learn the $(x + 1)$th bit of $f_v$. For each $(x + 1)$-bit binary string $s$, we do SR-comm with $(\mathcal{S}', \mathcal{R}')$, where $\mathcal{S}'$ is the set of vertices $u \in \mathcal{S}$ such that $k_u = s$, and $\mathcal{R}'$ is the set of vertices $v \in \mathcal{R}$ such that the first $x$ bits of $f_v$ equals the first $x$ bits of $s$.

Despite the fact that we perform $2^{x+1}$ times of SR-comm, each vertex only participates in at most 3 of them (join $\mathcal{S}'$ at most once, join $\mathcal{R}'$ at most twice). Thus, the procedure takes $O(2^x \log \Delta \log n)$ time and $O(\log \Delta \log n)$ energy.

If we use the above approach to solve SR-comm$^\text{min}$, the time complexity is $\sum_{x=0}^{\log K - 1} O(2^x \log \Delta \log n) = O(K \log \Delta \log n)$; and the energy cost is $\sum_{x=0}^{\log K - 1} O(\log \Delta \log n) = O(\log K \log \Delta \log n)$.

Note that for the last SR-comm that a vertex $v \in \mathcal{R}$ participates, the message $m_u$ that successfully sent to $v$ in that SR-comm must comes from a vertex $u \in N^+(v) \cap \mathcal{S}$ such that $k_u = f_v = \min_{w \in N^+(v) \cap \mathcal{S}} k_w$. 23
**Special Case.** Consider the special case of \( S \cap R = \emptyset \) and \(|R \cap N(u)| \leq 1\) for each \( u \in S \). For each \( v \in R \), we simply let the set \((S \cap N(v)) \cup \{v\}\) jointly conduct a binary search to determine all bits of \( f_v = \min_{u \in N(v) \cap S} k_u \). Note that the family of sets \((S \cap N(v)) \cup \{v\}\) for all \( v \in R \) are disjoint.

Suppose at some moment, for each vertex \( v \in R \), all vertices in the set \((S \cap N(v)) \cup \{v\}\) already know the first \( x \) bits of \( f_v \). We present an algorithm that allows all vertices in the set \((S \cap N(v)) \cup \{v\}\) to learn the \((x+1)\)th bit of \( f_v \).

**Step 1.** Do \( SR\text{-comm} \) with \((S', R')\), where \( R' = R \), and \( S' \) is the subset of \( S \) that contains vertices \( u \in S \) such that (i) first \( x \) bits of \( k_u \) equals the first \( x \) bits of \( f_v \), where \( v \) is the unique vertex in \( R \cap N(u) \), and (ii) the \((x+1)\)th bit of \( k_u \) is 0. This step allows each \( v \in R \) to learn the \((x+1)\)th bit of \( f_v \).

**Step 2.** Do \( SR\text{-comm} \) with \((S', R')\), where \( R' = S \) and \( S' = R \). The purpose of this \( SR\text{-comm} \) is to let each \( v \in R \) send the \((x+1)\)th bit of \( f_v \) (learned in the previous step) to all vertices in \( S \cap N(v) \).

The runtime for this algorithm is asymptotically the same as the runtime of \( SR\text{-comm} \), which is \( O(\log \Delta \log n) \). To determine all \( \log K \) bits of binary representation of \( f_v \), for each \( v \in R \), we run the above algorithm for \( O(\log K) \) times, and so the total time is \( O(\log K \log \Delta \log n) \).

We need the following auxiliary lemma (note that \( \epsilon \) can be either positive or negative).

**Lemma 30.** There exist three universal constants \( 0 < \epsilon_0 < 1 \), \( N_0 \geq 1 \), and \( c_0 \geq 1 \) such that the following is true. For any pair of numbers \((N, \epsilon)\) such that \( N \geq N_0 \) and \( \epsilon \geq \epsilon_0 \sqrt{N} \), we have:

\[
e^{-1}(1 - 0.51 \epsilon^2) \leq (1 + \epsilon)(1 - (1 + \epsilon)/N)^{N-1} \leq e^{-1}(1 - 0.49 \epsilon^2).
\]

Before we present our algorithm for \( SR\text{-comm}^{\text{apx}} \), we first consider a special case where \( W = 1 \), which corresponds to the “approximate counting” problem whose goal is to let each \( v \in R \) learn the number \(|N^+(v) \cap S|\) up to a small \((1 + \epsilon)\)-factor error.

**Lemma 31.** For the case of \( W = 1 \), \( SR\text{-comm}^{\text{apx}} \) can be solved in \( O(\log^{1/5} \Delta \log n) \) time and energy.

**Proof.** In this proof we solve a slightly different task of estimating \(|N(v) \cap S|\) for each \( v \in R \). If each \( v \in R \) knows an estimate of \(|N(v) \cap S|\) up to an \((1 + \epsilon)\)-factor error, then \( v \) can locally calculate an estimate of \(|N^+(v) \cap S|\) with the same approximation ratio.

**Basic Setup:** Let \( C \) be a sufficiently large constant. Let \( \epsilon_0, N_0 \), and \( c_0 \) be the constants in Lemma 30. We assume that \( \epsilon \leq \epsilon_0 \) (if not, then we reset \( \epsilon = \epsilon_0 \)).

The algorithm consists of two phases. The algorithm for the first phase achieves the following. For each \( v \in R \), either (i) \( v \) learns the number \(|N(v) \cap S|\) exactly, or (ii) \( v \) detects that \( \epsilon \geq 10c_0/\sqrt{|N(v) \cap S|} \). The algorithm for the second phase then solves \( SR\text{-comm}^{\text{apx}} \) for all remaining \( v \in R \) (each of them satisfies \( \epsilon \geq 10c_0/\sqrt{|N(v) \cap S|} \)).

**First Phase:** We let \( Z = (10c_0/\epsilon)^2 \). The algorithm consists of \( CZ \log n \) rounds. In each round, each vertex \( u \in S \cup R \) flips a biased coin that produces head with probability \( 1/Z \). Each \( u \in S \) sends \( \text{ID}(u) \) if and only if its result is head; each vertex \( v \in R \) listens if and only if its result is tail.

For each \( v \in R \), if there is a vertex \( u \in N(v) \cap S \) such that the number of messages \( v \) receive from \( u \) is less than \( 0.5 \cdot (C \log n)/\epsilon \), then \( v \) decides that \( \epsilon \geq 10c_0/\sqrt{|N(v) \cap S|} \) (and proceed to the second phase); otherwise, we will later see that \( v \) must have learned the list of all IDs of vertices in \( N(v) \cap S \) w.h.p.

In what follows, we prove the correctness of the algorithm. Let \( e = \{u, v\} \) be any edge where \( u \in S \) and \( v \in R \). In one round of the above algorithm, \( u \) successfully sends a message to \( v \) if and only if (i) the outcome of \( u \)'s coin flip is head, and (ii) all vertices in \( (N(v) \cap S) \cup \{v\} \setminus \{u\} \) get tails. This occurs with probability \( p^* = (1 - 1/Z)^{|N(v) \cap S|}(1/Z) \). Let \( X \) be the number of times \( v \) receives a message from \( u \). To prove the correctness of the algorithm, it suffices to show the following:

- If \( v \in R \) satisfies \( \epsilon \leq 10c_0/\sqrt{|N(v) \cap S|} \), then \( \Pr[X \geq 0.8 \cdot (C \log n)/\epsilon] = 1 - n^{-\Omega(C)} \).
- If \( v \in R \) satisfies \( \epsilon \geq 20c_0/\sqrt{|N(v) \cap S|} \), then \( \Pr[X \leq 0.2 \cdot (C \log n)/\epsilon] = 1 - n^{-\Omega(C)} \).
- If \( v \in R \) satisfies \( \epsilon \leq 20c_0/\sqrt{|N(v) \cap S|} \), then \( \Pr[X \geq 1] = 1 - n^{-\Omega(C)} \).
Case 1: Suppose that the vertex \( v \in R \) satisfies \( \epsilon \leq 10c_0/\sqrt{|N(v) \cap S|} \). We show that in this case the amount of messages \( v \) receive from \( u \in N(v) \cap S \) is at least \( 0.8 \cdot (C \log n)/\epsilon \), with probability \( 1 - n^{-\Omega(C)} \). In this case \( Z \geq |N(v) \cap S| \), and also \( p^* \geq 0.9/(e^2Z) \). The expected value of \( X \) is \( \mu \geq 0.9(C \log n)/\epsilon \). By a Chernoff bound, \( \Pr[X \leq 0.8 \cdot (C \log n)/\epsilon] \leq \exp(-\Omega(C \log n)) = n^{-\Omega(C)} \).

Case 2: Suppose that the vertex \( v \in R \) satisfies \( \epsilon \geq 20c_0/\sqrt{|N(v) \cap S|} \). We show that in this case the amount of messages \( v \) receive from \( u \in N(v) \cap S \) is at most \( 0.2 \cdot (C \log n)/\epsilon \), with probability \( 1 - n^{-\Omega(C)} \). In this case \( Z \leq |N(v) \cap S|/4 \), and also \( p^* \leq 1.1/(e^2Z) \). The expected value of \( X \) is \( \mu \leq 1.1(C \log n)/e^4 < 0.1(C \log n)/\epsilon \). By a Chernoff bound, \( \Pr[X \geq 0.2 \cdot (C \log n)/\epsilon] \leq \exp(-\Omega(C \log n)) = n^{-\Omega(C)} \).

Case 3: Suppose that the vertex \( v \in R \) satisfies \( \epsilon \leq 20c_0/\sqrt{|N(v) \cap S|} \). We show that in this case the amount of messages \( v \) receive from \( u \in N(v) \cap S \) is at least \( 1 \), with probability \( 1 - n^{-\Omega(C)} \). In this case \( Z \geq |N(v) \cap S|/4 \), and also \( p^* \geq 0.9/(e^4Z) \). We have \( \Pr[X < 1] \leq (1 - 0.9/(e^4Z))^{CZ \log n} = n^{-\Omega(C)} \).

Second Phase: In what follows, we assume \( \epsilon \geq 10c_0/\sqrt{|N(v) \cap S|} \) for each \( v \in R \). We consider the sequence of sending probabilities: \( p_1 = 2/\Delta \), and \( p_i = \max\{1, p_{i-1} \cdot (1 + \epsilon)\} \). We let \( i^* = O((1/\epsilon) \log \Delta) \) be the first index \( i \) such that \( p_i = 1 \).

The algorithm is as follows. For \( i = 1 \) to \( i^* \), we do the following for \( C \log n/\epsilon^4 \) rounds. In each round, each vertex \( v \in S \cup R \) flips a fair coin. If the outcome of \( v \) is head and \( v \in S \), then \( v \) sends with probability \( p_i \). If the outcome of \( v \) is tail and \( v \in R \), then \( v \) listens to the channel. After finishing the above procedure, each vertex \( v \in R \) finds an index \( i' \) such that the number of messages \( v \) successfully received during the \( i' \)th iteration is the highest. Then \( v \) decides that the estimate of \( |N(v) \cap S| \) is \( 2/p_{i'} \). Note that the runtime of the algorithm is \( O((1/\epsilon^2) \log \Delta \log n) \).

Next, we analyze the correctness of the above algorithm. Consider a vertex \( v \in R \) in the \( i \)th iteration. We say that \( i \) is good for \( v \) if \( 2/p_i \) is within an \((1 \pm 0.6\epsilon)\)-factor of \( |N(v) \cap S| \); and we say that \( i \) is bad for \( v \) if \( 2/p_i \) is not within an \((1 \pm \epsilon)\)-factor of \( |N(v) \cap S| \). Note that there must be at least one good index \( i \) for each \( v \in R \).

We write \( p_{i \text{ suc}}^v \) to be the probability that \( v \) successfully receives a message in one round of the \( i \)th iteration. It is clear that \( p_{i \text{ suc}}^v = (1/2) \cdot |N(v) \cap S|/(p_i/2) \cdot (1 - (p_i/2))^{p_i \cap S}^{-1} \cdot 1 \). By Lemma 30, we have:

\[
\begin{align*}
p_i \text{ suc}^v &\geq p_{\text{good}}^v \overset{\text{def}}{=} (1/2) \cdot e^{-1}(1 - 0.51(0.6\epsilon)^2) & \text{if } i \text{ is good for } v. \\
p_i \text{ suc}^v &\leq p_{\text{bad}}^v \overset{\text{def}}{=} (1/2) \cdot e^{-1}(1 - 0.49(\epsilon)^2) & \text{if } i \text{ is bad for } v.
\end{align*}
\]

Consider the \( i \)th iteration. Let \( X \) be the number of messages that \( v \) receives in the \( i \)th iteration. The following probability bounds follow by a Chernoff bound.

\[
\begin{align*}
\Pr[X \leq (1 - 0.01\epsilon^2)p_{\text{good}} \cdot C \log n/\epsilon^4] &= \exp(-\Omega(\epsilon^4 \cdot C \log n/\epsilon^4)) = n^{-\Omega(C)} & \text{if } i \text{ is good for } v. \\
\Pr[X \geq (1 + 0.01\epsilon^2)p_{\text{bad}} \cdot C \log n/\epsilon^4] &= \exp(-\Omega(\epsilon^4 \cdot C \log n/\epsilon^4)) = n^{-\Omega(C)} & \text{if } i \text{ is bad for } v.
\end{align*}
\]

Since \((1 - 0.01\epsilon^2)p_{\text{good}} > (1 + 0.01\epsilon^2)p_{\text{bad}}\), with high probability the index \( i' \) selected by \( v \) must be good, and so the estimate calculated by \( v \) is within an \((1 \pm \epsilon)\)-factor of \(|N(v) \cap S|\).

Lemma 32. SR-comm\textsuperscript{apx} can be solved in \( O((1/\epsilon^6) \log W \log \Delta \log n) \) time and energy.

Proof. We let \( \epsilon' = \Theta(\epsilon) \) be chosen such that \((1 + \epsilon')^2 < 1 + \epsilon \) and \((1 - \epsilon')^2 > 1 - \epsilon \). We consider the following sequence: \( w_0 = 0, w_1 = 1, \) and \( w_i = \min\{W, (1 + \epsilon')w_i-1\} \). Let \( i^* \) be the smallest index \( i \) such that \( w_i = W \).

For \( i = 1 \) to \( i^* \), we run SR-comm\textsuperscript{apx} [using the algorithm of Lemma 30] with the following setting. We set \( S' \) as the vertices \( u \in S \) with \( m_u \in (w_i-1, w_i] \); and the message for each \( v \in S' \) is 1. We set \( R' \) as \( R \). The maximum allowed error for SR-comm\textsuperscript{apx} is \( \epsilon' \).

For each \( v \in R \), denote \( N_i \) as the number of vertices \( u \in N^+(v) \cap S \) such that \( m_u \in (w_{i-1}, w_i] \); and we write \( \hat{N}_i \) to denote the result from the \( i \)th iteration. Note that \( \hat{N}_i \) is an \((1 + \epsilon')\)-factor approximation of \( N_i \), and \( \sum_{i=1}^{i^*} \hat{N}_i \) is an \((1 + \epsilon')\)-factor approximation of \( \sum_{u \in N^+(v) \cap S} m_u \). Thus, \( \sum_{i=1}^{i^*} \hat{N}_i \) (which can be calculated locally at \( v \)) is an \((1 + \epsilon')\)-factor approximation of \( \sum_{u \in N^+(v) \cap S} m_u \), as desired.

We calculate the cost of the algorithm. The time complexity for each iteration is \( O(\log \Delta \log n/\epsilon^5) \). The number of iterations is \( i^* = O((1/\epsilon) \log W) \). Thus, the total runtime is \( O((1/\epsilon^6) \log W \log \Delta \log n) \).