Unbiased Statistical Estimation and Valid Confidence Intervals Under Differential Privacy

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Abstract

We present a method for producing unbiased parameter estimates and valid confidence regions/intervals under the constraints of differential privacy, a formal framework for limiting individual information leakage from sensitive data. Prior work in this area is limited in that it is tailored to calculating confidence intervals for specific statistical procedures, such as mean estimation or simple linear regression. While other recent work can produce confidence intervals for more general sets of procedures, they either yield only approximately unbiased estimates, are designed for one-dimensional outputs, or assume significant user knowledge about the data-generating distribution. Our method induces distributions of mean and covariance estimates via the bag of little bootstraps (BLB) (Kleiner et al., 2014) and uses them to privately estimate the parameters’ sampling distribution via a generalized version of the CoinPress estimation algorithm (Biswas et al., 2020). If the user can bound the parameters of the BLB-induced parameters and provide heavier-tailed families, the algorithm produces unbiased parameter estimates and valid confidence intervals which hold with arbitrarily high probability. These results hold in high dimensions and for any estimation procedure which behaves nicely under the bootstrap.

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1 Introduction

1.1 Overview

The dramatic expansion of data collection and analysis has led to growing concerns around the role of privacy and security in the modern world. Our particular focus will be on statistical analysis and how it can leak information about individuals in a data set being analyzed.

Statistical agencies, in particular, have been concerned with statistical disclosure limitation, a broad set of techniques used to limit the leakage of sensitive information from statistical analyses. Duncan and Lambert (1986, 1989) and Reiter (2005) work on identifying and quantifying disclosure risk under different assumptions. There has also been substantial work developing various privacy definitions such as k-anonymity (Sweeney, 2002), t-closeness (Li et al., 2007), and l-diversity (Machanavajjhala et al., 2007) among others.

Dinur and Nissim (2003) developed a polynomial data reconstruction algorithm and used it to prove a result later coined in Dwork and Roth (2014) as the Fundamental Law of Information Recovery. Roughly, this law states that an attacker can reconstruct a data set by asking a sufficiently large number of cleverly chosen queries of the data. This inspired the invention of a more formal privacy definition called differential privacy (DP) (Dwork et al., 2006). DP is a definition which requires that, for any two data sets differing in one row, the change in the distribution of answers to any possible query between the two data sets is minimal. We loosely define a “DP Algorithm” as a procedure that takes a statistical estimator and converts it into an estimator that satisfies DP (i.e. a “DP Estimator”); we define what it means to “satisfy DP” later.

DP has become a popular tool in many corners of industry but has not been widely applied to research in many fields that often analyze sensitive data (social sciences, medicine, etc.). We suggest that this is, in part, because of a lack of DP algorithms that effectively meet the needs of these fields. First, DP algorithms typically require the user to, without looking at the data, specify a domain to which the data will be censored/clipped. This is because privacy mechanisms often take the form of additive noise, with a variance parameter that scales with the sensitivity of the function being privatized (see Definition A.3 and Lemma A.4 for more details, and note that any theorem, algorithm, section, etc. whose number is preceded by an S can be found in the online supplement). Many functions of interest, e.g. means and variances, are unbounded (and thus have infinite sensitivity) when defined over unbounded data domains, so we achieve finite sensitivity by bounding the data domain, i.e. clipping/censoring the data.

We claim that bounding the data domain effectively is a difficult problem in general, and potentially introduces substantial error into the DP pipeline which is difficult to account for. In particular, without assumptions that the data bounds are set well, DP estimators do not typically yield basic statistical guarantees that many applied researchers desire, namely unbiasedness and valid confidence intervals. Our goal is to provide a framework (i.e. a DP algorithm) for converting non-private estimators to DP estimators in a way that jointly addresses both of these concerns.

1.2 Problem Demonstration

Many DP mechanisms take the form “non-private statistic plus zero-mean noise” (e.g. the Gaussian mechanism defined in Lemma A.4) and thus look like they ought to be unbiased and yield easily characterizable confidence regions. However, this doesn’t take into account the necessary step of choosing a bounded data domain. In practice, analysts are required to specify this domain without looking at the data. The data are then projected into the domain, and analysis proceeds on the projected data. This introduces a potential trade-off in the analyst’s decision calculus. The analyst generally wants to specify a small domain, as this generally requires less noise addition to
satisfy DP. However, if the analyst’s domain is too small, they risk clipping the data, which can lead to biased estimates.

Consider the case where $X = \{X_1, \ldots, X_n\}$ with $X_i \sim N(0, 1)$ and $y = X\beta + \varepsilon$ for $\beta = 100, \varepsilon \sim N(0, 10^2)$. We estimate $\beta$ and get associated 95% confidence intervals using OLS and test the effect of various levels of clipping on the estimates and confidence intervals. Specifically, we leave $X$ unclipped and clip the top $\{0, 0.1, 1, 5\}$ percent of $y$.

![Coefficient estimates](image1.png) ![Coefficient estimates](image2.png) ![Coefficient estimates](image3.png)

(a) Bias: 0.1% clipping  (b) Bias: 1% clipping  (c) Bias: 5% clipping

![Centered coefficient estimates](image4.png) ![Centered coefficient estimates](image5.png) ![Centered coefficient estimates](image6.png)

(d) CI coverage: 0.1% clipping  (e) CI coverage: 1% clipping  (f) CI coverage: 5% clipping

Figure 1: Distribution of OLS coefficient estimates (a-c) and 95% confidence intervals (d-f) under different levels of clipping of $y$. Non-clipped distribution in green, clipped distribution in orange.

Figure 1 shows results from 10,000 simulations of this process. The top plots show coefficient distributions under each level of clipping and compare it to the condition of no clipping. Note that at even a moderate level of 1% clipping, the distributions of estimates are completely non-overlapping. The bottom plots show the absolute error of the estimates, arranged in increasing order on the x-axis, with vertical bars representing the 95% confidence interval for that estimate. The black dotted vertical lines on the left and right sides of each plot show the 0.025 and 0.975 quantiles, where we expect perfectly calibrated confidence intervals to cross the x-axis. At 0.1% clipping, approximately half of our confidence intervals do not contain the true parameter value; at higher levels of clipping, none of them do.

We chose a very simple regime for the experiments above: one-dimensional OLS with a Gaussian covariate, Gaussian error, clipping in only the outcome variable, and no attempt to satisfy DP (i.e. no noise addition). In more complex settings, the effect of clipping on the coefficients could be larger, in addition to being harder to predict and reason about. Thus, we argue that the data bounding step that precedes so many DP algorithms has potentially immense consequences for practical analysis and ought to be seriously considered.
1.3 Related Work

Differential privacy has grown in popularity in recent years, as has the literature exploring the intersection between statistics and DP. Dwork and Lei (2009) point out how a handful of common robust statistical estimators could be extended to satisfy differential privacy. Wasserman and Zhou (2010) compare DP mechanisms via convergence rates of distributions and densities from DP releases and frame DP in statistical language more broadly. Lei et al. (2016) explores model selection under DP, while Peña and Barrientos (2021) explores model uncertainty. Vu and Slavkovic (2009); Wang et al. (2015); Gaboardi et al. (2016); Canonne et al. (2019), and Awan and Slavkovic (2019) propose methods for DP hypothesis testing in various domains. Sheffet (2017); Wang (2018); Barrientos et al. (2019), and Alabi et al. (2020) all address the problem of differentially private linear regression.

Karwa and Vadhan (2017) gives nearly optimal confidence intervals for univariate Gaussian mean estimation with finite sample guarantees. Du et al. (2020) proposes their own algorithms for the same problem and finds superior practical performance in some domains, and Biswas et al. (2020) develop an algorithm that works well at reasonably small sample sizes and without strong assumptions on user knowledge, while also scaling well to high dimensions. Drechsler et al. (2021) explores non-parametric confidence intervals for calculating medians. D’Orazio et al. (2015) gives confidence intervals for a difference of means. Avella-Medina et al. (2021) shows how to construct DP version of M-estimators, as well as associated confidence regions.

Our work continues in a line of recent work for constructing confidence intervals for more general classes of differentially private estimators. Brawner and Honaker (2018) shows how to combine estimates from additive functions that satisfy zCDP to get confidence intervals at no additional cost. Wang et al. (2019) provides confidence intervals for models trained with objective or output perturbation algorithms. These algorithms are quite general, but require solving the non-private ERM sub-problem optimally. Ferrando et al. (2021) presents a very general approach based on privately estimating parameters of the data-generating distribution and bootstrapping confidence intervals by repeatedly running the model of interest on samples from a distribution parameterized by the privately estimated parameters. This method is efficient with respect to its use of the privacy budget, but relies on significant knowledge of the structure of the data-generating process. With the exception of Karwa and Vadhan (2017), these works either ignore the issue of bounding the data domain $\mathcal{X}$ effectively, or attempt to address it through bias-correction strategies which we believe are unlikely to work for complex problems. Barrientos et al. (2021) tests a variety of DP algorithms for various tasks (ranging from tabular statistics to OLS regression) on real data and argue that existing methods for performing DP regression, “would struggle to produce accurate regression estimates and confidence intervals” (Barrientos et al. (2021), p. 1).

Evans et al. (2019) is the closest existing work to ours. They also start with the Sample & Aggregate framework, and BLB algorithm to get $k$ estimates of the parameters of the sampling distribution of their non-private estimator. The $k$ estimates are then aggregated via a differentially private mean and confidence intervals are calculated using a differentially private variance estimate and CLT assumption. Because the estimates are projected into a bounded parameter domain to control the sensitivity of the mean, the resulting private mean is not necessarily unbiased. Evans et al. (2019) attempts to address this issue by privately estimating the proportion of the $k$ estimates that are clipped by the projection and adjusting the private mean by the estimated clipping proportions. This method has the advantage of allowing users to specify overly tight clipping bounds in order to decrease the global sensitivity of their estimator, but is sensitive to how well the clipping proportions are estimated and, to our knowledge, has no means of generalizing to multivariate parameter estimation.
1.4 Contributions

We introduce a general-purpose meta-algorithm that allows an analyst to take any estimator that is (1) unbiased in the non-private setting and (2) has “nice” properties under the bootstrap and produce a version that satisfies DP and, with high probability, is unbiased and produces valid confidence intervals or a valid confidence region. Our results hold under the central (or trusted curator) model of DP.

Our algorithm can be split into three distinct steps, each of which we explain in detail in Section 2. First, we use the Bag of Little Bootstraps (BLB) algorithm (Algorithm 2) developed by Kleiner et al. (2014) to produce estimates of the mean and covariance of the sampling distribution of the non-private estimator.

Second, we privately estimate the mean of each set of BLB estimates (both the means and covariances) using a modified version of the CoinPress private mean estimation algorithm (Algorithm 3 (Biswas et al., 2020)). For both the mean and covariance distributions induced by the BLB, the analyst must provide a distribution that is heavier-tailed (Definition C.1 and Assumption 2.5), as well as give bounds on the mean (Assumption 2.6) and covariance (Assumption 2.7) of the induced distribution. Under these conditions, this step produces parameter estimates which are unbiased and follow a multivariate Gaussian distribution with known covariance (Theorem 2.8). We argue that these conditions are natural and demonstrate how the properties of the CoinPress algorithm allow the analyst to get good performance even when they set the aforementioned bounds very conservatively.

Although our use of CoinPress in this way guarantees that DP is satisfied with respect to the BLB estimates, this guarantee also holds for the underlying sensitive data over which the BLB estimates were calculated. This fact follows from noting that combining the BLB with an aggregation step that satisfies DP (such as CoinPress) falls under the purview of the Sample & Aggregate framework developed in Nissim et al. (2007).

Third, we combine the estimated parameters using precision weighting to produce final mean and covariance estimates (Theorems 2.11 and 2.10), which are used to calculate a valid confidence region/intervals (Theorem 2.12). We do so via a multivariate extension of the precision-weighting technique to improve CoinPress’ estimates (Theorem 2.9). While precision-weighting is a well-known technique in the meta-analysis literature Cochran (1954), we give, to the best of our knowledge, the first proof of multivariate optimality, which may be of independent interest. This step maintains the privacy guarantees of the CoinPress algorithm because differential privacy is preserved under postprocessing (Lemma 2.3).

We believe that our framework is a promising step toward making differential privacy more practical for applied research. The problem of choosing good data bounds is significant in practice in that it is both generally difficult and that many DP algorithms are sensitive to poor choices. There is also currently an asymmetry in the failure modes, in that bounds that are too wide typically yield answers which are unbiased but very noisy, while bounds that are too narrow risk “silent failure”, where the DP result looks precise but is not actually representative of the non-private answer. Our framework, through use of the CoinPress algorithm, gives users more leeway to err on the side of conservatism without introducing bias, thus mitigating the possibility of getting DP results that appear precise but are systematically incorrect.

Moreover, our framework is general enough to be applied to any estimator for which the BLB does a “good” job approximating the sampling distribution of the estimator. The bootstrap is broadly familiar to applied statisticians, and its properties for any particular estimator an analyst wishes to use can, in principle, be tested on non-sensitive data. Thus, answering the question of whether or not our algorithm will be useful in a particular setting does not require significant
2 Algorithm Overview
2.1 DP Preliminaries
Throughout this work we use a particular notion of DP called zero-concentrated DP. Suppose we have a data domain $X$ and data set $X \in X^n$.

Definition 2.1 (Zero-concentrated differential privacy (zCDP) (Bun and Steinke, 2016)). Let $M: X^n \to \Omega$ be a randomized algorithm where $(\Omega, \Sigma, P)$ is a probability space and $\rho \geq 0$. We say that $M$ satisfies $\rho$-zCDP with respect to a data set $X$ if, for all $(X', X^*) \in X^n$ and $\alpha \in (1, \infty)$:

$$H_\alpha (M(X')\|M(X^*)) \leq \rho \alpha,$$

where $H_\alpha$ is the $\alpha$-Rényi divergence.

The parameter $\rho$ represents an upper bound on the amount of information $M$ leaks about the underlying data $X$. Larger $\rho$ implies more information leakage, or privacy loss, but also allows for the statistics returned by $M$ to be more accurate.

zCDP (like other standard notions of DP) has two properties which are very useful for reasoning about how DP guarantees operate within a full data analysis pipeline.

Lemma 2.2 (Composition of zCDP (Bun and Steinke, 2016)). Let $M: X^n \to \mathcal{Y}$ and $M': X^n \to \mathcal{Z}$ such that $M$ satisfies $\rho$-zCDP and $M'$ satisfies $\rho'$-zCDP. Define $M'': X^n \to \mathcal{Y} \times \mathcal{Z}$ by $M''(x) = (M(x), M'(x))$. Then $M''$ satisfies $(\rho + \rho')$-zCDP.

We use zCDP because its privacy parameters $\rho$ compose additively, which is convenient for algorithms, like CoinPress, that contain multiple private releases. zCDP implies the more familiar notion of $(\varepsilon, \delta)$-DP (see Proposition 3 of Bun and Steinke (2016)), so any zCDP guarantees in this paper can be converted to $(\varepsilon, \delta)$-DP if an analyst prefers.

Lemma 2.3 (Postprocessing of zCDP (Bun and Steinke, 2016)). Let $M: X^n \to \mathcal{Y}$ and $f: \mathcal{Y} \to \mathcal{Z}$ such that $M$ satisfies $\rho$-zCDP. Define $M': X^n \to \mathcal{Z}$ such that $M'(x) = f(M(x))$. Then $M'$ satisfies $\rho$-zCDP.

This postprocessing property of zCDP states that if some output satisfies DP with respect to some data $X$, functions of that output also satisfy DP with respect to $X$ (provided that the functions do not take $X$ as input).

2.2 Algorithm Step 1: Bag of Little Bootstraps and Sample & Aggregate
In Step 1 of our algorithm, the algorithm takes the analyst’s non-private estimator $\hat{\theta}$ and uses the Bag of Little Bootstraps to try to approximate the sampling distribution of $\hat{\theta}$. In particular, the BLB partitions the data into $k$ disjoint subsets, repeatedly runs the estimator over each subset, and produces $k$ estimates of its mean, $\{\hat{\theta}_{BLB}^i\}_{i \in [k]}$, and covariance, $\{\hat{\Sigma}_{BLB}^i\}_{i \in [k]}$.

Let $\mathcal{X}$ be our data domain, $\mathcal{D}$ a distribution over the domain, and $X = \{x_1, \ldots, x_n\}$ be our sensitive data set where $x_i$ are drawn i.i.d. from $\mathcal{D}$. For shorthand, we say that $X \in X^n$ and $X \sim \mathcal{D}^n$. We say that the analyst wants to run some model, which has an associated parameter vector $\theta \in \mathbb{R}^d$. The analyst specifies the estimator they would have liked to run in the non-private
setting \( \hat{\theta} : \mathcal{X}^n \to \mathbb{R}^d \). Our goal is eventually to approximate \( \hat{\theta} \) in a manner that satisfies DP with respect to \( X \). We assume that \( n \) is “public knowledge” and does not need to be privately estimated. If this is not true, we can generate a DP estimate of \( n \), call it \( \tilde{n} \), and then \( X \) could be subsampled or augmented with rows of synthetic data until it has \( \tilde{n} \) rows, creating a new data set \( \tilde{X} \) over which we can apply our algorithm.

As stated earlier, DP algorithms typically require specification of the global sensitivity of the function whose outputs are being privatized (see Definition A.3 and Lemma A.4). This can become arbitrarily complex for complicated models, even after assuming a bounded input domain. The Sample & Aggregate framework introduced in Nissim et al. (2007) provides a strategy for estimating such functions without specifying the global sensitivity. First run the function of interest non-privately over \( k \) disjoint subsets of the data, bound the outputs, and then aggregate the results using a function with a sensitivity that is easier to reason about (for our purposes, we assume the aggregation function is the mean). We then privately estimate the mean of these \( k \) results. Because each element in the original data contributes to one subset of the partition, its effect on the aggregation is localized to one of its \( k \) inputs, and so a mean estimation algorithm that satisfies DP with respect to those \( k \) elements also satisfies DP with respect to the underlying data. A more thorough treatment of this framework can be found in Nissim et al. (2007) and Chapter 7 of Dwork and Roth (2014).

The Bag of Little Bootstraps algorithm (developed in (Kleiner et al., 2014) and reproduced in Section B) randomly partitions the data \( X \) into \( k \) disjoint subsets \( \{X_1, \ldots, X_k\} \), scales the subset back up to an effective sample size that matches that of the original data (via multinomial sampling), and runs \( \hat{\theta} \) \( r \) times, producing estimates \( \{\hat{\theta}_{i,a}\}_{a \in [r]} \). It then aggregates these into an arbitrary assessment of estimator quality. We use the mean and covariance, so for each \( i \in [k] \) we get \( \hat{\theta}_{i}^{\text{BLB}} = \frac{1}{r} \sum_{a=1}^{r} \hat{\theta}_{i,a} \) and \( \hat{\Sigma}_{i}^{\text{BLB}} = \text{Cov} \left( \{\hat{\theta}_{i,a}\}_{a \in [r]} \right) \). Our approach allows us to find a single confidence region for the entire parameter vector jointly. However, we acknowledge that many analysts will prefer separate confidence intervals for each element in their parameter vector. This preference is advantageous from a privacy perspective, as confidence intervals will require less noise to privatize than a full confidence region and will thus be more accurate.

We present methods and results for the general case of finding a confidence region, but also show how this could be adapted to instead find confidence intervals. We include the setup for finding confidence intervals in the main text with results in the supplement.

These sets of estimates are now empirical approximations to the theoretical distributions of the BLB estimates, which we assume are themselves good approximations of the actual parameters of interest. We make this last assumption explicit as follows.

**Assumption 2.4.** Let the estimator \( \hat{\theta} \sim G(\theta, \Sigma) \) where the marginals of \( G \) each belong to a location-scale family. For \( \hat{\theta}_{i}^{\text{BLB}} \) and \( \hat{\Sigma}_{i}^{\text{BLB}} \) generated by applying BLB to our estimator \( \hat{\theta} \), let \( \hat{\theta}^{\text{BLB}} = \frac{1}{k} \sum_{i=1}^{k} \hat{\theta}_{i}^{\text{BLB}} \) and \( \hat{\Sigma}^{\text{BLB}} = \frac{1}{k} \sum_{i=1}^{k} \hat{\Sigma}_{i}^{\text{BLB}} \). We assume that

\[
\mathbb{E} \left( \hat{\theta}^{\text{BLB}} \right) = \mathbb{E} \left( \hat{\theta} \right) = \theta
\]

\[
\mathbb{P} \left( \hat{\Sigma}^{\text{BLB}} \succeq \hat{\Sigma} \right) = 1.
\]

We take \( \succeq \) to mean “greater/equal in L"owner order”. In particular, \( A \succeq B \iff A - B \) is a PSD matrix, in which case we call \( A \) a L"owner upper bound on \( B \).
2.3 Algorithm Step 2: Generating Private Parameter Estimates With CoinPress

Now that we have BLB estimates \( \{\hat{\theta}^{BLB}_i\}_{i \in [k]} \) and \( \{\hat{\Sigma}^{BLB}_i\}_{i \in [k]} \), we can privately estimate their empirical means \( \hat{\theta}^{BLB} \) and \( \hat{\Sigma}^{BLB} \) using the CoinPress mean estimation algorithm (Section C.1). We try to provide intuition here for how and why CoinPress works, but interested readers should consult Biswas et al. (2020) for a more complete treatment.

For generality, we say that we have data \( \{y_i\}_{i \in [k]} \) with empirical mean \( \hat{\mu} \), where each \( y_i \) is an i.i.d. instantiation of a random variable \( Y \) with mean \( \mu_Y \) and covariance \( \Sigma_Y \). Our goal is to estimate \( \mu_Y \) in a manner that satisfies DP. In order to state these results generally, we will assume that \( Y \in \mathbb{R}^{d'} \) for some \( d' \). When \( y_i \) stands in for \( \hat{\theta}^{BLB}_i \), we have \( d' = d \), the dimension of our parameter vector of interest. The same is true when \( y_i \) stands in for \( \hat{\Sigma}^{BLB}_i \) and we care only about univariate confidence intervals such that \( \hat{\Sigma}^{BLB}_i = \hat{V}_i^{BLB}I_d \) for some \( \hat{V}_i^{BLB} \). When we care about a joint confidence region, we need to use the entire upper triangular of \( \hat{\Sigma}^{BLB}_i \), so \( d' = \frac{d(d+1)}{2} \). In the actual implementation (and proofs), these covariance estimates are flattened into a vector of the appropriate dimension before estimation and unflattened at the end of estimation to produce full covariance matrices again. We assume that this is going on behind the scenes and, for notational clarity, keep calling them \( \hat{\Sigma}^{BLB}_i \).

The high-level idea behind CoinPress is to make a series of mean estimates, where each estimate (probabilistically) improves upon the last, making only a few requirements of the analyst.

**Assumption 2.5.** The analyst provides a location-scale family of distributions \( Q_Y(\mu, \Sigma) \) with heavier tails than the distribution of \( Y \) as described in Definition C.1.

**Assumption 2.6.** The analyst provides \( \hat{\mu}_0 \in \mathbb{R}^d \) and \( r_0 \in \mathbb{R} \) such that \( \mu_Y \in B_2(\hat{\mu}_0, r_0) \), the \( \ell_2 \) ball centered at \( \hat{\mu}_0 \) with radius \( r_0 \).

**Assumption 2.7.** The analyst provides \( \Sigma^U_Y \in \mathbb{R}^{d \times d} \) such that \( \Sigma_Y \preceq \Sigma^U_Y \).

These three assumptions provide the backbone of the iterative improvement in CoinPress. Recall that the noise added to privatize an estimator defined over an input domain generally scales with the size of the domain. Over a series of \( t \) steps, CoinPress attempts to find a small domain that, with high probability, contains all the \( \{y_i\}_{i \in [k]} \). Assumption 2.6 ensures that the algorithm starts with sufficiently conservative bounds that contain \( \mu_X \). Assumptions 2.5 and 2.7 then allow the algorithm to convert bounds on \( \mu_X \) to high-probability bounds on the individual data points \( y_i \). For the remainder of the section, we assume that our three assumptions hold.

At step \( m \in [t] \) of the algorithm, CoinPress takes the \( \ell_2 \) ball and expands it outward based on the assumed distribution. If you have a distribution of a known family with bounded mean and covariance, you can, with high probability, upper bound the \( \ell_2 \) norm of an arbitrary number of draws from said distribution. In the context of our case, it specifies a ball that, with high probability, contains all of the \( \{y_i\}_{i \in [k]} \). In this case, using the ball as our data domain and projecting our data into this domain will not clip any of the data. The global sensitivity of the mean is calculated using this ball and CoinPress adds noise scaled to the sensitivity using the Gaussian mechanism (Lemma A.4), which adds zero-mean noise from a multivariate Gaussian with diagonal covariance to get a private estimate \( \hat{\mu}_m \). Because we are, with high probability, not clipping any of the \( \{y_i\}_{i \in [k]} \), our use of the Gaussian mechanism implies that the form of our private estimator is “empirical mean + zero-mean noise”. Thus, based on the scale of the noise, we can produce a new \( \ell_2 \) ball which, with high probability, contains the true empirical mean. This ball becomes the bounding set for the mean at the next step and the process continues. The
guarantee that CoinPress does not clip any points ensures (with high probability) that for all \( m \in [t] \), the \( \hat{\mu}_m \) are unbiased estimates of the empirical mean of the \( y_i \).

Moreover, assuming no clipping, the variances (induced by clipping and our use of the Gaussian mechanism) of \( \hat{\mu}_m \) are just the diagonal of the covariance parameter used in the Gaussian mechanism, which we call \( \hat{\Sigma}_{\mu,m} \in \mathbb{R}^t_+ \). In other words, under no clipping, the additional error incurred by the general privacy mechanism is simply the error from the Gaussian mechanism.

In general, applying the CoinPress algorithm to a data set \( \{y_i\}_{i \in [k]} \) satisfies \( \rho \)-zCDP with respect to \( \{y_i\}_{i \in [k]} \) (Biswas et al., 2020). In our case \( \{y_i\}_{i \in [k]} \) stands in for either of the sets of BLB estimates \( \{\hat{\theta}_i^{BLB}\}_{i \in [k]} \) and \( \{\hat{\Sigma}_i^{BLB}\}_{i \in [k]} \). So, CoinPress satisfies zCDP with respect to each set of BLB estimates and, by the extension implied by our use of the Sample & Aggregate framework, also satisfies zCDP with respect to the original data \( X \). Moreover, CoinPress comes with a high-probability guarantee on the form of the private estimates it produces.

**Theorem 2.8.** Under Assumptions 2.5, 2.6, and 2.7, CoinPress produces \( t \) mean estimates \( \{\hat{\mu}_m\}_{m \in [t]} \) and associated privacy variances \( \{\hat{\Sigma}_{\mu,m}\}_{m \in [t]} \) such that

\[
\mathbb{P} \left[ \forall m \in [t] : \hat{\mu}_m \sim N(\hat{\mu}_m, \sigma^2_{\mu,m} I_d) \right] \geq 1 - \beta^t.
\]

We now recall that \( \hat{\mu} \) is the empirical mean of the \( \{y_i\}_{i \in [k]} \), where \( y_i \) stands in for either \( \hat{\theta}_i^{BLB} \) or \( \hat{\Sigma}_i^{BLB} \). That is, we use CoinPress to privately estimate the mean of the distributions induced by the BLB. This yields \( t \) estimates of both the parameter means and covariances (both with associated privacy variances) \( \{\hat{\theta}_m, \sigma^2_{\theta,m}\}_{m \in [t]} \) and \( \{\hat{\Sigma}_m, \sigma^2_{\Sigma,m}\}_{m \in [t]} \).

### 2.4 Algorithm Step 3: Get Final Parameter Estimates and Confidence Intervals Via Postprocessing

We now use our \( t \) DP mean and covariance estimates \( \{\hat{\theta}_m, \hat{\Sigma}_m\}_{m \in [t]} \) to get a single DP estimate of the mean and covariance, which we’ll call \( \tilde{\theta} \) and \( \tilde{\Sigma} \).

**Final Parameter Estimates** We start by presenting a multivariate version of the precision weighting argument which gives the minimal covariance way to combine a set of unbiased estimators.

**Theorem 2.9.** For a parameter \( \tau \), say we are given a series of independent estimates \( \{\hat{\tau}_m\}_{m \in [t]} \) such that \( \mathbb{E}(\hat{\tau}_m) = \tau \) and \( \text{Cov}(\hat{\tau}_m) = S_m \) for some positive definite \( S_m \). Then the minimum covariance unbiased linear weighting of the \( \{\hat{\tau}_m\}_{m \in [t]} \) is given by

\[
\hat{\tau} = \left( S_{m=1}^{-1} \right)^{-1} \left( S_{m=1}^{-1} \tau_m \right),
\]

which has \( \mathbb{E}(\hat{\tau}) = \tau \) and \( \text{Cov}(\hat{\tau}) = \left( S_{m=1}^{-1} \right)^{-1} \).

Specifically, by “minimum covariance” we mean that any other unbiased linear weighting \( \hat{\tau}' \) of the \( \{\hat{\tau}_m\}_{m \in [t]} \) will have \( \text{Cov}(\hat{\tau}) \preceq \text{Cov}(\hat{\tau}') \).

Recall from Theorem 2.8 that, with high probability, both our \( \{\hat{\theta}_m\}_{m \in [t]} \) and \( \{\hat{\Sigma}_m\}_{m \in [t]} \) are sets of estimates which are independent, unbiased, and have a known covariance structure. Thus, they both meet the criteria to be combined into precision-weighted estimators in the style of Theorem 2.9. Because the precision weighting step is a function of the DP estimates, it retains the privacy guarantees from Step 2 by the postprocessing property of zCDP (Lemma 2.3).
We start by using the precision-weighting idea to find an estimator for the covariances, $\hat{\Sigma}$. Unlike the mean estimation setting, where our goal is to produce an unbiased private estimator, our goal for our private covariance estimator is to find a private estimator that will reliably overestimate the empirical covariance, and thus yield valid confidence intervals. To get a sense for why this is necessary, consider the one-dimensional case where we have a sample variance $s$ and a privatized version $\tilde{s}$. In the non-private setting, we simply use the estimate $s$ to calculate confidence intervals, but in order to use $\tilde{s}$ for confidence intervals we need to understand the relationship between $s$ and $\tilde{s}$. An important first step is to ensure that there is no clipping in the construction of $\tilde{s}$, so we know that $\tilde{s}$ equals $s$ plus zero-mean noise; this is the same motivation as in the mean estimation case. An extra complication for variances is that zero-mean noise addition has an asymmetric impact on confidence interval coverage; if we were to use $\tilde{s}$ for confidence intervals, we would underestimate the true variance 50% of the time. We can avoid this problem by increasing $\tilde{s}$ to the point that we know, with high probability, that it is at least as large as $s$.

**Theorem 2.10.** Given covariance estimates and privacy variances $\{\hat{\Sigma}_m, \hat{\sigma}^2_{\Sigma,m}\}_{m \in [t]}$, let $\check{S}_m \in \mathbb{R}^d$ be the flattened version of $\hat{\Sigma}_m$. We can construct a precision-weighted estimator $\check{S}$:

$$\check{S} := \frac{\sum_{m=1}^t \hat{\Sigma}_m / \hat{\sigma}^2_{\Sigma,m}}{\sum_{m=1}^t 1 / \hat{\sigma}^2_{\Sigma,m}}.$$

Let $\check{\Sigma}'$ be the unflattened $d \times d$ version of $\check{S}$ and $b$ be the unflattened $d \times d$ matrix where $b_{ij} = \text{Var}(\hat{\Sigma}_{ij})$ (i.e. the diagonal values of the covariance matrix of the flattened precision-weighted estimator). For $\beta^{ub} \in (0, 1)$, define

$$c = \min_{\varepsilon \in (0, 1/2]} (1 + \varepsilon) \left( 2 \max_{i \in [d]} \|b_i\|_2 + \frac{6\sqrt{\log d}}{\log(1 + \varepsilon)} \max_{i,j \in [d] \times [d]} |b_{ij}| \right) + \sqrt{\frac{\ln(1/\beta^{ub})}{4\max_{ij} b^2_{ij}}}.$$

Then, for $\hat{\Sigma} = \check{\Sigma}' + cI_d$ we have $\mathbb{P}\left( \hat{\Sigma} \preceq \check{\Sigma} \right) \geq 1 - \beta^{\hat{\Sigma}} - \beta^{ub}$.

We use a similar strategy to convert our private parameter estimates $\{\hat{\theta}_m\}_{m \in [t]}$ into a final private parameter estimate $\hat{\theta}$. Because we want an unbiased estimator of $\hat{\theta}$, we don’t need to be conservative like we did with the covariance; we simply use the precision-weighted estimator.

**Theorem 2.11.** Given parameter estimates and privacy variances $\{\hat{\theta}_m, \hat{\sigma}^2_{\theta,m}\}_{m \in [t]}$, we define the precision-weighted estimator $\hat{\theta}$ as

$$\hat{\theta} := \frac{\sum_{m=1}^t \hat{\theta}_m / \hat{\sigma}^2_{\theta,m}}{\sum_{m=1}^t 1 / \hat{\sigma}^2_{\theta,m}}.$$

This estimator has expectation $\hat{\theta}$ and covariance $\Sigma_{\hat{\theta}} = \sum_{m=1}^t 1 / \hat{\sigma}^2_{\theta,m} I_d$. In particular, we say that

$$\mathbb{P}\left( \hat{\theta} \sim N \left( \hat{\theta}, \Sigma_{\hat{\theta}} \right) \right) \geq 1 - \beta^{\hat{\Sigma}} - \beta^{ub} - \beta^{\hat{\theta}}.$$
Theorem 2.12 (Confidence Region (valid with high probability)). Let Z be a d-dimensional random variable such that \( Z \sim Q_\theta \left( \hat{\theta} + N(0, \Sigma_\theta), \hat{\Sigma} \right) \). Suppose C is a d-dimensional ellipsoid such that \( \mathbb{P}(Z \in C) \geq 1 - \alpha \) for some \( \alpha \in (0, 1) \). Then, with probability \( 1 - \beta^z - \beta^u - \beta^\theta \):

\[
\mathbb{P}(\theta \in C) \geq 1 - \alpha.
\]

It is always trivial to find such an ellipsoid \( C \) (e.g. take \( C = \mathbb{R}^d \), but finding one with coverage close to \( 1 - \alpha \) analytically could be difficult in general. However, in typical scenarios it is likely to be much more nicely behaved. Most notable is the case where \( Q_\theta \) is multivariate Gaussian (e.g. the analyst is comfortable assuming that the CLT has kicked in for the BLB estimates). The resulting compound distribution is \( N \left( \hat{\theta}, \hat{\Sigma} + \Sigma_\theta \right) \); a Gaussian random variable with a Gaussian random variable as its location parameter is still Gaussian. This becomes even simpler if the analyst is interested in univariate confidence intervals, in which case they can use the fact that \( \forall \theta \in [d] : \hat{\theta}_j \sim N \left( \hat{\theta}_j, \Sigma_{\hat{\theta}_j} + \Sigma_{\theta_j} \right) \), and can calculate confidence intervals directly from the CDF of the univariate Gaussian. In more complicated scenarios, an analyst can always get approximate quantiles using Monte Carlo methods.

2.5 Full Algorithm Statement

In Algorithm 1, we finally present our algorithm in whole. We omit some hyperparameters in the subroutines to make it easier to focus on the core pieces that change between them. Let \( \xi : a_{1:k} \mapsto \frac{1}{k} \sum_{i=1}^k a_i \) and either \( \xi' : a_{1:k} \mapsto \mathsf{Cov} \left( \{a_i\}_{i \in [k]} \right) \) or \( \xi' : a_{1:k} \mapsto \mathsf{diag} \left( \mathsf{Cov} \left( \{a_i\}_{i \in [k]} \right) \right) \), depending on whether the analyst desires a joint confidence region or separate confidence intervals.

Algorithm 1 General Valid DP (GVDP)

Input: data set \( X \in \mathbb{R}^{n \times m} \), estimator \( \hat{\theta} : X^n \to \mathbb{R}^d \) families of distributions \( Q_{\theta}, Q_{\Sigma} \); privacy budgets \( \rho^\theta, \rho^\Sigma > 0 \), failure probabilities \( \beta^\theta, \beta^\Sigma, \beta^u \in (0, 1) \)

Output: parameter estimate \( \hat{\theta} \) and associated confidence intervals/region \( C \) which satisfy \( (\rho^\theta + \rho^\Sigma) \)-zCDP and have desired unbiased/validity properties with probability \( 1 - \beta^\theta - \beta^\Sigma - \beta^u \)

1: procedure GVDP(\( X, \hat{\theta}, Q_{\theta}, Q_{\Sigma}, \rho^\theta, \rho^\Sigma \))

2: \( \{\Sigma_i^{BLB}\}_{i \in [\ell]} = \text{BLB} \left( X, \hat{\theta}, \xi, \ldots \right) \) \hspace{1cm} \triangleright Algorithm 2 – get BLB estimates of parameter covariance

3: \( \{\hat{\theta}^{BLB}_i\}_{i \in [\ell]} = \text{BLB} \left( X, \hat{\theta}, \xi, \ldots \right) \) \hspace{1cm} \triangleright Algorithm 2 – get BLB estimates of parameter means

4: \( \{\hat{T}_m\}_{m \in [\ell]} = \text{MVMRec} \left( \{\Sigma_i^{BLB}\}_{i \in [\ell]}, Q_{\Sigma}, \ldots, \rho^\Sigma, \beta^\Sigma \right) \) \hspace{1cm} \triangleright Algorithm 3 – privately estimate parameter covariance at \( \rho^\Sigma \)-zCDP level

5: Combine \( \{\Sigma_m\}_{m \in [\ell]} \) via precision-weighting to get \( \hat{\Sigma} \) \hspace{1cm} \triangleright Theorem 2.9

6: \( \{\hat{\theta}_m\}_{m \in [\ell]} = \text{MVMRec} \left( \{\hat{T}_m\}_{m \in [\ell]}, \ldots, Q_{\theta}, \ldots, \rho^\theta, \beta^\theta \right) \) \hspace{1cm} \triangleright Algorithm 3 – privately estimate parameter means at \( \rho^\theta \)-zCDP level

7: Combine \( \{\hat{\theta}_m\}_{m \in [\ell]} \) via precision-weighting to get \( \hat{\theta} \) \hspace{1cm} \triangleright Theorem 2.9

8: Use \( \hat{\theta}, \hat{\Sigma}, Q_{\theta} \) and \( \beta^u \) to get confidence intervals/region \( C \). \hspace{1cm} \triangleright Theorem 2.12

9: return \( \{\hat{\theta}, C\} \)

In summary, our private estimates \( \hat{\theta} \) and \( \hat{\Sigma} \) have statistical guarantees relative to the true parameters of the sampling distribution of \( \hat{\theta} \sim G(\theta, \Sigma) \) via the following lines of reasoning:

\[
\mathbb{E} \left( \hat{\theta} \right) \overset{w.h.p.}{\geq} \text{CoinPress + postprocessing} \quad \mathbb{E} \left( \hat{\theta}^{BLB} \right) \overset{\text{Assumption 2.4}}{=} \mathbb{E} \left( \hat{\theta} \right) = \theta
\]

\[
\hat{\Sigma} \overset{w.h.p.}{\geq} \text{CoinPress + postprocessing} \quad \hat{\Sigma}^{BLB} \overset{\text{Assumption 2.4}}{\geq} \Sigma
\]
We provide empirical demonstrations of our core result, showing that we can produce unbiased parameter estimates and valid confidence intervals when the requisite assumptions hold. For every evaluation, we aim to get valid confidence intervals for each element of the parameter vector rather than a single valid confidence region, as we expect this to be the dominant use case in practice.

All results satisfy zCDP at the $\rho = 0.1$ level and, inside the GVDP algorithm, we always run CoinPress for $t = 5$ iterations. Additionally, we assume that the analyst chooses bounds that satisfy Assumptions 2.6 and 2.7, but are larger than the tightest possible bounds by a factor of $\approx 100$. For example, if the analyst had an estimator with a $N(\mu = 1, \sigma^2 = 1)$ sampling distribution we assume their prior knowledge to be that $\mu \in [-100, 100]$ and $\sigma^2 \leq 100$. Additional empirical results, including comparisons to existing methods, are described in Section F.

**OLS Regression Demonstration** We begin by testing parameter estimation for a properly specified OLS model with $d = 5$ parameters of interest. In a single iteration of our experiment, we generate data from a linear model $y = X\beta + \varepsilon$ with Gaussian covariates, Gaussian error, and correlation structure such that the effective rank of the resulting data is $\approx d - 1$. We increase the underlying noise in the data as $n$ increases such that the non-private confidence intervals are essentially constant across values of $n$. This allows us to better demonstrate the effects of changes in $k$ and $n$ on our algorithm’s performance. We then privately estimate the values of the $d$ coefficients and their associated standard errors. We run this entire experiment 100 times. We assume the sampling distribution of the coefficients is multivariate Gaussian and imagine that the user sets all upper bounds $\approx 100$ times larger than the tightest possible upper bounds. We present these results in Figure 2.

![Figure 2: OLS: Distribution of coefficient estimates and 95% confidence intervals.](image)

Each plot consists of coefficient estimates centered around their true values and presented in increasing order, with vertical bars representing the 95% confidence interval for that estimate.
We expect properly calibrated confidence intervals to cross the x-axis at the vertical dotted black lines, placed at the 2.5\textsuperscript{th} and 97.5\textsuperscript{th} quantiles, which is the behavior we observe in each plot.

Additionally, Figure 2 demonstrates the principle that the noise due to privacy in our algorithm scales with $k$ rather than $n$. The private confidence intervals are significantly tighter in plot (b) than in plot (a), while plots (c) and (d) are essentially identical.

One complicating factor to this story is that plot (c) looks better than plot (b), even though they use the same $k$. This is because the variance of the BLB estimates will tend to decrease as $\frac{n}{k}$ increases, up to the point where $\frac{n}{k}$ is large enough that the BLB estimates have converged to the sampling distribution of the estimator. Figure 3 shows the distribution of the BLB estimates of two of our estimated coefficients at the different levels of $\frac{n}{k}$ used in plots (b) and (c).

![Figure 3: OLS: BLB estimates from a single run](image)

Ideally, the analyst should attempt to choose $k$ to be as large as possible, subject to the constraint that the BLB estimates, calculated on subsets of size $\frac{n}{k}$, converge to the true sampling distribution of the estimator.

One possible strategy for doing this is for the analyst to test the BLB estimation for their estimator of interest on non-sensitive data and use a $k$ that performs well on the non-sensitive data. In particular, an analyst could start with $k = 1$ and increase $k$ until the distribution of BLB estimates starts to look substantially different (i.e. different beyond Monte Carlo error). Of course, this approach works only if the analyst can find or generate non-sensitive data that are similar enough to their sensitive data that the BLB will perform similarly on each. We discuss the issue further in Section G.4.

### 3.1 Comparison with UnbiasedPrivacy (Evans et al., 2019)

In Table 1, we compare our algorithm to the UnbiasedPrivacy (UP) algorithm from Evans et al. (2019). UP is designed for estimators with a univariate Gaussian sampling distribution, so we focus on that setting here.

While the goal of GVDP is to let users set very conservative bounds (which the algorithm improves upon) and not clip any points in the aggregation step, UP requires that users set reasonably good bounds upfront. UP never tightens bounds that are too loose, but it will attempt to bias-correct the results if the analyst’s chosen bounds clip some of the data (which our algorithm does not).

We test UP against GVDP across five scenarios using the implementation provided in the original paper. As suggested in Evans et al. (2019), we split the privacy budget evenly between the mean estimation task and the estimation of the proportion of points that are clipped. We consider two cases in which we expect UP to perform bias correction, clipping the top 20% and
Table 1: Comparison of UP and GVDP: Average Coefficient Estimate, Average Standard Error, and Empirical Coverage of 95% Confidence Intervals

| Bounds                        | Method | Avg Coef Err | Avg SE  | CI Cov |
|-------------------------------|--------|--------------|---------|--------|
| Top 20% Clipped               | UP     | 0.222        | 0.0872  | 0.237  |
|                               | GVDP   | 0.007        | 0.207   | 0.970  |
| Top 10% Clipped               | UP     | 0.080        | 0.108   | 0.932  |
|                               | GVDP   | 0.003        | 0.207   | 0.970  |
| Tightest bounds (no clipping) | UP     | 0.129        | 0.145   | 0.973  |
|                               | GVDP   | 0.001        | 0.208   | 0.975  |
| 3 times too large             | UP     | 0.004        | 0.370   | 0.968  |
|                               | GVDP   | 0.004        | 0.218   | 0.973  |
| 1,000 times too large         | UP     | 3.239        | 121.090 | 0.950  |
|                               | GVDP   | 0.005        | 0.701   | 0.961  |

10% of the BLB estimates, and three in which we don’t; clipping bounds set as tightly as possible with no clipping, bounds set three times larger than the true values of the parameter, and bounds set 1,000 times larger than the true value.

We run 1,000 simulations, each of which involves generating \( n = 50,000 \) data points \( Y_i \sim N(0, 250) \) and using UP and GVDP to generate a private OLS estimator, using \( k = 500 \) as the number of subsets for the BLB for each method. In Table 1, we provide the average \( \ell_1 \) error in the coefficient estimate, average standard error, and empirical 95% confidence interval coverage for each method with various levels of clipping bound.

In the 10% clipping and tight bound (no clipping) settings, UP mostly delivers as advertised; it gives (approximately) valid confidence intervals and does so with smaller standard errors than GVDP does under any bounds. However, it is unable to achieve this when the top 20% of BLB estimates are clipped, yielding highly biased coefficient estimates and poor CI coverage. Moreover, even in the 10% and tight bound settings, UP does not appear to achieve truly unbiased coefficient estimates. We believe this is because of the error in the bias correction step of UP, created when privately estimating the proportion of clipped data points.

In the cases where the bounds are too conservative, we see that GVDP outperforms UP, as GVDP is designed to improve conservative bounds whereas UP is not. It’s notable that in the 10% and 20% clipping where GVDP gives no guarantees, it appears to provide unbiased coefficient estimates and valid CI coverage. This is because of GVDP’s variance estimates are conservative by design (to ensure valid coverage with high probability), so even when the initial data bounds are set too narrowly it is possible that GVDP’s overly conservative variances compensate inside of CoinPress such that no BLB estimates end up being clipped.

4 Discussion

We believe that whether or not our method (GVDP) is effective relative to other approaches will generally come down to a few different factors.

First, we suggest that GVDP be considered primarily when the analyst is not confident in their ability to set “good” bounds on their underlying data domain \( \mathcal{X} \) for their given estimator. This is a function of both analyst knowledge of \( \mathcal{X} \) and the properties of their estimator, as some estimators will be robust even if the analyst sets bounds that clip small proportions of the data, while others will not be (see our demonstration in Section 1.2). Second, GVDP is likely to work well only for reasonably large \( n \). Recall that GVDP partitions the data \( X \) into \( k \) subsets of size \( \frac{n}{k} \) and bootstrapping the estimator over each subset. This creates a tradeoff between the plausibility of our assumptions and the required noise addition to satisfy...
DP. As $k$ increases, the sensitivity of our aggregator decreases and so too does the variance of the noise in our privacy mechanism.

However, we require that the BLB estimates be good estimates of the sampling distribution of our estimator in the non-private setting (Assumption 2.4). Similar to other bootstrap methods, the BLB’s guarantees are asymptotic (see Section 3 of Kleiner et al. (2014)), and so it is difficult to know how reasonable Assumption 2.4 is when $\frac{n}{k}$ is small.

Finally, we believe GVDP can be useful in settings where the dimension of the estimand of interest is significantly lower than the dimension of the data. Higher dimensional domains create two potential problems for differentially private estimation that are not present in non-private estimation. First, setting clipping bounds, generally speaking, gets more difficult as the dimension increases. Second, the function sensitivity, and thus variance of the noise in our privacy mechanism, increases with the dimension of the function’s input domain (again see Lemma A.4). The function being privatized in GVDP takes the BLB parameter estimates as input rather than the full data, so these two problems are minimized if the estimand is low-dimensional relative to the original data.

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Appendices

A Definitions

A.1 Differential Privacy

We begin with an introduction to the core definitions of DP.

**Definition A.1** (Neighboring data sets). Let $\mathcal{X}$ be a data universe and $D, D' \in \mathcal{X}^n$. We say that $D, D'$ are neighboring if

$$\max(|D \setminus D'|, |D' \setminus D|) = 1.$$ 

We also define the set of all neighboring data sets as

$$\mathcal{D}_n = \{(D, D') \in \mathcal{X}^n \times \mathcal{X}^n : D, D' \text{ are neighbors}\}.$$ 

**Definition A.2** (Rényi divergence (Rényi, 1961)). Let $P, Q$ be probability measures over a measurable space $(\Omega, \Sigma)$. Then we define the $\alpha$-Rényi divergence between $P, Q$ as

$$H_\alpha (P||Q) = \frac{1}{\alpha - 1} \ln \int_{\Omega} P(x)P(x)^{1-\alpha} dx.$$ 

**Definition A.3** (Global Function Sensitivity). Let $X$ be a data domain, $\gamma : X^n \to \mathbb{R}^d$, and $\mathcal{D}_n$ be the set of all neighboring data sets as in Definition A.1. Then we write the global sensitivity of $\gamma$ with respect to a distance metric $d$ as

$$GS_d(\mathcal{X}^n, \gamma) = \max_{D, D' \in \mathcal{D}_n} d(\gamma(D), \gamma(D')).$$

Algorithms can be made to respect DP in a variety of ways, but the most common way (as well as the approach we use in this work) is via an additive noise mechanism. This just entails running the algorithm as one would normally, and then adding random noise scaled relative to the algorithm’s sensitivity.

Throughout this work, we use a popular additive noise mechanism called the Gaussian mechanism.

**Lemma A.4** (Gaussian Mechanism). Let $f : \mathcal{X}^n \to \mathbb{R}^d$ have global $\ell_2$ sensitivity $GS_{\ell_2}(\mathcal{X}^n, f)$. Then the Gaussian mechanism

$$\mathcal{M}_f(D) = f(D) + N\left(0, \left(\frac{GS_{\ell_2}(\mathcal{X}^n, f)}{\sqrt{2\rho}}\right)^2 I_d\right)$$

satisfies $\rho$-zCDP.

Note that it is often necessary to bound the data domain $\mathcal{X}$ to ensure that $GS_{\ell_2}(\mathcal{X}^n, f) < \infty$. For example, let $\mathcal{X} = \mathbb{R}^d$, $D = (D_1, \ldots, D_n)$ with $D_i \in \mathcal{X}$, and $f : \mathbb{R}^{n \times d} \to \mathbb{R}^d$ be such that $f(D) = n^{-1} \sum_{i=1}^n D_i$. If we let $D' = (\infty, D_2, \ldots, D_n)$, then $D, D'$ are neighbors (they differ only in the first element), but $\|f(D) - f(D')\|_2 = \infty$. If instead $\mathcal{X} = [0, 1]^d$, then the $D, D'$ that induce the largest difference in $f$ are $D = (\bar{1}, D_2, \ldots, D_n)$ and $D' = (\bar{0}, D_2, \ldots, D_n)$. In this scenario, $\|f(D) - f(D')\|_2 = \|n^{-1}(\bar{1} - \bar{0})\|_2 = n^{-1/2}$, and thus $GS_{\ell_2}(\mathcal{X}^n, f) = n^{-1/2}$.

These bounds must be set without looking at the particular $D_i$, and are generally chosen by a data analyst based on public metadata and/or their beliefs about the data-generating process.
A.2 Statistical Inference

This need to bound \( X \) introduces complications for doing statistical inference under DP, while maintaining the types of guarantees we often want from non-private estimators. We focus specifically on unbiased estimators and valid confidence sets.

**Definition A.5 (Unbiased Estimator).** Let \( \theta \in \mathbb{R}^d \) be a model parameter we wish to estimate. We collect data \( D \sim \mathcal{D} \) and estimate \( \theta \) with a random variable \( \hat{\theta} : \mathcal{D} \to \mathbb{R}^m \). We say that \( \hat{\theta} \) is an unbiased estimator of \( \theta \) if

\[
E(\hat{\theta}(D)) = \theta,
\]

with randomness taken over the sampling of \( D \sim \mathcal{D} \), as well as any other randomness in \( \hat{\theta} \).

Many applied statisticians, particularly those interested in estimating causal effects using linear models, prize unbiased parameter estimation and are willing to sacrifice on other fronts to achieve it. For example, the standard OLS estimator (which is the minimum-variance unbiased estimator under the assumptions of the Gauss-Markov theorem) is used for estimating parameters of a linear regression model in favor of other biased estimators, such as the James-Stein estimator (Stein, 1956; Stein and James, 1961), which dominate it in terms of \( \ell_2 \) error of the parameter estimates.

**Definition A.6 (Confidence Set).** Let \( \theta \in \mathbb{R}^d \) be a model parameter we wish to estimate using data \( D \sim \mathcal{D} \). For arbitrary \( \alpha \in [0, 1] \), a \((1 - \alpha)\)-level confidence set for \( \theta \) is a random set \( S \subseteq \mathbb{R}^d \) such that

\[
P(\theta \in S) = 1 - \alpha,
\]

with randomness taken from the sampling of \( D \) and any other randomness in the construction of \( S \).

Ideally, we would be able to find a perfectly-calibrated confidence set, where the coverage probability (i.e. \( P(\theta \in S) \)) is exactly \( 1 - \alpha \). However, this is often impossible to compute exactly and so practitioners tend to default to being overly conservative instead. In this setting, we require \( P(\theta \in S) \geq 1 - \alpha \) and call such an \( S \) a valid confidence set. In this work, we focus on confidence regions, which are contiguous confidence sets, and occasionally confidence intervals, which are univariate confidence regions.

We can simplify the general problem of constructing confidence sets by restricting our attention to estimators whose sampling distribution belongs to a symmetric multivariate location-scale family.

**Definition A.7 (Location-Scale Family).** A set of probability distributions is a location-scale family if any density \( f(x; \mu, \Sigma) \) in the set is written as

\[
f(x; \mu, \Sigma) = c|\Sigma|^{-1/2} \exp \left( -\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu) \right),
\]

for some normalization constant \( c \).

Our restriction to location-scale families ensures that estimating the mean and (co)variance of the estimator is sufficient to characterize its distribution.

B Step 1: Bag of Little Bootstraps

This algorithm statement is adapted and simplified for our purposes; readers interested in the original version should consult Kleiner et al. (2014). We say that \( \mathcal{X} \) is our data universe, \( \mathcal{D} \) is a distribution over \( \mathcal{X} \), and our realized data \( X \in \mathbb{R}^{n \times m} \) are drawn from \( \mathcal{D}^n \). For an arbitrary estimator \( \hat{\theta} : \mathcal{X}^n \to \mathbb{R}^d \), we define \( \hat{\theta}(\mathcal{D}) = E_{X \sim \mathcal{D}^n}(\hat{\theta}(X)) \).
Algorithm 2 Bag of little bootstraps (BLB)

Input: data set $X \in \mathbb{R}^{n \times m}$, estimator $\hat{\theta} : X^n \rightarrow \mathbb{R}^d$, estimator quality assessment $\xi$, $k$ number of subsets of partition, $r$ number of bootstrap simulations

Output: $k$ estimates of $\hat{\theta}(D)$

1: procedure BLB($X, \hat{\theta}, k, r$)
2: Randomly partition $X$ into $k$ subsets $\{X_i\}_{i \in [k]}$
3: for $i \in [k]$ do
4: $b = |X_i|$
5: $\{\hat{\theta}_{i,c}\}_{c \in [r]} = \emptyset$
6: for $c \in [r]$ do
7: sample $(n_1, \ldots, n_b) \sim \text{Multinomial}(n, 1/b)$
8: create $X_i^c \in \mathbb{R}^{n \times m}$ by including the $j^{th}$ element of $X_i$ $n_j$ times
9: $\hat{\theta}_{i,c} = \hat{\theta}(X_i^c)$
10: $\hat{\theta}_i = \xi(\{\hat{\theta}_{i,c}\}_{c \in [r]})$
11: return $\{\hat{\theta}_i\}_{i \in [k]}$

C Step 2: Differentially Private Estimation

Definition C.1. Let $\mathcal{B}(\mu, \Sigma)$ and $\mathcal{C}(\mu, \Sigma)$ be families of distributions and $B, C$ be random variables drawn from each such that $\mathbb{E}(B) = \mathbb{E}(C) = \mu$ and Cov$(B) = \text{Cov}(C) = \Sigma$. Let PSD$_d$ be the set of all $d \times d$ PSD matrices. We say that $\mathcal{B}$ is heavier-tailed than $\mathcal{C}$ if for all $\mu \in \mathbb{R}^d, \Sigma \in \text{PSD}_d$, and $v \in \mathbb{R}^d$ such that $\|v\|_2 = 1$, then

$$\mathbb{P}(v^T(B - \mu) \leq z) \leq \mathbb{P}(v^T(C - \mu) \leq z),$$

for all $z > 0$.

C.1 Modified CoinPress Algorithm

Algorithm 3 Modified CoinPress

Input: $X = (x_1, \ldots, x_k)$ from a distribution $D$ with mean $\mu$ and covariance $\Sigma$, $\hat{\Sigma}$ such that $\Sigma \preceq \hat{\Sigma}, B_2(\mu_0, r_0)$ containing $\mu$, family of distributions $Q_X(\cdot, \Sigma)$ with heavier tails than $D$, number of iterations $t \in \mathbb{N}^+$, $z$CDP privacy losses parameter $\rho > 0$, failure probability $\beta > 0$

Output: $t$ estimates of $\mu$ that jointly respect $\rho$-zCDP

1: procedure MVMRec($X, \mu_0, r_0, \hat{\Sigma}, Q, t, \rho, \beta$)
2: $S = \Sigma^{1/2}$
3: $\tilde{\mu}_0 = S^{-1}\mu_0$
4: $r_0 = \max(\text{diag}(S^{-1})) \cdot r_0$
5: Define $\tilde{X} \in \mathbb{R}^{k \times d}$ such that $\forall j \in [d], \forall m \in [k]: \tilde{X}_{m,j} = \frac{1}{k} \sum_{m' = 1}^{k} x_{m',j}$. Note that each row $\tilde{X}_{m,:}$ is equal to the $d$-dimensional empirical mean of $X$
6: $X' = (X - \tilde{X}) S^{-1}$
7: for $m \in [t - 1]$ do
8: $(\mu_m, r_m, \sigma_m) = \text{MVM}(X', \tilde{\mu}_{m-1}, r_{m-1}, Q_X(0, I_d), \frac{\rho}{(t-1)!}, \frac{\beta}{T})$ \hspace{1cm} $\triangleright$ Algorithm 4
9: $(\tilde{\mu}_t, r_t, \sigma_t) = \text{MVM}(X', \tilde{\mu}_{t-1}, r_{t-1}, Q_X(0, I_d), \frac{\rho}{2}, \frac{\beta}{T})$ \hspace{1cm} $\triangleright$ convert mean estimates to proper scale
10: $\forall m \in [t]: \mu_m = (S\tilde{\mu}_m) + \tilde{\mu}_1$\hspace{1cm} $\triangleright$ convert private noise variances to proper scale
11: $\forall m \in [t]: \sigma_m^2 = \text{diag}(S\sigma_m)^2$
12: return $\{\hat{\mu}_m, \sigma_m^2\}_{m \in [t]}$
C.2 Modified CoinPress Algorithm - One Step Improvement

Algorithm 4 One Step Private Improvement of Mean Ball

Input: $x = (x_1, \ldots, x_k)$ from a distribution with mean 0 and covariance with smaller Löwner order than $I_d$, $B_2(\mu, r)$ containing 0, family of distributions $Q_X (\cdot, I_d)$, zCDP privacy loss parameter $\rho_m > 0$, failure probability $\beta_m > 0$

Output: A $\rho$-zCDP ball $B_2(\mu', r')$ and scale of the privatizing noise $\sigma$

\[
\begin{align*}
1: & \text{procedure MVM}(M, \mu, r, Q_X, \rho_m, \beta_m) \\
2: & \beta_s = \beta_m / 2 \\
3: & \text{Let } R \sim Q_X (0, I_d) \\
4: & \text{Set } \gamma_1 \text{ such that } \mathbb{P} (\|R\|_2 > \gamma_1) \leq \frac{\beta_s}{k} \\
5: & \text{Set } \gamma_2 \text{ such that } \mathbb{P} (\|R\|_2 > \gamma_2) \leq \beta_s \\
6: & \text{Project each } x_i \text{ into } B_2(\mu, r + \gamma_1). \\
7: & \Delta = 2(r + \gamma_1)/k. \\
8: & \sigma = \frac{\sqrt{2\rho_s}}{k}. \\
9: & \text{Compute } \tilde{\mu}' = \frac{1}{k} \sum_{i} x_i + Y, \text{ where } Y \sim N (0, \sigma^2 I_d). \\
10: & r' = \gamma_2 \sqrt{\frac{1}{k} + \frac{2(r + \gamma_1)}{k\rho_s}}. \\
11: & \text{return } (\mu', r', \sigma).
\end{align*}
\]

C.3 Privacy Analysis of Algorithm 3

Theorem C.2 (Modified CoinPress Privacy Statement). Algorithm 3 produces $t$ estimates of $\mu$ that jointly respect $\rho$-zCDP.

Proof. Algorithm 3 begins and ends by scaling the data to have empirical mean 0 and covariance which is Löwner upper bounded by $I_d$. The covariance scaling parameter is chosen independently of the data and the rest of the steps in the algorithm are invariant under location shift. So, our privacy analysis rests on the application of Algorithm 4 in lines 8 and 9 of Algorithm 3.

Algorithm 4 interacts with the raw data only in line 9, so satisfying DP reduces to correct specification of $\Delta$ (the $\ell_2$ sensitivity of the mean) and application of the Gaussian mechanism. The data are projected into $B_2(\tilde{\mu}, r + \gamma_1)$, and so the most a single data point can be changed in $\ell_2$ norm is $2(r + \gamma_1)$. Because neighboring data sets $X, Y$ differ in only one point (call it $z$), the $\ell_2$ norm of the $k - 1$ other points remains the same and so

\[
\left\| \frac{1}{k} \sum_{x \in X} x - \frac{1}{k} \sum_{y \in Y} y \right\|_2 = \left\| \frac{1}{k} z \right\|_2 = \frac{1}{k} \| z \|_2 \leq \frac{2(r + \gamma_1)}{k}
\]

as desired. Thus, each step of CoinPress satisfies zCDP at the stated level of its privacy parameter $\rho$. For each step $m \in [t - 1]$, we see in line 8 that the privacy parameter is $\frac{\rho}{2(t-1)}$. For step $t$, we see in line 9 that the privacy parameter is $\frac{\rho}{2}$. Because zCDP parameters compose additively, the zCDP parameter for the entire CoinPress algorithm is $(t - 1) \frac{\rho}{2(t-1)} + \frac{\rho}{2} = \rho$. \qed

C.4 Proof of Theorem 2.8

Proof. We start with Assumption 2.6 so we have $\mu \in B_2 (\tilde{\mu}_0, r_0)$. Note that the clipping bounds, parameterized by $\gamma_1$, in line 4 of Algorithm 4 are set such that, with probability $1 - \beta_s$, no points are affected by the bounding; this follows because any given point is affected only if it falls outside the clipping ball, which occurs with probability $\leq \frac{\beta_s}{k}$ and so, by the union bound, every point is unaffected with probability $\geq 1 - \beta_s$. Thus, with probability $\geq 1 - \beta_s$:

\[
\mu' \sim \frac{1}{k} \sum_{i} \tilde{\mu}_i + Y
\]
\[ \sim \hat{\mu} + Y \quad \text{(definition of } \hat{\mu} \text{)} \]
\[ \sim N \left( \hat{\mu}, \sigma^2 I_d \right). \]

We now consider \( \gamma_2 \), which is set as a \( 1 - \beta_s \) probability upper bound on the \( \ell_2 \) norm of the privatized mean of \( k \) draws from \( Q(0, \Sigma) \). Conditional on no points being clipped so that \( \tilde{\mu}' = \sum_{i=1}^{k} \hat{\mu}_i + Y \), we have

\[
1 - \beta_s \leq \mathbb{P} \left( \left\| \frac{1}{k} \sum_{i=1}^{k} \hat{\mu}_i - \mu + Y \right\|_2 \leq \gamma_2 \right)
\]
\[= \mathbb{P} \left( \| \tilde{\mu}' - \mu \|_2 \leq \gamma_2 \right). \tag{1}
\]

So, having \( \mu \in B_2(\tilde{\mu}_0, r_0) \) implies that \( \mathbb{P} (\mu \in B_2(\tilde{\mu}', r')) \geq 1 - 2\beta_s = 1 - \beta_m \). Using the fact that \( \sum_{m}^{t} \beta_m = \beta^m \) and a union bound, we proceed by induction over the \( t \) steps of the algorithm and see that with probability \( 1 - \beta^m \) we have

\[ \forall m \in [t] : \mu \in B_2(\tilde{\mu}_m, r_m) \]

and

\[ \forall m \in [t] : \mu'_m \sim N \left( \hat{\mu}_m, \sigma_m^2 I_d \right). \]

Scaling \( \mu'_m, \sigma_m^2 \) back up as in Lines 10 and 11 give the desired result.

**C.5 Setting \( \gamma_1, \gamma_2 \)**

This section is concerned with how to set \( \gamma_1, \gamma_2 \) in lines 4, 5 of Algorithm 4 for various \( Q_{\tilde{\mu}} \). We start with a general statement that works for arbitrary \( Q_{\tilde{\mu}} \).

**Fact C.3 (Chebyshev’s Inequality).** If \( X \) is a \( d \)-dimensional random vector with expected value \( \mu = \mathbb{E}(X) \) and covariance \( \Sigma = \mathbb{E} \left( (X - \mu)(X - \mu)^T \right) \), then

\[ \mathbb{P} \left( \sqrt{(X - \mu)^T \Sigma^{-1} (X - \mu)} > t \right) \leq \frac{d}{t^2}, \]

provided that \( \Sigma \) is positive definite.

**Corollary C.4.** For any \( R \) in Algorithm 4, \( \mathbb{P} \left( \| R \|_2 > \sqrt{d/\beta} \right) \leq \beta. \)

**Proof.** By construction of \( R \), we know that \( \mu = 0 \) and \( \Sigma = I_d \). Let \( R_j \) be the \( j \)th element of \( R \). Then we can write

\[ \sqrt{(R - \mu)^T \Sigma^{-1} (R - \mu)} = \sqrt{R^T R} = \left( \sum_{j=1}^{d} R_j^2 \right)^{1/2} = \| R \|_2 \]

We can set \( t = \sqrt{d/\beta} \) and rewrite Chebyshev’s Inequality as

\[ \mathbb{P} \left( \| R \|_2 > \sqrt{d/\beta} \right) \leq \beta. \]

\]
In practice, it is beneficial to set tighter bounds based on the specified $Q_{\tilde{\mu}}$. This can hypothetically be done via Monte Carlo sampling and empirical CDF inequalities. However, this can be computationally expensive for $\gamma_1$ in particular, as you need at least $k/\beta$ draws (and often far more) from the random variable to get a proper upper bound.

Some $Q_{\tilde{\mu}}$ also admit analytical bounds, which avoid the need for the costly computation. If $Q_{\tilde{\mu}}$ is multivariate Gaussian, we can use the following:

**Fact C.5** (Lemma 1 of Laurent and Massart (2000)). Let $Q_{\tilde{\mu}}$ be multivariate Gaussian such that $Q_{\tilde{\mu}}(\mu, \Sigma) = N(\mu, \Sigma)$. Then if $R \sim Q_{\tilde{\mu}}(0, I_d)$, we know that

$$
\forall \beta \in (0, 1] : \Pr \left( \|R\|_2 > \sqrt{d + \sqrt{d \log(1/\beta)} + 2 \log(1/\beta)} \right) \leq \beta.
$$

We present a similar bound for when $Q_{\tilde{\mu}}$ is multivariate Laplace, based heavily on a result from Corollary 3.1 from Vladimirova et al. (2020).

**Theorem C.6.** Let $Q_{\tilde{\mu}}$ be multivariate Laplace with mean $\mu = 0$ and covariance $\Sigma = I_d$. Then if $R \sim Q_{\tilde{\mu}}(0, I_d)$, we know that

$$
\forall \beta \in (0, 1] : \Pr \left( \|R\|_2 > \sqrt{e \cdot d \log(\beta)} \right) \leq \beta,
$$

where $e \approx 2.718$ is Euler’s number.

*Proof.* We start by noting that $\|R\|_2 = \left( \sum_{j=1}^d R_j^2 \right)^{1/2}$. We know that the $R_j$ are Laplace with mean 0 and variance 1, and thus $\forall j \in [d] : R_j^2 \sim \text{Weibull}(\lambda = 1/2, k = 1/2)$. For ease of notation, we’ll call $X_j = R_j^2$.

We now define sub-Weibull random variables, as is done in Vladimirova et al. (2020). We call a random variable $X_j$ sub-Weibull with tail parameter $\theta$ if there exists $\theta, a, b > 0$ such that

$$
\forall x > 0 : \Pr (|X_j| \geq x) \leq a \exp \left( -bx^{1/\theta} \right).
$$

For context, sub-Gaussian random variables are sub-Weibull with $\theta = 1/2$, sub-Exponentials are sub-Weibull with $\theta = 1$, and Weibull random variables themselves are sub-Weibull with $\theta = 2$.

We can state an alternative condition, also from Vladimirova et al. (2020), that $X_j$ is sub-Weibull with tail parameter $\theta$ if

$$
\exists c > 0 \text{ s.t. } \forall t \geq 1 : \|X_j\|_t \leq ct^\theta.
$$

Our goal is to find the smallest $c$ that holds for Weibull random variables in particular. We recall that $X_j \sim \text{Weibull}(\lambda = 1/2, k = 1/2)$ and $\theta = 2$. Thus, for all $t \geq 1$:

$$
\|X_j\|_t \leq ct^\theta
\iff (\mathbb{E}(|X_j|^t))^{1/t} \leq ct^\theta
\iff \lambda \Gamma \left( \frac{t}{k} + 1 \right)^{1/t} \leq ct^\theta
\iff \frac{1}{2t^2} \Gamma(2t + 1)^{1/t} \leq c.
$$
Line 3 follows by using the MGF of a Weibull random variable, and line 4 follows by plugging in the parameter values.

Our goal is to find the smallest $c$ such that $\|X\|_t \leq ct^\theta$ for all $t \geq 1$. The lefthand side of line 4 is decreasing in $t$ for $t \geq 1$, so finding the smallest possible $c$ for $t = 1$ will be sufficient for all $t \geq 1$. Plugging in $t = 1$, we get $c = 1$.

We can finally appeal to Corollary 3.1 from Vladimirova et al. (2020), which states that if $X_1, \ldots, X_d$ are i.i.d. Weibull random variables with tail parameter $\theta$, then for all $x \geq dK_\theta$ we have

$$P\left(\sum_{j=1}^d X_j \geq x\right) \leq \exp\left(-\left(\frac{x}{K_\theta d}\right)^{1/\theta}\right)$$

for $K_\theta = ec$. Plugging in the $c = 1$ we found for Weibull random variables yields

$$P\left(\sum_{j=1}^d X_j \geq x\right) \leq \exp\left(-\left(\frac{x}{e \cdot d}\right)^{1/\theta}\right).$$

We want the probability to be less than $\beta$, so we sub this in and get

$$P\left(\sum_{j=1}^d X_j \geq e \cdot d \log^2(\beta)\right) \leq \beta.$$

We note that $\|X\|_2 = \sqrt{\sum_{j=1}^d X_j}$, so setting the bound at $\sqrt{e \cdot d \log^2(\beta)}$ gives our desired result.

\[\Box\]

C.6 Trick for setting $\hat{\Sigma}$ for $\hat{\theta}^{BLB}$ estimation

In our GVDP algorithm, we independently estimate the means of both the $\{\hat{\theta}_{iBLB}\}_{i \in [k]}$ and $\{\hat{\Sigma}_{iBLB}\}_{i \in [k]}$, each requiring (among other things) that the analyst specify $\hat{\Sigma}$, a Löwner upper bound on the sample covariance of the BLB samples. If we estimate the mean of $\{\hat{\Sigma}_{iBLB}\}_{i \in [k]}$ and do our post-processing to find a private covariance estimate $\hat{\Sigma}$ prior to estimating the mean of $\{\hat{\theta}_{iBLB}\}_{i \in [k]}$, we can actually leverage some extra information that will generally improve our estimates with a small cost to the theoretical guarantee. All the experimental results in the paper use this trick.

Although we are scaling up our subsets to the original data size within the BLB to get correct overall covariance estimates, this does not imply that the covariance of the $\{\hat{\theta}_{iBLB}\}_{i \in [k]}$ match this correct scaling. In fact, this covariance will often be roughly the same if $\hat{\theta}$ were simply run on subsets of size $\frac{n}{k}$. So, the covariance of the $\{\hat{\theta}_{iBLB}\}_{i \in [k]}$ should be roughly $\frac{r(n/k)}{r(n)}\hat{\Sigma}$, where $r$ is the convergence rate of the estimator in question. For example, the covariance of OLS coefficients decays with $\frac{1}{n}$, so if $\hat{\theta}$ represents OLS estimation we would say the covariance is $\frac{1/(n/k)}{1/n}\hat{\Sigma} = k\hat{\Sigma}$. We upper bound this with $k\hat{\Sigma}$.

Under Assumption 2.4, this strategy gives us a $1 - \beta\hat{\Sigma}$ probability guarantee that $k\hat{\Sigma}$ will Löwner upper bound the empirical covariance of the $\{\hat{\theta}_{iBLB}\}_{i \in [k]}$. So, by using $k\hat{\Sigma}$ as the upper covariance bound for our mean estimation for $\{\hat{\theta}_{iBLB}\}_{i \in [k]}$, we generally start with a pretty tight bound and can dramatically improve the accuracy of our estimates. This does lose a bit of theoretical strength in the results; we generally assume that the analyst’s upper bound is an actual upper bound on the empirical covariance with probability 1, whereas this trick provides a guarantee with probability $1 - \beta\hat{\Sigma}$. 

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Figure 4: Distribution of coefficient estimates and 95% confidence intervals for $k = 5,000, d = 10, \rho = 0.1$ for multivariate Laplace distribution

C.7 Generalizing CoinPress beyond multivariate sub-Gaussians

In Figure 4 we provide evidence that our generalization of CoinPress beyond sub-Gaussian distributions delivers on its promises. We pretend as if the estimator and data were such that the distributions induced by the BLB were are dominated by the multivariate Laplace (i.e. they are sub-Exponential), and the analyst overestimated the relevant parameters by a factor of 100. We show results corresponding to two different methods for calculating the clipping parameters at each step of CoinPress. The analytic solution calculates the bound using a theoretical bound given in Theorem C.6, while the approximate solution calculates an approximate upper bound using Monte Carlo sampling.

D Step 3: Postprocessing

D.1 Proof of Theorem 2.9

Proof. Our goal is to find weights $\{A_m\}_{m \in [t]}$ with $A_m \in \mathbb{R}^{d \times d}$ such that the Löwner order of $\text{Cov} \left( \sum_{m=1}^{t} A_m \hat{\tau}_m \right)$ is minimized. Because we want our weighted estimator to remain unbiased, we restrict ourselves to sets of $A_m$ such that $\sum_{m=1}^{t} A_m = I_d$.

We note that the $A_m$ are constants and $\hat{\tau}_m$ are independent, so

$$\text{Cov} \left( \sum_{m=1}^{t} A_m \hat{\tau}_m \right) = \sum_{m=1}^{t} \text{Cov} (A_m \hat{\tau}_m)$$

$$= \sum_{m=1}^{t} A_m^T \text{Cov} (\hat{\tau}_m) A_m.$$ 

Assume $\hat{\tau}_m \in \mathbb{R}^d$ and let $\{B_m\}_{m \in [t]}$ with $B_m \in \mathbb{R}^{d \times d}$ be an arbitrary weighting. Then we can write

$$\text{Cov} \left( \sum_{m=1}^{t} A_m \hat{\tau}_m \right) \preceq \text{Cov} \left( \sum_{m=1}^{t} B_m \hat{\tau}_m \right)$$

$$\iff \forall v \in \mathbb{R}^d \setminus \{0\} : v^T \text{Cov} \left( \sum_{m=1}^{t} A_m \hat{\tau}_m \right) v \leq v^T \text{Cov} \left( \sum_{m=1}^{t} B_m \hat{\tau}_m \right) v.$$
Note that the quantities on the righthand side of the statement above are scalars, so we have translated the problem of finding a minimal Löwner bound into minimizing a one-dimensional quantity.

Let \( v \in \mathbb{R}^d \setminus \{0\} \) be arbitrary. We now have a one-dimensional constrained optimization problem; we want to find \( \{A_m\}_{m \in [t]} \) which minimizes \( v^T \text{Cov} \left( \sum_{m=1}^t A_m \hat{\tau}_m \right) v \) subject to \( \sum_{m=1}^t A_m = I_d \). We can solve this using a Lagrange multiplier.

We write
\[
\mathcal{L} \left( \{A_m\}_{m \in [t]}, \lambda \right) = v^T \text{Cov} \left( \sum_{m=1}^t A_m \hat{\tau}_m \right) v - \lambda v^T \left( \sum_{m=1}^t A_m - I_d \right) v
\]
and differentiate with respect to \( A_m \). Recall that \( \text{Cov} (\hat{\tau}_m) = S_m \). Then we have
\[
\frac{\partial \mathcal{L} \left( \{A_m\}_{m \in [t]}, \lambda \right)}{\partial A_m} = \partial v^T \text{Cov} \left( \sum_{m=1}^t A_m \hat{\tau}_m \right) v - \lambda v^T \left( \sum_{m=1}^t A_m - I_d \right) v
\]
\[
= \frac{\partial}{\partial A_m} \left( \sum_{m=1}^t v^T A_m^T \text{Cov} (\hat{\tau}_m) A_m v \right) - \lambda v^T \left( \sum_{m=1}^t A_m - I_d \right) v
\]
\[
= S_m A_m v v^T + S_m^T A_m v v^T - \lambda vv^T
\]
\[
= 2 \left( S_m A_m - \lambda I_d \right) vv^T
\]
(5) comes from a matrix calculus identity that for vectors \( a, b \) and matrix \( C \) all independent of \( X \),
\[
\frac{\partial (Xa)^T C (Xb)}{\partial X} = CXba^T + C^T Xab^T
\]
and noting that the partial with respect to \( A_m \) influences the sum only in the \( m \)th term.

We set this to 0 to find a stationary point.
\[
0 = 2 \left( S_m A_m - \lambda I_d \right) vv^T
\]
\[
\lambda I_d vv^T = S_m A_m vv^T
\]
\[
A_m = \lambda S_m^{-1} I_d vv^T (vv^T)^{-1}
\]
\[
= \lambda S_m^{-1}.
\]

We know from our constraint that \( \sum_{m=1}^t A_m = I_d \), so
\[
\sum_{m=1}^t \lambda S_m^{-1} = I_d
\]
\[
\lambda = \left( \sum_{m=1}^t S_m^{-1} \right)^{-1},
\]
and thus our stationary point is achieved at \( A_m = \left( \sum_{m=1}^t S_m^{-1} \right)^{-1} S_m^{-1} \).

We have shown that choosing \( A_m \) in this way achieves a stationary point, but we want to show that it is a global minimum. For that, we need to check the second partial derivative test, which states that our stationary point is a global minimum if \( \frac{\partial^2 \mathcal{L} \left( \{A_m\}_{m \in [t]}, \lambda \right)}{\partial^2 A_m} \) is PD.

We first note that
\[
\frac{\partial^2 \mathcal{L} \left( \{A_m\}_{m \in [t]}, \lambda \right)}{\partial^2 A_m} = \frac{\partial}{\partial A_m} 2 \left( S_m A_m - \lambda I_d \right) vv^T
\]

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where $\otimes$ is the Kronecker product.

We know $vv^T$ is PD, because $\forall z \in \mathbb{R}^d \setminus \{0\}$ we get $z^Tvv^Tz = (z^Tv)(v^Tz) = (v^Tz)^T(v^Tz) > 0$. The strict inequality comes because we know that both $v$ and $z$ are non-zero. We know $S_m$ is PD by assumption and that, in general, if a matrix $Y$ is PD then so is $2Y$. Finally, the Kronecker product of PD matrices is also PD, so $2(vv^T) \otimes S_m$ is PD and our second partial derivative condition is met. So $L$ is convex and our local minimum is also a global minimum. Thus, our choice of $A_m$ achieves the $\text{Cov} \left( \sum_{m=1}^t A_m \hat{\tau}_m \right)$ with minimal Löwner order.

\[ \square \]

**D.2 Proof of Theorem 2.10**

**Proof.** From Theorem 2.8, we know that, with probability $\geq 1 - \beta^{\hat{\Sigma}}$:

\[ \forall m \in [t]: \hat{S}_m \sim N \left( \hat{S}^{BLB}, \sigma^{2}_{\Sigma,m} I_d \right), \]

where $\hat{S}^{BLB}$ is the flattened form of $\hat{\Sigma}^{BLB}$. Assumption 2.4 then let’s us substitute in $\hat{S}$ for $\hat{S}^{BLB}$.

For the rest of the proof, we assume that this condition is met.

Thus, by Theorem 2.9, we know that a precision-weighted $\hat{S}$ will have mean $E[\hat{S}] = \hat{S}$ and covariance $\text{Cov} \left( \hat{S} \right) = \left( \Sigma_{m=1}^{t} \sigma_{\Sigma,m}^{2} I_d \right)^{-1}$. Moreover, this $\hat{S}$ is itself multivariate Gaussian because it is a linear combination of multivariate Gaussians. That is, we can write

\[ \hat{S} \sim N \left( \hat{S}, \left( \sum_{m=1}^{t} \sigma_{\Sigma,m}^{2} I_d \right)^{-1} \right) = N \left( \hat{S}, \sigma_{\hat{S}}^{2} I_d \right). \]

Let $\hat{\Sigma}'$ be the unflattened matrix constructed from $\hat{S}$. Then we can write $\hat{\Sigma}'_{i,j} \sim N \left( \hat{\Sigma}_{i,j}^{u}, b_{i,j}^{2} \right)$, where $b_{i,j} = \text{unflatten} \left( \hat{\Sigma}_{i,j}^{u} \right)$. Then, by Theorem 1.1 from Bandeira and Van Handel (2016), we know that

\[ E\| \hat{\Sigma}' - \hat{\Sigma} \|_2 \leq (1 + \varepsilon) \left( 2 \max_{i \in [d]} b_{i,i} \| b_{i,i} \|_2 + \frac{6 \sqrt{\log d}}{\log (1 + \varepsilon)} \max_{i,j \in [d] \times [d]} | b_{i,j} | \right), \]

for any $\varepsilon \in (0, 1/2]$, where $\| . \|_2$ is the spectral norm. Moreover, by Corollary 3.9 from Bandeira and Van Handel (2016) we have that, for any $t \geq 0$:

\[ \| \hat{\Sigma}' - \hat{\Sigma} \|_2 \leq (1 + \varepsilon) \left( 2 \max_{i \in [d]} b_{i,i} \| b_{i,i} \|_2 + \frac{6 \sqrt{\log d}}{\log (1 + \varepsilon)} \max_{i,j \in [d] \times [d]} | b_{i,j} | \right) + t \]

with probability $\geq 1 - \exp \left( \frac{-t^2}{4 \max_{i,j} b_{i,j}^2} \right)$. Setting $t = \sqrt{\frac{\ln (1/\beta^{ub})}{4 \max_{i,j} b_{i,j}^2}}$ yields a $1 - \beta^{ub}$ probability bound.

Now define

\[ c = \min_{\varepsilon \in (0, 1/2]} \left( 1 + \varepsilon \right) \left( 2 \max_{i \in [d]} b_{i,i} \| b_{i,i} \|_2 + \frac{6 \sqrt{\log d}}{\log (1 + \varepsilon)} \max_{i,j \in [d] \times [d]} | b_{i,j} | \right) + \sqrt{\frac{\ln (1/\beta^{ub})}{4 \max_{i,j} b_{i,j}^2}}. \]

The spectral norm $\| . \|_2$ of a matrix is its largest singular value (or equivalently, the square root of the absolute value of its largest magnitude eigenvalue). So, if $\| \hat{\Sigma}' - \hat{\Sigma} \|_2 \leq c$, we know that the smallest eigenvalue of $\hat{\Sigma}' - \hat{\Sigma}$ is necessarily at least $-c$. Therefore, the smallest eigenvalue of $\hat{\Sigma}' + cI_d - \hat{\Sigma}$ is at least 0 or, equivalently, $\hat{\Sigma}' + cI_d \succeq \hat{\Sigma}$. This statement holds with probability $1 - \beta^{ub}$. Combining this with the initial $1 - \beta^{\hat{\Sigma}}$ probability guarantee on the form of our estimator completes the proof.

\[ \square \]
The statement for \( c \) simplifies significantly in the case where the \( \{\tilde{\Sigma}_m\}_{m\in[t]} \) are diagonal matrices (which occurs if we care only about confidence intervals for each parameter rather than a joint confidence region). Let \( q(p, \mu, \sigma^2) := \sqrt{2\sigma} \text{erf}^{-1}(2p - 1) + \mu \) be the quantile function for a \( N(\mu, \sigma^2) \) distribution where \( \text{erf}^{-1}(\cdot) \) is the inverse error function.

**Corollary D.1.** Given diagonal covariance estimates and privacy variances \( \{\tilde{\Sigma}_m, \tilde{\sigma}_m^2\}_{m\in[t]} \), let \( \tilde{S}_m \in \mathbb{R}^d \) be the flattened version of \( \tilde{\Sigma}_m \). We can construct a precision-weighted estimator: \( \tilde{S} := \frac{\tilde{S}_m}{\tilde{\sigma}_m^2} \).

Let \( \tilde{\Sigma}' \) be the diagonal \( d \times d \) matrix created by unflattening \( \tilde{S} \) and \( b \) be the unflattened \( d \times d \) diagonal matrix where \( b_{j,j} = V \left( \tilde{\Sigma}'_{i,i} \right) \) (i.e. the diagonal values of the covariance matrix of the flattened precision-weighted estimator). For \( \beta^{ub} \in (0, 1) \), define \( \tilde{c} = \{c_j\}_{j\in[d]} \) where

\[
c_j = q \left( 1 - \frac{\beta^{ub}}{d}, 0, b_{j,j}^2 \right).
\]

Then, for \( \tilde{\Sigma} = \tilde{\Sigma}' + \tilde{c}I_d \) we have

\[
\mathbb{P} \left( \forall j \in [d] : \tilde{\Sigma}_{j,j} \leq \tilde{\Sigma}_{j,j} \right) \geq 1 - \beta^{\tilde{\Sigma}} - \beta^{ub}.
\]

**Proof.** We start as in the proof in Section D.2, but we know additionally that \( \tilde{\Sigma}'_{i,j} = 0 \) for \( i \neq j \). We know that if our assumptions hold, which happens with probability \( \geq 1 - \beta^{\tilde{\Sigma}} \), we can write \( \tilde{\Sigma}'_{j,j} \sim N \left( \hat{\Sigma}_{j,j}, b_{j,j}^2 \right) \) where \( b_{j,j} = (\tilde{\sigma}_j)^2 \).

By definition of the quantile function, we know then that, for arbitrary \( j \in [d] \):

\[
1 - \frac{\beta^{ub}}{d} = \mathbb{P} \left( \tilde{\Sigma}'_{j,j} \leq q \left( 1 - \frac{\beta^{ub}}{d}, \hat{\Sigma}_{j,j}, b_{j,j}^2 \right) \right)
\]

\[
= \mathbb{P} \left( \tilde{\Sigma}'_{j,j} \leq \hat{\Sigma}_{j,j} + q \left( 1 - \frac{\beta^{ub}}{d}, 0, b_{j,j}^2 \right) \right) \quad \text{(definition of } c_j \text{)}
\]

\[
= \mathbb{P} \left( \tilde{\Sigma}'_{j,j} + c_j \leq \hat{\Sigma}_{j,j} \right) \quad \text{(symmetry of the Gaussian)}.
\]

Applying a union bound over the \( d \) failure probabilities and combining with the \( 1 - \beta^{\tilde{\Sigma}} \) probability that our required assumptions hold yields the desired result.

**D.3 Proof of Theorem 2.11**

**Proof.** From Theorem 2.8, we know that, with probability \( \geq 1 - \beta^{\hat{\theta}} \):

\[
\forall m \in [t] : \tilde{\theta}_m \sim N \left( \hat{\theta}^{BLB}, \hat{\sigma}_m^2 I_d \right),
\]

where \( \tilde{S} \) is the flattened form of \( \hat{\Sigma} \). Assumption 2.4 then lets us substitute in \( \hat{\theta} \) for \( \hat{\theta}^{BLB} \).

However, we opt to use the trick from Section C.6 to avoid having to make Assumption 2.7. Theorem 2.11 gives us a covariance bound that, after appropriate scaling, satisfies Assumption 2.7 with probability \( 1 - \beta^{\tilde{\Sigma}} - \beta^{ub} \), so we fold this into our failure probability. Our result then follows directly from the precision-weighting procedure in Theorem 2.9.
E Confidence Region/Intervals

E.1 Proof of Theorem 2.12

Proof. We assume that our estimation of $\hat{\theta}$ and $\hat{\Sigma}$ worked as described at the top of Section 2.4, which comes with a $1 - \beta \hat{\Sigma} - \beta^{ub} - \beta \hat{\theta}$ probability guarantee. This means that $E(\hat{\theta}) = E(\hat{\theta}) = \theta$ and $\hat{\Sigma} \succeq \Sigma \succeq \Sigma$ where our non-private estimator $\hat{\theta} \sim G(\theta, \Sigma)$. Furthermore, we assume that $Q_{\hat{\theta}}$ is heavier-tailed than $G$.

Summarizing this, we have a high-probability guarantee that three conditions hold:

1. $E(Z) = E(\hat{\theta}) = \theta$
2. $\Sigma \preceq \tilde{\Sigma}$
3. $Q_{\hat{\theta}}$ is heavier-tailed than $G$.

Under these conditions, $Z$ and $\hat{\theta}$ have the same mean and $\hat{\theta}$ is more concentrated than $Z$, so a confidence region that is valid for $Z$ is valid for $\hat{\theta}$. In other words,

$$\forall \alpha \in (0, 1) : P(Z \in C) \geq 1 - \alpha \implies P(\theta \in C) \geq 1 - \alpha.$$  

The result for confidence intervals, rather than a single region, follows trivially by noting that a confidence interval is a 1-dimensional confidence region.

**Corollary E.1** (Confidence Intervals (valid with high probability)). Let $Z$ be a $d$-dimensional random variable such that $Z \sim Q_{\hat{\theta}} \left( \hat{\theta} + N(0, \Sigma_{\hat{\theta}}), \tilde{\Sigma} \right)$. Suppose $\{ (c_{ij}^{l}, c_{ij}^{u}) \}_{j \in [d]}$ is a set of intervals such that

$$\forall j \in [d] : P(Z_j \in (c_{ij}^{l}, c_{ij}^{u})) \geq 1 - \alpha_j,$$

for some $\{ \alpha_j \}_{j \in [d]}$ with $\alpha_j \in (0, 1)$. Then, with probability $1 - \beta \tilde{\Sigma} - \beta^{ub} - \beta \hat{\theta}$,

$$\forall j \in [d] : P(\hat{\theta}_{BBL}^{j} \in (c_{ij}^{l}, c_{ij}^{u})) \geq 1 - \alpha_j.$$

Likewise, with probability $1 - \beta \tilde{\Sigma} - \beta^{ub} - \beta \hat{\theta}$,

$$\forall j \in [d] : P(\hat{\theta}_{BBL}^{j} \in (c_{ij}^{l}, c_{ij}^{u})) \geq 1 - \sum_{j}^{d} \alpha_j.$$

**Proof.** All but the last statement is a trivial application of Theorem 2.12 to the 1-dimensional case. The last statement follows via a union bound.

F Empirical Results

F.1 Logistic regression with imbalanced class output

In Figure 5 we show qualitatively similar results for a more challenging setting; logistic regression with imbalanced classes.

We generate data as we did for Figure 2 but run the outcome variable $y$ through a scaled logistic function to get a new outcome variable $y' \in \{0, 1\}^n$. Specifically, for $p_i = \frac{1}{1 + \exp(-X_i \beta)}$ and $\bar{p} = \frac{1}{n} \sum_{i=1}^{n} p_i$, we have $P(y_i = 1) = \frac{\bar{p}}{\bar{p}} \cdot 0.05$. This induces a minority class that occurs with probability $\approx 0.05$. Having such imbalanced classes introduces a practical problem in choosing a good $k$ for GVDP. If $\frac{n}{k}$ is small, it becomes likely that we will see only the majority class in any given subset and the model will not be able to be fit. Thus, our experiments here have larger $n$ than we used for the OLS experiments.
Figure 5: Logistic Regression: Distribution of coefficient estimates and 95% confidence intervals

F.2 Logistic regression with fully sparse data

The requisite bootstrap assumptions do not hold for all estimators and data distributions. We make our setting more difficult again for Figure 6, by making both the outcome and covariates a sparse binary vector/matrix respectively. It is unlikely that an analyst would want to use GVDP in this setting, because knowing that the data are binary (which we assume an analyst would know) immediately provides tight clipping bounds for the data. Nevertheless, we include it as an example because it’s the most natural scenario we found where our method fails because of poor performance of the BLB.

We create a new set of covariates $X'$ such that $\forall j \in [d] : X'_{ij} = \mathbb{1}(X_{ij} \geq z_j)$ where $z_j = \min_{r \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(X_{ij} > r) \leq 0.05$. That is, $X'$ is itself now a binary matrix with highly imbalanced classes. The rightmost plot shows distributions of the $d$ coefficient estimates induced by BLB, with a black dotted line at the value of the non-private coefficient. Note that at $n = 100,000$, the BLB distributions are essentially point masses at two extreme points, and our algorithm yields biased estimates and confidence intervals with insufficient coverage. For $n = 1,000,000$, the BLB distributions are much closer to being a symmetric distribution about the true coefficient value, and our algorithm yields the promised guarantees.

The left plots show poor confidence interval coverage because the BLB distribution is not a good approximation of the non-private sampling distribution. The right plots show better confidence interval coverage because the BLB approximation is successful.

F.3 Explanation of Table 2

The GVDP and AdaSSP algorithms differ in a few key ways. First, AdaSSP does not attempt to do unbiased parameter estimation or give valid confidence intervals; instead, it is trying to estimate OLS coefficients with minimal $\ell_2$ error. For purposes of comparison, we will ignore confidence intervals altogether and focus only on the parameter estimates. Second, AdaSSP assumes only bounds on the data, assuming that we can specify data domains $X, Y$ for our covariates and outcome, respectively, such that $\|X\| = \sup_{x \in X} \|x\|_2$ where $x \in \mathbb{R}^m$ and $\|Y\| = \sup_{y \in Y} |y|$. Ignoring the assumptions needed for confidence intervals for now, GVDP requires Assumptions 2.6, and 2.7 on the distribution of covariances induced by the bag of little bootstraps, as well as Assumption 2.6 on the distribution of means. It’s worth noting the qualitative difference between these methods; AdaSSP requires the user to bound the data, GVDP requires the user to bound moments of the parameter distribution. For most analyses, we expect the AdaSSP bounds to be easier to specify tightly than those of GVDP. However, GVDP is designed to scale more gracefully under overly conservative bounds.

We generate data just as we did for our OLS demonstration and compare AdaSSP and GVDP
across a number of what we call “overestimation factors”. Say we have realized data $X \in \mathbb{R}^{n \times d}, y \in \mathbb{R}^d$. For an overestimation factor of $c$, we set the bounds for AdaSSP to $c \cdot \sup_{x \in X} \|x\|_2$ and $c \cdot \sup_{y \in Y} \|y\|$. For GVDP, we perform the BLB step to get our $\{\hat{\theta}_{BLB}^i\}_{i \in [k]}$, which we’ll say has empirical mean $\hat{\mu} \in \mathbb{R}^d$ and empirical covariance $\hat{\Sigma} \in \mathbb{R}^{d \times d}$. We set our $\ell_2$ bounding ball for the mean of the distribution as $B_2(\hat{\mu}, c (\max_{j \in [d]} \hat{\mu}_j))$ and our Löwner upper bound on the covariance as $c \left(\text{diag}(\hat{\Sigma}) \mathbb{I}_d\right)$. Runs of non-private OLS are included for comparison, but the overestimation factor does not affect them.

The experiment in the body of the paper was run with $n = 500,000, k = 2,500, d = 10$, and $\rho = 0.1$. We run each method over 100 simulations, estimating $d$ coefficients at each iteration, so each method produces 1,000 coefficient estimates overall.

**Comparison with AdaSSP** We again consider OLS, but now compare GVDP’s performance to that of the Adaptive Sufficient Statistic Perturbation (AdaSSP) algorithm from Wang (2018), one of the best-performing algorithms for DP OLS. AdaSSP assumes bounds on the underlying data and attempts to estimate the OLS coefficients with minimal $\ell_2$ error. We consider the performance of AdaSSP vs. GVDP as a function of the “overestimation factor” (OF), which is a multiplicative factor by which we overestimate the bounds of the data (for AdaSSP) or parameters (for GVDP).
Despite our presentation above, AdaSSP and GVDP can’t really be directly compared because the OFs have qualitatively different meanings. However, the general comparison is still useful; AdaSSP performs well with slightly overestimated bounds but scales poorly with overly conservative bounds, while GVDP performs a bit less well at low overestimation factors but scales much better when the bounds are poorly set. More information can be found in Section F.3.

F.4 Replication of Card (1999)

Inspired by the tests of Barrientos et al. (2021), we attempt to replicate the core analysis of Card (1999) under the constraints of DP. We use CPS ASEC data from 1994 to 1996 (Ruggles et al., 2021) and run OLS to estimate the following model:

$$\log(\text{inc\_wage}) = \beta_0 + \beta_1 \text{educ} + \beta_2 \text{PE} + \beta_3 \text{PE}^2 + \beta_4 \text{PE}^3 + \beta_5 \text{white} + \varepsilon,$$

where \(\text{inc\_wage}\) is an individual’s total pre-tax wage and salary income, \(\text{educ}\) is years of education, \(\text{PE}\) is potential years of work experience, and \(\text{white}\) indicated whether or not the individual identifies as white. The effect of education on income is our question of interest, with the other variables serving as controls. This allows us to use the full OLS model within the bootstrap, but release and privately estimate only the estimated mean/variance of \(\beta_1\).

Card (1999) runs this model separately for males and females; in Figure 7 we report the female results (\(n = 95,177\)) as well as for males and females combined (\(n = 197,756\)). To be consistent with Barrientos et al. (2021), we report results in approximate DP with \((\varepsilon, \delta) = (5, 1/k)\), which translates to \(\rho \approx \{0.879, 1.06, 1.23\}\) for \(k = \{1000, 500, 250\}\). All results are run with \(t = 5\) CoinPress iterations and an overestimation factor of 100, and we run the GVDP estimation algorithm 200 times to show the long-run performance of the algorithm.

We note that our bootstrapped means do not always equal to the non-private mean in expectation, so although our algorithm’s guarantees with respect to the bootstrapped distribution are met, they do not imply guarantees relative to the non-private answer as we hope they would. We see in these plots a clear trade-off. At \(k = 1,000\), our confidence intervals are fairly tight, but are essentially centered around the upper end of the non-private confidence interval rather than the the true coefficient estimate. At \(k = 250\) we minimize bias by generating more a representative bootstrap distribution, but do so at the cost of wider confidence intervals.

G Notes on Assumptions and Analyst Choices

G.1 Assumption 2.4

Assumption 2.4 essentially has three distinct pieces; we speak to the plausibility of each below.

First, we assume that the sampling distribution of the estimator is a member of a symmetric multivariate location-scale family, which we require because we want to be able to fully characterize the distribution by its mean and covariance. Given that \(X\) is of a reasonable sample size, we can appeal to the central limit theorem and argue that this assumption ought to hold. If the analyst cares only about confidence intervals for each element of the parameter, rather than a joint confidence region, it is sufficient for the marginal sampling distribution of each element in the parameter vector to belong to a symmetric univariate location-scale family.

Second, we assume that the BLB estimator in unbiased with respect to the estimand of interest in the non-private setting. The BLB shares many of the statistical properties of the traditional bootstrap, including asymptotic consistency (as both \(n \to \infty\) and \(\frac{n}{k} \to \infty\)), but also no finite-sample guarantee of unbiasedness. As such, this assumption may not hold in practice. However, if the BLB estimator exhibits low bias relative to the potential bias induced by poorly chosen
clipping bounds, our method could still be an effective way to produce private estimates with lower bias than existing methods.

Third, we assume that the BLB estimates of the covariance are, with probability 1, a L"owner upper bound on the true covariance of the sampling distribution. This condition on $\hat{\Sigma}^{BLB}$ is onerous (especially in high dimensions) and seems unlikely to hold in general. In practice however, this condition can be dropped at the cost of a bit of extra fuzziness in the results. As stated above, we later make claims about our private estimator relative to $\hat{\Sigma}^{BLB}$, which under Assumption 2.4 also hold relative to $\hat{\Sigma}$. This generalization to $\hat{\Sigma}$ is a higher bar than is typically set in applications of the bootstrap, where the bootstrap approximation is simply treated as a “good-enough” approximation of the sampling distribution. Moreover, if we care only about getting confidence intervals, rather than a confidence region, we can replace the L"owner condition with the condition that each element of the diagonal of $\hat{\Sigma}^{BLB}$ is at least as large as the corresponding element of $\hat{\Sigma}$.

G.2 Assumption 2.5

Assumption 2.5 essentially follows from the first part of Assumption 2.4 where we assume the sampling distribution is from a symmetric location-scale family. If we can identify the location-scale family of the sampling distribution (again, we often appeal to the central limit theorem and say this is Multivariate Gaussian), then this same family trivially satisfies Assumption 2.5.

G.3 Assumptions 2.6 and 2.7

Assumptions 2.6 and 2.7 state that the analyst can set bounds on the mean and covariance on the BLB estimates of both the mean and covariance of the sampling distribution. We believe that
setting tight bounds would be very difficult in general, often more difficult than setting tight bounds on the data (the requirement we’re trying to avoid). However, we suggest that the analyst aim to set very conservative bounds, unless they are very confident in their knowledge of the parameters. Because we use the CoinPress mean estimation algorithm to iteratively improve the bounds the analyst provides, the performance of the algorithm degrades slowly with more conservative bounds; e.g. see our results from Section 3 where the analyst’s bounds are too conservative by a factor of 100 or Section F.3 where we show performance when the analyst’s bounds are too conservative by a factor of 10,000.

G.4 Choosing $k$

Recall from our explanation of Algorithm 1 that $k$ is the number of subsets into which we partition our original data, which in turn becomes the number of elements fed into our private mean estimation algorithm, Algorithm 3. This presents a trade-off for the user; when $k$ is large, the sensitivity of our aggregator decreases (Line 7 of Algorithm 4) and thus so does the variance of the noise we need to add for privacy. On the other hand, we assume that the mean and covariance estimates we get from the BLB reasonably approximate the mean and covariance of the true sampling distribution of the parameters, which is provably true only as $n \to \infty$ and $\frac{n}{k} \to \infty$ for Hadamard differentiable estimators (Kleiner et al., 2014).

In the body of the paper, we argued that once $\frac{n}{k}$ is large enough that the BLB estimates have converged, there is no use in further increasing the ratio of $n$ to $k$; we are better served by increasing $k$ and reducing the noise needed for privacy. So, the best possible case for an analyst is that they choose the largest $k$ such that the BLB estimates, operating over subsets of size $\frac{n}{k}$, approximates the true parameters of the sampling distribution.

As a final note, recall from Section 2.2 that we assume that $n$ is public knowledge or has been privately estimated. Thus, we can choose $k$ in a way that depends on $n$ with no extra privacy cost; if $n$ is public knowledge then there's no dependence on the data at all and if it was privately estimated then this falls under the postprocessing property of zCDP.