Nonlinear Relativistic Invariance For Quadrahyperbolic Numbers

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Abstract

One may ask whether an extended group of invariance can naturally be attributed to the space of associative commutative Quadrahyperbolic Numbers? To search for a rigorous and positive answer to the question, we shall focus on the method of derivation of the respective invariance. The outcome that there exist 3-parametric nonlinear transformations which leave invariant the scalar product chosen appropriately for The Quadrahyperbolic Numbers, is the main result of the present publication.
1. INTRODUCTION

In the previous work [1] we raised several urgent problems which solution is doomed to elucidate the geometric as well as relativistic–geometric ingredients of the Space of Quadrahyperbolic Numbers (QH–Space for brevity). In the present paper we make due attempts to explain, from first ideas, how the group of nonlinear invariance should supplement the linear dilatation invariance operative trivially in the QH–Space. As matter stands, to proceed successfully in this direction, one should propose a convenient scalar product for pairs of the Quadrahyperbolic Numbers. In this respect, the available and attractive possibility is to exploit respectively the concept of transversality (introduced in Sec. 4 of [1]). Namely, it is the symmetrized metric form (1.7) for pairs of the Quadrahyperbolic Numbers that is a handy and convenient scalar product. Adhering to this choice, we prove that the nonlinear invariance transformations exist which do leave our scalar product invariant. In the dimension $N = 4$ used, the linear unimodular dilatations involve 3 parameters, and the nonlinear transformations, – as being the transformations which act independently in each of three basic sectors of the scalar product under study, – involve also three parameters in general. Therefore, the fact that the total invariance group is 6–parametrical is not violated under the transition from the Lorentz invariance of the pseudoeuclidean space to the proposed invariance of the QH-Space (!)

Below in Section 2, the required form of the nonlinear transformations will explicitly be derived and presented. A short discussion of involved aspects will be given in the last Section 3.

Generally, when attempting to propose a necessary definition of the scalar product $(X, Y)$ of two vectors related to the QH–numbers, it is worth setting forth the conditions of ordinary meaning and current applicability. They should include:

$C_1$: the symmetry

$$(X, Y) = (Y, X);$$  \hspace{1cm} (1.1)

$C_2$: the normalization

$$(X, X)^2 = ||X||;$$  \hspace{1cm} (1.2)

$C_3$: the homogeneity

$$(cX, Y) = (X, cY) = c(X, Y),$$  \hspace{1cm} (1.3)

where $c$ is any constant; $cX$ means the set \{cx$^0$, cx$^1$, cx$^2$, cx$^3$\} and $cY$ means the set \{cy$^0$, cy$^1$, cy$^2$, cy$^3$\};

$C_4$: the positivity

$$(X, X) > 0$$  \hspace{1cm} (1.4)

(over all the sector $V^\text{time-like}_4$);

$C_5$: the dilatation invariance

$$(kX, kY) = (X, Y),$$  \hspace{1cm} (1.5)

where $kX$ and $kY$ stay for \{k$^0$x$^0$, k$^1$x$^1$, k$^2$x$^2$, k$^3$x$^3$\} and \{k$^0$y$^0$, k$^1$y$^1$, k$^2$y$^2$, k$^3$y$^3$\}, respectively.

Under these conditions, the required scalar product can be proposed as follows:

$$S(X, Y) = S(A, B)$$  \hspace{1cm} (1.6)
with
\[ S(A, B) \overset{\text{def}}{=} \frac{1}{2 F^2(B)}(A, B, B, B) + \frac{1}{2 F^2(A)}(A, A, A, B), \] (1.7)
where \( A \) and \( B \) relate to \( X \) and \( Y \), respectively;
\[ F(A) = \sqrt[4]{a^1 a^2 a^3 a^4}, \quad F(B) = \sqrt[4]{b^1 b^2 b^3 b^4}, \] (1.8)
such that
\[ S(A, A) = (F(A))^2. \] (1.9)
In an explicit coordinate way, the definition (1.7)–(1.8) reads
\[ S(A, B) = \frac{1}{8 F^2(B)}(a^1 b^2 b^3 b^4 + a^2 b^1 b^3 b^4 + a^3 b^1 b^2 b^4 + a^4 b^1 b^2 b^3) \]
\[ + \frac{1}{8 F^2(A)}(b^1 a^2 a^3 a^4 + b^2 a^1 a^3 a^4 + b^3 a^1 a^2 a^4 + b^4 a^1 a^2 a^3), \] (1.10)
or
\[ S(A, B) = \frac{1}{8} F(A) F(B) \left[ \frac{F(B)}{F(A)} \left( \frac{a^1}{b^1} + \frac{a^2}{b^2} + \frac{a^3}{b^3} + \frac{a^4}{b^4} \right) + \frac{F(A)}{F(B)} \left( \frac{b^1}{a^1} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + \frac{b^4}{a^4} \right) \right]. \] (1.11)
All the conditions (1.1)–(1.5) are fulfilled.
In terms of the new variables
\[ d^1 = \frac{b^1}{a^1}, \quad d^2 = \frac{b^2}{a^2}, \quad d^3 = \frac{b^3}{a^3}, \quad d^4 = \frac{b^4}{a^4}, \] (1.12)
we get
\[ S = (F(A))^2 F(D) L(D) \] (1.13)
with
\[ F(D) = \sqrt[4]{d^1 d^2 d^3 d^4} \] (1.14)
and
\[ L(D) \overset{\text{def}}{=} \frac{1}{8} F(D) \left[ F(D) \left( \frac{1}{d^1} + \frac{1}{d^2} + \frac{1}{d^3} + \frac{1}{d^4} \right) + \frac{1}{F(D)} \left( d^1 + d^2 + d^3 + d^4 \right) \right]. \] (1.15)
On comparing the formulae (1.6)–(1.12) with the formulae (1.13)–(1.15), we may raise the conjecture that the invariance of the associated function (1.15) may entail the invariance of the initial scalar product (1.7). In the next section, the conjecture will be confirmed by special calculations.

2. The Method of Calculations and The Main Result
To our aims it is very convenient to adopt the parametrization
\[ d^1 = \exp(\delta + \alpha + \beta + \gamma), \] (2.1)
\[ d^2 = \exp(\delta - \alpha + \beta - \gamma), \quad (2.2) \]
\[ d^3 = \exp(\delta + \alpha - \beta - \gamma), \quad (2.3) \]
\[ d^4 = \exp(\delta - \alpha - \beta + \gamma), \quad (2.4) \]

where \( \{\alpha, \beta, \gamma\} \) are the exponent variables. We get
\[ F(D) = \exp(\delta). \quad (2.5) \]

The inverse transition reads as
\[ 4\delta = \ln d^1 + \ln d^2 + \ln d^3 + \ln d^4, \quad (2.6) \]
\[ 4\alpha = \ln d^1 - \ln d^2 + \ln d^3 - \ln d^4, \quad (2.7) \]
\[ 4\beta = \ln d^1 + \ln d^2 - \ln d^3 - \ln d^4, \quad (2.8) \]
\[ 4\gamma = \ln d^1 - \ln d^2 - \ln d^3 + \ln d^4. \quad (2.9) \]

In this way, the following representation is obtained for the function (1.15):
\[ \mathcal{L}/F(D) = \frac{1}{4} \left( \cosh(\alpha + \beta + \gamma) + \cosh(-\alpha + \beta - \gamma) + \cosh(\alpha - \beta - \gamma) + \cosh(-\alpha - \beta + \gamma) \right). \quad (2.10) \]

The subsequent use of the known hyperbolic identity
\[ \cosh \lambda - 1 = 2(\sinh \frac{\lambda}{2})^2 \quad (2.11) \]
yields
\[ \mathcal{L}/F(D) = 1 + \frac{1}{2} \left[ (\sinh \frac{\alpha + \beta + \gamma}{2})^2 + (\sinh \frac{-\alpha + \beta - \gamma}{2})^2 \right. \]
\[ + (\sinh \frac{\alpha - \beta - \gamma}{2})^2 + (\sinh \frac{-\alpha - \beta + \gamma}{2})^2 \]. \quad (2.12) \]

A careful consideration of the last function (2.12), and of Eqs. (2.1)–(2.11), suggests to apply the transformations indicated below.

*The \{\alpha, \beta\}–side turn acts as*
\[ \delta = \tilde{\delta}, \quad \gamma = \tilde{\gamma}, \quad (2.13) \]
\[ d^1 = e^{\tilde{\delta} + \tilde{\gamma}} \exp \left[ 2 \arcsinh \left( \frac{\tilde{\alpha} + \tilde{\beta}}{2} \cos \mu - \sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \sin \mu \right) \right], \quad (2.14) \]

\[ d^2 = e^{\tilde{\delta} - \tilde{\gamma}} \exp \left[ -2 \arcsinh \left( \frac{\tilde{\alpha} - \tilde{\beta}}{2} \cos \mu + \sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \sin \mu \right) \right], \quad (2.15) \]

\[ d^3 = e^{\tilde{\delta} - \tilde{\gamma}} \exp \left[ 2 \arcsinh \left( \frac{\tilde{\alpha} - \tilde{\beta}}{2} \cos \mu + \sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \sin \mu \right) \right], \quad (2.16) \]

\[ d^4 = e^{\tilde{\delta} + \tilde{\gamma}} \exp \left[ -2 \arcsinh \left( \frac{\tilde{\alpha} + \tilde{\beta}}{2} \cos \mu - \sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \sin \mu \right) \right]. \quad (2.17) \]

By the help of the known equality
\[ \arcsinh(x) = \ln \left( x + \sqrt{x^2 + 1} \right) \]
we arrive at new convenient representations:

\[ d^1 = e^{\tilde{\delta} + \tilde{\gamma}} \left[ (\sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \cos \mu - \sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \sin \mu) + \sqrt{\left( \sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \cos \mu - \sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \sin \mu \right)^2 + 1} \right]^2, \]

\[ d^2 = e^{\tilde{\delta} - \tilde{\gamma}} \left[ (\sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \cos \mu + \sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \sin \mu) + \sqrt{\left( \sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \cos \mu + \sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \sin \mu \right)^2 + 1} \right]^{-2}, \]

\[ d^3 = e^{\tilde{\delta} - \tilde{\gamma}} \left[ (\sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \cos \mu + \sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \sin \mu) + \sqrt{\left( \sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \cos \mu + \sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \sin \mu \right)^2 + 1} \right]^2, \]

\[ d^4 = e^{\tilde{\delta} + \tilde{\gamma}} \left[ (\sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \cos \mu - \sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \sin \mu) + \sqrt{\left( \sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} \cos \mu - \sinh \frac{\tilde{\alpha} - \tilde{\beta}}{2} \sin \mu \right)^2 + 1} \right]^{-2}. \]

After that, we apply the pull-back substitutions

\[ 4\tilde{\delta} = \ln d^1 + \ln d^2 + \ln d^3 + \ln d^4, \quad (2.18) \]

\[ 4\tilde{\alpha} = \ln d^1 - \ln d^2 + \ln d^3 - \ln d^4, \quad (2.19) \]

\[ 4\tilde{\beta} = \ln d^1 + \ln d^2 - \ln d^3 - \ln d^4, \quad (2.20) \]
\[ 4\tilde{\gamma} = \ln \tilde{d}^1 - \ln \tilde{d}^2 - \ln \tilde{d}^3 + \ln \tilde{d}^4, \]  

which entails

\[
\sinh \frac{\tilde{\alpha} + \tilde{\beta}}{2} = \frac{1}{2} \left( \sqrt{\frac{\tilde{d}^1}{d^1}} - \sqrt{\frac{\tilde{d}^4}{d^1}} \right), \quad \sinh \frac{\tilde{\beta} - \tilde{\alpha}}{2} = \frac{1}{2} \left( \sqrt{\frac{\tilde{d}^2}{d^3}} - \sqrt{\frac{\tilde{d}^3}{d^2}} \right) \tag{2.22}
\]

and

\[ e^{\tilde{\delta} + \tilde{\gamma}} = \sqrt{d^1 d^4} \tag{2.23} \]

together with

\[ e^{\tilde{\delta} - \tilde{\gamma}} = \sqrt{d^2 d^3}. \tag{2.24} \]

Thus we find eventually the nonlinear turn given by

\[
d^1(\mu; \tilde{d}^1, \tilde{d}^2, \tilde{d}^3, \tilde{d}^4) = \frac{1}{4} \sqrt{d^2 d^3} \left[ \left( \sqrt{\frac{\tilde{d}^1}{d^1}} - \sqrt{\frac{\tilde{d}^4}{d^1}} \right) \cos \mu + \left( \sqrt{\frac{\tilde{d}^2}{d^3}} - \sqrt{\frac{\tilde{d}^3}{d^2}} \sin \mu \right) \right] \tag{2.25}
\]

\[
d^2(\mu; \tilde{d}^1, \tilde{d}^2, \tilde{d}^3, \tilde{d}^4) = 4\sqrt{d^2 d^3} \left[ - \left( \sqrt{\frac{\tilde{d}^2}{d^3}} - \sqrt{\frac{\tilde{d}^3}{d^2}} \right) \cos \mu + \left( \sqrt{\frac{\tilde{d}^1}{d^4}} - \sqrt{\frac{\tilde{d}^4}{d^1}} \sin \mu \right) \right] \tag{2.26}
\]

\[
d^3(\mu; \tilde{d}^1, \tilde{d}^2, \tilde{d}^3, \tilde{d}^4) = \frac{1}{4} \sqrt{d^2 d^3} \left[ - \left( \sqrt{\frac{\tilde{d}^2}{d^3}} - \sqrt{\frac{\tilde{d}^3}{d^2}} \right) \cos \mu + \left( \sqrt{\frac{\tilde{d}^1}{d^4}} - \sqrt{\frac{\tilde{d}^4}{d^1}} \sin \mu \right) \right] \tag{2.27}
\]
\begin{align*}
d^4(\mu; d^1, d^2, d^3, d^4) &= 4 \sqrt{d^1 d^4} \left[ \left( \sqrt{\frac{d^1}{d^1}} - \sqrt{\frac{d^4}{d^4}} \right) \cos \mu + \left( \sqrt{\frac{d^2}{d^3}} - \sqrt{\frac{d^3}{d^2}} \right) \sin \mu \right] \\
&\quad + \sqrt{\left( \left( \sqrt{\frac{d^1}{d^1}} - \sqrt{\frac{d^4}{d^4}} \right) \cos \mu + \left( \sqrt{\frac{d^2}{d^3}} - \sqrt{\frac{d^3}{d^2}} \right) \sin \mu \right)^2 + 4}.
\end{align*}

The nearest implications

\begin{align*}
d^1 d^2 d^3 d^4 &= \tilde{d}^1 \tilde{d}^2 \tilde{d}^3 \tilde{d}^4, \\
=d^1_{\mu=0} = \tilde{d}^1, \quad d^2_{\mu=0} = \tilde{d}^2, \quad d^3_{\mu=0} = \tilde{d}^3, \quad d^4_{\mu=0} = \tilde{d}^4,
\end{align*}

and

\begin{align*}
d^1 d^4 &= \tilde{d}^1 \tilde{d}^4, \quad d^2 d^3 = \tilde{d}^2 \tilde{d}^3
\end{align*}

can readily be verified.

Also, on inserting (2.25)–(2.28) in (1.15), we arrive at the invariance

\begin{align*}
\mathcal{L}(D) &= \mathcal{L}(\tilde{D})
\end{align*}

after straightforward calculations.

Finally, we can return to the initial variables in accordance with

\begin{align*}
d^p &= \frac{b^p}{a^p}, \quad \tilde{d}^p = \frac{\tilde{b}^p}{\tilde{a}^p},
\end{align*}

and

\begin{align*}
a^p &= \tilde{a}^p,
\end{align*}

obtaining

\begin{align*}
b^1(\mu; a^1, a^2, a^3, a^4; \tilde{b}^1, \tilde{b}^2, \tilde{b}^3, \tilde{b}^4) &= \\
&= \sqrt{\frac{a^1}{a^4}} \sqrt{\frac{1}{16} \tilde{b}^1 \tilde{b}^4} \left[ \left( \sqrt{\frac{a^4 \tilde{b}^1}{a^1 b^4}} - \sqrt{\frac{a^1 \tilde{b}^4}{a^4 b^1}} \right) \cos \mu + \left( \sqrt{\frac{a^3 \tilde{b}^2}{a^1 b^3}} - \sqrt{\frac{a^1 \tilde{b}^3}{a^3 b^2}} \right) \sin \mu \right] \\
&\quad + \sqrt{\left( \left( \sqrt{\frac{a^4 \tilde{b}^1}{a^1 b^4}} - \sqrt{\frac{a^1 \tilde{b}^4}{a^4 b^1}} \right) \cos \mu + \left( \sqrt{\frac{a^3 \tilde{b}^2}{a^1 b^3}} - \sqrt{\frac{a^1 \tilde{b}^3}{a^3 b^2}} \right) \sin \mu \right)^2 + 4}.
\end{align*}

\begin{align*}
b^2(\mu; a^1, a^2, a^3, a^4; \tilde{b}^1, \tilde{b}^2, \tilde{b}^3, \tilde{b}^4) &= 
\end{align*}
\[
\begin{align*}
&= \sqrt{\frac{a^2}{a^3}} \sqrt{16\tilde{b}^2\tilde{b}^3} \left[ - \left( \frac{\sqrt[4]{a^2\tilde{b}^2}}{a^2\tilde{b}^2} - \frac{\sqrt[4]{a^2\tilde{b}^3}}{a^2\tilde{b}^3} \right) \cos \mu + \left( \frac{\sqrt[4]{a^4\tilde{b}^4}}{a^4\tilde{b}^4} - \frac{\sqrt[4]{a^5\tilde{b}^4}}{a^5\tilde{b}^4} \right) \sin \mu \right] \\
&\quad + \sqrt{\left( - \left( \frac{\sqrt[4]{a^2\tilde{b}^2}}{a^2\tilde{b}^2} - \frac{\sqrt[4]{a^2\tilde{b}^3}}{a^2\tilde{b}^3} \right) \cos \mu + \left( \frac{\sqrt[4]{a^4\tilde{b}^4}}{a^4\tilde{b}^4} - \frac{\sqrt[4]{a^5\tilde{b}^4}}{a^5\tilde{b}^4} \right) \sin \mu \right)^2 + 4}^{-2}, \quad (2.36)
\end{align*}
\]

\[
\begin{align*}
b^3(\mu; a^1, a^2, a^3; \tilde{b}^1, \tilde{b}^2, \tilde{b}^3, \tilde{b}^4) &= \\
&= \sqrt{\frac{a^3}{a^2}} \sqrt{\frac{1}{16\tilde{b}^2\tilde{b}^3}} \left[ - \left( \frac{\sqrt[4]{a^3\tilde{b}^2}}{a^2\tilde{b}^2} - \frac{\sqrt[4]{a^3\tilde{b}^3}}{a^2\tilde{b}^3} \right) \cos \mu + \left( \frac{\sqrt[4]{a^4\tilde{b}^4}}{a^4\tilde{b}^4} - \frac{\sqrt[4]{a^5\tilde{b}^4}}{a^5\tilde{b}^4} \right) \sin \mu \right] \\
&\quad + \sqrt{\left( - \left( \frac{\sqrt[4]{a^3\tilde{b}^2}}{a^2\tilde{b}^2} - \frac{\sqrt[4]{a^3\tilde{b}^3}}{a^2\tilde{b}^3} \right) \cos \mu + \left( \frac{\sqrt[4]{a^4\tilde{b}^4}}{a^4\tilde{b}^4} - \frac{\sqrt[4]{a^5\tilde{b}^4}}{a^5\tilde{b}^4} \right) \sin \mu \right)^2 + 4}^{-2}, \quad (2.37)
\end{align*}
\]

\[
\begin{align*}
b^4(\mu; a^1, a^2, a^3; \tilde{b}^1, \tilde{b}^2, \tilde{b}^3, \tilde{b}^4) &= \\
&= \sqrt{\frac{a^4}{a^1}} \sqrt{\frac{1}{16\tilde{b}^1\tilde{b}^4}} \left[ - \left( \frac{\sqrt[4]{a^4\tilde{b}^1}}{a^1\tilde{b}^1} - \frac{\sqrt[4]{a^1\tilde{b}^4}}{a^1\tilde{b}^4} \right) \cos \mu + \left( \frac{\sqrt[4]{a^3\tilde{b}^2}}{a^2\tilde{b}^2} - \frac{\sqrt[4]{a^3\tilde{b}^3}}{a^3\tilde{b}^3} \right) \sin \mu \right] \\
&\quad + \sqrt{\left( - \left( \frac{\sqrt[4]{a^4\tilde{b}^1}}{a^1\tilde{b}^1} - \frac{\sqrt[4]{a^1\tilde{b}^4}}{a^1\tilde{b}^4} \right) \cos \mu + \left( \frac{\sqrt[4]{a^3\tilde{b}^2}}{a^2\tilde{b}^2} - \frac{\sqrt[4]{a^3\tilde{b}^3}}{a^3\tilde{b}^3} \right) \sin \mu \right)^2 + 4}^{-2}. \quad (2.38)
\end{align*}
\]

We have

\[
b^1_{\mu=0} = \tilde{b}^1, \quad b^2_{\mu=0} = \tilde{b}^2, \quad b^3_{\mu=0} = \tilde{b}^3, \quad b^4_{\mu=0} = \tilde{b}^4, \quad (2.39)
\]

\[
b^1_{\mu=0} = \tilde{b}^1, \quad b^2_{\mu=0} = \tilde{b}^2, \quad b^3_{\mu=0} = \tilde{b}^3, \quad b^4_{\mu=0} = \tilde{b}^4, \quad (2.40)
\]
and
\[ b_1^4 b_4 = \bar{b}_1 \bar{b}_4, \quad b_2^3 = \bar{b}_2^3. \]  \hspace{1cm} (2.41)

Similarly to (2.32), the invariance
\[ S(A, B) = S(A, \bar{B}) \]  \hspace{1cm} (2.42)
holds fine.

3. Conclusions

The associative commutative Quadrahyperbolic Numbers come from several sources (see, e.g., [1–22]). The fact is known, however, that the applied capacity of relativistic theories is supported by their powerful invariance. In the preceding section we have obtained the 1-parametric transformations which are remarkable in that they act as nonlinear rotations around a fixed vector \( A \) (see Eqs. (2.33)–(2.34)), leave invariant the respective scalar product (defined by Eq. (1.7)), and simultaneously retain the length of rotated vectors \( B \) unchanged (as shown by Eq. (2.39)). One can think of them as defining

The \( \{ \alpha, \beta \} \)-turn of an arbitrary vector \( B \) by the angle \( \mu \) around a fixed vector \( A \).

To show this, we have used specific calculations in terms of associated exponents (2.6)–(2.9). Just similar calculations go through if the \( \{ \alpha, \gamma \} \)-turn, or \( \{ \beta, \gamma \} \)-turn, is dealt with.

The composition of these three transformations do form a 3–parametrical group of isotropy of the fixed vector \( A \). Whence we have gained the new relativistic framework in which the 6–parametrical group of hypercomplex nonlinear invariance should be substituted with the ordinary linear pseudo-Euclidean 6–parametrical Lorentz group.

It is hoped that the rigorous facts reported above may favour due applications of the associative commutative Quadrahyperbolic Numbers to various modern physical–relativistic subjects that may be very different from what we are used to in bilinear (Euclidean and pseudo–Euclidean) theories.

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