STRONGLY DIVISIBLE LATTICES AND CRYSSTALLINE
COHOMOLOGY IN THE IMPERFECT RESIDUE FIELD CASE

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Abstract. Let $k$ be a perfect field of characteristic $p \geq 3$, and let $K$ be a finite totally ramified extension of $K_0 = W(k)[p^{-1}]$. Let $L_0$ be a complete discrete valuation field over $K_0$ whose residue field has a finite $p$-basis, and let $L = L_0 \otimes_{K_0} K$. For $0 \leq r \leq p - 2$, we classify $\mathbb{Z}_p$-lattices of semistable representations of $\text{Gal}(\overline{L}/L)$ with Hodge-Tate weights in $[0, r]$ by strongly divisible lattices. This generalizes the result of [20]. Moreover, if $X$ is a proper smooth formal scheme over $\mathcal{O}_L$, we give a cohomological description of the strongly divisible lattice associated to $H^i_{\text{ét}}(X_L, \mathbb{Z}_p)$ for $i \leq p - 2$, under the assumption that the crystalline cohomology of the special fiber of $X$ is torsion-free in degrees $i$ and $i + 1$. This generalizes a result in [10].

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1. Introduction

Let $k$ be a perfect field of characteristic $p \geq 3$. Let $K$ be a finite totally ramified extension of $K_0 = W(k)[p^{-1}]$, and denote by $\mathcal{O}_K$ its ring of integers. In this article, we consider a complete discrete valuation field $L$ over $K$ whose residue field has a finite $p$-basis, and study lattices of semistable representations of $G_L := \text{Gal}(\overline{L}/L)$. More precisely, let $k'$ be a field extension of $k$ having a finite $p$-basis, and let $\mathcal{O}_{L_0}$ be a Cohen ring for $k'$. Let $\mathcal{O}_L = \mathcal{O}_{L_0} \otimes_{W(k)} \mathcal{O}_K$, and denote $L_0 = \mathcal{O}_{L_0}[p^{-1}]$ and $L = \mathcal{O}_L[p^{-1}]$. For semistable representations of $G_L$, it is proved in [15] (cf. also [8]) that weakly admissibility implies admissibility (which generalizes the result for the
perfect residue field case in [11], thereby giving a complete classification of semistable \( \mathbb{Q}_p \)-representations of \( G_L \).

It is natural to ask further for a classification of \( \mathbb{Z}_p \)-lattices of semistable representations of \( G_L \), since such an integral (or torsion) theory is very useful for studying semistable deformation rings as in the work of Liu on Fontaine’s conjecture [19]. Let \( r < p - 1 \) be a non-negative integer. In the case of perfect residue field, Breuil conjectured such a classification of lattices of semistable representations with Hodge-Tate weights in \([0, r]\) by strongly divisible lattices [7], which is proved by Liu in [20].

We generalize the result in [20] and classify lattices of semistable representations of \( G_L \) with Hodge-Tate weights in \([0, r]\). Let \( E(u) \in W(k)[u] \) be the Eisenstein polynomial for a uniformizer \( \pi \in \mathcal{O}_K \), and let \( S \) be the \( p \)-adically complete divided power envelope of \( \mathcal{O}_{L_0}[u] \) with respect to \( (E(u)) \). A strongly divisible lattice \( \mathcal{M} \) is a finite free \( S \)-module equipped with filtration, Frobenius, monodromy, and an integrable connection \( \nabla: \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{L_0}} \widehat{\Omega}_{\mathcal{O}_{L_0}}/W(k) \) satisfying certain conditions (cf. Definition 4.3). We naturally associate a lattice \( T_{\text{cris}}(\mathcal{M}) \) of semistable \( G_L \)-representation with Hodge-Tate weights in \([0, r]\), and prove:

**Theorem 1.1** (Theorem 4.22). The functor \( T_{\text{cris}}(\cdot) \) gives an anti-equivalence from the category of strongly divisible lattices of weight \( r \) to the category of lattices of semistable \( G_L \)-representations with Hodge-Tate weights in \([0, r]\).

We follow a similar strategy as in [20] to prove above theorem. The main new ingredient is the study of an integrable connection \( \nabla: \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{L_0}} \widehat{\Omega}_{\mathcal{O}_{L_0}}/W(k) \), which is crucial in understanding semistable representations of \( G_L \). Such a connection is trivial in the case of perfect residue field. We adapt the idea in [13] to study how this connection is precisely related to the Galois action.

As an application, we consider in Section 5 a proper smooth formal scheme \( \mathcal{X} \) over \( \mathcal{O}_L \) and study its étale cohomology and crystalline cohomology. We give a cohomological description of the strongly divisible lattice associated to \( H_{\text{dR}}^i(\mathcal{X}_L, \mathbb{Z}_p) \) for \( i \leq p - 2 \), under the assumption that the crystalline cohomology of the special fiber of \( \mathcal{X} \) is torsion-free in degrees \( i \) and \( i + 1 \). Denote the generic fiber of \( \mathcal{X} \) by \( \mathcal{X} \), and let \( \mathcal{X}_0 := \mathcal{X} \times_{\mathcal{O}_L} \mathcal{O}_L/(p) \) and \( \mathcal{X}_{k'} := \mathcal{X} \times_{\mathcal{O}_L} k' \).

**Theorem 1.2** (Theorem 5.1). Let \( i \leq p - 2 \). Suppose that \( H_{\text{cris}}^i(\mathcal{X}_k'/\mathcal{O}_{L_0}) \) and \( H_{\text{cris}}^{i+1}(\mathcal{X}_{k'}/\mathcal{O}_{L_0}) \) are \( p \)-torsion free. Then

(1) \( T_i := H_{\text{dR}}^i(\mathcal{X}_L, \mathbb{Z}_p) \) is torsion free.

(2) \( \mathcal{M}^i := H_{\text{cris}}^i(\mathcal{X}_0/S) \) is a strongly divisible lattice of weight \( i \).

(3) \( T_{\text{cris}}(\mathcal{M}^i) \cong (T_i)^\vee \) as \( G_L \)-representations.

This generalizes a result of Cais-Liu [10]. We remark that the proof in [10] is based on \( A_{\text{inf}} \)-cohomology theory in [4]. As we do not have yet the analogous \( A_{\text{inf}} \)-cohomology theory when the residue field is imperfect, our argument is rather indirect and uses certain base change compatibilities.

A main motivation for studying crystalline representations in the case of imperfect residue field (with a finite \( p \)-basis) is that it provides essential results for understanding crystalline local systems on any \( p \)-adic formal scheme with a smooth integral model over \( \text{Spf} \mathcal{O}_K \). For simplicity, consider the small affine case: let \( R \) be the \( p \)-adic
completion of an étale extension of $O_K[T_1, \ldots, T_d]$. By the construction of de Rham and crystalline period rings in [9], we have well-defined notions of de Rham étale local systems and crystalline étale local systems on the generic fiber of $\text{Spf}R$. If we consider the \textit{p}-adic completion $\hat{R}_{(\pi)}$ of the localization $R_{(\pi)}$ and let $L_1 = \hat{R}_{(\pi)}[p^{-1}]$, then $L_1$ is a complete discrete valuation field whose residue field has a finite \textit{p}-basis. Let $L$ be an isogeny $\mathbb{Z}_p$-local system on the generic fiber of $\text{Spf}R$. Suppose $L|_{\text{Gal}(L_1/L_1)}$ is crystalline. Then by [23] (see also [21]), $L$ is crystalline. Thus, studying a crystalline étale local system on the generic fiber of $\text{Spf}R$ amounts to studying an étale local system which is crystalline as a representation of $\text{Gal}(L_1/L_1)$. For example, some results established in Section 4 in this article for crystalline case are used in proving the main theorem in [12].

This paper is organized as follows. In Section 2, we recall some preliminary results from [8] and [15] on crystalline representations and semistable representations in the imperfect residue field. In Section 3, we review the theory of étale $\varphi$-modules. In both Section 2 and Section 3, we also consider the compatibility of the constructions under certain base change maps, which will be used in the following sections. In Section 4, we define the category of strongly divisible lattices and construct a functor to lattices of semistable representations, and prove Theorem 1.1. In Section 5, we give a cohomological description of strongly divisible lattices for certain geometric situations and prove Theorem 1.2.

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2. Crystalline and semistable representations in imperfect residue field case

2.1. Crystalline representations. We follow the same notations as in Section 1. As a preliminary, we review some results on crystalline $G_L$-representations over $\mathbb{Q}_p$ from [8]. By a representation of $G_L$, we always mean a finite continuous representation. Choose a lifting $T_1, \ldots, T_d \in \mathcal{O}_{L_0}$ of a $\textit{p}$-basis of $k'$. By [11] Cor. 1.2.7(ii), there exists a Frobenius endomorphism $\varphi$ on $\mathcal{O}_{L_0}$ such that $\varphi(T_i) = T_i^p$, and we fix such a Frobenius. Note that there exists a unique ring map $W(k) \to \mathcal{O}_{L_0}$ lifting $k \to k'$. By unicity, this map is compatible with Frobenius. Denote by $\hat{\Omega}_{\mathcal{O}_{L_0}}$ the $\textit{p}$-adically continuous Kahler differential of $\mathcal{O}_{L_0}$ relative to $\mathbb{Z}_p$, i.e. the $\textit{p}$-adic completion of $\Omega^1_{\mathcal{O}_{L_0}/\mathbb{Z}_p}$. We have

$$\hat{\Omega}_{\mathcal{O}_{L_0}} \cong \bigoplus_{i=1}^d \mathcal{O}_{L_0} \cdot d\log T_i.$$
We briefly recall crystalline period rings constructed in [8]. Denote by $O_L^\dagger$ the $p$-adic completion of $O_L$, and let $O_L^\dagger = \lim \downarrow \quad O_L/pO_L$ be its tilt. There exists a natural $W(k)$-linear surjective map $\theta: W(O_L^\dagger) \to O_L^\dagger$ which lifts the projection onto the first factor. Let $A_{\text{cris}}(O_L)$ be the $p$-adic completion of the PD-envelope of $W(O_L^\dagger)$ with respect to ker $\theta$. $A_{\text{cris}}(O_L)$ is equipped with $G_L$-action and Frobenius extending those on $W(O_L^\dagger)$, and is equipped with a natural PD-filtration. Let $\theta_O : O_L \otimes W(k) W(O_L^\dagger) \to O_L^\dagger$ be the $O_L$-linear extension of $\theta$. The integral crystalline period ring $O A_{\text{cris}}(O_L)$ is defined to be the $p$-adic completion of the PD-envelope of $O_L \otimes W(k) W(O_L^\dagger)$ with respect to ker $\theta_O$. $O A_{\text{cris}}(O_L)$ is equipped with $G_L$-action, Frobenius, and PD-filtration compatible with those on $A_{\text{cris}}(O_L)$. Moreover, we have a natural integrable connecton

$$\nabla : O A_{\text{cris}}(O_L) \to O A_{\text{cris}}(O_L) \otimes O_{L_0} \hat{\Omega}_{O_{L_0}}$$

such that $\varphi$ is horizontal and $O A_{\text{cris}}(O_L) \nabla = 0 = A_{\text{cris}}(O_L)$.

Choose compatibly $\epsilon_n \in O_L$ such that $\epsilon_0 = 1$, $\epsilon_n = \epsilon_{n+1}^p$ with $\epsilon_1 \neq 1$, and let $\epsilon = (\epsilon_n)_{n \geq 0} \in O_L$. Then $t := \log [\epsilon] \in A_{\text{cris}}(O_L)$. The crystalline period ring is defined to be $O B_{\text{cris}}(O_L) := O A_{\text{cris}}(O_L)[p^{-1}, t^{-1}]$. We also denote $B_{\text{cris}}(O_L) := A_{\text{cris}}(O_L)[p^{-1}, t^{-1}]$. For each $i = 1, \ldots, d$, we choose compatibly $T_{i,n} \in O_L$ such that $T_{i,0} = T_i, T_{i,n} = T_{i,n+1}^p$, and write $T_i = (T_{i,n})_{n \geq 0} \in O_L^d$. The following result on the structure of period rings is proved in [8].

**Lemma 2.1** (cf. [8 Prop. 2.39]). The map $X_i \mapsto T_i \otimes 1 - 1 \otimes [T_i]$ induces a $A_{\text{cris}}(O_L)$-linear isomorphism

$$A_{\text{cris}}(O_L) \{X_1, \ldots, X_d\} \cong O A_{\text{cris}}(O_L),$$

where the former ring denotes the $p$-adically completed divided power polynomial with variables $X_i$ and coefficients in $A_{\text{cris}}(O_L)$.

**Remark 2.2.** We use different notations for crystalline period rings from [8], where $A_{\text{cris}}(O_L)$, $O A_{\text{cris}}(O_L)$, $B_{\text{cris}}(O_L)$, $O B_{\text{cris}}(O_L)$ are denoted by $A_{\nabla}$, $A_{\text{cris}}$, $B_{\nabla}$, $B_{\text{cris}}$ respectively.

For a $G_L$-representation $V$ over $Q_p$, let

$$D_{\text{cris}}(V) := \text{Hom}_{G_L}(V, O B_{\text{cris}}(O_L)).$$

Writing $V^\vee := \text{Hom}_{Q_p}(V, Q_p)$ for the dual representation of $V$, the natural morphism

$$\alpha_{\text{cris}} : D_{\text{cris}}(V) \otimes L_0 \ O B_{\text{cris}}(O_L) \to V^\vee \otimes Q_p \ O B_{\text{cris}}(O_L)$$

is injective. We say $V$ is crystalline if $\alpha_{\text{cris}}$ is an isomorphism.

**Definition 2.3.** A filtered $(\varphi, \nabla)$-module over $L_0$ is a finite $L_0$-module $D$ equipped with:

- a $\varphi$-semi-linear injective endomorphism $\varphi : D \to D$;
- a decreasing filtration $\text{Fil}^i D_L$ on $D_L := D \otimes L$ by $L$-submodules such that $\text{Fil}^i D_L = D_L$ for $i < 0$ and $\text{Fil}^i D_L = 0$ for $i \gg 0$;
• a topologically quasi-nilpotent integrable connection \( \nabla : D \to D \otimes_{\mathcal{O}_{L_0}} \hat{\Omega}_{L_0} \) which satisfies Griffiths transversality and such that \( \varphi \) is horizontal (cf. \[8\] Def. 4.11, 4.4).

Denote by \( \text{MF}(\varphi, \nabla) \) the category of filtered \((\varphi, \nabla)\)-modules over \( L_0 \), whose morphisms are \( L_0 \)-linear maps compatible with all structure.

By \[8\] Prop. 4.19, \( D_{\text{cris}}(\cdot) \) gives a functor from the category of \( G_L \)-representations over \( \mathbb{Q}_p \) to \( \text{MF}(\varphi, \nabla) \). A filtered \((\varphi, \nabla)\)-module is said to be \textit{weakly admissible} if it satisfies the usual conditions as in \[8\] Def. 4.21. A main result in \[8\] is the following:

**Theorem 2.4** (cf. \[8\] Cor. 4.37]). The functor \( D_{\text{cris}}(\cdot) \) gives an anti-equivalence from the category of crystalline \( G_L \)-representations over \( \mathbb{Q}_p \) to the category of weakly admissible filtered \((\varphi, \nabla)\)-modules over \( L_0 \).

**Remark 2.5.** We use the contravariant functor for \( D_{\text{cris}}(\cdot) \) whereas \[8\] uses the covariant functor. A quasi-inverse of \( D_{\text{cris}}(\cdot) \) in the above theorem is given by

\[
V_{\text{cris}}(D) := \text{Hom}_{\text{Fil}, \varphi, \nabla}(D, \mathcal{O}_{L_0}^{\text{cris}}(\mathcal{T})),
\]

where the filtration compatibility means that the composite map

\[
D_L \to \mathcal{O}_{L_0}^{\text{cris}}(\mathcal{T}) \otimes_{L_0} L \leftarrow \mathcal{O}_{L_0}^{\text{dR}}(\mathcal{T})
\]

is compatible with filtrations. See \[8\] Sec. 2.1 for the construction of the de Rham period ring \( \mathcal{O}_{L_0}^{\text{dR}}(\mathcal{T}) \) (denoted by \( B_{\text{dR}} \) in loc. cit.), and also see \[8\] Prop. 2.47.

For later use, we consider a certain base change map from \( \mathcal{O}_L \) and functoriality of \( D_{\text{cris}}(\cdot) \). Let \( R_{0g} \) be the \( p \)-adic completion of the ring \( \varprojlim_{\to} \mathcal{O}_{L_0} \), and let \( k_g := R_{0g}/(p) \). The Frobenius on \( \mathcal{O}_{L_0} \) extends uniquely to \( R_{0g} \), and we have a \( \varphi \)-compatible isomorphism \( R_{0g} \cong W(k_g) \). Let \( b_g : \mathcal{O}_{L_0} \to W(k_g) \) be the induced map, and denote also by \( b_g : L \to K_g := W(k_g) \otimes_{W(k)} K' \) the \( K \)-linear extension. Choose an embedding \( \mathcal{T} \to K_g \) extending \( b_g \), which induces a map of Galois groups \( G_{K_g} := \text{Gal}(K_g/K) \to G_L \). We then have an induced map of crystalline period rings \( \mathcal{O}_{L_0}^{\text{cris}}(\mathcal{T}) \to \mathcal{O}_{K_g}^{\text{cris}}(\mathcal{T}) \cong \mathcal{B}_{\text{cris}}(\mathcal{O}_{K_g}) \) compatible with all structures. In this way, if \( V \) is a crystalline \( G_L \)-representation, then it can also be considered as a crystalline \( G_{K_g} \)-representation. Furthermore, for \( D_{\text{cris}, K_g}(V) := \text{Hom}_{G_{K_g}}(V, \mathcal{B}_{\text{cris}}(\mathcal{O}_{K_g})) \), we have

\[
D_{\text{cris}}(V) \otimes_{L_0, b_g} W(k_g)[p^{-1}] \cong D_{\text{cris}, K_g}(V)
\]

compatible with Frobenius and Frobenius.

### 2.2. Semistable representations.

We have a natural notion of semistable representations of \( G_L \) as in [15]. Choose a uniformizer \( \pi \in \mathcal{O}_K \). For integers \( n \geq 0 \), compatibly choose \( \pi_n \in K \) such that \( \pi_0 = \pi \) and \( \pi_{n+1} = \pi_n \), and let \( \overline{\pi} := (\pi_n)_{n \geq 0} \in \mathcal{O}_{\mathcal{T}}^{\text{st}} \). Write \( u := \text{log}(\overline{\pi}) \in \mathcal{B}_{\text{dR}}(\mathcal{T}) \cong \mathcal{B}_{\text{dR}}(\mathcal{O}_{K_g}) \), and let \( \mathcal{B}_{\text{st}}(\mathcal{T}) := \mathcal{B}_{\text{cris}}(\mathcal{T})[u] \subset \mathcal{B}_{\text{dR}}(\mathcal{T}) \).

The semistable period ring is defined to be \( \mathcal{B}_{\text{st}}(\mathcal{T}) := \mathcal{B}_{\text{cris}}(\mathcal{T})[u] \), equipped with \( \varphi \) given by \( \varphi(u) = pu \) and \( \mathcal{B}_{\text{cris}}(\mathcal{T}) \)-linear derivation \( N \) given by \( N(u) = -1 \).

For a \( G_L \)-representation \( V \) over \( \mathbb{Q}_p \), let

\[
D_{\text{st}}(V) := \text{Hom}_{G_L}(V, \mathcal{B}_{\text{st}}(\mathcal{T})).
\]
The natural morphism
\[ \alpha_{st}: D_{st}(V) \otimes_{\mathcal{O}_L} \mathring{\mathcal{O}}_{st}(\mathcal{O}_T) \rightarrow V^\vee \otimes_{\mathbb{Q}_p} \mathring{\mathcal{O}}_{st}(\mathcal{O}_L) \]
is injective. We say \( V \) is semistable if \( \alpha_{st} \) is an isomorphism.

**Definition 2.6.** A filtered \((\varphi, N, \nabla)\)-module over \( L_0 \) is a filtered \((\varphi, \nabla)\)-module \( D \) equipped with \( L_0 \)-linear map \( N: D \rightarrow D \) which commutes with \( \nabla \) and satisfies \( N \varphi = p \varphi N \). Denote by \( \text{MF}(\varphi, N, \nabla) \) the category of filtered \((\varphi, N, \nabla)\)-modules over \( L_0 \) whose morphisms are \( L_0 \)-linear maps compatible with all structure.

Note that \( \text{MF}(\varphi, \nabla) \) is a full subcategory of \( \text{MF}(\varphi, N, \nabla) \) consisting of objects with \( N = 0 \). Similarly as in [8, Prop. 4.19 Pf.], \( D_{st}(\cdot) \) gives a functor from the category of \( G_L \)-representations over \( \mathbb{Q}_p \) to \( \text{MF}(\varphi, N, \nabla) \). Weakly admissibility is defined as in the usual sense (cf. [15, Def. 2.4.8]), and following is proved in [15].

**Theorem 2.7** (cf. [15, Thm. 2.4.9]). The functor \( D_{st}(\cdot) \) gives an anti-equivalence from the category of semistable \( G_L \)-representations over \( \mathbb{Q}_p \) to the category of weakly admissible filtered \((\varphi, N, \nabla)\)-modules over \( L_0 \). A quasi-inverse of \( D_{st}(\cdot) \) is given by
\[ V_{st}(D) := \text{Hom}_{\text{Fil}, \varphi, N, \nabla}(D, \mathring{\mathcal{O}}_{st}(\mathcal{O}_T)). \]

As in the crystalline case, we have analogous functoriality for semistable representations under the base change \( b_g: L \rightarrow K_g \).

### 3. Étale \( \varphi \)-modules

We briefly recall some necessary facts on étale \( \varphi \)-modules from [16, Sec. 7] (cf. also [15, Sec. 4.2]). Let \( \mathcal{S} := \mathcal{O}_{L_0}[[u]] \) equipped with Frobenius given by \( \varphi(u) = u^p \). Let \( \mathcal{O}_\mathcal{E} \) be the \( p \)-adic completion of \( \mathcal{S}[u^{-1}] \), equipped with the Frobenius extending that on \( \mathcal{S} \).

**Definition 3.1.** An étale \((\varphi, \mathcal{O}_\mathcal{E})\)-module is a pair \((\mathcal{M}, \varphi_\mathcal{M})\) where \( \mathcal{M} \) is a finite free \( \mathcal{O}_\mathcal{E} \)-module and \( \varphi_\mathcal{M}: \mathcal{M} \rightarrow \mathcal{M} \) is a \( \varphi \)-semilinear endomorphism such that \( 1 \otimes \varphi_\mathcal{M}: \mathcal{M} \rightarrow \mathcal{M} \) is an isomorphism. Let \( \text{Mod}_{\mathcal{O}_\mathcal{E}} \) denote the category of étale \((\varphi, \mathcal{O}_\mathcal{E})\)-modules whose morphisms are \( \mathcal{O}_\mathcal{E} \)-linear maps compatible with Frobenius.

We use étale \((\varphi, \mathcal{O}_\mathcal{E})\)-modules to study certain Galois representations. Let \( L_\infty := \bigcup_{n \geq 0} L(\pi_n) \subset \overline{T} \) and \( \tilde{L}_\infty := \bigcup_{n \geq 0} L_\infty(T_{1,n}, \ldots, T_{d,n}) \subset \overline{T} \). Denote by \( G_{L_\infty} := \text{Gal}(\overline{T}/L_\infty) \) and \( G_{\tilde{L}_\infty} := \text{Gal}(\overline{T}/\tilde{L}_\infty) \) the corresponding Galois subgroups of \( G_L \). Let \( \text{Rep}_{\mathbb{Z}_p}(G_{\tilde{L}_\infty}) \) be the category of finite free \( \mathbb{Z}_p \)-representations of \( G_{\tilde{L}_\infty} \).

There exists a unique \( W(k) \)-linear map \( \mathcal{O}_{L_0} \rightarrow W(\mathcal{O}_L) \) which maps \( T_i \) to \( [T_i] \) and is compatible with Frobenius. We have a \( \varphi \)-equivariant embedding \( \mathcal{S} \rightarrow W(\mathcal{O}_L) \) given by \( u \mapsto [u] \), which extends to an embedding \( \mathcal{O}_\mathcal{E} \rightarrow W(\overline{T}) \). Let \( \mathcal{O}_\mathcal{E}^{ur} \) be the ring of integers of the maximal unramified extension of \( \mathcal{O}_\mathcal{E}[p^{-1}] \) inside \( W(\overline{T})[p^{-1}] \), and let \( \hat{\mathcal{O}}_\mathcal{E}^{ur} \) be its \( p \)-adic completion. We define \( \hat{\mathcal{S}}^{ur} := \hat{\mathcal{O}}_\mathcal{E}^{ur} \cap W(\mathcal{O}_L) \subset W(\overline{T}) \). The following is proved in [15].
Lemma 3.2. (cf. [16] Lem. 7.5, 7.6) We have \((\hat{\mathcal{O}}^\text{ur})^{G_{L,\infty}} = \mathcal{O}_E\). Furthermore, there exists a unique \(G_{L,\infty}\)-equivariant ring endomorphism \(\varphi\) on \(\hat{\mathcal{O}}^\text{ur}\) lifting the \(p\)-th power map on \(\hat{\mathcal{O}}^\text{ur} / (p)\) and extending \(\varphi\) on \(\mathcal{O}_E\). The inclusion \(\hat{\mathcal{O}}^\text{ur} \hookrightarrow W(\mathcal{T})\) is \(\varphi\)-equivariant.

For \(\mathcal{M} \in \text{Mod}_{\mathcal{O}_E}\) and \(T \in \text{Rep}_{\mathbb{Z}_p}(G_{L,\infty})\), define
\[
T(\mathcal{M}) := \text{Hom}_{\mathcal{O}_E,\varphi}(\mathcal{M}, \hat{\mathcal{O}}^\text{ur}), \quad \mathcal{M}(T) := \text{Hom}_{G_{L,\infty}}(T, \hat{\mathcal{O}}^\text{ur}).
\]

The following equivalence is proved in [16].

Proposition 3.3. (cf. [16] Prop. 7.7) The assignments \(T(\cdot)\) and \(\mathcal{M}(\cdot)\) are exact rank-preserving anti-equivalences (quasi-inverse of each other) of \(\otimes\)-categories between \(\text{Mod}_{\mathcal{O}_E}\) and \(\text{Rep}_{\mathbb{Z}_p}(G_{L,\infty})\). Moreover, \(T(\cdot)\) and \(\mathcal{M}(\cdot)\) commute with taking duals.

We remark on the compatibility with respect to \(b_g : L \rightarrow K_g\). Let \(\mathfrak{S}_g := W(k_g)[[u]]\) and \(\mathcal{O}_{E,g}\) be the \(p\)-adic completion of \(\mathfrak{S}_g[u^{-1}]\). Define \(\hat{\mathcal{O}}^\text{ur}_{E,g} \subset W(K_g^\flat)\) similarly as above and \(\hat{\mathfrak{S}}^\text{ur}_g := \hat{\mathcal{O}}^\text{ur}_{E,g} \cap W(\mathcal{O}^\phi_{K_g^\flat})\). Let \(K_{g,\infty} := \bigcup_{n \geq 0} K_g(\pi_n) \subset K_g\), and write \(G_{K_g,\infty} = \text{Gal}(K_g^\flat/K_{g,\infty})\). By [15] Prop. 4.2.5 Pf., under a suitable choice of embedding \(\mathcal{T} \hookrightarrow K_g\) extending \(b_g\) so that \(T_{i,n} \in K_g\) for each \(i = 1, \ldots, d\) and \(n \geq 0\), the map \(G_{K_g} \rightarrow G_L\) induces an isomorphism \(G_{K_g,\infty} \cong G_{L,\infty}\). We fix such a choice from now on. Note that the \(p\)-adic completions of \(\mathcal{T}\) and \(K_g\) are isomorphic, i.e. \(\hat{\mathcal{T}} = \hat{K}_g\), and thus \(\mathcal{T}^\flat = \mathcal{O}^\phi_{K_g^\flat}\). If \(\mathcal{M} \in \text{Mod}_{\mathcal{O}_E}\), then \(\mathcal{M}_g = \mathcal{M} \otimes_{\mathcal{O}_E} \mathcal{O}_{E,g}\) with the induced tensor-product Frobenius \(\varphi_{\mathcal{M}} \otimes \varphi_{\mathcal{O}_{E,g}}\) is an étale \((\varphi, \mathcal{O}_{E,g})\)-module. Furthermore, we have a natural isomorphism
\[
T(\mathcal{M}) \cong T(\mathcal{M}_g) := \text{Hom}_{\mathcal{O}_{E,g},\varphi}(\mathcal{M}_g, \hat{\mathcal{O}}^\text{ur}_{E,g})
\]
of \(G_{K_g,\infty}\)-representations.

Lemma 3.4. We have
\[
\hat{\mathcal{O}}^\text{ur} \cap \hat{\mathfrak{S}}^\text{ur}_g = \hat{\mathfrak{S}}^\text{ur}
\]
as subrings of \(\hat{\mathcal{O}}^\text{ur}_{E,g}\).

Proof. Note that \(\hat{\mathcal{O}}^\text{ur}_E \subset W(\mathcal{T}^\flat)\) and \(\hat{\mathfrak{S}}^\text{ur}_g \subset W(\mathcal{O}^\phi_{K_g^\flat}) = W(\mathcal{O}^\phi_{T})\). So
\[
\hat{\mathcal{O}}^\text{ur}_{E} \cap \hat{\mathfrak{S}}^\text{ur}_g = \hat{\mathcal{O}}^\text{ur}_E \cap W(\mathcal{O}^\phi_{T}) = \hat{\mathfrak{S}}^\text{ur}.
\]

\[\square\]

4. Strongly divisible lattices and semistable representations

4.1. Strongly divisible lattices. Fix a positive integer \(r \leq p - 2\). We will classify lattices of semistable \(G_L\)-representations with Hodge-Tate weights in \([0, r]\) via strongly divisible lattices, generalizing [20] Thm. 2.3.5]. Denote by \(\text{MF}^w(r)(\varphi, N, \nabla)\) the full subcategory of \(\text{MF}(\varphi, N, \nabla)\) consisting of weakly admissibly modules \(D\) such that \(\text{Fil}^0 D_L = D_L\) and \(\text{Fil}^{r+1} D_L = 0\).
Let $E(u) \in W(k)[u]$ be the Eisenstein polynomial for $\pi$. Let $S$ be the $p$-adically completed divided power envelope of $\mathfrak{S}$ with respect to $(E(u))$. The Frobenius on $\mathfrak{S}$ extends uniquely to $S$. For each integer $i \geq 0$, let $\text{Fil}^i S$ be the $p$-adically completed ideal of $S$ generated by the divided powers $\gamma_j(E(u)) = \frac{E(u)^j}{j!}$, $j \geq i$. Note that $\varphi(\text{Fil}^i S) \subset p^i S$ if $i \leq p - 1$. Let $N: S \to S$ be a $\mathcal{O}_{L_0}$-linear derivation given by $N(u) = -u$. We also have a natural integrable connection

$$\nabla: S \to S \otimes_{\mathcal{O}_{L_0}} \widehat{\mathcal{O}}_{L_0},$$

given by the universal derivation on $\mathcal{O}_{L_0}$, which commutes with $N$. Note that the embedding $\mathfrak{S} \to W(\mathcal{O}_{\mathfrak{L}})$ extends to $S \to A_{\text{cris}}(\mathcal{O}_{\mathfrak{T}})$, which is compatible with $\varphi$, filtration, and $G_{L_0}^\text{cris}$-action. Similarly as in [20, Lem. 3.2.2], we see that the induced map $S/(p^n) \to A_{\text{cris}}(\mathcal{O}_{\mathfrak{T}})/(p^n)$ is faithfully flat for each $n \geq 1$.

**Lemma 4.1.** Let $x \in S$ (resp. $A_{\text{cris}}(\mathcal{O}_{\mathfrak{T}})$), and let $i, j \geq 0$ be any integers. If $E(u)^j x \in \text{Fil}^{i+1} S$ (resp. $E(\gamma_i)^j x \in \text{Fil}^{i+1} A_{\text{cris}}(\mathcal{O}_{\mathfrak{T}})$), then $x \in \text{Fil}^i S$ (resp. $x \in \text{Fil}^i A_{\text{cris}}(\mathcal{O}_{\mathfrak{T}})$).

**Proof.** This follows from a similar argument as in [20, Lem. 3.2.2].

For $D \in \text{MF}_{w,r}(\varphi, N, \nabla)$, we define $\mathcal{D}(D) := S[p^{-1}] \otimes_{L_0} D$ equipped with the tensor product Frobenius. Let $N: \mathcal{D}(D) \to \mathcal{D}(D)$ be the $L_0$-linear derivation given by $N_S \otimes 1 + 1 \otimes N_D$, and let

$$\nabla: \mathcal{D}(D) \to \mathcal{D}(D) \otimes_{\mathcal{O}_{L_0}} \widehat{\mathcal{O}}_{L_0}$$

be the connection given by $\nabla_S \otimes 1 + 1 \otimes \nabla_D$. For each $j = 1, \ldots, d$, let $N_{T_j}: \mathcal{D}(D) \to \mathcal{D}(D)$ be the derivation given by $\nabla: \mathcal{D}(D) \to \mathcal{D}(D) \otimes_{\mathcal{O}_{L_0}} \widehat{\mathcal{O}}_{L_0} \cong \bigoplus_{j=1}^d \mathcal{D}(D) \cdot d \log T_j$ composed with the projection to the $j$-th factor, and let $\partial_{T_j}: \mathcal{D}(D) \to \mathcal{D}(D)$ be the derivation given by $\partial_{T_j} = T_j^{-1} N_{T_j}$. Define a decreasing filtration on $\mathcal{D}(D)$ by $S[p^{-1}]$-submodules $\text{Fil}^i \mathcal{D}(D)$ inductively by $\text{Fil}^0 \mathcal{D}(D) = \mathcal{D}(D)$ and

$$\text{Fil}^{i+1} \mathcal{D}(D) = \{ x \in \mathcal{D}(D) \mid N(x) \in \text{Fil}^i \mathcal{D}(D), \quad q_{\pi}(x) \in \text{Fil}^{i+1} D_L \}$$

where $q_\pi: \mathcal{D}(D) \to D_L$ is the map induced by $S[p^{-1}] \to L$, $u \mapsto \pi$.

We thank Tong Liu for the following lemma on Griffiths transversality.

**Lemma 4.2.** The connection $\nabla$ on $\mathcal{D}(D)$ satisfies Griffiths transversality:

$$\nabla(\text{Fil}^{i+1} \mathcal{D}(D)) \subset \text{Fil}^i \mathcal{D}(D) \otimes_{\mathcal{O}_{L_0}} \widehat{\mathcal{O}}_{L_0}.$$

**Proof.** We need to show $N_{T_j}(\text{Fil}^{i+1} \mathcal{D}(D)) \subset \text{Fil}^i \mathcal{D}(D)$. We induct on $i$. The case $i = 0$ is clear. For $i \geq 1$, suppose $N_{T_j}(\text{Fil}^i \mathcal{D}(D)) \subset \text{Fil}^{i-1} \mathcal{D}(D)$, and let $x \in \text{Fil}^{i+1} \mathcal{D}(D)$. We have

$$N(N_{T_j}(x)) = N_{T_j}(N(x)) \in N_{T_j}(\text{Fil}^i \mathcal{D}(D)) \subset \text{Fil}^{i-1} \mathcal{D}(D).$$

Furthermore, since $N_{T_j}: S \to S$ is $W(k)[u]$-linear, we have $N_{T_j} \circ q_\pi = q_\pi \circ N_{T_j}$ as maps from $\mathcal{D}(D)$ to $D_L$. Thus,

$$q_\pi(N_{T_j}(x)) = N_{T_j}(q_\pi(x)) \in N_{T_j}(\text{Fil}^{i+1} D_L) \subset \text{Fil}^i D_L.$$

Hence, $N_{T_j}(x) \in \text{Fil}^i \mathcal{D}(D)$. \qed
Definition 4.3. Let \( \text{Mod}_S^r \) be a category whose objects are finite free \( S \)-modules \( \mathcal{M} \) with \( \mathcal{M}[p^{-1}] \cong \mathcal{D}(D) \) for some \( D \in \text{MF}^{u,r}(\varphi, N, \nabla) \), such that:

- \( \mathcal{M} \) is stable under \( \varphi_{\mathcal{D}(D)} \) and \( N_{\mathcal{D}(D)} \);
- \( \mathcal{M} \) is stable under \( \nabla_{\mathcal{D}(D)} \), i.e. \( \nabla_{\mathcal{D}(D)} \) induces a connection \( \nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{L}_0}} \mathcal{O}_{\mathcal{L}_0} \);
- Let \( \text{Fil}^r \mathcal{M} := \mathcal{M} \cap \text{Fil}^r \mathcal{D}(D) \). Then \( \varphi(\text{Fil}^r \mathcal{M}) \subset p^r \mathcal{M} \). Denote \( \varphi_r = \frac{\varphi}{p^r} : \text{Fil}^r \mathcal{M} \to \mathcal{M} \).

Morphisms in \( \text{Mod}_S^r \) are \( S \)-linear maps compatible with \( \varphi, N, \nabla, \text{Fil}^r \). In above situation, we say \( \mathcal{M} \) is a strongly divisible lattice of weight \( r \) in \( \mathcal{D}(D) \).

Lemma 4.4. Let \( D \in \text{MF}^{u,r}(\varphi, N, \nabla) \), and let \( \mathcal{M} \in \text{Mod}_S^r \) be a strongly divisible lattice of weight \( r \) in \( \mathcal{D}(D) \). Then \( \mathcal{M} \) is generated by \( \varphi_r(\text{Fil}^r \mathcal{M}) \) as \( S \)-modules.

Proof. Since \( D \) is weakly admissible, this follows from the same argument as in [6, Prop. 2.1.3 Pf.]. \( \square \)

We now construct a functor \( T_{\text{cris}}(\cdot) \) from \( \text{Mod}_S^r \) to the category of finite free \( \mathbb{Z}_p \)-representations of \( G_L \). Let \( \mathcal{M} \in \text{Mod}_S^r \) and \( D \in \text{MF}^{u,r}(\varphi, N, \nabla) \) such that \( \mathcal{M} \) is a strongly divisible lattice in \( \mathcal{D}(D) \). We have a natural \( \mathbf{A}_{\text{cris}}(\mathcal{O}_L) \)-semi-linear \( G_{L_\infty} \)-action on \( \mathbf{A}_{\text{cris}}(\mathcal{O}_L) \otimes_S \mathcal{M} \) given by the \( G_{L_\infty} \)-action on \( \mathbf{A}_{\text{cris}}(\mathcal{O}_L) \) and the trivial \( G_{L_\infty} \)-action on \( \mathcal{M} \). We extend this to a \( G_L \)-action using differential operators. For each \( i = 1, \ldots, d \), let \( N_{T_i} : \mathcal{M} \to \mathcal{M} \) be the derivation given by \( \nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{L}_0}} \mathcal{O}_{\mathcal{L}_0} \cong \bigoplus_{i=1}^d \mathcal{M} \cdot d \log T_i \) composed with the projection to the \( i \)-th factor. For any \( \sigma \in G_L \), write

\[
\ell(\sigma) := \frac{\sigma([\mathcal{M}])}{[\mathcal{M}]} \quad \text{and} \quad \mu_i(\sigma) := \frac{\sigma([T_i])}{[T_i]}, \quad i = 1, \ldots, d.
\]

Note that \( \log(\ell(\sigma)) \) and \( \log(\mu_i(\sigma)) \) lie in \( \text{Fil}^1 \mathbf{A}_{\text{cris}}(\mathcal{O}_L) \). For any \( a \otimes x \in \mathbf{A}_{\text{cris}}(\mathcal{O}_L) \otimes_S \mathcal{M} \), define

\[
(4.1) \quad \sigma(a \otimes x) = \sum \sigma(a) \gamma_{i_0}(-\log(\ell(\sigma))) \gamma_{i_1}(\log(\mu_1(\sigma))) \cdots \gamma_{i_d}(\log(\mu_d(\sigma))) \cdot N_{T_1}^{i_1} \cdots N_{T_d}^{i_d}(x)
\]

where the sum goes over the multi-index \( (i_0, i_1, \ldots, i_d) \) of non-negative integers and \( \gamma_i \) denotes divided powers. Note that if \( \sigma \in G_{L_\infty} \), then \( \sigma(a \otimes x) = \sigma(a) \otimes x \). Since \( \nabla_{\mathcal{M}} \) and \( N \) are topologically quasi-nilpotent and \( \gamma_j(-\log(\ell(\sigma))), \gamma_j(\log(\mu_i(\sigma))) \to 0 \) \( p \)-adically as \( j \to \infty \), above sum converges. It is standard to check that this gives a well-defined \( \mathbf{A}_{\text{cris}}(\mathcal{O}_L) \)-semi-linear \( G_L \)-action, which is compatible with \( \varphi \) since \( N \varphi = p \varphi N \) and \( \varphi \) is horizontal with respect to \( \nabla_{\mathcal{M}} \). Furthermore, this \( G_L \)-action is compatible with filtration since \( N \) and \( \nabla \) satisfy Griffiths transversality by definition and Lemma 4.2.

Let

\[
T_{\text{cris}}(\cdot) := \text{Hom}_{S, \text{Fil}, \varphi}(\cdot, \mathbf{A}_{\text{cris}}(\mathcal{O}_L)).
\]

Note that \( T_{\text{cris}}(\cdot) \cong \text{Hom}_{\mathbf{A}_{\text{cris}}(\mathcal{O}_L), \text{Fil}, \varphi}(\mathbf{A}_{\text{cris}}(\mathcal{O}_L) \otimes_S \mathcal{M}, \mathbf{A}_{\text{cris}}(\mathcal{O}_L)) \). Define the \( G_L \)-action on \( T_{\text{cris}}(\mathcal{M}) \) by

\[
\sigma(f)(x) = \sigma(f(\sigma^{-1}(x))) \quad \text{for any} \quad x \in \mathbf{A}_{\text{cris}}(\mathcal{O}_L) \otimes_S \mathcal{M}.
\]
To study $T_{\text{cris}}(\mathcal{M})$, we consider $b_g: \mathcal{O}_{L_0} \to W(k_g)$. Let $S_g$ be the $p$-adically completed divided power envelope of $\mathcal{O}_{g}$ with respect to $(E(u))$, equipped with $\varphi$, filtration, and $N$ similarly as above. Note that $D_g := D \otimes_{L_0,b_g} W(k_g)[p^{-1}]$ is a weakly admissible filtered $(\varphi,N)$-module over $W(k_g)[p^{-1}]$, and $\mathcal{M}_g := \mathcal{M} \otimes_{S,b_g} S_g$ with the induced tensor product $\varphi$, filtration, $N$ is a strongly divisible lattice of $S_g[p^{-1}] \otimes W(k_g)[p^{-1}] D_g$ as in [20]. We have a natural injective map

$$T_{\text{cris}}(\mathcal{M}) \to T_{\text{cris}}(\mathcal{M}_g) := \text{Hom}_{S_g,\text{Fil}}(\mathcal{M}_g, A_{\text{cris}}(O_{K_g})).$$

Since the $G_{K_g}$-action on $T_{\text{cris}}(\mathcal{M}_g)$ is given by [20, Eq. (5.1.1)], this map is $G_{K_g}$-equivariant. Moreover, $T_{\text{cris}}(\mathcal{M}_g)$ is finite free over $\mathbb{Z}_p$ by [20], so $T_{\text{cris}}(\mathcal{M})$ is finite free over $\mathbb{Z}_p$. We first study how this construction relates with étale $\varphi$-modules.

**Definition 4.5.** A quasi-Kisin module over $\mathcal{O}$ of height $r$ is a finite free $\mathcal{O}$-module $\mathcal{M}$ equipped with $\varphi$-semi-linear endomorphism $\varphi: \mathcal{M} \to \mathcal{M}$ such that the cokernel of $1 \otimes \varphi: \varphi^* \mathcal{M} = \mathcal{O} \otimes_{\varphi,\mathcal{O}} \mathcal{M} \to \mathcal{M}$ is killed by $E(u)^r$.

By Lemma 4.4 and [15, Thm. 5.1.3], there exists a quasi-Kisin module $\mathcal{M}$ of height $r$ such that $S \otimes_{\varphi,\mathcal{O}} \mathcal{M} \cong \mathcal{M}$ compatible with Frobenius and

$$\text{Fil}^r \mathcal{M} = \{ x \in S \otimes_{\varphi,\mathcal{O}} \mathcal{M} \mid (1 \otimes \varphi_{\mathcal{M}})(x) \in \text{Fil}^r S \otimes_{\mathcal{O}} \mathcal{M} \}$$

if we identify $S \otimes_{\varphi,\mathcal{O}} \mathcal{M} = \mathcal{M}$. Let $\mathcal{M} = \mathcal{O}_\varepsilon \otimes_{\mathcal{O}} \mathcal{M}$ with the induced tensor product Frobenius. Then $\mathcal{M}$ is an étale $\varphi$-module.

**Lemma 4.6.** The natural $G_{L_{\infty}}$-equivariant map

$$\text{Hom}_{\mathcal{O},\varphi}(\mathcal{M}, \widehat{\mathcal{O}}^\text{ur}) \to T(\mathcal{M}) = \text{Hom}_{\mathcal{O}_\varepsilon,\varphi}(\mathcal{M}, \widehat{\mathcal{O}}^\text{ur})$$

is an isomorphism.

**Proof.** By [14, Sec. B, Prop. 1.8.3], the map

$$\text{Hom}_{\mathcal{O},\varphi}(\mathcal{M}, \widehat{\mathcal{O}}^\text{ur}) \to \text{Hom}_{\mathcal{O}_\varepsilon,\varphi}(\mathcal{M}, \widehat{\mathcal{O}}^\text{ur}_{\varepsilon,g})$$

induced by $b_g: \mathcal{O}_{L_0} \to W(k_g)$ is an isomorphism. Since $T(\mathcal{M}) \cong \text{Hom}_{\mathcal{O}_\varepsilon,\varphi}(\mathcal{M}, \widehat{\mathcal{O}}^\text{ur}_{\varepsilon,g})$, the assertion follows from Lemma 3.4. \hfill $\square$

Note that the embedding $\varphi: \widehat{\mathcal{O}}^\text{ur} \to A_{\text{cris}}(O_T)$ induces a natural $G_{L_{\infty}}$-equivariant injective map

$$\text{Hom}_{\mathcal{O},\varphi}(\mathcal{M}, \widehat{\mathcal{O}}^\text{ur}) \to T_{\text{cris}}(\mathcal{M}).$$

**Lemma 4.7.** The natural maps $\text{Hom}_{\mathcal{O},\varphi}(\mathcal{M}, \widehat{\mathcal{O}}^\text{ur}) \to T_{\text{cris}}(\mathcal{M})$ and $T_{\text{cris}}(\mathcal{M}) \to T_{\text{cris}}(\mathcal{M}_g)$ given above are $G_{L_{\infty}}$-equivariant isomorphisms.

**Proof.** We have a commutative diagram

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{O},\varphi}(\mathcal{M}, \widehat{\mathcal{O}}^\text{ur}) & \longrightarrow & T_{\text{cris}}(\mathcal{M}) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{O},\varphi}(\mathcal{M}, \widehat{\mathcal{O}}^\text{ur}_g) & \longrightarrow & T_{\text{cris}}(\mathcal{M}_g)
\end{array}$$

The left vertical map is an isomorphism by Lemma 4.6 and \( T(\mathcal{M}) \cong T(\mathcal{M}_g) \), and the bottom horizontal map is an isomorphism by [20] Lem. 3.3.4. Since the other two maps are injective, they are isomorphisms.

In the remainder of this subsection, we show that \( T_{\text{cris}}(\cdot) \) gives a fully faithful functor from \( \text{Mod}^d_S \) to the category of lattices in semistable \( G_L \)-representations with Hodge-Tate weights in \([0, r]\). We first show that \( T_{\text{cris}}(\mathcal{M})[p^{-1}] \) is a semistable \( G_L \)-representation. By Theorem 2.7 it suffices to show that

\[
T_{\text{cris}}(\mathcal{M})[p^{-1}] \cong V_{\text{st}}(D) = \text{Hom}_{\text{Fil}, \varphi, N, V}(D, \text{OB}_{\text{st}}(\mathcal{O}_T))
\]

as \( G_L \)-representations.

By [20] Thm. 2.2.1, Prop. 3.5.1, Sec. 5.1, we have a natural \( G_{K_g} \)-equivariant isomorphism

\[
T_{\text{cris}}(\mathcal{M}_g)[p^{-1}] \cong V_{\text{st}}(D_g) = \text{Hom}_{\text{Fil}, \varphi, N}(D_g, \text{B}_{\text{st}}(\mathcal{O}_{K_g})).
\]

Note that this isomorphism is induced by the natural maps

\[
(4.2) \quad D_g \otimes_{W(k_g)} \text{B}_{\text{st}}(\mathcal{O}_{K_g}) \to D_g \otimes_{W(k_g)} \hat{\text{B}}_{\text{st}} \cong \mathcal{M}_g \otimes_{S_g} \text{B}_{\text{st}} \to \mathcal{M}_g \otimes_{S_g} \text{B}_{\text{cris}}(\mathcal{O}_{K_g}),
\]

which is \( G_{K_g} \)-equivariant by [20] Lem. 5.2.1. Here, \( \hat{\text{B}}_{\text{st}} \) denotes the period ring constructed in [5], so that we have a natural embedding \( \text{B}_{\text{st}}(\mathcal{O}_{K_g}) \hookrightarrow \hat{\text{B}}_{\text{st}} \) and projection \( \hat{\text{B}}_{\text{st}} \to \text{B}_{\text{cris}}(\mathcal{O}_{K_g}) \) which give above maps. Let \( t_2 : L_0 \to \text{B}_{\text{st}}(\mathcal{O}_T) \) be the map given by \( T_i \mapsto [T_i] \). Since \( \text{B}_{\text{cris}}(\mathcal{O}_T) = \text{B}_{\text{cris}}(\mathcal{O}_{K_g}) \) and \( \mathcal{M} \otimes_S \text{B}_{\text{cris}}(\mathcal{O}_T) \cong \mathcal{M} \otimes_S \text{B}_{\text{cris}}(\mathcal{O}_{K_g}) \), the composite of the map (4.2) with \( D \otimes_{L_0, t_2} \text{B}_{\text{st}}(\mathcal{O}_T) \to D_g \otimes_{W(k_g)} \text{B}_{\text{st}}(\mathcal{O}_{K_g}) \) induces a \( G_{K_g} \)-equivariant isomorphism

\[
(4.3) \quad T_{\text{cris}}(\mathcal{M})[p^{-1}] \cong \text{Hom}_{\text{B}_{\text{st}}(\mathcal{O}_T), \text{Fil}, \varphi, N}(D \otimes_{L_0} \text{B}_{\text{st}}(\mathcal{O}_T), \text{B}_{\text{st}}(\mathcal{O}_T)).
\]

We define \( G_L \)-action on \( D \otimes_{L_0} \text{B}_{\text{st}}(\mathcal{O}_T) \) by

\[
(4.4) \quad \sigma(x \otimes a) = \sum \sigma(a) \gamma_{i_1}(\log(\mu_1(\sigma))) \cdots \gamma_{i_d}(\log(\mu_d(\sigma))) \cdot N_{T_i}^{i_1} \cdots N_{T_d}^{i_d}(x)
\]

for \( \sigma \in G_L \) and \( x \otimes a \in D \otimes_{L_0} \text{B}_{\text{st}}(\mathcal{O}_T) \), where the sum goes over the multi-index \((i_1, \ldots, i_d)\) of non-negative integers. Then the map \( D \otimes_{L_0} \text{B}_{\text{st}}(\mathcal{O}_T) \to \mathcal{M} \otimes_S \text{B}_{\text{cris}}(\mathcal{O}_T) \) giving the isomorphism (4.3) is also compatible with \( G_{L_{\infty}} \)-actions, since the \( G_L \)-action on \( \mathcal{M} \otimes_S \text{B}_{\text{cris}}(\mathcal{O}_T) \) is given by Equation (4.1). Note that the Galois subgroups \( G_{L_{\infty}} \) and \( G_{K_g} \) generate \( G_L \) by [15] Lem. 4.4.1. Thus, the isomorphism (4.3) is \( G_L \)-equivariant (where \( \text{Hom}_{\text{B}_{\text{st}}(\mathcal{O}_T), \text{Fil}, \varphi, N}(D \otimes_{L_0} \text{B}_{\text{st}}(\mathcal{O}_T), \text{B}_{\text{st}}(\mathcal{O}_T)) \) is equipped with the \( G_L \)-action via \( \sigma(f)(x) = \sigma(f(\sigma^{-1}(x))) \)).

**Lemma 4.8.** We have a natural \( G_L \)-equivariant isomorphism

\[
\text{Hom}_{\text{B}_{\text{st}}(\mathcal{O}_T), \text{Fil}, \varphi, N}(D \otimes_{L_0} \text{B}_{\text{st}}(\mathcal{O}_T), \text{B}_{\text{st}}(\mathcal{O}_T)) \cong V_{\text{st}}(D).
\]

**Proof.** By Lemma 2.1 we have a \( \text{B}_{\text{st}}(\mathcal{O}_T) \)-linear isomorphism

\[
\text{B}_{\text{st}}(\mathcal{O}_T)\{X_1, \ldots, X_d\} \cong \text{OB}_{\text{st}}(\mathcal{O}_T)
\]

given by \( X_i \mapsto T_i \otimes 1 - 1 \otimes [T_i] \). Consider the projection

\[
pr : \text{OB}_{\text{st}}(\mathcal{O}_T) \to \text{B}_{\text{st}}(\mathcal{O}_T)
\]
given by $X_i = T_i - \lfloor T_i \rfloor \mapsto 0$, which is compatible with filtration, $\varphi$, and $N$. This induces the projection

$$ pr: D \otimes_{L_{0}, \mathbf{i}_{1}} O_{B}^{\text{st}}(O_{\mathcal{T}}) \rightarrow D \otimes_{L_{0}, \mathbf{i}_{2}} B_{\text{st}}(O_{\mathcal{T}}) $$

compatible with $\varphi, N$ and filtration (after tensoring with $L$ over $L_0$). Here, $\mathbf{i}_{1}: L_{0} \rightarrow O_{B}^{\text{st}}(O_{\mathcal{T}})$ is the natural map given by $T_i \mapsto T_i \otimes 1$. We define a $B_{\text{st}}(O_{\mathcal{T}})$-linear section $s$ to $pr$ as follows. For $x \in D$, let

$$ s(x) = \sum \gamma_{i_{1}}([T_{1}] - T_{1}) \cdots \gamma_{i_{d}}([T_{d}] - T_{d}) \cdot \partial_{T_{1}}^{i_{1}} \cdots \partial_{T_{d}}^{i_{d}}(x) $$

where the sum goes over the multi-index $(i_{1}, \ldots, i_{d})$ of non-negative integers. Since $[T_{i}] - T_{i} \in \text{Fil}^{1} O_{A, \text{cris}}(O_{\mathcal{T}})$ and $\partial_{T_{i}}$ is topologically quasi-nilpotent, this indeed defines a $B_{\text{st}}(O_{\mathcal{T}})$-linear section $s: D \otimes_{L_{0}} B_{\text{st}}(O_{\mathcal{T}}) \rightarrow D \otimes_{L_{0}} O_{B}^{\text{st}}(O_{\mathcal{T}})$ to $pr$. By [18] Sec. 8.1, we have

$$ s(x) = \sum (-1)^{i_{1} + \cdots + i_{d}} \gamma_{i_{1}}(\log \frac{T_{1}}{T_{1}}) \cdots \gamma_{i_{d}}(\log \frac{T_{d}}{T_{d}}) \cdot N_{T_{1}}^{i_{1}} \cdots N_{T_{d}}^{i_{d}}(x). \tag{4.5} $$

It follows from standard computations that $\nabla(s(x)) = 0$ and $\varphi(s(x)) = s(\varphi(x))$ for any $x \in D$. Moreover, $s$ is compatible with filtration since $\nabla_{D}$ satisfies Griffiths transversality. Note that $D$ has a $O_{L_{0}}$-lattice stable under $\nabla_{D}$ on which the connection is topologically quasi-nilpotent. If $\{e_{1}, \ldots, e_{m}\}$ is a $O_{L_{0}}$-basis of such a lattice, then $\{s(e_{1}), \ldots, s(e_{m})\}$ generates $D \otimes_{L_{0}} O_{B}^{\text{st}}(O_{\mathcal{T}})$ as $O_{B}^{\text{st}}(O_{\mathcal{T}})$-modules. Thus, $s$ induces an isomorphism

$$ s: D \otimes_{L_{0}} B_{\text{st}}(O_{\mathcal{T}}) \xrightarrow{\cong} (D \otimes_{L_{0}} B_{\text{st}}(O_{\mathcal{T}}))^{
abla = 0} $$

of $B_{\text{st}}(O_{\mathcal{T}})$-modules compatible with filtration and $\varphi, N$.

By Equation (4.4) and (4.5), we have

$$ pr(\sigma(s(x \otimes a))) = \sigma(x \otimes a) $$

for $\sigma \in G_{L}$ and $x \otimes a \in D \otimes_{L_{0}} B_{\text{st}}(O_{\mathcal{T}})$. Thus, $s$ induces a $G_{L}$-equivariant isomorphism

$$ \text{Hom}_{B_{\text{st}}(O_{\mathcal{T}}), \text{Fil}, \varphi, N}(D \otimes_{L_{0}} B_{\text{st}}(O_{\mathcal{T}}), B_{\text{st}}(O_{\mathcal{T}})) \cong V_{\text{st}}(D). \tag{4.6} $$

Hence, from the isomorphism (4.3) and Lemma 4.8, we obtain $T_{\text{cris}}(\mathcal{M})[p^{-1}] \cong V_{\text{st}}(D)$ as $G_{L}$-representations. In particular, $T_{\text{cris}}(\mathcal{M})$ is a lattice of the semistable representation $V_{\text{st}}(D)$.

**Proposition 4.9.** $T_{\text{cris}}(\cdot)$ gives a fully faithful contravariant functor from $\text{Mod}_{S}^{\varphi}$ to the category of lattices in semistable $G_{L}$-representations with Hodge-Tate weights in $[0, r]$.

**Proof.** It remains to show fully faithfulness. Let $\mathcal{M}_{1}, \mathcal{M}_{2} \in \text{Mod}_{S}^{\varphi}$ and $D_{1}, D_{2} \in \text{MF}_{\text{w}, \varphi}(\varphi, N, \nabla)$ such that $\mathcal{M}_{i}[p^{-1}] \cong \mathcal{D}(D_{i})$, $i = 1, 2$. By constructions above, we have a canonical isomorphism $T_{\text{cris}}(\mathcal{M}_{i})[p^{-1}] \cong V_{\text{st}}(D_{i})$ of $G_{L}$-representations. Identify $T_{\text{cris}}(\mathcal{M}_{i})$ as a lattice of $V_{\text{st}}(D_{i})$. \hfill $\Box$
Suppose we have a $G_L$-equivariant map $f: T_{\text{cris}}(\mathcal{M}_1) \to T_{\text{cris}}(\mathcal{M}_2)$. By Theorem 27, we have a morphism $h: D_2 \to D_1$ in $\text{MF}((\varphi, N, \nabla))$ such that $V_{st}(h) = f[p^{-1}]$. Consider the induced map

$$h: \mathcal{D}(D_2) = \mathcal{M}_2[p^{-1}] \to \mathcal{D}(D_1) = \mathcal{M}_1[p^{-1}].$$

Note that $h$ is also compatible with filtration, since the induced map $\mathcal{D}(D_2) \otimes_{S,b} S_g \to \mathcal{D}(D_1) \otimes_{S,b} S_g$ is compatible with filtration by a main result in [5].

We need to show $h(\mathcal{M}_2) \subset \mathcal{M}_1$. By [20, Cor. 3.5.2], we have $h(\mathcal{M}_2) \subset \mathcal{M}_1 \otimes_S S_g$. So

$$h(\mathcal{M}_2) \subset (\mathcal{M}_1 \otimes_S S[p^{-1}]) \cap (\mathcal{M}_1 \otimes_S S_g) = \mathcal{M}_1 \otimes_S (S[p^{-1}] \cap S_g) = \mathcal{M}_1$$

since $\mathcal{M}_1$ is free over $S$ and $S[p^{-1}] \cap S_g = S$. □

4.2. Cartier Dual. To show the essential surjectivity of the functor $T_{\text{cris}}(\cdot)$ constructed in Section 4.1, we first consider Cartier dual for strongly divisible lattices as in [20, Sec. 4]. The results we need will follow essentially by the same arguments as in loc. cit. It is convenient to study Cartier dual for objects in a category larger than $\text{Mod}^r_S$. Let $c_1 = \frac{1}{p}\varphi(E(u)) \in S^\times$.

**Definition 4.10.** Let $\text{Mod}^r_S$ be the category whose objects are tuples $(\mathcal{M}, \text{Fil}^r \mathcal{M}, \varphi_r)$ such that

- $\mathcal{M}$ is a finite free $S$-module;
- $\text{Fil}^r \mathcal{M}$ is a $S$-submodule of $\mathcal{M}$ such that $\text{Fil}^r S \cdot \mathcal{M} \subset \text{Fil}^r \mathcal{M}$ and $\mathcal{M}/\text{Fil}^r \mathcal{M}$ is $p$-torsion free;
- $\varphi_r: \text{Fil}^r \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semi-linear map such that for any $s \in \text{Fil}^r S$ and $x \in \mathcal{M}$, we have $\varphi_r(sx) = (c_1)^{-r} \varphi_r(s) \varphi_r(E(u)^r x)$;
- $\varphi_r(\text{Fil}^r \mathcal{M})$ generates $\mathcal{M}$ as $S$-modules.

Morphisms in $\text{Mod}^r_S$ are $S$-linear maps compatible with all structures.

**Lemma 4.11.** Let $\mathcal{M} \in \text{Mod}^r_S$. There exist $\alpha_1, \ldots, \alpha_m \in \text{Fil}^r \mathcal{M}$ such that

1. $\text{Fil}^r \mathcal{M} = \bigoplus_{i=1}^m S \alpha_i + \text{Fil}^r S \cdot \mathcal{M}$,
2. $E(u)^r \mathcal{M} \subset \bigoplus_{i=1}^m S \alpha_i$ and $\{\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m)\}$ is a $S$-basis of $\mathcal{M}$,
3. If $A$ is the $m \times m$ matrix such that $(\alpha_1, \ldots, \alpha_m) = (\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m))A$, then there exists a $m \times m$ matrix $B$ with coefficients in $S$ such that $AB = E(u)^r I$.

**Proof.** This follows from the same argument as in [20, Prop. 4.1.2 Pf.], where the argument is valid without the assumption that the residue field of $L$ is perfect. □

For $\mathcal{M} \in \text{Mod}^r_S$, define its cartier dual $\mathcal{M}^* := \text{Hom}_S(\mathcal{M}, S)$, $\text{Fil}^r \mathcal{M}^* := \{f \in \mathcal{M}^* \mid f(\text{Fil}^r \mathcal{M}) \subset \text{Fil}^r S\}$, and $\varphi_r: \text{Fil}^r \mathcal{M}^* \to \mathcal{M}^*$ by $\varphi_r(f)(\varphi_r(x)) = \varphi_r(f(x))$ for any $x \in \text{Fil}^r \mathcal{M}$. Note that $\varphi_r$ is well-defined since $\varphi_r(\text{Fil}^r \mathcal{M}) = \mathcal{M}$ as $S$-modules.

**Lemma 4.12.** The assignment $\mathcal{M} \to \mathcal{M}^*$ defines a functor from $\text{Mod}^r_S$ to itself, which is an exact anti-equivalence such that $(\mathcal{M}^*)^* = \mathcal{M}$.

**Proof.** This follows directly from the construction of the Cartier dual and Lemma 4.11. □
Example 4.13. For the cotangent dual $S^*$ of $S$, we have $\text{Fil}^r S^* = S^*$ and $\varphi_r(1) = 1$.

For $\mathcal{M} \in \text{Mod}^{r,g}_S$, let

$\text{Fil}^r \mathcal{M} := \text{Hom}_S, \varphi, \mathcal{M}, A_{\text{cris}}(O_T)).$

As in Section 4.1, there exists a quasi-Kisin module $\mathfrak{M}$ of height $r$ such that $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \cong \mathcal{M}$ by [15] Thm. 5.1.3, and the natural maps $\text{Fil}^r \mathcal{M}$ is compatible with filtration and $\varphi$. Taking Cartier dual on both sides, we obtain a map

$S^* \to \mathcal{M}^* \times (\mathcal{M}^*)^* \cong \mathcal{M}^* \times \mathcal{M}$

by Lemma 4.12. This induces a pairing

$\text{Fil}^r \mathcal{M} \times \text{Fil}^r \mathcal{M}^* \to \text{Hom}(S^*, A_{\text{cris}}(O_T)).$

Lemma 4.14. The above pairing induces a perfect pairing of $\mathbb{Z}_p$-representations of $G_{\infty}$:

$\text{Fil}^r \mathcal{M} \times \text{Fil}^r \mathcal{M}^* \to \text{Hom}(S^*, A_{\text{cris}}(O_T)).$

Proof. It follows from the construction of Cartier dual that the image of the above pairing lies in $T_{\text{cris}}(S^*)$. We have canonical $G_{\infty}$-equivariant isomorphisms

$T_{\text{cris}}(\mathcal{M}) \cong T_{\text{cris}}(\mathcal{M}^*_g), T_{\text{cris}}(\mathcal{M}^*) \cong T_{\text{cris}}(\mathcal{M}^*_g), T_{\text{cris}}(S^*) \cong T_{\text{cris}}(S^*_g).$

Thus, the assertion follows by [20] Lem. 4.3.1.

We will use the perfect pairing in above lemma to construct natural maps to relate $\mathcal{M}$ and $T_{\text{cris}}(\mathcal{M})$. For $D \in \text{MF}^{w, r}(\varphi, N, \nabla)$, define

$\text{Fil}^r(\mathcal{M}) := \bigoplus_{i=0}^r \text{Im}(\text{Fil}^{-i} A_{\text{cris}}(O_T) \otimes S \text{Fil}^i D(D)).$

Note that $\text{Fil}^{-i} S \cdot \text{Fil}^i D(D) \subset \text{Fil}^i D(D)$. Since the map $\text{Fil}^r. \mathcal{M} / p \text{Fil}^r. \mathcal{M} \to \mathcal{M} / p. \mathcal{M}$ is injective and $S/[p] \to A_{\text{cris}}(O_T)/(p)$ is flat, we deduce that $\text{Fil}^r(\mathcal{M})$ is equal to the $p$-adic completion of $\text{Im}(\mathcal{M}) \otimes S \text{Fil}^r D(D)$. Let $\alpha_1, \ldots, \alpha_m \in \text{Fil}^r \mathcal{M}$ as in Lemma 4.11. We then have

$\text{Fil}^r(\mathcal{M}) = \bigoplus_{i=1}^m A_{\text{cris}}(O_T) \cdot \alpha_i + \text{Fil}^p A_{\text{cris}}(O_T) \otimes S \cdot \mathcal{M}.$

In particular, $\varphi_r : \text{Fil}^r \mathcal{M} \to \mathcal{M}$ extends $\varphi A_{\text{cris}}(O_T)$-semi-linearly to $\varphi_r : \text{Fil}^r(\mathcal{M}) \to A_{\text{cris}}(O_T) \otimes S \mathcal{M}.$
Note that in above situation, $D^* := D_{st}(\text{Vst}(D)^r(r))$ is in $\text{MF}^{w, r}(\varphi, N, \nabla)$, and $\mathcal{M}^*[p-1] \cong \mathcal{D}(D^*)$ compatibly with $\varphi$ and Fil$^r$. We denote by $A_{\text{cris}}(\mathcal{O}_T)^*$ the ring $A_{\text{cris}}(\mathcal{O}_T)$ equipped with non-canonical filtration Fil$^r$ $A_{\text{cris}}(\mathcal{O}_T)^* = A_{\text{cris}}(\mathcal{O}_T)^*$ and Frobenius $\varphi_r(1) = 1$.

**Lemma 4.15.** We have natural $G_{L_\infty}$-equivariant isomorphisms

$$\text{Hom}_{A_{\text{cris}}(\mathcal{O}_T), \text{Fil}, \varphi}(A_{\text{cris}}(\mathcal{O}_T)^*, A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M}^*) \cong \text{Fil}^r(A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M}^*)_{\varphi_r=1}$$

$$\cong \text{Hom}_{S, \text{Fil}, \varphi}(\mathcal{M}, A_{\text{cris}}(\mathcal{O}_T))$$

**Proof.** The first isomorphism is clear. For the second isomorphism, note that we have a natural isomorphism $A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M}^* \cong \text{Hom}_{S}(\mathcal{M}, A_{\text{cris}}(\mathcal{O}_T))$. Showing that this isomorphism induces the second isomorphism is based on Lemma 4.14 and Equation (4.16), and it follows by the same argument as in [20, Lem. 4.3.2 Pf.]

By above lemma, we obtain natural maps

$$T_{\text{cris}}(\mathcal{M}) \cong \text{Fil}^r(A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M}^*)_{\varphi_r=1} \hookrightarrow A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M}^*,$$

and also a natural map $T_{\text{cris}}(\mathcal{M}^*) \hookrightarrow A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M}$ similarly. Let $t$ be any $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(1) = (\text{Fil}^1 A_{\text{cris}}(\mathcal{O}_T))^{\varphi_r=1} \subset A_{\text{cris}}(\mathcal{O}_T)$. We obtain a diagram

$$\begin{array}{ccc}
T_{\text{cris}}(\mathcal{M}) \times T_{\text{cris}}(\mathcal{M}^*) & \rightarrow & A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M}^* \times A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M} \\
\mathbb{Z}_p(r) \downarrow & & \downarrow \\
A_{\text{cris}}(\mathcal{O}_T) & \rightarrow & A_{\text{cris}}(\mathcal{O}_T)
\end{array}$$

where the top horizontal map is given the map (4.7) and its dual, left vertical map by Lemma 4.14 and the map (4.7), we get a canonical map

$$\mathbb{Z}_p \to 1 \to T_{\text{cris}}(\mathcal{M}) \otimes \mathbb{Z}_p A_{\text{cris}}(\mathcal{O}_T)^* \hookrightarrow A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M}$$

This induces a canonical map

$$t : A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M} \rightarrow T_{\text{cris}}(\mathcal{M}) \otimes \mathbb{Z}_p A_{\text{cris}}(\mathcal{O}_T)^*$$

which is compatible with $G_{L_\infty}$-action, $\varphi$ and filtration. On the other hand, by Lemma 4.14 and the map (4.7), we get a canonical map

$$t^* : T_{\text{cris}}(\mathcal{M})^r(\mathcal{M}^*) \otimes \mathbb{Z}_p A_{\text{cris}}(\mathcal{O}_T)^* \to A_{\text{cris}}(\mathcal{O}_T) \otimes S \cdot \mathcal{M}$$
compatible with all structures.

We obtain the following diagram

$$T_{\text{cris}}(\mathcal{M}) \times T_{\text{cris}}(\mathcal{M}^*) \otimes_{\mathbb{Z}_p} A_{\text{cris}}(\mathcal{O}_T)^1 \xrightarrow{\text{Id} \otimes \iota} T_{\text{cris}}(\mathcal{M}) \times A_{\text{cris}}(\mathcal{O}_T) \otimes_S \mathcal{M}$$

$$\mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} A_{\text{cris}}(\mathcal{O}_T)^* \xrightarrow{1 \otimes \iota^r} A_{\text{cris}}(\mathcal{O}_T)$$

where the left vertical map is induced by Lemma 4.14. The above diagram is commutative, since the map (4.7) is injective and the diagram (4.8) is commutative.

**Lemma 4.17.** Let $\alpha_1, \ldots, \alpha_m \in \text{Fil}'\mathcal{M}$ as in Lemma 4.11, and let $\{\epsilon_1, \ldots, \epsilon_m\}$ be a basis of $T_{\text{cris}}(\mathcal{M})^\vee$. Let $C$ be the $m \times m$ matrix with coefficients in $\text{Fil}'A_{\text{cris}}(\mathcal{O}_T)$ such that $\nu(\alpha_1, \ldots, \alpha_m) = (\epsilon_1, \ldots, \epsilon_m)C$ given by Theorem 4.16. Then there exists an $m \times m$ matrix $C'$ with coefficients in $A_{\text{cris}}(\mathcal{O}_T)$ such that the coefficients of $C'C - \iota^r\text{Id}$ are in $\text{Fil}'A_{\text{cris}}(\mathcal{O}_T)$.

**Proof.** This follows from the same argument as in [20, Lem. 4.3.6 Pf.], based on Theorem 4.16, Equation (4.6), and Lemma 4.11 and 4.14.

### 4.3. Essential Surjectivity of $T_{\text{cris}}(\cdot)$

Let $V$ be a semistable $\mathbb{Q}_p$-representation of $G_L$ with Hodge-Tate weights in $[0, r]$, and let $T \subset V$ be a $G_L$-stable $\mathbb{Z}_p$-lattice. Denote $D = D_{st}(V) \in \text{MF}_{ur}(\varphi, N, \nabla)$. By [15, Thm. 3.2.3], there exists a quasi-Kisin module $\mathcal{M}^\prime$ over $\mathcal{S}$ of height $r$ such that denoting $\varphi^*\mathcal{M}^\prime = \mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}^\prime$, we have $(\varphi^*\mathcal{M}^\prime/\varphi^*\mathcal{M}^\prime)[p^{-1}] \cong D$ compatibly with $\varphi$, and that $\mathcal{M}^\prime_g = \mathcal{M}^\prime \otimes_{\mathcal{S}, \varphi} \mathcal{S}_g$ with the induced $\varphi$ is a Kisin module over $\mathcal{S}_g$ associated with $D_g := D \otimes_{\mathcal{S}, \varphi} W(k_g)[p^{-1}] = D_{st}(V|_{G_{K_g}})$. By [13, Lem. 4.2.9] and Lemma 4.6,

$$\mathcal{M} := \mathcal{M}^\prime[p^{-1}] \cap \mathcal{M}(T)$$

is a quasi-Kisin module of height $r$ such that $\text{Hom}_{\mathcal{S}, \varphi}(\mathcal{M}, \mathcal{S}_{ur}) \cong T$ as $G_{L_{\infty}}$-representations.

Let $\mathcal{M} := S \otimes_{\varphi, \mathcal{S}} \mathcal{M}$ with the induced $\varphi$.

**Remark 4.18.** If $\mathcal{M}_1 \in \text{Mod}_S^r$ such that $T_{\text{cris}}(\mathcal{M}_1) = T$ as $\mathbb{Z}_p$-representations of $G_L$, then the corresponding quasi-Kisin module of height $r$ given in Section 4.1 agrees with $\mathcal{M}$ above by [15, Lem. 4.2.9] and Lemma 4.7.

Consider the $\varphi$-compatible projection $q: S \to \mathcal{O}_{L_0}$ given by $u \mapsto 0$, and let $I_0 \subset S$ be its kernel and $M := \mathcal{M}/I_0\mathcal{M}$. We have the induced projection $q: \mathcal{M}[p^{-1}] \to M[p^{-1}] \cong D$.

**Lemma 4.19.** $q$ has a unique $\varphi$-compatible section $s: D \to \mathcal{M}[p^{-1}]$. Furthermore, $1 \otimes s: S[p^{-1}] \otimes_{L_0} D \to \mathcal{M}[p^{-1}]$ is an isomorphism.

**Proof.** Since $\mathcal{M}$ is a quasi-Kisin module of height $r$, the map $(1 \otimes \varphi)[p^{-1}]: \varphi^*M[p^{-1}] = \mathcal{O}_{L_0} \otimes_{\varphi, \mathcal{O}_{L_0}} M[p^{-1}] \to M[p^{-1}]$ is an isomorphism, and the preimage of $M$ is contained in $p^{-r}(\varphi^*M)$. Thus, it follows from the standard argument as in [16, Lem. 3.14 Pf.] that there exists a unique $\varphi$-compatible section $s: D \cong M[p^{-1}] \to \mathcal{M}[p^{-1}]$. 


Consider the map $1 \otimes s : \mathcal{S}[p^{-1}] \otimes_{\mathcal{O}} D \to \mathcal{M}[p^{-1}]$, which is a map of projective $\mathcal{S}[p^{-1}]$-modules of the same rank. By Nakayama’s lemma, there exists an element $a \in I_0 \mathcal{S}[p^{-1}]$ such that the induced map

$$S[p^{-1}] \otimes_{\mathcal{O}} \mathcal{M}[p^{-1}][(1 + a)^{-1}] \to \mathcal{M}[p^{-1}][(1 + a)^{-1}]$$

is an isomorphism. On the other hand, by \cite[Lem. 1.2.6]{17} (see also \cite[Sec. 3.2]{20}), the map

$$S_g[p^{-1}] \otimes_{\mathcal{O}_g} \mathcal{M}[p^{-1}] \to \mathcal{M} \otimes_{\mathcal{O}} S_g[p^{-1}]$$

induced by $b_g : \mathcal{O}_g \to W(k_g)$ is an isomorphism. We have

$$S[p^{-1}][(1 + a)^{-1}] \cap S_g[p^{-1}] = S[p^{-1}],$$

since $L_0[u] \cap S_g[p^{-1}] = S[p^{-1}]$ as subrings of $(W(k_g)[p^{-1}])[u]$. Hence, the map $1 \otimes s : S[p^{-1}] \otimes_{\mathcal{O}} \mathcal{M}[p^{-1}] \to \mathcal{M}[p^{-1}]$ is an isomorphism. \hfill $\square$

By above lemma, we can identify $\mathcal{M}[p^{-1}] = \mathcal{D}(D)$ compatibly with $\varphi$. Define $\Fil'' \mathcal{M} = \mathcal{M} \cap \Fil'' \mathcal{D}(D)$. By \cite[Sec. 3.4]{20}, $\mathcal{M}_g := \mathcal{M} \otimes_{\mathcal{O}} S_g$ is a quasi-strongly divisible lattice of $D_g \otimes_{W(k_g)} S_g[p^{-1}]$ as in \cite[Def. 2.3.3]{20}. Since $S[p^{-1}] \cap S_g = S$ as submodules of $S_g[p^{-1}]$, we have

$$\varphi_r(\Fil'' \mathcal{M}) \subset \mathcal{M}[p^{-1}] \cap \mathcal{M}_g = \mathcal{M} \otimes_{\mathcal{O}} (S[p^{-1}] \cap S_g) = \mathcal{M},$$

i.e. $\varphi(\Fil'' \mathcal{M}) \subset \Fil'' \mathcal{M}$. Since $D$ is weakly admissible, we have $\varphi_r(\Fil'' \mathcal{M}) = \mathcal{M}$ as in Lemma \cite[4.3]{13}. Thus, $\mathcal{M} \in \Mod^r_{\mathcal{S}}$.

Define the $G_L$-action on $\mathbf{A}_{\text{cris}}(\mathcal{O}_L) \otimes_{\mathcal{S}} \mathcal{D}(D)$ by the analogous formula as Equation \cite[4.1]{14}, and let $V_{\text{cris}}(\mathcal{D}(D)) := \text{Hom}_{\mathcal{S}, \Fil'' \mathcal{D}(D)}(\mathcal{A}_{\text{cris}}(\mathcal{O}_L)[p^{-1}])$ be equipped with the induced $G_L$-action. Similarly as in Section \cite[4.1]{14} (by considering the corresponding maps induced via $b_g$), we have a natural $G_{L_{\infty}}$-equivariant isomorphism

$$T_{\text{cris}}(\mathcal{M})[p^{-1}] \cong V_{\text{cris}}(\mathcal{D}(D)).$$

Furthermore, we have the natural $G_L$-equivariant isomorphism $V_{\text{cris}}(\mathcal{D}(D)) \cong V$ by Section \cite[4.1]{14}. Consider the commutative diagram

$$\begin{array}{ccc}
\mathbf{A}_{\text{cris}}(\mathcal{O}_L) \otimes_{\mathcal{S}} \mathcal{D}(D) & \xrightarrow{\iota \otimes \varphi} & \mathbf{V}_{\text{cris}}(\mathcal{D}(D))^\vee \otimes_{\mathbb{Z}_p} \mathbf{A}_{\text{cris}}(\mathcal{O}_L) \\
\downarrow & & \downarrow \\
\mathbf{A}_{\text{cris}}(\mathcal{O}_L) \otimes_{\mathcal{S}} \mathcal{M} & \xrightarrow{\iota} & T_{\text{cris}}(\mathcal{M})^\vee \otimes_{\mathbb{Z}_p} \mathbf{A}_{\text{cris}}(\mathcal{O}_L)
\end{array}$$

Note that the top horizontal map is $G_L$-equivariant, and the bottom map is $G_{L_{\infty}}$-equivariant. Moreover, $T_{\text{cris}}(\mathcal{M})^\vee \otimes_{\mathbb{Z}_p} \mathbf{A}_{\text{cris}}(\mathcal{O}_L)$ is stable under $G_L$-action on $V_{\text{cris}}(\mathcal{D}(D))^\vee \otimes_{\mathbb{Z}_p} \mathbf{A}_{\text{cris}}(\mathcal{O}_L)$ since $T_{\text{cris}}(\mathcal{M})^\vee = T^\vee$.

**Lemma 4.20.** Let $\sigma$ be any element of $G_L$. Suppose we have a commutative diagram

$$\begin{array}{ccc}
\mathbf{A}_{\text{cris}}(\mathcal{O}_L) \otimes_{\mathcal{S}} \mathcal{M} & \xrightarrow{\iota} & T_{\text{cris}}(\mathcal{M})^\vee \otimes_{\mathbb{Z}_p} \mathbf{A}_{\text{cris}}(\mathcal{O}_L) \\
\downarrow f & & \downarrow f_1 \\
\mathbf{A}_{\text{cris}}(\mathcal{O}_L) \otimes_{\mathcal{S}} \mathcal{M} & \xrightarrow{\iota} & T_{\text{cris}}(\mathcal{M})^\vee \otimes_{\mathbb{Z}_p} \mathbf{A}_{\text{cris}}(\mathcal{O}_L)
\end{array}$$


where \( f \) and \( f_1 \) are either \( A_{\text{cris}}(O_T) \)-linear or \( \sigma \)-semi-linear morphisms compatible with \( \varphi \) and filtration. If \( p \mid f_1 \), then \( p \mid f \).

**Proof.** Since \( A_{\text{cris}}(O_T) = A_{\text{cris}}(O_{K_g}) \), this follows from [20] Lem. 5.3.1 for \( M_g = M \otimes_S S_g \). □

**Corollary 4.21.** In the diagram \((\ref{eq:4.9})\), \( A_{\text{cris}}(O_T) \otimes_S M \) is stable under the \( G_L \)-action on \( A_{\text{cris}}(O_T) \otimes_S \mathcal{D}(D) \).

**Proof.** Let \( \sigma \) be any element of \( G_L \). Let \( n \geq 0 \) be an integer such that

\[
p^n \sigma(A_{\text{cris}}(O_T) \otimes_S M) \subset A_{\text{cris}}(O_T) \otimes_S M.
\]

Since \( T_{\text{cris}}(M)^\vee \otimes_{\mathcal{Z}} A_{\text{cris}}(O_T) \) is \( G_L \)-stable, we can apply Lemma 4.20 if \( n \geq 1 \) to deduce the assertion. □

Now, to show that the functor \( T_{\text{cris}}(\cdot) \) in Section 4.1 is essentially surjective, it suffices to show that \( M \) is stable under \( N \) and \( N_{T_j} \)'s on \( \mathcal{D}(D) \). By [20] Lem. 3.5.3, \( N(M_g) \subset M_g \). So as above, we have

\[
N(M) \subset \mathcal{D}(D) \cap M_g = M.
\]

It remains to show the stability under \( N_{T_j} \) for each \( j = 1, \ldots, d \). Recall that we chose an embedding \( L \hookrightarrow K_g \) so that \( G_{K_g} \) acts trivially on \([T_j]\) for each \( j \). Denote \( L_{p^{\infty}} = \bigcup_{n \geq 0} L(\epsilon_n) \) and \( K_{g,p^{\infty}} = \bigcup_{n \geq 0} K_g(\epsilon_n) \). As explained in [8] Not. 2.14], there exists \( \tau_j \in \text{Gal}(L/L_{p^{\infty}}) \) such that \( \tau_j([T_j]) = [\epsilon][T_j] \) (so \( \log (\mu_j(\tau_j)) \) is a generator of \( (\text{Fil}^1 A_{\text{cris}}(O_T))^{\psi_1=1} \)) and that \( \tau_j \) acts trivially on \([T_i]\) if \( i \neq j \). Since \( p \geq 3 \), by [20] Lem. 5.1.2 applied to \( G_{K_g} \), there exists \( \sigma \in \text{Gal}(K_g/K_{g,p^{\infty}}) \) such that \( \sigma([\tau]) = \tau_j([\tau]) \). Replacing \( \tau_j \) by \( \sigma^{-1} \tau_j \) if necessary, we may further assume that \( \tau_j \) acts trivially on \([\tau]\). Let \( t = \log (\mu_j(\tau_j)) \).

Note that \( \tau_j \) acts trivially on \( t \). For any \( x \in \mathcal{D}(D) \) and \( n \geq 0 \), we deduce by induction on \( n \) that

\[
(\tau_j - 1)^n(x) = \sum_{m=n}^{\infty} \sum_{i_1 + \cdots + i_n = m, \; i_q \geq 1} \frac{m!}{i_1! \cdots i_n!} \gamma_m(t) \otimes N_{T_j}^m(x).
\]

We have \( (\tau_j - 1)^n(x) \in \text{Fil}^n A_{\text{cris}}(O_T)[p^{-1}] \otimes_S \mathcal{D}(D) \), and \( \frac{1}{n}(\tau_j - 1)^n(x) \to 0 \) \( p \)-adically as \( n \to \infty \). So

\[
\log (\tau_j)(x) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}(\tau_j - 1)^n(x) \in A_{\text{cris}}(O_T)[p^{-1}] \otimes_S \mathcal{D}(D).
\]

By direct computation, we get

\[
\log (\tau_j)(x) = t \otimes N_{T_j}(x).
\]

We can now proceed similarly as in [20] Sec. 5.3]. We first claim that \( t \otimes N_{T_j}(M) \subset A_{\text{cris}}(O_T) \otimes_S M \), for which it suffices to show \( \frac{1}{n}(\tau_j - 1)^n(M) \subset A_{\text{cris}}(O_T) \otimes_S M \) for
each \( n \geq p \). Let \( \alpha_1, \ldots, \alpha_m \in \text{Fil}^n \mathcal{M} \) as in Lemma 4.11 so that \( \{ \varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m) \} \) is a basis of \( \mathcal{M} \). Since \( \tau_j(\mathcal{M}) \subset A_{\text{cris}(O_L)} \otimes_S \mathcal{M} \), Equation (4.10) gives
\[
(\tau_j - 1)^n(\alpha_1, \ldots, \alpha_m) \in (\text{Fil}^n A_{\text{cris}(O_L)} \cdot (A_{\text{cris}(O_L)} \otimes_S \mathcal{M}))^m.
\]
Thus,
\[
(\tau_j - 1)^n(\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m)) \in (\varphi_r(\text{Fil}^n A_{\text{cris}(O_L)}) \cdot (A_{\text{cris}(O_L)} \otimes_S \mathcal{M}))^m.
\]
By [20] Lem. 5.3.2, \( \varphi_r(E(u)^l) \in S \) for any integers \( l \geq n \geq p \). So \( \frac{1}{n}\varphi_r(\text{Fil}^n A_{\text{cris}(O_L)}) \subset A_{\text{cris}(O_L)} \), which proves the claim.

Let \( W \) be the \( m \times m \) matrix with coefficients in \( S^{[\frac{1}{p}]} \) such that
\[
N_{T_j}(\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m)) = (\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m))W,
\]
and let \( n \geq 0 \) be the smallest integer such that the coefficients of \( p^nY \) lie in \( S \). Then \( p^nN_{T_j}(\mathcal{M}) \subset \mathcal{M} \). Since \( E(u)N_{T_j}(\text{Fil}^r D) \subset \text{Fil}^r D \) by Lemma 4.12, we can write
\[
E(u)p^nN_{T_j}(\alpha_1, \ldots, \alpha_m) = (\alpha_1, \ldots, \alpha_m)X + (\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m))Y
\]
for some \( m \times m \) matrices \( X \) and \( Y \) with coefficients in \( S \) and \( \text{Fil}^pS \) respectively by Lemma 4.11. On the other hand, since \( t \otimes N_{T_j}(\mathcal{M}) \subset A_{\text{cris}(O_L)} \otimes_S \mathcal{M} \), we have
\[
t \otimes N_{T_j}(\text{Fil}^r \mathcal{M}) \subset (A_{\text{cris}(O_L)} \otimes_S \mathcal{M}) \cap \text{Fil}^r(A_{\text{cris}(O_L)} \otimes_S D) = \text{Fil}^r(A_{\text{cris}(O_L)} \otimes_S D).
\]
So
\[
tN_{T_j}(\alpha_1, \ldots, \alpha_m) = (\alpha_1, \ldots, \alpha_m)X' + (\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m))Y'
\]
for some \( m \times m \) matrices \( X' \) and \( Y' \) with coefficients in \( A_{\text{cris}(O_L)} \) and \( \text{Fil}^pA_{\text{cris}(O_L)} \) respectively. Let \( A, B \) be matrices as given in Lemma 4.11. Then we obtain
\[
E(u)^r(tX - E(u)p^nX') = -tBY + E(u)p^nBY'.
\]
The matrix on the right hand side has coefficients in \( \text{Fil}^{p+1}A_{\text{cris}(O_L)} \). So by Lemma 4.11, \( E(u)^{-1}(tX - E(u)p^nX') \) has coefficients in \( \text{Fil}^{p}A_{\text{cris}(O_L)} \).

Suppose \( n \geq 1 \). Then the coefficients of \( E(u)^{-1}tX \) lie in \( \text{Fil}^{p}A_{\text{cris}(O_L)} + pA_{\text{cris}(O_L)} \). By a similar argument as in [20] Lem. 5.3.1 Pf. using Lemma 4.11, the coefficients of \( X \) lie in \( \text{Fil}^{1}A_{\text{cris}(O_L)} + pA_{\text{cris}(O_L)} \). In particular, the coefficients of \( \varphi(X) \) lie in \( pA_{\text{cris}(O_L)} \). Since \( S/(p) \rightarrow A_{\text{cris}(O_L)}/(p) \) is faithfully flat and so injective, the coefficients of \( \varphi(X) \) lie in \( pS \). Note that
\[
c1p^nN_{T_j}(\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m)) = p^n\varphi_r(E(u)N_{T_j}(\alpha_1, \ldots, \alpha_m)) = \varphi_r(\alpha_1, \ldots, \alpha_m)\varphi(X) + \varphi(\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_m))\varphi(Y).
\]
Since the coefficients of \( \varphi(X) \) and \( \varphi_r(Y) \) lie in \( pS \), this contradicts the choice of \( n \) to be minimal. Hence, \( n = 0 \).

This concludes the proof of the following theorem.

**Theorem 4.22.** Let \( 0 \leq r \leq p - 2 \). The functor \( T_{\text{cris}}(\cdot) \) in Section 4.1 gives an anti-equivalence from \( \text{Mod}^r_S \) to the category of \( \mathbb{Z}_p \)-lattices in semistable representations of \( GL \) with Hodge-Tate weights in \([0, r] \).
Let \( \text{Mod}^{\text{cris}}_S \) be the full subcategory of \( \text{Mod}^\circ \) consisting of \( \mathcal{M} \) such that \( \mathcal{M}[p^{-1}] \cong \mathcal{D}(D) \) for \( D \in \text{MF}^{ur}(\varphi, N, \nabla) \) such that \( N_D = 0 \), satisfying the same conditions as in Definition 4.3. Above arguments also imply the following for crystalline case.

**Theorem 4.23.** Let \( 0 \leq r \leq p - 2 \). The functor \( T_{\text{cris}}(\cdot) \) in Section 4.1 gives an anti-equivalence from \( \text{Mod}^{\text{cris}}_S \) to the category of \( \mathbb{Z}_p \)-lattices in crystalline representations of \( G_L \) with Hodge-Tate weights in \([0, r]\).

5. **Crystalline cohomology and strongly divisible lattices**

Let \( \mathcal{X} \) be a proper smooth formal scheme over \( \mathcal{O}_L \). In this section, under certain assumptions, we study a cohomological description of the strongly divisible lattice associated with the \( \text{étale} \) cohomology \( H^i_{\text{ét}}(\mathcal{X}_{T^r}, \mathbb{Z}_p) \) for \( i \leq p - 2 \). More precisely, denote the generic fiber of \( \mathcal{X} \) by \( \mathcal{X} \), and let \( \mathcal{X}_0 := \mathcal{X} \times_{\mathcal{O}_L} \mathcal{O}_L/(p) \) and \( \mathcal{X}_k := \mathcal{X} \times_{\mathcal{O}_L} k \). Let \( T^i := H^i_{\text{ét}}(\mathcal{X}_{T^r}, \mathbb{Z}_p)/\text{tors} \), and let \( M^i := H^i_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_L) \) and \( \mathcal{M}^i := H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{S}) \).

**Theorem 5.1.** Let \( i \leq p - 2 \). Suppose that \( H^i_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_L) \) and \( H^{i+1}_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_L) \) are \( p \)-torsion free. Then

1. \( H^i_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_L) \) is torsion free (so \( T^i = H^i_{\text{ét}}(\mathcal{X}_{T^r}, \mathbb{Z}_p) \)),
2. \( \mathcal{M}^i \in \text{Mod}_S \),
3. \( T_{\text{cris}}(\mathcal{M}^i) \cong (T^i)^\vee \) as \( G_L \)-representations.

From now, we assume \( M^i \) and \( M^{i+1} \) are \( p \)-torsion free. To prove above theorem, we consider \( b_g : L \rightarrow K_g \) as before. Since \( b_g : \mathcal{O}_L \rightarrow W(k_g) \) is flat, we have \( H^j_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_L) \otimes_{\mathcal{O}_{S_0}} W(k_g) \cong H^j_{\text{cris}}(\mathcal{X}_k/W(k_g)) \) for any \( j \) by crystalline base change. So \( H^i_{\text{cris}}(\mathcal{X}_k/W(k_g)) \) and \( H^{i+1}_{\text{cris}}(\mathcal{X}_k/W(k_g)) \) are \( p \)-torsion free. In particular, we can apply [10, Thm. 5.4] for \( \mathcal{X}_{\mathcal{O}_{K_g}} \).

**Proposition 5.2.** \( H^i_{\text{ét}}(\mathcal{X}_{T^r}, \mathbb{Z}_p) \) is torsion free.

**Proof.** Since \( H^i_{\text{cris}}(\mathcal{X}_k/W(k_g)) \) is \( p \)-torsion free, we have \( H^i_{\text{ét}}(\mathcal{X}_{T^r}, \mathbb{Z}_p) \) is torsion free by [1, Thm. 14.5]. On the other hand, by smooth and proper base change theorems, we have \( H^i_{\text{ét}}(\mathcal{X}_{T^r}, \mathbb{Z}_p) \cong H^i_{\text{ét}}(\mathcal{X}_{K_g}, \mathbb{Z}_p) \). Thus, \( H^i_{\text{ét}}(\mathcal{X}_{T^r}, \mathbb{Z}_p) \) is torsion free.

Recall that \( q : S \rightarrow \mathcal{O}_{L_0} \) is the projection given by \( u \mapsto 0 \). This induces a natural map \( q : H^i_{\text{cris}}(\mathcal{O}_0/\mathcal{S}) \rightarrow H^i_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_L) \).

**Proposition 5.3.** There exists a unique section \( s : H^i_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_L)[p^{-1}] \rightarrow H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{S})[p^{-1}] \) of \( q[p^{-1}] \) such that \( s \) is \( \varphi \)-equivariant. Furthermore, the induced map

\[
S \otimes_{\mathcal{O}_{L_0}} H^i_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_L)[p^{-1}] \rightarrow H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{S})[p^{-1}]
\]

of \( S[p^{-1}] \)-modules is an isomorphism.

**Proof.** For each integer \( n \geq 0 \), set \( u_n \) such that \( u_0 = u \) and \( u_n = u^{p^{-1}} \). Recall that we choose \( T_{i,n} \in \mathcal{O}_L \) satisfying \( T_{i,n+1} = T_i \). Let \( \mathcal{O}_{L_0,(n)} := \mathcal{O}_L[T_{1,n}, \ldots, T_{d,n}] \) equipped with Frobenius given by \( \varphi(T_{i,n}) = T_{i,n}^p \), which extends the Frobenius on \( \mathcal{O}_{L_0} \). Equip \( \mathcal{O}_{L_0,(n)}[u_n] \) with Frobenius given by \( \varphi(u_n) = u_n^p \), and let \( S_n \) be the \( p \)-adically completed PD-envelope of \( \mathcal{O}_{L_0,(n)}[u_n] \) with respect to \( (E(u_n)) \). The Frobenius
on $\mathcal{O}_{L_0,[n]}$ extends naturally to $S_{(n)}$. Let $\mathcal{O}_{L,(n)} := \mathcal{O}_{L_0,(n)} \otimes W(k) \mathcal{O}_n$ where $K_n := K[\pi_n]$. We have a natural inclusion $S \hookrightarrow S_{(n)}$.

Let $e = [K : W(k) [p^{-1}]]$ be the ramification index. Consider the PD-thickenings $S_{(n)} \rightarrow \mathcal{O}_{L,(n)}/(\pi_n^e)$ given by $u_n \mapsto \pi_n$ and $S \rightarrow \mathcal{O}_L/(\pi^e) = \mathcal{O}_L/(p)$. Note that $\varphi^n: \mathcal{O}_{L_0,(n)} \rightarrow \mathcal{O}_L$ is an isomorphism, since $\{T_1, \ldots, T_d\}$ gives a $p$-basis of $\mathcal{O}_{L_0}/(p) = k'$ so the induced map $\varphi^n: \mathcal{O}_{L_0/(n)}/(p) \rightarrow \mathcal{O}_L/(p)$ is an isomorphism. Thus, $\varphi^n: S_{(n)} \rightarrow S$ is an isomorphism which is compatible with the isomorphism $\varphi^n: \mathcal{O}_{L_0,(n)}/(\pi_n^e) \rightarrow \mathcal{O}_L/(p)$. Denote $\mathcal{X}_{(n)} := \mathcal{X} \times_\mathcal{O}_L \mathcal{O}_{L,(n)}/(\pi_n^e)$. By crystalline base change theorem,

$$H^i_{\text{cris}}(\mathcal{X}/S) \otimes_{\mathcal{O}_L} \mathcal{O}_n \cong H^i_{\text{cris}}(\mathcal{X} \times_\mathcal{O}_L \mathcal{O}_{L,(n)}/(\pi_n^e) \mathcal{O}_L/(p)/S).$$

Choose $n$ such that $p^n \geq e$. Then

$$\mathcal{X}_{(n)} \cong \mathcal{X} \times_\mathcal{O}_L (\mathcal{O}_{L,(n)}/(\pi_n^e) = \times_\mathcal{O}_L \mathcal{O}_{L,(n)}/(\pi_n^e).$$

The natural inclusion $\mathcal{O}_{L_0} \hookrightarrow S_{(n)}$ is a PD-morphism over $\mathcal{O}_{L_0}/(p) \rightarrow \mathcal{O}_{L,(n)}/(\pi_n^e)$. Since $\mathcal{O}_{L_0} \hookrightarrow S_{(n)}$ is flat, we have by crystalline base change theorem that

$$H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L}) \otimes_{\mathcal{O}_{L_0}} S_{(n)} \cong H^i_{\text{cris}}(\mathcal{X}_{(n)}/S_{(n)}).$$

From the above isomorphisms,

$$H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L}) \otimes_{\mathcal{O}_{L_0}} S \cong H^i_{\text{cris}}(\mathcal{X}_0 \times_\mathcal{O}_L/(p), \varphi^n) \mathcal{O}_L/(p)/S).$$

Thus, we obtain a morphism

$$(5.1) \ H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L}) \otimes_{\mathcal{O}_{L_0}} S \rightarrow H^i_{\text{cris}}(\mathcal{X}_0 \times_\mathcal{O}_L/(p), \varphi^n) \mathcal{O}_L/(p)/S) \rightarrow H^i_{\text{cris}}(\mathcal{X}_0/S)$$

where the second map is the $n$-th iteration of relative Frobenius. Note that by [3 Thm. 1.1], $H^i_{\text{cris}}(\mathcal{X}_{k'/W(k)})[p^{-1}]$ is a filtered $\varphi$-module over $W(k)[p^{-1}]$, and so $\varphi^n: H^i_{\text{cris}}(\mathcal{X}_{k'/W(k)})[p^{-1}] \rightarrow H^i_{\text{cris}}(\mathcal{X}_{k'/W(k)})[p^{-1}]$ is injective. Since $H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L}) \otimes_{\mathcal{O}_{L_0}} W(k)$ is $H^i_{\text{cris}}(\mathcal{X}_{k'/W(k)})$, we have that $\varphi^n: H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L})[p^{-1}] \rightarrow H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L})[p^{-1}]$ is injective. Thus, the map

$$\varphi^n \otimes 1: H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L})[p^{-1}] \otimes_{\mathcal{O}_{L_0}} S \rightarrow H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L})[p^{-1}]$$

is an isomorphism. Composing the inverse of this isomorphism with the map $\mathbf{5.1}$ with $p$ inverted, we obtain a map

$$s: H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L})[p^{-1}] \rightarrow H^i_{\text{cris}}(\mathcal{X}_0/S)[p^{-1}].$$

It is clear from the constructions that $s$ is a $\varphi$-equivariant section of $q[p^{-1}]$. The uniqueness follows from the standard argument. By [3 Thm. 1.6], the second map in $\mathbf{5.1}$ with $p$ inverted is an isomorphism. Hence, the induced map

$$S \otimes_{\mathcal{O}_{L_0}} H^i_{\text{cris}}(\mathcal{X}_{k'/\mathcal{O}_L})[p^{-1}] \rightarrow H^i_{\text{cris}}(\mathcal{X}_0/S)[p^{-1}]$$

of $S[p^{-1}]$-modules is an isomorphism.

We now consider the natural connection on crystalline cohomology. Let $\mathcal{O}_{L_0}(1)$ be the $p$-adically completed PD-envelope of $\mathcal{O}_{L_0} \widehat{\otimes}_{W(k)} \mathcal{O}_{L_0}$ with respect to the kernel of the natural map $\mathcal{O}_{L_0} \widehat{\otimes}_{W(k)} \mathcal{O}_{L_0} \rightarrow \mathcal{O}_{L_0}/(p)$, and let $S(1)$ be the $p$-adically completed PD-envelope of $\mathcal{O}_{L_0} \widehat{\otimes}_{W(k)} \mathcal{S}$ with respect to the kernel of the map $\mathcal{O}_{L_0} \widehat{\otimes}_{W(k)} \mathcal{S} \rightarrow$
$\mathcal{O}_L/(p)$ given by $u \mapsto \pi$. Here, $\hat{\otimes}$ denotes the $p$-adically completed $\otimes$-product. Note that the kernel of $\mathcal{O}_L \hat{\otimes} W(k) \mathcal{O}_L \twoheadrightarrow \mathcal{O}_L/(p)$ is generated by $p$ and $a \otimes 1 - 1 \otimes a$, $a \in \mathcal{O}_L$, and the kernel of $\mathcal{O}_L \hat{\otimes} W(k) \mathfrak{S} \twoheadrightarrow \mathcal{O}_L/(p)$ is generated by $p$, $E(u)$, and $a \otimes 1 - 1 \otimes a$, $a \in \mathcal{O}_L$. The map $q: S \to \mathcal{O}_L$ naturally extends to $S(1) \to \mathcal{O}_L(1)$ via $u \mapsto 0$.

Let $p_1, p_2: \mathcal{O}_L \to \mathcal{O}_L(1)$ be the maps given by

$$p_1(a) = a \otimes 1, \ p_2(a) = 1 \otimes a, \ a \in \mathcal{O}_L.$$ 

We also denote by $p_1, p_2: \mathfrak{S} \to S(1)$ the maps given by

$$p_1 \left( \sum_{n \geq 0} a_n u^n \right) = \sum_{n \geq 0} a_n \otimes u^n, \ p_2 \left( \sum_{n \geq 0} a_n u^n \right) = \sum_{n \geq 0} 1 \otimes a_n u^n, \ a_n \in \mathcal{O}_L,$$

which extends to $p_1, p_2: S \to S(1)$. Since $\mathcal{O}_L(1)$ is $p$-torsion free, the maps $p_1, p_2: \mathcal{O}_L \to \mathcal{O}_L(1)$ are flat. So $p_1, p_2: S/(p^n) \to S(1)/(p^n)$ are flat for each $n \geq 1$ by [22, Tag 07HD].

By crystalline base change, we have isomorphisms

$$H^i_{\text{cris}}(X^k/\mathcal{O}_L) \otimes_{\mathcal{O}_L,p_1} \mathcal{O}_L(1) \cong H^i_{\text{cris}}(X^k/\mathcal{O}_L(1)) \cong H^i_{\text{cris}}(X^k/\mathcal{O}_L) \otimes_{\mathcal{O}_L,p_2} \mathcal{O}_L(1)$$

and

$$H^i_{\text{cris}}(X^0/S) \otimes_{S,p_1} S(1) \cong H^i_{\text{cris}}(X^0/S(1)) \cong H^i_{\text{cris}}(X^0/S) \otimes_{S,p_2} S(1).$$

By the standard construction, these naturally give connections $\nabla_{\mathcal{M}^i}: M^i \to M^i \otimes_{\mathcal{O}_L} \hat{\Omega}_{\mathcal{O}_L}$, $\nabla_{\mathcal{M}^i}: \mathcal{M}^i \to \mathcal{M}^i \otimes_{\mathcal{O}_L} \hat{\Omega}_{\mathcal{O}_L}$. Furthermore, it follows from above constructions that $\nabla_{\mathcal{M}^i}$ is compatible with $\nabla_{\mathcal{M}^i[p^{-1}]}$ via the isomorphism $\mathcal{M}^i[p^{-1}] \cong \mathfrak{D}(M^i[p^{-1}]) := S \otimes_{\mathcal{O}_L} M^i[p^{-1}]$ given in Proposition 5.3.

Let $\tilde{S}(1)$ be the $p$-adically completed PD-envelope of $\mathcal{O}_{L_0[x]} \hat{\otimes} W(k) \mathcal{O}_{L_0[y]}$ with respect to the kernel of the map $\mathcal{O}_{L_0[x]} \hat{\otimes} W(k) \mathcal{O}_{L_0[y]} \twoheadrightarrow \mathcal{O}_L/(p)$ given by $x, y \mapsto \pi$. Note that the kernel is generated by $p$, $E(x)$, $x - y$, and $a \otimes 1 - 1 \otimes a$, $a \in \mathcal{O}_L$. Similarly as above, we have the maps $p_1, p_2: S \to \tilde{S}(1)$ given by $p_1(u) = x$ and $p_2(u) = y$. For each $n \geq 1$, the induced map $p_1, p_2: S/(p^n) \to \tilde{S}(1)/(p^n)$ is flat by [22, Tag 07HD], since the $p$-adically completed PD-envelope of $\mathcal{O}_{L_0}(1)[z]$ with respect to $(z)$ (for $z = x - y$) is flat over $\mathcal{O}_L$. So by crystalline base change, we have isomorphisms

$$H^i_{\text{cris}}(X^0/S) \otimes_{S,p_1} \tilde{S}(1) \cong H^i_{\text{cris}}(X^0/\tilde{S}(1)) \cong H^i_{\text{cris}}(X^0/S) \otimes_{S,p_2} \tilde{S}(1).$$

For $\mathfrak{S} = \mathcal{O}_{L_0}[u]$, write $\hat{\Omega}_{\mathfrak{S}}$ for the $p$-adic completion of $\Omega^1_{\mathfrak{S}/\mathbb{Z}_p}$. By [1, Prop. 1.3.1], we have $\hat{\Omega}_{\mathfrak{S}} \cong \left( \bigoplus_{j=1}^d \mathfrak{S} \cdot dT_j \right) \hat{\otimes} \mathfrak{S} \cdot du$. The universal derivation $d: \mathfrak{S} \to \hat{\Omega}_{\mathfrak{S}}$ naturally extends to $d: S \to S \hat{\otimes} \hat{\Omega}_{\mathfrak{S}}$. By [2, Thm. 6.6], the HPD-stratification given by the isomorphisms in (5.2) induces a connection

$$\nabla: \mathcal{M}^i \to \mathcal{M}^i \hat{\otimes} \hat{\Omega}_{\mathfrak{S}} \cong \left( \bigoplus_{j=1}^d \mathcal{M}^i \cdot dT_j \right) \hat{\otimes} \mathcal{M}^i \cdot du,$$

which is compatible with $\nabla_{\mathcal{M}^i}$ above. For $1 \leq j \leq d$, let $\partial_{T_j}: \mathcal{M}^i \to \mathcal{M}^i$ be the derivation given by $\nabla$ composed with the projection to the $dT_j$-component, and let
\[ \partial_k : \mathcal{M}^i \to \mathcal{M}^i \] be the derivation given by \( \nabla \) composed with the projection to the \( du \)-component. Write \( N_{T_j} := T_j \partial_{T_j} \) and \( N := -u \partial_u \) for the corresponding derivations on \( \mathcal{M}^i \).

**Proposition 5.4.** We have a natural isomorphism

\[
H^i_{\text{ét}}(X^\flat, \mathcal{O}_p \otimes \mathbb{Q}_p \mathbf{B}_{\text{cris}}(\mathcal{O}_T)) \cong H^i_{\text{cris}}(X^{\flat}/\mathcal{O}_L) \otimes \mathcal{O}_{L_0} \mathbf{B}_{\text{cris}}(\mathcal{O}_T)
\]

compatible with filtration, \( \varphi, \nabla, \) and \( G_L \)-actions.

**Proof.** Let \( C_p := \widehat{T} = \widehat{K}_g \), and let \( \mathcal{O}_C \) its ring of integers. By [4, Thm. 1.1], since \( H^i_{\text{cris}}(X^{k_0}/\mathcal{O}_{L_0}) \otimes \mathcal{O}_{L_0} W(k_g) \cong H^i_{\text{cris}}(X^{k_0}/W(k_g)) \), we have a natural isomorphism

\[ \alpha : H^i_{\text{ét}}(X^\flat, \mathcal{O}_p \otimes \mathbb{Q}_p \mathbf{B}_{\text{cris}}(\mathcal{O}_C)) \cong H^i_{\text{cris}}(X^{\flat}/\mathcal{O}_L) \otimes \mathcal{O}_{L_0} \mathbf{B}_{\text{cris}}(\mathcal{O}_C) \]

compatible with filtration, \( \varphi, \) and \( G_K \)-actions. Consider the associated isomorphism

\[ \beta : H^i_{\text{ét}}(X^\flat, \mathcal{O}_p \otimes \mathbb{Q}_p \mathbf{B}_{\text{cris}}(\mathcal{O}_C)) \cong H^i_{\text{cris}}(X^{\flat}/\mathcal{O}_L) \otimes \mathcal{O}_{L_0} \mathbf{B}_{\text{cris}}(\mathcal{O}_C) \]

as in [4, Thm. 1.8]. Since \( \beta \) is functorial on \( X \) which is defined over \( \mathcal{O}_L \), it is compatible with \( G_L \)-actions. Note that

\[ H^i_{\text{cris}}(X^{\flat}/\mathcal{O}_L) \otimes \mathcal{O}_{L_0} \mathbf{B}_{\text{cris}}(\mathcal{O}_C) \cong H^i_{\text{cris}}(X^{\flat}/\mathcal{O}_L) \otimes \mathcal{O}_{L_0} \mathbf{B}_{\text{cris}}(\mathcal{O}_C) \]

by crystalline base change, since \( S/(p^n) \to \mathbf{A}_{\text{cris}}(\mathcal{O}_C)//(p^n) \) is flat for each \( n \geq 1 \). Let \( \sigma \in G_L \), and consider two maps \( S \to \mathbf{A}_{\text{cris}}(\mathcal{O}_C) \) and the composite \( S \to \mathbf{A}_{\text{cris}}(\mathcal{O}_C) \). Via \( p_1, p_2 : S \to \bar{S}(1) \), these two maps induce \( h_p : \bar{S}(1) \to \mathbf{A}_{\text{cris}}(\mathcal{O}_C) \). Then \( \sigma \)-action on \( H^i_{\text{cris}}(X^{\flat}/\mathcal{O}_L) \otimes \mathcal{O}_{L_0} \mathbf{B}_{\text{cris}}(\mathcal{O}_C) \) is induced by

\[ H^i_{\text{cris}}(X^{\flat}/\mathcal{O}_L) \otimes \mathcal{O}_{L_0} \mathbf{B}_{\text{cris}}(\mathcal{O}_C) \cong H^i_{\text{cris}}(X^{\flat}/\mathcal{O}_L) \otimes \mathcal{O}_{L_0} \mathbf{B}_{\text{cris}}(\mathcal{O}_C) \]

as in Section [5.2] above implies that \( G_L \)-action on \( M^i[p^{-1}] \otimes \mathcal{O}_{L_0} \mathbf{B}_{\text{cris}}(\mathcal{O}_C) = M^i[p^{-1}] \otimes \mathcal{O}_{L_0, t_2} \mathbf{B}_{\text{cris}}(\mathcal{O}_T) \) is given by Equation (4.4). On the other hand, consider the projection

\[ pr : M^i[p^{-1}] \otimes \mathcal{O}_{L_0, t_2} \mathbf{B}_{\text{cris}}(\mathcal{O}_T) \to M^i[p^{-1}] \otimes \mathcal{O}_{L_0, t_2} \mathbf{B}_{\text{cris}}(\mathcal{O}_T) \]

as in Section [5.1] Equation (5.6) induces

\[ s : M^i[p^{-1}] \otimes \mathcal{O}_{L_0, t_2} \mathbf{B}_{\text{cris}}(\mathcal{O}_T) \cong (M^i[p^{-1}] \otimes \mathcal{O}_{L_0, t_1} \mathbf{B}_{\text{cris}}(\mathcal{O}_T))^\nabla = 0 \]

compatible with filtration, \( \varphi, \) and \( G_L \)-actions, where the \( G_L \)-action on \( M^i[p^{-1}] \otimes \mathcal{O}_{L_0, t_1} \mathbf{B}_{\text{cris}}(\mathcal{O}_T) \) is given by the trivial action on \( M^i[p^{-1}] \). Thus, \( \alpha \) gives an isomorphism

\[ \alpha : T^i[p^{-1}] \otimes \mathbb{Q}_p \mathbf{B}_{\text{cris}}(\mathcal{O}_T) \cong (M^i[p^{-1}] \otimes \mathcal{O}_{L_0, t_1} \mathbf{B}_{\text{cris}}(\mathcal{O}_T))^\nabla = 0 \]

compatible with filtration, \( \varphi, \) and \( G_L \)-actions.
Note that $M^i[p^{-1}] \in \text{MF}(\varphi, \nabla)$ is weakly admissible since $M^i[p^{-1}] \otimes_{L_o} W(k_g)[p^{-1}] = H^i_{\text{cris}}(X_{k_g}/W(k_g))[p^{-1}]$ is weakly admissible. From the isomorphism $\alpha$, we obtain $V_{\text{cris}}(M^i[p^{-1}]) = (T^i[p^{-1}])^\vee$. Hence, we deduce from Theorem 2.4 that $H^i_{\text{et}}(X_T, Q_p) \otimes_{Q_p} \text{OB}_{\text{cris}}(O_T) \simeq H^i_{\text{cris}}(X_{k'/O_{L_0}}) \otimes_{O_{L_0},+1} \text{OB}_{\text{cris}}(O_T)$ compatibly with filtration, $\varphi, \nabla$, and $G_L$-actions.

By above proposition and its proof, $T^i[p^{-1}]$ is a crystalline representation of $G_L$ with Hodge-Tate weights in $[-i, 0]$, and we have $D_{\text{cris}}(T^i[p^{-1}]) \simeq M^i[p^{-1}]$. By Proposition 5.3, we can identify $M^i[p^{-1}] = \mathcal{M}^i = \mathcal{M}^i[p^{-1}]$.

**Proposition 5.5.** Under the conditions of Theorem 5.1, $\mathcal{M}^i$ is a strongly divisible lattice in $\mathcal{D}(D^i)$.

**Proof.** Note that $H^i_{\text{cris}}(X_{k_g}/W(k_g))$ for $j = i$, $i + 1$ are $p$-torsion free. So by Thm. 5.4, $\mathcal{M}^i_g := H^i_{\text{cris}}(X_{O_{k_g}, 0}/S_g)$ is finite free over $S_g$ of rank $m := \text{rk}_{W(k_g)[p^{-1}]} D_{\text{cris}, k_g}(T^i[p^{-1}])^\vee = \text{rk}_{W(k_g)[p^{-1}]} D_{\text{cris}}(T^i[p^{-1}])$.

Since $b_g: O_{L_0}[[u]] \to W(k_g)[[u]]$ is flat, the induced map $S/(p^n) \to S_g/(p^n)$ is flat for each integer $n \geq 1$ by Tag 07HD. Moreover, $S/(p^n) \to S_g/(p^n)$ is a map of local rings, so it is faithfully flat. By crystalline base change, we have

$$\mathcal{M}^i/p^n \mathcal{M}^i \otimes_S S_g \simeq \mathcal{M}^i_g/p^n \mathcal{M}^i_g.$$  

In particular, $\mathcal{M}^i_p \mathcal{M}^i$ is free over $S/(p)$ of rank $m$ by faithfully flat descent. Let $\{e_1, \ldots, e_m\}$ be a basis for $\mathcal{M}^i/p^n \mathcal{M}^i$ over $S/(p)$, and choose a lifting $\hat{e}_1, \ldots, \hat{e}_m \in \mathcal{M}^i$. By Nakayama’s lemma, $\mathcal{M}^i$ is generated by $\hat{e}_1, \ldots, \hat{e}_m$ as $S$-module. Since $\mathcal{M}^i[p^{-1}]$ is free over $S[p^{-1}]$ of rank $m$ by Proposition 5.3, we conclude that $\mathcal{M}^i$ is free over $S$ of rank $m$. Furthermore, the map $\mathcal{M}^i \otimes_S S_g \to \mathcal{M}^i_g$ is an isomorphism.

Let $\text{Fil}^i \mathcal{M} = \mathcal{M}^i \cap \text{Fil}^i \mathcal{D}(M^i[p^{-1}])$. Since $\mathcal{M}^i_g$ is a strongly divisible lattice of weight $i$ in $S_g \otimes_{W(k_g)} D_{\text{cris}, k_g}(T^i[p^{-1}])^\vee$ by Thm. 5.4, we have

$$\varphi(\text{Fil}^i \mathcal{M}_g) \subset p^i \mathcal{M}^i_g.$$  

Since $\text{Fil}^i \mathcal{M} \subset \text{Fil}^i \mathcal{M}_g$ and $S \cap p^n S_g = p^n S$, $\varphi(\text{Fil}^i \mathcal{M}) \subset \mathcal{M}^i \cap p^n \mathcal{M}^i = p^i \mathcal{M}^i$.

It remains to study the differential operators on $\mathcal{M}^i$. We observed above that $\mathcal{M}^i$ is stable under $\nabla_{\mathcal{D}(M^i[p^{-1}])}$. Furthermore, since $\mathcal{M}^i_g$ is a strongly divisible lattice in $S_g \otimes_{W(k_g)} D_{\text{cris}, k_g}(T^i[p^{-1}])^\vee$, we have

$$N_{\mathcal{D}(M^i[p^{-1}])}(\mathcal{M}^i) \subset \mathcal{M}^i_g \cap \mathcal{D}(M^i[p^{-1}]) = \mathcal{M}^i.$$  

**Proof of Theorem 5.1.** It remains to show that $T_{\text{cris}}(\mathcal{M}^i) \simeq (T^i)^\vee$ as $G_L$-representations.

Let $\mathcal{M}^i \in \text{Mod}_S^d$ such that $T_{\text{cris}}(\mathcal{M}^i) \simeq (T^i)^\vee$ given by Theorem 4.23. Note that $\mathcal{M}^i$ is constructed explicitly in Section 4.3 compatibly with the construction in 20 via $b_g: S \to S_g$. We identify $\mathcal{M}^i[p^{-1}] = \mathcal{D}(M^i[p^{-1}]) = \mathcal{D}(M^i[p^{-1}])$.

Since $\mathcal{M}^i_g := \mathcal{M}^i \otimes_S S_g$ is the strongly divisible lattice of $S_g \otimes_{W(k_g)} D_{\text{cris}, k_g}(T^i[p^{-1}])^\vee$ such that $T_{\text{cris}}(\mathcal{M}^i_g) = (T^i)^\vee$ as $G_{K_g}$-representations, it is shown in 10 Pf. of Thm.
5.4] that $\mathcal{M}_g = N_g^i$ as $S_g$-submodules of $S_g \otimes_{W(k_g)} D_{\text{cris}, k_g}(T^i[p^{-1}]^\vee) = \mathcal{O}(M^i[p^{-1}]) \otimes S_g$. Thus,

$$\mathcal{M}^i = \mathcal{M}^i[p^{-1}] \cap \mathcal{M}_g = N^i[p^{-1}] \cap N_g^i = N^i.$$ 

□

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