Relativistic hydrodynamics and non-equilibrium steady states

Michael Spillane and Christopher P Herzog

Department of Physics and Astronomy, C. N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, NY 11794, USA
E-mail: michael.spillane@stonybrook.edu

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Abstract. We review recent interest in the relativistic Riemann problem as a method for generating a non-equilibrium steady state. In the version of the problem under consideration, the initial conditions consist of a planar interface between two halves of a system held at different temperatures in a hydrodynamic regime. The new double shock solutions are in contrast with older solutions that involve one shock and one rarefaction wave. We use numerical simulations to show that the older solutions are preferred. Briefly we discuss the effects of a conserved charge. Finally, we discuss deforming the relativistic equations with a nonlinear term and how that deformation affects the temperature and velocity in the region connecting the asymptotic fluids.

Keywords: AdS/CFT correspondence, Compressible flow, Exact results
1. Introduction

The Riemann problem in hydrodynamics is an initial value problem where two equilibrium fluids are joined by a discontinuity. The solution for the case of relativistic fluids was first solved in 1948 \[\text{[1]}\]. It has since been studied in other papers including \([2, 3]\). The key feature of these solutions is a rarefaction (adiabatic) region joined to a boosted constant temperature region and then followed by a shock discontinuity\(^1\).

The problem has seen renewed interest as an example of a steady state system which is not in thermal equilibrium (NESS). This type of NESS was studied recently in \(1 + 1\) dimensional CFT\(\text{'s}\) \([5]\) and later extended to hydrodynamical descriptions of higher dimensional CFT\(\text{'s}\) \([11, 12]\). Finally, it was considered for a CFT deformed by a relevant operator \([13]\). For older results, see \([6–10]\).

The papers \([11–13]\) miss the important rarefaction region and as a result their values for the NESS temperature as well as the rate of growth of the NESS are incorrect. In \([11, 13]\), a mistake was made in the assumption that two shocks exist. It is less straightforward to see what went wrong in \([12]\). The authors do not use the word shock. In section 3.2, they find two possible solutions to the Riemann problem, one of which is identical to the two shock solution found in \([11, 13]\). The other solution does not correspond to a rarefaction wave, nor to any numerical simulations we have done. (It would be interesting to see if an appropriate physical context can be found where this second solution is the preferred solution of a Riemann problem.) The reason \([12]\) fail to find a rarefaction region is their assumption that the waves separating the steady state region from the asymptotic regions are either purely left moving or purely right moving while a rarefaction wave is neither.

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\(^{1}\) For a discussion of these issues in the context of the quark-gluon plasma, see for example \([4]\).
In this paper we review both the older solution with the rarefaction region as well as the more recent solution which we call the two shock solution. Using a numerical simulation of relativistic hydrodynamics we show that the rarefaction solution matches the numerical simulation better than the two shock solution. We also consider a CFT deformed by a relevant operator. We calculate the change in pressure in the boosted region as a function of operator dimension and reservoir pressure. The result again differs from the two shock solution.

2. Ideal hydrodynamics

We begin with the stress tensor of a perfect fluid and conserved currents

\[ T^{\mu \nu} = (e + p)u^\mu u^\nu + p\eta^{\mu \nu}, \]  
\[ J^\mu_i = q_i u^\mu. \]

We mean perfect in the sense of having no dissipation. Here \( e \) is the energy density, \( q_i \) a charge density, and \( p \) the pressure. We have introduced a four velocity \( u^\mu \) such that \( u^2 = g^{\mu \nu}u_\mu u_\nu = -1 \). At rest the fluid is described by \( u^\mu = (1, 0) \). We work in mostly plus signature with the Minkowski metric tensor \( \eta^{\mu \nu} = (-, +, \ldots, +) \). The conservation equations of energy and momentum are given by

\[ \partial_\mu T^{\mu \nu} = 0, \]  
\[ \partial_\mu J^\mu_i = 0. \]

The conservation equations are combined with an equation of state \( e = e(p) \). In this paper we will largely focus on a linear equation of state

\[ p = c_s^2 e, \]

where \( c_s \) is the speed of sound. One important example for us is a conformal fluid in \( d \) spatial dimensions, where \( c_s^2 = 1/d \).

We are interested in flows depending only on a single spatial variable, arbitrarily chosen to be \( x \), and time. We will perform a change of variables to the local fluid velocity \( v_i = u'^i u'^i \). In these variables the stress tensor conservation equations become

\[ \partial_t [(e + p)\gamma^2 - p] + \partial_x [(e + p)\gamma^2 v_x] = 0, \]  
\[ \partial_t [(e + p)\gamma^2 v_x] + \partial_x [(e + p)\gamma^2 v_x^2 + p] = 0, \]  
\[ \partial_t [(e + p)\gamma^2 v_T] + \partial_x [(e + p)\gamma^2 v_x v_T] = 0. \]

We have introduced \( \gamma = 1/\sqrt{1 - v_x^2 - v_T^2} \) and also \( v_T \), the fluid velocity in the spatial directions perpendicular to \( x \).
When there is a shock discontinuity we use the Rankine–Hugoniot jump condition to determine the relationship between conserved quantities on alternate sides of the shock [14]. For a conservation law of the form
\[ \partial_t Q(t, x) + \partial_x F(t, x) = 0, \]
\[ u_s[Q] = [F], \]
\[ [Q] = Q_L - Q_R, \]
\[ [F] = F_L - F_R, \]
where \( Q_L \) (\( Q_R \)) and \( F_L \) (\( F_R \)) is the value of \( Q \) and \( F \) to the left (right) of the shock and \( u_s \) is the velocity of the shock. For the case of a perfect fluid, the jump conditions are given by
\[ u_s[T^{it}] = [T^{tx}], \]
\[ u_s[T^{xt}] = [T^{xz}], \]
\[ u_s[J^t_i] = [J^t_i]. \]
Equations like (15) can exhibit what are known as contact discontinuities. These have a jump in the conserved quantities. However, there is no transportation of particles across the discontinuity. In the case of (15), such a contact discontinuity can occur when \( u_s/u_t = u_s \). The jump condition is trivially satisfied and the change in the conserved quantity can be arbitrary across the shock.

3. Double shock solution

We are interested in solving the Riemann problem, where two semi-infinite fluids of different temperatures are brought into contact. An interesting feature of the resulting fluid flow is a non equilibrium steady state (NESS) that forms in the expanding region between the two semi-infinite fluids. Recently three papers [11–13] have presented a solution that is not completely correct. As it is more straightforward to identify the mistake in [11, 13], let us quickly review their work to see where the problem occurs.

They start with the EOS \( p = c_s^2 e \). The initial conditions are that of two systems (with energies \( e_L \) and \( e_R \)) brought into thermal contact. They assume that two shocks propagate away from each other, leaving the NESS in between. The central region is assumed to have a constant fluid velocity \( v \). We will suppress the \( x \) subscript on the fluid velocity \( v \) in what follows and set \( v_T = 0 \). This latter choice is both for simplicity and to compare with [11, 13]. A nonzero \( v_T \) will make a brief appearance again in our expression for a rarefaction wave (27). The conservation laws (13)–(15) imply
\[ u_L = -c_s \sqrt{\frac{c_s^2 \chi + 1}{\chi + c_s^2}}, \quad u_R = c_s \sqrt{\frac{\chi + c_s^2}{c_s^2 \chi + 1}}, \]
\[ \text{do}i:10.1088/1742-5468/2016/10/103208 \]
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\[ e = \sqrt{c_L c_R}, \quad \frac{\chi}{c_s} = \frac{\chi - 1}{\sqrt{c_s^2 \chi + 1}(\chi + c_s^2)}, \]  

where \( \chi = \sqrt{e_L / e_R} \). The velocity of the left and right moving shocks are \( u_L \) and \( u_R \) respectively.

This solution however is invalid because it violates the entropy condition [14]. This condition is most easily stated when the conservation conditions are written in characteristic form,

\[ \partial_t \tilde{u} + B(\tilde{u}) \partial_x \tilde{u} = 0, \]  

where here \( \tilde{u}(x, t) = (p(x, t), v(x, t)) \). The eigenvalues \( \lambda \) of \( B \) are

\[ \lambda = \frac{v \pm c_s}{1 \pm vc_s^2}. \]  

These characteristics correspond to the local right and left moving speeds of sound at a given space-time point in the fluid. (Reassuringly, \( \lambda \to \pm c_s \) in the limit where the background fluid velocity vanishes, \( v = 0 \).)

The entropy condition requires that for solutions involving a shock, characteristics end on a shock discontinuity rather than begin on it. By ending on the shock, information is lost and entropy should increase. In contrast, in order for characteristics to begin on a shock, boundary conditions need to be specified, decreasing the entropy. More precisely, consider a right moving shock, \( u > 0 \). Let \( \lambda_R \) and \( \lambda_L \) be eigenvalues of \( B \) immediately to the right and left of the shock, respectively. We should take the eigenvalues corresponding to right moving characteristics. For the characteristics to end on the shock, it is necessary that

\[ \lambda_L > u > \lambda_R. \]  

This condition is true for \( u_R \in (c_s, 1) \), which is true for \( \chi > 1 \). Therefore a shock is a valid solution for the wave moving into the colder medium. However for \( u_R \in (c_s^2, c_s) \), which is true for \( \chi < 1 \), neither inequality holds. Thus the entropy condition rules out a shock moving into the hotter region. We could also analyze separately the left moving shock, but the physics is invariant under parity.\(^2\)

While we know from this analysis that the double shock solution (figure 1) is unphysical, it turns out to be very close to the actual solution in some situations. Given the simplicity of the double shock solution, it is interesting to consider adding

\[^2\text{When } c_s = 1, \text{ which holds for a conformal field theory in } 1+1 \text{ dimensions, } u_R = \lambda_R = 1 \text{ as well, and the entropy condition is satisfied (and saturated) for both the left and right moving shocks. We will see below that in this degenerate case, the two shock solution becomes identical with the solution involving a rarefaction wave.}\]
a conserved charge. This addition requires us to include a contact discontinuity. A contact discontinuity is a discontinuity in one variable that travels at the local fluid velocity. For such a discontinuity the Rankine–Hugoniot jump condition is trivially satisfied and the change in the variable can be arbitrary. In this case the discontinuity is in the conserved charge. The result is the splitting of the NESS region into two NESS with distinct charges but equal velocities and pressures. The resulting charge densities, which follow from the RH relation (15), are

\begin{align*}
q_1 &= \frac{q_L \sqrt{c_s^2 \chi + 1}}{\sqrt{\chi} \sqrt{c_s^2 + \chi}}, \\
q_2 &= \frac{q_R \sqrt{\chi} \sqrt{c_s^2 + \chi}}{\sqrt{c_s^2 \chi + 1}}.
\end{align*}

where \(q_1\) is the charge density in the region adjacent to \(q_L\), \(q_2\) is the charge density in the region adjacent to \(q_R\).\(^3\)

4. Adiabatic flow

We now need to replace the shock that does not satisfy the entropy condition with a smooth solution. The solution can be found in previous papers. As can be concluded by considering characteristics, the shock solution should be replaced with a fan of characteristics [2, 3, 14]. There is a characteristic for each value of the dimensionless ratio \(\xi = x/t\). Therefore we will search for a solution that depends only on \(\xi = x/t\). Such a solution would correspond to an adiabatic expansion.

It is simple to check that for an entropy-like quantity \(s = p^{1/(1+c)}\), equations (6)–(8) imply

\[
\partial_t(s\gamma) + \partial(s\gamma v_x) = 0.
\]

After switching to coordinates \(\xi = x/t\), we can combine equations (8) and (24) and obtain

\(^3\) The entropy condition is trivially satisfied for shock discontinuities.
where we have defined
\[ \kappa \equiv \frac{c_s^2}{1 + c_s^2}. \]  

Solving the conservation equations (3) and (4) with the ansatz that the currents are functions of \( x \) and \( t \) only through the combination \( \xi = x/t \), the solution in the adiabatic region is then
\begin{align*}
v_T &= \frac{\alpha \sqrt{1 - v_s(\xi)^2}}{\sqrt{\alpha^2 + p^2\kappa}}, \\
v_x &= \frac{\pm(\xi^2 - 1)c_s p^\kappa \sqrt{\alpha^2(1 - c_s^2) + p^2\kappa} + \xi(c_s^2 - 1)(\alpha^2 + p^2\kappa)}{\alpha^2(c_s^2 - 1) + (\xi^2 c_s^2 - 1)p^2\kappa}, \\
p^\kappa &= \frac{\alpha^2(c_s^2 - 1)\left(1 - \frac{\xi}{\xi + 1}\right)^{\pm\frac12}}{4c_1} + c_1 \left(1 - \frac{\xi}{\xi + 1}\right)^{\pm\frac{c_s^2}{2}}, \\
q &= \exp \left\{ \frac{\alpha^2 + p^2\kappa}{1 - \xi^2} + \frac{\alpha^2 \kappa \partial_\xi p}{p} + \frac{\alpha^2 \kappa \partial_\xi q_p}{\sqrt{p^2\kappa - \alpha^2(c_s^2 - 1)}} \right\}.
\end{align*}

There are five ‘integration’ constants associated with these expressions. Two are obvious: \( \alpha \) and \( c_1 \). A third is associated with the integral for \( q \). The fourth and fifth are the left and right endpoints of the rarefaction wave \( \xi_L \) and \( \xi_R \). We can use four of these integration constants to match \( v_T, v_x, p \) and \( q \) at one edge of the rarefaction wave to their values in an asymptotic region. We then have one degree of freedom left, which we may take to be \( \xi_R \), to match onto the boosted region and the shock.

For simplicity, we take the limit of zero tangential velocity \( (v_T = 0 \text{ or equivalently } \alpha = 0) \). Trading \( c_1 \) for \( p \) and making explicit the integration constant associated with \( q \), there are two solutions:
\begin{align*}
v_\pm(\xi) &= \frac{c_s \pm \xi}{c_s \xi + 1}, \\
p_\pm(\xi) &= \frac{1 - \xi^{1 + c_s^2}}{1 + \xi}. \quad (32)
\end{align*}
To solve the Riemann problem we need to match this adiabatic region onto a NESS region and a shock (figure 2). Without loss of generality we can choose \( p_L > p_R \). We then have a shock moving to the right at speed \( u_s \). In the left region, the disturbance moves at the speed of sound as can be seen by setting \( v_0 = \pm u_s \) to the right of the shock \( v = 0 \) and \( p = p_R \). To the left of the shock \( v = V \) and \( p = p_L \). Then the jump conditions are

\[
p_0(c_s^2 V(u_s V + 1) + u_s - V) + p_R u_s(V^2 - 1) = 0, \tag{34}
\]

\[
p_0(c_s^2 + V(c_s^2 u_s + u_s - V)) + p_R c_s^2(V^2 - 1) = 0. \tag{35}
\]

Ideally when we put everything together we would get \( V \) and \( u_s \) as functions of \( \chi \). The best we were able to achieve were parametric expressions of \( V \) and \( \chi \) as functions \( u_s \):

\[
V = \frac{u_s^2 - c_s^2}{u_s(1 - c_s^2)}, \tag{36}
\]

\[
\chi(u_s)^2 = \frac{(u_s - c_s^2)(c_s^2 + u_s)}{c_s^2(1 - u_s^2)} \left( \frac{1 + u_s}{1 - u_s} \right)^{\frac{c_s^2}{2c_s}} \left( \frac{u_s - c_s^2}{c_s^2 + u_s} \right)^{\frac{c_s^2}{2c_s}}, \tag{37}
\]

\[
p_0 = p_L \left( \frac{(1 - u_s)(u_s + c_s^2)}{(1 + u_s)(u_s - c_s^2)} \right)^{\frac{c_s^2}{2c_s}}, \tag{38}
\]

where \( \chi = \sqrt{p_L/p_R} \). We can then add in the charge which has a contact discontinuity.

\[
q_1 = q_L \left( \frac{(1 - u_s)(u_s + c_s^2)}{(1 + u_s)(u_s - c_s^2)} \right)^{\frac{1}{2c_s}}, \tag{39}
\]
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\[ q_2 = \frac{q_R u_R \sqrt{u_R^2 - c_s^4}}{c_s^4 \sqrt{1 - u_R^2}}. \]  

(40)

One way of deriving the entropy condition is to look at ideal hydrodynamics as a limit in which higher gradient corrections to the equations of motion are set to zero. Ideal hydrodynamics allows for several \textit{ab initio} equivalent solutions to the conservation equations—two shocks, two rarefactions, a shock-rarefaction, and a rarefaction-shock. If we think of ideal hydrodynamics as a limit in which a positive viscosity is reduced to zero, the rarefaction-shock solution is preferred because positive viscosity always leads to entropy production. If we were to take the limit in which viscosity approached zero from below, then an entropy decreasing shock-rarefaction solution would be preferred. Indeed, we could consider even more perverse situations where viscosity depends on position, allowing for a two-shock or two-rarefaction solution in the limit in which viscosity is reduced to zero.

While we have only analyzed the entropy condition for the simplest case where \( \nu_T = 0 \), we expect that the rarefaction-shock solution is preferred for the family of examples that we consider in this paper.

Viscous and other gradient corrections thus play an important role in selecting the physical solution from among a discrete set of fluid profiles. There is then a second order question of how they correct these fluid profiles as we move away from the ideal limit. We will return to this question in the discussion.

### 4.1. Simple limits

While we only have a parametric solution in general, various limits take a simpler form. Consider the first the limit \( c_s \to 1 \). As \( c_s^2 = 1/d \) for a conformal field theory, we can think about this limit as perturbing the number of spatial dimensions away from one, \( d = 1 + \varepsilon \). In this case, the width of the rarefaction fan scales as

\[ \delta \xi = \frac{1}{2} (\chi - 1) \varepsilon + O(\varepsilon^2), \]  

(41)

which vanishes as \( \varepsilon \to 0 \). Indeed, the difference between the rarefaction solution and the two-shock solution is controlled by \( \varepsilon \), with the differences

\[ \delta v = 2f(\chi) \varepsilon^2 + O(\varepsilon^3), \]  

(42)

\[ \delta u_R = f(\chi) \varepsilon^3 + O(\varepsilon^4), \]  

(43)

\[ \delta p = \frac{p_L}{\chi^2(1 + \chi)^3} f(\chi) \varepsilon^2 + O(\varepsilon^3), \]  

(44)

where we have defined the function of \( \chi \), positive for \( \chi > 1 \),

\[ f(\chi) = \frac{\chi}{8(1 + \chi)^3} \left( 1 - \chi + (1 + \chi) \tanh^{-1} \frac{\chi - 1}{\chi + 1} \right). \]

The quantities \( \delta v \), \( \delta p \) and \( \delta u_R \) are the differences between the rarefaction solution and the two shock solution in velocity, pressure and shock speed respectively, e.g.
\( \delta v = v(\text{2-shock}) - v(\text{rarefaction}) \). In this limit, the two-shock solution slightly overestimates the pressure and fluid velocity in the NESS, and also the right moving shock speed.

The linear response regime \( \chi \to 1 \) of the rarefaction solution also approaches the two-shock solution. Let \( p_L = p_R(1 + \epsilon) \), \( \epsilon \ll 1 \), in which case

\[
\delta \xi = \frac{(1 - c_s^2) c_s}{2(1 + c_s^2)} \epsilon + O(\epsilon^2),
\]

\[
\delta v = \frac{(1 - c_s^2) c_s^3}{384(1 + c_s^2)^3} \epsilon^3 + O(\epsilon^4),
\]

\[
\delta p = p_L \frac{(1 - c_s^2)^2 c_s}{384(1 + c_s^2)^2} \epsilon^3 + O(\epsilon^4),
\]

\[
\delta u_R = \frac{(1 - c_s^2)^2 c_s}{768(1 + c_s^2)^3} \epsilon^3 + O(\epsilon^4),
\]

where \( \delta \xi \), \( \delta v \), \( \delta p \) and \( \delta u_R \) are as before. In this limit, the two-shock solution again slightly overestimates the velocity and pressure in the NESS and the speed of the right moving shock.

While the two shock solution and the solution with a rarefaction region quickly approach each other in the limit \( p_L / p_R \), in the opposite limit where \( \chi \gg 1 \), the solutions have qualitatively different behavior. In both cases, the speed of the right moving shock approaches one, \( u_s \), \( u_R \to 1 \), but at a different rate. Let us assume a large \( \chi \) ansatz where \( u_s = 1 - \delta \). In the rarefaction case

\[
\chi = \left( \frac{1}{\delta} \right)^{c_s^2 + 1} \left( \frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{1 - c_s^2}} \left( -2 + \frac{4}{1 + c_s^2} \right)^{\frac{1 + c_s^2}{2c_s}} (1 + O(\delta)),
\]

\[
p = p_L \delta \frac{c_s^2 + 1}{2c_s} \left( -2 + \frac{4}{1 + c_s^2} \right)^{-\frac{1 + c_s^2}{2c_s}} (1 + O(\delta)),
\]

\[
v = 1 - \frac{1 + c_s^2}{1 - c_s^2} \delta + O(\delta^2).
\]

In contrast, for the two shock solution \( \chi \sim \frac{1}{\delta} \) and \( p \sim \sqrt{\delta} \).

Another important difference is that the size of the rarefaction region grows in this limit \( \chi \gg 1 \). The rightmost characteristic of the rarefaction fan approaches the location of the right moving shock:

\[
\xi_R = 1 - \frac{1 + c_s^2}{(1 - c_s^2)^2} \delta + O(\delta^2).
\]
Thus the size of the boosted region between the shock and rarefaction is correspondingly reduced. Indeed, for all practical purposes, the boosted region probably disappears in this limit. An initial condition that is a step function is an idealization, and slightly smoothing the step function will destroy the boosted region, as will viscous corrections, which smear out the shock.

In this limit $\chi \gg 1$, it is unclear to these authors at least what region is most appropriately called the NESS. For the authors of [15], which appeared concurrently with this work, the NESS is the region near and at the interface $x = 0$. There would then be a sort of phase transition associated with the size of $\chi$. For small $\chi$, the NESS is a boosted plasma, while for large $\chi$, it’s a point with some nice properties inside the rarefaction wave. (The dependence of the rarefaction wave on $\xi = x/t$ guarantees that the pressure and energy current are stationary as a function of $x$ at $x = 0$. This dependence also guarantees that the energy, pressure, and current instantaneously obtain their large $t$ values at $x = 0$.4) Another option, which seems of similar validity to us given the underlying boost invariant nature of the problem, is to declare the NESS is the boosted region between the rarefaction and shock wave, even if for large $\chi$, it does not overlap the interface point $x = 0$. One can boost to a frame where there are currents in the asymptotic regions and the interface point is again inside the boosted region.

5. Non-linear equation of state

We can also consider the Riemann problem for nonlinear equations of state. For simplicity we will assume that $v_T = 0$. We consider a perturbation to the CFT by a relevant operator as in [13, 16],

$$S_{\text{QFT}} = S_{\text{CFT}} + \lambda' \int d^{d+1}x \mathcal{O}(x)$$

in the limit $\lambda' T^{d+1-\Delta} \ll 1$. For relevance and unitarity, we require $\frac{d-1}{2} \leq \Delta < d + 1$. Such a perturbation should affect the equilibrium pressure and energy density at second order in $\lambda'$. Following [16], we assume an ansatz where

$$p = c_T T^{d+1} \left(1 - \frac{\lambda^2}{T^{2(d+1-\Delta)}}\right),$$

$$e = c d T^{d+1} \left(1 - \alpha \frac{\lambda^2}{T^{2(d+1-\Delta)}}\right),$$

where $\lambda$ and $\lambda'$ are proportional\(^5\). The Gibbs–Duhem relation $e + p = sT$ along with $s = \frac{dp}{dT}$ then imply that $\alpha = \frac{1}{d} (2\Delta - d - 2)$ (where $s$ is the entropy density). Eliminating $T$, we can write an equation of state for $e$ as a function of $p$:

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\(^4\) We thank Koenraad Schalm and Amos Yarom for discussion on these points.

\(^5\) We expect that a relevant perturbation should decrease the effective number of degrees of freedom of the theory and thus further decrease the entropy at low temperatures, explaining the minus sign in front of the correction to the pressure.
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\[ e = pd + \epsilon \lambda^2 p^n, \]  

(56)

where

\[ \epsilon = 2c \frac{2d(1-\Delta)}{d+1}(d+1 - \Delta), \]  

(57)

\[ n = \frac{2\Delta}{d+1} - 1. \]  

(58)

Note that \( \epsilon > 0. \)

As in the case of a linear equation of state, we find an adiabatic solution and match it onto a NESS region and a shock. The equations of motion remain (6) and (7). We can solve these equations in a perturbative expansion in \( \lambda \) as follows:

\[ p = p_0 + p_1 \lambda^2, \quad v_x = v_0 + v_1 \lambda^2. \]  

(59)

Again there are two solutions

\[ \begin{align*}
  p_0^\pm &= c_{0p} \left( \frac{1 - \xi}{1 + \xi} \right)^{\frac{1+c_s^2}{2c_s}}, \\
  v_0^\pm(\xi) &= \frac{c_s \pm \xi}{\xi c_s \pm 1}, \\
  p_1^\pm &= -\epsilon c_s^2(2 - 2c_s^2 + n(1 + c_s^2)(n + nc_s^2 - 2))c_p^{n-1} \left( \frac{1 - \xi}{1 + \xi} \right)^{\frac{1+c_s^2}{2c_s}}, \\
  v_1^\pm &= \pm \epsilon n(\xi^2 - 1)c_s^3 c_p^{n-1} \left( \frac{1 - \xi}{1 + \xi} \right)^{\frac{1+c_s^2}{2c_s}} \\
  \xi_L &= -c_s + \frac{n p_L^{n-1} c_s^3}{2} \epsilon \lambda^2, \\
  c_{0p} &= p_L \left( \frac{1 + c_s}{1 - c_s} \right)^{-\frac{(1+c_s^2)/c_s}{2c_s}} + \epsilon \lambda^2 p_L^{n-1} c_s^2(n c_s^2 + n - 2) \left( \frac{1 + c_s}{1 - c_s} \right)^{-\frac{(1+c_s^2)/c_s}{2c_s}}.
\end{align*} \]  

(60-65)

Unfortunately, analytic solutions for the correction to the NESS pressure are not available. However, one can still do the calculation numerically with an interesting result. Unlike for the double shock solution presented in [13] the change to the NESS temperature is dependent on both \( \Delta \) and \( p_L (p_L > p_R = 1) \) as opposed to only \( \Delta \) in [13].
We do not have an analytic expression for the dependence of $P_{NESS}$ (the pressure in the NESS region) on $p_L$ and $\Delta$, but a graphical representation of our results is presented in figure 3 for two different spacetime dimensions. There is a nontrivial curve in the $\Delta - p_L$ plane along which the pressure in the NESS region remains unchanged from its CFT value.

In the two shock solution we can find the corrections analytically. We find

$$\delta v = \frac{(d+1)(\chi^n + \chi)((d+\chi)\chi^n - \chi(d\chi + 1))}{4\sqrt{d}\chi(d + \chi)^{3/2}(d\chi + 1)^{3/2}} \epsilon \lambda^2,$$

$$\delta p = \frac{\chi^n - \chi)((d+\chi)\chi^n - \chi(d\chi + 1))}{2d(d + 1)\chi(\chi + 1)} \epsilon \lambda^2. \tag{66}$$

The quantities $\delta v$ and $\delta p$ are the differences between the QFT and the CFT values of velocity and pressure respectively, e.g. $\delta v = v_{QFT} - v_{CFT}$. We note that $\delta p \geq 0$ for all relevant operators independent of $\chi$, where the inequality is saturated for $\Delta = d + 1$.\(^6\)

6. Numerical check

We want to implement a numerical scheme to check our results. To do this we use the same hydrodynamic scheme as [17] which employed spectral methods. For simplicity we check only the simplest case, with no transverse velocity $v_T = 0$, no charge density $q = 0$, and no deformation by a relevant operator. It would of course be interesting to perform more general numerical simulations. Given the characteristic and entropy arguments in the earlier part of the paper, the extensive literature on the Riemann problem in the hydrodynamics literature, and the agreement with numerics we find in this simplest case, we expect no surprises. While ideally we should check each case individually, we anticipate that the rarefaction-shock solution is the preferred solution of the Riemann problem for the equations of state and initial conditions considered in this paper.

\(^6\) This result is not at odds with [13] because they consider temperature rather than pressure.
We start with
\[ T^{\mu \nu} = (e + p)u^\mu u^\nu + p\eta^{\mu \nu} + \Pi^{\mu \nu} \] (68)
where we define \( \Pi^{\mu \nu} \) recursively\(^7\) in a gradient expansion
\[ \Pi^{\mu \nu} = -\eta\sigma^{\mu \nu} - \eta \left[ (D \Pi)^{(\mu \nu)} + \frac{3}{2} \Pi^{\mu \nu} (\nabla \cdot u) \right] - \frac{\lambda^2}{\eta} \Pi^{\mu \alpha} \Omega^{\nu \alpha} \] (69)
where \( D \equiv u^\nu \nabla_\nu \).

Conformality implies tracelessness of \( T^{\mu \nu} \) which in turn yields a relationship \( e = d \, p \) between the energy density \( e \) and pressure \( p \) in \( d \) spatial dimensions. While \( \mu \) in principle takes values from 0 to \( d \), we let only \( u_0 \) and \( u^1 \) be nonzero. The vorticity is
\[ \Omega^{\mu \nu} = \frac{1}{2} \Delta^{\mu \alpha} \Delta^{\nu \beta} (\nabla_\alpha u_\beta - \nabla_\beta u_\alpha), \] (70)
where we have defined a projector onto a subspace orthogonal to the four velocity:
\[ \Delta^{\mu \nu} \equiv \eta^{\mu \nu} + u^\mu u^\nu. \] (71)
The shear stress tensor is
\[ \sigma^{\mu \nu} \equiv 2\nabla^{(\mu} u^{\nu)}. \] (72)
The angular brackets \( \langle \rangle \) on the indices indicate projection onto traceless tensors orthogonal to the velocity
\[ A^{(\mu \nu)} \equiv \frac{1}{2} \Delta^{\mu \alpha} \Delta^{\nu \beta} (A_{\alpha \beta} + A_{\beta \alpha}) - \frac{1}{d} \Delta^{\mu \alpha} \Delta^{\nu \beta} A_{\alpha \beta}. \] (73)

Note that with these definitions, both \( \Pi^{\mu \nu} \) and \( \Omega^{\mu \nu} \) are traceless and orthogonal to the velocity
\[ u_\mu \Pi^{\mu \nu} = u_\mu \Omega^{\mu \nu} = 0, \quad \Omega^{\mu \nu} = \Pi^{\mu \nu} = 0. \]
Using \( v_T = 0 \) and flows that only depend on \( x \) and \( t \), we know \( \Pi^{xy} = 0 \) and \( \Omega^{xy} = 0 \). The remaining one independent component of \( \Pi^{xy} \) we choose to be \( B = \Pi^{xx} - \Pi^{yy} \). We then use the two conservation equations and the implicit definiton of \( B \) to propagate forward in time.

We give the equation of state in terms of the energy density \( e \) and temperature \( T \),
\[ e = \left( \frac{4\pi T}{3} \right)^{d+1}, \] (74)
and we start with the initial condition
\[ u_x = 0, \] (75)
\[ T = \frac{T_R - T_L}{2} \tanh(\beta x) + \frac{T_R + T_L}{2}. \] (76)

\(^7\) The implicit definition of \( \Pi^{\mu \nu} \) is Israel–Stewart like. Formally, higher than second order gradient corrections are present in the definition of \( \Pi^{\mu \nu} \).
As can be seen in figure 4 our solution with an adiabatic region matches well with the numerics. We accurately match the speed of the shock as well as the position of the adiabatic region. The matching is not perfect in the adiabatic region because the initial profile was not a perfect step function. The results were insensitive to the values of the dissipative coefficients \( \eta, \tau, \Pi \) and \( \lambda_2 \). While the difference cannot be seen in the figures, we also calculated \( T_{\text{NESS}}^{\text{tx}} \) for the two analytic results and the numerics. The results are presented in table 1.

### Table 1

| \( d \) | Two shock | Adiabatic | Numerics |
|-------|------------|-----------|----------|
| 3     | 0.176587   | 0.176545  | 0.176551 ± 0.000006 |
| 5     | 0.14948    | 0.14936   | 0.149344 ± 0.000009 |

Note: The adiabatic solution presented in this paper is a better fit than the two shock solution.

As can be seen in figure 4 our solution with an adiabatic region matches well with the numerics. We accurately match the speed of the shock as well as the position of the adiabatic region. The matching is not perfect in the adiabatic region because the initial profile was not a perfect step function. The results were insensitive to the values of the dissipative coefficients \( \eta, \tau, \Pi \) and \( \lambda_2 \). While the difference cannot be seen in the figures, we also calculated \( T_{\text{NESS}}^{\text{tx}} \) for the two analytic results and the numerics. The results are presented in table 1.

### 7. Conclusion

We presented a review of previous work on the Riemann problem for a linear equation of state. The solution has the interesting feature of a steady state region with a momentum flux. We contrast this solution with recent solutions [11–13] that have appeared which incorrectly solved the Riemann problem by using a two shock ansatz. These papers had failed to consider the entropy condition and missed the adiabatic expansion region (rarefaction region) which exists between the hotter reservoir and the NESS. We also showed that for a conserved charge, the NESS region is actually two regions with different charges with a contact discontinuity separating them.

In the linearized regime (\( \chi \to 1 \)) and the small dimension limit (\( c_\star \to 1 \)), the two shock solution leads to small errors in the value for the momentum flux of the NESS as well as in the velocity of the shock wave. We were able to show using a numerical simulation that the solution with the adiabatic region is preferred over the two shock solution. The adiabatic region was also a better match for the fluid profile than the shock propagating toward the high temperature reservoir.
Finally, we considered a CFT which had been perturbed by a relevant operator. The perturbation leads to a non-linear correction to the equation of state. Again we found the adiabatic solution which should be matched onto the NESS. Our solution gave a correction to the NESS temperature that depended on $p_L$ and the operator dimension $\Delta$. This diagram contrasted with the two shock solution where the correction to the NESS temperature depends only on the operator dimension.

It would be interesting to move away from the ideal limit considered here and investigate the effects of viscosity and other dissipative corrections. We discussed already the role of viscosity in the entropy condition, in particular in choosing the rarefaction-shock solution from among a discrete family of solutions in the ideal limit. However, we left open how these ideal solutions are corrected as one gradually turns the viscosity back on. The numerical simulations we described in the second half of the paper seem to show that the rarefaction-shock solution is very weakly affected by viscosity. Indeed, there is strong reason to believe that the precise value of the viscosity changes the results in this paper little if at all. Viscosity functions like an irrelevant perturbation in the language of the renormalization group. The behavior we look at here is associated with long times and large distances, in other words ‘infrared’ physics that should be insensitive to irrelevant perturbations. Moreover, there are several textbook analyses of the effect of viscosity on rarefaction and shock waves. See for example [20]. Shocks typically form in finite time, and the width of the shock, while dependent on viscosity, is typically independent of time at large times. The rarefaction region approaches the ideal limit in general at worst as $1/\sqrt{t}$. We hope to make a more detailed analysis of these viscous corrections, tailored to the situation at hand, in the future.

Perhaps the most interesting question is a rephrasing of the problem using the fluid-gravity correspondence [19] as a question about black-hole dynamics. In this context, the Riemann problem considered here maps to a solution to Einstein’s equations in an asymptotically anti-de Sitter space-time. The temperature of the fluid can be re-interpreted as the location of an apparent horizon of a black hole. Is there a gravitational counter-part of the entropy condition that we considered in this paper? What is the gravity dual of the adiabatic region, and why is it required for consistency of the theory? For recent numerical results addressing some of these questions, see [18].

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Note added. While we were putting the finishing touches on this paper, we became aware of related and overlapping work [15].
Relativistic hydrodynamics and non-equilibrium steady states

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