Punctured holomorphic curves and Lagrangian embeddings

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Abstract

We use a neck stretching argument for holomorphic curves to produce symplectic disks of small area and Maslov class with boundary on Lagrangian submanifolds of nonpositive curvature. Applications include the proof of Audin’s conjecture on the Maslov class of Lagrangian tori in linear symplectic space, the construction of a new symplectic capacity, obstructions to Lagrangian embeddings into uniruled symplectic manifolds, a quantitative version of Arnold’s chord conjecture, and estimates on the size of Weinstein neighbourhoods. The main technical ingredient is transversality for the relevant moduli spaces of punctured holomorphic curves with tangency conditions.

1 Introduction

In this paper we use punctured holomorphic curves to establish some new restrictions on Lagrangian embeddings. We will denote by manifold of nonnegative curvature a manifold which admits a Riemannian metric of nonpositive sectional curvature. In fact, the only property that enters the proofs is the existence of a metric for which all closed geodesics are noncontractible and have no conjugate points.

Complex projective space. Consider the complex projective space $\mathbb{CP}^n$ with its standard symplectic structure $\omega$, normalized such that a complex line has symplectic area $\pi$. Let $D$ denote the closed unit disk. The following are the two main results of this paper.

Theorem 1.1. Let $L \subset \mathbb{CP}^n$ be a closed Lagrangian submanifold which admits a metric of nonpositive curvature. Then there exists a smooth map $f : (D, \partial D) \to (\mathbb{CP}^n, L)$ with $f^*\omega \geq 0$ whose symplectic area satisfies

$$0 < \int_D f^*\omega \leq \frac{\pi}{n+1}.$$ 

Theorem 1.2 (Audin’s conjecture). The Maslov index of the map $f$ in Theorem 1.1 satisfies:
(a) \( \mu(f) \in \{1, 2\} \) if \( L \) is monotone;
(b) \( \mu(f) = 2 \) if \( L \) is a (not necessarily monotone) torus.

In particular, Theorem 1.2 (b) answers a question posed by M. Audin in 1988: *Every Lagrangian torus in \( \mathbb{C}^n \) admits a disk of Maslov index 2.* This question was answered earlier for \( n = 2 \) by Viterbo and Polterovich, in the monotone case for \( n \leq 24 \) by Oh, and in the monotone case for general \( n \) by Buhovsky and Fukaya, Oh, Ohta and Ono. A different approach has been outlined by Fukaya.

The scheme to prove Audin’s conjecture using punctured holomorphic curves was suggested by Y. Eliashberg around 2001. The reason it took over 10 years to complete this paper are transversality problems in the non-monotone case. We solve these problems using techniques from \[18\].

The proof of Theorem 1.1 actually yields \( n + 1 \) maps \( f_0, \ldots, f_n : (D, \partial D) \to (\mathbb{C}P^n, L) \), each of positive area (and Maslov index 1 or 2 in the situation of Theorem 1.2), such that \( \sum_{i=0}^{n} \int_D f_i^* \omega \leq \pi \). We illustrate this with two examples.

(1) The *Clifford torus* is the monotone torus in \( \mathbb{C}P^n \) defined by

\[
T^n_{\text{Clifford}} := \left\{ \left[ z_0 : \cdots : z_n \right] \mid |z_0| = \cdots = |z_n| \right\}.
\]

It admits \( n + 1 \) holomorphic maps \( f_0, \ldots, f_n : (D, \partial D) \to (\mathbb{C}P^n, T^n_{\text{Clifford}}) \) of area \( \pi/(n+1) \) and Maslov index 2 given by \( f_i(z) := [1 : \ldots : 1 : z : 1 \ldots 1] \) with \( z \) in the \( i \)-th component. Note that the boundary loops \( f_i(\partial D) \) for \( 0 \leq i \leq n \) generate the first homology \( H_1(T^n) \).

(2) The *Chekanov torus* is an exotic monotone 2-torus \( T^2_{\text{Chekanov}} \) in \( \mathbb{C}P^2 \) described in [14]. We show in Appendix A that for the disks \( f_0, f_1, f_2 \) obtained in Theorem 1.1 all boundary loops \( f_i(\partial D) \) represent multiples of the same homology class. This answers in the negative Viterbo’s question whether for \( L \cong T^n \) the boundary loops \( f_i(\partial D) \) for \( 0 \leq i \leq n \) always generate the first homology \( H_1(T^n) \).

**A new symplectic capacity.** To explore the implications of Theorem 1.1, we define a new symplectic capacity, following a suggestion by J. Etnyre. Define the *minimal symplectic area* of a Lagrangian submanifold \( L \) of a symplectic manifold \( (X, \omega) \) by

\[
A_{\min}(L) := \inf \left\{ \int_\sigma \omega \mid \sigma \in \pi_2(X, L), \int_\sigma \omega > 0 \right\} \in [0, \infty].
\]

Define the *Lagrangian capacity* \( c_L(X, \omega) \) of \( (X, \omega) \) to be

\[
c_L(X, \omega) := \sup \{ A_{\min}(L) \mid L \subset X \text{ embedded Lagrangian torus} \} \in [0, \infty].
\]

The Lagrangian capacity satisfies

\[
\text{The “L” in } c_L \text{ just stands for “Lagrangian” and does not refer to a particular Lagrangian submanifold } L.
\]
(Monotonicity) \( c_L(X, \omega) \leq c_L(X', \omega') \) if there exists a symplectic embedding \( \iota : (X, \omega) \rightarrow (X', \omega') \) with \( \pi_2(X', \iota(X)) = 0 \);

(Conformality) \( c_L(X, \alpha \omega) = |\alpha| c_L(X, \omega) \) for \( 0 \neq \alpha \in \mathbb{R} \);

(Nontriviality) \( 0 < c_L(B^{2n}(1)) \) and \( c_L(Z^{2n}(1)) < \infty \).

In particular, \( c_L \) is a (generalized) capacity in the sense of \([37, 16]\) on the class of symplectic manifolds \((X, \omega)\) with \( \pi_1(X) = \pi_2(X) = 0 \). Here \( B^{2n}(r) \) is the open unit ball in \( \mathbb{C}^n \) and \( Z^{2n}(r) = B^2(r) \times \mathbb{C}^{n-1} \); open subsets of \( \mathbb{C}^n \) are always equipped with the canonical symplectic structure. The first two properties are obvious. The property \( 0 < c_L(B^{2n}(1)) \) holds because the unit ball contains a small standard torus \( T^n(r) := S^1(r) \times \cdots \times S^1(r) \), where \( S^1(r) \subset \mathbb{C} \) is the sphere of radius \( r \). For the last property, recall Chekanov’s result \([13]\) that \( A_{\text{min}}(L) \leq d(L) \) for every closed Lagrangian submanifold \( L \) in \( \mathbb{C}^n \). Here \( d(A) \) is the displacement energy of a subset \( A \subset \mathbb{C}^n \), i.e., the minimal Hofer energy of a compactly supported Hamiltonian diffeomorphism that displaces \( A \) from itself (see \([37]\)). This implies \( c_L(U) \leq d(U) \) for every open subset \( U \subset \mathbb{C}^n \). Since the symplectic cylinder \( Z^{2n}(1) \) has displacement energy \( \pi \) and contains the standard torus \( T^n(1) \) of minimal symplectic area \( \pi \), it follows that \( c_L(Z^{2n}(1)) = \pi \).

Most known symplectic capacities (exceptions being the higher Ekeland–Hofer capacities \([24]\) for subsets of \( \mathbb{C}^n \) and the ECH capacities \([39]\) in dimension 4) take the same value on the unit ball and on the cylinder. Surprisingly, this is not the case for the Lagrangian capacity:

**Corollary 1.3.**

\( c_L(B^{2n}(1)) = \frac{\pi}{n} \).

**Proof.** The standard torus \( T^n(r) \) is contained in the unit ball if and only if \( r < 1/\sqrt{n} \). This proves one inequality. For the converse inequality, suppose that the unit ball contains a Lagrangian torus \( L \) with \( A_{\text{min}}(L) \geq \pi/n \). After replacing \( L \) by \( \alpha L \) for a suitable \( 0 < \alpha \leq 1 \) we may assume that \( A_{\text{min}}(L) = \pi/n \). Compactifying the closed unit ball to \( \mathbb{C}P^n \), \( L \) gives rise to a Lagrangian torus \( L' \) in \( \mathbb{C}P^n \) with the property that all disks with boundary on \( L' \) have symplectic area a multiple of \( \pi/n \). (This property is clear for disks contained in the open ball \( \mathbb{C}P^n \setminus \mathbb{C}P^{n-1} \); for a disk passing through \( \mathbb{C}P^{n-1} \) it follows by gluing it along its boundary with a disk in the ball to a sphere whose area is a multiple of \( \pi \).) But this property contradicts Theorem 1.1. \( \square \)
For an application of this result, consider a polydisk $P^{2n}(r) := B^2(r) \times \cdots \times B^2(r)$. It contains the standard torus $T^n(r)$ and is contained in the cylinder $Z^{2n}(r)$, hence

$$c_L(P^{2n}(r)) = \pi r^2.$$  

Monotonicity of the Lagrangian capacity now implies the following non-squeezing result due to Ekeland and Hofer.

Monotonicity of the Lagrangian capacity now implies the following non-squeezing result due to Ekeland and Hofer.

Corollary 1.4 (Ekeland-Hofer [24]). The polydisk $P^{2n}(r)$ can be symplectically embedded into the ball $B^{2n}(1)$ if and only if

$$r \leq \frac{1}{\sqrt{n}}.$$  

We end this paragraph with a conjecture for the Lagrangian capacity of an ellipsoid

$$E(a_1, \ldots, a_n) := \left\{ z \in \mathbb{C}^n \mid \frac{|z_1|^2}{a_1} + \cdots + \frac{|z_n|^2}{a_n} \leq 1 \right\}$$

with $0 < a_1 \leq \cdots \leq a_n < \infty$.

Conjecture 1.5. The Lagrangian capacity of an ellipsoid is given by

$$c_L\left(E(a_1, \ldots, a_n)\right) = \frac{\pi}{1/a_1 + \cdots + 1/a_n}.$$  

It is shown in [16] that this conjecture would imply $c_L = \lim_{k \to \infty} \frac{1}{c_k^{EH}}$ on all ellipsoids, where $c_k^{EH}$ denotes the $k$-th Ekeland–Hofer capacity [24]. It would
be interesting to see whether this relation continues to hold on more general subsets of $\mathbb{R}^{2n}$, e.g. on convex sets. The relation may also be compared to a similar formula recovering the volume of a 4-dimensional Liouville domain from its ECH capacities [21].

Remark 1.6. If we define the Lagrangian capacity $c_L$ using arbitrary closed Lagrangian submanifolds instead of just tori we still obtain a capacity with the property $c_L(U) \leq d(U)$ for $U \subset \mathbb{C}^n$. The Lagrangian capacity of the unit ball is still $\pi/n$, but we do not know whether equality holds for $n > 2$. For $n = 2$, equality holds in view of Theorem 1.1 (and the proof of Corollary 1.3) because all closed Lagrangian submanifolds of $\mathbb{C}^2$ admit a metric of nonpositive curvature.

**Extremal Lagrangian tori.** Let us call a Lagrangian torus $L$ in a symplectic manifold $(X, \omega)$ **extremal** if it maximizes $A_{\min}$, i.e., $A_{\min}(L) = c_L(X, \omega)$. Recall that $L$ is **monotone** if its Maslov class is positively proportional to the symplectic area class on $\pi_2(X, L)$, i.e., $\mu = 2a[\omega]$ for some $a > 0$. This implies that $(X, \omega)$ is monotone with $c_1(TX) = a[\omega]$ on $\pi_2(X)$ (see e.g. [15]). In particular, for $X = \mathbb{CP}^n$ with its standard symplectic form we have $a = (n + 1)/\pi$, so for a monotone Lagrangian torus $L \subset \mathbb{CP}^n$ the values of $[\omega]$ on $\pi_2(\mathbb{CP}^n, L)$ are integer multiples $\pi/(n + 1)$ (since all Maslov indices are even). Therefore, Theorem 1.1 implies

**Corollary 1.7.** Every monotone Lagrangian torus in $\mathbb{CP}^n$ is extremal.

**Conjecture 1.8.** Every extremal Lagrangian torus in $\mathbb{CP}^n$ is monotone.

Turning to a different manifold, the following conjecture is motivated by a question of L. Lazzarini.

**Conjecture 1.9.** Every extremal Lagrangian torus in the unit ball $B^{2n}(1)$ is entirely contained in the boundary $\partial B^{2n}(1)$.

**Remark 1.10.** Note that the standard torus $T^n(1/\sqrt{n}) \subset \partial B^{2n}(1)$ is extremal. The following fact, which is proved in Appendix B, lends some plausibility to the conjecture. It establishes a phenomenon which is called “non-removable intersection” in [53]:

For a closed Lagrangian submanifold $L \subset \partial B^{2n}(1)$ no point can be pushed into the interior by a Hamiltonian isotopy without making $L$ exit the closed ball at some other point.

**Remark 1.11.** One motivation for our interest in monotone Lagrangian submanifolds is the observation (see e.g. [13]) that minimal (i.e., of zero mean curvature) Lagrangian submanifolds in $\mathbb{CP}^n$ are monotone.

**The chord conjecture.** Another application concerns Arnold’s chord conjecture [1]. Let $U \subset \mathbb{C}^n$ be a bounded star-shaped domain (with respect to the origin) with smooth boundary $S = \partial U$. The $1$-form

$$
\lambda := \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)
$$
Contact structures arise in this way (see Section 2 for the basic definitions concerning contact manifolds). Every closed Legendrian submanifold $\Lambda$ in $S = \partial U$ possesses a Reeb chord of length $T \leq d(U)$. If $\Lambda$ admits a metric of nonpositive curvature the above argument yields the following sharper estimate.

**Corollary 1.12.** Every closed Legendrian submanifold $\Lambda$ of nonpositive curvature in the boundary $S$ of a star-shaped domain $U \subset \mathbb{C}^n$ possesses a Reeb chord of length

$$T \leq c_L(U).$$

Let us compare the two estimates for the unit sphere $S$ in $\mathbb{C}^n$. Here all Reeb orbits are closed of length $\pi$ and form the fibres of the Hopf fibration $S^{2n-1} \to \mathbb{CP}^{n-1}$. The estimate in [50] yields a Reeb chord of length $T \leq \pi$, which may just be a closed Reeb orbit meeting $\Lambda$ once. The estimate in Corollary 1.12 combined with Corollary 1.3 yields

**Corollary 1.13.** Every closed Legendrian submanifold $\Lambda$ of nonpositive curvature in the unit sphere $S \subset \mathbb{C}^n$, $n \geq 2$, possesses a Reeb chord of length

$$T \leq \frac{\pi}{n}.$$

In particular, $\Lambda$ meets some fibre of the Hopf fibration at least twice.

In a recent preprint [60], F. Ziltener shows that the last statement actually holds without the assumption of nonpositive curvature. This follows from the proof in [50] described above and the observation that the displacement energy of the Lagrangian submanifold $L \cong S^1 \times \Lambda$ is strictly smaller than $\pi$.

The last statement can be rephrased as follows. Call a Lagrangian submanifold $L \subset \mathbb{CP}^n$ exact if every disk with boundary on $L$ has symplectic area an integer
multiple of \( \pi \). Every exact Lagrangian submanifold \( L \subset \mathbb{CP}^n \) lifts to a Legendrian submanifold \( \Lambda \subset S^{2n+1} \) that intersects each fibre of the Hopf fibration at most once. (A horizontal lift of \( L \) exist locally since \( L \) is Lagrangian; the obstructions to a global horizontal lift are the holonomies of the connection form \( 2\lambda \) along loops in \( L \), which vanish because the integral of the curvature \( 2\omega \) over each disk with boundary on \( L \) is an integer multiple of \( 2\pi \).) Hence the preceding discussion implies

**Corollary 1.14** ([60]). *There exists no exact closed Lagrangian submanifold in \( \mathbb{CP}^n \).*

For \( n = 1 \) this is obvious (and was the motivation for Arnold’s conjecture in [1]). For \( n > 1 \) it generalizes (for manifolds of nonpositive curvature) Gromov’s result [31] that there is no closed exact Lagrangian submanifold in \( \mathbb{C}^n \). However, Gromov’s original technique does not seem applicable here.

The definition of a Reeb chord given above includes periodic orbits intersecting \( \Lambda \). Let us call a Reeb chord *honest* if it is not a closed Reeb orbit meeting \( \Lambda \) just once. Arnold’s original conjecture asks for the existence of honest Reeb chords. Corollary 1.12 can be used to prove this stronger conjecture. For example, if \( S \subset \mathbb{C}^n \) is \( C^1 \)-close to the unit sphere, then all closed Reeb chords have length at least \( \pi - \varepsilon \), whereas Corollary 1.12 yields a Reeb chord of length at most \( \pi/n + \varepsilon \). Hence we have

**Corollary 1.15.** Let \( S \subset \mathbb{C}^n, \ n \geq 2, \) be sufficiently \( C^1 \)-close to the unit sphere. Then every closed Legendrian submanifold \( \Lambda \) of nonpositive curvature in \( S \) possesses an honest Reeb chord.

### Lagrangian submanifolds and symplectic balls

P. Biran [6] has shown that a symplectically embedded ball \( B \subset \mathbb{CP}^n \) of radius \( r > 1/\sqrt{2} \) must intersect \( \mathbb{RP}^n \subset \mathbb{CP}^n \). This turns out to be related to a quite general intersection phenomenon.

**Theorem 1.16.** Let \( L \subset \mathbb{CP}^n \) be a closed Lagrangian submanifold which admits a metric without contractible geodesics (e.g. one of nonpositive curvature). Let \( B \subset \mathbb{CP}^n \) be a symplectically embedded ball of radius \( r \). If \( L \cap B = \emptyset \), then

\[
A_{\min}(L) + \pi r^2 \leq \pi.
\]

Note that \( A_{\min}(\mathbb{RP}^n) = \pi/2 \), so the estimate in Theorem 1.16 would imply Biran’s result. However, the theorem is not applicable to \( L = \mathbb{RP}^n \) because every metric on \( \mathbb{RP}^n \) has contractible geodesics. Combined with Corollary 1.7, Theorem 1.16 implies that every monotone Lagrangian torus in \( \mathbb{CP}^n \) must intersect every embedded ball of radius \( r > \sqrt{n}/n+1 \); see Biran and Cornea [7] for generalizations of this result.

### Uniruled symplectic manifolds

Let us call a closed symplectic manifold \((\mathcal{X}, \omega)\) *uniruled* if it has some nonvanishing Gromov-Witten invariant of holomorphic spheres passing through a point (plus additional constraints). Ruan [50]
has proved that on a Kähler manifold this is equivalent to the algebro-geometric definition that $X$ is covered by rational curves. For example, every Fano manifold is uniruled (see [42]). The following result is due to Viterbo [58] for “strongly Fano manifolds”, and to Eliashberg, Givental and Hofer [25] in the general case. We include its proof in Section 4.

**Theorem 1.17 ([58] [25]).** Let $L$ be a closed manifold of dimension $\geq 3$ which carries a metric of negative curvature. Then $L$ admits no Lagrangian embedding into a uniruled symplectic manifold.

In dimension 2 the result is not true because of Givental’s nonorientable Lagrangian surfaces in $\mathbb{C}^2$ (see [3]). But we still obtain the following restriction.

**Theorem 1.18.** A closed orientable surface of genus $\geq 2$ admits no Lagrangian embedding into a uniruled symplectic 4-manifold.

**Remark 1.19.** The proof of Theorem 1.1 yields for every closed Lagrangian surface $L \subset \mathbb{C}P^2$ which carries a metric of negative curvature the estimate $A_{\text{min}}(L) \leq \pi/6$.

In [51], J. Nash proved that every compact smooth manifold is the real part of a real algebraic manifold (i.e., a smooth projective variety defined by real equations in $\mathbb{C}P^N$). In the same paper he conjectured that the real algebraic manifold can be chosen to be birational to $\mathbb{C}P^n$. Now the real part of a real algebraic manifold is Lagrangian (see [58]), and an algebraic manifold birational to $\mathbb{C}P^n$ is uniruled (it is even rationally connected, see [44]) and simply connected (see [29]). Hence Theorems 1.17 and 1.18 imply

**Corollary 1.20 ([19] [43] [58] [25]).** The Nash conjecture is false in all dimensions $\geq 2$. More precisely, we have:

(a) For $n \geq 3$, the real part of a real algebraic manifold birational to $\mathbb{C}P^n$ carries no metric of negative curvature.

(b) The real part of a real algebraic surface birational to $\mathbb{C}P^2$ is either the sphere, the torus, or a nonorientable surface.

For $n = 2$ the result is due to Comessatti [19] and sharp: the sphere and the torus occur in quadrics $x^2 + y^2 \pm z^2 = t^2$, and the nonorientable surfaces occur in blow-ups of $\mathbb{C}P^2$ at real points. Already for $n = 3$ the result is far from optimal. In fact, Kollár has derived a short list of possible topological types for real parts of real algebraic 3-folds [43] [44].

Suppose now that the values of the symplectic form $\omega$ on $\pi_2(X)$ are given by $ka$, $k \in \mathbb{Z}$, for some $a > 0$. Let us call $(X, \omega)$ minimally uniruled if the holomorphic spheres in the definition of “uniruled” have symplectic area $a$. Call a Lagrangian embedding $L \hookrightarrow X$ exact if every disk with boundary on $L$ has symplectic area an integer multiple of $a$. With these notations, Corollary 1.14 can be generalized as follows.
Theorem 1.21. Let $L$ be a closed manifold which carries a metric without contractible geodesics. Then $L$ admits no exact Lagrangian embedding into a minimally uniruled symplectic manifold.

Examples of minimally uniruled symplectic manifolds are $\mathbb{C}P^n$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ with their standard symplectic structures. The real locus $\mathbb{R}P^n \subset \mathbb{C}P^n$ is not exact, while the antidiagonal in $\mathbb{C}P^1 \times \mathbb{C}P^1$ is exact.

Size of Weinstein neighbourhoods. Weinstein’s Lagrangian neighbourhood theorem states that any Lagrangian submanifold $L \subset (X, \omega)$ has a tubular neighbourhood which is symplectomorphic to a neighbourhood of the zero section in the cotangent bundle $T^*L$ with the canonical symplectic form. We are interested in the maximal “size” of such a neighbourhood, which can be quantified as follows. For a Riemannian manifold $(Q^n, g)$ let $D^* (Q, g) := \{(q, p) \in T^* Q \mid \|p\|_g \leq 1\}$ be the unit codisk bundle. Fix a symplectic manifold $(X^{2n}, \omega)$ and define the embedding capacity

$$c^{(X, \omega)}(Q, g) := \inf \{ \alpha > 0 \mid D^* (Q, g) \hookrightarrow (X, \alpha \omega) \},$$

where $\hookrightarrow$ means symplectic embedding with respect to the canonical symplectic structure on $T^* Q$. This is the embedding capacity of $D^* (Q, g)$ into $(X, \omega)$ in the sense of [16]. It has the following elementary properties with respect to the metric:

1. **(Invariance)** $c^{(X, \omega)}(Q, g) = c^{(X, \omega)}(Q, g)$ if $(Q, g)$ and $(Q', g')$ are isometric;
2. **(Monotonicity)** $c^{(X, \omega)}(Q, g) \leq c^{(X, \omega)}(Q, h)$ if $g \leq h$;
3. **(Conformality)** $c^{(X, \omega)}(Q, \lambda g) = \lambda c^{(X, \omega)}(Q, g)$ for $\lambda > 0$;
4. **(Nontriviality)** $c^{(X, \omega)}(Q, g) < \infty$ if $Q$ admits Lagrangian embeddings into $(X, \omega)$, and $c^{(X, \omega)}(Q, g) > 0$ if $(X, \omega)$ has finite volume.

For last property, note that a symplectic embedding $D^* (Q, g) \hookrightarrow (X, \alpha \omega)$ yields

$$\text{vol}(Q, g) \text{vol} B^n(1) = \text{vol} D^* (Q, g) \leq \text{vol}(X, \alpha \omega) = \alpha^n \text{vol}(X, \omega)$$

and hence the lower estimate

$$c^{(X, \omega)}(Q, g) \geq \sqrt{\frac{\text{vol}(Q, g) \text{vol} B^n(1)}{\text{vol}(X, \omega)}}. \quad (1)$$

Let $\ell_{\text{min}}(Q, g)$ denote the minimal length of a closed geodesic on $(Q, g)$. Then the methods of this paper yield the following lower estimate.

Theorem 1.22. The embedding capacity of a Riemannian manifold $(Q, g)$ of nonpositive curvature into $\mathbb{CP}^n$ with its standard symplectic form satisfies

$$c^{\mathbb{CP}^n}(Q, g) \geq \frac{(n + 1) \ell_{\text{min}}(Q, g)}{\pi}.$$
Some explicit computations in Appendix C show:

1) For the flat torus $T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, the estimate in Theorem 1.22 is better than the volume estimate \((1)\) for each $n > 1$. Moreover, the volume bound grows as $\sqrt{n}$, while the bound in Theorem 1.22 grows linearly in $n$.

2) Explicit embeddings of the flat torus $T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ into $\mathbb{C}P^n$ yield an upper bound $c_{\mathbb{C}P^n}(T^n) \leq 2(n + \sqrt{n})$, which is strictly larger than the lower bound $2(n + 1)$ from Theorem 1.22 for each $n > 1$. This suggests that the estimate in Theorem 1.22 may not be sharp. Note, however, that the quotient of the upper and lower bound tends to 1 as $n \to \infty$.

Remark 1.23. (a) The techniques of this paper work well for Lagrangian submanifolds which admit metrics of nonpositive curvature, or at least without contractible geodesics. On the other hand, traditional techniques such as Gromov’s holomorphic disks and Floer homology work well for simply connected Lagrangians. Can one combine these techniques? A test case could be the product of a manifold of positive with a manifold of negative curvature.

(b) Neck stretching techniques were also used in [59, 45] to study Lagrangian embeddings in dimensions 4 and 6.

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2 Punctured holomorphic curves

In this section we recall the definitions and basic properties of punctured holomorphic curves, see [25, 10, 17] for details.

Contact manifolds. A contact form on a $(2n - 1)$-dimensional manifold $M^{2n-1}$ is a 1-form $\lambda$ for which $\lambda \wedge (d\lambda)^{n-1}$ is a volume form. Its Reeb vector field is the unique vector field $R_\lambda$ satisfying $i_{R_\lambda}d\lambda = 0$ and $\lambda(R_\lambda) = 1$. The distribution $\xi := \ker \lambda$ is called a (cooriented) contact structure, and the pair $(M, \xi)$ a contact manifold.

Consider a closed orbit $\gamma$ of the Reeb vector field. The linearized Reeb flow along $\gamma$ preserves the contact distribution $\xi$, and the maps $\Psi_t : \xi_{\gamma(0)} \to \xi_{\gamma(t)}$ preserve $d\lambda$. Let $\Phi : \xi|_{\gamma} \cong S^1 \times \mathbb{C}^{n-1}$ be a trivialization of $\xi$ along $\gamma$. Then $\Phi_t \Psi_t \Phi_0^{-1}$ is a path of symplectic matrices which has a Conley-Zehnder index (defined in [20] if $\gamma$ is nondegenerate, i.e., the linearized return map on a transverse section has no eigenvalue 1, and in [57] in general) and a Robbin-Salamon index (defined in [55]). We call

$$CZ(\gamma, \Phi) := CZ(\Phi_t \Psi_t \Phi_0^{-1}) \in \mathbb{Z}, \quad RS(\gamma, \Phi) := RS(\Phi_t \Psi_t \Phi_0^{-1}) \in \frac{1}{2} \mathbb{Z}$$

the Conley-Zehnder (resp. Robbin-Salamon) index of $\gamma$ with respect to the trivialization $\Phi$. 

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Following [9, 10], we say that \((M, \lambda)\) is Morse-Bott if the following holds: For all \(T > 0\) the set \(N_T \subset M\) formed by the \(T\)-periodic Reeb orbits is a closed submanifold, the rank of \(d\lambda|_{N_T}\) is locally constant, and \(T_p N_T = \ker(T_p \phi_T - 1)\) for all \(p \in N_T\), where \(\phi_t\) is the Reeb flow. Dividing out time reparametrization, we obtain the orbifold of unparametrized Reeb orbits \(\Gamma_T := N_T/S^1\). Note that the case \(\dim \Gamma_T = \dim N_T - 1 = 0\) corresponds to nondegeneracy of closed Reeb orbits. We will refer to this case as the Morse case. In the Morse-Bott case, the Conley-Zehnder and Robbin-Salamon indices are constant along each connected component \(\Gamma\) of \(\Gamma_T\) and we denote them by \(\text{CZ}(\Gamma)\) resp. \(\text{RS}(\Gamma)\). They are related by

\[
\text{CZ}(\Gamma) = \text{RS}(\Gamma) - \frac{1}{2} \dim \Gamma.
\]

An important special case is the geodesic flow on the unit cotangent bundle \(M = S^* L\) of a Riemannian manifold \((L, g)\). This is the Reeb flow of the contact form \(\lambda = p dq\) in canonical coordinates \((q, p)\) on \(T^* L\). A closed geodesic \(q\) on \(L\) lifts to the closed Reeb orbit \(\gamma(t) = (q(t), \dot{q}(t))\). A trivialization \(\Phi\) of the contact distribution along \(\gamma\) is equivalent to a trivialization of \(T(T^* L)\) as a symplectic vector bundle along \(q\) which is still denoted by \(\Phi\). Note that the zero section \(L \subset T^* L\) is Lagrangian for \(d\lambda\). Let

\[
\mu(\gamma, \Phi) := \mu(q, \Phi) := \mu(\Phi_t T_{q(t)} L)
\]

be the Maslov index of the induced loop of Lagrangian subspaces of \(\mathbb{C}^n\) (see e.g. [3]). Now \(\text{CZ}(\gamma, \Phi)\) and \(\mu(\gamma, \Phi)\) transform in the same way under change of the trivialization. So their difference is independent of \(\Phi\), and in fact is a well-known quantity:

**Lemma 2.1** (Viterbo [57]). For the lift \(\gamma\) of a (possibly degenerate) closed geodesic \(q\),

\[
\text{CZ}(\gamma, \Phi) + \mu(\gamma, \Phi) = \text{ind}(\gamma),
\]

where \(\text{ind}(\gamma) := \text{ind}(q)\) is the Morse index of \(q\).

We will need the following lemma about Morse indices:

**Lemma 2.2.** Let \(L^n\) be a closed manifold which admits a metric of nonpositive sectional curvature. Then for every \(c > 0\) there exists a metric on \(L\) such that every closed geodesic \(q\) of length \(\leq c\) is noncontractible and nondegenerate and satisfies

\[
0 \leq \text{ind}(q) \leq n - 1.
\]

In a metric of negative curvature every closed geodesic is noncontractible and nondegenerate of index zero.

**Proof.** By the Morse Index Theorem for closed geodesics (see e.g. [11], Theorem 2.5.14),

\[
\text{ind}(q) = \text{ind}_1(q) + \text{concav}(q),
\]
where \( \text{ind}_\Omega(q) \) equals the number of conjugate points along \( q \), and the nullity and concavity of \( q \) satisfy \( \text{null}(q) + \text{concav}(q) \leq n - 1 \). Now a metric \( g_0 \) of nonpositive curvature has no conjugate points along any geodesic (see e.g. [11], Theorem 2.6.2), hence \( \text{ind}(q) + \text{null}(q) \leq n - 1 \) for every closed \( g_0 \)-geodesic.

Consider the space of all closed \( g_0 \)-geodesics of length \( \leq c \), parametrized with constant speed over the time interval \([0, 1]\). Since this space is compact (say, with the \( C^2 \) topology), there exists a \( C^2 \)-neighbourhood \( U \) of \( g_0 \) in the space of metrics such that, for every \( g \in U \) and every closed \( g \)-geodesic \( q \) of length \( \leq c \), we still have \( \text{ind}(q) + \text{null}(q) \leq n - 1 \). Now there exist metrics \( g \in U \) for which all closed geodesics \( q \) of length \( \leq c \) are nondegenerate (see e.g. [10], Theorem 3.3.9), and therefore \( \text{ind}(q) \leq n - 1 \).

By Hadamard’s Theorem (see e.g. [11], Theorem 2.6.6), all closed \( g_0 \)-geodesics are noncontractible; this persists for \( g \)-geodesics of length \( \leq c \) for \( g \in U \). Finally, if \( g_0 \) has negative curvature, then the nullity and concavity are zero, so every closed \( g_0 \)-geodesic is nondegenerate of index 0.

**Symplectic cobordisms.** The *symplectization* of a contact manifold \((M, \xi)\) is the symplectic manifold \((\mathbb{R} \times M, d(e^r \lambda))\), where \( r \) is the coordinate on \( \mathbb{R} \). A *symplectic cobordism* is a symplectic manifold \((X, \omega)\) whose ends are the positive or negative halves of symplectizations. This means that for some compact subset \( K \subset X \) and contact manifolds \((M, \lambda)\) \((\overline{M}, \overline{\lambda})\), we have a symplectomorphism

\[
(X \setminus K, \omega) \cong (\mathbb{R}_+ \times \overline{M}, d(e^r \overline{\lambda})) \amalg (\mathbb{R}_- \times \overline{M}, d(e^r \overline{\lambda})).
\]

Here we allow one or both of \( \overline{M}, \overline{M} \) to be empty. The ends modelled over \( \overline{M}, \overline{M} \) are called the *positive and negative ends*, respectively. Obvious examples of symplectic cobordisms are closed symplectic manifolds and symplectizations.

**Example 2.3.** Cotangent bundles are symplectic cobordisms as follows. Let \( M = S^*L \) be the unit cotangent bundle of a Riemannian manifold \((L, g)\), with the contact form \( \lambda = p dq \big|_M \). We identify \( L \) with the zero section in \( T^*L \). Then the map \( \mathbb{R} \times M \to T^*L \setminus L \), \((r, q, p) \mapsto (q, e^r p)\) yields a symplectomorphism

\[
(\mathbb{R} \times M, d(e^r \lambda)) \cong (T^*L \setminus L, dp dq).
\]

This shows that the cotangent bundle \((T^*L, dp dq)\) is a symplectic cobordism with one positive end modelled over \((M, \lambda)\).

**Almost complex structures.** An almost complex structure \( J \) on a symplectization \((\mathbb{R} \times M, d(e^r \lambda))\) is called *compatible with \( \lambda \)* if it is translation invariant in the \( \mathbb{R} \)-direction, leaves the contact structure \( \xi \) invariant, maps \( \partial / \partial r \) to the Reeb vector field \( R_\lambda \), and \( d(e^r \lambda)(\cdot, J\cdot) \) defines a Riemannian metric. An almost complex structure on a symplectic cobordism \((X, \omega)\) is called *compatible* if it is compatible with the contact forms \( \overline{\lambda}, \overline{\lambda} \) outside some compact set and \( \omega(\cdot, J\cdot) \) defines a Riemannian metric. Compatible almost complex structures exist and form a contractible space \( \mathcal{J} \) (with the \( C^\infty \) topology).
Punctured holomorphic curves. Consider a symplectization \((\mathbb{R} \times M, d(e^\lambda))\) with a compatible almost complex structure \(J\). Let \(\gamma : [0, T] \to M\) be a (not necessarily simple) closed orbit of the Reeb vector field \(R_\lambda\) of period \(T\). A \(J\)-holomorphic map \(f = (a, u) : D \setminus 0 \to \mathbb{R} \times M\) of the punctured unit disk is called positively (resp. negatively) asymptotic to \(\gamma\) if \(\lim_{\rho \to 0} a(\rho e^{i\theta}) = \infty\) (resp. \(-\infty\)) and \(\lim_{\rho \to 0} u(\rho e^{i\theta}) = \gamma(\frac{T \theta}{2\pi})\) (resp. \(\gamma(-\frac{T \theta}{2\pi})\)) uniformly in \(\theta\).

A punctured holomorphic curve in \((X, J)\) consists of the following data:

- A Riemann surface \((\Sigma, j)\) with distinct positive and negative points \(\tilde{\Sigma} := (\Sigma_1, ..., \Sigma_p), \tilde{z} := (z_1, ..., z_p)\). We denote by \(\hat{\Sigma} := \Sigma \setminus \{\Sigma_1, ..., \Sigma_p\}\) the corresponding punctured Riemann surface.
- Corresponding vectors \(\Gamma := (\gamma_1, ..., \gamma_p), \Gamma^* := (\gamma_1^*, ..., \gamma_p^*)\) of closed Reeb orbits in \(\overline{M}, \overline{M}\).
- A \((j, J)\)-holomorphic map \(f : \hat{\Sigma} \to X\) which is positively (resp. negatively) asymptotic to \(\gamma_j\) (resp. \(\gamma_j^*\)) at the punctures \(\tilde{z}_j\) (resp. \(\tilde{z}_j^*\)).

In a symplectization \(\mathbb{R} \times M\), a cylinder over a \(T\)-periodic Reeb orbit \(\gamma\),

\[
f : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \times M, \quad (s, t) \mapsto (T s, \gamma(T t)),
\]

is a punctured holomorphic curve with one positive and one negative puncture. Denote by \(\hat{\Sigma}\) the compactification of the punctured surface \(\Sigma\) by adding a circle at each puncture, and by \(\hat{X}\) the compactification of the manifold \(X\) by adding a copy of \(\overline{M}\) resp. \(\overline{M}\) at the positive resp. negative end. In view of the behaviour near the punctures, the holomorphic map \(f : \hat{\Sigma} \to \hat{X}\) above extends to a continuous map \(\hat{f} : \hat{\Sigma} \to \hat{X}\). This extension represents a relative homology class

\[
[f] \in H_2(\hat{\Sigma}, \Gamma \cup \Sigma).
\]

Example 2.4. The cotangent bundle \(T^* L\) of a Riemannian manifold \((L, g)\) carries a canonical complex structure \(J_g\) induced by the Riemannian metric. This structure leaves the cylinders \(\{(q(t), s \dot{q}(t)) \mid s, t \in \mathbb{R}\}\) over closed geodesics \(q\) invariant. Now \(J_g\) is not compatible according to the definition above. But it can be deformed to a compatible almost complex structure \(J\) which still leaves all the cylinders over closed geodesics invariant. Then these cylinders, appropriately parametrized, are punctured holomorphic curves in \(T^* L\) with two positive punctures.

Neck stretching. Consider a closed connected symplectic manifold \((X, \omega)\) and a closed hypersurface \(M \subset X\) of contact type. This means that \(M\) carries a contact form \(\lambda\) such that \(d\lambda = \omega|_M\). Then a neighbourhood of \(M\) is symplectomorphic to \(\left([-\varepsilon, \varepsilon] \times M, d(e^\lambda)\right)\). Assume that \(X \setminus M = \bigcup X_0^+ \bigcup X_0^-\) consists of two components with \(\{\pm \varepsilon\} \times M \subset X_0^\pm\). Pick a compatible almost complex structure \(J\) on \((X, \omega)\) whose restriction \(J_M\) to \([-\varepsilon, \varepsilon] \times M\) is compatible with \(\lambda\).
Define a 1-parameter family of symplectic manifolds as follows. For a real number $k \geq 0$ let

$$X_k := X_0^+ \cup_M [-k, 0] \times M \cup_M X_k^+.$$  

This manifold is, of course, canonically diffeomorphic to $X$, but $X_k$ is a more convenient domain for describing the deformed structures. Define an almost complex structure on $X_k$ by

$$J_k := \begin{cases} J & \text{on } X_0^+, \\ J_M & \text{on } [-k, 0] \times M. \end{cases}$$

This almost complex structure is compatible with the symplectic form on $X_k$ given by

$$\omega_k := \begin{cases} \omega & \text{on } X_0^+, \\ d(e^r \lambda) & \text{on } [-k, 0] \times M, \\ e^{-k} \omega & \text{on } X_0^- \cup \{0\}. \end{cases}$$

In the limit $k \to \infty$ we obtain three symplectic cobordisms: $X^+ := X_0^+ \cup_M \mathbb{R}_- \times M$ (with one negative end), $X^- := X_0^- \cup_M \mathbb{R}_+ \times M$ (with one positive end), and the symplectization $\mathbb{R} \times M$. Here $\mathbb{R} \times M$ is equipped with the symplectic form $d(e^r \lambda)$ and almost complex structure $J_M$, and $X^\pm$ are equipped with the symplectic forms $\omega^\pm$ and almost complex structures $J^\pm$ satisfying $\omega^\pm = \omega$ on $X_k^\pm$, $\omega^\pm = d(e^r \lambda)$ on $\mathbb{R}_\pm \times M$, $J^\pm = J$ on $X_k^\pm$, $J^\pm = J_M$ on $\mathbb{R}_\pm \times M$.

**Example 2.5.** We will apply the neck stretching construction in the following situation. Let $L$ be a closed Lagrangian submanifold of $(X, \omega)$. Pick a Riemannian metric on $L$. After rescaling the metric by a small constant, a tubular neighbourhood of $L$ is symplectomorphic to $W := \{(q, p) \in T^*L \mid |p| \leq 1\}$ with the canonical symplectic form $dp \wedge dq$. The boundary $M := \partial W$ is a contact type hypersurface in $X$ with contact form $\lambda := p dq|_M$. It decomposes $X$ into two components $X_0^- = \text{int } W$ and $X_0^+ = X \setminus W$. So we can stretch the neck along $M$ as described above. In this case, the limiting manifolds can be identified as $(X^+, \omega^+) \cong (X \setminus L, \omega)$, $(X^-, \omega^-) \cong (T^*L, dp \wedge dq)$, and $(\mathbb{R} \times M, d(e^r \lambda)) \cong (T^*L \setminus L, dp \wedge dq)$.

**Lemma 2.6.** Consider a Riemannian manifold $(L, g)$ and a symplectic embedding $D^*L \hookrightarrow X$ of its unit cotangent bundle into a closed symplectic manifold $(X, \omega)$. Let $J$ be a compatible almost complex structure on $X \setminus L$ which is cylindrical on $D^*L \setminus L$, and $f : \mathbb{C} \to X \setminus L$ be a holomorphic plane asymptotic to the closed Reeb orbit $(\gamma, \dot{\gamma})$ corresponding to a closed geodesic $\gamma$. Then the length of $\gamma$ satisfies

$$\ell(\gamma) \leq \int_f \omega.$$  

**Proof.** The map $(r, q, p) \mapsto (q, e^r p)$ defines a symplectomorphism $(\mathbb{R} \times S^*L, d(e^r \lambda)) \to (D^*L \setminus L, \omega)$ with the contact form $\lambda = p dq|_{S^*L}$. Denote by $\dot{\gamma}$ the lift of the
closed geodesic $\gamma$ to $S^*L$. Then positivity of $\omega$ and $d\lambda$ on $J$-holomorphic curves implies
\[
\int f \omega \geq \int_{f \cap D^*L} d(e^r\lambda) = \int_{f \cap S^*L} \lambda \geq \int_{\hat{\gamma}} \lambda = \ell(\gamma).
\]

**Broken holomorphic curves.** We will now describe the limiting objects of sequences of $J_k$-holomorphic curves in the neck stretching procedure. We retain the setup of the preceding section. For an integer $N \geq 2$ set
\[
(X^{(\nu)}, \omega^{(\nu)}, J^{(\nu)}) := \begin{cases} 
(X^-, \omega^-, J^-) & \text{for } \nu = 1, \\
(\mathbb{R} \times M, d(e^r\lambda), J_M) & \text{for } \nu = 2, \ldots, N-2, \\
(X^+, \omega^+, J^+) & \text{for } \nu = N.
\end{cases}
\]

Define the *split symplectic cobordism*
\[
X^* := \prod_{\nu=1}^N X^{(\nu)},
\]
equipped with the symplectic and almost complex structures $\omega^*, J^*$ induced by the $\omega^{(\nu)}, J^{(\nu)}$. Glue the positive boundary component of the compactification $X^{(\nu)}$ (by copies of $M$) to the negative boundary component of $X^{(\nu+1)}$ to obtain a compact topological space
\[
\bar{X} := \overline{X^{(1)}} \cup_M \cdots \cup_M \overline{X^{(N)}}.
\]
Note that $\bar{X}$ is naturally homeomorphic to $X$ (see the proof of Lemma 2.8 below for a particular homeomorphism), so we can identify homology classes in $X$ and $\bar{X}$.

Let $\Sigma$ be a closed oriented surface and $\Delta \subset \Sigma$ a collection of finitely many disjoint simple loops. Write
\[
\Sigma^* := \Sigma \setminus \Delta =: \bigcup_{\nu=1}^N \Sigma^{(\nu)},
\]
as a disjoint union of (not necessarily connected) components $\Sigma^{(\nu)}$. Let $j$ be a conformal structure on $\Sigma^*$ such that $(\Sigma^*, j)$ is a punctured Riemann surface. A *broken holomorphic curve (with $N$ levels)*
\[
F = (F^{(1)}, \ldots, F^{(N)}): (\Sigma^*, j) \to (X^*, J^*)
\]
is a collection of punctured holomorphic curves $F^{(\nu)}: (\Sigma^{(\nu)}, j) \to (X^{(\nu)}, J^{(\nu)})$ such that $F: \Sigma^* \to X^*$ extends to a continuous map $\hat{F}: \Sigma \to \bar{X}$. Moreover, we require that each level is *stable*, i.e., $F^{(\nu)}$ contains a component which is not a cylinder over a closed Reeb orbit.
Note that, by continuity of $\tilde{F}$, the number of positive punctures of $F^{(\nu)}$ agrees with the number of negative punctures of $F^{(\nu+1)}$, and the asymptotic Reeb orbits at the punctures agree correspondingly: $T^{(\nu)} = T^{(\nu+1)}$.

The definition of a stable broken holomorphic curve only differs from this as follows. All Riemann surfaces are allowed to have nodes disjoint from the punctures, and the maps are holomorphic outside the nodes and continuous at the nodes.

**Remark 2.7.** Every stable broken holomorphic curve has an underlying graph: Its vertices are the connected components of $\Sigma^*$, and each asymptotic Reeb orbit defines an edge between the corresponding components. Note that if $\Sigma$ has genus zero the underlying graph is a tree.

We have the following easy

**Lemma 2.8 ([17]).** The homology class $A := [\tilde{F}] \in H_2(X; \mathbb{Z})$ of a nonconstant stable broken holomorphic curve $F : (\Sigma^*, j) \to (\mathbb{X}^*, J^*)$ satisfies $\omega(A) > 0$.

**Compactness.** For $R \in \mathbb{R}$ denote by

$$X_k \to X_0^- \cup_M [-k + R, R] \times M \cup_M X_0^+, \quad x \mapsto x + R$$

the map which equals the identity on $X_0^- \cup X_0^+$ and is given by $(r, x) \mapsto (r + R, x)$ on $[-k, 0] \times M$. The following theorem collects the compactness properties that we need in this paper. A more precise statement is proved in [17].

**Theorem 2.9 (Compactness).** Let $(X_k, J_k)$ be as above and assume that $(M, \lambda)$ is Morse-Bott. Let $f_k : (\Sigma_k, j_k) \to (X_k, J_k)$ be a sequence of holomorphic curves of the same genus and in the same homology class $[f_k] = A \in H_2(X; \mathbb{Z})$. After passing to a subsequence, there exist a stable broken holomorphic curve $F : (\Sigma^*, J^*) \to (\mathbb{X}^*, J^*)$, orientation preserving diffeomorphisms $\phi_k : \Sigma_k \to \Sigma$, and numbers $-k = r_k^{(1)} < r_k^{(2)} < \cdots < r_k^{(N)} = 0$ with $r_k^{(\nu+1)} - r_k^{(\nu)} \to \infty$ such that the following holds:

1. $(\phi_k)_* j_k \to j$ in $C_\infty^* \text{on } \Sigma^*$.
2. $f_k^{(\nu)} \circ \phi_k^{-1} \to F^{(\nu)}$ in $C^\infty_\text{loc}$ on $\Sigma^{(\nu)}$, where $f_k^{(\nu)}$ is the shifted map $z \mapsto f_k(z) - r_k^{(\nu)}$.
3. $\int_{\Sigma_k} f_k^* \omega_k \to \int_{\Sigma(N)} (F^{(N)})^* \omega^+$. 
4. $[\tilde{F}] = A \in H_2(X; \mathbb{Z})$.

Here on each component $\Sigma^{(\nu)}$ the convergence statements (i-ii) have to be understood in the sense of nodal holomorphic curves, see [17].

**Corollary 2.10.** In the situation of Theorem 2.9, assume in addition that the genus is zero and the homology class $A$ cannot be written as $A = B + C$ with $B, C \in H_2(X; \mathbb{Z})$ satisfying $\omega(B), \omega(C) > 0$. Then $F$ is a broken holomorphic curve without nodes and the convergence statements (i) and (ii) can be understood literally.
Proof. Suppose $F$ has a node. Since the genus is zero, the node decomposes the domain $\Sigma$ into two connected components $\Sigma_0, \Sigma_1$. The restrictions of $F$ to these components define nonconstant stable broken holomorphic curves $F_0, F_1$ representing homology classes $A_0, A_1 \in H_2(X; \mathbb{Z})$ with $A_0 + A_1 = A$. Lemma 2.8 yields $\omega(A_0), \omega(A_1) > 0$, contradicting the assumption on $A$. \hfill \Box

Moduli spaces. Now we turn to moduli spaces of punctured holomorphic curves. From now on we restrict to genus zero. Consider a symplectic cobordism $(X, \omega)$ of dimension $2n$ with ends modeled over the contact manifolds $(\overline{M}, \lambda)$ and $(\overline{M}, \lambda)$. Suppose first that all closed Reeb orbits on $\overline{M}$ and $\overline{M}$ are nondegenerate.

Fix collections of closed Reeb orbits 
$$
\overline{\Gamma} = (\overline{\gamma}_1, \ldots, \overline{\gamma}_p), \quad \overline{\Gamma} = (\overline{\gamma}_1, \ldots, \overline{\gamma}_p)
$$
on $\overline{M}$ and $\overline{M}$, respectively. Fix also a relative homology class $A \in H_2(X, \overline{\Gamma} \cup \overline{\Gamma})$. Denote by $\tilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \overline{\Gamma})$ the space of punctured holomorphic spheres asymptotic to $\overline{\Gamma}, \overline{\Gamma}$ in the homology class $A$. More precisely, elements of $\tilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \overline{\Gamma})$ are triples $(f, z, \tilde{z})$, where $z = (z_1, \ldots, z_p)$ and $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_p)$ are collections of distinct points on the 2-sphere $S = S^2$, $\dot{S} := S^2 \setminus \{z_i, \tilde{z}_i\}$, and $f : \dot{S} \to X$ is a $J$-holomorphic map that is positively asymptotic to the closed Reeb orbit $\overline{\gamma}_i$ at the puncture $\overline{z}_i$, negatively asymptotic to $\overline{\gamma}_j$ at $\overline{\tilde{z}}_j$, and represents the relative homology class $A$.

The space $\tilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \overline{\Gamma})$ is equipped with the topology induced by weighted Sobolev norms as in [11],[13]. Denote its quotient by the action of the Mobius group by $\mathcal{M}^{A,J}(\overline{\Gamma}, \overline{\Gamma}) := \tilde{\mathcal{M}}^{A,J}(\overline{\Gamma}, \overline{\Gamma}) / \text{Aut}(S)$.

Its expected dimension is (see [11])
$$
\dim \mathcal{M}^{A,J}(\overline{\Gamma}, \overline{\Gamma}) = (n - 3)(2 - p - \overline{p}) + 2c_1(A) + \sum_{i=1}^{\overline{p}} \text{CZ}(\overline{\gamma}_i) - \sum_{j=1}^{p} \text{CZ}(\gamma_j).
$$

Here the Conley-Zehnder indices are computed with respect to some fixed trivializations of the contact distribution $\xi$ over $\overline{\gamma}_i$ and $\gamma_j$. These trivializations induce trivializations of the tangent bundle $TX$ over the asymptotic orbits, and $c_1(A)$ is the relative first Chern class of $f^*TX$ with respect to these trivializations for any representative $f$ of $A$.

If $\overline{M}$ and $\overline{M}$ are only Morse-Bott, then instead of collections of closed Reeb orbits we fix collections
$$
\overline{\Gamma} = (\overline{\Gamma}_1, \ldots, \overline{\Gamma}_p), \quad \overline{\Gamma} = (\overline{\Gamma}_1, \ldots, \overline{\Gamma}_p)
$$
of components of the orbifolds $\overline{\Gamma}_T$ and $\overline{\Gamma}_T$ of $T$-periodic orbits (for varying $T$). The preceding discussion carries over to this case, with the only difference that
the expected dimension becomes (see [9, Corollary 5.4])
\[
\dim \mathcal{M}^{A,J}(\bar{T}, \Gamma) = (n-3)(2 - p - p' + 2c_1(A))
+ \sum_{i=1}^{p'} (\text{RS}(\Gamma_i) + \frac{1}{2} \dim \bar{\Gamma}_i) - \sum_{j=1}^{p} (\text{RS}(\Gamma_j) - \frac{1}{2} \dim \bar{\Gamma}_j),
\]
or in view of equation (2),
\[
\dim \mathcal{M}^{A,J}(\bar{T}, \Gamma) = (n-3)(2 - p - p') + 2c_1(A)
+ \sum_{i=1}^{p'} (\text{CZ}(\Gamma_i) + \dim \bar{\Gamma}_i) - \sum_{j=1}^{p} \text{CZ}(\Gamma_j).
\] (3)

3 Tangency conditions

In this section we introduce holomorphic curves satisfying tangency conditions to a complex hypersurface, and we compute a specific such invariant on $\mathbb{CP}^n$. The discussion follows Section 6 in [18].

Consider a complex hypersurface in $Z = h^{-1}(0) \subset \mathbb{C}^n$ defined by a holomorphic function $h: \mathbb{C}^n \to \mathbb{C}$ with $h(0) = 0$ and $dh(0) \neq 0$. We say that a holomorphic map $f: \Sigma \to X$ from a Riemann surface with $f(0) = 0$ is **tangent of order $\ell$ to $Z$ at $0$** if
\[
(h \circ f)'(0) = \cdots = (h \circ f)^(\ell)(0) = 0.
\]
This condition clearly only depends on the germs of $f$ and $Z$ at 0, and it is preserved under the action $(f, Z) \mapsto (\Phi \circ f \circ \phi^{-1}, \Phi(Z))$ by local biholomorphisms of $(\mathbb{C}, 0)$ and $(\mathbb{C}^n, 0)$. Note that, after applying a local biholomorphism of $(\mathbb{C}^n, 0)$, we may assume that $h(z_1, \ldots, z_n) = z_n$ and the tangency condition becomes the condition $f'(0), \ldots, f^{(\ell)}(0) \in \mathbb{C}^{n-1}$ used in [18].

More generally, fix an integrable almost complex structure $J_0$ in a neighborhood of a point $x \in X$ and the germ of a complex hypersurface $Z$ through $x$. Consider an almost complex structure $J$ which near $x$ coincides with $J_0$ and a holomorphic map $f: \Sigma \to X$ from a Riemann surface with $f(z) = x$. We say that $f$ is **tangent of order $\ell$ to $Z$ at $z$**, and write $d^\ell f(z) \in T_xZ$, if $\Phi \circ f \circ \phi^{-1}$ is tangent of order $\ell$ to $\Phi(Z)$ at 0 for local holomorphic coordinates $\phi: (\Sigma, z) \to (\mathbb{C}, 0)$ and $\Phi: (X, x) \to (\mathbb{C}^n, 0)$.

Suppose now that $(X, J)$ has cylindrical ends and fix collections $\bar{T}, \Gamma$ of closed Reeb orbits. Denote by $\hat{\mathcal{M}}^{A,J}(\bar{T}, \Gamma; x, Z, \ell)$ the space of punctured $J$-holomorphic spheres which are tangent of order $\ell$ to $Z$ at $x$, i.e.,
\[
\hat{\mathcal{M}}^{A,J}(\bar{T}, \Gamma; x, Z, \ell) := \{(f, \bar{\tau}, \bar{z}, z) \mid (f, \bar{\tau}, \bar{z}) \in \hat{\mathcal{M}}^{A,J}(\bar{T}, \Gamma), z \in \hat{S}, f(z) = x, d^\ell f(z) \in T_xZ\}.
\]
Denote the quotient by the action of the M"{o}bius group by
\[
\mathcal{M}^{A,J}(\bar{T}, \Gamma; x, Z, \ell) := \hat{\mathcal{M}}^{A,J}(\bar{T}, \Gamma; x, Z, \ell)/\text{Aut}(S).
\]
The following result is a straightforward extension of [18, Proposition 6.9] to punctured holomorphic curves, with a slight modification concerning transversality of the evaluation map at \( z \) to the point \( x \) rather than \( Z \).

**Proposition 3.1.** For a generic almost complex structure \( J \) as above the moduli space \( M_{s,J}^A(\Gamma; x, Z, \ell) \subset M_{s,J}^A(\Gamma; x, Z, \ell) \) of simple \( J \)-holomorphic spheres tangent of order \( \ell \) to \( Z \) at \( x \) is a manifold of dimension

\[
\dim M_{s,J}^A(\Gamma; x, Z, \ell) = (n - 3)(2 - p - p) + 2c_1(A) + \sum_{i=1}^{p} CZ(\gamma_i) - \sum_{j=1}^{p} CZ(\gamma_j) - (2n - 2) - 2\ell.
\]

**Tangency conditions in cotangent bundles.** We apply this proposition to the cotangent bundle \( T^*L \) of a manifold \( L \) which admits a metric of nonpositive curvature. For given \( c > 0 \), equip \( L \) with the Riemannian metric provided by Lemma 2.2 such that all closed geodesics of length \( \leq c \) are noncontractible and nondegenerate and have Morse index \( \leq n - 1 \). Let \( \Gamma = (\gamma_1, \ldots, \gamma_k) \) be a collection of (lifts of) closed geodesics of length \( \leq c \). Fix a point \( x \in T^*L \) and the germ of a complex hypersurface \( Z \) through \( x \). Consider the moduli space \( M_J(\Gamma; x, Z, \ell) \) of holomorphic spheres tangent of order \( \ell \) to \( Z \) at \( x \) which are asymptotic at the punctures to geodesics of length \( \leq c \) and tangent of order \( \ell \) to \( Z \) at \( x \). For a punctured holomorphic sphere \( f : \tilde{S} \to T^*L \) trivialize the pullback bundle \( f^*T^*L \to \tilde{S} \) and denote by \( CZ(\gamma_i) \) the Conley-Zehnder indices in this trivialization. Since the sum of the \( \gamma_i \) is null-homologous, the Maslov indices in this trivialization sum up to zero. Hence by Lemma 2.1, we can replace the Conley-Zehnder index by the Morse index in the dimension formula and obtain

\[
\dim M_J(\Gamma; x, Z, \ell) = (n - 3)(2 - k) + \sum_{i=1}^{k} CZ(\gamma_i) - (2n - 2) - 2\ell \\
\leq (n - 3)(2 - k) + k(n - 1) - (2n - 2) - 2\ell \\
= 2k - 4 - 2\ell.
\]

Since by Proposition 3.1 this dimension is nonnegative for generic \( J \), we obtain

**Lemma 3.2.** For generic \( J \) every simple \( J \)-holomorphic sphere in \( T^*L \) which is asymptotic at the punctures to geodesics of length \( \leq c \) and tangent of order \( \ell \) to \( Z \) at \( x \) must have at least \( \ell + 2 \) punctures.

Next we want to derive a version of this lemma for non-simple spheres. So consider \( \tilde{f} = f \circ \phi \) for a simple sphere \( f : \tilde{S} \to T^*L \) as above and a \( d \)-fold branched covering \( \phi : S^2 \to S^2 \). Thus \( f \) has \( k \) positive punctures asymptotic to geodesics \( \gamma_1, \ldots, \gamma_k \) and \( \tilde{f} \) has \( \tilde{k} \geq k \) positive punctures asymptotic to multiples of the \( \gamma_i \). We assume that \( \tilde{f} \) is tangent of order \( n - 1 \) to \( Z \) at a point \( \tilde{p} \in S^2 \). In suitable holomorphic coordinates near \( \tilde{p} \) and \( p = \phi(\tilde{p}) \) we have \( \phi(z) = z^b \),
where \( b = \text{ord}(\tilde{p}) \) is the branching order of \( \tilde{p} \). In holomorphic coordinates near \( x \in T^*L \) in which \( Z = \mathbb{C}^{n-1} \subset \mathbb{C}^n \) we have \( f_n(z) = a_1z + a_2z^2 + \ldots \) and
\[
\tilde{f}_n(z) = a_1z^b + a_2z^{2b} + \cdots = O(z^n),
\]
hence \( a_1 = \cdots = a_\ell = 0 \) where \( \ell \) is the smallest integer \( \geq n/b - 1 \). By the Riemann-Hurwitz formula we have
\[
2d - 2 = \sum_{z \in S^2} (\text{ord}(z) - 1) = \sum_{i=1}^k (\text{ord}(z_i) - 1) + \sum_{z \in \tilde{S}} (\text{ord}(z) - 1).
\]
Since \( \phi \) maps all punctures \( \tilde{z}_i \) of \( \tilde{f} \) to punctures of \( f \), we have \( \sum_{i=1}^k \text{ord}(z_i) = kd \). The sum over \( z \in \tilde{S} \) is estimated below by the contribution \( b - 1 \) coming from \( \tilde{p} \) and we obtain \( 2d - 2 \geq kd - \tilde{k} + b - 1 \). Combining this with the estimates \( k \geq \ell + 2 \) from Lemma 3.2, \( d \geq b \) and \( \ell \geq n/b - 1 \) we find
\[
\tilde{k} \geq (k - 2)d + b + 1 \geq (k - 1)b + 1 \geq (\ell + 1)b + 1 \geq n + 1.
\]
So we have shown

**Corollary 3.3.** Let \( L^n \) be a manifold which admits a metric of nonpositive curvature. For \( c > 0 \), equip \( L \) with a metric such that all closed geodesics of length \( \leq c \) are noncontractible and nondegenerate and have Morse index \( \leq n-1 \). Pick a point \( x \in T^*L \) and the germ of a complex hypersurface \( Z \) through \( x \). Then for generic \( J \) every (not necessarily simple) \( J \)-holomorphic sphere in \( T^*L \) which is asymptotic at the punctures to geodesics of length \( \leq c \) and tangent of order \( n-1 \) to \( Z \) at \( x \) must have at least \( n+1 \) punctures.

**Tangency conditions in \( \mathbb{C}P^n \).** The second situation we consider are non-punctured holomorphic spheres in \( \mathbb{C}P^n \). Let \( A = [\mathbb{C}P^1] \) and fix a point \( x \in \mathbb{C}P^n \) and the germ of a complex hypersurface \( \Sigma \) through \( x \). Since \( c_1(A) = n+1 \), the dimension of the moduli space in Proposition 3.1 with \( \ell = n-1 \) is
\[
\dim \mathcal{M}^{A,J}(x, Z, \ell) = 2n - 6 + 2n + 2 - (2n - 2) - (2n - 2) = 0.
\]
(Note that all holomorphic spheres in class \( A \) are simple). By Gromov compactness, the zero-dimensional manifold \( \mathcal{M}^{A,J}(x, Z, n-1) \) is compact. By the usual cobordism argument, the signed count of its points is independent of the regular almost complex structure \( J \), the point \( x \), and the complex hypersurface \( Z \).

**Proposition 3.4.** The signed count of points in \( \mathcal{M}^{A,J}(x, Z, n-1) \) equals \( (n-1)! \).

By Gromov compactness, this implies

**Corollary 3.5.** For every point \( x \in \mathbb{C}P^n \), germ of complex hypersurface \( \Sigma \) through \( x \), and compatible almost complex structure \( J \), there exists a \( J \)-holomorphic sphere in class \( [\mathbb{C}P^1] \) which is tangent to order \( n-1 \) to \( Z \) at \( x \).
The proof of Proposition 3.4 is based on two lemmas.

**Lemma 3.6.** For all positive real numbers $a_1, \ldots, a_n$ the system of $n$ equations for $n$ complex variables $z_1, \ldots, z_n$

\[
\begin{align*}
    a_1 z_1 + \cdots + a_n z_n &= 0 \\
    a_1 z_1^2 + \cdots + a_n z_n^2 &= 0 \\
    \cdots \\
    a_1 z_1^n + \cdots + a_n z_n^n &= 0 
\end{align*}
\]  

has only the trivial solution $z_1 = \cdots = z_n = 0$.

**Proof.** We prove the lemma by induction on $n$. The case $n = 1$ is clear. Suppose the lemma holds for $n - 1$ but not for $n$, so equation (4) has a nontrivial solution $z = (z_1, \ldots, z_n)$. Writing (4) as

\[
\begin{pmatrix}
    a_1 & \cdots & a_n \\
    a_1 z_1 & \cdots & a_n z_n \\
    \cdots & \cdots & \cdots \\
    a_1 z_1^{n-1} & \cdots & a_n z_n^{n-1}
\end{pmatrix}
\begin{pmatrix}
    z_1 \\
    z_2 \\
    \cdots \\
    z_n
\end{pmatrix} = 0,
\]

this implies

\[
\det
\begin{pmatrix}
    1 & \cdots & 1 \\
    z_1 & \cdots & z_n \\
    \cdots & \cdots & \cdots \\
    z_1^{n-1} & \cdots & z_n^{n-1}
\end{pmatrix} = \prod_{i<j} (z_j - z_i) = 0.
\]

So after reordering the $z_i$ we may assume $z_{n-1} = z_n$. Inserting this in (4) and deleting the last equation, we obtain

\[
\begin{align*}
    a_1 z_1 + \cdots + (a_{n-1} + a_n) z_{n-1} &= 0 \\
    a_1 z_1^2 + \cdots + (a_{n-1} + a_n) z_{n-1}^2 &= 0 \\
    \cdots \\
    a_1 z_1^{n-1} + \cdots + (a_{n-1} + a_n) z_{n-1}^{n-1} &= 0.
\end{align*}
\]

By hypothesis this has only the trivial solution $0 = z_1 = \cdots = z_{n-1} = z_n$, contradicting the assumption $z \neq 0$.

**Lemma 3.7.** For all positive real numbers $a_0, \ldots, a_n$ the $n$ complex hypersurfaces in $\mathbb{CP}^n$ defined by the homogeneous equations

\[
\begin{align*}
    a_0 z_0 + \cdots + a_n z_n &= 0 \\
    a_0 z_0^2 + \cdots + a_n z_n^2 &= 0 \\
    \cdots \\
    a_0 z_0^n + \cdots + a_n z_n^n &= 0
\end{align*}
\]  

intersect transversally in $n!$ points.
Remark 3.8. It follows that the hypersurfaces defined by (5) intersect transversally for all complex numbers \(a_0, \ldots, a_n\) outside an algebraic subvariety in \(\mathbb{C}^{n+1}\).

**Proof.** Suppose \([z_0 : \cdots : z_n]\) is a non-transverse intersection point. Then the matrix of linearized equations

\[
\begin{pmatrix}
a_0 & \cdots & a_n \\
a_0 z_0 & \cdots & a_n z_n \\
\vdots & & \vdots \\
a_0 z_0^{n-1} & \cdots & a_n z_n^{n-1}
\end{pmatrix}
\]

has rank \(< n\), which implies

\[
\det\begin{pmatrix}
1 & \cdots & 1 \\
z_0 & \cdots & z_0^{n-1} \\
\vdots & \cdots & \vdots \\
z_0^{n-1} & \cdots & z_0^{n-1}
\end{pmatrix} = \prod_{i<j} (z_j - z_i) = 0.
\]

So after reordering the \(z_i\) we may assume \(z_0 = z_1\). Inserting this in (5), we obtain

\[
(a_0 + a_1)z_1 + \cdots + a_n z_n = 0 \\
(a_0 + a_1)z_1^2 + \cdots + a_n z_n^2 = 0 \\
\vdots \\
(a_0 + a_1)z_0^n + \cdots + a_n z_n^n = 0.
\]

By Lemma 3.6 this system has only the trivial solution \(z_0 = z_1 = \cdots = z_n = 0\), contradicting the assumption \(z \neq 0\).

By Bezout’s theorem, the number of intersection points equals the product \(1 \cdot 2 \cdots n\) of the degrees of the equations in (5).

**Proof of Proposition 3.4.** Consider the standard complex structure on \(\mathbb{CP}^n\). Denote by \(\mathcal{M}\) the space of holomorphic maps \(f: \mathbb{CP}^1 \to \mathbb{CP}^n\) of degree 1 mapping \([1 : 0]\) to \([1 : 0 : \cdots : 0]\) and \([0 : 1]\) to the hyperplane \([z_0 = 0]\) at infinity. Each such map is of the form

\[
f_p([z_0 : z_1]) = [z_0 : p_1 z_1 : \cdots : p_n z_1] \in \mathbb{CP}^n,
\]

or in affine coordinates

\[
f_p(z) = (p_1 z, \ldots, p_n z) \in \mathbb{C}^n,
\]

with \(p = (p_1, \ldots, p_n) \in \mathbb{C}^n \setminus 0\). The correspondence \(p \mapsto f_p\) thus gives a diffeomorphism from \(\mathbb{C}^n \setminus 0\) to \(\mathcal{M}\). Reparametrizing \(f\) as \(f(\lambda z)\) for \(\lambda \in \mathbb{C}^*\) corresponds to replacing \(p\) by \(\lambda p\), so the correspondence \(p \mapsto f_p\) induces a diffeomorphism from \(\mathbb{C}^{n-1}\) to the moduli space \(\mathcal{M} = \mathcal{M}/\mathbb{C}^*\) of degree 1 holomorphic spheres passing through 0 in \(\mathbb{CP}^n\). It is well known (see [49]) that this moduli space is
transversely cut out and its orientation agrees with the complex orientation of $\mathbb{C}P^{n-1}$.

Let $Z \subset \mathbb{C}P^n$ be a complex hypersurface of degree $n - 1$ defined in affine coordinates $z = (z_1, \ldots, z_n)$ by a homogeneous equation

$$h(z) = h_1(z) + \cdots + h_{n-1}(z) = 0,$$

where $h_k(z) = a_1 z_1^k + \cdots + a_n z_n^k$ with positive real numbers $a_0, \ldots, a_n$. Note that $h_k \circ f_p(z) = z^k h_k(p)$ and thus $(h \circ f_p)^{(k)}(0) = k! h_k(p)$. So the jet evaluation map at 0

$$j : \mathcal{M} \to \mathbb{C}^{n-1}; \quad f_p \mapsto \left((h \circ f_p)'(0), \ldots, (h \circ f_p)^{(n-1)}(0)\right)$$

corresponds to the map

$$\mathbb{C}^n \setminus 0 \to \mathbb{C}^{n-1}; \quad p \mapsto \left(h_1(p), \ldots, (n-1)! h_{n-1}(p)\right).$$

By Lemma 3.7, this map is transverse to $0 \in \mathbb{C}^{n-1}$ and its zero set in $\mathcal{M} \cong \mathbb{C}P^{n-1}$ consists of $(n-1)!$ points. Since all spaces are equipped with their complex orientations, all these points count positively and yield the signed count $(n-1)!$ of points in the space $\mathcal{M}^{A,J}(x, Z, n-1)$ in Proposition 3.4.

4 Proofs requiring only standard transversality

In this section we prove all theorems from the Introduction except Theorem 1.2 (b) and Theorem 1.18.

Proof of Theorem 1.1. Equip $L$ with a Riemannian metric $g_0$ of nonpositive curvature such that the set $\{(q, p) \in T^*L \mid |p| \leq 2\}$ (with the symplectic form $dp \wedge dq$) embeds symplectically into $\mathbb{C}P^n$. Such a metric exists by the Lagrangian neighbourhood theorem and rescaling the metric. Perturb $g_0$ to a metric $g$ as in Lemma 2.2 such that all closed geodesics of length $\leq \pi$ are noncontractible and nondegenerate and have Morse index $\leq n-1$. Clearly we can achieve that the unit ball cotangent bundle $W := \{(q, p) \in T^*L \mid |p| \leq 1\}$ with respect to $g$ still embeds symplectically into $\mathbb{C}P^n$. Denote by $M := \partial W$ the unit cotangent bundle of $L$. We identify $W$ and $M$ with their images in $\mathbb{C}P^n$. Pick a compatible almost complex structure $J$ on $\mathbb{C}P^n$ with $J = J_M$ near $M$. Let $(J_k)_{k \in \mathbb{N}}$ be the sequence of almost complex structures on $\mathbb{C}P^n$ obtained by the neck stretching procedure described before Theorem 2.4.

Fix a point $x$ on $L$ and the germ of a complex hypersurface $Z \subset T^*L$ through $x$. Choose $J$ such that the conclusion of Corollary 3.3 holds (with $c = \pi$). By Corollary 3.5 there exists for each $k$ a $J_k$-holomorphic sphere $f_k : S^2 \to \mathbb{C}P^n$ in the class $[\mathbb{C}P^1]$ which is tangent of order $n - 1$ to $Z$ at $x$. See Figure 2.
Let $z_k \in S^2$ be such that
$$f_k(z_k) = x, \quad d^{n-1}f_k(z_k) \in T_xZ.$$ 

By Corollary 2.10 after passing to a subsequence, $f_k$ converges to an $N$-level broken holomorphic curve $F = (F^{(1)}, \ldots, F^{(N)}): (\Sigma^*, j) \to (X^*, J^*)$. In particular, there exist diffeomorphisms $\phi_k: S^2 \to \Sigma$ such that $f_k^{(\nu)} \circ \phi_k^{-1}$ converges in $C^\infty_{\text{loc}}$ on $\Sigma^{(\nu)}$ to $F^{(\nu)}: \Sigma^{(\nu)} \to X^{(\nu)}$. Recall that $X^{(1)} = T^*L$, $X^{(N)} = \mathbb{CP}^n \setminus L$, and $X^{(\nu)} = \mathbb{R} \times M$ for $\nu = 2, \ldots, N - 1$.

Note that $x \in X_0^- \subset X^{(1)}$, and therefore $y_k := \phi_k(z_k) \in \Sigma^{(1)}$ with $f_k^{(1)} \circ \phi_k^{-1}(y_k) = x$ for all $k$. By the asymptotics of $F^{(1)}$ and the $C^\infty_{\text{loc}}$-convergence $f_k^{(1)} \to F^{(1)}$, after passing to a subsequence we obtain $y_k \to y \in \Sigma^{(1)}$ with $F^{(1)}(y) = x$ and $d^{n-1}F^{(1)}(y) \in T_xZ$. Denote by $C$ the component of $F^{(1)}$ containing $y$ (see Figure 3).

Since $\Sigma$ has genus zero, the graph underlying $F$ is a tree (see the remark preceding Lemma 2.8). Let us replace each subtree emanating from the node $C$ by one node which corresponds to a broken holomorphic curve with components in $\mathbb{CP}^n \setminus L$, $T^*L$ and $M \times \mathbb{R}$ and formally one negative puncture. (All other asymptotics appear in pairs of a negative and a positive puncture asymptotic to

Figure 2: A $J_k$–holomorphic sphere through $x \in L$ and tangent of order $(n-1)$ to $Z$. 

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the same closed Reeb orbit). So we have a component $C$ in $T^*L$ with $m$ positive punctures asymptotic to geodesics $\gamma_1, \ldots, \gamma_m$ and $m$ (broken) holomorphic planes $C_i$, $i = 1, \ldots, m$, asymptotic to $\gamma_i$ at their negative punctures. Since the $\gamma_i$ are not contractible in $L$, each $C_i$ must have a component in $\mathbb{CP}^n \setminus L$ and hence satisfies $\int_{C_i} \omega > 0$.

By Lemma 2.6 (applied to the broken holomorphic planes $C_i$), all the geodesics $\gamma_i$ have length $\leq \pi$. So by Corollary 3.3 the number of punctures of $C_0$ satisfies $m \geq n + 1$. So the $C_i$ give rise to $m \geq n + 1$ nonconstant disks $g_i : D \to \mathbb{CP}^n$ with boundary on $L$. Since $C_i$ is holomorphic, $g_i^* \omega \geq 0$ and $\int_D g_i^* \omega > 0$. Since $\sum_{i=1}^m \int_D g_i^* \omega = \pi$, the $\omega$-area of one of the disks must be $\leq \pi/(n + 1)$. 

Proof of Theorem 1.2 (a). We retain the notation from the preceding proof. As-
sume that $L$ is monotone. Then, since $C_i$ has positive symplectic area, it must have positive Maslov index $\mu(C_i) \geq 1$. On the other hand, the sum of the Maslov indices equals the value of the first Chern class of $\mathbb{CP}^n$ on $[\mathbb{CP}^1]$,

$$\sum_{i=1}^{m} \mu(C_i) = 2n + 2.$$  

Since $m \geq n + 1$, this implies that some $\mu(C_i) \leq 2$. If in addition $L$ is orientable, then all Maslov indices are even, hence $m = n + 1$ and $\mu(C_i) = 2$ for all $i$. 

**Proof of Theorem 1.16.** We keep the notation of the proof of Theorem 1.1. Let $W$ be a tubular neighbourhood of $L$ as above with $W \cap B = \emptyset$ and let $M = \partial W$. Let $J$ and the sequence $J_k$ be as above. Moreover, choose $J$ to be standard on the ball $B$. Let $f_k$ be $J_k$-holomorphic spheres in the class of a complex line passing through the center $p$ of $B$ and a point $x$ on $L$. In the limit $k \to \infty$ we find an $N$-level broken holomorphic curve $F = (F^{(1)}, \ldots, F^{(N)})$ passing through $x$ and $p$. Since $L$ has no contractible geodesics, the component of $F^{(1)}$ passing through $x$ must have at least 2 positive punctures. Hence $F^{(N)}$ has at least two components $C_1, C_2$. One of them, say $C_1$, passes through the point $p$. By the classical isoperimetric inequality (see [31]), $C_1$ has symplectic area at least $\pi r^2$. Therefore we have $0 < \int_{C_2} \omega \leq \pi - \pi r^2$.

**Proof of Theorem 1.17.** Suppose $L$ admits a Lagrangian embedding into a uniruled symplectic manifold $(X, \omega)$. Pick a tubular neighbourhood of $L$ and almost complex structures $J_1, J_2$ as in the proof of Theorem 1.1. Let $f_k$ be $J_k$-holomorphic spheres passing through a point $x$ on $L$ (they exist because $X$ is uniruled). In the limit $k \to \infty$ we find an $N$-level broken holomorphic curve $F = (F^{(1)}, \ldots, F^{(N)})$ of total area $a$ passing through $x$. It belongs to some moduli space of $J$-holomorphic spheres in $T^*L$ with $m$ punctures passing through $x$. Since $L$ has no contractible geodesics, we must have $m \geq 2$. Since all geodesics have Morse index zero, the moduli space is (if $J$ is regular for simple curves passing through $x$) a manifold of dimension $(n-3)(2-m) - (2n-2)$. If $n \geq 3$ this dimension is negative and we have a contradiction.

**Proof of Theorem 1.21.** Consider a Lagrangian embedding of $L$ into a minimally uniruled symplectic manifold $(X, \omega)$. Construct an $N$-level broken holomorphic curve $F = (F^{(1)}, \ldots, F^{(N)})$ of total area $a$ passing through a point $x \in L$ as in the proof of Theorem 1.1. Since $L$ has no contractible geodesics, the component of $F^{(1)}$ passing through $x$ has at least 2 positive punctures. Hence $F^{(N)}$ has at least two components $C_1, C_2$. Since the total symplectic area of $F$ equals $a$, both components must have area $0 < \int_{C_i} \omega < a$. So the Lagrangian embedding is not exact.

**Proof of Theorem 1.22.** Consider $\alpha > 0$ such that the unit codisk bundle $D^*(Q, g)$ embeds symplectically into $(\mathbb{CP}^n, \alpha \omega)$. Pick an almost complex structure $J$.
as in Lemma 2.6. The proof of Theorem 1.1 yields a $J$-holomorphic plane $f : C \to \mathbb{CP}^n \setminus L$ asymptotic to a closed geodesic $\gamma$ whose symplectic area satisfies $\int f \omega \leq \frac{\alpha \pi}{n+1}$. Since $\ell(\gamma) \leq \int f \omega$ by Lemma 2.6, we conclude

$$\alpha \geq \frac{(n+1)\ell(\gamma)}{\pi} \geq \frac{(n+1)\ell_{\min}(Q, g)}{\pi}.$$

\[\square\]

5 Proofs requiring full transversality

The remainder of the paper is devoted to the proof of Theorem 1.2 (b) and Theorem 1.18 which require stronger transversality. Let us first explain their proofs assuming the necessary transversality.

Proof of Theorem 1.2 (b). Suppose that $L \subset \mathbb{CP}^n$ is a Lagrangian torus. We equip $L$ with the standard flat metric, rescaled such that the unit ball cotangent bundle $W := \{(q, p) \in T^*L \mid |p| \leq 1\}$ embeds symplectically into $\mathbb{CP}^n$. Note that for this metric all closed geodesics occur in $(n-1)$-dimensional Morse-Bott families of Morse index zero.

Construct $C, C_1, \ldots, C_m$ as in the proof of Theorem 1.1. Thus $C$ is a holomorphic sphere in $T^*L$ with $m$ positive punctures asymptotic to families of geodesics $\Gamma_1, \ldots, \Gamma_m$ and tangent of order $n-1$ to $Z$ at $x \in L$, and $C_i$ is a broken holomorphic plane negatively asymptotic to $\Gamma_i$. Assume that all components of $C_i$ are regular and $C$ is regular subject to the tangency condition to $Z$, and the evaluation maps at adjacent punctures are transverse to each other. It is shown in Section 9 that $C_i$ and $C$ belong to moduli spaces $\mathcal{M}_i$ resp. $\mathcal{M}$ of dimensions

$$\dim \mathcal{M}_i = (n-3) - CZ(\Gamma_i), \quad i = 1, \ldots, m,$$

$$\dim \mathcal{M} = (n-3)(2m) + (2n+2) + \sum_{i=1}^{m} (CZ(\Gamma_i) + \dim \Gamma_i) - (4n-4).$$

Here the Conley-Zehnder indices are defined using trivializations over $C_i$ resp. $C$. The $(2n+2)$ in the last line corresponds to the value of the first Chern class of $\mathbb{CP}^n$ on a complex line. The $-(4n-4)$ arises from $2n-2$ conditions for passing through $x$ and $2n-2$ conditions for order $n-1$ tangency to $Z$. Note that the dimension formulae agree with the ones resulting from equation (3) for non-broken curves.

Denote by $\mu_i$ the Maslov index of $C_i$. Using $\dim \Gamma_i = n-1$ as well as $\sum_{i=1}^{m} \mu_i = 2n+2$ and the relation

$$CZ(\Gamma_i) = \text{ind}(\Gamma_i) - \mu_i = -\mu_i.$$
from Lemma 2.1 we obtain

\[ \dim M_i = (n - 3) + \mu_i, \quad i = 1, \ldots, m, \]
\[ \dim M = (n - 3)(2 - m) + \sum_{i=1}^{m} \dim \Gamma_i - (4n - 4) \]
\[ = (n - 3)(2 - m) + m(n - 1) - (4n - 4) \]
\[ = 2m - 2n - 2. \]

Note that nonnegativity of \( \dim M \) yields \( m \geq n + 1 \). According to our regularity assumptions, the evaluation maps at the punctures

\[ \text{ev}_1 \times \cdots \times \text{ev}_m : \mathcal{M}_1 \times \cdots \times \mathcal{M}_m \to \Gamma := \Gamma_1 \times \cdots \times \Gamma_m, \quad \text{ev} : \mathcal{M} \to \Gamma \]

are transverse to each other. Hence the images of the linearized evaluation maps at any point must satisfy

\[ \sum_{i=1}^{m} \dim \text{im}(T_{\text{ev}_i}) + \dim \text{im}(T_{\text{ev}}) \geq \dim \Gamma = m(n - 1). \]

Inserting

\[ \dim \text{im}(T_{\text{ev}}) \leq \dim M = 2m - 2n - 2, \]
\[ \dim \text{im}(T_{\text{ev}_i}) \leq \min\{\dim \Gamma_i, \dim M_i\} = \min\{n - 1, n - 3 + \mu_i\} \]
\[ = \frac{1}{2}(2n - 4 + \mu_i - |2 - \mu_i|) \]

as well as \( \sum \mu_i = 2n + 2 \) we infer

\[ 0 \leq \sum_{i=1}^{m} \dim \text{im}(T_{\text{ev}_i}) + \dim \text{im}(T_{\text{ev}}) - \dim \Gamma \]
\[ \leq \frac{1}{2} \sum_{i=1}^{m} (2n - 4 + \mu_i - |2 - \mu_i|) + 2m - 2n - 2 - m(n - 1) \]
\[ = \frac{1}{2} \sum_{i=1}^{m} (\mu_i - |2 - \mu_i|) + m - (2n + 2) \]
\[ = \frac{1}{2} \sum_{i=1}^{m} (-\mu_i - |2 - \mu_i|) + m, \]

hence

\[ \sum_{i=1}^{m} (\mu_i + |2 - \mu_i|) \leq 2m. \]

Now note that \( \mu_i + |2 - \mu_i| \geq 2 \) for any number \( \mu_i \), with equality if \( \mu_i \leq 2 \). It follows that \( \mu_i \leq 2 \) for all \( i \). Since \( \sum_{i=1}^{m} \mu_i = 2n + 2 \) and all \( \mu_i \) are even, at least \( n + 1 \) of the \( \mu_i \) must be equal to 2.
The proof of Theorem 1.18 uses the following immediate consequence of results by McDuff in [46, 47, 48]. Here a symplectic 2-manifold is called rational if it is symplectomorphic to $\mathbb{CP}^2$ with a multiple of the Fubini-Study form, and ruled if it is the total space of a symplectic fibration over a closed oriented surface.

**Proposition 5.1.** Every uniruled symplectic 4-manifold $(X, \omega)$ is the blow-up of a rational or ruled symplectic 4-manifold $(\bar{X}, \bar{\omega})$. In the rational case, $(X, \omega)$ possesses a nonzero Gromov-Witten invariant $GW^{X,A}(pt, pt)$ of holomorphic spheres passing through two prescribed points (with no further constraints), and in the ruled case it possesses a nonzero Gromov-Witten invariant $GW^{X,A}(pt)$ of holomorphic spheres passing through one prescribed point.

**Proof.** By definition of uniruledness, there exists a nonzero Gromov-Witten invariant $GW^{X,A}(\alpha_1, \ldots, \alpha_r)$ of $J$-holomorphic spheres in $X$ in the homology class $A$ passing through a prescribed point $\alpha_1$ and cycles (in singular homology) $\alpha_2, \ldots, \alpha_r$. The corresponding moduli space $\mathcal{M}$ is a manifold of expected dimension

$$\dim \mathcal{M} = -2 + 2c_1(A) - \sum_{i=1}^{r} (\text{codim} \alpha_i - 2) = 0.$$  

If codim $\alpha_i < 2$ for some $i$, then the dimension of the moduli space with the $i$-th marked point removed would be negative and thus $\mathcal{M}$ would be empty. Hence codim $\alpha_i \geq 2$ for all $i$. Since codim $\alpha_1 = 4$, it follows that

$$2c_1(A) = 2 + \sum_{i=1}^{r} (\text{codim} \alpha_i - 2) \geq 4,$$

and thus $c_1(A) \geq 2$. Consider a $J$-holomorphic sphere $C \in \mathcal{M}$. According to [47, Proposition 1.2], after a $C^1$-small perturbation of $C$ and a $C^0$-small perturbation of $J$ we may assume that $C$ is immersed. Then $C$ is an immersed symplectic sphere with positive transverse self-intersections and Chern number $c_1([C]) \geq 2$, and [48, Theorem 1.4] yields that $(X, \omega)$ is the blow-up of a rational or ruled symplectic 4-manifold $(\bar{X}, \bar{\omega})$.

In the rational case, we have the Gromov-Witten invariant $GW^{\mathbb{CP}^2,[\mathbb{CP}^1]}(pt, pt) = 1$ of complex lines passing through two prescribed points $p \neq q$, and blowing up away from the unique complex line $C$ through $p, q$ we obtain a nonvanishing two-point invariant of $(X, \omega)$. (To see this, note that any other holomorphic sphere $C'$ in $X$ in the homology class $A$ passing through $p, q$ would have at least two intersection points with $C$ and homological intersection number $C \cdot C' = A \cdot A = 1$, so $C' = C$ by positivity of intersections.) In the ruled case, we have the Gromov-Witten invariant $GW^{X,[\text{fibre}]}(pt) = 1$ of holomorphic spheres in the class of a fibre through a prescribed point $p$, and blowing up away from the fibre through $p$ we obtain a nonvanishing one-point invariant of $(X, \omega)$. \hfill \Box

**Proof of Theorem 1.18.** Let $L$ be a closed orientable surface with a metric of negative curvature. Suppose $L$ admits a Lagrangian embedding into a uniruled symplectic 4-manifold $(X, \omega)$. By Proposition 5.1 there exists a nonzero
Gromov-Witten invariant $GW_{X,A}(\alpha_1, \ldots, \alpha_r)$ of holomorphic spheres passing through prescribed points $\alpha_1, \ldots, \alpha_r$ (where $r$ equals 1 or 2). Pick a tubular neighbourhood of $L$ and almost complex structures $J_k$ as in the proof of Theorem 1.1. Let $f_k$ be $J_k$-holomorphic spheres representing the Gromov-Witten invariant, where we choose $\alpha_1$ on $L$ and $\alpha_2$ (if it appears) outside the tubular neighbourhood of $L$. In the limit $k \to \infty$ we find an $N$-level broken holomorphic curve $F = (F^{(1)}, \ldots, F^{(N)})$.

Assume that every component of $F$ in $X \setminus L$ is regular. Hence the index (expected dimension, including possibly the constraint $\alpha_2$) of each such component is nonnegative. As all closed geodesics have Morse index zero, the index of a component with $k$ punctures and no constraint $\alpha_1$ in $T^*L$ or the symplectization $\mathbb{R} \times M$ of the unit cotangent bundle equals $k - 2 \geq 0$. The component $C_1$ in $T^*L$ carrying the point constraint $\alpha_1$ has index $k - 4$, where $k$ is the number of (positive) punctures. If $k = 2$, then $C_1$ is a holomorphic cylinder in $T^*L$ projecting onto a closed geodesic $\gamma$ with two positive punctures asymptotic to $\pm \gamma$ (this follows from a maximum principle argument similar to the proof of Lemma 7.3 below). If $k = 3$, then $C_1$ is either simple (and thus doesn’t exist for generic $J$) or a 2-fold branched cover of a cylinder over a closed geodesic. So if we choose the point $\alpha_1$ not to lie on a closed geodesic, then the cases $k = 2$ and $k = 3$ cannot occur and $C_1$ has index $\geq 0$. Since the total index of $F$, i.e., the sum of the indices of all components, equals zero and all indices are nonnegative, each component must have index zero.

Consider now a component $C$ of $F$ in $X \setminus L$ with just one negative puncture asymptotic to a closed geodesic $\gamma$. Its index equals either $-1 + \mu(\gamma, \Phi)$ or $-1 + \mu(\gamma, \Phi) - 2$ (if it carries the point constraint $\alpha_2$), where $\mu(\gamma, \Phi)$ is the Maslov index of $\gamma$ in the trivialization of $T^*X$ that extends over $C$. Now any loop of oriented Lagrangian subspaces has even Maslov index. Hence the index of $C$ is odd, contradicting the fact that it must be zero.

We see that both proofs work provided that we can achieve regularity for all punctured holomorphic spheres (with suitable point and tangency constraints) in $T^*L$, $\mathbb{R} \times M$ and $X \setminus L$ resulting from the neck stretching procedure. (In the case of Theorem 1.18 the proof actually only uses regularity in $X \setminus L$). There are three known techniques one could invoke to achieve such regularity: Kuranishi structures [28], polyfolds [30], and domain dependent perturbations [18]. In the remainder of this paper we will carry out the third approach.

6 Coherent perturbations

In this section we define a suitable class of perturbations of the Cauchy-Riemann equation parametrized by the Deligne-Mumford space $\mathcal{M}_{k+1}$. The discussion closely follows [18], see also [30].
Nodal curves. Let us first describe the Deligne-Mumford space $\overline{M}_k$ of stable curves of genus zero with $k$ marked points. We adopt the approach and notation of [19], see also [18].

A $k$-labelled tree is a triple $T = (T, E, \Lambda)$, where $(T, E)$ is a (connected) tree with set of vertices $T$ and edge relation $E \subset T \times T$, and $\Lambda = \{\Lambda_\alpha\}_{\alpha \in T}$ is a decomposition of the index set $\{1, \ldots, k\} = \bigsqcup_{\alpha \in T} \Lambda_\alpha$. We write $\alpha E \beta$ if $(\alpha, \beta) \in E$. Note that the labelling $\Lambda$ defines a unique map $\{1, \ldots, k\} \to T$, $i \mapsto \alpha_i$ by the requirement $i \in \Lambda_{\alpha_i}$. Let

$$e(T) = |T| - 1$$

be the number of edges. A tree homomorphism $\tau : T \to \tilde{T}$ is a map which collapses some subtrees of $T$ to vertices of $\tilde{T}$. A tree homomorphism $\tau$ is called tree isomorphism if it is bijective and $\tau^{-1}$ is a tree homomorphism.

A tree $T$ is called stable if for each $\alpha \in T$,

$$n_\alpha := \# \Lambda_\alpha + \# \{\beta \mid \alpha E \beta\} \geq 3.$$ 

Note that for $k < 3$ every $k$-labelled tree is unstable. For $k \geq 3$, a $k$-labelled tree $T$ can be canonically stabilized to a stable $k$-labelled tree $st(T)$ by deleting the vertices with $n_\alpha < 3$ and modifying the edges in the obvious way.

A nodal curve of genus zero with $k$ marked points modelled over the tree $T = (T, E, \Lambda)$ is a tuple $z = (\{z_{\alpha \beta}\}_{\alpha E \beta}, \{z_i\}_{1 \leq i \leq k})$ of points $z_{\alpha \beta}, z_i \in S^2$ such that for each $\alpha \in T$ the special points

$$SP_\alpha := \{z_{\alpha \beta} \mid \alpha E \beta\} \cup \{z_i \mid \alpha_i = \alpha\}$$

are pairwise distinct. Note that $n_\alpha = \# SP_\alpha$. For $\alpha \in T$ and $i \in \{1, ..., k\}$ denote by $z_{\alpha i}$ either the point $z_i$ if $i \in \Lambda_\alpha$, or the point $z_{\alpha \beta_1}$ if $z_i \in \Lambda_{\beta_1}$ and $(\alpha, \beta_1), (\beta_1, \beta_2), ..., (\beta_{r-1}, \beta_r) \in E$. We associate to $z$ the nodal Riemann surface

$$\Sigma_z := \bigsqcup_{\alpha \in T} S_\alpha \bigg/ z_{\alpha \beta} \sim z_{\beta \alpha},$$

obtained by gluing a collection of standard spheres $\{S_\alpha\}_{\alpha \in T}$ at the points $z_{\alpha \beta}$ for $\alpha E \beta$, with marked points $z_i \in S_{\alpha_i}$, $i = 1, \ldots, k$. Note that $z$ can be uniquely recovered from $\Sigma_z$, so we will sometimes not distinguish between the two. A nodal curve $z$ is called stable if the underlying tree is stable, i.e., every sphere $S_\alpha$ carries at least 3 special points. Stabilization of trees induces a canonical stabilization of nodal curves $z \mapsto st(z)$.

We will usually omit the genus zero from the notation. Denote the space of all nodal curves (of genus zero) with $k$ marked points modelled over $T$ by

$$\mathcal{M}_T \subset (S^2)^E \times (S^2)^k.$$
Note that this is an open subset of the product of spheres. A morphism between nodal curves \( z, \tilde{z} \) modelled over trees \( T, \tilde{T} \) is a tuple

\[
\phi = (\tau, \{ \phi_\alpha \}_{\alpha \in T}),
\]

where \( \tau : T \to \tilde{T} \) is a tree homomorphism and \( \phi_\alpha : S^2 \cong S_\alpha \to S_{\tau(\alpha)} \cong S^2 \) are (possibly constant) holomorphic maps such that

\[
\tilde{z}_{\tau(\alpha)\tau(\beta)} = \phi_\alpha(z_{\alpha\beta}) \quad \text{if} \quad \tau(\alpha) \neq \tau(\beta),
\]

\[
\phi_\alpha(z_{\alpha\beta}) = \phi_{\beta}(z_{\beta\alpha}) \quad \text{if} \quad \tau(\alpha) = \tau(\beta),
\]

\[
\tilde{\alpha}_i = \tau(\alpha_i), \quad \tilde{z}_i = \phi_{\alpha_i}(z_i)
\]

for \( i = 1, \ldots, k \) and \( \alpha E \beta \). A morphism \( \phi : z \to \tilde{z} \) induces a natural holomorphic map \( \Sigma_z \to \Sigma_{\tilde{z}} \) (i.e., a continuous map that is holomorphic on each component \( S_\alpha \)). A morphism \( (\tau, \{ \phi_\alpha \}) \) is called isomorphism if \( \tau \) is a tree isomorphism and each \( \phi_\alpha \) is biholomorphic.

Isomorphisms from \( z \) to itself are called automorphisms. If \( z \) is stable its only automorphism is the identity (see [49], discussion after Definition D.3.4). Thus for a stable tree \( T \) we have a free and proper holomorphic action

\[
G_T \times \tilde{M}_T \to \tilde{M}_T
\]

of the group \( G_T \) of isomorphisms fixing \( T \). Hence the quotient

\[
\mathcal{M}_T := \tilde{M}_T/G_T
\]

is a complex manifold of dimension

\[
\dim_{\mathbb{C}} \mathcal{M}_T = k + 2e(T) - 3|T| = k - 3 - e(T).
\]

For \( k \geq 3 \), denote by \( \mathcal{M}_k = \tilde{M}_k/G \) the moduli space of stable curves modelled over the \( k \)-labelled tree with one vertex. As a set, the Deligne-Mumford space (of genus zero) with \( k \) marked points is given by

\[
\mathcal{M}_k := \prod_T \mathcal{M}_T,
\]

where the union is taken over the (finitely many) isomorphism classes of stable \( k \)-labelled trees. However, \( \mathcal{M}_k \) is equipped with the topology of Gromov convergence which makes it a compact connected metrizable space (see [49]). As the notation suggests, \( \mathcal{M}_k \) is the compactification of \( \mathcal{M}_k \) in the Gromov topology. For a stable \( k \)-labelled tree \( T \), the closure of \( \mathcal{M}_T \) in \( \mathcal{M}_k \) is given by

\[
\overline{\mathcal{M}}_T = \prod_T \mathcal{M}_T,
\]

where the union is taken over all isomorphism classes of stable \( k \)-labelled trees \( \tilde{T} \) for which there exists a surjective tree homomorphism \( \tau : \tilde{T} \to T \) with \( \tau(\tilde{\alpha}_i) = \alpha_i \) for \( i = 1, \ldots, k \).

We have the following result of Knudsen (cf. [19]).
Theorem 6.1. For \( k \geq 3 \), the Deligne-Mumford space \( \bar{M}_k \) is a compact complex manifold of dimension \( \dim_{\mathbb{C}} \bar{M}_k = k - 3 \). Moreover, for each stable \( k \)-labelled tree \( T \), the space \( \bar{M}_T \subset \bar{M}_k \) is a compact complex submanifold of codimension \( \text{codim}_{\mathbb{C}} \bar{M}_T = e(T) \).

**Stable decompositions.** Consider the Deligne-Mumford space \( \bar{M}_{k+1} \) with \( k + 1 \) marked points \( z_0, \ldots, z_k \). In the following discussion, the point \( z_0 \) plays a special role (it will be the variable for holomorphic maps in later sections).

We have a canonical projection \( \pi: \bar{M}_{k+1} \rightarrow \bar{M}_k \) by forgetting the marked point \( z_0 \) and stabilizing. The map \( \pi \) is holomorphic and the fibre \( \pi^{-1}(z) \) is naturally biholomorphic to \( \Sigma_z \).

The projection \( \pi: \bar{M}_{k+1} \rightarrow \bar{M}_k \) is called the *universal curve*.

Given a stable \( (k + 1) \)-labelled tree \( T \), we define an equivalence relation on \( \{0, \ldots, k\} \) by \( i \sim j \) iff \( z_{\alpha_0 i} = z_{\alpha_0 j} \). The equivalence classes yield a decomposition \( \{0, \ldots, k\} = I_0 \cup \cdots \cup I_\ell \).

Note that the marked points on \( S_{\alpha_0} \) correspond to equivalence classes consisting of one element; in particular, we may put \( I_0 = \{0\} \). Stability implies \( \ell + 1 = n_{\alpha_0} \geq 3 \). Conversely, we call a decomposition \( I = (I_0, \ldots, I_\ell) \) of \( \{0, \ldots, k\} \) *stable* if \( I_0 = \{0\} \) and \( |I| := \ell + 1 \geq 3 \). We will always order the \( I_j \) such that the integers \( i_j := \min \{i \mid i \in I_j\} \) satisfy

\[
0 = i_0 < i_1 < \cdots < i_\ell.
\]

Denote by \( \mathcal{M}_I = \mathcal{M}_{(I_0, \ldots, I_\ell)} \subset \bar{M}_{k+1} \) the union over those stable trees that give rise to the stable decomposition \( I = (I_0, \ldots, I_\ell) \). The \( \mathcal{M}_I \) are submanifolds of \( \bar{M}_{k+1} \) with

\[
\bar{M}_{k+1} = \bigcup_I \mathcal{M}_I,
\]

and the closure of \( \mathcal{M}_I \) is a union of certain strata \( \mathcal{M}_I \) with \( |I| \leq |J| \). The above ordering of the \( I_j \) determines a projection

\[
p_I: \mathcal{M}_I \rightarrow \mathcal{M}_{|I|},
\]

sending a stable curve \( z \) to the special points on the component \( S_{\alpha_0} \).

We call the union of the strata \( \mathcal{M}_I \) with \( |I| = 3 \) the *set of special points* of \( \bar{M}_{k+1} \). On its complement \( \bar{M}_{k+1}^* \), the projection

\[
\pi: \bar{M}_{k+1}^* \rightarrow \bar{M}_k
\]

is a fibration. We denote by

\[
T^* \bar{M}_{k+1}^* := \ker(d\pi) \rightarrow \bar{M}_{k+1}^*
\]

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the vertical tangent bundle.

**Coherent perturbations.** Let now \((E, J) \to X\) be a complex vector bundle. The pullbacks under the obvious projections give complex vector bundles \((p_1^* T\nu \mathcal{M}_{k+1}^*, j)\) and \((p_2^* E, J)\) over the product \(\mathcal{M}_{k+1} \times X\). We denote by

\[
\text{Hom}^{0,1}(p_1^* T\nu \mathcal{M}_{k+1}^*, p_2^* E) \to \mathcal{M}_{k+1}^* \times X
\]

the bundle of complex antilinear bundle homomorphisms. Note that for every stable decomposition \(I\) with \(|I| \geq 4\), the restriction of this bundle to the stratum \(\mathcal{M}_I\) is the pullback bundle

\[
\text{Hom}^{0,1}(p_1^* T\nu \mathcal{M}_I, p_2^* E) \longrightarrow \text{Hom}^{0,1}(p_1^* T\nu \mathcal{M}_I, p_2^* E)
\]

\[
\mathcal{M}_I \times X \quad p_1 \times \text{id} \quad \mathcal{M}_I \times X
\]

**Definition.** A coherent perturbation on \((E, J)\) is a continuous section \(K\) of the bundle \((6)\) satisfying the following two conditions:

(a) \(K\) has compact support;

(b) for every stable decomposition \(I\) with \(|I| \geq 4\), the restriction \(K|_{\mathcal{M}_I}\) is the pullback under \(p_1 \times \text{id}\) of a smooth section \(K_I\) of the right-hand bundle in \((7)\).

In view of condition (a), we may view \(K\) as being extended by zero to all of \(\mathcal{M}_{k+1}\) (although the bundle \((6)\) is not defined at the special points). We denote the space of coherent perturbations on \((E, J)\) by

\[
\mathcal{K}(\mathcal{M}_{k+1}, E).
\]

It is equipped with the \(C^0\)-topology on \(\mathcal{M}_{k+1}\), and the \(C^\infty\)-topology on each \(\mathcal{M}_I\) via the pullback diagram \((7)\).

To better understand this definition, fix a stable curve \(z \in \mathcal{M}_k\), modelled over the \(k\)-labelled tree \(T\). Recall that \(\pi^{-1}[z]\) is naturally identified with the nodal Riemann surface \(\Sigma_z\) which is a union of \(|T|\) copies \(S_\alpha\) of \(S^2\), glued together at the points \(z_{\alpha\beta}\). Denote by \(\Sigma_z^* \subset \Sigma_z\) the complement of the special points, which is a smooth (possibly disconnected) punctured Riemann surface. Restriction of \(K \in \mathcal{K}(\mathcal{M}_{k+1}, E)\) to \(\pi^{-1}[z]\) yields a smooth section \(K_z\) with compact support in the bundle

\[
\text{Hom}^{0,1}(p_1^* T\Sigma_z^*, p_2^* E) \to \Sigma_z^* \times X.
\]

In other words, \(K_z\) is a \((0,1)\)-form on \(\Sigma_z^*\) with values in the sections of \(E\). If \(E = T\nu X\) is the tangent bundle of an almost complex manifold \((X, J)\), then \(K_z\) is a \((0,1)\)-form on \(\Sigma_z^*\) with values in the vector fields on \(X\).

In general, for any (possibly noncompact and disconnected) Riemann surface \(\Sigma\), we denote by

\[
\mathcal{K}(\Sigma, E)
\]

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the space of smooth sections with compact support in the bundle
\[ \text{Hom}^{0,1}(p_1^*T\Sigma, p_2^*E) \rightarrow \Sigma \times X. \]  
(9)

The proof of the following lemma is analogous to the proof of [18, Lemma 3.10].

Lemma 6.2. For \( z \in \bar{M}_k \), let \( K_z \) be a smooth section with compact support in the bundle \( E \). Then there exists a \( K \in \mathcal{K}(\bar{M}_{k+1}, E) \) whose restriction to \( \pi^{-1}[z] \) equals \( K_z \). □

\( K \)-holomorphic maps. Let \((X, J)\) be an almost complex manifold, \( \Sigma \) a Riemann surface, and \( K \in \mathcal{K}(\Sigma, \mathcal{T}X) \). Then to a map \( f : \Sigma \rightarrow X \) we associate the smooth section
\[ \partial J, K f := df + J(f) \circ df \circ j + K(f) \]
of the bundle \( \text{Hom}^{0,1}(T\Sigma, f^*\mathcal{T}X) \rightarrow \Sigma \) whose value at \( z \in \Sigma \) is the complex antilinear homomorphism
\[ dzf + J(f(z)) \circ dzf \circ jz + K(z, f(z)) : Tz\Sigma \rightarrow Tf(z)X. \]

A map \( f \) satisfying \( \partial J, K f = 0 \) will be called \( K \)-holomorphic.

More globally, a coherent perturbation \( K \in \mathcal{K}(\bar{M}_{k+1}, \mathcal{T}X) \) associates to each \( z \in \bar{M}_k \) a coherent perturbation \( K_z \in \mathcal{K}(\Sigma_z, \mathcal{T}X) \), and thus to a map \( f : \Sigma_z \rightarrow X \) the smooth section \( \partial J, K_z f \) of the bundle \( \text{Hom}^{0,1}(T\Sigma_z, f^*\mathcal{T}X) \rightarrow \Sigma_z \).

Taming conditions. Let \((X, J)\) be an almost complex manifold, \( \Sigma \) a Riemann surface, and \( K \in \mathcal{K}(\Sigma, \mathcal{T}X) \). Then a \( K \)-holomorphic map \( \Sigma \rightarrow X \) can be viewed as a holomorphic section for the almost complex structure \( \hat{J} \) on \( \Sigma \times X \) defined by the matrix
\[ \hat{J} := \begin{pmatrix} j & 0 \\ -K \circ j & J \end{pmatrix} \]
with respect to the splitting \( \mathcal{T}(\Sigma \times X) = T\Sigma \oplus \mathcal{T}X \). Here \( \hat{J}^2 = -\mathbb{I} \) follows from \( K \circ j = -J \circ K : T\Sigma \rightarrow T\Sigma \), and a short computation shows that \( f : \Sigma \rightarrow X \) is \( K \)-holomorphic iff \( \hat{f} := \text{id} \times f : \Sigma \rightarrow \Sigma \times X \) is \( \hat{J} \)-holomorphic. Another short computation shows that the projection \( \Sigma \times X \rightarrow \Sigma \) is holomorphic.

For an area form \( \sigma \) on \( \Sigma \), we obtain a symplectic form \( \hat{\omega} := \sigma \oplus \omega \) on \( \Sigma \times X \). The following computation shows that \( \hat{\omega} \) tames \( \hat{J} \) if the \( C^0 \)-norm of \( K \), measured with respect to the metrics \( \sigma(\cdot, j\cdot) \) and \( \omega(\cdot, J\cdot) \), satisfies \( \|K\| \leq 1 \):
\[
\hat{\omega} \left( \left( \begin{array}{c} v \\ w \end{array} \right), \hat{J} \left( \begin{array}{c} v \\ w \end{array} \right) \right) = \omega \left( \left( \begin{array}{c} v \\ w \end{array} \right), \left( \begin{array}{c} jv \\ Jw - Kjv \end{array} \right) \right) \\
= \sigma(v, jv) + \omega(w, Jw) - \omega(w, Kjv) \\
\geq |v|^2 + |w|^2 - \|K\| |v| |w| \\
\geq \frac{1}{2} (|w|^2 + |w|^2).
\]
Next, consider an oriented manifold $M^{2n-1}$ with a *stable Hamiltonian structure* $(\omega, \lambda)$, i.e., a closed 2-form $\omega$ and a 1-form $\lambda$ satisfying

$$\ker(\omega) \subset \ker(d\lambda), \quad \lambda \wedge \omega^{n-1} > 0.$$ 

It determines a *Reeb vector field* $R$ by the conditions $i_R \lambda = 1$ and $i_R \omega = 0$. Note that a contact form $(\omega, \lambda)$ induces a stable Hamiltonian structure $(d\lambda, \lambda)$. As in the contact case, an almost complex structure $J$ on $\mathbb{R} \times M$ is called *compatible* with $(\omega, \lambda)$ if $J$ is $\mathbb{R}$-invariant, preserves $\xi = \ker \lambda$, maps $\partial_r$ to the Reeb vector field $R$, and $J|_\xi$ is compatible with $\omega$. We say that $J$ is *tamed* by $(\omega, \lambda)$ if the compatibility condition on $J|_\xi$ is relaxed to the taming condition $\omega(v, Jv) > 0$ for all $0 \neq v \in \xi$.

For a Riemann surface $\Sigma$ with area form $\sigma$, the pair $(\tilde{\omega} = \sigma \oplus \omega, \tilde{\lambda} = \lambda)$ defines a stable Hamiltonian structure on $\Sigma \times M$. Let $J$ be an almost complex structure on $\mathbb{R} \times M$ tamed by $(\omega, \lambda)$. For $K \in K((\Sigma, T(\mathbb{R} \times M))$, we define an almost complex structure $\tilde{J}$ on $\Sigma \times (\mathbb{R} \times M)$ by $\tilde{J}(\partial_r) = R$ and the matrix

$$\tilde{J} := \begin{pmatrix} j & 0 \\ -K \circ j & J \end{pmatrix}$$

on $T\Sigma \oplus \xi$. This $\tilde{J}$ is tamed by $(\tilde{\omega}, \tilde{\lambda})$ provided that $K$ is $\mathbb{R}$-invariant and $\|K\| \leq 1$.

**Remark 6.3.** A map $f = (a, u) : \Sigma \to \mathbb{R} \times M$ defines a $\tilde{J}$-holomorphic section $\tilde{f} = \text{id} \times f$ in $\Sigma \times (\mathbb{R} \times M)$ iff it satisfies the equations

$$da - u^* \lambda \circ j = 0, \quad \pi_\xi du + J(u) \circ \pi_\xi du \circ j + K(z, u) = 0,$$

where $\pi_\xi : TM \to \xi$ is the projection along $R$. Note that the second equation is not satisfied by branched covers of orbit cylinders provided that $K(z, u)$ does not vanish identically. This suggests that the class of coherent perturbation is large enough to get rid of such non-regular branched covers, which will be made precise in the next section.

**Globalization and compactness.** To globalize this construction, we fix a Kähler form $\sigma$ on the Deligne-Mumford space $\mathcal{M}_{k+1}$. Such a Kähler form can, for example, be obtained from the embedding of $\mathcal{M}_{k+1}$ into a product of $\mathbb{CP}^1$’s constructed in [19]. Now every component $S_\alpha$ of a fibre $\pi^{-1}(z)$ of the projection $\pi : \mathcal{M}_{k+1} \to \mathcal{M}_k$ is an embedded holomorphic sphere, so $\sigma|_{S_\alpha}$ defines a positive area form on $S_\alpha$. For an almost complex manifold $(X, J)$ tamed by $\omega$ and a coherent perturbation $K \in K(\mathcal{M}_{k+1}, TX)$, the construction above yields a complex structure $\tilde{J}$ on the vertical tangent bundle (outside the special points) of the projection $\mathcal{M}_{k+1} \times X \to \mathcal{M}_k$. This complex structure is tamed by the restriction of $\tilde{\omega} = \sigma \oplus \omega$ to the fibres provided that the $C^0$-norm of $K$, measured with respect to the metrics $\sigma(\cdot, j \cdot)$ and $\omega(\cdot, J \cdot)$, satisfies $\|K\| \leq 1$.

If $(X, J)$ is a symplectization $\mathbb{R} \times M$ of a manifold with a stable Hamiltonian structure, then we require in addition that $K$ is $\mathbb{R}$-invariant. If $(X, J)$ has
cylindrical ends of this type, then we require that $K$ is $\mathbb{R}$-invariant outside a compact subset $X_0 \subset X$. If $X^* = X^+ \amalg (\mathbb{R} \times M) \amalg X^-$ is a split manifold, then $K$ is a triple $(K^+, K_M, K^-)$ where $K^\pm \in \mathcal{K}(\mathcal{M}_{k+1}, TX^\pm)$ and $K_M \in \mathcal{K}(\mathcal{M}_{k+1}, T(\mathbb{R} \times M))$, and the $K^\pm$ agree with $K_M$ outside compact subsets $X^\pm_0 \subset X^\pm$.

The Compactness Theorem continues to hold for $K$-holomorphic maps by formally applying it (in the setting of stable Hamiltonian structures) to the graphs of the holomorphic maps $f_k$ in $\Sigma_k \times X_k$, using the convergence of a subsequence of the domains $(\Sigma_k, j_k)$ in the Deligne-Mumford space.

**Remark 6.4.** In the preceding discussion we have used the setting of stable Hamiltonian structures because this is most natural for the graph construction. If the cylindrical ends are contact manifolds (which is the case we are interested in), one could avoid the use of stable Hamiltonian structures as follows. Consider a contact manifold $(M, \lambda)$. For a fixed open Riemann surface $\Sigma$, one considers a 1-form $\beta$ on $\Sigma$ such that $d\beta = \sigma$ is an area form and a family of contact forms $\lambda_z$ on $M$ depending with compact support on $z \in \Sigma$. Then $\hat{\lambda} = \lambda_z + \beta$ defines a contact form on $\Sigma \times M$ provided the derivatives of $\lambda_z$ with respect to $z$ are sufficiently small. Now one considers almost complex structures $\hat{J}$ on $\Sigma \times (\mathbb{R} \times M)$ compatible with (or tamed by) $\hat{\lambda}$. To globalize the construction, one picks a primitive $\beta$ of the restriction of the Kähler form $\sigma$ on $\mathcal{M}_{k+1}$ to the open stratum $\mathcal{M}_{k+1}$ (which exists because the codimension two homology of $\mathcal{M}_{k+1}$ is generated by boundary divisors). The corresponding holomorphic curves will again satisfy Gromov-Hofer compactness. However, the computations in the following section would be more involved for this class of perturbations.

## 7 Linearization at punctured $K$-holomorphic maps

In this section we study the linearization of the universal Cauchy-Riemann operator at punctured $K$-holomorphic maps into almost complex manifolds with cylindrical ends. Our goal is to derive conditions under which this linearization is surjective. Let $(X, J)$ be an almost complex manifold with cylindrical ends $\mathbb{R} \times \overline{M}, \mathbb{R} \times \overline{M}$ modelled over contact manifolds (or manifolds with stable Hamiltonian structures) $\overline{M}, \overline{M}$. We assume that $\overline{M}$ and $\overline{M}$ are of Morse-Bott type.

**Fredholm setup.** We adapt the Fredholm setup for moduli spaces of punctured holomorphic curves in [11] to our situation. We consider first the case of a fixed domain and fix the following data:

- distinct points $\mathcal{Z} := (\overline{z}_1, \ldots, \overline{z}_p)$, $\mathcal{Z} := (\overline{z}_1, \ldots, \overline{z}_p)$ be on the Riemann sphere $S = S^2$, denoting by $\tilde{S} := S \setminus \{\overline{z}_1, \overline{z}_2\}$ the corresponding punctured sphere;
- holomorphic cylindrical coordinates $\mathbb{R}_+ \times S^2$ near the (positive resp. negative) punctures of $\tilde{S}$;
• vectors $\Gamma := (\Gamma_1, \ldots, \Gamma_p)$, $\underline{\Gamma} := (\underline{\Gamma}_1, \ldots, \underline{\Gamma}_q)$ of components $\Gamma_i = N_i/S^1$, $\underline{\Gamma}_i = \mathcal{N}_i/S^1$ in the manifolds of closed Reeb orbits in $\overline{M}$, $\overline{M}$.

• $m \in \mathbb{N}$ and $p > 1$ with $mp > 2$ and a small $\delta > 0$.

We denote by

$$\mathcal{B} := \mathcal{B}^{m,p,\delta}(X, \Gamma, \underline{\Gamma})$$

the space of all maps $f : \hat{S} \to X$ of Sobolev class $W^{m,p}_{\text{loc}}$ which converge near each positive puncture $\pi_i$ to a closed Reeb orbit $\pi_i$ in the family $\Gamma_i$ in the following sense (and similarly near the negative punctures): Suppose first that $\pi_i$ is simple. Pick a tubular neigbourhood $U \cong \mathbb{R}^{\dim \Gamma_i} \times \mathbb{R}/T \times \mathbb{R}^{2n-\dim \Gamma_i}$ of $\pi_i \cong \{0\} \times \mathbb{R}/T \times \{0\}$ in $\overline{M}$ such that $\pi_i \cap U \cong \mathbb{R}^{\dim \Gamma_i} \times \mathbb{R}/T \times \{0\}$. Then $f$ maps a neighbourhood $[\rho, \infty) \times S^1$ of $\pi_i$ to $\mathbb{R} \times U$ such that in the cylindrical coordinates $(s, t) \in [\rho, \infty) \times S^1$, the map $f = (a, v, \vartheta, w) : [\rho, \infty) \times S^1 \to \mathbb{R} \times \mathbb{R}^{\dim \Gamma_i} \times \mathbb{R}/T \times \mathbb{R}^{2n-\dim \Gamma_i}$ satisfies

$$\left( a(s, t) - \bar{\pi}_i, \bar{\vartheta}_i - T_i s, v(s, t), \vartheta(s, t) - \bar{\vartheta}_i - T_i t, w(s, t) \right) \in W^{m,p,\delta}([\rho, \infty) \times S^1, \mathbb{R} \times \mathbb{R}^{\dim \Gamma_i} \times \mathbb{R} \times \mathbb{R}^{2n-\dim \Gamma_i})$$

for constants $\bar{\pi}_i \in \mathbb{R}$, $\bar{\vartheta}_i \in S^1$. Here $W^{m,p,\delta}([\rho, \infty) \times S^1, \mathbb{R}^{2n})$ denotes the weighted Sobolev space of functions $g$ such that $e^{\delta s} g(s, t)$ belongs to $W^{m,p}$. An open neighborhood of $f$ is given by maps $f' : \hat{S} \to X$ of Sobolev class $W^{m,p}_{\text{loc}}$, which map a neighborhood $[\rho, \infty) \times S^1$ of $\pi_i$ to $\mathbb{R} \times U$ such that

$$\left( a'(s, t) - \bar{\pi}_i, \bar{\vartheta}_i - T_i s, v'(s, t) - \bar{\vartheta}_i - T_i t, \vartheta'(s, t) - \bar{\vartheta}_i - T_i t, w'(s, t) \right) \in W^{m,p,\delta}([\rho, \infty) \times S^1, \mathbb{R} \times \mathbb{R}^{\dim \Gamma_i} \times \mathbb{R} \times \mathbb{R}^{2n-\dim \Gamma_i})$$

for constants $\bar{\pi}_i \in \mathbb{R}$, $\bar{\vartheta}_i \in S^1$, and $\bar{\vartheta}_i \in \mathbb{R}^{\dim \Gamma_i}$. The case of $d$-fold covered $\pi_i$ is reduced to the previous one by passing to the $d$-fold cover of the neighbourhood $U$, see [10]. Note that $\mathcal{B}$ is a Banach manifold with a bundle projection

$$\text{Ev} : \mathcal{B} \to \prod_{i=1}^{p} (\mathbb{R} \times \mathcal{N}_i) \times \prod_{j=1}^{p} (\mathbb{R} \times \mathcal{N}_j), \quad f \mapsto \{\bar{\pi}_i, \bar{\vartheta}_i, \bar{\pi}_j, \bar{\vartheta}_j\}.$$

**Remark 7.1.** The Banach manifold $\mathcal{B}$ does not depend on the chosen cylindrical coordinates near the punctures, but the bundle projection $\text{Ev}$ does. Its composition with the projections to the $\mathcal{N}_i$ and $\mathcal{N}_j$ depends on the choice of asymptotic directions at the punctures, and its composition with the projections to the $\Gamma_i$ and $\underline{\Gamma}_j$ is also independent of the asymptotic directions.

We will fix $J$ (and often suppress it from the notation), and vary $K$ in the space

$$\mathcal{K}_S \subset \mathcal{K}(\hat{S}, TX)$$
of perturbations as defined above satisfying \( \|K\| < 1 \). Recall that \( K \) is required to be \( \mathbb{R} \)-invariant outside a compact subset \( X_0 \subset X \) (and on all of \( X = \mathbb{R} \times M \) in the cylindrical case). Since \( K \) has compact support in \( \hat{S} \) by definition, \( K \)-holomorphic maps \( \hat{S} \to X \) are just \( J \)-holomorphic near the punctures. In view of the asymptotics in [34], each punctured \( K \)-holomorphic map \( f : \hat{S} \to X \) asymptotic to \( \Gamma, \Sigma \) belongs to \( B^{m,p,\delta} \) for \( \delta \) sufficiently small. Let

\[
\mathcal{E} := \mathcal{E}^{m-1,p,\delta} \to \mathcal{B}
\]

be the Banach bundle whose fibre at \( f \in \mathcal{B} \) is given by

\[
\mathcal{E}_f = W^{m-1,p,\delta}(\hat{S}, \Omega^{0,1}(f^*TX)).
\]

Then for \( K \in \mathcal{K}_{\hat{S}} \) the Cauchy-Riemann operator defines a smooth section

\[
\bar{\partial}_{J,K} : \mathcal{B} \to \mathcal{E}, \quad f \mapsto \bar{\partial}_{J,K} f.
\]

We denote its linearization by

\[
D_f : T_f \mathcal{B} \to \mathcal{E}_f.
\]

More generally, the universal Cauchy-Riemann operator is the section

\[
\bar{\partial} : \mathcal{B} \times \mathcal{K}_{\hat{S}} \to \mathcal{E}, \quad (f, K) \mapsto \bar{\partial}_{J,K} f.
\]

Its linearization

\[
T_f \mathcal{B} \oplus T_K \mathcal{K}_{\hat{S}} \to \mathcal{E}_f
\]

can be described as follows. Recall that \( f \) maps a neighbourhood \( [\rho, \infty) \) of the puncture \( \pi_i \) into \( \mathbb{R} \times U \) with \( U \) as defined above. Let \( \zeta_i \in \mathbb{R}^{\dim T_{\pi_i}} \times \mathbb{R}/T_{\pi_i}Z \) at \( (\pi_i, \pi_i, \pi_i) \). Pick a cutoff function \( \beta : \mathbb{R} \to [0, 1] \) with \( \beta(s) = 1 \) for \( s \geq \rho + 1 \) and \( \beta(s) = 0 \) for \( s \leq \rho \). Then \( \zeta_i \) extends to a section \( \beta(s) \zeta_i \) in \( f^*TX \) and we get an isomorphism

\[
\bigoplus_{i=1}^{\rho} \mathbb{R}^{\dim T_{\pi_i}+2} \oplus \bigoplus_{j=1}^{p} \mathbb{R}^{\dim \mathcal{L}_j+2} \oplus W^{m,p,\delta}(f^*TX) \to T_f \mathcal{B}, \quad (\zeta, g) \mapsto \hat{\zeta} + g,
\]

where for \( \zeta = (\zeta_i, \zeta_j) \) we have set

\[
\hat{\zeta} := \sum_i \beta(s) \zeta_i + \sum_j \beta(-s) \zeta_j.
\]

Under this isomorphism, the linearization of the universal Cauchy-Riemann operator at \((f, K)\) is given by

\[
\bigoplus_{i=1}^{\rho} \mathbb{R}^{\dim T_{\pi_i}+2} \oplus \bigoplus_{j=1}^{p} \mathbb{R}^{\dim \mathcal{L}_j+2} \oplus W^{m,p,\delta}(f^*TX) \oplus T_K \mathcal{K}_{\hat{S}} \to \mathcal{E}_f,
\]

\[
(\zeta, g, Y) \mapsto D_f g + D_f \hat{\zeta} + Y(z, f).
\]

The following lemma is an easy adaptation of the corresponding result in [35].
Lemma 7.2. Let \((X, J)\) be an almost complex manifold with cylindrical ends. Let \(f \in \mathcal{B} = \mathcal{B}^{m, p, \delta}(X, \Gamma, \Gamma)\) and \(K \in \mathcal{K}_S\) satisfy \(\bar{\partial}_J K f = 0\). Then the linearization of 
\[
(\bar{\partial}, \text{Ev}): \mathcal{B} \times \mathcal{K}_S \to \mathcal{E} \times \prod_{i=1}^{\eta} (\mathbb{R} \times \mathbb{N}_i) \times \prod_{j=1}^{\rho} (\mathbb{R} \times \mathbb{N}_j)
\]
at \((f, K)\) is surjective. More precisely, for any nonempty open subset \(U \subset \mathcal{S}\), the restriction of the linearization to the subspace
\[
T_f \mathcal{B} \oplus T_K K \subset T_f \mathcal{B} \oplus T_K K \mathcal{S}
\]
of sections with support in \(U\) is surjective.

Proof. Note that under the isomorphism above, the linearized evaluation map is simply the projection \((\zeta, g, Y) \mapsto \zeta\). Hence, in view of equation (10), the assertion of the lemma is equivalent to surjectivity of the map
\[
L : W^{m, p, \delta}(f^* TX) \oplus T_K K \subset T_f \mathcal{B} \oplus T_K K \mathcal{S}
\]
of sections with support in \(U\) is surjective.

The proof of surjectivity of \(L\) is completely analogous to the proof of [18, Lemma 4.1]. For the convenience of the reader, let us recall the argument. We first consider the case \(m = 1\). Suppose that \(\eta\) is an \(L^\delta\)-section orthogonal to the image of \(L\). Then \(\eta\) is smooth and satisfies the equations
\[
D^* f \eta = 0,
\]
\[
\langle Y, \eta \rangle_{L^2} = 0 \text{ for all } Y \in T_K K \mathcal{S}.
\]
The second equation implies \(\eta|_U \equiv 0\). Note that this also holds if \(f(U)\) is entirely contained in the cylindrical part (in which case we must choose \(Y\) to be \(\mathbb{R}\)-invariant) because we can still arbitrarily prescribe \(Y\) at one point and then cut it off in the domain \(U\). Since \(\rho\) satisfies unique continuation by the first equation, it must vanish identically on \(\mathcal{S}\). This proves surjectivity of \(L\) for \(m = 1\), and the general case follows by elliptic regularity.

Finally, let us record a lemma for (unperturbed) holomorphic cylinders that we will need in the proof of Theorem [122 (b)]. Let \(T^* T^n\) be the cotangent bundle of the \(n\)-torus, equipped with an almost complex structure \(J_\rho\) induced by the standard flat metric on \(T^n\),

\[
J_\rho : \frac{\partial}{\partial q_i} \mapsto -\rho(|p|) \frac{\partial}{\partial p_i}, \quad \frac{\partial}{\partial p_i} \mapsto \frac{\partial}{\partial q_i}.
\]

Here \((q_i, p_i), i = 1, \ldots, n\), are standard coordinates on \(T^* T^n \cong T^n \times \mathbb{R}^n\) and \(\rho : [0, \infty) \to (0, \infty)\) is a smooth function satisfying \(\rho(r) = 1\) near \(r = 0\) and \(\rho(r) = r\) for large \(r\).
Lemma 7.3. Let \( \sigma \in H_1(T^n; \mathbb{Z}) \) be the \( k \)-fold multiple of a primitive class in \( H_1(T^n; \mathbb{Z}) \). Then the moduli space \( \mathcal{M}_\sigma \) of \( J_\rho \)-holomorphic cylinders in \( T^*T^n \) with two positive punctures asymptotic to closed geodesics in the classes \( \pm \sigma \) is a smooth manifold of dimension \( 2n-2 \) whose image under the evaluation map is a \( k \)-fold covering of \( T^*T^n \).

Proof. Write \( \sigma = k\bar{\sigma} \) for a primitive class \( \bar{\sigma} \in H_1(T^n; \mathbb{Z}) \cong \mathbb{Z}^n \) and \( k \geq 1 \). Note that a linear coordinate change \( (q,p) \mapsto (Aq,Ap) \) does not change the form of \( J_\rho \) (only the norm \(|p|\) looks different in the new coordinates). So after such a coordinate change we may assume that \( \bar{\sigma} = (1,0,\ldots,0) \).

Note that for each constant \((\bar{q}',\bar{p}') = (\bar{q}_2,\ldots,\bar{q}_n,\bar{p}_2,\ldots,\bar{p}_n)\) the orbit cylinder

\[ C_{(\bar{q}',\bar{p}')} := \{ q_i = \bar{q}_i, p_i = \bar{p}_i, i = 2,\ldots,n \} \subset T^*T^n \]

is \( J_\rho \)-invariant, and these orbit cylinders foliate \( T^*T^n \). Each orbit cylinder is the image of a \( J_\rho \)-holomorphic \( k \)-fold covering map \( f: \mathbb{R} \times S^1 \to C_{(\bar{q}',\bar{p}')} \), which is unique up to automorphism of the domain. To see this, let us write out the equations for \( J_\rho \)-holomorphicity of a map \( f = (q,p): \mathbb{R} \times S^1 \to T^*T^n \):

\[
\partial_t q_i = \frac{1}{\rho} \partial_s p_i, \quad \partial_s q_i = -\frac{1}{\rho} \partial_t p_i, \quad i = 1,\ldots,n.
\]

So we obtain a \( J_\rho \)-holomorphic \( k \)-fold covering map \( f = (q,p): \mathbb{R} \times S^1 \to C_{(\bar{q}',\bar{p}')} \) by setting \( q_i = \bar{q}_i, p_i = \bar{p}_i \) for \( i = 2,\ldots,n \), \( q_1(s,t) := kt + \bar{q}_1 \), and defining \( p_1(s,t) = p_1(s) \) to be the solution of \( \partial_t p_1 = \rho(|(p_1(s),\bar{p}')|) \) with \( p_1(0) = \bar{p}_1 \).

The conditions on \( \rho \) ensure that \( p_1 \) defines a diffeomorphism \( \mathbb{R} \to \mathbb{R} \). Hence \( f \) is a \( J_\rho \)-holomorphic cylinder with two positive punctures asymptotic to the closed geodesics \( t \mapsto (\pm kt, \bar{q}') \) in the classes \( \pm \sigma \). Varying the constants \((\bar{q}',\bar{p}')\), these cylinders form a smooth manifold of dimension \( 2n-2 \) whose image under the evaluation map is a \( k \)-fold covering of \( T^*T^n \). So we are done once we show that the moduli space \( \mathcal{M}_\sigma \) consists only of the \( k \)-fold orbit cylinders.

Consider an arbitrary \( J_\rho \)-holomorphic cylinder \( f = (q,p): \mathbb{R} \times S^1 \to T^*T^n \) belonging to \( \mathcal{M}_\sigma \). It is asymptotic at its punctures to closed geodesics \( t \mapsto (\pm kt, q^\pm) \) for constants \( q^\pm = (q^\pm_1,\ldots,q^\pm_n) \). Suppose first that \( q^+ \neq q^- \), say \( q^+_n \neq q^-_n \). Pick a point \( s^* \in \mathbb{R} \) at which \( f(s^*,0) = (q^*,p^*) \) with \( q^*_n \neq q^+_n \). The codimension 2 hypersurface

\[ Y := \{ q_n = q^*_n, p_n = p^*_n \} = \bigcup_{q_n=q^*_n,p_n=p^*_n} C_{(\bar{q}',\bar{p}')} \]

is a union of orbit cylinders and thus \( J_\rho \)-invariant. Since \( Y \) lies over the set \( \{ q_n = q^*_n \} \) and \( f \) lies over the set \( \{ q_n = q^\pm_n \} \) at infinity, the image of \( f \) is disjoint from \( Y \) at infinity and the signed count of intersection points gives a well-defined homological intersection number \([f] \cdot [Y] \in \mathbb{Z} \). Since \( f \) and \( Y \) intersect by construction and are both \( J_\rho \)-holomorphic, positivity of intersection implies \([f] \cdot [Y] > 0 \). On the other hand, we can deform \( Y = Y_0 \) through the hypersurfaces \( Y_r := \{ q_n = q^*_n, p_n = p^*_n + r \} \) to \( Y_R \) for large \( R \). The preceding
argument shows that intersections with $f$ remain in a compact region during this deformation, hence $[f] \cdot [Y_R] = [f] \cdot [Y] > 0$. Now on each compact subset $K \subset \mathbb{R} \times S^1$, the component $p_n$ of $f$ is uniformly bounded and thus not equal to $p_n + R$ for large $R$. Hence for large $R$, intersections of $f$ with $Y_R$ can only occur near infinity in $\mathbb{R} \times S^1$, where they are excluded by the difference of the components $q_n$. So we conclude $[f] \cdot [Y_R] = 0$ for large $R$, contradicting our previous assertion. This proves that the constants $q^\pm$ must coincide, i.e., $f$ is asymptotic at its punctures to closed geodesics $t \mapsto (\pm kt, \bar{q}^i)$ for a constant $\bar{q}^i = (\bar{q}_2, \ldots, \bar{q}_n)$.

The asymptotic behaviour shows that for $i = 2, \ldots, n$ the component $q_i$ of $f$ defines a function $q_i : \mathbb{R} \times S^1 \to \mathbb{R}$. Adding up partial derivatives of equations (11), we see that $q_i$ satisfies the equation

$$\Delta q_i + \frac{1}{\rho}(\partial_s \rho \partial_t q_i + \partial_t \rho \partial_t q_i) = 0.$$ 

Since $q_i(s, t) \to \bar{q}_i$ as $s \to \pm \infty$, the maximum principle implies that $q_i$ is constant equal to $\bar{q}_i$. Equation (11) shows that $p_i$ is constant as well, hence $f$ defines a $k$-fold covered parametrization of an orbit cylinder $C(\bar{q}, \bar{\rho})$.

**Broken punctured $K$-holomorphic maps.** Next we generalize the preceding discussion to broken punctured $K$-holomorphic maps. We fix the following data:

- a split manifold $X^* = X^+ \amalg (\mathbb{R} \times M) \amalg X^-$ with almost complex structure $J^* = (J^+, J_M, J^-)$, where $J_M$ is $\mathbb{R}$-invariant and $J^\pm$ agree with $J_M$ outside compact subsets $X^\pm_0 \subset X^\pm$;

- a stable nodal curve (of genus zero) $z$ modelled over the $k$-labelled tree $T$;

- for each vertex $\alpha \in T$ a component $X_\alpha$ of $X^*$;

- asymptotic data $\Gamma^* = (\{\Gamma_{\alpha}\}_{\alpha \in \beta}, \{\Gamma_i\}_{1 \leq i \leq k})$, where each $\Gamma_{\alpha\beta}$ (resp. $\Gamma_i$) is either a Morse-Bott component of closed Reeb orbits in $M$ (case i), or the manifold $X_\alpha$ (resp. $X_\alpha$) (case ii).

We assume that all the Morse-Bott families $\Gamma_T$ of closed Reeb orbits in $M$ are manifolds (not orbifolds). We denote by $B_T(X^*, \Gamma^*) = \prod_{\alpha \in T} B_\alpha$ the space of collections $f = (f_\alpha)_{\alpha \in T}$ of maps $f_\alpha : \tilde{S}_\alpha \to X_\alpha$ of Sobolev class $W^{m,p,\delta}$ which near each puncture are asymptotic to the corresponding Morse-Bott component of closed Reeb orbits in case (i), or extend smoothly over the puncture in case (ii). Let $\mathcal{K}(\mathcal{M}_{k+1}, TX^*)$ be the space of coherent perturbations in the split manifold $X^*$ as defined in Section [6] that satisfy $\|K\| < 1$. Then we get a universal Cauchy-Riemann operator

$$\bar{\partial} : B_T(X^*, \Gamma^*) \times \mathcal{K}(\mathcal{M}_{k+1}, TX^*) \to \mathcal{E}, \quad (f, K) \mapsto \bar{\partial}_{J,K}\alpha f = (\bar{\partial}_{J,K}\alpha f_\alpha)_{\alpha \in T}$$

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with values in a suitable bundle $E_T = \prod_{\alpha \in T} E_\alpha \to B_T = \prod_{\alpha \in T} B_\alpha$. Moreover, we have an obvious evaluation map

$$\text{Ev} : B_T(X^*, \Gamma^*) \to \prod_{\alpha \in E_\beta} \Gamma_{\alpha \beta} \times \prod_{1 \leq i \leq k} \Gamma_i =: \prod_{\alpha \in T} \Gamma_\alpha,$$

where $\Gamma_\alpha$ denotes the products of the factors associated to the special points on $\alpha$.

**Proposition 7.4.** Let $X^*$ and $z$ be as above. Then the linearization of $\overline{\partial} \times \text{Ev} : B_T(X^*, \Gamma^*) \times K(\overline{M}_{k+1}, TX^*) \to E_T \times \prod_{\alpha \in E_\beta} \Gamma_{\alpha \beta} \times \prod_{1 \leq i \leq k} \Gamma_i$

at each $(f, K)$ with $\overline{\partial}_{J,K} f = 0$ is surjective.

**Proof.** Removing the special points from the nodal surface $\Sigma_z$ yields the disconnected punctured surface $\Sigma_z^* = \bigsqcup_{\alpha \in T} \dot{S}_\alpha$. So the space of sections with compact support over $\Sigma_z^*$ is the direct product $K(\Sigma_z^*, TX^*) = \prod_{\alpha \in T} K(\dot{S}_\alpha, TX^*)$.

Combined with the preceding discussion, this shows that the linearization at $(f, K)$ of the map

$$\overline{\partial} \times \text{Ev} : B_T(X^*, \Gamma^*) \times K(\Sigma_z^*, TX^*) \to E_T \times \prod_{\alpha \in E_\beta} \Gamma_{\alpha \beta} \times \prod_{1 \leq i \leq k} \Gamma_i$$

splits as a product of factors corresponding to $\alpha \in T$ and is therefore surjective by Lemma 7.2. In view of Lemma 6.2, this proves the proposition. \qed

**Remark 7.5.** Proposition 7.4 continues to hold if we replace the evaluation map at some removable puncture by an $\ell$-jet evaluation map (picking the Sobolev class so high that it embeds into $C^\ell$). This follows by combining the preceding proof with the proof of [18, Proposition 6.9].

Proposition 7.4 leads to regularity of moduli spaces by a standard argument. We keep the fixed tree $T$ and data $X_\alpha, \Gamma^*$, but let $z$ vary in the stratum $M_T$ of Deligne-Mumford space. We denote by $\Delta^E \subset \prod_{\alpha \in E_\beta} \Gamma_{\alpha \beta}$ the product of the diagonals in $\Gamma_{\alpha \beta} \times \Gamma_{\beta \alpha}$ for $\alpha \in E_\beta$. Moreover, we fix a submanifold $Z \subset \prod_{1 \leq i \leq k} \Gamma_i$.

By Proposition 7.4, the map

$$\overline{\partial} \times \text{Ev} : M_T \times B(X^*, \Gamma^*) \times K(M_{k+1}, TX^*) \to E \times \prod_{\alpha \in E_\beta} \Gamma_{\alpha \beta} \times \prod_{1 \leq i \leq k} \Gamma_i$$

is transverse to the submanifold $0 E \times \Delta^E \times Z$, where $0 E$ denotes the zero section of $E$. The implicit function theorem implies that $(\overline{\partial} \times \text{Ev})^{-1}(0 E \times \Delta^E \times Z)$ is a Banach manifold. Pick $K \in K(M_{1+\ell+1}, TX^*)$ to be a regular value of
the projection onto the third factor for this space (this exists by the Sard-Smale theorem). Moreover, we can pick $K$ to be a regular value also for the space with $Z$ replaced by $\prod_{1 \leq i \leq k} \Gamma_i$, as well as for all the corresponding spaces associated to individual components of $T$. Then all components of punctured $K$-holomorphic spheres in the corresponding moduli space $\mathcal{M}^{A,K}_T(X^*, \Gamma^*)$ appear in manifolds of the expected dimensions, and the evaluation maps at the edges are transverse to the diagonal. Hence the total moduli space (taking preimages of the diagonals under the edge evaluation maps) is a smooth manifold of the expected dimension, and the evaluation map at the marked points is transverse to $Z$. To work out the dimensions, we write

$$
k = \overline{p} + \underline{p} + m,
$$

$$
n_\alpha = \overline{p}_\alpha + \underline{p}_\alpha + m_\alpha,
$$

where $\overline{p}, \underline{p}, m$ is the number of marked points corresponding to positive/negative punctures and marked points, respectively, and similarly for the number $n_\alpha$ of special points on the component $\alpha$. We also write

$$
|T| - 1 = c(T) = p(T) + m(T),
$$

where $p(T), m(T)$ is the number of edges corresponding to punctures and nodes, respectively. We have the obvious relations

$$
\sum_{\alpha \in T} (\overline{p}_\alpha + \underline{p}_\alpha) = \overline{p} + \underline{p} + 2p(T), \quad \sum_{\alpha \in T} m_\alpha = m + 2m(T), \quad \sum_{\alpha \in T} A_\alpha = A,
$$

where $A_\alpha$ denotes the homology class of the map $f_\alpha$. According to equation (3), the dimension of the moduli space $\mathcal{M}_\alpha$ of punctured holomorphic spheres associated to the component $\alpha$ is now (in the obvious notation) given by

$$
\dim \mathcal{M}_\alpha = (n - 3)\left(2 - \overline{p}_\alpha - \underline{p}_\alpha\right) + 2c_1(A_\alpha) + 2m_\alpha
$$

$$
+ \sum_{i=1}^{\overline{p}} (\text{CZ}(\overline{\Gamma}_i) + \dim \overline{\Gamma}_i) - \sum_{j=1}^{\underline{p}} \text{CZ}(\underline{\Gamma}_j).
$$

Taking preimages of the diagonals at the edges yields the dimension formula

$$
\dim \mathcal{M}^{A,K}_T(X^*, \Gamma^*) = \sum_{\alpha \in T} \dim \mathcal{M}_\alpha - \frac{1}{2} \sum_{\alpha \in E} \dim \Gamma_{\alpha\beta}
$$

$$
= (n - 3)(2|T| - 2p(T) - \overline{p} - \underline{p}) + 2c_1(A) + 2m + 4m(T)
$$

$$
+ \sum_{i=1}^{\overline{p}} (\text{CZ}(\overline{\Gamma}_i) + \dim \overline{\Gamma}_i) - \sum_{j=1}^{\underline{p}} \text{CZ}(\underline{\Gamma}_j) - 2nm(T)
$$

$$
= (n - 3)(2 - \overline{p} - \underline{p}) + 2c_1(A) + 2m
$$

$$
+ \sum_{i=1}^{\overline{p}} (\text{CZ}(\overline{\Gamma}_i) + \dim \overline{\Gamma}_i) - \sum_{j=1}^{\underline{p}} \text{CZ}(\underline{\Gamma}_j) - 2m(T).
$$

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Here in the first line the factor $1/2$ appears because each edge contributes two terms to the sum over $\alpha E \beta$. For the second equality we use the dimension formula for $M_\alpha$ and the relations above. Here the terms $\text{CZ}(\Gamma_i^\alpha)$ and $\text{CZ}(\Gamma_j^\alpha)$, as well as the dimensions of the Bott manifolds, corresponding to edges associated to punctures drop out and we have

$$\sum_{\alpha \in T} \sum_{i=1}^{\bar{p}} \dim \Gamma_i^\alpha - \frac{1}{2} \sum_{\alpha E \beta} \dim \Gamma_{\alpha \beta} = \sum_{i=1}^{\bar{p}} \dim \Gamma_i - 2nm(T).$$

The third equality follows from the computation

$$(n-3)(2|T| - 2p(T)) + 4m(T) - 2nm(T)$$

$$= (n-3)(2 + 2m(T)) + 4m(T) - 2nm(T) = 2(n-3) - 2m(T).$$

So we have shown

**Corollary 7.6.** For generic $K \in \mathcal{K}(\mathcal{M}_{1+\ell+1}, TX^*)$, all components of punctured $K$-holomorphic spheres in the moduli space $\mathcal{M}_T^{A,K}(X^*, \Gamma^*)$ are manifolds of the expected dimensions, and the evaluation maps at the edges are transverse to the diagonal. Hence the total moduli space is a manifold of the expected dimension

$$\dim \mathcal{M}_T^{A,K}(X^*, \Gamma^*) = (n-3)(2 - \bar{p} - \bar{q}) + 2c_1(A) + 2m$$

$$+ \sum_{i=1}^{\bar{p}} (\text{CZ}(\Gamma_i) + \dim \Gamma_i) - \sum_{j=1}^{\bar{p}} \text{CZ}(\Gamma_j) - 2m(T).$$

Moreover, we can arrange for the evaluation map on this moduli space at the marked points to be transverse to any given submanifold $Z \subset \prod_{1 \leq i \leq k} \Gamma_i$. \[\square\]

8 **Symplectic hypersurfaces in the complement of Lagrangian submanifolds**

Consider a closed Lagrangian submanifold $L$ in a closed symplectic manifold $(X^{2n}, \omega)$. Suppose that $\omega$ represents an integral relative cohomology class $[\omega] \in H^2(X, L; \mathbb{Z})$. Denote by $\text{Tor}H_1(L; \mathbb{Z})$ the torsion subgroup of $H_1(L; \mathbb{Z})$ and by $|\text{Tor}H_1(L; \mathbb{Z})| \in \mathbb{N}$ its order. Fix a compatible almost complex structure $J$ on $X$. We say that a submanifold $Y \subset X$ is *approximately holomorphic* if its tangent space at each point differs from a complex subspace by an angle at most $CD^{-1/2}$, where $C$ is a constant independent of $D$. We say that two submanifolds $Y_0, Y_1 \subset X$ intersect $\varepsilon$-*transversally* if they intersect transversally and at each intersection point their tangent spaces form an angle at least $\varepsilon$ for an $\varepsilon > 0$ independent of $D$. The following theorem is an easy adaptation of the main result in [5].
Theorem 8.1 (Auroux–Gayet–Mohsen [5]). For each sufficiently large integer multiple $D$ of $|\text{Tor}H_1(L;\mathbb{Z})|$, there exists an approximately $J$-holomorphic closed codimension two submanifold $Y \subset X \setminus L$ whose Poincaré dual in $H^2(X,L;\mathbb{Z})$ equals $D[\omega]$.

Remark 8.2. It follows directly from the definitions that the submanifold $Y$ constructed in Theorem 8.1 is symplectic, and $\bar{J}$-holomorphic for a compatible almost complex structure $\bar{J}$ arbitrarily $C^0$-close to $J$.

The proof of Theorem 8.1 is based on the following observation. Consider a compact oriented surface $\Sigma$ with (possibly empty) boundary $\partial \Sigma$. Let $E \to \Sigma$ be a Hermitian line bundle together with a unitary trivialization $\tau: \partial \Sigma \times \mathbb{C} \to E|_{\partial \Sigma}$ over the boundary. We identify $\tau$ with the induced section $z \mapsto \tau(z,1)$ of the unit circle bundle $P \subset E$ over $\partial \Sigma$. Define the relative degree of $E$ with respect to the trivialization $\tau$ as

$$\deg(E, \tau) := \sum_{p \in s^{-1}(0)} \text{ind}_p(s)$$

for any section $s$ in $E$ which is transverse to the zero section and coincides with $\tau$ along $\partial \Sigma$. The following relative version of the Chern-Weil theorem is proved exactly like the relative Gauss-Bonnet theorem (see e.g. [23]), which is the special case $E = T\Sigma$.

Lemma 8.3. Let $A \in \Omega^1(P, i\mathbb{R})$ be a unitary connection 1-form on $E$ with curvature $F_A \in \Omega^2(\Sigma, i\mathbb{R})$. Then

$$\deg(E, \tau) = \frac{i}{2\pi} \left( \int_{\Sigma} F_A + \int_{\partial \Sigma} \tau^* A \right).$$

Note that the quantity $\int_{\partial \Sigma} \tau^* A$ measures the total deviation of the section $\tau$ from being parallel. In particular, if $\tau$ is parallel we have

$$\deg(E, \tau) = \frac{i}{2\pi} \int_{\Sigma} F_A.$$

Proof of Theorem 8.1. We follow the proof in [5] (for compatibility with the notation in [5], we will replace $D$ by $k$). Fix a Hermitian line bundle $E \to X$ with first Chern class $[\omega]$ and and a Hermitian connection $A$ on $E$ with curvature form $F_A = -2\pi i\omega$. Let $k$ be an integer multiple of $|\text{Tor}H_1(L;\mathbb{Z})|$. The induced connection $A_k$ on $E^{\otimes k}$ has curvature $F_{A_k} = -2\pi ik\omega$.

We first refine Lemma 2 in [5]. Let $\tau_k$ be unitary section of $E^{\otimes k}|_L$. For any map $f: (\Sigma, \partial \Sigma) \to (X, L)$ from a compact surface $\Sigma$, Lemma 8.3 yields

$$\deg(f^* E^{\otimes k}, f^* \tau_k) = k \int_{\Sigma} \omega + \frac{i}{2\pi} \int_{\partial \Sigma} f^* \tau_k^* A_k.$$

In particular, the holonomy $\frac{i}{2\pi} \int_{\partial \Sigma} f^* \tau_k^* A_k$ is integral, so we can make it zero by a suitable change of $\tau_k$. Applying this to all such maps $f$, we find a unitary
section \( \tau_k \) such that the closed 1-form \( \tau_k^* A_k \) vanishes on the image of the map \( \partial : H_2(X, L; \mathbb{Z}) \to H_1(L; \mathbb{Z}) \). Arguing as in [5], we arrange for \( \tau_k \) to satisfy the estimates in Lemma 2.

With this modification, the proofs of Lemmas 3 and 4, and thus of Theorem 2, in [5] work as before. Thus we find asymptotically holomorphic sections \( s_k \) of \( E^{\otimes k} \) that are uniformly transverse to zero and do not vanish on \( L \). Moreover, as remarked in Section 3.3 of [5], the sections satisfy \( |\arg(s_k/\tau_k)| \leq \pi/3 \) on \( L \). Hence we can deform \( s_k \) near \( L \) to arrange \( s_k = \tau_k \) on \( L \), without affecting its zero set \( Y_k = s^{-1}(0) \). But then for any map \( f : (\Sigma, \partial \Sigma) \to (X, L) \) from a compact surface we have

\[
\deg(f^* E^{\otimes k}; f^* \tau_k) = \sum_{p \in s_k^{-1}(0)} \text{ind}_p(s_k) = [Y_k] \cdot [f]
\]

by definition of the degree. On the other hand, by the choice of \( \tau_k \) above and Lemma 8.3, this degree equals \( k \int_{\Sigma} \omega \), and Theorem 8.1 is proved. \( \square \)

9 Proof of transversality

The proof of transversality now follows the scheme in [13]. Consider a closed symplectic manifold \( (X^{2n}, \omega) \) and a closed Lagrangian submanifold \( L \subset X \). After a perturbation of \( \omega \), we may assume that it represents a rational relative cohomology class \([\omega] \in H^2(X, L; \mathbb{Q})\), so \( D_0[\omega] \in H^2(X, L; \mathbb{Z}) \) for some positive integer \( D_0 \). We fix an energy \( E > 0 \), larger than the symplectic area of the holomorphic curves we want to consider (\( \pi \) in the case of Theorem 1.1 (b), and the area of the relevant Gromov-Witten invariant in the case of Theorem 1.18).

We fix some \( \omega \)-compatible almost complex structure \( J_0 \) on \( X \). Since the taming condition is open with respect to the \( C^0 \)-norm \( \| \cdot \| \) on \( X \), there exists a constant \( \theta_1 > 0 \) with the following property: for every symplectic form \( \bar{\omega} \) and every almost complex structure \( J \) satisfying \( \| \bar{\omega} - \omega \| < \theta_1 \) and \( \| J - J_0 \| < \theta_1 \), the symplectic form \( \bar{\omega} \) tames \( J \).

Let us pick a tubular neighbourhood \( U \) of \( L \). Since \( H^2(X, L; \mathbb{R}) \cong H^2(X, U; \mathbb{R}) \), we can represent a basis of \( H^2(X, L; \mathbb{R}) \) by closed forms with compact support in \( X \setminus U \). It follows that there exists an open neighbourhood \( \Omega \) of \( [\omega] \) in \( H^2(X, L; \mathbb{R}) \) represented by symplectic forms \( \bar{\omega} \) which agree with \( \omega \) on \( U \) and satisfy \( \| \bar{\omega} - \omega \| < \theta_1 \). Thus for every almost complex structure \( J \) with \( \| J - J_0 \| < \theta_1 \) and every \( J \)-holomorphic map \( f : (\Sigma, \partial \Sigma) \to (X, L) \) from a compact Riemann surface with boundary,

\[
\langle [\bar{\omega}], [f] \rangle = \int_{\Sigma} f^* \bar{\omega} \geq 0 \text{ for all } [\bar{\omega}] \in \Omega.
\]

This condition together with an energy bound \( \langle [\omega], [f] \rangle < E \) defines the bounded intersection of a cone with a half-space. This shows that there is only a finite set \( A_E \subset H_2(X, L; \mathbb{Z}) \) of homology classes that can be represented by such \( J \)-holomorphic maps \( f \) with \( \| J - J_0 \| < \theta_1 \) and \( \langle [\omega], [f] \rangle < E \).
Note that (12) continues to hold for every $J^+$-holomorphic map $f: \Sigma \to X \setminus L$ from a closed Riemann surface (of genus $g$) with $p$ punctures asymptotic to closed geodesics $\gamma_1, \ldots, \gamma_p$ on $L$, where $\|J - J_0\| < \theta_1$ and $J^+$ is obtained by deforming $J$ inside $U$ to a cylindrical almost complex structure tamed by $\omega$. Such holomorphic curves appear in moduli spaces of expected dimension
\[
(n - 3)(2 - 2g - p) - \sum_{i=1}^{p} CZ(\gamma_i) = (n - 3)(2 - 2g - p) + \mu([f]) - \sum_{i=1}^{p} \text{ind}(\gamma_i)
\leq (n - 3)(2 - 2g - p) + \mu([f]).
\]
Since $[f]$ belongs to the finite set $A_E$ from above, restricting to $g = 0$ and $p = 1$ we have shown

**Lemma 9.1.** There exists a constant $D_E > 0$ with the following property: for every $\omega$-tamed cylindrical almost complex structure $J^+$ satisfying $\|J^+ - J_0\| < \theta_1$ outside $U$, every nonempty moduli space of $J^+$-holomorphic planes in $X \setminus L$ asymptotic to $L$ of energy $< E$ has expected dimension $\leq D_E$. □

By Theorem 8.1 for each sufficiently large integer multiple $D$ of $D_0|\text{Tor}H_1(L; \mathbb{Z})|$, there exists an approximately $J_0$-holomorphic closed codimension two submanifold $Y \subset X \setminus L$ whose Poincaré dual in $H^2(X, L; \mathbb{Z})$ equals $D[\omega]$. By choosing $D$ large, we can make the maximal angle between a tangent space to $Y$ and a $J_0$-complex subspace (the “Kähler angle”) smaller than any given constant $\theta_2 > 0$. As shown in [18], for $\theta_2$ sufficiently small there exists an almost complex structure $J$ with $\|J - J_0\| < \theta_1$ such that $Y$ is $J$-complex.

We fix a Riemannian metric on $L$ such that the unit disk cotangent bundle $D^*L$ embeds symplectically into $(X \setminus Y) \cap U$. We deform the $\omega$-compatible almost complex structure $J$ inside $U$ to make it cylindrical near $M = \partial D^*L$, so it induces asymptotically cylindrical almost complex structures $J^+$ on $X^+ = X \setminus L$, $J^-$ on $X^- = T^*L$, and $J_M$ on $\mathbb{R} \times M$.

We fix a point $x \in L$ and the germ of a $J$-complex hypersurface $Z$ at $x$. We perturb $J$ such that all moduli spaces of simple punctured holomorphic curves for $J^\pm$ and $J_M$ (including tangency conditions to $Y$ and $Z$) as well as $J|_Y$ are transversely cut out. By [18] Proposition 6.9], tangency of order $\ell$ to $Y$ (at some point which is not fixed) is a condition of codimension $2\ell$ for simple curves. In view of Lemma 9.1 the maximal order of tangency to $Y$ of simple $J^+$-holomorphic planes in $X \setminus L$ of energy $< E$ is therefore bounded by $D_E/2$.

Since the homology classes of $J^+$-holomorphic planes in $X \setminus L$ belong to the finite set $A_E \subset H_2(X, L; \mathbb{Z})$ from above, their multiplicity is bounded by some constant $M_E$ and we have shown

**Lemma 9.2.** The maximal order of tangency to $Y$ of (not necessarily simple) $J^+$-holomorphic planes in $X \setminus L$ of energy $< E$ is bounded by $(M_E + D_E)/2$. □

Since the first Chern class of the normal bundle $N(Y, X)$ is represented by $D[\omega]|_Y$, the adjunction formula
\[
c_1(TY) = c_1(TX|_Y) - c_1(N(Y, X)) = c_1(TX|_Y) - D[\omega]|_Y
\]

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shows that raising the degree $D$ of $Y$ lowers the expected dimension of moduli spaces of nonconstant $J$-holomorphic spheres in $Y$. So for $D$ sufficiently large and energy bounded by $E$, these do not occur (because $J|_{Y}$ is regular on simple curves). Next, observe that the intersection number with $Y$ of a nonconstant punctured $J$-holomorphic curve $f : \Sigma \to X \setminus L$ satisfies

$$[f] \cdot [Y] = D \int_{\Sigma} f^* \omega \geq D/D_0$$

because $D_0[\omega] \in H_2(X,L;\mathbb{Z})$. So the intersection number is positive and increases with $D$. Since a point of tangency of order $\ell$ to $Y$ contributes $(\ell + 1)$ to the intersection number (see [18]), and the maximal order of tangency for holomorphic planes is bounded by Lemma 9.2, we have shown

**Lemma 9.3.** For $D$ sufficiently large (depending only on $\omega$, $J_0$ and $E$) and $J^+$ as above the following holds:

(i) all $J^+$-holomorphic spheres of energy $< E$ in $Y$ are constant;

(ii) all nonconstant punctured $J^+$-holomorphic curves in $X \setminus L$ intersect $Y$;

(iii) all nonconstant $J^+$-holomorphic planes of energy $< E$ in $X \setminus L$ intersect $Y$ in at least 2 distinct points in the domain.

Consider now the homology class $A \in H_2(X;\mathbb{Z})$ of holomorphic spheres we want to split along $M$, where $\omega(A) < E$. Thus $A$ is the class of a complex line in $\mathbb{CP}^n$ in the case of Theorem 1.2 (b), and a class with a nonvanishing genus zero Gromov-Witten invariant through a point (plus additional constraints) in the case of Theorem 1.18. Note that the intersection number of $J$-holomorphic spheres in the class $A$ with the hypersurface $Y$ equals

$$\ell := [Y] \cdot A = D \omega(A).$$

Consider the space of coherent perturbations $K(\tilde{M}_{k+\ell+1},TX^*)$ in the bundle $(TX^*,J)$ over the split manifold $X^* = X^+ \amalg (\mathbb{R} \times M) \amalg X^-$, as defined in Section 6. Here $k = 1$ in the first case, and $k$ is the number of constraints $(Z_1 = x, Z_2, \ldots, Z_k)$ for the Gromov-Witten invariant in the second case. Recall that $K \in K(\tilde{M}_{k+\ell+1},TX^*)$ is required to satisfy $\|K\| < 1$, where the norm of $K$ is defined with respect to the metrics $\sigma(\cdot,j)$ and $\omega(\cdot,J)$ as in Section 6. In the first case, let

$$K(\tilde{M}_{1+\ell+1},TX^*;Y, x) \subset K(\tilde{M}_{1+\ell+1},TX^*)$$

be the subset of $K$ which vanish near $Y$ as well as near the point $x$. In the second case, let

$$K(\tilde{M}_{k+\ell+1},TX^*;Y) \subset K(\tilde{M}_{k+\ell+1},TX^*)$$

be the subset of $K$ which vanish near $Y$. Pick such a $K$. In the first case, let

$$\mathcal{M} = \mathcal{M}_{1+\ell}([\mathbb{CP}^1],J,K;x,Z,Y)$$

be the space of coherent perturbations...
be the moduli space of broken \((J, K)\)-holomorphic spheres in \(X^*\) in the class \([\mathbb{CP}^1]\) with \(\ell\) marked points mapped to \(Y\), and 1 marked point passing through \(x\) and tangent of order \(n - 1\) to \(Z\) at \(x\). In the second case, let

\[
\tilde{M} = \tilde{M}_{k+\ell}(A, J, K; Z_1, \ldots, Z_k, Y)
\]

be the moduli space of broken \((J, K)\)-holomorphic spheres in \(X^*\) in the class \(A\) with \(\ell\) marked points mapped to \(Y\), and \(k\) marked points mapped to \(Z_1, \ldots, Z_k\).

Note that \(M\) has the same expected dimension as the corresponding moduli space without the \(\ell\) additional marked points mapped to \(Y\). By an argument using positivity of intersections as in [18], Lemma 9.3 implies

**Proposition 9.4.** For \(D\) sufficiently large (depending only on \(\omega, J_0\) and \(E\)), \((J, K)\) as above, and any broken holomorphic curve \(f \in M\) the following holds:

(i) all components of \(f\) contained in \(Y\) are constant;

(ii) all components of \(f\) in \(X \setminus L\) have a stable domain.

Now we can conclude the transversality part of the proofs of Theorems 1.2(b) and 1.18. In view of the preceding discussion, it suffices to show that all the components of broken holomorphic spheres in \(\tilde{M}\) belong to moduli spaces that are manifolds of the expected dimensions, and in the first case the asymptotic evaluation maps are transverse to the diagonal.

**Proof of transversality for Theorem 1.2(b).** Here \(L\) is an \(n\)-dimensional torus with a multiple of the standard flat metric. Hence there are no holomorphic planes in \(T^*L\) and \(\mathbb{R} \times M\), and the only holomorphic cylinders in \(T^*L\) and \(\mathbb{R} \times M\) are orbit cylinders in \(\mathbb{R} \times M\), and holomorphic cylinders in \(T^*L\) with two positive punctures. According to Proposition 9.4 all other punctured holomorphic spheres in \(T^*L, \mathbb{R} \times M\) or \(T^*L\) of energy \(< E\) have a stable domain.

Consider a stable \((J, K)\)-holomorphic map \((z, f) \in \tilde{M}\). Thus \(z\) is a nodal curve modelled over a (not necessarily stable) \((1 + \ell)\)-labelled tree \(T\) such that the map \(f_\alpha : \hat{S}_\alpha \to X^*\) is nonconstant over each unstable component \(\alpha \in T\). As in Section 7 we consider the linearization \(L\) of

\[
\bar{\partial} \times \text{Ev} : B(X^*, \Gamma^*) \times \mathcal{K}(\tilde{M}_{1+\ell+1}, TX^*; Y, x) \to \mathcal{E} \times \prod_{\alpha \in \beta} \Gamma_{\alpha\beta} \times \prod_{1 \leq i \leq 1+\ell} \Gamma_i
\]

at \((f, K)\). Here the normal \((n - 1)\)-jet evaluation map at \(z_1\) takes values in \(\Gamma_1 = (T_xT^*L/T_xZ)^n\), while the evaluation maps at the points \(z_i\) take values in \(\Gamma_i = X \setminus L\) for \(i = 2, \ldots, \ell + 1\). The points \(z_{\alpha\beta}\) can correspond to punctures, in which case \(\Gamma_{\alpha\beta}\) is a component of the space of closed geodesics, or to nodes, in which case \(\Gamma_{\alpha\beta}\) is a component of \(X^*\). In contrast to Proposition 7.4 we cannot conclude that \(L\) is surjective for two reasons:

(1) The perturbations in \(\mathcal{K}(\tilde{M}_{1+\ell+1}, TX^*; Y, x)\) are required to vanish near \(Y\) and near \(x\), so they are of no use to achieve transversality of the evaluation map.
on components $f_\alpha$ that are mapping entirely into $Y \cup \{x\}$. This can be remedied as follows.

Note first that no component can be mapped entirely into the point $x$. Indeed, such a component would be constant and carry at most one marked point (because the points $z_2, \ldots, z_{\ell+1}$ are mapped to $Y$ and $x \notin Y$). In view of stability, the component must carry at least two nodal points and thus split the tree into at least two subtrees. Each of these subtrees must contain a component in $\mathbb{C}P^n \setminus L$ and thus represent a nontrivial homology class in $\mathbb{C}P^n$, which is impossible because the total homology class $[\mathbb{C}P^1]$ is indecomposable.

Next consider a component $f_\alpha$ that is mapped entirely into $Y$. By Proposition 9.4(a), the map $f_\alpha$ is constant. Following [18], we denote by $T_1 \subset T$ the maximal subtree containing $\alpha$ consisting of constant components in $Y$ (a “ghost tree”). Indecomposability of the homology class $[\mathbb{C}P^1]$ implies that $T \setminus T_1$ must contain at least two of the marked points $z_2, \ldots, z_{\ell+1}$. According to [18] Proposition 7.1 and Lemma 7.2, the nonconstant map $f_\beta$ adjacent to $T_1$ is tangent to $Y$ at the point $z_{\beta\alpha}$. We replace $T$ by $T \setminus T_1$ and consider the remaining moduli space with an additional tangency condition to $Y$ imposed at the new marked point $z_{\beta\alpha}$. The new evaluation map (still denoted by $\text{Ev}$) contains the 1-jet evaluation map at $z_{\alpha\beta}$. This modification is performed for all ghost trees that are mapped to $Y$. Note that this decreases the number of marked points mapped to $Y$, but we keep denoting it by $\ell$.

(2) The tree $T$ need not be stable, and by construction the perturbations in $\mathcal{K}(\bar{M}_{1+\ell+1}, TX^*; Y, x)$ are domain independent on unstable components. Now by the discussion above, unstable components can only correspond to cylinders in $T^*L$ with two positive punctures (orbit cylinders can be removed from a stable map). Thus no two such components can be adjacent. Moreover, by Lemma 7.3 they appear in regular moduli spaces of the expected dimension $2(n-1)$ whose image under the evaluation map is a covering of $T^*L$. So we can use surjectivity of the linearized evaluation map at the adjacent components to achieve transversality to the diagonal at the punctures of each such component.

In view of this discussion and Proposition 7.4 after the modification in (1), the linearized operator $L$ is transverse to the manifold

$$Z := \{0_\varepsilon\} \times \prod_{\alpha \in E} \Delta_{\alpha\beta} \times \{0_{\Gamma_1}\} \times \prod_{2 \leq i \leq 1+\ell} Y_i.$$

Here $\Delta_{\alpha\beta}$ denotes the diagonal in $\Gamma_{\alpha\beta} \times \Gamma_{\beta\alpha}$ at the edge $\alpha E \beta$, and $Y_i$ denotes $TY$ at the new marked points $z_i$ where we take the 1-jet evaluation map, and $Y$ at the other ones.

Now the proof is finished by the standard argument as in the proof of Corollary 7.6. For each tree $T$ as above, the implicit function theorem implies that $(\partial \times \text{Ev})^{-1}(Z)$ is a Banach manifold. Pick $K \in \mathcal{K}(\bar{M}_{1+\ell+1}, TX^*; Y, x)$ to be a regular value of the projection onto the second factor for each of these spaces (this exists by the Sard-Smale theorem). Then all components of broken holo-
morphic spheres in the corresponding moduli space $\mathcal{M}$ appear in manifolds of the expected dimensions, and the evaluation maps at the edges are transverse to the diagonal. Hence the total moduli space (taking preimages of the diagonals under the edge evaluation maps) is a smooth manifold. According to Corollary 7.6, its dimension is given by

$$\dim \mathcal{M} = (2n - 6) + (2n + 2) - (4n - 4) + 2\ell - \sum_{i=2}^{\ell+1} \text{codim}(Y_i) - 2m(T)$$

Here the term $(2n+2)$ comes from $c_1(\mathbb{CP}^n)$ evaluated on $[\mathbb{CP}^1]$, $(4n-4)$ from the tangency condition to $Z$ at $x$ (see Section 5), codim$(Y_i)$ equals 2 if $Y_i = Y$ and 4 if $Y_i = TY$, and $m(T)$ is the number of nodes (i.e., edges of $T$ over which the maps extend continuously). It follows that the moduli space can be nonempty (and thus the dimension nonnegative) only if $m(T) = 0$ and $Y_i = Y$ for all $i$, so nodes and components mapped into $Y$ as discussed in (1) above cannot occur. (Nodes are actually also excluded by indecomposability of the homology class $[\mathbb{CP}^1].$)

The same discussion can be applied to the vertex containing the marked point $z_1$ (which is mapped to $x$ with tangency to $Z$) and the $m$ subtrees obtained by removing this vertex. Again by Corollary 7.6 their dimensions are given by

$$\dim \mathcal{M}_i = (n - 3) - CZ(\Gamma_i), \quad i = 1, \ldots, m,$$

$$\dim \mathcal{M} = (n - 3)(2 - m) + (2n + 2) + \sum_{i=1}^{m} (CZ(\Gamma_i) + \dim \Gamma_i) - (4n - 4).$$

This concludes the proof of transversality for Theorem 1.2(b).

**Proof of transversality for Theorem 1.18.** Here $L$ is a closed orientable surface with a metric of negative curvature. Thus there are no holomorphic planes in $T^*L$ and $\mathbb{R} \times M$, and the only holomorphic cylinders in $T^*L$ and $\mathbb{R} \times M$ are

(i) orbit cylinders in $\mathbb{R} \times M$, and

(ii) holomorphic cylinders in $T^*L$ projecting onto a closed geodesic $\gamma$ with two positive punctures asymptotic to $\pm \gamma$.

(This follows from a maximum principle argument similar to the proof of Lemma 7.3.) Orbit cylinders can be removed in the Gromov–Hofer compactification. The cylinders in (ii) appear in regular moduli spaces of the expected dimension 0. We choose the point $x \in L$ so that it does not meet these moduli spaces. According to Proposition 9.4 all other punctured holomorphic spheres in $T^*L$, $\mathbb{R} \times M$ or $T^*L$ of energy $< E$ have a stable domain. Now the proof is concluded by the same arguments as in the preceding proof of Theorem 1.2(b), which
are easier here because we have to consider neither Bott manifolds of closed geodesics, nor a higher jet evaluation map at $x$.  

\[Q \subseteq \mathbb{CP}^2\]

\[A \subseteq \mathbb{CP}^2\]

\[Q \subseteq \mathbb{CP}^2\]

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\[A \subseteq \mathbb{CP}^2\]

\[\begin{align*}
\text{Lemma A.1 (Chekanov–Schlenk [13]).} & \quad \text{(a) The Chekanov torus} \quad & \text{is a monotone Lagrangian 2-torus in} \quad & \mathbb{CP}^2 \quad & \text{with} \quad & \int_A \omega = \frac{2}{7} \mu(A) \quad \text{for the Maslov class} \quad & \mu \quad \text{and all} \quad & A \in H_2(\mathbb{CP}^2, L). \\
\text{(b) Let} \quad & \text{$J$ be an almost complex structure on} \quad & \mathbb{CP}^2 \quad & \text{with a negative cylindrical} \quad & \text{end for which} \quad & S_0, S_1, S_2 \quad & \text{and} \quad & Q \quad \text{are complex submanifolds. Then the only relative}
\end{align*}\]

\[\begin{align*}
&\text{A The Chekanov torus} \\
&\text{In this section we prove the “No Go” result mentioned in the Introduction:} \\
&\text{For the Chekanov torus} \quad T_{\text{Chekanov}} \subseteq \mathbb{CP}^2, \quad \text{the boundary loops of the three disks} \\
&\text{produced in Theorem 1.1 will in general represent multiples of the same homology} \\
&\text{class and thus fail to generate the first homology of the torus. More precisely,} \\
&\text{we will see that this occurs whenever we choose the almost complex structure} \\
&\text{in the proof of Theorem 1.1 from a suitable nonempty open set.} \\
&\text{We begin by recalling the construction of the Chekanov torus} \quad T_{\text{Chekanov}} \subseteq \mathbb{CP}^2 \\
&\text{following [14]. Let} \quad \gamma \subseteq \{ z \in \mathbb{C} \mid \Re z > 0 \} \quad \text{be a closed embedded curve in the} \\
&\text{right half-plane, bounding a closed disk} \quad D_\gamma \text{ of area } \pi/3 \text{ which contains} \quad 1 \in \mathbb{C} \\
&\text{in its interior. Denote their images under the diagonal embedding} \\
&\text{z \in \mathbb{C} \mapsto [1 : z : z] \in \mathbb{CP}^2 \text{ by}} \\
&\quad \Gamma := \{ [1 : z : z] \mid z \in \gamma \} \subseteq D_\Gamma := \{ [1 : z : z] \mid z \in D_\gamma \} \subseteq \mathbb{CP}^2. \\
&\text{Consider the symplectic} \quad S^1 \text{-action on} \quad \mathbb{CP}^2 \text{ given by} \quad e^{i\theta} \cdot [z_0 : z_1 : z_2] := [z_0 : e^{i\theta} z_1 : e^{-i\theta} z_2]. \quad \text{The Chekanov torus} \quad \text{is the union of the orbits of this action} \\
&\text{through} \quad \Gamma, \\
&\quad L := T_{\text{Chekanov}} := \{ [1 : e^{i\theta} z : e^{-i\theta} z] \mid z \in \gamma, \ e^{i\theta} \in S^1 \} \subseteq \mathbb{CP}^2. \\
&\text{Note that} \quad \partial D_\Gamma = \Gamma \subseteq L. \quad \text{We pick a point} \quad v \in \gamma \text{ with} \quad v \notin \mathbb{R} \text{ and set} \\
&\quad \tau := \{ [1 : e^{i\theta} v : e^{-i\theta} v] \mid e^{i\theta} \in S^1 \} \subseteq D_\tau := \{ [1 : \zeta v : \overline{\zeta} v] \mid \zeta \in D \} \subseteq \mathbb{CP}^2, \\
&\text{where} \quad D \subseteq \mathbb{C} \quad \text{denotes the closed unit disk. Thus} \quad \tau \text{ is the orbit of the point} \\
&\quad [1 : v : v] \text{ under the} \quad S^1 \text{-action and} \quad \partial D_\tau = \tau \subseteq L. \quad \text{We equip} \quad D_\Gamma \text{ and} \quad D_\tau \text{ with the orientations induced from} \quad D_\gamma \text{ resp.} \quad \Delta \text{ via the embeddings. Finally, let us} \\
&\text{introduce the projective lines} \quad S_0, S_1, S_2 \text{ and the quadric} \quad Q \text{ in} \quad \mathbb{CP}^2, \\
&\quad S_i := \{ [z_0 : z_1 : z_2] \mid z_i = 0 \}, \ i = 0, 1, 2, \quad Q := \{ [z_0 : z_1 : z_2] \mid z_0^2 = z_1 z_2 \}. \\
&\text{It follows directly from the defining equations that} \quad S_0, S_1, S_2, Q \subseteq \mathbb{CP}^2 \setminus L. \\
&\text{Since} \quad H_1(\mathbb{CP}^2) = 0 \text{ and} \quad [L] = 0 \in H_2(\mathbb{CP}^2), \text{ the long exact sequence of the pair} \quad (\mathbb{CP}^2, L) \text{ shows that} \quad H_2(\mathbb{CP}^2, L) \cong \mathbb{Z}^3 \text{ is generated by the relative homology} \\
&\text{classes represented by} \quad D_\Gamma, \ D_\tau \text{ and} \quad S_0 \text{ (which we will denote by the same} \\
&\text{symbols).} \\
\end{align*}\]

\[\begin{align*}
&\text{Lemma A.1 (Chekanov–Schlenk [13]).} \quad \text{(a) The Chekanov torus} \quad L \subseteq \mathbb{CP}^2 \quad \text{is a monotone Lagrangian 2-torus in} \quad \mathbb{CP}^2 \quad \text{with} \quad \int_A \omega = \frac{2}{7} \mu(A) \quad \text{for the Maslov class} \quad \mu \quad \text{and all} \quad A \in H_2(\mathbb{CP}^2, L). \\
&\text{(b) Let} \quad J \quad \text{be an almost complex structure on} \quad \mathbb{CP}^2 \setminus L \quad \text{with a negative cylindrical} \\
&\text{end for which} \quad S_0, S_1, S_2 \quad \text{and} \quad Q \quad \text{are complex submanifolds. Then the only relative} \\
\end{align*}\]

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homology classes in $H_2(\mathbb{C}P^2, L)$ which may contain punctured $J$-holomorphic spheres of Maslov number 2 are

$$D_\Gamma, S_0 - 2D_\Gamma, S_0 - 2D_\Gamma \pm D_\tau.$$ 

Apart from the doubles of these classes, the only classes which may contain punctured $J$-holomorphic spheres of Maslov number 4 are

$$S_0 - D_\Gamma, S_0 - D_\Gamma \pm D_\tau, 2S_0 - 4D_\Gamma \pm D_\tau.$$ 

**Proof.** Part (a) is proved in [14]. Part (b) is carried out in [14] for $S^2 \times S^2$ and the arguments easily carry over to $\mathbb{C}P^2$; the $\mathbb{C}P^2$ case is also treated in [4]. For convenience, we include the argument. The following table shows the relevant intersection and Maslov numbers in $\mathbb{C}P^2$. Here the Maslov numbers can be taken from [14] since they are the same in $S^2 \times S^2$ and $\mathbb{C}P^2$, and the intersection numbers follow easily from the defining equations and inspection of orientations at intersection points.

| $D_\Gamma$ | $D_\tau$ | $S_0$ | $A = a_\Gamma D_\Gamma + a_\tau D_\tau + bS_0$ |
|-----------|-----------|-------|---------------------------------------------|
| $S_0$     | 0         | 0     | $b \geq 0$                                  |
| $S_1$     | 0         | $-1$  | $-a_\tau + b \geq 0$                       |
| $S_2$     | 0         | 1     | $a_\tau + b \geq 0$                       |
| $Q$       | 1         | 0     | $a_\Gamma + 2b \geq 0$                    |
| $\mu$     | 2         | 0     | $2a_\Gamma + 6b = 2$ or 4                  |

Now suppose that a class $A = a_\Gamma D_\Gamma + a_\tau D_\tau + bS_0 \in H_2(\mathbb{C}P^2, L)$ contains a punctured $J$-holomorphic sphere. Since $S_0, S_1, S_2$ and $Q$ are $J$-complex submanifolds, all the intersection numbers in the last column of the table must be nonnegative. Together with the condition on the Maslov number this yields the result. 

**Corollary A.2.** Let $L = T_{\text{Chekanov}} \subset \mathbb{C}P^2$ be the Chekanov torus equipped with a flat metric. Let $C_0, \ldots, C_k \subset \mathbb{C}P^2 \setminus L$ be the punctured $J$-holomorphic spheres resulting from degenerating holomorphic spheres in the class $[\mathbb{C}P^1]$ along the boundary of a tubular neighbourhood of $L$, where $J$ is an almost complex structure on $\mathbb{C}P^2 \setminus L$ satisfying the conditions in Lemma A.1(b). Then the boundary loops of the $C_i$ all represent multiples of the same homology class $[\Gamma] \in H_1(L)$.

**Proof.** Let $F = (F^{(1)}, \ldots, F^{(N)})$ be a broken holomorphic sphere in $\mathbb{C}P^2$ with $N$ obtained as the limit of holomorphic spheres in the class $[\mathbb{C}P^1]$ in the neck-stretching procedure. Thus the components $C_0, \ldots, C_k$ of $F^{(N)}$ are punctured $J$-holomorphic spheres in $\mathbb{C}P^2 \setminus L$, where we assume that the almost complex structure $J$ on $\mathbb{C}P^2 \setminus L$ satisfies the conditions in Lemma A.1(b). Note that all components $C_i$ have positive symplectic area and the sum of the areas equals $\pi$. By monotonicity of $L$, the Maslov number of each $C_i$ equals $6/\pi$ times its
area. So all Maslov numbers are positive even integers and their sum equals 6, which leaves only the combinations $(2, 2, 2), (4, 2)$ and $(2, 4)$.

Since the total intersection number of $F$ with $S_0$ equals 1, there is a unique component, say $C_0$, which intersects $S_0$. According to Lemma A.1, the components which do not intersect $S_0$ are homologous to either $D_{Γ}$ or $2D_{Γ}$ in $H_2(\mathbb{CP}^2, L)$. Since the sum of the relative homology classes of all the components of $F^{(N)}$ equals $S_0$, this leaves only the following possibilities for the relative homology classes of $C_0, \ldots, C_k$:

$$(S_0 - 2D_{Γ}, D_{Γ}, D_{Γ}) \text{ or } (S_0 - 2D_{Γ}, 2D_{Γ}) \text{ or } (S_0 - D_{Γ}, D_{Γ}).$$

In particular, the boundary of each component $C_i$ is homologous to a multiple of $Γ$ (where the boundary of $C_i$ is the sum of the asymptotic geodesics at its punctures). Hence either all asymptotic geodesics of such a component are homologous to multiples of $Γ$, or at least two of them are not. The same holds for the components of the broken holomorphic curve $F$ in $T^*L$ and $\mathbb{R} \times S^*L$. Recall that one may assign to $F$ a tree whose nodes correspond to the components of $F$, and whose edges correspond to common asymptotic geodesics between two nodes resulting from the limit process. Hence the ends of the tree correspond to components with exactly one puncture which, by the preceding discussion, has to be asymptotic to a multiple of $Γ$. Remove all ends together with their adjacent edges. For each punctured holomorphic sphere corresponding to a node of the remaining tree it is still true that the asymptotics of the punctures corresponding to the remaining nodes are either all homologous to multiples of $Γ$, or at least two of them are not. (This holds since all edges we removed correspond to punctures asymptotic to multiples of $Γ$.) By induction over the number of nodes in the tree (since it holds for one node), it follows that all asymptotic geodesics are homologous to multiples of $Γ$ and the corollary follows.

**Remark A.3.**

(a) For the Chekanov torus, the proof of Theorem 1.1 (degenerating holomorphic spheres with a tangency condition) produces precisely three holomorphic planes $C_0, C_1, C_2$ in $\mathbb{CP}^2 \setminus L$, which by the proof of Corollary A.2 must represent the relative homology classes $(S_0 - 2D_{Γ}, D_{Γ}, D_{Γ})$.

(b) Corollary A.2 continuous to hold (with the same proof) for a tame almost complex structure $J$ for which there exist complex submanifolds in $\mathbb{CP}^2 \setminus L$ that are homologous (in the complement of $L$ to $S_0, S_1, S_2$ and $Q$. The set of these almost complex structures is open.

(c) Carrying over arguments from [14] to the $\mathbb{CP}^2$ case, it is shown in [3] that each of the relative homology classes of Maslov number 2 listed in Lemma A.1 contains a unique holomorphic disk passing through a given point on $L$. Neck stretching at the boundary of a tubular neighbourhood of $L$ yields holomorphic planes in $\mathbb{CP}^2 \setminus L$ representing these classes. This shows that there exist collections of holomorphic planes, e.g. three planes representing the classes $(S_0 - 2D_{Γ} + D_{τ}, S_0 - 2D_{Γ} - D_{τ}, 4D_{τ})$, whose asymptotic geodesics do generate $H_1(L)$. According to Corollary A.2 such collections of planes do not arise in degenerations of holomorphic spheres in the class $[\mathbb{CP}^1]$. 

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B  A non-removable intersection

Here we prove the claim in Remark 1.10 from the introduction, which we restate as follows.

**Proposition B.1.** Let \((L_t)_{t \in [0,1]}\) be a Hamiltonian isotopy of closed Lagrangian submanifolds in the closed unit ball \(B \subset \mathbb{C}^n\). If \(L_0 \subset \partial B\), then \(L_t \subset \partial B\) for all \(t \in [0,1]\).

**Proof.** The set \(\{t \in [0,1] \mid L_t \subset \partial B\}\) is clearly nonempty and closed, so we only need to show it is open. So let \(L := L_t \subset \partial B\). Let \(v\) be the vector field on \(\partial B\) generating the Hopf fibration (which is the characteristic foliation). Then \(\omega(v, w) = 0\) for all \(w \in TL \subset T(\partial B)\), so \(v \in (TL)^{\perp_\omega} = TL\). This shows that \(L\) is invariant under the Hopf fibration, hence it descends to a Lagrangian submanifold \(\bar{L} \subset \mathbb{C}P^{n-1}\). By the Lagrangian neighbourhood theorem and symplectic reduction, a neighbourhood of \(L\) in \(\mathbb{C}^n\) is symplectomorphic to a neighbourhood \(U\) of the zero section in 
\[
(T^*\bar{L} \times T^*S^1, dp \wedge dq).
\]

Here \((q_1, \ldots, q_{n-1}, p_1, \ldots, p_{n-1})\) are canonical coordinates on \(T^*\bar{L}\) and \((q_n, p_n)\) on \(T^*S^1\). Moreover, the variable \(p_n\) corresponds to the moment map \(z \mapsto \pi(|z|^2-1)\) of the standard circle action on \(\mathbb{C}^n\), so \(p_n > 0\) outside \(B\) and \(p_n \leq 0\) in \(B\).

For \(s\) sufficiently close to \(t\) we can write \(L_s\) as the graph of an exact 1-form in \(U\), i.e.,
\[
L_s = \{(q, p) \mid p_i = \frac{\partial f}{\partial q_i}\}
\]
for a function \(f : \bar{L} \times S^1 \to \mathbb{R}\). Fix a point \(\bar{q} = (q_1, \ldots, q_{n-1})\). Since \(L_s\) is contained in \(B\), we have \(\frac{\partial f}{\partial q_n}(\bar{q}, q_n) = p_n \leq 0\) for all \(q_n\). On the other hand,
\[
\int_0^{2\pi} p_n dq_n = \int_0^{2\pi} \frac{\partial f}{\partial q_n}(\bar{q}, q_n) dq_n = 0.
\]

So \(p_n\) must vanish identically, which means that \(L_s \subset \partial B\). \(\square\)

C  The embedding capacity for the flat torus

Let \(T^n = (\mathbb{R}/2\pi\mathbb{Z})^n\) be the standard flat torus, normalized such that each \(S^1\)-factor has length \(2\pi\). Equip \(\mathbb{C}P^n\) with the standard symplectic form such that a complex line has area \(\pi\), so \(\text{vol}(\mathbb{C}P^n) = \text{vol}B^{2n}(1)\). Then the volume estimate (11) becomes
\[
c_{\mathbb{C}P^n}(T^n) \geq \sqrt[n]{\frac{(2\pi)^n\text{vol}B^n(1)}{\text{vol}B^{2n}(1)}} =: C_n,
\]
(13)
whereas the estimate from Theorem 1.22 reads
\[ c_{\mathbb{CP}^n} (T^n) \geq 2(n + 1). \]

In view of the well-known formulae
\[ \text{vol}B^{2k}(1) = \frac{\pi^k}{k!}, \quad \text{vol}B^{2k+1}(1) = \frac{2^{2k+1} \pi^k k!}{(2k+1)!}, \]
the right hand side of equation (13) can be written as
\[ C_{2k} = 2\sqrt{\pi} 2^{k}\sqrt{(2k)!}k!, \quad C_{2k+1} = 4^{2k+1}\sqrt{\pi}k!. \]

Note that the estimates (13) and (14) agree for \( n = 1 \).

**Lemma C.1.** For every integer \( n \geq 2 \) we have \( C_n < 2(n + 1) \), so the estimate (14) is better than the volume estimate (13).

**Proof.** For \( n = 2k \geq 2 \) even we use \( 2\pi k < (2k+1)^2 \) to estimate
\[ C_{2k} \leq 2\sqrt{\pi} 2^{k}\sqrt{(2k)!}k! = 2\sqrt{2\pi k} < 2(2k+1) = 2(n + 1). \]
Similarly, for \( n = 2k + 1 \geq 3 \) odd we use \( \pi k < (k+1)^2 \) to estimate
\[ C_{2k+1} = 4^{2k+1}\sqrt{\pi}k^k < 4(k+1) = 2(n + 1). \]

Finally, we wish to compare the estimates to the values realized by the obvious symplectic embeddings
\[ \phi : \left((-1, 1) \times \mathbb{R}/2\pi \mathbb{R}\right)^n \hookrightarrow \mathbb{CP}^n, \quad (s_j, t_j) \mapsto z_j := \sqrt{2 + 2s_j e^{it_j}}. \]

**Lemma C.2.** The embedding \( \phi \) yields the upper bound
\[ c_{\mathbb{CP}^n} (T^n) \leq 2(n + \sqrt{n}), \]
which agrees with the lower bound (14) iff \( n = 1 \).

**Proof.** In the unit codisk bundle \( D^* (T^n) \) we have \( \sum_{j=1}^n s_j^2 \leq 1 \). The maximum of the squared norm \( |z|^2 = \sum_{j=1}^n (2 + 2s_j) \) on the image of \( \phi \) under this constraint is attained for \( s_1 = \cdots = s_n = 1/\sqrt{n} \) and given by
\[ \max |z|^2 = \sum_{j=1}^n (2 + 2/\sqrt{n}) = 2(n + \sqrt{n}). \]
Rescaling the symplectic form by this factor yields the upper bound. \( \square \)
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