Evolution of Curves and Surfaces by Mean Curvature*

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Abstract

This article describes the mean curvature flow, some of the discoveries that have been made about it, and some unresolved questions.

2000 Mathematics Subject Classification: 53C44.
Keywords and Phrases: Mean curvature flow, Singularities.

1. Introduction

Traditionally, differential geometry has been the study of curved spaces or shapes in which, for the most part, time did not play a role. In the last few decades, on the other hand, geometers have made great strides in understanding shapes that evolve in time. There are many processes by which a curve or surface can evolve, but among them one is arguably the most natural: the mean curvature flow. This article describes the flow, some of the discoveries that have been made about it, and some unresolved questions.

2. Curve-shortening flow

The simplest case is that of curves in the plane. Here the flow is usually called the “curvature flow” or the “curve-shortening flow”. Consider a smooth simple closed curve in the plane, and let each point move with a velocity equal to the curvature vector at that point. What happens to the curve?

The evolution has several basic properties. First, it makes the curve smoother. Consider a portion of a bumpy curve as in figure 1(a). The portions that stick up move down and the portions that stick down move up, so the curve becomes smoother or less bumpy as in figure 1(b). The partial differential equation for the motion is a parabolic or heat-type equation, and such smoothing is a general feature.

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of solutions to such equations. Thus, for example, even if the initial curve is only $C^2$, as it starts moving it immediately becomes $C^\infty$ and indeed real analytic.

![Figure 1: Smoothing](image)

However, there is an important caveat: the smoothing may only be for a short time. If the curve is $C^2$ at time 0, it will be real-analytic for times $t$ in some interval $(0, \epsilon)$. But because the equation of motion is nonlinear, the general theory of parabolic equations does not preclude later singularities. And indeed, as we shall see, any curve must eventually become singular under the curvature flow.

The simplest closed curve is of course a circle. The flow clearly preserves the symmetry, so in this case it is easy to solve the equation of motion explicitly. One finds that a circle of radius of radius $r$ at time 0 shrinks to a circle of radius $\sqrt{r^2 - 2t}$ at time $t$, so that time $t = r^2 / 2$ the circle has collapsed to a point and thereby become singular.

The second fundamental property is that arclength decreases. The proof is as follows. For any evolution of curves,

$$\frac{d}{dt}(\text{length}) = -\int k \cdot v \, ds,$$

where $k$ is the curvature vector, $v$ is the velocity, and $ds$ is arclength. For the curvature flow, $v = k$, so the right hand side of the equation is clearly negative. Indeed, the proof shows that this flow is, in a sense, the gradient flow for the arclength functional. Thus, roughly speaking, the curve evolves so as to reduce its arclength as rapidly as possible. This explains the name “curve-shortening flow”, though many other flows also reduce arc-length.

The third property is that the flow is collision-free: two initially disjoint curves must remain disjoint. The idea of the proof is as follows. Suppose that two initially disjoint curves, one inside the other, eventually collide. At the first time $T$ of contact, they must touch tangentially. At the point of tangency, the curvature of the inner curve is greater than or equal to the curvature of the outer curve. Suppose for simplicity that strict inequality holds. Then (at the point of tangency) the inner curve is moving inward faster than the outer curve is. But then at a slightly earlier time $T - \epsilon$, the curves would have to cross each other. But that contradicts the choice of $T$ (as first time of contact), proving that contact can never occur.

This collision avoidance is a special case of the maximum principle for parabolic differential equations. The maximum principle also implies in the same way that an initially embedded curve must remain embedded.
The fourth fundamental property is that every curve $\Gamma$ has a finite lifespan. To understand why, consider a circle $C$ that contains $\Gamma$ in its interior. Let $\Gamma$ and $C$ both evolve. The circle collapses in a finite time. Since the curves remain disjoint, $\Gamma$ must disappear before the circle collapses.

There is another nice way to see that a curve must become singular in a finite time:

**Theorem 1.** If $A(t)$ is the area enclosed by the curve at time $t$, then $A'(t) = -2\pi$ until the curve becomes singular. Thus a singularity must develop within time $A(0)/2\pi$.

**Proof.** For any evolution,

$$A'(t) = \int_{\Gamma(t)} v \cdot n \, ds$$

where $\Gamma(t)$ is the curve at time $t$, $v$ is the velocity, $n$ is the outward unit normal, and $ds$ is arclength. For the curvature flow, $v = k$, so

$$A'(t) = \int_{\Gamma} k \cdot n \, ds$$

which equals $-2\pi$ by the Gauss-Bonnet theorem. □

The first deep theorem about curvature flow was proved by Mike Gage and Richard Hamilton in 1986 [GH]:

**Theorem 2.** Under the curvature flow, a convex curve remains convex and shrinks to a point. Furthermore, it becomes asymptotically circular: if the evolving curve is dilated to keep the enclosed area constant, then the rescaled curve converges to a circle.

This theorem is often summarized by stating that convex curves shrink to round points.

The proof is too involved to describe here, but I will point out that the result is not at all obvious. Consider for example a long thin ellipse, with the major axis horizontal. The curvature is greater at the ends than at the top and bottom, so intuitively it should become rounder. But the ends are much farther from the center than the top and bottom are, so it is not clear that they all reach the center at the same time. Thus it is not obvious that the curve collapses to a point rather than a segment.

Indeed, there are natural flows that have all the above-mentioned basic properties of curvature flow but for which the Gage-Hamilton theorem fails. Consider for example a curve moving in the direction of the curvature vector but with speed equal to the cube root of the curvature. Under this flow, any ellipse remains an ellipse of the same eccentricity and thus does not become circular. For this flow, Ben Andrews [A1] has proved that any convex curve shrinks to an elliptical point. (See also [AST, SaT].) For other flows (e.g. if “cube root” is replaced by $r$th root for any $r > 3$), a convex curve must shrink to a point but in a very degenerate
way: if the evolving curve is dilated to keep the enclosed area constant, then length of the rescaled curve tends to infinity [A3]. More generally, Andrews has studied the existence and nonexistence of asymptotic shapes for convex curves for rather general classes of flows [A2, A3].

Shortly after Gage and Hamilton proved their theorem, Matt Grayson proved what is still perhaps the most beautiful theorem in the subject:

**Theorem 3.** [G1] Under the curvature flow, embedded curves become convex and thus (by the Gage-Hamilton theorem) eventually shrink to round points.

Again, the proof is too complicated to describe here, but let me indicate why the result is very surprising. Consider the annular region between a two concentric circles of radii and \( r = 1 \) and \( R = 2 \). Form a curve in this annular region by spiraling inward \( n \) times, and then back out \( n \) times to make a closed embedded curve. Figure 2 shows such a curve with \( n = 3/2 \), but think of \( n \) being very large, say \( 10^{100} \). Recall that the curve exists for a time at most \( R^2 / 2 = 2 \). By Grayson’s theorem, the curve manages, amazingly, to unwind itself and become convex in this limited time. Incidentally, notice that initially, except for two very small portions, the curve is not even moving fast: its curvature is no more than that of the inner circle.

![Spiral](image)

**Figure 2: Spiral**

As a corollary to Grayson’s theorem, one gets an exact formula for the lifespan of any curve. Recall that the area enclosed by a curve decreases with constant speed \(-2\pi\) as long as the curve is smooth. By Grayson’s theorem, the curve remains smooth until its area becomes 0. Thus the lifespan of any embedded curve must be exactly equal to the initial area divided by \( 2\pi \).

Grayson later generalized his theorem by proving that a closed curve moving on a compact surface by curvature flow must either collapse to a round point in a finite time or else converge to a simple closed geodesic as \( t \to \infty \) [G2].

### 3. Mean curvature flow for surfaces

We now leave curves and consider instead moving surfaces. Recall that at each point of an \( n \)-dimensional hypersurface in \( \mathbb{R}^{n+1} \), there are \( n \)-principal curvatures
The basic properties of curvature flow also hold for mean curvature flow:

1. Surfaces become smoother for a short time.
2. The area decreases. Indeed, mean curvature flow may be regarded as gradient flow for the area functional.
3. Disjoint surfaces remain disjoint, and embedded surfaces remain embedded.
4. Compact surfaces have limited lifespans.

The analog of the Gage-Hamilton theorem also holds, as Gerhard Huisken [H1] proved:

**Theorem 4.** For \( n \geq 2 \), an \( n \)-dimensional compact convex surface in \( \mathbb{R}^{n+1} \) must shrink to a round point.

Oddly enough, Huisken’s proof does not apply to the case of curves \((n = 1)\) considered by Gage and Hamilton. Huisken’s proof shows that the asymptotic shape is totally umbilic: at each point \( x \), the principal curvatures are all equal (though \textit{a priori} they may vary from point to point). For \( n \geq 2 \), the only totally umbilic surfaces are spheres, but for \( n = 1 \), the condition is vacuous.

The analog of Grayson’s theorem, however, is false for surfaces. Consider for example two spheres joined by a long thin tube. The spheres and the tube both shrink, but the mean curvature along the tube is much higher than on the spheres, so the middle of the tube collapses down to a point, forming a singularity. The surface then separates into two components, which eventually become convex and collapse to round points.

Thus, unlike a curve, a surface can develop singularities before it shrinks away. This raises various questions:

1. How do singularities affect the subsequent evolution of the surface?
2. How large can the set of singularities be?
3. What is the nature of the singularities? What does the surface look like near a singular point?

In the rest of this article I will describe some partial answers to these questions.

A great many results about mean curvature flow have been proved using only techniques of classical differential geometry and partial differential equations. However, most of the proofs are valid only until the time that singularities first occur. Once singularities form, the equation for the flow does not even make sense classically, so analyzing the flow after a singularity seems to require other techniques.

Fortunately, using non-classical techniques, namely the geometric measure theory of varifolds and/or the theory of viscosity or level-set solutions, one can define notions of weak solutions for mean curvature flow and one can prove existence of a
solution (with a given initial surface) up until the time that the surface disappears.

The definitions are somewhat involved and will not be given here. The different definitions are equivalent to each other (and to the classical definition) until singularities form, but are not completely equivalent in general. For the purposes of this article, the reader may simply accept that there is a good way to define mean curvature flow of possibly singular surfaces and to prove existence theorems. The notion of mean curvature flow most appropriate to this article is Ilmanen’s “enhanced Brakke flow of varifolds” [I].

4. Non-uniqueness or fattening

If a surface is initially smooth, classical partial differential equations imply that there is a unique solution of the evolution equation until singularities form. However, once a singularity forms, the classical uniqueness theorems do not apply. In the early 1990’s various researchers, including De Giorgi, Evans and Spruck, and Chen, Giga, and Goto, asked whether uniqueness could in fact break down after singularity formation. They already knew that uniqueness did fail for certain initially singular surfaces, but they did not know whether an initially regular surface could later develop singularities that would result in non-uniqueness.

A technical aside: the above-named people did not phrase the question in terms of uniqueness but rather in terms of “fattening”. They were all using a level set or viscosity formulation of mean curvature flow, in which solutions are almost by definition unique. But non-uniqueness of the enhanced varifold solutions corresponds to fattening of the viscosity solution in the following sense. If a single initial surface $M$ gives rise to different enhanced varifold solutions $M_1^t, M_2^t, \ldots, M_k^t$, then the viscosity solution “surface” at time $t$ will consist of the various $M_i^t$’s together with all the points in between. Thus if $k > 1$, the viscosity surface will in fact have an interior. Since the surface was initially infinitely thin, in developing an interior it has thereby “fattened”.

Recently Tom Ilmanen and I settled this question [IW]:

**Theorem 5.** There is a compact smooth embedded surface in $\mathbb{R}^3$ for which uniqueness of (enhanced varifold solutions of) mean curvature evolution fails. Equivalently, the viscosity (or level set) solution fattens.

The idea of the proof is as follows. Consider a solid torus of revolution about the $z$-axis centered at the origin, a ball centered at the origin that is disjoint from the torus, and $n$ radial segments in the $xy$-plane joining the ball to the torus. Call their union $W$. Now consider a nested one-parameter family of smooth surfaces $M^\epsilon$ ($0 < \epsilon < 1$) as follows. When $\epsilon$ is small, the surface should be a smoothed version of the set of points at distance $\epsilon$ from $W$. This $M^\epsilon$ looks like a wheel with $n$ spokes. The portion of the $xy$-plane that is not contained in $M^\epsilon$ has $n$ simply-connected components, which we regard as holes between the spokes of the wheel. As $\epsilon$ increases, the spokes get thicker and the holes between the spokes get smaller. When $\epsilon$ is close to 1, the holes should be very small, and near each hole the surface
should resemble a thin vertical tube.

Now let $M^\epsilon$ flow by mean curvature. If $\epsilon$ is small, the spokes are very thin and will quickly pinch off, separating the surface into a sphere and a torus. If $\epsilon$ is large, the holes between the spokes are very small and will quickly pinch off, so the surface becomes (topologically) a sphere. By a continuity argument, there is an intermediate $\epsilon$ such that both pinches occur simultaneously.

For this particular $\epsilon$, we claim that the simultaneous pinching immediately results in non-uniqueness, at least if $n$ is sufficiently large. Indeed, at the moment of simultaneous pinching, the surface will resemble a sideways figure 8 curve revolved around the $z$-axis as indicated in figure 3(a). Of course the surface will not be fully rotationally symmetric about the axis, but it will have $n$-fold rotational symmetry, and here I will be imprecise and proceed as though it were rotationally symmetric.

\begin{center}
\includegraphics[width=0.5\textwidth]{figure3.png}
\end{center}

Figure 3: Non-uniqueness

There is necessarily one evolution in which the surface then becomes topologically a sphere as in figure 3(b). If the angle $\theta$ in figure 6 is sufficiently small, there is also another evolution, in which the surface detaches itself from the $z$-axis and thereby becomes a torus as in figure 3(c).

One can show that as $n \to \infty$, the angle $\theta$ tends to 0. Thus if $n$ is large enough, the angle will be very small and both evolutions will occur.

The proof unfortunately does not give any bound on how large an $n$ is required. Numerical evidence [AIT] suggests that $n = 4$ suffices. The case $n = 2$ seems to be borderline and the case $n = 3$ has not been investigated numerically.

However, the argument completely breaks down for $n < 2$. Indeed, I would conjecture that if the initial surface is a smooth embedded sphere or torus, then uniqueness must hold.

It is desirable to know natural conditions on the initial surface that guarantee
uniqueness. As just mentioned, genus \( \leq 1 \) may be such a condition. Mean convexity (described below) and star-shapedness are known to guarantee uniqueness. The latter is interesting because the surface will typically cease to be star-shaped after a finite time, but its initially starry shape continues to ensure uniqueness [So].

Fortunately, uniqueness is known to hold generically in a rather strong sense. If a family of hypersurfaces foliate an open set in \( \mathbb{R}^{n+1} \), then uniqueness will hold for all except countably many of the leaves. Of course any smooth embedded surface is a leaf of such a foliation, so by perturbing the surface slightly, we get a surface for which uniqueness holds.

5. The size of singular sets

For general initial surfaces, our knowledge about singular sets is very limited. Concerning the size of the singular set, Tom Ilmanen [I], building on earlier deep work of Ken Brakke [BK], proved the following theorem. (See also [ES IV].)

**Theorem 6.** For almost every initial hypersurface \( M_0 \) and for almost every time \( t \), the surface \( M_t \) is smooth almost everywhere.

This theorem reminds me of Kurt Friedrichs, who used to say that he did not like measure theory because when you do measure theory, you have to say “almost everywhere” almost everywhere.

Aside from objections Friedrichs might have had, the theorem is unsatisfactory in that a much stronger statement should be true. But it is a tremendous achievement and it is the best result to date for general initial surfaces.

However, for some classes of initial surfaces, we now have a much better understanding of singularities. In particular, this is the case when the initial surface is mean-convex. The rest of this article is about such surfaces. For simplicity of language, only two-dimensional surfaces in \( \mathbb{R}^3 \) will be discussed, but the results all have analogs for \( n \)-dimensional surfaces in \( \mathbb{R}^{n+1} \) or, more generally, in \( (n+1) \)-dimensional riemannian manifolds.

Consider a compact surface \( M \) embedded in \( \mathbb{R}^3 \) and bounding a region \( \Omega \). The surface is said to be “mean-convex” if the mean curvature vector at each point is a nonnegative multiple of the inward unit normal (that is, the normal that points into \( \Omega \).) This is equivalent to saying that under the mean curvature flow, \( M \) immediately moves into \( \Omega \). Mean convexity is a very natural condition for mean curvature flow:

1. If a surface is initially mean convex, then it remains mean convex as it evolves.
2. Uniqueness (or non-fattening) holds for mean convex surfaces.

Mean convexity, although a strong condition, does not preclude interesting singularity formation. For example, one can connect two spheres by a thin tube as described earlier in such a way that the resulting surface is mean convex. Thus neck pinch singularities do occur for some mean convex surfaces.

**Theorem 7.** [W1] A mean convex surface evolving by mean curvature flow in \( \mathbb{R}^3 \) must be completely smooth (i.e., with no singularities) at almost all times, and
at no time can the singular set be more than 1 dimensional.

This theorem is in some ways optimal. For example, consider a torus of revolution bounding a region $\Omega$. If the torus is thin enough, it will be mean convex. Because the symmetry is preserved and because the surface always remains in $\Omega$, it can only collapse to a circle. Thus at the time of collapse, the singular set is one-dimensional.

However, in other ways the result is probably not optimal. In particular, the result should hold without the mean convexity hypothesis, and singularities should occur at only finitely many times. Indeed, I would conjecture that at each time, the surface can only have finitely many singularities unless one or more connected components have collapsed to curves. That is, the surface should consist of finitely many connected components, each of which either is a curve or has only finitely many singularities.

6. Nature of mean-convex singularities

Recall that when a mean convex surface evolves, it starts moving inward. Because mean convexity is preserved, it must continue to move inward. Consequently, the surface at any time lies strictly inside the region bounded by the surface at any previous time. Since the motion is continuous and since the surface collapses in a finite time, this implies that region $\Omega$ bounded by $M_0$ is the disjoint union of the $M_t$’s for $t > 0$. It is convenient and suggestive to speak of the $M_t$’s forming a foliation of $\Omega$, although it is not quite a foliation in the usual sense because some of the leaves are singular. Figure 4 shows the foliation when the initial surface is two spheres joined by a thin tube. (The entire foliation is rotationally symmetric about an axis, so suffices to show the intersection of the foliation with a plane containing that axis.)

\[ \text{Theorem 8. [W2]} \]

Consider a mean convex surface $M_t$ ($t \geq 0$) in $\mathbb{R}^3$ evolving by mean curvature flow. Let $p$ be any singular point in the region $\Omega$ bounded by the initial surface. If we dilate about $p$ by a factor $\lambda$ and then let $\lambda \to \infty$, the
dilated foliation must converge subsequentially to a foliation of \( \mathbb{R}^3 \) consisting of

1. parallel planes, or
2. concentric spheres, or
3. coaxial cylinders.

Let us call such a subsequential limit a **tangent foliation** at \( p \). The tangent foliation consists of parallel planes if and only if \( p \) is a regular point (i.e., \( p \) has a neighborhood \( U \) such that the \( M_t \cap U \) smoothly foliate \( U \).) A tangent foliation of concentric spheres corresponds to \( M_t \) (or a component of \( M_t \)) becoming convex and collapsing as in Huisken’s theorem (theorem 4) to the round point \( p \).

In figure 4, the tangent foliation at the “neck pinch” point \( A \) is a foliation by coaxial cylinders. The tangent foliations at points \( B \) and \( C \) are by concentric spheres. All other points are regular points and thus give rise to tangent foliations consisting of parallel planes.

Incidentally, in cases (1) and (2), the tangent foliation is unique. That is, we have convergence and not just subsequential convergence in the statement of theorem 8. However, this is not known in case (3). If one tangent foliation at \( p \) consists of cylinders, then so does any other tangent foliation at \( p \) [S]. But it is conceivable that different sequences of \( \lambda \)'s tending to infinity could give rise to cylindrical foliations with different axes of rotational symmetries. Whether this can actually happen is a major unsolved problem, exactly analogous to the long-standing uniqueness of tangent cone problem in minimal surface theory.

Although the tangent foliation at a singular point carries much information about the singularity, there are features that it misses. For example, consider the neck pinch in figure 4, located at the point \( A \). At a time just after the neck pinch, the two points \( D \) and \( E \) on the surface that are nearest to \( A \) have very large mean curvature and are therefore moving away from \( A \) very rapidly. However, such behavior cannot be seen in tangent foliations: the tangent foliation at any point near \( A \) consists of parallel planes, and the tangent foliation at \( A \) consists of coaxial cylinders.

To capture behavior such as the rapid motion away from \( A \), rather than dilating about a fixed point as in theorem 8, one needs to track a moving point.

**Theorem 9.** Consider a mean convex surface \( M_t \) (\( t \geq 0 \)) in \( \mathbb{R}^3 \) evolving by mean curvature flow. Let \( p_i \) be a sequence of points converging to a point \( p \) in the region bounded by \( M_0 \), and let \( \lambda_i \) be a sequence of numbers tending to infinity. Translate the \( M_t \)'s by \(-p_i\) and then dilate by \( \lambda_i \). Then the resulting sequence of foliations must converge subsequentially to a foliation of \( \mathbb{R}^3 \) by one of the following:

1. compact convex sets, or
2. coaxial cylinders, or
3. parallel planes, or
4. non-compact strictly convex surfaces, none of which are singular.

The convergence is locally smooth away from the limit foliation’s singular set (a point in case (1), a line in case (2), and the empty set in the other two cases.)
A foliation obtained in this way is called a **blow-up foliation** at $p$. Of course if all the $p_i$’s are equal to $p$, we get a tangent foliation at $p$.

It follows from the smooth convergence that a blow-up foliation is invariant under mean-curvature flow. That is, if we let a leaf flow for a time $t$, the result will still be a leaf. Consequently, except for the case (3) of parallel planes (which do not move under the flow), given any two leaves, one will flow to the other in finite time. Thus we can index the leaves as $M'_t$ in such a way that when $M'_t$ flows for a time $s$, it becomes $M'_{t+s}$. In the cases (1) and (2) of compact leaves and cylindrical leaves, the indexing interval may be taken to be $(-\infty, 0]$. In case (3), the indexing interval is $(-\infty, \infty)$. Thus, except in the case of parallel planes, the blow-up foliation corresponds to a semi-eternal or eternal flow of convex sets that sweep out all of $\mathbb{R}^3$.

As pointed out earlier, blow-up foliations (1), (2), and (3) already occur as tangent foliations. (If (2) occurs as a tangent foliation, then the compact convex sets must all be spheres, but in general blow-up foliations, other compact convex sets might conceivably occur.)

Thus the new case is (4). To see how that case arises, consider in figure 4 a sequence of points $E_i$ on the axis of rotational symmetry that converge to the neck pinch point $A$. Let $h_i$ be the mean curvature at $E_i$ of the leaf of the foliation that passes through $E_i$.

Now if we translate the foliation by $-E_i$ and then dilate by $\lambda_i = h_i$, then the dilated foliations converge to a blow-up foliation of $\mathbb{R}^3$ by convex non-compact surfaces. Each leaf qualitatively resembles a rotationally symmetric paraboloid $y = x^2 + z^2$. Furthermore, the leaves are all translates of each other. In other words, if we let one of the leaves evolve by mean curvature flow, then it simply translates with constant speed.

Incidentally, the same points $E_i$ with different choices of $\lambda_i$’s can give rise to different blow-up foliations. For if the dilation factors $\lambda_i \to \infty$ quickly compared to $h_i$ (that is, if $\lambda_i/h_i \to \infty$), then the resulting blow-up foliation consists of parallel planes. If $\lambda_i \to \infty$ slowly compared to $h_i$ (so that $\lambda_i/h_i \to 0$), then the resulting foliation consists of coaxial cylinders.

So what is the “right” choice of $\lambda_i$? In a way, it depends on what one wants to see. But this example does illustrate a general principle:

**Theorem 10.** [W2] *Let $p_i \in M_{t(i)}$ be a sequence of regular points converging to a singular point $p$. Translate $M_{t(i)}$ by $-p_i$ and the dilate by $h_i$ (the mean curvature of $M_{t(i)}$ at $p_i$) to get a new surface $M'_i$. Then a subsequence of the $M'_i$ will converge smoothly on bounded subset of $\mathbb{R}^3$ to a smooth strictly convex surface $M'$.*

Of course $M'$ is one leaf of the corresponding blow-up foliation.

Theorems 8, 9, and 10 give a rather precise picture of the singular behavior, but they raise some problems that have not yet been answered:

1. Classify all the eternal and semi-eternal mean-curvature evolutions of convex sets that sweep out all of $\mathbb{R}^3$. 
(2) Classify those associated with blow-up foliations.

The strongest conjecture for (2) is that a blow-up foliation can only consist of planes, spheres, cylinders, or the (unique) rotationally symmetric translating surfaces.

Many more eternal and semi-eternal evolutions of convex sets are known to exist. For instance, given any three positive numbers \(a, b,\) and \(c,\) there is a semi-eternal evolution of compact convex sets, each of which is symmetric about the coordinate planes and one of which cuts off segments of lengths \(a, b,\) and \(c,\) from the \(x,\) \(y,\) and \(z\) axes, respectively. (This can be proved by a slight modification of the proof given for example 3 in the “conclusions” section of [W2].) The case \(a = b = c\) is that of concentric spheres, which of course do occur as a blow-up foliation. Whether the other cases occur as blow-up foliations is not known.

A very interesting open question in this connection is: must every eternal evolution of convex sets consist of leaves that move by translating? Tom Ilmanen has recently shown that there is a one parameter family of surfaces that evolve by translation. At one extreme is the rotationally symmetric one, which does occur in blow-up foliations. At the other extreme is the Cartesian product of a certain curve with \(\mathbb{R}\):

\[
\{(x, y, z) : y = -\ln \cos x, \quad -1 < x < 1\}.
\]

This case does not occur in blow-up foliations ([W2].)

7. Further reading

Three distinct approaches have been very fruitful in investigating mean curvature: geometric measure theory, classical PDE, and the theory of level-set or viscosity solutions. These were pioneered in [BK]; [H1] and [GH]; and [ES] and [CGG] (see also [OS]), respectively. Surveys emphasizing the classical PDE approach may be found in [E1] and [H3]. A very readable and rather thorough introduction to the classical approach, including some new results (as well as some discussion of geometric measure theory), may be found in [E2]. An introduction to the geometric measure theory and viscosity approaches is included in [I]. See [G] for a more extensive introduction to the level set approach.

Theorems 7, 8, 9, and 10 about mean convex surfaces are from my papers [W1] and [W2]. These papers rely strongly on earlier work, for instance on Brakke’s regularity theorem and on Huisken’s monotonicity formula. Huisken proved theorem 8 much earlier under a hypothesis about the rate at which the curvature blows up [H2]. Huisken and Sinestrari [HS 1,2] independently proved results very similar to theorems 8, 9, and 10, but only up to the first occurrence of singularities.

Much of the current interest in curvature flows stems from Hamilton’s spectacular work on the Ricci flow. For survey articles about Ricci flow, see [CC] and [Ha]. For discussions of some other interesting geometric flows, see the articles by Andrews [A4] and by Bray [BH] in these Proceedings.
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