The complex AGM, periods of elliptic curves over $\mathbb{C}$ and complex elliptic logarithms

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Abstract

We present a method for computing period lattices and elliptic logarithms for elliptic curves defined over $\mathbb{C}$, using the complex Arithmetic-Geometric Mean (AGM) first studied by Gauss. Earlier authors have only considered the case of elliptic curves defined over the real numbers; here, the multi-valued nature of the complex AGM plays an important role. Our method, which we have implemented in both MAGMA and Sage, is illustrated with several examples.

1 Introduction

Let $E$ be an elliptic curve defined over $\mathbb{C}$, given by a Weierstrass equation

$$E: \quad Y^2 = 4(X - e_1)(X - e_2)(X - e_3),$$

where the roots $e_j \in \mathbb{C}$ are distinct. As is well known, there is an isomorphism (of complex analytic Lie groups) $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, where $\Lambda$ is the period lattice of $E$; specifically, we take $\Lambda$ to be the lattice of periods of the invariant differential $dX/Y$ on $E$. It is a discrete rank 2 subgroup of $\mathbb{C}$, spanned by a $\mathbb{Z}$-basis $\{w_1, w_2\}$ with $w_2/w_1 \notin \mathbb{R}$. The isomorphism is given by the map

$$z \ (\text{mod } \Lambda) \mapsto P = (\wp_\Lambda(z), \wp'_\Lambda(z)) \in E(\mathbb{C})$$

(with $0 \ (\text{mod } \Lambda) \mapsto O \in E(\mathbb{C})$, the base point at infinity) where $\wp_\Lambda$ denotes the classical elliptic Weierstrass function associated to the lattice $\Lambda$. The inverse of this map,

$$P \mapsto z \ (\text{mod } \Lambda),$$

from $E(\mathbb{C})$ to $\mathbb{C}/\Lambda$, is called the elliptic logarithm, and we say that any $z \in \mathbb{C}$ representing its class modulo $\Lambda$ is an elliptic logarithm of $P$. Two natural questions are:

1. How can we compute a basis for the period lattice $\Lambda$ of $E$, given a Weierstrass equation?

2. Given a point $P = (x, y) \in E(\mathbb{C})$, how can we compute its elliptic logarithm $z \in \mathbb{C}$?
For elliptic curves over $\mathbb{R}$, these questions have been answered satisfactorily and are well-known. Algorithms for computing $\mathbb{Z}$-bases for period lattices of elliptic curves defined over $\mathbb{R}$, and elliptic logarithms of real points on such curves, may be found in the literature (see, for example, [3, Algorithm 7.4.8] or [5, §3.7]). These use the real arithmetic-geometric mean (AGM), and allow one to compute both values rapidly with a high degree of precision. The theory behind this method is described succinctly by Mestre in [2]. The situation for elliptic curves over $\mathbb{C}$, however, is less satisfactory.

In this paper, we will give a complete method for computing period lattices and elliptic logarithms for elliptic curves over $\mathbb{C}$, by generalising the real algorithm. To this end, we will first explain the connection between the following three classes of objects:

- Complex AGM sequences, as first studied by Gauss and explored in depth more recently by Cox [4];
- Chains of lattices in $\mathbb{C}$;
- Chains of 2-isogenies between elliptic curves defined over $\mathbb{C}$.

These will be defined precisely below. This connection will allow us to derive an explicit formula (see Theorems 19 and 21 below), based on so-called optimal complex AGM values, for a $\mathbb{Z}$-basis of the period lattice of any elliptic curve defined over $\mathbb{C}$. We then develop our method further to give an iterative method (Algorithm 28) for computing elliptic logarithms of complex points.

Our approach to the computation of periods follows closely that of Bost and Mestre [2] in the real case. However, in that case there is only a single chain of 2-isogenies which needs to be considered, and a unique AGM sequence, while over $\mathbb{C}$ we find it convenient to consider a whole class of such sequences. The connection between these three types of sequence has some independent interest.

We note that the recent paper by Dupont [6] also presents related methods for evaluating modular functions using the complex AGM, including explicit complexity results. The results in [6] may also be used to compute complex periods, as these are given by elliptic integrals with complex parameters.

In the next three sections of the paper we consider in turn complex AGM sequences (as are described well in Cox [4]), then lattice chains and finally chains of 2-isogenies. Then we give the first application, to the computation of a basis for the period lattice (see Theorem 21). The following section gives a new proof of a result about the complete set of values of the (multi-valued) complex AGM, slightly more general than the version in [4]. Then in Section 8 we develop the elliptic logarithm algorithm (Algorithm 28). The paper ends with a set of illustrative examples.

Our algorithms have been implemented by the authors both in Sage (see [8]) and in MAGMA (see [1], code available from the second author).

The results of this paper form part of the PhD thesis [9] of the second author. The proofs are in some cases different: in [9] both the periods and elliptic logarithms are expressed more traditionally, as integrals over the Riemann surface $E(\mathbb{C})$; however the resulting iterative algorithms are identical. The second author acknowledges the support of the Development and Promotion of Science and Technology Talent Project (DPST) of the Ministry of Education, Thailand.
2 AGM Sequences

Let \((a, b) \in \mathbb{C}^2\) be a pair of complex numbers satisfying
\[
a \neq 0, \quad b \neq 0, \quad a \neq \pm b. \tag{1}
\]

We say that \((a, b)\) is **good** if \(\Re(b/a) \geq 0\), or equivalently,
\[
|a - b| \leq |a + b|; \tag{2}
\]
otherwise the pair is said to be **bad**. Clearly, only one of the pairs \((a, b), (a, -b)\)
is good, unless \(\Re(b/a) = 0\) (or equivalently, \(|a - b| = |a + b|\)), in which case both
are good.

An **arithmetic-geometric mean** (AGM) **sequence** is a sequence \(\((a_n, b_n)\)\)\(\in\mathbb{C}^2\)
whose pairs \((a_n, b_n)\) satisfy the relations
\[
2a_{n+1} = a_n + b_n, \quad b_{n+1}^2 = a_nb_n
\]
for all \(n \geq 0\). It is easy to see that if any one pair \((a_n, b_n)\) in the sequence satisfies (1) then all do, and we will make this restriction henceforth.

From any given starting pair \((a_0, b_0)\) there are uncountably many AGM sequences, obtained by iterating the procedure of replacing \((a_n, b_n)\) by the arithmetic mean \(a_{n+1} = (a_n + b_n)/2\) and the geometric mean \(b_{n+1} = \sqrt{a_nb_n}\), with a choice of the square root for \(b_{n+1}\) at each step. However, we usually prefer to consider the entire sequence as a whole. We say that an AGM sequence is **good** if the pairs \((a_n, b_n)\) are good for all but finitely many \(n\). A good AGM sequence in which \((a_n, b_n)\) are good for all \(n > 0\) is said to be **optimal**, and **strongly optimal** if in addition \((a_0, b_0)\) is good. If an AGM sequence is not good, then we say that it is **bad**.

It is easy to check that \((a_{n+1}, \pm b_{n+1})\) are both good if and only if \(a_n/b_n\) is real and negative, in which case \((a_n, b_n)\) is certainly bad. In an optimal sequence, this situation can only occur for \(n = 0\). In consequence, for every starting pair \((a_0, b_0)\) there is exactly one optimal AGM sequence, unless \(a_0/b_0\) is real and negative, in which case there are two, with different signs of \(b_1\), with the property that the ratios \(a_n/b_n\) in one of the sequences are the complex conjugates of those in the other.

The following proposition is from Cox (see [4]); the proof of parts (1) and (2) is elementary, and we refer the reader to [4]; part (3) appears deeper, and we will give a proof below after relating the different AGM values to a certain set of periods of an elliptic curve. Note that Cox defines the notion of “good” more strictly than above (when \(\Re(a/b) = 0\) he requires \(\Im(a/b) > 0\), so that exactly one of \((a, \pm b)\) is good in every case), but in view of the preceding remarks this does not affect the following result.

**Proposition 1.** Given a pair \((a_0, b_0) \in \mathbb{C}^2\) satisfying (1), every AGM sequence \(\((a_n, b_n)\)\(\in\mathbb{C}^2\) starting at \((a_0, b_0)\) satisfies the following:
1. \(\lim_{n \to \infty} a_n\) and \(\lim_{n \to \infty} b_n\) exist and are equal;
2. The common limit, say \(M\), is non-zero if and only if the sequence is good;
3. \(|M|\) attains its maximum (among all AGM-sequences starting at \((a_0, b_0)\)) if and only if the sequence is optimal.
For an AGM sequence \(((a_n, b_n))_{n=0}^\infty\) starting at \((a_0, b_0)\), we will denote the common limit \(\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n\) by \(M_S(a_0, b_0)\), where \(S \subseteq \mathbb{Z}_{>0}\) is the set of all indices \(n\) for which the pair \((a_n, b_n)\) is bad. For example, \(M_\emptyset(a_0, b_0)\) denotes the common limit for the optimal AGM sequence. To avoid ambiguities when \(a_0/b_0\) is negative real, we may agree to choose \(b_1\) so that \(\Im(a_1/b_1) > 0\) in that case, though this choice will not affect our results below. Note that the AGM sequence is good if and only if \(S\) is a finite set. To ease notation, we shall write \(M_\emptyset(a_0, b_0)\) simply as \(M(a_0, b_0)\).

### 3 Lattice Chains

In this paper, a lattice will always be a free \(\mathbb{Z}\)-module of rank 2, embedded as a discrete subgroup of \(\mathbb{C}\). Elements of lattices will often be called periods, since in our application the lattices will arise as period lattices of elliptic curves defined over \(\mathbb{C}\).

The following definition, as well as Lemma 2, only depend on the algebraic structure of lattices. We define a chain of lattices (of index 2) to be a sequence of lattices \((\Lambda_n)_{n=0}^\infty\) which satisfies the following conditions:

1. \(\Lambda_n \supset \Lambda_{n+1}\) for all \(n \geq 0\);
2. \([\Lambda_n : \Lambda_{n+1}] = 2\) for all \(n \geq 0\);
3. \(\Lambda_0/\Lambda_n\) is cyclic for all \(n \geq 1\); equivalently, \(\Lambda_{n+1} \neq 2\Lambda_{n-1}\) for all \(n \geq 1\).

Thus for each \(n \geq 1\) we have

\[\Lambda_{n+1} = \langle w \rangle + 2\Lambda_n\] \hspace{1cm} (3)

for some \(w \in \Lambda_n \setminus 2\Lambda_{n-1}\). Given an initial lattice \(\Lambda_0\), there are three possibilities for \(\Lambda_1\). When \(n \geq 1\), one of the three sublattices of index 2 is excluded, since it is contained in \(2\Lambda_{n-1}\) (which would contradict the last condition in the definition), and so there are only two possible choices for \(\Lambda_{n+1}\). The number of such chains starting with \(\Lambda_0\) is uncountable; we will distinguish a countable subset of these as follows. Let

\[\Lambda_\infty = \bigcap_{n=0}^\infty \Lambda_n.\]

Then \(\Lambda_\infty\) is free of rank at most 1; the rank cannot be 2, since for all \(n\),

\([\Lambda_0 : \Lambda_\infty] \geq [\Lambda_0 : \Lambda_n] = 2^n,\]

so \([\Lambda_0 : \Lambda_\infty]\) is infinite. We say that the chain is good if \(\Lambda_\infty\) has rank 1; in this case a generator for \(\Lambda_\infty\) will be called a limiting period of the chain. We will first show that the limiting period is primitive, i.e. not in \(m\Lambda_0\) for any \(m \geq 2\).

**Lemma 2.** Let \((\Lambda_n)_{n=0}^\infty\) be a good chain with \(\Lambda_\infty = \langle w_\infty \rangle\). Then

1. \(w_\infty\) is primitive; equivalently, \(\Lambda_0/\Lambda_\infty\) is free of rank 1;
2. \(\Lambda_n = \langle w_\infty \rangle + 2^n\Lambda_0\) for all \(n \geq 0\).
Proof. Suppose that \( w_\infty = mw \) for some \( m \geq 1 \) and \( w \in \Lambda_0 \). If \( m \) is odd, then since \( \Lambda_0/\Lambda_n \) has order \( 2^n \) which is prime to \( m \), we see that

\[
mw \in \Lambda_n \implies w \in \Lambda_n
\]

for all \( n \), so that \( w \in \Lambda_\infty \). Hence (by definition of \( w_\infty \)), \( m = 1 \).

Next suppose that \( w_\infty = 2w \) for some \( w \in \Lambda_0 \). By definition of \( w_\infty \), we then have \( w \notin \Lambda_\infty \), and hence there exists \( n > 0 \) such that \( w \notin \Lambda_n \). This implies that \( w_\infty \in \Lambda_n \setminus 2\Lambda_n \). But since \( w_\infty \in \Lambda_{n+1} \), we have

\[
\Lambda_{n+1} = \langle w_\infty \rangle + 2\Lambda_n = \langle 2w \rangle + 2\Lambda_{n+1} \subseteq 2\Lambda_0,
\]

which contradicts the definition of a chain. This proves the first statement.

The second statement follows from the fact that \( \Lambda_n/2^n\Lambda_0 \) is cyclic of order \( 2^n \), and is generated by \( w_\infty \) modulo \( 2^n\Lambda_0 \), since \( w_\infty \) is primitive. \( \square \)

So far, our notion of a good chain has been defined as a property of the chain as a whole, and only used the abstract structure of lattices as free \( \mathbb{Z} \)-modules. Using the next definition, we will see that this property can also be seen in terms of the individual steps \( \Lambda_n \supset \Lambda_{n+1} \), when the lattices are embedded in \( \mathbb{C} \). In view of \( \mathbb{R} \), the choice of \( \Lambda_{n+1} \) is determined by the class of \( w \) modulo \( 2\Lambda_n \).

For \( n \geq 1 \), we say that \( \Lambda_{n+1} \subset \Lambda_n \) is the right choice of sublattice of \( \Lambda_n \) if \( \Lambda_{n+1} = \langle w \rangle + 2\Lambda_n \) where \( w \) is a minimal element in \( \Lambda_n \setminus 2\Lambda_{n-1} \) (with respect to the usual complex absolute value).

**Lemma 3.** Let \( (\Lambda_n)_{n=0}^\infty \) be a good chain with \( \Lambda_\infty = \langle w_\infty \rangle \). Then \( w_\infty \) is minimal in \( \Lambda_n \) for all but finitely many \( n \geq 0 \).

**Proof.** Since \( \Lambda_0 \) is discrete, the number of periods \( w \in \Lambda_0 \) with \( 0 < |w| < |w_\infty| \) is finite. Each of these periods lies in only finitely many \( \Lambda_n \) by minimality of \( w_\infty \) in \( \Lambda_\infty \), so there exists \( n_0 \) such that \( w_\infty \) is minimal in \( \Lambda_{n_0} \) and hence also in \( \Lambda_n \) for all \( n \geq n_0 \). \( \square \)

The following proposition yields an alternative notion of a good chain. For now we remark that this is analogous to the definition of a good AGM sequence in the previous section; more of its analogues will be seen in later sections.

**Proposition 4.** A chain of lattices \( (\Lambda_n)_{n=0}^\infty \) is good if and only if \( \Lambda_{n+1} \subset \Lambda_n \) is the right choice for all but finitely many \( n \geq 1 \).

**Proof.** Let \( (\Lambda_n)_{n=0}^\infty \) be a good chain with \( \Lambda_\infty = \langle w_\infty \rangle \). Then by Lemma 3 there exists an integer \( n_0 \) such that \( w_\infty \) is minimal in \( \Lambda_n \) for all \( n \geq n_0 \). Since \( \Lambda_{n+1} = \langle w_\infty \rangle + 2\Lambda_n \) for all \( n \), then by definition, \( \Lambda_{n+1} \subset \Lambda_n \) is the right choice for all \( n \geq n_0 \).

Conversely, suppose that \( \Lambda_{n+1} \subset \Lambda_n \) is the right choice for all \( n \geq n_0 \) (where \( n_0 \geq 1 \)). Without loss of generality, we may suppose that \( n_0 = 1 \). Let \( w_1 \in \Lambda_1 \) be a minimal element. Then \( w_1 \) is certainly primitive (as an element of \( \Lambda_1 \), though not necessarily in \( \Lambda_0 \)). We claim that \( w_1 \in \Lambda_n \) for all \( n \geq 1 \), so that the chain is good with limiting period \( w_1 \).

To prove the claim, suppose that \( w_1 \in \Lambda_j \) for all \( j \leq n \). Then \( \Lambda_n = \langle w_1 \rangle + 2^{n-1}\Lambda_1 \), since the latter is contained in the former and both have index \( 2^{n-1} \) in \( \Lambda_1 \). Hence \( \Lambda_n = \langle w_1, 2^{n-1}w_2 \rangle \), where \( w_2 \in \Lambda_1 \) is such that \( \Lambda_1 = \langle w_1, w_2 \rangle \). The right sublattice of \( \Lambda_{n+1} \) is clearly \( \langle w_1 \rangle + \Lambda_n \), by minimality of \( w_1 \) (which is a candidate since \( w_1 \in \Lambda_n \setminus 2\Lambda_{n-1} \)); in particular, \( w_1 \in \Lambda_{n+1} \), as required. \( \square \)
3.1 Optimal Chains and Rectangular Lattices

Let us define a lattice chain to be **optimal** if $\Lambda_{n+1} \subset \Lambda_n$ is the right choice for all $n \geq 1$. We will see that there is in general just one optimal chain for each of the three choices of $\Lambda_1 \subset \Lambda_0$. In order to make the statement more precise, however, some preparation is necessary.

We say that a lattice $\Lambda \subset \mathbb{C}$ is **rectangular** if it has an “orthogonal” $\mathbb{Z}$-basis $\{w_1, w_2\}$, i.e. one which satisfies $\Re(w_2/w_1) = 0$. For example, the period lattice of an elliptic curve defined over $\mathbb{R}$ with positive discriminant is rectangular, where an orthogonal basis is given by the least real period and the least imaginary period. In general, rectangular lattices are homothetic to the period lattices of this family of elliptic curves.

If $\{w_1, w_2\}$ is any $\mathbb{Z}$-basis for a lattice $\Lambda$, the three non-trivial cosets of $2\Lambda$ in $\Lambda$ are $C_j = w_j + 2\Lambda$ for $j = 1, 2, 3$, where $w_3 = w_1 + w_2$. By a minimal coset representative in $\Lambda$ we mean a minimal element of one of these cosets. Minimal coset representatives are always primitive; for they are certainly not in $2\Lambda$, and if $w = mw'$ with $m \geq 3$ odd, then $|w'| < |w|$ while $w'$ is in the same coset as $w$.

**Lemma 5.** In each coset $C_j$ the minimal coset representative is unique up to sign, except in the case of a rectangular lattice with orthogonal basis $\{w_1, w_2\}$ where the coset $C_3$ has four minimal vectors, $\pm(w_1 \pm w_2)$.

**Proof.** For a rectangular lattice with orthogonal basis $\{w_1, w_2\}$, it is easy to see that the minimal coset representatives are as stated. Conversely, suppose that the lattice $\Lambda$ has a coset $C$ with at least two pairs of minimal elements, $\pm w$ and $\pm w'$. Then $w_1, w_2 = (w \pm w')/2 \in \Lambda$ are easily seen to be orthogonal.

If $w_1 \equiv 0 \pmod{2\Lambda}$, then $w_2 \equiv w \pmod{2\Lambda}$. But then $|w_2| < |w_1 + w_2| = |w|$, contradicting minimality of $w$ in its coset. Hence $w_1 \not\equiv 0 \pmod{2\Lambda}$. Similarly, $w_2 \not\equiv 0 \pmod{2\Lambda}$. Moreover, $w_1 \not\equiv w_2 \pmod{2\Lambda}$ since $w = w_1 + w_2 \equiv w_1 - w_2 \not\equiv 0 \pmod{2\Lambda}$. Therefore, $w_1, w_2, w$ do represent the three non-trivial cosets modulo $2\Lambda$. Now if $\{w_1, w_2\}$ was not a $\mathbb{Z}$-basis, there would exist a non-zero period $w_0 = \alpha w_1 + \beta w_2$ with $0 \leq \alpha, \beta < 1$. But then one of $w_0, w_0 - w_1, w_0 - w_2, w_0 - w$ is in the same coset as $w$, and all are smaller, contradiction. □

Our algorithm for computing periods of elliptic curves will in fact compute minimal coset representatives. Although these are individually primitive, to ensure that we thereby obtain a $\mathbb{Z}$-basis for the lattice, the following lemma is required.

**Lemma 6.** For $j = 1, 2, 3$, let $w_j$ be minimal coset representatives for a non-rectangular lattice $\Lambda \subset \mathbb{C}$; that is, minimal elements of the three non-trivial cosets of $2\Lambda$ in $\Lambda$. Then any two of the $w_j$ form a $\mathbb{Z}$-basis for $\Lambda$, and $w_3 = \pm(w_1 \pm w_2)$.

**Proof.** We may assume that $|w_1| \leq |w_2| \leq |w_3|$. Then $w_1$ is minimal in $\Lambda$ and $w_2$ is minimal in $\Lambda \setminus \langle w_1 \rangle$. Hence (replacing $w_2$ by $-w_2$ if necessary) $\tau = w_2/w_1$ is in the standard fundamental region for $\text{SL}_2(\mathbb{Z})$ acting on the upper half-plane, $\{w_1, w_2\}$ is a $\mathbb{Z}$-basis, and $w_3 = w_2 \pm w_1$; the sign depends on that of $\Re(\tau)$. □

The following proposition shows that the limiting period of an optimal chain is closely related to minimal coset representatives.
Proposition 7. A good chain of lattices \((\Lambda_n)_{n=0}^{\infty}\) with \(\Lambda_\infty = \langle w_\infty \rangle\) is optimal if and only if \(w_\infty\) is a minimal coset representative of \(2\Lambda_0\) in \(\Lambda_0\).

Proof. Suppose that \(w_\infty\) is a minimal coset representative. Then it is clear that \(\Lambda_{n+1} = \langle w_\infty \rangle + 2\Lambda_n \subset \Lambda_n\) is the right sublattice for all \(n \geq 1\), since \(w_\infty\) is certainly minimal in \(\Lambda_n \setminus 2\Lambda_{n-1}\).

Conversely, suppose that the sequence is optimal. Let \(w\) be a minimal element of \(\Lambda_1 \setminus 2\Lambda_0\), so that \(w\) is a minimal coset representative for the unique non-trivial coset of \(2\Lambda_0\) which is contained in \(\Lambda_1\). Note that \(w\) is unique up to sign, unless \(\Lambda_0\) is rectangular in which case (for one of the cosets) there will be two possibilities for \(w\) up to sign. By optimality, the sublattice \(\Lambda_2 \subset \Lambda_1\) is the right choice. In particular, if \(\Lambda_0\) is not rectangular, then we must therefore have \(\Lambda_2 = \langle w \rangle + 2\Lambda_1\). This, however, may not hold in the rectangular case, but it will hold if we replace \(w\) by the other choice of minimal coset representative.

Now we claim that \(\Lambda_n = \langle w \rangle + 2\Lambda_{n-1}\) for all \(n \geq 2\). We already know this for \(n = 2\). If the claim is true for \(n\), then certainly \(w \in \Lambda_n \setminus 2\Lambda_{n-1}\) (since \(w \notin 2\Lambda_0\)), so the (unique) good choice of sublattice of \(\Lambda_n\) is \(\langle w \rangle + 2\Lambda_n\). By optimality, this is \(\Lambda_{n+1}\), and so the claim holds for \(n + 1\). Thus \(w \in \bigcap_{n=0}^{\infty} \Lambda_n = \langle w_\infty \rangle\), and indeed, \(w = \pm w_\infty\), since \(w\) is primitive.

Combining Lemma 8 with Proposition 7 we have the following conclusion.

Corollary 8. Every non-rectangular lattice \(\Lambda\) has precisely three optimal sub-lattice chains, whose limiting periods are the minimal coset representatives in each of the three non-zero cosets of \(2\Lambda\) in \(\Lambda\). Every rectangular lattice \(\Lambda\) has precisely four optimal sublattice chains.

4 Short lattice chains and level 4 structures

In this section we establish a link between AGM sequences and lattice chains. The first step is to associate a pair of nonzero complex number \((a,b)\) (with \(a \neq \pm b\)) to each “short” lattice chain \(\Lambda_0 \supset \Lambda_1 \supset \Lambda_2\) in such a way that \((a,b)\) is good in the sense of Section 2 if and only if \(\Lambda_2\) is the right choice of sublattice of \(\Lambda_1\), in the sense of Section 3.

We establish bijections between the following sets:

1. “short” lattice chains \(\Lambda_0 \supset \Lambda_2\) with \(\Lambda_0/\Lambda_2\) cyclic of order 4;

2. triples \((E, \omega, H)\) where \(E\) is an elliptic curve defined over \(\mathbb{C}\), \(\omega\) a holomorphic differential on \(E\), and \(H \subset E(\mathbb{C})\) a cyclic subgroup of order 4;

3. unordered pairs of nonzero complex numbers \(a,b\) with \(a^2 \neq b^2\), where the pairs \(a,b\) and \(-a,-b\) are identified.

For each short lattice chain \(\Lambda_0 \supset \Lambda_2\), if we set \(\Lambda_1 = \Lambda_2 + 2\Lambda_0\) then \((\Lambda_0,\Lambda_1,\Lambda_2)\) satisfy the conditions for the first three terms in a lattice sequence as defined earlier. Hence we will usually think of a short lattice chain as a triple \(\Lambda_0 \supset \Lambda_1 \supset \Lambda_2\), even though \(\Lambda_1\) is uniquely determined by the other two.

To each short lattice chain we associate the elliptic curve \(E = \mathbb{C}/\Lambda_0\) with differential \(\omega = dz\) and subgroup \(H = (1/4)\Lambda_2/\Lambda_0\). Conversely, to a triple \((E, \omega, H)\) we associate the chain \(\Lambda_0 \supset \Lambda_2\) where \(\Lambda_0\) is the lattice of periods of \(\omega\).
(so that $E(\mathbb{C}) \cong \mathbb{C}/\Lambda_0$), and $\Lambda_2$ is the sublattice such that $H \cong (1/4)\Lambda_2/\Lambda_0$ under this isomorphism.

Each triple $(E, \omega, H)$ has a model of the form

$$E_{(a,b)} : \quad Y^2 = 4X(X + a^2)(X + b^2),$$

for some unordered pair $a, b \in \mathbb{C}^*$ such that $a^2 \neq b^2$, where $\omega = dX/Y$, and $H$ is the subgroup generated by the point

$$P_{(a,b)} = (ab, 2ab(a + b)).$$

The four points $P_{(\pm a, \pm b)}$ are the solutions to $2P = T = (0, 0) \in E_{(a,b)}(\mathbb{C})[2]$. Interchanging $\{a, b\}$ and $\{-a, -b\}$ does not affect the curve and interchanges $P_{(a,b)}$ and $P_{(-a,-b)} = -P_{(a,b)}$ so does not change $H$. On the other hand, changing the sign of just one of $a, b$ changes $H$ to the other cyclic subgroup of order 4 containing $T$. Hence the pair $\{a, b\}$ has the properties stated and is well-defined up to changing the signs of both $a$ and $b$.

Conversely, given $\{a, b\}$ with $ab \neq 0$ and $a \neq \pm b$, we recover the triple $(E_{(a,b)}, \omega, H)$, which is unchanged by either interchanging $a$ and $b$ or negating both.

If we only consider elliptic curves up to isomorphism, we may ignore the differential $\omega$, scale the equations arbitrarily, and consider lattices only up to homothety. Now we can identify pairs $\{a, b\}$ and $\{ua, ub\}$ for all $u \in \mathbb{C}^*$. The equation for $E_{(a,b)}$ can be scaled so that $ab = 1$, giving the homogeneous form

$$E_f : \quad Y^2 = 4X(X^2 + fX + 1),$$

where

$$f = \frac{a^2 + b^2}{ab} = \frac{a}{b} + \frac{b}{a} \neq \pm 2.$$

In this model, the points $\pm P_{(a,b)}$ generating the distinguished subgroup $H$ now have coordinates $(1, \pm 2\sqrt{2} \mp f)$. Thus the pair $E, H$ uniquely determines a complex number $f \in \mathbb{C} \setminus \{\pm 2\}$. We call this $f$ the modular parameter for the level 4 structure, since (as we will see below) it is in fact the value of a modular function for the congruence subgroup $\Gamma_0(4)$.

**Proposition 9.** The above constructions give a bijection between these sets:

1. “short” lattice chains $\Lambda_0 \supset \Lambda_2$ up to homothety;

2. pairs $(E, H)$ where $E$ is an elliptic curve defined over $\mathbb{C}$ with $H \subset E(\mathbb{C})$ a distinguished cyclic subgroup of order 4, up to isomorphism (where isomorphisms preserve the distinguished subgroups);

3. complex numbers $f \in \mathbb{C} \setminus \{\pm 2\}$.

4. points in the open modular curve $Y_0(4) = \Gamma_0(4)\backslash \mathbb{H}$, where $\mathbb{H}$ denotes the upper half-plane.

**Remark.** It would appear that considering pairs $(E, P)$, where $P$ is a point of exact order 4 in $E(\mathbb{C})$, would give a refinement to the level 4 structure, corresponding to points on the modular curve $Y_1(4) = \Gamma_1(4)\backslash \mathbb{H}$, since $[\Gamma_0(4) : \Gamma_1(4)] = 2$. However, this is an illusion: since every $E$ has an automorphism $[-1]$ which takes $P$ to $-P$, the set of pairs $(E, H)$ (up to isomorphism) may be identified with the space of pairs $(E, P)$ (also up to isomorphism). Similarly, since $\Gamma_0(4) = (\pm I)\Gamma_1(4)$, we may identify $Y_1(4)$ and $Y_0(4)$.
Proof. Bijections between all sets except the last have already been established.

Up to homothety, the lattice $\Lambda_0$ is determined by $\tau \in \mathcal{H}$ modulo the action of the modular group $\Gamma = \text{SL}(2, \mathbb{Z})$: for any oriented basis $w_1, w_2$ of $\Lambda_0$ (where “oriented” means $w_2/w_1 \in \mathcal{H}$) we associate $\tau = w_2/w_1 \in \mathcal{H}$. All oriented bases $w'_1, w'_2$ of $\Lambda_0$ have the form

$$w'_2 = aw_2 + bw_1$$
$$w'_1 = cw_2 + dw_1$$

with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and $\tau' = w'_2/w'_1 = (a\tau + b)/(c\tau + d)$. To allow for the additional level 4 structure, we restrict to oriented bases $w_1, w_2$ such that $\Lambda = \langle w_1 \rangle + 4\Lambda_0$, so that $w'_1, w'_2$ is only admissible if $w_1 \equiv \pm w'_1 \pmod{4\Lambda_0}$, or equivalently $(c, d) \equiv (0, \pm 1) \pmod{4}$. This uniquely determines the $\Gamma_0(4)$-orbit of $\tau$ and not just its $\Gamma$-orbit.

Let $Y_0(4) = \Gamma_0(4)/\mathcal{H}$ denote the open modular curve associated to $\Gamma_0(4)$, and $X_0(4)$ its completion, obtained by including the three cusps (represented by $\infty$, 0 and $1/2 \in \mathbb{P}^1(\mathbb{Q})$), which has genus 0. Hence the function field of $X_0(4)$ is generated by a single function. Since the $j$-invariant of $E_f$ is $256(f^2 - 3)^3/(f^2 - 4)$, we see that $f = f(\tau)$ is a suitable function. This establishes the claim concerning $f$, and completes the proof of the proposition.

Remark. There is an involution on each of these sets, which preserves the level 2 structure but interchanges the two possible associated level 4 structures. In each of the sets this takes the following forms: replace $\Lambda_2$ by the other sublattice $\Lambda'_2$ of index 2 in $\Lambda_1$ such that $\Lambda_0/\Lambda'_2$ is cyclic; replace $H$ by the other subgroup $H'$ which is cyclic of order 4 and contains $T = (0, 0)$; change the sign of one of $a, b$; or change $f$ to $-f$. This involution comes from the nontrivial automorphism of the cover $X_0(4) \rightarrow X_0(2)$ of degree 2; the function field of $X_0(2)$ is generated by $f^2$.

Remark. Since $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(4) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \Gamma(2)$, the function field of $X_0(4)$ may also be generated by $\lambda(2\tau)$ where $\lambda(\tau)$ is the classical Legendre elliptic function which generates the function field of $X(2)$. A calculation shows that $f(\tau) = 2(1 + \lambda(2\tau))/(1 - \lambda(2\tau))$. One interpretation of this is that instead of parametrizing short lattice chains by the parameter $\tau \in Y_0(4)$ corresponding to $\Lambda_0$ with $\Gamma_0(4)$-structure, we could instead have used the parameter $2\tau \in Y(2)$ to parametrize the middle lattice $\Lambda_1$ with full level 2-structure given by the sublattices $\frac{1}{2}\Lambda_0$ and $\Lambda_2$.

We now state the main result of this section.

**Theorem 10.** Let $\Lambda_0 \supset \Lambda_1 \supset \Lambda_2$ be a short lattice chain corresponding to the unordered pair $\{a, b\}$ and modular parameter $f$. Then the following are equivalent:

1. $\Lambda_2$ is the right choice of sublattice of $\Lambda_1$;
2. the pair $(a, b)$ is good;
3. $\Re(f) \geq 0$.

1The reason for ordering bases this way is to maintain consistency with other sections.
Proof. Equivalence of the second and third conditions is immediate from \( f = a/b + b/a \) since \((a, b) \) is good if and only if \( \Re(a/b) \geq 0 \). (In terms of the Legendre function, the equivalent condition is that \(|\lambda(2\tau)| \leq 1\).

For equivalence of the first condition, we need to work harder. Recall that \( \Lambda_2 \) is the right choice of sublattice if \( \Lambda_2 = \langle w_1 \rangle + 2\Lambda_1 = \langle w_1 \rangle + 4\Lambda_0 \), where \( w_1 \) is the minimal period in its coset modulo \( 2\Lambda_1 \). We can characterize this condition in terms of the imaginary part of \( \tau \in \mathcal{H} \).

**Lemma 11.** Let \( w_1, w_2 \) be any oriented basis for \( \Lambda_0 \) such that \( \Lambda_2 = \langle w_1 \rangle + 4\Lambda_0 \), and let \( \tau = w_2/w_1 \). The following are equivalent:

1. \( \Im \tau \) is maximal, over all \( \tau \) in its \( \Gamma_0(4) \)-orbit;
2. \( |ct + d| \geq 1 \) for all coprime \( c, d \in \mathbb{Z} \) such that \( (c, d) \equiv (0, \pm 1) \pmod{4} \);
3. \( |w_1| \) is minimal, over all primitive periods of \( \Lambda_0 \) such that \( \Lambda_2 = \langle w_1 \rangle + 4\Lambda_0 \);
4. \( |\tau + d/4| \geq 1/4 \) for all odd \( d \in \mathbb{Z} \).

Proof. Equivalence of the first two statements follows from \( \Im(\gamma \tau) = \Im\tau/|ct+d|^2 \) for \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \). Since \( |ct+d| \geq 1 \iff |cw_2+dw_1| \geq |w_1| \), the third statement is also equivalent to these. For the last statement, consider the geometry of the upper half-plane: the region given by these conditions is the same: \(4\) states that \( \tau \) lies on or above all the semicircles centred on rationals with denominator 4 and with radius 1/4, while \(2\) says that \( \tau \) lies above all semicircles centred at rationals \( -d/c \) with radius 1/c for which \((c, d) \equiv (0, \pm 1) \pmod{4}\); as the semicircles for \( c > 4 \) lie strictly under those for \( c = 4 \), this is no stronger.

We denote by \( \mathcal{F}(4) \) the set of \( \tau \) which satisfy these conditions; that is, those for which \( \Im \tau \) is maximal in a \( \Gamma_0(4) \)-orbit. The subset of \( \mathcal{F}(4) \) consisting of \( \tau \) such that \( 0 \leq \Re \tau \leq 1 \) is a (closed) fundamental region for \( \Gamma_0(4) \). Since \( \Gamma_0(4) \) has index 2 in \( \Gamma_0(2) \), the region \( \mathcal{F}(4) \) decomposes into two components, which we denote \( \mathcal{F}^\pm(4) \): the first is \( \mathcal{F}^+(4) = \mathcal{F}(2) \), consisting of all \( \tau \) lying on or above all semicircles of radius 1/2 centred at rationals with denominator 2; a closed fundamental region for \( \Gamma_0(2) \) is the subset of \( \mathcal{F}(2) \) consisting of \( \tau \) such that \( 0 \leq \Re \tau \leq 1 \). Secondly, \( \mathcal{F}^-(4) = \mathcal{F}(4) \setminus \mathcal{F}(2) \). The boundary between these is \( \mathcal{F}^+(4) \cap \mathcal{F}^-(4) \), which consists of the union of the semicircles \(|\tau + d/2| = 1/2 \) for all odd \( d \in \mathbb{Z} \).

The lemma above shows that \( \Lambda_2 \) is the right choice if and only if the \( \tau \) for which \( \Im \tau \) is maximal over all \( \tau \) in its \( \Gamma_0(4) \)-orbit is also maximal in the larger \( \Gamma_0(2) \)-orbit; in other words, given that \( \tau \in \mathcal{F}(4) \), we in fact have \( \tau \in \mathcal{F}^+(4) \).

The following lemma then completes the proof of the theorem.

**Lemma 12.** Let \( \tau \in \mathcal{F}(4) \). Then

\[ \Re f(\tau) \geq 0 \iff \tau \in \mathcal{F}^+(4). \]

Proof. This will follow by continuity from the following facts, for \( \tau \in \mathcal{F}(4) \) with \( 0 \leq \Re \tau \leq 1 \):

1. \( \Re f(\tau) = 0 \) if and only if \( \tau \) lies on the semicircle \(|2\tau-1| = 1 \) which separates the interiors of \( \mathcal{F}^\pm(4) \).
2. $\Re f(\tau) > 0$ for at least one $\tau \in F^+(4)$.

For the first fact implies that $\Re f(\tau)$ has constant nonzero sign on the two interiors. Since $f(\gamma \tau) = -f(\tau)$ for $\gamma \in \Gamma_0(2) \setminus \Gamma_0(4)$, the signs are different in the two interiors; and the second fact establishes that $\Re f(\tau)$ is positive in the interior of $F^+(4)$.

Let $f = f(\tau)$. Let the roots of $X(X^2 + fX + 1)$ be $e_1 = 0, e_2, e_3$. Since $e_2 e_3 = 1$, we have $\Re f = 0$ if and only if $e_2 = e_3 = 0$ with the imaginary parts $\Im e_2, \Im e_3$ of opposite sign; so $e_1, e_2, e_3$ collinear, with $0$ in between the other two roots. Conversely, if the $e_j$ are collinear with $0$ in the middle then (since $e_2 e_3 = 1$) it follows that $\Re f = 0$. However, this alignment of the roots happens precisely when the period lattice $\Lambda_0$ is rectangular, with orthogonal basis $w_2, w_1 + w_2$, which is when $\tau$ lies on the semicircle as claimed. This establishes the first fact.

Finally, one can check that for $i \in F^+(4)$ we have $f(i) = 3/\sqrt{2} > 0$ (equivalently, $\lambda(2i) = (3 - 2\sqrt{2})/(3 + 2\sqrt{2})$ so that $|\lambda(2i)| < 1$).

5 Chains of 2-Isogenies

From now on we will use standard Weierstrass models of elliptic curves rather than the special forms $E_{(a,b)}$ used above. Thus, let $E_0$ be an elliptic curve over $\mathbb{C}$ given by a Weierstrass equation

$$E_0: \quad Y^2 = 4(X_0 - e_1^{(0)})(X_0 - e_2^{(0)})(X_0 - e_3^{(0)}),$$

where the roots $e_j^{(0)}$ are distinct, and $\sum_{j=1}^3 e_j^{(0)} = 0$. We consider the ordering of the roots $e_j^{(0)}$ as fixed, with the point $T_0 = (e_1^{(0)}, 0)$ of order $2$ as distinguished. Let $a_0 = \pm \sqrt{e_1^{(0)} - e_3^{(0)}}, \quad b_0 = \pm \sqrt{e_1^{(0)} - e_2^{(0)}},$

the choice of signs will be discussed below. Via a shift of the $X$-coordinate we have $E_0 \cong E_{(a_0, b_0)}$, and the choice of signs determines the point $P_0 = (e_1^{(0)} + a_0 b_0, 2a_0 b_0 (a_0 + b_0))$ of order $4$ such that $2P_0 = T_0$.

Now consider arbitrary AGM sequences $((a_n, b_n))_{n=0}^\infty$ starting from $(a_0, b_0)$. As in [2], for $n \geq 1$ we let

$$e_1^{(n)} = \frac{a_1^2 + b_n^2}{3}, \quad e_2^{(n)} = \frac{a_2^2 - 2b_n^2}{3}, \quad e_3^{(n)} = \frac{b_n^2 - 2a_n^2}{3}. \quad (5)$$

These equalities also hold for $n = 0$, and for all $n \geq 0$ the $e_j^{(n)}$ are distinct, and satisfy $\sum_{j=1}^3 e_j^{(n)} = 0$. Hence each AGM sequence determines a sequence $(E_n)_{n=0}^\infty$ of elliptic curves defined over $\mathbb{C}$, where $E_n$ is given by the Weierstrass equation

$$E_n: \quad Y_n^2 = 4(X_n - e_1^{(n)})(X_n - e_2^{(n)})(X_n - e_3^{(n)}). \quad (6)$$

Each has a distinguished 2-torsion point $T_n = (e_1^{(n)}, 0)$. 

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For \( n \geq 1 \), define a 2-isogeny \( \varphi_n : E_n \to E_{n-1} \) via \((x_n, y_n) \mapsto (x_{n-1}, y_{n-1})\), where

\[
\begin{align*}
x_{n-1} &= x_n + \frac{(e_3^{(n)} - e_1^{(n)})(e_3^{(n)} - e_2^{(n)})}{x_n - e_3^{(n)}}, \\
y_{n-1} &= y_n \left(1 - \frac{(e_3^{(n)} - e_1^{(n)})(e_3^{(n)} - e_2^{(n)})}{(x_n - e_3^{(n)})^2}\right).
\end{align*}
\] (7)

Now \( \ker(\varphi_n) = \langle (e_3^{(n)}, 0) \rangle \), and

\[
\varphi_n(T_n) = T_{n-1} = \varphi_n((e_2^{(n)}, 0)).
\]

The dual isogeny \( \hat{\varphi}_n : E_{n-1} \to E_n \) has kernel \( \langle T_{n-1} \rangle \neq \ker(\varphi_{n-1}) \), so is distinct from \( \varphi_{n-1} \). Each composite \( \varphi_n \circ \varphi_{n+1} \) has cyclic kernel, since

\[
\varphi_n(\varphi_{n+1}(T_{n+1})) = \varphi_n(T_n) = T_{n-1} \neq O.
\]

Similarly, by tracing the images of \( T_n \), we see that all composites of the \( \varphi_n \) have cyclic kernels.

This chain of 2-isogenies may be depicted thus:

\[
\cdots \to E_n \xrightarrow{\varphi_n} E_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \to E_1 \xrightarrow{\varphi_1} E_0
\]

The number \( j \) next to each arrow originating from \( E_n \) denotes the point \( (e_j^{(n)}, 0) \) which generates the kernel of an associated 2-isogeny.

The construction of this isogeny chain from the original curve \( E^{(0)} \) depends on many choices. The definition of \( a_0, b_0 \) depends first on which root is labelled \( e_1^{(0)} \) (which determines \( T_0 \) and hence \( E_1 \)), and the order of labelling of \( e_2^{(0)} \) and \( e_3^{(0)} \). Secondly, the signs for \( a_0, b_0 \) were arbitrary; changing just one of them changes \( P_0 \) and hence \( E_2 \). So, as in the previous section, the unordered pair \( \{a_0, b_0\} \) determines the short isogeny chain \( E_2 \to E_1 \to E_0 \), with \( \{-a_0, -b_0\} \) determining the same short chain. Finally, for each \( a_0, b_0 \), there are many AGM sequences, which determine the rest of the chain.

Note that we can rewrite \( e_j^{(n+1)} \) given by (5) as

\[
e_1^{(n+1)} = \frac{e_1^{(n)} + 2a_nb_n}{4}, \quad e_2^{(n+1)} = \frac{e_1^{(n)} - 2a_nb_n}{4}, \quad e_3^{(n+1)} = -\frac{e_1^{(n)}}{2}.
\]

If \( (a_n, b_n) \) is replaced by \( (a_n, -b_n) \) for \( n \geq 1 \), we see that this interchanges \( e_1^{(n+1)} \) and \( e_2^{(n+1)} \) but leaves \( e_3^{(n+1)} \) unchanged. This does not change the curve \( E_{n+1} \); it only changes the labelling of its roots, which then changes \( E_{n+2} \).

Hence we have established a bijection between

- The set of all AGM sequences starting at \( (a_0, b_0) \), and
- The set of all isogeny chains starting with the short chain \( E_2 \to E_1 \to E_0 \).
We now consider what happens when \( n \to \infty \). From (5), we have

\[
\lim_{n \to \infty} e_1^{(n)} = \frac{2M^2}{3}, \quad \lim_{n \to \infty} e_2^{(n)} = \lim_{n \to \infty} e_3^{(n)} = -\frac{M^2}{3},
\]

where \( M = M_S(a_0, b_0) \) for some set \( S \subseteq \mathbb{Z}_{>0} \). The “limiting curve” \( E_\infty \) for the sequence \( \left( E_n^{(\infty)} \right)_{n=0}^\infty \) is the singular curve

\[
E_\infty : Y^2_\infty = 4 \left( X_\infty - \frac{2}{3}M^2 \right) \left( X_\infty + \frac{1}{3}M^2 \right)^2.
\]

Proposition 13 implies the following.

**Proposition 13.** The singular point of \( E_\infty \) is a node if and only if the AGM sequence \( (a_n, b_n) \) is good.

### 5.1 The associated lattice chain

For each \( n \geq 0 \), we have \( E_n(\mathbb{C}) \cong \mathbb{C}/\Lambda_n \), where \( \Lambda_n \) is the lattice of periods of the differential \( dX_n/Y_n \). From the definition of \( \varphi_n \) (see (2)), it can be verified that each \( \varphi_n \) is normalised, in the sense that

\[
\varphi_n^* \left( \frac{dX_{n-1}}{Y_{n-1}} \right) = \frac{dX_n}{Y_n}
\]

for all \( n \geq 1 \). Hence \( \varphi_n \) corresponds to the map \( \mathbb{C}/\Lambda_n \to \mathbb{C}/\Lambda_{n-1} \) induced from the identity map \( \mathbb{C} \to \mathbb{C} \). Since each \( \varphi_n \) is a 2-isogeny and the composites of the \( \varphi_n \) have cyclic kernels, it is clear that \( (\Lambda_n) \) is a lattice chain in the sense of Section 3.

This establishes the commutativity of the diagram (Figure 1), which shows the relationship between chains of lattices and chains of 2-isogenies. For brevity we denote \( z \mapsto (\varphi_1(z), \varphi_2(z)) \) by \( z \mapsto \varphi_n(z) \).

![Figure 1: A chain of isogenies linked with a chain of lattices](image)

Conversely, given any lattice chain \( (\Lambda_n) \) starting from \( \Lambda_0 \), we may recover the sequence of curves \( E_n \) and the chain of 2-isogenies linking them: first, \( \Lambda_1 \) determines which root of \( E_0 \) is labelled \( e_1^{(0)} \); then \( \Lambda_2 \) determines the choice of signs in the definition of \( a_0, b_0 \); and finally the AGM sequence starting with \( (a_0, b_0) \) is determined by the \( \Lambda_n \) for \( n \geq 2 \).

Thus we have a third set in bijection with both the set of all AGM sequences starting at \( (a_0, b_0) \), and the set of all isogeny chains starting with the short chain \( E_2 \to E_1 \to E_0 \); namely, the set of all lattice chains starting with the short chain \( \Lambda_0 \supset \Lambda_1 \supset \Lambda_2 \).
Proposition 14. With the above notation, for all $n \geq 0$,
1. $E_n \cong E(a_n, b_n)$;
2. $\Lambda_n \supset \Lambda_{n+1} \supset \Lambda_{n+2}$ is a short chain in the sense of Section 3;
3. $\Lambda_{n+2}$ is the right choice of sublattice of $\Lambda_{n+1}$ if and only if $(a_n, b_n)$ is a good pair;
4. the lattice chain $(\Lambda_n)$ is good (respectively, optimal) if and only if the sequence $((a_n, b_n))$ is good (respectively, optimal).

Proof. For (1), replace $x_n$ by $x_n + e_1^{(n)}$ in the equation for $E_n$ to obtain the equation for $E(a_n, b_n)$. The rest is then clear, using Theorem 10.

6 Period Lattices of Elliptic Curves

6.1 General Case

Let $E_0$ be an elliptic curve over $\mathbb{C}$ of the form (14). We keep the notation of the preceding section; in particular, the period lattice of $E_0$ is $\Lambda_0$. Each primitive period $w_1 \in \Lambda_0$ determines a good lattice chain $(\Lambda_n)$ where $\Lambda_n = \langle w_1 \rangle + 2^n \Lambda_0$, and conversely, since $\cap_n \Lambda_n = \langle w_1 \rangle$. So we have a bijection between the set of primitive periods of $\Lambda_0$ (up to sign) and good lattice chains. Each good lattice chain in turn determines a good AGM sequence $((a_n, b_n))$ starting at a pair $(a_0, b_0)$ such that $E_0 \cong E(a_0, b_0)$.

We now show that the period $w_1$ may be expressed simply in terms of the limit of the associated AGM sequence. It will follow that every primitive period $w_1$ of $E_0$ may be obtained from the limit of an appropriately chosen good AGM sequence. Conversely, we may express the set of all limits of AGM sequences starting at $(a_0, b_0)$ in terms of periods of $E_0$. We will also show that optimal AGM sequences give periods which are minimal in their coset modulo $4\Lambda_0$, and super-optimal sequences (where the initial pair $(a_0, b_0)$ also good) give periods which are minimal modulo $2\Lambda_0$. By Lemma 8, we will be then able to express a $\mathbb{Z}$-basis for $\Lambda_0$ in terms of specific AGM values.

Proposition 15. Let $(\Lambda_n)$ be a good lattice sequence with limiting period $w_1$ (generating $\cap \Lambda_n$, and defined up to sign). Then for all $z \in \mathbb{C} \setminus \Lambda_0$ we have

$$\lim_{n \to \infty} \wp_{\Lambda_n}(z) = \left( \frac{\pi}{w_1} \right)^2 \left( \frac{1}{\sin^2(\pi z/w_1)} - \frac{1}{3} \right),$$

$$\lim_{n \to \infty} \wp'_{\Lambda_n}(z) = -2 \left( \frac{\pi}{w_1} \right)^3 \left( \frac{\cos(\pi z/w_1)}{\sin^3(\pi z/w_1)} \right).$$

Proof. Since $w_1$ is primitive, there exists $w_2 \in \mathbb{C}$ such that $\Lambda_n = \langle w_1, 2^n w_2 \rangle$ for all $n \geq 0$. In the standard series expansion

$$\wp_{\Lambda_n}(z) = \frac{1}{z^2} + \sum_{0 \neq w \in \Lambda_n} \left( \frac{1}{(z - w)^2} - \frac{1}{w^2} \right),$$
we set \( w = m_1 w_1 + m_2 2^n w_2 \) with \( m_1, m_2 \) not both zero. As \( n \to \infty \) all terms with \( m_2 \neq 0 \) tend to zero, leaving

\[
\lim_{n \to \infty} \wp_{A_n}(z) = \sum_{m \in \mathbb{Z}} \frac{1}{(z - mw_1)^2} - \frac{1}{3} \left( \frac{\pi}{w_1} \right)^2.
\]

Using the expansion \( \frac{\pi^2}{\sin^2(\pi z)} = \sum_{m \in \mathbb{Z}} \frac{1}{(z - m)^2} \), this simplifies to the formula given.

For \( \lim_{n \to \infty} \wp'_{A_n}(z) \), we may either differentiate this, or apply the same argument to the series expansion of \( \wp'_{A_n}(z) \).

Corollary 16. In the above notation, let \((A_n)\) be a (good) lattice chain, with limiting period \( w_1 \), associated to the elliptic curve \( E_0 \) and the (good) AGM sequence \( ((a_n, b_n)) \) with non-zero limit \( M = M_S(a_0, b_0) \). Then \( M = \pm \pi / w_1 \), so that the period \( w_1 \) may be determined up to sign by

\[
w_1 = \pm \pi / M_S(a_0, b_0).
\]

Proof. For all \( n \geq 0 \) we have \( \wp_{A_n}(w_1/2) = e_1^{(n)} \). Letting \( n \to \infty \) and using the proposition we find that

\[
\frac{2}{3} M^2 = \lim_{n \to \infty} e_1^{(n)} = \frac{2}{3} \left( \frac{\pi}{w_1} \right)^2,
\]

from which the result follows.

The ambiguity of sign in this result will not matter in practice: changing the sign of \( w_1 \) does not change the lattice chain, and neither does changing the signs of both \( a_0, b_0 \) (and hence the sign of \( M_S(a_0, b_0) \)).

For fixed \((a_0, b_0)\), the value of \( M_S(a_0, b_0) \) depends on the set \( S \) of indices \( n \) for which \((a_n, b_n)\) is bad. Changing \( S \), we obtain different AGM sequences, and different lattice chains, but these all start with the same short chain \((A_n)_{n=0}^2\), and the periods given by \( \pi / M_S(a_0, b_0) \) are all in the same coset modulo \( 4 \Lambda_0 \).

We may now establish the result stated above as Proposition 1(3):

Corollary 17. \(|M_S(a_0, b_0)|\) attains its maximum (among all AGM-sequences starting at \((a_0, b_0)\)) if and only if the sequence is optimal.

Proof. By Corollary 16, \( M_S(a_0, b_0) \) is maximal (in absolute value) if and only if the limiting period \( w_1 = \pi / M_S(a_0, b_0) \) is minimal. By Proposition 7 this is if and only if the lattice chain is optimal. By Proposition 14(4), this in turn is if and only if the AGM sequence is optimal.

Corollary 18. 1. The optimal value \( M = M(a_0, b_0) \) gives a period \( w_1 = \pi / M \) which is minimal in its coset modulo \( 4 \Lambda_0 \).

2. \(|M(a_0, b_0)| \geq |M(a_0, -b_0)| \iff |a_0 - b_0| \leq |a_0 + b_0|\).

3. If \((a_0, b_0)\) is good, then \( \pi / M(a_0, b_0) \) is minimal in its coset modulo \( 2 \Lambda_0 \).
Proof. Changing the sign of \( b_0 \) (only) does not change \( \Lambda_1 \) (or \( E_1 \)), but does change \( \Lambda_2 \). The effect on \( w_1 \), therefore, is to change its coset modulo \( 4\Lambda_0 \) while not affecting its coset modulo \( 2\Lambda_0 \). By Proposition \[10] \( \Lambda_2 \) is the right choice if and only if \((a_0, b_0) \) is good. Hence, to obtain a period minimal in its coset modulo \( 2\Lambda_0 \), and not just modulo \( 4\Lambda_0 \), we choose the sign of \( b_0 \) so that the pair \((a_0, b_0) \) is good, and then take an optimal AGM sequence.

**Theorem 19** (Periods of Elliptic Curves over \( \mathbb{C} \), first version). Let \( E \) be an elliptic curve over \( \mathbb{C} \) given by the Weierstrass equation

\[
Y^2 = 4(X - e_1)(X - e_2)(X - e_3),
\]

with period lattice \( \Lambda \). Set \( a_0 = \sqrt{e_1 - e_3} \) and \( b_0 = \sqrt{e_1 - e_2} \), where the signs are chosen so that \((a_0, b_0) \) is good, i.e., \( |a_0 - b_0| \leq |a_0 + b_0| \), and let

\[
w_1 = \frac{\pi}{M(a_0, b_0)},
\]

using the optimal value of the AGM. Then \( w_1 \) is a primitive period of \( E \), and is a minimal period in its coset modulo \( 2\Lambda \).

Define \( w_2, w_3 \) similarly by permuting the \( e_j \); then any two of \( w_1, w_2, w_3 \) form a \( \mathbb{Z} \)-basis for \( \Lambda \).

Proof. Everything has been established except the last part. Letting \( e_2, e_3 \) in turn play the role of \( e_1 \) gives minimal periods in each of the cosets modulo \( 2\Lambda \), so Lemma \[6\] applies.

**Algorithm 20** (Computation of a period lattice basis). 

**Input:** An elliptic curve \( E \) defined over \( \mathbb{C} \), and roots \( e_j \in \mathbb{C} \) for \( j = 1, 2, 3 \). 

**Output:** Three primitive periods of \( E \), which are minimal coset representatives, any two of which form a \( \mathbb{Z} \)-basis for the period lattice of \( E \).

1. Label one of the roots as \( e_1 \), and the other two arbitrarily as \( e_2, e_3 \);
2. Set \( a_0 = \sqrt{e_1 - e_3} \) with arbitrary sign, and then \( b_0 = \pm \sqrt{e_1 - e_2} \) with the sign chosen such that \( |a_0 - b_0| \leq |a_0 + b_0| \).
3. Output \( w = \pi/M(a_0, b_0) \), using the optimal value of the AGM.
4. Repeat with each root \( e_j \) in turn playing the role of \( e_1 \).

Instead of computing \( w_2, w_3 \) by permuting the \( e_j \) as in Theorem \[19\] we may alternatively obtain all \( w_j \) by using a single ordering of the roots and three different AGM computations.

Starting with an arbitrary ordering of the roots, say \((e_1, e_2, e_3)\), define \( a \) and \( b \) as before, up to sign, by \( a^2 = e_1 - e_3 \) and \( b^2 = e_1 - e_2 \); and also define \( c \) (up to sign) by \( c^2 = e_2 - e_3 \), so that \( a^2 = b^2 + c^2 \). We would like to determine the signs of \( a, b, c \) so that all three of the following conditions hold:

\[
|a - b| \leq |a + b|, \quad |c - ib| \leq |c + ib|, \quad |a - c| \leq |a + c|.
\]

(11)

We claim that this is always possible. To see this, first choose the sign of \( a \) arbitrarily. Then choose the signs of \( b \) and \( c \) so that the first and the third
conditions in \( \text{II} \) hold. Finally, if the second condition fails, one can easily check that if \( e_1 \) and \( e_3 \) are interchanged and \( a, b, c \) replaced (in order) by \( ia, ic, ib \), then all three inequalities will hold.

We can now state an alternative theorem for obtaining a \( \mathbb{Z} \)-basis for the period lattice \( \Lambda \) of \( E \).

**Theorem 21** (Periods of Elliptic Curves over \( \mathbb{C} \), second version). Let \( E \) be an elliptic curve over \( \mathbb{C} \) given by the Weierstrass equation

\[
Y^2 = 4(X - e_1)(X - e_2)(X - e_3),
\]

with period lattice \( \Lambda \). Order the roots \((e_1, e_2, e_3)\) of \( E \), so that the signs of \( a = \sqrt{e_1 - e_2}, \ b = \sqrt{e_1 - e_3}, \ c = \sqrt{e_2 - e_3} \) may be chosen to satisfy all the conditions of \( \text{II} \). Define

\[
w_1 = \frac{\pi}{M(a, b)}, \quad w_2 = \frac{\pi}{M(c, ib)}, \quad w_3 = \frac{i\pi}{M(a, c)}.
\]

Then each \( w_j \) is a primitive period, minimal in its coset modulo \( 2\Lambda \), and any two of the \( w_j \) form a \( \mathbb{Z} \)-basis for \( \Lambda \).

**Proof.** Let \((e_1, e_2, e_3)\) be an order of the roots of \( E_0 \). Interchanging \( e_1 \) and \( e_3 \) if necessary, define \( a = \sqrt{e_1 - e_3}, \ b = \sqrt{e_1 - e_2}, \ c = \sqrt{e_2 - e_3} \), with the signs chosen so that all three inequalities in \( \text{II} \) hold.

Now \( w_1 = \pi/M(a, b) \) is primitive and minimal in its coset as before, since \((a, b)\) is good. Using \((e_1', e_2', e_3') = (e_2, e_1, e_3)\), we find that \((a', b') = (c, ib)\) is good, and set \( w_2 = \pi/M(a', b') = \pi/M(c, ib)\); and using \((e_1'', e_2'', e_3'') = (e_3, e_2, e_1)\), we see that \((a'', b'') = (ia, ic)\) is good, and set \( w_3 = \pi/M(a'', b'') = \pi i/M(a, c) \).

We complete this section by considering two special cases, which arise when considering elliptic curves defined over the real numbers, separating the cases of positive discriminant (rectangular period lattice) and negative discriminant.

### 6.2 Special Case I: Rectangular Lattices

For the rest of this section, we set \( i = \sqrt{-1} \). Recall that if \(|a_0 - b_0| = |a_0 + b_0|\), then both \((a_0, \pm b_0)\) are good and \( \Re(b_0/a_0) = 0 \). Then

\[
\frac{e_2 - e_1}{e_3 - e_1} = (b_0/a_0)^2
\]

is real and negative. Geometrically, this means that the \( e_j \) are collinear on the complex plane with \( e_1 \) in the middle.

To see what the associated period lattice looks like, let \( w = \pi/M(a_0, b_0) \) and \( w' = \pi/M(a_0, -b_0) \). Then \( w, w' \) are both minimal elements in the same coset modulo \( 2\Lambda_0 \). By Lemma [3] the periods \( w_1, w_2 = (w \pm w')/2 \) form an orthogonal \( \mathbb{Z} \)-basis for \( \Lambda_0 \), and the period lattice is rectangular. Alternatively, we could obtain a \( \mathbb{Z} \)-basis for \( \Lambda_0 \) by computing two periods (as in Theorem [19]) using the two other roots of \( E \) which are not “in the middle” in the role of \( e_1 \).

Finally, we note that whenever the \( e_j \) are collinear, we can “rotate” them by a multiplying by a suitable constant in \( \mathbb{C}^* \) so that the scaled roots \( e_j' \) are all
real. Then one could use an algorithm for computing period lattices of elliptic curves over \( \mathbb{R} \) (e.g. [3, Algorithm 7.4.7]) to compute the period lattice of the elliptic curve \( (Y')^2 = 4(X' - c'_1)(X' - c'_2)(X' - c'_3) \). The period lattice of our original elliptic curve is then obtained after suitable scaling. This may be more efficient in practice, since only real arithmetic would be needed in the AGM iteration.

If the \( e_j \) are all real (as is the case for an elliptic curve defined over \( \mathbb{R} \) with positive discriminant), we may order them so that \( e_1 > e_2 > e_3 \) and obtain a rectangular basis for the period lattice by setting

\[
w_1 = \pi/M(\sqrt{e_1 - e_2}, \sqrt{e_1 - e_3}), \quad w_2 = \pi i/M(\sqrt{e_2 - e_3}, \sqrt{e_1 - e_3})
\]

with all square roots positive; then \( w_1 \) and \( w_2/i \) are both real and positive. These familiar formulas may be found in [3, Algorithm 7.4.7] or [5, (3.7.1)].

### 6.3 Special Case II

If the roots of \( E \) are such that

\[
\frac{|e_1 - e_2|}{|e_1 - e_3|} = 1 \quad \text{with} \quad e_1 - e_2 \neq \pm(e_1 - e_3),
\]

then geometrically the \( e_j \) lie on an isosceles triangle having \( e_1 \) as the vertex where the sides of equal length intersect. As before, one can rotate this triangle by a suitable constant in \( \mathbb{C}^* \) so that \( e_1 \in \mathbb{R} \), and \( e_2, e_3 \) are complex conjugates. This yields a new elliptic curve \( E' \), defined over \( \mathbb{R} \), whose Weierstrass equation has only one real root.

Again, one could use an algorithm for computing period lattices of elliptic curves over \( \mathbb{R} \) (e.g. [3, Algorithm 7.4.7]) to compute the period lattice of \( E' \). This is of the form \( \Lambda = \langle w'_1, w'_2 \rangle \), for some \( w'_1, w'_2 \) satisfying

\[
w'_1 \in \mathbb{R}, \quad \Re(w'_2) = \frac{w'_1}{2}.
\]

The period lattice \( \Lambda = \langle w_1, w_2 \rangle \) of \( E \), with \( \Re(w_2/w_1) = 1/2 \), can then be obtained by a suitable scaling of \( w'_1, w'_2 \). This will be illustrated in Example 3.

For real curves with negative discriminant, we present here a simplification of the purely real algorithm given in [3]. Let \( e_1 \) be real and \( e_2, e_3 \) complex conjugates, ordered so that \( \Re e_2 > 0 \). Set \( a_0 = \sqrt{e_1 - e_3} = x + yi \); since \( e_1 - e_3 \) lies in the upper half-plane, we may choose the sign of \( a_0 \) so that \( x, y > 0 \). Set \( r = \sqrt{x^2 + y^2} > 0 \) and \( b_0 = \sqrt{e_1 - e_2} = -x - yi \). Now we may obtain a real period \( w_+ \) from

\[
w_+ = \pi/M(a_0, b_0) = \pi/M(x + yi, x - yi) = \pi/M(x, r),
\]

and an imaginary period \( w_- \) from

\[
w_- = \pi/M(-a_0, b_0) = \pi i/M(y - xi, y + xi) = \pi i/M(y, r).
\]

Note that both AGMs appearing here, \( M(x, r) \) and \( M(y, r) \), are classical (real and positive). These periods span a sublattice of index 2 in the period lattice, for which a \( \mathbb{Z} \)-basis may be taken to be \( w_1 = w_+ \) and \( w_2 = (w_+ + w_-)/2 \), where \( \Re(w_2/w_1) = 1/2 \).
7 The complete set of AGM values

In 1800, Gauss described the complete set of values of $M_S(a,b)$ as $S$ ranges through all finite sets. The proof given by Cox in [4, Theorem 2.2] uses theta and modular functions related to the modular functions which appeared earlier in this paper. Other proofs are also available in the literature, for example by Geppert [7].

We will give here a slightly more general form of the result than that stated in [4], and give an alternative proof which brings out clearly the relation with period lattices of elliptic curves.

In the following statement, we set $P_S(a,b) = \pi/M_S(a,b)$ (for any finite $S \subseteq \mathbb{Z}_{>0}$) and $P(a,b) = \pi/M(a,b)$.

**Theorem 22.** For $a, b \in \mathbb{C}^*$ with $a \neq \pm b$, let $E_{(a,b)}$ be the elliptic curve over \( \mathbb{C} \) given by the Weierstrass equation

\[ E_{(a,b)} : \quad Y^2 = 4X(X + a^2)(X + b^2), \]

and let $\Lambda$ be its period lattice. Let $c = \sqrt{a^2 - b^2}$, with the sign chosen so that the pair $(a,c)$ is good, and set

\[ w_1 = P(a,b), \quad w_3 = iP(a,c). \]

Then $\Lambda = zw_1 + zw_3$, and the set of values of $P_S(a,b)$ is precisely the set of primitive elements of the coset $w_1 + 4\Lambda$. More precisely, we have the following:

\[ \{P_S(a,b)\} = \{w \in w_1 + 4\Lambda, \quad w \text{ primitive}\}; \]
\[ \{P_S(a,-b)\} = \{w \in w_1 + 2w_3 + 4\Lambda, \quad w \text{ primitive}\}; \]
\[ \{P_S(-a,-b)\} = \{w \in -w_1 + 4\Lambda, \quad w \text{ primitive}\}; \]
\[ \{P_S(-a,b)\} = \{w \in -w_1 + 2w_3 + 4\Lambda, \quad w \text{ primitive}\}. \]

Thus, the complete set of all values of $P_S(\pm a,\pm b)$ is the set of primitive elements of the coset $w_1 + 2\Lambda$.

**Proof.** Since $\Lambda$ is invariant under translations of the $X$-coordinate, we may apply Theorem [21] to see that $\Lambda = Zw_1' + Zw_3$ where $w_3$ (as given) is a minimal coset representative, and either

- $(a,b)$ is good and $w_1' = w_1$; or
- $(a,b)$ is bad and $w_1' = w_1 \pm 2w_3$.

In either case, $\Lambda = Zw_1 + Zw_3$. Now the values of $P_S(a,b)$ are precisely the primitive periods in the same coset as $w_1 = P(a,b)$ modulo $4\Lambda$. Secondly, $P_S(-a,-b) = -P_S(a,b) = -w_1$, so the values of $P_S(-a,-b)$ are the primitive periods in the coset $-w_1$ (mod $4\Lambda$), as required. Next, $P(a,-b)$ is the minimal period in the coset $w_1 + 2w_3 + 4\Lambda$, since this is the other coset modulo $4\Lambda$ contained in $w_1 + 2\Lambda$, so the values of $\pm P_S(a,-b)$ are also as stated.

**Corollary 23.** Let $a, b, c \in \mathbb{C}^*$ satisfy $a^2 = b^2 + c^2$. Define $w = \pi/M(a,b)$ and $w' = \pi i/M(a,c)$, where $(a,c)$ is a good pair. Then
1. \( \Lambda = \mathbb{Z}w + \mathbb{Z}w' \) is a lattice in \( \mathbb{C} \);

2. the set of values of \( \pi/M_S(a, b) \) is the set of primitive elements of the coset \( w + 4\Lambda \); that is, the set

\[
\{uw + vw' \mid u, v \in \mathbb{Z}, \gcd(u, v) = 1, u - 1 \equiv v \equiv 0 \pmod{4}\};
\]

3. the set of values of \( \pi/M_S(\pm a, \pm b) \) is the set of primitive elements of the coset \( w + 2\Lambda \); that is, the set

\[
\{uw + vw' \mid u, v \in \mathbb{Z}, \gcd(u, v) = 1, u - 1 \equiv v \equiv 0 \pmod{2}\}.
\]

8 Elliptic Logarithms

We now extend the method for computing periods of elliptic curves in Section 6 to give a method for computing elliptic logarithms of points on elliptic curves.

Let \( E \) be an elliptic curve over \( \mathbb{C} \) given by a Weierstrass equation as before, and \( \Lambda \) the lattice of periods of the differential \( dx/y \) on \( E \), so that \( E(\mathbb{C}) \cong \mathbb{C}/\Lambda \). An elliptic logarithm of \( P \in E(\mathbb{C}) \) is a value \( z_P \in \mathbb{C} \) such that \( P = (\varphi_\Lambda(z_P), \varphi'_\Lambda(z_P)) \). Note that \( z_P \) is only well-defined modulo \( \Lambda \). We wish to have an algorithm which can compute the numerical value of the complex number \( z_P \), to any required precision, from the coefficients of \( E \) and the coordinates of \( P \) (which we assume are given exactly, or are available to arbitrary precision).

Construct as before an isogeny chain \( (E_n) \) with \( E_0 = E \), with associated lattice chain \( (\Lambda_n) \) (with \( \Lambda_0 = \Lambda \)) and AGM sequence \( (a_n, b_n) \). We will assume that the chain is super-optimal with \( |a_n - b_n| < |a_n + b_n| \) for all \( n \geq 0 \). (This is possible except when \( \Lambda_0 \) is rectangular, and even then is possible for two of the three super-optimal sequences). Let \( w_1, w_2 \) be a \( \mathbb{Z} \)-basis for \( \Lambda \) such that \( \Lambda_n = \langle w_1, 2^n w_2 \rangle \) for all \( n \geq 0 \). We have 2-isogenies \( \varphi_n : E_n \to E_{n-1} \) for \( n \geq 1 \), induced by the natural maps \( \mathbb{C}/\Lambda_n \to \mathbb{C}/\Lambda_{n-1} \).

8.1 Coherent point sequences

Consider sequences of points \( (P_n)_{n=0}^{\infty} \) where \( P_n \in E_n(\mathbb{C}) \) satisfy \( \varphi_n(P_n) = P_{n-1} \) for all \( n \geq 1 \). Such a sequence will be called coherent if there exists \( z \in \mathbb{C} \) such that \( P_n = \varphi_n(z) \) for all \( n \geq 0 \); here, as above, we write \( \varphi_n(z) \) for \((\varphi_{\Lambda_n}(z), \varphi'_{\Lambda_n}(z)) \). If such a \( z \) exists, it is uniquely determined modulo \( \cap \Lambda_n = \Lambda_\infty = \langle w_1 \rangle \).

In general there are uncountably many point sequences with a fixed starting point \( P_0 \), since for each \( P_n \in E_n(\mathbb{C}) \) there are two points \( P_{n+1} \in E_{n+1}(\mathbb{C}) \) with \( \varphi_{n+1}(P_{n+1}) = P_n \). However, only countably many of these are coherent, since \( \varphi_0^{-1}(P_0) \) is a coset of \( \Lambda_0 \) in \( \mathbb{C} \), and hence countable.

For example, taking \( z = 0 \) shows that the trivial sequence \( (O_n) \), where \( O_n \) is the base point on \( E_n \), is coherent. Also, the sequence with \( P_n = T_n = (\varepsilon_1(n), 0) \) is coherent, via \( z = w_1/2 \).

Given a point sequence \( (P_n) \), for each \( n \) let \( C_n = \varphi_n^{-1}(P_n) \subset \mathbb{C} \) be the complete set of all the elliptic logarithms of \( P_n \), which is a coset of \( \Lambda_n \) in \( \mathbb{C} \). Since \( \Lambda_{n+1} \) has index 2 in \( \Lambda_n \), each \( C_n \) is the disjoint union of two cosets of \( \Lambda_{n+1} \),

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one of these being $C_{n+1}$; the other is the set of elliptic logarithms of the second point $P'_{n+1} \in E_{n+1}(\mathbb{C})$ such that $\varphi_{n+1}(P'_{n+1}) = P_n$. Thus we have

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset C_{n+1} \supset \ldots.$$ 

The point sequence is coherent if and only if $C_\infty = \cap_{n=0}^\infty C_n \neq \emptyset$, in which case $C_\infty$ is a coset of $\Lambda_\infty$ in $\mathbb{C}$.

An argument similar to that used above for Proposition 3 shows the following.

Lemma 24. The sequence $(P_n)$ is coherent if and only if $C_{n+1}$ contains the smallest element of $C_n$ for almost all $n \geq 0$.

8.2 The elliptic logarithm formula

Proposition 25. With notation as above, let $(P_n)$ be a coherent point sequence determined by $z \in \mathbb{C}$. Assume that $2z \not\in \Lambda_\infty$. Then for $n$ sufficiently large, we have $P_n \neq O_n$, and write $P_n = (x_n, y_n)$. Let $P_\infty = (x_\infty, y_\infty) \in E_\infty(\mathbb{C})$ be the limit point, defined by $(x_\infty, y_\infty) = \lim_{n \to \infty}(x_n, y_n)$. Set $M = \pi/w_1$, and

$$t_\infty = -\frac{1}{2}y_\infty/(x_\infty + M^2/3).$$

Then $t_\infty \neq 0, \infty$, and (modulo $\Lambda_\infty$) we have

$$z = \frac{1}{M} \arctan \left( \frac{M}{t_\infty} \right) = \frac{w_1}{\pi} \arctan \left( \frac{\pi}{w_1 t_\infty} \right).$$

(13)

Proof. Since $z \not\in \Lambda_\infty$, for all $n \gg 0$ we have $z \not\in \Lambda_n$, so that $P_n \neq O_n$. Proposition 13 gives expressions for the coordinates of $P_\infty = (x_\infty, y_\infty) \in E_\infty(\mathbb{C})$ in terms of $M$, $s = \sin(z\pi/w_1)$ and $c = \cos(z\pi/w_1)$:

$$x_\infty = M^2 \left( \frac{1}{s^2} - \frac{1}{3} \right); \quad y_\infty = -2M^3 \frac{c}{s^3}. $$

Note that $s \neq 0$, since $z \not\in \Lambda_\infty$; also, $s \neq \pm 1$ (and $c \neq 0$) since $2z \not\in \Lambda_\infty$. Thus $x_\infty + M^2/3 = M^2/s^2 \neq 0$, and $t_\infty = -\frac{1}{2}y_\infty/(x_\infty + M^2/3) = Mc/s \neq 0$, giving formula (13). Taking different values of the multiple-valued function arctan changes $z$ by integer multiples of $w_1$; so this formula gives a well defined value for $z$ modulo $\Lambda_\infty$, as desired.

This result does also apply when $z = \pm w_1/2 \pmod{\Lambda_\infty}$, for then $s = \pm 1$ and $c = 0$, so $x_\infty + M^2/3 = M^2$ and $y_\infty = 0$, giving $t_\infty = 0$ and $z = w_1$; this is the case we used above to compute periods.

Proposition 25 and in particular formula (13), is the key to our elliptic logarithm algorithm, in which we will compute a sequence $(t_n)$ iteratively such that $\lim_{n \to \infty} t_n = t_\infty$. However, we derived (12) by starting from a value of $z \in \mathbb{C}$, rather than from the coordinates of a point $P = \varphi(z) \in E(\mathbb{C})$. In order to produce an algorithm for computing $z$ from the coordinates of $P$, we must show how to construct inductively a suitable coherent sequence of points, so that the limits $x_\infty, y_\infty$ and $t_\infty$ exist. We will do this in the next subsection.

Remark. Our formula (13) is similar to the one used in Cohen’s algorithm [3, Algorithm 7.4.8] for computing elliptic logarithms of real points on elliptic curves.
defined over \( \mathbb{R} \). The variable denoted \( c_n \) in [3] is related to our \( t_n \) (defined below) by \( c_n^2 = t_n^2 + a_n^2 \); setting \( c_\infty = \lim_{n \to \infty} c_n \), so that \( c_\infty^2 = t_\infty^2 + M^2 \), we can rewrite \( z_P \) as

\[
z_P = \pm \frac{1}{M} \arcsin \left( \frac{M}{c_\infty} \right),
\]

which is similar (up to sign) to the output of Cohen’s algorithm. This approach leaves an ambiguity of the sign of \( z_P \), which is resolved in [3] by considering the sign of \( y_0 \) at the end, something which is only possible in the real case. Using \( t_\infty \) instead of \( c_\infty \) avoids the ambiguity.

### 8.3 The elliptic logarithm iteration

Let \( P = (x, y) \in E(\mathbb{C}) \), where as above \( E \) is the elliptic curve with equation

\[
E : \quad Y^2 = 4(X - e_1)(X - e_2)(X - e_3).
\]

In order to compute the elliptic logarithm \( z_P \) of \( P \) using [13], we need to find a suitable coherent point sequence \( (P_n) \) starting at \( P_0 = P \). We iteratively compute \( P_1, P_2, \ldots \), using the explicit formulas for the isogenies \( \varphi_n \); at each stage there are two possible choices for \( P_n \), determined by choosing a specific sign for a square root. The main issue is how to make these choices in such a way that the sequences converge.

It is simpler in practice to use alternative models for the elliptic curves in the sequence, in which the isogeny formulas are simpler. We introduce these now. Let \( E_1' \) be the curve with equation

\[
E_1' : \quad R^2 = (T^2 + a^2)/(T^2 + b^2).
\]

We regard \( E_1' \) as a projective curve in \( \mathbb{P}^1 \times \mathbb{P}^1 \), with points at infinity given by \( (t, r) = (\infty, \pm 1), (\pm bi, \infty) \).

Define a map \( \alpha : E_1' \to E \) by \( (t, r) \mapsto (x, y) = (t^2 + e_1, -2rt(t^2 + b^2)) \), where as usual \( a^2 = e_1 - e_3 \) and \( b^2 = e_1 - e_2 \). This map is unramified and has degree 2; it sends \( (\infty, \pm 1) \mapsto O_E, (\pm bi, \infty) \mapsto (e_2, 0), (0, \pm a/b) \mapsto (e_1, 0) \) and \( (\pm ai, 0) \mapsto (e_3, 0) \).

Write \( a_1, b_1 \) for the arithmetic and geometric means of \( a, b \) as usual, set

\[
\begin{align*}
\epsilon_1 &= (a_1^2 + b_1^2)/3 = (a^2 + 6ab + b^2)/12, \\
\epsilon_2 &= (a_1^2 - 2b_1^2)/3 = (a^2 - 6ab + b^2)/12, \\
\epsilon_3 &= (b_1^2 - 2a_1^2)/3 = -(a^2 + b^2)/6,
\end{align*}
\]

so that \( E_1 \) has with Weierstrass equation

\[
E_1 : \quad Y_1^2 = 4(X_1 - \epsilon_1')(X_1 - \epsilon_2')(X_1 - \epsilon_3').
\]

Now \( E_1' \cong E_1 \) via the isomorphism \( \theta \) given by \( (t, r) \mapsto (x_1, y_1) \) where

\[
(x_1, y_1) = \left( \frac{1}{2}(t^2 + r(t^2 + a^2) + \frac{1}{6}(a^2 + b^2)), t(t^2 + r(t^2 + a^2) + \frac{1}{2}(a^2 + b^2)) \right).
\]

\(^2\)The sign of \( y \) here is chosen to avoid a minus sign in the elliptic logarithm formula [13].
with inverse
\[
(x_1, y_1) \mapsto (t, r) = \left( \frac{3y_1}{6x_1 + a^2 + b^2}, \frac{12x_1 + 5a^2 - b^2}{12x_1 + 5b^2 - a^2} \right).
\]
The composite \( \alpha \circ \theta^{-1} : E_1 \to E'_1 \to E \) is the 2-isogeny denoted \( \varphi \) in Section 5.

Given a complete 2-isogeny chain \( (E_n)_{n \geq 0} \) with \( E_0 = E \), as in Section 5 we define for each \( n \geq 1 \) a curve \( E'_n \) with equation \( R_2^{n} = (T_2^{n} + a_{n-1}^2)/(T_2^{n} + b_{n-1}^2) \), isomorphic to \( E_n \) via \( \theta_n \) (defined as for \( \theta = \theta_1 \) as above); these fit into a commutative diagram

\[
\begin{array}{ccc}
\cdots & \longrightarrow & E'_n \\
\varphi' & & \varphi' \\
\downarrow \theta_n & & \downarrow \theta_{n-1} \\
\cdots & \longrightarrow & E'_{n-1} \\
\end{array}
\]

where \( \varphi' : E'_n \to E'_{n-1} \) is the 2-isogeny which makes the diagram commute. A little algebra shows that \( \varphi_n \) is given by
\[
r_n = \frac{t_n^2 + a_{n-1}a_{n-2}}{t_n^2 + a_{n-1}b_n - 2} = \frac{a_{n-2}r_n^2 - a_{n-1}}{-b_n - 2r_n + a_{n-1}}, \quad t_n = \frac{r_n}{r_n - n}
\]

For any point sequence \( (P_n) \) (with \( P_n \in E_n(\mathbb{C}) \) and \( \varphi_{n+1}(P_{n+1}) = P_n \) for all \( n \geq 0 \)) we set \( P'_n = (r_n, t_n) = \theta_n^{-1}(P_n) \in E'_n(\mathbb{C}) \) for \( n \geq 1 \). Since \( \alpha(P'_1) = P_0 \), we have
\[
r^2 = \frac{x_0 - \epsilon_3}{x_0 - \epsilon_2}, \quad \text{and} \quad t_1 = -\frac{y_0}{2r_1(x_0 - \epsilon_2)} = \sqrt{x_0 - \epsilon};
\]

note that these equations determine \( r_1 \) (and then \( t_1 \)) up to sign. Next, from \( \varphi_n(P'_n) = P'_{n-1} \) for \( n \geq 2 \), we have
\[
r_n = \frac{a_{n-1}(r_{n-1} + 1)}{b_{n-2}r_{n-1} + a_{n-2}}, \quad \text{and} \quad t_n = r_nt_{n-1};
\]

again, these determine \( (r_n, t_n) \) up to sign.

Hence we may construct all possible point sequences \( (P'_n) \) with \( P'_n \in E'_n(\mathbb{C}) \) for \( n \geq 1 \), starting from \( P_0 = (x_0, y_0) \in E_0(\mathbb{C}) \) with \( y_0 \neq 0 \), by initialising
\[
r_1 = \sqrt{\frac{x_0 - \epsilon_3}{x_0 - \epsilon_2}}, \quad \text{and} \quad t_1 = -\frac{y_0}{2r_1(x_0 - \epsilon_2)}
\]
to determine \( P'_1 = (r_1, t_1) \), and then iterating the following to obtain \( P' = (r_n, t_n) \) for \( n \geq 2 \):
\[
r_n = \sqrt{\frac{a_{n-1}(r_{n-1} + 1)}{b_{n-2}r_{n-1} + a_{n-2}}}, \quad \text{and} \quad t_n = r_nt_{n-1}.
\]

Suitable choices of signs of \( r_n \) will be discussed below, which will ensure that these sequences converge. Then we will have \( r_\infty = \lim r_n = 1 \) and \( t_\infty = \lim t_n \) satisfying
\[
x_\infty = t_\infty^2 + \frac{2}{3}M^2, \quad y_\infty = -2t_\infty(t_\infty^2 + M^2),
\]
Both $R$ for all $R$ and $z \in \mathbb{N}$ lattice $\Lambda$ and $f$ we have $z$ is the value at $M$ we may regard $f$ of the left half-plane under $\mathbb{R}$ for all (or all but finitely many) $n$. We now show that we do obtain coherent, convergent sequences, provided that $8.4$ Choice of signs in the iteration

We now show that we do obtain coherent, convergent sequences, provided that for all (or all but finitely many) $n$ we choose the sign of $r_n$ so that $\Re(r_n) \geq 0$; always assuming that the isogeny sequence itself is optimal.

**Proposition 26.** With the notation of the previous section, assume that the AGM sequence satisfies $\Re(a_n/b_n) > 0$ for all $n \geq 0$.

If $\Re(r_n) \geq 0$ for all $n \geq 1$, then the point sequence $(P_n) = (\theta_n(r_n, t_n))$ determined by the iteratively defined sequence of pairs $(r_n, t_n)$ is coherent.

The same conclusion holds if $\Re(r_n) \geq 0$ for all but finitely many $n \geq 1$.

**Proof.** Recall that $\Lambda_n$ is the period lattice of $E_n$ for $n \geq 0$, with $\mathbb{Z}$-basis $w_1, w_2$ such that $w_1 = \pi/M(a_0, b_0)$ generates $\cap_n \Lambda_n$, and $\Lambda_n = (w_1, 2^n w_2)$ for all $n \geq 0$.

So for each $n$ there exists $z_n \in \mathbb{C}$, uniquely determined modulo $\Lambda_n$, such that $x_n = \varphi_{\Lambda_n}(z_n)$ and $y_n = \varphi'_{\Lambda_n}(z_n)$. We wish to show that the $z_n$ may be chosen independently of $n$.

Since

$$r_n = \frac{12x_n + 5a_n^2 - b_n^2}{12x_n + 5b_n^2 - a_n^2},$$

we may regard $r_n$ as the value at $z_n$ of an elliptic function $f_n$ of degree $2$ with respect to $\Lambda_n$. Similarly its square,

$$r_n^2 = \frac{x_n - e^{(n-1)}_1}{x_n - e^{(n-1)}_2},$$

is the value at $z_n$ of $f_n^2$, which is an elliptic function with respect to the larger lattice $\Lambda_{n-1}$. It follows that

$$f_n(z + 2^{n-1} w_2) = -f_n(z)$$

for all $z \in \mathbb{C}$ and all $n \geq 1$.

Since

$$\varphi_n(0) = O_E = \theta_n((\infty, 1))$$

and

$$\varphi_n(w_1/2) = (e_1^{(n)}, 0) = \theta_n((0, a_{n-1}/b_{n-1})),$$

we have $f_n(0) = 1$ and $f_n(w_1/2) = a_{n-1}/b_{n-1}$ for all $n \geq 1$.

We now consider the preimage $\mathcal{R}_n$ of the right half-plane under $f_n$, for $n \geq 1$. Since $f_n(w_1/2) = a_{n-1}/b_{n-1}$ and $\Re(a_{n-1}/b_{n-1}) > 0$, this contains $w_1/2$ for all $n$. Let $\mathcal{R}_n^0$ denote the connected component of $\mathcal{R}_n$ which contains $w_1/2$.

Both $\mathcal{R}_n$ and $\mathcal{R}_n^0$ are invariant under translation by $w_1$ (by periodicity of $f_n$), and $\mathcal{R}_n$ is the union of all translates of $\mathcal{R}_n^0$ by multiples of $2^n w_2$. The preimage of the left half-plane under $f_n$ is $\mathcal{L}_n = \mathcal{R}_n + 2^{n-1} w_2$, which is the union of the translates of $\mathcal{R}_n^0$ by odd multiples of $2^{n-1} w_2$. 

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Consider a point \( P_n = \varphi_n(z_n) \in E_n(\mathbb{C}) \), where \( z_n \in \mathbb{R}^n \). Its preimages in \( E_{n+1}(\mathbb{C}) \) are \( \varphi_{n+1}(z_n) \) and \( \varphi_{n+1}(z'_n) \), where \( z'_n = z_n + 2^n w_2 \). One of \( z_n, z'_n \) lies in \( \mathbb{R}^n \), the other in \( L_{n+1} \). Since \( w_1/2 \in \mathbb{R}^n \) for all \( k \), one can show that \( \mathbb{R}^n \subset \mathbb{R}^n_{n+1} \) (see Lemma 27 below). Hence, in fact, \( z_n \in \mathbb{R}^n \) and \( z'_n \in L_{n+1} \).

Hence, by choosing the sign of each \( r_n \) for \( n \geq 1 \) so that it lies in the right half-plane (for all \( n \geq 1 \)), we ensure that each \( P_n = \varphi_n(z_n) \), where \( z_n \in \mathbb{R}^n \) does not depend on \( n \). Hence the associated point sequence is coherent, as required.

For the last part, if \( \Re r_n > 0 \) only for \( n > n_0 \geq 0 \), then we simply apply the above argument to \( E_{n_0} \) and \( (P_n)_{n \geq n_0} \), noting that \( P_{n_0} \) is a lift of \( P_0 \) to \( E_{n_0}(\mathbb{C}) \), and that every elliptic logarithm of \( P_{n_0} \) is also one of \( P_0 \).

**Lemma 27.** In the notation of Proposition 27, \( \mathbb{R}^n \subset \mathbb{R}^n_{n+1} \) for all \( n \geq 1 \).

**Proof.** It suffices to show that \( \Re f_{n+1}(z) \) has constant sign for \( z \in \mathbb{R}^n \), since this sign is positive for \( z = w_1/2 \in \mathbb{R}^n \). If not, then there exists \( z \in \mathbb{R}^n \) such that \( \Re f_{n+1}(z) < 0 \), so \( f_{n+1}(z)^2 \) is real and negative. We show this to be impossible.

We have

\[
\begin{align*}
    r_{n-1} &= \frac{a_n - 2r_n^2}{b_n - 2r_n} + a_{n-1} = h_n(r_{n-1}^2) = g_n(r_{n-1}), \\

    \text{say, where } h_n \text{ is the linear fractional transformation} \\
    z &\mapsto \frac{a_n - 2z^2 - a_{n-1}}{-b_n - 2z + a_{n-1}}, \\

    \text{and } g_n(z) &= h_n(z^2). \text{ This implies that} \\
    f_n(z) &= g_{n+1}(f_{n+1}(z)) = h_{n+1}(f_{n+1}(z)^2).
\end{align*}
\]

To complete the proof we show that the image of the negative real axis under \( h_n \) is contained in the left half-plane, for all \( n \geq 1 \). Let \( t \in \mathbb{R} \) be negative, and set \( s = 2t - 1 < -1 \), and \( \alpha = a_n - 2/b_n - 2 \); then

\[
h_n(t) = \frac{s \alpha - 1}{\alpha - s},
\]

and we leave it to the reader to check that this has negative real part when \( s < -1 \) and \( \Re \alpha > 0 \).

We remark that this lemma implies that we always have \( \Re r_n > 0 \) for \( n \geq 2 \).

It is possible to have \( \Re r_1 = 0 \); this occurs if and only if \( x_0 \) lies on the open line segment between \( e_2 \) and \( e_3 \).

**8.5 The elliptic logarithm algorithm**

We summarise this section with the following algorithm.

**Algorithm 28** (Complex Elliptic Logarithm). Given an elliptic curve \( E \) defined over \( \mathbb{C} \) by the Weierstrass equation \( Y^2 = 4(X - e_1)(X - e_2)(X - e_3) \), and a non-2-torsion point \( P \in E(\mathbb{C}) \), compute an elliptic logarithm of \( P \).

**Input:** \( E \), with roots \( e_1, e_2, e_3 \), and \( P = (x_0, y_0) \in E(\mathbb{C}) \), with \( y_0 \neq 0 \).
1. Set \( a_0 = \sqrt{e_1 - e_3} \) and \( b_0 = \sqrt{e_1 - e_2} \), choosing the numbering of the roots (if necessary) and the signs so that \( |a_0 - b_0| < |a_0 + b_0| \).

2. Set \( r = \sqrt{(x_0 - e_3)/(x_0 - e_2)} \), with \( \Re r \geq 0 \).

3. Set \( t = -y_0/(2r(x_0 - e_2)) \) (so \( t^2 = x_0 - e_1 \)).

4. Repeat the following, for \( n = 1, 2, \ldots \):
   
   (a) set
   \[
   a_n = \frac{a_{n-1} + b_{n-1}}{2}, \quad b_n = \sqrt{a_{n-1}b_{n-1}},
   \]
   choosing the sign of \( b_n \) so that \( |a_n - b_n| < |a_n + b_n| \); 
   
   (b) set \( r \leftarrow \sqrt{a_n(r + 1)/(b_{n-1}r + a_{n-1})} \), with \( \Re r > 0 \).
   
   (c) set \( t \leftarrow rt \).

   until \( |a_n/b_n - 1| \) and \( |r - 1| \) are sufficiently small. Set \( M = \lim a_n \).

**Output:**

\[
zp = \frac{1}{M} \arctan \left( \frac{M}{t} \right)
\]

Note that the output value of \( z_P \) may not be in the fundamental parallelogram of the period lattice \( \Lambda \). However, assuming that the usual range for the \( \arctan \) function is used, where \( -\pi/2 < \Re \arctan(x) \leq \pi/2 \), we will have \( z_P = xw_1 + iyw_2 \) with \( x, y \in \mathbb{R} \) and \( -1/2 < x \leq 1/2 \).

For points \( P \) of order 2, choose the labelling of the roots so that \( P = (e_1, 0) \) and then take \( z_P = w_1/2 = \pi/(2M) \) where \( M = M(\sqrt{e_1 - e_3}, \sqrt{e_1 - e_2}) \).

### 8.6 The real case

For elliptic curves defined over \( \mathbb{R} \) there is some advantage in adapting the algorithm to use real arithmetic where possible, even though the algorithm as given above works perfectly well in this situation. We divide into cases as in sections 8.2 and 8.3 above.

#### 8.6.1 Curves with positive discriminant

Order the roots, which are all real, as in section 8.2 so that \( e_1 > e_2 > e_3 \); the real and imaginary periods \( w_1, w_2 \) are then given by \( 12 \).

Let \( P = (x_0, y_0) \in E(\mathbb{R}) \) with \( 2P \neq 0 \) (so \( y_0 \neq 0 \)). If \( P \) is in the connected component of the identity of \( E(\mathbb{R}) \) then \( x_0 > e_1 \), and it is immediate from the formulae given above that as well as all \( a_n, b_n \) being real and positive, so too are all \( r_n \), and the \( t_n \) are real and with constant sign (opposite to that of \( y_0 \)). Hence \( z_P \), the output of the algorithm, is real and in the interval \( |z_P| < w_1/2 \).

Now suppose that \( e_2 > x_0 > e_3 \), so that \( P \) is in the other real component. Now \( z_P = x_P + w_2/2 \) where \( x_P \) is real, and it suffices to compute \( x_P \). To do this we may replace \( P \) by \( P' = P + (e_3, 0) \) which is in the identity component and has elliptic logarithm equal to \( x_P \). A short calculation shows that we may compute \( x_P \) using the usual iteration, with the positive real initial values

\[
r' = a_0/\sqrt{e_1 - x_0}; \quad t' = r'y_0/2(x_0 - e_3).
\]
8.6.2 Curves with negative discriminant

As in 6.3, we order the roots so that $e_1$ is real and $\Re e_2 > 0$. Set $a_0 = \sqrt{e_1 - e_3} = x + yi$ where $x, y > 0$. The real period is $w_1 = \pi/M(a_0, b_0) = \pi/M(x, R)$ where $R = |a_0|$. Now let $\sqrt{x_0 - e_3} = u + iv$ with $u, v > 0$, and then set the initial values of $r$ and $t$ to $r_1 = (u + iv)/(u - iv)$ and $t_1 = -y_0/2(u^2 + v^2)$.

Applying the first step in the iteration, we find that $a_1 = x$ and $b_1 = R$, and also that $r_2 = \sqrt{ux/(ux + vy)}$, where the quantity inside the square root is real and positive, so we may take $r_2 > 0$ also, and $t_2 = r_2 t_1$ which is also real and with the opposite sign to $y_0$. Now the rest of the iteration may be carried out using real values for all quantities, and again the output value $z_P$ is real and satisfies $|z_P| < w_1/2$.

9 Examples

In the following examples, we will illustrate our method for computing the period lattices of elliptic curves over $\mathbb{C}$, and the elliptic logarithms of complex points. These examples were computed using the MAGMA implementation by the second author. All complex numbers in our examples are computed correctly up to 100 decimal places, but only the first 20 decimal places will be shown. Note that we implemented our own function for computing optimal AGM values, since the standard function in MAGMA does not always return an optimal one.

Example 1. Let $E$ be the elliptic curve over $\mathbb{C}$ given by the Weierstrass equation

$$E: \quad Y^2 = 4(X - e_1)(X - e_2)(X - e_3)$$

with

$$e_1 = 3 - 2i, \quad e_2 = 1 + i, \quad e_3 = -4 + i.$$ 

Observe that $\sum_{j=1}^{3} e_j = 0$. We will compute the period lattice of $E$ using the method described in Theorem 21. To do this, first we let $E_0 = E$ and calculate

$$a_0 = \sqrt{e_1 - e_3}, \quad b_0 = \sqrt{e_1 - e_2}, \quad c_0 = \sqrt{a_0^2 - b_0^2},$$

where the signs of $a_0, b_0, c_0$ are chosen so that (11) holds, i.e.

$$|a_0 - b_0| \leq |a_0 + b_0|, \quad |a_0 - c_0| \leq |a_0 + c_0|, \quad |c_0 - ib_0| \leq |c_0 + ib_0|.$$ 

In this example, one can verify that such $a_0, b_0, c_0$ are

$$a_0 = 2.70331029534753078867 \ldots - i0.55487525889334275023 \ldots$$

$$b_0 = 1.67414922803554004044 \ldots - i0.89597747612983812471 \ldots$$

$$c_0 = 2.23606797749978969640 \ldots .$$

In fact, all conditions in (11) are strictly inequalities in this case, so the period lattice of $E$ is non-rectangular. By Theorem 21 we have

$$w_1 = 1.29215151748713051904 \ldots + i0.44759218107818896608 \ldots$$

$$w_2 = 1.42661373451784507587 \ldots - i0.80963848056301882107 \ldots$$

$$w_3 = -0.13446221703071455682 \ldots + i1.25723066164120778715 \ldots$$

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and any two of $w_j$ form a $\mathbb{Z}$-basis for $\Lambda$ (where $\Lambda$ is the period lattice of $E$). In our computation, we also have $|w_1 - w_2 - w_3| \approx 10^{-100}$. As expected, these $w_j$ are minimal coset representatives of $2\Lambda$ in $\Lambda$.

Next, we wish to compute an elliptic logarithm of the point

\[ P = (2 - i, 8 + 4i) \in E(\mathbb{C}) \]

(which has infinite order). Using $a_0, b_0$ as above, Algorithm 28 gives

\[ z_P = -0.72212997914002299126 + i0.01717122412650902249 \ldots \]

Note that $z_P$ is only well-defined modulo $\Lambda$. Depending on the basis for $\Lambda$, the value $z_P$ obtained using Algorithm 28 may not lie in the fundamental parallelogram spanned by that basis. In our case, one can check that

\[ z_P = \ldots \]

and so $z_P$ is not in the fundamental parallelogram spanned by $\{w_1, w_2\}$. Finally, one may verify that, to the given precision, we have, as expected,

\[ \wp_\Lambda(z_P) = x(P), \quad \wp'_\Lambda(z_P) = y(P), \]

and

\[ \wp_\Lambda(w_j/2) = e_1, \quad \wp_\Lambda(w_2/2) = e_2, \quad \wp_\Lambda(w_3/2) = e_3, \]

and $\wp'_\Lambda(w_j/2) \approx 0$ for all $j = 1, 2, 3$.

**Example 2 (Rectangular Lattice).** Let $E$ be the elliptic curve over $\mathbb{C}$ given the Weierstrass equation

\[ E : \quad Y^2 = 4(X - e_1)(X - e_2)(X - e_3) \]

with

\[ e_1 = 1 + 3i, \quad e_2 = -4 - 12i, \quad e_3 = 3 + 9i. \]

Observe that $\sum_{j=1}^3 e_j = 0$ and all $e_j$ are collinear. By letting $E_0 = E$ and computing $a_0, b_0, c_0$ as before, we have

\[ a_0 = 1.47046851723128684330 \ldots - i2.04016608641756892919 \ldots \]

\[ b_0 = -3.22578581905571472955 \ldots - i2.32501487101070997214 \ldots \]

\[ c_0 = 2.75099469475848456460 \ldots - i3.81680125374499001591 \ldots \]

This time, however, we have $|a_0 - b_0| = |a_0 + b_0|$, while the other two conditions in (11) are strictly inequalities. Hence we have two minimal elements (up to sign) in one coset of $2\Lambda$ in $\Lambda$ (where $\Lambda$ is the period lattice of $E$), and so $\Lambda$ is rectangular.

To derive an orthogonal basis for $\Lambda$, first we let $w, w' = \pi/M(a_0, \pm b_0)$. In this example, we have

\[ w = -0.29920293143872535713 \ldots + i1.10940038117892953702 \ldots \]

\[ w' = 1.14708588706988127437 \ldots + i0.06697438037476960963 \ldots \]
One can check $|w| = |w'|$. Let $w_1 = (w + w')/2$ and $w_2 = (w - w')/2$. Then $w_1, w_2$ form an orthogonal basis for $\Lambda$, as proved in Lemma 5. Here, we have

$$w_1 = 0.42394147781557795862 \ldots + i0.5881873807768495733 \ldots$$
$$w_2 = -0.72314440925430331575 \ldots + i0.52121300040207996369 \ldots$$

Note that $\Re(w_2/w_1) = 0$, so both $\{w_1, w_2\}$ is indeed an orthogonal basis for $\Lambda$.

Let $z_P$ be an elliptic logarithm of the point $P = (3 + 2i, 28 - 14i) \in E(\mathbb{C})$ (note that $P$ has infinite order). Algorithm 28 shows that

$$z_P = -0.42599662534207481578 \ldots - i0.02491254923738153924 \ldots$$
$$\equiv (0.62858224538977667533 \ldots)w_1 + (0.37134662195976180031 \ldots)w_2.$$

Finally, we verify that

$$|\wp_A(z_P) - \pi(P)| \approx 10^{-98}, \quad |\wp_A'(z_P) - y(P)| \approx 10^{-97}.$$

Moreover, we have

$$|\wp_A(w_1/2) - e_2| \approx 10^{-99}$$
$$|\wp_A(w_2/2) - e_3| \approx 0$$
$$|\wp_A(w_2/2) - e_1| \approx 10^{-99},$$

and

$$|\wp_A'(w_1/2)| \approx 10^{-99}, \quad |\wp_A'(w_2/2)| \approx 10^{-99}, \quad |\wp_A'(w_2)| \approx 10^{-100}.$$

**Example 3.** Let $K = \mathbb{Q}(\theta)$ where $\theta$ is a root of the polynomial $x^3 - 2$. Let $E$ be the elliptic curve defined over $K$ given by the Weierstrass equation

$$E : \quad Y^2 = 4(X - \theta)(X - 1)(X + 1 + \theta).$$

Note that $K$ has one real embedding and one pair of complex embeddings. Let $E_1, E_2$ be the real and complex embedding of $E$ respectively, i.e.

$$E_1 : \quad Y^2 = 4(X - \sqrt[3]{2})(X - 1)(X + 1 + \sqrt[3]{2})$$
$$E_2 : \quad Y^2 = 4(X - \omega\sqrt[3]{2})(X - 1)(X + 1 + \omega\sqrt[3]{2})$$

where $\omega = \exp(2\pi i/3)$ is a cube root of unity. Since $E_1$ has three real roots, then the period lattice of $E_1$ is rectangular. In fact, by letting $e_1^{(0)} = \sqrt[3]{2}, e_2^{(0)} = 1, e_3^{(0)} = -1 - \sqrt[3]{2}$, we can compute $a_0, b_0, c_0$ satisfying (11) as

$$a_0 = 1.8761242291002530767 \ldots$$
$$b_0 = 0.5098245285339589808 \ldots$$
$$c_0 = 1.80552514518487755254 \ldots.$$

One can then verify that $|c_0 - ib_0| = |c_0 + ib_0|$. As before, we compute

$$w = \frac{\pi}{M(c_0, ib_0)} = 2.9013042594481764366 \ldots - i1.70677932803214980295 \ldots$$
$$w' = \frac{\pi}{M(c_0, -ib_0)} = \bar{w},$$

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and let \( w_1, w_2 = (w \pm w')/2 \). Then \( w_1, w_2 \) form an orthogonal basis for the period lattice of \( E_1 \). In this example, we have \( w_1 = \Re(w) \) and \( w_2 = i\Im(w) \).

Nevertheless, the period lattice of \( E_2 \) is non-rectangular, since all roots of \( E_2 \) are not collinear. In fact, by letting \( e_1^{(0)} = -1 - \omega \sqrt{2}, e_2^{(0)} = 1, e_3^{(0)} = \omega \sqrt{2} \) (here we must ensure that \( a_0, b_0, c_0 \) satisfy (11)), we have

\[
\begin{align*}
a_0 &= 1.10851094368231305521 \ldots - i0.98431471713501219051 \ldots \\
b_0 &= 0.436695170242855334726 \ldots - i1.24929666083200513980 \ldots \\
c_0 &= 1.34004098848655674756 \ldots - i0.40712323180652750769 \ldots \\
\end{align*}
\]

One can check that all conditions in (11) are strictly inequalities, hence this also confirms that the period lattice of \( E_2 \) is non-rectangular. By Theorem 21 we finally obtain

\[
\begin{align*}
w_1 &= 1.28194824894788708942 \ldots + i1.88277404359595361782 \ldots \\
w_2 &= 2.365576533380849535471 \ldots - i0.03808700290170419307 \ldots \\
w_3 &= -1.08362828486060826529 \ldots + i1.92086104649765781090 \ldots \\
\end{align*}
\]

with \( |w_1 - w_2 - w_3| \approx 10^{-100} \).

**Example 4.** Let \( E \) be the elliptic curve over \( \mathbb{C} \) given by the Weierstrass equation

\[ E : Y^2 = 4(X - e_1)(X - e_2)(X - e_3) \]

with \( e_1 = -1 - 3i, \ e_2 = 3 + i, \ e_3 = -2 + 2i \).

Observe that \( \sum_{j=1}^{3} e_j = 0 \) and \( |e_1 - e_3| = |e_2 - e_3| \). Thus \( e_1, e_2, e_3 \) are on an isosceles triangle. By letting \( E_0 = E \) and computing \( a_0, b_0, c_0 \) as before, we have

\[
\begin{align*}
a_0 &= 1.74628455779589152702 \ldots - i1.43161089573822132705 \ldots \\
b_0 &= 0.91017972112445468260 \ldots - i2.19736822693561993207 \ldots \\
c_0 &= 2.24711142509587014360 \ldots - i0.22250788030178260411 \ldots \\
\end{align*}
\]

Hence by Theorem 21 we obtain

\[
\begin{align*}
w_1 &= 0.81646689790312614904 \ldots + i1.1077333340066743861 \ldots \\
w_2 &= 1.36061503191563570645 \ldots - i0.20595647167234558716 \ldots \\
w_3 &= -0.54414813401250955741 \ldots + i1.31368980507301302578 \ldots \\
\end{align*}
\]

with \( |w_1 - w_2 - w_3| \approx 10^{-100} \). In addition, one can check that \( \Re(w_1/w_3) = 1/2 \) as claimed in Section 5.3. Let \( \Lambda \) be the period lattice of \( E \). We finally verify that \( |\varphi_\Lambda(w_j/2) - e_j| \approx 10^{-100} \) for all \( j = 1, 2, 3 \), and

\[
|\varphi_\Lambda'(w_1/2)| \approx 10^{-99}, \quad |\varphi_\Lambda'(w_2/2)| \approx 10^{-100}, \quad |\varphi_\Lambda'(w_3/2)| \approx 10^{-99}.
\]

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