We revisit, improve and complete some recent estimates of the $0^{++}$ and $1^{-}$ open charm ($\bar{c}\bar{d}(us)$) tetraquarks and the corresponding molecules masses and decay constants from QCD spectral sum rules (QSSR) by using QCD Laplace sum rule (LSR) within stability criteria where the factorised perturbative NLO corrections and the contributions of quark and gluon condensates up to dimension-6 in the OPE are included. We confront our results with the $DK$ invariant mass recently reported by LHCb from $B^+ \rightarrow D^+ (D^0 K^+)$ decays. We expect that the bump near the $DK$ threshold can be originated from the $0^{++}(D^0 K^+)$ molecule and/or $DK$ scattering. The prominent $X_0(2900)$ scalar peak and the bump $X_J(3150)$ (if $J = 0$) can emerge from a mixing between a scalar Tetramole ($T_{M0}$) (superposition of nearly degenerated hypothetical molecules and compact tetraquarks states having the same quantum numbers) and the first radial excitation of the $D^0 K^+$ molecule with a tiny mixing angle $\theta_0 \simeq (5.6 \pm 2.1)^0$. The $X_1(2900)$ and the $X_J(3350)$ (if $J = 1$) could be a mixture between the vector Tetramole $T_{M1}$ and its first radial excitation with an angle $\theta_1 \simeq (8.6 \pm 0.6)^0$.

QCD Spectral Sum Rules, Perturbative and Non-perturbative QCD, Exotic hadrons, Masses and Decay constants.

1 Introduction

QCD spectral sum rules (QSSR) à la SVZ [1, 2, 3] have been applied since 41 years to study successfully the hadron properties (masses, couplings and widths) and to extract some fundamental QCD parameters ($\alpha_s$, quark masses, quark and gluon condensates,...).

Beyond the successful quark model of Gell-Mann [18] and Zweig [19], Jaffe [20, 21] has introduced the four-quark states within the framework of the bag models for an attempt to explain the complex structure of the $I = 1, 0$ light scalar mesons (see also [22, 23, 24]).

In earlier papers, QSSR has been used to estimate the $I = 0$ light scalar mesons ($\sigma, f_0$) masses and widths [25, 26] assumed to be four-quark states. However, the true nature of these states remains still an open question as they can be well interpreted as glueballs / gluonia [27, 28, 29, 30, 31, 32].

More recently, after the recent discovery of many exotic states beyond the quark model found in different accelerator experiments [33, 34, 35] and references quoted therein.

1 For reviews, see e.g. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

2 For recent reviews, see e.g. [33, 34, 35] and references quoted therein.
In previous papers [36, 37, 38, 39], we have systematically studied the masses and couplings of the open-charm and -beauty molecules and tetraquark states using QSSR with the inclusion of factorised contributions at next-to-next-to leading order (N2LO) of perturbation theory (PT) and of the quark and gluon condensates up to dimension 5-7 using the inverse Laplace transform (LSR) [40, 41, 42, 43] of QCD spectral sum rules (QSSR). More recently, we have extended the analysis to the fully hidden scalar molecules and tetraquark states [44]. We have emphasized the importance of these PT corrections for giving a meaning on the input heavy quark mass which plays an important role in the heavy quark sector analysis. However, these corrections are numerically small in the $\overline{MS}$-scheme as there is a partial compensation of the radiative corrections in the ratio of sum rules used to extract these masses. This property (a posteriori) justifies the uses of the $\overline{MS}$ running masses in different channels at lowest order (LO) [33]. In this paper, we attempt to estimate,

![Figure 1](image-url)

Figure 1: LHCb preliminary results for the $D^- K^+$-invariant mass from $B \rightarrow D^+ D^- K^+$-decays.

from LSR, the masses and couplings of the $0^{++}$ and $1^-$ molecules and compact tetraquark states for interpreting the recent LHCb data from $B \rightarrow D^+ (D^- K^+)$ decays where in the $D^- K^+$-invariant mass shown in Fig.1 [45, 46], one finds two prominent peaks (units of MeV):

$$
\begin{align*}
M_{X_0}(0^+) &= (2866.3 \pm 6.5 \pm 2.0), \\
\Gamma_{X_0} &= (57.2 \pm 12.9), \\
M_{X_1}(1^-) &= (2904.1 \pm 4.8 \pm 1.3), \\
\Gamma_{X_1} &= (110.3 \pm 11.5).
\end{align*}
$$

(1)

We have studied in Ref. [36] the masses and couplings of the $D^0 K^0(0^{++})$ molecule and of the corresponding tetraquark states decaying into $D^0 K^0$ but not into $D^- K^+$ and found the lowest ground state masses:

$$
\begin{align*}
M_{DK} &= 2402(42) \text{ MeV}, \\
M_{\bar{c}\bar{d}us} &= 2395(68) \text{ MeV}, \\
f_{DK} &= 254(48) \text{ keV}, \\
f_{\bar{c}\bar{d}us} &= 221(47) \text{ keV}.
\end{align*}
$$

(2)

where the LSR parameters at which one obtains the previous optimal results are:

$$
\tau \simeq 0.7 \text{ GeV}^{-2}, \quad t_c \simeq (12 \sim 18) \text{ GeV}^2.
$$

(3)

We have used this result to interpret the nature of the $D^*_s(2317)$ compiled by PDG [47] where the existence of a $DK$ pole at this energy has been recently confirmed from lattice calculations of scattering amplitudes [48].

For the molecule state, we can interchange the $u$ and $d$ quarks in the interpolating current and deduce from $SU(2)$ symmetry that the $D^- K^+(0^{++})$ molecule mass is degenerated with the $D^0 K^0$ one. Compared with the LHCb data, one may invoke that this charged molecule can be responsible of the bump near the DK threshold around 2.4 GeV but is too light to explain the $X_{0.1}$ peaks.

For the tetraquark state, one may not use a simple $SU(2)$ symmetry (rotation of $u$ and $d$ quarks) to deduce the ones decaying into $D^- K^+$ due to our present ignorance of the diquark dynamics (for some attempts see [49, 50]).

Therefore, recent analysis based on QSSR at lowest order (LO) of perturbation theory (PT) using some specific tetraquarks and molecules configurations appear in the literature [51, 52, 53] (see also [54, 55] which appeared after the completion of this work) where masses around $(2.7 \sim 2.9)$ GeV have been derived.
However, due to the complexity of the QCD calculations, to some other possible configurations and to the different ways for extracting these predictions, we think that it is important to revisit and to improve these LO results by adding the NLO perturbative contributions and by using an optimization procedure where the values of the external LSR parameters [sum rule inverse energy variable (τ), QCD continuum threshold (t_c) and subtraction scale (μ)] are left as free parameters. This is the aim of the present paper.

2 The Laplace sum rule (LSR)

We shall work with the Finite Energy version of the QCD Inverse Laplace sum rules (LSR) and their ratios:

\[
\mathcal{L}_n^c(\tau, \mu) = \int_{(M_c + m_s)^2}^{t_c} dt^n \frac{e^{-\tau t}}{t} \text{Im} \Pi_{M, T}(t, \mu), \quad R_n^c(\tau) = \frac{\mathcal{L}_n^{c+1}}{\mathcal{L}_n^c},
\]

where \(M_c\) and \(m_s\) (we shall neglect \(u, d\) quark masses) are the on-shell / pole charm and running strange quark masses, \(\tau\) is the LSR variable, \(n = 0, 1\) is the degree of moments, \(t_c\) is the threshold of the “QCD continuum” which parametrizes, from the discontinuity of the Feynman diagrams, the spectral function \(\text{Im} \Pi_{M, T}(t, m^2, \mu^2)\) where \(\Pi_{M, T}(t, m^2, \mu^2)\) is the scalar correlator defined as:

\[
\Pi_{M, T}(q^2) = i \int d^4x \ e^{-iqx} \langle 0| \mathcal{O}^J_{M, T}(x) (\mathcal{O}^J_{M, T}(0))^{\dagger} | 0 \rangle,
\]

where \(\mathcal{O}^J_{M, T}(x)\) are the interpolating currents for the tetraquarks \(T\) and molecules \(M\) states. The superscript \(J\) refers to the spin of the particles.

3 The interpolating operators

- Scalar states \((0^+)\)
  - Tetraquarks

We shall work with the currents:

\[
\mathcal{O}_{SS}^0 = \epsilon_{ijk} \epsilon_{mnk} \left( u_i^T C \gamma_5 d_j \right) \left( \bar{c}_m \gamma_5 C \bar{s}_n^T \right), \quad \mathcal{O}_{PP}^0 = \epsilon_{ijk} \epsilon_{mnk} \left( u_i^T C \gamma_5 d_j \right) \left( \bar{c}_m C \bar{s}_n^T \right),
\]

\[
\mathcal{O}_{VV}^0 = \epsilon_{ijk} \epsilon_{mnk} \left( u_i^T C \gamma_5 \gamma_\mu d_j \right) \left( \bar{c}_m \gamma_\mu C \bar{s}_n^T \right), \quad \mathcal{O}_{AA}^0 = \epsilon_{ijk} \epsilon_{mnk} \left( u_i^T C \gamma_\mu d_j \right) \left( \bar{c}_m \gamma_\mu C \bar{s}_n^T \right), \quad (6)
\]

respectively for the Scalar-Scalar, Pseudoscalar-Pseudoscalar, Vector-Vector and Axial-Axial configurations.

- Molecules

We shall consider the following molecule currents:

\[
\mathcal{O}_{DK}^0 = (\bar{c} \gamma_5 d)(\bar{s} \gamma_5 u), \quad \mathcal{O}_{DK^*}^0 = (\bar{c} \gamma^\mu d)(\bar{s} \gamma_\mu u),
\]

\[
\mathcal{O}_{D_1 K_1}^0 = (\bar{c} \gamma_5 \gamma_\mu d)(\bar{s} \gamma_\mu \gamma_5 u), \quad \mathcal{O}_{D_0^* K_0^*}^0 = (\bar{c} d)(\bar{s} u). \quad (7)
\]

- Vector states \((1^-)\)
  - Tetraquarks

We shall work with the currents:

\[
\mathcal{O}_{AP}^1 = \epsilon_{mnk} \epsilon_{ijk} \left( \bar{c}_m \gamma_\mu C \bar{s}_n^T \right) \left( u_i^T C \gamma_5 d_j \right), \quad \mathcal{O}_{PA}^1 = \epsilon_{mnk} \epsilon_{ijk} \left( \bar{c}_m C \bar{s}_n^T \right) \left( u_i^T C \gamma_5 d_j \right),
\]

\[
\mathcal{O}_{SV}^1 = \epsilon_{ijk} \epsilon_{mnk} \left( u_i^T C \gamma_5 d_j \right) \left( \bar{c}_m \gamma_\mu \gamma_5 C \bar{s}_n^T \right), \quad \mathcal{O}_{YS}^1 = \epsilon_{ijk} \epsilon_{mnk} \left( u_i^T C \gamma_5 \gamma_\mu d_j \right) \left( \bar{c}_m \gamma_5 C \bar{s}_n^T \right). \quad (8)
\]
– Molecules

The corresponding currents are:

$$O_{D_{1 K}}^1 = (\bar{c}\gamma_\mu \gamma_5 d)(\bar{s}\gamma_5 u), \quad O_{D_{1 K_1}}^1 = (\bar{c}\gamma_5 d)(\bar{s}\gamma_\mu \gamma_5 u),$$

$$O_{D_{2 K_0}}^1 = (\bar{c}\gamma_\mu d)(\bar{s} u), \quad O_{D_{1 K_2}}^1 = (\bar{c} d)(\bar{s}\gamma_\mu u). \quad (9)$$

The lowest order (LO) perturbative (PT) QCD expressions including the quark and gluon condensates contributions up to dimension-six condensates of the corresponding two-point spectral functions are given in the Appendix.

• Higher Orders PT corrections to the Spectral functions

We extract the NLO PT corrections by considering that the molecule/tetraquark two-point spectral function is the convolution of the two ones built from two quark bilinear currents (factorization) which is justified because we have seen for the LO that the non-factorized part of the QCD diagrams gives negligible contribution and behaves like $1/N_c$ where $N_c$ is the number of colours (see some explicit examples in [39, 44]), while at order $\alpha_s$, this feature has been shown from the analysis of the four-quark correlator governing the $B^0 - B^0$ mixing [57, 58].

$$J^{P,S}(x) \equiv \bar{Q}[i\gamma_5, 1]Q \rightarrow \frac{1}{\pi} \text{Im} \psi^{P,S}(t),$$

$$J^{V,A}(x) \equiv \bar{Q}[\gamma_\mu, \gamma_\mu \gamma_5]Q \rightarrow \frac{1}{\pi} \text{Im} \psi^{V,A}(t).$$

(10)

In this way, we obtain the convolution integral [57, 56]:

\[ \frac{1}{\pi} \text{Im} \Pi_{M,\gamma}(t) = \theta(t - (M_c + m_s + m_d)^2), \quad \left(1 - \frac{t}{4\pi}\right)^{\frac{1}{2}} \int dt_1 \int dt_2 \sqrt{t_1 - t_2} \lambda^{1/2} K^{M}, \quad (11) \]

– Molecules:

\[ K^{S,P} = \left[ \frac{t_1 + t_2}{t} - 1 \right]^2 \times \frac{1}{\pi} \text{Im} \psi^{S,P}(t_1) \frac{1}{\pi} \text{Im} \psi^{S,P}(t_2), \]

\[ K^{V,A} = \left[ \frac{t_1 + t_2}{t} - 1 \right]^2 \times 8 \frac{t_1 t_2}{t^2} \times \frac{1}{\pi} \text{Im} \psi^{V,A}(t_1) \frac{1}{\pi} \text{Im} \psi^{V,A}(t_2), \quad (12) \]

with the phase space factor:

\[ \lambda = \left( \frac{1 - (\sqrt{t_1} - \sqrt{t_2})^2}{t} \right) \left( 1 - \frac{(\sqrt{t_1} + \sqrt{t_2})^2}{t} \right), \quad (13) \]

and $M_Q$ is the on-shell/pole perturbative heavy quark mass.

– The NLO perturbative expressions of the spectral functions built from bilinear quark-antiquark currents are known in the literature [4, 5, 12, 42, 59, 60, 61].

– We estimate the N2LO contributions assuming a geometric growth of the numerical coefficients [62, 63, 64, 65, 66]. We consider this contribution as an estimate of the error due to the truncation of the PT series.

• QCD input parameters

We shall use the QCD inputs in Table 1. The Renormalization Group Invariant parameters are defined as [4, 5]:

\[ \bar{m}_s(\tau) = \hat{m}_s(\bar{\beta}_1 a_s)^{-2/\beta_1}, \quad \langle \bar{q}q \rangle(\tau) = -\hat{\mu}_q^3(\bar{\beta}_1 a_s)^{2/\beta_1}, \]

\[ \langle \bar{q}Gq \rangle(\tau) = -M_0^2 \hat{\mu}_q^3(\bar{\beta}_1 a_s)^{1/3\beta_1}, \quad (14) \]

4
Table 1: QCD input parameters estimated from QSSR (Moments, LSR and ratios of sum rules). The running masses $\bar{m}_q$ are quoted by PDG [47].

| Parameters | Values | Hadron sources | Ref. |
|------------|--------|----------------|------|
| $\alpha_s(M_Z)$ | 0.1181(16)(3) | $M_{K^{0}, \pi^-}$ | [67] |
| $\bar{m}_c$ [MeV] | 1286(16) | $B_c \oplus J/\psi$ | [68, 69] |
| $\bar{m}_q$ [MeV] | 253(6) | Light | [4, 70, 71] |
| $\bar{m}_s$ [MeV] | 114(6) | Light | [4, 70, 71] |
| $\kappa \equiv \langle \bar{s}s \rangle / \langle \bar{d}d \rangle$ | 0.74 ± 0.06 | Light & heavy | [4, 70, 72, 73] |
| $M_0^2$ [GeV$^2$] | 0.8 ± 0.2 | Light & heavy | [67] |
| $\langle \alpha_s G^2 \rangle$ [GeV$^4$] | (6.35 ± 0.35) $\times 10^{-2}$ | Light & heavy | [4, 17, 74, 75, 76, 77, 78, 79] |
| $\langle g^3 G^2 \rangle / \langle \alpha_s G^2 \rangle$ | (8.2 ± 1.0) GeV$^2$ | $J/\psi$ | [80, 81, 82] |
| $\rho \alpha_s \langle \bar{q}q \rangle^2$ [GeV$^6$] | (5.8 ± 0.9) $\times 10^{-4}$ | Light, $\tau$-decays | [17, 83, 84, 77, 85] |

where $\beta_1 = -(1/2)(11 - 2n_f/3)$ is the first coefficient of the $\beta$ function for $n_f$ flavours: $\alpha_s \equiv \alpha_s(\tau)/\pi$; $\hat{\mu}_q$ is the spontaneous RGI light quark condensate [86]. The running charm mass is related to the on-shell (pole) mass used to compute the two-point correlator from the NLO relation [87, 88, 89, 90, 91]:

$$M_c(\mu) = \bar{m}_c(\mu) \left[1 + \frac{4}{3} a_s + \ln \left(\frac{\mu}{\bar{m}_c}\right) a_s + \mathcal{O}(a_s^2)\right]$$

(15)

The QCD condensates entering in the analysis are the light quark condensate $\langle \bar{q}q \rangle$, the gluon condensates $\langle \alpha_s G^2 \rangle \equiv \langle \alpha_s G_{\mu \nu} G^{\mu \nu} \rangle$ and $\langle g^3 G^2 \rangle \equiv \langle g^3 f_{abc} G_{\mu \nu} G^{\mu \nu} G_{\rho \sigma}^{abc} \rangle$, the mixed quark-gluon condensate $\langle \bar{q} G q \rangle \equiv \langle \bar{q} \sigma^{\mu \nu} (\lambda_s/2) G^{\mu \nu} q \rangle = M_0^2 \langle \bar{q}q \rangle$ and the four-quark condensate $\rho \alpha_s \langle \bar{q}q \rangle^2$, where $\rho \approx (3 \sim 4)$ indicates the deviation from the four-quark vacuum saturation.

4 Extracting the lowest ground state mass and coupling

In Ref. [36], we have extracted the lowest ground state mass by using the minimal duality ansatz:

$$\text{Im} \Pi_M \simeq f_M^2 M_M^8 \delta(t - M_M^2) + \Theta(t - t_c) \text{“Continuum”},$$

(16)

where the decay constant $f_M$ (analogue of $f_{\pi}$) is defined as:

$$\langle 0| O_{DK}^{\dagger} | DK \rangle = f_{DK} M_{DK}^4, \quad \langle 0| O_{D^* K}^{\dagger} | D^* K \rangle = \epsilon^n f_{D^* K} M_{D^* K}^5.$$

(17)

Interpolating currents constructed from bilinear (pseudo)scalar currents are not renormalization group invariants such that the corresponding decay constants possess anomalous dimension:

$$f_{DK}(\mu) = \tilde{f}_{DK} \left(-\beta_1 a_s\right)^{4/\beta_1} (1 - k_f a_s), \quad f_{D^* K} = \tilde{f}_{D^* K} \left(-\beta_1 a_s\right)^{2/\beta_1} (1 - k_f a_s/2)$$

(18)

where $\tilde{f}_M$ is the renormalization group invariant coupling and $-\beta_1 = (1/2)(11 - 2n_f/3)$ is the first coefficient of the QCD $\beta$-function for $n_f$ flavours. $a_s \equiv (\alpha_s/\pi)$ is the QCD coupling and $k_f = 2.028(2.352)$ for $n_f = 4(5)$ flavours.

Within a such parametrization, one obtains:

$$\mathcal{R}_6 \equiv \mathcal{R} \simeq M_M^7,$$

(19)

indicating that the ratio of moments appears to be a useful tool for extracting the mass of the hadron ground state as shown in the original SVZ papers [1, 2], different books, reviews and papers [4, 5, 6, 9, 11, 12, 13, 14, 15, 16, 17].

As $\tau$, $t_c$, and $\mu$ are free external parameters, we shall use stability criteria (minimum sensitivity on the variation of these parameters) to extract the lowest ground state mass and coupling (see more details discussions in the previous books and reviews).

Within the approach, one has obtained the masses of the lowest ground state $D^0 K^0$ molecule and of its $\bar{c}uds$ tetraquark states analogue quoted in Eq. 2.
5 The $0^{++} SS$ and $\bar{A}A$ tetraquarks

The two channels present similar features. Then, we show only explicitly the analysis of the $SS$ channel for a better understanding on the extraction of our numbers.

• $\tau$- and $t_c$-stabilities

We show in Fig.2a) the $\tau$- and $t_c$- dependence of the mass obtained from ratio of moments $R_0$. We have used $\mu=2.25$ GeV obtained in [36] which we shall check later on. The analysis of the coupling from the moment $L_0$ is shown in Fig. 2b). The results stabilize at $\tau \simeq 0.5$ GeV$^{-2}$ (inflexion point for the mass and minimum for the coupling). These results are compiled in Table 3 together with the different sources of errors.

![Figure 2](image1)

Figure 2: $f_{SS}$ and $M_{SS}$ as function of $\tau$ at NLO for different values of $t_c$, for $\mu=2.25$ GeV and for values of the QCD parameters given in Table 1.

• $\mu$-stability

We show in Fig.3 the $\mu$-dependence of the results for given $t_c=18$ GeV$^2$ and $\tau=0.49$ GeV$^{-2}$. One finds stability for :

\[ \mu = (2.25 \pm 0.25) \text{ GeV}, \]

which confirms the result in Ref. [36].

![Figure 3](image2)

Figure 3: $M_{SS}$ and $f_{SS}$ as function of $\mu$ at NLO for fixed values of $t_c$ and $\tau$ and for the values of the QCD parameters given in Table 1.
LO versus NLO results

We compare in Fig. 4 the $\tau$-behaviour of the mass and coupling for fixed $\tau$ and $\mu$ at LO and NLO of perturbative QCD in the $\overline{\text{MS}}$-scheme. One can notice that the NLO corrections are relatively small. At the stability point, the radiative corrections decreases the SS (rep. $AA$) mass by 46 (resp. 22) MeV and increases the coupling by 7 (resp. 8) keV.

6 The $0^{++}$ $\bar{P}P$ and VV tetraquarks

The two channels present similar features. Then, it suffices to show explicitly the analysis for the $PP$ channel.

• $\tau$- and $t_c$-stabilities

The analysis is shown in Fig. 5. Compared to the previous cases of SS and $AA$ configurations, one can notice that the stabilities are reached for smaller values of $\tau \simeq (0.15 \sim 0.20)$ GeV$^{-2}$ and for larger values of $t_c \geq 45$ GeV$^2$. This peculiar feature can be understood from the QCD expression of the corresponding correlators, where the $\langle \bar{\psi}\psi \rangle$ and $\langle \bar{\psi}\psi \rangle^2$ contribute largely and in a negative way which necessitates to work at higher energies for having a positive QCD expression of the spectral function and a convergence of the OPE.

As a consequence of the duality between the QCD and experimental sides, the resulting value of the lowest resonance mass becomes relatively high (see Table 3). Notice that working only with the ratio of moments
\( \mathcal{R}_0 \) to extract the meson mass without inspecting the moment \( \mathcal{L}_0 \) leads to misleading results as one can obtain a lower mass at larger values of \( \tau \) but one does not find that this low mass comes from the ratio of imaginary decay constants from \( \mathcal{L}_0 \).

- \( \mu \)-stability

The \( \mu \)-behaviour of the mass and coupling is shown in Fig. 6 where one can see inflexion points at \( \mu \simeq (2.25 \sim 2.35) \text{ GeV} \) which are consistent with the one for the SS and AA discussed previously.

![Figure 6: \( M_{PP} \) and \( f_{PP} \) as function of \( \mu \) at NLO for fixed values of \( t_c \) and \( \tau \) and for the values of the QCD parameters given in Table 1.](image)

- LO versus NLO results

We compare in Fig. 7 the \( \tau \)-behaviour of the mass and coupling for fixed \( t_c \) and \( \mu \) at LO and NLO of perturbative QCD in the \( \overline{\text{MS}} \)-scheme. One can notice that the \( \alpha_s \) corrections are large for \( PP \) which decrease the mass by 495 MeV while increase the coupling by 137 keV. On the contrary, the NLO corrections for \( VV \) are relatively small which decrease the mass by 20 MeV and increase the coupling by 52 MeV.

![Figure 7: \( M_{PP} \) and \( f_{PP} \) as function of \( \tau \) at LO and NLO for fixed values of \( t_c \) and \( \mu \) and for the values of the QCD parameters given in Table 1.](image)

7 The 0\(^{++} \) molecules

The behaviours of the different curves are similar to the previous cases.

- \( D^*K^* \)

The curves for the \( D^*K^* \) molecule are similar to the cases of SS and AA tetraquarks. Here the NLO corrections are \(-50 \text{ MeV} \) for the mass and \(+16 \text{ keV} \) for the coupling.
• $D_1K_1$, $D_0^*K_0^*$

The curves for the $D_1K_1$, $D_0^*K_0^*$ molecules are similar to the cases of the $PP$ and $VV$ tetraquarks. The NLO corrections are $-394$ (resp. $+36$) MeV for the mass and $+46$ (resp. $-106$) keV for the coupling of the $D_1K_1$ (resp. $D_0^*K_0^*$) molecules.

| States | Scalars ($0^+$) | Vectors ($1^-$) |
|--------|----------------|----------------|
| Parameters | $SS$ | $AA$ | $PP$ | $VV$ | $DK$ | $D^*K^*$ | $D_1K_1$ | $D_2K_2^*$ | $AP$ | $PA$ | $SV$ | $VS$ | $DK$ | $D^*K^*$ | $D_1K_1$ | $D_2K_2^*$ |
| $t_c$ [GeV$^2$] | 14.18 | 14.18 | 59.65 | 40.55 | 12.18 | 14.18 | 40.55 | 50.66 | 40.55 | 12.18 | 14.18 | 40.55 | 50.66 | 40.55 | 12.18 | 14.18 | 40.55 |
| $\tau$ [GeV$^{-2}$] | 45.52 | 47.53 | 20.21 | 19.22 | 73.77 | 44.52 | 26.27 | 13.15 | 20.23 | 34.49 | 32.48 | 17.21 | 32.47 | 22.24 | 40.48 | 20.22 |

Table 2: Values of the LSR parameters ($t_c$, $\tau$) used to deduce the optimal results given in Table 3. The results will be commented later on.

• Results

We show in Table 2 the different values of the LSR parameters ($t_c$, $\tau$) used to deduce the optimal results given in Table 3. The results will be commented later on.

| Observables | $\Delta t_c$ | $\Delta \tau$ | $\Delta \mu$ | $\Delta \alpha_s$ | $\Delta m_s$ | $\Delta m_c$ | $\Delta \psi \psi$ | $\Delta \kappa$ | $\Delta \alpha_s G^2$ | $\Delta M_0^2$ | $\Delta \psi \psi^*$ | $\Delta G^2$ | $\Delta M_G^2$ | Values |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------|
| 0$^+$ States | $I_G$ [keV] | | | | | | | | | | | | | |
| Tetraquark | | | | | | | | | | | | | | |
| SS | 15.00 | 0.40 | 7.70 | 3.95 | 0.65 | 5.05 | 0.85 | 2.99 | 0.28 | 0.80 | 38.60 | 0.09 | 0.68 | 336(43) |
| PP | 11.27 | 0.37 | 2.57 | 0.79 | 1.01 | 7.65 | 3.55 | 13.93 | 0.30 | 0.59 | 29.73 | 0.06 | 73.66 | 563(82) |
| VV | 41.57 | 13.45 | 5.59 | 1.61 | 0.04 | 10.14 | 4.35 | 13.90 | 0.28 | 1.04 | 22.24 | 0.13 | 92.29 | 728(106) |
| AA | 8.87 | 0.83 | 11.07 | 6.12 | 1.49 | 7.37 | 0.90 | 5.65 | 0.12 | 1.03 | 63.10 | 0.12 | 1.71 | 490(66) |
| Molecule | | | | | | | | | | | | | | |
| DK | 254(48) (Ref. [36]) | | | | | | | | | | | | | | |
| $D_1K_1$ | 11.08 | 4.44 | 1.56 | 1.07 | 9.30 | 0.45 | 5.64 | 24.52 | 0.20 | 2.87 | 30.61 | 0.01 | 116.39 | 656(124) |
| M$G$ [MeV] | | | | | | | | | | | | | | |
| Tetraquark | | | | | | | | | | | | | | |
| SS | 2.66 | 42.35 | 11.80 | 4.29 | 1.13 | 5.73 | 2.65 | 11.17 | 0.25 | 0.48 | 7.28 | 0.01 | – | 2733(47) |
| PP | 61.29 | 53.40 | 35.37 | 7.79 | 2.40 | 14.60 | 15.89 | 65.16 | 1.45 | 4.43 | 181.95 | 0.33 | – | 5920(214) |
| VV | 131.38 | 39.11 | 42.75 | 3.35 | 1.54 | 13.04 | 13.91 | 55.90 | 1.01 | 4.60 | 154.39 | 0.55 | – | 5674(219) |
| AA | 2.39 | 41.86 | 12.11 | 3.22 | 2.78 | 5.87 | 2.35 | 14.45 | 0.21 | 0.46 | 12.69 | 0.04 | – | 2659(48) |
| Molecule | | | | | | | | | | | | | | |
| DK | 2402(42) (Ref. [36]) | | | | | | | | | | | | | | |
| $D_1K_1$ | 53.01 | 26.05 | 33.95 | 8.92 | 12.11 | 3.15 | 18.09 | 178.09 | 0.66 | 21.33 | 168.60 | 0.07 | – | 5281(257) |
| $D_0^*K_0^*$ | 126.02 | 56.53 | 41.01 | 10.42 | 8.41 | 15.88 | 42.45 | 141.15 | 5.69 | 4.26 | 147.88 | 0.91 | – | 6285(255) |

Table 3: Sources of errors and predictions from LSR at NLO and for the decay constants and masses of the $(0^+)$ scalar molecules and tetraquark states. The errors from the QCD input parameters are from Table 1. $\Delta \mu$ is given in Eq. 20. We take $|\Delta \tau| = 0.02$ GeV$^{-2}$.

8 The 1$^-$ Vector states

• $AP$, $VS$ tetraquarks and $DK_1$, $D^*K^*_0$ molecules

Their corresponding curves behave like the ones of the $PP$, $VV$ ($0^{++}$) tetraquarks and of the $D_1K_1$, $D_0^*K_0^*$ ($0^{++}$) molecules. The $AP$ (resp. $VS$) mass decreases by 164 (resp. 117) MeV while the coupling increases by
Table 4: Same as in Table 3 but for the $1^−$ vector states.

29 (resp. 63) keV. For the $DK_1$ (resp. $D^*K^*_0$) molecules, the mass decreases by 351 (resp. 48) MeV while the coupling increases by 17 (resp. decreases by 50) keV.

- **PA, SV tetraquarks and $D_1K$, $D_0^*K^*$ molecules**

Their corresponding curves behave like the ones of SS, AA ($0^{++}$) tetraquarks and of the $DK$, $D^*K^*$ ($0^{++}$) molecules. The $PA$ (resp. $SV$) mass changes by $-3$ (resp. $+26$) MeV while the coupling changes by $+2$ (resp. $-2$) keV. For the $D_1K$ (resp. $D_0^*K^*$) molecules, the mass increases by 56 (resp. 46) MeV while the coupling decreases by 1 (resp. increases by 9) keV.

- **Results**

The results are shown in Table 4 and will be commented later on. The different values of the LSR parameters $(t_c, \tau)$ used to deduce the optimal results are shown in Table 2.

9 The first radial excitation ($\bar{D}K)_1^{0^{++}}$ of the $0^{++} (\bar{D}^−K^+)$ molecule

For this purpose, we extend the analysis in Ref. [36] by using a “Two resonances + $\theta(t−t_c)$ “QCD continuum” parametrization of the spectral function. To enhance the contribution of the 1st radial excitation [hereafter called $(D\bar{K})_1$], we shall also work with the ratio of moments $R_1$ in addition to $R_0$ for getting the mass of $(D\bar{K})_1$.

- **$\tau$- and $t_c$-stabilities**

We show in Fig 8 the $\tau$- and $t_c$-behaviours of the coupling from $\mathcal{L}_0$ and in Fig 9 the ones of the mass from $\mathcal{R}_0$ and $\mathcal{R}_1$ using as input the values of the lowest ground state mass and coupling obtained in Eq. 2. – One can notice that the coupling from $\mathcal{L}_0$ stabilizes for $\tau \simeq (0.55 \sim 0.65) \text{ GeV}^{-2}$ which is slightly lower than the value $\tau = 0.7 \text{ GeV}^{-2}$ corresponding to the one-resonance parametrization. The corresponding values of
Scalars (044, 46 48, 50 55-65 35-45 (52-56 30, 38 28, 36 36, 42 36, 40 36, 40 50 40 28-36 28-36 18-24 32-40 (20-32 28-36 28-36 28-36 45-48 38, 42 34, 38 34, 38

M\_\text{D(K)}\_1 \text{GeV} \begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{States} & \text{Scalars (0)} & \text{Vectors (1)} \\
\hline
\text{Parameters} & \text{(SN)}_1 & \text{(AA)}_1 & \text{(DK)}_1 & \text{(D\_K)} & \text{(P\_A)}_1 & \text{(SV)}_1 & \text{(D\_K)}_1 \\
\hline
\text{t}_c \text{[GeV]}^-2 & 28-36 & 28-36 & 18-24 & 32-40 & 20-32 & 28-36 & 28-36 \\
\text{\tau} \text{[GeV]}^-2 & 10^1 & 44, 46 & 48, 50 & 55-65 & 35-45 & 52-56 & 30, 38 \\
\text{M\_\text{C(K)}} & 36, 40 & 36, 40 & 50 & 40 & 45-48 & 38, 42 & 34, 38 \\
\hline
\end{array}

Table 5: Values of the LSR parameters (t_c, \tau) at the optimization region where the masses and couplings of the 1st radial excitations are obtained for the PT series up to NLO and for the OPE truncated at \langle g_s^3 G^3 \rangle.

- \mu-stability

We study in Fig. 10 the \mu-stability fixing t_c = 24 \text{GeV}^2 and for \tau \approx (0.3 \sim 0.6) \text{GeV}^{-2} depending on the value of \mu where the \tau-stability is reached.


10 The first radial excitation $(\bar{D}^*K^*)_1$ of the $0^{++}(\bar{D}^*K^*)$ molecule

- $\tau$- and $t_c$-stabilities

We show in Fig. 11a) the $\tau$- and $t_c$-behaviours of the coupling from $\mathcal{L}_0$ and in Fig 11b) the ones of the mass from $\mathcal{R}_1$ using as input the values of the lowest ground state mass and coupling obtained in Table 3. The optimal results are obtained for the $(t_c, \tau)$ values given in Table 5.

- $\mu$-stability

The $\mu$-behaviours of the coupling and mass are shown in Fig. 12.

We shall extract the mass and coupling of the 1st radial excitation $(\bar{D}^*K^*)_1$. We shall show the analysis.
explicitly as it may (a priori) differ from the one of \((DK)_1\) (position of the optimal \(\tau\) and value of \(t_c\)) as the mass of the \(\bar{D}K^*\) molecule is higher than that of \(\bar{D}K\).

11 The first radial excitation \((\bar{P}A)_1\) of the \(1^- (\bar{P}A)\) tetraquark

- \(\tau\)- and \(t_c\)-stabilities

![Graphs](image)

Figure 13: a) \(f_{(PA)_1}\) from the first moment \(\mathcal{L}_0\) and b) \(M_{(PA)_1}\) from the 2nd ratio of moments \(\mathcal{R}_1\) as function of \(\tau\) at NLO for different values of \(t_c\), for \(\mu=2.25\) GeV and for values of the QCD parameters given in Table 1.

We show in Fig. 13 the \(\tau\)- and \(t_c\)-behaviour of the coupling from \(\mathcal{L}_0\) and in Fig 14 the one of the mass from \(\mathcal{R}_1\) using as input the values of the lowest ground state mass and coupling obtained in Table 3. The behaviour of the curves for the coupling differs slightly from the previous cases.

- \(\mu\)-stability

The \(\mu\)-behaviour of the mass and coupling is shown in Fig. 14.

![Graphs](image)

Figure 14: \(\mu\)-behaviour of the \((PA)_1\) mass and coupling.

12 The \((D_0^*K^*)_1\), \((AA)_1\), \((SS)_1\) and \((D_1K)_1\), \((SV)_1\) radial excitations

These radial excitations correspond to the Low Mass ground states. The analysis of their \(\tau\), \(t_c\) and \(\mu\)-behaviours shows that they behave like the \((D^*K^*)_1\) and \((DK)_1\) studied explicitly in previous sections. We quote the results of the analysis in Tables 6 and 7.

13 Comments on the results

The results of the analysis are compiled in Tables 3, 4, 6 and 7.
tetraquark states with the same quantum numbers $J$ previously. Degenerated in masses which can be understood from the properties of the QCD spectral functions discussed splittings of masses and couplings given in Tables 3 and 4.

$\gamma \cdot D$ tetraquarks are respectively the same as for $SS, PP, VV, AA$ $SV, VS, AP, PA$ channels:

$SS, PP, VV, AA$ chiral condensate contributions for feature occurs for the vector $AP, PA, SV, VS$ of the non-factorized part of the LO PT expressions including quarks and gluon condensates in some other quark-antiquark currents for the estimate of the NLO corrections is a good approximant. The smallness vector) states. This feature indicates that the use of the convolution of two spectral functions of bilinear in the chiral limit $m = 0$ corrections are relatively negligible.

The estimate of uncalculated HO corrections using a geometric growth of the series also shows that these corrections are relatively negligible.

One can notice that:

- The NLO corrections are relatively small ($\leq 10\%$) which indicate a good convergence of the PT series. The estimate of uncalculated HO corrections using a geometric growth of the series also shows that these corrections are relatively negligible.

- To LO of PT there are no non-factorised contributions as one has only a product of two traces, while in the chiral limit $m_s = 0$, the PT expressions of the spectral functions are all the same for the scalar (resp. vector) states. This feature indicates that the use of the convolution of two spectral functions of bilinear quark-antiquark currents for the estimate of the NLO corrections is a good approximation. The smallness of the non-factorized part of the LO PT expressions including quarks and gluon condensates in some other channels (if any) has been also checked in several examples [36, 37, 38, 39, 44].

- The contributions of the gluon condensates $\langle \alpha_s G^2 \rangle$ and $\langle g^2 G^3 \rangle$ are negligible, while the ones of the chiral condensates $\langle \bar{\psi} \psi \rangle$, $\langle \bar{\psi} G \psi \rangle$, $\langle \bar{\psi} \psi \rangle^2$ are important in this open-charm channel.

- In the chiral limit $m_s = 0$ and for $\langle \bar{s}s \rangle = \langle \bar{q}q \rangle$ and in the scalar channels, the coefficients of the chiral condensate contributions for $SS, PP, VV, AA$ tetraquarks are opposite of the ones for (respectively) $D_0^* K_0^*$, $DK$, $D^* K^*$, $D_1 K_1$ molecules modulo some trivial factors. The same feature is observed in the vector channels: $SV, VS, AP, PA$ versus $D^* K_0^*, D_0^* K^*, D_1 K, DK_1$, which is due to the $\gamma_5$ property.

- In the chiral limit $m_s = 0$ and for $\langle \bar{s}s \rangle = \langle \bar{q}q \rangle$, the coefficients of the chiral condensate contributions for $SS, PP, VV, AA$ tetraquarks are respectively the same as for $DK, D_0^* K_0^*, D_1 K_1, D^* K^*$ molecules. The same feature occurs for the vector $AP, PA, SV, VS$ compared respectively to $D^* K_0^*, D_0^* K^*, D_1 K, DK_1$.

**QCD corrections and the spectral functions**

One can notice that:

- The NLO corrections are relatively small ($\leq 10\%$) which indicate a good convergence of the PT series. The estimate of uncalculated HO corrections using a geometric growth of the series also shows that these corrections are relatively negligible.

- To LO of PT there are no non-factorised contributions as one has only a product of two traces, while in the chiral limit $m_s = 0$, the PT expressions of the spectral functions are all the same for the scalar (resp. vector) states. This feature indicates that the use of the convolution of two spectral functions of bilinear quark-antiquark currents for the estimate of the NLO corrections is a good approximation. The smallness of the non-factorized part of the LO PT expressions including quarks and gluon condensates in some other channels (if any) has been also checked in several examples [36, 37, 38, 39, 44].

- The contributions of the gluon condensates $\langle \alpha_s G^2 \rangle$ and $\langle g^2 G^3 \rangle$ are negligible, while the ones of the chiral condensates $\langle \bar{\psi} \psi \rangle$, $\langle \bar{\psi} G \psi \rangle$, $\langle \bar{\psi} \psi \rangle^2$ are important in this open-charm channel.

- In the chiral limit $m_s = 0$ and for $\langle \bar{s}s \rangle = \langle \bar{q}q \rangle$ and in the scalar channels, the coefficients of the chiral condensate contributions for $SS, PP, VV, AA$ tetraquarks are opposite of the ones for (respectively) $D_0^* K_0^*$, $DK$, $D^* K^*$, $D_1 K_1$ molecules modulo some trivial factors. The same feature is observed in the vector channels: $SV, VS, AP, PA$ versus $D^* K_0^*, D_0^* K^*, D_1 K, DK_1$, which is due to the $\gamma_5$ property.

- In the chiral limit $m_s = 0$ and for $\langle \bar{s}s \rangle = \langle \bar{q}q \rangle$, the coefficients of the chiral condensate contributions for $SS, PP, VV, AA$ tetraquarks are respectively the same as for $DK, D_0^* K_0^*, D_1 K_1, D^* K^*$ molecules. The same feature occurs for the vector $AP, PA, SV, VS$ compared respectively to $D^* K_0^*, D_0^* K^*, D_1 K, DK_1$.

**Comparison of the molecules and tetraquarks states**

- The flip of signs of the chiral condensate contributions due to $\gamma_5$ in the chiral multiplets explains the large splittings of masses and couplings given in Tables 3 and 4.

- Our results indicate that the molecules and tetraquark states leading to the same final states are almost degenerated in masses which can be understood from the properties of the QCD spectral functions discussed previously.

- Therefore, we expect that the “physical state” is a combination of almost degenerated molecules and tetraquark states with the same quantum numbers $J^{PC}$ which we shall call: Tetramole ($T_{M,J}$).
\begin{table*}[h]
\centering
\begin{tabular}{ccccccccccccccccccc}
\hline
Observables & $\Delta t_c$ & $\Delta t_\tau$ & $\Delta t_\mu$ & $\Delta t_{\Delta m_c}$ & $\Delta m_c$ & $\Delta m_\tau$ & $\Delta m_\mu$ & $\Delta m_{\Delta G^2}$ & $\Delta m_{G^2}$ & $\Delta m_{G_0}$ & $\Delta f_G$ & $\Delta M_{(G)}$ & Values \\
\hline
\hline
$\Delta t_{\Delta m_c}$ & 4 & 1 & 8 & 6 & 8 & 2 & 7 & 31 & 1 & 6 & 30 & 0 & 9 & 29 & 37 & 149(67) \\
$\Delta t_{\Delta m_\tau}$ & 4 & 1 & 17 & 13 & 18 & 1 & 9 & 57 & 1 & 0 & 23 & 1 & 14 & 43 & 52 & 217(97) \\
$\Delta t_{\Delta m_\mu}$ & 7 & 1 & 8 & 6 & 8 & 1 & 3 & 3 & 0 & 2 & 24 & 0 & 10 & 28 & 37 & 251(55) \\
$\Delta t_{\Delta m_{\Delta G^2}}$ & 8 & 1 & 9 & 8 & 11 & 2 & 5 & 18 & 0 & 6 & 43 & 0 & 11 & 37 & 76 & 234(99) \\
$\Delta t_{\Delta m_{G^2}}$ & 65 & 9 & 53 & 33 & 38 & 3 & 46 & 85 & 8 & 65 & 375 & 2 & 115 & 436 & – & 4602(606) \\
$\Delta t_{\Delta m_{G_0}}$ & 73 & 9 & 40 & 27 & 34 & 1 & 28 & 175 & 2 & 1 & 271 & 3 & 89 & 339 & – & 4671(486) \\
$\Delta t_{\Delta f_G}$ & 177 & 8 & 79 & 56 & 60 & 12 & 24 & 26 & 2 & 21 & 277 & 1 & 116 & 405 & – & 4241(548) \\
$\Delta t_{\Delta M_{(G)}}$ & 59 & 7 & 63 & 43 & 45 & 12 & 28 & 118 & 2 & 49 & 375 & 1 & 110 & 373 & – & 4593(566) \\
\hline
\end{tabular}
\caption{The same as in Table 6 but for the $(1^-)$ vector molecules and tetraquarks states.}
\end{table*}

• Mass hierarchies

From our results, one can notice three classes of spectra:

- **The Low Mass ground states**
  These states are around 2.4 to 2.8 GeV. They are the $0^{++}$ $DK$ and $D^*K^*$ molecules and the $SS$ and $PP$ tetraquarks. For the $1^-$ states, we have the $D_1K$ and $D_0^*K^*$ molecules and $PA$ and $SV$ tetraquark states.

- **The High Mass ground states**
  These states are in the region above 4.5 GeV. For the $0^{++}$ states, they are the $D_1K_1$, $D_0^*K_0^*$ molecules and $PP$ and $VV$ tetraquark states, while for the $1^-$ states, they are the $DK_1$, $D^*K_0^*$ molecules and the $AP$, $VS$ tetraquarks. We have noticed that the shift of the results to higher masses is due to the positivity of the spectral function which is violated by working at lower energy scale due to the large negative contributions of chiral $\langle \bar{\psi}\psi \rangle$ and $\langle \bar{\psi}\psi \rangle^2$ in the OPE.

- **The First Radial excitations**
  The masses of the 1st radial excitations are compiled in Tables 6 and 7, where the large errors in their determinations have been induced by the ones of the ground state couplings.

\section{14 Comparison with existing results}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure15.png}
\caption{a) $d = 4 \langle \alpha_s G^2 \rangle$ gluon condensate; b) $d = 5 \langle \bar{\psi}G\psi \rangle$ mixed quark-gluon condensates; c) $d = 6 \langle g^3 G^3 \rangle$ triple gluon condensates.}
\end{figure}

• **QCD expressions of the spectral functions**

Among the four papers mentioned above [51, 52, 53, 54], only the one in [51] gives explicit QCD expressions of the $0^{++}$ $SS$ and $AA$ configurations of the tetraquarks. Comparing the results step by step, we realize that the contributions from the gluon in external fields are systematically missing (see Fig. 15). Hopefully,
the contributions of these diagrams as well as of the total $\langle \alpha_s G^2 \rangle$ and $\langle G^3 \rangle$ condensates are small which do not affect the numerical results within the precision of the approach.

- **Results of the analysis**

Though some of our lowest order (LO) PT results agree within the errors with the recent estimates from QSSR [51, 52, 53, 54], we emphasize that the inclusion of the NLO PT corrections are mandatory for making a sense on the use of the value of the charm quark $M_{\bar{C}}$ mass value in these analyses.

### 15 Confrontation with the LHCb data

From our previous results given in Table 3, one can notice that high mass states corresponding to the $(0^+)$ $D_1 K_1$, $D_s^0 K_0^*$ molecules and $PP$ and $VV$ tetraquark states and to the $(1^-)$ $D K_1$, $D^* K_0^*$ molecules and the $AP$, $VS$ tetraquarks states are above 5.5 GeV which are too far to contribute to the LHCb $D K$ invariant mass shown in Fig. 1.

- **The 2400 MeV bump around the $D K$ threshold**

This bump coincides with the $D^+ K^-$ mass 2400 MeV of the chiral partner of the $D^0 K^0$ obtained in [36]. Then, in addition to the $D K$ scattering process which can occur around the $D K$ threshold, we also expect that the $D K$ molecule may participate to this bump.

- **The $X_0(2866)$ state and the bump $X_J(3150)$**

Taking literally our results in Table 3, one can see that we have three (almost) degenerate states:

\[ M_{SS} = 2733(47) \text{ MeV}, \quad M_{AA} = 2659(48) \text{ MeV}, \quad \text{and} \quad M_{D-K*} = 2835(65) \text{ MeV}, \]  

and their couplings to the corresponding operators / currents are almost the same:

\[ f_{SS} = 336(43) \text{ keV}, \quad f_{AA} = 249(28) \text{ keV}, \quad \text{and} \quad f_{D-K*} = 394(58) \text{ keV}, \]  

We assume that the physical state, hereafter called Tetramole ($T_{M0}$), is a superposition of these nearly degenerated hypothetical states having the same quantum numbers. Taking its mass and coupling as (quadratic) means of the previous numbers, we obtain:

\[ M_{T_{M0}} \simeq 2726(30) \text{ MeV}, \quad f_{T_{M0}} \simeq 292(22) \text{ keV}. \]  

The ($T_{M0}$) tetramole is a good candidate for explaining the $X_0(2866)$ though its mass is slightly lighter. One can also see from Table 6 that the radial excitation $(D K)_1$ mass and coupling are:

\[ M_{(DK)_1} \simeq 3686(355) \text{ MeV}, \quad f_{(DK)_1} \simeq 199(62) \text{ keV}, \]  

which is the lightest $0^{++}$ first radial excitation. Assuming that the $X_J(3150)$ bump is a scalar state ($J=0$), we attempt to use a two-component mixing between the Tetramole ($T_{M0}$) and the $(D K)_1$ radially excited molecule:

\[
|X_0(2866)\rangle = \cos \theta_0 |T_{M0}\rangle + \sin \theta_0 |(DK)_1\rangle \\
|X_0(3150)\rangle = -\sin \theta_0 |T_{M0}\rangle + \cos \theta_0 |(DK)_1\rangle.
\]  

We reproduce the data with a tiny mixing angle:

\[ \theta_0 \simeq (5.6 \pm 2.1)^0. \]
• The $X_1(2904)$ state and the $X_J(3350)$ bump

  From our result in Table 3, one can see that there are four degenerate states with masses:

  $$M_{PA} = 2701(45) \text{ MeV}, \quad M_{SV} = 2616(49) \text{ MeV}$$
  $$M_{D1K} = 2701(48) \text{ MeV}, \quad M_{D*K^-} = 2771(57) \text{ MeV},$$

  and couplings:

  $$f_{PA} = 276(49) \text{ keV}, \quad f_{SV} = 244(43) \text{ keV}$$
  $$M_{D1K} = 186(44) \text{ keV}, \quad f_{D*K^-} = 210(49) \text{ keV}.$$  \hspace{1cm} (27)

  Like previously, we assume that the (unmixed) physical state is a combination of these hypothetical states. We take the mass and coupling of this Tetramole as the (geometric) means:

  $$M_{T_{M1}} = 2693(25) \text{ MeV}, \quad f_{T_{M1}} \simeq 228(23) \text{ keV},$$ \hspace{1cm} (29)

  where one may notice that it can contribute to the $X_1(2904)$ state but its mass is slightly lower.

  One can also notice from Tables 6 and 7 that the radial excitations are almost degenerated around 4.5 GeV from which one can extract the masses and couplings (geometric mean) of the spin 0 and 1 Tetramoles:

  $$M_{(T_{M0})1} \simeq 4542(209) \text{ MeV}, \quad M_{(T_{M1})1} \simeq 4532(273) \text{ MeV},$$
  $$f_{(T_{M0})1} \simeq 467(87) \text{ keV}, \quad f_{(T_{M1})1} \simeq 214(36) \text{ keV}.$$  \hspace{1cm} (30)

  Then, we may consider a two-component mixing of the spin 1 Tetramole $T_{M1}$ with its 1st radial excitation $(T_{M1})_1$ to explain the $X_1(2904)$ state and the $X_J(3350)$ bump assuming that the latter is a spin 1 state. The data can be fitted with a tiny mixing angle:

  $$\theta_1 \simeq (8.6 \pm 0.6)^0.$$  \hspace{1cm} (31)

16 Summary and conclusions

• Motivated by the recent LHCb data on the $D^- K^+$ invariant mass from $B \rightarrow D^+ D^- K^+$ decay (see Fig 1), we have systematically calculated the masses and couplings of some possible configurations of the molecules and tetraquarks states using QCD Laplace sum rules (LSR) within stability criteria where we have added to the LO perturbative term, the NLO radiative corrections which are essential for giving a meaning on the input value of the charm quark which plays an important role in the analysis. We consider our results as improvement and a completion of the results obtained to LO from QCD spectral sum rules [51, 52, 53, 54, 55].

• We have added to the PT contributions the ones of quark and gluon condensates up to dimension-6 in the OPE. We have noted that in some channels, these condensates contributions are large and negative which pushes to work at higher values of energy $s$ for respecting the positivity of the QCD spectral functions. By duality, the resulting values of the corresponding resonances masses are high (see Table 3) which are outside the region reached by LHCb.

• Therefore, we have used the results of the Low Mass resonances for an attempt to understand the whole range of $DK$ invariant mass found by LHCb:
  - The bump around the $DK$ threshold can be due to $DK$ scattering amplitude $\oplus$ the $DK(2400)$ lowest mass molecule.
  - The $(0^{++})$ $X_0(2866)$ and $X_J(3150)$ (if it is a $0^{++}$ state) can e.g result from a mixing of the Tetramole $(T_{M0})$ with the 1st radial excitation $(DK)_1$ of the molecule state $(DK)$ with a tiny mixing angle $\theta_0 \simeq 5.6^0$.
  - The $(1^-)$ $X_1(2904)$ and $X_J(3350)$ (if it is a $1^-$ state) can result from a mixing of the Tetramole $(T_{M1})$ with its 1st radial excitation $(T_{M1})_1$ with a tiny mixing angle $\theta_1 \simeq 8.6^0$.

• In addition to the QSSR approaches, some alternative explanations using other models are given in the literature [92, 93, 94, 95, 96, 97, 98, 99, 100, 101]. However, to our knowledge, the discussions in the existing papers are limited to the interpretation of the two resonances $X_0(2866)$ and $X_1(2904)$. We expect that more data on the precise quantum numbers of the $X_J(3150)$ and $X_J(3350)$ states are helpful for testing our model proposal. For completing our study, we plan to estimate the widths of the previous states in a future publication.
A Scalar Tetraquarks \((0^+)\)

Hereafter, we define: \(\langle G^2 \rangle \equiv \langle g^2 G^2 \rangle\), and \(\langle G^3 \rangle \equiv \langle g^3 G^3 \rangle\).

- **Scalar-Scalar configuration (SS)**

  \[
  \rho_0^{\text{pert}}(s) = \frac{m_5^6}{5 \cdot 3 \cdot 2^{12} \pi^6} \left[ 4x + 155 - 60 \left( 1 + \frac{4}{x} + \frac{2}{x^2} \right) \log(x) + \frac{80}{x} - \frac{220}{x^2} - \frac{20}{x^3} + \frac{1}{x^4} \right] \\
  - \frac{m_s m_{\pi}^2}{3 \cdot 2^{10} \pi^6} \left[ x + 28 - 12 \left( 1 + \frac{3}{x} + \frac{1}{x^2} \right) \log(x) - \frac{28}{x^2} - \frac{1}{x^3} \right] 
  \]

  \[
  \rho_0^{(g\bar{q})}(s) = \frac{m_5^5 \langle s \bar{s} \rangle}{3 \cdot 2^{9} \pi^4} \left[ x + 9 - 6 \left( 1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right] + \frac{m_s m_{\pi}^2 \langle s \bar{s} \rangle}{3 \cdot 2^{7} \pi^4} \left[ 2x + 3 - 6 \log(x) - \frac{6}{x} + \frac{1}{x^2} \right] 
  \]

  \[
  \rho_0^{(G^2)}(s) = \frac{m_5^4 \langle G^2 \rangle}{3^{2} \cdot 2^{12} \pi^6} \left[ 5x - 6 \left( 2 - \frac{1}{x} \right) \log(x) - \frac{9}{x} + \frac{4}{x^2} \right] - \frac{m_s m_{\pi}^2 \langle G^2 \rangle}{3 \cdot 2^{10} \pi^6} \left[ 7x + 15 - 3 \left( \frac{7}{3} + \frac{3}{x} \right) \log(x) - \frac{21}{x} - \frac{1}{x^2} \right] 
  \]

  \[
  \rho_0^{(Gq\bar{q})}(s) = - \frac{m_5^3 \langle \bar{q} G s \rangle}{2^{8} \pi^4} \left[ 3x + 4 - 2 \left( 4 + \frac{1}{x} \right) \log(x) - \frac{7}{x} \right] - \frac{m_s m_{\pi}^2 \langle \bar{q} G s \rangle}{3 \cdot 2^{8} \pi^4} \left[ 5x - 4 - 6 \log(x) - \frac{1}{x} \right] 
  \]

  \[
  \rho_0^{(Gq)}(s) = \frac{m_5^2 \langle \bar{q} q \rangle}{3 \cdot 2^{9} \pi^2} \left[ x - 2 + \frac{1}{x} \right] + \frac{m_s m_{\pi}^2 \langle \bar{q} q \rangle}{3 \cdot 2^{7} \pi^2} \left( 1 - x \right) 
  \]

  \[
  \rho_0^{(Gq)}(s) = - \frac{m_5^2 \langle \bar{q} q \rangle}{5 \cdot 3^{3} \cdot 2^{12} \pi^6} \left[ 76x - 3 - 150 \log(x) - \frac{72}{x} - \frac{1}{x^2} \right] 
  \]

- **Pseudoscalar-Pseudoscalar configuration (PP)**

  \[
  \rho_0^{\text{pert}}(s) = \frac{m_5^6}{5 \cdot 3 \cdot 2^{12} \pi^6} \left[ 4x + 155 - 60 \left( 1 + \frac{4}{x} + \frac{2}{x^2} \right) \log(x) + \frac{80}{x} - \frac{220}{x^2} - \frac{20}{x^3} + \frac{1}{x^4} \right] \\
  + \frac{m_s m_{\pi}^2}{3 \cdot 2^{10} \pi^6} \left[ x + 28 - 12 \left( 1 + \frac{3}{x} + \frac{1}{x^2} \right) \log(x) - \frac{28}{x^2} - \frac{1}{x^3} \right] 
  \]

  \[
  \rho_0^{(g\bar{q})}(s) = - \frac{m_5^5 \langle s \bar{s} \rangle}{3 \cdot 2^{9} \pi^4} \left[ x + 9 - 6 \left( 1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right] + \frac{m_s m_{\pi}^2 \langle s \bar{s} \rangle}{3 \cdot 2^{7} \pi^4} \left[ 2x + 3 - 6 \log(x) - \frac{6}{x} + \frac{1}{x^2} \right] 
  \]

  \[
  \rho_0^{(G^2)}(s) = \frac{m_5^4 \langle G^2 \rangle}{3^{2} \cdot 2^{12} \pi^6} \left[ 5x - 6 \left( 2 - \frac{1}{x} \right) \log(x) - \frac{9}{x} + \frac{4}{x^2} \right] + \frac{m_s m_{\pi}^2 \langle G^2 \rangle}{3 \cdot 2^{10} \pi^6} \left[ 7x + 15 - 3 \left( \frac{7}{3} + \frac{3}{x} \right) \log(x) - \frac{21}{x} - \frac{1}{x^2} \right] 
  \]

  \[
  \rho_0^{(Gq\bar{q})}(s) = \frac{m_5^3 \langle \bar{q} G s \rangle}{2^{8} \pi^4} \left[ 3x + 4 - 2 \left( 4 + \frac{1}{x} \right) \log(x) - \frac{7}{x} \right] - \frac{m_s m_{\pi}^2 \langle \bar{q} G s \rangle}{3 \cdot 2^{8} \pi^4} \left[ 5x - 4 - 6 \log(x) - \frac{1}{x} \right] 
  \]

  \[
  \rho_0^{(Gq)}(s) = - \frac{m_5^2 \langle \bar{q} q \rangle}{3 \cdot 2^{9} \pi^2} \left[ x - 2 + \frac{1}{x} \right] + \frac{m_s m_{\pi}^2 \langle \bar{q} q \rangle}{3 \cdot 2^{7} \pi^2} \left( 1 - x \right) 
  \]

  \[
  \rho_0^{(Gq)}(s) = \frac{m_5^2 \langle \bar{q} q \rangle}{5 \cdot 3^{3} \cdot 2^{12} \pi^6} \left[ 76x - 3 - 150 \log(x) - \frac{72}{x} - \frac{1}{x^2} \right] 
  \]
• Vector-Vector configuration (VV)

\[
\rho_0^{\text{pert}}(s) = \frac{m^8}{5 \cdot 3 \cdot 210 \pi^6} \left[ 4x + 155 - 60 \left( 1 + \frac{4}{x} + \frac{2}{x^2} \right) \log(x) + \frac{80}{x} - \frac{220}{x^2} - \frac{20}{x^3} + \frac{1}{x^4} \right] \\
+ \frac{m_s m_c^7}{3 \cdot 2^9 \pi^6} \left[ x + 28 - 12 \left( 1 + \frac{3}{x} + \frac{1}{x^2} \right) \log(x) - \frac{28}{x^2} - \frac{1}{x^3} \right]
\]

\[
\rho_0^{(qq)}(s) = -\frac{m_s^5 \langle \bar{s}s \rangle}{3 \cdot 2^9 \pi^4} \left[ x + 9 - 6 \left( 1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right] + \frac{m_s m_c^3 \langle \bar{s}s \rangle}{3 \cdot 2^9 \pi^4} \left[ 2x + 3 - 6 \log(x) - \frac{6}{x} + \frac{1}{x^2} \right]
\]

\[
\rho_0^{(G^2)}(s) = \frac{m_s^4 (G^2)}{3^2 \cdot 211 \pi^6} \left[ 7x + 18 - 6 \left( 4 + \frac{1}{x} \right) \log(x) - \frac{27}{x} + \frac{2}{x^2} \right] - \frac{m_s m_c^2 \langle G^2 \rangle}{3 \cdot 2^{11} \pi^6} \left[ 17x - 24 - 6 \left( 4 - \frac{3}{x} \right) \log(x) + \frac{3}{x} + \frac{4}{x^2} \right]
\]

\[
\rho_0^{(Gq)}(s) = \frac{m_s^3 \langle sG \rangle}{2^7 \pi} \left[ x - 2 \log(x) - \frac{1}{x} \right] - \frac{m_s m_c \langle \bar{s}q \rangle^2}{3 \cdot 2^7 \pi^4} \left[ x - 2 + \frac{1}{x} \right]
\]

\[
\rho_0^{(Gq)}(s) = \frac{m_s^2 \langle G^3 \rangle}{5 \cdot 3^3 \cdot 2^{10} \pi^6} \left[ 31x - 3 - 60 \log(x) - \frac{27}{x} - \frac{1}{x^2} \right]
\]

• Axial-Axial configuration (AA)

\[
\rho_0^{\text{pert}}(s) = \frac{m^8}{5 \cdot 3 \cdot 210 \pi^6} \left[ 4x + 155 - 60 \left( 1 + \frac{4}{x} + \frac{2}{x^2} \right) \log(x) + \frac{80}{x} - \frac{220}{x^2} - \frac{20}{x^3} + \frac{1}{x^4} \right] \\
- \frac{m_s m_c^7}{3 \cdot 2^9 \pi^6} \left[ x + 28 - 12 \left( 1 + \frac{3}{x} + \frac{1}{x^2} \right) \log(x) - \frac{28}{x^2} - \frac{1}{x^3} \right]
\]

\[
\rho_0^{(qq)}(s) = \frac{m_s^5 \langle \bar{s}s \rangle}{3 \cdot 2^9 \pi^4} \left[ x + 9 - 6 \left( 1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right] + \frac{m_s m_c^3 \langle \bar{s}s \rangle}{3 \cdot 2^9 \pi^4} \left[ 2x + 3 - 6 \log(x) - \frac{6}{x} + \frac{1}{x^2} \right]
\]

\[
\rho_0^{(G^2)}(s) = \frac{m_s^4 (G^2)}{3^2 \cdot 211 \pi^6} \left[ 7x + 18 - 6 \left( 4 + \frac{1}{x} \right) \log(x) - \frac{27}{x} + \frac{2}{x^2} \right] + \frac{m_s m_c^2 \langle G^2 \rangle}{3 \cdot 2^{11} \pi^6} \left[ 17x - 24 - 6 \left( 4 - \frac{3}{x} \right) \log(x) + \frac{3}{x} + \frac{4}{x^2} \right]
\]

\[
\rho_0^{(Gq)}(s) = -\frac{m_s^3 \langle sG \rangle}{2^7 \pi} \left[ x - 2 \log(x) - \frac{1}{x} \right] - \frac{m_s m_c \langle \bar{s}q \rangle^2}{3 \cdot 2^7 \pi^4} \left[ x - 2 + \frac{1}{x} \right]
\]

\[
\rho_0^{(Gq)}(s) = \frac{m_s^2 \langle G^3 \rangle}{5 \cdot 3^3 \cdot 2^{10} \pi^6} \left[ 31x - 3 - 60 \log(x) - \frac{27}{x} - \frac{1}{x^2} \right]
\]
• $D_0^* K^*$ molecule configuration

\[
\rho_{0}^{\scriptscriptstyle \text{pert}}(s) = \frac{m_c^8}{5 \cdot 2^{14} \pi^6} \left[ 4x+155-60 \left(1 + \frac{4}{x} + \frac{2}{x^2} \right) \log(x) + \frac{80}{x} - \frac{220}{x^2} - \frac{20}{x^3} + \frac{1}{x^4} \right]
\]

\[
\rho_{0}^{\langle \bar{q}q \rangle}(s) = -\frac{m_c^8 \langle \bar{q}q \rangle}{2^{28} \pi^4} \left[ x+9-6 \left(1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right] + \frac{m_c^8 m_s^2}{2^{9} \pi^4} \left(2 \langle \bar{q}q \rangle + \langle \bar{s}s \rangle \right) \left[2x+3-6 \log(x) - \frac{6}{x} + \frac{1}{x^2} \right]
\]

\[
\rho_{0}^{\langle G^2 \rangle}(s) = \frac{m_c^8 \langle G^2 \rangle}{3 \cdot 2^{13} \pi^6} \left[ 4x-9-6 \left(1 - \frac{2}{x} \right) \log(x) + \frac{5}{x^2} \right]
\]

\[
\rho_{0}^{\langle \bar{q}Gq \rangle}(s) = \frac{3m_c^8 \langle \bar{q}Gq \rangle}{2^{28} \pi^4} \left[ x+2 - \left(3 + \frac{1}{x} \right) \log(x) - \frac{3}{x} \right] - \frac{m_c^8 m_s^2}{2^{9} \pi^4} \left(3 \langle \bar{q}Gq \rangle - 2 \langle \bar{s}Gs \rangle \right) \left[x-2 + \frac{1}{x} \right]
\]

\[
\rho_{0}^{\langle \bar{q}\bar{q} \rangle^2}(s) = -\frac{m_c^8 \langle \bar{q}\bar{q} \rangle \langle \bar{s}s \rangle}{2^{15} \pi^2} \left[ x-2 + \frac{1}{x} \right] + \frac{m_c^8 m_s^2}{2^{9} \pi^4} \left(2 \langle \bar{q}\bar{q} \rangle^2 + \langle \bar{q}s \rangle \langle \bar{s}s \rangle \right) (1-x)
\]

\[
\rho_{0}^{\langle G^3 \rangle}(s) = \frac{m_c^8 \langle G^3 \rangle}{5 \cdot 3^{2} \cdot 2^{14} \pi^6} \left[ 166x-3-330 \log(x) - \frac{162}{x} - \frac{1}{x^2} \right]
\]

• $D K$ molecule configuration (see Re. [36])

\[
\rho_{0}^{\scriptscriptstyle \text{pert}}(s) = \frac{m_c^8}{5 \cdot 2^{14} \pi^6} \left[ 4x+155-60 \left(1 + \frac{4}{x} + \frac{2}{x^2} \right) \log(x) + \frac{80}{x} - \frac{220}{x^2} - \frac{20}{x^3} + \frac{1}{x^4} \right]
\]

\[
\rho_{0}^{\langle \bar{q}q \rangle}(s) = \frac{m_c^8 \langle \bar{q}q \rangle}{2^{8} \pi^4} \left[ x+9-6 \left(1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right] - \frac{m_c^8 m_s^2}{2^{7} \pi^4} \left(2 \langle \bar{q}q \rangle - \langle \bar{s}s \rangle \right) \left[2x+3-6 \log(x) - \frac{6}{x} + \frac{1}{x^2} \right]
\]

\[
\rho_{0}^{\langle G^2 \rangle}(s) = \frac{m_c^8 \langle G^2 \rangle}{3 \cdot 2^{13} \pi^6} \left[ 4x-9-6 \left(1 - \frac{2}{x} \right) \log(x) + \frac{5}{x^2} \right]
\]

\[
\rho_{0}^{\langle \bar{q}Gq \rangle}(s) = -\frac{3m_c^8 \langle \bar{q}Gq \rangle}{2^{28} \pi^4} \left[ x+2 - \left(3 + \frac{1}{x} \right) \log(x) - \frac{3}{x} \right] + \frac{m_c^8 m_s^2}{2^{9} \pi^4} \left(3 \langle \bar{q}Gq \rangle + 2 \langle \bar{s}Gs \rangle \right) \left[x-2 + \frac{1}{x} \right]
\]

\[
\rho_{0}^{\langle \bar{q}\bar{q} \rangle^2}(s) = \frac{m_c^8 \langle \bar{q}\bar{q} \rangle \langle \bar{s}s \rangle}{2^{8} \pi^2} \left[ x-2 + \frac{1}{x} \right] + \frac{m_c^8 m_s^2}{2^{9} \pi^4} \left(2 \langle \bar{q}\bar{q} \rangle^2 - \langle \bar{q}s \rangle \langle \bar{s}s \rangle \right) (1-x)
\]

\[
\rho_{0}^{\langle G^3 \rangle}(s) = \frac{m_c^8 \langle G^3 \rangle}{5 \cdot 3^{2} \cdot 2^{14} \pi^6} \left[ 166x-3-330 \log(x) - \frac{162}{x} - \frac{1}{x^2} \right]
\]

• $D^* K^*$ molecule configuration

\[
\rho_{0}^{\scriptscriptstyle \text{pert}}(s) = \frac{m_c^8}{5 \cdot 2^{12} \pi^6} \left[ 4x+155-60 \left(1 + \frac{4}{x} + \frac{2}{x^2} \right) \log(x) + \frac{80}{x} - \frac{220}{x^2} - \frac{20}{x^3} + \frac{1}{x^4} \right]
\]

\[
\rho_{0}^{\langle \bar{q}q \rangle}(s) = \frac{m_c^8 \langle \bar{q}q \rangle}{2^{7} \pi^4} \left[ x+9-6 \left(1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right] - \frac{m_c^8 m_s^2}{2^{7} \pi^4} \left(\langle \bar{q}q \rangle - \langle \bar{s}s \rangle \right) \left[2x+3-6 \log(x) - \frac{6}{x} + \frac{1}{x^2} \right]
\]

\[
\rho_{0}^{\langle G^2 \rangle}(s) = \frac{m_c^8 \langle G^2 \rangle}{3 \cdot 2^{11} \pi^6} \left[ x+9-6 \left(1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right]
\]

\[
\rho_{0}^{\langle \bar{q}Gq \rangle}(s) = -\frac{3m_c^8 \langle \bar{q}Gq \rangle}{2^{28} \pi^4} \left[ x-2 \log(x) - \frac{1}{x} \right] + \frac{m_c^8 m_s^2}{2^{8} \pi^4} \left(3 \langle \bar{q}Gq \rangle - 2 \langle \bar{s}Gs \rangle \right) \left[x-2 + \frac{1}{x} \right]
\]

\[
\rho_{0}^{\langle \bar{q}\bar{q} \rangle^2}(s) = \frac{m_c^8 \langle \bar{q}\bar{q} \rangle \langle \bar{s}s \rangle}{2^{4} \pi^2} \left[ x-2 + \frac{1}{x} \right] + \frac{m_c^8 m_s^2}{2^{4} \pi^2} \left(4 \langle \bar{q}\bar{q} \rangle^2 - \langle \bar{q}s \rangle \langle \bar{s}s \rangle \right) (1-x)
\]

\[
\rho_{0}^{\langle G^3 \rangle}(s) = -\frac{m_c^8 \langle G^3 \rangle}{5 \cdot 3^{2} \cdot 2^{12} \pi^6} \left[ 14x+3-30 \log(x) - \frac{18}{x} + \frac{1}{x^2} \right]
\]

20
• $D_1 K_1$ molecule configuration

$$
\rho_0^{(\text{pert})}(s) = \frac{m_c^8}{5 \cdot 212 \pi^6} \left[ 4x + 155 - 60 \left( 1 + \frac{4}{x} + \frac{2}{x^2} \right) \log(x) + \frac{80}{x} - \frac{220}{x^2} + \frac{20}{x^3} + \frac{1}{x^4} \right]
$$

$$
\rho_0^{(\bar{q}q)}(s) = -\frac{m_c^5(\bar{q}q)}{2 \pi^4} \left[ x + 9 - 6 \left( 1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right] + \frac{m_s m_c^4}{2 \pi^4} \left( \langle \bar{q}q \rangle + \langle \bar{s}s \rangle \right) \left[ 2x - 3 - 6 \log(x) - \frac{6}{x} - \frac{1}{x^2} \right]
$$

$$
\rho_0^{(G^2)}(s) = \frac{m_c^4(G^2)}{3 \cdot 2^{11} \pi^6} \left[ x + 9 - 6 \left( 1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right]
$$

$$
\rho_0^{(G^3)}(s) = \frac{3m_c^2(\bar{q}Gq)}{28 \pi^4} \left[ x - 2 \log(x) - \frac{1}{x} \right] - \frac{m_s m_c^2}{28 \pi^4} \left( 3 \langle \bar{q}Gq \rangle + 2 \langle \bar{s}Gs \rangle \right) \left[ x - 2 - \frac{1}{x} \right]
$$

$$
\rho_0^{(\bar{q}q)^2}(s) = -\frac{m_c^2(\bar{q}q \langle \bar{s}s \rangle)}{2^{13} \pi^4} \left[ x - 2 + \frac{1}{x} \right] + \frac{m_s m_c}{2^{13} \pi^4} \left( 4 \langle \bar{q}q \rangle^2 + \langle \bar{q}q \rangle \langle \bar{s}s \rangle \right) \left( 1 - x \right)
$$

$$
\rho_0^{(G^3)}(s) = -\frac{m_c^2(G^3)}{5 \cdot 3^2 \cdot 2^{12} \pi^6} \left[ 14x + 3 - 30 \log(x) - \frac{18}{x} + \frac{1}{x^2} \right]
$$

\[ B \] Vector Tetraquark and Molecule States \($1^-$\)

• Axial-Pseudoscalar tetraquark configuration (AP)

$$
\rho_1^{(\text{pert})}(s) = \frac{m_c^8}{5 \cdot 3^2 \cdot 2^{13} \pi^6} \left[ x^2 + 555 - 60 \left( 3 + \frac{16}{x} + \frac{9}{x^2} \right) \log(x) + \frac{480}{x} \right] - \frac{945 - 96}{x^3} + \frac{5}{x^4}
$$

$$
\rho_1^{(\bar{q}q)}(s) = \frac{m_c^5(\bar{q}q \langle \bar{s}s \rangle)}{2^{13} \pi^4} \left[ x + 28 - 12 \left( 1 + \frac{3}{x} + \frac{1}{x^2} \right) \log(x) - \frac{28}{x^2} - \frac{1}{x^3} \right]
$$

$$
\rho_1^{(G^2)}(s) = \frac{m_c^4(G^2)}{3^2 \cdot 2^{13} \pi^6} \left[ 5x^2 + 12x + 198 - 48 \left( 3 + \frac{2}{x} \right) \log(x) - \frac{212}{x^2} - \frac{3}{x^3} \right]
$$

$$
\rho_1^{(G^3)}(s) = -\frac{m_c^3(\bar{s}Gs)}{2^7 \pi^4} \left[ x - 2 \log(x) - \frac{1}{x} \right] - \frac{m_s m_c^2(\bar{s}Gs)}{3^2 \cdot 2^9 \pi^4} \left[ 5x^2 - 12x - 9 + 18 \log(x) + \frac{16}{x} \right]
$$

$$
\rho_1^{(\bar{q}q)^2}(s) = -\frac{m_c^2(\bar{q}q \langle \bar{s}s \rangle)}{3 \cdot 2^4 \pi^4} \left[ x^2 - 3 + \frac{2}{x} \right] - \frac{m_s m_c (\bar{q}q)^2}{3 \cdot 2^4 \pi^4} \left( 1 - x \right)
$$

$$
\rho_1^{(G^3)}(s) = \frac{m_c^2(G^3)}{5 \cdot 3^3 \cdot 2^{14} \pi^6} \left[ 11x^2 - 240x + 24 + 420 \log(x) + \frac{208}{x} - \frac{3}{x^2} \right]
$$

21
• Pseudoscalar-Axial tetraquark configuration (PA)

\[
\rho_1^{pert}(s) = \frac{m_s^8}{5 \cdot 3^2 \cdot 2^{14} \pi^6} \left[ x^2 + 555 - 60 \left( 3 + \frac{16}{x} + \frac{9}{x^2} \right) \log(x) + \frac{480}{x^2} - \frac{945}{x^3} - \frac{96}{x^4} + \frac{5}{x^5} \right]
+ \frac{m_s m_c}{5 \cdot 3 \cdot 2^{12} \pi^6} \left[ x^2 + 340 - 60 \left( 2 + \frac{8}{x} + \frac{3}{x^2} \right) \log(x) + \frac{80}{x^2} - \frac{405}{x^3} - \frac{16}{x^4} \right]
\]

\[
\rho_1^{(\bar{s}s)}(s) = \frac{-m_s^5 \langle \bar{s}s \rangle}{3^2 \cdot 2^8 \pi^4} \left[ x^2 + 72 - 12 \left( 3 + \frac{4}{x} \right) \log(x) - \frac{64}{x^2} + \frac{9}{x^3} - \frac{3}{x^4} \right]
+ \frac{m_s m_c^4 \langle \bar{s}s \rangle}{3 \cdot 2^9 \pi^4} \left[ x^2 + 12 - 12 \log(x) - \frac{16}{x} + \frac{3}{x^2} \right]
\]

\[
\rho_1^{(G^2)}(s) = \frac{m_s^4 \langle G^2 \rangle}{3^3 \cdot 2^{13} \pi^6} \left[ x^2 - 24x - 54 + 24 \left( 3 + \frac{1}{x} \right) \log(x) + \frac{80}{x} - \frac{3}{x^2} \right]
+ \frac{m_s m_c^3 \langle G^2 \rangle}{3^2 \cdot 2^{12} \pi^6} \left[ 2x^2 - 12x + 45 - 6 \left( 1 + \frac{4}{x} \right) \log(x) - \frac{32}{x} - \frac{3}{x^2} \right]
\]

\[
\rho_1^{(\bar{s}Gs)}(s) = \frac{m_s^3 \langle \bar{s}Gs \rangle}{3^2 \cdot 2^9 \pi^4} \left[ x^2 + 9 - 3 \left( 3 + \frac{1}{x} \right) \log(x) - \frac{10}{x} \right]
- \frac{m_s m_c^2 \langle \bar{s}Gs \rangle}{3 \cdot 2^9 \pi^4} \left[ 7x^2 - 3 - 18 \log(x) - \frac{4}{x} \right]
\]

\[
\rho_1^{(\bar{q}q)^2}(s) = \frac{m_s^2 \langle \bar{q}q \rangle^2}{3 \cdot 2^3 \pi^2} \left[ x^2 - 2 + \frac{1}{x} \right]
- \frac{m_s m_c \langle \bar{q}q \rangle^2}{3 \cdot 2^2 \pi^2} \left[ 1 - x \right]
\]

\[
\rho_1^{(G^3)}(s) = \frac{m_s^2 \langle G^3 \rangle}{5^2 \cdot 3^3 \cdot 2^{14} \pi^6} \left[ 8x^3 + 235x^2 + 760 - 1500 \log(x) - \frac{1000}{x} + \frac{3}{x^2} - \frac{4m_c^2 x^2}{3} \right]
- \frac{1}{(1 - x)^5}
\]

• Scalar-Vector tetraquark configuration (SV)

\[
\rho_1^{pert}(s) = \frac{m_s^8}{5 \cdot 3^2 \cdot 2^{14} \pi^6} \left[ x^2 + 555 - 60 \left( 3 + \frac{16}{x} + \frac{9}{x^2} \right) \log(x) + \frac{480}{x^2} - \frac{945}{x^3} - \frac{96}{x^4} + \frac{5}{x^5} \right]
+ \frac{m_s m_c}{3 \cdot 2^{12} \pi^6} \left[ x^2 + 28 - 12 \left( 1 + \frac{3}{x} + \frac{1}{x^2} \right) \log(x) - \frac{28}{x^2} - \frac{1}{x^3} \right]
\]

\[
\rho_1^{(\bar{q}q)}(s) = \frac{m_s^5 \langle \bar{q}q \rangle}{3 \cdot 2^9 \pi^4} \left[ x^2 + 9 - 6 \left( 1 + \frac{1}{x} \right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right]
+ \frac{m_s m_c^4 \langle \bar{q}q \rangle}{3 \cdot 2^9 \pi^4} \left[ x^2 + 12 - 12 \log(x) - \frac{16}{x} + \frac{3}{x^2} \right]
\]

\[
\rho_1^{(G^2)}(s) = \frac{m_s^4 \langle G^2 \rangle}{3^3 \cdot 2^{13} \pi^6} \left[ 5x^2 + 12x + 198 - 48 \left( 3 + \frac{2}{x} \right) \log(x) - \frac{212}{x^2} - \frac{3}{x^3} \right]
+ \frac{m_s m_c^3 \langle G^2 \rangle}{3^2 \cdot 2^{12} \pi^6} \left[ 7x^2 + 15 - 3 \left( 7 + \frac{3}{x} \right) \log(x) - \frac{21}{x^2} - \frac{1}{x^3} \right]
\]

\[
\rho_1^{(\bar{q}Gs)}(s) = \frac{m_s^3 \langle \bar{q}Gs \rangle}{2^7 \pi^4} \left[ -2 \log(x) - \frac{1}{x} \right]
- \frac{m_s m_c^2 \langle \bar{q}Gs \rangle}{3^2 \cdot 2^9 \pi^4} \left[ 5x^2 - 12x - 9 + 18 \log(x) + \frac{16}{x} \right]
\]

\[
\rho_1^{(\bar{q}q)^2}(s) = \frac{m_s^2 \langle \bar{q}q \rangle^2}{3 \cdot 2^3 \pi^2} \left[ x^2 - 3 + \frac{2}{x} \right]
- \frac{m_s m_c \langle \bar{q}q \rangle^2}{3 \cdot 2^2 \pi^2} \left[ 1 - x \right]
\]

\[
\rho_1^{(G^3)}(s) = \frac{m_s^2 \langle G^3 \rangle}{5 \cdot 3^3 \cdot 2^{14} \pi^6} \left[ 11x^2 - 240x + 24 + 420 \log(x) + \frac{208}{x} - \frac{3}{x^2} \right]
\]
• Scalar-Vector tetraquark configuration (VS)

\[ \rho_1^\text{pert}(s) = \frac{m_c^8}{5 \cdot 3^2 \cdot 2^{13} \pi^6} \left[ x^2 + 555 - 60 \left(3 + \frac{16}{x} + \frac{9}{x^2}\right) \log(x) + \frac{480}{x} - \frac{945}{x^2} - \frac{96}{x^3} + \frac{5}{x^4} \right] \]

\[ \rho_1^{(\bar{q}q)}(s) = \frac{m_{c}^2(\bar{q}q)}{3^2 \cdot 2^{10} \pi^6} \left[ x^2 + 72 - 12 \left(3 + \frac{4}{x}\right) \log(x) - \frac{64}{x} - \frac{9}{x^2} \right] + \frac{m_s m_{c}^4(\bar{s}s)}{3 \cdot 2^9 \pi^4} \left[ x^2 + 12 - 12 \log(x) - \frac{16}{x} + \frac{3}{x^2} \right] \]

\[ \rho_1^{(G^2)}(s) = \frac{m_{c}^4(G^2)}{3^3 \cdot 2^{13} \pi^6} \left[ x^2 - 24x - 54 + 24 \left(3 + \frac{1}{x}\right) \log(x) + \frac{80}{x} - \frac{3}{x^2} \right] \]

\[ \rho_1^{(\bar{q}Gq)}(s) = \frac{m_{c}^2(\bar{q}Gq)}{3^2 \cdot 2^9 \pi^4} \left[ 7x^2 - 23 - 18 \log(x) - \frac{4}{x} \right] - \frac{m_s m_{c}^2(\bar{s}Gq)}{3^2 \cdot 2^9 \pi^2} \left[ 7x^2 - 3 - 18 \log(x) - \frac{4}{x} \right] \]

\[ \rho_1^{(\bar{q}q)^2}(s) = \frac{m_{c}^2(\bar{q}q)}{3 \cdot 2^5 \pi^2} \left[ x - 2 + \frac{1}{x} \right] - \frac{m_s m_{c}^4(\bar{q}q)}{3 \cdot 2^2 \pi^2} (1 - x) \]

\[ \rho_1^{(G^2)}(s) = \frac{m_{c}^2(G^2)}{5^2 \cdot 3^3 \cdot 2^{14} \pi^6} \left[ 8x^3 + 235x^2 + 760 - 1500 \log(x) - \frac{1000}{x} - \frac{3}{x^2} - \frac{4m_{c}^2 \tau}{x^3} (1 - x)^5 \right] \]

• \( D_{1}K \) vector molecule configuration

\[ \rho_1^\text{pert}(s) = \frac{m_c^8}{5 \cdot 3^2 \cdot 2^{15} \pi^6} \left[ x^2 + 555 - 60 \left(3 + \frac{16}{x} + \frac{9}{x^2}\right) \log(x) + \frac{480}{x} - \frac{945}{x^2} - \frac{96}{x^3} + \frac{5}{x^4} \right] \]

\[ \rho_1^{(\bar{q}q)}(s) = -\frac{m_{c}^2(\bar{q}q)}{2^8 \pi^4} \left[ x + 9 - 6 \left(1 + \frac{1}{x}\right) \log(x) - \frac{9}{x} - \frac{1}{x^2} \right] - \frac{m_s m_{c}^4(\bar{q}q)}{2^8 11 \pi^4} (2(\bar{q}q) - (\bar{s}s)) \left[ x^2 + 12 - 12 \log(x) - \frac{16}{x} + \frac{3}{x^2} \right] \]

\[ \rho_1^{(G^2)}(s) = \frac{m_{c}^4(G^2)}{3 \cdot 2^{15} \pi^6} \left[ 3x^2 + 8x + 108 - 12 \left(7 + \frac{4}{x}\right) \log(x) - \frac{120}{x} + \frac{1}{x^2} \right] \]

\[ \rho_1^{(\bar{q}Gq)}(s) = \frac{3m_{c}^2(\bar{q}Gq)}{2^9 \pi^4} \left[ x - 2 \log(x) - \frac{1}{x} \right] + \frac{m_s m_{c}^2(\bar{q}Gq)}{3 \cdot 2^9 \pi^4} (3(\bar{q}Gq) + 2(\bar{s}Gq)) \left[ x^2 - 3 + \frac{2}{x} \right] \]

\[ \rho_1^{(\bar{q}q)^2}(s) = \frac{m_{c}^2(\bar{q}q)}{3 \cdot 2^5 \pi^2} \left[ x^2 - 3 + \frac{2}{x} \right] - \frac{m_s m_{c}^4(\bar{q}q)}{3 \cdot 2^2 \pi^2} (2(\bar{q}q)^2 - (\bar{q}q)(\bar{s}s)) (1 - x) \]

\[ \rho_1^{(G^2)}(s) = \frac{m_{c}^2(G^2)}{5 \cdot 3^3 \cdot 2^{16} \pi^6} \left[ 31x^2 - 480x + 84 + 780 \log(x) + \frac{368}{x} - \frac{3}{x^2} \right] \]
• **DK$_1$ vector molecule configuration**

\[
\rho_{1\text{pert}}(s) = \frac{m_{\tau}^6}{5 \cdot 3^\frac{1}{2} \cdot 2^{15} \pi^6} \left[ x^2 + 555 - 60 \left( 3 + \frac{16}{x} + \frac{9}{x^2} \right) \log(x) + \frac{480}{x} - \frac{945}{x^2} - \frac{96}{x^3} + \frac{5}{x^4} \right]
\]

\[
\rho_{1\langle \bar{q}q \rangle}(s) = \frac{m_{\tau}^5 \langle \bar{q}q \rangle}{3 \cdot 2^{10} \pi^4} \left[ x^2 + 72 - 12 \left( 3 + \frac{4}{x} \right) \log(x) - \frac{64}{x} - \frac{9}{x^2} \right] + \frac{m_{s} m_{\tau}^6 \langle \bar{s}s \rangle}{2^8 \pi^4} \left[ \frac{2x + 3 - 6 \log(x) - \frac{6}{x} + \frac{1}{x^2}}{2^{11} \pi^4} \right] \left[ x^2 + 12 - 12 \log(x) - \frac{16}{x} + \frac{3}{x^2} \right] + \frac{m_{s} m_{\tau}^3 \langle \bar{s}s \rangle}{2^9 \pi^4} \left[ x^2 + 12 - 12 \log(x) - \frac{16}{x} + \frac{3}{x^2} \right]
\]

\[
\rho_{1\langle G^2 \rangle}(s) = \frac{m_{\tau}^4 \langle G^2 \rangle}{3^2 \cdot 2^{15} \pi^6} \left[ x^2 - 48x - 180 + 12 \left( 15 + \frac{8}{x} \right) \log(x) + \frac{224}{x} + \frac{3}{x^2} \right] + \frac{m_{s} m_{\tau}^6 \langle \bar{s}s \rangle}{2^{12} \pi^4} \left[ x^2 + 12 - 12 \log(x) - \frac{16}{x} + \frac{3}{x^2} \right]
\]

\[
\rho_{1\langle \bar{q}Gq \rangle}(s) = \frac{-m_{\tau}^3 \langle \bar{q}Gq \rangle}{3 \cdot 2^{10} \pi^4} \left[ 5x^2 + 63 - 6 \left( 9 + \frac{4}{x} \right) \log(x) - \frac{68}{x} \right] - \frac{m_{s} m_{\tau}^6 \langle \bar{s}Gq \rangle}{2^9 \pi^4} \left[ 3 \langle \bar{q}Gq \rangle + 2 \langle \bar{s}Gq \rangle \right] \left[ x - 2 + \frac{1}{x} \right]
\]

\[
\rho_{1\langle \bar{q}q \rangle^2}(s) = -\frac{m_{\tau}^2 \langle \bar{q}q \rangle \langle \bar{s}s \rangle}{2^6 \pi^2} \left[ x - 2 + \frac{1}{x} \right] - \frac{m_{s} m_{\tau}^4 \langle \bar{q}q \rangle^2}{2^4 \pi^2} \left( 1 - x \right) - \frac{m_{s} m_{\tau}^6 \langle \bar{s}s \rangle}{2^6 \pi^2} \left( 1 - x^2 \right)
\]

\[
\rho_{1\langle G^3 \rangle}(s) = \frac{m_{\tau}^4 \langle G^3 \rangle}{5^2 \cdot 3^2 \cdot 2^{16} \pi^6} \left[ 8x^3 + 535x^2 + 1660 - 3300 \log(x) - \frac{2200}{x} - \frac{3}{x^2} + 4m_{\tau}^2 \left( x^2 - 5x + 10 - \frac{10}{x} + \frac{5}{x^2} - \frac{1}{x^3} \right) \right]
\]

• **D$_k^\ast$K$^*$ vector molecule configuration**

\[
\rho_{1\text{pert}}(s) = \frac{m_{\tau}^6}{5 \cdot 3^\frac{1}{2} \cdot 2^{15} \pi^6} \left[ x^2 + 555 - 60 \left( 3 + \frac{16}{x} + \frac{9}{x^2} \right) \log(x) + \frac{480}{x} - \frac{945}{x^2} - \frac{96}{x^3} + \frac{5}{x^4} \right]
\]

\[
\rho_{1\langle \bar{q}q \rangle}(s) = -\frac{m_{\tau}^5 \langle \bar{q}q \rangle}{3 \cdot 2^{10} \pi^4} \left[ x^2 + 72 - 12 \left( 3 + \frac{4}{x} \right) \log(x) - \frac{64}{x} - \frac{9}{x^2} \right] - \frac{m_{s} m_{\tau}^6 \langle \bar{s}s \rangle}{2^8 \pi^4} \left[ \frac{2x + 3 - 6 \log(x) - \frac{6}{x} + \frac{1}{x^2}}{2^{11} \pi^4} \right] \left[ x^2 + 12 - 12 \log(x) - \frac{16}{x} + \frac{3}{x^2} \right] + \frac{m_{s} m_{\tau}^3 \langle \bar{s}s \rangle}{2^9 \pi^4} \left[ x^2 + 12 - 12 \log(x) - \frac{16}{x} + \frac{3}{x^2} \right]
\]

\[
\rho_{1\langle G^2 \rangle}(s) = \frac{m_{\tau}^4 \langle G^2 \rangle}{3^2 \cdot 2^{15} \pi^6} \left[ x^2 - 48x - 180 + 12 \left( 15 + \frac{8}{x} \right) \log(x) + \frac{224}{x} + \frac{3}{x^2} \right] + \frac{m_{s} m_{\tau}^6 \langle \bar{s}s \rangle}{2^{12} \pi^4} \left[ 3 \langle \bar{q}Gq \rangle + 2 \langle \bar{s}Gq \rangle \right] \left[ x - 2 + \frac{1}{x} \right]
\]

\[
\rho_{1\langle \bar{q}Gq \rangle}(s) = \frac{m_{\tau}^3 \langle \bar{q}Gq \rangle}{3 \cdot 2^{10} \pi^4} \left[ 5x^2 + 63 - 6 \left( 9 + \frac{4}{x} \right) \log(x) - \frac{68}{x} \right] + \frac{m_{s} m_{\tau}^6 \langle \bar{s}Gq \rangle}{2^9 \pi^4} \left[ 3 \langle \bar{q}Gq \rangle + 2 \langle \bar{s}Gq \rangle \right] \left[ x - 2 + \frac{1}{x} \right]
\]

\[
\rho_{1\langle \bar{q}q \rangle^2}(s) = \frac{m_{\tau}^2 \langle \bar{q}q \rangle \langle \bar{s}s \rangle}{2^6 \pi^2} \left[ x - 2 + \frac{1}{x} \right] - \frac{m_{s} m_{\tau}^4 \langle \bar{q}q \rangle^2}{2^4 \pi^2} \left( 1 - x \right) - \frac{m_{s} m_{\tau}^6 \langle \bar{s}s \rangle}{2^6 \pi^2} \left( 1 - x^2 \right)
\]

\[
\rho_{1\langle G^3 \rangle}(s) = \frac{m_{\tau}^4 \langle G^3 \rangle}{5^2 \cdot 3^2 \cdot 2^{16} \pi^6} \left[ 8x^3 + 535x^2 + 1660 - 3300 \log(x) - \frac{2200}{x} - \frac{3}{x^2} + 4m_{\tau}^2 \left( x^2 - 5x + 10 - \frac{10}{x} + \frac{5}{x^2} - \frac{1}{x^3} \right) \right]
\]
\* $D^* K_0^*$ vector molecule configuration

$$
\begin{align*}
\rho_1^{\text{pert}}(s) &= \frac{m^8_s}{5 \cdot 3\cdot 2^{15} \pi^6} \left[ x^2 + 555 - 60 \left( 3 + \frac{16}{x} + \frac{9}{x^2} \right) \left( \frac{3}{x} \right) \log(x) + \frac{480}{x} - \frac{945}{x^2} + \frac{96}{x^3} + \frac{5}{x^4} \right] \\
\rho_1^{\langle \bar{q}q \rangle}(s) &= \frac{m_5^2}{2^8 \pi^4} \left[ x + 9 - \frac{6}{x} \right] \left( \frac{1}{x} + \frac{4}{x^2} \right) \log(x) - \frac{120}{x} + \frac{12}{x^2} \right] \\
\rho_1^{\langle G^2 \rangle}(s) &= \frac{m_5^2}{3 \cdot 2^{15} \pi^6} \left[ 3x^2 + 8x + 108 - 12 \left( \frac{7}{x} + \frac{4}{x^2} \right) \log(x) - \frac{120}{x} + \frac{12}{x^2} \right] \\
\rho_1^{\langle \bar{q}q \rangle^2}(s) &= -\frac{3m_c^2}{2^9 \pi^4} \left[ x - 2 \log(x) - \frac{1}{x} \right] - \frac{m_s m_c^2}{3 \cdot 2^8 \pi^4} \left( 3\langle \bar{q}Gq \rangle - 2\langle \bar{s}Gs \rangle \right) \left[ x^2 - 3 + \frac{2}{x} \right] \\
\rho_1^{\langle \bar{q}Gq \rangle}(s) &= -\frac{m_5^2}{3 \cdot 2^9 \pi^2} \left[ x^2 - 3 + \frac{2}{x} \right] - \frac{m_s m_c^2}{2^8 \pi^2} \left( 3\langle \bar{q}Gq \rangle - 2\langle \bar{s}Gs \rangle \right) \left[ 1 - x \right] \\
\rho_1^{\langle G^3 \rangle}(s) &= \frac{m_5^2}{5 \cdot 3^2 \cdot 2^{16} \pi^6} \left[ 31x^2 - 480x + 84 + 780 \log(x) + \frac{368}{x} - \frac{3}{x^2} \right]
\end{align*}
$$

References

[1] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys.* **B147** (1979) 385.

[2] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys.* **B147** (1979) 448.

[3] V.I. Zakharov, talk given at the Sakurai’s Price, *Int. J. Mod. Phys.* **A14**, (1999) 4865.

[4] S. Narison, *QCD spectral sum rules, World Sci. Lect. Notes Phys.* **26** (1989) 1.

[5] S. Narison, *QCD as a theory of hadrons, Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol.* **17** (2004) 1-778 [hep-ph/0205006].

[6] S. Narison, *Phys. Rept.* **84** (1982) 263;

[7] S. Narison, *Nucl. Part. Phys. Proc.* **258-259** (2015) 189;

[8] S. Narison, *Nucl. Part. Phys. Proc.* **207-208** (2010) 315.

[9] S. Narison, *Acta Phys. Pol. B* **26**(1995) 687;

[10] S. Narison, *Riv. Nuovo Cim.* **10N2** (1987) 1.

[11] B.L. Ioffe, *Prog. Part. Nucl. Phys.* **56** (2006) 232.

[12] L. J. Reinders, H. Rubinstein and S. Yazaki, *Phys. Rept.* **127** (1985) 1.

[13] E. de Rafael, les Houches summer school, hep-ph/9802448 (1998).

[14] R.A. Bertlmann, *Acta Phys. Austriaca* **53**, (1981) 305.

[15] F.J Yndurain, *The Theory of Quark and Gluon Interactions*, 3rd edition, Springer (1999).

[16] P. Pascual and R. Tarrach, *QCD: renormalization for practitioner*, Springer 1984.

[17] H.G. Dosch, *Non-pertubative Methods*, ed. Narison, World Scientific (1985).

[18] M. Gell-Mann, *Phys. Lett.* **8** (1964) 214.
[19] G. Zweig, CERN-TH-401 and TH-412 (1964) in developments in quark theory of hadrons, Vol. 1, 1964/1978, ed. D.B. Lichtenberg and S.P Rosen, Hadronic Press, MA, (1980)

[20] R. L. Jaffe, Phys. Rev. D15 (1977) 267.

[21] R. L. Jaffe, Phys. Rept. 409 (2005) 1.

[22] J. D. Weinstein and N. Isgur, Phys. Rev. D 27 (1983) 588.

[23] N.N. Achasov, S.A. Devyanin and G.N. Shestakov, Sov. J. Nucl. Phys. 32 (1980) 566.

[24] G. ’t Hooft et al., Phys. Lett. B 662 (2008) 424.

[25] J. I. Latorre and P. Pascual, J. Phys. G11 (1985) 231.

[26] S. Narison, Phys. Lett. B 175 (1986) 88.

[27] S. Narison and G. Veneziano, Int. J. Mod. Phys. A 4, 11 (1989) 2751.

[28] A. Bramon and S. Narison, Mod. Phys. Lett. A 4 (1989) 1113.

[29] S. Narison, Nucl. Phys. B 509 (1998) 312.

[30] P. Minkowski and W. Ochs, Eur. Phys. J. C 9 (1999) 283.

[31] G. Mennessier, S. Narison, W. Ochs, Phys. Lett. B 665 (2008) 205.

[32] G. Mennessier, S. Narison, X.-G. Wang, Phys. Lett. B 696 (2011) 40.

[33] R.M. Albuquerque et al., J. Phys. G46 (2019) 9, 093002.

[34] A. Ali, L. Maiani, A. D. Polosa, Cambridge Univ. Press, ISBN 9781316761465 (2019).

[35] J.-M. Richard, Few Body Syst. 57 (2016) 12, 1185.

[36] R. M. Albuquerque et al., Int. J. Mod. Phys. A 31 (2016) 17, 1650093.

[37] R.M. Albuquerque et al., Int. J. Mod. Phys. A 33 (2018) 16, 1850082.

[38] R. M. Albuquerque et al., Nucl. Part. Phys. Proc. 282-284 (2017) 83.

[39] R. M. Albuquerque et al., Int. J. Mod. Phys. A 31 (2016) 36, 1650196.

[40] J.S. Bell and R.A. Bertlmann, Nucl. Phys. B177, (1981) 218.

[41] J.S. Bell and R.A. Bertlmann, Nucl. Phys. B187, (1981) 285.

[42] C. Becchi et al., Z. Phys. C8 (1981) 335.

[43] S. Narison, E. de Rafael, Phys. Lett. B103 (1981) 57.

[44] R. M. Albuquerque et al., arXiv 2008.01569 [hep-ph] (2020).

[45] [LHCb collaboration], R. Aaij et al., arXiv: 2009.00025v1 [hep-ex] (2020).

[46] [LHCb collaboration], R. Aaij et al., arXiv: 2009.00026v1 [hep-ex] (2020).

[47] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98 (2018) 030001 and 2019 update.

[48] G.K.C. Cheung et al., arXiv: 2008.06432v1 [hep-lat] (2020).

[49] H. G. Dosch, M. Jamin and B. Stech, Z. Phys. C42 (1989) 167.

[50] M. Jamin and M. Neubert, Phys. Lett. B238 (1990) 387.

[51] J.-R. Zhang, arXiv: 2008.07295 [hep-ph] (2020).
[52] H.-X. Chen et al., arXiv: 2008.07516 [hep-ph] (2020).
[53] Z.-W. Zhang, arXiv: 2008.07833 [hep-ph] (2020).
[54] S.S. Agaev, K. Azizi and H. Sundu, arXiv: 2008.13027 [hep-ph] (2020).
[55] H. Mutuk, arXiv: 2009.02492 [hep-ph] (2020).
[56] A. Pich and E. de Rafael, Phys. Lett. B158 (1985) 477.
[57] S. Narison and A. Pivovarov, Phys. Lett. B327 (1994) 341.
[58] K. Hagiwara, S. Narison and D. Namura, Phys. Lett. B540 (2002) 233.
[59] D.J. Broadhurst, Phys. Lett. B101 (1981) 423.
[60] K.G. Chetyrkin and M. Steinhauser, Phys. Lett. B 502, 104 (2001).
[61] P. Gelhausen et al. Phys Rev. D 88, 014015 (2013) [Erratum: ibid. D89, 099901 (2014); ibid. D 91, 099901 (2015)].
[62] S. Narison and V.I. Zakharov, Phys. Lett. B679 (2009) 355.
[63] K.G. Chetyrkin, S. Narison and V.I. Zakharov, Nucl. Phys. B550 (1999) 353.
[64] S. Narison and V.I. Zakharov, Phys. Lett. B522 (2001) 266.
[65] For reviews, see e.g.: V.I. Zakharov, Nucl. Phys. Proc. Suppl. 164 (2007) 240.
[66] S. Narison, Nucl. Phys. Proc. Suppl. 164 (2007) 225.
[67] S. Narison, Int. J. Mod. Phys. A33 (2018) no.10, 1850045, Addendum: Int. J. Mod. Phys. A33 (2018) no.10, 1850045 and references therein.
[68] S. Narison, Phys. Lett. B802 (2020) 135221.
[69] S. Narison, Phys. Lett. B784 (2018) 261.
[70] S. Narison, Int. J. Mod. Phys. A30 (2015) no.20, 1550116.
[71] S. Narison, Phys. Lett. B738 (2014) 346.
[72] R.M. Albuquerque, S. Narison, Phys. Lett. B694 (2010) 217.
[73] R.M. Albuquerque, S. Narison, M. Nielsen, Phys. Lett. B684 (2010) 236.
[74] B.L. Ioffe, Nucl. Phys. B188 (1981) 317.
[75] B.L. Ioffe, Nucl. Phys. B191 (1981) 591.
[76] Y. Chung et al., Z. Phys. C25 (1984) 151.
[77] H.G. Dosch, M. Jamin and S. Narison, Phys. Lett. B220 (1989) 251.
[78] A.A.Ovchinnikov and A.A.Pivovarov, Yad. Fiz. 48 (1988) 1135.
[79] S. Narison, Phys. Lett. B605 (2005) 319.
[80] S. Narison, Phys. Lett. B693 (2010) 559; Erratum ibid 705 (2011) 544;
[81] S. Narison, Phys. Lett. B706 (2011) 412;
[82] S. Narison, Phys. Lett. B707 (2012) 259.
[83] S. Narison, Phys. Lett. B673 (2009) 30.
[84] G. Launer, S. Narison and R. Tarrach, *Z. Phys.* C26 (1984) 433.

[85] R.A. Bertlmann, G. Launer and E. de Rafael, *Nucl. Phys.* B250, (1985) 61.

[86] E.G. Floratos, S. Narison and E. de Rafael, *Nucl. Phys.* B155, (1979) 155.

[87] R. Tarrach, *Nucl. Phys.* B183 (1981) 384.

[88] R. Coquereaux, *Annals of Physics* 125 (1980) 401.

[89] P. Binetruy and T. Süsser, *Nucl. Phys.* B178 (1981) 293.

[90] S. Narison, *Phys. Lett.* B197 (1987) 405.

[91] S. Narison, *Phys. Lett.* B216 (1989) 191.

[92] M. Karliner and J.L. Rosner, arXiv:2008.05993 [hep-ph] (2020).

[93] M. W. Hu et al., arXiv:2008.06894 [hep-ph] (2020).

[94] X. G. He, W. Wang and R. Zhu, arXiv:2008.07145 [hep-ph] (2020).

[95] X. H. Liu et al., arXiv:2008.07190 [hep-ph] (2020).

[96] Q. F. Lu, D. Y. Chen and Y. B. Dong, arXiv:2008.07340 [hep-ph] (2020)

[97] M.-Z. Liu, J.-J. Xie and L.-S. Geng, arXiv: 2008.07389 [hep-ph] (2020).

[98] J. He and D.-Y. Chen, arXiv:2008.07782 [hep-ph] (2020).

[99] Y. Huang et al., arXiv:2008.07959 [hep-ph] (2020).

[100] R. Molina, E. Oset, arXiv:2008.11171 [hep-ph] (2020).

[101] T.J. Burns and E.S. Swanson, arXiv: 2008.12838 [hep-ph] (2020).