NONARCHIMEDEAN COALGEBRAS AND COADMISSIBLE MODULES

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Abstract. We show that basic notions of locally analytic representation theory can be reformulated in the language of topological coalgebras (Hopf algebras) and comodules. We introduce the notion of admissible comodule and show that it corresponds to the notion of admissible representation in the case of compact $p$-adic group.

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INTRODUCTION

The study of $p$-adic locally analytic representation theory of $p$-adic groups seems to start in 1980s, with the first examples of such representations studied in the works...
of Y. Morita [M1, M2, M3] (and A. Robert, around the same time), who considered locally analytic principal series representations for $p$-adic $SL_2$. Morita determined when those representations are irreducible by analyzing them as topological modules over the corresponding Lie algebra with the action of a finite group. Among the other work on $p$-adic (non-analytic) representations, relevant to this paper, we would like to mention the works of B. Diarra, who was focusing on special classes of representations of compact groups. Namely, Diarra studied nonarchimedean Banach Hopf algebras and his major applications were to the classes of representations, whose Hopf algebra of matrix elements admit an invariant functional. Diarra also considered Banach Hopf algebra $C(Z_p, K)$ of continuous functions on $p$-adic integers $Z_p$ and showed that the category of continuous representations of $Z_p$ in $p$-adic Banach spaces is equivalent to the category of Banach comodules over $C(Z_p, K)$. Diarra also constructed examples of irreducible infinite-dimensional representations of $Z_p$, thus showing that even in the case of compact group, the picture over $Q_p$ is a lot more complicated than the one over $C$.

The major development of the theory was made in a series of papers of P. Schneider and J. Teitelbaum[ST1, ST2, ST3, ST4]. Their approach to continuous or locally analytic representations of a locally analytic group $G$ is algebraic and, instead of analyzing representation $V$ as topological $K$-vector space, they work with its strong dual space $(V)_1'$ as with a module over distribution algebra $D\alpha(G,K)$ ($D(G,K)$ for continuous representations). They were able to obtain results similar to Morita’s for principal series representations of $p$-adic $GL_2$ and formulated good finiteness conditions on representations, called admissibility. Admissible representations rule out pathologies of $p$-adic case, like infinite-dimensional irreducible representations of $Z_p$, and form a broad enough category of representations, which include all important examples and provide a framework for developing a general theory. By definition, a representation is admissible if its strong dual is a coadmissible module over $D\alpha(G,K)$ and the later means it is a coadmissible module over $D\alpha(H,K)$ for some (and thus any) open compact subgroup $H < G$. For a compact subgroup $H < G$ the algebra $A = D\alpha(H,K)$ have an additional structure, called Fréchet-Stein structure, which means it is a locally convex projective limit $A = \lim A_n$ of Noetherian Banach algebras $A_n$ with flat transition maps. In [ST3] Schneider and Teitelbaum prove basic results for general Fréchet-Stein algebras and define coadmissible modules w.r.t. Fréchet-Stein structure $\{A_n\}$ as modules that are projective limits $M = \lim M_n$ of finitely-generated $A_n$-modules $M_n$, such that we also have isomorphisms $M_n \cong A_n \otimes_{A_{n+1}} M_{n+1}$. They prove that $D\alpha(H,K)$ is a Fréchet-Stein algebra which is additionally is a nuclear space. In [EM] the notion of coadmissible module was generalized to the case of weak Fréchet-Stein algebras, which can be thought of as projective limits of Banach algebras.

In this paper we develop a corresponding notion for comodules over certain topological coalgebras. Namely, we define the category of compact type (CT) $\hat{\otimes}$-coalgebras and show that this category is antiequivalent to the category of nuclear Fréchet (NF) algebras (however, our terminology is a little different from [ST3] and [EM]). We give a definition of an admissible comodule over a CT-$\hat{\otimes}$-coalgebra and of a coadmissible module over an NF-$\hat{\otimes}$-algebra and we show that categories of admissible comodules and coadmissible modules are antiequivalent. In case of CT-$\hat{\otimes}$-coalgebra $C$ with $(C)'_b$ being a nuclear Fréchet-Stein algebra, our definition of coadmissible module over $(C)'_b$ is equivalent to the one of Schneider and Teitelbaum.
Thus for a compact \( p \)-adic group \( H \) our admissible comodules over CT-\( \hat{\otimes} \)-coalgebra \( C^{la}(H, K) \) provide an explicit description of the category of admissible representations of \( H \), since the category of locally analytic representations of \( H \) in CT-spaces is equivalent to the category of CT-\( \hat{\otimes} \)-comodules over \( C^{la}(H, K) \). Hopf algebra and comodule formalism has proven to be useful in representation theory and provide a natural language for treatment of certain questions. In particular, it is the language of quantum groups (for example, see \([PW]\)) and many statements of the geometric representation theory of algebraic groups are formulated and proven in quantum case. Besides applications to quantization (see \([SO]\)), which was one of the original motivations for this work, it is our belief that a similar thing can be done in \( p \)-adic representation theory and thus Hopf algebra formalism can be an alternative to geometric methods. Finally, comodule and Hopf algebra formalism provide the most convenient framework for discussing Tannaka reconstruction (the main motivation for this work, with nonarchimedean version being the subject of \([L2]\)), which, in its turn, is the key element in the proof of the geometric Satake correspondence. This paper, and most importantly the description of admissible representations of a compact \( p \)-adic group in terms of comodules, provides the foundation for further work in all these directions.

Let us outline the content of this paper.

In the first two sections we develop the basic notions of the theory of Banach \( \hat{\otimes} \)-coalgebras (bialgebras, Hopf algebras). Some of the results there have been worked out in \([D1, D2, D3, D4, D5]\) for Hopf \( \hat{\otimes} \)-algebras, some are more or less straightforward generalizations of the corresponding notions and statements of the algebraic theory of coalgebras and comodules \([DNR, SW]\). We also formulate some concepts and prove results, foundational for the theory of comodules, like Frobenius reciprocity and tensor identity, which will not be used along the proof of our main result. Some of the traditional questions of Hopf algebra theory, like the description of rational modules (\([SW, 2.1]\), for more general version see \([L3]\)) have very simple answers in \( p \)-adic case. Since our main result is a “module-comodule” duality, we focus our attention on coalgebras and most of the time are sketchy on bialgebra and Hopf algebra case.

The last three sections are devoted to the topological theory. Although some basic results share the same proofs in Banach and more general topological case, we think it is better to consider those two cases separately for the sake of clarity. Our main objects of study are a \( \hat{\otimes} \)-coalgebras \( C \), such that \( C \) is topologically isomorphic to a compact locally convex inductive limit of a sequence of Banach \( \hat{\otimes} \)-coalgebras \( C_n \), and a \( \hat{\otimes} \)-comodules \( V \), such that \( V \) is topologically isomorphic to a compact locally convex inductive limit of a sequence of Banach \( C_n - \hat{\otimes} \)-comodules \( V_n \). We call such objects CT-coalgebras and CT-comodules correspondingly. However, these notions are not sufficient to formulate our notion of admissibility for CT-comodule, for which one need to consider \( \hat{\otimes} \)-comodules of the form \( V \hat{\otimes} C_n \). The cotensor product \( V \hat{\otimes} C_n \) is a closed subspace of the complete tensor product \( V \otimes C_n \). The problem is that \( V \otimes C_n \) is not a compact type space, but only an LB-space, (not compact) locally convex inductive limit of Banach spaces, and \( V \hat{\otimes} C_n \) does not even have to be an LB-space. This forces us to consider more general topological comodules, which we call LB- and LS-comodules. Since CT-spaces are also called LS-spaces in functional analysis, we use this name for topological algebraic structures, which
are compact type only as a space, i.e., LS-comodule \( V \) is a topological comodule, such that \( V \cong \lim V_n \) is a compact locally convex inductive limit of Banach spaces \( V_n \), but \( V_n \) are only spaces, not comodules themselves.

Since our intended audience has various background (Hopf algebras, quantum groups and \( p \)-adic representation theory), we moved proofs of two key technical results from functional analysis into appendix. We hope it makes this paper a better reading.

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1. Banach coalgebras

Throughout this paper \( K \) means a discretely valued field with norm \( |x|_K \). \( K^0 := \{ x \in K \mid |x|_K \leq 1 \} \) is its ring of integers, \( K^{00} := \{ x \in K \mid |x|_K < 1 \} \) and \( \bar{K} = K/K^{00} \) is the residue field. The linear dual space of a linear space \( V \) we denote as \( V^* \). For a linear map \( A : V \to W \) of vector spaces we denote by \( A^* \) its adjoint map.

A Banach space over \( K \) (or \( K \)-Banach space) is a normed space \( (V, \| \cdot \|_V) \) such that \( V \) is complete w.r.t. \( \| \cdot \|_V \). When no confusion can occur, we will just write \( V \).

\( K \)-Banach spaces form a category \( \text{Ban}_K \) with bounded linear maps as morphisms. For \( V, W \in \text{Ban}_K \), \( \text{Ban}_K (V, W) \) is the Banach space itself and we write \( C(V, W) \subset \text{Ban}_K (V, W) \) for the closed subspace of compact maps. \( V^0 := \{ x \in V \mid \|x\|_V < 1 \} \) is a closed unit ball in \( V \), \( V^{00} := \{ x \in V \mid \|x\|_V < 1 \} \) and \( \bar{V} := V^0/V^{00} \). \( \bar{V} \) is a vector space over \( \bar{K} \).

For convenience we always suppose that the norm \( \| \cdot \|_V \) is solid, i.e. \( \|V\|_V \subset |K|_K \). From [PGS, 2.1.9] it is known that this assumption is not a restriction.

We will say that a \( K \)-Banach space \( V \) is free if it is of the form \( c_0(E_V, K) = (\sum_{n=0}^{\infty} a_n e_n \mid e_n \in E_V, a_n \in K : a_n \rightarrow 0) \). Over discretely valued fields any Banach space \( V \) is free, with a possible choice of \( E_V \) being a \( K \)-vector space basis of \( \bar{V} \).

The continuous dual of a \( K \)-Banach space \( V \) will be denoted by \( V' \). The adjoint map for a continuous linear map \( f \) will also be denoted by \( f' \).

For a subset \( U \subset V \) its annihilator is denoted by \( U^\perp = \{ \phi \in V' : \phi(x) = 0 \ \forall x \in U \} \).

The completion of a linear (sub)space \( U \) w.r.t. a given norm (or topology \( \tau \)) will be denoted by \( U^{\perp \tau} \).

For two \( K \)-Banach spaces \( V \) and \( W \) one defines a norm \( \| \cdot \|_{V \otimes W} \) on the space \( V \otimes_K W \) as \( \|u\|_{V \otimes W} = \inf \{ \sum \|v_i\|_V \|w_i\|_W \mid u = \sum v_i \otimes w_i \} \), where infimum is taken over all decompositions of \( u = \sum v_i \otimes w_i \). We denote the completion of the space \( V \otimes_K W \) w.r.t. \( \| \cdot \|_{V \otimes W} \) by \( V \overline{\otimes} W \) and call it the complete(d) tensor product of \( V \) and \( W \). It is known that the topology, induced by \( \| \cdot \|_{V \overline{\otimes} W} \) on \( V \overline{\otimes} W \) coincide with projective and inductive tensor product topologies [NFA, section 17]. \( \text{Ban}_K \) has a symmetric monoidal category structure w.r.t. \( \otimes \). We also use the notation \( \bar{\otimes} \) to denote the composition of the tensor product (\( \otimes \) or \( \bar{\otimes} \)) with the canonical isomorphism \( K \overline{\otimes} V \cong V (V \overline{\otimes} K \cong V) \), i.e. we write \( a \bar{\otimes} b \) when either \( a \) or \( b \) belongs to \( K \) and for maps \( f \bar{\otimes} g \) when the range of either \( f \) or \( g \) is \( K \).

In a few places we use Sweedler’s notation, so we briefly recall it. For comultiplication on a coassociative coalgebra \( C \), \( \Delta (c) = \sum_i a_i \otimes b_i \) in order to avoid introducing
new letters one writes $\Delta(c) = \sum_i c(1)_i \otimes c(2)_i$. In Sweedler’s notation one does not think of the summation symbol $i$, so we just write $\Delta(c) = \sum c(1) \otimes c(2)$ or, in sumless notation, simply $\Delta(c) = c(1) \otimes c(2)$. The coassociativity $\Delta \otimes \text{id} \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ of the comultiplication $\Delta$ allows us to write $c(1) \otimes c(2) \otimes c(3) = c(1) \otimes c(2) \otimes c(2)(2)$. One can also use the notation for coactions $\rho^\ast : V \to V \otimes C$, $\rho_C(v) = v(0) \otimes v(1)$. In our setting (Banach or locally convex) the sum in Sweedler’s notation is always assumed to be countable and convergent.

In this section we provide a summary of basic facts about Banach coalgebras (Hopf algebras). Some of them were worked out in [D1, D2, D3, D4, D5] for Hopf algebras.

1.1. Banach $\otimes$-Coalgebras.

**Definition 1.** A (coassociative and counital) $K$-Banach $\otimes$-coalgebra (or simply Banach coalgebra) $C$ is a triple $(C, \Delta_C, \epsilon_C)$, where $C$ is a $K$-Banach space, $\Delta_C : C \to C \otimes C$ is a comultiplication and $\epsilon_C : C \to K$ is a counit, continuous morphisms, such that $(C, \Delta_C, \epsilon_C)$ is a coalgebra object in $\text{Ban}_K$.

Taking continuous duals give us the map

$$\Delta_C' : (C \otimes C)' \to C'.$$

Since we have an embedding $C' \otimes C' \to (C \otimes C)'$, we get on $C'$ the structure of a Banach $\otimes$-algebra, with the product given by

$$\alpha \ast \beta = \Delta_C' \circ (\alpha \otimes \beta) = (\alpha \otimes \beta) \circ \Delta_C.$$

The adjoint map of counit $\epsilon'_C : K \to C'$ is the unit of this algebra.

**Definition 2.** $C'$ is called dual Banach $\otimes$-algebra for $C$. The $(-\ast-)$ product is called convolution product.

**Remark 3.** If we take $C$ - finite-dimensional, then $(C \otimes C)' = C' \otimes C'$, and thus, in case of Hopf algebra $C$, $C'$ also becomes a Hopf algebra by standard argument.

**Definition 4.** Let $C$ and $B$ be Banach $\otimes$-coalgebras. A bounded linear map $f : C \to B$ is a (counital) morphism of Banach $\otimes$-coalgebras if $\Delta_B \circ f = (f \otimes f) \circ \Delta_C$ and $\epsilon_B \circ f = \epsilon_C$

The dual map $f' : B' \to C'$ would satisfy

$$f' \circ \Delta_B' = \Delta_C' \circ (f' \otimes f')$$

and

$$f' \circ \epsilon_B' = \epsilon'_C,$$

i.e. $f'$ is a (unital) homomorphism of the Banach $\otimes$-algebras $B'$ and $C'$.

Denote the category of Banach $\otimes$-coalgebras over $K$ by $\text{BCoalg}_K$ and the category of Banach $\otimes$-algebras over $K$ by $\text{BAlg}_K$. For $C, B \in \text{BCoalg}_K$, $\text{BCoalg}_K(C, B)$ is a *-weakly closed subset of $\text{Ban}_K(C, B)$. A similar statement is true for homomorphisms of Banach $\otimes$-algebras.

Again, in finite-dimensional setting, the dual of the $\otimes$-algebra homomorphism is a $\otimes$-coalgebra morphism.
Remark 5. Since for a $K$-Banach space $C$ we have $C \otimes_K C \subset C \hat{\otimes} C$, Banach $\hat{\otimes}$-algebras form a subcategory of usual associative algebras. This is no longer true for $\hat{\otimes}$-coalgebras. Neither Banach $\hat{\otimes}$-coalgebras are a subcategory of coassociative coalgebras, nor coassociative coalgebras form a subcategory of Banach $\hat{\otimes}$-coalgebras. For the first statement, the reason is that comultiplication on a Banach $\hat{\otimes}$-coalgebra $C$ acts from $C$ into the completed tensor product $C \hat{\otimes} C$, which is larger than the algebraic tensor product $C \otimes C$, which is larger than the algebraic tensor product $C \otimes K C$. For the second, the comultiplication on a coassociative coalgebra may not be continuous.

1.2. Constructions in the category of Banach $\hat{\otimes}$-coalgebras.

Definition 6. Let $A$ be a subspace of $C$. $A$ is a $\hat{\otimes}$-subcoalgebra, if $\Delta_C (A) \subseteq A \hat{\otimes} A$. In case $A$ is closed, we will say that $A$ is a closed (or Banach) $\hat{\otimes}$-subcoalgebra.

We will say that Banach $\hat{\otimes}$-coalgebra $C$ is simple if it does not have a proper nonzero closed $\hat{\otimes}$-subcoalgebra.

Let $S \subset C$ be a subset in a Banach $\hat{\otimes}$-coalgebra $C$. Denote by $C_S$ the intersection of all closed $\hat{\otimes}$-subcoalgebras containing $S$. Although the proof uses some notions that will be given later, we formulate the following fact.

Proposition 7. $\forall c \in C$: $C_c$ is a Banach space of countable type. Thus any simple $\hat{\otimes}$-coalgebra is of countable type.

Proof. Similar to [SW, 2.2.1], see also [D2, I.2.1.ii].

Let $(C, \| \cdot \|_C)$ be a normed vector space over $K$. Then the norm $\| \cdot \|_C$ induces the norm $\| \cdot \|_{C \hat{\otimes} C}$ on $C \otimes C$. Let $C$ be a coassociative counital coalgebra over $K$, such that the coaction $\Delta_C$ and counit $\epsilon_C$ are continuous with respect to the norm $\| \cdot \|_C$ on $C$ and corresponding norm $\| \cdot \|_{C \hat{\otimes} C}$ on $C \otimes C$. Then for the completion $C^\wedge$ of $C$, from the universal property of completion and continuity of $\Delta_C$, one can extend the coaction and counit and thus get a $K$-Banach coalgebra $(C^\wedge, \Delta_C^\wedge, \epsilon_C^\wedge)$.

1.2.1. Dual coalgebra. We review the notion of the dual Banach coalgebra from [D5].

Let $A$ be a unital Banach algebra, $m_A : A \hat{\otimes} A \to A$ is the multiplication and $\| m_A \| \leq 1$. Let $A' := (A)_1'$ be the dual Banach space and $m_A' : A' \to (A \hat{\otimes} A)'$ be the dual map, which is necessarily isometric. We have an isometric embedding $A \hat{\otimes} A' \hookrightarrow (A \hat{\otimes} A)'$ [VANR, 4.34].

Let $A^\circ := (m_A')^{-1} (A' \hat{\otimes} A')$. Recall that $A'$ is an $A$-bimodule with the left action $(a \cdot a') (x) = a' (xa)$ and the right action $(a' \cdot a) (x) = a' (ax)$ for $a' \in A'$, $a \in A$ ($\gamma_a' (a)$ for $a \rightarrow a'$ and $\delta_a' (a)$ for $a' \rightarrow a$ in [D5]). For a fixed $a'$ these define a continuous linear maps $\delta : A \to A'$ ($\gamma_a$ for $a \rightarrow a'$ and $\delta_a$ for $a' \rightarrow a$, with $\delta_a = \| a \|$ in [D5]), $\delta_a' (a) = a \rightarrow a'$ and $\delta_a' (a) = a' \rightarrow a$, with $\| a \|$ in [D5].

Proposition 8. Let $a' \in A^\circ$. For $u : A \to B$ a homomorphism of unital Banach algebras, denote by $u^\circ$ the restriction $u|_{B^\circ}$.

1. $\delta, \delta_a' : A \to A'$ are completely continuous (compact) operators [D5, lemma 1].
2. $\delta_a : A^\circ$ and $\delta_a' : A^\circ$ for all $a \in A$, and thus $a', a' \rightarrow : A \to A^\circ$ [D5, lemma 2].
3. $m_A' (A^\circ) \subset A^\circ \hat{\otimes} A^\circ$, [D5, Theorem 1].
4. If $u : A \to B$ is a homomorphism of unital Banach algebras. Then $u^\circ (B^\circ) \subset A^\circ$ [D5, lemma 3]. Thus we have a functor $(\cdot)^\circ$. 
For two unital Banach algebras $A$ and $B$ we have $(A \otimes B)^{\circ} = A^\circ \otimes B^\circ$ [D5, Theorem 2].

(6) For a Banach algebra $(A, m_A, u_A)$ its dual $(A^\circ, m_A^\circ, u_A^\circ)$ is a Banach coalgebra [D5, Theorem 3].

**Remark 9.** For a Banach Hopf algebra $(H, m_H, u_H, \Delta_H, \epsilon_H)$ (see section 1.3) one gets a dual Banach Hopf algebra $(H^\circ, \Delta_H^\circ, \epsilon_H^\circ, m_H^\circ, u_H^\circ)$ [D5, Theorem 4].

**Definition 10.** We call $A^\circ$ the dual Banach coalgebra of $A$.

If $A = C'$ for some Banach coalgebra $C$, then, since $K$ is discretely valued field, $C$ is pseudo-reflexive Banach space and thus $C \to C'^\circ$ is a strict injection. Similar to [DNR, 1.5.12], one can show that $C \to C'^\circ$ is an embedding of Banach coalgebras.

Clearly, if $u : A \to B$ is a homomorphism of unital Banach algebras then $u^\circ$ is a Banach coalgebra morphism (or see [DNR, 1.5.4]). Thus we have a functor $(-)^\circ : BAlg_K \to BCoalg_K$ [DNR, 1.5.5]. Similar to [DNR, 1.5.22], one can show that $(-)^\circ$ is the left adjoint functor to the dual functor $(-)' : BCoalg_K \to BAlg_K$.

One can also define an analog of Sweedler’s finite dual of an algebra. Recall [DNR, sec. 1.5] that for any $K$-algebra $A$ one has a notion of finite dual $A^0$. It is a set of linear functionals $f \in \text{Hom}_K (A, K)$, such that $\text{Ker} (f)$ contains an ideal of finite codimension. $A^0$ is a coalgebra and if $A$ is a bialgebra (Hopf algebra) then $A^0$ is also a bialgebra (Hopf algebra).

In Banach case one can consider the set $A^{0*} = A^0 \cap A'$. $A^{0*}$ can be described as a set of continuous functionals, who’s kernel contains a closed ideal of finite codimension. Also denote by $A_\circ$ the topological closure of $A^{0*}$ in $A'$.

**Proposition 11.** Properties of $A^0$ imply the following properties of $A_\circ$:

1. If $f : A \to B$ is a continuous morphism of Banach $\otimes$-algebras and $I$ is a closed ideal of finite codimension in $B$, then $f^{-1} (I)$ is a closed finite-codimensional $\otimes$-ideal [DNR, 1.5.1];
2. $f^* (B^\circ) \subset A^\circ$ [DNR, 1.5.2.i];
3. If $\phi : A \otimes B' \to (A \otimes B)'$ is a canonical (isometric) embedding, then we have $\phi (A^0 \otimes B^0) = (A \otimes B)^0$ and $\phi (A_\circ \otimes B_\circ) = (A \otimes B)^0$ [DNR, 1.5.2.ii];
4. If $m : A \otimes A \to A$ is a multiplication on $A$, then $m^* (A^{0*}) \subset A^{0*} \otimes A^{0*}$ and $m^* (A_\circ) \subset A_\circ \otimes A_\circ$ (same as in [DNR, 1.5.2.iii]);
5. $(A^0, m^* \circ \phi^{-1}, \epsilon)$ is a coalgebra and $(A_\circ, m^* \circ \phi^{-1}, \epsilon)$ is a Banach $\otimes$-coalgebra, with $\epsilon (a^*) = a^* (1)$ (same as in [DNR, 1.5.3]);
6. Let $f : A \to B$ be a Banach $\otimes$-algebra morphism. Then $f^{0*} = f^*|_{B^{0*}} : B^{0*} \to A^{0*}$ is a coassociative coalgebra morphism and $f_\circ = f^*|_{B_\circ} : B^\circ \to A^\circ$ is a Banach $\otimes$-coalgebra morphism [DNR, 1.5.4]).

Thus we have a $A^{0*}$ is a coassociative, counital coalgebra and $A_\circ$ is its completion, which we will call complete finite (c-finite for short) dual. For a map of Banach $\otimes$-algebras $f : A \to B$ we define $f_\circ : B^\circ \to A^\circ$ as the restriction of the dual map $f' : B' \to A'$. Thus we have a contravariant functor $(-)_\circ : BAlg_K \to BCoalg_K$.

One has embeddings $A^{0*} \subset A_\circ \subset A^\circ$. If $A = C'$ for some Banach coalgebra $C$, then $C \subset C^{0*}$ only if $C$ is a completion of a coassociative coalgebra. In general
one only has $C \subset C^\circ$ and the general relation between $A^0$ and $A^\circ$ is currently unknown.

We have the following description

$$A^\circ = \left\{ f \in A \mid \exists f_i, g_i \in A', i \in \mathbb{N}: f(xy) = \sum_{i=1}^{\infty} f_i(x)g_i(y) \ \forall x, y \in A \right\}.$$ 

1.2.2. $\otimes$-coideals.

**Definition 12.** Let $C$ be a Banach $\otimes$-coalgebra and $V$ be a closed subspace of $C$.

- $V$ is called right closed $\otimes$-coideal if we have $\Delta(V) \to V \otimes C$;
- $V$ is called left closed $\otimes$-coideal if we have $\Delta(V) \to C \otimes V$;
- $V$ is called a (two-sided) closed $\otimes$-coideal if we have $\Delta(V) \to V \otimes C + C \otimes V$ and $\epsilon(V) = 0$

Completed sums and intersections of (left, right, two-sided) (closed) $\otimes$-coideals is again a (left, right, two-sided) closed $\otimes$-coideal.

If $V$ is a closed $\otimes$-coideal in a Banach $\otimes$-coalgebra $C$, one can consider the projection map $\pi : C \to C/V$ in $\text{Ban}_K$.

**Proposition 13.** [SW, 1.4.7]

1. $C/V$ is a Banach $\otimes$-coalgebra and $\pi$ is a Banach $\otimes$-coalgebra map;
2. if $f : C \to D$ is a Banach $\otimes$-coalgebra map, then $\text{Ker}(f)$ is a closed $\otimes$-coideal;
3. if $f$ is surjective, than there is a canonical topological isomorphism $C/\text{Ker}(f) \cong D$;
4. universal property of quotient holds

**Proposition 14.** Let $f \in \text{BCoalg}_K(C, D)$. Then

1. Preimage of a closed $\otimes$-coideal in $D$ is a closed $\otimes$-coideal in $C$;
2. Closure of the image of $f$ is a Banach $\otimes$-subcoalgebra of $D$.

**Proof.**

1. Consider a $\otimes$-coideal $I$ in $D$. Then the kernel of the composition $\pi \circ f : C \to D/I$ is exactly $f^{-1}(I)$.

2. Since $f$ is a morphism of $\otimes$-coalgebras, we have $\Delta_D(f(C)) \subset f(C) \otimes f(C)$. Since we have $f(C) \otimes f(C) = f(C)^\wedge \otimes f(C)^\wedge$ and since $\Delta_D$ is continuous, we have $\Delta_D(f(C)^\wedge) \subset f(C)^\wedge \otimes f(C)^\wedge$.

\[\square\]

1.3. Banach $\otimes$-bialgebras and Hopf $\otimes$-algebras.

**Definition 15.** A Banach $\otimes$-bialgebra is a tuple $(B, m_B, u_B, \Delta_B, \epsilon_B)$ that is a bialgebra object in $\text{Ban}_K$.

A map is a morphism of Banach $\otimes$-bialgebras if it is a Banach $\otimes$-algebra and Banach $\otimes$-coalgebra homomorphism.

**Remark 16.** As in the algebraic case, there is no $\{0\}$ $\otimes$-bialgebra! (since in this case $\epsilon(1) \neq 1$)

**Remark 17.** If $B$ is a finite-dimensional $\otimes$-bialgebra, then $B'$ is a $\otimes$-bialgebra.
Recall that in general morphism sets in $\text{BA}_K$ and $\text{BCom}_K$ don’t have additional algebraic structure. For any Banach $\hat{\mathcal{B}}$-bialgebra $B$ the space of endomorphisms $\text{Ban}_K(B)$ is an algebra with respect to the convolution product $$\forall \phi, \psi \in \text{Ban}_K(B) : \phi \ast \psi := m_B \circ (\phi \hat{\otimes} \psi) \circ \Delta_B.$$ It is clear that the convolution is a bounded map, since $\|\phi \ast \psi\| \leq \|m_B\| \|\phi\| \|\psi\| \|\Delta_B\|$, and thus $\text{Ban}_K(B)$ is a $K$-Banach algebra w.r.t. convolution, with the unit $u_B \circ \epsilon_B$.

**Definition 18.** The morphism $S$ is called the antipode of the Banach $\hat{\mathcal{B}}$-bialgebra $H$, if $S \ast \text{id}_H = \text{id}_H \ast S = u_H \circ \epsilon_H$.

The antipode of a bialgebra is unique, if it exists.

**Definition 19.** A Banach $\hat{\mathcal{B}}$-bialgebra $H$ with antipode is called Banach Hopf $\hat{\mathcal{B}}$-algebra. It is a Hopf algebra object in $\text{Ban}_K$.

Any $\hat{\mathcal{B}}$-bialgebra map between two Hopf $\hat{\mathcal{B}}$-algebras is also a Hopf $\hat{\mathcal{B}}$-algebra map, i.e. commutes with antipodes.

1.4. **Constructions in the category of Banach $\hat{\mathcal{B}}$-bialgebras and Hopf $\hat{\mathcal{B}}$-algebras.** Banach $\hat{\mathcal{B}}$-subbialgebra (Hopf $\hat{\mathcal{B}}$-subalgebra) is a closed subspace, which is a Banach $\hat{\mathcal{B}}$-bialgebra (Hopf $\hat{\mathcal{B}}$-algebra) with induced operations.

A closed subspace is a closed $\hat{\mathcal{B}}$-biideal (Hopf $\hat{\mathcal{B}}$-ideal) if it is a $\hat{\mathcal{B}}$-ideal and $\hat{\mathcal{B}}$-coideal (and is invariant under the antipode map). A kernel of a Banach $\hat{\mathcal{B}}$-bialgebra (Hopf $\hat{\mathcal{B}}$-algebra) morphism is a closed $\hat{\mathcal{B}}$-biideal (Hopf $\hat{\mathcal{B}}$-ideal). The quotient by a closed $\hat{\mathcal{B}}$-biideal (Hopf $\hat{\mathcal{B}}$-ideal) carries the structure of a Banach $\hat{\mathcal{B}}$-bialgebra (Hopf $\hat{\mathcal{B}}$-algebra). Preimage of a closed $\hat{\mathcal{B}}$-biideal (Hopf $\hat{\mathcal{B}}$-ideal) under a $\hat{\mathcal{B}}$-bialgebra (Hopf $\hat{\mathcal{B}}$-algebra) morphism is a closed $\hat{\mathcal{B}}$-biideal (Hopf $\hat{\mathcal{B}}$-ideal).

For a Banach $\hat{\mathcal{B}}$-bialgebra (Hopf $\hat{\mathcal{B}}$-algebra) $H$ its dual Banach $\hat{\mathcal{B}}$-coalgebra $H^\hat{\mathcal{B}}$ is a Banach $\hat{\mathcal{B}}$-bialgebra (Hopf $\hat{\mathcal{B}}$-algebra) ([D5, Theorem 4]).

2. **Banach comodules**

2.1. **Basic definitions.**

**Definition 20.** Let $C$ be a Banach $\hat{\mathcal{B}}$-coalgebra and $V$ is a Banach space. We say that $V$ is a right (Banach) $\hat{\mathcal{B}}$-comodule over $C$ ($V \in \text{BCom}_C$) if exists $\rho_V : V \to V \hat{\otimes} C$ a $K$-linear continuous map such that

$$(id_V \hat{\otimes} \epsilon_C) \circ \rho_V = id_V$$

$$(\rho_V \hat{\otimes} id_C) \circ \rho_V = (id_V \otimes \Delta_C) \circ \rho_V .$$

Similarly one defines left Banach $\hat{\mathcal{B}}$-comodules. We denote the corresponding category by $\text{C}_{\text{BCom}}$.

**Definition 21.** Let $V \in \text{BCom}_C$. Then $V'$ is a Banach space which is a right $C^\prime - \hat{\mathcal{B}}$-module. We call $V'$ a dual $\hat{\mathcal{B}}$-module for $V$. The action is given by convolution

$$v' \cdot c' = (v' \hat{\otimes} c') \circ \rho_V .$$

On $V$ there is also a left $C^\prime - \hat{\mathcal{B}}$-module structure

$$m : C' \hat{\otimes} V \to V,$$

$$\lambda \hat{\otimes} v \mapsto \lambda \cdot v = (id_V \hat{\otimes} \lambda) \circ \rho_V (v).$$

We call this $\hat{\mathcal{B}}$-module structure induced.
In general, if $W$ is a Banach space, one can give a structure of a continuous right $C^\prime$-$\bowtie$-module to the space Ban$_K(V,W)$, similar to the dual $\bowtie$-module.

If $M, N \in$ BComod$_C$ and $f \in$ Ban$_K(M, N)$, we say it is a $\bowtie$-comodule morphism, if $(f \circ \text{id}) \circ \rho_M = \rho_N \circ f$. The set of Banach $\bowtie$-comodule morphisms BComod$_C(M, N)$ is a closed $K$-linear subspace of Ban$_K(M, N)$.

2.2. Constructions in the category of Banach $\bowtie$-comodules.

**Definition 22.** Let $M \in$ BComod$_C$. We say that a closed subspace $N \subset M$ is a Banach $\bowtie$-subcomodule (or just closed $\bowtie$-subcomodule) if $\rho_M(N) \subset N \bowtie C$.

We say that $\bowtie$-comodule $M$ is simple, if it does not have proper nonzero closed $\bowtie$-subcomodules.

**Proposition 23.** Let $C \in$ BCoalg$_K$ and $M \in$ BComod$_C$. Then

1. a closed subspace $N$ of $M$ is a right closed $C$-$\bowtie$-submodule iff $N$ is a closed left $C$-$\bowtie$-submodule of $M$;
2. let $x \in M$. Then the closure of $C'x$ in $M$ is a closed $\bowtie$-subcomodule of $M$ and is a space of countable type;
3. if $M$ is simple, then it is of countable type.

**Proof.** [D2, I.1].

**Lemma 24.** [D2, II.1] Let $f : M \rightarrow N$ be a morphism of Banach $\bowtie$-comodules. Then

1. Preimage of a Banach $\bowtie$-subcomodule of $N$ is a Banach $\bowtie$-subcomodule of $M$. In particular, $\text{Ker}(f)$ is a closed $\bowtie$-subcomodule.
2. The closure of the image of $f$ is a Banach $\bowtie$-subcomodule of $N$.

**Proposition 25.** [SW, 2.0.1] Let $f$ be a morphism of Banach $\bowtie$-comodules.

1. If $L \subset M$ is a Banach $\bowtie$-subcomodule, then $M/L$ is a Banach $\bowtie$-comodule and projection is a Banach $\bowtie$-comodule morphism;
2. universal property of quotient holds.

Intersection of two (or many) Banach $\bowtie$-subcomodules is again Banach $\bowtie$-subcomodule. For a Banach $\bowtie$-algebra $(A, \phi_A, u_A)$ we have defined its dual Banach $\bowtie$-coalgebra $(A^\circ, \phi_A^\circ, u_A^\circ)$, which is the largest Banach subspace of $A^\prime$, which is a Banach $\bowtie$-coalgebra w.r.t. $\phi_A$. One can give a similar definition for a Banach $A$-$\bowtie$-module.

For a right Banach $\bowtie$-module $(M, \phi_M)$ with multiplication $\phi_M : M \bowtie A \rightarrow M$, let $M^\circ := (\phi_M)^{-1}(M^\prime \bowtie A^\circ)$.

Similar to the section 1.2.1, recall that $M'$ is a left $A$-module with left action $(a \mapsto m')(m) = m'(ma)$ for $m' \in M'$, $m \in M$, $a \in A$. For a fixed $m'$ it defines a continuous linear map $- : m' : A \rightarrow M'$, $- : m'(a) = a - m'$ with $\| - : m' \|= \|m'\|_{M'}$.

**Proposition 26.** Let $m' \in M^\circ$. For $u : M \rightarrow N$ a homomorphism of right unital Banach $\bowtie$-modules, denote by $u^\circ$ the restriction $u|_{N^\circ}$.

1. $m' \in M^\circ$ iff $- : m' : A \rightarrow M'$ is a completely continuous (i.e. compact) operator and $- : m' \in M^\circ \bowtie A'$;
2. $a - m' \in M^\circ$ for all $a \in A$, and thus $- : m' : A \rightarrow M^\circ$;
3. $\phi_M' (M^\circ) \subset M^\circ \bowtie A^\circ$;
4. if $u : M \rightarrow N$ is a homomorphism of right unital Banach $\bowtie$-modules, then $u^\circ (N^\circ) \subset M^\circ$;
(5) for a right Banach $A - \mathfrak{S}$-module $(M, \phi_M)$ the pair $(M^\circ, \phi_M^\circ)$ is a (right) Banach $A^\circ - \mathfrak{S}$-comodule.

Proof. (1) is similar to [D5, lemma 1]

Let $m' \in M^\circ$ and $\phi'_M(m') = \sum m_j^\ell \otimes a_j^\ell$, $m'_j \in M'$, $a'_j \in A^\circ$. For all $a \in A$, $m \in M$ we have

$$(a \cdot m') (m) = (m', \phi_M (m \otimes a)) = (\phi'_M (m' \cdot m), m \otimes a) = \sum m'_j (m) \otimes a'_j (a).$$

Thus we have $a \cdot m' = \sum m'_j \otimes a'_j (a)$ and $m' = \sum m'_j \otimes a'_j \in M^\circ \otimes A^\circ$.

Conversely, let $\rightarrow m'$ be completely continuous, $\rightarrow m' = \sum m'_j \otimes a'_j \in M^\circ \otimes A^\circ$. For all $a \in A$, $m \in M$

$$(a \cdot m') (m) = \sum m'_j (m) \otimes a'_j (a) = m' (ma) = m' (\phi_M (m \otimes a)) = (\phi'_M (m'), m \otimes a).$$

We get $\phi'_M (m') = \sum m'_j \otimes a'_j \in M^\circ \otimes A^\circ$ and thus $\phi'_M (m') \in M^\circ$.

(2) is similar to [D5, lemma 2]; One can check that $b \rightarrow (a \cdot m') = ba \rightarrow m'$. Let $m' \in M^\circ$ and $a \rightarrow m' = \sum m_j^\ell \otimes a_j^\ell \in M^\circ \otimes A^\circ$. Then $ba \rightarrow m' = \sum m_j^\ell \otimes a_j^\ell (ba) = \sum m_j^\ell (b) \otimes a_j^\ell$ and thus $\rightarrow (a \cdot m') = \sum m_j^\ell \otimes a_j^\ell$. Since $a_j^\ell \in A^\circ$, by proposition 8.2, $a \rightarrow a_j^\ell \in A^\circ$. Thus $\rightarrow (a \cdot m') \in M^\circ \otimes A^\circ$ and by part 1 of this proposition $a \rightarrow m' \in M^\circ$.

(3) follows from (1) and (2); (4) direct check; (5) is similar to [D5, Theorem 3].

\begin{definition}
\textbf{Definition 27.} We call $M^\circ$ the dual $\mathfrak{S}$-comodule of $M$. It is a the largest Banach subspace of $M'$, which is a Banach $A^\circ - \mathfrak{S}$-comodule w.r.t. $m^\prime_M.M$.

For details, which are similar to the algebraic case, we refer to [HF]. For a morphism $f : M \rightarrow N$ of Banach $\mathfrak{S}$-modules $f^\circ : N^\circ \rightarrow M^\circ$ is a morphism of Banach comodules and thus we get a functor $(-)^\circ : \text{BMod}_A \rightarrow \text{BComod}_{A^\circ}$. If $A = C'$, then the functor $(-)^\circ$ is the right adjoint to the functor $(-)': \text{BComod}_C \rightarrow \text{BMod}_{C'}$ and preserves finite direct sums. If $M \in \text{BComod}_C$ then $M \subset M^\circ$.

Since $\text{BComod}_C (M, N)$ is a Banach space, for the pair of morphisms $f, g \in \text{BComod}_C (M, N)$ the kernel $\text{Ker} (f - g)$ and the closure of the image $\text{Im} (f - g)$ are equalizer and coequalizer.

Over a Banach $\mathfrak{S}$-bialgebra $H$ one can define the tensor product of two right Banach $\mathfrak{S}$-comodules $M$ and $N$ in the usual way ([D1, I.I.1.1]). Namely, it is $(M \otimes N, \rho_{M \otimes N})$ with the coaction

$$\rho_{M \otimes N} = (\text{Id}_M \otimes \text{Id}_N \otimes m_H) \circ (\text{Id}_M \otimes \tau \otimes \text{Id}_H) \circ (\rho_M \otimes \rho_N),$$

where $\tau (a \otimes b) = b \otimes a$.

\end{definition}

2.3. \textbf{Induction.} In the algebraic groups or quantum groups one can describe an induction functor in terms of cotensor product of comodules over coordinate algebra (for quantum case see, for example, [Li, PW]). Similar definitions work for Banach $\mathfrak{S}$-coalgebras (bi-, Hopf $\mathfrak{S}$-algebras).

Let $C$ and $B$ be a Banach $\mathfrak{S}$-coalgebra. Let $(M, \rho_M)$ be a right Banach $C - \mathfrak{S}$-comodule and let $(N, \rho_N)$ be a left Banach $C - \mathfrak{S}$-comodule.

\begin{definition}
\textbf{Definition 28.} The space $M \otimes_C N = \text{Ker} (\rho_M \otimes \text{id}_N - \text{id}_M \otimes \rho_N)$ is called cotensor product of $M$ and $N$ over $C$.
\end{definition}
Since $M \bar{\otimes} N$ is a kernel of a continuous map, it is a closed subspace of $M \bar{\otimes} N$. Thus if $M$ and $N$ are both Banach spaces, then $M \bar{\otimes} N$ is a Banach space. It is an equalizer of $\rho_M \circ id_N$ and $id_M \otimes N\rho$.

If $N$ is also a right Banach $B$-$\bar{\otimes}$-comodule, then $M \bar{\otimes} N$ is also a right Banach $B$-$\bar{\otimes}$-comodule.

If $f : M \to L$ is a Banach $\bar{\otimes}$-comodule morphism, one has a morphism $f \bar{\otimes} N : M \bar{\otimes} N \to L \bar{\otimes} N$ of Banach spaces, which is defined as a restriction $(f \bar{\otimes} id_N)_{|M \bar{\otimes} N}$. Thus cotensoring with a left Banach $\bar{\otimes}$-comodule $N$ gives a functor $\left(-\bar{\otimes} N\right) : BComod_C \to BComod_B$.

**Definition 29.** Let $A$ be a Banach $\bar{\otimes}$-algebra. For $M \in BMod_A$ and $N \in_A BMod$ define $M \bar{\otimes}_A N$ as the quotient Banach space of $M \bar{\otimes} N$ by the closure of the linear hull of the set $\{ma \otimes n - m \otimes an\}$ for all $m \in M, n \in N, a \in A$.

$M \bar{\otimes}_A N$ is an equalizer of the maps $\phi_M \otimes id_N$ and $id_M \otimes \phi_N$, i.e. we have a diagram $M \bar{\otimes} A \bar{\otimes} N \Rightarrow M \bar{\otimes} N \Rightarrow M \bar{\otimes}_A N \Rightarrow 0$ in $Ban_K$. $M \bar{\otimes}_A N$ is also a completion of the algebraic tensor product $M \bar{\otimes}_A N$ w.r.t. cross-norm $\|\cdot\|_M \otimes \|\cdot\|_N$.

**Proposition 30.** For $M \in BComod_C$, $N \in_C BComod$ the space $M' \bar{\otimes}_C N'$ is a closed subspace of $\left(M \bar{\otimes} C N\right)'$.

**Proof.** Consider the defining exact sequence for $\bar{\otimes}$-cotensor product

$$0 \to M \bar{\otimes} N \xrightarrow{\phi} M \bar{\otimes} N \xrightarrow{\psi} M \bar{\otimes} C \bar{\otimes} N,$$

where $\psi = \rho_M \bar{\otimes} id_N - id_M \bar{\otimes} \rho_N$.

Taking duals gives us the following sequence

$$0 \leftarrow (M \bar{\otimes} N)'_b \xrightarrow{\phi'} (M \bar{\otimes} N)'_b \xleftarrow{\psi'} (M \bar{\otimes} C \bar{\otimes} N)'_b.$$

The space $(M \bar{\otimes} N)'$ contains $M' \bar{\otimes} N'$ as a closed subspace [VANR, 4.34]. Since $M \bar{\otimes} N$ is a complemented subspace of $M \bar{\otimes} N = \left(M \bar{\otimes} N\right) \oplus W$, we have a decomposition of the dual space

$$(M \bar{\otimes} N)' = \left(M \bar{\otimes} N\right)' \oplus W'$$

and clearly

$$W' \cong \left(M \bar{\otimes} N\right)'^\perp = \left\{\phi \in (M \bar{\otimes} N)': \phi(x) = 0 \forall x \in M \bar{\otimes} N\right\}.$$
of dual bases, \( M \mathbb{B} N = \bigcap \ker (e'_m e'_c \otimes e'_n - e'_m \otimes e'_e e'_n) \). Thus \( \left(M \mathbb{B} N\right)^\perp\) is equal to the \(^*\)-weak closure \( I^{\perp\perp} \) ([R, 4.7], which relies on [R, 3.5], which can be easily seen via [PGS, 7.3.1] and [PGS, 5.1.4, 5.1.6] or [NFA, 9.5]) of the linear space
\[
I = \{ \sum (m' e' \otimes n' - m' \otimes e' n'), m' \in M', n' \in N', e' \in C' \}
\]
(actually, just of the elements \( (e'_m e'_c \otimes e'_n - e'_m \otimes e'_e e'_n) \)). Thus we have a topological isomorphism \( (M \mathbb{B} N)' = \left(M \mathbb{B} N\right)^\perp \otimes I^{\perp\perp} \).

Since \( M' \otimes N' \) is a free normed space, we can complete an orthogonal basis \( E_I \) of \( I \) to an orthogonal basis \( E_{M' \mathbb{B} N'} = E' \cup E_I \) of \( M' \otimes N' = (M' \mathbb{B} C, N') \otimes I \). Taking norm completions we get \( M' \mathbb{B} N' = (M' \mathbb{B} C, N') \) and taking \(^*\)-weak completion we get \( (M \mathbb{B} N)' = (M' \mathbb{B} C, N')^{\perp\perp} \otimes I^{\perp\perp} \). Thus we see that inside \( (M \mathbb{B} N)' \) we have \( (M' \mathbb{B} N') \cap I^{\perp\perp} = I^{\perp} \) and that under the quotient map \( (M \mathbb{B} N)' / \left(M \mathbb{B} N\right)^\perp \otimes I^{\perp\perp} \). Thus we have an embedding \( M' \mathbb{B} N' / I^{\perp} = M' \mathbb{B} C, N' \rightarrow (M \mathbb{B} N)' / \left(M \mathbb{B} N\right)^\perp \otimes I^{\perp\perp} \). \( \square \)

Let \( \phi : C \rightarrow B \) be a morphism of Banach \( \mathbb{B} \)-coalgebras. Then \( C \) is a left and right Banach \( B-\mathbb{B} \)-comodule via maps
\[
\rho_{l\phi} = (\phi \otimes \text{id}_C) \circ \Delta_C
\]
and
\[
\rho_{r\phi} = (\text{id}_C \otimes \phi) \circ \Delta_C.
\]
Denote those \( \mathbb{B} \)-comodules by \( C_{\phi} := (C, \rho_{l\phi}) \) and \( C_{\phi} := (C, \rho_{r\phi}) \).

**Definition 31.** Let \( M \in B\text{Comod}_B \). Then \( M \mathbb{B} C_B \in B\text{Comod}_C \) is called the induced \( \mathbb{B} \)-comodule. The other notation is \( M^\phi \). The functor \((-)\mathbb{B} \) is called induction.

**Definition 32.** For \( M \in B\text{Comod}_C \), \( M_{\phi} \) will denote the \( B-\mathbb{B} \)-comodule \( M \) with coaction \( M \stackrel{\rho_{M\phi}}{\rightarrow} M \mathbb{B} C \). The functor \((-)\mathbb{B} \) is called restriction.

**Lemma 33.** The map \( \text{id}_M \mathbb{B} e_C : M \mathbb{B} e_C \rightarrow M \) such that \( (\text{id}_M \mathbb{B} e_C)(m \otimes c) = e(c)m \) is a morphism of Banach \( B-\mathbb{B} \)-comodules \( (M^\phi)_{\phi} \) and \( M \). If \( C \subset B \) then the image of \( (\text{id}_M \mathbb{B} e_C)(M^\phi) \) is the maximal closed subspace of \( M \), which is an \( C-\mathbb{B} \)-comodule.

**Proof.** the proof is the same as in algebraic case. \( \square \)

**Corollary 34.** \( M \mathbb{B} B \geq M \) (in fact, \( M \mathbb{B} B = \rho_M(M) \)).

For a Banach \( \mathbb{B} \)-coalgebra \( C \) a finite direct product \( C^n = \{ \sum_{i=1}^n c_i e_i | c_i \in C \} \) has a right (and left) Banach \( C-\mathbb{B} \)-comodule structure with both right coaction
\[
\rho_{rC^n} \left( \sum_{i=1}^n c_i e_i \right) = \sum (c_i)_{(0)} e_i \otimes (c_i)_{(1)}
\]
and left coaction
\[
\rho_{lC^n} \left( \sum_{i=1}^n c_i e_i \right) = \sum (c_i)_{(0)} \otimes (c_i)_{(1)} e_i
\]
defined coordinate-wise.
**Definition 35.** A (right) Banach $C$-$\circledS$-comodule $M$ is called finitely cogenerated if it is a closed $\circledS$-subcomodule of $C^n$.

**Proposition 36.** Let $M \in \text{BComod}_B$ and $N \in \text{BComod}_C$.

1. If $M$ is a finitely cogenerated Banach $B$-$\circledS$-comodule, then $M^\circ$ is a finitely cogenerated Banach $C$-$\circledS$-comodule;
2. If $\phi$ is injective and $N$ is finitely cogenerated, then $N_\phi$ is a finitely cogenerated Banach $B$-$\circledS$-comodule.
3. If $M$ is finitely generated Banach $\circledS$-module over a Banach $\circledS$-algebra $A$, then $M^\circ$ is finitely cogenerated over $A^\circ$.

**Proof.** We have an embedding $M \hookrightarrow B^n$ for some $n \in \mathbb{N}$. Then $M^\circ = M\circledS B^\circ C \hookrightarrow B^n B\circledS C \cong C^n$, which proves 1). 2) is obvious. 3) is similar to [HF]. □

**Definition 37.** A Banach $C$-$\circledS$-comodule $M$ is called cofree if it is of the form $M \cong V\circledS C$ for a Banach space $V$ with $\circledS$-comodule action $\rho = id_V \circledS \Delta_C$.

Cofree Banach $\circledS$-comodules are cofree objects in the category of right Banach $C$-$\circledS$-comodules over the Banach space $V$ with the covering map $p : V\circledS C \rightarrow V$ being $p = id_V \circledS \epsilon_C$. If $V$ is finite-dimensional, we have a $\circledS$-comodule isomorphism $V\circledS C \cong \text{Cspan}_K V$.

If $M$ is free finitely generated Banach $\circledS$-module over a Banach $\circledS$-algebra $A$, then $M^\circ$ is cofree finitely cogenerated over $D^\circ$ (similar to [HF]).

As in the algebraic case, our induction and restriction functors are related by Frobenius reciprocity.

**Proposition 38.** (Frobenius reciprocity)

Let $\pi \in \text{BCoalg}_K (C,B)$, be a morphism of $K$-Banach $\circledS$-coalgebras, $M \in \text{BComod}_B$ and $N \in \text{BComod}_C$.

There is a topological isomorphism

$$
\text{BComod}_C \left( N, M\circledS_B C \right) \cong \text{BComod}_B (N, M).
$$

**Proof.** The morphisms of $\circledS$-comodules

$$
\phi \mapsto (\text{id}_M \circledS \epsilon_C) \circ \phi = \tilde{\phi} = (\text{id}_M \circledS (\epsilon_B \circ \pi)) \circ \phi
$$

are inverse to each other [DOI, Prop.6].

Since the topology on spaces $\text{BComod}_C \left( N, M\circledS_B C \right)$ and $\text{BComod}_B (N, M)$ is induced from $L_b (N, M\circledS C)$ and $L_b (N, M)$, the continuity of our linear bijections follows from the argument same as in [NFA, sec.18]. Namely, composition with linear continuous map $W \rightarrow U$ is a linear continuous map $L_b (V,W) \rightarrow L_b (V,U)$.

Since our (') maps are compositions of continuous maps, they are continuous. □

### 2.4. Rational $\circledS$-modules.

Let $C \in \text{BCoalg}_K$ be a Banach $\circledS$-coalgebra and $M$ be a left Banach $\circledS$-module over $C'$. Every element $m \in M$ defines a map from $C'$ to $M$ through the cyclic $\circledS$-module, generated by $m$. Following [SW, 2.1], taken altogether for all $m \in M$, this defines a continuous map

$$
\rho : M \rightarrow \text{Ban}_K (C', M)
$$

$$
m \mapsto \rho (m) : \rho (m) (c') = c' m.
$$
There is a natural embedding
\[ M \otimes C \hookrightarrow \text{Ban}_K (C', M) \]
\[ m \otimes c \mapsto f(m \otimes c) : f(m \otimes c)(c') = c'(c) \cdot m . \]

Let \( M \in \text{BComod}_C \) and consider the induced \( C' - \widehat{\otimes} \)-module structure on \( M \). In this case \( c' \cdot m = (\text{id}_M \otimes c') \circ \rho_M(m) \). Then our map \( \rho : M \to \text{Ban}_K (C', M) \) acts on \( m \) exactly as \( \rho_M \) (since the results coincide on every element of \( C' \)), and thus \( \rho(m) \in M \otimes C \).

**Definition 39.** In the notations above
- \( M \) is called \( t \)-rational if \( \rho(M) \subset M \otimes C \);
- \( M \) is called rational if \( \rho(M) \subset M \otimes C \).

In both cases \( \rho(M) \) satisfy axioms of a \( \widehat{\otimes} \)-comodule action.

In other words, a \( \widehat{\otimes} \)-module is \( t \)-rational if its \( \widehat{\otimes} \)-module structure is induced from a Banach \( \widehat{\otimes} \)-comodule structure and rational if it is induced from a \( \otimes \)-comodule.

**Remark 40.** Since for any Banach spaces \( M \) and \( C \) by [NFA, 18.11] we have the inclusion \( M \otimes C \subset M \otimes C'' \cong C(C', M) \). Thus if \( M \) is \( t \)-rational (rational) then \( \rho(M) \subset C(C', M) \), i.e. \( \rho(m) \) is a compact (finite rank) map for every \( m \in M \).

**Proposition 41.** [SW, 2.1.3] Let \( L, M, N \in C' \mod \) and \( M, N \) being \( t \)-rational (rational).

1. if \( N \subset M \) is a \( \widehat{\otimes} \)-subcomodule (\( \otimes \)-submodule), then \( N \) is also a \( \widehat{\otimes} \)-submodule;
2. every cyclic \( \widehat{\otimes} \)-submodule is countable type (finite dimensional);
3. any quotient of a \( t \)-rational (rational) module is \( t \)-rational (rational);
4. \( L \) has a unique maximal rational submodule
   \[ L^{\text{rat}} = \sum \text{"all rational submodules"} = \rho^{-1}_L (L \otimes C) \]
   and a unique maximal \( t \)-rational \( \widehat{\otimes} \)-submodule
   \[ L^{t\text{-rat}} = \sum \text{"all } t \text{-rational } \widehat{\otimes} \text{-submodules"} = \rho^{-1}_L (L\widehat{\otimes}C) ; \]
5. any homomorphism of \( t \)-rational (rational) modules is a \( \widehat{\otimes} \)-comodule (\( \otimes \)-comodule) morphism.

**Corollary 42.** Any finitely generated rational module is finite dimensional.

2.5. Tensor identities.

**Proposition 43.** (Tensor identities) Let \( \pi \in BHopf_K (H, B) \), \( W \in \text{BComod}_B \) and \( V \in \text{BComod}_H \). Then we have the following isomorphisms of Banach \( H \widehat{\otimes} \)-comodules:

1. \( V \widehat{\otimes} \left( W \otimes_B H \right) \cong (V \widehat{\otimes} W) \otimes_B H \)
2. \( \left( W \otimes_B H \right) \otimes V \cong (W \otimes V) \otimes_B H \)

**Proof.** The maps
\[ \phi : V \widehat{\otimes} \left( W \otimes_B H \right) \to (V \widehat{\otimes} W) \otimes_B H \]
\[ \sum v \otimes w \otimes h \mapsto \sum v_{(0)} \otimes w \otimes v_{(1)} h . \]
and
\[ \psi : (V \circledast W) \circledast H \longrightarrow V \circledast \left( W \circledast H \right) \]
\[ \sum v \otimes w \otimes h \quad \mapsto \quad \sum v(0) \otimes w \otimes S_H (v(1)) h. \]

are morphisms of right Banach \( H \circledast \)-comodules that are inverse to each other[DOI].

Since all involved maps are continuous, \( \phi \) and \( \psi \) are continuous.

The proof of (2) is similar. \( \square \)

**Corollary 44.** If \( B = K, \pi = \varepsilon_H \) and \( W = K \) then we have a \( H \)-comodule isomorphism
\[ V \circledast H \cong V \circledast H, \]
where \( V \) means the underlying vector space of \( V \) with trivial \( H \)-comodule structure \( (\rho_v, v) = v \otimes 1 \).

3. **Locally convex \( \mathcal{O} \)-coalgebras**

**Preliminaries.**

**Basic notions.** For the background on nonarchimedean functional analysis we refer mostly to [NEA] and [PGS].

Let \( V \) be Topological Vector Space (TVS) over \( K \).

A subset of \( V \) is called **absolutely convex** if it is an \( K^0 \)-submodule of \( V \).

A subset of \( V \) is called **convex** if it is of the form \( x + A, x \in V \) and \( A \) is an absolutely convex set.

An absolutely convex subset \( L \subset V \) is called a **absorbing** if \( \forall x \in V \exists \lambda \in K^* \) s.t. \( x \in \lambda L \).

A **lattice** is an absolutely convex absorbing set.

A TVS is Locally Convex (LCTVS) if its topology is generated by lattices. We use LCTVS both as abbreviation and as the notation for the corresponding category.

The strong dual of an LCTVS \( V \) is denoted by \( (V)' \).

**Inductive and projective limits.** Let \( \mathcal{F} = \{ f_h : V_h \rightarrow V \}_{h \in H} \) be a family of maps of LCTVSs. A **final locally convex topology** \( \tau_{fin} \) on \( V \) w.r.t. \( \mathcal{F} \) is the strongest locally convex topology that makes all \( f_h \in \mathcal{F} \) continuous. Such a topology always exists.

Let \( (V_n, \phi_{nm})_{n \in \mathbb{N}} \) be an inductive sequence of LCTVSs. A locally convex inductive limit of \( (V_n, \phi_{nm})_{n \in \mathbb{N}} \) is its algebraic inductive limit \( V = \lim \rightarrow V_n \), equipped with final locally convex topology w.r.t. \( \phi_n : V_n \rightarrow V. \) Any family of morphisms \( f_n : V_n \rightarrow U \) on \( (V_n, \phi_{nm})_{n \in \mathbb{N}} \) in LCTVS, compatible with \( \phi_{nm} \), factors through \( V \).

\( V \) can also be described as a quotient of the locally convex direct sum \( \bigoplus V_n \) by the subspace \( D \), generated by vectors \( \{ (v_n, \phi_{n,n+1} (v_n), \ldots ) | v_n \in V_n \} \). We denote by \( \pi : \bigoplus V_n \rightarrow V = \bigoplus V_n / D \) the quotient map. The universal property of \( V \) follows from the universal property of \( \bigoplus V_n \).

A map \( f : V \rightarrow U \) in LCTVS is continuous iff maps \( f \circ \phi_n \) are continuous. Taking \( f = \text{id}_V \) one get that subset \( F \subset V \) is open (closed) if and only if \( \phi_n^{-1} (F) \) is open (closed) [GKPS, 1.1.6] for all \( n \).

Similarly one can define locally convex projective limits. Let \( \mathcal{F} = \{ f_h : V \rightarrow V_h \}_{h \in H} \) be a family of maps of LCTVSs. An **initial locally convex topology** \( \tau_{inf} \) on \( V \) w.r.t. \( \mathcal{F} \) is the weakest locally convex topology that makes all \( f_h \in \mathcal{F} \) continuous.
Let \((V_n, \phi_{nm})_{n \in \mathbb{N}}\) be a projective sequence of LCTVSs. A locally convex projective limit of \((V_n, \phi_{nm})_{n \in \mathbb{N}}\) is its algebraic projective limit \(V = \varprojlim V_n\), equipped with final locally convex topology w.r.t. \(\phi_n : V \to V_n\). \(V\) can be described as a subspace of the locally convex direct product \(\prod V_n\), consisting of vectors \(\{(\ldots, v_n, v_{n+1}, \ldots), v_n \in V_n\} = \{v_n + 1_n\} = v_n\).

Limits of sequences of spaces.

**Definition 45.** Let \(V\) be an LCTVS.
- \(V\) is a **LB-space**, if it is an locally convex inductive limit of a sequence \((V_n, \phi_{nm})\) of Banach spaces.
- \(V\) is a **Fréchet** if it is a locally convex projective limit of a sequence \((V_n, \phi_{nm})\) of Banach spaces.
- \(V\) is an **LS-space** if it is an LB-space and the transition maps \(\phi_n\) are compact.
- \(V\) is an **FS (Fréchet-Schwartz) space**, if it is a Fréchet space and the transition maps \(\phi_n\) are compact.

In another terminology, LS-spaces are called compact type (CT) spaces, and FS-spaces are called nuclear Fréchet (NF) spaces.

**Proposition 46.** Any LS-space \(V\) is Hausdorff, complete and reflexive [ST4, Theorem 1.1].
Every Fréchet space is Hausdorff and complete [NFA, Proposition 8.2]. FS-spaces are also reflexive.
A closed subspace of an LS (FS) space is an LS (FS) space. The quotient of an LS (FS) space by a closed subspace is an LS (FS) space [ST4, 1.2].
The strong dual of an LS-space is an FS-space and vice versa [ST4, 1.3]. Thus the categories of LS and FS spaces are antiequivalent.

The following version of the Open Mapping theorem follows from the general theorem of De Wilde [MV, 24.30].

**Theorem 47.** (Open Mapping theorem) Let \(f : E \to U\) be a continuous surjective map of LCTVS \(E\) and \(U\). If \(E\) is either LB- or Fréchet space, or their closed subspace or quotient, and if \(U\) is an LB- or Fréchet space (in particular, Banach), then \(f\) is an open map.

**Tensor products.** The category of LCTVS has several natural topologies on the tensor product of two LCTVS, most commonly used are the inductive \(\otimes_i, K\), injective \(\otimes_e, K\) and projective \(\otimes_p, K\) tensor product topologies. Over discretely valued fields, injective and projective tensor products coincide for all LCTVS [PGS, 10.2]. For Fréchet or sequentially complete LB spaces (and possibly in more general cases too) inductive and projective tensor products coincide [NFA, 17.6], [EM, 1.1.31]. Thus in our cases of interest (LS and FS spaces) we have only one reasonable tensor product topology, which after completion give rise to the completed tensor product \(\hat{\otimes}\).

The categories of LS and FS spaces are tensor categories with respect to the \(\hat{\otimes}\).

**Proposition 48.** Let \(V \cong \varprojlim V_n, U \cong \varprojlim U_n\) be LS-spaces, \(F = \varprojlim F_n\), \(H = \varprojlim H_n\) be FS-spaces and \(W\) be a Banach space.

1. \(V \hat{\otimes} U = \varprojlim V_n \hat{\otimes} U_n\) [EM, 1.1.32];
Definition 50. The category of projective systems of Banach spaces.

Continuous. In this case, it is a monoid in the tensor category \((\text{LCTVS}, \otimes)\).

Definition 49. Topological Coalgebras.

1. An LCTVS algebra \(A\) is a LCTVS with a continuous multiplication map.
2. An F-algebra \(A\) is a topological algebra, which is a Fréchet space.
3. An FS-algebra \(A\) is an F-algebra, which is a nuclear Fréchet space.
4. A Fréchet algebra \(A\) is an F-algebra, whose topology can be given by a (countable) system of submultiplicative seminorms. In this case it is a locally convex projective limit of Banach \(\otimes\)-algebras with transition maps being \(\otimes\)-algebra morphisms.
5. A Nuclear Fréchet (NF) algebra \(A\) is a Fréchet algebra, which is a nuclear space.

One can require the multiplication on LCTVS algebra only to be separately continuous. In this case, it is a monoid in the tensor category \((\text{LCTVS}, \otimes_{i,K})\) with the tensor structure given by inductive tensor product. In our definition, it is a monoid w.r.t. projective tensor product \(\otimes_{\pi,K}\). For Fréchet spaces these two notions coincide, since any separately continuous map in this case is (jointly) continuous and an F-algebra is an algebra in the tensor category of Fréchet spaces with the tensor structure given by \(\otimes\).

Every Fréchet space \(A\) can be presented as a locally convex projective limit of Banach spaces, i.e. \(A = \lim A_n\). If \(A\) is a Fréchet algebra, \(A_n\) can be chosen to be Banach algebras (Arens-Michael presentation) and if \(A\) is an NF-algebra, one can chose \((A_n, \phi_{nm})\) such that transition maps \(\phi_{nm}\) are compact. In general, for multiplication to be continuous only a family form of submultiplicativity is required, i.e. \(\|x \cdot y\|_n \leq \|x\|_{n+k} \cdot \|y\|_{x+k}\) (see [G]) and thus for an F-algebra an Arens-Michael presentation might not exist.

Our main object of interest is the category, opposite to the category of NF-algebras. For an NF-algebra we will call a corresponding Arens-Michael system an \(\text{NF-structure}\). It is known that any two NF-structures are equivalent [EM, 1.2.7] in the category of projective systems of Banach spaces.

Definition 50. Let \(C\) be an LCTVS.

1. For a tensor structure \(\otimes (\otimes_{i,K}, \otimes_{\pi,K})\) on LCTVS, \(C\) is an \text{LCTVS} \(\otimes\)-coalgebra if it is a \(\otimes\)-coalgebra object in the tensor category \((\text{LCTVS}, \otimes)\).
2. \(C\) is an \(\text{LS(LB)}\)-coalgebra if it is an \text{LCTVS} \(\otimes\)-coalgebra and an \(\text{LS(LB)}\)-space.
3. \(C\) is a \(\text{CT (Compact Type)}\) \(\otimes\)-coalgebra, if it is an LS-coalgebra, that is topologically isomorphic to a compact locally convex inductive limit of a
compact inductive system \((C_n, \phi_{nm})\) of Banach \(\bar{\otimes}\)-coalgebras with transition maps \(\phi_{nm}\) being \(\bar{\otimes}\)-coalgebra morphisms. We say that \((C_n, \phi_{nm})\) gives \(C\) a CT-structure.

If \(\{C_n, \phi_n\}\) be an inductive system of K-Banach \(\bar{\otimes}\)-coalgebras \(C_n\) with injective transition maps \(\phi_n : C_n \to C_{n+1}\), s.t. \(\phi_n\) are morphisms of \(K\)-Banach \(\bar{\otimes}\)-coalgebras, then \(C = \lim\ C_n\) is a \(\bar{\otimes}\)-coalgebra in the category of locally convex K-vector spaces, with \(\bar{\otimes}\)-coalgebra maps \((\Delta, \epsilon)\) defined by the corresponding maps \((\Delta_n, \epsilon_n)\). Thus \(C_n\) indeed define an LB-\(\bar{\otimes}\)-coalgebra structure in \(C\). We remind again that for sequentially complete (equivalently, regular) LB-spaces \(\bar{\otimes}_{i,K}\) and \(\bar{\otimes}_{\pi,K}\) coincide. In particular, it is true for LS-spaces.

Since for any inductive system one can construct the one with the same inductive limit and injective transition maps, without loss of generality we can only consider CT-structures with injective transition maps.

**Definition 51.** We say that two CT-structures \(C = \lim\ C_n\) and \(D = \lim\ D_n\) are equivalent if they are isomorphic in the category of inductive systems of Banach \(\bar{\otimes}\)-coalgebras.

**Lemma 52.** For a CT-\(\bar{\otimes}\)-coalgebra \(C\) any two CT-structures with injective transition maps are equivalent.

**Proof.** Let \((C_n, \phi_{nm})\) and \((D_n, \psi_{nm})\) be two CT-structures for \(C\). Since \(C = \lim\ C_n\) and \(C = \lim\ D_n\), \((C_n, \phi_{nm})\) and \((D_n, \psi_{nm})\) are equivalent as inductive systems of Banach spaces. Since \(\phi_{nm}\) and \(\psi_{nm}\) are injective, the embeddings \(\phi_n : C_n \to C\) and \(\psi_n : D_n \to D\) are also injective and therefore the above equivalence maps must be Banach \(\bar{\otimes}\)-coalgebra morphisms. \(\square\)

**Remark 53.** Every LS-\(\bar{\otimes}\)-coalgebra is a \(\bar{\otimes}\)-coalgebra object in the category of LS-spaces. Since the dual of a completed tensor product \(C \bar{\otimes} D\) of two LS-spaces \(C\) and \(D\) is the completed tensor product \(C'_b \bar{\otimes} D'_b\) of their strong duals, the duality functor maps the commutative diagrams, defining \(\bar{\otimes}\)-coalgebra structure for \(C\), into the diagrams, which satisfy \(\bar{\otimes}\)-algebra axioms for \(C'_b\). Thus the dual of a \(\bar{\otimes}\)-coalgebra object in the category of LS-spaces is an \(\bar{\otimes}\)-algebra object in the category of FS-spaces and we have an antiequivalence of categories of LS-\(\bar{\otimes}\)-coalgebras and FS-\(\bar{\otimes}\)-algebras.

Clear that if \(C\) is a CT-\(\bar{\otimes}\)-coalgebra \(C = \lim\ C_n\), then \(C'_b\) is an NF-\(\bar{\otimes}\)-algebra with NF-structure \((C'_n, \phi'_{nm})\).

**Proposition 54.** Let \(A = (A, m, u)\) be an NF-\(\bar{\otimes}\)-algebra with NF-structure \((A_n, \phi_{nm})\), \(A \cong \lim\ A_n\). Then \(A'_b\) is a CT-\(\bar{\otimes}\)-coalgebra \((A'_b, m'_b, u'_b)\) with a CT-structure \((A'_n, \phi'_{nm}, \psi'_{nm})\).

**Proof.** The only thing one needs to prove is that \(\lim\ A^\otimes_n = (A)'_b\). Recall that \(A^\otimes_n = (m'_b A_n)^{-1}(A'_n \bar{\otimes} A'_n)\). Since \(\phi'_{nm}\) are Banach \(\bar{\otimes}\)-coalgebra morphisms, \(A^\otimes_n\) form a compact inductive system of Banach \(\bar{\otimes}\)-coalgebras, with the locally convex inductive limit \(\lim\ A^\otimes_n\) being the preimage of \((m')^{-1}\) \(\big((A)'_b \bar{\otimes} (A)'_b\big)\). But since \(\big((A)'_b \bar{\otimes} (A)'_b\big) = (A \bar{\otimes} A)'_b\), the preimage of the coaction is the whole \((A)'_b\), which is itself an LS-\(\bar{\otimes}\)-coalgebra. \(\square\)
Since spaces of compact type are reflexive, the above proposition gives us an antiequivalence of categories
\[
\{\text{CT-}\hat{\otimes}\text{-coalgebras}\} \leftrightarrow \{\text{NF-}\hat{\otimes}\text{-algebras}\}
\]
with morphisms being continuous topological \(\hat{\otimes}\)-coalgebra and \(\hat{\otimes}\)-algebra morphisms.

The notions of \(\hat{\otimes}\)-subcoalgebra, \(\hat{\otimes}\)-left coideal, \(\hat{\otimes}\)-right coideal and \(\hat{\otimes}\)-coideal (2-sided) for topological \(\hat{\otimes}\)-coalgebras are defined similarly to the Banach case.

**Proposition 55.** Let \(V\) be a CT-\(\hat{\otimes}\)-coalgebra.

1. If \(U\) is a closed \(\hat{\otimes}\)-subcoalgebra, then \(U\) is also of compact type.
2. If \(I\) is a closed \(\hat{\otimes}\)-coideal of \(V\), then \(V/I\) is a CT-\(\hat{\otimes}\)-coalgebra.
3. If \(f : V \to W\) is a morphism of (topological) CT-\(\hat{\otimes}\)-coalgebras then \(\text{Ker}(f)\) is closed \(\hat{\otimes}\)-coideal.
4. If \(J\) is a closed \(\hat{\otimes}\)-coideal of \(W\) then \(f^{-1}(J)\) is a closed \(\hat{\otimes}\)-coideal.
5. The closure of \(f(V)\) is a closed CT-\(\hat{\otimes}\)-subcoalgebra.

**Proof.** \(U \subset V\) is closed iff \(U_n = U \cap V\) is closed \(\forall n\). Thus \(U_n\) are Banach subspaces of \(V_n\) and, since \(U\) is a \(\hat{\otimes}\)-subcoalgebra, are K-Banach \(\hat{\otimes}\)-subcoalgebras of \(V_n\). So \(U = \lim U_n\) is a CT-\(\hat{\otimes}\)-coalgebra.

The proofs of the rest statements are the same as in Banach case. □

### 3.2. Topological Bialgebras and Hopf Algebras

We have similar definitions of an LCTVS-, LS-, FS-, CT- and NF-\(\hat{\otimes}\)-bialgebras.

Similar to the remark 53, the categories of LS- and FS-\(\hat{\otimes}\)-bialgebras are antiequivalent under the duality map \(V \mapsto (V)^{\hat{\otimes}}\). To see if the categories of CT-\(\hat{\otimes}\)-bialgebras and NF-\(\hat{\otimes}\)-bialgebras are equivalent, we will first establish an auxiliary result.

**Lemma 56.** Let \(F\) and \(H\) be FS-spaces with presentations \(F = \lim F_n\) and \(H = \lim H_n\). Let \(f : F \to H\) be a morphism, which is defined by a morphism of projective systems \(\{f_n : F_n \to H_n\}\). Let \(U = \lim U_n\), \(U_n \subset H_n\), be a closed subspace of \(H\). Then \(f^{-1}(U) = \lim f_n^{-1}(U_n)\).

**Proof.** Follows from definitions unpacking. □

**Proposition 57.** For a CT-\(\hat{\otimes}\)-bialgebra \(B \cong \lim B_n\) with CT-structure \((B_n, \rho_{nm})\), the dual system \((B^{\hat{\otimes}}, \rho_{nm}^{\hat{\otimes}})\) forms an NF-structure for \((B)^{\hat{\otimes}}_b\). Thus \((B)^{\hat{\otimes}}_b\) is an NF-\(\hat{\otimes}\)-bialgebra.

**Proof.** Clear that \((B^{\hat{\otimes}}, \rho_{nm}^{\hat{\otimes}})\) is an NF-system of Banach \(\hat{\otimes}\)-bialgebras. The proof that its projective limit is \((B)^{\hat{\otimes}}_b\) is similar to the Proposition 54. The maps \(m^{\hat{\otimes}}_{B_n} : B_n \to (B_n \otimes B_n)^{\hat{\otimes}}\) define a morphism of projective systems \(\{B'_n\}\) and \(\{(B_n \otimes B_n)^{\hat{\otimes}}\}\).

The tensor products \(B'_n \otimes B'_n \subset (B_n \otimes B_n)^{\hat{\otimes}}\) form a projective subsystem and thus define a closed subspace of \((B \otimes B)^{\hat{\otimes}}_b\). By preceding lemma 56, \((m^{\hat{\otimes}}_B)^{-1} (B \otimes B)^{\hat{\otimes}}_b = (m^{\hat{\otimes}}_B)^{-1} \left( \lim_n B'_n \otimes B'_n \right) = \lim_n (B'_n \otimes B'_n)^{\hat{\otimes}}_b\). Since, by nuclearity, \((B \otimes B)^{\hat{\otimes}}_b = (B)^{\hat{\otimes}}_b \otimes (B)^{\hat{\otimes}}_b\), we have \((B)^{\hat{\otimes}}_b = (m^{\hat{\otimes}}_B)^{-1} (B^{\hat{\otimes}}_b \otimes (B)^{\hat{\otimes}}_b)\). All together it gives us

\[
B^{\hat{\otimes}}_b = (m^{\hat{\otimes}}_B)^{-1} ((B \otimes B)^{\hat{\otimes}}_b) = (m^{\hat{\otimes}}_B)^{-1} ((B)^{\hat{\otimes}}_b \otimes (B)^{\hat{\otimes}}_b) = (m^{\hat{\otimes}}_B)^{-1} \left( \lim_n B'_n \otimes B'_n \right) =
\]
Thus we have an equivalence of categories
\[ \{ \text{CT-}\mathcal{H}-\text{bialgebras} \} \leftrightarrow \{ \text{NF-}\mathcal{H}-\text{bialgebras} \}. \]

LCTVS-, LS-, FS-, CT- and NF-Hopf \( \mathcal{H} \)-algebras are defined similarly. One can easily check that, similar to algebraic and Banach case, the antipode on a topological bialgebra is unique if exists, and that the proposition 57 still holds. Thus we also have an equivalence of categories
\[ \{ \text{CT-Hopf} \mathcal{H}-\text{algebras} \} \leftrightarrow \{ \text{NF-Hopf} \mathcal{H}-\text{algebras} \}. \]

The notions of \( \mathcal{H} \)-subbialgebra (Hopf \( \mathcal{H} \)-subalgebra), left \( \mathcal{H} \)-biideal (Hopf \( \mathcal{H} \)-ideal), right \( \mathcal{H} \)-biideal (Hopf \( \mathcal{H} \)-ideal) and \( \mathcal{H} \)-biideal (Hopf \( \mathcal{H} \)-ideal) (2-sided) can be defined similar to the Banach case.

The quotient of a (CT-) \( \mathcal{H} \)-bialgebra (Hopf \( \mathcal{H} \)-algebra) by a closed \( \mathcal{H} \)-biideal (Hopf \( \mathcal{H} \)-ideal) is a (CT-) \( \mathcal{H} \)-bialgebra (Hopf \( \mathcal{H} \)-algebra).

The kernel of the morphism of \( \mathcal{H} \)-bialgebras (Hopf \( \mathcal{H} \)-algebras) is closed \( \mathcal{H} \)-biideal (Hopf \( \mathcal{H} \)-ideal).

Preimage of a closed \( \mathcal{H} \)-biideal (Hopf \( \mathcal{H} \)-ideal) is a closed \( \mathcal{H} \)-biideal (Hopf \( \mathcal{H} \)-ideal).

Closure of the image of a morphism of \( \mathcal{H} \)-bialgebras (Hopf \( \mathcal{H} \)-algebras) is a closed \( \mathcal{H} \)-subbialgebra (Hopf \( \mathcal{H} \)-subalgebra). The proof is the same as in Banach case.

Example 58. For any open compact subgroup \( G \) of a locally analytic \( K \)-group \( \mathbb{G} \) the algebra of locally analytic \( K \)-valued functions \( C^{la}(G,K) \) is a commutative CT-Hopf \( \mathcal{H} \)-algebra. Its is known that \( G \) possesses a system of neighborhoods of zero \( \{ H_i \} \) consisting of open normal compact subgroups \( H_i \). The CT-structure is given by Banach Hopf \( \mathcal{H} \)-algebras \( C^{la}_{H_i}(G,K) \) of locally \( H_i \)-analytic functions, i.e. the functions \( f \in C^{la}(G,K) \) s.t. \( f|_{H_i} \) is (rigid) analytic for all \( g \in G \). The strong dual space \( D^{la}(G,K) = (C^{la}(G,K))' \) is a cocommutative NF-Hopf \( \mathcal{H} \)-algebra.

For a worked out noncommutative noncocommutative example we refer to [L1].

4. Modules and Comodules

4.1. Definitions.

Definition 59. Let \( C \) be a CT-coalgebra and \( V \) is an LCTVS. Then we say that

1. \( V \) is a right \( LCTVS \mathcal{H}_\pi \)-comodule over \( C \) (\( V \in \text{LCVSComod}_C \)) if there exists a \( K \)-linear continuous map \( \rho_V : V \to V \mathcal{H}_\pi C \) such that

\[ (id_V \otimes \epsilon_C) \circ \rho_V = id_V \]
\[ (\rho_V \otimes id_C) \circ \rho_V = (id_V \otimes \Delta_C) \circ \rho_V . \]

2. \( V \) is an \( LS(\text{LB}) \mathcal{H} \)-comodule over \( C \) if \( V \) is an LCTVS comodule and an \( LS(\text{LB}) \)-space.

3. \( V \) is a \( CT \mathcal{H} \)-comodule over \( C \) if \( V \) is an LS-comodule over \( C \) and is isomorphic to the locally convex inductive limit \( V \cong \lim_{\leftarrow} V_n \) of a compact inductive system \((V_n,\psi_{n,m})\) of Banach comodules \((V_n,\rho_{V_n})\) over \( C_n \), where \((C_n,\phi_{nm})\) is a CT-structure for \( C \). For a fixed \((C_n,\phi_{nm})\) we will also say that \( V \) is a \((C_n,\phi_{nm})\)-comodule.
As morphisms of LCTVS-comodules we take continuous comodule maps.

**Definition 60.** Let $A$ be an NF-$\mathfrak{A}$-algebra and $V$ be an LCTVS. Then we say that

1. $V$ is a right LCTVS-$\mathfrak{A}$-module over $A$ if it is a $\mathfrak{A}$-module over $A$.
2. $V$ is a right FS-$\mathfrak{A}$-module over $A$, if, in addition, $V$ is a FS-space.
3. $V$ is an NF-$\mathfrak{A}$-module over $A$, if it is a projective limit of a compact projective system $(V_n)_{n \in \mathbb{N}}$ of Banach $A_n$-$\mathfrak{A}$-modules $V_n$, where $(A_n)$ is an NF-structure for $A$.

By duality, for an LCTVS-$\mathfrak{A}$-comodule $V$, $V' = (V)_b'$ is an LCTVS which is a right $(C)_b'$-module.

**Definition 61.** We will say that $V'$ is a $\mathfrak{A}$-module, dual to $\mathfrak{A}$-comodule $V$.

Note that if $C$ is a CT-Hopf $\mathfrak{A}$-algebra, then $(C)_b'$ is an NF-Hopf $\mathfrak{A}$-algebra.

Clearly the dual of an LS-$\mathfrak{A}$-module is a FS-$\mathfrak{A}$-module and vice versa. The duality map gives an anti-equivalence of categories

$$\text{LSComod}_C \sim \text{FSMod}_{(C)_b'}.$$

Since CT- and NF-spaces are reflexive, one can prove similarly to proposition 57 that dual of NF-$\mathfrak{A}$-module is a CT-$\mathfrak{A}$-comodule. Thus we also have an anti-equivalence of categories

$$\text{CTComod}_C \sim \text{NMod}_{(C)_b'}.$$

**Definition 62.** On $V$ there is also a left topological $(C)_b'$-$\mathfrak{A}$-module structure

$$m: (C)_b' \mathfrak{A}V \to V$$

$$\lambda \otimes v \mapsto \lambda \cdot v = (id_V \otimes \lambda) \circ \rho_V (v).$$

Such $\mathfrak{A}$-modules (with $\mathfrak{A}$-module structure coming from $\mathfrak{A}$-comodule structure) are called induced.

4.2. **Rationality.** Similar to the section 2.4, one can define a notion of a t-rational left LS-$\mathfrak{A}$-module over $(C)_b'$. If $V$ is a left $(C)_b'$-$\mathfrak{A}$-module and a compact type LCTVS, then, by proposition 48.4, we have the map $2.1 \rho_V : V \to V \mathfrak{A}C \simeq L_b((C)_b', V)$ (where $\rho_V (v)(\lambda) = \lambda \cdot v$), such that the $(C)_b'$-module structure on $V$ is exactly $\lambda \cdot v = (id_V \otimes \lambda) \circ \rho_V (v)$. The $\mathfrak{A}$-module axioms for $V$ imply that $\rho_V$ satisfy right $\mathfrak{A}$-comodule axioms ([SW, 2.1.1]). Thus we have

**Proposition 63.** All continuous $(C)_b'$-$\mathfrak{A}$-modules structures on LS-space are t-rational.

Similarly to the Banach case, morphisms of t-rational modules are comodule morphisms. Thus the above proposition gives an equivalence of categories

$$\text{LSComod}_C \sim (C)_b' \text{LSMod}.$$

4.3. **Quotients, subobjects and simplicity.**

**Proposition 64.** Let $V$ be a right (CT-) LS-$\mathfrak{A}$-comodule over $C$ and $U$ be a closed $\mathfrak{A}$-subcomodule of $C$. Then $U$ and $V/U$ are right (CT-) LS-$\mathfrak{A}$-comodules over $C$ and the exact sequence

$$0 \to U \to V \to V/U \to 0$$

give rise to the exact sequence of right (NF-) FS-$\mathfrak{A}$-modules over $(C)_b'$

$$0 \to (V/U)_b' \to V_b' \to U_b' \to 0$$
with the morphisms in this exact sequence being strict.

Proof. Similar to [ST4, 1.2].

Proposition 65. Let \( M \in \text{NFMod}_A \) and \( \{M_n\} \) is an \( \{A_n\}\)-structure on \( M \). Then each \( M_n \) is a topological \( \widehat{\otimes} \)-module and

1. Each \( (M_n)_b' \) is an LB \( (A)_b' \)-\( \widehat{\otimes} \)-comodule;
2. \( M_n^\circ \cong (M_n)'_b \widehat{\otimes} A_n^\circ \).

Proof. Since \( M_n \) is an \( A \)-module via \( \pi_n : A \to A_n \) with the module structure given by the map \( m_{M_n} \circ (\text{id}_{M_n} \otimes \pi_n) : M_n \widehat{\otimes} A \to M_n \), it follows from proposition 48.7 that taking duals gives us the map \( (m_{M_n} \circ (\text{id}_{M_n} \otimes \pi_n))' = \Delta_n : (M_n)'_b \to (M_n)_b' \widehat{\otimes} (A)'_b \).

\( \Delta_n \) satisfies \( \widehat{\otimes} \)-comodule axioms, which proves the first statement. The image of \( (M_n)_b \widehat{\otimes} A_n^\circ \) in \( (M_n)'_b \) under \( \text{id}_{M_n} \otimes \pi_A \) is the closure of all elements \( m \) such that \( \rho_{(M_n)_b}(m) \in (M_n)'_b \widehat{\otimes} A_n^\circ \), which is exactly the definition of \( M_n^\circ \).

\( \square \)

Definition 66. We call a topological \( \widehat{\otimes} \)-comodule \( V \) simple, if it does not have any closed \( \widehat{\otimes} \)-subcomodules.

Proposition 67. Let \( V \in \text{LSComod}_C \). Then

1. \( V \) is simple if and only if \( V'_b \) is simple as \( C'-\widehat{\otimes} \)-module;
2. \( V \) is simple if and only if \( V \) is simple as \( C'-\widehat{\otimes} \)-module.

Proof. (1) follows from Proposition 64. (2) is proved similarly to Banach case.

The following is the analog of [ST3, lemma 3.9]

Proposition 68. Let \( V \) be a \( (C_n)_b \)-\( \widehat{\otimes} \)-comodule over a CT-\( \widehat{\otimes} \)-coalgebra \( C \) and let \( \rho_V|_{V_n} = (\rho_{V_n})_b|_{V_n} = \rho_{V_n} \). Suppose there exists \( N>0 \) such that \( V_n \) are simple for all \( n>N \). Then \( V \) is simple.

Proof. Let \( U \subset V \) be a proper closed \( \widehat{\otimes} \)-subcomodule. Since \( V \) is of compact type, \( U = \lim_n \overline{U_n} \), such that \( U_n \) are closed subspaces of \( V_n \). Since \( U \) and \( V_n \) are \( \widehat{\otimes} \)-subcomodules of \( V \), \( U_n \) are \( C_n \)-\( \widehat{\otimes} \)-subcomodules of \( V_n \). Since \( U \) is a proper subspace of \( V \), \( U_n \) must be proper subcomodules of \( V_n \) for all \( n>N \) for some \( N>0 \), and this is a contradiction with the simplicity of \( V_n \).

\( \square \)

Remark 69. [ST3, lemma 3.9] is proved for coadmissible \( \widehat{\otimes} \)-modules over Fréchet-Stein algebra, i.e. the duals of \( V_n \) are supposed to be finitely generated over \( C_n' \) and \( C_n \) are supposed to be Noetherian. These assumptions are not required in our result. On the other hand, our result is for CT-\( \widehat{\otimes} \)-comodules, which on the dual side mean a nuclear Fréchet \( \widehat{\otimes} \)-module, and nuclearity is not required in [ST3, lemma 3.9].

4.4. Cotensor product. Let \( A \) and \( B \) be LS-\( \widehat{\otimes} \)-coalgebras. Let \( (M, \rho_M) \) be a right LS \( A \)-\( \widehat{\otimes} \)-comodule and let \( (N, N\rho) \) be a left LS \( A \)-\( \widehat{\otimes} \)-comodule. Similar to the Banach case, one can give the following definition.

Definition 70. The space \( M \widehat{\otimes}_A N = Ker (\rho_M \otimes id_N - id_M \otimes N\rho) \) is called cotensor product of \( M \) and \( N \) over \( A \).

Since \( M \widehat{\otimes}_A N \) is a kernel of a continuous map, it is a closed subspace of \( M \widehat{\otimes} N \), which is a LS-space, and thus is a LS-space. It is an equalizer of \( \rho_M \otimes id_N \) and \( id_M \otimes N\rho \).
If $N$ is also a right LS $B\mathcal{O}_A$-comodule, then $M\mathcal{O}_A N$ is also a right LS $B\mathcal{O}_A$-comodule.

Since under antiequivalences of categories equalizers and coequalizers are dual to each other, we have the following result.

**Proposition 71.** $(M\mathcal{O}_A N)\overset{\rho_b}{\longleftarrow} = (M)\overset{\rho}{\longleftarrow} \mathcal{O}_A (N)\overset{\rho}{\longleftarrow}, \quad ((M)\overset{\rho}{\longleftarrow} \mathcal{O}_A (N)\overset{\rho}{\longleftarrow})\overset{\rho_b}{\longleftarrow} = M\mathcal{O}_A N.$

Similarly one defines induction and restriction functors. Lemma 33 and Proposition 38 (Frobenius reciprocity) remain true in LS case.

If $M = \lim \limits_n M_n$ and $N = \lim \limits_n N_n$ are CT-$\mathcal{O}_A$-comodules, then $M\mathcal{O}_A N = \lim \limits_n M_n\mathcal{O}_A N_n$. Since $\rho_M$ and $\rho_N$ are defined by $\rho_{M_n}$ and $\rho_{N_n}$, we have $M\mathcal{O}_A N = \lim \limits_n M_n\mathcal{O}_A N_n$.

Similar to the case of Banach $\mathcal{O}_A$-comodules, one defines induction and restriction functors. Frobenius reciprocity and Tensor identity remain true for CT-$\mathcal{O}_A$-comodules.

5. **Admissibility**

**Definition 72.** Let $A = \lim \limits_n A_n$ be an NF-$\mathcal{O}_A$-algebra. An NF $\{A_n\}$-module $M = \lim \limits_n M_n$ is called $\{A_n\}$-coadmissible if

1. Each $M_n$ is finitely generated;
2. For all $n$ we have a topological isomorphism $M_n \cong M\mathcal{O}_A A_n$.

**Remark 73.** Recall the definitions from [EM]. A weak Fréchet-Stein $\mathcal{O}_A$-algebra is a projective limit of locally convex hereditary complete topological $\mathcal{O}_A$-algebras, s.t. transition maps factor through Banach spaces (BH-maps), thus this notion is slightly more general than our notion of Fréchet $\mathcal{O}_A$-algebra.

A module $M$ over a weak Fréchet-Stein $\mathcal{O}_A$-algebra $A$ is called coadmissible (w.r.t. a fixed nuclear Fréchet structure $A_n$ on $A$) [EM, 1.2.8], if we have the following data:

1. a sequence of finitely generated Banach $\mathcal{O}_A$-modules $M_n$ over $A_n$;
2. an isomorphism of Banach $A_n\mathcal{O}_A$-modules $M_{n+1}\mathcal{O}_A A_n \cong M_n$;
3. an isomorphism of topological $A\mathcal{O}_A$-modules $M \cong \lim \limits_n M_n$ (projective limit is taken w.r.t. transition maps $M_{n+1} \longrightarrow M_n$, induced by (2)).

Condition (3) is already contained in our definition of NF-$\mathcal{O}_A$-module. It is easy to see that our condition (2) imply condition (2) as in loc. cit. [ST3, Cor. 3.1] states that over a Fréchet-Stein algebra $A$ there is an isomorphism $M \mathcal{O}_A A_n \cong M_n$.

Since $M_{n+1}\mathcal{O}_A A_n \cong M_{n+1} \mathcal{O}_A A_n$, one can see that $M \mathcal{O}_A A_n \cong M_n$ is equivalent to the topological isomorphism $M\mathcal{O}_A A_n \cong M_n$ (see also (the proof of) [EM, 1.2.11]).

Thus over Fréchet-Stein $\mathcal{O}_A$-algebra these two conditions are equivalent and these two definitions coincide. In general, our condition (2) is stronger and results in existence of a duality between $\mathcal{O}_A$-modules and $\mathcal{O}_A$-comodules.

It is known that if $M$ is coadmissible w.r.t. one nuclear Fréchet structure on $A$, then it is coadmissible w.r.t. any [EM, 1.2.9].

We want to study an analog of this notion for $\mathcal{O}_A$-comodules.

**Definition 74.** Let $C$ be a CT-coalgebra with CT-structure $\{C_n\}$. Let $V$ be a right $\{C_n\}$-comodule with CT-structure $\{V_n\}_{n\in\mathbb{N}}$. Suppose that
(1) $V_n$ is a finitely cogenerated Banach right $\mathfrak{S}$-comodule over $C_n$;
(2) we have a topological isomorphism of $C_n$-$\mathfrak{S}$-comodules $V_C^n C_n \simeq V_n$ ($C_n$ is a left and right $C$-$\mathfrak{S}$-comodule, so we can take completed cotensor product).

We call $V$ admissible w.r.t. CT-structure $\{(C_n, \phi_{n,m})\}_{n \in \mathbb{N}}$ (or $\{C_n\}$-admissible).

We call $\{V_n\}_{n \in \mathbb{N}}$ a $\{C_n\}$-admissible structure.

The second condition essentially means that the only elements $x \in V$, such that $\rho_V(x) \in V \mathfrak{S} C_n$, are the elements of $V_n$.

**Proposition 75.** Let $\phi : A \to B$ be a morphism of CT-$\mathfrak{S}$-coalgebras, given by a morphism of CT-structures $\{\phi_n : A_n \to B_n\}$ with CT-structure $\{B_n\}_{n \in \mathbb{N}}$ for $A$ and CT-structure $\{B_n\}_{n \in \mathbb{N}}$ for $B$. Then:

1. If $W$ is $\{B_n\}_{n \in \mathbb{N}}$-admissible $B$-comodule, then $W^\phi$ is an $\{A_n\}_{n \in \mathbb{N}}$-admissible $A$-comodule;
2. If $V$ is $\{A_n\}_{n \in \mathbb{N}}$-admissible $A$-comodule and $\{A_n\}_{n \in \mathbb{N}}$ is $\{B_n\}_{n \in \mathbb{N}}$-admissible, then $V^\phi$ is an $\{B_n\}_{n \in \mathbb{N}}$-admissible $B$-comodule.

**Proof.** In both cases we have to check the three conditions of admissibility.

- $W^\phi$ case:
  1. (1) Proposition 36.
  2. We have $W^\phi = W \mathfrak{S}_A B \simeq \lim_{\to} W_n \mathfrak{S}_A B_n$. Since $W$ is admissible, we have a topological isomorphism

$$W_n^\phi \simeq W_B \mathfrak{S}_n A_n \simeq \left( W_B B_n \right) \mathfrak{S}_n A_n \simeq W_B^\phi A_n.$$

We would like to show that $W^\phi B_A A_n \simeq W^\phi_n$, which follows from identity

$$W^\phi B_A A_n \simeq W^\phi B_A A_n \simeq W^\phi B_A A_n \simeq W^\phi_n.$$

- $V^\phi$ case
  1. If $V_n$ is embedded into $A_{k_n}$ and $A_n$ is embedded into $B_{m_n}$, then we have an embedding $V_n \to B_{m_n}$.
  2. We have $V \mathfrak{S}_A A_n \simeq V_n$ and $(A) \mathfrak{S}_B B_n \simeq (A_n)_{\phi}$. Then

$$V^\phi B_n \simeq \left( V^\phi A \right)_{\phi} B_n \simeq V^\phi A_{\phi} B_n =$$

$$= V^\phi A (A_{\phi} B_n) = V^\phi A (A_n)_{\phi} \simeq \left( V^\phi A_{\phi} \right) \simeq (V_n)_{\phi}$$

$\square$

**Corollary 76.** If a CT-$\mathfrak{S}$-comodule is admissible w.r.t. one CT-structure, then it is admissible w.r.t. any.

**Proof.** Similar to [EM, 1.2.9]. Let $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ be two different CT-structures on a CT-$\mathfrak{S}$-coalgebra $A$ and let $M$ be $\{B_n\}$-admissible with admissible structure $\{M_n\}$. Let $\psi : \mathbb{N} \to \mathbb{N}$ be the increasing map giving the morphisms $\Psi : \{A_n\} \to \{B_n\}$, $\Psi(n) : A_n \to B_{\psi(n)}$. Then we have an automorphism $\Psi : A \to A$, the system $\{B_{\psi(n)}\}$ is also a CT-structure, $M$ is $\{B_{\psi(n)}\}$-admissible and $M^\Psi = M$ is $\{A_n\}$-admissible by Proposition 75.1. $\square$
Next we show that coadmissible \( \mathfrak{S} \)-modules and admissible \( \mathfrak{S} \)-comodules are dual to each other.

**Theorem 77.** Let \( C \cong \lim C_n \) be a CT-\( \mathfrak{S} \)-coalgebra. If \( \{V_n\} \) is an \( \{C_n\} \)-admissible structure for CT-\( \mathfrak{S} \)-comodule \( V \) over \( C \), then \( (V_n)'_b \) is a \( \{C_n\}'_b \)-coadmissible structure for the NF-module \( (V)'_b \) over \( (C)'_b \).

**Proof.** Let \( \{V_n\} \) be \( \{C_n\}\)-admissible structure on \( V \). We have \( V_n \cong V \mathfrak{S} C_n \), which is by definition the kernel

\[
0 \to V \mathfrak{S} C_n \xrightarrow{\phi} V \mathfrak{S} C_n \xrightarrow{\psi} V \mathfrak{S} C \mathfrak{S} C_n,
\]

where \( \psi = \rho_V \mathfrak{S} \text{id}_{C_n} - \text{id}_V \mathfrak{S} \rho_{C_n} \).

Taking duals gives us the following sequence

\[
0 \leftarrow \left(V \mathfrak{S} C_n\right)'_b \xrightarrow{\phi'} \left(V \mathfrak{S} C_n\right)'_b \xrightarrow{\psi'} \left(V \mathfrak{S} C \mathfrak{S} C_n\right)'_b.
\]

By proposition 48.7 this sequence is equal to

\[
0 \leftarrow \left(V \mathfrak{S} C_n\right)'_b \xrightarrow{\phi'} \left(V \mathfrak{S} C_n\right)'_b \cong \left(V \mathfrak{S} C_n\right)'_b \xrightarrow{\psi'} \left(V \mathfrak{S} C \mathfrak{S} C_n\right)'_b
\]

and \( \psi' = m_{(V)'_b} \mathfrak{S} \text{id}_{C_n} - \text{id}_V \mathfrak{S} m_{C_n}' \).

Since \( \phi \) is an embedding, \( \phi' \) is surjective and its kernel is a \( * \)-weak closure of the elements of the form \( v^e c'_n - v^e \otimes c'_n \). Thus we have a linear surjection \( (V)'_b \mathfrak{S} (C)'_b \cong \left(V \mathfrak{S} C_n\right)'_b \), which is continuous by the universal property of the complete tensor product. The embeddings \( V_m \mathfrak{S} C_m C_n \to \left(V \mathfrak{S} C_n\right)'_b \), from proposition 30, give rise to the embedding \( (V)'_b \mathfrak{S} (C)'_b \to \left(V \mathfrak{S} C_n\right)'_b \) and thus the above surjection is a continuous isomorphism.

Since \( \left(V \mathfrak{S} C_n\right)'_b \cong (V_n)'_b \) is, in fact, a Banach space and \( (V)'_b \mathfrak{S} (C)'_b \) is a quotient of Fréchet space \( (V)'_b \mathfrak{S} (C)'_b \), by Open Mapping theorem 47, we have the following topological isomorphism

\[
(V_n)'_b \cong \left(V \mathfrak{S} C_n\right)'_b \cong (V)'_b \mathfrak{S} (C)'_b,
\]

which is the 2nd axiom of coadmissibility. The check of the axiom 1 is trivial. \( \square \)

Before proving the inverse statement we prove an auxiliary result.

**Lemma 78.** Let \( f : V \to U \) be an open surjective morphism of LCTVS. Then we have a continuous isomorphism \( (U)'_b \cong (\ker f)^\perp \).

**Proof.** We have a dual map \( f' : (U)'_b \to (V)'_b \), which is continuous and, clearly, \( \text{im}(f') \in (\ker f)^\perp \). Consider a map \( \tilde{f} : (\ker f)^\perp \to (U)'_b \) defined as \( \tilde{f}(\phi)(u) = \phi(v) \) for \( u = f(v) \). If \( u = f(v') \), then \( v - v' \in \ker f \) and \( \tilde{f}(\phi)(0) = \phi(v - v') = 0 \).
Thus $\tilde{f}(\phi)$ and the map $\tilde{f}$ are well defined. For an open set $U \subset K$, $\tilde{f}(\phi)^{-1}(U) = f(\phi^{-1}(U))$ is open, since $\phi$ is continuous and $f$ is open. Thus $\tilde{f}(\phi) \in (U)'.$ Clearly $\tilde{f}$ and $f'$ are inverse to each other and thus $f'$ is a bijection onto $(\ker f)^{\perp}$.

**Theorem 79.** Let $A \cong \varinjlim A_n$ be an NF-$\mathfrak{S}$-algebra and $M \cong \varinjlim M_n$ is a coadmissible $A$-$\mathfrak{S}$-module with $\{A_n\}$-coadmissible structure $\{M_n\}$. Then $(M)^{\perp}_b$ is an admissible $CT$-$\mathfrak{S}$-comodule over the $CT$-$\mathfrak{S}$-coalgebra $(A)^{\perp}_b$ with $\{A^\circ_n\}$-admissible structure $\{M^\circ_n\}$.

**Proof.** Consider the defining exact sequence of Fréchet spaces for $M \mathfrak{S}_A A_n \cong M_n$,

$$
M \mathfrak{S}_A A_n \xrightarrow{\psi} M \mathfrak{S} A_n \xrightarrow{\phi} M \mathfrak{S}_A A_n \rightarrow 0.
$$

where $\psi = m_M \mathfrak{S} \id_{A_n} - \id_M \mathfrak{S} m_{A_n}$. Taking duals and applying proposition 48.7 gives us the sequence

$$
M^\prime_b \mathfrak{S} A'_n \mathfrak{S} (A_n)'_b \xrightarrow{\psi^\prime} M^\prime_b \mathfrak{S} (A_n)'_b \xrightarrow{\phi^\prime} (M \mathfrak{S}_A A_n)'_b \rightarrow 0.
$$

with $\psi^\prime = \rho_{M^\prime_b} \mathfrak{S} \id_{(A_n)_b} - \id_{M^\prime_b} \mathfrak{S} \rho_{(A_n)_b}$. By definition, $M \mathfrak{S}_A A_n = M \mathfrak{S} A_n / I^\perp$, where $I^\perp$ is the closure of the linear hull of the set of elements of the form $mc \mathfrak{S} a - m \mathfrak{S} ca$. By lemma 78, $(M \mathfrak{S}_A A_n)'_b \cong (I^\perp)^{\perp}$. For any linear map $f : V \rightarrow U$ we have $\ker f^\perp = (\im f)^{\perp}$ and $\im I^\perp$ is the closure of the image of the map $m_M \mathfrak{S} \id_{A_n} - \id_M \mathfrak{S} m_{A_n} : M \mathfrak{S} A A_n \rightarrow M \mathfrak{S} A_n$, we have $(M \mathfrak{S}_A A_n)'_b \cong (I^\perp)^{\perp} = \ker (m_M \mathfrak{S} \id_{A_n} - \id_M \mathfrak{S} m_{A_n})^{\perp}$ and since

$$(m_M \mathfrak{S} \id_{A_n} - \id_M \mathfrak{S} m_{A_n})^\perp = \rho_{(M^\prime_b)\mathfrak{S} \id_{(A_n)'_b} - \id_{(M^\prime_b)\mathfrak{S} \rho_{(A_n)'_b}}$$

we get a continuous isomorphism $(M \mathfrak{S}_A A_n)'_b \cong (M^\prime_b \mathfrak{S} (A_n)'_b)$. Since $(M^\prime_b \mathfrak{S} (A_n)'_b)$ is a Banach space and $(M^\prime_b \mathfrak{S} (A_n)'_b)$ is an LB-space (in fact, regular LB-space), $\phi^\prime((M \mathfrak{S}_A A_n)'_b)$ is embedded into $(M^\prime_b \mathfrak{S} (A_n)'_b)$ for some $k$. Thus both $(M \mathfrak{S}_A A_n)'_b$ and $(M^\prime_b \mathfrak{S} (A_n)'_b)$ are Banach spaces and by Open Mapping theorem 47, we have the topological isomorphism $(M \mathfrak{S}_A A_n)'_b \cong (M^\prime_b \mathfrak{S} (A_n)'_b)$.

Note that, since $A_n$ is a Banach $A - A$-$\mathfrak{S}$-bimodule (via projective limit structure map $\psi_n : A \rightarrow A_n$), $(A_n)'_b$ is a Banach $(A)'_b - (A)'_b$-$\mathfrak{S}$-bicomodule by proposition 48.7.

To prove the claim, first we remark that $A^\circ_n$ is also a Banach $(A)'_b - (A)'_b$-$\mathfrak{S}$-bimodule, since $(A)'_b = \varinjlim A^\circ_n$. Consider $N_n = (M^\prime_b \mathfrak{S} (A_n)'_b)$. Each $(M^\prime_b \mathfrak{S} (A_n)'_b)$ is a closed subspace of $(M^\prime_b \mathfrak{S} (A_n)'_b)$, and thus $N_n$ is a closed subspace of $(M \mathfrak{S}_A A_n)'_b \cong (M_n)'_b$. We have $N_n = (M^\prime_b \mathfrak{S} A^\circ_n) \cong (M^\prime_b \mathfrak{S} (A_n)'_b \mathfrak{S} A^\circ_n) \cong (M^\prime_b \mathfrak{S} (A_n)'_b \mathfrak{S} A^\circ_n)$ and by proposition 65 $N_n \cong M^\circ_n$. Since $(-)^\circ$ preserves finite direct sums, $M^\circ_n$ is a finitely cogenerated $A^\circ_n$-$\mathfrak{S}$-comodule, which gives us the 1st axiom of admissibility. The 2nd axiom is in the definition of $N_n$. 

Corollary 80. Let $C$ be a CT-$\otimes$-coalgebra and $(C)'_b$ its dual NF-$\otimes$-algebra. Then the categories of admissible $C$-$\otimes$-comodules and coadmissible $(C)'_b$-$\otimes$-modules are antiequivalent. In particular, if $(C)'_b$ is Fréchet-Stein, then by [ST3, 3.5] the category of admissible $C$-$\otimes$-comodules is abelian.

Remark 81. One can define admissible $\otimes$-comodules in analogy with the definition [EM, 1.2.8], i.e. to require that

1. $V_n$ is a finitely cogenerated Banach right $\otimes$-comodule over $C_n$;
2. we have an isomorphism of Banach $C_n$-$\otimes$-comodules $V_{n+1} \otimes C_n \cong V_n$ ($C_n$ is a left and right $C_n$-$\otimes$-comodule, so we can take completed cotensor product).

With this definition admissibility will still be preserved by induction and restriction functors and will not depend on CT-structure. However for duality “admissible $\otimes$-comodules” - “coadmissible $\otimes$-modules” one needs the subspaces $V \otimes C_n \subset V \otimes C_n$ to be (ultra)bornological. For general admissible $\otimes$-comodule in the sense of this remark, it is currently unknown to the author if it is always true.

Appendix

Here we prove two technical results, stated in the proposition 48.

Lemma 82. Let $V$ be an LS-space and $W$ be a Banach space. Then for any bounded set $B$ in $V \otimes W$ there exists a bounded set $B_V \subset V$ and a bounded set $B_W \subset W$ such that $B \subset (B_V \otimes B_W)^{(V \otimes W)}$.

Proof. Let $0 < t < 1$. Similar to the proof of [PGS, 10.4.6], for any element $z \in V \otimes W$ one has a presentation $z = \sum_{k=0}^{\infty} v_k \otimes w_k$ with $\{w_k\} \in W \setminus \{0\}$ being a $t$-orthogonal base in $W$ (follows from 48.6 and [VANR, 4.30.ii]). The rest of the proof goes the same way as in [PGS, 10.4.6]. □

Remark. The result is true for more general $V$. However the proof is more complicated, since instead of sums (or sequences) one has to work with nets. The above version is sufficient for our purposes.

Remark. One can make $B_V$ and $B_W$ into $K^0$-submodules by considering their convex hulls.

We will use the notation $B_V \hat{\otimes} B_W := (B_V \otimes B_W)^{(V \hat{\otimes} W)}$.

Proof of proposition 48.7. We first note that, since $V$ is bornological and reflexive [ST4, 1.1] and $W$ is Banach space, by [NFA, 18.8] we have a topological isomorphism $$(V)'_b \hat{\otimes} (W)'_b \cong L_b \left(V, (W)'_b\right).$$

Since $V$ is reflexive, by [NFA, 15.5] it is barrelled and by [EM, 1.1.35] (via putting $W = K$ there) we have a continuous bijection $$\phi : (V \hat{\otimes} W)'_b \cong L_b \left(V, (W)'_b\right)$$

and, similar to the proof of [NFA, 20.13], a correspondence of open lattices $$\phi : L \left(l_V \hat{\otimes} l_W, K^0\right) \mapsto L \left(l_V, L \left(l_W, K^0\right)\right),$$
where $l_V$ and $l_W$ are bounded $K^0$-submodules in $V$ and $W$. The open lattices $L\left(l_V^0, L\left(l_W^0, K^0\right)\right)$ generate the strong topology on $(V, W)_{0}$ by definition. Our splitting lemma implies that the collection of bounded sets of the form $\{l_{V} \otimes l_{W}\}$ and of all bounded sets in $V \otimes W$ satisfy the assumptions from [NFA, 6.5] for families $\mathcal{B}$ and $\mathcal{B}$ correspondingly. Thus [NFA, 6.5, 6.4] imply that open lattices $L\left(l_{V} \otimes l_{W}, K^0\right)$ generate the strong topology on $(V \otimes W)_{0}$ and thus $\phi$ is a topological isomorphism.

The second identity is [NFA, 20.13].

□

Recall that an LCTVS $V$ is called bornological if its topology $\tau_V$ is the final topology for a system (not necessarily a sequence) of maps $(E_i \rightarrow E)_{i \in I}$ from normed vector spaces $E_i$. It is called ultrabornological if the spaces $E_i$ are Banach spaces. Clearly a complete bornological space is ultrabornological and locally convex inductive limits of Banach spaces are ultrabornological. Our reference for (ultra)bornological spaces is [MV, Chapter 24], with nonarchimedean analogs of most statements there being easy exercises (also see [NFA]). For any unfamiliar term in the following proof one should follow the references in [MV, Chapter 24].

Proof of proposition 48.6. Let $Z$ be a Banach space and $f : V \otimes W \rightarrow Z$ be a locally bounded map and $\phi : V \times W \rightarrow V \otimes W$ be the canonical bilinear map.

For every $v \in V$ consider the map $f \circ \phi (v, -) : W \rightarrow Z$. For every bounded subset $B \subset W$ the set $v \otimes W$ is bounded in $V \otimes W$ and thus $f \circ \phi (v, -)$ is a bounded map of Banach spaces, which imply it is continuous for every $v \in V$. Similarly, for every $w \in W$, $f \circ \phi (-, w) : V \rightarrow Z$ is a locally bounded map, which imply it is continuous since $V$, as an LS-space, is ultrabornological [MV, 24.10.4]. Thus $f \circ \phi$ is a separately continuous map. By the universal property of inductive tensor product, $f$ is continuous.

Thus we get that each locally bounded map $f$ is continuous, which imply that $V \otimes W$ is bornological [MV, 24.10.4]. Since it is also complete, it is ultrabornological [MV, 24.15.b]. $\lim_{\rightarrow} (V_n \otimes W)$ is a webbed space as a locally convex inductive limit of webbed spaces and thus by Open Mapping theorem [MV, 24.30] the canonical continuous bijection $\lim_{\rightarrow} (V_n \otimes W) \approx V \otimes W$ is open, which imply it is a topological isomorphism. □

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