A FAMILY OF NON-INJECTIVE SKINNING MAPS
WITH CRITICAL POINTS

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Abstract. Certain classes of 3-manifolds, following Thurston, give rise to a 'skinning map', a self-map of the Teichmüller space of the boundary. This paper examines the skinning map of a 3-manifold $M$, a genus-2 handlebody with two rank-1 cusps. We exploit an orientation-reversing isometry of $M$ to conclude that the skinning map associated to $M$ sends a specified path to itself and use estimates on extremal length functions to show non-monotonicity and the existence of a critical point. A family of finite covers of $M$ produces examples of non-immersion skinning maps on the Teichmüller spaces of surfaces in each even genus, and with either 4 or 6 punctures.

1. Introduction

Thurston introduced the skinning map as a tool for locating hyperbolic structures on certain closed 3-manifolds, an integral part of his celebrated proof of the Hyperbolization Theorem for Haken manifolds (see [28], [27], [18]). Apart from a class of simple cases, explicit examples of skinning maps remain unexplored. One reason for this is that the skinning map uses the deformation theory of Ahlfors and Bers to pass back and forth from a conformal structure on the boundary of a 3-manifold to a hyperbolic structure on its interior. Like the uniformization theorem of Riemann surfaces, an explicit formula for the resulting map is typically out of reach.

Let $M$ be an orientable compact 3-manifold with non-empty incompressible boundary $\Sigma := \partial M$ such that the interior of $M$ admits a geometrically finite hyperbolic structure. Given a point $X \in \mathcal{T}(\Sigma)$ in the Teichmüller space of $\Sigma$, the 3-manifold $M$ has a quasi-Fuchsian cover with $X$ on one side and $Y \in \mathcal{T}(\Sigma)$ on the other, where $\tau$ indicates a reversal of orientation. The skinning map of $M$, $\sigma_M$, is given by $\sigma_M(X) = Y$. Thurston’s key result about skinning maps is the Bounded Image Theorem: if $M$ is acylindrical, then $\sigma_M(\mathcal{T}(\Sigma))$ is contained in a compact set.

Thurston described how to locate hyperbolic structures on a class of closed Haken 3-manifolds by iteration of $\tau \circ \sigma_M$, where $\tau$ is the ‘gluing map’, an isometry from $\mathcal{T}(\Sigma)$ to $\mathcal{T}(\Sigma)$. There are further questions, and results, about ‘how effectively’ iteration of $\tau \circ \sigma_M$ locates fixed points. [20] shows that the diameter of the image of $\sigma_M$ is controlled by constants depending only on the volume of $M$. If $M$ is without rank-1 cusps [10] shows that $\sigma_M$ is non-constant. This result has an improvement in [8], which shows that $\sigma_M$ is open and finite-to-one.
When $M$ is an interval bundle over a surface, $\sigma_M$ is a global diffeomorphism, easily described as a switch of coordinates using the Ahlfors-Bers parametrization. However, in all other cases $\sigma_M$ is not directly accessible to analysis. Thus it is consistent with the current literature to ask:

- Are skinning maps necessarily diffeomorphisms onto their images?
- Are they necessarily immersions?

In this paper, we present a negative answer to these questions. The reader should note that we work in the category of pared 3-manifolds and hyperbolic structures with rank-1 cusps. This allows low-dimensional calculations, in which $\sigma_M$ is ‘simply’ a holomorphic map $\sigma_M : \mathbb{H} \to \mathbb{H}$.

**Theorem 1.1.** There exists a pared 3-manifold $M = (H_2, P)$, where $H_2$ is a genus-2 handlebody and $P$ consists of two rank-1 cusps, whose skinning map $\sigma_M$ is not one-to-one and has a critical point.

Briefly, the proof of Theorem 1.1 uses an orientation-reversing isometry of $M$ to conclude that the skinning map sends a certain real 1-dimensional submanifold of $\mathcal{T}(\Sigma)$ to itself. Analysis of this submanifold together with extremal length estimates show that this restricted map is not one-to-one, and hence the skinning map has a critical point. That is, this proof produces a (real) 1-parameter path of quasi-Fuchsian groups $\{Q(X_t, \sigma_M(X_t))\}$, where $X_t$ and $\sigma_M(X_t)$ are both contained in a line in $\mathcal{T}(\Sigma)$ such that $t \mapsto X_t$ is injective and $t \mapsto \sigma_M(X_t)$ is not.

By passing to finite covers of the manifold from Theorem 1.1 this example implies a family of examples of skinning maps with critical points on Teichmüller spaces of arbitrarily high dimension. Below, $\Sigma_{g,n}$ indicates a topological surface of genus $g$ with $n$ punctures.

**Corollary 1.2.** There exists a family of pared 3-manifolds $\{(H_n, P_n)\}_{n=2}^{\infty}$, each admitting a geometrically finite hyperbolic structure, satisfying:

- For $n$ even, the paring locus $P_n$ consists of two rank-1 cusps and the boundary has topological type $\Sigma_{n-2,4}$.
- For $n$ odd, the paring locus $P_n$ consists of three rank-1 cusps and the boundary has topological type $\Sigma_{n-3,6}$.

For each integer $n \geq 2$, the associated skinning map has a critical point.

This work owes a debt of inspiration to numerical experiments developed by Dumas-Kent [9] which suggest the presence of a critical point of the skinning map associated to the manifold in Theorem 1.1. Their work also examines some other closely related manifolds and skinning maps, at least one of which appears to be a diffeomorphism onto its image. It would be interesting to have a unified understanding of these different behaviors.

The reader may note that the proof of Theorem 1.1 is one only of existence. It does not identify a critical point, determine the number of critical points, or determine any local degrees of $\sigma_M$ near critical points. However, for this skinning map there is some evidence of a unique simple critical point, both in the experiments of Dumas-Kent and in numerical tests conducted by the author. The latter tests further allow analysis of the critical point itself. Curiously, the evidence indicates that the critical point occurs where $M$ has hexagonal convex core boundary (that is, as a point in moduli space the convex core boundary is isomorphic to the Poincaré
metric on $\mathbb{C} \setminus \{0, 1, e^{\pi i/3}\}$. This would seem to imply some geometrical significance to the critical point, which for the moment remains elusive.

In §2, we introduce background and notation. The proof that critical points of skinning maps persist under certain finite covers is found in §3. In §4 and §5 we introduce the 3-manifold relevant to Theorem 1.1 and a path of geometrically finite structures on it, noting that this path maintains some important symmetry. In §6 we study the implications of this symmetry on the geometry of a four-punctured sphere. We apply some of these implications in §7 to the convex core boundaries of our path. This information is used in §8 which collects the main ideas for the proof of Theorem 1.1, deferring some computations to §9 and the Appendix. Finally, §10 presents a family of finite covers of the example from Theorem 1.1, which proves Corollary 1.2.

2. Background and notation

Let $\Sigma = \Sigma_{g,n}$ be a smooth surface of genus $g$ with $n$ punctures.

Let $\mathcal{S}$ denote the set of non-peripheral unoriented simple closed curves on $\Sigma$ and $ML(\Sigma)$ the space of measured geodesic laminations. Recall the natural inclusion $\mathbb{R}_+ \times \mathcal{S} \to ML(\Sigma)$, which has dense image. The quotient of $ML(\Sigma)$ under the action of multiplication by positive weights will be written $\mathbb{P}ML(\Sigma)$. See [6] and [12] for details.

Recall that the Teichmüller space of $\Sigma$, denoted $T(\Sigma)$, is the set of marked complex structures (equivalently, marked hyperbolic structures, via uniformization) on $\Sigma$ up to marking equivalence. Recall that $T(\Sigma)$ is naturally a complex manifold homeomorphic to $\mathbb{C}^{6g-6+2n}$.

For a smooth manifold $N$, possibly with boundary, the extended mapping class group of $N$ is $\text{MCG}^*(N) := \text{Diff}(N)/\text{Diff}_0(N)$. Pre-composition of a marking with a diffeomorphism of the surface $\Sigma$ descends to an action of $\text{MCG}^*(\Sigma)$ on $T(\Sigma)$. Note that we allow orientation-reversing diffeomorphisms. When $\Sigma$ is the boundary of a 3-manifold $M$, there is a restriction map $r : \text{MCG}^*(M) \to \text{MCG}^*(\Sigma)$. See [21], [11], and [12] for details about $T(\Sigma)$ and $\text{MCG}^*(\Sigma)$.

A Kleinian group $\Gamma$ is a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. The action of $\Gamma$ on $\mathbb{CP}^1$ has a domain of discontinuity $\Omega_\Gamma$, and its complement is the limit set, $\Lambda_\Gamma$. One component of the complement of a totally geodesic plane in $\mathbb{H}^3$ is a supporting half-space for a connected component of $\Omega_\Gamma$—or, less specifically, a supporting half-space for $\Gamma$—if it meets the conformal boundary $\mathbb{CP}^1$ in that component, and the closure of the intersection with $\mathbb{CP}^1$ intersects $\Lambda_\Gamma$ in at least two points. The totally geodesic boundary of a supporting half-plane for $\Gamma$ is a support plane for $\Gamma$. The convex hull of $\Gamma$ is the complement of the union of all of its supporting half-spaces, and the convex core of $\Gamma$ is the quotient of the convex hull by $\Gamma$. When an $\epsilon$-neighborhood of the convex core is finite volume, $\Gamma$ is geometrically finite. See [23] and [4] for details.

When $N$ is a topological space with finitely generated fundamental group, the set $\text{Hom}(\pi_1 N, \text{PSL}_2 \mathbb{C})$ naturally has the structure of an algebraic variety. There is a conjugation action of $\text{PSL}_2 \mathbb{C}$ on $\text{Hom}(\pi_1 N, \text{PSL}_2 \mathbb{C})$, and the quotient, interpreted in the sense of geometric invariant theory, is the $\text{PSL}_2 \mathbb{C}$-character variety of $N$. We will denote the $\text{PSL}_2 \mathbb{C}$-character variety of $N$ by $X(N)$. Recall that there is a natural holomorphic structure that $X(N)$ inherits from $\text{PSL}_2 \mathbb{C}$. See [15] and [16] for details.
A Kleinian group $\Gamma$ is Fuchsian if it is conjugate, in $\text{PSL}_2 \mathbb{C}$, into $\text{PSL}_2 \mathbb{R}$. In this case, $\Gamma$ stabilizes a totally geodesic plane in $\mathbb{H}^3$, and $\Lambda \Gamma$ is contained in a round circle on $\mathbb{CP}^1$. If $\Lambda \Gamma$ is a quasi-circle and $\Gamma$ stabilizes each of the components of $\Omega_{\Gamma}$, then $\Gamma$ is quasi-Fuchsian. In this case, the hyperbolic manifold $\mathbb{H}^3 / \Gamma$ has a conformal compactification $(\mathbb{H}^3 \sqcup \Omega_{\Gamma}) / \Gamma$, with two conformal structures on finite-type Riemann surfaces ‘at infinity’. We denote by $\mathcal{Q}(\Sigma)$ the locus of $\mathcal{X}(\Sigma)$ consisting of faithful representations with quasi-Fuchsian image. Recall Bers’ Theorem [24]:

**Theorem 2.1** (Simultaneous uniformization). There exists an identification $AB_{\Sigma} : \mathcal{Q}(\Sigma) \cong \mathcal{F}(\Sigma) \times \mathcal{F}(\Sigma)$, a biholomorphism of complex manifolds.

When $\rho$ is faithful, we identify $[\rho] \in \mathbb{X}(N)$ with the conjugacy class of its image $\rho(\pi_1 N)$ in $\text{PSL}_2 \mathbb{C}$. We write $Q$ for the map $AB_{\Sigma}^{-1}$ so that, for $X, Y \in \mathcal{F}(\Sigma)$, we have $Q(X, Y) \in \mathcal{Q}(\Sigma)$.

A pared 3-manifold is a pair, $(H_0, P)$, where $H_0$ is a compact, atoroidal, reducible 3-manifold with boundary and $P$ is a disjoint union of incompressible annuli and tori in $\partial H_0$ such that:

- $P$ contains all of the tori in $\partial H_0$.
- Any embedded cylinder with boundary components in $P$ is homotopic, relative to the boundary, into $P$.

By the ‘boundary’ of a pared 3-manifold, $\partial(H_0, P)$, we mean $\partial H_0 \setminus P$. See [24, p. 434] or [5, §5] for details. The pared 3-manifolds we consider will satisfy a slightly stronger condition, so that the skinning map will take values in $\mathcal{F}(\Sigma)$ [24, p. 435]:

- Any embedded cylinder with one boundary component in $P$ is homotopic, relative to the boundary, into $P$.

This latter condition is motivated by consideration of accidental parabolics, which we discuss briefly below. An accidental parabolic of $\mathbb{H}^3 / \Gamma$ is a curve homotopic into a cusp of $\mathbb{H}^3 / \Gamma$ but which descends to a non-boundary parallel curve on a component of the boundary $\Omega_{\Gamma} / \Gamma$.

When $M = (H_0, P)$ is a pared 3-manifold, we denote by $\mathbb{X}(M) \subset \mathbb{X}(H_0)$ the locus of points for which the image of curves homotopic into $P$ are parabolic. The subset $\mathcal{Q}(M) \subset \mathbb{X}(M)$ consists of $[\rho] \in \mathbb{X}(M)$ such that $\rho(\pi_1 H_0)$ is a geometrically finite Kleinian group, the image $\rho(\Gamma)$ is parabolic if and only if $\gamma$ is homotopic into $P$, and such that $\rho$ is induced by a homeomorphism $H_0 \cong \mathbb{H}^3 / (\rho(\pi_1 H_0))$. It is a theorem of Thurston that a pared 3-manifold $M$ is hyperbolizable, i.e. $\mathcal{Q}(M) \neq \emptyset$, so that $M$ carries a complete geometrically finite hyperbolic structure (see [5, p. 106]). Let $\text{Diff}(H_0, P)$ be the diffeomorphisms of $H_0$ that preserve the set $P$ and $\text{MCG}^*(M) := \text{Diff}(H_0, P)/\text{Diff}_0(H_0, P)$. Note that $\text{MCG}^*(M)$ acts on $\mathbb{X}(M)$ preserving $\mathcal{Q}(M)$.

Since $\mathcal{Q}(\Sigma)$ identifies with $\mathcal{Q}(\Sigma \times \mathbb{R})$, it is natural to seek a generalization of Bers’ Uniformization Theorem to 3-manifolds other than $\Sigma \times \mathbb{R}$. Such a theorem is provided by the deformation theory developed by Ahlfors, Bers, Marden, Sullivan, and others. We refer to this identification as the ‘Ahlfors-Bers’ parameterization:

**Theorem 2.2** (Ahlfors-Bers parameterization). For a pared 3-manifold $M = (H_0, P)$ with incompressible boundary, there exists a biholomorphism of complex manifolds $AB_M : \mathcal{Q}(M) \cong \mathbb{F}(\partial M)$.

Note that $\partial M$ may be disconnected, in which case $\mathcal{F}(\partial M)$ is the product of the Teichmüller spaces of the components.
For the remainder of the section, let $M = (H_0, P)$ be a hyperbolizable pared 3-manifold, satisfying $(\ast)$, such that $\Sigma = \partial M$ is non-empty, incompressible, and connected. Let $[\hat{\Gamma}] \in \mathcal{SF}(M)$. Fix a choice of basepoints for $M$ and $\Sigma$, and a path connecting them, so that inclusion induces the well-defined homomorphism $\iota_* : \pi_1 \Sigma \hookrightarrow \pi_1 M$ and the restriction morphism $\iota^*$ on representations. Let $\Gamma = \iota^* \hat{\Gamma}$.

**Lemma 2.3.** Under the assumptions above, we have $[\Gamma] \in \mathcal{QF}(\Sigma)$.

**Proof.** The choices above fix an identification of universal covers $\tilde{M} \cong \mathbb{H}^3$ and $\tilde{\Sigma} \cong U_0$, for some component $U_0 \subset \Omega_{\hat{\Gamma}}$. In this case, $\Gamma = \text{Stab}_{\hat{\Gamma}}(U_0)$, and [22 Corollary 6.5.] shows that $\Gamma$ must be geometrically finite. Because $\Sigma$ is incompressible in $M$, the domain $U_0$ must be simply-connected.

The condition $(\ast)$ implies that $\rho(\pi_1 \Sigma)$ has no accidental parabolics: Suppose a curve $\gamma$ on $\Sigma$ has parabolic image but is not homotopic into a cusp in the boundary. Because $\gamma$ is homotopic into $P$, the Annulus Theorem would yield a cylinder between a non-pared curve and a pared curve in $\partial H_0$, violating $(\ast)$. □

Note that the assumption that $\partial M$ is non-empty rules out a fibered hyperbolic 3-manifold, where the Kleinian group corresponding to the fiber is a geometrically infinite surface subgroup of a geometrically finite Kleinian group, and the conclusion of Lemma 2.3 fails.

**Definition 2.4.** For $M$ as above and $[\hat{\Gamma}] \in \mathcal{SF}(M)$ with $\Gamma = \iota^* \hat{\Gamma}$, the domains $\Omega_{\Gamma}$ and $\Omega_{\hat{\Gamma}}$ share a connected component, which we refer to as the top of $\Gamma$. By Lemma 2.3, the domain $\Omega_{\Gamma}$ has one other component, which we refer to as the bottom. When clear from context, we may refer to the quotient of the top component of $\Omega_{\Gamma}$ as the top of $\Gamma$.

Lemma 2.3 ensures the skinning map will take values in $\mathcal{T}(\Sigma)$. Below, we assume the connectedness of $\Sigma$ (for $\Sigma$ disconnected, see [27]).

**Definition 2.5.** The skinning map $\sigma_M : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ fits into the following commutative diagram:

$$
\begin{array}{cccc}
\mathcal{T}(\Sigma) & \xrightarrow{\text{AB}_M} & \mathcal{SF}(M) & \xrightarrow{\iota^*} & \mathcal{QF}(\Sigma) & \xrightarrow{\text{AB}_\Sigma} & \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \\
\sigma_M & & & & \downarrow \text{proj}_2 & & \\
& & & & \mathcal{T}(\Sigma) \\
\end{array}
$$

In words, $\iota^* \circ \text{AB}_M : \mathcal{T}(\Sigma) \to \mathcal{QF}(\Sigma)$ associates to $X \in \mathcal{T}(\Sigma)$ a quasi-Fuchsian structure $Q(X, \overline{Y}) \in \mathcal{QF}(\Sigma)$, where the top of $Q(X, \overline{Y})$ is $X$, and the bottom is $\overline{Y}$. The skinning map is given by $\sigma_M(X) = \overline{Y}$.

Because each of the maps above is holomorphic, $\sigma_M$ is holomorphic. As a result of the naturality of the Ahlfors-Bers identifications $\sigma_M$ is equivariant for the actions of $\text{MCG}^*(M)$, $\text{MCG}^*(\Sigma)$, and the restriction map $r : \text{MCG}^*(M) \to \text{MCG}^*(\Sigma)$. As a consequence,

**Proposition 2.6.** For each $\phi \in \text{MCG}^*(M)$, the fixed point set $\text{Fix } r(\phi)$ is preserved by the skinning map, i.e. $\sigma_M(\text{Fix } r(\phi)) \subset \text{Fix } r(\phi)$.

The two copies of $\text{Fix } r(\phi)$ in Proposition 2.6 technically lie in Teichmüller spaces of surfaces with opposite orientations. Which one is intended will be clear from
context. Because there is a canonical anti-holomorphic isometry \( \gamma : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) \), this distinction is not crucial.

The reader should note that the existence of any non-trivial mapping class of the 3-manifold \( M \) is not assured. In the 3-manifold from Theorem 3.1 this existence allows a reduction of dimension, an essential tool in our analysis.

### 3. Skinning maps of finite covers of \( M \)

Let \( p : (M', \Sigma') \to (M, \Sigma) \) be a finite covering of manifolds with boundary. We will denote the restriction \( p|_{\Sigma'} \) by \( p \) as well.

**Proposition 3.1.** Suppose that \( \sigma_M \) has a critical point at \( X \in \mathcal{F}(\Sigma) \). Then \( \sigma_{M'} \) has a critical point at \( p^*X \in \mathcal{F}(\Sigma') \).

**Proof.** Denote the image \( \mathcal{E}_M := \iota^*\mathcal{F}(M) \subset \mathcal{Q}\mathcal{F}(\Sigma) \), and the Bers slice through \( Y \in \mathcal{F}(\Sigma) \) by \( \mathcal{B}_Y := \{Q(X, Y) \mid X \in \mathcal{F}(\Sigma)\} \subset \mathcal{Q}\mathcal{F}(\Sigma) \) (that is, quasi-Fuchsian manifolds with \( \overline{Y} \in \mathcal{F}(\overline{\Sigma}) \) on the bottom). Note that \( \mathcal{E}_M \) is the graph of \( \sigma_M \) in the Ahlfors-Bers coordinates for \( \mathcal{Q}\mathcal{F}(\Sigma) \). From this point of view, \( \mathcal{B}_Y \) is a 'vertical line', and \( \overline{Y} = \sigma_M(X) \) if and only if \( \mathcal{E}_M \) and \( \mathcal{B}_Y \) intersect at \( Q(X, \overline{Y}) \). If \( \overline{Y} = \sigma_M(X) \), then \( \sigma_M \) has a critical point at \( X \) if and only if \( \mathcal{E}_M \) and \( \mathcal{B}_Y \) have a tangency at \( Q(X, \overline{Y}) \).

As \( \Sigma' \) is incompressible in \( M' \), we may consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{F}(\Sigma') & \xrightarrow{\text{AB}_{M'}} & \mathcal{Q}\mathcal{F}(M') \\
p^* & & p^* \\
\mathcal{F}(\Sigma) & \xrightarrow{\text{AB}_M} & \mathcal{Q}\mathcal{F}(M)
\end{array}
\]

The maps \( p^* \) above each correspond to lifting structures via \( p \). For example, the natural map induced by restriction \( \mathcal{Q}\mathcal{F}(M) \to \mathcal{Q}\mathcal{F}(M') \) can be interpreted as lifting hyperbolic structures via \( p \).

In the center of the diagram, each map is given by restricting a representation to a subgroup. This is induced by the commutative diagram of subgroups:

\[
\pi_1 \Sigma' \subset \pi_1 M' \\
\cap \\
\pi_1 \Sigma \subset \pi_1 M
\]

and hence the center of diagram 3.1 commutes.

The outside pieces of diagram 3.1 commute because the map \( \text{AB}_M \) is natural with respect to passing to finite covers: Given a point \([\Gamma] \in \mathcal{Q}\mathcal{F}(M)\), let \( \Gamma' := p^*(\Gamma) \), let \( \text{AB}^{-1}_M[\Gamma] = X \), and let \( X' := \text{AB}^{-1}_M[\Gamma'] \). Because \( \Gamma' < \Gamma \) is finite-index, the tops of the domains of discontinuity of \( \Gamma \) and \( \Gamma' \) are the same set. Thus \( X' \) holomorphically covers \( X \), with compatible markings so that \( p^*(X) = X' \).

Consider a critical point \( X \in \mathcal{F}(\Sigma) \), with \( \overline{Y} = \sigma_M(X) \), i.e. a tangency between \( \mathcal{B}_Y \) and \( \mathcal{E}_M \) in \( \mathcal{Q}\mathcal{F}(\Sigma) \). By commutativity of the diagram, \( p^*\mathcal{B}_Y \subset \mathcal{B}_{Y'} \), where \( Y' = p^*(Y) \), and \( p^*\mathcal{E}_M \subset \mathcal{E}_{M'} \). Thus, in order to see a critical point of \( \sigma_{M'} \) at \( p^*X \), it is enough to observe that \( p^* : \mathcal{Q}\mathcal{F}(\Sigma) \to \mathcal{Q}\mathcal{F}(\Sigma') \) is an immersion: A tangency between \( \mathcal{B}_Y \) and \( \mathcal{E}_M \) at \( Q(X, \overline{Y}) \) lifts to a tangency between \( \mathcal{B}_{Y'} \) and \( \mathcal{E}_{M'} \) at \( p^*Q(X, \overline{Y}) = Q(p^*X, p^*\overline{Y}) \).
In order to see that \( p^*: \mathcal{QF}(\Sigma) \to \mathcal{QF}(\Sigma') \) is an immersion, we must consider the smooth structure on \( X(\Sigma) \). One interpretation of this structure identifies \( T[\rho]X(\Sigma) \) with \( H^1(\Sigma, (\text{sl}_2\mathbb{C})_\rho) \), the vector space of 1-forms with values in the flat \( \text{sl}_2\mathbb{C} \)-bundle on \( \Sigma \) associated to \( \rho \). (See [15] for details about the infinitesimal deformation theory of \( X(\Sigma) \)). The restriction map \( p^*: X(\Sigma) \to X(\Sigma') \) naturally induces the pullback map \( p^*: H^1(\Sigma, (\text{sl}_2\mathbb{C})_\rho) \to H^1(\Sigma', (\text{sl}_2\mathbb{C})_{p^*\rho}) \) on tangent spaces.

Finite covering maps induce injective pullback maps on cohomology: If the pullback \( p^*\phi \) is a coboundary \( df \), one may average \( f \) over the finite sheets of the cover to obtain a form that descends, showing that \( \phi \) was also a coboundary. Hence \( d[\rho]|p^* : T[\rho]|X(\Sigma) \to T[p^*\rho]|X(\Sigma') \) is injective, \( p^*: X(\Sigma') \to X(\Sigma) \) is an immersion, and \( \sigma_M^o \) has a critical point at \( p^*X \).

4. The example 3-manifold

Consider the following pared 3-manifold \( M = (H_2, P) \): Let \( H_2 \) denote the closed genus-2 handlebody and \( P \) the union of the annuli obtained as regular neighborhoods of the curves pictured in Figure 1. Fix a choice of basepoint \( x \in \partial H_2 \setminus \overline{P} \), and the presentation \( \pi_1(\Sigma_2, x) = \langle A, B \rangle \) (see Figure 2). For a natural choice of basis for \( \pi_1(\Sigma_2, x) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle \) the conjugacy classes of the core curves of \( P \) are \( \{[b_1a_1b_1b_2, a_1, b_1], [b_2a_2b_2b_1] \} \).\(^1\)

\(^1\)The reader is warned of the notational offense that the choices above imply that \( \pi_1(\Sigma_2, x) \to \pi_1(M, x) \) will be given by \( b_1 \mapsto A \) and \( b_2 \mapsto B \). Though inconvenient, \( a_i \) and \( b_i \) will play no further role in our analysis, and the reader may ignore \( \pi_1(\Sigma_2, x) \).
Since the annuli in $P$ are disk-busting in the genus-2 surface, [26, Lemma 1.15.] guarantees that the boundary is incompressible and $(H_2, P)$ is acylindrical. Since the annuli in $P$ are non-separating, $\Sigma = \partial M$ is a 4-holed sphere. Fix the presentation $\pi_1(\Sigma, x) = \langle \delta_1, \delta_2, \delta_3, \delta_4 \mid \prod \delta_i = 1 \rangle$, with the $\delta_i$ as pictured in Figure 3; $\delta_i$ takes the path drawn to a boundary component and winds around it counterclockwise. Deleting $P$ from $\partial H_2$ we arrive at a topological picture of $\Sigma$, shown in Figure 4.

![Figure 4. Another view of $\pi_1(\Sigma, x)$.](image)

Choosing the constant path from $x$ to $x$, we record the homomorphism induced by inclusion $\iota_* : \pi_1(\Sigma, x) \hookrightarrow \pi_1(M, x)$ in the chosen generators:

$$\iota_*(\delta_1) = A^{-2}B,$$

$$\iota_*(\delta_2) = B^{-2}A^{-1},$$

$$\iota_*(\delta_3) = AB^{-1}A = A \iota_*(\delta_1)^{-1} A^{-1},$$

$$\iota_*(\delta_4) = A^{-1}B^2A^2 = A^{-2} \iota_*(\delta_2)^{-1} A^2.$$ 

We suppress the basepoint $x$ and the notation $\iota_*$ in what follows, and view $\pi_1(\Sigma)$ as a subgroup of $\pi_1(M)$. In everything that follows, we fix the notation $M = (H_2, P)$ and $\Sigma$ for the 4-holed sphere boundary. We now describe a geometrically finite hyperbolic structure on $M$. Details about similar structures (gluings of regular right-angled ideal polyhedra) can be found in [7].

For $v, w \in (\mathbb{CP}^1)^3$, each with pairwise distinct entries, let $m(v, w)$ be a lift to $\text{SL}_2 \mathbb{C}$ (which one is immaterial) of the unique Möbius transformation taking the triple $v$ to the triple $w$. Fix an identification $\partial_{\infty} \mathbb{H}^3 \cong \mathbb{CP}^1$, and let $\mathcal{O}$ be the ideal octahedron in $\mathbb{H}^3$ with totally geodesic triangular faces whose vertices are $\{1, 0, -1, i, -i, \infty\}$. 
Consider the following representation of $\pi_1 H_2$:

\[ \tilde{\rho}_1 : \langle A, B \rangle \rightarrow \text{SL}_2 \mathbb{C} \]

\[ A \mapsto m((-1, i, 0), (0, -i, 1)) = \frac{1 + i}{2} \begin{pmatrix} 1 & 1 \\ 1 + 2i & 1 \end{pmatrix} \]

\[ B \mapsto m((-1, -i, \infty), (1, \infty, i)) = -\frac{1 + i}{2} \begin{pmatrix} i & -1 + 2i \\ 1 & i \end{pmatrix} \]

Let $\rho_1 : \langle A, B \rangle \rightarrow \text{PSL}_2 \mathbb{C}$ be the representation induced by $\tilde{\rho}_1$.

Geometrically, $\rho_1(A)$ and $\rho_1(B)$ perform face identifications on $\mathcal{O}$ with hyperbolic isometries (see Figure 5), and so $\rho_1$ is faithful by a standard Ping-Pong Lemma argument. Let $\hat{\Gamma}_1 = \rho_1(\langle A, B \rangle)$. By the Poincaré Polyhedron Theorem [23, p. 75], $\hat{\Gamma}$ is discrete, and $\mathbb{H}^3/\hat{\Gamma}$ is a hyperbolic structure on the genus-2 handlebody. Since all of the dihedral angles of $\mathcal{O}$ are $\frac{\pi}{2}$, the non-paired faces meet flush. Thus the quotient $\mathcal{O}/\hat{\Gamma}_1$ is a hyperbolic structure on the genus-2 handlebody, with totally geodesic boundary, homotopy equivalent to $\mathbb{H}^3/\hat{\Gamma}_1$. Horoball neighborhoods of the six ideal vertices glue to two rank-1 cusp neighborhoods in the complete hyperbolic manifold $\mathbb{H}^3/\hat{\Gamma}_1$.

Denote the convex core of $\hat{\Gamma}_1$ by $\mathcal{C}$. Since $\mathcal{O}$ is the convex hull of points that are in the limit set, $\mathcal{C} \supset \mathcal{O}/\hat{\Gamma}_1$. Since $\mathcal{C}$ is minimal among convex subsets of $\mathbb{H}/\hat{\Gamma}_1$ for which inclusion is a homotopy equivalence, we see that $\mathcal{C} = \mathcal{O}/\hat{\Gamma}_1$. Since $\mathcal{O}$ has finite volume and non-empty interior, $\hat{\Gamma}_1$ is geometrically finite.

Figure 5. The octahedron $\mathcal{O}$ with side identifications labelled by $\alpha$ and $\beta$. 

Consider the following representation of $\pi_1 H_2$: 

\[ \tilde{\rho}_1 : \langle A, B \rangle \rightarrow \text{SL}_2 \mathbb{C} \]

\[ A \mapsto m((-1, i, 0), (0, -i, 1)) = \frac{1 + i}{2} \begin{pmatrix} 1 & 1 \\ 1 + 2i & 1 \end{pmatrix} \]

\[ B \mapsto m((-1, -i, \infty), (1, \infty, i)) = -\frac{1 + i}{2} \begin{pmatrix} i & -1 + 2i \\ 1 & i \end{pmatrix} \]

Let $\rho_1 : \langle A, B \rangle \rightarrow \text{PSL}_2 \mathbb{C}$ be the representation induced by $\tilde{\rho}_1$.

Geometrically, $\rho_1(A)$ and $\rho_1(B)$ perform face identifications on $\mathcal{O}$ with hyperbolic isometries (see Figure 5), and so $\rho_1$ is faithful by a standard Ping-Pong Lemma argument. Let $\hat{\Gamma}_1 = \rho_1(\langle A, B \rangle)$. By the Poincaré Polyhedron Theorem [23, p. 75], $\hat{\Gamma}$ is discrete, and $\mathbb{H}^3/\hat{\Gamma}$ is a hyperbolic structure on the genus-2 handlebody. Since all of the dihedral angles of $\mathcal{O}$ are $\frac{\pi}{2}$, the non-paired faces meet flush. Thus the quotient $\mathcal{O}/\hat{\Gamma}_1$ is a hyperbolic structure on the genus-2 handlebody, with totally geodesic boundary, homotopy equivalent to $\mathbb{H}^3/\hat{\Gamma}_1$. Horoball neighborhoods of the six ideal vertices glue to two rank-1 cusp neighborhoods in the complete hyperbolic manifold $\mathbb{H}^3/\hat{\Gamma}_1$.

Denote the convex core of $\hat{\Gamma}_1$ by $\mathcal{C}$. Since $\mathcal{O}$ is the convex hull of points that are in the limit set, $\mathcal{C} \supset \mathcal{O}/\hat{\Gamma}_1$. Since $\mathcal{C}$ is minimal among convex subsets of $\mathbb{H}/\hat{\Gamma}_1$ for which inclusion is a homotopy equivalence, we see that $\mathcal{C} = \mathcal{O}/\hat{\Gamma}_1$. Since $\mathcal{O}$ has finite volume and non-empty interior, $\hat{\Gamma}_1$ is geometrically finite.
The $\rho_1$-images of the core curves of $P$ are distinct maximal parabolic conjugacy classes in $\hat{\Gamma}_1$. Any parabolics in $\hat{\Gamma}_1$ must be conjugate to parabolics stabilizing the equivalence class of a vertex of $O$ [23, p. 131]. Since $P$ consists of two annuli, and since the vertices split into two equivalence classes, $\rho_1$ determines a one-to-one correspondence between components of $P$ and conjugacy classes of maximal parabolic subgroups of $\hat{\Gamma}_1$. This means we are in the setting of [7, Lemma 2.6.] and $\rho_1$ is induced by a homeomorphism from the convex core of $\hat{\Gamma}_1$, $O/\hat{\Gamma}_1$, to $H_2 \setminus P$.

We denote this point $[\rho_1] \in \mathcal{G}(P) \subset \mathcal{X}(M)$. The octahedron has an evident self-map which respects the gluings: Reflect through an equatorial plane and rotate by $90^\circ$ around the axis perpendicular to the plane. In the coordinates chosen in Figure 3 this is the anti-Möbius map $z \mapsto i/z$. This self-map of $O$ thus descends to an orientation-reversing isometry of the hyperbolic 3-manifold $\mathbb{H}^3/\hat{\Gamma}_1$ (cf. [5]).

Before analyzing this symmetry of $(M, \Sigma)$ (cf. [9]), we examine deformations of the above hyperbolic structure. In particular, the octahedron $O$ provides a ‘hands-on’ method to deform the representation $\rho_1$.

5. DEFORMING THE EXAMPLE

Consider now the path of representations:

\[ \tilde{\rho}_t : (A, B) \to \text{SL}_2 \mathbb{C} \]

\begin{align*}
A & \mapsto m((-1, it, 0), (0, -it, 1)) = \frac{1}{t(t-i)} \left( \begin{array}{cc} t^2 & t^2 \\ 1 + 2it & t^2 \end{array} \right) \\
B & \mapsto m((-1, -it, \infty), (1, \infty, it)) = \frac{-i}{t+i} \left( \begin{array}{cc} it & -1 + 2it \\ 1 & it \end{array} \right)
\end{align*}

Once again, let $\rho_t : (A, B) \to \text{PSL}_2 \mathbb{C}$ be the representations induced by $\tilde{\rho}_t$. Taking $t = 1$, we recover the representation $\rho_1$ from [11]. For $t \neq 1$, $\rho_t$ performs identifications of the triangular totally geodesic faces of an ideal octahedron with vertices \(\{1, 0, -1, it, -it, \infty\}\). Note that the octahedron is not regular, and the dihedral angles are no longer $90^\circ$.

For each $t \in \mathbb{R}$, let $\tilde{\Gamma}_t := \rho_t((A, B))$ and $\Gamma_t := \rho_t(\pi_1 \Sigma)$.

Lemma 5.1. The maps $[\rho_t] : (0, \infty) \to \mathcal{X}(M)$ and $[\rho_t|_{\pi_1(\Sigma)}] : (0, 1] \to \mathcal{X}(\Sigma)$ are injective. For $t > 0$, $[\rho_t|_{\pi_1(\Sigma)}]$ is in the real locus of $\mathcal{X}(\Sigma)$ if and only if $t = 1$.

Proof. Let $A_t := \rho_t(A)$ and $B_t := \rho_t(B)$. Noting that the square of the trace is a well-defined function on $\text{PSL}_2 \mathbb{C}$, for $i \in \{1, 2, 3, 4\}$ we have

\[ tr^2 \rho_t(\delta_i) = 4. \]

Thus $[\rho_t] \in \mathcal{X}(M) \subset \mathcal{X}(H_0)$.

In order to show injectivity of the paths $[\rho_t]$ and $[\rho_t|_{\pi_1 \Sigma}]$ on $(0, \infty)$ and $(0, 1)$, respectively, we show that $tr^2 A_t$ and $tr^2 \rho_t(\delta_1 \delta_2)$ are one-to-one functions of $t$ on the respective intervals.

We compute

\[ tr^2 A_t = \left( \frac{2t}{t-i} \right)^2. \]

Note that $\frac{2t}{t-i} - 1 = \frac{t+i}{t-i}$. For $t > 0$, the quantity $\frac{t+i}{t-i}$ is evidently a parameterization of the upper hemisphere of the unit circle centered at 0. Thus $\{tr^2 A_t : t > 0\}$ is a
parameterization of the square of the upper hemisphere of the unit circle centered at 1. In particular, it is one-to-one on \((0, \infty)\).

For brevity, let \(f(t) := tr\rho_t(\delta_1\delta_2)\). A computation shows that
\[
f(t) = \frac{2t^2(t^4 - 22t^2 - 7)}{(1 + t^2)^3} + t\frac{(t^2 - 1)(5t^2 + 1)^2}{t(1 + t^2)^3}.
\]

For \(t \in (0, 1)\), the quantity \(\text{Im } f(t) < 0\). Thus \(f(t)^2\) is one-to-one on \((0, 1)\) if and only if \(f(t)\) is one-to-one on \((0, 1)\). We compute
\[
\frac{d}{dt}\text{Im } f(t) = \frac{-(1 + 5t^2)(5t^6 - 35t^4 + 7t^2 - 1)}{t^2(1 + t^2)^4}.
\]

We now estimate, for \(t \in (0, 1)\),
\[
5t^6 - 35t^4 + 7t^2 - 1 = 5(t^2 - 1)^3 - 20(t^2 - 1)^2 - 48(t^2 - 1) - 64 < 0 + 0 + 48 - 64 < 0.
\]

Thus \(\frac{d}{dt}\text{Im } f(t) > 0\), and \(\text{Im } f(t)\) is monotone increasing for \(t \in (0, 1)\). In particular, the function \(tr^2 \rho_t(\delta_1\delta_2) : (0, 1) \to \mathbb{C}\) is one-to-one.

Finally, since the Kleinian group \(\hat{\Gamma}_t\) has totally geodesic boundary, the boundary subgroup \([\hat{\Gamma}_t]\) is in the real locus of \(X(\Sigma)\). As well, it is clear that \(\text{Im } f(t) = 0\), for \(t > 0\), if and only if \(t = 1\). Thus \([\hat{\Gamma}_t]\) is in the real locus of \(X(\Sigma)\) if and only if \(t = 1\). \(\square\)

The ‘symmetry’ of \(\hat{\Gamma}_t\) (cf. \(\S\)) persists along the path \(t \mapsto [\rho_t]\). (In fact, this path parametrizes the full fixed subset of \(X(M)\) preserved by this symmetry, but this is not relevant to our analysis). In Lemma \(\S\) we check that conjugation by the anti-Möbius map \(\Psi_t(z) = it/\bar{z}\) descends to an isometry of \(\mathbb{H}^3/\hat{\Gamma}_t\). This isometry will be instrumental in our analysis of the path \([\rho_t]\) (cf. \(\S\) and \(\S\)).

**Lemma 5.2.** For all \(t\), we have \(\Psi_t\hat{\Gamma}_t\Psi_t^{-1} = \hat{\Gamma}_t\), and \([\Psi_t\Gamma_t\Psi_t^{-1}] = [\Gamma_t]\). Thus \(\Psi_t\)
induces a mapping class \(\Psi \in MCG^*(M)\), with \([\hat{\Gamma}_t] \in \text{Fix } \Psi\) and \([\Gamma_t] \in \text{Fix } \Psi|_\Sigma\). The action of \(\Psi|_\Sigma\) on the punctures is an order four permutation, and \(\Psi|_\Sigma\) preserves the two simple closed curves \(\delta_1\delta_3\) and \(\delta_2\delta_4\).

**Proof.** A calculation using the definition of \(\rho_t\) (equation \(\S\)) shows:
\[
\begin{align*}
\cdot \Psi_t A_t \Psi_t^{-1} &= B_t, \\
\cdot \Psi_t B_t \Psi_t^{-1} &= A_t^{-1}.
\end{align*}
\]

Thus \(\Psi_t\hat{\Gamma}_t\Psi_t^{-1} = \hat{\Gamma}_t\), and \(\Psi_t\) induces a mapping class \(\Psi \in MCG^*(M)\) with \([\hat{\Gamma}_t] \in \text{Fix } \Psi\). We will drop the restriction map notation and consider \(\Psi\) as simultaneously an element of \(MCG^*(M)\) and \(MCG^*(\Sigma)\). We compute:
\[
\begin{align*}
\cdot \Psi_t \rho_t(\delta_1) \Psi_t^{-1} &= A_t^2 \cdot \rho_t(\delta_1^{-1}) \cdot A_t^{-2}, \\
\cdot \Psi_t \rho_t(\delta_2) \Psi_t^{-1} &= A_t^2 \cdot \rho_t(\delta_2^{-1}) \cdot A_t^{-2}, \\
\cdot \Psi_t \rho_t(\delta_3) \Psi_t^{-1} &= A_t^2 \cdot \rho_t(\delta_1\delta_3\delta_4) \cdot A_t^{-2}, \\
\cdot \Psi_t \rho_t(\delta_4) \Psi_t^{-1} &= A_t^2 \cdot \rho_t(\delta_1\delta_2\delta_4) \cdot A_t^{-2}.
\end{align*}
\]

Thus \(\Psi_t\Gamma_t\Psi_t^{-1}\) is conjugate to \(\Gamma_t\), and \(\Psi\) is an orientation-reversing mapping class of \(\Sigma\) with \([\Gamma_t] \in \text{Fix } \Psi\).

Because \(\delta_1\delta_2\delta_4 = \delta_1\delta_2^{-1}\delta_1^{-1} \sim \delta_2^{-1}\) (and similarly \(\delta_1\delta_2\delta_4 \sim \delta_3^{-1}\)), we see that the action of \(\Psi\) on the conjugacy classes of punctures is a cyclic permutation. The fact
that $Ψ$ preserves the geodesic representatives of the simple closed curves $δ_1δ_3$ and $δ_2δ_4$ is immediate. □

Note that, by Proposition 2.6, the fixed set of the mapping class $Ψ$ is preserved by the skinning map. Namely,

(5.2) $σ_M(\text{Fix } Ψ|_Σ) \subset \text{Fix } Ψ|_Σ$.

Since $GF(M) \subset X(M)$ is open (see [22, Theorem 10.1.]), there is some open interval about 1 so that $[\rho_t] \in GF(M)$ and, by Lemma 2.3, $[\rho_t|_π(Σ)] ∈ QF(Σ)$. Denote the maximal such open interval around 1 by $U$.

The path of quasi-Fuchsian groups $t → \Gamma_t$, for $t ∈ U$, induces two paths in $T(Σ)$ corresponding to the top and bottom of $Γ_t$. For $t ∈ U$, fix notation $[Γ_t]=Q(X_t,Z_t)$, so that $σ_M(X_t) = Z_t$ and $X_t,Z_t ∈ \text{Fix } Ψ|_Σ$. Lemma 5.1 implies, in particular, that $\{X_t \mid t ∈ U\}$ is an injective path in $T(Σ)$. In fact, the remainder of the paper is devoted to checking that $\{Z_t \mid t ∈ U\}$ is a non-injective path whose image is confined to the real 1-dimensional submanifold $\text{Fix } Ψ|_Σ ⊂ T(Σ)$. Our strategy will be to examine the convex core boundary surfaces and bending laminations of $Γ_t$.

First, we need to better understand the set $\text{Fix } Ψ|_Σ$. This is the subject of §6.

6. Rhombic 4-punctured spheres

In this section, we examine the symmetries of 4-punctured spheres and collect some useful facts about a special symmetrical set in $T(Σ)$.

It is well-known that for all $X ∈ T(Σ)$ there exists a Klein 4-group of conformal automorphisms that acts trivially on $S$, whose non-trivial elements are involutions that exchange punctures in pairs. Lemma 6.2 characterizes a more restrictive orientation-reversing symmetry of a subset of $T(Σ)$.

Definition 6.1. If $X ∈ T(Σ)$ may be written as the complement of the vertices of a Euclidean rhombus in $CP^1$, with $ξ$ and $η$ as pictured (see Figure 6), then $X$ is $\{ξ,η\}$-rhombic.

Let $R{ξ,η} := \{X ∈ T(Σ) \mid X$ is $\{ξ,η\}$-rhombic$\}$. Recall Fenchel-Nielsen coordinates for $T(Σ)$: Let $ξ$ be the pants curve. Using the coordinate pictured in Figure 6, for any $s < 0$, let the transversal $δ$ be the homotopy class of the curve $\text{Im } z = s \cdot \text{Re } z$ (see Figure 7). By the classical work of Fenchel and Nielsen, we obtain a diffeomorphism $T(Σ) \cong \{(ℓ, θ) \mid ℓ ∈ R^+, θ ∈ R\}$.

Lemma 6.2. Let $X ∈ T(Σ), ξ, η ∈ S$. The following are equivalent:

1. $X$ is $\{ξ,η\}$-rhombic.
2. In the Fenchel-Nielsen coordinates above, $X ∈ \{θ = π/2\}$.
3. There exists an orientation-reversing $ψ ∈ \text{MCG}^+(Σ)$ that acts as an order four cyclic permutation of the punctures, preserves $ξ$ and $η$, and $X ∈ \text{Fix } ψ$.

Proof. (1) ⇒ (2): In order to measure the twisting coordinate relative to the choice of transversal $δ$, we observe that the four points of intersection of $ξ$ with the real and imaginary axes divide $ξ$ into four arcs. These arcs are cyclically permuted by the isometry $z → \frac{z}{2}$, and hence are of equal length. This shows that the twisting coordinate is in $\mathbb{Z} \cdot \frac{π}{2}$, and the choice of $δ$ guarantees the twisting coordinate is $\frac{π}{2}$.

(2) ⇒ (3): There is a reflection/twist symmetry of $X$ that preserves $ξ$ and $η$ and acts on punctures as desired.
Figure 6. Coordinates on a \{\xi, \eta\}-rhombic \(X \in \mathcal{T}(\Sigma)\).

Figure 7. The transversal \(\delta\).

(3) \(\Rightarrow\) (1): By the existence of holomorphic involutions exchanging the punctures in pairs, there exists an automorphism \(\alpha\) such that \(\phi := \alpha \circ \psi\) is a simple transposition of the punctures that preserves \(\xi\) and \(\eta\). Without loss of generality, suppose \(\phi\) interchanges the punctures enclosed by \(\eta\). With an appropriate Möbius transformation, take these to 1 and \(-1\). Anti-conformal involutions of \(\mathbb{CP}^1\) are inversions through a circle, and their fixed point sets are the circle they involve through. Since \(\phi\) fixes the other two punctures, they lie on the fixed circle for \(\phi\). We may thus apply another Möbius transformation, which fixes 1 and \(-1\) and takes this fixed circle to the imaginary axis. Apply an elliptic Möbius transformation fixing 1 and \(-1\) and centering the imaginary punctures about 0, and \(X\) has the desired form.

\[\square\]

Following Lemma 6.2, we refer to the defining symmetry \(\psi\) as \{\xi, \eta\}-rhombic as well. Though it is not involved in our analysis, the reader may recall that there is a natural identification \(\mathcal{T}(\Sigma) \cong \mathbb{H}\). Each element of \(\mathcal{T}(\Sigma)\) is covered by \(\mathbb{C} \setminus \mathbb{Z} \oplus \tau \mathbb{Z}\) for some \(\tau \in \mathbb{H}\), with covering group \(\langle z \mapsto z + 2, z \mapsto z + 2\tau, z \mapsto -z \rangle\), such that the line \(\{\text{Im } z = \frac{1}{2} \text{Im } \tau\}\) projects to \(\xi\) and the line \(\{\text{Im } z(\text{Re } \tau - \frac{1}{2}) = \text{Im } \tau(\text{Re } z - \frac{1}{4})\}\) projects to \(\eta\). Relative to this parametrization the reader may identify \(\mathcal{R}_{\{\xi, \eta\}}\) as the line \(\{\text{Re } z = \frac{1}{2}\}\): The orientation-reversing symmetry \(z \mapsto 1 - \bar{z}\) preserves the lattices \(\mathbb{Z} \oplus \mathbb{Z}(\frac{1}{2} + yi)\) and preserves \(\xi\) and \(\eta\), fulfilling (3).

We look to the action of \(\text{MCG}^*(\Sigma)\) on \(\mathbb{P}\mathcal{M}\mathcal{L}(\Sigma)\). As it turns out, the fixed points of the action of \{\xi, \eta\}-rhombic isometries are easy to characterize.

Lemma 6.3. A \{\xi, \eta\}-rhombic mapping class \(\psi \in \text{MCG}^*(\Sigma)\), acting on \(\mathbb{P}\mathcal{M}\mathcal{L}(\Sigma)\), has \(\text{Fix } \psi = \{\xi, \eta\}\).

Proof. There is an identification \((\mathbb{P}\mathcal{M}\mathcal{L}(\Sigma), S) \cong (\mathbb{RP}^1, \mathbb{QP}^1)\) which is equivariant with respect to a homomorphism \(\text{MCG}^*(\Sigma) \rightarrow \text{PGL}_2 \mathbb{Z}\) (cf. \cite[p. 60]{11}). For the four-punctured sphere there is an \(\text{MCG}^*(\Sigma)\)-equivariant map from rays in \(\mathcal{M}\mathcal{L}(\Sigma)\) to lines in \(H_1(\Sigma, \mathbb{R})\), which sends rays of laminations supported on simple closed curves to lines in \(H_1(\Sigma, \mathbb{Q}) \subset H_1(\Sigma, \mathbb{R})\). Because the action of an element of \(\text{MCG}^*(\Sigma)\) on \(\mathbb{P}\mathcal{M}\mathcal{L}(\Sigma)\) is given by the action of an element of \(\text{PGL}_2 \mathbb{Z}\) on \(\mathbb{RP}^1\),
each element of $MCG^*(\Sigma)$ which acts non-trivially on $\mathcal{PML}(\Sigma)$ has at most two fixed points.

For a curve $\gamma \in \mathcal{S}$, recall the real-valued functions $\ell_\gamma$ and $\text{Ext}_\gamma$ on $\mathcal{T}(\Sigma)$: For $X \in \mathcal{T}(\Sigma)$, the quantity $\ell_\gamma(X)$ is the hyperbolic length of the geodesic representative of $\gamma$ and the quantity $\text{Ext}_\gamma(X)$ is the extremal length of the family of curves homotopic to $\gamma$. See [11] and [1] for details.

Lemma 6.4. The maps $\ell_\xi$, $\ell_\eta$, $\text{Ext}_\xi$, and $\text{Ext}_\eta$ each provide a diffeomorphism from $\mathcal{R}_{\{\xi,\eta\}}$ to $\mathbb{R}^+$. 

Proof. Fix pants decomposition with pants curve $\xi$ and transversal $\delta$, as in Figures 6 and 7. By Lemma 6.2 $\ell_\xi|_{\mathcal{R}_{\{\xi,\eta\}}}$ is injective. Since one may construct $X \in \mathcal{R}_{\{\xi,\eta\}}$ with pants curve of specified hyperbolic length, and since length functions are smooth, the lemma is clear for $\ell_\xi$. The proof for $\ell_\eta$ is the same.

Along a Teichmüller geodesic, the extremal lengths of the vertical and horizontal foliations each provide diffeomorphisms to $\mathbb{R}^+$ (this follows from [14, Lemma 5.1], Gardiner’s formula [13, Theorem 8], and the inverse function theorem). Thus it is enough to check:

(1) The set $\mathcal{R}_{\{\xi,\eta\}}$ is a Teichmüller geodesic.

(2) The projective classes of its foliations are given by $[\xi], [\eta] \in \mathcal{PML}(\Sigma)$.

The first follows from the fact that fixed point sets of isometries, in uniquely geodesic spaces such as $(\mathcal{T}(\Sigma), d_T)$, are convex, while the second follows because the two foliations must be preserved by the rhombic symmetry, and Lemma 6.3 applies.

In fact, it will be more direct to deal with a conformal invariant closely related to extremal length, the modulus of a quadrilateral. Recall that the modulus of a quadrilateral is the extremal length of the family of curves that connect the vertical sides (see [21], [1] for details).

For $X \in \mathcal{R}_{\{\xi,\eta\}}$, consider the natural coordinate on $X$ provided by Lemma 6.2. Let $Q_X$ be the quadrilateral given by the first quadrant in $\mathbb{C}\mathbb{P}^1$, with vertices $\{i\tau, 0, 1, \infty\}$ and vertical sides given by the arcs $(0, 1)$ and $(i\tau, \infty)$ (see Figure 8). Denote the modulus of a quadrilateral or annulus by $\text{Mod}(\cdot)$.

Lemma 6.5. We have $\text{Ext}_\xi X = 4 \text{Mod}(Q_X)$.

Proof. The quantity $\text{Ext}_\xi X$ can also be computed as the modulus of the unique maximal modulus annulus containing $\xi$ as its core curve. Consider this annulus $A_\xi$. Since $\xi$ is preserved by the anti-holomorphic maps $z \mapsto \bar{z}$ and $z \mapsto -\bar{z}$, the annulus $A_\xi$ is preserved by these maps. This is enough to ensure that $A_\xi = \mathbb{C} \setminus ((-\infty, -1) \cup (1, \infty) \cup (-i\tau, i\tau))$. (Alternatively, in [11, p. 23, II], there is a classical extremal length problem whose solution—due to Teichmüller—implies that $A_\xi$ has the given form.)

Since the curves

- $\{i(s + \tau) \mid \tau \in (0, \infty)\}$,
- $\{-i(s + \tau) \mid \tau \in (0, \infty)\}$,
- $\{\tau \mid \tau \in (0, 1)\}$,
- $\{-\tau \mid \tau \in (0, 1)\}$

are all fixed by $z \mapsto \bar{z}$ and $z \mapsto -\bar{z}$, they must be vertical geodesics in the Euclidean metric for $A_\xi$. By [11, p. 16], the modulus $\text{Mod}(A_\xi)$ is equal to the sum of the
moduli of the four quadrilateral pieces obtained after cutting along these vertical geodesics. Since these four quadrilateral pieces are each congruent to \( Q_X \), we have \( \text{Mod}(A_\xi) = 4 \text{Mod}(Q_X) \).

\[ \square \]

**7. The convex core boundaries of \( \Gamma_t \)**

We return to the goal of understanding \( \Gamma_t \) via its convex core boundary. Recall from Lemma 5.2 that \( \Psi \) is an order four orientation-reversing isometry of \( \Gamma_t \), cyclically permuting the punctures and preserving the simple closed curves \( \delta_1 \delta_3 \) and \( \delta_2 \delta_4 \). From here on we fix the the notation \( \xi := \delta_1 \delta_3 \) and \( \eta := \delta_2 \delta_4 \). The reader may notice that the isometry \( \Psi \) is \( \{\xi, \eta\} \)-rhombic. Equation (5.2) now becomes

\[
(7.1) \quad \sigma_M \left( \mathcal{R}_{\{\xi, \eta\}} \right) \subset \mathcal{R}_{\{\xi, \eta\}}.
\]

Below, we present the computation of the bending angles for the convex core of \( \Gamma_t \). We choose the branch \([0, \pi]\) for \( \cos^{-1} \) and \([0, 2\pi]\) for arg. Note that the existence of a ‘top’ and ‘bottom’ of \( \Gamma_t \), relative to \( \hat{\Gamma}_t \), implies that there is a ‘top’ and ‘bottom’ bending lamination. The computation of the bending measure for the top lamination is computable by the same methods presented as for the bottom. Recall that \( U \) is the maximal open interval, containing 1, so that \( \Gamma_t \) is quasi-Fuchsian for \( t \in U \) (see §5).

**Lemma 7.1.** For \( t \in U \), the top and bottom surfaces of the convex core boundary of \( \Gamma_t \) are both in \( \mathcal{R}_{\{\xi, \eta\}} \). The bending lamination on the bottom of the convex core boundary is \( \theta(t) \cdot \xi \), for

\[
(7.2) \quad \theta(t) = \cos^{-1} \left( \frac{2t^3(3 - t^2)}{(1 + t^2)^2} \right).
\]

**Proof.** By Lemma 5.2, \( \Psi_t \) is a \( \{\xi, \eta\} \)-rhombic isometry of \( \Gamma_t \). It thus sends the convex core to itself, so preserves the convex core boundary pleated surfaces, which are hence in \( \mathcal{R}_{\{\xi, \eta\}} \). Moreover, \( \Psi_t \) preserves the bending laminations \( \{\lambda^+, \lambda^-\} \),
where $\lambda^+$ is on the top and $\lambda^-$ is on the bottom. By Lemma 6.3, $\{[\lambda^+], [\lambda^-]\} \subset \{\xi, \eta\}$, where $[\cdot]$ denotes the projective class of the measured lamination. Before computing $\theta(t)$ and determining that $\xi = [\lambda^-]$, we describe the method in words.

To compute the bending angle $\theta(t)$, it is necessary to find a pair of distinct maximal support planes that intersect along the axis of the hyperbolic Möbius map $\rho_t(\xi)$, i.e. two half-planes that meet along the axis of $\rho_t(\xi)$. This can be achieved by considering the two other lifts of $\xi$ which are the axes of $\rho_t(\delta_1^{-1}\xi\delta_1)$ and $\rho_t(\delta_4\xi\delta_4^{-1})$. One can check easily that $[\delta_1^{-1}\cdot \xi]$ and $[\delta_4 \cdot \xi]$—the homotopy classes of the concatenations—are simple closed curves. Thus, for the pair $\{\xi, \delta_1^{-1}\cdot \xi\}$, for example, there is a connected fundamental domain for the action of $\pi_1\Sigma$ that has the lifts of these curves on its boundary (motivating the term ‘neighbors’ for this pair; see Figure 9). The same is true, of course, for $\{\xi, \delta_4 \cdot \xi\}$.

As we know that $\xi$ is the support of a bending lamination for $\Gamma_t$, the four fixed points of each of the pairs $\{\xi, \delta_1^{-1}\cdot \xi\}$ and $\{\xi, \delta_4 \cdot \xi\}$ lie on a round circle which intersects $\Lambda_{\Gamma_t}$ at these four points. Each such circle is the frontier of a maximal support plane for $\Gamma_t$. The two pairs $\{\xi, \delta_1^{-1}\cdot \xi\}$ and $\{\xi, \delta_4 \cdot \xi\}$ thus determine two maximal support planes intersecting in $\mathbb{H}^3$ along the axis of $\rho_t(\xi)$, as desired.

We thus consider three pairs of fixed points, corresponding to the three chosen lifts of $\xi$: Fix $\rho_t(\delta_1^{-1}\xi\delta_1)$, Fix $\rho_t(\xi)$, and Fix $\rho_t(\delta_4\xi\delta_4^{-1})$. The first four points determine one half-plane in $\mathbb{H}^3$, and the last four determine a second. We may now compute the bending angle between these half-planes using the argument of a cross-ratio.

There is an ambiguity in our calculation of support planes: we do not know which side of the totally geodesic plane through a pair of neighboring axes is actually a supporting half-space for either of the groups $\Gamma_t$ or $\hat{\Gamma}_t$. In order, then, to distinguish between the top and the bottom laminations we will use the following consequence of the definitions: if there exists a support plane for a component $\Omega_0 \subset \Omega_{\Gamma_t}$ which is not simultaneously a support plane for $\hat{\Gamma}_t$, then $\Omega_0$ is the bottom. Such a support plane is produced explicitly below at $t = \frac{1}{2}$ for the domain facing the convex core boundary component with bending supported on $\xi$.

In fact, for our purposes, it is enough to distinguish the top lamination from the bottom lamination at $t = \frac{1}{2}$: The bending lamination map from $\mathcal{MF}(M)$ to $\mathcal{ML}(\Sigma)$ is continuous [19, Theorem 4.6.], and its image along the path $\{[\rho_t] \mid t \in U\}$, by the
argument above, is confined to the subset of $\mathcal{ML}(\Sigma)$ supported on either $\xi$ or $\eta$. This subset is a pair of rays, intersecting only at the ‘zero’ lamination. Thus, for instance, if $\xi$ switches at some point from the support for the bottom lamination of $\Gamma_t$ to the support for the top lamination of $\Gamma_t$, the Kleinian group $\hat{\Gamma}_t$ must have the ‘zero’ bending lamination, i.e. totally geodesic boundary. In this case, $\Gamma_t$ would be Fuchsian at this value of $t$. By Lemma 5.1 $[\rho_t]$ is not Fuchsian for $t < 1$; hence the lamination on top at $t = \frac{1}{2}$ must remain on top for all $t < 1$.

We proceed with the calculation of $\theta(t)$. Denote the fixed points of $\rho_t(\xi)$ by $p_t^\pm$, where the choice is fixed by asking that the root in the expression of the fixed points be positive for $p^+$ and negative for $p^-$. The fixed points of $\rho_t(\delta_4 \delta_4^{-1})$ and $\rho_t(\delta_4 \delta_4^{-1})$ are thus, respectively, $\rho_t(\delta_1^{-1}) \cdot p_t^\pm$ and $\rho_t(\delta_4) \cdot p_t^\pm$. The root below refers to the positive one.

$$\rho_t(\delta_4) \cdot p_t^- \rho_t(\delta_1^{-1}) \cdot p_t^+ \rho_t(\delta_4) \cdot p_t^+ \rho_t(\delta_1^{-1}) \cdot p_t^-$$

**Figure 10.** Maximal support planes for $\Gamma_t$, and the bending angle $\theta(t)$.

We use the notation for the cross-ratio $[a : b : c : d] = \frac{d-a}{d-c} \cdot \frac{b-c}{b-a}$.

$$[p_t^+ : \rho_t(\delta_4) \cdot p_t^+ : p_t^- : \rho_t(\delta_1^{-1}) \cdot p_t^+] = \frac{1}{2t(-i+t)(i+t)^3}$$

$$\cdot \left( -i - 2t - 2it^2 - 13it^4 + 2t^5 + 4it^6 
+ \left( -i - 2t - it^2 + 2t^3 \right) \sqrt{1 + 6t^2 + 17t^4 - 4t^6} \right)$$

$$= \frac{1 + 5t^2 + \sqrt{1 + 6t^2 + 17t^4 - 4t^6}}{2t(t^2 + 1)^3}$$

$$\cdot \left( 2t^3(t^2 - 3) + i (1 - t^2) \sqrt{1 + 6t^2 + 17t^4 - 4t^6} \right).$$

It is easy to see that the imaginary part of this cross-ratio, for $t \in (0, 1)$, is positive (cf. [7.4]). Thus $\theta(t) = \pi - \arg [p_t^+ : \rho_t(\delta_4) \cdot p_t^+ : p_t^- : \rho_t(\delta_1^{-1}) \cdot p_t^+]$. Taking the argument of the expression above gives

$$\theta(t) = \cos^{-1} \left( \frac{2t^3(3 - t^2)}{(1 + t^2)^2} \right).$$
It remains to distinguish the top of the convex core boundary from the bottom. Following the outline described above, we fix some notation at the point \( t = \frac{1}{2} \).

Let \( S \) be the circle through the points \( p_1^{+}, p_1^{-} \), and \( \rho_{1/2}(\delta_1^{-1}) \cdot p_1^{+} \), let its center be \( c \) and its radius \( r \). Noting that \( B_{1/2} \) is loxodromic, let its fixed points be \( p_1, p_2 \in \Lambda_{\hat{\Gamma}_{1/2}} \). The following computations are straightforward. The root is chosen with positive imaginary part.

\[
c = -\frac{1}{2} - i; \quad r = \frac{\sqrt{5}}{2}; \quad p_1 = \left( \frac{1 + 2i}{5} \right) \sqrt{7 + i}; \quad p_2 = -\left( \frac{1 + 2i}{5} \right) \sqrt{7 + i};
\]

\[
|c - p_1|^2 = \frac{5}{4} + \sqrt{2} + \text{Re} \sqrt{7 + i} > \frac{5}{4} = r^2;
\]

\[
|c - p_2|^2 = \frac{5}{4} + \sqrt{2} - \text{Re} \sqrt{7 + i} < \frac{5}{4} = r^2.
\]

(Note that \( \text{Re} \sqrt{7 + i} = \sqrt{\frac{7 + 5\sqrt{2}}{2}} > \sqrt{2} \).)

\[\text{Figure 11. A support plane for } \Gamma_{1/2} \text{ fails to be a support plane for } \hat{\Gamma}_{1/2}.\]

A support plane for \( \hat{\Gamma}_{1/2} \) must be the boundary of a supporting half-space, which must intersect \( \mathbb{C}P^1 \) in the domain of discontinuity. Because there are points \( p_1, p_2 \in \Lambda_{\hat{\Gamma}_{1/2}} \) in the interior and exterior of the round disk \( S \), the geodesic plane which meets \( \mathbb{C}P^1 \) in \( S \) cannot be the boundary of a supporting half-space for \( \hat{\Gamma}_{1/2} \). Because \( S \) is a support plane for \( \Gamma_{1/2} \) but not for \( \hat{\Gamma}_{1/2} \), \( S \) is a support plane for the bottom of \( \Gamma_{1/2} \). (See Figure 11 for a schematic where the exterior of \( S \) is a supporting half-space for the bottom of \( \Gamma_{1/2} \).) Since \( S \) is on the side of \( \Gamma_{1/2} \) with bending lamination \( \xi \), this implies that \( \xi \) is the support of the bending lamination on the bottom. \( \square \)
For $\gamma \in \pi_1(M)$, let $\ell(\gamma, \Gamma_t)$ denote the hyperbolic translation length of $\rho_t(\gamma)$ in $\mathbb{H}^3$. Using Lemma 7.1 we may now find $\inf U$ explicitly.

**Lemma 7.2.** The bottom (resp. top) convex core boundary surface of $\Gamma_t$ is the element of $\mathbb{R}\{\xi, \eta\}$ determined by $\ell(\xi, \Gamma_t)$ (resp. $\ell(\eta, \Gamma_t)$), where

\begin{align}
\ell(\xi, \Gamma_t) &= 2 \cosh^{-1} \left( \frac{1 + 8t^2 + 21t^4 - 2t^6}{2t^2(1 + t^2)^2} \right), \\
\ell(\eta, \Gamma_t) &= 2 \cosh^{-1} \left( \frac{-1 + 4t^2 + 74t^4 + 196t^6 - t^8}{(1 + t^2)^4} \right).
\end{align}

Moreover, $\frac{1}{2} \left( 5 + 3\sqrt{3} - \sqrt{44 + 26\sqrt{3}} \right) = \inf U$.

**Proof.** For a hyperbolic isometry $A \in \text{PSL}_2 \mathbb{C}$, its translation length is given by $\ell(A) = 2 \cosh^{-1} \left( \frac{|\text{tr}A|}{2} \right)$. Thus equations (7.3) can be computed directly. For $t \in U$, the hyperbolic length of the curve which supports the bending lamination of $\Gamma_t$, on the hyperbolic surface on the top or the bottom of the convex core boundary, is precisely the length of the curve in the quasi-Fuchsian manifold $\mathbb{H}^3/\Gamma_t$. By Lemma 6.4, this quantity determines the convex core boundary surface.

Let $t_0 := \inf U$. Because $\Sigma$ is incompressible in $M$, by [3, Theorem A] the hyperbolic manifold $\mathbb{H}^3/\hat{\Gamma}_{t_0}$ is geometrically tame. In particular, it has an end invariant $\lambda$, a geodesic lamination which is the union of limits of simple closed geodesics exiting the end and/or simple closed curves that are parabolic in $\hat{\Gamma}_{t_0}$. Since $\hat{\Gamma}_{t_0} \in \text{Fix } \Psi$, the invariant $\lambda$ must also be preserved by the rhombic symmetry. Because the natural map from $\text{PMC}(\Sigma)$ to geodesic laminations is equivariant for the action of $\text{MCG}^*(\Sigma)$, Lemma 6.3 implies that $\lambda \in \{\xi, \eta\}$.

Thus $\hat{\Gamma}_{t_0}$ is geometrically finite with end invariant either $\xi$ or $\eta$.

We check below the following computational facts:

1. $|\text{tr} \tilde{\rho}_t(\xi)| > 2$ for all $t \in (0, 1]$.
2. For $1 > t > \frac{1}{2} \left( 5 + 3\sqrt{3} - \sqrt{44 + 26\sqrt{3}} \right)$, we have $|\text{tr} \tilde{\rho}_t(\eta)| > 2$ and $\text{tr}^2 \rho_t(\eta) = 4$ at $t = \frac{1}{2} \left( 5 + 3\sqrt{3} - \sqrt{44 + 26\sqrt{3}} \right)$.

As a result of (1), the end invariant of $\hat{\Gamma}_{t_0}$ must be $\eta$. As a result of (2), we may compute explicitly the lower bound $t_0 = \frac{1}{2} \left( 5 + 3\sqrt{3} - \sqrt{44 + 26\sqrt{3}} \right)$.

(1) For all $t \in (0, 1]$,

\begin{align}
2 + \text{tr} \tilde{\rho}_t(\xi) &= \frac{-1 - 8t^2 - 21t^4 + 2t^6}{t^2(1 + t^2)^2} + 2 \\
&= \frac{-1 - 6t^2 - 17t^4 + 4t^6}{t^2(1 + t^2)^2} \\
&= \frac{-1 - 6t^2 - 13t^4 - 4t^4(1 - t^2)}{t^2(1 + t^2)^2} < 0.
\end{align}
(2) For all $t \in (t_0, 1]$,
\[
2 + tr \tilde{\rho}_t(\eta) = \frac{2(1 - 4t^2 - 74t^4 - 196t^6 + t^8)}{(1 + t^2)^4} + 2
\]
\[
= \frac{4(1 - 34t^4 - 96t^6 + t^8)}{(1 + t^2)^4}
\]
\[
= \frac{4 ((1 + t^2)^2 - 2t(1 + 5t^2)) ((1 + t^2)^2 + 2t(1 + 5t^2))}{(1 + t^2)^4}.
\]

One may check directly (using the explicit equation for the zeros of a quartic) that $(1 + t^2)^2 - 2t(1 + 5t^2)$ has two real zeros, the lesser of which is $t_0$. The larger zero, $\frac{1}{2} \left(5 + 3\sqrt{3} + \sqrt{44 + 26\sqrt{3}}\right)$, is greater than 1, and for $t = 1$, $(1 + t^2)^2 - 2t(1 + 5t^2)$ is negative. Thus $2 + tr \tilde{\rho}_t(\eta) < 0$ for all $t \in (t_0, 1]$, and $tr \rho_{t_0}(\eta) = 4$.

For $t \in (t_0, 1]$, recall the notation $[\Gamma_t] = Q(X_t, Z_t)$, so that in particular $\sigma_{\mathcal{M}}(X_t) = Z_t$. Denote the bottom surface of the convex core boundary of $\Gamma_t$ by $Y_t$. That is, recalling Lemma 6.4
\[
Y_t = (\ell_\xi |_{\mathcal{R}(\xi, \eta)})^{-1}(\ell(\xi, \Gamma_t)) .
\]

8. Non-injectivity of $t \mapsto Z_t$

In this section we use a description of $Z_t$ as a grafted surface to show the non-monotonicity of the function $Ext_\xi Z_t : (t_0, 1] \to \mathbb{R}^+$. Since $Z_t \in \mathcal{R}(\xi, \eta)$ by Lemma 5.2 and since $Ext_\xi : \mathcal{R}(\xi, \eta) \to \mathbb{R}^+$ is a diffeomorphism by Lemma 6.4 the non-monotonicity of $Ext_\xi Z_t$ implies the non-injectivity of $t \mapsto Z_t$, for $t \in (t_0, 1]$.

Recall that grafting is a map (see 17 and 25 for details)
\[
gr : \mathcal{ML}(\Sigma) \times \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma).
\]
Briefly, for $\gamma \in \mathcal{S}$, the Riemann surface $gr(\gamma \cdot \gamma, X)$ is obtained by cutting open $X$ along the geodesic representative for $\gamma$ and inserting a Euclidean annulus of height $\tau$.

For a geometrically finite hyperbolic 3-manifold $N$, with conformal boundary $X_\infty \in \mathcal{T}(\partial N)$, convex core boundary $X_{cc} \in \mathcal{T}(\partial N)$, and bending lamination $\lambda \in \mathcal{ML}(\partial N)$, grafting provides the description:
\[
X_\infty = gr(\lambda, X_{cc}).
\]

Employing Lemmas 6.1 and 7.2 we thus have
\[
Z_t = gr(\theta(t) \cdot \xi, Y_t).
\]

We note that there is also a projective version of grafting, $Gr$, that ‘covers’ conformal grafting 17. This provides the natural (quasi-Fuchsian) projective structure $Z_t = Gr(\theta(t) \cdot \xi, Y_t)$ which the surface $Z_t$ inherits as the boundary of $\mathbb{H}^3 / \Gamma_t$.

To each $X \in \mathcal{R}(\xi, \eta)$, recall the quadrilateral $Q_X$ (see 10). For brevity, let $Q_t := Q_{Z_t}$ (see Figure 12).

In Lemma 1.1 we present the natural projective structure on $Q_t$, given by the projective grafting description of the projective structure $Z_t$ on $Z_t$. Let $D(z, r)$ be the open disk in $\mathbb{C}$ centered at $z$ of radius $r$, let $E^c$ indicate the complement of a set $E \subset \mathbb{C}$, and let $L(t) := \frac{1}{4} \ell(\xi, \Gamma_t)$. 
Figure 12. The quadrilateral $Q_t \subset Z_t$ and the curve $\xi$.

Figure 13. A picture of the quasi-Fuchsian projective structure on $Q_t$. The horizontal sides are labelled $h_1$ and $h_2$.

Lemma 8.1. The projective structure on the quadrilateral $Q_t$ is given by the region (see Figure 13)

$$Q_t \cong D(c_1, r_1) \cap \left( \bigcap_{i=2}^{4} D(c_i, r_i)^c \right)$$
with ‘vertical’ sides given by the two concentric arcs, such that:

1. \((c_1, r_1) = (0, e^{L(t)})\),
2. \((c_2, r_2) = (0, 1)\),
3. \((c_3, r_3) = (e^{L(t)} \cosh(L(t)), e^{L(t)} \sinh(L(t)))\),
4. \((c_4, r_4) = (e^{(\pi + \theta(t))} \cosh(L(t)), \sinh(L(t)))\).

Proof. By \(8.1\), the projective structure induced by \(\Gamma_t\) on \(Z_t\) is given by the projective grafting of the uniformized hyperbolic surface \(Y_t\) along the measured lamination \(\theta(t) \cdot \xi\). We form the picture presented in Figure 13 by developing \(Y_t\) onto \(\mathbb{H}^2\) as follows: Choose an orientation for \(\xi\). Consider the intersection of the geodesic representative of \(\xi\) with \(Q_Y\). Develop this arc onto the imaginary axis of \(\mathbb{H}^2\) so that its endpoints lift to \(i\) and \(e^{L(t)}i\). In this case, the sides of \(Q_Y\) are contained in the boundaries of the disks \(D(0, 1), D(\cosh(L(t)), e^{L(t)} \sinh(L(t)))\), \(D(\cosh(L(t)), \sinh(L(t)))\), and \(D(0, e^{L(t)})\). With the chosen coordinate, the process of projective grafting along \(\theta(t) \cdot \xi\) inserts a wedge of angle \(\theta(t)\).

We now have a function, \(t \mapsto \text{Mod}(Q_t)\), for \(t \in (t_0, 1]\), whose non-monotonicity would suffice to show the non-injectivity of \(t \mapsto Z_t\). As a result of Lemma \(8.1\) it is possible to produce numerical estimates of the modulus that suggest non-monotonicity: One may use conformal maps to ‘open’ the two punctures to right angles and a carefully scaled logarithm to make the quadrilateral look ‘nearly rectangular’. These produce the rough estimates:

- \(\text{Mod}(Q_1) \approx 2.3\),
- \(\text{Mod}(Q_{52}) \approx 1.8\),
- \(\text{Mod}(Q_{39}) \approx 2.0\).

(Recall that \(t_0 = \frac{1}{2} \left( 5 + 3\sqrt{3} - \sqrt{44 + 26\sqrt{3}} \right)\), and note that \(.39 > t_0\).) Unfortunately, controlling the error in these approximations quickly becomes very delicate, so we pursue a different approach. In Lemma \(8.2\) we apply a normalizing transformation to the conformal structure given in Lemma \(8.1\) which allows a direct comparison of moduli of quadrilaterals.

While there remain some involved calculations, this method reduces the computational difficulties considerably. After Lemma \(8.2\) we have a reasonable list of verifications to check, involving integers and simple functions. We defer this list of verifications to \(8.2\) and to the Appendix, as they contribute no new ideas to the proof.

Choose a branch of the logarithm \(\arg(z) \in (0, 2\pi)\). Applying \(z \mapsto \log z\) takes \(Q_t\) into a vertical strip, with its vertical sides sent into a pair of vertical lines. Its horizontal sides can be described as the graphs of two explicit functions over the interval between the vertical sides. Following this map by \(z \mapsto \frac{1}{L(t)}(z - \frac{\pi + \theta(t)}{2}i)\) we arrive at \(R_t\), a certain conformal presentation of \(Q_t\) (see Figure 14). The proof of the following lemma is a straightforward calculation.

Lemma 8.2. The image under the map \(z \mapsto \frac{1}{L(t)}(\log z - \frac{\pi + \theta(t)}{2}i)\) of \(Q_t\) is given by \(R_t\) (see Figure 14). The quadrilateral \(R_t\) has vertical sides in the vertical lines \(\{\text{Re } z = 0\}, \{\text{Re } z = 1\}\), and horizontal sides given as the graphs of the functions \(x \mapsto F(x, t)\) and \(x \mapsto -F(1 - x, t)\) over \(x \in [0, 1]\). The function \(F(x, t)\) is given by

\[
F(x, t) = \alpha(t) - \beta(x, t)
\]
Figure 14. The normalizing transformation $Q_t \rightarrow R_t$.

where $\alpha$ and $\beta$ are given by

\begin{align}
\alpha(t) &= \frac{\pi + \theta(t)}{2L(t)}, \\
\beta(x, t) &= \frac{1}{L(t)} \cos^{-1} \left( \frac{\cosh (xL(t))}{\cosh (L(t))} \right).
\end{align}

The useful aspect of Lemma 8.2 is that $\text{Mod}(Q_t)$ now depends only on the graph of the function $x \mapsto F(x, t)$ over $x \in [0, 1]$, which forms a kind of ‘profile’ for the quadrilateral $Q_t$. There is some geometric intuition to the two terms $\alpha(t)$ and $\beta(x, t)$ of this profile function. Note that $F(x, t)$ depends on $x$ only through $\beta$, as $\alpha$ is independent of $x$. Let $\tilde{Z}_t$ denote the cover of $Z_t$ corresponding to the subgroup $\langle \xi \rangle < \pi_1(\Sigma)$. Since the maximal modulus annulus with core curve homotopic to $\xi$ can be lifted to $\tilde{Z}_t$, one obtains an estimate relating $\text{Ext}_\xi \tilde{Z}_t$ and $\text{Ext}_\xi Z_t$ (cf. [25, p. 21]). The reader may interpret the $\alpha(t)$ term in $F(x, t)$ as determining $\text{Ext}_\xi \tilde{Z}_t$, while the $\beta(x, t)$ term determines the error in this estimate.

Precisely, one has

$$\text{Ext}_\xi \tilde{Z}_t = \frac{4L(t)}{\pi + \theta(t)} = \frac{2}{\alpha(t)}$$

and

$$\text{Ext}_\xi Z_t \leq \text{Ext}_\xi \tilde{Z}_t.$$ 

In other words,

\begin{equation}
\text{Mod}(Q_t) \leq \frac{1}{2\alpha(t)}.
\end{equation}

Since $\frac{1}{2\alpha(t)}$ is the modulus of the rectangle sharing its vertical sides with $Q_t$ and its horizontal sides contained in the lines $\{\text{Im } z = \alpha(t)\}$ and $\{\text{Im } z = -\alpha(t)\}$, $\beta(x, t)$ accounts for the discrepancy in inequality (8.5).
Some computational details, deferred to §9, provide the following:

**Lemma 8.3.** We have the containment of quadrilaterals $R_1 \subset R_{1,2}$, with the vertical sides of $R_1$ contained in the vertical sides of $R_{1,2}$.

**Lemma 8.4.** We have the containment of quadrilaterals $R_{2,5} \subset R_{1,2}$, with the vertical sides of $R_{2,5}$ contained in the vertical sides of $R_{1,2}$.

These lemmas directly imply non-monotonicity of $t \mapsto \text{Mod}(Q_t)$.

**Proposition 8.5.** The function $t \mapsto \text{Mod}(Q_t)$ is not monotone on the interval $(t_0, 1]$.

**Proof.** Note that $t_0 < \frac{2}{5}$, by Lemma 11.1. By Lemmas 8.3 and 8.4 (see Figure 15), we have

\[
\text{Mod}(Q_1) \leq \text{Mod}(Q_{1/2}),
\]
\[
\text{Mod}(Q_{2/5}) \leq \text{Mod}(Q_{1/2}).
\]

Figure 15. The non-monotonicity of $\text{Mod}(Q_t)$. The quadrilateral $R_t$ is pictured rotated so that its vertical sides appear horizontal.

The non-monotonicity of Proposition 8.5 implies the existence of a critical point for the skinning map that is the subject of Theorem 1.1, so we are now ready to give the proof.

**Proof of Theorem 1.1.** Composing $\sigma_M$ with a reversal of orientation, one obtains the anti-holomorphic map $\overline{\sigma_M} : \mathbb{H} \to \mathbb{H}$. This map, by equation (8.1), sends the real 1-dimensional submanifold $\mathcal{R} := \mathcal{R}_{[\xi, \eta]}$ to itself. By Lemmas 6.4 and 8.5, $(\text{Ext}_\xi \circ \overline{\sigma_M})_{|\mathcal{R}}$ is a non-monotonic continuously differentiable function, and thus $(\text{Ext}_\xi \circ \overline{\sigma_M})_{|\mathcal{R}}$ has a critical point. By Lemma 6.4, $\text{Ext}_\xi|_{\mathcal{R}}$ is a diffeomorphism, and thus $\sigma_M$ has a critical point.

9. Computational lemmas

The goal of this section is to prove Lemmas 8.3 and 8.4 which together imply non-monotonicity of $\text{Mod}(Q_t)$ (see Proposition 8.5). Since the vertical sides of $R_t$ are contained in the lines $\{\text{Re } z = 0\}$ and $\{\text{Re } z = 1\}$, and the horizontal sides are graphs of functions over $x \in [0, 1]$, the containments in these lemmas are a consequence of inequalities involving the functions $x \mapsto F(x, t)$. In particular, we
seek uniform estimates such as $F(x, t_1) < F(x, t_2)$, for $t_1, t_2 \in (t_0, 1]$, and for all $x \in [0, 1]$. Essentially, it is the non-monotonicity of $\alpha(t)$ that accounts for such uniform estimates. However, while $\alpha$ is independent of $x$, $F$ is not, and in fact we may have $\alpha(t_1) > \alpha(t_2)$ while $F(x_0, t_1) < F(x_0, t_2)$, for some $x_0 \in [0, 1]$. We must therefore take more care in estimates on $F(x, t_1) - F(x, t_2)$ by accounting for the effect of $\beta$.

While $\beta$ does depend on $x$, we can obtain control over $\max_{x \in [0, 1]} |\beta(x, t_1) - \beta(x, t_2)|$. This allows comparisons between $F(x, t_1)$ and $F(x, t_2)$ over all $x$. Some computational pieces that are easily checked with a computer are collected in the Appendix, where a Mathematica notebook containing a demonstration of these verifications is also indicated.

For ease in presentation, we introduce some notation for functions that will be used throughout this section:

(9.1) $p_1(t) = 1 + 8t^2 + 21t^4 - 2t^6,$
(9.2) $p_2(t) = 2t^2(1 + t^2)^2,$
(9.3) $p_3(t) = 4t^5(3 - t^2).$

In terms of these polynomials, equations (7.2–7.3) become:

(9.4) $\theta(t) = \cos^{-1}\left(\frac{p_3(t)}{p_2(t)}\right),$
(9.5) $L(t) = \frac{1}{2} \cosh^{-1}\left(\frac{p_1(t)}{p_2(t)}\right).$

And recall (8.3–8.4):

(9.6) $\alpha(t) = \frac{\pi + \theta(t)}{2L(t)},$
(9.7) $\beta(x, t) = \frac{1}{L(t)} \cos^{-1}\left(\frac{\cosh(xL(t))}{\cosh(L(t))}\right).$

The interested reader can refer to equations (9.1–9.7) to check the computations below. The computational strategy is to first turn inequalities involving the $\cos^{-1}$ and $\cosh^{-1}$ functions into inequalities involving logarithms and constants consisting of algebraic numbers and $\pi$. These statements are then reduced to explicit inequalities involving large powers of algebraic numbers, which can be checked with a computer. Indeed, for the patient reader, they could all be checked by hand.

**Lemma 9.1.** We have $\alpha\left(\frac{1}{2}\right) > \alpha(1)$.

**Proof.** Calculating $\alpha(1)$ and $\alpha\left(\frac{1}{2}\right)$ and estimating $\cos^{-1}\left(\frac{11}{25}\right)$ using Lemma 11.2, we have

$$\alpha(1) = \frac{\pi}{\cosh^{-1}\left(\frac{7}{2}\right)},$$
$$\alpha\left(\frac{1}{2}\right) = \frac{\pi + \cos^{-1}\left(\frac{11}{25}\right)}{\cosh^{-1}\left(\frac{137}{25}\right)} > \frac{27\pi}{20} \cosh^{-1}\left(\frac{137}{25}\right).$$
Now referring to Lemma 11.6
\[
\left( \frac{7 + 3\sqrt{5}}{2} \right)^{27} > \left( \frac{137 + 36\sqrt{14}}{25} \right)^{20}.
\]
Taking logarithms and rearranging, we find
\[
\frac{27}{20} \log \left( \frac{137 + 36\sqrt{14}}{25} \right) > \frac{1}{20} \log \left( \frac{7 + 3\sqrt{5}}{2} \right).
\]
Since, for \( x \geq 1 \), \( \cosh^{-1} x = \log \left( x + \sqrt{x^2 - 1} \right) \), we have
\[
\frac{27}{20} \cosh^{-1} \left( \frac{137}{25} \right) > \frac{1}{20} \cosh^{-1} \left( \frac{7}{2} \right).
\]
This implies \( \alpha \left( \frac{1}{2} \right) > \alpha (1) \), as desired. \( \square \)

The proof of Lemma 9.2 proceeds along similar lines, so we suppress commentary.

**Lemma 9.2.** We have \( \alpha \left( \frac{1}{2} \right) - \alpha \left( \frac{2}{5} \right) > \beta (0, \frac{1}{2}) - \beta (0, \frac{2}{5}). \)

**Proof.**
\[
\alpha \left( \frac{1}{2} \right) - \alpha \left( \frac{2}{5} \right) > \frac{27\pi}{20} \cosh^{-1} \left( \frac{137}{25} \right) - \cosh^{-1} \left( \frac{137}{25} \right) \quad \text{(Lemmas 11.2 and 11.3),}
\]
\[
\beta \left( 0, \frac{1}{2} \right) - \beta \left( 0, \frac{2}{5} \right) = \frac{2 \cosh^{-1} \left( \frac{5}{2} \right) - 2 \cosh^{-1} \left( \frac{3}{5} \right)}{\cosh^{-1} \left( \frac{137}{25} \right) - \cosh^{-1} \left( \frac{137}{25} \right) \quad \text{(Lemmas 11.4 and 11.5),}
\]
\[
\left( \frac{137 + 36\sqrt{14}}{25} \right)^{46} < \left( \frac{137 + 36\sqrt{14}}{25} \right)^{43} \quad \text{(Lemma 11.7)}
\]
\[
\implies 46 = \frac{27\pi}{20} - \frac{17\pi}{12} < \frac{27\pi}{20} - \frac{17\pi}{12}. \quad \text{(Lemmas 11.1, 11.2, and 11.5).}
\]
When \( t^2 > t^4 \) for \( t \in (0, 1) \), it is clear that \( p_1(t) > p_2(t) > 0 \). As \( \cosh^{-1} \) is monotone increasing on \( (1, \infty) \), the lemma will follow from checking that \( \frac{p_1(t)}{p_2(t)} \) is monotone.

**Lemma 9.3.** \( L \) is monotone decreasing on \( (0, 1] \).

**Proof.** Recall that \( L(t) = \frac{1}{2} \cosh^{-1} \left( \frac{p_1(t)}{p_2(t)} \right) \) (see equations (9.1), (9.2), and (9.5)).
decreasing on $(0, 1)$. We have
\[
\frac{d}{dt} \left( \frac{p_1(t)}{p_2(t)} \right) = \frac{-1}{t^3(1 + t^2)^3} (1 + 5t^2)(1 - 2t^2 + 5t^4).
\]
As $t^4 - t^2 \geq -\frac{1}{4}$, it is straightforward to check that $\frac{d}{dt} \left( \frac{p_1(t)}{p_2(t)} \right) < 0$ for $t > 0$. □

**Lemma 9.4.** For $t_1, t_2 \in (0, 1)$ such that $t_2 > t_1$ and for all $x \in (0, 1)$, we have
\[
0 < \beta(x, t_2) - \beta(x, t_1) < \beta(0, t_2) - \beta(0, t_1).
\]

**Proof.** We compute
\[
\frac{\partial^2 \beta}{\partial x \partial t}(x, t) = \left[ \frac{-L'(t) \sinh(2L(t)) \cosh(xL(t))}{2 (\cosh^2 L(t) - \cosh^2(xL(t)))^{\frac{3}{2}}} \right] \cdot [x \tanh L(t) - \tanh(xL(t))].
\]
By Lemma 9.3, $L'(t) < 0$ for all $t \in (0, 1)$ and the first bracketed term is positive. The function $x^{-1} \tanh x$ is decreasing for $x > 0$, so for $x \in (0, 1)$ we have
\[
\frac{\tanh(xL(t))}{xL(t)} > \frac{\tanh L(t)}{L(t)},
\]
and the second bracketed term is negative. We conclude that $\frac{\partial^2 \beta}{\partial x \partial t}(x, t) < 0$ for all $t, x \in (0, 1)$.

Thus on the domain $(0, 1) \times (0, 1)$, we have:

(i) For fixed $x$, the function $\frac{\partial \beta}{\partial t}(x, t)$ is decreasing in $t$.

(ii) For fixed $t$, the function $\frac{\partial \beta}{\partial t}(x, t)$ is decreasing in $x$.

A straightforward computation shows that
\[
\lim_{x \to 1} \frac{\partial \beta}{\partial t}(x, t) = 0.
\]
By (ii), this limit is the infimum of $\frac{\partial \beta}{\partial t}(x, t)$ over $x \in (0, 1)$. Thus $\frac{\partial \beta}{\partial t}(x, t) > 0$ and $\beta(x, t)$ is increasing in $t$.

Suppose that $t_2 > t_1$. By (i),
\[
\frac{\partial}{\partial x} (\beta(x, t_2) - \beta(x, t_1)) < 0
\]
for all $x \in (0, 1)$. We now have that $\beta(x, t_2) - \beta(x, t_1)$ is positive and monotone decreasing in $x$. The result follows. □

**Proof of Lemma 8.3.** For any $x \in (0, 1)$, combining Lemmas 9.1 and 9.4
\[
F \left( x, \frac{1}{2} \right) - F(x, 1) = \left( \alpha \left( \frac{1}{2} \right) - \alpha(1) \right) + \left( \beta(x, 1) - \beta \left( x, \frac{1}{2} \right) \right)
\]
\[
> \left( \alpha \left( \frac{1}{2} \right) - \alpha(1) \right) > 0.
\]
Referring to the description of $R_t$ in Lemma 8.2, this inequality implies the containment $R_1 \subset R_\frac{1}{2}$ with containment of vertical sides as desired. □
Proof of Lemma 8.4. For any \( x \in (0, 1) \), combining Lemmas 9.2 and 9.4,

\[
F\left(x, \frac{1}{2}\right) - F\left(x, \frac{2}{5}\right) = \left(\alpha\left(\frac{1}{2}\right) - \alpha\left(\frac{2}{5}\right)\right) - \left(\beta\left(x, \frac{1}{2}\right) - \beta\left(x, \frac{2}{5}\right)\right) > \alpha\left(\frac{1}{2}\right) - \alpha\left(\frac{2}{5}\right) - \beta\left(0, \frac{1}{2}\right) - \beta\left(0, \frac{2}{5}\right).
\]

The containment \( R_2^2 \subset R_1^2 \) follows as above.

\[\square\]

10. A FAMILY OF FINITE COVERS OF \( M \)

Proof of Corollary 1.2. Recall that \( H_g \) indicates the closed genus \( g \) handlebody, and recall the identification \( \pi_1 H_2 \cong \langle A, B \rangle \) as in Figure 2. For each genus \( g \geq 2 \), consider the cyclic cover \( H_g \rightarrow H_2 \) corresponding to the subgroup \( \langle A, B^{g-1} \rangle \). The family \( \{(H_n, P_n)\}_{n=2}^{\infty} \) is generated from this family of covers, as we now describe.

The annuli in \( P \) lift to annuli \( P_g \subset \Sigma_g \). Denote \( \Sigma' := \Sigma_g \setminus P_g \) (and recall \( \Sigma = \Sigma_2 \setminus P \)). We choose a fundamental domain for the covering \( \Sigma' \rightarrow \Sigma \), as pictured in Figure 16, and denote it \( F \). Precisely, choose a two-holed torus fundamental domain for \( \Sigma_g \rightarrow \Sigma_2 \), include only one of the two boundary components, and take the complement of \( P_g \). Because \( F \) is connected and \( \Sigma' \) is the union of finitely many copies of \( F \), it is clear that the cover \( \Sigma' \) is connected.

\[\text{Figure 16. The fundamental domain } F \text{ for the covering } \Sigma' \rightarrow \Sigma.\]

\[\text{Figure 17. The cover for } g = 4, \text{ the pared 3-manifold } (H_4, P_4). \text{ Note that } P_4 \text{ has two connected components.}\]

The intersection of \( P_g \) with \( F \) consists of three connected components. The gluing pattern of \( g-1 \) copies of \( F \), in order to build \( \Sigma' \), matches these components in a straightforward fashion. Upon inspection it is clear that \( P_{2g} \) has two components and \( P_{2g+1} \) has three. (See Figure 17 for a picture of \( P_4 \).

We consider the pared 3-manifolds \( M_n := (H_n, P_n) \). The boundary \( \partial M_{2g} \) is a genus \( 2g \) surface, with two non-separating disk-busting annuli deleted, and is thus homeomorphic to \( \Sigma_{2g-2,4} \). Similarly, \( \partial M_{2g+1} \cong \Sigma_{2g-2,6} \). Applying Proposition 3.1, the corollary follows.

\[\square\]
11. Appendix

Lemmas 11.1 through 11.7 as they can clearly be verified by a finite list of computations involving integers and standard functions, are available in a Mathematica notebook.[2]

Lemma 11.1. \( \frac{1}{2} \left( 5 + 3\sqrt{3} - \sqrt{44 + 26\sqrt{3}} \right) < \frac{2}{3} \).

Lemma 11.2. \( \frac{11}{25} < \cos \left( \frac{7\pi}{20} \right) \).

Lemma 11.3. \( \frac{1136}{4205} > \cos \left( \frac{5\pi}{12} \right) \).

Lemma 11.4. \( \frac{5}{9} > \cos \left( \frac{19\pi}{60} \right) \).

Lemma 11.5. \( \frac{116}{225} < \cos \left( \frac{13\pi}{40} \right) \).

Lemma 11.6. \( 25^{20} (7 + 3\sqrt{5})^{27} > 2^{27} (137 + 36\sqrt{14})^{20} \).

Lemma 11.7. \( 6728^{43} (137 + 36\sqrt{14})^{46} < 25^{46} (43897 + 225\sqrt{37169})^{43} \).

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