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A remark on Sarnak’s conjecture

Régis de la Bretèche & Gérard Tenenbaum

Abstract. We investigate Sarnak’s conjecture on the Möbius function in the special case
when the test function is the indicator of the set of integers for which a real additive function
assumes a given value.

Keywords: Sarnak’s conjecture, Möbius function, complexity, additive functions, concentration of additive functions, Halász mean value theorem, mean values of multiplicative functions.

1. Introduction and statements of results

According to a general pseudo-randomness principle related to a famous conjecture of
Chowla [1] and recently considered by Sarnak [7], the Möbius function \( \mu \) does not correlate
with any function \( \varphi \) of low complexity. In other words,

\[
\sum_{n \leq x} \mu(n) \xi(n) = o \left( \sum_{n \leq x} |\xi(n)| \right) \quad (x \to \infty).
\]

There are many ways of constructing functions of low complexity. Sarnak and others use
return times of sampling sequences of a dynamical system, which leads to a natural measure
of the complexity. Here we propose to follow another path by selecting the test-function
as the indicator of the set of those integers where a real additive function assumes a given
value. It is known since Halász [5] that

\[
Q(x; f) := \sup_{m \in \mathbb{R}} \sum_{n \leq x \atop f(n) = m} 1 \ll \frac{x}{\sqrt{1 + E(x)}}
\]

where we have put

\[
E(x) := \sum_{p \leq x \atop f(p) \neq 0} \frac{1}{p}.
\]

Here and in the sequel, the letter \( p \) denotes a prime number.

The estimate (1-2) is known to be optimal in this generality since the two sides achieve
the same order of magnitude when \( f(n) \) is equal to the total number of prime factors of \( n \),
counted with or without multiplicity.

As a first investigation of the above described problem, we would like to show that

\[
Q(x; f, \mu) := \sup_{m \in \mathbb{R}} \left| \sum_{n \leq x \atop f(n) = m} \mu(n) \right|
\]

is generically smaller than the right-hand side of (1-2). Of course we have to avoid the case
when \( f(p) \) is constant, for then \( \mu(n) \) does not oscillate on the set of squarefree integers \( n \)
with \( f(n) = m \). Therefore we seek an estimate which coincides with (1-2) when \( f(p) \) is close
to a constant and which has smaller order of magnitude otherwise.

When \( f(p) \) is restricted to assume the values 0 or 1 only, we thus expect a significant
improvement over (1-2) when

\[
F(x) := \sum_{p \leq x} \frac{1 - f(p)}{p}
\]

is large. Indeed, in this simple case we obtain the following estimate.
Theorem 1.1. Let $f$ denote a real additive arithmetic function such that $f(p) \in \{0,1\}$ for all $p$. Then, with the above notation and $c = (2\pi - 4)/(3\pi - 2) \approx 0.30751$, we have

\begin{equation}
Q(x; f, \mu) \ll \frac{x(1 + F(x)) e^{-F(x)}}{\sqrt{1 + E(x)}}.
\end{equation}

For simplicity, let us retain in the sequel the hypothesis $f(p) \in \{0,1\}$. (1) Under the assumption that $F(x)$, as defined in (1.3) above, grows sufficiently slowly, we may prove an estimate that is valid for each $m$ in a large range around the mean, and so may be stated in the exact frame of Sarnak’s conjecture.

Moreover, Halász announced (see [2], p. 312) the possibility to obtain, in the same range for $p$, an asymptotic formula for $N_m(x; f)$, a result which actually follows, as shown in [10], from a general effective mean value estimate for multiplicative functions established in the same work—see below.

This supports the hope to obtain an asymptotic formula for

\[ N_m(x; f, \mu) := \sum_{n \leq x \atop f(n) = m} \mu(n) \]

which directly compares to (1.5). In view of (1.1), we may assume with no loss of generality that $f$ is strongly additive. We obtain the following result. Here and in the sequel we let $\log_k$ denote the $k$-fold iterated logarithm.

Theorem 1.2. Let $\kappa \in [0,1]$ and let $f$ denote a strongly additive function such that $f(p) \in \{0,1\}$ for all primes $p$. Assume furthermore that

\begin{equation}
F(x) := \sum_{p \leq x} \frac{1 - f(p)}{p} \ll \log_3 x \quad (x \to \infty)
\end{equation}

\begin{equation}
\sum_{\exp((\log x)/\log_2 x)^D < y \leq x} \frac{\{1 - f(p)\} \log p}{p} \ll \frac{(\log y)}{(\log_2 x)^{c_0}} \quad (x^{1/(\log_2 x)^D} < y \leq x)
\end{equation}

where $D$ and $c_0$ are positive constants. Provided $D$ is sufficiently large and uniformly in the range $\kappa E(x) \leq m \leq E(x)/\kappa$, we have

\begin{equation}
N_m(x; f, \mu) = (-1)^m N_m(x; f) \left( \lambda_f e^{-2F(x)} + O \left( \frac{1}{(\log_2 x)^6} \right) \right),
\end{equation}

with

\begin{equation}
\lambda_f := \prod_{f(p) = 0} \frac{1 - 1/p}{1 + 1/p} e^{2/p}, \quad b := \frac{1}{2} \min\{1, c_0\kappa/(4 - \kappa)\}.
\end{equation}

1. All our results could be straightforwardly adapted to case when $f(p)$ is restricted to a fixed, finite set, or even to a set of moderate size depending on $x$. 


To fix ideas, note that a strongly additive function $f$ such that $f(p) \in \{0, 1\}$ satisfies hypotheses (1-6) and (1-7) as soon as

$$\sum_{p \leq y} \{1 - f(p)\} \log p \ll \frac{y}{(\log y)^{\max(1, c_n)}}.$$ 

The proof of Theorem 1.2 rests on the following recent result of the second author [10] (theorem 1.4), for the statement of which we introduce further notation. We let $M(A, B)$ designate the class of those complex-valued multiplicative functions $g$ such that

(1-10) $$\max_p |g(p)| \leq A, \quad \sum_{p, \nu \geq 2} \frac{|g(p^\nu)| \log p^\nu}{p^\nu} \leq B,$$

and, for $b \in \mathbb{R}$, we write

(1-11) $$\beta_0 = \beta_0(b, A) := 1 - \frac{\sin(2\pi b/A)}{2\pi b/A}.$$ 

Moreover, given a complex-valued function $g$, we put $w_g := 1$ if $g$ is real, $w_g := \frac{1}{2}$ otherwise, and write

$$M(x; g) := \sum_{n \leq x} g(n), \quad Z(x, g) := \sum_{p \leq x} \frac{g(p)}{p}.$$ 

**Theorem 1.3** ([10]). Let

$$a \in [0, \frac{1}{4}], \quad b \in [0, \frac{1}{2}], \quad b := (1 - b)/a, \quad A \geq 2b, \quad B > 0, \quad \beta := \beta_0(b, A),$$

and let the multiplicative functions $g$, $r$, such that $r \in M(x; 2A, B)$, $|g| \leq r$, satisfy the conditions

(1-12) $$\sum_{p \leq x} \frac{r(p) - \Re g(p)}{p} \leq \frac{1}{2} \beta b \log(1/\varepsilon),$$

(1-13) $$\sum_{x^\varepsilon < p \leq y} \frac{(r(p) - \Re g(p))^b \log p}{p} \ll \varepsilon^b \log y \quad (x^\varepsilon < y \leq x),$$

with $\delta \in [a, \frac{1}{2} \beta b]$, and

(1-14) $$\min_{x^\varepsilon < p \leq x} r(p) \geq 4b.$$ 

We then have

(1-15) $$M(x; g) = M(x; r) \prod_p \sum_{p^\nu \leq x} \frac{g(p^\nu)}{p^\nu}^{\nu} \prod_{p^\nu \leq x} \frac{r(p^\nu)}{p^\nu}^{\nu} + O\left(\frac{x^{\varepsilon^a \delta \mu Z(x; r) - \varepsilon Z(x; |g| - g)}}{\log x}\right)$$

where $\varepsilon := b/A$. The implicit constant in (1-15) depends at most upon $A$, $B$, $a$, and $b$.

**2. Proof of Theorem 1.1**

As noted by Halász [5], we may assume that $f$ is integer-valued. (Note, however, that a slight modification of his construction is needed to ensure that changing the range of $f$ does not create new coincidences.) With this reduction, we plainly have

$$Q(x; f, \mu) \leq \int_{-1/2}^{1/2} |M(x; \vartheta)| d\vartheta$$

with

$$M(x; \vartheta) := \sum_{n \leq x} \mu(n)e^{2\pi i \vartheta f(n)}.$$
From Corollary III.4.12 in [8], we get, uniformly for $\vartheta \in \mathbb{R}$, $T \geq 1$, $x \geq 1$,

\begin{equation}
M(x; \vartheta) \ll \frac{x \{1 + m(x; \vartheta, T)\}}{e^{m(x; \vartheta, T)}} + \frac{x}{T},
\end{equation}

where we have put

\[ m(x; \vartheta, T) := \min_{|\tau| \leq T} \sum_{p \leq x} \frac{1 + \cos(2\pi \vartheta f(p) - \tau \log p)}{p}. \]

We select $T := \log x$, so that the second term on the right of (2.1) is negligible compared to the upper bound in (1.4). Let $h_{\vartheta}$ defined by

\[ h_{\vartheta}(t) := 1 + \min\{\cos(t), \cos(2\pi \vartheta - t)\} \quad (t \in \mathbb{R}), \]

so that

\[ s_{\vartheta} := \frac{1}{2\pi} \int_{-\pi}^{\pi} h_{\vartheta}(t) \, dt = 1 - \frac{2}{\pi} \sin(\pi \vartheta) \quad (\vartheta \in [-\frac{1}{2}, \frac{1}{2}]), \]

and, for suitable $\tau \in [-T, T]$,

\[ m(x; \vartheta, T) \geq \sum_{p \leq x} \frac{h_{\vartheta}(\tau \log p)}{p}. \]

The right-hand side may be estimated via partial summation as made explicit in lemma III.4.13 of [8]. For any $w \in [2, x]$ and $\vartheta \in [-\frac{1}{2}, \frac{1}{2}]$, we have

\begin{equation}
\sum_{w < p \leq x} \frac{h_{\vartheta}(\tau \log p)}{p} = s_{\vartheta} \log \left( \frac{\log x}{\log w} \right) + O\left( \frac{1}{w \log x} + \frac{1 + |\tau|}{e^{\sqrt{\log w}}} \right).
\end{equation}

If $1 \leq |\tau| \leq T$, we select $w := (\log_2 x)^2$ to obtain

\[ m(x; \vartheta, T) \geq s_{\vartheta} \log_2 x + O(\log_3 x). \]

Next, set

\[ \log v := (\log x) \exp \left\{ -\frac{2 \cos^2(\pi \vartheta) E(x) + 2F(x)}{2 + s_{\vartheta}} \right\}. \]

If $1/\log v < |\tau| \leq 1$, we put $w := v$ in (2.2) and get

\[ \sum_{w < p \leq x} \frac{h_{\vartheta}(\tau \log p)}{p} \geq \frac{2s_{\vartheta} \cos^2(\pi \vartheta)}{2 + s_{\vartheta}} E(x) + \frac{2s_{\vartheta}}{2 + s_{\vartheta}} F(x) + O(1). \]

And finally, if $|\tau| \leq 1/\log v$, we have trivially

\[ \sum_{w < p \leq x} \frac{1 + \cos(2\pi \vartheta f(p) - \tau \log p)}{p} = \sum_{w < p \leq v} \frac{1 + \cos(2\pi \vartheta f(p))}{p} + O(1) \]

\[ = (1 + \cos(2\pi \vartheta)) \sum_{f(p) = 1} \frac{1}{p} + 2 \sum_{f(p) = 0} \frac{1}{p} + O(1) \]

\[ \geq 2 \cos^2(\pi \vartheta) E(x) + 2F(x) - 2 \log \left( \frac{\log x}{\log v} \right) + O(1) \]

\[ \geq \frac{2s_{\vartheta} \cos^2(\pi \vartheta)}{2 + s_{\vartheta}} E(x) + \frac{2s_{\vartheta}}{2 + s_{\vartheta}} F(x) + O(1). \]

Therefore, we get in all cases

\begin{equation}
(2.3) \quad m(x; \vartheta, T) \geq \frac{2s_{\vartheta} \cos^2(\pi \vartheta)}{2 + s_{\vartheta}} E(x) + \frac{2s_{\vartheta}}{2 + s_{\vartheta}} F(x) + O(1)
\end{equation}

\[ \geq c \cos^2(\pi \vartheta) E(x) + cF(x) + O(1). \]

Integrating over $\vartheta$ immediately yields the result stated. \qed
3. Proof of Theorem 1.2

Let us introduce the multiplicative function \( g(n) := \mu(n)z^{f(n)} \) with \( z := -\phi e^{2\pi i \theta}, |\theta| \leq \frac{1}{2}, \kappa \leq \phi \leq 1/\kappa. \) Put \( r(n) := \mu(n)^2 \phi^{f(n)}. \) From (2.3), we see that, with \( c \) as in the statement of Theorem 1.1,

\[
\sum_{p \leq x} \frac{r(p) - \Re (g(p)/p^\tau)}{p} \geq ce^\sin^2(\pi \theta) E(x) + c\phi F(x) + O(1) \quad (|\tau| \leq T := \log x).
\]

We may therefore apply Corollary 2.1 of [10] to get

\[
M(x; g) \ll M(x; r) e^{-c_0 h/2} (\log x)^{-c_1 h/2} e^{-c_1 \phi^2 E(x) - c_1 F(x)},
\]

where \( h \) is a large constant—actually any \( h > 1/\sqrt{4\kappa c} \) will do. We have

\[
\sum_{p \leq x} \frac{r(p) - \Re g(p)}{p} = \phi(1 - \cos 2\pi \theta) E(x) + 2\phi F(x) \leq 2\phi \pi^2 \theta^2 + 2\phi F(x),
\]

hence condition (1.12) is plainly fulfilled with \( \varepsilon := |\theta|^{2/\delta} + (\log x)^{-c_0/(h\delta)} \) provided \( \delta \) is chosen sufficiently small in terms of \( b, \kappa \) and \( K \). Next, for \( x^\varepsilon < y \leq x \), we have

\[
\sum_{x^\varepsilon < p \leq y} \frac{(r(p) - \Re g(p)) h \log p}{p} \ll \phi^{2h} \log y + \sum_{x^\varepsilon < p \leq y} \frac{\log p}{p} = \phi^{2h} \log y \ll \kappa \left\{ \varepsilon^{2h} + (\log x)^{-c_0} \right\} \log y,
\]

so hypothesis (1.13) is also verified. Since (1.14) holds trivially on selecting \( b := \kappa/4 \), and hence \( b = 4/\kappa - 1 \), we conclude that (1.15) is valid. We obtain, with \( \varepsilon := \kappa b \),

\[
M(x; g) = M(x; r) \prod_{p \leq x} \frac{1 - z/p}{1 + e^{z/p}} \prod_{f(p) = 0} \frac{1 - 1/p}{1 + 1/p} + O\left( \frac{x e^{\delta/2} e^{2(x; r-g)}}{\log x} \right).
\]

Now, appealing for instance to theorem 1.1 of [9], we observe that

\[
M(x; r) \asymp \frac{x e^{2(x; r)}}{\log x}
\]

and so we may rewrite (3.2) as

\[
M(x; g) = M(x; r) \left\{ \lambda_1 e^{-(z+\varepsilon) E(x) - 2F(x)} + O\left( (|\theta| + (\log x)^{-c_0 h/2}) e^{-(z+\varepsilon) E(x) - e_1 F(x)} \right) \right\},
\]
valid for $|\theta| \leq \vartheta_0$ and some constant $c_1 > 0$. Integrating on the circle $|z| = \varrho := m/E(x)$ and taking (3.1) into account, we readily obtain in the stated range for $m$,

$$N_m(x; f, \mu) = (-1)^m \int_{-1/2}^{1/2} e^{-2i\pi \varrho \theta} \varrho^{-m} M(x; g) \, d\theta$$

(3.3)

$$= (-1)^m \lambda_f M(x; r) \frac{E(x)^m}{m! e^m} \left\{ e^{-2F(x)} + O\left( \frac{e^{-c_2 F(x)}}{(\log_2 x)^b} \right) \right\},$$

with $c_2 := \min(c_1, c\kappa)$. Since, by a straightforward variant of corollary 2.4 of [10],

$$N_m(x; f) = M(x; r) \frac{E(x)^m}{m! e^m} \left\{ 1 + O\left( \frac{1}{\sqrt{\log_2 x}} \right) \right\},$$

we reach the required conclusion.

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2. Applied to $\omega(n; E)$ instead of $\Omega(n; E)$ with the notation of [10].