ABSTRACT. Curved algebras are a generalization of differential graded algebras which have found numerous applications recently. The goal of this foundational article is to introduce the notion of a curved operad, and to develop the operadic calculus at this new level. The algebraic side of the curved operadic calculus provides us with universal constructions: using a new notion of curved operadic bimodules, we construct curved universal enveloping algebras. Since there is no notion of quasi-isomorphism in the curved context, we develop the homotopy theory of curved operads using new methods. This approach leads us to introduce the new notion of a curved absolute operad, which is the notion Koszul dual to counital cooperads non-necessarily conilpotent, and we construct a complete Bar-Cobar adjunction between them. We endow curved absolute operads with a suitable model category structure. We establish a duality square of duality functors which intertwines this complete Bar-Cobar construction with the Bar-Cobar adjunction between unital operads and conilpotent curved cooperads. This allows us to compute minimal cofibrant resolutions for various curved absolute operads. Using the complete Bar construction, we show a general Homotopy Transfer Theorem for curved algebras. Along the way, we construct the non-necessarily conilpotent cofree cooperad.

CONTENTS

Introduction 1
1. Recollections on different types of (co)operads 6
2. Curved operads and their curved algebras 14
3. Curved bimodules and universal functors 18
4. Curved cooperads and curved partial cooperads 24
5. The groupoid-colored level 26
6. Curved twisting morphisms and Bar-Cobar adjunctions at the operadic level 39
7. Counital partial cooperads up to homotopy and transfer of model structures 50
8. Duality functors and Koszul duality 55
9. Application: Homotopy transfer theorem for curved algebras 61
Appendix A: What is an absolute partial operad ? 64
References 72

INTRODUCTION

Global picture. The theory of operads shifts the point of view of universal algebra: instead of working by hand with specific types of algebras, one works with the operads that encode them. Examples of algebraic structures encoded by operads include associative algebras, Lie algebras, Poisson algebras, Batalin–Vilkovisky algebras, etc. One calls operadic calculus the set of techniques that allow us to work with operads themselves. There is a purely algebraic side to the operadic calculus. For instance, one might consider morphisms between operads. A morphism between operads induces a structure of an operadic bimodule. Via the theory of operadic bimodules, one can construct universal functors between the categories of algebras...
over operads. Thus this theory gives, for any morphism of operads, a universal adjunction between their respective categories of algebras. The universal enveloping algebra of a Lie algebra can be recovered in this way. This approach allows vast generalizations, and new universal enveloping algebras can be constructed for other types of algebraic structures in this way. It can also generalize well-known theorems such as the Poincaré–Birkhoff–Witt theorem in a functorial way, see [DT20].

The operadic calculus has also an homotopical side. When operads themselves live in the category of differential graded $S$-modules, they admit a notion of weak equivalence given by arity-wise quasi-isomorphisms. Understanding the homotopy theory of operads gives results on the homotopy theory in the category of algebras it encodes. Two weakly equivalent operads encode the same homotopy category of algebras. For any operad $\mathcal{P}$, algebras over a cofibrant resolution of $\mathcal{P}$ provide us with a suitable notion of a $\mathcal{P}$-algebra up to homotopy. This recovers the seminal notions of $A_\infty$-algebras, $L_\infty$-algebras, $C_\infty$-algebras as particular examples. These resolutions also allow one to construct universal André-Quillen cohomology theories for algebras over an operad, see [Mil11]. The main tools for studying the homotopy theory of operads and computing cofibrant resolutions are the Bar-Cobar adjunction and the Koszul duality theory. Combining both the algebraic and the homotopical aspects of the operadic calculus provides us with new tools to solve problems, see [LV12]. For example, in [CPRNW20], R. Campos, D. Petersen, D. Robert-Nicoud, and F. Wierstra show that a nilpotent Lie algebra is completely characterized up to isomorphism by its universal enveloping algebra, seen as an associative algebra. The proof of this purely algebraic result requires a combination of both of the aforementioned methods. Another example is given by [RNV20], where D. Robert-Nicoud and B. Vallette develop the integration theory of $L_\infty$-algebras using methods coming from the operadic calculus. One of the main motivations of this article was to lay down the operadic tools required to generalize the results of loc.cit to the case of curved $L_\infty$-algebras. See [RiL22].

In the mist of algebraic structures, there are the so-called curved algebras. The prototype of curved algebras are curved associative algebras. These are graded associative algebras $(A, \mu_A)$, endowed with a derivation $d$ of degree $-1$ and a distinguished element $\Theta$ of degree $-2$ called the curvature, such that
\[
d^2(-) = \mu_A(\Theta, -) - \mu_A(-, \Theta).
\]
Here, the element $\Theta$ is the obstruction for the derivation $d$ to square to zero. When $\Theta$ is non-trivial, this type of algebras do not have underlying homology groups, thus there is no notion of quasi-isomorphism for them in general. In these examples, the distinction between homological algebra and the more general concept of homotopical algebra becomes apparent. Other examples include curved Lie algebras, curved $A_\infty$-algebras, curved $L_\infty$-algebras and so on. These types of algebras are playing an increasingly important role in various areas of mathematics: curved $A_\infty$-algebras in Floer cohomology in symplectic geometry [FOOO09], curved $L_\infty$-algebras in derived differential geometry [BLX21], derived deformation theory [CCN21] and $L_\infty$-spaces [Cos11].

**Main results.** The goal of this foundational article is to settle the operadic calculus for curved operads. Curved operads appear naturally when one tries to encode types of curved algebras with operad-like structures: to encode the curvature relation $(*)$ at the algebra level, one needs to add a curvature on the operad level. Working with curved operads forces the underlying category to be the category of pre-differential graded (pdg) modules, which are given by graded modules with a degree $-1$ endomorphism. So we start by settling the basic properties related to this new framework. In order to develop the algebraic side of the curved operadic calculus, we introduce the notion of a curved operadic bimodule and we develop the subsequent theory. There are obstructions to this generalization: the “free algebra” over a curved operad does not, in general, satisfy the curvature relation. This implies that a curved operad is not a “naive” curved left module over herself. Nevertheless, we develop the theory of curved
bimodules for curved operads. This allows us to construct universal functors between the categories of curved algebras. For instance, we construct for the first time the universal curved enveloping algebra for curved Lie algebras and the universal curved enveloping $A_\infty$-algebra for curved $L_\infty$-algebras.

Once this algebraic framework is established, the rest of the article is devoted to generalizing the homotopical side of the operadic calculus to the curved setting. We start ab initio from a conceptual point of view. We use the groupoid-colored formalism developed by B. Ward in [War19]. We introduce the unital groupoid-colored operad $uO$ that encodes unital partial operads as its algebras and counital partial cooperads as its coalgebras. By partial operad, we mean the definition of operads introduced by M. Markl in [Mar96] in terms of partial composition maps $\{\circ_i\}$. The notion of partial cooperads is the dual notion, defined in terms of partial decomposition maps $\{\Delta_i\}$. We generalize the main point of the inhomogeneous Koszul duality of Hirsh–Milles in [HM12] to extend them to the groupoid-colored framework. This allows us to compute the conilpotent curved groupoid-colored dual cooperad $uO^\vee$ (up to suspension). We show that (curved) coalgebras over $cO^\vee$ correspond to conilpotent curved partial cooperads.

**Theorem 1** (Theorem 5.57). The unital groupoid-colored operad $uO$ is a Koszul operad and its Koszul dual curved groupoid-colored cooperad is given by $cO^\vee$ up to suspension.

In particular, this result implies that there is a Koszul curved twisting morphism

$$\kappa : cO^\vee \rightarrow uO.$$ 

From this curved twisting morphism, one obtains a first Bar-Cobar adjunction using the classical methods of [LV12] and [Gri19]. This adjunction recovers the Bar-Cobar adjunction between unital partial operads and conilpotent curved partial cooperads constructed by B. Le Grignou in [Gri21]. In loc.cit., the author endows the category of unital operads with a model structure where weak-equivalences are given by arity-wise quasi-isomorphisms, and then transfers it along this adjunction to conilpotent curved coaugmented cooperads. This endows conilpotent curved coaugmented cooperads with a meaningful model structure, and the Bar-Cobar adjunction is shown to be a Quillen equivalence. The above theorem provides a new proof that the Bar-Cobar adjunction recovered is indeed a Quillen equivalence.

Our goal here is to construct another adjunction using the curved twisting morphism $\kappa$, in order to obtain a Koszul dual notion of counital partial cooperads. In order so, we generalize the results of D. Lejay and B. Le Grignou in [GL18] to the groupoid-colored case. In op.cit, from a curved twisting morphism the authors construct a "complete" Bar-Cobar adjunction between the category of coalgebras over an operad and the category of curved algebras over a curved cooperad. The idea is that the Koszul dual of non-conilpotent types of coalgebras are absolute types of algebras, which are algebraic structures endowed with well-defined infinite sums of operations without any having an underlying topology.

In our case, we get an adjunction between the category of coalgebras over the groupoid-colored operad $uO$ and the category of curved algebras over the curved groupoid-colored cooperad $cO^\vee$, which we call the complete Bar-Cobar adjunction. Coalgebras over $uO$ are simply counital partial cooperads. Whereas curved algebras $cO^\vee$ give rise to new objects in the operadic calculus, which we call curved absolute partial operads. There is also a more basic notion of absolute partial operad. Both of them can be though as partial operads where infinite sums of partial compositions have a well-defined image. These notions are further characterized in Appendix 9.2. Note that (curved) absolute partial operads appear precisely when arity 0 and 1 phenomena are taken into account, and coincide with usual notion of a partial operad otherwise.

Summarizing, we obtain a complete Bar-Cobar adjunction

$$\text{dg upCoop} \xrightarrow{\delta} \text{curv abs pOp}.$$
Notice first that this adjunction interrelates "counits" with "curvature", which are known to be Koszul dual. See [PP05]. But its main novelty lies in the fact that it also lifts the usual assumption of conilpotency on the cooperad side. Lifting conilpotency on one side of the Koszul duality is what makes infinite sums appear in the other side. This is the conceptual explanation for the existence of these curved absolute partial operads. Absolute partial operads admit a canonical filtration, dual to the coradical filtration for partial cooperads. And like the coradical filtration, this filtration comes from the structure of the object. By definition, infinite sums of partial compositions are well-defined. But the topology induced by this filtration might not be Hausdorff. We restrict ourselves to the sub-category of those which are complete. Notice that unlike many other approaches to curved objects, where one changes the base category from graded modules to filtered/complete graded modules in order to deal with the infinite sums that appear, here we introduce new types of algebraic structures which admit these infinite sums without enriching further the underlying category of graded modules. And the complete filtrations that appear, come from the structure of these new objects, and are therefore canonical.

Considering these new algebraic objects, namely complete curved absolute partial operads, is what allows us to obtain a well-behaved homotopy theory. Since the notion of a quasi-isomorphism does not exist in the curved context, our approach is to obtain a notion of weak equivalences via a transfer theorem from a Koszul dual category. For this purpose, the category of counital partial cooperads is too narrow. We define the notion of counital partial cooperads up to homotopy. Then, we endow the category of counital partial cooperads up to homotopy with strict morphisms with a model category structure where weak equivalences are given by arity-wise quasi-isomorphisms. Finally, we construct another complete Bar-Cobar adjunction and transfer this model structure to the category of complete curved absolute partial operads.

**Theorem 2** (Theorem 7.16). The category of complete curved absolute partial operads admits a model structure transferred along the adjunction

\[
\text{upCoo}_\infty ^\text{op} \xleftarrow{\Omega_\text{c}} \xrightarrow{\hat{\Omega}} \text{curv abs pOp}^{\text{comp},\text{op}},
\]

where the model category structure considered on the left-hand side has arity-wise quasi-isomorphisms as weak-equivalences and monomorphisms as cofibrations.

We then relate the "classical" Bar-Cobar adjunction with the complete Bar-Cobar adjunction constructed here. For this purpose, we construct a pair of duality adjunctions that relate these Bar-Cobar adjunctions. First we do this in the algebraic context, where we prove the following result.

**Theorem 3** (Theorem 8.9). The following square of adjunction

\[
\begin{array}{ccc}
\text{dg upOp}^{\text{op}} & \xleftarrow{B^{\text{op}}} & \left(\text{curv pCoo}^{\text{conil},\text{op}}\right)\
\downarrow{(-)^*} & & \downarrow{(-)^\vee} \\
\text{upCoo} & \xrightarrow{\Omega} & \text{curv abs pOp}^{\text{comp},\text{op}}
\end{array}
\]

commutes in the following sense: right adjoints going from top right corner to bottom left corner are naturally isomorphic.

Here the functor \((-)^\circ\) is a direct generalization of the Sweedler duality functor defined in [Swe69] for associative algebras, and the functor \((-)^\vee\) can be thought of as a topological dual. Then, by replacing (co)unital partial (co)operads on the left-hand side of the square with their
"up to homotopy" counterparts, we construct another commuting square of duality adjunctions which are, in this case, all Quillen adjunctions. Using this duality square, we construct explicit cofibrant resolutions for a vast class of complete curved absolute partial operads. For instance, we construct a minimal cofibrant resolution of complete curved absolute partial operads encoding curved Lie algebras, which provides us with a suitable notion of curved Lie algebra up to homotopy. Applications of these results can be found in [Ril22], where we develop the integration theory of curved absolute $\mathcal{L}_\infty$-algebras, and apply it to deformation theory and rational homotopy theory.

Finally, using our complete Bar construction, we extend the classical Homotopy Transfer Theorem to the case of algebras over a general class of cofibrant complete curved absolute operads. Let $V$ and $H$ be two complete pdg modules. A homotopy contraction amounts to the data of

$$h \subset V \xrightarrow[p]{\cdot} H,$$

where $p$ and $i$ are two morphisms of filtered pdg modules and $h$ is a morphism of filtered graded modules of degree $-1$, which satisfy standard conditions. This data allows us to construct a Van der Laan morphism between the complete Bar constructions of the curved endomorphisms operads of $V$ and $H$. See [LV12, Chapter 10] for the classical Van der Laan morphism. Using this new morphism, we obtain the following result.

**Theorem 4 (Theorem 9.8).** Let $\mathcal{C}$ be a dg counital partial cooperad. Let $H$ be a homotopy retract of $V$ and let

$$\varphi : \hat{\Omega}\mathcal{C} \longrightarrow \text{end}_V$$

be a curved algebra structure on $V$. There is a transferred curved $\hat{\Omega}\mathcal{C}$-algebra on $H$ constructed from the Van der Laan morphism.

In the particular case of complete curved $\mathcal{L}_\infty$-algebras, the transferred structure coincides with that constructed by K. Fukaya in [Fuk02].

Although (curved) absolute partial operads appear naturally when one considers the groupoid-colored point of view, there are still many open questions related to them. A crucial one is to have a meaningful notion of a (curved) absolute algebra over a (curved) absolute partial operad. One can forget the "absolute" structure and only consider (curved) algebras on the underlying (curved) partial operad, or one can define them using filtered objects like we do Section 9. These answers are not fully satisfactory, and expanding this theory should be the subject of future work.

**Layout.** This article is structured as follows: in Section 1, we recall the different definition of operads and cooperads, and we define their respective categories of algebras and coalgebras. This includes the notion of an algebra over a cooperad introduced by Le Grignou–Lejay which will play an important role in the subsequent sections. This section contains non-standard definitions even for experts in the operadic literature. In Section 2, we introduce the notion of a curved operad in the underlying category of pre-differential modules and establish basic properties. In Section 3, we develop the theory of curved bimodules over curved operads and we prove that they induce universal functors between categories of curved algebras. In Section 4, we recall different results on curved cooperads and their categories of curved coalgebras and curved algebras.

In Section 5, we explain the formalism of algebras and coalgebras over operads and cooperads in the $S$-colored case. We extend the inhomogeneous Koszul duality to this framework in order to prove that the unital $S$-colored operad $u\mathcal{O}$, which encodes unital partial operads as its algebras, is a Koszul $S$-colored operad. In Section 6, we use this $S$-colored inhomogeneous Koszul duality to construct a new complete Bar-Cobar adjunction. In Section 7, we endow the category of complete curved absolute partial operads with a transferred model category structure from counital partial cooperads up to homotopy, considered with strict morphisms.
In Section 8, we construct the duality adjunctions that intertwine the Bar-Cobar adjunction constructed before with the Bar-Cobar adjunction constructed by Le Grignou. We use this duality square to obtain explicit cofibrant resolutions. In Section 9, we apply the complete Bar-Cobar adjunction to obtain a general Homotopy Transfer Theorem for complete curved absolute partial operads. In the appendix, we define and characterize the key notion of absolute partial operads and their curved counterparts.

Acknowledgments. This work was carried out during my PhD at the Université Paris 13, and I would like to thank my former advisor Bruno Vallette for numerous discussions which led to this paper as well as for the careful readings of this paper. I would also like to thank Brice Le Grignou for the time he spent discussing with me, as well as the interesting ideas he shared with me, like the suggestion to use of counital partial cooperads up to homotopy in Section 7. I thank Mathieu Anel, Joan Bellier-Millès, Ricardo Campos, Damien Calaque, Geoffroy Horel, and all the members of the Séminaire Roberta for interesting discussions. Finally, I would also like to thank the referee of this paper for the numerous comments, remarks and corrections that helped greatly improve this paper.

Conventions. Let \( K \) be a ground field of characteristic 0. The ground category is the symmetric monoidal category \((\text{gr-mod}, \otimes, K)\) of graded \( K \)-modules, with the tensor product \( \otimes \) of graded modules given by

\[
(A \otimes B)_n := \bigoplus_{p+q=n} A_p \otimes B_q.
\]

The tensor is taken over the base field \( K \), which will be implicit from now on. The isomorphism \( \tau_{A,B} : A \otimes B \to B \otimes A \) is given by the Koszul sign rule \( \tau(a \otimes b) = (-1)^{|a||b|} b \otimes a \) on homogeneous elements. We work with the homological degree convention, that is, the degree of the (pre)-differentials considered is \(-1\). The suspension of a graded module \( V \) is denoted by \( sV \), given by \((sV)_p \coloneqq V_{p-1}\). A graded \( S \)-module \( M \) is a collection \( \{M(n)\}_{n \in \mathbb{N}} \) of graded \( K[S_n]\)-modules. This category is denoted by \( \text{gr-mod} \). We denote \((\text{gr}\ S\text{-mod}, \circ, 1)\) the monoidal category of graded \( S \)-modules endowed with the composition product \( \circ \). We adopt the conventions of [LV12], unless stated otherwise.

1. Recollections on different types of (co)operads

In this section, we recall briefly the (co)monoidal definition of (co)operads. Operads were introduced to encode types of algebras and cooperads to encode types of conilpotent coalgebras. We explain briefly the dual point of view developed in [GL18], where operads encode non-necessarily conilpotent coalgebras, and where cooperads encode a new type of algebras. This type of algebras have well-defined infinite sums of operations by definition. We review the partial definitions of (co)operads and compare them with the (co)monoidal definitions. The partial definitions are extremely useful to study operads themselves. These categories are the "natural habitat" of the operadic Bar-Cobar constructions defined later on. The material presented here contains non-standard results, thus we encourage experts in the field to read it as well.

1.1. (Co)operads as (co)monoids and their algebras and coalgebras. We recall the basic definition of operads and cooperads as monoids and comonoids in \((\text{gr}\ S\text{-mod}, \circ, 1)\). We omit the epithet "graded" before each definition, although it is implicit that all our objects are graded.

Definition 1.1 (Operad). An operad \((P, \gamma_P, \eta)\) is the data of a unital monoid in \((\text{gr}\ S\text{-mod}, \circ, 1)\).

Definition 1.2 (Cooperad). A cooperad \((\mathcal{C}, \Delta, \epsilon)\) is the data of a counital comonoid in \((\text{gr}\ S\text{-mod}, \circ, 1)\).

Notation. If there is no ambiguity, we will denote the data of an operad \((P, \gamma_P, \eta)\) simply by \(P\). Likewise, we will denote the data of a cooperad \((\mathcal{C}, \Delta, \epsilon)\) simply by \(\mathcal{C}\).
Theses definitions are well suited for defining algebras and coalgebras because of their “monadic” nature. To any graded $S$-module one can associate an endofunctor in the category of graded modules via the Schur realization functor

\[ \mathcal{S} : \text{gr } S\text{-mod} \rightarrow \text{End}(\text{gr mod}) \]

\[ M \rightarrow \mathcal{S}(M)(-) := \bigoplus_{n \geq 0} M(n) \otimes S_n (-)^{\otimes n}. \]

**Lemma 1.3.** The Schur realization functor defines a strong monoidal functor between the monoidal categories \((\text{gr } S\text{-mod}, \circ, 1)\) and \((\text{End}(\text{gr mod}), \circ, \text{Id})\), where the monoidal product on endofunctors is given by the composition of endofunctors.

This implies that if $P$ is an operad, then $\mathcal{S}(P)$ is a monad and that if $\mathcal{C}$ is a cooperad, then $\mathcal{S}(\mathcal{C})$ is comonad.

**Definition 1.4** ($P$-algebras). A $P$-algebra $A$ amounts to the data \((A, \gamma_A)\) of an algebra over the monad $\mathcal{S}(P)$.

**Definition 1.5** ($\mathcal{C}$-coalgebras). A $\mathcal{C}$-coalgebra $C$ amounts to the data \((C, \Delta_C)\) of a coalgebra over the comonad $\mathcal{S}(\mathcal{C})$.

**Remark 1.6.** The notion of a coalgebra over a cooperad can only encapsulate types of coalgebras which are conilpotent in some sense. The reason is that the structural map

\[ \Delta_C : C \rightarrow \bigoplus_{n \geq 0} \mathcal{C}(n) \otimes S_n C^{\otimes n}, \]

lands on the direct sum of a non-completed tensor product. Thus decompositions of elements in $C$ are supported in finitely many arities and in finitely many steps of the coradical filtration of $\mathcal{C}$.

There is another kind of Schur realization functor, which was introduced in [GL18]. The dual Schur realization functor $\hat{\mathcal{S}}$ is given by

\[ \hat{\mathcal{S}} : \text{gr } S\text{-mod}^{\text{op}} \rightarrow \text{End}(\text{gr mod}) \]

\[ M \rightarrow \hat{\mathcal{S}}(M)(-) := \prod_{n \geq 0} \text{Hom}_{S_n}(M(n), (-)^{\otimes n}). \]

**Lemma 1.7** ([GL18, Corollary 3.4]). The functor $\hat{\mathcal{S}}(-) : (\text{S-mod}, \circ, 1)^{\text{op}} \rightarrow (\text{End}(\text{K-mod}), \circ, \text{Id})$ can be endowed with a lax monoidal structure, that is, there exists a natural transformation

\[ \varphi_{M,N} : \hat{\mathcal{S}}(M) \circ \hat{\mathcal{S}}(N) \rightarrow \hat{\mathcal{S}}(M \circ N). \]

which satisfies associativity and unitality compatibility conditions with respect to the monoidal structures. Furthermore, $\varphi_{M,N}$ is a monomorphism for all graded $S$-modules $M, N$.

This implies that if $\mathcal{C}$ is a cooperad, then $\hat{\mathcal{S}}(\mathcal{C})$ is monad. Thus one can define algebras over a cooperad in this way.

**Definition 1.8** ($\mathcal{C}$-algebra). A $\mathcal{C}$-algebra $B$ amounts to the data \((B, \gamma_B)\) of an algebra over the monad $\hat{\mathcal{S}}(\mathcal{C})$.

**Remark 1.9.** The notion of an algebra over a cooperad defines a new type of algebraic structures. The reason is that the structural map

\[ \gamma_B : \prod_{n \geq 0} \text{Hom}_{S_n}(\mathcal{C}(n), B^{\otimes n}) \rightarrow B \]
associates an element in $B$ to any infinite series of operations. Thus algebras over a cooperad are endowed with a notion of infinite summation without presupposing any underlying topology.

The notion of an algebra over a cooperad admits a further description in the case where the cooperad is conilpotent. Our definition of a conilpotent cooperad is explained in Section 1.3.

Let $(C_u, \Delta, \epsilon)$ be a conilpotent cooperad. Each term of the coradical filtration $R_\omega C$ defines a sub-cooperad, and there is a short exact sequence of $S$-modules

$$0 \to R_\omega C \to C \to \pi_\omega C / R_\omega C \to 0.$$  

**Definition 1.10** (Canonical filtration on a $C$-algebra). Let $(C_u, \Delta, \epsilon)$ be a conilpotent cooperad and let $(B, \gamma_B)$ be a $C$-algebra. The canonical filtration of $B$ is the decreasing filtration of given by

$$W_\omega B := \text{Im} \left( \gamma_B \circ \hat{\mathcal{F}}(\pi_\omega B) : \hat{\mathcal{F}}(C / R_\omega C)(B) \to B \right)$$

where $R_\omega C$ denotes the $\omega$-th term of the coradical filtration, for all $\omega \geq 0$. Notice that we have

$$B = W_0 B \supseteq W_1 B \supseteq W_2 B \supseteq \cdots \supseteq W_\omega B \supseteq \cdots.$$  

**Definition 1.11** (Completion of a $C$-algebra). Let $(B, \gamma_B)$ be a $C$-algebra. Its completion is given by

$$\hat{B} := \lim_{\omega \in \mathbb{N}} B / W_\omega B,$$

where the limit is taken in the category of $C$-algebras.

It comes equipped with a canonical morphism of $C$-algebras $\varphi_B : B \to \hat{B}$.

**Definition 1.12** (Complete $C$-algebra). The $C$-algebra $(B, \gamma_B)$ is said to be complete if $\varphi_B$ is an isomorphism of $C$-algebras.

**Proposition 1.13** ([GL18, Proposition 4.24]). Let $(B, \gamma_B)$ be a $C$-algebra. The canonical morphism

$$\varphi_B : B \to \hat{B}$$

is an epimorphism.

**Remark 1.14.** Conceptually, this comes from the fact that any $C$-algebras already carries a meaningful notion of infinite summation. Thus "nothing is added" when one applies the completion functor. On the other hand, the topology induced by the canonical filtration of a $C$-algebra might not be Hausdorff. Meaning that the canonical morphism $\varphi_B$ might not be a monomorphism. This completion functor should be considered a sifting functor.

**Proposition 1.15** ([GL18, Proposition 4.21, Proposition 4.25]). Let $(C_u, \Delta, \epsilon)$ be a conilpotent cooperad. Any free $C$-algebra is complete. Furthermore, the category of complete $C$-algebras forms a reflexive subcategory of the category of $C$-algebras, where the reflector is given by the completion.

**Remark 1.16.** Contrary to $C$-coalgebras, which are always conilpotent, there are examples of $C$-algebras which are not complete. See for instance [GL18, Section 4.5].

Even though that, for an operad $P$, its dual Schur functor $\hat{\mathcal{F}}(P)$ fails to be a comonad, one can still define a notion of a coalgebra over an operad.

**Definition 1.17** ($P$-coalgebra). A $P$-coalgebra $D$ amounts to the data $(D, \Delta_D)$ of a graded module $D$ endowed with a structural map

$$\Delta_D : D \to \prod_{n \geq 0} \text{Homs}_n(P(n), D^\otimes n),$$
such that the following diagram commutes

\[
\begin{array}{c}
\Delta_D : D \xrightarrow{} \hat{\mathcal{F}}(\mathcal{P})(D) \\
\Downarrow \Delta_D
\end{array}
\xrightarrow{\hat{\mathcal{F}}(\id_D)}
\begin{array}{c}
\hat{\mathcal{F}}(\id_D)(\Delta_D) : \hat{\mathcal{F}}(\mathcal{P})(D) \xrightarrow{} \hat{\mathcal{F}}(\mathcal{P})(D)
\end{array}
\xrightarrow{\varphi_{\mathcal{P},\mathcal{P}}(D)}
\begin{array}{c}
\hat{\mathcal{F}}(\mathcal{P})(D) \\
\Downarrow \hat{\mathcal{F}}(\gamma)(\id)
\end{array}
\]

Remark 1.18. The data of a $\mathcal{P}$-coalgebra $D$ amounts to the data of a morphism of operads $\mathcal{P} \rightarrow \text{Coend}_D$, where Coend$_D$ stands for the coendomorphisms operads of $D$.

Notation. Let $f : X \rightarrow Y$ be a map of degree 0 and $g : X \rightarrow Y$ be a map of degree $p$. We denote

\[
\Pi_n(f, g) := \sum_{i=0}^{n} f^{\otimes i-1} \otimes g \otimes f^{\otimes n-i} : X^{\otimes n} \rightarrow Y^{\otimes n}
\]

the resulting $S_n$-equivariant map of degree $p$. Let $M$ be an graded $S$-module. It induces first a map of degree $p$

\[
\bigoplus_{n \geq 0} M(n) \otimes_{S_n} X^{\otimes n} \rightarrow \bigoplus_{n \geq 0} M(n) \otimes_{S_n} Y^{\otimes n}
\]

by applying $\id_M \otimes \Pi_n(f, g)$ at each arity. By a slight abuse of notation, this map will be denoted by $\hat{\mathcal{F}}(\id_M)(\Pi_1(f, g))$. Likewise, it induces a second map of degree $p$

\[
\prod_{n \geq 0} \text{Hom}_{S_n}(M(n), X^{\otimes n}) \rightarrow \prod_{n \geq 0} \text{Hom}_{S_n}(M(n), Y^{\otimes n})
\]

by applying $\text{Hom}(\id_M, \Pi_1(f, g))$ at each arity. By a slight abuse of notation, this map will be denoted by $\hat{\mathcal{F}}(\id_M)(\Pi_2(f, g))$.

1.2. Partial definition of (co)operads. In order to work with operads and cooperads themselves, it is convenient to use another definition.

Definition 1.19. Let $n, k \in \mathbb{N}$, for all $1 \leq i \leq n$, we define maps:

\[
\circ_i : S_n \times S_k \rightarrow S_{n+k-1}.
\]

For $(\tau, \sigma) \in S_n \times S_k$, $\sigma \circ_i \tau$ is given by the unique permutation in $S_{n+k-1}$ which first acts as $\tau$ on the set $\{1, \ldots, n+k-1\} - \{i, \ldots, i+k-1\}$ and sends $\{i, \ldots, i+k-1\}$ to $\{\tau(i), \ldots, \tau(i+k-1)\}$; then acts as $\sigma$ on the set $\{\tau(i), \ldots, \tau(i+k-1)\}$.

Definition 1.20 (Partial operad). A partial operad $(\mathcal{P}, \{\circ_i\})$ is the data of a graded $S$-module $\mathcal{P}$ endowed with a family of partial composition maps of degree 0:

\[
\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(k) \rightarrow \mathcal{P}(n+k-1)
\]

for $1 \leq i \leq n$, subject to the following conditions.

(1) It satisfies the sequential axiom: for $1 \leq i \leq n, 1 \leq j \leq k$, the following diagram commutes

\[
\begin{array}{c}
\mathcal{P}(n) \otimes \mathcal{P}(k) \otimes \mathcal{P}(m) \xrightarrow{\id_{\mathcal{P}(n)} \otimes \circ_i} \mathcal{P}(n+k-1) \otimes \mathcal{P}(m) \\
\downarrow \circ_{i+j-1} \\
\mathcal{P}(n) \otimes \mathcal{P}(k+m-1) \xrightarrow{\circ_i} \mathcal{P}(n+k+m-2).
\end{array}
\]
(2) It satisfies the parallel axiom: for $1 \leq i < j \leq n$, the following diagram commutes

$$
P(n) \otimes P(k) \otimes P(m) \xrightarrow{o_i \otimes \text{id}_{P(k)}} P(n+m-1) \otimes P(k)
$$

$$
\xrightarrow{o_i \otimes \text{id}_{P(m)}} P(n+k-1) \otimes P(m) \xrightarrow{o_{i+k-1}} P(n+k+m-2).
$$

(3) The maps

$$
o_i : P(n) \otimes P(k) \rightarrow P(n+k-1)
$$

satisfy the following condition: let $(\tau, \sigma) \in S_n \times S_k$, then the following diagram commutes

$$
P(n) \otimes P(k) \xrightarrow{o_i} P(n+k-1)
$$

$$
\tau \otimes \sigma \xrightarrow{\tau \circ \sigma} P(n+k-1).
$$

**Definition 1.21** (Unital partial operad). A unital partial operad $(P, \{\circ_i\}, \eta)$ is the data of a partial operad $(P, \{\circ_i\})$ together with a morphism $\eta : I \rightarrow P$, such that $\circ_1(\eta(\text{id}), -)$ and $\circ_1(-, \eta(\text{id}))$ are the identity on $P(n)$.

**Remark 1.22.** Unital partial operads are sometimes called Markl operads in the literature, see [Mar96].

Let $T$ denote the tree monad on the category of graded $S$-modules and let $\overline{T}$ denote the reduced tree monad on the same category. For an explicit construction, see [LV12, Section 5.6].

**Proposition 1.23.** There are isomorphisms of categories between:

1. The category of algebras over the tree monad $T$ and the category of unital partial operads.
2. The category of algebras over the reduced tree monad $\overline{T}$ and the category of partial operads.

**Proposition 1.24.** The category of operads defined as monoids is equivalent to the category of unital partial operads.

**Remark 1.25.** Given a partial operad $(P, \{\circ_i\})$, one can freely adjoin a unit by setting $P^\uparrow := P \oplus I$ and considering the inclusion $\eta : I \rightarrow P^\uparrow$. This allows us to associate to any partial operad an augmented operad defined as a monoid.

**Definition 1.26** (Partial cooperad). A partial cooperad $(C, \{\Delta_i\})$ is the data of a graded $S$-module $C$ together with partial decomposition maps:

$$
\Delta_i : C(n+k-1) \rightarrow C(n) \otimes C(k)
$$

for $1 \leq i \leq n$, subject to the following conditions.

1. It satisfies the sequential axiom: for $1 \leq i \leq n$, $1 \leq j \leq k$, the following diagram commutes

$$
C(n+k+m-2) \xrightarrow{\Delta_{i+j-1}} C(n+k-1) \otimes C(m)
$$

$$
\xrightarrow{\Delta_i} C(n) \otimes C(k+n-1) \xrightarrow{id_{C(n)} \otimes \Delta_i} C(n) \otimes C(k) \otimes C(m).
$$
(2) It satisfies the parallel axiom: for $1 \leq i < j \leq n$, the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{C}(n+k+m-2) & \xrightarrow{\Delta_{i+k-1}} & \mathcal{C}(n+k-1) \otimes \mathcal{C}(m) \\
\Delta_i & & \Delta_i \otimes \operatorname{id}_{\mathcal{C}(m)} \\
\mathcal{C}(n+m-1) \otimes \mathcal{C}(k) & \xrightarrow{\Delta_i \otimes \operatorname{id}_{\mathcal{C}(k)}} & \mathcal{C}(n) \otimes \mathcal{C}(k) \otimes \mathcal{C}(m) .
\end{array}
$$

(3) The maps

$$
\Delta_i : \mathcal{C}(n+k-1) \longrightarrow \mathcal{C}(n) \otimes \mathcal{C}(k)
$$

satisfy the following condition: let $(\tau, \sigma) \in S_n \times S_k$, then the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{C}(n+k-1) & \xrightarrow{\Delta_{i-1}(1)} & \mathcal{C}(n) \otimes \mathcal{C}(k) \\
\tau \circ_i \sigma & & \tau \otimes \sigma \\
\mathcal{C}(n+k-1) & \xrightarrow{\Delta_i} & \mathcal{C}(n) \otimes \mathcal{C}(k) .
\end{array}
$$

**Definition 1.27** (Counital partial cooperad). A **counital partial cooperad** $(\mathcal{C}, \{\Delta_i\}, \epsilon)$ is the data of a partial cooperad $(\mathcal{C}, \{\Delta_i\})$ and a morphism of $S$-modules $\epsilon : \mathcal{C} \longrightarrow I$ such that:

$$
\begin{array}{ccc}
\mathcal{C}(n) & \xrightarrow{\Delta_i} & \mathcal{C}(n) \otimes \mathcal{C}(1) \\
\mathcal{C}(n) & \xrightarrow{\epsilon} & \mathcal{C}(1) \otimes \mathcal{C}(n) \\
\mathcal{C}(n) \cong \mathcal{C}(n) \otimes \mathbb{K} & & \mathcal{C}(n) \cong \mathbb{K} \otimes \mathcal{C}(n) .
\end{array}
$$

1.3. **Filtrations on partial (co)operads.** In order to compare the notion of a (possibly counital) partial cooperad with the notion of a cooperad defined as a comonoid, one needs to introduce filtrations. The tree monad has a natural weight grading $\mathcal{T}^{(\omega)}$ given by the number of internal edges $\omega$ of the rooted trees. Notice that:

$$
\mathcal{T}^{(-1)}(M) = 1, \quad \mathcal{T}^{(0)}(M) = M, \quad \text{and} \quad \mathcal{T}^{(1)}(M) = M \circ_{(1)} M .
$$

Recall that the tree monad on a graded $S$-module $\mathcal{T}(M)$ is given by the direct sum over rooted trees where vertices are labeled by elements of $M$ in the following way: if $\tau$ is a rooted tree and $v$ is one of its vertices with $k$ incoming edges, then $v$ must be labeled with an element of $M(k)$. See for more details [LV12, Section 5.6.1].

**Definition 1.28** (Reduced completed tree monad). The **reduced completed tree endofunctor** $\mathcal{T}^{\wedge}$ is the endofunctor of the category of graded $S$-modules given by

$$
\mathcal{T}^{\wedge}(M) := \lim_{\omega} \mathcal{T}(M) / \mathcal{T}^{(\omega)}(M) ,
$$

that is, the completion of the tree monad with respect to this weigh filtration.

**Remark 1.29.** The reduced complete tree endofunctor is isomorphic to the product over rooted trees where vertices are labeled by elements of $M$ in the following way: if $\tau$ is a rooted tree and $v$ is one of its vertices with $k$ incoming edges, then $v$ must be labeled with an element of $M(k)$. In particular, if $M(0) = 0$ and $M(1) = 0$, then there are only a finite number of rooted trees in each arity and one has that

$$
\mathcal{T}^{\wedge}(M) \cong \mathcal{T}(M) .
$$

**Lemma 1.30.** Let $(\mathcal{C}, \{\Delta_i\})$ be a partial cooperad. The partial decomposition maps $\{\Delta_i\}$ induce a morphism of graded $S$-modules

$$
\Delta_e : \mathcal{C} \longrightarrow \mathcal{T}^{\wedge}(\mathcal{C}) .
$$
1.30 The factors through $\left(\begin{array}{c}1.35 \\ \text{onto} \end{array}\right)$ from the category of conilpotent partial cooperads to the category of cooperads defined as comonoids.

If it is in the essential image of the functor $\text{grCoop}$, then it can be projected onto the sub-$S$-module $C \circ C$ of $\text{grCoop}^\circ$. The axioms of a partial cooperad ensure that this projection is coassociative. □

Corollary 1.35. Let $(\mathcal{C}, (\Delta_i))$ be a conilpotent partial cooperad. Let $\mathcal{C}^\circ := \mathcal{C} \circ \mathcal{I}$ and $c$ be the projection onto $\mathcal{I}$. Then $(\mathcal{C}^\circ, (\Delta_i \circ c))$ forms a cooperad defined as a comonoid. It defines a functor

$$\text{Conil} : \text{grCoop}^{\text{conil}} \rightarrow \text{gr Coop}$$

from the category of conilpotent partial cooperads to the category of cooperads defined as comonoids.

Remark 1.36. Counital partial cooperads do not, in general, induce a cooperad defined as a comonoid. For example, the graded $S$-module $u\mathbb{C}om^*$ given by the linear dual of the operad $u\mathbb{C}om$ admits a counital partial cooperad structure but does not admit a cooperad structure.

Definition 1.37 (Conilpotent cooperad). Let $(\mathcal{C}, \Delta, c)$ be a cooperad. It is said to be conilpotent if it is in the essential image of the functor Conil defined in Corollary 1.35.
Definition 1.40 (Canonical filtration on a partial operad). Let $(\mathcal{P},\{\alpha_i\})$ be a partial operad and let $\gamma_\mathcal{P} : \mathcal{T}(\mathcal{P}) \rightarrow \mathcal{P}$ be its $\mathcal{T}$-algebra structure. Its canonical filtration is the decreasing filtration given by

$$\mathcal{F}_\omega \mathcal{P} := \text{Im}\left(\gamma_\mathcal{P}^{(\geq \omega)} : \mathcal{T}^{(\geq \omega)}(\mathcal{P}) \rightarrow \mathcal{P}\right)$$

for all $\omega \geq 0$, where $\mathcal{T}^{(\geq \omega)}(\mathcal{P})$ denotes the elements of weight greater or equal to $\omega$. Each $\mathcal{F}_\omega \mathcal{P}$ defines an operadic ideal of $\mathcal{P}$ since they are stable under partial composition. Notice that:

$$\mathcal{P} = \mathcal{F}_0 \mathcal{P} \supseteq \mathcal{F}_1 \mathcal{P} \supseteq \cdots \supseteq \mathcal{F}_\omega \mathcal{P} \supseteq \cdots.$$  

Definition 1.41 (Nilpotent partial operad). Let $(\mathcal{P},\{\alpha_i\})$ be a partial operad. It is said to be nilpotent if there exists an $\omega \geq 1$ such that

$$\mathcal{P}/\mathcal{F}_\omega \mathcal{P} \cong \mathcal{P}.$$  

The partial operad is said to be $\omega_0$-nilpotent if $\omega_0$ is the smallest integer such that the above isomorphism exits.

Definition 1.42 (Completion of a partial operad). Let $(\mathcal{P},\{\alpha_i\})$ be a partial operad, its completion $\hat{\mathcal{P}}$ is given by the following limit

$$\hat{\mathcal{P}} := \lim_{\omega} \mathcal{P}/\mathcal{F}_\omega \mathcal{P}$$

taken in the category of partial operads.

Any morphism of partial operads $f : (\mathcal{P},\{\alpha_i\}) \rightarrow (\mathcal{Q},\{\alpha_i\})$ is continuous with respect to the canonical filtration (i.e $f(\mathcal{F}_\omega \mathcal{P}) \subseteq \mathcal{F}_\omega \mathcal{Q}$). Thus the completion of partial operads is functorial. For every $\omega \geq 1$, there are projection morphisms of partial operads $\varphi_\omega : \mathcal{P} \rightarrow \mathcal{P}/\mathcal{F}_\omega \mathcal{P}$ which induce a canonical morphism of partial operads:

$$\varphi : \mathcal{P} \rightarrow \hat{\mathcal{P}}.$$  

Definition 1.43 (Complete partial operad). Let $(\mathcal{P},\{\alpha_i\})$ be a partial operad. It is complete if the canonical morphism $\varphi : \mathcal{P} \rightarrow \hat{\mathcal{P}}$ is an isomorphism of partial operads.

Example 1.44. Any nilpotent partial operad is complete. Any complete partial operad is the limit of a tower of nilpotent partial operads.

1.4. Convolution partial operad. Finally, we introduce the convolution partial operad and recall what a twisting morphism is.

Definition 1.45 (Convolution partial operad). Let $(\mathcal{C},\{\Delta_i\})$ be a partial cooperad and let $(\mathcal{P},\{\alpha_i\})$ be a partial operad. The convolution partial operad of $\mathcal{C}$ and $\mathcal{P}$ is given by the graded $S$-module

$$\text{Hom}(\mathcal{C},\mathcal{P})(n) := \text{Hom}_{\text{gr mod}}(\mathcal{C}(n),\mathcal{P}(n)),$$

with its natural $S_n$-action. It can be endowed with the partial composition given by

$$\alpha \circ_1 \beta : \mathcal{C}(n+k-1) \xrightarrow{\Delta_1} \mathcal{C}(n) \otimes \mathcal{C}(k) \xrightarrow{\alpha \otimes \beta} \mathcal{P}(n) \otimes \mathcal{P}(k) \xrightarrow{\circ_1} \mathcal{P}(n+k-1).$$

Notation. We denote by $\text{OrPar}(i_1,\cdots,i_k)$ the set of ordered partitions of $\{1,\cdots,n\}$, where $n = i_1 + \cdots + i_k$, whose $j$-th part has $i_j$ elements. Any partition $\mathcal{P}$ in $\text{OrPar}(i_1,\cdots,i_k)$ defines a $\{i_1,\cdots,i_k\}$-unshuffle, see [LV12, Section 5.4.2].
**Definition 1.46** (Totalization of a partial operad). Let \((\mathcal{P}, \{\circ_i\})\) be a partial operad, the totalization of \(\mathcal{P}\) given by
\[
\prod_{n \geq 0} \mathcal{P}(n)^{S_n}.
\]
It can be endowed with a pre-Lie algebra structure by setting
\[
\mu \star \nu := \sum_{i=1}^{n} \sum_{\sigma_\mathcal{P}} (\mu \circ_i \nu)^{\sigma_\mathcal{P}},
\]
where \(\mu\) is in \(\mathcal{P}(n)\), \(\nu\) is in \(\mathcal{P}(m)\), and where the second sum ranges over unshuffles \(\sigma_\mathcal{P}\) associated to all ordered partitions \(\mathcal{P}\) in \(\text{OrPar}(1, \cdots, 1, n-i+1, 1, \cdots, 1)\).

**Remark 1.47.** One can check by direct computation that the axioms of a partial operad make the associator of \(\star\) right symmetric. We refer to [LV12, Section 5.4.3] for more details on the totalization of a partial operad.

**Definition 1.48** (Twisting morphism). Let \((\mathcal{C}, \{\Delta_i\})\) be a partial cooperad and let \((\mathcal{P}, \{\circ_i\})\) be a partial operad. Let
\[
g_{\mathcal{C}, \mathcal{P}} := \prod_{n \geq 0} \text{Hom}_{S_n}(\mathcal{C}(n), \mathcal{P}(n))
\]
be the pre-Lie algebra given by the totalization of the convolution partial operad of \(\mathcal{C}\) and \(\mathcal{P}\). A twisting morphism is a Maurer-Cartan element \(\alpha\) of \(g_{\mathcal{C}, \mathcal{P}}\), that is, a morphism \(\alpha : \mathcal{C} \to \mathcal{P}\) graded \(S\)-modules of degree \(-1\) satisfying:
\[
\alpha \star \alpha = 0.
\]
The set of twisting morphisms between \(\mathcal{C}\) and \(\mathcal{P}\) is denoted by \(\text{Tw}(\mathcal{C}, \mathcal{P})\).

**Remark 1.49.** Everything stated in this section can be generalized *mutatis mutandis* to the context of differential graded modules.

## 2. Curved Operads and their Curved Algebras

In the section, we introduce a new type of operad-like structure, called curved operads. They naturally encode curved algebras. The proper framework for this theory will be the underlying category of pre-differential graded modules. This framework, and the existence of the curved endomorphisms operad, was discovered independently of [DCBM20].

**Definition 2.1** (Pre-differential graded module). A pre-differential graded module (pdg module for short) \((V, d_V)\) is the data of a graded module \(V\) together with a linear map \(d_V : V \to V\) of degree \(-1\). A morphism \(f : (V, d_V) \to (W, d_W)\) is a morphism of graded modules \(f : V \to W\) that commutes with the pre-differentials.

Pre-differential graded modules form a symmetric monoidal category \((\text{pdg mod}, \otimes, \mathbb{K})\), where the pre-differential on the graded tensor product \(A \otimes B\) is given by
\[
d_{A \otimes B}(a \otimes b) := d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b).
\]

The symmetric monoidal category \(\text{pdg mod}\) will be our underlying category. Notice that we do not ask the condition \(d_V^2 = 0\). The category of dg modules is a full subcategory of the category of pdg modules.
2.1. First definitions. The data of a pdg operad is the data $(\mathcal{P}, \gamma, \eta, d_\mathcal{P})$ of a graded operad $(\mathcal{P}, \gamma, \eta)$ together with a derivation $d_\mathcal{P}$ of degree $-1$. Similarly, a pdg partial operad amounts to a graded partial operad $(\mathcal{P}, \{\alpha_i\})$ equipped with a degree $-1$ derivation $d_\mathcal{P}$. Again, a pdg cooperad $(\mathcal{C}, \Delta, e, d_\mathcal{C})$ amounts to a graded cooperad $(\mathcal{C}, \Delta, e)$ with a coderivation $d_\mathcal{C}$ of degree $-1$ and a pdg partial cooperad $(\mathcal{C}, \{\Delta_i\}, d_\mathcal{C})$ amounts to a graded partial cooperad $(\mathcal{C}, \{\Delta_i\})$ with a coderivation $d_\mathcal{C}$ of degree $-1$. The notions of conilpotency for partial cooperads or completeness for partial operads are the same in this framework.

**Notation.** Since we already use $\circ$ for the composition product of $S$-modules, we will sometimes denote the composition of two morphisms by $\cdot$ when the context might be ambiguous.

**Definition 2.2 (Curved operad).** A curved operad $(\mathcal{P}, \gamma, \eta, d_\mathcal{P}, \Theta_\mathcal{P})$ amounts to the data of a pdg operad $(\mathcal{P}, \gamma, \eta, d_\mathcal{P})$ together with a morphism of pdg $S$-modules $\Theta : (1, 0) \to (\mathcal{P}, d_\mathcal{P})$ of degree $-2$, such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{P} & \overset{\text{diag}}{\longrightarrow} & \mathcal{P} \oplus \mathcal{P} \cong (1 \circ \mathcal{P}) \oplus (\mathcal{P} \circ 1) \\
 & & \downarrow \gamma(1) \\
 & & \mathcal{P}
\end{array}
\]

where $\text{diag}$ is given by $\text{diag}(\mu) := (\mu, \mu)$.

The data of the morphism of pdg $S$-modules $\Theta_\mathcal{P}$ is equivalent to an element $\Theta_\mathcal{P}(\text{id}) := \theta$ in $\mathcal{P}(1)_{-2}$ such that $d_\mathcal{P}(\theta) = 0$. The commutativity of the diagram amounts to the following condition: for every $\mu$ in $\mathcal{P}(n)$, we ought to have

\[
d_\mathcal{P}^2(\mu) = 1 \cdots n - \sum_{i=1}^{n} 1 \cdots \theta \cdots n \]

This states that the square of the pre-differential $d^2_\mathcal{P}$ is equal to the operadic Lie bracket $[\theta, -]$ on the totalization of $\mathcal{P}$. By a slight abuse of notation, we denote this equality by

\[
d_\mathcal{P}^2 = \gamma(1) \cdot (\Theta_\mathcal{P} \circ \text{id} - \text{id} \circ \Theta_\mathcal{P}),
\]

forgetting diag and the identifications made.

**Notation.** A curved operad $(\mathcal{P}, \gamma, \eta, d_\mathcal{P}, \Theta_\mathcal{P})$ will be denoted by $(\mathcal{P}, d_\mathcal{P}, \Theta_\mathcal{P})$ for short, making the composition map and the unit implicit. When regarded as an element of $\mathcal{P}(1)$, the curvature is denoted $\theta$. Likewise for pdg operads, which will be sometimes denoted by $(\mathcal{P}, d_\mathcal{P})$ when the context is clear.

**Definition 2.3.** A morphism of curved operads $f : (\mathcal{P}, d_\mathcal{P}, \Theta_\mathcal{P}) \to (\mathcal{D}, d_\mathcal{D}, \Theta_\mathcal{D})$ amounts to the data of a pdg operad morphism $f : (\mathcal{P}, d_\mathcal{P}) \to (\mathcal{D}, d_\mathcal{D})$ that preserves the curvatures $f \circ \Theta_\mathcal{P} = \Theta_\mathcal{D}$.

Let $(V, d_V)$ be a pdg module, we consider the endomorphism pdg $S$-module given by:

\[\text{End}_V(n) := \text{Hom}(V^\otimes n, V).\]

Endowed with the composition of functions, and the pre-differential $\partial := [d_V, -]$ , it forms a pdg operad, called the endomorphism operad of $V$.

**Lemma 2.4.** Equipped with the curvature $\Theta_V : 1 \to \text{End}_V$ given by $\Theta_V(\text{id}) = d_V^2$, the data $(\text{End}_V, \partial, \Theta_V)$ forms a curved operad, called the curved endomorphism operad associated to $(V, d_V)$.
Let \( \gamma \) be curved by definition.

**Definition 2.5 (pdg \( \mathcal{P} \)-algebra).** Let \( (\mathcal{P}, d_\mathcal{P}) \) be a pdg operad. A pdg \( \mathcal{P} \)-algebra \( (A, \gamma_A, d_A) \) amounts to the data of a graded \( \mathcal{P} \)-algebra structure \( \gamma_A : \mathcal{J}(\mathcal{P})(A) \rightarrow A \) on a pdg module \( (A, d_A) \) such that

\[
\gamma_A \cdot (\mathcal{J}(d_\mathcal{P})(\text{id}) + \mathcal{J}(\text{id})(\text{III}(\text{id}, d_A))) = d_A \cdot \gamma_A.
\]

A morphism of pdg \( \mathcal{P} \)-algebras \( g : (A, \gamma_A, d_A) \rightarrow (B, \gamma_B, d_B) \) amounts to a morphism of pdg modules \( f : (A, d_A) \rightarrow (B, d_B) \) that commutes with the \( \mathcal{P} \)-algebra structures. The category of pdg \( \mathcal{P} \)-algebras is denoted by pdg \( \mathcal{P} \)-alg.

**Remark 2.6.** The notation \( \mathcal{J}(\text{id})(\text{III}(\text{id}, d_A)) \) is explained in 1.1.

**Definition 2.7 (Curved \( \mathcal{P} \)-algebra).** Let \( (\mathcal{P}, d_\mathcal{P}, \Theta_{\mathcal{P}}) \) be a curved operad. A curved \( \mathcal{P} \)-algebra \( (A, \gamma_A, d_A) \) amounts to the data of a pdg \( \mathcal{P} \)-algebra structure \( (A, \gamma_A, d_A) \) such that the following diagram commutes:

\[
\mathcal{J}(1)(A) \xrightarrow{\mathcal{J}(\Theta_{\mathcal{P}})(\text{id})} \mathcal{J}(\mathcal{P})(A) \xrightarrow{d_A} A.
\]

Otherwise stated, we have that \( \gamma_A(\theta, a) = d_A^2(a) \) for any \( a \) in \( A \). Morphisms of curved \( \mathcal{P} \)-algebras are just morphisms of pdg \( \mathcal{P} \)-algebras. The category of curved \( \mathcal{P} \)-algebras will be denoted curv \( \mathcal{P} \)-alg.

**Remark 2.8.** Notice that a curved operad \( (\mathcal{P}, d_\mathcal{P}, \Theta_{\mathcal{P}}) \) is the data of a pdg operad \( (\mathcal{P}, d_\mathcal{P}) \) together with extra structure given by the curvature \( \Theta_{\mathcal{P}} \). On the other hand, the data of a curved algebra structure over a curved operad is the same as the data of a pdg algebra structure over its underlying pdg operad, but this structure has to satisfy an extra property given by diagram 3.

There is an obvious inclusion functor \( \text{inc} : \text{curv} \mathcal{P}-\text{alg} \hookrightarrow \text{pdg} \mathcal{P}-\text{alg} \) which is fully faithful.

**Proposition 2.9 ([DCBM20, Proposition C.31]).** Let \( (\mathcal{P}, d_\mathcal{P}, \Theta_{\mathcal{P}}) \) be a curved operad. The inclusion functor has a left adjoint

\[
\text{Curv} : \text{pdg} \mathcal{P}-\text{alg} \rightarrow \text{curv} \mathcal{P}-\text{alg}.
\]

Hence \( \text{curv} \mathcal{P}-\text{alg} \) is a reflexive subcategory of \( \text{pdg} \mathcal{P}-\text{alg} \). For a pdg \( \mathcal{P} \)-algebra \( (A, \gamma_A, d_A) \), its image under this functor is given by the following quotient:

\[
\text{Curv}(A) := \frac{A}{\text{Im}(\gamma_A(\theta, -) - d_A^2(-))}.
\]

where \( \text{Im}(\gamma_A(\theta, -) - d_A^2(-)) \) is the ideal generated by \( \text{Im}(\gamma_A(\theta, -) - d_A^2(-)) \). Its pdg \( \mathcal{P} \)-algebra structure is induced by \( \gamma_A \) and \( d_A \).

**Proof.** First, lets show that \( \text{Im}(\gamma_A(\theta, -) - d_A^2(-)) \) is stable under the pre-differential \( d_A \). We have that \( d_A(\gamma_A(\theta, a) - d_A^2(a)) = \gamma_A(d_A(\theta), a) + \gamma_A(\theta, d_A(a)) - d_A^3(a) \), which is again in \( \text{Im}(\gamma_A(\theta, -) - d_A^2(-)) \) since \( \gamma_A(d_A(\theta), a) = 0 \). Hence the quotient forms a pdg \( \mathcal{P} \)-algebra, which is curved by definition.

Let \( A \) be a pdg \( \mathcal{P} \)-algebra and \( B \) be a curved \( \mathcal{P} \)-algebra. Since the pdg \( \mathcal{P} \)-algebra structure on \( B \) satisfies diagram 3, any pdg \( \mathcal{P} \)-algebra morphism \( f : A \rightarrow B \) factors through \( \text{Curv}(A) \). This gives the adjunction isomorphism by the universal property of the quotient.

**Corollary 2.10.** Let \( (\mathcal{P}, d_\mathcal{P}, \Theta_{\mathcal{P}}) \) be a curved operad. The category of curved \( \mathcal{P} \)-algebras is complete and cocomplete.

**Proof.** The category pdg \( \mathcal{P} \)-alg is both complete and cocomplete, since curv \( \mathcal{P} \)-alg is a reflexive subcategory of pdg \( \mathcal{P} \)-alg by Proposition 2.9.
Lemma 2.11. Let \((\mathcal{P}, d_p, \Theta_p)\) be a curved operad and let \(\langle A, d_A \rangle\) be a pdg module. The data of a curved \(\mathcal{P}\)-algebra structure \(\gamma_A\) on \(\langle A, d_A \rangle\) is equivalent to the data of a morphism of curved operads \(\Gamma_A : (\mathcal{P}, d_p, \Theta_p) \longrightarrow (\text{End}_A, \partial, \Theta_A)\).

Proof. The data of a graded \(\mathcal{P}\)-algebra structure \(\gamma_A : \mathcal{P}(\langle A \rangle) \longrightarrow A\) is equivalent to the data of a morphism of graded operads \(\Gamma_A : \mathcal{P} \longrightarrow \text{End}_A\). One can check that condition of equation 2 is equivalent to \(\Gamma_A\) being a morphism of pdg operads. The commutativity of the diagram 3 means that \(\gamma_A(\mathcal{P}(\langle d_p \rangle)(\langle \Theta_p \rangle)) = d_A^2\), which is equivalent to \(\Gamma_A(\mathcal{P}) = \Theta_A\).

Given a pdg module \((V, d_V)\), we can endow the *coendomorphism operad* \(\text{Coend}_V\) with a curved operad structure using the same curvature \(\Theta_V(\langle \text{id} \rangle) := d_V^2\).

Definition 2.12 (Curved \(\mathcal{P}\)-coalgebra). A curved \(\mathcal{P}\)-coalgebra structure on \((V, d_V)\) is the data of a morphism of curved operads \(\Delta_V : (\mathcal{P}, d_p, \Theta_p) \longrightarrow (\text{Coend}_V, \partial, \Theta_V)\).

Remark 2.13. The notion of a curved \(\mathcal{P}\)-algebra can encode types of curved coalgebras which are not necessarily conilpotent.

Remark 2.14. It is straightforward to introduce the analogous notion of a curved partial operad, since the condition imposed on the pre-differential involves the partial compositions of the operad and not the total compositions. The category of curved partial operads is denoted by \(\text{curv pOp}\). Furthermore, it immediate to see that the category of curved unital partial operads is equivalent to the category of curved operads. See Proposition 1.24 for the non-curved statement.

2.2. First examples. Here are some classical examples of curved algebras that one can encode with curved operads. See for instance [Pos11] for curved associative algebras. See for instance [CLM16] for curved Lie algebras. For a first treatment of curvature using operadic methods, see [HM12].

Definition 2.15 (Curved Lie algebra). A curved Lie algebra \((g, [-,-], d_g, \theta)\) is the data of a graded Lie algebra \((g, [-,-])\) together with a pre-differential \(d_g\) of degree \(-1\) which is a derivation with respect to the bracket \([-,-]\), and a morphism of graded modules \(\theta : \mathbb{K} \longrightarrow g\) of degree \(-2\) such that:

\[d_g^2 = [\theta(1), -], \quad \text{and} \quad d_g(\theta(1)) = 0.\]

A morphism \(f : (g, d_g, \theta_g) \longrightarrow (h, d_h, \theta_h)\) is the data of a graded Lie algebra morphism \(f : g \longrightarrow h\) that commutes with the pre-differentials and such that \(f(\theta_g) = \theta_h\).

Recall that the classical partial operad \(\mathcal{L}\text{ie}\), encoding Lie algebras, can be defined as the free partial operad generated by one binary skew-symmetric operation, modulo the operadic ideal generated by the Jacobi relation. See [LV12, Section 13.2] for a complete account.

Let \(M\) be the pdg \(\mathcal{S}\)-module given by \((\mathbb{K}, \zeta, 0, \mathbb{K}, \beta, 0, \cdots)\) with zero pre-differential, where \(\zeta\) is an arity 0 operation of degree \(-2\), and \(\beta\) is an arity 2 operation of degree 0, basis of the signature representation of \(S_2\).

Definition 2.16 (\(c\mathcal{L}\text{ie}\) operad). The curved partial operad \(c\mathcal{L}\text{ie}\) is given by the free pdg partial operad generated by \(M\) modulo the operadic ideal generated by the Jacobi relation on the generator \(\beta\). It is endowed with the curvature \(\Theta_{\mathcal{L}}\) given by \(\Theta_{\mathcal{L}}(\langle \text{id} \rangle) := \beta \circ_1 \zeta\).

Lemma 2.17. The data \((c\mathcal{L}\text{ie}, 0, \Theta_{\mathcal{L}})\) forms a curved partial operad. Furthermore, the category of curved \(c\mathcal{L}\text{ie}\)-algebras is isomorphic to the category of curved Lie algebras.

Proof. First, let us check that \((c\mathcal{L}\text{ie}, 0, \Theta_{\mathcal{L}})\) is indeed a curved partial operad. The pre-differential is 0, hence \(d(\theta) = 0\). Now we need to check that \([\theta, -] = 0\). Since \([\theta, -]\) is a derivation, it is enough to check it on the two generators of \(c\mathcal{L}\text{ie}\). The result \([\theta, \theta] = 0\) follows from the antisymmetry of \(\beta\). A straightforward computation gives that \([\theta, \beta] = 0\), using the Jacobi relation.

A pdg algebra over the curved partial operad \(c\mathcal{L}\text{ie}\) amounts to a pdg-module \((A, d_A)\) endowed with a graded Lie bracket \([-,-] := \gamma_A(\beta)\) and a morphism \(\theta := \gamma_A(\zeta) : \mathbb{K} \longrightarrow g\) such that
\(d_A(\theta(1)) = 0\). They form a curved c\(\mathcal{L}\)ie-algebra if and only if Diagram 3 commutes, which equivalent to \([0, -] = d_A^2(-)\). Morphisms of curved c\(\mathcal{L}\)ie-algebras must commute with the structure, i.e: the bracket and curvature \(\vartheta\); and with the underlying pre-differentials. □

**Definition 2.18 (Curved associative algebra).** A curved associative algebra \((A, \mu_A, d_A, \vartheta)\) amounts to the data a non-unital graded associative algebra \((A, \mu_A)\) together with a pre-differential \(d_A\) of degree \(-1\) which is a derivation with respect to the associative product \(\mu_A\), and a morphism of graded modules \(\vartheta : K \to A\) of degree \(-2\) such that:

\[d_A^2 = \mu_A(\theta(1), -) - \mu_A(-, \theta(1)), \quad \text{and} \quad d_A(\theta(1)) = 0.\]

A morphism \(f : (A, \mu_A, d_A, \vartheta_A) \to (B, \mu_B, d_B, \vartheta_B)\) is the data of a morphism of graded non-unital associative algebras \(f : (A, \mu_A) \to (B, \mu_B)\) that commutes with the pre-differentials and preserves the curvatures \(f(\vartheta_A) = \vartheta_B\).

Let \(N\) be the pdg \(S\)-module given by \((K, \phi, 0, K[S_2], \mu, 0, \cdots)\) with zero pre-differential, where \(\phi\) is an arity 0 operation of degree \(-2\), and \(\mu\) is an binary operation of degree 0, basis of the regular representation of \(S_2\).

**Definition 2.19 (c\(\mathcal{A}\)ss operad).** The curved partial operad c\(\mathcal{A}\)ss is given by the free pdg partial operad generated by \(N\) modulo the operadic ideal generated by the associativity relation on the generator \(\mu\). It is endowed with the curvature \(\Theta_A\) given by \(\Theta_A(\text{id}) := \mu \circ_1 \phi - \mu \circ_2 \phi\).

**Lemma 2.20.** The data \((c\(\mathcal{A}\)ss, 0, \Theta_A)\) forms a curved partial operad. Furthermore, the category of curved c\(\mathcal{A}\)ss-algebras is isomorphic to the category of curved associative algebras.

**Proof.** The proof of this lemma follows the same steps as the proof of Lemma 2.17. □

**Remark 2.21.** See Appendix 9.2 for the "absolute analogues" of these curved operads.

**Proposition 2.22.** The classical morphism of partial operads \(\mathcal{L}\)ie \(\to\) \(Ass\) induced by \(\beta \mapsto \mu - \mu^{(12)}\), extends to a morphism of curved partial operads c\(\mathcal{L}\)ie \(\to\) c\(\mathcal{A}\)ss by sending \(\zeta\) to \(\vartheta\).

**Proof.** It is straightforward to check that the latter morphism commutes with the respective curvatures. □

**Corollary 2.23.** The morphism c\(\mathcal{L}\)ie \(\to\) c\(\mathcal{A}\)ss induces a functor

\[
\begin{align*}
\text{Skew} : \text{curv c\(\mathcal{A}\)ss-alg} & \longrightarrow \text{curv c\(\mathcal{L}\)ie-alg} \\
(A, \mu_A, d_A, \vartheta) & \mapsto (A, [-, -] := \mu_A - \mu_A^{(12)}, d_A, \vartheta).
\end{align*}
\]

given by the skew-symmetrization of the associative product.

**Proof.** Given a morphism of curved partial operads \(\Gamma_A : c\(\mathcal{A}\)ss \to \text{End}_A\), one pulls back along c\(\mathcal{L}\)ie \(\to\) c\(\mathcal{A}\)ss. □

**Remark 2.24.** In the next section we will construct the left adjoint of this functor using curved bimodules.

3. **Curved bimodules and universal functors**

The goal of this section is to define the "curved" generalization of modules over operads. Their appeal comes from the fact that operadic modules encode universal functors relating categories of algebras over operads. The generalization is not immediate since in a curved operad, the curvature condition intertwines the underlying category with the operadic structure. Nevertheless, there exists a good notion of bimodule that encodes universal functors between the categories of curved algebras over curved operads. Note that in order for a bimodule to define a functor between categories of algebras over operads, we need to use the monoidal definition of operads, like in [Fre09]. If one is working with curved partial operads, one can simply add an augmented unit to fit into this framework.
3.1. Curved bimodules. One major difference between curved operads and classical operads is the following fact. Let \((P, d_P, \Theta_P)\) be a curved operad and \((V, d_V)\) be a pdg module. Then \(\mathcal{S}(P)(V)\) endowed with its canonical \(P\)-algebra structure does not form a curved \(P\)-algebra in general. The pre-differential on \(\mathcal{S}(P)(V)\) is given by:

\[
d_{\mathcal{S}(P)(V)} = \mathcal{S}(d_P)(d_V) + \mathcal{S}(id_P)(\text{Id}(d_V, d_V))
\]

Hence we have that

\[
d_{\mathcal{S}(P)(V)}^2 = \mathcal{S}(d_P)(d_V) + \mathcal{S}(id_P)(\text{Id}(d_V, d_V))
\]

because of the Koszul sign convention. Since \(P\) is a curved operad:

\[
d_{\mathcal{S}(P)(V)} = \gamma(1) \cdot (\Theta_P \circ id_P - id_P \circ \Theta_P).
\]

Therefore

\[
d_{\mathcal{S}(P)(V)}^2 = \mathcal{S}(\gamma)(d_V) \cdot \left(\mathcal{S}(\Theta_P) \circ \mathcal{S}(id_P)(d_V) - \mathcal{S}(id_P) \circ \mathcal{S}(\text{Id}(id_P, \Theta_P))(d_V)\right)
\]

and does not satisfy the condition imposed by the diagram \(3\). In general

\[
d_{\mathcal{S}(P)(V)}^2 \neq \mathcal{S}(\gamma)(d_V) \cdot \left(\mathcal{S}(\Theta_P) \circ \mathcal{S}(id_P)(d_V)\right),
\]

even if we impose the extra condition that \(d_V^2 = 0\).

**Proposition 3.1** ([DCBM20, Proposition C.31]). Let \((P, d_P, \Theta_P)\) be a curved operad and \((V, d_V)\) be a pdg module. The free curved \(P\)-algebra is given by:

\[
F(P)(V) := \frac{\mathcal{S}(P)(A)}{\text{Im}(\mathcal{S}(id_P)(d_V^2) - \mathcal{S}(\Theta_P)(d_V))}.
\]

**Proof.** One can check that \(F(P)(V)\) is equal to the composition of the free pdg \(P\)-algebra functor \(\mathcal{S}(P)(V)\) followed by the reflector \(\text{Curv}\) constructed in Proposition 2.9. \(\square\)

This type of construction is encoded by a left \(P\)-module in the classical case. The above Proposition shows that in the curved case, modules over operads are more intricate, since it is not obvious that this quotient can be interpreted as a "curved" left \(P\)-module. We begin by recalling standard definitions.

**Definition 3.2** (Left and right \(P\)-modules). Let \((P, \gamma, \eta, d_P)\) be a pdg operad.

1. A left pdg \(P\)-module \((N, \lambda, d_N)\) is the data of a pdg \(S\)-module \((N, d_N)\) together with a morphism of pdg \(S\)-modules \(\lambda : P \circ N \rightarrow N\), such that the following diagram

\[
\begin{array}{ccc}
P \circ P \circ N & \xrightarrow{\gamma_P \circ id} & P \circ N \\
\text{id} \circ \lambda & \downarrow & \downarrow \lambda \\
P \circ N & \xrightarrow{\lambda} & N
\end{array}
\]

commutes and such that \(\lambda \cdot (d_N \circ d_N) = d_N\).

2. A right pdg \(P\)-module \((M, \rho, d_M)\) is the data of a pdg \(S\)-module \((M, d_M)\) together with a morphism of pdg \(S\)-modules \(\rho : M \circ P \rightarrow P\), such that the following diagram

\[
\begin{array}{ccc}
M \circ P \circ P & \xrightarrow{id \circ \gamma} & M \circ P \\
\rho \circ id & \downarrow & \downarrow \rho \\
M \circ P & \xrightarrow{\rho} & M
\end{array}
\]

commutes and such that \(\rho \cdot (id_M \circ \rho) = id_M\).
Remark 3.3. One can identify left pdg $\mathcal{P}$-modules concentrated in arity 0 with the category of pdg $\mathcal{P}$-algebras.

**Definition 3.4 (Relative composition product).** Let $(\mathcal{P}, d_{\mathcal{P}})$ be a pdg operads. Let $(N, \lambda_N, d_N)$ be a left pdg $\mathcal{P}$-module and let $(M, \rho_M, d_M)$ be a right pdg $\mathcal{P}$-module, the relative composition product $M \circ_{\mathcal{P}} N$, given by the following coequalizer

$$M \circ_{\mathcal{P}} N := \text{Coeq} \left( M \circ \mathcal{P} \circ N \xrightarrow{\rho_M \circ \text{id}} M \circ N \right),$$

in the category of pdg $S$-modules.

**Definition 3.5 (Operadic bimodule).** Let $(\mathcal{P}, d_{\mathcal{P}})$ and $(\mathcal{Q}, d_{\mathcal{Q}})$ be two pdg operads. A pdg $(\mathcal{P}, \mathcal{Q})$-bimodule is the data $(M, \lambda, \rho, d_M)$ of a pdg $S$-module $(M, d_M)$ together with two morphisms of pdg $S$-modules $\lambda : \mathcal{P} \circ M \to M$ and $\rho : M \circ \mathcal{Q} \to M$ which endow $M$ with a left $\mathcal{P}$-module structure and a right $\mathcal{P}$-module structure. Those structures are compatible with each other in the following sense:

$$\begin{array}{ccc}
\mathcal{P} \circ M \circ \mathcal{Q} & \xrightarrow{\lambda \circ \text{id}} & M \circ \mathcal{Q} \\
\text{id} \circ \rho & & \rho \\
\mathcal{P} \circ M & \xrightarrow{\lambda} & M.
\end{array}$$

Operadic bimodules encode functors between the categories of algebras over operads, see [Fre09, Chapter 9] for a detailed account.

**Definition 3.6 (Relative Schur functor of a bimodule).** Let $(M, \lambda, \rho, d_M)$ be a pdg $(\mathcal{P}, \mathcal{Q})$-bimodule. The relative Schur functor $\mathcal{S}_\mathcal{Q}(M)(-)$ associated to $M$ is given, for $(A, \gamma_A, d_A)$ a pdg $\mathcal{Q}$-algebra, by the following coequalizer:

$$\mathcal{S}_\mathcal{Q}(M)(A) := \text{Coeq} \left( \mathcal{S}(M) \circ \mathcal{S}(\mathcal{Q})(A) \xrightarrow{\mathcal{S}(\rho) \circ \text{id}} \mathcal{S}(\mathcal{Q})(A) \right).$$

It defines a functor:

$$\mathcal{S}_\mathcal{Q}(M)(-): \text{pdg } \mathcal{Q}\text{-alg } \to \text{ pdg } \mathcal{P}\text{-alg}.$$

Under the identification of left pdg $\mathcal{Q}$-modules concentrated in arity 0 with pdg $\mathcal{Q}$-algebras, $\mathcal{S}_\mathcal{Q}(M)(A)$ is in fact given by the relative composition product $M \circ_{\mathcal{Q}} A$ of the left pdg $\mathcal{Q}$-module $(A, \gamma_A, d_A)$ with the $(\mathcal{P}, \mathcal{Q})$-bimodule $(M, \lambda, \rho, d_M)$.

**Definition 3.7 (Curved Bimodules).** Let $(\mathcal{P}, d_{\mathcal{P}}, \Theta_{\mathcal{P}})$ and $(\mathcal{Q}, d_{\mathcal{Q}}, \Theta_{\mathcal{Q}})$ be two curved operads. A curved $(\mathcal{P}, \mathcal{Q})$-bimodule is the data of a pdg $(\mathcal{P}, \mathcal{Q})$-bimodule $(M, \lambda_M, \rho_M, d_M)$ such that the following diagram commutes:

$$\begin{array}{ccc}
M & \xrightarrow{\text{diag}} & (I \circ M) \oplus (M \circ I) \\
\downarrow{d_M} & & \downarrow{(\Theta_{\mathcal{P}} \circ \text{id})-(\text{id} \circ o' \Theta_{\mathcal{Q}})} \\
M & \xrightarrow{\lambda+\rho_{(1)}} & (M \circ (1)) \oplus \mathcal{Q}.
\end{array}$$

This equality is denoted by $d_M^2 = \lambda \cdot (\Theta_{\mathcal{P}} \circ \text{id}) - \rho_{(1)} \cdot (\text{id} \circ o' \Theta_{\mathcal{Q}})$.

Remark 3.8. A curved generalization of infinitesimal bimodules for curved partial operads is immediate since the curvature is an infinitesimal notion. For instance, curved infinitesimal bimodules are the coefficients for the André-Quillen cohomology of a curved operad, which can be defined using the same methods as for operads and standard bimodules. See [MV09] for more details.
Proposition 3.9. Any curved \((P, Q)\)-bimodule \((M, \lambda, \rho, d_M)\) induces a functor
\[
\mathcal{S}_Q(M)(-): \text{curv } Q\text{-alg} \rightarrow \text{curv } P\text{-alg},
\]
given by the relative Schur functor associated to M as a pdg \((P, Q)\)-bimodule.

Proof. Let \((A, \gamma_A, d_A)\) be a curved \(Q\)-algebra. Let \(\pi_A: \mathcal{S}_Q(M)(A) \rightarrow \mathcal{S}_Q(M)(A)\) be the projection map. The pre-differential of \(\mathcal{S}_Q(M)(A)\) is given by the image in the quotient of
\[
\mathcal{S}_Q(d_M)(\text{id}_A) + \mathcal{S}_Q(\text{id}_M)(\text{id}(\text{id}_A, d_A)),
\]
which we will denote \(d\) for simplicity. The pdg \(P\)-algebra structure on \(\mathcal{S}_Q(M)(A)\) is given by the map \(\pi_A \cdot (\mathcal{S}(\lambda)(\text{id}))\). We need to check that \(\mathcal{S}_Q(M)(A)\) is indeed a curved \(P\)-algebra, i.e.: that the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{S}_Q(I)(\mathcal{S}_Q(M)(A)) & \xrightarrow{\mathcal{S}(\Theta_P)(\text{id})} & \mathcal{S}_Q(\mathcal{S}_Q(M)(A)) \\
d^2 & & \downarrow \pi_A \cdot (\mathcal{S}(\lambda_M)(\text{id})) \\
\mathcal{S}_Q(M)(A) & & \\
\end{array}
\]

We know that \(d^2\) will be induced by \(\mathcal{S}_Q(d_M^2)(\text{id}_A) + \mathcal{S}_Q(\text{id}_M)(\text{id}(\text{id}_A, d_A^2))\). On one hand, since \(M\) is a curved \((P, Q)\)-bimodule, \(d^2_M = \lambda \cdot (\Theta_P \circ \text{id}) - \rho_{(1)} \cdot (\text{id} \circ \Theta_Q)\). On the other hand, since \(A\) is a curved \(Q\)-algebra, \(d^2_A = \gamma_A \cdot (\mathcal{S}(\Theta_Q)(\text{id}))\). Therefore:
\[
d^2 = \pi_A \cdot (\mathcal{S}(\lambda)(\text{id}_A)) \cdot (\mathcal{S}(\Theta_P) \circ \mathcal{S}(\text{id}_M)(\text{id}_A)) - \pi_A \cdot (\mathcal{S}(\rho)(\text{id}_A)) \cdot (\mathcal{S}(\text{id}_M, \Theta_Q)(\text{id})(\text{id}_A)) + \pi_A \cdot (\mathcal{S}(\text{id}_M)(\text{id}(\text{id}_A, \gamma_A \cdot (\mathcal{S}(\Theta_Q)(\text{id}))))).\]

By definition of \(\mathcal{S}_Q(M)(A)\), we have that the right action of \(Q\) on \(M\) given by \(\rho\) is equal to the action of \(Q\) on \(A\) given by \(\gamma_A\), hence the last two terms are equal and therefore cancel each other. As a result:
\[
d^2 = \pi_A \cdot (\mathcal{S}(\lambda)(\text{id}_A)) \cdot (\mathcal{S}(\Theta_P) \circ \mathcal{S}(\text{id}_M)(\text{id}_A)),\]
which is precisely the condition imposed by the above diagram.

\(\Box\)

Example 3.10. Given a curved operad \((P, d_P, \Theta_P)\), its underlying pdg \(S\)-module \((P, d_P)\) can be endowed with a canonical curved \((P, P)\)-bimodule structure given by the composition of \(P\). This curved bimodule encodes the identity endofunctor of curved \(P\)-algebras.

Theorem 3.11. Let \((P, d_P, \Theta_P)\) and \((Q, d_Q, \Theta_Q)\) be two curved operads, and let \(f: P \rightarrow Q\) be a morphism of curved operads. The morphism \(f\) induces an adjunction at the level of curved algebras:
\[
\begin{array}{ccc}
\text{Ind}_f: \text{curv } P\text{-alg} & \xrightarrow{\perp} & \text{curv } Q\text{-alg} \\
\end{array}
\]
given by the restriction and the induction functors.

Proof. The right adjoint is given by the restriction along \(f\) as follows. Given a curved \(Q\)-algebra structure on a pdg module \((A, d_A)\), that is, a morphism of curved operads \(\Gamma_A: Q \rightarrow \text{End}_A\), one can always pre-compose \(\Gamma_A\) with \(f\). This curved operadic morphism \(\Gamma_A \cdot f: P \rightarrow \text{End}_A\) endows \((A, d_A)\) with a curved \(P\)-algebra structure, denoted by \(\text{Res}_f(A)\).

On the other hand, the morphism \(f\) allows us to endow \(Q\) with a curved \((Q, P)\)-bimodule structure, where right action of \(P\) is given by:
\[
\rho: Q \circ P \xrightarrow{id_Q \circ f} Q \circ Q \xrightarrow{\gamma_Q} Q,\]
and the left action of \(Q\) is simply given by \(\gamma_Q\). Let us check that this endows \(Q\) with a curved bimodule structure. We have that:
\[
d^2_Q = \gamma_Q \cdot (\Theta_Q \circ \text{id}) - (\gamma_Q)_{(1)} \cdot (\text{id} \circ \Theta_Q) = \gamma_Q \cdot (\Theta_Q \circ \text{id}) - \rho_{(1)} \cdot (\text{id} \circ \Theta_P)
\]

since $f \cdot \Theta_P = \Theta_Q$. Thus, the curved $(\Omega, \mathcal{P})$-bimodule $\Omega$ induces a functor
\[ \text{Ind}_f(-) := \mathcal{P}(\Omega)(-): \text{curv \, \mathcal{P}alg} \to \text{curv \, \Omegaalg}. \]
Since morphism of curved algebras are morphisms of pdg algebras, this general operadic construction is still an adjunction.

One important application of this new result is the construction of the curved universal enveloping algebra of a curved Lie algebra. To the best of our knowledge, this construction is new. The morphism $cLie \to cAss$, defined in Proposition 2.22, induces the following adjunction.

**Corollary 3.12 (Curved universal enveloping algebra).** There an adjunction between the category of curved Lie algebras and the category of curved associative algebras
\[ \mathcal{U}: \text{curv \, cLie-\text{alg}} \rightleftarrows \text{curv \, cAss-\text{alg}}: \text{Skew}, \]
where $\mathcal{U}$ denotes the curved universal enveloping algebra and $\text{Skew}$ denotes the functor obtained by the skew-symmetrization of the associative product of Corollary 2.23.

**Proof.** Simply apply Theorem 3.11 to the morphism $f: cLie \to cAss$, and set $\mathcal{U} := \text{Ind}_f$ and $\text{Anti} := \text{Res}_f$. 

**Proposition 3.13.** Let $g$ be a curved Lie algebra. Its universal enveloping algebra is isomorphic to
\[ \mathcal{U}(g) \cong \overline{T}(g) / (x \otimes y - (-1)^{|x|} y \otimes x - [x, y]), \]
where $\overline{T}(-)$ denotes the non-unital tensor algebra, endowed with the curvature $\vartheta: K \to g \hookrightarrow \mathcal{U}(g)$.

**Proof.** One can check this adjunction by hand. Indeed, the adjunction holds between between pdg $cLie$-algebras and pdg $cAss$-algebras. It is straightforward to check that $\mathcal{U}(g)$ endowed with the curvature $\vartheta$ forms a curved $cLie$-algebra. 

**Remark 3.14 (PBW property).** In [DT20], the authors develop a general theory of PBW properties for monads: a morphism $f: M \to N$ has the PBW property if $N$ is free as a right $M$-module, where the structure is induced by the morphism $f$. This encompasses the classical PBW theorem, since we have an isomorphism
\[ \text{Ass} \cong \text{Com} \circ \text{Lie} \]
of right $Lie$-modules. This gives back the isomorphism of vector spaces between the universal enveloping algebra of a Lie algebra and the symmetric algebra of a Lie algebra.

The same result hold in the curved setting, where there is an isomorphism
\[ cAss \cong \text{Com} \circ cLie \]
of pdg right $cLie$-modules. This can be deduced from the classical PBW, since the morphism $cLie \to cAss$ preserves the arity zero component; grading by the number of corks, the map induced map in the associated graded is an isomorphism, therefore the map was an isomorphism to begin with. This PBW property implies that the universal enveloping algebra of a curved Lie algebra is also isomorphic to the symmetric algebra of the curved Lie algebra as graded vector spaces.

This machinery can also be applied to curved mixed $L_\infty$-algebras. These are curved $L_\infty$-algebras with two pre-differentials, one coming from the underlying pdg module and another one from the structure, such that their difference satisfies the axioms of a classical curved $L_\infty$-algebra. See [RiL22, Appendix] for more details. These algebras are exactly curved algebras over $\Omega(u\text{Com}^*)$, where $\Omega$ is the complete Cobar construction of Section 6.
Corollary 3.15 (Curved $A_\infty$ universal enveloping algebra). There is the following adjunction between the category of curved mixed $L_\infty$-algebras and the category of curved mixed $A_\infty$-algebras given by:

$$
\Omega : \text{curv.mix } L_\infty\text{-alg} \quad \dashv \quad \text{curv.mix } A_\infty\text{-alg} : \text{Anti},
$$
where $\Omega$ denotes the curved $A_\infty$ universal enveloping algebra.

Proof. There is a morphism of counital partial cooperads $\iota : u\text{Com}^* \to u\text{Ass}^*$ given by the linear dual of the classical morphism of unital partial operads $u\text{Ass} \to u\text{Com}$. Therefore, using the complete Cobar construction of Section 6, we get a morphism of curved partial operads $\bar{\Omega}(\iota) : \bar{\Omega}(u\text{Com}^*) \to \bar{\Omega}(u\text{Ass}^*)$. By Theorem 3.11 it induces an adjunction. \hfill \Box

3.2. The analogues of left and right modules. One may ask whether there exists a natural notion of a left (resp. right) curved $\mathcal{P}$-module. This is in fact not obvious. Since every algebra over an operad is a particular case of left module in the classical setting, a naive generalization could be: a left pdg $\mathcal{P}$-module $(M, \lambda, d_M)$ is a left curved $\mathcal{P}$-module if the following diagram commutes

$$
\begin{array}{ccc}
I \circ M & \xrightarrow{\Theta_\mathcal{P} \circ \text{id}} & \mathcal{P} \circ M \\
\downarrow d_M & & \downarrow \lambda \\
M. & & M.
\end{array}
$$

But in this case $\mathcal{S}(M)(-) \to \text{curv } \mathcal{P}\text{-alg}$, since $\mathcal{S}(M)(A)$ is not, in general, a curved $\mathcal{P}$-algebra. One can notice by doing a straightforward computation that for this notion of curved left $\mathcal{P}$-module, $\mathcal{S}(M)(A)$ is a curved $\mathcal{P}$-algebra if and only if $d_A^2 = 0$.

The conceptual explanation is that the data of a dg module is equivalent to the data of a curved algebra over the curved operad $(I, 0, 0)$, where $I$ is the trivial operad with the zero pre-differential and the zero curvature. The above-mentioned naive definition of a left curved $\mathcal{P}$-module is in fact a curved $(\mathcal{P}, I)$-bimodule, which is coherent with the fact that this notion encodes functors $\text{dg mod} \to \text{curv } \mathcal{P}\text{-alg}$. Likewise, a symmetric naive definition of a right curved $\mathcal{P}$-module would in fact be a $(I, \mathcal{P})$-bimodule that would encode functors $\text{curv } \mathcal{P}\text{-alg} \to \text{dg mod}$.

Remark 3.16. Let $(\mathcal{P}, \gamma, \eta, d_{\mathcal{P}}, \Theta_{\mathcal{P}})$ be a curved operad. The unit $\eta : I \to \mathcal{P}$ is not a morphism of curved operads. Otherwise, we could endow $\mathcal{P}$ with a curved $(\mathcal{P}, I)$-bimodule structure and $\mathcal{S}(\mathcal{P})(A)$ would be a curved $\mathcal{P}$-algebra for $(A, d_A)$ a dg module, which is not the case.

The natural question is then to find a curved operad that plays the same role as the unit operad $I$ plays for classical operads in the theory of bimodules. Otherwise stated, a curved operad that encodes pdg modules as its curved algebras.

Definition 3.17 ($\mathcal{J}\mathcal{C}$ operad). The pdg operad $\mathcal{J}\mathcal{C}$ is the free pdg operad generated by an operation of arity $1$ and degree $-2$. It is given by the pdg $S$-module $(0, K.\text{id} \oplus S(\theta), 0, \cdots)$ with zero pre-differential, where $S(\theta)$ denotes the free non-unital commutative algebra generated by $\theta$.

Lemma 3.18. The pdg operad $\mathcal{J}\mathcal{C}$, endowed the curvature $\Theta_{\mathcal{J}\mathcal{C}}(\text{id}) := \theta$ forms a curved operad. The category of curved $\mathcal{J}\mathcal{C}$-algebras is isomorphic to the category of pdg modules.

Proof. The commutator with $\theta$ is zero, hence $\mathcal{J}\mathcal{C}$ is a curved operad. Let $(V, d_V)$ be a pdg module and let $\Gamma : \mathcal{J}\mathcal{C} \to \text{End}_V$ be a morphism of curved operads. The image of $\theta$ determines $\Gamma$, since it is a morphism of curved operads then $\Gamma(\theta) = d_V^2$. \hfill \Box

For any curved operad $(\mathcal{P}, d_{\mathcal{P}}, \Theta_{\mathcal{P}})$, there is an unique morphism of curved operads $\varphi_{\mathcal{P}} : \mathcal{J}\mathcal{C} \to \mathcal{P}$ given by $\varphi_{\mathcal{P}}(\theta) = \theta_{\mathcal{P}}$, where $\theta_{\mathcal{P}} = \Theta_{\mathcal{P}}(\text{id})$. Therefore the pdg $S$-module $\mathcal{P}$ can be endowed canonically with a curved $(\mathcal{J}\mathcal{C}, \mathcal{P})$-bimodule structure and with a curved $(\mathcal{P}, \mathcal{J}\mathcal{C})$-bimodule structure.
Proposition 3.19. Let \((\mathcal{P}, \delta_{\mathcal{P}}, \Theta_{\mathcal{P}})\) be a curved operad and let

\[
F(\mathcal{P}) : \text{pdg mod} \xleftrightarrow{\perp} \text{curv } \mathcal{P}\text{-alg} : U,
\]

be the free-forgetful adjunction of Proposition 3.1. The free functor \(F(\mathcal{P})\) is naturally isomorphic to the functor \(\mathcal{A}_{\mathcal{C}}(\mathcal{P})\) given by the canonical curved \((\mathcal{P}, \mathcal{C})\)-bimodule structure on \(\mathcal{P}\). The forgetful functor \(U\) is naturally isomorphic to \(\mathcal{A}_{\mathcal{P}}(\mathcal{P})\) given by the canonical curved \((\mathcal{C}, \mathcal{P})\)-bimodule structure on \(\mathcal{P}\).

Proof. Let \((V, d_V)\) be a pdg module, \(\mathcal{A}_{\mathcal{C}}(\mathcal{P})(V)\) is defined by the same quotient as \(F(\mathcal{P})(V)\): the right action of \(\mathcal{C}\) on \(\mathcal{P}\) is given by \(\Theta_{\mathcal{P}}\) and the left action of \(\mathcal{C}\) on \((V, d_V)\) is given by \(d_V^{\mathcal{C}}\), which are identified in \(\mathcal{A}_{\mathcal{P}}(\mathcal{P})(V)\). It is straightforward to check that \(\mathcal{A}_{\mathcal{P}}(\mathcal{P})(V)\) amounts to \((V, d_V)\) endowed with its pdg module structure.

This indicates that curved \((\mathcal{P}, \mathcal{C})\)-bimodules are a good analogue to left \(\mathcal{P}\)-modules over operads in the case of curved operads; likewise, curved \((\mathcal{C}, \mathcal{P})\)-bimodules are a good analogue to right \(\mathcal{P}\)-modules.

Remark 3.20. To the best of our knowledge, these definitions also provide new notions of left and right modules for curved associative algebras when we consider them as curved operads concentrated in arity one.

4. CURVED COOPERADS AND CURVED PARTIAL COOPERADS

In this section, we briefly recall the notion of a curved cooperad which first appeared in [HM12]. Conilpotent curved partial cooperads will play the role of the Koszul dual of unital partial operads. The notions of coalgebras and algebras over cooperads also extend to the case of curved cooperads.

Definition 4.1 (Curved cooperad). A curved cooperad \((\mathcal{C}, \Delta, \epsilon, d_{\mathcal{C}}, \Theta_{\mathcal{C}})\) amounts to the data of a pdg cooperad \((\mathcal{C}, \Delta, \epsilon, d_{\mathcal{C}})\) and a morphism of pdg \(S\)-modules \(\Theta_{\mathcal{C}} : (\mathcal{C}, d_{\mathcal{C}}) \rightarrow (1, 0)\) of degree \(-2\), such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Delta_{(1)}} & \mathcal{C} \\
\downarrow{d_{\mathcal{C}}} & & \downarrow{\text{proj}} \\
\mathcal{C} & \xrightarrow{\epsilon \circ (1) - (\Theta_{\mathcal{C}} \circ (1) \cdot \text{id})} & \mathcal{C} \oplus (1 \circ \mathcal{C}) \cong \mathcal{C} \oplus \mathcal{C}
\end{array}
\]

where proj is given by proj\((\mu, \nu) := \mu + \nu\). A morphism of curved cooperads \(f : (\mathcal{C}, d_{\mathcal{C}}, \Theta_{\mathcal{C}}) \rightarrow (\mathcal{D}, d_{\mathcal{D}}, \Theta_{\mathcal{D}})\) is the data of a morphism of pdg cooperads \(f : (\mathcal{C}, d_{\mathcal{C}}) \rightarrow (\mathcal{D}, d_{\mathcal{D}})\) such that \(\Theta_{\mathcal{D}} \circ f = \Theta_{\mathcal{C}}\).

By a slight abuse of notation, we denote this equality by

\[
d_{\mathcal{C}}^2 = (\text{id} \circ (1) \cdot \Theta_{\mathcal{C}} - \Theta_{\mathcal{C}} \circ \text{id}) \cdot \Delta_{(1)},
\]

forgetting proj and the identifications made.

Remark 4.2. One also defines curved partial cooperads and curved counital partial cooperads as one would expect, since the condition on the curvature only involves partial decompositions. In this context, the comparison results of Section 1 between different types of cooperads extend without any difficulty to the curved case, mutatis mutandis.

Any curved cooperad \((\mathcal{C}, d_{\mathcal{C}}, \Theta_{\mathcal{C}})\) induces a comonad structure on its Schur functor \(\mathcal{A}_{\mathcal{C}}(\mathcal{C})\), and a monad structure on its dual Schur functor \(\mathcal{A}_{\mathcal{C}}(\mathcal{C})^\vee\). Hence we can define curved coalgebras and curved algebras over any given curved cooperad.
Definition 4.3 (Curved $\mathcal{C}$-coalgebra). Let $(\mathcal{C}, d_{\mathcal{C}}, \Theta_{\mathcal{C}})$ be a curved cooperad and let $(C, \Delta_C, d_C)$ be a pdg $\mathcal{C}$-coalgebra. It is a **curved $\mathcal{C}$-coalgebra** if the following diagram commutes:

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta_C} & \mathcal{F}(\mathcal{C})(C) \\
d_{\mathcal{C}} & & \downarrow \mathcal{F}(\Theta_{\mathcal{C}})(id) \\
C & \cong & \mathcal{F}(I)(C).
\end{array}
$$

A morphism of curved conilpotent $\mathcal{C}$-coalgebras $f : (C, \Delta_C, d_C) \to (C', \Delta_{C'}, d_{C'})$ is the data of a morphism of pdg $\mathcal{C}$-coalgebras.

Thus the category of curved $\mathcal{C}$-coalgebras is a full subcategory of the category of pdg $\mathcal{C}$-coalgebras. It is not obvious to us whether the category of curved $\mathcal{C}$-coalgebras admits a cofree object or not. It would be interesting to have a comonadicity result like in the case of curved algebras over a curved operad. Nevertheless, the following result guarantees the existence of limits and colimits.

**Theorem 4.4 ([Gri19]).** The category curved $\mathcal{C}$-coalgebras is a presentable category. In particular, it is both complete and cocomplete.

Definition 4.5 (Curved algebra over a cooperad). Let $(\mathcal{C}, d_{\mathcal{C}}, \Theta_{\mathcal{C}})$ be a curved cooperad and let $(B, \gamma_B, d_B)$ be a pdg $\mathcal{C}$-algebra. It is a **curved $\mathcal{C}$-algebra** if the following diagram commutes:

$$
\begin{array}{ccc}
B & \cong & \mathcal{F}(I)(B) \\
\gamma_B & \downarrow & \mathcal{F}(\mathcal{C})(B) \\
B & \xrightarrow{-d_B^2} & B.
\end{array}
$$

A morphism of curved $\mathcal{C}$-algebras $f : (B, \gamma_B, d_B) \to (B', \gamma_{B'}, d_{B'})$ is the data of a morphism of pdg $\mathcal{C}$-algebras.

Thus the category of curved $\mathcal{C}$-algebras is also a full subcategory of the category of pdg $\mathcal{C}$-algebras. The following proposition gives a reflector.

**Proposition 4.6 ([GL18, Theorem 7.5]).** Let $(\mathcal{C}, d_{\mathcal{C}}, \Theta_{\mathcal{C}})$ be a curved cooperad. The inclusion functor

$$
\text{Inc} : \text{curv} \mathcal{C}\text{-alg} \hookrightarrow \text{pdg} \mathcal{C}\text{-alg}
$$

has a left adjoint

$$
\text{Curv} : \text{pdg} \mathcal{C}\text{-alg} \to \text{curv} \mathcal{C}\text{-alg}.
$$

Hence curv $\mathcal{C}$-alg is a reflexive subcategory of pdg $\mathcal{C}$-alg. For a pdg $\mathcal{C}$-algebra $(B, \gamma_B, d_B)$, its image under this functor is given by the following quotient:

$$
\text{Curv}(B) \coloneqq \frac{B}{\gamma_B \cdot \mathcal{F}(\mathcal{C})(\Theta_{\mathcal{C}})(id) + d_B^2(-)}.
$$

where $\left(\gamma_B \cdot \mathcal{F}(\mathcal{C})(\Theta_{\mathcal{C}})(id) + d_B^2(-)\right)$ denoted the ideal generated by $\gamma_B \cdot \mathcal{F}(\mathcal{C})(\Theta_{\mathcal{C}})(id) + d_B^2(-)$. Its pdg $\mathcal{C}$-algebra structure is induced by $\gamma_B$ and $d_B$.

**Remark 4.7.** Let $B$ be a $\mathcal{C}$-algebra, an ideal is a subobject $1 \hookrightarrow B$ such that the quotient $B/I$ is a $\mathcal{C}$-algebra, and such that the projection morphism is a morphism of $\mathcal{C}$-algebras. See [GL18, Definition 4.1].

**Corollary 4.8.** The category of curved $\mathcal{C}$-algebras is presentable. In particular it is complete and cocomplete.

The notion of a conilpotent cooperad also generalizes to the case of curved cooperads.
Definition 4.9 (Conilpotent curved partial cooperad). Let \((\mathcal{E}, \{\Delta_i\}, d_\mathcal{E}, \Theta_\mathcal{E})\) be a curved partial cooperad. It is conilpotent if its underlying partial pdg cooperad \((\mathcal{E}, \{\Delta_i\}, d_\mathcal{E})\) is a conilpotent partial cooperad.

There is a functor

\[
\text{Conil} : \text{curv pdgCoop} \xrightarrow{\text{conil}} \text{curv Coop},
\]

from the category of conilpotent curved partial cooperads to the category of curved cooperads defined as comonoids given by adding a counit as in Corollary 1.35.

Definition 4.10 (Conilpotent curved cooperad). A curved cooperad \((\mathcal{E}, d_\mathcal{E}, \Theta_\mathcal{E})\) is said to be conilpotent if it is in the essential image of the functor \text{Conil}.

In the case of a conilpotent curved cooperad, one can induce a canonical filtration on the category of curved \(\mathcal{E}\)-algebras.

Definition 4.11 (Complete curved algebra over a conilpotent cooperad). Let \((\mathcal{E}, d_\mathcal{E}, \Theta_\mathcal{E})\) be a conilpotent curved cooperad. A complete curved \(\mathcal{E}\)-algebra \(B\) amounts to the data of a curved \(\mathcal{E}\)-algebra \((B, \gamma_B, d_B)\) such that the canonical morphism of curved \(\mathcal{E}\)-algebras

\[
\varphi_B : B \to \lim_{\omega} B / W_\omega B
\]

is an isomorphism, where \(W_\omega B\) is the canonical filtration of the \(\mathcal{E}\)-algebra \(B\).

Remark 4.12. These notions were defined in Section 1.

Proposition 4.13. Let \((\mathcal{E}, d_\mathcal{E}, \Theta_\mathcal{E})\) be a conilpotent curved cooperad. The category of complete curved \(\mathcal{E}\)-algebras is a reflexive subcategory of pdg \(\mathcal{E}\)-algebras. It is thus presentable, and bicomplete.

Proof. Its reflector is given by the composition of the previous two reflectors: the functor Curv and the completion functor with respect to the canonical filtration. \(\square\)

5. The Groupoid-Colored Level

In this section, we develop the formalism of groupoid-colored (co)operads introduced in [War19] in order to encode the different (co)operadic structures that we have encountered so far. More precisely, we construct a unital groupoid-colored operad, denoted by \(u\mathcal{O}\), that encodes unital partial operads as its algebras and counital partial cooperads as its coalgebras. The goal of this section is to compute its Koszul dual conilpotent curved groupoid-colored cooperad.

In order to do this, we need to extend to groupoid-colored operads the inhomogeneous Koszul duality for operads introduced in [HM12] (by inhomogeneous we mean relations that involve constant-linear-quadratic terms in this case). Extending the quadratic Koszul duality for groupoid-colored operads of [War19] to the inhomogeneous case is conceptually quite straightforward. Since this theory will only be applied to the case that interests us, we give an overview of the main results, without entering in full generality.

\[
\text{Koszul duality}
\]

inhomogeneous operads

[HM12]

Koszul duality
homogeneous operads \([GK95],[GJ94]\)

[War19]

Koszul duality
groupoid-colored inhomogeneous operads

Koszul duality
groupoid-colored homogeneous operads

This new Koszul duality gives us a conilpotent curved groupoid-colored cooperad \((u\mathcal{O})^!\), which encodes conilpotent curved partial cooperads as its curved coalgebras. Its category of curved algebras provides us with a new notion of operads which we call curved absolute partial operads. See the Appendix 9.2 for a detailed description. Using the Koszul curved twisting morphism given by this theory, we will construct two Bar-Cobar adjunctions that interrelate these objects in the next section.
5.1. S-colored (co)operads. We begin by applying the formalism of [War19] to the specific case that interests us. We consider the groupoid

\[
S := \coprod_{n \geq 0} BS_n
\]

where \(BS_n\) is the classifying space of \(S_n\) (the category with one object \(*\) and \(Aut(* ) = S_n\)). We denote by \(n\) the single object in \(BS_n\). In this context, an S-color scheme as defined in [War19, Definition 2.2.1] amounts to the following definition:

**Definition 5.1** (pdg S-color scheme). An pdg S-color scheme \(E\) amounts to the data of a family \(\{E(n_1, \cdots, n_r; n)\}\) of pdg modules for all \(r\)-tuples of natural numbers \((n_1, \cdots, n_r)\) in \(\mathbb{N}^r\) and all \(n\) in \(\mathbb{N}\). Each pdg module \(E(n_1, \cdots, n_r; n)\) comes equipped with an action of \(S_{n_1} \times \cdots \times S_{n_r}\) on the right and an action of \(S_n\) on the left as part of the structure. There is a supplementary action of the symmetric groups on this family given by permuting the entries. For any \(r\)-tuples of natural numbers \((n_1, \cdots, n_r)\), there is an isomorphism

\[
\varphi_\sigma : E(n_1, \cdots, n_r; n) \to E(n_{\sigma^{-1}(1)}, \cdots, n_{\sigma^{-1}(r)}; n)
\]

of right pdg \(K[S_{n_1} \times \cdots \times S_{n_r}] \otimes K[S_n]^{op}\)-module for all \(\sigma\) in \(S_r\) which defines a group action. This endows each \(E(n_1, \cdots, n_r; n)\) with an left action of \((S_{n_1} \times \cdots \times S_{n_r})\) \(\wr\ S_r\) where \(\wr\) denotes the wreath product of groups.

A morphism \(f : E \to F\) is the data of a morphism of pdg \(K[S_{n_1} \times \cdots \times S_{n_r}] \otimes K[S_n]^{op}\)-modules

\[
f(n_1, \cdots, n_r; n) : E(n_1, \cdots, n_r; n) \to F(n_1, \cdots, n_r; n),
\]

for all \(r\)-tuples \((n_1, \cdots, n_r)\) in \(\mathbb{N}^r\) which commutes with the permutation maps \(\varphi_\sigma\). Pdg S-color schemes form a category denoted by pdg Col\(_S\).

Let \(E\) and \(F\) be two pdg S-color schemes, their composition product \(E \circ F\) is given by:

\[
E \circ F(n_1, \cdots, n_r; n) := \bigoplus_{j \geq 1} \bigoplus_{(a_1, \cdots, a_j) \in \mathbb{N}^j} \left( E(a_1, \cdots, a_j; n) \otimes (S_{a_1} \times \cdots \times S_{a_j}) \right) \circ \bigoplus_{i_1 + \cdots + i_j = r} \text{Ind}_{S_{i_1} \times \cdots \times S_{i_j}}^S \left( F(n_{i_1}, \cdots, n_{i_j}; a_1) \otimes \cdots \otimes F(n_{r-i_1}, \cdots, n_{r}; a_j) \right) \right) \circ S_j
\]

where the second sum runs over all \(j\)-tuples \((a_1, \cdots, a_j)\) and the third sum runs over all \(j\)-tuples \((i_1, \cdots, i_j)\) such that \(i_1 + \cdots + i_j = r\). It is the analogue of the composition product \(\circ\) of S-modules. In this case, leaves are colored by elements of the symmetric groups, hence the extra requirement that the colors have to match. The unit of this product is given by the S-color scheme \(I_S\), defined by \(I_S(n; n) := K[S_n]\) as a bimodule over itself and 0 elsewhere, endowed with the trivial pre-differential.

**Lemma 5.2.** The data (pdg Col\(_S\), \(\circ\), \(I_S\)) forms a monoidal category.

**Proof.** A straightforward computation analogous to the standard case. \(\square\)

**Definition 5.3** (S-colored operad). A pdg S-colored operad \(\mathfrak{g}\) is the data of a monoid \((\mathfrak{g}, \gamma_\mathfrak{g}, \eta_\mathfrak{g}, d_\mathfrak{g})\) in the monoidal category (pdg Col\(_S\), \(\circ\), \(I_S\)).

**Example 5.4** (S-colored endomorphism operad). Let \(M\) be a pdg S-module. The collection

\[
\text{End}_M(n_1, \cdots, n_r; n) := \text{Hom}(M(n_1) \otimes \cdots \otimes M(n_r), M(n)) \quad \text{for} \quad (n_1, \cdots, n_r) \in \mathbb{N}^r
\]

has a natural structure of pdg S-color scheme where the right pdg \(K[S_{n_1} \times \cdots \times S_{n_r}] \otimes K[S_n]^{op}\)-module structure comes from the S-module structure of \(M\) and where the permutation morphism \(\varphi_\sigma\) are given by the natural action of the symmetric group \(S_r\) on the tensor product \(M(n_1) \otimes \cdots \otimes M(n_r)\). The composition of morphisms and the identity \(\text{id}_n\) in \(\text{End}_M(n; n)\) endow this pdg S-color scheme with an S-colored operad structure, called the S-colored endomorphism operad of \(M\).
Definition 5.6 (S-colored cooperad). A pdg S-colored cooperad $V$ is the data of a comonoid $(V, \Delta_V, \varepsilon, d_V)$ in the monoidal category $(\text{Col}_S, \diamond, I_S)$.

To any pdg $S$-color scheme $E$ one can associate an endofunctor in the category of pdg $S$-modules via the Schur realization functor:

$$\mathcal{S} : \text{pdg Col}_S \longrightarrow \text{End}(\text{pdg S-mod})$$

$$E \longrightarrow \mathcal{S}(E)(-).$$

The Schur endofunctor $\mathcal{S}(E)$ is given, for $M$ a pdg $S$-module, by:

$$\mathcal{S}(E)(M)(n) := \bigoplus_{(n_1, \ldots, n_r) \in \mathbb{N}^r, \ r \geq 1} E(n_1, \ldots, n_r; n) \otimes (s_{n_1} \times \cdots \times s_{n_r}) \cdot s_r(M(n_1) \otimes \cdots \otimes M(n_r)).$$

Lemma 5.7. The colored Schur realization functor $\mathcal{S}$ is a strong monoidal functor between the monoidal categories $(\text{pdg Col}_S, \diamond, I_S)$ and $(\text{End}(\text{pdg S-mod}), \diamond, \text{Id})$.

Proof. The computation is analogous to the classical case. \qed

Corollary 5.8. Let $E$ be an $S$-color scheme. Any $S$-colored operad structure on $E$ induces a monad structure on $\mathcal{S}(E)$ and any $S$-colored cooperad structure on $E$ induces a comonad structure on $\mathcal{S}(E)$.

Definition 5.9 (Algebra over an $S$-colored operad). Let $S$ be a pdg $S$-colored operad. A $S$-algebra $M$ amounts to the data $(M, \gamma_M, d_M)$ of an algebra over the monad $\mathcal{S}(S)$.

Lemma 5.10. Let $S$ be a pdg $S$-colored operad. The data of a pdg $S$-algebra $\gamma_M$ on a pdg $S$-module $(M, d_M)$ is equivalent to a morphism of pdg $S$-colored operads $\Gamma_M : S \longrightarrow \text{End}_M$.

Proof. Straightforward generalization of the standard case. \qed

Definition 5.11 (Coalgebra over an $S$-colored cooperad). Let $V$ be a pdg $S$-colored cooperad. A $V$-coalgebra $K$ amounts to the data $(K, \Delta_K, d_K)$ of a coalgebra over the comonad $\mathcal{S}(V)$.

The dual Schur realization functor $\mathcal{S}^c$ also associates to any pdg $S$-color scheme $E$ an endofunctor in the category of pdg $S$-modules

$$\mathcal{S}^c : \text{pdg Col}_S^{\text{op}} \longrightarrow \text{End}(\text{pdg S-mod})$$

$$E \longrightarrow \mathcal{S}^c(E)(-).$$

The dual Schur endofunctor $\mathcal{S}^c(E)$ is given, for $M$ a pdg $S$-module, by:

$$\mathcal{S}^c(E)(M)(n) := \prod_{(n_1, \ldots, n_r) \in \mathbb{N}^r, \ r \geq 1} \text{Hom}(s_{n_1} \times \cdots \times s_{n_r}) \cdot s_r(E(n_1, \ldots, n_r; n) \cdot M(n_1) \otimes \cdots \otimes M(n_r)).$$

Lemma 5.12. The functor $\mathcal{S}^c : (\text{pdg Col}_S, \diamond, I_S)^{\text{op}} \longrightarrow (\text{End}(\text{pdg S-mod}), \diamond, \text{Id})$ can be endowed with a lax monoidal structure. That is, there exists a natural transformation

$$\varphi_{E,F} : \mathcal{S}^c(E) \circ \mathcal{S}^c(F) \longrightarrow \mathcal{S}^c(E \circ F),$$

which satisfies associativity and unitality compatibility conditions with respect to the monoidal structures. Furthermore, this natural transformation is a monomorphism for all pdg $S$-color schemes $E, F$. \hfill 28
Proof. The construction of ϕ_{E,F} and the proof are completely analogous to [GL18, Corollary 3.4].

Corollary 5.13. Let E be an S-color scheme. Any S-colored cooperad structure on E induces a monad structure on \( \mathcal{F}_S^c(E) \).

Definition 5.14 (Algebra over an S-colored cooperad). Let V be an S-colored cooperad. A V-algebra L amounts to the data \((L, γ_L, d_L)\) of an algebra over the monad \( \mathcal{F}_S^c(V) \).

Definition 5.15 (Coalgebra over an S-colored operad). Let S be a pdg S-colored operad. A S-coalgebra N amount to the \((N, Δ_N, d_N)\) of a morphism of pdg S-modules such that the following diagram commutes:

\[
\begin{array}{ccc}
N & \xrightarrow{Δ_N} & \mathcal{F}_S^c(S)(N) \\
\downarrow & & \downarrow \\
\mathcal{F}_S^c(S)(N) & \xrightarrow{Δ_N} & \mathcal{F}_S^c(S)(N) \circ \mathcal{F}_S^c(P)(D)
\end{array}
\]

Remark 5.16. Let S be a pdg S-colored operad. The data of a pdg S-coalgebra structure \( Δ_N \) on a pdg S-module \((N, d_N)\) is equivalent to a morphism of pdg S-colored operads \( δ_N : S \rightarrow \text{Coend}_N \).

5.2. Partial S-colored (co)operads. As the careful reader might expect by now, all the other definitions of Section 1 generalize to the S-colored setting mutatis mutandis.

Definition 5.17 (Partial S-colored operad). An pdg partial S-colored operad amounts to the data of \((S, \{o_i\}, d_S)\) a pdg S-colored scheme S endowed with partial composition operations

\[
o_i : S(n_1, \cdots, n_i, \cdots, n_r; n) ⊗ S_{n_i} S(p_1, \cdots, p_i; n_1) \rightarrow S(n_1, \cdots, n_{i-1}, p_1, \cdots, p_i, n_{i+1}, \cdots, n_r; n).
\]

This family of partial compositions maps \( \{o_i\} \) satisfies sequential and parallel axioms analogous to those of a partial operad. They satisfy an equivariance condition with respect to the permutations of the entries \( φ_o \) which is also completely analogous to Definition 1.20.

Definition 5.18 (Unital partial S-colored operad). A pdg unital partial S-colored operad amounts to the data of \((S, \{o_i\}, 1, d_S)\) a partial pdg S-colored operad \((S, \{o_i\}, d_S)\) together with a morphism of pdg S-color schemes \( ι_S : I_S \rightarrow S \) which acts as a unit for the partial compositions maps.

The data of a unit \( ι_S : I_S \rightarrow S \) amounts to a family of elements \( \{id_n\} \) in \( S(n; n) \).

Lemma 5.19. The category of pdg unital partial S-colored operads is equivalent to the category of pdg S-colored operads defined as monoids.

Proof. Straightforward generalization of Proposition 1.24.

Thus, given a pdg partial S-colored operad \((S, \{o_i\}, d_S)\) one can obtain a pdg S-colored operad defined as a monoid by freely adding a unit to it \( S^u := S ⊕ I_S \).

Theorem 5.20 ([War19, Theorems 2.10, 2.11 and 2.24]). Let V be a groupoid.

1. The exists a monad \( \mathcal{F}_V \) in the category of pdg V-color schemes, called the V-colored tree monad, such that the category of \( \mathcal{F}_V\)-algebras is equivalent to the category of pdg unital partial V-colored operads.

29
(2) The exists a monad $\mathcal{T}_V$ in the category of pdg $V$-color schemes, called the reduced $V$-colored tree monad, such that the category of $\mathcal{T}_V$-algebras is equivalent to the category of pdg partial $V$-colored operads.

**Definition 5.21** (Partial $S$-colored cooperad). A pdg partial $S$-colored cooperad amounts to the data of $(\mathcal{V}, \{\Delta_i\}, d_V)$ a pdg $S$-color scheme $V$ endowed with partial decomposition operations

$$\Delta_i : \mathcal{V}(n_1, \cdots, n_i-1, p_1, \cdots, p_i, n_{i+1}, \cdots, n_r, n) \to \mathcal{V}(n_1, \cdots, n_i, \cdots, n_r, n) \otimes_{s_{n_i}} \mathcal{V}(p_1, \cdots, p_i; n_i).$$

This family of partial decompositions maps $\{\Delta_i\}$ satisfies cosequential and coparallel axioms analogous to those of a partial cooperad. They satisfy an equivariance condition with respect to the permutations of the entries $\varphi_\sigma$ which is also completely analogous to Definition 1.26.

**Definition 5.22** (Counital partial $S$-colored cooperad). A pdg counital partial $S$-colored cooperad amounts to the data of $(\mathcal{V}, \{\Delta_i\}, c, d_V)$ a pdg partial $S$-colored cooperad $(\mathcal{V}, \{\Delta_i\}, d_V)$ together with a morphism of pdg $S$-color schemes $c : \mathcal{V} \to I_N$ which acts as a counit for the partial decompositions maps.

Any pdg partial $S$-colored cooperad structure induces a morphism of pdg $S$-color schemes

$$\Delta_V : \mathcal{V} \to \mathcal{T}_S^\wedge(\mathcal{V}),$$

where $\mathcal{T}_S^\wedge$ is the endofunctor given by the completion of the reduced $V$-colored tree monad with respect to its canonical weight filtration given by the number of internal edges of the rooted trees. Using this morphism one defines an analogous version of the coradical filtration of a partial cooperad and an analogous definition of a conilpotent pdg partial $S$-colored cooperad. *Mutatis mutandis* the same characterization of this type of partial cooperads still holds. See Section 1.3.

**Theorem 5.23** ([War19, Section 2.4.1]). Let $V$ be a groupoid such that for any $v$ in $V$, the set $\text{Aut}(v)$ is finite.

1. The exists a comonad structure on the underlying endofunctor of the reduced tree monad $\mathcal{T}_V$ on the category of $V$-colored schemes.
2. The category of $\mathcal{T}_V$-coalgebras is equivalent to the category of conilpotent pdg partial $S$-colored cooperads.

Let $(\mathcal{V}, \{\Delta_i\}, d_V)$ be a conilpotent pdg partial cooperad. Then one can cofreely add a counit to it by setting $V^u := V \oplus I_V$. The pdg $S$-color scheme has an unique cooperad structure induced by the partial decompositions maps $\{\Delta_i\}$. It defines a functor

$$\text{Conil} : \text{pdg pCoop}_{S}^{\text{conil}} \to \text{pdg Coop}_{S}.$$ 

**Definition 5.24** (Conilpotent $S$-colored cooperad). A pdg $S$-colored cooperad $\mathcal{V}$ is said to be conilpotent if its in the essential image of the functor defined above.

**Remark 5.25.** For any conilpotent pdg $S$-colored cooperad $\mathcal{V}$, there is a canonical filtration on $\mathcal{V}$-algebras and a notion of complete $\mathcal{V}$-algebra analogous to the case studied in Section 1.

5.3. **Koszul duality for quadratic $S$-colored operads.** We now state the quadratic Koszul duality of [War19] in the context of $S$-colored operads. In order to do this, we briefly restrict from the underlying category of pdg modules to the underlying category of dg modules.

**Proposition 5.26.** Let $(\mathcal{G}, \{\cdot_1\}, d_3)$ be a dg partial $S$-colored operad. The totalization of $\mathcal{G}$ is given by

$$\bigoplus_{(n_1, \cdots, n_r, n) \in \mathbb{N}^{r+1}} \mathcal{G}(n_1, \cdots, n_r; n)(s_{n_1} \times \cdots \times s_{n_r}) \otimes s_{n}.$$
It can be endowed with a dg pre-Lie algebra structure. Let \( \mu \) be in \( \mathcal{S}(n_1, \ldots, n_r; n) \) and \( \nu \) be in \( \mathcal{S}(n_1, \ldots, n_r; n) \), where \( \mu \) is colored by \( \sigma_1, \ldots, \sigma_r, \sigma \) in \( S_{n_1} \times \cdots \times S_{n_r} \times S_n \) and where \( \nu \) is colored by \( \tau_1, \ldots, \tau_l; \tau \) in \( S_{p_1} \times \cdots \times S_{p_l} \times S_p \). The pre-Lie bracket of \( \mu \) and \( \nu \) is given by
\[
\mu \ast \nu := \sum_{\{ (n_i, \sigma_i) = (p, \tau) \}} \sum_{\sigma_p} (\mu \circ_1 \nu)^{\sigma_p},
\]
where the first sum runs over all pairs \((n_i, \sigma_i)\) which are equal to \((p, \tau)\) (if there are none, then \( \mu \ast \nu \) is 0), and where the second sum ranges over unshuffles \(\sigma_p\) associated to all ordered partitions \(P\) in \(\text{OrPar}(1, \ldots, l - i + 1, 1, \cdots, 1)\).

\[i\]-th position

**Proof.** The associator of the product \( \star \) is right symmetric like in the case of partial operads, and is compatible with the differentials by definition. \(\square\)

**Definition 5.27** (Convolution partial \(\mathcal{S}\)-colored operad). Let \((\mathcal{V}, \{\Delta_l\}, d_\mathcal{V})\) be a conilpotent dg partial \(\mathcal{S}\)-colored cooperad and let \((\mathcal{G}, \{\circ_l\}, d_\mathcal{G})\) be a dg partial \(\mathcal{S}\)-colored operad. The dg \(\mathcal{S}\)-color scheme
\[
\mathcal{G}(\text{om}(\mathcal{V}, \mathcal{G}))(n_1, \cdots, n_r; n) := \text{Hom}_{\mathcal{G}}(\mathcal{V}(n_1, \cdots, n_r; n), \mathcal{G}(n_1, \cdots, n_r; n))
\]
endowed with the differential
\[
\partial(\alpha) := d_\mathcal{G} \circ \alpha - (-1)^{|\alpha|} \alpha \circ d_\mathcal{V}.
\]
forms a dg partial \(\mathcal{S}\)-colored operad structure where the partial compositions maps are given by
\[
\alpha \circ_1 \beta := \circ_1 \cdot (\alpha \otimes \beta) \cdot \Delta_l.
\]

**Definition 5.28** (Twisting morphism). Let
\[
\mathcal{G}_{\mathcal{V},\mathcal{G}} := \prod_{(n_1, \cdots, n_r; n) \in \mathbb{N}^{r+1}} \text{Hom}_{\mathcal{S}_{\text{col}}}(\mathcal{V}(n_1, \cdots, n_r; n), \mathcal{G}(n_1, \cdots, n_r; n))
\]
be the dg pre-Lie algebra given by the totalization of the convolution operad of \(\mathcal{V}\) and \(\mathcal{G}\). A **twisting morphism** is a Maurer-Cartan element \(\alpha\) of \(\mathcal{G}_{\mathcal{V},\mathcal{G}}\), that is, a morphism \(\alpha : \mathcal{V} \rightarrow \mathcal{G}\) of dg \(\mathcal{S}\)-color schemes of degree \(-1\) satisfying:
\[
\partial(\alpha) + \alpha \ast \alpha = 0.
\]
The set of twisting morphism between \(\mathcal{V}\) and \(\mathcal{G}\) is denoted by \(\text{Tw}(\mathcal{V}, \mathcal{G})\).

Twisting morphisms induce Bar-Cobar adjunctions relative to them.

**Proposition 5.29** (Bar-Cobar adjunction relative to \(\alpha\)). Let \(\alpha : \mathcal{V} \rightarrow \mathcal{G}\) be a twisting morphism between a conilpotent partial \(\mathcal{S}\)-colored cooperad and a partial \(\mathcal{S}\)-colored operad. It induces a Bar-Cobar adjunction
\[
dg \mathcal{G}^u_{\text{alg}} \xleftarrow{\Omega_\alpha} \underbrace{\text{dg} \mathcal{V}^u_{\text{coalg}_\mathcal{G}^\text{conil}}}_{B_\alpha}
\]
relative to the twisting morphism \(\alpha\).

**Proof.** This adjunction is constructed using the free \(\mathcal{G}^u\)-algebra given by \(\mathcal{G}^u(\mathcal{G}^u)\) and the cofree conilpotent \(\mathcal{V}^u\)-coalgebra given by \(\mathcal{G}^u(\mathcal{V}^u)\). They are endowed with differentials which are analogous to those constructed in the standard case [LV12, Section 11.2]. See [War19, Section 2.7.1] for a detailed exposition of this construction. \(\square\)

**Definition 5.30** (Koszul twisting morphism). Let \(\alpha : \mathcal{V} \rightarrow \mathcal{G}\) be a twisting morphism between a conilpotent partial \(\mathcal{S}\)-colored cooperad and a partial \(\mathcal{S}\)-colored operad. It is a **Koszul twisting morphism** if the twisted complex \(\mathcal{V}^u \circ_\alpha \mathcal{G}^u\) is acyclic or if the twisted complex \(\mathcal{G}^u \circ_\alpha \mathcal{V}^u\) is acyclic.
exists a quadratic presentation \( (S, \{o_i\}, d) \) be a quadratic presentation of \( G \). It comes equipped with a canonical twisting morphism \( \kappa : G \twoheadrightarrow G \) such that there is an isomorphism of partial \( S \)-colored operads

\[
\mathcal{G} \cong \mathcal{T}_S(E)/(R),
\]

where \( \mathcal{T}_S(E) \) denotes the free partial \( S \)-colored operad on the \( S \)-color scheme \( E \) modulo the quadratic relations \( R \subset \mathcal{T}_S(E)[2] \).

**Definition 5.34** ([War19, Section 2.5.4]). Let \( (G, \{o_i\}, d) \) be a partial \( S \)-colored operad \( (E, R) \) be a quadratic presentation of \( G \). The Koszul dual conilpotent partial \( S \)-colored cooperad \( G^! \) of \( G \) is given by the conilpotent partial \( S \)-colored cooperad

\[
G^! := \mathcal{T}_S(sE, s^2R),
\]

where \( \mathcal{T}_S(sE, s^2R) \) denotes the cofree conilpotent partial \( S \)-colored cooperad cogenerated by the \( S \)-color scheme \( sE \), with \( s^2R \) as the corelations. (Smallest sub-cooperad of \( \mathcal{T}_S(sE) \) containing \( s^2R \).) Here \( s \) denotes the suspension of graded \( S \)-color schemes.

It comes equipped with a canonical twisting morphism \( \kappa : G^! \twoheadrightarrow G \) given by

\[
G^! \twoheadrightarrow sE \cong E \twoheadrightarrow G.
\]

**Definition 5.35** (Koszul partial \( S \)-colored operad). Let \( (G, \{o_i\}, d) \) be a partial \( S \)-colored operad. The partial \( S \)-colored operad \( G \) is said to be a Koszul quadratic \( S \)-colored operad if there exists a quadratic presentation \( (E, R) \) of \( G \) such that the canonical twisting morphism

\[
\kappa : G^! \twoheadrightarrow sE \cong E \twoheadrightarrow G
\]

is a Koszul twisting morphism.

**5.4. The partial \( S \)-colored operad encoding partial (co)operads and its Koszul dual.** We now define the partial \( S \)-colored operad \( 0 \) that encodes partial (co)operads as its (co)algebras. This is a direct generalization of the set-colored operad encoding non-symmetric partial (co)operads defined in [VdL03, Definition 4.1]. The ground category is still dg modules in this section.

**Definition 5.36** (The partial \( S \)-colored operad encoding partial (co)operads). The partial \( S \)-colored operad \( 0 \) is given by the following quadratic presentation:

\[
0 := \mathcal{T}_S(E)/(R),
\]

where \( \mathcal{T}_S(E) \) denotes the free partial \( S \)-colored operad on the \( S \)-color scheme \( E \) modulo the quadratic relations \( R \subset \mathcal{T}_S(E)[2] \). The \( S \)-color scheme \( E \) is the generated by:

\[
\gamma_{n,k}^i \in E(n, k; n + k - 1) \quad \text{for} \quad 1 \leq i \leq n,
\]
as a right \( K[S_n \times S_k] \otimes_K K[S_{n+k-1}]^{op} \)-module, endowed with the zero differential.

The right action of the symmetric groups \( S_n \) and \( S_k \) can be written as in both cases as the left action of an element \( S_{n+k-1} \). For \( \sigma \) in \( S_k \), its action is equal to

\[
\sigma \cdot (\gamma^{n,k}_i) = (\gamma^{n,k}_i) \cdot \sigma' \quad \text{for} \quad 1 \leq i \leq n ,
\]

where \( \sigma' \in S_{n+k-1} \) is the unique permutation that acts as the identity everywhere except for \( \{i, \cdots, i, i+k-1\} \), where it acts as \( \sigma \). For \( \tau \) in \( S_n \), its action is equal to

\[
\tau \cdot (\gamma^{n,k}_i) = (\gamma^{n,k}_{\tau(i)}) \cdot \tau' \quad \text{for} \quad 1 \leq i \leq n ,
\]

where \( \tau' \) is the unique permutation of \( S_{n+k-1} \) that acts as \( \tau \) on the block \( \{1, \cdots, n+k-1\} \) with values in \( \{1, \cdots, n+k-1\} \) and as the identity on \( \{i, \cdots, i+k-1\} \) with values in \( \{\tau(i), \cdots, \tau(i)+k-1\} \). The \( S_2 \) action that permutes the entries is simply given by \( \varphi_{\{12\}}(\gamma^{n,k}_i) = \gamma^{n,k}_{i,1} \).

The relations \( R \) are the given by:

\[
\begin{cases}
\gamma^{n+k-1,1}_{i,1} \circ_1 \gamma^{n,k}_1 = \gamma^{1+n-k,1}_1 \circ_1 \gamma^{n,k}_1 & \text{if} \quad 1 \leq i \leq n, 1 \leq j \leq k , \quad (1) \\
\gamma^{n+k-1,1}_{j,1} \circ_1 \gamma^{n,k}_1 = \gamma^{n+k-1,1}_j \circ_1 \gamma^{n,k}_1 & \text{if} \quad 1 \leq i < j \leq n \quad (2) .
\end{cases}
\]

The relation (1) is the parallel axiom and (2) is the sequential axiom.

**Remark 5.37.** The equivariance relations of the partial (de)compositions of partial (co)operads are not coded as relations in the partial \( S \)-colored operad \( \mathcal{O} \). They are encoded by the action of the symmetric groups on the generators \( \{\gamma^{n,k}_i\} \).

Although similar in spirit, it does not coincide with the approach of [DV21], where the authors encode partial operads using a set-colored operad, and where the action of the symmetric groups is given by operations in their set-colored operad.

By adding freely a unit to \( \mathcal{O} \) one obtains an \( S \)-colored operad \( \mathcal{O}^U \) defined as a monoid.

**Proposition 5.38.** The category of dg \( \mathcal{O}^U \)-algebras is equivalent to the category of dg partial operads.

**Proof.** Let \( (P, d_P) \) be a dg-S-module and let \( \Gamma_P : \mathcal{O}^U \to \text{End}_P \) be a morphism of dg \( S \)-colored operads. It is determined by the image of the generators \( \gamma^{n,k}_i \) in \( \mathcal{O}(n, k; n+k-1) \). Let’s denote \( c^{n,k}_i := \Gamma_P(\gamma^{n,k}_i) : P(n) \otimes P(k) \to P(n+k-1) \). The family \( \{c^{n,k}_i\} \) satisfies the equivariance axiom of a partial operad since \( \Gamma_P \) is \( K[S_n \times S_k] \otimes K[S_{n+k-1}] \)-equivariant; it satisfies the parallel and sequential axioms by definition of \( \mathcal{O} \). Since \( \Gamma_P \) commutes with the pre-differentials, \( d_P \) is a derivation with respect to the partial composition maps. Hence it endows \( (P, d_P) \) with a dg partial operad structure. \( \square \)

**Proposition 5.39.** The category of dg \( \mathcal{O}^U \)-coalgebras is equivalent to the category of dg partial cooperads.

**Proof.** The proof is analogous to the proof of Proposition 5.38. One replaces the endomorphism operad with the coendomorphism operad. \( \square \)

**Lemma 5.40.** The Koszul dual conilpotent partial \( S \)-colored cooperad \( \mathcal{O}^! \) is isomorphic to the operadic suspension of the linear dual of \( \mathcal{O} \). The operad \( \mathcal{O} \) is Koszul autodual.

**Proof.** First, since each \( \mathcal{O}(n_1, \cdots, n_r; n) \) is finite dimensional over \( K \) and since it is reduced in the sense of [War19, Definition 2.35], the arity-wise linear dual of \( \mathcal{O} \) is a conilpotent partial \( S \)-colored operad, and we have that \( (\mathcal{O}^!)^* \cong \mathcal{O} \). Therefore it is only necessary to compute its Koszul dual partial \( S \)-colored operad, which is given by the operadic suspension of \( (\mathcal{O}^!)^* \).

We adapt the proof of [VdL03, Theorem 4.3] to the groupoid-colored case. The Koszul dual operad \( \mathcal{O}^! \) is given by \( \mathcal{T}_G(E^*)/(R^\perp) \), where \( \mathcal{T}_G(E) \) is the free partial \( S \)-colored partial operad.
generated by $E^*$, and $(R^\perp)$ denotes the operadic ideal generated by the orthogonal of $R$ inside $\mathcal{T}_S(E^*)^{(2)}$. Using the same arguments as in the set-colored case, one can show that the dimension of the relations $R$ as a $S$-color scheme is exactly half of the dimension over $K$ of $\mathcal{T}_S(E^*)^{(2)}$. Since, the relations $R$ must be contained in the ideal generated by the orthogonal, concludes by a dimension argument that the relations in $R$ form a basis of $R^\perp$. □

**Lemma 5.41.** The following statements hold:

1. There is an isomorphism of monads in the category of $S$-modules between the monad $\mathcal{S}(O^u)$ and the reduced tree monad $\mathcal{T}$ encoding partial operads.

2. There is an isomorphism of comonads in the category of $S$-modules between the comonad $\mathcal{S}(O^\ast)^u$ and the reduced tree comonad $\mathcal{T}^c$ encoding conilpotent partial cooperads.

**Proof.** The first result follows immediately from the fact that the category of $O$-algebras is equivalent to the category of partial operads. Nevertheless, for any $S$-module $M$, there is an explicit bijection between the $S$-modules $\mathcal{S}(O^u)(M)$ and $\mathcal{T}(M)$ that identifies the partial operad structures. For any element $\psi; m_1, \ldots, m_r$ in $O(n_1, \ldots, n_r; n) \otimes (S_{n_1} \times \cdots \times S_{n_r}) \otimes S_r M(n_1) \otimes \cdots \otimes M(n_r)$, one can represent $\psi$ as equivalence class of binary trees with vertices labeled by the generators $\gamma_{n,k}$ and with edges labeled by elements of the symmetric groups that match the coloring of the edge. To any $\psi; m_1, \ldots, m_r$, one associates the rooted tree $\tau_\psi$ obtained by composing the $n_i$-corollas labeled by $m_i$ in the way the binary tree $\psi$ indicates, applying at each step the permutations that label the edges of $\psi$. Pictorially, the bijection is given by:

This type of bijections can be found in [DV15, Section 1.3]). It is straightforward to check that is bijection identifies the partial operad structures of $\mathcal{S}(O^u)(M)$ and $\mathcal{T}(M)$ that identifies the partial cooperad structures as well. □

**Proposition 5.42.** Let $O^\ast$ be the Koszul dual conilpotent partial $S$-colored cooperad of $O$.

1. The category of dg $(O^\ast)^u$-coalgebras is equivalent to the category of shifted conilpotent dg partial cooperads.

2. The category of complete dg $(O^\ast)^u$-algebras is equivalent to the category of shifted complete dg absolute partial operads.

**Proof.** The first statement follows from the previous Lemma. For a definition of absolute partial operads, as well as their characterization, we refer to the Appendix 9.2. □

**Proposition 5.43.** Let $\kappa : O^\ast \rightarrow O$ be the canonical twisting morphism. The adjunction induced by the twisting morphism $\kappa$

$$
\begin{array}{ccc}
dg pOp & \xrightarrow{\Omega_\kappa} & dg pCoop^{conil} \\
\perp & \Downarrow B_\kappa & \end{array}
$$
between dg partial operads and conilpotent dg partial cooperads is naturally isomorphic to the classical Bar-Cobar adjunction of [LV12, Section 6.5].

**Proof.** In order to check that two adjunctions are isomorphic, it is only necessary to check that there exists a natural isomorphism between left adjoints, since the mate of this natural isomorphism will also induce a natural isomorphism of right adjoints. The isomorphism between the reduced tree monad \( \mathcal{T} \) and the monad \( \mathcal{S}(\mathcal{O}) \) induces a natural isomorphism of partial operads between \( \Omega \) and \( \Omega_\kappa \). It is straightforward to check that this bijection commutes with the respective differentials of the two Cobar constructions. \( \square \)

**Theorem 5.44.** The canonical twisting morphism \( \kappa : \mathcal{O}^! \to \mathcal{O} \) is a Koszul twisting morphism. Hence the partial S-colored operad \( \mathcal{O} \) is Koszul.

**Proof.** We extend the arguments of [LV12, Theorem 11.3.3] to this specific case. Indeed, we know that for any dg partial operad \( (\mathcal{P}, (\mathcal{O}_i, d_\mathcal{P})) \), the counit \( \epsilon_\kappa : \Omega_\kappa \mathcal{B}_\kappa \mathcal{P} \to \mathcal{P} \) of the adjunction induced by \( \kappa \) is a quasi-isomorphism of dg partial operads. In particular, any dg S-module can be endowed with the trivial dg partial operad structure given by the family of partial composition maps \( \{0\} \). Therefore we conclude that \( \mathcal{O} \circ_\kappa \mathcal{O}^! \) and \( \mathcal{O}^! \circ_\kappa \mathcal{O} \) are both acyclic. \( \square \)

**Remark 5.45.** Generalizing the formalism of [GL18], one can construct a complete Bar-Cobar adjunction relative to the twisting morphism \( \kappa : \mathcal{O}^! \to \mathcal{O} \) between complete dg absolute partial operads and non-necessarily conilpotent dg partial cooperads.

5.5. **Inhomogeneous S-colored Koszul duality.** We introduce the unital partial S-colored operad \( u\mathcal{O} \) which encodes (co)unital partial (co)operads as its (co)algebras. In order to compute its Koszul dual conilpotent curved S-colored, we generalize the inhomogeneous Koszul duality of [HM12] to the S-colored case. This provides us with a Koszul dual conilpotent curved partial S-colored cooperad \( c\mathcal{O}^! \). It encodes conilpotent curved partial cooperads as its curved coalgebras and complete curved absolute partial operads as its complete curved algebras. Absolute partial operads are a new type of operad-like structure, for which we provide useful descriptions in the Appendix 9.2. The ground category considered is once again the category of pdg modules.

Let \( \mathcal{U} \) be the S-colored scheme given by \( \mathcal{U}(0; 1) := k.u \), where \( u \) is an element of degree 0, and zero elsewhere.

**Definition 5.46** (The unital partial S-colored operad encoding (co)unital partial (co)operads). Let \( (\mathcal{E}, R) \) be the quadratic presentation of the partial S-colored operad \( \mathcal{O} \). The unital partial S-colored operad \( u\mathcal{O} \) is given by

\[
u\mathcal{O} := \mathcal{R}(\mathcal{E} \oplus \mathcal{U})/(R'),
\]

where \( \mathcal{R}(\mathcal{E} \oplus \mathcal{U}) \) is the free unital partial S-colored operad generated by the S-colored scheme \( \mathcal{E} \oplus \mathcal{U} \), and where \( (R') \) is the operadic ideal generated by \( R' \). Here \( R' \) is the sub-S-color scheme of \( \mathcal{R}(\mathcal{E} \oplus \mathcal{U})^{(\leq 2)} \) given by \( R \), together with the additional relations

\[
\begin{align*}
\gamma_1^{l,n} \circ_1 u &= |_n \quad \text{for} \quad n \in \mathbb{N}, \\
\gamma_1^{n,1} \circ_2 u &= |_{1 \leq i \leq n} \quad \text{and} \quad n \in \mathbb{N},
\end{align*}
\]

where \( |_n \in \mathcal{R}(\mathcal{E} \oplus \mathcal{U})^{(0)}(n; n) \) is the trivial tree.

**Lemma 5.47.** We have that:

1. The category of dg \( u\mathcal{O}-\)algebras is equivalent to the category of dg unital partial operads.
2. The category of dg \( u\mathcal{O}\)-coalgebras is equivalent to the category of dg counital partial cooperads.

**Proof.** It follows immediately from Proposition 5.38 and Proposition 5.39. \( \square \)
**Definition 5.48** (Inhomogeneous quadratic presentation). Let \( (\mathcal{G}, \{\alpha_i\}, \eta) \) be a unital partial \( S \)-colored operad and let
\[
\mathcal{G} \cong \mathcal{F}_S(V)/(S)
\]
be a presentation of \( \mathcal{G} \), where \( S \subset \{I_S \oplus V \oplus \mathcal{F}_S(V)^{(2)}\} \). The presentation is an *inhomogeneous quadratic presentation* if the \( S \)-color scheme of relations \( S \) satisfies the following conditions.

1. That the space of generators is *minimal*, that is \( S \cap \{I_S \oplus V\} = \{0\} \).
2. That the space of relations is *maximal*, that is \( (S) \cap \{I_S \oplus V \oplus \mathcal{F}_S(V)^{(2)}\} = S \).

**Lemma 5.49.** The presentation of the unital partial \( S \)-colored operad \( uO \) given in its definition is an inhomogeneous quadratic presentation.

**Proof.** It is straightforward to check given the aforementioned presentation. \( \square \)

**Definition 5.50** (Curved partial \( S \)-colored cooperad). A curved partial \( S \)-colored cooperad \( (V, \{\Delta_i\}, d_V, \Theta_V) \) amounts to the data of a partial pdg \( S \)-colored operad \( (V, \{\Delta_i\}, d_V) \) and a morphism of pdg \( S \)-color schemes \( \Theta_V : (V, d_V) \to (I_S, 0) \) of degree \(-2\), such that the following diagram commutes:
\[
\begin{array}{ccc}
V & \xrightarrow{\Delta} & V \circ \{i\} \oplus \{I_S \circ V\} \\
\rotatebox{90}{\text{proj}} & & \downarrow \text{proj} \\
& & V,
\end{array}
\]
where \( \text{proj} \) is given by \( \text{proj}(\mu, v) := \mu + v \).

**Remark 5.51.** The definitions developed in Section 4 generalize *mutatis mutandis* to the \( S \)-colored case.

We extend the formalism of semi-augmented operads of [HM12] to the \( S \)-colored case in order to compute the Koszul dual conilpotent curved partial \( S \)-colored cooperad of \( uO \).

**Definition 5.52** (Semi-augmented unital partial \( S \)-colored operad). A semi-augmented unital partial \( S \)-colored operad \( (\mathcal{G}, \{\alpha_i\}, \eta, i) \) is the data of a unital partial \( S \)-colored operad \( (\mathcal{G}, \{\alpha_i\}, \eta, d_\mathcal{G}) \) together with a morphism of \( S \)-color schemes \( i : \mathcal{G} \to I_S \) of degree \( 0 \) such that \( i \cdot \eta = \text{id}_{I_S} \).

**Remark 5.53.** The unital partial \( S \)-colored operad \( uO \) is canonically semi-augmented by the identity morphism of \( I_S \), we denote this semi-augmentation by \( i_{uO} \).

**Proposition 5.54.** Let \( (\mathcal{G}, \{\alpha_i\}, \eta, i) \) be a semi-augmented unital partial \( S \)-colored operad that admits an inhomogeneous quadratic presentation \( (V, S) \). Let \( qS := S \cap \mathcal{F}_S(V)^{(2)} \). Let \( \varphi : qS \to I_S \oplus V \) be the linear map that gives \( S \) as its graph. The Koszul dual conilpotent partial \( S \)-colored cooperad of \( \mathcal{G} \) is given by
\[
\mathcal{G}^! := \mathcal{F}_S^C(sV, s^2qR).
\]
It is endowed with a coderivation of degree \(-1\) \( d_{\mathcal{G}^!} \) given by the unique extension of:
\[
\begin{array}{ccc}
\mathcal{G}^! & \xrightarrow{s^2qR} & s^2qR \\
\xrightarrow{s^{-1}\varphi_1} & & sE,
\end{array}
\]
and with a curvature \( \Theta_{\mathcal{G}^!} \) given by the degree \(-2\) map:
\[
\begin{array}{ccc}
\mathcal{G}^! & \xrightarrow{s^2qR} & s^2qR \\
\xrightarrow{s^{-2}\varphi_0} & & I_S.
\end{array}
\]
The data of \( (\mathcal{G}^!, d_{\mathcal{G}^!}, \Theta_{\mathcal{G}^!}) \) forms a conilpotent curved partial \( S \)-colored cooperad.

**Proof.** The proof is completely analogous to the non-\( S \)-colored case developed in [HM12, Section 4]. \( \square \)
Definition 5.55 (Koszul unital partial S-colored operad). Let $(S, \{o_i\}, \eta, \iota)$ be a semi-augmented unital partial $S$-colored operad that admits an inhomogeneous quadratic presentation $(V, S)$. Let $qS = S \cap \mathcal{R}(V)^{(2)}$ and let

$$qS := \mathcal{F}(V)/(qS)$$

be the quadratic partial $S$-colored operad associated to $S$. The semi-augmented unital partial $S$-colored operad $S$ is said to be Koszul if the quadratic operad $qS$, endowed with the quadratic presentation $(V, qS)$, is a Koszul quadratic partial $S$-colored in the sense of Definition 5.35.

Let $J$ be the graded $S$-color scheme given by $J(0; 1) := \mathbb{K}\theta$, where $\theta$ is an element of degree $-2$, and zero elsewhere.

Definition 5.56 (Curved partial $S$-colored $cO^\vee$). Let $(E, R)$ be the quadratic presentation of the partial $S$-colored operad $O$. The curved partial $S$-colored cooperad $cO^\vee$ is given by the presentation

$$cO^\vee := \mathcal{F}_S(E \oplus J, R),$$

where $E \oplus J$ are the cogenerators and $R$ are the corelations. It is endowed with the curvature $\Theta_{cO^\vee} : cO^\vee \rightarrow I_S$ defined on $cO^\vee(n; n)$ by the following map

$$\begin{cases}
\gamma_1^{1,n} \circ_1 \theta - \sum_{i=0}^n \gamma_i^{n,1} \circ_2 \theta & \mapsto \text{id}_n, \\
\mu & \mapsto 0,
\end{cases}$$

if $\mu$ is not contained in the sub-$S_n$-module generated by $\gamma_1^{1,n} \circ_1 \theta - \sum_{i=0}^n \gamma_i^{n,1} \circ_2 \theta$.

Theorem 5.57. Let $uO$ be the unital partial $S$-colored operad encoding (co)unital partial (co)operads.

1. The Koszul dual conilpotent curved partial $S$-colored cooperad $(uO)^!$ is isomorphic to the suspension of $cO^\vee$.

2. The unital partial $S$-colored operad $uO$ is a Koszul, meaning that $q(uO)$ is a Koszul quadratic partial $S$-colored operad.

Proof. The Koszul dual cooperad $(uO)^!$ is given by $\mathcal{F}_S(s(E \oplus U), s^2qR')$. It is cogenerated by a degree 1 operation $u \in sU(0; 1)$ and the suspension of the family $\{\gamma_i^{n,k}\}$. The corelations $s^2qR'$ are given by $s^2R$.

Let us compute the rest of the structure. The projection of $R'$ into $E \oplus U$ is zero, hence $(uO)^!$ has a zero pre-differential. The projection of $R'$ into $I_S$ will be non zero only on the relations that involve the element $u \in U(0; 1)$. It is straightforward to compute that the pre-image of $\text{id}_n \in I_S$ by the projection $s^{-2}q_0$ is given by

$$\sum_{i=0}^n s\gamma_i^{1,n} \circ_2 su - s\gamma_1^{1,n} \circ_1 su.$$ 

This completely characterizes the curvature of $(uO)^!$. A direct inspection identifies $(uO)^!$ with the operadic suspension of $cO^\vee$.

Let us prove the second assertion. The space $qR'$ is in fact $R$, and therefore $q(uO)$ is given by the coproduct $0 \coprod (u)$. By Theorem 5.44, we know that $0$ is a quadratic Koszul $S$-colored operad. Therefore the Koszul complex of $q(uO)$ is also acyclic by an argument analogous to that of [HLM12, Proposition 6.16].

The last point of this section is to understand what curved coalgebras and complete curved algebras over the conilpotent curved partial $S$-colored cooperad $cO^\vee$ are.

Proposition 5.58. The category of curved $(cO^\vee)^u$-coalgebras is isomorphic to the category of conilpotent curved partial cooperads.
Proof. First, notice that, for any pdg $S$-module $M$, there is an isomorphism of pdg $S$-modules
\[ \mathcal{S}(cO^\vee)(M) \cong \mathcal{T}^c(M \oplus \nu), \]
where $\nu$ is an arity 1 and degree $-2$ generator. Indeed, this can be shown by extending the bijection given in the proof of Lemma 5.41. This extension is defined by sending $\theta$ in $cO^\vee(0; 1)$ to the generator $\nu$. Pictorially it is given by
\[
\begin{array}{c}
\psi = m_1 \\
\downarrow m_2 \ (123) \\
4 \\
\downarrow \text{id} \ (1234) \\
\theta \end{array} \quad \rightarrow \quad \begin{array}{c}
\tau_\psi = m_2 \\
5 \quad 4 \quad 6 \\
\downarrow \nu \\
1 \quad 2 \quad 3 \\
\end{array}
\]
This isomorphism is natural in $M$. Therefore the endofunctor $\mathcal{T}^c(- \oplus \nu)$ in the category of pdg $S$-modules as a comonad structure. A direct computation shows that this comonad structure coincides with the reduced tree comonad structure.

Let $(C, \{\Delta_i\}, d_C, \Theta_C)$ be a conilpotent curved partial cooperad. Let $\Delta_C : C \rightarrow \mathcal{T}^c(C)$ be its structure map as a coalgebra over the reduced tree comonad. We construct an extension $\Delta_C^+ : C \rightarrow \mathcal{T}^c(C) \times \mathcal{T}^c(\nu) \cong \mathcal{T}^c(C \oplus \nu)$, using $\Theta_C : C \rightarrow \nu$.I and its unique extension to $\mathcal{T}^c(\nu)$.

The data $(C, \Delta_C^+, d_C)$ forms a pdg $(cO^\vee)^u$-coalgebra. It is in fact a curved $(cO^\vee)^u$-coalgebra. Indeed, the diagram
\[
\begin{array}{c}
C \xrightarrow{\Delta_C^+} \mathcal{S}(cO^\vee)(C) \\
\downarrow d_C^+ \\
C \cong \mathcal{S}(\Theta_CcO^\vee)(\text{id}) \\
\end{array}
\]
commutes since $(C, \{\Delta_i\}, d_C, \Theta_C)$ forms a curved partial cooperad. The other way around, given a curved $(cO^\vee)^u$-coalgebra $(C, \Delta_C^+, d_C)$, one can compose the structural map $\Delta_C^+ : C \rightarrow \mathcal{T}^c(C \oplus \nu)$ with the projection $\mathcal{T}^c(C \oplus \nu) \rightarrow \mathcal{T}^c(C)$, which endows $C$ with a conilpotent partial cooperad structure $\{\Delta_i\}$. By composing $\Delta_C^+$ with the projection $\mathcal{T}^c(C \oplus \nu) \rightarrow I.\nu$, one obtains a map $\Theta_C : C \rightarrow I$ of pdg $S$-modules of degree $-2$. The data $(C, \{\Delta_i\}, d_C, \Theta_C)$ forms a conilpotent curved partial cooperad, since $(C, \Delta_C^+, d_C)$ is a curved $(cO^\vee)^u$-coalgebra. □

**Proposition 5.59.** The category of complete curved $(cO^\vee)^u$-algebras is isomorphic to the category of complete curved absolute partial operads.

Proof. For the proof of this statement, see Proposition 9.36. For an explicit description of these objects, we refer to the Appendix 9.2. □
6. Curved twisting morphisms and Bar-Cobar adjunctions at the operadic level

The curved Koszul duality established in the previous section at the groupoid-colored level gives a curved twisting morphism $\kappa$ between the groupoid-colored curved cooperad encoding curved partial (co)operads $cO^\vee$ and the groupoid-colored operad encoding unital partial (co)operads $uO$. This curved twisting morphism induces two different Bar-Cobar adjunctions. The first Bar-Cobar adjunction induced by $\kappa$ is between dg unital partial operads and conilpotent curved partial cooperads. It will be shown to be isomorphic to the Bar-Cobar adjunction defined in [Gri19, Section 4.1]:

$$\text{curv pCoop}^{\text{conil}} \cong \begin{array}{c} \text{dg upOp} \cong \end{array}$$

On the other hand, using the generalization of [GL18] to the groupoid-colored setting, we obtain a new complete Bar-Cobar adjunction:

$$\text{dg upCoop} \cong \begin{array}{c} \text{comp} \end{array} \cong \text{curv abs pOp}^{\text{comp}},$$

between complete curved absolute partial operads and counital partial cooperads. Constructing this second adjunction is the main goal of this section.

6.1. Curved twisting morphisms and curved pre-Lie algebras. In order to defined curved twisting morphisms in the first place, we introduce a new type of structure, called curved pre-Lie algebras. In this type of algebras, the curvature has to satisfy a special condition, called the left-nucleus condition. This condition comes from the deformation theory. More precisely, it appears in [DSV18, Proposition 1.1, Chapter 4] for the following reason: one can show that a dg pre-Lie algebra is twistable by a Maurer-Cartan element if and only if this element satisfies the extra condition of being left-nucleus.

**Definition 6.1** (Curved pre-Lie algebra). A curved pre-Lie algebra $(g, \{-,-\}, d_g, \vartheta)$ amounts to the data of a pre-Lie algebra $(g, \{-,-\})$, a derivation $d_g$ with respect to $\{-,-\}$ of degree $-1$, and a morphism of pdg modules of degree $-2$ $\vartheta : K \rightarrow g$. The data of this morphism is equivalent to the data of an element $\vartheta(1) := \vartheta$ in $g_{-2}$. They are subject to the following conditions.

1. The element $\vartheta$ has to be left-nucleus, that is, for all $\mu, \nu$ in $g$:

$$\{\vartheta, \{\mu, \nu\}\} = \{\{\vartheta, \mu\}, \nu\}.$$

2. Moreover, for all $\mu$ in $g$:

$$d_g(\vartheta(1)) := d_g(\vartheta) = \{\vartheta, \mu\} - \{\mu, \vartheta\}.$$

3. And finally, $d_g(\vartheta) = 0$.

In the same spirit as for curved Lie algebras and curved associative algebras, one can define a curved partial operad, $\text{cpLie}$, that encodes curved pre-Lie algebras. Let $H$ be the pdg $S$-module $(K, d, 0, K[S_2], \nu, 0, \cdots)$ endowed with the zero pre-differential.

**Definition 6.2** (Curved operad encoding curved pre-Lie algebras). The curved partial operad $\text{cpLie}$ is given by the presentation:

$$\text{cpLie} := \mathcal{F}(H)/(D),$$

where $(D)$ is the operadic ideal generated by the following relations:

(1) The right pre-Lie relation, already present in the classical pre-Lie operad, given by:

1 2 3 1 2 3 1 3 2 1 3 2
$$\nu \quad \nu \quad \nu \quad \nu \quad \nu \quad \nu$$
Proposition 6.3. The data \( (\text{cpLie}, 0, \Theta_{\text{cpLie}}) \) forms a curved partial operad. The category of curved \( \text{cpLie} \)-algebras is equivalent to the category of curved \( \text{pre-Lie} \) algebras.

Furthermore, the morphism of curved partial operads \( \text{cLie} \to \text{cAss} \) given by the skew-symmetrization of the Lie bracket factors through the curved partial operad \( \text{cpLie} \).

Proof. Let us show that the data forms a curved partial operad. The proof is quite similar to that of Lemma 2.17. In order to check that \([\nu \circ_1 \vartheta - \nu \circ_2 \vartheta, -] = 0\), since it is a derivation with respect partial compositions, it is enough to test this equality on the generators. First, \([\nu \circ_1 \vartheta - \nu \circ_2 \vartheta, 0] = 0\) is evident. Second, expanding \([\nu \circ_1 \vartheta - \nu \circ_2 \vartheta, \nu]\) makes six terms that appear: it is straightforward to check that four of them cancel because of the right pre-Lie relation and the last two because of the left-nucleus relation. Proving that this curved partial operad encodes curved pre-Lie algebras is completely analogous to Lemma 2.17. One checks that the map \( \text{cLie} \to \text{cpLie} \) given on generators by \( \beta \mapsto \nu - \nu^{(12)} \) and \( \zeta \mapsto \vartheta \) is indeed a morphism of curved partial operads. Furthermore, there is a morphism of curved partial operads \( \text{cpLie} \to \text{cAss} \) simply given by \( \nu \mapsto \mu \) and \( \vartheta \mapsto \phi \). Their composition gives back the morphism constructed in Lemma 2.22. \(\square\)

Remark 6.4. Contrary to the case of Lie algebras and associative algebras, we had to introduce a new relation into the curved partial operad that encodes curved pre-Lie algebras. In the previous cases, this meant that the curved homotopy version of them was given simply by adding a curvature to the classical homotopy version. For instance, a curved \( \mathcal{A}_\infty \)-algebra is just an \( \mathcal{A}_\infty \)-algebra with an added structure of a curvature; the relations of a curved \( \mathcal{A}_\infty \)-algebra are clearly analogous to those of an \( \mathcal{A}_\infty \)-algebra. In the curved pre-Lie case, more structure will appear when one resolves the left-nucleus relation up to homotopy. Nevertheless, since the left-nucleus relation is not even quadratic, one would first need an expanded Koszul duality in order to treat this case.

We leave it to the reader to generalize the definition of curved partial operads to the \( S \)-colored setting. (See Definition 5.50 for a similar definition).

Lemma 6.5 (Totalization of a curved partial \( S \)-colored operad). Let \( \mathcal{G}, \{\circ_i\}, d_\mathcal{G}, \Theta_\mathcal{G} \) be a curved partial \( S \)-colored operad. The totalization of \( \mathcal{G} \) given by

\[
\prod_{(n_1, \ldots, n_r, n) \in \mathbb{N}^{r+1}} \left( \mathcal{G}(n_1, \ldots, n_r, n) (s_{n_1} \times \cdots \times s_{n_r}) \right)_{s_r} s_n
\]

forms a curved pre-Lie algebra, where the bracket is defined like in Proposition 5.26 and where the curvature is given by

\[
\vartheta(1) = \sum_{n \geq 0} \theta_n .
\]

Proof. Let \( g \) be in \( \mathcal{G}(n_1, \ldots, n_r, n) \), we have that \( d_\mathcal{G}^2(g) = \theta_n \circ_1 g - \sum_{i=0}^r g \circ_i \theta_{n_i} \) since \( \mathcal{G} \) is a curved partial \( S \)-colored operad. The bracket \( \vartheta(1) \ast g = \theta_n \circ_1 g \) since only \( \theta_n \) matches the appropriate color. Similarly, \( g \ast \vartheta(1) = \sum_{i=0}^r g \circ_i \theta_{n_i} \). Since each \( \theta_n \) is an arity one operation, by the sequential and parallel axioms of a partial \( S \)-colored operad, the curvature satisfies the left-nucleus relation of a curved pre-Lie algebra. \(\square\)
Lemma 6.6 (Curved convolution partial S-colored operad). Let \((\mathcal{S}, \{\phi_i\}, \eta, d_\mathcal{S})\) be a dg unital partial S-colored operad and let \((\mathcal{V}, \{\Delta_i\}, d_\mathcal{V}, \Theta)\) be a curved partial S-colored cooperad. The convolution partial S-colored operad \(\text{Hom}(\mathcal{V}, \mathcal{S})\) forms a curved partial S-colored operad endowed with the curvature:

\[ \Theta_{\text{Hom}} : \mathcal{V} \xrightarrow{\Theta} I_2 \xrightarrow{\eta} \mathcal{S} \]

Proof. Let \(\alpha\) be in \(\text{Hom}(\mathcal{V}, \mathcal{S})(n_1, \ldots, n_r; n)\), we have that:

\[ \partial^2(\alpha) = -(-1)^{2|\alpha|} \alpha \circ d_\mathcal{V}^2 = \Theta_{\text{Hom}} \circ_1 \alpha - \sum_{i=0}^r \alpha \circ_i \Theta_{\text{Hom}} \]

since \(d_\mathcal{S}^2 = 0\) and \(d_\mathcal{V}^2 = (\text{id}_{(1)} \Theta_{\mathcal{V}} - \Theta_{\mathcal{V}} \circ \text{id}) \cdot \Delta_{(1)} \).

Definition 6.7 (Maurer-Cartan of a curved pre-Lie). Let \((\mathfrak{g}, \{-, -\}, d_\mathfrak{g}, \emptyset)\) be a curved pre-Lie algebra. A Maurer-Cartan element \(\alpha\) is a degree \(-1\) element that of \(\mathfrak{g}\) that satisfies the following equation:

\[ d_\mathfrak{g}(\alpha) + \{\alpha, \alpha\} = 0 \]

Remark 6.8. The set of Maurer-Cartan elements of a curved pre-Lie (or similarly a curved Lie) algebra can be empty. A curvature term \(\emptyset \neq 0\) stops 0 in \(\mathfrak{g}_{-1}\) from being a canonical Maurer-Cartan element in \(\mathfrak{g}\).

Definition 6.9 (Curved S-colored twisting morphism). Let \((\mathcal{S}, \{\phi_i\}, \eta, d_\mathcal{S})\) be a unital partial dg S-colored operad and let \((\mathcal{V}, \{\Delta_i\}, d_\mathcal{V}, \Theta)\) be a curved partial S-colored cooperad. A curved twisting morphism \(\alpha\) between \(\mathcal{V}\) and \(\mathcal{S}\) is a morphism of pdg S-color schemes \(\alpha : \mathcal{V} \rightarrow \mathcal{S}\) of degree \(-1\) that satisfies the Maurer-Cartan equation in the totalization of the curved convolution S-colored operad \(\text{Hom}(\mathcal{V}, \mathcal{S})\). Otherwise stated, \(\alpha\) satisfies:

\[ \partial(\alpha) + \alpha \star \alpha = \Theta_{\text{Hom}} \]

Lemma 6.10. The S-color scheme morphism \(\kappa : \mathcal{S} \otimes (c\mathcal{O})^\vee \rightarrow u\mathcal{O}\) given by

\[ \kappa : \mathcal{S} \otimes (c\mathcal{O})^\vee \rightarrow \mathcal{S} \oplus \mathcal{S} \cong \mathcal{E} \oplus \mathcal{U} \rightarrow u\mathcal{O} \]

is a curved twisting morphism. Here \(\mathcal{S}\) denotes the operadic suspension of \(c\mathcal{O}^\vee\).

Proof. Notice that \(\partial(\kappa) = 0\) since the pre-differentials are null. Let us show that

\[ \kappa \star \kappa = \Theta_{\text{Hom}} \]

The morphism of S-color schemes \(\kappa \star \kappa\) is non-zero only on elements of weight two in \(c\mathcal{O}^\vee\). These are elements which can informally be written as \(\gamma_i^{n+1} \circ_1 \gamma_i^n \) and \(\gamma_i^{n+k+m-1} \circ_2 \gamma_i^{k,m}\), or as \(\gamma_i^{1,n} \circ_1 \emptyset\) and \(\gamma_i^{n,1} \circ_2 \emptyset\). One computes that \(\kappa \star \kappa\) of the first kind of weight two elements is zero because of the Koszul signs, by using the sequential and parallel relations. On the second kind of weight two elements, \(\kappa \star \kappa\) is equal to \(\gamma_i^{1,n} \circ_1 \emptyset\) and \(\gamma_i^{n,1} \circ_2 \emptyset\). Using the unital relation in \(u\mathcal{O}\), these are both equal to \(\text{id}_{n_i}\), the operadic unit of the S-colored operad \(u\mathcal{O}\). Hence \(\kappa \star \kappa\) is equal to \(\Theta_{\text{Hom}}\).

Notation. Let \(f : M \rightarrow N\) be a morphism of graded S-modules of degree 0 and \(g : M \rightarrow N\) be a morphism of graded S-modules degree \(p\). We denote \(\Pi_\mathcal{S}(n_1, \ldots, n_r)(f, g)\) the map

\[ \sum_{i=1}^r f(n_1) \otimes \cdots \otimes f(n_{i-1}) \otimes g(n_i) \otimes f(n_{i+1}) \otimes \cdots \otimes f(n_r) : M(n_1) \otimes \cdots \otimes M(n_r) \rightarrow N(n_1) \otimes \cdots \otimes N(n_r) \]

which is an \(S_{n_1} \times \cdots \times S_{n_r}\) \(S_r\)-equivariant morphism of degree \(p\). Let \(E\) be a graded S-color scheme. The family of maps \(\Pi_\mathcal{S}(n_1, \ldots, n_r)(f, g)\) induces a morphism of graded S-modules of degree \(p\):

\[ \mathcal{H}_\mathcal{S}(E)(M) \rightarrow \mathcal{H}_\mathcal{S}(E)(N) \]

by applying \(\text{id}_E \otimes \Pi_\mathcal{S}(n_1, \ldots, n_r)(f, g)\) to each component. By a slight abuse of notation, this morphism will be denoted by \(\mathcal{H}_\mathcal{S}(\text{id}_E)(\Pi_\mathcal{S}(f, g))\). Likewise, it induces a morphism of graded S-modules of degree \(p\):
by applying \( \text{Hom}(\text{id}_E, i\text{III}_S(n_1, \ldots, n_r)) (f, g) \) to each component. By a slight abuse of notation, this morphism will be denoted by \( \hat{\mathcal{R}}_S^c(id_E)(\text{III}_S(f, g)) \).

### 6.2. Classical Bar-Cobar adjunction relative to \( \kappa \)

The first adjunction induced by \( \kappa \) will be an adjunction between dg \( u\mathcal{O} \)-algebras and curved \( S \otimes (c\mathcal{O}^\vee)^u \)-coalgebras. That is, between dg unital partial operads and shifted conilpotent curved partial cooperads.

**Definition 6.11** (Bar-Cobar constructions relative to \( \kappa \)). Using \( \kappa \), one can define two functor:

1. Let \( (\mathcal{P}, \gamma_\mathcal{P}, d_\mathcal{P}) \) a dg \( u\mathcal{O} \)-algebra. Its **Bar construction relative to \( \kappa \)**, denoted by \( B_\kappa \mathcal{P} \), is given by the cofree \( S \otimes (c\mathcal{O}^\vee)^u \)-coalgebra \( \mathcal{R}_S(S \otimes (c\mathcal{O}^\vee)^u)(\mathcal{P}) \). Its pre-differential \( d_{\text{bar}} \) is given by the sum of two terms \( d_1 \) and \( d_2 \). The first term is given by 
   \[
   d_1 := \mathcal{R}_S(\text{id})(\text{III}_S(\text{id}, d_\mathcal{P})) .
   \]
   The second term \( d_2 \) is the unique coderivation extending:
   \[
   \mathcal{R}_S(S \otimes (c\mathcal{O}^\vee)^u)(\mathcal{P}) \xrightarrow{\mathcal{R}_S(\kappa)(\text{id})} \mathcal{R}_S(u\mathcal{O})(\mathcal{P}) \xrightarrow{\gamma_\mathcal{P}} \mathcal{P} .
   \]

2. Let \( (\mathcal{E}, \Delta_\mathcal{E}, d_\mathcal{E}) \) be a curved \( S \otimes (c\mathcal{O}^\vee)^u \)-coalgebra. Its **Cobar construction relative to \( \kappa \)**, denoted by \( \Omega_\kappa \mathcal{E} \), is given by the free \( u\mathcal{O} \)-algebra \( \mathcal{R}_S(u\mathcal{O})(\mathcal{E}) \). Its differential \( d_{\text{cobar}} \) is the sum of two terms \( d_1 \) and \( d_1 \). The first term is given by 
   \[
   d_1 := -\mathcal{R}_S(\text{id})(\text{III}_S(\text{id}, d_\mathcal{E})) .
   \]
   The second term \( d_2 \) is the unique derivation extending:
   \[
   \mathcal{E} \xrightarrow{\Delta_\mathcal{E}} \mathcal{R}_S(S \otimes (c\mathcal{O}^\vee)^u)(\mathcal{E}) \xrightarrow{\mathcal{R}_S(\kappa)(\text{id})} \mathcal{R}_S(u\mathcal{O})(\mathcal{E}) .
   \]

**Lemma 6.12.** There is an adjunction

\[
\text{curv } S \otimes (c\mathcal{O}^\vee)^u \text{-coalg} \xrightarrow{\Omega_\kappa} \text{dg uO-alg} .
\]

**Proof.** The proof is a minor generalization of a the adjunction induced by a curved twisting morphism at the level of algebras in \([HM12, \text{Section 5}]\). \( \square \)

This adjunction relative to \( \kappa \) is in fact isomorphic to the adjunction between conilpotent curved coaugmented cooperads and operads introduced in \([Gri19, \text{Section 4.1}]\). Before proving this result, we briefly recall the definition of this adjunction.

**Definition 6.13** (\([Gri19, \text{Definition 58}]\)). Let \( (\mathcal{P}, \{\gamma_1\}, \eta, d_\mathcal{P}) \) be a dg unital partial operad. The **Bar construction** \( B\mathcal{P} \) of \( \mathcal{P} \) is given by:

\[
B\mathcal{P} := \left( \mathcal{R}_c(s\mathcal{P} \oplus \nu), d_{\text{bar}} = d_1 + d_2, \Theta_{\text{bar}} \right) ,
\]

where \( \mathcal{R}_c(s\mathcal{P} \oplus \nu) \) is the cofree conilpotent partial pdg cooperad generated by the dg \( S \)-module \( s\mathcal{P} \oplus \emptyset \). Here \( \nu \) is an arity 1 and degree \(-2\) generator. It is endowed the pre-differential \( d_{\text{bar}} \), given by the sum of \( d_1 \) and \( d_2 \). The term \( d_1 \) is the unique coderivation extending

\[
\mathcal{R}_c(s\mathcal{P} \oplus \nu) \xrightarrow{s\gamma_\mathcal{P}} s\mathcal{P} .
\]

The term \( d_2 \) comes from the **structure** of a unital partial operad on \( \mathcal{P} \), it is given by the unique coderivation extending

\[
\mathcal{R}_c(s\mathcal{P} \oplus \nu) \xrightarrow{1.\nu + s\gamma_\mathcal{P} \circ \{1\} s\mathcal{P}} s\mathcal{P} .
\]
It is also endowed with the following curvature:

\[ \Theta_{\text{bar}} : \mathcal{T}(s\mathcal{P} \oplus \nu) \to 1. \nu \to s^{-2} \to 1. \]

The resulting Bar construction of \( \mathcal{P} \) forms a conilpotent curved partial cooperad.

**Definition 6.14 ([Gri19, Definition 6.0]).** Let \( (\mathcal{C}, \{\Delta_i\}, d, \Theta_{\mathcal{C}}) \) be a curved partial cooperad. The Cobar construction \( \Omega \mathcal{C} \) of \( \mathcal{C} \) is given by:

\[ \Omega \mathcal{C} : = \left( \mathcal{T}(s^{-1}\mathcal{C}), d_{\text{cobar}} = d_1 - d_2 \right), \]

where \( \mathcal{T}(s^{-1}\mathcal{C}) \) is the free unital partial operad generated by the pdg \( S \)-module \( s^{-1}\mathcal{C} \). It is endowed with the differential \( d_{\text{cobar}} \) given by the difference of \( d_1 \) and \( d_2 \). The term \( d_1 \) is the unique derivation extending \( s^{-1}\mathcal{C} \to s^{-1}\mathcal{C} \mathcal{T}(s^{-1}\mathcal{C}) \).

The resulting Cobar construction of \( \mathcal{C} \) forms a dg unital partial operad.

**Remark 6.15.** The Cobar constructions \( \Omega \mathcal{C} \) is not augmented because the canonical morphism \( \mathcal{T}(s^{-1}\mathcal{C}) \to 1 \) does not commute with the differentials in general. Indeed, \( d_{\text{cobar}}(\Theta_{\mathcal{C}}(\text{id})) \) is the trivial tree. It is therefore augmented if and only if the curvature \( \Theta_{\mathcal{C}}(\text{id}) \) is zero.

The Bar-Cobar constructions described above also form an adjunction

\[ \text{curv pCoop}^{\text{conil}} \xleftarrow{\Omega} \text{upOp} \xrightarrow{B} \text{dg upOp}. \]

**Proposition 6.16** (Mise en abîme). The Bar-Cobar adjunction relative to \( \kappa \) given by \( \Omega \kappa \dashv B \kappa \) is naturally isomorphic to the Bar-Cobar adjunction \( \Omega \dashv B \) constructed in [Gri19].

**Proof.** Let \( (\mathcal{P}, \{\circ_i\}, \eta, d) \) be a dg unital partial operad. One uses the bijection given in the proof of Proposition 5.58 to construct a natural isomorphism \( B \mathcal{P} \cong B \kappa \mathcal{P} \). \( \square \)

**Corollary 6.17.** This gives another proof that the Bar-Cobar adjunction \( \Omega \dashv B \) constructed in [Gri19] is a Quillen equivalence.

**Proof.** Since we consider a transferred structure on curved conilpotent partial cooperads, proving that this adjunction is a Quillen equivalence simply amounts to show that for any unital partial operad \( \mathcal{P} \), the unit

\[ \mathcal{P} \to \Omega B \mathcal{P}, \]

is a quasi-isomorphism. This follows from the fact that the Koszul complex of the curved twisting morphism \( \kappa : S \otimes (c\mathcal{O}^{\circ}) \to \mathcal{U}^{\circ} \) that induces this Bar-Cobar adjunction is acyclic, using analogous arguments to [Val20, Theorem 2.6]. \( \square \)

The notion of curved twisting morphism between conilpotent curved partial cooperad and dg unital partial operad is again encoded by a curved pre-Lie algebra.

**Lemma 6.18.** Let \( (\mathcal{P}, \{\circ_i\}, d, \Theta_{\mathcal{P}}) \) be a curved partial operad. The totalization of \( \mathcal{P} \) given by

\[ \prod_{n \geq 0} \mathcal{P}(n)^{S_n}, \]

together with its pre-Lie bracket and endowed with the curvature \( \partial(1) := \Theta_{\mathcal{P}}(\text{id}) \) forms a curved pre-Lie algebra.

**Proof.** The proof is completely analogous to Lemma 6.5. \( \square \)
Lemma 6.19 (Curved convolution operad). Let \((\mathcal{C}, \{\circ_i\}, \eta, d_\mathcal{C})\) be a dg unital partial operad and let \((\mathcal{E}, (\Delta_i), d_\mathcal{E}, \Theta)\) be a curved partial cooperad. The convolution partial dg operad \((\text{Hom}(\mathcal{C}, \mathcal{P}), \{\circ_i\}, \partial)\) forms a curved partial operad endowed with the curvature given by

\[
\Theta_{\text{Hom}}(\text{id}) : \mathcal{C} \rightarrow \mathcal{P}.
\]

Proof. The proof is completely analogous to Lemma 6.6. □

Definition 6.20 (Curved twisting morphism). Let \((\mathcal{P}, \{\circ_i\}, \eta, d_\mathcal{P})\) be a dg unital partial operad and let \((\mathcal{C}, \{\Delta_i\}, d_\mathcal{C}, \Theta)\) be a curved partial cooperad. A curved twisting morphism \(\alpha\) is a Maurer-Cartan element curved pre-Lie algebra given by the curved convolution operad:

\[
g_{\mathcal{C}, \mathcal{P}} := \prod_{n \geq 0} \text{Hom}(\mathcal{C}(n), \mathcal{P}(n)).
\]

This is the data of a morphism of \(S\)-modules \(\alpha : \mathcal{C} \rightarrow \mathcal{P}\) of a degree \(-1\) such that:

\[
\partial(\alpha) + \alpha \star \alpha = \Theta_{\text{Hom}}(\text{id}).
\]

The set of twisting morphism between \(\mathcal{C}\) and \(\mathcal{P}\) will be denoted \(\text{Tw}(\mathcal{C}, \mathcal{P})\).

The set of curved twisting morphism between conilpotent curved partial cooperads and unital partial dg operads defines a bifunctor

\[
\text{Tw}(\mathcal{C}, -) : (\text{curv pCoop}^{\text{conil}})^{\text{op}} \times \text{dg upOp Set} \rightarrow \text{Set},
\]

which is represented on both sides by the Bar-Cobar construction defined before.

Proposition 6.21 ([Gri19, Proposition 63]). Let \((\mathcal{P}, \{\circ_i\}, \eta, d_\mathcal{P})\) be a dg unital partial operad and let \((\mathcal{C}, \{\Delta_i\}, d_\mathcal{C}, \Theta)\) be a conilpotent curved partial cooperad. There are isomorphisms:

\[
\text{Hom}_{\text{dg upOp}}(\Omega \mathcal{C}, \mathcal{P}) \cong \text{Tw}(\mathcal{C}, \mathcal{P}) \cong \text{Hom}_{\text{curv pCoop}^{\text{conil}}}(\mathcal{C}, \mathcal{B}\mathcal{P}),
\]

which are natural in \(\mathcal{C}\) and \(\mathcal{P}\).

6.3. Complete Bar-Cobar adjunction relative to \(\kappa\). We introduce a new adjunction, which we call the complete Bar-Cobar adjunction, between complete curved absolute partial operads and dg counital partial cooperads. This adjunction is again induced by the curved twisting morphism \(\kappa\), using the techniques developed in [GL18]. In order to construct it, we need a mild generalization of the results in [Ane14], in which the author only considers set-colored operads.

Theorem 6.22. Let \((\mathcal{S}, \gamma, \eta, d_\mathcal{S})\) be a dg \(S\)-colored operad. The category of dg \(\mathcal{S}\)-coalgebras is comonadic, that is, there exists a comonad \((\mathcal{L}(\mathcal{S}), \omega, \xi)\) in the category of dg \(S\)-modules such that the category of dg \(\mathcal{L}(\mathcal{S})\)-coalgebras is equivalent to the category of dg \(\mathcal{S}\)-coalgebras.

The endofunctor \(\mathcal{L}(\mathcal{S})\) is given by the following pullback:
Lemma 6.23. Let \( G \) be a morphism of graded \( S \)-modules of degree 0 and \( \varphi : G \to \varphi \) a morphism of graded \( S \)-modules of degree \( p \). Then the degree \( p \) map:

\[
\mathcal{L}(\{g\} \circ \varphi)(M) \xrightarrow{\varphi(M)} \mathcal{L}(\{g\})(M) \xrightarrow{\mathcal{F}_S(\{g\})(M)} \mathcal{F}_S(M)
\]

restricts to a degree \( p \) morphism \( \mathcal{L}(\{\varphi\})(M) \to \mathcal{L}(\varphi)(M) \to \mathcal{L}(\varphi)(N) \).

Proof. The proof is essentially the same as [GL18, Lemma 6.20]. It is a consequence of the universal property of pullbacks: it suffices to construct a morphism of graded \( S \)-modules

\[
\mathcal{L}(\{g\})(M) \to \mathcal{F}_S(\{g\})(M) \to \mathcal{F}_S(M)
\]

that coincides with the morphism \( \mathcal{F}_S(\{g\})(M) \to \mathcal{F}_S(M) \). One can check that this morphism is given by

\[
\mathcal{F}_S(\{\varphi\})(M) \circ \mathcal{F}_S(\{\varphi\})(M)
\]

\( \square \)
Proposition 6.24 (Coderivations on the cofree construction). Let $M$ be a graded $S$-module and let $\mathcal{L}(u\mathcal{O})(M)$ be the cofree graded $u\mathcal{O}$-coalgebra generated by $M$. There is a natural bijection between maps

$$\varphi : \mathcal{L}(u\mathcal{O})(M) \rightarrow M$$

of degree $p$ and coderivations

$$d_\varphi : \mathcal{L}(u\mathcal{O})(M) \rightarrow \mathcal{L}(u\mathcal{O})(M)$$

of degree $p$. This bijection sends $\varphi$ to the degree $p$ coderivation given by

$$d_\varphi := \mathcal{L}(\text{id})(\Pi(\pi_M, \varphi)) \cdot \omega(M),$$

where $\omega(M)$ is the comonad structure map of $\mathcal{L}(u\mathcal{O})$.

Proof. For any $\varphi$, the morphism $d_\varphi$ is an endomorphism of $\mathcal{L}(u\mathcal{O})(M)$ by Lemma 6.23. One has to check that it is a coderivation. The main point is to use the fact that $p_1$ is a monomorphism to be able to check it directly on the dual Schur functors: the same diagrams as in [GL18, Proposition 6.23] commute, thus $d_\varphi$ is a coderivation. \qed

We first present the formal constructions before giving a more explicit description of this adjunction.

Definition 6.25 (Complete Bar construction relative to $\kappa$). Let $(\mathcal{P}, \gamma_p, d_\mathcal{P})$ be a complete curved $S \otimes c\mathcal{O}^\vee$-algebra. The complete Bar construction relative to $\kappa$, denoted by $\bar{\mathcal{B}}\kappa\mathcal{P}$, is given by:

$$\bar{\mathcal{B}}\kappa\mathcal{P} := (\mathcal{L}(u\mathcal{O})(\mathcal{P}), d_{\text{bar}} = d_1 + d_2)$$

where $\mathcal{L}(u\mathcal{O})$ is the cofree dg $u\mathcal{O}$-coalgebra generated by the pdg $S$-module $\mathcal{P}$. It is endowed with the pre-differential $d_{\text{bar}}$ given by the sum of two terms $d_1$ and $d_2$. The term $d_1$ is of the pre-differential is the unique coderivation extending the following map

$$\phi_1 : \mathcal{L}(u\mathcal{O})(\mathcal{P}) \xrightarrow{\mathcal{L}(u\mathcal{O})(d_\mathcal{P})} \mathcal{L}(u\mathcal{O})(\mathcal{P}) \xrightarrow{\pi_\mathcal{P}} \mathcal{P}.$$ 

It is given by

$$d_1 = \mathcal{L}(\text{id})(\Pi_S(\text{id}, d_\mathcal{P})).$$

The term $d_2$ is given by the unique coderivation that extends the map:

$$\phi_2 : \mathcal{L}(u\mathcal{O})(\mathcal{P}) \xrightarrow{p_1} \mathcal{R}_S^\kappa(\mathcal{P}) \xrightarrow{\mathcal{R}_S^\kappa(\text{id})} \mathcal{R}_S^\kappa(S \otimes c\mathcal{O}^\vee)(\mathcal{P}) \xrightarrow{\gamma_\mathcal{P}} \mathcal{P}.$$

Proposition 6.26. Let $(\mathcal{P}, \gamma_p, d_\mathcal{P})$ be a complete curved $S \otimes c\mathcal{O}^\vee$-algebra. The complete Bar construction $\bar{\mathcal{B}}\kappa\mathcal{P}$ forms a dg counital partial cooperad. Meaning that

$$d_{\text{bar}}^2 = 0.$$ 

Proof. The proof is very similar to the proof of [GL18, Proposition 9.2]. In order to check that $d_{\text{bar}}^2 = 0$, by a straightforward generalization of [GL18, Proposition 6.25] to the $S$-colored case, it is enough to check that $(\phi_1 + \phi_2) \cdot d_{\text{bar}} = 0$. We have that:

$$d_{\text{bar}} = \mathcal{R}_S^\kappa(\text{id})(\Pi_S(\text{id}, d_\mathcal{P})) + \mathcal{R}_S^\kappa(\text{id}_{u\mathcal{O}}) \circ \mathcal{R}_S^\kappa(\Pi(\pi_p, \gamma_p \cdot \mathcal{R}_S^\kappa(\text{id}_p))(\text{id}_p)) \cdot p_2(\mathcal{P}),$$

where the first term is $d_1$ and the second is $d_2$. One computes that:

$$(\phi_1 + \phi_2) \cdot d_{\text{bar}} = \gamma_p \cdot \mathcal{R}_S^\kappa(\kappa \star \kappa - \Theta_{S \otimes c\mathcal{O}^\vee} \cdot \eta_{u\mathcal{O}})(\text{id}_p) = 0.$$

since $\kappa$ is a curved twisting morphism. Therefore $d_{\text{bar}}^2 = 0$ and $\bar{\mathcal{B}}\kappa\mathcal{P}$ is a dg counital partial cooperad. \qed
Derivations on $\hat{\mathcal{F}}_S^c(S \otimes cO^\vee)(M)$, the free pdg $S \otimes cO^\vee$-algebra on an pdg $S$-module $M$, are completely characterized by their restrictions to the generators $M$. See [GL18, Proposition 6.15] for an analogous statement, which holds \textit{mutatis mutandis} in our case. Let $M$ be an pdg $S$-module, we denote $t_M$ be the following map:

$$t_M : M \xrightarrow{\hat{\mathcal{F}}_S^c((1\text{id}_M)} \xrightarrow{\mathcal{F}_S^c(S \otimes cO^\vee)(M).$$

**Definition 6.27** (Complete Cobar construction relative to $\kappa$). Let $(\mathcal{C}, \{\Delta \mathcal{C}, d_\mathcal{C}\})$ be a dg $uO$-coalgebra. The complete Cobar construction relative to $\kappa$, denoted by $\widehat{\Omega}_\kappa \mathcal{C}$, is given by:

$$\widehat{\Omega}_\kappa \mathcal{C} := \left( \mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C}), d_\text{cobar} = d_1 - d_2 \right)$$

where $\mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C})$ is the free pdg $S \otimes cO^\vee$-algebra. It is endowed with the pre-differential $d_\text{cobar}$ which is the difference of two terms $d_1$ and $d_2$. The term $d_1$ is the unique derivation that extends the map:

$$\psi_1 : \mathcal{C} \xrightarrow{t_\mathcal{C}} \mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C}) \xrightarrow{\mathcal{F}_S^c((\text{id}(d_\mathcal{C})) \xrightarrow{\mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C}).$$

It is given by

$$d_1 = \mathcal{F}_S^c((\text{id})(\text{III}(\text{id}, d_\mathcal{C})).$$

The term $d_2$ is given by the unique derivation that extends the map:

$$\psi_2 : \mathcal{C} \xrightarrow{\Delta_\mathcal{C}} \mathcal{F}_S^c(uO)(\mathcal{C}) \xrightarrow{\mathcal{F}_S^c((\kappa(d_\mathcal{C})) \xrightarrow{\mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C}).$$

**Proposition 6.28.** Let $(\mathcal{C}, \{\Delta \mathcal{C}, d_\mathcal{C}\})$ be a dg $uO$-coalgebra. The complete Cobar construction relative to $\kappa \widehat{\Omega}_\kappa \mathcal{C}$ forms a complete curved $S \otimes cO^\vee$-algebra. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{F}_S^c(I_S) \circ \mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C}) & \xrightarrow{\mathcal{F}_S^c(\Theta_{S \otimes cO^\vee}) \circ \mathcal{F}_S^c((\text{id}(d_\mathcal{C})) \circ \mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C} \xrightarrow{-d_\text{cobar}} \mathcal{F}_S^c(S \otimes cO^\vee)) \circ \mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C})}) \\
& \xrightarrow{-d_\text{cobar}} \mathcal{F}_S^c(S \otimes cO^\vee) \circ \mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C}) \xrightarrow{\mathcal{F}_S^c(S \otimes cO^\vee)} \mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C}) \xrightarrow{\mathcal{F}_S^c(S \otimes cO^\vee)} \mathcal{F}_S^c(S \otimes cO^\vee)(\mathcal{C})
\end{array}
$$

**Proof.** The proof is again very similar to the proof of [GL18, Proposition 9.1]. In order to check that

$$-d_\text{cobar} = \mathcal{F}_S^c(\Delta_{S \otimes cO^\vee}) \cdot \varphi_{S \otimes cO^\vee} \cdot \mathcal{F}_S^c(\Theta_{S \otimes cO^\vee}) \circ \mathcal{F}_S^c((\text{id}(d_\mathcal{C})),$$

it is in fact enough to check that

$$-d_\text{cobar} \cdot (\psi_1 - \psi_2) = \mathcal{F}_S^c(\Theta_{S \otimes cO^\vee})(\text{id}(d_\mathcal{C})).$$

This comes from a straightforward generalization of [GL18, Proposition 7.4]. The pre-differential $d_\text{cobar}$ is given by

$$d_\text{cobar} = \mathcal{F}_S^c((\text{id})(\text{III}(t_\mathcal{C}, d_\mathcal{C})) - \mathcal{F}_S^c(\Delta_{S \otimes cO^\vee})(\text{id}(d_\mathcal{C})) \cdot \mathcal{F}_S^c(\text{III}(t_\mathcal{C}, \mathcal{F}_S^c((\kappa(d_\mathcal{C})) \cdot \Delta_\mathcal{C}))(\text{id}(d_\mathcal{C})),
$$

where the first term is $d_1$ and the second is $d_2$. One computes that:

$$-d_\text{cobar} \cdot (\psi_1 - \psi_2) = \mathcal{F}_S^c(\kappa \star \kappa)(\text{id}(d_\mathcal{C})).$$

Since $\kappa$ is a curved twisting morphism, this concludes the proof. □

Let us give explicit descriptions of these functors.
Definition 6.29 (Complete Cobar construction). Let \((\mathcal{C}, \Delta_1), e_\mathcal{C}, d_\mathcal{C}\) be a dg counital partial cooperad. The complete Cobar construction \(\hat{\Omega}_{\mathcal{C}}\) of \(\mathcal{C}\) is given by:

\[
\hat{\Omega}_{\mathcal{C}} := \left( \mathcal{T}^\wedge (s^{-1}e \oplus \nu), d_{\text{cobar}} = d_1 - d_2, \Theta_{\text{cobar}} \right),
\]

where \(\mathcal{T}^\wedge (s^{-1}e \oplus \nu)\) is the completed reduced tree monad applied to the dg \(S\)-module \(s^{-1}e \oplus \nu\). Here \(\nu\) is an arity 1 degree \(-2\) generator. It is endowed with the differential \(d_{\text{cobar}}\) given by the difference of \(d_1\) and \(d_2\). The term \(d_1\) is the unique derivation extending

\[
s^{-1}e \xrightarrow{s^{-1}d_\mathcal{C}} s^{-1}e \xrightarrow{s^{-1}d_\mathcal{C}} \mathcal{T}^\wedge (s^{-1}e \oplus \nu).
\]

The term \(d_2\) comes from the structure of a dg counital partial cooperad \(\mathcal{C}\), it is given by the unique derivation extending

\[
s^{-1}e \xrightarrow{s^{-2}\Delta_1 + s^{-1}e_\mathcal{C}} (s^{-1}e_\mathcal{C} \circ_1 s^{-1}e_\mathcal{C}) \oplus 1.\nu \xrightarrow{} \mathcal{T}^\wedge (s^{-1}e \oplus 1.\nu).
\]

Its curvature \(\Theta_{\text{cobar}}\) is given by the following map

\[
\Theta_{\text{cobar}} : I \xrightarrow{s^{-2}} 1.\nu \xrightarrow{} \mathcal{T}^\wedge (s^{-1}e \oplus 1.\nu).
\]

The resulting complete Cobar construction of \(\mathcal{C}\) forms a complete curved absolute partial operad.

Notation. We denote by \(\mathcal{T}^\vee(-)\) the cofree counital partial cooperad endofunctor in the category of pdg \(S\)-modules. It is given by the cofree graded \(u\mathcal{O}\)-coalgebra \(\mathcal{L}(\mathcal{O})(-\))

Definition 6.30 (Complete Bar construction). Let \((\mathcal{P}, \gamma_\mathcal{P}, d_\mathcal{P}, \Theta_\mathcal{P})\) a complete curved absolute partial operad. The complete Bar construction \(\hat{B}_{\mathcal{P}}\) of \(\mathcal{P}\) is given by:

\[
\hat{B}_{\mathcal{P}} := \left( \mathcal{T}^\vee (s\mathcal{P}), d_{\text{bar}} = d_1 + d_2 \right),
\]

where \(\mathcal{T}^\vee (s\mathcal{P})\) is the cofree counital partial cooperad generated by the pdg \(S\)-module \(s\mathcal{P}\). It is endowed the pre-differential \(d_{\text{bar}}\) given by the sum of \(d_1\) and \(d_2\). The term \(d_1\) is the uniquecoderivation extending

\[
\mathcal{T}^\vee (s\mathcal{P}) \xrightarrow{} s\mathcal{P} \xrightarrow{s\gamma_\mathcal{P}} s\mathcal{P}.
\]

The term \(d_2\) comes from the structure of \(\mathcal{P}\), it is given by the unique coderivation extending

\[
\mathcal{T}^\vee (s\mathcal{P}) \xrightarrow{} I \oplus (s\mathcal{P} \circ_1 s\mathcal{P}) \xrightarrow{s\Theta_\mathcal{P} + s^2\gamma_1} s\mathcal{P}.
\]

The resulting complete Bar construction of \(\mathcal{P}\) forms a dg counital partial cooperad.

Remark 6.31. The complete Bar construction of a complete curved absolute partial operad \(\mathcal{P}\) is, in general, not coaugmented. Indeed, one has that

\[
d_{\text{bar}}(\mathcal{P}) = d_2(\mathcal{P}) = \Theta_\mathcal{P}(\text{id}),
\]

where \(\mathcal{P}\) denotes the coaugmented counit of \(\mathcal{T}^\vee (s\mathcal{P})\).

Proposition 6.32. The following holds:

1. Let \((\mathcal{C}, \Delta_1), e_\mathcal{C}, d_\mathcal{C}\) be a dg counital partial cooperad. There is a natural isomorphism

\[
\hat{\Omega}_\mathcal{C} \cong \hat{\Omega}_{\mathcal{C}}.
\]

2. Let \((\mathcal{P}, \gamma_\mathcal{P}, \Theta_\mathcal{P}, d_\mathcal{P})\) a complete curved absolute partial operad. There is a natural isomorphism

\[
\hat{B}_\mathcal{P} \cong \hat{B}_\mathcal{P}.
\]

Proof. This can be shown by direct inspection, extending the isomorphism of Lemma 9.34. □
Lemma 6.33 (Curved convolution operad). Let $(\mathcal{P}, \gamma_{\mathcal{P}}, d_{\mathcal{P}}, \Theta_{\mathcal{P}})$ be a complete curved absolute partial operad and let $(\mathcal{C}, \{\Delta_i\}, \epsilon, d_{\mathcal{C}})$ be a dg counital partial cooperad. The convolution pdg partial operad $(\mathcal{H}om(\mathcal{C}, \mathcal{P}), \{\circ_i\}, \partial)$ forms a curved partial operad endowed with the curvature given by

$$\Theta_{\mathcal{H}om} : \epsilon : \mathcal{C} \to \mathcal{P}$$

Proof. The proof is completely analogous to Lemma 6.6. □

Remark 6.34. We only used the operations $\{\circ_i\}$ on $\mathcal{P}$ in order to define this convolution operad. These operations are obtained by restricting the structural morphism $\gamma_{\mathcal{P}}$ to finite sums inside the completed reduced tree monad. See Appendix B 9.2 for more details. In fact, using the morphism $\gamma_{\mathcal{P}}$, one can show that this convolution curved partial operad is in fact an curved absolute partial operad. The idea is that infinite sums of convolution operations have a well-defined image in $\mathcal{P}$ because $\mathcal{P}$ is an absolute operad.

Moreover, the convolution operad between a conilpotent cooperad $\mathcal{C}$ and an operad is also an absolute operad. Indeed, the decompositions in $\mathcal{C}$ always produce finite sums, therefore infinite sums of convolution operations are still well-defined. For a more detailed discussion about how convolution structures always produce absolute types of algebraic structures, see [RiL22, Section 4.1].

Once we have the curved convolution partial operad, we can define the notion of curved twisting morphism between complete curved absolute partial operads and dg counital partial cooperads.

Definition 6.35 (Curved twisting morphism). Let $(\mathcal{P}, \gamma_{\mathcal{P}}, d_{\mathcal{P}}, \Theta_{\mathcal{P}})$ be a complete curved absolute partial operad and let $(\mathcal{C}, \{\Delta_i\}, \epsilon, d_{\mathcal{C}})$ be a dg counital partial cooperad. A curved twisting morphism $\alpha$ is a Maurer-Cartan element curved pre-Lie algebra given by the curved convolution operad:

$$g_{\mathcal{C}, \mathcal{P}} := \prod_{n \geq 0} \text{Hom}_{S}(\mathcal{C}(n), \mathcal{P}(n))$$

This is the data of a morphism of graded $S$-modules $\alpha : \mathcal{C} \to \mathcal{P}$ of a degree $-1$ such that:

$$\partial(\alpha) + \alpha \star \alpha = \Theta_{\mathcal{H}om}(\text{id})$$

The set of twisting morphism between $\mathcal{C}$ and $\mathcal{P}$ will be denoted $\text{Tw}(\mathcal{C}, \mathcal{P})$.

The set of curved twisting morphism between between complete curved partial operads and counital partial dg cooperads

$$\text{Tw}(-, -) : (\text{dg upCoop})^{op} \times \text{curv abs pOp}^{\text{comp}} \to \text{Set},$$

which is represented on both sides by the complete Bar-Cobar constructions.

Proposition 6.36 (Complete Bar-Cobar adjunction). Let $(\mathcal{P}, \gamma_{\mathcal{P}}, d_{\mathcal{P}}, \Theta_{\mathcal{P}})$ be a complete curved partial operad and let $(\mathcal{C}, \{\Delta_i\}, \epsilon, d_{\mathcal{C}})$ be a dg counital partial cooperad. There are isomorphisms:

$$\text{Hom}_{\text{curv abs pOp}^{\text{comp}}}(\mathcal{C}, \mathcal{P}) \cong \text{Tw}(\mathcal{C}, \mathcal{P}) \cong \text{Hom}_{\text{dg upCoop}}(\mathcal{C}, \overline{\mathcal{B}}_{\mathcal{P}}),$$

which are natural in $\mathcal{C}$ and $\mathcal{P}$.

Proof. This can be shown by a straightforward computation. □

Corollary 6.37. There is a pair of adjoint functors:

$$\text{dg upCoop} \dashv \text{curv abs pOp}^{\text{comp}}.$$
In this section, we introduce the notion of a counital partial cooperads up to homotopy. This notion is dual to that of unital partial operads up to homotopy developed in [Gri21]. In fact, there is a S-colored operad which encodes unital partial operads up to homotopy as its algebras and counital partial cooperads up to homotopy as its coalgebras. The advantage of counital partial cooperads up to homotopy with respect to counital partial cooperads is that the admit a canonical cylinder object. Thus, using a left transfer theorem, we endow them with a model structure where weak equivalences are arity-wise quasi-isomorphisms. Notice that here, the category of counital partial cooperads up to homotopy with strict morphisms is the one being considered. Afterwards, we transfer this model structure to the category of complete curved absolute partial operads via a complete Bar-Cobar adjunction.

**Notation.** We denote by $\text{RT}_n^\omega$ the set of rooted trees of arity $n$ with $\omega$ internal edges. There is no restriction on the number of incoming edges of the vertices considered.

**Definition 7.1 (Counital partial cooperad up to homotopy).** Let $(C, d_C)$ be a dg $S$-module. A counital partial cooperad up to homotopy structure on $C$ amounts to the data of a derivation of degree $-1$

$$d : \mathcal{F}^\wedge (s^{-1}C \oplus \nu) \to \mathcal{F}^\wedge (s^{-1}C \oplus \nu)$$

such that the restriction of $d$ to $s^{-1}C$ is given by $s^{-1}d_C$ and such that for any series of rooted trees of arity $n$ labeled by elements of $s^{-1}C$:

$$d^2 \left( \sum_{\omega \geq 1} \sum_{\tau \in \text{RT}_n^\omega} \tau \right) = \sum_{\omega \geq 1} \sum_{\tau \in \text{RT}_n^\omega} \left( \nu \circ_1 \tau - \sum_{i=0}^{n} \tau \circ_1 \nu \right).$$

In order words, the data of $(\mathcal{F}^\wedge (s^{-1}C \oplus \nu), d)$ forms a complete curved absolute partial operad.

**Remark 7.2.** The morphism

$$\begin{align*}
\mathcal{C} \xrightarrow{\cong} s^{-1}C \xrightarrow{d} \mathcal{F}^\wedge (s^{-1}C \oplus \nu) \xrightarrow{\delta} I.\nu
\end{align*}$$

endows $\mathcal{C}$ with a counit $\epsilon_C : \mathcal{C} \to I$ which satisfies the counital axiom of counital partial cooperad only up to higher homotopies. The morphism

$$\begin{align*}
\mathcal{C} \xrightarrow{\cong} s^{-1}C \xrightarrow{d} \mathcal{F}^\wedge (s^{-1}C \oplus \nu) \xrightarrow{\mathcal{F}^\wedge (2)} (s^{-1}C \oplus \nu)
\end{align*}$$

endows $\mathcal{C}$ with a family of partial decompositions maps $\{\Delta_i\}$ which satisfy the coparallel and cosequential axioms of a counital partial cooperad only up to higher homotopies. All the higher homotopies are given by the morphism

$$\begin{align*}
\mathcal{C} \xrightarrow{\cong} s^{-1}C \xrightarrow{d} \mathcal{F}^\wedge (s^{-1}C \oplus \nu) \xrightarrow{\mathcal{F}^\wedge (\geq 3)} (s^{-1}C \oplus \nu).
\end{align*}$$

**Example 7.3.** Any dg counital partial cooperad $(\mathcal{C}, \{\Delta_i\}, \epsilon, d_C)$ is an example of counital partial cooperad up to homotopy via its complete Cobar construction $\hat{\Omega} \mathcal{C}$.

**Definition 7.4 (Strict morphisms).** Let $(\mathcal{C}, d_C)$ and $(\mathcal{D}, d_D)$ be two counital partial cooperads up to homotopy. A morphism $f : \mathcal{C} \to \mathcal{D}$ amounts to the data of a morphism of graded
S-modules \( f : \mathcal{C} \to \mathcal{D} \) such that the following diagram commute

\[
\begin{array}{c}
\mathcal{C} & \xrightarrow{\cong} & s^{-1}\mathcal{C} & \xrightarrow{d_1} & \mathcal{T}^\wedge (s^{-1}\mathcal{C} \oplus \nu) \\
\downarrow f & & \downarrow \mathcal{T}^\wedge (s^{-1}f \oplus \nu) \\
\mathcal{D} & \xrightarrow{\cong} & s^{-1}\mathcal{D} & \xrightarrow{d_2} & \mathcal{T}^\wedge (s^{-1}\mathcal{D} \oplus \nu) \\
\end{array}
\]

**Remark 7.5** (∞-morphisms). Let \( (\mathcal{C}, d_1, d_\nu) \) and \( (\mathcal{D}, d_2, d_\nu) \) be two counital partial cooperads up to homotopy. An ∞-morphism \( f : \mathcal{C} \to \mathcal{D} \) amounts to the data of a morphism of complete curved absolute partial operads

\[
f : \left( \mathcal{T}^\wedge (s^{-1}\mathcal{C} \oplus \nu), d_1 \right) \to \left( \mathcal{T}^\wedge (s^{-1}\mathcal{D} \oplus \nu), d_2 \right).
\]

These ∞-morphisms of counital partial cooperads up to homotopy should have all the expected properties of ∞-morphism between unital partial operads described in [Gri21]. In particular, ∞-quasi-isomorphisms of counital partial cooperads admit inverses in homology, and their set describes the hom-sets of the homotopy category. For the sake of brevity, we do not explore this path here.

Counital partial cooperads up to homotopy are coalgebras over a dg S-colored operad, which is given by the Cobar construction on the conilpotent curved S-colored partial cooperad \( S \otimes c0^\vee \). This Cobar construction is the S-colored analogous of the Cobar construction of Definition 6.14. (We swear, dear reader, this is the last Cobar construction of the article!)

**Definition 7.6** (The dg S-colored operad encoding (co)unital partial (co)operads up to homotopy). The dg S-colored operad \( \Omega_S(S \otimes c0^\vee) \) is given by

\[
\Omega_S \left( S \otimes c0^\vee \right) := \left( \mathcal{K}_S \left( s^{-1}(S \otimes c0^\vee) \right), d_{\text{cobar}} = -d_2 \right),
\]

where \( \mathcal{K}_S(s^{-1}(S \otimes c0^\vee)) \) is the free unital partial S-colored operad generated by the desuspension of the pdg S-colored scheme \( S \otimes c0^\vee \). It is endowed with a differential \( d_2 \), which is given by the unique derivation extending the map:

\[
s^{-1}(S \otimes c0^\vee) \xrightarrow{s^{-2} \Delta_{(1)} \otimes s^{-1} \Theta_S \otimes c0^\vee} (s^{-1}(S \otimes c0^\vee) \circ (s^{-1}(S \otimes c0^\vee))) \oplus I_S \hookrightarrow \mathcal{K}_S \left( s^{-1}(S \otimes c0^\vee) \right).
\]

**Proposition 7.7.** The following holds:

1. The category of unital partial operads up to homotopy with strict morphisms is equivalent to the category of dg algebras over \( \Omega_S \left( S \otimes c0^\vee \right) \).
2. The category of counital partial cooperads up to homotopy with strict morphism is equivalent to the category of dg coalgebras over \( \Omega_S \left( S \otimes c0^\vee \right) \).

**Proof.** For the first statement, the proof is mutatis mutandis the same as the proof of [Gri21, Proposition 36]. For the second statement, the proof is mutatis mutandis the same as the proof of [GL18, Theorem 12.1].

**Corollary 7.8.** There forgetful functor from counital partial cooperads up to homotopy to the category of dg S-modules admits a right adjoint

\[
\begin{array}{c}
\text{upCoop}_\infty & \cong & \text{dg S-mod}, \\
\downarrow U & & \downarrow \mathcal{K} \left( \Omega_S \left( S \otimes c0^\vee \right) \right)
\end{array}
\]

which is the cofree counital partial cooperad up to homotopy functor.

**Proof.** This is an immediate consequence of Theorem 6.22.
Using this adjunction, we want to transfer the standard model structure on dg $S$-modules via a left-transfer theorem. The key ingredient to apply the left-transfer theorem in this situation is the construction of a functorial cylinder object in the category of counital partial cooperads up to homotopy.

Let $I$ be the commutative Hopf monoid given by the cellular chains on the interval $[0,1]$, see [BM03, Example 3.3.3]. It is a commutative algebra with a compatible coassociative coalgebra structure is given by the choice of a cellular diagonal on the topological interval. It is an interval object in the category of dg modules, meaning that there is a factorization

$$K \oplus K \to I \to K$$

of the codiagonal map $id \oplus id : K \oplus K \to K$.

**Definition 7.9** (Interval dg $S$-modules). The interval dg $S$-module $Int$ is the dg $S$-module given by

$$Int(n) := I$$

for all $n \geq 0$.

**Remark 7.10.** If we endow $Int$ with the partial decomposition maps $\Delta_i : Int(n + k - 1) \to Int(n) \otimes Int(k)$ given by the coalgebra structure of $I$

$$\Delta : I \to I \otimes I,$$

then $Int$ almost forms a dg counital partial cooperad. It satisfies the cosequential axiom but, since $\Delta$ is not cocommutative, it does not satisfy the coparallel axiom. If there existed a cocommutative interval object in the category of dg module, we could directly transfer the standard model structure of dg $S$-module to the category of counital partial cooperads, in a dual version of [BM03, Theorem 3.2]. This is one reason to work with counital partial cooperads up to homotopy.

**Proposition 7.11.** Let $(C, d_C, e_C)$ be a counital partial cooperad up to homotopy. The Hadamard product $C \otimes Int$ has a canonical structure of counital partial cooperad up to homotopy.

**Proof.** The proof is completely dual to the proof of [Gri21, Proposition 31], where given a unital partial operad up to homotopy, the author builds a unital partial operad up to homotopy structure on $P \otimes Int$.

Let us sketch this construction: the idea to build the map that induces the coderivation

$$\Delta : s^{-1}(C \otimes Int) \to \overline{T}^\wedge(s^{-1}(C \otimes Int) \oplus v)$$

is to do so by induction on the weight of rooted trees in $\overline{T}^\wedge$. Suppose there is a map

$$\Delta_{m-1} : s^{-1}(C \otimes Int) \to \overline{T}^\leq(m-1)(s^{-1}(C \otimes Int) \oplus v)$$

such that

$$d_{m-1}^2\left(\sum_{\omega \geq 1} \sum_{\tau \in RT^n_{\omega}} \tau\right) = \sum_{\omega \geq 1} \sum_{\tau \in RT^n_{\omega}} v \circ_1 \tau - \sum_{i=0}^n \tau \circ_i v,$$

where $d_{m-1}$ is the induced coderivation of $\overline{T}^\leq(m-1)$ by $\Delta_{m-1}$. Furthermore, we suppose that the following equalities hold

$$\Delta e \left(s\tau \otimes Id_C\right) = (s\tau \otimes Id_C \oplus Id_V)^{(m-1)} \Delta_{m-1},$$

$$(s\tau \otimes Id_C \oplus Id_V)^{(m-1)} \Delta e = \Delta_{m-1} \left(s\tau \otimes Id_C\right),$$

where $p$ is the projection $C \otimes Int \to C$ and $\iota_i : C \to C \otimes Int$, for $i = 1, 2$ are the two inclusions. Notice that these two conditions imply the functoriality of this construction. Extending $\Delta_{m-1}$ to $\Delta_m$ can be shown to be equivalent to finding a lift in a commutative square in the underlying
category of dg modules. This lift because in the square, the left hand side is a cofibration and the right hand side is an acyclic fibration.

Corollary 7.12 (Functorial cylinder object). Let \((\mathcal{C}, d_1, d_\mathcal{C})\) be a counital partial cooperad up to homotopy. Then the codiagonal morphism \(\text{id} \oplus \text{id} : \mathcal{C} \oplus \mathcal{C} \rightarrow \mathcal{C}\) factors as follows

\[
\begin{array}{cccc}
\mathcal{C} \oplus \mathcal{C} & \longrightarrow & \mathcal{C} \otimes \text{Int} & \sim & \mathcal{C}
\end{array}
\]

where the first arrow is a degree-wise monomorphism and the second arrow is an arity-wise quasi-isomorphism. Thus \(\mathcal{C} \otimes \text{Int}\) is a cylinder object which is functorial in \(\mathcal{C}\).

Theorem 7.13 (Transferred model structure). There is a cofibrantly generated model structure on the category of counital partial cooperads up to homotopy defined by the following classes of morphisms:

1. The class of weak-equivalences \(W\) is given by strict morphisms of counital partial cooperads up to homotopy \(f : \mathcal{C} \rightarrow \mathcal{D}\) such that \(f(n) : \mathcal{C}(n) \rightarrow \mathcal{D}(n)\) is a quasi-isomorphism for all \(n \geq 0\).

2. The class of cofibrations is \(\text{Cof}\) is given by strict morphisms of counital partial cooperads up to homotopy \(f : \mathcal{C} \rightarrow \mathcal{D}\) such that \(f(n) : \mathcal{C}(n) \rightarrow \mathcal{D}(n)\) is a degree-wise monomorphism for all \(n \geq 0\).

3. The class of fibrations \(\text{Fib}\) is given by strict morphisms of counital partial cooperads up to homotopy which have the right lifting property against all morphism in \(W \cap \text{Cof}\).

Proof. In order to apply the left-transfer theorem such as [HKRS17], essentially need a functorial cofibrant replacement functor and a functorial cylinder object. Since every object is cofibrant in this model structure, we only need a functorial cylinder object. This follows from Corollary 7.12.

Now our aim is to transfer this structure to category of complete curved absolute partial operads, replicating the methods of Section 10 in [GL18]. For this purpose, we induce a complete Bar-Cobar adjunction using a curved twisting morphism between the \(\mathcal{S}\)-colored objects encoding these categories. (Yes, dear reader, we lied). For that, we consider the universal curved twisting morphism \(\iota\):

Lemma 7.14. The \(\mathcal{S}\)-color scheme map given by the inclusion

\[
\iota : \mathcal{S} \otimes \mathcal{O}^\vee \hookrightarrow \Omega_{\mathcal{S}} \left( \mathcal{S} \otimes \mathcal{O}^\vee \right)
\]

is a curved twisting morphism.

Proof. This is immediate to check from the definition.

Proposition 7.15 (Complete Bar-Cobar adjunction relative to \(\iota\)). There is a pair of adjoint functors

\[
\begin{array}{cccc}
\text{upCoop}_\infty & \stackrel{\Omega_{\mathcal{B}}}{\longleftarrow} & \text{curv abs pOp}^\text{comp}
\end{array}
\]

Proof. The constructions of the complete Bar and the complete Cobar functors are mutatis mutandis the same as those described in the previous section, where we constructed the complete Bar-Cobar relative to \(\kappa\). The proof that these functors form an adjunction is also mutatis mutandis the same.

This allows us in turn to transfer the above model structure from counital partial cooperads up to homotopy to complete curve absolute partial operads using a right transfer theorem.

Theorem 7.16. There is a cofibrantly generated model structure on the category of complete curved absolute partial operads defined by the following classes of morphisms:
The class of weak-equivalences $W$ is given by morphisms of complete curved absolute partial operads $f : \mathcal{P} \rightarrow \Omega$ such that $\bar{B}_1(f)(n) : \bar{B}_1(\mathcal{P})(n) \rightarrow \bar{B}_1(\Omega)(n)$ is a quasi-isomorphism for all $n \geq 0$.

(2) The class of fibrations is $\text{Fib}$ is given by morphisms of complete curved absolute partial operads $f : \mathcal{P} \rightarrow \Omega$ such that $\bar{B}_1(f) : \bar{B}_1(\mathcal{P}) \rightarrow \bar{B}_1(\Omega)$ is a fibration of counital partial cooperads up to homotopy.

(3) The class of cofibrations $\text{Cof}$ is given by morphisms of complete curved absolute partial operads which have the left lifting property against all morphism in $W \cap \text{Fib}$.

Proof. In order to apply the right transfer theorem, we need to check that the acyclicity conditions of [BM03, Section 2.5] are satisfied. Thus these conditions need to be checked by direct computation, as it is done in [GL18, Section 10.5]. The key point is to show that acyclic cofibrations of complete curved partial cooperads are precisely given by graded quasi-isomorphisms of Definition 7.18. Then the result follows, as it is clear that the complete Cobar functor preserve small objects, and since these trivial cofibrations are stable under pushouts and sequential colimits. In is a straightforward but tedious exercise to check that the arguments used in loc.cit generalize to our framework.

Proposition 7.17. Let $f : (\mathcal{P}, \gamma_\mathcal{P}, d_\mathcal{P}, \Theta_\mathcal{P}) \rightarrow (\Omega, \gamma_\Omega, d_\Omega, \Theta_\Omega)$ be a morphism of complete curved absolute partial operads. If $f(n) : \mathcal{P}(n) \rightarrow \Omega(n)$ is a degree-wise epimorphism, then it is a fibration. Thus all complete curved absolute partial operads are fibrant.

Proof. The arguments of Section 10.2 in [GL18] generalize to the operadic setting mutatis mutandis.

Definition 7.18 (Graded quasi-isomorphism). Let $f : (\mathcal{P}, \gamma_\mathcal{P}, d_\mathcal{P}, \Theta_\mathcal{P}) \rightarrow (\Omega, \gamma_\Omega, d_\Omega, \Theta_\Omega)$ be a morphism of complete curved absolute partial operads. It is a graded quasi-isomorphism if the induced morphism of $dg\ S$-modules

$$\text{gr}(f) : \text{gr}(\mathcal{P}) \cong \bigoplus_{\omega \geq 1} \mathcal{P}_{\omega} / \mathcal{P}_{\omega + 1} \mathcal{P} \rightarrow \text{gr}(\Omega) \cong \bigoplus_{\omega \geq 1} \Omega_{\omega} / \mathcal{Q}_{\omega + 1} \Omega$$

is an arity-wise quasi-isomorphism. Here $\mathcal{P}_{\omega}$ denotes the $\omega$-term of the canonical filtration of an absolute partial operad defined in Appendix 9.2.

Remark 7.19. One checks easily that since $d_\mathcal{P}^2$ raises the weight of an operation in $\mathcal{P}$ by one, it is equal to zero in the associated graded. Therefore $\text{gr}(\mathcal{P})$ forms a $dg\ S$-module.

Proposition 7.20. Let $f : (\mathcal{P}, \gamma_\mathcal{P}, d_\mathcal{P}, \Theta_\mathcal{P}) \rightarrow (\Omega, \gamma_\Omega, d_\Omega, \Theta_\Omega)$ be a graded quasi-isomorphism between two complete curved absolute partial operads. Then it is a weak-equivalence, meaning that $\bar{B}_1(f) : \bar{B}(\mathcal{P}) \rightarrow \bar{B}_1(\Omega)$ is an arity-wise quasi-isomorphism.

Proof. The proof is completely analogous to that of [GL18, Theorem 10.23]. It is essentially done by induction. The cofree counital cooperad functor preserves quasi-isomorphisms. Thus it sends graded quasi-isomorphisms between curved partial operads endowed with the trivial operad structure to quasi-isomorphisms. Then one shows a weak version of the five-lemma [GL18, Lemma 10.20] which allows us to go from weight $\omega$ to weight $\omega + 1$. See Section 10.3 for more details on that matter.

Remark 7.21. The above proposition implies that the weak-equivalences of complete curved operads as defined in [DCBM20] by graded quasi-isomorphisms should be strictly included inside the weak-equivalences of our model structure. Nevertheless, the objects and the underlying categories considered are quite different in nature, so we do not attempt a precise comparison result.

There is a morphism of $dg$ unital partial $S$-colored operads $f_\kappa : \Omega_S (S \otimes cO^\vee) \rightarrow uO$, which in turn induces a morphism between the associated comonads in the category of $dg\ S$-modules.
Thus one has an adjunction
\[ \text{dg upCoop} \cong \text{upCoop}\_\infty, \]
where the forgetful functor \( \text{Res}_{f\kappa} \) is fully faithful and preserves quasi-isomorphisms.

**Proposition 7.22.** There following triangle of adjunctions

\[ \text{upCoop}\_\infty \]
\[ \text{curv abs pOp}\_\text{comp}, \]
\[ \text{curv abs pOp}\_\text{comp}, \]

commutes up to natural isomorphism.

**Proof.** It is immediate to check that the left-adjoints are naturally isomorphic. Thus by mates of adjunction, the right adjoints are also naturally isomorphic. \( \square \)

**Remark 7.23.** We believe that the complete Bar-Cobar adjunction relative to \( \iota \) is a Quillen equivalence. One could generalize the arguments in Section 11 of [GL18] to the S-colored framework in order to prove this.

On the other hand, one could try to induce a model structure on counital partial cooperads using the forgetful-cofree adjunction and transfer it to the category of complete curved absolute partial operads. It is not clear to us that these model structures should coincide. Indeed, this would amount to proving that the adjunction

\[ \text{dg upCoop} \cong \text{upCoop}\_\infty, \]

is a Quillen equivalence. But this adjunction restricts to the adjunction

\[ \text{dg } \mathcal{U}\text{-Ass-coalg} \cong \text{uA}\_\infty\text{-coalg}, \]

when one restricts to dg S-modules concentrated in arity one. It is not clear at all that this adjunction is a Quillen equivalence, see [GL18, Conjecture 8.10].

8. Duality functors and Koszul duality

In this section, we build duality functors that intertwine the two Bar-Cobar adjunctions defined in Section 6, forming an algebraic duality square of commuting adjunctions. On the homotopical side of things, we build a second duality square of commuting Quillen adjunctions when (co)unital partial (co)operads are replaced by their "up to homotopy" counterparts. Using these squares, we compute minimal cofibrant resolutions for complete curved absolute partial operads in certain cases of interest.

8.1. Algebraic duality square square. Our goal is to construct left adjoint functors to the linear dual functor that sends "coalgebraic objects" into "algebraic objects".

**Lemma 8.1.** The linear duality functor

\[ \text{dg upCoop}^\text{op} \cong \text{dg upOp}, \]

admits a left adjoint.
Proof. Consider the following square of functors

\[
\begin{array}{ccc}
\text{dg upCoop}^{\text{op}} & \xrightarrow{(-)^*} & \text{dg upOp} \\
\text{dg S-mod}^{\text{op}} & \xleftarrow{\top} & \text{dg S-mod} \\
\end{array}
\]

First notice that the adjunction on the left hand side is monadic, since we consider the opposite of a comonadic adjunction. Furthermore, all categories involved are complete and cocomplete. It is absolutely clear that \((-)^* \cdot U^{\text{op}} \cong U \cdot (-)^*\). Thus we can apply the Adjoint Lifting Theorem \([\text{Joh75}, \text{Theorem 2}]\) to this situation, which concludes the proof.

\[\square\]

**Definition 8.2** (Sweedler dual). The Sweedler duality functor

\[
\begin{array}{ccc}
\text{dg upOp} & \xrightarrow{(-)^*} & \text{dg upCoop}^{\text{op}} \\
\end{array}
\]

is defined as the left adjoint of the linear dual functor.

**Remark 8.3.** The proof of the Adjoint Lifting Theorem \([\text{Joh75}, \text{Theorem 2}]\) gives an explicit construction of this left adjoint. Let \((\mathcal{P}, \{\circ_i\}, \eta, d_P)\) be a dg unital partial operad, and let

\[\gamma_P : \mathcal{T}(\mathcal{P}) \longrightarrow \mathcal{P}\]

be its structural morphism as an algebra over the tree monad. The Sweedler dual dg counital partial cooperad \(\mathcal{P}^o\) is given by the following equalizer:

\[\text{Eq}\left(\mathcal{T}^\vee(\mathcal{P}^*) \xrightarrow{\rho} \mathcal{T}^\vee((\mathcal{T}(\mathcal{P}))^*))\right),\]

where \(\rho\) is an arrow constructed using the comonadic structure of \(\mathcal{T}^\vee\) and the canonical inclusion of a dg \(S\)-module into its double linear dual.

**Remark 8.4.** If we restrict to unital partial operads concentrated in arity 1, that is, unital associative algebras, the Sweedler dual functor we have constructed is naturally isomorphic with the original Sweedler dual of \([\text{Swe69}]\).

**Remark 8.5.** Let \((\mathcal{P}, \{\circ_i\}, \eta, d_P)\) be a dg unital partial operad such that \(\mathcal{P}(n)\) is degree-wise finite dimensional and bounded above or below. Its Sweedler dual \(\mathcal{P}^o\) is simply given by \((\mathcal{P}^*, \{\circ_i^*\}, \eta^*, d_P^*)\). The adjunction constructed restricts to an anti-equivalence of categories between dg counital partial cooperads which are arity-wise degree-wise finite dimensional and bounded above or below and dg unital partial operads which are arity-wise degree-wise finite dimensional and bounded above or below.

Let’s turn to the other side of the Koszul duality. We postpone the following proofs and constructions to the Appendix 9.2, where there is a detailed discussion of absolute partial operads and their properties.

**Lemma 8.6.** Let \((\mathcal{E}, \{\Delta_i\}, d_\mathcal{E}, \Theta_\mathcal{E})\) be a conilpotent curved partial cooperad. Then its linear dual \(\mathcal{E}^*\) has inherits a structure of a complete curved absolute partial operad. This defines a functor

\[
\begin{array}{ccc}
\text{curv pCoop}^{\text{conil}} & \xrightarrow{(-)^*} & \text{curv abs pOp}^{\text{comp}} \\
\end{array}
\]

**Proof.** See Lemma 9.37. \[\square\]
Proposition 8.7. The linear duality functor admits a left adjoint. There is an adjunction
\[
\begin{array}{ccc}
\text{curv abs pOp}^\text{comp} & \xleftarrow{(-)^\vee} & \left(\text{curv pCoop}^\text{conil}\right)^\text{op} \\
\downarrow & & \downarrow \\
(-)^* & \xrightarrow{\sim} & (-)^{\vee}
\end{array}
\]\n
Proof. See Proposition 9.38. \hfill \square

Definition 8.8 (Topological dual functor). The topological dual functor
\[
\text{curv abs pOp}^\text{comp} \xrightarrow{(-)^\vee} \left(\text{curv pCoop}^\text{conil}\right)^\text{op}
\]
is defined as the left adjoint of the linear dual functor.

This allows us to construct the first duality square of commuting functors.

Theorem 8.9 (Duality square). The following square of adjunction
\[
\begin{array}{ccc}
\text{dg upOp}^\text{op} & \xleftarrow{\text{B}^\text{op}} & \left(\text{curv pCoop}^\text{conil}\right)^\text{op} \\
\downarrow & & \downarrow \\
\text{curv abs pOp}^\text{comp} & \xrightarrow{\text{dg upCoop}} & \text{dg upCoop}^\text{op} \\
\downarrow & & \downarrow \\
\hat{\Omega} & \xrightarrow{\sim} & \hat{\Omega}
\end{array}
\]
commutes in the following sense: right adjoints going from the top right to the bottom left are naturally isomorphic.

Proof. Let us show the commutativity of this square. We have, for any graded $S$-module $M$, a natural isomorphism of graded counital partial cooperads
\[
\mathcal{T}^\vee(sM^*) \cong \left(\mathcal{T}(s^{-1}M)\right)^\circ.
\]
This isomorphism is obtained as the mate of the obvious isomorphism $(-)^* \cdot U \cong U^\text{op} \cdot (-)^*$. Let $(\mathcal{E}, \{\Delta_i\}, \epsilon, \Theta_\mathcal{E})$ be a conilpotent curved partial cooperad. One can show by direct inspection that the isomorphism of graded counital partial cooperads
\[
\mathcal{T}^\vee(s\mathcal{E}^*) \cong \left(\mathcal{T}(s^{-1}\mathcal{E})\right)^\circ
\]
extends to an isomorphism of dg counital partial cooperads
\[
\hat{\mathcal{B}}(\mathcal{E}^*) \cong (\Omega(\mathcal{E}))^\circ.
\]

Remark 8.10. Let $(\mathcal{E}, \{\Delta_i\}, \epsilon, d_\mathcal{E})$ be a dg counital partial cooperad. Then, by the above theorem, there is an isomorphism
\[
\mathcal{B}(\mathcal{E}^*) \cong \left(\hat{\Omega}(\mathcal{E})\right)^\vee,
\]
which is natural in $\mathcal{E}$.

Proposition 8.11. Let $(\mathcal{P}, \{o_i\}, \eta, d_\mathcal{P})$ be a dg unital partial operad which is arity-wise degree-wise finite dimensional and bounded above or bounded below. There is an isomorphism
\[
\hat{\Omega}(\mathcal{P}^*) \cong (\mathcal{B}(\mathcal{P}))^*
\]
of complete curved absolute partial operads.
Proof. Let $M$ be a pdg $S$-module which is arity-wise degree-wise finite dimensional and bounded above or bounded below. There is an isomorphism of complete pdg absolute partial operads:

$$\overrightarrow{T} \left( s^{-1} M^* \oplus \nu \right) \cong \left( \overrightarrow{T}(sM \oplus \nu) \right)^*.$$ 

One can show that this isomorphism extends to an isomorphism of complete curved absolute partial operads

$$\widehat{\Omega}(P^*) \cong (B(P))^*$$

by direct inspection, looking at the morphisms that induce the pre-differentials on each of those constructions. □

Remark 8.12 (Beck-Chevalley condition). In fact, one can show that there is a monomorphism of complete curved absolute partial operads

$$\lambda_P : \widehat{\Omega}(P^\circ) \hookrightarrow (B(P))^*$$

which is natural in $P$. And $\lambda_P$ is an isomorphism if $P$ is arity-wise degree-wise finite dimensional and bounded above or below. The sub-category of dg unital partial operad which satisfy the Beck-Chevalley condition contains the sub-category of arity-wise degree-wise finite dimensional and bounded above or below dg unital partial operads.

Example 8.13. Let $uCom$ be the unital partial operad encoding dg unital commutative algebras. There is an isomorphism of complete curved absolute partial operads:

$$(BuCom)^* \cong \widehat{\Omega}uCom^*.$$ 

The complete curved absolute partial operad $\widehat{\Omega}uCom^*$ encodes the notion of mixed curved $L_\infty$-algebras. See Section 9 for more details on this.

8.2. Homotopical duality square. We now construct the homotopical version of the duality square by replacing (co)unital partial (co)operads by their "up to homotopy" counterparts. Before making those constructions, we state results that are a direct consequence of the results stated in [Gri21]. They give Quillen equivalence between unital partial operads up to homotopy and conilpotent curved partial cooperads.

Theorem 8.14 ([Hin97]). There is a cofibrantly generated model structure on the category of unital partial operads up to homotopy with strict morphisms defined by the following classes of morphisms:

1. The class of weak-equivalences $W$ is given by strict morphisms of unital partial operads up to homotopy $f : P \longrightarrow \Omega$ such that $f(n) : P(n) \longrightarrow \Omega(n)$ is a quasi-isomorphism for all $n \geq 0$.

2. The class of fibrations is $\text{Fib}$ is given by strict morphisms of unital partial operads up to homotopy $f : P \longrightarrow \Omega$ such that $f(n) : P(n) \rightarrow \Omega(n)$ is a degree-wise epimorphism for all $n \geq 0$.

3. The class of cofibrations $\text{Cof}$ is given by strict morphisms of unital partial operads up to homotopy which have the left lifting property against all morphism in $W \cap \text{Fib}$.

Proof. It is obtained by right transfer theorem using the free-forgetful adjunction. □

The author of [Gri21] endows the category of conilpotent curved partial cooperads with a transferred structure from unital partial operads, using the Bar-Cobar construction of Subsection 6.2.

Theorem 8.15 ([Gri21, Theorem 5]). There is a cofibrantly generated model structure on the category of conilpotent curved partial cooperads defined by the following classes of morphisms:

1. The class of weak-equivalences $W$ is given by morphisms of conilpotent curved partial cooperads $f : C \longrightarrow D$ such that $\Omega(f)(n) : \Omega(C)(n) \longrightarrow \Omega(D)(n)$ is a quasi-isomorphism for all $n \geq 0$.

Proof. It is obtained by right transfer theorem using the free-forgetful adjunction. □
(2) The class of cofibrations is \( \text{Cof} \) is given by morphisms of conilpotent curved partial cooperads \( f : C \longrightarrow D \) such that \( \Omega(f) : \Omega(C) \longrightarrow \Omega(D) \) is a cofibration of unital partial operads up to homotopy.

(3) The class of fibrations \( \text{Fib} \) is given by morphisms of conilpotent curved partial cooperads which have the right lifting property against all morphism in \( W \cap \text{Cof} \).

**Proposition 8.16** ([Gri21, Proposition 13]). *In the model structure of conilpotent curved partial cooperads, the class of cofibrations is given by morphisms of conilpotent curved partial cooperads \( f : C \longrightarrow D \) such that \( f(n) : C(n) \longrightarrow D(n) \) is a monomorphism for all \( n \geq 0 \).*

The following result is a straightforward consequence of the above.

**Proposition 8.17.** There is a Bar-Cobar adjunction

\[
\Omega \vee \iota \dashv \text{curv pCoop}_{\text{conil}}
\]

which is a Quillen equivalence.

**Proof.** This adjunction is induced by the curved twisting morphism \( \iota : S \otimes cO^\vee \longrightarrow \Omega_S (S \otimes cO^\vee) \) using the same methods that in Subsection 6.2.

Using analogous arguments as in Theorem 5.57, one can show that this curved twisting morphism is Koszul. In turn, using the same arguments as in the proof of Corollary 6.17, one can deduce that the Bar-Cobar adjunction relative to \( \iota \) is a Quillen equivalence. \qed

**Lemma 8.18.** The linear duality functor

\[
(\text{upCoop}_\infty)^{\text{op}} \longrightarrow (\text{upOp}_\infty)^* \text{admits a left adjoint.}
\]

**Proof.** Recall that there exists a dg unital partial S-colored operad \( \Omega_S (S \otimes cO^\vee) \) which encodes unital partial operads up to homotopy as its algebras and counital partial cooperads up to homotopy as its coalgebras. Thus the first category is monadic and the second is comonadic. The rest of the proof is *mutatis mutandis* the same as the proof of Lemma 8.1. \qed

**Definition 8.19** (Homotopical Sweedler dual). The *homotopical Sweedler duality functor*

\[
\text{upOp}_\infty \xrightarrow{(-)^\vee_h} (\text{upCoop}_\infty)^{\text{op}}
\]

is defined as the left adjoint of the linear dual functor.

**Lemma 8.20.** The adjunction

\[
\text{upCoop}_\infty \xleftarrow{(-)^*} (\text{upOp}_\infty)^{\text{op}}
\]

is a Quillen adjunction.

**Proof.** The left adjoint \( (-)^* \) sends degree-wise monomorphisms to degree-wise epimorphisms. Thus it preserves cofibrations. It also preserves quasi-isomorphisms. Therefore we have a Quillen adjunction. \qed
**Theorem 8.21 (Homotopical duality square).** The following square of adjunction

\[
\begin{array}{ccc}
\text{upOp}_\infty^\text{op} & \xrightarrow{B_i^\text{op}} & \text{curv pCoop}^\text{conil}^\text{op} \\
\downarrow & & \downarrow \\
\text{upCoop}_\infty & \xrightarrow{\Omega_i} & \text{curv abs pOp}^\text{comp}, \\
\end{array}
\]

commutes in the following sense: right adjoints going from the top right to the bottom left are naturally isomorphic. Furthermore, this square is a square of Quillen adjunctions.

**Proof.** The commutativity of this square of functors can be shown using the same arguments as in Theorem 8.9. In order to check that it is a square of Quillen adjunctions, we need to check that the adjunction

\[
\text{curv pOp}^\text{comp} \xleftrightarrow{\Omega_i} \text{curv pCoop}^\text{conil},
\]

is a Quillen adjunction. The right adjoint \((-)^*\) sends monomorphisms to epimorphisms, thus preserves fibrations. Let \(f : C \to D\) be a weak equivalence between two conilpotent curved partial cooperads. This is equivalent to \(\Omega_i(f) : \Omega_iC \to \Omega_iD\) being an arity-wise quasi-isomorphism. Since \((-)^h_i\) is right Quillen, it preserves quasi-isomorphisms between cofibrant unital partial operads up to homotopy (fibrant in the opposite category). Thus \(\Omega_i(f)^h_i : \Omega_i(C)^h_i \to \Omega_i(D)^h_i\) is an arity-wise quasi-isomorphism. Using the commutativity of this adjunction, we get that

\[
\hat{B}_i(f^*) : \hat{B}_i(C^*) \to \hat{B}_i(D^*),
\]

is an arity-wise quasi-isomorphism. Which is equivalent to \(f^* : C^* \to D^*\) being a weak equivalence in the transferred model structure. Thus \((-)^*\) is a right Quillen functor. This concludes the proof. \(\square\)

The above homotopical square allows us, using the curved Koszul duality established in [HM12], to compute explicit cofibrant resolutions for a vast class of complete curved absolute partial operads.

**Proposition 8.22.** Let \((P, \{\circ_i\}, \eta)\) be a unital partial operad, given by some quadratic data \((V, R)\), and which is arity-wise degree-wise finite dimensional and bounded above or below. Let \(P^!\) be its Koszul dual conilpotent curved partial cooperad. If \(P\) is Koszul, then

\[
\hat{\Omega}(P^!) \Rightarrow (P^!\!^* )
\]

is a cofibrant resolution of \((P^!\!^*)\) in the model category of complete curved absolute partial operads.

**Proof.** Recall that \(\kappa : P^! \to P\) is Koszul if and only if \(f_\kappa : \Omega(P^!) \to P\) is an arity-wise quasi-isomorphism. This is equivalent to

\[
g_\kappa : P^! \to B\!P
\]

being a weak-equivalence in the category of conilpotent curved partial cooperads, since the two model categories are Quillen equivalent. Every conilpotent curved partial cooperad is cofibrant in the model structure of Theorem 8.15 (thus fibrant in the opposite category), therefore the linear dual functor \((-)^*\) preserves all weak equivalences. This implies that

\[
(g_\kappa)^* : (B\!P)^* \to (P^!\!^*)
\]
is a weak equivalence of complete curved absolute partial operads. Using Proposition 8.11, we know that there is an isomorphism
\[ \hat{\Omega}(\mathcal{P}^\ast) \cong (B\mathcal{P})^\ast \]
of complete curved absolute partial operads, which concludes the proof. \[ \square \]

**Example 8.23** (Cofibrant resolution of \(c\text{Lie}^\wedge\)). Let \(c\text{Lie}^\wedge\) be the complete curved absolute partial operad encoding curved Lie algebras of Definition 9.39. There is a weak equivalence
\[ \hat{\Omega}(u\text{Com}^\ast) \to c\text{Lie}^\wedge \]
of complete curved absolute partial operads and \(\hat{\Omega}(u\text{Com}^\ast)\) is the minimal cofibrant resolution of \(c\text{Lie}^\wedge\) in this model structure.

**Example 8.24** (Cofibrant resolution of \(c\text{Ass}^\wedge\)). Let \(c\text{Ass}^\wedge\) be the complete curved absolute partial operad encoding curved associative algebras of Definition 9.42. There is a weak equivalence
\[ \hat{\Omega}(u\text{Ass}^\ast) \to c\text{Ass}^\wedge \]
of complete curved absolute partial operads and \(\hat{\Omega}(u\text{Ass}^\ast)\) is the minimal cofibrant resolution of \(c\text{Ass}^\wedge\) in this model structure.

9. **Application: Homotopy Transfer Theorem for Curved Algebras**

The Homotopy Transfer Theorem is a fundamental tool in homological algebra, as it allows to transfer algebraic structures up to homotopy using contractions of dg modules. The operadic reformulation of this theorem was given in [LV12, Section 10.3] in terms of a morphism between the Bar constructions of the endomorphisms operads associated to the contraction. This morphism is called the Van der Laan morphism in *loc.cit*. The notion of a contraction is homotopical not homological, and extends easily to the setting of pdg modules. In this section, we prove a version of the Homotopy Transfer Theorem for curved algebraic structures up to homotopy, extending the Van der Laan morphism to the complete Bar construction of Section 6. We recover, in the case of "pro-nilpotent" curved \(\mathcal{L}_\infty\)-algebras in the sense of [Get18], the Homotopy Transfer Theorem constructed by Fukaya using fixed-point equations in [Fuk02].

9.1. **Complete filtrations on pdg modules and complete curved absolute endomorphisms operad.** A complete filtration on a pdg module allows to endow its curved endomorphisms operad with a structure of complete curved absolute partial operad.

**Definition 9.1** (Filtered pdg module). A filtered pdg module \(V\) amounts to the data \((V,F,V,d_V)\) of a pdg module \((V,d_V)\) together with a degree-wise decreasing filtration:
\[ V = F_0V \supset F_1V \supset F_2V \supset \cdots \supset F_nV \supset \cdots , \]
such that \(d_V(F_nV) \subset F_{n+1}V\) for all \(n\) in \(\mathbb{N}\). Morphisms \(f: V \to W\) of filtered pdg modules are morphisms of pdg modules which are compatible with the filtrations, that is, \(f(F_nV) \subset F_nW\) for all \(n\).

**Remark 9.2.** We ask that \(d_V\) raises the filtration degree by one for simplicity. These are not the optimal assumptions. In fact, we only need to impose that \(d_V^2(F_nV) \subset F_{n+1}V\). These would correspond to "gr-dg filtered modules" in [DCBM20]. In any case, notice that the associated graded of these filtrations are dg modules.

The category of filtered pdg modules can be endowed with a closed symmetric monoidal structure by considering the following filtration on the tensor product. For \(V\) and \(W\) two filtered pdg modules, their tensor product \(V \otimes W\) is endowed with
\[ F_n(V \otimes W) := \sum_{p+q=n} \text{Im}(F_pV \otimes F_qW \to V \otimes W) . \]
The internal hom functor, denoted \( \text{hom}(A, B) \), is given by the internal hom of pdg modules endowed with the filtration

\[
F_n(\text{hom}(V, W)) := \left\{ f \in \text{hom}(V, W) \mid f(F_k V) \subset F_{k+n} W \right\}.
\]

**Definition 9.3 (Complete pdg module).** Let \( V \) be a filtered pdg module. It is a *complete* pdg module if the canonical morphism

\[
\pi_V : V \longrightarrow \lim_n V / F_n V
\]

is an isomorphism of filtered pdg modules.

Complete pdg modules form a reflexive subcategory of filtered pdg modules. The reflector is simply given by the completion functor, which sends a filtered pdg module \( V \) to its completion

\[
\hat{V} := \lim_n A / F_n A.
\]

By setting \( V \hat{\otimes} W := \hat{V} \otimes W \), the category of complete pdg modules is endowed with a closed symmetric monoidal structure, the internal hom being the same as the one defined above. One can define operads in this context, [DSV18, Section 2] for more details. Here, we will only use the following construction.

**Definition 9.4 (Complete curved endomorphisms partial operad).** Let \( V \) be a complete pdg module. Its *complete curved endomorphisms partial operad* is given by the complete pdg \( S \)-module

\[
\text{end}_V(n) := \text{hom}^{(\geq 1)}(V \otimes^n, V),
\]

where we only consider here morphisms of pdg modules which raise the filtration degree by at least one. The operad structure is given by the partial composition of functions. Its pre-differential is given by \( \partial := [d_V, -] \) and its curvature determined by \( \Theta_V(id) := d^2_V \).

**Lemma 9.5.** Let \( V \) be a complete pdg module. Its complete curved endomorphisms partial operad is a complete curved absolute partial operad.

**Proof.** We postpone this proof to Lemma 9.48 in the Appendix 9.2, where there is a detailed discussion of complete curved absolute partial operads. \( \square \)

9.2. **Homotopy contractions and Van der Laan morphisms.** The data of a homotopy contraction between two complete pdg modules which is compatible with their underlying filtrations induces a ”Van der Laan morphism” between the complete Bar construction of their complete curved endomorphisms operads. From this, one recovers a Homotopy Transfer theorem for curved algebras over a vast class of cofibrant complete curved absolute partial operads.

**Definition 9.6 (Homotopy contraction).** Let \( V \) and \( H \) be two complete pdg modules. A *homotopy contraction* amounts to the data of

\[
\begin{array}{ccc}
\hat{V} & \equiv & \hat{H} \\
p & \downarrow & 0 \\
\iota & \downarrow & \iota \\
V & \longrightarrow & H,
\end{array}
\]

where \( p \) and \( \iota \) are two morphisms of filtered pdg modules and \( h \) is a morphism of filtered graded modules of degree \(-1\). This data satisfies the following conditions:

\[
p \iota = \text{id}_H, \quad \iota p - \text{id}_V = d_V h + h d_V.
\]

**Notation.** We say that \( H \) is a homotopy retract of \( V \) if there exists a homotopy contraction as in the definition above.

**Proposition 9.7 (Van der Laan morphism).** Let \( V \) and \( H \) be two complete pdg modules such \( H \) is a homotopy retract of \( V \). The data of this homotopy contraction induces a morphism of dg counital partial cooperads

\[
\text{VdL} : \hat{B}(\text{end}_V) \longrightarrow \hat{B}(\text{end}_H).
\]

This morphism is called the Van der Laan morphism associated to the contraction.
Proof. Let 

\[ \text{vdL} : T^\vee(\text{end}_V) \rightarrow \text{end}_H \]

be the morphism of pdg $S$-modules which sends a series of rooted trees labeled by elements of $\text{end}_V$ to the converging series in $\text{end}_H$ given by applying the Van der Laan map to each rooted tree of the series. For a recollection on the standard Van der Laan map, see [LV12, Section 10.3.2]. One can restrict $\text{vdL}$ along the inclusion of the cofree counital partial cooperad inside the completed tree endofunctor. By the universal property of the cofree counital partial cooperad, this induces a morphism of counital partial cooperads:

\[ \text{VdL} : T^\vee(\text{end}_V) \rightarrow T^\vee(\text{end}_H). \]

Let us show that $\text{VdL}$ commutes with the differentials. It equivalent to the following diagram commuting

\[\xymatrix{T^\vee(\text{end}_V) \ar[r]^-{\text{VdL}} \ar[d]_{\text{d}_{\text{bar}}} & T^\vee(\text{end}_H) \ar[d]^{\psi_1 + \psi_2} \\
T^\vee(\text{end}_V) \ar[r]^-{\text{vdL}} & \text{end}_H,}\]

where $\psi_1 + \psi_2$ is the map that induces the differential of the complete Bar construction as its unique coderivation extending it. One can show that this diagram commutes for elements in $T^\vee(\text{end}_V)$, which viewed as infinite sums of trees, have a trivial coefficient in front of the trivial tree $|$, extending the same computations as in [LV12, Section 10.3.2] to infinite sums of rooted trees. For the trivial tree $|$, a small computation shows that:

\[\text{vdL} \cdot \text{d}_{\text{bar}}(|) = p \cdot d^2_V \cdot i = d_H \cdot p \cdot i \cdot d_H = d_H^2 = (\psi_1 + \psi_2) \cdot \text{VdL}(|),\]

using the fact that $p$ and $i$ are morphisms of pdg modules. □

**Theorem 9.8** (Curved Homotopy Transfer Theorem). Let $\mathcal{C}$ be a dg counital partial cooperad. Let $V$ and $H$ be two complete pdg modules such $H$ is a homotopy retract of $V$. Then any curved $\hat{\Omega}(\mathcal{C})$-algebra structure on $V$ can be transferred along the homotopy contraction to a curved $\hat{\Omega}(\mathcal{C})$-algebra structure on $H$.

**Proof.** One has that

\[\text{Hom}_{\text{curv abs pOp}^{\text{simp}}} (\hat{\Omega}(\mathcal{C}), \text{end}_V) \cong \text{Tw}(\mathcal{C}, \text{end}_V) \cong \text{Hom}_{\text{dg upCoop}} (\mathcal{C}, \hat{B}(\text{end}_V)),\]

thus a curved $\hat{\Omega}(\mathcal{C})$-algebra structure on $V$ is equivalent to a morphism of dg counital partial cooperads

\[\phi : \mathcal{C} \rightarrow \hat{B}(\text{end}_V).\]

Any such morphism can be pushed forward by the Van der Laan morphism

\[\text{VdL} \cdot \phi : \mathcal{C} \rightarrow \hat{B}(\text{end}_H),\]

thus inducing a curved $\hat{\Omega}(\mathcal{C})$-algebra structure on $H$. □

**Remark 9.9.** All cofibrant complete curved absolute partial operads can be written as $\hat{\Omega}(\mathcal{C})$, where $\mathcal{C}$ is a counital partial cooperad up to homotopy. By restricting to the case where $\mathcal{C}$ is a dg counital partial cooperad, we are restricting to the case of Koszul resolutions. See Proposition 8.22. In order to prove the statement for all cofibrant complete curved absolute partial operad, one should prove that a homotopy contraction induces a morphism between the complete Bar constructions relative to $\iota$ of Proposition 7.15.

Let us conclude with some examples of how this framework can be applied. Let $\hat{\Omega}(u\mathcal{C}om^*)$ be the complete curved absolute partial operad of Example 8.23.

**Example 9.10.** Let $V$ be a complete dg modules. Then a curved $\hat{\Omega}(u\mathcal{C}om^*)$-algebra structure on $V$ corresponds to a pro-nilpotent curved $\mathcal{L}_\infty$-algebra structure on $V$ in the sense of [Get18].
EXAMPLE 9.11 (Fukaya’s Homotopy Transfer Theorem). Let $V$ and $H$ be two complete dg modules and let

$$h \hookrightarrow V \overset{p}{\twoheadleftarrow} H,$$

be homotopy contraction. Given a pro-nilpotent curved $\L_{\infty}$-algebra structure on $V$, one can transfer it to $H$ using a version of the Homotopy Transfer Theorem given in [Fuk02]. The transferred structure is given as the solution of a fixed-point equation. If one computes the solution of this fixed point equation using the methods of [RNV20, Appendix A], one obtains the same formulae for the transferred structure as the ones obtained from Theorem 9.8.

APPENDIX A: WHAT IS AN ABSOLUTE PARTIAL OPERAD?

The notion of an absolute partial operad is an example of a vast class of algebraic structures that emerge as algebras over a cooperad. See Subsection 1.1 for an introduction to this notion. In our particular case, absolute partial operads appear as algebras over the conilpotent partial $S$-colored cooperad $0^*$. Here $0^*$ denotes the linear dual of the partial $S$-colored operad $0$ which encodes partial operads as its algebras. The goal of this Appendix is to give a somewhat explicit definition of what absolute partial operads are and how to characterize them. Then to compare them with standard partial operads. Afterwards, we will generalize these results to the curved case.

9.3. Absolute partial operads. We work in the underlying category of $S$-modules for simplicity. This subsection admits a straightforward generalization to dg $S$-modules or pdg $S$-modules.

Lemma 9.12. There is an isomorphism of endofunctors in the category of $S$-modules:

$$\tilde{\mathcal{R}}^e(0^*)(\cdot) \cong \mathcal{F}^\wedge(\cdot),$$

where $\mathcal{F}^\wedge(\cdot)$ is the reduced completed tree endofunctor.

Proof. We have that:

$$\tilde{\mathcal{R}}^e(0^*)(M)(n) \cong \prod_{(n_1, \ldots, n_r) \in \mathbb{N}^r} \text{Hom}_{S}(s_r(0^*(n_1, \ldots, n_r; n), M(n_1) \otimes \cdots \otimes M(n_r))) \cong \prod_{(n_1, \ldots, n_r) \in \mathbb{N}^r} \text{Hom}_{S}(s_r(0^*(n_1, \ldots, n_r; n) \otimes (S_{n_1} \times \cdots \times S_{n_r}) \otimes s_r M(n_1) \otimes \cdots \otimes M(n_r)),$$

since each $0^*(n_1, \ldots, n_r; n)$ is finite dimensional over $K$ and that we are working over a field of characteristic zero, thus invariants and coinvariants turn out to be canonically isomorphic. Using the same bijection as in the proof of Lemma 5.41, we identify this last term with the reduced completed tree endofunctor. □

Corollary 9.13. There is a monad structure on the reduced completed tree endofunctor $\mathcal{F}^\wedge(\cdot)$. Its structural morphism

$$\text{Sub}^\wedge : \mathcal{F}^\wedge \circ \mathcal{F}^\wedge(\cdot) \longrightarrow \mathcal{F}^\wedge(\cdot)$$

is given by the substitution of infinite series of rooted trees.

Proof. Let $M$ be an $S$-module. An element of $\mathcal{F}^\wedge \circ \mathcal{F}^\wedge(M)$ amounts to the data of a series of rooted trees which are labeled in the following way: if $v$ is a vertex of with $k$ inputs of a rooted tree $\tau$, then it is labeled by a series of rooted trees of arity $k$ which are themselves labeled by elements of $M$ in the usual way. Let us describe the monad structure transported via the isomorphism of Lemma 9.12. This monad structure is given by first distributing all the labeling series and then by substituting each of the node by the rooted tree it is labeled with. Pictorially, the substitution is given by

64
where one should substitute each node by the corresponding labeling rooted tree $\tau_{\omega_i}$. □

Thus we can rewrite the definition of an $O^*$-algebra in terms of the reduced completed tree monad.

**Definition 9.14** (Absolute partial operad). An absolute partial operad $\mathcal{O}$ amounts to the data $(\Omega, \gamma_{\mathcal{O}})$ of an algebra structure over the reduced completed tree monad $\gamma_{\mathcal{O}} : \mathcal{T}^\wedge(\Omega) \to \Omega$.

The goal of this section is to make sense of this definition.

**Remark 9.15.** The structural morphism $\gamma_{\mathcal{O}}$ of an absolute partial operad gives a well-defined composition of any infinite sums for rooted trees labeled by elements of $\mathcal{O}$, without presupposing any underlying topology on $\mathcal{O}$. Let

$$\sum_{n \geq 0} \sum_{\omega \geq 1} \sum_{\tau \in \mathcal{RT}_n^w} \tau$$

be a infinite sum of rooted trees with vertices labeled by elements of $\mathcal{O}$. Notice that, in general:

$$\gamma_{\mathcal{O}} \left( \sum_{n \geq 0} \sum_{\omega \geq 1} \sum_{\tau \in \mathcal{RT}_n^w} \tau \right) = \sum_{n \geq 0} \gamma_{\mathcal{O}} \left( \sum_{\omega \geq 1} \sum_{\tau \in \mathcal{RT}_n^w} \tau \right) \neq \sum_{n \geq 0} \sum_{\omega \geq 1} \sum_{\tau \in \mathcal{RT}_n^w} \gamma_{\mathcal{O}}(\tau),$$

as the later sum is not even well-defined in $\mathcal{O}$ in general.

**Proposition 9.16.** Let $M$ be an $S$-module such that $M(0) = M(1) = 0$, then the data of a partial operad structure on $M$ is equivalent to the data of an absolute partial operad structure on $M$.

**Proof.** In this case, there are only a finite amount of rooted tree labeled by elements of $M$ for each possible arity, thus the reduced completed tree monad coincides with the reduced tree monad. □

**Lemma 9.17.** Let $(\mathcal{O}, \gamma_{\mathcal{O}})$ be an absolute partial operad. The structure map $\gamma_{\mathcal{O}}$ induces a partial operad structure on $\mathcal{O}$. This defines a faithful functor

$$\text{Res} : \text{abs pOp} \longrightarrow \text{pOp}.$$

**Proof.** It is straightforward to check that the inclusion of endofunctors

$$\mathcal{T}(-) \hookrightarrow \mathcal{T}^\wedge(-)$$

is in fact an inclusion of monads. Thus, by pulling back along this inclusion, any absolute partial operad structure gives a partial operad structure. This pullback amounts to restrict $\gamma_{\mathcal{O}}$ to finite sums of rooted trees inside the reduced completed tree monad. □

**Remark 9.18.** Let $(\mathcal{O}, \gamma_{\mathcal{O}})$ be an absolute partial operad. It is in particular endowed with partial composition maps $\{\alpha_l\}$ which satisfy the usual axioms of a partial operad. These maps are part of the structure of an absolute partial operad.

**Proposition 9.19** (Absolute envelope of a partial operad). There is an adjunction

$$\text{pOp} \leftrightarrow \text{abs pOp} \quad \text{Abs} \quad \text{Res}.$$

We call the left adjoint the absolute envelope of a partial operad, and we denote it by $\text{Abs}$. 115
Proof. Notice that both categories are categories of algebras over accessible monads, thus are presentable. Since Res is accessible and preserves all limits, it has a left adjoint, by [AR94, Theorem 1.66].

There is a canonical topology on absolute partial operads induced by the structural morphism.

**Definition 9.20** (Canonical filtration on an absolute partial operad). Let \((Q, \gamma_Q)\) be an absolute partial operad. The canonical filtration is the decreasing filtration given by

\[ F_\omega Q := \text{Im} \left( \gamma_Q^{(\geq \omega)} : F^{(\geq \omega)}(Q) \to Q \right) \]

for all \(\omega \geq 0\), where \(F^{(\geq \omega)}(Q)\) denotes the possibly infinite sums of rooted trees that have terms of at least \(\omega\) internal edges. Each \(F_\omega Q\) defines an ideal of \(Q\). Notice that:

\[ Q = F_0 Q \supseteq F_1 Q \supseteq \cdots \supseteq F_\omega Q \supseteq \cdots \]

**Definition 9.21** (Nilpotent absolute partial operad). Let \((Q, \gamma_Q)\) be an absolute partial operad. It is said to be nilpotent if there exists an \(\omega \geq 1\) such that

\[ Q/F_\omega Q \cong Q \]

The absolute partial operad is said to be \(\omega_0\)-nilpotent if \(\omega_0\) is the smallest integer such that the above isomorphism exists.

**Definition 9.22** (Completion of an absolute partial operad). Let \((Q, \gamma_Q)\) be an absolute partial operad, its completion \(\hat{Q}\) is given by the following limit

\[ \hat{Q} := \lim_{\omega} Q/F_\omega Q \]

taken in the category of absolute partial operads.

The completion is functorial. There is a canonical morphism of absolute partial operads

\[ \psi : Q \to \hat{Q}, \]

which is always an epimorphism. See Remark 1.14 as to why this is the case in the general setting.

**Definition 9.23** (Complete absolute partial operad). Let \((Q, \gamma_Q)\) be an absolute partial operad. It is complete if the canonical morphism \(\psi : Q \to \hat{Q}\) is an isomorphism of absolute partial operads.

**Example 9.24.** Any nilpotent absolute partial operad is also a complete absolute partial operad. Any complete absolute partial operad is the limit of a tower of nilpotent absolute partial operads. Any free absolute partial operad is also complete.

**Remark 9.25.** An absolute partial operad is complete if and only if the topology induced by its canonical filtration is Hausdorff.

**Remark 9.26.** Let \((Q, \gamma_Q)\) be a complete absolute partial operad and let

\[ \sum_{n \geq 0} \sum_{\omega \geq 1} \sum_{\tau \in RT_n^\omega} \tau \]

be a infinite sum of rooted trees with vertices labeled by elements of \(Q\). In this case we have that

\[ \gamma_Q \left( \sum_{n \geq 0} \sum_{\omega \geq 1} \sum_{\tau \in RT_n^\omega} \tau \right) = \sum_{n \geq 0} \sum_{\omega \geq 1} \sum_{\tau \in RT_n^\omega} \gamma_Q(\tau), \]

since \(\gamma_Q\) commutes with the sum over the weight of rooted trees and since there are only a finite amount of rooted trees of arity \(n\) and of weight \(\omega\).
Proposition 9.27. There is an isomorphism of categories between the category of $\omega_0$-nilpotent absolute partial operads and the category of $\omega_0$-nilpotent partial operads, for all $\omega_0 \geq 1$.

Proof. Let $\mathcal{T}^{(\leq \omega_0)}$ denote the reduce tree monad truncated at rooted trees of with more than $\omega_0$ internal edges. The data of an $\omega_0$-nilpotent partial operad amounts to the data of an algebra over the reduced $\omega_0$-truncated tree monad. The data of an $\omega_0$-nilpotent absolute partial operad also amounts to the data of an algebra over the reduced $\omega_0$-tree monad. Indeed, since there are only a finite number of rooted trees of weight less than $\omega_0$ at any given arity, the direct sum and the product coincide.

The notion of a complete absolute partial operad appears naturally when one takes the linear dual of a conilpotent partial cooperad.

Proposition 9.28. Let $(\mathcal{C}, \{\Delta_i\})$ be a conilpotent partial cooperad. Then its linear dual $\mathcal{C}^*$ has an induced absolute partial operad structure. Furthermore, since $\mathcal{C}$ is conilpotent, then $\mathcal{C}^*$ is a complete absolute partial operad. It defines a functor $(\mathcal{C}, \{\Delta_i\})^{\mathcal{C}^*} \rightarrow \text{abs pOp}^{\text{comp}}$.

Proof. Let $\Delta_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{T}^{\mathcal{C}}$ denote the structural map of $\mathcal{C}$ as a coalgebra over the reduced tree comonad, given by Proposition 1.33. By taking the linear dual we obtain a map $\Delta_\mathcal{C}^* : \left(\mathcal{T}^{\mathcal{C}}\right)^* \rightarrow \mathcal{C}^*$. There is a canonical inclusion $\iota_\mathcal{C} : \mathcal{T}^{\mathcal{C}}(\mathcal{C}^*) \rightarrow \left(\mathcal{T}^{\mathcal{C}}\right)^*$ which is an isomorphism if and only if $\mathcal{C}$ is arity-wise finite dimensional. Thus by pulling back along $\iota_\mathcal{C}$, we obtain a map $\gamma_\mathcal{C} := \Delta_\mathcal{C}^* \cdot \iota_\mathcal{C}$. One can check that this endows $\mathcal{C}^*$ with the structure of an algebra over the reduced complete tree monad. Since $\mathcal{C}$ is conilpotent, it can be written as

$$\mathcal{C} \cong \text{colim}_\omega \mathcal{R}_\omega \mathcal{C},$$

where $\mathcal{R}_\omega \mathcal{C}$ is the sub-cooperad given by operations which admit non-trivial $\omega$-iterated partial decompositions. Thus

$$\mathcal{C}^* \cong \text{lim}_\omega (\mathcal{R}_\omega \mathcal{C})^*,$$

where one can check that

$$(\mathcal{R}_\omega \mathcal{C})^* \cong \mathcal{C}^* / \mathcal{T}_\omega \mathcal{C}^*.$$ 

Therefore the resulting absolute partial operad $\mathcal{C}^*$ is indeed complete.

Proposition 9.29. The linear functor admits a left adjoint

$$(-)^* : \text{abs pOp}^{\text{comp}} \rightarrow \left(\mathcal{C}, \{\Delta_i\}\right)^{\mathcal{C}^*}.$$ 

We denote its left adjoint $(-)^\vee$ and call it the topological dual functor.
**Proof.** Consider the following square of functors

\[
\begin{array}{c}
\begin{array}{c}
(p\text{Coop}_{\text{conil}})^{\text{op}} \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(-)^* \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{abs} \text{pOp}_{\text{comp}}^{\text{op}} \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
S\text{-mod}_{\text{op}}^{\text{op}} \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
S\text{-mod} \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow
\end{array}
\end{array}
\end{array}
\]

The adjunction on the left hand side is monadic. All these categories are bicomplete. It is clear that \((-)^* \cdot \text{U}^{\text{op}} \cong \text{U} \cdot (-)^*\). Thus we can apply the Adjoint Lifting Theorem [Joh75, Theorem 2], which concludes the proof.

**Remark 9.30.** There is an explicit description of this left adjoint. Let \((\Omega, \gamma_\Omega)\) be a complete absolute partial operad, its topological dual \(\Omega^\vee\) is given by the equalizer

\[
\text{Eq}(\mathcal{F}(\Omega^*)) \xrightarrow{\rho} \mathcal{F}(\mathcal{F}(\gamma_\Omega^*))
\]

where \(\rho\) is a map constructed using the comonad structure of \(\mathcal{F}\) and the canonical inclusion into the double linear dual.

Lastly, one can compare the canonical filtrations induced by the structure of an absolute partial operad and by the structure of a partial operad. In general, they do not agree.

**Lemma 9.31.** Let \((\Omega, \gamma_\Omega)\) be an absolute partial operad. Let \((\text{Res}(\Omega), \text{Res}(\gamma_\Omega))\) the partial operad given by its restriction. Then there is an inclusion of \(S\)-modules:

\[
\mathcal{F}_\omega \text{Res}(\Omega) \subseteq \mathcal{F}_\omega \Omega
\]

for all \(\omega \geq 0\), between its canonical filtration as a partial operad and its canonical filtration as an absolute partial operad.

**Proof.** Elements in \(\mathcal{F}_\omega \Omega\) are given by the images of the structural morphism \(\gamma_\Omega\) of series of rooted trees of at least weight \(\omega\). Since any finite sum of at least weight \(\omega\) is a particular example of this, any element in \(\mathcal{F}_\omega \text{Res}(\Omega)\) is also included in \(\mathcal{F}_\omega \Omega\). \(\square\)

**Corollary 9.32.** Let \((\Omega, \gamma_\Omega)\) be an absolute partial operad. Then the canonical topology of \((\text{Res}(\Omega), \text{Res}(\gamma_\Omega))\) as a partial operad is Hausdorff.

**Proof.** We have that

\[
\bigcap_{\omega \in \mathbb{N}} \mathcal{F}_\omega \text{Res}(\Omega) \subset \bigcap_{\omega \in \mathbb{N}} \mathcal{F}_\omega \Omega = 0,
\]

hence the canonical topology on \(\text{Res}(\Omega)\) is indeed Hausdorff. \(\square\)

**Counter-Example 9.33.** Let \(\mathcal{F}^\wedge (M)\) be the free complete absolute partial operad generated by an \(S\)-module \(M\). Then

\[
\text{Res}(\mathcal{F}^\wedge (M))
\]

is not complete as a partial operad.
9.4. Complete curved absolute partial operads. Recall Definition 5.56 of the conilpotent curved S-colored partial cooperad $cO^\vee$ which encodes conilpotent curved partial cooperads as its coalgebras. The goal of this subsection is to understand what algebras over this S-colored cooperad are. For this purpose, we now consider pdg S-modules as our ground category.

**Lemma 9.34.** Let $M$ be an pdg $S$-module. There is an isomorphism of pdg $S$-modules

$$\mathcal{F}_S^c(cO^\vee)(M) \cong \mathcal{F}^\wedge(M \oplus \nu),$$

where $\nu$ is an arity 1 generator of degree $-2$. This isomorphism is natural in $M$.

**Proof.** This is a straightforward extension of Lemma 9.12, using a similar bijection to the one defined in the proof of Proposition 5.58. \(\square\)

**Definition 9.35 (Complete curved absolute partial operad).** A complete curved absolute partial operad $(Q, \gamma_Q, d_Q, \Theta_Q)$ amounts to the data of $(Q, \gamma_Q, d_Q)$ a complete pdg absolute partial operad, as defined in the preceding subsection, endowed with a morphism of pdg $S$-modules $\Theta_Q: I \to \mathcal{F}_1 \Omega$ of degree $-2$ such that the following diagram commutes

$$Q \xrightarrow{\text{diag}} Q \oplus Q \cong (I \circ Q) \oplus (Q \circ I) \xrightarrow{(\Theta_Q \circ \text{id}) - (\text{id} \circ \Theta_Q)} Q \circ (Q \oplus \nu) \xrightarrow{(d_Q)^2} Q,$$

where diag is given by diag($\mu$) := ($\mu$, $\mu$).

**Proposition 9.36.** The category of complete curved $cO^\vee$-algebras is equivalent to the category of complete curved absolute partial operads.

**Proof.** Let $(\Omega, \gamma_\Omega, d_\Omega, \Theta_\Omega)$ be a complete curved absolute partial operad. One can extend $\gamma_\Omega$ into a morphism of pdg $S$-modules $\gamma_\Omega^+: \mathcal{F}^\wedge(\Omega \oplus \nu) \to \Omega$ as follows

1. It sends $\nu$ to $\Theta_\Omega(\text{id})$ in $\mathcal{F}_1 \Omega$.
2. It sends a rooted tree $\tau$ with vertices labeled by elements of $\Omega$ and possibly to containing some unary vertices labeled by $\nu$ to the corresponding compositions of operations in $\Omega$ where the unary vertices labeled by $\nu$ are replaced by $\Theta_\Omega$.
3. It defined on infinite sums of rooted trees in $\mathcal{F}^\wedge(\Omega \oplus \nu)$ as follows

$$\sum_{n \geq 0} \sum_{\omega \geq 1} \sum_{\tau \in \text{RT}_n(\omega)} \gamma_\Omega^+(\tau),$$

which is well-defined since $\Theta_\Omega(\text{id})$ is in $\mathcal{F}_1 \Omega$ and since $\Omega$ is a complete absolute partial operad.

This endows $\Omega$ with the structure of a pdg $cO^\vee$-algebra. Furthermore, one checks that it is complete for its canonical filtration as a $cO^\vee$-algebra, since both filtrations are the same. It is straightforward to check that

$$\Omega \cong \mathcal{F}_S^c(\Omega) \xrightarrow{\sigma} \mathcal{F}_S^c(cO^\vee)(\Omega) \xrightarrow{\gamma_\Omega} \Omega$$

where $\sigma$ is given by $\sigma(m) := \Theta_\Omega(\text{id})(m)$.
commutes, making it a complete curved $cO^\vee$-algebra. The other way around, let $(Q, γ_Q^+, d_Q)$ be a complete curved $cO^\vee$-algebra. Restricting $γ_Q^+$ along the obvious inclusion

$$\mathcal{T}^\wedge(Q) \hookrightarrow \mathcal{T}^\wedge(Q \oplus \nu),$$

endows $Q$ with an absolute partial operad structure. Furthermore, since its canonical filtration as an absolute partial operad is contained in its canonical filtration as a $cO^\vee$-algebra, this implies that $Q$ is a complete absolute partial operad. Define $Θ_Q(id) := γ_Q^+(ν)$, and the commutative of the above triangle implies that $Q$ forms indeed a complete curved absolute partial operad.

□

Lemma 9.37. Let $(C, [Δ_i], d_C, Θ_C)$ be a conilpotent curved partial cooperad. Then its linear dual $C^*$ has inherits a structure of a complete curved absolute partial operad. This defines a functor

$$\left( \text{curv pCoop} \conil \right)^{\text{op}} \xrightarrow{(-)^*} \text{curv abs pOp}^{\text{comp}}.$$

Proof. We know that $C^*$ is a complete absolute partial operad from Proposition 9.28. Therefore $Θ_C^*$ endows $C^*$ with a complete curved absolute partial operad structure. □

Proposition 9.38. There is an adjunction

$$\text{curv pOp}^{\text{comp}} \xleftrightarrow{(-)^\vee} \left( \text{curv pCoop}^{\text{conil}} \right)^{\text{op}}.$$

Proof. This is a particular case of Proposition 9.29. The only thing to prove is that if $(Q, γ_Q^+, d_Q, Θ_Q)$ is a complete curved absolute partial operad, its topological dual $Q^\vee$ conilpotent partial cooperad is indeed a conilpotent curved partial operad. Notice that the square constructed in the proof of loc.cit induces a natural monomorphism of pdg $S$-modules

$$(-)^\vee \hookrightarrow (-)^*,$$

thus $Q^\vee$ is a sub-object of $Q^*$ and its pre-differential is induced by restriction. This in turn implies that $Θ_Q^\vee$ endows $Q^\vee$ with a conilpotent curved partial cooperad structure. □

9.5. Examples. Here are some examples of complete curved absolute partial operads. As first examples, let us construct the “absolute analogues” of the curved operads $cLie$ and $cAss$ constructed in Section 2.

Let $M$ be the pdg $S$-module given by $(K,ζ,0,K,β,0,\cdots)$ with zero pre-differential, where $ζ$ is an arity 0 operation of degree $-2$, and $β$ is an arity 2 operation of degree 0, basis of the signature representation of $S_2$.

Definition 9.39 ($cLie^\wedge$ absolute operad). The complete curved absolute partial operad $cLie^\wedge$ is given by the free pdg absolute partial operad generated by $M$ modulo the ideal generated by the Jacobi relation on the generator $β$. It is endowed with the curvature $Θ$ given by $Θ(id) := β \circ_1 ζ$.

Lemma 9.40. The data $(cLie^\wedge, 0, Θ)$ forms a complete curved absolute partial operad.

Proof. One can show by direct computation that this absolute partial operad is complete. The rest of the proof is analogous to Lemma 2.17. □

Proposition 9.41. Let $uCom$ be the unital partial operad encoding unital commutative algebras and let $uCom^\vee$ be its Koszul dual conilpotent curved partial cooperad. There is an isomorphism of complete curved partial operads

$$(uCom^\vee)^* \cong cLie^\wedge.$$

Proof. By direct inspection. □
Let $N$ be the pdg $S$-module given by $(K, \phi, 0, K[S_2], \mu, 0, \cdots)$ with zero pre-differential, where $\phi$ is an arity 0 operation of degree $-2$, and $\mu$ is an binary operation of degree 0, basis of the regular representation of $S_2$.

**Definition 9.42** (c.\textit{Ass}^\wedge absolute operad). The complete curved absolute partial operad c.\textit{Ass}^\wedge is given by the free pdg absolute partial operad generated by $N$ modulo the ideal generated by the associativity relation on the generator $\mu$. It is endowed with the curvature $\Theta$ given by $\Theta(\text{id}) := \mu \circ_1 \phi - \mu \circ_2 \phi$.

**Lemma 9.43.** The data $(c.\text{Ass}^\wedge, 0, \Theta)$ forms a complete curved absolute partial operad.

**Proof.** Analogous to the proof of the previous lemma. \hfill \Box

**Proposition 9.44.** Let $u.\text{Ass}$ be the unital partial operad encoding unital associative algebras and let $u.\text{Ass}^!$ be its Koszul dual conilpotent curved partial cooperad. There is an isomorphism of complete curved partial operads $(u.\text{Ass}^!)^* \cong c.\text{Ass}^\wedge$.

**Proof.** By direct inspection. \hfill \Box

**Proposition 9.45.** There is a morphism of complete curved absolute partial operads $c.\text{Lie}^\wedge \to c.\text{Ass}^\wedge$ which is determined by sending $\beta \mapsto \mu - \mu^{(12)}$, and $\zeta$ to $\phi$.

**Proof.** The proof is identical to the proof of Proposition 2.22. \hfill \Box

**Remark 9.46.** As one can guess by the above examples, one can construct "absolute analogues" of well-known operads when they are given by generators and relations. Furthermore, they show that if $P$ is a unital partial operad, then its Koszul dual operad $P^!$ is in fact an absolute operad.

Another class of examples is given by object which already carry an underlying filtration that makes them "complete". Let $V$ be a complete pdg module as defined in Definition 9.3, its curved endomorphism operad carries a natural structure of a complete curved absolute partial operad.

**Definition 9.47** (Complete curved endomorphisms partial operad). Let $V$ be a complete pdg module. Its complete curved endomorphisms partial operad is given by the complete pdg $S$-module $\text{end}_V(n) := \text{hom}^{[21]}(V^\otimes n, V)$, where we only consider here morphisms of pdg modules which raise the filtration degree by at least one. The operad structure is given by the partial composition of functions. Its pre-differential is given by $\partial := [d_V, -]$ and its curvature determined by $\Theta_V(\text{id}) := d^2_V$.

**Lemma 9.48.** Let $V$ be a complete pdg module. Its complete curved endomorphisms partial operad is a complete curved absolute partial operad.

**Proof.** Forgetting the filtration, $\text{end}_V$ is in particular a partial operad, thus an algebra over the reduced tree monad. Let $\gamma_V : T(\text{end}_V) \to \text{end}_V$ be its structural morphism. The morphism $\gamma_V$ can be extended to the completed reduced tree monad using the completeness of the underlying filtration of $\text{end}_V$. Notice that it is crucial to restrict ourselves to operations in the endomorphisms operad which raise the degree by at least one in order for this to be true. One can check that the extension of $\gamma_V$ to the completed reduced tree monad satisfies the axioms of an algebra over this monad. It is thus an absolute partial operad; moreover it forms a curved absolute partial operad endowed with its curvature, see Lemma 2.4.

Let us show it is a complete as a curved absolute partial operad. Let $F_* \text{end}_V$ be its underlying filtration and let $\mathcal{T}_* \text{end}_V$ the canonical filtration induced by its curved absolute partial operad.
structure. See Definition 9.20 for more details. There is an obvious inclusion $\mathcal{F}_n \text{end}_V \subset F_n \text{end}_V$, since any operation which can be written as partial compositions iterated $n$ times also must raise the filtration degree by at least $n$. Thus
\[
\bigcap_{n \in \mathbb{N}} \mathcal{F}_n \text{end}_V \subset \bigcap_{n \in \mathbb{N}} F_n \text{end}_V = 0,
\]
which implies that $\text{end}_V$ is a complete curved absolute partial operad. \qed

**Remark 9.49.** This provides for a natural setting that allows the canonical filtration of a complete curved absolute partial operad to be reflected upon its algebras. See Section 9 for an application in this direction.

### References

[Ane14] Mathieu Anel. Cofree coalgebras over operads and representative functions, arXiv:1409.4688, 2014.
[AR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189. Cambridge: Cambridge University Press, 1994.
[BLX21] Kai Behrend, Hsuan-Yi Liao, and Ping Xu. Derived differentiable manifolds; arXiv:2006.01376, 2021.
[BM03] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.
[CCCN21] Damien Calaque, Ricardo Campos, and Joost Nuiten. Lie algebroids are curved Lie algebras, arXiv:2103.10728, 2021.
[CLM16] Joseph Chuang, Andrey Lazarev, and W. H. Mannan. Cocommutative coalgebras: homotopy theory and Koszul duality. *Homology Homotopy Appl.*, 18(2):303–336, 2016.
[Cos11] Kevin J. Costello. A geometric construction of the Witten genus, ii, arXiv:1112.0816, 2011.
[CR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189. Cambridge: Cambridge University Press, 1994.
[BM03] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.
[CCCN21] Damien Calaque, Ricardo Campos, and Joost Nuiten. Lie algebroids are curved Lie algebras, arXiv:2103.10728, 2021.
[CLM16] Joseph Chuang, Andrey Lazarev, and W. H. Mannan. Cocommutative coalgebras: homotopy theory and Koszul duality. *Homology Homotopy Appl.*, 18(2):303–336, 2016.
[Cos11] Kevin J. Costello. A geometric construction of the Witten genus, ii, arXiv:1112.0816, 2011.
[CR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189. Cambridge: Cambridge University Press, 1994.
[BLX21] Kai Behrend, Hsuan-Yi Liao, and Ping Xu. Derived differentiable manifolds; arXiv:2006.01376, 2021.
[BM03] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.
[CCCN21] Damien Calaque, Ricardo Campos, and Joost Nuiten. Lie algebroids are curved Lie algebras, arXiv:2103.10728, 2021.
[CLM16] Joseph Chuang, Andrey Lazarev, and W. H. Mannan. Cocommutative coalgebras: homotopy theory and Koszul duality. *Homology Homotopy Appl.*, 18(2):303–336, 2016.
[Cos11] Kevin J. Costello. A geometric construction of the Witten genus, ii, arXiv:1112.0816, 2011.
[CR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189. Cambridge: Cambridge University Press, 1994.
[BLX21] Kai Behrend, Hsuan-Yi Liao, and Ping Xu. Derived differentiable manifolds; arXiv:2006.01376, 2021.
[BM03] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.
[CCCN21] Damien Calaque, Ricardo Campos, and Joost Nuiten. Lie algebroids are curved Lie algebras, arXiv:2103.10728, 2021.
[CLM16] Joseph Chuang, Andrey Lazarev, and W. H. Mannan. Cocommutative coalgebras: homotopy theory and Koszul duality. *Homology Homotopy Appl.*, 18(2):303–336, 2016.
[Cos11] Kevin J. Costello. A geometric construction of the Witten genus, ii, arXiv:1112.0816, 2011.
[CR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189. Cambridge: Cambridge University Press, 1994.
[BLX21] Kai Behrend, Hsuan-Yi Liao, and Ping Xu. Derived differentiable manifolds; arXiv:2006.01376, 2021.
[BM03] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.
[CCCN21] Damien Calaque, Ricardo Campos, and Joost Nuiten. Lie algebroids are curved Lie algebras, arXiv:2103.10728, 2021.
[CLM16] Joseph Chuang, Andrey Lazarev, and W. H. Mannan. Cocommutative coalgebras: homotopy theory and Koszul duality. *Homology Homotopy Appl.*, 18(2):303–336, 2016.
[Cos11] Kevin J. Costello. A geometric construction of the Witten genus, ii, arXiv:1112.0816, 2011.
[CR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189. Cambridge: Cambridge University Press, 1994.
[BLX21] Kai Behrend, Hsuan-Yi Liao, and Ping Xu. Derived differentiable manifolds; arXiv:2006.01376, 2021.
[BM03] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.
[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2012.

[Mar96] Martin Markl. Models for operads. *Commun. Algebra*, 24(4):1471–1500, 1996.

[Mil11] Joan Millès. André-Quillen cohomology of algebras over an operad. *Adv. Math.*, 226(6):5120–5164, 2011.

[MV09] Sergei Merkulov and Bruno Vallette. Deformation theory of representations of prop(erad)s. I. *J. Reine Angew. Math.*, 634:51–106, 2009.

[Pos11] Leonid Positselski. *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence*, volume 996. Providence, RI: American Mathematical Society (AMS), 2011.

[PP05] A. Polishchuk and L. Positselski. *Quadratic algebras*, volume 37 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2005.

[RiL22] Victor Roca i Lucio. The integration theory of curved absolute l-infinity algebras, *arxiv:2209.10282*, 2022.

[RNV20] Daniel Robert-Nicoud and Bruno Vallette. Higher Lie theory, *arXiv:2010.10485*, 2020.

[Swe69] M. E. Sweedler. Hopf algebras. New York: W.A. Benjamin, Inc. 1969, 336 p. (1969), 1969.

[Val20] Bruno Vallette. Homotopy theory of homotopy algebras. *Ann. Inst. Fourier*, 70(2):683–738, 2020.

[VdL03] P. Van der Laan. Coloured Koszul duality and strongly homotopy operads. *arXiv:math.QA/0312147*, 2003.

[War19] Benjamin C Ward. Massey products for graph homology. *arXiv:1903.12055*, 2019.

**Victor Roca i Lucio, Ecole Polytechnique Fédérale de Lausanne, EPFL, CH-1015 Lausanne, Switzerland**

*Email address: victor.rocalucio@epfl.ch*