On the $\ell$-adic Fourier transform and the determinant of the middle convolution

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October 20, 2023

Abstract

We study the relation of the middle convolution to the $\ell$-adic Fourier transformation in the étale context. Using Katz’ work and Laumon’s theory of local Fourier transformations we obtain a detailed description of the local monodromy and the determinant of Katz’ middle convolution functor $\text{MC}_\chi$ in the tame case. The theory of local $\epsilon$-constants then implies that the property of an étale sheaf of having an at most quadratic determinant up to Tate twist is often preserved under $\text{MC}_\chi$ if $\chi$ is quadratic.

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Introduction

Consider the addition map $\pi : \mathbb{A}^n_k \times \mathbb{A}^n_k \to \mathbb{A}^n_k$ for $k$ a finite field. If $K$ and $L$ are objects in the derived category $D^b_c(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$ then one may consider two kinds of convolutions, exchanged by Verdier
duality (cf. [10]):

\[ K \ast_s L = R\pi_*(K \boxtimes L) \quad \text{and} \quad K \ast_l L := R\pi_l(K \boxtimes L). \]

It is convenient to restrict the above construction to the abelian category of perverse sheaves \( \text{Perv}(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell) \subseteq D^b_c(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell) \) (see [1]). Under some restrictions (e.g., if \( n = 1 \) and if \( K \) is geometrically irreducible and not geometrically translation invariant, [10], Lem. 2.6.9) the above defined convolutions are again perverse and one can define the \textit{middle convolution} of \( K \) and \( L \in \text{Perv}(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell) \) as

\[ K \ast_{\text{mid}} L = \text{Im} (K \ast_l L \rightarrow K \ast_s L), \]

cf. [10], Chap. 2.6.

One reason why one is interested in the middle convolution is that \( K \ast_{\text{mid}} L \) is often irreducible, while the convolutions \( K \ast_s L \) and \( K \ast_l L \) are usually mixed and hence not irreducible. A striking application of the concept of middle convolution is Katz’ existence algorithm for irreducible rigid local systems ([10], Chap. 6).

The aim of this article is the determination of the behaviour of the Frobenius determinants under the middle convolution with Kummer sheaves.

Our main results are:

(i) Using Laumon’s theory of local Fourier transformation [12] and the principle of stationary phase ([12], [7]) we derive in Thm. 3.2.2 an explicit description of the local monodromy (the structure of Frobenius elements on the vanishing cycle spaces at the singularities)

\[ \text{MC}_\chi(K) := K \ast_{\text{mid}} (j_* \mathcal{L}_\chi[1]), \]

for \( K \) an irreducible nontrivial tame middle extension sheaf and \( \mathcal{L}_\chi \) a Kummer sheaf.

(ii) Building on Laumon’s product formula expressing the epsilon constant in terms of local epsilon constants [12], we obtain a formula for the determinant of \( \text{MC}_\chi(K) \) in the tame case (Cor. 4.2.4).

(iii) From Thm. 3.2.2 and Cor. 4.2.4 we conclude in Thm. 4.2.5 that, under certain natural restrictions, the property for a tame middle extension sheaf of having an at most quadratic determinant up to Tate-twist is preserved under middle convolution with quadratic Kummer sheaves.

In a companion article to this work, written jointly with Stefan Reiter [6], we use these methods in order to prove the following: \textit{Let} \( \mathbb{F}_q \) \textit{be the finite field of order} \( q = \ell^k \), \textit{where} \( \ell \) \textit{is an odd prime number and} \( k \in \mathbb{N} \). \textit{Then the special linear group} \( \text{SL}_n(\mathbb{F}_q) \) \textit{occurs regularly as Galois group over} \( \mathbb{Q}(t) \) \textit{if} \( n > 8\varphi(q - 1) + 11 \) \textit{and if} \( q \) \textit{is odd.}

1 General notation and conventions

1.1 General notation. If \( F \) is any field, then \( \overline{F} \) denotes an algebraic closure of \( F \). Let in the following \( k \) be an either finite or algebraically closed field of characteristic \( p \geq 0 \) and let \( \ell \) be a prime \( \ell \neq p \). Let us recall the setup used in Laumon’s work on the \( \ell \)-adic Fourier transform, see
If $X$ is a variety over $k$ (meaning that $X$ is separated of finite type over $k$), then $|X|$ denotes the set of closed points of $X$. For $x \in |X|$, the residual field is denoted $k(x)$ and the degree of $k(x)$ over $k$ is denoted by $\deg(x)$. The symbol $\mathfrak{P}$ always denotes the geometric point extending $x$ using the composition $\text{Spec}(k) \to \text{Spec}(k) \to X$. If $x$ is a point of $X$ (not necessarily closed) then $\dim(x)$ denotes the dimension of the closure of $x$. A $\mathbb{Q}_\ell$-sheaf always is by definition an étale constructible $\mathbb{Q}_\ell$-sheaf on $X$ and the associated derived category with bounded cohomology sheaves is denoted $D^b_c(X, \mathbb{Q}_\ell)$. If $x$ is a point of $X$ (not necessarily closed) and if $F$ is a $\mathbb{Q}_\ell$-sheaf on $X$, then $F_x$ denotes the restriction of $F$ to $x$ and $F_x^\mathfrak{P}$ denotes the stalk of $F$, viewed as a $\text{Gal}(k(x)/k(x))$-module.

By our assumptions on $k$, the category $D^b_c(X, \mathbb{Q}_\ell)$ is triangulated and supports Grothendieck’s six operations, with internal tensor product $\otimes$ and $\text{Rhom}$, external product $\boxtimes$, and Verdier dual $\mathbf{D} : D^b_c(X, \mathbb{Q}_\ell)^{\text{opp}} \to D^b_c(X, \mathbb{Q}_\ell)$ ([5]). For $S$ a regular scheme of dimension $\leq 1$ over $k$ and for a morphism of finite type $f : X \to Y$ of $S$-schemes one has the usual functors

$$RF_*, RF_! : D^b_c(X, \mathbb{Q}_\ell) \to D^b_c(Y, \mathbb{Q}_\ell) \quad \text{and} \quad f^*, RF^1 : D^b_c(Y, \mathbb{Q}_\ell) \to D^b_c(X, \mathbb{Q}_\ell)$$

with $\mathbf{D}$ interchanging $RF_*$ and $RF_!$ (resp. $f^*$ and $RF^1$). Often one writes $f_*, f_!$, and $f^!$ instead of $RF_*, RF_!$ and $RF^1$ (resp.). The category of smooth (lisse) $\mathbb{Q}_\ell$-sheaves on $X$ is denoted by $\text{Lisse}(X, \mathbb{Q}_\ell)$.

### 1.2 Remarks on perverse sheaves.

Recall that $D^b_c(X, \mathbb{Q}_\ell)$ contains the abelian subcategory of perverse sheaves $\text{Perv}(X, \mathbb{Q}_\ell)$ with respect to the autodual (middle) perversity [1]. An object $K \in D^b_c(X, \mathbb{Q}_\ell)$ is perverse if and only if the following conditions hold for any point $x \in X$ (see [1], (4.0)): if $i$ denotes the inclusion of $x$ into $X$ then

$$\mathcal{H}^\nu((i^*K)_\mathfrak{P}) = 0 \quad \text{for} \quad \nu > -\dim(x) \quad \text{and} \quad \mathcal{H}^\nu((i^!K)_\mathfrak{P}) = 0 \quad \text{for} \quad \nu < -\dim(x).$$

#### 1.2.1 Remark.

An object $K \in D^b_c(X, \mathbb{Q}_\ell)$ is perverse if and only if $K|_{X \otimes \mathbb{F}} \in D^b_c(X \otimes \mathbb{F}, \mathbb{Q}_\ell)$ is perverse. (This is a tautology given $i^1 = \mathbf{D} \circ i^* \circ \mathbf{D}$ and the compatibility of $\mathbf{D}$ with respect to base change to $\mathbb{F}$, cf. [8], Prop. 1.1.7; [1], Prop. 5.1.2.)

Let $j : U \to X$ be an open immersion with complement $i : Y \to X$. If $K$ is a perverse sheaf on $U$ then there is a unique extension $j_*K \in \text{Perv}(X, \mathbb{Q}_\ell)$ of $K$ to $X$ which has neither subobjects nor quotients of the form $i_\ast \text{Perv}(Y, \mathbb{Q}_\ell)$ ([1]). This extension is called the intermediate extension or middle extension.

Let $X$ be a smooth and geometrically connected curve over $k$, let $j : U \to X$ be a dense open subscheme, and let $F$ be a smooth sheaf on $U$. Then the shifted sheaf $F[1]$ (concentrated at $-1$) is a perverse sheaf on $U$ and the middle extension $j_*F[1]$ is a perverse sheaf which coincides with $(j_*F)[1]$ (here we mean the usual sheaf extension $j_*F$ shifted by 1, see [11], Chap. III.5). A middle extension sheaf on $X$ ($X$ a smooth geometrically connected curve) is by definition a perverse sheaf of the form $(j_*F)[1]$ as above, cf. [10], Chap. 5.1.

### 1.3 Further notions ([12])

Let here $X$ denote a scheme of finite type over $k = \mathbb{F}_q$ and let $K \in D^b_c(X, \mathbb{Q}_\ell)$. The geometric Frobenius morphism relative to $k$ will be denoted by $\text{Frob}_q$ or $\text{Frob}_k$. 


In the associated Grothendieck group \( \mathcal{K}(X, \overline{Q}_\ell) \), one has an equality
\[
(1.2) \quad [K] = \sum_j (-1)^j [H^j(K)],
\]
with constructible cohomology sheaves \( H^j(K) \). Recall that for any closed point \( x \in |X| \) and any constructible sheaf \( F \) on \( X \), the stalk \( F_x \) has a natural action of the geometric Frobenius element \( \text{Frob}_x = \text{Frob}_{\text{deg}(k(x)/k)} \), leading to the well defined characteristic polynomial \( \det(1 - t \cdot \text{Frob}_x, F) \). One defines \( \text{trace}(\text{Frob}_x, F) \), resp. \( \det(\text{Frob}_x, F) \), to be the coefficient of \(-t\), resp. \((-t)^n (n = \dim(F_X))\), in \( \det(1 - t \cdot \text{Frob}_x, F) \). Using (1.2) we obtain homomorphisms of groups
\[
\text{det}(1 - t \cdot \text{Frob}_x, -) : \mathcal{K}(X, \overline{Q}_\ell) \to \overline{Q}_\ell(t)^\times
\]
\[
\text{trace}(\text{Frob}_x, -) : \mathcal{K}(X, \overline{Q}_\ell) \to \overline{Q}_\ell
\]
\[
\text{det}(\text{Frob}_x, -) : \mathcal{K}(X, \overline{Q}_\ell) \to \overline{Q}_\ell
\]
by additivity/multiplicativity (cf. [12], Section 0.9). This notion extends to \( D^b_c(X, \overline{Q}_\ell) \) by setting
\[
\text{det}(1 - t \cdot \text{Frob}_x, K) = \text{det}(1 - t \cdot \text{Frob}_x, [K]).
\]

Let \( X \) be a curve and let \( F \) be a smooth \( \overline{Q}_\ell \)-sheaf on a dense open subset \( j : U \hookrightarrow X \). If \( x \in |X| \) then \( X_x \) (resp. \( X_{(x)} \)) denotes the Henselization of \( X \) with respect to \( x \) (resp. \( \mathfrak{p} \)) and \( \eta_x \) (resp. \( \eta_{(x)} \)) denotes the generic point of \( X_x \) (resp. \( X_{(x)} \)), cf. [5]. One defines the generic rank \( r(F) = r(j_! F) \) of \( F \) as \( \text{rk}(F_{\eta_x})(x \in X) \) and extends this notion to \( K \in D^b_c(X) \) by additivity, cf. [12], 2.2.1.

### 1.4 Artin-Schreier and Kummer sheaves

Recall the construction of Artin-Schreier and Kummer sheaves: Let \( k \) be the finite field \( \mathbb{F}_q \) and let \( G \) be a commutative connected algebraic group of finite type over \( k \).

#### 1.4.1 Remark.
If \( f : X \to G \) is a morphism of schemes, and if \( \mathcal{L} \) is a sheaf on \( G \) then we set \( \mathcal{L}(f) = f^* \mathcal{L} \). Sometimes we simply \( \mathcal{L} \) instead of \( \mathcal{L}(f) \), especially if \( f \) is an obvious change of base.

The Lang isogeny of \( G \) is the extension of \( G \) by \( G(k) \)
\[
1 \to G(k) \to G \xrightarrow{L} G \to 1
\]
where \( L(x) = \text{Frob}_q(x) \cdot x^{-1} \). Hence \( L \) exhibits \( G \) as a \( G(k) \)-torsor over itself, the Lang torsor. To a character \( \chi : G(k) \to \overline{Q}_\ell^\times \) one then associates a smooth rank-one sheaf \( \mathcal{L}_\chi \) on \( G \) by pushing out the Lang torsor by \( \chi^{-1} : G(k) \to \overline{Q}_\ell^\times \).

If \( G = \mathbb{G}_{m,k} \), then \( \mathcal{L}_\chi \) is called a Kummer sheaf. If \( G = \mathbb{A}^1_k \), then a nontrivial character \( A^1(k) \to \overline{Q}_\ell^\times \) is usually denoted by \( \psi \) and the resulting sheaf \( \mathcal{L}_\psi \) is called an Artin-Schreier sheaf.

Consider the multiplication map
\[
x \cdot x' : A^1 \times_k A^1 \to A^1, \quad (x, x') \mapsto x \cdot x'.
\]
Then, for a closed point \( s \) of \( A^1 \), the restriction of \( \mathcal{L}_\psi(x \cdot x') \) to \( s \times_k A^1 \) is denoted by \( \mathcal{L}_\psi(s \cdot x') \).

If \( k \) is a field of odd order then the unique quadratic character \( \mathbb{G}_m(k) = \mathbb{K}^\times \to \overline{Q}_\ell^\times \) is denoted \( -1 \). The trivial character \( \mathbb{G}_m(k) = k^\times \to \overline{Q}_\ell^\times \) is denoted \( 1 \).
1.4.2 Remark. Recall from [12], (1.1.3.7), an alternative construction of Kummer sheaves: let $k$ be momentarily allowed to be any field which contains a primitive $N$-th root of unity. Consider the exact sequence

$$1 \to \mu_N(k) \to \mathbb{G}_{m,k} \to \mathbb{G}_{m,k} \to 1,$$

where $[N]$ denotes the $N$-th power morphism and $\mu_N(k)$ is the group of $N$-th roots of unity in $k$. This is a Galois cover with Galois group $\mu_N(k)$ and pushing out the resulting $\mu_N(k)$-torsor via $\chi^{-1}$ (where $\chi : \mu_N(k) \to \mathbb{Q}_\ell^\times$ is a character) one obtains a Kummer sheaf $\mathcal{K}_\chi$ on $\mathbb{G}_{m,k}$. Then the sheaf $\mathcal{K}_\chi$ coincides with the above Kummer sheaves $\mathcal{L}_\chi$ if $k = \mathbb{F}_q$ and $N = q - 1$ ([12], (1.1.3.7)).

2 Convolution in characteristic $p$

2.1 Basic definitions. In this section $k$ denotes either a finite field of characteristic $p \not= \ell$ or the algebraic closure of such a field. Let us recall the definitions and basic results of [10], Section 2.5. For $G$ a smooth $k$-group, denote the multiplication map by $\pi : G \times G \to G$. Let $K$ and $L$ be two objects of $D^b_{c}(G, \mathbb{Q}_\ell)$ and let $K \boxtimes L$ denote the external tensor product of $K$ and $L$ on $G \times G$ with respect to the two natural projections. Then one may form the $！$-convolution $K *！ L := R\pi_!(K \boxtimes L)$
as well as the $∗$-convolution $K * L := R\pi_*(K \boxtimes L)$
with duality interchanging both types of convolution. Under the shearing transformation $\sigma : \mathbb{A}^2_{x,y} \to \mathbb{A}^2_{x,t}, (x,y) \mapsto (x, t = x + y)$,
the above convolutions can be written as

$$K *！ L := R\text{pr}_{2!}(K \boxtimes L), \quad K * L := R\text{pr}_{2*}(K \boxtimes L),$$

where the external tensor product is now formed with respect to the first projection $\text{pr}_1 : \mathbb{A}^1_x \times \mathbb{A}^1_t \to \mathbb{A}^1_x$ and the difference map

$$\delta : \mathbb{A}^1_x \times \mathbb{A}^1_t \to \mathbb{A}^1_y, (x,t) \mapsto y = t - x.$$

An object $K$ of $\text{Perv}(G, \mathbb{Q}_\ell)$ has property $\mathcal{P}$ by definition if for any perverse sheaf $L \in \text{Perv}(G, \mathbb{Q}_\ell)$ the convolutions $L *！ K$ as well as $L * L$ are again perverse. If either $K$ or $L$ has the property $\mathcal{P}$ then one can define the middle convolution of $K$ and $L$ as the image of $L *！ K$ in $L * L$ under the natural forget supports map

$$L *\text{mid} K := \text{Im}(L *！ K \to L * L),$$

2.1.1 Lemma. Let $K$ be a perverse sheaf on $\mathbb{A}^1_k$ which is geometrically irreducible and not geometrically translation invariant. Then $K$ has the property $\mathcal{P}$.
Proof: This follows from [10], Cor. 2.6.10 together with Rem. 1.2.1.

Using the previous result we obtain for each Kummer sheaf $L_\chi$, associated to a nontrivial character $\chi$, a functor

$$\MC_\chi : \Perv(\mathbb{A}^1_{k,\overline{\mathbb{Q}_\ell}}) \to \Perv(\mathbb{A}^1_{k,\overline{\mathbb{Q}_\ell}}), K \mapsto K *_{\text{mid}} L_\chi,$$

with $L_\chi = j_*L_\chi[1]$, where $j: \mathbb{G}_m \to \mathbb{A}^1$ denotes the natural inclusion.

Let now $S$ be any $k$-variety, let $f: X \to S$ be proper, let $j: U \to X$ be an affine open immersion over $S$, let $D = X \setminus U$, and suppose that $\overline{f}|_D: D \to S$ is affine. Suppose that $K$ is an object in $\Perv(U, \mathbb{Q}_\ell)$ such that both $Rf!K$ and $Rf_*K$ are perverse. Then Prop. 2.7.2 of [10] states that $Rf_*(j_*K)$ is again perverse and that

$$Rf_*(j_*K) = \text{Im}(Rf!K \to Rf_*K).$$

Let us take $S = \mathbb{A}^1_t$, $X = \mathbb{P}^1 \times \mathbb{A}^1_t$, $U = \mathbb{A}^2_{x,t}$, and let $f = \text{pr}_2: \mathbb{A}^2_{x,t} \to \mathbb{A}^1_t$, and $\overline{f} = \text{pr}_2: \mathbb{P}^1 \times \mathbb{A}^1_t \to \mathbb{A}^1_t$. Then Eq. (2.3) implies:

2.1.2 Lemma. Let $K \in \Perv(\mathbb{A}^1, \mathbb{Q}_\ell)$ have property $\mathcal{P}$ and let $L \in \Perv(\mathbb{A}^1, \mathbb{Q}_\ell)$. Then $K *_{\text{mid}} L$ is a perverse sheaf with

$$K *_{\text{mid}} L = R\text{pr}_{2*}(j_*(K \boxtimes L)) \quad \text{with} \quad K \boxtimes L = \text{pr}_1^*K \otimes \delta^*L.$$

2.1.3 Proposition. (Katz) The middle convolution has the following properties:

(i) Let $K \in \Perv(\mathbb{A}^1_x, \mathbb{Q}_\ell)$ and $L \in \Perv(\mathbb{A}^1_y, \mathbb{Q}_\ell)$ be irreducible middle extensions which are not geometrically translation invariant. If $K$ and $L$ are tame at $\infty$ then there is a short exact sequence of perverse sheaves on $\mathbb{A}^1_t$

$$0 \to H \to K \boxtimes L \to K *_{\text{mid}} L \to 0,$$

where $H$ is the constant sheaf $\text{pr}_{2*}(j_*(K \boxtimes L)_{\infty \times \mathbb{A}^1_t})$ on $\mathbb{A}^1_t$ ([10], 2.9.4).

(ii) If $F, K, L \in D^b_c(\mathbb{A}^1, \mathbb{Q}_\ell)$ have all property $\mathcal{P}$ then

$$F *_{\text{mid}} (K *_{\text{mid}} L) = (F *_{\text{mid}} K) *_{\text{mid}} L,$$

cf. [10], 2.6.5.

(iii) For each nontrivial Kummer sheaf $L_\chi$ and for each $K \in \Perv(\mathbb{A}^1, \mathbb{Q}_\ell)$ having the property $\mathcal{P}$, the following holds:

$$\MC_{\chi^{-1}}(\MC_\chi(K)) = K(-1).$$

This follows from (i) using $L_{\chi^{-1}} *_{\text{mid}} L_\chi = \delta_0(-1)$ with $L_{\chi^{-1}} = j_*L_\chi[1]$ and with $\delta_0$ denoting the trivial sheaf supported at 0, cf. [10], Thm. 2.9.7.

(iv) Formation of $K *_{\text{mid}} L$ is compatible with arbitrary change of base (see [10] 4.3.8–4.3.11).
2.2 Fourier transformation and convolution. In this section, we fix a finite field $k = \mathbb{F}_q (q = p^m)$ and an additive $\mathbb{G}_a^\times$-character $\psi$ of $\mathbb{A}^1 (\mathbb{F}_p)$, inducing for all $k \in \mathbb{N}$ an additive character $\psi_{\mathbb{F}_q^k} = \psi \circ \text{trace}_{\mathbb{F}_q^k}$.

By the discussion in Section 1.4 we have the associated Artin-Schreier sheaf $\mathcal{L}_\psi$ on $\mathbb{A}_k^1$. Let $\mathbb{A} = \text{Spec}(k[x])$ and $\mathbb{A}' = \text{Spec}(k'[x'])$ be two copies of the affine line and let

$$x \cdot x' : \mathbb{A} \times \mathbb{A}' \longrightarrow \mathbb{G}_a, \quad (x, x') \mapsto x \cdot x'.$$

The two projections of $\mathbb{A} \times \mathbb{A}'$ to $\mathbb{A}$ and $\mathbb{A}'$ are denoted $\text{pr}$ and $\text{pr}'$, respectively. Following Deligne and Laumon [12], we can form the Fourier transform as follows:

$$\mathcal{F}_\psi = \mathcal{F} : D^b_c(\mathbb{A}, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(\mathbb{A}', \overline{\mathbb{Q}}_\ell), \quad K \longmapsto R\text{pr}'_* (\text{pr}^* K \otimes \mathcal{L}_\psi(x \cdot x')) [1].$$

By exchanging the roles of $\mathbb{A}$ and $\mathbb{A}'$, one obtains the Fourier transform

$$\mathcal{F}'_\psi = \mathcal{F}' : D^b_c(\mathbb{A}', \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(\mathbb{A}, \overline{\mathbb{Q}}_\ell), \quad K \longmapsto R\text{pr}_* (\text{pr}'^* K \otimes \mathcal{L}_\psi(x \cdot x')) [1].$$

Consider the automorphism $a : \mathbb{A} \to \mathbb{A}, x \mapsto -x$. By [12], Cor. 1.2.2.3 and Thm. 1.3.2.3, the Fourier transform is an equivalence of triangulated categories $D^b_c(\mathbb{A}, \overline{\mathbb{Q}}_\ell) \to D^b_c(\mathbb{A}', \overline{\mathbb{Q}}_\ell)$ and $\text{Perv}(\mathbb{A}, \overline{\mathbb{Q}}_\ell) \to \text{Perv}(\mathbb{A}', \overline{\mathbb{Q}}_\ell)$ with quasi-inverse $a^* \mathcal{F}'(-)(1)$. Especially, it maps simple objects to simple objects.

2.2.1 Definition. Let $\text{Fourier}(\mathbb{A}, \overline{\mathbb{Q}}_\ell) \subset \text{Perv}(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ and $\text{Fourier}(\mathbb{A}', \overline{\mathbb{Q}}_\ell) \subset \text{Perv}(\mathbb{A}', \overline{\mathbb{Q}}_\ell)$ be the categories of simple middle extension sheaves on $\mathbb{A}_k$ and $\mathbb{A}'_k$ (resp.) which are not geometrically isomorphic to a translated Artin-Schreier sheaf $\mathcal{L}_\psi(s \cdot x)$ with $s \in \mathbb{F}_q^\times$ (cf. [12], (1.4.2)). We call the objects in $\text{Fourier}(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ irreducible Fourier sheaves.

In [9], (7.3.6), the sheaves $\mathcal{H}^{-1}(K)$ with $K \in \text{Fourier}(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ are called irreducible Fourier sheaves, justifying the nomenclature (up to a shift). By Thm. 1.4.2.1 and Thm. 1.4.3.2 in [12], the following holds:

2.2.2 Proposition. (Deligne, Laumon)

(i) The functor $\mathcal{F}$ induces a categorical equivalence from $\text{Fourier}(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ to $\text{Fourier}(\mathbb{A}', \overline{\mathbb{Q}}_\ell)$.

(ii) If $H = V \otimes \mathcal{L}_\psi(s \cdot x)$, $(s \in \mathbb{A}^1)$ with $V$ constant, then $\mathcal{F}_\psi(H)$ is the punctual sheaf $V_s$ supported at $s$.

(iii) Let $\chi$ be a nontrivial character of $\mathbb{G}_m(k)$ and let $j : \mathbb{G}_m \to \mathbb{A}$, resp. $j' : \mathbb{G}'_m \to \mathbb{A}'$ denote the canonical inclusions. Then

$$\mathcal{F}(j_* \mathcal{L}_\chi[1]) = j'_* \mathcal{L}_{\chi^{-1}}[1] \otimes G(\chi, \psi),$$

where $G(\chi, \psi)$ is the geometrically constant sheaf on $\mathbb{A}'$ on which the Frobenius acts via the Gauß sum

$$g(\chi, \psi) = - \sum_{x \in \mathbb{K}^\times} \chi(x) \psi(x)$$

(as a $\text{Frob}_q$-module, $G(\chi, \psi) = \mathcal{H}_c^1(\mathbb{A}_k \setminus 0, \mathcal{L}_\chi \otimes (\mathcal{L}_\psi|_{\mathbb{A}^1 \setminus 0})))$.
2.2.3 Remark. An irreducible perverse sheaf $K \in \Perv(A, \overline{\mathbb{Q}_\ell})$ has the property $\mathcal{P}$ if and only if $\mathcal{F}(K)$ is a middle extension (cf. [10], 2.10.3). Note that the trivial rank-one sheaf $\mathcal{L}_e(0 \cdot x')$. It follows hence from by Lem. 2.1.1 that any object in Fourier$(A, \overline{\mathbb{Q}_\ell})$ and in Fourier$(A', \overline{\mathbb{Q}_\ell})$ has the property $\mathcal{P}$.

The relation of the Fourier transform to the convolution is expressed as follows ([12], Prop. 1.2.2.7):

$$\mathcal{F}(K_1 \ast K_2) = (\mathcal{F}(K_1) \otimes \mathcal{F}(K_2))[-1].$$

Applying Fourier inversion yields

$$K_1 \ast K_2 = a^* \mathcal{F}'(\mathcal{F}(K_1) \otimes \mathcal{F}(K_2))[-1](1).$$

2.2.4 Proposition. Let $K, L \in \Perv(A, \overline{\mathbb{Q}_\ell})$ be tame middle extensions in Fourier$(A, \overline{\mathbb{Q}_\ell})$. Suppose that for $j : \mathbb{G}_m \hookrightarrow A'$ the inclusion one has

$$\mathcal{F}(K) = j_* F[1] \in \text{Fourier}(A', \overline{\mathbb{Q}_\ell}) \quad \text{and} \quad \mathcal{F}(L) = j_* G[1] \in \text{Fourier}(A', \overline{\mathbb{Q}_\ell})$$

for smooth sheaves $F, G$ on $\mathbb{G}_m$. Then the following holds:

(i) $$\mathcal{F}(K \mid \mid L) = j_*(F \otimes G)[1].$$

(ii) if $K$ is not a translate of $j_* \mathcal{L}_{\chi^{-1}}[1]$, then $\mathcal{F}(K \mid \mid j_* \mathcal{L}_{\chi^{-1}}[1])$ is an object in Fourier$(A', \overline{\mathbb{Q}_\ell})$.

Proof: This is [10], 2.10.8 (the proof generalizes to our context). The second claim holds since, under the given assumptions on $K$ and $L$, the sheaf $j_*(F \otimes G)[1]$ is irreducible and not an Artin-Schreier sheaf.

2.2.5 Corollary. Under the assumptions of Prop. 2.2.4:

$$K \mid \mid L = a^* \mathcal{F}'(j_*(F \otimes G)[1])(1).$$

Moreover, if $L = j_* \mathcal{L}_{\chi}[1]$ and if $L$ is not a translate of $j_* \mathcal{L}_{\chi^{-1}}[1]$, then $K \mid \mid L \in \text{Fourier}(A, \overline{\mathbb{Q}_\ell})$.

Proof: This follows from Fourier inversion and from Prop. 2.2.4 (ii).

3 Local Fourier transform and local monodromy of the middle convolution.

3.1 Local Fourier transform. As before, we fix a finite field $k = \mathbb{F}_q (q = p^m)$, a prime $\ell \neq p$, and an additive $\overline{\mathbb{Q}_\ell}$-character $\psi$ of $\mathbb{A}^1(\mathbb{F}_q)$, see Section 2.2. In the following we summarize Laumon’s construction of the local Fourier transform [12] and the stationary phase decomposition:

Recall the notion of a henselian trait $T = (T, \eta, t)$ (see [5], [12]): it is the spectrum of a henselian discrete valuation ring $R$ having generic point $\eta$ and closed point $t = \text{Spec}(k)$. The usual geometric point over $t$ is denoted $\overline{t} = \text{Spec}(\overline{k})$ and the generic point of the strict henselization $T^\flat$ of $T$
is denoted $\mathcal{P}$ ([5], (0.6)). Note that any henselian trait $T$ (resp. $R$) has a uniformizer, usually denoted $\pi$.

Below we will consider two henselian traits $T = (T, \eta, t)$ and $T' = (T', \eta', t')$ in equiconstant characteristic $p$ with given uniformizers $\pi$, resp. $\pi'$, having $k$ as residue field. The fundamental groups $\pi_1(\eta, \mathcal{P})$ and $\pi_1(\eta', \mathcal{P})$ are denoted $G$ and $G'$, respectively. The inertia subgroups of $G, G'$ are denoted $I, I'$ (resp.).

For $X$ a variety over $k$ and $x$ a closed point of $X$ one has the henselian trait $X_\langle x \rangle = (X_\langle x \rangle, \eta_x, x)$ given by the spectrum of the henselization $R_x$ of the local ring $O_{X,x}$, its generic point $\eta_x$ and the point $x$. We define $G_x = \pi_1(\eta_x, \mathcal{P}_x)$ and $I_x \leq G_x$ denotes the inertia subgroup of $G_x$.

The category of smooth $\mathcal{Q}_\mathbb{L}$-sheaves on $\eta$ (which will be identified with the category of continuous $\mathcal{Q}_\mathbb{L}$-representations of finite rank of $G$ via the fibre functor in $\mathcal{P}$, cf. [12], Rem. 2.1.2.1) is denoted $\mathcal{M}$. Similarly we define the category $\mathcal{M}'$ of smooth sheaves on $\eta'$. For $V \in \text{ob} \mathcal{M}$, denote by $V_t$ the extension by zero to $T$, similarly for $V' \in \mathcal{M}'$. The subcategory of $\mathcal{M}$, resp. $\mathcal{M}'$, formed by objects whose inertial slopes are in $[0, 1]$ are denoted $\mathcal{M}_{[0,1]}$, resp. $\mathcal{M}'_{[0,1]}$, cf. [12], Section 2.1. Recall that an object of $\mathcal{M}$ is tamely ramified if and only if it is pure of slope 0 (loc.cit., 2.1.4). If $V$ (resp. $V'$) is an object of $\mathcal{M}$ (resp. of $\mathcal{M}'$) then its extension by zero to $T$ (resp. $T'$) is denoted $V_t$ (resp. $V'_t$).

One has the $\mathcal{Q}_\mathbb{L}$-sheaves $\mathcal{L}_\psi(\pi/\pi')$, $\mathcal{L}_\psi(\pi'/\pi)$ and $\mathcal{L}_\psi(1/\pi')$ on $T \times_k \eta'$, $\eta \times_k T'$ and $\eta \times \eta'$ (resp.) and the respective extensions by zero to $T \times_k T'$ are denoted $\mathcal{L}_\psi(\pi/\pi')$, $\mathcal{L}_\psi(\pi'/\pi)$ and $\mathcal{L}_\psi(1/\pi')$. For any $V \in \text{ob} \mathcal{M}$ one may form the vanishing cycles

$$R\Phi_{\eta'}(\text{pr}^*V_t \otimes \mathcal{L}_\psi(\pi/\pi')), \quad R\Phi_{\eta'}(\text{pr}^*V_t \otimes \mathcal{L}_\psi(\pi'/\pi)), \quad R\Phi_{\eta'}(\text{pr}^*V_t \otimes \mathcal{L}_\psi(1/\pi'))$$

as objects in $D^b_c(T \times_k \eta', \mathcal{Q}_\mathbb{L})$ with respect to $\text{pr}' : T \times_k T' \to T'$ ($\text{pr} : T \times_k T' \to T$ denoting the first projection), see [2], (2.1.1).

This leads to three functors, called local Fourier transforms,

$$\mathcal{F}^{(0,\infty)}, \mathcal{F}^{(\infty,0)}, \mathcal{F}^{(\infty,\infty)} : \mathcal{M} \to \mathcal{M}'$$

defined by

$$\mathcal{F}^{(0,\infty)}(V) = R^1\Phi_{\mathcal{P}}(\text{pr}^*V_t \otimes \mathcal{L}_\psi(\pi/\pi'))_{(\mathcal{P},\mathcal{P})},$$

$$\mathcal{F}^{(\infty,0)}(V) = R^1\Phi_{\mathcal{P}}(\text{pr}^*V_t \otimes \mathcal{L}_\psi(\pi'/\pi))_{(\mathcal{P},\mathcal{P})},$$

$$\mathcal{F}^{(\infty,\infty)}(V) = R^1\Phi_{\mathcal{P}}(\text{pr}^*V_t \otimes \mathcal{L}_\psi(1/\pi'))_{(\mathcal{P},\mathcal{P})},$$

see [12], 2.4.2.3. By interchanging the roles of $T$ and $T'$ one obtains the functors

$$\mathcal{F}^{(0,\infty)}, \mathcal{F}^{(\infty,0)}, \mathcal{F}^{(\infty,\infty)} : \mathcal{M}' \to \mathcal{M}'.$$

3.1.1 Remark. Note that we have neither fixed $T$ nor $T'$ so that the local Fourier transform may be formed with respect to any pair of henselian traits in equiconstant characteristic $p$ having $k$ as residue field.

We will need the following properties of the local Fourier transform below:
3.1.2 Theorem. (Laumon)

(i) The three local Fourier transforms are exact functors. Moreover, \( \mathcal{F}^{(0,\infty')} : \mathcal{G} \to \mathcal{G}_{[0,1]} \) is an equivalence of categories quasi-inverse to \( a^* \mathcal{F}^{(\infty',0)}(-)(1) \), where \( a : T \to T \) is the automorphism defined by \( \pi \mapsto -\pi \) and (1) denotes a Tate-twist.

(ii) If \( W \) denotes an unramified \( G \)-module, then

\[
\mathcal{F}^{(0,\infty')}(W) = W, \quad \mathcal{F}^{(\infty,0')}(W) = W(-1), \quad \mathcal{F}^{(\infty,\infty')}(W) = 0.
\]

(iii) For a non-trivial Kummer sheaf \( \mathcal{K}_\chi \) on \( \mathbb{G}_m = \text{Spec}(k[u,u^{-1}]) \), denote \( V_\chi \), resp. \( V'_\chi \) the \( G \)-module \( \mathcal{K}_\chi(\pi) \) (resp. the \( G' \)-module \( \mathcal{K}''_\chi(\pi') \)) on \( T \) (resp. \( T' \)), where \( \pi : \eta \to \mathbb{G}_m \) (resp; \( \pi' : \eta' \to \mathbb{G}_m \)) is the morphism which maps \( \pi \) to \( u \) (resp. \( \pi' \to u \)). Then

\[
\mathcal{F}^{(0,\infty')}(V_\chi) = V'_\chi \otimes G(\chi, \psi),
\]

\[
\mathcal{F}^{(\infty,0')}(V_\chi) = V'_\chi \otimes G(\chi^{-1}, \psi),
\]

where \( G(\chi, \psi) \) denotes the unramified \( G \)-module \( H^1_c(\mathbb{G}_m, \mathcal{K}_\chi \otimes \mathcal{L}_\psi) \) whose Frobenius trace is the Gauss sum

\[
\text{trace}(\text{Frob}_k, G(\chi, \psi)) = g(\chi, \psi) = - \sum_{a \in k^\times} \chi(a) \psi_k(a).
\]

(iv) If the restriction of the representation \( V \) to the inertia subgroup \( I \) is unipotent indecomposable (resp. tame), then \( \mathcal{F}^{(0,\infty')}(V) \), resp. \( \mathcal{F}^{(\infty,0')}(V) \), is unipotent and indecomposable (resp. tame) of the same rank.

(v) If \( W \) is an unramified \( G \)-module and if \( W' \) denotes the unique \( G' \)-module corresponding to the same \( \text{Gal}(\overline{k}/k) \)-module then

\[
\mathcal{F}^{(0,\infty')}(V_\chi \otimes W) = \mathcal{F}^{(0,\infty')} (V_\chi) \otimes W'.
\]

(vi) Let \( T_1 = T \otimes_k k_1 \) with \( k_1 \) a finite extension of \( k \), let \( \eta_1 \) denote the generic point of \( T_1 \) and let \( G_1 = \text{Gal}(\overline{k_1}/\eta_1) \). Let \( f : T_1 \to T \) denote the étale map given by the canonical projection. If \( V \) is a tamely ramified irreducible \( G \)-module of the form \( V = \text{Ind}_{G_1}^G(V_1) \), for \( V_1 \) a rank-1 module of \( G_1 \) then the following holds:

\[
\mathcal{F}^{(0,\infty')}(V) = \text{Ind}_{G_1}^G(\mathcal{F}^{(0,\infty')}(V_1)).
\]

Proof: The assertions (i)–(iii) are contained in [12] Thm. 2.4.3, Prop. 2.5.3.1. Assertion (iv) is proven in [7], Lemma 5. Assertion (v) can be found in [12], (3.5.3.1). Assertion (vi) follows from proper base change ([12], (2.5.2), (3.5.3.1)). \( \square \)

3.1.3 Remark. The following formula for Gauss sums will be used below:

\[
g(\chi, \psi)g(\chi^{-1}, \psi) = \chi(-1) \cdot q.
\]
The following result is one version of Laumon’s principle of stationary phase [12]. We use the formulation of Katz in [9], Cor. 7.4.2. Although in loc. cit. the result is stated for $k = \mathbb{F}_q$, this is not essential (as remarked in [9], beginning of Section 7.4, this is only for notational reasons) and the transition to $k = \mathbb{F}_q$ can be made using the arguments of [12], preuve de (3.4.2). See Rem. 3.1.1 for an explanation of the modules $\mathcal{F}^{(0, \infty')}(F_{\eta_\infty} / F_{\eta_0}^{I_0})$ below (using the uniformizer $\pi = x - s$).

3.1.4 Theorem. (Laumon) Let $K = j_* F[1] \in \text{Fourier}(\mathbb{A}^1_k, \mathbb{Q}_\ell)$ be a middle extension of a smooth sheaf $F$ on $U = \mathbb{A}^1 \setminus S$ which is tamely ramified at $S \cup \infty$ and let $F'[1] = \mathcal{F}(K)$. Then there is an isomorphism of $G_{\infty'}$-modules

\[
F_{\eta_\infty}' \simeq \bigoplus_{s \in S} \text{Ind}^G_{G_{s \times k, \infty'}} \left( \mathcal{F}^{(0, \infty')}(F_{\eta_\infty} / F_{\eta_0}^{I_0}) \otimes \mathcal{L}_\psi(s \cdot x') \eta_{\infty'} \right).
\]

\[\blacksquare\]

3.1.5 Corollary. Let $K \in \text{Fourier}(\mathbb{A}^1_k, \mathbb{Q}_\ell)$ be a middle extension of a smooth sheaf $F$ on $U \rightarrow \mathbb{A}^1_k$, tamely ramified at $S \cup \infty$. Let $K' := \mathcal{F}(K) = F'[1]$. Write the stationary phase decomposition in (3.2) as

\[
F_{\eta_\infty}' \simeq \bigoplus_{s \in S} \text{Ind}^G_{G_{s \times k, \infty'}} \left( \left( V'_s \otimes \mathcal{L}_\psi(s \cdot x') \eta_{\infty'} \right) \right).
\]

For each $s \in S$ there is an isomorphism of $G_s$-modules

\[
F_{\eta_s} / F_{\eta_0}^{I_0} \simeq a^* \mathcal{F}^{(\infty', 0)}(V'_s)(1).
\]

**Proof:** Using Thm. 3.1.2(vi) one reduces to the case where $k(s) = k$. Suppose first that $s = 0$. Then [9], Cor. 7.4.3.1, states that

\[
\mathcal{F}(F')_{\eta_0} / \mathcal{F}(F')_{\eta_0}^{I_0} \simeq \mathcal{F}^{(\infty', 0)}(F_{\eta_\infty}').
\]

Since in (3.2) the summand belonging to $s = 0$ is uniquely determined by its slope being equal to 0 (see [9], Cor. 7.4.1.1) and since the other summands in have slope equal to 1 (loc.cit.) it follows from $\mathcal{F}^{(\infty', 0)}(W) = 0 \forall W \in \mathcal{G}_{[1, \infty]}$ (see [12], Thm. (2.4.3)(ii)b)) that

\[
\mathcal{F}^{(\infty', 0)}(F'_{\eta_\infty}) = \mathcal{F}^{(\infty', 0)}(V'_0).
\]

The claim follows now from Fourier inversion (see Prop. 2.2.2(i) and Thm. 3.1.2(i)) for $s = 0$. If $s \neq 0$ then (as in [9], 7.4.1) one can use the formula

\[
\mathcal{F}(F) \simeq \mathcal{F}((\text{Add}(s)^x(F)) \otimes \mathcal{L}_\psi(sx')), \]

where $\text{Add}(s) : x \mapsto x + s$, in order to reduce to the case $s = 0$. \[\blacksquare\]
3.2 Local monodromy of the middle convolution with Kummer sheaves. Let $T$ be a henselian trait with residue field $k = \mathbb{F}_q$, with uniformizer $\pi$, generic point $\eta$, closed point $t$, and with fraction field $K_t$. Then the tame quotient $G'$ of the fundamental group $G = \pi_1(\eta, \eta)$ is a semidirect product of the procyclic tame inertia group

\[
I^t \simeq \hat{\mathbb{Z}}(1)(\overline{k}) := \lim_{\longrightarrow} \mu_N(\overline{k}),
\]

and the absolute Galois group $\text{Gal}(\overline{k}/k)$ of the residue field of $T$, cf. [12], Section 2.1.

For $l \in \mathbb{N}_{>1}$ let $k_l = \mathbb{F}_{q^l}$, let $T_l := T \times_k k_l$, with $T_l$ having residue field $k_l$ and generic point $\eta_l$. Let $G'_l$ denote the tame quotient of $\pi_1(\eta_l, \eta_l) \simeq \text{Gal}(\eta/\eta_l)$, semidirect product of $I^t$ and $\text{Gal}(k/k_l)$. We view $G'_l$ as a subgroup of $G'$ in the obvious way.

Moreover, for $s$ a closed point of $\mathbb{A}^1$ with henselian trait $(T_s = \mathbb{A}^1_{(s)}, \eta_s, s)$ we set $G_s := \pi_1(\eta_s, \eta_s)$ and $G'_s$ denotes the tame quotient of $G_s$. Note that since $s \simeq \text{Spec}(\mathbb{F}_{q^l})$ for some $l$ there is an isomorphism between $G'_s$ and this $G'_l$.

By the theorem of Krull, Remak and Schmidt, any $G'$-module $W$ decomposes into a direct sum of indecomposable summands $V_1 \oplus \cdots \oplus V_k$, unique up to renumeration. In the following we suppose that each indecomposable summand $V$ of $W$ is of the form

\[
V = \mathcal{J}_n \otimes \text{Ind}_{G'_l}^{G'}(V_{\chi'} \otimes F)
\]

with

(i) $V_{\chi'}$ denoting the rank one $G'_l$-module of the Kummer sheaf belonging to $\chi' : k_l^\times \to \mathbb{Q}_l^\times$ (as in Thm 3.1.2(iii)),

(ii) $F$ an unramified $G'_l$-module of rank 1,

(iii) $\mathcal{J}_n$ an indecomposable $G'$-module of rank $n \geq 1$ on which the group $I^t$ acts unipotently such that the following holds:

\[
\text{Gr}^M(\mathcal{J}_n) = \bigoplus_{j=0}^{n-1} \mathbb{Q}_l(-j),
\]

where $M$ denotes the weight filtration (see [5], (1.7.5)). Since multiplication with the logarithm of the usual topological generator of the tame inertia lowers the weight by 2 (loc. cit.), the Frobenius operation on the $I^t$-invariants is trivial.

The defining properties of $\mathcal{J}_n$ do not determine it as a $G'$-module up to isomorphism if $n > 1$. Hence it would be more accurate to speak about a class of representations satisfying these properties. Since for us only the indecomposability and the triviality of the fixed space under the Frobenius operation is important, we neglect this ambiguity notationally.

The restriction of $\mathcal{J}_n$ to some $G'_l$ is denoted by the same symbol.

3.2.1 Remark. (i) Since $\text{Gal}(\overline{k}/k)$ (resp. $\text{Gal}(\overline{k}/k_l)$) is abelian any unramified irreducible representation of $G'$ (resp. $G'_l$) has rank one.
(ii) Any representation of rank one of $G^t$ (resp. $G^t_1$) is of the form $V_{\chi} \otimes F$ with $F$ an unramified rank-one representation of $G^t$ (resp. $G^t_1$), see [12], (3.5.3.1).

(iii) Any irreducible $\overline{\mathbb{Q}}_\ell$-module $V$ of rank $l$ of $G^t$ is of the above used form $\text{Ind}^{G^t}_{G^t_1}(V_{\chi} \otimes F)$ (loc. cit.).

(iv) Let $G^t_1 \subseteq G^t_\ell \subseteq G^t$ be a chain of proper inclusions such that $V_{\chi}$ is defined over the intermediate field $F^t_\ell$, meaning that the order of the image of the tame inertia divides $q^t - 1$. Then $\text{Ind}^{G^t}_{G^t_1}(V_{\chi} \otimes F)$ decomposes into several rank-one factors since, by the operation of the Frobenius on the tame inertia determined by Formula (3.4), the image of $G^t_\ell$ under $\text{Ind}^{G^t}_{G^t_1}(V_{\chi} \otimes F)$ is abelian. It hence follows from the functoriality of the induction that $\text{Ind}^{G^t}_{G^t_1}(V_{\chi} \otimes F)$ is not irreducible.

3.2.2 Theorem. For $k = \mathbb{F}_q$ and $\ell \neq \text{Char}(k)$ let $F$ be an irreducible smooth $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_k \setminus S$ tamely ramified at $S \cup \infty$, such that $K = j_* F[1] \in \text{Fourier}(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$. Let $\mathcal{L}_\chi$ be a nontrivial Kummer sheaf and suppose that $F$ is geometrically not isomorphic to a translate of the Kummer sheaf $\mathcal{L}_{\chi^{-1}}$. For fixed $s \in S$ with $k(s) = k_i$, let

$$F_{\eta_s}/F^{I_s}_{\eta_s} = \bigoplus_{i=1}^{r_s} \mathcal{J}_{n_i} \otimes \text{Ind}^{G^t_1}_{G^t_1}(V_{\chi} \otimes F_i)$$

be the decomposition into indecomposable $G^t_1$-modules as in (3.5), where we identify $G^t_1$ with $G^t_i$ and where $l \leq l_i$.

Then $MC_{\chi}(K)$ is an object in Fourier$(\mathbb{A}^1_k, \overline{\mathbb{Q}}_\ell)$ of the form $j_* H[1]$ with $H$ irreducible and smooth on $\mathbb{A}^1_k \setminus S$ such one has a decomposition of $H_{\eta_s}/H^{I_s}_{\eta_s}$ into indecomposable $G^t_1$-modules

$$H_{\eta_s}/H^{I_s}_{\eta_s} = \bigoplus_{i=1}^{r_s} H_i,$$

where $H_i$ is as follows:

(i) If $\chi_i \neq \chi^{-1}, 1$ then

$$H_i = \mathcal{J}_{n_i} \otimes \text{Ind}^{G^t_1}_{G^t_1}(V_{\chi_i} \otimes (F_{\chi_i}(-1) \otimes G((\chi_i)^{-1}, \psi) \otimes G(\chi, \psi) \otimes G(\chi, \psi) \otimes F_i))$$

where $F_{\chi_i}(-1)$ stands for the geometrically constant rank-one $G^t_\ell$-module whose Frobenius trace is $\chi(1)$ (note the convention of Rem. 1.4.1 for the restriction of representations/sheaves, so that we interpret in the above formula $\mathcal{J}_{n_i}$ as the $G^t_\ell$-module $(\mathcal{J}_{n_i})_{\eta_s}$ and $\chi$ as the character $\chi \circ Nm^{k_i}_{k_i}$), and where the (1) of the right hand side of the formula denotes a Tate twist.

(ii) If $\chi_i = 1$, then $G^t_1 = G^t_i$ and

$$H_i = \mathcal{J}_{n_i} \otimes V_{\chi} \otimes F_i.$$

(iii) If $\chi_i = \chi^{-1}$, then $G^t_1 = G^t_i$ and

$$H_i = \mathcal{J}_{n_i} \otimes F_{\chi}(-1) \otimes F_i(-1),$$

where $F_{\chi}(-1)$ stands for the geometrically constant rank-one $G^t_\ell$-module whose Frobenius trace is $\chi(-1)$, and where the (−1) of the right hand side of the formula denotes a Tate twist.
3.2.3 Remark.  (i) For $k$ an algebraically closed field this is proven in [10], Cor. 3.3.6. For its proof we use similar arguments, further refined by the results in Thm. 3.1.2.

(ii) It follows from Rem. 3.2.1 (iii) that any $F$ with finite monodromy satisfies the conditions on $F_{\eta_1}/F_{\eta_2}^{l_s}$ of the above theorem.

(iii) In the dissertation of Tenzler [13], a variant of Thm. 3.2.2 is used which provides an algorithm to successively compute Frobenius traces for the construction of rigid local systems under the Katz algorithm [10]. As an application, Frobenius traces of certain $G_2$-motives are computed in [13].

Proof: Write $\mathcal{F}(K) = j'_*F'[1] \in \text{Fourier}(A', \overline{\mathcal{C}}_{\ell})$ (where $A'$ is as in Section 2.2) for $F'$ the restriction of $\mathcal{F}(K)$ to $j' : \mathbb{G}_m \to A'$ and let

$$\mathcal{F}(j_*(\mathcal{L}_\chi[1]) = j'_*(\mathcal{L}_\chi \otimes G(\chi, \psi))[1] = j'_*H'[1] \in \text{Fourier}(A', \overline{\mathcal{C}}_{\ell}).$$

By Prop. 2.2.4

$$\mathcal{F}(\text{MC}_\chi(K)) = j'_*(F' \otimes H')[1] \iff \text{MC}_\chi(K) = a^*\mathcal{F}'(j_*(F' \otimes H')[1])[1](1)$$

by Fourier inversion. Note that by our assumption on $F$, the sheaf $j_* (F' \otimes H')[1]$ is a Fourier sheaf which implies that $\text{MC}_\chi(K)$ is again in Fourier$(A, \overline{\mathcal{C}}_{\ell})$. Therefore Cor. 3.1.5 and Thm. 3.1.2 imply that

$$(3.7) \quad H_{\eta_1}/H_{\eta_2}^{l_s} = a^*\mathcal{F}((\mathcal{L}_\chi)^{-1}) \left( \mathcal{F}((0, \infty'))(F_{\eta_1}/F_{\eta_2}^{l_s}) \otimes V_\chi \otimes G(\chi, \psi) \right)(1),$$

where we write $V_\chi$ instead of $(\mathcal{L}_\chi)^{-1}$ (resp. $G(\chi, \psi)$ instead of $G(\chi, \psi)\eta_2$) so that

$$(3.8) \quad F_{\eta_1}/F_{\eta_2}^{l_s} = \bigoplus_i \mathcal{J}_{n_i} \otimes \text{Ind}_{G_{l_i}}^G(V_{\chi_i} \otimes F_i)$$

by assumption. We consider the individual contribution of each summand $\mathcal{J}_{n_i} \otimes \text{Ind}_{G_{l_i}}^G(V_{\chi_i} \otimes F_i)$ of (3.8) to (3.7):

Assume first that $\chi_i \neq \chi^{-1}, 1$. Let

$$H_i := a^*\mathcal{F}((\mathcal{L}_\chi)^{-1}) \left( \mathcal{F}((0, \infty'))(\mathcal{J}_{n_i} \otimes \text{Ind}_{G_{l_i}}^G(V_{\chi_i} \otimes F_i)) \otimes V_\chi \otimes G(\chi, \psi) \right)(1).$$

It follows from the definition of $\mathcal{J}_n$ and from the exactness of the local Fourier transform that tensoring tame irreducible $G_{l_i}$-modules with $\mathcal{J}_{n_i}$ commutes with local Fourier transformations (use an induction on $n$, cf. [13] Prop. 4.22(vi)). Hence

$$\mathcal{F}((0, \infty'))(\mathcal{J}_{n_i} \otimes \text{Ind}_{G_{l_i}}^G(V_{\chi_i} \otimes F_i)) = \mathcal{J}_{n_i} \otimes \mathcal{F}((0, \infty'))(\text{Ind}_{G_{l_i}}^G(V_{\chi_i} \otimes F_i))$$

$$= \mathcal{J}_{n_i} \otimes \text{Ind}_{G_{l_i}}^G(V_{\chi_i} \otimes G(\chi, \psi) \otimes F_i)$$

is again indecomposable. Hence $H_i$ is indecomposable since tensor products with rank-one sheaves and $\mathcal{F}((\mathcal{L}_\chi)^{-1})$ preserve indecomposability. The push-pull formula for the induction and restriction of representations implies

$$H_i = a^*\mathcal{F}((\mathcal{L}_\chi)^{-1}) \left( \mathcal{J}_{n_i} \otimes \text{Ind}_{G_{l_i}}^G(V_{\chi_i} \otimes G(\chi, \psi) \otimes F_i) \otimes V_\chi \otimes G(\chi, \psi) \right)(1)$$

$$= a^*\mathcal{F}((\mathcal{L}_\chi)^{-1}) \left( \mathcal{J}_{n_i} \otimes \text{Ind}_{G_{l_i}}^G(V_{\chi_{XX_i}} \otimes G(\chi, \psi) \otimes F_i \otimes G(\chi, \psi)) \right)(1),$$

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where in the square bracket after the induction sign we have restricted $V_{\chi} \otimes G(\chi, \psi)$ to $G_{l_i}$. This gives

$$H_i = a^* \left( \mathcal{J}_{n_i} \otimes \text{Ind}_{G_{l_i}}^{G} \left[ V_{\chi x_i} \otimes G(\chi^{-1}, \psi) \otimes G(\chi, \psi) \otimes G(\chi, \psi) \otimes F_{l_i} \right] \right) (1)$$

and

$$= \mathcal{J}_{n_i} \otimes \text{Ind}_{G_{l_i}}^{G} (V_{\chi x_i} \otimes F_{x_i} \otimes G(\chi^{-1}, \psi) \otimes G(\chi, \psi) \otimes G(\chi, \psi) \otimes F_{l_i})(1),$$

using Thm. 3.1.2(iii). Note for the last equation, via the trace function of Kummer sheaves (cf. Section 1.4), the effect of $a^*$ on the associated Frobenius trace in the above formula for $H_i$ amounts to a multiplication with $\chi^{x_i-1}$.

If $\chi_i = 1$, then we have $G_{l_i} = G^i_i$ by indecomposability (see Rem. 3.2.1 (iv)) and Thm. 3.1.2 implies that

$$\mathcal{F}^{(0, \infty)}(\mathcal{J}_{n_i} \otimes F_{l_i}) = \mathcal{J}_{n_i} \otimes F_{l_i}.$$

Hence, with

$$H_i := a^* \mathcal{F}^{(\infty, 0)} \left( \mathcal{F}^{(0, \infty)}(\mathcal{J}_{n_i} \otimes F_{l_i}) \otimes V_{\chi} \otimes G(\chi, \psi) \right) (1)$$

we obtain from Thm. 3.1.2

$$H_i = a^* \mathcal{F}^{(\infty, 0)} \left( \mathcal{J}_{n_i} \otimes V_{\chi} \otimes G(\chi, \psi) \otimes F_{l_i} \right) (1)$$

$$= a^* \left( \mathcal{J}_{n_i} \otimes V_{\chi} \otimes G(\chi, \psi) \otimes F_{l_i}(1) \right)$$

$$= \mathcal{J}_{n_i} \otimes V_{\chi} \otimes F_{l_i},$$

where we used $\chi = \chi_i^{-1}$ and Formula (3.1).

If $\chi_i = \chi^{-1}$, then again have $G_{l_i} = G^i_i$ and with

$$H_i := a^* \mathcal{F}^{(\infty, 0)} \left( \mathcal{F}^{(0, \infty)}(\mathcal{J}_{n_i} \otimes V_{\chi^{-1}} \otimes F_{l_i}) \otimes V_{\chi} \otimes G(\chi, \psi) \right) (1)$$

we obtain

$$H_i = a^* \mathcal{F}^{(\infty, 0)} \left( \mathcal{J}_{n_i} \otimes V_{\chi^{-1}} \otimes G(\chi^{-1}, \psi) \otimes V_{\chi} \otimes G(\chi, \psi) \otimes F_{l_i} \right) (1)$$

$$= \mathcal{J}_{n_i} \otimes F_{\chi^{-1}} \otimes F_{l_i}(1),$$

where we used the identity (3.1) as well as Thm. 3.1.2, (ii), (iv) and (v).

4 The determinant of the étale middle convolution.

4.1 Local epsilon constants, local Fourier transform, and Frobenius determinants.

As in the previous sections, we fix a finite field $k = \mathbb{F}_q (q = p^n)$ and an additive $\mathbb{Q}_{\ell}^\times$-character $\psi$ of $\mathbb{A}_1(k) \otimes \mathbb{F}_q (\ell \neq p)$. By composing with the trace function, the character $\psi$ uniquely determines additive $\mathbb{Q}_{\ell}^\times$-characters of any extension $\mathbb{A}_1(\mathbb{F}_q)$. Again denoted $\psi$.

Recall the theory of local epsilon constants for $X$ a connected smooth projective curve over $k$, see [3] and [12], Section 3: The $L$-function of $K \in D^b_c(X, \mathbb{Q}_{\ell})$ is defined as

$$L(X, K; t) = \prod_{x \in |X|} \frac{1}{\det(1 - t^{\deg x} \cdot \text{Frob}_x, K)}.$$
By the work of Grothendieck, this \( L \)-function is the product expansion of
\[
\det(1-t.Frob_q, R\Gamma(X \otimes_k \mathbb{F}, K))^{-1}
\]
and it satisfies the following functional equation:
\[
(4.1) \quad L(X,K;t) = \epsilon(X,K) \cdot t^{a(X,K)} \cdot L(X,D(K)),
\]
where \( D(K) \) denotes the Verdier dual of \( K \) and where \( a(X,K) \) and \( \epsilon(X,K) \) are defined as follows:
\[
(4.2) \quad a(X,K) := -\chi(X,K) \quad \text{(Euler characteristic as defined in loc.cit., Section 0.8)}
\]
\[
\epsilon(X,K) := \det(-\text{Frob}_q, R\Gamma(X, \mathbb{F}, K))^{-1},
\]

By the work of Deligne [3], there is a unique map \( \epsilon \) which, depending on a fixed character \( \psi \) as above, associates to a triple \((T,K,\omega)\) \((T=(T,\eta,t)\) a henselian trait with residue field \(k\), \(K \in D^b_c(T, \mathbb{Q}_\ell)\), \(\omega\) a nontrivial meromorphic 1-form on \(T\)) a local epsilon constant \(\epsilon(T,K,\omega) \in \mathbb{Q}_\ell \times \) such that the following axioms hold (see [3], Thm. 4.1, and the reformulation in [12], Thm. (3.1.5.4)):

\textbf{4.1.1 Proposition.} \quad (i) The association \((T,K,\omega) \mapsto \epsilon(T,K,\omega)\) depends only on the isomorphism class of the triple \((T,K,\omega)\).

(ii) For any distinguished triangle \(K' \to K \to K'' \to K'[1]\) in \(D^b_c(T, \mathbb{Q}_\ell)\) one has
\[
(4.3) \quad \epsilon(T,K,\omega) = \epsilon(T,K',\omega) \cdot \epsilon(T,K'',\omega).
\]

(iii) If \(K\) is supported on the closed point \(t\) of \(T\) then
\[
(4.4) \quad \epsilon(T,K,\omega) = \det(-\text{Frob}_t, K)^{-1}.
\]

(iv) If \(\eta\) denotes the generic point of \(T\), if \(\eta_1/\eta\) is a finite separable extension of \(\eta\) and if \(f : T_1 \to T\) denotes the normalization of \(T\) inside \(\eta_1\), then for any \(K_1 \in D^b_c(T_1, \mathbb{Q}_\ell)\) such that the generic rank \(r(K_1)\) of \(K_1\) (extended from the usual generic rank by additivity, so that in odd cohomological degree the rank is counted negative) is equal to 0 ([3]), one has
\[
(4.5) \quad \epsilon(T, f_*K_1, \omega) = \epsilon(T_1, K_1, f^*\omega).
\]
(The condition on the rank is rather mild because one can replace \(K_1\) by \(\tilde{K}_1 := K_1 \oplus (\mathbb{Q}_\ell e_1[e_2])\) such that \(r(\tilde{K}_1) = 0\), see [3] and [12], (3.3.2).)

(v) If \(V\) denotes a rank-one local system on \(\eta\) corresponding to a character \(\mu : K_2^\times \to \mathbb{Q}_\ell^\times\) via reciprocity and if \(j : \eta \hookrightarrow T\) denotes the obvious inclusion, then \(\epsilon(T,j_*V,\omega)\) coincides with Tate’s local constant associated to \(\mu\) (cf. [12], (3.1.3.2)).

For \(V\) a smooth sheaf on \(\eta\) and \(j : \eta \hookrightarrow T\) the inclusion (with \(V\) viewed as a \(G\)-module via the fibre functor at \(\eta\) and with inertial invariants \(V^I \simeq (j_*V)_{\overline{\eta}}\)) one defines
\[
\epsilon(T,V,\omega) := \epsilon(T,j_*V,\omega)
\]
and

\[(4.6) \quad \epsilon_0(T, V, \omega) := \epsilon(T, j_! V, \omega) = \epsilon(T, V, \omega) \cdot \det(-\text{Frob}_t, V^I),\]

where the last equality follows from Prop. 4.1.1(ii),(iii), using that in $D^c_b(T)$, the extension by zero $j_! V$ is an extension of $j_* V$ (placed in cohomological degree zero) and the shifted invariants $V^I[-1]$, supported at the closed point $t$, cf. [3], Chapitre (5).

If $x$ is a closed point of $X$, then $X_{(x)}$ denotes the Henselization of $X$ at $x$ (cf. [12], Section 0.4). By the work of Laumon ([12], Thm. 3.2.1.1), the epsilon constant decomposes into a product of local epsilon constants, depending on a nontrivial meromorphic differential 1-form $\omega$ on $X$, as follows:

\[(4.7) \quad \epsilon(X, K) = q^{C(1-g(X))r(K)} \prod_{x \in |X|} \epsilon(X_{(x)}, K|_{X_{(x)}}, \omega|_{X_{(x)}}),\]

where $C$ denotes the cardinality of connected components of $X \times \overline{k}$ and where $g(X)$ is the genus of some component of $X \times \overline{k}$. Another variant of (4.7) is

\[(4.8) \quad \epsilon(X, j_* F) = q^{C(1-g(X))r(F)} \prod_{x \in |X|} \epsilon(X_{(x)}, F|_{\eta_\tau}, \omega|_{X_{(x)}}),\]

where $F$ is smooth on an dense open $U \subseteq X$ with $j : U \hookrightarrow X$ ([12], (3.2.1.6)).

The local epsilon constants satisfy the following additional properties:

**4.1.2 Proposition. (Laumon)**

(i) If

\[0 \to V' \to V \to V'' \to 0\]

is a short exact sequence of $G$-modules then

\[(4.9) \quad \epsilon_0(T, V, \omega) = \epsilon_0(T, V', \omega) \cdot \epsilon_0(T, V'', \omega),\]

see [12], (3.1.5.7).

(ii) Let $K_t$ denote the completion of the function field of the generic point $\eta$ of $T$ and let $\nu_t : K_t^\times \to \mathbb{Z}$ its natural valuation. By [12], 3.1.5.6, if $K \in D^b_c(T, \overline{\mathbb{Q}}_\ell)$ and if $F$ is a smooth sheaf on $T$ then

\[(4.10) \quad \epsilon(T, K \otimes F, \omega) = \epsilon(T, K, \omega)^{r(F)} \cdot \det(\text{Frob}_t, F)^{a(T, K, \omega)},\]

where $a(T, K, \omega)$ is defined as follows (cf. [12], (3.1.5.1), (3.1.5.2)):

\[a(T, K, \omega) = r(K_\tau) + s(K_\tau) - r(K_\tau) + r(K_\tau) \nu_t(\omega),\]

where $s(K_\tau)$ is the Swan conductor of $K_\tau$ (which vanishes if and only if $K_\tau$ is tame, cf. [12], (2.1.4)) and where $\nu_t(a \cdot db) = \nu_t(a)$ for $a \cdot db \in \Omega^1_{K_t} \setminus 0$ and $\nu_t(b) = 1$. 

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(iii) Let $k_1$ be a finite extension of $k$. Let $V$ be an irreducible $G$-module of the form $f_*V_1$ with $f : T_1 = T \otimes_k k_1 \to T$ and with $V_1$ tame. Let $G_1 = \text{Gal}(\overline{\eta}/\eta_1)$ where $\eta_1$ denotes the generic point of $T_1$. Then

\begin{equation} \tag{4.11}
\epsilon_0(T, V, d\pi) = \epsilon_0(T_1, V_1, d\pi_1),
\end{equation}

where $\pi_1$ is a uniformizer of $T_1$ induced by $\pi$, cf. [12], 3.5.3.1.

(iv) If the character $\chi$ is nontrivial, then $j_!V_\chi = j_!V_\chi$. Hence if $K = j_*V_\chi$ then

\begin{equation} \tag{4.12}
\epsilon(T, K, d\pi) = \epsilon_0(T, V_\chi, d\pi) = \chi(-1)g(\chi, \psi)
\end{equation}

with $g(\chi, \psi)$ the Gauß sum occurring in Prop. 2.2.2 ([12], (3.5.3.1)). If $\chi$ is trivial then also

$$\epsilon_0(T, V_\chi, d\pi) = -1 \quad \text{and} \quad \epsilon(T, V_\chi, d\pi) = 1$$

(v) For $a \in k(\eta)\times$ one has

$$\epsilon(T, K, a\omega) = \chi_K(a)q^{(\text{detr}(-\chi)(a))} \epsilon(T, K, \omega)$$

where $\chi_K : K_1^\times \to \overline{\mathbb{Q}}_\ell^\times$ is the character induced by $\text{det}(K_1)$ via reciprocity ([12], (3.1.5.5)).

(vi) The behaviour of local epsilon constants under Tate twists is given as follows ([12] (3.2.1.4)):

$$\epsilon(\chi(x, K(m)|x(x), \omega|_{x(x)}) = q^{-m \cdot a(x, K|x(x), \omega|_{x(x)})} \epsilon(\chi(x, K)|x(x), \omega|_{x(x)}),$$

\[ \square \]

4.1.3 Proposition. Let $G = \pi_1(\eta, \overline{\eta})$ be the Galois group of the generic point of a henselian trait $T = (T, \eta, t), t = \text{Spec}(k), k = \mathbb{F}_q$, which uniformizer $\pi$ as above. Let

$$T_l = T \otimes_k k_l = (T_l, \eta_l, t_l) (k_l = \mathbb{F}_q)$$

and $G_l = \pi_1(\eta_l, \overline{\eta})$. Suppose that $V$ is a tame indecomposable $G$-module which, in the notation of Eq. (3.5), can be written as

\begin{equation} \tag{4.13}
V = \mathcal{J}_n \otimes \text{Ind}_{G_1}^{G_l}(V_{\chi_1} \otimes F_l),
\end{equation}

where $V_{\chi_1}$ is the $G_l$-module derived from the Kummer sheaf attached to $\chi_1 : G_m(k_l) \to \overline{\mathbb{Q}}_\ell^\times$ (see Thm. 3.1.2(iii)) and where $F_l$ is an unramified $G_l^\prime$-module. Then the following holds:

(i) If $\chi \neq 1$ then

$$\epsilon(T, V, d\pi) = \epsilon_0(T, V, d\pi) = q^{(n-1)/2} \cdot (\chi(-1)g(\chi, \psi) \cdot \text{det}(\text{Frob}_l, F))^n \cdot .$$

(ii) If $\chi = 1$ then $l = 1$ and we can write

$$V = \mathcal{J}_n \otimes F$$

with $F$ an unramified $G^\prime$-module. Then

$$\epsilon_0(T, V, \omega) = (-q)^{n(n-1)/2} \det(-\text{Frob}_l, F)^n$$

and

$$\epsilon(T, V, \omega) = (-q)^{n(n-1)/2} \det(-\text{Frob}_l, F)^{(n-1)} = \epsilon_0(T, V/V^l, \omega).$$

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Proof: Let us first treat the case where $\chi_l \neq 1$: By definition of $\mathcal{F}_n$ (see Eq. (3.6)),

$$\text{Gr}^M(V) = \bigoplus_{j=0}^{n-1} \text{Ind}_{G_l}^G(V_{\chi_l} \otimes F_l)(-j).$$

Therefore Prop. 4.1.2 (i) implies that

$$(4.14) \quad \epsilon_0(T, V, d\pi) = \prod_{j=0}^{n-1} \epsilon_0(T, \text{Ind}_{G_l}^G(V_{\chi_l} \otimes F_l)(-j), d\pi).$$

Since $\chi_l$ is nontrivial,

$$j_*(\text{Ind}_{G_l}^G(V_{\chi_l} \otimes F_l)) = j_!(\text{Ind}_{G_l}^G(V_{\chi_l} \otimes F_l)),$$

and therefore

$$\epsilon(T, \text{Ind}_{G_l}^G(V_{\chi_l} \otimes F_l), d\pi) = \epsilon_0(T, \text{Ind}_{G_l}^G(V_{\chi_l} \otimes F_l), d\pi),$$

as well as

$$\epsilon_0(T, V, d\pi) = \epsilon(T, V, d\pi).$$

It follows from Prop. 4.1.2(iv) that

$$(4.15) \quad \epsilon_0(T_l, \text{Ind}_{G_l}^G(V_{\chi_l} \otimes F_l), d\pi) = \epsilon_0(T_l, V_{\chi_l} \otimes F_l, d\pi).$$

Using Prop. 4.1.2(ii), especially Eq. (4.10), for the first equality (using that the valuation of $d\pi_l$ is zero and $F_l$ is smooth on $T_l$), and Prop. 4.1.2(iv) for the second equality, one obtains

$$(4.16) \quad \epsilon_0(T_l, V_{\chi_l} \otimes F_l, d\pi_l) = \epsilon_0(T_l, V_{\chi_l}, d\pi_l) \cdot \det(\text{Frob}_l, F_l) = \chi_l(-1)g(\chi_l, \psi) \cdot \det(\text{Frob}_l, F_l).$$

Therefore, taking Tate twists into account,

$$(4.17) \quad \epsilon_0(T, V, d\pi) = \epsilon(T, V, d\pi) = \prod_{j=0}^{n-1} q^j \epsilon(T, \text{Ind}_{G_l}^G(V_{\chi_l} \otimes F_l(-j)), d\pi)$$

$$(4.18) \quad = q^{n(n-1)/2} (\chi_l(-1)g(\chi_l, \psi) \cdot \det(\text{Frob}_l, F_l))^n.$$

Let now $\chi = 1$: it follows from indecomposability that $G_l = G$, see Rem. 3.2.1(iv). Hence we have

$$V = \mathcal{F}_n \otimes F$$

with $F$ smooth on $T$ of rank one. As above we obtain

$$(4.19) \quad \epsilon_0(T, V, d\pi) = \prod_{j=0}^{n-1} \epsilon_0(T, F(-j), d\pi)$$

since $\epsilon_0$ is multiplicative on short exact sequences of $G$-modules. It follows from Prop. 4.1.2(ii),(iv) that

$$\epsilon(T, F(-j), d\pi) = \epsilon(T, \overline{\mathcal{F}_l} \otimes F(-j), d\pi) = \epsilon(T, V_1, d\pi)^1 \cdot \det(\text{Frob}_l, F(-j))^0 = 1$$

since the valuation of $d\pi$ is zero and $F(-j)$ is smooth (note that $V_1 = \overline{\mathcal{F}_l}$). Since, by Eq. (4.6), for general $G$-modules $W$

$$\epsilon_0(T, W, \omega) = \epsilon(T, W, \omega) \cdot \det(-\text{Frob}_l, W),$$

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we obtain
\[ \epsilon_0(T, F(-j), d\pi) = \det(-\text{Frob}_t, F(-j)) = (-q)^j \det(-\text{Frob}_t, F). \]

Consequently,
\[ \epsilon_0(T, V, \omega) = (-q)^{(n-1)/2} \det(-\text{Frob}_t, F)^n \]
and
\[ \epsilon(T, V, \omega) = (-q)^{(n-1)/2} \det(-\text{Frob}_t, F)^{(n-1)}. \]

The last equation follows from
\[ \epsilon_0(T, V, \omega) = \epsilon_0(T, V^t, \omega) \epsilon_0(T, V/V^t, \omega) \]
(see Prop. 4.1.2(i)) and \( \epsilon_0(T, V^t, \omega) = \det(-\text{Frob}_t, F). \)

\[ \square \]

4.2 The Frobenius determinant of the middle convolution. Let \( k = \mathbb{F}_q (q = p^n) \), let \( U \rightarrow \mathbb{A}_k^1 \) be a dense open subscheme, and let \( S = \mathbb{A}_k^1 \setminus U \). Let \( V = j_* F[1] \in \text{Fourier}(\mathbb{A}_k^1, \overline{\mathbb{Q}_l}) \) be an irreducible middle extension sheaf, where \( F \) is an irreducible smooth sheaf of rank \( r(F) \) on \( U \) which is tamely ramified in \( S \cup \infty \). Consider the following conditions on \( F \) with respect to a nontrivial character \( \chi : \mathbb{G}_m(k) \rightarrow \overline{\mathbb{Q}_l}^\times \):

(i) \( F \) has scalar inertial local monodromy at \( \infty \), whose restriction to the tame inertia group is given by the (restriction of the) Kummer sheaf associated to a character \( \chi : k^\times \rightarrow \overline{\mathbb{Q}_l} \), see Section 1.4.

(ii) \( F \) is geometrically not isomorphic to a translated Kummer sheaf of the form \( \mathcal{L}_{\chi^{-1}}(y - x) \), where \( x \) denotes the coordinate of \( \mathbb{A}_1 \) and \( y \in |\mathbb{A}_1| \).

4.2.1 Definition. If \( F \) as above satisfies the above conditions (i),(ii) we say that \( F \) is in standard form with respect to \( \chi \).

4.2.2 Remark. We remark that if \( F \), is in standard form with respect to \( \chi \), then \( \text{MC}_\chi(V) \) is in standard form with respect to \( \chi^{-1} \) (see [10], Cor. 3.3.6).

For \( y \in \mathbb{A}_1(k) \setminus S \), let \( U_y = U \setminus y \), and let
\[ F_y(\chi) = F|_{U_y} \otimes \mathcal{L}_\chi(y - x)|_{U_y}. \]

Note that \( j_* F_y(\chi) \) is smooth in \( \infty \), where \( j : U_y \hookrightarrow \mathbb{P}^1 \) is the natural inclusion. We define \( F_y(\chi) = j_* F_y(\chi) \). We assume further that for each \( s \in S \) with \( k(s) = k_{l_s} \) one has a decomposition of \( F_{\eta_s}/F_{\eta_s}^{T_{l_s}} \) into indecomposable \( G_s^t = G_{l_s}^t \) modules (where \( G_s^t \) is the tame quotient of \( G_s = \pi_1(\eta_s/\eta_s) \) with tame inertia subgroup \( T_{l_s} \) and where \( G_{l_s} = \pi_1(\eta_s/\eta_s) \) with \( \eta_s \) the generic point of \( T_{l_s} = T \otimes_k k_{l_s} \) with \( T = \text{Spec}(k{\pi}) \) the trait of the henselization of \( k[\pi](\pi) \))

(4.20)
\[ F_{\eta_s}/F_{\eta_s}^{T_{l_s}} \cong \bigoplus_{i_s=1}^{r_s} \mathcal{F}_{n_{l_s}} \otimes \text{Ind}_{G_{l_s}^{t}}^{G_{l_s}} (V_{\chi_{l_s}} \otimes F_{l_s}). \]
with $F_{i_s}$ unramified (using the notation of Section 3.2). Since the sheaf $\mathcal{L}_x(y - x)$ is smooth at $s$, the latter formula implies that $F_y(\chi)_{\pi_s}/F_y(\chi)_{\pi_s}^{I_1}$ is of the form

$$F_y(\chi)_{\pi_s}/F_y(\chi)_{\pi_s}^{I_1} \simeq \mathfrak{f}_{n_{i_s}} \otimes \text{Ind}^G_{G_{i_{i_s}}} (V_{\chi_{i_s}} \otimes F'_{i_s})$$

with $F'_{i_s}$ unramified.

4.2.3 Theorem. Under the above assumptions, let $F$ be in standard form with respect $\chi$ and let $\omega_0 := dx$. Then

$$\det(-\text{Frob}_q, H^1(\mathbb{P}^1_{\bar{k}}, j_\ast F_y(\chi))) = q^{-r(F)} \det(\text{Frob}_\infty, F_y(\chi)_\infty)^{-2} \prod_{s \in S \cup y} \epsilon_0(\mathbb{P}^1_{(s)}, F_y(\chi)_{\eta_s}/F_y(\chi)_{\eta_s}^{I_1}, \omega_0|_{\mathbb{P}^1_{(s)}}),$$

where for $s = y$,

$$\epsilon_0(\mathbb{P}^1_{(y)}, F_y(\chi)_{\eta_y}/F_y(\chi)_{\eta_y}^{I_1}, \omega_0|_{\mathbb{P}^1_{(y)}}) = \epsilon_0(\mathbb{P}^1_{(y)}, F_y(\chi)_{\eta_y}, \omega_0|_{\mathbb{P}^1_{(y)}}) = (\chi(-1)g(\chi, \psi))^{r(F)} \cdot \det(\text{Frob}_y, F_y(\chi)_{\pi_y}),$$

and where for $s \in S$,

$$\epsilon_0(\mathbb{P}^1_{(s)}, F_y(\chi)_{\eta_s}/F_y(\chi)_{\eta_s}^{I_1}, \omega_0|_{\mathbb{P}^1_{(s)}}) = \prod_{i_s=1}^{r_s} \epsilon_0(\mathbb{P}^1_{(s)}, \mathfrak{f}_{n_{i_s}} \otimes \text{Ind}^G_{G_{i_{i_s}}} (V_{\chi_{i_s}} \otimes F'_{i_s}), \omega_0|_{\mathbb{P}^1_{(s)}}),$$

with

$$\epsilon_0(\mathbb{P}^1_{(s)}, \mathfrak{f}_{n_{i_s}} \otimes \text{Ind}^G_{G_{i_{i_s}}} (V_{\chi_{i_s}} \otimes F'_{i_s}), \omega_0|_{\mathbb{P}^1_{(s)}}) =

\begin{cases}
q^{l_{i_s} \cdot n_{i_s}}(n_{i_s} - 1)/2 \cdot \chi_{l_{i_s}}(-1)g(\chi_{l_{i_s}}, \psi) \cdot \det(\text{Frob}_{l_{i_s}}, F'_{l_{i_s}})^{n_{i_s}} & \text{if } \chi_{l_{i_s}} \neq 1 \\
(-q)^{l_{i_s} \cdot n_{i_s}}(n_{i_s} - 1)/2 \cdot \det(-\text{Frob}_{s}, F'_{i_s})^{n_{i_s}} & \text{if } \chi_{l_{i_s}} = 1.
\end{cases}$$

Proof: Let $X = \mathbb{P}^1_{\bar{k}}$. The cohomology groups $H^1(X_{\bar{k}}, j_\ast F_y(\chi)) (i \neq 1)$ vanish by the irreducibility assumption on $F$. Hence with the conventions of Section 1.3 we obtain

$$\det(-\text{Frob}_q, R\Gamma(X_{\bar{k}}, j_\ast F_y(\chi)))^{-1} = \det(-\text{Frob}_q, H^1(X_{\bar{k}}, j_\ast F_y(\chi))).$$

Laumon’s product formula for local epsilon constants (Formula (4.8)) implies therefore that

$$\det(-\text{Frob}_q, H^1(X_{\bar{k}}, j_\ast F_y(\chi))) = q^{r(F)} \prod_{x \in [X]} \epsilon(x, F_x, \omega_0|_{x}).$$

We next use arguments similar to [12], Prop. 3.3.2: Since the valuation of $\omega_0|_{x}$ is zero for $x \in |\mathbb{A}^1|$ it follows from Prop. 4.1.2(ii),(iv) that for all $x \in |\mathbb{A}^1 \setminus S|$ one has

$$\epsilon(x, F_y(\chi)|_{\eta_x}, \omega_0|_{x}) = 1$$

and therefore

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\[ \det(-\text{Frob}_q, H^1(X_{\mathfrak{T}}, j_* F_y(\chi))) = q^{r(F)} \epsilon(X_{(\infty)}, F_y(\chi)_{\eta_{\infty}}, \omega_0|_{X_{(\infty)}}) \prod_{s \in S \cup y} \epsilon(X_s, F_y(\chi)_{\eta_s}, \omega_0|_{X_s}). \]

The sheaf \( F_y(\chi) \) is smooth at \( \infty \) since \( F \) is in standard form w.r. to \( \chi \). Since the valuation of \( \omega_0|_{X_{(\infty)}} \) is equal to \(-2 \) it follows from Prop. 4.1.2(ii),(iv),(v) that

\[ \epsilon(X_{(\infty)}, F_y(\chi)_{\eta_{\infty}}, \omega_0|_{X_{(\infty)}}) = q^{-2r(F)} \cdot \det(\text{Frob}_{\infty}, F_y(\chi)_{\infty})^{-2}. \]

To determine the other local epsilon constants, let \( k\{\pi\} \) the henselization of \( k[\pi](\pi) \) and \( T = \text{Spec}(k\{\pi\}) = (T, \eta, t) \) the corresponding trait with uniformizer \( \pi \) and tame fundamental group \( G^t \). Let \( s \in S \cup y \) with \( k(s) = k_l, \pi_l = F_q \). Let \( T_l := T \otimes k_l, \pi_l \) be as in Prop. 4.1.2(iii). For each \( s \in S \) we have a canonical uniformizer \( \pi_s \) of the trait \( X_s \) as follows: for
\[
\begin{align*}
\kappa_{s} = \prod_{x \in \text{Hom}_k(k(s), \kappa)} s_{x} \quad \text{define} \quad 
\pi_s := \prod_{x}(x - s_x).
\end{align*}
\]

Then one obtains a finite étale \( k \)-morphism \( f_s : X_s \rightarrow T \) induced by \( \pi_s \mapsto \pi \). In this case we then have for any \( G^t_k \)-module \( V \)
\[
\epsilon_0(X_s, V, \omega_0|_{X_s}) = \epsilon_0(T, \pi_{s,*}(V), d\pi),
\]

since we have \( \omega_0|_{X_s} = f_s^*(d\pi) \) because \( f_s \) is basically the localization of the inclusion \( X_s \rightarrow X \) (Eq. (4.24) itself can be deduced using the last remark in Prop. 4.1.1(iv) as in [12] (3.5.2), (3.5.3.1)). Moreover, by Prop. 4.1.2(iii),
\[
\epsilon_0(T, \pi_{s,*}(V), d\pi) = \epsilon_0(T_l, V, \omega|_{T_l}) \cdot d\pi_l.
\]

This implies one the one hand that
\[
\epsilon_0(\mathbb{P}^1_{(y)}, F_y(\chi)_{\eta_{\infty}}, \omega_0|_{\mathbb{P}^1_{(y)}}) = (\chi(-1)g(\chi, \psi))^r(F) \cdot \det(\text{Frob}_y, F_y(\chi)_{\eta_{\infty}})
\]
by Prop. 4.1.3 (or directly from Prop. 4.1.1 and 4.1.2).

On the other hand, the same argument as used for Eq. (4.22), the local epsilon constant of the inertia invariants \( F_y(\chi)_{\eta_{(s)}} \) is equal to 1 for all \( s \in S \). Using the multiplicativity of \( \epsilon_0 \) on short exact sequences together with the last statement in Prop. 4.1.3 one sees that
\[
\epsilon(X_s, F_y(\chi)_{\eta_s}, \omega_0|_{X_s}) = \prod_{i_s=1}^{r_s} \epsilon_0(\mathbb{P}^1_{v_i}, \mathcal{F}_{n_{i_s}} \otimes \text{Ind}_{G_{s,l_{i_s}}}^{G_{s}}(V_{l_{i_s}} \otimes F_{l_{i_s}}), \omega|_{\mathbb{P}^1_{v_i}}).
\]
Combining (4.24) with (4.25) and using Prop. 4.1.3 we obtain for \( i_s = 1, \ldots, r_s \)
\[
\epsilon_0(X_s, \mathcal{F}_{n_{i_s}} \otimes \text{Ind}_{G_{s,l_{i_s}}}^{G_{s}}(V_{l_{i_s}} \otimes F_{l_{i_s}}), \omega|_{\mathbb{P}^1_{v_i}}) =
\begin{cases}
q_{l_{i_s}}^{\chi_{l_{i_s}}(n_{i_s} - 1)/2} \cdot (\chi_{l_{i_s}}(-1)g(\chi_{l_{i_s}}, \psi) \cdot \det(\text{Frob}_{l_{i_s}}, F_{l_{i_s}}))^{n_{i_s}} & \text{if } \chi_{l_{i_s}} \neq 1 \\
(-q)^{\chi_{l_{i_s}}(n_{i_s} - 1)/2} \cdot \det(\text{Frob}_{l_{i_s}}, F_{l_{i_s}})^{n_{i_s}} & \text{if } \chi_{l_{i_s}} = 1.
\end{cases}
\]

By combining the formulas (4.22), (4.23), (4.27), and (4.28), one obtains the claimed result.

\[ \square \]
4.2.4 Corollary. Under the above assumptions,
\[
\det(-\text{Frob}_q, MC\chi(j_*(F[1]))_\mathfrak{y}) = \det(-\text{Frob}_q, H^1(\mathbb{P}^1_{\mathbb{F}_q}, j_*(F_y(\chi))))^{-1}
\]
\[
= q^{-r(F)} \det(\text{Frob}_\infty, F_y(\chi)_{\mathbb{Q}_\ell})^{-2} \prod_{s \in S \cup \mathfrak{y}} \epsilon(\mathbb{P}^1_{(s)}, F_y, \eta_s, \omega_0|_{\mathbb{F}^1_{(s)}},)
\]
as in Thm. 4.2.3.

Proof: Only the first equality has to shown. Using the definition of $MC\chi(F)$ (see Eq. (2.2) and Lemma 2.1.2) and Prop. 2.1.3(iii) one sees that
\[
MC\chi(j_*(F[1]))_\mathfrak{y} = H^1(\mathbb{P}^1_{\mathbb{F}_q}, j_*(F_y(\chi)))_{\mathfrak{y}}.
\]

4.2.5 Theorem. Let $k = \mathbb{F}_q$, be a finite field of odd order, let $U \rightarrow A^1_k$ be a dense open subscheme, and let $S = A^1_k \setminus U$. Let $V = j_*(F[1]) \in \text{Fourier}(A^1_k, \mathbb{Q}_\ell)$ be an irreducible nonconstant tame middle extension sheaf, where $F$ is smooth on $U$ as above. Assume that $F$ satisfies the following conditions:

(i) The sheaf $F$ is in standard form with respect to the quadratic character $-1: k^\times \rightarrow \mathbb{Q}_\ell$.

(ii) The $I^*_s$-module $F_{\eta_s}$ is self-dual for all $s \in S$.

(iii) For each $y \in |A^1_k|$ there exists an integer $m$ so that
\[
det(\text{Frob}_y, (j_*(F)_\mathfrak{y})) = \pm q^m.
\]

Then $MC_{-1}(V) = j_*(G[1]) \in \text{Fourier}(A^1_k, \mathbb{Q}_\ell)$ with $G$ smooth on $U$ and $G$ satisfies the conditions (i)–(iii) with $F$ replaced by $G$.

Proof: It follows from Rem. 4.2.2 that $G$ is again in standard form w.r. to $-1$ and hence fulfills the respective condition (i). The effect of $MC_{-1}$ on the geometric local monodromy ([10], Thm. 3.3.5 and Cor. 3.3.6; cf. Thm. 3.2.2) implies that also (ii) holds for $G$.

The self-duality assumption in (ii) for $F$ implies that the contribution of the Gauss-sums to the determinant of $G$ (by local epsilon constants as in Thm. 4.2.3 and Cor. 4.2.4) which arise from nontrivial characters $\chi_i \neq -1$ cancel out to $\pm q^{k_1} (k_1 \in \mathbb{Z})$ because also $\chi_i^{-1}$ appears in this local monodromy (using Rem. 3.1.3).

The product formula for the (determinant of the) geometric monodromy implies that the character $-1$ appears in an even number. Hence also the contribution of these Gauss-sums amounts to $\pm q^{k_2} (k_2 \in \mathbb{Z})$, again by Rem. 3.1.3.

It follows from condition (iii) for $F$ that for each $s \in S$,
\[
det(\text{Frob}_s, F_y(-1)_{\eta_s}/F_y(-1)_{\eta_s}^{I_s}) = \pm q^{k_3}
\]
for some $k_3 \in \mathbb{Z}$ since tensoring by the quadratic Kummer sheaf $\mathcal{L}_{-1}$ does not change this property. Hence, by the usual rules for the behaviour of determinants under induction,
\[
\prod_{i=1}^{r_s} \det(-\text{Frob}_s, F_i^{n_s}) = \pm q^{k_4}
\]

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for some $k_4 \in \mathbb{Z}$, where $F_i$ is as in Thm. 4.2.3. Finally, the contribution of $\det(\text{Frob}_\infty, F_y(\chi))^{-2}$ to the determinant of $G$ is $\pm q^{k_5}$ ($k_5 \in \mathbb{Z}$) by assumption (iii) for $F$ and Cebotarev’s density theorem. Via Thm. 4.2.3 this proves that the condition (iii) holds for $G$ for any closed point $y \in |\mathbb{A}^1 \setminus S|$.

One has $\det(\text{Frob}_s, F_{\eta_s}) = \pm q^{k_6}$ for some $k_6 \in \mathbb{Z}$ by Cebotarev’s density theorem ($s \in S$). It follows therefore from condition (iii) for $F$ together with $F_{\eta_s} \simeq (j_* F)_\pi$ that

$$\det(\text{Frob}_s, F_{\eta_s}/F_{\eta_s}^{T_1}) = \pm q^{k_7}$$

for some $k_7 \in \mathbb{Z}$.

Hence Thm. 3.2.2 implies that

$$\det(\text{Frob}_s, G_{\eta_s}/G_{\eta_s}^{T_1}) = \pm q^{k_8}$$

for some $k_8 \in \mathbb{Z}$, implying that the condition (iii) holds for $G$ also at each point $s \in S$ (again by Cebotarev’s theorem). □

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