NEW PROOFS OF THE TORELLI THEOREMS FOR RIEMANN SURFACES

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ABSTRACT. In this paper, by using the Kuranishi coordinates on the Teichmüller space and the explicit deformation formula of holomorphic one-forms on Riemann Surface, we give an explicit expression of the period map and derive new differential geometric proofs of the Torelli theorems, both local and global, for Riemann Surfaces.

1. INTRODUCTION

The theme of this paper is to present a new differential geometric understanding of the Torelli problems of Riemann Surfaces, which are central topics in the study of the complex structures of Riemann Surface. The Torelli problems are usually divided into two types: local Torelli and Torelli problems of Riemann Surfaces, which are central topics in the study of the complex structures of Riemann Surface. These two problems are about the immersion and injectivity of the period map from the moduli space of Riemann Surfaces to the moduli space of principally polarized abelian varieties, respectively.

Two key points of this paper are the use of the Kuranishi coordinates on the Teichmüller space \( \mathcal{T}_g \) of Riemann Surface of genus \( g \) and the explicit deformation formula of holomorphic one-forms in Section 2. Roughly speaking, the Kuranishi coordinate chart of \( \mathcal{T}_g \) is given by

\[
(B, b_0) \rightarrow \mathcal{T}_g, \quad t \rightarrow [X_t, [F_t]],
\]

where the triple \((\varpi, \varphi, F)\) is the Kuranishi family of Riemann Surface with the Teichmüller structure of \((X_0, [F_0])\). Let us write \((B, b_0)\) as \(\Delta_{p,e}\), where \(p\) denotes the point \([X_0, [F_0]] \in \mathcal{T}_g\).

Then, given a global holomorphic one-form \(\theta \in H^0(X_p, \Omega^1_{X_p})\) on \(X_p\), we have the following deformation formula \(\theta(t)\) of \(\theta\) for small \(t\) on \(X_t\):

\[
\theta(t) = \theta + \sum_{i=1}^n t_i \left( \mathbb{H}(\mu_i, \theta) + df_i \right) + \sum_{|I| \geq 2} t^I \left( \sum_{j=1}^n \mathbb{H}(\mu_{j-I}, \eta_{i_1, \ldots, i_j-1, \ldots, i_n}) + df_{j, (i_1, \ldots, i_j-1, \ldots, i_n)} \right),
\]

where \(\eta_{i_1, \ldots, i_n}\) is a sequence of \((1, 0)\)-forms on \(X_p\), \(f_{j, (i_1, \ldots, i_j-1, \ldots, i_n)} \in C^\infty(X_p)\) and \(|I| = \sum_{j=1}^{3g-3} i_j\). Here and henceforth \(\mathbb{H}\) denotes the harmonic projection on \((X_p, \omega_p)\), where \(\omega_p\) is the Poincaré metric on \(X_p\), and \(n = 3g - 3\). An application of this to the canonical basis \(\{\theta_p^\alpha\}_{\alpha=1}^g\) of \(H^0(X_p, \Omega^1_{X_p})\) with respect to the symplectic basis \(\{A_\gamma, B_\gamma\}_{\gamma=1}^g\) for \(\Delta_{p,e}\) tells us that

\[
\theta_p^\alpha(t) = \theta_p^\alpha + \sum_{i=1}^n t_i \left( \mathbb{H}(\mu_i, \theta_p^\alpha) + df_i^\alpha \right) + \sum_{|I| \geq 2} t^I \left( \sum_{j=1}^n \mathbb{H}(\mu_{j-I}, \eta_{i_1, \ldots, i_j-1, \ldots, i_n}) + df_{j, (i_1, \ldots, i_j-1, \ldots, i_n)}^\alpha \right).
\]

The \(g \times g\) matrix \(A(t)\) is naturally defined as:

\[
\sum_{|I| \geq 1} t^I \left( \sum_{j=1}^n \mathbb{H}(\mu_{j-I}, \eta_{i_1, \ldots, i_j-1, \ldots, i_n}) \right) = A(t)_{\alpha\beta}^\alpha \theta_p^\beta.
\]

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Meanwhile, let \( \pi_p \) be the \( B \) period matrix of \( \{ \theta^\alpha \}_{\alpha=1}^g \). Then the period map \( \Pi : \mathcal{T}_g \to \mathcal{H}_g \) to the Siegel upper half space can be written down explicitly:

\[
\Pi(t) = \begin{pmatrix} \pi_p & \pi_p \\ I_g & I_g \end{pmatrix} \circ A(t)^T,
\]

where the action \( \circ \) is given by

\[
\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}_{2g \times 2g} \circ Z = (C_1 Z + C_2)(C_3 Z + C_4)^{-1};
\]

The transition formula between \( \Pi(t) \) and \( \Pi(\tau) \) of two adjacent Kuranishi coordinates is

\[
\Pi(t) = L_{pq} \circ \Pi(\tau),
\]

where \( L_{pq} \) is defined at the end of the proof of Theorem 2.3.

Let \( \Gamma_g \) be the mapping class group of Riemann Surface of genus \( g \), which has a natural representation in the symplectic group \( \text{Sp}(g, \mathbb{Z}) \) with integral coefficients, written as \( \rho : \Gamma_g \to \text{Sp}(g, \mathbb{Z}) \). The moduli space \( \mathcal{M}_g \) of Riemann Surfaces of genus \( g \) is the quotient space of \( \mathcal{T}_g \) by \( \Gamma_g \), while \( \mathcal{A}_g = \mathcal{H}_g/\text{Sp}(g, \mathbb{Z}) \) is known as the moduli space of principally polarized abelian varieties. In Section 3 we first give a proof of the following two well-known local Torelli theorems by our deformation method.

**Theorem 1.1.** a) (Local Torelli Theorem 1) The period map \( \Pi : \mathcal{T}_g \to \mathcal{H}_g \) is an immersion on the non-hyperelliptic locus and also when restricted to the hyperelliptic locus for \( g \geq 3 \); while for \( g = 2 \), \( \Pi \) is an immersion on the whole \( \mathcal{T}_g \).

b) (Local Torelli Theorem 2) For \( g \geq 2 \), the period map \( \mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g \) is an immersion.

Write the quotient space of the Teichmüller space \( \mathcal{T}_g \) by the Torelli group \( T_g \) as \( \mathcal{T}_{\text{or}_g} \), which has a natural \( \mathbb{Z}_2 \) action. Recall that the Torelli group \( T_g \) is the kernel of the representation \( \rho : \Gamma_g \to \text{Sp}(g, \mathbb{Z}) \). Then we will present a new proof of the following global Torelli theorem in Section 4.

**Theorem 1.2.** \( \mathcal{J}^{\text{tor}} : \mathcal{T}_{\text{or}_g}/\mathbb{Z}_2 \to \mathcal{H}_g \) is an embedding for \( g \geq 3 \).

We also prove that the period map \( \Pi \) maps the \( \Gamma_g \) orbit of \( \Delta_{p,e} \) onto the \( \text{Sp}(g, \mathbb{Z}) \) orbit of its image in \( \mathcal{H}_g \). More precisely, let \( \Delta_{p,e} \) be a Kuranishi coordinate chart on \( \mathcal{T}_g \) and \( \Delta_{p,e}^{[\phi]} := [\phi] \Delta_{p,e} \) for \( [\phi] \in \Gamma_g \). Set

\[
\rho([\phi]) = \begin{pmatrix} U & V \\ R & S \end{pmatrix} \in \text{Sp}(g, \mathbb{Z}).
\]

Then on \( \Delta_{p,e}^{[\phi]} \), the period map \( \tilde{\Pi}(t) \) has the following relation with \( \Pi(t) \):

\[
\tilde{\Pi}(t) = \begin{pmatrix} S & R \\ V & U \end{pmatrix} \circ \Pi(t).
\]

Based on these, we prove that two \( \Gamma_g \) orbits of \( \mathcal{M}_g \), if mapped to the same \( \text{Sp}(g, \mathbb{Z}) \) orbit by \( \mathcal{J} \), must coincide, and thus prove the main result of this paper:

**Theorem 1.3** (Torelli Theorem). The period map \( \mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g \) is injective for \( g \geq 2 \).

The maps considered in this paper can be summarized in the following diagram:
It is well-known that the global Torelli theorem holds by R. Torelli’s result \cite{22} and also the modern proofs \cite{11,23} while the local Torelli holds due to the work of \cite{19}. A more complete list of the history about Torelli problems is contained in the bibliographical notes on Page 261 of \cite{3}.

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2. Kuranishi Coordinates On $\mathcal{T}_g$

We first recall some basics of the construction of Kuranishi coordinate charts, which is based on \cite{2}. Fix a compact topological surface $\Sigma$ of genus $g$ with $g \geq 2$. The pair $(C, [f])$ is a Riemann Surface $C$ with the Teichmüller structure $[f]$, where $f$ is an orientation-preserving homeomorphism from $C$ to $\Sigma$ and $[f]$ denotes the isotopic class represented by $f$. An isomorphism between Riemann Surfaces with the Teichmüller structures, $(C, [f])$ and $(C', [f'])$, is a biholomorphic map $\phi$ from $C$ to $C'$ such that $[f] = [f']$. The equivalence classes of all compact Riemann Surfaces of genus $g$ with the Teichmüller structure, modulo the isomorphism equivalences, actually constitute the Teichmüller space $\mathcal{T}_g$ of Riemann Surfaces of genus $g$. Thus an isomorphism class of $(C, [f])$ is a point in $\mathcal{T}_g$.

From the construction of Hilbert scheme, the existence of the Kuranishi family of Riemann Surfaces follows. To be more precise, for every Riemann Surface $C$, there exists a holomorphic deformation $(\varpi, \varphi)$

$$\varpi : \mathcal{X} \to B, \quad \varphi : C \simeq X_{b_0}$$

of $C$ parametrized by a pointed base $(B, b_0)$, a complex manifold with $\dim_C B = 3g - 3$, and this deformation is universal at $b_0$, actually universal at every point $b$ of $B$. The pair $(\varpi, \varphi)$ is called the Kuranishi family of $C$. For any other deformation $(t, \psi)$

$$t : \mathcal{X}' \to B', \quad \psi : C \simeq X'_{b_0}$$

of $C$, there exists a unique map $(\phi, \Phi)$ in a small neighborhood of $b'_0$ such that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\Phi} & \mathcal{X} \\
\downarrow{t} & & \downarrow{\varpi} \\
(B', b'_0) & \xrightarrow{\phi} & (B, b_0),
\end{array}
$$

where $\varphi^{-1} \Phi \psi = \mathbb{1}_C$ and $\mathcal{X}'$ is isomorphic to the pullback family $\Phi^* \mathcal{X}$ on the small neighborhood of $b'_0$. Accordingly, we also have a family of Riemann Surfaces with the Teichmüller structure $(X_b, [f_b])$, i.e., $\varpi : \mathcal{X} \to B$ together with local topological trivialization $F^\alpha : \mathcal{X}|_{U_\alpha} \to \Sigma \times U_\alpha$, where $\bigcup_\alpha U_\alpha$ is an open covering of $B$ such that $[F^\alpha_b] = [f_b]$ with $b \in U_\alpha$. For any Riemann Surface with the Teichmüller structure $(C, [f])$, Kuranishi family also exists and satisfies exactly analogous universal properties to the one without the Teichmüller structure above. Possibly shrinking $B$, we can describe the Kuranishi family of $(C, [f])$ as a triple $(\varpi, \varphi, F)$ given by

$$\varpi : \mathcal{X} \to B, \quad \varphi : C \simeq C_{b_0}, \quad F : \mathcal{X} \to \Sigma \times B,$$

where $F$ is a topological trivialization such that $F_{b_0} \varphi = f$.

A Kuranishi coordinate trivialization of $\mathcal{T}_g$ is given by

$$
(B, b_0) \to \mathcal{T}_g, \quad t \to [X_t, [F_t]],
$$
where the triple \((\varpi, \varphi, F)\) is the Kuranishi family of \((C, [f])\). From the classical Ehresmann’s theorem, there is a natural diffeomorphism \(\Psi : \Sigma \times B \rightarrow \mathcal{X}\); all the fibers of \(\varpi : \mathcal{X} \rightarrow B\)

\[
\Sigma \times B \xrightarrow{F} \mathcal{X} \xrightarrow{\Psi} \Sigma \times B
\]

share the same differential structure as \(\Sigma \times B\). From this point of view, for every \(b \in B\), the map \(F_b \Psi_b^{-1}\) can be deformed to \(F_b \Psi_{b,0}^{-1}\), i.e. \([F_b \Psi_b^{-1}] = [F_b \Psi_{b,0}^{-1}]\). Let \(\omega : H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}) \rightarrow \mathbb{Z}\) be the intersection pairing on \(\Sigma\). The symplectic basis of \(H_1(\Sigma, \mathbb{Z})\) on \((\Sigma, \omega)\) gives, from the map \(\Psi F^{-1}\), one such basis on \(\Sigma_b\), which is enjoyed by the whole Kuranishi family \(\mathcal{X}\) over the Kuranishi coordinate chart \(B\). Later on we will write \((B, b_0)\) as \(\Delta_{p, \epsilon}\), where \(p\) denotes the point \([C, [f]]\) in \(T_g\), and \(\Delta_{p, \epsilon} = \{ t \in \mathbb{C}^n \| t \| < \epsilon, t(p) = 0 \}\) with \(n = 3g - 3\).

Fix the representation \(\rho : \Gamma_g \rightarrow \text{Sp}(g, \mathbb{Z})\), where \(\Gamma_g\) is the mapping class group, namely the isotopic classes of orientation preserving homeomorphisms of \(\Sigma\), and \(\text{Sp}(g, \mathbb{Z})\) is actually \(\text{Aut}(H_1(\Sigma, \mathbb{Z}), \omega)\). Now we have two Kuranishi coordinate charts \(\Delta_{p, \epsilon}\) and \(\Delta_{q, \epsilon'}\) with \(\Delta_{p, \epsilon} \cap \Delta_{q, \epsilon'} \neq \emptyset\). Let \((\mathcal{X}, F)\) and \((\mathcal{Y}, G)\) denote the two Kuranishi families with Teichmüller structures over \(\Delta_{p, \epsilon}\) and \(\Delta_{q, \epsilon'}\), respectively. Let \(r \in \Delta_{p, \epsilon} \cap \Delta_{q, \epsilon'}\). The definition of Kuranishi coordinates tells us that \([X_t(r), F_t(r)] = [Y_t(r), G_t(r)]\). Then we have a biholomorphic map \(\phi : X_t(r) \rightarrow Y_t(r)\) such that \([F_t(r)] = [G_t(r)]\phi\). It is described in the following picture

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{F_b} & \Sigma \\
\downarrow \quad \Psi_X & \quad & \quad \downarrow \quad \Psi_Y \\
X_{t(r)} & \xrightarrow{\phi} & Y_{t(r)} \\
\end{array}
\]

that \([G_0 \Psi_Y \phi \Psi_X F_0^{-1}]\) gives us an element of \(\Gamma_g\). From the representation \(\rho\), a matrix in \(\text{Sp}(g, \mathbb{Z})\) is obtained, linking the two symplectic bases of the two Kuranishi coordinates.

2.1. Small Deformation Of Holomorphic One-Forms. Let \(\Delta_{p, \epsilon}\) be a Kuranishi coordinate chart centered at \(p \in T_g\) as above. Denote the corresponding Kuranishi family on \(\Delta_{p, \epsilon}\) by \(\varpi : \mathcal{X} \rightarrow \Delta_{p, \epsilon}\) with the central fiber \(\varpi^{-1}(p) = X_p\). Let \(\theta \in H^0(X_p, \Omega^1_{X_p})\) be a global holomorphic one-form on \(X_p\). We will construct an explicit formula \(\theta(t) \in H^0(X_t, \Omega^1_{X_t})\), the holomorphic deformation of \(\theta\).

Denote the Poincaré metric on \(X_p\) by \(\omega_p\). Fix \(\{\mu_i\}_{i=1}^n\) as a basis of harmonic \(T_{X_p}(1,0)\)-valued \((0,1)\)-form on \((X_p, \omega_p)\), namely \(\mathbb{H}^{0,1}(X_p, T^1_{X_p})\). And let \(\mu(t) = \sum_{i=1}^n t_i \mu_i\) denote the Beltrami differential of the Kuranishi family \(\varpi : \Sigma \xrightarrow{\Phi} \Delta_{p, \epsilon}\).

**Theorem 2.1.** Given \(\theta \in H^0(X_p, \Omega^1_{X_p})\), there exists a unique \((1,0)\)-form \(\eta(t)\) on \(X_p\), which is holomorphic in \(t\) for sufficiently small \(t\), satisfying

1. \(\mathbb{H}(\eta(t)) = \theta\), where \(\mathbb{H}\) is the harmonic projection on \((X_p, \omega_p)\),
2. \(\theta(t) = (1 + \mu(t)) \cdot \eta(t) \in H^0(X_t, \Omega^1_{X_t})\)

and \(\theta(t)\) is the desired deformation of \(\theta\).

**Proof.** The formal power series of \(\eta(t) \in A^{1,0}(X_p)\) can be written out as

\[
\eta(t) = \theta + \sum_{i=1}^n t_i \eta_i + \sum_{|I| \geq 2} t^I \eta_I,
\]

where \(I = (i_1, \ldots, i_n), t^I = t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}\) and \(|I| = \sum_{j=1}^n i_j\).

Condition (1) implies

\[
\begin{align*}
\mathbb{H}(\eta_I) &= 0, \\
\mathbb{H}(\eta_I) &= 0, \quad |I| \geq 2.
\end{align*}
\]
Then, one has
\[
\theta(t) = (1 + \mu(t)) \eta(t)
\]
\[= \left(1 + \sum_{i=1}^{n} t_i \mu_i \right) \left(\theta + \sum_{j=1}^{n} t_j \eta_j + \sum_{|J| \geq 2} t^J \eta_J \right)
\]
\[= \theta + \sum_{i=1}^{n} t_i (\eta_i + \mu_i \theta) + \sum_{|J| \geq 2} t^J \left(\eta(i_1, \ldots, i_n) + \sum_{k=1}^{n} \mu_k \eta(i_1, \ldots, i_{k-1}, \ldots, i_n)\right).
\]
Since \(\theta(t)\) is a holomorphic one-form on \(X_t\) from Condition (2), i.e., \(d\theta(t) = 0\), which implies
\[
\begin{cases}
    d(\eta_i + \mu_i \theta) = 0, \\
    d(\eta(i_1, \ldots, i_n) + \sum_{k=1}^{n} \mu_k \eta(i_1, \ldots, i_{k-1}, \ldots, i_n)) = 0,
\end{cases}
\]
we see that
\[
\begin{align*}
\bar{\partial} \eta_i + \partial(\mu_i \theta) & = 0, \\
\bar{\partial} \eta(i_1, \ldots, i_n) + \partial(\sum_{k=1}^{n} \mu_k \eta(i_1, \ldots, i_{k-1}, \ldots, i_n)) & = 0.
\end{align*}
\]
Combining with (2.1) and solving the \(\bar{\partial}\)-equation, we get
\[
\begin{align*}
\eta_i & = -G\bar{\partial} \partial(\mu_i \theta), \\
\eta(i_1, \ldots, i_n) & = -G\bar{\partial} \partial(\sum_{k=1}^{n} \mu_k \eta(i_1, \ldots, i_{k-1}, \ldots, i_n)).
\end{align*}
\]
Here \(G\) denotes the Green operator in the Hodge decomposition with respect to the operator \(\bar{\partial}\) and \(1 = H + (\bar{\partial} \partial + \bar{\partial} \partial) G\). Thus we have proved the uniqueness of \(\eta(t)\), which is fixed by conditions (1) and (2).

Now let us discuss the convergence of the power series constructed above. By the standard estimates of elliptic operators \(G, \bar{\partial}\) and \(\partial\), such as in [17], we easily have
\[
\|\eta_I\|_{m+\alpha} \leq C^{|I|} \|\theta\|_{m+\alpha},
\]
where the constant \(C\) depends on \(m, \alpha\) and \(X_p\), and \(\| \cdot \|_{m+\alpha}\) is the Hölder norm. Consequently the estimates of \(\eta(t)\) yield
\[
\|\eta(t)\|_{m+\alpha} \leq \|\theta\|_{m+\alpha} + \|\theta\|_{m+\alpha} \sum_{|I| \geq 1} C^{|I|} \epsilon^{|I|}
\]
\[= \|\theta\|_{m+\alpha} + \|\theta\|_{m+\alpha} \sum_{k \geq 1} \sum_{|I| \geq k} C^{|I|} \epsilon^{|I|}
\]
\[= \|\theta\|_{m+\alpha} + \|\theta\|_{m+\alpha} \sum_{k \geq 1} C^k \epsilon^k C_{n+k-1}^k
\]
\[\leq \|\theta\|_{m+\alpha} + \|\theta\|_{m+\alpha} \sum_{k \geq 1} C^k \epsilon^k n^k,
\]
where \(C_{n+k-1}^k\) is the common combinatorial number. By taking \(\epsilon\) smaller than \(\frac{1}{2nC}\), we are done.

**Corollary 2.2.** The deformation formula of \(\theta\), with \(t\) small, is given by
\[
\theta(t) = \theta + \sum_{i=1}^{n} t_i (\mathbb{H}(\mu_i \theta) + df_i) + \sum_{|J| \geq 2} t^J \left(\sum_{j=1}^{n} \mathbb{H}(\mu_j \eta(i_1, \ldots, i_{j-1}, \ldots, i_n)) + df_{j, (i_1, \ldots, i_{j-1}, \ldots, i_n)}\right)
\]
where \(f_{j, (i_1, \ldots, i_{j-1}, \ldots, i_n)} \in C^\infty(X_p)\).
Proof. From Theorem 2.1 we can easily write out
\[
\theta(t) = (1 + \mu(t)) \cdot \eta(t)
\]
\[
= \left(1 + \sum_{i=1}^{n} t_i \eta_i \right) \left(\theta - \sum_{j=1}^{n} t_j \left(\overline{G^*} \partial (\mu_j \theta) + \sum_{|I| \geq 2} t^I \eta_I\right)\right)
\]
\[
= \theta + \sum_{|I| \geq 1} \left(\sum_{j=1}^{n} \left(\overline{G^*} \partial (\mu_j \eta_{(i_1, \ldots, i_{j-1}, \ldots, i_n)}) + \overline{f}_{j, (i_1, \ldots, i_{j-1}, \ldots, i_n)}\right)ight)
\]
\[
= \theta + \sum_{|I| \geq 1} \left(\sum_{j=1}^{n} \left(\overline{G^*} \partial (\mu_j \eta_{(i_1, \ldots, i_{j-1}, \ldots, i_n)}) + \overline{f}_{j, (i_1, \ldots, i_{j-1}, \ldots, i_n)}\right) df_{j, (i_1, \ldots, i_{j-1}, \ldots, i_n)}\right).
\]
The convergence follows from Theorem 2.1. q.e.d.

Remark 2.3. The iteration method to construct canonical forms on the deformation space of Riemann Surfaces is essentially contained in [10] and [24, Theorem 2.1]. Note that [15] contains a generalization to Kähler manifolds while our proof emphasizes on the uniqueness of the construction.

Denote the canonical basis of $H^0(X_p, \Omega^1_{X_p})$ by $\{\theta_p^\alpha\}_{\alpha=1}^g$ with respect to the symplectic basis $\{A^\alpha, B^\alpha\}_{\alpha=1}^g$ on the Kuranishi coordinate chart $\Delta_{p,c}$. Here a canonical basis means the unique basis of $H^0(X_p, \Omega^1_{X_p})$ such that its A period matrix is $\mathbb{I}_g$. Let $\sigma_p$ and $\pi_p$ be the A and B period matrices of $\{\theta_p^\alpha\}_{\alpha=1}^g$, respectively, and $M_p = \text{Im}(\pi_p)$. Applying the deformation formula above, we get the holomorphic one-forms $\theta_p^\alpha(t)$ on $X_t$, starting with $\theta_p^\alpha$, given by
\[
(2.4)
\]
\[
\theta_p^\alpha(t) = \theta_p^\alpha + \sum_{i=1}^{n} t_i \left(\overline{H} (\mu_i \theta_p^\alpha) + df_i^\alpha\right) + \sum_{|I| \geq 2} \left(\sum_{j=1}^{n} \left(\overline{H} (\mu_j \eta_{(i_1, \ldots, i_{j-1}, \ldots, i_n)}) + df^\alpha_{j, (i_1, \ldots, i_{j-1}, \ldots, i_n)}\right)\right).
\]

Definition 2.4. Let $A(t)$ be a $g \times g$ matrix and $E(t)$ a $g \times 1$ vector given by:
\[
\begin{align*}
\sum_{|I| \geq 1} t^I \left(\sum_{j=1}^{n} \overline{H} (\mu_j \eta_{(i_1, \ldots, i_{j-1}, \ldots, i_n)})\right) &= A(t) \cdot \theta_p^\beta, \\
\sum_{|I| \geq 1} t^I \left(\sum_{j=1}^{n} df^\alpha_{j, (i_1, \ldots, i_{j-1}, \ldots, i_n)}\right) &= E^\alpha(t).
\end{align*}
\]
Also the homogeneous part of order $N$ of $A(t)$ is written as $A_N(t) = \sum_{|I| = N} t^I A_I$,
\[
\sum_{j=1}^{n} \left(\overline{H} (\mu_j \eta_{(i_1, \ldots, i_{j-1}, \ldots, i_n)})\right) = A_{ij} \theta_p^\beta.
\]
In particular, $\overline{H} (\mu_i \theta_p^\alpha) = A_{ij} \theta_p^\beta$.

Set
\[
\Theta_p(t) = \begin{pmatrix} \theta^1_p(t) \\ \vdots \\ \theta^g_p(t) \end{pmatrix} \quad \text{and} \quad \Theta_p = \begin{pmatrix} \theta^1_p \\ \vdots \\ \theta^g_p \end{pmatrix}.
\]
Thus by use of $A(t)$ and $E(t)$, we rewrite (2.4) as
\[
(2.5) \quad \Theta_p(t) = \left(1_g - A(t)\right) \Theta_p + E(t).
\]
Since a holomorphic one-form on Riemann Surfaces is uniquely determined by its integration on \( A \) cycles, it is clear that \( \{ \theta^\alpha_p(t) \}_{\alpha=1}^g \) being a frame of \( H^0(X_t, \Omega^1_{X_t}) \) on \( X_t \), is equivalent to non-degeneration of the \( A \) period matrix \( \sigma_{\alpha\beta}(t) \) on \( X_t \), i.e.,

\[
\text{det} (\sigma_{\alpha\beta}(t)) = \text{det} \left( \int A_t \theta^\alpha_p(t) \right) \neq 0 \iff \text{det} (I_g + A(t)^T) \neq 0, \tag{2.6}
\]

where \( A(t)^T \) is the transpose of \( A(t) \). And when \( \{ \theta^\alpha_p(t) \}_{\alpha=1}^g \) becomes a frame, we have the Hodge-Riemann bilinear relations on \( X_t \)

\[
\begin{cases}
0 = \frac{1}{2} \int_{\tilde{X}_t} \theta^\alpha_p(t) \wedge \theta^\beta_p(t), \\
0 < \frac{1}{2} \int_{\tilde{X}_t} \theta^\alpha_p(t) \wedge \bar{\theta}^\beta_p(t),
\end{cases}
\]

which and also (2.5) imply that

\[
\begin{cases}
0 = \frac{1}{2} \int_{X_p} (\theta^\alpha_p + A(t)\bar{\theta}^\gamma_p + E^\alpha(t)) \wedge (\theta^\beta_p + A(t)\bar{\theta}^\lambda_p + E^\beta(t)), \\
0 < \frac{1}{2} \int_{X_p} (\theta^\alpha_p + A(t)\bar{\theta}^\gamma_p + E^\alpha(t)) \wedge (\bar{\theta}^\beta_p + A(t)\bar{\theta}^\lambda_p + E^\beta(t)),
\end{cases}
\]

and thus

\[
\begin{cases}
0 = \Delta_{\alpha\gamma} M_p \gamma \Delta_{\alpha\gamma} A(t)_\gamma, \\
0 < \Delta_{\alpha\beta} - \Delta_{\alpha\gamma} A(t)_\gamma \Delta_{\alpha\gamma}.
\end{cases}
\]

The matrix forms of these are given by

\[
\begin{cases}
A(t) M_p = (A(t) M_p)^T, \\
M_p - A(t) M_p A(t)^T > 0.
\end{cases}
\]

As our deformation formula is local, \( \{ \theta^\alpha_p(t) \}_{\alpha=1}^g \) is always a frame when \( t \in \Delta_{p,\epsilon} \) with \( \epsilon \) sufficiently small. Therefore, (2.6) and (2.7) hold.

### 2.2. Transition Formulas Between the Kuranishi Coordinates.

**Theorem 2.5.** Assume that the two Kuranishi coordinate charts \( \Delta_{p,\epsilon} \) and \( \Delta_{q,\epsilon'} \) have a non-empty intersection containing those two centers \( p \) and \( q \), and let \( t \) and \( \tau \) denote the corresponding Kuranishi coordinates. Then \( A(t) \) and \( A(\tau) \) are related by the following equality:

\[
A(t)^T = \begin{pmatrix} \pi_p & \pi_p \\ \pi_q & \pi_q \end{pmatrix}^{-1} L_{pq} \begin{pmatrix} \pi_p & \pi_p \\ \pi_q & \pi_q \end{pmatrix} \circ A(\tau)^T, \tag{2.8}
\]

where \( L_{pq} \in \text{Sp}(g, \mathbb{Z}) \) denotes the transition matrix between the symplectic bases of the two Kuranishi coordinates in terms of transformations in \( \text{Sp}(g, \mathbb{Z}) \) of \( \mathcal{H}_g \), and the action \( \circ \) is given by

\[
\left( \begin{array}{cc}
C_1 & C_2 \\
C_3 & C_4
\end{array} \right) \circ Z = (C_1 Z + C_2)(C_3 Z + C_4)^{-1}.
\]

Observe that the transition matrix linking \( A(t) \) and \( A(\tau) \) depends only on \( p \) and \( q \), but not the coordinates \( t \) and \( \tau \).

**Proof.** From (2.3), it yields

\[
[\Theta_p(q)] = (I_g \ A(t(q))) \left( \begin{array}{c} \Theta_p \\ \Theta_p \end{array} \right), \tag{2.9}
\]

where \( [\Theta_p(q)] \) denotes the cohomology class represented by \( \Theta_p(q) \). The frames given by the deformation formula \( [\Theta_p(q)] \) and the canonical one \( [\Theta_q] \) at \( q \) are different by a multiple of a nonsingular matrix \( C \):

\[
[\Theta_q] = C \ [\Theta_p(q)]. \tag{2.10}
\]
Let \( \{ A_{\gamma}, B_{\gamma} \}_{\gamma=1}^g \) and \( \{ A'_{\gamma}, B'_{\gamma} \}_{\gamma=1}^g \) be the symplectic bases on \( \Delta_{p,\epsilon} \) and \( \Delta_{q,\epsilon'} \), respectively. Set

\[
\begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
A_1 \\
\vdots \\
A_g \\
B_1 \\
\vdots \\
B_g
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
A' \\
B'
\end{pmatrix} = \begin{pmatrix}
A'_1 \\
\vdots \\
A'_g \\
B'_1 \\
\vdots \\
B'_g
\end{pmatrix}.
\]

Denote the matrix linking these two bases by \( \begin{pmatrix} U & V \\ R & S \end{pmatrix} \in \text{Sp}(g, \mathbb{Z}) \), i.e.,

\[
(2.11) \quad \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} U & V \\ R & S \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix}.
\]

By \( (2.11) \), we integrate over \( A \) cycles and \( B \) cycles on \( (2.10) \) to get

\[
\begin{align*}
U + V \pi_q &= \left( \mathbb{1}_g + A(t(q))^T \right) C^T, \\
R + S \pi_q &= \left( \pi_p + \bar{\pi}_p A(t(q))^T \right) C^T,
\end{align*}
\]

which imply that

\[
(2.12) \quad \begin{cases}
C^T = \left( \mathbb{1}_g + A(t(q))^T \right)^{-1} (U + V \pi_q), \\
\left( \pi_p + \bar{\pi}_p A(t(q))^T \right) \left( \mathbb{1}_g + A(t(q))^T \right)^{-1} = (R + S \pi_q) (U + V \pi_q)^{-1}.
\end{cases}
\]

By \( (2.9) \) and \( (2.10) \), we have

\[
[\Theta_q] = C \left( \mathbb{1}_g \ A(t(q)) \right) \begin{pmatrix} [\Theta_q] \\ [\Theta_p] \end{pmatrix} = \begin{pmatrix} [\Theta_q] \\ [\Theta_p] \end{pmatrix}, \tag{2.13}
\]

Let \( r \in \Delta_{p,\epsilon} \cap \Delta_{q,\epsilon'} \). Then one has

\[
[\Theta_p(r)] = \left( \mathbb{1}_g \ A(t) \right) \begin{pmatrix} [\Theta_q] \\ [\Theta_p] \end{pmatrix} ;
\]

while by \( (2.13) \), one also has

\[
[\Theta_p(r)] = C_r [\Theta_q(r)] = C_r \left( \mathbb{1}_g \ A(\tau) \right) \begin{pmatrix} [\Theta_q] \\ [\Theta_p] \end{pmatrix} = C_r \left( \mathbb{1}_g \ A(\tau) \right) \begin{pmatrix} C \ C A(t(q)) \ C A(t(q)) \ C \end{pmatrix} \begin{pmatrix} [\Theta_q] \\ [\Theta_p] \end{pmatrix},
\]

where the two frames \( [\Theta_p(r)] \) and \( [\Theta_q(r)] \) at the point \( r \) are related by a nonsingular matrix \( C_r \). These give us the following identities:

\[
\begin{pmatrix}
\mathbb{1}_g \\
A(t)
\end{pmatrix} = C_r \left( \mathbb{1}_g \ C A(t(q)) + A(\tau) \bar{C} \right) .
\]

Combine with \( (2.12) \) to simplify the computation as follows:

\[
A(t)^T = \left( \begin{pmatrix} \bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} \pi_p + \bar{\pi}_p A(t(q))^T \mathbb{1}_g + A(t(q))^T \bar{C} \end{pmatrix} \right) \left( \begin{pmatrix} \bar{\pi}_q & \pi_q \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix} \right) A(\tau)^T \]

\[
= \left( \begin{pmatrix} \bar{\pi}_p & \pi_p \\ \mathbb{1}_g & \mathbb{1}_g \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} S & R \\ V & U \end{pmatrix} \right) \left( \begin{pmatrix} \pi_q & \bar{\pi}_q \end{pmatrix} \right) A(\tau)^T .
\]
where \( \begin{pmatrix} S & R \\ V & U \end{pmatrix} \) also belongs to \( \text{Sp}(g, \mathbb{Z}) \), denoted by \( L_{pq} \).

q.e.d.

On our Kuranishi coordinate chart \( \Delta_{p,e} \), the period map \( \Pi : \mathcal{T}_g \to \mathcal{H}_g \) can be written out quite explicitly:

\[
\Pi(t)_{\alpha\beta} = \int_{B_n} \sigma(t) T_{a\beta} (t) = \int_{B_n} \sigma(t)(T_{a\gamma} + A(t)T_{b\beta}) = \pi_{p,a\gamma}(t) T_{b\beta} + \pi_{p,a\delta} A(t)T_{b\beta},
\]

where \( \sigma(t)^{a\beta} \) is the inverse matrix of \( \sigma(t)_{a\beta} \). \((2.6)\) gives us

\[
\sigma_{a\beta}(t) = \int_{A_n} \theta_{a\beta} = (1 + A(t)^T)_{a\beta}.
\]

Now we can formulate these into the matrix form:

\[
\Pi(t) = (\pi_p + \pi_p A(t)^T) (1 + A(t)^T)^{-1}
\]

\[(2.14)\]

\[
\Pi(t) = L_{pq} \Pi(\tau).
\]

\textbf{Corollary 2.6.} The period maps \( \Pi(t) \) and \( \Pi(\tau) \) on the intersection of the two Kuranishi coordinate charts \( \Delta_{p,e} \) and \( \Delta_{q,e'} \) have the following transition formula:

\[(2.15)\]

\[
\Pi(t) = L_{pq} \Pi(\tau).
\]

Proof. By \[(2.14)\] and Theorem 2.5, we have

\[
\Pi(t) = \left( \begin{array}{cc} \pi_p & \pi_p \\ 1_g & 1_g \end{array} \right) \Pi(\tau)
\]

\[
= \left( \begin{array}{cc} \pi_p & \pi_p \\ 1_g & 1_g \end{array} \right) \left( \begin{array}{cc} \pi_q & \pi_q \\ 1_g & 1_g \end{array} \right)^{-1} L_{pq} \left( \begin{array}{cc} \pi_q & \pi_q \\ 1_g & 1_g \end{array} \right)^{-1} \Pi(\tau)
\]

\[
= L_{pq} \Pi(\tau).
\]

q.e.d.

\section{3. Local Torelli Theorems and Matrix Model}

\textbf{Theorem 3.1.} (Local Torelli Theorem 1) For \( g \geq 3 \), the period map \( \Pi : \mathcal{T}_g \to \mathcal{H}_g \) is an immersion on the non-hyperelliptic locus \( \mathcal{T}_g \setminus \mathcal{HET}_g \) and also on the hyperelliptic locus \( \mathcal{HET}_g \). In the case \( g = 2 \), \( \Pi \) is an immersion on the whole \( \mathcal{T}_g \).

Proof. From \[(2.14)\], the period map can be written as \( \Pi(t) = (\pi_p A(t)^T + \pi_p) (A(t)^T + 1_g)^{-1} \) via Kuranishi coordinates. By use of \( A(t) \), we expand it to obtain the first order part \( \Pi^{(1)}(t) \) of \( \Pi(t) \):

\[
\Pi^{(1)}(t) = \pi_p A_{T}^T (t) - \pi_p A_{T}^T (t)
\]

\[
= -2\sqrt{-1} M_{p,a\gamma} A_{b\gamma}
\]

\[(3.1)\]

\[
= \sum_{i=1}^{n} t_i \int_{X_p} \theta_{p\gamma} \wedge H_{(\mu_i, \theta_p^\gamma)}
\]

\[
= \sum_{i=1}^{n} t_i \int_{X_p} \theta_{p\gamma} \wedge (\mu_i, \theta_p^\gamma).
\]
It is a well-known fact that the pairing \( \langle \theta^0, \theta^1 \rangle \) is non-degenerate. The matrices \( \{ J_X \} \) are linearly dependent if and only if there exists a nonzero vector \( t = (t_1, \ldots, t_n) \) such that the matrix

\[
\int_{X_p} \theta^0 \wedge \left( \sum_{i=1}^n t_i \mu_i \right) \partial \theta^1 = 0.
\]

This is equivalent to the multiplication map \( H^0(X_p, K_{X_p}) \times H^0(X_p, K_{X_p}) \to H^0(X_p, 2K_{X_p}) \) is surjective.

A well-known theorem by Max Noether in [4, P. 117] tells us the multiplication map \( H^0(X_p, K_{X_p}) \times H^0(X_p, K_{X_p}) \to H^0(X_p, 2K_{X_p}) \) is always surjective when \( X_p \) is non-hyperelliptic. Thus the period map \( \Pi \) is an immersion when restricted to \( T_g - HE T_g \) for \( g \geq 3 \). As to the hyperelliptic case described in [7, P. 104], the image of the multiplication map is exactly the vector space \( H^0(X_p, 2K_{X_p}) \) when \( g = 2 \) and the above multiplication map is surjective since \( 2g - 1 = 3g - 3 \) when \( g = 2 \). Consequently, \( \Pi \) is an immersion on \( T_g \) for \( g = 2 \).

**Definition 3.2.** \( T_g, \bar{T}_g \) and \( \text{Tor}_g \).

\( T_g \), called the Torelli group, is the kernel of the representation \( \rho : \Gamma_g \to \text{Sp}(g, \mathbb{Z}) \) while the extended Torelli group \( \bar{T}_g \) is defined to be \( \rho^{-1}(\{1 \}) \) where \( \{1 \} \) is the subgroup of \( \text{Sp}(g, \mathbb{Z}) \) generated by \( -1_{2g} \). The Torelli space \( \text{Tor}_g \) is the quotient space of the Teichmüller space \( T_g \) by \( T_g \).

**Definition 3.3.** \( \Gamma_g(n) \) and \( \mathcal{M}_g^{(n)} \).

\( \Gamma_g(n) \), the level \( n \) subgroup of the mapping class group \( \Gamma_g \), is the kernel of the representation \( \Gamma_g \to \text{Sp}(g, \mathbb{Z}) \) via \( \text{Aut}(X_p) \). \( \mathcal{M}_g^{(n)} \) is the moduli space of Riemann Surfaces of genus \( g \) with level \( n \) structure, which is defined as the quotient space of the Teichmüller space \( T_g \) by the group action of \( \Gamma_g(n) \). And we identify \( \Gamma_g(1) \) with \( T_g \).

As we know, the action of the mapping class group \( \Gamma_g \) on the Teichmüller space \( T_g \) is properly discontinuous. From the construction of the Kuranishi coordinate of \( T_g \) in [2], we know that the isotropy group \( \Gamma_p^g \) of \( \Gamma_g \) at \( p = [X_p, [f_p]] \) on \( T_g \) is \( \text{Aut}(X_p) \) if we fix the injective homomorphism

\[
\text{Aut}(X_p) \to \Gamma_g^p \to [f_p, f_p^{-1}].
\]

Moreover, we can choose \( \epsilon \) and \( \epsilon' \) sufficiently small such that the points \( p \) and \( p' \) in different \( \Gamma_g \) orbits have disjoint Kuranishi coordinates, i.e., \( \Delta_{p, \epsilon} \cap \Delta_{p', \epsilon'} = \emptyset \), and

\[
\{ \gamma \in \Gamma_g^p | \gamma \Delta_{p, \epsilon} \cap \Delta_{p', \epsilon'} \neq \emptyset \} = \Gamma_g^p.
\]

**Proposition 3.4.** The action of \( T_g \) and \( \Gamma_g(n) \) with \( n \geq 3 \) on \( T_g \) is fixed point free.

This proposition implies that \( \text{Tor}_g \) and \( \mathcal{M}_g^{(n)} \) with \( n \geq 3 \) are complex manifolds of complex dimension \( 3g - 3 \).

**Proof.** We just need to show that \( T_g \cap \Gamma_g^p = \{1\} \) and \( \Gamma_g(n) \cap \Gamma_g^p = \{1\} \). But we can identify \( \Gamma_g^p \) with \( \text{Aut}(X_p) \). It follows from the theory of automorphism groups of Riemann Surfaces in [7, Chapter V] that the representation of \( \text{Aut}(X_p) \) in \( H_1(X_p, \mathbb{Z}) \) and \( H_1(X_p, \mathbb{Z}_n) \) with \( n \geq 3 \) are faithful, i.e., the homomorphisms \( \text{Aut}(X_p) \to \text{Sp}(g, \mathbb{Z}) \) and \( \text{Aut}(X_p) \to \text{Sp}(g, \mathbb{Z}_n) \) are injective. Now the isotropy group \( \Gamma_g^p \) embeds into \( \text{Sp}(g, \mathbb{Z}) \) by the representation \( \rho : \Gamma_g \to \text{Sp}(g, \mathbb{Z}) \) if we view \( \Gamma_g^p \) as \( f_p^{-1} \text{Aut}(X_p) f_p \) while \( T_g \) is the kernel of \( \rho \). Thus \( T_g \cap \Gamma_g^p = \{1\} \). Similarly, \( \Gamma_g^p \) embeds...
into $\text{Sp}(g,\mathbb{Z})$ by the representation $\Gamma_g \overset{p}{\rightarrow} \text{Sp}(g,\mathbb{Z}) \overset{\pi}{\rightarrow} \text{Sp}(g,\mathbb{Z}_n)$, and $\Gamma_g(n)$ is the kernel of the representation $\pi p$. Finally we have $\Gamma_g(n) \cap \Gamma_g^g = \{1\}$. q.e.d.

From the discussion above, we can shrink our Kuranishi coordinate chart $\Delta_{p,\epsilon}$ on $T_g$ such that $\gamma \Delta_{p,\epsilon} \cap \Delta_{p,\epsilon} = \emptyset$ for any $\gamma \in T_g$ and $\gamma \neq 1$. Naturally, the Kuranishi coordinate chart $\Delta_{p,\epsilon}$ descends to $\mathcal{T}_{\text{tor}} g$. Let $\mathbb{Z}_2 \cong \widetilde{T_g}/T_g$ and then $\mathcal{T}_{\text{tor}} g$ has a natural $\mathbb{Z}_2$ action. There is a commutative diagram

$$
\begin{array}{ccc}
\Delta_{p,\epsilon} & \overset{\Pi}{\rightarrow} & T_g \\
\downarrow & & \downarrow \\
\mathcal{T}_{\text{tor}} g & \underset{\phi_f}{\rightarrow} & \mathcal{H}_g.
\end{array}
$$

**Lemma 3.5.** Let $X$ be a compact Riemann Surface with genus $g \geq 2$ and $J$ an involution on $X$, which does not fix any element of $H^0(X,K_X)$. Then $X$ is hyperelliptic and $J$ must be a hyperelliptic involution.

**Proof.** Since $J^2 = 1$, the automorphism $J^* : H^0(X,K_X) \rightarrow H^0(X,K_X)$ has two eigenvalue $\pm 1$. As $J^*$ fixes no element of $H^0(X,K_X)$, $J^* = -\mathbb{1}_2 g$ on $H^0(X,K_X)$. Consider the quotient map $\pi : X \rightarrow X/J$, a $2 : 1$ branched covering map, and $\pi = J \pi$. We claim that $g(X/J) = 0$. If not, there exists a nonzero holomorphic one-form $\theta \in H^0(X/J,K_X)$. Pulling it back, we derive a nonzero holomorphic one-form $\pi^* \theta \in H^0(X,K_X)$. But $\pi^* \theta$ is invariant under $J^*$, which is a contradiction. Thus $X/J$ is the Riemann sphere and $\pi$ is a degree 2 meromorphic function on $X$, which implies that $X$ is hyperelliptic and $J$ is a hyperelliptic involution. q.e.d.

**Proposition 3.6.** $\mathbb{Z}_2$ acts freely on the non-hyperelliptic locus $\mathcal{T}_{\text{tor}} g - \mathcal{H}ET_{\text{tor}} g$ of $\mathcal{T}_{\text{tor}} g$ and fixes every point in the hyperelliptic locus $\mathcal{H}ET_{\text{tor}} g$ for $g \geq 3$. In the case $g = 2$, $\mathbb{Z}_2$ acts trivially on $\mathcal{T}_{\text{tor}} g$.

**Proof.** Let $\{[\phi]\}$ be the non-unit element in $\mathbb{Z}_2$, where $[\phi] \in \Gamma_g$ is a representative of the class $\{[\phi]\}$ and $\rho([\phi]) = -\mathbb{1}_2 g$. Then that $\{[\phi]\}$ fixes a point $\{[X_p,[f_p]]\}$ in $\mathcal{T}_{\text{tor}} g$ is equivalent to that there exists some element $[\psi] \in T_g$ such that $[X_p,[\phi f_p]] = [X_p,[\psi f_p]]$. We have a commutative diagram up to isotopy

$$
\begin{array}{ccc}
X_p & \overset{h}{\rightarrow} & X_p \\
\downarrow & & \downarrow \\
\Sigma & \overset{\phi f_p}{\rightarrow} & \psi f_p
\end{array}
$$

where $h$ is an automorphism of $X_p$. Hence $h \simeq f_p^{-1} \psi^{-1} \phi f_p$. As $\rho([\psi]) = \mathbb{1}_2 g$ and $\rho([\phi]) = -\mathbb{1}_2 g$, $h : H_1(X_p,\mathbb{Z}) \rightarrow H_1(X_p,\mathbb{Z})$ is nothing but $-\mathbb{1}_2 g$. Since a holomorphic one-form is uniquely determined by its integration on $A$ cycles and

$$
\int_{A_n} h^* \theta = \int_{h(A_n)} \theta = -\int_{A_n} \theta,
$$

$h^* : H^0(X_p,K_{X_p}) \rightarrow H^0(X_p,K_{X_p})$ is $-\mathbb{1}_g$. Also the representation of $\text{Aut}(X_p)$ to $H_1(X_p,\mathbb{Z})$ is faithful and hence $h$ is an involution. From Lemma 3.5, $h$ is a hyperelliptic involution and $X_p$ is hyperelliptic. q.e.d.

It is easy to check that the $\mathbb{Z}_2$ orbit of $\mathcal{T}_{\text{tor}} g$ has the same image under $\mathcal{J}_{\text{tor}}$, since we also have Kuranishi coordinate on $\mathcal{T}_{\text{tor}} g$ and use (2.11). Consequently, $\mathcal{J}_{\text{tor}}$ factors through $\mathcal{T}_{\text{tor}} g/\mathbb{Z}_2$:

$$
\begin{array}{ccc}
\mathcal{T}_{\text{tor}} g & \overset{\mathcal{J}_{\text{tor}}}{\rightarrow} & \mathcal{H}_g \\
\downarrow & & \downarrow \\
\mathcal{T}_{\text{tor}} g/\mathbb{Z}_2 & \overset{\mathcal{J}_{\text{tor}}}{\rightarrow} & \mathcal{H}_g.
\end{array}
$$

From Proposition 3.6, $\mathcal{T}_{\text{tor}} g \rightarrow \mathcal{T}_{\text{tor}} g/\mathbb{Z}_2$ is a $2 : 1$ branched covering map branching over the hyperelliptic locus $\mathcal{H}ET_{\text{tor}} g$ for $g \geq 3$. Meanwhile, the Kuranishi coordinate chart $\Delta_{p,\epsilon}$,
$p \in Tor_g - \mathcal{HET} Tor_g$, also descends to $\mathcal{T} \mathcal{O}r_g/\mathbb{Z}_2$. When $p \in \mathcal{HET} Tor_g$, we can view the Kuranishi coordinate $\Delta_{p,\theta}$ on $\mathcal{T} \mathcal{O}r_g$ as follows: $\mathcal{X}_{3g-3}$ decomposes into $\Delta^{2g-1} \times \Delta^{2g-2}$ where $\Delta^{2g-1}$ indicates the direction of $\mathcal{T}_p(\mathcal{HET} Tor_g)$ and $\Delta^{2g-2}$ is the normal direction in which the period map $\mathcal{J}_{\mathcal{T} \mathcal{O}r_g}^*$ vanishes. The $\mathbb{Z}_2$ action fixes $\Delta^{2g-1}$ but acts as the multiplication of $-1$ on $\Delta^{2g-2}$. Thus $\mathcal{T} \mathcal{O}r_g/\mathbb{Z}_2$ locally looks like $\Delta^{2g-1} \times (\Delta^{2g-2}/\mathbb{Z}_2)$ around the hyperelliptic locus.

**Theorem 3.7.** (Local Torelli Theorem 2) $\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ is an immersion for $g \geq 2$.

This local Torelli Theorem was first proved by F. Oort and J. Steenbrink [19] and then by Y. Karpishpan [13] under his framework of understanding higher order derivatives of period maps in terms of Čech cohomology. We approach it by our deformation method.

**Proof.** From the local Torelli Theorem 3.1 the tangent map $\Pi_\ast$, restricted to $\mathcal{T}_g - \mathcal{HET} \mathcal{T}_g$, is injective for $g \geq 3$ and everywhere injective for $g = 2$. Thus it suffices to show that the tangent map of $\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ at hyperelliptic locus $\mathcal{H} \mathcal{E}_g$ is injective for $g \geq 3$. To this end, we lift the period map to $\mathcal{J}^{tor} : \mathcal{T} \mathcal{O}r_g/\mathbb{Z}_2 \to \mathcal{H}_g$. Fix $p \in \mathcal{HET} Tor_g$ which descends to $\bar{p}$ in $\mathcal{T} \mathcal{O}r_g/\mathbb{Z}_2$. From Proposition 3.7, $\bar{p}$ is a double point. Moreover, the dimension of the Zariski tangent space at $\bar{p}$ is $g(g+1)/2$. In fact, as $\mathcal{T} \mathcal{O}r_g$ is a complex manifold of complex dimension $3g - 3$ and $p$ is a smooth point, we can choose local parameters $(t_1, t_2, \cdots, t_{3g-3})$ such that $\hat{\mathcal{O}}_{\mathcal{T} \mathcal{O}r_g,p} = \mathbb{C}[[t_1, t_2, \cdots, t_{3g-3}]]$ and $\mathbb{Z}_2$ action is given by

$$\mathbb{Z}_2 t_i = t_i, \quad 1 \leq i \leq 2g - 1,$$

$$\mathbb{Z}_2 t_i = -t_i, \quad 2g \leq i \leq 3g - 3,$$

where $\{t_i\}_{i=1}^{2g-1}$ indicates the tangent directions of $\mathcal{HET} Tor_g$ and $\{t_i\}_{i=2g}^{3g-3}$ is the normal directions in which $\mathcal{J}_{\mathcal{T} \mathcal{O}r_g}^*$ vanishes. Clearly,

$$\mathcal{O}_{\mathcal{T} \mathcal{O}r_g/\mathbb{Z}_2,\bar{p}} = \left(\mathcal{O}_{\mathcal{T} \mathcal{O}r_g,p}\right)_{\mathbb{Z}_2} = \mathbb{C}[[t_1, \cdots, t_{2g-1}, t_2^2, t_2 t_{2g+1}, \cdots, t_{3g-3}]].$$

It is exactly the $g(g+1)/2$ parameters that give the basis of the Zariski tangent space at $\bar{p}$. We denote these directions by $\{(D_k, D_{ij})\}_{1 \leq k \leq 2g-1, 2g \leq i \leq j \leq 3g-3}$ respectively. Also by (2.14), we know that $\mathcal{J}^{tor}$ can also be written as $(\bar{\pi}_p A(t)^T + \pi_p) (A(t)^T + I_g)^{-1}$. The first and second order parts of $\mathcal{J}^{tor}$ are given by

$$\left(\mathcal{J}^{tor}\right)^{(1)} + \left(\mathcal{J}^{tor}\right)^{(2)}$$

$$= -2 \sqrt{-1} M_p A_1(t)^T + 2 \sqrt{-1} M_p \left( A_1(t)^T \right)^2 - 2 \sqrt{-1} M_p A_2(t)^T$$

$$= \sum_{i=1}^{n} t_i \int_{X_p} \theta^\alpha_{p} \wedge H(\mu_i, \theta^\beta_{p}) - \sum_{i,j=1}^{n} \sqrt{-1} t_i t_j \int_{X_p} \theta^\alpha_{p} \wedge H(\mu_i, \theta^\beta_{p}) M_p^{\beta_1} \int_{X_p} \theta^\beta_{p} \wedge H(\mu_j, \theta^\beta_{p})$$

$$+ \sum_{i,j=1}^{n} t_i t_j \int_{X_p} \theta^\alpha_{p} \wedge H(\mu_i, \theta^\beta_{p}),$$

where $n^\alpha_{p} = -\partial_t \partial_t (\mu_i, \theta^\alpha_{p})$ and $M_p^{\alpha\beta}$ is the inverse matrix of $M_p$. From the choice of $t_i$ above, for any $1 \leq \alpha, \beta \leq g$, we have

$$\int_{X_p} \theta^\alpha_{p} \wedge H(\mu_i, \theta^\beta_{p}) = 0,$$

where $2g \leq i \leq 3g - 3$. Hence we can write out the image of $\{(D_k, D_{ij})\}_{1 \leq k \leq 2g-1, 2g \leq i \leq j \leq 3g-3}$ under $\mathcal{J}_{\mathcal{T} \mathcal{O}r_g}^*$ by using the expansion formula of $\mathcal{J}^{tor}$.

$$\mathcal{J}^{tor}_{\mathcal{T} \mathcal{O}r_g} (D_k) = \int_{X_p} \theta^\alpha_{p} \wedge H(\mu_i, \theta^\beta_{p}),$$

$$1 \leq k \leq 2g - 1,$$

$$\mathcal{J}^{tor}_{\mathcal{T} \mathcal{O}r_g} (D_{ij}) = \int_{X_p} \theta^\alpha_{p} \wedge H(\mu_i, \theta^\beta_{p}) \left( \mu_j, \theta^\beta_{p} \right),$$

$$2g \leq i = j \leq 3g - 3,$$

$$\mathcal{J}^{tor}_{\mathcal{T} \mathcal{O}r_g} (D_{ij}) = \int_{X_p} \theta^\alpha_{p} \wedge H(\mu_i, \theta^\beta_{p}) \left( \mu_j, \theta^\beta_{p} \right) + \mu_j, \theta^\beta_{p} \left( \mu_i, \theta^\beta_{p} \right),$$

$$2g \leq i < j \leq 3g - 3.$$

Finally we need to show that $\{\mathcal{J}^{tor}_{\mathcal{T} \mathcal{O}r_g} (D_k), \mathcal{J}^{tor}_{\mathcal{T} \mathcal{O}r_g} (D_{ij})\}$ are linearly independent. Since $X_p$ is a hyperelliptic Riemann Surface, these Čech cohomology groups, such as $H^0(\Omega^1_{X_p})$, $\check{H}^1(\mathcal{O}_{X_p})$ and
$H^1(T_{X_p})$, have explicit bases just as described in [13] [19]. Moreover, these papers have showed that these directions are linearly independent in terms of Čech cohomology. We give a proof in Appendix 5 that our directions are actually the same as theirs, which completes the proof of this theorem.

q.e.d.

Local Torelli Theorems 3.1 and 3.7 tell us that the period map gives a local embedding of the Kuranishi coordinate chart $\Delta_{p,\mathbb{Z}}$ when $p$ lies in the nonhyperelliptic locus, and of $\Delta_{p,\mathbb{Z}/2}$ when $p$ lies in the hyperelliptic locus. This local embedding induces a matrix model for the local Kuranishi coordinates.

**Definition 3.8.** Matrix Model for the Kuranishi coordinate charts.

The image of the Kuranishi coordinate chart under the period map is called the matrix model when the local Torelli theorems hold. Here we identify the Kuranishi coordinate chart with its matrix model, which lies in $H_g \subseteq \mathbb{C}^{g(n+1)/2}$.

4. **Proof of the global Torelli Theorems**

This section is devoted to the proof of the global Torelli theorem for Riemann Surfaces.

We recall some basic facts of Riemann Surface $S$ and its Jacobian $\text{Jac}S$. Fix one point $p_0$ on $S$. There is a natural map

$$
\chi_{p_0} : S^{(d)} \rightarrow \text{Jac}S
$$

where $S^{(d)} = \underbrace{S \times \cdots \times S}_{d \text{ times}} / S_d$ with $S_d$ the symmetric group of order $d$ and $\{\theta^\alpha\}_{\alpha=1}^g$ is the basis of $H^0(S, K_S)$.

**Definition 4.1.** Define $W^g_d$ in $\text{Jac}S$ associated to $S$:

$$
W^g_d := \{\chi_{p_0}(D) \in \text{Jac}S | D \geq 0, \deg D = d \text{ and } \dim |D| \geq r\}
$$

and set $W^1_d$ as $W^1_d$.

The polarization class determines the line bundles associated to it up to translations on abelian varieties. Theta functions are holomorphic sections of those line bundles. On principally polarized abelian varieties $(A, [\omega])$, $\dim H^0(A, L) = 1$ where $L$ is associated to the polarization class $[\omega]$. Theta divisor $\Theta$ is the zero locus of the generator section of $L$. Thus the polarization class $[\omega]$ determines theta divisor up to translations. Moreover, we have the following famous theorem on Jacobians.

**Theorem 4.2.** (Riemann’s Theorem. See P. 338 of [9]) $\Theta = W_{g-1} - \frac{K}{2}$, where $K$ is the image of canonical divisor under $\chi_{p_0}$.

Riemann’s Theorem tells us that the polarization class on $\text{Jac}S$ determines $W_{g-1}$, which reflects the complex structure on $S$ to some extent, up to translations.

Also the intersection number $\theta(W_1 \cdot \Theta) = g$ and we have another theorem.

**Theorem 4.3.** (See P. 336 — P. 339 of [9]) For $\lambda \in \text{Jac}S$ such that $W_1 \not\subseteq \{\Theta + \lambda\}$, $W_1$ and $\Theta + \lambda$ have $g$ intersection points, denoted by $p_1(\lambda), \cdots, p_g(\lambda)$. And the equality

$$
\sum_{i=1}^g p_i(\lambda) = \lambda + \frac{K}{2}
$$

holds. Moreover, $W_1 \subseteq \{\Theta + \lambda\}$ if and only if $\lambda + \frac{K}{2} \in W^1_g$. Equivalently, when $W_1 \not\subseteq \{W_{g-1} + b\}$, there are $p_1(b), \cdots, p_g(b)$ in $W_1 \cap \{W_{g-1} + b\}$ and $\sum_{i=1}^g p_i(b) = b + K$. Meanwhile, $W_1 \subseteq \{W_{g-1} + b\}$ if only if $b + K \in W^1_g$.  

Define an operation of two sets \( A, B \) in JacS:

\[
A \ominus B := \bigcap_{x \in B} \{ A - x \}.
\]

**Proposition 4.4.** For 0 ≤ \( r \leq g - 1 \) and \( a, b \in \text{JacS} \), \( \{ W_t + a \} \ominus \{ W_r + b \} = W_{t-r} + a - b \).

**Corollary 4.5.** For 0 ≤ \( r \leq g - 1 \),

1. \( \{ W_{g-1} + a \} \ominus \{ W_r + b \} = W_{g-1-r} + a - b \).
2. \( \{ W_{g-1} + a \} \ominus \{ -W_r + b \} = -W_{g-1-r} + a - b + K \).

These two proofs can be found on P. 155, P. 156, and P. 161 of [7]. And we give a sketch of proof to the following proposition.

**Proposition 4.6.** For 0 ≤ \( r \leq g - 2 \), fix \( a \in \text{JacS} \) and let \( b = a + x - y \) where \( x \in W_1 \) and \( y \in W_{g-1-r} \). Then either \( \{ W_{r+1} + a \} \subseteq \{ W_{g-1} + b \} \) or

\[
\{ W_{r+1} + a \} \cap \{ W_{g-1} + b \} = \{ W_r + a + x \} \bigcup T,
\]

where \( T = \{ W_{r+1} + a \} \cap \{ -W_{g-2} - y + a + K \} \).

**Sketch of proof.** \( W_{r+1} + a \) and \( W_{g-1} + b \) are two irreducible subvarieties of JacS. If one is not contained in another, then we will have intersection. Thus we assume \( \{ W_{r+1} + a \} \nsubseteq \{ W_{g-1} + b \} \).

Let \( x = \chi_{p_0}(F) \) and \( y = \chi_{p_0}(D) \) with effective divisors \( F \) and \( D \) of degree 1 and \( g - 1 - r \) respectively. \( D \) can’t contain \( F \), otherwise \( W_{r+1} + a = W_{r+1} + b + y - x = W_{r+1} + b + \chi_{p_0}(F + D') - \chi_{p_0}(F) = W_{r+1} + b + \chi_{p_0}(D') \subseteq W_{g-1} + b \).

Let \( u \in \{ W_{r+1} + a \} \cap \{ W_{g-1} + b \} \). Then there are two effective divisors \( P \) and \( Q \) of degree \( r+1 \) and \( g - 1 \) respectively, such that \( u = \chi_{p_0}(P) + a = \chi_{p_0}(Q) + b \). This implies \( \chi_{p_0}(P + D) = \chi_{p_0}(Q + F) \).

By Abel Theorem, \( P + D \equiv Q + F \).

1. \( \dim |P + D| = 0 \). Then \( P + D = Q + F \), implying that \( F \) is contained in \( P \). Hence \( u = \chi_{p_0}(P) + a = \chi_{p_0}(P' + F) + a = \chi_{p_0}(P') + x + a \subseteq W_r + x + a \).

2. \( \dim |P + D| > 0 \). For any \( F \in S \), there exists an effective divisor \( Q \) such that \( P + D \equiv \tilde{F} + \tilde{Q} \). Then \( u = \tilde{F} - \chi_{p_0}(Q) + b = \chi_{p_0}(\tilde{F}) + \chi_{p_0}(\tilde{Q}) - \chi_{p_0}(F) + b = \chi_{p_0}(\tilde{F}) + \chi_{p_0}(\tilde{Q}) + a - y \).

implying \( u \in \bigcap_{F \in S} \{ W_{g-1} + \chi_{p_0}(F) \} + a - y = \{ W_{g-1} \} + a - y = W_{g-2} + K + a - y \).

Hence \( \{ W_{r+1} + a \} \cap \{ W_{g-1} + b \} \subseteq \{ W_r + a + x \} \bigcup T \). The reverse inclusion is clear. q.e.d.

**Theorem 4.7.** \( J^{tor} : \text{Tor}_g / \mathbb{Z}_2 \rightarrow \mathcal{H}_g \) is an embedding for \( g \geq 3 \).

**Proof.** From the discussion of Section 3, \( \text{Tor}_g / \mathbb{Z}_2 \) is a complex orbifold of complex dimension \( 3g - 3 \). For every point \( p \) in the non-hyperelliptic locus, we have the Kuranishi coordinate chart \( \Delta_{p,e} \) centered at \( p \), which descends from \( \text{Tor}_g \). As to the hyperelliptic locus, we denote by \( \Delta_{p,e} / \mathbb{Z}_2 \) the local coordinate chart around the hyperelliptic point according to the local behavior of the hyperelliptic locus. From the local Torelli theorems [3,1] and [3,7], \( J^{tor} \) gives a local embedding on both of these two kinds of coordinate charts. All we need to show is that \( J^{tor} \) is injective. It is easy to see that \( \text{Tor}_g / \mathbb{Z}_2 \cong T_g / \tilde{T}_g \). Thus the proof of one-to-one correspondence between \( \tilde{T}_g \) orbit and its Jacobian is our ultimate, which is equivalent to say that two points in \( T_g \) with the same Jacobian must be related by some element in \( \tilde{T}_g \).

According to [1] and [12], \( \mathcal{H}_g \) can be viewed as the isomorphism classes of principally polarized abelian varieties together with a symplectic basis \( (A, \gamma) \), where \( \gamma : H_1(\Sigma, \mathbb{Z}) \rightarrow H_1(A, \mathbb{Z}) \) preserves the intersection paring on \( \Sigma \) and the principally polarized form on \( A \). And the identification is given from \( (A, \gamma) \) to its period matrix with respect to this symplectic basis. By changing the symplectic basis, we have the natural \( \text{Sp}(g, \mathbb{Z}) \) action on \( \mathcal{H}_g \). However, the kernel of the \( \text{Sp}(g, \mathbb{Z}) \) action is \( \pm \mathbb{I}_{2g} \). That is to say, for every principally polarized abelian variety \( A \),

\[
(A, \gamma) \cong (A, -\gamma).
\]
Also \( \mathcal{T}_{\text{or}_g} \) can be identified with the isomorphism classes of Riemann Surfaces together with a symplectic basis \((C, \gamma)\), where \( \gamma : H_1(\Sigma, \mathbb{Z}) \to H_1(C, \mathbb{Z}) \) preserves the intersection paring on \( \Sigma \) and \( C \), since \( \Gamma_g/\mathcal{T}_g = \text{Sp}(g, \mathbb{Z}) \). Moreover, the period map \( J^\text{tor} : \mathcal{T}_{\text{or}_g} \to \mathcal{H}_g \) is given by

\[
\mathcal{T}_{\text{or}_g} \longrightarrow \mathcal{H}_g \\
(C, \gamma) \mapsto (\text{Jac}C, \gamma),
\]

where we have the natural isomorphism \( H_1(C, \mathbb{Z}) \cong H_1(\text{Jac}C, \mathbb{Z}) \).

Now assume that two points \([C, [f]]\) and \([C', [f']]\) on \( \mathcal{T}_g \) are mapped to the same Jacobian, namely \((A, \gamma)\). Write \((C, \gamma)\) and \((C', \gamma')\) on \( \mathcal{T}_{\text{or}_g} \) as the corresponding two points descended from \([C, [f]]\) and \([C', [f']]\), respectively. As \((C, \gamma)\) and \((C', \gamma')\) are mapped to the same Jacobian \((A, \gamma)\), their symplectic bases will be the same up to a change of the sign. Without loss of generality, we may assume that \((C, \gamma)\) and \((C', \gamma')\) share the same symplectic basis after changing the sign.

Going back to the two corresponding points on \( \mathcal{T}_g \), the following picture appears since we can see \( \mathcal{T}_g \) from the deformation theoretic point of view:

\[
\begin{array}{c}
\Sigma \\
\downarrow \quad \downarrow \\
C & \phi & C' \\
\end{array}
\]

In fact we will prove that \( \phi \) is biholomorphic. Hence \([C, [f]]\) and \([C', [f']]\) are related by \( \tilde{T}_g \).

To see this, we first recall the definition of the Jacobian. The Jacobian of a Riemann Surface \( X \) is nothing but \( \mathbb{C}^g/\Lambda \) and

\[
\Lambda = \mathbb{Z} \left\{ \int_{A_1} \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^g \end{array} \right), \ldots, \int_{A_g} \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^g \end{array} \right), \int_{B_1} \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^g \end{array} \right), \ldots, \int_{B_g} \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^g \end{array} \right) \right\},
\]

where \( \{\theta^\alpha\}_{\alpha=1}^g \) is a basis of \( H^0(X, K_X) \). Since \([C, [f]]\) and \([C', [f']]\) are mapped to the same Jacobian by the period map, their symplectic bases \((A_\alpha, B_\alpha)\) and \((A'_\alpha, B'_\alpha)\) are related by \( \phi \) together with

\[
\int_{A_\alpha} \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^g \end{array} \right) = \int_{A'_\alpha} \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^g \end{array} \right), \quad \int_{B_\alpha} \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^g \end{array} \right) = \int_{B'_\alpha} \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^g \end{array} \right).
\]

Let

\[
\Omega = \left( \begin{array}{c} \theta^1 \\ \vdots \\ \theta^g \end{array} \right), \quad \Omega' = \left( \begin{array}{c} \theta'^1 \\ \vdots \\ \theta'^g \end{array} \right).
\]

Set \( \phi^\ast[\theta^\alpha] = \sum_{\beta=1}^g x_{\alpha\beta}[\theta^\beta] + \sum_{\beta=1}^g y_{\alpha\beta}[\theta^\beta] \) where \( \phi^\ast : H^1_{\text{dR}}(C', \mathbb{C}) \to H^1_{\text{dR}}(C, \mathbb{C}) \) and \([\theta^\beta]\) is the de Rham class represented by \( \theta^\beta \). Thus

\[
\phi^\ast[\Omega'] = (X \ Y) \left( \begin{array}{c} \Omega \\ \Omega' \end{array} \right).
\]
Put (4.1) and (4.2) together to get
\[
\int_{A_\alpha} [\Omega] = \int_{A'_\alpha} [\Omega'] = \int_{A_\alpha} \phi^* [\Omega'] = (X, Y) \int_{A_\alpha} \left( \frac{[\Omega]}{[\Omega']} \right),
\]
\[
\int_{B_\alpha} [\Omega] = \int_{B'_\alpha} [\Omega'] = \int_{B_\alpha} \phi^* [\Omega'] = (X, Y) \int_{B_\alpha} \left( \frac{[\Omega]}{[\Omega']} \right).
\]
Reformulating these two equalities into matrix form, we get
\[
(4.3) \quad \left( \begin{array}{c}
\int_{A_\alpha} [\Omega] \\
\int_{B_\alpha} [\Omega]
\end{array} \right) = \left( \begin{array}{cc}
X & Y \\
Y & X
\end{array} \right) \left( \begin{array}{c}
\int_{A_\alpha} \left( \frac{[\Omega]}{[\Omega']} \right) \\
\int_{B_\alpha} \left( \frac{[\Omega]}{[\Omega']} \right)
\end{array} \right).
\]
Observe that \( \det \left( \begin{array}{c}
\int_{A_\alpha} [\Omega] \\
\int_{B_\alpha} [\Omega]
\end{array} \right) \neq 0 \), since \( \left( \frac{[\Omega]}{[\Omega']} \right) \) is the basis of \( H^1_{dR}(C, \mathbb{C}) \) and \( (A_\alpha, B_\alpha) \) is the one of \( H_1(C, \mathbb{Z}) \). These imply that
\[
(4.4) \quad X = 1_g, \ Y = 0.
\]
Hence \( \phi^*(\theta^\alpha) = \theta^\alpha + df^\alpha \) for some \( f^\alpha \).

Now we consider \( \chi_{p_0} : C \rightarrow \text{Jac} C \)
\[
p \rightarrow \left[ \left( \int_{p_0}^p \theta^1, \ldots, \int_{p_0}^p \theta^g \right) \right]
\]
embeds Riemann Surface \( C \) into its Jacobian. The following diagram shows that two images of \( C \) and \( C' \), denoted by \( W_1 \) and \( V_1 \) respectively, are related by \( \phi \):

\[
\begin{array}{ccc}
(C, p_0) & \xrightarrow{\phi} & (C', \phi(p_0)) \\
\xrightarrow{\chi_{p_0}} & \downarrow & \xrightarrow{\chi_{\phi(p_0)}} \\
(\mathbb{C}^g / \Lambda) & \rightarrow & (\mathbb{C}^g / \Lambda)
\end{array}
\]

More precisely, let one smooth curve \( \tau \) on \( C \) connect \( p_0 \) and \( p \) with \( \phi(\tau) \) connecting \( \phi(p_0) \) and \( \phi(p) \). Then we have
\[
\left[ \left( \int_{\phi(p_0)}^{\phi(p)} \theta^1, \ldots, \int_{\phi(p_0)}^{\phi(p)} \theta^g \right) \right] = \left[ \left( \int_{\phi(\tau)}^{\phi(\tau)} \theta^1, \ldots, \int_{\phi(\tau)}^{\phi(\tau)} \theta^g \right) \right]
\]
\[
= \left[ \left( \int_{\tau} \phi^*(\theta^1), \ldots, \int_{\tau} \phi^*(\theta^g) \right) \right] = \left[ \left( \int_{\tau} (\theta^1 + df^1), \ldots, \int_{\tau} (\theta^g + df^g) \right) \right]
\]
\[
= \left[ \left( f^1(p) - f^1(p_0) + \int_{p_0}^p \theta^1, \ldots, f^g(p) - f^g(p_0) + \int_{p_0}^p \theta^g \right) \right] = \left[ \left( f^1(p) + \int_{p_0}^p \theta^1, \ldots, f^g(p) + \int_{p_0}^p \theta^g \right) \right]
\]
\[
= \left[ \left( f^1(p), \ldots, f^g(p) \right) \right] + \left( \int_{p_0}^p \theta^1, \ldots, \int_{p_0}^p \theta^g \right).
\]
Hence \( W_1 \) and \( V_1 \) are different by a varying vector \( \left[ (f^1(p), \ldots, f^g(p)) \right] \) and here we normalize \( f^i, 1 \leq i \leq g \) such that \( f^i(p_0) = 0 \). We would like to use the same polarization (actually the same theta divisor) in the Jacobian to show that varying vector to be constant. Afterwards we will associate \( W'_d \) and \( V'_d \) to \( C \) and \( C' \) through the mappings \( \chi_{p_0} \) and \( \chi_{\phi(p_0)} \) respectively just as Definition 4.11.

Consider the smallest integer \( r \) such that
\[
V_1 \subseteq W_{r+1} + a \text{ or } V_1 \subseteq -W_{r+1} + a
\]
for some \( a \in \mathbb{C}^g / \Lambda \). It is easy to see that \( r \leq g - 2 \). Actually from Theorem 4.2
\[
(4.5) \quad V_{g-1} - \frac{K'}{2} = W_{g-1} - \frac{K}{2},
\]
Together with Theorem 4.3 and (4.5), it will happen that $V_1 \subseteq W_{g-1} + b$ for $b + \frac{K + K'}{2} \in V^1_g$.

**Case 1:** \( r = 0 \). Set $b = -W_1 + a$ or $V_1 = W_1 + a$. But $V_1$ and $W_1$ start through the origin of $\mathbb{C}^g / \Lambda$. Thus $a = 0$. $V_1 = -W_1$ which means that $\phi_*$ reverses the symplectic basis of these two Riemann Surfaces. Contradict with our assumption on $\phi$ ahead. Thus $V_1 = W_1$, which forces all $f_i$ to be zero.

**Case 2:** \( r > 0 \). Suppose $V_1 \subseteq W_{r+1} + a$. Set $b = a + x - y$ where $x \in W_1$ and $y$ varies in $W_{g-1-r}$. For fixed $x$, $V_1$ can’t always lie in $W_{g-1} + b$ when $y$ runs through $W_{g-1-r}$. If so, then we would have

$$V_1 \subseteq \bigcap_{y \in W_{g-1-r}} \{W_{g-1} + a + x - y\} = \{W_{g-1} \oplus W_{1-r}\} + a + x = W_r + a + x$$

by Corollary 4.5, contradicting to the minimality of $r$. Hence we have two following results

1. For any fixed $x \in W_1$, $V_1 \cap \{W_{g-1} + b\}$ will be $g$ points for generic $y \in W_{g-1-r}$.
2. There exists some $y \in W_{g-1-r}$ such that $V_1 \cap \{W_{g-1} + b\}$ will be $g$ points for generic $x \in W_1$. Because it is impossible that for any fixed $y \in W_{g-1-r}$, $V_1 \subseteq \{W_{g-1} + a + x - y\}$ when $x$ runs through $W_1$.

Under the circumstance of the result (1), we have, by Proposition 4.6

$$V_1 \bigcap \{W_{g-1} + b\} = V_1 \bigcap \{W_{g-1} + b\} \bigcap \{W_{r+1} + a\}$$

$$= \left( V_1 \bigcap \{W_r + a + x\} \right) \bigcup \left( V_1 \bigcap T \right),$$

where $V_1 \bigcap \{W_r + a + x\}$ depends on $x$, while $V_1 \bigcap T$ on $y$. Write the $g$ intersection points as $p_1(b), \ldots, p_g(b)$. From (4.5) and Theorem 4.3 we have

$$\sum_{i=1}^{g} p_i(b) = b + \frac{K + K'}{2} = a + x - y + \frac{K + K'}{2}.$$

**Claim:** For any fixed $x \in W_1$, $V_1 \bigcap \{W_r + a + x\}$ has at most one point.

In fact, if there are two points in $V_1 \bigcap \{W_r + a + x\}$ for some $x \in W_1$, fixing that $x$, we know that equality (4.7) holds for generic $y \in W_{g-1-r}$, leaving $p_1(b)$ and $p_2(b)$ fixed, which implies that $a + x - W_{g-1-r} \subseteq \{V_{g-2} + c\}$ with $c$ a constant. Hence $-W_{g-1-r} \subseteq \{V_{g-2} + c'\}$ with $c'$ a constant. By Corollary 4.5 we get

$$V_1 = V_{g-1} \oplus V_{g-2} \subseteq \{W_{g-1} + \frac{K' - K}{2}\} \bigcap \{W_{g-1-r} + c'\} = -W_r + \frac{K + K'}{2} + c',$$ contradicting to the minimality of $r$. Thus our claim is proved.

As $V_1$ and $W_r + x + a$ are subvarieties in $W_{r+1} + a$ with complementary dimensions, $\#(V_1 \cdot \{W_r + x + a\}) \leq 1$ for all $x \in W_1$ from our claim. And $W_r + x + a$ and $W_r + a$ have the same homology class. In fact, let us denote the origin of the Jacobian by $x_0$. Consider a $C^\infty$ curve $\gamma(t), t \in [0, 1]$ between $x_0$ and $x$ on $W_1$ such that $\gamma(0) = x_0$ and $\gamma(1) = x$. Then we have $\partial(\bigcup_{y \in \gamma(t)} \{W_r + y + a\}) = \{W_r + x + a\} - \{W_r + a\}$. Hence $\#(V_1 \cdot \{W_r + x + a\}) = \#(V_1 \cdot \{W_r + a\})$. But the constant $\#(V_1 \cdot \{W_r + x + a\})$ for all $x \in W_1$ can’t be zero since $V_1 = V_1 \bigcap \{W_{r+1} + a\} = \bigcup_{x \in W_1} (V_1 \bigcap \{W_r + x + a\})$. Thus $\#(V_1 \cdot \{W_r + x + a\}) = 1$ for all $x \in W_1$, namely $V_1 \bigcap \{W_r + x + a\}$ has one point for all $x \in W_1$.

Apply the result (2). Fix that $y$ and we still have (4.6). Equality (4.7) holds for generic $x \in W_1$, leaving $p_2(b), \ldots, p_g(b)$ fixed. Hence $V_1 + c'' \subseteq a + W_1 - y + \frac{K + K'}{2}$ with $c''$ a constant. This contradicts to the minimality of $r$. Therefore the case $r > 0$ is impossible.

Now we have proved that $f_i \equiv 0$, $1 \leq i \leq g$. Hence $\phi^*$ preserves holomorphic one forms from $C'$ to $C$. Choose coordinates centered at $p$ and $\phi(p)$, which are denoted by $(z, p)$ and $(w, \phi(p))$. Pick $\Xi \in H^0(C', K_{C'})$ with $\Xi(\phi(p)) \neq 0$. Locally $\Xi$ can be written as $\Xi = g(w) dw$.

Pull $\Xi$ back by $\phi$, then we get holomorphic one form on $C$. However,

$$\phi^*\Xi = g(\phi(z)) \frac{\partial \phi}{\partial z} dz + g(\phi(z)) \frac{\partial \phi}{\partial \bar{z}} \bar{d}z.$$
Then $g(\phi(z)) \frac{\partial \phi}{\partial z} = 0$. Restricting to the point $p$ and $g(\phi(p)) \neq 0$, we have $\frac{\partial \phi}{\partial z} \bigg|_{z=0} = 0$. At last, $\phi$ is holomorphic, finishing the proof of the theorem. q.e.d.

**Remark 4.8.** In the discussion of the case $r > 0$, if we suppose $V_1 \subseteq -W_{r+1} - a$, take $-W_{g-1} - b$ to intersect with $V_1$. Following the same method, we will get $V_1 \subseteq -W_1 + c$ with $c$ a constant, contradicting the minimality of $r$.

**Corollary 4.9.** For the case of $g = 2$, $\mathcal{J}^{tor} : \text{Tor}_g \to \mathcal{H}_g$ is an open embedding.

**Proof.** Theorem 3.1 tells us that $\mathcal{J}^{tor} : \text{Tor}_g \to \mathcal{H}_g$ is an immersion everywhere when $g = 2$. Besides, $\mathcal{J}^{tor}$ is an open map from the fact $\dim \mathcal{C} \text{Tor}_g = \dim \mathcal{C} \mathcal{H}_g = 3$. Moreover, Proposition 3.6 implies that $\mathbb{Z}_2$ is a trivial action on $\text{Tor}_g$ since any Riemann Surface with $g = 2$ is hyperelliptic, indicating that $\check{T}_g$ orbit is the same as $T_g$ orbit on $T_g$. The proof of Theorem 4.4 implies that $\mathcal{J}^{tor}$ is an open embedding. q.e.d.

**Corollary 4.10.** $\mathcal{J}^{tor} : \text{Tor}_g \to \mathcal{H}_g$ is a $2 : 1$ branched covering map branched over $\mathcal{H} \setminus \text{Tor}_g$ onto its image for $g \geq 3$.

**Proof.** This is a direct consequence of Theorem 4.7 q.e.d.

**Proposition 4.11.** Let $\Delta_{p, \varepsilon}$ be the Kuranishi coordinate chart on $T_g$. The period map $\Pi$ maps the $\Gamma_g$ orbit of $\Delta_{p, \varepsilon}$ onto the $\text{Sp}(g, \mathbb{Z})$ orbit of its image in $\mathcal{H}_g$.

**Proof.** Recall that the Kuranishi coordinate chart $\Delta_{p, \varepsilon}$ is given by

$$\Delta_{p, \varepsilon} \quad \text{to} \quad \phi_{T_g} \quad \text{to} \quad \phi_{[X_t, [F_i]]},$$

where $(X, F)$ is the Kuranishi family with the Teichmüller structure of $(X_p, [F_0])$ over $\Delta_{p, \varepsilon}$, while the coordinate map of $\Delta_{p, \varepsilon}^{[\phi]}$ can be written as

$$\Delta_{p, \varepsilon}^{[\phi]} \quad \text{to} \quad \phi_{T_g} \quad \text{to} \quad \phi_{[X_t, [\phi F_i]]},$$

where $\Delta_{p, \varepsilon}^{[\phi]} = [\phi] \Delta_{p, \varepsilon}$. Now the Kuranishi family becomes $(X, (\phi \times 1) F)$, where $\phi \times 1 : \Sigma \times \Delta_{p, \varepsilon}^{[\phi]} \to \Sigma \times \Delta_{p, \varepsilon}$, the same family as $(X, F)$ up to a different Teichmüller structure. Two symplectic bases are linked by $\rho([\phi])$, denoted by \( \begin{pmatrix} U & V \\ R & S \end{pmatrix} \) $\in$ $\text{Sp}(g, \mathbb{Z})$, i.e.,

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} U & V \\ R & S \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix},$$

where \( \begin{pmatrix} A \\ B \end{pmatrix} \) and \( \begin{pmatrix} A \\ B \end{pmatrix} \) are the symplectic bases on $\Delta_{p, \varepsilon}^{[\phi]}$ and $\Delta_{p, \varepsilon}$, respectively. As we have seen, the matrix model of $\Delta_{p, \varepsilon}$ is

$$\Pi(t) = \begin{pmatrix} \tilde{\sigma}_p & \tau_p \\ \Pi_g & \Pi_g \end{pmatrix} \tau(t) T.$$

While on $\Delta_{p, \varepsilon}^{[\phi]}$, one has

$$\bar{\Pi}(t)_{\alpha \beta} = \int_{B_{a\lambda}} \tilde{\sigma}(t)^{\gamma \beta} \theta_\alpha \gamma(t)$$

$$= \int_{B_{a\lambda}} \tilde{\sigma}(t)^{\gamma \beta} \left( \theta_\alpha \gamma + A(t)^{\gamma \beta} \right)$$

$$= \int_{R} \tilde{\sigma}(t)^{\gamma \beta} \left( \theta_\alpha \gamma + A(t)^{\gamma \beta} \right)$$

$$= (R_{\alpha \gamma} + S_{a\lambda \pi_p, \lambda \gamma} + R_{a\delta} A(t)^{\gamma \beta} + S_{a\lambda \pi_p, \lambda \delta} A(t)^{\gamma \beta}) \tilde{\sigma}(t)^{\gamma \beta},$$
where $\tilde{\sigma}(t)^{\alpha\beta}$ is the inverse matrix of $\tilde{\sigma}(t)_{\alpha\beta}$. And $\tilde{\sigma}(t)_{\alpha\beta}$ is given by

$$\tilde{\sigma}(t)_{\alpha\beta} = \int_{A_{\alpha}} \theta_{p}^{\beta}(t)$$

$$= \int_{U_{\alpha\lambda}A_{\lambda} + V_{\alpha\lambda}B_{\lambda}} \theta_{p}^{\beta} + A(t)^{\beta\gamma} \tilde{\theta}_{p}^{\gamma}$$

$$= U_{\alpha\beta} + V_{\alpha\gamma} \pi_{p,\gamma} + U_{\alpha\gamma} A(t)^{\beta\gamma} + V_{\alpha\gamma} \pi_{p,\lambda\gamma} A(t)^{\beta}_{\gamma}.$$  

Then we formulate all these into the matrix form:

$$\tilde{\Pi}(t) = (R(1_{g} + A(t)^{T}) + S(\pi_{p} + \tilde{\pi}_{p} A(t)^{T})) (U(1_{g} + A(t)^{T}) + V(\pi_{p} + \tilde{\pi}_{p} A(t)^{T}))^{-1}$$

$$= \left( \begin{array}{cc} S & R \\ V & U \end{array} \right) \left( \begin{array}{cc} \tilde{\pi}_{p} & \pi_{p} \\ 1_{g} & 1_{g} \end{array} \right) \left( \begin{array}{cc} S & R \\ V & U \end{array} \right)^{-1} \wedge A(t)^{T}$$

$$= \left( \begin{array}{cc} S & R \\ V & U \end{array} \right) \hat{\Pi}(t).$$

Thus the $\Gamma_{g}$ orbit of $\Delta_{p,\epsilon}$ is mapped, by the period map, onto the $\text{Sp}(g, \mathbb{Z})$ orbit of its matrix model $\hat{\Pi}(t)$ in $\mathcal{H}_{g}$, since the representation $\rho : \Gamma \to \text{Sp}(g, \mathbb{Z})$ is surjective. q.e.d.

Denote by $\nu$ the transformation of $\text{Sp}(g, \mathbb{Z})$

$$\text{Sp}(g, \mathbb{Z}) \to \text{Sp}(g, \mathbb{Z})$$

$$\left( \begin{array}{cc} U & V \\ R & S \end{array} \right) \to \left( \begin{array}{cc} S & R \\ V & U \end{array} \right)$$

and it is obvious that $\nu^{2} = 1$.

**Theorem 4.12.** (Global Torelli Theorem on moduli space) $\mathcal{J} : \mathcal{M}_{g} \to \mathcal{A}_{g}$ is injective for $g \geq 2$.

**Proof.** As we have seen from Corollary 4.10, $\mathcal{J}$ is a 2 : 1 branched covering map onto its image, branching over $\mathcal{H}\mathcal{E}\mathcal{T} \mathcal{r}_{g}$ for $g \geq 3$. That is to say that the $\hat{T}_{g}$ orbits on $\mathcal{T}_{g}$ have one-to-one correspondence to their Jacobian given by the period map $\hat{\Pi}$. This is also true for $g = 2$, from the proof of Corollary 4.10.  

From Proposition 4.11, the $\Gamma_{g}$ orbits are mapped onto $\text{Sp}(g, \mathbb{Z})$ orbits. Assume that two $\Gamma_{g}$ orbits $[\rho]$ and $[\varrho]$ of $\mathcal{M}_{g}$ are mapped to the same $\text{Sp}(g, \mathbb{Z})$ orbit by $\mathcal{J}$. We lift these to $\hat{\Pi} : \mathcal{T}_{g} \to \mathcal{H}_{g}$ and thus have $\hat{\Pi}(p) = L \cap \hat{\Pi}(q)$ for some $L \in \text{Sp}(g, \mathbb{Z})$. There is the following exact sequence

$$1 \to T_{g} \to \Gamma_{g} \xrightarrow{\rho} \text{Sp}(g, \mathbb{Z}) \to 1.$$  

Pick $[\phi] \in \rho^{-1}(\nu(L))$. Then $\hat{\Pi}([\phi] [q]) = L \cap \hat{\Pi}(q)$ by Proposition 4.11. Hence $p$ and $[\phi] [q]$ are in the same $\hat{T}_{g}$ orbit, which implies that $p$ and $q$ are in the same $\Gamma_{g}$ orbit. q.e.d.

**5. Appendix**

Recall that the natural isomorphism between the Čech cohomology $\hat{H}^{1}(T_{X})$ and the Dolbeault cohomology $H^{0,1}_{\partial} (T_{X})$, and isomorphism between $\hat{H}^{1}(\mathcal{O}_{X})$ and $H^{0,1}_{\partial}$ follows similarly. Assume that there is an open covering $\bigcup_{\alpha} U_{\alpha}$ on $X$ and then the natural isomorphism $\Psi$ is given by

$$\Psi : \hat{H}^{1}(T_{X}) \to H^{0,1}_{\partial} (T_{X})$$

$$[\theta_{\alpha\beta}] \to [\partial \xi^{\alpha}]$$

where $\xi^{\alpha} \in A^{0,0}(U_{\alpha}, T_{X})$ and $\xi^{\beta} - \xi^{\alpha} = \theta_{\alpha\beta}$.

Now we return to the proof of the Theorem 4.17 that is, $X$ is a hyperelliptic Riemann Surface, covered by two affine charts $U_{0}$ and $U_{1}$ as described in [13, P. 568]. The first derivative of the period map in the direction $D_{k}, 1 \leq k \leq 2g - 1$ in terms of Čech cohomology is given by

$$H^{0}(X, \Omega^{1}) \to \hat{H}^{1}(X, \mathcal{O}_{X})$$

$$\omega \to [\partial \phi_{k}, \omega]$$

where $\phi_{k}$ is the $k$-th coordinate function of $X$. 


where \([\theta_k] \in \hat{H}^1(T_X)\) corresponds to \([\mu_k] \in H^{0,1}(T_X)\). It is obvious that \([\theta_k, \omega]\) is mapped to \([\mu_k, \omega]\) by the natural isomorphism from \(\hat{H}^1(\mathcal{O}_X)\) to \(H^{0,1}_0\). The second derivative of period map in the direction of \(D_{ij}, 2g \leq i < j \leq 3g - 3\) in terms of \(\check{\text{C}}\check{e}h\) cohomology is given by

\[
\begin{align*}
H^0(X, \Omega^1) & \quad \rightarrow \quad H^1(X, \mathcal{O}_X) \\
\omega & \quad \rightarrow \quad [\theta_j, \omega]_L^	heta ,
\end{align*}
\]

where \(L_\theta\) denotes Lie derivative along \(\theta_i\) and \([\theta_i] \in \hat{H}^1(T_X)\) corresponds to \([\mu_i] \in H^{0,1}(T_X)\). It is easy to see that \(\theta_j, \theta_0, \omega = \theta_j, \omega(\theta_i, \omega)\). Hence we need to show that \([\theta_j, \omega(\theta_i, \omega)]\) is mapped to \([\mu_j, \partial G(\mu_j, \omega) + \mu_j, \partial G(\mu_i, \omega)]\) by the natural isomorphism from \(\hat{H}^1(\mathcal{O}_X)\) to \(H^{0,1}_0\). The \(i = j\) case follows from almost the same method as below. By the natural isomorphism between \(\hat{H}^1(X, T_X)\) and \(H^{0,1}_0(X, T_X)\), we get \(\xi^1_i \in A^{0,0}(U_1, T_X)\) and \(\xi^0_i \in A^{0,0}(U_0, T_X)\) such that

\[
\begin{align*}
\xi^1_i - \xi^0_i &= \theta_i, \\
\bar{\partial}\xi^1_i &= \mu_i + \bar{\partial} f_i, 
\end{align*}
\]

where \(f_i \in A^{0,0}(X, T_X)\). If \(\mu_i\) can change in the Dolbeault cohomology class, we can assume that \(\mu_i|_{U_0 \cap U_1} = 0\). In fact, \(\mu_i\) is \(\bar{\partial}\)-closed and thus locally \(\bar{\partial}\)-exact, i.e., \(\mu_i = \bar{\partial}h_i\) on \(U_0 \cap U_1\). The desired representative can be chosen as \(\mu_i - \bar{\partial}(\rho h_i)\), where \(\rho\) is the suitable cut-off function. Moreover we can choose \(f_i\) such that \(f_i = \xi^0_i\) on \(U_0 \cap U_1\), for example \(f_i := \rho \xi^0_i|_{U_0 \cap U_1}\). By use of (5.1), on \(U_1\), we have

\[
\begin{align*}
\mu_j, \partial G(\mu_j, \omega) + \mu_j, \partial G(\mu_i, \omega) \\
= \bar{\partial}(\xi^1_i - f_i), \partial((\xi^1_i - f_j), \omega) + \bar{\partial}(\xi^1_i - f_j), \partial((\xi^1_i - f_i), \omega).
\end{align*}
\]

Similarly, we have an analogous equality on \(U_0\).

Now we shall identify the \(\check{\text{C}}\check{e}h\) and Dolbeault cohomology classes above. This question is equivalent to finding \(\phi^1_{ij}\) and \(\phi^0_{ij}\) belonging to \(A^{0,0}(U_1)\) and \(A^{0,0}(U_0)\), satisfying the following equations

\[
\begin{align*}
\begin{cases}
\bar{\partial}\phi^1_{ij} = \bar{\partial}(\xi^1_i - f_i), \partial((\xi^1_i - f_j), \omega) + \bar{\partial}(\xi^1_i - f_j), \partial((\xi^1_i - f_i), \omega), \\
\bar{\partial}\phi^0_{ij} = \bar{\partial}(\xi^0_i - f_i), \partial((\xi^0_i - f_j), \omega) + \bar{\partial}(\xi^0_i - f_j), \partial((\xi^0_i - f_i), \omega), \\
\phi^1_{ij} - \phi^0_{ij} = \theta_j, \partial(\theta_i, \omega).
\end{cases}
\end{align*}
\]

It is obvious that the solutions of the first two equalities of (5.2) always exist since the right hand sides of these two equalities are \((0, 1)\)-forms and clearly \(\bar{\partial}\)-closed. As

\[
\begin{align*}
\bar{\partial}(\xi^1_i - f_i), \partial((\xi^1_i - f_j), \omega) + \bar{\partial}(\xi^1_i - f_j), \partial((\xi^1_i - f_i), \omega) \\
= \bar{\partial}(\xi^1_i - f_i), \partial((\xi^1_i - f_j), \omega) - (\xi^1_i - f_i), \partial(\bar{\partial}(\xi^1_i - f_j), \omega) \\
+ \bar{\partial}(\xi^1_i - f_j), \partial((\xi^1_i - f_i), \omega) - (\xi^1_i - f_j), \partial(\bar{\partial}(\xi^1_i - f_i), \omega),
\end{align*}
\]

we can write \(\phi^1_{ij}\) as

\[
\begin{align*}
(\xi^1_i - f_i), \partial((\xi^1_i - f_j), \omega) + (\xi^1_i - f_j), \partial((\xi^1_i - f_i), \omega) \\
- \bar{\partial}^{-1}(\xi^1_i - f_i), \partial(\bar{\partial}(\xi^1_i - f_j), \omega) - \bar{\partial}^{-1}(\xi^1_i - f_j), \partial(\bar{\partial}(\xi^1_i - f_i), \omega),
\end{align*}
\]

where \(\bar{\partial}^{-1}(\xi^1_i - f_i), \partial(\bar{\partial}(\xi^1_i - f_j), \omega)\) stands for some solution \(g\) satisfying

\[
\bar{\partial}g = (\xi^1_i - f_i), \partial(\bar{\partial}(\xi^1_i - f_j), \omega).
\]
This notation is reasonable as the solution always exists. Thus
\[\phi_{ij}^1 - \phi_{ij}^0 = (\xi_j^1 - f_j) \partial ((\xi_j^1 - f_j) \omega) - (\xi_j^0 - f_i) \partial ((\xi_j^0 - f_j) \omega) + (\xi_j^0 - f_j) \partial ((\xi_j^0 - f_i) \omega) - (\xi_j^0 - f_i) \partial ((\xi_j^1 - f_j) \omega) \]
\[= (\xi_j^1 - \xi_j^0) \partial ((\xi_j^1 - f_j) \omega) + (\xi_j^0 - f_i) \partial ((\xi_j^0 - \xi_j^0) \omega) + (\xi_j^0 - f_j) \partial ((\xi_j^0 - f_j) \omega) \]
\[= (\xi_j^1 - \xi_j^0) \partial ((\xi_j^1 - f_j) \omega) + (\xi_j^0 - f_i) \partial ((\xi_j^0 - \xi_j^0) \omega) + (\xi_j^0 - f_j) \partial ((\xi_j^0 - f_j) \omega) \]
\[= (\xi_j^1 - \xi_j^0) \partial ((\xi_j^1 - f_j) \omega) + (\xi_j^0 - f_i) \partial ((\xi_j^0 - \xi_j^0) \omega) + (\xi_j^0 - f_j) \partial ((\xi_j^0 - f_j) \omega) \]
\[= (\xi_j^1 - \xi_j^0) \partial ((\xi_j^1 - f_j) \omega) + (\xi_j^0 - f_i) \partial ((\xi_j^0 - \xi_j^0) \omega) + (\xi_j^0 - f_j) \partial ((\xi_j^0 - f_j) \omega) \]
\[= (\xi_j^1 - \xi_j^0) \partial ((\xi_j^1 - f_j) \omega) + (\xi_j^0 - f_i) \partial ((\xi_j^0 - \xi_j^0) \omega) + (\xi_j^0 - f_j) \partial ((\xi_j^0 - f_j) \omega) \]
\[= (\xi_j^1 - \xi_j^0) \partial ((\xi_j^1 - f_j) \omega) + (\xi_j^0 - f_i) \partial ((\xi_j^0 - \xi_j^0) \omega) + (\xi_j^0 - f_j) \partial ((\xi_j^0 - f_j) \omega) \]
\[= \theta_j \partial (\theta_i \omega) + (\xi_j^0 - f_i) \partial ((\xi_j^0 - \xi_j^0) \omega) + (\xi_j^0 - f_j) \partial ((\xi_j^0 - f_j) \omega) \]
\[= \theta_j \partial (\theta_i \omega).\]

The penultimate equality results from
\[\dd (\xi_j^0 - f_i) \partial ((\xi_j^0 - f_i) \omega) = (\xi_j^0 - f_i) \partial (\dd (\xi_j^0 - f_i) \omega),\]
\[\dd (\xi_j^0 - f_j) \partial ((\xi_j^0 - f_j) \omega) = (\xi_j^0 - f_j) \partial (\dd (\xi_j^0 - f_j) \omega),\]
since we observe \(\dd \xi_j^0 = \dd \xi_j^1\) on \(U_0 \cap U_1\), and the last step stems from our choice of \(f_i\).

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