Nonequilibrium dynamics of closed interacting quantum systems.

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This colloquium gives an overview of recent theoretical and experimental progress in the area of nonequilibrium dynamics of isolated quantum systems. We particularly focus on quantum quenches: the temporal evolution following a sudden or slow change of the coupling constants of the system Hamiltonian. We discuss several aspects of the slow dynamics in driven systems and emphasize the universality of such dynamics in gapless systems with specific focus on dynamics near continuous quantum phase transitions. We also review recent progress on understanding thermalization in closed systems through the eigenstate thermalization hypothesis and discuss relaxation in integrable systems. Finally we overview key experiments probing quantum dynamics in cold atom systems and put them in the context of our current theoretical understanding.

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I. INTRODUCTION

In the past two decades the outlook of condensed matter physics has been deeply and unexpectedly revolutionized by a few experimental breakthroughs in atomic physics, quantum optics and nanoscience. In synthesis, crucial advances in these fields have made it possible to realize artificial systems (e.g. optical lattices, quantum dots, nanowires) that are described to a very high degree of accuracy by models (e.g. Hubbard, Kondo, and Luttinger models) whose physics has been a subject of intense investigation in various contexts ranging from high temperature superconductivity to low temperature transport in metals. It is fair to say that this experimental progress has changed the way theory and experiment look at each other. In the past, effective models were largely devised to explain the low energy physics of highly complex systems. The situation has now been reversed so that one can experimentally realize and simulate the physics of such models. On one hand, the design and realization of interacting many-body systems could in principle be used to perform practical tasks, such as quantum information processing [Farhi et al., 2001]. On the other hand, direct simulations of simple models could help resolving important problems in condensed matter physics. But most importantly, the availability of experimental controllable systems whose properties can be accurately described by simple models provides unprecedented opportunity to explore several new frontiers of condensed matter physics including the nonequilibrium dynamics in closed interacting quantum systems.

Equilibrium systems can often be understood using a combination of a mean field theory, renormalization group, and universality. This allows us to understand low temperature experimental data obtained in complex systems, such as interacting electrons in solids, in terms of simple effective models containing a few relevant parameters. Away from equilibrium the situation is much less clear. While some progress was made in the past for classical systems [Schmittmann and Zia, 1995], there are no rigorously justified generalizations of any of these approaches to generic quantum nonequilibrium systems. It is thus not obvious that the theoretical study of the dynamics of simplified models would accurately describe experiments of more complex systems. In addition there are fewer available tools for analyzing dynamics of even simple interacting models. In this respect cold atomic gases and nanostructures make possible what would be arduous otherwise: a fruitful comparison between nonequilibrium theories based on simple models and carefully designed experiments with tunable system parameters.

Finding systematic ways to understand the nonequilibrium physics of interacting systems is not only of fundamental importance, but could also be crucial for future technologies. A quantum computer, for example, will
definitely require the capability of performing real time manipulations of interacting quantum systems. Though large scale quantum computers are yet to be on the horizon, it is evident that quantum coherent dynamics will be one of the focus points of various experimental systems and of future technologies.

Nonequilibrium dynamics is a potentially a vast field: there are many ways to take a system out of equilibrium, such as applying a driving field or pumping energy or particles in the system through external reservoirs as in transport problems. It is thus of utter importance to focus on simple, yet fundamental protocols. In this Colloquium we will concentrate on the simplest paradigm: the study of the nonequilibrium dynamics of closed interacting quantum systems following a change in one of the system parameters (quantum quench). Such a change, which could be either fast or slow, is particularly interesting when it takes the system through a quantum phase transition involving macroscopic changes in the state of the many-body system at the initial and the final point. Seminal work in this direction includes groundbreaking experiments (Greiner et al., 2002a,b) showing both the feasibility of observing a quantum phase transition in cold atoms and the possibility of observing quantum coherent dynamics. Following this work, a number of different experiments explored the dynamics of cold atom systems driven across BCS-BEC crossover (Regal et al., 2006), polar and ferromagnetic phases of spinor condensates (Sadler et al., 2006), insulating and superfluid phases of ultracold bosons (Tuchman et al., 2006) and many others (see Ref. Bloch et al., 2008 for a review).

These experiments stimulated an active theoretical research in the relatively unexplored area of quantum dynamics in closed interacting systems. An interesting characteristic common to these systems is that despite of the absence of energy exchange with an environment and of the consequent global relaxation, it nevertheless frequently possible to look at the long time dynamics and characterize it in terms of an asymptotic state attained by physical (measurable) observables (Cramer et al., 2008; Flesch et al., 2008; Gogolin et al., 2011; Lin den et al., 2009; Reimann, 2008; Rigol et al., 2008). In connection to this, it is possible to categorize recent research on the subject of this Colloquium in two main questions:

- What is universal in the dynamics of a system following a quantum quench?

- What are the characteristics of the asymptotic, steady state reached after a quench? When is it thermal?

In this Colloquium we will discuss both of these questions extensively. We shall outline our current level of understanding of these issues and chart out the outstanding open questions in the field. In Sec. II we will focus on the first question and describe, from various points of view, the universal aspects of nearly adiabatic dynamics near quantum critical points as well as in generic gapped and gapless systems. We will argue that the proximity to the adiabatic limit allows us to make specific universal predictions of scaling of various quantities such as the defect density and heat with the quench rate.

In Sec. III we will discuss recent progress in understanding thermalization of a quantum system following a quench. In classical systems active interest in this topic was stimulated by the celebrated work of Fermi Pasta and Ulam on the dynamics of a one-dimensional (1D) anharmonic chain (Campbell et al. 2005; Fermi et al. May 1955; Porter et al. 2009) which demonstrated the absence of such thermalization. It was realized much later that the nonlinearity of the interaction is not sufficient for thermalization which occurs, in this system, only if the initial amplitude of interaction exceeds a certain threshold (Zirrailev and Chirikov 1966). Below this threshold, the solution splits into solitons and retains its quasi-periodic nature (Zabusky and Kruskal 1965) which is a consequence of the Kolmogorov-Arnold-Moser (KAM) theorem (Tabor 1989). In quantum systems, the question of sufficient criteria for thermalization has remained largely unaddressed so far. Some experimental progress in this direction has been made by a recent experiment from Kinoshita et. al. (Kinoshita et al. 2006) on non-thermalizing dynamics of 1D bosons with short range interactions. This experiment constitutes the first clear demonstration of the fact that a nearly integrable quantum interacting many-particle system does not thermalize for a very long time. Currently, the question of extension of KAM theorem to quantum systems is a subject of active theoretical debate (see e.g. Ref. Olshanii and Yurovsky, 2009).

Finally, we note that many important topics concerning the physics of closed interacting systems did not find space in this Colloquium. Most important among these are the tools that are being developed to describe theoretically the physics of interacting systems out of equilibrium. Among such methods we mention density-matrix renormalization group (DMRG) and time-evolving block decimation (TEBD) for analyzing equilibrium and nonequilibrium 1D systems (Schollwock 2005; Schollwock and White, 2006; Vidal, 2003; 2004; White 1992) and higher dimensional ones (Verstraete et al., 2008), the Keldysh technique (Kamenev and Levchenko, 2009) which is particularly helpful for deriving quantum kinetic equations, and closely related functional integral methods (Gasenzer, 2009; Plimak et al., 2001; Rey et al., 2005). Cold atom experiments also prompted rapid developments in phase space methods, where quantum dynamics is represented as an evolution in the classical phase space (Blakie et al., 2008; Polkovnikov, 2010). These methods were originally developed and applied to various problems in single-particle dynamics (Hillery et al., 1984; Zurek, 2003) and independently in quantum optics in the context of coherent states (Gardiner and Zoller, 2004; Walls and Milburn, 1994). There are other reviews available in literature
(see the references above) which specifically target these areas.

II. NEARLY ADIABATIC DYNAMICS IN QUANTUM SYSTEMS

A. Universality (in a nutshell)

Universality (or insensitivity to microscopic details) is one of the crucial concepts of modern condensed matter physics. It naturally emerged from one of the milestones of modern physics: the renormalization group. In condensed matter physics universality is a very powerful tool for the understanding of continuous (second order) phase transitions, both classical (Chaikin and Lubensky, 1995; Landau and Lifshitz, 1980) and quantum (Sachdev, 1999; Sondhi et al., 1997; Vojta, 2003). As a consequence of the divergence of the correlation length, a system undergoing such a continuous phase transition is typically scale invariant in the vicinity of the critical point and can be characterized by relatively simple massless field theories, which permit a classification of perturbations driving the system away from the critical point. Consequently, universality manifests itself in the scaling behavior of various quantities such as the order parameter, (free) energy, susceptibilities and correlations functions near the critical point. In this review we will focus mostly on quantum phase transitions occurring at zero temperature upon the variation of a control parameter $\lambda$ through a critical point $\lambda_c$. A standard example of universality is the fact that the exponent $\nu$ characterizing the divergence of the correlation length $\xi \sim 1/|\lambda - \lambda_c|^\nu$ near the quantum critical point (QCP) is insensitive to the microscopic details of the system and depends only on the universality class of the transition, determined by the dimensionality, overall symmetries and range of the interactions. For classical (thermal) phase transitions similar universality manifests in the divergence of the relaxation time $\tau_{rel} \sim 1/(|\lambda - \lambda_c|^{z\nu})$, where $z$ is the dynamical critical exponent. For quantum phase transitions the exponent $z$ can be associated with a vanishing energy scale $\Delta \sim |\lambda - \lambda_c|^{z\nu}$, which can be either a gap or a crossover scale where the spectrum changes qualitatively. By the uncertainty principle this energy scale corresponds to a divergent time scale, which typically describes the crossover in the scaling behavior of unequal time correlation functions. Phase transitions can be also characterized by singular susceptibilities, which are in turn connected through the fluctuation-dissipation theorem to the correlation functions of conjugate variables (e.g., the magnetic susceptibility is related to the correlation function of the magnetization). At critical points these correlation functions have often power law scaling behavior at long distances, e.g. $(m(x)m(x')) \approx 1/|x - x'|^{2\alpha}$. The exponent $\alpha$ sets the scaling dimension of the corresponding operator $m(x)$: $\dim[m(x)] = \alpha$. Because similar correlation functions can enter different susceptibilities not all the scaling exponents are independent but must satisfy scaling relations (Chaikin and Lubensky, 1995; Vojta, 2003).

As mentioned in the introduction, the idea of universality makes it possible to interpret experimental data obtained in real systems in terms of effective models with a few parameters. Universality can be ultimately understood using the renormalization group, which shows that as a system is coarse grained to lower energies and longer length scales, more and more parameters of its original, ab initio description become unimportant (irrelevant), while the remaining few (relevant) parameters define an effective low energy model. A standard example of universality in this context is the scaling relation between energy and momentum of elementary excitations, $\epsilon \propto k^z$, controlled by the dynamical exponent $z$ which depends on the symmetries of the system. In particular, $z = 1$ in most phases characterized by a continuous broken symmetry (crystals, superfluids, ferromagnets), $z = 2$ in ferromagnets, where there is an additional conservation law of the order parameter.

Universality is well established and understood in equilibrium. It is, however, crucial for many experimentally relevant situations to understand the extent to which this concept can be extended to out of equilibrium physics. Can irrelevant interactions turn out to be important away from equilibrium? Since there are many ways to take a system out of equilibrium, for which specific protocols will universality emerge and which details of the protocol are potentially important? Below we will focus on recent studies addressing these important issues in closed interacting quantum systems, and in particular on the dynamics of a system whose parameters are dynamically tuned either through a quantum critical point, or in general within a gapless/gapped phase.

B. Universal dynamics near quantum critical points

Let us start by considering the simplest nonequilibrium protocol (Dziarmaga, 2005; Polkovnikov, 2005; Zurek et al., 2005): the system is prepared in its ground state and is then driven through a QCP by changing an external parameter $\lambda$ in time. As long as the rate change of the gap in the spectrum $\Delta$ caused by changing $\lambda$ is smaller than the square of the gap one can expect the system to approximately follow the ground state adiabatically (we will revisit this statement in the next section). However, the vanishing of the gap at $\lambda = \lambda_c$ implies that the system will always violate adiabaticity close to the quantum critical point, no matter how slowly the parameters are changed. It is then natural to ask how many excitations

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1 We use the word “gap” for brevity. However, the system can be gapless on one or both sides of the transition (e.g., superfluid-insulator transition). Then $\Delta$ would denote a crossover energy scale vanishing at the QCP.
A similar question known under the name of the Kibble-Zurek mechanism (KZ) (Kibble 2007, 1976; Zurek, 1985, 1996) has been addressed in the past decades for classical phase transitions. In that case the excitation density of defects is described by a simple scaling argument (Zurek, 1985, 1996). Suppose that the tuning parameter, for example external temperature $T$, slowly decreases in time across the critical value $T_c$: $T = T_c - \nu t$. The system will respond adiabatically (quasi-statically if the system is not thermally insulated) up to some close vicinity of the critical point, where adiabaticity will be violated as a result of the divergence of the relaxation time ($\tau_{rel} \sim 1/(T - T_c)^{z\nu}$) and the dynamics will become diabatic (sudden). The adiabatic response is once again resumed after the system moves out of the vicinity of the critical point. Zurek suggested a very simple criterion for separating such adiabatic and diabatic (impulse) regimes: the time to reach the critical point $t = |T - T_c|/\nu$ should be equal to the relaxation time. This immediately introduces the time and length scales characterizing the adiabatic to diabatic crossover: $t^* \sim 1/|\nu|^z/(z\nu+1)$, $\xi^* \sim 1/|\nu|^z/(z\nu+1)$. The violation of adiabaticity implies that order can not form on distances larger than $\xi^*$ leading to the formation of a domain structure with a characteristic distance $\xi^*$ between the domain boundaries. In two and three dimensional systems, when the order parameter is characterized by a continuous broken symmetry, the points where several domains meet correspond to vortices or vortex rings. These are robust topological excitations with a very long life time (see Fig. 1). Since $\xi^*$ determines the average distance between the defects their density is given by a simple universal expression

$$n_{ex} \sim (\xi^*)^d \sim |\nu|^{dz/(z\nu+1)}. \tag{1}$$

The universality of the KZ prediction above is manifest in the appearance of the universal critical exponents $z$ and $\nu$ in the scaling law. This scaling was confirmed in experiments in liquid crystals (Ducci et al., 1999). Experiments in other systems (superconductors (Maniv et al., 2003), arrays of Josephson junctions (Monaco et al., 2006)) observed the production of topological defects with a power law scaling on the quench rate but gave a different exponent. The KZ scaling was also confirmed theoretically using stochastic dynamics (Ginzburg-Landau dynamics with Langevin noise or Glauber dynamics) where temperature changes in time (Krapivsky, 2010; Laguna and Zurek, 1997; Rivers, 2001; Yates and Zurek, 1998), though there are also works suggesting various modifications (Biroli et al., 2010; Hindmarsh and Rajantie, 2000). One can also interpret Eq. (1) as a measure of non-adiabaticity near the critical point. It is naturally expected that other measures like non-adiabatic energy production and entropy generation will display similar universality. These measures might be preferable over $n_{ex}$ in situations where it is difficult to identify defects.

Finding the scaling of these quantities remains an open question.

The arguments above were recently generalized to the crossing of quantum phase transitions (Dziarmaga, 2005, 2006; Polkovnikov, 2005; Zurek et al., 2005) for recent reviews on this subject see Refs. Dziarmaga, 2010; Gritsev and Polkovnikov, 2010. As discussed before in the quantum case the parameter to be varied is not temperature $T$ but rather the coupling $\lambda$ tuning the system through the quantum critical point. In order to obtain the scaling for the number of excitations produced in the quantum
case let us first recall the Landau-Zener analysis of the crossover between adiabatic and nonadiabatic dynamics in a simple driven two-level system:

$$H_{\text{LZ}} = g(t)\sigma_z + \Delta\sigma_x,$$

where $g(t) = vt$. If the system was initially prepared in the ground state at $t \to -\infty$, the probability of transition to the excited state at $t \to +\infty$ is (Landau 1932, Majorana 1932, Stückelberg 1932, Zener 1932)

$$p_{\text{ex}} = \exp[-\pi \gamma],$$

where we introduced the Landau-Zener parameter $\gamma = \Delta^2/\nu$. Notice that the limit $\gamma \gg 1$ corresponds to the adiabatic limit with an exponentially suppressed transition probability while $\gamma \ll 1$ corresponds to the diabatic limit where the transition happens with probability close to unity. Hence when the rate of change of the energy splitting between two levels becomes larger or comparable to the energy splitting squared one observes a crossover from adiabatic to diabatic dynamics. An alternative qualitative explanation of this result has been recently formulated (Damski and Zurek 2006).

The Landau-Zener argument can be straightforwardly extended to the crossing of a QCP. The characteristic energy scale which changes in time is now the gap $\Delta$. As we discussed earlier this gap universally depends on the tuning parameter $\lambda$: $\Delta(\lambda) \sim |\lambda - \lambda_c|^{2\nu} \sim |vt|^{2\nu}$, where we assumed that the dependence $\lambda(t)$ can be linearized near the QCP: $\lambda(t) \approx \lambda_c + vt$. Comparing the rate of change of the gap with its square, i.e. solving the equation $d\Delta/dt \approx \Delta^2$, we find the energy scale at which adiabaticity breaks down is $\Delta^* \sim |v|^{2\nu/(2\nu+1)}$. At this point the system is characterized by the length scale $\xi^* \sim |v|^{-(2\nu+1)/(2\nu+1)}$, which can be interpreted as the typical length scale of fluctuations of the order parameter. Beyond this point the adiabatic approximation breaks down and fluctuations at longer scales cannot adiabatically follow the ground state. This results in the creation of defects with typical distance $\xi$ between them and density $n_{\text{ex}} \sim |\xi|^{2\nu} \sim |v|^{2\nu/(2\nu+1)}$. This scaling is identical to the classical one, Eq. (1), with $\lambda \to T$, and was proposed independently in Refs. (Polkovnikov 2005, Zurek et al. 2005). There is a simple quasi-particle interpretation for this scaling: assuming that the excitations in the system are characterized by isolated quasi-particles then their density can be found from $n_{\text{ex}} \approx \int_0^\Delta d\epsilon \rho(\epsilon)$, where $\rho(\epsilon)$ is the single-particle density of states near the QCP. In uniform $d$-dimensional systems $\rho(\epsilon) \sim \epsilon^{d/2-1}$, which again reproduces Eq. (1).

The scaling in Eq. (1) was verified in a series of exact solutions of the dynamics across the QCP in integrable systems whose dynamics can be mapped into a series of Landau-Zener transitions of a few quasi-particle modes. In particular, it has been verified for various spin models in one and two dimensions which can be mapped to noninteracting fermions (Mukherjee et al. 2007), for models where low energy excitations near phase transitions can be described by bosonic Goldstone modes (Dziarmaga et al. 2008, Lamacraft 2007, Polkovnikov 2005), the sine-Gordon model, where elementary excitations are solitons and breathers with fractional statistics (De Grandi et al. 2008, 2010a), graphene (Dóra et al. 2010, Dóra and Moessner 2010). This scaling was also extended to disordered systems, like a disordered Ising spin chain, where it was found that $n_{\text{ex}} \sim 1/\log^2(v)$ (Caneva et al. 2007, Dziarmaga 2006), as expected from Eq. (1) due to the divergence of the exponent $z$ near the critical point (Fisher 1995).

The scaling in Eq. (1) can be generalized to the case of nonlinear dependence of the tuning parameter on time, $\lambda(t) \sim \lambda_c \pm \nu|vt|^r$, where considerations similar to those leading to Eq. (1) give (Barankov and Polkovnikov 2008, De Grandi et al. 2010a, Sen et al. 2008):

$$n_{\text{ex}} \sim |v|^{d\nu/(z\nu+1)},$$

In all cases $v$ in Eq. (4) plays the role of the adiabatic parameter: the limit $v \to 0$ corresponds to the adiabatic limit (this interpretation is valid even for instantaneous quenches $r = 0$, where $v$ plays the role of the quench amplitude). This suggests the dynamics can be systematically analyzed using adiabatic perturbation theory (De Grandi and Polkovnikov 2010, Polkovnikov 2005, Rigolin et al. 2008), i.e. expanding the transition amplitudes to the instantaneous eigenstates of the system in powers of $v$. Using such analysis in Refs. (De Grandi et al. 2010a,b) it was shown that the scaling (4) can be derived from the scaling of the adiabatic fidelity, defined as the overlap of the time-dependent wave function with the instantaneous ground state:

$$F(t) = |\langle \psi(t) | \psi_0(0) \rangle|.$$ (5)

In particular, for $\lambda(t) = \lambda_c + vt^r/\theta(t)$, where $\theta(t)$ is a step function,

$$P_{\text{ex}}(v) = 1 - F(t)^2 \approx L^d |\psi|^2 \chi_{2r+2}(\lambda_c),$$

where

$$\chi_{2r+2}(\lambda) = \frac{1}{L^d} \sum_{n \neq 0} |\langle n | V | 0 \rangle|^2$$

is the adiabatic (fidelity) susceptibility of the order $2r+2$ ($\chi_2$ is the conventional fidelity susceptibility (Gu and Lin 2009)). Here $E_n$ are the eigenenergies and $V$ is the operator coupled to the parameter $\lambda$: $V = \partial_\lambda H(\lambda)|_{\lambda = \lambda_c}$. If the perturbation is local and spatially uniform , i.e. $V = \int dx u(x)$, then the scaling dimension of the adiabatic fidelity susceptibility is obtained from a straightforward generalization of the result of Refs. (Gu and Lin 2009, Vemuri and Zanardi 2007), i.e. $\text{dim}[\chi_{2r+2}] = 2\Delta_u - 2z(r + 1) - d$, where $\Delta_u \equiv \text{dim}[u]$ is the scaling dimension of $u(x)$.

Let us now discuss from this general perspective the arguments leading to the generalized scaling relation...
Eq. (4). If the scaling dimension of the susceptibility is negative, this implies that $\chi_{2z+2}$ diverges at the critical point. In this case from Eq. (6) we find that asymptotically at $v \to 0$

$$P_{\text{ex}}(v) \sim |v|^2 L^{2d+2z(r+1)-2\Delta_u}.$$  (8)

From Eq. (8) we see that the probability of exciting the system becomes of order one when $L \sim 1/|v|^{1/(d+z(r+1)-\Delta_u)}$. This length scale can be interpreted as the typical distance between elementary excitations (defects) and thus we find that instead of Eq. (8) we get

$$n_{\text{ex}} \sim |v|^{d/(d+z(r+1)-\Delta_u)}.$$  (9)

This expression reduces to Eq. (4) if $u(x)$ is a relevant operator driving the system to the new phase. Indeed, in this case $\lambda \int d^d x u(x)$ should have the same scaling dimension as the gap, i.e. $z$, which immediately implies that the scaling dimension of $u(x)$ is $\Delta_u = d - z - 1/\nu$ and that $\dim u = d - 2z - 2/\nu$ [De Grandi et al. 2010] [Schwandt et al. 2009]. Notice finally that if the scaling dimension of $\chi_{2z+2}$ is positive then the asymptotics in Eq. (8) gives a subleading correction to the regular analytic part, $P_{\text{ex}} \approx \text{const} \, L^d \, v^2$, coming from the high energy (ultra-violet) contribution to the susceptibility. We will discuss its importance in the next section.

Other possible generalizations of the scaling law Eq. (4) involve studies of defect production in systems where the dynamics describes the passage through quantum critical lines. A concrete example of such a situation occurs in the transverse-field XY model [Divakaran et al. 2008] [Mukherjee et al. 2008]. Here the quench takes one through a gapless line where the critical point occurs at the same momenta ($\hat{k} = 0$ for the present case) at each point on the line. A detailed analysis shows that in such cases, for critical lines with $z = 1$, the defect density still obeys a universal scaling law albeit with a different power: $n \sim |v|^{1/2}$ [Divakaran et al. 2008] [Mondal et al. 2009] [Mukherjee et al. 2008]. The second situation involves the 2D Kitaev model [Sengupta et al. 2008], where a quench once again involves the passage through a gapless line with an energy gap vanishing for different momenta at different points on the line. It can be shown that in such a case, the defect density scales as $\sqrt{|v|}$ for 2D Kitaev model instead of the expected $n \sim |v|$ behavior for 2D systems with $z = \nu = 1$ [Sengupta et al. 2008]. A generalization of these results for linear quenches through critical lines with arbitrary $z$ and $\nu$ has also been worked out [Mondal et al. 2008] [2009] [Mukherjee et al. 2008] [Sengupta et al. 2008]. Many other situations involving anisotropic phase transitions and quenching through multi-critical points were analyzed in literature leading to various deviations from the scaling [4] (Bermudez et al. 2009) [Deng et al. 2008] [Sen and Vishveshwara 2010].

In order to detect experimentally the density of excitations generated by passing through a QCP, it is evident that one should distinguish between situations where such excitations are long-lived quasi-particles (as for nearly integrable systems) or decay after being created (as for non-integrable systems). In the first case, the presence of excitations above the ground state could for example be detected by measuring correlation functions long after the quench. This has been shown for a Quantum Ising chain linearly tuned through its quantum critical point [Cherng and Levitov 2006]. The presence of defects with respect to the ferromagnetic ground state lead to exponentially decaying correlations of the order parameter superimposed, for slow enough quenches, to characteristic oscillations with period scaling with the quench velocity. This second feature is observed for abrupt quenches as well [Sengupta et al. 2004] and is a consequence of the integrability of the model [Rossini et al. 2010]. In turn we consider a generic non-integrable system, it is necessary to express deviations from adiabaticity in terms of quantities, such as the excess energy or the entropy generated by passing through the QCP, which are not sensitive to the decay of quasi-particles, but can still be related to the density of excitations created close to the quantum critical point. Energy can be unambiguously determined both experimentally and numerically for both integrable and non-integrable systems and its scaling with the rate of quench can be used to differentiate between different nonadiabatic regimes [Eckstein and Kollar 2010] [Moeckel and Kehrein, 2010] [Polkovnikov and Gritsev, 2008]. The excess energy or equivalently heat ($Q$ [Polkovnikov, 2008]) generated during the quench process is in general universal if the process ends near the critical point. The scaling of $Q$ is associated with the singularity of the susceptibility $\chi_{2z+1}$ at the critical point [De Grandi and Polkovnikov, 2010], which implies that for relevant perturbations in the thermodynamic limit

$$Q \sim |v|^{(d+z)/2d(zr+1)}.$$  (10)

Unless one considers cyclic processes, the drawback of using heat as a measure of non-adiabaticity is that it is hard to separate it from the adiabatic part of the energy change, corresponding to the limit $v \to 0$. Moreover, if the position of the QCP is not exactly known, the heat becomes sensitive to the nonuniversal details of the spectrum at the final point of the evolution. A way out could be to measure the higher moments of the energy or the whole distribution function of the energy, connected to the statistics of the work [Silva, 2008] [Talkner et al. 2007] in finite size systems. In particular, in the case of abrupt quenches close to critical points the statistics of the work is characterized by sharp edge singularities [Paraan and Silva, 2009] [Silva, 2008]. A related natural measure of non-adiabaticity is obtained by focusing on the entropy since entropy is conserved only for slow (adiabatic) processes, while is expected to increase as the system passes through the QCP. Moreover entropy production can be detected experimentally in certain systems, e.g. in cold atoms by driving the system to the weakly interacting regime, where the relation between entropy and energy
is known (Luo et al. 2007). Theoretically, the quantification of entropy production in a closed quantum system is rather subtle. Indeed, the von Neumann’s entropy of the entire system, being conserved throughout unitary evolution (Landau and Lifshitz 1980), cannot be a good characterization of deviations from adiabatic dynamics. However, the concept of diagonal-entropy (Polkovnikov 2008a), defined as $S_d = -\sum_n \rho_{nn} \ln(\rho_{nn})$, where $\rho_{nn}$ are the diagonal matrix elements of the density matrix in the instantaneous basis, avoids this difficulty. In stationary systems, the diagonal entropy is nothing but the von Neumann’s entropy of the time averaged density matrix, also called diagonal ensemble. It is clear that the diagonal entropy is generated only due to nonadiabatic transitions and thus satisfies the key requirement of the thermodynamic entropy: it is conserved for adiabatic processes, and can only increase or stay constant in closed systems if the initial state is stationary (Polkovnikov 2008a). For initial equilibrium states the diagonal entropy also satisfies fundamental thermodynamic relation: $dE = TdS - Fd\lambda$, where $F = -\langle \partial_\lambda H \rangle$ is the generalized force. For particular noninteracting models, the scaling of the diagonal entropy was found to be the same as that of the density of quasi-particles (De Grandi et al. 2010a,b) and Mukherjee et al. (2008). It is also possible to analyze the entanglement entropy (Calabrese and Cardy 2004, 2005; Rafael and Moore 2004; Vidal et al. 2003), i.e. the von Neumann’s entropy of the reduced density matrix of a part of the system, and in particular at its time evolution following a quench (Cincio et al. 2007; Pollmann et al. 2010; Sengupta and Sen 2009). For specific 1D spin systems it was found that the entanglement entropy scales logarithmically with the quench time (Cincio et al. 2007; Pollmann et al. 2010). Notice however that, at the moment, it is unclear how one can measure entanglement in many body systems and the entanglement entropy in particular (see some suggestions in Refs. (Klich and Levitov 2009; Klich et al. 2006)) and what its relation with the thermodynamic entropy is.

Finally, another interesting question that has received attention is the connection between microscopic dynamics and thermodynamics in the semiclassical limit. In general, in classical systems there is no simple analogue to the instantaneous energy levels, the key concept in analysis of quantum systems. Such analogue, however, does exist in the case of periodic motion. Then in the semiclassical limit the stationary levels are found from the Bohr quantization (or more accurately from the WKB approximation (Landau and Lifshitz 1981)), which states that the reduced action in the stationary orbit should be quantized. In classical mechanics it is known that the reduced action is an adiabatic invariant, i.e. it is conserved for the adiabatic evolution (Landau and Lifshitz 1982). From the previous discussion applied to quantum systems we can deduce that near singularities like second order phase transitions, conservation of adiabatic invariants should be violated and this is indeed the case (Landau and Lifshitz 1982). In Refs. (Altland et al. 2009; Itin and Torma 2009a,b) slow dynamics was analyzed for a particular many-body generalization of the Landau-Zener model (closely related to the Dicke model) in the semiclassical limit. It was found that the nontrivial power law scaling of the number of excitations in this system (similar to Eq. (1)) follows from the changes of adiabatic invariants near the singularity, which in turn corresponds to a quantum critical point in the thermodynamic limit. It is interesting that quantum fluctuations in this problem entered only through the initial distribution of the adiabatic invariants but not through the equations of motion. The corresponding truncated Wigner approximation turned out to be very accurate in all regimes of the dynamics (Altland et al. 2009; Kiefer 2009). It would be very important to understand precise connections between transitions among microscopic energy levels in the quantum case and changes of suitable generalizations of adiabatic invariants in the classical limit.

C. Slow dynamics in gapped and gapless systems.

Up to now we have discussed the universal dynamics resulting from the variation of a control parameter $\lambda$ through a quantum critical point. However the dynamics of interacting quantum systems has interesting regimes even when the system is fully gapped or gapless for the entire duration of the protocol. The classification of these regimes is important in order to understand dissipation and to develop optimal protocols minimizing non-adiabatic effects. Many of these questions are currently a subject of intense theoretical research in different contexts, from quantum computation to transport.

The general formulas Eq. (6)-(7), which describe the density of excitations $P_{\text{ex}}(v)$ generated by a variation of the control parameter $\lambda$ through a quantum critical point, tell us that if the system remains fully gapped throughout the evolution, then $P_{\text{ex}}$ and $n_{\text{ex}}$ will have a quadratic scaling with $v$ whenever the susceptibility $\chi_{2r+2}$ evaluated at the initial and final couplings is finite (De Grandi and Polkovnikov 2010; Rigolin et al. 2008). A similar argument shows that the heat $Q$ is also quadratic in $v$ if $\chi_{2r+1}$ is finite. This quadratic scaling is characteristic of any quantum system. Let us point that in the standard Landau-Zener problem in the slow limit $\Delta^2 \gg v$ if we start in the ground state at $t \rightarrow -\infty$ and let the system evolve up $t \rightarrow +\infty$, the transitions to excited states are exponentially suppressed as a result of destructive interference between multiple transitions (Vitanov 1999; Vitanov and Garraway 1996). At the same time in the intermediate stages of the evolution the transition probability reaches much higher values which scale only quadratically with the rate $v$. For example, if one considers a process which starts at $t_0 \rightarrow -\infty$ the transition probability to the instantaneous excited state at the moment $t$ in the slow limit $v/\Delta^2 \ll 1$ can be approximated by (Vitanov 1999):

$$p_{\text{ex}} \approx \frac{v^2 \Delta^2}{16(g(t))^2 + \Delta^2)^3},$$

(11)
If \( t > 0 \) there is an additional exponential term which leads to Eq. (3) at \( t \to +\infty \). If the process starts at \( t_0 = 0 \) exactly in the symmetric point, where \( g(0) = 0 \) the quadratic asymptotics (11) is also recovered [Cucchi et al. 2007]. This scaling occurs as a result of the discontinuity of the first derivative of \( g(t) \) at the moment where the process starts or ends or following a discontinuity in any other point of the evolution (see e.g. Refs. [Damski and Zurek, 2006; Divakaran et al. 2010] for particular cases). Likewise if there is a discontinuity in the second order derivative of \( g(t) \) asymptotically the transition probability in the LZ problem scales quadratically with acceleration. More generally for the protocol \( g(t) = g_0 + v(t - t_0)^r/r! \theta(t - t_0) \), where \( \theta(t) \) is the step function, one can show that [De Grandi and Polkovnikov 2010]

\[
p_{\text{ex}}(t \to \infty) \approx \frac{v^2\Delta^2}{16(g_0^2 + \Delta^2)^{2r+1}}.
\]

As we discussed in the previous section this formula applies even to sudden transitions \((r = 0)\) where it reduces to the result of the ordinary perturbation theory. The same expression applies to the reverse process. If both the initial and final couplings are finite then the resulting transition probability is asymptotically determined by the sum of probabilities associated with discontinuities of derivatives of \( g(t) \) at the initial and final times of the evolution plus additional interference terms which highly oscillate in the slow limit. Let us point out that in the LZ problem (and in general in gapped systems) one can suppress power law asymptotics of the transition probability by starting and ending the protocol infinitely smoothly, e.g. \( g(t) \sim g_0 + g_1 \exp[-r/(t - t_0)]\theta(t - t_0) \). In this case only the non-analytic term in the transition probability survives and we are back to Eq. (3) where \( v \) is the time derivative of \( g(t) \) near the symmetric point where \( g(t) = 0 \).

In gapless systems the situation becomes qualitatively different. In this case the adiabatic susceptibilities can diverge leading to non-adiabatic dependence of the corresponding quantities on \( v \), as in the case of the crossing of a quantum critical point. For example, it is straightforward to see that for marginal perturbations in a generic gapless phase the scaling dimension of the adiabatic susceptibility \( \chi_{2r+2} \) equals \( d - 2zr \). It becomes negative in low dimensions \( d < 2zr \) leading to a non-adiabatic scaling of the density of excitations with \( v \). Thus depending on dimensionality in gapless systems one expects at least two different regimes of the response of the system to a slow external perturbation: analytic and non-adiabatic. These regimes were first suggested in Ref. [Polkovnikov and Gritsev 2008] together with a third regime where adiabaticity is violated in the thermodynamic limit and \( Q \) or \( n_{\text{ex}} \) become proportional to a power of the system size or some other large length scale associated with some irrelevant operator. In this regime, which can be realized in low-dimensional bosonic systems, the scaling Eq. (1) is violated. At the moment it is unclear how generic it is and what sets the scaling of various quantities.

A close inspection of the adiabatic (quadratic) susceptibility shows that in general the analytic (quadratic) part of the heat and energy of excitations on \( v \) comes from the high energy (or ultra-violet) part of the spectrum, while the non-adiabatic part comes from low energies. This was indeed shown to be the case in several situations, from the sine-Gordon model [De Grandi et al. 2010b], to the Falicov-Kimball model [Eckstein and Kollar 2010] and the turning on interactions in a Fermi liquid [Moeckel and Kehrein 2010]. As we pointed above the ultra-violet transitions can be suppressed by avoiding discontinuities in \( \lambda(t) \) and its derivatives. However, this is not necessarily the case for the low energy non-adiabatic contribution.

To see this we need to reexpress the excess energy (or density of excitations) in terms of the total time of the process \( \tau \). Doing this it was found that in an insulating, gapped phase, the details of the protocol are important and smoother protocols lead to a suppression of non-adiabatic effects, while in a gapless phase making \( \lambda(t) \) smoother does not affect the heating [Eckstein and Kollar 2010]. This result can be again understood by analyzing the scaling dimension of the susceptibility \( \chi_{2r+1} \). According to our discussion for generic gapless systems its scaling dimension is negative when \( d + z < 2zr \). Then \( Q \sim v^{(d+z)/zr} \) (this result immediately follows from Eq. (10) by taking the limit \( v \to \infty \)). On the other hand, \( v \) is related to the total quench time as \( v \sim 1/\tau^r \). Thus we see that in this case \( Q \sim 1/\tau^{(d+z)/z} \), i.e. indeed independent on \( r \). On the other hand for positive scaling dimension of \( \chi_{2r+1} \) which is the case for \( d + z > 2zr \) and which is always true in gapped systems we have \( Q \sim v^2 \sim 1/(\tau^2) \). Since in the adiabatic limit \( \tau \) is large we see that indeed the heat can be suppressed by increasing \( r \) and making \( \lambda(t) \) smoother. An interesting open question is finding an optimal protocol for minimizing the non-adiabatic effects within given time \( \tau \). It is plausible that the optimal power is determined by a vanishing scaling dimension of the corresponding adiabatic susceptibility \( \chi_{2r+1} \). The questions of finding protocols minimizing non-adiabatic effects for gapped systems (with possibility of crossing isolated quantum critical points) were also addressed by approximately minimizing the transition probability and identifying the Riemannian metric tensor underlying the adiabatic evolution [Rezakhani et al. 2010, 2009]. Studying the optimization of the protocol taking a system through a QCP it was found the optimal exponent of \( \lambda(t) \sim |t|^{d/zr} \) near the QCP scales logarithmically with the quench time, \( r \propto \ln(\tau) \) [Barankov and Polkovnikov 2008]. This result was also extended to systems with external confining potential [Collura and Karevskii 2010].

**D. Effects of finite temperature.**

In the discussion above we always implicitly assumed that the system is initially prepared in the ground state.
An interesting and genuine question is how finite temperature effects modify the picture. In isolated systems temperature enters through initial conditions: the system is prepared in the initial finite temperature equilibrium state and is then dynamically driven out of equilibrium. How is the response of the system affected by the initial thermal fluctuations? One naturally expects that while the transitions to high energy states (quadratic in \( \upsilon \)) will not be affected by small temperatures in the system, the transitions to the low energy states, which determine the non-analytic contribution to heat and density of excitations, will be very sensitive to temperature. In Refs. (De Grandi et al., 2010a,b) (see also Ref. (Gritsev and Polkovnikov, 2010)) studying a particular Sine-Gordon model in the two limits where it could be mapped to free bosons and free fermions, it was shown that the statistics of quasi-particles enters the scaling of both \( Q \) and \( n_{\text{ex}} \) making dynamics more adiabatic (compared to the zero temperature case) for fermions due to Pauli blocking and less adiabatic for bosons due to Bose enhancement. These results were not yet extended to generic interacting systems.

Another aspect of thermalization, the influence of the coupling to an environment setting the temperature on the slow dynamics near quantum critical points, has been studied in Refs. (Fubini et al., 2007; Mostame et al., 2007; Patanè et al., 2008; Patanè et al., 2009a; Patanè et al., 2009b). This setup allows one to analyze the effects of thermal smearing and of dephasing/dissipation on the dynamics of a quantum critical system. Using a combination of kinetic equations and scaling arguments it was found that in this situation the excess energy has two universal contributions, one still given by Eq. (10), while the second involving a universal power of temperature replacing the universal power of \( \upsilon \) (Patanè et al., 2008).

### E. Open problems

While the physics described above is definitely an important example of the emergence of universality in the dynamics of interacting quantum systems, it is evidently a piece, albeit important, of the puzzle that has to be composed in order to understand to which extent the standard concepts of statistical physics can be applied to nonequilibrium problems. Understanding the meaning of relevance or irrelevance of a perturbation in generic nonequilibrium processes, extending the notion of universality to nonequilibrium systems, as well as the concept of renormalization group, is a task (or dream) that certainly requires the solution of many specific problems, and a close comparison between experiments and theory.

So far most of the theoretical research focused on analyzing slow dynamics for global quenches, where the external perturbation couples to the whole system. How these results can be extended to local or spatially nonuniform perturbations is an open question. At one extreme limit, one can imagine performing a quench only locally.
As an effective thermal bath? And if this is not possible, are there some observable effects on the system dynamics? While these questions are definitely connected to quantum ergodicity [Goldstein et al., 2010], a topic with a long history dating back to the early days of quantum mechanics [Deutsch, 1991; Mazur, 1968; von Neumann, 1929; Pauli and Fierz, 1937; Peres, 1984; Srednicki, 1994; Suzuki, 1971], the past few years have brought a great deal of progress in the context of closed many-body systems. The main motivation came from recent experiments on low dimensionnal cold atomic gases described in some detail in Sec. [IV] in this Colloquium [Greiner et al., 2002b] Kinoshita et al. et al., 2006). The experimental availability of essentially closed (on the time scales of experiments) strongly correlated systems together with the awareness of the conceptual importance of these issues in a number of areas (e.g. transport problems, many-body localization, integrable and non-integrable dynamics) have stimulated a lot of interest on quantum thermalization. Below we will give a synthetic view on a number of recent important developments on this subject, starting with the discussions of the general concepts of ergodicity and thermalization, and then moving to the discussion of many-body systems and integrability.

A. Quantum and classical ergodicity.

While the idea of ergodicity is well defined in classical mechanics, the concept of quantum ergodicity is somewhat less precise and intuitive. Classically, an interacting system of \( N \) particles in \( d \) dimensions is described by a point \( X \) in a \( (2^d N) \)-dimensional phase space. The intuitive content of the word "ergodic", i.e. the equivalence of phase space and time averages, can be then formalized by requiring that if we select an initial condition \( X_0 \) having initial energy \( H(X_0) = E \), where \( H \) is the Hamiltonian of the system, then

\[
\delta(X - X(t)) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \delta(X - X(t)) = \rho_{mc}(E),
\]

where \( \rho_{mc}(E) \) is the microcanonical density of the system on the hyper-surface of the phase space of constant energy \( E \), and \( X(t) \) is the phase space trajectory with initial condition \( X_0 \). Of course if this condition is satisfied by all trajectories, then it is also true for every observable. We immediately see that in order to have ergodicity, the dynamics cannot be arbitrary: the trajectories \( X(t) \) have to cover uniformly the energy hyper-surface for almost every initial condition \( X_0 \).

The most obvious quantum generalization of this notion of ergodicity is arduous [von Neumann, 1929]. Let us first of all define a quantum microcanonical density matrix: given a Hamiltonian with eigenstates \( | \Psi_\alpha \rangle \) of energies \( E_\alpha \), a viable definition of the microcanonical ensemble is obtained by coarse graining the spectrum on energy shells of width \( \delta E \), sufficiently big to contain many states but small on macroscopic scales. Noting by \( \mathcal{H}(E) \) the set of eigenstates of \( H \) having energies between \( E \) and \( E + \delta E \), we define \( \hat{\rho}_{mc}(E) = \sum_{\alpha \in \mathcal{H}(E)} \frac{1}{N} | \Psi_\alpha \rangle \langle \Psi_\alpha | \), where \( N \) is the total number of states in the micro-canonical shell. Let us now ask the most obvious question: given a generic initial condition made out of states in a microcanonical shell, \( | \Psi_0 \rangle = \sum_{\alpha \in \mathcal{H}(E)} c_\alpha | \Psi_\alpha \rangle \), is the long time average of the density matrix of the system given by the micro-canonical density matrix? The answer to this question for a quantum system is, unlike in the classical case, almost always no, as J. von Neumann realized already in 1929 [von Neumann, 1929]. More precisely, if we assume the eigenstates of the system not to be degenerate, the time average is

\[
\langle \Psi(t) | \langle \Psi(t) | = \sum_\alpha | c_\alpha |^2 | \Psi_\alpha \rangle \langle \Psi_\alpha | = \hat{\rho}_{diag},
\]

where \( | \Psi(t) \rangle \) is the time evolved of \( | \Psi_0 \rangle \). This object is known in the modern literature as the diagonal ensemble [Rigol, 2009; Rigol et al., 2008; 2007]. Notice now that the most obvious definition of ergodicity, i.e. the requirement \( \rho_{mc} = \hat{\rho}_{diag} \) implies that \( | c_\alpha |^2 = 1/N \) for every \( \alpha \), a condition that can be satisfied only for a very special class of states. Quantum ergodicity in the strict sense above is therefore almost never realizable [Goldstein et al., 2010] [von Neumann, 1929].

Our common sense and expectations, which very frequently fail miserably in the quantum realm, make us nevertheless believe that, in contrast with the arguments above, macroscopic many-body systems should behave ergodically almost always, unless some very special conditions are met (e.g. integrability). The key to understand ergodicity in therefore to look at quantum systems in a different way, shifting the focus on observables rather than on the states themselves [Mazur, 1968; von Neumann, 1929; Peres, 1984]. Given a set of macroscopic observables \( \{ M_\beta \} \) a natural expectation from an ergodic system would be for every \( | \Psi_0 \rangle \) on a microcanonical shell \( \mathcal{H}(E) \)

\[
\langle \Psi(t) | M_\beta(t) | \Psi(t) \rangle \to_{t \to +\infty} \text{Tr}[M_\beta \hat{\rho}_{mc}] = \langle M_\beta \rangle_{mc}(15)
\]

i.e. that looking at macroscopic observables long after the time evolution started makes the system appear ergodic for every initial condition we may choose in \( \mathcal{H}(E) \). One needs a certain care in defining the infinite time limit here, since literally speaking it does not exist in finite systems because of quantum revivals. A proper way to understand this limit is to require that Eq. [15] holds in the long time limit at almost all times. Mathematically this means that the mean square difference between the LHS and RHS of Eq. [15] averaged over long times is vanishingly small for large systems [Reimann, 2008]. To avoid dealing with these issues ergodicity can be defined using the time average, i.e. requiring that

\[
\langle \Psi(t) | M_\beta(t) | \Psi(t) \rangle = \text{Tr}[M_\beta \hat{\rho}_{diag}] = \langle M_\beta \rangle_{mc}. \quad \text{(16)}
\]

Notice that if the expectation value of \( M_\beta \) relaxes to a well defined state in the sense described above, this state
will coincide with the time averaged state and the two definitions of ergodicity Eq. (15)-(16) will be equivalent. If the conditions above are satisfied then in loose terms \( \rho_{nc} \) can be considered as equivalent to \( \hat{\rho}_{\text{diag}} \). J. von Neumann proved that if the system satisfies some very natural requirements (e.g. absence of resonances), and the set \( \{M_\beta\} \) is constructed in such a way as to define macrostates of the system, which obviously requires the observables to be coarse grained on the various microcanonical shells \( \mathcal{H}(E) \) and mutually commuting, then a form of ergodicity is observed (sometimes referred to a normal typicality). In particular, for every \( |\Psi_0\rangle \) and almost every set \( \{M_\beta\} \) the diagonal and microcanonical ensembles are equivalent (Goldstein et al. 2010 [von Neumann 1929]). More recently it was proven that the whole density matrix of a small subsystem of a bigger system which is placed initially in a typical eigenstate is described by the canonical ensemble (Pопescu et al. 2006). In Ref. Gogolin et al. 2011 these results were further extended to the problem of measurement and decoherence. Particular care is nevertheless needed in relating these statements to the dynamics and thermalization of actual many body systems, since physical initial conditions in quenched system almost never correspond to eigenstates of a new Hamiltonian.

B. Nonergodic behavior in integrable systems: the generalized Gibbs ensemble

While the statements above are very general, their application to specific systems is not at all straightforward. Looking at a concrete many-body system, it is of primary interest not just to find out whether in principle a set of macroscopic observables that behave ergodically exists, but whether specific and natural observables, such as the magnetization for spin chains, density for cold atomic gases, or various correlation functions behave ergodically or not. In this respect, experiments tell us that ergodicity is not at all guaranteed (Kinoshita et al. 2006) if the closed system is integrable or nearly integrable. While this fact was expected (Barouch et al. 1970: Giraud 1969:1970: Mazur 1968: Suzuki 1971) recent research on the dynamics of integrable systems has focused on finding ways to predict the asymptotic states taking into account integrability, i.e. the presence of many constants of motion.

Let us discuss how this can be done qualitatively using the simplest example of integrable system, a periodic harmonic chain of finite length described by the Hamiltonian

\[
H = \sum_{j=1}^{M-1} \left[ \frac{p_j^2}{2m} + \frac{m\nu^2}{2} (x_j - x_{j+1})^2 \right],
\]

where \( x_j \) are deviations of particles from the equilibrium positions and \( p_j \) are their momenta; we use the identification \( x_M \equiv x_0 \). Let us imagine that initially we deform the system in a particular way and ask how this deformation evolves with time. We note that since this is a harmonic system described by linear equations of motion the following analysis also applies to quantum systems. From elementary physics we know that the initial deformation splits into normal modes characterized by the quasi-momenta \( q_n = 2\pi n/M \), where \( n \) is integer \( n \in [0, M-1] \), and the dispersion \( \omega_q = 2\nu \sin q/2 \). This system obviously does not thermalize even at long times because there is no energy exchange between modes. This does not imply though that it can not reach a well defined asymptotic state in the long time limit (Barthele and Schollwöck 2008; Cramer and Eisert 2010). To illustrate this point consider for example the displacement of the \( j \)-th atom at time \( t \) after some initial displacement:

\[
x_j(t) = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} x_{q_n}(t)e^{iq_jn}.
\]

\( x_{q_n}(t) = A_q \cos[\omega_n t] \), where \( A_q \) is a complex amplitude determined by initial conditions (for simplicity we assumed an initial stationary state). Let us now analyze qualitatively the dynamics of this system. At short times, provided that the initial modulation is smooth and only modes with small momenta (\( q \ll \pi \)) are excited we can linearize the spectrum \( \omega_q \approx \nu q \). Clearly in this case we recover periodic motion of the wave-packet with a period equal to the ratio of the system size and sound velocity: \( T = M/\nu \). This persistent motion is characteristic of the absence of any relaxation. However, as time gets longer deviations of the dispersion from linear become more important. In particular, when \( t^*(\omega_{\pi n+1} + \omega_{\pi n-1} - 2\omega_{\pi n}) \sim 1 \), where \( \omega_{\pi n} \) is the central frequency of the wavepacket, correlations between phases of different modes are lost and they can be treated as essentially random numbers. For our model \( t^* \sim M^2/\omega_{\pi n} \). At long times \( t \gg t^* \) the different momentum modes become uncorrelated and the system reaches the asymptotic stationary state in a sense we defined earlier (it can be found close to that state at almost all times). For this asymptotic state the only relevant information about the initial conditions is encoded in the \( M \) mode amplitudes \( |A_q|^2 \) or equivalently in their squares \( |A_q|^2 \) proportional to the occupancies of the modes of energy \( E_q \), which are the integrals of motion. Note though that there are special modes corresponding to momenta \( q \) and \( -q \) which are exactly degenerate. The correlations between \( A_q \) and \( A_{-q} = A_q^* \) thus never disappear and in general one needs to fix \( M \) additional constraints representing the relative phases of the complex amplitudes. For example, if the initial configuration is symmetric \( x_j = x_{-j} \) then \( A_q \) is real, meaning that the phases of all modes are identical. Then it is easy to check that with this constraint \( (x_j^2(t)) \) acquires spatial dependence on \( j \) even in the long time limit. This dependence can not be recovered by fixing only mode occupancies. Only when these phases are unimportant e.g. they average to zero or if there are no degeneracies between the normal modes the asymptotic state is fully fixed by the integrals of motion. Thus in contrast
with ergodic systems where only the energy needs to be fixed, the long time behavior of our integrable model can be reproduced by fixing the \( M \) integrals of motion and possibly \( \sim M \) other constraints if there are degeneracies. While the number of commuting (local) integrals of motion is large, equal to the system size \( M \), it is vastly smaller than the total number of states which scales with \( M \) exponentially.

Let us now see how these considerations are transposed in many-body systems, focusing on another simple integrable model, the Quantum Ising chain (Sachdev 1999) described by the Hamiltonian: \( H_0 = -\sum_i \sigma_i^x \sigma_{i+1} + g \sigma_i^z \). Here \( \sigma_i^{x,z} \) are the spin operators at site \( i \) and \( g \) is the strength of the transverse field. This model gives one of the simplest examples of quantum phase transition, with a quantum critical point at \( g_c = 1 \) separating two mutually dual gapped phases, a quantum paramagnet (\( g > g_c \)) and a ferromagnet (\( g < g_c \)).

In the Quantum Ising chain the local transverse magnetization, \( M^z = \sum \sigma_i^z \), is a non-ergodic operator (Barouch et al. 1970; Girardeau 1969, 1970; Mazur 1968). To see this, it is useful to employ a Jordan-Wigner transformation that reduces the problem to a free fermion model (Sachdev 1999). In terms of the fermionic operators \( c_k \) relative to modes of momentum \( k = 2\pi n/L \) the Hamiltonian takes the form

\[
H = 2 \sum_{k>0} (g - \cos(k))(c_k^\dagger c_{-k} - c_{-k}^\dagger c_k) + i \sin(k)(c_k^\dagger c_{-k} - c_{-k}^\dagger c_k),
\]

(19)

Under this mapping the transverse magnetization becomes \( M^z = -2 \sum_{k>0} (c_k^\dagger c_{-k}^\dagger + c_k c_{-k}) \). The eigenmodes \( \gamma_k \) of energy \( E_k = 2\sqrt{(g - \cos(k))^2 + \sin(k)^2} \) diagonalizing the Hamiltonian are related to the fermionic operators \( c_k \) by a Bogoliubov rotation, \( c_k = \cos(\theta_k) \gamma_k - i \sin(\theta_k) \gamma_k^\dagger \), with tan(2\( \theta_k \)) = \( \sin(k)/(g - \cos(k)) \). In Heisenberg representation the operators \( \gamma_k \) acquire simple time dependence: \( \gamma_k(t) = \gamma_k(0) \exp[-i E_k t] \). As in the previous problem of a harmonic chain if the energies \( E_k \) are incommensurate, at sufficiently long times different momentum modes become statistically independent from each other. This statement does not apply to modes with opposite momenta \( k \) and \(-k \) which have identical energies. However, if these correlations are not important then in the long time limit (see below) each mode can be characterized by the conserved quantity \( n_k = \langle \gamma_k^\dagger \gamma_k \rangle \). Let us now consider unitary dynamics of the transverse magnetization starting with a generic initial condition \( | \Psi_0 \rangle \). The time evolution of the operator \( M^z(t) \) expressed in terms of the eigenmodes of the Hamiltonian is

\[
M^z(t) = -2 \sum_{k>0} \cos(2\theta_k)(\gamma_k^\dagger \gamma_k - \gamma_{-k}^\dagger \gamma_{-k}) + i \sin(2\theta_k)(\gamma_{-k}^\dagger \gamma_k e^{-2iE_k t} - \gamma_k^\dagger \gamma_{-k} e^{2iE_k t}).
\]

(20)

In the long time limit only the diagonal terms in the sum survive, while the off diagonal ones, describing creation or destruction of two fermions average to zero. Therefore for any initial condition \( | \Psi_0 \rangle \) the asymptotic value attained by the transverse magnetization is

\[
\langle M^z(t) \rangle = -2 \sum_{k>0} \cos(2\theta_k)\langle \gamma_k^\dagger \gamma_k - \gamma_{-k}^\dagger \gamma_{-k} \rangle.
\]

(21)

This asymptotic value is therefore perfectly described by the set of the occupation numbers \( n_k \).

The result above leads one to conjecture that the asymptotic state is described by a Gibbs-like statistical ensemble of the type (Rigol et al. 2007)

\[
\rho_G = e^{-\sum_k \lambda_k \gamma_k^\dagger \gamma_k}/Z,
\]

(22)

where the Lagrange multipliers \( \lambda_k \) are fixed by requiring that \( n_k \equiv \langle \Psi_0 | \gamma_k^\dagger \gamma_k | \Psi_0 \rangle = \text{Tr}[\rho_G \gamma_k^\dagger \gamma_k] = \langle \gamma_k^\dagger \gamma_k \rangle_G \). The ensemble defined above in Eq. (22) can be seen as a particular case of the ensemble

\[
\hat{\rho}_G = e^{-\sum \alpha \lambda_\alpha I_\alpha}/Z,
\]

(23)

known as the generalized Gibbs ensemble (GGE) or the maximum entropy ensemble and introduced by Jaynes (Jaynes 1957) to describe the equilibrium state of a system possessing \( N \) constant of motions \( I_k \). A recent conjecture (Rigol et al. 2007) proposed to use the GGE to describe the asymptotic state of a generic quantum integrable model. This proposal had however to face two obvious subtleties. It is first of all to be specified how to choose the \( I_k \) in Eq. (23). Indeed, if all constants of motion would be admissible, including non-local ones, then one would obviously and tautologically describe the asymptotic state (both for integrable and non-integrable systems), as one can easily see by choosing as \( I_k \) the projectors onto the eigenstates of the Hamiltonian. The way out comes however by observing that in standard thermodynamics the Gibbs ensemble emerges for small sub-systems from the assumption of statistical independence between sufficiently big subsystems. In this derivation the additivity of a conserved quantity - energy - plays a crucial role. This is the reason why the probability of a given configuration is exponential in energy and not e.g. in energy squared (Kardar 2007). Similar arguments apply to any additive integrals of motion so that statistical independence and invariance of the ensemble to the choice of a subsystem of an integrable system puts strong constraints on the choice of the integrals of motion in GGE when the latter is applied to subsystems of an integrable system. In this respect the average occupation numbers of different momentum modes used in Eq. (22) become approximately additive for small subsystems. This approximate additivity of integrals of motion was recently discovered in Ref. (Cassidy et al. 2011) for another integrable system of one-dimensional hard-core bosons. In particular, it was noticed that the integrals of motion \( I_\alpha \) and the lagrange multipliers \( \lambda_\alpha \) in that case
can be written as a smooth functions of $\alpha/N$ implying that in large systems the argument of the exponent entering Eq. (23) can be written in the extensive (additive) form: $\sum_{\alpha} \lambda_{\alpha} I_{\alpha} \approx L \int_{0}^{1} d\xi, I(\xi) \lambda(\xi)$, where $\xi = \alpha/L$. This suggests that GGE can be defined through a smooth function $\lambda(\xi)$, which replaces the temperature in the ergodic systems.

There is a second subtlety in applying the GGE to quantum systems. Here the most natural definition of integrability is based on requirement that the system has well defined quasi-particles that maintain their identity upon scattering (see Ref. (Caux and Mossel, 2011) for a more detailed discussion), i.e. scattering is purely elastic and there is no production of particles or dissipation associated to it [Mussardo 2009 Sutherland 2004]. This notion can be made precise in continuum integrable models, such as the Luttinger liquid or the Sinh-Gordon model, which can emerge as low energy descriptions of other integrable models, such as the critical XXZ chain and the Lieb-Liniger gas. In these systems it is natural to associate the $I_{\alpha}$ to the occupation numbers of the quasi-particle states. More specifically, considering a generic one dimensional relativistically invariant integrable system with say a single species of quasi-particles of mass $m$, energy $E = m \cosh(\theta)$ and momentum $p = m \sinh(\theta)$ ($\theta$ is the rapidity), the quasi-particles can be described by annihilation operators $A(\theta)$ satisfying the algebra $A(\theta_1)A(\theta_2) = S(\theta_1 - \theta_2)A(\theta_1)A(\theta_2)$, where $S$ is the S-matrix of the two particle scattering. Similar relations are valid for the products of creation, and creation-annihilation operators [Mussardo 2009].

Since the Hamiltonian is by definition diagonal in $A(\theta)$, $H = \int d\theta E(\theta)A^\dagger(\theta)A(\theta)$, and every eigenstate can be written as $| \theta_1, \ldots, \theta_n \rangle = A^\dagger(\theta_1) \cdots A^\dagger(\theta_n) | 0 \rangle$, with $\theta_1 > \cdots > \theta_n$, in this case it is rather natural to postulate the form

$$\hat{\rho}_G = e^{-\int d\theta A^\dagger(\theta)A(\theta)} Z, \quad (24)$$

for the generalized Gibbs ensemble [Fioretto and Mussardo 2010]. This ensemble is a direct generalization of the GGE for the Quantum Ising Model, where $S = -1$. For this general class of integrable systems and a specific class of translationally invariant initial states it was indeed shown the long-time limit of the average of local operators is well described by this ensemble [Fioretto and Mussardo 2010]. Such initial states can be written as

$$| \Psi_0 \rangle = N e^{-\int d\theta K(\theta) A^\dagger(\theta)A(-\theta)}, \quad (25)$$

which in turn are similar to the so-called integrable boundary states in statistical field theory [Ghoshal and Zamolodchikov 1994]. Such states naturally emerge in experimentally relevant systems, for example when studying dephasing in split quasi-1d condensates [Gritsev et al. 2007] or in the Quantum Ising model, when studying a quantum quench from a transverse field $\gamma_0$ to a transverse field $\gamma_f$ [Silva 2008].

A very interesting idea related to the GGE was suggested by Gurarie [Gurarie 1995] to explain the steady state of a driven nearly integrable system. It was shown that the steady state distribution of the wave amplitudes corresponding to different momenta (see Ref. [Zakharov et al. 1992] for details) can be obtained by taking the probability density $\rho \propto \exp(-F)$, where $F$ is a (complex) combination of the approximate integrals of motion found perturbatively. In terms of this ensemble one recovers the correct power law distribution of the amplitudes of waves with the momentum and other observables.

Another view towards elucidating the validity of the generalized Gibbs ensemble has been pursued for special quenches in a 1D Bose-Hubbard model [Cramer et al. 2008] and in integrable systems with free quasiparticles (Barthal and Schollwöck 2008). It was shown that, upon tracing all degrees of freedom of the system outside a small region of space and under specific conditions, the local density matrix tends asymptotically to $\hat{\rho}_G$. More recently a series of recent theoretical [Flesch et al. 2008] and experimental [Trotzky et al. 2011] works on the dynamics of Bose-Hubbard models has proven the relaxation of local observables in these systems to a maximum entropy ensemble consistent with the constraints of the dynamics. A hint towards the generalization of $\hat{\rho}_G$ for Bethe Ansatz integrable systems was proposed in Ref. [Barthal and Schollwöck 2008]. The GGE was also tested in a number of models, from Luttinger liquids [Cazalilla 2006 Fucci and Cazalilla 2009] and free bosonic theories [Calabrese and Cardy 2007], to integrable hard-core boson models [Rigol et al. 2007] and Hubbard-like models [Eckstein and Kollar 2008 Kol and Eckstein 2008]. In all cases, it was shown to correctly predict the asymptotic momentum distribution functions for a variety of systems and quantum quenches.

At this point is should be stressed that as we discussed before the GGE does not always give complete description of the asymptotic state of the system. In the simple example of the harmonic chain we saw that for generic initial conditions it is necessary to specify $2N$ real constants or $N$ complex amplitudes in order to correctly describe the asymptotic state even if we focus exclusively on local observables. For a quantum Ising chain, moreover, $\rho_G$ can be interpreted as a grand-canonical distribution with an energy dependent chemical potential $\mu_k = E_k - \lambda_k$. It is evident now that if we consider the correlations of $\delta n_k = \gamma_k^\dagger \gamma_k - \langle \gamma_k^\dagger \gamma_k \rangle$, the occupation numbers of different eigenmodes, the GGE necessarily predicts $\langle \delta n_k \delta n_{k'} \rangle = 0$. Likewise the GGE predicts the correlators of the type $\langle \gamma_k^\dagger(\tau) \gamma_{-k}(\tau) \rangle$ are always equal to zero. For a generic initial state $| \Psi_0 \rangle$ both statements are not necessarily true: by e.g. breaking translational invariance in the initial state one could have $\langle \Psi_0 | \delta n_k \delta n_{k'} | \Psi_0 \rangle \neq 0$ and $\langle \Psi_0 | \gamma_k^\dagger \gamma_{-k} | \Psi_0 \rangle \neq 0$. Notice that the mere survival of off-diagonal correlations of this type when the evolution starts with a non-translationally invariant state signals in a sense the integrability of the model, i.e. the existence of well defined quasiparticles $\gamma_k$. Indeed, follow-
ing the argument of Gangardt and Pustilnik (Gangardt and Pustilnik, 2008), if the Hamiltonian of the system is translationally invariant but integrability is broken, the off-diagonal correlators are expected to decay to zero for any initial condition, thereby restoring the translational invariance in the asymptotic state. Finally, notice that off-diagonal correlations might influence the asymptotics of physically relevant observables: a simple example is the asymptotic value of $(\langle M^2(t) \rangle)^2$, which for a generic non-translationally invariant condition $\langle \Psi_0 \rangle$ cannot be predicted using the GGE.

A very important open question is to understand under which general circumstances the GGE can be applied. For free fermionic and bosonic systems the GGE was argued to hold for local observables (Barthel and Schollwöck, 2008; Cramer et al., 2008). For more general integrable systems this is not evident at all. For example in the case of the Quantum Ising chain the two significant observables, the transverse magnetization $\sigma^z_i$ and the order parameter $\sigma^z$, are local in the spin representation. However, this locality does not translate directly to their representation in terms of the quasiparticles of the model: while $\sigma^z_i$ retains a local character in terms of $\gamma_i$, $\sigma^z$ does not. Will the asymptotic dynamics of any local operator be represented by the GGE, or just that of local operators in the quasi-particle fields? Do the symmetries of the initial state play any role in this? Answering these questions appears to be crucial to understand the role of integrability in the dynamics of many-body systems.

Another very important question is whether all natural observables of an integrable system behave necessarily nonthermally. The answer to it appears to be no, as pointed out recently (Rossini et al., 2009). The key to understand this issue seems to be again locality with respect to the quasi-particles diagonalizing the model. Thus in the Quantum Ising Model it was shown that while the transverse magnetization is non-ergodic, the correlators of the order parameter $\sigma^z$ following a quench of the transverse field relax as in a thermal state with an effective temperature $T_{\text{eff}}$ set by the initial energy of the system $E = \langle \Psi_0 | H | \Psi_0 \rangle$. At low $T_{\text{eff}}$ this relaxation appears to be universal, i.e. determined only by the low energy scattering properties of quasiparticles (Rossini et al., 2009). Analogous studies for an XXZ chain hint towards a different behavior of local and non-local operators with respect to quasi-particles (Canovi et al., 2011). The situation is much less clear for quenches with high effective temperatures, where the universal character of the low energy theory is lost (Barmettler et al., 2009).

C. Breaking integrability: eigenstate thermalization.

When integrability is explicitly broken with a strong enough perturbation one naturally expects ergodic behavior to emerge for all observables (Kollath et al., 2007; Manmana et al., 2007; Rigol, 2009; Rigol et al., 2008; Roux, 2009, 2010). The quest for the necessary conditions for thermalization to occur (i.e. how strongly should integrability be broken, which spectral properties should the system display) is an important problem in many different fields, from mathematical and statistical physics to quantum chaos (Deutsch, 1991; Peres, 1984; Rigol et al., 2008; Srednicki, 1994, 1999). In classical systems the intense research on this subject was stimulated by the study of dynamics of a nonlinear chain of coupled oscillators by Fermi Pasta and Ulam (FPU) (Fermi et al., 1955), where instead of thermalization regular quasi-periodic oscillations were observed. Later it was realized that the FPU problem is nearly integrable and that there is a finite threshold for the chaotic behavior (Campbell et al., 2005). In quantum systems the situation was far less clear: while different views on this issue emerged from time to time, the key towards a clear understanding of quantum thermalization appears to be linked to the emergence of quantum chaotic behavior (Peres, 1984). In particular, it has been proposed that the emergence of thermal behavior is linked to the pseudo-random form of natural observables once represented in the eigenbasis of the Hamiltonian (Peres, 1984). This observation has been made more precise by conjecturing that thermalization in quantum chaotic systems occurs eigenstate-by-eigenstate, i.e. the expectation value of a natural observable $\langle \Psi_\alpha | A | \Psi_\alpha \rangle$ on an eigenstate $| \Psi_\alpha \rangle$ is a smooth function of its energy $E_\alpha$ being essentially constant on each microcanonical energy shell (Deutsch, 1991; Srednicki, 1994, 1999). If this happens, then ergodicity and thermalization in the asymptotic state follow for every initial condition sufficiently narrow in energy (e.g. localized in a microcanonical shell), as one can easily understand using the diagonal ensemble. This hypothesis is known as eigenstate thermalization (ETH).

In order to understand how eigenstate thermalization can emerge, let us consider a quantum gas of $N$ particles of mass $m$ with hard-core interactions (Srednicki, 1994). Srednicki pointed out that in the time evolution of this system starting with an initial condition $| \Psi_0 \rangle$ sufficiently narrow in energy, the momentum distribution will always relax to the Maxwell-Boltzmann distribution $f_{MB}(p)$ as long as the eigenstates of the system $| \Psi_\alpha \rangle$ can be considered as pseudo-random superpositions of plane waves, i.e. have a diffusive nature in phase space. This requirement should be satisfied as a result of the chaoticity of the system, the so-called Berry’s conjecture. Calling $X = (x_1, x_2, \ldots, x_N)$ the coordinates of the particles and $P = (p_1, p_2, \ldots, p_N)$ their momenta, Berry’s conjecture states that the eigenstates have the form

$$\Psi_\alpha(X) = N \int dPA_\alpha(P) \delta(P^2 - 2mE_\alpha)e^{iP\cdot X},$$

(26)

with $A_\alpha(P)$ being pseudo-random variables with Gaussian statistics, $\langle A_\alpha(P)A_\beta(P') \rangle = \delta_\alpha^\beta\delta(3N)(P + P')/\delta(P^2 - P'^2)$. Notice that we are disregarding the symmetrization of the wave function, see (Srednicki, 1994) for a discussion of this aspect. If the properties above are assumed it is easy to prove that on average in the thermodynamic
limit the momentum distribution function is

\[ \langle f(p) \rangle = \int d^3p \, e^{-\frac{p^2}{2n^2}} \sum \langle \Psi_\alpha(p, p_2, \ldots) |^2} = \frac{e^{-\frac{p^2}{2n^2}}}{(2\pi n^2)^{3/2}} = f_{MB}(p), \tag{27} \]

where the temperature is set by the equipartition law as \( E_\alpha = 3/2kT \). Notice that this is expected to happen for every eigenstate of energy close to \( E_\alpha \), as required by the ETH. Hence thermal behavior will follow for every initial condition sufficiently narrow in energy.

For generic many-body systems, such as Hubbard-like models and spin chains, the close relation between breaking of integrability and quantum chaotic behavior is a known fact (Polcal et al. 1993). In particular, finite size many-body integrable systems are characterized by the Poisson spectral statistics while the gradual breaking of integrability by a perturbation leads to a crossover to the Wigner-Dyson statistics. The latter is typically associated, in mesoscopic systems or billiards, with diffusive behavior and can be taken as a signature of quantum chaos (Imry 1997). In many-body disordered systems the emergence of the Wigner-Dyson statistics was argued to be an indicator of the transition between metallic (ergodic) and insulating (non-ergodic) phases (Mukerjee et al. 2006; Oganianyan and Huse 2007). Inspired by these close analogies, recent studies gave a boost to our understanding of the crossover from non-ergodic to thermal behavior as integrability is gradually broken and of the origin of ergodicity/thermalization in systems sufficiently far from integrability (Biroli et al. 2006; Rigol 2009; Rigol et al. 2008). In particular, a careful study of the asymptotics of density-density correlators and momentum distribution function for hard-core bosons in 1d showed that the transition from non-thermal to thermal behavior in finite size systems takes the form of a crossover controlled by the strength of the integrability breaking perturbation and the system size (Rigol 2009). Moreover, there is a universality in state to state fluctuations of simple observables in this crossover regime (Neuenhahn and Marquardt 2010), which goes hand-in-hand with an analogous transition from Poisson to Wigner-Dyson level statistics (Rigol and Santos 2010; Santos and Rigol 2010a). When integrability is broken by sufficiently strong perturbation ergodic behavior emerges (Neuenhahn and Marquardt 2010; Rigol 2009; Rigol and Santos 2010), which in turn appears to be related to the validity of the ETH (Rigol et al. 2008). In this context, the anomalous, non-ergodic behavior of integrable models has been reinterpreted as originating from wide fluctuations of the expectation value of natural observables around the microcanonical average (Biroli et al. 2010).

All these statements apply to the asymptotic (or time averaged) state. So far the relaxation in time, in particular in the thermodynamic limit, has received much less attention. In a series of studies of relaxation in fermionic Hubbard models subject to quenches in the interactions strength it has been argued that for sufficiently rapid quenches relaxation towards thermal equilibrium occurs through a pre-thermalized phase (Moekel and Kehrein 2008; 2010). Similar two step dynamics occurs in quenches of coupled superfluids where initial fast “light cone” dynamics leads to a pre-thermalized steady state, which then slowly decays to the thermal equilibrium through the vortex-antivortex unbinding (Mathey and Polkovnikov 2010). In Ref. (Burkov et al. 2007) a very unusual sub-exponential in time decay of correlation functions was predicted and later observed experimentally (Hofferberth et al. 2007) for relaxational dynamics of decoupled 1d bosonic systems.

D. Outlook and open problems: quantum KAM threshold as a many-body delocalization transition?

The arguments above clearly pointed to the connection between thermalization in strongly correlated systems and in chaotic billiards. This analogy however, rather than being the end of a quest, opens an entire new kind of questions, which are a current focus of both theoretical and experimental research. In particular, we do know that in a number of models of strongly correlated particles eigenstate thermalization is at the root of thermal behavior (Rigol et al. 2008). What is the cause of eigenstate thermalization in a generic many-body system, i.e. the analogue of the diffusive eigenstates in phase space of Berry’s conjecture? And most importantly, while in a finite size system the transition from non-ergodic to ergodic behavior takes the form of a crossover, what happens in the thermodynamic limit? Is the transition from ergodic to non-ergodic behavior still a crossover or is it sharp (a quantum KAM threshold?)?

At present, research on these questions has just started. An interesting idea that has recently emerged is that the study of the transition from integrability to non-integrability in quantum many-body systems is deeply connected to another important problem at the frontier of condensed matter physics: the concept of many-body localization (Altshuler et al. 1997; Basko et al. 2006), which extends the original work of Anderson on single-particle localization (Anderson 1958). We note that related ideas were put forward in studying energy transfer in interacting harmonic systems in the context of large organic molecules (Leitner and Wolynes 1996; Logan and Wolynes 1990). More specifically, it has been noticed that a transition from localized to delocalized states either in real space (Pal and Huse 2010), or more generally in quasi-particle space (Canovi et al. 2011) is closely connected to a corresponding transition from thermal to non-thermal behavior in the asymptotics of significant observables. For weakly perturbed integrable models, the main characteristic of the observables to display such transition appears to be again their locality with respect
to the quasiparticles (Canovi et al. 2011). This connection with many-body localization becomes more clear on the basis of a recently proposed way to quantify the transition from non-ergodic to ergodic behavior in many body systems (Olshanii and Yurovsky 2009). The authors consider an integrable Hamiltonian $H_0$ with a weak non-integrable perturbation $\lambda V$. Formulating essentially a generalization of the Berry’s conjecture and making some additional assumptions they showed that the deviations from thermal behavior in the expectation value of observables can be quantified according to the formula

$$
\langle \Psi(t) | A | \Psi(t) \rangle - \langle A \rangle_{\text{mc}} \approx \eta \{ \langle \Psi(t) | A | \psi_0(t) \rangle - \langle A \rangle_{\text{mc}} \},
$$

where $| \psi_0(t) \rangle = \exp[-iH_0 t] | \psi_0 \rangle$ is the time evolved state with respect to the integrable Hamiltonian, while $| \Psi(t) \rangle = \exp[-i(\mathcal{H}_0 + \lambda V) t] | \psi_0 \rangle$ evolves with the non-integrable one. The key ingredient in this formula is the parameter $\eta$, defined as the average over the microcanonical shell of the inverse participation ratio

$$
\eta = \sum_n \langle \phi_n | \psi_0 \rangle^4,
$$

where $| \phi_n \rangle$ are the eigenstates of the integrable Hamiltonian $\mathcal{H}_0$ and $| \psi_0 \rangle$ are the eigenstates of $\mathcal{H}_0 + \lambda V$. Notice that when the system is close to integrability $\eta \approx 1$ but as the strength of $V$ increases, $\eta$ is roughly proportional to the inverse of the number of states $N$ hybridized by the perturbation.

Using this formula it is now possible to understand how many-body localization enters the scenario (Canovi et al. 2011, Pal and Huse 2010): an abrupt transition at a certain $\lambda_c$ from localized to delocalized states in quasi-particle space corresponds to a sharp decrease of the inverse participation ratio $\eta$ from a value $O(1)$ to a value negligibly small and tending to zero in the thermodynamic limit, essentially $O(1/N(\lambda))$, where $N(\lambda)$ is the total number of states in an energy window of width of the order of the matrix elements of the perturbation. This would lead to an abrupt transition from non-thermal to thermal behavior at $\lambda_c$, a transition qualitatively corresponding to the physics of a quantum KAM threshold. Notice that on the delocalized side of this transition the eigenstates are expected to be of diffusive nature (in quasi-particle space), i.e. a natural generalization of the form postulated by Berry’s conjecture. Such transition have been studied extensively in confined electronic system, following a seminal paper by Altschuler, Kamenev, Levitov and Gefen (Altschuler et al. 1997) and on interacting electron systems with localized single particle states (Basko et al. 2006). While the dependence $\eta(\lambda)$ was analyzed numerically in certain small systems (Canovi et al. 2011, Neuenhahn and Marquardt 2010, Santos and Rigol 2010a,b), eventual emergence of a sharp KAM-like threshold in the thermodynamic limit remains an open question.

The ETH also suggested a new way on looking at quantum relaxational dynamics as dephasing in the many-body basis. In particular, the information about the asymptotic state is fully contained in the diagonal elements of the density matrix, which do not change in time if the Hamiltonian is constant. So the process of time evolution in this picture is equivalent to averaging of oscillating off-diagonal elements of the density matrix to zero. In a way this picture is different from conventional thinking based on kinetic theory of thermalization through collisions between quasi-particles and the time evolution of their distribution function. This apparent difference is hidden in the complicated structure of the many-body eigenstates. Our intuition is based on thinking about dynamics in the approximate basis, e.g. a basis of independent quasi-particles. The precise relation between the many-body and kinetic approaches to thermalization is still an open question. Another potentially intriguing possibility is understanding thermalization as a renormalization group process, where time evolution results in averaging over high-energy degrees of freedom. If one deals with approximate noninteracting basis then because of interactions the process of eliminating high energy states affects dynamics of low energy modes and hence in renormalization of the low energy dynamics. In Ref. (Mathey and Polkovnikov 2010) it was shown that such renormalization process indeed can explain real time dynamics in a two-dimensional sine Gordon model and the emerging nonequilibrium Kosterlitz-Thouless transition. In Ref. (Moeckel and Kehrein 2009) similar ideas were put forward to analyze dynamics of interacting fermions using the flow equation method. At the moment it is unclear whether using such real time renormalization group one can analyze relaxational long time dynamics in generic interacting systems.

### IV. Experimental Progress in Quantum Dynamics in Cold Atoms and Other Systems

As we mentioned before, the study of nonequilibrium dynamics of quantum many-body systems has been increasingly motivated by a series of advances in the field of ultracold atomic and molecular gases. Due to the convergence of various features, these mesoscopic quantum systems are in many ways near-ideal systems for the study of nonequilibrium quantum phenomena.

Firstly, quantum gases can exhibit a remarkably high degree of isolation from environmental sources of decoherence and dissipation. Thus, to an excellent approximation, during duration of experiments they can be regarded as closed quantum systems. Further, the dilute nature of these gases and exceptionally low temperatures result in long timescales of dynamical effects (typically on the order of milliseconds or longer) allowing for time-resolved studies of nonequilibrium processes resulting from phase-coherent many-body dynamics. Such studies are hardly possible in conventional condensed matter systems.

Secondly, an array of techniques have been developed to dynamically tune various parameters of the Hamiltonian governing these quantum gases. This has made it possible to realize various prototypical nonequilibrium processes such as quantum quenches discussed above.
Quenches across phase transitions have been realized to investigate the onset and formation of long range order and the mechanism underlying the spontaneous formation of topological defects. The latter is closely related to the KZ mechanism described earlier in the text. A quantitative experimental study of the defect, entropy and energy production resulting from such quantum quenches should allow for an accurate comparison with the theoretical predictions.

Lastly, the ability to engineer and experimentally realize model Hamiltonians of archetypal correlated systems coupled with a detailed knowledge of the microscopic interactions make ultracold atomic gases a tantalizing system for applications ranging from the quantum simulation of strongly correlated systems to the adiabatic quantum computation. In addition to the form of the model Hamiltonian, experimental control can also be achieved over the effective dimensionality of the ultracold gas making it possible to investigate the nontrivial interplay between fluctuations, interactions and dimensionality.

From a technological perspective, there is an increasing thrust towards engineering ultracold atomic many-body systems for applications in quantum metrology (Appel et al., 2009; Estève et al., 2008; Leroux et al., 2010; Meiser et al., 2008; Riedel et al., 2010; Vengalattore et al., 2007). A deeper understanding of the dynamics of interacting many-body systems and the mechanisms of decoherence and dissipation in these systems is of crucial importance in this context.

Motivated by these factors, a number of experiments have been performed in recent years using ultracold quantum gases to investigate topics including quantum coherent dynamics in optical lattices, quenches across quantum phase transitions and thermalization in low dimensional systems. For the purposes of this colloquium, we distinguish between classes of nonequilibrium experiments both in terms of the general protocol as well as the questions being addressed by these experiments. (i) Nonequilibrium states of many body atomic systems wherein the high degree of isolation of the atomic system from the environment allows for the creation of metastable or highly excited many-body states with long lifetimes, (ii) Quantum quench experiments in which one or more parameters of the Hamiltonian are changed rapidly to create an out-of-equilibrium state of the many-body system and (iii) Dynamical tuning of the Hamiltonian in order to study quantum coherent dynamics of an interacting many-body system. These experimental advances have stimulated a very active theoretical research in the area of nonequilibrium quantum dynamics in interacting many-body systems. Among the issues most debated in recent literature is the relation between thermalization in isolated quantum systems and quantum integrability. In this regard, a recent pioneering study on thermalization in 1D Bose gases was performed in Ref. [Kinoshita et al., 2006]. In this experiment, a blue detuned 2D optical lattice was used to create arrays of tightly confined tubes of ultracold $^{87}$Rb atoms. The depth of the lattice potential far exceeded the energy of the ultracold gas ensuring negligible tunneling among the tubes. The array of tubes was then placed in a superposition of states of momentum $\pm 2p_0$ by the application of a transient optical phase grating. The imparted kinetic energy was small compared to the energy required to excite the atoms to the higher transverse states and the gases remained one-dimensional. This out-of-equilibrium system was then allowed to evolve for variable durations before the momentum distribution was probed by absorption imaging of the gas (see Fig. 1).

It was found that, while the initial momentum distributions exhibit some dephasing on account of trap anharmonicities, the dephased distribution remains non-gaussian even after thousands of collisions. This is in distinct contrast to the gaussian distributions observed when the 2D optical lattice is adiabatically imposed on an equilibrium 3D Bose gas. This remarkable observation that the nonequilibrium Bose gases do not equilibrate on the timescales of the experiment appears consistent with the fact that this system is a very close experimental realization of a Lieb-Liniger gas with point-like collisional interactions - an integrable quantum system in which only elastic pairwise interactions can occur. Apparently the experimental technicalities such as anharmonicities or the axial potential are insufficient to sufficiently lift integrability in this system.

In addition to unambiguously showing the absence of thermalization within experimental timescales in this model realization of the Lieb-Liniger gas, this study also points the way towards addressing more general questions on integrability and ergodicity. Starting from an integrable system, modifications such as the addition of finite range interactions, tunneling between the 1D tubes or the imposition of axial potentials one can tuneably lift integrability and analyze emergence of irreversibility and thermalization. This experiment largely motivated much of the theoretical work discussed in the previous section.

Another issue that has attracted a lot of attention is the search for universal effects either in the nonequilibrium dynamics following a quantum quench or in the adiabatic dynamics near a quantum critical point which we described earlier in the text. In particular, the issue of non-adiabatic dynamics near quantum phase transitions has been the focus of recent experimental studies on condensate formation in a dilute, weakly interacting Bose gas that is rapidly cooled past the BEC phase transition (Weiler et al., 2008). This process was found to be accompanied by the spontaneous formation of topological defects, i.e. vortices, in the nascent superfluid. This can be phenomenologically understood as being due to the formation of isolated superfluid regions of a characteristic size $\xi$, each with a random relative phase. These isolated regions then gradually merge to give rise to global phase coherence. In this process, regions which enclose phase loops of $2\pi$ are constrained by the nature of the superfluid, i.e. the continuity of the wavefunction, to have a vanishing superfluid density at the core. Thus, the KZ
mechanism predicts a density of vortices that scales as \(1/\xi^2\).

In this experiment, a magnetically trapped thermal gas of \(^{87}\text{Rb}\) atoms was cooled by radiofrequency (rf) evaporation to temperatures below the BEC transition temperature. The quench rate, i.e. the rate of cooling, was controlled by varying the rate at which the rf frequency was ramped down. Following a brief duration of equilibration, vortices are detected by absorption imaging of the gas after ballistic expansion. Allowing for some uncertainty in the ability to discern a vortex due to line-of-sight integration in these images, it was found that about a quarter of the images showed at least one vortex core.

The rate of cooling during the quench was limited by the collision rate between atoms in the trapped gas during evaporative cooling. This resulted in a limited dynamic range for the quench rate. Also, the rapid decrease of the thermal fraction following the formation of the condensate led to a low damping rate for the vortices. A faster quench rate, realized through a trap with tighter confinement or increased density or via sympathetic cooling with another species, could result in the observation of an increased number of vortices during the quench. In turn, this would potentially allow for quantitative tests of the predicted scaling of vortex number with the quench rate and the extraction of dynamic critical exponents.

While the formation of a superfluid by quenching the temperature is seeded by thermal fluctuations, ultracold atomic gases also potentially allow for the realization of phase transitions initiated purely by quantum fluctuations (Greiner et al. 2002a; Sadler et al. 2006).

A particularly intriguing study of a quench past such quantum phase transitions was carried out in a degenerate \(F = 1\) spinor Bose gas of \(^{87}\text{Rb}\) (Sadler et al. 2006). These gases, with a spin degree of freedom arising from a non-zero hyperfine spin \(F\), are quantum fluids that may simultaneously exhibit the phenomena of magnetism and superfluidity, both of which result from symmetry breaking and long range order. Owing to rotational symmetry, the contact interactions between two atoms can be characterized by the total spin of the colliding pair. In the case of a \(F = 1\) spinor gas, these interactions give rise to a mean field energy given by \(n(c_0 + c_2(\mathbf{F})^2)\) where the coupling strengths \(c_{0,2}\) are related to the \(s\)-wave scattering lengths in the total spin \(f = 0, 2\) channels (Ohmi and Machida 1998). In addition to the mean field interactions, a finite external magnetic field \(B\) imposes a quadratic Zeeman energy (QZE) that scales as \(q(\mathbf{F}^2)\) with \(q = (\mu_B B)^2/4\Delta_{hf}\) where \(\mu_B\) is the Bohr magneton and \(\Delta_{hf}\) is the energy splitting between the ground state hyperfine manifolds.

For a \(F = 1\) condensate of \(^{87}\text{Rb}\), the competing influences of the spin-dependent interaction and the QZE give rise to a continuous quantum phase transition between a 'polar' and a ferromagnetic phase. Rapidly tuning the external magnetic field from large values \((q \gg |c_2n|)\) to small values \((q \ll |c_2n|)\) quenches the spinor gas from the polar phase to the ferromagnetic phase. The ensuing growth of ferromagnetic domains was directly detected by \textit{in situ} imaging (see Fig. 1). It was found that the resulting texture of ferromagnetic domains was spatially inhomogeneous and characterized by a typical length scale that was related to the spin healing length \(\xi = \hbar/\sqrt{2m|c_2n|}\). Concurrent with the appearance of these domains, the spin textures revealed the spontaneous formation of polar-core spin vortices. These topological defects are characterized by a non-zero spin current but no mass current. The origin of these spin vortices is also rooted in the KZ mechanism. It was shown that, for slow quenches, the number of such vortices is expected to scale as \(\tau^{-1/6}\) where \(\tau\) is the time over which the spinor gas is swept into the ferromagnetic state (Saito et al. 2007).

The weak spin-dependent interactions inherent to this spinor gas also allow for nondestructive detection of the vortices and studies of their dynamics. In addition, the weak coupling between the spin and mass degrees of freedom make it straightforward to realize extremely low spin temperatures to examine the role of quantum fluctuations.

FIG. 2 Time-of-flight absorption images of an ensemble of 1D Bose gases - Ultracold atoms are confined in arrays of 1D optical traps. Optical pulses are used to place the atoms in a superposition of ±2\(\hbar k\) momentum states. The gas is then allowed to evolve for variable durations before being released from the trap and photographed to reveal the momentum distribution. The false color in each image is rescaled to show detail. The non-gaussian nature of the momentum distribution clearly indicates an absence of thermalization. (Adapted from Kinoshita et al. 2006)
tations in seeding this phase transition [Klempt et al., 2010]. These features make spinor quantum fluids a rich system to investigate the quench dynamics and KZ mechanism past quantum phase transitions between different magnetically ordered phases. In addition, corrections to the KZ scaling imposed by long range interactions [Ven-galattore et al. 2008], conservation laws and finite temperature effects can also be studied.

Yet another range of experimental studies is made possible by the tunability of atomic interactions using a Feshbach resonance. This technique allows the rapid dynamic control of the s-wave scattering length by means of a time-varying external magnetic field. This ability was utilized in a recent study of a strongly interacting two-component Fermi mixture [Jo et al., 2009]. Starting from an initially weak, repulsive interaction between the two Fermionic species, the interactions were rapidly increased by tuning the magnetic field to the vicinity of the Feshbach resonance. The subsequent decrease in the atomic loss rate, the increase in the size of the trapped gas and the increase in kinetic energy as measured in time-of-flight images were interpreted as an indication of the Stoner transition to a ferromagnetic state. However, in a later theoretical work this interpretation was questioned and an alternative explanation based on rapid molecule formation was suggested (Ref. [Babadi et al., 2010]). Thus, a direct in-situ measurement of local magnetization is necessary to understand whether ferromagnetism plays a role in this experiment.

In addition to the thermalization dynamics across phase transitions, the long coherence times inherent to ultracold gases also makes it possible to study the quantum coherent dynamics of many-particle systems. A particularly dramatic instance of such coherent many-body dynamics was illustrated in the collapse and revival of the matter wave field of a Bose condensate [Greiner et al., 2002]. Here, the interaction-induced dynamical evolution of a matter wave field was clearly revealed in the multiple matter wave interference patterns obtained after releasing the gas from the lattice. This work has also been extended to the time-resolved observation of superexchange processes in optical ‘superlattice’ potentials [Trotzky et al., 2008]. Similar demonstrations of collisional coherence have also been shown in spinor Bose gases [Chang et al., 2005; Kronjäger et al., 2005]. Due to the internal degrees of freedom in a spinor gas, the dynamics in this fluid is due to coherent spin-mixing collisions. In a trapped gas that is well described by the single mode approximation (SMA), these coherent collisions can lead to the periodic and reversible formation of condensates in initially unpopulated spin states.

Further, in certain situations, coherent interactions can also lead to quantum correlations [Sorensen et al., 2001]. Schemes that might realize such entangled many-particle states have received attention due to potential applications in quantum information processing and metrology. The dynamical evolution of such entangled states in the presence of quantum or thermal fluctuations is obviously of great interest. A recent experiment investigated this evolution in low dimensional two-component Bose gases with adjustable interactions [Widera et al., 2008], finding that quantum fluctuations play a crucial role in the phase diffusion dynamics of low dimensional systems. More recently, the dynamical control of a Bose-Einstein condensate confined in a strongly driven optical lattice was demonstrated [Lignier et al., 2007]. By periodically modulating the lattice potential, the tunneling parameter J was shown to be suppressed in a phase coherent manner opening the possibility of driving quantum phase transitions using this technique.

The isolation of ultracold atomic gases from external sources of dissipation also makes it possible to study relaxation dynamics driven purely by intrinsic mechanisms. Such mechanisms should set the timescales for adiabatic quantum computing or the simulation of strongly correlated lattice models. A recent experiment along these lines investigated the evolution of excited states of the repulsive Fermi-Hubbard system [Strohmaier et al., 2010]. Here, doubly occupied lattice sites (doublons) were created by modulating the lattice and the subsequent decay of the system to thermal equilibrium was monitored over time. It was shown that the lifetime of these doublons scales exponentially with the ratio of the interaction energy to kinetic energy, in fair agreement with theoretical predictions. It was argued that the dominant mechanism driving this relaxation was a high-order scattering process involving several fermions [Sensarma et al., 2010].

While this colloquium places an emphasis on experiments involving ultracold atomic gases, there is a range of other mesoscopic quantum systems which also lend themselves to studies on quantum nonequilibrium dynamics. For the sake of completeness, we briefly review a few of these systems here. Defect formation following a quench was first studied in the context of vortices in liquid crystals [Chuang et al., 1991]. This has since been followed by similar studies in various mesoscopic systems including isolated superconducting loops where the defects assume the form of spontaneous fluxoids [Monaco et al., 2009], superconducting thin films [Maniv et al., 2003] and multi-Josephson junction loops [Monaco et al., 2006, 2002]. A cumulative view of these studies would indicate that the influence of finite size effects, thermal fluctuations and dimensionality on the production of topological defects by the KZ mechanism is as yet unclear and a topic that warrants further study.

Another potential system for the study of nonequilibrium dynamics of many-particle states arises from rapid advances in the field of photonics. There have been several proposals [Chang et al., 2008; Greentree et al., 2006] for the dynamical creation of strongly correlated photonic states using photon-photon interactions mediated by a nonlinear optical medium. The realization of states such as a Tonks gas of photons have been proposed using hollow-core optical fibers, tapered optical fibers, photons in coupled cavities and surface plasmons on conducting nanowires. Such strongly correlated photon states should
have applications in metrology, sub-shot noise interferometry and the quantum emulation of exotic spin models.

V. OUTLOOK

One of the ultimate goals of the new field of quantum dynamics is to develop a systematic understanding of nonequilibrium phenomena in strongly interacting quantum many-body systems. A few of the most significant open questions along this avenue are readily identified: How can we classify nonequilibrium behavior in closed many-body systems? What is the general relation between integrability and dynamics? What is the dynamical effect of a gradual breaking of integrability? What are the effects of dissipation on these nonequilibrium processes? Can we understand time evolution of interacting systems through the renormalization group? Answering these and other questions allied with systematic, quantitative studies of possible nonequilibrium quantum phase transitions and the extraction of dynamical critical exponents, are just a few of the many tantalizing programs to be pursued. The rapidly developing sophistication and precision of ultracold atomic experiments and other experimental systems should allow for close and direct comparison between theoretical predictions and ad hoc experiments.

The realization of robust techniques for the experimental study of such systems and the development of theoretical tools to describe nonequilibrium many-body processes should bode for tantalizing opportunities in this nascent field, potentially leading to a deeper understanding of the principles governing nonequilibrium many-body phenomena and establishing robust connections between microscopic dynamics and statistical physics.

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