Probability distribution function for systems driven by superheavy-tailed noise

S.I. Denisov\textsuperscript{1,2,a} and H. Kantz\textsuperscript{1}

\textsuperscript{1} Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, D-01187 Dresden, Germany
\textsuperscript{2} Sumy State University, Rimsky-Korsakov Street 2, UA-40007 Sumy, Ukraine

Received:

Abstract. We develop a general approach for studying the cumulative probability distribution function of localized objects (particles) whose dynamics is governed by the first-order Langevin equation driven by superheavy-tailed noise. Solving the corresponding Fokker-Planck equation, we show that due to this noise the distribution function can be divided into two different parts describing the surviving and absorbing states of particles. These states and the role of superheavy-tailed noise are studied in detail using the theory of slowly varying functions.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.) – 02.50.-r Probability theory, stochastic processes, and statistics

1 Introduction

The Langevin approach, i.e., incorporation of the noise terms into deterministic equations of motion to describe the stochastic dynamics of particles, is one of the most effective tools for studying the effects induced by fluctuating environment. If these terms arise from the time derivative of the noise generating processes, i.e., random processes with independent and identically distributed increments, then the solution \( x(t) \) of the corresponding Langevin equation is a Markov process \([1,2]\). The main feature of Markov processes is that their future behavior depends only on the present state, or in other words, the correlation time of Markov processes equals zero. Although physical processes are usually characterized by nonzero correlation times, the Markov processes can be considered as a first approximation of physical ones if the correlation time is the smallest time scale of the system.

In the Markovian case, the statistical properties of the particle position \( x(t) \) are controlled by the distribution of increments of the noise generating process. If this distribution is described by a scaled probability density \( p(y) \) (see the next section) then, depending on the asymptotic behavior of the tails of \( p(y) \), three qualitatively different cases can be distinguished. The Gaussian density represents the first case in which the process \( x(t) \) is continuous and the probability density \( P(x,t) \) of \( x(t) \) satisfies the ordinary Fokker-Planck equation \([3,4]\). This equation describes a wide variety of noise phenomena, including particle diffusion in external potentials \([3,4]\), stochastic resonance \([5]\), noise-induced transport \([6,7,8]\), noise-enhanced stability \([9,10]\), cross-correlation effects \([11,12]\), and many others. The Lévy stable densities represent the second case in which the probability density \( P(x,t) \) satisfies the fractional Fokker-Planck equation \([13,14,15,16,17,18]\) that describes a special class of discontinuous processes \( x(t) \), the so-called Lévy flights \([19,20,21,22]\). It has been recently shown \([23]\) that the ordinary Fokker-Planck equation holds for all probability densities \( p(y) \) with finite variance, and the fractional one for all heavy-tailed \( p(y) \), i.e., probability densities with power-law tails and infinite second moment.

Finally, the third case corresponds to the superheavy-tailed \( p(y) \), i.e., probability densities whose all fractional moments are infinite. Because of this unusual feature, the distributions with superheavy tails are rarely used in physics. Nevertheless, the examples of ultraslow diffusion \([24,25,26,27]\) demonstrate the utility of these distributions in modeling the systems with an extremely anomalous behavior. Very recently, the usefulness of superheavy-tailed distributions has also been demonstrated for the Langevin approach \([28]\). Specifically, using the generalized Fokker-Planck equation \([18]\), it has been shown that superheavy-tailed noise, i.e., noise arising from a generating process whose independent increments are distributed with superheavy tails, induces two probabilistic states of a particle, surviving and absorbing. Due to this feature, the Langevin equation driven by superheavy-tailed noise can be used to describe some randomly interrupted processes. From a physical point of view, the interruption can be associated with the transition of a particle to a qualitatively new

\textsuperscript{a} e-mail: stdenis@pks.mpg.de
state. In particular, such a situation occurs when the particle dynamics is accompanied by the absorption of these particles. It should be stressed, however, that the Langevin equation driven by superheavy-tailed noise and the corresponding generalized Fokker-Planck equation describe the simplest situation when the absorption rate does not depend on the spatial and temporal variables. It has been shown in Ref. [18] that the probability density of a special class of superheavy-tailed densities. Finally, our results confirm our previous results, derive the probability distribution function of the random variable \( \eta(1) \) as a particle position, 

\[ \eta(1) = \lim_{\tau \to 0} \sum_{j=0}^{[1/\tau]-1} \Delta \eta(j\tau) \]  

([1/\tau] is the integer part of 1/τ). For example, if the probability density of the increments \( \Delta \eta(j\tau) \) is given by 

\[ p(\Delta \eta, \tau) = \frac{1}{a(\tau)} p\left( \frac{\Delta \eta}{a(\tau)} \right), \]  

where the probability density \( p(\eta) \) satisfies the condition 

\[ \lim_{|y| \to \infty} \frac{p(y/\epsilon)}{\epsilon^\chi} = \delta(y) \]  

and the scale function approaches zero as \( \tau \to 0 \), then, depending on the asymptotic behavior of \( p(y) \) at \( |y| \to \infty \), the ordinary or generalized central limit theorem can be applied to determine \( S_k \) [18,31] (see also Sect. 3.2.1). It has been shown in particular that if \( p(y) \) has heavy tails then \( S_k \) is the characteristic function of Lévy stable distributions and, as a consequence, equation (2) reduces to the fractional Fokker-Planck equation. On the other hand, since \( \chi > 0 \), the probability density \( p(\eta) \) is assumed to be slowly varying at infinity, i.e., 

\[ p(\eta) \sim \frac{1}{y} b(y) \quad (y \to \infty), \]  

with \( p_{ka(\tau)} = \int_{-\infty}^{\infty} dy e^{-i\eta y} p(y) \), the parameters of the last equation can be expressed through the asymptotic behavior of the probability density \( p(y) \) at \( |y| \to \infty \) and the scale function \( a(\tau) \) at \( \tau \to 0 \) [23].

In this paper we focus on the study of the cumulative distribution function \( F(x, t) = \int_{-\infty}^{x} dx P(x, t) \) of particles subjected to superheavy-tailed noises. These noises arise from the noise generating processes characterized by the superheavy-tailed densities \( p(y) \) whose fractional moments 

\[ M_\chi = \int_{-\infty}^{\infty} dy |y|^\chi p(y) \]  

are infinite for all \( \chi > 0 \) (the normalization of \( p(y) \) implies that \( M_0 = 1 \)). Next we restrict ourselves to the symmetric superheavy-tailed densities with the following asymptotic behavior:

\[ p(y) \sim \frac{1}{y} b(y) \quad (y \to \infty), \]  

where a positive function \( b(y) \) is assumed to be slowly varying at infinity, i.e., 

\[ b(y) \sim b(y) \quad (y \to \infty) \]  

for all \( \mu > 0 \). Since for all slowly varying functions the condition \( y^\chi b(y) \to \infty \) holds as \( y \to \infty \) [32], the probability densities characterized by the asymptotic behavior (6) are indeed superheavy-tailed, i.e., \( M_\chi > 0 = \infty \). It should be noted, however, that the slowly varying functions \( b(y) \) are not arbitrary because the condition \( M_0 = 1 \) implies that \( b(y) = o(1/\ln y) \) as \( y \to \infty \).

The main advantages of equation (2) are that it (i) is valid for all noise generating processes \( \eta(t) \), (ii) accounts for the noise action in a unified way, namely through the characteristic function \( S_k \) of \( \eta(1) \), and (iii) reproduces all presently known Fokker-Planck equations associated with the Langevin equation [18,31]. The possible forms of this equation, which are determined by the possible forms of the characteristic function \( S_k \), are restricted by the condition that the random variable \( \eta(1) \) is represented as an infinite sum of independent and identically distributed increments \( \Delta \eta(j\tau) = \eta(j\tau + \tau) - \eta(j\tau) \) when the split time \( \tau \) tends to zero, i.e.,

\[ \eta(1) = \lim_{\tau \to 0} \sum_{j=0}^{[1/\tau]-1} \Delta \eta(j\tau) \]  

satisfies the generalized Fokker-Planck equation 

\[ \frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} f(x, t) P(x, t) + F^{-1}\{ P_k(t) \ln S_k \}. \]  

Here, we interpret \( x(t) \) \( x(0) = 0 \) as a particle position, \( f(x, t) \) as a deterministic force and \( \eta(t) \) as a noise term. The noise generating process \( \eta(t) \) \( \eta(0) = 0 \) is assumed to have independent and identically distributed (for a given \( dt \)) increments \( \eta(t) = \eta(t + dt) - \eta(t) \). It is also assumed that the solution of equation (2) obeys the initial condition 

\[ P(x, 0) = \delta(x) \]  

where \( \delta(x) \) is the Dirac \( \delta \) function, and the normalization condition \( \int_{-\infty}^{\infty} dx P(x, t) = 1 \). The last term in (2) contains the direct and inverse Fourier transforms defined as 

\[ F\{ u_k \} \equiv u_k = \int_{-\infty}^{\infty} dx e^{-ikx} u(x) \]  

and 

\[ F^{-1}\{ u_k \} \equiv u(x) = (1/2\pi) \int_{-\infty}^{\infty} dk e^{ikx} u_k, \]  

respectively. Specifically, \( P_k(t) = \int_{-\infty}^{\infty} dx e^{-ikx} P(x, t) \) is the characteristic function of \( x(t) \), i.e., 

\[ P_k(t) = \langle e^{-ikx(t)} \rangle, \]  

where the angular brackets denote averaging over the sample paths of the noise generating process \( \eta(t) \), and \( S_k = \langle e^{-ikx(0)} \rangle \) is the characteristic function of the random variable \( \eta(1) \), i.e., noise generating process at \( t = 1 \).
3 General results

3.1 Characteristic function $S_k$

For finding the characteristic function $S_k$ of the random variable $\eta(1)$ we use the basic relation \(5\), which with the symmetry property $p(-y) = p(y)$ and the normalization condition $2\int_0^\infty dy p(y) = 1$ can be rewritten as

$$\ln S_k = -2 \lim_{\tau \to 0} \frac{1}{\tau} Y \left(\frac{1}{k|a(\tau)|}\right),$$

where

$$Y(\lambda) = \int_0^\infty dy [1 - \cos(y/\lambda)] p(y)$$

is a positive function of $\lambda$ that approaches zero as $\lambda \to \infty$. In those cases when the indefinite integral $\int dp(y)$ is known (see, e.g., the example in Section 4), it is convenient to use an alternative representation of $Y(\lambda)$:

$$Y(\lambda) = \int_0^\infty dx \sin x \int_x^\infty dy \, p(y).$$

Since the function $h(y)$ in the asymptotic formula \(6\) is slowly varying, $Y(\lambda)$ also varies slowly. Indeed, using \(8\), one can easily verify that

$$Y(\mu \lambda) = \mu \lambda \int_0^\infty dx (1 - \cos x) p(\mu \lambda x)$$

$$\sim \int_0^\infty dx \left(1 - \frac{\cos x}{x}\right) h(\mu \lambda x) \sim Y(\lambda)$$

($\lambda \to \infty$). Therefore, if $k \neq 0$ then $Y(1/k|a(\tau)|)$ in \(7\) can be replaced by $Y(1/a(\tau))$ yielding $\ln S_k = -q$, where

$$q = 2 \lim_{\tau \to 0} \frac{1}{\tau} Y \left(\frac{1}{a(\tau)}\right)$$

is a non-negative parameter that does not depend on $k$ (its physical meaning will be discussed in Sect. 6). In contrast, if $k = 0$ then $p_0 = \int_0^\infty dy p(y) = 1$ and so $\ln S_0 = 0$. Combining these results and introducing the Kronecker delta $\delta_{k0}$, we obtain \(8\)

$$\ln S_k = -q(1 - \delta_{k0}).$$

Thus, for all symmetric probability densities $p(y)$ with superheavy tails the characteristic function of $\eta(1)$ has the form $S_k = e^{-q(1-\delta_{k0})}$. It should be noted that although $S_k$ differs from $e^{-q}$ only in one point $k = 0$, the Kronecker delta cannot be neglected because it provides a correct normalization of the distribution of $\eta(1)$ (see Sect. 6.2).

3.1.1 Representations of $q$ and $a(\tau)$

As it follows from \(11\), the parameter $q$, which accounts for the influence of superheavy-tailed noise on the system, depends on the asymptotic behavior of the slowly varying function $Y(\lambda)$. In general, according to \(8\), this behavior is controlled by a given probability density $p(y)$. However, because $Y(\lambda)$ varies slowly, there is a possibility to write $Y(\lambda)$ at $\lambda \to \infty$ in a quite general form. Such a possibility is provided by the Karamata representation theorem \(52\), which states that for some $l > 0$ and all $\lambda$ satisfying the condition $\lambda \geq l$ every slowly varying function $L(\lambda)$ can be written in the form

$$L(\lambda) = g(\lambda) \exp \left(\int_l^\infty du \frac{\epsilon(u)}{u}\right),$$

where $g(\lambda) \to g \in (0, \infty)$ and $\epsilon(\lambda) \to 0$ as $\lambda \to \infty$. Since the defining property of $L(\lambda)$, i.e., $L(\mu \lambda) \sim L(\lambda)$, involves only the asymptotic behavior of this function, the representation \(13\) is essentially non-unique. Using this fact, it can be shown \(52\) that $L(\lambda)$ at $\lambda \to \infty$ can always be represented in a simpler form

$$L(\lambda) \sim g \exp \left(\int_{l1}^{l2} du \epsilon(u)\right).$$

Thus, assuming that $\int du \epsilon(u) = -\Phi(v)$, for the slowly varying function $Y(\lambda)$ we obtain

$$Y(\lambda) \sim c \exp[-\Phi[\ln \lambda]],$$

($\lambda \to \infty$) with $c = g \exp[\Phi[\ln 1]]$. An explicit form of the function $\Phi(v)$ depends on $p(y)$, but since $Y(\lambda)$ tends to zero when $\lambda$ increases, the condition $\Phi(v) \to \infty$ as $v \to \infty$ must hold in all cases. Using \(14\) and \(15\), for the parameter $q$ we find the desired general representation

$$q = \frac{2c}{\tau},$$

where $r$ is a time scale parameter defined as

$$r = \lim_{\tau \to 0} \tau \exp \left[\Phi \left(\ln \frac{1}{a(\tau)}\right)\right].$$

Depending on how fast the function $a(\tau)$ approaches zero as $\tau \to 0$, the parameter $r$ can take the values from the whole interval $[0, \infty]$. If $a(\tau)$ tends to zero so rapidly that $r = \infty$ then such noise does not affect the system at all ($q = 0$). In the opposite case, when $r = 0$, the noise action is so strong that the system reaches the final state at $t = 0^+$, i.e., immediately ($q = \infty$).

In what follows we are interested in a non-trivial action of superheavy-tailed noise. This case is specified by the condition $0 < r < \infty$ and occurs only if the scale function $a(\tau)$ has a proper asymptotic behavior at $\tau \to 0$. In order to find this behavior, we assume that for large enough $\tau$ there exists the inverse function $v = \Phi^{-1}(\varphi)$ of the function $\varphi = \Phi(v)$ (since $\Phi(v) \to \infty$ as $v \to \infty$, in this case $\Phi^{-1}(\varphi) \to \infty$ as $\varphi \to \infty$). Then the limit in \(17\) is equal to a given value $r$ if the asymptotic behavior of $a(\tau)$ at $\tau \to 0$ is given by

$$a(\tau) \sim \exp \left[-\Phi^{-1}\left(\ln \frac{r}{\tau}\right)\right].$$
We note that the asymptotic formula \[ F(\eta; \kappa) = e^{-\eta} F_{\delta}(\eta) + (1 - e^{-\eta}) \frac{\kappa}{\pi} \text{sech} \eta \] is of no importance. Therefore, without loss of generality, the main features of the probability density function of \( k \) and the corresponding distribution function of \( \eta \) can always be chosen as \( b(\tau) = 1 \) without loss of generality.

### 3.2 Distribution function of \( \eta(1) \)

In general, using (2) and (12), it is possible to establish the main features of the probability density function \( P(x, t) \) arising from the action of superheavy-tailed noise. However, because the random variable \( \eta(1) \) is responsible for all these features, it is reasonable to study the statistical properties of \( \eta(1) \) in more detail. This will also permit us to obtain the probability distribution function of the particle position, \( F(x, t) \), in a consistent way.

In order to gain more insight into the properties of the random variable \( \eta(1) \), we calculate its distribution function \( F(\eta) \). To this end it is convenient to temporarily replace the Kronecker delta \( \delta_{\kappa 0} \) in the characteristic function \( S_k = e^{-q(1 - \Delta_k)} \) of \( \eta(1) \) by a symmetric and smooth function of \( k \), \( \Delta_k(k) \), which depends on a positive parameter \( \kappa \) in such a way that \( \lim_{\kappa \to 0} \Delta_k(k) = \delta_{\kappa 0} \). We assume that \( S_k(k) = e^{-q(1 - \Delta_k(k))} \) is the characteristic function of the random variable \( \eta(1; k) \) whose probability density \( S(\eta; k) \) and the corresponding distribution function \( F(\eta; k) \) are given by \( S(\eta; k) = (1/2\pi) \int_0^\infty dk e^{i\eta k} S_k(k) \) and \( F(\eta; k) = \int_0^\infty d\eta S(\eta; k) \), respectively. If the distribution function \( F(\eta; k) \) is known then the desired distribution function can be determined as \( F(\eta) = \lim_{\kappa \to 0} F(\eta; k) \).

The choice of the function \( \Delta_k(k) \) is, of course, not unique. For example, we can choose as

\[
\Delta_k(k) = \frac{1}{q} \ln \left[ 1 + (e^q - 1) \text{sech} \left( \frac{\pi k}{2\kappa} \right) \right] \tag{20}
\]

or

\[
\Delta_k(k) = \frac{1}{q} \ln \left[ 1 + (e^q - 1) e^{-|k|/\kappa} \right]. \tag{21}
\]

However, in the limit \( \kappa \to 0 \) the explicit form of \( \Delta_k(k) \) is of no importance. Therefore, without loss of generality, we consider \( \Delta_k(k) \) from (20) for which the characteristic function of \( \eta(1; k) \) has the form

\[
S_k(k) = e^{-q} + (1 - e^{-q}) \text{sech} \left( \frac{\pi k}{2\kappa} \right). \tag{22}
\]

The probability density and the distribution function that correspond to this characteristic function are given by

\[
S(\eta; \kappa) = e^{-\eta} \delta(\eta) + (1 - e^{-\eta}) \frac{\kappa}{\pi} \text{sech} \eta \tag{23}
\]

and

\[
F(\eta; \kappa) = e^{-\eta} F_{\delta}(\eta) + (1 - e^{-\eta}) \tilde{F}(\kappa \eta), \tag{24}
\]

respectively. Here,

\[
F_{\delta}(\eta) = \begin{cases} 0, & \eta < 0 \\ 1, & \eta \geq 0 \end{cases} \tag{25}
\]

is the step function describing the degenerate distribution localized in the point \( \eta = 0 \), and

\[
\tilde{F}(\kappa \eta) = \frac{2}{\pi} \arctan(e^{\kappa \eta}) \tag{26}
\]

is the distribution function which describes the hyperbolic secant distribution.

According to (24), the distribution of the random variable \( \eta(1; \kappa) \) is the mixing of the degenerate and hyperbolic secant distributions. In contrast to the former, which does not depend on the parameter \( \kappa \), the latter strongly depends on this parameter. In particular, the maximum height of the distribution, \( d\tilde{F}(\kappa \eta)/d\eta|_{\eta=0} \), and the root square of the variance, which characterizes the width of this distribution, are equal to \( \kappa/\pi \) and \( \pi/(2\kappa) \), respectively. It is clear, therefore, that the probability density \( d\tilde{F}(\kappa \eta)/d\eta \) of the hyperbolic secant distribution tends to zero as \( \kappa \to 0 \). On the other hand, as it follows from (26), this occurs in such a way that \( \tilde{F}(\kappa \eta) \to 0 \) and \( \tilde{F}(\kappa \eta) \to 1 \) for all \( \kappa > 0 \). Assuming that at \( \kappa \to 0 \) these conditions hold as well, for the limiting distribution function \( F(\eta) = \lim_{\kappa \to 0} F(\kappa \eta) \) we obtain

\[
F_1(\eta) = \begin{cases} 0, & \eta = -\infty \\ 1/2, & |\eta| < \infty \\ 1, & \eta = \infty. \end{cases} \tag{27}
\]

This particular distribution function describes a normalized random variable whose probability density equals zero for all \( |\eta| < \infty \). From a formal point of view, it can also be considered as a discrete variable that takes only two values, \( \eta = -\infty \) and \( \eta = \infty \), with probability 1/2 each. Thus, the probability distribution of \( \eta(1) \) characterized by the distribution function \( F(\eta) = \lim_{\kappa \to 0} F(\eta; \kappa) \) is the mixing of the degenerate distribution (25) taken with the weight \( e^{-q} \) and the limiting distribution (27) taken with the weight \( 1 - e^{-q} \), i.e.,

\[
F(\eta) = e^{-q} F_{\delta}(\eta) + (1 - e^{-q}) F_1(\eta). \tag{28}
\]

### 3.2.1 Connection with limit theorems

Representing the increments \( \Delta \eta(j \tau) \) of the noise generating process \( \eta(t) \) as \( \Delta \eta(j \tau) = \eta_j / c_n \) with \( c_n = 1/a(\tau) \) and \( n = [1/\tau] + 1 \), from (3) one obtains

\[
\eta(1) = \lim_{n \to \infty} \sum_{j=1}^n \frac{\eta_j}{c_n}. \tag{29}
\]
where \( y_j \) are the independent random variables distributed with the same probability density \( p(y) \). We note that in probability theory the partial sums of a more general form, i.e., \( \sum_{j=1}^{n} y_j/c_n - d_n \), is usually considered. But in our situation \( d_n = 0 \) because a class of probability densities \( p(y) \) is restricted by the condition \( \lim_{n \to 0} p(y/e) = \delta(y) \). If the variance of \( p(y) \) is finite then, according to the central limit theorem \([33]\), \( c_n \approx n^{1/2} \) and the distribution of \( \eta(1) \) is Gaussian. In this case \( a(\tau) \approx \tau^{1/2} \) and for all such \( p(y) \) the generalized Fokker-Planck equation \([22]\) reduces to the ordinary one \([23]\). In contrast, if \( p(y) \) with heavy tails belongs to the domain of normal attraction of a given stable distribution characterized by an index of stability \( \alpha \in (0, 2) \) then, as the generalized central limit theorem suggests \([33]\), \( n \approx n^{1/\alpha} \) and \( \eta(1) \) is described by this stable distribution. It has been shown \([22]\) that in this case \( a(\tau) \approx \tau^{1/\alpha} \) [in special situations, there also exist two different forms of the scale function \( a(\tau) \)] and equation \([22]\) reduces to the fractional Fokker-Planck equation.

Finally, the above derived results show that for all symmetric super-heavy-tailed densities \( p(y) \) characterized by the asymptotic behavior \([9]\) the distribution function of the random variable \( \eta(1) \) has the form \([25]\). It is important to emphasize, however, that in contrast to densities with finite variance and densities with heavy tails there is no universal scale function \( a(\tau) \) for super-heavy-tailed \( p(y) \). On the contrary, as is clear from \([9, 15] \) and \([15]\), in this case different \( p(y) \) lead in general to different \( a(\tau) \). This is in accordance with the well-known fact \([33]\) that only stable distributions have domains of attraction. Thus, the result \([25]\) can be interpreted as follows: For every symmetric super-heavy-tailed the scale function \( a(\tau) \) can always be chosen so that the distribution function of \( \eta(1) \) is given by \([25]\) with \( q \in (0, \infty) \).

### 3.3 Solution of the generalized Fokker-Planck equation

Our next step is to solve the generalized Fokker-Planck equation \([22]\) for the characteristic function \( S_k = \exp(-q(1-\delta_{\alpha})) \) that corresponds to super-heavy-tailed noise. To this end we first solve this equation for the characteristic function \([22]\) and then take the limit \( \kappa \to 0 \). In order to clarify our approach, let us designate the solution of equation \([22]\) in which \( S_k \) is replaced by \( S_k(\kappa) = P(x, t; \kappa) \) as \( \bar{A}(x, t; \kappa) \) governed by the equations

\[
\frac{\partial}{\partial t} P(x, t; \kappa) = - \frac{\partial}{\partial x} f(x, t) P(x, t; \kappa) - q P(x, t),
\]

\[
\frac{\partial}{\partial t} A(x, t; \kappa) = - \frac{\partial}{\partial x} f(x, t) A(x, t; \kappa) - q A(x, t; \kappa) + qF^{-1}\{P_k(t; \kappa) + A_k(t; \kappa)\}A_k(t; \kappa)
\]

and satisfy the initial conditions \( P(x, 0) = \delta(x) \) and \( A(x, 0; \kappa) = 0 \). We also assume that the normalization condition for \( P(x, t; \kappa) \), i.e., \( \int_{-\infty}^{\infty} P_k(t; \kappa) = 0 \) holds for all \( t \) and \( \kappa \). Then, with these assumptions, the solution of equation \([22]\) can be represented as follows:

\[
A(x, t; \kappa) = P(x, t) + \lim_{\kappa \to 0} A(x, t; \kappa). \tag{32}
\]

An explicit form of the first term in the right-hand side of \([22]\) can easily be determined from equation \([21]\). Indeed, assuming that \( P(x, t) = e^{-q(x - z(t))} \), where \( \Phi(x) \) is an arbitrary function and \( z(t) \) is the solution of the equation \( z(t) = f(z(t), t) \) (we assume that \( z(0) = 0 \) and \( |z(t)| < \infty \) for all finite \( t \)). Using the initial condition \( A(x, 0; \kappa) = 0 \), one finds \( \Phi(x) = \delta(x) \) and so

\[
P(x, t) = e^{-q(x - z(t))}. \tag{33}
\]

As it follows from this result, the normalization condition for \( P(x, t) \), \( \int_{-\infty}^{\infty} P(x, t) = 1 \), reads \( P \to e^{-q(x - z(t))} \). According to \([31]\) and \([32]\), the term \( A(x, t; \kappa) \) obeys a closed integro-differential equation

\[
\frac{\partial}{\partial t} \bar{A}(x, t; \kappa) = - \frac{\partial}{\partial x} f(x, t) \bar{A}(x, t; \kappa) - q \bar{A}(x, t; \kappa) + q \int_{-\infty}^{\infty} dy \Delta(x - y; \kappa) \bar{A}(y, t; \kappa)
\]

\[
+ q \int_{-\infty}^{\infty} dy \Delta(x - z(t); \kappa)
\]

\[
\Delta(x; \kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \Delta_k(k) \text{ is the kernel function. Using the fact that } \int_{-\infty}^{\infty} dk \Delta(x; \kappa) = 1 \text{ which follows from the condition } \Delta_0(k) = 1 \text{ [see } \theta], \text{ and the integral representation of the } \delta \text{ function } \delta(x) = (1/2\pi) \int_{-\infty}^{\infty} dk e^{ikx} \text{, from equation } \int_{-\infty}^{\infty} dk \delta(x; \kappa) \text{ we obtain a simple equation } \partial \bar{A}/\partial \kappa \partial \kappa = ge^{-q(x - z(t))} \text{ for } \bar{A}_0(t; \kappa) = \int_{-\infty}^{\infty} dx \Delta_k(x, t; \kappa). \text{ The solution of this equation satisfying the initial condition } \bar{A}_0(t; \kappa) = 0 \text{ yields } \bar{A}_0(t; \kappa) = 1 - e^{-q(x - z(t))}. \text{ Therefore, like } P(x, t), \text{ the normalization condition of } \bar{A}(x, t; \kappa), \text{ i.e., } \bar{A}_0(t; \kappa) = 1 - e^{-q(x - z(t))}, \text{ does not depend on the parameter } \kappa. \text{ Because of its complexity, we are not able to solve equation } \int_{-\infty}^{\infty} dk \delta(x; \kappa) \text{ for an arbitrary } \kappa. \text{ However, in the case of our interest, when } \kappa \text{ approaches zero, the dependence of the solution of this equation on } \kappa \text{ can easily be found. The key point is that the kernel function } \Delta(x; \kappa) \text{ at } |x|/|\kappa| \ll 1 \text{ can be approximated as } \Delta(x; \kappa) = \xi \kappa \text{ with }
\]

\[
\xi = \frac{2}{\pi^2 q} \int_{0}^{\infty} dy \ln[1 + (e^y - 1)
\]

\[
\times \text{ sech} y].
\]

In this approximation, the integral term in \([35]\) equals \( q \xi \kappa \) and \( \bar{A}(x, t; \kappa) \) at \( |x| < |\kappa| \ll 1 \) is governed by a linear differential equation

\[
\frac{\partial}{\partial t} \bar{A}(x, t; \kappa) = - \frac{\partial}{\partial x} f(x, t) \bar{A}(x, t; \kappa) - q \bar{A}(x, t; \kappa) + q \xi \kappa.
\]

\[
\text{Its general solution is written as } \bar{A}(x, t; \kappa) = \bar{A}_0(x, t; \kappa) + \bar{A}_1(x, t; \kappa), \text{ where } \bar{A}_0(t; \kappa) \propto \kappa \text{ is a particular solution of}
this equation and $\mathcal{A}_0(x, t; \kappa)$ is the general solution of the corresponding homogeneous equation (when $\kappa = 0$). Since $\mathcal{A}(x, 0; \kappa)$, the latter is also proportional to $\kappa$ and so $\mathcal{A}(x, t; \kappa) \propto \kappa$. Put differently, the term $\mathcal{A}(x, t; \kappa)$ tends to zero linearly as $\kappa \to 0$. In particular, if the external force is a constant, i.e., $f(x, t) = f$, then $\mathcal{A}(x, t; \kappa) = \xi(1 - e^{-qt})\kappa$ at $|x - f|\kappa \ll 1$. We note also that while $\mathcal{A}(x, t; \kappa)$ approaches zero in the limit $\kappa \to 0$, the spatial region of localization of $\mathcal{A}(x, t; \kappa)$, which is determined by the condition $|x - z(t)| \sim 1/\kappa$, tends to infinity keeping the normalization condition of $\mathcal{A}(x, t; \kappa)$ fixed.

Thus, according to (22), (34) and the above properties of $\mathcal{A}(x, t; \kappa)$ at $\kappa \to 0$, the distribution function $F(x, t) = \int_{-\infty}^{x} dx P(x, t)$ ($t < \infty$) of particles driven by superheavy-tailed noise has the form

$$F(x, t) = e^{-qt}F_0[x - z(t)] + (1 - e^{-qt})F_1(x).$$

(38)

It shows that this noise generates two time-dependent probabilistic states of each particle. The first one is described by the degenerate distribution function $F_0$ and is realized with the probability $P_0(t) = e^{-qt}$. Since the particle trajectory $z(t)$ is not influenced by the noise [see (23)], following the terminology of [28] we call this state the surviving state. The second state is associated with the limiting distribution function $F_1(x)$ and is realized with the probability $\mathcal{A}_0(t) = 1 - e^{-qt}$. In this state the probability to find the particle in any finite interval equals zero. The transition to this state implies that the particle jumps to plus or minus infinity and, in fact, it is excluded from future consideration. Therefore, like in [28], we refer to this state as the absorbing state. Because of these features, superheavy-tailed noise acts on particles as an absorbing medium characterized by the rate of absorption $q$. Hence, the Langevin equation (11) driven by this noise and the corresponding generalized Fokker-Planck equation (2) describe the overdamped motion of a particle in an absorbing medium. Moreover, one expects that these equations can also be used to describe some other processes whose duration is random.

### 3.4 Compound noise

According to the above results, superheavy-tailed noise is characterized by two main features which at first glance seem to be contradictory. On the one hand, this noise is so strong that the transition into the absorbing state occurs so that a particle is immediately transferred to infinity. But, on the other hand, before this transition the noise does not affect the particle position. In other words, before absorption which is random in time the particle motion is deterministic and is governed by the equation of motion $\ddot{z}(t) = f(z(t), t)$ [see (24)]. From a mathematical point of view, these features result from both superheavy tails of the probability density $p(y)$ and appropriate choice of the scale function $a(\tau)$.

The case when before absorption the particle motion is random can also be incorporated into the proposed approach. For this purpose we consider the compound noise generating process $\eta(t) = \eta_1(t) + \eta_2(t)$, where $\eta_1(t)$ generates superheavy-tailed noise, which models an absorbing medium, and $\eta_2(t)$ generates any other noise (e.g., Lévy stable noise), which causes the random motion of a particle. Assuming that these random processes are independent, we obtain $\mathcal{S}_k = \mathcal{S}_{1k} + \mathcal{S}_{2k}$, where $\mathcal{S}_{1k}$ and $\mathcal{S}_{2k}$ are the characteristic functions of $\eta_1(t)$ and $\eta_2(t)$, respectively. Then, proceeding as before, we replace $\mathcal{S}_{1k}$ by the characteristic function (22) and write the solution of equation (2) at $\kappa \neq 0$ as $P(x, t; \kappa) = \mathcal{P}(x, t) + \mathcal{A}(x, t; \kappa)$, where $\mathcal{P}(x, t)$ is the solution of the equation

$$\frac{\partial}{\partial t}\mathcal{P}(x, t) = -\frac{\partial}{\partial x}f(x, t)\mathcal{P}(x, t) - q\mathcal{P}(x, t)$$

$$+ F^{-1}\{\mathcal{P}(\kappa) ln S_{2k}\}$$

(39)

satisfying the initial condition $\mathcal{P}(x, 0) = \delta(x)$. This solution can be written in the form $\mathcal{P}(x, t) = e^{-qt}\mathcal{W}(x, t)$, where the probability density $\mathcal{W}(x, t)$ is governed by the generalized Fokker-Planck equation

$$\frac{\partial}{\partial t}\mathcal{W}(x, t) = -\frac{\partial}{\partial x}f(x, t)\mathcal{W}(x, t) + F^{-1}\{\mathcal{W}(\kappa) ln S_{2k}\}$$

(40)

and obeys the initial condition $\mathcal{W}(x, 0) = \delta(x)$ and the normalization condition $\mathcal{W}(0) = 1$. It is also not difficult to verify that, like in the previous case, $\mathcal{A}_0(t; \kappa) = 1 - e^{-qt}$ and $\mathcal{A}(x, t; \kappa) \to 0$ as $\kappa \to 0$. Therefore, for all finite times the distribution function of particles subjected to this compound noise takes the form

$$F(x, t) = e^{-qt}\int_{-\infty}^{x} dx \mathcal{W}(x, t) + (1 - e^{-qt})F_1(x).$$

(41)

Using the fact that superheavy-tailed noise plays the role of an absorbing medium, the above result can be interpreted as the distribution function of particles whose random motion before absorption is described by the following Langevin equation: $\ddot{x}(t) = f(x(t), t)dt + \sqrt{t}d\eta(t)$. The second term in the right-hand side of (11) relates to the absorbing state and does not depend on the character of the particle motion. In contrast, the particle dynamics strongly influences the first term which describes the surviving state. As is clear from (10) and (11), this influence is determined by the deterministic force $f(x, t)$ and the characteristic function $\mathcal{W}_0$ of $\eta_2(t)$. If $\eta_2(t) = 0$, i.e., the particle dynamics before absorption is deterministic, then $\mathcal{S}_{2k} = 1$, $\mathcal{W}(x, t) = \delta(x - z(t))$ and so the distribution function (11) reduces to (25). The random motion of particles occurs if the noise generating process $\eta_2(t)$ depends on time. As it was discussed in Sect. 3.2.2, in this case the possible distributions of $\eta_2(t)$ follow from the fact that $\eta_2(t)$ is represented as an infinite sum of independent and identically distributed increments. For example, if the distribution of the increments $\Delta\eta_2(\tau)$ is symmetric and has heavy tails then the random variable $\eta_2(1)$ is distributed with the symmetric Lévy stable distribution whose characteristic function is given by $S_{2k} = \exp(-|\gamma|\kappa^\alpha)$, where $\alpha$ and $\gamma$ are the index of stability and scale parameter, respectively. Defining the Riesz
derivative as $\partial^\alpha h(x)/\partial|x|^\alpha = -F^{-1}\{\nu^\alpha h_k\}$ [13], in this particular case equation (10) reduces to the fractional Fokker-Planck equation [13,14,15,16,17,18]

$$\frac{\partial}{\partial t}W(x,t) = -\frac{\partial}{\partial x}f(x,t)W(x,t) + \gamma \frac{\partial^\alpha}{\partial|x|^\alpha}W(x,t),$$

(42)

which together with the distribution function [11] describes Lévy flights (if $0 < \alpha < 2$) and Brownian motion of particles (if $\alpha = 2$) in an absorbing medium.

Equation (42) can be solved analytically in some simple cases. In particular, if $f(x,t)$ is a linear restoring force, i.e., $f(x,t) = -bx$ ($b > 0$), then the solution of this equation is the Lévy stable density [13]

$$W(x,t) = F^{-1}\{\exp(-c^2(t)|k|^\alpha)\}$$

(43)

classified by the same index of stability and the time-dependent scale parameter $c^2(t) = \gamma(1-e^{-\alpha b t})/ab$. At $\alpha = 2$ it reduces to the Gaussian density

$$W(x,t) = \frac{1}{\sqrt{2\pi c(t)}} \exp\left(-\frac{x^2}{4c(t)}\right),$$

(44)

and the distribution function [11] for Brownian particles in an absorbing medium takes the form

$$F(x,t) = \frac{e^{-qt}}{2}\text{erfc}\left(-\frac{x}{2c(t)}\right) + (1-e^{-qt})F_1(x),$$

(45)

where erfc$(x) = (2/\sqrt{\pi}) \int_x^\infty dy e^{-y^2}$ is the complementary error function.

### 4 Particular class of superheavy-tailed distributions

As is shown in the previous section, the distribution of particles subjected to superheavy-tailed noise is the same as their distribution in an absorbing medium characterized by the rate of absorption $q$. According to [11], this rate is determined by both the asymptotic behavior of the probability density $p(y)$ at $y \to \infty$ and the asymptotic behavior of the scale function $a(\tau)$ at $\tau \to 0$. Since $p(y)$ is assumed to be given, the noise action is non-trivial, i.e., $0 < q < \infty$, if the asymptotic behavior of $a(\tau)$ is given by [13]. It should be noted that finding the scale function $a(\tau)$ at small $\tau$ is important not only for determining the absorption rate $q$, but also for the numerical simulations of the Langevin equation [28].

In order to illustrate how our general results can be used for the determination of $q$ and $a(\tau)$ at $\tau \to 0$, we consider a two-parametric class of superheavy-tailed distributions that is described by the probability density $p(y) = (\nu - 1)/(\ln s)^{\nu-1}/2(s + |y|)|\ln(s + |y|)|^\nu$. Since $p(y)$ is assumed to be normalized and non-negative, the parameters $\nu$ and $s$ must satisfy the conditions $\nu > 1$ and $s > 1$. According to (10), the probability density $p(y)$ is symmetric, has a single maximum located at $y = 0$, and its asymptotic behavior at $y \to \infty$ is given by

$$h(y) \sim \frac{(\nu - 1)(\ln s)^{\nu-1}}{2(\ln y)^\nu},$$

(47)

For the probability density (46) the inner integral in (11) can easily be calculated yielding

$$Y(\lambda) = \frac{(\ln s)^{\nu-1}}{2} \int_0^{\infty} \frac{dx}{(\ln(s + \lambda x))^{\nu-2}} \sin \frac{x}{\ln(s + \lambda x)}. $$

(48)

To find the leading term of the asymptotic expansion of $Y(\lambda)$ as $\lambda \to \infty$, we use the identity

$$\frac{1 - e^{-\sigma \rho}}{\sigma} = \int_0^\rho d\rho \ e^{-\rho \sigma},$$

(49)

with $\sigma = [\ln(s + \lambda x)]^{\nu-1}$ and $\rho = (\ln \lambda)^{2-\nu}$. Introducing the variable of integration $z = \rho/\sigma$ and taking into account that $\sigma \rho \sim \ln \lambda \sim \ln \lambda$ if $\lambda \to \infty$ and $0 < \sigma < \infty$, from (49) we obtain

$$\frac{1}{\ln(s + \lambda x)^{\nu-1}} \sim \frac{1}{(\ln \lambda)^{\nu-2}} \int_0^1 dz e^{-z \ln \lambda} \Gamma(1 - z) \cos \frac{\pi z}{2}.$$  

(50)

Then, substituting (50) into (48) and using the integral formula $\int_0^\infty dx e^{-x^2} \sin x = \Gamma(1 - z) \cos \pi z/2$, where $\Gamma(z)$ is the gamma function, we arrive to

$$Y(\lambda) \sim \frac{(\ln s)^{\nu-1}}{2(\ln \lambda)^{\nu-2}} \int_0^1 dz e^{-z \ln \lambda} \Gamma(1 - z) \cos \frac{\pi z}{2}.$$  

(51)

Since $\lambda \to \infty$, the main contribution to the integral in (51) comes from a small vicinity of the lower limit of integration. Making use of this fact, one can easily verify that this integral is asymptotically equal to $1/\ln \lambda$, and so the asymptotic formula (51) takes the form

$$Y(\lambda) \sim \frac{1}{2} (\frac{\ln s}{\ln \lambda})^{\nu-1}.$$  

(52)

We are now in a position to find the asymptotic representation of the functions $\Phi(v)$ and $\Phi^{-1}(\varphi)$ in the reference case [we remind that according to (16) - (18) these functions determine the absorption rate $q$ and the scale function $a(\tau)$. By comparing (52) with (15), one obtains $c = (\ln s)^{\nu-1}/2$ and $\Phi(v) \sim (\nu - 1) \ln v$ ($v \to \infty$). The first result leads to the absorption rate

$$q = \frac{(\ln s)^{\nu-1}}{\gamma},$$  

(53)

and the second permits us to find an explicit asymptotic formula for the inverse function: $\varphi^{-1}(\varphi) \sim \exp[\varphi/(\nu - 1)]$ ($\varphi \to \infty$). Using this formula, from (18) we get

$$a(\tau) \sim \exp\left[-\exp\left(\frac{1}{\nu - 1} \ln \frac{\tau}{\gamma}\right)\right].$$  

(54)
as $\tau \to 0$. Thus, the noise action is specified not only by the parameters $\nu$ and $s$, which characterize the superheavy-tailed probability density $f(t)$, but also by the parameter $r$, which characterizes the asymptotic behavior (54) of the scale function $a(\tau)$. We note also that at $\nu = 2$ the formulas (55) and (54) are reduced to those derived in Ref. [28], i.e., $q = (\ln s)/r$ and $a(\tau) \sim \exp(-r/\tau)$.

We complete our analysis by comparing the sample paths of the solutions $x(t)$ of the Langevin equation (1) driven by different noises. If the transition probability density of the noise generating process is given by (4) then, depending on the probability density $p(y)$, the sample paths can show three qualitatively different behaviors. Specifically, if the variance of $p(y)$ is finite then the sample paths of $x(t)$ are random and continuous (e.g., as in the case of the Wiener process). In contrast, the sample paths which correspond to the probability density $p(y)$ with heavy tails are random and discontinuous (e.g., as in the case of the Lévy process). Finally, if $p(y)$ has superheavy tails then the corresponding noise does not influence $x(t)$ up to a random time $t_{tr}$, whose average value, $t_{tr} = \int_0^\infty tdA_0(t)$, is expressed through the absorption rate as $t_{tr} = 1/q$ (here $dA_0(t)/dt = qe^{-qt}$ is the probability density of the surviving time). At $t = t_{tr}$ an infinite jump of $x(t)$ occurs in the positive or negative direction, which is interpreted as the transition of a particle to the absorbing state (see also Ref. [28]). Thus, in this case the sample paths are the randomly interrupted realizations of the same deterministic function $z(t)$; $z(0) = 0$ satisfying the motion equation $\dot{z}(t) = f(z(t), t)$ (in the case of compound noise the sample paths are the randomly interrupted random processes, see Sect. 5.4). It is this property of the sample paths which makes it possible to model a number of physical processes interrupted at random times by the use of the Langevin equation driven by superheavy-tailed noise.

Examples of these processes can be found, e.g., in systems where the particle dynamics displays a qualitative change at some random time. One of them, the particle propagation in an absorbing medium characterized by a constant absorption rate, is studied in this paper. We note that because all superheavy-tailed noises generate the absorbing state, for the numerical simulations of the Langevin equation (1) driven by this noise the probability density $p(y)$ can always be chosen as in (4). The other example is the position of a fluid particle in a boiling liquid. In this case, the interruption of the process corresponds to the transition of a particle to the vapor state. One more example is the position of a free electron in a semiconductor. Here the interruption of the process occurs due to the recombination of the electron-hole pair.

5 Conclusions

We have developed a general approach for studying the statistical properties of particles driven by superheavy-tailed noises. These noises arise from the noise generating processes whose independent increments are characterized by the absence of finite fractional moments. Due to this feature, the distribution of particles has been studied in detail. Our approach is based on the generalized Fokker-Planck equation (2) which corresponds to the generalized Langevin equation (1). Within this framework we have established that the distribution of particles is the same as in the case of particles moving in an absorbing medium characterized by the rate of absorption $q$. In other words, superheavy-tailed noise plays the role of an absorbing medium.

By analyzing the solution of the generalized Fokker-Planck equation, we have shown that the distribution function can be divided into two parts which describe two probabilistic states of each particle. These states, surviving and absorbing, are induced by superheavy-tailed noise and are realized with the probabilities $e^{-qt}$ and $1 - e^{-qt}$, respectively. In the surviving state the particle dynamics occurs in such a way as if this noise is absent. Therefore, the distribution of particles in this state is non-degenerate or degenerate depending on that is the driving noise compound or not. We have derived the corresponding distribution functions for both these cases. An infinite jump of a particle, which occurs at a random time under the action of superheavy-tailed noise, is interpreted as the transition of this particle to the absorbing state. We have shown explicitly that the distribution of particles in this state is characterized by the condition that the probability to find these particles in any finite interval equals zero.

Using the theory of slowly varying functions, we have derived a general representation for the main characteristic of superheavy-tailed noise, i.e., the absorption rate $q$. This representation depends on the asymptotic behavior of the superheavy-tailed probability density $p(y)$ at $y \to \infty$ and the scale function $a(\tau)$ at $\tau \to 0$. The asymptotic behavior of $a(\tau)$ is controlled by the asymptotic behavior of $p(y)$, which is assumed to be known. Finally, as an illustration of our results, we have found the absorption rate and the scale function for a wide class of superheavy-tailed densities.

References

1. W. Horsthemke, R. Lefever, Noise-Induced Transitions (Springer-Verlag, Berlin, 1984)
2. W.T. Coffey, Yu.P. Kalmykov, J.T. Waldron, The Langevin Equation, 2nd edn. (World Scientific, Singapore, 2004)
3. H. Risken, The Fokker-Planck Equation, 2nd edn. (Springer-Verlag, Berlin, 1989)
4. C.W. Gardiner, Handbook of Stochastic Methods, 2nd edn. (Springer-Verlag, Berlin, 1990)
5. L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, Rev. Mod. Phys. 70, 223 (1998)
6. F. Jülicher, A. Ajdari, J. Prost, Rev. Mod. Phys. 69, 1269 (1997)
7. P. Reimann, Phys. Rep. 361, 57 (2002)
8. P. Hänggi, F. Marchesoni, Rev. Mod. Phys. 81, 387 (2009)
9. N.V. Agudov, B. Spagnolo, Phys. Rev. E 64, 035102(R) (2001)
10. A.A. Dubkov, N.V. Agudov, B. Spagnolo, Phys. Rev. E 69, 061103 (2004)
11. S.I. Denisov, A.N. Vitrenko, W. Horsthemke, Phys. Rev. E 68, 046132 (2003)
12. S.I. Denisov, A.N. Vitrenko, W. Horsthemke, P. Hänggi, Phys. Rev. E 73, 036120 (2006)
13. S. Jespersen, R. Metzler, H.C. Fogedby, Phys. Rev. E 59, 2736 (1999)
14. P.D. Ditlevsen, Phys. Rev. E 60, 172 (1999)
15. D. Schertzer, M. Larchevêque, J. Duan, V.V. Yanovsky, S. Lovejoy, J. Math. Phys. 42, 200 (2001)
16. D. Brockmann, I.M. Sokolov, Chem. Phys. 284, 409 (2002)
17. A. Dubkov, B. Spagnolo, Fluct. Noise Lett. 5, L267 (2005)
18. S.I. Denisov, W. Horsthemke, P. Hänggi, Phys. Rev. E 77, 061112 (2008)
19. Lévy Flights and Related Topics in Physics, edited by M.F. Shlesinger, G.M. Zaslavsky, U. Frisch (Springer-Verlag, Berlin, 1995)
20. R. Metzler, J. Klafter, Phys. Rep. 339, 1 (2000)
21. A.V. Chechkin, V.Y. Gonchar, J. Klafter, R. Metzler, Adv. Chem. Phys. 133, 439 (2006)
22. A. Dubkov, B. Spagnolo, V.V. Uchaikin, Int. J. Bifurcat. Chaos 18, 2649 (2008)
23. S.I. Denisov, P. Hänggi, H. Kantz, Europhys. Lett. 85, 40007 (2009)
24. S. Havlin, H. Weissman, Phys. Rev. B 37, 487 (1988)
25. S. Havlin, G.H. Weiss, J. Stat. Phys. 58, 1267 (1990)
26. J. Dräger, J. Klafter, Phys. Rev. Lett. 84, 5998 (2000)
27. S.I. Denisov, H. Kantz, Europhys. Lett. 92, 30001 (2010)
28. S.I. Denisov, H. Kantz, P. Hänggi, J. Phys. A: Math. Theor. 43, 285004 (2010)
29. M. Chaichian, A. Demichev, Path Integrals in Physics (Institute of Physics Publishing, Bristol, 2001), Vol. 1.
30. P. Del Moral, A. Doucet, Stoch. Anal. Appl. 22, 1175 (2004)
31. S.I. Denisov, W. Horsthemke, P. Hänggi, Eur. Phys. J. B 68, 567 (2009)
32. N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation (Cambridge University Press, Cambridge, 1987)
33. W. Feller, An Introduction to Probability Theory and its Applications, 2nd edn. (Wiley, New York, 1971), Vol. 2
34. S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications (Gordon & Breach, New York, 1993)