A SYSTEM OF HYPERGEOMETRIC DIFFERENTIAL EQUATIONS IN TWO VARIABLES OF RANK 9

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Abstract. We study a hypergeometric function in two variables and a system of hypergeometric differential equations associated with this function. This is a regular holonomic system of rank 9. We give a fundamental system of solutions to this system in terms of this hypergeometric series. We give circuit matrices along generators of the fundamental group of the complement of its singular locus with respect to our fundamental system.

1. Introduction

There are several generalizations of the original hypergeometric function $\mathbf{2F}_{1}$ and differential equation. For examples, we have the generalized hypergeometric function $_{p}F_{p-1}$, Appell’s functions $F_{1}, \ldots, F_{4}$, Lauricella’s functions $F_{A}, \ldots, F_{D}$, and the differential equations associated with them.

In this paper, we study a hypergeometric function $F \left( \begin{matrix} a \\ B \end{matrix} ; x \right)$ in two variables $x_1$ and $x_2$ with parameters $a = (a_1, a_2, a_3)$, $B = \begin{pmatrix} b_1, b_2, 1 \\ b_3, b_4, 1 \end{pmatrix}$ defined in (2.1), which is one of generalizations of the hypergeometric series introduced by Kampé de Fériet as mentioned in [5][§1.5]. Our function can be regarded as an extension of $3F_2 \left( \begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \\ x \end{matrix} \right)$ just like Appell’s $F_4 \left( \begin{matrix} a_1, a_2, x_1, x_2 \\ b_1, b_2 \end{matrix} \right)$ as that of the original $2F_1 \left( \begin{matrix} a_1, a_2 \\ b_1 \\ x \end{matrix} \right)$.

We give a system $\mathcal{F} \left( \begin{matrix} a \\ B \end{matrix} \right)$ of differential equations satisfied by our function in Proposition 2.3. Our function and this system admit the $S_3 \times D_4$-symmetry with respect to parameter and variable changes, where $S_3$ is the symmetric group of degree 3 and $D_4$ is the dihedral
group of order 8. This symmetry helps us to find solutions and their integral representations, and to check some formulas. We show in Theorem 2.1 that the system $F \left( \frac{a}{B} \right)$ is holonomic of rank 9 and that its singular locus $S$ consists of the coordinate axes and a nodal cubic curve $R(x) = 0$ triply tangent to them. Under a $(\mathbb{Z}/(3\mathbb{Z}))^2$ covering map, the pull back of $S$ is decomposed into twelve lines in the complex projective plane $\mathbb{P}^2$. These lines together with nine intersection points of them form the Hesse configuration, in which lines and points satisfies that three points per line and four lines through each point. In other words, we also have a holonomic system of rank 9 defined on the complement of the Hesse configuration.

We investigate in [9] the structure of the fundamental group $\pi_1(X, \dot{x})$, where $X$ is the complement of the singular locus $S$. We show that it is generated by three loops $\rho_1, \rho_2$ and $\rho_3$ turning around $x_1 = 0, x_2 = 0$ and $R(x) = 0$, respectively, and that its is isomorphic to a group generated by three elements with four relations among them.

To study the monodromy representation of $F \left( \frac{a}{B} \right)$ in this paper, we use the generating loops $\rho_1, \rho_2$ and $\rho_3$ and three relations among them, which characterize an Artin group of infinite type.

We construct a fundamental system of solutions to the system $F \left( \frac{a}{B} \right)$ in Theorem 2.2 by using the series $F \left( \frac{a'}{B'}; x \right)$ with different parameters $a', B'$. We give their integral representations of Euler type in §3 by following results in §4 in [6]. We consider the monodromy representation of our system in §5. The problem is the computation of the circuit matrix along the loop $\rho_3$. We show that it has an eigenvector $v$ of eigenvalue $\lambda \neq 1$ and the 8-dimensional eigenspace of eigenvalue 1 under some non-integral conditions on the parameters. We express it as a reflection of root $v$ with respect to an indeterminate form $H$.

By normalizing $v$ to $(1, \ldots, 1)$ and restricting the system $F \left( \frac{a}{B} \right)$ to $x_i = 0 \ (i = 1, 2)$, we determine the eigenvalue $\lambda$ and the form $H$ from the monodromy representation of the generalized hypergeometric differential equation $3F_2$ as the method for Appell’s system $F_C$ in [13].

Note also that the system $F \left( \frac{a}{B} \right)$ is regular singular by the behavior of its solutions around each component of the singular locus $S$. 
Finally, we compute intersection numbers of twisted cycles given in §3. By these computations, we can conclude that the fundamental system of solutions used in the monodromy representation is given by the integral representations without Gamma factors. As in [7], [8], [12] and [14] for Appell-Lauricella’s systems, we express in Theorem 6.2 the circuit transforms in terms of intersection form independent of the choice of fundamental systems of solutions. It is studied in [9] that the monodromy representation of $\mathcal{F}(aB)$ is irreducible under some non-integral conditions of parameters.

There is a further generalization $\mathcal{F}_m(aB; x_1, \ldots, x_m)$, which can be regarded as an $m$-variable version of $pF_{p-1}$. This function satisfies a system $\mathcal{F}_m(aB)$ of differential equations of rank $p^m$. We study this function and system in the forthcoming paper [10].

2. A SYSTEM OF HYPERGEOMETRIC DIFFERENTIAL EQUATIONS

We define a hypergeometric series of two variables $x_1$ and $x_2$ with parameters $a = (a_1, a_2, a_3)$, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} b_{11}, b_{12}, b_{13} \\ b_{21}, b_{22}, b_{23} \end{pmatrix} = \begin{pmatrix} b_1, b_2, 1 \\ b_3, b_4, 1 \end{pmatrix}$ as

$$F \begin{pmatrix} a \\ B \end{pmatrix}; x = \sum_{n \in \mathbb{N}^2} \prod_{j=1}^{3} \frac{(a_j, n_1 + n_2)}{(b_{1j}, n_1)(b_{2j}, n_2)} \frac{(a_3, n_1 + n_2)}{(b_{3j}, n_1)(b_{4j}, n_2)} x_1^{n_1} x_2^{n_2},$$

where $b_1, \ldots, b_4 \notin -\mathbb{N} = \{0, -1, -2, \ldots \}$. Note that this series reduces to $3F_2 \begin{pmatrix} a_1, a_2, a_3 \\ b_1, b_2 \end{pmatrix}; x_1$ and to $3F_2 \begin{pmatrix} a_1, a_2, a_3 \\ b_3, b_4 \end{pmatrix}; x_2$ when it is restricted to $x_2 = 0$ and to $x_1 = 0$, respectively. The following proposition is a direct consequence of the formula (29) in §1.3 in [15].

**Proposition 2.1.** If $x$ belongs to the domain

$$\mathcal{D} = \{(x_1, x_2) \in \mathbb{C}^2 \mid \sqrt[3]{|x_1|} + \sqrt[3]{|x_2|} < 1\},$$

then the hypergeometric series (2.1) absolutely converges. If $x$ does not belong to the closure of $\mathcal{D}$, then it diverges.
Proposition 2.2. The hypergeometric series \(2.1\) admits the symmetry:

\[
F \left( \frac{a}{B}; x_1, x_2 \right) = F \left( \frac{a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}}{b_{\tau(1), \sigma(1)}; b_{\tau(2), \sigma(2)}}; 1; x_{\tau(1)}, x_{\tau(2)} \right),
\]

where \(\sigma\) belongs to the symmetric group \(S_3\) of degree 3, \(\sigma_1, \sigma_2, \tau\) belong to the symmetric group \(S_2\) of degree 2. This symmetry is isomorphic to the direct product \(S_3 \times D_4\) of \(S_3\) and the dihedral group \(D_4\) of order 8.

Proof. Since the numerator of the right hand side of \(2.1\) is symmetric with respect to the parameters \(a_1, a_2, a_3\), \(F \left( \frac{a}{B}; x \right)\) is invariant under the action \(\sigma \in S_3\). We see the action on the parameters \(b_1, \ldots, b_4\) and the variables \(x_1, x_2\). Note that

\[
\begin{align*}
\sigma_{100} \cdot (b_1, b_2, b_3, b_4; x_1, x_2) &= (b_2, b_1, b_3, b_4; x_1, x_2), \\
\sigma_{010} \cdot (b_1, b_2, b_3, b_4; x_1, x_2) &= (b_1, b_2, b_4, b_3; x_1, x_2), \\
\sigma_{001} \cdot (b_1, b_2, b_3, b_4; x_1, x_2) &= (b_3, b_4, b_1, b_2; x_2, x_1),
\end{align*}
\]

for three elements \(\sigma_{100}, \sigma_{010}, \sigma_{001}\) of the triple of \(S_2\) given by \((\sigma_1, \sigma_2; \tau) = ((12); \text{id}; \text{id}), (\text{id}; (12); \text{id}), (\text{id}; \text{id}; (12))\). By the symmetry of the right hand side of \(2.1\), \(F \left( \frac{a}{B}; x \right)\) is also invariant under the actions generated by them. We consider the group structure of the triple of \(S_2\). It is clear that \(\sigma_{100}\) and \(\sigma_{010}\) are commutative. By definition, we have

\[
\begin{align*}
(\sigma_{100}\sigma_{001}) \cdot (b_1, b_2, b_3, b_4; x_1, x_2) &= \sigma_{100} \cdot (b_3, b_1, b_4, b_2; x_1, x_2) \\
= (b_4, b_3, b_1, b_2; x_2, x_1), \\
(\sigma_{001}\sigma_{100}) \cdot (b_1, b_2, b_3, b_4; x_1, x_2) &= \sigma_{001} \cdot (b_2, b_1, b_3, b_4; x_1, x_2) \\
= (b_3, b_4, b_2, b_1; x_2, x_1),
\end{align*}
\]

These imply that

\[
\sigma_{101} = ((12); \text{id}; (12)) = \sigma_{100}\sigma_{001} \neq \sigma_{001}\sigma_{100} = (\text{id}; (12); (12)) = \sigma_{011}.
\]

Since \(\sigma_{101}\) and \(\sigma_{011}\) are of order 4, this group is isomorphic to the dihedral group \(D_4\) of order 8. By this action on \(b_1, \ldots, b_4\), we have an inclusion of \(D_4\) into the symmetric group \(S_4\) as in Table 1. Elements of \(S_4\) are expressed in terms of cyclic permutations. It is easy to see that the action of \(S_3\) commutes with that of \(D_4\). \(\square\)
Table 1. Inclusion $D_4 \hookrightarrow S_4$

| $D_4$ | $\sigma_{000}$ | $\sigma_{100}$ | $\sigma_{010}$ | $\sigma_{110}$ | $\sigma_{001}$ | $\sigma_{101}$ | $\sigma_{011}$ | $\sigma_{111}$ |
|-------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $S_4$ | id            | (12)          | (34)          | (12)(34)      | (13)(24)      | (1423)        | (1324)        | (14)(23)      |

Proposition 2.3. The function $F\left(\frac{a}{B}; x\right)$ satisfies hypergeometric differential equations

\[
\left[ \prod_{j=1}^{3}(b_{ij} - 1 + \theta_i) \right] \cdot f(x) = \left[ x_i \prod_{j=1}^{3}(a_j + \theta_1 + \theta_2) \right] \cdot f(x) \quad (i = 1, 2),
\]

where $f(x)$ is an unknown function, and $\theta_i = x_i \partial_i$, $\partial_i = \frac{\partial}{\partial x_i}$ ($i = 1, 2$).

Proof. We write down the differential equations as

\[
\begin{align*}
\theta_1(b_1 - 1 + \theta_1)(b_2 - 1 + \theta_1) f(x) \\
= x_1(a_1 + \theta_1 + \theta_2)(a_2 + \theta_1 + \theta_2)(a_3 + \theta_1 + \theta_2) f(x),
\end{align*}
\]

\[
\begin{align*}
\theta_2(b_3 - 1 + \theta_2)(b_4 - 1 + \theta_2) f(x) \\
= x_2(a_1 + \theta_1 + \theta_2)(a_2 + \theta_1 + \theta_2)(a_3 + \theta_1 + \theta_2) f(x).
\end{align*}
\]

Let us show (2.2). Since

\[
\begin{align*}
\theta_1(b_1 - 1 + \theta_1)(b_2 - 1 + \theta_1)x_1^{n_1}x_2^{n_2} &= n_1(b_1 - 1 + n_1)(b_2 - 1 + n_1)x_1^{n_1}x_2^{n_2}, \\
x_1 \prod_{i=1}^{3}(a_i + \theta_1 + \theta_2)x_1^{n_1}x_2^{n_2} &= \prod_{i=1}^{3}(a_i + n_1 + n_2)x_1^{n_1+1}x_2^{n_2},
\end{align*}
\]

we have

\[
\begin{align*}
\theta_1(b_1 - 1 + \theta_1)(b_2 - 1 + \theta_1)F\left(\frac{a}{B}; x\right) \\
= \sum_{n_1=1, n_2=0}^{\infty} \frac{(a_1, n_1 + n_2)(a_2, n_1 + n_2)(a_3, n_1 + n_2)}{(b_1, n_1 - 1)(b_2, n_1 - 1)(b_3, n_2)(b_4, n_2)(1, n_2)} x_1^{n_1}x_2^{n_2}, \\
x_1 \prod_{i=1}^{3}(a_i + \theta_1 + \theta_2)F\left(\frac{a}{B}; x\right) \\
= \sum_{n_1=1, n_2=0}^{\infty} \frac{(a_1, n_1 + n_2 + 1)(a_2, n_1 + n_2 + 1)(a_3, n_1 + n_2 + 1)}{(b_1, n_1)(b_2, n_1)(1, n_1)(b_3, n_2)(b_4, n_2)(1, n_2)} x_1^{n_1+1}x_2^{n_2}.
\end{align*}
\]

Note that these series coincide. Similarly, we can show (2.3). \qed
We study the system $F\left(\frac{a}{B}\right) = F\left(\frac{a}{B}; x\right)$ generated by the differential equations (2.2) and (2.3).

**Proposition 2.4.** The system $F\left(\frac{a}{B}\right)$ admits the $S_3 \times D_4$-symmetry:

$$F\left(\frac{a}{B}; x_1, x_2\right) = F\left(\begin{array}{ccc} a_{\sigma(1)}, & a_{\sigma(2)}, & a_{\sigma(3)} \\ b_{\tau(1), \sigma(1)(1)}, & b_{\tau(1), \sigma(1)(2)}, & 1 \\ b_{\tau(2), \sigma(2)(1)}, & b_{\tau(2), \sigma(2)(2)}, & 1 \end{array}; x_{\tau(1)}, x_{\tau(2)}\right),$$

for any $\sigma \in S_3$ and $\sigma_1, \sigma_2, \tau \in S_2$.

**Proof.** It is easy to see that the differential equations (2.2) and (2.3) are invariant under the action of $S_3$. They are also invariant under the actions of $\sigma_{100}$ and $\sigma_{010}$, and exchanged by the action of $\sigma_{001}$, where $\sigma_{100}$, $\sigma_{010}$ and $\sigma_{001}$ are given in the proof of Proposition 2.2. \qed

**Theorem 2.1.** The system $F\left(\frac{a}{B}\right)$ is of rank 9. Its singular locus $S$ is

$$S = \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1x_2R(x_1, x_2) = 0\},$$

where

$$R(x) = R(x_1, x_2) = \prod_{k_1, k_2=0}^2 \left(1 - \omega^{k_1} \sqrt[3]{x_1} - \omega^{k_2} \sqrt[3]{x_2}\right)$$

$$= (1 - x_1 - x_2)^3 - 27x_1x_2,$$

and $\omega = \frac{-1 + \sqrt{-3}}{2}$.

**Proof.** We can regard the system as a left ideal $I$ in the Weyl algebra $W_2 = \mathbb{C}\langle x_1, x_2, \partial_1, \partial_2 \rangle$. Any element of $W_2$ has a canonical form

$$\sum_{(i_1, i_2)} c_{i_1, i_2}(x_1, x_2)\partial_1^{i_1}\partial_2^{i_2}, \quad c_{i_1, i_2}(x_1, x_2) \in \mathbb{C}[x_1, x_2].$$

By replacing $\partial_j$ with $\xi_j$ ($j = 1, 2$) in canonical forms, we have a natural map from $W_2$ to the polynomial ring $\mathbb{C}[x_1, x_2, \xi_1, \xi_2]$. Then the initial ideal $\text{in}_w(I)$ with respect to the weight vector $w = (0, 0, 1, 1)$ is defined as an ideal of $\mathbb{C}[x_1, x_2, \xi_1, \xi_2]$. The zero set $V(\text{in}_w(I))$ of the initial ideal in $\mathbb{C}^4$ is called the characteristic variety of $I$. The singular locus $\text{Sing}(I)$ of the left ideal $I$ is defined by the Zariski closure of the image of $V(\text{in}_w(I)) \setminus \{(x, \xi) \mid \xi_1 = \xi_2 = 0\}$ under the projection $\mathbb{C}^4 \ni (x, \xi) \mapsto (x, \xi)$.
$x \in \mathbb{C}^2$. It is shown in §1.4 of [16] that $\text{Sing}(I)$ coincides with the zero set $V(J)$ of the polynomial ideal

$$J = (\text{in}_w(I): \langle \xi_1, \xi_2 \rangle^\infty) \cap \mathbb{C}[x_1, x_2].$$

Generators of the initial ideal $\text{in}_w(I)$ are calculated from the Gröbner basis of $I$ with respect to the weight vector $w$. We can execute saturation and elimination of the ideal $I$ by using Risa/Asir (a computer algebra system). By this computation, it turns out that $J = \langle x_1^4x_2^4R(x) \rangle$.

Thus we have

$$\text{Sing}(I) = V(J) = V(\sqrt{J}) = V(x_1x_2R(x)),$$

where $\sqrt{J}$ is the radical of $J$.

The holonomic rank of $I$ is defined by

$$\text{rank}(I) = \dim_{\mathbb{C}(x_1,x_2)}(W_2/W_2I),$$

where $W_2 = \mathbb{C}(x_1, x_2)\langle \partial_1, \partial_2 \rangle$, and the quotient $W_2/W_2I$ is regraded as a vector space over $\mathbb{C}(x_1, x_2)$. With respect to the graded lexicographic order in $W_2$, a left $W_2$-ideal $W_2I$ has a Gröbner basis whose initial terms are given by $\{\xi_1^5, \xi_2^3, \xi_1\xi_2^3, \xi_2^5\}$. Then the vector space $W_2/W_2I$ has a basis $\{1, \partial_1, \partial_2, \partial_1^2, \partial_2^2, \partial_2^3\}$ consisting of 9 monomials. Thus the holonomic rank of $I$ is 9.

\[ \square \]

Remark 2.1.

1. Since the affine curve $R(x) = 0$ has a nodal singular point $(-1, -1)$, it is rational. In fact, it is expressed by a complex parameter as

$$(x, y) = (t^2, (1 - t)^3), \quad t \in \mathbb{C}.$$  

Via this parametrization, $t = -\omega$ and $t = -\omega^2$ correspond to the singular point $(-1, -1)$.

2. Under the covering map $\mathbb{C}^2 \ni (z_1, z_2) \mapsto (x_1, x_2) = (z_1^3, z_2^3) \in \mathbb{C}^2$, the pull back of the cubic curve $R(x) = 0$ is decomposed into nine lines

$$1 - \omega^{k_1}z_1 - \omega^{k_2}z_2 = 0, \quad (0 \leq k_1, k_2 \leq 2).$$

Thus the pull back of the singular locus $S$ consists of these lines together with $z_1 = 0, z_2 = 0$ and the line at infinity in the projective plane $\mathbb{P}^2$. Note that there are nine intersection points $[\zeta_0, \zeta_1, \zeta_2] = [0, 1, -\omega^{k_1}], [1, 0, \omega^{k_1}], [1, \omega^{k_1}, 0] (0 \leq k_1 \leq 2)$ of two lines of them, where $[\zeta_0, \zeta_1, \zeta_2]$ are the projective coordinates with $(z_1, z_2) = (\zeta_1/\zeta_0, \zeta_2/\zeta_0)$. These twelve lines and the nine points form the Hesse configuration, in which lines and points satisfies that three points per line and four lines through each point.
We set
\[ X = \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1 x_2 R(x_1, x_2) \neq 0\}, \]
which is the complement of the singular locus \(S\) of the system \(F\left(\begin{bmatrix} a \\ B \end{bmatrix}\right)\).

We select a base point \(x = (\varepsilon_1, \varepsilon_2)\) in \(\mathbb{D} \cap X \cap \mathbb{R}^2\) with \(0 < \varepsilon_2 < \varepsilon_1\), and take a small neighborhood \(U\) of \(x\) in \(\mathbb{D} \cap X\).

**Theorem 2.2.** Suppose that
\[ b_1, b_2, b_3, b_4, b_1 - b_2, b_3 - b_4, \notin \mathbb{Z}. \]

A fundamental system of solutions to \(F\left(\begin{bmatrix} a \\ B \end{bmatrix}\right)\) on \(U\) is given by
\[ (2.4) \quad F_{jk}(x) = x_1^{1-b_1} x_2^{1-b_2} F\left(\begin{bmatrix} a + (2 - b_1 j - b_2 k)(e_1 + e_2 + e_3) \\ B_1 + (1 - b_1 j)(e_j + e_1 + e_2) \\ B_2 + (1 - b_2 k)(e_k + e_1 + e_2) \end{bmatrix} ; x\right) \]
for \(1 \leq j, k \leq 3\), where \(e_j\) is the \(j\)-th unit row vector of \(\mathbb{R}^3\), and the indices \(j\) and \(k\) in \(F_{jk}(x)\) are regarded as elements of \(\{0, 1, 2\} = \mathbb{Z}/(3\mathbb{Z})\). They are arrayed in the order
\[ (2.5) \quad (jk) = (00), (10), (20), (01), (11), (21), (02), (12), (22) \]
as
\[ F\left(\begin{bmatrix} a_1, a_2, a_3 \\ b_1, b_2, 1 \\ b_3, b_4, 1 \end{bmatrix} ; x\right), \]
\[ x_1^{1-b_1} F\left(\begin{bmatrix} a_1 - b_1 + 1, & a_2 - b_1 + 1, & a_3 - b_1 + 1 \\ 2 - b_1, & b_2 - b_1 + 1, & 1 \\ b_3, & b_4, & 1 \end{bmatrix} ; x\right), \]
\[ x_1^{1-b_2} F\left(\begin{bmatrix} a_1 - b_2 + 1, & a_2 - b_2 + 1, & a_3 - b_2 + 1 \\ b_1 - b_2 + 1, & 2 - b_2, & 1 \\ b_3, & b_4, & 1 \end{bmatrix} ; x\right), \]
\[ x_2^{1-b_3} F\left(\begin{bmatrix} a_1 - b_3 + 1, & a_2 - b_3 + 1, & a_3 - b_3 + 1 \\ b_1, & b_2, & 1 \\ 2 - b_3, & b_4 - b_3 + 1, & 1 \end{bmatrix} ; x\right), \]
\[ x_1^{1-b_1} x_2^{1-b_3} F\left(\begin{bmatrix} a_1 - b_1 - b_3 + 2, & a_2 - b_1 - b_3 + 2, & a_3 - b_1 - b_3 + 2 \\ 2 - b_1, & b_2 - b_1 + 1, & 1 \\ 2 - b_3, & b_4 - b_3 + 1, & 1 \end{bmatrix} ; x\right), \]
\[ x_1^{1-b_2} x_2^{1-b_3} F\left(\begin{bmatrix} a_1 - b_2 - b_3 + 2, & a_2 - b_2 - b_3 + 2, & a_3 - b_2 - b_3 + 2 \\ b_1 - b_2 + 1, & 2 - b_2, & 1 \\ 2 - b_3, & b_4 - b_3 + 1, & 1 \end{bmatrix} ; x\right), \]
Proof. We can show that these functions are solutions to $\mathcal{F}\left(\begin{array}{c}a \\ B\end{array}\right)$ by using properties

$$\prod_{i=1}^{3}(a_{i}+\theta_{1}+\theta_{2})x_{1}^{1-b_{1j}}x_{2}^{1-b_{2k}} = x_{1}^{1-b_{1j}}x_{2}^{1-b_{2k}}\prod_{i=1}^{3}(a_{i}-b_{1j}-b_{2k}+2+\theta_{1}+\theta_{2}),$$

$$\prod_{i=1}^{3}(b_{1i}-1+\theta_{1})x_{1}^{1-b_{1j}}x_{2}^{1-b_{2k}} = x_{1}^{1-b_{1j}}x_{2}^{1-b_{2k}}\prod_{i=1}^{3}(b_{1i}-b_{1j}+\theta_{1}),$$

$$\prod_{i=1}^{3}(b_{2i}-1+\theta_{2})x_{1}^{1-b_{1j}}x_{2}^{1-b_{2k}} = x_{1}^{1-b_{1j}}x_{2}^{1-b_{2k}}\prod_{i=1}^{3}(b_{2i}-b_{2k}+\theta_{2}),$$

as elements of the (extended) Weyl algebra. Since the power functions $x_{1}^{1-b_{1j}}x_{2}^{1-b_{2k}}$ are mutually different under our assumption, these functions are linearly independent. \[\square\]

Recall that the series $F\left(\begin{array}{c}a \\ B\end{array};x\right)$ and the system $\mathcal{F}\left(\begin{array}{c}a \\ B\end{array};x\right)$ are invariant under the action of $S_3 \times D_4$.

**Proposition 2.5.** The group $S_3 \times D_4$ acts on the fundamental system (2.4) of solutions to $\mathcal{F}\left(\begin{array}{c}a \\ B\end{array}\right)$ via the change of parameters and variables in Proposition 2.2. There are three orbits $\{F_{00}(x)\}$,

$$\{F_{10}(x), F_{20}(x), F_{01}(x), F_{02}(x)\}, \quad \{F_{11}(x), F_{21}(x), F_{12}(x), F_{22}(x)\}.$$  

Proof. We have only to consider actions of $D_4$ on $B$ and $x$. It is easy to see that every $F_{jk}(x)$ ($0 \leq j, k \leq 2$) is changed into one of (2.4). By Proposition 2.2, $F_{00}(x)$ is invariant under this action. By the actions $\sigma_{100}, \sigma_{001}$ and $\sigma_{101}$ on $F_{10}(x)$, it changes into $F_{20}(x), F_{01}(x)$ and $F_{02}(x)$, respectively; see Table II. For the orbit of $F_{11}(x)$, act $\sigma_{100}, \sigma_{010}$ and $\sigma_{110}$ on $F_{11}$. \[\square\]
Corollary 2.1. We set \( y = (y_1, y_2) = (\frac{-x_1}{x_2}, \frac{1}{x_2}) \), and suppose that 
\[ b_1, b_2, b_1 - b_2, a_1 - a_2, a_1 - a_3, a_2 - a_3 \notin \mathbb{Z}. \]

There are nine solutions to \( \mathcal{F} \left( \frac{a}{B} \right) \) around a point \( y \in X \) near to \( y = (0, 0) \) as follows:

\[
y_2^{a_1} \mathcal{F} \left( \begin{array}{c} a_i, & a_i - b_3 + 1, & a_i - b_4 + 1 \\ b_1, & b_2, & 1 \\ a_i - a_j + 1, & a_i - a_k + 1, & 1 \end{array} \right); y),
\]

\[
y_1^{1-b_1} y_2^{a_1} \mathcal{F} \left( \begin{array}{c} a_i - b_1 + 1, & a_i - b_1 - b_3 + 2, & a_i - b_1 - b_4 + 2 \\ 2 - b_1, & b_2 - b_1 + 1, & 1 \\ a_i - a_j + 1, & a_i - a_k + 1, & 1 \end{array} \right); y),
\]

\[
y_1^{1-b_2} y_2^{a_1} \mathcal{F} \left( \begin{array}{c} a_i - b_2 + 1, & a_i - b_2 - b_3 + 2, & a_i - b_2 - b_4 + 2 \\ b_1 - b_2 + 1, & 2 - b_2, & 1 \\ a_i - a_j + 1, & a_i - a_k + 1, & 1 \end{array} \right); y),
\]

where \( i = 1, 2, 3 \) and \( \{j, k\} = \{1, 2, 3\} - \{i\} \) as a set.

Proof. Since \( y_1 = -x_1/x_2, y_2 = 1/x_2 \), we have
\[
x_1 \partial_1 = x_1 \left( \frac{\partial y_1}{\partial x_1} \frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_1} \frac{\partial}{\partial y_2} \right) = -\frac{x_1}{x_2} \frac{\partial}{\partial y_1},
\]
\[
x_2 \partial_2 = x_2 \left( \frac{\partial y_1}{\partial x_2} \frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_2} \frac{\partial}{\partial y_2} \right) = \frac{x_1}{x_2} \frac{\partial}{\partial y_1} - \frac{1}{x_2} \frac{\partial}{\partial y_2}.
\]

Thus the operators \( \theta_{y_1} = y_1 \frac{\partial}{\partial y_1} \) and \( \theta_{y_2} = y_2 \frac{\partial}{\partial y_2} \) relate \( \theta_1 \) and \( \theta_2 \) as
\[
\theta_1 = \theta_{y_1}, \quad \theta_2 = -\left( \theta_{y_1} + \theta_{y_2} \right).
\]

Note that the system \( \mathcal{F} \left( \frac{a}{B} \right) \) is generated by the equation (2.3) and
\[
x_2 \theta_1 (b_1 - 1 + \theta_1) (b_2 - 1 + \theta_2) f(x) = x_1 \theta_2 (b_3 - 1 + \theta_2) (b_4 - 1 + \theta_2) f(x).
\]

We express these differential equations in terms of \( \theta_{y_1} \) and \( \theta_{y_2} \):
\[
(-a_1 + \theta_{y_2})(-a_2 + \theta_{y_2})(-a_3 + \theta_{y_2}) f(y) = y_2 (\theta_{y_1} + \theta_{y_2}) (-b_3 + 1 + \theta_{y_1} + \theta_{y_2}) (-b_4 + 1 + \theta_{y_1} + \theta_{y_2}) f(y),
\]
\[
\theta_{y_1} (b_1 - 1 + \theta_{y_1}) (b_2 - 1 + \theta_{y_1}) f(y) = y_1 (\theta_{y_1} + \theta_{y_2}) (-b_3 + 1 + \theta_{y_1} + \theta_{y_2}) (-b_4 + 1 + \theta_{y_1} + \theta_{y_2}) f(y).
\]
Substitute $f(y) = y_d^\mu g(y)$ into these differential equations. Then we have
\[
(\mu - a_1 + \theta_{yz})(\mu - a_2 + \theta_{yz})(\mu - a_3 + \theta_{yz})g(y)
= y_2(\mu + \theta_{yz})(\mu - b_3 + 1 + \theta_{yz})g(y),
\]
\[
\theta_{yz}(b_1 - 1 + \theta_{yz})(b_2 - 1 + \theta_{yz})g(y)
= y_1(\mu + \theta_{yz})(\mu - b_3 + 1 + \theta_{yz})g(y),
\]
then $y_d^\mu g(y)$ satisfies these differential equations. To obtain the rests, consider the series with exponents $y_1^{1-b_1}$ and $y_1^{1-b_2}$ in Theorem 2.2.

3. Integral representations of Euler type

**Theorem 3.1.** The hypergeometric series $F(a, b; x)$ admits the integral representation of Euler type:

\[
C_{00} \cdot \int_{\Delta_{00}} u(t, x) dt,
\]

where $u(t, x)$ is

\[
\left(\prod_{j=1}^{4} t_j^{-b_j}\right)(1-t_1-t_3)^{b_1+b_3-a_1-2}(1-t_2-t_4)^{b_2+b_4-a_2-2}\left(1 - \frac{x_1}{t_1t_2} - \frac{x_2}{t_3t_4}\right)^{-a_3},
\]

\[dt = dt_1 \wedge dt_3 \wedge dt_2 \wedge dt_4,\]

and the gamma factor $C_{00}$ is

\[
\frac{\Gamma(1-a_1)\Gamma(1-a_2)}{\Gamma(1-b_1)\Gamma(1-b_3)\Gamma(b_1+b_3-a_1-1)\Gamma(1-b_2)\Gamma(1-b_4)\Gamma(b_2+b_4-a_2-1)}.
\]

Here $\Delta_{00}$ is a 4-chain reg$^c_{00}(\Delta_1 \times \Delta_2)$ of the regularization of the direct product of triangles

$\Delta_1 = \{(t_1, t_3) \in \mathbb{R}^2 \mid t_1 > 0, \ t_3 > 0, \ 1 - t_1 - t_3 > 0\}$,

$\Delta_2 = \{(t_2, t_4) \in \mathbb{R}^2 \mid t_2 > 0, \ t_4 > 0, \ 1 - t_2 - t_4 > 0\}$,

associated with

\[u(t, 0, 0) = t_1^{-b_1}t_3^{-b_3}(1-t_1-t_3)^{b_1+b_3-a_1-2}t_2^{-b_2}t_4^{-b_4}(1-t_2-t_4)^{b_2+b_4-a_2-2},\]

and $x = (x_1, x_2)$ is supposed to be so close to $(0, 0)$ that the hypersurface

\[Q = \{t \in \mathbb{C}^4 \mid t_1t_2t_3t_4 - t_3t_4x_1 - t_1t_2x_2 = 0\}\]

does not intersect with any component of $\Delta_{00}$. 
Remark 3.1. 

(1) We set 
\[
\ell_{ij}(t) = 1 - t_i - t_j \quad (1 \leq i < j \leq 4), \quad q(t) = 1 - \frac{x_1}{t_1t_2} - \frac{x_2}{t_3t_4},
\]
and 
\[
(3.2) \quad T = \{ t \in \mathbb{C}^4 \mid t_1t_2t_3t_4\ell_{13}(t)\ell_{24}(t)q(t) \neq 0 \},
\]
which depends on \( x \in \mathbb{C}^2 \). For any fixed \( x \in X \), a locally constant sheaf \( \mathcal{L}_u \) on \( T \) is given by \( u(t, x) \). Twisted homology groups \( H_k(T, \mathcal{L}_u) \) are defined from a complex of chains with sections of \( \mathcal{L}_u \). Similarly, locally finite ones \( H_{lf}^k(T, \mathcal{L}_u) \) are defined. It is known that the natural map from \( H_4(T, \mathcal{L}_u) \) to \( H_{lf}^4(T, \mathcal{L}_u) \) becomes isomorphic under some non-integral conditions on the parameters. In this case, the regularization \( \text{reg} \) is defined by its inverse. It is different from the regularization \( \text{reg}_{00} \) in the construction of \( \Delta_{00} \), since the function \( u(t, 0, 0) \) is used. However, by loading a branch of \( u(t, x) \) on every component of \( \Delta_{00} \), we have an element \( \Delta^u_{00} \) of \( H_4(T, \mathcal{L}_u) \) for any \( x \) near to \((0, 0)\). We can also make the continuation of any element \( \Delta^u \in H_4(T, \mathcal{L}_u) \) along a path in \( X \) starting from \( \dot{x} = (\varepsilon_1, \varepsilon_2) \).

(2) We can construct \( \text{reg}_{00}^c(\Delta_1 \times \Delta_2) \) as the direct product of 2-chains given by the regularizations of the triangles 
\[
\Delta_1 = \{ (t_1, t_3) \in \mathbb{R}^2 \mid t_1 > 0, \ t_3 > 0, \ 1 - t_1 - t_3 > 0 \},
\]
\[
\Delta_2 = \{ (t_2, t_4) \in \mathbb{R}^2 \mid t_2 > 0, \ t_4 > 0, \ 1 - t_2 - t_4 > 0 \},
\]
associated with 
\[
u_1 = t_1^{-b_1}t_3^{-b_1}(1-t_1-t_3)^{b_1+b_3-a_1-2},
\]
\[
u_2 = t_2^{-b_2}t_4^{-b_4}(1-t_2-t_4)^{b_2+b_4-a_2-2},
\]
see Figure 1. Refer to §3.2.4 of [1] and §2 in [8] for an explicit construction of the regularization. If \( x \) is close to \((0, 0)\), then the hypersurface \( Q \) passes through the tuber neighborhood of the hyperplanes \( t_1 = 0, \ldots, t_4 = 0 \), which is made in the construction of the regularization of \( \Delta_1 \times \Delta_2 \). Thus we can assume that 
\[
\left| \frac{x_1}{t_1t_2} + \frac{x_2}{t_3t_4} \right| < 1
\]
for any \( t \) in each component of \( \Delta_{00} \).
Proof. Since $x$ is close to $(0,0)$, the factor including $x_1$ and $x_2$ admits the expansion
\[
\left(1 - \frac{x_1}{t_1 t_2} - \frac{x_2}{t_3 t_4}\right)^{-a_3} = \sum_{N=0}^{\infty} \frac{(a_3, N)}{N!} \left(\frac{x_1}{t_1 t_2} + \frac{x_2}{t_3 t_4}\right)^N
\]
\[
= \sum_{N=0}^{\infty} \frac{(a_3, N)}{N!} \sum_{n_1+n_2=N} \binom{N}{n_1} \left(\frac{x_1}{t_1 t_2}\right)^{n_1} \left(\frac{x_2}{t_3 t_4}\right)^{n_2}
\]
\[
= \sum_{n_1,n_2=0}^{\infty} \frac{(a_3, n_1+n_2)}{n_1!n_2!} (t_1 t_2)^{-n_1} (t_3 t_4)^{-n_2} \cdot x_1^{n_1} x_2^{n_2}.
\]

Thanks to Remark 3.1 (2), this expansion is valid on each component of $\Delta_{00}$. Change the order of the summation and the integration. The coefficient of $x_1^{n_1} x_2^{n_2}$ in the integration (without the gamma factor $C_{00}$) is the product of
\[
\int_{\Delta_1} t_1^{-b_1-n_1} t_3^{-b_3-n_2} (1 - t_1 - t_3)^{b_1+b_3-a_1-2} dt_1 \wedge dt_3
\]
\[
\Gamma(1 - b_1 - n_1) \Gamma(1 - b_3 - n_2) \Gamma(b_1 + b_3 - a_1 - 1) \\
\Gamma(1 - a_1 - n_1 - n_2)
\]

and
\[
\int_{\triangle_2} t_2^{-b_2 - n_1} t_4^{b_4 - n_2} (1 - t_2 - t_4)^{b_2 + b_4 - a_2 - 2} dt_2 \wedge dt_4 \\
= \frac{\Gamma(1 - b_2 - n_1) \Gamma(1 - b_4 - n_2) \Gamma(b_2 + b_4 - a_2 - 1)}{\Gamma(1 - a_2 - n_1 - n_2)}.
\]

By using the formulas \((\alpha, m) \Gamma(\alpha) = \Gamma(\alpha + m)\), and \(\Gamma(\alpha) \Gamma(1 - \alpha) = \frac{\pi}{\sin(\pi \alpha)}\), we rewrite \(\Gamma(1 - b_1 - n_1)\) to
\[
\Gamma(1 - b_1 - n_1) = \frac{\pi}{(-1)^{n_1} \sin(\pi b_1) \Gamma(b_1)(b_1, n_1)} = \frac{\Gamma(1 - b_1)}{(-1)^{n_1}(b_1, n_1)}.
\]

Similarly we have
\[
\Gamma(1 - b_3 - n_2) = \frac{\Gamma(1 - b_3)}{(-1)^{n_2}(b_3, n_2)},
\]
\[
\Gamma(1 - a_1 - n_1 - n_2) = \frac{\Gamma(1 - a_1)}{(-1)^{n_1 + n_2}(a_1, n_1 + n_2)},
\]

which yield
\[
\frac{\Gamma(1 - b_1 - n_1) \Gamma(1 - b_3 - n_2) \Gamma(b_1 + b_3 - a_1 - 1)}{\Gamma(1 - a_1 - n_1 - n_2)} = \frac{\Gamma(1 - b_1) \Gamma(1 - b_3) \Gamma(b_1 + b_3 - a_1 - 1)}{\Gamma(1 - a_1)} \cdot \frac{(a_1, n_1 + n_2)}{(b_1, n_1)(b_3, n_2)}.
\]

Since the other gamma factor is
\[
\frac{\Gamma(1 - b_2) \Gamma(1 - b_4) \Gamma(b_2 + b_4 - a_2 - 1)}{\Gamma(1 - a_2)} \cdot \frac{(a_2, n_1 + n_2)}{(b_2, n_1)(b_4, n_2)},
\]
the integral representation is obtained. \(\Box\)

**Remark 3.2.** By acting the group \(S_3 \times D_4\) on \([3, 1]\), we have 48 integral representations of \(F\left(\frac{a}{B}; x\right)\).

**Corollary 3.1.** The Euler number \(\chi(T)\) is nine for any \(x \in X\).

**Proof.** Thanks to Theorem 1 in \([3]\), only the 4-th twisted homology \(H_4(T, \mathcal{L}_a)\) survives. Thus \(\chi(T)\) is equal to the rank of \(H_4(T, \mathcal{L}_a)\). There is a natural isomorphism from this homology group to the vector space of local solutions to \(F\left(\frac{a}{B}\right)\) around \(x\). Hence we have \(\chi(T) = 9\) for any \(x \in X\) by Theorem 2.1. \(\Box\)
Corollary 3.2. The solution $F_{10}(x)$ in Theorem 2.2 is expressed as

$$C_{10} \int_{\Delta_{10}} u(t, x) dt,$$

where the gamma factor $C_{10}$ is

$$\frac{\Gamma(b_1 - a_1) \Gamma(b_1 - a_2) \Gamma(b_1 - a_3)}{\Gamma(b_1 + b_3 - a_1 - 1) \Gamma(b_2 + b_4 - a_2 - 1) \Gamma(1 - a_3)} \frac{1}{\Gamma(b_1 - 1) \Gamma(b_1 - b_2) \Gamma(1 - b_3) \Gamma(1 - b_4)}',$$

and $\Delta_{10}$ is the image of $-\text{reg}^c_{10}(\square_1 \times \triangle_2)$ under the map

$$t_{10} : (s_1, s_2, s_3, s_4) \mapsto (t_1, t_2, t_3, t_4) = \left( \frac{x_1}{s_1s_2}, s_2, s_3, s_4 \right).$$

Here $\text{reg}^c_{10}(\square_1 \times \triangle_2)$ is a 4-chain of the regularization of the direct product of

$$\square_1 = \{ (s_1, s_3) \in \mathbb{R}^2 \mid 0 < s_1 < 1, 0 < s_3 < 1 \}$$

and $\triangle_2$ associated with

$$s_1^{b_1-2}(1-s_1)^{-a_3}s_3^{b_3}(1-s_3)^{b_3-b_1-a_1-2}s_2^{b_1-b_2-1}s_4^{b_1-1}(1-s_2-s_4)^{b_2+b_4-a_2-2},$$

and the point $x = (x_1, x_2)$ is supposed to be so close to $(0, 0)$ that the pull back hypersurface $t_{10}^*Q$ does not intersect with any component of $\text{reg}^c_{10}(\square_1 \times \triangle_2)$.

**Proof.** Apply the variable change $t = t_{10}(s)$ to $\int_{\Delta_{10}} u(t, x) dt$ with paying attention to the signs in $\Delta_{01} = t_{10}(-\text{reg}^c(\square_1 \times \triangle_2))$ and $t_{10}^*(dt)$. Then it changes into

$$x_1^{b_1-1} \int_{\text{reg}^c(\square_1 \times \triangle_2)} s_1^{b_1-2}s_2^{b_1-b_2-1}s_3^{b_3-b_4}(1 - s_1 - \frac{x_2}{s_3s_4})^{-a_3} \cdot (1 - \frac{x_1}{s_1s_2} - s_3)^{b_1+b_3-a_1-2}(1 - s_2 - s_4)^{b_2+b_4-a_2-2} ds,$$

where $ds = ds_1 \wedge ds_3 \wedge ds_2 \wedge ds_4$. Expand the factors including $x_1$ and $x_2$. Then we have

$$(1 - s_3 - \frac{x_1}{s_1s_2})^{b_1+b_3-a_1-2} = \{(1 - s_3) \cdot (1 - \frac{x_1}{(1-s_3)s_1s_2})\}^{b_1+b_3-a_1-2}$$

$$= (1 - s_3)^{b_1+b_3-a_1-2} \sum_{n_1=0}^{\infty} \frac{(a_1 - b_1 - b_3 + 2, n_1)}{n_1!} \frac{x_1^{n_1}}{(1-s_3)^{n_1}x_1^{n_1}}.$$
\[(1 - s_1 - \frac{x_2}{s_3 s_4})^{-a_3} = (1 - s_1)^{-a_3} \cdot \left(1 - \frac{x_2}{(1 - s_1)s_3 s_4}\right)^{-a_3}\]

\[= (1 - s_1)^{-a_3} \sum_{n_2=0}^{\infty} \frac{(a_3, n_2)}{n_2!} \frac{x_2^{n_2}}{(1 - s_1)^{n_2} s_3 s_4 s_2^{n_2}}.
\]

Thus the integrand is

\[\sum_{(n_1, n_2) \in \mathbb{N}^2} x_1^{n_1} x_2^{n_2} \frac{(a_1 - b_1 - b_3 + 2, n_1) (a_3, n_2)}{n_1! n_2!} \cdot s_1^{-b_1-2-n_1} (1 - s_1)^{-a_3-n_2} s_3^{-b_3-n_2} (1 - s_3)^{-b_1+b_3-a_1-2-n_1} \cdot s_2^{-b_2-1-n_1} s_4^{-b_4-n_2} (1 - s_2 - s_4)^{-b_2+b_4-a_2-2}.
\]

Integrate this over \(\text{reg}^c_{10}(\square_1 \times \triangle_2)\), then we have

\[\sum_{(n_1, n_2) \in \mathbb{N}^2} x_1^{n_1} x_2^{n_2} \frac{(a_1 - b_1 - b_3 + 2, n_1) (a_3, n_2)}{n_1! n_2!} \cdot \frac{\Gamma(b_1 - 1 - n_1) \Gamma(-a_3 - n_2 + 1)}{\Gamma(b_1 - a_3 - n_1 - n_2)} \cdot \frac{\Gamma(-b_3 - n_2 + 1) \Gamma(b_1 + b_3 - a_1 - n_1 - 1)}{\Gamma(b_1 - a_1 - n_1 - n_2)} \cdot \frac{\Gamma(b_1 - b_2 - n_1) \Gamma(-b_4 - n_2 + 1) \Gamma(b_2 + b_4 - a_2 - 1)}{\Gamma(b_1 - a_2 - n_1 - n_2)}.
\]

By rewriting each Gamma factor including \(-n_1, -n_2, -n_1 - n_2\) into the product of the Gamma factor and Pochhammer’s symbol, we have the second solution in Theorem 2.2. Here note that Pochhammer’s symbols \((a_1 - b_1 - b_3 + 2, n_1)\) and \((a_3, n_2)\) are canceled with \(\Gamma(b_1 + b_3 - a_1 - 1 - n_1)\) and \(\Gamma(-a_3 - n_2 + 1)\).

**Remark 3.3.**  
(1) A twisted cycle \(\Delta_{10}^u\) is defined by the pair of \(\Delta_{10}\) and a branch of \(u(t, x)\) on it.  
(2) We can construct \(\text{reg}^c_{10}(\square_1 \times \triangle_2)\) as the direct product of 2-chains given by the regularizations of \(\square_1\) and \(\triangle_2\) associated with

\[u_1' = s_1^{-b_1-2} (1 - s_1)^{-a_3} s_3^{-b_3} (1 - s_3)^{b_1+b_3-a_1-2}, \]

\[u_2' = s_2^{-b_2-1} s_4^{-b_4} (1 - s_2 - s_4)^{b_2+b_4-a_2-2},\]

see Figure 2 for the regularization of \(\square_1\).  
(3) By acting \((12, \sigma_{110}) \in S_3 \times D_4\) on Corollary 3.2 and using the variable change \((t_1, t_2, t_3, t_4) \mapsto (t_2, t_1, t_4, t_3)\), we have an Euler
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\[ s_3 = 1 - \frac{x_1}{s_1 s_2} \]

\[ s_3 = 1 \]

\[ s_1 = 1 - \frac{x_2}{s_3 s_4} \]

\[ s_1 = 1 \]

\[ s_1 = 1 - \frac{x_2}{s_3 s_4} \]

Figure 2. Regularization of \( \square_1 \)

type integral

\[ C_{20} \int_{\Delta_{20}} u(t, x) dt \]

of the solution \( F_{20}(x) \) in Theorem 2.2. A twisted cycle \( \Delta_{20}^u \) is defined by the pair of \( \Delta_{20} \) and a branch of \( u(t, x) \) on it.

(4) We have Euler type integrals

\[ C_{0k} \int_{\Delta_{0k}} u(t, x) dt \quad (k = 1, 2) \]

of the solutions \( F_{0k}(x) \) in Theorem 2.2 and twisted cycles \( \Delta_{0k}^u \) by the action \( (\text{id}, \sigma_{001}) \in S_3 \times D_4 \) and the variable change \( t \mapsto (t_3, t_4, t_1, t_2) \) to those of \( F_{k0}(x) \).

Let \( \iota_{11} \) be an involution given by

\[ \iota_{11} : T \ni (s_1, s_2, s_3, s_4) \mapsto t = \left( \frac{x_1}{s_1 s_2}, s_2, \frac{x_2}{s_3 s_4}, s_4 \right) \in T. \]

Note that

\[ \iota_{11}^*(\ell_{13}(t)) = q(s), \quad \iota_{11}^*(\ell_{24}(t)) = \ell_{24}(s), \quad \iota_{11}^*(q(t)) = \ell_{13}(s). \]
We define a 4-chain $\Delta_{11}$ as the image of $\text{reg}^c_{11}(\Delta_1 \times \Delta_2)$ associated with
\[
s_1^{-b_2-b_1-1}(1-s_1)^{b_1+b_3-a_2-1}s_3^{-a_3} \cdot s_2^{-b_2-b_1-1} \cdot s_4^{-b_3-b_4-1}(1-s_2-s_4)^{b_2+b_4-a_2-2}
\]
under the involution $\iota_{11}$.

**Corollary 3.3.** The solution $F_{11}(x)$ in Theorem 2.2 is expressed as
\[
C_{11} \cdot \int_{\Delta_{11}} u(t, x) dt,
\]
where $C_{11}$ is
\[
C_{11} = \frac{\Gamma(b_1+b_3-a_3-1)\Gamma(b_1+b_3-a_2-1)}{\Gamma(b_1-1)\Gamma(b_3-1)\Gamma(1-a_3)\Gamma(b_1-b_2)\Gamma(b_3-b_4)\Gamma(b_2+b_4-a_2-1)}.
\]

**Proof.** By the variable change $\iota_{11}$, the integral changes to
\[
C_{11} \cdot \int_{\text{reg}^c_{11}(\Delta_1 \times \Delta_2)} x_1^{1-b_1} x_2^{1-b_1} \
\cdot \int_{\text{reg}^c_{11}(\Delta_1 \times \Delta_2)} s_1^{-b_1-b_2-1} s_2^{-b_2-b_1-1} s_3^{-b_3-b_4-1} s_4^{-b_4-b_3-1} \
\cdot \ell_{13}(s)^{-a_3} q(s)^{b_1+b_3-a_1-2} \ell_{24}(s)^{b_2+b_4-a_2-2} ds.
\]
We have only to consider the change of parameters. Here note that $F\left(\frac{a}{B}; x\right)$ is symmetric with respect to $a_1, a_2, a_3$. □

**Remark 3.4.** (1) A twisted cycle $\Delta_{11}^*$ is defined by the pair of $\Delta_{11}$ and a branch of $u(t, x)$ on it.

(2) By using the action of $\sigma_{110} \in D_4$ on Corollary 3.3, we have an Euler type integral of the solution $F_{22}(x)$ in Theorem 2.2. Its integrand can be transformed into $u(t, x)$ by the variable change $t \mapsto (t_3, t_4, t_1, t_2)$, and a twisted cycle $\Delta_{22}^*$ is obtained.

Let $\iota_{21}$ be the variable change
\[
\iota_{21} : (s_1, s_2, s_3, s_4) \mapsto t = \left( s_1, \frac{x_1}{s_1 s_2}, \frac{x_2}{s_3 s_4}, s_3 \right) \in T.
\]

We define a 4-chain $\Delta_{21}$ as the image of $-\text{reg}^c_{21}(\square_1 \times \Delta_2)$ associated with
\[
s_1^{b_2-b_1-1}(1-s_1)^{b_1+b_3-a_2-1}s_3^{b_3-b_4-1}(1-s_3)^{b_2+b_4-a_2-2} \cdot s_2^{b_2-2} s_4^{b_3-2} (1-s_2-s_4)^{-a_3}
\]
under the map $\iota_{21}$.

**Corollary 3.4.** The solution $F_{21}(x)$ in Theorem 2.2 is expressed as
\[
C_{21} \cdot \int_{\Delta_{21}} u(t, x) dt,
\]
where the gamma factor \( C_{21} \) is
\[
\frac{\Gamma(b_2 + b_3 - a_1 - 1) \Gamma(b_2 + b_3 - a_1 - b_1) \Gamma(b_2 + a_2 - 1) \Gamma(b_2 + a_2 - 1)}{\Gamma(b_1 + b_3 - a_1 - 1) \Gamma(b_2 + b_4 - a_2 - 1) \Gamma(b_2 - b_4)}.
\]

Proof. Apply the variable change \( \nu_{21} \) to the integral with paying attention to the signs in \( \Delta_{21} = \nu_{21}(-\text{reg}_{21}(\square_1 \times \triangle_2)) \) and \( \nu_{21}^*(dt) \). Then it becomes
\[
x_1^{1-b_2} x_2^{1-b_3} \int_{\text{reg}_{21}(\square_1 \times \triangle_2)} s_1^{b_2-b_1-1} s_2^{b_2-2} s_3^{b_3-b_4-1} s_4^{b_4-2} (1 - s_1 - \frac{b_2}{s_3 s_4})^{b_1+b_3-a_1-2} \cdot (1 - \frac{x_1}{s_1 s_2} - s_3)^{b_2+b_4-a_2-2} (1 - s_2 - s_4)^{-a_3} ds.
\]
To express this integral by the hypergeometric function, follow the proof of Corollary \ref{cor:hypergeometric_integral} \( \square \)

**Remark 3.5.**

1. A twisted cycle \( \Delta^u_{21} \) is defined by the pair of \( \Delta_{21} \) and a branch of \( u(t, x) \) on it.

2. By using the action of \( \sigma_{110} \in D_4 \) on Corollary \ref{cor:hypergeometric_integral} we have an Euler type integral of the solution \( F_{12}(x) \) in Theorem \ref{thm:euler_integral}. Its integrand can be transformed into \( u(t, x) \) by the variable change \( t \mapsto (t_3, t_4, t_1, t_2) \), and a twisted cycle \( \Delta^u_{12} \) is obtained.

4. **Fundamental group of the complement of the singular locus**

Recall that the base point \( \dot{x} = (\varepsilon_1, \varepsilon_2) \in X \) is chosen so that \( \varepsilon_1 \) and \( \varepsilon_2 \) are small positive real numbers satisfying \( 0 < \varepsilon_2 < \varepsilon_1 \). Let \( \rho_1 \) and \( \rho_2 \) be loops given by
\[
\rho_1 : [0, 1] \ni r \mapsto (\varepsilon_1 \exp(2\pi \sqrt{-1}r), \varepsilon_2) \in X,
\rho_2 : [0, 1] \ni r \mapsto (\varepsilon_1, \varepsilon_2 \exp(2\pi \sqrt{-1}r)) \in X.
\]
Let \( L \) be the complex line passing through \( (0, 0) \) and \( \dot{x} \), which is expressed as
\[
L = \{(\varepsilon_1 r, \varepsilon_2 r) \in \mathbb{C}^2 \mid r \in \mathbb{C}\}.
\]
We regard \( r \in \mathbb{C} \) as a coordinate of \( L \). The intersection of \( L \) and the curve \( S_3 : R(x) = 0 \) consists of three points \( P_1, P_2, P_3 \). Let \( P_1 \) be the point whose coordinate \( r_1 \in \mathbb{C} \) is real. Let \( \rho_3 \) be a loop in \( L \) starting from \( \dot{x} \), approaching \( P_1 \) along the real axis, turning once around \( P_1 \) positively, and tracing back to \( \dot{x} \).

We use the same symbol \( \rho \) for the element of the fundamental group \( \pi_1(X, \dot{x}) \) represented by a loop \( \rho \) with terminal \( \dot{x} \). The structure of
$\pi_1(X, \dot{x})$ is studied in [9]. It is shown that these loops generate $\pi_1(X, \dot{x})$, and that it is isomorphic to a group generated by three elements with four relations among them. In this paper, we use a relation

$$\rho_1 \cdot \rho_2 = \rho_2 \cdot \rho_1$$

and that in the following proposition.

**Proposition 4.1.** The element

$$\rho_3 \cdot (\rho_2 \cdot \rho_3 \cdot \rho_2^{-1}) \cdot (\rho_2^2 \cdot \rho_3 \cdot \rho_2^{-2})$$

commutes with $\rho_2$ as elements of $\pi_1(X, \dot{x})$, where $\rho_2 \cdot \rho_3$ is the loop joining $\rho_2$ to $\rho_3$. Moreover, this loop is homotopic to the loop $\rho_2': [0, 1] \ni r \mapsto (1 + (\varepsilon_1 - 1) \exp(2\pi \sqrt{-1} r), 0) \in \{(x_1, x_2) \mid x_2 = 0\} \subset \mathbb{C}^2$ in the space $\{(x_1, x_2) \in \mathbb{C}^2 \mid x_2 R(x_1, x_2) \neq 0\}$.

**Proof.** Let the line $L$ move along the loop $\rho_2$. By tracing the deformation of $\rho_3$, we can show that $\rho_3$ is changed into a loop turning around once the point $P_2$ positively. Since the base point $\dot{x}$ is moved along $\rho_2$, this loop is equal to $\rho_2 \cdot \rho_3 \cdot \rho_2^{-1}$. Similarly, we can show that $\rho_2 \cdot \rho_3 \cdot \rho_2^{-2}$ is a loop turning around once the point $P_3$ positively. Thus the loop $\rho_3 \cdot (\rho_2 \cdot \rho_3 \cdot \rho_2^{-1}) \cdot (\rho_2^2 \cdot \rho_3 \cdot \rho_2^{-2})$ turns around once the all points $P_1, P_2$ and $P_3$ positively. If we deform the line $L$ to $x_2 = 0$, then $P_1, P_2$ and $P_3$ confluent to the point $(1, 0)$. Hence we can see that this loop is homotopic to $\rho_2'$. It is clear that $\rho_2'$ commutes with $\rho_2$ since $\rho_2'$ is in the line $x_2 = 0$. \(\Box\)

**Remark 4.1.** The relation

$$\rho_2 \cdot [\rho_3(\rho_2 \rho_3 \rho_2^{-1})(\rho_2^2 \rho_3 \rho_2^{-2})] = [\rho_3(\rho_2 \rho_3 \rho_2^{-1})(\rho_2^2 \rho_3 \rho_2^{-2})] \cdot \rho_2$$

is equivalent to

$$(\rho_2 \rho_3)^3 = (\rho_3 \rho_2)^3$$

as elements of $\pi_1(X, \dot{x})$. We also have

$$(\rho_1 \rho_3)^3 = (\rho_3 \rho_1)^3.$$

An Artin group of infinite type is defined by three generators $\varrho_1$, $\varrho_2$ and $\varrho_3$ together with three relations among them:

$$\varrho_2 \varrho_1 = \varrho_1 \varrho_2, \quad (\varrho_3 \varrho_1)^3 = (\varrho_1 \varrho_3)^3, \quad (\varrho_3 \varrho_2)^3 = (\varrho_2 \varrho_3)^3.$$
5. Monodromy representation

In this section, we assume that
\begin{equation}
(5.1) \quad a_i, b_j, b_k, b_1 - b_2, b_3 - b_4, a_i - b_j, a_i - b_k, \notin \mathbb{Z}
\end{equation}
for \(i = 1, 2, 3\) and \(j = 1, 2\) and \(k = 3, 4\). Note that the condition (5.1) is stable under the action of \(S_3 \times D_4\).

We array the fundamental system of solutions to \(F(aB)\) on \(U\) in Theorem 2.2 as a column vector
\[F(x) = t(g_{00}F_{00}(x), \ldots, g_{jk}F_{jk}(x), \ldots)\]
in the order (2.5), where \(g_{jk}\) are non-zero constants. Let \(M_g\) be the circuit matrix along \(\rho_i\) with respect to this vector valued function. That is, this vector valued function is transformed into \(M_gF(x)\) by the analytic continuation along the loop \(\rho_i\).

**Lemma 5.1.** We have
\[M_g^1 = \text{diag}(1, \beta_1^{-1}, \beta_2^{-1}, 1, \beta_1^{-1}, \beta_2^{-1}),\]
\[M_g^2 = \text{diag}(1, 1, 1, \beta_3^{-1}, \beta_3^{-1}, \beta_4^{-1}, \beta_4^{-1}),\]
for any \(g = (g_{00}, g_{10}, \ldots, g_{22})\), where \(\beta_j = e^{2\pi \sqrt{-1} b_j} (j = 1, \ldots, 4)\) and \(\text{diag}(c_1, \ldots, c_m)\) denotes the diagonal matrix of size \(m\) with diagonal entries \(c_1, \ldots, c_m\).

**Proof.** It is a direct consequence of Theorem 2.2 that the circuit matrices \(M_1\) and \(M_2\) take the forms in this lemma. \(\square\)

The essential problem is the determination of the circuit matrix \(M_3^g\). At first, we study eigenspaces of \(M_3^g\).

**Lemma 5.2.** Suppose that the circuit matrix \(M_3^g\) is diagonalizable. Then its eigenvalues are 1 and \(\lambda(\neq 1)\). The \(\lambda\)-eigenspace is 1 dimensional and the 1-eigenspace is 8 dimensional.

**Proof.** We construct a vanishing cycle \(O\) when \(x = (x_1, x_2) \in X\) approaches to \(P_1 = (x_1', x_2') \in L \cap S\) along a part of the loop \(\rho_3\). The integral \(\int_0 u(t, x) dt\) becomes a \(\lambda\)-eigenvector.

We regard the space \(T\) in (3.2) as a fiber bundle over the base \((t_1, t_3)\)-space with the fiber \((t_2, t_4)\)-space. At first, we fix \(x\) and a generic \((t_1, t_3)\). Let \(O_{24}\) be the real 2-dimensional chamber surrounded by \(\ell_{24}(t) = 0\) and \(q(t) = 0\) in the \((t_2, t_4)\)-space and including \(\triangle\) in Figure \[.\] By
moving \((t_1,t_3)\) in a 2-chain, we construct a locally finite 4-chain as a family of \(\mathcal{O}_{24}\) over \((t_1,t_3)\). Note that \(\ell_{24}(t) = 0\) is tangent to \(q(t) = 0\) if and only if \(R_2(x_1/t_1,x_2/t_3) = 0\), where

\[
R_2(y_1,y_2) = (1 - y_1 - y_2)^2 - 4y_1y_2 = y_1^2 + y_2^2 + 1 - 2y_1y_2 - 2y_1 - 2y_2.
\]

Since \(t_1 \neq 0\) and \(t_3 \neq 0\) in the base space, the condition \(R_2(x_1/t_1,x_2/t_3) = 0\) is equivalent to

\[
q_4(t_1,t_3,x) = (t_1t_3 - x_1t_3 - x_2t_1)^2 - 4x_1x_2t_1t_3 = 0.
\]

Note that the chamber \(\mathcal{O}_{24}\) vanishes on this curve for the fixed \((x_1,x_2)\) in the base space \((t_1,t_3)\). It is easy to see that this quartic curve in \(\mathbb{C}^2\) has a cusp singular point at \((t_1,t_3) = (0,0)\) for any \(x \in X\) and that this curve intersects the line \(t_i = 0\) \((i = 1,3)\) only at \((0,0)\) in \(\mathbb{C}^2\). We consider the intersection of \(q_4(t_1,t_3,x) = 0\) and line \(L_{13} : t_1 + t_3 = 1\).

By eliminating \(t_3\) from \(q_4(t_1,t_3,x) = 0\) and \(t_1 + t_3 = 1\), we have

\[
t_1^4 - 2(x_1 - x_2 + 1)t_1^3 + (x_1^2 + x_2^2 + 1 + 2x_1x_2 + 4x_1 - 2x_2)t_1^2
\]

\[
- 2x_1(x_1 + x_2 + 1)t_1 + x_1^2 = 0.
\]

Since its discriminant with respect to \(t_1\) is

\[
256x_1^3x_2^3R(x_1,x_2),
\]

\(q_4(t_1,t_3,x) = 0\) and \(t_1 + t_3 = 1\) intersect at four distinct points for any \(x \in X\); see Figure 3.

We set

\[
\mathcal{O} = \bigcup_{(t_1,t_3) \in \mathcal{O}_{13}} \mathcal{O}_{24},
\]

where the region \(\mathcal{O}_{13}\) in the \((t_1,t_3)\)-space is surrounded by the line \(t_1 + t_3 = 0\) and \(q_4(t_1,t_3,x) = 0\), see Figure 3. We claim that this is a cycle as a locally finite chain. Its boundary consists of

\[
\bigcup_{(t_1,t_3) \in \partial \mathcal{O}_{13}} \mathcal{O}_{24}, \quad \bigcup_{(t_1,t_3) \in \partial \mathcal{O}_{13}} \partial \mathcal{O}_{24}.
\]

Since \(\mathcal{O}_{24}\) is a locally finite cycle, \(\partial \mathcal{O}_{24} = 0\). The boundary component of \(\mathcal{O}_{13}\) in \(t_1 + t_3 = 1\) vanishes as a locally finite chain. Since \(\mathcal{O}_{24}\) vanishes over the boundary component of \(\mathcal{O}_{13}\) in \(q_4(t_1,t_3,x) = 0, \bigcup_{(t_1,t_3) \in \partial \mathcal{O}_{13}} \mathcal{O}_{24} = 0\). Hence \(\mathcal{O}\) is a locally finite cycle.

We consider the limit as \(x\) to \(P_1 = (x_1',x_2')\). If \(x = x'\) then the line \(t_1 + t_3 = 1\) tangents to \(q_4(t_1,t_3,x') = 0\) and we can show that \(\mathcal{O}_{13}\) vanishes; see Figure 4. Hence \(\mathcal{O}\) is a required cycle.

By tracing the movement of \(\mathcal{O}\) along the loop \(\rho_3\), we see that the deformed \(\mathcal{O}\) coincides with the initial \(\mathcal{O}\). Since the branch of \(u(t,x)\)
Figure 3. \( q_4(t_1, t_3, x) = 0 \) in \((t_1, t_3)\) space

changes by the continuation along \( \rho_3 \), the integral \( \int_G u(t, x)dt \) is multiplied \( \lambda \neq 1 \) under suitable non-integrable conditions on the parameters \( a \) and \( B \). Thus the circuit matrix \( M_3^a \) has an eigenvalue \( \lambda \) different from 1.

Finally, we show that the 1-eigenspace of \( M_3^a \) is 8 dimensional. Let \( \text{Sol}(U_{P_1}) \) be the vector space of holomorphic solutions to \( F \left( \begin{array}{c} a \\ B \end{array} \right) \), which can be extended to a neighborhood \( U_{P_1} \) of \( P_1 = (x'_1, x'_2) \) in \( \mathbb{C}^2 \). It is sufficient to show that \( \text{Sol}(U_{P_1}) \) is 8 dimensional. By following the proof of Corollary 3.1, we can show that \( \dim \text{Sol}(U_{P_1}) \) is equal to the Euler number \( \chi(T') \) of the space \( T' \) for \( x = (x'_1, x'_2) \). On the other hand, we have \( \chi(T) = 9 \) for any \( x \in X \) by Corollary 3.1. Since the space \( T' \) loses a 4-chain \( \emptyset \) homeomorphic to \( \mathbb{R}^4 \) from the space \( T \), we have \( \chi(T') = 9 - 1 = 8 \).

Remark 5.1. For a generic point \( x \) of \( R(x) = 0 \), \( q_4(t_1, 1 - t_1, x) = 0 \) has a double root and two simple roots. At the node \( (x_1, x_2) = (-1, -1) \) of \( R(x) = 0 \), \( q_4(t_1, 1 - t_1, x) = 0 \) has two double roots, and
Figure 4. $q_4(t_1, t_3, x') = 0$ in $(t_1, t_3)$ space

$t_1 + t_3 = 1$ becomes a bi-tangent of $q_4(t_1, t_3, x) = 0$ with tangent points $(t_1, t_3) = (-\omega, -\omega^2), (-\omega^2, -\omega)$. For $x = (1, 0)$ or $(0, 1)$, $q_4(t_1, t_3, x) = 0$ degenerates to a product of duplicate lines $(t_1 - 1)^2 t_3^2$ or $t_1^2 (t_3 - 1)^2$, and it intersects the line $t_1 + t_3 = 1$ at the quadruple point $(t_1, t_3) = (1, 0)$ or $(0, 1)$, respectively.

Next we normalize $g$ so that $M^g_3$ admits a simple expression. For this purpose, we introduce the intersection pairing $I$ between the twisted homology groups $H_4(T, \mathcal{L}_u)$ and $H_4^{lf}(T, \mathcal{L}_u^\vee)$, where $\mathcal{L}_u^\vee$ is the dual local system of $\mathcal{L}_u$. Here $\vee$ denotes the sign change operator

$$\nu(a_1, a_2, a_3; b_1, b_2, b_3, b_4)^\vee = \nu(-a_1, -a_2, -a_3; -b_1, -b_2, -b_3, -b_4)$$

for any function $\nu$ of parameters. For example, we have $\alpha_i^\vee = 1/\alpha_i$ and $u(t, x)^\vee = 1/u(t, x)$. Under our assumption (5.1), the natural map from $H_4(T, \mathcal{L}_u)$ to $H_4^{lf}(T, \mathcal{L}_u^\vee)$ is isomorphic, and its inverse reg is defined. We can regard the intersection pairing as defined between $H_4(T, \mathcal{L}_u)$ and $H_4(T, \mathcal{L}_u^\vee)$. There exist twisted cycles $\Delta^{g,u}_{jk}$ given by 4-chain $\Delta^g_{jk}$.
with a branch of \((u, t)\) such that

\[
\int_{\Delta_{jk}} u(t, x) dt = g_{jk} F_{jk}(x)
\]

for any \(0 \leq j, k \leq 2\). We define a \(9 \times 9\) matrix \(H^g\) by

\[
H^g = \frac{1}{\mathcal{I}(\Delta_{jk}^u, \Delta_{jk}^{u, \vee})} \left( \mathcal{I}(\Delta_{jk}^u, \Delta_{jk}^{u, \vee}) \right)_{(jk), (j'k')} ,
\]

where \((jk)\) and \((j'k')\) are arrayed in the order \((2.5)\). We compute \(\mathcal{I}(\Delta^u_{jk}, \Delta_{jk}^{u, \vee})\) in §6. In this section, we treat the entries \(h_{10}^g, h_{20}^g, \ldots, h_{22}^g\) of \(H^g\) as indeterminate for the moment.

**Lemma 5.3.** The matrix \(H^g\) satisfies

\[
M_i^g H^g t(M_i^g)^\vee = H^g \quad (i = 1, 2, 3),
\]

and takes the form \(\text{diag}(1, h_{10}^g, \ldots, h_{22}^g)\).

**Proof.** By the local triviality of the intersection form, we can show that the relations is satisfied by any circuit matrices \(M_i^g\) along loops \(\rho\) with terminal \(\dot{x}\); refer to the proof of Lemma 4 in [14] for details. By the property

\[
M_i^g H^g t(M_i^g)^\vee = H^g \quad (i = 1, 2),
\]

for \(M_1^g\) and \(M_2^g\) given in Lemma 5.1, \(H^g\) should be diagonal under the assumption (5.1). It is clear that its top-left entry is 1. \(\square\)

**Lemma 5.4.** Suppose that the eigenvalue \(\lambda\) of \(M_3^g\) is different from 1 and that \(v = (v_0, \ldots, v_2)\) is a \(\lambda\)-eigenvector of \(M_3^g\).

1. The eigenspace of \(M_3^g\) of eigenvalue 1 is characterized as

\[
\{ w \in \mathbb{C}^9 \mid w H^g t v^\vee = 0 \}.
\]

2. The vector \(v\) satisfies \(v H^g t v^\vee \neq 0\).

3. The circuit matrix \(M_3^g\) is expressed as

\[
M_3^g = I_9 - \frac{(1 - \lambda)}{v H^g t v^\vee} H^g t v^\vee v.
\]

4. No entries of \(v\) vanish.

**Proof.** (1) Let \(w\) be a 1-eigenvector of \(M_3^g\). Then we have

\[
w H^g t v^\vee = w [M_3^g H^g t (M_3^g)^\vee] t v^\vee = (w M_3^g) H^g t (v M_3^g)^\vee = \lambda^\vee w H^g t v^\vee, \]

\[
(1 - \lambda^\vee) w H^g t v^\vee = 0.
\]

Since \((\lambda^\vee)^\vee = \lambda\) and \(\lambda \neq 1\), the factor \(1 - \lambda^\vee\) does not vanish. Hence we have \(w H^g t v^\vee = 0\).
(2) It is known that the intersection form is a perfect pairing between twisted homology groups. Since the matrix $H$ corresponds to the intersection matrix, it is non-degenerate. If $vH^t v^\vee = 0$ then $uH^t v^\vee = 0$ for any $u \in \mathbb{C}^9$ by the result (1) and Lemma 5.2. This means that $H$ degenerates.

(3) Put
\[ M' = I_9 - \frac{1 - \lambda}{vH^t v^\vee}H^t v^\vee. \]
Since we have
\[ vM' = v + \frac{1 - \lambda}{vH^t v^\vee}vH^t v^\vee v = \lambda v, \quad wM' = w + \frac{1 - \lambda}{vH^t v^\vee}wH^t v^\vee v = w \]
for any $w$ in $\{ w \in \mathbb{C}^9 \mid wH^t v^\vee = 0 \}$, the eigenvalues and eigenspaces of $M'$ and $M_3^g$ coincide. Hence $M'$ is equal to $M_3^g$.

(4) Assume that an entry $v_i$ of $v$ vanishes. Then the unit vector $e_i$ is a $1$-eigenvector of $M_3^g$ since $e_i H^t v^\vee = 0$. Moreover, $e_i$ is an eigenvector of $M_1^g$ and that of $M_2^g$. Hence the space spanned by $e_i$ is invariant subspace of the monodromy representation. By removing the power function $x_1^{a_1-b_1}x_2^{a_2-b_2}$ from corresponding solution to $e_i$ and restricting it to $x_2 = 0$, we have the differential equation $3F_2$ associated with it. This has an invariant subspace of the monodromy representation. By Proposition 3.3 in \cite{2}, some of $a_i, a_i - b_1, a_i - b_2$ belong to $\mathbb{Z}$. It contradicts to our assumption (5.1). \hfill $\Box$

We normalize $g = (g_{00}, g_{10}, \ldots, g_{22})$ so that the $\lambda$-eigenvector $v$ of $M_3^g$ becomes $1 = (1, \ldots, 1)$. From now on, we fix $g$, and we use symbols $F(x), H = (1, h_{10}, \ldots, h_{22}), M_i (i = 1, 2, 3)$ for this fixed $g$. Note that $M_1$ and $M_2$ are given in Lemma 5.1 and that
\[ M_3 = I_9 - \frac{1 - \lambda}{1H1}H11, \]
where we regard $\lambda$ and entries of $H$ as indeterminate.

Finally, we determine them by considering the restrictions $F(x)$ to $x_i = 0$ ($i = 1, 2$).

**Proposition 5.1.** The eigenvalue $\lambda$ and the diagonal entries of $H$ are
\[
\lambda = \frac{\beta_1 \beta_2 \beta_3 \beta_4}{\alpha_1 \alpha_2 \alpha_3},
\]
\[
h_{10} = \frac{(\alpha_1 - \beta_1)(\alpha_2 - \beta_1)(\alpha_3 - \beta_1)(\beta_2 - 1)}{(\alpha_1 - 1)(\alpha_2 - 1)(\alpha_3 - 1) \beta_1 (\beta_1 - \beta_2)},
\]
\[
h_{20} = \frac{(\alpha_1 - \beta_2)(\alpha_2 - \beta_2)(\alpha_3 - \beta_2)(\beta_1 - 1)}{(\alpha_1 - 1)(\alpha_2 - 1)(\alpha_3 - 1) \beta_2 (\beta_2 - \beta_1)},
\]
Remark 5.2. The group \( S_3 \times D_4 \) naturally acts on the entries \( h_{jk} \) of \( H \). This is compatible with that on the fundamental solutions \( F_{jk}(x) \).

**Proof.** Consider the loop \( \rho_3 \cdot (\rho_2 \cdot \rho_1 \cdot \rho_2^{-1}) \cdot (\rho_2^2 \cdot \rho_3 \cdot \rho_2^{-2}) \). By Proposition 4.1 it commutes with \( \rho_2 \) and homotopic to a loop \( \rho'_2 \) turning \( x_1 = 1 \) once positively in the space \( \{(x_1, 0) \in \mathbb{C}^2 \mid x_1 \neq 0, 1\} \). Let \( N_2 \) be the circuit matrix along this loop with respect to \( F(x) \). Since it commutes with \( M_2 \), it is a block diagonal matrix with respect to the \((3, 3, 3)\)-partition. Since the restrictions of \( F_{j0}(x) \) \((j = 0, 1, 2)\) to \( x_2 = 0 \) reduce to a fundamental system \( 3F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2} \right) \), the top-left block of \( N_2 \) coincides with the circuit matrix of \( \rho'_2 \) with respect to their restrictions. This \( 3 \times 3 \) circuit matrix is expressed in Proposition 3.1 of \([13]\) as

\[
I_3 - \frac{1 - \lambda'}{1'H' \cdot 1'H'} \cdot 1' \cdot 1',
\]

where \( I_3 \) is the unit matrix of size 3, \( \lambda' = \frac{\beta_1 \beta_2}{\alpha_1 \alpha_2 \alpha_3} \), \( 1' = (1, 1, 1) \) and \( H' = \text{diag}(1, h_{10}, h_{20}) \) for given \( h_{10} \) and \( h_{20} \) in this proposition. By following the proof of Proposition 4.1 of \([13]\), we can show the coincidence of two fundamental systems. By the uniqueness of the matrix \( H \), we determine the values of \( h_{10} \) and \( h_{20} \) as in this proposition.

We consider the middle block of \( N_2 \) and the ratios of the solutions \( F_{j1}(x) \) \((j = 0, 1, 2)\). By taking out the factor \( x_2^{a_2-b_3} \) from them, we have the circuit matrix of \( \rho'_2 \) with respect to a fundamental system of
Thus we have

\[ \lambda^3 = \left( \frac{\beta_1 \beta_2 \beta_3 \beta_4}{\alpha_1 \alpha_2 \alpha_3} \right)^3. \]

Note that \((1, 1, 1, 0, \ldots, 0)\) is an eigenvector of \(N_2\). This property yields that \(\lambda\) is as in this proposition. 

We conclude results in this section as the following theorem.
Theorem 5.1. Suppose the condition (5.1). Then we have

\[ M_1 = \text{diag}(1, \beta_1^{-1}, \beta_2^{-1}, 1, \beta_1^{-1}, \beta_2^{-1}, 1, \beta_1^{-1}, \beta_2^{-1}), \]
\[ M_2 = \text{diag}(1, 1, 1, \beta_3^{-1}, \beta_3^{-1}, \beta_4^{-1}, \beta_4^{-1}, \beta_4^{-1}), \]
\[ M_3 = I_9 - \frac{1 - \lambda}{1H\ 1} H^t 1, \]

where \( I_9 \) is the unit matrix of size 9, \( \alpha_i = \exp(2\pi\sqrt{-1}a_i), \beta_j = \exp(2\pi\sqrt{-1}b_j), 1 = (1, \ldots, 1) \in \mathbb{N}^9, \lambda = (\beta_1\beta_2\beta_3\beta_4)/(\alpha_1\alpha_2\alpha_3), H = \text{diag}(1, h_{10}, h_{20}, \ldots, h_{22}) \) whose entries are given in Proposition 5.1.

Corollary 5.1. The system \( F(a \ B) \) is regular singular.

Proof. We have only to consider the behavior of its solutions around each component of the singular locus \( S \). \( \square \)

Remark 5.3. (1) To study the monodromy of \( a F_2(a_1, a_2, a_3; b_1, b_2) \) in [13], we assume

\[ a_i - a_{i'} \notin \mathbb{Z}, \quad (1 \leq i < i' \leq 3) \]

so that the eigen-polynomial \( f(t) \) of a circuit matrix implies two independent linear equations by the substitution of \( t = \alpha_i \) (\( i = 1, 2, 3 \)). Since we can remove these conditions by considering \( f'(t) \) and \( f''(t) \) in the case of \( \alpha_i = \alpha_{i'} \), we do not need these conditions for Theorem 5.1.

(2) Note that we can cancel the factor \( \alpha_1\alpha_2\alpha_3 - \beta_1\beta_2\beta_3\beta_4 \) from \( \frac{1 - \lambda}{1H\ 1} \) in the expression of \( M_3 \) in Theorem 5.1. Thus we can remove the condition

\[ a_1 + a_2 + a_3 - b_1 - b_2 - b_3 - b_4 \notin \mathbb{Z} \]

from (5.1).

6. Intersection numbers

In this section, we compute the intersection numbers of twisted cycles corresponding to fundamental solutions. We start from the following lemma, which is a consequence of Lemma 5.3.

Lemma 6.1. If \( (j, k) \neq (j', k') \) for \( 0 \leq j, j', k, k' \leq 2 \) then

\[ I(\Delta_{jk}^u, \Delta_{j'k'}^{u \lor}) = 0, \]

where \( \Delta_{jk}^u \) are given in Remarks 2.1, 3.3, 3.4, and 3.5 and \( \Delta_{jk}^{u \lor} \) denotes the image of \( \Delta_{jk}^u \) under the map \( \lor \).
We compute intersection numbers by the two-dimensional reduction technique as follows.

**Lemma 6.2.** The intersection numbers $\mathcal{I}(\Delta_{00}^u, \Delta_{00}^v)$, $\mathcal{I}(\Delta_{10}^u, \Delta_{10}^v)$, $\mathcal{I}(\Delta_{11}^u, \Delta_{11}^v)$ and $\mathcal{I}(\Delta_{21}^u, \Delta_{21}^v)$ are

$$\frac{(\alpha_1 - 1)\beta_1\beta_3}{(1 - \beta_1)(1 - \beta_3)(\alpha_1 - \beta_1\beta_3)} \cdot \frac{(\alpha_2 - 1)\beta_2\beta_4}{(1 - \beta_2)(1 - \beta_4)(\alpha_2 - \beta_2\beta_4)},$$

$$\frac{(\alpha_1 - \beta_1)(\alpha_3 - \beta_1)\beta_3}{(1 - \beta_1)(1 - \beta_3)(\alpha_3 - \beta_1)(1 - \beta_3)} \cdot \frac{(\alpha_2 - \beta_1)\beta_2\beta_4}{(1 - \beta_2)\beta_4(\beta_2 - \beta_1)(1 - \beta_4)},$$

$$\frac{\alpha_3 - \beta_1\beta_3}{(1 - \beta_1)(1 - \beta_3)(\alpha_3 - \beta_1)(1 - \beta_3)} \cdot \frac{(\alpha_2 - \beta_1\beta_3)\beta_2\beta_4}{(\beta_2 - \beta_1)(\beta_4 - \beta_3)(\alpha_2 - \beta_2\beta_4)},$$

$$\frac{(\alpha_1 - \beta_1\beta_3)\beta_2\beta_4(\beta_1 - \beta_2)\beta_4(\beta_3 - \beta_4)}{(\alpha_3 - \beta_1\beta_3)(\alpha_2 - \beta_2\beta_4)(\beta_1 - \beta_2)(\beta_4 - \beta_3)} \cdot \frac{(1 - \beta_3)(1 - \beta_2)(1 - \beta_3)}{(1 - \beta_1)(1 - \beta_2)(1 - \beta_3)},$$

respectively.

**Proof.** By Remark 3.1(2), we can compute $\mathcal{I}(\Delta_{00}^u, \Delta_{00}^v)$ as the product of the intersection numbers $\mathcal{I}(\Delta_{i1}^u, \Delta_{i1}^v)$ and $\mathcal{I}(\Delta_{i2}^u, \Delta_{i2}^v)$, where $\Delta_{i1}^u$ ($i = 1, 2$) is a twisted cycle given by $\Delta_i$ and $u_i$ in the $(t_i, t_{i+2})$-space. By using results in §3.1 of Chapter VIII in [17], we have

$$\mathcal{I}(\Delta_{i1}^u, \Delta_{i1}^v) = \frac{(1 - \alpha_i^{-1})}{(1 - \beta_1^{-1})(1 - \beta_3^{-1})(1 - \alpha_i^{-1}\beta_1\beta_3)},$$

$$\mathcal{I}(\Delta_{i2}^u, \Delta_{i2}^v) = \frac{(1 - \alpha_i^{-1})}{(1 - \beta_2^{-1})(1 - \beta_4^{-1})(1 - \alpha_i^{-1}\beta_2\beta_4)}.$$

The intersection number $\mathcal{I}(\Delta_{10}^u, \Delta_{10}^v)$ can be computed in the $s$-space by the involution $t_{13}$. It reduces to the product of $\mathcal{I}(\square_{1}^{u_i}, \square_{1}^{v_i})$ and $\mathcal{I}(\Delta_{22}^u, \Delta_{22}^v)$, where $\square_1^{u_i}$ and $\Delta_{22}^v$ are twisted cycles in the $(s_i, s_{i+2})$-space. We can similarly compute them as

$$\mathcal{I}(\square_{1}^{u_i}, \square_{1}^{v_i}) = \frac{1 - \alpha_3^{-1}\beta_1}{(1 - \beta_1)(1 - \alpha_3^{-1})} \cdot \frac{1 - \alpha_1^{-1}\beta_1}{(1 - \beta_3^{-1})(1 - \alpha_1^{-1}\beta_1\beta_3)},$$

$$\mathcal{I}(\Delta_{22}^u, \Delta_{22}^v) = \frac{1 - \alpha_3^{-1}\beta_1}{(1 - \beta_1\beta_3^{-1})(1 - \beta_4^{-1})(1 - \alpha_3^{-1}\beta_2\beta_4)}.$$

The rests also can be similarly computed. \qed

**Proposition 6.1.** We have

$$\mathcal{I}(\Delta_{jk}^u, \Delta_{jk}^v) = h_{jk} \cdot \mathcal{I}(\Delta_{00}^u, \Delta_{00}^v),$$

for any $0 \leq j, k \leq 2$, where $h_{jk}$ are given in Proposition 5.1.
Proof. Lemma 6.2 yields that this proposition holds for the twisted cycles $\Delta^u_{jk} ((jk) = (10), (11), (21))$. Recall that the other twisted cycles are constructed by these twisted cycles and some actions of $S_3 \times D_4$ on the parameters. By acting them on the obtained identities and using Remark 5.2 we have this proposition. For example, by the action of $((12), \sigma_{110}) \in S_3 \times D_4$ on the identity for $\Delta^u_{10}$, the intersection number $I(\Delta^u_{10}, \Delta^u_{10} \vee)$ can be computed as

$$I(\Delta^u_{10}, \Delta^u_{10} \vee) = \frac{(\alpha_2 - \beta_2)(\alpha_3 - \beta_2)\beta_4}{(\alpha_2 - \beta_2\beta_4)(\alpha_3 - 1)(1 - \beta_2)(1 - \beta_4)} \cdot \frac{(\alpha_1 - \beta_2)\beta_1\beta_3}{(\alpha_1 - \beta_1\beta_3)(\beta_1 - \beta_2)(1 - \beta_3)},$$

and it is equal to $h_{20}I(\Delta^u_{00}, \Delta^u_{00} \vee)$, where $\sigma_{110}$ is given in Table 1. □

Remark 6.1. The intersection number $I(\Delta^u_{00}, \Delta^u_{00} \vee)$ is not invariant under the action of $S_3 \times D_4$ on the parameters. In the constructions of $\Delta_{jk} ((jk) = (20), (01), (02), (21)$ and (12), we use only actions which keep it invariant. By Proposition 6.1 and Remark 5.2, the action of $S_3 \times D_4$ on the ratio

$$\frac{I(\Delta^u_{jk}, \Delta^u_{jk} \vee)}{I(\Delta^u_{00}, \Delta^u_{00} \vee)}$$

is compatible with that on the fundamental solutions $F_{jk}(x)$. This property yields that

$$I(\Delta^u_{\sigma(jk)}, \Delta^u_{\sigma(jk) \vee}) = \frac{\sigma \cdot I(\Delta^u_{jk}, \Delta^u_{jk} \vee)}{\sigma \cdot I(\Delta^u_{00}, \Delta^u_{00} \vee)} \times I(\Delta^u_{00}, \Delta^u_{00} \vee),$$

where $\sigma \in D_4$ and $\Delta^u_{\sigma(jk)}$ is the twisted cycle corresponding to the fundamental solution $\sigma \cdot F_{jk}$. For example, $I(\Delta^u_{20}, \Delta^u_{20} \vee)$ can be computed as

$$\frac{\sigma_{100} \cdot I(\Delta^u_{10}, \Delta^u_{10} \vee)}{\sigma_{100} \cdot I(\Delta^u_{00}, \Delta^u_{00} \vee)} \times I(\Delta^u_{00}, \Delta^u_{00} \vee),$$

where $\sigma_{100}$ is given in Table 1.

Theorem 6.1. The fundamental system $F(x)$ is given by the integrals

$$\Delta(x) = ^t(\ldots, \int_{\Delta_{jk}} u(t, x)dt, \ldots),$$

where $(jk)$ are arrayed in the order $(00), (10), (20), \ldots, (22)$.

Proof. By Theorem 3.1 and Corollaries 3.2, 3.3 and 3.4 there exists a diagonal matrix $g = \text{diag}(\ldots, g_{jk}, \ldots) \in GL_9(\mathbb{C})$ such that

$$F(x) = g\Delta(x).$$

Moreover, it turns out by Proposition 6.1 that

$$gHg^\vee = H,$$
which is equivalent to \( g_{jk}g_{jk} = 1 \) for any \((jk)\). We show that \( g \) is a scalar matrix. By our normalization of \( F(x) \), it is sufficient to show that the \( \lambda \)-eigenvector of the circuit matrix of \( \rho_3 \) is \( \mathbf{1} = (1, \ldots, 1) \) with respect to \( \Delta(x) \). As in proved in Lemma 5.8 in \([7]\), we can show that the key identity

\[
(6.1) \quad \mathbb{O}^u = \sum_{0 \leq j, k \leq 2} c_{jk} \Delta_{jk}^u,
\]

of \( \Delta_{jk}^u \in H_4(T, \mathcal{L}_u) \), where \( \mathbb{O}^u \) is the twisted cycle defined by the 4-cycle \( \mathbb{O} \) in (5.2) and a branch of \( u(t, x) \) on it. By following the proof of Proposition 5.8 in \([7]\), we can show that the key identity

\[
\mathcal{I}(\mathbb{O}^u, \Delta_{jk}^u) = \mathcal{I}(\Delta_{jk}^u, \Delta_{jk}^u)
\]

for any \((jk)\). By considering the intersection numbers of both sides of (6.1) and \( \Delta_{jk}^u \), we see that \( c_{jk} = 1 \) by the key identity and Lemma 6.1. Hence we have

\[
\mathbb{O}^u = \sum_{0 \leq j, k \leq 2} \Delta_{jk}^u = (1, \ldots, 1)^t((\ldots, \Delta_{jk}^u, \ldots)),
\]

which shows the \( \lambda \)-eigenvector of the circuit matrix of \( \rho_3 \) is \( \mathbf{1} \) with respect to \( \Delta(x) \) by the linearity of the integration. \( \square \)

Recall that the \((j, k)\)-entry of \( \Delta(x) \) is \( \int_{\Delta_{jk}} u(t, x) dt = F_{jk}(x)/C_{jk} \).

Note that \( C_{jk} \) times

\[
\Gamma(1 - a_3) \Gamma(b_1 + b_3 - a_1 - 1) \Gamma(b_2 + b_4 - a_2 - 1)
\]

becomes

\[
C'_{jk} = \frac{\prod_{i=1}^{3} \Gamma(b_{1j} + b_{2k} - a_i - 1)}{\prod_{1 \leq i \leq 3} \Gamma((b_{1j} - b_{1i}) \prod_{1 \leq i \leq 3} \Gamma(b_{2k} - b_{2i})}.
\]

Thus we consider

\[
\frac{F_{jk}(x)}{C'_{jk}} \Gamma(1 - a_3) \Gamma(b_1 + b_3 - a_1 - 1) \Gamma(b_2 + b_4 - a_2 - 1);
\]

its \((j, k)\)-entry \( G_{jk}(x) \) is

\[
(6.2) \quad \frac{F_{jk}(x)}{C'_{jk}} = \frac{\prod_{i \neq j}^{3} \sin(\pi(a_i - b_{1j} - b_{2k} + 2))}{\prod_{1 \leq i \leq 3} \sin(\pi(b_{1j} - b_{1i})) \prod_{1 \leq i \leq 3} \sin(\pi(b_{2k} - b_{2i}))} \cdot \sum_{n'_1, n'_2} \left[ \prod_{i=1}^{3} \frac{\Gamma(a_i + n'_1 + n'_2)}{\Gamma(b_{1i} + n'_1) \Gamma(b_{2i} + n'_2)} \right] x_1^{n'_1} x_2^{n'_2};
\]
Remark 6.2. In \[9\], we give a linear transformation of them so that

\[ G \]

respectively. Note that \( G_{jk}(x) \) is defined under conditions

\[ b_{ij} - b_{ij} \ (i \in \{1, 2, 3\} - \{j\}), \ b_{ik} - b_{ik} \ (i \in \{1, 2, 3\} - \{k\}) \notin \mathbb{Z}. \]

**Remark 6.2.** In \[9\], we give a linear transformation of them so that

\[
(\beta_2 - \beta_1)(\beta_2 - 1)(\beta_1 - 1)(\beta_4 - \beta_3)(\beta_4 - 1)(\beta_3 - 1) = 0,
\]

and study the irreducibility of the monodromy representation of \( \mathcal{F} \left( \begin{array}{c} a \\ B \end{array} \right) \).

We can regard the twisted homology group \( H_4(T, \mathcal{L}_u) \) as the representation space of the monodromy of \( \mathcal{F} \left( \begin{array}{c} a \\ B \end{array} \right) \). By the deformation of elements in \( H_4(T, \mathcal{L}_u) \) along a loop \( \rho \), we have a homomorphism

\[ \mathcal{M} : \pi_1(X, \dot{x}) \ni \rho \mapsto \mathcal{M}_\rho \in GL(H_4(T, \mathcal{L}_u)). \]

We express the circuit transformations \( \mathcal{M}_i = \mathcal{M}_{\rho_i} \) along the loops \( \rho_i \) (\( i = 1, 2, 3 \)) in terms of the intersection form.

**Theorem 6.2.** We have

\[
\mathcal{M}_1(\Delta^u) = \\
\Delta^u - (1 - \beta_1^{-1})(\mathcal{I}(\Delta^u, \Delta^u_{10}), \mathcal{I}(\Delta^u, \Delta^u_{11}), \mathcal{I}(\Delta^u, \Delta^u_{12})) \mathcal{H}_1^{-1} \left( \begin{array}{c} \Delta^u_{10} \\ \Delta^u_{11} \\ \Delta^u_{12} \end{array} \right),
\]

\[
\mathcal{M}_2(\Delta^u) = \\
\Delta^u - (1 - \beta_2^{-1})(\mathcal{I}(\Delta^u, \Delta^u_{20}), \mathcal{I}(\Delta^u, \Delta^u_{21}), \mathcal{I}(\Delta^u, \Delta^u_{22})) \mathcal{H}_2^{-1} \left( \begin{array}{c} \Delta^u_{20} \\ \Delta^u_{21} \\ \Delta^u_{22} \end{array} \right),
\]

\[
\mathcal{M}_3(\Delta^u) = \Delta^u - \left(1 - \frac{\beta_1\beta_2\beta_3\beta_4}{\alpha_1\alpha_2\alpha_3} \right) \mathcal{I}(\Delta^u, \mathcal{O}^u) \mathcal{O}^u,
\]

where \( n'_1 \) and \( n'_2 \) run over the sets

\[
1 - b_{ij} + N = \{1 - b_{ij}, 2 - b_{ij}, 3 - b_{ij}, \ldots \},
\]

\[
1 - b_{2k} + N = \{1 - b_{2k}, 2 - b_{2k}, 3 - b_{2k}, \ldots \},
\]

and

\[
1 - b_{ij}, 2 - b_{ij}, 3 - b_{ij}, \ldots \}
\]

respectively.
where $\Delta^u \in H_4(T, \mathcal{L}_u)$ and
\[
\mathcal{H}_{1j} = \text{diag}\left(\mathcal{I}(\Delta^u_{j0}, \Delta^u_{j0} \vee), \mathcal{I}(\Delta^u_{j1}, \Delta^u_{j1} \vee), \mathcal{I}(\Delta^u_{j2}, \Delta^u_{j2} \vee)\right),
\]
\[
\mathcal{H}_{2j} = \text{diag}\left(\mathcal{I}(\Delta^u_{0j}, \Delta^u_{0j} \vee), \mathcal{I}(\Delta^u_{1j}, \Delta^u_{1j} \vee), \mathcal{I}(\Delta^u_{2j}, \Delta^u_{2j} \vee)\right).
\]

Proof. Let $\mathcal{M}'_i(\Delta^u)$ be the right hand side of $\mathcal{M}_i(\Delta^u)$ in this theorem. We show that $\mathcal{M}_i$ in Theorem 5.1 appears as the representation matrix of $\mathcal{M}'_i$ with respect to the column vector
\[
\Delta^u \equiv \{\Delta^u_{00}, \Delta^u_{10}, \ldots, \Delta^u_{22}\}.
\]
By Lemma 6.1 it is easy to see that $\mathcal{M}'_1(\Delta^u_{0j}) = \Delta^u_{0j} (j = 1, 2, 3)$. Note that
\[
\mathcal{M}'_1(\Delta^u_{1j}) = \Delta^u_{1j} - (1 - \beta_1^{-1})(\delta_{0j}, \delta_{1j}, \delta_{2j}) \begin{pmatrix} \Delta^u_{10} \\ \Delta^u_{11} \\ \Delta^u_{12} \end{pmatrix} = \beta_1^{-1} \Delta^u_{1j},
\]
\[
\mathcal{M}'_1(\Delta^u_{2j}) = \Delta^u_{2j} - (1 - \beta_2^{-1})(\delta_{0j}, \delta_{1j}, \delta_{2j}) \begin{pmatrix} \Delta^u_{20} \\ \Delta^u_{21} \\ \Delta^u_{22} \end{pmatrix} = \beta_2^{-1} \Delta^u_{2j},
\]
where $\delta_{ij}$ is Kronecker’s symbol. Thus we have the representation matrix $\mathcal{M}_i$ from $\mathcal{M}'_i$. Similarly we have the representation matrix $\mathcal{M}_2$ from $\mathcal{M}'_2$. By the definition of $\mathcal{M}'_3$, we have
\[
\mathcal{M}'_3(0^u) = \frac{\beta_1 \beta_2 \beta_3}{\alpha_1 \alpha_2 \alpha_3} 0^u, \quad \mathcal{M}'_3(\Delta^u) = \Delta^u
\]
for any $\Delta^u \in H_4(T, \mathcal{L}_u)$ satisfying $\mathcal{I}(\Delta^u, 0^u \vee) = 0$. We have only to note that $0^u = (1, \ldots, 1) \Delta^u$ and
\[
\mathcal{I}(\Delta^u, 0^u \vee) = 0 \iff (w_{00}, \ldots, w_{22})H^1(1, \ldots, 1) = 0
\]
for $\Delta^u = (w_{00}, \ldots, w_{22}) \Delta^u$. \hfill $\square$

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