Delooping level of Nakayama algebras

EMRE SENGÜL

Abstract. We give another proof of the recent result of Ringel, which asserts equality between the finitistic dimension and delooping level of Nakayama algebras. The main tool is the syzygy filtration method introduced in our earlier work.

Mathematics Subject Classification. 16E05, 16G20.

Keywords. Nakayama algebras, Finitistic dimension, Delooping level.

1. Introduction. Let $A$ be an Artin algebra. The delooping level of a finitely generated $A$-module $M$ is defined by Gelinas in [1] as:

$$\text{del } M = \min \{ d \mid \Omega^d(M) \in \text{add } (A \oplus \Omega^{d+1}(M')) , M' \in \text{mod-}A \}$$

and $\text{del } M = \infty$ if such a $d$ does not exist. The delooping level of an algebra $A$ is the maximum of the delooping levels of simple $A$-modules:

$$\text{del } A = \max \{ \text{del } S \mid S \text{ is simple } A\text{-module} \} .$$

The finitistic dimension of an algebra $A$ is given by $\text{fin.dim } A = \sup \{ \text{pdim } M \mid M \in \text{mod-}A \}$. An open problem in representation theory of Artin algebras is the finitistic dimension conjecture which states that the finitistic dimension of $A$ is finite. The relationship between the finitistic dimension and the delooping level is: $\text{fin.dim } A^{op} \leq \text{del } A$ which makes the delooping level an interesting homological measure, where $A^{op}$ is the opposite algebra [1]. Ringel proved the following theorem in [3]:

Theorem 1.1. The finitistic dimension and the delooping level of a Nakayama algebra are equal.

In this note, we give a shorter proof of the theorem by using the syzygy filtration method. The keystone is the following reduction: $\text{del } \Lambda = \text{del } \varepsilon(\Lambda) + 2$ which we show in Proposition 2.1, where $\Lambda$ is a cyclic Nakayama algebra and $\varepsilon(\Lambda)$ is its syzygy filtered algebra (see Definition 1.2). This helps us to make
such that they are indexed by one cyclically larger indices of $S$.

1.1. Preliminaries on the Syzygy filtration. We recall basic definitions and the
construction of the syzygy filtered algebra $\varepsilon(\Lambda)$. Details can be found in [4,5].

Consider the irredundant system of relations $\alpha_{k_2} \cdots \alpha_{k_{n-1}} = 0$ where
$1 \leq i \leq r$ and $k_f \in \{1, 2, \ldots, n\}$ for a cyclic oriented quiver $Q$ where each
arrow $\alpha_i$, $1 \leq i \leq n - 1$, starts at the vertex $i$ and ends at the vertex $i + 1$ and
$\alpha_n$ starts at vertex $n$ and ends at vertex 1. It follows that the bound
quiver algebra $\Lambda = kQ/I$ where $I$ is an irredundant system of relations, is a
connected cyclic Nakayama algebra. Explicitly, $I$ is generated by the relations:
$\langle \alpha_{k_2} \cdots \alpha_{k_1+1}\alpha_{k_1} = 0, \alpha_{k_4} \cdots \alpha_{k_3+1}\alpha_{k_3} = 0, \ldots, \alpha_{k_2r-1+1}\alpha_{k_2r-1} = 0 \rangle$
where $\cdots < k_1 < k_3 < \cdots < k_{2r-1} < k_1 < \cdots$ is cyclically ordered [4].

Let $S(\Lambda)$ be the complete set of representatives of socles of indecomposable
projective modules over $\Lambda$:

$$S(\Lambda) = \{S_{k_2}, S_{k_4}, \ldots, S_{k_2r}\}.$$  \hfill (1.3)

Similarly, let $S'(\Lambda)$ be the complete set of representatives of simple modules
such that they are indexed by one cyclically larger indices of $S(\Lambda)$:

$$S'(\Lambda) = \{S_{k_2+1}, S_{k_4+1}, \ldots, S_{k_2r+1}\}.$$  \hfill (1.4)

Now we define the following base set $B(\Lambda)$ and its $\Lambda^{op}$ analogue $\nabla(\Lambda)$:

$$B(\Lambda) := \left\{ \Delta_1 \cong S_{k_{2r+1}}, \ldots, \Delta_2 \cong S_{k_{2r}}, \ldots, \Delta_{j-1} \cong S_{k_{2j-1}}, \ldots, \Delta_r \cong S_{k_{2r-2+1}} \right\},$$

$$\nabla(\Lambda) := \left\{ \nabla_1 \cong S_{k_{1+1}}, \ldots, \nabla_2 \cong S_{k_{3}}, \ldots, \nabla_{j-1} \cong S_{k_{2j-1}}, \ldots, \nabla_r \cong S_{k_{2r-1}} \right\}.$$  \hfill (1.5)

**Definition 1.2** ([5]). Let $\Lambda$ be a cyclic Nakayama algebra. The syzygy filtered
algebra $\varepsilon(\Lambda)$ is:

$$\varepsilon(\Lambda) := \text{End}_\Lambda(\oplus_{S \in S'(\Lambda)} P(S))$$  \hfill (1.7)

where $P(S)$ is the projective cover of the simple $\Lambda$-module $S$. Similarly, the
d$th$ syzygy filtered algebra $\varepsilon^d(\Lambda)$ is:

$$\varepsilon^d(\Lambda) := \text{End}_{\varepsilon^{d-1}(\Lambda)}(\oplus_{S \in S'(\varepsilon^{d-1}(\Lambda))} P(S))$$  \hfill (1.8)

provided that $\varepsilon^{d-1}(\Lambda)$ is a cyclic non-selfinjective Nakayama algebra and $P(S)$
is the projective cover of the simple $\varepsilon^{d-1}(\Lambda)$-module $S$.

For cyclic Nakayama algebras of infinite global dimension, a nice homolog-
ical measure is the $\varphi$-dimension which attains only even numbers (see [4, Theorem A]). Here we briefly recall the definition of the $\varphi$-dimension:
Definition 1.3. For a given finitely generated $A$-module $M$, $\varphi(M)$ is defined in [2] as:

$$\varphi(M) := \min \{ t \mid \text{rank} \left( L^t(\text{add}M) \right) = \text{rank} \left( L^t+j(\text{add}M) \right) \text{ for } \forall j \geq 1 \}$$

where $L[M] := [\Omega M]$ in the Grothendieck group of $A$-modules modulo projective summands. The $\varphi$-dimension of an algebra $A$ is:

$$\varphi \text{dim}(A) := \sup \{ \varphi(M) \mid M \text{ is an } A\text{-module} \}.$$  

Remark 1.4. We summarize some results from [5]:

(1) The second and higher syzygies of $\Lambda$-modules have a unique $B(\Lambda)$ filtration. $\varepsilon(\Lambda)$ is a Nakayama algebra and the category of $B(\Lambda)$-filtered $\Lambda$-modules is equivalent to the category of $\varepsilon(\Lambda)$-modules.

(2) If the global dimension of $\Lambda$ is infinite, then there exists $d$ such that $\varepsilon^{d+1}(\Lambda)$ is a selfinjective Nakayama algebra and $\varphi \text{dim } \varepsilon^d(\Lambda) = 2$.

(3) We have $\varphi \text{dim } \Lambda - \text{fin.dim } \Lambda \leq 1$. In particular, $\text{fin.dim } \Lambda = 1$ or $\text{fin.dim } \Lambda = 2$ imply $\varphi \text{dim } \Lambda = 2$.

2. Proof of the main theorem.

Proposition 2.1. Let $\Lambda$ be a cyclic Nakayama algebra with $\varphi \text{dim } \Lambda \geq 3$. Then

$$\text{del } \Lambda = \text{del } \varepsilon(\Lambda) + 2.$$  

(2.1)

Proof. Let $S$ be a simple $A$-module with $\text{del } S = d \geq 3$. By Definition 1.1, there exists a $\Lambda$-module $M$ satisfying $\Omega^d(S) \in \text{add}(\Lambda \oplus \Omega^{d+1}(M))$, provided that $d$ is minimal. By Remark 1.4 (1), we get: $\Omega^{d-2}(\varepsilon(\Lambda) \oplus \Omega^{d-1}(\Omega^2(M)))$ and $\Omega^{d-2}(S') \in \text{add}(\varepsilon(\Lambda) \oplus \Omega^{d-1}(M'))$ where $S' = \Omega^2(M)$, $M' = \Omega^2(M)$ are $\varepsilon(\Lambda)$-modules. $d - 2$ is minimal otherwise $d$ would not be the delooping level of simple $\Lambda$-module $S$.

Corollary 2.2. Let $\Lambda$ be a cyclic Nakayama algebra of infinite global dimension which is not selfinjective. Then we have $\text{del } \Lambda = \text{del } \varepsilon^k(\Lambda) + 2k$ where $1 \leq k \leq \frac{\varphi \text{dim } \Lambda}{2}$.

Proof. Since the global dimension of $\Lambda$ is infinite, we can apply the $\varepsilon$ reduction to $\Lambda$ iteratively because $\varepsilon^k(\Lambda) \cong \varepsilon^{k+1}(\Lambda)$ holds for any $k$. In particular, if a cyclic Nakayama algebra is selfinjective, then $\varepsilon$ induces an isomorphism, i.e., in our set up $\varepsilon^{\frac{\varphi \text{dim } \Lambda}{2}}(\Lambda)$ is selfinjective. Therefore, for any $j > \frac{\varphi \text{dim } \Lambda}{2}$, we have $\varepsilon^j(\Lambda) \cong \varepsilon^{\frac{\varphi \text{dim } \Lambda}{2}}(\Lambda)$. Therefore we can apply the above proposition iteratively to get the following equalities:

$$\text{del } \Lambda = \text{del } \varepsilon(\Lambda) + 2 = \text{del } \varepsilon^2(\Lambda) + 4 = \cdots = \text{del } \varepsilon^k(\Lambda) + 2k$$  

(2.2)

where $k$ is at most $\frac{\varphi \text{dim } \Lambda}{2}$.

Remark 2.3. From the definitions, we can deduce that $\text{del } \Lambda \leq \varphi \text{dim } \Lambda$. Before proceeding, we give the definition of a periodic module: $M$ is called periodic if there exists $t$ such that $\Omega^t(M) = M$. In details, there are two types of simple modules: either the projective dimension of $S$ is finite, so $\text{pdim } S \leq \varphi \text{dim } \Lambda$ or the projective dimension of $S$ is unbounded, then $\Omega^{\varphi \text{dim } \Lambda}(S)$ has to be a periodic module, so there exists another periodic module $M$ such that $\Omega^{\varphi \text{dim } \Lambda}(S) \cong \Omega(M)$. 

Vol. 117 (2021)  Delooping level of Nakayama algebras  143
Proposition 2.4. If $\Lambda$ is a cyclic Nakayama algebra with $\varphi \dim \Lambda = 2$, then $\text{fin.dim} \Lambda = 1$ if and only if $\text{del} \Lambda = 1$.

Proof. First we show the implication $\text{fin.dim} \Lambda = 1 \implies \text{del} \Lambda = 1$. Assume that $\text{fin.dim} \Lambda = 1$. By Remark 2.3, the possibilities are:

(i) The projective dimension of $S$ is one, therefore $\text{del} S = 1$.
(ii) $S$ itself is a periodic module which makes $\text{del} S = 0$.
(iii) $S$ is not periodic but $\Omega(S)$ is a periodic module. Therefore $\text{del} \Omega(S) = 0$ and by using Definition 1.1, we conclude that $\text{del} S = 1$.

As a result, the maximum of $\text{del} S$ is one. Now, we consider the other direction. Since $\text{del} \Lambda = \max_S \{\text{del} S\}$, we have two cases:

(i) $\text{pdim} S = 1$,
(ii) $\Omega(S)$ is a periodic module.

We call an indecomposable projective $\Lambda$-module minimal if its radical is not projective. If the projective dimension of a simple module is one, it cannot be the top of a minimal projective. The second item forces that the first syzygies of tops of minimal projectives have $B(\Lambda)$ filtrations. Therefore they are the indices of simple modules of the set $S'(\Lambda)$. On the other hand, the indices of the tops of the first syzygies are $\{k_1 + 1, k_3 + 1, \ldots, k_{2r-1} + 1\}$ in this case. We get: $\{k_2 + 1, k_4 + 1, \ldots, k_{2r} + 1\} = \{k_1 + 1, k_3 + 1, \ldots, k_{2r-1} + 1\}$ which is equivalent to $B(\Lambda) = \nabla(\Lambda)$. Therefore the finitistic dimension is one (see [5, Proof of Proposition 5.14]). This finishes the proof. □

In the proof, we do not need to analyze the case $\text{del} \Lambda = 2$ because both $\text{del} \Lambda$ and $\text{fin.dim} \Lambda$ are bounded by 2 which follows from Remarks 1.4 (3) and 2.3. Actually, $\text{del} \Lambda = 2$ is equivalent to the existence of a simple module $S$ which is not the socle of any projective module and in particular $\Omega^2(S)$ is periodic but $\Omega(S)$ is not.

It is clear that if the global dimension of $\Lambda$ is finite, then $\text{del} \Lambda = \text{fin.dim} \Lambda$. So we consider the case that $\Lambda$ has infinite global dimension. Moreover, we can exclude the equalities of the left and the right finitistic dimensions since we proved them in [5]. Now we can use the induction to prove the main result:

Proof of Theorem 1.1. Let $\Lambda$ be a cyclic not selfinjective Nakayama algebra of infinite global dimension. We consider the following reductions similar to (2.2) [5]

$$\text{fin.dim} \Lambda = \text{fin.dim} \varepsilon(\Lambda) + 2 = \text{fin.dim} \varepsilon^2(\Lambda) + 4 = \cdots = \text{fin.dim} \varepsilon^d(\Lambda) + 2d$$

(2.3)

where $d+1 = \frac{\varphi \dim \Lambda}{2}$. Hence $\varepsilon^d(\Lambda)$ is a cyclic Nakayama algebra of $\varphi$-dimension two and its finitistic dimension satisfies

$$1 \leq \text{fin.dim} \varepsilon^d(\Lambda) \leq 2.$$  

(2.4)

Because $\varepsilon^d(\Lambda)$ is a cyclic Nakayama algebra (see Remark 1.4 (1)), we can use Proposition 2.4 to get

$$1 \leq \text{fin.dim} \varepsilon^d(\Lambda) = \text{del} \varepsilon^d(\Lambda) \leq 2.$$  

(2.5)
Now, by Corollary 2.2, we obtain the desired equality \( \text{fin. dim } \Lambda = \text{del } \Lambda \) because

\[
1 \leq \text{fin. dim } \varepsilon^d(\Lambda) = \text{del } \varepsilon^d(\Lambda) \leq 2 \iff \\
3 \leq \text{fin. dim } \varepsilon^{d-1}(\Lambda) = \text{del } \varepsilon^{d-1}(\Lambda) \leq 4 \iff \\
\vdots \\
2d - 1 \leq \text{fin. dim } \varepsilon(\Lambda) = \text{del } \varepsilon(\Lambda) \leq 2d \iff \\
2d + 1 \leq \text{fin. dim } \Lambda = \text{del } \Lambda \leq 2d + 2.
\]

\[\square\]

**Remark 2.5.** In [3], Ringel shows that: (1) if \( X \) is a submodule of a finitely generated module \( Y \), then \( \text{del } X \leq \text{pdim } Y \), (2) any simple module is a submodule of an indecomposable module with finite projective dimension, and a factor module of an indecomposable module with finite injective dimension. Those are his main tools to prove the theorem.

The result can be obtained as another application of the syzygy filtration method: any homological measure based on the supremum of the projective dimensions of the modules such as (left-right) finitistic, dominant, Gorenstein dimensions, delooping level etc. can be studied in a unified way.

**Acknowledgements.** We are deeply thankful to Prof. Ringel for devoting Appendices B and C in his work to our \( \varepsilon \)-construction and elucidating its difference from the other methods as well as to Prof. Igusa and Prof. Todorov for encouraging us to post this paper. We also thank the referee for suggestions to improve the text.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

**References**

[1] Gélinas, V.: The depth, the delooping level and the finitistic dimension. arXiv:2004.04828 (2020)

[2] Igusa, K., Todorov, G.: On the finitistic global dimension conjecture for Artin algebras. In: Representations of Algebras and Related Topics, 201–204. Fields Inst. Commun., 45. Amer. Math. Soc., Providence, RI (2005)

[3] Ringel, C.M.: The finitistic dimension of a Nakayama algebra. J. Algebra 576, 95–145 (2021)

[4] Sen, E.: The \( \varphi \)-dimension of cyclic Nakayama algebras. Comm. Algebra 49(6), 2278–2299 (2021)

[5] Sen, E.: Syzygyfiltrations of cyclic Nakayama algebras. arXiv:1903.04645 (2019)
Emre Sen
Department of Mathematics
University of Iowa
Iowa City, IA
USA
e-mail: emre-sen@uiowa.edu

Received: 14 December 2020

Revised: 12 April 2021

Accepted: 26 April 2021.