Abstract

We present a novel space-efficient graph coarsening technique for \( n \)-vertex planar graphs \( G \), called cloud partition, which partitions the vertices \( V(G) \) into disjoint sets \( C \) of size \( O(\log n) \) such that each \( C \) induces a connected subgraph of \( G \). Using this partition \( \mathcal{P} \) we construct a so-called structure-maintaining minor \( F \) of \( G \) via specific contractions within the disjoint sets such that \( F \) has \( O(n/\log n) \) vertices. The combination of \((F, \mathcal{P})\) is referred to as a cloud decomposition.

For planar graphs we show that a cloud decomposition can be constructed in \( O(n) \) time and using \( O(n) \) bits. Given a cloud decomposition \((F, \mathcal{P})\) constructed for a planar graph \( G \) we are able to find a balanced separator of \( G \) in \( O(n/\log n) \) time. Contrary to related publications, we do not make use of an embedding of the planar input graph. We generalize our cloud decomposition from planar graphs to \( H \)-minor-free graphs for any fixed graph \( H \). This allows us to construct the succinct encoding scheme for \( H \)-minor-free graphs due to Blelloch and Farzan (CPM 2010) in \( O(n) \) time and \( O(n) \) bits improving both runtime and space by a factor of \( \Theta(\log n) \).

As an additional application of our cloud decomposition we show that, for \( H \)-minor-free graphs, a tree decomposition of width \( O(n^{1/2+\epsilon}) \) for any \( \epsilon > 0 \) can be constructed in \( O(n) \) bits and a time linear in the size of the tree decomposition. A similar result by Izumi and Otachi (ICALP 2020) constructs a tree decomposition of width \( O(k\sqrt{n}\log n) \) for graphs of treewidth \( k \leq \sqrt{n} \) in sublinear space and polynomial time.

Finally, we implemented our cloud decomposition algorithm and experimentally verified its practical effectiveness on both randomly generated graphs and real-world graphs such as road networks. The obtained data shows that a simplified version of our algorithms suffices in a practical setting, as many of the theoretical worst-case scenarios are not present in the graphs we encountered.

Introduction

Graphs are used to model a multitude of systems that can be expressed via entities and relationships between these entities. Many real-world problems operate on very large graphs for which standard algorithms and data structures use too much space. This has spawned an area of research with the aim of reducing the required space. Examples include large road-networks [56] or social graphs arising from interactions between users of large internet communities [22]. Therefore, it is of high interest to find compact representations of such graphs, space-efficient algorithms and other space-efficient or succinct data structures. In the following we denote by \( n \) the number of vertices of a graph under consideration and \( m \) the number of edges. An algorithm is called space-efficient if it has (almost) the same asymptotic runtime as a standard algorithm for the same problem, but uses asymptotically
fewer bits. Examples include space-efficient graph searching algorithms such as depth-first search and breadth-first search, which run in linear time, but use \( O(n \log n) \) bits with standard methods. Space-efficient solutions lower the space requirement to \( O(n) \) bits and keep the runtime asymptotically (almost) the same. For a data structure or algorithm to be called *succinct* the space used must be \( Z + o(Z) \) bits with \( Z \) being the information theoretic minimum to store the data. One of the most researched topics regarding space-efficiency are graph-traversal algorithms such as depth-first search (DFS). It is still an open research question if a linear-time DFS exists that uses \( O(n) \) bits, with the current best bound of \( O(n + m + \min\{n, m\} \log^* n) \) due to Hagerup [29], with \( \log^* n \) being the iterated logarithm. This result is the work of gradual improvements over the span of seven years spanning multiple publications by multiple research groups, with some of the first results providing a runtime \( \Omega(n \log n) \) time due to Asano et al. [9] and a runtime of \( O((n + m) \log \log n) \) due to Elmasry et al. [20]. Other typical problems that can be easily solved in settings where space is of no concern, such as storing spanning trees or simple mappings between vertices, require new problem-specific strategies when considered in the space-efficient setting.

Another interesting setting is aiming for arbitrary polynomial time runtime, but using only \( o(n) \), or even \( O(\log n) \) bits. In such settings information can be recomputed in polynomial time, and for problems such as mappings this is often trivial, while other problems such as deciding if two vertices are connected by a path in a graph requires the very involved algorithm due to Reingold [53], but is quite easy in a space-efficient setting. Thus, our space-efficient setting differentiates itself quite strongly from both, the well-researched sublinear-space settings and the common settings that do not regard space as limited.

Many large graphs that arise in practice have some known structural property that allows the design of specialized techniques that make use of these properties. A common such structural property is the existence of small separators. Such a separator is a small subset of vertices whose removal disconnects the graph if it was connected previously, or increases the number of connected components if it was not connected. For the graphs of interest to this work the separators are so-called balanced separators. For now the intuition suffices that such balanced separators split the graph in somewhat equally parts. Section 2 contains precise definitions of these terms and also defines what we consider small in regard to separators. Arguably the most well-known graphs that contain such small balanced separators are planar graphs, which are graphs that can be drawn in the plane without overlapping edges. All planar graphs contain a balanced separator of size \( O(\sqrt{n}) \) [47]. Similar separator theorems exist for almost-planar graphs [23] (which includes road networks) and well-formed meshes such as nearest neighbor graphs [49] and \( H \)-minor-free graphs for some fixed graph \( H \) [41]. For arbitrary separable graphs there exists a polylogarithmic approximation algorithm for finding balanced separators due to Leighton and Rao [45].

We present a novel partitioning scheme called *cloud partition* for planar graphs that partitions the vertices of a connected input graph \( G \) into connected subsets called *clouds* that induce a connected subgraph and are of size \( O(\log n) \). For a cloud partition \( \mathcal{P} \) constructed for \( G \) we construct a so-called *structure-maintaining minor* \( F \) of \( G \) induced by \( \mathcal{P} \). For easier reading comprehension we call vertices of such a minor *nodes*. Intuitively, a node \( v \in V(F) \) is mapped to one or more clouds \( C \in \mathcal{P} \) such that \( |V(F)| = O(n/\log n) \)—the exact mapping is outlined in Section 3. A solution for some problems such as finding small separators can be found in \( F \) and then translated to an approximate solution for \( G \). As \( F \) contains \( O(n/\log n) \) nodes, the time and space bounds of linear or superlinear algorithms can be decreased by a factor of \( \Omega(\log n) \) when being executed on \( F \) instead of \( G \). We show how this speedup is especially helpful for recursive algorithms such as the recursive separator search used during
the computation of the succinct representation of separable graphs due to Blelloch and Farzan [14]. Additionally, we show that small modifications of this recursive separator search can be used to find a tree decomposition of width $O(n^{1/2+\epsilon})$ for any $\epsilon > 0$ for planar graphs in $O(n)$ bits and a time linear in the size of the tree decomposition. Finally, we generalize our partitioning scheme from planar graphs to $H$-minor-free graphs for any fixed graph $H$.

One of the key points of our novel scheme is that we do not make use of a (planar) embedding of the input graph, as there is no known way to construct such an embedding with $O(n)$ bits and in linear time. It is often implied that an embedding is given alongside a planar graph in many of the publications regarding planar graphs, as it is computable in linear time [33] when space is of no concern. Additionally, it is often required that the planar graph is maximal. Both of these properties are, for example, required for the major result of the $O(\sqrt{n})$-separator theorem [47] or the so-called $r$-partition of Klein et al. [44], which similarly to our result partitions the input graph into regions of size $O(n/r)$. Again, if the graph is not maximal, it can be easily made maximal in linear time with the help of an embedding. In a space-efficient context we can not make use of either of these properties and are thus quite limited. For sublinear space settings there exist algorithms that produce an embedding in polynomial time with a rather large polynomial degree [6, 19]. Thus, the $O(n)$-bit setting provides a unique challenge due to the additional goal of matching the runtime of non space-efficient algorithms.

Succinct and space-efficient representations of planar graphs is a highly researched topic partially due to the practical applications. For compressing planar graphs without regards to providing fast access operations refer to the work of Keeler et al. [42] for an $O(n)$ bits representation and for a compression within the information theoretic lower bound refer to He et al. [31]. Due to Munro and Raman [50] there exists an encoding using $O(n)$ bits that allows constant-time queries which has subsequently been improved by Chiang et al. [17] to use a constant factor less space. We use the succinct representation due to Blelloch and Farzan, which allows encoding arbitrary separable graphs and subsequently allows constant-time adjacency-, neighborhood- and degree-queries [14], which builds on the work of Blanford et al. [13]. For $H$-minor-free graphs for a fixed graph $H$, their encoding takes $\Theta(n \log n)$ time and $\Theta(n \log n)$ bits using their described technique of recursive separator searches. We are able to improve it to $O(n)$ time and $O(n)$ bits. For planar graphs we only improve the space-requirement from $\Theta(n \log n)$ to $O(n)$ due to the algorithm of Goodrich [27] which can be used instead of the generic recursive separator search described by Blelloch and Farzan. Note that all mentioned publications above regarding succinct representations of planar graphs use $\Theta(n \log n)$ bits during the construction. Additionally, they assume a planar embedding is given explicitly or implicitly by making direct use of it, or referring to other publications as sub-routines that require them. For maximal planar graphs there exist special encodings. Some major research in this field is due to Aleardi et al. [2, 3, 4, 5], which deals with practical and theoretical representations of such maximal planar graphs, typically arising in applications making use of geometric meshes or computer graphics. Note that these space-efficient encodings of maximal planar graphs also make use of and store an embedding, as navigation of the faces is a vital feature in these applications.

The general technique of graph coarsening has been heavily used in the past for practical and theoretical algorithms [16, 21, 32, 39, 55]. A typical approach is to find a matching $E' \subset E(G)$ for a graph $G$ and contract the edges in $E'$ to find a minor $F$ of $G$. The problem at hand is then solved on $F$ and the solution is translated to an approximate solution for $G$ [39]. Highly specialized practical algorithms using sophisticated data structures are implemented in the METIS [40] and SCOTCH [51] library. Other approaches such as graph coarsening on
the GPU [10, 11] or graph coarsening via neural networks [15] have been developed. Note that the use of sophisticated data structures that focus on runtime optimization do not lend themselves to modifications to make them space-efficient. Thus, there was a need to develop a novel partitioning scheme for this work, in particular a scheme that does not require an embedding when dealing with planar graphs.

For finding tree decompositions in a space-efficient manner refer to the work of Kammer et al. [37] and the work of Izumi and Otachi [35]. Izumi and Otachi showed that for a given graph $G$ with treewidth $k \leq \sqrt{n}$ there exists an algorithm that obtains a tree decomposition of width $O(k\sqrt{n}\log n)$ in polynomial time and $O(k\sqrt{n}\log^2 n)$ bits. Note that the polynomial degree in the runtime is rather large due to the use of the well known $s$-$t$ reachability algorithm of Reingold [53]. Izumi and Otachi additionally mention that for planar graphs there exists a polynomial-time and sublinear-space algorithm for finding a tree decomposition of a planar graph due to a simple recursive separator search with the result of Imai et al. [34], which present an algorithm for finding balanced separators of size $O(\sqrt{n})$ in planar graphs with sublinear space and polynomial runtime. Note that the algorithm of Imai et al. makes use of an embedding of the input graph and also uses the algorithm of Reingold. In contrast, we present an algorithm that computes a tree decomposition of width $O(n^{1/2+\epsilon})$ in time linear in the size of the tree decomposition using $O(n)$ bits. As the tree decomposition has $O(n)$ bags of size $O(n^{1/2+\epsilon})$ this results in a runtime of $O(n^{3/2+\epsilon})$.

In Section 2 we present concepts needed to understand the subsequent sections. In Section 3 we present our space-efficient graph-coarsening framework. Following that, in Section 4 and 5 we show how this scheme is used for finding separators, tree decompositions and constructing succinct encodings of planar graphs. In Section 6 we generalize our work to $H$-minor-free graphs. Finally, in Section 7 we analyze the practical effectiveness of our decomposition algorithm. We implemented the cloud partition algorithm and tested it on both randomly generated graphs, and real-world data such as street networks.

2 Background

We operate in the standard word RAM model of computation with word size $w = \Omega(\log n)$. This assumes the existence of read-only input memory, read/write working memory and write-only output memory. When we talk about space-usage we focus on bits used in the read/write working memory. Sometimes this model is also referred to as the register input model [24]. This is a common setting for space-efficient algorithms.

We make use of common graph theoretic notations and terminology. Refer to Diestel [18] for more information. When using space-efficient graph algorithms the exact representation of the input graph is important. We assume that any input graph is given via adjacency arrays or an equivalent interface. Given a vertex $u$ and index $i \leq \text{degree}(u)$ this allows us to determine in constant time the $i$th edge $\{u, v\}$ out of $u$. All our input graphs are assumed to be undirected with the vertices labeled from $1, \ldots, n$. Such a labeling is given implicitly by the order of the adjacency arrays. Given the use of adjacency arrays we store each undirected edge $\{u, v\}$ as two directed arcs $uv$ and $vu$. For each arc $uv$ we call $vu$ the co-arc of $uv$. We assume the existence of so-called crosspointers that allow us to find the co-arc of an arc $uv$ in constant time. Typically, this is realized by storing an index $i$ in addition to the vertex $v$ in $u$’s adjacency array. The index $i$ then indicates the position of $u$ in $v$’s adjacency array. We assume w.l.o.g. that every given graph $G$ is connected as otherwise all our techniques can be done iteratively for each connected component of $G$. We assume that a given input graph is in read-only memory.
A separator $S$ of a graph $G = (V, E)$ is a subset of $V$ such that its removal from $V$ divides $V$ into non-empty sets $A \subset V$ and $B \subset V$ so that $\{A, S, B\}$ is a partition of $V$ with the constraint that all paths from a vertex $u \in A$ to a vertex $v \in B$ contain at least one vertex of $S$. If $|A| < \alpha n$ and $|B| < \alpha n$ for some $\alpha < 1$, then $S$ is called an $\alpha$-balanced separator or simply balanced separator.

A family of graphs $\mathcal{G}$ that is closed under taking vertex-induced subgraphs satisfies the $f(\cdot)$-separator theorem exactly if, for constants $\alpha < 1$ and $\beta > 0$, each member $G \in \mathcal{G}$ has an $\alpha$-separator $S$ of size $|S| < \beta f(n)$. We say a family of graphs $\mathcal{G}$ is separable exactly if it satisfies the $n^c$-separator theorem for some $c < 1$. We say a graph is separable if it belongs to a separable family of graphs. For planar graphs there exist an $O(\sqrt{n})$-separator theorem with $\alpha = 2/3$ that runs in linear time [47], later extended to graphs of bound genus with the same runtime [26]. For minor-closed graph classes excluding the complete graph $K_t$ on $t$ vertices for some constant $t$ there exists an $O(n^{(2-\epsilon)/3})$-separator theorem with runtime $O(n^{1+\epsilon} + m)$ [41]. Note that this includes $H$-minor-free graphs for any fixed graph $H$.

The definition of the separable graph families are not closed under taking minors, but vertex-induced subgraphs [14]. As we specifically construct minors $F$ of a separable graph $G$ at various points in this publication, we require the more restrictive property that $F$ also belongs to the same family of separable graphs as $G$, which holds for all graph classes explicitly mentioned in the previous paragraph.

Due to Lipton et al. [46] it is known that a separable graph class $\mathcal{G}$ has bound density $d$ for some constant $d$. This means that each $G \in \mathcal{G}$ contains at most $d n$ edges. In particular, we use the well-known fact, by Euler’s Formula, that for a planar graph $G = (V, E)$ it holds that $|E| \leq 3|V| - 6$. For planar bipartite graphs a stronger bound $|E| \leq 2|V| - 4$ holds.

A tree decomposition of a graph $G = (V, E)$ consists of a tree $T$ and a family $X$ of subsets $X_w$ (called bags) of $V$, one for each $w \in V(T)$, such that:

1. $\bigcup_{w \in V(T)} X_w = V$
2. for all $\{u, v\} \in E$ there exists $w \in V(T)$ such that $u, v \in X_w$
3. for all $w_1, w_2, w_3 \in V(T)$, if there is a path from $w_1$ to $w_3$ that contains $w_2$, then $X_{w_1} \cap X_{w_3} \subseteq X_{w_2}$.

The width of a tree decomposition is the size of the largest bag minus one. The treewidth of a graph $G$ is the smallest width amongst all possible tree decompositions of $G$.

We make use of indexable dictionaries, which is a structure that supports constant-time rank-select queries over a bitvector. The rank($i$) operation counts the number of occurrences of bits set to 1 before the $i$th index and the select($i$) operation returns the index of the $i$th bit set to 1.

**Lemma 1** ([52]). Given a bitvector $S$ of length $\ell$ there is an indexable dictionary on $S$ that requires $O(\ell)$ additional bits, supports rank-select queries in constant time and can be constructed in $O(\ell)$ time.

What can be thought of a dynamic alternative to indexable dictionaries is the so-called choice dictionary [28]. A choice dictionary is initialized for a universe $1, \ldots, \ell$ and supports constant time insert, delete, and contains operations, with the latter returning true exactly if an element of the universe is contained in the choice dictionary. Additionally, the choice dictionary offers the operation choose that returns an arbitrary member and iteration over its members. The iteration outputs all members of a choice dictionary in constant time per member. All operations run in constant time and the iteration over its members is linear in the number of members. The following lemma has been adapted from [38], with some minor rephrasing to make it more clear in the context of this paper.
Lemma 2. There is a succinct choice dictionary initialized for the universe 1, \ldots, \ell that occupies \( \ell + o(\ell) \) bits and provides constant-time insert, delete, contains and choice operations and constant-time (per member) iteration. The choice dictionary can be initialized in \( O(\ell) \) time.

We make use of a folklore technique called static space allocation allowing us to store \( \ell \) items of varying size compactly. The following description is adapted from [36]. Each item \( B_k \) occupies \( d_k \) bits for \( k \in \{1, \ldots, \ell\} \). Denote by \( L \) the amount of bits all these items totally occupy. We want to store all these items with \( L + o(L) \) bits such that we can access each item \( B_k \) in \( O(1) \) time for \( k \in \{1, \ldots, \ell\} \). In \( O(\ell + L) \) time compute the sums \( s_k = k + \sum_{j=1}^{k-1} d_j \) and an indexable dictionary over a bitvector \( S \) of size \( \ell + L \) with \( S[i] = 1 \) exactly if \( i = s_k \) for \( k \in \{1, \ldots, \ell\} \). The location of \( B_k \) is then equivalent to \( \text{select}_B(k) - k \). Refer to [12, 30, 36] for more information and detailed descriptions.

For traversing graphs we use standard breadth-first search (BFS), which puts a start vertex in a first queue, then the unprocessed neighbors of the first queue in a second queue, swaps the queues and repeats. It can easily be seen that space requirement is \( O(n \log n) \) bits. More precise, the BFS uses \( O(\ell \log n) \) bits where \( \ell \) is the maximum number of vertices in a queue at any point of the BFS.

In this work we need to store subgraphs \( G' \) of a given graph \( G = (V, E) \). Similar to the techniques used by Hagerup et al. [30] we are able to store \( G' \) using \( O(n + m) \) bits. We store a subgraph \( G' \) of \( G \) via \( n \) choice dictionaries \( D_v \) for each vertex \( v \in V(G) \). Each \( D_v \) has length \( d_v \) with \( d_v \) being the degree of \( v \) in \( G \). The choice dictionaries are stored using static space allocation. A member \( i \) in \( D_v \) then indicates the existence of the \( i \)th arc out of \( v \)'s adjacency array. Another choice dictionary \( D' \) of length \( n \) can be used to mark vertices of degree \( > 0 \). Iterating over the neighborhood of a vertex \( v \in V(G') \) can be done in linear time of the degree of \( v \) in \( G' \). This additionally allows dynamic insertion and deletion of edges as long as \( G' \) remains a subgraph of \( G \), i.e., no new edges can be added. Note that we do not directly allow the deletion of vertices. Instead, when a vertex has degree 0 in \( G' \) (due to the deletion of all incident edges) it is marked in \( D' \). This allows to iterate over all vertices \( V' = \{v \in V(G')|d_v > 0\} \) in \( O(|V'|) \) time with \( d_v \) being the degree of \( v \) in \( G' \). Additionally, an arbitrary vertex \( v \in V(G') \) with \( d_v > 0 \) can be obtained via \( D'.\text{choice}() \). We refer to such a structure as a dynamic subgraph \( G' \) of \( G \). Note that a dynamic subgraph \( G' \) can be used to direct an edge \( \{u, v\} \in E(G') \) by deleting only the arc \( uv \) or \( vu \) in \( G' \). Using this we are able to implement the next lemma in \( O(n) \) time and \( O(n) \) bits using the same algorithm as described in [13].

Lemma 3. Let \( G \) be a separable graph. We can obtain the directed graph \( G' \) of \( G \) such that each vertex of \( G' \) has bounded in-degree (out-degree) in \( O(n) \) time and \( O(n) \) bits.

Proof. We implement a space-efficient algorithm of the algorithm outlined by Blandford et al. [13]. Note that for all separable graph classes there exist some constant \( b \) such that at least half of the vertices have degree \( \leq 2b \) [13] and each separable graph \( G = (V, E) \) has \( E = O(|V|) \) [46]. The algorithm due to Blandford et al. takes as input a separable graph \( G = (V, E) \), obtains the vertices \( V' \subseteq V \) such that each \( v \in V' \) has degree \( \leq 2b \). All edges of \( E \) with an endpoint in \( V' \) are then directed towards \( V' \). Then all vertices of \( V' \) are removed, and the process is repeated until all edges are handled this way. The runtime is \( O(n) \) as \( > 1/2|V| \) vertices are removed in each step. We now describe the technical details of how to obtain such a directed graph in \( O(n) \) time and \( O(n) \) bits.

Let \( G \) be a separable graph and \( G_{n, \ell}, G' \) dynamic subgraphs of \( G \), both initially containing all edges of \( G \). Note that each undirected edge \( \{u, v\} \in E \) is represented by two arcs \( uv \) and
Then we use $G_h$ as a helping structure during the construction. As an additional structure we use a choice dictionary $Q$ which functions as a queue for vertices of degree $\leq 2b$. Iterate over all vertices $v$ of degree $> 0$ in $G_h$ using the choice dictionary $D'$ (Section 2). For each $v$ obtain the degree $d_v$ of $v$ by iterating over the adjacency list of $v$ in $G_h$. If $d_v \leq 2b$, then add $v$ to $Q$. Finding and adding all these vertices to $Q$ takes $O(n + m) = O(n)$ time. In $G'$ remove all arcs $uv$ with $u \in Q$ and $v \notin Q$. In the case that both $u$ and $v$ are in $Q$ remove the arc $uv$ with $u < v$. Removing all these arcs in $O(n)$ time. For each $v \in Q$ remove all incident edges from $G_h$ and continue the iteration in $G_h$ until all edges are handled. The runtime is the same as due to Blanford et al. while the use of dynamic subgraphs uses $O(n)$ bits.

3 Graph Coarsening Framework for Planar Graphs

In this section we outline a strategy for coarsening a planar graph $G = (V, E)$. The idea is to create a specific type of partition $\mathcal{P}$ of $V$, which we call cloud partition that induces a unique minor $F$ of $G$, defined later. We refer to each $C \in \mathcal{P}$ as a cloud. Thus, a cloud is a set of vertices. In the following we describe the exact specifications that define such a cloud partition. For each $C \in \mathcal{P}$, $C$ induces a connected subgraph of $G$ and $|C| \leq [c \log n]$ for an arbitrary, but fixed constant $c$. We differentiate between different types of clouds $C \in \mathcal{P}$, which we define after introducing some terminology. Let $C_1, C_2 \in \mathcal{P}$ with $C_1 \neq C_2$. We call an edge $\{u, v\} \in E$ a border edge exactly if $u \in C_1$ and $v \notin C_2$. We then refer to $C_1$ and $C_2$ as adjacent or neighbors and as incident to $\{u, v\}$. Furthermore, we call $C$ a big cloud if $|C| = [c \log n]$ and a small cloud otherwise. We call a small cloud $C$ a leaf cloud if $C$ is adjacent to one cloud, a bridge cloud if it is adjacent to two clouds and a critical cloud if it is adjacent to at least three clouds. We call two clouds $C_1, C_2$ adjacent to a bridge cloud $B$ connected by $B$. A cloud partition is created by starting with the following scheme: Initially, mark all vertices as unvisited. Run a BFS from an arbitrary unvisited vertex $v$. The BFS only traverses vertices marked unvisited. Each time an unvisited vertex is traversed, it is marked visited. The BFS runs until either $[c \log n]$ vertices are marked visited or no unvisited vertices can be reached by the BFS. In the first case a big cloud is found and in the second case a small cloud is found. Repeat this until no unvisited vertices remain, always starting a new BFS at an arbitrary unvisited vertex. Only a partition created in this way is referred to as a cloud partition. By fixing the search for unvisited vertices and the BFS algorithm, each graph has a fixed cloud partition. The next two observations are derived directly from the process of creating a cloud partition.

▶ Observation 4. Let $\mathcal{P}$ be an arbitrary cloud partition created for a graph $G$. Then no two small clouds $C_1, C_2 \in \mathcal{P}$ are adjacent.

▶ Observation 5. Let $\mathcal{P}$ be an arbitrary cloud partition created for a graph $G$. Since each big cloud has size $[c \log n]$ for a constant $c$, $\mathcal{P}$ contains at most $n/[c \log n]$ big clouds.

For planar graphs, the number of critical clouds can be bounded similarly. The proof is based on the fact that a planar graph has $O(n)$ edges.

▶ Lemma 6. Let $\mathcal{P}$ be a cloud partition created for a planar graph $G$ containing $k$ big clouds. Then $\mathcal{P}$ contains $O(k)$ critical clouds.

Proof. Consider the graph $F = (V', E')$ constructed by contracting the vertices of each cloud $C \in \mathcal{P}$ to a single vertex $v \in C$. The graph $F$ is a minor of $G$ with $|V'| = |\mathcal{P}|$ and each
vertex \( v \in V' \) represents a single cloud \( C_v \in \mathcal{P} \). Two vertices \( v, u \in V' \) are adjacent in \( F \) exactly if \( C_v \) and \( C_u \) are adjacent. For brevity’s sake we refer to vertices of \( F \) introduced for critical clouds as critical vertices, and vertices introduced for big clouds as big vertices. Assume that all vertices that are not big vertices or critical vertices are removed in vertex \( v \). 

Corollary 7. 

Assume that all vertices that are not big vertices or critical vertices are removed in vertex \( v \). 

Corollary 7. 

Let \( \mathcal{P} \) be a cloud partition created for a planar graph \( G \). Then \( \mathcal{P} \) contains \( O(n/\log n) \) critical clouds.

Note that for leaf or bridge clouds no such bound exists as a cloud partition can contain \( O(n) \) leaf and bridge clouds. Extreme examples include a cloud partition \( \mathcal{P} \) with one big cloud \( C \) and \( n - |C| \) leaf clouds, all adjacent to \( C \) and containing only one vertex. For bridge clouds a similar example partition \( \mathcal{P} \) can be constructed, with two big clouds \( C_1, C_2 \in \mathcal{P} \) and \( n - (|C_1| + |C_2|) \) bridge clouds, all adjacent to both \( C_1 \) and \( C_2 \).

Next we focus on the construction of a specific weighted minor \( F \) of \( G \) with weights \( w(v) \) assigned to each node \( v \in V(F) \). Intuitively, \( F \) represents a minor of \( G \) that is constructed by repeatedly contracting the vertices in one or more clouds, and the weights keep track of the number of vertices that have been contracted. We call such an \( F \) a structure-maintaining minor of \( G \), with the formal definition outlined shortly. As each such minor is constructed specifically for a cloud partition \( \mathcal{P} \) we say that \( F \) is induced by \( \mathcal{P} \). We now define the properties of such a minor and follow with a sketch of a construction. Let \( F \) be a structure-maintaining minor induced by a cloud partition \( \mathcal{P} \). Denote by \( \mathcal{C}_u \) the set of clouds contracted to a node \( u \in V(F) \). Each node \( u \in V(F) \) is assigned a weight \( w(u) = \sum_{C \in \mathcal{C}_u} |C| \). For each \( u \) one of the following two properties holds: (1) \( |\mathcal{C}_u| = 1 \), \( G[C] \) is connected for \( C \in \mathcal{C}_u \) and \( w(u) = [c \log n] \) for some fixed constant \( c \) or (2) each \( C \in \mathcal{C}_u \) is adjacent to exactly the same clouds, \( G[\bigcup_{C \in \mathcal{C}_u} C] \) contains \( |\mathcal{C}_u| \) connected components. Additionally, \( \{v, w\} \in E(F) \) exactly if there exist adjacent clouds \( C_v \) and \( C_w \) with \( C_v \in \mathcal{C}_v \) and \( C_w \in \mathcal{C}_w \).

We now outline the construction of \( F \). Initially, \( F = (V' = \emptyset, E' = \emptyset) \). For each cloud \( C \) that is big or critical, add a node \( v \) to \( V' \) with \( w(v) = |C| \). Add edges between \( u, v \in V' \) to \( E' \) exactly if the clouds \( u \) and \( v \) represent are adjacent. We call \( v \) a big node if it was added to \( V' \) for a big cloud and critical node if it was added for a critical cloud. For each pair of big clouds \( C_1, C_2 \in \mathcal{P} \) connected via one or more bridge clouds, let \( B_{C_1, C_2} \subseteq \mathcal{P} \) be the set of all such bridge clouds. Add a single node \( v \) to \( V' \) adjacent to the nodes \( u, w \) added to \( V' \) for \( C_1, C_2 \) and set \( w(v) = \sum_{B \in B_{C_1, C_2}} |B| \). We refer to such a node \( v \) as a meta-bridge node. For each big cloud \( C \in \mathcal{P} \) adjacent to one or more leaf clouds, denote by \( L_C \subseteq \mathcal{P} \) the set of all leaf clouds adjacent to \( C \). For each such \( C \) add a single node \( v \) to \( V' \) adjacent to the node \( u \) added to \( V' \) for \( C \) with \( w(v) = \sum_{L \in L_C} |L| \). We call such a node \( v \) a meta-leaf node.

Since we have only \( O(n/\log n) \) big nodes, we also have only \( O(n/\log n) \) critical, meta-bridge and meta-leaf nodes, expressed explicitly in the following lemma. Note that the weight
of a meta-bridge or meta-leaf node can be bounded only by $n$. We next bound the number of nodes and edges of $F$.

Lemma 8. Let $P$ be a cloud partition constructed for a planar graph $G$ and $F = (V', E')$ the structure-maintaining minor induced by $P$. Then $|V'| = O(n/\log n)$ and $O(|E'|) = O(|V'|)$.

Proof. As $F$ is a minor of $G$ it is planar as well and so it holds that $O(|E'|) = O(|V'|)$. From Obs. 5 we know that the number of big nodes in $F$ is $k = O(n/\log n)$. From the construction of meta-leaf nodes it directly follows that each meta-leaf node is adjacent to exactly one big node and vice-versa. Therefore there are at most $k$ meta-leaf nodes in $V'$. For meta-bridge nodes the proof follows analogous, consider two big nodes $u, v$ that are adjacent to the same meta-bridge node $w$. Then there is no meta-bridge node $w \neq w$ adjacent to both $u$ and $v$, as per the construction. By the fact that $F$ is planar it follows that there are at most $O(k) = O(n/\log n)$ meta-bridge nodes in $V'$. Putting it all together we arrive at $|V'| = O(n/\log n)$.

In the following we describe a data structure for a cloud partition and show how it can be constructed and stored in $O(n)$ time and $O(n)$ bits. We refer to this data structure as $cp$-structure. A $cp$-structure constructed for a planar connected graph $G$ admits the following operations:

- **type**($v$): Given a vertex $v$ outputs the type of cloud $v$ belongs to (big, small, critical, bridge or leaf) in $O(1)$ time.
- **cloud**($v$): Given a vertex $v$ returns all vertices of the cloud $C$ in which $v$ is contained in $O(|C|)$ time and $O(|C| \log n)$ space.
- **border**($v$, $k$): Outputs if the $k$-th arc out of $v$’s adjacency array is part of a border edge in $O(1)$ time, with $v \in V(G)$.

Additionally, the structure allows access to the subgraph $G'$ of $G$ induced by all non-border edges of $P$. The graph $G'$ admits adjacency array access to its vertices and edges. The following lemma describes the runtime and space usage of constructing a cloud partition and a $cp$-structure.

Lemma 9. Let $G$ be a planar connected graph. We can compute a cloud partition $P$ of $G$ and a $cp$-structure of $P$ in $O(n)$ time and $O(n)$ bits.

Proof. In the following we construct a variety of data structures that are combined to a final $cp$-structure. For each type of cloud, construct a bitvector of the same name of length $n$, i.e., **big**, **small**, **critical**, **bridge** or **leaf**. During the following procedure a bit at index $i$ will be set to 1 in the respective bitvector to indicate that the vertex $i \in V(G)$ is of the respective type, with all bits set to 0 initially. To track border edges of $P$ we maintain a dynamic subgraph $G'$ of $G$, initially containing all edges of $G$ as outlined in Section 2. A final bitvector $start$ of length $n$ is used during the construction, containing a 1 at index $i$ exactly if $i$ was the first vertex discovered in a cloud during the initial construction of the cloud partition $P$.

The first step of the construction is computing $P$ by finding all clouds. Starting at an arbitrary unvisited vertex $v$ set $start[v] = 1$ and run a BFS until $\lceil c \log n \rceil$ unvisited vertices are visited, or no unvisited vertices can be reached. Denote by $A$ the set of all vertices found during such a BFS. If $|A| = \lceil c \log n \rceil$ we have found a big cloud. In this case, iterate over the neighborhood of each vertex $u \in A$ and mark all edges that are incident to a vertex $w \notin A$ as border edge by removing it in $G'$. Additionally, for all vertices $u \in A$ set $big[u] = 1$. In
the case that $|A| < [c \log n]$ we have found a small cloud for which we set $\text{small}[u] = 1$. In both cases we start the same procedure again from an unvisited vertex $v$. If no such vertices remains, we have successfully found $\mathcal{P}$ and have constructed the bitvectors that allows us to differentiate between small and big clouds. Additionally, the graph $G'$ is exactly the subgraph of $G$ induced by non-border edges.

The next step is to construct the bitvectors for critical, bridge and leaf clouds. For each vertex $v$ that is a start vertex of a big cloud $C$, identifiable as $\text{big}[v] = 1$ and $\text{start}[v] = 1$, run a BFS in $G$. During this BFS explore all small clouds $S$ adjacent to $C$. If the vertices $w \in S$ are marked as part of a critical cloud during previous iterations of this step, identifiable via $\text{critical}[w] = 1$, ignore $S$ completely. I.e., at no point traverse edges that are already identified as leading to a critical cloud. Otherwise, for each $w \in S$ set $\text{leaf}[w] = 1$ during the first exploration of $S$, $\text{leaf}[w] = 0$ and $\text{bridge}[w] = 1$ during the second exploration and $\text{bridge}[w] = 0$ and $\text{critical}[w] = 1$ during the third. By this each small cloud and each border edge is accessed at most a constant number of time. Note that each big cloud is traversed once. Afterwards the exact type of cloud a vertex belongs to is accessible via the respective bitvectors in constant time.

The operation $\text{type}(v)$ is supported in constant time by checking if the bit at index $v$ is set to 1 in the respective bitvectors constructed for the cloud types in $O(1)$ time. The operation $\text{cloud}(v)$ can be accessed by running a BFS from $v$ in $G'$ until no vertex can be reached. As $G'$ contains no border edges it contains each cloud as a connected component. The runtime is linear in the number of vertices of the cloud $C$ the vertex $v$ is contained in. The space requirement in terms of bits of a BFS run in a cloud $C$ is $O(|C| \log n) = O((\log n)^2)$ as this equals the maximum size of the queue of the BFS. The operation $\text{border}(v, k)$ can be accessed by simply checking if the edge is present or not in $G'$.

Next we show how to construct a structure-maintaining minor $F$ induced by a cloud partition $\mathcal{P}$ of a graph $G$ in $O(n)$ time and $O(n)$ bits of space. By Lemma 8, $F$ can be stored in $O(n)$ bits by adjacency lists. We additionally store a bi-directional mapping from each big/critical cloud $C$ to the node $v \in V(F)$ added to $V(F)$ to represent $C$. This mapping is stored by a pointer at $v$ to a single vertex $v' \in C$ and a pointer at $v'$ to $v$. As there are $O(n/\log n)$ pointers to store for this bi-directional mapping we use standard pointers of size $\Theta(\log n)$. To store the pointers from the direction of a vertex $v \in V(G)$ we use static-space allocation. Note that the choice of the vertex to store these pointers (per cloud) is arbitrary, but should be fixed. We choose the vertex with the lowest label. For meta-bridge and meta-leaf nodes $v \in V(F)$ we store a pointer from $v$ to the lowest labeled vertex $v'$ amongst all clouds represented by the meta-bridge or meta-leaf node. The details are described in the proof of the next lemma. Next, we define a special operation on $F$.

\begin{itemize}
  \item $\text{expand}(v)$: Given a node $v \in V(F)$ with weight $w(v)$, first determine the cloud $C$ mapped to $v$ and then return iteratively all vertices part of the clouds $C \in \mathcal{C}$.
\end{itemize}

Later we make use of the $\text{expand}$ operation when translating solutions for problems such as finding separators from $F$ to a solution to $G$. The key difficulty is implementing the $\text{expand}$ operation for a meta bridge $v$. As the clouds represented by $v$ may induce up to $n - O(\log n)$ connected components in $G$ and we store only a pointer to the lowest labeled vertex $u \in V(G)$ amongst all such clouds represented by $v$, we are unable to simply output each cloud without additional information or we must traverse (in the worst case) the entire graph $G$. The idea is to store a spanning tree that spans all vertices in clouds represented by $v$. As mentioned, since these clouds each induces a connected component, a spanning tree with these clouds does not exist. Therefore, we create a spanning tree by additionally
using the vertices of one adjacent big cloud. Traversing all vertices represented by \( v \) can now be done by traversing the respective spanning tree. The key observation for this is that we can direct the edges of \( F \) in such a way that each vertex has up to \( c = O(1) \) outgoing edges (Lemma 3). By this we are able to store all spanning trees in \( O(n) \) bits due to the fact each big cloud is used only by \( c \) spanning trees. Concerning the problem of having \( c \) spanning trees that use the vertices of one big cloud, i.e., that overlap, our approach is to assign each spanning tree a color from a set of \( c \) colors. These spanning trees are then stored in \( c \) dynamic subgraphs, with each subgraph storing all spanning trees of one color.

**Lemma 10.** Let \( G \) be a connected planar graph and \( \mathcal{P} \) a cloud partition of \( G \) given as a cp-structure. We can construct the structure-maintaining minor \( F \) induced by \( \mathcal{P} \) in \( O(n) \) time and bits such that the expand operation on \( v \in V(F) \) runs in \( O(w(v) + \log n) \) time.

**Proof.** We construct \( F \) iteratively. Initially, we add all big and critical nodes \( v \) and the bi-directional mapping from \( v \) to one vertex \( v' \) in the cloud \( v \) represents. During this process, we maintain a bitvector to mark vertices of \( G \) as visited. An additional bitvector \( X \) is maintained for marking the lowest numbered vertex in each big or critical cloud. Starting at the lowest labeled vertex \( v \), obtain the cloud \( C \) with \( v \in C \) via the \( \mathcal{P}.\text{cloud}(v) \) operation. Mark all vertices of \( C \) as visited, add a node \( v' \) to \( V(F) \) and store a pointer at \( v' \) to \( v \) and store a pointer from \( v \) to \( v' \) in a standard linked list \( L \) and set \( X[v] = 1 \). Simply iterate this process until all big and critical nodes are added to \( V(F) \). Note that by this construction the lowest labeled vertex of a cloud is always the vertex for which the bi-directional mapping is constructed, as we iterate over the bitvector for visited vertices in order. Constructing an indexable dictionary for \( X \) and transforming \( L \) into an array \( \mathcal{P}.\text{cloud}(v) \) (in \( O(|L|) \) time and \( O(|L| \log n) \) bits) gives us exactly the static space allocation for pointers from vertices in \( V(G) \) to nodes in \( V(F) \).

Now we add all edges between big and critical nodes. We maintain two bitvectors \( \text{complete} \) and \( \text{discovered} \) for vertices, initially containing only 0 bits. In \( \text{complete} \) we mark processed big clouds and in \( \text{discovered} \) we mark clouds discovered from a big cloud during the iteration to ensure that these clouds are only visited once. Iteratively, start at an arbitrary vertex \( v \) that is not marked complete and is contained in a big or critical cloud. Obtain the cloud \( C \) via \( \mathcal{P}.\text{cloud}(v) \). Mark all vertices of \( C \) as complete. For each border edge \( \{u,w\} \) with \( u \in C \) and \( w \in \text{critical or big clouds} \) adjacent to \( C \) and for which \( w \) is not marked as discovered obtain \( C' \) via \( \mathcal{P}.\text{cloud}(w) \). Mark all vertices in \( C' \) as discovered and obtain the node \( w' \in V(F) \) that represents \( C' \) via the bi-directional mapping, stored for the lowest labeled vertex of \( C' \). Add the arc \( v'w' \) to the adjacency list of \( v' \). Note that the arc \( w'v' \) is added during a different iteration, i.e., when we start at \( C' \) and find \( C \) as an adjacent cloud. Once this process has been completed for all border edges incident to \( C \) mark all adjacent big or critical clouds as no longer discovered by an analogous process. The entire process is iterated until no vertices \( v \) in critical or big clouds remain that are not marked complete. The weights for each node \( v \) added to \( V(F) \) during this step can be determined and stored during the run by simply counting the vertices in the cloud that \( v \) represents. We so have added \( O(n/\log n) \) nodes and edges to \( F \) during this process. For each edge \( \{u,v\} \in E(F) \) (as the two arcs \( vw \) and \( vu \)) we have explored the clouds of each endpoint a constant number of times via the \( \text{cloud} \) operation, which runs in linear time. As each such cloud has size \( O(\log n) \) and \( |E(F)| = O(n/\log n) \) the overall runtime is \( O(n) \).

In the next step we add all meta-leaf nodes to \( F \). As in the previous step we maintain the same bitvectors and re-initialize them. Iteratively, start at an arbitrary vertex \( v \) that is not marked complete and is contained in a big cloud \( C \), obtainable via \( \mathcal{P}.\text{cloud}(v) \). For each border edge \( \{u,w\} \) with \( u \in C \) that is incident to a leaf cloud and for which \( w \) is not marked
as discovered obtain the cloud $C'$ adjacent to $C$ via $\mathcal{P}\text{.cloud}(w)$ and mark all vertices of $C$ as discovered. Once all border edges have been iterated this way, add a node $u$ to $V(F)$ incident to $v'$. The weight of $u$ is set to the number of vertices in all leaf clouds adjacent to $C$. Additionally, we store a pointer from $u$ to the lowest labeled vertex $v'$ amongst all leaf clouds which are represented by $u$. Repeat the entire process until no (unvisited vertices in) big clouds remain. As in the previous step, for each edge $\{u, v\}$ added to $E(F)$ we iterate over the clouds of $u$ and $v$ a constant number of time and therefore have a runtime of $O(n)$.

Finally, we need to add all meta-bridge nodes to $F$. Similar as to the previous steps we start at a big cloud $C$ and explore all neighboring bridge clouds $B$ and the big clouds connected to $C$ via the bridge clouds. Assume the same bitvectors as in the previous steps are available. We work on two dynamic subgraphs $G_a, G_b$ of $G$, with $G_a$ initially being a copy of $G$ and $G_b$ being the empty graph. In $G_a$ we remove border edges incident to vertices in bridge clouds for which we have already added a meta-bridge node to $F$ to avoid unnecessary explorations. During the process we start at an arbitrary big cloud $C$ not marked complete. We then construct $G_b$ as the graph induced by $C$, all bridge clouds $B$ adjacent to $C$ and all big clouds connected to $C$ via $B$. In $G_b$ we start at an arbitrary bridge cloud $B$ adjacent to $C$, remove the edges incident to vertices of $B$ and vertices of $C$, and traverse to the big cloud $C'$ connected to $C$ via $B$. From $C'$ we then explore all adjacent bridge clouds $B$, removing all traversed edges and all edges adjacent to vertices of $B$ in $G_b$ and $G_a$. Once finished, we add a meta-bridge node $v$ to $F$ and the edges $\{v, w\}$ and $\{v, u\}$ with $w$ being the node added to $V(F)$ for $C$ and $u$ being the node added to $V(F)$ for $C'$. The weight of $v$ is simply the number of vertices traversed in the bridge clouds represented by $v$. Additionally, we store a pointer from $v$ to the lowest labeled vertex $v'$ amongst all bridge clouds which are represented by $v$. Finally, we mark all vertices of $C$ as complete and repeat the process from the next big cloud not yet marked complete. For each meta-bridge node $v$ added to $V(F)$ we explore the two clouds $C, C'$ represented by the two big nodes adjacent to $v$ once. Additionally, we explore all vertices in bridge clouds represented by $v$ a constant number of time. As we add $O(n/\log n)$ meta-bridge nodes overall the runtime is $O(n)$.

It remains to show how to implement the expand operation. For big and critical nodes $v \in V(F)$ the expand operation is trivially solved by obtaining the vertex $v'$ mapped to $v$ and calling $\mathcal{P}\text{.cloud}(v')$. We therefore now focus only on meta-bridge and meta-leaf nodes. We first give a non-technical overview of our technique followed by a technical implementation and analysis. Denote by $C_u$ the set of clouds represented by a meta-leaf or meta-bridge node $u \in V(F)$. Intuitively, our goal is to create a tree $T_u$ spanning over all vertices in $\cup_{C \in C_u} C$. As the clouds in $C_u$ induce disconnected components in $G$, we must add vertices of one cloud $C_0$ to make $T_u$ a tree where $C_0 \in \mathcal{P}$ is one of the clouds adjacent to all $C \in C_u$ in $F$. Once such a spanning tree is available for all meta-bridge and meta-leaf nodes $u$, the expand operation can simply be achieved by traversing $T_u$ via a depth-first search. It is easy to see that the height of each $T_u$ is $O(\log n)$ since $F[C_0 \cup C_u]$ is a star so that we can easily execute a standard DFS and only need $O(n)$ bits.

For meta-leaf nodes storing these spanning trees can easily be achieved in a single dynamic subgraph $G_{\text{leaf}}$ as all $T_u$ are pairwise disjoint.

For meta-bridge nodes we make use of the fact that any separable graph can be directed in such a way that each vertex has at most in-degree $d$ for some graph class dependent constant $d$ as per Lemma 3. We first create a minor $F'$ of $F$ that contains only big nodes and an edge between two big nodes $u, v$ exactly if there is a meta-bridge node adjacent to both $u$ and $v$. Finally, we split each edge by adding the corresponding meta-bridge node in the middle of the edge and direct the two split edges in the direction of the original edge.
Recall that each \( T_u \) constructed for the bridge clouds \( C_u \) represented by some meta-bridge node \( u \) spans only one big cloud \( C_u \). We choose \( C_v \) as the cloud such that there is a directed edge \( uv \) in \( F' \) between the nodes \( u \) and \( v \) corresponding to \( C_u \) and \( C_v \), respectively. By this there are at most \( d \) meta-bridge nodes that use the same big cloud. Observe that all spanning trees that do not use the same big cloud are disjoint. From this, we are able to store all spanning trees in \( d \) dynamic subgraphs \( G_1, \ldots, G_d \). In intuitive terms, we can color all spanning trees with \( d \) colors such that any two spanning trees \( T, T' \) are disjoint if they are assigned the same color. The spanning trees can be constructed in intermediate computations while repeating the same process for adding the meta-bridge nodes \( v \) to \( V(F) \) and their incident edges. Analogously (and simpler) the same is true for meta-leaf nodes. For each meta-bridge node \( v \) we store a pointer to the one dynamic subgraph \( G_i \in G_1, \ldots, G_d \) in which \( T_u \) is contained. To traverse \( T_u \) we then obtain the vertex \( v' \in V(G) \) mapped to \( v \) and start a depth-first first search from \( v' \) in \( G_i \). For each \( T_u \) it holds \(|V(T_u)| = w(u) + O(\log n)\). Constructing each \( T_u \) can be done in \( O(w(u) + \log n) \) time, and there are \( O(n/\log n) \) spanning trees overall, as each node in \( F \) results in up to \( d = O(1) \) spanning trees and there are \( O(n/\log n) \) nodes in \( F \). As such, it takes \( O(n) \) time and uses \( O(n) \) bits.

We refer to the combined data structure of Lemma 9 and Lemma 10 as a cloud decomposition \((F, \mathcal{P})\) of \( G \).

4 Applications: Succinctly Encoded Planar Graphs

In this section we show how a cloud decomposition \((F, \mathcal{P})\) constructed for a planar graph \( G \) can be used to find a balanced separator of size \( O(\sqrt{n \log n}) \) in \( O(n/\log n) \) time and \( O(n) \) bits, which in turn can be used to construct a succinct encoding of planar graphs, described later. Afterwards we show how such a cloud decomposition can be used together with the search for balanced separators to compute a tree decomposition with width \( O(n^{1/2+\epsilon}) \) for any \( \epsilon > 0 \) of a planar graph \( G \). We make use of an \( O(n) \)-time \( O(n \log n) \)-bits algorithm for finding \( 2/3 \)-balanced separators of size \( O(\sqrt{n}) \) in weighted planar graphs as long as no vertex has a weight more than \( 1/3 \) times the total weight \([43, 47, 48]\). The next theorem shows how a balanced separator can be constructed for a planar graph with the help of a cloud decomposition. The idea is to run a slightly modified standard algorithm for finding balanced separators \( S \) on the structure-maintaining minor \( F \) and translating \( S \) to a separator \( S' \) of \( G \).

\[ \textbf{Theorem 11.} \text{Let } G = (V, E) \text{ be a planar graph and } (F = (V', E'), \mathcal{P}) \text{ a cloud decomposition of } G. \text{ We can construct a balanced separator } S \text{ of } G \text{ with size } O(\sqrt{n \log n}) \text{ in } O(n/\log n) \text{ time using } O(n) \text{ bits.} \]

\[ \textbf{Proof.} \text{Start to search for a separator } S \subseteq V' \text{ on the weighted graph } F \text{ in time } O(|V'|) = O(n/\log n) \text{ and } O(|V'| \log n) = O(n) \text{ bits. Denote by } S \subseteq \mathcal{P} \text{ the clouds of } \mathcal{P} \text{ represented by nodes of } S \text{ and by } w(S) \text{ the sum of the weights } w(v) \text{ of all } v \in S. \text{ We can construct a separator } S' \text{ for } G \text{ via } S' = \bigcup_{C \in S} C. \text{ We store } S' \text{ in a bitvector of length } n, \text{ initialized to all 0-bits in } O(n/\log n) \text{ time. For each } v \in S \text{ we obtain an iterator over the clouds } v \text{ represents via } F.\text{expand}(v) \text{ and simply add the vertices returned by the iterator to } S' \text{ in } O(w(s)) \text{ time. Note that } |S'| = O(|S| \log n) = O(\sqrt{n/\log n} \log n) = O(\sqrt{n \log n}). \]

A balanced separator \( S \) of \( F \) can only be found if no node \( v \in V' \) has weight larger than \( 1/3 \) times the total weight. Only meta-leaf or meta-bridge nodes can have such large weights. If such a node exists, we split it into up to three nodes by distributing its clouds as equally as possible to the three nodes and run the search on the modified graph. \[ \]
Due to Blelloch and Farzan [14] there exists a succinct encoding of an arbitrary separable graph $G = (V, E)$ that provides constant-time adjacency, degree and neighborhood queries. The encoding is constructed via recursive application of a separator algorithm such that $V$ is split into two sets $A$ and $B$ via a separator $S$. This recursion is continued for $G_a = G[A \cup S]$ and $G_b = G[B \cup S]$ until the remaining graphs are of size at most $\log^\delta n$ for a graph class specific constant $\delta$, which they refer to as mini graphs. In several papers, what is constructed here is referred to as a separator hierarchy. For technical reasons the edges between vertices of $S$ are only included in $G_a$ and not $G_b$. The graph $G$ is then encoded by a combination of these mini graphs, which in turn are further decomposed by the same recursive separator search into micro graphs, which are then small enough to be handled via lookup tables encoding every possible micro graph and in turn are used to encode the mini graphs. Constructing these mini and micro graphs can be done in $O(n)$ time for a planar graph $G$ using the techniques described by Blelloch and Farzan combined with the algorithm of Goodrich for constructing a separator hierarchy [27], but we want to use only $O(n)$ bits and maintain the runtime. The main result of [14] is summed up by the next theorem.

\textbf{Theorem 12 ([14])}. Any family of separable graphs with entropy $\mathcal{H}(n)$ can be succinctly encoded in $\mathcal{H}(n) + o(n)$ bits such that adjacency, neighborhood, and degree queries are supported in constant time.

One important point of the construction is the fact that the number of duplicate vertices in each mini graph is bounded. A duplicate is a vertex that is contained in more than one mini graph. A vertex becomes a duplicate any time it is part of a separator $S$, as it is then contained in both $G_a$ and $G_b$. The exact bound is defined by the next lemma.

\textbf{Lemma 13 ([14])}. The number of mini graphs is $\Theta(n/\log^\delta n)$. The total number of duplicates among mini graphs is $O(n/\log^2 n)$. The sum of number of vertices of mini graph together is $n + O(n/\log^2 n)$.

Our goal is to execute the recursive decomposition of the graph $G$ via a cloud decomposition $(F, \mathcal{P})$ by recursively searching for separators in $F$ until the entire weight of the remaining graphs is less than $\log^\delta n$. These small subgraphs of $F$ then are expanded to be exactly the mini graphs of $G$ we aim to find. For each mini graph a new cloud decomposition is constructed, and the recursive separator search is repeated for each mini graph to construct the micro graphs.

\textbf{Lemma 14}. Let $G$ be a planar graph and $(F, \mathcal{P})$ a cloud decomposition of $G$. We can output all mini graphs of $G$ in $O(n)$ time.

\textbf{Proof}. The idea is to search for a balanced separator $S$ of $F$ that partitions $V(F)$ into three parts $\{A, S, B\}$. Continue recursively for $F' = F[A \cup S]$ and $F'' = F[B \cup S]$ (removing edges between nodes of $S$ from one of these instances). Stop the recursion once $F'$ has an overall weight $< \log^\delta n$. The nodes $v \in V(F')$ are then expanded to their respective clouds $C \in \mathcal{P}$. The vertex induced subgraph of $G$ over the vertices in all these clouds $C$ is then a mini graph that we output. In the following, let $F'$ be the input graph of the current recursive call of our procedure. We denote by $w(v)$ the weight of a node and by $w(F')$ the total weight of all nodes of $V(F')$. Assume for now that $V(F')$ contains no node of weight $w(v) > 1/3w(F')$. In this case a balanced separator $S$ of $F'$ of weight $O(\sqrt{w(F')})$ can easily be found in $O(\sqrt{w(F')})$ time [43, 47, 48], and we can progress recursively as outlined. If we do not encounter a problem with nodes of too large weight at any point, the entire recursive algorithm takes $O(|V(F)|\log n) = O(n)$ time.
Analogous to the proof of Theorem 11 we need to consider the special case of a leaf and bridge node $v$ exceeding the weight limit of $\beta w(F')$ (other nodes can not exceed this limit). In this case the neighborhood $S$ of $v$ in $F'$ is a separator of $V(F')$ into $\{A, S, B\}$ with $A = \{v\}$ and $B = V(G) \setminus (A \cup S)$. Denote by $S$ the set of all clouds represented by nodes of $S$ and by $C_v$ the set of all clouds represented by $v$. Note that $S$ contains one cloud if $v$ is a meta-leaf node, and two clouds if $v$ is a meta-bridge node. We can now output mini graphs $G_m$ directly, with each $G_m$ being the vertex induced subgraph over $S$ and some $C \subseteq C_v$. We collect the vertices of each $G_m$ as follows. Start with a set $M = \bigcup_{S \in S} S$, i.e., the separator. Now iterate over $C \subseteq C_v$ and set $M = M \cup C$ if $|M \cup C| < \log^\delta n$. If the threshold is reached or the iteration is over, we have found a mini graph $G_m = G[M]$. In this case reset $M = \bigcup_{S \in S} S$ and continue. This entire process takes $O(|S| + |C_v|) = O(\log n + w(v)) = O(w(F'))$ time. The recursion now only has to continue for $F'[V(F') \setminus v]$, which contains less than $2/3$ times the total weight of $F'$.

In any case the recursion is stopped once the input graph has weight $< \log^\delta n$ at which point we have found a mini graph, which can be output by translating nodes of the input graph $F'$ to the respective clouds of $G$, which induce the mini graph $G_m$ that we want to output. Note that the vertices $M$ of a mini graph $G_m$ can easily be achieved in $O(|V(G_m)|)$ time by translating the nodes of $F'$ to their respective clouds in $O(\log^\delta n)$ time. For obtaining the edges between vertices of $M$ we need to create a graph $G_d$ which is a directed version of $G$ such that each vertex has bounded in-degree (Lemma 3). Using $G_d$ we can translate the vertices for each mini graph linear in the size of the mini graph. Note that the graph $G_d$ needs to be created only once as a global structure in $O(n)$ time and $O(n)$ bits by using a dynamic subgraph to store $G_d$. The overall runtime of all recursive calls is then $O(|V(F)| \log n) + O(n) = O((n / \log n) \log n) = O(n)$. ▶

The bound on the number of duplicates due to Lemma 13 can be upheld with some care.

\section*{Lemma 15.}
Let $G$ be a planar graph and $(F, \mathcal{P})$ a cloud decomposition of $G$. The total number of duplicate vertices among the mini graphs of the cloud decomposition is $O(n / \log n)$.

\textbf{Proof.} As mentioned by Blelloch et al. [14] it suffices if the separator that is found is a polylogarithmic approximation when encoding an arbitrary separable graph, i.e., if the graph that is to be encoded admits to an $O(n^c)$-separator theorem for some $c < 1$, it suffices to find a separator of size $O(n^c \log^k n) = O(n^c)$ for some $k > 1$ and some $c' > c$. When using the standard method of finding recursive separators, the size of each separator is dependent on the input graph of each recursive call to the separator algorithm. As an example consider an input graph $G'$ of size $\log^\delta n$, which is the largest graph that can occur as an input during the recursion (any smaller graph is a mini graph). When searching for a separator for $G'$ a separator of size $O((\log^k n)^c)$ is found for some $c < 1$. Consider our space-efficient separator search that operates on a cloud decomposition $(F, \mathcal{P})$ constructed for $G$, with a recursive separator search on $F$ instead of $G$. In this case, the largest graph is defined by the total weight of the input graph $F'$ of the recursion. If the total weight is $\log^k n$, then a separator $S$ of $F'$ of size $O((\log^{k-1} n)^c)$ with total weight $O((\log^{k-1} n)^c \log n)$ is found in $F'$. The key observation is that the log $n$ factor at the end is independent of the size of the input graph $F'$. As outlined, the vertices that are contained in a separator are exactly those that are duplicate vertices in the mini graphs of the encoding. By using our scheme to output the mini graphs increases the number of duplicate vertices by such a large factor that it invalidates Lemma 13. We therefore adjust the constant $\delta$ specific to the graph class—see below.

Intuitively, the encoding due to Blelloch and Farzan can encode arbitrary separable graphs. For these separable graphs, the $\delta$ values used can become arbitrarily large to
adhere to Lemma 13. Therefore, we are able to increase it. In Lemma 13 Blelloch and Farzan mention a constant $\delta$. In their publication they set $\delta = 1/(1-c)$ when encoding a graph $G$ that admits to an $O(n^c)$-separator theorem. For planar graphs $c = 1/2$ and therefore $\delta = 4$. When using a recursive separator algorithm until mini graphs of size $< \log^\delta n = \log^4 n$ are found, the number of total mini graphs is $O(n/\log^4 n)$, which is equal to the number of leaf nodes of the recursion tree. This tree has depth $O(\log n)$ and as it is binary, $O(n/\log^4 n)$ total nodes. The number of duplicate vertices can then be bounded by $O(n/\log^4 n \cdot (\log^4 n)^{1/2}) = O(n/\log^2 n)$ as per Blelloch and Farzan. When using our space-efficient approach we have to increase the $\delta$ value to adhere to this bound, as the number of duplicates is now bounded by $O(n/\log^4 n \cdot (\log^4 n)^{1/2} \log n)$ due to the constant $O(\log n)$ factor that is independent of the smaller input graphs of the recursive separator search and only dependent on the size of the first input graph $G$.

This would mean a total number of duplicate vertices among mini graphs of $O(n/\log n)$. To mitigate this, we replace $\delta$ with $\delta + 2$ to bound the number of duplicates by $O(n/\log^6 n \cdot (\log^4 n)^{1/2} \log n) = O(n/\log^2 n)$. Note that arbitrary large $\delta$ values can be chosen for the encoding as separable graphs can be encoded with arbitrarily small $c$ values [14].

The full encoding in $O(n)$ time can be done by using the algorithm of Lemma 14 to output all mini graphs. For every mini graph $G_m$ we again construct a cloud decomposition in linear time and use the algorithm of Lemma 14 to find the micro graphs of $G_m$. All micro graphs are encoded using the table lookup scheme outlined in [14]. All other structures needed for the encoding can be constructed in $O(n)$ time and $O(n)$ bits easily, as they are simple indexable dictionaries over vertices of the micro graphs and mini graphs or small lookup tables which can be initialized in time linear in their size, which is $O(n)$. From this description the next theorem follows.

► **Theorem 16.** A planar graph $G$ with entropy $\mathcal{H}(n)$ can be succinctly encoded in $\mathcal{H}(n) + o(n)$ bits such that adjacency, neighborhood, and degree queries are supported in constant time. The encoding can be constructed in $O(n)$ time using $O(n)$ bits.

## 5 Applications: Planar Tree Decompositions

We next present a simple modification of the recursive separator search of Lemma 14 using standard techniques to output a tree decomposition of $G$. Let $(F, \mathcal{P})$ be a cloud decomposition of $G$. Each balanced separator $S$ of $F$ induces a bag in a tree decomposition of $F$ via the following recursive relation. Let $F'$ be the input graph used in the recursive calls, initially $F' = F$. Additionally, maintain a set $X$ containing vertices contained in separators found in previous recursive calls, initially $X = \emptyset$. When a separator $S$ is found for $F$ that partitions $V(F)$ into three sets $\{A, S, B\}$ output the next bag of the tree decomposition of $F$ as $S \cup X$. Continue the recursion for the input $F' \setminus (S \cup A)$ and $X = (X \cup S) \cap A$ and the input $F' \setminus (S \cup B)$ and $X = (X \cup S) \cap B$. Note that this tree decomposition of $F$ has width $O(\sqrt{|V(F)|})$. This tree decomposition can easily be expanded to a tree decomposition of $G$ by expanding all vertices of a bag $B$ to their respective clouds. The width of this expanded tree decomposition is $O(\sqrt{|V(F)|} \log n) = O(n^{1/2+\epsilon})$ for any $\epsilon > 0$. Each bag can be output in $O(n^{1/2+\epsilon})$ time and there are $O(n)$ bags. This description allows the next corollary.

► **Corollary 17.** Let $G$ be a planar graph. We can compute and output the bags of a tree decomposition with width $O(n^{1/2+\epsilon})$ for any $\epsilon > 0$ in time linear in the size of the tree decomposition $O(n^{3/2+\epsilon})$ using $O(n)$ bits.
6 Generalizing to $H$-Minor-Free Graphs

To generalize the previous results to $H$-minor-free graphs from planar graphs we must generalize cloud partitions and their induced minors. Recall that a cloud partition $P$ for a planar graph $G$ contains $O(n/\log n)$ critical clouds. Critical clouds are exactly those of size $< c \log n$ and with $\geq 3$ adjacent big clouds. We define a $\phi$-critical cloud as a small cloud adjacent to $\geq \phi$ big clouds. The idea of the proof is the same as for the proof of Lemma 6, but now we have a bound $\phi$ that depends on the graph class. The existence of the bound $\phi$ is due to the fact that a separable graph $G = (V, E)$ has bound density $d$, i.e., $|E| \leq d|V|$ for some fixed constant $d$. In particular we show that setting $\phi$ as $d + 1$ gives us the desired properties outlined in the following.

$\blacktriangleright$ Lemma 18. Let $G$ be a $H$-minor-free family of graphs for some fixed graph $H$ and let $d$ be the maximal density of a graph in $G$. There exists a cloud partition $P$ for each graph $G \in G$ such that $P$ contains $O(n/\log n)$ $\phi$-critical clouds for $\phi = d + 1$.

Proof. Let $P$ be a cloud partition of a graph $G \in G$ and let $F = (V', E')$ be the graph constructed by contracting the vertices of each cloud $C \in P$ to a single vertex $v \in V'$. Two vertices $u, v \in V'$ are adjacent in $F$ exactly if $C_u$ and $C_v$ are adjacent. We refer to the vertices of $F$ introduced for $\phi$-critical vertices and vertices introduced for big clouds as big vertices. Recall that a $\phi$-critical cloud is a cloud of size $< c \log n$ for some chosen constant $c$ with $\geq \phi$ adjacent big clouds. Remove all vertices that are not big vertices or $\phi$-critical vertices and all edges incident to two big vertices from $F$. As $G$ is minor-closed and $F$ is a minor of $G$, $F$ must be part of $G$ as well. It follows that $|E'| \leq d|V'|$ with $d$ being the maximal density of any graph part of $G$. Denote by $h$ the number of $\phi$-critical vertices and $k$ the number of big vertices in $F$. As the degree of each $\phi$-critical vertex is $\geq \phi$ it holds that $|E'| \geq \phi h$. Using this we can bound the number of $\phi$-critical vertices in $F$ via $\phi h \leq d|V'| = dh + dk$ and thus $h \leq dk/(\phi - d)$. For $\phi = d + 1$ it then follows $h \leq dk$. As there are $O(n/\log n)$ big clouds in $P$ it follows that there are $O(n/\log n)$ $\phi$-critical clouds in $P$. $\blacktriangleright$

It remains to show how small clouds adjacent to $< \phi$ big clouds are to be handled. For planar graphs such clouds are exactly the bridge and leaf clouds. To generalize from planar graphs to $H$-minor-free graphs we must in turn generalize bridge clouds similarly to the generalization from critical to $\phi$-critical clouds. We call such generalized bridge clouds $\phi$-bridge clouds, which we define as small clouds adjacent to $< \phi$, but $> 2$ big clouds. Analogous to regular bridge clouds, there can be $O(n)$ such clouds in a cloud partition for an $H$-minor-free graph. We refer to cloud partitions with the additional labeling of clouds as $\phi$-critical and $\phi$-bridge as a generalized cloud partition $P$.

To construct the structure-maintaining minor $F$ induced by a generalized cloud partition $P$ we can use the same strategy for bridge clouds and leaf clouds. For $\phi$-critical clouds the strategy also remains the same as for critical clouds in regular cloud partitions. For $\phi$-bridge clouds we introduce a generalized version of the meta-bridge node to $F$, which we call $\phi$-meta-bridge node. Note that each $\phi$-meta-bridge node has degree $< \phi$ in $F$. For adding all $\phi$-meta bridge nodes we iterate over $i \in (3, \ldots, \phi - 1)$. For each $i$ we add all $\phi$-meta bridges with degree $i$ to $V(F)$. Adding all degree-$i$ $\phi$-meta bridge nodes $v$ takes $O(in)$ time as for each neighbor of $v$ the respective clouds must be explored. Thus, an overall time of $O(\phi^2 n)$ is needed during the construction. As described above Lemma 10, we store spanning trees for meta-bridge nodes and meta-leaf nodes. We store and construct the exact same spanning trees for the $\phi$-meta-bridge nodes. These spanning trees are stored in $c$ different
dynamic subgraphs, with \( c \) being some constant, as described in Lemma 3. Storing these dynamic subgraphs takes \( O(cn) = O(\phi n) \) bits of space.

**Lemma 19.** Let \( G \) be a \( H \)-minor-free graph for some fixed graph \( H \) with a generalized cloud partition \( \mathcal{P} \). We can construct the minor induced by \( \mathcal{P} \) in \( O(\phi^2 n) \) time and \( O(\phi n) \) bits with \( \phi \) being a graph class dependent constant.

Analogous for planar graphs, we call the combination of a generalized cloud partition \( \mathcal{P} \) and a minor induced by \( \mathcal{P} \) a \( \phi \)-cloud decomposition. This allows us to generalize Theorem 11, Theorem 12 and Corollary 17 to \( H \)-minor-free graphs using such generalized cloud partitions. For generalizing the proofs of these theorems and the corollary we only must handle the new case of \( \phi \)-meta-bridge nodes exceeding the weight threshold during the separator search, as all other cases work analogous. Recall that we must handle the case when a node \( v \in V(F) \) has too large of a weight, and thus no balanced separator can be found. A new additional case now arises when a \( \phi \)-meta-bridge node \( v \) exceeds the weight threshold which is handled exactly the same as meta-bridge and meta-leaf nodes. In detail, the neighborhood \( S \) of \( v \) in \( F \) is a separator that separates \( v \) from \( V(F) \setminus S \). Expanding \( S \) to a separator \( S' \) of \( G \) then separates all vertices in clouds represented by \( v \) from the rest of the graph. The balanced separator \( S' \) then contains \( O(\phi \log n) \) vertices. For the following theorems we make use of the linear time \( O(n^{2/3}) \)-separator theorem for \( H \)-minor-free graphs [41].

**Theorem 20.** Let \( G \) be a \( H \)-minor-free graph for some fixed graph \( H \), let \( \phi \) be a graph class dependent constant and \( (F, \mathcal{P}) \) a generalized cloud partition of \( G \). We can compute a balanced separator \( S \) of \( G \) with size \( O(n^{2/3 + \epsilon}) \) for any \( \epsilon > 0 \) in \( O(n/\log n) \) time using \( O(n) \) bits.

**Theorem 21.** A \( H \)-minor-free graph for some fixed graph \( H \) with entropy \( \mathcal{H}(n) \) can be succinctly encoded in \( \mathcal{H}(n) + o(n) \) bits such that adjacency, neighborhood, and degree queries are supported in constant time. The construction of the encoding takes \( O(\phi^2 n) \) time and uses \( O(\phi n) \) bits with \( \phi \) a graph class dependent constant.

**Corollary 22.** Let \( G \) be a \( H \)-minor-free graph for some fixed graph \( H \), \( \phi \) a graph class dependent constant and \( (F, \mathcal{P}) \) a generalized cloud partition of \( G \). We can compute a tree decomposition of \( G \) with width \( O(n^{2/3 + \epsilon}) \) for any \( \epsilon > 0 \) in \( O(n^{5/3 + \epsilon}) \) time using \( O(n) \) bits.

## 7 Experimental Analysis of Cloud Decompositions

In this section we present experimental results on the distribution of cloud types found via the cloud partition algorithm of Section 3. The source code of our implementation is available on GitHub [1]. Our implementation has been tested on randomly generated planar graphs, geometric graphs, and \( G(n, p) \) graphs, as well as real-world road networks, biological datasets, and social network graphs. For all graphs used in our experimental analysis we considered only the largest connected component, resulting in minor fluctuations in the number of vertices.

To generate random planar graphs, we first constructed random maximum planar graphs using a tool from Fuentes and Navarro [25] and then removed a random number of edges between 0 and \( 2n \) to obtain planar graphs having between \( n \) and \( 3n - 6 \) edges. The hyperbolic graphs and the \( G(n, p) \) graphs were generated using networkit [8]. For each graph type mentioned above we generated 100 graphs each with \( 10^5 \), \( 5 \cdot 10^6 \), and \( 10^7 \) vertices, respectively. The \( G(n, p) \) graphs were generated using \( 1/n \leq p \leq 20/n \). From the network repository [54], we obtained a dataset of 11 planar road networks, each with between 114599 and 221668...
vertices. For non-planar graphs, we tested 13 biological datasets, with between 827766 and 969582 vertices [7, 54] and 11 social media graphs with between 513969 and 4033137 vertices [54].

We begin by discussing our experimental results obtained for the random planar graphs. Since critical clouds are treated exactly like big clouds when coarsening the graph as described in Section 3, they do not require special attention. Therefore, instead of focussing on only small clouds when discussing planar graphs, we give special consideration to the exact cloud type, i.e., critical, leaf or bridge. Looking at the distribution of small clouds in random planar graphs, i.e., critical-, bridge- and leaf clouds, we see that among the small clouds, the fraction of critical clouds exceeds the fraction of leaf clouds in graphs with a density of at least 2.25, and the fraction of bridge clouds in graphs with a density of at least 2.35. Figure 1a shows the distribution among small clouds and Figure 2a shows the distribution big and small clouds. However, even when we encounter many leaf and bridge clouds, the total number of clouds in the random planar graphs is still only a small constant factor of the theoretical lower bound of $\lceil n/\log n \rceil$. On average the total number of clouds is 2.3 times of the theoretical minimum of $\lceil n/\log n \rceil$ clouds, at most the total number of clouds is about 3 times the theoretical lower bound.

![Figure 1](image1.png)

(a) Planar graphs with about 1M vertices. (b) Hyperbolic graphs.

**Figure 1** Distribution of critical, bridge and leaf clouds among small clouds in tested graph types.

Looking at the results obtained for hyperbolic graphs, we can see that the distribution of cloud types looks similar to the distribution in random planar graphs. Analogous to planar graphs, Figure 1b shows the distribution among small clouds and Figure 2b shows the distribution big and small clouds. In hyperbolic graphs with a density of 3 the amount of big clouds lies at around 20%. Around a density of 6.5 the amount of big clouds gets larger than the amount of small clouds. At a density of 9 the amount of big clouds lies at around 70%. The total amount of clouds is on average about 2 times the lower bound and at worst 3 times the lower bound.

A similar distribution can be seen for the $G(n, p)$ graphs (Figure 2c). At a density of 4 there are about 30% big clouds, at about a density of 7 there are as many big clouds as small clouds, and at a density of 10 there are about 60% big clouds. The total number of clouds in this graph type lies on average at about 2 times the lower bound, in the worst case about 4 times the lower bound. The biological datasets consist of about 90% big clouds (Figure 2d). The total number of clouds lies on average 13% and at most 18.1% above the
lower bound. The number of small clouds in the road networks (Figure 2e) varies greatly. We can see that 8 of the 11 test graphs have more small clouds than big clouds. Despite the small number of big clouds in some cases, the average total number of clouds is about 2 to 2.5 times the lower bound.

In contrast to the previously mentioned graph classes, the social media graphs behave quite differently, as we encounter only few big clouds (Figure 2f). Comparing these social media graphs with previously presented graphs of similar density, we can observe more than 45% big clouds among those other graphs, while most social media graphs of this density have less than 8% big clouds, excluding two outliers with 13% and 28% big clouds. Among the small clouds, leaf clouds are the most common, with a share of 54% - 78% as can be seen in Table 1. The total number of clouds is far from the theoretical lower bound of \(\lceil n / \log n \rceil\), with between 6 and 17 times this lower bound—a factor too large to consider a constant. We explain this behavior by the high variance of vertex degrees in these graphs, coupled with an extreme distribution of the edges. Most edges are incident to a very small set of vertices, resulting in few vertices with very high degrees, leaving the vast majority of vertices with low degrees. On average, 43% of the vertices in these graphs have a degree of 1. The graph in which this distribution is most extreme is soc-FourSquare, which has a median degree of 1 and a maximum degree of 106218. 51% of all vertices have degree 1, 16% of all vertices have degree 2. Only about the top 0.1% of nodes have a degree above 100. We presume that the large number of small clouds is due to this extreme distribution of the vertex degrees, with most vertices contributing very few edges.

To confirm this hypothesis, we randomly added vertices of degree 1 to random maximal planar graphs. When we add \(n\) such vertices, we get planar graphs with a density of 2. When computing a cloud decomposition on these graphs, we get roughly 87% bridge and leaf clouds. Planar graphs with density 2, but no extra degree-1 vertices, have only 48% bridge and leaf clouds. When we add \(n\) degree-2 vertices (by subdividing randomly selected edges) in random maximal planar graphs, we get graphs with a density of 2.5. These have about 77% bridge and leaf clouds while other planar graphs with a similar density have only about 31% bridge and leaf clouds. Even when adding only \(\lfloor \log n \rfloor\) vertices with degree 1, the share of bridge and leaf clouds lies at around 33% with a density of 2.9. Other planar graphs with this density have only about 16% bridge and leaf clouds. When adding \(\log n\) degree-2 vertices we get graphs with a density of 3 and roughly 12% bridge and leaf clouds. This is comparable to the number of bridge and leaf clouds found in other planar graphs of similar density.

In summary, small clouds are quite rare in most graph classes, especially in graphs with a high density and graph types where the degree distribution is fairly well-balanced. But even for graphs with a low density, such as the road networks and some randomly generated planar graphs and road networks, the highest number of clouds among our test graph types is 3 times the lower bound of \(\lceil n / \log n \rceil\). We conclude that in practice it may not be necessary to use the complex strategies that introduce meta-leaves and meta-bridges, as there may not be enough leaf and bridge clouds to justify it.

On the other hand, graphs with a high variance in vertex degrees perform much worse than graphs with a low variance in vertex degrees, and for these graphs specialized strategies may be of interest, e.g., for the social-media graphs. One such area of research would be to design preprocessing steps that low degree vertices in a problem-specific way. Consider the problem of finding separators. For vertices of degree 1, it is trivial to find separators since these are simply the sets of neighbors, and similar strategies can be used for degree 2, 3, ..., \(k\) up to some small constant \(k\). If one were able to preprocess all such vertices with degree
≤ k, it is feasible that social-media graphs shrink significantly, possibly by a non-constant factor. At this point, standard techniques may be applicable while remaining space efficient.

| Name                  | Big  | Small | Critical | Bridge | Leaf  |
|-----------------------|------|-------|----------|--------|-------|
| soc-livejournal       | 8.5% | 91.5% | 28.3%    | 18.2%  | 53.6% |
| soc-delicious         | 2.5% | 97.5% | 11.2%    | 14.1%  | 74.7% |
| soc-wiki-Talk-dir     | 0.2% | 99.8% | 9.1%     | 14.4%  | 76.5% |
| soc-orkut             | 28.3%| 71.7% | 72.7%    | 9.9%   | 17.4% |
| soc-flickr            | 4.9% | 95.1% | 12.8%    | 9.5%   | 77.8% |
| soc-pokec             | 12.6%| 87.4% | 48.7%    | 17.5%  | 33.8% |
| soc-digg              | 1.8% | 98.2% | 25.6%    | 16.5%  | 57.8% |
| soc-youtube-snap      | 2.5% | 97.5% | 15.2%    | 14.2%  | 70.6% |
| soc-FourSquare        | 3.5% | 96.5% | 14.0%    | 17.1%  | 68.9% |
| soc-flixster          | 0.9% | 99.1% | 20.7%    | 15.5%  | 63.7% |
| soc-lastfm             | 1.5% | 98.5% | 26.0%    | 18.3%  | 55.7% |
| *_M87117093           | 87.5%| 12.5% | 62.1%    | 13.8%  | 24.1% |
| *_M87128519           | 84.4%| 15.6% | 62.7%    | 14.3%  | 23.0% |
| *_M87127667           | 88.0%| 12.0% | 67.4%    | 11.2%  | 21.4% |
| *_M87102575           | 89.1%| 10.9% | 68.8%    | 11.6%  | 19.6% |
| *_M87126525           | 90.9%| 9.1%  | 68.7%    | 12.1%  | 19.2% |
| *_M87104300           | 84.9%| 15.1% | 63.7%    | 13.9%  | 22.4% |
| *_M87117515           | 83.5%| 16.5% | 59.1%    | 15.2%  | 25.7% |
| *_M87113679           | 82.6%| 17.4% | 61.2%    | 14.0%  | 24.8% |
| *_M87115834           | 83.2%| 16.8% | 60.8%    | 14.3%  | 24.9% |
| *_M87125089           | 80.8%| 19.2% | 56.8%    | 15.8%  | 27.5% |
| *_M87128519           | 84.4%| 15.6% | 62.7%    | 14.3%  | 23.0% |
| *_M87123456           | 80.7%| 19.3% | 59.7%    | 15.3%  | 25.1% |
| *_M87105849           | 85.8%| 14.2% | 66.1%    | 13.4%  | 20.5% |
| road-netherlands-osm  | 35.9%| 64.1% | 5.8%     | 42.8%  | 51.4% |
| road-usroads          | 56.2%| 43.8% | 10.1%    | 60.1%  | 29.8% |
| road-great-britain-osm| 36.2%| 63.8% | 2.6%     | 29.6%  | 67.8% |
| road-luxembourg-osm   | 54.7%| 45.3% | 3.2%     | 46.1%  | 50.7% |
| road-belgium-osm      | 43.8%| 56.2% | 5.5%     | 50.7%  | 43.8% |
| road-italy-omes       | 56.9%| 43.1% | 4.5%     | 49.6%  | 46.0% |
| road-usroads-48       | 56.2%| 43.8% | 10.1%    | 60.1%  | 29.8% |
| road-germany-osm      | 38.4%| 61.6% | 3.7%     | 38.2%  | 58.0% |
| road-euroroad         | 34.5%| 65.5% | 8.3%     | 37.6%  | 54.1% |
| road-roadNet-PA       | 31.6%| 68.4% | 10.2%    | 23.7%  | 66.1% |
| road-roadNet-CA       | 31.9%| 68.1% | 10.1%    | 24.7%  | 65.2% |

**Table 1** Analyzed social networks, biological datasets and road networks. The share of big and small clouds is shown in relation to all clouds, the share of critical, bridge and leaf clouds in relation to all small clouds.
Figure 2 Distribution of big and small clouds in tested graph types.

(a) Planar graphs with about 10M vertices,

(b) Hyperbolic graphs with about 10M vertices.

(c) G(n, p) graphs with about 10M vertices.

(d) Biological datasets.

(e) Road networks.

(f) Social-media graphs.
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