On some Chern-Simons forms of the Bott-Shulman-Stasheff forms

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Abstract

We exhibit the Chern-Simons forms of some characteristic classes in the simplicial de Rham complex.

1 Introduction

In the framework of differential geometry on the simplicial manifold, the author exhibited some cocycles in $\Omega^*(NG(*))$ which represent classical characteristic classes of the universal bundle $EG \to BG$\cite{9}\cite{10} on the basis of Dupont’s work\cite{5}. Here $NG$ is a simplicial manifold called nerve of $G$ and it is well-known that the cohomology ring of $\Omega^*(NG(*))$ is isomorphic to $H^*(BG)$ for any Lie group $G$. These cocycles in $\Omega^*(NG(*))$ are called the Bott-Shulman-Stasheff forms.

On the other hand, there is a simplicial manifold $PG$ which play the role of $EG$. Since $H^*(EG)$ is trivial, any cocycle in $\Omega^*(PG)$ is exact. So if we pullback the Bott-Shulman-Stasheff form to $\Omega^*(PG)$, there exists a cochain $\Omega^{*-1}(PG)$ which hits the cocycle by a coboundary operator. These forms can be called the Chen-Simons forms of the BSS forms.

In this paper, we exhibit the Chen-Simons forms of the BSS forms which represent some classical characteristic classes.

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2 Review of the BSS forms

In this section we recall the universal Chern-Weil theory following [6]. For any Lie group $G$, we define simplicial manifolds $NG$, $PG$ and simplicial $G$-bundle $\gamma : PG \to NG$ as follows:

$$NG(q) = \underbrace{G \times \cdots \times G}_{q\text{-times}} \ni (h_1, \cdots, h_q) :$$
face operators $\varepsilon_i : NG(q) \to NG(q-1)$

$$\varepsilon_i(h_1, \cdots, h_q) = \begin{cases} (h_2, \cdots, h_q) & i = 0 \\ (h_1, \cdots, h_i h_{i+1}, \cdots, h_q) & i = 1, \cdots, q-1 \\ (h_1, \cdots, h_{q-1}) & i = q. \end{cases}$$

$$PG(q) = \underbrace{G \times \cdots \times G}_{q+1\text{-times}} \ni (g_0, \cdots, g_q) :$$
face operators $\bar{\varepsilon}_i : PG(q) \to PG(q-1)$

$$\bar{\varepsilon}_i(g_0, \cdots, g_q) = (g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_q) \quad i = 0, 1, \cdots, q.$$  

We define $\gamma : PG \to NG$ as $\gamma(g_0, \cdots, g_q) = (g_0 g_1^{-1}, \cdots, g_{q-1} g_q^{-1})$ then $\| \gamma \|$ is a model of the universal bundle $EG \to BG$ [8].

There is a double complex associated to a simplicial manifold.

**Definition 2.1.** For any simplicial manifold $\{X_\ast\}$ with face operators $\{\varepsilon_\ast\}$, we define a double complex as follows:

$$\Omega^{p,q}(X) := \Omega^q(X_p).$$

Derivatives are:

$$d' := \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*, \quad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).$$

For $NG$ and $PG$ the following holds [2] [6] [7].

**Theorem 2.1.** There exist ring isomorphisms

$$H(\Omega^*(NG)) \cong H^*(BG), \quad H(\Omega^*(PG)) \cong H^*(EG).$$

Here $\Omega^*(NG)$ and $\Omega^*(PG)$ mean the total complexes.
There is another double complex associated to a simplicial manifold.

**Definition 2.2** ([5]). A simplicial $n$-form on a simplicial manifold $\{X_p\}$ is a sequence $\{\phi^{(p)}\}$ of $n$-forms $\phi^{(p)}$ on $\Delta^p \times X_p$ such that

$$(\varepsilon^i \times id)^* \phi^{(p)} = (id \times \varepsilon_i)^* \phi^{(p-1)}.$$  

Here $\varepsilon^i$ is the canonical $i$-th face operator of $\Delta^p$.

Let $A^{k,l}(X)$ be the set of all simplicial $(k+l)$-forms on $\Delta^p \times X_p$ which are expressed locally of the form

$$\sum a_{i_1 \cdots i_k j_1 \cdots j_l} (dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l})$$

where $(t_0, t_1, \cdots, t_p)$ are the barycentric coordinates in $\Delta^p$ and $x_j$ are the local coordinates in $X_p$. We define its derivatives as:

$$d':= \text{the exterior differential on } \Delta^p,$$

$$d'':= (-1)^k \times \text{the exterior differential on } X_p.$$

Then $(A^{k,l}(X), d', d'')$ is a double complex.

**Theorem 2.2** ([5]). Let $A^*(X)$ denote the total complex of $A^{*,*}(X)$. A map

$I_\Delta(\alpha) := \int_{\Delta^p}(\alpha|_{\Delta^p \times X_p})$ induces a natural ring isomorphism $I_\Delta^* : H(A^*(X)) \cong H(\Omega^*(X))$.

Let $G$ denote the Lie algebra of $G$. A connection on a simplicial $G$-bundle $\pi : \{E_p\} \rightarrow \{M_p\}$ is a sequence of 1-forms $\{\theta\}$ on $\{E_p\}$ with coefficients $G$ such that $\theta$ restricted to $\Delta^p \times E_p$ is a usual connection form on a principal $G$-bundle $\Delta^p \times E_p \rightarrow \Delta^p \times M_p$. There is a canonical connection $\theta \in A^1(PG)$ on $\gamma : PG \rightarrow NG$ defined as $\theta|_{\Delta^p \times NG(p)} := t_0 \theta_0 + \cdots + t_p \theta_p$. Here $\theta_i$ is defined by $\theta_i = \text{pr}_i^* \theta$ where $\text{pr}_i : \Delta^p \times PG(p) \rightarrow G$ is the projection into the $i$-th factor of $PG(p)$ and $\theta$ is the Maurer-Cartan form of $G$. We also obtain its curvature $\Omega \in A^2(PG)$ on $\gamma$ as: $\Omega|_{\Delta^p \times PG(p)} = d\theta|_{\Delta^p \times PG(p)} + \frac{1}{2} [\theta|_{\Delta^p \times PG(p)}, \theta|_{\Delta^p \times PG(p)}]$.

Let $I^*(G)$ denote the ring of $G$-invariant polynomials on $G$. For $P \in I^*(G)$, we restrict $P(\Omega) \in A^*(PG)$ to each $\Delta^p \times PG(p) \rightarrow \Delta^p \times NG(p)$ and apply the usual Chern-Weil theory then we have a simplicial $2k$-form $P(\Omega)$ on $NG$. 

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Now we have the universal Chern-Weil homomorphism $w : I^*(G) \rightarrow H^*(NG)$ which maps $P \in I^*(G)$ to $w(P) = [I_\Delta(P)]$. The images of this homomorphism in $H^*(NG(\ast))$ are called the Bott-Shulman-Stasheff (BSS) forms.

**Example 2.1.** In the case that $G = U(n)$, the BSS form which represents the $p$-th power of the first Chern class $c_1^p (= \text{ch}_1^p) \in H^{2p}(BU(n))$ is given as follows:

$$
\left(\frac{1}{2\pi i}\right)^p (-1)^{\frac{p(p-1)}{2}} \text{tr}(h_1^{-1}dh_1)\text{tr}(h_2^{-1}dh_2) \cdots \text{tr}(h_p^{-1}dh_p) \in \Omega^p(NG(p)).
$$

**Remark 2.1.** The cocycle in the example 2.1 is given as the image of $(\text{tr} \left(-X \over 2\pi i\right))^p$ under the universal Chern-Weil homomorphism. On the other hand, there is a product $\cup$ on $\Omega^*(NG)$ defined as follows:

$$
\alpha \cup \beta = (-1)^{qr} \varepsilon_{s+t}^{\ast} \cdots \varepsilon_{s+1}^{\ast} \alpha \wedge \varepsilon_0^{\ast} \cdots \varepsilon_0^{\ast} \beta, \quad (\alpha \in \Omega^q(NG(s)), \beta \in \Omega^r(NG(t))).
$$

We can see $c_1 \cup \cdots \cup c_1$ coincides with the cocycle in this example as a cochain.

**Example 2.2.** In the case that $G = U(n)$, the BSS form which represents the 3rd Chern class $c_3 (= 2\text{ch}_3 - \text{ch}_2\text{ch}_1 + \frac{1}{6}\text{ch}_1^3) \in H^6(BU(n))$ in $\Omega^6(NG)$ is the sum of the following $c_{1,5}$, $c_{2,4}$ and $c_{3,3}$:

$$
c_{1,5} = \frac{1}{6} \left(\frac{1}{2\pi i}\right)^3 \frac{1}{5} \text{tr}(h^{-1}dh)^5,
$$
\[ c_{2,4} = \frac{-1}{6} \left( \frac{1}{2\pi i} \right)^3 (\text{tr}(dh_1^{-1}h_1^{-1}dh_1^{-1}dh_2^{-1}h_1^{-1}) \]
\[ \quad + \frac{1}{2} \text{tr}(dh_1dh_2^{-1}h_1^{-1}dh_1dh_2^{-1}h_1^{-1}) \]
\[ \quad + \text{tr}(dh_1dh_2^{-1}dh_2^{-1}dh_2^{-1}h_1^{-1})) \]
\[ - \frac{1}{12} \left( \frac{1}{2\pi i} \right)^3 (\text{tr}(h_2^{-1}dh_2^{-1}dh_2^{-1}dh_2 + dh_1dh_2^{-1}dh_2^{-1}h_1^{-1}) \]
\[ \quad + dh_1h_1^{-1}dh_1dh_2^{-1}h_1^{-1})\text{tr}(h_1^{-1}dh_1) \]
\[ - \text{tr}(h_1^{-1}dh_1^{-1}dh_1^{-1}dh_1 + dh_1h_1^{-1}dh_1dh_2^{-1}h_1^{-1}) \]
\[ \quad + dh_1dh_2^{-1}dh_2^{-1}h_1^{-1})\text{tr}(h_2^{-1}dh_2)), \]

\[ c_{3,3} = \frac{-1}{6} \left( \frac{1}{2\pi i} \right)^3 (\text{tr}(dh_1dh_2dh_3^{-1}h_2^{-1}h_1^{-1}) \]
\[ \quad - \text{tr}(dh_1dh_2dh_3^{-1}h_2^{-1}dh_2^{-1}h_1^{-1})) \]
\[ + \frac{1}{6} \left( \frac{1}{2\pi i} \right)^3 (\text{tr}(dh_1dh_2^{-1}h_1^{-1})\text{tr}(h_3^{-1}dh_3) + \text{tr}(dh_2dh_3h_3^{-1}h_2^{-1})\text{tr}(h_1^{-1}dh_1) \]
\[ \quad - \text{tr}(dh_1dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1})\text{tr}(h_2^{-1}dh_2)) \]
\[ - \frac{1}{6} \left( \frac{1}{2\pi i} \right)^3 \text{tr}(h_1^{-1}dh_1)\text{tr}(h_2^{-1}dh_2)\text{tr}(h_3^{-1}dh_3). \]

Remark 2.2. The cocycle in the example 2.2 is given as the image of \( \frac{1}{3} \text{tr} \left( \left( -\frac{X}{2\pi i} \right)^3 \right) - \)
\[ \frac{1}{2} \text{tr} \left( \left( -\frac{X}{2\pi i} \right)^2 \right) \text{tr}(\frac{X}{2\pi i}) + \frac{1}{6} (\text{tr}(\frac{X}{2\pi i}))^3 \] under the universal Chern-Weil homomorphism. Unfortunately, the cocycle \( 2\text{ch}_3 - \text{ch}_2 \cup \text{ch}_1 + \frac{1}{6} \text{ch}_1 \cup \text{ch}_1 \cup \text{ch}_1 \) does not coincide with the one in this example as a cochain.
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Since $H^*(EG)$ is trivial, any cocycle in $\Omega^*(PG)$ is exact. So if we pullback the Bott-Shulman-Stasheff form to $\Omega^*(PG)$, there exists a cochain $\Omega^{*-1}(PG)$ which hits the cocycle by a coboundary operator. These forms can be called the Chen-Simons forms of the BSS forms.

In this section, we exhibit the Chen-Simons forms of the BSS forms which represent some classical characteristic classes.

3.1 The Chern-Simons form of the universal torus bundle

In this subsection we exhibit the Chern-Simons form of the Chern characters and Chern classes of the universal torus bundle.

We write the $s$-th factor of $PT^n(p)$ as:

$$g_s = \begin{pmatrix} \exp(i\theta_1^s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \exp(i\theta_n^s) \end{pmatrix} \in T^n, \quad (s = 0, \ldots, p).$$

**Theorem 3.1** ([11]). The cocycle $\bar{\omega}_p$ in $\Omega^p(PT^n(p))$ which corresponds to the $p$-th Chern character of the universal torus bundle is given as follows:

$$\bar{\omega}_p = (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!} \left( \frac{1}{2\pi} \right)^p \sum_{k=1}^{n} \left( \prod_{s=0}^{p-1} (d\theta_k^s - d\theta_k^{s+1}) \right).$$

**Theorem 3.2.** The Chern-Simons form $T\bar{\omega}_{p-1}$ of the $p$-th Chern character of the universal torus bundle in $\Omega^p(PT^n(p-1))$ is given as follows:

$$T\bar{\omega}_{p-1} = (-1)^{\frac{p(p+1)}{2}} \frac{1}{p!} \left( \frac{1}{2\pi} \right)^p \sum_{k=1}^{n} (d\theta_k^0 \wedge d\theta_k^1 \wedge \cdots \wedge d\theta_k^{p-1}).$$

**Proof.**

$$\prod_{s=0}^{p-1} (d\theta_k^s - d\theta_k^{s+1}) = \sum_{j=0}^{p} (-1)^{p-j} d\theta_k^0 \wedge \cdots \wedge d\theta_k^{j-1} \wedge d\theta_k^{j+1} \wedge \cdots \wedge d\theta_k^{p} $$

$$= (-1)^{p} \sum_{j=0}^{p} \varepsilon_j^s (d\theta_k^0 \wedge d\theta_k^1 \wedge \cdots \wedge d\theta_k^{p-1}).$$

So we can see that $(d' + d'')T\bar{\omega}_{p-1} = \bar{\omega}_p$. \qed
Theorem 3.3 (III). The cocycle $\tilde{\mu}_p$ in $\Omega^p(PT^n(p))$ which corresponds to the $p$-th Chern class of the universal torus bundle is given as follows:

$$\tilde{\mu}_p = (-1)^{\frac{p(p-1)}{2}} \frac{1}{p!} \left( \frac{1}{2\pi} \right)^p \sum_{1 \leq k_1 < \cdots < k_p \leq n} \det (d\theta_{k_1}^s - d\theta_{k_1}^{s+1})_{0 \leq s, t \leq p-1}.$$ 

Theorem 3.4. The Chern-Simons form $T\tilde{\mu}_{p-1}$ of the $p$-th Chern class of the universal torus bundle in $\Omega^p(PT^n(p-1))$ is given as follows:

$$T\tilde{\mu}_{p-1} = (-1)^{\frac{p(p+1)}{2}} \frac{1}{p!} \left( \frac{1}{2\pi} \right)^p \sum_{1 \leq k_1 < \cdots < k_p \leq n} \det (d\theta_{k_1}^s)_{0 \leq s, t \leq p-1}.$$ 

Proof. We write:

$$d\theta^j = \begin{pmatrix} d\theta_{k_0}^j \\ \vdots \\ d\theta_{k_{p-1}}^j \end{pmatrix}, \quad (j = 0, \cdots, p-1),$$

$$(d\theta_{k_1}^s)_{0 \leq s, t \leq p-1} = (d\theta^0 \ d\theta^1 \cdots d\theta^{p-1}).$$

Then the following equations hold.

$$\varepsilon_0 \det (d\theta^0 \ d\theta^1 \cdots d\theta^{p-1}) - \varepsilon_1 \det (d\theta^0 \ d\theta^1 \cdots d\theta^{p-1})$$

$$= \det (d\theta^1 \ d\theta^2 \cdots d\theta^{p-1}) - \det (d\theta^0 \ d\theta^2 \cdots d\theta^{p-1})$$

$$= \det ((d\theta^1 - d\theta^0) \ d\theta^2 \cdots d\theta^{p}) + \varepsilon_2 \det (d\theta^0 \ d\theta^1 \cdots d\theta^{p-1})$$

$$= \det ((d\theta^1 - d\theta^0) \ d\theta^2 \ d\theta^3 \cdots d\theta^{p}) + \det (d\theta^0 \ d\theta^1 \ d\theta^3 \cdots d\theta^{p-1})$$

$$= \det ((d\theta^1 - d\theta^0) \ d\theta^2) (d\theta^3 \cdots d\theta^{p-1})$$

Repeating this argument, we can see that $(d' + d'') T\tilde{\mu}_{p-1} = \tilde{\mu}_p$. \qed

Remark 3.1. In the case that $n$ is equal to $p$, the cochain in theorem 3.4 is written as follows:

$$T\tilde{\mu}_{p-1} = (-1)^{\frac{p(p+1)}{2}} \frac{1}{p!} \left( \frac{1}{2\pi} \right)^p \det (d\theta_{k_1}^s)_{0 \leq s, t \leq p-1}.$$
3.2 The Chern-Simons form of the 3rd Chern character

In this subsection we exhibit the Chern-Simons form of the 3rd Chern character in $\Omega^5(PG)$. Throughout this subsection, $G = GL(n; \mathbb{C})$.

We first recall the cocycle in $\Omega^{p+q}(PG(p-q))(0 \leq q \leq p-1)$ which corresponds to the $p$-th Chern character.

**Theorem 3.5** ([9]). We set:

$$\bar{S}_{p-q} = \sum_{\sigma \in \Theta_{p-q-1}} (\text{sgn}(\sigma))(\theta_{\sigma(1)} - \theta_{\sigma(1)+1}) \cdots (\theta_{\sigma(p-q-1)} - \theta_{\sigma(p-q-1)+1}).$$

Then the cocycle in $\Omega^{p+q}(PG(p-q))(0 \leq q \leq p-1)$ which corresponds to the $p$-th Chern character $\text{ch}_p$ is

$$\frac{1}{p!} \left( \frac{1}{2\pi i} \right)^p (-1)^{(p-q)(p-q-1)/2} \times$$

$$\text{tr} \sum \left( (p(\theta_0 - \theta_1)) \wedge \bar{H}_q(\bar{S}_{p-q}) \times \int_{\Delta_{p-q}} \prod_{i<j} (t_it_j)^{a_{ij}(\bar{H}_q(\bar{S}_{p-q}))} dt_1 \wedge \cdots \wedge dt_{p-q} \right).$$

Here $\bar{H}_q(\bar{S}_{p-q})$ means the terms that $(\theta_i - \theta_j)^2$ $(0 \leq i < j \leq p-q)$ are put $q$ times between $(\theta_{k-1} - \theta_k)$ and $(\theta_l - \theta_{l+1})$ in $\bar{S}_{p-q}$ permitting overlaps; $a_{ij}(\bar{H}_q(\bar{S}_{p-q}))$ means the number of $(\theta_i - \theta_j)^2$ in it. $\sum$ means the sum of all such terms.

As a corollary of this theorem, we obtain the cocycle which corresponds to the 3rd Chern character in $\Omega^6(PG)$.

**Corollary 3.1.** The cocycle which corresponds to the 3rd Chern character
in $\Omega^6(PG)$ is the sum of the following $\bar{C}_{1,5}, \bar{C}_{2,4}$ and $\bar{C}_{3,3}$:

\[
\begin{align*}
\bar{C}_{1,5} \in \Omega^5(PG(1)) & \xrightarrow{d''} \Omega^5(PG(2)) \\
\bar{C}_{2,4} \in \Omega^4(PG(2)) & \xrightarrow{d''} \Omega^4(PG(3)) \\
\bar{C}_{3,3} \in \Omega^3(PG(3)) & \xrightarrow{d''} 0 \\
\end{align*}
\]

\[
\begin{align*}
\bar{C}_{1,5} &= \frac{1}{3!} \left( \frac{1}{2\pi i} \right)^3 \frac{1}{10} \text{tr}((\theta_0 - \theta_1)^5, \\
\bar{C}_{2,4} &= -\frac{1}{3!} \left( \frac{1}{2\pi i} \right)^3 \left( \frac{1}{2} \text{tr}((\theta_0 - \theta_1)^3(\theta_1 - \theta_2) \\
&\quad+ \frac{1}{4} \text{tr}((\theta_0 - \theta_1)(\theta_1 - \theta_2)(\theta_0 - \theta_1)(\theta_1 - \theta_2) \\
&\quad+ \frac{1}{2} \text{tr}((\theta_0 - \theta_1)(\theta_1 - \theta_2)^3), \\
\bar{C}_{3,3} &= -\frac{1}{3!} \left( \frac{1}{2\pi i} \right)^3 \left( \frac{1}{2} \text{tr}((\theta_0 - \theta_1)(\theta_1 - \theta_2)(\theta_2 - \theta_3) \\
&\quad- \frac{1}{2} \text{tr}((\theta_0 - \theta_1)(\theta_2 - \theta_3)(\theta_1 - \theta_2)).
\end{align*}
\]

**Theorem 3.6.** The Chern-Simons form of the 3rd Chern character in $\Omega^5(PG)$
is the sum of the following $TC_{0,5}, TC_{1,4}$ and $TC_{2,3}$:

\[
\begin{align*}
0 \\
\uparrow d'' = d \\
TC_{0,5} \in \Omega^5(G) \xrightarrow{d'} \tilde{C}_{1,5} \\
\uparrow d'' = -d \\
TC_{1,4} \in \Omega^4(PG(1)) \xrightarrow{d'} \tilde{C}_{2,4} \\
\uparrow d'' = d \\
TC_{2,3} \in \Omega^3(PG(2)) \xrightarrow{d'} \tilde{C}_{3,3}
\end{align*}
\]

\[
TC_{0,5} = \frac{-1}{3!} \left( \frac{1}{2 \pi i} \right)^3 \frac{1}{10} \text{tr}(\theta_0^5),
\]

\[
TC_{1,4} = \frac{-1}{3!} \left( \frac{1}{2 \pi i} \right)^3 \left( \frac{1}{2} \text{tr}(\theta_0^3\theta_1) - \frac{1}{4} \text{tr}(\theta_0\theta_1\theta_0\theta_1) + \frac{1}{2} \text{tr}(\theta_0\theta_1^3) \right),
\]

\[
TC_{2,3} = \frac{1}{3!} \left( \frac{1}{2 \pi i} \right)^3 \left( \frac{1}{2} \text{tr}(\theta_0\theta_1\theta_2) - \frac{1}{2} \text{tr}(\theta_0\theta_2\theta_1) \right).
\]

**Proof.**

\[
d'(\text{tr}(\theta_0^5)) = \text{tr}(\theta_1^5 - \theta_0^5),
\]

\[
d'(\frac{1}{2} \text{tr}(\theta_0^3\theta_1) - \frac{1}{4} \text{tr}(\theta_0\theta_1\theta_0\theta_1) + \frac{1}{2} \text{tr}(\theta_0\theta_1^3))
\]

\[
= -\frac{1}{2} \text{tr}(\theta_0^3\theta_1 - \theta_0^3\theta_1^3) + \frac{1}{2} \text{tr}(\theta_0^3\theta_1\theta_0\theta_1 - \theta_0^3\theta_1\theta_0\theta_1) - \frac{1}{2} \text{tr}(\theta_0^3\theta_0 - \theta_0\theta_1^4).
\]

So we can see that $d'TC_{0,5} + d''TC_{1,4} = \tilde{C}_{1,5}$. 

\[
d'(\frac{1}{2} \text{tr}(\theta_0^3\theta_1) - \frac{1}{4} \text{tr}(\theta_0\theta_1\theta_0\theta_1) + \frac{1}{2} \text{tr}(\theta_0\theta_1^3))
\]

\[
= \frac{1}{2} \text{tr}(\theta_0^3\theta_2 - \frac{1}{4} \text{tr}(\theta_1\theta_2\theta_0) + \frac{1}{2} \text{tr}(\theta_1^2\theta_2^2) - \frac{1}{2} \text{tr}(\theta_0\theta_2\theta_0\theta_2) + \frac{1}{2} \text{tr}(\theta_0\theta_2^3))
\]

\[
+ \frac{1}{2} \text{tr}(\theta_0^3\theta_1 - \frac{1}{4} \text{tr}(\theta_0\theta_1\theta_0\theta_1) + \frac{1}{2} \text{tr}(\theta_0\theta_1^3),
\]
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\[ d \left( \frac{1}{2} \text{tr}(\theta_0 \theta_1 \theta_2) - \frac{1}{2} \text{tr}(\theta_0 \theta_2 \theta_1) \right) \]

\[ = -\frac{1}{2} \text{tr}(\theta_0^2 \theta_1 \theta_2 - \theta_0 \theta_2^2 \theta_1 + \theta_0 \theta_1 \theta_2^2) + \frac{1}{2} \text{tr}(\theta_0^2 \theta_2 \theta_1 - \theta_0 \theta_1 \theta_2^2). \]

So we can see that \( d' T \bar{C}_{1,4} + d'' T \bar{C}_{2,3} = \bar{C}_{2,4} \). We can check also that \( d'' T \bar{C}_{0,5} = 0 \) and \( d' T \bar{C}_{2,3} = \bar{C}_{3,3} \). \( \square \)

3.3 The Chern-Simons form of the Euler class

In this subsection we take \( G = SO(4) \) and exhibit the Chern-Simons form of the Euler class in \( \Omega^3(PSO(4)) \).

For the Pfaffian \( Pf \in \Pi^2(SO(4)) \), we put the canonical simplicial connection into

\[ TPf(\theta) = 2 \int_0^1 Pf(\theta \wedge (t\Omega + \frac{1}{2}t(t - 1)[\theta, \theta])) dt \]

and integrate it along the standard simplex \( \Delta^* \), then we obtain the Chern-Simons form in \( \Omega^3(PSO(4)) \).

**Theorem 3.7.** The Chern-Simons form of the Euler class of \( ESO(4) \to BSO(4) \) in \( \Omega^3(PSO(4)) \) is the sum of the following \( T \tilde{E}_{0,3} \) and \( T \tilde{E}_{1,2} \):

\[ T E_{0,3} \in \Omega^3(SO(4)) \xrightarrow{d''} \tilde{E}_{1,3} \]

\[ T E_{1,2} \in \Omega^2(SO(4) \times SO(4)) \xrightarrow{d'} \tilde{E}_{2,2} \]

\[ T \tilde{E}_{0,3} = \frac{-1}{96\pi^2} \sum_{\tau \in S_4} \text{sgn}(\tau) \left( (\theta_0)^{\tau(1)}(\theta_0)^{\tau(2)}(\theta_0^2)^{\tau(3)}(\theta_0^2)^{\tau(4)} \right) \]

\[ T \tilde{E}_{1,2} = \frac{-1}{64\pi^2} \sum_{\tau \in S_4} \text{sgn}(\tau) \left( (\theta_0)^{\tau(1)}(\theta_0)^{\tau(2)}(\theta_1)^{\tau(3)}(\theta_1)^{\tau(4)} + (\theta_0)^{\tau(3)}(\theta_0)^{\tau(4)}(\theta_1)^{\tau(1)}(\theta_1)^{\tau(2)} \right) \]

**Proof.** Since \( \Omega|_{\Delta^0 \times PG(0)} = 0 \) and \([\theta_0, \theta_0] = 2\theta_0^2\),

\[ T E_{0,3} = \int_0^1 t(t - 1) dt \frac{1}{16\pi^2} \sum_{\tau \in S_4} \text{sgn}(\tau) \left( (\theta_0)^{\tau(1)}(\theta_0)^{\tau(2)}(\theta_0^2)^{\tau(3)}(\theta_0^2)^{\tau(4)} \right) \]
On some Chern-Simons forms of the BSS forms

\[ = \frac{-1}{96\pi^2} \sum_{\tau \in S_4} \text{sgn}(\tau) ((\theta_0)_{\tau(1)\tau(2)}(\theta_0^2)_{\tau(3)\tau(4)}). \]

Since \(\Omega|_{\Delta^1 \times PG(1)} = -dt_1 \wedge (\theta_0 - \theta_1) - t_0 t_1 (\theta_0 - \theta_1)^2\),

\[ TE_{1,2} = \int_0^1 t dt \left( \frac{(-1)}{16\pi^2} \sum_{\tau \in S_4} \text{sgn}(\tau) \left( \int_0^1 (t_0 \theta_0 + t_1 \theta_1)_{\tau(1)\tau(2)} \wedge dt_1 \wedge (\theta_0 - \theta_1)_{\tau(3)\tau(4)} \right) \right) \]

\[ = \frac{-1}{64\pi^2} \sum_{\tau \in S_4} \text{sgn}(\tau) ((\theta_0)_{\tau(1)\tau(2)}(\theta_1)_{\tau(3)\tau(4)} + (\theta_0)_{\tau(3)\tau(4)}(\theta_1)_{\tau(1)\tau(2)}). \]

\[ \square \]

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