LOCAL SOLUTION TO AN ENERGY CRITICAL 2-D STOCHASTIC WAVE EQUATION WITH EXPONENTIAL NONLINEARITY IN A BOUNDED DOMAIN

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ABSTRACT. We prove the existence and the uniqueness of a local maximal solution to an $H^1$-critical stochastic wave equation with multiplicative noise on a smooth bounded domain $\mathcal{D} \subset \mathbb{R}^2$ with exponential nonlinearity. First, we derive the appropriate deterministic and stochastic Strichartz inequalities in suitable spaces and, then use them in arguments based on fixed point method to show the local well-posedness result. We also present an explosion result for the constructed unique local maximal solution.

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1. Introduction

In this paper we consider a nonlinear wave equation subject to random forcing, called the stochastic nonlinear wave equation (SNLWE). Due to its numerous applications to physics, relativistic quantum mechanics and oceanography, SNLWEs have been thoroughly studied under various sets of assumptions, see for example [12]-[13], [20]-[29], [41]-[54], [56]-[59], [63]-[64] and references therein. The case that has so far attracted the most attention seems to be of the stochastic wave equation with initial data belonging to the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. For such equations, the nonlinearities can be of polynomial type, for instance the following SNLWE

$$u_{tt} - \Delta u = -u|u|^{p-1} + |u|^q W, \quad \text{s.t.} \quad u(0) = u_0, \quad \partial_t u(0) = u_1,$$

with the suitable exponents $p, q \in (0, \infty)$; see a series of papers by Ondreját [54], [56]-[59]. Another extensively studied important case is when the initial data belongs to $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ (possibly with weights), see [63], [64] for more details. Similar problems on a bounded domain have been investigated in [15], [26] and [54].

In the deterministic case, see for instance [69], the question of solvability of (1.1) without noise, when the initial data belongs to $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, has been investigated in the following three cases: (i) subcritical, i.e. $p < p_c$; (ii) critical, i.e. $p = p_c$; and (iii) supercritical, i.e. $p > p_c$, where $p_c = \frac{d+2}{d-2}$. In particular, for $d = 2$, any polynomial nonlinearity is subcritical. Therefore, an exponential nonlinearity is a legitimate choice for a critical one. Nonlinearities of exponential type have been studied in many physical models, e.g. a model of self-trapped beams in plasma, see [10], and mathematically in [3], [24], [41]-[42] and [53]. With the help of suitable Strichartz estimates, the existence of global solutions have been proved, in [41]-[42], in the cases when the initial energy is strictly below or at the threshold given by the sharp Moser-Trudinger inequality. Moreover, an instability result has been shown when the initial energy is strictly above the threshold.

Our aim here is to extend the existing studies to the wave equation with exponential nonlinearity subject to randomness. In this way, we generalise the above mentioned results of Ondreját for two dimensional domains, by allowing the exponential nonlinearities, as well as the results of Ibrahim, Majdoub, and Masmoudi and others by allowing randomness. To be precise, we are interested in the following stochastic nonlinear wave equation on a smooth bounded domain $\Omega \subset \mathbb{R}^2$,

$$\begin{cases}
    u_{tt} + Au + F(u) = G(u)W & \text{in } [0, \infty) \times \Omega, \\
    u(0) = u_0, \quad u_t(0) = u_1 & \text{on } \partial \Omega,
\end{cases}$$

where $A$ is either $-\Delta_D$ or $-\Delta_N$, i.e. $-A$ is the Laplace-Beltrami operator with the Dirichlet or the Neumann boundary conditions, respectively; $(u_0, u_1) \in \mathcal{D}(A_\Omega^2) \times L^2(\Omega)$; $W = \{W(t) : t \geq 0\}$ is a $K$-cylindrical Wiener process for a real separable Hilbert space $K$; $F$ and $G$ are locally Lipschitz maps with some growth properties. In particular, the functions $F(u)$ and $G(u)$ are allowed to be of the form $\pm u(\varepsilon^4 u^2 - 1)$ and hence our results cover the recent results obtained in [42]. Detailed and precise assumptions on the model are stated in the subsequent sections. We would like to stress that, to the best of our knowledge, the present paper is the first one to study the wave equations in two dimensional domain with an exponential nonlinearity and an additive or multiplicative noise. As compared with the deterministic paper [42] we prove a counterpart of its Theorem 1.4, i.e. we prove the existence of a unique local solution with a smallness condition on the gradient of initial position. The difficulty we encounter in the present paper is that although the nonlinear function is defined on the whole space $\mathcal{D}(A_\Omega^2)$, it’s values belong to the space $L^2$ only for the elements of the space $\mathcal{D}(A_\Omega^2)$ whose
norm is sufficiently small. We emphasize that our proof of stochastic Strichartz estimate, see Theorem 4.7, simplifies and clarifies the one from [12]. Since the proof of the existence and the uniqueness presented here is obtained by means of appropriate Strichartz estimates and these estimate are different for the full domain case, we will address the question of solvability of (1.2) on $\mathbb{R}^2$ in a forthcoming paper.

The organization of the present paper is as follows. In Section 2 we introduce our notation and provide the required definitions used in the paper. In Sections 3 and 4, we derive the required inhomogeneous and stochastic Strichartz estimates, respectively, by the methods introduced in [18]-[19] and [12]. Section 5 is devoted to the estimates which are sufficient to apply the Banach Fixed Point Theorem in a suitable space. We study the approximated version of problem (1.2) in Section 6. The proof of the existence and uniqueness of a local maximal solution is given in Section 7. Here we also formulate the results about the explosion for the constructed unique local maximal solution. In Appendix A, we provide a rigorous justification of our definition of a local mild solution. In Appendix B, we formulate a result about pointwise evaluation of $L^p$-valued Bochner integrals. In Appendix C, we state, without proof, an equivalence of two natural definitions of a mild solution for stochastic PDE (1.2). We conclude the paper with Appendix D and E in which we prove, respectively, a slight simplification of the stochastic Gronwall Lemma [37, Lemma 5.3] and a generalization of an existence of a Lipschitz extension result [5, Corollary 3].

2. Notation and conventions

In this section we introduce notation and some basic estimates that we use throughout the paper. We write $a \lesssim b$ if there exists a universal constant $c > 0$, independent of $a, b$, such that $a \leq cb$, and we write $a \simeq b$ when $a \lesssim b$ and $b \lesssim a$. In case we want to emphasize the dependence of $c$ on some parameters $a_1, \ldots, a_k$, then we write, respectively, $\lesssim_{a_1, \ldots, a_k}$ and $\simeq_{a_1, \ldots, a_k}$. For any two Banach spaces $X, Y$, we denote by $\mathcal{L}(X, Y)$ the space of linear bounded operators $L : X \to Y$.

To state the definitions of required spaces here, we denote by $E$ and $H$ a separable bounded and Hilbert space, respectively.

2.1. Function spaces and interpolation theory. In the next few basic definitions and remarks, which are included here for the reader’s convenience, from function spaces and interpolation theory we borrow the notation from [71].

By $L^q(\Omega)$, for $q \in [1, \infty)$ and a bounded smooth domain $\Omega$ of $\mathbb{R}^2$, we denote the classical real Banach space of all (equivalence classes of) $\mathbb{R}$-valued $q$-integrable Lebesgue measurable functions on $\Omega$. The norm in $L^q(\Omega)$ is given by

$$
\|u\|_{L^q(\Omega)} := \left( \int_\Omega |u(x)|^q \, dx \right)^{\frac{1}{q}}, \quad u \in L^q(\Omega).
$$

By $L^\infty(\Omega)$ we denote the real Banach space of all (equivalence classes of) Lebesgue measurable essentially bounded $\mathbb{R}$-valued functions defined on $\Omega$ with the norm

$$
\|u\|_{L^\infty(\Omega)} := \text{esssup} \{ |u(x)| : x \in \Omega \}, \quad u \in L^\infty(\Omega).
$$

For any $T > 0$, the space $C([0, T]; H)$ of all $H$-valued continuous functions $u : [0, T] \to H$ endowed with the norm

$$
\|u\|_{C([0, T]; H)} := \sup_{t \in [0, T]} \|u(t)\|_H, \quad u \in C([0, T]; H),
$$

is the real Banach space.
For every $p \in [1, \infty)$, we define the space $L^p(0, T; E)$ as the vector space of all (equivalence classes of) $E$-valued strongly measurable functions $u : [0, T] \rightarrow E$ such that $\int_0^T \|u(t)\|_E^p \, dt < \infty$. The space $L^p(0, T; E)$ endowed with the norm

$$\|u\|_{L^p(0, T; E)} := \left( \int_0^T \|u(t)\|_E^p \, dt \right)^{\frac{1}{p}}, \quad u \in L^p(0, T; E),$$

is a real Banach space.

For $\alpha \in (0, 1)$, by $C^\alpha(\partial \Omega)$, the homogeneous $\alpha$-Hölder space, we mean the set of continuous functions $u$ whose Hölder coefficient

$$\|u\|_{C^\alpha(\partial \Omega)} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

is finite. Note that the Hölder coefficient serves as a seminorm. The inhomogeneous $\alpha$-Hölder space is $C^\alpha(\Omega) = C^\alpha(\partial \Omega) \cap L^\infty(\Omega)$ endowed with the norm $\|u\|_{C^\alpha(\partial \Omega)} + \|u\|_{L^\infty(\Omega)}$.

For any $s \in \mathbb{R}$ and $q \in (1, \infty)$, the Sobolev space $H^{s,q}(\Omega)$ is defined as the restriction of $H^{s,q}(\mathbb{R}^2)$ (see e.g. [74], Definition 2.3.1/1]) to $\Omega$ with norm

$$\|f\|_{H^{s,q}(\Omega)} := \inf_{g \in H^{s,q}(\mathbb{R}^2)} \|g\|_{H^{s,q}(\mathbb{R}^2)}, \quad f \in H^{s,q}(\Omega). \quad (2.1)$$

Here $g \mid \partial \Omega$ is the restriction in the sense of distribution. We denote the closure of $C^\infty_c(\Omega)$, the set of smooth functions defined over $\Omega$ with compact support, in $H^{s,q}(\Omega)$ by $H^{s,q}_0(\Omega)$.

Throughout the whole paper, we denote by $A$ the Dirichlet or the Neumann—Laplacian on Hilbert space $L^2(\Omega)$ with domain, respectively,

$$\mathcal{D}(-\Delta_D) = H^{2,2}(\Omega) \cap H^1_{0,\Omega}(\partial \Omega),$$

and

$$\mathcal{D}(-\Delta_N) = \{ f \in H^{2,2}(\Omega) : \partial_\nu f \mid \partial \Omega = 0 \}.$$ 

Here $\nu$ denotes the outward normal unit vector to $\partial \Omega$. It is well-known, see e.g. [67], that the Dirichlet Laplacian $(-\Delta_D, \mathcal{D}(-\Delta_D))$ is a positive self-adjoint operator on $L^2(\Omega)$ and there exists an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ of $L^2(\Omega)$ which consists of eigenvectors of $-\Delta_D$. If we denote the corresponding eigenvalues by $\{\lambda^2_j\}_{j \in \mathbb{N}}$, then we have

$$-\Delta_D e_j = \lambda^2_j e_j; \quad e_j \in \mathcal{D}(-\Delta_D), \forall j \geq 1; \quad 0 < \lambda^2_1 \leq \lambda^2_2 \leq \cdots \text{ and } \lambda^2_n \xrightarrow{n \to \infty} \infty.$$ 

In the case of the Neumann Laplacian, $(-\Delta_N, \mathcal{D}(-\Delta_N))$ is a non-negative self-adjoint operator on $L^2(\Omega)$ and there exists an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ of $L^2(\Omega)$ which consists of eigenvectors of $-\Delta_N$. Moreover, if we denote the corresponding eigenvalues by $\{\lambda^2_j\}_{j \in \mathbb{N}}$, then we have

$$-\Delta_N e_j = \lambda^2_j e_j; \quad e_j \in \mathcal{D}(-\Delta_N), \forall j \geq 1; \quad \lambda^2_0 \xrightarrow{n \to \infty} \infty,$$

and

$$0 = \lambda^2_1 = \lambda^2_2 = \cdots = \lambda^2_{m_0} < \lambda^2_{m_0+1} \leq \lambda^2_{m_0+2} \leq \cdots,$$

for some $m_0 \in \mathbb{N}$. Since we work with both the operators simultaneously, we denote the pair of operator and its domain by $(A, \mathcal{D}(A))$ and make the distinction where required.

From the functional calculus of self-adjoint operators, see for instance [73], it is known that, the power $A^s$ of operator $A$, for every $s \in \mathbb{R}$, is well-defined and self-adjoint. It is also known that, for any $s \in \mathbb{R}$, $\mathcal{D}(A^{s/2})$, where $A = -\Delta_D$ or $A = -\Delta_N$, with the following norm

$$\|u\|_{\mathcal{D}(A^{s/2})} := \left( \sum_{j \in \mathbb{N}} (1 + \lambda^2_j)^s \|u(e_j)\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}},$$


is a Hilbert space. For \( s \in (0, 2) \) the space \( \mathcal{D}(A^{s/2}) \) is equal to the following complex interpolating space, refer [71, 2.5.3/(13)],

\[
\mathcal{D}(A^{s/2}) = [L^2(\mathcal{O}), \mathcal{D}(A)]_{s/2}.
\]

To derive the Strichartz estimate in suitable spaces, we also need to consider the Dirichlet or the Neumann–Laplacian on Banach space \( L^q(\mathcal{O}) \), \( q \in (1, \infty) \), denoted by \( A_{D,q} \) and respectively \( A_{N,q} \), with domains, respectively,

\[
\mathcal{D}(A_{D,q}) = H^2 \cap H_0^{1,q}(\mathcal{O}), \quad \mathcal{D}(A_{N,q}) = \{ f \in H^2 : \partial_\nu f \mid \partial \mathcal{O} = 0 \}.
\]

Note that \( A_{D,2} = -\Delta_D \) and \( A_{N,2} = -\Delta_N \).

Under some reasonable assumptions on the regularity of the domain \( \mathcal{O} \), one can show that both of these operators have very nice analytic properties. In particular both have bounded imaginary powers with exponent strictly less than \( \pi \) (and thus both \( -A_{D,q} \) and \( -A_{N,q} \) generate analytic semigroups on the space \( L^q(\mathcal{O}) \)). As in [71], one can define the fractional powers \( A_{B,q}^{\alpha/2} \) where as below \( B = D \) or \( B = N \). The domains \( \mathcal{D}(A_{B,q}^{\alpha/2}) \) of these operators can be identified as certain subsets of the Sobolev spaces \( H^{s,q}(\mathcal{O}) \), see Lemma 2.2 below.

Next, we fix the notation for the required subspaces of \( H^{s,q}(\mathcal{O}) \) which are determined by differential operators. Fix \( k \in \mathbb{N} \) and let \( B_j, j = 1, \ldots, k \), be differential operators on \( \partial \mathcal{O} \) defined by

\[
B_j f(x) = \sum_{|\alpha| \leq m_j} b_{j,\alpha}(x) D^\alpha f(x), \quad b_{j,\alpha} \in C^\infty(\partial \mathcal{O}).
\]

Then \( \{B_j\}_{j=1}^k \) is said to be a normal system if and only if

\[
0 \leq m_1 < m_2 < \cdots < m_k,
\]

and for every vector \( \nu_x \) which is normal to \( \partial \mathcal{O} \) at \( x \) the following holds

\[
\sum_{|\alpha| = m_j} b_{j,\alpha}(x) \nu_x^\alpha \neq 0, \quad j = 1, \ldots, k,
\]

where for \( \alpha \in \mathbb{N}^d \) and \( y \in \mathbb{R}^d \), \( g^\alpha = \Pi_i y_i^\alpha \).

**Definition 2.1.** Let \( \{B_j\}_{j=1}^k \) be a normal system as defined above for some \( k \in \mathbb{N} \). For \( s > 0, q \in (1, \infty) \), we set

\[
H^{s,q}_{\{B_j\}}(\mathcal{O}) := \left\{ f \in H^{s,q}(\mathcal{O}) : B_j f \mid \partial \mathcal{O} = 0 \text{ whenever } m_j < s - \frac{1}{q} \right\}.
\]

By taking the suitable choice of normal system \( \{B_j\} \) in the Definition 2.1 for \( s > 0 \) and \( q \in (1, \infty) \), we define

\[
H^s_D(\mathcal{O}) := \left\{ f \in H^{s,q}(\mathcal{O}) : f \mid \partial \mathcal{O} = 0 \text{ if } s > \frac{1}{q} \right\},
\]

and

\[
H^s_N(\mathcal{O}) := \left\{ f \in H^{s,q}(\mathcal{O}) : \nu_x \cdot \nabla f \mid \partial \mathcal{O} = 0 \text{ if } s > 1 + \frac{1}{q} \right\}.
\]

Since the \( H^{1,q}_0(\mathcal{O}) \) space can also be defined by using the \( f \mid \partial \mathcal{O} = 0 \) condition appearing in (2.2) and the Neumann boundary condition appearing in (2.3) can be written as \( \nu_x \cdot \nabla f \mid \partial \mathcal{O} = 0 \), we expect to have some relation between the spaces \( H^{s,q}_B(\mathcal{O}) \) and \( \mathcal{D}(A_{B,q}^{\alpha/2}) \) where
A = −∆_B with B = D or B = N. The next stated result, which is standard in the theory of interpolation spaces, see [11 Theorem 4.3.3], provides a suitable range of s for which the function spaces \( H^{s,q}_N(0) \) and \( \mathcal{D}(A^{s/2}_{B,q}) \) are equivalent.

**Lemma 2.2.** With our notation from this section, we have the following

1. For \( s \in (0, 2) \setminus \left\{ 1 + \frac{1}{q} \right\} \),
   \[
   H^{s,q}_N(0) = \mathcal{D}(A^{s/2}_{N,q}).
   \]

2. For \( s \in (0, 2) \setminus \left\{ \frac{1}{q} \right\} \),
   \[
   H^{s,q}_D(0) = \mathcal{D}(A^{s/2}_{D,q}).
   \]

We close this subsection with the following well-known identity

\[
\mathcal{D}(\sqrt{-\Delta_D}) = H^{1,2}_D(0) \quad \text{and} \quad \mathcal{D}(\sqrt{-\Delta_N}) = H^{1,2}(0).
\]

2.2. **Stochastic analysis.** Now we state a few required definitions from the theory of stochastic analysis, refer [4] and [16] for more details. Throughout the whole paper we assume that \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\), where \(\mathbb{F} := \{\mathcal{F}_t : t \geq 0\}\), is a filtered probability space which satisfies the usual hypothesis, that is, the filtration \(\mathbb{F}\) is right continuous and the \(\sigma\)-field \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets of \(\mathcal{F}\), see [49] Definition I.1.1.

As the noise, we consider a cylindrical \(F\)-Wiener process on a real separable Hilbert space \(K\), see [16] Definition 4.1. Let us recall that \(E\) is a separable Banach space. We denote by \(L^p(\Omega, \mathcal{F}, \mathbb{P}; E)\), for \(p \in [1, \infty)\), the Banach space of all (equivalence classes of) \(E\)-valued random variables equipped with the norm

\[
\|X\|_{L^p(\Omega)} = \left( \mathbb{E} \left[ \|X\|_E^p \right] \right)^{\frac{1}{p}}, \quad X \in L^p(\Omega, \mathcal{F}, \mathbb{P}; E),
\]

where \(\mathbb{E}\) is the expectation operator w.r.t. \(\mathbb{P}\).

**Definition 2.3.** For any \(K\), a separable Hilbert space, the set \(\gamma(K, E)\) of all \(\gamma\)-radonifying operators consists of all bounded operators \(\Lambda : K \rightarrow E\) such that the series \(\sum_{j=1}^{\infty} \beta_j \Lambda(f_j)\) converges in \(L^2(\Omega, \mathcal{F}, \mathbb{P}; E)\) for some (or any) orthonormal basis \(\{f_j\}_{j \in \mathbb{N}}\) of \(K\) and some (or any) sequence \(\{\beta_j\}_{j \in \mathbb{N}}\) of i.i.d. \(N(0, 1)\) real random variables on probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We set

\[
\|\Lambda\|_{\gamma(K, E)} := \left( \mathbb{E} \left[ \sum_{j \in \mathbb{N}} \beta_j \Lambda(f_j)^2 \right]_E^2 \right)^{\frac{1}{2}}.
\]  

One may prove that \(\cdot \|_{\gamma(K, E)}\) is a norm, and \((\gamma(K, E), \| \cdot \|_{\gamma(K, E)})\) is a separable Banach space, see [52] Section 4 in Chapter 1. Note that if \(K = \mathbb{R}\), then \(\gamma(\mathbb{R}, E)\) can be identified with \(E\).

2.3. **Various types of measurability.** Before we continue we need to recall some basic definitions about various notions of measurability. For this purpose let us fix \((Z, \mathcal{Z})\) be a measurable space and \(X\) be a Banach space. Let us denote the Borel \(\sigma\)-algebra of \(X\) by \(\mathcal{B}(X)\). The names we use are slightly untypical, since we use the letter \(\mathcal{Z}\) to underline the \(\sigma\)-field in question.

**Definition 2.4.** [19 Sec 1.1., p. 2] Suppose that \(X\) is a \(\sigma\)-field of subsets of \(X\). A function \(f : Z \rightarrow X\) is \(\mathcal{Z}/X\)-measurable, iff the pre-image \(f^{-1}(B)\) belongs to \(\mathcal{Z}\) for every set \(B \in X\). In particular, a function \(f : Z \rightarrow X\) is \(\mathcal{Z}\)-Borel iff it is \(\mathcal{Z}/\mathcal{B}(X)\)-measurable.
Definition 2.5. 39 Defn 1.1.3] A function \( f : Z \to X \) is called \( \mathcal{Z} \)-simple iff it is of the form \( f = \sum_{i=1}^{N} 1_{A_i} \otimes x_i \) for some \( N \in \mathbb{N} \), \( A_i \in \mathcal{Z} \) and \( x_i \in X \) for all \( i = 1, \ldots, N \). Here \((1_{A_i} \otimes x_n)(s) = 1_{A_{n_i}}(s)x_n\).

Definition 2.6. 39 Defn 1.1.4] A function \( f : Z \to X \) is said to be strongly \( \mathcal{Z} \)-measurable iff there exists a sequence of \( \mathcal{Z} \)-simple functions \( f_n : Z \to X \) such that \( \lim_{n \to \infty} f_n = f \) pointwise on \( Z \).

Remark 2.7. 39 Corollary 1.1.10] If \( X \) is a separable Banach space, then a function \( f : Z \to X \) is strongly \( \mathcal{Z} \)-measurable iff it is \( \mathcal{Z}/\mathcal{B}(X) \) measurable.

When \( X \) is a space of linear operators, the situation becomes more involved. In particular, the pronoun “strongly” can be used in two different ways. Hence we have to be careful. It is well-known, see [52, Lemma 44, p. 51], that many classical \( \sigma \)-fields of subsets of the space \( \gamma(K,E) \) coincide. In particular, the Borel \( \sigma \)-field \( \mathcal{B}(\gamma(K,E)) \), i.e. the \( \sigma \)-field generated by \( \text{top}(\gamma(K,E)) \), i.e. the topology on \( \gamma(K,E) \) which is induced by the norm \( (2.4) \) on \( \gamma(K,E) \), is equal to the strong \( \sigma \)-field \( \mathcal{S}(\gamma(K,E)) \), i.e. the \( \sigma \)-field generated by the following family of strong open sets

\[
\text{strongtop}(\gamma(K,E)) := \{ \{ L \in \gamma(K,E) : Lk \in A \} : k \in K, A \in \text{top}(E) \},
\]

where \( \text{top}(E) \) is the topology on \( E \) which is induced by the norm on \( E \). In other words,

\[
\mathcal{B}(\gamma(K,E)) = \mathcal{S}(\gamma(K,E)).
\]

To formulate our main result in this part of our work we need the following definitions of strong and Borel measurability of an \( \gamma(K,E) \)-valued function.

Definition 2.8. 39 Defn 1.1.27] A function \( f : Z \to \gamma(K,E) \) is called \( \mathcal{Z} \)-double-strongly measurable if for all \( k \in K \) the \( E \)-valued function \( f(\cdot)k : Z \ni z \mapsto f(z)(k) \in E \) is strongly \( \mathcal{Z} \)-measurable according to Definition [2.7].

Definition 2.9. A function \( f : Z \to \gamma(K,E) \) is called strongly \( \mathcal{Z} \)-Borel measurable if for all \( k \in K \) the \( E \)-valued function \( f(\cdot)k : Z \ni z \mapsto f(z)(k) \in E \) is \( \mathcal{Z} \)-Borel according to Definition [2.4].

Proposition 2.10. For a function \( f : Z \to \gamma(K,E) \) the following conditions are equivalent.

(i) \( f \) is \( \mathcal{Z}/\mathcal{B}(\gamma(K,E)) \) measurable, see Definition [2.4].
(ii) \( f \) is strongly \( \mathcal{Z} \)-measurable, see Definition [2.6].
(iii) \( f \) is strongly \( \mathcal{Z} \)-Borel measurable, see Definition [2.9].
(iv) \( f \) is \( \mathcal{Z} \)-double strongly measurable, see Definition [2.8].
(v) \( f \) is \( \mathcal{Z}/\mathcal{S}(\gamma(K,E)) \) measurable, see Definition [2.4].

It is well-known, that such a result does not hold if the space \( \gamma(K,E) \) is replaced by the space \( \mathcal{L}(K,E) \) of all bounded and linear operators from \( K \) to \( E \).

Proof of Proposition 2.10. Since \( X := \gamma(K,E) \) and \( E \) are separable Banach spaces, by Remark 2.7 we infer that (i)\( \Leftrightarrow \) (ii) and (iii)\( \Leftrightarrow \) (iv). Moreover, (iv)\( \Leftrightarrow \) (v). For this, let us observe that \( k \in K \), then \( f(\cdot)k = f \circ i_k \), where \( i_k : \gamma(K,E) \ni L \mapsto L(k) \in E \) is the evaluation map. Thus, for every \( A \in \text{top}(E) \) and every \( k \in K \), \( f^{-1}(\{ L \in \gamma(K,E) : L \circ i_k \in A \}) = (f(\cdot)k)^{-1}(A) \). Hence, the equivalence (iv)\( \Leftrightarrow \) (v) follows. We conclude the proof by observing that the equivalence (i)\( \Leftrightarrow \) (v) is a consequence of Neidhardt’s [52] result (2.6). This completes the proof. □
Let us recall that the progressive σ-field $\mathcal{B}(\mathbb{R}_+ \times \Omega)$ consists of all sets $A \subseteq \mathbb{R}_+ \times \Omega$ such that for any $t \in \mathbb{R}_+$, the set $A \cap ([0, t] \times \Omega) \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t$, see [66] Definition I.4.7 and [75] 6.0.4 and 6.0.5. Similarly we can define the σ-field $\mathcal{B}(\mathbb{R}_+ \times \Omega)$.

Specializing Proposition 2.10 to $Z = [0, T] \times \Omega$ and $\mathcal{Z} = \mathcal{B}([0, T] \times \Omega)$, we infer that all five possible definitions of $\mathcal{B}(\mathbb{R}_+ \times \Omega)$-measurability of processes taking values in $\gamma(K, E)$ are equivalent. In what follows we will simply use words “a progressively measurable $\gamma(K, E)$-valued process”.

Let us point out that there are even more versions of this definition. For instance, a process $\eta : [0, T] \times \Omega \to \gamma(K, E)$ is progressively Borel measurable iff $\eta^{-1}(A) \in \mathcal{B}([0, T] \times \Omega)$ for every $A \in \mathcal{B}(\gamma(K, E))$.

In a more general case, when $\gamma(K, E)$ is replaced by a separable Banach space $X$, we can only use Remark 2.7 to assert that a process $\xi : [0, T] \times \Omega \to X$ is Borel progressively measurable iff it is strongly progressively measurable.

In addition to measurable and strongly measurable functions, one can consider a notion of $\mathcal{Z}_\mu$-measurable functions with respect to a measure. More precisely, we have, see [74] and [39] Definitions 1.1.13 and 1.1.14.

**Definition 2.11.** Assume that $(Z, \mathcal{Z}, \mu)$ is a measure space and $X$ is a Banach space.

(i) A function $f : Z \to X$ is called $\mu$-simple iff it is of the form $f = \sum_{i=1}^N I_{A_i} \otimes x_i$ for some $N \in \mathbb{N}$, $A_i \in \mathcal{Z}$ with $\mu(A_i) < \infty$ and $x_i \in X$ for all $i = 1, \ldots, N$.

(ii) A function $f : Z \to X$ is said to be strongly $\mu$-measurable iff there exists a sequence of $\mu$-simple functions $f_n : Z \to X$ such that $f_n \to f$, $\mu$-almost everywhere.

The next result is borrowed from [39] Proposition 1.1.16 and Remark 1.1.18.

**Proposition 2.12.** Assume that $(Z, \mathcal{Z}, \mu)$ is a σ-finite measure space and $X$ is a separable Banach space. Then the following three assertions are equivalent.

(i) $f$ is strongly $\mu$-measurable;

(ii) there exists a strongly $\mathcal{Z}_\mu$-measurable function $\tilde{f} : Z \to X$ such that $f = \tilde{f}$, $\mu$-almost everywhere;

(iii) $f$ is strongly $\mathcal{Z}_\mu$-measurable, where $\mathcal{Z}_\mu$ is the completion of $\mathcal{Z}$ with respect to $\mu$.

Let us also recall the following definition of Bochner spaces.

**Definition 2.13.** Assume that $(Z, \mathcal{Z}, \mu)$ is a measure space, $X$ is a Banach space and $p \in [1, \infty)$. By $L^p(Z, \mathcal{Z}; X) = L^p(Z; X)$ we denote the vector space of all strongly $\mu$-measurable functions $f : Z \to X$ such that

$$\int_Z |f(z)|_X^p \mu(dz) < \infty.$$  \hfill (2.7)

By $\tilde{L}^p(Z, \mathcal{Z}; X) = \tilde{L}^p(Z; X)$ we denote the vector space of all strongly $\mathcal{Z}$-measurable functions $f : Z \to X$ such that condition (2.7) holds.

By $L^p(Z, \mathcal{Z}; X) = L^p(Z; X)$ we denote the vector space of all equivalence classes of functions from $L^p(Z; X)$. By $L^p(Z, \mathcal{Z}; X) = \tilde{L}^p(Z; X)$ we denote the vector space of all equivalence classes of functions from $\tilde{L}^p(Z; X)$.

Proposition 2.12 implies that the spaces $L^p(Z, \mathcal{Z}; X)$ and $\tilde{L}^p(Z, \mathcal{Z}; X)$ are naturally isometrically isomorphic. In particular, for every element $g$ of $L^p(Z, \mathcal{Z}; X)$, there exists a strongly $\mathcal{Z}$-measurable function $\tilde{f} : Z \to X$ such that $|f| = g$. Endowed with the classical norm, the Bochner space $L^p(Z; X)$ is a Banach space.
When $X = \gamma(K, E)$, where $K$ and $E$ are separable Hilbert and, respectively, Banach spaces, it follows from the above and Proposition 2.10 that the space $L^p(Z; \mathbb{F})$ can be defined as the space of all equivalence classes of $L_2(\Omega; \mathcal{B}(\gamma(K, E)))$-measurable functions such that
\[
\int_Z |f(z)|^p d\mu(z) < \infty. \tag{2.8}
\]
In fact, the $L_2(\Omega; \mathcal{B}(\gamma(K, E)))$-measurability can be replaced by each of the five versions of measurability listed in Proposition 2.10.

### 2.4. Stopping times.

In this section, as throughout the whole paper, we assume that $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F} := \{\mathcal{F}_t : t \geq 0\}$, is a filtered probability space which satisfies the **usual hypothesis**, that is, the filtration $\mathbb{F}$ is right continuous and the $\sigma$-field $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$, see [49, Definition I.1.1]. According to [49, Definition I.4.1], a function $\tau : \Omega \to [0, \infty]$ is an $\mathbb{F}$-stopping time iff for every $t \in \mathbb{R}_+$, the set $\{\tau \leq t\} := \{\omega \in \Omega : \tau(\omega) \leq t\}$ belongs to the $\sigma$-field $\mathcal{F}_t$.

**Definition 2.14.** A stopping time $\tau$ is called accessible, see e.g. [45, section 2.1, p. 45], iff there exists an increasing sequence of stopping times $\{\tau_n\}_{n \in \mathbb{N}}$ with the following properties:

1. $\lim_{n \to \infty} \tau_n = \tau$, $\mathbb{P}$-a.s.,
2. for every $n$, $\tau_n < \tau$, $\mathbb{P}$-a.s. on $\{\tau > 0\}$.

For such sequence we write $\tau_n \nearrow \tau$. Such a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ will be called an announcing sequence for the accessible stopping time $\tau$.

Let us point out that in [49, Definition IV.5.4] a process which we call accessible is called predictable. On the other hand, Metivier in [49, Definition I.4.9] gives a different definition of a predictable stopping time. Fortunately, according to [49, Theorem I.6.6], $\tau$ is a predictable stopping time according to [49, Definition I.4.9] if and only if $\tau$ is an accessible according to our definition, provided the usual hypothesis satisfies, see [49, Definition I.1.1]. Let us point out that our standing assumption is that the the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ satisfies the usual hypothesis. Therefore, in our paper, the notions of accessible and predictable stopping times are equivalent and this allows us to use later on [49, Proposition I.4.14].

For any given stopping time $\tau$, we set
\[
\Omega_t(\tau) := \{\omega \in \Omega : t < \tau(\omega)\}, \tag{2.9}
\]
\[
[0, \tau) \times \Omega := \{(t, \omega) \in [0, \infty) \times \Omega : 0 \leq t < \tau(\omega)\}, \tag{2.10}
\]
\[
[0, \tau] \times \Omega := \{(t, \omega) \in [0, \infty) \times \Omega : 0 \leq t \leq \tau(\omega)\}. \tag{2.11}
\]
Note that the sets $[0, \tau) \times \Omega$ and $[0, \tau] \times \Omega$ are sometimes denoted by $[[0, \tau))$ and $[[0, \tau]]$ respectively. They are also sometimes called "stochastic intervals". It is useful to observe that for two stopping times $\tau_1$ and $\tau_2$ and $t > 0$, the following equality holds:
\[
\Omega_t(\tau_1) \cap \Omega_t(\tau_2) = \Omega_t(\tau_1 \wedge \tau_2),
\]
where $(\tau_1 \wedge \tau_2)(\omega) := \min\{\tau_1(\omega), \tau_2(\omega)\}, \omega \in \Omega$.

To prove the uniqueness of a local solution we need the following criteria of equivalent processes.

**Definition 2.15.** Assume that $X$ is a separable Banach space. A local $X$-valued stochastic process is a function $\xi : [0, \tau) \times \Omega \to X$, where $\tau$ is an accessible stopping time. Suppose that $\tau$ is an accessible stopping time with an announcing sequence $\{\tau_n\}_{n \in \mathbb{N}}$. Then a
local stochastic process \( \xi : [0, \tau) \times \Omega \to X \) is called \( \mathcal{F} \)-progressively measurable, see e.g. [10], iff for every \( n \), the stopped \( X \)-valued process
\[
\xi(\cdot \wedge \tau_n) := [0, \infty) \times \Omega \ni (t, \omega) \mapsto \xi(t \wedge \tau_n(\omega), \omega) \in X
\]
is \( \mathcal{F} \)-progressively measurable. If the filtration \( \mathcal{F} \) is unambiguous from the context, then we often skip it from using.

Two local stochastic processes \( \xi_i : [0, \tau_i) \times \Omega \to X \), \( i = 1, 2 \) are called equivalent, we will write \((\xi_1, \tau_1) \sim (\xi_2, \tau_2)\), if and only if \( \tau_1 = \tau_2 \), \( \mathbb{P} \)-a.s. and for any \( t > 0 \) the following holds
\[
\xi_1(\cdot, \omega) = \xi_2(\cdot, \omega) \quad \text{on} \ [0, t],
\]
for almost all \( \omega \in \Omega_i(\tau_1) \cap \Omega_i(\tau_2) \).

Assume that \( p \in [1, \infty) \). By \( M^p_{\text{loc}}(\mathbb{R}^+, E) \), we denote the space of all progressively measurable \( E \)-valued processes \( \xi : \mathbb{R}^+ \times \Omega \to E \) for which there exists a sequence \( \{\tau_n\}_{n \in \mathbb{N}} \) of bounded stopping times such that \( \tau_n \nearrow \infty \), \( \mathbb{P} \)-a.s. and
\[
\mathbb{E} \left[ \int_0^{\tau_n} \|\xi(t)\|_E^p \, dt \right] < \infty, \quad \forall n \in \mathbb{N}.
\]

Assume that \( p \in [1, \infty) \), \( q \in [1, \infty) \), and \( T \geq 0 \). By \( M^p(\mathcal{L}^q([0, T], E)) \) we denote the space of all progressively measurable \( E \)-valued processes \( \xi : [0, T] \times \Omega \to E \) such that
\[
\mathbb{E} \left[ \left( \int_0^T \|\xi(t)\|_E^q \, dt \right)^\frac{p}{q} \right] < \infty.
\]

The space \( M^p(\mathcal{L}^\infty([0, T], E)) \) is defined analogously with the “norm” \( \left( \int_0^T \|\xi(t)\|_E^q \, dt \right)^\frac{p}{q} \) being replaced by \( \text{esssup}_{t \in [0,T]} \|\xi(t)\|_E \).

As usual, see e.g. [66, Definition IV.2.1], by \( M^p(\mathcal{L}^q([0, T], E)) \) we denote the space of equivalence classes of elements of \( M^p(\mathcal{L}^q([0, T], E)) \). Let us note that \( M^p(\mathcal{L}^q([0, T], E)) \) is a closed subspace of, typically not equal to, \( L^p(\Omega, \mathcal{F}; \mathcal{L}^q([0, T], E)) \). To simplify the notation, we will often use the notation \( M^p([0, T], E) \) to denote \( M^p(\mathcal{L}^p([0, T], E)) \).

## 3. Inhomogeneous Strichartz estimates

In this section we will prove the deterministic Strichartz type estimate, see Theorem 3.2 below, which is a generalization of [42, Theorem 1.2] and is essential to tackle, both, the Dirichlet and the Neumann boundary case.

Recall that in our setting, the operator \( (A, \mathcal{D}(A)) \) possesses a complete orthonormal system of eigenvectors \( \{e_j\}_{j \in \mathbb{N}} \) in \( L^2(\Omega) \). We have denoted the corresponding eigenvalues by \( \lambda_j^2 \). From the functional calculus of self-adjoint operators, it is known that \( \{(e_j, \lambda_j)\}_{j \in \mathbb{N}} \) is a sequence of the associated eigenvector and eigenvalue pair for \( \sqrt{A} \). For any integer \( \lambda \geq 0 \), \( \Pi_\lambda \) is defined as the spectral projection of \( L^2(\Omega) \) onto the subspace spanned by \( \{e_j\}_{j \in \mathbb{N}} \) for which \( \lambda_j \in [\lambda, \lambda + 1) \), i.e.
\[
\Pi_\lambda u = \sum_{j=1}^{\infty} \mathbb{1}_{[\lambda, \lambda+1)}(\lambda_j) (u, e_j)_{L^2(\Omega)} e_j, \quad u \in L^2(\Omega).
\]

At this juncture, it is relevant to note that the proof of the Strichartz estimate in deterministic setting, see e.g. [18] and [19], is based on the following estimate in the Lebesgue spaces of the spectral projector \( \Pi_\lambda \), refer [70] for the proof.
\textbf{Theorem 3.1.} For any smooth bounded domain $\mathcal{O} \subset \mathbb{R}^2$, the following estimate holds for all $u \in L^2(\mathcal{O})$
\[ \| \Pi_{\lambda} u \|_{L^q(\mathcal{O})} \leq C \lambda^\rho \| u \|_{L^2(\mathcal{O})}, \] (3.1)
where
\[ \rho := \begin{cases} 
\frac{2}{3} \left( \frac{1}{2} - \frac{1}{q} \right) & \text{if } 2 \leq q \leq 8, \\
2 \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{1}{2} & \text{if } 8 \leq q \leq \infty.
\end{cases} \] (3.2)

Since the Strichartz estimates below derived, for the homogeneous and inhomogeneous wave equation, holds for both the Dirichlet and the Neumann case, from now onwards, to shorten the notation, we denote $A_{B,q}$ and $A_{B,2}$, respectively, by $A_q$ and $A$.

\textbf{Theorem 3.2 (Deterministic Strichartz Estimates).} Let us assume that $T > 0$. Then there exists a positive constant $C_T$, which is increasing w.r.t. $T$ and may also depend on $p, q, r$, such that the following holds: if $u$ satisfies the following linear inhomogeneous wave equation
\[ \begin{cases} 
u \partial_t^2 u - \Delta u = f & \text{in } (0, T) \times \mathcal{O} \\
u u(0, \cdot) = u_0(\cdot), \quad \nu u_t(0, \cdot) = u_1(\cdot),
\end{cases} \] (3.3)
with either boundary condition
- Dirichlet: $u \upharpoonright (0, T) \times \partial \mathcal{O} = 0$,
- Neumann: $\partial_t u \upharpoonright (0, T) \times \partial \mathcal{O} = 0$,
where $\nu$ is the outward normal unit vector to $\partial \mathcal{O}$ and $f \in L^1(0, T; L^2(\mathcal{O}))$, then
\[ \| u \|_{L^p(0, T; \mathcal{D}(A_{\rho}^{\frac{1}{2}}))} \leq C_T \left[ \| u_0 \|_{\mathcal{D}(A_{\rho}^{\frac{1}{2}})} + \| u_1 \|_{L^2(\mathcal{O})} + \| f \|_{L^1(0, T; L^2(\mathcal{O}))} \right], \] (3.4)
for all $(p, q, r)$ which satisfy
\[ 2 \leq q \leq p < \infty, \quad \text{and } r = \begin{cases} 
\frac{5}{6} - \frac{1}{p} - \frac{2}{3q} & \text{if } 2 \leq q \leq 8, \\
1 - \frac{1}{q} & \text{if } 8 \leq q < \infty.
\end{cases} \] (3.5)

\textbf{Remark 3.3.} In addition to the Strichartz estimates (3.4) one also has the classical (including those on the velocity $u_t$) estimates, see [2] for the inhomogeneous part,
\[ \| u \|_{C([0, T]; \mathcal{D}(A_{\rho}^{\frac{1}{2}}))} + \| u_t \|_{C([0, T]; H)} \leq \tilde{C}_T \left[ \| u_0 \|_{\mathcal{D}(A_{\rho}^{\frac{1}{2}})} + \| u_1 \|_{L^2(\mathcal{O})} + \| f \|_{L^1(0, T; L^2(\mathcal{O}))} \right]. \] (3.6)
We can assume that the constants $\tilde{C}_T$ increase w.r.t. $T$. In a standard way, this inequality can be lifted to more regular data as follows, for any $k \geq 0$,
\[ \| u \|_{C([0, T]; \mathcal{D}(A_{\rho}^{k+\frac{1}{2}}))} + \| u_t \|_{C([0, T]; \mathcal{D}(A^k))} \leq \tilde{C}_T \left[ \| u_0 \|_{\mathcal{D}(A^{k+\frac{1}{2}})} + \| u_1 \|_{\mathcal{D}(A^k)} + \| f \|_{L^1(0, T; \mathcal{D}(A^k))} \right]. \] (3.7)

\textbf{Remark 3.4.} Let us observe that if for $T > 0$, $C_T$ denotes the smallest constant for which inequality (3.4) holds for all data $u_0, u_1$ and $F$ from appropriate spaces, then the function
\[ (0, \infty) \ni T \mapsto C_T \in (0, \infty), \]
is non-decreasing (or weakly increasing as some people call). Similar results hold true for $\tilde{C}_T$. 

Remark 3.5. Ibrahim and Jrad proved, see inequality (3.5) in the proof of [42, Theorem 1.2], that

\[
\|u\|_{L^8(0,T;H^s_{\frac{1}{8}})} \leq C_T \left[ \|u_0\|_{2(A)} + \|u_1\|_{L^2(0)} + \|F\|_{L^1(0,T;L^2(\Omega))} \right],
\]

(3.8)

for \(A = -\Delta_D\). Since the substitution of \(p = q = 8\) into (3.4) gives (3.8), our result generalizes [42, Theorem 1.2]. Note that in [42] the space \(H^{s_{\frac{1}{8}}}(\Omega)\) is denoted, inconsistently with current approach, by \(W^{s_{\frac{1}{8}}}(\Omega)\). Finally, let us point out that the inequality (1.8) with the Hölder space \(\mathcal{C}^\tau(\Omega)\) in [42, Theorem 1.2] is a consequence of the Sobolev embedding, see e.g. [71, Theorem 2.8.1 (c) and Definition 1 (d) in 2.3.1].

Proof of Theorem 3.2. Without loss of generality we assume that \(T = 2\pi\). The proof is divided into two cases. In the first case, we derive the Strichartz estimate for the homogeneous problem (i.e. \(F = 0\)) and then, in second case, we prove the inhomogeneous one (i.e. \(F \neq 0\)) by using the homogeneous estimate from first case.

The first case: Estimate for the homogeneous problem. In this case, the Duhamel formula gives

\[
u(t) = \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1,
\]

(3.9)

where, from the functional calculus for self-adjoint operators, for each \(t\), \(\cos(t\sqrt{A})\) and \(\sin(t\sqrt{A})\) are well-defined bounded operators on \(L^2(\Omega)\). Moreover, we have

\[
\cos(t\sqrt{A}) = e^{it\sqrt{A}} + e^{-it\sqrt{A}}.
\]

Let \(\mathcal{L}_\pm(t)u_0 := e^{\pm it\sqrt{A}}u_0\) be the solution \(u\) of \(\partial_t u = \pm i\sqrt{A}u\) such that \(u(0) = u_0\). In other words, \(\mathcal{L}_\pm = (\mathcal{L}_\pm(t))_{t \geq 0}\) is \(C_0\)-group with the generator \(\pm i\sqrt{A}\). Using the Minkowski’s inequality we get

\[
\|u\|_{L^p(0,T;D(A^{\frac{1}{2}}))} \lesssim \|e^{it\sqrt{A}}u_0\|_{L^p(0,T;D(A^{\frac{1}{2}}))} + \|e^{-it\sqrt{A}}u_0\|_{L^p(0,T;D(A^{\frac{1}{2}}))} + \left\| \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 \right\|_{L^p(0,T;D(A^{\frac{1}{2}}))}.
\]

(3.10)

Therefore, it is enough to estimate, as done in the following Steps 1-4, the \(L^p(0,T;D(A^{\frac{1}{2}}))\)-norm of \(e^{it\sqrt{A}}u_0\) and \(\frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1\). We will write the variables in subscript, wherever required, to avoid any confusion.

Step 1. Here we show that

\[
\|e^{itB}u_0\|_{L^p(0,2\pi;L^2(\Omega))} \leq C\|u_0\|_{D(A^{r/2})},
\]

(3.11)

where \(B\) is the following “modification” of \(\sqrt{A}\) operator by considering only the integer eigenvalues, i.e.

\[
B(e_j) = [\lambda_j]e_j, \quad j \in \mathbb{N}.
\]

The notation \([\cdot]\) stands for the integer part and \(e_j\) is an eigenfunction of \(A\) associated to the eigenvalue \(\lambda_j^2\). Before moving further we prove the boundedness property of the operator \(B - \sqrt{A}\).

Lemma 3.6. For every \(r \in [0,1]\), the operator \(B - \sqrt{A}\) is bounded on \(D(A^{r/2})\).
Proof of Lemma 3.6. Let us fix $r \in [0, 1]$. Observe that by definition of $B$ we have for every $u \in \mathcal{D}(A^\sharp)$,
\[
(B - \sqrt{A})u = \sum_{j \in \mathbb{N}} \{\lambda_j\} \langle u, e_j \rangle_{L^2(0)} e_j,
\]
where $\{\lambda_j\} := \lambda_j - [\lambda_j]$ is the fractional part of $\lambda_j$. Then
\[
\|B - \sqrt{A}\|_{L^2(0)}^2 \leq \sum_{j \in \mathbb{N}} (\lambda_j)^2 \|\langle u, e_j \rangle_{L^2(0)}\|^2 \leq \|u\|_{L^2(0)}^2 \leq \|u\|^2_{\mathcal{D}(A^\sharp)}.
\]
Moreover,
\[
\|A^\sharp(B - \sqrt{A})u\|_{L^2(0)}^2 = \sum_{j \in \mathbb{N}} \lambda_j^2 \{\lambda_j\}^2 \|\langle u, e_j \rangle_{L^2(0)}\|^2 \leq \sum_{j \in \mathbb{N}} \lambda_j^2 \|\langle u, e_j \rangle_{L^2(0)}\|^2 = \|A^\sharp u\|_{L^2(0)}^2.
\]
Hence, by the definition of norm in $\mathcal{D}(A^\sharp)$ we have
\[
\|B - \sqrt{A}\|_{\mathcal{D}(A^\sharp)}^2 = \|B - \sqrt{A}\|_{L^2(0)}^2 + \|A^\sharp(B - \sqrt{A})u\|_{L^2(0)}^2 \lesssim \|u\|^2_{\mathcal{D}(A^\sharp)}.
\]

In continuation of the proof of (3.12), since $u_0 \in L^2(0)$, we can write
\[
u_0 = \sum_{j \in \mathbb{N}} \langle u_0, e_j \rangle_{L^2(0)} e_j =: \sum_{j \in \mathbb{N}} u_j e_j.
\]
By functional calculus for self-adjoint operators,
\[
e^\xi tB u_0(t) = \sum_{j \in \mathbb{N}} e^\xi t\lambda_j u_j e_j(t) =: \sum_{k \in \mathbb{N}} u_k(t, x),
\]
where, since $[\lambda_j] = k$,
\[
u_k(t, x) = \sum_{j \in \mathbb{N}} 1_{[k, k+1)}(\lambda_j) e^\xi t k u_j e_j(t) = e^\xi t k \Pi_k u_0(t).
\]
Assume that $0 < \theta < \frac{1}{2}$. Let $C := -D_x + I$ on $L^2(0, 2\pi)$ with the periodic boundary conditions. Then, see [11], $H^{\theta, 2} = \mathcal{D}(C^{\theta/2})$ with equivalent norms. The norm $\mathcal{D}(C^{\theta/2})$ is Hilbertian with the corresponding inner product
\[
\langle u, v \rangle_{\mathcal{D}(C^{\theta/2})} := \langle C^{\theta/2} u, C^{\theta/2} v \rangle_{L^2(0, 2\pi)}.
\]
We claim that the sequence $\{e^{i k}\}_{k \in \mathbb{N}}$, after normalization, is an orthonormal basis (ONB) in $\mathcal{D}(C^{\theta/2})$. Indeed, since the sequence $\{e^{i k}\}_{k \in \mathbb{N}}$, consisting of eigenvectors of $C$, is an ONB of $L^2(0, 2\pi)$, we infer that for all $j, k \in \mathbb{N}$,
\[
\langle e^{i j}, e^{i k} \rangle_{\mathcal{D}(C^{\theta/2})} = \langle C^{\theta/2} e^{i j}, C^{\theta/2} e^{i k} \rangle_{L^2(0, 2\pi)} = (1 + |j|^{2\theta/2}) (1 + |k|^{2\theta/2}) \langle e^{i j}, e^{i k} \rangle_{L^2(0, 2\pi)}.
\]
Hence the claim follows.
Moreover, there exists $C_\theta > 0$ such that
\[
\|e^{i k}\|_{H^{\theta, 2}(0, 2\pi)} \leq C_\theta (1 + k)^\theta, \quad k \in \mathbb{N}.
\]
Therefore, for a fixed $x \in D$, we have
\[
e^\xi tB u_0(t) = \sum_{k \in \mathbb{N}} \frac{e^{i k}}{\|e^{i k}\|} \|e^{i k}\| \Pi_k u_0(x), \quad t \in (0, 2\pi).
\]
Thanks to the 1D Sobolev embedding and Lemma 2.2, we have
\[ \| e^{itB}u_0(x) \|_{D(C^0/2)}^2 = \sum_{k \in \mathbb{N}} \| e^{itk} \|_{D(C^0/2)}^2 \| \Pi_k u_0(x) \|^2. \]

Therefore,
\[ \| e^{itB}u_0(x) \|_{D(C^0/2)}^2 \leq \sum_{k \in \mathbb{N}} \| e^{itk} \|_{D(C^0/2)}^2 \| \Pi_k u_0(x) \|^2 \leq C^2_\theta \sum_{k \in \mathbb{N}} (1 + k)^{2\theta} \| \Pi_k u_0(x) \|^2. \]

Using the equivalence of the two norms we deduce that
\[ \| e^{itB}u_0(x) \|_{H^{0',2}}^2 \leq C' \sum_{k \in \mathbb{N}} (1 + k)^{2\theta} \| \Pi_k u_0(x) \|^2. \]

Thanks to the 1D Sobolev embedding and Lemma 2.2 we have
\[ H^{1 - \frac{1}{p} - \frac{1}{2}}(0, 2\pi) \hookrightarrow L^p(0, 2\pi) \text{ for all } p \in [2, \infty), \]
where the space \( H^{\theta,2}(0, 2\pi), \theta > 0, \) is defined by formula (2.11), with set \( \mathcal{O} \) replaced by the interval \((0, 2\pi)\). Consequently we argue as follows:
\[ \| e^{itB}u_0 \|_{L^2(0; L^p(0,2\pi))}^2 = \left( \int_0^1 \| e^{itB}u_0(x) \|^q_{L^p(0,2\pi)} \, dx \right)^{\frac{2}{q}} \]
\[ \lesssim \left( \int_0^1 \| e^{itB}u_0(x) \|^q_{H^1_{\theta} + \frac{1}{p} - \frac{1}{2}}(0,2\pi) \, dx \right)^{\frac{2}{q}} = \left\| \| e^{itB}u_0(x) \|^2_{H^1_{\theta} + \frac{1}{p} - \frac{1}{2}}(0,2\pi) \right\|_{L^2(\mathcal{O})}. \] (3.13)

Hence, by applying the last Claim with \( \theta = \frac{1}{q} - \frac{1}{p} \) in view of the Parseval formula we deduce that, for fixed \( x, \)
\[ \| e^{itB}u_0(x) \|^2_{H^1_{\theta} + \frac{1}{p} - \frac{1}{2}}(0,2\pi) \lesssim \sum_{k \in \mathbb{N}} (1 + k)^{1 - \frac{2}{p}} \| u_k(t, x) \|^2_{L^2(0,2\pi)}, \] (3.14)

Combining the estimate (3.14) and (3.13) followed by Minkowski’s inequality and Theorem 3.1 we obtain
\[ \| e^{itB}u_0 \|^2_{D(A^r/2)} \lesssim \left\| \sum_{k \in \mathbb{N}} (1 + k)^{1 - \frac{2}{p}} \| u_k \|^2_{L^2(0,2\pi)} \right\|_{L^2(\mathcal{O})} \]
\[ \lesssim \sum_{k \in \mathbb{N}} (1 + k)^{1 - \frac{2}{p}} \| \Pi_k u_0 \|^2_{L^2(\mathcal{O})} \lesssim \sum_{k \in \mathbb{N}} (1 + k)^{1 - \frac{2}{p} + 2\rho} \| \Pi_k u_0 \|^2_{L^2(\mathcal{O})} \]
\[ \lesssim \sum_{j \in \mathbb{N}} (1 + k)^{1 - \frac{2}{p} + 2\rho} \| \Pi_k u_0 \|^2_{L^2(\mathcal{O})} \lesssim \sum_{j \in \mathbb{N}} (1 + k)^{1 - \frac{2}{p} + 2\rho} \sum_{j \in \mathbb{N}} \Pi_{[k,k+1]}(\Lambda_j) \langle u_0, e_j \rangle_{L^2(\mathcal{O})}^2 \]
\[ = \sum_{j \in \mathbb{N}} (1 + k)^{1 - \frac{2}{p} + 2\rho} \| \Pi_k u_0 \|^2_{D(A^r/2)} = \| u_0 \|^2_{D(A^r/2)} \lesssim \| u_0 \|^2_{D(A^r/2)}, \] (3.15)

where, from \( \rho \) in Theorem 3.1 we have\(^1\)
\[ r := \frac{1}{2} - \frac{1}{p} + \rho = \begin{cases} \frac{5}{6} - \frac{1}{p} - \frac{2}{3q} & \text{if } 2 \leq q \leq 8, \\ 1 - \frac{1}{p} - \frac{2}{q} & \text{if } 8 \leq q < \infty. \end{cases} \]

Here it is important to highlight that, the equivalence \( \| u_0 \|_{D(B^r)} \simeq \| u_0 \|_{D(A^r/2)} \) holds in the last step of (3.15), because \( D(A) = D(B^2) \) and the spaces \( D(B^r) \) and \( D(A^r/2), \) for \( r \in [0, 1], \)

\(^1\)Note that \( r < \frac{2}{q} \) in the case \( 2 \leq q \leq 8 \) and \( r < 1 \) in the complimentary case \( 8 \leq q < \infty. \)
are equal to the complex interpolation spaces, between \( L^2(\mathcal{O}) \) and, respectively, \( \mathcal{D}(B^2) \) and \( \mathcal{D}(A) \), see [71, Theorem 4.3.3].

Next, since \( p \geq q \), by the Minkowski inequality we obtain the following desired result:

\[
\|e^{itB}u_0\|_{L^p_t(0,2\pi;L^q_x(\mathcal{O}))} \lesssim \|u_0\|_{\mathcal{D}(A^{r/2})},
\]

what also implies that the operator \( e^{itB} \) is continuous from \( \mathcal{D}(A^{r/2}) \) to \( L^p_t(0,2\pi;L^q_x(\mathcal{O})) \).

**Step 2.** In this step we extend inequality (3.11) to operator \( \mathcal{L}_+ \), i.e. we show that

\[
\|\mathcal{L}_+(\cdot)u_0\|_{L^p_t(0,2\pi;L^q_x(\mathcal{O}))} \leq C\|u_0\|_{\mathcal{D}(A^{r/2})}, \quad (3.16)
\]

Let \( v(t) = e^{itB}u_0 \). It is clear that \( v \) satisfies

\[
\begin{cases}
\left( \partial_t - iB \right)v = (-iB + i\sqrt{A})v \\
v_{|t=0} = u_0,
\end{cases}
\]

and, therefore, according to the Duhamel formula

\[
v(t) = e^{itB}u_0 + \int_0^t e^{i(t-s)B}(-iB + i\sqrt{A})v(s) \, ds. \quad (3.17)
\]

If we denote \( e^{i(t-s)B}(-iB + i\sqrt{A})v(s,x) \) by \( z(s,t,x) \) and \( (-iB + i\sqrt{A})v(s,x) \) by \( w(s,x) \), then using the Minkowski inequality, followed by estimate (3.11) and Lemma 3.6, we argue as follows:

\[
\left\| \left( \int_0^t \left| z(s,t,x) \right|^2 \, ds \right)^{p/2} \right\|_{L^q_x(\mathcal{O})} \leq \left( \int_0^t \left\| z(s,t,x) \right\|_{L^q_x(0,2\pi)}^q \, ds \right)^{1/q} \leq \int_0^t \left\| z(s,t,x) \right\|_{L^q_x(0,2\pi)}^q \, ds \leq \int_0^t \left\| v(s,x) \right\|_{\mathcal{D}(A^{r/2})}^q \, ds. \quad (3.18)
\]

By putting together (3.17) and (3.18) we obtain

\[
\|v(t,x)\|_{L^q_x(0;L^p_t(0,2\pi))} \leq \|e^{itB}u_0(x)\|_{L^2_x(0;L^p_t(0,2\pi))} + \int_0^{2\pi} \|v(s,x)\|_{\mathcal{D}(A^{r/2})} \, ds
\]

\[
\leq \|u_0\|_{\mathcal{D}(A^{r/2})} + \int_0^{2\pi} \|v(s,x)\|_{\mathcal{D}(A^{r/2})} \, ds. \quad (3.19)
\]

Now, from the boundedness of \( e^{it\sqrt{A}} \) on \( \mathcal{D}(A^{r/2}) \), we infer that

\[
\sup_{t \in [0,2\pi]} \left\| e^{it\sqrt{A}}u_0 \right\|_{\mathcal{D}(A^{r/2})} \leq C\|u_0\|_{\mathcal{D}(A^{r/2})}. \quad (3.20)
\]

Combining (3.20) and (3.19) we get

\[
\|v(t,x)\|_{L^q_x(0;L^p_t(0,2\pi))} \leq \|u_0\|_{\mathcal{D}(A^{r/2})} + \int_0^{2\pi} \|u_0\|_{\mathcal{D}(A^{r/2})} \, ds \lesssim \|u_0\|_{\mathcal{D}(A^{r/2})}. \quad (3.21)
\]

Hence, again, as an application of the Minkowski inequality we get (3.16) and finish with the proof of Step 2.

**Step 3.** Here, by using the well-known consequence of Agmon-Douglis-Nirenberg regularity results for the elliptic operators, refer [1], we prove the required estimate of the first term in (3.10), in particular, we show

\[
\|\mathcal{L}_+(\cdot)u_0\|_{L^p_t(0,2\pi;\mathcal{D}(A^{r/2}))} \lesssim \|u_0\|_{\mathcal{D}(A^{r/2})}. \quad (3.21)
\]
We start the proof by recalling the following consequence of the Agmon-Douglis-Nirenberg regularity results for the elliptic operators. The operators
\[-\Delta_D + I : H^{2,q}(\mathcal{O}) \cap H^{1,q}_0(\mathcal{O}) = H^{2,q}(\mathcal{O}) \cap H^{1,q}_0(\mathcal{O}) \rightarrow L^q(\mathcal{O}),\]
and
\[-\Delta_N + I : H^{2,q}(\mathcal{O}) \cap H^{1,q}_0(\mathcal{O}) \rightarrow L^q(\mathcal{O}),\]
are isomorphisms. These operators will, respectively, be denoted by \(A_{D,q} + I\) and \(A_{N,q} + I\), or simply by \(A_q + I\). Suppose that \(u_0 \in D(A^k)\) for sufficiently large \(k \in \mathbb{N}\) so that \(Au_0 \in \mathcal{D}(A^{r/2})\). Then, since the operators \(A\) and \(\mathcal{L}_+\) commute, we infer that for all \(t \in [0, T]\),
\[
\|\mathcal{L}_+(t)u_0\|_{H^{2,q}(\mathcal{O})} \simeq \|(A + I)\mathcal{L}_+(t)u_0\|_{L^q(\mathcal{O})} = \|\mathcal{L}_+(t)((A + I)u_0)\|_{L^q(\mathcal{O})}.
\]
Consequently by (3.10) we get
\[
\|\mathcal{L}_+(\cdot)u_0\|_{L^p(0,2\pi;H^{2,q}(\mathcal{O}))} \lesssim \|(A + I)u_0\|_{\mathcal{D}(A^{r/2})} \sim \|u_0\|_{\mathcal{D}(A^{(r+2)/2})}.
\] (3.22)
Thus, complex interpolation between (3.10) and (3.22) with \(\theta = \frac{1-r}{2}\) gives the desired following estimate
\[
\|\mathcal{L}_+(\cdot)u_0\|_{L^p(0,2\pi;\mathcal{D}(A^{1/(r+2)}))} \lesssim \|u_0\|_{\mathcal{D}(A^{1/2})}.
\]
Hence we have completed the proof of Step 3.

**Step 4:** Here we incorporate the term with \(u_1\), in (3.9), and complete the proof of the homogeneous Strichartz estimate.

Recall that \(\lambda_1 = 0\) for the Neumann condition and \(\lambda_1 > 0\) in the Dirichlet case. As mentioned before, we denote by \(m_0\) the dimension of eigenspace corresponding to zero eigenvalue. It is known that \(m_0 = 0\) for \(A = -\Delta_D\) and a positive finite integer when \(A = \Delta_N\). To proceed with the proof of this Step, as in [19], we single out the contribution of zero eigenvalue and decompose \(L^2(\mathcal{O})\) into the direct sum of a finite dimensional space \(\ker A\) and the space orthogonal to \(\ker A\), which we denote by \(L^{2,+}(\mathcal{O})\). Let us observe that if \(\mathcal{O}\) is connected, then \(\ker A\) is a one dimensional vector space consisting of constant functions. Mathematically, it means, for all \(u_1 \in L^2(\mathcal{O})\),
\[
u_1 = \sum_{j=1}^{m_0} \langle u_1, e_j \rangle_{L^2(\mathcal{O})} e_j + \sum_{k>m_0} \langle u_1, e_k \rangle_{L^2(\mathcal{O})} e_k =: \Pi u_1 + (1 - \Pi)u_1.
\]
Note that the term \(\Pi u_1\) does not exist in the Dirichlet condition. Then we argue as follows:
\[
\frac{\sin(t\sqrt{A})}{\sqrt{A}} u_1 = \frac{\sin(t\sqrt{A})}{\sqrt{A}} \Pi u_1 + \frac{\sin(t\sqrt{A})}{\sqrt{A}} (1 - \Pi)u_1 = t\Pi u_1 + \frac{\sin(t\sqrt{A})}{\sqrt{A}} (1 - \Pi)u_1,
\] (3.23)
where the last step holds due to the following argument
\[
\frac{\sin(t\sqrt{A})}{\sqrt{A}} u_1 = \sum_{j \in \mathbb{N}} \frac{\sin(t\lambda_j)}{\lambda_j} \langle u_1, e_j \rangle_{L^2(\mathcal{O})} e_j
\]
\[
= t \sum_{j \in \mathbb{N}} \mathbb{1}_{(0)}(\lambda_j) \langle u_1, e_j \rangle_{L^2(\mathcal{O})} e_j + \sum_{j \in \mathbb{N}} \mathbb{1}_{(0,\infty)}(\lambda_j) \frac{\sin(t\lambda_j)}{\lambda_j} \langle u, e_j \rangle_{L^2(\mathcal{O})} e_j
\]
\[
= t\Pi u_1 + \frac{\sin(t\sqrt{A})}{\sqrt{A}} (1 - \Pi)u_1.
\] (3.24)
Now, since \((\sqrt{A})^{-1}\) is isometry from \(L^2(\mathbb{Q})\) into \(\mathcal{D}(A^{1/2})\), by invoking (3.21) on \((\sqrt{A})^{-1}((1 - \Pi)u_1))\) we get
\[
\|L_+((\sqrt{A})^{-1}((1 - \Pi)u_1))\|_{L^p(0,2\pi;\mathcal{D}(A_q^{1/2}))} \lesssim \|((\sqrt{A})^{-1}((1 - \Pi)u_1))\|_{\mathcal{D}(A^{1/2})} = \|(1 - \Pi)u_1\|_{L^2(\mathbb{Q})}.
\] (3.25)

We mention that all the computations we have done so far in Steps 1-4 would work if we replace \(L_+\) by \(L_-\). Combining (3.23) and (3.25) we obtain
\[
\left\|\frac{\sin(t\sqrt{A})}{\sqrt{A}} u_1\right\|_{L^p(0,2\pi;\mathcal{D}(A_q^{1/2}))} \lesssim \|\Pi u_1\|_{L^p(0,2\pi;\mathcal{D}(A_q^{1/2}))}
+ \left\|L_+((\sqrt{A})^{-1}((1 - \Pi)u_1))\right\|_{L^p(0,2\pi;\mathcal{D}(A_q^{1/2}))}
\lesssim \|\Pi u_1\|_{L^p(0,\pi;\mathcal{D}(A_q^{1/2}))} + \|u_1\|_{L^2(\mathbb{Q})} \lesssim \|u_1\|_{L^2(\mathbb{Q})}.
\]

This finishes the proof of Step 4 and, in particular, the first case.

**Second case: when** \(L^1(0,2\pi;L^2(\mathbb{Q})) \ni F \neq 0\): Due to the Duhamel formula
\[
u(t) = \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}} u_1 + \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} F(s) \, ds.
\]

Applying the first case and using the calculation of (3.19) and (3.24) we get
\[
\|u\|_{L^p(0,2\pi;\mathcal{D}(A_q^{1/2}))} \lesssim \|u_0\|_{\mathcal{D}(A^{1/2})} + \|u_1\|_{L^2(\mathbb{Q})}
+ \int_0^{2\pi} \left\|\frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} F(s)\right\|_{L^p(0,2\pi;\mathcal{D}(A_q^{1/2}))} \, ds
\lesssim \|u_0\|_{\mathcal{D}(A^{1/2})} + \|u_1\|_{L^2(\mathbb{Q})} + \int_0^{2\pi} \|F(s)\|_{L^2(\mathbb{Q})} \, ds.
\]

Hence we have proved the Theorem 3.2.

4. *Stochastic Strichartz estimates*

This section is devoted to prove a stochastic Strichartz inequality, which is sufficient to apply the Banach Fixed Point Theorem in the proof of a local well-posedness result for problem (1.2), see Theorem 7.1 in Section 7.

4.1. **Main assumptions.** Here we describe the main assumptions we consider in Sections 1-7. Let us set
\[
H = L^2(\mathbb{Q}), \quad H_A = \mathcal{D}(A^{1/2}); \quad E = \mathcal{D}(A_q^{1/2}),
\]
(4.1)
where \(q \in [2,\infty)\), \(r \in [0,1]\) and \(p = p(q,r)\) satisfies the equality (3.23). Let us define the following Banach space. For fixed \(T > 0\), we put
\[
X_T := C([0,T]; H_A),
\]
(4.2)
\[
Y_T := L^p(0,T; E),
\]
(4.3)
\[
Z_T := C([0,T]; H_A) \cap L^p(0,T; E),
\]
(4.4)
\[
Q_T := C([0,T]; H)
\]
(4.5)
Obviously, all these four spaces are (separable) Banach spaces with naturally defined norms, i.e.

$$\|\xi\|_{X_T} := \sup_{t \in [0,T]} \|u(t)\|_{H_A},$$

$$\|\xi\|_{Y_T} := \left( \int_0^T |\xi(s)|^p_{E} \, ds \right)^{1/p},$$

$$\|\xi\|_{Z_T} := \left( \sup_{t \in [0,T]} \|u(t)\|_{H_A}^p + \int_0^T |\xi(s)|^p_{E} \, ds \right)^{1/p},$$

$$\|\xi\|_{Q_T} := \sup_{t \in [0,T]} \|u(t)\|_{H}.$$  \hfill (4.9)

By $M^p(Z_T)$ we denote the Banach space of (equivalence classes of all E-valued progressively measurable processes $\{u(t), t \in [0,T]\}$ having a continuous $H_A$-valued modification and satisfying

$$\|\xi\|_{M^p(Z_T)} := E[\|\xi\|_{Z_T}^p] = E[\|\xi\|_{C([0,T];H_A)}^p + \|\xi\|_{L^p([0,T];E)}^p] < \infty. \hfill (4.10)$$

4.2. Martingales. In order to define the Itô type integrals for a Banach space valued stochastic process, we restrict ourself to, the so called, $M$-type 2 Banach spaces which are defined as follows.

**Definition 4.1.** A Banach space $E$ is of $M$-type 2 iff there exists a constant $L := L_2(E) > 0$ such that for any $E$-valued martingale $\{M(n), n \in \mathbb{N}\}$ the following holds:

$$\sup_n E(\|M(n)\|_E^2) \leq L \sum_{n=0}^\infty E(\|M(n) - M(n-1)\|_E^2),$$

where, as usual, we set $M(-1) = 0$.

For an interval $I \subseteq \mathbb{R}$, we say that an $E$-valued process $\{M(t), t \in I\}$ is an $E$-valued martingale iff $M(t) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ for $t \in I$ and

$$E(M(t)|\mathcal{F}_s) = M(s), \quad \mathbb{P}\text{-a.s., for all } s \leq t \in I.$$  

4.3. Burkholder inequality. To prove the main result of this section we need the following consequence of the Kahane-Khintchin inequality and the Itô-Nisio Theorem, see [39]. For any $\Lambda \in \gamma(K,E)$, by the Itô-Nisio Theorem, the series $\sum_{j=1}^\infty \beta_j A(f_j)$ is $\mathbb{P}$-a.s. convergent in $E$, where $\{f_j\}_{j \in \mathbb{N}}$ and $\{\beta_j\}_{j \in \mathbb{N}}$ are as in Definition 2.3. Then, as an application of the Kahane-Khintchin inequality, X. Fernique [35] proved that, for any $p \in [1, \infty)$, there exists a positive constant $C(p, E)$ such that,

$$(C(p, E))^{-1} \|\Lambda\|_{\gamma(K,E)} \leq \left( E \left\| \sum_{j \in \mathbb{N}} \beta_j A(f_j) \right\|_E^p \right)^{\frac{1}{p}} \leq C(p, E) \|\Lambda\|_{\gamma(K,E)}.$$  \hfill (4.11)

This inequality tells that the convergence in $L^2(\Omega, \mathcal{F}, \mathbb{P}; E)$ can be replaced by a condition of convergence in $L^p(\Omega, \mathcal{F}, \mathbb{P}; E)$ for some (or any) $p \in [1, \infty)$. Furthermore, we need the following version of the Burkholder inequality which holds in our setting, refer [55] for the proof.

**Theorem 4.2** (Burkholder inequality). Let $E$ be a $M$-type 2 Banach space and $p \in [1, \infty)$. If $\xi \in M^p_{loc}(\mathbb{R}_+, \gamma(K,E))$, then the following conditions hold.
There exists an $E$-valued continuous process $\{\Xi(t), t \in [0, T]\}$, such that

$$\Xi(t) = \int_0^t \xi(s) \, dW(s) \text{ $\mathbb{P}$-a.s., for every } t \in [0, T].$$

(4.12)

The random variable $\Xi(t)$ will be denoted, unless a danger of ambiguity, by $\int_0^t \xi(s) \, dW(s)$.

(ii) There exists a constant $B_p(E) > 0$, independent of $\xi$, such that for each stopping time $\tau > 0$,

$$\mathbb{E} \left[ \sup_{t \geq 0} \left\| \int_0^{t \wedge \tau} \xi(s) \, dW(s) \right\|^p_E \right] \leq B_p(E) \mathbb{E} \left[ \int_0^\tau \|\xi(t)\|^2_{(K,E)} \, dt \right]^\frac{p}{2}. \quad (4.13)$$

Finally, (4.13) holds provided only that $\xi = \{\xi(t), 0 \leq t < \tau\}$ is a local progressively measurable process.

Remark 4.3. /Warning!/ The Itô integral $\int_0^t \xi(s) \, dW(s)$ is, by definition, an element of $L^p(\Omega, \mathcal{F}_t; E)$, thus an equivalence class of a certain class of $\mathcal{F}_t$/$\mathcal{B}(E)$-measurable functions. In what follows we shall use a formally imprecise formulation as $\Xi(t) = \int_0^t \xi(s) \, dW(s) \text{ $\mathbb{P}$-a.s.}$ used in (4.12), instead of the correct, but awkward, one that is incorrect as from the proof in [12] it is clear that the result only holds true if $p \geq 2$.

We ask the reader to refer [12, Corollary 3.7] for a proof of the following result. It is important to mention here that the range of $p$ assumed in the statement of [12, Corollary 3.7] is incorrect as from the proof in [12] it is clear that the result only holds true if $p \geq 2$.

Corollary 4.4. Let $E$ be a $M$-type 2 Banach space and $p \in [2, \infty)$. Then there exists a constant $\tilde{B}_p(E)$, depending on $E$, such that for every $T \in (0, \infty]$ and every $L^p(0, T; E)$-valued progressively measurable process $\{\xi(s), s \in [0, T]\}$,

$$\mathbb{E} \left[ \left\| \int_0^T \xi(s) \, dW(s) \right\|^p_{L^p(0,T;E)} \right] \leq \tilde{B}_p(E) \mathbb{E} \left[ \int_0^T \|\xi(s)\|^2_{(K,L^p(0,T;E))} \, ds \right]^\frac{p}{2}. \quad (4.14)$$

For a $\gamma(K,H)$-valued random variable $\xi$, let us define a $\gamma(K,L^p(0,T;E))$-valued process $\Xi = \{\Xi_r : r \in [0, T]\}$ by

$$\Xi_r := \left\{0, T] \ni t \mapsto \mathbb{1}_{[r, T]}(t) \frac{\sin((t-r)\sqrt{A})}{\sqrt{A}} \xi(r) \right\} \in \gamma(K,L^p(0,T;E)), \ r \in [0, T]. \quad (4.15)$$

We will need the following auxiliary result.

Lemma 4.5. Assume that $T > 0$. Then there exists a positive constant $C(p,T,H)$ such that for every $\gamma(K,H)$-valued progressively measurable process $\xi$, the $\gamma(K,L^p(0,T;E))$-valued process $\{\Xi_r : r \in [0, T]\}$, defined by formula (4.15), is progressively measurable and,

$$\|\Xi_r\|_{(K,L^p(0,T;E))} \leq C(p,T,H) \|\xi(r)\|_{(K,H)}, \text{ for each } r \in [0, T]. \quad (4.16)$$

Proof of Lemma 4.5. Let us consider a sequence $\{\beta_j\}_{j \in \mathbb{N}}$ of i.i.d. $N(0,1)$ random variables on probability space $([\Omega, \mathcal{F}, \mathbb{P})$, and an orthonormal basis $\{f_j\}_{j \in \mathbb{N}}$ of the separable Hilbert space $K$. We first observe that the random variable $\Xi_r$ is well-defined because by Theorem 3.2 for each $r \in [0, T]$ and $x \in H$, the solution of the following homogeneous wave equation

$$\begin{cases}
  u_{tt} - \Delta u = 0 \text{ on } [r, r + T] \\
  u(r) = 0, \ u_t(r) = x,
\end{cases}$$

belongs to $L^p(r, r + T; E)$. In particular,

$$\mathbb{1}_{[r,T]}(\cdot) \frac{\sin((\cdot-r)\sqrt{A})}{\sqrt{A}} x \in L^p(0,T;E),$$
and the map

\[ \Lambda_r : H \ni x \mapsto 1_{[r,T]}(\cdot) \frac{\sin((t-r)\sqrt{A})}{\sqrt{A}} x \in L^p(0,T;E), \] (4.17)

is linear and continuous. Moreover, we have \( \sup_{r \in [0,T]} \| \Lambda_r \| < \infty \). By the above argument and (4.15), we infer that

\[ \Xi_r(\omega) = \Lambda_r \circ [\xi(r,\omega)], \quad (r,\omega) \in [0,T] \times \Omega. \]

Next, for each \( r \in [0,T] \) and (4.15), we infer that, by property (3.7) for \( \xi \), because

\[ \text{Lemma 4.6.} \]

we have the following result.

\[ \| \Lambda_r x \|_{L^p(0,T;E)} = \left( \int_r^T \left\| \frac{\sin((t-r)\sqrt{A})}{\sqrt{A}} x \right\|_E^p \, dt \right)^{\frac{1}{p}} \leq C_T \| x \|_H, \] (4.18)

with the RHS being independent of \( r \). Consequently, we have

\[ \| \Lambda_r \|_{\mathcal{L}(H,L^p(0,T;E))} = \sup_{x \in H} \| \Lambda_r x \|_{L^p(0,T;E)} \leq C_T. \] (4.19)

To move further with the proof let us set the following useful notation:

\[ f_r := 1_{[r,T]}, \quad r \in [0,T] \quad \text{and} \quad g(t,x) := \frac{\sin(t\sqrt{A})}{\sqrt{A}} x, \quad (t,x) \in [0,T] \times H. \] (4.20)

We have the following result.

**Lemma 4.6.** If \( x \in H \), then the function

\[ [0,T] \ni r \mapsto \Lambda_r x \in L^p(0,T;E), \]

is continuous.

**Proof of Lemma 4.6.** We first assume that \( x \in \mathcal{D}(A^{k-\frac{1}{2}}) \), where \( k \in \mathbb{N} \) is such that \( \mathcal{D}(A^k) \hookrightarrow E \). Since \( \mathcal{D}(A^{k-\frac{1}{2}}) \) is dense in \( H \), it is sufficient to prove the lemma for \( x \in \mathcal{D}(A^{k-\frac{1}{2}}) \). Note that, by property (3.7) for \( x \in \mathcal{D}(A^{k-\frac{1}{2}}) \) we have

\[ g(\cdot,x) \in C([0,T];\mathcal{D}(A^k)) \subset C([0,T];E). \]

Since \( \| f_r - f_{r_0} \|_{L^p(0,T;\mathbb{R})}^p = r - r_0 \), we infer that the function

\[ [0,T] \ni r \mapsto f_r \in L^p(0,T;\mathbb{R}), \]

is continuous. Next we claim that if \( g(\cdot,x) \in C([0,T];E) \), then the map

\[ h : [0,T] \ni r \mapsto f_r g(\cdot-r,\cdot) =: h_r \in L^p(0,T;E), \]

is continuous. For this we note that for \( 0 \leq r < s \leq T \) we have

\[ \| h_r - h_s \|_{L^p(0,T;E)}^p = \int_r^s \| h_r(t) \|_E^p \, dt + \int_s^T \| h_r(t) - h_s(t) \|_E^p \, dt, \] (4.21)

because

\[ (h_r - h_s)(t) = \begin{cases} 0 & \text{if } t \in [0,r] \\ h_r(t) & \text{if } t \in [r,s] \\ h_r(t) - h_s(t) & \text{if } t \geq s \end{cases}. \]

Concerning the first integral on the RHS of (4.21) we have

\[ \int_r^s \| h_r(t) \|_E^p \, dt = \int_0^{s-r} \| g(\tau,x) \|_E^p \, d\tau = \int_0^{s-r} \| g(\tau,x) \|_E^p \, d\tau \leq (s-r) \| g(\cdot,x) \|_{C([0,T];E)}^p. \]
For the second integral in the RHS of (4.21), we have
\[
\int_s^T \|h_r(t) - h_s(t)\|_E^p \, dt = \int_0^{T-s} \|g(s - r + \tau, x) - g(\tau, x)\|_E^p \, dt \leq T \Delta(s - r),
\]
where, by the uniform continuity of function \(g(\cdot, x) : [0, T] \to E\),
\[
\Delta(\delta) := \sup_{s, t \in [0, T], \|t - s\| \leq \delta} \|g(t, x) - g(s, x)\|_E \to 0 \text{ as } \delta \to 0.
\]
Hence the continuity of function \(h\) follows and we are done with the proof of Lemma 4.6. \(\square\)

Thus, by coupling the Lemma 4.6 with Proposition 2.10 and \([39, \text{Corollary } 1.1.29]\), we infer that the process \(\Xi\) is progressively measurable.

It only remains to prove the inequality (4.16). For this aim let us fix \(r \in [0, T]\). By invoking the inhomogeneous Strichartz estimate from Theorem 3.2 and (4.11), followed by (4.19), we obtain
\[
\|\Lambda r \circ \xi_{(K, L^p(0, T; E))}\| \lesssim \left(\mathbb{E} \left[ \left\| \sum_{j \in \mathbb{N}} \beta_j \Lambda_r(\xi_{e_j}) \right\|_{L^p(0, T; E)}^p \right] \right)^{\frac{1}{p}} \leq \|\Lambda_r\|_{L^p(H, L^p(0, T; E))} \left(\mathbb{E} \left[ \left\| \sum_{j \in \mathbb{N}} \beta_j \xi_{e_j} \right\|_H^p \right] \right)^{\frac{1}{p}} \leq C(p, H) C_T \|\xi\|_{\gamma(K, H)}.
\]
Hence the proof of Lemma 4.5 is complete. \(\square\)

The following main result of this section is one of the most important ingredient in the proof of the local existence theorem in Section 7.

**Theorem 4.7** (Stochastic Strichartz Estimates). **Let us assume that** \(T > 0\) and \(p \in [2, \infty)\). Then there exist constants \(\tilde{K}_T := K(p, T, H) > 0\) and \(\tilde{C}_T := C(p, T, E, H) > 0\) such that if a process \(\xi\) belongs to \(\mathbb{M}^p(\mathcal{L}^2([0, T], \gamma(K, H)))\), then the following assertions hold.

(I) **There exists an** \(\mathbb{F}\)-adapted, and \(H_A \times H\)-valued\(\text{II}\) continuous process \((\tilde{u}, \tilde{v})\) such that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|\tilde{u}(t)\|_{H_A}^p + \sup_{t \in [0, T]} \|\tilde{v}(t)\|_H^p \right] \leq \tilde{K}_T \mathbb{E} \left[ \int_0^T \|\xi(t)\|_{\gamma(K, H)}^2 \, dt \right]^{\frac{p}{2}}, \tag{4.22}
\]
\[
\tilde{u}(t) = \int_0^t \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}} \xi(s) \, dW(s), \ \text{\(\mathbb{P}\)-a.s., for every } t \in [0, T], \tag{4.23}
\]
\[
\tilde{u}(t) = \tilde{u}(0) + \int_0^t \tilde{v}(s) \, ds \text{ in } H, \ \text{\(\mathbb{P}\)-a.s., for every } t \in [0, T]. \tag{4.24}
\]

(II) **There exists an** \(E\)-valued progressively measurable process \(\tilde{u}\) such that
\[
\mathbb{E} \left[ \int_0^T \|\tilde{u}(t)\|_E^p \, dt \right] \leq \tilde{C}_T \mathbb{E} \left[ \int_0^T \|\xi(t)\|_{\gamma(K, H)}^2 \, dt \right]^{\frac{p}{2}}, \tag{4.25}
\]
\[
j(\tilde{u}(t, \omega)) = i(\tilde{u}(t, \omega)) \text{ for Leb } \otimes \mathbb{P}\text{-almost all } (t, \omega) \in [0, T] \times \Omega, \tag{4.26}
\]

\(\text{II}\)Let us recall that \(H_A = \mathbb{D}(A^\frac{1}{2}).\)
where \( i : H_A \to H \) and \( j : E \to H \) are the natural embeddings.

(III) Moreover, if processes \( \xi_1 \) and \( \xi_2 \) are equivalent, then so are the corresponding processes \( \tilde{u}_1 \) and \( \tilde{u}_2 \). In particular, the map
\[
\mathcal{M}^p(L^2([0,T], \gamma(K,H))) \ni \xi \mapsto \tilde{u} \in L^p(0,T; E),
\]
eextends in a unique way to the following bounded and linear map
\[
\mathcal{M}^p(L^2([0,T], \gamma(K,H))) \ni [\xi] \mapsto [\tilde{u}] \in L^p(0,T; E).
\]

(4.27)

Remark 4.8. Suppose that \( \tau \) is a stopping time such that \( \tau \leq T \) for some \( T > 0 \) and \( \xi(s), s \in [0,\tau) \) is an \( \gamma(K,H) \)-valued progressively measurable process. Since the process \( \mathbb{1}_{[0,\tau)}(s), s \in [0,\infty) \) is well-measurable, see [49, Proposition 4.2] and, see [49, Theorem 1.6], the \( \sigma \)-field of well-measurable sets is smaller than the \( \sigma \)-field of progressively measurable sets, it follows that the process \( \mathbb{1}_{[0,\tau)}(s), s \in [0,\infty) \) is progressively measurable. In particular, the process \( \{\mathbb{1}_{[0,\tau)}(s), s \in [0,\infty)\} \) is progressively measurable. Hence, by applying inequalities (4.22) and (4.23) to the process \( \mathbb{1}_{[0,\tau)} \xi \) we infer the following stopped versions of those inequalities.

(i) There exists an \( \mathbb{F} \)-adapted, \( H_A \times H \)-valued continuous local process \( (\tilde{u}, \tilde{v}) = ((\tilde{u}(t), \tilde{v}(t)), t \in [0,T]) \), such that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|\tilde{u}(t \wedge \tau)\|_{H_A}^p + \sup_{t \in [0,T]} \|\tilde{v}(t \wedge \tau)\|_{H}^p \right] \leq \tilde{K}_T \mathbb{E} \left[ \int_0^\tau \|\xi(t)\|_{\gamma(K,H)}^2 dt \right]^\frac{p}{2},
\]
(4.28)

\[
\tilde{u}(t) = \int_0^t \mathbb{1}_{[0,t \wedge \tau)}(s) \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}} \xi(s) \, dW(s), \quad \mathbb{P}\text{-a.s., for every } t \in [0,T],
\]
(4.29)

\[
\tilde{u}(t) = \tilde{u}(0) + \int_0^t \tilde{v}(s) \, ds \text{ in } H, \quad \mathbb{P}\text{-a.s., for every } t \in [0,\tau).
\]
(4.30)

(ii) There exists an \( E \)-valued progressively measurable local process \( \tilde{u}(t), t \in [0,\tau) \), such that
\[
\mathbb{E} \left[ \int_0^\tau \|\tilde{u}(t)\|_E^p dt \right] \leq \tilde{C}_T \mathbb{E} \left[ \int_0^\tau \|\xi(t)\|_{\gamma(K,H)}^2 dt \right]^\frac{p}{2},
\]
(4.31)

\[
j(\tilde{u}(t,\omega)) = i(\tilde{u}(t,\omega)) \text{ for } \text{Leb} \otimes \mathbb{P}\text{-almost all } (t,\omega) \in [0,\tau] \times \Omega.
\]
(4.32)

Remark 4.9. It follows from the proof that \( \tilde{K}_T \leq M_1 e^{mT} B_p(\mathcal{H}) \) for some constants \( m \geq 0 \) and \( M_1 \geq 1 \) and \( \tilde{C}_T \leq C_T C(p,H) B_p(E) \).

Proof of part (I) of Theorem 4.7. In what follows we fix either the Dirichlet or the Neumann boundary conditions. Let us fix \( p \in [2,\infty) \). To prove the first assertion, let us consider the Hilbert space \( \mathcal{H} := H_A \times H = \mathcal{D}(A^\frac{1}{2}) \times H \) and a linear operator \( \mathcal{A} \) on \( \mathcal{H} \) defined by
\[
\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{D}(A^\frac{1}{2}),
\]
\[
\mathcal{A}(x,y) = (y, -Ax), \quad (x,y) \in \mathcal{D}(\mathcal{A}).
\]
(4.33)

It is well known that since \( \mathcal{A} \) is non-negative and self-adjoint in \( L^2(\mathcal{O}) \), one may prove that \( \mathcal{A} \) generates a \( C_0 \)-group on \( \mathcal{H} \), denoted by \( \{S(t)\}_{t \geq 0} \). Moreover, for \( t \geq 0 \),
\[
S(t)(x,y) = (\cos(t\sqrt{A})x + \sin(t\sqrt{A})/\sqrt{A}y, -\sqrt{A}\sin(t\sqrt{A})x + \cos(t\sqrt{A})y), \quad (x,y) \in \mathcal{H}.
\]
(4.34)

i.e., using a matrix notation,
Applying identity (4.35) to the two previous equalities (4.37)-(4.38) we infer that
\[
\pi_1(S(t)(0, y)) = \sin(t \sqrt{A})/\sqrt{A}y, \quad y \in H. \tag{4.35}
\]
It follows, with being \(\pi_1 : \mathcal{H} \to H_A\) the natural projection, that for \(t \geq 0\),
\[
\pi_1(S(t)(0, y)) = \sin(t \sqrt{A})/\sqrt{A}y, \quad y \in H.
\]
Let us introduce the following auxiliary \(\gamma(K, \mathcal{H})\)-valued process \(\tilde{\xi}\)
\[
\tilde{\xi}(t)[k] = (0, \xi(t)[k]), \quad k \in K, \ t \in [0, T]. \tag{4.36}
\]
Our argument now is based on \[38\]. We begin by observing, see e.g. [75, Theorem 12.2], that there exists an \(\mathcal{H}\)-valued continuous process \(\tilde{\eta} = \{\tilde{\eta}(t), t \in [0, T]\}\) such that
\[
\tilde{\eta}(t) = \int_0^t S(-s) \tilde{\xi}(s) dW(s), \quad \mathbb{P}\text{-a.s., for every } t \in [0, T].
\]
Since \(\{S(t)\}_{t \geq 0}\) is a \(C_0\)-group, we infer that the process \(\tilde{u}\) defined by
\[
\tilde{u}(t) := S(t)\tilde{\eta}(t), \quad t \in [0, T],
\]
is a continuous \(\mathcal{H}\)-valued and that
\[
\tilde{u}(t) = \int_0^t S(t-s) \tilde{\xi}(s) dW(s), \quad \mathbb{P}\text{-a.s., for every } t \in [0, T]. \tag{4.37}
\]
With \(\pi_2 : \mathcal{H} \to H\) being the natural projection, we define continuous \(H_A\) and \(H\)-valued processes \(\tilde{u}\) and \(\tilde{\nu}\), respectively, by
\[
\tilde{u}(t) := \pi_1(\tilde{u}(t)), \quad \text{and} \quad \tilde{\nu}(t) := \pi_2(\tilde{u}(t)), \ t \in [0, T]. \tag{4.38}
\]
Applying identity (4.34) to the two previous equalities (4.37)-(4.38) we infer that
\[
\tilde{u}(t) = \int_0^t \sin((t-s)\sqrt{A})/\sqrt{A}\xi(s) dW(s), \quad \mathbb{P}\text{-a.s. for all } t \in [0, T].
\]
Thus, we proved that \(\tilde{u}\) is \(H_A\)-valued continuous and \(\mathbb{F}\)-adapted process satisfying equality (4.22). Moreover, using the Burkholder inequality (1.13) and the bound property of \(C_0\)-group, we get the following train of inequalities:
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|\tilde{u}(t)\|_{D(A \frac{1}{2})}^p \right] \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \|	ilde{u}(t)\|_{\mathcal{H}}^p \right] = \mathbb{E} \left[ \sup_{t \in [0,T]} \|S(t)\int_0^t S(-s)\tilde{\xi}(s) dW(s)\|_{\mathcal{H}}^p \right] \leq K_T^p B_p(\mathcal{H}) \mathbb{E} \left[ \int_0^T \|S(-s)\tilde{\xi}(s)\|_{\gamma(K,\mathcal{H})}^2 ds \right]^{\frac{p}{2}} \leq K_T B_p(\mathcal{H}) \mathbb{E} \left[ \int_0^T \|\tilde{\xi}(s)\|_{\gamma(K,\mathcal{H})}^2 ds \right]^{\frac{p}{2}} = K_T B_p(\mathcal{H}) \mathbb{E} \left[ \int_0^T \|\xi(s)\|_{\gamma(K,H)}^2 ds \right]^{\frac{p}{2}},
\]
where \(K_T \leq M_1 e^{mT}\) for some constants \(m \geq 0\) and \(M_1 \geq 1\). This yields inequality (4.22) and, in particular, assertion of part (i).

Equality (4.34) follows from Proposition [C.1] and equalities (4.38).

Proof of part (II) of Theorem 4.7. We split the proof into two steps. First we will prove the theorem for more regular processes. Then we will transfer the results to the right class of processes by employing a suitable approximation.
Step 1: We begin by observing that by the classical Sobolev embedding theorem there exists natural number $k$ such that the Hilbert space

$$D(A^{k+\frac{1}{2}})$$

is continuously embedded into the Banach space $E = D(A_0^{\frac{1}{2}-})$. (4.39)

Let us fix $p \in [2, \infty)$. Let us assume that a process $\xi$ belongs to $M^p(\mathcal{L}^2([0, T], \gamma(K, D(A^k))))$. By assertion (i), we infer that there exists an $\mathcal{F}$-adapted $D(A^{k+\frac{1}{2}})$-valued continuous process $\tilde{u}$ which satisfies condition (4.23) and the following inequality

$$\mathbb{E}\left[ \sup_{t \in [0, T]} \|\tilde{u}(t)\|^p_{D(A^{k+\frac{1}{2}})} \right] \leq K(p, T, H) \mathbb{E}\left[ \int_0^T \|\xi(t)\|^2_{\gamma(K, D(A^k))} dt \right]^\frac{p}{2} < \infty.$$

Also, let us note that, in view of our additional assumption (4.39), the process $\tilde{u}$ is an $\mathcal{F}$-adapted $H_A$-valued continuous (hence progressively measurable), and

$$\mathbb{E}\left[ \|\tilde{u}\|^p_{C([0, T]; H_A)} \right] < \infty.$$

Next, we define an $\gamma(K, L^p(0, T; D(A^k)))$-valued process $\Xi = \{\Xi_r : r \in [0, T]\}$ by formula (4.15). This process, in view of Lemma 4.5, is progressively measurable and, by the Burkholder inequality (4.14) together with inequality (4.16), it satisfies the following inequality

$$\mathbb{E}\left[ \left\| \int_0^T \Xi_r dW(r) \right\|^p_{L_p(0, T; E)} \right] \leq C_T C(p, H) \hat{B}_p(E) \mathbb{E}\left[ \int_0^T \|\xi(r)\|^2_{\gamma(K, H)} dr \right]^\frac{p}{2}. \tag{4.40}$$

Let us choose an $\mathcal{F}_T / \mathcal{B}(L^p(0, T; E))$-measurable function $\tilde{u} : \Omega \to L^p(0, T; E)$ such that, see Remark 4.3

$$\tilde{u} = \int_0^T \Xi_r dW(r) \text{ in } L^p(0, T; E), \quad \mathbb{P}\text{-a.s..} \tag{4.41}$$

By the first part of Proposition 4.1, there exists an $\mathcal{B}([0, T]) \otimes \mathcal{F}_T / \mathcal{B}(E)$-measurable function

$$\tilde{u} : [0, T] \times \Omega \to E,$$

and a set $\Omega' \subset \Omega$ such that $\mathbb{P}(\Omega') = 1$ and for every $\omega \in \Omega'$,

$$\tilde{u}(\cdot, \omega) = \tilde{u}(\omega) \text{ in } L^p(0, T; E).$$

Later on we will show that $\tilde{u}$ is progressively measurable process. Then, by inequality (4.40) we infer that process $\tilde{u}$ satisfies inequality (4.23).

Let us recall that $i : H_A \hookrightarrow H$ and $j : E \hookrightarrow H$ are the natural embeddings. We define corresponding Nemyski type embeddings $I$ and $J$ by

$$I : C([0, T]; H_A) \ni f \mapsto i \circ f \in L^2(0, T; H), \quad \tag{4.42}$$

$$J : L^p(0, T; E) \ni f \mapsto j \circ f \in L^2(0, T; H), \quad \tag{4.43}$$

and observe that both $I$ and $J$ are continuous. Therefore we deduce that $J \circ \tilde{u}$ is an $L^2(0, T; H)$-valued random variable and

$$J \circ \tilde{u} = \int_0^T (J \circ \Xi_r) dW(r) \text{ in } L^2(0, T; H), \quad \mathbb{P}\text{-a.s..}$$

Since the process $\tilde{u}$ has $\mathbb{P}$-almost surely continuous $H_A$-valued trajectories, by [32] Proposition 3.18 it induces, in a natural way, an $\mathcal{F}_T / \mathcal{B}(C([0, T]; H_A))$-measurable function $\tilde{u} : \Omega \to \mathbb{R}^{\mathbb{R}_+}$.
C([0, T]; H_A). Because the map \( I : C([0, T]; H_A) \to L^2(0, T; H) \) is continuous, \( I(\tilde{u}) : \Omega \to L^2(0, T; H) \) is \( \mathcal{F}_T / \mathcal{B}(L^2(0, T; H)) \)-measurable. We claim that

\[
I(\tilde{u}) = \int_0^T (J \circ \Xi_r) dW(r) \quad \text{in} \quad L^2(0, T; H), \quad \mathbb{P}\text{-a.s.} \tag{4.44}
\]

**Proof of equality (4.44).** Since \( H \) is a separable Hilbert space, the Banach space \( C([0, T]; H) \) is also separable. Let us choose a dense subset \( \{ h_n : n \in \mathbb{N} \} \) in \( C([0, T]; H) \). Let us choose and fix \( n \in \mathbb{N} \) and define a linear and bounded operator

\[
H_n : L^2(0, T; H) \ni x \mapsto \langle x, h_n \rangle_{L^2(0, T; H)} \in \mathbb{R}. \tag{4.45}
\]

Thus, we infer that

\[
H_n(\int_0^T (J \circ \Xi_r) dW(r)) = \langle \int_0^T (J \circ \Xi_r) dW(r), h_n \rangle_{L^2(0, T; H)} \quad \text{on} \quad \Omega, \tag{4.46}
\]

and

\[
H_n(\int_0^T (J \circ \Xi_r) dW(r)) = \int_0^T (H_n \circ J \circ \Xi_r) dW(r) \quad \text{in} \quad \mathbb{R}, \quad \mathbb{P}\text{-a.s.}
\]

Let us note that by definitions (4.45) of \( H_n \), (4.43) of \( J \) and (4.15) of \( \Xi_r \) we have, for all \( k \in K \) and \( r \in [0, T] \), the following equality

\[
(H_n \circ J \circ \Xi_r)[k] = H_n(\langle J \circ \Xi_r \rangle[k]) = \langle \langle J \circ \Xi_r \rangle, h_n \rangle_{L^2(0, T; H)}
\]

\[
= \int_0^T \langle (\Xi_r[k](t)), h_n(t) \rangle_H dt = \int_0^T \mathbf{1}_{[r,T]}(t) \frac{\sin((t-r)\sqrt{A})}{\sqrt{A}} \langle \Xi(r)[k], h_n(t) \rangle_H dt.
\]

Therefore, we infer that \( \mathbb{P} \)-almost surely

\[
H_n(\int_0^T (J \circ \Xi_r) dW(r)) = \int_0^T \int_0^T \mathbf{1}_{[r,T]}(t) \frac{\sin((t-r)\sqrt{A})}{\sqrt{A}} \langle \Xi(r)[k], h_n(t) \rangle_H dt dW(r)
\]

\[
= \int_0^T \int_0^T \mathbf{1}_{[0,t]}(r) \frac{\sin((t-r)\sqrt{A})}{\sqrt{A}} \langle \Xi(r), h_n(t) \rangle_H dW(r) dt, \tag{4.47}
\]

where the last equality is a consequence of the stochastic Fubini Theorem \[8\], which is a generalization of \[23\] Theorem 2.4.16 and \[72\] Theorem 2.2.

On the other hand, by the definition (4.42) of the map \( I \), we have

\[
\langle I(\tilde{u}), h_n \rangle_{L^2(0, T; H)} = \int_0^T \langle I(\tilde{u})(t), h_n(t) \rangle_H dt \tag{4.48}
\]

\[
= \int_0^T \langle \tilde{u}(t), h_n(t) \rangle_H dt = \int_0^T \langle \tilde{u}(t), h_n(t) \rangle_H dt \tag{4.48}
\]

\[
= \int_0^T \langle \tilde{u}(t), h_n(t) \rangle_H dt = \int_0^T \mathbf{1}_{[0,t]}(r) \frac{\sin((t-r)\sqrt{A})}{\sqrt{A}} \langle \Xi(r), h_n(t) \rangle_H dW(r) dt.
\]

Thus, from (4.46), (4.48) and (4.47), we infer that for every \( n \in \mathbb{N} \),

\[
\langle I(\tilde{u}), h_n \rangle_{L^2(0, T; H)} = \int_0^T (J \circ \Xi_r) dW(r), h_n \rangle_{L^2(0, T; H)}, \quad \mathbb{P}\text{-a.s.}
\]

By the density of the countable set \( \{ h_n \} \) in \( L^2(0, T; H) \) we deduce (4.44). \( \square \)
From the just proven equation (4.41) and equality (4.41) we infer that
\[ I \circ \tilde{u} = J \circ \tilde{u} \text{ in } L^2(0, T; H), \quad \mathbb{P}\text{-a.s..} \]
Hence, by the second part of Proposition B.1, we infer that the \( H \)-valued processes \( \tilde{i}(\tilde{u}) : [0, T] \times \Omega \to H \) and \( j(\tilde{u}) : [0, T] \times \Omega \to H \), are \( \text{Leb} \otimes \mathbb{P} \) equal. Since, the former is \( H \)-valued progressively measurable, by the Kuratowski Theorem, see e.g. [61, Corollary I.3.3] and the argument in the proof of [9, Proposition A.1] we infer that process \( \tilde{u} \) is \( E \)-valued progressively measurable. This concludes the proof of Step 1.

**Step 2:** The result follows by applying Step 1. Let \( \xi \) be a progressively measurable process from the space \( M^p(L^2([0, T], \gamma(K, H))) \), and \( k \in \mathbb{N} \) as in Step 1.

We choose a sequence \( \{\xi_n\}_{n \in \mathbb{N}} \) of processes from \( M^p(L^2([0, T], \gamma(K, D(A^k))) \) s.t.
\[ ||\xi_n - \xi||_{M^p(L^2([0, T], \gamma(K, H)))} \to 0 \text{ sufficiently fast as } n \to \infty. \] (4.49)

We denote the corresponding processes for \( \xi_n \), from the previous step, by \( \tilde{u}_n \) and \( \tilde{u}_n \). By Step 1, for each \( n \), the processes \( \tilde{u}_n \) and \( \tilde{u}_n \) satisfy the condition (4.20), the process \( \tilde{u}_n \) satisfies inequality (4.22) and the process \( \tilde{u}_n \) satisfies inequality (4.25). Thus, both sequences are Cauchy in the appropriate Banach spaces \( M^p(L^\infty([0, T], H_A)) \) and \( M^p([0, T], E) \), respectively. Hence, there exist unique elements in those spaces, whose representatives, respectively, we denote by \( \tilde{u} \) and \( \tilde{u} \). Because the convergence (4.49) is sufficiently fast, we deduce that \( \tilde{u} \) is \( H_A \)-valued \( \mathbb{F} \)-adapted and continuous process and \( \tilde{u} \) is an \( E \)-valued progressively measurable process. Moreover, the processes \( \tilde{u} \) and \( \tilde{u} \) satisfy the condition (4.20). Hence we are done with the proof of part (ii) of Theorem 4.7.

\[ \square \]

**Proof of part (III) of Theorem 4.7** This part follows straightforwardly from the second part of Proposition B.1.

\[ \square \]

5. LOCAL WELL-POSEDNESS - PRELIMINARY RESULTS

The aim of this section is to formulate and prove some preliminary results which will be helpful in Section 7 where we show the existence and uniqueness of solutions to the stochastic wave equation (1.2). Recall that we are working in the setting mentioned in the Subsection 4.1.

Let us also recall the following definitions. For any \( T > 0 \), we put
\[ X_T := C([0, T]; H_A), \quad Y_T := L^p(0, T; E), \quad Q_T := C([0, T]; H) \text{ and } Z_T := X_T \cap Y_T. \] (5.1)

Obviously, \( X_T, Y_T, Q_T \) and \( Z_T \) are (separable) Banach spaces with naturally defined norms, see e.g. (1.4) - (5.4).

By \( M^p(Z_T) \) we denote the Banach space of (equivalence classes) of all \( E \)-valued progressively measurable processes \( \{u(t) : t \in [0, T]\} \) having a continuous \( H_A \)-valued modification and satisfying
\[ \|u\|_{M^p(Z_T)} := E[\|u\|_{L^p(0, T; E)}^p] = E[\|u\|_{L^p([0, T]; H_A)}^p + \|u\|_{L^p(0, T; E)}^p] < \infty. \] (5.2)

Analogously, by \( M^p(Q_T) \) we denote the Banach space of all \( H \)-valued continuous and progressively measurable processes \( \{u(t), t \in [0, T]\} \) satisfying
\[ \|u\|_{M^p(Q_T)} := E[\|u\|_{C([0, T]; H)}^p] < \infty. \] (5.3)

If \( \tau \) is a bounded stopping time, by \( M^p(Z_\tau) \) we mean the Banach space of (equivalence classes of) all progressively measurable processes \( u : [0, \tau] \times \Omega \to E \) which have a continuous
\(H_A\)-valued modification such that for each \(\omega \in \Omega\), \(u(\cdot, \omega) \in Z_{\tau(\omega)}\) and
\[
\int_\Omega \left[ \|u(\omega)\|^p_{C([0,\tau(\omega)]; H_A)} + \|u(\omega)\|^p_{L^p(0,\tau(\omega); E)} \right] dP(\omega) < \infty.
\]

5.1. SNLWE and assumptions. Here we recall the stochastic nonlinear wave equation we consider here and state the assumptions on the drift and diffusion terms. To be precise, we consider the following Cauchy problem for stochastic nonlinear wave equation with the Dirichlet or the Neumann boundary condition
\[
\begin{cases}
    u_{tt} + Au + F(u) = G(u)\dot{W}, \\
    u(0) = u_0, \quad u_t(0) = u_1,
\end{cases}
\]
where \(A\) is either \(-\Delta_D\) or \(-\Delta_N\); \((u_0, u_1) \in H_A \times L^2(\Omega)\) and \(W = \{W(t), t \geq 0\}\) is a cylindrical Wiener process on some real separable Hilbert space \(K\) such that \(K \hookrightarrow L^\infty(\Omega)\) and for some orthonormal basis \(\{f_j\}_{j \in \mathbb{N}}\) of \(K\),
\[
\sum_{j \in \mathbb{N}} \|f_j\|_{L^\infty(\Omega)}^2 < \infty.
\]

We assume the following hypotheses for the nonlinearity \(F\) and the diffusion coefficient \(G\) in equation (5.4).

A.1 Assume that a map
\[F : B_{H_A}(0,1) \cap E \to H,\]
where for \(R > 0\), \(B_{H_A}(0,R)\) is the open ball in the space \(H_A\), centered at the origin and of radius \(R\), i.e.
\[B_{H_A}(0,R) := \{u \in H_A : \|u\|_{H_A} < R\},\]
is such that
\[F(0) = 0,\]
and there exists a \(\gamma_F > 0\) such that for every \(M \in (0,1)\) there exists a positive real number \(C_F := C_F(M)\), such that the following holds
\[
\|F(u) - F(v)\|_H \leq C_F(M) [1 + \|u\|_E + \|v\|_E]^{\gamma_F} \|u - v\|_{H_A},
\]
provided
\[u, v \in B_{H_A}(0,M) \cap E.\]

A.2 Assume that a map
\[G : B_{H_A}(0,1) \cap E \to \gamma(K,H),\]
is such that
\[G(0) = 0,\]
and there exists a \(\gamma_G > 0\) such that for every \(M \in (0,1)\) there exists a positive real number \(C_G(M) := C_G(M,\gamma_G)\) such that if \(u, v\) satisfy (5.8), then
\[
\|G(u) - G(v)\|_{\gamma(K,H)} \leq C_G(M) [1 + \|u\|_E + \|v\|_E]^{\gamma_G} \|u - v\|_{H_A}.
\]

Remark 5.1. Without loss of generality we will assume that \(\gamma_F = \gamma_G =: \gamma\).
Lemma 5.3. Let us assume that Remark 5.1, there exists a map \( F = \tilde{F}_M \) and \( G = \tilde{G}_M \), defined on \( H_A \cap E \), taking values in spaces \( H \) and \( \gamma(K,H) \), respectively, such that
\[
\tilde{F}_M(u) = F(u) \quad \text{and} \quad \tilde{G}_M(u) = G(u) \quad \text{if} \quad u \in \tilde{B}_H(0, M) \cap E,
\]
and the inequalities (5.7) and (5.9) hold true for every \( u, v \in H_A \cap E \). In particular, with Remark 5.1 there exists a \( \gamma > 0 \) such that for every \( u, v \in H_A \cap E \)
\[
\|\tilde{F}_M(u) - \tilde{F}_M(v)\|_H \leq 3C_F(M) [1 + \|u\|_E + \|v\|_E]^{\gamma} \|u - v\|_{H_A},
\]
and
\[
\|\tilde{G}_M(u) - \tilde{G}_M(v)\|_{\gamma(K,H)} \leq 3C_G(M) [1 + \|u\|_E + \|v\|_E]^{\gamma} \|u - v\|_{H_A}.
\]

The next two lemmata are a straightforward but important consequences of Remark 5.2 and Assumptions [A.1] and [A.2].

Lemma 5.4. Let us assume that \( M \in (0,1) \). Let us assume that the function \( F : \tilde{B}_H(0, M) \cap E \rightarrow H \) satisfies Assumption [A.1] with \( \gamma = \gamma_F \) and let \( T > 0 \) and \( p \geq \gamma \). Let \( \tilde{F} = \tilde{F}_M \) be the extension of \( F \) as introduced in Remark 5.2 above. Then there exists a number \( K(\gamma) > 0 \) such that the following inequality holds, provided \( u_1, u_2 \in Z_T \),
\[
\|\tilde{F}_M(u_1) - \tilde{F}_M(u_2)\|_{L^1(0,T;H)} \leq 3K(\gamma)C_F(M) \left[ T + T^{1-\frac{2p}{\gamma}} \left( \int_0^T \|u_1(t)\|_E^p dt \right)^\frac{1}{p} + T^{1-\frac{2p}{\gamma}} \left( \int_0^T \|u_2(t)\|_E^p dt \right)^\frac{1}{p} \right].
\]

Hence Lemma 5.4 follows.

Lemma 5.5. Let us assume that \( M \in (0,1) \). Let us assume that the function \( G : \tilde{B}_H(0, M) \cap E \rightarrow \gamma(K,H) \) satisfies Assumption [A.2] with \( \gamma = \gamma_G \) and let \( T > 0 \) and \( p \geq 2\gamma \). Let \( \tilde{G} \) be the extension of \( G \) as in Remark 5.2. Then there exists a real number \( K(2\gamma) > 0 \) such that the following inequality holds, provided \( u_1 \) and \( u_2 \) belong to \( Z_T \),
\[
\|\tilde{G}_M(u_1) - \tilde{G}_M(u_2)\|^2_{L^2(0,T;\gamma(K,H))} \leq 9K(2\gamma)C_G^2(M) \left[ T + T^{1-\frac{2p}{2\gamma}} \left( \int_0^T \|u_1(t)\|_{E'}^p dt \right)^\frac{1}{p} + T^{1-\frac{2p}{2\gamma}} \left( \int_0^T \|u_2(t)\|_{E'}^p dt \right)^\frac{1}{p} \right].
\]

Hence the proof of Lemma 5.5 is complete.
We set the notation \( C_{F,M} := 3C_F(M) \) and \( C_{G,M} := 3C_G(M) \) and use them in Section 6. The Lemmata 5.3 and 5.5 have been formulated in the language of the extensions \( \tilde{F} \) and \( \tilde{G} \) of functions \( F \) and \( G \). The following corollary is written in the language of the original functions \( F \) and \( G \).

**Corollary 5.6.** Let \( T > 0 \) and \( M \in (0,1) \). Let us assume that the function \( F : B_{H_A}(0,1) \cap E \to H \) satisfies Assumption \( \text{A.1} \) with \( \gamma = \gamma_F \) and \( p \geq 2 \gamma \). Then there exists \( \gamma = \gamma_F \) such that the following inequality holds

\[
\| F(u_1) - F(u_2) \|_{L^1(0,T;H)} \leq K(\gamma)C_F(M) \left[ T + T^{1 - \frac{2\gamma}{p}} \left( \frac{\| u_1 \|_{Y_T}^{2\gamma}}{M^{2\gamma}} + \frac{\| u_2 \|_{Y_T}^{2\gamma}}{M^{2\gamma}} \right) \right] \| u_1 - u_2 \|_{X_T},
\]

for all \( u_1, u_2 \in Z_T \) satisfying

\[
\sup_{t \in [0,T]} \| u(t) \|_{H_A} \leq M \quad \text{and} \quad \sup_{t \in [0,T]} \| v(t) \|_{H_A} \leq M. \tag{5.13}
\]

Let us assume that the function \( G : B_{H_A}(0,1) \cap E \to \gamma(K,H) \) satisfies Assumption \( \text{A.2} \) with \( \gamma = \gamma_G \) and \( p \geq 2 \gamma \). Then there exists \( \gamma = \gamma_G \) such that the following inequality holds

\[
\| G(u_1) - G(u_2) \|_{L^2(0,T;\gamma(K,H))} \leq K(\gamma)C_G(M) \left[ T + T^{1 - \frac{2\gamma}{p}} \left( \frac{\| u_1 \|_{Y_T}^{2\gamma}}{M^{2\gamma}} + \frac{\| u_2 \|_{Y_T}^{2\gamma}}{M^{2\gamma}} \right) \right] \| u_1 - u_2 \|_{X_T},
\]

for all \( u_1, u_2 \in Z_T \) satisfying condition \( (5.13) \).

To prove the main result of this Section 5 we need the following known results. The first one is from [60].

**Theorem 5.7.** [Moser-Trudinger Inequality]
Let \( \mathcal{O} \subseteq \mathbb{R}^2 \) be a domain (bounded or unbounded) and \( \alpha \leq 4\pi \). Then

\[
C(\alpha, \mathcal{O}) := \sup_{u \in H^{1,2}(\mathcal{O}) \cap \| u \|_{H^{1,2}(\mathcal{O})} \leq 1} \int_{\mathcal{O}} \left( e^{\alpha(u(x))^2} - 1 \right) dx < +\infty. \tag{5.14}
\]

Moreover, this result is sharp in the sense that if \( \alpha > 4\pi \) then \( C(\alpha, \mathcal{O}) = \infty \).

The next required result is the well-known Logarithmic inequality from [60].

**Theorem 5.8.** Let \( \mathcal{O} \) be a domain in \( \mathbb{R}^d \). Let \( p, q, m \in \mathbb{R} \) satisfy \( 1 < p < \infty, 1 \leq q < \infty \), and \( m > \frac{d}{q} \). Then there exists a positive constant \( L \) such that for all \( u \in H^{\frac{d}{q},p}(\mathcal{O}) \cap H^{m,q}(\mathcal{O}) \) the following holds,

\[
\| u \|_{L^\infty(\mathcal{O})} \leq L \| u \|_{H^{\frac{d}{q},p}(\mathcal{O})} \left[ 1 + \log(1 + \| u \|_{H^{m,q}(\mathcal{O})}) \right]^{\frac{1}{p}}, \tag{5.15}
\]

In the next two results we provide an example of functions \( f \) and \( g \) such that the corresponding \( F \) and \( G \) satisfy the assumptions \( \text{A.1} \) and \( \text{A.2} \), respectively. The example below comes from [41] and [42] when \( E \) is a suitable Hölder space. We will prove the next result in detail because we need a slightly more general version of the Moser-Trudinger inequality and the Logarithmic estimate, respectively, see Theorem 5.7 and 5.8 than those used in [41] and [42].
**Lemma 5.9.** Assume that \( \mathcal{O} \subseteq \mathbb{R}^2 \) is a bounded domain. Let \( L \) be a constant from the Theorem 5.6. Let \( h : \mathbb{R} \to \mathbb{R} \) be a function defined by \( h(x) = \pm x(e^{4\pi x^2} - 1) \) for \( x \in \mathbb{R} \). Assume that a pair \((q, r)\) of positive numbers satisfies
\[
0 < r + \frac{2}{q} < 1.
\]
(5.16)
Then for every \( M \in (0, 1) \) there exists a constant \( C_{M,L} > 0 \), which depending only on \( M \) and \( L \), such that
\[
\|h \circ u - h \circ v\|_{L^2(\mathcal{O})} \leq C_{M,L} \left[1 + \|u\|_{H^{1-r,q}(\mathcal{O})} + \|v\|_{H^{1-r,q}(\mathcal{O})}\right]^{2\pi L^2} \|u - v\|_{H^{1,2}(\mathcal{O})},
\]
(5.17)
provided \( u, v \in H^{1,2}(\mathcal{O}) \cap H^{1-r,q}(\mathcal{O}) \) satisfy the following condition
\[
\|u\|_{H^{1,2}(\mathcal{O})} \leq M \quad \text{and} \quad \|v\|_{H^{1,2}(\mathcal{O})} \leq M.
\]
(5.18)
In the next result, which is about a generalized Nemytskii operator \( G \) associated with function \( g \), \( \{f_j\}_{j \in \mathbb{N}} \) is an ONB of a Hilbert space \( K \).

**Lemma 5.10.** Assume that condition (5.6) holds. Assume that \( g(x) = x(e^{4\pi x^2} - 1), x \in \mathbb{R} \) and a pair \((q, r)\) exists which satisfies condition (5.16). Let \( L \) be a constant from the Theorem 5.7 and \( G \) be defined by
\[
G(u) := \{K \ni k \mapsto (g \circ u) \cdot k \in L^2(\mathcal{O})\}, \quad u \in B_{H^{1,2}(\mathcal{O})}(0,1) \cap H^{1-r,q}(\mathcal{O}).
\]
Then for every \( M \in (0, 1) \) the following inequality holds
\[
\|G(u) - G(v)\|_{\gamma(K,L^2(\mathcal{O}))} \leq \tilde{C}_G \left[1 + \|u\|_{H^{1-r,q}(\mathcal{O})} + \|v\|_{H^{1-r,q}(\mathcal{O})}\right]^{2\pi L^2} \|u - v\|_{H^{1,2}(\mathcal{O})},
\]
where
\[
\tilde{C}_G := C_{M,L} \times \left(\sum_{j \in \mathbb{N}} \|f_j\|^2_{L^\infty(\mathcal{O})}\right)^{\frac{1}{2}},
\]
for all \( u, v \in H^{1,2}(\mathcal{O}) \cap H^{1-r,q}(\mathcal{O}) \) satisfying condition (5.18).

**Remark 5.11.** Both Lemmata 5.9 and 5.10 are applicable to spaces defined in (4.11) because \( H_A \subset H^{1,2}(\mathcal{O}) \) and \( E \subset H^{1-r,q}(\mathcal{O}) \).

**Proof of Lemma 5.5.** Let \( u \) and \( v \) belong to \( H^{1,2}(\mathcal{O}) \cap H^{1-r,q}(\mathcal{O}) \) satisfying condition (5.18). By assumption (5.6) and Lemma 5.9 (applied to \( h = g \)) we infer that
\[
\|G(u) - G(v)\|^2_{\gamma(K,L^2(\mathcal{O}))} = \sum_{j \in \mathbb{N}} \|G(u)f_j - G(v)f_j\|^2_{L^2(\mathcal{O})}
\]
\[
= \sum_{j \in \mathbb{N}} \|(g \circ u)f_j - (g \circ v)f_j\|^2_{L^2(\mathcal{O})} \leq \|g \circ u - g \circ v\|^2_{L^2(\mathcal{O})} \sum_{j \in \mathbb{N}} \|f_j\|^2_{L^\infty(\mathcal{O})},
\]
as required. Hence the result follows by applying the inequality (5.17). \( \square \)

**Proof of Lemma 5.9.** We only prove the result for \( h(x) = x(e^{4\pi x^2} - 1) \), since the proof for the function \( -x(e^{4\pi x^2} - 1) \) is analogous. We begin here with the following observation which is a consequence of the Mean Value Theorem. If \( u, v \in \mathbb{R} \), then the following equality holds
\[
h(u) - h(v) = (u - v) \int_0^1 \left[(1 + 8\pi u^2 e^{4\pi v^2} - 1]\right] d\theta.
\]
(5.19)
Let us now fix \( M \in (0, 1) \) and choose \( \zeta \in (0, 1) \) and \( \varepsilon > 0 \), such that
\[
(1 + \varepsilon)(1 + \zeta)M^2 \leq 1.
\]
(5.18)
Let us take arbitrary \( u, v \in H^{1,2}(\Omega) \cap H^{1-\gamma, q}(\Omega) \) satisfying condition (5.18). Applying the above for \( u = u(x) \) and \( v = v(x) \), for a fixed \( x \in \Omega \) we get, with \( u_\theta(x) := (1 - \theta)u(x) + \theta v(x) \),

\[
h(u(x)) - h(v(x)) = (u(x) - v(x)) \int_0^1 \left[(1 + 8\pi u_\theta^2(x))e^{4\pi u_\theta^2} - 1\right] \, d\theta.
\]

Thus, we infer that

\[
\| h \circ u - h \circ v \|_{L^2(\Omega)} \leq \left\| (u - v) \int_0^1 \left[(1 + 8\pi u_\theta^2) e^{4\pi u_\theta^2} - 1\right] \, d\theta \right\|_{L^2(\Omega)}.
\]

Applying the Minkowski inequality gives

\[
\left\| (u - v) \int_0^1 \left[(1 + 8\pi u_\theta^2) e^{4\pi u_\theta^2} - 1\right] \, d\theta \right\|_{L^2(\Omega)} \leq \int_0^1 \left\| (u - v) \left[(1 + 8\pi u_\theta^2) e^{4\pi u_\theta^2} - 1\right] \right\|_{L^2(\Omega)} \, d\theta.
\]

Then due to the Hölder inequality, the Sobolev embedding and the following basic inequality

\[
(1 + 2\alpha)e^a - 1 \leq 2\left(1 + \frac{1}{\varepsilon}\right)(e^{1+\varepsilon}a - 1), \quad \alpha, \varepsilon > 0,
\]

we infer that

\[
\left\| \left(1 + 8\pi u_\theta^2\right) e^{4\pi u_\theta^2} - 1 \right\|_{L^2(\Omega)}^2 \leq 4\left(1 + \frac{1}{\varepsilon}\right)^2 \left\| (u - v)(e^{4\pi(1+\varepsilon)u_\theta^2} - 1) \right\|_{L^2(\Omega)}^2
\]

\[
\leq 4\left(1 + \frac{1}{\varepsilon}\right)^2 \left\| u - v \right\|_{L^{2+\frac{2}{\varepsilon}}(\Omega)}^2 \left\| e^{4\pi(1+\varepsilon)u_\theta^2} - 1 \right\|_{L^{1+\frac{2}{\varepsilon}}(\Omega)}^2
\]

\[
\leq 4C_{10}\left(1 + \frac{1}{\varepsilon}\right)^2 \left\| u - v \right\|_{D(A^2)}^2 e^{4\pi(1+\varepsilon)\|u_\theta\|_{L^\infty(\Omega)}} e^{4\pi(1+\varepsilon)\|u_\theta\|_{L^1(\Omega)}} - 1 \right\|_{L^{1+\frac{2}{\varepsilon}}(\Omega)}.
\]

Moreover, since \( \theta \in (0, 1) \), \( \|u_\theta\|_{H^{1,2}(\Omega)} \leq M \) and therefore (5.19) holds. Thus, the Moser-Trudinger inequality from Theorem 5.7 gives

\[
\| e^{4\pi(1+\varepsilon)\|u_\theta\|_{L^\infty(\Omega)}} - 1 \right\|_{L^{1+\frac{2}{\varepsilon}}(\Omega)} \leq \| e^{4\pi(1+\varepsilon)(1+\varepsilon)\|u_\theta\|_{L^1(\Omega)}} - 1 \right\|_{L^{1+\frac{2}{\varepsilon}}(\Omega)} \leq C(4\pi, 0).
\]

Invoking the estimate from Theorem 5.8 which is possible due to (5.19) and Lemma 2.2, we obtain

\[
e^{4\pi(1+\varepsilon)\|u_\theta\|_{L^\infty(\Omega)}} \leq \exp \left[4\pi L^2(1 + \varepsilon)\|u_\theta\|_{H^{1,2}(\Omega)}^2 \left\{1 + \log \left(1 + \frac{\|u_\theta\|_{H^{1-\gamma, q}(\Omega)}}{\|u_\theta\|_{H^{1,2}(\Omega)}}\right)\right\}\right].
\]

Using the fact that if \( B_1 > 1, B_2 > 0 \), then the function \( x \mapsto x^2 \left(1 + \log \left(B_1 + \frac{B_2}{x}\right)\right) \) is non-decreasing, we deduce that,

\[
e^{4\pi(1+\varepsilon)\|u_\theta\|_{L^\infty(\Omega)}} \leq \left[ e\left(1 + \frac{\|u_\theta\|_{H^{1-\gamma, q}(\Omega)}}{M}\right)\right]^{4\pi L^2(1+\varepsilon)M^2}.
\]

Let us put

\[
\gamma := 2\pi L^2(1 + \varepsilon)M^2.
\]

Note that, since \((1+\varepsilon)M^2 < 1\), from (5.19) we get that \(\gamma < 2\pi L^2\). Next, from (5.20), (5.21), (5.22), and (5.23), we infer that

\[
\| h \circ u - h \circ v \|_{L^2(\Omega)} \leq 2\sqrt{C_{10}}\left(1 + \frac{1}{\varepsilon}\right)\left(\frac{\varepsilon}{M}\right)^{2\pi L^2(C(4\pi, 0))\frac{1}{1+\varepsilon}}\| u - v \|_{D(A^2)} \times...
\]
variable which satisfies the following condition

\[ \lim_{u \in H^{1,2}(0) \cap H^{1-r,q}(0), \|u\|_{H^{1,2}(0)} \downarrow 1} \|h \circ u\|_{L^2(0)} = \infty? \]  

(5.25)

\[ \lim_{r \not\in M} \|h \circ (rv)\|_{L^2(0)} = \infty? \]  

(5.26)

5.2. Definition of a local mild solution. In this subsection we introduce the definitions of local and maximal local solutions we adopt in this paper. They are modifications of definitions used in earlier papers, see e.g. [11].

**Definition 5.13.** Let \( p \in [2, \infty) \). Assume that \((u_0, u_1)\) is an \( \mathcal{F}_0 \)-measurable \( H_A \times H \) random variable which satisfies the following condition

\[ \|u_0\|_{H_A} < 1, \quad \mathbb{P}\text{-almost surely,} \]  

(5.27)

**A.** A local mild solution to problem \((5.4)-(5.5)\) is a \( \mathcal{D}(A^2) \)-valued continuous and \( \mathbb{F} \)-adapted process \( u = \{u(t) : t \in [0, \tau]\} \) satisfying the following conditions

1. \( \tau \) is an accessible stopping time,
2. the condition \((5.27)\) is preserved, i.e.

\[ \|u(t)\|_{H_A} < 1, \quad \text{for } t \in [0, \tau), \quad \mathbb{P}\text{-a.s.,} \]  

(5.28)

3. there exists an announcing sequence \( \{\tau_k\}_{k \geq 1} \) of the stopping times for \( \tau \), such that

\[ u \text{ belongs to } \mathbb{M}^p(Z_{1 \wedge \tau_k}), \text{ for all } t \geq 0 \text{ and every } k, \]

and, for all \( t \geq 0 \) and \( k \in \mathbb{N} \),

\[ u(t \wedge \tau_k) = \cos((t \wedge \tau_k) \sqrt{A})u_0 + \frac{\sin((t \wedge \tau_k) \sqrt{A})}{\sqrt{A}} u_1 \]

\[ + \int_0^{t \wedge \tau_k} \frac{\sin((t \wedge \tau_k - s) \sqrt{A})}{\sqrt{A}} F(u(s))\, ds + I(\tau_k, G)(t \wedge \tau_k), \quad \mathbb{P}\text{-a.s.,} \]  

(5.29)

where \( I(\tau_k, G) \) is a process defined by

\[ I(\tau_k, G)(t) = \int_0^t 1_{[0, \tau_k]}(s) \frac{\sin((t - s) \sqrt{A})}{\sqrt{A}} G(u(s))\, dW(s), \quad t \geq 0. \]  

(5.30)

**B.** A local mild solution \( u = \{u(t) : t \in [0, \tau]\} \) to problem \((5.4)-(5.5)\) is unique iff for any other local solution \( \tilde{u} = \{\tilde{u}(t) : t \in [0, \tilde{\tau}]\} \) to problem \((5.4)-(5.5)\), the restricted processes \( u|_{[0, \tau \wedge s) \times \Omega} \) and \( \tilde{u}|_{[0, \tau \wedge s) \times \Omega} \) are equivalent.

**C.** A local mild solution \( u = \{u(t) : t \in [0, \tau]\} \) to problem \((5.4)-(5.5)\) is not maximal iff there exists a local solution \( \hat{u} = \{\hat{u}(t) : t \in [0, \hat{\tau}]\} \) to problem \((5.4)-(5.5)\) such that \( \mathbb{P}(\tau \leq \hat{\tau}) = 1, \mathbb{P}(\hat{\tau} < \hat{\tau}) > 0 \) and processes \( u \) and \( \hat{u}|_{[0, \tau) \times \Omega} \) are equivalent.
Remark 5.14. The definition of the process $I_{\tau_n}(G)$ is explained in Lemma A.7 of Appendix A. The use of processes $I_{\tau_n}(G)$ was first introduced for the SPDEs of parabolic type in [7] and [23] and in [11] for the hyperbolic SPDEs. The definition we use above is only in terms of the process $u$ and thus it is different from the one used in [11] which is in terms of pair processes $(u, u_t)$. In Appendix C we discuss the equivalence between these two approaches.

Remark 5.15. It can be shown, see Remark 2.22 in [6], that if an $\mathbb{F}$-adapted and $\mathcal{D}(A^{\frac{1}{2}})$-valued continuous process $u = \{ u(t) : t \in [0, \tau) \}$ is a local mild solution to problem (5.4)-(5.5), then equality (5.29) with (5.30) hold for any stopping time of the form $\tau_k \wedge \sigma$, where $\sigma$ is an accessible stopping time.

It can be shown that the concept of a local maximal solution introduced in part C. of Definition 5.13 is equivalent to the following set theoretical one. A natural continuation of this new definition is the so called “Amalgamation Lemma”, see [34, Lemma III 6A and 6B] and Definition 3.11 in [10].

Definition 5.16. Let us denote the set of all local solutions $(u, \tau)$ to the problem (5.4)-(5.5) by $\mathcal{L}S$. For any two elements $(u, \tau), (v, \sigma) \in \mathcal{L}S$ we write that $(u, \tau) \preceq (v, \sigma)$ iff $\tau \leq \sigma$ $\mathbb{F}$-a.s. and $v_{|[0, \tau) \times \Omega} \sim u$, see Definition 2.12 for the notation $\sim$. We write $(u, \tau) \prec (v, \sigma)$ iff $(u, \tau) \preceq (v, \sigma)$ and $(u, \tau) \not\prec (v, \sigma)$. It is straightforward to show that $\preceq$ is a partial order on $\mathcal{L}S$.

We say that $(u, \tau)$ is a maximal element of $(\mathcal{L}S, \preceq)$ iff there is no $(v, \sigma) \in (\mathcal{L}S, \preceq)$ such that $(u, \tau) \prec (v, \sigma)$. Each maximal element $(u, \tau)$ in the set $(\mathcal{L}S, \preceq)$ is called a maximal local solution to the problem (5.4)-(5.5).

6. The approximation problem

In this section we will study the following approximated version of problem (5.4)-(5.5), for $n \in \mathbb{N} \setminus \{0\}$,

\[
\begin{align*}
&\begin{cases}
&\begin{aligned}
&u_t(t) + Au(t) + \theta_n(\|u\|_{L^2})F_n(u(t)) = \theta_n(\|u\|_{L^2})G_n(u(t))\dot{W}(t), \\
&u(0) = u_0, \quad u_t(0) = u_1,
\end{aligned}
\end{cases} \\
&(6.1)
\end{align*}
\]

where $\theta : \mathbb{R}_+ \rightarrow [0, 1]$ be an auxiliary smooth function with compact support such that

\[
\inf_{x \in \mathbb{R}_+} \theta'(x) \geq -2, \quad 1_{[0,1]} \leq \theta \leq 1_{[0,2]},
\]

and for $n \geq 1$ set $\theta_n(\cdot) = \theta\left(\frac{\cdot}{n}\right)$.

The following lemma states the basic properties of $\theta_n$.

Lemma 6.1. The functions $\theta_n$ is Lipschitz and bounded and, for all $x, y \in [0, \infty)$,

\[
|\theta_n(x) - \theta_n(y)| \leq \frac{2}{n}|x - y|.
\]

Moreover, if $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non decreasing function, then for all $x \in [0, \infty)$,

\[
\theta_n(x)h(x) \leq h(2n).
\]

Remark 6.2. Let us point out that there is a typo in the lower bound of $\inf_{x \in \mathbb{R}_+} \theta(x)$ in [12, condition (4.10)] and the value of this lower bound should be strictly smaller than $-1$. Indeed, we can easily show that there does not exist a smooth function satisfying [12, condition (4.10)]. Consequently, the Lipschitz constant in [12, Lemma 4.3] should be strictly greater than $\frac{1}{n}$.
To move ahead let us consider a numerical sequence \( M = (M_n)_{n=1}^{\infty} = (1 - \frac{1}{n+1})_{n=1}^{\infty} = (\frac{n}{n+1})_{n=1}^{\infty} \).

The approximating problem (6.1) approximates the original problem (5.3) in a two-fold way. Firstly, the very bad non-linearities (of possibly exponential growth) \( F \) and \( G \) are replaced by nonlinearities \( \tilde{F}_n \) and \( \tilde{G}_n \) depending on \( M_n \) which are globally Lipschitz and bounded in the sense of Remark 5.2 and Theorem 5.3. Secondly, these new nonlinearities are multiplied by a suitable cut-off function \( \theta_n(\|u\|_{Z_t}) \) which depends on the whole history up to time \( t \) of the solution. Note however that this cut-off function takes values in the interval \([0,1]\). The second modification makes the coefficients of the approximating problem (6.1) not only time dependent but also random. This two-fold modification makes the coefficients in problem (6.1) enough Lipschitz so that the corresponding fixed point problem has a unique solution and problem (6.1) has a unique global solution.

**Definition 6.3.** Assume that \((u_0,u_1)\) is an \( H_A \times H \)-valued \( \mathcal{F}_0 \)-measurable random variable. A mild solution to problem (6.1)-(6.2) on time interval \([0,T]\), where \( T > 0 \), is a process \( u \) belonging to space \( \mathcal{M}^p(Z_T) \) such that for every \( t \in [0,T] \), the following equality holds \( \mathbb{P} \)-almost surely,

\[
\begin{align*}
    u(t) &= \cos(t\sqrt{A})u_0 + \sin(t\sqrt{A})u_1 + \int_0^t \theta_n(\|u\|_{Z_s}) \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \tilde{F}_n(u(s)) \, ds \\
    &\quad + \int_0^t (\theta_n(\|u\|_{Z_s})) \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \tilde{G}_n(u(s)) \, dW(s).
\end{align*}
\]

A global mild solution to problem (6.1)-(6.2) on time interval \([0,T]\).

The following result is a consequence of Proposition 6.1.

**Lemma 6.4.** Assume that \( T > 0 \) and \((u_0,u_1)\) is an \( H_A \times H \)-valued \( \mathcal{F}_0 \)-measurable random variable satisfying

\[
\mathbb{E}(\|u_0\|^2_{H_A} + \|u_1\|^2_{H_A}) = \mathbb{E}(\|(u_0,u_1)\|_{\mathcal{M}^p(H_A \times H)}^p) < \infty.
\]

If a process \( u \in \mathcal{M}^p(Z_T) \) mild solution to problem (6.1)-(6.2) on time interval \([0,T]\), then the \( \mathbb{P} \)-almost surely the trajectories of the process \( u \) belong to the space \( C([0,T]; H_A) \cap C^1([0,T]; H) \) and the velocity process \( u' = u_t \) belongs to the class \( \mathcal{M}^p(Q_T) \), see (5.3).

The main aim of this section is to prove that problem (6.1)-(6.2) has a unique global solution, i.e. the following result.

**Theorem 6.5.** Let \( n \in \mathbb{N} \setminus \{0\} \) and \( M_n \in (0,1) \) be the corresponding number. Let us assume that a triple \((p,q,r)\) satisfies condition (5.5). Let \( H, H_A \) and \( E \) be Hilbert and respectively Banach spaces defined in (A.1). Let us assume that the maps \( F : E \cap B_{H_A}(0,1) \to H \) and \( G : E \cap B_{H_A}(0,1) \to \gamma(K,H) \), where \( K \) is a separable Hilbert space, satisfy assumptions [A.1] and [A.2] with \( \gamma \), independent of \( M_n \), satisfying

\[
0 < 2\gamma < p.
\]

Assume that \( \tilde{F}_n, \tilde{G}_n \) and \( \theta_n \) are as above. Assume that \((u_0,u_1)\) is an \( H_A \times H \)-valued \( \mathcal{F}_0 \)-measurable random variable satisfying condition (6.4). Then the problem (6.1)-(6.2) has a unique global solution, denoted by \( u_n \), in the sense of Definition 6.3. In particular, for every \( T > 0 \), \( u_n \) belongs to the space \( \mathcal{M}^p(Z_T) \).

Moreover, if \( 0 < T_1 < T_2 < \infty \) and \( u_i, \ i = 1,2 \) is a solution to problem (6.1)-(6.2) on respectively interval \([0,T_1]\) and \([0,T_2]\) (or \([0,\infty)\) if \( T_2 = \infty \)), then \( u_2 \) restricted to the smaller interval \([0,T_1]\) coincides with \( u_1 \).
For the remainder of this section we choose and fix a natural number \( n \in \mathbb{N} \setminus \{0\} \) and a number \( M_n \in (0,1) \). Our proof of Theorem 6.5 will use a modified fixed point argument.

Now, if \( S \geq 0 \) and \( T \geq 0 \), let us denote by \( M^p(Z_{[S,S+T]}) \) the Banach space of (equivalence classes) of all \( E \)-valued progressively measurable processes \( \{u(t), t \in [S,S+T]\} \) having a continuous \( H_A \)-valued modification and satisfying

\[
||u||^p_{M^p(Z_{[S,S+T]})} := \mathbb{E}[||u||^p_{C([S,S+T];H_A)}] < \infty. \tag{6.6}
\]

Let us make a trivial observation that the space \( Z_{[0,0]} \) is isomorphic with the space \( H_A \) and the space \( M^p(Z_{[0,0]}) \) is isomorphic with the space \( L^p(\Omega, \mathcal{F}_0; H_A) \). In case \( S = 0 \), we will use both the notation \( M^p(Z_T) \) and \( M^p(Z_{[0,T]}) \) interchangeably. We will use similar notational convention even if we have \( X, Y \) or \( Q \) in place of \( Z \).

**Definition 6.6.** Assume that \( S \geq 0 \) and \( T > 0 \). If processes \( v \in M^p(Z_{[0,S]}) \) and \( u \in M^p(Z_{[S,S+T]}) \) agree at time \( S \), i.e., satisfy the following compatibility condition

\[
v(S) = u(S), \quad \mathbb{P}\text{-almost surely,} \tag{6.7}
\]

then we define the concatenation process \( \bar{u} \) by

\[
\bar{u} = v \cup u,
\]

and note that \( \bar{u} \in M^p(Z_{[0,S+T]}) \). The set of all processes \( \bar{u} \) obtained in such a way will be denoted by \( M^p(Z_{[0,S+T]}) \). This set is a closed affine subspace of \( M^p(Z_{[0,S+T]}) \) and we consider the distance on \( M^p(Z_{[0,S+T]}) \) induced by the norm on \( M^p(Z_{[0,S+T]}) \).

In particular, if \( u_0 \) is an \( H_A \)-valued \( \mathcal{F}_S \)-measurable random variable satisfying the following condition

\[
\mathbb{E}([||u_0||_{H_A}^p]) < \infty, \tag{6.8}
\]

then by \( M^p(Z_{[0,S+T]}^{u_0}) \) we denote the closed subspace of \( M^p(Z_{[S,S+T]}) \) consisting of all processes \( u \in M^p(Z_{[S,S+T]}) \) which satisfy

\[
u(S) = u_0, \quad \mathbb{P}\text{-almost surely.} \tag{6.9}
\]

**Definition 6.7.** Assume that \( S \geq 0 \) and \( T > 0 \) and that a process \( v \in M^p(Z_{[0,S]}) \), is such that \( \mathbb{P}\text{-almost surely it’s trajectories belong to the space } C([0,S]; H_A) \cap C^1([0,S]; H) \) and the velocity process \( v_t \) belongs to the class \( M^p(Q_{[0,S]}) \).

A mild solution to equation (6.1) on time interval \([S,S+T]\) with the history process \( v \) is a process \( u \) belonging to space \( M^p(Z_{[S,S+T]}) \) such that for every \( t \in [S,S+T] \), the following equality holds \( \mathbb{P}\text{-almost surely},

\[
u(t) = \cos((t-S)\sqrt{A})v(S) + \frac{\sin((t-S)\sqrt{A})}{\sqrt{A}}v_t(S) + \frac{\sin((t-S)\sqrt{A})}{\sqrt{A}}(\theta_n(||\bar{u}||_{Z_{[0,s]}})\tilde{F}_n(u(s))) ds
\]

\[
+ \frac{\sin((t-S)\sqrt{A})}{\sqrt{A}}(\theta_n(||\bar{u}||_{Z_{[0,s]}})\tilde{G}_n(u(s))) dW(s), \tag{6.10}
\]

where \( \bar{u} = v \cup u \) is the concatenation process of processes \( v \) and \( u \) and \( || : Z_{[0,s]} \) is the norm in the space \( Z_{[0,s]} \).

In the framework of Definition 6.7, let us define a map

\[
\Psi_n^{u}: M^p(Z_{[S,S+T]}^v) \ni u \mapsto \Psi_n^{u}(u) \in M^p(Z_{[S,S+T]}^v), \tag{6.11}
\]
by the following formula, for every \( u \in \mathcal{M}^p(Z_{[S,S+T]}) \) and all \( t \in [S, S + T], \)

\[
[\Psi_n^{p,p}(u)](t) = \cos((t - S)\sqrt{A})v(S) + \frac{\sin((t - S)\sqrt{A})}{\sqrt{A}}v_t(S) + \int_s^t \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}}(\theta_n(||Z_{[0,s]}||)\hat{F}_n(u(s))) \, ds \]

\[+ \int_s^t \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}}(\theta_n(||Z_{[0,s]}||)\hat{G}_n(u(s))) \, dW(s), \] \text{ P-a.s..} \tag{6.12}

We will show in the following pages that the map \( \Psi_n^{p,p} \) is well defined. Let us observe that if \( T > 0 \), a process \( v \in \mathcal{M}^p(Z_{[0,S]}) \), is such that \( \mathcal{F} \)-almost surely it’s trajectories belong to the space \( C([0,S];H_A) \cap C^1([0,S];H) \) and the velocity process \( v_t \) belongs to the class \( \mathcal{M}^p(Q_{[0,S]}) \) and \( u \in \mathcal{M}^p(Z_{[S,S+T]}) \), then the concatenation process \( \bar{u} \) is well defined and belongs to \( \mathcal{M}^p(Z_{[0,S+T]}) \). Moreover, it follows from Proposition 6.16 that if a process \( u \) belongs to \( \mathcal{M}^p(Z_{[S,S+T]}) \), \( \Psi_n^{p,p}(u) \) belongs to the same subspace \( \mathcal{M}^p(Z_{[S,S+T]}) \).

We continue with the following a’priori estimates and the uniqueness results for the solutions of the approximating problem (6.1). The proof of this result is based on some estimates proven in the following assertions.

**Theorem 6.8.** Let \( T > 0 \) and assume that the conditions in the statement of the Theorem 6.3 hold. Let \( u \in \mathcal{M}^p(Z_T) \) be a mild solution to problem (6.1) - (6.2) on the time interval \([0,T]\) in the sense of Definition 6.3. Then, there exists two constants \( C_1(T,p) \) and \( C_2(T,n,p) \) such that

\[
\|u\|_{\mathcal{M}^p(Z_{[0,T]})}^p = E[\|u\|_{Z_{[0,T]}}^p] \leq C_1 E\left[\|u_0\|_{H_A}^p + \|u_1\|_{H}^p\right] + C_2(T,n,p). \tag{6.13}
\]

The constants \( C_1(T,p) \) and \( C_2(T,n,p) \) can be chosen in such a way that they are increasing functions w.r.t. \( T. \)

**Theorem 6.9.** Assume that \( S \geq 0 \) and \( T > 0 \). Let \( v \in \mathcal{M}^p(Z_{[0,S]}) \) be a process such that \( \mathcal{F} \)-almost surely it’s trajectories belong to the space \( C([0,S];H_A) \cap C^1([0,S];H) \) and the velocity process \( v_t \) belongs to the class \( \mathcal{M}^p(Q_{[0,S]}) \). Then there exists at most one mild solution to equation (6.1) on time interval \([S,S+T]\) with the history process \( v \), in the sense of Definition 6.7.

**Proof of Theorem 6.8.** Recall that \( n \in \mathbb{N} \) is fixed. Let us choose and fix \( T > 0 \) and \( u \) a mild solution to problem (6.1) - (6.2) on the time interval \([0,T]\). Then, by definition, \( u \) satisfies (6.3) and therefore,

\[
\|u\|_{\mathcal{M}^p(Z_{[0,T]})}^p \leq 3^p\|t \mapsto \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1\|_{\mathcal{M}^p(Z_{[0,T]})}^p
\]

\[+ 3^p\|t \mapsto \int_0^t \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}}(\theta_n(||Z_s||)\hat{F}_n(u(s))) \, ds\|_{\mathcal{M}^p(Z_{[0,T]})}^p
\]

\[+ 3^p\|t \mapsto \int_0^t \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}}(\theta_n(||Z_s||)\hat{G}_n(u(s))) \, dW(s)\|_{\mathcal{M}^p(Z_{[0,T]})}^p. \tag{6.14}
\]

The first term in the right hand side is equal to \( \|T^T_1(u_0, u_1)\|_{\mathcal{M}^p(Z_{[0,T]})}^p \) as in Lemma 6.11 with \( S = 0 \), and thus we have

\[
\|t \mapsto \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1\|_{\mathcal{M}^p(Z_{[0,T]})}^p \leq (C_T^p + C_T^p)^{2^{p-1}} \left[ E\|u_0\|_{H_A}^p + E\|u_1\|_{H}^p \right].
\]
Proof of Lemma 6.11 and fix $T > I$ respectively, with $S = 0$. In particular we obtain
\begin{align*}
||t \mapsto \int_0^t \sin((t-s)\sqrt{A}) (\theta_n(||u||_Z) F_n(u(s)) + \sin((t-s)\sqrt{A}) (\theta_n(||u||_Z) \tilde{F}_n(u(s)) \, ds||_{MP(Z_T)}^p \leq (2n)^p (C_T^n + K_T^n) \left(T + T^{1-\frac{2}{p}} (2n)^\gamma\right)^p,
\end{align*}
where $\gamma$ satisfies $6.3$ as in the statement of Theorem 6.5. By setting the sum of right hand sides of last two estimates as $C_2(T,n,p)$ we complete the proof of Theorem 6.8. \hfill $\square$

We skip the proof of Theorem 6.9 because it can be done in a similar (in fact much simpler) way to the proof of Theorem 7.3.

Remark 6.10. It is important to note that the cut-off function $\theta_n$ plays essential role in the fixed point argument we display here because of the quasi-Lipschitz properties of $F_n$ and $\tilde{F}_n$, respectively.

Let us recall that the $Y_t$ norm has been defined in (4.7). We will show that there exists $T_n > 0$ such that $\Psi_{|S,S+T_n]}^n$ is a strict contraction. We divide our argument in a couple of lemmata.

Lemma 6.11. Let $(u_0, u_1)$ be an $H_A \times H$-valued $\mathcal{F}_S$-measurable random variable such that (6.8) holds true. If $S \geq 0$ and $T > 0$, then the map
\begin{align*}
\mathcal{J}_n^T : H_A \times H \ni (u_0, u_1) \mapsto \left\{[S, S + T] \ni t \mapsto \cos((t - S)\sqrt{A}) u_0 + \sin((t - S)\sqrt{A}) u_1\right\} \in MP(Z_n^{u_0, u_1})
\end{align*}
is well-defined and
\begin{align*}
||\mathcal{J}_n^T (u_0, u_1)||_{MP(Z_{|S,S+T|})}^p \leq \left(C_T^n + \tilde{C}_T^n\right) 2^{p-1} \left[E||u_0||^p_{H_A} + E||u_1||^p_{H}ight]. \tag{6.16}
\end{align*}

Proof of Lemma 6.11. It is obvious to see that $\mathcal{J}_n^T (u_0, u_1) (S) = u_0, P$-a.s. Let us choose and fix $T > 0$. It is known that, see [2], $w := \mathcal{J}_n^T (u_0, u_1)$ is the unique solution, for each $\omega \in \Omega$, of the homogeneous wave equation, with the Dirichlet or the Neumann boundary condition,
\begin{align*}
\left\{
\begin{array}{l}
\partial_t w - \Delta w = 0, \quad t \geq S \\
w(S, \cdot) = u_0(\cdot), \quad \partial_t w(S, \cdot) = u_1(\cdot).
\end{array}
\right.
\end{align*}
Moreover, for each $\omega \in \Omega$, see Remark 3.3 $w$ belongs to $C([S, S + T]; H_A) = X_{|S,S+T|}$ with
\begin{align*}
||w||_{X_{|S,S+T|}} \leq \tilde{C}_T \left(||u_0||_{H_A} + ||u_1||_{H}\right),
\end{align*}
and, by Theorem 3.2 $w$ belongs to $Y_{|S,S+T|}$ and satisfy
\begin{align*}
||w||_{Y_{|S,S+T|}} \leq C_T \left(||u_0||_{H_A} + ||u_1||_{H}\right).
\end{align*}
So, for every $\omega \in \Omega$, $\mathcal{J}_n^T (u_0, u_1) \in Y_{|S,S+T|} \cap X_{|S,S+T|}$ and, in view of [32], we have
\begin{align*}
||\mathcal{J}_n^T (u_0, u_1)||_{MP(Z_{|S,S+T|})}^p \leq \left(C_T^n + \tilde{C}_T^n\right) 2^{p-1} \left[E||u_0||^p_{H_A} + E||u_1||^p_{H}\right].
\end{align*}
Furthermore, since the process \( J^0_2(u_0, u_1) \) is \( \mathbb{F} \)-adapted and continuous, it is progressively measurable and, hence Lemma 6.11 follows.

In the remaining part of this section, let us fix \( v \in \mathbb{M}^p(Z^{[0,S]}) \) a process such that \( \mathbb{F} \)-almost surely it’s trajectories belong to the space \( C([0, S]; H_A) \cap C^1([0, S]; H) \) and the velocity process \( v_t \) belongs to the class \( \mathbb{M}^p(Q_{[0,S]}) \). Moreover, we will write \( \tilde{u} = v \cup u \) for \( u \in \mathbb{M}^p(Z^{[S,S+T]}) \), see Definition 6.6. Furthermore, the assumptions and notation of the Theorem 6.5 hold true in the rest of the section unless otherwise stated.

**Lemma 6.12.** If \( S \geq 0 \) and \( T > 0 \), then the map

\[
J^0_2 : \mathbb{M}^p(Z^{[S,S+T]}) \ni u \mapsto \tilde{u} = \left\{ [S, S + T] \ni t \mapsto \int_S^t \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}} \left( \theta_n(\|\tilde{u}\|_{Z_{[0,s]}}) \tilde{F}_n(u(s)) \right) ds \right\} \in \mathbb{M}^p(Z^{[S,S+T]}),
\]

is well-defined and satisfies

\[
\mathbb{E}\left[ \|\tilde{u}\|_{Z^{[S,S+T]}}^p \right] \leq (2n)^p (C^{p}_F + K^p) (T + T^{1 - \gamma}(2n)^p).
\]

**Proof of Lemma 6.12.** Let us choose and fix \( T > 0 \). The computation is pathwise and thus we will not write \( \omega \) explicitly unless any confusion arises. Take an arbitrary \( u \in \mathbb{M}^p(Z^{[S,S+T]}) \) and \( \tilde{u} = J^0_2(u) \). Then, by definition (6.17) of the map \( J^0_2 \), we have

\[
\|\tilde{u}\|_{X^{[S,S+T]}} \leq \sup_{t \in [S, S + T]} \left\| \int_S^t \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}} \left( \theta_n(\|\tilde{u}\|_{Z_{[0,s]}}) \tilde{F}_n(u(s)) \right) ds \right\|_{H_A}
\]

\[
\leq K_T \int_S^{S+T} \theta_n(\|\tilde{u}\|_{Z_{[0,s]}}) \tilde{F}_n(u(s)) ds.
\]

Note that, in the last step we used the following bound, which is a consequence of 6.2 - Lemma 2.2] applied to \( C_v \)-group associated to the wave operator \( \partial_t - \Delta \),

\[
\left\| \frac{\sin((t - S)\sqrt{A})}{\sqrt{A}} \right\|_{L(H,H_A)} \leq K_T, \quad t \in [S, S + T],
\]

where \( K_T := \bar{M} e^{mT} \) for some constants \( m \geq 0 \) and \( \bar{M} \geq 1 \). Let \( T^* \) be stopping time defined by

\[
T^* := \inf\{ t \in [0, T] : \|\tilde{u}\|_{Z_{[0,S+t]}} \geq 2n \}.
\]

If the set in the definition of \( T^* \) is empty, then we set \( T^* = T \). Returning to (6.19), by applying (6.21) we get

\[
\int_S^{S+T} \theta_n(\|\tilde{u}\|_{Z_{[0,s]}}) \|\tilde{F}_n(u(s))\|_H ds \leq \|\tilde{F}_n(u)\|_{L^1([S,S+T^*];H)}. \tag{6.22}
\]

Moreover, since in view of (6.21), \( \|u\|_{X^{[S,S+T^*]}} \|u\|_{Y^{[S,S+T^*]}} \leq 2n \) and since \( \tilde{F}_n(0) = 0 \), by Lemma 6.3 we infer the following inequality

\[
\|\tilde{F}_n(u)\|_{L^1([S,S+T^*];H)} \leq C^p_{F_n} \left( T^* + (T^*)^{1 - \frac{\gamma}{2}} \|u\|_{Y^{[S,S+T^*]}} \right) \|u\|_{X^{[S,S+T^*]}} \leq (2n) C^p_{F_n} \left( T + T^{1 - \frac{\gamma}{2}} (2n)^\gamma \right), \tag{6.23}
\]
where $C'_{F_n} := K(\gamma)C_{F_n}$. Combining (6.19), (6.22) and (6.23) followed by taking the expectation gives
\[
\mathbb{E} \left[ \|\tilde{u}\|_{Y_{[S,S+T]}}^p \right] \leq (2n)^p (C'_{F_n})^p K_T^p \left( T + T^{1 - \frac{2}{p}} (2n)\gamma \right)^p.
\] (6.24)

By definition (6.17) of the map $\mathcal{J}^p_2$, invoking the inhomogeneous Strichartz estimates from Theorem 3.2 followed by (6.23) we get
\[
\|\tilde{u}\|_{Y_{[S,S+T]}} \leq C_T \|\tilde{F}_n(u)\|_{L^1([S,S+T];H)} \leq (2n) C'_T \left( T + T^{1 - \frac{2}{p}} (2n)\gamma \right).
\]

Which consequently, after taking the expectation, gives,
\[
\mathbb{E} \left[ \|\tilde{u}\|_{Y_{[S,S+T]}}^p \right] \leq (2n)^p (C'_{F_n})^p C_T^p \left( T + T^{1 - \frac{2}{p}} (2n)\gamma \right)^p.
\] (6.25)

Hence, by estimates (6.24) and (6.25) we get the Lemma 6.12. □

The next result establishes the Lipschitz property of $\mathcal{J}^p_2$ as a map acting on $\mathbb{M}^p(Z_{[S,S+T]})$.

**Lemma 6.13.** Let $S \geq 0$ be given. For every $T > 0$, there exists a constant $L^p_2(T) > 0$ such that the following assertions are true:

(i) for every $n \in \mathbb{N}$, the function $L^p_2(\cdot)$ is non-decreasing;
(ii) for every $n \in \mathbb{N}$, $\lim_{T \to 0} L^p_2(T) = 0$;
(iii) for all $u_1, u_2 \in \mathbb{M}^p(Z_{[S,S+T]})$, the following inequality holds,
\[
\|\mathcal{J}^p_2(u_1) - \mathcal{J}^p_2(u_2)\|_{\mathbb{M}^p(Z_{[S,S+T]})} \leq L^p_2(T) \|u_1 - u_2\|_{\mathbb{M}^p(Z_{[S,S+T]})}.
\]

**Proof of Lemma 6.13.** Let us choose and fix two arbitrary elements $u_1, u_2 \in \mathbb{M}^p(Z_{[S,S+T]})$.

As in before, we will not write $\omega$ explicitly unless any confusion arises. Since $\mathcal{J}^p_2$ is well-defined, we denote $\tilde{u}_1 := \mathcal{J}^p_2(u_1), \tilde{u}_2 := \mathcal{J}^p_2(u_2) \in \mathbb{M}^p(Z_{[S,S+T]})$. As in the proof of Lemma 6.12 we define the following stopping times $T_1, T_2$ and $T^*$
\[
T_i := \inf\{t \in [0,T] : \|\tilde{u}_i\|_{Z_{[0,s+i]}} \geq 2n, \ i = 1, 2, \ \text{and} \ \ T^* := \max\{T_1, T_2\}.
\]

Let us observe that for $t \in [S, S + T]$, $\mathbb{P}$-a.s.,
\[
\tilde{u}_2(t) - \tilde{u}_1(t) = \int_s^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \left[ \theta_n(\|\tilde{u}_2\|_{Z_{[0,s]}}(\tilde{F}_n(u_2(s)))) - \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}}(\tilde{F}_n(u_1(s)))) \right] ds.
\]

Moreover, similarly to inequality (6.23) we get the following inequality
\[
\int_S^{S+T_2} \|\tilde{F}_n(u_2(t))\|_{H} dt \leq C'_{F_n} \left( T_2 + T_2^{1 - \frac{2}{p}} \|u_2\|_{Y_{[S,S+T_2]}} \right) \|u_2\|_{X_{[S,S+T_2]}} \leq 2n C'_{F_n} \left( T + T^{1 - \frac{2}{p}} (2n)\gamma \right).
\]

Therefore, by invoking the inhomogeneous Strichartz estimates from Theorem 3.2 we get the following inequality
\[
\mathbb{E} \left[ \|\tilde{u}_1 - \tilde{u}_2\|_{Y_{[S,S+T]}}^p \right] = \mathbb{E} \left[ \int_S^{S+T} \|\tilde{u}_1(t) - \tilde{u}_2(t)\|_{E}^p dt \right] \leq C_T^p \mathbb{E} \left[ \int_S^{S+T} \left| \theta_n(\|\tilde{u}_2\|_{Z_{[0,s]}}(\tilde{F}_n(u_2(s)))) - \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}}(\tilde{F}_n(u_1(s)))) \right|_{H} ds \right]
\]
\[
= C_T^p \mathbb{E} \left[ \int_S^{S+T^*} \left| \theta_n(\|\tilde{u}_2\|_{Z_{[0,s]}}(\tilde{F}_n(u_2(s)))) - \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}}(\tilde{F}_n(u_1(s)))) \right|_{H} ds \right].
\]
\[
\begin{align*}
\leq \bar{C}_T^p & \mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \int_{S}^{S+T^*} \theta_n(\|\tilde{u}_2\|_{Z_{[0,s]}})(\tilde{F}_n(u_2(s))) - \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}})(\tilde{F}_n(u_1(s))) | H \, ds \right]^p \\
+ \bar{C}_T^p & \mathbb{E} \left[ \mathbf{1}_{\{T_2 \leq T_1\}} \int_{S}^{S+T^*} \theta_n(\|\tilde{u}_2\|_{Z_{[0,s]}})(\tilde{F}_n(u_2(s))) - \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}})(\tilde{F}_n(u_1(s))) | H \, ds \right]^p,
\end{align*}
\]

where \(\bar{C}_T^p := 2^{p-1}C_T^p\).

Next, since \(\theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}}) = 0\) for \(s \geq S + T_1\) and \(\|\tilde{u}_2\|_{Z_{[0,t]}} \leq 2n\) for \(t \in [S, S + T_2]\), \(\mathbb{P}\)-a.s., by using the Lemmata 3.3 and 6.1 we estimate the first of the two integrals in the right hand side above as

\[
\mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \int_{S}^{S+T^*} \theta_n(\|\tilde{u}_2\|_{Z_{[0,s]}})(\tilde{F}_n(u_2(s))) - \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}})(\tilde{F}_n(u_1(s))) | H \, ds \right]^p \\
\leq 2^{p-1} \mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \int_{S}^{T_2} \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}})(\tilde{F}_n(u_2(s))) - \tilde{F}_n(u_1(s)) | H \, ds \right]^p \\
+ 2^{p-1} \mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \int_{T_2}^{T_1} \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}})(\tilde{F}_n(u_2(s))) - \tilde{F}_n(u_1(s)) | H \, ds \right]^p \\
= 2^{p-1} \mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \int_{S}^{S+T^*} \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}})(\tilde{F}_n(u_2(s))) - \tilde{F}_n(u_1(s)) | H \, ds \right]^p \\
+ 2^{p-1} \mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \int_{T_2}^{T_1} \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}})(\tilde{F}_n(u_2(s))) - \tilde{F}_n(u_1(s)) | H \, ds \right]^p \\
\leq 2^{p-1}(C_{\tilde{F}_n}^p)^p \mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \|u_1 - u_2\|_{S,S+T_1} \left( T_1 + T_1^{1-\frac{n}{p}}\|u_1\|_{Y_{S,S+T_1}^\gamma} + T_1^{1-\frac{n}{p}}\|u_2\|_{Y_{S,S+T_1}^\gamma} \right)^p \right] \\
+ \frac{2^{p-1}}{n^p} \mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \|\tilde{u}_1 - \tilde{u}_2\|_{Z_{[0,s]}} \int_{S}^{T_2} \|\tilde{F}_n(u_2(t))\|_H \, dt \right]^p \\
= 2^{p-1}(C_{\tilde{F}_n}^p)^p \mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \|u_1 - u_2\|_{S,S+T_1} \left( T_1 + T_1^{1-\frac{n}{p}}\|u_1\|_{Y_{S,S+T_1}^\gamma} + T_1^{1-\frac{n}{p}}\|u_2\|_{Y_{S,S+T_1}^\gamma} \right)^p \right] \\
+ \frac{2^{p-1}}{n^p} \mathbb{E} \left[ \mathbf{1}_{\{T_1 \leq T_2\}} \|u_1 - u_2\|_{Z_{[0,s]}} \int_{S}^{T_2} \|\tilde{F}_n(u_2(t))\|_H \, dt \right]^p \\
\leq 2^{p-1}(C_{\tilde{F}_n}^p)^p \left( T_1 + 2T_1^{1-\frac{2}{p}}(2n)^\gamma \right)^p \mathbb{E} \left[ \|u_1 - u_2\|^p_{K_{S,S+T_1}} \right] \\
+ \frac{2^{p-1}}{n^p}(C_{\tilde{F}_n}^p)^p (2n)^p \left( T_2 + T_2^{1-\frac{2}{p}}(2n)^\gamma \right)^p \mathbb{E} \left[ \|u_1 - u_2\|^p_{Z_{[S,S+T_2]}} \right] \\
\leq 2^{2p}(C_{\tilde{F}_n}^p)^p \|u_1 - u_2\|^p_{M^p(Z_{S,S+T_1})} \left( T + 2T_1^{1-\frac{2}{p}}(2n)^\gamma \right)^p. \tag{6.27}
\end{align*}
\]

Swapping between \(u_2\) and \(u_1\) we can analogously show the following estimate the second of the two integrals in the RHS of (6.26), i.e.

\[
\mathbb{E} \left[ \mathbf{1}_{\{T_2 \leq T_1\}} \int_{S}^{S+T^*} \theta_n(\|\tilde{u}_2\|_{Z_{[0,s]}})(\tilde{F}_n(u_2(s))) - \theta_n(\|\tilde{u}_1\|_{Z_{[0,s]}})(\tilde{F}_n(u_1(s))) | H \, ds \right]^p \\
\leq 2^{2p}(C_{\tilde{F}_n}^p)^p \|u_1 - u_2\|^p_{M^p(Z_{S,S+T_1})} \left( T + 2T_1^{1-\frac{2}{p}}(2n)^\gamma \right)^p, \tag{6.28}
\]
Thus, by combining the computation from (6.26)–(6.28) we obtain

\[ E \left[ \| \tilde{u}_1 - \tilde{u}_2 \|^p_{\mathcal{M}^{p}(\mathbb{R}^2)} \right] 
\leq 2^{2p+1}C^p_T \left( C^p_F \right)^p \| u_1 - u_2 \|^p_{\mathcal{M}^{p}(\mathbb{R}^2)} (T + 2T^{1-\frac{2}{p}} (2n)^\gamma)^p. \]

Next, using the inequality (6.20), followed by repeating calculations as in (6.26) and (6.27), we obtain

\[ E \left[ \| \tilde{u}_1 - \tilde{u}_2 \|^p_{\mathcal{M}^{p}(\mathbb{R}^2)} \right] 
\leq 2^{2p+1}K^p_T \left( C^p_F \right)^p \| u_1 - u_2 \|^p_{\mathcal{M}^{p}(\mathbb{R}^2)} (T + 2T^{1-\frac{2}{p}} (2n)^\gamma)^p. \]

Hence, in combination with the estimate (6.20) we get

\[ \| \tilde{u}_1 - \tilde{u}_2 \|^p_{\mathcal{M}^{p}(\mathbb{R}^2)} \leq 2^{2p+1} \left( C^p_F + K^p_T \right) \left( C^p_F \right)^p \| u_1 - u_2 \|^p_{\mathcal{M}^{p}(\mathbb{R}^2)} (T + T^{1-\frac{2}{p}} (2n)^\gamma)^p. \]

Since \( \gamma < p \), by definition of \( L^p_2(T) \), it is clear that, for each \( n \in \mathbb{N} \), \( \lim_{T \to 0} L^p_2(T) = 0 \). Thus, the proof of Lemma 6.13 is complete.

Next, we set

\[ \xi^n_u(t) := \theta_n(\| \tilde{u} \|_{L^p([0,t])}) \tilde{G}_n(u(t)), \quad t \in [S, S + T]. \]

Then we can write

\[ \int_S^t \sin((t-r)\sqrt{A}) (\theta_n(\| \tilde{u} \|_{L^p([0,t])}) \tilde{G}_n(u(r))) dW(r) =: \int_S^t \sin((t-r)\sqrt{A}) \xi^n_u(r) dW(r) =: [J\xi^n_u](t), \quad t \in [S, S + T]. \]

In the next result, we show that \( T^3 \) maps \( \mathcal{M}^{p}(Z_{[S, S + T]}^\gamma) \) into \( \mathcal{M}^{p}(Z_{[S, S + T]}^\gamma) \).

**Lemma 6.14.** For any \( S \geq 0 \) and \( T > 0 \), the map

\[ T^3 : \mathcal{M}^{p}(Z_{[S, S + T]}^\gamma) \ni u \mapsto J\xi^n_u \in \mathcal{M}^{p}(Z_{[S, S + T]}^\gamma), \]

where \( J\xi^n_u \) is as (6.29), is well-defined and satisfies

\[ \| J\xi^n_u \|_{\mathcal{M}^{p}(Z_{[S, S + T]}^\gamma)} \leq (2n)^p \left( C^p_{\tilde{G}_n} \right)^p \left( \tilde{C}_T + \tilde{K}_T \right) \left[ T + T^{1-\frac{2}{p}} (2n)^\gamma \right]^{\frac{p}{2}}. \]

**Proof of Lemma 6.14.** Take any \( u \in \mathcal{M}^{p}(Z_{[S, S + T]}^\gamma) \) and set \( \tilde{u} := J\xi^n_u \). Observe that from (6.25), we have

\[ E \left[ \| J\xi^n_u \|^p_{\mathcal{M}^{p}(L^p([S, S + T], E))} \right] \leq \tilde{C}_T E \left[ \int_S^{S+T} \| \xi^n_u(t) \|^2_{\gamma(K,H)} dt \right]^{\frac{p}{2}}. \]

Let us define

\[ T^* := \inf \{ t \in [0, T] : \| \tilde{u} \|_{Z_{[0,t]}} \geq 2n \}, \]

as a stopping time. Since \( \nu \in \mathcal{M}^{p}(Z_{[0,T]}) \) and \( u \in \mathcal{M}^{p}(Z_{[0,T]}^\gamma) \), the map \( t \mapsto \| \tilde{u} \|_{Z_{[0,t]}} \) is non-decreasing and continuous. Consequently, we infer that \( \theta_n(\| \tilde{u} \|_{Z_{[0,t]}}) = 0 \) for all \( t \in (S + T^*, S + T] \). Moreover, since \( \| u \|_{Z_{[t,t]}} \leq \| \tilde{u} \|_{Z_{[0,t]}} \) for every \( t \in [S, S + T] \), we deduce that

\[ \| u \|_{Z_{[S, S + T]}} \leq 2n. \]

Thus, invoking Lemma 5.5, followed by the Hölder inequality give

\[ \int_S^{S+T} \| \xi^n_u(t) \|^2_{\gamma(K,H)} dt \leq \int_S^{S+T^*} \| \tilde{G}_n(u(t)) \|^2_{\gamma(K,H)} dt \]

\[ \leq (C^p_{\tilde{G}_n})^2 \| u \|^2_{\mathcal{M}^{p}(Z_{[S, S + T]}^\gamma)} \left[ T^* + (T^*)^{1-\frac{2}{p}} \| u \|^2_{\mathcal{M}^{p}(Z_{[S, S + T]}^\gamma)} \right]. \]
4.7, followed by (6.33), we get

\[
\mathbb{E} \left[ \int_S^{S+T} \| J_{\xi_n}^u(t) \|^p_{\mathcal{E}} \, dt \right] \leq (2n)^p \left( C_{G_n}' \right)^p \tilde{C}_T \left[ T + T^{1 - \frac{2p}{p}} (2n)^{2\gamma} \right]^\frac{p}{2}. \tag{6.34}
\]

Next, to estimate \( \mathbb{E} \left[ \| J_{\xi_n}^u \|^p_{\mathcal{E}([S,S+T])} \right] \), using the stochastic Strichartz estimates from Theorem 4.7, followed by (6.33), we get

\[
\mathbb{E} \left[ \sup_{t \in [S,S+T]} \| J_{\xi_n}^u(t) \|^p_{H_A} \right] \leq (2n)^p \left( C_{G_n}' \right)^p \tilde{K}_T \left[ T + T^{1 - \frac{2p}{p}} (2n)^{2\gamma} \right]^\frac{p}{2}. \tag{6.35}
\]

Combining (6.31) and (6.35) completes the proof of Lemma 6.14. \( \square \)

The next result establishes the Lipschitz property of the map \( T_3^n \) defined in (6.30).

**Lemma 6.15.** Let \( S \geq 0 \) be given. Assume that a Hilbert space \( K \) satisfies Assumption (5.6). If \( T > 0 \), then there exists a constant \( L_3^n(T) > 0 \) such that the following assertions are true:

- \( L_3^n(\cdot) \) is non-decreasing;
- for every \( n \in \mathbb{N} \), \( \lim_{T \to 0} L_3^n(T) = 0; \)
- for \( u_1, u_2 \in \mathbb{M}^p(Z_{[S,S+T]}), T_3^n \) satisfy,

\[
\| T_3^n(u_1) - T_3^n(u_2) \|_{\mathbb{M}^p(Z_{[S,S+T]})} \leq L_3^n(T) \| u_1 - u_2 \|_{\mathbb{M}^p(Z_{[S,S+T]})}. \tag{6.36}
\]

**Proof of Lemma 6.15.** Let us choose and fix \( T > 0 \). We set

\[
\xi_i^n(t) = \theta_n(\| \bar{u}_i \|_{Z_{[0,t]}})G(u_i(t)), \quad i = 1, 2 \quad \text{and} \quad t \in [S,S+T].
\]

Since \( T_3^n \) is well-defined, we denote \( J_{\xi_1}^n := J_{\xi_1}^u, J_{\xi_2}^n := J_{\xi_2}^u \in \mathbb{M}^p(Z_{[S,S+T]}). \) Then, note that since for \( t \in [S,S+T], \mathbb{P} \)-a.s.,

\[
J_{\xi_1}^n(t) - J_{\xi_2}^n(t) = \int_{S}^{t} \sin((t-s)\sqrt{A}) \sqrt{A} \left[ \theta_n(\| \bar{u}_1 \|_{Z_{[0,s]}})(G_n(u_1(s))) - \theta_n(\| \bar{u}_2 \|_{Z_{[0,s]}})(G_n(u_2(s))) \right] \, dW(s).
\]

Thus, applying (4.15) from Theorem 4.7 gives

\[
\mathbb{E} \left[ \| J_{\xi_1}^n - J_{\xi_2}^n \|^p_{L^p(0,T;\mathcal{E})} \right] \leq \tilde{C}_T \mathbb{E} \left[ \int_S^{S+T} \| \xi_1^n(t) - \xi_2^n(t) \|^2_{\gamma(K,H)} \, dt \right]^\frac{p}{2}. \tag{6.37}
\]

Now, we define the following stopping times

\[
T_i := \inf \{ t \in [0,T] : \| \bar{u}_i \|_{Z_{[0,s+2n]}} \geq 2n, \quad i = 1, 2, \quad \text{and} \quad T^* := \max(T_1, T_2). \]

By applying the stochastic Strichartz estimate (4.22) from Theorem 4.7, we get

\[
\mathbb{E} \left[ \sup_{t \in [S,S+T]} \| J_{\xi_1}^n(t) - J_{\xi_2}^n(t) \|^p_{H_A} \right] \leq \tilde{K}_T \mathbb{E} \left[ \int_S^{S+T} \| \xi_1^n(t) - \xi_2^n(t) \|^2_{\gamma(K,H)} \, dt \right]^\frac{p}{2}. \tag{6.38}
\]

Next, observe that computation similar to (6.26) gives

\[
\begin{align*}
\mathbb{E} \left[ \int_S^{S+T} \| \xi_1^n(t) - \xi_2^n(t) \|^2_{\gamma(K,H)} \, dt \right]^\frac{p}{2} & \leq 2^{p-1} \mathbb{E} \left[ \mathbb{1}_{\{ T_1 \leq T_2 \}} \int_S^{S+T} \| \xi_1^n(t) - \xi_2^n(t) \|^2_{\gamma(K,H)} \, dt \right]^\frac{p}{2} \\
& \quad + 2^{p-1} \mathbb{E} \left[ \mathbb{1}_{\{ T_2 \leq T_1 \}} \int_S^{S+T} \| \xi_1^n(t) - \xi_2^n(t) \|^2_{\gamma(K,H)} \, dt \right]^\frac{p}{2}. \tag{6.39}
\end{align*}
\]
As in the proof of Lemma 6.13, we will only estimate the first of the two integrals in the right hand side above. Since $\theta_n(\|\bar{u}_1\|_{Z_{[0,\bar{t}]}}) = 0$ for $t \geq S + T_1$ and $\|\bar{u}_2\|_{Z_{[0,\bar{t}]}} \leq 2n$ for $t \in [S, S + T_2]$, P-a.s., by following the computation of (6.27) and using the Lemmata 5.5 and 6.1, we estimate the first integral in the right hand side of (6.39) as

\[
E \left[ \mathbb{I}_{\{T_1 \leq T_2\}} \int_S^{S + T} \|\xi_1^u(t) - \xi_2^u(t)\|_{\gamma(K, H)}^2 dt \right]^{\frac{\gamma}{2}}
\]

\[
= E \left[ \mathbb{I}_{\{T_1 \leq T_2\}} \int_S^{S + T} \|\theta_n(\|\bar{u}_2\|_{Z_{[0,\bar{t}]}})(\bar{G}_n(u_2(s))) - \theta_n(\|\bar{u}_1\|_{Z_{[0,\bar{t}]}})(\bar{G}_n(u_1(s)))\|_{\gamma(K, H)}^2 ds \right]^{\frac{\gamma}{2}}
\]

\[
\leq 2^{p-1} E \left[ \mathbb{I}_{\{T_1 \leq T_2\}} \int_S^{S + T_1} \|\theta_n(\|\bar{u}_2\|_{Z_{[0,\bar{t}]}})\|_{\gamma(K, H)}^2 ds \right]^{\frac{\gamma}{2}}
\]

\[
\leq 2^{p-1} (C_{G_n}^\prime)^p E \left[ \mathbb{I}_{\{T_1 \leq T_2\}} \|u_1 - u_2\|_{L^p(Z[S,S+T_1])}^2 \right]^{\frac{\gamma}{2}}
\]

\[
\leq 2^{p-1} (C_{G_n}^\prime)^p \|u_1 - u_2\|_{L^p(Z[S,S+T_1])}^2.
\]

(6.40)

Swapping between $u_2$ and $u_1$ we can analogously show the following estimate the second of the two integrals in the RHS of (6.39), i.e.

\[
E \left[ \mathbb{I}_{\{T_2 \leq T_1\}} \int_S^{S + T} \|\xi_1^u(t) - \xi_2^u(t)\|_{\gamma(K, H)}^2 dt \right]^{\frac{\gamma}{2}}
\]

\[
= E \left[ \mathbb{I}_{\{T_2 \leq T_1\}} \int_S^{S + T} \|\theta_n(\|\bar{u}_1\|_{Z_{[0,\bar{t}]}})(\bar{G}_n(u_1(s))) - \theta_n(\|\bar{u}_2\|_{Z_{[0,\bar{t}]}})(\bar{G}_n(u_1(s)))\|_{\gamma(K, H)}^2 ds \right]^{\frac{\gamma}{2}}
\]

\[
\leq 2^{p-1} (C_{G_n}^\prime)^p \|u_1 - u_2\|_{L^p(Z[S,S+T_1])}^2.
\]

(6.41)

Thus, substituting (6.39), (6.41) into (6.38) and and (6.38) yield, respectively,

\[
E \left[ \sup_{t \in [S, S + T]} \| [J\xi_1^u(t)] - [J\xi_2^u(t)] \|_{H_A^p}^p \right] \leq 2^{p-1} (C_{G_n}^\prime)^p \tilde{K}_T (T + 2T^{1-\frac{n}{p}}(2n)^{2\gamma})^{\frac{\gamma}{2}} \|u_1 - u_2\|_{L^p(Z[S,S+T_1])}^p
\]

and

\[
E \left[ \|J\xi_1^u - J\xi_2^u\|_{L^p(0,T;E)}^p \right] \leq 2^{p-1} (C_{G_n}^\prime)^p \tilde{C}_T (T + 2T^{1-\frac{n}{p}}(2n)^{2\gamma})^{\frac{\gamma}{2}} \|u_1 - u_2\|_{L^p(Z[S,S+T_1])}^p.
\]

Hence, we get

\[
\|J\xi_1^u - J\xi_2^u\|_{L^p(Z[S,S+T_1])}^p \leq 2^{p-1} (C_{G_n}^\prime)^p (\tilde{K}_T + \tilde{C}_T) (T + 2T^{1-\frac{n}{p}}(2n)^{2\gamma})^{\frac{\gamma}{2}} \|u_1 - u_2\|_{L^p(Z[S,S+T_1])}^p.
\]
problem (6.1) on the interval $[0, S, S + T_0]$.

Since $2\gamma < p$, by definition of $L_p^\alpha(T)$, it is clear that $\lim_{T\to 0} L_p^\alpha(T) = 0$ for every $n$. Thus, we have proved (6.6) and hence the proof of Lemma (6.5) is complete.

Let us recall that $n \in \mathbb{N} \setminus \{0\}$ and the positive number $M_n = \frac{n}{n+1}$ are fixed. Next we prove the following auxiliary result which gives the existence of $T_n > 0$ such that the map $\Psi_n^{[S, S + T_n]}$ is contractive.

**Proposition 6.16.** There exists a positive number $T_n > 0$ such that the following assertion holds. If $S \geq 0$ and a process $v \in M^p(Z_{[0, S]})$ is such that $\mathbb{P}$-almost surely it’s trajectories belong to the space $C([0, S], H_A) \cap C^1([0, S], H)$ and the velocity process $v_t$ belongs to the class $M^p(Q_{[0, S]})$, then there exists a unique $u \in M^p(Z_{[S, S + T_n]})$ which is a unique mild solution to equation (6.1) on time interval $[S, S + T_n]$ with the history process $v$, in the sense of Definition 6.4.

**Proof of Proposition 6.16** Let us choose and fix $S \geq 0$ and a process $v \in M^p(Z_{[0, S]})$ is such that $\mathbb{P}$-almost surely it’s trajectories belong to the space $C([0, S], H_A) \cap C^1([0, S], H)$ and the velocity process $v_t$ belongs to the class $M^p(Q_{[0, S]})$. We will show that there exits $T_n > 0$, independent of $v$, such that the map $\Psi_n^{[S, S + T_n]}$ defined in (6.11)-(6.12) is a $\frac{1}{2}$-contraction in the space $M^p(Z_{[S, S + T_n]}^{v(S)})$.

Let us define, for $T > 0$, $L_n(T) := L_2^\alpha(T) + L_0^\alpha(T)$. By Lemmata 6.13 and 6.15, we infer that the function $T \mapsto L_n(T)$ is nondecreasing and that $\lim_{T \to 0} L_n(T) = 0$. Hence we can find $T_n > 0$ such that $L_n(T_n) \leq \frac{1}{2}$.

Since also by Lemmata 6.11, 6.15 we infer that the map $\Psi_n^{[S, S + T_n]}$ is well-defined on $M^p(Z_{[S, S + T_n]}^{v(S)})$ and, for all $u_1, u_2 \in M^p(Z_{[S, S + T_n]}^{v(S)})$,

$$||\Psi_n^{[S, S + T_n]}(u_1) - \Psi_n^{[S, S + T_n]}(u_2)||_{M^p(Z_{[S, S + T_n]}^{v(S)})} \leq L_n(T)||u_1 - u_2||_{M^p(Z_{[S, S + T_n]}^{v(S)})},$$

we infer that $\Psi_n^{[S, S + T_n]}$ is a $\frac{1}{2}$-contraction. Thus, by the Banach Fixed Point Theorem there exists a unique fixed point $u_n \in M^p(Z_{[S, S + T_n]}^{v(S)})$ of the map $\Psi_n^{[S, S + T_n]}$. Hence the proof of Proposition 6.16 is complete by observing that a fixed point of the map $\Psi_n^{[S, S + T_n]}$ is a mild solution of problem (6.1) on $[S, S + T_n]$ with the history process $v$.

Finally we are ready to embark on the proof of Theorem 6.5.

**Proof of Theorem 6.5** Let us choose and fix $n \in \mathbb{N} \setminus \{0\}$ and an $H_A \times H$-valued $\mathcal{F}_0$-measurable random variable $(u_0, u_1)$ which satisfies condition (6.4).

By Proposition 6.16 there exists a positive number $T_n > 0$ such that for every $S \geq 0$ and for every process $v \in M^p(Z_{[0, S]})$, which has $\mathbb{P}$-almost surely it’s trajectories belong to the space $C([0, S], H_A) \cap C^1([0, S], H)$ and the velocity process $v_t$ belongs to the class $M^p(Q_{[0, S]})$, there exists a unique $u \in M^p(Z_{[S, S + T_n]}^{v(S)})$ which is a unique mild solution of problem (6.1) on $[S, S + T_n]$, i.e., (6.10) satisfies.

We will construct by induction a sequence $(u^k)_{k=1}^\infty$ such that $u^k \in M^p(Z_{[0, kT_n]})$ such that $\mathbb{P}$-almost surely its trajectories belong to the space $C^1([0, kT_n], H)$ and the velocity process $u_t^k$ belongs to the class $M^p(Q_{[0, kT_n]})$, and $u^k$ is a solution, in the sense of Definition 6.3 of problem (6.1) on the interval $[0, kT_n]$ with initial condition (6.2).
For $k = 1$ we take as $u^1$ the unique process mentioned above with $S = 0$ and the initial data $(u_0, u_1) = (u_0, u_1)$. This process satisfies the listed conditions because of Proposition 6.16 and Lemma 6.4.

Suppose next that $k \in \mathbb{N} \setminus \{0\}$ and we have constructed a process $u^k \in \mathcal{M}^p(Z_{[0,kT_n]})$ with prescribed conditions.

Then by Proposition 6.16 with $S = kT_n$ and the history process $v = u^k$, there exists a unique process $u \in \mathcal{M}^p(Z_{[kT_n,(k+1)T_n]}^{uk^k(kT_n)})$ which solves problem (6.1) on the interval $[kT_n, (k + 1)T_n]$ and satisfying initial condition $(u^k(kT_n), u^k_T(kT_n))$. Then we observe that the concatenation process $u^k \cup u$ makes sense and it belongs to $\mathcal{M}^p(Z_{[0,(k+1)T_n]}^{uk^k(kT_n)})$. It is sufficient to prove that $u^{k+1} := u^k \cup u$ is a solution of problem (6.1) on the interval $[0, (k + 1)T_n]$ with initial condition (6.2). In view of Definition 6.4 because $u^k$ solves equation (6.1) on the interval $[0, kT_n]$ with initial condition (6.2), it is sufficient to show that $u^{k+1}$ satisfies identity (6.3) for every $t \in (kT_n, (k + 1)T_n)$, i.e.

\[
\begin{align*}
u^{k+1}(t) &= \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 \\
&\quad + \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}(\theta_n(\|u^{k+1}\|_{L^2})\tilde{F}_n(u^{k+1}(s))) \, ds \\
&\quad + \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}(\theta_n(\|u^{k+1}\|_{L^2})\tilde{G}_n(u^{k+1}(s))) \, dW(s) \mathbb{P}\text{-a.s.} .
\end{align*}
\]

Recall that $\| \cdot \|_{L^2} = \| \cdot \|_{Z_{[0,s]}}$. Let us choose and fix $t \in ((kT_n, (k + 1)T_n)$. Then, on one hand by Definition 6.7 of a solution on time interval $[kT_n, (k + 1)T_n)$ with history process $u^k$ and, since by definition of the concatenation $u^{k+1} := u^k \cup u$, the process $u = u^{k+1}$ on the interval $[kT_n, (k + 1)T_n)$, we have that

\[
\begin{align*}
u^{k+1}(t) &= \cos((t - kT_n)\sqrt{A})u^k(kT_n) + \frac{\sin((t - kT_n)\sqrt{A})}{\sqrt{A}}u^k(kT_n) \\
&\quad + \int_{kT_n}^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}(\theta_n(\|u^{k+1}\|_{L^2})\tilde{F}_n(u^{k+1}(s))) \, ds \\
&\quad + \int_{kT_n}^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}(\theta_n(\|u^{k+1}\|_{L^2})\tilde{G}_n(u^{k+1}(s))) \, dW(s) , \mathbb{P}\text{-a.s.} .
\end{align*}
\]

On the other hand, since $u^k$ and $u^{k+1}$ are equal on $[0, kT_n]$ we have

\[
\begin{align*}
u^k(kT_n) &= \cos(kT_n\sqrt{A})u_0 + \frac{\sin(kT_n\sqrt{A})}{\sqrt{A}}u_1 \\
&\quad + \int_0^{kT_n} \frac{\sin((kT_n-s)\sqrt{A})}{\sqrt{A}}(\theta_n(\|u^{k+1}\|_{L^2})\tilde{F}_n(u^{k+1}(s))) \, ds \\
&\quad + \int_0^{kT_n} \frac{\sin((kT_n-s)\sqrt{A})}{\sqrt{A}}(\theta_n(\|u^{k+1}\|_{L^2})\tilde{G}_n(u^{k+1}(s))) \, dW(s) , \mathbb{P}\text{-a.s.} ,
\end{align*}
\]

and

\[
\begin{align*}
u^k_T(kT_n) &= -\sqrt{A}\sin(kT_n\sqrt{A})u_0 + \cos(kT_n\sqrt{A})u_1 \\
&\quad + \int_0^{kT_n} \cos((kT_n-s)\sqrt{A})(\theta_n(\|u^{k+1}\|_{L^2})\tilde{F}_n(u^{k+1}(s))) \, ds .
\end{align*}
\]
Thus, in light of (6.43)-(6.44), in order to prove (6.42) it is sufficient to prove equality (6.46) below, i.e.,

\[ u^{k+1}_t(t) = -\sqrt{A} \sin(t\sqrt{A}) u_0 + \cos(t\sqrt{A}) u_1 + \int_0^t \cos((t-s)\sqrt{A}) (\theta_n(\|u^{k+1}\|_{Z_s}) \widehat{G}_n(u^{k+1}(s))) \, ds \]

\[ + \int_0^t \cos((t-s)\sqrt{A}) (\theta_n(\|u^{k+1}\|_{Z_s}) \widehat{G}_n(u^{k+1}(s))) \, dW(s), \quad \mathbb{P}\text{-a.s..} \]

(6.46)

However, using notation from Appendix C (6.46) is equivalent to the following equality

\[ u^{k+1}(t) = e^{At} u(0) + \int_0^t e^{A(t-s)} (\theta_n(\|u^{k+1}\|_{Z_s}) \widehat{G}_n(u^{k+1}(s))) \, ds \]

\[ + \int_0^t e^{A(t-s)} (\theta_n(\|u^{k+1}\|_{Z_s}) \widehat{G}_n(u^{k+1}(s))) \, dW(s), \quad \mathbb{P}\text{-a.s..} \]

(6.47)

On the other hand, identities (6.44)-(6.45) are equivalent to the following equality

\[ u^k(T_n) = e^{AkT_n} u(0) + \int_0^{kT_n} e^{A(kT_n-s)} (\theta_n(\|u^{k+1}\|_{Z_s}) \widehat{G}_n(u^{k+1}(s))) \, ds \]

\[ + \int_0^{kT_n} e^{A(kT_n-s)} (\theta_n(\|u^{k+1}\|_{Z_s}) \widehat{G}_n(u^{k+1}(s))) \, dW(s), \quad \mathbb{P}\text{-a.s..} \]

(6.48)

Finally, we notice that equality (6.43) implies

\[ u^{k+1}(t) = e^{A(t-kT_n)} u^k(T_n) + \int_{kT_n}^t e^{A(t-s)} (\theta_n(\|u^{k+1}\|_{Z_s}) \widehat{G}_n(u^{k+1}(s))) \, ds \]

\[ + \int_{kT_n}^t e^{A(t-s)} (\theta_n(\|u^{k+1}\|_{Z_s}) \widehat{G}_n(u^{k+1}(s))) \, dW(s). \]

(6.49)

Now, by using the semigroup property of the \(C_0\)-group \(e^{tA}\), we can easily show that (6.49) together with (6.48) imply (6.47). Thus we proved that the process \(u^{k+1}\) satisfies all the required conditions.

Hence we constructed the sequence \((u^{k})_{k=1}^{\infty}\). Let us also note that \(u^k\) and \(u^{k+1}\) are equal on \([0, kT_n]\), for every \(k \geq 1\). This implies that the infinite concatenation process

\[ u_n := \bigcup_{k=1}^{\infty} u^k \]

is a global mild solution to the problem (6.1) satisfying the initial conditions (6.2). The proof is complete.

**Alternative proof of Theorem 6.5** Let us choose and fix \(n \in \mathbb{N} \setminus \{0\}\) and an \(H_A \times H\)-valued \(\mathcal{F}_0\)-measurable random variable \((u_0, u_1)\) which satisfies condition (6.4). Then by applying Proposition 6.16 with \(S = 0\) we find a time \(T_n > 0\) and process \(u_* \in \mathcal{M}^p(Z_{[0,T_n]}^\mathbb{R})\) which is the (unique) solution to our auxiliary problem (6.1)-(6.2) on \([0,T_n]\). Let \(\mathcal{U}\) be a family of pairs \((T, u)\), where \(T > 0\) and \(u \in \mathcal{M}^p(Z_{[0,T]}^\mathbb{R})\) is a (unique) solution to our auxiliary problem (6.1)-(6.2) on \([0,T_*]\). By what we have just proven this family is non-empty, because \((T_*, u_*)\) belongs to it.

Suppose first that the set \(\{T : (T, u) \in \mathcal{U}\}\) is unbounded from above. Then it is easy to show that it is equal to the open interval \((0, \infty)\) and therefore we can find an \(\mathcal{U}\)-valued sequence \(\{(k, u_k)\}_{k=1}^{\infty}\). By the uniqueness part of Theorem 6.5 we infer that the restriction of
the process $u_{k+1}$ to the interval $[0, k]$ coincides with the process $u_k$. Therefore by a standard construction there exists a unique process $u(t), t \in [0, \infty)$ such that for every $k \in \mathbb{N}$, that the restriction of the process $u$ to the interval $[0, k]$ coincides with the process $u_k$. Finally, because $u_k$ is a solution to our auxiliary problem (6.1)-(6.2) on $[0, k]$ we easily can show that the process $u$ is a global solution problem (6.1)-(6.2).

We will prove that the set $\{ T : (T, u) \in \mathcal{U} \}$ is unbounded from above. Suppose by contradiction that the set $\{ T : (T, u) \in \mathcal{U} \}$ is bounded from above. Put

$$ T := \sup \{ T : (T, u) \in \mathcal{U} \} < \infty. \tag{6.50} $$

Then it is easy to show that it is equal to the open interval $(0, T)$ is contained in $\{ T : (T, u) \in \mathcal{U} \}$ and therefore we can find an $\mathcal{U}$-valued sequence $\{ (T - \frac{1}{k}, u_k) \}_{k=1}^\infty$. Then, as in the first part above, there exists a unique process $u(t), t \in [0, T)$ such that for every $k \in \mathbb{N}$, that the restriction of the process $u$ to the interval $[0, T - \frac{1}{k}]$ coincides with the process $u_k$. Moreover, because $u_k$ is a solution to our auxiliary problem (6.1)-(6.2) on $[0, T - \frac{1}{k}]$, we infer that $u$ is a solution to problem (6.1)-(6.2) on $[0, T)$. Moreover, by Theorem 6.8

$$ C_5 := \sup_k \| u \|_{L^p(Z[0,T-\frac{1}{k}])}^p < \infty. \tag{6.51} $$

Let now take $T_n = T_n(\| (u_0, v_1) \|_{L^p(0,H_A \times H)}^p, C_5) > 0$ from Proposition 6.16 and $k \in \mathbb{K}$ such that $\frac{2}{k} < T_n$. Then by Proposition 6.16 with $S = -T + \frac{1}{k}$ we can find a unique solution $v$ to the (6.1) on the time interval $[T - \frac{1}{k}, T - \frac{1}{k} + T_n] \supset [T - \frac{1}{k}, T + \frac{1}{k}]$ such that

$$ v(T - \frac{1}{k}) = u(T - \frac{1}{k}), \quad v(t - \frac{1}{k}) = u(t - \frac{1}{k}). $$

Moreover, by the uniqueness part of Theorem 6.5 the union of $u$ and $v$ is a well defined process which belongs $\mathcal{M}(Z[0,T+\frac{1}{k}])$ and is as well a (unique) solution to problem (6.1)-(6.2) on $[0, T + \frac{1}{k}]$. Therefore $T$ does not satisfy (6.50). This contradicts the assumption that the set $\{ T : (T, u) \in \mathcal{U} \}$ is bounded from above. Theorem 6.5 is thus proved. \qed

7. The main results

The main result of the present paper, i.e. the existence of an unique local maximal solution to the problem (5.1)-(5.5), will be proved in this section. Let $H, H_A$ and $E$ be Hilbert and respectively Banach spaces defined in (4.1).

**Theorem 7.1.** Let us assume that a triple $(p, q, r)$ satisfies condition (5.5). Let us assume that the maps $F : E \cap B_{H_A}(0,1) \to H$ and $G : E \cap B_{H}(0,1) \to \gamma(K, H)$, where $K$ is a separable Hilbert space, satisfy assumptions $A.7$ and $A.2$ with $\gamma$ independent of $M \in (0, 1)$ satisfying

$$ 0 < 2\gamma < p. \tag{7.1} $$

Then, for every $H_A \times H$-valued $\mathcal{F}_0$-measurable random variable $(u_0, u_1)$ satisfying condition (5.27), there exists a unique local maximal mild solution $u = \{ u(t) : t \in [0, \tau) \}$, an announcing sequence $\{ \tau_k \}_{k \geq 1}$ of the stopping time $\tau$, to the problem (5.1)-(5.5), in the sense of Definition 5.13. Moreover, if $\mathbb{P}(\{ \tau < \infty \}) > 0$, then $\mathbb{P}$-almost surely on the set $\{ \tau < \infty \}$, the following explosion condition is satisfied

$$ \text{if } \lim_{t \to \tau} \| u \|_{Z_t} < \infty \text{ then } \limsup_{t \to \tau} \| u(t) \|_{H_A} = 1. \tag{7.2} $$
Proof of Theorem 7.1. The proof is divided into 3 parts. We prove the existence part in the first and the uniqueness result in the second part. We introduce the concept of maximality with its proof in our setting in the last part. For the purpose of the proof, as before we consider a numerical sequence $\left( M_n \right)_{n=2}^{\infty} = \left( \frac{n}{n+1} \right)_{n=1}^{\infty}$.

Let us choose and fix a triple $(p,q,r)$ satisfies condition (8.5) and a number $\gamma \in (0, \frac{q}{p})$. Let $H, H_A$ be Hilbert spaces and let $E$ be a Banach space defined in (4.1). Let us consider maps $F : E \cap B_{H_A}(0,1) \rightarrow H$ and $G : E \cap B_{H_A}(0,1) \rightarrow \gamma(K, H)$, where $K$ is a separable Hilbert space, which satisfy assumptions $A.1$ and $A.2$. Let us recall that by Remark 5.2 for every $n \in \mathbb{N} \setminus \{0,1\}$, by Theorem 5.3 there exist maps $F_n$ and $G_n$ (we do write the explicit dependency on $n$ for accuracy), corresponding to $M = M_n$, defined on the whole $H_A \cap E$ and taking values in $H$ and $\gamma(K, H)$, respectively, such that the inequalities (5.11) and (5.12) hold true and satisfying the equality (5.10) with $M = M_n$.

7.1. A proof of the existence of a local solution. The proof of the existence of a local mild solution to problem (5.4)-(5.5) in the sense of Definition 5.13 is carried out here.

To prove the existence of a local mild solution to problem (5.4)-(5.5) in the sense of Definition 5.13 let us choose and fix the initial data $(u_0, u_1) \in H_A \times H$ satisfying condition (5.27).

Let $\left\{ u_{0,n}, u_{1,n} \right\}_{n \in \mathbb{N}}$ be a sequence of initial data such that (1) $u_{0,n} = u_0$ for every $n$; (2) $u_{1,n} \rightarrow u_1$ as $n \rightarrow \infty$ in $H$ and, for each $n \in \mathbb{N}$ and $\omega \in \Omega$, $u_{1,n}(\omega)$ is equal to $u_1(\omega)$ if $|u_1(\omega)|_H \leq n$ and is equal to 0, otherwise.

Let a process $u_n$ be the unique solution of the approximating equation (6.1) with initial data $(u_{0,n}, u_{1,n})$ whose existence is guaranteed by Theorem 6.5. Let us recall that for every $T > 0$, $u_n \in M^p(\mathcal{F}T)$. Let $\tau_n$ be a stopping time defined by the following formula

$$\inf \left\{ t \in [0, \infty) : \max \{ \|u_n\|_{X_t}, \psi(\|u_n\|_{Z_t}) \} \geq \frac{n}{n+1} \right\}, \quad (7.3)$$

where $\psi$ is an increasing homeomorphism defined by the following formula

$$\psi : [0, \infty) \ni x \mapsto \frac{x}{1+x} \in [0, 1). \quad (7.4)$$

In view of the definition (4.6) of the $\| \cdot \|_{X_t}$-norm we deduce the following equivalent definition of the stopping time $\tau_n$,

$$\tau_n := \inf \{ t \in [0, \infty) : \max \{ \|u_n(t)\|_{H_A}, \psi(\|u_n\|_{Z_t}) \} \geq \frac{n}{n+1} \}. \quad (7.5)$$

Let us observe that $\tau_n = 0$ can be equal to 0, for instance when $\|u_0\|_{H_A} \geq \frac{n}{n+1}$.

Moreover, we use the convention that $\inf(J) = \infty$ if $J = \emptyset$ and the function $[0, \infty) \ni t \mapsto \max \{ \|u_n\|_{X_t}, \psi(\|u_n\|_{Z_t}) \} \in [0, \infty)$ is continuous we deduce that

$$\tau_n = \infty \text{ iff } \max \{ \|u_n\|_{X_t}, \psi(\|u_n\|_{Z_t}) \} < n \text{ for every } t \in [0, \infty), \quad (7.6)$$

if $\tau_n < \infty$ then $\max \{ \|u_n\|_{X_t}, \psi(\|u_n\|_{Z_t}) \} < \frac{n}{n+1} \text{ for every } t \in [0, \tau_n), \quad (7.7)$

and $\max \{ \|u_n\|_{X_t}, \psi(\|u_n\|_{Z_t}) \} = \frac{n}{n+1}$.

Since our filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ satisfies the usual hypothesis, the notions of accessible and predictable stopping times are equivalent, see [49] Theorem 1.6.6 and therefore by [49] Proposition I.4.14, the stopping time $\tau_n$ is accessible. Thus, we can find an announcing sequence $\left\{ \tau_{n_k} \right\}_{k \in \mathbb{N}}$ for $\tau_n$.
Arguing as in the proofs of [12] Lemma 5.2 and [3] Theorem 4.10, we can prove that for every \( n \in \mathbb{N} \),
\[
\tau_n \leq \tau_{n+1} \quad \mathbb{P}\text{-a.s.}
\] (7.8)
and
\[
u_n(t) = u_{n+1}(t) \quad \text{if } t \in [0, \tau_n) \text{ and } n \in \mathbb{N}, \quad \mathbb{P}\text{-a.s.}
\] (7.9)
Moreover, one can easily show, by using once again the continuity of functions \( t \mapsto \|u_n\|_{X_1} \) and \( t \mapsto \|u_n\|_{Z_1} \), that the inequality (7.8) is strict when \( \tau_n \in (0, \infty) \), i.e.
\[
\tau_n < \tau_{n+1} \quad \mathbb{P}\text{-a.s. on } \{ \tau_n \in (0, \infty) \}.
\] (7.10)
By taking appropriate modifications we can assume that (7.8) is satisfied on the whole space \( \Omega \). Hence, the following limit exists
\[
\tau_\infty(\omega) = \lim_{n \to \infty} \tau_n(\omega), \quad \omega \in \Omega.
\] (7.11)
Note that the above definition implies that if for some \( n \in \mathbb{N} \), \( \tau_n(\omega) = \infty \), then \( \tau_\infty(\omega) = \infty \). Thus we infer that if there exists \( n \in \mathbb{N} \) such that \( \sup \{ \max \{ \|u_n(t)\|_{H_A}, \psi(\|u_n\|_{Z_1}) : t \in [0, \infty) \} \} < 1 \), then \( \tau_\infty(\omega) = \infty \). However, \( \tau_\infty(\omega) \) can be equal to \( \infty \) even if for every \( n \in \mathbb{N} \) the previous supremum is equal to \( \infty \). Finally, let us also observe that
\[
\tau_n < \tau_\infty \quad \mathbb{P}\text{-a.s. on } \{ \tau_n \in (0, \infty) \}.
\] (7.12)
On the other hand, the first assertion we observed above is equivalent to the following. If \( \tau_\infty(\omega) < \infty \) then for every \( n \in \mathbb{N} \), \( \tau_n(\omega) < \infty \).

At this juncture it is important to observe that, since \( \|u_n(0)\|_{H_A} < 1 \), \( \mathbb{P}\)-almost surely, by condition (6.27), we infer that there exists \( n \in \mathbb{N} \) such that \( \|u_n(0)\|_{H_A} < M_n \), \( \mathbb{P}\)-almost surely. Moreover, \( \|u_n\|_{Z_0} = \|u_n(0)\|_{H_A} < 1 < n \). Therefore since the functions \( t \mapsto \|u_n\|_{Z_1} \) and \( t \mapsto \|u_n\|_{X_1} \) are continuous, we deduce that the stopping time \( \tau_n > 0 \) and hence \( \tau_\infty > 0 \). Hence we deduce that \( \tau_\infty \) is strictly positive \( \mathbb{P}\)-almost surely.

Arguing as before, since our probability basis satisfies the usual hypothesis, \( \tau_\infty \) is an accessible stopping time, see [43] Proposition 2.3 and Lemma 2.11 with \( \{ \tau_n \wedge n \}_{n \in \mathbb{N}} \) being an announcing sequence for \( \tau_\infty \). By using the claims (7.5) and (7.10) we define a local process \((u, \tau_\infty)\) in the following way
\[
u(t, \omega) = u_n(t, \omega) \quad \text{if } t < \tau_n(\omega), \quad \omega \in \Omega.
\] (7.13)
We will show later on that \((u, \tau_\infty)\) is a local solution to problem (5.4)-(5.5). Before embarking on this objective let us point out that because of the definitions (7.3) and (7.11) of respectively stopping times \( \tau_n \) and \( \tau_\infty \) and the process \( u \) we deduce that
\[
\tau_n = \inf \{ t \in [0, \tau_\infty) : \max \{ \|u\|_{X_1}, \psi(\|u\|_{Z_1}) \} \geq \frac{n}{n+1} \}.
\] (7.14)
In view of the definition (4.6) of the \( \| \cdot \|_{X_1} \)-norm we deduce the following equivalent form of identity (7.14)
\[
\tau_n = \inf \{ t \in [0, \tau_\infty) : \max \{ \|u(t)\|_{H_A}, \psi(\|u\|_{Z_1}) \} \geq \frac{n}{n+1} \}.
\] (7.15)
The above implies the following result.

**Corollary 7.2.** If \((u, \tau_\infty)\) is a local process defined above, then, \( \mathbb{P}\)-almost surely the following assertion holds true.
If \( \tau_\infty(\omega) < \infty \) and
\[
\sup \{ \|u\|_{Z_1} : t \in [0, \tau_\infty) \} < \infty,
\] (7.16)
then
\[
\limsup_{t \to \tau_\infty(\omega)} \|u(t, \omega)\|_{H_A} = 1. \tag{7.17}
\]

**Proof of Corollary 7.2.** Indeed, \(\tau_\infty(\omega) < \infty\) and condition (7.16) is satisfied, then there exists \(n_0 \in \mathbb{N}\), such that (i) \(\|u(t)\|_{Z_t} < n_0\) for every \(t \in [0, \tau_\infty]\), and (ii) \(0 < \tau_{n_0}(\omega) < \tau_\infty(\omega)\) and the sequence \(\tau_k(\omega)\) is strictly increasing and converges to \(\tau_\infty(\omega)\). Therefore, since by (7.15)
\[
\max_{k \geq n_0} \bigl\{\|u(\tau_k(\omega))\|_{H_A}, \psi(\|u||Z_{\tau_k(\omega)})\bigr\} = \frac{k}{k+1}, k \geq n_0.
\]
This implies that
\[
\lim_{k \to \infty} \|u(\tau_k(\omega))\|_{H_A} = 1. \tag{7.18}
\]
This concludes the proof of Corollary 7.2. Note however, that we proved a more precise assertion (7.18) than the claimed assertion (7.2).

Let us also notice that from the above Corollary 7.2 we infer that the local process \((u, \tau_\infty)\) constructed above satisfies that last part of our main Theorem 7.1.

Now we present the promised proof of the assertion that \((u, \tau_\infty)\) is a local solution to problem (5.4)–(5.5). Since the accessibility of \(\tau_\infty\) has already been established, we need to prove properties (2) and (3) of Definition 5.13 for \(\{u(t) : t \in [0, \tau_\infty]\}\). The property \(\|u(t)\|_{H_A} < 1\) for \(t \in [0, \tau_\infty)\), \(\mathbb{P}\)-a.s. directly follows by definition of \(\tau_\infty\) and the map \(\psi\). Next, since \(u_n \in \mathcal{M}^p(Z_T)\) for every \(T > 0\), due to (7.13) we infer that \(u\) belongs to \(\mathcal{M}^p(Z_{t \land \tau_n})\), for all \(t \geq 0\).

Since, from Theorem 6.8 \(u_n \in \mathcal{M}^p(Z_T)\) for every \(T > 0\), expression (6.3) gives that the following holds, for every \(t \geq 0\),
\[
u_n(t) - \cos(t\sqrt{A}) u_0 - \frac{\sin(t\sqrt{A})}{\sqrt{A}} u_1 - \int_0^t \frac{\sin((t-s)(\sqrt{A})}{\sqrt{A}} (\theta_n(\|u_n\|_{Z_s}) \tilde{F}_n(u_n(s))) ds
\]
\[
= \int_0^t \frac{\sin((t-s)(\sqrt{A})}{\sqrt{A}} (\theta_n(\|u_n\|_{Z_s}) \tilde{G}_n(u_n(s))) dW(s), \ \mathbb{P}\text{-a.s.} \tag{7.19}
\]
Observe that, since from the definition of \(\mathcal{M}^p(Z_T)\), the processes on both sides of equality (7.19) are \(\mathcal{D}(A^2)\)-valued continuous, and by [65] Theorem I.2, any two modified stochastic processes are indistinguishable, we conclude that the equality even holds when the fixed deterministic time is replaced by the random one, in particular, (7.19) holds for \(t \land \tau_n\). Moreover, from definition of \(\tau_n\) we infer that \(\|u_n\|_{X_{t \land \tau_n}} \leq \frac{n}{n+1}\) for all \(n \in \mathbb{N}\). Therefore, \(\|u_n\|_{X_{t \land \tau_n}} \leq 1 - \frac{1}{n} = M_n\) for all \(n \in \mathbb{N}\). Thus, the equality (7.13) gives that
\[
F(u(s)) = \tilde{F}_n(u_n(s)) \quad \text{and} \quad G(u(s)) = \tilde{G}_n(u_n(s)), \quad \forall s \in [0, t \land \tau_n) \quad \text{and} \quad n \in \mathbb{N}.
\]
Consequently, since by the definition of \(\theta_n\) and \(\tau_n\) we have \(\theta_n(\|u_n\|_{Z_{t \land \tau_n}}) = 1\) for all \(n, k \in \mathbb{N}\), we deduce that for every \(t \geq 0\),
\[
\int_0^{t \land \tau_n} \frac{\sin((t \land \tau_n - s)(\sqrt{A})}{\sqrt{A}} (\theta_n(\|u_n\|_{Z_s}) \tilde{F}_n(u_n(s))) ds
\]
\[
= \int_0^{t \land \tau_n} \frac{\sin((t \land \tau_n - s)(\sqrt{A})}{\sqrt{A}} (F(u(s))) ds, \ \mathbb{P}\text{-a.s.}
\]
Moreover, by invoking Lemma A.1, which is a generalization of [14] Lemma A.1, we deduce that for every \(t \geq 0\),
\[
I(t \land \tau_n) = \int_0^t \frac{\sin((t-s)(\sqrt{A})}{\sqrt{A}} (\theta_n(\|u_n\|_{Z_{s \land \tau_n}}) \mathbb{I}_{[0, \tau_n]}(s) \tilde{G}_n(u_n(s)) ) dW(s)
\]
where \( I_{\tau_n}(G(u))(t) \) is defined in \((5.30)\). This proves that the equality \((5.29)\) is satisfied by the process \( u \). Hence, we have completed the existence of a local solution part of Theorem 4.1.

### 7.2. A proof of the uniqueness of a local solution.

The proof below is based on the proof of \([10, \text{Theorem 5.14}]\).

**Theorem 7.3.** Assume that \((u_0, u_1) \in H_A \times H\) and that condition \((5.27)\) is satisfied. Assume that \((u, \tau)\) and \((v, \sigma)\) are two local solutions of problem \((5.4)-(5.5)\), with the same initial data \((u_0, u_1)\). Then,

\[
(u_{[0,\tau \wedge \sigma) \times \Omega}, \sigma \wedge \tau) \sim (v_{[0,\tau \wedge \sigma) \times \Omega}, \sigma \wedge \tau).
\]

**Proof of Theorem 7.3.** Let us choose and fix two local solutions \((u, \tau)\) and \((v, \sigma)\) of problem \((5.4)-(5.5)\), respectively, with the same fixed initial data \((u_0, u_1) \in H_A \times H\) satisfying condition \((5.27)\). Without loss of generality, we can assume that \(\tau \vee \sigma \leq T\) for some \(T > 0\) which we fix for the remaining of the proof.

Let \(\{\tau_n\}_{n \in \mathbb{N}}\) and \(\{\sigma_n\}_{n \in \mathbb{N}}\) be the announcing sequences of \(\tau\) and \(\sigma\), respectively. Since the considered probability space satisfies the usual hypothesis, by \([19, \text{Propositions 4.3 and 4.11, and Theorem 6.6}]\) the stopping time \(\varrho := \tau \wedge \sigma\) is accessible and it is easy to show that \(\{\varrho_n\}_{n \in \mathbb{N}} := \{\tau_n \wedge \sigma_n\}_{n \in \mathbb{N}}\) is an announcing sequence of \(\varrho\).

Let us fix \(n \in \mathbb{N}\). Since \((v, \sigma)\) is a local solution to the problem \((5.4)-(5.5)\) we have that, for all \(t \in [0, T]\),

\[
v(t \wedge \sigma_n) = \cos((t \wedge \sigma_n)\sqrt{A})u_0 + \frac{\sin((t \wedge \sigma_n)\sqrt{A})}{\sqrt{A}} u_1 + \int_0^{t \wedge \sigma_n} \frac{\sin((t \wedge \sigma_n - s)\sqrt{A})}{\sqrt{A}} F(v(s)) \, ds + I_{\sigma_n}(G(v))(t \wedge \sigma_n), \text{\[\mathbb{P}\text{-a.s.,}\]}
\]

where

\[
I_{\sigma_n}(G(v))(t) = \int_0^t \mathbf{1}_{(0,\sigma_n)}(s) \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}} G(v(s)) \, dW(s), \quad t \in [0, T].
\]

Since the above holds true for all \(t \in [0, T]\), by replacing \(t\) by \(t \wedge \varrho_n\) we get, \(\mathbb{P}\)-almost surely,

\[
v(t \wedge \varrho_n \wedge \sigma_n) = \cos((t \wedge \varrho_n \wedge \sigma_n)\sqrt{A})u_0 + \frac{\sin((t \wedge \varrho_n \wedge \sigma_n)\sqrt{A})}{\sqrt{A}} u_1 + \int_0^{t \wedge \varrho_n \wedge \sigma_n} \frac{\sin((t \wedge \varrho_n \wedge \sigma_n - s)\sqrt{A})}{\sqrt{A}} F(v(s)) \, ds + I_{\sigma_n}(G(v))(t \wedge \varrho_n \wedge \sigma_n).
\]

Consequently, since \(\varrho_n \leq \sigma_n\), Lemma \((A.3)\) yields, for every \(t \in [0, T]\), \(\mathbb{P}\)-almost surely

\[
v(t \wedge \varrho_n) = \cos((t \wedge \varrho_n)\sqrt{A})u_0 + \frac{\sin((t \wedge \varrho_n)\sqrt{A})}{\sqrt{A}} u_1 + \int_0^{t \wedge \varrho_n} \frac{\sin((t \wedge \varrho_n - s)\sqrt{A})}{\sqrt{A}} F(v(s)) \, ds + I_{\varrho_n}(G(v))(t \wedge \varrho_n).
\]  

(7.20)

This proves that \((v, \varrho)\) is a local solution to problem \((5.4)-(5.5)\). In a similar way, we can also prove that \((u, \varrho)\) is a local solution to problem \((5.4)-(5.5)\).

Let us choose and fix an arbitrary \(M \in (0, 1)\) such that \(\|u_0\|_{H_A} < M\) and fix \(k \in \mathbb{N}\). We define the following six stopping times

\[
\hat{\tau}_k := \inf\{t \in [0, T]\, : \|u\|_{Z_t} \geq k\} \wedge \tau, \quad \hat{\sigma}_k := \inf\{t \in [0, T]\, : \|v\|_{Z_t} \geq k\} \wedge \sigma,
\]

\[
\hat{\tau}_M := \inf\{t \in [0, T]\, : \|u\|_{X_t} \geq M\} \wedge \tau, \quad \hat{\sigma}_M := \inf\{t \in [0, T]\, : \|v\|_{X_t} \geq M\} \wedge \sigma,
\]
\(\varrho_{k,M} := \tilde{\tau}_k \wedge \hat{\tau}_k \wedge \tilde{\tau}_M \wedge \hat{\sigma}_M,\)
\(\varrho_{n,k,M} := \tau_n \wedge \sigma_n \wedge \varrho_{k,M}.\)

Arguing as in the proof of (1.20) we can show that for all \(t \in [0, T],\) \(\mathbb{P}\text{-a.s.}\)

\[v(t \wedge \varrho_{n,k,M}) = \cos((t \wedge \varrho_{n,k,M})\sqrt{A})u_0 + \frac{\sin((t \wedge \varrho_{n,k,M})\sqrt{A})}{\sqrt{A}}u_1 + \int_0^{t \wedge \varrho_{n,k,M}} \sin((t \wedge \varrho_{n,k,M} - s)\sqrt{A}) F(v(s)) \, ds + I_{\varrho_{n,k,M}}(G(v)(t \wedge \varrho_{n,k,M})).\]

Similarly, we can prove that the above identity holds with \(v\) replaced by \(u.\) Hence, by setting \(w = u - v\) we obtain, for all \(t \in [0, T],\) \(\mathbb{P}\text{-a.s.},\)

\[w(t \wedge \varrho_{n,k,M}) = \int_0^{t \wedge \varrho_{n,k,M}} \frac{\sin((t \wedge \varrho_{n,k,M} - s)\sqrt{A})}{\sqrt{A}} [F(u(s)) - F(v(s))] \, ds + I_{\varrho_{n,k,M}}(G(u) - G(v))(t \wedge \varrho_{n,k,M}),\]

where

\[I_{\varrho_{n,k,M}}(G(u) - G(v))(t) := \int_0^t \mathbf{1}_{[0, \varrho_{n,k,M}]}(s) \frac{\sin((t - s)\sqrt{A})}{\sqrt{A}} [G(u(s)) - G(v(s))] \, dW(s).\]

In order to move forward, we set \(w_0 := u(0) - v(0),\) \(w_1 := u_t(0) - v_t(0).\)

It is obvious that \(w_0 = w_1 = 0.\) Observe that

\[\sup_{t \in [0, T]} \|u(t \wedge \varrho_{n,k,M})\|_{H_\lambda} \leq M \quad \text{and} \quad \sup_{t \in [0, T]} \|v(t \wedge \varrho_{n,k,M})\|_{H_\lambda} \leq M.\]

Invoking the inequality (3.6) and the stochastic Strichartz estimates (4.28), with \(p = 2,\) and observing that the inequalities (3.6) and (4.28) hold for derivative also, followed by the Assumptions [A.1] and [A.2] we infer that

\[
\mathbb{E} \left[ \|w\|^2_{C([0, t \wedge \varrho_{n,k,M}]; H_\lambda)} + \|w_t\|^2_{C([0, t \wedge \varrho_{n,k,M}]; H_\lambda)} \right] \lesssim \mathbb{E} \left[ \bar{C}_T \left( \|w_0\|_{H_\lambda} + \|w_1\|_{H_\lambda} \right)^2 \right. \\
\left. + \mathbb{E} \bar{C}_T \int_0^{t \wedge \varrho_{n,k,M}} \|F(u(r)) - F(v(r))\|_{H_\lambda} \, dr \right]^2 \\
\left. + \mathbb{E} \bar{K}_T \int_0^{t \wedge \varrho_{n,k,M}} \|G(u(r)) - G(v(r))\|^2_{(K, H)} \, dr \right] \\
\lesssim \mathbb{E} \left[ \bar{C}_T \left( \|w_0\|_{H_\lambda} + \|w_1\|_{H_\lambda} \right)^2 \right. \\
\left. + \mathbb{E} \left[ \bar{C}_F^2 \|C_F^2 + \bar{K}_T \|C_G \|^2 \right \right] \\
\times \int_0^{t \wedge \varrho_{n,k,M}} \left\{ \sup_{s \in [0, r]} \|w(s)\|^2_{H_\lambda} + \sup_{s \in [0, r]} \|w_t(s)\|^2_{H_\lambda} \right\} \left( 1 + \|u(r)\|_E + \|v(r)\|_E \right)^{2\gamma} \, dr \right].
\]

Now we can apply Lemma [D.1] with the following choice of processes.

\[X(t) := \|w\|^2_{C([0, t \wedge \varrho_{n,k,M}]; H_\lambda)} + \|w_t\|^2_{C([0, t \wedge \varrho_{n,k,M}]; H_\lambda)}, \quad t \geq 0,
\]

\[R(s) := 1 + \|u(s)\|^{2\gamma}_E + \|v(s)\|^{2\gamma}_E, \quad s \geq 0,
\]

\[C_0 := \bar{C}_F^2 \|C_F^2 + \bar{K}_T \|C_G \|^2.
\]

Note that by the definition of the stopping time \(\varrho_{n,k,M}\) the following inequality holds and therefore Assumption [D.1] from Lemma [D.1] is satisfied.

\[
\int_0^{\varrho_{n,k,M}} R(s) \, ds - \varrho_{n,k,M} = \int_0^{\varrho_{n,k,M}} \left( \|u(r)\|^{2\gamma}_E + \|v(r)\|^{2\gamma}_E \right) \, ds \leq 2k^{2\gamma}.
\]
Finally, inequality (7.21) holds for all possible pairs \((\tau_n, \tau_0)\) of accessible stopping times such that \(0 \leq \tau_n \leq \tau_0 \leq \tau_{n,k,M}\). Thus, since \(w(0) = 0\), by Lemma D.1 we infer that

\[
\mathbb{E} \left[ \sup_{s \in \{0, \tau_n, \tau_M\}} \|w(s)\|_{H^1}^2 \right] = 0.
\]

This implies that there exists \(\Omega_{n,k,M} \in \mathcal{F}\) such that \(\mathbb{P}(\Omega_{n,k,M}) = 1\) and

\[
\sup_{s \in \{0, \tau_n, \tau_M\}} \|w(s)\|_{H^1}^2 = 0, \quad \text{if } \omega \in \Omega_{n,k,M}.
\]

Put \(\Omega_{k,M} = \bigcup \Omega_{n,k,M}\). Note that \(\mathbb{P}(\Omega_{k,M}) = 1\). Since \(\tau_n \wedge \sigma_n \not\supset \tau \wedge \sigma\) we infer that

\[
\sup_{s \in \{0, \tau_n, \tau_M\}(\omega)} \|w(s)\|_{H^1}^2 = 0, \quad \text{if } \omega \in \Omega_{k,M}.
\]

Put \(\Omega_0 := \bigcup_{\mathcal{F}(0,1)\cap \Omega} \bigcup \Omega_{k,M}\). Note that \(\mathbb{P}(\Omega_0) = 1\). Since \(\lim_{M \to \infty} \lim_{k \to \infty} \theta_{k,M} = \tau \wedge \sigma\), \(\mathbb{P}\)-almost surely, we infer that there exists \(\Omega_1 \in \mathcal{F}\) such that \(\mathbb{P}(\Omega_1) = 1\) and \(\theta_{k,M} \not\supset \tau(\omega) \wedge \sigma(\omega)\) for each \(\omega \in \Omega_1\). Hence we infer that

\[
\sup_{s \in \{0, \tau \wedge \sigma(\omega)\}} \|w(s)\|_{H^1}^2 = 0, \quad \omega \in \Omega_0 \cap \Omega_1.
\]

Because \(\mathbb{P}(\Omega_0 \cap \Omega_1) = 1\), this completes the proof of uniqueness.

### 7.3. Constructed solution \((u, \tau_\infty)\) is a local maximal solution.

Let \((u, \tau_\infty)\) is the local solution to problem (5.4)-5.5 constructed in the earlier parts of this section.

We will prove that \((u, \tau_\infty)\) is a local maximal solution to problem (5.4)-5.5 such that \(\mathbb{P}(\tau_\infty \leq \hat{\tau}) = 1\). By the uniqueness of local solutions, see Theorem 7.3, we deduce that the processes \(u\) and \(\hat{u}\) are equivalent. Hence, it is sufficient to show that \(\mathbb{P}(\tau_\infty = \hat{\tau}) = 1\).

Firstly we observe that \(\tau_\infty = \hat{\tau}\), on the set \(\{\tau_\infty = \infty\}\), \(\mathbb{P}\)-almost surely. So it is enough to prove that \(\mathbb{P}(\{\tau_\infty < \hat{\tau}\}) = 0\).

Let us introduce two auxiliary events \(\Omega_1\) and \(\Omega_2 = \Omega \setminus \Omega_1\) defined by

\[
\Omega_1 := \left\{ \omega \in \Omega : \sup_{t \in [0, \tau_\infty(\omega))] \|u(t, \cdot, \omega)\|_{L^2} < \infty \right\}, \quad (7.22)
\]

Suppose that \(\mathbb{P}(\Omega_1 \cap \{\tau_\infty < \hat{\tau}\}) > 0\). Let \((\hat{\tau}_n)\) be the announcing sequence of \(\hat{\tau}\). Since \(\hat{\tau}_n \not\supset \hat{\tau}\) \(\mathbb{P}\)-a.s., we infer that there exists \(n \in \mathbb{N}\) such that \(\mathbb{P}(\Omega_1 \cap \{\tau_\infty < \hat{\tau}_n < \hat{\tau}\}) > 0\). Let us observe that if \(\omega \in \Omega_1 \cap \{\tau_\infty < \hat{\tau}_n < \hat{\tau}\}\), then

\[
\sup_{t \in [0, \tau_\infty(\omega))] \|u(t, \cdot, \omega)\|_{H^1} = \sup_{t \in [0, \tau_\infty(\omega))} \|\hat{u}(t, \cdot, \omega)\|_{H^1} \leq \sup_{t \in [0, \tau_\infty(\omega))} \|\hat{u}(t, \cdot, \omega)\|_{H^1} < 1.
\]

This contradicts assertion of the Corollary 7.2 and therefore we infer that \(\mathbb{P}(\Omega_1 \cap \{\tau_\infty < \hat{\tau}\}) = 0\).

Now, we suppose that \(\mathbb{P}(\Omega_2 \cap \{\tau_\infty < \hat{\tau}\}) > 0\). Let \((\hat{\tau}_n)\) be the announcing sequence of \(\hat{\tau}\). Since \(\hat{\tau}_n \not\supset \hat{\tau}\) \(\mathbb{P}\)-a.s., we infer that there exists \(n \in \mathbb{N}\) such that \(\mathbb{P}(\Omega_2 \cap \{\tau_\infty < \hat{\tau}_n < \hat{\tau}\}) > 0\). Let us observe that if \(\omega \in \Omega_2 \cap \{\tau_\infty < \hat{\tau}_n < \hat{\tau}\}\), then

\[
\sup_{t \in [0, \tau_\infty(\omega))] \|u(t, \cdot, \omega)\|_{L^2} = \sup_{t \in [0, \tau_\infty(\omega))} \|\hat{u}(t, \cdot, \omega)\|_{L^2} \leq \sup_{t \in [0, \tau_\infty(\omega))] \|\hat{u}(t, \cdot, \omega)\|_{L^2} < \infty.
\]

This is impossible in view of the definition of the set \(\Omega_2\) and therefore we infer that \(\mathbb{P}(\Omega_2 \cap \{\tau_\infty < \hat{\tau}\}) = 0\).
Summing up, we proved that $\mathbb{P}(\{\tau_\infty < \hat{\tau}\}) = 0$. Thus, $(u, \tau_\infty)$ is a local maximal solution to problem (5.4)-(5.4).

Hence, the proof of Theorem 7.1 is complete.

**APPENDIX A. STOPPED PROCESSES**

In this appendix we justify the choice of $I_r(G)$ process in the Definition 5.13. The proof of the next result to some extent is analogous to the proof of [11, Lemma A.1], where this is formulated in terms of a semigroup.

**Lemma A.1.** Assume that a process $\xi$ belongs to $\mathcal{M}^p(\mathcal{L}^2([0, T], \gamma(K, H)))$. Set

$$I(t) := \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \xi(s) \, dW(s),$$

and

$$I_r(t) := \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} (\mathbb{1}_{[0, \tau]}(s)\xi(s)) \, dW(s).$$  \hspace{1cm} (A.1)$$

For any stopping time $\tau$ and for all $t \in [0, T]$, the following holds

$$I(t \land \tau) = I_r(t \land \tau), \quad \mathbb{P}\text{-a.s.}$$  \hspace{1cm} (A.2)$$

**Remark A.2.** Let us note that, since $\tau$ is a stopping time, due to [49, Theorem 1.6 and Proposition 4.2] the stochastic process $\mathbb{1}_{[0, \tau]}$ is progressively measurable. In particular, the integrand in (A.1) is progressively measurable.

Note that, it follows from Lemma A.1 that if $\xi = 0$ on $[0, \tau)$, then $I(t \land \tau) = 0$ for all $t \in [0, T]$, $\mathbb{P}$-a.s. It is relevant to mention that the importance of such results goes back to [14, Proposition B.4] and [23]. The next result is useful in the proof of Theorem 7.3. We ask the reader to see [10, Corollary A.2] for the proof.

**Lemma A.3.** Let $\xi \in \mathcal{M}^p(\mathcal{L}^2([0, T], \gamma(K, H)))$ and $\tau, \sigma$ be two stopping times such that $\sigma \leq \tau$; then

$$I_r(t \land \sigma) = I_\sigma(t \land \sigma) \text{ for all } t \in [0, T], \quad \mathbb{P}\text{-a.s.}$$  \hspace{1cm} (A.3)$$

**APPENDIX B. ON POINTWISE EVALUATION**

Let us now formulate a special case of [39, Proposition 1.2.25]. This result is a converse to [32, Proposition 3.19] and is closely related to [11, Proposition B.4].

**Proposition B.1.** Assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$ is a filtered probability space. Assume also that $E$ is a separable Banach space, $T > 0$ and $p \in [1, \infty)$. Assume finally that

$$F : \Omega \to L^p(0, T; E) \text{ is Bochner integrable on } (\Omega, \mathcal{F}_T, \mathbb{P}).$$

Then, there exists a $\mathbb{B}([0, T]) \otimes \mathcal{F}_T / \mathbb{B}(E)$-measurable function

$$f : [0, T] \times \Omega \to E,$$

and there exists $\Omega' \in \mathcal{F}_T$ such that $\mathbb{P}(\Omega') = 1$ and for every $\omega \in \Omega'$, the following equality holds:

$$f(\cdot, \omega) = F(\omega) \text{ in } L^p(0, T; E).$$

This equality is again imprecise. Rigorously, one should replace it by $[f(\cdot, \omega)] = F(\omega)$ in $L^p(0, T; E)$, where $[\cdot]$ is the equivalence class w.r.t. the Lebesgue measure $\text{Leb}$. But, see Remark [14] it is standard to use this imprecise formulation.
Moreover, if \( g : [0, T] \times \Omega \to E \) is another \( \mathcal{B}([0, T]) \otimes \mathcal{F}_T/\mathcal{B}(E) \)-measurable function such that the above assertion holds, then \( f = g \), \( \mathbb{L} \otimes \mathbb{F} \)-almost everywhere.

**Appendix C. About the definition of a solution**

Here we state an equivalence, without proof, between two natural definitions of a mild solution for SPDE (1.2). We begin by recalling the framework from Section 5. In particular, we set

\[
H = L^2(\Omega); \quad H_A = \mathcal{D}(A^{\frac{1}{2}}); \quad E = \mathcal{D}(A^{\frac{1}{2}}),
\]

where \((p,q,r)\) is any suitable triple which satisfy \((3.3)\). Let us recall that the linear (unbounded) operator \( A \) in the space \( \mathcal{H} := H_A \times H \) and the \( C_0 \)-group on \( \mathcal{H} \) generated by it have been defined in formulae \((4.33)\) and \((4.34)\), respectively. Let us also recall that the space \( \mathcal{M}^p(Z_T) \) has been defined in \((5.2)\). We assume that the maps \( F \) and \( G \) satisfy \([A.1]\) and \([A.2]\), respectively.

**Proposition C.1.** Suppose that \( u_0 \in H_A, u_1 \in H, \text{ and } T > 0 \).

- If an \( H_A \)-valued process \( u = \{u(t) : t \in [0, T]\} \) such that \( u \in \mathcal{M}^p(Z_T) \), is a mild to problem \((5.4)-(5.5)\), i.e. for all \( t \in [0, T] \), \( \mathbb{P} \)-a.s

\[
u(t) = \cos (t \sqrt{A}) u_0 + \sin (t \sqrt{A}) u_1 + \int_0^t \frac{\sin((t-s) \sqrt{A})}{\sqrt{A}} F(u(s)) \, ds + \int_0^t \frac{\sin((t-s) \sqrt{A})}{\sqrt{A}} G(u(s)) \, dW(s).
\]  

\((C.1)\)

Then, \( \mathbb{P} \)-almost surely, the process \( u \) is differentiable as \( H \)-valued process and the \( H_A \times H \)-valued \( \mathbb{F} \)-adapted process \( u \) defined by

\[
u(t) = (u(t), v(t)), \quad t \in [0, T],
\]

where \( v(t) = u'(t), t \in [0, T], \) is a continuous \( H \)-valued process and satisfies, for all \( t \in [0, T] \), \( \mathbb{F} \)-almost surely, the following equation,

\[
u(t) = e^{At} u(0) + \int_0^t e^{A(t-s)} \mathfrak{F}[u(s)] \, ds + \int_0^t e^{A(t-s)} \mathfrak{G}[u(s)] \, dW(s),
\]  

\((C.2)\)

where \( u(0) = (u_0, u_1) \in H_A \times H \), and

\[
\mathfrak{G}[u] = (0, G(u)), \quad \mathfrak{F}[u] = (0, F(u)), \quad u = (u,v) \in H_A \times H.
\]  

\((C.3)\)

- Conversely, if an \( \mathbb{F} \)-adapted continuous \( H_A \times H \)-valued process

\[
u(t) = (u(t), v(t)), \quad t \in [0, T],
\]

such that \( u \in \mathcal{M}^p(Z_T) \), is a solution to \((C.2)\) with notation \((C.3)\), then the process \( u \) is a solution to \((C.1)\).

**Proposition C.2.** Suppose that \( u_0 \in H_A, u_1 \in H, \text{ and } \tau \) is an accessible stopping time with an announcing sequence \( \{\tau_k\}_{k \in \mathbb{N}} \).

- If an \( H_A \)-valued local process \( u = \{u(t) : t \in [0, \tau]\} \) is a local mild to problem \((5.4)-(5.5)\), then, \( \mathbb{P} \)-almost surely, the trajectories of \( u \) are differentiable and the \( H_A \times H \)-valued \( \mathbb{F} \)-adapted local process \( u \) defined by

\[
u(t) = (u(t), v(t)), \quad t \in [0, \tau],
\]  

\((C.4)\)
where \( v(t) = u'(t), t \in [0, \tau), \) is continuous and, for all \( t \geq 0, \) satisfies \( \mathbb{P} \)-almost surely, the following equations, for every \( k \in \mathbb{N}, \)
\[
\begin{align*}
u(t \wedge \tau_k) &= e^{A(t \wedge \tau_k)} u(0) + \int_0^{t \wedge \tau_k} e^{A(t-s)} \mathcal{F}[u(s)] \, ds + \mathcal{I}(\tau_k, \mathcal{G})(t \wedge \tau_k), \tag{C.5}
\end{align*}
\]
where \( u(0) = (u_0, u_1) \in H_A \times H \) and the process \( \mathcal{I}(\tau_k, \mathcal{G}) \) is defined by
\[
\mathcal{I}(\tau_k, \mathcal{G})(t) := \int_0^t e^{A(t-s)} \mathbf{1}_{[0,\tau_k)}(s) \mathcal{G}[u](s) \, dW(s), \quad t \geq 0, \tag{C.6}
\]
and \( \mathcal{F} \) and \( \mathcal{G} \) are defined in (C.3).

- Conversely, if an \( \mathcal{F} \)-adapted continuous \( H_A \times H \)-valued local process \( u(t) = \{(u(t), v(t)), t \in [0, \tau)\}, \) such that
\[
u \text{ belongs to } \mathcal{M}(Z_{\tau \wedge \tau_k}), \text{ for all } t \geq 0 \text{ and every } k,
\]
satisfies, for all \( t \geq 0, \) \( \mathbb{P} \)-almost surely, for every \( k \in \mathbb{N}, \) equation (C.5) with the notation (C.3) and (C.6), then the process \( u = \{u(t) : t \in [0, \tau)\} \) is a solution to (C.4).

The next result corresponds to Remark 5.15.

**Remark C.3.** It can be shown, see Remark 2.22 in [6], that if an \( \mathcal{F} \)-adapted and \( \mathcal{D}(A^\tau)^{1/2} \)-valued continuous process \( u = \{u(t) : t \in [0, \tau)\} \) is a local mild solution to problem (B.4)-(B.5), then equality (C.5) with (C.6), hold for any stopping time of the form \( \tau_k \wedge \sigma, \) where \( \sigma \) is an accessible stopping time.

**Appendix D. Stochastic Gronwall lemma**

The following result is a slight simplification of [37, Lemma 5.3] which in turn is a generalization of [33, Lemma 3.9].

**Lemma D.1.** Let us assume that \( \tau \) is an accessible bounded stopping time. Let \( X \) and \( R \) be real valued non-negative local processes defined on \([0, \tau)\) such that for some \( \kappa > 0 \)
\[
\int_0^\tau R(s) \, ds \leq \kappa, \quad \mathbb{P} \text{-almost surely,} \tag{D.1}
\]
and \( \mathbb{E} \int_0^\tau R(s)X(s) \, ds < \infty. \) Suppose also that there exists \( C_0 > 0 \) such that for all pairs \((\tau_a, \tau_b)\) of accessible stopping times such that \( 0 \leq \tau_a < \tau_b \leq \tau, \)
\[
\mathbb{E} \left[ \sup_{t \in [\tau_a, \tau_b]} X(t) \right] \leq C_0 \mathbb{E} \left[ X(\tau_a) + \int_{\tau_a}^{\tau_b} R(s)X(s) \, ds \right]. \tag{D.2}
\]
Then, there exists a constant \( C = C(C_0, T, \kappa) \) such that
\[
\mathbb{E}[X(\tau)] \leq C \mathbb{E}[X(0)]. \tag{D.3}
\]

**Proof of Lemma D.1.** The proof is almost identical to the proof of [37, Lemma 5.3]. Since our assumptions are weaker, as we only consider pairs of accessible stopping times, we only need to observe that in view of the assumption (D.1) there exist a natural number \( N \) and a sequence \((\tau_i)_{i=0}^N\) of accessible stopping times defined as follows: \( \tau_0 = 0 \) and
\[
\tau_1 = \inf \left\{ t \in [\tau_0, T] : \int_{\tau_0}^t R(s) \, ds \geq \frac{1}{2C_0} \right\}.
\]
Then for all \( x, x \in X \) and \( \|C \| \geq 0 \).

In particular, the range of \( \psi \) that a map \( g : X \to X \) through the formula

\[
\psi_R(x) = \begin{cases}  
  x, & \text{if } \|x\|_X \leq R, \\
  Rx\|x\|_{X}^{-1}, & \text{if } \|x\|_X > R.
\end{cases}
\]

Then for all \( x, x_1, x_2 \in X \cap E \)

\[
\|\psi_R(x_2) - \psi_R(x_1)\|_X \leq 3\|x_2 - x_1\|_X, \quad (E.2)
\]

\[
\|\psi_R(x)\|_X \leq R \text{ for every } x \in X \cap E. \quad (E.3)
\]

\[
\|\psi_R(x)\|_E \leq \|x\|_E \text{ for every } x \in X \cap E. \quad (E.4)
\]

In particular, the range of \( \psi_R \) is contained in the set \( B_X(0, R) \cap E \).

The following result is a slight generalization of a result that has been proved in [5].

**Lemma E.1.** Let \( X \) and \( E \) be normed vector spaces with norms denoted respectively by \( \| \cdot \|_X \) and \( \| \cdot \|_E \). For \( R > 0 \) define \( \psi_R : X \cap E \to X \cap E \) through the formula

\[
\psi_R(x) = \begin{cases}  
  x, & \text{if } \|x\|_X \leq R, \\
  Rx\|x\|_{X}^{-1}, & \text{if } \|x\|_X > R.
\end{cases}
\]

The next result was also formulated and proved in [5].

**Corollary E.2.** Let \( X \) and \( Y \) be normed vector spaces with norms denoted by \( \| \cdot \| \). Suppose that a map \( g : X \to Y \) is Lipschitz on the closed ball \( \bar{B}(0, R), R > 0, \) with Lipschitz constant \( C \). Then, there exists a bounded map \( \tilde{g} : X \to Y \) such that \( \tilde{g} = g \) on \( \bar{B}(0, R) \) and \( \tilde{g} \) is Lipschitz on \( X \), with Lipschitz constant \( 3C \).

We conclude this section with a new result which tells about the existence of a quasi-Lipschitz extension with the Lipschitz constant being 3 times the Lipschitz constant of the original quasi-Lipschitz map. This atypical extension type theorem is a generalization of [5, Corollary 3] and to be best knowledge of the authors, this is a new result.

**Theorem E.3.** Let \( X, E \) and \( Y \) be normed vector spaces with norms denoted respectively by \( \| \cdot \|_X, \| \cdot \|_E \) and \( \| \cdot \|_Y \). Suppose that a map

\[
g : \bar{B}_X(0, R) \cap E \to Y,
\]

where \( \bar{B}_X(0, R) \) is the closed ball in \( X \) centered at the origin and of radius \( R \), satisfies the following “Lipschitz” property. There exists \( C > 0 \) and \( p \geq 0 \) such that

\[
\|g(x_2) - g(x_1)\|_Y \leq C(1 + \|x_2\|_E + \|x_1\|_E)^p \|x_2 - x_1\|_X, \quad x_1, x_2 \in \bar{B}(0, R) \cap E. \quad (E.5)
\]

Then, there exists a map

\[
\tilde{g} : X \cap E \to Y
\]

such that \( \tilde{g} = g \) on \( \bar{B}_X(0, R) \cap E \) and

\[
\|\tilde{g}(x_2) - \tilde{g}(x_1)\|_Y \leq 3C(1 + \|x_2\|_E + \|x_1\|_E)^p \|x_2 - x_1\|_X, \quad x_1, x_2 \in X \cap E. \quad (E.6)
\]

**Proof of Theorem E.3.** Set

\[
\tilde{g} := g \circ \psi_R.
\]

Since the range of function \( \psi_R \) is contained in the set \( B_X(0, R) \cap E \), the function \( \tilde{g} : X \cap E \to Y \) is well defined and coincides with \( g \) on the set \( \bar{B}_X(0, R) \cap E \).
Let us choose and fix for $x_1, x_2 \in X \cap E$. Then the generalized Lipschitz property (E.5) of $g$, the Lipschitz, w.r.t. the $X$-norm, property (E.2) of the function $\psi_R$ and the linear growth, w.r.t. the $E$-norm, property (E.4) of the function $\psi_R$ imply that
\[
\|\tilde{g}(x_2) - \tilde{g}(x_1)\|_Y = \|g(\psi_R(x_2)) - g(\psi_R(x_1))\|_Y \\
\leq C(1 + \|\psi_R(x_2)\|_E + \|\psi_R(x_1)\|_E)^p\|\psi_R(x_2) - \psi_R(x_1)\|_X \\
\leq 3C(1 + \|\psi_R(x_2)\|_E + \|\psi_R(x_1)\|_E)^p\|x_2 - x_1\|_X \\
\leq 3C(1 + \|x_2\|_E + \|x_1\|_E)^p\|x_2 - x_1\|_X.
\]
This implies the inequality (E.6) and hence completes the proof of Theorem E.3.

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