SEGAL-TYPE ALGEBRAIC MODELS OF $n$-TYPES

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Abstract. For each $n \geq 1$, we introduce two new Segal-type models of $n$-types of topological spaces: weakly globular $n$-fold groupoids, and a lax version of these. We show that any $n$-type can be represented up to homotopy by such models via an explicit algebraic fundamental $n$-fold groupoid functor.

We compare these models to Tamsamani’s weak $n$-groupoids, and extract from them a model for $(k-1)$-connected $n$-types.

1. Introduction and summary

Many homotopy invariants of a topological space $T$, such as its homotopy, homology, and cohomology groups, are graded by dimension, so that we do not need to know all of $T$ to determine $\pi_n T$, $H_n T$, or $H^n(T; G)$, but only a skeleton or Postnikov section of $T$. Thus, for many purposes a good first approximation to homotopy theory is the study of $n$-types: that is, spaces $T$ whose homotopy groups $\pi_k(T, t_0)$ vanish for $k > n$.

One advantage of such approximations is that they have algebraic models: the classical example is the homotopy category of connected 1-types, which is equivalent to the category of groups. More generally, all 1-types are modelled by groupoids via the fundamental groupoid functor $\hat{\pi}_1 : \text{Top} \to \text{Gpd}$.

The arrows of $\hat{\pi}_1 T$ are homotopy classes of paths, so higher order approximations should encode higher homotopies (see [34]), and thus involve higher categorical structures.

Many such structures have been shown to model the homotopy category $\text{ho} \text{P}^n \text{Top}$ of $n$-types of topological spaces: in the path-connected case, these include the cat$^n$-groups of [40], the crossed $n$-cubes of [30] and [45], the $n$-hyper-crossed complexes of [23], and the weakly globular cat$^n$-groups of [44]. Special models exist for $n = 2, 3$, starting with the crossed modules of [41], and including the homotopy double groupoids of [16], the homotopy bigroupoids of [33], the strict 2-groupoids of [43], the double groupoids of [24], the double groupoids with connections of [19], the Gray groupoids of [39, 11, 38], and the quadratic modules of [14]. In the general case, such models include Batanin’s higher groupoids (see [3, 25]), the $n$-hypergroupoids of [32], and Tamsamani’s weak $n$-groupoids (see [51, 50]).

In this paper we discuss three algebraically defined categories of Segal-type objects which can be used to model all $n$-types of topological spaces. All three are full subcategories of the category $[\Delta^{n-1}\text{-P}, \text{Gpd}]$ of $(n-1)$-fold simplicial objects in groupoids.
1.1. The three models. The first is the known category \( \mathbf{Tam}^n \) of Tamsamani weak \( n \)-groupoids. The second is a new category \( \mathbf{Gpd}^n_{wg} \) of weakly globular \( n \)-fold groupoids. This is a full subcategory of the category \( \mathbf{Gpd}^n \) of \( n \)-fold groupoids (iteratively defined as groupoids internal to \( \mathbf{Gpd}^{n-1} \)). The third is another new category \( \mathbf{PsGpd}^n_{wg} \) of weakly globular pseudo \( n \)-fold groupoids.

To grasp the idea behind these notions, it is useful to consider another higher categorical structure which embeds in all three of the above, the category \( n\mathbf{-Gpd} \) of strict \( n \)-groupoids (iteratively defined as groupoids enriched in \( (n-1)\mathbf{-Gpd} \)).

There are full and faithful inclusions:

\[
\begin{array}{c}
\text{PsGpd}^n_{wg} \\
\downarrow \\
\mathbf{Tam}^n \\
\downarrow \\
\mathbf{Gpd}^n_{wg} \\
\downarrow \\
n\mathbf{-Gpd}
\end{array}
\]

The category \( n\mathbf{-Gpd} \) admits a multi-simplicial description as the full subcategory of those \( (n-1) \)-fold simplicial objects \( X \in [\Delta^{n-1\mathbf{-Gpd}}, \mathbf{Gpd}] \) satisfying:

(i) \( X^{(1)}_0 \in [\Delta^{n-2\mathbf{-Gpd}}, \mathbf{Gpd}] \) and \( X^{(1,\ldots,r+1)}_1 \in [\Delta^{n-r-2\mathbf{-Gpd}}, \mathbf{Gpd}] \) are discrete – that is, constant multi-simplicial sets – for all \( 1 \leq r \leq n-2 \). Here we use the notation of §2.6(b).

(ii) The Segal maps (see §2.3 below) in all directions are isomorphisms.

In addition, we require that after applying \( \pi_0: \mathbf{Gpd} \to \mathbf{Set} \) in each simplicial dimension we obtain a strict \( (n-1) \)-groupoid.

The sets in (i) corresponds to the set of \( r \)-cells \( (1 \leq r \leq n-2) \) in the strict \( n \)-groupoid. By (ii), their composition is associative and unital.

Condition (i) is also called the globularity condition, since it determines the globular shape of the cells in a strict \( n \)-groupoid. For instance, when \( n = 2 \) we can picture the 2-cells as globes:

Although strict \( n \)-groupoids have applications in homotopy theory, especially in their equivalent form of crossed \( n \)-complexes (cf. [18]), they cannot model all \( n \)-types of topological spaces (see [49, §5] for a counterexample in dimension 3). Therefore, we must relax the strict structure in order to recover all \( n \)-types.

We consider three approaches to this:

(a) In the first approach, we preserve condition (i) and relax (ii), by allowing the Segal maps to be suitable iteratively-defined equivalences. The composition
of cells is then no longer strictly associative and unital. This leads to the category $\text{Tam}^n$ of Tamsamani weak $n$-groupoids (Definition 5.1).

(b) In this paper we offer a second approach, in which condition (ii) is preserved, while (i) is replaced by weak globularity, so that the multi-simplicial objects in (i) are no longer required to be discrete, but only “homotopically discrete” (in a way that allows iteration). This leads to the category $\text{Gpd}^n_{\text{wg}}$ of weakly globular $n$-fold groupoids (Definition 3.19).

(c) We also describe a third approach, in which both (i) and (ii) are relaxed. This yields the category $\text{PsGpd}^n_{\text{wg}}$ of weakly globular pseudo $n$-fold groupoids (Definition 6.4).

Moreover, we have a realization functor $B: \text{PsGpd}^n_{\text{wg}} \to \text{Top}$ — the composite:

\[(1.3) \text{PsGpd}^n_{\text{wg}} \hookrightarrow [\Delta^{n-1}\text{op}, \text{Gpd}] \xrightarrow{N} [\Delta^{n-1}\text{op}, \text{Set}] \xrightarrow{\text{Diag}(n)} [\Delta^{n-1}\text{op}, \text{Set}] \xrightarrow{\|\|} \text{Top},\]

where $\text{Diag}(n)$ is the $n$-fold diagonal. The same is therefore true of the two subcategories $\text{Tam}^n$ and $\text{Gpd}^n_{\text{wg}}$. In all three categories, maps which induce weak homotopy equivalences on realizations are called geometric weak equivalences.

The precise definitions of these three categories appear as cited above; here we will only highlight some key features common to all three:

1) The construction of each category is by induction, starting in all three cases from the category of groupoids for $n = 1$. A weakly globular pseudo $n$-fold groupoid is in particular a simplicial object $X$ in $\text{PsGpd}^{n-1}_{\text{wg}}$, (and similarly for the other two categories).

2) Moreover, $X_0$ is a homotopically discrete weakly globular pseudo $(n - 1)$-fold groupoid (Definition 6.1) and similarly for $\text{Gpd}^n_{\text{wg}}$, while in the case of a Tamsamani weak $n$-groupoid $X \in [\Delta^{op}, \text{Tam}^{n-1}]$, $X_0$ is actually discrete.

3) Since $\text{PsGpd}^n_{\text{wg}}$ is a subcategory of $[\Delta^{n-1}\text{op}, \text{Gpd}]$, we can apply the functor $\pi_0$ to each groupoid of any weakly globular pseudo $n$-fold groupoid $X$ to obtain $\pi_0^n(X) \in [\Delta^{n-1}\text{op}, \text{Set}]$. In each of our three categories the functor $\pi_0^n$ lifts to functors

\[\Pi_0^n: \text{PsGpd}^n_{\text{wg}} \to \text{PsGpd}^{n-1}_{\text{wg}}, \quad \Pi_0^n: \text{Gpd}^n_{\text{wg}} \to \text{Gpd}^{n-1}_{\text{wg}},\]

and

\[\Pi_0^n: \text{Tam}^n \to \text{Tam}^{n-1}.\]

These serve as algebraic $(n - 1)$-Postnikov section functors, so we have a natural Postnikov tower:

\[\text{PsGpd}^n_{\text{wg}} \xrightarrow{\Pi_0^n} \text{PsGpd}^{n-1}_{\text{wg}} \xrightarrow{\Pi_0^{n-1}} \cdots \text{Gpd} \xrightarrow{\pi_0} \text{Set},\]

and similarly for the other two categories.

4) In all three categories, let $\gamma: X_0 \to X_0^d$ denote the weak equivalence from the homotopically discrete object $X_0$ to its discretization (so $\gamma$ is the identity for
For each \( k \geq 2 \) the composite of the maps

\[
X_k \xrightarrow{\mu_k} X_1 \times_{X_0} X_{k-1} \times_{X_0} \cdots \times_{X_0} X_1 \xrightarrow{\gamma^*} X_1 \times_{X_0} \cdots \times_{X_0} X_1
\]

(cf. \( \footnotesize{\text{§2.3}} \)) is called the \( k \)-th induced Segal map. We require that these maps be geometric weak equivalences.

When \( X \in \text{Tam}^n \), the second map in (1.4) is the identity, while when \( X \in \text{Gpd}_{wg}^n \), the first map is an isomorphism.

1.5. Main results. The process of discretizing the homotopically discrete subobjects in \( \text{Gpd}_{wg}^n \) and \( \text{PsGpd}_{wg}^n \) gives rise to discretization functors \( D_n \) making the following diagram commute

\[
\begin{array}{ccc}
\text{PsGpd}_{wg}^n & \xrightarrow{D_n} & \text{Gpd}_{wg}^n \\
\downarrow & & \downarrow \\
\text{Tam}^n & \xrightarrow{D_n} & \text{Gpd}_{wg}^n
\end{array}
\]

All three categories \( \text{PsGpd}_{wg}^n \), \( \text{Gpd}_{wg}^n \), and \( \text{Tam}^n \) share some useful features:

First, the realization functor \( B : \text{PsGpd}_{wg}^n \to \text{Top} \) actually lands in the category \( \text{P}^n_{\text{Top}} \) of \( n \)-types, so the same is true of \( \text{Gpd}_{wg}^n \) and \( \text{Tam}^n \).

Furthermore, all three models have algebraic homotopy groups \( \omega_k(X,x) \) (cf. \( \footnotesize{\text{§3.26}} \)), which allow one to extract \( \pi_k BX \) directly from the model \( X \). In addition, we have higher-dimensional analogues of the categorical notion of an equivalence of groupoids. Together, these two notions allow to define algebraic weak equivalences in each of the categories, and show that these are the same as the geometric weak equivalences (see Corollary \( \footnotesize{\text{4.3}} \) [T, \( \footnotesize{\text{§6}} \), \( \footnotesize{\text{§6.35}} \) and \( \footnotesize{\text{§6.37}} \)). Thus each of these models is entirely algebraic.

Our main results are as follows:

**Theorem A.** For each \( n \geq 1 \):

(a) The functor \( \tilde{Q}_{(n)} \) induces a faithful embedding

\[
\text{ho P}^n_{\text{Top}} \hookrightarrow \text{ho Gpd}_{wg}^n,
\]

so for each \( T \in \text{P}^n_{\text{Top}} \) there is an isomorphism in \( \text{ho P}^n_{\text{Top}} \) between \( T \) and \( B\tilde{Q}_{(n)} T \).

(b) There is a functor \( \Pi_0^{(n)} : \text{Gpd}_{wg}^n \to \text{Gpd}_{wg}^{n-1} \) with a natural isomorphism

\[
\Pi_0^{(n)} \tilde{Q}_{(n)} \cong \tilde{Q}_{(n-1)}
\]

so we can extract the model for the \((n-1)\)-st Postnikov section \( P^{n-1} T \) from \( \tilde{Q}_{(n)} T \) algebraically.

(c) There are algebraic homotopy group functors \( \omega_k : \text{Gpd}_{wg}^n \to \text{Gp} \) such that

\[
\pi_k(BG; x_0) \cong \omega_k(G; x_0) \quad (0 \leq k \leq n).
\]

[See Theorem \( \footnotesize{\text{4.32}} \) Proposition \( \footnotesize{\text{4.28}} \) and Theorem \( \footnotesize{\text{4.46}} \).]
Theorem B. The functors \( \hat{Q}_{(n)} \) and \( B \) induce equivalences of categories

\[ \text{ho} P^n \text{Top} \cong \text{ho} \text{PsGpd}_{\text{wg}}^n. \]

[See Theorem 6.28.]

Furthermore, every object of \( \text{PsGpd}_{\text{wg}}^n \) is weakly equivalent through a zig-zag to an object of \( \text{Gpd}_{\text{wg}}^n \) as well as to an object of \( \text{Tam}^n \) (see Remark 6.32). Thus we can regard \( \text{Tam}^n \) and \( \text{Gpd}_{\text{wg}}^n \) as two different types of partial strictifications of the category \( \text{PsGpd}_{\text{wg}}^n \) which preserve the homotopy type. The passage from \( \text{PsGpd}_{\text{wg}}^n \) to \( \text{Tam}^n \) strictifies the globularity condition, while the passage from \( \text{PsGpd}_{\text{wg}}^n \) to \( \text{Gpd}_{\text{wg}}^n \) strictifies the Segal maps.

The fundamental \( n \)-fold groupoid functor \( \hat{Q}_{(n)} : \text{Top} \to \text{Gpd}_{\text{wg}}^n \) provides an explicit form for the algebraic model of an \( n \)-type. This is a desirable feature of an algebraic model, especially in view of applications.

In Subsection 7.A we discuss an application to the modelling of \((k-1)\)-connected \( n \)-types. For this purpose we identify suitable subcategories \( \text{PsGpd}_{\text{wg}}^{(n,k)} \) and \( \text{Gpd}_{\text{wg}}^{(n,k)} \) of \( \text{PsGpd}_{\text{wg}}^n \) and \( \text{Gpd}_{\text{wg}}^n \) respectively, which are algebraic models of \((k-1)\)-connected \( n \)-types, and we also establish a connection with iterated loop spaces (Proposition 7.9).

1.7. Organization of the paper.

In Section 2 we describe the construction of the fundamental \( n \)-fold groupoid functor \( \hat{Q}_{(n)} \): to obtain a multi-simplicial algebraic model from a space, we first take a fibrant simplicial set model using the singular functor \( S : \text{Top} \to [\Delta^{\text{op}}, \text{Set}] \). We can associate to any simplicial set \( X \) an “\( n \)-fold resolution” \( \text{Or}_{(n)} X \), which is an object of \( [\Delta^{\text{op}}, \text{Set}] \) representing the same homotopy type (Lemma 2.13). We then obtain an \( n \)-fold groupoid by applying the left adjoint \( P_{(n)} : [\Delta^{\text{op}}, \text{Set}] \to \text{Gpd}^n \) to the \( n \)-fold nerve \( N_{(n)} : \text{Gpd}^n \to [\Delta^{\text{op}}, \text{Set}] \). Thus \( \hat{Q}_{(n)} \) is the composite:

\[ \text{Top} \xrightarrow{S} [\Delta^{\text{op}}, \text{Set}] \xrightarrow{\text{Or}_{(n)}} [\Delta^{\text{op}}, \text{Set}] \xrightarrow{P_{(n)}} \text{Gpd}^n \]

(cf. Definition 2.30).

For a general \( n \)-fold simplicial set \( Y \), \( P_{(n)} Y \) does not have a simple and explicitly computable expression. However, we show that the fibrancy of \( ST \) induces a property of \( \text{Or}_{(n)} ST \), which we call \((n, 2)\)-fibrancy (see Definition 2.31 and Proposition 2.39). We then show that to apply \( P_{(n)} \) to an \((n, 2)\)-fibrant \( n \)-simplicial set, we need only apply the usual fundamental groupoid functor in each of the \( n - 1 \) simplicial directions. We thus have

\[ \hat{Q}_{(n)} T = \hat{x}_1^{(1)} \hat{x}_1^{(2)} \ldots \hat{x}_1^{(n)} \text{Or}_{(n)} ST \]

(cf. Theorem 2.40).

In Section 3 we describe certain features of those \( n \)-fold groupoids which are in the image of the functor \( \hat{Q}_{(n)} \) (and thus will be used to represent \( n \)-types). These are encoded in the notion of weakly globular \( n \)-fold groupoids. As explained in 3.1.1 we first identify a suitable subcategory of homotopically discrete objects (Subsection 3.A), which are needed for the weak globularity condition in the definition of weakly globular \( n \)-fold groupoid (Subsection 3.B).
In Section 4 we show that the $n$-Postnikov section $P^n T$ and $B \hat{Q}_{(n)} T$ have the same homotopy type (Proposition 4.28), so $\text{Gpd}_{\text{wg}}^n$ represents all $n$-types. In Subsection 4.A we show that the realization of a weakly globular $n$-fold groupoid is an $n$-type (an alternative proof using a comparison with Tamsamani’s model is given in Section 5). Subsection 4.B provides a new iterative description of the fundamental $n$-fold groupoid functor $Q_{(n)}$. This is used in Proposition 4.28 of Subsection 4.C, where we show that the functor $\hat{Q}_{(n)}$ lands in the category $\text{Gpd}_{\text{wg}}^n$. This leads to one of our main results, Theorem 4.32, saying that $B$ and $\hat{Q}_{(n)}$ induce functors

$$\text{ho } P^n \text{Top} \xleftarrow{B} \text{ho } \text{Gpd}_{\text{wg}}^n,$$

with $B \circ \hat{Q}_{(n)} \cong \text{Id}$.

In Section 5 we provide an equivalent definition of Tamsamani’s weak $n$-groupoids (see Subsection 5.A), and in Subsection 5.B we construct a discretization functor $D_n : \text{Gpd}_{\text{wg}}^n \to \text{Tam}^n$, which replaces a weakly globular $n$-fold groupoid $G \in \text{Gpd}_{\text{wg}}^n$ by a Tamsamani weak $n$-groupoid $D_n G$ of the same homotopy type (Theorem 5.19).

In Section 6 we consider the wider context of weakly globular pseudo $n$-fold groupoids. These are defined in Subsection 6.A, and compared to Tamsamani’s model in Section 6.B, where we again construct a discretization functor $D_n : \text{PsGpd}_{\text{wg}}^n \to \text{Tam}^n$, and in Theorem 6.23 we show that for any $X \in \text{PsGpd}_{\text{wg}}^n$, there is a zig-zag of weak equivalences in $\text{PsGpd}_{\text{wg}}^n$ between $X$ and $D_n X$. Our main Theorem 6.28 then follows.

In Section 7 we describe an application, and indicate some directions for future work: in Subsection 7.A, we show how to extract from our results an algebraic model for $(k - 1)$-connected $n$-types (Proposition 7.7), and thus for iterated loop spaces. In Subsection 7.B we define $n$-track categories (one of the original motivations for our work), with possible future applications.

An appendix proves some technical facts about $\text{Or}_{(n)}$ needed in Section 2.
INDEX OF TERMINOLOGY AND NOTATION

\begin{itemize}
  \item \textbf{BG} realization of an \(n\)-fold (pseudo) groupoid \(G\) \hspace{1cm} [2.21]
  \item \textbf{c}\(X\) discrete groupoid on a set \(X\) \hspace{1cm} [2.23]
  \item \textbf{\(\pi^{(n)}, c^{(n)}\)} discrete groupoid functor applied to an \((n-1)\)-fold simplicial groupoid or an \((n-1)\)-fold groupoid \hspace{1cm} [3.12] [3.13]
  \item \textbf{\(D_n\)} discretization functors for \(\text{Gpd}_{\text{wg}}^n\) and \(\text{PsGpd}_{\text{wg}}^n\) \hspace{1cm} [5.18] [6.22]
  \item \textbf{\(\text{Disc}_0\)} 0-discretization functor on weakly globular \(n\)-fold groupoids \hspace{1cm} [5.13]
  \item \textbf{\text{Dec, Dec}'} décalage functors on simplicial sets \hspace{1cm} [2.7]
  \item \textbf{\(\text{Diag}_{(n)}\)} \(n\)-fold diagonal functor \hspace{1cm} [2.6]
  \item \textbf{\([\Delta^{op}, C]\)} category of \(n\)-fold simplicial objects in \(C\) \hspace{1cm} [2.10]
  \item \textbf{\(F^n_{\text{Tm}}\)} Tamsamani’s Poincaré \(n\)-groupoid functor \hspace{1cm} [5.7]
  \item \textbf{\(\text{Gpd}\)} category of groupoids \hspace{1cm} [2.12]
  \item \textbf{\(\text{n-Gpd}\)} category of strict \(n\)-groupoids \hspace{1cm} [1.1]
  \item \textbf{\(\text{Gpd}(\mathcal{V})\)} category of internal groupoids in \(\mathcal{V}\) \hspace{1cm} [2.16]
  \item \textbf{\(\text{Gpd}^n\)} category of \(n\)-fold groupoids \hspace{1cm} [2.16]
  \item \textbf{\(\text{Gpd}^n_{\text{wg}}\)} category of weakly globular \(n\)-fold groupoids \hspace{1cm} [3.19]
  \item \textbf{\(\text{Gpd}^{(n,k)}_{\text{wg}}\)} category of \((n,k)\)-weakly globular pseudo \(n\)-fold groupoids \hspace{1cm} [7.3]
  \item \textbf{\(\text{Gpd}^n_{\text{hd}}\)} category of homotopically discrete \(n\)-fold groupoids \hspace{1cm} [3.3]
  \item \textbf{\(L_k X\)} simplicial “bar-path construction” \hspace{1cm} [4.10]
  \item \textbf{\(\mu_k\)} \(k\)-th Segal map \hspace{1cm} [2.4]
  \item \textbf{\(\tilde{\mu}_k\)} \(k\)-th induced Segal map \hspace{1cm} [1.3]
  \item \textbf{\(N(i)\)} nerve functor of an \(n\)-fold groupoid in the \(i\)-th direction \hspace{1cm} [2.18]
  \item \textbf{\(N_{(n)}\)} multinerve functor on \(n\)-fold groupoids \hspace{1cm} [2.19]
  \item \textbf{\(\text{Or}_{(n)}\)} \(n\)-fold ordinal sum of a simplicial set \hspace{1cm} [2.19]
  \item \textbf{\(P(i)\)} left adjoint to \(N(i)\) \hspace{1cm} [2.38]
  \item \textbf{\(P_{(n)}\)} left adjoint to \(N_{(n)}\) \hspace{1cm} [2.30]
  \item \textbf{\(\text{PsGpd}^n_{\text{hd}}\)} category of homotopically discrete pseudo \(n\)-fold groupoids \hspace{1cm} [6.1]
  \item \textbf{\(\text{PsGpd}^{(n,k)}_{\text{wg}}\)} category of \((n,k)\)-weakly globular pseudo \(n\)-fold groupoids \hspace{1cm} [7.3]
  \item \textbf{\(\text{P}^n\text{Top}\)} full subcategory of \(n\)-Postnikov sections in \(\text{Top}\) \hspace{1cm} [3.25]
  \item \textbf{\(\tilde{\pi}_1\)} fundamental groupoid of a topological space \hspace{1cm} [2.17]
  \item \textbf{\(\Pi_{(n)}^0\)} algebraic \((n-1)\)-st Postnikov section functor \hspace{1cm} [3.15] [4.19] [5.1] [6.1]
  \item \textbf{\(Q_{(n)}, \tilde{Q}_{(n)}\)} fundamental \(n\)-fold groupoid functors \hspace{1cm} [2.30]
  \item \textbf{\(T^{\text{wg}}_{(n)}\)} fundamental groupoid functor for \(\text{Gpd}_{\text{wg}}^n\) \hspace{1cm} [5.15]
  \item \textbf{\(T^{\text{Tm}}_{(n)}\)} Tamsamani fundamental groupoid functor \hspace{1cm} [5.2]
\end{itemize}
\[ T_{(n)} \text{ fundamental groupoid functor for } \text{PsGpd}^n_{\text{wg}} \]

\[ \text{Tam}^n, \text{Tam}^n \text{ two equivalent formulations of the category of Tamsamani weak } n\text{-groupoids} \]

\[ \text{Top} \text{ category of topological spaces.} \]

\[ W_{(n,k)} \text{ } k\text{-fold object of arrows of an } n\text{-fold groupoid} \]

\[ \omega_k(G; x_0) \text{ } k\text{-th algebraic homotopy group} \]

See also the list of special notations for \( n\)-fold simplicial objects in \([2.6]\) in particular for the notation \( \text{F} \) for any functor \( F \).

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2. **THE FUNDAMENTAL \( n\)-FOLD GROUPOID OF A SPACE**

As noted above, the fundamental groupoid \( \hat{\pi}_1 T \) of a (not necessarily connected) space \( T \) is an algebraic model for its 1-type. We now show how the notion of the fundamental 2-typical double groupoid defined in [12, \S 2.21] generalizes to all \( n \).

We consider the standard model structure on \( \text{Top} \), so that \( \text{ho } \text{Top} \) means its localization with respect to the class of weak homotopy equivalences.

**2.A. Simplicial constructions**

Given a topological space \( T \), we construct its fundamental \( n\)-fold groupoid from a fibrant simplicial set model for \( T \), such as its singular set \( X := ST \in [\Delta^{\text{op}}, \text{Set}] \). We therefore first recall some notation and constructions for simplicial sets.

2.1. **Definition.** For any category \( \mathcal{C} \), \( [\Delta^{\text{op}}, \mathcal{C}] \) is the category of simplicial objects in \( \mathcal{C} \), where \( \Delta \) denotes the category of finite ordered sets: \([0], [1], \text{ and so on.} \) As usual, we write \( X_n \) for \( X([n]) \). If \( \mathcal{C} \) is concrete, the \( n\text{-skeleton} \) \( \text{sk}_n X \in [\Delta^{\text{op}}, \mathcal{C}] \) of any \( X \in [\Delta^{\text{op}}, \mathcal{C}] \) is generated under the degeneracy maps by \( X_0, \ldots, X_n \). The \( n\text{-coskeleton} \) functor \( \text{csk}_n : [\Delta^{\text{op}}, \mathcal{C}] \to [\Delta^{\text{op}}, \mathcal{C}] \) is left adjoint to \( \text{sk}_n \). We say that \( X \) is \( n\text{-coskeletal} \) if the natural map \( X \to \text{csk}_n X \) is an isomorphism.

2.2. **Remark.** There is an order-reversing involution \( I : \Delta \to \Delta \), which induces a functor \( I^* : [\Delta^{\text{op}}, \mathcal{C}] \to [\Delta^{\text{op}}, \mathcal{C}] \) (sending \( d_i : X_n \to X_{n-1} \) to \( d_{n-i} \)). This functor \( I^* \) is not generally an isomorphism, but for a Kan complex \( X \in [\Delta^{\text{op}}, \text{Set}] \) we have a natural isomorphism of fundamental groupoids \( (\hat{\pi}_1 X)^{\text{op}} \cong \hat{\pi}_1 X \) (cf. [33, I.8]).
2.3. **Definition.** Let \( X \in \Delta^{op}, C \) be a simplicial object in any category \( C \) with pullbacks. For each \( 1 \leq j \leq k \), let \( \nu_j : X_k \to X_1 \) be induced by the map \([1] \to [k]\) in \( \Delta \) sending 0 to \( j - 1 \) and 1 to \( j \). Then the following diagram commutes:

\[
\begin{array}{cccccc}
X_k & \rightarrow & X_1 & \rightarrow & \cdots & \rightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_1 & \rightarrow & X_0 & \rightarrow & \cdots & \rightarrow & X_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_0 & \rightarrow & X_0 & \rightarrow & \cdots & \rightarrow & X_0 \\
\end{array}
\]

If we let \( X_1 \times_{X_0} \cdots \times_{X_0} X_1 \) denote the limit of the lower part of Diagram (2.4), the \( k \)-th Segal map for \( X \) is the unique map

\[ \mu_k : X_k \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1 \]

such that \( \text{pr}_j \mu_k = \nu_j \), where \( \text{pr}_j \) is the \( j \)-th projection (see \([48]\)).

Note that \( X \) is the nerve of an internal category in \( C \) if and only if all the Segal maps are isomorphisms.

2.5. **n-fold simplicial objects.** An \( n \)-fold simplicial object in \( C \) is a functor \( \Delta^{op^n} \rightarrow C \), and we denote the category of such by \( [\Delta^{op^n}, C] \). Thus \( X \in [\Delta^{op^n}, C] \) consists of objects \( X_{i_1i_2\cdots i_n} \) in \( C \) for each \( n \)-fold multi-index \( i_1, i_2, \ldots, i_n \in \mathbb{N} \), along with face and degeneracy maps in each of the \( n \) directions, satisfying the usual simplicial identities. We assume a fixed ordering of these directions as first, second, and so on.

2.6. **Notation and conventions.**

(a) We can identify \( [\Delta^{op}, C] \) with \( [\Delta^{op}, [\Delta^{op-1}, C]] \) in \( n \) different ways: thus, given an \( n \)-fold simplicial object \( X \in [\Delta^{op}, C] \), for each \( 1 \leq i \leq n \) we write \( X^{(i)} \in [\Delta^{op}, [\Delta^{op-1}, C]] \) to indicate that the primary simplicial direction is the \( i \)-th one of the original \( X \).

(b) More generally, if we choose \( k \) of the \( n \) directions \( 1 \leq j_1 < j_2 < \cdots < j_k \leq n \), we obtain a \( k \)-fold simplicial object \( X^{(j_1,j_2,\ldots,j_k)} \) in \( [\Delta^{op-k}, C] \). Thus

\[ X^{(j_1,j_2,\ldots,j_k)} \in [\Delta^{op-k}, [\Delta^{op-1}, C]] \]

is a diagram of objects \( X^{(j_1,j_2,\ldots,j_k)}_{i_1 \cdots i_k} \) in \( \Delta^{op-k} \times [\Delta^{op-1}, C] \). For example, \( X^{(i_1 \cdots i_k)} = X([i_1, \ldots, [i_k]], -) \), in the notation of \([2,4] \).

Equivalently, for each object \( a \in \Delta^{op-k} \), \( X^{(j_1,j_2,\ldots,j_k)}(a) \in [\Delta^{op-k}, C] \) is a \( k \)-fold simplicial object in \( C \), natural in \( a \).

(c) In particular,

\[ X^{(i)} = X^{(1,\ldots,i-1,i+1,\ldots,n)} \in [\Delta^{op}, [\Delta^{op-1}, C]] \]

is an \((n-1)\)-fold simplicial object in \( [\Delta^{op}, C] \) (in the \( i \)-th direction).

(d) Given \( X \in [\Delta^{op}, C] \) and a functor \( F : [\Delta^{op}, C] \rightarrow C \), we denote by \( F^{(k)}X \in [\Delta^{op-1}, C] \) the object obtained by applying \( F \) “objectwise” to \( X^{(k)} \) (thought of as a \( \Delta^{op-1} \)-indexed diagram in \( [\Delta^{op}, C] \)).
Thus for every $i_1, \ldots, i_{n-1} \in \mathbb{N}$, we have
\[(F^{(k)} X)_{i_1, \ldots, i_{n-1}} = FX_{i_1, \ldots, i_{n-1}}^{(1, \ldots, k-1, k+1, \ldots, n)}.
\]

(e) The composite $F^{(1)} F^{(2)} \ldots F^{(n-1)} F^{(n)}$ will be denoted by $F^{(n)} : [\Delta^{op}, \mathcal{C}] \to \mathcal{C}$.

(f) In particular, the $n$-fold diagonal functor $\text{Diag}_{(n)} : [\Delta^{op}, \mathcal{C}] \to [\Delta^{op}, \mathcal{C}]$ is given by $(\text{Diag}_{(n)} X)_m := X_{m, \ldots, m}$. (In this case, the order does not matter.)

(g) For any functor $F : \mathcal{C} \to \mathcal{D}$, the prolongation of $F$ to simplicial objects is denoted by $\mathcal{F} : [\Delta^{op}, \mathcal{C}] \to [\Delta^{op}, \mathcal{D}]$.

(h) In particular, for a functor $G : [\Delta^{n-1^{op}}, \mathcal{C}] \to \mathcal{D}$, the result of applying $G$ to an $n$-fold simplicial object $X \in [\Delta^{n^{op}}, \mathcal{C}]$ in each simplicial dimension in the $k$-th direction will be denoted by $G^{(k)} X \in [\Delta^{op}, \mathcal{D}]$. Thus for every $j \in \mathbb{N}$ we have
\[(G^{(k)} X)_j = GX_j^{(k)}.
\]

2.7. Décalage. Recall from [27] \S 2.6 the comonad $\text{Dec} : [\Delta^{op}, \mathcal{Set}] \to [\Delta^{op}, \mathcal{Set}]$ on simplicial sets, where $(\text{Dec} X)_n = X_{n+1}$, forgetting the last face and degeneracy operators in each dimension (see also [36]). The counit $\varepsilon : \text{Dec} X \to X$ is given by $d_n : X_{n+1} \to X_n$ in simplicial dimension $n$. It has a section $\sigma : X \to \text{Dec} X$, given by $s_n : X_n \to X_{n+1}$.

There is also a version forgetting the first face and degeneracy operators, which we denote by $\text{Dec} : [\Delta^{op}, \mathcal{Set}] \to [\Delta^{op}, \mathcal{Set}]$. In the notation of \S 2.2 $\text{Dec} X := I^* \text{Dec} I^* X$.

The comonad $\text{Dec}$ yields a simplicial resolution $Y_* \in [\Delta^{op}, [\Delta^{op}, \mathcal{Set}]]$ for any $X \in [\Delta^{op}, \mathcal{Set}]$, with
\[Y_{k-1} := \text{Dec}^k X := \text{Dec}(\text{Dec} \ldots \text{Dec} X \ldots) \quad \text{in } [\Delta^{op}, \mathcal{Set}],\]

and the counit $\varepsilon$ for $\text{Dec}$ induces a map of bisimplicial sets $\varepsilon : Y \to c^{(2)} X$, where $c^{(2)} X$ is the constant simplicial object on $X$ in $[\Delta^{op}, [\Delta^{op}, \mathcal{Set}]]$ (thinking of the outer simplicial direction of $[\Delta^{op}, [\Delta^{op}, \mathcal{Set}]]$ as second). The bisimplicial set $Y_*$ is depicted in Figure I viewed as a horizontal simplicial object over $[\Delta^{op}, \mathcal{Set}]$ (degeneracy maps and $\varepsilon$ are not shown).

The corresponding resolution using $\text{Dec}'$ is also depicted in Figure I viewed as a vertical simplicial object over $[\Delta^{op}, \mathcal{Set}]$.

2.8. Remark. Note that if $X$ is a fibrant simplicial set, then so is $\text{Dec} X$, and the augmentation $\varepsilon : \text{Dec} X \to X$ is a fibration (with section $\sigma : X \to \text{Dec} X$). Similarly for $\text{Dec}'$.

2.9. Ordinal sum. In order to produce an $n$-fold simplicial set out of a Kan complex $X \in [\Delta^{op}, \mathcal{Set}]$, with the same homotopy type (that is, an $n$-fold resolution of $X$), we shall use the functor $\text{Or}_{(n)} := \alpha^n_* : [\Delta^{op}, \mathcal{Set}] \to [\Delta^{n^{op}}, \mathcal{Set}]$, induced by the ordinal sum $\alpha_n : \Delta^n \to \Delta$ (cf. [29] \S 2). Thus
\[(\text{Or}_{(n)} X)_{p_1, \ldots, p_n} := X_{n-1+p_1+\ldots+p_n},\]

(2.10)
If we define \( \text{Or}^{(i)}_{(n-1)}: [\Delta^{op}, \text{Set}] \rightarrow [\Delta^{op}, \text{Set}] \) for a bisimplicial set \( X \) by applying \( \text{Or}^{(i)}_{(n-1)} \) to \( X \) in each simplicial dimension in the \( i \)-th direction \((i = 1, 2)\) (cf. §2.6(h)), we have:

\[
\text{Or}(n) X = \text{Or}^{(2)}_{(n-1)} \text{Or}(2) X.
\]

See Figure 2 for a depiction of \( \text{Or}^{(3)} X \), where the vertical direction is first, the diagonal is second, and the horizontal is third.

The bisimplicial set \( \text{Or}^{(2)} X \) appears in Figure 1: this means that if we choose the vertical direction to be first and the horizontal to be second, then

\[
(\text{Or}(2) X)^{(1)} = \text{Dec}^{i+1} X \quad \text{and} \quad (\text{Or}(2) X)^{(2)} = (\text{Dec}')^{i+1} X.
\]

2.13. Lemma. For any simplicial set \( X \in [\Delta^{op}, \text{Set}] \), there is a natural weak equivalence \( \varepsilon_{(n)}: \text{Diag}_{(n)} \text{Or}(n) X \rightarrow X \).

**Proof.** By induction on \( n \geq 2 \).

For \( n = 2 \), as noted in §2.7 the counit \( \varepsilon: \text{Dec} X \rightarrow X \) induces a map of bisimplicial sets \( \varepsilon: \text{Or}^{(2)} X \rightarrow c^{(2)} X \) which is a weak equivalence of horizontal simplicial sets \( (\text{Dec}')^{i+1} X \rightarrow c X_i \) (where \( c X_i \) is the constant simplicial set on the set \( X_i \)), using (2.12). Thus by [27] §2.6 it induces a weak equivalence

\[
\varepsilon_{(2)}: \text{Diag}_{(2)} \text{Or}^{(2)} X \rightarrow \text{Diag}_{(2)} c^{(2)} X = X.
\]

In the induction stage we have a weak equivalence

\[
\varepsilon_{(n-1)}: \text{Diag}_{(n-1)} \text{Or}^{(n-1)} Y \rightarrow Y,
\]

natural in \( Y \). Using (2.11), and applying \( \varepsilon_{(n-1)} \) to \( \text{Or}(n) X \) in each simplicial dimension (in direction 2), we obtain a map of bisimplicial sets:

\[
\text{Diag}^{(2)}_{(n-1)} \text{Or}(n) X = \text{Diag}^{(2)}_{(n-1)} \text{Or}^{(2)}_{(n-1)} \text{Or}(2) X \xrightarrow{\varepsilon_{(n-1)}} \text{Or}(2) X
\]

which is a weak equivalence in each simplicial dimension in direction 2, by the induction hypothesis. Therefore, after applying \( \text{Diag}_{(2)} \) we obtain a weak equivalence
of simplicial sets

\[ \text{Diag}(2) \xrightarrow{(n-1)} : \text{Diag}(n) \text{Or}(n) X \to \text{Diag}(2) \text{Or}(2) X. \]

Post-composing with \( \varepsilon(2) : \text{Diag}(2) \text{Or}(2) X \to X \) yields the required weak equivalence \( \varepsilon(n) : \text{Diag}(n) \text{Or}(n) X \to X. \)

\[ \square \]

**2. B. n-Fold groupoids**

Recall that a *groupoid* is a small category \( G \) in which all morphisms are isomorphisms. It can thus be described by a diagram of sets:

\[
\begin{array}{ccc}
G_0 & \xrightarrow{s_0} & G_1 \\
\downarrow & & \downarrow \\
G_1 & \xrightarrow{G_1 \times_{G_0} G_1} & G_0 \\
\downarrow & & \downarrow \\
s_1 & \xrightarrow{m} & t \\
\end{array}
\]

where \( G_0 \) is the set of objects of \( G \) and \( G_1 \) the set of arrows. Here \( s \) and \( t \) are the source and target functions, \( i \) associates to an object its identity map, \( d_0 \) and \( d_2 \) are the respective projections, with sections \( s_0 \) and \( s_1 \), and \( m \) is the composition – all satisfying appropriate identities. Let \( \text{Gpd} \) denote the category of small groupoids (a full subcategory of the category \( \text{Cat} \) of small categories).

We can think of \( (2.14) \) as the 2-skeleton of a simplicial set (with \( G_2 := G_1 \times_{G_0} G_1 \), and \( d_1 = m : G_2 \to G_1 \)). The *nerve* functor \( N : \text{Gpd} \to [\Delta^{op}, \text{Set}] \) (cf. [48]) assigns to \( G \) the corresponding 2-coskeletal simplicial set \( NG \), so:

\[
(NG)_n := G_1 \times_{G_0} G_1 \cdots G_1 \times_{G_0} G_1
\]

for all \( n \geq 2 \), with face and degeneracy maps determined by the associativity and unit laws for the composition \( m \).

**2.16. Definition.** If \( V \) is any category with pullbacks, an *internal groupoid* in \( V \) is a diagram in \( V \) of the form \( (2.14) \), satisfying the same axioms (see [13], I, [8.1]). The category of internal groupoids in \( V \) is denoted by \( \text{Gpd}(V) \). Thus an (ordinary) groupoid is an internal groupoid in \( \text{Set} \).

When \( V \) is locally finitely presentable, the nerve functor \( N : \text{Gpd}(V) \to [\Delta^{op}, V] \) has a left adjoint, the fundamental internal groupoid (see [13], II, \$5.5-5.6\]).

For each \( n \geq 1 \), an *\( n \)-fold groupoid* is defined inductively to be an internal groupoid in the category \( V = \text{Gpd}^{n-1} \) of \((n-1)\)-fold groupoids (where \( \text{Gpd}^0 := \text{Set} \)), so

\[
\text{Gpd}^n := \text{Gpd}(\text{Gpd}^{n-1}).
\]

**2.17. Definition.** Let \( \hat{\pi}_1 : [\Delta^{op}, \text{Set}] \to \text{Gpd} \) denote the *fundamental groupoid* functor. See [33], [L.8] and [2.16] when \( V = \text{Set} \). When \( X \) is fibrant, \( \hat{\pi}_1 X \) has the simple form described in [33], [L.8]. If \( X \in [\Delta^{op}, \text{Set}] \) is an \( n \)-fold simplicial set, then for each \( 1 \leq i \leq n \), \( \hat{\pi}_1(i) X \) is the \((n-1)\)-fold simplicial object in \( \text{Gpd} \) obtained by applying the fundamental groupoid functor \( \hat{\pi}_1 \) in the \( i \)-th direction – that is, objectwise to the \( \Delta^{n-1,i} \)-indexed diagram \( X(i) \).

**2.18. Notation.** As in [2.14(d)], for each \( 1 \leq i \leq n \), let \( N^{(i)} : \text{Gpd}^n \to [\Delta^{op}, \text{Gpd}^{n-1}] \) denote the nerve functor in the \( i \)-th direction. More generally, for any \( k \) of the \( n \)
indices \(1 \leq i_1 < i_2 < \ldots < i_k \leq n\), \(N^{(i_1,i_2,\ldots,i_k)}: \Gpd^n \to [\Delta^{i_1\ldots i_k}, \Gpd^{n-k}]\) takes an \(n\)-fold groupoid \(G\) to a \(k\)-fold simplicial object in \((n-k)\)-fold groupoids by applying the nerve functor in the indicated \(k\) directions. In particular, \(N^{(i)}\) means that we take nerves in all but the \(i\)-th direction.

2.19. **Definition.** The *multinerve* 

\[
N_{(n)}: \Gpd^n \to [\Delta^n, \Set]
\]

is defined by applying \(N^{(i)}\) for \(1 \leq i \leq n\) to obtain the \(n\)-fold simplicial set \(N_{(n)}G := N^{(1)}N^{(2)} \ldots N^{(n)}G\). We say that an \(n\)-fold groupoid \(G\) is *discrete* if \(N_{(n)}G\) is a constant \(n\)-fold simplicial set. It is readily verified that we have an adjoint pair \(P_{(n)} \dashv N_{(n)}\):

\[
\begin{array}{ccc}
[\Delta^n, \Set] & \xrightarrow{N_{(n)}} & \Gpd^n \\
P_{(n)} & \xleftarrow{\sim} & \end{array}
\]

where \(P_{(n)}\) is the left adjoint to \(N_{(n)}\) as in (2.10) with \(\mathbb{V} = \Gpd^{n-1}\).

2.21. **Definition.** The composite of \(N_{(n)}\) with \(\text{Diag}_{(n)}\) (cf. §2.10) yields the *diagonal nerve* functor \(dN := \text{Diag}_{(n)} N_{(n)}\), and its geometric realization \(BG := \|dNG\| \in \Top\) is called the *classifying space* of \(G\).

A map of \(n\)-fold groupoids \(f: G \to G'\) is called a *geometric weak equivalence* if it induces a weak equivalence of simplicial sets \(dNf: dNG \to dNG'\) (that is, a homotopy equivalence of topological spaces on geometric realizations \(BF: BG \to BG'\)).

2.22. **Remark.** Since the diagonal of a bisimplicial set is its homotopy colimit, a map \(f: X \to Y\) in \([\Delta^{2^{\geq0}}, \Set]\) which is a weak equivalence \(f_k: X_k \to Y_k\) in each simplicial dimension \(k \geq 0\) is a geometric weak equivalence (cf. [33, IV, Proposition 1.7]). Thus by induction the same is true for a map \(f: X \to Y\) in \([\Delta^n, \Set]\) which is a geometric weak equivalence in each simplicial dimension.

2.23. **Definition.** If \(G \in \Gpd^{n-1}\) is an \((n-1)\)-fold groupoid, then \(c^{(n)}G\) denotes the \(n\)-fold groupoid which, as a groupoid object in \(\Gpd^{n-1}\), is discrete on \(G\). In particular, if \(A\) is a set, \(A^{d}_{(n)}\) denotes the discrete \(n\)-fold groupoid \(c^{(1)} \ldots c^{(n)}A\) on \(A\). For an \(n\)-fold groupoid \(G\) we let \(G^d\) denote the discrete \(n\)-fold groupoid \((\pi_0BG)^{d}_{(n)}\).

2.24. **Notation.** If \(G \in \Gpd^n\) is an \(n\)-fold groupoid for \(n \geq 2\), it is a groupoid object in \((n-1)\)-fold groupoids (cf. (2.10)): that is, it is described by a diagram \(G_1^{(1)} \to G_0^{(1)}\) in \(\Gpd^{n-1}\), as in (2.14). Thus it has an \((n-1)\)-fold groupoid of objects denoted by \(G_0^{(1)}\), in the notation of (2.6(a)) (which in turn has its \((n-2)\)-fold groupoid of objects \(G_0^{(1,2)}\) and \((n-2)\)-fold groupoid of morphisms \(G_0^{(1,2)}\)). Similarly, the \((n-1)\)-fold groupoid of morphisms of \(G\) (in the first direction) is denoted by \(G_1^{(1)}\).
More explicitly, $G$ may be described by a diagram in $\mathsf{Gpd}^{n-2}$ of the form:

\begin{equation}
\begin{aligned}
G_{11} \times_{G_{10}} G_{11} & \longrightarrow G_{01} \times_{G_{00}} G_{01} \\
\downarrow^{c_1} & \quad \downarrow^{c_2} \\
G_{11} & \longrightarrow G_{01} \\
\downarrow^{d_1} & \quad \downarrow^{d_2} \\
G_{10} \times_{G_{10}} G_{10} & \longrightarrow G_{00}.
\end{aligned}
\end{equation}

Here we omit throughout the upper index $(1,2)$, which indicates that we are showing only the first two directions of $G$.

More generally, for each $i \geq 2$ we let

\begin{equation}
G_{i1} := G_{11} \times_{G_{10}} \cdots \times_{G_{10}} G_{11} \quad \text{and} \quad G_{i0} := G_{10} \times_{G_{00}} \cdots \times_{G_{00}} G_{10}
\end{equation}

as limits of $(n-2)$-fold groupoids, with $d_{i0}^*, d_{i1}^* : G_{i1} \to G_{i0}$ induced by the source and target maps.

2.27. **Remark.** Using this convention, an $n$-fold groupoid $G$ may be thought of as a diagram of sets with objects $G_{i_1, \ldots, i_n}$ for each $(i_1, \ldots, i_n) \in \mathbb{N}^n = \text{Obj}(\Delta^n)$, where all the maps in the diagram are induced by those of (2.14) and the structure maps for the limits (2.26) (in each of the $n$ directions).

The following technical fact about $\text{Or}_{(n)}$ will be used in Subsection 4.1 below:

2.28. **Lemma.** For any fibrant simplicial set $X \in [\Delta^{op}, \text{Set}]$ and $n \geq 2$, we have:

\begin{equation}
\text{Or}_{(n-1)} N^\pi(2) \times^{(2)} \text{Or}_{(2)} X = N^\pi(2) \times^{(2)} \text{Or}_{(n)} X.
\end{equation}

**Proof.** By induction on $n \geq 2$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{corner_of_or3_x}
\caption{Corner of $\text{Or}_{(3)} X$}
\end{figure}
When $n = 2$, $\mathrm{Or}_{(n-1)}$ is the identity, so both sides of (2.29) are the same.

For $n = 3$, $\hat{\pi}_1^{(3)} \mathrm{Or}_{(3)} X$ is obtained from Figure 2 by replacing the left-hand square by

$$
\begin{array}{ccc}
X_5/\sim & \xrightarrow{d_3} & X_4/\sim \\
\downarrow d_0 & & \downarrow d_1 \\
X_4/\sim & \xrightarrow{d_2} & X_3/\sim
\end{array}
$$

and from Figure 1 we see this is the same as first applying $N\hat{\pi}_1$ to $\mathrm{Or}_{(2)} X$ horizontally in each vertical dimension (which is $N^{(2)}\hat{\pi}_1^{(2)}$), and then taking $\mathrm{Or}_{(2)}$ vertically in each horizontal dimension (which is $\mathrm{Or}_{(2)}^{(2)}$).

For $n \geq 4$, we see that

$$
N^{(n)}\hat{\pi}_1^{(n)} \mathrm{Or}_{(n)} X = N^{(n)}\hat{\pi}_1^{(n)} \mathrm{Or}_{(n-1)}^{(n-1)} \mathrm{Or}_{(2)} X = N^{(n-1)}\hat{\pi}_1^{(n-1)} \mathrm{Or}_{(n-1)}^{(n-1)} \mathrm{Or}_{(2)} X
$$

using (2.11) and the convention of §2.6(d). Applying the induction hypothesis (2.29) for $n - 1$, we see this is equal to:

$$
\frac{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}}{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}} \mathrm{Or}_{(2)} X = \frac{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}}{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}} \mathrm{Or}_{(2)} X
$$

and using (2.11) for $n = 3$, we see this is

$$
\frac{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}}{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}} \mathrm{Or}_{(2)} X = \frac{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}}{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}} \mathrm{Or}_{(2)} X
$$

where for a 3-fold simplicial object $Z$, we have

$$
\frac{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}}{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}} Z = \frac{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}}{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}} Z
$$

by our indexing convention (2.6(h)).

Now applying (2.29) for $n = 3$, we see this equals:

$$
\frac{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}}{\mathrm{Or}_{(n-2)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}} \mathrm{Or}_{(2)} X = \frac{\mathrm{Or}_{(n-1)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}}{\mathrm{Or}_{(n-1)}^{(2)} N^{(2)}\hat{\pi}_1^{(2)} \mathrm{Or}_{(2)}} \mathrm{Or}_{(2)} X
$$

using (2.11) once more.

\[\square\]

2.C. The fundamental $n$-fold groupoid of a space

We now introduce the central construction of our paper. Its internal analogue in the category of groups is the fundamental cat$^n$-group of a space, due to Bullejos, Cegarra, and Duskin (see [20, §2]).

2.30. Definition. We define $Q_{(n)} : [\Delta^\text{op}, \text{Set}] \to \text{Gpd}^n$ to be the composite

$$
[\Delta^\text{op}, \text{Set}] \xrightarrow{\text{Or}_{(n)}} [\Delta^{n\text{op}}, \text{Set}] \xrightarrow{P_{(n)}} \text{Gpd}^n,
$$

for $P_{(n)}$ the left adjoint to $N_{(n)}$ of (2.20). We define $\hat{Q}_{(n)} : \text{Top} \to \text{Gpd}^n$ to be the composite

$$
\text{Top} \xrightarrow{S} [\Delta^\text{op}, \text{Set}] \xrightarrow{Q_{(n)}} \text{Gpd}^n,
$$

where $S : \text{Top} \to [\Delta^\text{op}, \text{Set}]$ is the singular set functor (cf. [33] I, §1), and call $\hat{Q}_{(n)}T$ the fundamental $n$-fold groupoid of $T \in \text{Top}$. 
We shall show that if $Y \in [\Delta^{op}, \text{Set}]$ satisfies certain fibrancy conditions, then $P_{(n)}Y$ has a particularly simple form. These require that a certain 2-dimensional notion of fibrancy (introduced in [12, §2]) hold in every bisimplicial bidirection. They hold for $Y = Q_{(n)}X$ when $X$ is fibrant (in particular, for $X = ST$), leading to a simple expression for $Q_{(n)}T$ in Theorem 2.41 below.

2.31. **Definition.** Let $n \geq 2$. An $n$-fold simplicial set $X \in [\Delta^{op}, \text{Set}]$ is called $(n, 2)$-fibrant if for each $1 \leq i \neq j \leq n$ and $a \in \Delta^{n-2}$, the bisimplicial set $Y$ obtained by applying the 2-coskeleton functor to each vertical simplicial set $X^{(i,j)}(a)$ is a Kan complex for $k = 0, 1, 2$, and the horizontal face map $d_0 : Y_{1 \bullet} \to Y_{0 \bullet}$ is a fibration in $[\Delta^{op}, \text{Set}]$.

2.32. **Definition.** Let $G \in [\Delta^{op}, \text{Gpd}^{n-m}]$ be an $m$-fold simplicial object in $(n-m)$-fold groupoids (cf. [2.16]). We say that $G$ is $(n, 2)$-fibrant if, after applying the nerve functor in each of the $n-m$ groupoid directions, the resulting $n$-fold simplicial set $N(1, 2, \ldots, n-m)G \in [\Delta^{op}, \text{Set}]$ is $(n, 2)$-fibrant in the sense of Definition 2.31.

We recall the following results from [12] where the left adjoint $P^{(1)} : [\Delta^{op}, \text{Gpd}] \to \text{Gpd}^2$ to the nerve $N^{(1)} : \text{Gpd}^2 \to [\Delta^{op}, \text{Gpd}]$, is as in (2.16) with $V = \text{Gpd}$.

2.33. **Proposition** ([12] Prop. 2.10). The left adjoint $P^{(1)} : [\Delta^{op}, \text{Gpd}] \to \text{Gpd}^2$ to the nerve $N^{(1)} : \text{Gpd}^2 \to [\Delta^{op}, \text{Gpd}]$, when applied to a $(2, 2)$-fibrant simplicial groupoid $G_\bullet$, is given by $\tilde{\pi}_1G_\bullet$ (the functor $\tilde{\pi}_1$ applied in the simplicial direction).

2.34. **Proposition** ([12] Prop. 2.11). If $X \in [\Delta^{op}, \text{Set}]$ is a $(2, 2)$-fibrant bisimplicial set, then $\tilde{\pi}_1(X)$ is a $(2, 2)$-fibrant simplicial groupoid.

2.35. **Lemma.** If $G_\bullet$ is a $(2, 2)$-fibrant simplicial groupoid (with simplicial sets of objects $G_{0 \bullet}$ and morphisms $G_{1 \bullet}$), then $N(2)^{\tilde{\pi}_1}G_\bullet = \tilde{\pi}_1N(2)G_\bullet$.

*Proof.* It suffices to show that for each $k \geq 2$:

\[
\tilde{\pi}_1(G_{1 \bullet} \times G_{0 \bullet} \times \cdots \times G_{0 \bullet}G_{1 \bullet}) \cong \tilde{\pi}_1(G_{1 \bullet}) \times \cdots \times \tilde{\pi}_1(G_{0 \bullet})\tilde{\pi}_1(G_{1 \bullet})
\]

Since both sides are groupoids, we evidently have equality on objects, and (2.36) holds on morphisms by [12, App. A, following (8.13)].

2.37. **Lemma.** If $X \in [\Delta^{op}, \text{Set}]$ is $(n, 2)$-fibrant, then $\tilde{\pi}_1^{(k)}X$ is $(n, 2)$-fibrant.

*Proof.* By definition of $(n, 2)$-fibrancy, for each $a \in \Delta^{n-2}$ and $1 \leq i \neq j \leq n$, the bisimplicial set $X^{(i,j)}(a)$ satisfies the hypotheses of Proposition 2.33. Hence, applying $\tilde{\pi}_1^{(k)}$ to it yields an $(n, 2)$-fibrant object of $[\Delta^{n-2}, [\Delta^{op}, \text{Gpd}]]$. □

2.38. **Proposition.** For each $1 \leq i \leq n$, $P^{(i)} : [\Delta^{op}, \text{Gpd}^{n-1}] \to \text{Gpd}^n$, the left adjoint to $N^{(i)} : \text{Gpd}^n \to [\Delta^{op}, \text{Gpd}^{n-1}]$ of (2.20), when applied to an $(n, 2)$-fibrant simplicial $(n-1)$-fold groupoid $X$, is given by $P^{(i)}X = \tilde{\pi}_1^{(i)}X$.

*Proof.* We think of the simplicial direction of $X$ as being the $i$-th, and let $1 \leq j \leq n$ be one of the groupoidal directions (so $i \neq j$). Applying the $(n-2)$-fold iterated nerve functor

\[
N^{(\tilde{\pi}_1)} : [\Delta^{op}, \text{Gpd}^{n-1}] \to [\Delta^{op}, [\Delta^{n-2}, [\Delta^{op}, \text{Gpd}]]] \cong [\Delta^{n-2}, [\Delta^{op}, \text{Gpd}]]
\]
of \(\mathbb{2.18}\) (in all but the \(i\) and \(j\) directions) to \(X\) yields an \((n-2)\)-fold simplicial object in simplicial groupoids \(\tilde{X}\). Since \(X\) is \((n,2)\)-fibrant, for each \(a \in \Delta^{n-2}\), the simplicial groupoid \(\tilde{X}(a)\) (see \(\mathbb{2.36}\) b)) satisfies the hypotheses of Proposition \(\mathbb{2.33}\) where the simplicial direction is the original \(i\) and the groupoid direction is the original \(j\). Using \([12\, (8.12)]\), we can therefore define a composition map:

\[
(N(i)\tilde{\pi}_1(i)\tilde{X}(a))_1 \times (N(i)\tilde{\pi}_1(i)\tilde{X}(a))_1 \to (N(i)\tilde{\pi}_1(i)\tilde{X}(a))_1.
\]

As the construction is functorial in \(a \in \Delta^{n-2}\), it defines a map in \(\text{Gpd}^{n-1}\), since it consists of maps in sets commuting with compositions in each of the different directions (see \([12\, Appendix\ A]\)). Thus \(\tilde{\pi}_1(i)X\) is a groupoid object in \(\text{Gpd}^{n-1}\) – that is, \(\tilde{\pi}_1(i)X \in \text{Gpd}^n\).

It remains to show that \(\tilde{\pi}_1(i)X = P(i)X\). Since the (iterated) nerve functor is fully faithful, again using Proposition \(\mathbb{2.33}\) we see that for any \(n\)-fold groupoid \(Y\) we have natural isomorphisms

\[
\text{Hom}_{\text{Gpd}^n}(\tilde{\pi}_1(i)\tilde{X}, Y) \cong \text{Hom}_{\{\Delta^{n-2}\op, [\Delta^{op}, \text{Gpd}]\}}(\tilde{\pi}_1(i)\tilde{X}, Y) = \text{Hom}_{\{\Delta^{op}, \text{Gpd}^{n-1}\}}(\tilde{X}, N(i)Y) = \text{Hom}_{\{\Delta^{op}, \text{Gpd}^{n-1}\}}(X, N(i)Y).
\]

Hence \(\tilde{\pi}_1(i)\) is left adjoint to \(N(i)\), as required. \(\square\)

2.39. Proposition. If \(X \in [\Delta^{op}, \text{Set}]\) is a Kan complex, then \(\text{Or}_{(n)}X\) (cf. \(\mathbb{2.9}\)) is \((n,2)\)-fibrant.

See Appendix for the proof.

2.40. Theorem. The functor \(Q_{(n)}\) of \(\mathbb{2.36}\) applied to a Kan complex \(X \in [\Delta^{op}, \text{Set}]\), is:

\[Q_{(n)}X = \tilde{\pi}_1(1)\tilde{\pi}_1(2)\ldots\tilde{\pi}_1(n)\text{Or}_{(n)}X\,.
\]

Proof. We prove the Theorem by induction on \(n \geq 2\). For \(n = 2\), see \([12\, Corollary 2.12]\). Suppose the claim holds for \(n - 1\). The left adjoint \(P_{(n)}: [\Delta^{op}, \text{Set}] \to \text{Gpd}^n\) to \(N_{(n)}\) is the composite

\[
[\Delta^{n-2}, \text{Set}] \cong [\Delta^{op}, [\Delta^{n-1,op}, \text{Set}]] \xrightarrow{\mathcal{P}_{(n-1)}^{(1)}} [\Delta^{op}, \text{Gpd}^{n-1}] \xrightarrow{P^{(1)}} \text{Gpd}^n,
\]

where \(\mathcal{P}_{(n-1)}^{(1)}\) is induced by applying \(P_{(n-1)}\) in each dimension in the first simplicial direction, and \(P^{(1)}\) is left adjoint to the nerve \(N^{(1)}: \text{Gpd}^n \to [\Delta^{op}, \text{Gpd}^{n-1}]\). By the induction hypothesis and \(\mathbb{2.11}\),

\[
\mathcal{P}_{(n-1)}^{(1)}\text{Or}_{(n)}X = \mathcal{P}_{(n-1)}^{(1)}\text{Or}_{(n-1)}(\text{Or}_{(2)}X) = \tilde{\pi}_1(1)\tilde{\pi}_1(2)\ldots\tilde{\pi}_1(n)\text{Or}_{(n-1)}(\text{Or}_{(2)}X) = \tilde{\pi}_1(2)\ldots\tilde{\pi}_1(n)\text{Or}_{(n)}X.
\]

Since \(X\) is a Kan complex, \(\text{Or}_{(n)}X\) is \((n,2)\)-fibrant by Proposition \(\mathbb{2.39}\). Therefore, by Lemma \(\mathbb{2.37}\), \(\tilde{\pi}_1(2)\ldots\tilde{\pi}_1(n)\text{Or}_{(n)}X\) is \((n,2)\)-fibrant. It follows by Proposition \(\mathbb{2.38}\) that

\[P^{(1)}\tilde{\pi}_1(2)\ldots\tilde{\pi}_1(n)\text{Or}_{(n)}X = \tilde{\pi}_1(1)\tilde{\pi}_1(2)\ldots\tilde{\pi}_1(n)\text{Or}_{(n)}X.
\]
Therefore,
\[ Q_n X = P_n \text{Or}_n X = P^{(1)}T_{n-1} \text{Or}_n X = P^{(1)}\hat{\pi}_1^{(2)} \ldots \hat{\pi}_1^{(n)} \text{Or}_n X = \hat{\pi}_1^{(1)}\hat{\pi}_1^{(2)} \ldots \hat{\pi}_1^{(n)} \text{Or}_n X, \]
which concludes the induction step. □

2.41. Remark. The functor \( \text{Or}_n : [\Delta^{op}, \text{Set}] \to [\Delta^{n^{op}}, \text{Set}] \) has a right adjoint, a generalized Artin-Mazur codiagonal (cf. [1, §III] and [20, 22]), so both \( \text{Or}_n \) and \( P_n \) – and thus \( Q_n \) – preserve colimits, and in particular coproducts.

On the other hand, clearly \( \text{Or}_n \) and \( \hat{\pi}_1 \) preserve products when applied to Kan complexes, so \( Q_n \) does, too. Therefore, \( Q_n \) preserves fiber products over discrete simplicial sets.

3. Weakly globular \( n \)-fold groupoids

We now introduce the central notion of this paper: that of a weakly globular \( n \)-fold groupoid. We will show in the next Section that the fundamental \( n \)-fold groupoid \( \hat{\pi}_1 T \) of a space \( T \) (see §2.30) is such an object.

3.1. Definition. Let \( f : A \to B \) be a morphism in a category \( \mathcal{C} \) with finite limits. The diagonal map defines a unique section \( s : A \to A \times B A \) (so that \( p_1 s = \text{Id}_A = p_2 s \), where \( A \times B A \) is the pullback of \( A \xrightarrow{f} B \xleftarrow{f} A \) and \( p_1, p_2 : A \times B A \to A \) are the two projections). The commutative diagram

\[
\begin{array}{ccc}
A \times B A & \xrightarrow{p_1} & A \\
\downarrow{p_2} & \downarrow{f} & \downarrow{p_1} \\
A & \xrightarrow{f} & B \\
\end{array}
\]

defines a unique morphism \( m : (A \times B A) \times_A (A \times B A) \to A \times B A \) such that \( p_2 m = p_2 \pi_2 \) and \( p_1 m = p_1 \pi_1 \), where \( \pi_1 \) and \( \pi_2 \) are the two projections. We denote by \( A^f \) the following object of \( \text{Cat}(\mathcal{C}) \):

\[
(3.2) \quad (A \times B A) \times_A (A \times B A) \xrightarrow{m} A \times B A \xrightarrow{p_1} A
\]

It is easy to see that \( A^f \) is an internal groupoid.

3.3. Definition. We define a full subcategory \( \text{Gpd}_{\text{hd}}^n \subset \text{Gpd}^n \) of homotopically discrete \( n \)-fold groupoids by induction on \( n \geq 1 \):
A groupoid is called *homotopically discrete* if $G \cong A^I$ for some surjective map of sets $f: A \to B$. In general, an $n$-fold groupoid $G \in \text{Gpd}^n$ is *homotopically discrete* if $G \cong A^I$ for some map $f: A \to B$ in $\text{Gpd}^{n-1}$ with a section $f': B \to A$ (that is, $f \circ f' = \text{Id}_B$).

As noted above, for an (ordinary) groupoid $G$ this just means that $\pi_1(BG, x) = 0$ for any $x \in G_0$.

### 3.4. Remark
Note that the category $\text{Gpd}_{\text{hd}}^n$ is closed under pullbacks. We show this by induction on $n$. When $n = 1$, let $f: A \to B$, $f': A' \to B'$, and $g: C \to D$ be surjections in $\text{Set}$. Then

$$A^I \times_{C^I} A'^I = (A \times_C A')^{(f, f')}$$

where $(f, f'): A \times_C A \to A' \times_{C'} A'$ is a surjection in $\text{Set}$. Thus $A^I \times_{C^I} A'^I \in \text{Gpd}_{\text{hd}}^n$.

Suppose the statement holds for $n - 1$, and let $f': A' \to B'$ and $g: C \to D$ be maps with sections in $\text{Gpd}_{\text{hd}}^{n-1}$. Then $(f, f'): A \times_C A \to A' \times_{C'} A'$ is a map in $\text{Gpd}_{\text{hd}}^{n-1}$ with a section, by the inductive hypothesis, and *(3.5)* holds, showing that $A^I \times_{C^I} A'^I \in \text{Gpd}_{\text{hd}}^n$.

### 3.6. Example
Given a commuting (inner) square of sets:

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{h} \\
C & \xrightarrow{\ell} & D
\end{array}$$

with $ff' = \text{Id}_B$, $gg' = \text{Id}_C$, $hh' = \ell\ell' = \text{Id}_D$, and $fg' = h'\ell$, we obtain a morphism of homotopically discrete groupoids $v: A^I \to C^I$. The homotopically discrete double groupoid $G$ associated to $v$ is described in Figure 3 where we abbreviate $(A \times_B A) \times_{(C \times_D C)} (A \times_B A)$ by $(A \times_B A) \times_{(g, g)} (A \times_B A)$, and so on.

$$
\begin{array}{c}
(A \times_B A) \times_{(g, g)} (A \times_B A) \\
\cong (A \times_B A) \times_B A \\
\cong A \times_B A
\end{array}
$$

$$
\begin{array}{c}
(A \times_B A) \times_{(g, g)} (A \times_B A) \\
\cong (A \times_B A) \times_{(g, g)} (A \times_B A) \\
\cong A \times_B A
\end{array}
$$

Figure 3. A homotopically discrete double groupoid

Note that

$$(A \times_B A) \times_{(g, g)} (A \times_B A) \cong (A \times_C A) \times_{(f, f)} (A \times_C A)$$

via the map $(a, b, c, d) \mapsto (a, c, b, d)$, and more generally

$$(A \times_B A) \times_{(g, g)} \cdots (A \times_B A) \cong (A \times_C A) \times_{(f, f)} \cdots (A \times_C A)$$
for each $k \geq 2$. It follows that
\begin{equation}
(N^{(1)}G)_{k-1} = \begin{cases} 
A^f, & \text{if } k = 1; \\
(A \times C A)(f, \ldots, f) & \text{if } k \geq 2 
\end{cases}
\end{equation}
and
\begin{equation}
(N^{(2)}G)_{k-1} = \begin{cases} 
A^g, & \text{if } k = 1; \\
(A \times B A)(g, \ldots, g) & \text{if } k \geq 2 
\end{cases}
\end{equation}
Therefore $(N^{(1)}G)_k$ and $(N^{(2)}G)_k$ are homotopically discrete groupoids for all $k \geq 0$.
Moreover, applying $\pi_0$ vertically to each column in Figure \ref{fig:figure3} yields the groupoid $B^h$, that is:
\begin{equation}
(3.10) \quad B \times_D B \times_D B \to B \times_D B \to B
\end{equation}
Similarly, applying $\pi_0$ horizontally in each row yields $C^\ell$.

3.11. **Remark.** The construction of \ref{eq:fig3} makes sense in any category with enough limits. Conversely, any map $v: A^f \to C^\ell$ with a section $v'$ has a map of objects $g: A \to C$ and induces a map $h: B \to D$ on $\pi_0$, which fits into a commuting square as in \ref{eq:square}.

3.12. **Definition.** Recall from \ref{lem:26} that if $X \in [\Delta^n]^{op}$, $\Gamma_0^{(n)}$ is an $(n-1)$-fold simplicial object in groupoids, $\pi_0^{(n)} X$ is the $(n-1)$-fold simplicial set obtained by applying $\pi_0$ (the coequalizer of the source and target maps of the groupoid) in each $(n-1)$-fold simplicial dimension of $X$. If $cX$ denotes the discrete groupoid on a set $X$ (cf. \ref{lem:22}), $c: \text{Set} \to \text{Gpd}$ is right adjoint to $\pi_0$, and the unit of the adjunction $\gamma: \text{Id} \to c \pi_0$ induces a natural transformation of $(n-1)$-simplicial groupoids
\[ \gamma : X \to \pi_0^{(n)} \pi_0^{(n)} X. \]

3.13. **Remark.** Let $G \in \text{Gpd}^n$ be an $n$-fold groupoid and
\[ X = N^{(n-1)} \ldots N^{(1)} G \in [\Delta]^{op} \text{ Gpd}. \]
Let us suppose that $\pi_0^{(n)} X$ is the multinerve of an $(n-1)$-fold groupoid, denoted by $\Pi_0^{(n)} G$, so that
\[ \pi_0^{(n)} X = N^{(n-1)} \Pi_0^{(n)} G. \]
Then $\pi_0^{(n)} \pi_0^{(n)} X$ is the multinerve of an $n$-fold groupoid $c^{(n)} \Pi_0^{(n)} G$ (discrete in the new $n$-th direction) and
\[ \gamma = N^{(n-1)} \ldots N^{(1)} \gamma^{(n)} \]
for a map of $n$-fold groupoids $\gamma^{(n)} : G \to c^{(n)} \Pi_0^{(n)} G$.

3.14. **Remark.** Since $\pi_0 : \text{Gpd} \to \text{Set}$ preserves products and coproducts, it preserves fiber products over discrete groupoids. Therefore, the same is true of $\pi_0^{(n)}$.

3.15. **Lemma.** Let $G \in \text{Gpd}^n$ be a homotopically discrete $n$-fold groupoid. Then:
\begin{enumerate}
\item[(a)] If $N^{(i)} : \text{Gpd}^n \to [\Delta]^{op}, \text{Gpd}^{n-1}$ for some $1 \leq i \leq n$ is as in \ref{lem:27}, then $(N^{(i)}G)_k$ is homotopically discrete for all $k \geq 0$. 
\end{enumerate}
(b) The \((n - 1)\)-simplicial set \(\pi_0^{(n)} N^{(n-1)} \cdots N^{(1)}G\) is the multinerve of a homotopically discrete \((n - 1)\)-fold groupoid \(\Pi_0^{(n)} G\), and there is a commutative diagram

\[
\begin{array}{c}
\text{Gpd}_{\text{bd}}^{n-1} \xrightarrow{N^{(n-1)} \cdots N^{(1)}} [\Delta^{n-1}\text{op}, \text{Gpd}] \\
\Pi_0^{(n)} \downarrow \quad \quad \quad \downarrow \tau_0^{(n)} \\
\text{Gpd}_{\text{bd}}^{n-1} \xrightarrow{N^{(n-1)}} [\Delta^{n-1}\text{op}, \text{Set}] \\
\end{array}
\]

\(\gamma^{(n)}\) is the nerve of the \((n - 1)\)-fold homotopically discrete \((n - 1)\)-fold groupoid \(\Pi_0^{(n)} G\), as in \((3.7)\), by Remark \(3.11\).

(c) The map of \(n\)-fold groupoids \(\gamma^{(n)}: G \to c^{(n)} \Pi_0^{(n)} G\) of \((3.16)\) is a geometric weak equivalence \((2.21)\).

(d) The set \(\Pi_0^{(1)} \cdots \Pi_0^{(n)} G\) is isomorphic to \(\pi_0 BG\) (cf. \((2.21)\)).

(e) If we let \(\gamma^{(n)}\) denote the composite

\[
G \xrightarrow{\gamma^{(n)}} c^{(n)} \Pi_0^{(n)} G \xrightarrow{c^{(n)} \gamma^{(n-1)}} c^{(n-1)} c^{(n)} \Pi_0^{(n-1)} \Pi_0^{(n)} G \quad \cdots \quad c^{(1)} \cdots c^{(n)} \Pi_0^{(1)} \cdots \Pi_0^{(n)} G,
\]

it induces a geometric weak equivalence:

\[(3.16) \quad \hat{\mu}_k : G_1 \times_{G_0} \cdots \times_{G_0} G_1 \to G_1 \times_{G_0} \cdots \times_{G_0} G_1 \quad \text{for all} \quad k \geq 2
\]

(where \(G_0^d\) is as in \((2.2)\)).

Proof. By Definition \(3.13\) \(G\) (as an object of \(\text{Gpd}^2(\text{Gpd}^{n-2})\)) has the form of Figure 3 for some commuting square:

\[
\begin{array}{c}
A \\
\xrightarrow{f} B \\
\xleftarrow{g} C \\
\xrightarrow{h} D
\end{array}
\]

of \((n - 2)\)-fold groupoids, as in \((3.7)\), by Remark \(3.11\).

(a) By \((3.8)\) and \((3.9)\), the statement holds for \(n = 2\). Suppose by induction that it holds for \(n - 1\); then \((N^{(1)}G)_0 = A\) is in \(\text{Gpd}_{\text{bd}}^{n-1}\). Also \((N^{(k)}G)_{k-1} = (A \times_{C} \cdots \times_{C} A)_{(f, \ldots, f)}\) for \(k \geq 2\). By definition and the induction hypothesis,

\[
(f, \ldots, f): A \times_{C} \cdots \times_{C} A \to B \times_{D} \cdots \times_{D} B
\]

is a morphism with a section in \(\text{Gpd}_{\text{bd}}^{n-1}\). Hence, by definition, \((N^{(1)}G)_{k-1} \in \text{Gpd}_{\text{bd}}^{n-1}\). Similarly for any \(N^{(i)} G\).

(b) By \((3.17)\) and (a), \(N^{(1)}\tau_0^{(n)} G\) is the nerve of the \((n - 1)\)-fold homotopically discrete groupoid \(\Pi_0^{(n)} G := B^h\), and the map \(\tau^{(n)}\) lifts to a map of \(n\)-fold groupoids.

(c)-(e) By induction on \(n \geq 2\). For \(n = 2\), we saw that \(\Pi_0^{(2)} G = B^h\), and since each column in Figure 3 is homotopically discrete, we see from \((3.10)\) that the rightmost column is equivalent to \(B\), the next to \(B \times D\), and so on. Thus
\( N^{(1)}g^{(2)} : N^{(1)}G \to N^{(1)}c\Pi_0^{(2)}G \) induces dimensionwise weak equivalences of simplicial spaces, so a weak equivalence of classifying spaces. Since \( B^h \) is a homotopically discrete groupoid, it is weakly equivalent to \( cD \) (in the notation of (3.17)), which is \( (\pi_0BG)^d \).

By (3.8) for each \( k \geq 2 \):
\[
G_1 \times_{G_0} \cdots \times_{G_0} G_1 = (N^{(1)}G)_{k-1} = (A \times C \times \cdots \times C)_{(f, \ldots, f)} \cong B \times \times_{D} \times_{D} B,
\]
while since \( G_1 \) is homotopically discrete and \( G_0^d \) is discrete, \( G_1 \times_{G_0} \cdots \times_{G_0} G_1 \) is homotopically discrete (see Remark 5.3), so it is also weakly equivalent to \( B \times D \times D \). Thus (3.16) holds for \( n = 2 \).

In the induction step, \( N^{(1)}G \) is a simplicial \( (n-1) \)-fold homotopically discrete groupoid (by (3.8) again), and thus by the induction hypothesis for \( n - 1 \) we have a weak equivalence
\[
(N^{(1)}g^{(n-1)})_r : (N^{(1)}G)_r \to (e^{(2)} \ldots e^{(n)}N^{(1)}\Pi_0^{(2)} \ldots \Pi_0^{(n)}G)_r =: P_r
\]
in each simplicial dimension \( r \geq 0 \). Applying the \( (n-1) \)-fold nerve \( N_{n-1} \) to both sides, we obtain a map of \( n \)-fold simplicial sets \( N_nG \to \Pi_n \) which is a weak equivalence in each simplicial dimension, so induces a weak equivalence
\[
\text{Diag}_n N_nG \to \text{Diag}_n \Pi_n.
\]

However, \( \Pi_n \) is discrete in all but the first simplicial direction, where it is (the nerve of) a homotopically discrete groupoid \( H := \Pi_0^{(2)} \ldots \Pi_0^{(n)}G \). In fact, \( H = (B^d)^{h^d} \), in the notation of (3.1) where \( h^d : B^d \to D^d \) is the discretization of the map \( h : B \to D \) in (3.17).

Therefore, \( \text{Diag}_n \Pi_n = BH \) has \( \pi_0 BH = \pi_0 H^d = \pi_0 BG \) while \( \pi_i BH = 0 \) for \( i \geq 1 \), and the map \( \gamma(n) = \gamma^{(1)} \circ \gamma^{(n-1)} \) induces the requisite weak equivalence. Since also \( \gamma(n) = e^{(n)} \gamma^{(n-1)} \circ \gamma^{(n)} \), we deduce by induction that \( \gamma^{(n)} \) is a geometric weak equivalence, too.

To show (3.16), note that by (3.10) we have:
\[
(\Pi_0^{(n)}G)_2 = \Pi_0^{(n-1)}(G_1 \times_{G_0} G_1) = B \times_D B \times_D B = (B \times_D B) \times_B (B \times_D B),
\]
which by the induction hypothesis (3.16) and Remark 3.14 equals:
\[
\Pi_0^{(n-1)}G_1 \times_{\Pi_0^{(n-1)}G_0} \Pi_0^{(n-1)}G_1 \cong \Pi_0^{(n-1)}G_1 \times_{(\Pi_0^{(n-1)}G_0)^d} \Pi_0^{(n-1)}G_1 = \Pi_0^{(n-1)}(G_1 \times_{G_0} G_1).
\]

That is, we have a commuting square
\[
\begin{array}{ccc}
G_1 \times_{G_0} G_1 & \xrightarrow{\gamma^{(n-1)}} & \Pi_0^{(n-1)}(G_1 \times_{G_0} G_1) \\
\mu_2 \downarrow & & \downarrow \simeq \\
G_1 \times_{G_0^d} G_1 & \xrightarrow{\gamma^{(n-1)}} & \Pi_0^{(n-1)}(G_1 \times_{G_0^d} G_1)
\end{array}
\]
in which three of the maps are geometric weak equivalences, so \( \mu_2 \) is, too.

Similarly for all \( k > 2 \).

From (d) of the Lemma we see:
3.18. Corollary. If $G$ is a homotopically discrete $n$-fold groupoid, the map $\gamma_{(n)}: G \to c^{(1)} \cdot c^{(n)} \Pi_0^{(1)} \cdots \Pi_0^{(n)} G$ is a geometric weak equivalence, so $BG$ is homotopically trivial (i.e., $\pi_iBG = 0$ for all $i \geq 1$).

3B. Weakly globular $n$-fold groupoids

We are now in a position to define the main notion of this section. At first glance, it does not appear to be fully algebraic, since it uses the concept of a geometric weak equivalence. However, as we shall show in Corollary 13, below, this concept has an equivalent purely algebraic description.

3.19. Definition. For each $n \geq 1$, the full subcategory $\text{Gpd}_n^{\text{wg}}$ of $\text{Gpd}^n$, whose objects are called weakly globular $n$-fold groupoids, is defined by induction on $n$, as follows:

For $n = 1$, any groupoid is weakly globular; suppose we have defined $\text{Gpd}_n^{\text{wg}}$. We say that an $n$-fold groupoid $G = (G_1^{(1)} \xrightarrow{\gamma_1} G_0^{(1)})$ is weakly globular if:

(i) $G_0 := G_0^{(1)}$ is in $\text{Gpd}^n_{\text{wg}}$;

(ii) $G_1 := G_1^{(1)}$ is in $\text{Gpd}^n_{\text{wg}}$, and for each $k \geq 2$, $G_1 \times_{G_0^{\cdot k}} \cdots \times_{G_0^{\cdot 1}} G_1$ is in $\text{Gpd}^n_{\text{wg}}$;

(iii) The $(n-1)$-simplicial set $\pi_0^{(n)} N^{(n-1)} \cdots N^{(1)} G$ is the nerve of a weakly globular $(n-1)$-fold groupoid $\Pi_0^{(n)} G$ such that $N^{(n-1)} \Pi_0^{(n)} G = \pi_0^{(n)} N^{(n-1)} \cdots N^{(1)} G$.

(iv) The map of $(n-1)$-fold groupoids

$$G_1 \times_{G_0^{\cdot k}} \cdots \times_{G_0^{\cdot 1}} G_1 \xrightarrow{\mu_k} G_1 \times_{G_0^{\cdot k}} \cdots \times_{G_0^{\cdot 1}} G_1$$

induced by $\gamma_{(n)}: G_0 \to G_0^{\text{ed}}$ is a geometric weak equivalence for all $k \geq 2$.

Note the special role played by the first of the $n$-directions in this definition. Also, note that we have a functor $\Pi_0^{(n)}$ making the following diagram commute:

$$\begin{array}{ccc}
\text{Gpd}_n^{\text{wg}} & \xrightarrow{N^{(n-1)} \cdots N^{(1)}} & [\Delta^{n-1}, \text{Gpd}] \\
\Pi_0^{(n)} \downarrow & & \downarrow \pi_0^{(n)} \\
\text{Gpd}_n^{\text{wg}} & \xrightarrow{N^{(n-1)}} & [\Delta^{n-1}, \text{Set}].
\end{array}$$

3.20. Remark. For $n = 2$, the above definition is slightly more general than [12, Definition 2.21]. In fact, in [12] $G$ is required to be symmetric, and both maps $G_1 \xrightarrow{} G_0$ are required to be fibrations of groupoids; the latter implies conditions (iii) and (iv).

Note also that if $G \in \text{Gpd}_n^{\text{wg}}$, not only is $G_1 \times_{G_0^{\cdot k}} \cdots \times_{G_0^{\cdot 1}} G_1 \in \text{Gpd}_n^{\text{wg}}$ (by Definition 13), but also $G_1 \times_{G_0^{\cdot k}} \cdots \times_{G_0^{\cdot 1}} G_1 \in \text{Gpd}_n^{\text{wg}}$. We show this for $k = 2$, the general case being similar. In fact we observe more generally that the pullback $P$ of $G \to H \leftarrow G'$ with $G, G'$ in $\text{Gpd}_n^{\text{wg}}$ and $H$ discrete is an object of $\text{Gpd}_n^{\text{wg}}$. 
We proceed by induction on $n$. For $n = 1$ the statement is clear, since $Gpd^1_{\text{wg}} = Gpd$. Suppose it is true for $n - 1$. We have $P_0 = G_0 \times H_0 G'_0 \in Gpd^{n-1}_{\text{hd}}$. Since $G_0, G'_0 \in Gpd^{n-1}_{\text{hd}}$, and $H_0$ is discrete (using Remark 3.4). Furthermore, $P_1 = G_1 \times H_1 G'_1 \in Gpd^{n-1}_{\text{hd}}$ by the induction hypothesis.

Likewise, since $H$ is discrete,

$$P_1 \times_{P_0} P_1 \cong (G_1 \times_{G_0} G_1) \times (H_1 \times_{H_0} H_1)(G'_1 \times_{G'_0} G'_1) = (G_1 \times_{G_0} G_1) \times_{H_0} (G'_1 \times_{G'_0} G'_1).$$

Thus $P_1 \times_{P_0} P_1 \in Gpd^{n-1}_{\text{wg}}$ by the induction hypothesis. For the same reason, $P_1 \times_{P_0} \cdots \times_{P_0} P_1$ is in $Gpd^{n-1}_{\text{wg}}$. Since $\pi_0$ commutes with fiber products over discrete objects, we have $\pi_0^n P = \Pi_{0}^{n} G \times H \Pi_{0}^{n} G$, and this is in $Gpd^{n-1}_{\text{wg}}$ by the induction hypothesis.

Finally,

$$P_1 \times_{P_0} P_1 \to P_1 \times_{P_0} P_1.$$ 

Similarly, one shows that for each $k \geq 2$, there is a geometric weak equivalence

$$P_1 \times_{P_0} \cdots \times_{P_0} P_1 \to P_1 \times_{P_0} \cdots \times_{P_0} P_1.$$ 

This completes the proof that $P \in Gpd_{\text{wg}}^n$.

3.23. Definition. For any $n$-fold groupoid $G$ and $1 \leq k \leq n$, we define its $k$-fold object of arrows to be the $(n-k)$-fold groupoid:

$$W_{(n,k)}G := G_{1,\ldots,k}^{(1)}(n),$$

using the indexing conventions of (2.6(b)).

3.24. Remark. Note that by Definition 3.19(ii), if $G$ is weakly globular, so is $W_{(n,1)}G$, so by induction we have a functor $W_{(n,k)} : Gpd_{\text{wg}}^n \to Gpd_{\text{wg}}^{n-k}$, since

$$W_{(n,k)} = W_{(n-k+1,1)}W_{(n-k+2,1)} \cdots W_{(n-1,1)}W_{(n,1)}.$$

3.26. Algebraic homotopy groups and algebraic weak equivalences. For any weakly globular $n$-fold groupoid $G$, we define the $k$-th algebraic homotopy group of $G$ at $x_0 \in G_{0,-0}$ to be:

$$\omega_k(G;x_0) \cong \begin{cases} W_{(n,n)}G(x_0,x_0) & \text{if } k = n \\ W_{(n-k,n-k)}(\Pi_{0}^{(k+1)} \cdots \Pi_{0}^{(n)})G(x_0,x_0) & \text{if } 0 < k < n \end{cases}$$

with the 0-th algebraic homotopy set of $G$ defined:

$$\omega_0(G) := \Pi_{0}^{(1)} \cdots \Pi_{0}^{(n)}G.$$ 

Here $W_{(n,n)}G(a,b)$ (cf. (3.23)) is the set of morphisms from $a$ to $b$ in the groupoid $W_{(n,n-1)}G$ (in the $n$-th direction), so in particular $W_{(n,n)}G(a,a)$ is the group of automorphisms of $a$ (which is abelian for $n \geq 2$).
A map \( f: G \to G' \) of weakly globular \( n \)-fold groupoids is called an algebraic weak equivalence if it induces bijections on the \( k \)-th algebraic homotopy groups (set) for all \( x_0 \in G_{0_0} \) and \( 0 \leq k \leq n \).

3.28. Definition. For each \( n \geq 0 \), let \( P^n \text{Top} \) denote the full subcategory of \( \text{Top} \) consisting of spaces \( X \) for which the natural map \( X \to P^n X \) is a weak equivalence (that is, \( \pi_r(X,x) = 0 \) for all \( x \in X \) and \( i > n \)). An \( n \)-type is an object in \( P^n \text{Top} \) (or in the corresponding full subcategory \( \text{ho}(P^n \text{Top}) \) of \( \text{ho} \text{Top} \)).

We use similar notation for \( n \)-Postnikov simplicial sets (where for a Kan complex \( X \) (cf. [33 I.3]), we can use \( \text{csk}_{n+1} X \) as a model for the \( n \)-th Postnikov section \( P^n X \)).

For any \( n \geq 0 \), a map \( f: X \to Y \) in \( [\Delta^{\text{op}}, \text{Set}] \) (or in \( \text{Top} \)) is called an \( n \)-equivalence if it induces isomorphisms \( f_*: \pi_0 X \to \pi_0 Y \) (of sets), and \( f_*: \pi_i(X,x) \to \pi_i(Y,f(x)) \) for every \( 1 \leq i \leq n \) and \( x \in X_0 \).

We recall the following notion and fact from [12]:

3.29. Definition. A map \( f: W \to V \) of bisimplicial sets is called a diagonal \( n \)-equivalence if \( f^k_0: W_k^k \to V_k^k \) is an \( (n-k) \)-equivalence for each \( k \leq n \).

3.30. Proposition ([12 Prop. 3.9]). If \( f: W \to V \) is a diagonal \( n \)-equivalence, then the induced map \( \text{Diag} f: \text{Diag} W \to \text{Diag} V \) is an \( n \)-equivalence.

3.31. Lemma. For any \( G \in \text{Gpd}^n_{\text{wg}} \), the map \( \overline{\gamma} \) of Definition 3.12 corresponds to a map of \( n \)-fold groupoids \( \gamma^{(n)}: G \to c^{(n)} \Pi_0 G \) with \( \overline{\gamma} = N^{(n-1)} \cdots N^{(1)} \gamma^{(n)} \), which induces an \( (n-1) \)-equivalence \( B\gamma^{(n)}: BG \to B c^{(n)} \Pi_0 G \) on classifying spaces.

Proof. By Definition 3.19 and Remark 3.13 the map \( \overline{\gamma} \) corresponds to a map of \( n \)-fold groupoids as stated. We show that this is an \( (n-1) \)-equivalence by induction on \( n \). It is clear for \( n = 1 \). Suppose, inductively, it holds for \( n - 1 \).

By construction we have

\[
(\Pi_0^{(n)} G)^r := (N^{(n)} \Pi_0^{(n)} G)^r = \Pi_0^{(n-1)} (N^{(n)} G)^r ,
\]

and therefore, for each \( r \geq 0 \) there is a map

\[
(N^{(n)} \gamma^{(n-1)})_r: (\Pi_0^{(n)} G)_r \to (c^{(n)} \Pi_0^{(n-1)} G)_r .
\]

By taking realizations, we obtain a map of simplicial spaces \( B\gamma^{(n-1)} \). We claim that the corresponding map of bisimplicial sets is a diagonal \( (n-1) \)-equivalence (cf. 3.29). In fact, since \( G_0 = (N^{(n)} G)_0^{(n)} \) is homotopically discrete, by Lemma 3.15 \( (B\gamma^{(n-1)})_0 \) is a weak equivalence, hence in particular an \( (n-1) \)-equivalence. By the induction hypothesis \( (B\gamma^{(n-1)})_r \) is a \( (n-2) \)-equivalence for all \( r \geq 1 \). Hence \( B\gamma^{(n-1)} \) is an \( (n-1) \)-equivalence by Proposition 3.30.

3.32. Remark. From Lemmas 3.15 and 3.31 we see that a homotopically discrete \( n \)-fold groupoid is weakly globular.

4. \( n \)-Types

In this section we prove one of the main result of this paper, Theorem 4.32 which asserts that all \( n \)-types are modelled by weakly globular \( n \)-fold groupoids.
**4.1. The homotopy type of a weakly globular n-fold groupoid**

We start by showing that if $G \in \mathbf{Gpd}^n_{\text{wg}}$, then its classifying space $BG$ (cf. [2.21]) is an $n$-type; that is, $\pi_i(BG, x) = 0$ for all $x \in BG$ and $i > n$. We prove this using a spectral sequence computation of $\pi_i(BG, x)$. In Section 2 we give an alternative proof using a comparison with Tamsamani’s weak $n$-groupoids.

In [47], Quillen constructed a spectral sequence for a bisimplicial group, which was generalized in [14, Appendix B] to define the Bousfield-Friedlander spectral sequence of a bisimplicial set $X_{\bullet \bullet} \in [\Delta^{op}, \mathbf{Set}]$, with

$$E^2_{s,t} = \pi^h_s \pi^v_t X_{\bullet \bullet} \Rightarrow \pi_{s+t} \text{Diag} X_{\bullet \bullet}.$$  

(4.1)

See [28, §8.4] for an alternative construction when $X_{\bullet \bullet}$ is connected in each simplicial dimension. The spectral sequence need not converge otherwise; however, we have the following sufficient condition for convergence (cf. [14, B.3]):

**4.2. Definition.** Think of a bisimplicial set $X_{\bullet \bullet} \in [\Delta^{op}, \mathbf{Set}]$ as a (horizontal) simplicial object in $[\Delta^{op}, \mathbf{Set}]$ (with the simplicial direction inside $[\Delta^{op}, \mathbf{Set}]$ thought of as being vertical). In this notation, a $k$-$\pi_i$-matching collection at $a \in X_{n,0}$ (for $0 \leq k \leq n$) is a set of elements $x_i \in \pi_i(X_{n-i}, d^h_i a)$ $(0 \leq i \leq n, i \neq k)$, such that:

$$ (d^h_i)_* x_j = (d^h_{i-1})_* x_i $$

for every $0 \leq i < j \leq n$ $(i, j \neq k)$.

We say that $X_{\bullet \bullet}$ satisfies the $\pi_\ast$-Kan condition if for every $n, t \geq 1$, $0 \leq k \leq n$, $a \in X_{n,0}$, and $k$-$\pi_t$-matching collection $(x_i)_{i \neq k}^n$ at $a$, there is a fill-in $w \in \pi_t^v(X_{\bullet \bullet}, a)$ such that $(d^h_i)_* w = x_i$ for all $0 \leq i \leq n$ $(i \neq k)$.

By [14, Theorem B.5], if $X_{\bullet \bullet}$ satisfies the $\pi_\ast$-Kan condition – for example, if each $X_{n, \bullet}$ is connected – then the spectral sequence (4.1) converges.

**4.4. Notation.** For any simplicial set $X$ and $t \geq 1$, the $t$-th homotopy group $\pi_t(Y, y)$, as $y \in Y$ varies, constitutes a semi-discrete groupoid, in the sense of [12, §1] – that is, a disjoint union of groups (abelian, if $t \geq 2$). We denote it by $\hat{\pi}_t Y$.

**4.5. Lemma.** Let $G_* \in \mathbf{Gpd}([\Delta^{op}, \mathbf{Set}])$ be a groupoid in $[\Delta^{op}, \mathbf{Set}]$, such that

$$ G_1 \times_{G_0} \cdots \times_{G_0} G_1 \rightarrow G_1 \times_{c\pi_0 G_0} \cdots \times_{c\pi_0 G_0} G_1 $$

is a weak equivalence of simplicial sets for all $k \geq 2$, with $G_0$ a homotopically trivial simplicial set. Then the bisimplicial set $X_{\bullet \bullet} := NG_*$ satisfies the $\pi_\ast$-Kan condition, and for each $t \geq 1$, $\hat{\pi}_t X_{\bullet \bullet}$ is a groupoid object in semi-discrete groupoids, so is 2-coskeletal.

**Proof.** We think of the simplicial direction as vertical. Let $X_k = (NG_*)_k$. Since $X_0 = G_0$ is homotopically trivial (that is, a disjoint union of contractible spaces), the groupoid $\hat{\pi}_0 X_0$ is discrete on $\pi_0 G_0$, so any $k$-$\pi_t$-matching collection for $n = 1$ is trivial.

For $n = 2$, note that $X_2 = X_1 \times_{X_0} X_1$, so any $a \in X_{2,0}$ is of the form $a = (a', a'')$, where $d_2 a' = d_0 a'' = b$ Moreover, $d_0 a = a'$, $d_1 a = a' \ast a''$ (where $\ast$ denotes the groupoid composition), and $d_2 a = a''$.

Thus for $t \geq 1$, there are three cases for a $k$-$\pi_t$-matching collection $(x_i \in \pi_t^v(X_1, d_0 a))_{i \neq k}$ at $a$:
(i) When \( k = 1 \), the fill-in \( w \in \pi^v_i(X_2, a) \) for \( x_0 \) and \( x_2 \) is the pull-back pair \((x_0, x_2)\) in
\[
\pi^v_i(X_2, a) = \pi^v_i(X_1, a') \times \pi^v_i(X_0, b) \pi^v_i(X_1, a'').
\]

(ii) When \( k = 0 \), the fill-in \( w = (y, x_2) \) for \( x_1 \) and \( x_2 \) should satisfy \( x_1 = d_1 w = y * x_2 \), so \( y = x_1 * (x_2)^{-1} \), using the groupoid structure on \( \pi^v_i X_1 \).

(iii) The case \( k = 2 \) is similar.

For \( n > 2 \) the proof of the \( \pi \)-Kan condition is analogous; however, because \( \pi^v_i X_{\bullet \bullet} \) is 2-coskeletal, we do not even need to verify it, since the spectral sequence \((4.1)\) from the \( E^2 \)-term on then depends only on the 2-truncation of \( X_{\bullet \bullet} \) in the horizontal direction.

In order to study the homotopy groups of the \( n \)-fold diagonal \( dNG \) of an \( n \)-fold groupoid, we think of it as an iterative construction in which we take diagonals in successive bisimplicial bidirections. The weak globularity allows us to iteratively apply Lemma \((4.5)\) and thus the Bousfield-Friedlander spectral sequence.

**Theorem.** For any weakly globular \( n \)-fold groupoid \( G \in Gpd^n_{aw} \), \( BG \) is an \( n \)-type, and for each base point \( x_0 \in G_0^{n,0} \) we have natural isomorphisms
\[
(4.7) \quad \pi_k(BG; x_0) \cong \omega_k(G; x_0) \text{ for } 0 < k \leq n \text{ and } \pi_0 BG \cong \omega_0(G)
\]
(see \((\ref{3.27})\)).

**Proof.** Since \( BG \) is the geometric realization of \( dNG \), we prove the Theorem simplicially, for \( dNG \), by induction on \( n \).

Using the convention of \((\ref{2.27})\) for each \( a \in \Delta^{n-2} \) we have a double groupoid \( G^{(1,2)}(a) \in Gpd^2 \) (in the notation of \((\ref{2.0})\)). Assuming that the first of the \( n \) directions of \( G \) is not among those of \( \Delta^{n-2} \), \( N^{(1)} G^{(1,2)}(a) \in [\Delta^{op}, Gpd] \) satisfies the hypotheses of Lemma \((\ref{4.5})\) by Definition \((\ref{3.19})\). Therefore, the Bousfield-Friedlander spectral sequence for the bisimplicial set
\[
X(a) := N^{(1,2)} G^{(1,2)}(a)
\]
converges to \( \pi_* \operatorname{Diag} X(a) \). Moreover, \( \pi^v_i X(a) \) is 2-coskeletal for each \( t \geq 1 \), by the Lemma, as is \( \pi^v_0 X(a) \) (by Definition \((\ref{3.19})\) again). Thus in the \( E^2 \)-term of the spectral sequence only the two right columns of two bottom rows can be non-zero, so that \( \operatorname{Diag} X(a) \) is a 2-type. In fact, the rightmost column is zero (except at the bottom), so we can read off the homotopy groups of \( \operatorname{Diag} X(a) \) from those of \( X(a) \).

Since \( \operatorname{Diag} \) is functorial in \( a \in \Delta^{n-2} \), we see that the resulting object \( Y := \operatorname{Diag} N^{(1,2)} G^{(1,2)} \) is in \([\Delta^{op}, Gpd^{n-2}]\), with each \( Y(a) \in [\Delta^{op}, \text{Set}] \) a simplicial 2-type. Since \( G_0 \) was a homotopically discrete \((n-1)\)-fold groupoid, the object \( Y^v_0 \) (in dimension 0 in the first (simplicial) direction) is a homotopically discrete \((n-2)\)-fold groupoid. Moreover, for any choice of a third (groupoid) direction \( i \), and each \( b \in \Delta^{n-3} \), by Definition \((\ref{3.19})\) we have a bisimplicial groupoid
\[
Z_{\bullet \bullet} := N^{(1,2)} G^{(1,2,i)}(b)
\]
(where the third index is the groupoid direction). This has a weak equivalence of bisimplicial sets
\[
Z_{\bullet \bullet k} = Z_{\bullet \bullet 1} \times Z_{\bullet \bullet 0} \cdots \times Z_{\bullet \bullet 0} \xrightarrow{\sim} Z_{\bullet \bullet 1} \times_{G_0^i} Z_{\bullet \bullet 0} \cdots \times_{G_0^i} Z_{\bullet \bullet 1}
\]
for each \( k \geq 2 \), natural in \( b \) (note that \( G^d \) is independent of \( b \)). This map therefore induces a weak equivalence in the bisimplicial direction (cf. \( \S 2.22 \)). Thus each simplicial groupoid \( Y(b) = \text{Diag} Z_{\bullet \bullet} \) satisfies the hypotheses of Lemma 4.5.

Now assume by descending induction on \( 2 \leq k < n \) that we have \( Y \in [\Delta^{op}, \text{Gpd}^{n-k}] \), with \( Y(a) \in [\Delta^{op}, \text{Set}] \) a \( k \)-type for each \( a \in \Delta^{n-k} \), with \( Y_0^w \) a homotopically discrete \( (n-k) \)-fold groupoid. Here the first (vertical) direction is simplicial.

For any choice of a second (groupoid) direction, and each \( b \in \Delta^{n-k-1} \), the simplicial groupoid \( Y^{(1, 2)}(b) \in [\Delta^{op}, \text{Gpd}] \) satisfies the hypotheses of Lemma 4.5.

Therefore, \( \Pi \) converges, with only the two right columns of the bottom \( k \) rows non-zero, and \( \text{Diag} Y(a) \) is thus a \((k+1)\)-type. When \( k = n-1 \), \( Y \) is a simplicial groupoid which is an \((n-1)\)-type in the simplicial direction, with \( BG \) appearing as the realization of \( \text{Diag} Y \).

For any weakly globular double groupoid \( G \), the \( E^2 \)-term of the Bousfield-Friedlander spectral sequence for the bisimplicial set \( X_{\bullet \bullet} = N^h N^w G \) survives to \( E^\infty \). Moreover, because \( G_0 \) is homotopically trivial, \( E^2_{1,0} = \pi_1 \pi_0 X_{\bullet \bullet} = 0 \), so in fact by Lemma 4.5

\[
\pi_i(\text{Diag} X_{\bullet \bullet}, x_0) = \begin{cases} 
E^2_{0,0} = \pi_0 \pi_0(X_{\bullet \bullet}, x_0) & \text{if } i = 0 \\
E^2_{0,1} = \pi_0 \pi_1(X_{\bullet \bullet}, x_0) & \text{if } i = 1 \\
E^2_{1,1} = \pi_1 \pi_1(X_{\bullet \bullet}, x_0) & \text{if } i = 2 ,
\end{cases}
\]

for each choice of a base-point \( x_0 \) in \( G_{00} \). Actually, \( \pi_1 \pi_1(X_{\bullet \bullet}, x_0) \) is just the automorphism group of \( G_1 \), i.e., \( W_{(2, 2)} G(x_0, x_0) \).

Therefore, given a weakly globular \( n \)-fold groupoid \( G \), by what we have shown above we see that

\[
\pi_n(BG; x_0) \cong \omega_n(G; x_0)
\]

for each \( x_0 \in G_{0 \ldots 0} \). Moreover, by Lemma 3.31 we have

\[
\pi_i(BG, x_0) \cong \pi_i(B\Pi_0^{(n-k+1)} \ldots \Pi_0^{(n)} G, x_0)
\]

for all \( 0 \leq i \leq n-k \), and \( \Pi_0^{(n-k+1)} \ldots \Pi_0^{(n)} G \) is an \((n-k)\)-weakly globular \((n-k)\)-fold groupoid, so in particular \( \Pi_0 \) holds for each \( 0 \leq k \leq n \).

Observe that Theorem 4.6 provides an intrinsic algebraic definition of the notion of geometric weak equivalences among weakly globular \( n \)-fold groupoids, since we have:

4.8. **Corollary.** (a) A map of weakly globular \( n \)-fold groupoids is a geometric weak equivalence (\( \S 2.22 \)) if and only if it is an algebraic weak equivalence (\( \S 3.22 \)).

(b) The notion of a weakly globular \( n \)-fold groupoid \( G \) is purely algebraic.

4.9. **Remark.** It follows from above that the functor \( \Pi_0^{(n)} : \text{Gpd}^{n}_{\text{wg}} \to \text{Gpd}^{n-1}_{\text{wg}} \) preserves geometric weak equivalences and serves as an algebraic \((n-1)\)-Postnikov section functor.

4.10. **An iterative description of** \( Q_{(n)} \)

We now use the notions of the previous section to provide a more transparent iterative description of the fundamental \( n \)-fold groupoid functor \( Q_{(n)} X \) (Definition 2.30) for a Kan complex \( X \).
4.10. **Definition.** For any simplicial set $X$, let

$$L_kX := \begin{cases} 
\text{Dec } X & \text{if } k = 0 \\
\text{Dec } X \times_X \cdots \times_X \text{Dec } X & \text{if } k \geq 1.
\end{cases}$$

4.12. **Remark.** If $X$ is a Kan complex, we have a natural fibration of simplicial sets $u: \text{Dec } X \to X$ (cf. §2.5), yielding the internal groupoid $(\text{Dec } X)^u \in \text{Gpd}[\Delta^{op}, \text{Set}]$ of §3.1. We see that

$$N(\text{Dec } X)^u_k = L_kX = L_1X \times_{\text{Dec } X} \cdots \times_{\text{Dec } X} L_1X$$

for all $k \geq 1$, so we may denote the bisimplicial set $N(\text{Dec } X)^u$ by $L_*X$. This is depicted in Figure 4, where the vertical maps are induced by those indicated in the rightmost column, and the horizontal maps are structure maps for the pullbacks, as in (3.2).

[Diagram of bisimplicial set]

If $X$ is reduced, $\text{Dec } X$ is contractible, so $L_1X$ models the loop space $\Omega X$. In general, $L_1X$ is homotopy equivalent to the “path object” $PX$ of §2.2.

4.14. **Lemma.** Let $X$ be a Kan complex, and $cX$ the corresponding bisimplicial set, constant in the horizontal direction.

(a) There is a natural map of bisimplicial sets $\phi: cX \to L_*X$, which is a dimensionwise weak equivalence (as horizontal simplicial sets, in each vertical dimension – see Figure 4), so induces a weak equivalence $\text{Diag } \phi: X \to \text{Diag } L_*X$.

(b) We have $N^{(n)}Q^{(n)}(cX) = Q^{(n)}(n-1)L_*X$ – i.e., for each $k \geq 0$:

$$N^{(n)}Q^{(n)}(cX)_k = Q^{(n)}(n-1)L_kX.$$  

Thus for each $k \geq 1$:

$$Q^{(n)}(n-1)L_kX \simeq Q^{(n)}(n-1)L_1X \times Q^{(n)}(n-1)\text{Dec } X \times_{Q^{(n)}(n-1)\text{Dec } X} \cdots \times_{Q^{(n)}(n-1)\text{Dec } X} Q^{(n)}(n-1)L_1X.$$  

(c) If $X$ is homotopically trivial, then for $k \geq 1$:

$$Q^{(n)}L_kX \simeq Q^{(n)}\text{Dec } X \times Q^{(n)}\text{Dec } X \times_{Q^{(n)}\text{Dec } X} \cdots \times_{Q^{(n)}\text{Dec } X} Q^{(n)}\text{Dec } X.$$
Proof. (a) The section $\sigma: X \to \text{Dec } X$ to the augmentation $\varepsilon = \overline{a}_*: \text{Dec } X \to X$, given in dimension $i$ by the degeneracy $s_i: X_i \to X_{i+1}$ (cf. [2.7]), fits into a diagram of vertical arrows in $[\Delta^{op}, \text{Set}]$:

$$
\begin{array}{ccc}
X & \xrightarrow{\sigma} & \text{Dec } X \\
\downarrow{=} & \downarrow{=} & \downarrow{=} \\
X & \xrightarrow{\varepsilon} & X
\end{array}
$$

(4.18)

(where the horizontal composite is the identity). Applying the construction of [3.1] to each vertical arrow we obtain $cX \xrightarrow{\varphi} L\bullet X \xrightarrow{\tau} cX$. Here the map of simplicial sets $\phi_i: c(X_i) \to (L\bullet X)_i$ in each internal simplicial dimension $i$ is given by the vertical maps in:

$$
\begin{array}{ccc}
\cdots & X_i & \cdots \\
\downarrow{(s_i,s_i,s_i)} & \downarrow{(s_i,s_i)} & \downarrow{s_i} \\
\cdots & X_{i+1} \times X_i, X_{i+1} \times X_i & \cdots
\end{array}
$$

(4.19)

Since the lower row in (4.19) is the nerve of a homotopically discrete groupoid, the vertical map is a weak equivalence (with inverse induced by the right square in (4.18)).

(b) We will show that for $n \geq 2$:

$$
N^{(n)}Q_{(n)}X = \overline{Q}^{(2)}_{(n-1)}N^{(2)}\hat{\pi}_1^{(2)} \text{Or}_{(2)}X,
$$

(4.20)

where $\overline{Q}^{(2)}_{(n-1)}$ is obtained by applying $Q_{(n-1)}$ in each simplicial dimension in the second direction to the bisimplicial object $N^{(2)}\hat{\pi}_1^{(2)} \text{Or}_{(2)}$.

By Lemma 2.28, we have

$$
\overline{Q}^{(2)}_{(n-1)}N^{(2)}\hat{\pi}_1^{(2)} \text{Or}_{(2)}X = N^{(n)}\hat{\pi}_1^{(n)} \text{Or}_{(n)}X,
$$

(4.21)

and since by definition of $Q_{(n-1)}$:

$$
\overline{Q}^{(2)}_{(n-1)}N^{(2)}\hat{\pi}_1^{(2)} \text{Or}_{(2)}X = \hat{\pi}_1^{(1)} \cdots \hat{\pi}_1^{(n-1)} \overline{Q}^{(2)}_{(n-1)}N^{(2)}\hat{\pi}_1^{(2)} \text{Or}_{(2)}X
$$

we deduce that:

$$
\overline{Q}^{(2)}_{(n-1)}N^{(2)}\hat{\pi}_1^{(2)} \text{Or}_{(2)}X = \hat{\pi}_1^{(1)} \cdots \hat{\pi}_1^{(n-1)} N^{(n)}\hat{\pi}_1^{(n)} \text{Or}_{(n)}X.
$$

(4.22)

Since $Q_{(n)}X := \hat{\pi}_1^{(1)} \cdots \hat{\pi}_1^{(n)} \text{Or}_{(n)}X$ and $\text{Or}_{(n)}X$ is $(n,2)$-fibrant, in order to show (4.20) it suffices to show by induction on $n \geq 2$ that

$$
N^{(n)}\hat{\pi}_1^{(1)} \cdots \hat{\pi}_1^{(n)} Y = \hat{\pi}_1^{(1)} \cdots \hat{\pi}_1^{(n)} N^{(n)}\hat{\pi}_1^{(n)} Y
$$

(4.23)

for any $(n,2)$-fibrant $n$-fold simplicial set $Y$. For $n = 2$,

$$
N^{(2)}\hat{\pi}_1^{(1)} \hat{\pi}_1^{(2)} Y = \hat{\pi}_1^{(1)} N^{(2)}\hat{\pi}_1^{(2)} Y
$$

by Lemma 2.35 and Proposition 2.34.

In the induction step, let $G_{\bullet}$ be the simplicial $(n-1)$-fold groupoid $\hat{\pi}_1^{(2)} \cdots \hat{\pi}_1^{(n)} Y$. By Lemma 2.37 $G_{\bullet}$ is $(n-1,2)$-fibrant, so for each $a \in \Delta^{n-2}$, the simplicial
groupoid $G_\bullet(a)$ (in the first groupoid direction of $G_\bullet$) is $(2,2)$-fibrant. Thus by Lemma 3.29 we have $N^{(n)}\hat{\pi}_1 G_\bullet(a) = \hat{\pi}_1 N^{(n)}G_\bullet(a)$, so

$$N^{(n)}\hat{\pi}_1 \ldots \hat{\pi}_1 Y = N^{(n)}\hat{\pi}_1 G_\bullet = \hat{\pi}_1 N^{(n)}G_\bullet = \hat{\pi}_1 N^{(n)}\hat{\pi}_1 \ldots \hat{\pi}_1 Y.$$  

If we think of $Y$ as a simplicial $(n-1,2)$-fibrant $(n-1)$-fold simplicial set $Y_\bullet^{(1)}$ (in the first direction), by the induction hypotheses

$$N^{(n)}\hat{\pi}_1 \ldots \hat{\pi}_1 Y_m^{(1)} = \hat{\pi}_1 \ldots \hat{\pi}_1 N^{(n)}\hat{\pi}_1 Y_m^{(1)}$$

for each $m \geq 0$, so (4.23) holds for $Y$, too. This concludes the proof of (4.20).

Observe that

$$(4.24) \quad N^{(2)}\hat{\pi}_1 \text{Or}_2 X = NA^f$$

for $A^u \in \text{Gpd}(\Delta^{op}, \text{Set})$ as in (5.2), where $u$ is the map of simplicial sets $u : \text{Dec} X \to X$. In fact, $\hat{\pi}_1 \text{Or}_2 X$, thought of as a simplicial object in $\text{Gpd}$, has $(\hat{\pi}_1 \text{Or}_2 X)_k = \hat{\pi}_1 \text{Dec}^k X$ in simplicial dimension $k$. This is isomorphic to the homotopically discrete groupoid $(X_k)^{u_k}$ (where $u_k : X_k \to X_{k-1}$ is a map of sets). Hence from (4.20) and (4.24) we conclude that

$$N^{(n)}Q(n)X = \overline{Q}_{(n-1)}NA^u.$$  

Since $(NA^u)_k = L_k X$ for each $k \geq 0$, (4.15) follows.

In particular, since $Q(n)X \in \text{Gpd}^{n}_{\text{wg}}$, we have, for $k \geq 2$,

$$Q(n-1)L_k X = (N^{(n)}Q(n)X)^{(n)}_k \cong (N^{(n)}Q(n)X)^{(n)}_1 \times (N^{(n)}Q(n)X)^{(n)}_2 \times \cdots \times (N^{(n)}Q(n)X)^{(n)}_k$$

so by (4.15) we have:

$$Q(n-1)L_k X \cong (Q(n-1)L_1 X)^{\gamma} \times (Q(n-1)\text{Dec} X)^{\gamma} \times \cdots \times (Q(n-1)\text{Dec} X)^{\gamma} \times (Q(n-1)L_1 X)^{\gamma}.$$  

(c) By induction on $n$. For $n = 1$, $Q(1) = \hat{\pi}_1$. Since by hypothesis $X$ is homotopically trivial and $u : \text{Dec} X \to X$ is a fibration, $L_1 X = \text{Dec} X \times X \text{Dec} X$ is also homotopically trivial; hence $\hat{\pi}_1 L_1 X$ is a homotopically discrete groupoid, and is therefore isomorphic to $A^f$ where $f : A \to B$ is the obvious map.

$$X_1 \times X_0 X_1 \to X_0 \times \pi_0 X X_0.$$  

On the other hand, $\hat{\pi}_1 \text{Dec} X \cong (X_1)^{d_0}$ and $\hat{\pi}_1 X = (X_0)^\gamma$ (for $\gamma : X_0 \to \pi_0 X$), so:

$$\hat{\pi}_1 L_1 X \cong \hat{\pi}_1 \text{Dec} X \times \pi_0 X \hat{\pi}_1 \text{Dec} X.$$  

In the induction step, applying $N^{(n)}$ to both sides of (4.17), we must show that for each $k \geq 1$ and $i \geq 1$ we have

$$N^{(n)}Q(n)L_k X)^{(n)}_{i-1} \cong (N^{(n)}Q(n)\text{Dec} X)^{(n)}_{i-1} \times (N^{(n)}Q(n)X)^{(n)}_{i-1} \times \cdots \times (N^{(n)}Q(n)\text{Dec} X)^{(n)}_{i-1}$$

or equivalently (after applying (b)), that:

$$(4.25) Q(n-1)L_i(L_k X) \cong Q(n-1)L_i \text{Dec} X \times Q(n-1)L_i X \times Q(n-1)L_i X \times \cdots \times Q(n-1)L_i X$$
Since $X$ is homotopically trivial, so are $\text{Dec} \ X$ and $L_k X$ (since $u \colon \text{Dec} \ X \to X$ is a fibration), so we can apply induction hypothesis (c) for $(n-1)$ to replace the left hand side of (4.25) by:

$$Q_{(n-1)} \text{Dec}(L_k X) \times Q_{(n-1)} L_k X \rightarrow Q_{(n-1)} L_k X \cdot \cdot \cdot \times Q_{(n-1)} L_k X,$$

and since $\text{Dec}$ commutes with fiber products, and thus with $L_k$, this equals:

$$(Q_{(n-1)}(\text{Dec}^2 X \times_{\text{Dec} k+1} \text{Dec}^2 X)) \times (Q_{(n-1)} \text{Dec} X \times X \cdot \cdot \cdot \times (Q_{(n-1)} \text{Dec} X \times X \cdot \cdot \cdot \times (Q_{(n-1)} (\text{Dec}^2 X \times_{\text{Dec} k+1} \text{Dec}^2 X))).$$

If we write $A := Q_{(n-1)} \text{Dec}^2 X$, $B := Q_{(n-1)} \text{Dec} X$, and $C := Q_{(n-1)} X$, applying (b) for $n-1$ to this last expression yields:

$$Q_{(n-1)} A \times B \cdot \cdot \cdot X A = Q_{(n-1)} A \times (\text{Dec} C) \cdot \cdot \cdot X (\text{Dec} C).$$

Similarly, (c) applied to the right hand side of (4.25) yields

$$Q_{(n-1)} A \times (\text{Dec} C) \cdot \cdot \cdot X (\text{Dec} C) = Q_{(n-1)} A \times B \cdot \cdot \cdot X A,$$

and the two limits (4.26) and (4.27) are evidently equal, proving (4.25). □

### 4. C. Modelling n-types

In the last part of this section we finally show that weakly globular $n$-fold groupoids indeed model $n$-types.

#### 4.28. Proposition

Let $X$ be a Kan complex. Then:

(a) There is a natural $n$-equivalence $\psi^X_{(n)} : X \to dNQ_{(n)} X$.

(b) $Q_{(n)}$ preserves weak equivalences of Kan complexes.

(c) If $X$ is homotopically trivial (i.e., all higher homotopy groups vanish), then $Q_{(n)} X$ is a homotopically trivial $n$-fold groupoid.

(d) $Q_{(n)} X$ is a weakly globular $n$-fold groupoid, and $\Pi_{(n)}^0 Q_{(n)} X$ is isomorphic to $Q_{(n-1)} X$.

**Proof.** By induction on $n$. The claim is immediate for $n = 1$ (with $Q_{(0)} X := \pi_0 X$ and $\psi^X_{(1)} : X \to N\pi_1 X \simeq P^1 X$ the Postnikov structure map).

(a) We assume that we have a map

$$\psi^X_{(n-1)} : X \to \text{Diag}_{(n-1)} N_{(n-1)} Q_{(n-1)} X,$$

natural in $X$. Applying this to the simplicial object $L_\bullet X \in [\Delta^{op}, [\Delta^{op}, \text{Set}]]$ (which is fibrant in each simplicial dimension, by (2.8), we obtain a map of bisimplicial sets

$$\psi^{L_\bullet X}_{(n-1)} : L_\bullet X \to \text{Diag}_{(n-1)} N_{(n-1)} Q_{(n-1)} L_\bullet X,$$

which is an $(n-1)$-equivalence in each simplicial dimension.

However, in simplicial dimension 0 we have $L_0 X = \text{Dec} X$, which is homotopically trivial, while $Q_{(n-1)} \text{Dec} X$ is a homotopically discrete $(n-1)$-fold groupoid by induction assumption (c) for $n - 1$, so $dNQ_{(n-1)} \text{Dec} X$ is homotopically trivial by Corollary [3.13]. Thus $\psi^{L_0 X}_{(n-1)}$ is actually a geometric weak equivalence, and thus $\psi^{L_X}_{(n-1)}$ is a diagonal $n$-equivalence (cf. [3.22]), which implies that
(4.29) \[ \text{Diag } \psi_{(n-1)}^{L_X} : \text{Diag } L_X \to \text{Diag}_{(n)} N_{(n-1)} Q_{(n)} X \]

is an \( n \)-equivalence by Proposition 3.30.

Now by (4.15), \( N^{(n)} Q_{(n)} X = \overline{Q}_{(n-1)} L_X \), so together with the map \( \phi : cX \to L_X \) of Lemma 4.11(a) we have maps of bisimplicial sets:

\[ cX \xrightarrow{\phi} L_X X \xrightarrow{\psi_{(n-1)}^{L_X}} \text{Diag}_{(n)} N_{(n-1)} Q_{(n)} L_X X = \text{Diag}_{(n)} N^{(n)} Q_{(n)} X . \]

Applying \( \text{Diag} \) to both maps we see that the first is a weak equivalence, while the second is an \( n \)-equivalence, because (4.29) is such. We define the composite to be \( \psi_{(n)}^X : X \to dNQ_{(n)} X \), which is therefore an \( n \)-equivalence.

(b) Let \( f : X \to Y \) be a weak equivalence of Kan complexes. Since by (a), \( X \to \text{Diag}_{(n)} Q_{(n)} X \) and \( Y \to \text{Diag}_{(n)} Q_{(n)} Y \) are \( n \)-equivalences, it follows that \( \text{Diag}_{(n)} Q_{(n)} f \) is an \( n \)-equivalence. By Theorem 4.10 \( \text{Diag}_{(n)} Q_{(n)} X \) and \( \text{Diag}_{(n)} Q_{(n)} Y \) are \( n \)-types. Hence \( \text{Diag}_{(n)} Q_{(n)} f \) is a weak equivalence.

(c) Since \( X \) is homotopically trivial, by Lemma 4.11 for each \( k \geq 1 \) we have:

\( (N^{(n)} Q_{(n)} X)_k = Q_{(n-1)} L_k X = Q_{(n-1)} \text{Dec } X \times_{Q_{(n-1)} X} Q_{(n-1)} \text{Dec } X \).

Therefore \( Q_{(n)} X = A^f \), where \( A = Q_{(n-1)} \text{Dec } X \) and by induction

\[ f := Q_{(n-1)} \varepsilon : Q_{(n-1)} \text{Dec } X \to Q_{(n-1)} X \]

is a map of homotopically discrete \((n-1)\)-fold groupoids with a section \( Q_{(n-1)} \sigma \) (cf. 2.7). Hence \( Q_{(n)} X \) is homotopically discrete, by definition.

(d) To show that \( Q_{(n)} X \) is weakly globular (Definition 3.19), we think of it as a groupoid in \( \text{Gpd}_{\text{wg}}^{n-1} \), with \( (n-1) \)-fold groupoid of objects \( (Q_{(n)} X)_0 \) and \( (n-1) \)-fold groupoid of arrows \( (Q_{(n)} X)_1 \). Note that \( (Q_{(n)} X)_0 = Q_{(n-1)} \text{Dec } X \) by (4.15) for \( k = 0 \), and since \( \text{Dec } X \) is homotopically discrete, \( (Q_{(n)} X)_0^{(n)} \) is homotopically discrete, by (c).

Similarly, \( (N^{(n)} Q_{(n)} X)_1^{(n)} = Q_{(n-1)} L_1 X \in \text{Gpd}_{\text{wg}}^{n-1} \), and by (4.10):

\( (N^{(n)} Q_{(n)} X)_k = (Q_{(n)} X)_1 \times_{(Q_{(n)} X)_0} \cdots \times_{(Q_{(n)} X)_0} (Q_{(n)} X)_1 = Q_{(n-1)} (L_1 X \times_{\text{Dec } X} \cdots \times_{\text{Dec } X} L_1 X) \),

so \( (N^{(n)} Q_{(n)} X)_k \) is weakly globular for each \( k \geq 0 \).

If we apply \( \Pi_0^{(n-1)} \) in each simplicial dimension in the \( n \)-th direction, by Lemma 4.11(b) and the induction hypothesis:

\( (\Pi_0^{(n-1)} N^{(n)} Q_{(n)} X)_k^{(n)} = \Pi_0^{(n-1)} Q_{(n-1)} L_k X = Q_{(n-2)} L_k X = (Q_{(n-1)} X)_k \),

where \( (N^{(n-1)} Q_{(n-1)} X)_k^{(n-1)} \) is abbreviated to \( (Q_{(n-1)} X)_k \).

This shows that \( \Pi_0^{(n)} G \) lands in weakly globular \((n-1)\)-fold groupoids, and that:

\( \Pi_0^{(n)} Q_{(n)} X \cong Q_{(n-1)} X \).
To prove that $Q_n X \in \text{Gpd}^n_{\text{wg}}$, it remains to show that in each simplicial dimension $k \geq 2$ (in the $n$-th direction), the map:

$$\tag{4.30} (Q_n X)_1 \times (Q_n X)_0 \cdots \times (Q_n X)_0 (Q_n X)_1 \to (Q_n X)_1 \times (Q_n X)_0 \cdots \times (Q_n X)_0 (Q_n X)_1$$

is a geometric weak equivalence. By Lemma [4.14 (b)] we have:

$$\tag{4.30} (Q_n X)_1 \times (Q_n X)_0 \cdots \times (Q_n X)_0 (Q_n X)_1 = (\mathcal{N}(n) Q_n X)_k$$

$$= Q_{(n-1)} L_k X \cong Q_{(n-1)} (L_1 X \times_{\text{Dec} X} \cdots \times_{\text{Dec} X} L_1 X)$$

where the second equality is [4.15] and the third is [4.16].

Since $\text{Dec} X$ is homotopically trivial, $Q_{(n-1)} \text{Dec} X$ is homotopically discrete by (c), so

$$(Q_n X)_0^d = (Q_{(n-1)} \text{Dec} X)^d = Q_{(n-1)} c(\pi_0 \text{Dec} X) = Q_{(n-1)} c(X_0)$$

by (a) and Lemma [2.41 (d)], where $c(X_0)$ is the constant simplicial set on $X_0$.

Since $X$ is a Kan complex and $Q_{(n-1)}$ commutes with fiber products over discrete objects, by Remark [2.41] we have:

$$\tag{4.30} (Q_n X)_1 \times (Q_n X)_0 \cdots \times (Q_n X)_0 (Q_n X)_1 = Q_{(n-1)} L_1 X \times_{\text{Dec} X} \cdots \times_{\text{Dec} X} L_1 X$$

$$\cong Q_{(n-1)} (L_1 X \times c(X_0) \cdots \times c(X_0) L_1 X).$$

Since $\text{Dec} X \to X$ is a fibration, so is $L_1 X \to \text{Dec} X$, and $\text{Dec} X \to c(X_0)$ is a weak equivalence; thus the map

$$L_1 X \times_{\text{Dec} X} \cdots \times_{\text{Dec} X} L_1 X \to L_1 X \times c(X_0) \cdots \times c(X_0) L_1 X$$

is a weak equivalence of Kan complexes. Therefore, by (b), [4.30] is a weak equivalence, as required. \qed

Recall from [3.28] that $P^n \text{Top}$ denotes the full subcategory of $\text{Top}$ consisting of spaces $T$ for which the natural map $T \to P^n T$ is a weak equivalence, and $\text{ho} P^n \text{Top}$ is the corresponding full subcategory of the homotopy category $\text{ho} \text{Top}$ of topological spaces.

4.31. Definition. Let $\text{ho} \text{Gpd}^n_{\text{wg}}$ denote the localization of the category $\text{Gpd}^n_{\text{wg}}$ with respect to the (algebraic) weak equivalences (see Corollary [3.8] and compare [31]).

4.32. Theorem. The functors $\hat{Q}(n): \text{Top} \to \text{Gpd}^n_{\text{wg}}$ and $B: \text{Gpd}^n_{\text{wg}} \to \text{Top}$ induce functors:

$$\tag{4.33} \text{ho} P^n \text{Top} \xrightarrow{\hat{Q}(n)} \text{ho} \text{Gpd}^n_{\text{wg}},$$

with $B \circ \hat{Q}(n) \cong \text{Id}_{\text{ho} P^n \text{Top}}$ so $\hat{Q}(n): \text{ho} P^n \text{Top} \to \text{ho} \text{Gpd}^n_{\text{wg}}$ is a faithful embedding.
Proof. By Theorem 4.6 and Proposition 4.28 both functors $\hat{Q}_n = Q(n)S$ and $B$ preserve weak equivalences, and therefore induce corresponding functors on the homotopy categories. Also, for any $T \in P^n\text{Top}$, by Theorem 4.6 and Proposition 4.28 there is a span

$$\begin{align*}
B\hat{Q}_{(n)}T &\leftarrow S|T| \rightarrow T,
\end{align*}$$

where the map on the left is a homotopy equivalence and the map on the right is a weak homotopy equivalence. It follows that $T$ and $B\hat{Q}_{(n)}T$ are weakly equivalent in $P^n\text{Top}$; that is, $B \circ \hat{Q}_{(n)} \cong \text{Id}_{P^n\text{Top}}$. □

4.35. Weakly globular double groupoids. For $n = 2$, we can strengthen Theorem 4.32 to obtain an equivalence $\text{ho} P^2\text{Top} \approx \text{ho} \text{Gpd}^\text{wg}_2$, where on the right hand side we use the (internally defined) algebraic weak equivalences of $\text{Gpd}^\text{wg}_2$ itself:

As in [20, Theorem 2.5], for any double groupoid $G$ one can construct a map $\varepsilon_\bullet : \text{Or}_2(dNG) \rightarrow N_2G$. By [24, Theorem 8], if $G$ is weakly globular (and therefore $(2,2)$-fibrant), $dNG$ is a Kan complex. Therefore, $P_2\text{Or}_2(dNG)$ and $P_2N_2G = G$ are weakly globular double groupoids. Since we have a homotopy equivalence of Kan complexes $\xi : dNG \rightarrow S\|dNG\| = SBG$, we also have a geometric weak equivalence of weakly globular double groupoids:

$$\begin{align*}
Q_2dNG &\xrightarrow{Q_2\xi} Q_2SBG = \hat{Q}_2BG.
\end{align*}$$

Therefore, the algebraic homotopy groups $\omega_*Q_2dNG$ are isomorphic by (3.27) to

$$\begin{align*}
\pi_*BQ_2dNG &\cong \pi_*\hat{Q}_2BG \cong \pi_*BG
\end{align*}$$

(using (4.34) for $T := BG$). By Theorem 4.6, $\omega_*G \cong \pi_*BG$, and since $\omega_*G \cong \pi_*dNG$, also by (3.27), we conclude that $\omega_*Q_2dNG \cong \omega_*G$. One can verify that this isomorphism is induced by the map

$$P_2\varepsilon_\bullet : P_2\text{Or}_2(dNG) = Q_2dNG \rightarrow P_2N_2G = G,$$

which is therefore a geometric weak equivalence of double groupoids.

Together with a map of double groupoids induced by (4.36), we obtain a zig-zag of geometric weak equivalences:

$$\begin{align*}
\hat{Q}_2BG &\xleftarrow{Q_2\xi} Q_2dNG \xrightarrow{P_2\varepsilon_\bullet} G.
\end{align*}$$

This implies that (4.33) is an equivalence of localized categories when $n = 2$.

5. Tamsamani’s model and weakly globular $n$-fold groupoids

In this section we construct a comparison functor from weakly globular $n$-fold groupoids to Tamsamani’s weak $n$-groupoids, which preserves homotopy types.

5A. Tamsamani’s weak $n$-groupoids

We begin with a brief recapitulation of the notion of a Tamsamani weak $n$-groupoid, starting with a modified definition. This differs somewhat from the original definition in [51, §5] (compare [50, §15.2] and [14, §8]), which was motivated by the goal of modelling higher categories, rather than groupoids.
5.1. **Definition.** The category $\text{Tam}^n$ of Tamsamani weak $n$-groupoids is a full subcategory of $[\Delta^{n-1}\text{op}, \text{Gpd}]$, defined inductively as follows:

(a) $\text{Tam}^1 := \text{Gpd}$ is the category of groupoids.
(b) Each $X \in \text{Tam}^n$ is a simplicial object in $\text{Tam}^{n-1}$ (in the first simplicial direction). We therefore have an inclusion functor $J_n: \text{Tam}^n \to [\Delta^{n-1}\text{op}, \text{Gpd}]$.
(c) The 0-th Tamsamani weak $(n-1)$-groupoid $X_0$ is discrete (that is, a constant $(n-1)$-fold simplicial set).
(d) The Segal maps $\mu_k: X_k \to X_1 \times X_0 \cdots \times X_0 X_1$ (Definition 2.3) are geometric weak equivalences of Tamsamani weak $(n-1)$-groupoids for each $k \geq 2$: that is, $B\mu_k: BX_k \to B(X_1 \times X_0 \cdots \times X_0 X_1)$ is a weak equivalence of topological spaces, where $B: \text{Tam}^n \to \text{Top}$ is the realization functor of (1.3).
(e) The $(n-1)$-simplicial set $\pi^{(n)}_0 J_n X$ is the nerve of a Tamsamani weak $(n-1)$-groupoid $\Pi^{(n)}_0 X$, and we have a commutative diagram

$$
\begin{array}{ccc}
\text{Tam}^n & \xrightarrow{J_n} & [\Delta^{n-1}\text{op}, \text{Gpd}] \\
\Pi^{(n)}_0 \downarrow & & \downarrow \pi^{(n)}_0 \\
\text{Tam}^{n-1} & \xrightarrow{J_{n-1}} & [\Delta^{n-2}\text{op}, \text{Gpd}] \\
\end{array}
$$

Furthermore, $\Pi^{(n)}_0$ preserves geometric weak equivalences.

5.2. **Tamsamani’s original definition.** Tamsamani’s original approach (as described in [44, §8]) gave an inductive definition of the category $\text{Tam}^n \subseteq [\Delta^{n-1}\text{op}, \text{Gpd}]$ equipped with a class of maps called $n$-equivalences for each $n \geq 1$. The following assumptions must be satisfied:

(a) $\text{Tam}^1 := \text{Gpd}$ (with 1-equivalences being equivalences of groupoids).
(b) Each $X \in \text{Tam}^n$ is a simplicial object in $\text{Tam}^{n-1}$.
(c) $X_0$ is discrete.
(d) The Segal maps $\mu_k: X_k \to X_1 \times X_0 \cdots \times X_0 X_1$ are $(n-1)$-equivalences in $\text{Tam}^{n-1}$ for each $k \geq 2$.
(e) The functor $\pi_0^{(1)} \pi_0^{(2)} \cdots \pi_0^{(n)} : \text{Tam}^n \to \text{Set}$, (cf. [3.12]) takes $n$-equivalences to bijections and preserves fiber products over discrete objects.

Note that (d) and (e) together imply that the Tamsamani fundamental groupoid

$$
T^{(n)}_\text{Tm} := \frac{\pi_0^{(2)} \cdots \pi_0^{(n-1)} \pi_0^{(n)}}{\pi_0^{(1)} \cdots \pi_0^{(n)}},
$$

when applied to $X \in \text{Tam}^n$, lands in groupoids.

(f) For every $a$ and $b$ in the set $X_0$, the fiber of $X_{(a,b)}$ of $(d_0, d_1): X_1 \to X_0 \times X_0$ is a Tamsamani weak $(n-1)$-groupoid.

(g) A map $f: X \to Y$ in $\text{Tam}^n$ is an $n$-equivalence if and only if:

i. The map $T^{(n)}_\text{Tm} f: T^{(n)}_\text{Tm} X \to T^{(n)}_\text{Tm} Y$ is an equivalence of groupoids.

ii. $f_{(a,b)}: X_{(a,b)} \to Y_{(a,b)}$ is an $(n-1)$-equivalence for every $(a, b) \in X_0 \times X_0$. 


5.3. **Remark.** Note that if \( g: X \to Y \) is a morphism in \( \text{Tam}^n \) with \( Y \) discrete, then \( X \) is isomorphic to \( \coprod_{y \in Y} g^{-1}(y) \), where the coproduct is taken in \( \text{Tam}^n \) (compare Lemma 6.7 below).

This implies that if \( X \in \text{Tam}^n \), then \( X_1 \) is isomorphic to the coproduct over all \( a, b \in X_0 \) of \( X_1(a, b) \in \text{Tam}^{n-1} \) (where \( X_1(a, b) \) is the fiber of \( (d_0, d_1): X_1 \to X_0 \times X_0 \)).

From this and from (e) we deduce that if \( X \in \text{Tam}^n \), then \( X_1 \) is isomorphic to the coproduct over all \( a, b \in X_0 \) of \( X_1(a, b) \in \text{Tam}^{n-1} \), where the coproduct is taken in \( \text{Tam}^n \).

Further, \( \Pi_0^n \) takes \( n \)-equivalences to \( (n-1) \)-equivalences, and one can therefore replace \( (g) \) in the definition above by:

i. The map \( \Pi_0^n f: \Pi_0^n X \to \Pi_0^n Y \) is an \( (n-1) \)-equivalence in \( \text{Tam}^{n-1} \).

ii. \( f(a,b): X(a,b) \to Y(a,b) \) is an \( (n-1) \)-equivalence for every \( (a,b) \in X_0 \times X_0 \).

We recall the following fact from [44, Lemma 10.1]:

5.4. **Lemma.** A map \( f: X \to Y \) in \( \text{Tam}^n \) is an \( n \)-equivalence if and only if it is a geometric weak equivalence.

5.5. **Proposition.** The categories \( \text{Tam}^n \) and \( \text{Tam}^n \) are identical.

**Proof.** By induction on \( n \geq 1 \), starting with \( \text{Tam}^1 = \text{Gpd} = \text{Tam}^1 \). The fact that \( \text{Tam}^n \) is contained in \( \text{Tam}^n \) is immediate (by the induction hypothesis and Lemma 5.4), while the other direction follows from Remark 5.3 and Lemma 5.4 again. \( \square \)

5.6. **Definition.** Let \( \text{ho} \text{Tam}^n \) denote the localization of the category \( \text{Tam}^n \) with respect to the \( n \)-equivalences.

5.7. **Theorem** ([51, Theorem 8.0]). There is a Poincaré \( n \)-groupoid functor \( F^n_{\text{Tm}}: \text{Top} \to \text{Tam}^n \) which, together with \( B: \text{Tam}^n \to \text{Top} \), induces equivalences of categories:

\[
\text{ho} \text{P}^n \text{Top} \xrightarrow{F^n_{\text{Tm}}} \text{ho} \text{Tam}^n.
\]

For every \( T \in \text{Top} \), there is a zigzag of weak equivalences in \( \text{P}^n \text{Top} \) between \( BF^n_{\text{Tm}}T \) and \( P^n TX \), and for every \( X \in \text{Tam}^n \), there is a natural weak equivalence \( X \to F^n_{\text{Tm}}BX \).

5.B. **Comparison with weakly globular \( n \)-fold groupoids**

We construct iteratively a discretization functor \( D_n: \text{Gpd}^n_{\text{wg}} \to \text{Tam}^n \), which preserves the homotopy type.
5.9. Two simplicial constructions. Let $\mathcal{C}$ be a (co)complete category, $X \in [\Delta^{op}, \mathcal{C}]$ a simplicial object, and $\gamma: X_0 \to W$ a map in $\mathcal{C}$. In this context we mimic the construction of a new simplicial object $Y \in [\Delta^{op}, \mathcal{C}]$ described in [12, §3], as follows:

Consider the pushout in $\mathcal{C}$

$$
\begin{array}{c}
X_0 \xrightarrow{s_{(n)}} X_n \\
\gamma \downarrow \quad \downarrow f_n \\
W \xrightarrow{\sigma_{(n)}} Y_n
\end{array}
$$

(5.10)

where $s_{(n)}$ is induced by the unique morphism $[0] \to [n]$ in $\Delta^{op}$. For any morphism $\phi: [m] \to [n]$ in $\Delta^{op}$, $\phi s_{(n)} = s_{(m)}$ by the uniqueness, so that

$$
f_m \phi s_{(n)} = f_m s_{(m)} = \sigma_{(m)} f_0 : X_0 \to Y_m .
$$

By the universal property of pushouts there exists a unique $\hat{\phi}: Y_n \to Y_m$ with $\hat{\phi} f_n = f_m \hat{\phi}$ and $\hat{\phi} \sigma_{(n)} = \sigma_{(m)}$. In particular, we have maps $d_i: Y_n \to Y_{n-1}$ for $0 \leq i \leq n$, and $\sigma_i: Y_{n-1} \to Y_n$ for $0 \leq i < n$. The maps $d_i$ and $\sigma_i$ satisfy the simplicial identities, so that $Y$ is a simplicial object in $\mathcal{C}$. In fact, if $[n] \xrightarrow{\phi} [m] \xrightarrow{\psi} [k]$ are morphisms in $\Delta^{op}$ and $\xi = \psi \circ \phi$, then

$$
\hat{\xi} \sigma_{(n)} = \sigma_{(k)} = \hat{\psi} \sigma_{(m)} = \hat{\psi} \hat{\phi} \sigma_{(n)}
$$

and

$$
\hat{\xi} f_n = f_k \hat{\xi} = f_k \hat{\psi} \hat{\phi} = \hat{\psi} f_m \hat{\phi} = \hat{\psi} \hat{\phi} f_n .
$$

It follows by universal property of pushouts that $\hat{\xi} = \hat{\psi} \hat{\phi}$. In particular, since the simplicial identities are satisfied by $d_i$ and $\sigma_i$, they are satisfied by $\hat{d}_i$ and $\hat{\sigma}_i$. So we have a map of simplicial objects $f : X \to Y$.

Note that if $\gamma': W \to X_0$ is a section for $\gamma$ (with $\gamma \gamma' = \text{Id}$), we may construct a new simplicial object $X^\gamma \in [\Delta^{op}, \mathcal{C}]$ by setting

$$
X^\gamma_n = \begin{cases} 
W & \text{if } n = 0 \\
X_n & \text{if } n > 0 
\end{cases}
$$

Let $d_0, d_1: X_1 \to X_0$ and $\sigma_0 = s_{(1)}: X_0 \to X_1$ be the face and degeneracy maps of $X$, and let $d'_0$, $d'_1: X_1 \to W$, and $\sigma'_0: W \to X_1$, respectively, denote $d'_i = \gamma d_i$ ($i = 0, 1$), and $\sigma'_0 = \sigma_0 \gamma'$. All other face and degeneracy operators of $X^\gamma$ are the same as those of $X$.

Finally, we define a map $h: X^\gamma \to Y$ in $[\Delta^{op}, \mathcal{C}]$ by setting $h_0 := \text{Id}$ and $h_n := f_n$ for $n > 0$. In fact, $d'_i = d_i f_1$; also, $f_1 \sigma_0 = \hat{\sigma}_0 \gamma'$, which implies $f_1 \sigma_0 \gamma' = \hat{\sigma}_0 \gamma' = \hat{\sigma}_0$. All other identities are the same as for $f$.

5.11. The functor $D$. Let $[\Delta^{op}, \text{Set}]^2_{\text{h}}$ be the full subcategory of bisimplicial sets $X$ such that the simplicial set $X_0$ is homotopically trivial, through a weak equivalence $\gamma: X_0 \to X^0_0$ with a section $\gamma': X^0_0 \to X_0$ with $\gamma \gamma' = \text{Id}$, where $X^0_0$ is the constant simplicial set on $\pi_0 X_0$. Let $[\Delta^{op}, \text{Set}]^2_{\text{d}}$ denote the full subcategory of bisimplicial sets $X$ such that the simplicial set $X_0$ is constant. We construct a functor

$$
D : [\Delta^{op}, \text{Set}]^2_{\text{h}} \to [\Delta^{op}, \text{Set}]^2_{\text{d}}
$$
by setting $DX := X^γ$ (in the notation of §5.9).

5.12. Lemma. Let $D: [Δ^{op}, Set]^2_h → [Δ^{op}, Set]^2_h$ be as above. Then for each $X ∈ [Δ^{op}, Set]^2_h$, $DX$ and $X$ have the same homotopy type.

Proof. We construct a bisimplicial set $Y$ and weak equivalences

$$X \xrightarrow{f} Y \xrightarrow{h} DX$$

using the construction of §5.9 for $C = [Δ^{op}, Set]$, $W := X^d_0$ and $γ: X_0 → X^d_0$ as above. Since $γ$ is a weak equivalence and $s(n)$ is a cofibration of simplicial sets, the right vertical map $f_n$ in (5.10) is a weak equivalence for each $n ≥ 0$ – that is, we have a map of bisimplicial sets $f: X → Y$ which is a levelwise weak equivalence. Thus $Bf$ is also a weak equivalence.

Since the map $h: DX → Y$ of (5.9) is a levelwise weak equivalence, $Bh$ is a weak equivalence. In conclusion, $f$ and $h$ are weak equivalences, so that $Diag X ≃ Diag DX$.

5.13. Definition. We define the $0$-discretization functor

$$Diag_0: Gpd^{n}_{WG} → [Δ^{op}, Gpd^{n-1}_{WG}]$$

on any weakly globular $n$-fold groupoid $G$ as follows: set

$$(Diag_0 G)_k :=\begin{cases} G^d_0 & \text{if } k = 0 \\ (N(1)G)_k & \text{if } k > 0 \end{cases}$$

(cf. 4.13). If $d_0, d_1: G_1 → G_0$ are the source and target maps, and $σ_0: G_0 → G_1$ is the degeneracy operator (all in $Gpd^{n-1}_{WG}$), we define $d_0', d_1': (Diag_0 G)_1 → (Diag_0 G)_0$ and $σ_0': (Diag_0 G)_0 → (Diag_0 G)_1$ by $d_i' = γd_i$ ($i = 0, 1$) and $σ_0' = σ_0γ'$. All other face and degeneracy operators of $Diag_0 G$ are those of $G$. Since $γγ' = Id$, all simplicial identities hold for $Diag_0 G$.

5.14. Lemma. For any weakly globular $n$-fold groupoid $G ∈ Gpd^{n}_{WG}$, $Diag(n) G$ and $Diag(n) Diag_0 G$ are weakly equivalent.

Proof. $Diag(n) G$ is the diagonal of the bisimplicial set $X$ with

$$X_k := Diag(n-1)(N(n)G)^{(n)}_k$$

for all $k ≥ 0$, while $Diag(n) Diag_0 G$ is the diagonal of the bisimplicial set $Y$ with $Y_0 := G^d_0$ and $Y_k := Diag(n-1)(N(n)G)^{(n)}_k$ for $k ≥ 1$. By construction, $X ∈ [Δ^{op}, Set]^2_h$ and $Y = DX$. Hence, by Lemma 5.12 $Diag(n) G = Diag Y ≃ Diag Y = Diag(n) Diag_0 G$.

5.15. Notation. Let $T^{wg}_{(n)}: Gpd^n_{WG} → Gpd$ denote the weakly globular fundamental groupoid functor – that is, the composite

$$T^{wg}_{(n)} := Π^{(2)}_0 \cdots Π^{(n-1)}_0 Π^{(n)}_0$$

(see Definitions 5.12 and 5.19).

By construction, for all $i ≥ 0$,

$$(T^{wg}_{(n)})_i = π_0 T^{wg}_{(n-1)} G_i.$$
5.18. **Definition.** For each \( n \geq 1 \), we define discretization functors

\[
D_n : \text{Gpd}^n_{\text{wg}} \to [\Delta^{n-1}]^{\text{op}}, \text{Gpd}
\]

by induction on \( n \), starting with \( D_1 := \text{Id} : \text{Gpd} \to \text{Gpd} \). For \( n \geq 2 \), we let \( D_n \) be the composite:

\[
\text{Gpd}^n_{\text{wg}} \xrightarrow{N(n)} [\Delta^{n}, \text{Gpd}^{n-1}_{\text{wg}}] \xrightarrow{\text{Disc}_0} [\Delta^{n}, \text{Gpd}^{n-1}] \xrightarrow{\mathfrak{T}_{n-1}} [\Delta^{n-1}]^{\text{op}}, \text{Gpd}
\]

where \( \mathfrak{T}_{n-1} \) is obtained by applying \( D_{n-1} \) in each simplicial dimension.

5.19. **Theorem.** The functor \( D_n \) lands in \( \text{Tam}^n \). Furthermore, \( T_{(n)}^{\text{Tm}} D_n = T_{(n)}^{\text{wg}} \)

and for each \( G \in \text{Gpd}^n_{\text{wg}} \), we have a natural weak equivalence

\[
\text{Diag}_{(n)} G \simeq \text{Diag}_{(n)} D_n G.
\]

**Proof.** By induction on \( n \geq 2 \). For \( n = 2 \), note that \( D_2 G = \text{Disc}_0 N(2) G \) is in \( \text{Tam}^2 \) for any weakly globular double groupoid \( G \), since for each \( k \geq 2 \) by Definition 5.19 iv) we have:

\[
(D_2 G)_k = G_1 \times G_0 \cdots \times G_0 G_1 \simeq G_1 \times G_0 \cdots \times G_0 G_1 \simeq (D_2 G)_1 \times (D_2 G)_0 \cdots \times (D_2 G)_1.
\]

Furthermore, \( T_{(2)}^{\text{Tm}} D_2 G = T_{(2)}^{\text{wg}} G = \Pi_0^2 G \) is a groupoid. Hence by definition, \( D_2 G \in \text{Tam}^2 \). By Lemma 5.14 \( BD_2 G \simeq dNG \) since \( G \in \text{Gpd}^2_{\text{wg}} \).

In the induction step, note that \( (D_n G)_0 = G_0^d \) is discrete. So to prove that \( D_n G \) is in \( \text{Tam}^n \), it remains to show:

(a) The Segal maps

\[
\mu_k : (D_n G)_k \to (D_n G)_1 \times (D_n G)_0 \cdots \times (D_n G)_0 (D_n G)_1
\]

are geometric weak equivalences.

(b) \( T_{(n)}^{\text{Tm}} D_n G \) is a groupoid.

Note that by Definition 5.18 and by the inductive hypothesis, for \( k \geq 2 \) we have:

\[
\text{Diag}_{(n-1)} (D_n G)_k = \text{Diag}_{(n-1)} D_{n-1} (G_1 \times G_0 \cdots \times G_0 G_1) \simeq dN(G_1 \times G_0 \cdots \times G_0 G_1),
\]

and by Definition 5.19 iv) and the inductive hypothesis again this is weakly equivalent to

\[
dN(G_1 \times G_0 \cdots \times G_0 G_1) \simeq \text{Diag}_{(n-1)} D_{n-1} G_1 \times dNG_0 \times \cdots \times dNG_0 \text{Diag}_{(n-1)} D_{n-1} G_1
\]

which is \( \text{Diag}_{(n)} ((D_n G)_1 \times (D_n G)_0 \cdots \times (D_n G)_0 (D_n G)_1) \) by Definition 5.18. Thus each Segal map \( \mu_k \) is a geometric weak equivalence. This proves (a).

To prove (b), note that by definition of \( T_{(n)}^{\text{Tm}} \), 5.16, and 5.17, we have:

\[
(T_{(n)}^{\text{Tm}} D_n G)_0 = \pi_0 T_{(n-1)}^{\text{Tm}} (D_n G)_0 = \pi_0 T_{(n-1)}^{\text{Tm}} G_0^d = G_0^d
\]

\[
= \pi_0 NT_{(n-1)}^{\text{wg}} G_0 = (T_{(n)}^{\text{wg}} G)_0.
\]

where \( \pi_0 T_{(n)}^{\text{Tm}} X = \Pi_0^{(1)} \Pi_0^{(2)} \cdots \Pi_0^{(n)} X \).

Furthermore:

\[
(T_{(n)}^{\text{Tm}} D_n G)_k = \pi_0 T_{(n-1)}^{\text{Tm}} (D_n G)_k = \pi_0 T_{(n-1)}^{\text{Tm}} D_{n-1} (\text{N}^{(n)} G)_k
\]
for \( k \geq 1 \). By induction we therefore have:
\[
π_0^{Tm}D_{n-1}(N^{(n)}G)_k = π_0^{Tm}T_{(n-1)}^{wg}(N^{(n)}G)_k = (T^{wg}_{(n)}G)_k
\]

It follows that \( T_{(n)}^{Tm}G = T_{(n)}^{wg}G \), as claimed. Since \( T_{(n)}^{wg}G \) is a groupoid, so is \( T_{(n)}^{Tm}G \). This concludes the proof that \( D_nG ∈ \text{Tam}^n \).

Finally, we show that \( \text{Diag}_{(n)}D_nG ∼= dNG \). Let \( Y = \text{Disc}_0 N^{(n)}G ∈ [\Delta^{op}, \text{Gpd}^{n-1}] \).

By Lemma 5.14 \( dNY ∼= dNG \). Furthermore, \( \text{Diag}_{(n)}D_nG \) is the realization of the bisimplicial set \( Z_k := \text{Diag}_{(n-1)}D_{n-1}Y_k \). By induction, \( Z_k ∼= \text{Diag}_{(n-1)}Y_k \), so that \( \text{Diag} Z ∼= dNY ∼= dNG \), as required.

5.20. Remark. Since by [51] [8], \( BD_nG \) is an \( n \)-type, Theorem 5.19 implies that the realization of a weakly globular \( n \)-fold groupoid is an \( n \)-type. This provides an alternative proof of the first statement in Theorem 4.6. Moreover, [51] [5] provides a formula for the homotopy groups:
\[
π_n(BD_nG, x) = \text{Aut}_{C_n(\text{D}_nG)}(\text{Id}_x)
\]
where \( C_n(\text{D}_nG) \) is the groupoid \( W_{(n,n-1)}G \). This matches 4.7.

6. WEAKLY GLOBULAR PSEUDO \( n \)-FOLD GROUPOIDS

We now introduce the category \( \text{PsGpd}^{n}_{wg} \) of weakly globular pseudo \( n \)-fold groupoids, and prove Theorem 6.23 stating that there is a zig-zag of weak equivalences between any \( X ∈ \text{PsGpd}^{n}_{wg} \) and \( \hat{Q}^{(n)}_0BX \). This implies our second main result (Theorem 6.28), stating that \( \hat{Q}^{(n)}_0 \) induces an equivalence \( \text{ho P}^n\text{Top} ∼= \text{ho PsGpd}^{n}_{wg} \).

6.1. Definition. For each \( n \), we introduce a full subcategory \( \text{PsGpd}^{n}_{hd} \) of \( [\Delta^{n-1^{op}}, \text{Gpd}] \), whose objects are called homotopically discrete pseudo \( n \)-fold groupoids. These categories are defined by induction on \( n \geq 1 \), as follows:
(a) \( \text{PsGpd}^{1}_{hd} = \text{Gpd}^{1}_{hd} \) consists of the homotopically discrete groupoids.
(b) If \( X ∈ \text{PsGpd}^{n}_{hd} \), then \( X_k ∈ \text{PsGpd}^{n-1}_{hd} \) for all \( k ≥ 0 \), where \( k \) is the simplicial dimension in the first direction (cf. 3.19).
(c) If \( X ∈ \text{PsGpd}^{n}_{hd} \), the \((n-1)\)-simplicial set \( π_0^{(n)}J_nX \) is the nerve of an object \( Π_0^{(n)}X \) of \( \text{PsGpd}^{n-1}_{hd} \) and the following diagram commutes (where \( J_n \) denotes the inclusion)

\[
\begin{array}{ccc}
\text{PsGpd}^{n}_{hd} & \xrightarrow{J_n} & [\Delta^{n-1^{op}}, \text{Gpd}] \\
\Pi_0^{(n)} & \downarrow & \pi_0^{(n)} \\
\text{PsGpd}^{n-1}_{hd} & \xrightarrow{J_{n-1}} & [\Delta^{n-2^{op}}, \text{Gpd}] \\
\end{array}
\]

Furthermore, the map \( \gamma \) of 3.12 induces a map \( γ^{(n)} : X → ϵ^{(n)}Π_0^{(n)}X \) in \( \text{PsGpd}^{n}_{hd} \) which is a weak equivalence of groupoids in each multi-simplicial dimension (and thus a geometric weak equivalence by Remark 2.22).
(d) For each $k \geq 2$, the induced Segal map:

$$X_k \xrightarrow{\tilde{\mu}_k} X_1 \times X_0 \times \cdots \times X_2 X_1$$

of (1.4) is a geometric weak equivalence.

Note that condition (c) implies that the composite $\gamma_{(n)}$ of

$$X \xrightarrow{\gamma_{(n)}} c^{(n)} \Pi_0^{(n)} X \xrightarrow{c^{(n)} \gamma_{(n-1)}} \cdots \xrightarrow{c^{(1)} \cdots c^{(n)} \Pi_0^{(1)} \cdots \Pi_0^{(n)} X},$$

is a geometric weak equivalence, so that $BX$ is a homotopically trivial simplicial set (i.e., a 0-type).

6.4. Definition. We now use Definition 6.1 to specify, for each $n \geq 1$, another full subcategory $\text{PsGpd}_{wg}^n$, of $[\Delta^{n-1}, \text{Gpd}]$, whose objects are called weakly globular pseudo $n$-fold groupoids, defined by induction on $n \geq 1$.

(a) $\text{PsGpd}_{wg}^1 := \text{Gpd}$.

(b) If $X \in \text{PsGpd}_{wg}^n$, then $X_0 \in \text{PsGpd}_{hd}^{n-1}$ and $X_k \in \text{PsGpd}_{wg}^{n-1}$ for all $k \geq 1$.

(c) If $X \in \text{PsGpd}_{hd}^n$, the $(n-1)$-simplicial set $\Pi_0^{(n)} J_n X$ is the nerve of an object $\Pi_0^{(n)} X$ of $\text{PsGpd}_{hd}^{n-1}$ and the following diagram commutes (where $J_n$ denotes the inclusion)

$$\begin{array}{ccc}
\text{PsGpd}_{wg}^n & \xrightarrow{J_n} & [\Delta^{n-1}, \text{Gpd}] \\
\Pi_0^{(n)} & \xrightarrow{J_{n-1}} & [\Delta^{n-2}, \text{Gpd}] \\
\text{PsGpd}_{wg}^{n-1} & \xrightarrow{J_n} & [\Delta^{n-2}, \text{Set}] \\
\end{array}$$

Furthermore, $\Pi_0^{(n)}$ preserves geometric weak equivalences.

(d) For each $k \geq 2$, the induced Segal map

$$X_k \xrightarrow{\tilde{\mu}_k} X_1 \times X_0 \times \cdots \times X_2 X_1$$

is a geometric weak equivalence.

6.5. Remark. Both $\text{Tam}^n$ and $\text{Gpd}_{wg}^n$ are full subcategories of $\text{PsGpd}_{wg}^n$, and $\text{Gpd}_{hd}^n$ is a full subcategory of $\text{PsGpd}_{hd}^n$.

6.6. Example. When $n = 2$, a weakly globular pseudo double groupoid is just a simplicial object in groupoids $X \in [\Delta^{2op}, \text{Gpd}]$ such that $X_0$ is a homotopically discrete groupoid, the simplicial set $\Pi^{(2)}_0 X$ is the nerve of a groupoid, and for each $k \geq 2$, the induced Segal map

$$X_k \xrightarrow{\tilde{\mu}_k} X_1 \times X_0 \times \cdots \times X_2 X_1$$

is an equivalence of groupoids.

6.7. Lemma. If $f : X \to Y$ is a map in $\text{PsGpd}_{wg}^n$, and $Y$ is discrete (that is, the constant $(n-1)$-fold simplicial object on a discrete groupoid), then $X$ is the coproduct in $\text{PsGpd}_{wg}^n$ of the fibers $X^{-1}(a)$, taken over all $a \in Y$. 
Proof. By induction on \( n \geq 1 \), where for \( n = 1 \), \( X \) is a groupoid, which is a coproduct of its connected components. The \( n \)-th step follows from the \( (n-1) \)-st, since coproducts in \( \text{PsGpd}^n_{\text{wg}} \) are those of \([\Delta^{n-1}, \text{Gpd}]\), namely, disjoint unions, which are therefore taken dimensionwise. \( \square \)

6.8. Corollary. If \( X \in \text{PsGpd}^n_{\text{wg}} \), then \( X_1 \) is isomorphic to the coproduct in \( \text{PsGpd}^{n-1}_{\text{wg}} \) of \( X_1(a,b) \) (the fiber of \((\gamma_n(d_0),\gamma_n(d_1)) : X_1 \to X_0^d \times X_0^d\), taken over all \((a,b) \in X_0^d \times X_0^d\)).

6.9. Definition. We now define the notion of \( n \)-equivalence for maps of weakly globular pseudo \( n \)-fold groupoids by induction on \( n \geq 1 \), where a \( 1 \)-equivalence is simply an equivalence of groupoids:

A map \( f : X \to Y \) in \( \text{PsGpd}^n_{\text{wg}} \) is an \( n \)-equivalence if:

(a) \( \Pi^{(n)}_0 f : \Pi^{(n)}_0 X \to \Pi^{(n)}_0 Y \) is an \((n-1)\)-equivalence in \( \text{PsGpd}^{n-1}_{\text{wg}} \);

(b) For every \( a,b \in X_0^d \), the map \( f(a,b) : X_1(a,b) \to Y_1(f(a),f(b)) \) is also an \((n-1)\)-equivalence in \( \text{PsGpd}^{n-1}_{\text{wg}} \).

6.B. Comparison with Tamsamani’s weak \( n \)-groupoids

We describe a procedure for transforming a weakly globular pseudo \( n \)-fold groupoid \( X \) into a Tamsamani weak \( n \)-groupoid, without altering the homotopy type. The construction is done in two stages:

In the first, we use the general construction of \([5.9]\) to produce \( \text{Disc}_0 X \in \text{PsGpd}^n_{\text{wg}} \), in which only \( X_0 \) is discretized (as in Subsection 5.B). This time we must proceed by induction on the \( n \) simplicial directions in order to obtain a zig-zag of intermediate objects (in Lemma 6.21), all weakly equivalent in \( \text{PsGpd}^n_{\text{wg}} \) (which was not possible in \( \text{Gpd}^n_{\text{wg}} \)).

In the second stage, we define the full discretization functors \( D_n : \text{PsGpd}^n_{\text{wg}} \to \text{Tam}^n \) by induction on \( n \geq 2 \), with \( D_2 := \text{Disc}_0 \), so as to make each \( X_k \) a Tamsamani weak \((n-1)\)-groupoid.

First, we need some technical facts about weakly globular pseudo \( n \)-fold groupoids:

6.10. Lemma. If \( f : X \to Y \) is a map in \( [\Delta^{op}, \text{PsGpd}^{n-1}_{\text{wg}}] \) which is a weak equivalence in each simplicial dimension, with \( Y_0 \in \text{PsGpd}^{n-1}_{\text{id}} \) and \( X \in \text{PsGpd}^n_{\text{wg}} \), then for each \( k \geq 2 \) the induced Segal maps of \([6.2]\) for \( Y \) are geometric weak equivalences.

Proof. First note that \( f \) induces an isomorphism \( X_0^d \cong Y_0^d \), so by Corollary 6.8 \( f_1 : X_1 \to Y_1 \) is the coproduct over \((a,b) \in X_0^d \times X_0^d\) of its restrictions \( f_1(a,b) : X_1(a,b) \to Y_1(a,b) \). Since the classifying space functor \( B : \text{PsGpd}^n_{\text{wg}} \to \text{Top} \) commutes with disjoint unions, the fact that \( f_1 \) is a weak equivalence implies that each \( f_1(a,b) \) is a geometric weak equivalence in \( \text{PsGpd}^{n-1}_{\text{wg}} \).

Moreover, since \( X_0^d \) is discrete,

\[
X_1 \times X_0^d \times X_0^d \times X_0^d \times \cdots \times X_0^d \cong \prod_{a_0,\ldots,a_k \in X_0^d} X_1(a_0,a_1) \times X_1(a_1,a_2) \times \cdots \times X_1(a_{k-1},a_k)
\]

and similarly for \( Y \).
Now consider the commutative diagram of induced Segal maps:

\[
\begin{array}{ccc}
X_k & \xrightarrow{\tilde{\mu}_k^X} & X_1 \times X_0^d \times \cdots \times X_0^d X_1 \\
\downarrow f_k & \simeq & \downarrow (f_1 \times \cdots \times f_1) \\
Y_k & \xrightarrow{\tilde{\mu}_k^Y} & Y_1 \times Y_0^d \times \cdots \times Y_0^d Y_1
\end{array}
\]

where the left vertical map is a geometric weak equivalence by assumption, as is the top horizontal induced Segal map (since \(X \in \text{PsGpd}^{\text{hd}}_n\)), while the right horizontal map is a geometric weak equivalence because of (6.11). Therefore, \(\mu_k^Y\) is a geometric weak equivalence, too. \(\square\)

6.13. **Lemma.** Consider a pushout diagram in \([\Delta^{n-1} \op, \text{Gpd}]\):

\[
\begin{array}{ccc}
A & \xrightarrow{j} & B \\
\downarrow \gamma^{(n)} & \simeq & \downarrow g \\
\Pi_0^{(n)} A & \xrightarrow{h} & C
\end{array}
\]

with \(A \in \text{PsGpd}^{\text{hd}}_n\) and \(j\) monic. Then:

(a) If \(B \in \text{PsGpd}^{\text{hd}}_n\), so is \(C\).

(b) If \(B \in \text{PsGpd}^{\text{hd}}_n\), so is \(C\).

**Proof.** By induction on \(n \geq 1\):

First note that for any \(n \geq 1\), \(g\) is a geometric weak equivalence, since \(f\) is, because \(\text{Diag}_{(n)}\) preserves pushouts, \(A\) is in \(\text{PsGpd}^{\text{hd}}_n\), and \(\text{Diag}_{(n)} j\) is a cofibration of simplicial sets.

When \(n = 1\), (6.14) is a diagram of groupoids, so (a) is clear, and (b) follows from [37, Corollary 3].

In general, since the pushout is taken in a diagram category, \(C_0\) is the pushout of the objects in simplicial dimension 0, which is therefore in \(\text{PsGpd}^{\text{hd}}_n\) by (b) for \(n - 1\), while for \(k \geq 1\), \(C_k\) is in \(\text{PsGpd}^{\text{hd}}_n\) by (a) for \(n - 1\).

Since the functor \(\Pi_0^{(n)}\) is defined by applying \(\pi_0\) to each groupoid, \(\pi_0 \circ \gamma\) commutes with pushouts of groupoids, and \(\pi_0 \gamma\) is an isomorphism, we see that \(\Pi_0^{(n)} C = \Pi_0^{(n)} B\) is in \(\text{PsGpd}^{\text{hd}}_n\) by (a) for \(n - 1\).

Finally, the Segal condition follows from Lemma 6.10 for \(g\), since \(g_k\) is a weak equivalence for each \(k \geq 0\), \(B \in \text{PsGpd}^{\text{hd}}_n\), and \(C_0 \in \text{PsGpd}^{\text{hd}}_n\).

This shows (a). (b) is immediate. \(\square\)

6.15. **Proposition.** Assume given a weakly globular pseudo \(n\)-fold groupoid \(X\), and let \(Y \in [\Delta^{n-1} \op, \text{Gpd}]\) be the result of applying the construction of \([57]\) to the map \(\gamma\): \(X_0 \to W\) for \(W := (c^{(n)} \Pi_0^{(n)} X)_0\) and \(\mathcal{C} = [\Delta^{n-1} \op, \text{Gpd}]\); then \(Y\) is actually in \(\text{PsGpd}^{\text{hd}}_n\). Moreover, the maps

\[
X \xrightarrow{f} Y \xleftarrow{h} X^\gamma
\]
are geometric weak equivalences in $\text{PsGpd}^n_{\text{wg}}$, where $X^\gamma$ is as in §5.9.

Proof. First, note that $Y_0 := W$ is in $\text{PsGpd}^n_{\text{hd}}$, by §6.4. Furthermore, for any $k \geq 1$ $Y_k$ is defined by the pushout square of (5.10):

$$
\begin{array}{ccc}
X_0 & \xrightarrow{s(k)} & X_k \\
\gamma & \downarrow & \downarrow f_k \\
(c^{(n)}\Pi_0^{(n)}X)_0 & \xrightarrow{\sigma(k)} & Y_k
\end{array}
$$

(6.17)

where $\gamma$ is a geometric weak equivalence since $X_0$ is in $\text{PsGpd}^n_{\text{hd}}$, and the iterated degeneracy map $s(k)$ is one-to-one since it has a left inverse $d(k)$. Thus by Lemma 6.13, $Y_k \in \text{PsGpd}^n_{\text{wg}}$.

The maps $f_k$ in (6.17) are geometric weak equivalences, since after applying $\text{Diag}_{(n)}$ we obtain a pushout of a weak equivalence along a cofibration in $[\Delta^{op}, \text{Set}]$. Therefore, by Lemma 6.10 applied to $f$, the induced Segal maps for $Y$ are weak equivalences.

Finally, $\Pi_0^{(n)}Y$ is obtained by applying $\pi_0$ to each groupoid of $Y \in [\Delta^{n-1}^{op}, \text{Gpd}]$, and since this commutes with pushouts and $\pi_0\gamma$ is an isomorphism, we see that $\Pi_0^{(n)}Y \cong \Pi_0^{(n)}X$, so in particular it is in $\text{PsGpd}^n_{\text{wg}}$. This shows that $Y \in \text{PsGpd}^n_{\text{wg}}$.

Since each $f_k$ is a geometric weak equivalence, as is $h_0 = \gamma$ and $h_k = \text{Id}$ for $k \geq 1$, the two maps $f$ and $h$ are geometric weak equivalences in $\text{PsGpd}^n_{\text{wg}}$.

6.18. Notation. Let $T^{(n)}_{ps}: \text{PsGpd}^n_{\text{wg}} \to \text{Gpd}$ denote the fundamental groupoid functor for $\text{PsGpd}^n_{\text{wg}}$ — that is, the composite

$$
T^{(n)}_{ps} := \Pi_0^{(2)} \cdots \Pi_0^{(n-1)}\Pi_0^{(n)}
$$

(6.19)

6.20. Definition. For each $n \geq 2$ we define a sequence of functors $\text{Disc}^{(k)}_0: \text{PsGpd}^n_{\text{wg}} \to \text{PsGpd}^n_{\text{wg}}$ ($1 \leq k \leq n$) by setting $\text{Disc}^{(k)}_0 X := X^{\gamma^{(k)}}$ (in the notation of §6.9), where

$$
\gamma^{(k)}: X_0 \to (c^{(k)} \cdots c^{(n)}\Pi_0^{(k)} \cdots \Pi_0^{(n)}X)_0
$$

is the composite of the first $k$ maps of (6.3) in dimension 0. We write $\text{Disc}_0$ for $\text{Disc}^{(1)}_0$.

6.21. Lemma. For each $X \in \text{PsGpd}^n_{\text{wg}}$ we have a sequence of natural geometric weak equivalences

$$
\begin{array}{ccc}
X & \xrightarrow{f^{(n)}} & \text{Disc}^{(n)}_0 X \\
\text{Disc}^{(n)}_0 Y & \xrightarrow{f^{(n-1)}} & \text{Disc}^{(n-1)}_0 Y \\
Y^{(n)} & \xrightarrow{h^{(n)}} & Y^{(n-1)} \\
\text{Disc}^{(n-1)}_0 Y & \xrightarrow{h^{(n-1)}} & \text{Disc}^{(n-1)}_0 Y \\
\cdots & \cdots & \cdots \\
Y^{(1)} & \xrightarrow{h^{(1)}} & Y^{(1)}
\end{array}
$$

(6.18)

Proof. Each $Y^{(k)}$ is obtained by applying Proposition 6.15 to $\text{Disc}^{(k+1)}_0 X$, where $\text{Disc}^{(n+1)}_0 X := X$, and using (6.10).
6.22. Definition. We now define discretization functors

\[
D_n : \text{PsGpd}^n_{\text{wg}} \to [\Delta^n_-, \text{Gpd}]
\]

for each \( n \geq 1 \) by induction on \( n \), starting with \( D_1 := \text{Id} : \text{Gpd} \to \text{Gpd} \). For \( n \geq 2 \), we define \( D_n \) inductively to be the composite:

\[
\text{PsGpd}^n_{\text{wg}} \to [\Delta^{op}, \text{PsGpd}^{n-1}_{\text{wg}}] \xrightarrow{\text{Disco}} [\Delta^{op}, \text{PsGpd}^{n-1}_{\text{wg}}] \to [\Delta^{n-1}_-, \text{Gpd}]
\]

where \( \overline{D}_{n-1} \) is obtained by applying \( D_{n-1} \) in each simplicial dimension.

Note that \( D_2 \) is simply \( \text{Disc}_0 : \text{PsGpd}^2_{\text{wg}} \to \text{Tam}^2 \).

6.23. Theorem. The functor \( D_n \) lands in \( \text{Tam}^n \) and preserves geometric weak equivalences and fiber products over discrete objects. Moreover, for every weakly globular pseudo \( n \)-fold groupoid \( X \in \text{PsGpd}^n_{\text{wg}} \), the groupoid \( T^{(n)} X \) is isomorphic to \( T^{\text{ps}}_n X \), and there is a zigzag of weak equivalences in \( \text{PsGpd}^n_{\text{wg}} \) between \( D_n X \) and \( X \).

Proof. By induction on \( n \geq 2 \). For \( n = 2 \), \( D_2 X = \text{Disc}_0 X \) is clearly in \( \text{Tam}^2 \) for any \( X \in \text{PsGpd}^2_{\text{wg}} \).

In the induction step, note that \( (D_n X)_0 = X_0^d \) is discrete and \( (D_n X)_k = D_{n-1} X_k \) is in \( \text{Tam}^{n-1} \), by induction. So to prove that \( D_n X \) is in \( \text{Tam}^n \), it remains to show:

(a) The Segal maps

\[
\mu_k : (D_n X)_k \to (D_n X)_1 \times_{(D_n X)_0} \cdots \times_{(D_n X)_0} (D_n X)_1
\]

are \( (n-1) \)-equivalences.

(b) \( \Pi_{(n)}^0 D_n X \) is in \( \text{Tam}^{n-1} \).

To show (a), note that since \( X \in \text{PsGpd}^n_{\text{wg}} \), the induced Segal maps

\[
X_k \xrightarrow{\sim} X \times_{X_0^d} \cdots \times_{X_0^d} X_1
\]

are geometric weak equivalences for all \( k \geq 2 \). Since by induction \( D_{n-1} \) preserves geometric weak equivalences, we have weak equivalences:

\[
D_{n-1} X_k \xrightarrow{\sim} D_{n-1} (X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1)
\]

Moreover, \( (D_n X)_1 = D_{n-1} X_1 \) and \( (D_n X)_0 = X_0^d \) is discrete, so the right hand side is an iterated fiber product over discrete objects, and thus (again by induction)

\[
D_{n-1} (X_1 \times_{X_0^d} \cdots \times_{X_0^d} X_1) = (D_n X)_1 \times_{(D_n X)_0} \cdots \times_{(D_n X)_0} (D_n X)_1
\]

which proves (a) for \( n \).

To show (b), by [5.2] and Proposition [5.5] it suffices to show that \( T^{(n)}_n D_n X \) is a groupoid, which we do by showing that it is isomorphic to \( T^{\text{ps}}_n X \). We have

\[
(T^{(n)}_n D_n X)_0 = \pi_0 T^{(n)}_n (D_n X)_0 = \pi_0 T^{(n)}_n X_0^d = X_0^d.
\]

and

\[
(T^{(n)}_n D_n X)_k = \pi_0 T^{(n)}(D_n X)_k = \pi_0 T^{(n)}_n D_{n-1} X_k
\]

\[= \pi_0 T^{(n)}_n X_k = (T^{\text{ps}}_n X)_k.\]
for $k \geq 1$, where we use the induction hypothesis for the equality before last.

It follows that $T_n^{Tm}D_nX = T_n^{ps}X$, and since the latter is a groupoid, so is $T_n^{Tm}D_nX$. This concludes the proof that $D_nX$ is in $\text{Tam}^n$.

Finally, we obtain the required natural zig-zag of geometric weak equivalences:

$$D_nX \rightarrow \ldots \leftarrow \text{Disc}_0X \rightarrow \ldots \leftarrow X,$$

by induction on $n \geq 1$, where the right-hand zig-zag is provided by Lemma 6.21.

For $n = 1$, we have $D_1X = X$, while for $n \geq 2$ we use Definition 6.22 to identify $(D_nX)_k$ with $(\overline{D_{n-1}X})_k$ for $k \geq 1$.

\[
\begin{array}{ccc}
D_nX & \cdots D_{n-1}X_2 & D_{n-1}X_1 & X^d_0 \\
\downarrow & \cdots & \downarrow & \downarrow \\
\downarrow & \cdots & \downarrow & \downarrow \\
\text{Disc}_0X & \cdots X_2 & X_1 & X^d_0
\end{array}
\]

using the induction to obtain the righthand vertical zig-zag in each simplicial dimension. □

6.25. **Remark.** The functor $D_n: \text{PsGpd}^n_{wg} \rightarrow \text{Tam}^n$ extends the functor $D_n: \text{Gpd}^n_{ho} \rightarrow \text{Tam}^n$ of [5.18]

6.26. **Remark.** It follows from Theorem 6.23 that if $X \in \text{PsGpd}^n_{wg}$, $BX$ is an $n$-type.

6.27. **Definition.** Let $\text{ho PsGpd}^n_{wg}$ denote the localization of the category $\text{PsGpd}^n_{wg}$ with respect to the geometric weak equivalences.

6.28. **Theorem.** The functors $\hat{Q}_{(n)}: \text{Top} \rightarrow \text{Gpd}^n_{wg}$ and $B: \text{PsGpd}^n_{wg} \rightarrow \text{Top}$, together with the inclusion $J: \text{Gpd}^n_{ho} \hookrightarrow \text{PsGpd}^n_{wg}$, induce equivalences of categories

\[
\text{ho P}^n\text{Top} \xleftarrow{B} \text{ho PsGpd}^n_{wg}.
\]

Moreover, for every $T \in \text{Top}$, there is a zigzag of weak equivalences in $\text{P}^n\text{Top}$ between $P^nT$ and $B\hat{Q}_{(n)}T$, and for $X \in \text{PsGpd}^n_{wg}$ there is a zig-zag of geometric weak equivalences in $\text{PsGpd}^n_{wg}$ between $X$ and $\hat{Q}_{(n)}BX$.

**Proof.** All three functors preserve weak equivalences, so we have induced functors as in (6.29). For any $n$-type $T$, we have an isomorphism in $\text{ho P}^n\text{Top}$ between $T$ and $B\hat{Q}_{(n)}T$ by Theorem 4.32 which also implies (see Remark 6.26) that for any $X \in \text{PsGpd}^n_{wg}$ we have a homotopy equivalence (of CW complexes) in $\text{Top}$:

\[
BX \xrightarrow{\cong} B\hat{Q}_{(n)}BX.
\]
By Theorem \ref{thm:zig-zag} we also have zig-zags of geometric weak equivalences in \( \text{PsGpd}^n_{\text{wg}} \):

\begin{equation}
\label{eq:zig-zag}
D_n X \rightarrow \ldots \leftarrow X \quad \text{and} \quad D_n \tilde{Q}_{(n)} BX \rightarrow \ldots \leftarrow \tilde{Q}_{(n)} BX
\end{equation}

Therefore, after applying \( B \) to \eqref{eq:zig-zag} we have homotopy equivalences of CW complexes:

\[
BD_n X \xrightarrow{\sim} BX \quad \text{and} \quad B\tilde{Q}_{(n)} BX \xrightarrow{\sim} BD_n \tilde{Q}_{(n)} BX .
\]

Combining these with \eqref{eq:weak-equivalence} yields a weak equivalence

\[
BD_n X \rightarrow BD_n \tilde{Q}_{(n)} BX
\]

in \( \text{Top} \), which by Theorem \ref{thm:equivalence} implies that \( D_n X \) and \( D_n \tilde{Q}_{(n)} BX \) are isomorphic in \( \text{ho Tam}^n \), and thus in \( \text{ho PsGpd}^n_{\text{wg}} \). By \eqref{eq:zig-zag}, we see that \( X \) and \( J\tilde{Q}_{(n)} BX \) are weakly equivalent through a zig-zag in \( \text{PsGpd}^n_{\text{wg}} \).

\begin{remark}
Note that Theorem \ref{thm:zig-zag} implies that the functor \( D_n \) induces an equivalence of categories

\[ \text{ho PsGpd}^n_{\text{wg}} \simeq \text{ho Tam}^n . \]

Together with Theorem \ref{thm:equivalence} and Theorem \ref{thm:equivalence3} this implies the equivalence of categories \eqref{eq:equivalence}. In the course of the proof of Theorem \ref{thm:zig-zag} we have further shown that any weakly globular pseudo \( n \)-fold groupoid \( X \in \text{PsGpd}^n_{\text{wg}} \) has two different functorial partial strictifications: the Tamsamani weak \( n \)-groupoid \( D_n X \), and the weakly globular \( n \)-fold groupoid \( \tilde{Q}_{(n)} BX \in \text{Gpd}^n_{\text{wg}} \), each equipped with zig-zags of weak equivalences in \( \text{PsGpd}^n_{\text{wg}} \) from \( X \):

\begin{equation}
\label{eq:zig-zag2}
D_n X \rightarrow \ldots \leftarrow X \rightarrow \ldots \leftarrow \tilde{Q}_{(n)} BX .
\end{equation}

\end{remark}

\begin{definition}
As in \ref{sec:algebraic-homotopy}, for any weakly globular pseudo \( n \)-fold groupoid \( X \) and \( 1 \leq k \leq n \), we define its \( k \)-fold object of arrows to be the pseudo \((n-k)\)-fold groupoid \( W_{(n,k)} \): \( X_{(1...k)}^{(1,...k)} \).

\begin{equation}
\omega_k(X; x_0) \cong \begin{cases} W_{(n,n)}(x_0, x_0) & \text{if } k = n \\
W_{(n-k,n-k)}(\Pi_0^{(k+1)} \ldots \Pi_0^{(n)} X)(x_0, x_0) & \text{if } 0 < k < n
\end{cases}
\end{equation}

with the 0-th algebraic homotopy set of \( X \) defined:

\[
\omega_0(X) := \Pi_0^{(1)} \ldots \Pi_0^{(n)} X .
\]

A map \( f : X \rightarrow Y \) of weakly globular pseudo \( n \)-fold groupoids is called an algebraic weak equivalence if it induces bijections on the \( k \)-th algebraic homotopy groups (set) for all \( x_0 \in X_{0...n} \) and \( 0 \leq k \leq n \).

\begin{remark}
As for weakly globular \( n \)-fold groupoids (see Remark \ref{rem:algebraic-homotopy}), our definition of algebraic homotopy groups for \( \text{PsGpd}^n_{\text{wg}} \) generalizes that of \cite[\S5]{algebraic}, and since \( D_n X \) and \( X \) by Remark \ref{rem:algebraic-homotopy2} have the same algebraic homotopy groups, by construction, both provide an algebraic way of calculating the homotopy groups of \( BX \), as in Theorem \ref{thm:algebraic}.\end{remark}
Using this fact, one can show that a map \( f: X \rightarrow Y \) in \( \text{PsGpd}_{\text{wg}}^n \) is an \( n \)-equivalence (Definition 6.5) if and only if it is a geometric weak equivalence.

7. Applications and Further Directions

In this section we provide an application for our model of \( n \)-types, and indicate some directions for future work.

7A. Modelling \((k-1)\)-Connected \(n\)-Types

We now provide an algebraic model of \((k-1)\)-connected \(n\)-types, and relate it to the homotopy types of iterated loop spaces. This was mentioned in [22] as a desirable feature for models of \( n \)-types (see also [11]).

Recall that a space \( X \) is \((k-1)\)-connected if \( \pi_0 X = 0 \) and \( \pi_i(X,x) = 0 \) for \( 1 \leq i \leq k-1 \), and all \( x \in X \). We denote the category of \((k-1)\)-connected pointed \( n \)-types by \( \text{PsGpd}_{\text{wg}}^n \).

7.1. Lemma. If \( X \) is a \( k \)-connected pointed Kan complex, \( X \) is naturally weakly equivalent to a \((k-1)\)-reduced Kan complex \( \hat{X} \) -- that is, \( \hat{X}_i = \{\ast\} \) for \( 1 \leq i \leq k-1 \).

Proof. See [33] III, §3.

7.2. Definition. For any \( k \)-connected pointed topological space \( T \in \text{Top}_* \), let \( \text{St}_T \) denote the canonical \( k \)-reduced version \( \text{St}(T) \) of the singular set \( S(T) \).

7.3. Definition. A homotopically discrete pseudo \( n \)-fold groupoid \( X \in \text{PsGpd}_{\text{hd}}^n \) is contractible if \( \pi_0 BX \) is trivial (so that \( BX \) is contractible).

More generally, a weakly globular pseudo \( n \)-fold groupoid \( X \in \text{PsGpd}_{\text{wg}}^n \) is called \((n,k)\)-weakly globular if for each \( 0 \leq r < k \), the homotopically discrete pseudo \((n-r-1)\)-fold groupoid \( X^{(1\ldots r+1)}_{1\ldots r+1} = (W_{(n,r),X}^{(r+1)})_0 \) is contractible. This is the pseudo \((n-r-1)\)-fold groupoid of objects of the pseudo \((n-r)\)-fold groupoid \( W_{(n,r),X} \in \text{PsGpd}_{\text{wg}}^{n-r} \) (see [6, 34]).

In particular, when \( r = 0 \), this just means that the pseudo \((n-1)\)-fold groupoid of objects \( X_0^{(n)} \) of \( X \) in the \( n \)-th direction (which is a homotopically discrete pseudo \((n-1)\)-fold groupoid) is in fact contractible.

We let \( \text{PsGpd}_{\text{wg}}^{(n,k)} \) denote the full subcategory of \((n,k)\)-weakly globular pseudo \( n \)-fold groupoids in \( \text{PsGpd}_{\text{wg}}^n \). Similarly, \( \text{Gpd}_{\text{wg}}^{(n,k)} \) is the full subcategory of \((n,k)\)-weakly globular pseudo \( n \)-fold groupoids in \( \text{Gpd}_{\text{wg}}^n \).

We now want to show that \( \text{PsGpd}_{\text{wg}}^{(n,k)} \) is an algebraic model of \((k-1)\)-connected \( n \)-types. For this, we need the following:

7.4. Lemma. If \( X \) is a \((k-1)\)-reduced Kan complex, then \( Q_{(n)}X \) is \((n,k)\)-weakly globular.

Proof. By Lemma 4.14(b), \( (Q_{(n)}X)^{(n)}_0 = Q_{(n-1)} \text{Dec} X \). Since \( \text{Dec} X \simeq c(X_0) = c(\ast) \) and \( Q_{(n-1)}X \) preserves weak equivalences of Kan complexes by Proposition 4.28(b), we have \( Q_{(n-1)} \text{Dec} X \simeq Q_{(n-1)}(\ast) = \ast \). Therefore, \( dV(Q_{(n)}X)^{(n)}_0 \) is contractible.
We now show by induction on \(1 \leq r < k\) that
\[
W_{\langle n, r \rangle} Q_{\langle n-1 \rangle} X := (N^{(n-r+1, \ldots, n)} Q_{\langle n \rangle} X)^{(n-r+1, \ldots, n)} = Q_{\langle n-r \rangle} L^r_1 X
\]
(in the notation of \(\text{(3.11)}\), where \(L^r_1 X := L^{r-1}_1 L^1_1 X\) for \(r \geq 2\), and \(L^1_1 X = L_1 X\)). The case \(r = 1\) is Lemma \(\text{(4.14)}\) for \(k = 1\), which implies that we have an isomorphism of \((n-1)\)-fold groupoids:
\[
W_{\langle n, 1 \rangle} Q_{\langle n-1 \rangle} X := (N^{(n)} Q_{\langle n \rangle} X)^{(n)} = Q_{\langle n-1 \rangle} L_1 X.
\]

In the induction step, since \(L_1 X\) is still a Kan complex, by \(\text{(3.25)}\) we can apply the induction hypothesis to the right hand side of \(\text{(7.0)}\) (using the fact that \(W_{\langle n, r \rangle} = W_{\langle n-1, r-1 \rangle} W_{\langle n, 1 \rangle}\), by \(\text{(3.25)}\), to deduce that:
\[
W_{\langle n, r \rangle} Q_{\langle n-1 \rangle} L_1 X \cong Q_{\langle n-r \rangle} L^{r-1}_1 (L_1 X),
\]
which yields \(\text{(7.5)}\). From this and Lemma \(\text{(4.14)}\) (for \(k = 0\)) we have
\[
(W_{\langle n, r \rangle} Q_{\langle n \rangle} X)_0 = Q_{\langle n-r \rangle} \Delta^0 \cong \text{Dec } L^r_1 X,
\]
and since \(\Delta^0 \cong \text{Dec } L^r_1 X\), we have \(dN(W_{\langle n, r \rangle} Q_{\langle n \rangle} X)_0 \cong c(L^r_1 X_0)\).

Note that since \(X\) is \((k-1)\)-reduced, \(\text{Dec } X\), and thus \(L_1 X\) are \((k-2)\)-reduced, so by induction \(L^r_1 X\) is \((k-r-1)\)-reduced. Thus as long as \(r < k\), \(L^r_1 X\) is 0-reduced, so \(dN(W_{\langle n, r \rangle} Q_{\langle n \rangle} X)_0\) is contractible.

\[\text{7.7. Proposition.}\] The functors \(Q_{\langle n \rangle}\) and \(B\) induce equivalences of categories:
\[
\text{ho } P^n_k \text{Top}_* \to \mathcal{S}^{\text{red}} T \to \text{ho } \text{PsGpd}^{(n,k)}_{\text{wg}}.
\]

\[\text{Proof:}\] If \(T \in P^n_k \text{Top}_*\), then \(\mathcal{S}^{\text{red}} T\) is \((k-1)\)-reduced, so by Lemma \(\text{7.5}\) \(Q_{\langle n \rangle} X \in \text{PsGpd}^{(n,k)}_{\text{wg}}\). The result follows immediately from Theorem \(\text{6.28}\). \(\square\)

\[\text{7.8. Remark.}\] Note that the composition of \(W_{\langle n, k \rangle}\) of \(\text{6.32}\) with the classifying space functor \(B\) lands in \(P^n_k \text{Top}_*\), by Theorem \(\text{4.32}\) so its restriction to \(\text{PsGpd}^{(n,k)}_{\text{wg}}\) lands in the category \(P^{n-k} \text{Top}\) of \((n-k)\)-types.

Moreover, if \(T = \Omega^k Y\) is an \((n-k)\)-type \(k\)-fold loop space, applying the \(k\)-fold delooping functor \(E_{\langle k \rangle} : P^{n-k}_\Omega \to P^n_k \text{Top}_*\) of \(\text{12. Theorem 13.1}\) yields the \((k-1)\)-connected \(n\) type \(Y = E_{\langle k \rangle} T \in P^n_k \text{Top}_*\). In fact:

\[\text{7.9. Proposition.}\] For any \((k-1)\)-connected \(n\) type \(Y \in P^n_k \text{Top}_*\), we have a zigzag of weak equivalences in \(P^{n-k} \text{Top}\) between \(BW_{\langle n, k \rangle} \hat{Q}_{\langle n \rangle} Y\) and \(\Omega^k Y\), so the weakly globular \((n-k)\)-fold groupoid \(W_{\langle n, k \rangle} \hat{Q}_{\langle n \rangle} Y\) is an algebraic model for \(\Omega^k Y\).

\[\text{Proof:}\] By induction on \(k\). Let \(G := \hat{Q}_{\langle n \rangle} Y \in \text{Gpd}^{(n,k)}_{\text{wg}}\), so \(BG \cong Y\) in \(\text{ho } P^n_k \text{Top}_*\).

For \(k = 1\), consider the simplicial \((n-1)\)-fold groupoid \(N^{(n)} G\). Applying the classifying space functor \(B : \text{Gpd}^{n-1} \to \text{Top}\) in each simplicial dimension yields a simplicial space \(Y_\bullet = (B^{(n)} N^{(n)} G)_\bullet\). Thus \(Y_0 = B(N^{(n)} G)^{(n)}_0\) is contractible, and the Segal maps for \(Y_\bullet\) are isomorphisms (since \(N^{(n)} G_0^{(n)}\) is the nerve of an internal groupoid), hence in particular geometric weak equivalences.
As $G$ is weakly globular, applying the functor $T_{\text{wg}}^{(n)}$ of §5.18 yields a groupoid, and $\pi_0 Y_\bullet = NT_{\text{wg}}^{(n)} G$. Since $Y_0$ is contractible, $\pi_0 Y_\bullet$ is the nerve of a group. Thus $Y_1$ has a homotopy inverse (cf. [26 (6.3,4)]), so it follows from [48 Proposition 1.5] that $Y_1 \simeq \Omega [Y_\bullet]$. That is,

$$BG^{(n)} = BW_{(n,1)} G \simeq \Omega BG.$$

Since $\Omega BG \cong \Omega Y$ in $\text{ho} P_{k-1}^{n-1} \text{Top}_*$, it follows that $BW_{(n,1)} G \cong \Omega Y$ in $\text{ho} P_{k-1}^{n-1} \text{Top}_*$. In the induction step, let

$$H := W_{(n,1)} G = (N^{(n)} G)^{(n)}_1$$

in $\text{Gpd}_{\text{wg}}^{(n-1,k-1)}$, where by the inductive hypothesis $BW_{(n-1,k-1)} H \cong \Omega^{k-1} BH$ in $\text{ho} P_{n-k}^{n-1} \text{Top}_*$. By what we have shown above for $k = 1$ we have $BH = B(N^{(n)} G)^{(n)}_1 \cong \Omega BG$. It follows that there are isomorphisms in $\text{ho} P^{n-k} \text{Top}_*$

$$BW_{(n,k)} G = BW_{(n-1,k-1)} H \cong \Omega^{k-1} BH \cong \Omega^{k-1} (\Omega BG) = \Omega^k BG.$$ 

\section*{7.10. n-Track categories.} For $n \geq 2$, an $n$-track category is a category enriched in weakly globular $n$-fold groupoids $(\text{Gpd}_{\text{wg}}^n, \times)$, with respect to the cartesian monoidal structure. The category of $n$-track categories is denoted by $\text{Track}_n$.

Since $Q_{(n)} \colon [\Delta^{op}, \text{Set}] \to \text{Gpd}_{\text{wg}}^n$ preserves products (see §2.11), it induces a functor

$$S_{(n)} \colon [\Delta^{op}, \text{Set}] \cdot \text{Cat} \to \text{Track}_n$$

from simplicial categories to $n$-track categories. Furthermore, the functors $\Pi_{(n)}^{(n)}_0 : \text{Gpd}_{\text{wg}}^n \to \text{Gpd}_{\text{wg}}^{n-1}$ giving the Postnikov decomposition of $\text{Gpd}_{\text{wg}}^n$ induce functors

$$P^{n-1} : \text{Track}_n \to \text{Track}_{n-1}$$

providing the Postnikov decomposition of simplicially enriched categories.

For $n = 1$, the corresponding $k$-invariant was described in [10] in terms of the Baues-Wirsching cohomology of categories, and a similar result was obtained in [12] for $n = 2$, using an algebraically-defined cohomology of track categories. The extension of this to general $n$ via an appropriate cohomology of $(n-1)$-track categories will be investigated in the future.
7.11. **Spectral sequences.** In [6], the authors introduced the notion of the Postnikov $n$-stem $\mathcal{P}[n]X$ of a topological space $X$—that is, the system of $(k-1)$-connected $(n+k)$-Postnikov sections $P^{n+k}X(k-1)$ ($k = 0, 1, \ldots$), with the natural maps between them.

They then show that the $E^{n+2}$-term of the homotopy spectral sequence of a (co)simplicial space $W_\bullet$ (respectively, $W^\bullet$) depends only on the simplicial $n$-stems $\mathcal{P}[n]W_\bullet$ or $\mathcal{P}[n]W^\bullet$. Thus we can in principle use the $(n+k)$-fold groupoid models of each $W_m$ or $W^m$, as in §7.3 to extract information about the $d^{n+1}$-differentials.

However, in many cases of interest—including the (stable or unstable) Adams spectral sequence, the Eilenberg-Moore spectral sequence, and others—a more “algebraic” approach can be used, using the notion of $n$-th order derived functors introduced in [8].

For example, the (unstable) $\mathbb{F}_p$-Adams spectral sequence for a (simply connected) space $X$ constructed in [15] is the homotopy spectral sequence of a cosimplicial space $W^\bullet$ obtained as a $\mathbb{F}_p$-resolution of $X$. It can be shown that the $E^{n+2}$-term of this spectral sequence depends only on the $n$-Postnikov sections of the mapping spaces $\text{map}(X, E)$ and $\text{map}(E, E')$ for various products of $\mathbb{F}_p$-Eilenberg-Mac Lane space $E$ and $E'$. Thus we do not need a full algebraic model for the $\mathcal{P}^\bullet[\Delta^\text{op}, \text{Set}]$-category $\text{Top}$, but only for the small subcategory with objects $X$ and $E$ as above. Since all mapping spaces in this category are themselves simplicial $\mathbb{F}_p$-vector spaces, the associated $n$-track category is correspondingly simplified. The case $n = 1$ was treated in great detail in [5], and some progress on the case $n = 2$ has been made in [9]. However, it is clear from [7] that a better conceptual framework, such as an algebraic model for such “linear” $n$-track categories, will be needed before any further progress can be made for $n \geq 2$. 


Appendix: Fibrancy conditions on \(n\)-fold simplicial sets

In this appendix we prove some technical facts about \(\text{Or}(n)\):

7.12. Remark. Given a map of simplicial sets \(f: A \to B\) and \(m \geq 2\), let \(P := \text{Or}(m)A\), \(Q := \text{Or}(m)B\), and \(F := \text{Or}(m)f: P \to Q\). From the description in (2.9) we see by induction on \(m\) (using (2.11)) that for every multi-index \((p_1 \ldots p_m)\) the map of sets \(F_{(p_1 \ldots p_m)}: P_{(p_1 \ldots p_m)} \to Q_{(p_1 \ldots p_m)}\) is simply \(f_\ell: A_\ell \to B_\ell\), for \(\ell := m - 1 + p_1 + \ldots + p_m\) (cf. (2.10)).

7.13. Lemma. If \(Y = \text{Or}(n)X \in [\Delta^{op}, \text{Set}]\) for some \(X \in [\Delta^{op}, \text{Set}]\) and \(n \geq 2\), then for any two of its \(n\) directions \(1 \leq p \neq q \leq n\), the lower right corner of the bisimplicial set \(Z = Y^{(p,q)} \in [\Delta^{op}, \text{Set}]\) has the form:

\[
\begin{array}{c}
X_{s+2} \\
\downarrow d_{k+2} \\
X_{s+1} \\
\downarrow d_i \\
X_s \\
\downarrow d_k \\
X_{s-1}
\end{array}
\]

(7.14)

for some \(s \geq n\) and \(0 \leq i < k < s\).

Proof. By induction on \(n \geq 2\), where the case \(n = 2\) is depicted in Figure 1 of Section 2. Using (2.11), we see that \(Y = (\text{Or}(n-1) \circ \text{Or}(2))X\), so if we number the \(n\) directions of \(Y\) so as to start with the horizontal direction of \(\text{Or}(2)X\), then for any \(1 < p \neq q \leq n\) the bisimplicial set \(Z = Y^{(p,q)} \in [\Delta^{op}, \text{Set}]\) is contained in the \((n-1)\)-fold simplicial set \(\text{Or}(n-1)Q_{1\bullet}\), for one of the vertical simplicial sets of \(Q := \text{Or}(2)X\). Thus the claim for such a \(Z\) follows by the induction hypothesis.

Thus it suffices to treat the case \(1 = p < q\). Since the corresponding vertical maps in each of the vertical simplicial sets \(Q_{1\bullet}\), for various \(t\), have the same labels (in terms of the original face maps of \(X\)), the same will be true after applying the functor \(\text{Or}(n-1)\) to each of them. This implies that the vertical maps in (7.14) are indeed both labelled \(d_i, d_{i+1}\), for some \(i < k + 1\). However, since each of the simplicial sets \(Q_{1\bullet}\) is obtained by repeated applications of \(\text{Dec}\) to \(X\) (see (2.12)), we must have omitted at least the maximally labelled face map \(d_{k+1}: X_{k+1} \to X_k\), by definition of \(\text{Dec}\). Therefore, among the various face maps of \(X\) appearing in \(\text{Or}(n-1)Q_{1\bullet}\), the map \(d_{k+1}: X_{k+1} \to X_k\) cannot appear. Thus, we actually have \(i < k\).

From Figure 1 (or from the fact that the bisimplicial set \(Q\), as a (vertical) simplicial object over \([\Delta^{op}, \text{Set}]\), is the resolution of \(X\) produced by the comonad \(\text{Dec}\)), we see that the horizontal maps in \(Q\) are always the face maps of \(X\). Thus the leftmost horizontal maps in Figure 1 are \(d_1, d_2: X_2 \to X_1\).

On the other hand, by Remark 7.12 (for \(m = n-1\)), the two pairs of horizontal maps in (7.13) are just those that appear in the rightmost sequence of horizontal maps in Figure 1, namely, \(d_k, d_{k+1}: X_{k+1} \to X_k\) and \(d_{k+1}, d_{k+2}: X_{k+2} \to X_{k+1}\).
Thus when \( p = 1 \), in fact \( k = s - 1 \) in (7.14) (as for the front and back squares in Figure 2).

\[ \square \]

**Proposition (2.39).** If \( X \in [\Delta^{\text{op}}, \text{Set}] \) is a Kan complex, then \( Y := \text{Or}_{(n)} X \) is \((n, 2)\)-fibrant.

**Proof.** For every \( 1 \leq p \leq n \), the simplicial set \( Y^{(p)} \) is obtained from \( X \) by repeated applications of \( \text{Dec} \) and \( \text{Dec}' \), so it is still a Kan complex, and the same is true of \( \text{csk}_2 Y^{(p)} \).

For each bisimplicial set of the form (7.14), denote by \( W \) and \( Z \) the middle and right vertical simplicial sets, respectively, with \( \phi: W \to Z \) the horizontal map in \([\Delta^{\text{op}}, \text{Set}]\) given by \( d_k: W_0 = X_s \to Z_0 = X_{s-1}, \ d_{k+1}: W_1 = X_{s+1} \to Z_1 = X_s \), and so on. Similarly, denote by \( U \) and \( V \) the middle and bottom horizontal simplicial sets, respectively, with \( \psi: U \to V \) the vertical map in \([\Delta^{\text{op}}, \text{Set}]\) given by \( d_i: U_j = X_{s+j} \to V_j = X_{s+j-1} \) for all \( j \geq 0 \).

By Definition 2.31 and Lemma 7.13 in order to verify that \( Y \) is \((n, 2)\)-fibrant, we must check that \( \text{csk}_2 \phi \) and \( \text{csk}_2 \psi \) are fibrations for any choice of (7.14) with \( i \leq k \). This means that we must show that a lifting \( \hat{g} \) exists for every solid commuting square of one of the two following forms:

\[
\begin{align*}
\begin{array}{c}
\Lambda^j[m] \\
\downarrow i_j \\
\Delta[m]
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\phi \\
\downarrow g
\end{array}
\begin{array}{c}
U \\
\downarrow \psi \\
V
\end{array}
\end{array}
\end{align*}
\]

\[ (a) \]

\[
\begin{align*}
\begin{array}{c}
\Lambda^j[m] \\
\downarrow i_j \\
\Delta[m]
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\phi \\
\downarrow g
\end{array}
\begin{array}{c}
W \\
\downarrow \psi \\
Z
\end{array}
\end{array}
\end{align*}
\]

\[ (b) \]

for \( m = 1, 2 \) and \( 0 \leq j \leq m \) (where \( \Lambda^j[m] \subseteq \partial\Delta[m] \) consists of all but the \( j \)-th face of \( \Delta[m] \), and \( i_j: \Lambda^j[m] \to \Delta[m] \) is the inclusion).

**Case I:** When \( m = 1 \) in (7.15) (a), the map \( f: \Lambda^j[1] \to U \quad \) (\( j = 0, 1 \)) corresponds to a 0-simplex \( \sigma \in U_0 \) – that is, an \( s \)-simplex \( \sigma \in X_s \) (since \( U_0 = X_s \) and \( \Lambda^j[1] \cong \Delta[0] \)), and the map \( g: \Delta[1] \to V \) corresponds to a 1-simplex \( \tau \in V_1 \) – that is, an \( s \)-simplex \( \tau \in X_s \).

Commutativity of the solid square in (7.15) (a) – that is, \( \psi \circ f = g \circ i_j \) – means that \( d_{k+j}^X(\tau) = \psi(\sigma) \), i.e.,

\[
d_{k+j}^X \tau = d_i^X \sigma.
\]

A lift \( \hat{g}: \Delta[1] \to U \) corresponds to a 1-simplex \( \bar{\omega} \in U_1 \) – that is, an \( (s + 1) \)-simplex \( \omega \in X_{s+1} \), and commutativity of the two triangles in (7.15) (a) translates into the two conditions \( d_j^U(\bar{\omega}) = \bar{\sigma} \) and \( \psi(\bar{\omega}) = \bar{\tau} \), that is:

\[
d_{k+j}^X \omega = \sigma \quad \text{and} \quad d_i^X \omega = \tau.
\]

Combining (7.16) and (7.17) yields the simplicial identity:

\[
d_i d_{k+1+j} \omega = d_{k+j} d_i \omega,
\]

since \( i < k \).
The two $s$-simplices $\sigma$ and $\tau$ satisfying (7.16) define a map from the following pushout $P$ in $[\Delta^{op}, \text{Set}]$:

Diagram:

Since $P$ is a union of two $s$-simplices along a common face, it is a contractible subspace of $\Delta[s+1]$ so $P \hookrightarrow \Delta[s+1]$ is an acyclic cofibration in $[\Delta^{op}, \text{Set}]$. Because $X$ is fibrant, a lift $\omega: \Delta[s+1] \to X$ for $(\sigma, \tau)$ - and thus $\hat{g}: \Delta[m] \to U$ - always exists.

**Case II:** When $m = 2$ in (7.15) (a), the map $f: \Lambda^j[2] \to U$ $(j = 0, 1, 2)$ corresponds to a pair of 1-simplices $\tilde{\alpha}, \tilde{\beta} \in U_1$ with $d_p \tilde{\alpha} = d_q \tilde{\beta}$, where

\[
(p, q) = \begin{cases} 
(1, 1) & \text{if } j = 0 \\
(0, 1) & \text{if } j = 1 \\
(0, 0) & \text{if } j = 2 
\end{cases}
\]  

(7.19)

This means that we have $\alpha, \beta \in X_{s+1} = U_1$ with

\[
d^{X_{s+1}}_{k+1+p} \alpha = d^{X_{s+1}}_{k+1+q} \beta.
\]  

(7.20)

The map $g: \Delta[2] \to V$ corresponds to $\sigma \in X_{s+1} = V_2$, and the map $\hat{g}: \Delta[2] \to U$ corresponds to $\omega \in X_{s+2}$.

Commutativity of the solid square in (7.15) (a) means that

\[
d^{X_{s+1}}_{k+p} \sigma = d^{X_{s+1}}_{k+q} \alpha \quad \text{and} \quad d^{X_{s+1}}_{k+1+p} \sigma = d^{X_{s+1}}_{k+1+q} \beta.
\]  

(7.21)

Commutativity of the upper triangle in (7.15) (a) means:

\[
d^{X_{s+2}}_{k+1+p} \omega = \alpha \quad \text{and} \quad d^{X_{s+2}}_{k+1+q} \omega = \beta,
\]  

(7.22)

and commutativity of the lower triangle in (7.15) (a) means:

\[
d^{X_{s+1}}_{i+1} \omega = \sigma.
\]  

(7.23)

Combining (7.21), (7.22), and (7.23) yields the two simplicial identities:

\[
d_i d_{k+1+p} \omega = d_{k+p} d_i \omega \quad \text{and} \quad d_i d_{k+1+q} \omega = d_{k+q} d_i \omega.
\]  

(7.24)

since $i < k$. The existence of $\omega$ follows as above.

The analogous cases for (7.15) (b) are obtained from these by applying the inversion $I^*$ of (7.22). □
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