THE MODULE CATEGORY WEIGHT OF COMPACT EXCEPTIONAL LIE GROUPS

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Abstract. We compute the lower bound estimate for the module category weight of compact exceptional Lie groups by analyzing several Eilenberg–Moore type spectral sequences.

1. Introduction

The Lusternik - Schnirelmann category \( \text{cat}(X) \) of a topological space \( X \) is the least integer \( n \) such that there exists an open cover \( X = U_1 \cup \cdots \cup U_{n+1} \) with each \( U_i \) contractible to a point in \( X \). There are other computable homotopy invariants such as cup length, category weight, and module category weight with the relation \[ \text{cup}(X; \mathbb{F}_p) \leq \text{wgt}(X; \mathbb{F}_p) \leq M\text{wgt}(X; \mathbb{F}_p) \leq \text{cat}(X). \]

Toomer introduced the explicit formula for the difference between the cup length and the category weight. Using the formula he calculated the difference \( \text{cup}(X; \mathbb{F}_p) - \text{wgt}(X; \mathbb{F}_p) \) of any simply connected compact simple Lie group \([11]\). In fact, it is precisely \( F_4, E_6, E_7, E_8 \) which yield a positive difference.

On the other hands, Iwase and Kono \([3]\) determined \( \text{cat}(\text{Spin}(9)) = 8 \) by computing the lower bound of the difference between the category weight and the module category weight of \( \text{Spin}(9) \), which is \( M\text{wgt}(\text{Spin}(9); \mathbb{F}_2) - \text{wgt}(\text{Spin}(9); \mathbb{F}_2) \geq 2. \)

In this paper we compute the lower bound estimate for the module category weight of exceptional compact simple Lie groups by studying the difference between the category weight and the module category weight.

This paper is organized as follows. In section 2, we collects some known facts, which will be used in next sections. In section 3, we compute the module category weight with respect to \( \mathbb{F}_2 \) coefficients of compact exceptional Lie groups by analyzing several Eilenberg–Moore type spectral sequences. In section 4, we compute the module category weight with respect to \( \mathbb{F}_3 \) coefficients of compact exceptional Lie groups by the similar method as in the case of \( \mathbb{F}_2 \) coefficients.

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2. Some Known Facts

Throughout this paper, the subscript of an element always means the degree of the element, for example, the degree of \( x_i \) is \( i \). Let \( E(x) \) be the exterior algebra on \( x \) and \( \mathbb{F}_2[x] \) be the

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polynomial algebra on $x$ and $\Gamma(x)$ be the divided power algebra on $x$ which is free over $\gamma_i(x)$ with coproduct

$$\Delta(\gamma_n(x)) = \sum_{i=0}^{n} \gamma_{n-i}(x) \otimes \gamma_i(x)$$

and the product

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x).$$

We define $\text{cup}(X; \mathbb{F}_p)$, the cup-length with respect to $\mathbb{F}_p$, by the least integer $m$ such that $x_1 \cdots x_{m+1} = 0$ for any $m + 1$ elements $x_j \in H^*(X; \mathbb{F}_p)$. Let $P^n(\Omega X)$ be the $m$ th projective space, in the sense of Stasheff [10], such that there is a homotopy equivalence $P^\infty(\Omega X) \simeq X$. Let $\epsilon_m : P^m(\Omega X) \to P^n(\Omega X) \simeq X$ be the inclusion map. Consider $(\epsilon_m)^* : H^*(X; \mathbb{F}_p) \to H^*(P^m(\Omega X); \mathbb{F}_p)$. Then we can define category weight $\text{wgt}(X; \mathbb{F}_p)$ and module category weight $\text{MWgt}(X; \mathbb{F}_p)$ as follows [3]:

$$\text{wgt}(X; \mathbb{F}_p) = \min \{ m \mid (\epsilon_m)^* \text{ is a monomorphism} \},$$

$$\text{MWgt}(X; \mathbb{F}_p) = \min \{ m \mid (\epsilon_m)^* \text{ is a split monomorphism of all Steenrod algebra module} \}.$$

Then we have the following relation [3].

$$\text{cup}(X; \mathbb{F}_p) \leq \text{wgt}(X; \mathbb{F}_p) \leq \text{MWgt}(X; \mathbb{F}_p) \leq \text{cat}(X).$$

Now we list the mod $p$ cohomology of the exceptional Lie groups. We refer [8] for the condensed treatment of these cohomology including Hopf algebra structure and the action of the Steenrod algebra.

**Theorem 2.1.** The mod 2 cohomology of the exceptional Lie groups $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$ are as follows.

\[
\begin{align*}
H^*(G_2; \mathbb{F}_2) &\cong \mathbb{F}_2[x_3]/(x_3^4) \otimes E(Sq^2 x_3) \\
H^*(F_4; \mathbb{F}_2) &\cong \mathbb{F}_2[x_3]/(x_3^4) \otimes E(Sq^2 x_3, Sq^{4,2} x_3, x_{15}, Sq^8 x_{15}) \\
H^*(E_6; \mathbb{F}_2) &\cong \mathbb{F}_2[x_3]/(x_3^4) \otimes E(Sq^2 x_3, Sq^{4,2} x_3, x_{15}, Sq^8, Sq^{8,4,2} x_3, Sq^8 x_{15}) \\
H^*(E_7; \mathbb{F}_2) &\cong \mathbb{F}_2[x_3, Sq^2 x_3, Sq^{4,2} x_3]/(x_3^4, (Sq^2 x_3)^4, (Sq^{4,2} x_3)^4) \\
&\quad \otimes E(x_{15}, Sq^{8,4,2} x_3, Sq^8 x_{15}, Sq^{4,8} x_{15}) \\
H^*(E_8; \mathbb{F}_2) &\cong \mathbb{F}_2[x_3]/(x_3^{16}) \otimes \mathbb{F}_2[Sq^2 x_3]/((Sq^2 x_3)^8) \\
&\quad \otimes E(Sq^{4,2} x_3, x_{15})/((Sq^{4,2} x_3)^4, x_{15}^4) \\
&\quad \otimes E(Sq^{8,4,2} x_3, Sq^8 x_{15}, Sq^{4,8} x_{15}, Sq^{8,4,8} x_{15}).
\end{align*}
\]
Theorem 2.2. The mod 3 cohomology of the exceptional Lie groups $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$ are as follows.

\[
\begin{align*}
H^*(G_2; \mathbb{F}_3) & \cong E(x_3, x_{11}) \\
H^*(F_4; \mathbb{F}_3) & \cong \mathbb{F}_3[\beta \mathcal{P}^1 x_3]/((\beta \mathcal{P}^1 x_3)^3) \otimes E(x_3, \mathcal{P}^1 x_3, x_{11}, \mathcal{P}^1 x_{11}) \\
H^*(E_6; \mathbb{F}_3) & \cong \mathbb{F}_3[\beta \mathcal{P}^1 x_3]/((\beta \mathcal{P}^1 x_3)^3) \otimes E(x_3, \mathcal{P}^1 x_3, x_9, x_{11}, \mathcal{P}^1 x_{11}, x_{17}) \\
H^*(E_7; \mathbb{F}_3) & \cong \mathbb{F}_3[\beta \mathcal{P}^1 x_3]/((\beta \mathcal{P}^1 x_3)^3) \\
& \quad \otimes E(x_3, \mathcal{P}^1 x_3, x_{11}, \mathcal{P}^1 x_{11}, \mathcal{P}^3 x_3, x_{27}, x_{35}) \\
H^*(E_8; \mathbb{F}_3) & \cong \mathbb{F}_3[\beta \mathcal{P}^1 x_3, \beta \mathcal{P}^3 x_3]/((\beta \mathcal{P}^1 x_3)^3, (\beta \mathcal{P}^3 x_3)^3) \\
& \quad \otimes E(x_3, \mathcal{P}^1 x_3, x_{15}, \mathcal{P}^3 x_3, \mathcal{P}^3 x_{15}, x_{35}, x_{39}, x_{47}).
\end{align*}
\]

3. Module Category Weight with respect to $\mathbb{F}_2$ coefficients

Let $\tilde{G}$ be the 3-connected cover of $G$ which is the homotopy fibre of the map $G \rightarrow K(Z, 3)$ where $\iota$ is the fundamental class of $H^3(G; Z)$. Then we have the following fibrations: $\mathbb{C}P^\infty \rightarrow \tilde{G} \rightarrow G$, $S^1 \rightarrow \Omega \tilde{G} \rightarrow \Omega G$.

Theorem 3.1. \[\text{[5, 7]}\] The mod 2 cohomology of the 3-connected covers of the exceptional Lie groups $\tilde{G}_2$, $\tilde{F}_4$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ are as follows.

\[
\begin{align*}
H^*(\tilde{G}_2; \mathbb{F}_2) & \cong \mathbb{F}_2[x_8] \otimes E(Sq^1 x_8, Sq^2 x_8) \\
H^*(\tilde{F}_4; \mathbb{F}_2) & \cong \mathbb{F}_2[x_8] \otimes E(Sq^1 x_8, Sq^2 x_8, Sq^4 x_8, Sq^4, Sq^8, Sq^{2,1}, x_8) \\
H^*(\tilde{E}_6; \mathbb{F}_2) & \cong \mathbb{F}_2[x_{32}] \otimes E(x_9, Sq^2 x_9, Sq^4 x_9, Sq^8 x_9, x_{23}, Sq^{16,8} x_9) \\
H^*(\tilde{E}_7; \mathbb{F}_2) & \cong \mathbb{F}_2[x_{32}] \otimes E(x_{11}, Sq^4 x_{11}, Sq^8 x_{11}, x_{23}, Sq^8 x_{11}, x_{32}, Sq^{16,8} x_{11}) \\
H^*(\tilde{E}_8; \mathbb{F}_2) & \cong \mathbb{F}_2[x_{15}]/(x^4_{15}) \otimes \mathbb{F}_2[x_{32}] \\
& \quad \otimes E(x_{23}, x_{27}, x_{29}, Sq^4 x_{32}, x_{35}, Sq^4 x_{35}, Sq^{8,4} x_{35}).
\end{align*}
\]

To get the module category weight of exceptional Lie groups $G$, we study the Eilenberg–Moore spectral sequence converging to $H^*(G)$ with $E_2 \cong \text{Cotor}_{H^*(\Omega G; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2)$. This is a spectral sequence of Hopf algebras but it depends on the coalgebra structure. So we should determine the coalgebra structure of $H^*(\Omega G; \mathbb{F}_2)$. To get the coalgebra structure of $H^*(\Omega G; \mathbb{F}_2)$, we consider the Eilenberg–Moore spectral sequence converging to $H^*(\Omega G; \mathbb{F}_2)$ with

$$E_2 \cong \text{Tor}_{H^*(G; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2)$$

Since $E_2$ concentrates in the even dimensions, the spectral sequence collapses at the $E_2$-term, i.e., $E_2 = E_\infty$. Then there is no coalgebra extension problem in such a spectral sequence \[4\]. We refer the reader to \[6\] for concise treatment of above Eilenberg Moore spectral sequence. So as a coalgebra we have the following.

\[
\begin{align*}
E_2 & \cong \text{Tor}_{H^*(G; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \\
E_\infty & \cong \text{Tor}_{H^*(G; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2).
\end{align*}
\]
The coalgebra structure of the mod 2 cohomology of the loop spaces of exceptional Lie groups $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$ are as follows.

\[
H^*(\Omega G_2; \mathbb{F}_2) \cong E(a_2) \otimes \Gamma(a_4, b_{10}) \\
H^*(\Omega F_4; \mathbb{F}_2) \cong E(a_2) \otimes \Gamma(a_4, b_{10}, a_{14}, a_{16}, a_{22}) \\
H^*(\Omega E_6; \mathbb{F}_2) \cong E(a_2) \otimes \Gamma(a_4, a_8, b_{10}, a_{14}, a_{16}, a_{22}) \\
H^*(\Omega E_7; \mathbb{F}_2) \cong E(a_2, a_4, a_8) \otimes \Gamma(b_{10}, a_{14}, a_{16}, b_{18}, a_{22}, a_{26}, b_{34}) \\
H^*(\Omega E_8; \mathbb{F}_2) \cong E(a_2, a_4, a_8, a_{14}) \otimes \Gamma(a_{16}, a_{22}, a_{26}, a_{28}, b_{34}, b_{38}, b_{46}, b_{58})
\]

especially we have $Sq^4 b_{10} = a_{14}$ and $Sq^8 b_{18} = a_{26}$ by Theorem [7.1].

Now we consider the Eilenberg–Moore spectral sequence converging to $H^*(G; \mathbb{F}_2)$ with

\[
E_2 \cong \text{Cotor}_{H^*}(\Omega G; \mathbb{F}_2)(\mathbb{F}_2, \mathbb{F}_2).
\]

Then we get the following theorem by the formal Cotor computation. We refer the reader to [9] for detail computation method of this spectral sequence.

**Theorem 3.4.** $\text{Cotor}_{H^*}(\Omega G; \mathbb{F}_2)(\mathbb{F}_2, \mathbb{F}_2)$ of the exceptional Lie groups $G$ for $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$ are as follows.

\[
\begin{align*}
\text{Cotor}_{H^*}(\Omega G_2; \mathbb{F}_2)(\mathbb{F}_2, \mathbb{F}_2) & \cong \mathbb{F}_2[x_3] \otimes E(x_5, z_{11}) \\
\text{Cotor}_{H^*}(\Omega F_4; \mathbb{F}_2)(\mathbb{F}_2, \mathbb{F}_2) & \cong \mathbb{F}_2[x_3] \otimes E(x_5, z_{11}, x_{15}, x_{23}) \\
\text{Cotor}_{H^*}(\Omega E_6; \mathbb{F}_2)(\mathbb{F}_2, \mathbb{F}_2) & \cong \mathbb{F}_2[x_3] \otimes E(x_5, x_9, z_{11}, x_{15}, x_{17}, x_{23}) \\
\text{Cotor}_{H^*}(\Omega E_7; \mathbb{F}_2)(\mathbb{F}_2, \mathbb{F}_2) & \cong \mathbb{F}_2[x_3, x_5, x_9] \otimes E(z_{11}, x_{15}, x_{17}, z_{19}, x_{23}, x_{27}, z_{35}) \\
\text{Cotor}_{H^*}(\Omega E_8; \mathbb{F}_2)(\mathbb{F}_2, \mathbb{F}_2) & \cong \mathbb{F}_2[x_3, x_5, x_9, x_{15}] \otimes E(x_{17}, x_{23}, x_{27}, x_{29}, z_{35}, z_{39}, z_{47}, z_{59})
\end{align*}
\]

especially we have $Sq^4 z_{11} = x_{15}$ and $Sq^8 z_{19} = x_{27}$.

Then from information Theorem [2.1] of $H^*(G; \mathbb{F}_2)$, we can analyze non trivial differentials of the Eilenberg–Moore spectral sequence (3.3) converging to $H^*(G; \mathbb{F}_2)$ as follows:

\[
\begin{align*}
d_3(z_{11}) & = x_3^4 & \text{for } G = G_2, F_4, E_6, E_7 \\
d_3(z_{19}) & = x_3^4 & \text{for } G = E_7 \\
d_3(z_{35}) & = x_3^4 & \text{for } G = E_7, E_8 \\
d_7(z_{39}) & = x_3^8 & \text{for } G = E_8 \\
d_{15}(z_{47}) & = x_3^{16} & \text{for } G = E_8 \\
d_3(z_{59}) & = x_3^4 & \text{for } G = E_8.
\end{align*}
\]

Next as in [2, 3], truncating the above computation with the same differential $d_i$ in (3.5), we can compute the spectral sequence of Stasheff’s type converging to $H^*(P^m(\Omega G); \mathbb{F}_2)$.

Let $A = H^*(G; \mathbb{F}_2)$ in Theorem [2.1]. Then like the result in [3, Proposition 2.1], for low $m$ such as $1 \leq m \leq 3$, we have the following:
Theorem 3.7. The module category weight is as follows:

\[
\begin{align*}
M\text{wgt}(G_2; \mathbb{F}_2) & \geq 4, \\
M\text{wgt}(F_4; \mathbb{F}_2) & \geq 8, \\
M\text{wgt}(E_6; \mathbb{F}_2) & \geq 10, \\
M\text{wgt}(E_7; \mathbb{F}_2) & \geq 15, \\
M\text{wgt}(E_8; \mathbb{F}_2) & \geq 32.
\end{align*}
\]

Proof. From Theorem 3.4, \( S^4z_{11} = x_{15} \) in \( H^*(P^1(\Omega G); \mathbb{F}_2) \) for \( G = F_4, E_6, E_7 \). Then from (3.6), \( S^4z_{11} = x_{15} \) modulo \( S_2 \) in \( H^*(P^2(\Omega G); \mathbb{F}_2) \) for \( G = F_4, E_6, E_7 \). Since \( S_2 \) is even-dimensional \([1, 3]\), the modulo \( S_2 \) is trivial so \( S^4z_{11} = x_{15} \) in \( H^*(P^2(\Omega G); \mathbb{F}_2) \). Therefore in \( H^*(P^7(\Omega F_4); \mathbb{F}_2), H^*(P^9(\Omega E_6); \mathbb{F}_2), \) and \( H^*(P^{14}(\Omega E_7); \mathbb{F}_2) \), we have

\[
\begin{align*}
S^4(x_3^3x_5z_{11}x_{23}) & = x_3^3x_5x_{15}x_{23} \text{ for } F_4, \\
S^4(x_3^3x_5x_9z_{11}x_{17}x_{23}) & = x_3^3x_5x_9x_{15}x_{17}x_{23} \text{ for } E_6, \\
S^4(x_3^3x_5^3x_9z_{11}x_{17}x_{23}x_{27}) & = x_3^3x_5^3x_9x_{15}x_{17}x_{23}x_{27} \text{ for } E_7.
\end{align*}
\]

By the definition in section 2, \( M\text{wgt}(X; \mathbb{F}_2) \) is the least \( m \) such that \((e_m)^*\) is a split monomorphism of all Steenrod algebra module. Let \( \phi_m : H^*(P^m(\Omega G); \mathbb{F}_2) \to H^*(G; \mathbb{F}_2) \) be a epimorphism which preserves all Steenrod actions and \( \phi_m \circ (e_m)^* \cong 1_{H^*(G; \mathbb{F}_2)} \). Suppose that there are epimorphisms

\[
\begin{align*}
\phi_7 : H^*(P^7(\Omega F_4); \mathbb{F}_2) & \to H^*(F_4; \mathbb{F}_2), \\
\phi_9 : H^*(P^9(\Omega E_6); \mathbb{F}_2) & \to H^*(E_6; \mathbb{F}_2), \\
\phi_{14} : H^*(P^{14}(\Omega E_7); \mathbb{F}_2) & \to H^*(E_7; \mathbb{F}_2).
\end{align*}
\]
Summarizing above results, we have the following results.

| X      | wgt(X; \mathbb{F}_2) | Mwgt(X; \mathbb{F}_2) | cat(X) |
|--------|----------------------|------------------------|--------|
| G_2    | 4                    | \geq 4                 | 4      |
| F_4    | 6                    | \geq 8                 | ?      |
| E_6    | 8                    | \geq 10                | ?      |
| E_7    | 13                   | \geq 15                | ?      |
| E_8    | 32                   | \geq 32                | ?      |

4. Module Category Weight with respect to \mathbb{F}_3 coefficients

Now we turn to the case of \mathbb{F}_3 coefficients.
Theorem 4.1. [3, 7] The mod 3 cohomology of the 3-connected covers of the exceptional Lie groups $\tilde{G}_2$, $\tilde{F}_4$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ are as follows.

\[
\begin{align*}
H^*(\tilde{G}_2; \mathbb{F}_3) & \cong \mathbb{F}_3[y_6] \otimes E(x_{11}, \beta y_6) \\
H^*(\tilde{F}_4; \mathbb{F}_3) & \cong \mathbb{F}_3[y_{18}] \otimes E(x_{11}, P^1 x_{11}, \beta y_{18}, \beta y_{18}) \\
H^*(\tilde{E}_6; \mathbb{F}_3) & \cong \mathbb{F}_3[y_{18}] \otimes E(x_{9}, x_{11}, P^1 x_{11}, x_{17}, \beta y_{18}, P^1 \beta y_{18}) \\
H^*(\tilde{E}_7; \mathbb{F}_3) & \cong \mathbb{F}_3[y_{54}] \otimes E(x_{11}, P^1 x_{11}, x_{19}, P^1 x_{19}, P^2 x_{19}, \beta y_{54}) \\
H^*(\tilde{E}_8; \mathbb{F}_3) & \cong \mathbb{F}_3[y_{54}] \otimes E(x_{15}, z_{23}, P^1 z_{23}, x_{35}, x_{39}, x_{47}, \beta y_{54}, y_{59}).
\end{align*}
\]

Following the similar method as in the case of $\mathbb{F}_2$ coefficients, we have the next theorem.

Theorem 4.2. The coalgebra structure of the mod 3 cohomology of the loop spaces of exceptional Lie groups $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$ are as follows.

\[
\begin{align*}
H^*(\Omega G_2; \mathbb{F}_3) & \cong \Gamma(a_2, a_{10}) \\
H^*(\Omega F_4; \mathbb{F}_3) & \cong \mathbb{F}_3[a_2]/(a_2^3) \otimes \Gamma(a_6, a_{10}, a_{14}, b_{22}) \\
H^*(\Omega E_6; \mathbb{F}_3) & \cong \mathbb{F}_3[a_2]/(a_2^3) \otimes \Gamma(a_6, a_8, a_{10}, a_{14}, a_{16}, b_{22}) \\
H^*(\Omega E_7; \mathbb{F}_3) & \cong \mathbb{F}_3[a_2]/(a_2^3) \otimes \Gamma(a_6, a_{10}, a_{14}, a_{18}, b_{22}, a_{26}, a_{34}) \\
H^*(\Omega E_8; \mathbb{F}_3) & \cong \mathbb{F}_3[a_2]/(a_2^3) \otimes \mathbb{F}_3[a_6]/(a_6^3) \otimes \Gamma(a_{14}, a_{18}, b_{22}, a_{26}, a_{34}, a_{38}, a_{40}, b_{58})
\end{align*}
\]

especially we have $P^1 b_{22} = a_{26}$ by Theorem 4.1 and $P^1 P^1 = 2 P^2$ and by change of generators.

Consider the Eilenberg–Moore spectral sequence converging to $H^*(G; \mathbb{F}_3)$ with

\[
E_2 \cong \text{Cotor}_{H^*(\Omega G; \mathbb{F}_3)}(\mathbb{F}_3, \mathbb{F}_3).
\]

Theorem 4.4. $\text{Cotor}_{H^*(\Omega G; \mathbb{F}_3)}(\mathbb{F}_3, \mathbb{F}_3)$ of the exceptional Lie groups $G$ for $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$ are as follows.

\[
\begin{align*}
\text{Cotor}_{H^*(\Omega G_2; \mathbb{F}_3)}(\mathbb{F}_3, \mathbb{F}_3) & \cong E(x_3, x_{11}) \\
\text{Cotor}_{H^*(\Omega F_4; \mathbb{F}_3)}(\mathbb{F}_3, \mathbb{F}_3) & \cong E(x_3) \otimes \mathbb{F}_3[\beta P^1 x_3] \otimes E(P^1 x_3, x_{11}, P^1 x_{11}, z_{23}) \\
\text{Cotor}_{H^*(\Omega E_6; \mathbb{F}_3)}(\mathbb{F}_3, \mathbb{F}_3) & \cong E(x_3) \otimes \mathbb{F}_3[\beta P^1 x_3] \otimes E(P^1 x_3, x_9, x_{11}, P^1 x_{11}, x_{17}, z_{23}) \\
\text{Cotor}_{H^*(\Omega E_7; \mathbb{F}_3)}(\mathbb{F}_3, \mathbb{F}_3) & \cong E(x_3) \otimes \mathbb{F}_3[\beta P^1 x_3] \otimes E(P^1 x_3, x_{11}, P^1 x_{11}, x_{19}, z_{23}, x_{27}, x_{35}) \\
\text{Cotor}_{H^*(\Omega E_8; \mathbb{F}_3)}(\mathbb{F}_3, \mathbb{F}_3) & \cong E(x_3) \otimes \mathbb{F}_3[\beta P^1 x_3] \otimes E(P^1 x_3) \otimes \mathbb{F}_3[\beta P^3 P^1 x_3] \\
& \quad \otimes E(x_{15}, x_{19}, z_{23}, x_{27}, x_{35}, x_{39}, x_{47}, z_{59})
\end{align*}
\]

especially we have $P^1 z_{23} = x_{27}$.

Then from information Theorem 2.2 of $H^*(G; \mathbb{F}_3)$, we can analyze non trivial differentials of the Eilenberg–Moore spectral sequence (4.3) converging to $H^*(G; \mathbb{F}_3)$ as follows:

\[
\begin{align*}
d_3(z_{23}) & = (\beta P^1 x_3)^3, \text{ for } G = F_4, E_6, E_7 \\
d_3(z_{59}) & = (\beta P^3 P^1 x_3)^3, \text{ for } G = E_8
\end{align*}
\]
Let $A = H^*(G; \mathbb{F}_3)$ in Theorem 2.2. Then like the result in (3.6), for low $m$ such as $1 \leq m \leq 3$, we have the following:

(4.6) \[ H^*(P^m(\Omega G); \mathbb{F}_3) = A^{[m]} \oplus \sum_{i} z_{4i+3} \cdot A^{[m-1]} \oplus S_m \]

as modules. Now we compute the module category weight using the same method in Theorem 3.7.

**Theorem 4.7.** The module category weight is as follows:

\[
\begin{align*}
Mwgt(G_2; \mathbb{F}_3) & \geq 2, \\
Mwgt(F_4; \mathbb{F}_3) & \geq 8, \\
Mwgt(E_6; \mathbb{F}_3) & \geq 10, \\
Mwgt(E_7; \mathbb{F}_3) & \geq 13, \\
Mwgt(E_8; \mathbb{F}_3) & \geq 18.
\end{align*}
\]

**Proof.** From Theorem 4.2, we get $P^1 z_{23} = x_{27}$ in $H^*(P^1(\Omega G); \mathbb{F}_3)$ for $G = E_7, E_8$. Then $P^1 z_{23} = x_{27}$ modulo $S_2$ in $H^*(P^2(\Omega G); \mathbb{F}_3)$ for $G = E_7, E_8$ from (4.6). Since $S_2$ is even-dimensional [11, 3], the modulo $S_2$ is trivial and $P^1 z_{23} = x_{27}$ in $H^*(P^2(\Omega G); \mathbb{F}_3)$. So in $H^*(P^{12}(\Omega E_7); \mathbb{F}_3)$ and $H^*(P^{27}(\Omega E_8); \mathbb{F}_3)$, we have

\[
\begin{align*}
\mathcal{P}^1((\beta P^1 x_3)^2 x_{23} x_{11} x_{15} x_{19} x_{27} x_{35}) &= (\beta P^1 x_3)^2 x_{23} x_{11} x_{15} x_{19} x_{27} x_{35}, \\
\mathcal{P}^1((\beta P^3 P^1 x_3)^2 x_{23} x_{11} x_{15} x_{19} x_{27} x_{35} x_{39} x_{47}) &= (\beta P^1 x_3)^2 (\beta P^3 P^1 x_3)^2 x_{23} x_{11} x_{15} x_{19} x_{27} x_{35} x_{39} x_{47}.
\end{align*}
\]

Note that the filtration lengths of $\beta P^1 x_3$ and $\beta P^3 P^1 x_3$ are both 2 by the result in [11]. Let $\phi_m : H^*(P^m(\Omega G); \mathbb{F}_p) \rightarrow H^*(G; \mathbb{F}_p)$ be a epimorphism which preserves all Steenrod actions and $\phi_m \circ (e_m)^* \cong 1_{H^*(G; \mathbb{F}_p)}$. Suppose that there are epimorphisms

\[
\begin{align*}
\phi_{12} : H^*(P^{12}(\Omega E_7); \mathbb{F}_3) & \rightarrow H^*(E_7; \mathbb{F}_3), \\
\phi_{17} : H^*(P^{17}(\Omega E_8); \mathbb{F}_3) & \rightarrow H^*(E_8; \mathbb{F}_3).
\end{align*}
\]

Then we have the following diagrams:

\[
\begin{array}{ccc}
H^*(P^{12}(\Omega E_7); \mathbb{F}_3) & \xrightarrow{\phi_{12}} & H^*(E_7; \mathbb{F}_3) \\
(\beta P^1 x_3)^2 x_{23} x_{11} x_{15} x_{19} x_{27} x_{35} & \rightarrow & (\beta P^1 x_3)^2 x_{23} x_{11} x_{15} x_{19} x_{27} x_{35} \\
\mathcal{P}^1 & \xrightarrow{\phi_{12}} & \mathcal{P}^1 \\
(\beta P^1 x_3)^2 x_{23} x_{11} x_{15} x_{19} x_{27} x_{35} & \rightarrow & 0
\end{array}
\]
Combined with Toomer’s result in [11], we have the following conclusion:

\[
\begin{align*}
H^*(P^{17}(\Omega E_{78}); \mathbb{F}_3) &\xrightarrow{\phi_{17}} H^*(E_8; \mathbb{F}_3) \\
(\beta P^1x_3)^2(\beta P^1P^1x_3)^2x_3x_7x_{15}x_{19}x_{27}x_{35}x_{39}x_{47} &\xrightarrow{P^1\uparrow} (\beta P^1x_3)^2(\beta P^1P^1x_3)^2x_3x_7x_{15}x_{19}x_{27}x_{35}x_{39}x_{47} \\
\mathcal{P}^1((\beta P^1x_3)^2(\beta P^1P^1x_3)^2x_3x_7x_{15}x_{19}x_{27}x_{35}x_{39}x_{47} &\xrightarrow{P^1\uparrow} 0
\end{align*}
\]

Obviously this is a contradiction. So \(\phi_{12}\) and \(\phi_{17}\) are not epimorphisms. This means that \((e_{12})^*\), and \((e_{17})^*\) can not be split monomorphisms of all Steenrod algebra module. Hence we obtain that

\[
Mwgt(E_7; \mathbb{F}_3) \geq 13, \quad Mwgt(E_8; \mathbb{F}_3) \geq 18.
\]

Now we consider the category weight. For \(G_2, x_3x_5 \in H^*(P^2(\Omega G_2); \mathbb{F}_3)\), so \((e_2)^*\) is a monomorphism, so \(wgt(G_2; \mathbb{F}_3) = 2\). For \(F_4\), \((\beta P^1x_3)^2x_3x_7x_{11}x_{15} \in H^*(P^8(\Omega F_4); \mathbb{F}_3)\), so \((e_8)^*\) is a monomorphism, so \(wgt(F_4; \mathbb{F}_3) = 8\). By the same way, \(wgt(E_0; \mathbb{F}_3) = 10\), \(wgt(E_7; \mathbb{F}_3) = 11\), and \(wgt(E_8; \mathbb{F}_3) = 16\). Here the category weight is the same as the the filtration length in [11], that is, \(wgt(G; \mathbb{F}_3) = f_3(G)\).

For the case of \(G_2, F_4\), and \(E_6\), by dimensional reason, any generator of type \(x_–\) can not be of the form \(Sq^i(z)\) for any \(i\) and for any generator of type \(z\). So we can not apply the method in [11]. Hence we do not obtain any positive difference between the category weight and the module category weight. Hence we have

\[
\begin{align*}
Mwgt(G_2; \mathbb{F}_3) &\geq wgt(G_2; \mathbb{F}_3) = 2, \quad Mwgt(F_4; \mathbb{F}_3) \geq wgt(F_4; \mathbb{F}_3) = 8, \\
Mwgt(E_6; \mathbb{F}_3) &\geq wgt(E_6; \mathbb{F}_3) = 10.
\end{align*}
\]

\[\square\]

**Remark.** Combined with Toomer’s result in [11], we have the following conclusion:

| G   | \(wgt(G; \mathbb{F}_3) – cup(G; \mathbb{F}_3)\) | \(Mwgt(G; \mathbb{F}_2) – wgt(G; \mathbb{F}_2)\) | \(Mwgt(G; \mathbb{F}_3) – wgt(G; \mathbb{F}_3)\) |
|-----|------------------------------------------------|---------------------------------|---------------------------------|
| \(G_2\) | 0 | \(\geq 0\) | \(\geq 0\) |
| \(F_4\) | 2 | \(\geq 2\) | \(\geq 0\) |
| \(E_6\) | 2 | \(\geq 2\) | \(\geq 0\) |
| \(E_7\) | 2 | \(\geq 2\) | \(\geq 2\) |
| \(E_8\) | 4 | \(\geq 0\) | \(\geq 2\) |

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