Topological $rf$–SQUID with a frustrating $\pi$– junction for probing the Majorana Bound State

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Majorana Bound States are predicted to appear as boundary states of the Kitaev model. Here we show that a $\pi$–Josephson Junction, inserted in a topologically non trivial model ring, sustains a Majorana Bound State, which is robust with respect to local and non local perturbations. The realistic structure could be based on a High Tc Superconductor tricrystal structure, similar to the one used to spot the d-wave order parameter. The presence of the Majorana Bound State changes the ground state of the topologically non trivial ring in a measurable way, with respect to that of a conventional one.

I. INTRODUCTION

After the pioneering theoretical proposal by Kitaev,$^1$ Majorana Fermions (MFs) have been predicted in a wide class of low-dimensional solid state devices. Being neutral excitations in Fractional Quantum Hall systems or hybrid superconducting devices, MFs are highly attractive for quantum computing gates, as well as for fundamental reasons. Despite the considerable theoretical and experimental efforts$^2$, challenges still remain before a real solid state device can be realized, allowing for isolation and manipulation of MFs. Among the promising systems, there are superconductors in contact with topological insulators (TIs) or quasi one-dimensional systems with strong spin orbit interactions$^3$, helical magnets$^4$ and other materials$^5$. However several issues have to be convincingly solved$^6$ and the recently announced transport measurements$^7$ still arise excitement and debate$^2$. Disappointingly, their zero bias anomalies can be fitted by both MF physics and Kondo or 0.7 anomaly physics$^8,9$.

Adopting a different point of view, we leave aside transport measurements, and in this paper we propose a magnetic flux measurement on a device combining the physics of topological insulators and superconductive d-wave systems. We show that the spontaneous flux generated in the ground state (GS) of a frustrated topological SQUID Josephson ring (we call it ”frustrated $\pi$–ring” (F$\pi$R ) in the following) can be unambiguously related to the presence of MFs.

Feasible realizations of such a system could be a semiconducting nano-wire with strong spin-orbit coupling (e.g. InAs or InSb), with spin polarization splitted by a Żeeman field, or, alternatively nano-wires made by 3D TIs (Bi2Se3 or Bi2Te3), in contact with a superconductor. Superconductors in contact with the edge states of a 2D topological insulator (HgTe) could be also considered$^{10}$. The frustration is obtained employing, as superconducting material, a high Tc tricrystal structure, realized by epitaxial growing of high Tc (HTS) cuprates, matching three differently oriented crystals. Rings built on tricrystals have provided the evidence for the d–wave pairing in HTS materials$^{11}$.

In Ref.$^{12}$ the experiment has been properly designed so that the ring contains an odd number of $\pi$-junctions and has a total critical current ($I_c$) and an inductance ($L$) to guarantee $I_cL >> \phi_0$ (here $\phi_0 = \hbar c/2e$ is the superconductive flux quantum). Its GS spontaneously breaks the time reversal symmetry with a current flowing in the ring and generating a spontaneous fractional flux $\phi = \pm \phi_0/2$ that can be measured by scanning SQUID microscopy$^{13}$.

Our proposed devices is sketched in Fig. 1. Nano wires, either made out of semiconductors of 3DTI, mimicks quasi 1D system, properly described by the Kitaev model$^2$.

![FIG. 1: (color online) Left Panel) A Ring made of an InAs or 3D TI nano wires is deposited onto a high Tc tricrystal superconductor. Right panel) Sketch of the Kitaev chain with a $\pi$–junction at the top point $U$. Red (blue) circles represent $\gamma_{A(B)}$ type MFs. Yellow ellipses signal strong coupling.](Image)
parities $\mathcal{P}$ at zero energy, which signals unambiguously the presence of the Majorana Bound State (MBS) in the spectrum.

As shown in Ref.28, $\pi$ junctions can display a MBS at a phase difference $\varphi = 0$, rather than $\varphi = \pi$. Here a corresponding state is found in the ring geometry, stabilized with respect to residual interactions.

The free energy describing a Josephson rf-SQUID ring shows minima at a phase difference $\varphi$ corresponding to the measurable spontaneous flux. In the conventional $\text{F}_\pi$ geometry, there is quite a high barrier separating the two minima at $\pm \phi_{o}/2$, which freezes the frustrated system in a time reversal symmetry broken GS. In our model, we find two minima close to $\pm \phi_{o}/2$ as well, as in the conventional $\text{F}_\pi$. However, they correspond to different parities. This marks a fundamental difference with respect to unfrustrated structures (conventional "0" rings), where the GS is non degenerate (at zero flux) and fixes the parity.

In our case, $\mathcal{P}$ conservation, would imply that changing the number of electrons (for instance by means of a quantum point contact) should affect the measured spontaneous flux, thus marking a strong difference w.r. to a conventional $\text{F}_\pi$. However, while the isolated model Hamiltonian conserves $\mathcal{P}$, this symmetry is rather unlikely to be maintained in a real device, where impurities, can trap and release charges. These events would correlate to flux quanta entering or leaving the ring, thus inducing tunneling of the system between the two minima, so that the expected average flux associated to the $\langle \Phi \rangle = 0$. Measuring zero flux at a $\text{F}_\pi$, would be the smoking gun evidence for the existence of the MBS in this device. It is a weird case that the breaking of the discrete symmetry ($\mathcal{P}$) enforces the time reversal symmetry to be restored.

The paper is structured as follows. In sec. II we introduce the model Hamiltonian. In sec. III we show the energy spectrum of a $\text{Pi}$ ring compared with the one of a $\text{Pi}$ Junction. In sec IV we study the free energy landscapes depending on the fermion parity and discuss the possible spontaneous fluxes threading the ring. Conclusions are summarized in sec. V.

II. MODEL HAMILTONIAN

We consider a $N$-sites Kitaev chain of unitary lattice spacing and full length $L = 2N$, with real inter site hopping $t$. When folded in the shape of a ring the system displays mirror symmetry across the vertical line connecting points U and D between the top and the bottom (see Fig.1).

Our device can be described as two $N$-sites Majorana wires (left ($\ell$) and right ($r$)), coupled at the top of the ring U by the weak electron tunneling of energy $\Gamma$. At chemical potential $\mu = 0$, in the presence of an electromagnetic vector potential $\vec{A}$, the gauge invariant Hamiltonian reads as $H = H_\ell + H_r$, with:

$$H_\alpha = \sum_{j=1}^{N-1} \left( -\frac{t}{2} e^{ig_{\alpha j} c_{\alpha j}^{\dagger} c_{\alpha j+1}} + \Delta e^{i(\phi_{\alpha j} + \phi_{\alpha j+1})/2} c_{\alpha j} c_{\alpha j+1} + h.c. \right).$$

(1)

Here $\alpha$ labels the $\ell$ and $r$ side of the ring, $c_{\alpha j}$ are spinless Dirac fermions at the site $j$ and $\Delta$ is the effective p-wave superconductive pairing. The phases $g_{\alpha j}$ acquired in the hopping between the $j$ site and its nearest neighbor and the gauge invariant phase $\phi_{\alpha j}$ are defined as:

$$g_{\alpha j} = -\frac{e}{\hbar c} \int_{1}^{\alpha j} \vec{A} \cdot d\vec{l},$$

(2)

$$\phi_{\alpha j} = \theta_{\alpha j} - \frac{2e}{\hbar c} \int_{1}^{\alpha j} \vec{A} \cdot d\vec{l},$$

(3)

where $\theta_{\alpha j}$ is the phase of the superconducting order parameter.

At the top point U of the ring (see Fig.1) there is a tunnel junction:

$$H_U = -\Gamma \left( c_{\ell N}^{\dagger} c_{r1} + h.c. \right),$$

(4)

($\Gamma \ll t$). For sake of further investigations, we explicitly consider also the hopping term at the bottom point in the ring D, where the $\ell$ and $r$ chains are matched:

$$H_D = -u t \left( c_{rN}^{\dagger} c_{\ell 1} + h.c. \right).$$

(5)

Here we will keep the dimensionless parameter $u$ (which may be complex) as a variable, to discuss also the trivial case of $u = 0$, which corresponds to the ring cut at D, with open ends.

The spinless Dirac fermions can be expressed in terms of two species of Majorana fermions at each site of the ring $\gamma_{\ell}/\gamma_{r}$, such that:

$$\gamma_{\ell}^{\alpha} = c_{\alpha j} e^{i\phi_{\alpha j}/2} + c_{\alpha j}^{\dagger} e^{-i\phi_{\alpha j}/2},$$

(6)

$$\gamma_{r}^{\alpha} = -i \left( c_{\alpha j} e^{i\phi_{\alpha j}/2} - c_{\alpha j}^{\dagger} e^{-i\phi_{\alpha j}/2} \right).$$

(7)

A $\pi$–Josephson Junction requires $\Delta$ having opposite signs at U, between $\ell 1$ and $r N$. In the gauge in which
\( \Delta \) is real, the OP \( \Delta \) has to vanish somewhere along the ring and we choose this point to be D with no loss of generality. As a first step, to make the approach as simple as possible, deep in the topological phase, we will adopt the Kitaev approximation, \(|\Delta| = \frac{\Delta}{2}\) all along the chain and we choose \( \Delta = t \) in the \( \ell \) region and \( \Delta = -t \) in the \( r \) region of the ring. Thus, the chain Hamiltonian becomes:

\[
H_\ell + H_r = -i \frac{\hbar}{2} \sum_{j=1}^{N-1} \left[ \gamma_{Bj}^\dagger \gamma_{Aj+1}^f - \gamma_{Aj}^f \gamma_{Bj+1}^\dagger \right].
\]

Pictorially, this kind of hybridization can be represented as in Fig[11]. Blue (red) circles represent B (A)-type MFs. The yellow ellipses denote effective strong coupling between nearest neighbor MFs. Were Eq.(8) the full Hamiltonian, the \( \ell \) and \( r \) chains would dimerize with opposite phases. Four unpaired MFs would appear: two (the red/A ones) located at U and two (the blu/B ones) at D.

To account for the extra interactions \( \Gamma \) and \( ut \), following Ref.22, we refermionize the Hamiltonian by including \( H_U + H_D \) and by rearranging the MFs at the boundaries. Three effective Dirac Fermions, \( d_A, d_B, d_{end} \) are required, located at the U weak link, and three more ones, \( f_A, f_B, f_{end} \), at the D boundary, according to22:

\[
H_{eff} = \frac{\hbar}{2} tu \left[ f_{end} \left(f_B - f_A^\dagger\right) - h.c. \right] + t \left[d_A^\dagger d_B + h.c.\right] - \frac{\hbar}{2} \left[\sin \frac{\Delta}{2} \left(2d_{end}d_{end}^\dagger - 1\right) - i\sqrt{2}\cos \frac{\Delta}{2} \left(d_{end}(d_B + d_A^\dagger) - h.c.\right) + \sin \frac{\Delta}{2} \left(d_Bd_A + d_A^\dagger d_B^\dagger + d_A^\dagger d_B - d_Bd_A^\dagger\right)\right],
\]

where

\[
\varphi = \phi_{\ell 1} - \phi_{r N} = \frac{2e}{\hbar c} \oint \vec{A} \cdot d\vec{r}. \tag{10}
\]

is the phase difference at the U weak link. The total energy only depends on the flux threading the ring, \( \Phi = \oint \vec{A} \cdot d\vec{r} \), in units of \( \phi_0 \). This Bogolubov-de-Gennes (BdG) Hamiltonian \( H_{eff} \) changes sign under the operation \( P = (-1)^n \) (\( n \) is the number of fermions at the weak links). It can be shown that the low energy spectrum only depends on the first term in the second line, which involves the Dirac fermion \( d_{end} = (\gamma_{A1} + i\gamma_{A N})/2 \), and on terms involving \( f_{end}, f_{end}^\dagger \) with \( f_{end} = (\gamma_{B1} + i\gamma_{B N})/2 \), which can be obtained by perturbation theory from the first term of the first line. In addition, residual interactions, not included in \( H_{eff} \), can account for finite size effects. In particular an extra coupling \( z_0 \Gamma \propto e^{-N} (\alpha = \ell, r) \) arises from realistic long range interactions between the edge MFs at each chain, when the simple \( t = |\Delta| \) limitation is relaxed.

We are led to the minimal 4X4 Hamiltonian in the Majorana representation, just involving the four relevant MFs:

\[
H_M = i\Gamma \begin{pmatrix}
0 & \sin \frac{\Delta}{2} z_\ell & 0 \\
-\sin \frac{\Delta}{2} & 0 & z_r \\
-z_\ell & 0 & 0 \\
0 & -z_r & -u 0
\end{pmatrix}, \tag{11}
\]

in the basis \( [\gamma_{A1}^f, \gamma_{A N}^f, \gamma_{B N}^f, \gamma_{B 1}^f] \), and we have redifined \( u = ut/\Gamma \).
A. Ring cut at the bottom point : $u = 0$

When the control parameter $u$ is set to zero, the ring is cut at the bottom point $D$ and the system is equivalent to a linear topological $\pi$ junction with phase difference $\varphi$ and open ends. In this case, if the finite size couplings $z_n$ are neglected (see Fig. 2 left panel), there is a crossing at zero energy and zero flux due to the MFs at the U point, signalling a change of parity in the GS when the flux changes sign. Together with it, two dispersionless zero energy modes appear, corresponding to the dangling MFs at the open ends. However, the system is expected to be unstable with respect to finite size interactions described by the $z_n$ couplings. Indeed as soon as one turns $z_n$ on, a gap opens and the zero energy MBS disappears (see Fig. 2b).

B. $\pi$–ring configuration: $u \neq 0$

In the ring configuration ($u \neq 0$), the situation is quite different (see Fig. 3). The zero energy MBS is always present, no matter how strong the finite size effects $z_n$ are. In the non-symmetric case ($z_L \neq z_R$), the location of the crossing occurs at non zero flux. With increasing of $u$, the flux of the crossing point drifts towards $\varphi = 0$ (see Fig. 3 bottom panels). For the physically relevant case $u >> z_L, z_R$, just the crossing Andreev bound states survive at low energy and the dispersion turns out to be approximately symmetric. The dispersion of the two crossing low lying energies tends to

$$E_{\pm} \sim \Gamma \sin\left(\frac{\varphi}{2}\right) (2n_{\text{end}} - 1) = \pm \Gamma \sin\left(\frac{\varphi}{2}\right).$$

The minimum is for $\varphi \approx \pm \pi$ (i.e. flux $\pm \phi_0/2$), depending on the occupancy observable $n_{\text{end}} = d^\dagger_{\text{end}} d_{\text{end}}$ of the MBS located at the Josephson Junction.

IV. MODEL FREE ENERGY AND STATIONARY CONDITIONS OF THE rf–SQUID

We have shown that, in the topologically non trivial $\pi$–ring structure, an unpaired zero energy MBS exists and it is robust with respect to perturbations. This shows up as a crossing of the particle and hole excitation dispersions at flux close to zero. Our $\pi$–ring modelizes a topologically non trivial rf–SQUID device and we now argue that the MBS characterizes in a measurable way the stationary conditions of the device.

If the ring has a small diameter, so that its self-inductance $L$ cannot be disregarded, the free energy is a function of the circulating current and of an external flux $\phi_{\text{ext}}$ which may be intentionally added. Its simplest form, arising from Eq.[12] is:

$$F_{\pm}(I, \phi_{\text{ext}}) = \frac{1}{2} LI^2 \pm \frac{\phi_0 I_c}{2\pi} \sin\left(\frac{\pi}{\phi_0}(\phi_{\text{ext}} + LI)\right).$$

We have disregarded charging effects that are always negligible in all HTS structures, unless the dimensions are scaled to a few hundred nanometers. Charging effects have been also considered here15, with no effect on the current periodicity, provided that the entire ring is in a topological nontrivial state. The free energy $F_{\pm}(I, \phi_{\text{ext}} = 0)$ at zero external flux is plotted in Fig. 4b). The first and second minimum, belonging to the same $\mathcal{P}$, differ in phase by $\approx 4\pi$. Fig.4b reports the change in shape of the free energy for one single $\mathcal{P}$, at different applied fluxes $\phi_{\text{ext}}$.

The similarity with the conventional YBCO F$\pi$R is only superficial. At first sight Fig. 4b could report the free energy plot of a conventional F$\pi$R . By fine-tuning the external flux the two energy minima can be made degenerate. If $\mathcal{P}$ is strictly conserved, this would be the end of the story and no difference would arise. However, in the topologically non trivial case, there is a corresponding set of curves belonging to the other $\mathcal{P}$, a switching between the two different minima e.g. at $\phi_{\text{ext}} = 0$ (see Fig. 4b) not only requires a flux quantum entering or exiting the ring but a simultaneous change of $\mathcal{P}$.

A tool to change the parity could be a side quantum point contact (QPC) controlling the charge tunneling. Addition of an electron on the ring would suddenly re-excite the system. The similarity with the conventional YBCO F$\pi$R is only superficial. At first sight Fig. 4b could report the free energy plot of a conventional F$\pi$R . By fine-tuning the external flux the two energy minima can be made degenerate. If $\mathcal{P}$ is strictly conserved, this would be the end of the story and no difference would arise. However, in the topologically non trivial case, there is a corresponding set of curves belonging to the other $\mathcal{P}$, a switching between the two different minima e.g. at $\phi_{\text{ext}} = 0$ (see Fig. 4b) not only requires a flux quantum entering or exiting the ring but a simultaneous change of $\mathcal{P}$.

A tool to change the parity could be a side quantum point contact (QPC) controlling the charge tunneling. Addition of an electron on the ring would suddenly require the switching of the whole device between the two possible GSs, corresponding to a jump in the trapped spontaneous flux. By contrast, a conventional F$\pi$R is expected to be widely insensitive to the in-out tunneling of induced charges. Operating with a side gate on the QPC allows to distinguish a topological non trivial ring from a conventional one.

However, in real life, the job of fixing $\mathcal{P}$ appears to
be rather hard. The ring is not expected to be isolated: background impurities could provide charge noise, by releasing or capturing charges. For a system open to the environment, the energy spectrum of the isolated ring looses meaning and a description of the state of affairs in terms of the statistical density matrix $\hat{\rho}(t)$ is required. The latter accounts for the transitions, with absorption and emission of the energy between the two parities and with simultaneous switching of the flux. Under these conditions, the expectation value of the observable corresponding to the flux, is likely to average to zero:

$$\lim_{t \to \infty} \langle \Phi \rangle = \text{Tr} \left\{ \hat{\rho} \Phi \right\} \sim 0 \quad (14)$$

V. CONCLUDING REMARKS

TIs or semiconducting nanowires with strong spin-orbit coupling in proximity with a singlet superconductor can host MBSs at their ends as predicted by the Kitaev model. There are various experiments involving charge transport, to provide evidence of the presence of the zero energy MBSs.

In our case, trenches between the crystals do not allow for Josephson coupling and no flux is trapped at the center of the structure until a ring is deposited on top of the HTS tricrystal. The three SNS weak links are determined by nanowire barrier and can sustain a supercurrent in the ring. Hence, again frustration can occur, provided that $LI_e >> \phi_0$. The top ring can be either made of a topological insulator nanowire or by a semiconducting nanowire with strong spin orbit coupling and magnetic field (topological non-trivial $\pi$ ring). The experiment we have suggested rests on the possibility of realizing control measurements on the same structure, where the topological insulator or semiconducting nanowire is replaced by a standard metallic nanowire (e.g. Au), thus which does not display any topological protection (see fig.5). The spontaneous and stable flux supported by a trivial tricrystal ring is in striking contrast with what expected in our case. The free energy of a topologically non trivial ring is depicted in Fig. 4]). It displays two minima, but it is qualitatively different from the the trivial case in that the two branches correspond to different fermion parities and the protected zero energy crossing corresponds to a MBS which can be occupied or empty. Two different scenarios are possible: if quasiparticle poisoning is negligible, parity is conserved and by changing the number of particles in the ring we can modify the flux state of the $\pi$ ring. By contrast, in the presence of quasiparticle poisoning, relaxation takes place and the expected flux is zero. An experiment comparing the behavior of the spontaneous fluxes of two tricrystal structures with different rings (one topological and the other trivial) would give an unambiguous proof of the presence of the MBS (see Fig. 4)). This would make the answer hopefully sharp and universal.

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FIG. 5: (color online) Comparison of the possible spontaneous flux states of a trivial and a nontrivial π-ring.

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30. Here the definition of the relevant fermions: $d_{end} = \frac{1}{2}(\gamma_LA + i\gamma_RAN)$, $d_1 = \frac{1}{2}(\gamma_LA + i\gamma_LBN)$, $f_{N-1} = \frac{1}{2}(\gamma_RBN + i\gamma_LAN - 1)$, $d_A = d_N + f_{N-1}$, $d_B = d_N - f_{N-1}$, $f_{end} = (\gamma_RBN + i\gamma_LBN)/2$, $d_{N-1} = \frac{1}{2}(\gamma_LAN + i\gamma_LBN - 1)$, $f_1 = \frac{1}{2}(\gamma_RBN + i\gamma_LAN)$, $f_2 = (f_1 + - d_{N-1})/\sqrt{2}$, $d_B = (f_1 + - d_{N-1})/\sqrt{2}$. 