Strongly non-quantitative classical information in quantum carriers

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A quantum state from which one can guess little about its underlying physical system may hide knowledge of the system which is revealed when the copy of quantum state is supplied. We give an example of two quantum states parameterized differently by the same random variable such that the first state alone offers a more accurate guess about the random variable in any figure of merit, while the two copies of second quantum state together do more in some figure of merit than the two copies of the original quantum state. The amount of information contained in quantum carriers does not behave quantitatively with respect to the number of simultaneously available carriers. Hidden information activated by copies implies the impossibility to specify the capability of quantum states to carry classical information from the single state.

When a complete description for carrier of information is given by a probability distribution or a quantum density operator, a single measurement of the carrier may not be sufficient to perfectly recover original information conveyed by the carrier. It is better to request multiple copies of the same carrier from the source if possible. If we have limitations on resources such as the number of copies, we have to optimize the measurement and guessing strategy for better information.

In this article, we show that the amount of information contained in quantum carriers may increase under copying so that it behaves non-quantitatively with respect to the number of copies to be measured together. Suppose we have two carriers $E_\rho$ and $E_\tau$, whose states are differently parameterized by a random variable of the underlying physical system. Carrier $E_\rho$ alone is assumed to offer better knowledge about the system than carrier $E_\tau$ does: that is, the reader can make a more accurate guess about the value of random variables from measurement results on $E_\rho$ than from on $E_\tau$, where the accuracy is measured by a certain figure of merit. Then the reader might guess that multiple copies of $E_\rho$ will give even better information than multiple copies of $E_\tau$, and would prefer to have carrier $E_\rho$ no matter whether copying is possible or not. Behind this guess is an intuition that information content is a quantity inherent to its carriers, and grows quantitatively along the number of identical carriers.

If the carriers are quantum entities, however, two copies of $E_\tau$ may offer better knowledge about the system. A series of analyses on entangled measurements [21, 20, 19] leads the existence of two carriers such that the first alone contains more information in a certain measure, while the two copies of second carrier get the benefit of entangled measurement and together offer more information in the same measure than the two copies of the first carrier. When the amount of information contained in these carriers are evaluated by a certain measure, it does not necessarily behave quantitatively with respect to the number of identically copied quantum carriers.

A question at this point is whether non-quantitative information (NQI) can be exhibited without employing particular measures of information. Even if carrier $E_\rho$ contains more information than $E_\tau$ does in a certain measure, it does not necessarily in an other measure [15, 11]. If a certain measure behaves non-quantitatively on a pair of carriers under copying, and if another measure evaluates their information content without copies differently, we say the pair exhibits weakly non-quantitative information (wNQI). The choice of measure is essential for wNQI.

Pairs of quantum carriers, if carefully chosen, may exhibit NQI independently to the measure evaluating information content of carriers without their copies. As we will show in the following, there are carriers $E_\rho$ and $E_\tau$ such that the former alone offers better information about the system in any measure, but with copies, the latter performs better in a certain measure. In contrast to wNQI, the measure only needs to be chosen on copied carriers, hence we say these carriers exhibit strongly non-quantitative information (sNQI) in this case. A quantum carrier less informative about the underlying physical system than another carrier on its own in any measure may still hide knowledge on the system and outperform the other one when multiple copies of them are compared.
The sNQI breaks the intuition that information content is a quantity inherent to its carriers. Besides its fundamental interest, further analysis on the pair of carriers exhibiting the sNQI leads to observations on quantum information theory that wNQI does not. Among these observations, we address quantum non-Markovianity exhibited by multiple uses of same channel sequences, incompleteness of what we call “single-carrier” measures, and a relationship between quantum information and hidden classical information potentially activated by copying.

To explain the NQI precisely, we employ the following abstract treatments of quantum carriers, their information content, and strategies to obtain the information. A quantum carrier refers to any physical system whose state is described by a density operator on a Hilbert space \( \mathcal{H} \). The density operator of the carrier is assumed to be parameterized by a random variable \( x \in X \) and denoted by \( \rho_x \in \mathcal{B}(\mathcal{H}) \); \( \mathcal{B}(\mathcal{H}) \) denotes the space of linear operators on \( \mathcal{H} \). Since the information theoretic character of a carrier investigated in this article is completely characterized by the ensemble \( \mathcal{E}_\rho = \{ \rho_x, p_x \}_{x \in X} \) of quantum states with probability \( p_x \) of the random variable, we sometimes use the symbol \( \mathcal{E}_\rho \) to refer also to the corresponding carrier.

When copies of the carrier is not available, the observer gets a supply of single carriers in state \( \rho_x \) with given probability \( p_x \), on which they perform a measurement represented by positive operator-valued measure (POVM) elements \( \{ E_y \in \mathcal{B}(\mathcal{H}) \}_{y \in Y} \) on \( \mathcal{H} \). They obtain result \( y \) with probability \( p(y|x) = \text{Tr}[\rho_x E_y] \) and guess the value \( x \) from \( y \). The guessing process is represented by the function \( g: Y \to X, \ y \to g_y \).

When copies of the carrier are available, the observer gets a supply of two carriers in the same state \( \rho_x \otimes \rho_x \) with probability \( p_x \), on which they perform a joint measurement represented by POVM elements \( \{ E_{xy} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \}_{y \in Y} \). They obtain result \( y \) with probability \( p(y|x) = \text{Tr}[\rho_x \otimes \rho_x E_y] \) and guess the value \( x \) from \( y \). A strategy by the observer is constituted of the POVM measurement and the function of the guessing process. The observer can optimize the strategy according to how density operators are parameterized by random variables, and to how the accuracy of guesses are estimated.

Since we consider information content obtainable by measurement strategies, its measures are real-valued functions of only measurement probabilities of single POVM measurements applied on carriers. For such a function \( \mathcal{M} \) to be a measure of information content obtainable without copies, it must satisfy the following: Let \( \mathcal{E}_\tau = \{ \tau_x \in \mathcal{B}(\mathcal{H}_1), p_x \}_{x \in X} \) and \( \mathcal{E}_\rho = \{ \rho_x \in \mathcal{B}(\mathcal{H}_2), p_x \}_{x \in X} \) be ensembles with the same random variable. If for any set of POVM elements \( \{ E_y \in \mathcal{B}(\mathcal{H}_1) \}_{y \in Y} \) there is a set of POVM elements \( \{ E'_y \in \mathcal{B}(\mathcal{H}_2) \}_{y \in Y} \) such that \( \text{Tr}[E_y \tau_x] = \text{Tr}[E'_y \rho_x] \) holds for any \( y \in Y \) and \( x \in X \), then \( \mathcal{M}(\mathcal{E}_\tau) \leq \mathcal{M}(\mathcal{E}_\rho) \). In words, if measurement results for ensemble \( \mathcal{E}_\tau \) can be reproduced by measurement results for \( \mathcal{E}_\rho \), information content of \( \mathcal{E}_\tau \) must be estimated to be lower than or equal to that of \( \mathcal{E}_\rho \). Conversely, any function of probabilities obtained by single POVM measurements with the above described condition is regarded as a measure of information content obtainable without copies, and we call them single-carrier (SC) measures.

The set of SC measures thus defined contains distinguishability measures such as maximum probabilities of correct hypothesis testing \([6]\) and unambiguous state discrimination \([11]\). These measures include maximization or minimization with regard to measurement probabilities on single carriers in their definition. If any measurement on \( \mathcal{E}_\tau \) can be simulated by those on \( \mathcal{E}_\rho \), \( \mathcal{E}_\rho \)'s distinguishability should be evaluated higher since \( \mathcal{E}_\rho \) has larger family of measurement probabilities over which optimization is taken. There are SC measures such as accessible information \([12]\) which are not considered as distinguishability measures.

When a SC measure \( \mathcal{M} \) is used to estimate information content of \( \mathcal{E}_\rho = \{ \rho_x, p_x \}_{x \in X} \) without copies, the corresponding measure of information content obtainable with the aid of single copy is \( \mathcal{M}_2(\mathcal{E}_\rho) := \mathcal{M}(\{ \rho_x \otimes \rho_x, p_x \}_{x \in X}) \). We call \( \mathcal{M}_2 \) a double-carrier (DC) measure. Measurements for DC measure may be jointly performed on the 2-copies of the same state from the ensemble.

NQI can be stated in a precise manner based on the presented setup. When a pair of quantum carriers, \( \mathcal{E}_\rho = \{ \rho_x, p_x \}_{x \in X} \) and \( \mathcal{E}_\tau = \{ \tau_x, p_x \}_{x \in X} \), satisfies the following two conditions:

\begin{align}
\mathcal{M}(\mathcal{E}_\rho) &> \mathcal{M}(\mathcal{E}_\tau), \\
\mathcal{M}_2(\mathcal{E}_\rho) &< \mathcal{M}_2(\mathcal{E}_\tau),
\end{align}

for some measure of information content \( \mathcal{M} \), the pair is said to exhibit wNQI. If the pair further satisfies

\[ \mathcal{M}'(\mathcal{E}_\rho) \geq \mathcal{M}'(\mathcal{E}_\tau), \]

for any SC measure \( \mathcal{M}' \), the pair is said to exhibit sNQI. In what follows we present a measure of information content and an example pair of carriers exhibiting sNQI.

The random variable in this article is a vector \( \mathbf{n} \) uniformly distributing over unit sphere \( S_2 \), which is called “spin direction” from its relevance to particle physics. For this random variable, averaged fidelity used in \([20][10][13]\) estimates information content of a carrier. Averaged fidelity \( F(\mathcal{E}_\rho) \) as a SC measure for a
carrier $E_\rho = \{\rho_n, \text{dn}\}_{n \in S_2}$ (dn represents the probability density for uniform distribution over unit sphere) is

$$F(E_\rho) := \max \int \rho(y|n) \frac{1 + n \cdot g_y}{2} \text{d}ng_y,$$

where the maximization is over strategies constituted of POVM elements $\{E_y\}_{y \in Y}$ and guessing process $g : y \mapsto g_y \in S_2$. The averaged fidelity estimates how much on average the observer can learn about the direction $n$ from a given carrier with state $\rho_n$, where the score of learning is $\cos^2(\alpha/2) = (1 + \mathbf{n} \cdot g_y)/2$ with $\alpha$ being the angle between $\mathbf{n}$ and guess $g_y$.

Dimensions of Hilbert spaces for our carriers $E_\rho = \{\rho_n, \text{dn}\}_{n \in S_2}$ and $E_\tau = \{\tau_{n,\delta}, \text{dn}\}_{n \in S_2}$ exhibiting sNQI are 2 and 4, respectively. For later convenience we denote the Hilbert space for $\rho_n$ by $\mathcal{H}$ and that for $\tau_{n,\delta}$ by $\mathcal{H} \otimes \mathcal{H}'$ where $\dim\mathcal{H} = \dim\mathcal{H}' = 2$. The density operators $\rho_n$ and $\tau_{n,\delta}$ are given by

$$\rho_n := \frac{I_{\mathcal{H}} + \sum_{i=1}^{3} n_i \sigma_i}{2},$$

$$\tau_{n,\delta} := \rho_{n,\delta} \otimes \frac{|0\rangle\langle 0|}{2} + \rho_{-n,\delta} \otimes \frac{|1\rangle\langle 1|}{2},$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$, $\sigma_i$ $(i=1,2,3)$ are unitary Pauli operators, $|0\rangle, 1\rangle \in \mathcal{H}'$ are orthonormal vectors, and state $\rho_{n,\delta}$ is defined by

$$\rho_{n,\delta} = (1 - \delta)\rho_n + \frac{\delta}{2} I_{\mathcal{H}}$$

with a constant $\delta \in [0,1]$.

To show sNQI, it remains to prove condition $[3]$. In the supplemental material [27] we construct a unital positive map $L_\delta : B(\mathcal{H} \otimes \mathcal{H}') \to B(\mathcal{H})$ such that

$$\text{Tr}[E\tau_{n,\delta}] = \text{Tr}[L_\delta(E)\rho_n] \quad (\forall n \in S_2),$$

for any operator $E \in B(\mathcal{H} \otimes \mathcal{H}')$. Existence of the map $L_\delta$ satisfying Eq. $[8]$ is sufficient for condition $[3]$. In fact any POVM measurement with elements $\{E_i\}_{i \in I}$ on ensemble $E_\tau$ is simulated by that with elements $\{L_\delta(E_i)\}_{i \in I}$ on $E_\rho$.

Remarkably, condition $[3]$, is satisfied with equality for any SC measure at $\delta = 0$. This can be observed by the inverse of relation Eq. $[8]$, namely, there is a unital positive map $J : B(\mathcal{H}) \to B(\mathcal{H} \otimes \mathcal{H}')$ such that

$$\text{Tr}[J(\mathcal{E}_\tau)\tau_{n,0}] = \text{Tr}[\mathcal{E}_\rho\rho_n] \quad (\forall n \in S_2),$$

for any operator $E \in B(\mathcal{H})$ [27]. Any SC measure is evaluated to be same for ensembles $\mathcal{E}_\rho$ and $\mathcal{E}_\tau$ at $\delta = 0$, since any POVM measurement on carrier $E_\rho$ can be simulated by that on $E_\tau$ and vice versa.

In summary, the pair of carriers $\mathcal{E}_\rho$ and $\mathcal{E}_\tau$ whose states defined by Eqs. $[5]$ and $[6]$, satisfies conditions $[1]$, $[2]$ and $[3]$ when $0 < \delta < 7 - 4\sqrt{3}$. These carriers exhibit sNQI: information content of these carriers reverses when copies are available. Without copies, the spin direction cannot be guessed more accurately by measurements on $\mathcal{E}_\tau$ than on $\mathcal{E}_\rho$ in any figure of merit. With copies, in other words when pairs of these carriers are compared, the averaged fidelity of $\mathcal{E}_\tau$ is higher than that of $\mathcal{E}_\rho$. 

TABLE 1: The averaged fidelity for carriers $\mathcal{E}_\rho$ and $\mathcal{E}_\tau$ with and without their copies. $F(\mathcal{E}_\rho)$ is well known and $F_2(\mathcal{E}_\rho)$ is obtained in [20]. Only lower bound is derived for $F_2(\mathcal{E}_\tau)$ ("l.b." stands for lower bound). See supplemental material [27] for derivations of $F_2(\mathcal{E}_\tau)$ and lower bound of $F_2(\mathcal{E}_\tau)$.

|                | $E_\rho$ | $E_\tau$ |
|----------------|---------|---------|
| without copies F | $\frac{5}{7}$ | $\frac{5}{7} - \frac{g}{7}$ |
| with a single copy F | $\frac{24}{25}$ | $\frac{24}{25} - \frac{3}{25} \delta$ |
Average fidelities calculated above for showing sNQI does not contradict values of mutual information \[9\]. In FIG. [1] we plot mutual information
\[
H(S_2; Y) := \int p(y|\mathbf{n}) \log_2 \frac{p(y|\mathbf{n})}{p(y)} \, d\mathbf{y},
\]
between spin direction $S_2$ of the underlying physical system and observers’ register $Y$ created by the measurements giving fidelities listed in TABLE. [1] With a single copy, mutual information of $\mathcal{E}_\rho$ is larger than that of $\mathcal{E}_\tau$ for small enough $\delta$. Averaged fidelity and mutual information share a region of $\delta$ in which the order of their value is reversed under copying.

Currently we are not sure if the carriers $\mathcal{E}_\rho$ and $\mathcal{E}_\tau$ exhibit sNQI with accessible information, namely, the maximally attainable mutual information. Under an assumption that the optimal strategy constitutes of covariant measurements \[13\], the values of mutual information plotted in FIG. [1] for $\mathcal{E}_\rho$ with and without its copies, and for $\mathcal{E}_\tau$ without its copies are maximum \[24\]. Accessible information demonstrates sNQI if the assumption is true.

Perhaps sNQI is against ones’ intuition if one knows classical information theory, because it is never demonstrated by any pair of probabilistic carriers, of which states are described by random variables. In terms of the difference between quantum and probabilistic carriers, sNQI is originating with the gap between positivity and complete positivity. The unital positive map $\mathcal{L}_\delta$ satisfying Eq. (8) immediately implies its extension for copied carriers since positive maps between random variables are automatically completely positive. Thus probabilistic carriers never exhibit sNQI.

The difference between Markov processes in classical \[9\] and quantum information theory \[1, 8, 5, 22, 2\] is highlighted by sNQI. Let us consider a sequence of classical-input quantum-output channels $(\Gamma_\rho : S_2 \to \mathcal{B}(\mathcal{H}), \Gamma_\tau : S_2 \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H}'))$ defined by $\Gamma_\rho(\mathbf{n}) = \rho_\mathbf{n}$ and $\Gamma_\tau(\mathbf{n}) = \tau_{\mathbf{n}, \delta}$. The existence of positive map $\mathcal{L}_\delta$ (regarded as a statistical morphism in \[3, 4\]) implies Markovianity of sequence $(\Gamma_\rho, \Gamma_\tau)$ in any of its classical snap-shots: for any POVM measurement $\{E_j\}_{j \in I}$ on $\mathcal{E}_\rho$ there exists a POVM measurement $\{F_i\}_{i \in I}$ on $\mathcal{E}_\rho$ and conditional probability $P(j|i)$ such that $\text{Tr}[E_j \tau_{\mathbf{n}, \delta}] = \sum_{i} P(j|i) \text{Tr}[F_i \rho_\mathbf{n}]$ holds for any $\mathbf{n} \in S_2$. Nevertheless sequence $(\Gamma_\rho, \Gamma_\tau)$ exhibits quantum non-Markovianity, since $\mathcal{L}_\delta$ is not completely positive \[4\]. Here, sNQI tells us that the increase of information content is simply demonstrated by the doubled sequence $(\Gamma_\rho \otimes \Gamma_\rho, \Gamma_\tau \otimes \Gamma_\tau)$. In this way, sNQI adds a new perspective that quantum non-Markovianity can be already observed when certain sequences are used in combination with the sequences themselves.

Comparison of carriers $\mathcal{E}_\rho$ and $\mathcal{E}_\tau$ in their information content leads to a consequence on quantum statistics which we call incompleteness of SC measures. According to sNQI, there is hidden information in ensembles of quantum states which cannot be witnessed by any SC measure. Moreover, even if the values of all SC

![Figure 1: Mutual information between spin direction and observers’ register obtained by measuring carriers $\mathcal{E}_\rho$ and $\mathcal{E}_\tau$ with and without their copies. The POVM elements $\{E_y\}_{y \in Y}$ of the observers’ measurement are those we used to obtain the values of fidelity listed in TABLE. [1] Mutual information for $\mathcal{E}_\tau$ with its single copy is higher than that for $\mathcal{E}_\rho$ when $0 \leq \delta \leq 0.0575$. See supplemental material \[24\] for derivations and analytic forms of these mutual information.](image)
measures are available at the same time, one cannot recognize the information hidden in $\mathcal{E}_\tau$ potentially activated by copying. In fact, at $\delta = 0$, any SC measure is evaluated to the same for $\mathcal{E}_\rho$ and $\mathcal{E}_\tau$, while at least one DC measure is evaluated higher for $\mathcal{E}_\tau$. In this sense the set of all SC measures is incomplete among all measures, since they are not sufficient for recognizing the hidden information potentially activated by copying.

If SC measures do not witness the hidden information, which measure effectively detects it without the use of measurements on copied systems? The incompleteness of SC measures tells us that such a measure does not estimate classical information extracted by measurements. Therefore, it is worth comparing the DC measures and measures of quantum information to see if classical information is hidden in a form of quantum information, while the notion of quantum information itself is ambiguous \cite{15}.

Among calculable functions of quantum information, optimal compression rate $R$ of blind compression task, in which the message sender has to compress a sequence of unidentified quantum states supplied from a source, is evaluated higher for ensemble $\mathcal{E}_\tau$ than for $\mathcal{E}_\rho$. It is given by von Neumann entropy for ensembles constituted only of pure states such as $\mathcal{E}_\rho$ \cite{24}, and can be calculated according to the prescription from \cite{14, 15} for ensembles of general mixed states. We have $R(\mathcal{E}_\rho) = 1$, and $R(\mathcal{E}_\tau)$ keeps constant value 2 for $0 \leq \delta < 1$. Thus the optimal blind compression rate witnesses the hidden information contained in $\mathcal{E}_\tau$.

This result on blind compression rate, together with the incompleteness of SC measures, extends a known discrepancy between von Neumann entropy and pairwise fidelity. We have $F_p(\rho_n, \rho_m) = F_p(\tau_n, 0, \tau_m, 0)$ for all the pairs of unit vectors $n, m \in S_2$, for pairwise fidelity $F_p(\rho_1, \rho_2) := \text{Tr}(\rho_1^{1/2} \rho_2^{1/2})^{1/2}$. However, the blind compression rate of ensembles $\mathcal{E}_\rho$ and $\mathcal{E}_\tau$ at $\delta = 0$ differ. Thus, it is possible to change the blind compression rate while keeping the values of all pairwise fidelity and all SC measures of the ensemble. The same discrepancy is previously known between von Neumann entropy and pairwise fidelity for pure state ensembles \cite{10}. Here we extend the discrepancy to mixed state ensembles where von Neumann entropy is generalized to blind compression rate, and under this generalized setting, answer to a question remain opened in \cite{15}: SC measures such as accessible information and minimum error probability does not help calculating blind compression rate for mixed state ensembles.

The presented sNQI reveals that the concept of “classical information” is independent of its “carrier” in quantum theory. When we say “classical information is conveyed by its carrier,” it is assumed that the carrier itself has an inherent ability to convey the information. It is already known that this inherent ability does not behave perfectly quantitatively when different carriers are combined (see e.g. \cite{14}). Still, we intuitively consider good carriers remain good when same copies of them can be used at the same time. This intuition lasting in a small way finally collapses by the discovery of sNQI. If carriers “contain” classical information, how is the carrier containing hidden information, potentially activated? Classical information requires a carrier when it is conveyed. However, the ability to convey classical information is not inherent in each carrier, but in the final structure of carriers at the message receiver.

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[24] See Supplemental Material for proofs of statements and analytical forms of mutual information.
Supplemental Materials

In this supplemental material we give our methods to obtain the results presented in the main text. In section 1 we construct two statistical morphisms used to compare $E_r$ and $E_\rho$ in its information content. In section 2 we calculate fidelity. Average fidelity of $E_r$ is derived with the optimal measurement strategy. While average fidelity of $E_r$ with its copy is not derived, we show a measurement strategy that gives a fidelity exceeding average fidelity of $E_\rho$ with its copy. In section 3 we calculate mutual information. Maximal mutual information between spin direction and observers’ register is derived under restriction to covariant measurements for three cases: $E_r$ with and without its copy, and $E_\rho$ without its copy. We show that the optimal covariant measurements rising maximum mutual information coincide with the optimal measurements for average fidelity. Finally we calculate mutual information of $E_r$ with its copy, obtained from the same measurement strategy we employed for calculating fidelity.

Here we give definitions of two ensembles $E_\rho = \{\rho_n, p(n)\}_{n\in S_2}$ and $E_r = \{\tau_{n,\delta}, p(n)\}_{n\in S_2}$ again. The vector $n$ on unit sphere $S_2$ is assumed to be completely unknown, so that $p(n)$ represents the uniform distribution over the unit sphere. Let $\mathcal{H}$ and $\mathcal{H}'$ both be two dimensional Hilbert spaces. Unitary Pauli matrices on $\mathcal{H}$ are denoted by $\sigma_i$ ($i = 0, 1, 2, 3$) with the first element $\sigma_0$ being the identity operator $I_\mathcal{H}$. Density operators $\rho_n \in B(\mathcal{H})$ and $\tau_{n,\delta} \in B(\mathcal{H} \otimes \mathcal{H}')$ are defined by

$$\rho_n := \frac{I_\mathcal{H} + \sum_{i=1}^3 n_i \sigma_i}{2},$$

$$\tau_{n,\delta} := \rho_{n,\delta} \frac{|0\rangle\langle 0|}{2} + \rho_{-n,\delta} \frac{|1\rangle\langle 1|}{2},$$

where $|0\rangle, |1\rangle \in \mathcal{H}'$ are orthonormal vectors, and state $\rho_{n,\delta}$ is defined by

$$\rho_{n,\delta} = (1 - \delta)\rho_n + \delta \frac{I_\mathcal{H}}{2},$$

with a constant $\delta \in [0, 1]$. For later convenience we denote the ensemble $\{\rho_n, p(n)\}_{n\in S_2}$ by $E_\rho'$.

1 Statistical transformation between $\rho_n$ and $\tau_{n,\delta}$

We construct unital positive maps $L_\delta : B(\mathcal{H} \otimes \mathcal{H}') \to B(\mathcal{H})$ and $J : B(\mathcal{H}) \to B(\mathcal{H} \otimes \mathcal{H}')$ such that

$$\text{Tr}[E_{\tau_{n,\delta}}] = \text{Tr}[L_\delta(E)\rho_n] \quad (\forall n \in S_2, \forall E \in B(\mathcal{H} \otimes \mathcal{H}')), \quad (4)$$

$$\text{Tr}[J(E)\tau_{n,0}] = \text{Tr}[E\rho_n] \quad (\forall n \in S_2, \forall E \in B(\mathcal{H})). \quad (5)$$

These maps are examples of statistical morphisms studied in [1]. The existence of $L_\delta$ is a proof that information content of $E_r$ never becomes larger than that of $E_\rho$ in any single-carrier measures. The existence of $J$ is a proof that information content of $E_r$ and $E_\rho$ is measured to be the same in any single-carrier measure.

We decompose $L_\delta$ into a sequence $D_0 \circ L_0$ of a statistical morphism $L_0$ from $E_r$ to $E_\rho'$ and the conjugate of depolarizing channel $D_0$. First define $L_0$ by

$$L_0(E) = \frac{1}{2} \text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |0\rangle\langle 0|)] + \frac{1}{2} \sigma_2 \{\text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |1\rangle\langle 1|)]\}^* \sigma_2, \quad (6)$$

where the complex conjugation is taken in the basis $|0\rangle, |1\rangle$. Then we have

$$\text{Tr}[L_0(E)\rho_{n,\delta}] = \frac{1}{2} \text{Tr}[\rho_{n,\delta} \text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |0\rangle\langle 0|)]] + \frac{1}{2} \text{Tr} [\rho_{n,\delta} \sigma_2 \{\text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |1\rangle\langle 1|)]\}^* \sigma_2]$$

$$= \frac{1}{2} \text{Tr}[\rho_{n,\delta} \text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |0\rangle\langle 0|)]] + \frac{1}{2} \text{Tr} [\sigma_2 \rho_{n,\delta} \text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |1\rangle\langle 1|)]]^* \sigma_2$$

$$= \frac{1}{2} \text{Tr}[\rho_{n,\delta} \text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |0\rangle\langle 0|)]] + \frac{1}{2} \text{Tr} [\rho_{n,\delta} \text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |1\rangle\langle 1|)]]$$

$$= \frac{1}{2} \text{Tr}[\rho_{n,\delta} \text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |0\rangle\langle 0|)]] + \frac{1}{2} \text{Tr} [\rho_{n,\delta} \text{Tr}_{\mathcal{H}'}[E(I_{\mathcal{H}} \otimes |1\rangle\langle 1|)]]$$

$$= \frac{1}{2} \text{Tr}[\epsilon(\rho_{n,\delta} \otimes |0\rangle\langle 0|)] + \frac{1}{2} \text{Tr}[\epsilon(\rho_{n,\delta} \otimes |1\rangle\langle 1|)]$$

$$= \text{Tr}[E_{\tau_{n,\delta}}], \quad (7)$$

$$1$$
where the second equality follows from

\[ (\sigma_2 \rho_{n, \delta} \sigma_2)^* = (1 - \delta) \sigma_2 \rho_{n, \delta} \sigma_2 + \delta \sigma_2 \left( \frac{I_H}{2} \right)^* \sigma_2 = (1 - \delta) \rho_{-n} + \delta \frac{I_H}{2}. \]

The linear map \( \mathcal{L}_0 \) is positive since it is a convex sum of two positive maps \( E \mapsto \text{Tr}_{H} [E (I_H \otimes |0\rangle \langle 0|)] \) and \( E \mapsto \sigma_2 \{ \text{Tr}_{H} [E (I_H \otimes |1\rangle \langle 1|)] \}^* \sigma_2 \), where the latter half is not completely positive.

Since \( \rho_{n, \delta} \) is obtained from \( \rho_n \) through depolarizing channel \( \rho \mapsto (1 - \delta) \rho + \delta \frac{I}{d} \), the conjugation of depolarizing channel

\[ \mathcal{D}_\delta(E) := (1 - \delta) E + \delta I_H \quad (\forall E \in \mathcal{B}(H)), \]

satisfies

\[ \text{Tr}[\mathcal{D}_\delta(E) \rho_n] = \text{Tr}[E \rho_{n, \delta}], \quad (8) \]

for any operator \( E \in \mathcal{B}(H) \). Combining equations (7) and (8) we have

\[ \text{Tr}[\mathcal{D}_\delta \circ \mathcal{L}_0(E) \rho_n] = \text{Tr}[\mathcal{L}_0(E) \rho_{n, \delta}] = \text{Tr}[E \tau_{n, \delta}], \]

for any operator \( E \in \mathcal{B}(H) \) as required.

When calculating averaged fidelity and mutual information of \( \mathcal{E}_\tau \) in Secs. 2 and 3, we use the fact that information content of \( \mathcal{E}_\tau \) and \( \mathcal{E}_{\rho'} \) are equivalent in any single-carrier measures. To see the equivalence, define a unital positive map \( \mathcal{J} : \mathcal{B}(H) \rightarrow \mathcal{B}(H \otimes H') \) by

\[ \mathcal{J}(E) = E \otimes |0\rangle \langle 0| + \sigma_2 E^* \sigma_2 \otimes |1\rangle \langle 1| \quad (\forall E \in \mathcal{B}(H)). \]

(9)

Then we have

\[ \text{Tr}[\mathcal{J}(E) \tau_{n, \delta}] = \frac{1}{2} \text{Tr}[E \rho_{n, \delta} \otimes |0\rangle \langle 0|] + \frac{1}{2} \text{Tr}[\sigma_2 E^* \sigma_2 \rho_{-n, \delta} \otimes |1\rangle \langle 1|] \]

\[ = \frac{1}{2} \text{Tr}[E \rho_{n, \delta} \otimes |0\rangle \langle 0|] + \frac{1}{2} \text{Tr}[E \rho_{n, \delta} \otimes |1\rangle \langle 1|] \]

\[ = \text{Tr}[E \rho_{n, \delta}] \quad (10) \]

for any operator \( E \in \mathcal{B}(H) \) and \( n \in S_2 \). While Eq. (7) implies that any measurement on carrier \( \mathcal{E}_\tau \) is simulated by a corresponding measurement on \( \mathcal{E}_{\rho'} \), Eq. (10) implies that the other way around is also true. Thus information content of carriers \( \mathcal{E}_\tau \) and \( \mathcal{E}_{\rho'} \) are estimated to be equal by any single-carrier measures.

2 Average fidelity

2.1 \( F(\mathcal{E}_\tau) \)

Instead of calculating the average fidelity \( F(\mathcal{E}_\tau) \) directly from \( \mathcal{E}_\tau \), we use equality

\[ F(\mathcal{E}_\tau) = F(\mathcal{E}_{\rho'}), \]

just noted at the end of Sec. 1 and search the optimal measurement strategy for maximizing \( f \) of ensemble \( \mathcal{E}_{\rho'} \). The explicit form of \( f \) is

\[ f(\mathcal{E}_{\rho'}, \{ E_g \}_{g \in Y}, g) = \int dY \text{Tr}[\rho_{n, \delta} E_g \frac{1 + n \cdot g}{2}], \]

where \( \{ E_g \}_{g \in Y} \) is a elements and \( g : Y \rightarrow S_2, \ y \mapsto g_y \) is a function. We first show that the optimal measurement can be assumed covariant. Then the form of POVM operators is reduced so that they depend only on the direction \( n \) and a single parameter. The optimization is completed by maximizing the fidelity along the parameter.

Let \( \{ E_g \}_{g \in Y}, g \) be an optimal strategy. Without loss of generality, we can assume that the POVM elements are labeled by \( m \) in image of function \( g \) because

\[ \int_{g_y = m} dY \text{Tr}[\rho_{n, \delta} E_y \frac{1 + n \cdot g_y}{2}] = \text{Tr} \left[ \rho_{n, \delta} \int_{g_y = m} dY E_y \frac{1 + n \cdot m}{2} \right]. \]
implies that integrating POVM elements up to have \( E_m = \int_{g_m=\mathbf{m}} dy E_y \) does not change the fidelity. Since POVM elements are labeled by the spin direction which is to be the guess, and since the spin direction is uniformly distributing over the sphere, we can assume that the measurement is covariant. That is, the label \( \mathbf{m} \) spreads all over the sphere \( S_2 \), and we have

\[
E_{R(m)} = UR_E m UR_R^\dagger.
\]

if \( R \) is a rotation on \( S_2 \), and \( UR \) is its representation on \( \mathcal{H} \).

Let us consider the POVM operator \( E_\uparrow \) for \( \uparrow := (0, 0, 1) \). Trace of \( E_\uparrow \) is

\[
\text{Tr}[E_\uparrow] = \int dn \text{Tr}[UR_n E_\uparrow UR_n^\dagger] = \text{Tr}[\int dn E_{R_n(\uparrow)}] = \text{Tr}[I_\mathcal{H}] = 2,
\]

where \( R_n \) is a rotation that turns vector \( \uparrow \) to \( \mathbf{n} \). Together with the positivity condition \( E_\uparrow \geq 0 \), this implies decomposition

\[
E_\uparrow = I_\mathcal{H} + \sum_{i=1}^3 r_i \sigma_i,
\]

with the Pauli matrices and a real vector \( \mathbf{r} = (r_1, r_2, r_3) \) satisfying \( |\mathbf{r}| \leq 1 \). Since vector \( \uparrow \) is invariant under rotations along \( z \)-axis,

\[
E_\uparrow = E_{R_z(\uparrow)} = UR_z E_\uparrow UR_z^\dagger,
\]

holds for any rotation \( R_z \) along \( z \)-axis, so that \( \sigma_1 \) and \( \sigma_2 \) components of \( E_\uparrow \) are eliminated. Finally we have the decomposition

\[
E_\uparrow = I_\mathcal{H} + r_3 \sigma_3 = (1 + r_3) |\uparrow\rangle\langle\uparrow| + (1 - r_3) |\downarrow\rangle\langle\downarrow|,
\]

with a real number \( r_3 \in [-1, 1] \). For this POVM operator, the probability (density) on state \( \rho_{\mathbf{n}, \delta} \) reduces to

\[
\text{Tr}[\rho_{\mathbf{n}, \delta} E_\uparrow] = (1 - \delta) \left\{ \left( 1 + r_3 \right) \frac{1 + \cos \theta}{2} + \left( 1 - r_3 \right) \frac{1 - \cos \theta}{2} \right\} + \delta = (1 - \delta)r_3 \cos \theta + 1,
\]

where \( \theta \) represents the angle between \( z \)-axis and \( \mathbf{n} \).

Now the integration for fidelity \( f \) can be calculated to have

\[
f(\mathcal{E}_\uparrow, \{ E_m \}_{m \in S_2}, g) = \int d\mathbf{m} d\mathbf{n} \text{Tr}[\rho_{\mathbf{n}, \delta} E_m] \frac{1 + \mathbf{n} \cdot \mathbf{m}}{2} = \int d\mathbf{m} d\mathbf{n} \text{Tr}[\rho_{\mathbf{n}, \delta} UR_n E_\uparrow UR_n^\dagger] \frac{1 + R_n^{-1}(\mathbf{n}) \cdot \uparrow}{2}
\]

\[
\qquad = \int d\mathbf{m} d\mathbf{n} \text{Tr}[\rho_{\mathbf{n}, \delta} E_\uparrow] \frac{1 + R_n^{-1}(\mathbf{n}) \cdot \uparrow}{2} = \int d\mathbf{m} \text{Tr}[\rho_{\mathbf{n}, \delta} E_\uparrow] \frac{1 + \mathbf{n} \cdot \uparrow}{2}
\]

\[
\qquad = \int \frac{d\theta d\phi \sin \theta}{4\pi} \left( 1 - \delta \right) \left\{ \left( 1 + r_3 \right) \frac{1 + \cos \theta}{2} + \left( 1 - r_3 \right) \frac{1 - \cos \theta}{2} \right\} + \frac{1 + \cos \theta}{2} - \frac{1}{2} + r_3(1 - \delta).
\]

The maximum value of \( f \) is \( \frac{2}{3} - \frac{\delta}{6} \) obtained for \( r_3 = 1 \). At the end, we have derived the average fidelity

\[
F(\mathcal{E}_\uparrow) = F(\mathcal{E}_\uparrow') = \frac{2}{3} - \frac{\delta}{6},
\]

accomplished by the optimal covariant measurement with POVM elements \( \{ 2|\mathbf{n}\rangle\langle\mathbf{n}| \}_{\mathbf{n} \in S_2} \).

### 2.2 Achievable averaged fidelity \( f_2(\mathcal{E}_\uparrow) \)

We make a rough outline of the measurement strategy for obtaining the fidelity \( f_2(\mathcal{E}_\uparrow) = \). On the second registers \( \mathcal{H}' \) of each side of the state \( \tau_{\mathbf{n}, \delta} \otimes \tau_{\mathbf{n}, \delta} \), first make measurement with operators \( \{ |0\rangle|0\rangle, |1\rangle|1\rangle \} \). Depending on the result \( 00, 11, 01, 10 \) which occurs with equal probability \( \frac{1}{4} \), the remaining state on the first registers \( \mathcal{H} \) is in either parallel or anti-parallel noisy spin state. Then we employ optimal measurement strategies for parallel spin states obtained in [4, 3] or for anti-parallel spin state obtained in [2, 3] depending on the result to guess the spin direction.
Let us first define vectors \( \mathbf{n}_i \) \((i = 0, 1, 2, 3)\) pointing to the summits of tetrahedron by

\[
\mathbf{n}_0 = (0, 0, 1), \quad \mathbf{n}_1 = \left( \frac{2\sqrt{2}}{3}, 0, -\frac{1}{3} \right), \quad \mathbf{n}_2 = \left( -\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3} \right), \quad \mathbf{n}_3 = \left( \frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3} \right).
\]

Following unit vectors in \( \mathcal{H} \otimes \mathcal{H} \) are employed from [3,2] for the measurement strategies:

\[
\begin{align*}
|\psi_{\text{para}^+\,i}\rangle &:= \frac{\sqrt{3}i}{2}|\mathbf{n}_i \otimes |\mathbf{n}_i \rangle + \frac{1}{2}|\Psi^\prime\rangle, \\
|\psi_{\text{para}^+\,i}\rangle &:= \frac{\sqrt{3}i}{2}|\mathbf{n}_i \otimes |\mathbf{n}_i \rangle + \frac{1}{2}|\Psi^\prime\rangle, \\
|\psi_{\text{anti}^+\,i}\rangle &:= \frac{3\sqrt{3} + 1}{4\sqrt{2}}|\mathbf{n}_i \otimes |\mathbf{n}_i \rangle - \frac{\sqrt{3} - 1}{4\sqrt{2}}\sum_{j \neq i}|\mathbf{n}_j \rangle \otimes |\mathbf{n}_j \rangle, \\
|\psi_{\text{anti}^+\,i}\rangle &:= \frac{3\sqrt{3} + 1}{4\sqrt{2}}|\mathbf{n}_i \otimes |\mathbf{n}_i \rangle - \frac{\sqrt{3} - 1}{4\sqrt{2}}\sum_{j \neq i}|\mathbf{n}_j \rangle \otimes |\mathbf{n}_j \rangle,
\end{align*}
\]

where \(|\Psi^\prime\rangle\) is the antisymmetric state.

In the reminder of this subsection, Hilbert spaces are lined in the order \( \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H}' \) on the notation. The POVM operators \( \{E_i\}_{i \in I_4} \) \((I_4 = \{0, 1, 2, 3\})\) for our strategy are given by

\[
E_i := E_{00i} + E_{01i} + E_{10i} + E_{11i} \quad (\forall i \in I_4),
\]

where

\[
\begin{align*}
E_{00i} &:= |\psi_{\text{para}^+\,i}\rangle \langle \psi_{\text{para}^+\,i}| \otimes |0\rangle \langle 0| \otimes |0\rangle \langle 0|, \\
E_{11i} &:= |\psi_{\text{para}^+\,i}\rangle \langle \psi_{\text{para}^+\,i}| \otimes |1\rangle \langle 1| \otimes |1\rangle \langle 1|, \\
E_{01i} &:= |\psi_{\text{anti}^+\,i}\rangle \langle \psi_{\text{anti}^+\,i}| \otimes |0\rangle \langle 0| \otimes |1\rangle \langle 1|, \\
E_{10i} &:= |\psi_{\text{anti}^+\,i}\rangle \langle \psi_{\text{anti}^+\,i}| \otimes |1\rangle \langle 1| \otimes |0\rangle \langle 0|.
\end{align*}
\]

The guessing direction \( g \) is defined by

\[
g_i = \mathbf{n}_i
\]

The fidelity \( f \) for \( \mathcal{E}_\tau \) with this measurement strategy is

\[
f(\mathcal{E}_\tau, E, g) = \frac{3}{2} \sum_{i = 0} \int \text{d}n \text{Tr}[\tau_{\mathbf{n}, \delta} \otimes \tau_{\mathbf{n}, \delta} E_i] \frac{1 + \mathbf{n} \cdot \mathbf{n}_i}{2}
\]

\[
= \frac{3}{4} \sum_{i = 0} \int \text{d}n \frac{\langle \psi_{\text{para}^+\,0} | \rho_{\mathbf{n}, \delta} \otimes \rho_{\mathbf{n}, \delta} | \psi_{\text{para}^+\,0} \rangle}{2} (1 + \mathbf{n} \cdot \mathbf{n}_i) + \frac{3}{4} \sum_{i = 0} \int \text{d}n \frac{\langle \psi_{\text{anti}^+\,0} | \rho_{\mathbf{n}, \delta} \otimes \rho_{\mathbf{n}, \delta} | \psi_{\text{anti}^+\,0} \rangle}{2} (1 + \mathbf{n} \cdot \mathbf{n}_i)
\]

\[
= \frac{3}{8} \sum_{i = 0} \int \text{d}n \langle \psi_{\text{para}^+\,0} | \rho_{\mathbf{n}, \delta} \otimes \rho_{\mathbf{n}, \delta} | \psi_{\text{para}^+\,0} \rangle (1 + \mathbf{n} \cdot \mathbf{n}_0) + \frac{3}{8} \sum_{i = 0} \int \text{d}n \langle \psi_{\text{anti}^+\,0} | \rho_{\mathbf{n}, \delta} \otimes \rho_{\mathbf{n}, \delta} | \psi_{\text{anti}^+\,0} \rangle (1 + \mathbf{n} \cdot \mathbf{n}_0),
\]

where the last equality comes from the symmetry of the measurement. Under the parameter \( \mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) we have

\[
\begin{align*}
\langle \psi_{\text{para}^+\,0} | \rho_{\mathbf{n}, \delta} \otimes \rho_{\mathbf{n}, \delta} | \psi_{\text{para}^+\,0} \rangle &= \frac{3(1 - \delta)^2 \cos^2 \theta + 6(1 - \delta) \cos \theta - (\delta + 1)(\delta - 3)}{16}, \quad (15) \\
\langle \psi_{\text{anti}^+\,0} | \rho_{\mathbf{n}, \delta} \otimes \rho_{\mathbf{n}, \delta} | \psi_{\text{anti}^+\,0} \rangle &= \frac{3(1 - \delta)^2 \cos^2 \theta + 2\sqrt{3}(1 - \delta) \cos \theta - (\delta^2 - 2\delta - 1)}{8},
\end{align*}
\]

which are substituted to the last line of Eq. (13) to yield

\[
\begin{align*}
\int \text{d}n \langle \psi_{\text{para}^+\,0} | \rho_{\mathbf{n}, \delta} \otimes \rho_{\mathbf{n}, \delta} | \psi_{\text{para}^+\,0} \rangle (1 + \mathbf{n} \cdot \mathbf{n}_0) &= \frac{3 - \delta}{8}, \\
\int \text{d}n \langle \psi_{\text{anti}^+\,0} | \rho_{\mathbf{n}, \delta} \otimes \rho_{\mathbf{n}, \delta} | \psi_{\text{anti}^+\,0} \rangle (1 + \mathbf{n} \cdot \mathbf{n}_0) &= \frac{3 + \sqrt{3} - \sqrt{3}\delta}{12}.
\end{align*}
\]
so we finally obtain

\[
f(E, E, g) = \frac{3 - \delta}{8} + \frac{3 + \sqrt{3} - \sqrt{3} \delta}{12} = \frac{2\sqrt{3} + 15}{24} - \frac{2\sqrt{3} + 3}{24} \delta.
\]

3 mutual information

In this section we calculate mutual information for carriers \( \rho_n \) and \( \tau_{n, \delta} \) obtained by the optimal measurements for average fidelity. We derive these values as maximum mutual information accomplished by optimal covariant measurements and then show that the covariant measurement is equivalent to the optimal measurement for average fidelity. Since average fidelity for \( E \) with its copy is not obtained, we calculate the corresponding mutual information for \( E \) with its copy from the measurement strategy described in Sec. [22]

In general, mutual information \( H_i(S_2; Y)_{E_{\nu}, (E_\nu) \in \{\nu\} \cup \{Y\}} \) between random variables \( S_2 \) and \( Y \) generated by POVM measurement \( \{E_y\}_{y \in \mathcal{Y}} \) on ensemble \( E_n = \{\mu_n, p(n)\}_{n \in S_2} \) is given by

\[
H(S_2; Y)_{E_{\nu}, (E_\nu) \in \{\nu\} \cup \{Y\}} = \int dny \ln \frac{\text{Tr}[\mu_n E_y]}{\text{Tr}[E_y \int d\mu_n]},
\]

\[
H_2(S_2; Y)_{E_{\nu}, (E_\nu) \in \{\nu\} \cup \{Y\}} = \int dny \ln \frac{\text{Tr}[\mu_n \otimes \mu_n E_y]}{\text{Tr}[E_y \int d\mu_n \otimes \mu_n]},
\]

If the measurements are assumed to be covariant measurement \( \{E_m\}_{m \in \mathcal{S}_2}, H(S_2; Y)_{E_{\nu}, (E_\nu) \in \{\nu\} \cup \{Y\}} \) further simplifies to

\[
H(S_2; S_2)_{E_{\nu}, (E_\nu) \in \{\nu\} \cup \{Y\}} = \int \text{d}m \text{Tr}[\mu_n E_m] \ln \frac{\text{Tr}[\mu_n E_m]}{\text{Tr}[E_m \int d\mu_n]},
\]

\[
= \int \text{d}m \text{Tr}[\mu_n^{-1}(E_m)] \ln \frac{\text{Tr}[\mu_n^{-1}(E_m)]}{\text{Tr} E_m \int d\mu_n},
\]

\[
= \int \text{d}n \text{Tr}[\mu_n E_n] \ln \frac{\text{Tr}[\mu_n E_n]}{\text{Tr} E_n \int d\mu_n},
\]

(17)

and \( H_2(S_2; Y)_{E_{\nu}, (E_\nu) \in \{\nu\} \cup \{Y\}} \) to

\[
H_2(S_2; S_2)_{E_{\nu}, (E_\nu) \in \{\nu\} \cup \{Y\}} = \int \text{d}n \text{Tr}[\mu_n \otimes \mu_n E_n] \ln \frac{\text{Tr}[\mu_n \otimes \mu_n E_n]}{\text{Tr} E_n \int d\mu_n \otimes \mu_n},
\]

(18)

by the same procedure.

3.1 Mutual information for \( E_\rho \) and \( E_\tau \) by optimal covariant measurements

As we have already noted in Sec. [1] any measurement on \( \tau_{n, \delta} \) can be simulated by a measurement on \( \rho_{n, \delta} \), and vice versa. This relationship is preserved by the restriction to covariant measurements, namely, any covariant measurement on \( \tau_{n, \delta} \) can be simulated by a covariant measurement on \( \rho_{n, \delta} \), and vice versa. In fact, if \( U \in B(\mathcal{H}) \) is a two dimensional representation of an element from \( SU(2) \) acting on \( \rho_{n, \delta} \), the action of the same element on state \( \tau_{n, \delta} \) is presented by \( U \otimes |0\rangle \langle 0| + \sigma_2 U \sigma_2 \otimes |1\rangle \langle 1| \in B(\mathcal{H} \otimes \mathcal{H}') \). These two representations are interchanged to each other by the statistic morphisms \( \mathcal{E}_0 \) and \( \mathcal{J} \) defined by Eqs. [9] and [10], respectively. This implies covariant measurements on \( \rho_{n, \delta} \) and \( \tau_{n, \delta} \) are also interchanged by these two statistic morphisms.

Thus optimization of the covariant measurement for mutual information of \( E_\tau \) can be replaced to that of \( E_\rho' \). Explicitly, we have

\[
\max_{\{E_n\}_{n \in S_2} \text{covariant}} H(S_2; S_2)_{E_{\nu}, (E_\nu) \in \{\nu\} \cup \{Y\}} = \max_{\{E_n\}_{n \in S_2} \text{covariant}} H(S_2; S_2)_{E_{\nu}, (E_\nu) \in \{\nu\} \cup \{Y\}},
\]

and shall consider optimal covariant measurement for \( E_\rho' \). The analysis includes the optimization of covariant measurement for mutual information of \( E_\rho \) as the special case \( \delta = 0 \).

As already derived, POVM operator \( E_1 \) for covariant measurement on \( \rho_{n, \delta} \) has a decomposition given by Eq. [13]. The denominator in logarithm of mutual information (17) is

\[
\text{Tr}[E_1 \int d\rho_{n, \delta}] = \text{Tr}[E_1 \frac{1}{2}] = 1,
\]

5
where we used $\int \text{d}n \rho_{n,\delta} = \frac{i}{2}$. Then the mutual information is

$$\int \text{d}n \text{Tr}[\rho_{n,\delta} E_{T}] \ln \text{Tr}[\rho_{n,\delta} E_{T}]$$

The integral of first term can be calculated by substituting Eq. (12) which yields

$$H(S_{2}; S_{2})_{E_{\rho}}; \{ E_{n} \}_{n \in S_{2}} = \frac{1}{2} \left( \frac{r}{2} + 1 + \frac{1}{2r} \right) \ln(1 + r) - \frac{1}{2} \left( \frac{r}{2} - 1 + \frac{1}{2r} \right) \ln(1 - r) - \frac{1}{2} \ln 2,$$

where $r = (1 - \delta)r_{3}$. Mutual information $H(S_{2}; S_{2})_{E_{\rho}}; \{ E_{n} \}_{n \in S_{2}}$ given by Eq. (19) is monotonically increasing according to $r$. Figure 1 presents $H(S_{2}; S_{2})_{E_{\rho}}; \{ E_{n} \}_{n \in S_{2}}$ as function (19) of $r$. The maximum value of (19) is obtained when $r_{3} = \pm 1$, and hence $r = 1 - \delta$, in which case the measurement is also optimal for average fidelity. We finally arrive at the values

$$\max_{\{ E_{n} \}_{n \in S_{2}}; \text{covariant}} H(S_{2}; S_{2})_{E_{\rho}}; \{ E_{n} \}_{n \in S_{2}} = 1 - \frac{1}{2 \log 2} \approx 0.279,$$

for ensemble $E_{\rho}$.

### 3.2 Mutual information for $E_{\rho}$ with its copy by optimal covariant measurement

The $+z$ component $E_{T}$ of POVM operators $\{ E_{n} \}_{n \in S_{2}}$ for covariant measurement on $\rho_{n} \otimes \rho_{n}$ is represented by

$$E_{T} = I_{H} \otimes I_{H} + \alpha (\sigma_{3} \otimes I_{H} + I_{H} \otimes \sigma_{3}) + \gamma (2\sigma_{3} \otimes \sigma_{3} - \sigma_{1} \otimes \sigma_{1} - \sigma_{2} \otimes \sigma_{2})$$

where two real parameters $\alpha$ and $\gamma$ satisfy

$$\alpha \leq \frac{\gamma}{2} + 1, \quad \alpha \geq -\frac{\gamma}{2} - 1, \quad \gamma \leq 1,$$

for operator $E_{T}$ to be positive [3]. We substitute this decomposition of $E_{T}$ to Eq. (15) to calculate the mutual information of $E_{\rho}$ obtained by covariant measurements on the 2 copies.

The denominator in logarithm of Eq. (15) is

$$\text{Tr}[E_{T} \int \text{d}n \rho_{n} \otimes \rho_{n}] = \text{Tr}[\int \text{d}n E_{R_{n}}^{-1} \rho_{T} \otimes \rho_{T}] = \text{Tr}[\rho_{T} \otimes \rho_{T}] = 1,$$

for ensemble $E_{\rho}$.

![Figure 1: Mutual information $H(S_{2}; S_{2})_{E_{\rho}}; \{ E_{n} \}_{n \in S_{2}}$ given by Eq. (19).](image_url)
\[ H_2(S_2; S_2)_{\varepsilon_p(E_n)_{n \in S^2}} = \int \ln \text{Tr}[\rho_n \otimes \rho_n E_1] \ln \text{Tr}[\rho_n \otimes \rho_n E_1]. \] (20)

The probability (density) \( \text{Tr}[\rho_n \otimes \rho_n E_1] \) is given by
\[ \text{Tr}[\rho_n \otimes \rho_n E_1] = \frac{3\gamma}{4} \cos^2 \theta + \alpha \cos \theta + 1 - \frac{\gamma}{4} \] (21)
where \( \theta \) is the angle between \( \uparrow \) and \( n \).

The analytical solution of the integral (20) needs case divergence based on the values of \( \alpha \) and \( \gamma \). Let \( D(\alpha, \gamma) \) be the discriminator
\[ D(\alpha, \gamma) = \alpha^2 + \frac{3}{4}\gamma^2 - 3\gamma \]
of Eq. (21) seen as a equation of order 2 of \( \cos \theta \). Then \( H_2(S_2; S_2)_{\varepsilon_p(E_n)_{n \in S^2}} \) is equal to
\[ \frac{(1 + \alpha)^2}{2\alpha} \ln(1 + \alpha) - \frac{(1 - \alpha)^2}{2\alpha} \ln(1 - \alpha) - 1, \]
when \( \gamma = 0 \), otherwise to
\[ h_0(\alpha, \gamma) := \left( \frac{\alpha}{3} - \frac{4\alpha^3}{27\gamma^2} + \frac{2\alpha}{3\gamma} + 1 \right) \ln \left( \frac{\gamma}{2} + 1 + \alpha \right) - \left( \frac{\alpha}{3} - \frac{4\alpha^3}{27\gamma^2} + \frac{2\alpha}{3\gamma} - 1 \right) \ln \left( \frac{\gamma}{2} + 1 - \alpha \right) + \frac{1}{\log 2} \left( \frac{\gamma}{3} + \frac{4\alpha^2}{9\gamma} - \frac{8}{3} \right), \]
when \( D(\alpha, \gamma) = 0 \) and
\[ h_0(\alpha, \gamma) + \frac{8D(\alpha, \gamma)}{27\gamma^2 \log 2} \left( \arctan \sqrt{\frac{\alpha + \frac{3\gamma}{2}}{\gamma D(\alpha, \gamma)}} - \arctan \sqrt{\frac{\alpha - \frac{3\gamma}{2}}{\gamma D(\alpha, \gamma)}} \right), \]
when \( D(\alpha, \gamma) < 0 \) and
\[ h_0(\alpha, \gamma) + \frac{4D(\alpha, \gamma)}{27\gamma^2} \ln \frac{1 - \gamma + \sqrt{D(\alpha, \gamma)}}{1 - \gamma - \sqrt{D(\alpha, \gamma)}}, \]
when \( D(\alpha, \gamma) > 0 \). Note that when \( D(\alpha, \gamma) > 0 \), the solution can be written in a different form
\[ \left( \frac{\alpha}{3} - \frac{4\alpha^3}{27\gamma^2} + \frac{2\alpha}{3\gamma} + 1 - \frac{4D(\alpha, \gamma)}{27\gamma^2} \right) \ln \left( \frac{\gamma}{2} + 1 + \alpha \right) + \left( \frac{\alpha}{3} - \frac{4\alpha^3}{27\gamma^2} - \frac{2\alpha}{3\gamma} + 1 - \frac{4D(\alpha, \gamma)}{27\gamma^2} \right) \ln \left( \frac{\gamma}{2} + 1 - \alpha \right) + \frac{1}{\log 2} \left( \frac{\gamma}{3} + \frac{4\alpha^2}{9\gamma} - \frac{8}{3} \right) + \frac{4D(\alpha, \gamma)}{27\gamma^2} \ln(1 - \gamma + \sqrt{D(\alpha, \gamma)})^2 \]
from which convergences along lines \( \alpha = \frac{3}{2} + 1 \) and \( \alpha = -\frac{3}{2} - 1 \) are easier to be seen. Figure 2 presents \( H_2(S_2; S_2)_{\varepsilon_p(E_n)_{n \in S^2}} \) as function of \( \alpha \) and \( \gamma \).

The maximum of mutual information is obtained for
\[ \alpha = \pm \frac{3}{2}, \gamma = 1, \]
at which the covariant measurement is also optimal for average fidelity on 2 copies. We finally obtain
\[ \max_{(E_n)_{n \in S^2; \text{covariant}}} H_2(S_2; S_2)_{\varepsilon_p(E_n)_{n \in S^2}} = \ln 3 - \frac{2}{3 \log 2} \approx 0.623. \] (22)
3.3 Mutual information for $\mathcal{E}_\tau$ with its copy

The measurement strategy we employed for deriving averaged fidelity of $\mathcal{E}_\tau$ exceeding that of $\mathcal{E}_\rho$ with their copies is not shown to be the optimal covariant measurement. Here we calculate mutual information of $\mathcal{E}_\tau$ obtained by the same measurement strategy.

The random variable of observer’s register is $I_4 := \{0, 1, 2, 3\}$, and the POVM operators $\{E_i\}_{i \in I_4}$ is presented in Eq. (13). Mutual information is given by

$$H_2(S_2; I_4)_{\mathcal{E}_\tau, \{E_i\}_{i \in I_4}} = \sum_{i=0}^{3} \int d\tau [\tau \otimes \tau] \ln \frac{\text{Tr}[\tau \otimes \tau E_i]}{\text{Tr}[\tau \otimes \tau E_0]}$$

where the second equality follows from symmetry of the POVM elements. The denominator in logarithm is

$$\text{Tr}[E_0 \int d\tau [\tau \otimes \tau]] = \frac{1}{2} (\psi^{\text{para}+}_0 [\rho_{n,\delta} \otimes \rho_{n,\delta}] \psi^{\text{para}+}_0 + \frac{1}{2} (\psi^{\text{anti}+}_0 [\rho_{n,\delta} \otimes \rho_{-n,\delta}] \psi^{\text{anti}+}_0 + 1)$$

The measurement probability (density) is

$$\text{Tr}[\tau \otimes \tau E_0] = \frac{1}{2} (\psi^{\text{para}+}_0 [\rho_{n,\delta} \otimes \rho_{n,\delta}] \psi^{\text{para}+}_0 + \frac{1}{2} (\psi^{\text{anti}+}_0 [\rho_{n,\delta} \otimes \rho_{-n,\delta}] \psi^{\text{anti}+}_0 + 1)$$

where we substitute Eqs. (15) and (16). Analytical solution of the integral (23) is given by

$$H_2(S_2; I_4)_{\mathcal{E}_\tau, \{E_i\}_{i \in I_4}} = \left( \frac{3 + 2\sqrt{3}}{24} q \left( 1 + \frac{87 - 12\sqrt{3}}{81q^2} \right) + \frac{1}{2} \right) \ln \left( \frac{3}{4} q^2 + \frac{3 + 2\sqrt{3}}{4} q + 1 \right)
- \left( \frac{3 + 2\sqrt{3}}{24} q \left( 1 + \frac{87 - 12\sqrt{3}}{81q^2} \right) - \frac{1}{2} \right) \ln \left( \frac{3}{4} q^2 - \frac{3 + 2\sqrt{3}}{4} q + 1 \right) + \frac{1}{4 \log 2} \left( q^2 + \frac{4\sqrt{3} - 41}{9} \right)
+ \frac{(51 - 12\sqrt{3} - 27q^2)}{972q \log 2} \left( \text{arctan} \frac{3 + 2\sqrt{3} + 9q}{\sqrt{51 - 12\sqrt{3} - 27q^2}} - \text{arctan} \frac{3 + 2\sqrt{3} - 9q}{\sqrt{51 - 12\sqrt{3} - 27q^2}} \right),$$

where $q = 1 - \delta$. When $\delta = 0$ the value of $H_2(S_2; I_4)_{\mathcal{E}_\tau, \{E_i\}_{i \in I_4}}$ reduces to

$$\frac{117 + 25\sqrt{3}}{162} \ln \frac{5 + \sqrt{3}}{2} - \frac{45 + 25\sqrt{3}}{162} \ln \frac{2 - \sqrt{3}}{2} + \frac{\sqrt{3} - 8}{9 \log 2} + \frac{2(6 - 3\sqrt{3})^2}{243 \log 2} \left( \text{arctan} \frac{6 + \sqrt{3}}{\sqrt{6 - 3\sqrt{3}}} - \text{arctan} \frac{-3 + \sqrt{3}}{\sqrt{6 - 3\sqrt{3}}} \right) \approx 0.718,$$

which is greater than mutual information (22) of $\mathcal{E}_\rho$ obtained by the optimal covariant measurement on 2 copies.

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Figure 2: Mutual information $H_2(S_2;S_2)\epsilon_{\rho\cdot\{E_n\}_{n\in S_2}}$ given as a function of $\alpha$ and $\gamma$ is plotted in region $\gamma \leq 1$, $\alpha \leq \frac{\gamma}{2} + 1$, and $\alpha \geq 0$. While the graph at negative $\alpha$ is omitted here, $H_2(S_2;S_2)\epsilon_{\rho\cdot\{E_n\}_{n\in S_2}}$ is symmetric with respect to the transformation $\alpha \rightarrow -\alpha$. 