TOPOLOGIZATION OF SETS ENDOowered WITH AN ACTION OF A MONOID

T. BANAKH, I. PROTASOV, O. SIPACHEVA

ABSTRACT. Given a set $X$ and a family $G$ of functions from $X$ to $X$ we pose and explore the question of the existence of a non-discrete Hausdorff topology on $X$ such that all functions $f \in G$ are continuous. A topology on $X$ with the latter property is called a $G$-topology. The answer will be given in terms of the Zariski $G$-topology $\zeta_G$ on $X$. This is the topology generated by the subbase consisting of the sets $\{x \in X : f(x) \neq g(x)\}$ and $\{x \in X : f(x) \neq c\}$ where $f, g \in G$, $c \in X$. We prove that for a countable submonoid $G \subseteq X^X$ the $G$-act $X$ admits a non-discrete Hausdorff $G$-topology if and only if the Zariski $G$-topology $\zeta_G$ is not discrete if and only if $X$ admits $2^c$ normal $G$-topologies.

1. Principal Problems

In this paper we consider the following general problem:

**Problem 1.1.** Given a set $X$ and a family $G$ of functions from $X$ to $X$ detect if $X$ admits a non-discrete Hausdorff (or normal) topology such that all functions $g \in G$ are continuous.

Since the composition of continuous functions is continuous, we lose no generality assuming that the family $G$ is a subsemigroup of the semigroup $X^X$ of all functions $X \to X$, endowed with the operation of composition. Also we can assume that $G$ contains the identity function $id_X$ of $X$ and hence $G$ is a submonoid of $X^X$. Thus it is natural to consider Problem 1.1 in the context of $G$-acts, i.e. sets endowed with an action of a monoid $G$. The two-sided unit of the monoid $G$ will be denoted by $1_G$. An action of a monoid $G$ on a set $X$ is a function $\alpha : G \times X \to X$, $\alpha : (g, x) \mapsto g(x)$ that has two properties:
- $1_G(x) = x$ for all $x \in X$ and
- $f(g(x)) = (fg)(x)$ for all $f, g \in G$ and $x \in X$.

A topology $\tau$ on a $G$-act $X$ is called a $G$-topology if for every $g \in G$ the shift $g : X \to X$, $g : x \mapsto g(x)$, is continuous. A $G$-act $X$ is called (normally) $G$-topologizable if $X$ admits a (normal) Hausdorff $G$-topology. A topology $\tau$ on a set $X$ is called normal if the topological space $(X, \tau)$ is normal in the sense that $X$ is a $T_1$-space such that any two disjoint closed subsets in $X$ have disjoint open neighborhoods.

In this terminology Problem 1.1 can be rewritten as follows.

**Problem 1.2.** Find necessary and sufficient conditions of (normal) $G$-topologizability of a given $G$-act $X$.

For $G$-acts endowed with an action of a group $G$ this problem has been considered in [1]. For countable monoids $G$, Problem 1.2 will be answered in Theorem 5.1 proved in Section 5. The answer will be given in terms of the Zariski $G$-topology on $X$, defined and studied in Section 2. In Section 5 we investigate largest $G$-topologies, generated by (special) filters.

2. The Zariski $G$-topology on a $G$-act

In this section we define the Zariski $G$-topology on a $G$-act $X$ and study this topology on some concrete examples of $G$-acts.

**Definition 2.1.** For a monoid $G$ and a $G$-act $X$ the Zariski $G$-topology $\zeta_G$ on $X$ is the topology generated by the subbase $\zeta_G$ consisting of the sets $\{x \in X : f(x) \neq g(x)\}$ and $\{x \in X : f(x) \neq c\}$ where $f, g \in G$ and $c \in X$.

The following easy fact follows immediately from the definition.

**Proposition 2.2.** For any $G$-act $X$ the Zariski $G$-topology $\zeta_G$ satisfies the separation axiom $T_1$ and lies in any Hausdorff $G$-topology on $X$.
Now let us introduce a cardinal characteristic \( \psi(x, \zeta_G) \) of the subbase \( \tilde{\zeta}_G \) of the Zariski \( G \)-topology \( \zeta_G \) called the pseudocharacter of \( \tilde{\zeta}_G \) at a point \( x \in X \). In fact, the pseudocharacter \( \psi(x, F) \) can be defined for any family \( F \) of subsets of \( X \). Given a point \( x \in X \) let
\[
F(x) = \{ X \} \cup \{ F \in F : x \in F \}
\]
and define the pseudocharacter of \( F \) at \( x \) as
\[
\psi(x, F) = \min \{ |U| : U \subset F(x) \text{ and } \cap U = \cap F(x) \}.
\]
It \( \tau \) is the topology on \( X \) generated by the subbase \( F \), then \( \tau(x) \) is the family of all open neighborhoods of \( x \) and \( \psi(x, \tau) \) is the usual pseudocharacter of the point \( x \) in the topological space \( (X, \tau) \). It is easy to see that \( \psi(x, \tau) = \psi(x, F) \) for any non-isolated point \( x \) in \( (X, \tau) \). If \( x \) is isolated in \( (X, \tau) \), then \( \psi(x, \tau) = 1 \) while \( 1 \leq \psi(x, F) < \aleph_0 \), so the pseudocharacter \( \psi(x, \tilde{\zeta}_G) \) carries more information than \( \psi(x, \zeta_G) \) in case of an isolated point \( x \) in \( (X, \zeta_G) \). If \( x \) is non-isolated, then \( \psi(x, \tilde{\zeta}_G) = \psi(x, \zeta_G) \).

In the algebraic language the pseudocharacter \( \psi(x, \tilde{\zeta}_G) \) equals the smallest number of inequalities of the form
\[
f(x) \neq g(x) \text{ or } f(x) \neq c \text{ where } f, g \in G, c \in X
\]
in a system of inequalities whose unique solution is \( x \).

Now let us consider the Zariski \( G \)-topology on some concrete examples of \( G \)-acts.

**Example 2.3.** Let \( X \) be an infinite set endowed with the natural action of the group \( G \) of all bijective functions \( f : X \to X \) that have finite support
\[
\text{supp}(f) = \{ x \in X : f(x) \neq x \}.
\]
It is easy to see that \( \psi(x, \zeta_G) = 1 \) and \( \psi(x, \tilde{\zeta}_G) = 2 \) for any point \( x \in X \). Consequently, the Zariski \( G \)-topology \( \zeta_G \) on \( X \) is discrete and the \( G \)-act \( X \) is not \( G \)-topologizable.

Each group \( G \) can be considered as an \( S \)-act for many natural actions of various submonoids \( S \) of the monoid \( G^G \). We define 6 such natural submonoids of \( G^G \):

- \( G_l \) is the subgroup of \( G^G \) that contains all left shifts \( l_a : x \mapsto ax \) of \( G \) for \( a \in G \);
- \( G_r \) is the subgroup of \( G^G \) that contains all right shifts \( r_a : x \mapsto xa \) of \( G \) for \( a \in G \);
- \( G_s \) is the subgroup of \( G^G \) that contains all two-sided shifts \( s_{a,b} : x \mapsto axb \) of \( G \) for \( a, b \in G \);
- \( G_q \) is the subgroup of \( G^G \) that contains all bijections of the form \( f : x \mapsto ax^r b \) where \( a, b \in G \) and \( r, s \in \{ 1, -1 \} \);
- \( G_m \) is the smallest submonoid of \( G^G \) that contains all functions of the form \( f : x \mapsto ax^m b \) where \( a, b \in G \) and \( m \in \mathbb{Z} \);
- \( G_p \) is the smallest submonoid that contains the subgroup \( G_q \) and together with any two functions \( f, g \in G_p \) contains their product \( f \cdot g : x \mapsto f(x) \cdot g(x) \).

Functions from the families \( G_m \) and \( G_p \) will be called monomials and polynomials on the group \( G \), respectively.

It is clear that
\[
G_l \cup G_r \subset G_s \subset G_q \subset G_m \subset G_p
\]
and hence
\[
\zeta_{G_l} \cup \zeta_{G_r} \subset \zeta_{G_s} \subset \zeta_{G_q} \subset \zeta_{G_m} \subset \zeta_{G_p}.
\]

A group \( G \) endowed with a \( G_l \)-topology (resp. \( G_r \)-topology, \( G_s \)-topology, \( G_q \)-topology) is called left-topological (resp. right-topological, semi-topological, quasi-topological).

Now for each monoid \( S \in \{ G_l, G_r, G_s, G_q, G_m, G_p \} \) we shall analyse the structure of the Zariski \( S \)-topology on the group \( G \). By the cofinite topology on a set \( X \) we understand the topology
\[
\tau_1 = \{ \emptyset \} \cup \{ X \setminus F : F \text{ is a finite subset of } X \}.
\]

The following remark can be easily derived from the definitions.

**Remark 2.4.** For any group \( G \) the Zariski topologies \( \zeta_{G_l} \) and \( \zeta_{G_r} \) on \( G \) coincide with the cofinite topology on \( G \). If \( G \) is infinite, then the topologies \( \zeta_{G_l} \) and \( \zeta_{G_r} \) are not Hausdorff.

**Remark 2.5.** For any infinite group \( G \) the Zariski topologies \( \zeta_{G_s} \) and \( \zeta_{G_q} \) are not discrete. This follows from a deep result of Y.Zelenyuk [14, 15] who proved that each infinite group \( G \) admits a non-discrete Hausdorff topology with continuous two-sided shifts and continuous inversion.
Remark 2.6. There is a countable infinite group $G$ whose Zariski $G_m$-topology $\zeta_{G_m}$ is discrete, see [12, p.70].

Remark 2.7. For a group $G$ endowed with the natural action of the monoid $G_p$ of all polynomial functions on $G$, the Zariski $G_p$-topology $\zeta_{G_p}$ coincides with the usual Zariski topology on the group $G$, studied in [2], [13], [3]. By the classical Markov’s result [9], a countable group $G$ is topologizable (which means that $G$ admits a non-discrete Hausdorff topology that turns $G$ into a topological group) if and only if the Zariski topology $\zeta_{G_p}$ is not discrete. Countable non-topologizable groups were constructed in [10] and [8]. For such groups $G$ the Zariski $G_p$-topology $\zeta_{G_p}$ is discrete.

Remark 2.8. The Markov’s topologizability criterion is not valid for uncountable groups: in [3] Hesse constructed an example of an uncountable non-topologizable group $G$ whose Zariski topology $\zeta_{G_p}$ is discrete.

Therefore, for an infinite group $G$, the Zariski $G_q$-topology $\zeta_{G_q}$ is always not discrete while the topology $\zeta_{G_m}$ can be discrete (for some countable non-topologizable groups).

If a group $G$ is abelian, then the Zariski topology $\zeta_{G_q}$ on $G$ coincides with the topologies $\zeta_{G_l}$ and $\zeta_{G_r}$ and hence is cofinite. However, for non-abelian groups $G$ the topology $\zeta_{G_q}$ can have rather unexpected properties.

Example 2.9 (Dicranjan-Toller). Let $H$ be a finite discrete topological group with trivial center (for example, let $H = \Sigma_3$ be the group of bijections of a 3-element set). For any cardinal $\kappa$ the Zariski topologies $\zeta_{G_\kappa}$, $\zeta_{G_q}$ and $\zeta_{G_p}$ on the group $G = H^\kappa$ coincide with the Tychonoff product topology $\tau$ on $G = H^\kappa$ and hence are compact, Hausdorff, and have pseudocomponent $\psi(x, \tau) = \kappa < 2^\kappa = |G_a| = |G_r| = |G_p| = |G|$ at each point $x \in G$.

Proof. Observe that the Tychonoff product topology $\tau$ on $G = H^\kappa$ turns the group $G$ into a compact topological group. Then each polynomial map on $G$ is continuous and each set $U \in \zeta_{G_q}$ is open in $X$. Consequently, $\zeta_{G_q} \subset \zeta_{G_\kappa} \subset \zeta_{G_p} \subset \tau$. The Tychonoff product topology $\tau$ is generated by the subbasic sets

$$U_{\alpha,h} = \{x \in X : \text{pr}_{\alpha}(x) = h\}$$

where $\alpha \in \kappa$, $h \in H$ and $\text{pr}_\alpha : H^\kappa \rightarrow H$ denotes the $\alpha$th coordinate projection. To prove that $\zeta_{G_\kappa} = \zeta_{G_q} = \zeta_{G_p} = \tau$ it suffices to check that each set $U_{\alpha,h}$ belongs to the topology $\zeta_{G_q}$.

Consider the embedding $i_\alpha : H \rightarrow H^\kappa$ that assigns to each element $x \in H$ the point $i_\alpha(x) \in H^\kappa$ such that $\text{pr}_\alpha \circ i_\alpha(x) = x$ and $\text{pr}_\beta \circ i_\alpha(x) = 1_H$ for all $\beta \neq \alpha$.

Given a point $h \in H$, consider the finite set $A_h = \{(a, b) \in H \times H : ah \neq hb, (a,b) \in A_h\}$ and observe that $\{h\} = \bigcap_{(a, b) \in A_h} \{x \in H : x^{-1}ax = b\}$. Indeed, by the triviality of the center of $H$, for any $x \in H \setminus \{h\}$ there is an element $a \in H$ such that $(xh^{-1})a \neq a(xh^{-1})$ and hence $h^{-1}ah \neq x^{-1}ax$. Put $b = x^{-1}ax$ and observe that $h^{-1}ah \neq b$ and hence $(a, b) \in A_h$.

For each pair $(a, b) \in A_h$ consider the left and right shifts $l_a : x \mapsto i_\alpha(a) \cdot x$ and $r_b : x \mapsto x \cdot i_\alpha(b)$ of the group $G = H^\kappa$. These shifts generate the subbasic set

$$U_{a,b} = \{x \in X : \text{pr}_{\alpha}(a) = x \neq x \cdot i_\alpha(b)\} = \{x \in X : l_a(x) \neq r_b(x)\} \in \zeta_{G_\kappa}.$$ 

It remains to observe that

$$\text{pr}_{\alpha}^{-1}(h) = \bigcap_{(a, b) \in A_h} U_{a,b} \in \zeta_{G_\kappa},$$

witnessing that $\tau = \zeta_{G_p} = \zeta_{G_q} = \zeta_{G_\kappa}$.

Remark 2.10. In fact, the Zariski topologies $\zeta_{G_l}, \zeta_{G_r}, \zeta_{G_q}$ can be defined on each semigroup $G$. If $G$ is commutative, then these Zariski topologies coincide. For the monoid $G = (\mathbb{N}, \max)$ the Zariski topology $\zeta_{G_q}$ is discrete. Indeed, for each $n \in \mathbb{N}$, the singleton $\{n\}$ belongs to the topology $\zeta_{G_q}$ as

$$\{n\} = \{x \in \mathbb{N} : \max\{x, n\} \neq n\} \cap \bigcap_{k < n} \{x \in \mathbb{N} : x \neq k\}.$$

This implies that the monoid $G = (\mathbb{N}, \max)$ is not $G_q$-topologizable.

3. $G$-topologies on $G$-acts, generated by special filters

In this section we describe and study $G$-topologies on $G$-acts, generated by filters.

A filter on a set $X$ is a family $\mathcal{F}$ of subsets of $X$ such that

- $\emptyset \notin \mathcal{F};$
- $A \cap B \in \mathcal{F}$ for any sets $A, B \in \mathcal{F};$
- $A \cup B \in \mathcal{F}$ for any sets $A \in \mathcal{F}$ and $B \subset X.$
By the pseudocharacter $\psi(\varphi)$ of a filter $\varphi$ we understand the smallest cardinality $|F|$ of a subfamily $F \subseteq \varphi$ such that $\bigcap F = \cap \varphi$. The character $\chi(\varphi)$ of a filter $\varphi$ equals the smallest cardinality of a subfamily $F \subseteq \varphi$ such that each set $F \subseteq \varphi$ contains some set $F \in F$. Observe that the character $\chi(x, \tau)$ of a topological space $(X, \tau)$ at a point $x$ can be defined as the character $\chi(x, \tau)$ of the neighborhood filter $\tau_x = \{U \in \tau : x \in U\}$.

For a filter $\varphi$ on $X$ consider the family

$$\varphi^+ = \{E \subseteq X : \forall F \in \varphi, F \cap E \neq \emptyset\}$$

equal to the union of all filters on $X$ that contain $\varphi$. It is easy to check that for each $A \subseteq X$ with $A \notin \varphi$, we get $X \setminus A \in \varphi^+$.

We shall say that a filter $\varphi$ on a topological space $X$ converges to a point $x_0$ if each neighborhood $U \subseteq X$ of $x_0$ belongs to the filter $\varphi$.

Now assume that $G$ is a monoid, $X$ is a $G$-act and $\varphi$ is a filter on $X$ such that $\bigcap \varphi = \{x_0\}$ for some point $x_0$. Then we can consider the largest $G$-topology $\tau_\varphi$ on $X$ for which the filter $\varphi$ converges to $x_0$. This topology admits the following simple description:

**Proposition 3.1.** The topology $\tau_\varphi$ consists of all sets $U \subseteq X$ such that for any $g \in G$ with $x_0 \in g^{-1}(U)$ the preimage $g^{-1}(U)$ belongs to the filter $\varphi$.

Now our strategy is to detect filters $\varphi$ on $X$ generating “nice” $G$-topology $\tau_\varphi$ on $X$.

**Definition 3.2.** Let $\kappa$ be a cardinal. An injective transfinite sequence $(x_\alpha)_{\alpha < \kappa}$ of points of a $G$-act $X$ is called special if there is an enumeration $G = \{g_\alpha\}_{\alpha < \kappa}$ of the monoid $G$ such that for all ordinals $\alpha < \kappa$ and $\beta, \gamma, \delta < \alpha$ we get

1. if $g_\beta(x_\alpha) \neq g_\gamma(x_\alpha)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\alpha)$;
2. if $g_\beta(x_\alpha) \neq g_\gamma(x_\beta)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\beta)$.

**Definition 3.3.** A filter $\varphi$ on a $G$-act $X$ is called special if for some cardinal $\kappa$ there is a special sequence $(x_\alpha)_{\alpha < \kappa}$ in $X$ such that $\bigcap \varphi = \{x_0\}$ and $\{x_0\} \cup \{x_\beta : \beta > \alpha\} \in \varphi$ for all ordinals $\alpha < \kappa$. In this case the set $X_0 = \{x_\alpha\}_{\alpha < \kappa}$ is called the special support of $\varphi$.

For a special filter $\varphi$ on $X$ the $G$-topology $\tau_\varphi$ has many nice properties.

**Theorem 3.4.** For any special filter $\varphi$ on a $G$-act $X$ with special support $X_0$ and the intersection $\bigcap \varphi = \{x_0\}$, the $G$-topology $\tau_\varphi$ has the following properties:

1. the topological space $(X, \tau_\varphi)$ is normal;
2. for any set $F \subseteq \varphi$ the set $G(F) = \{g(x) : g \in G, x \in F\}$ is closed-and-open in $(X, \tau_\varphi)$ and $X \setminus G(F)$ is discrete in $(X, \tau_\varphi)$;
3. $\{F \subseteq X_0 : F \subseteq \varphi\} = \{U \subseteq X_0 : x_0 \in U \subseteq \tau_\varphi\}$;
4. $\psi(x_0, \tau_\varphi) = \psi(\varphi)$ and $\chi(x_0, \tau_\varphi) \geq \chi(\varphi)$.

**Proof.** By definition, the special support $X_0$ of $\varphi$ admits an enumeration $X_0 = \{x_\alpha\}_{\alpha < \kappa}$ that has the properties (1), (2) from Definition 3.2 for some enumeration $G = \{g_\alpha\}_{\alpha < \kappa}$ of the monoid $G$. For every ordinal $\alpha < \kappa$ consider the set $X_{>\alpha} = \{x_\beta : \alpha < \beta < \kappa\}$ and observe that $\{x_0\} \cup X_{>\alpha} \in \varphi$ according to Definition 3.3. Now we shall prove the required properties of the $G$-topology $\tau_\varphi$.

**Claim 3.5.** The topology $\tau_\varphi$ satisfies the separation axiom $T_1$.

**Proof.** Given any point $x \in X$, we need to show that $X \setminus \{x\} \in \tau_\varphi$. Since the special filter $\varphi$ contains the sets $\{x_0\} \cup X_{>\alpha}$, $\alpha < \kappa$, it suffices for every map $g \in G$ with $g(x_0) \in X \setminus \{x\}$ to find $\alpha < \kappa$ such that $g(X_{>\alpha}) \subset X \setminus \{x\}$. If $x \notin G(X_0)$, then $g(X_{>\alpha}) \subset G(X_0) \subset X \setminus \{x\}$ and we are done.

So, assume that $x \in G(X_0)$ and find ordinals $\gamma, \delta < \kappa$ such that $x = g_\gamma(x_\delta)$. Also find an ordinal $\beta < \kappa$ such that $g_\beta = g$. Since $g_\beta(x_\delta) = g(x_\delta) \neq x = g_\gamma(x_\delta)$, the condition (2) of Definition 3.2 guarantees that $g(x_\alpha) = g_\beta(x_\alpha) \neq g_\gamma(x_\delta) = x$ for all $\alpha > \max\{\beta, \gamma, \delta\}$. Consequently, for the ordinal $\alpha = \max\{\beta, \gamma, \delta\}$ we get the required inclusion $g(X_{>\alpha}) \subset X \setminus \{x\}$.

**Claim 3.6.** The topology $\tau_\varphi$ is normal.

**Proof.** Let $A_0, B_0$ be two disjoint closed subsets in the topological space $(X, \tau_\varphi)$. Consider the sequences of sets $(A_n)_{n \in \omega}$ and $(B_n)_{n \in \omega}$ defined by the recursive formulas:

$$A_{n+1} = A_n \cup \{g_\alpha(x_\gamma) : \alpha < \gamma < \kappa, g_\alpha(x_\gamma) \in A_n, g_\alpha(x_\gamma) \notin B_n\}$$

and

$$B_{n+1} = B_n \cup \{g_\alpha(x_\gamma) : \alpha < \gamma < \kappa, g_\alpha(x_\gamma) \in B_n, g_\alpha(x_\gamma) \notin A_n\}$$

for each $n \in \omega$. Clearly, $A_0 \subseteq A_n$ and $B_0 \subseteq B_n$ for all $n \in \omega$ and $A_0 \cup B_0 \subseteq A_{n+1}$ and $A_0 \cup B_0 \subseteq B_{n+1}$.

Then $x_\alpha \in \tau_\varphi$ for some $\alpha < \kappa$. Since $x_\alpha \notin A_0$, it follows that $x_\alpha \notin A_n$ for all $n \in \omega$. Since $x_\alpha \notin B_0$, it follows that $x_\alpha \notin B_n$ for all $n \in \omega$. Hence $x_\alpha$ is not in any $g(\varphi)$, which is a contradiction. Therefore, $\tau_\varphi$ is normal.

**References:**

4. T. Banakh, I. Protasov, O. Sipachev, "A..."
and 

\[ B_{n+1} = B_n \cup \{ g_\beta(x_\delta) : \beta < \delta < \kappa, \ g_\beta(x_\delta) \in B_n, \ g_\beta(x_\delta) \notin A_n \}. \]

We claim that the sets \( A_x = \bigcup_{n \in \omega} A_n \) and \( B_x = \bigcup_{n \in \omega} B_n \) are open disjoint neighborhoods of the sets \( A_0 \) and \( B_0 \) in \((X, \tau_x)\). First we check that these sets are disjoint. Assuming the opposite, we can find numbers \( n, m \in \omega \) such that \( A_{n+1} \cap B_{m+1} = \emptyset \) but \( A_n \cap B_m = \emptyset = A_{n+1} \cap B_m \). Choose any point \( c \in A_{n+1} \cap B_{m+1} \). By the definitions of the sets \( A_{n+1} \) and \( B_{m+1} \), the point \( c \) is of the form \( g_\beta(x_\delta) = c = g_\beta(x_\delta) \) for some ordinals \( \alpha < \gamma < \kappa \) and \( \beta < \delta < \kappa \) such that \( g_\beta(x_\delta) \in A_n, \ g_\beta(x_\delta) \in B_m \). It follows from \( A_n \cap B_m = \emptyset \) that \( g_\beta(x_\delta) \neq g_\beta(x_\delta) \). The property (1) of Definition 3.2 guarantees that \( \gamma \neq \delta \). Without loss of generality, \( \delta > \gamma \).

Since \( g_\beta(x_\delta) \neq g_\beta(x_\delta) \), the property (2) of Definition 3.2 guarantees that \( g_\beta(x_\delta) \neq g_\beta(x_\delta) \) and this is the desired contradiction showing that \( A_x \cap B_x = \emptyset \).

Now let us show that the set \( A_x \) is open in \((X, \tau_x)\). Given an ordinal \( \alpha < \kappa \) with \( g_\alpha(x_\delta) \in A_x \), we should find a set \( F \in \varphi \) with \( g_\alpha(F) \in A_x \). Let \( n \in \omega \) be the smallest number such that \( g_\alpha(x_\delta) \in A_n \). We claim that the set \( F = \{ x_0 \} \cup \{ x \in X_{>\alpha} : g_\alpha(x) \notin B_n \} \) belongs to the filter \( \varphi \). Assuming that \( F \notin \varphi \), we conclude that the set \( X_{>\alpha} \setminus F \) belongs to the family \( \varphi^+ \). Then for \( k = n \) the set \( E_k = \{ x \in X_{>\alpha} : g_\alpha(x) \notin B_k \} \) belongs to the family \( \varphi^+ \). Let \( k \leq n \) be the smallest number such that \( E_k \in \varphi^+ \).

We claim that \( k > 0 \). Indeed, since \( B_0 \) is a closed subset in \((X, \tau_x)\), its complement \( X \setminus B_0 \) is an open neighborhood of the point \( g_\alpha(x_\delta) \in A_n \). Then the definition of the topology \( \tau_x \) yields a set \( F \in \varphi \) such that \( g_\alpha(F) \subset X \setminus B_0 \). Since \( F_0 \) intersects \( E_k \) and is disjoint with \( E_0 \), we conclude that \( k > 0 \).

Since \( \varphi^+ \notin E_{k-1} \subset E_k \in \varphi^+ \), the set \( E_k \setminus E_{k-1} \) is not empty and hence contains some point \( x_\gamma \), with \( \gamma > \alpha \). Then \( g_\alpha(x_\gamma) \notin B_k \setminus B_{k-1} \) and hence \( g_\alpha(x_\gamma) = g_\beta(x_\delta) \) for some ordinals \( \beta < \delta < \kappa \) with \( g_\beta(x_\delta) \in B_{k-1} \).

By the condition (1) of Definition 3.2 \( \delta \neq \gamma \) (as \( g_\alpha(x_\gamma) = g_\beta(x_\delta) \) and \( g_\alpha(x_\delta) \neq g_\beta(x_\delta) \)). If \( \delta > \gamma \), then the equality \( g_\alpha(x_\gamma) = g_\beta(x_\delta) \) is forbidden by the condition (2) of Definition 3.2 as \( B_{k-1} \neq g_\alpha(F) \neq g_\beta(x_\delta) \). If \( \gamma > \delta \), then the equality \( g_\alpha(x_\gamma) = g_\beta(x_\delta) \) also is forbidden by (2) because \( \alpha \neq g_\alpha(x_\delta) \neq g_\beta(x_\delta) \) belongs. The obtained contradiction shows that \( F \in \varphi \) and \( g_\alpha(F) \subset A_{<k} \subset A_x \), witnessing that the set \( A_x \) is open.

By analogy we may prove that the set \( B_x \) is open in \((X, \tau_x)\). Since \( A_x \) and \( B_x \) are disjoint open neighborhoods of the closed sets \( A_0, B_0 \), the topological \( T_1 \)-space \((X, \tau_x)\) is normal.

The definition of the topology \( \tau_x \) implies that for every set \( F \in \varphi \) the set \( G(F) = \{ g(x) : g \in G, x \in F \} \) is closed and-open in \((X, \tau_x)\) and \( X \setminus G(F) \) is discrete in \((X, \tau_x)\).

**Claim 3.7.** \( F \cap X_0 : F \in \varphi = \{ U \cap X_0 : x_0 \in U \in \tau_x \} \) and hence \( \chi(\varphi) \leq \chi(x_0, \tau_x) \).

**Proof.** The definition of the topology \( \tau_x \) guarantees that \( \{ U \in \tau_x : x_0 \in U \in \tau_x \} \subset \{ F \cap X_0 : F \in \varphi = \{ F \in \varphi : F \subset X_0 \} \} \). To prove the reverse inclusion, fix any subset \( F \in \varphi \) with \( F \subset X_0 \) and consider the set \( U = F \cup (X \setminus X_0) \). We claim that \( U \in \tau_x \). Given any ordinal \( \alpha < \kappa \) with \( g_\alpha(x_0) \in U \), we need to find a set \( E \in \varphi \) with \( g_\alpha(E) \subset U \). Find \( \beta < \kappa \) such that \( g_\beta = \text{id}_X \) and consider the set

\[ E = \{ x_0 \} \cup \{ x \in F : \text{max}\{\alpha, \beta\} < \gamma < \kappa \in \varphi \}. \]

We claim that \( g_\alpha(E) \subset U \). Assuming the converse, we could find an ordinal \( \gamma > \text{max}\{\alpha, \beta\} \) such that \( x_\gamma \in F \) and \( g_\alpha(x_\gamma) \in X_0 \setminus F \). Then \( g_\alpha(x_\gamma) = x_\delta = g_\beta(x_\delta) \) for some ordinal \( \delta < \kappa \). Since \( x_\gamma \in F \) and \( x_\delta \notin F \), the ordinals \( \gamma \) and \( \delta \) are distinct.

If \( \gamma < \delta \), then the inequality \( g_\beta(x_\delta) = x_\delta = g_\alpha(x_\gamma) \) and the condition (2) of Definition 3.2 guarantee that \( g_\beta(x_\delta) \neq g_\alpha(x_\gamma) \), which is a contradiction.

If \( \gamma > \delta \), then the inequality \( g_\alpha(x_\gamma) = x_\delta = g_\beta(x_\delta) \) and the condition (2) of Definition 3.2 imply that \( g_\alpha(x_\gamma) \neq g_\beta(x_\delta) = x_\delta \), which again leads to a contradiction.

**Claim 3.8.** The topology \( \tau_x \) has pseudocharacter \( \psi(x_0, \tau_x) = \psi(\varphi) \) at the point \( x_0 \).

**Proof.** The inequality \( \psi(\varphi) \leq \psi(x_0, \tau_x) \) follows from Claim 3.7. To show that \( \psi(x_0, \tau_x) \leq \psi(\varphi) \), fix a subfamily \( F \subset \varphi \) such that \( |F| = \psi(\varphi) \) and \( \cap F = \{ x_0 \} \). For every \( F \in F \) define an open neighborhood \( U^F \in \tau_x \) of \( x_0 \) as the union \( U^F = \bigcup_{n \in \omega} U^F_n \) of the sequence of sets \( \{ U^F_n \}_{n \in \omega} \) defined by the recursive formula: \( U^F_0 = \{ x_0 \} \) and

\[ U^F_{n+1} = U^F_n \cup \{ g_\alpha(x_\beta) : \alpha < \beta < \kappa, \ x_\beta \notin F, \ g_\alpha(x_\beta) \in U^F_n \} \]

for every \( n \in \omega \).

The definition of the topology \( \tau_x \) implies that \( U^F = \bigcup_{n \in \omega} U^F_n \) is an open neighborhood of the point \( x_0 \) in \( X \).

Let us show that \( \bigcap_{F \in \varphi} U^F = \{ x_0 \} \). Assume conversely that this intersection contains a point \( x \), distinct from \( x_0 \). For every \( F \in \varphi \) find the smallest number \( n_F \in \omega \) such that \( x \in U^F_{n_F} \). Since \( U^F_0 = \{ x_0 \} \neq \{ x \} \), we
conclude that \( n_F > 0 \) and hence \( x \notin U^F_{n_F - 1} \). By the definition of the set \( U_n^F \), there are ordinals \( \alpha_F < \beta_F < \kappa \) such that \( x_{\beta_F} \in F, x = g_{\alpha_F}(x_{\beta_F}) \neq g_{\alpha_F}(x_0) \in U^F_{n_F - 1} \).

Fix any set \( F \in \mathcal{F} \). Since \( x_{\beta_F} \in F \) and \( x_{\beta_F} \notin \{x_0\} = \cap \mathcal{F} \), there is a set \( E \in \mathcal{F} \) such that \( x_{\beta_E} \notin E \). Then \( \beta_F \neq \beta_E \). Without loss of generality, \( \beta_F < \beta_E \). Since \( \beta_E > \max\{\alpha_E, \beta_F, \alpha_F\} \) and \( g_{\alpha_E}(x_0) \neq x = g_{\alpha_F}(x_{\beta_F}) \), the condition (2) of Definition 3.2 guarantees that \( x = g_{\alpha_E}(x_{\beta_F}) \neq g_{\alpha_F}(x_{\beta_F}) = x \), which is a desired contradiction that proves the equality \( \bigcap_{F \in \mathcal{F}} U^F = \{x_0\} \) and the upper bound \( \psi(x_0, \tau_\varphi) \leq \psi(\varphi) \). \( \square \)

4. ZARISKI G-TOPOLOGY AND THE EXISTENCE OF SPECIAL FILTERS

In light of Theorem 3.4 it is important to detect G-acts X that possess special sequences and special filters.

**Proposition 4.1.** Let \( G \) be a monoid, \( X \) be a G-act, \( x_0 \in X \) be a point, and \( \lambda \) be an infinite cardinal.

(1) If \( |G| \leq \kappa \leq \psi(x_0, \zeta_G) \), then the G-act \( X \) contains a special sequence \((x_\alpha)_{\alpha<\kappa}\).

(2) If the G-act \( X \) contains a special sequence \((x_\alpha)_{\alpha<\kappa}\), then \( |G| \leq \kappa \) and \( \text{cf}(\kappa) \leq \psi(x_0, \zeta_G) \).

**Proof.** 1. Assume that \( |G| \leq \kappa \leq \psi(x_0, \zeta_G) \) and let \( G = \{g_\alpha : \alpha < \kappa \} \) be an enumeration of the monoid \( G \) such that \( g_0 = 1_G \). By induction we shall construct an injective transfinite sequence \((x_\alpha)_{\alpha<\kappa}\) of points of the set \( X \) such that for any \( \alpha < \kappa \) and \( \beta, \gamma, \delta < \alpha \):

(3) if \( g_\beta(x_0) \neq g_\gamma(x_0) \), then \( g_\beta(x_0) \neq g_\gamma(x_\alpha) \);

(4) if \( g_\beta(x_0) \neq g_\gamma(x_\alpha) \), then \( g_\beta(x_\alpha) \neq g_\gamma(x_\alpha) \).

Assume that for some ordinal \( \alpha < \kappa \) the points \( x_\beta, \beta < \alpha \), have been constructed. For any ordinals \( \beta, \gamma, \delta < \alpha \) consider the open neighborhoods

\[ U_{\beta,\gamma} = \{x \in X : g_\beta(x_0) \neq g_\gamma(x_0) \Rightarrow g_\beta(x) \neq g_\gamma(x)\} \]

and

\[ V_{\beta,\gamma,\delta} = \{x \in X : g_\beta(x_0) \neq g_\gamma(x_\alpha) \Rightarrow g_\beta(x) \neq g_\gamma(x_\delta)\} \]

of \( x_0 \) in the Zariski G-topology \( \zeta_G \). Since \( \psi(x_0, \zeta_G) \geq \kappa \), the intersection \( \bigcap_{\beta,\gamma,\delta<\alpha} U_{\beta,\gamma,\delta} \) has cardinality \( \geq \kappa \) and hence contains some point \( x_\alpha \in X \setminus \{x_\beta : \beta < \alpha\} \). It is clear that this point \( x_\alpha \) satisfies the conditions (3), (4).

2. Now assume that the G-act \( X \) contains a special sequence \( X_0 = \{x_\alpha\}_{\alpha<\kappa} \) for some infinite cardinal \( \kappa \). Let \( G = \{g_\alpha\}_{\alpha<\kappa} \) be an enumeration of the monoid \( G \) such that the conditions (1), (2) of Definition 3.2 are satisfied. Then \( |G| \leq \kappa \). We claim that \( \psi(x_0, \zeta_G) \geq \text{cf}(\kappa) \). Assuming the opposite, we can find a subfamily \( U \subset \zeta_G \) such that \( \cap U = \{x_0\} \) and \( |U| < \text{cf}(\kappa) \). For each set \( U \subset \zeta_G \) we can choose ordinals \( \alpha_U, \beta_U < \kappa \) and a point \( c_U \in X \) such that \( U = \{x \in X : g_{\alpha_U}(x) \neq g_{\beta_U}(x)\} \) or \( \{x \in X : g_{\alpha_U}(x) = c_U\} \). If \( c_U \in G(X_0) \), then we can find ordinals \( \gamma_U, \delta_U < \kappa \) such that \( c_U = g_{\gamma_U}(x_{\delta_U}) \). In the opposite case, put \( \gamma_U = \delta_U = 0 \).

Since the set \( A_U = \{\alpha_U, \beta_U, \gamma_U, \delta_U : U \in U\} \) has cardinality < \( \text{cf}(\kappa) \), there is an ordinal \( \alpha < \kappa \) such that \( \alpha > \sup A_U \). We claim that \( x_\alpha \in \cap U \). To prove this inclusion, take any set \( U \in \mathcal{U} \). If \( U = \{x \in X : g_{\alpha_U}(x) \neq g_{\beta_U}(x)\} \), then the inclusion \( x_0 \in U \) and the condition (1) of Definition 3.2 guarantee that \( x_\alpha \in U \). If \( U = \{x \in X : g_{\alpha_U}(x) = c_U\} \) and \( c_U \in G(X_0) \), then the equality \( c_U = g_{\gamma_U}(x_{\delta_U}) \), the inclusion \( x_0 \in U \) and the condition (2) of Definition 3.2 imply that \( x_\alpha \in U \). If \( c_U \notin G(X_0) \), then \( g_{\alpha_U}(x_\alpha) \neq c_U \) and hence \( x_\alpha \notin U \)

Therefore \( x_\alpha \cap \mathcal{U} = \{x_0\} \), which is a desired contradiction. \( \square \)

5. G-TOPOLOGIZABILITY OF G-ACTS

In this section we apply the results of the preceding sections and prove our main result:

**Theorem 5.1.** Let \( G \) be a monoid and \( X \) be a G-act. If \( |G| \leq \kappa \leq \psi(x_0, \zeta_G) \) for some point \( x_0 \in X \) and some infinite cardinal \( \kappa \), then for any infinite cardinal \( \lambda \leq \text{cf}(\kappa) \) the G-act \( X \) admits \( 2^{2^\kappa} \) normal G-topologies with pseudocharacter \( \lambda \) at the point \( x_0 \).

**Proof.** By Proposition 4.1 the space \( X \) contains a special sequence \( X_0 = \{x_\alpha\}_{\alpha<\kappa} \). Let \( \varphi_0 \) be the filter on \( X \) generated by the sets \( \{x_\alpha : \beta > \alpha\}, \alpha < \kappa \). Denote by \( \uparrow \varphi_0 \) the set of all filters \( \varphi \) on \( X \) that contain the filter \( \varphi_0 \) and have \( \cap \varphi = \{x_0\} \).

**Claim 5.2.** For any infinite cardinal \( \lambda \leq \text{cf}(\kappa) \) the set \( \mathcal{F}_\lambda = \{\varphi \in \uparrow \varphi_0 : \psi(\varphi) = \lambda\} \) has cardinality \( |\mathcal{F}| = 2^{2^\kappa} \).
Proof. First observe that the family of all filters on the set $X_0$ has cardinality $\leq 2^{2^\kappa}$. So, $|\mathcal{F}_\lambda| \leq 2^{2^\kappa}$. To prove the reverse inequality, we consider two cases.

1. $\lambda = \text{cf}(\kappa)$. Write the set $X_0$ as the disjoint union $X_0 = X_0' \cup X_0''$ of two sets of cardinality $|X_0'| = |X_0''| = \kappa$ such that $x_0 \in X_0'$. On the set $X_0'$ consider the filter $\varphi_0|X_0' = \{F \cap X_0' : F \in \varphi_0\}$. The Pospíšil Theorem [11] (see also [9]) implies that the family $\mathcal{U}_0$ of all ultrafilters on $X_0''$ that contain the filter $\varphi_0|X_0'' = \{F \cap X_0'' : F \in \varphi_0\}$ has cardinality $2^{2^\kappa}$. For any ultrafilter $u \in \mathcal{U}_0$ consider the filter $\varphi_u = \{A \subset X : A \cap X_0'' \in u, A \cap X_0' \in \varphi_0|X_0'\}$ and observe that $\psi(\varphi_u) = \psi(\varphi_0|X_0') = \text{cf}(\kappa)$. Since for distinct ultrafilters $u, v \in \mathcal{U}_0$ the filters $\varphi_u$, $\varphi_v$ are distinct, we conclude that $|\mathcal{F}_{\text{cf}(\kappa)}| \geq 2^{2^\kappa}$.

2. $\lambda < \text{cf}(\kappa)$. In this case the ordinal $\kappa$ can be identified with the product $\kappa \times \lambda$ endowed with the lexicographic order: $(\alpha, \beta) < (\alpha', \beta')$ iff $\alpha < \alpha'$ or $(\alpha = \alpha'$ and $\beta < \beta')$. Let $\xi : \kappa \times \lambda \to \kappa$ be the order isomorphism. On the cardinal $\lambda$ consider the filter $\varphi_\lambda$ of cofinite subsets. This filter has pseudocharacter $\psi(\varphi_\lambda) = \lambda$. By the preceding case, the family $\mathcal{F}_{\text{cf}(\kappa)}$ has cardinality $2^{2^\kappa}$. For any filter $u \in \mathcal{F}_{\text{cf}(\kappa)}$ consider the filter $\varphi_u$ on $X$ generated by the sets

$$\Phi_{U,L} = \{x_0\} \cup \{x_\xi(\alpha, \beta) : x_\alpha \in U, \beta \in L\}$$

where $U \in u$, $L \in \varphi_\lambda$. It can be shown that $\psi(\varphi_u) = \psi(\varphi_\lambda) = \lambda$ and for distinct filters $u, v \in \mathcal{F}_{\text{cf}(\kappa)}$ the filters $\varphi_u$ and $\varphi_v$ are distinct. Consequently, $|\mathcal{F}_\lambda| \geq |\mathcal{F}_{\text{cf}(\kappa)}| \geq 2^{2^\kappa}$.

For any filter $\varphi \in \mathcal{F}_\lambda$ the $G$-topology $\tau_\varphi$ on $X$ is normal and has pseudocharacter $\psi(x_0, \tau_\varphi) = \psi(\varphi) = \lambda$ at $x_0$ according to Theorem 5.4. Theorem 5.4(3) implies that for distinct filters $u, v \in \mathcal{F}_\lambda$ the topologies $\tau_u$ and $\tau_v$ are distinct. Consequently, $X$ admits at least $|\mathcal{F}_\lambda| = 2^{2^\kappa}$ normal $G$-topologies with pseudocharacter $\lambda$ at $x_0$.

Remark 5.3. Example 2.9 shows that Theorem 5.1 cannot be reversed: the group $G = H^\kappa$ is normally $G_\kappa$-topologizable but $\psi(x_0, \zeta_G) = \kappa < 2^\kappa = |G_\kappa|$ for any point $x_0$.

For countable monoids $G$, Theorem 5.4 implies the following characterization of $G$-topologizability that answers Problem 1.2.

Theorem 5.4. For countable monoid $G$ and a $G$-act $X$ the following conditions are equivalent:

1. $X$ admits a non-discrete Hausdorff $G$-topology;
2. the Zariski $G$-topology $\zeta_G$ on $X$ is not discrete;
3. $X$ admits $2^\kappa$ non-discrete normal $G$-topologies.

We do not know if this theorem holds for arbitrary $G$-acts.

Problem 5.5. Let $G$ be an uncountable monoid (group). Is a $G$-act $X$ $G$-topologizable if its Zariski $G$-topology $\zeta_G$ is not discrete?

It may happen that the results of [5] can help to give an answer to this problem.

References

[1] T. Banakh, I. Protasov, Topologization of $G$-spaces, preprint.
[2] R. Bryant, The verbal topology of a group, J. Algebra 48:2 (1977), 340–346.
[3] D. Dikranjan, D. Shakhmatov, The Markov-Zariski topology of an abelian group, J. Algebra 324 (2010), no. 6, 1125–1158.
[4] G. Hesse, Zur Topologisierbarkeit von Gruppen, Dissertation (Univ. Hannover, Hannover, 1979).
[5] N. Hindman, I. Protasov, D. Strauss, Topologies on $S$ determined by idempotents in $\beta S$, Topology Proc. 23 (1998), 155–190 (2000).
[6] T. Jech, Set Theory, Springer-Verlag, Berlin, 2003.
[7] M. Kilp, U. Knauer, A.V. Mikhalev, Monoids, Acts and Categories: with Applications to Wreath Products and Graphs, Expositions in Mathematics 29, Walter de Gruyter, Berlin, 2000.
[8] A.A. Klyachko, A.V. Trofimov, The number of non-solutions of an equation in a group, J. Group Theory 8:6 (2005), 747–754.
[9] A.A. Markov, On unconditionally closed sets, Mat. Sb. 18.1 (1946), 3–28.
[10] A.Yu. Ol’shanskii, A remark on a countable non-topologized group, Vestnik Moscow Univ. Ser. I. Mat. Mekh, no.3 (1980), p.103.
[11] D. Pospíšil, Remark on bicomplete spaces, Ann. of Math. (2) 38:4 (1937), 845–846.
[12] I. Protasov, E. Zelenyuk, Topologies on Groups Determined by Sequences, VNTL, L’viv, 1999.
[13] O. Sipacheva, Unconditionally $\tau$-closed and $\tau$-algebraic sets in groups, Topology Appl. 155:4 (2008), 335–341.
[14] E. Zelenyuk, On topologies on groups with continuous shifts and inversion, Visn. Kyiv. Univ. Ser. Fiz.-Mat. Nauki, no. 2 (2000), 252–256.
[15] Y. Zelenyuk, On topologizing groups, J. Group Theory 10:2 (2007), 235–244.
