ON SURFACES OBTAINED AS SINGULAR LOCI OF NORMAL CONGRUENCE OF FRONTALS WITH PURE-FRONTAL SINGULAR POINTS

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ABSTRACT. We study singularities and geometric properties of surfaces given by the singular loci of normal congruence of frontals with pure-frontal singular points. These surfaces consist of the normal ruled surface and focal surfaces of the initial frontal. For the normal ruled surface, we give characterizations of singularities in terms of geometric invariants of the initial frontal defined on the set of singular points. For focal surfaces, we show relation between certain singularities of them and geometric property of the given frontal. Moreover, we consider behavior of Gaussian curvature of focal surfaces of frontal with a 5/2-cuspidal edge.

1. Introduction

In differential geometry of surfaces in the Euclidean 3-space $\mathbb{R}^3$, focal surfaces are classical objects. Although initial surfaces do not have any singular points, their focal surfaces have singular points in general. Porteous [32] (see also [33]) studied focal surfaces of regular surfaces using singularity theory techniques. In the study, he defined a notion of (higher order) ridge points on a surface and showed relation between ridge points and types of singularities on focal surfaces. In contrast, Bruce and Wilkinson [2] investigated the folding map of surfaces and they introduced the notion of sub-parabolic points on surfaces, which correspond to parabolic points on focal surface (see also [29, 33]). Although definitions of these concepts are simple, they were not studied deeply before their investigations.

On the other hand, there are several articles treating surfaces with singular points from the differential geometric viewpoint recently (cf. [8–12, 16, 25, 26, 31, 35, 38]). In particular, the study of classes called frontals and fronts has been attractive. Although the first fundamental form (or the induced metric) of a surface in the Euclidean 3-space $\mathbb{R}^3$ is degenerate at singular points, frontals or fronts admit smooth unit normal vector even at singular points. Since a unit normal vector field of a given frontal can be taken smoothly, we can consider normal congruence of the frontal naturally. It is known that singular loci of normal congruence form focal surfaces (or caustics) of the initial surface ([1, 21]). As mentioned above, to investigate focal surfaces of frontals, it is expected that we might obtain new geometrical properties for frontals.

In this paper, we study singularities and geometric properties of surfaces, which are arose from the singular loci of normal congruence of frontals with pure-frontal singular points (see Section 2). For regular surfaces and fronts with certain singularities, the singular loci corresponds to only focal surfaces ([21, 40]). However, in the case of frontals, we obtain an additional surface (say the normal ruled surface, see Section 3). This might
be a characteristic phenomenon of a frontal but not a front. Normal ruled surfaces are ruled surfaces whose base curves are the singular loci of given frontals and the direction of generators is corresponding unit normal vector. We clarify relation between types of singularities of normal ruled surfaces and geometrical properties of given frontal surface (Theorems 3.6 and 3.7). Moreover, we investigate singularities and geometrical properties of focal surfaces of frontals in Section 4. We give geometric characterization for focal surfaces to be singular at the same singular point of the initial frontal (Proposition 4.3). By this result, we find that if the initial frontal has a $5/2$-cuspidal edge, then both focal surfaces are regular at that point (Corollary 4.4). When focal surfaces have singular points, we characterize those points to be of the first kind or of the second kind by geometric properties of the initial frontal (Proposition 4.9). In particular, if a singular point of focal surfaces is of the first kind, we show condition for the point to be a cuspiddal cross cap in terms of geometric invariants of the initial frontal (Theorem 4.10). Furthermore, we show that the singular set of the initial frontal surface consisting of pure-frontal singular points is also the set of pure-frontal singular points of corresponding focal surfaces under certain geometrical properties (Theorem 4.12). Finally, we investigate behavior of the Gaussian and the mean curvature of the focal surfaces at a $5/2$-cuspidal edge of the initial frontal (Theorem 4.14 and Proposition 4.15).

2. Frontal surfaces

We recall some notions and properties of frontal surfaces. For detailed explanations, see [1, 18, 19, 26, 34, 38].

Let $f: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a $C^\infty$ map germ. Then $f$ is said to be a frontal if there exists a $C^\infty$ map $\nu: (\mathbb{R}^2, 0) \rightarrow \mathbb{S}^2$ such that $\langle df_q(X), \nu(q) \rangle = 0$ holds for any $q \in (\mathbb{R}^2, 0)$ and $X \in T_q \mathbb{R}^2$, where $\mathbb{S}^2$ is the standard unit sphere in $\mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ is the canonical inner product of $\mathbb{R}^3$. Moreover, a frontal $f$ is called a front if the pair $(f, \nu): (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3 \times \mathbb{S}^2, (0, \nu(0)))$ gives an immersion. We call $\nu$ a unit normal vector field of $f$.

We fix a frontal $f: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ with a unit normal vector $\nu$. We set a function $\lambda: (\mathbb{R}^2, 0) \rightarrow \mathbb{R}$ by

$$
\lambda(u, v) = \det(f_u, f_v, \nu)(u, v) \quad (f_u = \partial f / \partial u, \ f_v = \partial f / \partial v),
$$

where $(u, v)$ is a local coordinate system on $(\mathbb{R}^2, 0)$ and det is the determinant. We call $\lambda$ the signed area density function. A function $\Lambda$ is called an identifier of singularities of $f$ if $\Lambda$ is a nowhere-vanishing function multiple of $\lambda$. Denoting by $S(f) = \{q \in (\mathbb{R}^2, 0) | \operatorname{rank} df_q < 2\}$ the set of singular points of $f$, we see that $\Lambda^{-1}(0) = S(f)$.

Assume that $0 \in S(f)$ in the following. Then 0 is said to be non-degenerate if $(\Lambda_u, \Lambda_v) \neq (0, 0)$ at 0. When 0 is a non-degenerate singular point of $f$, then there exists a regular curve $\gamma: (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ such that $\Lambda(\gamma) = 0$ holds. Moreover, in such a case, since $\operatorname{rank} df_0 = 1$, there exists a never vanishing vector field $\eta$ on $(\mathbb{R}^2, 0)$ such that $df(\eta) = 0$ along $\gamma$. We call $\gamma$ and $\eta$ a singular curve and a null vector field for $f$, respectively. The origin 0 is said to be a singular point of the first kind (resp. second kind) if $\gamma' = d\gamma / dt$ is linearly independent (resp. dependent) to $\eta$ at 0. We note that if the singular curve $\gamma$ through 0 consists of the first kind, then the singular locus $\hat{\gamma} = f \circ \gamma$ of $f$ is a regular spacial curve.

We assume that the origin 0 is a singular point of the first kind of a frontal $f: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$. We define a function $\psi: (\mathbb{R}, 0) \rightarrow \mathbb{R}$ by

$$
\psi(t) = \det(\gamma'(i), \nu(\gamma(t)), d\nu(\gamma(t))(\eta)),
$$

where $\gamma' = d\gamma / dt$ and $\nu(\gamma(t))$ is a unit normal vector of the front $f$ along $\gamma(t)$.
where $\dot{\gamma}' = d\dot{\gamma}/dt$ and $d\nu_{\gamma(t)}(\eta)$ is the directional derivative of $\nu$ in the direction of a null vector field $\eta$ along $\gamma$.

**Proposition 2.1** ([7, 26]). Let $f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal and $0$ a singular point of the first kind. Then $f$ is a front at $0$ if and only if $\psi(0) \neq 0$, where $\psi$ is a function defined by (2.2).

By this fact, we call the singular point of the first kind $0$ of a frontal $f$ a non-front singular point if $\psi(0) = 0$.

**Definition 2.2** ([36]). Let $f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal, $\nu$ its unit normal vector and $0$ is a singular point of the first kind of $f$. Let $\psi$ be a function in (2.2) defined along the singular curve $\gamma$ passing through $0$. Then $0$ is said to be a $k$-non-front singular point of $f$ if there exists a positive integer $k$ such that $\psi(0) = \psi'(0) = \cdots = \psi^{(k-1)}(0) = 0$ and $\psi^{(k)}(0) \neq 0$ hold; a point $0$ is said to be a pure-frontal singular point if $\psi = 0$ along the singular curve $\gamma$ through $0$.

**Definition 2.3.** Let $f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ a $C^\infty$ map germ and $0$ a singular point of $f$. Then

- $f$ at $0$ is a cuspidal edge if it is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3)$ at the origin;
- $f$ at $0$ is a cuspidal cross cap if it is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, uv^3)$ at the origin;
- $f$ at $0$ is a cuspidal $S^\pm_k$ singularity if it is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3(\pm u^{k+1} \pm v^2))$ at the origin;
- $f$ at $0$ is a 5/2-cuspidal edge (or a rhamphoid cuspidal edge) if it is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^5)$ at the origin;
- $f$ at $0$ is a fold singularity if it is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, 0)$ at the origin.

Here two map germs $f, g: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are said to be $\mathcal{A}$-equivalent if there exist diffeomorphism germs $S: (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ on the source and $T: (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ on the target such that $T \circ f = g \circ S$ holds.

**Figure 1.** From top left to bottom right: The images of a cuspidal edge, a cuspidal cross cap, a cuspidal $S^+_1$ singularity, a cuspidal $S^-_1$ singularity, a 5/2-cuspidal edge and a fold singularity.
Singular points in the above definition are all singular points of the first kind. A cuspidal edge is a singularity of a front, but other singular points are examples of non-front singularities. Although cuspidal cross caps and cuspidal \( S_0 \) singularities are \( k \)-non-front singular points for some integer \( k \geq 1 \), \( 5/2 \)-cuspidal edges and fold singular points are pure-frontal singularities of frontals ([7, 16, 34, 36]). We remark that a cuspidal \( S_0 \) singularity is \( \mathcal{A} \)-equivalent to a cuspidal cross cap. In the following, our main target is a frontal with pure-frontal singularities.

### 2.1. Geometrical properties

We recall geometrical properties of a frontal \( f: \mathbb{R}^2 \to \mathbb{R}^3 \) with pure-frontal singular point \( 0 \). Under this setting, we can take a local coordinate system \((u, v)\) on \( \mathbb{R}^2 \) ([26]) satisfying

- the \( u \)-axis gives a singular curve, that is, \( S(f) = \{ v = 0 \} \),
- \( \partial_u \) gives a null vector field \( \eta \),
- \( \{ f_u, f_v, v \} \) gives an orthonormal frame along the \( u \)-axis.

We call this coordinate system adapted.

When we take an adapted coordinate system \((u, v)\) on \( \mathbb{R}^2 \), then there exists a map \( h: \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( f_v = vh \) holds. Note that \( h \) is perpendicular to \( v \). Since \( \lambda_v = \det(f_u, f_v, v) = \det(f_u, h, v) \neq 0 \) at 0, the triple \( \{ f_u, h, v \} \) forms a moving frame along \( f \) on \( \mathbb{R}^2 \). Using this frame, we set the following functions:

\[
\begin{align*}
\tilde{E} &= \langle f_u, f_u \rangle, \quad \tilde{F} = \langle f_u, h \rangle, \quad \tilde{G} = \langle h, h \rangle, \\
\tilde{L} &= -\langle f_u, v_u \rangle, \quad \tilde{M} = -\langle h, v_u \rangle, \quad \tilde{N} = -\langle h, v_v \rangle.
\end{align*}
\]

If a frontal \( f \) has a pure-frontal singular point \( 0 \), then the function \( \psi \) as in (2.2) vanishes identically along the singular curve \( y \) through \( 0 \). Thus when we take an adapted coordinate system \((u, v)\), then \( v_v(u, 0) = 0 \). This implies that there exists a map \( v_1: \mathbb{R}^2 \to \mathbb{R}^3 \) such that \( v_v = v v_1 \). Moreover, setting \( \tilde{N}_1 = -\langle h, v_1 \rangle \), the function \( \tilde{N} \) as in (2.3) can be rewritten as \( \tilde{N} = v \tilde{N}_1 \).

**Lemma 2.4** ([36]). Let \((u, v)\) be an adapted coordinate system around a pure-frontal singular point \( 0 \) of a frontal \( f: \mathbb{R}^2 \to \mathbb{R}^3 \). Then

\[
\begin{align*}
v_u &= \frac{\tilde{F} M - \tilde{G} L}{\tilde{E}G - \tilde{F}^2} f_u + \frac{\tilde{F} L - \tilde{E} M}{\tilde{E}G - \tilde{F}^2} h, \quad v_v = v \frac{\tilde{F} \tilde{N}_1 - \tilde{G} M}{\tilde{E}G - \tilde{F}^2} f_u + v \frac{\tilde{F} M - \tilde{E} N_1}{\tilde{E}G - \tilde{F}^2} h.
\end{align*}
\]

We next consider curvatures of a frontal \( f: \mathbb{R}^2 \to \mathbb{R}^3 \) around a pure-frontal singular point \( 0 \). Let us take an adapted coordinate system \((u, v)\) on \( \mathbb{R}^2 \). Then the Gaussian curvature \( K \) and the mean curvature \( H \) are expressed as

\[
\begin{align*}
K &= \frac{\tilde{L} \tilde{N}_1 - M^2}{\tilde{E}G - \tilde{F}^2}, \quad H = \frac{\tilde{E} \tilde{N}_1 - 2 \tilde{F} M + \tilde{G} \tilde{L}}{2(\tilde{E}G - \tilde{F}^2)}
\end{align*}
\]

on the set of regular points of \( f \). Since \( \tilde{E}G - \tilde{F}^2 > 0 \), \( K \) and \( H \) can be defined on \( \mathbb{R}^2 \). In particular, these are bounded \( C^\infty \) functions on \( \mathbb{R}^2 \). By using \( K \) and \( H \), we define principal curvatures \( \kappa_j \) (\( j = 1, 2 \)) as follows:

\[
\kappa_1 = H + \sqrt{H^2 - K}, \quad \kappa_2 = H - \sqrt{H^2 - K}.
\]

We remark that \( H^2 - K \geq 0 \) holds. We also note that if the cuspidal torsion \( \kappa_t \) (see Subsection 2.2) does not vanish at 0, then 0 is a non-umbilic point, that is, \( \kappa_1 \neq \kappa_2 \) at 0 ([36, Theorem 3.1]). Assuming \( \kappa_t \neq 0 \) at 0, we can take principal vectors \( V_j \) (\( j = 1, 2 \)) relative to \( \kappa_j \). Using the functions (2.3), \( V_j \) can be written as

\[
V_j = (-v(M - \kappa_j \tilde{F}), \tilde{L} - \kappa_j \tilde{E}) \quad (j = 1, 2)
\]
Remark 2.5. By (2.5) and (2.6), the principal curvatures of a pure frontal can be extended as $C^\infty$ functions. If a frontal $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ has a $k$-non-front singular point ($k \geq 0$) at 0, then the principal curvatures cannot be extended as $C^\infty$ functions ([36, Theorem 4.1]). On the other hand, if $f$ is a front and 0 a cuspidal edge, then one of the two principal curvatures can be extended as a $C^\infty$ function and another is unbounded near 0 ([30, 41]).

Using Lemma 2.4, we have the following.

Lemma 2.6. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal with pure-frontal singular point 0 which is a non-umbilic point. Let $V_j$ be principal vectors with respect to $\kappa_j$. Then

$$d\nu(V_j) = -\kappa_j df(V_j).$$

Proof. We give a proof for $j = 1$. Take an adapted coordinate system $(u, v)$. Then the principal vector $V_1$ is given by

$$V_1 = (-v(\bar{M} - \kappa_1 \bar{F}), \bar{L} - \kappa_1 \bar{E})$$

(cf. (2.7)). Thus we have

$$df(V_1) = -v(\bar{M} - \kappa_1 \bar{F})f_u + v(\bar{L} - \kappa_1 \bar{E})h = v_x v_1.$$

On the other hand, by direct calculations with Lemma 2.4, we get

$$d\nu(V_1) = v\kappa_1 (\bar{M} - \kappa_1 \bar{F})f_u - v\kappa_1 (\bar{L} - \kappa_1 \bar{E})h = -\kappa_1 df(V_1) = -v\kappa_1 x_1.$$

Thus we obtain the assertion for $j = 1$. For the case of $j = 2$, one can show in a similar way.

We give definitions of ridge points and sub-parabolic points for frontals as follows.

Definition 2.7 (cf. [20, 39, 41]). Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal and 0 a pure-frontal singular point of $f$. Let $\kappa_j$ ($j = 1, 2$) be the principal curvatures of $f$ and $V_j$ the corresponding principal vectors. Then

- a point 0 is a $V_j$-ridge point if $V_j \kappa_j = 0$ holds at 0;
- a point 0 is a $k$-th order $V_j$-ridge point if $V_j \kappa_j = \cdots = V_j^k \kappa_j = 0$ and $V_j^{k+1} \kappa_j \neq 0$ hold at 0, where $V_j^m \kappa_j$ means the $m$-th order directional derivative of $\kappa_j$ in the direction $V_j$;
- a point 0 is a $V_j$-sub-parabolic point if $V_j \kappa_{j+1} = 0$ holds at 0, where we consider $\kappa_3$ as $\kappa_1$.

We shall see geometrical interpretations for these using geometric invariants in the next subsection. In the rest of this work, we use $\kappa_3 = \kappa_1$ as in the above definition.

2.2. Geometric invariants. We recall geometric invariants of a frontal defined along a singular curve. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal, $\nu$ a unit normal vector to $f$ and 0 a singular point of the first kind of $f$. Let $\gamma$ be a singular curve for $f$ passing through 0. Then along $\gamma$, we can define the following geometric invariants: the singular curvature $\kappa_s$ ([38]), the limiting normal curvature $\kappa_\nu$ ([26, 38]), the cuspidal torsion $\kappa_t$ ([25]), the cuspidal curvature $\kappa_c$ ([26]), the bias $r_b$ ([16, 31]) and the secondary cuspidal curvature $r_c$ ([16, 31]). If 0 is a pure-frontal singular point, then $\kappa_c$ vanishes along the singular curve $\gamma$ ([26]). Moreover, a pure-frontal singular point 0 is a 5/2-cuspidal edge of a frontal $f$ if and only if $r_c$ does not vanish ([14, Theorem 4.1] and [16, (3.12)]). For precise definitions and geometrical properties, see [15, 16, 25, 26, 31, 38]. Using functions as in (2.3), we have the following characterizations (see [36, Lemma 2.5 and Corollary 2.6]).
Lemma 2.8. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal and 0 a pure-frontal singular point of $f$. Take an adapted coordinate system $(u, v)$ on $(\mathbb{R}^2, 0)$. Then we have the following:
\begin{align*}
k_v(u) &= \tilde{L}(u, 0), \quad k_1(u) = \tilde{M}(u, 0), \quad k_c(u) = 2\tilde{N}(u, 0)(\equiv 0), \\
r_b(u) &= 3\tilde{N}_t(u, 0) = 3\tilde{N}_1(u, 0), \\
r_c(u) &= 12\left(\tilde{N}_{uv} - 4\tilde{F}_v\tilde{M} - 2\tilde{G}_v\tilde{N}_t \right)(u, 0) \\
&= 24\left(\tilde{N}_1 - 2\tilde{F}_v\tilde{M} - \tilde{G}_v\tilde{N}_1 \right)(u, 0),
\end{align*}
where $\tilde{N} = v\tilde{N}_1$. Moreover, $\tilde{E}(u, 0) = \tilde{G}(u, 0) = 1$ and $\tilde{F}(u, 0) = 0$.

We give the following characterizations of $V_j$-ridge and $V_{j+1}$-sub-parabolic point of a frontal by geometric invariants.

Proposition 2.9. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal, $v$ its unit normal vector and 0 a pure-frontal singular point. Suppose that $k_1 \neq 0$ holds at 0. Then for each $j = 1, 2$, a point 0 is both a $V_j$-ridge and $V_{j+1}$-sub-parabolic point of $f$ if and only if $r_c = 0$ at 0.

Proof. We first show the case of $j = 1$. Take an adapted coordinate system $(u, v)$. Then by (2.7) and Lemma 2.8, $V_1k_1$ and $V_2k_1$ can be written as
\begin{align*}
V_1k_1 &= (k_v - k_1)(k_1), \\
V_2k_1 &= (k_v - k_2)(k_1),
\end{align*}
at 0. We now remark that
\begin{align*}
k_v - k_j &= \frac{1}{2}\left( k_v - \frac{r_b}{3} \right) + (-1)^j \sqrt{\left( k_v - \frac{r_b}{3} \right)^2 + 4k_c^2} \neq 0
\end{align*}
at 0 for $j = 1, 2$ if $k_1 \neq 0 ([36, (3.8)])$. Thus $k_j \neq k_v$ at 0, and hence $V_1k_1 = V_2k_1 = 0$ at 0 is equivalent to $(k_1)_v = 0$ at 0. We write $k_1$ as $k_1 = H + \sqrt{\Gamma}$, where $\Gamma = H^2 - K$. Then we have
\begin{align*}
(k_1)_v &= H_v + \left( HH_v - \frac{K_v}{2} \right) \Gamma^{-1/2} \\
&= \frac{1}{\sqrt{\Gamma}} \left( H_vk_1 - \frac{K_v}{2} \right).
\end{align*}
By [16, Lemma 4.3], $H_v$ and $K_v$ are represented as
\begin{align*}
H_v &= \frac{r_c}{48}, \quad K_v = \frac{r_\Pi}{24} = \frac{k_vr_c}{24}
\end{align*}
at 0. Therefore it holds that
\begin{align*}
(2.9) \quad (k_1)_v &= \frac{r_c}{48\sqrt{\Gamma}}(k_1 - k_v)
\end{align*}
at 0. This completes the proof for this case. For the case of $j = 2$, we show in the similar way by using
\begin{align*}
(k_2)_v &= -\frac{1}{\sqrt{\Gamma}} \left( H_vk_2 - \frac{K_v}{2} \right) = -\frac{r_c}{48\sqrt{\Gamma}}(k_2 - k_v)
\end{align*}
at 0.

Corollary 2.10. Under the same assumptions in Proposition 2.9, a pure-frontal singular point 0 of a frontal $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is not a 5/2-cuspidal edge if and only if 0 is a $V_j$-ridge and $V_{j+1}$-sub-parabolic point of $f$ for each $j = 1, 2$.

Proof. Since $f$ at 0 is a 5/2-cuspidal edge if and only if $r_c \neq 0$, we have the assertion by Proposition 2.9.
This gives a geometrical interpretation for a pure-frontal singular point either a 5/2-cuspidal edge or not. We note that an intrinsic criterion for 5/2-cuspidal edge is given by [16, Corollary 4.5] (see also [15]).

3. Normal congruence and its singular value sets

We consider a normal congruence of a frontal $f$: $(\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$. For cases of a regular surface and a front, see [21, 40].

We assume that principal curvatures $\kappa_j (j = 1, 2)$ of $f$ do not vanish on $(\mathbb{R}^2, 0)$. A normal congruence $F: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ is given by

$$\mathcal{F}(u, v, w) = f(u, v) + w\nu(u, v),$$

(3.1)

where $\nu$ is a unit normal vector to $f$. Calculating the Jacobian $J_{\mathcal{F}}$ of $\mathcal{F}$, we have

$$\det J_{\mathcal{F}} = \det(\mathcal{F}_u, \mathcal{F}_v, \mathcal{F}_w) = (1 - w\kappa_1)(1 - w\kappa_2)\lambda,$$

where $\kappa_j (j = 1, 2)$ are principal curvatures of $f$ and $\lambda$ is the signed area density function of $f$. Thus the set of singular points of $\mathcal{F}$ is $S(\mathcal{F}) = (S(f) \times \mathbb{R}) \cup S_1 \cup S_2$, where

$$S_j = \{(u, v, w) \in (\mathbb{R}^3, 0) | 1 - w\kappa_j(u, v) = 0\} \quad (j = 1, 2),$$

and hence the singular locus $\mathcal{F}(S(\mathcal{F}))$ of $\mathcal{F}$ is the union $\mathcal{F}(S(f) \times \mathbb{R}) \cup \mathcal{F}(S_1) \cup \mathcal{F}(S_2)$.

Considering this, we define $NR: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ and $C_j: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) (j = 1, 2)$ by

$$NR(u, w) = \hat{\gamma}(u) + w\hat{\nu}(u), \quad C_j(u, v) = f(u, v) + \rho_j(u, v)\nu(u, v),$$

(3.2)

where $\hat{\nu} = \nu \circ \gamma$ and $\rho_j = 1/\kappa_j (j = 1, 2)$. We notice that the image of $NR$ coincides with $\mathcal{F}(S(f) \times \mathbb{R})$, and the image of $C_j$ coincides with $\mathcal{F}(S_j)$ $(j = 1, 2)$. We call $NR$ and $C_j (j = 1, 2)$ the normal ruled surface along $\hat{\gamma}$ and the (normal) focal surfaces or caustics of $f$ associated to $\kappa_j$, respectively. In the rest of this section, we study singularities of $NR$. We shall investigate singularities and certain geometric properties of $C_j$ in the next section.

3.1. Singularities of $NR$. We deal with the normal ruled surface $NR$ along the singular locus $\hat{\gamma}$ of a frontal $f$ with a pure-frontal singular point. For a frontal $f$: $(\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$, we take an adapted coordinate system $(u, v)$. We set $\hat{h}$ by $\hat{h}(u) = h(u, 0)$, where $h$ is a map satisfying $f_0 = oh$. In this case, $\{\hat{\gamma}', \hat{h}, \hat{\nu}\}$ is an orthonormal frame along $\hat{\gamma}$. Moreover, we have the following formula (cf. [22, Proposition 3.1] and [8, Lemma 1.3]):

$$\begin{pmatrix} \hat{\gamma}' \\ \hat{h} \\ \hat{\nu} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_x & \kappa_v \\ -\kappa_x & 0 & \kappa_l \\ -\kappa_v & -\kappa_l & 0 \end{pmatrix} \begin{pmatrix} \hat{\gamma}' \\ \hat{h} \\ \hat{\nu} \end{pmatrix}.$$

(3.3)

We assume that $NR$ is noncylindrical, that is, $\hat{\nu}'$ does not vanish identically. By (3.3), this condition is equivalent to $(\kappa_x, \kappa_l) \neq (0, 0)$ along $\gamma$.

**Lemma 3.1.** A point $(u_0, w_0)$ is a singular point of $NR$ as in (3.2) if and only if $\kappa_l(u_0) = 0$ and $w_0 = 1/\kappa_v(u_0)$ hold.

**Proof.** By (3.3), we have

$$NR_u = (1 - w\kappa_v)\hat{\gamma}' - w\kappa_l\hat{h}, \quad NR_w = \hat{\nu}.$$  

(3.4)

Thus it holds that

$$NR_u \times NR_w = -(1 - w\kappa_v)\hat{h} - w\kappa_l\hat{\gamma}'.$$

Hence we have the assertion. \(\square\)
We next consider $NR$ to be a developable surface or not. For this, we give the following characterization:

**Proposition 3.2.** The normal ruled surface $NR$ is developable if and only if $\kappa_1 = 0$ along $\gamma$.

*Proof.* Let us take an adapted coordinate system $(u, v)$. Then by (3.3), we see that

$$\det(\gamma', \nu, \nu') = \kappa_1$$

holds along the $u$-axis. Hence we have the assertion (see [4, Page 194]).

This implies that when the singular curve of an initial frontal is a line of curvature ([36, Proposition 3.3]), then $NR$ is a developable surface. Moreover, we have the following.

**Corollary 3.3.** If $NR$ is a developable surface, then the singular locus of $NR$ is $\hat{\gamma} + \nu/\kappa_v$.

*Proof.* By Lemma 3.1 and Proposition 3.2, the conclusion follows.

If $NR$ is developable, then we can take $\hat{h}$ as a unit normal vector field. Thus $NR$ is a frontal. Moreover, since $NR$ is noncylindrical, $\kappa_v \neq 0$ holds, and hence all singular points of $NR$ are non-degenerate. By (3.4), $\partial_u$ can be taken as a null vector field $\eta_{NR}$ of $NR$. Thus we have the following.

**Proposition 3.4.** Assume that the normal ruled surface $NR$ is developable. Then $NR$ is a frontal if and only if $\kappa_s \neq 0$.

*Proof.* To show this, it is sufficient to check the condition that $\eta_{NR}\hat{h} \neq 0$. By (3.3), we have $\eta_{NR}\hat{h} = -\kappa_s\gamma'$. This implies the conclusion.

**Corollary 3.5.** If the developable normal surface $NR$ is a frontal but not a front on $S(f) \times \mathbb{R}$, then the image of $NR$ is a part of a plane.

*Proof.* Let us take an adapted coordinate system $(u, v)$. Then by Proposition 3.4, if $NR$ is a frontal but not a front, then $\hat{h}'$ vanishes identically along the $u$-axis. Thus we have that $h$ is a constant vector. Thus we have the assertion.

By this corollary, $NR$ does not have a 5/2-cuspidal edge. When $\kappa_s$ does not vanish identically, there are possibilities for $NR$ to be a front or not. For the case that $NR$ is developable, we have the following characterizations of singularities.

**Theorem 3.6.** Let $NR$ be a normal ruled surface as in (3.2) of a frontal $f$ and $q = (u_0, w_0)$ a singular point of $NR$. Suppose that $NR$ is non-cylindrical and developable. Then

1. $NR$ has a cuspidal edge at $q$ if and only if $\kappa_s(u_0)\kappa_v'(u_0) \neq 0$;
2. $NR$ has a swallowtail at $q$ if and only if $\kappa_s(u_0) = 0$ and $\kappa_v''(u_0) \neq 0$;
3. $NR$ has a cuspidal cross cap at $q$ if and only if $\kappa_s(u_0) = 0$ and $\kappa_v'(u_0)\kappa_v''(u_0) \neq 0$;
4. $NR$ has a cuspidal $S_1^1$ singularity at $q$ if and only if $\kappa_s(u_0) = \kappa_v'(u_0) = 0$ and $\kappa_v''(u_0) \neq 0$.

Here a swallowtail is defined as a germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, 4v^3 + 2uv, 3v^4 + uv^2)$ at the origin (see Figure 2).

*Proof.* By Proposition 3.4, $NR$ at $q$ is a front if and only if $\kappa_s(u_0) \neq 0$. Moreover, an identifier of singularities $\Lambda$ of $NR$ is given as $\Lambda = 1 - wk_v(u)$, and $\eta_{NR} = \partial_u$ gives a null vector field of $NR$. Thus we have

$$\eta_{NR}\Lambda(q) = -w_0\kappa_v'(u_0), \quad \eta_{NR}\Lambda(q) = -w_0\kappa_v''(u_0).$$

Hence we have the first two assertions by the criteria for cuspidal edges and swallowtails ([37, Corollary 2.5]).
We next consider the case that $q$ is a non-front singular point of $NR$, that is, $q$ is a singular point of the first kind and $NR$ is not a front at $q$. Since $NR$ is non-cylindrical, $\kappa_N \neq 0$ holds, and hence the singular curve $\beta$ of $NR$ can be parametrized as $\beta(u) = (u, 1/\kappa_N(u))$. We set $\hat{\beta}(u) = NR \circ \beta(u)$ and $\psi(u) = \det(\hat{\beta}'(u), \hat{h}(u), \eta_{NR}\hat{h}(u))$. By (3.3), we have

$$\psi(u) = -\frac{\kappa_N(u)\kappa_N'(u)}{\kappa_N(u)^2}.$$ 

When $NR$ is not a front at $q$, then $\psi(u_0) = 0$. Moreover, $\psi'(u_0)$ is calculated as

$$\psi'(u_0) = -\frac{\kappa_N'(u_0)\kappa_N''(u_0)}{\kappa_N(u_0)^2}.$$ 

By the criterion for a cuspidal cross cap ([7, Corollary 1.5]), we have the third assertion. Finally, we assume that $NR$ at $q$ is not a cuspidal cross cap. Then $\psi(u_0) = \psi'(u_0) = 0$ holds. Since $q = (u_0, w_0)$ is of the first kind, $\kappa_N'(u_0) \neq 0$ and $\kappa_N''(u_0) = 0$ hold. Under these situations, we calculate $\psi''(u_0)$. By a direct computation with (3.3), we have

$$\psi''(u_0) = -\frac{\kappa_N''(u_0)\kappa_N'(u_0)}{\kappa_N(u_0)^2} (= B).$$

On the other hand, there exists a null vector field $\eta_{NR}$ such that

$$\langle \hat{\beta}'(u_0), \hat{\eta}_{NR}^2 NR(q) \rangle = \langle \hat{\beta}'(u_0), \hat{\eta}_{NR}^3 NR(q) \rangle = 0,$$ 

where $\hat{\eta}_{NR}^k$ means the $k$-time directional derivative of $NR$ in the direction $\hat{\eta}_{NR}$ ([14, 16, 31]). We note that $\hat{\eta}_{NR}^2 NR \neq 0$ at $q$. By the similar calculations in the proof of [36, Lemma 2.4], we have

$$\hat{\eta}_{NR} = \partial_u + \kappa_N'(u_0)(u - u_0)^2 \partial_u.$$ 

In particular, $\eta_{NR} = \hat{\eta}_{NR}$ holds at $q$. Using this vector field, we have

$$\hat{\eta}_{NR}^2 NR = -\frac{\kappa_N'}{\kappa_N} \hat{\gamma}', \quad \hat{\eta}_{NR}^3 NR = -\frac{\kappa_N''}{\kappa_N} \hat{\gamma}'$$ 

at $q$, and hence it holds that $\hat{\eta}_{NR}^3 NR(q) = (\kappa_N'(u_0)/\kappa_N'(u_0))\hat{\eta}_{NR}^2 NR(q)$. To characterize cuspidal $S_1$ singularities, we set $C = \kappa_N'(u_0)/\kappa_N'(u_0)$. The cross product of $\hat{\beta}' \times \hat{\eta}_{NR}^2 NR$ can be written as

$$\hat{\beta}' \times \hat{\eta}_{NR}^2 NR = (\kappa_N')^2 \hat{\gamma} \times \hat{\gamma}'$$

at $q$. We calculate the fourth and fifth order directional derivatives of $NR$ in the direction $\hat{\eta}_{NR}$. By straightforward computations, we have

$$\hat{\eta}_{NR}^4 NR \equiv 0, \quad \hat{\eta}_{NR}^5 NR \equiv -\frac{4k_N' k_N''}{k_N} \hat{h} \mod \langle \hat{\gamma}', \hat{\gamma}' \rangle_{\mathbb{R}}.$$
at \( q \) since \( \kappa_s = \kappa'_s = 0 \) at \( u_0 \). Here for \( a, b, x, y \in \mathbb{R}^3, a \equiv b \mod (x, y) \mathbb{R} \) means that there exist \( c_1, c_2 \in \mathbb{R} \) such that \( a - b = c_1 x + c_2 y \) holds. Thus we obtain

\[
A = \det(\beta', \eta_{NR}^2 NR, 3\eta_{NR}^5 NR - 10C\eta_{NR}^4 NR)(q) = -\frac{12\kappa'_c(u_0)^3\kappa''(u_0)}{\kappa_c(u_0)^4}.
\]

Hence the product of \( A \) and \( B \) is

\[
AB = \frac{12\kappa'_c(u_0)^4\kappa''(u_0)^2}{\kappa_c(u_0)^6} > 0.
\]

Therefore we obtain the conclusion by the criterion [34, Theorem 3.2].

We note that a developable normal ruled surface does not admit cuspidal \( S_1^- \) singularities. This comes from the general theory [17] (see also [34]). We note that duality of singularities on flat surfaces which contain developable surfaces are studied by Honda [13].

We turn to our consideration to nondevelopable case. It is known that generic singularities of such surfaces is a cross cap (or a Whitney umbrella) (see [23]), which is defined as a map germ \( \mathcal{A} \)-equivalent to \( (u, v) \mapsto (u, uv, v^2) \) at the origin. Moreover, \( S_1^{\pm} \) singularities (or Chen Matumoto Mond \( \pm \) singularities) defined by map germs \( \mathcal{A} \)-equivalent to \( (u, v) \mapsto (u, v^2, v(u^2 \pm v^2)) \) at the origin are known as the generic singularities of one-parameter families of 2-manifolds in \( \mathbb{R}^3 \) ([3, 28]). We give characterization of these

\[\text{Figure 3. The images of a cross cap (left), an } S_1^+ \text{ singularity (middle) and an } S_1^- \text{ singularity (right).}\]

singularities on a noncylindrical nondevelopable normal ruled surface in terms of geometric invariants of the initial frontal surface by using the criteria obtained in [34].

Let \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) be a frontal and 0 a pure-frontal singular point of \( f \). Let \( NR \) be the normal ruled surface on \((S(f) \times \mathbb{R}; u, w)\) as in (3.2). Assume that \( NR \) is noncylindrical and nondevelopable. Then the set of singular points of \( NR \) is

\[
(3.5) \quad S(NR) = \{(u_0, w_0) \in S(f) \times \mathbb{R} \mid \kappa_f(u_0) = 0, 1 - w_0 \kappa_c(u_0) = 0\}
\]

by Lemma 3.1.

**Theorem 3.7.** Let \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) be a frontal with pure-frontal singular point 0. Suppose that a normal ruled surface \( NR \) as in (3.2) of \( f \) is noncylindrical and nondevelopable. Then we have the following.

1. \( NR \) has a cross cap at \((u_0, w_0) \in S(NR)\) if and only if \( \kappa'_f(u_0) \neq 0 \).
2. \( NR \) has a \( S_1^- \) singularity at \((u_0, w_0) \in S(NR)\) if and only if \( \kappa'_f(u_0) = 0 \) and \( \kappa''_f(u_0)(2\kappa_f(u_0)\kappa'_f(u_0) + \kappa''_f(u_0)) < 0 \).
3. \( NR \) has a \( S_1^+ \) singularity at \((u_0, w_0) \in S(NR)\) if and only if \( \kappa'_f(u_0) = 0, \kappa''_f(u_0) \neq 0 \) and \( \kappa''_f(u_0)(2\kappa_f(u_0)\kappa'_f(u_0) + \kappa''_f(u_0)) > 0 \).

**Proof.** Let us take an adapted coordinate system \((u, v)\) for \( f \). Then \( S(f) = \{v = 0\} \) holds. Under this setting, we show the assertions. Putting \( \xi_{NR} = \partial_w \) and \( \eta_{NR} = \partial_u \) on
$S(f) \times \mathbb{R}$, the pair $(\xi_{NR}, \eta_{NR})$ satisfies that $dNR(\eta_{NR}) = \eta_{NR}N(R) = 0$ and $\xi_{NR}$, $\eta_{NR}$ are linearly independent at $q \in S(NR)$. Using this pair $(\xi_{NR}, \eta_{NR})$, we define a function $\varphi : S(f) \times \mathbb{R} \to \mathbb{R}$ by $\varphi = \det(\xi_{NR}NR, \eta_{NR}NR, \eta_{NR}N(R)) = \det(NR_w, NR_u, NR_{uu})$. Since $NR_w, NR_u$ and $NR_{uu}$ can be calculated as

$$NR_w = \hat{\gamma}, \quad NR_u = (1 - w\kappa_v)\hat{\gamma'}w\kappa_t \hat{h}$$

$$NR_{uu} = w(\kappa_s\kappa_t - \kappa'_v)\hat{\gamma} + (\kappa_s(1 - w\kappa_v) - w\kappa'_v)\hat{h} + *\hat{\gamma}$$

by (3.4), where * is some function of $u$ and $w$, we have

$$\varphi = w^2\left(\kappa_s\left(\kappa'_v + \kappa''_v\right) + \kappa_s\kappa'_v - \kappa'v\kappa_t\right) - w\left(2\kappa_s\kappa_v + \kappa'_v\right) + \kappa_s$$

In this case, $NR$ at $q = (u_0, u_0)$ is a cross cap if and only if $\varphi(q) = 0$ and $\xi_{NR}\varphi(q) = \varphi_w(q) \neq 0$ (see [43] and [34, Remark 2.3]). Thus we have the first assertion.

We next consider the case that $NR$ does not have a cross cap at $q$, namely $\kappa'_v(u_0) = 0$. The first order differentials of $\varphi$ satisfy $\varphi_u(q) = 0$ and $\varphi_w(q) = \kappa'_v(u_0)$. Thus $\varphi$ has a critical point at $q$. We consider the Hessian of $\varphi$ at $q$. By straightforward calculations, we have

$$\varphi_{uu} = \frac{\kappa'_v}{\kappa''_v}(\kappa''_v + 2\kappa_s\kappa'_v), \quad \varphi_{uw} = \kappa''_v + 2\kappa_s\kappa'_v, \quad \varphi_{ww} = 2\kappa_s\kappa''_v$$

at $q$. Therefore the Hessian $\varphi_{uu}\varphi_{ww} - \varphi_{uw}^2$ at $q$ is

$$\varphi_{uu}(q)\varphi_{ww}(q) - \varphi_{uw}(q)^2 = -\kappa''_v(u_0)(2\kappa_s(u_0)\kappa'_v(u_0) + \kappa''_v(u_0)).$$

Moreover, when $\kappa'_v(u_0) = 0$, then $NR_{uu} = -\left(\kappa'_v(u_0)/\kappa_v(u_0)\right)\hat{\gamma}'(u_0)$. Thus $NR_w(q)$ and $NR_{uu}(q)$ are linearly independent if and only if $\kappa'_v(u_0) \neq 0$, and hence we have the assertion for $S_1$ singularities by the criteria for these singularities [34, Theorem 2.2].

For characterizations of singularities of $NR$, we only need a singular point to be of the first kind. Thus same results hold for $NR$ of a frontal with such a singular point.

4. Focal surfaces of frontal surfaces with pure-frontal singular points

We investigate singularities and certain geometric properties of $C_j$ as in (3.2) of a frontal surface with pure-frontal singularities.

4.1. Singularities of $C_j$. We here consider singularities of focal surface $C_j$ of a frontal surface $f$ with pure-frontal singular point. Let $V_j$ be the principal vector relative to $\kappa_j$. If $\kappa_j \neq 0$, then it is known that there exists a never vanishing smooth map $x_j : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ such that $df(V_j) = \Lambda x_j$ ([36, Corollary 3.4]), where $\Lambda$ is the identifier of singularities.

**Lemma 4.1.** The map $x_j$ ($j = 1, 2$) is normal to $C_j$ at any $q \in (\mathbb{R}^2, 0)$.

**Proof.** Let us take an adapted coordinate system $(u, v)$. Then $V_j$ can be written as

$$V_j = (-v(\tilde{M} - \kappa_j\tilde{F}), \tilde{L} - \kappa_j\tilde{E}).$$

Thus $x_j$ is given by

$$x_j = -\left(\tilde{M} - \kappa_j\tilde{F}\right)f_u + \left(\tilde{L} - \kappa_j\tilde{E}\right)h$$

(cf. (2.8)). We note that $x_j$ is perpendicular to $\nu$. By direct calculation, we have

$$(C_j)_u \equiv f_u + \rho_j\nu_u, \quad (C_j)_v \equiv v(h + \rho_j\nu_1) \mod \langle \nu \rangle_{\mathcal{E}},$$

where $\rho_j = \frac{f_{xx}}{f_x}$.
where \( \nu_0 = \nu v_1 \), \( \mathcal{E} \) is the ring of \( C^\infty \) function germs on \(( \mathbb{R}^2, 0)\), and \( \alpha \equiv \beta \mod \langle \nu \rangle \mathcal{E} \) means that there exists \( a \in \mathcal{E} \) such that \( \alpha - \beta = av \). Thus we have

\[
\begin{align*}
\langle (C_j)_{u}, x_j \rangle &= (\tilde{M} - \kappa_j \tilde{F})(-\tilde{E} + \rho_j \tilde{L}) + (\tilde{L} - \kappa_j \tilde{E})(\tilde{F} - \rho_j \tilde{M}) = 0, \\
\langle (C_j)_{v}, x_j \rangle &= v(\tilde{M} - \kappa_j \tilde{F})(-\tilde{F} + \rho_j \tilde{M}) + v(\tilde{L} - \kappa_j \tilde{E})(\tilde{G} - \rho_j \tilde{N}_1) \\
&= -v(\tilde{E} \tilde{G} - \tilde{F}^2)(\kappa_j - 2H + K \rho_j) = 0,
\end{align*}
\]

where \( K \) and \( H \) are the Gaussian and mean curvatures (see (2.5)). This completes the proof. \( \square \)

By this lemma, the map \( e_j = x_j/|x_j| \) gives an unit normal vector to \( C_j \) \((j = 1, 2)\). Thus \( C_j \) is a frontal surface when \( \kappa_j \neq 0 \).

**Proposition 4.2.** Let \( C_j \) be a focal surface of a frontal \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) associated to \( \kappa_j \). Then the set of singular points \( S(C_j) \) of \( C_j \) coincides with the zero set of \( V_j \rho_j \), where \( V_j \) is the principal vector relative to \( \kappa_j \) and \( \rho_j = 1/\kappa_j \).

**Proof.** We give a proof for \( j = 1 \). Take an adapted coordinate system \((u, v)\) on \((\mathbb{R}^2, 0)\). Then the map \( e_1 = x_1/|x_1| \) is a unit normal vector to \( C_1 \), where \( x_1 \) is given by (4.1) satisfying \( df(V_1) = \nu x_1 \). We then calculate \( \det((C_1)_{u}, (C_1)_{v}, e_1) \). By direct calculations, one can see

\[
(C_1)_{u} = f_u + \rho_1 v_u + (\rho_1)_{u} \nu = (1 + \rho_1 X_1)f_u + \rho_1 X_2 h + (\rho_1)_{u} \nu = A_1 f_u + B_1 h + C_1 \nu,
\]

\[
(C_1)_{v} = f_v + \rho_1 v_v + (\rho_1)_{v} \nu = v \rho_1 Y_1 f_u + v(1 + \rho_1 Y_2) h + (\rho_1)_{v} \nu = A_2 f_u + B_2 h + C_2 \nu,
\]

where we set \( v_u = X_1 f_u + X_2 h \) and \( v_v = v Y_1 f_u + v Y_2 h \) (see Lemma 2.4). Thus we have

\[
(C_1)_{u} \times (C_1)_{v} = (A_1 B_2 - A_2 B_1) f_u \times h + (A_1 C_2 - A_2 C_1) f_u \times \nu + (B_1 C_2 - B_2 C_1) h \times \nu.
\]

We remark that \( f_u \times h \) is parallel to \( \nu \). Hence \( \det((C_1)_{u}, (C_1)_{v}, e_1) \) can be calculated as

\[
\det((C_1)_{u}, (C_1)_{v}, e_1) = - \frac{\det(f_u, h, \nu)}{|x_1|}.
\]

By Lemma 2.4, we have

\[
\left( \tilde{L} - \kappa_1 \tilde{E} \right)(A_1 C_2 - A_2 C_1) + (\tilde{M} - \kappa_1 \tilde{F})(B_1 C_2 - B_2 C_1)
\]

\[
= (1 - \rho_1 \kappa_2) \left( v (\tilde{M} - \kappa_1 \tilde{F})(\rho_1)_{u} - (\tilde{L} - \kappa_1 \tilde{E})(\rho_1)_{v} \right) = -(1 - \rho_1 \kappa_2)V_1 \rho_1.
\]

Since \( \kappa_1 \neq \kappa_2 \), one can notice that \( 1 - \rho_1 \kappa_2 = \rho_1 (\kappa_1 - \kappa_2) \neq 0 \) holds. Thus the zero set of \( \det((C_1)_{u}, (C_1)_{v}, e_1) \) coincides with the zero set of \( V_1 \rho_1 \). This implies that \( S(C_1) = (V_1 \rho_1)^{-1}(0) \). For the case of \( j = 2 \), one can prove similarly. \( \square \)

This result corresponds to the case of regular surfaces [33].

By the above proposition, a pure-frontal singular point 0 of a frontal \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) is also a singular point of \( C_j \) if and only if \( V_j \rho_j = 0 \) at 0. We characterize this condition in terms of geometric invariants.

**Proposition 4.3.** Let \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) be a frontal and 0 a pure-frontal singular point of \( f \). Suppose that \( \kappa_j \neq 0 \) at 0. Then the point 0 is also a singular point of \( C_j \) \((j = 1, 2)\) if and only if \( r_c = 0 \) holds at 0.
Proof. Take an adapted coordinate system \((u, v)\) on \((\mathbb{R}^2, 0)\). Then we have \(V_j \rho_j = (\bar{L} - \kappa_j \bar{E})(\rho_j)_v\) at 0. We note that \(\bar{L} - \kappa_j \bar{E} = \kappa_v - \kappa_j \neq 0\) at 0 for \(j = 1, 2\) if \(\kappa_j \neq 0\). Thus \(V_j \rho_j = 0\) at 0 if and only if \((\rho_j)_v = 0\) at 0 when \(\kappa_j \neq 0\). This is equivalent to \((\kappa_j)_v = 0\) at 0. By Proposition 2.9, we have the assertion. \(\square\)

This proposition means that if 0 is a \(V_j\)-ridge point (and also a \(V_{j+1}\)-sub-parabolic point) of \(f\), then 0 is a singular point of \(C_j\) (see Proposition 2.9). By Corollary 2.10 and Proposition 4.3, the following assertion follows immediately.

**Corollary 4.4.** If a frontal \(f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)\) has a pure-frontal singular point other than a 5/2-cuspidal edge at 0 and \(\kappa_j \neq 0\), then 0 is a singular point of both \(C_1\) and \(C_2\).

By this corollary, when a frontal \(f\) has a fold singular point at 0, then \(C_j\) has a singularity at 0 if \(\kappa_j \neq 0\). We remark that maxfaces introduced by Umehara and Yamada [42], that is, spacelike zero mean curvature surfaces in the Minkowski 3-space \(\mathbb{R}_1^3\) (or \(L^3\), with fold singularities (cf. [5]) are examples of such surfaces when we consider the ambient space as the Euclidean 3-space \(\mathbb{R}^3\) (cf. [27]). Moreover, the 7/2-cuspidal cross cap (see [33, Example 14.10]), which is defined as a map germ \(\mathcal{A}\)-equivalent to \((u, v) \mapsto (u, v^5, uv^4)\) at the origin, with non vanishing \(\kappa_j\) is other typical example (see Figure 4).

**Figure 4.** The image of a 7/2-cuspidal cross cap with non vanishing \(\kappa_j\).

**Lemma 4.5.** Let \(f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)\) be a frontal and 0 a pure-frontal singular point of \(f\). Suppose that \(\kappa_j \neq 0\) and \(r_c = 0\) hold at 0. Then \(C_j\) (\(j = 1, 2\)) satisfies \(\text{rank } dC_j(0) = 1\). Moreover, \(V_j\) is a null vector field of \(C_j\).

**Proof.** Take an adapted coordinate system \((u, v)\). Then we see that
\[
(C_j)_u = f_u + \rho_j v_u + (\rho_j)_u v, \quad (C_j)_v = v(h + \rho_j v_1) + (\rho_j)_v v
\]
hold. By the assumption, \((\rho_j)_v = 0\) holds at 0, and hence \((C_j)_v = 0\) holds at 0. On the other hand, by Lemmas 2.4 and 2.8, we have
\[
(C_j)_u = (1 - \rho_j \kappa_v)f_u - \rho_j \kappa_v h + (\rho_j)_u v \neq 0
\]
at 0. Thus \(\text{rank } dC_j = 1\) holds at 0. By this property, there exists a null vector field \(\eta^{C_j}\) (\(j = 1, 2\)) for \(C_j\) around 0. On the other hand, by Lemma 2.6, we see that
\[
dC_j(V_j) = df(V_j) + \rho_j dv(V_j) + (V_j \rho_j)v = (V_j \rho_j)v
\]
holds. Since \(S(C_j) = (V_j \rho_j)^{-1}(0)\) holds by Proposition 4.2, it follows that \(dC_j(V_j) = 0\) holds on \(S(C_j)\). Therefore one can take \(\eta^{C_j}\) as \(\eta^{C_j} = V_j\). This shows the conclusion. \(\square\)

This property also holds for cases of regular surfaces without umbilic points and fronts with non-degenerate singular points ([20, 33, 40]).

**Proposition 4.6.** If a point 0 is a pure-frontal singular point of a frontal \(f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)\) with \(\kappa_j \neq 0\) and also a singular point of \(C_j\) (\(j = 1\) or 2), then \(C_j\) is a frontal but not a front at 0.
Proof. By Lemma 4.5, it is sufficient to show $de_j(V_j) = 0$ at 0. Let us take an adapted coordinate system $(u, v)$. We then deal with the case of $j = 1$. Suppose that $\kappa_i \neq 0$ and 0 is a singular point of $C_1$. By a direct calculation, we have $de_1(V_1) = (\tilde{L} - \kappa_1 \tilde{E})(e_1)_v$ at 0. Since $\tilde{L} - \kappa_1 \tilde{E} \neq 0$, we consider $(e_1)_v$ at 0. Since $e_1 = x_1/|x_1|$, it follows that $(e_1)_v = ((|x_1|^3)((x_1)_v|x_1|^2 - x_1(1, (x_1)_v))$. Thus we evaluate $(x_1)_v|x_1|^2 - x_1(1, (x_1)_v)$ at 0. Since $x_1$ is given by $x_1 = -\kappa_1 f_u + (\kappa_v - \kappa_1)h$ at 0, and hence $|x_1|^2 = \kappa_1^2 + (\kappa_v - \kappa_1)^2$ at 0. Further, $(x_1)_v$ can be calculated as 
\begin{equation}
(x_1)_v = - (\tilde{M}_v - \kappa_1 \tilde{E}_v)f_u + (\tilde{L}_v - (\kappa_1)_v)h + \tilde{L} - \kappa_1 \tilde{E}_v h
\end{equation}
at 0 since $\tilde{E}(0) = \tilde{E}_v(0) = 0$ and $\tilde{E}(0) = 1$. We note that $h_v = \tilde{E}_v f_u + (\tilde{G}_v/2)h$ holds at 0 ([36, (2.9)]). Moreover, since $f_{uu} = v_v = 0$ at 0, we see that $\tilde{L}_v = 0$ and $\tilde{M}_v = \tilde{E}_v \tilde{L} + \tilde{G}_v \tilde{M}/2$ at 0. Thus $(x_1)_v$ can be written as 
\begin{equation}
(x_1)_v = - \frac{\kappa_1 \tilde{G}_v}{2} f_u - (\kappa_1 - \kappa_v) \left( \frac{r_c}{48 \sqrt{\Gamma}} + \frac{\tilde{G}_v}{2} \right) h
\end{equation}
at 0 by (2.9). Therefore we have 
\begin{equation}
\langle x_1, (x_1)_v \rangle = \frac{k_1^2 \tilde{G}_v}{2} + (\kappa_1 - \kappa_v)^2 \left( \frac{r_c}{48 \sqrt{\Gamma}} + \frac{\tilde{G}_v}{2} \right)
\end{equation}
at 0. Since 0 is a singular point of $C_i$, $r_c = 0$ holds at 0. Thus we have 
\begin{equation}
(x_1)_v = - \frac{\tilde{G}_v}{2} \left( \kappa_1 f_u + (\kappa_1 - \kappa_v)h \right)
\end{equation}
at 0 by (4.2). Thus we have 
\begin{equation}
(x_1)_v|x_1|^2 = - \frac{(\kappa_1^2 + (\kappa_1 - \kappa_v)^2) \tilde{G}_v}{2} (\kappa_1 f_u + (\kappa_1 - \kappa_v)h) = \langle x_1, (x_1)_v \rangle x_1
\end{equation}
at 0. This implies that $(e_1)_v = 0$ holds at 0. Therefore we get the conclusion for $j = 1$. For the case of $j = 2$, one can show in a similar way. 

We seek conditions that 0 is a singular point of the second kind for $C_j$.

**Lemma 4.7.** Let $f$ be a frontal and 0 a pure-frontal singular point of $f$ with $\kappa_i(0) \neq 0$. Take an adapted coordinate system $(u, v)$ and suppose that 0 is a $V_j$-ridge point $(j = 1, 2)$ of $f$, that is, $r_c = 0$ at 0. Then 0 is an at least second order $V_j$-ridge point of $f$ if and only if 
\begin{equation}
(\kappa_v - \kappa_j)(\kappa_j)_{uv} - \kappa_i(\kappa_j)_u = 0
\end{equation}
or, equivalently, 
\begin{equation}
\kappa_i(\kappa_j)_{uv} - \left( \frac{r_b}{3} - \kappa_j \right) (\kappa_j)_u = 0
\end{equation}
at 0.

**Proof.** Since 0 is a ridge point of $f$, $(\kappa_j)_v(0) = 0$ holds by Proposition 4.2. Thus we have 
\begin{equation}
V_j V_j = (\tilde{L} - \kappa_j \tilde{E})(\tilde{L} - \kappa_j \tilde{E})(\kappa_j)_{uv} - (\tilde{M} - \kappa_j \tilde{F})(\kappa_j)_u
\end{equation}
at the origin. Since $\kappa_i \neq 0$, we have $\tilde{L} - \kappa_j \tilde{E} \neq 0$. This implies that 0 is an at least second order $V_j$-ridge point of $f$ if and only if 
\begin{equation}
(\tilde{L} - \kappa_j \tilde{E})(\kappa_j)_{uv} - (\tilde{M} - \kappa_j \tilde{F})(\kappa_j)_u = 0.
\end{equation}
By Lemma 2.8, we have the first expression. Moreover, using the formula $\kappa_j^2 - 2H\kappa_j + K = 0$, we note that the vector field $\tilde{V}_j = (-u(N_1 - \kappa_j \tilde{G}), \tilde{M} - \kappa_j \tilde{F})$ is also a principal vector field which does not vanish at 0, and we have

$$V_jV_{jk} = (\tilde{M} - \kappa_j \tilde{G}) \left( (\tilde{M} - \kappa_j \tilde{F})(\kappa_j)_{uv} - (N_1 - \kappa_j \tilde{G})(\kappa_j)_{u} \right) = 0$$

at 0. By Lemma 2.8 again, we have the second expression. Since being a ridge point does not depend on the principal vector field, we get the conclusion.

**Lemma 4.8.** Let $f$ be a front and 0 a pure-frontal singular point of $f$ with $\kappa_i(0) \neq 0$. Suppose that 0 is a $V_j$-ridge point ($j = 1, 2$) of $f$. Then 0 is a non-degenerated singular point of $C_j$ ($j = 1, 2$) of $f$.

**Proof.** We show the case of $j = 1$. The case $j = 2$ is similar. Take an adapted coordinate system $(u, v)$. Let $\lambda^{C_1}$ be the signed area density function of $C_1$. Since 0 is a $V_1$-ridge point, $(V_1\rho_1)_u$ at 0 can be calculated as

$$V_1\rho_1)_{uv} = \frac{r'(\kappa_1^1 - \kappa_v)}{48(\kappa_1^1)}$$

at the origin, and hence it holds that

$$V_1\rho_1)_{uv} = \frac{r'(\kappa_1^1 - \kappa_v)^2}{48\kappa_1^1}$$

at the origin. Moreover, setting $V = V_{11}\partial_u + V_{12}\partial_x$, $V_1(V_1\rho_1)$ at 0 is computed as

$$V_1(V_1\rho_1) = V_{12}(V_1\rho_1)_v$$

since $V_{11}$ at 0. On the other hand, it follows that

$$V_1(V_1\rho_1) = -V_1\left( V_1\kappa_1 \right)_{\kappa_1^2} = -V_1\left( V_1\kappa_1^2 - 2\kappa_1 V_1^4 \right) = -V_1(V_1\kappa_1) \left( V_1(K_1) \right)_{\kappa_1^2}$$

holds at the origin because $V_1\kappa_1 = 0$ at 0. Thus $(V_1\rho_1)_{uv} = 0$ if and only if $V_1(V_1\kappa_1) = 0$ at 0, since $V_{12} = \kappa_v - \kappa_1 = 0$ at 0. Therefore we have $(V_1\rho_1)_{uv} = 0$ if and only if $V_1(V_1\kappa_1) = 0$ or the origin is a first order $V_1$-ridge point of $f$.

**Proposition 4.9.** Let $f$: $(\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a front and 0 a pure-frontal singularity of $f$ with $\kappa_i(0) \neq 0$. Suppose that 0 is a non-degenerated singular point of $C_j$ ($j = 1, 2$). Then 0 is an at least second order $V_j$-ridge point of $f$ if and only if 0 is a singular point of the second kind of $C_j$.

**Proof.** We show the case $j = 1$. Take an adapted coordinate system $(u, v)$. Suppose 0 is an at least second order $V_1$-ridge point of $f$, that is, $V_1\kappa_1 = V_1V_1\kappa_1 = 0$ holds at 0. Thus, by (4.8), we have $(V_1\rho_1)_{uv}(0) = 0$. Since 0 is a non-degenerated singularity of $C_1$, we have $(\kappa_1)_{uv}(0) \neq 0$ by (4.6). By the identity

$$(V_1\rho_1)_{uv} = -(\kappa_v - \kappa_1)(\kappa_1)_{uv} / \kappa_1^2 \neq 0$$
at 0 and the implicit function theorem, there exists a regular curve \((g_1(t), t)\) such that 

\[ V_1 \rho_1 (g_1(t), t) = 0. \]

We also have

\[ g_1'(t) = -\frac{(V_1 \rho)_{\nu}}{(V_1 \rho)_{\mu}} (g_1(t), t), \]

which vanishes at 0 by (4.8). In other words, the tangent vector to the singular curve of \(C_1\) at 0 is the vector \((0, 1)\), which is parallel to the null vector field of \(V_1\) at 0 (Lemma 4.5).

Therefore 0 is a singular point of the second kind of \(C_1\). The case \(j = 2\) can be shown similarly.

On the other hand, if 0 is a first order \(V_j\)-ridge point \((j = 1, 2)\) of a frontal \(f:\,(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\), 0 is a non-degenerate singular point of \(C_j\) (Lemma 4.8), in particular, \((V_j \rho_j)_{\nu}(0) \neq 0\) holds. Thus by the implicit function theorem, there exists a curve \(\beta_j(t) = (t, g_j(t))\) such that \(g_j(0) = 0\) and \(V_j \rho_j(t, g_j(t)) = 0\). The tangent vector of \((t, g_j(t))\) at \(t = 0\) is \((1, g_j'(0))\), where \(g_j'(0) = -((V_j \rho_j)_{\nu}(V_j \rho_j)_{\mu})(0)\). Since \((1, g_j'(0))\) is not parallel to the null direction \((0, 1)\) of \(C_j\), the origin 0 is a singular point of the first kind of \(C_j\).

By Proposition 4.9, \(C_j\) \((j = 1, 2)\) cannot be a 5/2-cuspidal edge or a cuspidal cross cap at \(q\) when \(q\) is a second order \(V_j\)-ridge point of \(f\). In the following, we assume that 0 is a singular point of the first kind of \(C_j\). Then we give conditions that \(C_j\) has a cuspidal cross cap at 0 in terms of geometrical properties of the initial frontal.

**Theorem 4.10.** Let \(f:\,(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\) be a frontal with pure-frontal singular point 0. Suppose that 0 is a first order \(V_j\)-ridge point of \(f\) and a singular point of the first kind of \(C_j\) \((j = 1, 2)\). Then \(C_j\) has a cuspidal cross cap at 0 if and only if \(r'_c \neq 0\) and

\[ \hat{\beta}'_j \left( \kappa^2 + 2 \kappa \kappa'_j (\kappa_j - \kappa_v) + \frac{r'_b}{3} (\kappa_j - \kappa_v)^2 \right) \]

\[ \neq (\kappa^2 + (\kappa_j - \kappa_v)^2)(\kappa_{j+1} - \kappa_j)(\kappa_v - \kappa_j) \]

hold at 0, where \(\hat{\beta}_j\) is the composition of \(\rho_j\) and the singular curve \(\beta(t)\) of \(C_j\) and \(\hat{\beta}'_j\) is its derivative.

**Proof.** We show the case for \(j = 1\). Since \(f\) is not a front, we may assume \(f\) as in

\[ f(u, v) = (u, u^2 a_2(u) + v^2/2, u^2 a_3(u) + v^2 c_2(u) + v^4 c_4(u) + v^5 c_5(u, v)), \]

with \(c_2(0) = 0\) (cf. [16, Proposition 3.9] and [31, Proposition 2.1]). Since 0 is a singular point of \(C_j\), 0 is not a 5/2-cuspidal edge of \(f\). Thus \(c_5\) as in (4.9) satisfies \(c_5(0, 0) = 0\) (cf. [14, 16]).

We note that, in this case, the singular set of \(f\) is the \(u\)-axis and \(\eta = \partial_{\nu}\) is a null vector field of \(f\). Moreover, the functions in (2.3) can be defined in the same way. Since 0 is a singular point of the first kind of \(C_j\), the singular curve \(\beta(t)\) of \(C_1\) can be represented as \(\beta(t) = (t, g(t))\) with \(g(0) = 0\) by the proof of Proposition 4.9. It is well known (see [7, Corollary 1.5], for example) that, with the notations we used so far, \(C_1\) is a cuspidal crosscap at 0 if and only if \(\psi_{C_1}(0) = 0\) and \(\psi'_{C_1}(0) \neq 0\), where

\[ \psi_{C_1}(t) = \det(\hat{\beta}'_1(t), \hat{e}_1(t), d e_{1\beta(t)}(V_1)), \]

\[ \hat{\beta}_1(t) = C_1(\beta(t)), \hat{e}_1(t) = e_1(\beta(t)) \]

and \(V_1\) is a null vector field of \(C_1\). Since \(d e_{1}(V_1)(0) = 0, \psi_{C_1}(0) = 0\) holds.

We calculate \(\psi'_{C_1}\). First we consider \(\hat{\beta}'_1 \times \hat{e}_1\) at 0. Along \(\beta(t)\), we have

\[ \hat{\beta}'_1 = (C_1)_u + (C_1)_v g'(t), \]
and hence \( \beta'_1(0) = (C_1)_u(0) \) holds since \( (C_1)_v = 0 \) at 0. Using Lemmas 2.4 and 2.8, it follows that
\[
(C_1)_u = (1 - \kappa_v \rho_1) f_u - \rho_1 \kappa_i h + (\rho_1)_u v = \rho_1((\kappa_1 - \kappa_v) f_u - \kappa_i h) + (\rho_1)_u v
\]
holds at 0. Moreover, \( \hat{e}_1 \) can be written as
\[
\hat{e}_1 = \frac{-\kappa_i f_u + (\kappa_v - \kappa_i) h}{\Delta_1}
\]
at 0, where \( \Delta_1 = \sqrt{\kappa_i^2 + (\kappa_1 - \kappa_v)^2} \), and hence we have
\[
\beta'_1 \times \hat{e}_1 = -\frac{1}{\Delta_1} \left( (\rho_1)_u((\kappa_v - \kappa_i) f_u + \kappa_i h) + (\rho_1)_u \Delta_1^2 v \right)
\]
at 0.
We next consider the derivative of the curve \( \alpha(t) = d e_{1 \beta(t)}(V_1) \). Since \( \alpha(t) \) can be represented as
\[
\alpha(t) = V_{11}(\beta(t))((e_1)_u(\beta(t)) + V_{12}(\beta(t))(e_1)_v(\beta(t))
\]
we have
\[
\alpha'(t) = ((V_{11})_u + (V_{11})_v g'(t))(e_1)_u(\beta(t)) + V_{11}((e_1)_{uu}(\beta(t)) + (e_1)_{uv}(\beta(t)) g'(t)) + ((V_{12})_u + (V_{12})_v g'(t))(e_1)_v(\beta(t)) + V_{12}(\beta(t))((e_1)_{uv}(\beta(t)) + (e_1)_{vv} g'(t))
\]
Here \( (e_1)_v = 0 \) and \( V_{11} = (V_{11})_u = 0 \) hold at 0. Hence it follows that
\[
\alpha' = g'(V_{11})_u(e_1)_u + V_{12}(e_1)_v + V_{12}(e_1)_{vu}
\]
holds at 0.
Since 0 is a pure-frontal singular point of \( f \) and \( f_v(u, 0) = 0 \), \( f_v(u, v) = vh(u, v) \) and
\( \nu_v(u, v) = v \nu_1(u, v) \), for smooth maps \( h \) and \( \nu_1 \). Thus we have \( \tilde{E}_v(u, 0) = \tilde{E}_v(u, 0) = 0 \). On the other hand, from (4.9), it holds
\[
h(u, v) = (0, 1, 2c_2(u) + v^2 (4c_4(u) + v (5c_5(u, v) + v(c_3)(u, v))))
\]
Therefore we get \( \tilde{h}_v(u, 0) = \tilde{h}_{uv}(u, 0) = 0 \) and consequently, \( \tilde{M}_v(u, 0) = \tilde{G}_v(u, 0) = 0 \). Thus we have \( (x_1)_v = 0 \) at 0 since \( (\kappa_1)_v(0, 0) = 0 \). We note that \( (V_{11})_v = -\kappa_i \) and \( V_{12} = \kappa_v - \kappa_1 \) hold at 0.
Since \( e_1 = x_1/|x_1| \), we have
\[
(e_1)_u = \frac{(x_1)_u}{|x_1|} - e_1 \left( \frac{|x_1|}{|x_1|} \right)_u, \quad (e_1)_v = 0,
\]
\[
(e_1)_{uv} = (e_1)_{uu} = \frac{(x_1)_{uu}}{|x_1|} - e_1 \left( \frac{|x_1|}{|x_1|} \right)_u,
\]
\[
(e_1)_{vv} = \frac{(x_1)_{vv}}{|x_1|} - e_1 \left( \frac{|x_1|}{|x_1|} \right)_v
\]
at 0. By a direct calculation, we get
\[
(x_1)_u = -(\kappa'_i + \kappa_i (\kappa_v - \kappa_1)) f_u + ((\kappa'_v - (\kappa_1)_u) - \kappa_i \kappa_i) h - \kappa_1 \kappa_i
\]
at 0. Thus we see that
\[
\det(\beta'_1, \hat{e}_1, (e_1)_u) = \frac{1}{\Delta_1^2} \left( (\rho_1)_u \left( \kappa_i \Delta_1^2 + \kappa'_v (\kappa_v - \kappa_1) + \kappa_i ((\kappa_1)_u - \kappa'_v) \right) + \kappa_i \Delta_1^2 \right)
\]
at 0.
By \((x_1)_v(0, 0) = 0\), we obtain \(\det(\beta'_1, \hat{e}_1, (x_1)_v) = 0\) at 0, and hence \(\det(\beta'_1, \hat{e}_1, (e_1)_v) = 0\) at 0. Moreover, since \(\tilde{E}_v, \tilde{F}_v, \tilde{G}_v, \tilde{L}_v\) and \(\tilde{M}_v\) vanish on the \(u\)-axis, then \(\tilde{E}_{uv}, \tilde{F}_{uv}, \tilde{G}_{uv}, \tilde{L}_{uv}\) and \(\tilde{M}_{uv}\) also vanish on the \(u\)-axis. Using these facts and \(f_{uvv}(0) = 0\), we have

\[
(x_1)_{uv} = -(k_1)_{uv} h
\]

holds at 0. Therefore we get

\[
\det(\beta'_1, \hat{e}_1, (x_1)_{uv}) = \frac{p_u(k_1)_{uv}}{\Delta_1} \tag{4.13}
\]
at 0. Since \(|x_1|_v = \langle x_1, (x_1)_v \rangle / |x_1| = 0\) at 0 and by (4.7), we have

\[
\det(\beta'_1, \hat{e}_1, (e_1)_{uv}) = \frac{(p_1)_u(\kappa, \nu' - \kappa)_{uv}}{48A^2\sqrt{\nu}}
\]
at 0.

We next consider \((x_1)_{uv}\). By using \(\tilde{F} = \tilde{F}_v = \tilde{L}_v = \tilde{E}_v = (k_1)_v = h_v = h_{uv} = 0\), \(\tilde{G} = \tilde{E} = 1\) and \(f_{uv} = 0\) at 0, we have

\[
(x_1)_{uv} = -(\tilde{M}_{uv} - \kappa_1 \tilde{F}_v) f_u - \tilde{M} f_{uvv} + (\tilde{L}_{uv} - (k_1)_{uv} - \kappa_1 \tilde{E}_v) h + (\tilde{L} - \kappa_1 \tilde{E} ) h_{uv}
\]
at 0. By \(f_0 = vh\), (4.11) and (4.9), we see that \(h_{uv} = \frac{r}{3} V\) and \(f_{uvv} = \nu u_v = \nu_1\) hold at 0, which implies that \(\tilde{F}_{uv} = 0\). Moreover, we have the following.

**Lemma 4.11.** Under the above setting, we have

\[
\tilde{L}_{uv} = \kappa' + \kappa_s \left( \kappa_v - \frac{r_b}{3} \right), \quad \tilde{M}_{uv} = \frac{r_b}{3} + 2\kappa_s \kappa_l
\]
at 0.

**Proof.** By definition of \(\tilde{L}\) and \(\tilde{M}\), and \(f_{uv} = \nu_{uv} = h_v = h_{uv} = 0\) at 0, we have

\[
\tilde{L}_{uv} = -\langle f_{uvv}, \nu_u \rangle - \langle f_u, \nu_{uvv} \rangle \quad \text{and} \quad \tilde{M}_{uv} = -\langle h, \nu_{uvv} \rangle
\]
at 0. By (4.9), \(f_{uvv} = \kappa_1 \nu_v\), which is orthogonal to \(v\). Thus, \(\tilde{L}_{uv} = -\langle f_u, \nu_{uvv} \rangle\). On the other hand, since \(\nu = \frac{\tilde{v}}{|\tilde{v}|}\) with \(\tilde{v} = f_u \times h\) and \(\tilde{v}_v(0) = 0\), it holds

\[
\nu_{uvv} = \frac{\tilde{v}_{uvv}}{|\tilde{v}|} = \frac{\tilde{v} \langle \tilde{v}, \nu_{uvv} \rangle}{|\tilde{v}|^3}.
\]

By (4.9) and direct calculations,

\[
\tilde{v} = \left(-2ua_3 + 2u^2c_2a_2' + 4u^2v^2c_4a_2' + 5u^2v^3c_5a_2' - u^2a_3' - v^2c_2' - v^4c_4'
+ u^2v^4a_2'(c_5)_v + 2u a_2(2c_2 + 4v^2c_4 + 5v^3c_5 + v^4(c_5)_v) - v^5(c_5)_v,
-2c_2 - v^2(4c_4 + v(5c_5 + v(c_5)_v), 1)
\right),
\]

where \(\nu' = d/du\). Thus we get

\[
\tilde{v}_{uvv}(0) = (16a_2(0)c_4(0) - 2c_2'(0), 0), \quad \tilde{v}_{uvu}(0) = (0, 0, 0)
\]

We note that \(2a_2 = \kappa_s, 2a_3 = \kappa_v, 2c_2'' = \kappa_s \kappa_v + \kappa'_s, 24c_4 = r_b\) and \(24c_4' = r'_b + 6\kappa_s \kappa_l\) hold at 0 (cf. [16, Page 508]). Hence we obtain

\[
\tilde{v}_{uvv} = \left(-\kappa'_s + \kappa_s \left( \frac{r_b}{3} - \kappa_v \right), -\frac{r'_b}{3} - 2\kappa_s \kappa_l, 0 \right) \quad \text{and} \quad \tilde{v}_{uvu} = 0
\]
at 0. Now, since \(f_u(0) = (1, 0, 0), h(0) = (0, 1, 0),\) and \(\nu(0) = (0, 0, 1)\), we have the result.
We proceed calculations. By Lemma 4.11, \((x_1)_{uv}\) as in (4.14) can be written as
\[
(x_1)_{uv} = -\left(\frac{r_h^b}{3} + 2\kappa_s\kappa_t\right)f_u + \left(k_s' + k_s\left(k_d - \frac{r_h^b}{3}\right) - (k_1)_{uv}\right)h
\]
\[
+ \left(\frac{r_b^b}{3}(k_d - k_1) - k_1^2\right)v
\]
at 0. Thus one can see that
\[
-\Delta_1 \det(\hat{\beta}_1', \hat{e}_1, (x_1)_{uv}) = -\Delta_1 \left(\hat{\beta}_1' \times \hat{e}_1, (x_1)_{uv}\right)
\]
\[
= (\rho_1)_u \left(2k_s\kappa_d + \frac{r_h^b}{3}\right) - k_s' \left(-k_s' + k_s\left(k_d + \frac{r_h^b}{3}\right) + (k_1)_{uv}\right)
\]
\[
+ \rho_1\Delta_1^2 \left(\frac{r_b^b}{3}(k_d - k_1) - k_1^2\right)
\]
holds at 0. Hence we have
\[
- \Delta_1^2 \det(\hat{\beta}_1', \hat{e}_1, (e_1)_{uv}) = (\rho_1)_u \left(2k_s\kappa_d + \frac{r_h^b}{3}\right) - k_s' \left(-k_s' + k_s\left(k_d + \frac{r_h^b}{3}\right) + (k_1)_{uv}\right)
\]
\[
+ \rho_1\Delta_1^2 \left(\frac{r_b^b}{3}(k_d - k_1) - k_1^2\right)
\]
at 0. By (4.12), and (4.15), we have
\[
g' \det(\hat{\beta}_1', \hat{e}_1, (V_{11})_u(e_1)_{uv} + V_{12}(e_1)_{uv})
\]
\[
= g'((V_{11})_u \det(\hat{\beta}_1', \hat{e}_1, (e_1)_{uv} + V_{12} \det(\hat{\beta}_1', \hat{e}_1, (e_1)_{uv}))
\]
\[
= g'(-k_1 \det(\hat{\beta}_1', \hat{e}_1, (e_1)_{uv} + (k_d - k_1) \det(\hat{\beta}_1', \hat{e}_1, (e_1)_{uv}))
\]
\[
= \frac{g'}{\Delta_1^2} (\rho_1)_u A_1 + \rho\Delta_1^2 A_2
\]
at 0, where
\[
A_1 = -k_1(V_1\rho_1)_u k_1^2 + k_s\kappa_d A_3 + \frac{r_h^b}{3}(k_1 - k_d) + k_1^2 k_s' + 2k_s\kappa_d(k_1 - k_d),
\]
\[
A_2 = k_1^2(k_d - 2k) - (k_1 - k_d)^2 \frac{r_h^b}{3}, \quad A_3 = -\Delta_1^2 - (k_d - k_1) \left(2k_1 - k_d - \frac{r_b^b}{3}\right),
\]
and we use the identity \((V_{1}\rho_1)_u k_1^2 = ((k_1)_{uv}(k_1 - k_d) + k_1(k_1)_{uv})\) at 0. Note that we can rewrite \(A_2\) and \(A_3\) as
\[
A_2 = \left(\frac{k_d r_b^b}{3} - k_1^2\right)(2k_1 - k_d) - \frac{k_1^2 r_b^b}{3} \quad \text{and} \quad A_3 = k_1 \left(k_1 - k_d - \frac{r_b^b}{3}\right) - k_1^2 + \frac{r_b^b k_d}{3}.
\]
Now, using the identities \(k_d + \frac{r_d^c}{3} = 2H = k_1 + k_2\) and \(-k_1^2 + \frac{r_b^b k_d}{3} = K = k_1 k_2\) at 0, we get \(A_3 = 0\) and \(A_2 = k_1(k_1 - k_2)(k_d - k_1)\) at 0.

On the other hand, we see that
\[
V_{12} \det(\hat{\beta}_1', \hat{e}_1, (e_1)_{uv}) = \frac{(\rho_1)_u k_d r_b^b(k_1 - k_d)^2}{48\Delta_1^2 \sqrt{\Gamma}}
\]
holds at 0. Since
\[
(V_1\rho_1)_u = \frac{r_b^b(k_1 - k_d)^2}{48k_1^2 \sqrt{\Gamma}}
\]
holds at 0 and \( g'(0) = \frac{(V_1 \rho_1) u}{(V_1 \rho_1) v} \), we have
\[
g'(\rho_1)_u k_1(V_1 \rho_1)_v k_1^2 - \frac{(V_1 \rho_1)_v (\rho_1)_u k_1^2}{\Delta_1^2} = -V_{12} \det(\hat{\beta}_1^1, \hat{\epsilon}_1, (e_1)_{uv})
\]
at 0. Therefore we obtain
\[
\psi'_{C_j} = \det(\hat{\beta}_1^1, \hat{\epsilon}_1, \alpha')
\]
\[
= g'(0) \left( (\rho_1)_u \left( k_1^2 \kappa'_v + 2 \kappa_i \kappa'_u (\kappa_1 - \kappa_v) + \frac{r''}{3} (\kappa_1 - \kappa_v)^2 \right) + \Delta_1^2 (\kappa_1 - \kappa_2) (\kappa_v - \kappa_1) \right)
\]
at 0 by (4.10), (4.13) and (4.16). Since \( \beta_1^1 = (\rho_1)_u + (\rho_1)_v g' \) and \( (\rho_1)_v = 0 \), we have the assertion by [7, Corollary 1.5].

**Theorem 4.12.** Let \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) be a frontal and 0 a pure-frontal singular point of \( f \). Suppose that the secondary cuspidal curvature \( r_c \) identically vanishes along the singular curve \( y \) for \( f \) through 0, and 0 is a first order \( V_j \)-ridge point of \( f \). Then \( y \) is also a singular curve for \( C_j \) \((j = 1, 2)\) consisting of pure-frontal singular points of \( C_j \). In addition, the Gaussian and the mean curvature of \( C_j \) are bounded near 0.

**Proof.** Let us take an adapted coordinate system \((u, v)\). Then we show that \( \psi_{C_j} \) \((j = 1, 2)\) vanishes along the singular curve of \( C_j \). By the definition of \( \psi_{C_j} \), it is sufficient to show \( de_j (V_j) = 0 \) along the singular curve for \( C_j \). We first remark that the \( u \)-axis is also a singular curve of \( C_j \) because \( (V_j \rho_j)_u = 0 \) along the \( u \)-axis by Proposition 2.9. Moreover, since 0 is a first order \( V_j \)-ridge point, \((V_j \rho_j)_u, (V_j \rho_j)_v \neq (0, 0)\) at 0. Thus we get \( S(f) = S(C_j) = \{v = 0\} \).

By (2.7), we have \( de_j(V_j) = (\kappa_v - \kappa_j)(e_j)_v \) along the \( u \)-axis, where \( \kappa_j(u) = \kappa_j(u, 0) \). By the proof of Proposition 4.6, it holds that
\[
(e_j)_u = \frac{(-1)^j + 1 r_c k_i (\hat{\kappa}_j - \kappa_v)}{48 \Delta_1^3 \sqrt{\Gamma}} \left( (\kappa_v - \kappa_j) \hat{\gamma}' - \kappa_j \hat{h} \right) \Delta_j = \sqrt{\kappa_i^2 + (\kappa_j - \kappa_v)^2}
\]
along the \( u \)-axis, where \( \Gamma(u) = \Gamma(u, 0) = H(u, 0)^2 - K(u, 0) \). Thus if \( r_c \) vanishes identically along the \( u \)-axis, then \( de_j(V_j) \) also vanishes. This implies that \( \psi_{C_j} \) vanishes along the \( u \)-axis, and hence the \( u \)-axis consists of pure-frontal singular points for \( C_j \). Boundedness of the Gaussian and the mean curvature for \( C_j \) follows from this result and [26, Propostion 3.8 and Theorem 3.9].

**Example 4.13.** Let \( f : \mathbb{R}_{>0} \times \mathbb{R} \rightarrow \mathbb{R}^3 \) be a \( C^\infty \) map given by
\[
f(u, v) = (-\cosh(\log u) \sin v, \cosh(\log u) \cos v, v),
\]
where \( \mathbb{R}_{>0} = \{a \in \mathbb{R} \mid a > 0\} \). When we consider the ambient space as the Minkowski 3-space \( \mathbb{R}^3_1 \) with signature \((++-), f \) is a parametrization of the maximal helicoid ([24]). It is known that the set of singular points of \( f \) is \( S(f) = \{u = 1\} \) and all singular points are fold singularities (cf. [5, 6]). In fact, if we change parameter by \( u = e^w \), then \( f \) can be written as
\[
f(w, v) = (-\cosh w \sin v, \cosh w \cos v, v).
\]
Further, \( S(f) = \{w = 0\} \) and \( f \) satisfies \( f(w, v) = f(-w, v) \).

We consider focal surfaces of \( f \). By direct calculations, we have
\[
f_u = \frac{u^2 - 1}{2u^2} (-\sin v, \cos v, 0)
\]
\[
f_v = \left( -\frac{(1 + u^2) \cos v}{2u^2}, -\frac{(1 + u^2) \sin v}{2u}, 1 \right).
\]
Thus one can take a unit normal vector \( v \) to \( f \) as
\[
v = \left( \frac{2u \cos v}{\sqrt{1 + 6u^2 + u^4}}, \frac{2u \sin v}{\sqrt{1 + 6u^2 + u^4}}, \frac{1 + u^2}{\sqrt{1 + 6u^2 + u^4}} \right).
\]
Using this \( v \), we have principal curvatures \( \kappa_1 \) and \( \kappa_2 \) as follows:
\[
\kappa_1 = -\frac{4u^2}{1 + 6u^2 + u^4}, \quad \kappa_2 = \frac{4u^2}{1 + 6u^2 + u^4}.
\]
These functions are of class \( C^\infty \) even on \( S(f) = \{u = 1\} \). Moreover, the reciprocals \( \rho_1, \rho_2 \) of them are also \( C^\infty \) functions. We note that \( \text{Im} f \) is the subset of the right helicoid in \( \mathbb{R}^3 \). So this can be considered as a minimal surface in \( \mathbb{R}^3 \) with fold singularities. Setting \( \delta = \sqrt{1 + 6u^2 + u^4}, \) focal surfaces \( C_1 \) and \( C_2 \) are written as
\[
C_1 = \left( -\frac{\delta \cos v + (1 + u^2) \sin v}{2u}, -\frac{\delta \sin v - (1 + u^2) \cos v}{2u}, -\frac{\delta}{4} \left( 1 + \frac{1}{u^2} \right) + v \right),
\]
\[
C_2 = \left( \frac{\delta \cos v - (1 + u^2) \sin v}{2u}, \frac{\delta \sin v + (1 + u^2) \cos v}{2u}, \frac{\delta}{4} \left( 1 + \frac{1}{u^2} \right) + v \right)
\]
(see Figure 5).

We focus on \( C_1 \). We note that the set \( S(f) = \{u = 1\} \) is also the set of singular points of \( C_1 \). In fact, one can see that \((C_1)_u \times (C_1)_v = (u^4 - 1)a(u, v)\) holds, where \( a(u, v) \) is a non-zero \( \mathbb{R}^3 \)-valued function on \( \mathbb{R}_{>0} \times \mathbb{R} \). Moreover, since \( C_1(1/u, v) = C_1(u, v) \) holds for any \((u, v) \in \mathbb{R}_{>0} \times \mathbb{R}\), it might hold that \( C_1 \) also has a fold singularity at \((1, v)\). (For \( C_2 \), we have the same property.) By direct calculations, a unit normal vector \( v^{C_1} \) to \( C_1 \) can be taken as
\[
v^{C_1} = \left( -\frac{(1 + u^2) \cos v - \delta \sin v}{\sqrt{2} \delta}, -\frac{(1 + u^2) \sin v - \delta \cos v}{\sqrt{2} \delta}, \frac{\sqrt{2} u}{\delta} \right).
\]
Thus the Gaussian curvature \( K^{C_1} \) and the mean curvature \( H^{C_1} \) can be calculated as
\[
K^{C_1} = \frac{4u^4}{1 + 6u^2 + u^4} > 0, \quad H^{C_1} = -\frac{u}{\sqrt{2}(1 + u^2)} (< 0).
\]
These are bounded \( C^\infty \) functions on the source.

**Figure 5.** The images of \( f \) (left), \( C_1 \) (middle) and both of them (right) in Example 4.13.

### 4.2. Curvatures of \( C_j \) for 5/2-cuspidal edges.
By Proposition 4.3, when a frontal \( f \) has a 5/2-cuspidal edge at 0, then focal surfaces \( C_j \) \((j = 1, 2)\) are regular at 0. Thus one can consider the Gaussian and mean curvature for \( C_j \) at 0. For the Gaussian curvature \( K^{C_j} \) for \( C_j \), we have the following assertion.
Theorem 4.14. Let $f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal and $0$ a $5/2$-cuspidal edge of $f$. Suppose that $\kappa_i \neq 0$ at $0$. Then the Gaussian curvatures $K^{C_1}$ and $K^{C_2}$ of $C_1$ and $C_2$ are written as

$$K^{C_j} = -\frac{\kappa_i^2 \kappa_j^4}{(\kappa_i^2 + (\kappa_v - \kappa_j)^2)^2}$$

at $0$ ($j = 1, 2$). In particular, these are strictly negative at $0$.

Proof. Let us take an adapted coordinate system $(u, v)$ on $(\mathbb{R}^2, 0)$. Then we consider differentials of caustics $C_j$ ($j = 1, 2$). By direct calculations, we have

$$(C_j)_u = (1 - \rho_j \kappa_v) f_u - \rho_j \kappa_i h + (\rho_j)_u \nu, \quad (C_j)_v = (\rho_j)_v \nu$$

at $0$. We note that $(\rho_j)_v = - \frac{(\kappa_v - \kappa_j)}{\kappa_j} \neq 0$ at $0$ (cf. (2.9)). Thus the coefficients of the first fundamental form of $C_j$ are

$$E^{C_j} = \rho_j^2 ((\kappa_v - \kappa_j)^2 + \kappa_j^2) + ((\rho_j)_u)^2, \quad F^{C_j} = (\rho_j)_u (\rho_j)_v, \quad G^{C_j} = ((\rho_j)_v)^2$$

at $0$. Hence we get

$$(4.18) \quad E^{C_j} G^{C_j} - (F^{C_j})^2 = ((\rho_j)_v)^2 \rho_j^2 ((\kappa_v - \kappa_j)^2 + \kappa_j^2)$$

at $0$. We next investigate the coefficients of the second fundamental form of $C_j$. By (4.17) and the above calculation, we have $\langle (C_j)_v, (e_j)_v \rangle = 0$ at $0$. Thus the quantity $N^{C_j} = - \langle (C_j)_v, (e_j)_u \rangle$ vanishes at $0$. This implies that the Gaussian curvatures $K^{C_1}$ and $K^{C_2}$ are non-positive at $0$. To obtain the explicit representation for $K^{C_j}$ ($j = 1, 2$), we consider the quantity $M^{C_j}$ given by $M^{C_j} = - \langle (C_j)_u, (e_j)_u \rangle$. Since $e_j = x_j |x_j|$, we have $(e_j)_u = (x_j)_u |x_j|^{-1} + x_j (|x_j|^{-1})_u$. Since $x_j \perp (C_j)_v$ hold, it follows $\langle (C_j)_v, (e_j)_u \rangle = \langle (C_j)_u, (x_j)_u \rangle |x_j|^{-1}$. Thus we first consider $(x_j)_u$. The map $x_j$ can be written as

$$x_j = -\kappa_i f_u + (\kappa_v - \kappa_j) h$$

along the $u$-axis. Therefore one can see that

$$(x_j)_u = -\kappa_i f_u - \kappa_i f_u + (\kappa_v - (\kappa_j)_u) h + (\kappa_v - \kappa_j) h_u$$

holds along the $u$-axis. By (3.3), we may write $f_{uu} = \kappa_s h + \kappa_v v$ and $h_u = -\kappa_s f_u + \kappa_i v$ along the $u$-axis. Hence $(x_j)_u$ can be expressed as

$$(x_j)_u = - (\kappa_i + \kappa_s (\kappa_v - \kappa_j)) f_u + (\kappa_v - (\kappa_j)_u - \kappa_s \kappa_i) h - \kappa_i \kappa_j v$$

along the $u$-axis, in particular at $0$. Thus it holds that

$$(4.19) \quad M^{C_j} = - \langle (C_j)_u, (e_j)_u \rangle = \frac{(\rho_j)_u \kappa_j}{\sqrt{\kappa_j^2 + (\kappa_v - \kappa_j)^2}} (\neq 0)$$

at $0$. Thus the Gaussian curvature $K^{C_j}$ of $C_j$ is calculated as

$$K^{C_j} = -\frac{(M^{C_j})^2}{E^{C_j} G^{C_j} - (F^{C_j})^2} = - \frac{\kappa_j^2 \kappa_i^4}{\rho_j^2 (\kappa_j^2 + (\kappa_v - \kappa_j)^2)^2}$$

at $0$ by (4.18) and (4.19). This is the desired one. In particular, $K^{C_j}$ is strictly negative at $0$ by $\kappa_i \neq 0$. \qed
This theorem implies that a 5/2-cuspidal edge corresponds to a hyperbolic point (cf. [20, Page 12]) of the focal surfaces $C_j$ when $\kappa_t$ of the initial frontal does not vanish at that point.

For the mean curvature, we have the following.

**Proposition 4.15.** The mean curvature $H^{C_j}$ ($j = 1, 2$) of $C_j$ for a frontal $f$ with a 5/2-cuspidal edge $0$ can be represented as
\[
H^{C_j} = -\frac{\kappa_j (\kappa_x^2 + (\kappa_y - \kappa_j)^2) + \kappa_x' (\kappa_y - \kappa_j) - \kappa_t \kappa_x'}{2(\kappa_x^2 + (\kappa_y - \kappa_j)^2)^{3/2}}
\]
at $0$.

**Proof.** We take an adapted coordinate system $(u, v)$. By the proof of Theorem 4.14, we have
\[
L^{C_j} = -\frac{\langle (C_j)_u, (x_j)_u \rangle}{\langle (C_j)_u, (x_j)_u \rangle} = -\rho_j \kappa_x \kappa_y \kappa_x' \kappa_y - \kappa_t \kappa_x \kappa_y (\kappa_y - \kappa_j) \kappa_x' \kappa_y - \kappa_t \kappa_x \kappa_y \kappa_x' \kappa_y - \kappa_t \kappa_x \kappa_y \kappa_x' \kappa_j
\]
at $0$. Since $N^{C_j} = 0$ at $0$ and relations $(\rho_j)_u = -(\kappa_j)_u \rho_j^2$ and $\kappa_j \rho_j = 1$ hold, we have
\[
E^{C_j}N^{C_j} - 2F^{C_j}M^{C_j} + G^{C_j}L^{C_j} = -2F^{C_j}M^{C_j} + G^{C_j}L^{C_j} = -\frac{(\kappa_x^2 + (\kappa_y - \kappa_j)^2)}{\sqrt{\kappa_x^2 + (\kappa_y - \kappa_j)^2}}
\]
at $0$. By (4.18) and the above expression, we have the assertion. \(\square\)

**Example 4.16.** Let $f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map given by
\[
f(u, v) = \left(u, u^2 + \frac{v^2}{2}, uv^2 + \frac{v^3}{3}\right).
\]
Then one can see that $S(f) = \{v = 0\}$ and $f$ has a 5/2-cuspidal edge at $0$. We can take a unit normal vector $\nu$ to $f$ as
\[
\nu(u, v) = \frac{(2u + v^3, -v, 2u - v^3, 1)}{\sqrt{1 + (2u + v^3)^2 + (v^2 - 2u^3)^2}}.
\]
We note that $\kappa_x = 2$, $\kappa_y = 0$, $\kappa_t = 2 \neq 0$, $r_b = 0$ and $r_c = 72$ hold at $0$. Thus principal curvatures $\kappa_1, \kappa_2$ take different values. In particular, $\kappa_1 = 2$ and $\kappa_2 = -2$ hold at $0$. Thus the Gaussian curvature $K$ and the mean curvature $H$ of $f$ are $K = -4 < 0$ and $H = 0$ at $0$, respectively. We consider the Gaussian curvatures and mean curvatures of $C_1$ and $C_2$. By direct calculations, we have
\[
K^{C_1} = K^{C_2} = -1 < 0
\]
at $0$. On the other hand,
\[
-\frac{\kappa_x^2 \kappa_y^4}{(\kappa_x^2 + (\kappa_y - \kappa_j)^2)} = -1
\]
holds at $0$ for $j = 1, 2$. This verifies Theorem 4.14. Moreover, we have
\[
H^{C_1} = -\frac{3}{2\sqrt{2}}, \quad H^{C_2} = \frac{3}{2\sqrt{2}}
\]
at 0 by direct calculations. On the other hand, since $\kappa_1' = 0$ and $\kappa_2' = -4$ hold at 0, it follows that

$$\frac{-k_1(k_3(k_2^2+(k_1-k_2)^2)+k_4'(k_2-k_1)-k_1k_4')}{2(k_2^2+(k_1-k_2)^2)^{3/2}} = \frac{3}{2\sqrt{2}},$$

$$\frac{-k_2(k_3(k_2^2+(k_1-k_2)^2)+k_4'(k_2-k_1)-k_1k_4')}{2(k_2^2+(k_1-k_2)^2)^{3/2}} = \frac{3}{2\sqrt{2}}$$

hold at 0. This verifies that Proposition 4.15 holds.

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References

[1] V. I. Arnol'd, *Singularities of caustics and wave fronts*, Mathematics and its Applications (Soviet Series), vol. 62, Kluwer Academic Publishers Group, Dordrecht, 1990. MR 1151185

[2] J. W. Bruce and T. C. Wilkinson, *Folding maps and focal sets*, Singularity theory and its applications, Part I (Coventry, 1988/1989), Lecture Notes in Math., vol. 1462, Springer, Berlin, 1991, pp. 63–72. MR 1129024

[3] X.-Y. Chen and T. Matumoto, *On generic 1-parameter families of $C^\infty$-maps of an n-manifold into a $(2n-1)$-manifold*, Hiroshima Math. J. 14 (1984), no. 3, 547–550. MR 772985

[4] M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976, Translated from the Portuguese. MR 0394451

[5] S. Fujimori, Y. W. Kim, S.-E. Koh, W. Rossman, H. Shin, M. Umehara, K. Yamada, and S.-D. Yang, *Zero mean curvature surfaces in Lorentz-Minkowski 3-space and 2-dimensional fluid mechanics*, Math. J. Okayama Univ. 57 (2015), 173–200. MR 3289302

[6] S. Fujimori, W. Rossman, M. Umehara, K. Yamada, and S.-D. Yang, *New maximal surfaces in Minkowski 3-space with arbitrary genus and their cousins in de Sitter 3-space*, Results Math. 56 (2009), no. 1-4, 41–82. MR 2575851

[7] S. Fujimori, K. Saji, M. Umehara, and K. Yamada, *Singularities of maximal surfaces*, Math. Z. 259 (2008), no. 4, 827–848. MR 2403743

[8] T. Fukui, *Local differential geometry of cuspidal edge and swallowtail*, Osaka J. Math. 57 (2020), no. 4, 961–992. MR 4160343

[9] T. Fukunaga and M. Takahashi, *Framed surfaces in the Euclidean space*, Bull. Braz. Math. Soc. (N. S.) 50 (2019), no. 1, 37–65. MR 3935057

[10] , *Framed surfaces and one-parameter families of framed curves in Euclidean 3-space*, J. Singul. 21 (2020), 30–49. MR 4084199

[11] M. Hasegawa, A. Honda, K. Naokawa, K. Saji, M. Umehara, and K. Yamada, *Intrinsic properties of surfaces with singularities*, Internat. J. Math. 26 (2015), no. 4, 1540008, 34. MR 3338072

[12] M. Hasegawa, A. Honda, K. Naokawa, M. Umehara, and K. Yamada, *Intrinsic invariants of cross caps*, Selecta Math. (N. S.) 20 (2014), no. 3, 769–785. MR 3217459

[13] A. Honda, *Duality of singularities for flat surfaces in Euclidean space*, J. Singul. 21 (2020), 132–148. MR 4084204

[14] A. Honda, M. Koiso, and K. Saji, *Fold singularities on spacelike CMC surfaces in Lorentz-Minkowski space*, Hokkaido Math. J. 47 (2018), no. 2, 245–267. MR 3815292

[15] A. Honda, K. Naokawa, K. Saji, M. Umehara, and K. Yamada, *Duality on generalized cuspidal edges preserving singular set images and first fundamental forms*, J. Singul. 22 (2020), 59–91. MR 4192691

[16] A. Honda and K. Saji, *Geometric invariants of S/2-cuspidal edges*, Kodai Math. J. 42 (2019), no. 3, 496–525. MR 4025756

[17] G. Ishikawa, *Determinacy of the envelope of the osculating hyperplanes to a curve*, Bull. London Math. Soc. 25 (1993), no. 6, 603–610. MR 1245089

[18] , *Singularities of frontals*, Singularities in generic geometry, Adv. Stud. Pure Math., vol. 78, Math. Soc. Japan, Tokyo, 2018, pp. 55–106. MR 3839942

[19] , *Recognition problem of frontal singularities*, J. Singul. 21 (2020), 149–166. MR 4084205

[20] S. Izumiya, M. C. Romero Fuster, M. A. S. Ruas, and F. Tari, *Differential geometry from a singularity theory viewpoint*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016. MR 3409029
SINGULAR LOCI OF NORMAL CONGRUENCE OF FRONTALS

[21] S. Izumiya, K. Saji, and N. Takeuchi, Singularities of line congruences, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003), no. 6, 1341–1359. MR 2027650

[22] ———, Flat surfaces along cuspidal edges, J. Singul. 16 (2017), 73–100. MR 3655304

[23] S. Izumiya and N. Takeuchi, Singularities of ruled surfaces in $\mathbb{R}^3$, Math. Proc. Cambridge Philos. Soc. 130 (2001), no. 1, 1–11. MR 1797727

[24] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space $L^3$, Tokyo J. Math. 6 (1983), no. 2, 297–309. MR 732085

[25] L. F. Martins and K. Saji, Geometric invariants of cuspidal edges, Canad. J. Math. 68 (2016), no. 2, 445–462. MR 3484374

[26] L. F. Martins, K. Saji, M. Umehara, and K. Yamada, Behavior of Gaussian curvature and mean curvature near non-degenerate singular points on wave fronts, Geometry and topology of manifolds, Springer Proc. Math. Stat., vol. 154, Springer, [Tokyo], 2016, pp. 247–281. MR 3555987

[27] Y. Matsushita, T. Nakashima, and K. Teramoto, Geometric properties near singular points of surfaces given by certain representation formulae, Publ. Math. Debrecen 99 (2021), no. 3-4, 331–354. MR 4333835

[28] D. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, Proc. Lond. Math. Soc. (3) 50 (1985), no. 2, 333–369. MR 772717

[29] R. Morris, The sub-parabolic lines of a surface, The mathematics of surfaces, VI (Uxbridge, 1994), Inst. Math. Appl. Conf. Ser. New Ser., vol. 58, Oxford Univ. Press, New York, 1996, pp. 79–102. MR 1430581

[30] S. Murata and M. Umehara, Flat surfaces with singularities in Euclidean 3-space, J. Differential Geom. 82 (2009), no. 2, 279–316. MR 2520794

[31] R. Oset Sinha and K. Saji, On the geometry of folded cuspidal edges, Rev. Mat. Complut. 31 (2018), no. 3, 627–650. MR 3847079

[32] I. R. Porteous, The normal singularities of a submanifold, J. Differential Geom. 5 (1971), 543–564. MR 292092

[33] ———, Geometric differentiation. For the intelligence of curves and surfaces, second ed., Cambridge University Press, Cambridge, 2001. MR 1871900

[34] K. Saji, Criteria for cuspidal $S_k$ singularities and their applications, J. Gökova Geom. Topol. GGT 4 (2010), 67–81. MR 2755994

[35] K. Saji and S. P. dos Santos, Geometry of bifurcation sets of generic unfoldings of corank two functions, Monatsh. Math. 196 (2021), no. 3, 553–575. MR 4320538

[36] K. Saji and K. Teramoto, Behavior of principal curvatures of frontals near non-front singular points and their applications, J. Geom. 112 (2021), no. 3, Paper No. 39, 25 pp. MR 4328072

[37] K. Saji, M. Umehara, and K. Yamada, $A_k$ singularities of wave fronts, Math. Proc. Cambridge Philos. Soc. 146 (2009), no. 3, 731–746. MR 2496355

[38] ———, The geometry of fronts, Ann. of Math. (2) 169 (2009), no. 2, 491–529. MR 2480610

[39] K. Teramoto, Parallel and dual surfaces of cuspidal edges, Differential Geom. Appl. 44 (2016), 52–62. MR 3433975

[40] ———, Focal surfaces of wave fronts in the Euclidean 3-space, Glasg. Math. J. 61 (2019), no. 2, 425–440. MR 3928646

[41] ———, Principal curvatures and parallel surfaces of wave fronts, Adv. Geom. 19 (2019), no. 4, 541–554. MR 4015189

[42] M. Umehara and K. Yamada, Maximal surfaces with singularities in Minkowski space, Hokkaido Math. J. 35 (2006), no. 1, 13–40. MR 2225080

[43] H. Whitney, The singularities of a smooth $n$-manifold in $(2n–1)$-space, Ann. of Math. (2) 45 (1944), 247–293. MR 10275

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