ON THE IRREDUCIBLE REPRESENTATION ALGEBRA OF
THE ALTERNATING GROUP OF DEGREE FOUR

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Abstract. We obtain a description of the irreducible representation algebra
of the alternating group of degree four over the ring of 2-adic integers.

Let $G$ be a finite group and let $K$ be a commutative principal ideal domain. We denote the module $M$ of a $K$-representation of the group $G$ by the symbol $[M]$. We assume that $[M] = [N]$ if and only if the $KG$-modules $M$ and $N$ of $K$-representations are isomorphic. The ring $a(KG)$ of $K$-representation of the group $G$ is a smallest ring which contains all symbols $[M]$ with the following operations

$[M] + [N] = [M ⊕ N]$,

$[M] · [N] = [M ⊗ N]$,

where $g(m ⊗ n) = g(m) ⊗ g(n), \ (g ∈ G, m ∈ M, n ∈ N)$.

The subring $b(KG) ⊂ a(KG)$ which is generated by $[M]$, where $M$ is a module of the irreducible $K$-representation of the group $G$ is called the irreducible $K$-representations ring of $G$.

The algebras $A(KG) = a(KG) ⊗ Z Q$ and $B(KG) = b(KG) ⊗ Z Q$ over the field of rational numbers $Q$ are called the $K$-representation algebra of $G$ and the irreducible $K$-representation algebra of $G$, respectively.

It is well known that $A(RG)$, where $R$ is the ring of $p$-adic integers, is finite dimensional if and only if the $p$-Sylow subgroups of $G$ are cyclic of order $p^r$, where $r ≤ 2$. This is a classical result of S.D. Berman, P.M. Gudivok, A. Heller and I. Reiner (see Theorem 33.6, [2], p.690).

Although the number of irreducible $R$-representations of the group $G$ is finite, the algebra $B(RG)$ may be infinite dimensional (see [13],[4]).

In the present paper we obtain a description of the irreducible $R$-representation algebra of the alternating group $A_4$ of degree four over the ring $R$ of 2-adic integers. Note that earlier it was known that this algebra is infinite dimensional (see Proposition 13.1, [4], p.108).

Our main result is the following.

Theorem. Let $R$ be the ring of 2-adic integers. The irreducible $R$-representation algebra $B(RG)$ of the alternating group $G$ of degree 4 isomorphic to

$\mathbb{Q}[x, x^{-1}] ⊕ \mathbb{Q}[x, x^{-1}] ⊕ \mathbb{Q}[x, x^{-1}] ⊕ \mathbb{Q}[x, x^{-1}]$.

Notation. Let $G$ be the alternating group of degree four. Put $a_1 = (1, 2)(3, 4)$, $a_2 = (1, 4)(2, 3)$, $b = (1, 2, 3) ∈ S_4$. Clearly

$G ≅ \langle a_1, b \rangle = H \rtimes \langle b \rangle ≅ (C_2 × C_2) \rtimes C_3$,

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Lemma 1. (see irreducible \( R \))

Let \( L \) be a free \( RG \)-module over the ring \( R \) with \( R \)-basis \( \{e_1, e_2, e_3, e_4\} \) in which \( G \) acts as a permutation group. Let \( d \) be a divisor of 4 and put

\[
L_d = \left\{ \sum_{j=1}^{4} \alpha_j e_j \mid \alpha_j \in R, \quad \sum_{j=1}^{4} \alpha_j \equiv 0 \pmod{d} \right\},
\]

\[
L_0 = R(e_1 + \cdots + e_4), \quad M_d = L_d/L_0.
\]

Obviously \( L_d, L_0 \) and \( M_d \) are \( RG \)-modules and the elements

\[
u_1 = de_1 + L_0, \quad \nu_2 = e_2 - e_1 + L_0, \quad \nu_3 = e_3 - e_2 + L_0
\]

form an \( R \)-basis of \( M_d \). Since \( e_4 \equiv -e_1 - e_2 - e_3 \pmod{L_0} \), it is not difficult to show that the following \( R \)-representation of \( G \)

\[
\Gamma_d : a_1 \mapsto \begin{pmatrix} 1 & 0 & -4d^{-1} \\ 0 & 0 & -1 \end{pmatrix}, \quad a_2 \mapsto \begin{pmatrix} -3 & 4d-1 & 0 \\ -2d & 3 & 0 \\ -d & 2 & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

is afforded by module the \( M_d \).

Note that \( \Gamma_2 \) is equivalent to the following monomial representation

\[
a_1 \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

We shall need the following result.

**Lemma 1.** (see [3]) The representations \( \Gamma_d \), where \( d = 1, 2, 4 \), are irreducible and nonequivalent \( R \)-representations of the group \( G \). Moreover, except these representations and the trivial one \( \tau_0 : g \mapsto 1 \ (g \in G) \), the group \( G \) has only one more irreducible \( R \)-representation:

\[
\tau : a_1 \mapsto E, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

**Lemma 2.** Let \( \Gamma = P_0 \oplus P_1 \), where \( P_0 = RGw_0 \), \( P_1 = RGw_1 \) are projective \( RG \)-modules and \( w_0 = \frac{1}{4}(1 + b + b^2) \), \( w_1 = 1 - w_0 \) are orthogonal idempotents. Then the following equations hold

\[
[P_0]^2 = 2[P_0] + [P_1], \quad [P_0][P_1] = 2[P_0] + 3[P_1], \quad [P_1]^2 = 6[P_0] + 5[P_1].
\]

**Proof.** It is easy to see that \( \chi_{P_0}(b) = 1 \) and \( \chi_{P_1}(b) = -1 \), where \( \chi_{P_1} \) is the character of \( P_1 \). Remember that \( \chi_{P_0 \oplus P_1} = \chi_{P_0} + \chi_{P_1} \), \( \chi_{P_0 \oplus P_1} = \chi_{P_0} \chi_{P_1} \).

If \( [P_0]^2 = s[P_0] + t[P_1] \), then \( s - t = 1 \) and \( 4s + 8t = 16 \). It follows that \( s = 2 \) and \( t = 1 \), so the first equation is true. The proof of second one is analogues. \( \square \)

**Corollary 1.** The elements

\[
f_1 = \frac{1}{12}[\Gamma], \quad f_2 = [P_0](1 - f_1), \quad f_3 = (1 - [P_0])(1 - f_1)
\]

are pairwise orthogonal idempotents of the algebra \( \mathcal{A}(RG) \) such that \( f_1 + f_2 + f_3 = 1 \).

Moreover,

\[
\mathcal{A}(KG) f_1 \cong \mathcal{A}(KG) f_2 \cong \mathbb{Q}.
\]

Let \( W \) be a finite group. Assume that for the \( K \)-representations of \( W \) the Krull-Schmidt-Azumaya theorem holds (see Theorem 36.0, [2], p.768). We assume that there exists at least one nonzero indecomposable, non-projective \( KW \)-module of the \( K \)-representation of the group \( W \). For each module \( B \) of the \( K \)-representation
of the group $W$ we consider the $K$-module $B^* = \text{Hom}_K(B, K)$. Clearly $B^*$ is a $KW$-module of the $K$-representation of the group $W$ such that
\[(g \cdot \varphi) \cdot b = \varphi(g^{-1}b), \quad (g \in W, \varphi \in B^*, b \in B).\]

The module $B^*$ is called the contragredient module of $B$.

Note that each module of the $K$-representation of $W$ is always a homomorphic image of a free module, and it can be considered as a submodule of a free module, too.

We shall use the following well known result.

**Lemma 3.** Let $0 \to B \to P \to A \to 0$ be an exact sequence of modules of $K$-representations of the group $W$. If $P$ is a projective $KW$-module and $A$ is an indecomposable non-projective $KW$-module, then the module $B$ contains only one unique indecomposable nonprojective direct summand $B_0$. Moreover the sequence $0 \to B_0 \to P_0 \to A \to 0$ is exact for some projective $KW$-module $P_0$. There exists a duality corresponding to the exchange of $A$ and $B$.

**Proof.** All modules in the lemma are free of finite rank over the ring $K$. There exists a projective $KW$-module $P$ and a sequence of indecomposable $KW$-modules $B_1, \ldots, B_s$, such that the sequence $0 \to B_1 \oplus \cdots \oplus B_s \to P \to A \to 0$ is exact.

Assume that $A_1, \ldots, A_s$ are $KW$-modules and $P_1, \ldots, P_s$ are projective $KW$-modules, such that the sequences $0 \to B_j \to P_j \to A_j \to 0$ are exact for each $1 \leq j \leq s$. Sum them over $j$ and apply Schanuel’s Lemma (Lemma 2.24, [2], p.30), then we obtain that
\[P \oplus A_1 \oplus \cdots \oplus A_s \cong A \oplus P_1 \oplus \cdots \oplus P_s.\]

This yields that $A$ is a direct summand only one of the $A_i$, says of $A_1$, so $A_2, \ldots, A_s$ are projective. Then the modules $B_2, \ldots, B_s$ are projective, too. Therefore we have the following exact sequence $0 \to B_0 \oplus P' \to P \to A \to 0$, where $P'$ is projective and $B_0$ is an indecomposable non-projective $KW$-module. The following isomorphism of $KW$-modules hold
\[A \cong P/(B_0 \oplus P') \cong (P/P')/B_0,\]
where $P/P'$ is a projective $KW$-module.

By the use of the contragredient module we obtain the dual statement. \qed

Obviously, each projective $RH$-module is free, where $H = Syl_2(G)$. Let $M$ be a module of $R$-representation of the group $H$ and assume that $M$ does not contain projective summands. By $\Theta(M)$ we denote the kernel (the Green operator, after J.A. Green) of the projective cover of $RH$-module $M$ (see Lemma 3). The following sequence of $RH$-modules
\[0 \to \Theta(M) \to F \to M \to 0\]
is exact, where $F$ is a free $RH$-module of smallest rank. If $M$ is an indecomposable $RH$-module, then $\Theta(M)$ is indecomposable, too.

Let $\Delta_0$ be the module of the trivial representation $h \mapsto 1$ $(h \in H)$ of $H$. Put
\[\Delta_n = \Theta(\Delta_{n-1}), \quad \Delta_{-n} = \Delta_n^*, \quad (n = 1, 2, \ldots)\]
where $\Delta_n^*$ is the contragredient module to $\Delta_n$. 
Lemma 4. The sequence \( 0 \to \Delta_n \to (RH)^n \to \Delta_{n-1} \to 0 \) is exact and \( \text{rank}_R(\Delta_n) = 2n + 1 \).

Proof. Let \( \Gamma^n \) be an \( n \)-dimensional module over the ring \( RH \), where \( H = \langle a_1, a_2 \rangle \). In the follows we will treat \( \Gamma^n \) considering as component columns. Denote by \( F_n \) the matrix over the ring \( RH \) with \( n \) rows and \( n + 1 \) columns, in which only the three upper diagonals (if we begin calculation from the main diagonal) are non-zero:

- the main diagonal of \( F_n \) is the first \( n \) element from the following sequence \( F_1 \cup F_1 \cup F_1, \ldots, \) where
  
  \[
  F_1 = \begin{cases} 
  \{a_1 - 1, a_2 - 1, -(a_1 + 1), -(a_2 + 1)\} & \text{if } n \text{ is odd}, \\
  \{a_1 + 1, a_2 + 1, -(a_1 - 1), -(a_2 - 1)\} & \text{if } n \text{ is even}.
  \end{cases}
  \]

- the second diagonal consist of zero elements except the last one which is equal to one of the following elements
  
  \( a_2 - 1, -(a_1 - 1), a_2 + 1, a_1 + 1 \)

according to the cases \( n \equiv \{1, 2, 3, 0\} \pmod{4} \), respectively.

- the third diagonal consist of \( n - 1 \) elements of the sequence \( F_2 \cup F_2 \cup F_2, \ldots, \) where \( F_2 = \{a_2 - 1, a_1 - 1, a_2 + 1, a_1 + 1\} \).

Example. \( F_4 = \begin{pmatrix} 
  a_1 + 1 & 0 & 0 & 0 \\
  0 & a_2 + 1 & a_1 - 1 & 0 \\
  0 & 0 & -(a_1 - 1) & 0 \\
  0 & 0 & 0 & -(a_2 - 1) a_1 + 1
  \end{pmatrix} \)

\( F_5 = \begin{pmatrix} 
  a_1 - 1 & 0 & a_2 - 1 & 0 & 0 \\
  0 & a_2 - 1 & 0 & a_1 - 1 & 0 \\
  0 & 0 & -(a_1 + 1) & 0 & a_2 + 1 \\
  0 & 0 & 0 & 0 & a_1 - 1 a_2 - 1
  \end{pmatrix} \).

Let \( V_n \) be a submodule in \( \Gamma^n \) generated by the columns of the matrix \( F_n \). Let \( \varepsilon : \Gamma^n \to V_{n-1} \) be the following epimorphism of modules:

\( \Gamma^n \ni x = (x_1, \ldots, x_n) \mapsto x_1 f_1 + \cdots + x_n f_n \in V_n \),

where \( f_j \) are columns of the matrix \( F_{n-1} \).

It is not difficult to check that \( F_{n-1} F_n = 0 \).

Moreover, any solution of \( \varepsilon(x) = 0 \) belongs to the linear combination of the columns of \( F_n \), so \( x \in V_n \). Therefore, we proved the exactness of the sequence

\( 0 \to V_n \to \Gamma^n \to V_{n-1} \to 0 \).

We construct a basis of the \( R \)-module \( V \) in the following way: the first \( n+1 \) elements of the basis are the columns of \( F_n \). Then we take the first \( n \) columns of \( F_n \), which multiply each of them by corresponding element of the sequence \( F_3 \cup F_3 \cup \cdots \cup F_3 \), where \( F_3 = \{a_2 - 1, a_1 - 1, a_2 + 1, a_1 + 1\} \). These columns form the remaining \( n \) elements of the basis.

Consequently we obtain a system of \( 2n + 1 \) basis elements of \( R \)-module \( V_n \). By Lemma 3 beginning with \( V_0 \) all \( RH \)-modules \( V_n \) are indecomposable. \( \square \)

Induce the exact sequence of Lemma 4 from the subgroup \( H \) to the group \( G \). In this exact sequence of \( RG \)-modules the middle modules are free, and the \( 2^{nd} \) and the \( 4^{th} \) terms of the sequence have a decomposition into the direct sums of indecomposable \( RG \)-modules

\( \Delta_n^G = \Delta_{0,n} \oplus \Delta_{1,n} \).
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where \( \text{rank}(\Delta_{0,n}) = \text{rank}(\Delta_n) \) and \( \text{rank}(\Delta_{1,n}) = 2 \cdot \text{rank}(\Delta_n) \). Moreover \( \Delta_{0,0} \) and \( \Delta_{1,0} \) are modules of the \( R \)-representations \( \pi_0, \pi \) (see Lemma 1) of \( G \), respectively. It is easy to check that \( \Delta_{0,1} \) and \( \Delta_{1,0} \) are modules of the \( R \)-representations of \( \Gamma_1, \Gamma_4 \) of \( G \), respectively (see Lemma 1). Since

\[
(\Delta^n_G)^H = (\Delta_n)^{(3)}.
\]

the \( RG \)-module \( \Delta_{0,n} \) is a lifting of the \( RH \)-module \( \Delta_n \).

We will use the notation \( \Delta^n = \Delta_{0,n} \).

Lemma 5. The sequences of \( RG \)-modules

\[
0 \to \Delta_{3k} \to \Gamma^k \to \Delta_{3k-1} \to 0,
0 \to \Delta_{3k+1} \to P_0 \oplus \Gamma^k \to \Delta_{3k} \to 0,
0 \to \Delta_{3k+2} \to P_1 \oplus \Gamma^k \to \Delta_{3k+1} \to 0
\]

are exact, where \( \Gamma = RG = P_0 \oplus P_1 \) (see Lemma 2). Moreover

\[
\Delta_{3k} \otimes \Delta_1 = \Delta_{3k+1} \oplus \Gamma^k,
\Delta_{3k+1} \otimes \Delta_1 = \Delta_{3k+2} \oplus P_0 \oplus \Gamma^k,
\Delta_{3k+2} \otimes \Delta_1 = \Delta_{3k+3} \oplus P_1 \oplus \Gamma^k.
\]

Proof. It is easy to see that

\[
\{(a_1 - 1)(a_2 - 1)w_0, (a_1 - 1)w_0, (a_2 - 1)w_0\}
\]

is an \( R \)-basis of the \( RG \)-submodule \( M \subset P_0 \). Obviously, \( P_0/M \cong \Delta_0 \), so we can assume that \( M = \Delta_1 \) and the following sequence

\[
(2) 0 \to \Delta_1 \to P_0 \to \Delta_0 \to 0
\]

is exact. Comparing the values of the characters at \( b \in G \), we get

\[
P_0 \otimes \Delta_1 \cong \Gamma = P_0 \oplus P_1.
\]

Multiplying (2) tensorially by \( \Delta_1 \), we obtain the exact sequence

\[
0 \to \Delta_1 \otimes \Delta_1 \to P_0 \oplus P_1 \to \Delta_1 \to 0,
\]

which is possible only if \( \Delta_1 \otimes \Delta_1 \cong \Delta_2 \oplus P_0 \), so the sequence

\[
0 \to \Delta_2 \to P_1 \to \Delta_1 \to 0.
\]

is exact, too. Multiplying the last sequence again tensorially by \( \Delta_1 \) and using the value of the characters (for example, \( \chi_{\Delta_2}(b) = -1 \)), we can verify the lemma for \( n = 2,3,\ldots \). \( \square \)

Using the contragredient modules we can obtain an analogue of Lemma 5 for the modules \( \Delta_m \) with negative \( m \). As a consequence we have

Corollary 2. In the algebra \( A(RG) \) the following equations hold

\[
[\Delta_n] \cdot [\Delta_m] = [\Delta_{n+m}] = 0 \quad [\Delta_{1,0}] \cdot [\Delta_{1,0}] = 2[\Delta_0] + [\Delta_{1,0}].
\]

Proof. These equations follows from Lemma 5 where \( [P_j] = 0 \) and \( f_3 \) from (1). \( \square \)
The group \( H = \langle a_1, a_2 \rangle \cong C_2 \times C_2 \) has the following four linear characters:

\[
\begin{align*}
\delta_0 &: \quad a_1 \mapsto 1, \quad a_2 \mapsto 1; \\
\delta_1 &: \quad a_1 \mapsto -1, \quad a_2 \mapsto 1; \\
\delta_2 &: \quad a_1 \mapsto 1, \quad a_2 \mapsto -1; \\
\delta_3 &: \quad a_1 \mapsto -1, \quad a_2 \mapsto -1.
\end{align*}
\]

It is easy to check that the induced representations \( \delta_G^1, \delta_G^2, \delta_G^3 \) of \( G \) are irreducible and equivalent to the representation \( \Gamma_2 \) (see Lemma 1).

**Lemma 6.** The following equations hold

\[
[\Delta_1, 0]^2 = [\Delta_1, 0] + [\Delta_0], \quad [L]^2 = [\Delta_0] + [\Delta_1, 0] + 2[L],
\]

where \( L \) is the module of the representation \( \Gamma_2 \) (see Lemma 1) and \( \Delta_0 \) is the trivial \( RG \)-module.

**Proof.** Using Mackey’s theorem (see Theorem 10.18, [2], p.240) we get

\[
\delta_2^G \otimes \delta_2^G \simeq \sum_{j=0}^{2} (\delta_2 \otimes \delta_2^j)^G,
\]

where \( \delta_2^j(h) = \delta_2(b^{-j}h b^j) \) for \( h \in H \). This yields that \( \delta_2 \otimes \delta_2 = \delta_0, \quad \delta_2 \otimes \delta_1 = \delta_1, \quad \delta_2 \otimes \delta_2^2 = \delta_3, \quad \text{and} \quad \delta_2^G \otimes \delta_2^G = \delta_0^G + \delta_1^G + \delta_3^G. \)

**Proof of the Theorem.** Let \( \mathfrak{B}'(RG) = \mathfrak{B}(RG) f_3 \). Put

\[
x = [\Delta_1], \quad y = [\Delta_1, 0], \quad z = [L].
\]

We identify \( f_3 \) with the unity 1 of the algebra \( \mathfrak{B}'(RG) \), where \( f_3 \) from [1]. Since \( [\Delta_1][\Delta_{-1}]f_3 = f_3 \), we can assume that \( [\Delta_{-1}]f_3 = \frac{1}{z} \). By Lemma 6 it follows that

\[
\mathfrak{B}'(RG) = \langle 1, x, \frac{1}{x}, y, z \mid y^2 = y + 2, \quad z^2 = 2z + y + 1 \rangle.
\]

Since \( y^2 - y - 2 = (y - 2)(y + 1) \), we have

\[
\langle 1, x, \frac{1}{x}, y \rangle \cong \mathbb{Q}[x, \frac{1}{x}]/(y^2 - y - 2) \cong \mathbb{Q}[x, \frac{1}{x}] \oplus \mathbb{Q}[x, \frac{1}{x}].
\]

Now the algebra \( \mathfrak{B}'(RG) \cong \mathbb{Q}[x, \frac{1}{x}] \oplus \mathbb{Q}[x, \frac{1}{x}] \oplus \mathbb{Q}[x, \frac{1}{x}] \oplus \mathbb{Q}[x, \frac{1}{x}], \) because

\[
z^2 - 2z - y - 1 = \begin{cases} (z + 1)(z - 3) & \text{for } y = 2; \\ (z - 2)z & \text{for } y = -1. \end{cases}
\]

Finally, since the algebra \( \mathfrak{B}(RG) \) has no projective summands, the map \( u \mapsto uf_3 \) gives the isomorphism \( \mathfrak{B}'(RG) \cong \mathfrak{B}(RG) \). \( \square \)
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