What do homotopy algebras form?

Vasily A. Dolgushev, Alexander E. Hoffnung, and Christopher L. Rogers

Abstract

In paper [4], we constructed a symmetric monoidal category \( \mathcal{S}\text{Lie}_{\infty}^{MC} \) whose objects are shifted (and filtered) \( L_{\infty} \)-algebras. Here, we prove that the category of homotopy algebras of a fixed type is naturally enriched over \( \mathcal{S}\text{Lie}_{\infty}^{MC} \). We “integrate” this \( \mathcal{S}\text{Lie}_{\infty}^{MC} \)-enriched category to a simplicial category \( \text{HoAlg}_C^\Delta \) whose mapping spaces are Kan complexes. The simplicial category \( \text{HoAlg}_C^\Delta \) gives us a particularly nice model of an \((\infty,1)\)-category of homotopy algebras. We show that the homotopy category \( \pi_0(\text{HoAlg}_C^\Delta) \) of \( \text{HoAlg}_C^\Delta \) is a correct homotopy category of homotopy algebras of a fixed type. Finally, we show that the Homotopy Transfer Theorem is a simple consequence of the Goldman-Millson theorem.

1 Introduction

This work is motivated by Dmitry Tamarkin’s answer [23] to Vladimir Drinfeld’s question “What do dg categories form?” [10], and by papers [1], [8], [12], [14], and [17]. Here, we give an answer to the question “What do homotopy algebras form?”

Homotopy algebras and their generalizations appear in constructions of string topology, in rational homotopy theory, symplectic topology, deformation quantization, and quantum field theory. For a gentle introduction to this topic, we refer the reader to paper [24]. For a more detailed exposition, a standard reference is book [18] by Jean-Louis Loday and Bruno Vallette.

Despite an important role of homotopy algebras, it is not clear what higher categorical structure stands behind the homotopy theory of homotopy algebras. A possible way to answer to this question is to use closed model categories and this approach is undertaken in paper [25] by Bruno Vallette. Here, we suggest a different approach which is based on the use of the convolution \( L_{\infty} \)-algebra and the Getzler-Hinich construction [12], [14].

By a homotopy algebra structure on a cochain complex of \( k \)-vector spaces \( A \) we mean a \( \text{Cobar}(\mathcal{C}) \)-algebra structure on \( A \), where \( \mathcal{C} \) is a differential graded (dg) cooperad satisfying some technical conditions. In this paper, we fix such a cooperad \( \mathcal{C} \) and show that \( \text{Cobar}(\mathcal{C}) \)-algebras form a \( \mathcal{S}\text{Lie}_{\infty}^{MC} \)-enriched category \( \text{HoAlg}_{\mathcal{C}}^\Delta \), where \( \mathcal{S}\text{Lie}_{\infty}^{MC} \) is the enhancement of the symmetric monoidal category of shifted \( L_{\infty} \)-algebras introduced in [4]. Then, using a generalization of the Getzler-Hinich construction [12], [14], we show that the \( \mathcal{S}\text{Lie}_{\infty}^{MC} \)-category of \( \text{Cobar}(\mathcal{C}) \)-algebras can be “integrated” to the category \( \text{HoAlg}_{\mathcal{C}}^\Delta \) enriched over \( \infty \)-groupoids (a.k.a. Kan complexes).

\footnote{In this paper, we assume that \text{char}(k) = 0.}
Our approach is conceptually similar to a standard construction of the simplicial category of chain complexes i.e., “trivial” homotopy algebras. There, instead of constructing the simplicial mapping spaces directly, one exploits the simple fact that chain complexes are naturally enriched over chain complexes i.e., abelian $L_\infty$-algebras. The simplicial category is then constructed by first truncating the mapping complexes and applying the Dold-Kan functor. It has been noted, for example by E. Getzler in [12], that the integration of a $L_\infty$-algebra is, up to homotopy, a non-abelian analogue of the Dold-Kan functor. Hence, to construct the simplicial category $\text{HoAlg}_C^{\Delta}$ of non-trivial homotopy algebras, we proceed via analogy by first considering a category of homotopy algebras enriched over $L_\infty$-algebras i.e., “non-abelian complexes”.

Furthermore, we prove that the homotopy category of $\text{HoAlg}_C^{\Delta}$ is the localization of the category of Cobar($C$)-algebras with respect to $\infty$ quasi-isomorphisms, and this is indeed the correct homotopy category of homotopy algebras (see for example [18, Thm. 11.4.12]). Thus the simplicial category $\text{HoAlg}_C^{\Delta}$ is the sought higher categorical structure which stands behind the homotopy category of homotopy algebras.

For simplicity of the exposition, we present all constructions in the “1-colored” setting. However, it is not hard to see the all the statements can be easily extended to the multi-colored setting.

Our paper is organized as follows. Section 2 is devoted to prerequisites on homotopy algebras. In this section, we describe the convolution $L_\infty$-algebra $\text{Conv}(V,A)$ associated to a $C$-coalgebra $V$ and a Cobar($C$)-algebra $A$, and recall [8] that, for every pair of Cobar($C$)-algebras $A, B$, Maurer-Cartan (MC) elements of $\text{Conv}(C(A),B)$ are in bijection with $\infty$-morphisms from $A$ to $B$. In Section 3 we construct the $\mathcal{G}_{\text{Lie}}^{\infty}$-enriched category $\text{HoAlg}_C^{\Delta}$ and the corresponding simplicial category $\text{HoAlg}_C^{\Delta}$. In Section 4 we show that $\pi_0(\text{HoAlg}_C^{\Delta})$ is the homotopy category of Cobar($C$)-algebras, i.e. every $\infty$ quasi-isomorphism of Cobar($C$)-algebras gives an invertible morphism in $\pi_0(\text{HoAlg}_C^{\Delta})$ and the category $\pi_0(\text{HoAlg}_C^{\Delta})$ has the desired universal property. Also in Section 4 we characterize $\infty$ quasi-isomorphisms as precisely those morphisms which induce homotopy equivalences between mapping spaces via (pre)composition. Finally, in Section 5 we give a very concise proof of the Homotopy Transfer Theorem for homotopy algebras. This proof is based on a construction from [7] and a version of the Goldman-Millson theorem from [2].

On a more general definition of a homotopy algebra. One could also define a homotopy algebra as an algebra over an operad $O$ which is freely generated by a collection $C$ when viewed as an operad in the category of graded vector spaces, where the collection $C$ and the differential on $O$ are subject to some technical conditions. It is not hard to generalize all the constructions of our paper to this more general setting. However, for simplicity of the exposition, we decided to present the whole story for the case when $O$ is the cobar construction applied to a fixed cooperad.

Is the $\mathcal{G}_{\text{Lie}}^{\infty}$-enriched category of $L_\infty$-algebras related to the rational homotopy theory? It is not surprising that the answer to this question is “yes”. However, a precise formulation of the answer requires some technical conditions on $L_\infty$-algebras and some amendments to the definition of the mapping space. So separate note [5] is devoted to this
question.

Related work on this subject

While this preprint was in preparation, paper [15] appeared on arXiv.org. In this paper, the authors constructed a quasi-category of Leibniz $\infty$-algebras also using the Getzler-Hinich construction [12], [14].

Our paper agrees in spirit with treatise [19] by Jacob Lurie. In this treatise, J. Lurie develops a systematic approach to algebraic structures on objects in a fixed symmetric monoidal $\infty$-category. This approach allows him to define $E_\infty$-ring as a commutative algebra object in the $\infty$-category of spectra. Even though, the framework of the presentation of [19] is very general, we could not find a particular statement in the current version of draft [19] from which all statements proved in our paper will follow. One of the reasons for this is that, in [19], J. Lurie usually assumes the all the (co)chain complexes under consideration satisfy some kind of boundedness condition whereas in our paper we do not impose this assumption.

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Notation and conventions

A big part of our conventions is borrowed from [4]. For example, the ground field $k$ has characteristic zero and $\text{Ch}_k$ denotes the category of unbounded cochain complexes of $k$-vector spaces. Any $\mathbb{Z}$-graded vector space $V$ is tacitly considered as the cochain complex with the zero differential. Following [4] we frequently use the ubiquitous abbreviation “dg” (differential graded) to refer to algebraic objects in $\text{Ch}_k$. The notation $sV$ (resp. by $s^{-1}V$) is reserved for the suspension (resp. the desuspension) of a cochain complex $V$, i.e. $(sV)^* = V^{*-1}$ and $(s^{-1}V)^* = V^{*-1}$. Finally, for a pair $V, W$ of $\mathbb{Z}$-graded vector spaces we denote by

$$\text{Hom}(V, W)$$

the corresponding inner-hom object in the category of $\mathbb{Z}$-graded vector spaces.

Following [4] we tacitly assume the Koszul sign rule. In particular,

$$(-1)^{\varepsilon(\sigma; v_1, \ldots, v_m)}$$

will always denote the sign factor corresponding to the permutation $\sigma \in S_m$ of homogeneous vectors $v_1, v_2, \ldots, v_m$. Namely,

$$(-1)^{\varepsilon(\sigma; v_1, \ldots, v_m)} := \prod_{(i < j)} (-1)^{|v_i||v_j|}, \quad (1.1)$$

where the product is taken over all inversions $(i < j)$ of $\sigma \in S_m$.  

3
For a finite group $G$ acting on a cochain complex (or a graded vector space) $V$, we denote by

$$V^G, \quad \text{and} \quad V_G,$$

respectively, the subcomplex of $G$-invariants in $V$ and the quotient complex of $G$-coinvariants. Using the advantage of the zero characteristic, we often identify $V_G$ with $V^G$ via this isomorphism

$$v \mapsto \sum_{g \in G} g(v) : V_G \to V^G. \quad (1.2)$$

For a graded vector space (or a cochain complex) $V$ the notation $S(V)$ (resp. $\underline{S}(V)$) is reserved for the underlying vector space of the symmetric algebra (resp. the truncated symmetric algebra) of $V$:

$$S(V) = \mathbb{k} \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \ldots,$$

$$\underline{S}(V) = V \oplus S^2(V) \oplus S^3(V) \oplus \ldots,$$

where

$$S^n(V) = (V^\otimes_n)_S.$$

We denote by $\text{As}$, $\text{Com}$, $\text{Lie}$ the operads governing associative, commutative (and associative), and Lie algebras, respectively. We set $\text{As}(0) = \text{Com}(0) = 0$. In other words, algebras over $\text{As}$ and $\text{Com}$ are non-unital. We denote by $\text{As}_\infty$, $\text{Com}_\infty$, and $\text{Lie}_\infty$ the dg operads which govern homotopy versions of the corresponding algebras. Furthermore, we denote by $\text{coAs}$, $\text{coCom}$, and $\text{coLie}$, the cooperads which are obtained from $\text{As}$, $\text{Com}$, and $\text{Lie}$ respectively, by taking the linear dual.

For an augmented operad $\mathcal{O}$, we denote by $\mathcal{O}_\circ$ the kernel of the augmentation. Dually, for a coaugmented cooperad $\mathcal{C}$ we denote by $\mathcal{C}_\circ$ the cokernel of the coaugmentation. Recall that for every augmented operad $\mathcal{O}$ (resp. coaugmented cooperad $\mathcal{C}$) the collection $\mathcal{O}_\circ$ (resp. $\mathcal{C}_\circ$) is naturally a pseudo-operad (resp. pseudo-cooperad) in the sense of [3, Section 3.2] (resp. [3, Section 3.4]).

For an operad (resp. a cooperad) $P$ and a cochain complex $V$ we denote by $P(V)$ the free $P$-algebra (resp. the cofree $P$-coalgebra) generated by $V$:

$$P(V) := \bigoplus_{n \geq 0} \left( P(n) \otimes V^\otimes_n \right)_{S_n}. \quad (1.3)$$

For example,

$$\text{Com}(V) = \underline{S}(V) \quad \text{and} \quad \text{coCom}(V) = \underline{S}(V).$$

For a cooperad $\mathcal{C}$, it is sometimes more convenient to work with the cofree $\mathcal{C}$-coalgebra defined as the direct sum of the space of invariants (instead of coinvariants):

$$\bigoplus_{n \geq 0} \left( \mathcal{C}(n) \otimes V^\otimes_n \right)^{S_n}. \quad (1.4)$$

\[\text{In this paper we only consider nilpotent coalgebras.}\]
For example, it is more natural to define a $C$-coalgebra structure on a graded vector space (or a cochain complex) $V$ as a collection of comultiplication maps

$$\Delta_n : V \to \left( C(n) \otimes V^\otimes n \right)^{S_n}$$

satisfying some natural coassociativity axioms which are obtained by dualizing the corresponding axioms for algebras over an operad.

We denote this direct sum by

$$C(V)^{inv} := \bigoplus_{n \geq 0} \left( C(n) \otimes V^\otimes n \right)^{S_n}$$

and keep in mind that $C(V)^{inv}$ is isomorphic to $C(V)$ via map (1.2).

Following [4], $\mathcal{S}$ and $\mathcal{S}^{-1}$ denote the underlying collections of the operads $\text{End}_{s^{-1}k}$, and $\text{End}_{sk}$, respectively. Furthermore, for a dg (co)operad $P$, we denote by $\mathcal{S}P$ (resp. $\mathcal{S}^{-1}P$) the dg (co)operad which is obtained from $P$ by tensoring with $\mathcal{S}$ (resp. $\mathcal{S}^{-1}$):

$$\mathcal{S}P := \mathcal{S} \otimes P, \quad \mathcal{S}^{-1}P := \mathcal{S}^{-1} \otimes P.$$ 

For example, $\mathcal{S}\text{Lie}_\infty$-algebras are algebra over the dg operad

$$\mathcal{S}\text{Lie}_\infty := \text{Cobar}(\text{coCom})$$

and $L_\infty$-algebras are algebras over the dg operad

$$L_\infty := \text{Cobar}(\mathcal{S}^{-1}\text{coCom})$$

Just as in [4], we often call $\mathcal{S}\text{Lie}_\infty$-algebras shifted $L_\infty$-algebras. Although a $\mathcal{S}\text{Lie}_\infty$-algebra structure on a cochain complex $V$ is the same thing as an $L_\infty$ structure on $sV$, working with $\mathcal{S}\text{Lie}_\infty$-algebras has important technical advantages. This is why we prefer to deal with shifted $L_\infty$ structures on $V$ versus original $L_\infty$ structures on $sV$.

Following [4], we denote the tensor product of $\infty$-morphisms of $\mathcal{S}\text{Lie}_\infty$-algebras by $\otimes$ even though the tensor product of the $\mathcal{S}\text{Lie}_\infty$-algebras is denoted by $\oplus$.

We often use the plain arrow $\rightarrow$ for $\infty$-morphisms of homotopy algebras. Of course, it should be kept in mind that in general such morphisms are maps of the corresponding coalgebras but not the underlying cochain complexes.

The abbreviation “MC” is reserved for the term “Maurer-Cartan”.

**Conventions about trees.** By a tree we mean a connected graph without cycles with a marked vertex called the root. In this paper, we assume that the root of every tree has valency 1 (such trees are sometimes called planted). The edge adjacent to the root is called the root edge. Non-root vertices of valency 1 are called leaves. A vertex is called internal if it is neither a root nor a leaf. We always orient trees in the direction towards the root. Thus every internal vertex has at least 1 incoming edge and exactly 1 outgoing edge. An edge
adjacent to a leaf is called external. A tree $t$ is called planar if, for every internal vertex $v$ of $t$, the set of edges terminating at $v$ carries a total order.

Let us recall [3, Section 2] that for every planar tree $t$ the set $V(t)$ of all its vertices is equipped with a natural total order such that the root is the smallest vertex of the tree.

For a non-negative integer $n$, an $n$-labeled planar tree $t$ is a planar tree equipped with an injective map

$$I : \{1, 2, \ldots, n\} \to L(t)$$

from the set $\{1, 2, \ldots, n\}$ to the set $L(t)$ of leaves of $t$. Although the set $L(t)$ has a natural total order we do not require that map (1.9) is monotonous.

The set $L(t)$ of leaves of an $n$-labeled planar tree $t$ splits into the disjoint union of the image $I(\{1, 2, \ldots, n\})$ and its complement. We call leaves in the image of $I$ labeled.

A vertex $x$ of an $n$-labeled planar tree $t$ is called nodal if it is neither the root, nor a labeled leaf. We denote by $V_{nod}(t)$ the set of all nodal vertices of $t$. Keeping in mind the canonical total order on the set of all vertices of $t$ we can say things like “the first nodal vertex”, “the second nodal vertex”, and “the $i$-th nodal vertex”.

It is convenient to talk about (co)operads and pseudo-(co)operads using the groupoid $\text{Tree}(n)$ of $n$-labeled planar trees. Objects of $\text{Tree}(n)$ are $n$-labeled planar trees and morphisms are non-planar isomorphisms of the corresponding (non-planar) trees compatible with labeling.

Following [3, Section 3.2, 3.4], for an $n$-labelled planar tree $t$ and pseudo-operad $P$ (resp. pseudo-cooperad $Q$) the notation $\mu_t$ (resp. the notation $\Delta_t$) is reserved for the multiplication map

$$\mu_t : P(r_1) \otimes P(r_2) \otimes \cdots \otimes P(r_k) \to P(n)$$

and the comultiplication map

$$\Delta_t : Q(n) \to Q(r_1) \otimes Q(r_2) \otimes \cdots \otimes Q(r_k).$$

respectively. Here, $k$ is the number of nodal vertices of the planar tree $t$ and $r_i$ is the number of edges (of $t$) which terminate at the $i$-th nodal vertex of $t$.

For example, if $t_{n,k,i}$ is the labeled planar tree shown on figure then the map

$$\mu_{t_{n,k,i}} : P(n) \otimes P(k) \to P(n + k - 1)$$

is precisely the $i$-th elementary insertion

$$\circ_i : P(n) \otimes P(k) \to P(n + k - 1).$$

Recall that a (co)operad $P$ is reduced if it satisfies this technical condition

$$P(0) = 0.$$  (1.14)

When we deal with reduced (co)operads, we may discard all labeled trees which have at least one nodal vertex with no incoming edges. In other words, one may safely assume that nodal vertices of a labeled tree are precisely its internal vertices, i.e. map (1.9) is a bijection.

On figures, small white circles are used for nodal vertices and small black circles are used for all the remaining vertices.
2 Prerequisites on homotopy algebras

Let $\mathcal{C}$ be a dg coaugmented cooperad satisfying the following technical condition:

**Condition 2.1** The pseudo-cooperad $\mathcal{C}_o$ carries an ascending filtration

$$0 = \mathcal{F}^0 \mathcal{C}_o \subset \mathcal{F}^1 \mathcal{C}_o \subset \mathcal{F}^2 \mathcal{C}_o \subset \mathcal{F}^3 \mathcal{C}_o \subset \ldots$$

which is compatible with the pseudo-cooperad structure in the following sense:

$$\Delta_t(\mathcal{F}^m \mathcal{C}_o(n)) \subset \bigoplus_{m_1 + \cdots + m_k = m} \mathcal{F}^{m_1} \mathcal{C}_o(r_1) \otimes \mathcal{F}^{m_2} \mathcal{C}_o(r_2) \otimes \cdots \otimes \mathcal{F}^{m_k} \mathcal{C}_o(r_k),$$

where $k$ is the number of nodal vertices of the planar tree $t$ and $r_i$ is the number of edges (of $t$) which terminate at the $i$-th nodal vertex of $t$. We also assume that $\mathcal{C}_o$ is cocomplete with respect to filtration (2.1), i.e.

$$\mathcal{C}_o = \bigcup_m \mathcal{F}^m \mathcal{C}_o.$$

We will use the following pedestrian definition of homotopy algebras of a given type:

**Definition 2.2** Homotopy algebras of type $\mathcal{C}$ are algebras in $\text{Ch}_k$ over the operad $\text{Cobar} (\mathcal{C})$.

Thus, $A_\infty$, $L_\infty$, and $\text{Com}_\infty$-algebras are examples of homotopy algebras. Indeed, $A_\infty$-algebras are algebras over the operad $\text{Cobar} (\mathcal{S}^{-1} \text{coAs})$, $L_\infty$-algebras are algebras over $\text{Cobar} (\mathcal{S}^{-1} \text{coCom})$, and $\text{Com}_\infty$-algebras are algebras over $\text{Cobar} (\mathcal{S}^{-1} \text{coLie})$.

Let $A$ be a cochain complex and let

$$\text{coDer}(\mathcal{C}(A))$$

be the dg Lie algebra of coderivations of the $\mathcal{C}$-coalgebra $\mathcal{C}(A)$.

It is known [3, Corollary 5.3], [13, Proposition 2.15], that $\text{Cobar} (\mathcal{C})$-algebra structures on $A$ are in bijection with MC elements of the dg Lie subalgebra

$$\text{coDer}^\prime(\mathcal{C}(A)) \subset \text{coDer}(\mathcal{C}(A))$$

of coderivations $Q$ satisfying the condition

$$Q\big|_A = 0.$$

Figure 1: The $(n+k-1)$-labeled planar tree $t_{n,k,i}$
Thus every homotopy algebra $A$ of type $\mathcal{C}$ gives us a (dg) $\mathcal{C}$-coalgebra

$$\left(\mathcal{C}(A), \partial + Q\right)$$

where $\partial$ is the differential on $\mathcal{C}(A)$ induced by the ones on $A$ and $\mathcal{C}$.

This observation is used to define a notion of $\infty$-morphism of homotopy algebras of type $\mathcal{C}$. In other words,

**Definition 2.3** Let $A$ and $B$ be homotopy algebras of type $\mathcal{C}$ and let $Q_A$ (resp. $Q_B$) be the MC element of $\text{coDer}(\mathcal{C}(A))$ (resp. $\text{coDer}(\mathcal{C}(B))$) corresponding to the homotopy algebra structure on $A$ (resp. on $B$). Then, an $\infty$-morphism from $A$ to $B$ is a homomorphism of dg $\mathcal{C}$-coalgebras

$$U : \left(\mathcal{C}(A), \partial + Q_A\right) \to \left(\mathcal{C}(B), \partial + Q_B\right).$$

(2.8)

Recall that any such homomorphism $U$ is uniquely determined by its composition

$$U' := p_B \circ U$$

(2.9)

with the canonical projection $p_B : \mathcal{C}(B) \to B$. In this paper, we often use this convention: $U'$ denotes composition (2.9) corresponding to a homomorphism of $\mathcal{C}$-coalgebras $U$ (2.8).

Given an $\infty$-morphism $U$ from $A_1$ to $A_2$ and an $\infty$-morphism $\tilde{U}$ from $A_2$ to $A_3$, their composition is defined, in the obvious way, as the composition of the corresponding homomorphisms of dg $\mathcal{C}$-coalgebras. We denote by

$$\text{Cat}_\mathcal{C}$$

the category whose objects are homotopy algebras of type $\mathcal{C}$ (a.k.a. Cobar($\mathcal{C}$)-algebras) and whose morphisms are $\infty$-morphisms with the above obvious composition.

**Remark 2.4** Due to [22, Proposition 38], Condition 2.1 implies that Cobar($\mathcal{C}$) is a cofibrant object in the closed model category of dg operads. The same condition also guarantees that homotopy algebras of type $\mathcal{C}$ enjoy the obvious version of the Homotopy Transfer Theorem (see [18, Theorem 10.3.2]). A very concise proof of this important theorem is given in Section 5 of this paper.

**Remark 2.5** Another way to define the notion of $\infty$-morphism of homotopy algebras is to resolve a 2-colored operad which governs pairs of algebras with a morphism between them. This different approach to the “higher category” of homotopy algebras is initiated in works [9], [20] of M. Doubek and M. Markl.

### 2.1 The convolution $\mathcal{S}\text{Lie}_\infty$-algebra

Let $V$ be a $\mathcal{C}$-coalgebra and $A$ be a homotopy algebra of type $\mathcal{C}$ (i.e. an algebra over Cobar($\mathcal{C}$)).

On the graded vector space

$$\text{Hom}(V, A)$$

(2.10)
we define the following multi-brackets:

\[
\{ f \} (v) = \partial_A f(v) - (-1)^{|f|} f(\partial_V v) + p_A \circ Q_A (1 \otimes f(\Delta_1(v))) \tag{2.11}
\]

\[
\{ f_1, \ldots, f_m \} (v) = p_A \circ Q_A (1 \otimes f_1 \otimes \cdots \otimes f_m(\Delta_m(v))), \quad m \geq 2, \tag{2.12}
\]

where \( \Delta_m \) is the \( m \)-th component of the comultiplication

\[
\Delta_m : V \to \left( \mathcal{C}(m) \otimes V^\otimes m \right)^{S_m}
\]

and \( p_A \) is the canonical projection

\[
p_A : \mathcal{C}(A) \to A.
\]

Note that, since \( Q_A \) has degree 1, each multi-bracket in (2.12) also carries degree 1.

We claim that

**Proposition 2.6** For every \( \mathcal{C} \)-coalgebra \( V \) and a Cobar(\( \mathcal{C} \))-algebra \( A \), multi-brackets (2.11), (2.12) equip the graded vector space \( \text{Hom}(V, A) \) with a structure of a \( \mathcal{S}\text{Lie}_\infty \)-algebra.

The proof of this proposition is given in Appendix A.

**Definition 2.7** Let \( V \) be a \( \mathcal{C} \)-coalgebra and \( A \) be a homotopy algebra of type \( \mathcal{C} \). Then \( \mathcal{S}\text{Lie}_\infty \)-algebra (2.10) is called the convolution algebra of the pair \( (V, A) \). We use the notation:

\[
\text{Conv}(V, A) := \text{Hom}(V, A).
\]

### 2.1.1 Convolution \( \mathcal{S}\text{Lie}_\infty \)-algebra and \( \infty \)-morphisms

For a pair \( A, B \) of homotopy algebras of type \( \mathcal{C} \), we consider the convolution \( \mathcal{S}\text{Lie}_\infty \)-algebra

\[
L = \text{Hom}(\mathcal{C}(A), B), \tag{2.13}
\]

where the \( \mathcal{C} \)-coalgebra \( \mathcal{C}(A) \) is considered with the differential \( \partial + Q_A \), and \( \partial \) comes from the differential on \( A \) and the differential on \( \mathcal{C} \).

We observe that the \( \mathcal{S}\text{Lie}_\infty \)-algebra carries the following descending filtration

\[
\mathcal{F}^\text{ari}_0 L \supset \mathcal{F}^\text{ari}_1 L \supset \mathcal{F}^\text{ari}_2 L \supset \cdots
\]

\[
\mathcal{F}^\text{ari}_n L = \{ f \in \text{Hom}(\mathcal{C}(A), B) \mid f\big|_{\mathcal{C}(m) \otimes {S_m} A^\otimes m} = 0 \quad \forall \ m < n \}. \tag{2.14}
\]

It is also easy to check that:

**Proposition 2.8** The convolution \( \mathcal{S}\text{Lie}_\infty \)-algebra structure given by (2.11) and (2.12) is compatible with filtration (2.14) i.e.

\[
\left\{ \mathcal{F}^\text{ari}_{i_1} L, \mathcal{F}^\text{ari}_{i_2} L, \ldots, \mathcal{F}^\text{ari}_{i_k} L \right\} \subseteq \mathcal{F}^\text{ari}_{i_1 + i_2 + \cdots + i_k} L \quad \forall \ k > 1,
\]

Moreover, the \( \mathcal{S}\text{Lie}_\infty \)-algebra \( L = \text{Hom}(\mathcal{C}(A), B) \) is complete with respect to this filtration, i.e.

\[
L = \lim_{\xleftarrow{k}} L / \mathcal{F}^\text{ari}_k L.
\]
The notion of the convolution $\mathfrak{S}\text{Lie}_\infty$-algebra is partially justified by the following lemma:

**Lemma 2.9** Let $A$ and $B$ be homotopy algebras of type $C$. If the cooperad $C$ satisfies condition

$$C(0) = 0$$

then

$$\text{Hom}(C(A), B) = F^1 \text{Hom}(C(A), B).$$

In particular, the $\mathfrak{S}\text{Lie}_\infty$-algebra $\text{Hom}(C(A), B)$ is pro-nilpotent. Furthermore, the assignment

$$U \mapsto U' := p_B \circ U$$

is a bijection between the set of $\infty$-morphisms from $A$ to $B$ and the set of MC elements of the $\mathfrak{S}\text{Lie}_\infty$-algebra $\text{Hom}(C(A), B)$.

Proof. The first statement of the lemma follows directly from the definition of filtration (2.14) on the $\mathfrak{S}\text{Lie}_\infty$-algebra $\text{Hom}(C(A), B)$. Since $\text{Hom}(C(A), B)$ is complete with respect to this filtration, we conclude that the $\mathfrak{S}\text{Lie}_\infty$-algebra $\text{Hom}(C(A), B)$ is pro-nilpotent.

This conclusion allows us to write the MC equation in the $\mathfrak{S}\text{Lie}_\infty$-algebra $\text{Hom}(C(A), B)$ for any degree zero element. According to Definition 2.3, an $\infty$-morphism from $A$ to $B$ is a homomorphism of dg $C$-coalgebras

$$U : \left( C(A), \partial + Q_A \right) \to \left( C(B), \partial + Q_B \right).$$

Since the $C$-coalgebra $C(B)$ is cofree (over graded vector spaces), the homomorphism $U$ is uniquely determined by its composition

$$U' := p_B \circ U : C(A) \to B.$$ (2.17)

Furthermore, the compatibility of $U$ with the differentials $\partial + Q_A$ and $\partial + Q_B$ is equivalent to the equation

$$\partial \circ U'(X; a_1, \ldots, a_m) + Q_B \circ (1 \otimes U') \circ \Delta_1(X; a_1, \ldots, a_m) - U' \circ (\partial + Q_A)(X; a_1, \ldots, a_m)$$

$$+ \sum_{k=2}^{\infty} \frac{1}{k!} Q_B \circ (1 \otimes (U')^{\otimes k}) \circ \Delta_k(X; a_1, \ldots, a_m) = 0,$$ (2.18)

where $(X; a_1, \ldots, a_m)$ represents a vector in $C(A)$ and the factor $1/k!$ in the last sum comes from the identification of $C(B)^{\text{inv}}$ with $C(B)$ via the inverse of isomorphism (1.2).

Using the definition of multi-brackets (2.11), (2.12) on $\text{Hom}(C(A), B)$, we see that (2.18) is precisely the MC equation for $U'$ in the $\mathfrak{S}\text{Lie}_\infty$-algebra $\text{Hom}(C(A), B)$. Thus

$$U \leftrightarrow U' := p_B \circ U$$

is a desired bijection between the set of $\infty$-morphisms from $A$ to $B$ and the set of MC elements of the $\mathfrak{S}\text{Lie}_\infty$-algebra $\text{Hom}(C(A), B)$.
Remark 2.10 A version of Lemma 2.9 is proved in [8]. See Proposition 3 in [8, Section 1.3].

Example 2.11 Let us recall that coAs is the cooperad which governs coassociative coalgebras without counit and the dg operad Cobar(\(S^{-1}\)coAs) governs (flat) A\(_\infty\)-algebras. It is easy to see that, for every A\(_\infty\)-algebra A,

\[
\text{Hom}(S^{-1}\text{coAs}(A), A)
\]

(2.19)
is the completed version of the truncated Hochschild cochain complex of A

\[
\prod_{n \geq 1} S^{n-1}\text{Hom}(A^\otimes n, A)
\]

(2.20)

and the shifted L\(_\infty\)-algebra on (2.20) is obtained by symmetrizing the cup product and its higher analogues. It is worthy of mentioning that the Hochschild differential on (2.20) is obtained by twisting the differential on (2.19) by the MC element corresponding to the identity map id : A → A.

3 The G\(\text{Lie}_{\infty}\)MC\(^{-}\)-enriched category HoAlg\(_C\) and the simplicial category HoAlg\(_C^\Delta\)

Given two Cobar\(_C\)-algebras A and B, we denote by

\[
\text{map}(A, B) := \text{Hom}(\text{C}(A), B)
\]

(3.1)

the convolution G\(\text{Lie}_{\infty}\)-algebra defined in Proposition 2.6.

In this section, we construct a G\(\text{Lie}_{\infty}\)MC\(^{-}\)-enriched category HoAlg\(_C\) whose objects are homotopy algebras A, B, . . . of type C and whose mapping spaces are G\(\text{Lie}_{\infty}\)-algebras (3.1).

We start with defining a degree 0 linear map

\[
\mathcal{U}' : S(\text{Hom}(\text{C}(A_2), A_3) \oplus \text{Hom}(\text{C}(A_1), A_2)) \to \text{Hom}(\text{C}(A_1), A_3)
\]

(3.2)

by using the identification

\[
S(\text{Hom}(\text{C}(A_2), A_3) \oplus \text{Hom}(\text{C}(A_1), A_2)) = S(\text{Hom}(\text{C}(A_2), A_3)) \oplus S(\text{Hom}(\text{C}(A_1), A_2))
\]

\[
\oplus \left(S(\text{Hom}(\text{C}(A_2), A_3)) \otimes S(\text{Hom}(\text{C}(A_1), A_2))\right)
\]

and the formulas:

\[
\mathcal{U}'(g \otimes (f_1 \ldots f_n))(X) = g(1 \otimes f_1 \otimes \cdots \otimes f_n (\Delta_n(X)))
\]

(3.3)

for all \(X \in \text{C}(A_1), g \in \text{Hom}(\text{C}(A_2), A_3), f_1, \ldots, f_n \in \text{Hom}(\text{C}(A_1), A_2)\), and

\[
\mathcal{U}'\big|_{S(\text{Hom}(\text{C}(A_2), A_3))} = \mathcal{U}'\big|_{S(\text{Hom}(\text{C}(A_1), A_2))} = 0,
\]

(3.4)

\[
\mathcal{U}'\big|_{S^{n \neq 1}(\text{Hom}(\text{C}(A_2), A_3)) \otimes S(\text{Hom}(\text{C}(A_1), A_2))} = 0.
\]

We claim that
Proposition 3.1  The vector $U'$ defined by the above formulas is a MC element of the $\mathcal{S}$Lie$_\infty$-algebra
\[
\text{Hom}\left(\mathcal{S}(\text{Hom}(C(A_2), A_3) \oplus \text{Hom}(C(A_1), A_2)), \text{Hom}(C(A_1), A_3)\right). \tag{3.5}
\]
Proof. Let us denote by $L_{ij}$ the $\mathcal{S}$Lie$_\infty$-algebra $\text{Hom}(C(A_i), A_j)$ and by $d_{ij}$ the differential on $L_{ij}$. We also denote by $Q_{ij}$ the degree 1 coderivation on the coCom-coalgebra
\[
\mathcal{S}(L_{ij}). \tag{3.6}
\]
corresponding to the $\mathcal{S}$Lie$_\infty$-algebra $L_{ij}$. By abuse of notation, $d_{ij}$ also denotes the differential on (3.6) coming from the one on $L_{ij}$.

In terms of this notation, our goal is to prove that $U'$ satisfies the equation\(^4\)
\[
d_{13} \circ U' - U' \circ ((d_{23} + Q_{23}) \otimes 1 + 1 \otimes (d_{12} + Q_{12})) + \sum_{m=2}^{\infty} \frac{1}{m!}\{U', U', \ldots, U'\}_m = 0. \tag{3.7}
\]

We will present the most bulky part of the proof of (3.7). Namely, we will show in detail that the sum
\[
\sum_{m=2}^{\infty} \frac{1}{m!}\{U', U', \ldots, U'\}_m \tag{3.8}
\]
cancels with the term $-U' \circ (Q_{23} \otimes 1)$ in (3.7). The remaining cancellations are much more straightforward and we leave them to the reader.

In the computations given below, we do not specify explicitly sign factors coming from the Koszul sign rule. We address this issue by a short comment at the end of the proof.

Let $g_1, \ldots, g_k \in L_{23}$ and $f_1, \ldots, f_n \in L_{12}$. Due to (3.4),
\[
\frac{1}{m!}\{U', U', \ldots, U'\}_m(g_1, \ldots, g_k, f_1, \ldots, f_n) = 0
\]
if $k \neq m$ or $n < k$.

Unfolding the term
\[
\frac{1}{m!}\{U', U', \ldots, U'\}_m(g_1, \ldots, g_m, f_1, \ldots, f_n) \tag{3.9}
\]
with $n \geq m$, we get
\[
\frac{1}{m!}\{U', U', \ldots, U'\}_m(g_1, \ldots, g_m, f_1, \ldots, f_n) =
\sum_{k_1 + \cdots + k_m = n} \sum_{\tau \in S_m} \sum_{\sigma \in S_{k_1, k_2, \ldots, k_m}} \frac{\pm 1}{m!}\{U'(g_{\tau(1)}, f_{\sigma(1)}, \ldots, f_{\sigma(k_1)}), U'(g_{\tau(2)}, f_{\sigma(k_1+1)}, \ldots, f_{\sigma(k_1+k_2)}), \ldots U'(g_{\tau(m)}, f_{\sigma(n-k_m+1)}, \ldots, f_{\sigma(n)})\}_m \tag{3.10}
\]
\(^4\)We sometimes use the subscript $m$ in $\{ , , \ldots , \}_m$ to denote the number of entries of the corresponding multi-bracket.
where \( \{ , \ldots , \}^{L_{13}}_m \) denotes the corresponding multi-bracket on \( L_{13} \) and the sign factors in the right hand side are determined by the Koszul rule.

Since \( \{ , \ldots , \}^{L_{13}}_m \) is (graded) symmetric in its argument, we can simplify term (3.9) further:

\[
\frac{1}{m!} \{ \mathcal{U}', \mathcal{U}', \ldots, \mathcal{U}' \}_m (g_1, \ldots, g_m, f_1, \ldots, f_n) =
\sum_{k_1+\cdots+k_m=n} \sum_{k_j \geq 1} \pm \{ \mathcal{U}'(g_1, f_{\sigma(1)}), \ldots, f_{\sigma(k_1)}) \mathcal{U}'(g_2, f_{\sigma(k_1+1)}, \ldots, f_{\sigma(k_1+k_2)}) \ldots
\]
\[
\ldots \mathcal{U}'(g_m, f_{\sigma(n-k_m+1)}, \ldots, f_{\sigma(n)}) \}_{L_{13}}. \quad (3.11)
\]

Let \( X \in \mathcal{C}(A_1) \) and

\[
\Delta_m(X) = \sum_{\alpha} (\gamma_{\alpha}; X_{\alpha,1}, X_{\alpha,2}, \ldots, X_{\alpha,m}) \in (\mathcal{C}(m) \otimes \mathcal{C}(A_1) \otimes m)^{S_m}. \quad (3.12)
\]

Then, applying equation (3.11) and using the definition of the multi-bracket on \( L_{13} = \text{Hom}(\mathcal{C}(A_1), A_3) \), we get

\[
\frac{1}{m!} \{ \mathcal{U}', \mathcal{U}', \ldots, \mathcal{U}' \}_m (g_1, \ldots, g_m, f_1, \ldots, f_n)(X) =
\sum_{k_1+\cdots+k_m=n} \sum_{k_j \geq 1} \pm Q_{A_3} (\gamma_{\alpha}; \mathcal{U}'(g_1, f_{\sigma(1)}), \ldots, f_{\sigma(k_1)})(X_{\alpha,1}),
\]
\[
\mathcal{U}'(g_2, f_{\sigma(k_1+1)}, \ldots, f_{\sigma(k_1+k_2)})(X_{\alpha,2}), \ldots, \mathcal{U}'(g_m, f_{\sigma(n-k_m+1)}, \ldots, f_{\sigma(n)})(X_{\alpha,m})), \quad (3.13)
\]

where \( Q_{A_3} \) denotes the coderivation of \( \mathcal{C}(A_3) \) corresponding to the Cobar(\( \mathcal{C} \))-algebra structure on \( A_3 \).

Using (3.3), we deduce that

\[
\frac{1}{m!} \{ \mathcal{U}', \mathcal{U}', \ldots, \mathcal{U}' \}_m (g_1, \ldots, g_m, f_1, \ldots, f_n)(X) =
\sum_{k_1+\cdots+k_m=n} \sum_{k_j \geq 1} \pm Q_{A_3} \circ (1 \otimes g_1 \otimes f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k_1)} \otimes \cdots \otimes g_m \otimes f_{\sigma(n-k_m+1)} \otimes \cdots \otimes f_{\sigma(n)})
\]
\[
\circ (1 \otimes \Delta_{k_1} \otimes \Delta_{k_2} \otimes \cdots \otimes \Delta_{k_m}) \circ \Delta_m(X). \quad (3.14)
\]

By the axioms of the \( \mathcal{C} \)-coalgebra structure on \( \mathcal{C}(A_1) \), we have

\[
(1 \otimes \Delta_{k_1} \otimes \Delta_{k_2} \otimes \cdots \otimes \Delta_{k_m}) \circ \Delta_m(X) = b \circ (\Delta_{t_{k_1,\ldots,k_m}}^n \otimes 1^n) \circ \Delta_n(X), \quad (3.15)
\]

where \( n = k_1 + k_2 + \cdots + k_m \), \( \Delta_{t_{k_1,\ldots,k_m}}^n \) is the cooperadic comultiplication

\[
\Delta_{t_{k_1,\ldots,k_m}}^n : \mathcal{C}(n) \rightarrow \mathcal{C}(m) \otimes \mathcal{C}(k_1) \otimes \mathcal{C}(k_2) \otimes \cdots \otimes \mathcal{C}(k_m)
\]
corresponding to the planar tree $t_{k_1,\ldots,k_m}^+$ shown on figure 2, and $b$ is the braiding isomorphism which “changes the positions” of tensor factors appropriately.

Unfolding $\Delta_n(X)$

$$\Delta_n(X) = \sum_\beta (\tilde{\gamma}_\beta; \tilde{X}_{\beta,1}, \ldots, \tilde{X}_{\beta,n})$$

and using the fact that $\Delta_n$ lands in the space of $S_n$-invariants, we rewrite the expression

$$(\Delta t_{k_1,\ldots,k_m}^+ \otimes 1^\otimes n) \circ \Delta_n(X)$$

as follows:

$$(\Delta t_{k_1,\ldots,k_m}^+ \otimes 1^\otimes n) \circ \Delta_n(X) =
\sum_\beta \pm (\Delta t_{k_1,\ldots,k_m}^+ \otimes 1^\otimes n)(\sigma^{-1}(\tilde{\gamma}_\beta); \tilde{X}_{\beta,\sigma(1)}, \ldots, \tilde{X}_{\beta,\sigma(n)}) =
\sum_\beta \pm (\Delta \sigma(t_{k_1,\ldots,k_m}^+) \otimes 1^\otimes n)(\tilde{\gamma}_\beta; \tilde{X}_{\beta,\sigma(1)}, \ldots, \tilde{X}_{\beta,\sigma(n)}) \quad (3.16)$$

for any $\sigma \in S_n$.

Combining this observation with (3.15), we conclude that

$$\frac{1}{m!}\{\mathcal{U}, \mathcal{U}', \ldots, \mathcal{U}'\}_m(g_1, \ldots, g_m, f_1, \ldots, f_n)(X)$$

$$= \sum_\beta \sum_{k_1+\cdots+k_m=n} \sum_{\sigma \in S_{k_1,k_2,\ldots,k_m}} \sum_{k_j \geq 1} \pm Q_{A_3} \circ (1 \otimes g_1 \otimes \cdots \otimes g_m)$$

$$\circ (\Delta \sigma(t_{k_1,\ldots,k_m}^+) \otimes 1^\otimes n)(\tilde{\gamma}_\beta; f_{\sigma(1)}(\tilde{X}_{\beta,\sigma(1)}), \ldots, f_{\sigma(n)}(\tilde{X}_{\beta,\sigma(n)})) \quad (3.17)$$

On the other hand,

$$\sum_\beta \sum_{k_1+\cdots+k_m=n} \sum_{\sigma \in S_{k_1,k_2,\ldots,k_m}} \pm (\Delta \sigma(t_{k_1,\ldots,k_m}^+) \otimes 1^\otimes n)(\tilde{\gamma}_\beta; f_{\sigma(1)}(\tilde{X}_{\beta,\sigma(1)}), \ldots, f_{\sigma(n)}(\tilde{X}_{\beta,\sigma(n)}))$$

$$= \Delta_m \left( \sum_\beta \pm (\tilde{\gamma}_\beta; f_1(\tilde{X}_{\beta,1}), \ldots, f_n(\tilde{X}_{\beta,n})) \right) \quad (3.18)$$
Therefore, by definition of the $\mathcal{G}$Lie$_\infty$-structure on $L_{23} = \text{Hom}(\mathcal{C}(A_2), A_3)$, we have

$$\frac{1}{m!} \{U', U', \ldots, U'\}_m(g_1, \ldots, g_m, f_1, \ldots, f_n)(X) = U'(\{g_1, \ldots, g_m\}_{L_{23}}^m, f_1, \ldots, f_n)(X). \quad (3.19)$$

Let us also observe that, due to (3.4),

$$(U' \circ (Q_{23} \otimes 1)(g_1, \ldots, g_m, f_1, \ldots, f_n))(X) = U'(\{g_1, \ldots, g_m\}_{L_{23}}^m, f_1, \ldots, f_n)(X).$$

Thus,

$$\frac{1}{m!} \{U', U', \ldots, U'\}_m(g_1, \ldots, g_m, f_1, \ldots, f_n)(X) = (U' \circ (Q_{23} \otimes 1)(g_1, \ldots, g_m, f_1, \ldots, f_n))(X) \quad (3.20)$$

which implies the desired cancellation of sum (3.8) with the term $-U' \circ (Q_{23} \otimes 1)$ in (3.7).

Let us now address the issue of sign factors. The sign factor in front of the term $Q_{A_3} \circ (1 \otimes g_1 \otimes \cdots \otimes g_m) \circ (\Delta_{\sigma(t_{k_1}^{\hat{n}}, \ldots, k_m)}^n \otimes 1^n)(\tilde{\gamma}_\beta; f_{\sigma(1)}(\tilde{X}_{\beta,\sigma(1)}), \ldots, f_{\sigma(n)}(\tilde{X}_{\beta,\sigma(n)}))$ (3.21) in (3.17) comes from rearranging the homogeneous vectors

$$f_1, f_2, \ldots, f_n, \tilde{\gamma}_\beta, \tilde{X}_{\beta,1}, \tilde{X}_{\beta,2}, \ldots, \tilde{X}_{\beta,n} \quad (3.22)$$

from their standard order in (3.22) to the order in which they appear in (3.21). It is easy to see that we get the same sign factors in front of the corresponding terms, when we unfold the right hand side of (3.19).

Proposition 3.1 is proved. \hfill \Box

Combining Lemma 2.9 with Proposition 3.1 we deduce that

**Corollary 3.2** The map $U'$ defined by equations (3.3) and (3.4) lifts to an $\infty$-morphism

$$U : \text{map}(A_2, A_3) \oplus \text{map}(A_1, A_2) \to \text{map}(A_1, A_3). \quad (3.23)$$

\hfill \Box

**Remark 3.3** We would like to remark that the map

$$U' : \mathcal{G}(\text{Hom}(\mathcal{C}(A_2), A_3) \oplus \text{Hom}(\mathcal{C}(A_1), A_2)) \to \text{Hom}(\mathcal{C}(A_1), A_3),$$

can be equivalently defined by the single formula:

$$U'((g_1 \oplus f_1), (g_2 \oplus f_2), \ldots, (g_n \oplus f_n)) = \sum_{i=1}^{n} \pm g_i \circ (1 \otimes f_1 \otimes \cdots \otimes \hat{f}_i \otimes \cdots \otimes f_n) \circ \Delta_{n-1} \quad (3.24)$$

where $(g_i \oplus f_i) \in \text{Hom}(\mathcal{C}(A_2), A_3) \oplus \text{Hom}(\mathcal{C}(A_1), A_2)$ and $\pm$ is the usual Koszul sign factor.
The following theorem shows that the composition in \( \HoAlg \) given by \( \infty \)-morphism (3.23) is associative.

**Theorem 3.4** Let \( A_1, \ldots, A_4 \) be Cobar(\( C \))-algebras and let
\[
U_{i_1i_2i_3} : \map(A_{i_2}, A_{i_3}) \oplus \map(A_{i_1}, A_{i_2}) \to \map(A_{i_1}, A_{i_3})
\]
be the \( \infty \)-morphism given in Corollary 3.2. Then the following diagram
\[
\begin{array}{ccc}
\map(A_3, A_4) \oplus (\map(A_2, A_3) \oplus \map(A_1, A_2)) & \cong & \map(A_1, A_4) \\
\downarrow & & \downarrow \underline{U_{134}} \\
(\map(A_3, A_4) \oplus \map(A_2, A_3)) \oplus \map(A_1, A_2) & \map(A_2, A_4) \oplus \map(A_1, A_2)
\end{array}
\]

commutes.

Proof. Let \( h \in \map(A_3, A_4) \), \( g_1, \ldots, g_m \in \map(A_2, A_3) \), and \( f_1, \ldots, f_n \in \map(A_1, A_2) \). Composing the lower arrows in (3.25) with the canonical projection
\[
p_{14} : S(\map(A_1, A_4)) \to \map(A_1, A_4),
\]
we get
\[
p_{14} \circ U_{124} \circ (U_{234} \otimes \text{id})(h, g_1, \ldots, g_m, f_1, \ldots, f_n) =
\]
\[
h \left( (1 \otimes g_1 \otimes \cdots \otimes g_m) \circ \Delta_m \circ (1 \otimes f_1 \otimes \cdots \otimes f_n) \circ \Delta_n \right).
\]

Similarly, composing the upper arrows in (3.25) with canonical projection (3.26), we get
\[
p_{14} \circ U_{134} \circ (\text{id} \otimes U_{123})(h, g_1, \ldots, g_m, f_1, \ldots, f_n) =
\]
\[
\sum_{k_1 + \cdots + k_m = n \atop \sigma \in \text{Sh}(k_1, \ldots, k_m)} \pm h \left( 1 \otimes g_1(1 \otimes f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \cdots \otimes f_{\sigma(k_1)}) \otimes g_2(1 \otimes f_{\sigma(k_1+1)} \otimes \cdots \otimes f_{\sigma(k_1+k_2)}) \otimes \cdots \otimes g_m(1 \otimes f_{\sigma(n-k_m+1)} \otimes \cdots \otimes f_{\sigma(n)}) \circ (1 \otimes \Delta_{k_1} \otimes \Delta_{k_2} \otimes \cdots \otimes \Delta_{k_m}) \right) \Delta_m,
\]
where the sign factors are determined by the Koszul rule.

Using equation (3.15), computation (3.16), and equation (3.18) from the proof of Proposition 3.1, we conclude that the left hand side of (3.27) coincides with the left hand side of (3.28). Since any \( \infty \)-morphism to \( \map(A_1, A_4) \) is uniquely determined by its composition with projection (3.26), we deduce that
\[
U_{124} \circ (U_{234} \otimes \text{id}) = U_{134} \circ (\text{id} \otimes U_{123}).
\]

Thus diagram (3.25) is indeed commutative. \( \square \)
3.1 The proof of the unit axiom

Let us recall that $0$ is the unit object in the category $\mathcal{S}\text{Lie}_{\infty}^{MC}$ and observe that for every Cobar$(\mathcal{C})$-algebra $A$ we have a canonical enhanced morphism

$$0 \xrightarrow{(\text{id}_A, 0)} \text{map}(A, A), \quad (3.29)$$

where $\text{id}_A$ is the MC element of $\text{map}(A, A) = \text{Hom}(\mathcal{C}(A), A)$ corresponding to the identity $\infty$-morphism $\text{id}_{\mathcal{C}(A)} : \mathcal{C}(A) \to \mathcal{C}(A)$ and $0$ is the unique $\mathcal{S}\text{Lie}_{\infty}$-morphism from $0$ to $\text{map}(A, A)$.

We claim that

**Proposition 3.5** For every pair $A, B$ of homotopy algebras of type $\mathcal{C}$, the diagrams

$$\xymatrix{ \text{map}(A, B) \oplus \text{map}(A, A) \ar[r]^U \ar[d]_{\text{id}_{\text{map}(A, B)} \otimes (\text{id}_A, 0)} & \text{map}(A, B) \ar[d] \cr \text{map}(A, B) \oplus 0 \ar[ru] } \quad (3.30)$$

$$\xymatrix{ \text{map}(B, B) \oplus \text{map}(A, B) \ar[r]^U \ar[d]_{(\text{id}_B, 0) \otimes \text{id}_{\text{map}(A, B)}} & \text{map}(A, B) \ar[d] \cr 0 \oplus \text{map}(A, B) \ar[ru] } \quad (3.31)$$

commute.

Proof. Let us denote by

$$\mathcal{K} : \text{map}(A, B) \oplus 0 \to \text{map}(A, B)$$

the composition of the vertical arrow and the horizontal arrow in (3.30) and let $\mathcal{K}'$ be the corresponding element in

$$\text{Hom}(S(\text{map}(A, B) \oplus 0), \text{map}(A, B)).$$

Unfolding the definition of composition of enhanced morphisms in $\mathcal{S}\text{Lie}_{\infty}^{MC}$ (see Proposition 3.4 in [4]), we get that

$$\mathcal{K}'(g_1, \ldots, g_m) = \sum_{n \geq 0} \frac{1}{n!} U'( (g_1, \ldots, g_m) \otimes \text{id}_A^n ), \quad (3.32)$$

where $g_1, \ldots, g_m \in \text{map}(A, B)$. 

17
Hence, using (3.3) and (3.4) we deduce that
\[ K'(g_1, \ldots, g_m) = 0 \quad \forall \ m \geq 2 \]
and, for every \( X \in \mathcal{C}(A) \),
\[ K'(g_1)(X) = \sum_{n \geq 0} \frac{1}{n!} g_1((1 \otimes \text{id}^n_A) \Delta_n(X)) = g_1(X), \tag{3.33} \]
where the last equality follows from the identification of \( \mathcal{C}(\mathcal{C}(A))^{\text{inv}} \) and \( \mathcal{C}(\mathcal{C}(A)) \) via the inverse of isomorphism (1.2). Thus diagram (3.30) indeed commutes.

The proof of the commutativity of (3.31) is easier. So we leave it to the reader. \( \square \)

### 3.2 The simplicial category \( \text{HoAlg}_{\mathcal{C}}^\Delta \) of homotopy algebras

Let \( \Omega_n = \Omega^*(\Delta^n) \) denote the polynomial de Rham complex on the \( n \)-simplex with coefficients in \( k \), and \( \{\Omega_n\}_{n \geq 0} \) the associated simplicial dg commutative \( k \)-algebra. Let us recall [4, Proposition 4.1] that for every filtered \( \mathcal{S}\text{Lie}_{\infty} \)-algebra \( L \) the simplicial set \( \mathcal{MC}_\bullet(L) \) with
\[ \mathcal{MC}_n(L) := MC(L \otimes \Omega_n) \]
is a Kan complex (a.k.a. an \( \infty \)-groupoid). We call the simplicial set \( \mathcal{MC}_\bullet(L) \) the Deligne-Getzler-Hinich (DGH) \( \infty \)-groupoid.

Let us also recall (see Theorem 4.2 in [4]) that applying the functor \( \mathcal{MC}_\bullet \) to mapping spaces of any \( \mathcal{S}\text{Lie}_{\mathcal{MC}}^{\infty} \)-enriched category, we get a simplicial category whose mapping spaces are Kan complexes. Thus, applying [4, Theorem 4.2] to the \( \mathcal{S}\text{Lie}_{\mathcal{MC}}^{\infty} \)-enriched category \( \text{HoAlg}_{\mathcal{C}}^\Delta \) and using Lemma 2.9 we deduce the following theorem:

**Theorem 3.6** Let \( \mathcal{C} \) be a coaugmented dg cooperad satisfying Conditions (2.1) and (2.15). Then the assignment
\[ (A, B) \in \text{Objects}(\text{Cat}_{\mathcal{C}}) \times \text{Objects}(\text{Cat}_{\mathcal{C}}) \mapsto \mathcal{MC}_\bullet(\text{map}(A, B)) \]
gives us a category enriched over \( \infty \)-groupoids (a.k.a. Kan complexes). Moreover, for every pair of Cobar(\( \mathcal{C} \))-algebras \( A, B \), the set \( \mathcal{MC}_0(\text{map}(A, B)) \) is in bijection with the set of \( \infty \)-morphisms from \( A \) to \( B \). \( \square \)

### 4 \( \pi_0(\text{HoAlg}_{\mathcal{C}}^\Delta) \) is a correct homotopy category of homotopy algebras

Let \( A \) and \( B \) be homotopy algebras of type \( \mathcal{C} \) and \( F \) be an \( \infty \)-morphism from \( A \) to \( B \):
\[ F : \left( \mathcal{C}(A), \partial + Q_A \right) \to \left( \mathcal{C}(B), \partial + Q_B \right). \]

\(^5\text{Recall that } \text{MC}(L) \text{ denotes the set of MC elements of a filtered } \mathcal{S}\text{Lie}_{\infty} \text{-algebras } L \text{ [4].} \)
Composing $F$ with a canonical projection $p_B : C(B) \to B$, and restricting this composition to $A \subset C(A)$, we get a map of cochain complexes:

$$p_B \circ F \big|_A : A \to B. \quad (4.1)$$

We refer to (4.1) as the linear term of the $\infty$-morphism $F$. Recall that an $\infty$-morphism $F$ is called an $\infty$ quasi-isomorphism if its linear term (4.1) is a quasi-isomorphism of cochain complexes.

Let us recall that $\text{Cat}_C$ is the category of Cobar($C$)-algebras with morphisms being $\infty$-morphisms, and observe that we have the obvious functor

$$\mathfrak{F} : \text{Cat}_C \to \pi_0(\text{HoAlg}_{\Delta}^C) \quad (4.2)$$

which acts by identity on objects and assigns to every $\infty$-morphism the isomorphism class of the corresponding MC element.

The goal of this section is to prove that the category $\pi_0(\text{HoAlg}_{\Delta}^C)$ is the homotopy category for $\text{Cat}_C$. Namely,

**Theorem 4.1** The functor $\mathfrak{F}$ sends $\infty$ quasi-isomorphisms to isomorphisms and it is a universal functor with this property. I.e., if $\mathfrak{G} : \text{Cat}_C \to \mathfrak{D}$ is a functor which sends $\infty$ quasi-isomorphisms to isomorphisms in $\mathfrak{D}$ then there exists a unique functor

$$\mathfrak{G}' : \pi_0(\text{HoAlg}_{\Delta}^C) \to \mathfrak{D}$$

such that $\mathfrak{G} = \mathfrak{G}' \circ \mathfrak{F}$.

### 4.1 “Inverting” $\infty$ quasi-isomorphisms

Let $A_1, A_2, A_3$ be Cobar($C$)-algebras, $F$ be an $\infty$-morphism from $A_1$ to $A_2$, and $F'$ be the MC element of $\text{map}(A_1, A_2)$ corresponding to $F$. Let us denote by $F' \oplus 0, 0 \oplus F'$ the corresponding MC elements of the $\mathfrak{S}\text{Lie}_{\infty}$-algebras

$$\text{map}(A_1, A_2) \oplus \text{map}(A_3, A_1)$$

and

$$\text{map}(A_2, A_3) \oplus \text{map}(A_1, A_2),$$

respectively.

Twisting the $\mathfrak{S}\text{Lie}_{\infty}$-morphisms

$$\mathcal{U} : \text{map}(A_1, A_2) \oplus \text{map}(A_3, A_1) \to \text{map}(A_3, A_2)$$

and

$$\mathcal{U} : \text{map}(A_2, A_3) \oplus \text{map}(A_1, A_2) \to \text{map}(A_1, A_3)$$

by the MC elements $F' \oplus 0, 0 \oplus F'$, respectively, and composing the resulting $\mathfrak{S}\text{Lie}_{\infty}$-morphisms with the canonical maps

$$f \mapsto 0 \oplus f : \text{map}(A_3, A_1) \to \text{map}(A_3, A_1) \oplus \text{map}(A_3, A_1)$$
Let \( f \mapsto f \oplus 0 : \text{map}(A_2, A_3) \to \text{map}(A_2, A_3) \oplus \text{map}(A_1, A_2) \)
we get two \( \mathcal{S}\text{Lie}_\infty \)-morphisms

\[
\mathcal{U}_{A_3, A_1, A_2} : \text{map}(A_3, A_1) \to \text{map}(A_3, A_2)
\]

and

\[
\mathcal{U}_{A_1, A_2, A_3} : \text{map}(A_2, A_3) \to \text{map}(A_1, A_3).
\]

The following proposition says that the induced maps of MC elements

\[
(\mathcal{U}_{A_3, A_1, A_2})_* : \text{MC}(\text{map}(A_3, A_1)) \to \text{MC}(\text{map}(A_3, A_2))
\]

and

\[
(\mathcal{U}_{A_1, A_2, A_3})_* : \text{MC}(\text{map}(A_2, A_3)) \to \text{MC}(\text{map}(A_1, A_3))
\]
correspond to the composition (resp. the pre-composition) of an \( \infty \)-morphism from \( A_3 \) to \( A_1 \) (resp. from \( A_2 \) to \( A_3 \)) with \( F \):

**Proposition 4.2** If \( G \) is an \( \infty \)-morphism from \( A_3 \) to \( A_1 \) and \( G' \) is the corresponding MC element of \( \text{map}(A_3, A_1) \) then the MC element \( (\mathcal{U}_{A_3, A_1, A_2})_* (G') \) of \( \text{map}(A_3, A_2) \) corresponds to the composition \( F \circ G \). Similarly, if \( G \) is an \( \infty \)-morphism from \( A_2 \) to \( A_3 \) and \( G' \) is the corresponding MC element of \( \text{map}(A_2, A_3) \) then the MC element \( (\mathcal{U}_{A_1, A_2, A_3})_* (G') \) of \( \text{map}(A_1, A_3) \) corresponds to the composition \( G \circ F \).

Proof. The proof of these statements is straightforward.

We also claim that

**Proposition 4.3** \( \mathcal{S}\text{Lie}_\infty \)-morphisms (4.3) and (4.4) are compatible with the filtrations \( \mathcal{F}^\text{ari}_m \) from (2.14). Furthermore, if \( F \) is an \( \infty \) quasi-isomorphism, then (4.3) and (4.4) give us \( \mathcal{S}\text{Lie}_\infty \) quasi-isomorphisms

\[
\mathcal{F}^\text{ari}_m \text{map}(A_3, A_1) \to \mathcal{F}^\text{ari}_m \text{map}(A_3, A_2) \quad \text{and} \quad \mathcal{F}^\text{ari}_m \text{map}(A_2, A_3) \to \mathcal{F}^\text{ari}_m \text{map}(A_1, A_3)
\]
respectively, for every \( m \geq 1 \).

Proof. Due to Remark 3.3 the composition

\[
\mathcal{U}'_{A_3, A_1, A_2} := p_{\text{map}(A_3, A_2)} \circ \mathcal{U}_{A_3, A_1, A_2} : \mathcal{S}(\text{map}(A_3, A_1)) \to \text{map}(A_3, A_2)
\]
is given by the formula

\[
\mathcal{U}'_{A_3, A_1, A_2}(g_1, \ldots, g_n)(X) = F'(1 \otimes g_1 \otimes \cdots \otimes g_n (\Delta_n(X))),
\]
where \( X \in \mathcal{C}(A_3) \) and \( g_1, \ldots, g_n \in \text{map}(A_3, A_1) \). Similarly, the composition

\[
\mathcal{U}'_{A_1, A_2, A_3} := p_{\text{map}(A_1, A_3)} \circ \mathcal{U}_{A_1, A_2, A_3} : \mathcal{S}(\text{map}(A_2, A_3)) \to \text{map}(A_1, A_3)
\]
is given by the formula
\[
\mathcal{U}'_{A_1 A_2 A_3}(h_1, \ldots, h_n)(Y) = \begin{cases} 
\sum_{m \geq 1} \frac{1}{m!} h_1 \left( 1 \otimes F' \otimes \cdots \otimes F' \left( \Delta_m(Y) \right) \right) & \text{if } n = 1, \\
0 & \text{otherwise},
\end{cases}
\] (4.9)
where \( Y \in \mathcal{C}(A_1) \) and \( h_1, \ldots, h_n \in \text{map}(A_2, A_3) \).

The compatibility of map \( \mathcal{U}'_{A_3 A_1 A_2} \) and \( \mathcal{U}'_{A_1 A_2 A_3} \) with the filtrations \( \mathcal{F}^\text{ari}_m \) can be checked directly by unfolding the right hand side of (4.8) and the right hand side of (4.9), respectively.

We also see that the linear terms
\[
\mathcal{U}_{1, A_3 A_1 A_2} : \text{map}(A_3, A_1) \to \text{map}(A_3, A_2) \tag{4.10}
\]
and
\[
\mathcal{U}_{1, A_1 A_2 A_3} : \text{map}(A_2, A_3) \to \text{map}(A_1, A_3) \tag{4.11}
\]
of the \( \mathfrak{S}\text{Lie}_\infty \)-morphisms \( \mathcal{U}_{A_3 A_1 A_2} \) and \( \mathcal{U}_{A_1 A_2 A_3} \) are given by the formulas:
\[
\mathcal{U}_{1, A_3 A_1 A_2}(g)(X) = F' \left( (1 \otimes g) \circ \Delta_1(X) \right)
\]
and
\[
\mathcal{U}_{1, A_1 A_2 A_3}(h)(Y) = \sum_{k \geq 1} \frac{1}{k!} h \left( 1 \otimes F' \otimes \cdots \otimes F' \left( \Delta_k(Y) \right) \right),
\]
respectively, where \( g \in \text{map}(A_3, A_1) \), \( h \in \text{map}(A_2, A_3) \), \( X \in \mathcal{C}(A_3) \) and \( Y \in \mathcal{C}(A_1) \).

Let \( \phi : A_1 \to A_2 \) be the linear term
\[
\phi := p_{A_2} \circ F \big|_{A_1} \tag{4.12}
\]
of the \( \infty \)-morphism \( F \) and assume that \( \phi \) is a quasi-isomorphism of cochain complexes.

To prove that \( \mathcal{U}_{1, A_1 A_2 A_3} \) induces an isomorphism
\[
H^\bullet \left( \mathcal{F}^\text{ari}_m \text{map}(A_2, A_3) \right) \to H^\bullet \left( \mathcal{F}^\text{ari}_m \text{map}(A_1, A_3) \right)
\]
for every \( m \), we observe that, for \( m \geq 2 \)
\[
\mathcal{F}^\text{ari}_m \text{map}(W, A_3) = \mathcal{F}^\text{ari}_m \text{Hom}(\mathcal{C}_o(W), A_3)
\]
and \( \text{map}(W, A_3) = \mathcal{F}^\text{ari}_1 \text{map}(W, A_3) \) splits into the direct sum of cochain complexes
\[
\text{map}(W, A_3) = \text{Hom}(W, A_3) \oplus \text{Hom}(\mathcal{C}_o(W), A_3), \tag{4.13}
\]
where \( W \) is either \( A_2 \) or \( A_1 \), and \( \mathcal{C}_o \) is the cokernel of the coaugmentation of \( \mathcal{C} \).

We also observe that the chain map \( \mathcal{U}_{1, A_1 A_2 A_3} \) is compatible with decomposition (4.13) and the corresponding chain map
\[
\text{Hom}(A_2, A_3) \to \text{Hom}(A_1, A_3)
\]
is a quasi-isomorphism because the functor \( \text{Hom}(-, -) \) preserves quasi-isomorphisms in \( \text{Ch}_k \).
So we should now prove that the chain map

$$U_{1,A_1A_2A_3}^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(A_2), A_3)} : \mathcal{F}_m^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(A_2), A_3)} \to \mathcal{F}_m^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(A_1), A_3)} \quad (4.14)$$

induces an isomorphism on cohomology for every $m \geq 1$.

For this purpose, we equip the cochain complex $\mathcal{F}_m^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)}$ with the following descending filtration

$$\mathcal{F}_m^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)} = \mathcal{F}_0^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)} \supset \mathcal{F}_1^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)} \supset \ldots$$

$$\mathcal{F}_q^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)} := \{ g \in \mathcal{F}_m^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)} \mid g(X) = 0 \ \forall X \in \mathcal{F}_q \mathcal{C}_o(W) \} \quad (4.15)$$

where $\mathcal{F}^\mathfrak{c} \mathcal{C}_o$ is the ascending filtration on the pseudo-cooperad $\mathcal{C}_o$ from (2.21) and $W$ is, as above, either $A_1$ or $A_2$.

Due to inclusion (2.2), the differential on $\text{map}(W, A_3)$ is compatible with the filtration. Moreover, condition (2.3) implies that $\mathcal{F}_m^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)}$ is complete with respect to this filtration, i.e.

$$\mathcal{F}_m^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)} = \lim_q \mathcal{F}_m^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)} / \mathcal{F}_q^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)} .$$

Let us denote by $E_{m,q}(\mathcal{C}, W)$ the following cochain complex

$$E_{m,q}(\mathcal{C}, W) := \bigoplus_{n \geq m} \left( \left( \mathcal{F}^q \mathcal{C}_o(n) / \mathcal{F}^{q-1} \mathcal{C}_o(n) \right) \otimes W^{\otimes n} \right)_S . \quad (4.16)$$

It is clear that, the associated graded complex

$$\text{Gr}_{\mathcal{F}^\mathfrak{c}} \mathcal{F}_m^{\mathfrak{ari} \text{Hom}(\mathcal{C}_o(W), A_3)}$$

is isomorphic to

$$\bigoplus_{q \geq 1} \text{Hom}(E_{m,q}(\mathcal{C}_o, W), A_3) . \quad (4.17)$$

Furthermore, the differential $\partial_{\text{Gr}}$ on (4.17) comes from those on $\mathcal{C}_o$ and $A_3$.

The map between the associated graded complexes

$$U_{1,A_1A_2A_3}^{\text{Gr}} : \text{Hom}(E_{m,q}(\mathcal{C}_o, A_2), A_3) \to \text{Hom}(E_{m,q}(\mathcal{C}_o, A_1), A_3) \quad (4.18)$$

induced by (4.14) is given by the formula

$$U_{1,A_1A_2A_3}^{\text{Gr}}(g)(\gamma; a_1, \ldots, a_n) = g(\gamma; \varphi(a_1), \ldots, \varphi(a_n)) , \quad (4.19)$$

where $\varphi$ is the linear term of the $\infty$-morphism $F$ and $\gamma \in \mathcal{F}^q \mathcal{C}_o(n) / \mathcal{F}^{q-1} \mathcal{C}_o(n)$.

Using the Künneth theorem, the fact that $\varphi$ induces an isomorphism $H^\bullet(A_1) \to H^\bullet(A_2)$, and $\text{char}(\mathbb{k}) = 0$ we conclude that the chain map

$$E_{m,q}(\mathcal{C}_o, A_1) \to E_{m,q}(\mathcal{C}_o, A_2)$$
induced by \( \varphi \) is a quasi-isomorphism.

Therefore, since the functor \( \text{Hom}(-, -) \) preserves quasi-isomorphisms in \( \text{Ch}_k \), we deduce chain map (4.18) between the associated graded complexes is a quasi-isomorphism.

Since \( \mathcal{F}_m \text{Hom}(\mathcal{C}_0(W), A_3) \) is complete with respect to filtration (4.15), and filtration (4.14) is bounded from the left, applying Lemma D.1 from [6] to the cone of chain map (4.14), we conclude that (4.14), and hence (4.11), is indeed a quasi-isomorphism of cochain complexes.

A similar argument shows that map (4.10) is a quasi-isomorphism of cochain complexes, provided so is \( \varphi \) (4.12).

Proposition 4.3 is proved. \( \square \)

Let \( A \) and \( B \) be \( \text{Cobar}(\mathcal{C}) \)-algebras. We will now prove that every \( \infty \) quasi-morphism \( F \) from \( A \) to \( B \) is “invertible” in the following sense:

**Corollary 4.4** Let \( F \) be an \( \infty \) quasi-isomorphism from \( A \) to \( B \). Then there exists an \( \infty \)-morphism \( G \) from \( B \) to \( A \) such that the MC element \((G \circ F)' \) (resp. \((F \circ G)' \)) of the \( \mathcal{S}\text{Lie}_{\infty} \)-algebra \( \text{map}(A, A) \) (resp. \( \text{map}(B, B) \)) corresponding to the composition \( G \circ F \) (resp. \( F \circ G \)) is isomorphic in \( \mathcal{M}_0(\text{map}(A, A)) \) (resp. in \( \mathcal{M}_0(\text{map}(B, B)) \)) to the MC element \( \text{id}_A' \) (resp. \( \text{id}_B' \)) corresponding to the identity morphism \( \text{id}_A \) (resp. \( \text{id}_B \)). If \( \tilde{G} \) is another \( \infty \)-morphism from \( B \) to \( A \) satisfying the above properties then the MC element \( \tilde{G}' \in \text{map}(B, A) \) corresponding to \( \tilde{G} \) is isomorphic to \( G' \) in \( \mathcal{M}_0(\text{map}(B, A)) \).

Proof. Let us start with the question of existence of \( G \).

Due to Proposition 4.2 it suffices to prove that there exists a MC element \( G' \) of \( \text{map}(B, A) \) such that the 0-cell

\[
(U_{ABA})_*(G') \in \mathcal{M}_0(\text{map}(A, A))
\]

is connected to \( \text{id}_A' \) and the 0-cell

\[
(U_{BAB})_*(G') \in \mathcal{M}_0(\text{map}(B, B))
\]

is connected to \( \text{id}_B' \).

Proposition 4.3 implies that the \( \mathcal{S}\text{Lie}_{\infty} \)-morphism

\[
U_{ABA} : \text{map}(B, A) \to \text{map}(A, A)
\]

is a quasi-isomorphism and, moreover, it satisfies the necessary conditions of Theorem 2.2 from [2]. This theorem, in turn, implies that \( U_{ABA} \) induces a bijection of sets

\[
\pi_0\left(\mathcal{M}_*(\text{map}(B, A))\right) \xrightarrow{\simeq} \pi_0\left(\mathcal{M}_*(\text{map}(A, A))\right).
\]

Therefore, there exists a MC element \( G' \in \text{map}(B, A) \) such that the 0-cell

\[
(G \circ F)' = (U_{ABA})_*(G')
\]

is connected to \( \text{id}_A' \).

To prove that the 0-cell \((F \circ G)' = (U_{BAB})_*(G') \) is connected to \( \text{id}_B' \), we consider the \( \mathcal{S}\text{Lie}_{\infty} \)-morphisms

\[
U_{ABB} : \text{map}(B, B) \to \text{map}(A, B),
\]

23
and
\[ U_{AAB} : \text{map}(A, A) \to \text{map}(A, B) \] (4.25)
constructed, as above, using the ∞-morphism F.

Proposition 4.2 implies that, if K′ is a MC element of \( \text{map}(B, B) \) (resp. \( \text{map}(A, A) \))
corresponding to an ∞-morphism K from B to B (resp. from A to A) then the MC element
\( (U_{AAB})_*(K') \) (resp. \( (U_{AAB})_*(K') \)) corresponds to the composition K \circ F (resp. F \circ K).

Let us now consider the composition F \circ G and denote by (F \circ G)' the 0-cell of \( \mathcal{MC}_*(\text{map}(B, B)) \)
corresponding to the ∞-morphism F \circ G.

Due to the above observation, the MC element (F \circ G \circ F)' ∈ \text{map}(A, B) corresponding
to the ∞-morphism F \circ G \circ F satisfies the equations
\[ (F \circ G \circ F)' = (U_{AAB})_*(F \circ G)' \] (4.26)
and
\[ (F \circ G \circ F)' = (U_{AAB})_*(G \circ F)' \] (4.27)

Therefore, since the 0-cell (G \circ F)' is connected to the 0-cell id′A, the 0-cell (F \circ G \circ F)' is connected to F' in \( \mathcal{MC}_*(\text{map}(A, B)) \).

On other hand, F' = (U_{AAB})*(id′B), and hence the 0-cells
\( (U_{AAB})*(id′B) \) and \( (U_{AAB})*(F \circ G)' \)
are connected in \( \mathcal{MC}_*(\text{map}(A, B)) \).

Since F is an ∞ quasi-isomorphism, Proposition 4.3 and [2, Theorem 2.2] imply that
\( (U_{AAB})_* \) induces a bijection
\[ π_0\left( \mathcal{MC}_*(\text{map}(B, B)) \right) \cong π_0\left( \mathcal{MC}_*(\text{map}(A, B)) \right) . \]

Thus the 0-cells (F\circ G)' and id′B are also connected in \( \mathcal{MC}_*(\text{map}(B, B)) \) and the existence
of a desired ∞-morphism G is proved.

Let \( \tilde{G} \) be another ∞-morphism from B to A such that 0-cells \( (4.23) \) and id′A are connected
in \( \mathcal{MC}_*(\text{map}(A, A)) \).

Therefore, since \( U_{ABA} \) induces bijection \( (4.22) \) and the 0-cells \( (U_{ABA})*G' \) and id′A are connected, we conclude that the 0-cells \( \tilde{G}' \) and G' are also connected. □

Thus we proved the first part of Theorem 4.1.

We would like to conclude this subsection with the observation that mapping spaces of
the simplicial category \( \text{HoAlg}_C^A \) enjoy the following remarkable property

Corollary 4.5 Let A₁, A₂, A₃ be Cobar(C)-algebras and F be an ∞ quasi-isomorphism from A₁ to A₂. Then the composition (resp. pre-composition) with F induces the weak equivalences of simplicial sets:
\[ \mathcal{MC}_*(\text{map}(A₃, A₁)) \to \mathcal{MC}_*(\text{map}(A₃, A₂)) , \]
\[ \mathcal{MC}_*(\text{map}(A₂, A₃)) \to \mathcal{MC}_*(\text{map}(A₁, A₃)) . \]

Proof. The desired statement is a direct consequence of Proposition 4.3 and [2, Theorem 2.2]. □
4.2 The functor $\mathfrak{F}$ from Theorem 4.1 has the desired universal property

The proof of the universal property of the functor $\mathfrak{F}$ is based on the following proposition:

**Proposition 4.6** Let $\mathfrak{G} : \text{Cat}_C \rightarrow \mathfrak{D}$ be a functor which sends $\infty$ quasi-isomorphisms to isomorphisms in $\mathfrak{D}$. Let $A, B$ be Cobar($C$)-algebras and $F, G$ be $\infty$-morphisms from $A$ to $B$. If the corresponding 0-cells $F'$ and $G'$ of $\mathfrak{MC}_\infty(\text{map}(A, B))$ are connected then

$$\mathfrak{G}(F) = \mathfrak{G}(G).$$

Proof. By the condition of the proposition, there exists a 1-cell

$$K' \in \text{Hom}(C(A), B) \otimes \mathbb{k}[t, dt]$$

such that

$$K' \bigg|_{t = dt = 0} = F'$$

and

$$K' \bigg|_{t = 1, dt = 0} = G'.$$

Since

$$C(A) = \bigoplus_n (C(n) \otimes A^{\otimes n})_{S_n}$$

and $\text{Hom}(C(A), B)$ is considered with the topology coming from filtration (2.14), we have the natural strict $\mathfrak{GLie}_\infty$-morphism

$$\text{Hom}(C(A), B) \otimes \mathbb{k}[t, dt] \rightarrow \text{Hom}(C(A), B \otimes \mathbb{k}[t, dt]).$$

Therefore the 1-cell $K'$ gives us an $\infty$-morphism $K$ from $A$ to $B \otimes \mathbb{k}[t, dt]$ which fits into the following commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{K} & B \\
\downarrow F & & \downarrow \text{id} \\
B \otimes \mathbb{k}[t, dt] & \xrightarrow{p_0} & B \\
\downarrow G & & \downarrow p_1 \\
B
\end{array}$$

(4.32)

where $B \otimes \mathbb{k}[t, dt]$ is considered with the differential $\partial_B + dt \partial_t$ and the natural Cobar($C$)-structure coming from the one on $B$. Moreover, $p_0$ and $p_1$ are the obvious (strict) morphisms of Cobar($C$)-algebras

$$p_0(v) := v \bigg|_{t = dt = 0} : B \otimes \mathbb{k}[t, dt] \rightarrow B$$

(4.33)

and

$$p_1(v) := v \bigg|_{t = 1, dt = 0} : B \otimes \mathbb{k}[t, dt] \rightarrow B.$$
Let us observe that the maps $p_0$ and $p_1$ fit into the commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{id_B} & B \\
\downarrow{i} & & \downarrow{p_0} \\
B \otimes k[t, dt] & \xrightarrow{p_1} & B,
\end{array}
\]

where $i : B \to B \otimes k[t, dt]$ is the natural embedding given by

\[i(v) := v \otimes 1\,.
\]

Applying the functor $G$ to (4.35), we get

\[G(p_0) \circ G(i) = G(p_1) \circ G(i) = \text{id}_{G(B)}\,.
\]

Hence, since $i$ is obviously a quasi-isomorphism, we deduce that

\[G(p_0) = G(p_1)\,.
\]

Finally, applying the functor $G$ to (4.32), we get

\[G(p_0) \circ G(K) = G(F), \quad G(p_1) \circ G(K) = G(G)
\]

which implies that $G(F) = G(G)$.

Proposition 4.6 motivates the following definition

**Definition 4.7** Let $A, B$ be Cobar$(C)$-algebras. We say that $\infty$-morphisms $F, G$ from $A$ to $B$ are homotopic if the corresponding 0-cells $F'$ and $G'$ are connected in $MC_\bullet(\text{map}(A, B))$.$^6$

We can now prove the universal property of functor (4.2).

Indeed, let $G$ be a functor from $\text{Cat}_C$ to some category $D$ which sends $\infty$ quasi-isomorphisms to isomorphisms.

For objects of $A, B, \ldots$ of $\pi_0(\text{HoAlg}_C)$ we set

\[G'(A) := G(A)\,.
\]

Next, given two Cobar($C$)-algebras $A, B$ and an isomorphism class

\[[F'] \in \pi_0(MC_\bullet(\text{map}(A, B)))
\]

of a MC element $F'$ in $\text{map}(A, B)$ we set

\[G([F']) := G(F)\,.
\]

$^6$For several other justifications of this definition, we refer the reader to paper [8] by V. Dotsenko and N. Poncin.
where $F$ is the $\infty$-morphism from $A$ to $B$ corresponding to the MC element $F'$.

Due to Proposition 4.6, the right hand side of (4.36) does not depend on the choice of
the MC element $F'$ in its isomorphism class $[F']$.

It is clear that, this way, we get a functor

$$\mathcal{G}' : \pi_0(\text{HoAlg}_{\Delta}^A) \to \mathcal{D}$$

satisfying $\mathcal{G}' \circ \mathfrak{f} = \mathcal{G}$.

It is also clear that such a functor $\mathcal{G}'$ is unique and the proof of Theorem 4.1 is complete.
We can use the above results to obtain the following converse to Cor. 4.5. Together they
give a recognition principal for $\infty$ quasi-isomorphisms, in analogy with the characterization
of weak equivalences via function complexes in a simplicial model category.

**Corollary 4.8** Let $F : A \to B$ be an $\infty$-morphism between Cobar($\mathcal{C}$) algebras. If for all
Cobar($\mathcal{C}$) algebras $W$, the $\mathfrak{S}\text{Lie}_{\infty}$-morphism

$$U_{WAB} : \text{map}(W, A) \to \text{map}(W, B)$$

induces a homotopy equivalence of simplicial sets

$$\mathcal{MC}_\bullet(\text{map}(W, A)) \xrightarrow{\sim} \mathcal{MC}_\bullet(\text{map}(W, B)),$$

then $F$ is an $\infty$-quasi-isomorphism.

Proof. Suppose, for all $W$ the morphism

$$U_{WAB} : \text{map}(W, A) \to \text{map}(W, B),$$

sending a MC element $G'$ in $\text{map}(W, A)$ to the MC element $(F \circ G)'$ in $\text{map}(W, B)$ induces
a homotopy equivalence of simplicial sets

$$\mathcal{MC}_\bullet(\text{map}(W, A)) \xrightarrow{\sim} \mathcal{MC}_\bullet(\text{map}(W, B)).$$

Then, in particular, we have an isomorphism

$$\pi_0\left(\mathcal{MC}_\bullet(\text{map}(B, A))\right) \cong \pi_0\left(\mathcal{MC}_\bullet(\text{map}(B, B))\right),$$

which implies that there exists an $\infty$-morphism $G : B \to A$ such that $(F \circ G)'$ and id$_B$ are
homotopic, in the sense of Def. 4.7 As in the proof of Prop. 4.6 this gives a commutative
diagram of Cobar($\mathcal{C}$) algebras:
By taking the linear terms of the $\infty$-morphisms, this diagram, in turn, gives a diagram of cochain complexes

![Diagram](image)

where $h$ and $f \circ g$ are the linear terms $p_B \otimes k[t, dt] \circ H|_B$, and $p_B \circ F \circ G|_B$, respectively. Let $I : B \otimes k[t, dt] \to B$ denote “integration over the fiber”, i.e. the degree $-1$ linear map

$$I\left(b \otimes q(t) + \tilde{b} \otimes \tilde{q}(t) dt\right) = (-1)^{|b|} \int_0^1 \tilde{q}(t) dt.$$

Then one can show that the map $s : B \to B$ defined as

$$s := I \circ h$$

is a chain homotopy

$$\text{id}_B - f \circ g = \partial_B s + s \partial_B.$$

Hence, the map $f$ is surjective on cohomology.

Now consider the $\mathfrak{S}\text{Lie}_\infty$-morphism

$$U_{ABB} : \text{map}(B, B) \to \text{map}(A, B)$$

which sends a MC element $K'$ to $(K \circ F)'$. This gives a map between sets

$$\pi_0\left(\mathcal{MC}_*(\text{map}(B, B))\right) \to \pi_0\left(\mathcal{MC}_*(\text{map}(A, B))\right),$$

which is not necessarily a bijection. But since $(F \circ G)'$ is homotopic to $\text{id}_B$, the above map gives $[F'] = [(F \circ G \circ F)']$. By assumption, the $\mathfrak{S}\text{Lie}_\infty$-morphism

$$U_{AAB} : \text{map}(A, A) \to \text{map}(A, B)$$

induces a bijection

$$\pi_0\left(\mathcal{MC}_*(\text{map}(A, A))\right) \cong \pi_0\left(\mathcal{MC}_*(\text{map}(A, B))\right),$$

sending $[\text{id}_A]$ to $[F']$, and $[(G \circ F)']$ to $[(F \circ G \circ F)'] = [F']$. Hence, we see that $\text{id}_A$ is homotopic to $(G \circ F)'$. We produce a chain homotopy $s : A \to A$ using the same construction as before, which shows that $f$ is also injective on cohomology. Hence, $F : A \to B$ is an $\infty$-quasi-isomorphism.

**Remark 4.9** A modification of the proof of Corollary 4.8 shows that $F : A \to B$ is also an $\infty$ quasi-isomorphism if the map $U_{ABW} : \text{map}(B, W) \to \text{map}(A, W)$ is a homotopy equivalence for all Cobar($C$) algebras $W$. 

28
5 The Homotopy Transfer Theorem is a simple consequence of the Goldman-Millson theorem

In this section we give an elegant proof of the Homotopy Transfer Theorem for Cobar($\mathcal{C}$)-algebras. This proof is based on a construction from [7] and a version of the Goldman-Millson theorem from [2].

Let us consider a dg cooperad $\mathcal{C}$ for which $\mathcal{C}_0$ carries ascending filtration (2.1) satisfying condition (2.3) and let $A$, $B$ be cochain complexes.

In [7, Section 3.1], we equipped the cochain complex

$$Cyl(\mathcal{C}, A, B) := s^{-1} \text{Hom}(\mathcal{C}_0(A), A) \oplus \text{Hom}(\mathcal{C}(A), B) \oplus s^{-1} \text{Hom}(\mathcal{C}_0(B), B)$$

(5.1)

with a natural $\mathcal{G}\text{Lie}_\infty$-algebra structure.

According to [7, Section 3.1], MC elements of $Cyl(\mathcal{C}, A, B)$ are triples:

- a Cobar($\mathcal{C}$)-algebra structure on $A$,
- a Cobar($\mathcal{C}$)-algebra structure on $B$, and
- an $\infty$-morphism from $A$ to $B$.

In particular, any chain map $\varphi : A \to B$ gives a MC element $Q_\varphi$ corresponding to the zero Cobar($\mathcal{C}$)-algebra structures on $A$, $B$, and the strict $\infty$-morphism from $A$ to $B$.

Let us twist the $\mathcal{G}\text{Lie}_\infty$-algebra structure on (5.1), and denote the new $\mathcal{G}\text{Lie}_\infty$-algebra by

$$Cyl(\mathcal{C}, A, B)^{Q_\varphi}.$$  (5.2)

It is not hard to see that the graded subspace

$$Cyl_0(\mathcal{C}, A, B)^{Q_\varphi} := s^{-1} \text{Hom}(\mathcal{C}_0(A), A) \oplus \text{Hom}(\mathcal{C}_0(A), B) \oplus s^{-1} \text{Hom}(\mathcal{C}_0(B), B)$$

(5.3)

is a $\mathcal{G}\text{Lie}_\infty$-subalgebra of (5.2) and filtration (2.1) on $\mathcal{C}_0$ allows us to equip (5.3) with a natural complete descending filtration $\mathcal{F}_\cdot Cyl_0(\mathcal{C}, A, B)^{Q_\varphi}$ (see Remark 2 in [7, Section 3.2]) such that

$$Cyl_0(\mathcal{C}, A, B)^{Q_\varphi} = \mathcal{F}_1 Cyl_0(\mathcal{C}, A, B)^{Q_\varphi}.$$  (5.4)

In other words, (5.3) is a filtered $\mathcal{G}\text{Lie}_\infty$-algebra.

Furthermore, according to [7, Section 3.2], MC elements of (5.3) are in bijection with triples:

- a Cobar($\mathcal{C}$)-algebra structure on $A$,
- a Cobar($\mathcal{C}$)-algebra structure on $B$, and
- an $\infty$-morphism $F$ from $A$ to $B$ whose linear term is $\varphi$.

The Homotopy Transfer Theorem can be now formulated as follows:

---

*In [7], we actually introduce an $L_\infty$-structure on the suspension of (5.1). But the latter is, of course, equivalent to introducing a $\mathcal{G}\text{Lie}_\infty$-algebra structure on (5.1).*
**Theorem 5.1** Let $B$ be a Cobar($C$)-algebra, $A$ be a cochain complex and $\varphi : A \to B$ be a quasi-isomorphism of cochain complexes. Then there exists a Cobar($C$)-algebra structure $Q_A$ on $A$, a Cobar($C$)-algebra structure $Q_B$ on $B$ (which is homotopy equivalent to the original one) and an $\infty$-morphism $F$ from $(A, Q_A)$ to $(B, Q_B)$ whose linear term is $\varphi$. If $(\tilde{Q}_A, \tilde{Q}_B, \tilde{F})$ is another triple satisfying the above properties then the MC elements corresponding to $(Q_A, Q_B, F)$ and $(\tilde{Q}_A, \tilde{Q}_B, \tilde{F})$ are isomorphic in $\mathcal{MC}(\text{Cyl}(C, A, B)^{Q_{\varphi}})$.

Proof. Let us identify the graded vector space of the convolution Lie algebra $\text{Conv}(C_{\circ}, \text{End}_B)$ with $\text{Hom}(C_{\circ}(B), B)$ and consider $s^{-1} \text{Hom}(C_{\circ}(B), B)$ with the corresponding $\mathcal{S}\text{Lie}_{\infty}$-algebra structure.

Due to [7, Proposition 3.2], the canonical projection $\pi_B : \text{Cyl}(C, A, B)^{Q_{\varphi}} \to s^{-1} \text{Hom}(C_{\circ}(B), B)$ (5.5) is a strict quasi-isomorphism of $\mathcal{S}\text{Lie}_{\infty}$-algebras which is obviously compatible with the descending filtrations coming from (2.1).

Using the same arguments, as in the proof of [7, Proposition 3.2], it is easy to see that $\pi_B : \mathcal{F}_m \text{Cyl}(C, A, B)^{Q_{\varphi}} \to s^{-1} \mathcal{F}_m \text{Hom}(C_{\circ}(B), B)$ is a quasi-isomorphism of cochain complexes for every $m \geq 1$.

Therefore, applying Theorem 1.1 from [2] to (5.5), we conclude that $\pi_B$ induces a weak equivalence of simplicial sets $\mathcal{MC}(\text{Cyl}(C, A, B)^{Q_{\varphi}}) \to \mathcal{MC}(s^{-1} \text{Hom}(C_{\circ}(B), B))$ and hence a bijection $\pi_0 \left( \mathcal{MC}(\text{Cyl}(C, A, B)^{Q_{\varphi}}) \right) \to \pi_0 \left( \mathcal{MC}(s^{-1} \text{Hom}(C_{\circ}(B), B)) \right)$.

Thus Theorem 5.1 is a simple consequence of the fact that MC elements of the $\mathcal{S}\text{Lie}_{\infty}$-algebra $s^{-1} \text{Hom}(C_{\circ}(B), B)$ (5.6) are in bijection with Cobar($C$)-algebra structures on $B$. Moreover, homotopy equivalent Cobar($C$)-algebra structures on $B$ correspond precisely to isomorphic MC elements of (5.6).

□

**Remark 5.2** In the usual version of the Homotopy Transfer Theorem (see [18, Theorem 10.3.2]) one constructs a Cobar($C$)-algebra structure on $A$ and an $\infty$ quasi-isomorphism $F$ from $A$ to $B$ with the original Cobar($C$)-algebra structure, while in the above theorem, $F$ lands in $(B, Q_B)$ where $Q_B$ is only homotopy equivalent to the original one. On the other hand, given an isomorphism connecting two MC elements $\tilde{Q}_B$ and $Q_B$ in $\mathcal{MC}(s^{-1} \text{Hom}(C_{\circ}(B), B))$, it is easy to construct an $\infty$-morphism $G$ from $(B, Q_B)$ to $(B, \tilde{Q}_B)$ whose linear term is $\text{id}_B$. So the “usual” version of the Homotopy Transfer Theorem follows from Theorem 5.1.
A Proof of Proposition 2.6

Although very similar claims to Proposition 2.6 appeared in the literature (see, for example, [1], [11], [15], [21]) we still decided to give its proof for convenience of the reader.

Let us denote by \( Q_A' \) the composition

\[
p_A \circ Q_A : \mathcal{C}(A) \to A.
\]

To prove that multi-bracket (2.12) is symmetric in its arguments, we let

\[
\Delta_m(v) = \sum_{\alpha} (\gamma_{\alpha}; v_{\sigma(1)}^\alpha, v_{\sigma(2)}^\alpha, \ldots, v_{\sigma(m)}^\alpha) \quad (A.1)
\]

and recall that \( \Delta_m(v) \) lands in \( S_m \)-invariants of \( \mathcal{C}(m) \otimes V^\otimes m \). In other words, for every \( \sigma \in S_m \) we have

\[
\sum_{\alpha} (\sigma^{-1}(\gamma_{\alpha}); v_{\sigma(1)}^\alpha, v_{\sigma(2)}^\alpha, \ldots, v_{\sigma(m)}^\alpha) = \sum_{\alpha} (\gamma_{\alpha}; v_{1}^\alpha, v_{2}^\alpha, \ldots, v_{m}^\alpha). \quad (A.2)
\]

Therefore, for every \( \sigma \in S_m \), we have

\[
\{f_{\sigma(1)}, \ldots, f_{\sigma(m)}\}(v) = \sum_{\alpha} Q_A' \circ (1 \otimes f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)}) (\sigma^{-1}(\gamma_{\alpha}); v_{\sigma(1)}^\alpha, v_{\sigma(2)}^\alpha, \ldots, v_{\sigma(m)}^\alpha) = \sum_{\alpha} \pm Q_A' \sigma^{-1}(\gamma_{\alpha}); f_{\sigma(1)}(v_{\sigma(1)}^\alpha), f_{\sigma(2)}(v_{\sigma(2)}^\alpha), \ldots, f_{\sigma(m)}(v_{\sigma(m)}^\alpha) \}
\]

Thus, since \( Q_A \) is compatible with the action of the symmetric group,

\[
\{f_{\sigma(1)}, \ldots, f_{\sigma(m)}\}(v) = \sum_{\alpha} \pm Q_A' (\sigma^{-1}(\gamma_{\alpha}); f_{\sigma(1)}(v_{\sigma(1)}^\alpha), f_{\sigma(2)}(v_{\sigma(2)}^\alpha), \ldots, f_{\sigma(m)}(v_{\sigma(m)}^\alpha)) = \sum_{\alpha} \pm Q_A'(\gamma_{\alpha}; f_1(v_1^\alpha), f_2(v_2^\alpha), \ldots, f_m(v_m^\alpha)) = \sum_{\alpha} \pm Q_A' (1 \otimes f_1 \otimes \cdots \otimes f_m)(\gamma_{\alpha}; v_1^\alpha, v_2^\alpha, \ldots, v_m^\alpha) = \pm \{f_1, \ldots, f_m\}(v).
\]

So multi-bracket (2.12) is indeed symmetric\(^8\) in its arguments.

Let us now prove that the operation

\[
f \mapsto \{f\} \quad (A.3)
\]

is a differential on \( \text{Hom}(V, A) \), i.e. \( \{\{f\}\} = 0 \) for every \( f \in \text{Hom}(V, A) \).

\(^8\)In the above calculations, the sign factors can be easily deduced from the Koszul rule of signs.
Indeed, using the identities $\partial_A^2 = 0, \partial_V^2 = 0$ and the compatibility of $\Delta_1$ with the differentials on $V$ and $C(1) \otimes V$, we get

\[
\{\{ f \}\}(v) = \partial_A \{ f \}(v) - (-1)^{|f|+1} \{ \partial_V v \} + Q'_A(1 \otimes \{ f \}(\Delta_1(v))) = \\
\quad \partial_A \circ Q'_A \circ (1 \otimes f) \circ \Delta_1(v) + (-1)^{|f|} Q'_A \circ (1 \otimes f) \left( (\partial_C + \partial_V) \Delta_1(v) \right) \\
\quad + Q'_A \circ (1 \otimes (\partial_A \circ f)) \left( \Delta_1(v) \right) - (-1)^{|f|} Q'_A \circ (1 \otimes (f \circ \partial_V)) \left( \Delta_1(v) \right) \\
\quad + Q'_A \circ (1 \otimes Q'_A) \circ (1 \otimes 1 \circ f) \left( (1 \otimes \Delta_1) \circ \Delta_1(v) \right) \tag{A.4}
\]

for every $v \in V$.

Using the axioms of a coalgebra over a cooperad, we rewrite the expression $(1 \otimes \Delta_1) \circ \Delta_1(v)$ in (A.4) as follows

\[
(1 \otimes \Delta_1) \circ \Delta_1(v) = (\Delta^C \otimes 1) \circ \Delta_1(v),
\]

where $\Delta^C$ is the cooperadic comultiplication

\[
\Delta^C : C(1) \to C(1) \otimes C(1).
\]

Therefore the expression $\{\{ f \}\}(v)$ can be rewritten as

\[
\{\{ f \}\}(v) = (\partial_A \circ Q'_A + Q'_A \circ (\partial_C + \partial_A) + Q'_A \circ Q_A) \circ (1 \otimes f) \circ \Delta_1(v). \tag{A.6}
\]

Thus the identity $\{\{ f \}\} = 0$ is a consequence of the MC equation for $Q'_A$.

Our goal now is to prove the relation

\[
\sum_{p=1}^{m} \sum_{\sigma \in \text{Sh}_{p,m-p}} (-1)^{\varepsilon(\sigma; f_1, \ldots, f_m)} \{\{f_{\sigma(1)}, \ldots, f_{\sigma(p)}, f_{\sigma(p+1)}, \ldots, f_{\sigma(m)}\}(v) = 0 \tag{A.7}
\]

for every $m \geq 2$ and $v \in V$, where the sign factor $(-1)^{\varepsilon(\sigma; f_1, \ldots, f_m)}$ is defined in (1.1).

In our calculations below, we often put $\pm$ instead of the precise sign factor. These sign factors can be easily deduced from the Koszul rule of signs.

Unfolding (A.7) we get

\[
\sum_{p=1}^{m} \sum_{\sigma \in \text{Sh}_{p,m-p}} (-1)^{\varepsilon(\sigma; f_1, \ldots, f_m)} \{\{f_{\sigma(1)}, \ldots, f_{\sigma(p)}, f_{\sigma(p+1)}, \ldots, f_{\sigma(m)}\}(v) = \\
\quad \partial_A(\{f_1, \ldots, f_m\}(v)) + (-1)^{|f_1|+\cdots+|f_m|} \{f_1, \ldots, f_m\}(\partial_V(v)) \\
\quad + \sum_{i=1}^{m} (-1)^{|f_1|+\cdots+|f_{i-1}|} \{f_1, \ldots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V), \ldots, f_m\}(v) \\
\quad + \sum_{1 \leq p \leq m} \sum_{\sigma \in \text{Sh}_{p,m-p}} (-1)^{\varepsilon(\sigma; f_1, \ldots, f_m)} Q'_A(1; Q'_A((1; f_{\sigma(1)}, \ldots, f_{\sigma(p)})(\Delta_p(v)), f_{\sigma(p+1)}, \ldots, f_{\sigma(m)}(\Delta_{m-p+1}(v)))) \tag{A.8}
\]
Expanding the expression $\partial_A(\{f_1, \ldots, f_m\}(v))$, we get

\[
\partial_A(\{f_1, \ldots, f_m\}(v)) = \partial_A Q'_A(1; f_1, \ldots, f_m(\Delta_m(v)))
\]

\[
= \sum_{\alpha} \partial_A Q'_A((1; f_1, \ldots, f_m(\gamma_\alpha; v_1^\alpha, v_2^\alpha, \ldots, v_m^\alpha))
\]

\[
= \sum_{\alpha} \pm \partial_A Q'_A(\gamma_\alpha; f_1(v_1^\alpha), \ldots, f_m(v_m^\alpha)).
\]

Using

\[
\Delta_m(\partial_V(v)) = \sum_{\alpha} (\partial_c(\gamma_\alpha); v_1^\alpha, v_2^\alpha, \ldots, v_m^\alpha) + \sum_{1 \leq i \leq m} \pm (\gamma_\alpha; v_1^\alpha, \ldots, \partial_V(v_i^\alpha), \ldots, v_m^\alpha),
\]

we expand $\{f_1, \ldots, f_m\}(\partial_V(v))$ obtaining

\[
\{f_1, \ldots, f_m\}(\partial_V(v)) = Q'_A(1; f_1, \ldots, f_m(\Delta_m(\partial_V(v))))
\]

\[
= \sum_{\alpha} Q'_A((1; f_1, \ldots, f_m(\partial_c(\gamma_\alpha); v_1^\alpha, v_2^\alpha, \ldots, v_m^\alpha))
\]

\[
+ \sum_{1 \leq i \leq m} \pm Q'_A((1; f_1, \ldots, f_m(\gamma_\alpha; v_1^\alpha, \ldots, \partial_V(v_i^\alpha), \ldots, v_m^\alpha))
\]

\[
= \sum_{\alpha} \pm Q'_A(\partial_c(\gamma_\alpha); f_1(v_1^\alpha), \ldots, f_m(v_m^\alpha))
\]

\[
+ \sum_{1 \leq i \leq m} \pm Q'_A(\gamma_\alpha; f_1(v_1^\alpha), \ldots, f_i(\partial_V(v_i^\alpha)), \ldots, f_m(v_m^\alpha)).
\]

Expanding the sum

\[
\sum_{i=1}^m (-1)^{|f_1|+\cdots+|f_i-1|} \{f_1, \ldots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V), \ldots, f_m\}(v)
\]

we obtain

\[
\sum_{i=1}^m (-1)^{|f_1|+\cdots+|f_i-1|} \{f_1, \ldots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V), \ldots, f_m\}(v) =
\]

\[
= \sum_{1 \leq i \leq m} \pm Q'_A(1; f_1, \ldots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V), \ldots, f_m)(\gamma_\alpha; v_1^\alpha, v_2^\alpha, \ldots, v_m^\alpha)
\]

\[
= \sum_{\alpha} \pm Q'_A(\gamma_\alpha; f_1(v_1^\alpha), \ldots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V)(v_i^\alpha), \ldots, f_m(v_m^\alpha))
\]

\[
= \sum_{\alpha} \pm Q'_A(\gamma_\alpha; f_1(v_1^\alpha), \ldots, \partial_A(f_i(v_i^\alpha)), \ldots, f_m(v_m^\alpha))
\]

\[
- \sum_{\alpha} \pm Q'_A(\gamma_\alpha; f_1(v_1^\alpha), \ldots, f_i(\partial_V v_i^\alpha), \ldots, f_m(v_m^\alpha)). \quad (A.9)
\]
The last sum in the R.H.S. of (A.8) is expanded as follows:

\[ \sum_{1 \leq p \leq m} (-1)^{\varepsilon(f_1, \ldots, f_m)} Q'(1; Q'((1; f_{\sigma(1)}), \ldots, f_{\sigma(p)})(\Delta_p(-)), f_{\sigma(p+1)}, \ldots, f_{\sigma(m)}(\Delta_{m-p+1}(v))) \]

\[ = \sum_{1 \leq p \leq m} \sigma \in Sh_{p,m-p} - p \cdot \varepsilon(\sigma; f_1, \ldots, f_m) Q'(1 \otimes Q'(1 \otimes 1 \otimes (m-p)))(1; (1; f_{\sigma(1)}), \ldots, f_{\sigma(p)}), f_{\sigma(p+1)}, \ldots, f_{\sigma(m)}) \]

\[ = \sum_{1 \leq p \leq m} \sigma \in Sh_{p,m-p} - p \cdot \varepsilon(\sigma; f_1, \ldots, f_m) Q'(1 \otimes 1 \otimes 1 \otimes (m-p)) \Delta_{m-p+1}(v). \] (A.10)

Using the axioms of \(C\)-coalgebra structure on \(V\), we can rewrite the term \((1 \otimes \Delta_p \otimes 1 \otimes (m-p)) \circ \Delta_{m-p+1}(v)\) as follows:

\[ (1 \otimes \Delta_p \otimes 1 \otimes (m-p)) \circ \Delta_{m-p+1}(v) = (\Delta^C_{t_{m,p}} \otimes 1 \otimes m) \circ \Delta_m(v), \] (A.11)

where \(t_{m,p}\) is the labeled planar tree depicted in Figure 3 and let \(\gamma_{\alpha,\beta,1}\) and \(\gamma_{\alpha,\beta,2}\) be the tensor factors in

\[ \Delta^C_{\sigma(t_{m,p})}(\gamma_\alpha) = \sum_{\beta} \gamma_{\alpha,\beta,1,1} \otimes \gamma_{\alpha,\beta,2}. \] (A.12)

where \(\sigma(t_{m,p})\) is the tree corresponding to the \((p, m - p)\)-shuffle \(\sigma\).

Using the axioms of the cooperadic comultiplication \(\Delta^C\), we get

\[ \text{Sum } (A.10) = \]

\[ \sum_{1 \leq p \leq m} \sigma \in Sh_{p,m-p} \pm Q'(1 \otimes Q'(1 \otimes 1 \otimes (m-p))) \]

\[ \left( \gamma_{\alpha,\beta,1}; \gamma_{\alpha,\beta,2}; f_{\sigma(1)}(v^\alpha_{\sigma(1)}), \ldots, f_{\sigma(p)}(v^\alpha_{\sigma(p)}), f_{\sigma(p+1)}(v^\alpha_{\sigma(p+1)}), \ldots, f_{\sigma(m)}(v^\alpha_{\sigma(m)}) \right), \]

which can be rewritten in terms of the bracket on the convolution Lie algebra \(\text{Conv}(C, \text{End}_A)\) (Prop. 4.1, [3]). Namely,

\[ \text{Sum } (A.10) = \]

\[ \frac{1}{2} \sum_{\alpha} [Q'_A, Q'_A](\gamma_\alpha; f_1(v^\alpha_1), \ldots, f_m(v^\alpha_m)) \], (A.14)
where we tacitly identify the composition $Q'_{A} = p_{A} \circ Q_{A}$ with the corresponding element in
\[ \text{Conv}(\mathcal{C}, \text{End}_{A}) = \prod_{n \geq 1} \text{Hom}_{S_{n}}(\mathcal{C}(n), \text{End}_{A}(n)). \] (A.15)

Collecting the expanded terms of the R.H.S. of (A.8) we obtain
\[
\sum_{\alpha} \pm \partial_{A} Q'_{A}(\gamma_{\alpha}; f_{1}(v_{1}^{\alpha}), \ldots, f_{m}(v_{m}^{\alpha})) + \sum_{\alpha} \pm Q'_{A}(\partial_{c}(\gamma_{\alpha}); f_{1}(v_{1}^{\alpha}), \ldots, f_{m}(v_{m}^{\alpha}))
\] (A.16)
\[
+ \sum_{1 \leq i \leq m} \pm Q'_{A}(\gamma_{\alpha}; f_{1}(v_{1}^{\alpha}), \ldots, f_{i}(\partial_{V}(v_{i}^{\alpha})), \ldots, f_{m}(v_{m}^{\alpha}))
\] (A.17)
\[
- \sum_{1 \leq i \leq m} \pm Q'_{A}(\gamma_{\alpha}; f_{1}(v_{1}^{\alpha}), \ldots, f_{i}(\partial_{V}v_{i}^{\alpha}), \ldots, f_{m}(v_{m}^{\alpha}))
\] (A.18)
\[
\sum_{1 \leq i \leq m} \pm Q'_{A}(\gamma_{\alpha}; f_{1}(v_{1}^{\alpha}), \ldots, \partial_{A}(f_{i}(v_{i}^{\alpha})), \ldots, f_{m}(v_{m}^{\alpha})) + \frac{1}{2} \sum_{\alpha} [Q'_{A}, Q'_{A}](\gamma_{\alpha}; f_{1}(v_{1}^{\alpha}), \ldots, f_{m}(v_{m}^{\alpha}))
\] (A.19)

Canceling terms (A.16) and (A.17), we see that the right hand side of (A.8) can be rewritten as
\[
\text{The R.H.S. of (A.8) = } \left( \partial_{A} \circ Q'_{A} + Q'_{A} \circ (\partial_{c} + \partial_{A}) + \frac{1}{2}[Q'_{A}, Q'_{A}] \right) (1; f_{1}, \ldots, f_{m})(\Delta_{m}(v)),
\] (A.20)
where, as above, we identify $Q'_{A}$ with the corresponding element in (A.15).

Thus desired equations (A.7) are satisfied due to the fact that $Q'_{A}$ is a MC element of dg Lie algebra (A.15).

We conclude this proof by a comment about the sign factors.

All sign factors in the above computations are subject to the usual Koszul rule. For example, to show explicitly that terms (A.16) and (A.17) cancel each other, we need to verify that the corresponding contributions from the term
\[
(-1)^{|f_{1}|+\cdots+|f_{m}|} \{f_{1}, \ldots, f_{m}\}(\partial_{V}(v))
\] (A.21)
matches with the contributions from
\[
(-1)^{|f_{i}|+\cdots+|f_{i}|} \{f_{1}, \ldots, f_{i} \circ \partial_{V}, \ldots, f_{m}\}(v).
\] (A.22)

It is easy to see that the contribution
\[
Q'_{A}(\gamma_{\alpha}; f_{1}(v_{1}^{\alpha}), \ldots, f_{i} \circ \partial_{V}(v_{i}^{\alpha}), \ldots f_{m}(v_{m}^{\alpha}))
\] (A.23)
from (A.19) and from (A.20) has the same sign factor
\[
(-1)^{\varepsilon + \varepsilon'},
\] (A.24)
where
\[
\varepsilon = |\gamma_{\alpha}|(1 + |f_{1}| + \cdots + |f_{m}|) + |v_{1}^{\alpha}| + \cdots + |v_{i-1}^{\alpha}| + |f_{1}| + \cdots + |f_{i}|
\] (A.25)
and
\[
\varepsilon' = |v_{i}^{\alpha}|(|f_{2}| + \cdots + |f_{m}|) + \cdots + |v_{m-1}^{\alpha}| |f_{m}|.
\] (A.26)

Proposition 2.6 is proved. \qed

35
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Department of Mathematics, Temple University,
Wachman Hall RM. 638
1805 N. Broad St.,
Philadelphia PA, 19122 USA
E-mail addresses: vald@temple.edu, hoffnung@temple.edu

Mathematics Institute
Georg-August Universität Göttingen,
Bunsenstrasse 3-5, D-37073 Göttingen, Germany
E-mail address: crogers@uni-math.gwdg.de