AN EXPLICIT EXPRESSION FOR THE MINIMAL POLYNOMIAL OF THE KRONECKER PRODUCT OF MATRICES. EXPLICIT FORMULAS FOR MATRIX LOGARITHM AND MATRIX EXPONENTIAL

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Abstract. Using $\mathcal{P}$-canonical forms of matrices, we derive the minimal polynomial of the Kronecker product of a given family of matrices in terms of the minimal polynomials of these matrices. This allows us to prove that the product $\prod_{i=1}^{m} L(P_i)$, $L(P_i)$ is the set of linear recurrence sequences over a field $F$ with characteristic polynomial $P_i$, is equal to $L(P)$ where $P$ is the minimal polynomial of the Kronecker product of the companion matrices of $P_i$, $1 \leq i \leq m$. Also, we show how we deduce from the $\mathcal{P}$-canonical form of an arbitrary complex matrix $A$, the $\mathcal{P}$-canonical form of the matrix function $e^{tA}$ and a logarithm of $A$.

1. Introduction

This paper is devoted to some applications of the main results of the paper [5] and the $\mathcal{P}$-canonical forms of matrices presented in [4]. The paper is organized as follows. In section 2, we show that an important application of the $\mathcal{P}$-canonical forms of matrices is to derive the minimal polynomial of the Kronecker product of a given family of matrices in terms of the minimal polynomials of these matrices. This result allows us to prove that, if $P_1, \ldots, P_m$ are monic polynomials over a field $F$, then the product $\prod_{i=1}^{m} L(P_i)$ is equal to $L(P)$ where $P$ is the minimal polynomial of the Kronecker product of the companion matrices of $P_i$, $1 \leq i \leq m$. In section 3, we investigate the $\mathcal{P}$-canonical form of the matrix function $e^{tA}$ of an arbitrary complex matrix $A$ and, we show also how we can simply obtain the matrix logarithm $\log(A)$ and its $\mathcal{P}$-canonical form if $A$ is nonsingular. More precisely, we describe a

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relationship between these \( \mathcal{P} \)-canonical forms which is helpful for determining two of them if the other one is known. Finally, in section 4 some theoretical and numerical examples are presented to verify the theoretical results.

Through this paper, we use the following notations which are similar to those used in the papers [4, 5].

- \( F \) is an arbitrary field.
- \( J_s(\alpha) \) denotes the Jordan block of order \( s \) associated to \( \alpha \in F \).
- \([a_1, \ldots, a_n]_n\) denotes the semicirculant matrix whose first row is \((a_1, \ldots, a_n)\).
- The symbol \( \sim \) denotes the similarity of matrices. It is well known that if \( A \sim B \) and \( C \sim D \) then \( A \otimes B \sim C \otimes D \).
- For all nonzero element \( \lambda \) of \( F \), \( \lambda \) denotes the geometric sequence \((\lambda^k)_{k \geq 0}\).
- For all nonegative integer \( n \), \( 0_n \) denotes the sequence \((\delta_{n,k})_{k \geq 0}\) where \( \delta_{n,k} \) is the Kronecker symbol.
- For all nonegative integer \( i \), \( \Lambda_i \) denotes the sequence \((\binom{k}{i})_{k \geq 0}\).
- For all nonzero element \( \lambda \) of \( F \) and all positive integer \( s \), \( \langle \lambda \rangle_s \) denotes the subspace of \( \mathcal{C}_F \) spanned by \( \{\lambda\Lambda_0, \ldots, \lambda\Lambda_{s-1}\} \), where \( \mathcal{C}_F \) is the set of all linear recurrence sequences over \( F \).
- For all positive integer \( s \), \( \langle 0 \rangle_s \) denotes the subspace of \( \mathcal{C}_F \) spanned by \( \{0_0, \ldots, 0_{s-1}\} \).
- \( \langle \lambda \rangle_0 = \langle 0 \rangle_0 = 0 \) the zero subspace of \( \mathcal{C}_F \).
- \( \Gamma \) denotes the sequence \((0, 1, 2, \ldots)\).
- \( S^* = \{\lambda = (\lambda^k)_{k \geq 0} / \lambda \in F, \lambda \neq 0\} \) denotes the set of all nonzero geometric sequences.
- \( F_{S^*} \) denotes the \( F \)-vector spaces spanned by \( S^* \).
- \( T = \{\Lambda_n / n \in \mathbb{N}\} \).
- \( \mathcal{H} = (\Gamma_n)_{n \geq 0} \).

Let \( A \) be a square matrix over a field \( F \) and let \( \mathcal{A} \) denote the sequence \((A^k)_{k \geq 0}\). Then there exist matrices \( \mathcal{V}_0, \ldots, \mathcal{V}_n \) with coefficients in \( F \) and matrices \( \mathcal{A}_0, \ldots, \mathcal{A}_l \) with coefficients in \( F_{S^*} \) such that

\[
\mathcal{A} = N(A) + \mathcal{A}_0\Lambda_0 + \cdots + \mathcal{A}_l\Lambda_l
\]

If the characteristic of \( F \) is 0, then there exist matrices \( \mathcal{A}'_0, \ldots, \mathcal{A}'_l \) with coefficients in \( F_{S^*} \) such that

\[
\mathcal{A} = N(A) + \mathcal{A}'_0\Gamma^0 + \cdots + \mathcal{A}'_l\Gamma^l
\]

where

\[
N(A) = \mathcal{V}_00_0 + \cdots + \mathcal{V}_n0_n.
\]
The matrices $V_0, \ldots, V_n, A_0, \ldots, A_l, A'_0, \ldots, A'_l$ are uniquely determined by $A$.

We say that

- $N(A)$ is the non-geometric part of $A$.
- $A - N(A)$ is the geometric part of $A$.
- $A = N(A) + A_0 \Lambda_0 + \cdots + A_l \Lambda_l$ is the $P$-canonical form (which we abbreviate by the $P$-cf) of $A$ relative to $(S^*, T)$.
- $A = N(A) + A'_0 \Gamma^0 + \cdots + A'_l \Gamma^l$ is the $P$-cf of $A$ relative to $(S^*, H)$.

2. Explicit expression of the minimal polynomial of the Kronecker product of matrices

In this section we obtain the minimal polynomial of the Kronecker product of the companion matrices of any finite family of monic polynomials $P_i$, $1 \leq i \leq m$ over a field $F$. As a consequence we obtain the monic polynomial $P$ such that $\prod_{i=1}^{m} L(P_i) = L(P)$, where $L(P)$ denotes the vector space of all sequences over $F$ generated by the linear recurrence corresponding to the polynomial $P$.

The following definition is from [2] and it is equivalent to ours proposed in [5].

**Definition 2.1.** For two positive integers $s$ and $t$, let $s \wedge t$ be the maximum value of $i + j + 1$ such that $\binom{i+j}{i} \neq 0$ (in $F$) where $0 \leq i \leq s - 1$ and $0 \leq j \leq t - 1$.

Let us recall two lemmas and a theorem proved in [5] that will be used in the proof of Theorem [2.6]

**Lemma 2.2.** Let $\lambda \in F$ and let $\wedge$ be the noncommutative binary operation on $\mathbb{N}$ defined by $t \wedge s = \min(t, s) \delta_{0, \lambda} s + t \delta_{0, \lambda} \delta_{c, t}$ where $\delta_{c,t} = 1 - \delta_{e,t}$ and $\delta_{e,t}$ is the Kronecker symbol. Then for all $s,t \in \mathbb{N}$ and all $\lambda \in F$ we have $\langle 0 \rangle_t^i(\lambda)s = \langle 0 \rangle_t^i(\lambda)\wedge s$.

**Theorem 2.3.** Let $i, j \in \mathbb{N}$, $\lambda, \mu \in F^* = F \setminus \{0\}$ and $\eta \in F$. Put $0 \wedge s = s \wedge 0 = 0$ for all $s \in \mathbb{N}$. Then

(i) $\dim(\langle \eta \rangle_i) = i$.
(ii) $\langle 1 \rangle_i^i(1) = \langle 1 \rangle_{i \wedge j}$.
(iii) $\langle \lambda \rangle_i^i(\mu) = \langle \lambda \mu \rangle_{i \wedge j}$.

**Lemma 2.4.** Let $i, j$ be positive integers and $s, s', t, t'$ non-negative integers.

1. If $F$ is a field with characteristic 0, then we have $i \wedge j = i + j - 1$.
2. If $s' \leq s$ and $t' \leq t$, then $s' \wedge t' \leq s \wedge t$ and $s' \wedge t' \leq s' \wedge t'$ for all $\lambda \in F$. 

The following lemma is proved in [3].

**Lemma 2.5.** Let $A$ be a square matrix over $F$. Then the minimal polynomial of $A$ is $M_A(X) = X^{t_0} \prod_{j=1}^{p} (X-\lambda_j)^{t_j}$, where $t_0$ and $t_j, j \neq 0$, are respectively the greatest integers such that $0_{t_0-1}$ and $\lambda_j \Lambda_{t_j-1}$ appear in the $P$-cf of $A$.

**Theorem 2.6.** Let $\alpha$ and $\beta$ be non zero elements of $F$. Let $A = \bigoplus_{i=1}^{m} J_{s_i}(\alpha)$, $B = \bigoplus_{i=1}^{n} J_{t_i}(\beta)$ and $C = \bigoplus_{i=1}^{m} J_{s_i}(0)$. Then

1. The minimal polynomial of the matrix $J_{s_i}(1) \otimes J_{t_i}(1)$ is $(X - 1)^{s_i t_i}$.
2. The minimal polynomial of the matrix $J_{s_i}(\alpha) \otimes J_{t_i}(\beta)$ is $(X - \alpha \beta)^{s_i t_i}$.
3. The minimal polynomial of the nilpotent matrix $J_{s_i}(0) \otimes J_{t_i}(0)$ is $X^{s_i t_i}$.
4. The minimal polynomial of the matrix $A \otimes B$ is $(X - \alpha \beta)^{\delta (s_i \wedge t_j)}$, where $\delta (s_i \wedge t_j)$ are the indexes of $\alpha$ and $\beta$ as eigenvalues of $A$ and $B$, respectively.
5. The minimal polynomial of the matrix $C \otimes A$ is $X^{\max_i(s_i \wedge t_j)}$.

**Proof.**

1. Observe first that the geometric part of $J_{s_i}(1) \otimes J_{t_i}(1)$ is $(J_{s_i}(1)^k \otimes J_{t_i}(1)^k)_{k \geq 0}$. Moreover, it is well-known that

   $$(J_{s_i}(1)^k)_{k \geq 0} = [\Lambda_0, \ldots, \Lambda_{s_i-1}]_s$$

(see, e.g., formula (18) of [3]). Then

   $$(J_{s_i}(1)^k \otimes J_{t_i}(1)^k)_{k \geq 0} = [\Lambda_0, \ldots, \Lambda_{s_i-1}]_s \otimes [\Lambda_0, \ldots, \Lambda_{t_i-1}]_t$$

Since all elements of this last matrix lie in its first row which is

   $$[\Lambda_0, \Lambda_0, \ldots, \Lambda_0, \Lambda_0, \ldots, \Lambda_0, \Lambda_0, \ldots, \Lambda_{s_i-1}, \Lambda_{s_i-1}, \ldots, \Lambda_{s_i-1} \Lambda_0, \ldots, \Lambda_{s_i-1} \Lambda_0]$$

the greatest integer $m$ such that $\Lambda_{m-1}$ appear in the geometric part of $J_{s_i}(1) \otimes J_{t_i}(1)$ is exactly the dimension of the vector space $(1)_s (1)_t$. Therefore in view of Lemma 2.5 we deduce that the minimal polynomial of the matrix $J_{s_i}(1) \otimes J_{t_i}(1)$ is $(X - 1)^{s_i t_i}$.

2. The proof is similar to that of (2).

3. Since $J_{s_i}(1) \sim [1, \lambda, 0, \ldots, 0]_s$ for all $\lambda \in F$ such that $\lambda \neq 0$, it follows that, for all $s, t \in \mathbb{N}^*$ we have $J_{s_i}(\alpha) \otimes J_{t_i}(\beta) \sim \alpha \beta J_{s_i}(1) \otimes J_{t_i}(1)$, and then the minimal polynomial of $J_{s_i}(\alpha) \otimes J_{t_i}(\beta)$ is $(X - \alpha \beta)^{s_i t_i}$.

4. Since $A \otimes B \sim \bigoplus_{j} (J_{s_i}(\alpha) \otimes J_{t_i}(\beta))$, it follows that the minimal polynomial of the matrix $A \otimes B$ is $(X - \alpha \beta)^{\max_i(s_i \wedge t_j)} = (X - \alpha \beta)^{s_i t_j}$, in view of the second assertion of Lemma 2.4.

5. The proof is similar to that of (4)
To deal with the general case of any finite number of arbitrary square matrices, we need the following notations used in [5]. Let $m$ be a positive integer and let $\mathcal{R}$ be the equivalence relation on $\overline{F}^m$, $F$ designates the algebraic closure of the field $F$, defined by

$$(\lambda_1, \ldots, \lambda_m) \mathcal{R} (\mu_1, \ldots, \mu_m) \text{ if and only if } \lambda_1 \cdots \lambda_m = \mu_1 \cdots \mu_m.$$ 

Let us partition $F^m$ into equivalence classes under the equivalence relation $\mathcal{R}$

$$F^m = \bigcup_{i \in T} \Omega_i$$

Let $\wedge_m$ denote the map defined as follows

$$\wedge_m : \mathbb{N}^m \rightarrow \mathbb{N}$$

$$(t_1, \ldots, t_m) \mapsto t_1 \wedge \cdots \wedge t_m$$

For any arbitrary monic polynomials $P_1, \ldots, P_m$, let us define the map $f = f_{P_1, \ldots, P_m}$ as follows

$$f : F^m \rightarrow \mathbb{N}^m$$

$$(\lambda_1, \ldots, \lambda_m) \mapsto (\lambda_1(P_1), \ldots, \lambda_m(P_m)),$$

where $\lambda(P_i)$ designates the multiplicity of $\lambda$ in $P_i$

let us use the following notations for simplicity

$$[\mu_1]_f \cdots [\mu_m]_f = [\mu_1]_{\mu_1(P_1)} \cdots [\mu_m]_{\mu_m(P_m)}$$

$$\overline{f} = \wedge_m \circ f$$

$$\overline{\Omega_i} = \mu_1 \cdots \mu_m \text{ if } (\mu_1, \ldots, \mu_m) \in \Omega_i$$

$$\Omega_i^f = \max_{(\mu_1, \ldots, \mu_m) \in \Omega_i} (\overline{f}(\mu_1, \ldots, \mu_m))$$

$$\Upsilon(P_1, \ldots, P_m) = \prod_{i \in T} (X - \overline{\Omega_i})^{\Omega_i^f}$$

Under the above notations and assumptions, we have the following theorem:

**Theorem 2.7.** Let $A_i, 1 \leq i \leq m$ be square matrices over $F$ with minimal polynomials $M_{A_i}(X) = X^{\iota_{A_i}(0)} Q_i$, and put $\Theta = \{ i/Q_i = 1 \}$. Then the minimal polynomial of the matrix $\otimes_{i=1}^m A_i$ is $X^\rho \Upsilon(Q_1, \ldots, Q_m)$, where

$$\rho = \begin{cases} \min \{ \iota_{A_i}(0) / i \in \Theta \} & \text{ if } \Theta \neq \emptyset \\ \max \{ \iota_{A_i}(0) / 1 \leq i \leq m \} & \text{ otherwise.} \end{cases}$$
Proof. There is no loss of generality to consider \( m = 2 \), as the general case follows by induction. Let

\[
J(0)_1 \oplus J(\alpha_1) \oplus \cdots \oplus J(\alpha_n)
\]

and

\[
J(0)_2 \oplus J(\beta_1) \oplus \cdots \oplus J(\beta_q)
\]

be respectively the Jordan canonical forms of \( A_1 \) and \( A_2 \) in \( \overline{F} \). Since

\[
A \otimes B \sim [J(0)_1 \oplus J(\alpha_1) \oplus \cdots \oplus J(\alpha_n)] \oplus [\oplus_{i=1}^n (J(\alpha_i) \otimes J(0)_2)] \oplus [\oplus_{i,j=1}^{n,q} (J(\alpha_i) \otimes J(\beta_j))],
\]

it follows, in view of Theorem 2.6, that the minimal polynomial of the matrix \( A \otimes B \) is

\[
X^s Q(X)
\]

where

\[
s = \max \{ \lambda_{A_1}(0) \wedge_0 \lambda_{A_2}(0), \lambda_{A_1}(\alpha_i) \wedge_{\beta_j} \lambda_{A_2}(\beta_j), \lambda_{A_2}(0) \wedge_{\alpha_i} \lambda_{A_1}(\alpha_i) \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq q \}
\]

\[
= \rho
\]

and \( Q \) is the least common multiple of the polynomials

\[
(X - \alpha_i \beta_j)^{\lambda_{A_1}(\alpha_i) \wedge_{\lambda_{A_2}(\beta_j)}}, \ 1 \leq i \leq n \text{ and } 1 \leq j \leq q
\]

which is exactly the polynomial \( \Upsilon(Q_1, Q_2) \) by definition of this last polynomial. The proof is finished. \( \Box \)

As an immediate consequence of Theorem 2.13 of [5], Theorem 2.7 above and the fact that every monic polynomial is the minimal polynomial of its companion matrix we have the following result

Theorem 2.8. Let \( P_i, 1 \leq i \leq m \) be any monic polynomials over \( F \). Then we have

\[
\prod_{i=1}^m L(P_i) = L(P)
\]

where \( P(X) \in F[X] \) is the minimal polynomial of the Kronecker product of the companion matrices of \( P_i, 1 \leq i \leq m \).

Remark 2.9. In their paper [1] U. Cerruti and F. Vaccarino proved that if \( P_i, 1 \leq i \leq m \) are monic polynomials over a commutative ring with identity \( R \), then \( \prod_{i=1}^m L(P_i) \subseteq L(H) \) where \( H(X) \in R[X] \) is the characteristic polynomial of the Kronecker product of the companion matrices of \( P_i, 1 \leq i \leq m \).
3. \( \mathcal{P} \)-CANONICAL FORM AND THE MATRIX EXPONENTIAL FUNCTION

The following result shows how to deduce the \( \mathcal{P} \)-cf relative to \( (S^*, \mathcal{H}) \) of the matrix exponential function \( e^{tA} \), \( A \) is a square matrix with complex elements, from the \( \mathcal{P} \)-cf of \( A \) relative to \( (S^*, \mathcal{T}) \).

**Theorem 3.1.** Let \( A \in M_q(\mathbb{C}) \) and let

\[
A = N(A) + A_0 \Lambda_0 + \cdots + A_m \Lambda_m
\]

be the \( \mathcal{P} \)-cf relative to \( (S^*, \mathcal{T}) \) of \( A \). Then for every \( t \in \mathbb{C} \), the \( \mathcal{P} \)-cf relative to \( (S^*, \mathcal{H}) \) of \( e^{tA} \) can be easily obtained by transforming the \( \mathcal{P} \)-cf relative to \( (S^*, \mathcal{T}) \) of \( A \) using the following substitutions:

\[
\lambda \Lambda_i \mapsto \frac{(t \lambda)^i e^{t \lambda} \Gamma^i}{i!} \\
0_i \mapsto \frac{t^i \Gamma^i}{i!}
\]

**Proof.** Let \( \mathcal{M}_A(X) = X^{t_0} \prod_{j=1}^{p} (X - \lambda_j)^{t_j} \) (possibly \( t_0 = 0 \)), be the minimal polynomial of \( A \) and let \( t \in \mathbb{C} \). Let \( \pi_0, \ldots, \pi_p \) be the projections of \( A \) at \( 0, \lambda_1, \ldots, \lambda_p \) respectively. From Proposition 3.1 of \([4]\), we have

\[
A = \sum_{i=0}^{t_0-1} 0_i A_i \pi_0 + \sum_{i=0}^{m} \sum_{j=1}^{p} \lambda_j \lambda_j^{-i} (A - \lambda_j I_q)^i \pi_j \Lambda_i
\]

Then

\[
tA = \sum_{i=0}^{t_0-1} 0_i tA_i \pi_0 + \sum_{i=0}^{m} \sum_{j=1}^{p} t \lambda_j \lambda_j^{-i} (A - \lambda_j I_q)^i \pi_j \Lambda_i
\]

\[
= \sum_{i=0}^{t_0-1} 0_i t^i A_i \pi_0 + \sum_{i=0}^{m} \sum_{j=1}^{p} t \lambda_j (t \lambda_j)^{-i} t^i (A - \lambda_j I_q)^i \pi_j \Lambda_i,
\]

where \( m = \max\{t_1 - 1, \ldots, t_p - 1\} \). Dividing by \( k! \), we obtain

\[
\frac{(tA)^k}{k!} = \sum_{i=0}^{t_0-1} \delta_{i,k} \frac{t^i}{k!} A_i \pi_0 + \sum_{i=0}^{m} \sum_{j=1}^{p} \frac{1}{k!} (t \lambda_j)^{k-i} \binom{k}{i} t^i (A - \lambda_j I_q)^i \pi_j.
\]

Therefore

\[
e^{tA} = \sum_{i=0}^{t_0-1} \sum_{k=0}^{\infty} \delta_{i,k} \frac{t^i}{k!} A_i \pi_0 + \sum_{i=0}^{m} \sum_{j=1}^{p} \sum_{k=0}^{\infty} \frac{1}{k!} (t \lambda_j)^{k-i} \binom{k}{i} t^i (A - \lambda_j I_q)^i \pi_j
\]

\[
= \sum_{i=0}^{t_0-1} \frac{t^i}{i!} A_i \pi_0 + \sum_{i=0}^{m} \sum_{j=1}^{p} \frac{t^i}{i!} (t \lambda_j)^{k-i} \binom{k}{i} (A - \lambda_j I_q)^i \pi_j
\]

\[
= \sum_{i=0}^{t_0-1} \frac{t^i}{i!} A_i \pi_0 + \sum_{i=0}^{m} \sum_{j=1}^{p} \frac{t^i}{i!} e^{t \lambda_j} (A - \lambda_j I_q)^i \pi_j
\]
which is a well known result. But since \((e^{tA})^k = e^{ktA}\) for all \(k \geq 0\), it follows that

\[
e^{tA} = \sum_{i=0}^{t_{0} - 1} \frac{t_{i}}{i!} A^{i} \pi_{0} + \sum_{j=1}^{p} e^{t\lambda_{j}} (A - \lambda_{j} I_{\eta})^{0} \pi_{j} \frac{t^{0} \Gamma_{0}}{0!} + \ldots
\]

\[(3.2)\]

So

\[
e^{tA} = \sum_{j=1}^{p} e^{t\lambda_{j}} (A - \lambda_{j} I_{\eta})^{m} \pi_{j} \frac{t^{m} \Gamma_{m}}{m!}.
\]

where \(d = \max\{m, t_{0} - 1\}\), which is the \(\mathcal{P}\)-cf relative to \((S^{*}, \mathcal{H})\) of \(e^{tA}\).

To conclude, all that remains is to compare equation \[3.2\] with the \(\mathcal{P}\)-cf of \(A\) relative to \((S^{*}, \mathcal{T})\).

The following Theorem is an interesting consequence of Theorem \[3.1\].

**Theorem 3.2.** Let \(A\) be a nonsingular complex matrix such that \(\chi_{A} \in \mathbb{R} [X]\). Let

\[
A = N(A) + A_{0} \lambda_{0} + \ldots + A_{m} \lambda_{m}
\]

be the \(\mathcal{P}\)-cf relative to \((S^{*}, \mathcal{T})\) of \(A\). Suppose that the sequences appearing in \(A_{0}, \ldots, A_{m}\) are

\[
\begin{align*}
\mu_{1} &= r_{1} e^{i \theta_{1}}, \ldots, \mu_{s} = r_{s} e^{i \theta_{s}}; \\
\mu_{1}^{\overline{1}} &= r_{1} e^{-i \theta_{1}}, \ldots, \mu_{s}^{\overline{1}} = r_{s} e^{-i \theta_{s}}; \\
\lambda_{1}, \ldots, \lambda_{l} &\text{ are the eigenvalues, not necessarily distinct, of } A.
\end{align*}
\]

Then we have

(1) \(\mu_{1} = r_{1} e^{i \theta_{1}}, \ldots, \mu_{s} = r_{s} e^{i \theta_{s}}; \mu_{i}^{\overline{1}} = r_{i} e^{-i \theta_{i}}, \ldots, \mu_{s}^{\overline{1}} = r_{s} e^{-i \theta_{s}};\lambda_{1}, \ldots, \lambda_{l}\) are the eigenvalues, not necessarily distinct, of \(A\).

(2) The \(\mathcal{P}\)-cf relative to \((S^{*}, \mathcal{H})\) of the matrix \(e^{tA}\) can be easily obtained by transforming the \(\mathcal{P}\)-cf relative to \((S^{*}, \mathcal{T})\) of \(A\) using the following substitutions:

\[
\begin{align*}
\Lambda_{j}(r_{i}^{k} \cos(k\theta_{i}))_{k} &\rightarrow \Re(f_{j}(tk\mu_{i}))_{k} \\
\Lambda_{j}(r_{i}^{k} \sin(k\theta_{i}))_{k} &\rightarrow \Im(f_{j}(tk\mu_{i}))_{k} \\
\lambda_{i} \Lambda_{j} &\rightarrow \frac{(tk\lambda_{i})^{j} e^{t\lambda_{i} \Gamma_{j}}}{j!}, \quad 1 \leq i \leq l \\
0_{j} &\rightarrow \frac{t^{j} \Gamma_{j}}{j!}
\end{align*}
\]
where for all $z \in \mathbb{C}$,

$$f_j(z) = \begin{cases} 
\frac{z^j e^z}{j!} & \text{if } j \neq 0 \\
\frac{e^z}{j!} & \text{if } j = 0
\end{cases}$$

Proof. The proof is a straightforward application of Theorem 3.1 and the fact that if $\mu = re^{i\theta}$ then

$$r^k \cos(k\theta) = \frac{\mu^k + \overline{\mu}^k}{2}$$

and

$$r^k \sin(k\theta) = \frac{\mu^k - \overline{\mu}^k}{2i}.$$ 

\[\Box\]

4. $\mathcal{P}$-canonical form of the logarithm of matrices

It is well known that a complex matrix $A$ has a logarithm if and only if $A$ is nonsingular. The following theorem shows that knowing the $\mathcal{P}$-cf relative to $(S^*, T)$ (or $(S^*, H)$) of a nonsingular matrix $A$, we can derive a logarithm of it.

Theorem 4.1. Let $A$ be a nonsingular matrix of order $q$ over $\mathbb{C}$ and let

$$\lambda_1 = e^{z_1}, \ldots, \lambda_s = e^{z_s}$$

its eigenvalues. Let us denote by $(A(k))_{k \geq 0}$ the $\mathcal{P}$-cf relative to $(S^*, T)$ of $A$, and consider the matrix complex-valued smooth function $A(t)$ of a real variable $t$ obtained by plugging the variable $t$ for $k$ in $A(k)$. Then we have

1. The derivative $A'(0)$ of $A(t)$ at $t = 0$ is a matrix logarithm of $A$.
2. The $\mathcal{P}$-cf relative to $(S^*, T)$ of $A'(0)$ can be easily obtained by transforming the $\mathcal{P}$-cf relative to $(S^*, H)$ of $A$ using the following substitutions:

$$\lambda_j \Gamma^i \leftrightarrow i! z_j^{-i} \Lambda_i \quad \text{if } z_j \neq 0$$

$$\lambda_j \Gamma^i \leftrightarrow i! 0_i \quad \text{if } z_j = 0$$

3. The eigenvalues of $A'(0)$ are $z_1, \ldots, z_s$. More precisely, if $\chi_A = (X - \lambda_1)^{m_1} \cdots (X - \lambda_s)^{m_s}$ is the characteristic polynomial of $A$, then $A'(0)$ is the unique logarithm of $A$ with characteristic polynomial $\chi_{A'(0)} = (X - z_1)^{m_1} \cdots (X - z_s)^{m_s}$.
4. If $A$ has nonnegative real eigenvalues, then the principal logarithm of $A$ is equal to the matrix $A'(0)$ obtained when we choose the $z_i$ to be in the strip $\{ z \in \mathbb{C} | -\pi < \text{Im}(z) < \pi \}$.

Proof.
(1) Let \( {t \choose i} = \frac{t(t-1)\cdots(t-i+1)}{i!} \) be the binomial polynomial of degree \( i \). Since, for \( i \geq 1 \), the coefficient of \( t \) in that polynomial is \( (-1)^{i-1} \frac{1}{i} \), it follows that the derivative of \( {t \choose i} \) at 0 is \( \frac{(-1)^{i-1}}{i} \) for all integer \( i \geq 1 \). On the other hand, we have

\[
\mathcal{A}_0'(0) = z_1 \pi_1 + \cdots + z_s \pi_s.
\]

Hence

\[
A'(0) = z_1 \pi_1 + \cdots + z_s \pi_s + \sum_{i=1}^{m} \frac{(-1)^{i-1}}{i} \mathcal{A}_i(0) = z_1 \pi_1 + \cdots + z_s \pi_s + \sum_{i=1}^{m} \frac{(-1)^{i-1}}{i} \sum_{j=1}^{s} \lambda_j^{-i} (A - \lambda_j I_q)^{i} \pi_j.
\]

Since the matrix \( (\lambda_j^{-1}A - I_q)\pi_j)^{m+1} = 0 \), it follows that

\[
C_j = \sum_{i=1}^{m} \frac{(-1)^{i-1}}{i} (\lambda_j^{-1}A - I_q)^{i} \pi_j
\]

is a logarithm of \( I_q + (\lambda_j^{-1}A - I_q)\pi_j \). Consequently, as the \( C_j \)'s are pairwise commuting matrices, we have \( C = \sum_{j=1}^{s} C_j \) is a logarithm of

\[
\prod_{j=1}^{s} (I_q + (\lambda_j^{-1}A - I_q)\pi_j) = I_q + \sum_{j=1}^{s} (\lambda_j^{-1}A - I_q)\pi_j
\]

\[
= A \sum_{j=1}^{s} \lambda_j^{-1} \pi_j
\]

\[
= Ae^{-\sum_{j=1}^{s} (z_1 \pi_1 + \cdots + z_s \pi_s)}.
\]

Hence

\[
A = e^C e^{-\sum_{j=1}^{s} (z_1 \pi_1 + \cdots + z_s \pi_s)}.
\]

Finally, since the matrix \( C \) commutes with \( z_1 \pi_1 + \cdots + z_s \pi_s \), it follows that \( A'(0) \) is a logarithm of \( A \).

(2) It is sufficient to use the argument of Proposition 3.1 in reverse direction.

(3) Follows from property (2) above and Corollary (3.6) of [4] which allows us to deduce the minimal polynomial, and in particular the eigenvalues, of a matrix from its \( \mathcal{P}-\text{cf} \)

(4) Follows directly from property (3) above.

As a consequence of Theorem 4.1 we have the following result.
Corollary 4.2. Let $A$ be a nonsingular matrix of order $q$ over $\mathbb{C}$ and let $\lambda_1 = e^{z_1}, \ldots, \lambda_s = e^{z_s}$ its eigenvalues. Let $P_{1,ij}(X), \ldots, P_{s,ij}(X) \in \mathbb{C}[X]$ such that 
\[
(A^k)_{ij} = P_{1,ij}(k)\lambda_1^k + \cdots + P_{s,ij}(k)\lambda_s^k
\]
for all $k \geq 0$. Here, $(A^k)_{ij}$ denotes the $(i,j)$-th entry of the matrix $A^k$. Assume 
\[
P_{1,ij}(X) = p_{1,ij,0} + p_{1,ij,1}X + \cdots \]
\[
P_{s,ij}(X) = p_{s,ij,0} + p_{s,ij,1}X + \cdots
\]
Then the matrix 
\[
\left(\sum_{\epsilon=1}^s p_{\epsilon,ij,0} + \sum_{\epsilon=1}^s p_{\epsilon,ij,1}z_\epsilon\right)_{ij}
\]
is a logarithm of $A$.

Proof. The proof follows immediately from property (1) of Theorem 4.1. □

The following theorem is also a consequence of Theorem 4.1.

Theorem 4.3. Let $A$ be a nonsingular complex matrix such that $\chi_A \in \mathbb{R}[X]$. Let 
\[
A = N(A) + A_0\Gamma^0 + \cdots + A_l\Gamma^l
\]
be the $\mathcal{P}$-cf relative to $(S^*, \mathcal{H})$ of $A$. Suppose that the sequences appearing in $A_0, \ldots, A_l$ are 
\[
r_i^k \cos(k\theta) \quad (\text{or} \quad r_i^k \sin(k\theta)), \ 1 \leq i \leq s, \quad \text{and} \quad \lambda_i, 1 \leq i \leq m
\]
Then we have 
(1) $\mu_1 = r_1e^{i\theta_1}, \ldots, \mu_s = r_1e^{i\theta_s}, \overline{\mu}_1 = r_1e^{-i\theta_1}, \ldots, \overline{\mu}_s = r_1e^{-i\theta_s}, \lambda_1, \ldots, \lambda_m$ are the eigenvalues, not necessarily distinct, of $A$.
(2) Let $z_1, \ldots, z_m$ be logarithms of $\lambda_1, \ldots, \lambda_m$, $w_1, \ldots, w_s$ be logarithms of $\mu_1, \ldots, \mu_s$ and $u_1, \ldots, u_s$ be logarithms of $\overline{\mu}_1, \ldots, \overline{\mu}_s$. Then the $\mathcal{P}$-cf relative to $(S^*, \mathcal{T})$ of the logarithm $A'(0)$ of $A$ can be easily obtained by transforming the $\mathcal{P}$-cf relative to $(S^*, \mathcal{H})$ of $A$ using the following substitutions:
\[
\begin{align*}
\Gamma^j(r_i^k \cos(k\theta_i))_k &\leftrightarrow \frac{1}{2i}(g_{(j,k)}(w_i) + g_{(j,k)}(u_i)) \\
\Gamma^j(r_i^k \sin(k\theta_i))_k &\leftrightarrow \frac{1}{2i}(g_{(j,k)}(w_i) - g_{(j,k)}(u_i)) \\
\lambda_j \Gamma^i &\leftrightarrow i!z_j z_j^{-1}\Lambda_i \quad \text{if} \quad z_j \neq 0 \\
\lambda_j \Gamma^i &\leftrightarrow i!0_i \quad \text{if} \quad z_j = 0
\end{align*}
\]
where for all \( z \in \mathbb{C} \),

\[
g_{j,k}(z) = \begin{cases} 
  j!\Lambda_j z^{-j} & \text{if } z \neq 0 \\
  j! & \text{if } z = 0 
\end{cases}
\]

(3) \( z_1, \ldots, z_m, w_1, \ldots, w_s, u_1, \ldots, u_s \) are the eigenvalues, not necessarily distinct, of \( A'(0) \).

**Proof.** The proof is a straightforward application of Theorem 4.1, and hence will be omitted. \( \blacksquare \)

5. **Illustration examples**

In this section we give some theoretical and numerical illustrations of our methods.

**Example 5.1.** We begin by the trivial and well known case of a nilpotent matrix. Suppose that \( A^q = 0 \). Then we have \( A^k = \sum_{i=0}^{q-1} 0_i A^i \) and hence \( e^{kA} = \sum_{i=0}^{q-1} \frac{k^i}{i!} A^i \) in virtue of Theorem 3.1.

**Example 5.2.** Let

\[
A = \begin{bmatrix}
  a_0 & a_1 & \cdots & a_{n-1} \\
  0 & a_0 & a_1 & \cdots & a_{n-2} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & a_1 & \vdots \\
  0 & \cdots & \cdots & 0 & a_0 
\end{bmatrix}
\]

a semicirculant matrix. Using the method given in [3] together with the methods obtained in this paper, we can compute easily the exponential and the logarithm of \( A \). For example let consider Example 3.4 of [3].

\( A = [2, 4, 2, 3] \). We have

\[A^k = [a_0(k), a_1(k), a_2(k), a_3(k)]\]

where

\[
a_0(k) = 2^k \binom{k}{0}
\]

\[
a_1(k) = 2^{k+1} \binom{k}{1}
\]

\[
a_2(k) = 2^{k+2} \binom{k}{2} + 2^k \binom{k}{1}
\]

\[
a_3(k) = 2^{k+3} \binom{k}{3} + 2^{k+2} \binom{k}{2} + 3 \times 2^{k-1} \binom{k}{1}
\]
Hence $e^{kA} = [b_0(k), b_1(k), b_2(k), b_3(k)]$ where
\[
b_0(k) = e^{2k} \\
b_1(k) = 4ke^{2k} \\
b_2(k) = (8k^2 + 2k)e^{2k} \\
b_3(k) = \left(\frac{25k^3}{3} + 8k^2 + 3k\right)e^{2k}
\]
On the other hand, we have
\[
a_0(k) = 2^k \\
a_1(k) = 2^{k+1}k \\
a_2(k) = 2^{k+1}k^2 - 2^kk \\
a_3(k) = \frac{2^{k+2}}{3} - 2^{k+1}k^2 + \frac{13}{6}2^kk
\]
Then $A'(0)^k = [a_0(k), a_1(k), a_2(k), a_3(k)]$ where
\[
a_0(k) = \ln(2)^k \\
a_1(k) = 2\ln(2)^{k-1}\binom{k}{1} \\
a_2(k) = 2\ln(2)^{k-2}\binom{k}{2} - \ln(2)^{k-1}\binom{k}{1} \\
a_3(k) = \frac{4}{3}\ln(2)^{k-3}\binom{k}{3} - 2\ln(2)^{k-2}\binom{k}{2} + \frac{13}{6}\ln(2)^{k-1}\binom{k}{1}
\]
$A'(0)$ is the logarithm of $A$ with characteristic polynomial $(X - \ln(2))^3$. Let $J_{n+1}(\lambda) = [\lambda, 1, 0, \ldots, 0]$ the Jordan block of order $n+1$ with eigenvalue $\lambda \neq 0$. It is well-known that for all nonnegative integer $k$
\[
J_{n+1}(\lambda)^k = [\lambda^k, \lambda^{k-1}\binom{k}{1}, \ldots, \lambda^{k-i}\binom{k}{i}, \ldots, \lambda^{k-n}\binom{k}{n}]
\]
Hence we get the following well-known result
\[
e^{kJ_{n+1}(\lambda)} = [e^{k\lambda}, ke^{k\lambda}, \ldots, \frac{e^{k\lambda}k^i}{i!}, \ldots, \frac{e^{k\lambda}k^n}{n!}]
\]
on the other hand, Since
\[
\binom{k}{i} = \sum_{m=0}^{i} \frac{s(i, m)}{i!}k^m
\]
where $s(m, k)$ are the Stirling numbers of the first kind (see, e.g. Quaintance and Gould [6]), it follows that
\[
\lambda^{k-i}\binom{k}{i} = \sum_{m=0}^{i} \frac{s(i, m)}{i!}\lambda^{-i}\lambda^kk^m
\]
Hence \( \log(J_{n+1}(\lambda))^k = [d_0(k), \ldots, d_n(k)] \)
where
\[
d_i(k) = \sum_{m=0}^{i} \frac{s(i, m)}{i!} m! \lambda^{-i} \ln(\lambda)^{k-m} \binom{k}{m}
\]
In particular
\[
D = \ln(J_{n+1}(\lambda)) = [\ln(\lambda), \lambda^{-1}, \ldots, (-1)^{i-1} \frac{\lambda^{-i}}{i}, \ldots, (-1)^{n-1} \lambda^{-n}]
\]
where we have used the fact that \( s(i, 1) = \frac{(-1)^{i-1}}{i} \) for all positive integer \( i \). It is possible to verify that \( J_{n+1}(\lambda) = e^D \) through direct calculation, see e.g. Appendix F in Dick.

**Example 5.3.** Let
\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & -1 \\
0 & 0 & -1 & 1 \\
0 & 0 & 1 & -1 \\
\end{pmatrix}
\]
and
\[
A(k) = \begin{pmatrix}
2^{-1+k} & 2^{-1+k} & \frac{1}{16} 2^k((-1)^{1+k} + 5) & \frac{1}{16} 2^k((-1)^{k} - 1) \\
2^{-1+k} & 2^{-1+k} & \frac{1}{16} 2^k((-1)^{1+k} + 1) & \frac{1}{16} 2^k(5((-1)^{k} - 1) \\
0 & 0 & (-1)^{k} 2^{-1+k} & (-1)^{1+k} 2^{-1+k} \\
0 & 0 & (-1)^{1+k} 2^{-1+k} & (-1)^{k} 2^{-1+k} \\
\end{pmatrix}
\]
Then we have
- The non-geometric part of \( A \) is \( (I_4 - A(0))0_0 + (A - A(1))0_1 \).
- The geometric part of \( A \) is
\[
\begin{pmatrix}
2^{-1}(2^k) & 2^{-1}(2^k) & \frac{5}{16}(2^k) - \frac{1}{16}((-2)^k) & \frac{1}{16}(2^k) + \frac{1}{16}((-2)^k) \\
2^{-1}(2^k) & 2^{-1}(2^k) & \frac{5}{16}(2^k) - \frac{1}{16}((-2)^k) & \frac{1}{16}(2^k) + \frac{1}{16}((-2)^k) \\
0 & 0 & 2^{-1}((-2)^k) & -2^{-1}((-2)^k) \\
0 & 0 & -2^{-1}((-2)^k) & 2^{-1}((-2)^k) \\
\end{pmatrix}
\]
- For all \( k \in \mathbb{Z} \)
\[
e^{ktA} = I_4 - A(0) + (A - A(1))kt + \begin{pmatrix}
\frac{e^{2kt}}{2} & \frac{e^{2kt}}{2} & \frac{5e^{2kt} - e^{-2kt}}{2} & \frac{-e^{2kt} + e^{-2kt}}{2} \\
\frac{2}{e^{2kt}} & \frac{2}{e^{2kt}} & \frac{16}{5e^{2kt} - 5e^{-2kt}} & \frac{16}{-e^{2kt} + 5e^{-2kt}} \\
0 & 0 & \frac{e^{2kt}}{2} & \frac{-e^{-2kt}}{2} \\
0 & 0 & \frac{-e^{-2kt}}{2} & \frac{e^{2kt}}{2}
\end{pmatrix}
\]
Since

\[
(A - A(1)) = \begin{pmatrix}
0 & 0 & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & -\frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
I_4 - A(0) = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{4} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

we have for all \( k \in \mathbb{Z} \)

\[
e^{ktA} = \begin{pmatrix}
e^{2kt/2} & e^{2kt/1} & 5e^{2kt/1}-e^{2kt/4+4kt/4} & -e^{2kt/1+e^{-2kt/4+4kt/4}} \\
e^{2kt/2} & e^{2kt/1} & 16 & -16 \frac{e^{2kt/1+e^{-2kt/4+4kt/4}}}{2} \\
0 & 0 & e^{4kt/1} & -16 \frac{e^{-2kt/1}}{2} \\
0 & 0 & -e^{-2kt/1} & e^{-2kt/1}
\end{pmatrix}
\]

**Example 5.4.** Let

\[
E = \begin{pmatrix}
2\sqrt{3} - 10 & 2\sqrt{3} - 23 & \sqrt{3} - 5 \\
4 & \sqrt{3} + 9 & 2 \\
-2\sqrt{3} + 2 & -4\sqrt{3} + 5 & -\sqrt{3} + 1
\end{pmatrix}
\]

and put

\[
E(k) = \begin{pmatrix}
e_{11}(k) & e_{12}(k) & e_{13}(k) \\
e_{21}(k) & e_{22}(k) & e_{23}(k) \\
e_{31}(k) & e_{32}(k) & e_{33}(k)
\end{pmatrix}
\]

where

\[
e_{11}(k) = 2^{k+1}(\cos\left(\frac{k\pi}{6}\right) - 5\sin\left(\frac{k\pi}{6}\right))
\]

\[
e_{12}(k) = 2^{k+1}(\cos\left(\frac{k\pi}{6}\right) - \frac{23}{2}\sin\left(\frac{k\pi}{6}\right))
\]

\[
e_{13}(k) = 2^k(\cos\left(\frac{k\pi}{6}\right) - 5\sin\left(\frac{k\pi}{6}\right))
\]

\[
e_{21}(k) = 2^{k+2}\sin\left(\frac{k\pi}{6}\right)
\]

\[
e_{22}(k) = 2^k(\cos\left(\frac{k\pi}{6}\right) + 9\sin\left(\frac{k\pi}{6}\right))
\]

\[
e_{23}(k) = 2^{k+1}\sin\left(\frac{k\pi}{6}\right)
\]

\[
e_{31}(k) = -2^{k+1}(\cos\left(\frac{k\pi}{6}\right) - \sin\left(\frac{k\pi}{6}\right))
\]
We have, for all $k \geq 1$, $E^k = E(k)$. Hence For all $k \in \mathbb{Z}$

$$e^{ktE} = I_3 - E(0) + \begin{pmatrix} f_{11}(k, t) & f_{12}(k, t) & f_{13}(k, t) \\ f_{21}(k, t) & f_{22}(k, t) & f_{23}(k, t) \\ f_{31}(k, t) & f_{32}(k, t) & f_{33}(k, t) \end{pmatrix}$$

where

$$f_{11}(k, t) = (1 + 5i)e^{2kte^{\pi i/6}} + (1 - 5i)e^{2kte^{-\pi i/6}}$$
$$f_{12}(k, t) = (1 + \frac{23}{2}i)e^{2kte^{\pi i/6}} + (1 - \frac{23}{2}i)e^{2kte^{-\pi i/6}}$$
$$f_{13}(k, t) = \frac{1 + 5i}{2}e^{2kte^{\pi i/6}} + \frac{1 - 5i}{2}e^{2kte^{-\pi i/6}}$$
$$f_{21}(k, t) = -2ie^{2kte^{\pi i/6}} + 2ie^{2kte^{-\pi i/6}}$$
$$f_{22}(k, t) = \frac{1 - 9i}{2}e^{2kte^{\pi i/6}} + \frac{1 + 9i}{2}e^{2kte^{-\pi i/6}}$$
$$f_{23}(k, t) = -ie^{2kte^{\pi i/6}} + ie^{2kte^{-\pi i/6}}$$
$$f_{31}(k, t) = (-1 - i)e^{2kte^{\pi i/6}} + (-1 + i)e^{2kte^{-\pi i/6}}$$
$$f_{32}(k, t) = \frac{-4 - 5i}{2}e^{2kte^{\pi i/6}} + \frac{-4 + 5i}{2}e^{2kte^{-\pi i/6}}$$
$$f_{33}(k, t) = \frac{-1 - i}{2}e^{2kte^{\pi i/6}} + \frac{-1 + i}{2}e^{2kte^{-\pi i/6}}$$

**Example 5.5.** Let $x$ be a nonzero complex number ant let

$$E = \begin{pmatrix} 2\sqrt{3} - x - 10 & 2\sqrt{3} - 2x - 23 & \sqrt{3} - x - 5 \\ 4 & \sqrt{3} + 9 & 2 \\ -2\sqrt{3} + 2x + 2 & -4\sqrt{3} + 4x + 5 & -\sqrt{3} + 2x + 1 \end{pmatrix}$$

Put

$$E(k) = \begin{pmatrix} e_{11}(k) & e_{12}(k) & e_{13}(k) \\ e_{21}(k) & e_{22}(k) & e_{23}(k) \\ e_{31}(k) & e_{32}(k) & e_{33}(k) \end{pmatrix}$$
where

\[ e_{11}(k) = 2^{k+1} \left( \cos\left(\frac{k\pi}{6}\right) - 5 \sin\left(\frac{k\pi}{6}\right) \right) - x^k \]

\[ e_{12}(k) = 2^{k+1} \left( \cos\left(\frac{k\pi}{6}\right) - \frac{23}{2} \sin\left(\frac{k\pi}{6}\right) \right) - 2x^k \]

\[ e_{13}(k) = 2^k \left( \cos\left(\frac{k\pi}{6}\right) - 5 \sin\left(\frac{k\pi}{6}\right) \right) - x^k \]

\[ e_{21}(k) = 2^{k+2} \sin\left(\frac{k\pi}{6}\right) \]

\[ e_{22}(k) = 2^k \left( \cos\left(\frac{k\pi}{6}\right) + 9 \sin\left(\frac{k\pi}{6}\right) \right) \]

\[ e_{23}(k) = 2^{k+1} \sin\left(\frac{k\pi}{6}\right) \]

\[ e_{31}(k) = -2^{k+1} \left( \cos\left(\frac{k\pi}{6}\right) - \sin\left(\frac{k\pi}{6}\right) \right) + 2x^k \]

\[ e_{32}(k) = -2^{k+2} \left( \cos\left(\frac{k\pi}{6}\right) - \frac{5}{4} \sin\left(\frac{k\pi}{6}\right) \right) + 4x^k \]

\[ e_{33}(k) = -2^k \left( \cos\left(\frac{k\pi}{6}\right) - \sin\left(\frac{k\pi}{6}\right) \right) + 2x^k \]

We have, for all \( k \geq 1 \), \( E^k = E(k) \). Then for all positive integer \( k \)

1. The eigenvalues of the matrix \( E \) are \( 2e^{\frac{i\pi}{6}}, 2e^{-\frac{i\pi}{6}} \) and \( x \).

2. Let \( B \) be the matrix such that

\[
B^k = \begin{pmatrix}
g_{11}(k, t) & g_{12}(k, t) & g_{13}(k, t) \\
g_{21}(k, t) & g_{22}(k, t) & g_{23}(k, t) \\
g_{31}(k, t) & g_{32}(k, t) & g_{33}(k, t)
\end{pmatrix}
\]

where

\[ g_{11}(k) = (1 + 5i)(\ln(2) + \frac{\pi i}{6})^k + (1 - 5i)(\ln(2) - \frac{\pi i}{6})^k - \ln(x)^k \]

\[ g_{12}(k) = (1 + \frac{23}{2}i)(\ln(2) + \frac{\pi i}{6})^k + (1 - \frac{23}{2}i)(\ln(2) - \frac{\pi i}{6})^k - 2\ln(x)^k \]

\[ g_{13}(k) = \frac{1 + 5i}{2} (\ln(2) + \frac{\pi i}{6})^k + \frac{1 - 5i}{2} (\ln(2) - \frac{\pi i}{6})^k - \ln(x)^k \]

\[ g_{21}(k) = -2i(\ln(2) + \frac{\pi i}{6})^k + 2i(\ln(2) - \frac{\pi i}{6})^k \]

\[ g_{22}(k) = \frac{1 - 9i}{2} (\ln(2) + \frac{\pi i}{6})^k + \frac{1 + 9i}{2} (\ln(2) - \frac{\pi i}{6})^k \]

\[ g_{23}(k) = -i(\ln(2) + \frac{\pi i}{6})^k + i(\ln(2) - \frac{\pi i}{6})^k \]

\[ g_{31}(k) = (-2 - i)(\ln(2) + \frac{\pi i}{6})^k + (-2 + i)(\ln(2) - \frac{\pi i}{6})^k + 2\ln(x)^k \]
\[ g_{32}(k) = \frac{-4 - 5i}{2} (\ln(2) + \frac{\pi i}{6})^k + \frac{4 + 5i}{2} (\ln(2) - \frac{\pi i}{6})^k + 4 \ln(x)^k \]

\[ g_{33}(k) = \frac{-2 - i}{2} (\ln(2) + \frac{\pi i}{6})^k + \frac{2 + i}{2} (\ln(2) - \frac{\pi i}{6})^k + 2 \ln(x)^k \]

Then \( B \) is a logarithm of \( E \).

3. The eigenvalues of the matrix \( B \) are \( \ln(2) + \frac{\pi i}{6} \), \( \ln(2) - \frac{\pi i}{6} \) and \( \ln(x) \).

4. Suppose that \( x \) is not a negative real number. Then if we replace the logarithm \( \ln(x) \) by the principal logarithm of \( x \) in the above expressions, we obtain that \( B \) is the principal logarithm of \( E \).

**Example 5.6.** Let

\[ C = \begin{pmatrix} 1 & 3 \\ -3 & -5 \end{pmatrix} \]

It is easily seen that for all nonnegative integer \( k \)

\[ C^k = (-2)^k \begin{pmatrix} \frac{-3k}{2} + 1 & \frac{-3k}{2} \\ \frac{3k}{2} & \frac{3k}{2} + 1 \end{pmatrix} \]

Then

\[ C(t) = e^{(\ln(2) + i\pi)t} \begin{pmatrix} \frac{-3t}{2} + 1 & \frac{-3t}{2} \\ \frac{3t}{2} & \frac{3t}{2} + 1 \end{pmatrix} \]

Thus

\[ C'(0) = \begin{pmatrix} \frac{-3}{2} + \ln(2) + i\pi & \frac{-3}{2} + \ln(2) + i\pi \\ \frac{3}{2} + \ln(2) + i\pi & \frac{3}{2} \end{pmatrix} \]

is a logarithm of \( C' \).

From Theorem [4.4] we we have also that

\[ C'(0)^k = (\ln(2) + i\pi)^k \begin{pmatrix} \frac{-3k}{2} + \ln(2) + i\pi & \frac{-3k}{2} + \ln(2) + i\pi \\ \frac{3k}{2} & \frac{3k}{2} \end{pmatrix} \]

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