Neutrino Optics and Oscillations in Gravitational Fields

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(Received: November 26, 2018)

We study the propagation of neutrinos in gravitational fields using wave functions that are exact to first order in the metric deviation. For illustrative purposes, the geometrical background is represented by the Lense-Thirring metric. We derive explicit expressions for neutrino deflection, helicity transitions, flavor oscillations and oscillation Hamiltonian.

PACS No.: 14.60.Pq, 04.62.+v, 95.30.Sf

I. INTRODUCTION

Because of their coexistence in a host of problems, the interaction of neutrinos with gravitational fields is of interest in astrophysics and cosmology. At the same time, neutrino detectors of increasing sensitivity and scope require a better understanding of propagation and oscillation properties of neutrinos in the neighborhood of massive astrophysical objects.

While in a few instances the treatment of gravity may necessitate the complete apparatus of general relativity, using a weak field approximation is sufficient in most problems that deal with neutrinos. Even in the case of black holes, a first order approximation in the metric deviation may still be adequate up to distances of a few Schwarzschild radii from the gravitational source. In the majority of instances, gravitation is treated as a quasi-classical external field. Even so, the calculations tend to be complex.

Useful, simplified treatments of the problem of neutrinos in gravitational fields can be found in the literature, notably in the heuristic approach of Cardall and Fuller [1] in which the authors construct an effective mechanical four-momentum operator that incorporates spin and matter effects.

A more direct approach would require solving the Dirac equation completely which is in general very difficult. It was however shown in [2,3] that the covariant Dirac equation can be solved exactly to first order in the metric deviation. This procedure is manifestly covariant and gauge-invariant and only requires the evaluation of path integrals that are trivial for most physically relevant metrics. It can also be extended to include electromagnetic fields and the effect of media on the propagation of the particles. These are all issues of principle one meets in the interaction of quantum systems with semiclassical gravitational fields [4]. They have been raised time and again in the literature [5]. We think it advantageous to address them within the context of an approach that is consistent and reproduces all gravitational effects that are known, or are supposed to apply to fermions [6,3,7]. We propose to apply this method to the study of neutrino optics, helicity transitions and flavor oscillations.

In Sec. II we briefly review the essential points of the solution found in [2]. The neutrino geometrical optics is given in Sec. III and applied in particular to the propagation of neutrinos in a Lense-Thirring background for the distinct cases of propagation parallel and orthogonal to the angular velocity of the gravitational source. Helicity transitions and flavor oscillations in the absence of matter are dealt with in Sec. IV and Sec. V for the same two directions of propagation. Sec. VI contains the conclusions.
II. THE COVARIANT DIRAC EQUATION

The behavior of spin-1/2 particles in the presence of a gravitational field \( g_{\mu \nu} \) is determined by the covariant Dirac equation

\[
[i \gamma^\mu(x) D_\mu - m \sigma^0 \Psi(x) = 0, \tag{II.1}
\]

where \( D_\mu = \nabla_\mu + i \Gamma_\mu(x), \nabla_\mu \) is the covariant derivative, \( \Gamma_\mu(x) \) the spin connection and the matrices \( \gamma^\mu(x) \) satisfy the relations \( \{ \gamma^\mu(x), \gamma^\nu(x) \} = 2 g^{\mu \nu} \). Both \( \Gamma_\mu(x) \) and \( \gamma^\mu(x) \) can be obtained from the usual constant Dirac matrices by using the vierbein fields \( e^\mu_\alpha \) and the relations

\[
\gamma^\mu(x) = e^\mu_\alpha(x) \gamma^\alpha, \quad \Gamma_\mu(x) = -\frac{1}{4} \sigma^{\alpha \beta} \epsilon^\alpha_\mu e^\beta_\nu, \tag{II.2}
\]

where \( \sigma^{\alpha \beta} = \frac{i}{2} [\gamma^\alpha, \gamma^\beta] \). A semicolon and a comma are frequently used as alternative ways to indicate covariant and partial derivatives respectively. We use units \( \hbar = 1 \) throughout the paper.

Eq. (II.1) can be solved exactly to first order in the metric deviation \( \gamma_{\mu \nu}(x) = g_{\mu \nu} - \eta_{\mu \nu}, \) where the Minkowski metric \( \eta_{\mu \nu} \) has signature -2. This is achieved by first transforming (II.1) into the equation

\[
[i \tilde{\gamma}^\mu(x) \nabla_\nu - m \sigma^0 \tilde{\Psi}(x) = 0, \tag{II.3}
\]

where

\[
\tilde{\Psi}(x) = S^{-1} \Psi(x), \quad S(x) = e^{-i \Phi_s(x)}, \quad \Phi_s(x) = \mathcal{P} \int^x_0 dz^\lambda \Gamma^\lambda(z), \quad \tilde{\gamma}^\mu(x) = S^{-1} \gamma^\mu(x) S . \tag{II.4}
\]

By multiplying (II.3) on the left by \( -i \tilde{\gamma}^\nu(x) \nabla_\nu - m \), we obtain the equation

\[
(g^{\mu \nu} \nabla_\mu \nabla_\nu + m^2) \tilde{\Psi}(x) = 0, \tag{II.5}
\]

whose solution

\[
\tilde{\Psi}(x) = e^{-i \Phi_G(x)} \Psi_0(x), \tag{II.6}
\]

is exact to first order. The operator \( \Phi_G(x) \) is defined as

\[
\Phi_G = -\frac{1}{4} \int^x_0 dz^\lambda [ \gamma_{\alpha \beta, \lambda}(z) - \gamma_{\beta \lambda, \alpha}(z) ] \hat{L}^{\alpha \beta}(z) + \frac{1}{2} \int^x_0 dz^\lambda \gamma_{\alpha \beta} \hat{k}^\alpha, \tag{II.7}
\]

\[
[\hat{L}^{\alpha \beta}(z), \Psi_0(x)] = \left( (x^\alpha - z^\alpha) \hat{k}^\beta - (x^\beta - z^\beta) \hat{k}^\alpha \right) \Psi_0(x), \quad [\hat{k}^\alpha, \Psi_0(x)] = i \partial^\alpha \Psi_0,
\]

and \( \Psi_0(x) \) satisfies the usual flat spacetime Dirac equation. \( \hat{L}^{\alpha \beta} \) and \( \hat{k}^\alpha \) are the angular and linear momentum operators of the particle. It follows from (II.6) and (II.4) that the solution of (II.1) can be written in the form

\[
\Psi(x) = e^{-i \Phi_s} (-i \tilde{\gamma}^\mu(x) \nabla_\mu - m) e^{-i \Phi_G} \Psi_0(x) , \tag{II.8}
\]

and also as

\[
\Psi(x) = -\frac{1}{2m} (-i \gamma^\mu(x) D_\mu - m) e^{-i \Phi_T} \Psi_0(x) , \tag{II.9}
\]

where \( \Phi_T = \Phi_s + \Phi_G \) is of first order in \( \gamma_{\alpha \beta}(x) \). The factor \( -1/2m \) on the r.h.s. of (II.9) is required by the condition that both sides of the equation agree when the gravitational field vanishes.

It is useful to re-derive some known results from the covariant Dirac equation. On multiplying (II.1) on the left by \( (-i \gamma^\nu(x) D_\nu - m) \) and using the relations

\[
\nabla_\mu \Gamma_\nu(x) - \nabla_\nu \Gamma_\mu(x) + i [\Gamma_\mu(x), \Gamma_\nu(x)] = -\frac{1}{4} \sigma^{\alpha \beta}(x) R_{\alpha \beta \mu \nu} , \tag{II.10}
\]

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and
\[ [\mathcal{D}_\mu, \mathcal{D}_\nu] = -\frac{i}{4} \sigma^{\alpha\beta}(x)R_{\alpha\beta\mu\nu}, \] (II.11)
we obtain the equation
\[ \left( g^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu - \frac{R}{4} + m^2 \right) \Psi(x) = 0. \] (II.12)

In (II.11) and (II.12) \( R_{\alpha\beta\mu\nu} \) is the Riemann tensor, \( R \) the Ricci scalar, and \( \sigma^{\alpha\beta}(x) = (i/2)[\gamma^\alpha(x), \gamma^\beta(x)] \).

By using Eq. (II.4), we also find
\[ (-i\gamma^\nu(x)\mathcal{D}_\nu - m) S(i\tilde{\gamma}^\mu \nabla_\mu - m) \tilde{\Psi}(x) = S(g^{\mu\nu} \nabla_\mu \nabla_\nu + m^2) \tilde{\Psi}(x) = 0. \] (II.13)

While Eq. (II.13) is mainly a re-statement of the fact that (II.8) is a solution of (II.1), Eq. (II.12) implies that the gyro-gravitational ratio of a massive Dirac particle is one when \( R \neq 0 \), as found by Oliveira and Tiomno [8], Audretsch [9] and Kannenberg [10].

It is known that the weak field approximation \( g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \) does not fix the reference frame completely. The transformations of coordinates \( x_\mu \rightarrow x_\mu + \xi_\mu \), with \( \xi_\mu(x) \) also small of first order, are still allowed and lead to the "gauge" transformations \( \gamma_{\mu\nu} \rightarrow \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \). It is therefore necessary to show that \( \Phi_T \) in (II.9) is gauge invariant.

In fact, on applying Stokes theorem to a closed spacetime path \( C \) and using (II.10), we find that \( \Phi_T \) changes by
\[ \Delta \Phi_T = \frac{1}{4} \int_\Sigma d\tau^{\mu\nu} J_{\alpha\beta} R_{\mu\nu\alpha\beta}, \] (II.14)

where \( \Sigma \) is a surface bound by \( C \) and \( J_{\alpha\beta} \) is the total momentum of the particle. Eq. (II.14) shows that (II.9) is gauge invariant and confirms that, to first order in the gravitational field, the gyro-gravitational ratio of a Dirac particle is one even when \( R = 0 \). Use of (II.9) automatically insure that particle spin and angular momentum are treated simultaneously and correctly.

### III. NEUTRINO GEOMETRICAL OPTICS

In this Section we study the propagation of a one-flavor Dirac neutrino in the Lense-Thirring gravitational field [11] represented, in its post-Newtonian form, by
\[ \gamma_{\mu\nu} = 2\phi, \quad \gamma_{ij} = 2\phi \delta_{ij}, \quad \gamma_{0i} = h_i = \frac{2}{r^3} (J \wedge \mathbf{r})_i, \] (III.1)

where
\[ \phi = -\frac{GM}{r}, \quad h = \frac{4GM^2\omega}{5r^3} (y, -x, 0), \] (III.2)
and \( M, R, \omega = (0, 0, \omega) \) and \( J \) are mass, radius, angular velocity and angular momentum of the source. This metric is particularly interesting because it describes a rotating source and has no Newtonian counterpart. By using the freedom allowed by local Lorentz transformations, the vierbein field to \( \mathcal{O}(\gamma_{\mu\nu}) \) is
\[ e_i^0 = 0, \quad e_i^0 = 1 - \phi, \quad e_i^i = h_i, \quad e_i^l = (1 + \phi) \delta_i^l. \] (III.3)

It is also useful to further isolate the gravitational contribution in (III.3) by writing \( e_i^\mu \simeq \delta_i^\mu + h_i^\mu \). The spin connection can be calculated using (II.2) and (III.3) and is
\[ \Gamma_0 = -\frac{1}{2} \phi_{\beta j} \sigma^{ij} - \frac{1}{8} (h_{i,j} - h_{j,i}) \sigma^{ij}, \] (III.4)
\[ \Gamma_i = -\frac{1}{4} (h_{i,j} + h_{j,i}) \sigma^{ij} - \frac{1}{2} \phi_{\beta j} \sigma^{ij}. \]

Explicit expressions for \( \Gamma_\mu \) are given in Eq.(A.4) of the Appendix.
In the geometrical optics approximation, valid whenever $|\partial \gamma_{\mu\nu}| \ll |k\gamma_{\mu\nu}|$, where $k$ is the momentum of the particle, the interaction between the angular momentum of the source and the particle’s spin vanishes. This interaction is quantum mechanical in origin. Then the geometrical phase $\Phi_G$ is sufficient to reproduce the classical angle of deflection, as it should, because $\Phi_G$ is common to the solutions of all wave equations when the spin is neglected.

The deflection angle $\varphi$ is defined by

$$\tan \varphi = \sqrt{-g_{00}p_0^0 p_0^3},$$

(III.5)

where $p_\perp$ and $p_\parallel$ are the orthogonal and parallel components of the momentum with respect to the initial direction of propagation. In the weak field approximation $\tan \varphi \simeq \varphi$ and (III.5) reduces to

$$\varphi \simeq \frac{p_\perp}{k_\parallel},$$

(III.6)

where $k_\parallel = p_\parallel$ is the unperturbed momentum and $|p_\perp| = \sqrt{-\eta_{ij}p_i^\perp p_j^\perp}$, for $p_i^\perp \sim \mathcal{O}(\gamma_{\mu\nu})$.

It is clear from (II.7) and (II.8) that, once $\Psi_0(x)$ is chosen to be a plane wave solution of the flat spacetime Dirac equation, the geometrical phase of a neutrino of four-momentum $k^\mu$ is given by

$$\psi(x) = -k_\alpha x^\alpha - \Phi_G(x),$$

(III.7)

where

$$\Phi_G(x) = -\frac{1}{4} \int_{-\infty}^{x} dz^\lambda \left[ \gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z) \right] ((x^\alpha - z^\alpha)k^\beta - (x^\beta - z^\beta)k^\alpha) + \frac{1}{2} \int_{-\infty}^{x} dz^\lambda \gamma_{\alpha\lambda}k^\alpha.$$ (III.8)

The components of $p_\perp$ can be determined from the equation

$$p_i = \frac{\partial \psi}{\partial x^i} = -k_i - \Phi_{G,i} =$$

(III.9)

$$= -k_i - \frac{1}{2} \gamma_{\alpha i}(x)k^\alpha + \frac{1}{2} \int_{-\infty}^{x} dz^\lambda (\gamma_{i\lambda,\beta}(z) - \gamma_{\beta\lambda,i}(z))k^\beta.$$

We consider below the two cases of propagation along the $z$-axis, which is parallel to the angular momentum of the source, and along the $x$-axis, orthogonal to it. In both instances, the neutrinos are assumed to be ultrarelativistic, i.e. $dz^0 \simeq dz(1 + m^2/2E^2), E \simeq k(1 + m^2/2E^2)$.

**A. Propagation along $z$**

Without loss of generality, we consider neutrinos starting from $z = -\infty$ with impact parameter $b \geq R$ and propagating along $x = b, y = 0$. From (III.9), (III.1) and (III.2), we find

$$p_1 = -\frac{1}{2} \left[ \int_{-\infty}^{x} dz^0 \gamma_{00,1}k^0 + \int_{-\infty}^{x} dz^3 \gamma_{33,1}k^3 \right]$$

(III.10)

$$= -2k \left( 1 + \frac{m^2}{2E^2} \right) \int_{-\infty}^{x} \phi_{13}dz,$$

$$p_2 = -\frac{1}{2} \gamma_{02}k^0 + \frac{1}{2} \int_{-\infty}^{x} dz^0 \gamma_{20,3}k^3 = 0$$

and

$$(p_\perp)^1 = g^{1\mu}p_\mu \simeq -p_1 = -\frac{2GMk}{b} \left( 1 + \frac{m^2}{2E^2} \right) \left( 1 + \frac{z}{r} \right),$$

(III.11)

$$(p_\perp)^2 = g^{2\mu}p_\mu \simeq h_2E = -\frac{4GM\omega^2b}{5r^3} \left( 1 + \frac{m^2}{2E^2} \right).$$
From (III.6), we finally obtain

$$\varphi = \frac{2GM}{b} \left( 1 + \frac{m^2}{2E^2} \right) \sqrt{\left( 1 + \frac{z}{r} \right)^2 + \left( \frac{2R^2b^2\omega}{5r^3} \right)^2}, \tag{III.12}$$

which is the deflection predicted by general relativity for photons, with corrections due to the neutrino mass and the rotation of the source. In the limit $z \to \infty$ Eq. (III.12) reduces to

$$\varphi = \frac{4GM}{b} \left( 1 + \frac{m^2}{2E^2} \right). \tag{III.13}$$

If, as in [1], we introduce an effective mechanical momentum

$$P'^{\alpha} = k_\alpha + \Phi G,\alpha + G_\alpha, \tag{III.14}$$

then there is a residual contribution of the real part of $\Gamma_2$ to $\varphi$. It is quantum mechanical, it amounts to

$$\frac{3GM^2\omega b^2z}{5r^5} \left( 1 - \frac{m^2}{E^2} \right)$$

and is smaller than all the other terms in $\varphi$. With this correction, we obtain

$$\varphi = \frac{2GM}{b} \left( 1 + \frac{m^2}{2E^2} \right) \sqrt{\left( 1 + \frac{z}{r} \right)^2 + \left[ \frac{2R^2b^2\omega}{5r^3} \right] \left( 1 - \frac{m^4}{E^4} \right)} \tag{III.15}.$$ 

**B. Propagation along $x$**

In this case neutrinos start from $x = -\infty$ with impact parameter $b$. For simplicity, we consider neutrinos propagating in the equatorial plane $z = 0$, $y = b$. We find

$$p_2 = -\frac{1}{2} \gamma_0 \gamma^1 k^0 + \frac{1}{2} \int_{-\infty}^{x} dz \left( \gamma_{01} k^1 - \gamma_{00} k^0 - \gamma_{10} k^1 \right) - \frac{1}{2} \int_{-\infty}^{x} dz \left( \gamma_{10} k^0 + \gamma_{11} k^1 \right) \tag{III.16}$$

$$= -k \left( 1 + \frac{m^2}{2E^2} \right) \int_{-\infty}^{x} (2\phi_2 + h_{12}) dx,$$

$$p_3 = 0 \tag{III.17}$$

and

$$(p_\perp)^2 = g^{2\mu} p_\mu \simeq -p_2 + b_2 E = -\frac{2GMk}{b} \left( 1 + \frac{m^2}{2E^2} \right) \left( 1 - \frac{2R^2\omega}{5b} \right) \left( 1 + \frac{z}{r} \right), \tag{III.18}$$

$$(p_\perp)^3 = g^{3\mu} p_\mu \simeq -p_3 = 0.$$ 

It then follows that

$$\varphi = \frac{2GM}{b} \left( 1 - \frac{2R^2\omega}{5b} \right) \left( 1 + \frac{m^2}{2E^2} \right) \left( 1 + \frac{x}{r} \right). \tag{III.19}$$ 

Contrary to the previous case, the contribution of the angular momentum of the source does not vanish in the limit $x \to \infty$. In fact, in this limit we get

$$\varphi = \frac{4GM}{b} \left( 1 - \frac{2R^2\omega}{5b} \right) \left( 1 + \frac{m^2}{2E^2} \right), \tag{III.20}$$

which coincides with the prediction of general relativity. Additional, smaller spin contributions can be obtained from $\Gamma_\mu$ as in Subsection (III.A).
IV. HELICITY TRANSITIONS

In what follows, it is convenient to write the left and right neutrino wave functions in the form

\[ \Psi_0(x) = \nu_{0L,R} e^{-ik_{\alpha}x^{\alpha}} = \sqrt{\frac{E+m}{2E}} \left( \frac{\nu_{L,R}}{E+m} \right) e^{-ik_{\alpha}x^{\alpha}}, \]  

where \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \) represents the Pauli matrices. \( \nu_{L,R} \) are eigenvectors of \( \sigma \cdot k \) corresponding to negative and positive helicity and \( \nu_{0L,R}(k) \equiv \nu_{0L,R}(k)\gamma^0, \nu_{0L,R}(k)\nu_{0L,R}(k) = 1 \). This notation already takes into account the fact that if \( \nu_{\pm} \) are the helicity states, then we have \( \nu_L \simeq \nu_- \), \( \nu_R \simeq \nu_+ \) for relativistic neutrinos.

In general, the spin precesses during the motion of the neutrino. This can be seen, for instance, from the contribution \( \Phi_s \) in \( \Phi_T \). The expectation value of the contribution of \( \Gamma_0 \) to the effective mechanical momentum (III.14) can in fact be re-written in the form

\[ \frac{1}{2} \Psi_0^\dagger \hat{\Omega} \cdot \vec{\sigma} \Psi_0, \]  

where \( \hat{\Omega} \equiv \frac{GM^2}{2r^3} \left( 1 - \frac{2}{r} \right) \hat{\omega}. \) Eq. (IV.2) represents the spin-rotation coupling, or Mashhoon term [12], for the Lense-Thirring metric. Here rotation is provided by the gravitational source, rather than by the particles themselves.

We now study the helicity flip of one flavor neutrinos as they propagate in the gravitational field produced by a rotating mass. The neutrino state vector can be written as

\[ |\psi(\lambda)\rangle = \alpha(\lambda) |\nu_R\rangle + \beta(\lambda) |\nu_L\rangle, \]  

where \( |\alpha|^2 + |\beta|^2 = 1 \) and \( \lambda \) is an affine parameter along the world-line. In order to determine \( \alpha \) and \( \beta \), we can write Eq. (II.8) as

\[ |\psi(\lambda)\rangle = \hat{T}(\lambda) |\psi(0)\rangle, \]  

where

\[ \hat{T} = -\frac{1}{2m} (-i\gamma^{\mu}(x)D_\mu - m) e^{-i\Phi_T}, \]  

and \( |\psi(0)\rangle \) is the corresponding solution in Minkowski spacetime. The latter can be written as

\[ |\psi(0)\rangle = e^{-ik_{\alpha}x^{\alpha}} [\alpha(0) |\nu_R\rangle + \beta(0) |\nu_L\rangle]. \]  

Strictly speaking, \( |\psi(\lambda)\rangle \) should also be normalized. However, it is shown below that \( \alpha(\lambda) \) is already of \( O(\gamma_{\mu\nu}) \), can only produce higher order terms and is therefore unnecessary in this calculation. From (IV.3), (IV.4) and (IV.6) we obtain

\[ \alpha(\lambda) = \langle \nu_R | \psi(\lambda) \rangle = \alpha(0) \langle \nu_R | \hat{T} | \nu_R \rangle + \beta(0) \langle \nu_R | \hat{T} | \nu_L \rangle. \]  

An equation for \( \beta \) can be derived in an entirely similar way.

If we consider neutrinos which are created in the left-handed state, then \( |\alpha(0)|^2 = 0, |\beta(0)|^2 = 1 \), and we obtain

\[ P_{L\rightarrow R} = |\alpha(\lambda)|^2 = \left| \langle \nu_R | \hat{T} | \nu_L \rangle \right|^2 = \left| \int_{\lambda_0}^{\lambda} \langle \nu_R | \hat{x}^{\mu} \partial_\mu \hat{T} | \nu_L \rangle d\lambda \right|^2, \]  

where \( \hat{x}^{\mu} = k^{\mu}/m \). As remarked in [1], \( \hat{x}^{\mu} \) need not be a null vector if we assume that the neutrino moves along an "average" trajectory. We also find, to lowest order,

\[ \partial_\mu \hat{T} = \frac{1}{2m} \left( -i2m\Phi_{G,\mu} - i(\gamma^{\hat{\alpha}}k_{\alpha} + m)\Phi_{s,\mu} + \gamma^{\hat{\alpha}}(h^{\beta}_{\alpha,\mu}k_{\beta} + \Phi_{G,\alpha\mu}) \right) \]  

\[ \Phi_{s,\Lambda} = \gamma^{\hat{\alpha}}, \quad \Phi_{G,\alpha\mu} = k_{\beta}\gamma^{\hat{\alpha}}_{\alpha\mu}, \quad \nu_{0}^{\hat{\alpha}}(\gamma^{\hat{\alpha}}k_{\alpha} + m) = 2E]\nu_{0}\gamma^{\hat{\alpha}}. \]

where \( \gamma^{\hat{\alpha}}_{\alpha\mu} \) are the usual Christoffel symbols, and

\[ \langle \nu_R | \hat{x}^{\mu} \partial_\mu \hat{T} | \nu_L \rangle = \frac{E}{m} \left[ -i \frac{k^{\lambda}}{m} \nu_{0} \Gamma_{\lambda} + \frac{k^{\lambda}k^{\mu}}{2mE} (h^{\mu}_{\alpha,\lambda} + \Gamma^{\mu}_{\alpha\lambda})\nu_{0}^{\hat{\alpha}}\gamma^{\hat{\alpha}} | \nu_L \rangle \right]. \]

In what follows, we compute the probability amplitude (IV.10) for neutrinos propagating along the \( z \) and the \( x \) directions explicitly.
A. Propagation along $z$

For propagation along the $z$-axis, we have $k^0 = E$ and $k^3 \equiv k \simeq E(1 - m^2/2E^2)$. As in Section III, we choose $y = 0$, $x = b$. With the help of Eq.(A.7), we get

$$-i\frac{k^\lambda}{m}\nu_R\Gamma_\lambda\nu_L = \frac{k}{m}\phi_{1,1} + \frac{i}{4E}h_{2,3},$$  \hspace{1cm} (IV.11)

$$\frac{k^\lambda k_\mu}{2mE}(h^\mu_{\alpha\lambda} + \Gamma^\mu_{a\lambda})\nu_R\hat{T}\nu_L = -\frac{k}{2m}\left(1 + \frac{k^2}{E^2}\right).$$

Summing up, and neglecting terms of $\mathcal{O}(m/E)^2$, Eq.(IV.10) becomes

$$\langle\nu_R|\hat{x}^\mu\partial_\mu\hat{T}|\nu_L\rangle = \frac{1}{2}\phi_{1,1} + \frac{i}{4}h_{2,3}.$$  \hspace{1cm} (IV.12)

The contributions to $\mathcal{O}((E/m)^2)$ vanish. As a consequence

$$\frac{d\alpha}{dz} \simeq \frac{m}{E}\frac{d\alpha}{d\lambda} = \frac{m}{E}\left(\frac{1}{2}\phi_{1,1} + \frac{i}{4}h_{2,3}\right),$$  \hspace{1cm} (IV.13)

and the probability amplitude for the $\nu_L \rightarrow \nu_R$ transition is of $\mathcal{O}(m/E)$, as expected. Integrating (IV.13) from $-\infty$ to $z$, yields

$$\alpha \simeq \frac{m}{E}\left[\frac{1}{2}\int_{-\infty}^{z}dz\phi_{1,1} + \frac{i}{4}h_{2,3}(z)\right] = \frac{m}{E}\frac{GM}{2b}\left[1 + \frac{z}{r} - i\frac{2\omega R^2b^2}{5r^3}\right].$$  \hspace{1cm} (IV.14)

It also follows that

$$P_{L \rightarrow R}(-\infty, z) \simeq \left(\frac{m}{E}\right)^2\left(\frac{GM}{2b}\right)^2\left[1 + \frac{z}{r}\right]^2 + \left(\frac{2\omega R^2b^2}{5r^3}\right)^2.$$  \hspace{1cm} (IV.15)

The first of the two terms in (IV.15) comes from the mass of the gravitational source. The second from the source’s angular momentum and vanishes for $r \rightarrow \infty$ because the contribution from $-\infty$ to 0 exactly cancels that from 0 to $+\infty$. In fact, if we consider neutrinos propagating from 0 to $+\infty$, we obtain

$$P_{L \rightarrow R}(0, +\infty) \simeq \left(\frac{m}{E}\right)^2\left(\frac{GM}{2b}\right)^2\left[1 + \left(\frac{2\omega R^2b^2}{5r}\right)^2\right].$$  \hspace{1cm} (IV.16)

According to semiclassical spin precession equations [13], there should be no spin motion when spin and $\vec{\omega}$ are parallel as in the present case. This is a hint that rotation of the source, rather than of the particles, should produce a similar effect. The probabilities (IV.15) and (IV.16) mark therefore a departure from expected results. They are however small of second order. Both expressions vanish for $m \rightarrow 0$, as it should for a stationary metric. In this case, in fact, helicity is conserved [14]. It is interesting to observe that spin precession also occurs when $\omega$ vanishes [15,16]. In the case of (IV.15) the mass contribution is larger when $b < (r/R)\sqrt{\frac{2E}{M}}$, which, close to the source, with $b \sim r \sim R$, becomes $R\omega < 5/2$ and is always satisfied. In the case described by (IV.16), the rotational contribution is larger if $b/R < 2\omega R/5$ which effectively restricts the region of dominance to a narrow strip about the $z$-axis in the equatorial plane, if the source is compact and $\omega$ is relatively large.

B. Propagation along $x$

In this case, we put $k^0 = E$, $k^1 \equiv k \simeq E(1 - m^2/2E^2)$. As in Section III, the calculation can be simplified by assuming that the motion is in the equatorial plane with $z = 0$, $y = b$. We then have
 Integrating (IV.19) from $-\infty$ to $\omega R < 0$, the contributions to between flavor eigenstates (Greek indices) and mass eigenstates (Latin indices) is given by the standard expression

$\Phi(\lambda,\mu) = \Phi(0,\mu) e^{-i \omega R \gamma^0}.$

The mass term is larger when $2 \frac{\omega R}{b} < 1$. At the poles $b \sim R$ and the mass term dominates because the condition $\omega R < 5/2$ is always satisfied. The angular momentum contribution prevails in proximity of the equatorial plane. The transition probability vanishes at $b = 2 \omega R^2/5$.

**V. FLAVOR OSCILLATIONS**

Eq. (II.9) can be recast into a form that contains only first order contributions. By using (II.9) we find

$$\Psi(x) = f(x) e^{-i \Phi(x)} \Psi_0,$$

where

$$f(x) = \frac{1}{2m} \left[ e^{\mu \gamma^0} (k_\mu + \Phi_{G, \mu}) + m \right]$$

and $\Psi_0$ now represents the phase independent part of (IV.1) that refers to mass eigenstate neutrinos. The relationship between flavor eigenstates (Greek indices) and mass eigenstates (Latin indices) is given by the standard expression

$$|\nu_\alpha(x)\rangle = \sum_j U_{\alpha j}(\theta) |\nu_j(x)\rangle,$$

into which (V.1) must now be substituted. We find

$$|\nu_\alpha(x(\lambda))\rangle = \sum_j U_{\alpha j}(\theta) f_j e^{-i \Phi_j(x) - i k_\mu \gamma^\mu x^\mu} |\nu_j\rangle = \sum_j U_{\alpha j} \phi(x(\lambda)) |\nu_j\rangle,$$

where
where

$$\dot{O}(\lambda) = \frac{1}{2m} [e^{\mu_\alpha} J(\mu_\alpha + \hat{\Phi}_{G,\mu}) + m] e^{-i(\hat{\Phi} + k \cdot x)}, \quad (V.5)$$

and \( U \) is the mixing matrix

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (V.6)$$

Restricting \( \alpha \) to the flavor \( e \) for simplicity, we define the column matrix

$$\chi = \begin{pmatrix} \langle \nu_e | \hat{O} \langle \nu_e(x) | \rangle \\ \langle \nu_\mu | \hat{O} \langle \nu_\mu(x) | \rangle \end{pmatrix} = \begin{pmatrix} \cos^2 \theta \hat{O}_{11} + \sin \theta \cos \theta \hat{O}_{12} + \sin \theta \cos \theta \hat{O}_{21} + \sin^2 \theta \hat{O}_{22} \\ -\sin \theta \cos \theta \hat{O}_{11} - \sin^2 \theta \hat{O}_{12} + \cos^2 \theta \hat{O}_{21} + \sin \theta \cos \theta \hat{O}_{22} \end{pmatrix} = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} U \phi, \quad (V.7)$$

$$= U^\dagger \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} U \phi, \quad (V.8)$$

where \( \hat{O}_{ij} = \langle \nu_i | \hat{O}(\lambda) | \nu_j \rangle \equiv \nu_i^\dagger \hat{O} \nu_j = \langle \zeta_i | \hat{O}(\lambda) | \zeta_j \rangle \delta_{ij}, \) \( i, j = 1, 2, \) and \( \phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \)

The differentiation with respect to the parameter \( \lambda \) gives

$$i \frac{d\chi}{d\lambda} = U^\dagger \begin{pmatrix} i \frac{dO_{11}}{d\lambda} & 0 \\ 0 & i \frac{dO_{22}}{d\lambda} \end{pmatrix} U \phi. \quad (V.9)$$

By keeping only terms of the first order in \( \gamma_{\alpha\beta} \), observing that \( \Phi_{G,\mu}\gamma^{\mu}_{\alpha\rho} \approx O(\gamma^2_{\mu\nu}) \), that \( \Phi_{G,\mu}\Gamma_{\rho} \approx O(\gamma^2_{\mu\nu}) \) and using (A.2) we find that the matrix elements \( i d\hat{O}_{ij} / d\lambda \) in (V.9) are of the form

$$i \frac{d\hat{O}_{ij}}{d\lambda} = A^{(i)} O_{ij} + \sum_k d^{(i)}_{ik} O_{kj} + C^{(i)}_{ij}, \quad (V.10)$$

where the index \( (i) \) refers to the \( i \)-th mass eigenstate, and

$$A^{(i)} = \hat{\chi}^{(i)} \left( k_p^{(i)} + \Phi_{G,\mu}^{(i)} \right), \quad d^{(i)}_{ik} = \langle \nu_i | \hat{\chi}^{(i)} \Gamma_{\rho} | \nu_j \rangle = \hat{\chi}^{(i)} \langle \zeta_i | \Gamma_{\rho} | \zeta_j \rangle \delta_{ij}, \quad (V.11)$$

$$C^{(i)}_{ij} = \frac{i}{2m} \hat{\chi}^{(i)} \left( k_{\rho}^{(i)} + \delta^{\rho}_{\alpha} \Phi_{G,\mu}^{(i)} \right) \langle \zeta_i | \Gamma_{\rho} | \zeta_j \rangle \delta_{ij} e^{-i\Phi^{(i)}_{G,\mu} k_{\rho}^{(i)} - i\hat{\chi}^{(i)} \cdot x}. \quad (V.11)$$

The equation of evolution therefore is

$$i \frac{d\chi}{d\lambda} = (A_f + d_f) \chi + C, \quad (V.12)$$

where

$$A_f = U^\dagger \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} U \phi, \quad (V.13)$$

and

$$C = U^\dagger \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix} U \phi. \quad (V.14)$$

A relation similar to (V.13) links \( d_f \) to \( d \).

Up to (V.14) the derivation has been general and can be applied to any metric. It is not possible to say more about the matrix \( C \) at this point, except that it may contain some dissipation terms.

In the case of the Lense-Thirring metric, however, \( C \) can be written as \( C = \tilde{C} \chi \). After some algebra, the matrix elements \( C_{ij} \) become, in fact,

$$C_{ij}^{(i)} \sim -i \frac{E}{4 \gamma_{00,3}} e^{-i\hat{\chi}^{(i)} \cdot (x - x_0) - i\Phi^{(i)}_{G,\mu}} \delta_{ij}. \quad (V.15)$$

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Starting from Eq. (V.5), we can write

\[ O_{ij} \simeq e^{-ik^{(i)}(x-x_0)-i\Phi^{(i)}_G(1 + A^{(i)} + i2J_0)\delta_{ij}}, \quad (V.16) \]

where

\[ A^{(i)} \equiv A = \frac{1}{2m_i} (k_a^{(i)} + \delta^{(i)}_a \Phi^{(i)}_G) (\nu_1 |\gamma^a| \nu_1) - \frac{1}{2} h_0^0 \quad (V.17) \]

\[ J_0 = \frac{GM\omega}{10} \frac{1}{x^2 + y^2} \left[ \left( \frac{z}{r} \right)^3 - \left( \frac{z_0}{r_0} \right)^3 \right] + \frac{GMxy}{4} \frac{1}{r} \left( \frac{1}{x^2 + y^2} - \frac{1}{y^2 + z^2} \right). \]

To \( O(\gamma_0 m^2/E^2) \), \( A^{(i)} = A \) is independent of neutrino mass and energy. It is, in fact a pure geometrical term. Using (V.16), Eq. (V.15) becomes

\[ C^{(i)}_{ij} = C_{ij} \sim -i \frac{E}{4} \frac{\gamma_{00,3}}{1 + A + i2J_0} O_{ij} = \bar{C} O_{ij}, \quad (V.18) \]

where

\[ \bar{C} \sim -i \frac{E}{4} \gamma_{00,3}. \quad (V.19) \]

To \( O(m^2/E^2) \), \( \bar{C} \) does not depend on the mass eigenstate index \( (i) \). Eq. (V.10) therefore becomes

\[ i \frac{dO_{ij}}{d\lambda} \simeq A^{(i)} O_{ij} + \sum_k d_{ik}^{(i)} O_{kj} + \bar{C} O_{ij}, \quad (V.20) \]

and the matrix \( C \) is

\[ C = U^\dagger \begin{pmatrix} \bar{C} O_{11} & 0 \\ 0 & \bar{C} O_{22} \end{pmatrix} U \phi = \bar{C} \chi. \quad (V.21) \]

In the equation of evolution (V.12) one can then carry out the transformation \( \chi \rightarrow e^{-i \int_0^{\lambda} \bar{C} d\lambda} \chi \) and remove the \( \bar{C} \) dependence by means of a suitable normalization.

The equation of evolution can be re-cast into a more traditional form by taking \( \dot{x}^\rho \) as the tangent vector to the null world line. Then \( \dot{x}^\rho \dot{x}_\rho = 0, k^\rho = (E, k^i) \sim (E, n^i E(1 - m^2/2E^2)), \) where \( n^i \) is a unit vector parallel to the neutrino three-momentum, \( k^0 = \dot{x}^0 \) and \( k^i \approx (1 - \varepsilon) \dot{x}^i \), with \( \varepsilon \ll 1 \). The matrix elements of (V.13) contain terms of the form \( \dot{x}^{(i)} \rho k^{(i)}_\rho \approx \frac{m^2}{2} + \varepsilon E^2 \). The term with \( E \) is diagonal in the matrix of evolution and does not therefore contribute to the oscillations. The equation of evolution (V.9) then becomes

\[ \dot{\chi} = \left( \frac{M_f^2}{2} + \Phi^{(f)}_G + \Gamma^{(f)} \right) \chi, \quad (V.22) \]

where the flavor mass matrix \( M_f \) is related to the vacuum mass matrix in the flavor base by

\[ M_f^2 = U^\dagger \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U, \quad (V.23) \]

while

\[ \Phi_G^{(f)} = U^\dagger \begin{pmatrix} \langle \xi_1 | k^\rho \Phi_{G, \rho} | \xi_1 \rangle \\ 0 \\ 0 \\ 0 \end{pmatrix} U, \quad (V.24) \]

and

\[ \Gamma^{(f)} = U^\dagger \begin{pmatrix} \langle \xi_1 | k^\rho \Gamma_{\rho} | \xi_1 \rangle \\ 0 \\ 0 \\ 0 \end{pmatrix} U. \quad (V.25) \]
To $O(\gamma_{\mu\nu} m_\ell^2/E^2)$, Eq. (V.22) becomes

$$i \frac{d\chi}{d\lambda} \simeq \left( \frac{M_f^2}{2} + k^\alpha (\Phi_{\alpha,\mu} + \Gamma_{\alpha}^{(f)}) \right) \chi. \tag{V.26}$$

The term in brackets on the r.h.s. of (V.26) is the effective Hamiltonian of evolution.

By writing the effective mechanical momentum in the form (III.14), squaring it and keeping only first order terms, we find

$$M^2 \simeq m^2 + 2k^\alpha \Phi_{\alpha,\mu}. \tag{V.27}$$

and

$$|P_{\alpha}^{eff}| = \sqrt{E^2 - M^2} \simeq E - \frac{M^2}{2E} = E - \frac{1}{E} \left[ \frac{m^2}{2E} - k^\alpha (\Phi_{\alpha,\mu} + \Gamma_{\alpha}) \right]. \tag{V.28}$$

The third and fourth terms of (V.28) represent the gravitational contributions (V.24) and (V.25) and provide an explicit expression for the term $\vec{p} \cdot \vec{A}_T \rho_T$ in Eq.(27) of [1] when matter effects are neglected. These can be easily incorporated in the diagonal element of (V.22). The effective potential $V_{\nu,\nu}^{\alpha}$ induced by matter, depends on neutrino flavors and is defined by [17]

$$V_{\nu,\nu}^{\alpha} = -V_{\nu,\nu}^{\alpha} = V_0 (3Y_e - 1 + 4Y_{\nu_e}) \tag{V.29},$$

$$V_{\nu,\nu}^{r,\nu} = -V_{\nu,\nu}^{r,\nu} = V_0 (6Y_e - 1 + 2Y_{\nu_e}) \tag{V.30},$$

where $Y_e (Y_{\nu_e})$ represents the ratio between the number density of electrons (neutrinos),

$$V_0 = \frac{G_F \rho}{\sqrt{2m_n}} = \frac{\rho}{10^{14} gr/cm^2} 3.8 eV, \tag{V.31}$$

$m_n = 938 MeV$ is the nucleon mass, $\rho$ the matter density, and $G_F$ the Fermi coupling constant.

**A. Transition amplitudes**

In order to calculate the transition amplitudes between states of different flavor, we follow a procedure analogous to that developed in Section IV for the helicity transitions. To discuss oscillations, we assume that neutrinos in different mass eigenstates have the same energy and keep in mind that energy is conserved in a stationary gravitational field. In order that different neutrinos interfere at the same final point, we require, as usual, that the relevant components of the wave function do not start from the same initial point. This however introduces negligible corrections in the final results if one assumes, as we do, that $\Delta m^2 = m_2^2 - m_1^2 \ll m^2$ and $\gamma_{\mu\nu} \ll 1$.

Let us consider, for simplicity, only two flavors and, in particular, the transition $\nu_e \rightarrow \nu_{\mu}$. The transition probability is given by

$$P_{\nu_e \rightarrow \nu_{\mu}} = \frac{|\langle \nu_{\mu} | \nu_e (\lambda) \rangle|^2}{\langle \nu_e (\lambda) | \nu_e (\lambda) \rangle}, \tag{V.32}$$

where $|\nu_e (\lambda) \rangle$ represents at $\lambda$ a neutrino initially created in the state $|\nu_e \rangle$. From (V.3) and (V.6) we obtain

$$\langle \nu_{\mu} | \nu_e (\lambda) \rangle = \sin \theta \cos \theta (\langle \nu_2 | \nu_2 (\lambda) \rangle - \langle \nu_1 | \nu_1 (\lambda) \rangle) \tag{V.33}$$

$$\langle \nu_e (\lambda) | \nu_e (\lambda) \rangle = \cos^2 \theta \langle \nu_1 (\lambda) | \nu_1 (\lambda) \rangle + \sin^2 \theta \langle \nu_2 (\lambda) | \nu_2 (\lambda) \rangle. \tag{V.34}$$

For a generic mass eigenstate, we find

$$\frac{d}{d\lambda} \langle \nu | \nu_e (\lambda) \rangle = \left\{ \nu | i \frac{d\nu (\lambda)}{d\lambda} \right\} = -i \hat{\pi}^\mu \hat{\pi}_\mu (\nu | \nu_e (\lambda) \rangle + \hat{\pi}^\mu (\nu_0 | \hat{\pi}_\mu | \nu_0 \rangle) e^{-i \hat{\pi}_\mu \hat{\pi}^\mu}. \tag{V.34}$$

Since $\langle \nu_0 | \hat{\pi}_\mu | \nu_0 \rangle$ is already $\sim O(\gamma_{\mu\nu})$, it is possible to carry out the substitution

$$e^{-i \hat{\pi}_\mu \hat{\pi}^\mu} = (\nu | \nu_0 (\lambda) \rangle) \simeq (\nu | \nu (\lambda) \rangle). \tag{V.35}$$
Eq. (V.34) has the first order solution

\[ \langle \nu | \nu(\lambda) \rangle \simeq \exp \left\{ \int_P^Q d\lambda \hat{\pi}^{\mu} \left( -ik_\mu + \langle \nu_0 | \partial_\mu \hat{T} | \nu_0 \rangle \right) \right\} , \]  

(V.36)

where \( P \) and \( Q \) are the points at which neutrinos are generated and detected.

We also obtain

\[ \frac{d}{d\lambda} \langle \nu(\lambda) | \nu(\lambda) \rangle = \left\{ \frac{d\nu(\lambda)}{d\lambda} \right\} \langle \nu(\lambda) | \nu(\lambda) \rangle + \left\{ \nu(\lambda) \right\} \left\{ \frac{d\nu(\lambda)}{d\lambda} \right\} = \hat{\pi}^{\mu} \langle \nu_0 | \partial_\mu (\hat{T} + \hat{T}^\dagger) | \nu_0 \rangle , \]

(V.37)

which has the first order solution

\[ \langle \nu(\lambda) | \nu(\lambda) \rangle \simeq \exp \left\{ \int_P^Q d\lambda \hat{\pi}^{\mu} \langle \nu_0 | \partial_\mu (\hat{T} + \hat{T}^\dagger) | \nu_0 \rangle \right\} . \]

(V.38)

From (IV.9) we get

\[ \langle \nu_0 | \partial_\mu \hat{T} | \nu_0 \rangle = -i\Phi_{G,\mu} - \frac{E}{m} \vec{p}_0 \Gamma_\mu \nu_0 + \frac{k_\nu}{2E} (h^\nu_{0,\mu} + \Gamma^\nu_{0,\mu}) \]

(V.39)

\[ \langle \nu_0 | \partial_\mu (\hat{T} + \hat{T}^\dagger) | \nu_0 \rangle = \frac{k_\nu}{E} (h^\nu_{0,\mu} + \Gamma^\nu_{0,\mu}) . \]

Substituting (V.39) into (V.32), we finally obtain

\[ P_{\nu_\mu \rightarrow \nu_\nu} = \left| \sin \theta \cos \theta \exp \left( -i \int_P^Q d\lambda \hat{\pi}^{\mu} \frac{E}{m} \vec{p}_0 \Gamma_\mu \nu_0 \right) \right|^2 . \]

(V.40)

\[ \cdot \left[ \exp \left( -i \int_P^Q d\lambda \hat{\pi}^{\mu} (k^{(2)}_{\mu,G,\mu} + \Phi^{(2)}_{G,\mu}) \right) - \exp \left( -i \int_P^Q d\lambda \hat{\pi}^{\mu} (k^{(1)}_{\mu} + \Phi^{(1)}_{G,\mu}) \right) \right]^2 \]

\[ = \sin^2 2\theta \sin^2 \left( \frac{1}{2} \int_P^Q d\lambda \hat{\pi}^{\mu} (\Delta k_{\mu} + \Delta \Phi_{G,\mu}) \right) . \]

In deriving (V.40) we have used the fact that the terms \( \bar{\nu}_0 \theta^{\mu\nu} \nu_0 \) are always proportional to \( m/E \), as shown in the Appendix, and that the contributions from the coefficients \( \Gamma_\mu \) are therefore independent of \( m, E \) and \( \hat{\pi} \). It follows from (V.40), that the effect of the gravitational field is simply obtained by the substitution \( k_\mu \rightarrow k_\mu + \Phi_{G,\mu} \equiv p_\mu \). This conforms to the semiclassical principle that action is the quantum phase of the particle.

Eq. (V.40) can be simply rewritten as

\[ P_{\nu_\mu \rightarrow \nu_\nu} = \sin^2 2\theta \sin^2 \left( \frac{1}{2} \int_P^Q dx^j (\Delta k_j + \Delta \Phi_{G,j}) \right) , \]

(V.41)

because the neutrinos have fixed energy, hence \( \Delta k_0 = 0 \), and \( \Phi_{G,\theta} = 0 \) for static gravitational fields.

If, for simplicity, we consider the motion to be parallel to the \( x^0 \)-axis, the first term can be written, to \( O(m/E) \), in the traditional form

\[ \frac{1}{2} \int_P^Q dx^j \Delta k_j = \frac{\Delta m^2}{4E} (z - z_0) . \]

(V.42)

Using (III.8), (III.9) and \( dz^0 = (E/k)dz^i \), we can write

\[ \Phi_{G,i}(z) = \frac{1}{2} \gamma_{i\alpha} k^\alpha - \frac{1}{2} \int_{z_0}^z dz^\lambda (\gamma_{i\lambda,\beta} - \gamma_{\beta\lambda,i}) k^\beta \]

(V.43)

\[ = E(2\phi(z) + h_i(z)) + \frac{m^4}{4E^2} \phi(z) + E \left( 1 + \frac{m^2}{2E^2} \right) \phi(z_0) , \]
where, as above, we have retained only the lowest non vanishing terms in $m/E$. We finally obtain
\[
\Delta \Phi_{G,i}(z) = \frac{m^2 \Delta m^2}{2E^3} \phi(z) + \frac{\Delta m^2}{2E} \phi(z_0) .
\] (V.44)

The first term of (V.44) is the well known gravitational contribution to the phase of oscillating neutrinos. The second term corresponds to the redefinition of the constant $E \rightarrow E \sqrt{g_{00}(z_0)}$ in the non-gravitational term of the oscillation for a neutrino created at $z_0$, at the desired order of approximation. With this redefinition, the phase that appears in (V.40) becomes
\[
\Omega = \frac{\Delta m^2}{4E^2} (z - z_0) + \frac{m^2 \Delta m^2}{2E^3} \int_{z_0}^{z} d\zeta \phi(\zeta) .
\] (V.45)

The coordinate difference $z - z_0$ is not, however, the physical distance between two points. This is in fact defined as
\[
l = \int_{z_0}^{z} \sqrt{-g_{ii}}dx^i \approx \int_{z_0}^{z} (1 - \phi)dx^i ,\]
where $E$ is the neutrino energy measured by an inertial observer at rest at infinity. Introducing $E_I = E \sqrt{g_{00}(z)}$, which is the energy measured by a locally inertial observer momentarily at rest in the gravitational field, the phase $\Omega$ can be entirely rewritten in terms of physical quantities as
\[
\Omega = \frac{\Delta m^2}{4} \int_{E_I}^{l} \frac{dl}{E_I} + \frac{m^2 \Delta m^2}{2E^3} \int_{z_0}^{z} d\zeta \phi(\zeta) .
\] (V.46)

Eq. (V.46) reflects the fact that the curvature of spacetime affects the oscillation probability through the gravitational red-shift of the local energy $E_I$ and the proper distance $dl$.

The oscillation amplitude does not depend on $\gamma_{0\nu}$. This means that the relevant oscillation parameters like the phase $\Omega$, depend quadratically (and not linearly) on the angular momentum of the source. This should be expected because, as noted by Wudka [18], the quantum mechanical phase is a scalar, whereas the angular momentum is a pseudovector.

VI. CONCLUSIONS

In this paper we have applied the weak field solution of [2] to the study of neutrinos in a Lense-Thirring field. There are advantages to using this solution. It is exact to first order, is covariant and gauge invariant. It is based on the Dirac equation and reproduces well all gravitational effects that have so far been observed [19,20], or have been predicted to exist for a spin-1/2 particle [6,7]. It also refers to fermions with unit gyro-gravitational ratio [3] and thus allows a unified treatment of spin-rotation and angular momentum-rotation coupling without requiring ad hoc procedures.

We have applied the solution to the propagation of single flavor neutrinos. Two cases have been considered in the geometrical optics approximation. In both instances the larger contribution to the deviation $\varphi$ comes from the mass of the source and is as predicted by Einstein’s theory, with corrections due to the neutrino mass. For propagation parallel to the axis of rotation of the source, the rotation corrections vanish at infinity. Not so for propagation perpendicular to the axis of rotation.

It may be argued that spin contributions exist in the approximation used by introducing an effective neutrino mechanical momentum given by (III.14) that incorporates a contribution from $\Gamma_\mu$ as in [1]. This definition is also supported by the form of the flavor oscillation Hamiltonian on the r.h.s. of (V.26)-(V.27). The contributions that come from taking the expectation values of $\Gamma_\mu$ are given in (III.15) and are much smaller than those of (III.12).

We have then calculated the helicity transition amplitudes of ultrarelativistic, single flavor neutrinos as they propagate in the Lense-Thirring field. These transitions are interesting because at high energies chirality states are predominantly helicity states and right-handed neutrinos do not interact. The transition probabilities are of $O(\gamma_{\mu\nu}^2)$. Two directions of propagation have again been selected and the results contain contributions from both mass and angular momentum of the source. The transitions also occur in the absence of rotation or with spin parallel to rotation, which is unexpected on semiclassical grounds. The mass contributions predominate when the neutrinos propagate from $r = 0$ to $r = \infty$ (and matter effects are neglected), provided $b > 2\omega R^2/5$. There is, however, a narrow region about the axis of propagation in the equatorial plane where the $\omega$ contribution is larger. The rotational contribution behaves differently in the two cases. It vanishes as $z \rightarrow \mp\infty$ for propagation along $z$, but not so as $x \rightarrow \infty$ in the second case. In addition, when the neutrinos propagate from $x = 0$ to $x = \infty$, the mass term dominates in the neighborhood of the poles, while the contribution of $\omega$ is larger close to the equator, with no attenuation at $b = 2\omega R^2/5$. 

We have also calculated gravity induced, two-flavor oscillations and derived the relative equation and effective Hamiltonian. A comparison with the simplified approach of [1] leads to explicit expressions for the term \( \tilde{p} \cdot \dot{A}_f \mathcal{P}_C \) of these authors. The expressions contain \( \Gamma_{\alpha} \), as expected, and also \( \Phi_{G,\mu} \). These terms do not exhaust, however, all possibilities if the metric is not stationary, because of the presence of the matrix \( C \) in (V.12). In the general case, therefore, the evolution of \( \chi \) is more complicated and the effective Hamiltonian does not comply with the simpler form given in [1], or in (V.26). Finally, the transition probabilities do indeed oscillate for the Lense-Thirring metric, and the curvature of spacetime enters the oscillation probability through the gravitational red-shift of the local energy \( E_l \) and the proper distance \( dl \).

The neutrino mass appears quadratically in all effects considered and, in particular, as \( \Delta m^2 \) in flavor oscillations. It therefore seems possible, in principle, to obtain the neutrino masses from an appropriate combination of measurements of these effects, without resorting to models that might depend on additional aspects of particle physics. The only ingredient used is in fact the covariant Dirac equation whose merits have been extolled at the beginning of this section.

The results presented in this paper agree with those of other authors, where appropriate. They can be applied to a number of problems in astroparticle physics and cosmology [21]. For instance, an interesting question is whether gravity induced helicity and flavor transitions could effect changes in the ratio \( \nu_e : \nu_\mu : \nu_\tau \) of the expected fluxes at Earth.

Lepton asymmetry in the Universe [21] also is an interesting problem. It is known that the active-sterile oscillation of neutrinos can generate a discrepancy in the neutrino and antineutrino number densities. The lepton number of a neutrino of flavor \( f \) is defined by \( L_f = (n_{\nu_f} - n_{\bar{\nu}_f})/n_{\gamma}(T) \), where \( n_{\nu_f} (n_{\bar{\nu}_f}) \) is the number density of neutrinos (antineutrinos) and \( n_{\gamma}(T) \) is the number density of photons at temperature \( T \). As discussed in Sec. IV, the gravitational field generates transitions from left-handed (active) neutrinos to right-handed (sterile) neutrinos. If, in primordial conditions, (IV.16) and (IV.22) become larger, then helicity transitions may contribute in some measure to lepton asymmetry.

**APPENDIX A: USEFUL FORMULAE**

In this Appendix, we collect useful formulae and results that have been extensively used in the paper.

**Derivatives of \( \Phi_G \):** The first derivative with respect to \( x^\mu \) gives

\[
\Phi_{G,\mu} = -\frac{1}{2} \int_{P}^{x} dz^\lambda (\gamma_\mu \lambda, \beta - \gamma_\beta \lambda, \mu) k^\beta + \frac{1}{2} \gamma_\alpha \mu k^\alpha, \tag{A.1}
\]

whereas the second derivative is

\[
\Phi_{G,\mu\nu} = k_\alpha \Gamma^\alpha_{\mu\nu}, \tag{A.2}
\]

where \( \Gamma^\alpha_{\mu\nu} \) are the Christoffel symbols of the second type.

**Christoffel symbols and spin connections.** To \( O(\gamma_{\mu\nu}) \), the Christoffel symbols for a Lense-Thirring metric are

\[
\Gamma^0_{00} = 0, \quad \Gamma^0_{0i} = \phi_i, \quad \Gamma^0_{ij} = \frac{1}{2} (h_{i,j} + h_{j,i}), \tag{A.3}
\]

\[
\Gamma^i_{00} = \phi_i, \quad \Gamma^i_{0j} = \frac{1}{2} (h_{j,i} - h_{i,j}), \quad \Gamma^i_{jk} = \delta^i_k \phi_j - \delta^i_j \phi_k - \delta^i_k \phi_j.
\]

The spin connection coefficients, already calculated in (III.4), have the explicit form

\[
\Gamma_0 = \frac{GM}{2r^3} \left( x\sigma^{01} + y\sigma^{02} + z\sigma^{03} \right) + \frac{GMR^2}{5r^5} \left[ (r^2 - 2z^2)\sigma^{12} + 3yz\sigma^{13} - 3xz\sigma^{23} \right], \tag{A.4}
\]

\[
\Gamma_1 = \frac{3GMR^2}{5r^5} \left[ 2x y\sigma^{01} + (y^2 - x^2)\sigma^{02} + yz\sigma^{03} \right] + \frac{GM}{2r^3} \left( y\sigma^{12} + z\sigma^{13} \right),
\]

\[
\Gamma_2 = \frac{3GMR^2}{5r^5} \left[ (y^2 - x^2)\sigma^{01} - 2xy\sigma^{02} - xz\sigma^{03} \right] + \frac{GM}{2r^3} \left( -x\sigma^{12} + z\sigma^{23} \right),
\]

\[
\Gamma_3 = \frac{3GMR^2}{5r^5} \left( yz\sigma^{01} - xz\sigma^{02} \right) + \frac{GM}{2r^3} \left( x\sigma^{13} + y\sigma^{23} \right).
\]
\( \gamma \) and \( \sigma \) matrices. We have used the Dirac representation. The neutrino mass eigenstates are

\[
\nu(x) = \nu_{0L} e^{-ik_ax^a} = \sqrt{E+m \left( \frac{\nu_{LR}}{2E + \nu_{LR}} \right)} e^{-ik_ax^a}.
\]  

(A.5)

Motion along \( z \) direction. In this case \( \nu_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) and \( \nu_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) (i.e. \( \nu_{L,R} \) are eigenstates of \( \sigma^3 \)). To calculate the helicity transitions, we need

\[
\langle \nu_R | \gamma^0 \sigma^0 | \nu_L \rangle = -i \frac{k}{E}, \quad \langle \nu_R | \gamma^0 \sigma^1 | \nu_L \rangle = \frac{k}{E}, \quad \langle \nu_R | \gamma^0 \sigma^3 | \nu_L \rangle = 0,
\]

(A.6)

\[
\langle \nu_R | \gamma^0 \sigma^{12} | \nu_L \rangle = 0, \quad \langle \nu_R | \gamma^0 \sigma^{13} | \nu_L \rangle = i, \quad \langle \nu_R | \gamma^0 \sigma^{23} | \nu_L \rangle = 1,
\]

\[
\langle \nu_R | \gamma^0 | \nu_L \rangle = \langle \nu_R | \gamma^1 | \nu_L \rangle = -\frac{k}{E}, \quad \langle \nu_R | \gamma^2 | \nu_L \rangle = i \frac{k}{E}.
\]

As a consequence of (A.6) and (III.3) we obtain

\[
\langle \nu_R | \gamma^0 \Gamma_0 | \nu_L \rangle = \frac{k}{2E} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \quad \text{and} \quad \langle \nu_R | \gamma^0 \Gamma_3 | \nu_L \rangle = \frac{k}{4E} \left( \begin{array}{c} 1 \\ 1 \end{array} \right)
\]

(i.e. \( \nu_{L,R} \) are eigenstates of \( \sigma^1 \)). We find

\[
\langle \nu_R | \gamma^0 \sigma^{11} | \nu_L \rangle = 0, \quad \langle \nu_R | \gamma^0 \sigma^{12} | \nu_L \rangle = \frac{k}{E}, \quad \langle \nu_R | \gamma^0 \sigma^{13} | \nu_L \rangle = -i \frac{k}{E},
\]

(A.8)

\[
\langle \nu_R | \gamma^0 \sigma^{12} | \nu_L \rangle = 1, \quad \langle \nu_R | \gamma^0 \sigma^{13} | \nu_L \rangle = -i, \quad \langle \nu_R | \gamma^0 \sigma^{23} | \nu_L \rangle = 0,
\]

\[
\langle \nu_R | \gamma^0 | \nu_L \rangle = \langle \nu_R | \gamma^1 | \nu_L \rangle = 0, \quad \langle \nu_R | \gamma^2 | \nu_L \rangle = i \frac{k}{E}, \quad \langle \nu_R | \gamma^3 | \nu_L \rangle = -\frac{k}{E}.
\]

We also obtain (see (III.3))

\[
\langle \nu_R | \gamma^0 \Gamma_0 | \nu_L \rangle = \frac{k}{2E} (\phi_2 + i\phi_3 - \frac{1}{4} [(h_{12} - h_{21}) - i(h_{13} - h_{31})])
\]

(A.9)

\[
\langle \nu_R | \gamma^0 \Gamma_1 | \nu_L \rangle = \frac{k}{4E} [-(h_{12} + h_{21}) + i(h_{13} + h_{31})] + \frac{1}{2} (-\phi_2 + i\phi_3).
\]

APPENDIX: ACKNOWLEDGMENTS

Research supported by MURST PRIN 2003.

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