EXTENSIONS OF INVARIANT RANDOM ORDERS ON GROUPS

YAIR GLASNER, YUQING FRANK LIN, AND TOM MEYEROVITCH

Abstract. In this paper we study the action of a countable group Γ on the space of orders on the group. In particular, we are concerned with the invariant probability measures on this space, known as invariant random orders. We show that for any countable group the space of random invariant orders is rich enough to contain an isomorphic copy of any free ergodic action, and characterize the non-free actions realizable. We prove a Glasner-Weiss dichotomy regarding the simplex of invariant random orders. We also show that the invariant partial order on SL₃(Z) corresponding to the semigroup of positive matrices cannot be extended to an invariant random total order. We thus provide the first example for a partial order (deterministic or random) that cannot be randomly extended.

1. Introduction

The origins of the theory of orderable groups goes back to the end of the nineteenth century and the beginning of the twentieth century. It continues to be an active area of research, mainly due to its connections with many different branches of mathematics. In this paper we extend a fruitful and relatively modern theme in this theory: The study of the space of orders on a group from a topological and a dynamical point of view. The space of left-invariant orders on Γ is a zero-dimensional compact Hausdorff topological space on which Γ acts by conjugation [22]. The study of this action from the point of view of topological dynamics proved to be a powerful tool, especially in the case where this action has no fixed points, i.e. the group is not bi-orderable. See for example [15]. For an exposition and further historical background on this point of view, see for instance [17]. Inspired by the success of this theory, we turn to investigate the action of Γ by left-multiplication on the space of all total orders on Γ. This is a much larger space, whose fixed points, if such exist, are the left invariant orders.

The central objects of this paper are invariant random orders, namely probability measures on the space of orders whose distribution is invariant with respect to multiplication (say, from the left). The term “invariant random orders” appeared in [2], and we refer the reader to [13, 23] for earlier applications, in particular in the context of entropy theory for actions of amenable groups. There are further recent applications of invariant random orders in the context of entropy theory [4, 8].

Our results can be considered as evidence for the dynamical and geometric richness of the space of invariant random orders.

The organization of the paper is as follows: In Section 2 we introduce some definitions and notation, as well as basic structural results on the space of orders on a group and the associated action. In Section 3, we show that for non-amenable groups it not always possible to extend a left-invariant partial order to a left-invariant random (total) order. This answers a question posed in [2], where extendability of partial invariant random orders in the amenable case was resolved. More specifically, we demonstrate non-extendability as above with respect to the left-invariant partial order on the group SL₃(Z) corresponding to the semi-group of matrices that preserve the positive octant. In Section 4 we establish the
uniform specification property for the space of orders on a countable group $\Gamma$. In Section 5 we further demonstrate the dynamical richness of the space of invariant random orders by proving that any free ergodic action of $\Gamma$ can be “realized” as an invariant measure on the space of total orders. Furthermore, we provide a sufficient condition for realization of a non-free ergodic action as invariant random orders in terms of equivariant orderability of the the invariant random subgroup corresponding to the stabilizer of the action. We observe that this necessary condition is “almost necessary”. In Section 6 we prove that for groups $\Gamma$ that do not admit property (T) the space of invariant random orders is the Poulsen simplex, thus establishing a Glasner-Weiss type dichotomy for the simplex of invariant random orders. We conclude with open questions and some additional remarks.

2. Definitions, and basic observations

Partial and total orders. Let $\Gamma$ be a countable group. We denote by $p\text{-Ord}(\Gamma)$ the set of partial orders on $\Gamma$, namely the set of binary relations on $\Gamma$ that are transitive, non-reflexive and antisymmetric.

- Antisymmetry: $x \preceq y$ implies that $y \not\preceq x$
- Non-reflexivity: $x \not\preceq x, \forall x \in \Gamma$
- Transitivity: $x \preceq y$ and $y \preceq z$ imply $x \preceq z$

$\text{Ord}(\Gamma) \subset p\text{-Ord}(\Gamma)$ will denote the subset of total orders, namely orders for which every two group elements are comparable as described below:

- Total antisymmetry: For every $x \neq y \in \Gamma$ either $x \prec y$ or $y \prec x$.

Both collections admit a natural compact metrizable topology, upon identifying them as closed subsets of the set of all binary relations $\{0, 1\}^{\Gamma \times \Gamma}$ endowed with the Tychonoff (product) topology.

The group $\Gamma \times \Gamma$ acts, by homeomorphisms, from the left, on $\text{Ord}(\Gamma)$ by:

$x \cdot (\gamma, \delta) \cdot \prec y$ if and only if $(\gamma^{-1}x\delta) \prec (\gamma^{-1}y\delta)$.

In this paper, whenever we refer to the action of $\Gamma$ on $\text{Ord}(\Gamma)$ it will be implicitly understood that $\Gamma$ is acting via its identification with $\Gamma \times \langle e \rangle < \Gamma \times \Gamma$. We denote the set of $\Gamma$-fixed points of $\text{Ord}(\Gamma)$, equivalently the set of left-invariant orders by $\text{Ord}(\Gamma)^{\Gamma} := \text{Ord}(\Gamma)^{\Gamma \times \langle e \rangle}$. If $\text{Ord}(\Gamma)^{\Gamma}$ is nonempty we say that $\Gamma$ is left-orderable. The left invariant orders are exactly the orders satisfying $x \prec y \iff \gamma x \prec \gamma y, \forall x, y, \gamma \in \Gamma$. A softer, more probabilistic notion of left invariance that will be the focus of this paper is the following:

Definition 2.1. A (left) invariant random order on $\Gamma$, or an IRO for short, is a $\Gamma$-invariant Borel probability measure on $\text{Ord}(\Gamma)$. We will denote by $\text{IRO}(\Gamma)$ the collection of all invariant random orders on $\Gamma$. Similarly $p\text{-IRO}(\Gamma)$ will denote the collection of invariant random partial orders, defined as $\Gamma$-invariant probability measures on $p\text{-Ord}(\Gamma)$.

Both $\text{IRO}(\Gamma)$ and $p\text{-IRO}(\Gamma)$, endowed with the $w^*$ topology become compact metrizable spaces. While many groups do not admit a left invariant order, every countable group admits an IRO. Here is one construction to have in mind. Consider the $\Gamma$ equivariant map $\Phi : [0, 1]^\Gamma \to \text{Ord}(\Gamma)$ sending $\{\omega_\gamma \mid \gamma \in \Gamma\}$ to the order on $\Phi(\omega)$ defined by the requirement that $x\Phi(\omega)y$ if and only if $\omega_x < \omega_y$. If $\lambda$ denotes the Lebesgue (or any other atomless probability) measure on $[0, 1]$ and $\Lambda = \lambda^\Gamma$ the corresponding product measure on $[0, 1]^\Gamma$, then
Φ(ω) is well defined for Λ-almost every ω and Φ∗(Λ) is an IRO on Γ. The above random order, which is sometimes called “the uniform random order”, is uniquely characterized by the property that for any finite set \( F \subseteq Γ \) the restriction of \( \prec \) to \( F \) is uniformly distributed among the \(|F|!\) possible permutations of \( F \). An early appearance of the uniform random order in the context of entropy theory is due to Kieffer [13], where it was used to prove an asymptotic equipartition theorem for amenable groups (see also [23] and the earlier paper [19]).

Extension of orders. Given a partial order \( \prec \in p-\text{Ord}(Γ) \) we denote by

\[ \text{Ext}(\prec) = \{ \ll \in \text{Ord}(Γ) \mid x \prec y \Rightarrow x \ll y, \forall x, y \in Γ \} \]

the collection of total orders that extend the given partial order. \( \text{Ext}(\prec) \) is a closed subset of \( \text{Ord}(Γ) \) and is Γ-invariant whenever \( \prec \) is. We refer to any \( \ll \in \text{Ext}(\prec) \) as an extension of \( \prec \) to a total invariant order. Again we will be interested in softer, more probabilistic, notions for extensions of orders.

**Definition 2.2.** Let \( \prec \in p-\text{Ord}(Γ) \). \( µ \in \text{IRO}(Γ) \) will be called a random extension of \( \prec \) if \( µ(\text{Ext}(\prec)) = 1 \). More generally if \( ν \in p-\text{IRO}(Γ) \), We will say that \( µ \in \text{IRO}(Γ) \) extends \( ν \) if there exists a joining \( (\text{Ord}(Γ) \times p-\text{Ord}(Γ), \theta) \) with the property that \( \ll \in \text{Ext}(\prec) \) for \( \theta \) almost every \( (\ll, \prec) \in \text{Ord}(Γ) \times p-\text{Ord}(Γ) \).

In some special classes of groups extensions always exist.

Deterministic random orders are “deterministically” extendable within the class of torsion-free locally nilpotent groups:

**Theorem 2.3.** (Rhemtulla [20], Formanek [9]) Every invariant partial order on a torsion-free locally nilpotent group can be extended to an invariant total order.

It is well known that the conclusion of the above theorem fails if we relax the assumption of torsion-free locally nilpotent group for instance to the class of finitely generated torsion free solvable groups. For some examples and further references, see [11]. In contrast, extending invariant random partial orders is possible under the much more general assumption of amenability.

**Proposition 2.4 ([23],[2]).** Any invariant random partial order on an amenable group can be extended to an invariant random (total) order.

A proof of the above proposition follows by observing that the space of (not necessarily invariant) extensions of a given random order is a non-empty simplex on which the group acts. See [2] for details.

It seems natural to wonder if the amenability assumption above is necessary [2, Question 2.2]. In the current paper, we will present a result showing that in general, extension of partial orders to IRO’s is not possible, at least for some non-amenable groups. In a different direction, special kinds of partial orders can be randomly extended in any group.

**Proposition 2.5.** Let \( Γ \) be a countable group and let \( Δ < Γ \) be a subgroup. Any IRO on \( Δ \) (viewed as an invariant random partial order on \( Γ \)) can be extended to an IRO on \( Γ \).

**Proof.** The proof generalizes our prior construction of the uniform random order. Let \( Λ = λ^{Γ/Δ} \) be Lebesgue measure on \([0, 1]^{Γ/Δ}\) (namely the product of the Lebesgue measure taken in each coordinate), and let \( X \subseteq [0, 1]^{Γ/Δ} \) denote the subspace of injective functions (so \( x_{gΔ} = x_{hΔ} \))
implies \(g^{-1}h \in \Delta\) for all \(x \in X, g, h \in \Gamma\). Then \(\lambda^{\Gamma/\Delta}(X) = 1\), so \(\lambda^{\Gamma/\Delta}\) can be regarded as a probability measure on \(X\). We define a function \(\Phi : \text{Ord}(\Delta) \times X \to \text{Ord}(\Gamma)\) as follows: 
\[\prec = \Phi(\prec, x)\] 
is given by
\[g \prec h \text{ if and only if } x_{g\Delta} < x_{h\Delta} \text{ or } (g < h \text{ and } g^{-1}h \in \Delta)\]
\[\prec \in \text{Ord}(\Gamma) \text{ for every } \prec \in \text{Ord}(\Delta) \text{ and } x \in X.\] Also it is easy to verify that \(\Phi\) is a \(\Gamma\)-equivariant map and that \(\prec\) extends \(\prec\).

Now define \(\tilde{\Phi} : \text{Ord}(\Delta) \times X \to \text{Ord}(\Delta) \times \text{Ord}(\Gamma)\) by \(\tilde{\Phi}(\prec, x) = (\prec, \Phi(\prec, x))\) and set 
\[\theta = \Phi_*(\mu_0 \times \Delta).\]
All the properties mentioned in the end of the last paragraph show that \(\theta\) is the joining needed in order to define and extension of \(\prec\) to an IRO on \(\Gamma\), according to definition 2.2.

**Dynamical pasts and semigroups.** We now recall a simple and well known correspondence between left-invariant orders and semigroups not containing the identity, and observe that this correspondence can be meaningfully extended to a \(\Gamma \times \Gamma\)-equivariant bijective correspondence between \(p\text{-Ord}(\Gamma)\) and a space we refer to as “dynamical pasts”.

A left (partial) order on \(\Gamma\) is uniquely determined by the semigroup of positive elements \(\Phi_\prec := \{x \in \Gamma \mid e < \gamma\}\). Conversely, any semigroup \(S\) in \(\Gamma\) that does not contain the identity gives rise to a partial \(\Gamma\)-invariant order \(\prec_S\), given by \(x <_S y \iff x^{-1}y \in S\). The above correspondences define a bijection between left invariant orders on \(\Gamma\) and semi-groups of \(\Gamma\) that do not contain the identity. A semigroup \(S\) not containing the identity corresponds to a left invariant total order if and only if it has the additional property that \(\Gamma = S \cup \{e\} \cup S^{-1}\). Such semigroups are known as algebraic pasts. A semigroup \(S\) not containing the identity corresponds to a bi-invariant order if and only if it the group \(\Gamma\) normalizes \(S\) in the sense that \(gsg^{-1} \in S\) for every \(s \in S\) and \(g \in \Gamma\).

There is a natural bijection \(\Psi : \{0,1\}^{\Gamma \times \Gamma} \to (\{0,1\}^\Gamma)^\Gamma\) between the space \(\{0,1\}^{\Gamma \times \Gamma}\) of binary relations on \(\Gamma\) and the space \((\{0,1\}^\Gamma)^\Gamma\) of functions from \(\Gamma\) to the space \(\{0,1\}^\Gamma\), which we naturally identify as the space of functions from \(\Gamma\) to the space of subsets of \(\Gamma\). This bijection is furthermore a homeomorphism (where the topology on \((\{0,1\}^\Gamma)^\Gamma\) is the product of \(\Gamma \times \Gamma\) copies of \(\{0,1\}\) with the discrete topology and the topology on \((\{0,1\}^\Gamma)^\Gamma\) is the “iterated” product topology). For \(\prec \in p\text{-Ord}(\Gamma)\) the image \(\Psi_\prec\) can be written as follows:
\[\Psi_\prec(x) = \{y \in \Gamma \mid x < y\}, \ x \in \Gamma.\]

For \(\prec \in p\text{-Ord}(\Gamma)\) and \(x \in \Gamma\), the set \(\Psi_\prec(x) \subset \Gamma\) can be thought of as the **past of** \(x\) with respect to \(\prec\). On the space \((\{0,1\}^\Gamma)^\Gamma\) we have a natural self-homeomorphism \(\phi \mapsto \tilde{\phi}\), given by 
\[\tilde{\phi}(x) := x^{-1}\phi(x) = \{x^{-1}y \in \Gamma \mid y \in \phi(x)\}.\]

The composition of \(\Psi\) and the self-homeomorphism above gives another bijection \(\Phi\) between \(\{0,1\}^{\Gamma \times \Gamma}\) and \((\{0,1\}^\Gamma)^\Gamma\): \(\Phi : \{0,1\}^{\Gamma \times \Gamma} \to (\{0,1\}^\Gamma)^\Gamma\). For \(\prec \in p\text{-Ord}(\Gamma)\) we can write:
\[\Phi_\prec(x) = \{\gamma \in \Gamma \mid x < x\gamma\}, \ x \in \Gamma.\]

For \(x \in \Gamma\) and \(\prec \in p\text{-Ord}(\Gamma)\), the set \(\Phi_\prec(x) \subset \Gamma\) can be thought of as “the directions pointing to the past from \(x\)”.

A function \(S : \Gamma \to \{0,1\}^\Gamma\) is in the image of \(p\text{-Ord}(\Gamma)\) under \(\Phi\) if and only if it satisfies the following conditions:

- **Antisymmetry**: For every \(x, \gamma \in \Gamma\) with \(e \neq \gamma\) at most one of the conditions \(\gamma \in S(x)\), \(\gamma^{-1} \in S(x\gamma)\) can hold.
• Non-reflexivity: \( e \not\in S(x), \forall x \in \Gamma \)
• Transitivity: \( \gamma S(x \gamma) \subset S(x), \forall \gamma \in S(x) \).

A function \( S : \Gamma \to \{0, 1\}^\Gamma \) in the image of \( p\text{-Ord}(\Gamma) \) under \( \Phi \) satisfies the following property in addition:

• Total antisymmetry: For every \( x, \gamma \in \Gamma \) with \( e \neq \gamma \) exactly one of the conditions \( \gamma \in S(x) \) or \( \gamma^{-1} \in S(x \gamma) \) holds.

For a left invariant (total) order \( \prec \), \( \Phi \prec(x) \) is independent of \( x \), and is precisely the semigroup (algebraic past) corresponding to \( x \).

**Example 2.6.** Let \( P = \mathbb{Z}_+^n \setminus \{0\} \subset \mathbb{Z}^n \) denote the semigroup corresponding to the positive orthant in \( \mathbb{Z}^n \) with zero removed. Then \( P \) defines an invariant partial order on \( \mathbb{Z}^n \), that we denote by \( \preceq \). We have

\[
\text{Ext}(\preceq) = \{ \ll \in \text{Ord}(\mathbb{Z}^n) \mid P \subset \Phi \ll(x), \forall x \in \mathbb{Z}^n \}. \quad (1)
\]

Given a function \( u : \mathbb{Z}^n \to \mathbb{R} \) which is injective and strictly monotone with respect to \( \preceq \) in the sense that \( u(x) < u(y) \) whenever \( y - x \in P \), we can define an element \( \prec_u \in \text{Ext}(\preceq) \) by \( x \prec_u y \) iff \( u(x) < u(y) \). In fact, this is true in general: For any countable group \( \Gamma \), any injective function \( u : \Gamma \to \mathbb{R} \) defines a total order \( \prec_u \in \text{Ord}(\Gamma) \) as above. Given \( \preceq \in p\text{-Ord}(\Gamma) \) we have that \( \prec_u \in \text{Ext}(\preceq) \) if and only if \( u : \Gamma \to \mathbb{R} \) is \( \preceq \)-monotone. Conversely, any \( \prec \in \text{Ext}(\preceq) \) is of the form \( \prec = \prec_u \) for some injective, \( \preceq \)-monotone function. Composing \( u \) from the left with a strictly increasing function from \( \mathbb{R} \) to \( \mathbb{R} \) does not change the resulting order \( \prec_u \). In the case \( \Gamma = \mathbb{Z}^n \) we can assume without loss of generality that the function \( u : \mathbb{Z}^n \to \mathbb{R} \) is the restriction of some continuous function \( \tilde{U} : \mathbb{R}^n \to \mathbb{R} \) (or even 1-Lipschitz, piecewise linear, smooth and so on) having the property that each level set of \( \tilde{u} \) intersects \( \mathbb{Z}^n \) in at most 1 point. The level lines of \( \tilde{u} \) (at least in the piece-wise smooth case), which are \( n - 1 \)-dimensional surfaces, uniquely determine \( \tilde{u} \). At least in the case \( \Gamma = \mathbb{Z}^n \) this point of view provides a method of visualizing elements of \( \text{Ext}(\preceq) \). For instance, **Figure 1** shows the level lines of a function corresponding to some element of \( \text{Ext}(\preceq) \), restricted to some bounded square region in \( \mathbb{R}^2 \), where the larger bold point represents the zero vector, and the smaller bold points represent other elements of \( \mathbb{Z}^2 \).

![Figure 1](image)

**Figure 1.** The level lines of a function corresponding to an element of \( \text{Ext}(\preceq) \), restricted to a finite window.

We briefly recall that the standard and well-known classification of invariant total orders on \( \mathbb{Z}^n \) and identify which of these are elements of \( \text{Ext}(\preceq) \): To any invariant total order on
there is an associated \((n - 1)\)-dimensional linear subspace of \(\mathbb{R}^n\). An \((n - 1)\)-dimensional linear subspace of \(\mathbb{R}^n\) which does not contain any non-zero integral points corresponds to exactly two invariant total orders, whose corresponding algebraic pasts are the two half-planes in the complement of \(V\), intersected with \(\mathbb{Z}^d\). Such an element corresponds to \(\text{Ext}(\sqsubseteq)\) if and only if the corresponding half-plane contains \(P\). The algebraic pasts corresponding to an \((n - 1)\)-dimensional linear subspace \(V\) of \(\mathbb{R}^n\) which contains a subgroup \(\Gamma_0\) of \(\mathbb{Z}^n\) of rank \(k\) for some \(k \leq n - 1\), are exactly the union of an algebraic past of \(\Gamma \cong \mathbb{Z}^k\) (which we can identify inductively) and one of the half-planes in the complement of \(V\).

In the case of an invariant total order \(\prec \in \text{Ext}(\sqsubseteq)\) corresponding to an irrational subspace \(V\), the “level surfaces” could be taken as affine spaces parallel to \(V\). In the case of a subspace \(V\) containing non-zero rational points, one has to perturb the level surfaces slightly so that each level surface contains at most one integral point.

**Example 2.7.** The discrete Heisenberg group is given by:

\[
H = \left\{ [a,b,c] := \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \bigg| a, b, c \in \mathbb{Z} \right\}.
\]

The discrete Heisenberg group \(H\) is generated by the three matrices

\[
x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}
\]

The matrix \(z\) is the commutator of \(x\) and \(y\), and it commutes with each of them. These relations give a presentation of the discrete Heisenberg group:

\[H = \langle x, y, z \mid [x, z] = [y, z] = [x, y]z^{-1} \rangle.\]

The following well known formula can be verified by a straightforward induction:

\[[x^k, y^l] = z^{kl}, \quad \forall k, l \in \mathbb{Z} \tag{3}\]

The vertices of the Cayley graph \(C = \text{Cay}(H, \{x, y, z\})\) of \(H\) are naturally identified with \(\mathbb{Z}^3 \subset \mathbb{R}^3\). Let \(P \subset H\) denote the semigroup of matrices in \(H\) whose entries are all non-negative, excluding the identity matrix. Then \(P\) is precisely the semigroup generated by \(x, y, z\). Let \(\sqsubseteq\) denote the partial left-invariant order corresponding to \(P\). For any \([A, B, C] \in H\), and \(\prec \in \text{Ext}(\sqsubseteq)\) we know that \(\Psi_\prec([A, B, C])\) is bound to contain \([A, B, C]P\), which assumes the form of a skewed cone

\[[A, B, C]P = \{[A + a, B + b, C + c + Ab] \mid a, b, c \in \mathbb{Z}_{\geq 0} \} \subset \Psi_\prec([A, B, C]).\]

Examples of such sheared cones, including \(P\) itself at the origin, appear in Figure 2.

3. A Non-extendable Partial Invariant Order on \(\text{SL}_3(\mathbb{Z})\)

In this section we prove the following:

**Theorem 3.1.** There does not exist an invariant random order on \(\Gamma := \text{SL}_3(\mathbb{Z})\) with the property that any \(e \neq A \in \text{SL}_3(\mathbb{Z})\) without negative entries is almost surely positive.

This shows that without the amenability assumption, it is not in general possible to extend a partial invariant order to a total invariant order, providing a negative answer to a question posed in [2]. The proof is an adaptation of Dave Morris’ proof that any finite index subgroup
of SLₙ(Z) with n ≥ 3 is non-orderable [25, Proposition 3.3] (see also the monograph [7], or the survey [16]).

Denote by ⊏ the left-invariant partial order on Γ whose semigroup of positive elements is given by Ψ ⊏ = {A ∈ SLₙ(Z) | e ≠ A, 0 ≤ aᵢ,ⱼ, ∀1 ≤ i, j ≤ 3}. Geometrically g ⊏ h if and only if hP ⊈ gP where P ⊂ R³ is the positive octant. As a semigroup Ψ ⊏ is generated by the following six matrices

\[
\begin{align*}
  a_1 &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & a_2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & a_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
  a_4 &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & a_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & a_6 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\end{align*}
\]

A straightforward verification shows that [aᵢ, aᵢ₊₁] = e and [aᵢ₋₁, aᵢ₊₁] = aᵢ, with all subscripts read modulo 6. Our goal is to show that Ext(⊏) admits no Γ-invariant Borel probability measure.

As in [25] we will be interested in the question when an element a ∈ Γ is much larger than another element b. Here are four natural definitions capturing this notion in some special cases:

**Definition 3.2.** Let ⊏ ∈ p-Ord(Γ). Let a, b ∈ Γ such that e ⊏ a, b, and ⟨a, b⟩ = Z².

\[
\begin{align*}
  \text{wml}^+(a, b) &:= \{≺ ∈ \text{Ext}(⊏) \mid ∀M > 0, ∃N > 0 \text{ such that } a^{-k}b^{-M} ≺ e, ∀k ≥ N\}, \\
  \text{wml}^-(a, b) &:= \{≺ ∈ \text{Ext}(⊏) \mid ∀M > 0, ∃N > 0 \text{ such that } e ≺ b^{-M}a^k, ∀k ≥ N\} \\
  \text{sml}^+(a, b) &:= \{≺ ∈ \text{Ext}(⊏) \mid ∃q > 0 \text{ such that } a^{-q}b^n ≺ e ∀n ∈ \mathbb{N}\}, \\
  \text{sml}^-(a, b) &:= \{≺ ∈ \text{Ext}(⊏) \mid ∃q > 0 \text{ such that } e ≺ b^{-n}a^q ∀n ∈ \mathbb{N}\},
\end{align*}
\]

The acronyms wml and sml stand for weakly and strongly much larger, respectively.
These definitions are tailored for our current proof. Of course, similar definitions can be given when \( a, b \) are not assumed to be a-priori positive, or when they fail to commute. Note also that this definition looks only at the dynamic past \( \Phi_\prec(e) \) at the identity.

That \( a \) be much larger than \( b \) should entail inside the \( a, b \)-plane that the line \( U \) separating the positive and negative elements has to come very close to the \( b \)-axis. The stronger notion above, requires \( U \) to be at a bounded distance from the \( b \)-axis. The weak notion requires \( U \) to be eventually closer to the \( b \)-axis than any linear line with a finite slope. The \( \pm \) superscript represents whether this closeness is measured along the positive and negative directions of the \( b \)-axis. It is quite possible for an order to exhibit completely different behavior in these two regions. Indeed if \( P \) is the positive quadrant without zero, then every order \( \prec \in \Ext(\sqsubset) \) with the property that \( P \supset \Psi_\prec(0) \subset P - v \) for some positive \( v \in \mathbb{Z}^2 \) will in fact satisfy \( \prec \in \text{sm}l_\prec^+(a, b) \cap \text{sm}l_\prec^+(b, a) \). So that this latter set is far from being empty. For invariant orders all four notions above coincide and contain exactly one order in \( \Ext(\sqsubset) \) - the lexicographic order.

\textbf{Lemma 3.3.} Let \( a, b \in \Gamma \) be such that \( e \sqsubset a, b \) and \( \langle a, b \rangle = \mathbb{Z}^2 \). Then

1. \( \text{sm}l_\prec^\pm(a, b) \subseteq \text{wl}l_\prec^\pm(a, b) \).
2. \( \text{sm}l_\prec^\pm(a, b) \subseteq \text{wl}l_\prec^\pm(b, a)^c \).

\textbf{Proof.} Let \( a, b \in \Gamma \) be as given.

1. Suppose \( \prec \in \text{sm}l_\prec^-(a, b) \) so that \( a^{-q}b^n \prec e \) for some \( q \) and all \( n \). Then for any \( 0 < M \) and \( k > q \), we have \( a^{-k}b^M \prec a^{-k}b^M a^k-q = a^{-q}b^M \prec e \), as required. The other statement follows similarly.

2. Suppose, by way of contradiction that \( \prec \in \text{sm}l_\prec^-(a, b) \cap \text{wl}l_\prec^+(b, a) \). In particular \( \prec \in \text{sm}l_\prec^-(a, b) \) yields \( q > 0 \) such that \( e \prec a^q b^{-m} \) for all \( n \), while \( \prec \in \text{wl}l_\prec^+(b, a) \) ensures that \( b^{-n} a^n \prec e \) for all large enough \( n \). By our hypotheses that \( a \) and \( b \) commute and \( a \) is positive, we obtain \( e \prec a^q b^{-n} \prec a^q b^n \prec e \), a contradiction to transitivity and non-reflexivity.

The following lemma is a slight elaboration of [25, Lemma 3.2].

\textbf{Lemma 3.4.} With \( a_i \in \text{SL}_3(\mathbb{Z}) \) the basic unipotent matrices defined in eq. (4), and all indexes taken modulo 6 we have:

1. \( \text{wl}l_\prec^+(a_i, a_{i+1})^c \subseteq \text{sm}l_\prec^-(a_{i+2}, a_{i+1}) \).
2. \( \text{wl}l_\prec^+(a_i, a_{i+1}) \subseteq \text{sm}l_\prec^-(a_{i-1}, a_i) \).

Both statements actually reflect some information about the geometry of the discrete Heisenberg group \( H = \langle a_i, a_{i+1}, a_{i+2} \rangle \). Before proceeding to the formal proof let us explain this geometric point of view:

As in example 2.7 we denote Heisenberg matrices by triplets of integers

\[
[a, b, c] \cong \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}
\]

Note that when we restrict ourselves to the \( \langle x, z \rangle \) and \( \langle z, y \rangle \) planes the group operations are well represented by the arithmetic in \( \mathbb{R}^3 \). The proof of the first claim is depicted in Figure 3, where \( a_i, a_{i+1}, a_{i+2} \) are represented by the \( x, z, y \) coordinates respectively. Suppose \( \prec \in \text{wl}l_\prec^+(a_i, a_{i+1})^c \). Then there exists \( M \in \mathbb{N} \) such that \( e \prec a_i^{-k} a_{i+1}^M \) for infinitely many
values of \( k \). Graphically this yields the green line \( L_1 \), of slope \(-M\) in the \( x, z \) plane, with the property that infinitely many integral points falling on it are \(<\)-positive. Two such points \( x_1, y_1 \) are depicted in the picture. As in Figure 2, together with each such positive point comes a sheared cone of points that are bound to be even bigger according to \( \square \). Following the red lines, along the boundary of these sheared cones, we obtain another a line \( L_2 \) parallel to the negative direction of the \( z \) axis in the \( y, z \)-plane, containing infinitely many positive points. Since we are extending \( \square \), all points directly above each positive point on \( L_2 \) is also positive, so that \( L_2 \) consists entirely of positive points, which is exactly what we need. Keeping the geometric interpretation in mind, we proceed to prove the statements algebraically:

**Proof.** As above suppose \(<\in wml^+_\C(a_i, a_{i+1})^c\). Then there exists \( M \in \mathbb{N} \) such that \( e < a_i^{-k}a_{i+1}^M \) for infinitely many values of \( k \). Because \(<\in \text{Ext}(\square)\) and \( e\in a_{i+2}, a_i \) for any \( q \in \mathbb{N} \) we have \( a_i^{-k}a_{i+1}^M < a_i^{-k}a_{i+1}^M a_i^q a_i^{-k} \). Using the commutation relations in the Heisenberg group (cf. (eq. (3)) at the end of Section 2) we obtain \( a_i^{-k}a_{i+1}^M a_i^q a_i^{-k} = a_{i+1}^{M-k} a_i^{q+k} \). Choose \( q = 2M \) with \( M \) as above. It follows that there are infinitely many \( k \)'s such that \( e < a_i^{-M}a_{i+1}^q \). By multiplying by positive powers of \( a_{i+1} \) we conclude that \( e < a_i^{-n}a_i^{q+n} \) for all \( n \), so \(<\in sml^\C_\C(a_{i+2}, a_{i+1})\) as required. This completes the proof of (item 1).

Suppose now that \(<\in wml^+_\C(a_i, a_{i+1}) \). Choosing \( M = 1 \) in the definition of that set, we can find \( N \in \mathbb{N} \) such that \( a_i^{-n}a_{i+1}^N < e \) for all \( n > N \). Multiplying from the right by negative powers of \( a_{i-1} \) and \( a_{i+1} \), we have \( a_i^{-n}a_{i+1}a_{i-1}a_i^{-n} < e \). Using (eq. (3)) again, we have \( a_i^{-2}a_i^{-n} = a_i^{-2}a_i^{-n} \), so that

\[
 a_i^{-2}a_i^{-n} = a_i^{-n}a_i^{-n}a_i^{-n}a_i^{-n} = a_i^{-n}a_i^{-n}a_i^{-2}a_i^{-n} = e
\]

Multiplying by negative powers of \( a_i \) if needed, we conclude that \( a_i^{-2}a_i^{-n} < e \). Let \( \text{sml}^\C_\C := \bigcap_{i=1}^6 \text{sml}^\C_\C(a_i, a_{i-1}) \) and \( \text{sml}^+_\C := \bigcap_{i=1}^6 \text{sml}^+_\C(a_i, a_{i+1}) \), where all the indices are considered modulo 6.
Lemma 3.5.

\[ \text{Ext}(\sqsubset) = \text{sml}_{\geq}^+ \cup \text{sml}_{\leq}^- \]

Proof. Take any \( \sqsubset \in \text{Ext}(\sqsubset) \).

Case 1: suppose \( \sqsubset \in \text{wml}_{\leq}^+(a_1, a_2) \). By Lemma 3.4 (item 1), \( \sqsubset \in \text{sml}_{\leq}^-(a_3, a_2) \). Then by Lemma 3.3 (item 2), \( \sqsubset \in \text{wml}_{\leq}^+(a_2, a_3) \). Repeat this argument five more times to obtain \( \sqsubset \in \cap_{i=1}^6 \text{sml}_{\leq}^-(a_i, a_{i-1}) \).

Case 2: suppose \( \sqsubset \in \text{wml}_{\leq}^+(a_1, a_2) \). By Lemma 3.4 (item 2), \( \sqsubset \in \text{sml}_{\leq}^+(a_6, a_1) \). Then by Lemma 3.3 (item 1), \( \sqsubset \in \text{wml}_{\leq}^+(a_6, a_1) \). Repeat this argument five more times to obtain \( \sqsubset \in \cap_{i=1}^6 \text{sml}_{\leq}^+(a_i, a_{i+1}) \). \( \square \)

The above lemma is an analogue of Dave Witte Morris’ proof of [25, Proposition 3.3]. In the deterministic setting, it is rather immediate that neither \( \text{sml}_{\geq}^+ \) nor \( \text{sml}_{\leq}^- \) contain an invariant order. In our random setting more work must be done to show neither support an invariant probability measure. It is interesting to note that all our results so far, and in particular Lemma 3.4 and Lemma 3.5 are deterministic in the sense that no probability is involved.

To show that \( \text{sml}_{\geq}^+, \text{sml}_{\leq}^- \) cannot support an invariant probability measure, we will eventually decompose them into wandering sets: Suppose \( \Gamma \) acts on a Borel space \( X \) by Borel automorphisms. We say \( A \subseteq X \) is wandering if there exists \( g \in \Gamma \) such that \( (g^n A)_{n \in \mathbb{Z}} \) are all pairwise disjoint. Let \( \mathcal{W} \) be the collection of all countable unions of wandering sets. We say that \( A \) is non-recurrent with respect to \( g \in \Gamma \) if for all \( x \in X \), \( \sum_{n=1}^{\infty} 1_A(g^n(x)) < +\infty \).

The following simple lemma collects some basic facts about wandering and non-recurrent sets. The arguments are all elementary set-theoretic, no assumptions at all are made about the algebraic structure of \( \Gamma \).

Lemma 3.6. The following are true:

1. \( \mathcal{W} \) is a \( \Gamma \)-invariant \( \sigma \)-ideal. That is, it is closed under countable unions and taking subsets.
2. Any \( A \in \mathcal{W} \) has zero measure with respect to any \( \Gamma \)-invariant probability measure.
3. If \( A \subset X \) is non-recurrent with respect to any \( g \in \Gamma \) then \( A \in \mathcal{W} \).
4. If \( A \subset X \) is such that there exists \( g \in \Gamma \) and \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( g^n A \cap A = \emptyset \), then \( A \in \mathcal{W} \).
5. If \( A \subset X \) is such that there exists \( g \in \Gamma \) and \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( g^n A \cap A \in \mathcal{W} \), then \( A \in \mathcal{W} \).

Proof. (1). It is straightforward that for every \( A \in \mathcal{W} \) and \( B \subset A \), \( B \in \mathcal{W} \); and that a countable union of elements of \( \mathcal{W} \) is also in \( \mathcal{W} \). To see that \( \mathcal{W} \) is \( \Gamma \)-invariant, observe that if \( B \) is wandering with respect to \( g \) and \( h \in \Gamma \), then \( h(B) \) is wandering with respect to \( hgh^{-1} \).

(2). For any \( \Gamma \)-invariant probability measure \( \mu \), if \( A \in \mathcal{W} \) has positive measure then there exists \( B \subset A \) with \( \mu(B) > 0 \) and \( g \in \Gamma \) with \( (g^n B)_{n \in \mathbb{Z}} \) pairwise disjoint, so \( \mu \) has infinite measure, a contradiction.

(3). Let \( A \) be non-recurrent with respect to \( g \). For \( k \in \mathbb{Z} \) let \( A_k = \{ x \in X : g^k x \in A, \forall l > k g^l x \notin A \} \). Notice that \( (A_k)_{k \in \mathbb{Z}} \) are pairwise disjoint, and \( A = \bigsqcup_{k \geq 0} (A_k \cap A) \). Also notice that for \( n, k \in \mathbb{Z} \), \( g^n A_k \subset A_{k-n} \), so \( A_k \cap A \) is wandering, so \( A \in \mathcal{W} \).

(4). Note that in this case for all \( x \in X \), \( \sum_{n=1}^{\infty} 1_A(g^n(x)) \leq N + 1 \), so \( A \) is non-recurrent. The result follows from (3).
(5). For $n \geq N$ let $A_n = g^n A \cap A$, and let $A_0 = A \setminus (\cup_{n \geq N} A_n)$. Each $A_n \in \mathcal{W}$ by assumption, and we claim that $A_0$ is also in $\mathcal{W}$. Indeed note that for $n \geq N$, $g^n A_0 \cap A_0 \subset g^n A \cap A$, but also $g^n A_0 \cap A_0 \subset A_0$, so by (4) $A_0 \in \mathcal{W}$. So $A \in \mathcal{W}$ being a countable union of sets in $\mathcal{W}$.

\[ \square \]

**Lemma 3.7.** If $\Gamma$ acts on $X$ and $A \subset X$ satisfies that there exists $N > 0$, $k \in \mathbb{N}$ and $g_1, \ldots, g_k \in \Gamma$ such that for any $n_1, \ldots, n_k \geq N$ we have $\bigcap_{i=1}^{k} g_i^{n_i} A = \emptyset$ then $A \in \mathcal{W}$.

*Proof.* We prove the following slightly stronger statement by induction on $k \in \mathbb{N}$: Suppose there exists $g_1, \ldots, g_k \in \Gamma$ and $N \in \mathbb{N}$ such that for every $n_1, \ldots, n_k \geq N$ $A \cap \left( \bigcap_{i=1}^{k} g_i^{n_i} A \right) \in \mathcal{W}$. Then $A \in \mathcal{W}$. The case $k = 1$ follows directly from Lemma 3.6(5). Now suppose there exists $g_1, \ldots, g_k, g_{k+1} \in \Gamma$ and $N \in \mathbb{N}$ such that for every $n_1, \ldots, n_k, n_{k+1} \geq N$ $A \cap \left( \bigcap_{i=1}^{k+1} g_i^{n_i} A \right) \in \mathcal{W}$. Then for all $n_{k+1} \geq N$ we have that $A_{n_1,\ldots,n_k} \cap g_{k+1} A_{n_1,\ldots,n_k} \subseteq A \cap \left( \bigcap_{i=1}^{k+1} g_i^{n_i} A \right)$. Then for all $n_{k+1} \geq N$ let $A_{n_1,\ldots,n_k} = A \cap \left( \bigcap_{i=1}^{k+1} g_i^{n_i} A \right)$. Then for all $n_{k+1} \geq N$ it follows that $A_{n_1,\ldots,n_k} \in \mathcal{W}$. Since this holds for all $n_1, \ldots, n_k \geq N$, it follows that $A \in \mathcal{W}$ by the induction hypothesis. \[ \square \]

**Lemma 3.8.** The sets $\text{sml}_-^+$ and $\text{sml}_-^-$ are in $\mathcal{W}$.

*Proof.* We will prove that $\text{sml}_-^+ \in \mathcal{W}$, as the proof that $\text{sml}_-^+ \in \mathcal{W}$ is completely analogous. For $q \in \mathbb{N}$ and $1 \leq u \leq 6$, denote $B(q,i) = \{ \preceq \in \text{Ext}(\Box) \mid e \prec a_i^q a_{i-1}^{-n}, \forall n \in \mathbb{N} \}$. Observe that

$$\text{sml}_-^+ = \bigcup_{q \in \mathbb{N}} \left( \bigcap_{i=1}^{6} B(q,i) \right).$$

Since $\mathcal{W}$ is closed under countable unions, it suffices to show that $\bigcap_{i=1}^{6} B(q,i) \in \mathcal{W}$ for every $q \in \mathbb{N}$. By Lemma 3.7 is suffices to show that for any $n_1, \ldots, n_6 > q$ we have $\bigcap_{i=1}^{6} a_i^{-n_i} B(q,i) = \emptyset$.

Indeed, for every $m > q$ and $\preceq \in a_i^{-m} B(q,i)$ we have

$$a_i^q a_{i-1}^{-m} \prec a_i^m a_{i-1}^{-m} a_i^{-m} a_{i-1}^m = a_i^q.$$

In particular, if $n_1, \ldots, n_6 > q$ and $\preceq \in \bigcap_{i=1}^{6} a_i^{-n_i} B(q,i)$, it follows that $a_i^q < a_2^q < \ldots < a_6^q < a_1^q$.

\[ \square \]

**Proposition 3.9.** The set $\text{Ext}(\Box) \in \mathcal{W}$, and in particular there is no invariant random total order on $\Gamma$ extending $\Box$.

*Proof.* That $\text{Ext}(\Box) \in \mathcal{W}$ follows directly from Lemma 3.5 together with Lemma 3.8. The “in particular” part follows by Lemma 3.6 (2). \[ \square \]

4. The Uniform Specification Property of $\text{Ord}(\Gamma)$

**Definition 4.1.** A topological dynamical system $\Gamma \curvearrowright X$ has the (uniform) specification property if for every $\epsilon > 0$ there exists a non-empty finite subset $F \subset \Gamma$ such that for every $x_1, x_2 \in X$ and any $K \subset \Gamma$ there exists $x \in X$ such that

$$d(g(x), g(x_1)) \leq \epsilon$$

for all $g \in K$, and

$$d(g(x), g(x_2)) \leq \epsilon$$

for all $g \in \Gamma \setminus (FK)$. 


Remark 1. It is routine to check that the specification property is independent of the particular metric $d$. If $X$ is a totally disconnected compact space then $\Gamma \bowtie X$ has the specification property if and only if for every partition of $P$ of $X$ into clopen sets there exists a finite set $F \subset \Gamma$ such that for every $x_1, x_2 \in X$ and any $K \subset \Gamma$ there exists $x \in X$ such that
\[ P(g(x)) = P(g(x_1)) \text{ for all } g \in K, \]
and
\[ P(g(x)) = P(g(x_2)) \text{ for all } g \in \Gamma \setminus (FK), \]
where $P(x)$ is the partition element of $P$ containing $x$.

Remark 2. If $A$ is a finite set and $X \subseteq A^\Gamma$ is a $\Gamma$-subshift then it is straightforward that the uniform specification property of $\Gamma \bowtie X$ is equivalent to $X$ being strongly irreducible: There exists a finite subset $F \subset \Gamma$ such that for any $x_1, x_2 \in X$ and any $K \subset \Gamma$ there exists $x \in X$ such that $x \mid_K = x_1 \mid_K$ and $x \mid_{\Gamma \setminus (KF)} = x_2 \mid_{\Gamma \setminus (KF)}$.

Remark 3. If $X$ is a totally disconnected compact metrizable space, than $\Gamma \bowtie X$ has the specification property if and only if any subshift factor of $\Gamma \bowtie X$ is strongly irreducible.

Definition 4.2. Given $D \subseteq \Gamma$ and $\prec \in \text{Ord}(\Gamma)$, denote
\[ [\prec]_D := \{ \tilde{\prec} \in \text{Ord}(\Gamma) : \tilde{\prec} \mid_{D \times D} = \prec \mid_{D \times D} \}. \]
Whenever $D \subset \Gamma$ is finite and $\prec \in \text{Ord}(\Gamma)$, we have that $[\prec]_D$ is a clopen subset of $\text{Ord}(\Gamma)$. The sets of the form $[\prec]_D$ for $\prec \in \text{Ord}(\Gamma)$ and $D$ a finite subset of $\Gamma$ are called cylinder sets.

For instance, for $\Gamma = \mathbb{Z}^2$, Figure 1 describes the collection “level lines” for elements of a specific cylinder set in $\text{Ord}(\Gamma)$.

Proposition 4.3. The action $\Gamma \bowtie \text{Ord}(\Gamma)$ has the specification property.

Proof. It suffices to show that for any finite $D \subset \Gamma$ there exists $F \subset \Gamma$ such that for any $\prec_1, \prec_2 \in \text{Ord}(\Gamma)$ and $K \subset \Gamma$ there exists $\prec \in \text{Ord}(\Gamma)$ that “$D$-shadows $\prec_1$ on $K$” and “$D$-shadows $\prec_2$ on $\Gamma \setminus (FK)$” in the following sense:
\[ [g(\prec)]_D = [g(\prec_1)]_D \text{ for all } g \in K, \]
and
\[ [g(\prec)]_D = [g(\prec_2)]_D \text{ for all } g \in \Gamma \setminus (FK). \]
Indeed, given a finite subset $D \subset \Gamma$, let $F = DD^{-1}$. Then for any $K \subset \Gamma$ and $\prec_1, \prec_2 \in \text{Ord}(\Gamma)$ one can define $\prec \in \text{Ord}(\Gamma)$ by declaring that $g_1 \prec g_2$ if and only if one of the following holds:
\begin{enumerate}
\item $g_1, g_2 \in K^{-1}D$ and $g_1 \prec_1 g_2$.
\item $g_1, g_2 \notin K^{-1}D$ and $g_1 \prec_2 g_2$.
\item $g_1 \in K^{-1}D$ and $g_2 \in \Gamma \setminus K^{-1}D$.
\end{enumerate}
Note that the condition that $\prec D$-shadows $\prec_1$ on $K$ is implied by $[\prec]_{K^{-1}D} = [\prec_2]_{K^{-1}D}$, which is given by (1). Similarly the condition that $\prec D$-shadows $\prec_2$ on $\Gamma \setminus (FK)$ is implied by $[\prec]_{(\Gamma \setminus (FK))^{-1}D} = [\prec_1]_{(\Gamma \setminus (FK))^{-1}D}$. Now we claim to have chosen $F$ so that $(\Gamma \setminus (FK))^{-1}D \cap K^{-1}D = \emptyset$, which is equivalent to $\Gamma \setminus (K^{-1}F^{-1}) \cap K^{-1}DD^{-1} = \emptyset$, which is true since $F = DD^{-1} = F^{-1}$. It follows that (2) implies $\prec D$-shadows $\prec_2$ on $\Gamma \setminus (FK)$, and (3) ensures that $\prec$ is total. \qed
Remark 4. The proof of Proposition 4.3 actually shows an a priori stronger property for \( \Gamma \cong \text{Ord}(\Gamma) \): For any finite subset \( D \subset \Gamma \) there exists \( F \subset \Gamma \) and an equivariant \(^1\) Borel function \( \Phi : \text{Ord}(\Gamma) \times \text{Ord}(\Gamma) \times 2^\Gamma \to \text{Ord}(\Gamma) \) such that \( \Phi(\prec_1, \prec_2, K) \) \( D \)-shadows \( \prec_1 \) on \( K \) and \( D \)-shadows \( \prec_2 \) on \( \Gamma \setminus (FK) \). We will refer to this property as the **Borel equivariant specification property**. In fact in the proof of Proposition 4.3 the map \( \Phi \) is continuous. We do not know if in general this property follows automatically from the “usual” specification property.

5. **Ergodic universality of \( \text{Ord}(\Gamma) \): Realizing ergodic systems via IROs**

The purpose of this section is to provide a characterization of ergodic \( \Gamma \)-actions that can be “realized as IROs”, in the ergodic theoretic sense. This turns out to be a rather general class, as it includes all essentially free ergodic actions. We briefly recall the notion of an invariant random subgroup, which turns out to be key in the characterization of ergodic \( \Gamma \)-actions that can be realized as IROs. Let \( \text{Sub}(\Gamma) \) be the set of all subgroups of \( \Gamma \). The space \( \text{Sub}(\Gamma) \) can be naturally viewed as a closed, hence compact subset of \( \{0,1\}^\Gamma \). The group \( \Gamma \) acts on \( \text{Sub}(\Gamma) \) by conjugation. Recall that a probability measure on \( \text{Sub}(\Gamma) \) that is invariant with respect to the action of \( \Gamma \) by conjugation is called an invariant random subgroup \([1]\), abbreviated by IRS.

Let \( \text{IRS}(\Gamma) \subset \text{Prob}(\text{Sub}(\Gamma)) \) denote the space of invariant random subgroups for \( \Gamma \). For any \( \Gamma \)-space \( X \), and any \( x \in X \), we denote by \( \text{stab}(x) \in \Gamma(x) \) the stabilizer subgroup of \( x \). A basic observation in \([1]\) is that any p.m.p action \( \Gamma \acts (X, \mathcal{B}, \mu) \), gives rise to a invariant random subgroup via the stabilizer map \( \text{stab} : X \to \text{Sub}(\Gamma) \). That is, \( \text{stab}_\ast(\mu) \in \text{IRS}(\Gamma) \). Conversely, it was shown in \([1]\) that any invariant random subgroup is realizable as the stabilizer of some p.m.p action. The invariant random subgroups arising from invariant random orders are subject to an obvious orderability constraint:

**Proposition 5.1.** For any \( \prec \in \text{Ord}(\Gamma) \), the subgroup \( \text{stab}(\prec) < \Gamma \) is left-orderable.

**Proof.** Fix \( \prec \in \text{Ord}(\Gamma) \). Let \( \Gamma_0 = \text{stab}(\prec) \). By definition, for any \( g \in \Gamma_0 \), \( g(\prec) = \prec \). In particular, \( g(\prec|_{\Gamma_0 \times \Gamma_0}) = \prec|_{\Gamma_0 \times \Gamma_0} \) - the restriction of \( \prec \) to \( \Gamma_0 \) is also \( g \)-invariant. Thus \( \prec|_{\Gamma_0 \times \Gamma_0} \) is a left-invariant order on \( \Gamma_0 \).

**Corollary 5.2.** For any invariant random order on \( \Gamma \) the stabilizer is almost surely an orderable subgroup. Namely, if \( \mu \in \text{IRO}(\Gamma) \) then

\[
\mu \left( \prec \in \text{Ord}(\Gamma) : \text{stab}(\prec) \text{ is left-orderable } \right) = 1.
\]

**Theorem 5.3.** Let \( \Gamma \) be a countable group, and \( \Gamma \acts (X, \mathcal{B}, \mu) \) an ergodic probability-preserving \( \Gamma \)-action not supported on a finite set. Then \( \Gamma \acts (X, \mathcal{B}, \mu) \) is measure-theoretically isomorphic to a \( \Gamma \)-invariant measure on \( \text{Ord}(\Gamma) \) if and only if there exists an equivariant measurable function \( \pi : X \to \text{p-Ord}(\Gamma) \) such that for almost every \( x \in X \), the restriction of \( \pi(x) \in \text{p-Ord}(\Gamma) \) to the stabilizer \( \text{stab}(x) \prec \Gamma \) is a left-invariant total order on \( \text{stab}(x) \).

**Proof.** One direction is trivial: Suppose \( \Gamma \acts (X, \mathcal{B}, \mu) \) is measure isomorphic to an IRO on \( \Gamma \) via \( \Phi : X \to \text{Ord}(\Gamma) \). Let \( x \in X \). Since \( \text{Ord}(\Gamma) \subseteq \text{p-Ord}(\Gamma) \), we can take \( \pi = \Phi \), then clearly the restriction of \( \pi(x) \) to \( \text{stab}(x) = \text{stab}(\Phi(x)) \) is a total order on \( \text{stab}(x) = \text{stab}(\Phi(x)) \), which is furthermore invariant (as explained in the proof of Proposition 5.1).

Let \( \Gamma \acts (X, \mathcal{B}, \mu) \) be an ergodic probability preserving \( \Gamma \)-action, and suppose \( \pi : X \to \text{p-Ord}(\Gamma) \) is an equivariant measurable function such that for almost every \( x \in X \), the

\(^1\)The equivariance is with respect to the action of \( \Gamma \) on subsets given by \( g.K = Kg^{-1} \).
restriction of \( \pi(x) \in \text{p-Ord}(\Gamma) \) to the stabilizer \( \text{stab}(x) < \Gamma \) is an left-invariant total order on \( \text{stab}(x) \). By ergodicity, since \( \mu \) is not supported on a finite orbit, it has no atoms, so \((X, \mathcal{B}, \mu)\) is a standard Lebesgue space with a non-atomic probability measure. We can assume that \( X = [0,1] \), that \( \mathcal{B} \) is the Lebesgue \( \sigma \)-algebra , and that \( \mu \) is Lebesgue measure. Given \( x \in X \), denote by \( \lesssim_x \in \text{Ord}(\text{stab}(x)) \) the restriction of \( \pi(x) \in \text{p-Ord}(\Gamma) \) to \( \text{stab}(x) \). Define \( \lesssim_x \in \text{Ord}(\Gamma) \) by \( g \lesssim_x h \) iff \( g(x) < h(x) \) or \( g(x) = h(x) \) and \( e \lesssim_x g^{-1} h \). Because \( g(x) = h(x) \) if and only if \( g^{-1} h \in \text{stab}(x) \), it follows that indeed \( \lesssim_x \in \text{Ord}(\Gamma) \). The map \( \Phi : X \to \text{Ord}(\Gamma) \) given by \( \Phi(x) := \lesssim_x \) is clearly Borel and equivariant. It remains to check that it is injective on a set of full measure. We do so by explicitly describing a Borel inverse.

We recall the notion of a pointwise ergodic sequence of measures for a group \( \Gamma \). A sequence of probability measures \( \{\nu_n\}_{n=1}^\infty \) on \( \Gamma \) is called pointwise ergodic for \( \Gamma \) if for any probability preserving \( \Gamma \)-action \( \Gamma \curvearrowright (X, \mathcal{B}, \mu) \) and any \( f \in L^1(X, \mathcal{B}, \mu) \) we have \( \int f(g(x))\nu_n(g) \to \int f d\mu \) for \( \mu \)-almost every \( x \in X \). Any countable group \( \Gamma \) admits a pointwise ergodic sequence of measures \([12, 18]\), for instance the convolution powers of a symmetric probability measure on \( \Gamma \) whose support generates \( \Gamma \). For background and a historical account see for instance \([3]\). Let \( \{\nu_n\}_{n=1}^\infty \) be a pointwise ergodic sequence of measures on \( \Gamma \). We claim that almost surely we have

\[
x = \lim_{n \to \infty} \nu_n(\{h \in \Gamma : h \lesssim_x e\}).
\]

Indeed, this is equivalent to showing that for every \( t \in [0,1] \cap \mathbb{Q} \), and almost every \( x \in [0,t] \) we have

\[
\lim_{n \to \infty} \nu_n(\{h \in \Gamma : h \lesssim_x e\}) \leq t.
\]

But by definition \( h \lesssim_x e \) implies that \( h(x) \leq x \). It follows that for any \( x \in [0,t] \)

\[
\nu_n(\{h \in \Gamma : h \lesssim_x e\}) \leq \nu_n(\{h \in \Gamma : h(x) \leq t\}).
\]

By pointwise ergodicity of \( \nu_n \), we have that for almost every \( x \),

\[
\lim_{n \to \infty} \nu_n(\{h \in \Gamma : h(x) \leq t\}) = \mu([0,t]) = t.
\]

This shows that the equivariant Borel map \( \Psi : \text{Ord}(\Gamma) \to [0,1] \) given by

\[
\Psi(\lesssim) = \lim inf_{n \to \infty} \nu_n(\{h \in \Gamma : h \lesssim e\})
\]

Satisfies \( \Psi(\Phi(x)) = x \) for almost every \( x \). \( \square \)

In particular we have:

**Corollary 5.4.** For any countable group \( \Gamma \), any essentially free, ergodic probability preserving \( \Gamma \)-action can be realized as an IRO.

**Proof.** If \( \Gamma \curvearrowright (X, \mathcal{B}, \mu) \) is ergodic and essentially free, the map \( \pi : X \to \text{p-Ord}(\Gamma) \) defined by letting \( \pi(x) \) be the identity relation, satisfies the required property that the restriction of \( \pi(x) \) to the trivial subgroup is an invariant total order. The fact that the action is essentially free means that \( \text{stab}(x) \) is almost surely equal to the trivial subgroup. \( \square \)

**Corollary 5.5.** If \( \Gamma \) is left-orderable, any infinite ergodic action of \( \Gamma \) on a space can be realized as an IRO.

**Proof.** Suppose \( \lesssim_0 \in \text{Ord}(\Gamma) \) is left-invariant order. The map \( \pi : X \to \text{p-Ord}(\Gamma) \) defined by \( \pi(x) := \lesssim_0 \) satisfies the required property that the restriction of \( \pi(x) \in \text{p-Ord}(\Gamma) \) to \( \text{stab}(x) \) is an invariant total order. \( \square \)
Remark 5. An inspection of the proof of Theorem 5.3 shows that the uniform total order is measure-theoretically isomorphic to the Bernoulli shift $[0,1]^\Gamma$, equipped with Haar measure.

Remark 6. As a particular consequence of Corollary 5.5 we see that that whenever $\Gamma$ is a countable sofic group, then the action of $\Gamma$ on $\text{Ord}(\Gamma)$ has infinite sofic topological entropy with respect to any sofic approximation sequence.

Remark 7. The ergodicity assumption in Theorem 5.3 is necessary. For instance, the group $\Gamma = \mathbb{Z}$ has only 2 invariant deterministic total orders so the trivial action on a space with more than 2 points cannot be realized as an IRO.

Remark 8. In the case $\Gamma = \mathbb{Z}^d$, it follows from [5] that any action of $\mathbb{Z}^d$ having the specification property and infinite topological entropy can realize any essentially free probability-preserving action of $\mathbb{Z}^d$ can be realized as an IRO on $\mathbb{Z}^d$, so the ergodicity assumption in Theorem 5.3 can be removed in this case. It is plausible that the arguments of [5] can be extended to more general amenable groups.

6. The structure of the simplex $\text{IRO}(\Gamma)$

Let us consider the space of all invariant random orders on $\Gamma$ as a compact convex set. One can ask what properties of the group $\Gamma$ can be detected by considering $\text{IRO}(\Gamma)$ as a metrizable Choquet simplex, up to affine homeomorphisms. A Poulsen simplex is a metrizable Choquet simplex whose extreme points are dense. Lindenstrauss, Olsen and Sternfeld [14] have shown that there is a unique Poulsen simplex up to affine homeomorphism. A Bauer simplex is a metrizable Choquet simplex whose extreme points are closed. In [10] Glasner and Weiss proved the following striking dichotomy for the simplex of invariant measures of the shift action $\Gamma \curvearrowright K^\Gamma$, for a compact set $K$: If $\Gamma$ has property $T$, then $\text{Prob}_{\Gamma}(K^\Gamma)$ is a Bauer simplex. If $\Gamma$ does not have property $T$, then $\text{Prob}_{\Gamma}(K^\Gamma)$ is a Poulsen simplex.

We now show a similar dichotomy holds for $\text{IRO}(\Gamma)$:

Theorem 6.1. Let $\Gamma$ be a countable group. If $\Gamma$ has property $T$, then $\text{IRO}(\Gamma)$ is a Bauer simplex. If $\Gamma$ does not have property $T$, then $\text{IRO}(\Gamma)$ is a Poulsen simplex.

Proof. If $\Gamma$ has property $T$ it follows directly form [10, Theorem 1] that $\text{IRO}(\Gamma)$ is a Bauer simplex. Now suppose $\Gamma$ does not have property $T$. In order to prove that $\text{IRO}(\Gamma)$ is a Poulsen simplex, we will show that for any two ergodic IROs $\nu_1, \nu_2 \in \text{IRO}(\Gamma)$, the average $\frac{1}{2}(\nu_1 + \nu_2)$ can be “well weak-* approximated” by an ergodic IRO. Specifically, it suffices to show that for any ergodic $\nu_1, \nu_2 \in \text{IRO}(\Gamma)$, any $\epsilon > 0$, any finite $D \subset \Gamma$ there exists an ergodic $\nu \in \text{IRO}(\Gamma)$ such that

$$\nu(\langle \prec \rangle_D) - \frac{1}{2}(\nu_1(\langle \prec \rangle_D) + \nu_2(\langle \prec \rangle_D)) < \epsilon \quad \forall \langle \prec \rangle \in \text{Ord}(\Gamma).$$

So fix $\epsilon$ and a finite $D \subset \Gamma$. By Glasner-Weiss [10], there exists a weakly mixing $\Gamma$ action on some space $(X, \mu)$ with asymptotically invariant sets. In particular there exists a measurable set $A \subset X$ with $\mu(A) = \frac{1}{2}$ such that $\mu(\bigcap_{g \in DD^{-1}} g(A)) > \frac{1}{2} - \frac{1}{2} \epsilon$ and $\mu(\bigcap_{g \in DD^{-1}} g(A^c)) > \frac{1}{2} - \frac{1}{2} \epsilon$.

Let $\Phi : \text{Ord}(\Gamma) \times \text{Ord}(\Gamma) \times 2^\Gamma \to \text{Ord}(\Gamma)$ be a Borel equivariant function witnessing the Borel equivariant specification property for $\Gamma \curvearrowright \text{Ord}(\Gamma)$ as in Remark 4 and Proposition 4.3. Let $\lambda$ be an ergodic joining of $\nu_1$ and $\nu_2$. Let $\varphi : X \to 2^\Gamma$ be the $\Gamma$-equivariant function such that
\( \varphi(x) \varepsilon = 1_A(x) \). Then \( \lambda \times \varphi \cdot \mu \) is an ergodic measure on \( \text{Ord}(\Gamma) \times \text{Ord}(\Gamma) \times \mathcal{X} \). Let \( \nu \) be the push-forward of \( \lambda \times \varphi \cdot \mu \) via \( \Phi \). Then for every \( \varepsilon \in \text{Ord}(\Gamma) \)

\[
\nu(\lfloor \varepsilon \rfloor_D) \cdot \left( 1 - \mu\left( \bigcap_{g \in D} g A^c \right) \right) + \nu(\lfloor \varepsilon \rfloor_D) \cdot (1 - \mu(A)) + \mu\left( X \setminus \left( A^c \cup \bigcap_{g \in D} g A^c \right) \right) \\
\geq \nu(\lfloor \varepsilon \rfloor_D) \geq \nu(\lfloor \varepsilon \rfloor_D) \cdot \mu(A) + \nu(\lfloor \varepsilon \rfloor_D) \cdot \mu\left( \bigcap_{g \in D} g A^c \right). \tag{5}
\]

The leftmost-hand-side is bounded from above by \( \frac{1}{2} \left( \nu_1(\lfloor \varepsilon \rfloor_D) + \nu_2(\lfloor \varepsilon \rfloor_D) \right) + \epsilon \) and the righthand-most-side is bounded from below by \( \frac{1}{2} \left( \nu_1(\lfloor \varepsilon \rfloor_D) + \nu_2(\lfloor \varepsilon \rfloor_D) \right) - \epsilon \) which completes the proof. \( \square \)

**Remark 9.** The proof of Theorem 6.1 actually shows that for any countable group \( \Gamma \) that does not have property \( T \), for any action \( \Gamma \rhd X \) with the Borel equivariant specification property the simplex of invariant measures is a Poulsen simplex.

The above remark motivates us to ask the following question:

**Question 6.2.** Let \( \Gamma \) be a countable group that does not have property \( T \), and let \( \Gamma \rhd X \) be an action with the specification property. Is the simplex of invariant probability measures a Poulsen simplex?

In this direction we have the following partial result:

**Proposition 6.3.** If \( \Gamma \) is a countable amenable group, \( X \) is compact metrizable, and \( \Gamma \rhd X \) has the specification property, then the simplex of \( \Gamma \)-invariant probability measures on \( X \) is a Poulsen simplex.

Although not directly in line with the central theme of this paper, we provide a short proof. Somewhat amusingly, our proof uses an auxiliary total order on a compact topological space (via the proof of Lemma 6.4 below).

**Proof.** Suppose \( \Gamma \) is an amenable group acting on a compact metrizable space \( X \). Assume that \( \Gamma \rhd X \) has the specification property.

As in the proof of Theorem 6.1, it suffices to show that for any ergodic \( \nu_1, \nu_2 \in \text{Prob}_\Gamma(X) \), any weak*-neighborhood of \( \frac{1}{2}(\nu_1 + \nu_2) \) contains an ergodic element of \( \text{Prob}_\Gamma(X) \). Fix \( \epsilon > 0 \). Consider the collection \( P_\epsilon \) of (not necessarily \( \Gamma \)-invariant) probability measures \( \nu \) on \( X \) that have the property that for every \( g \in \Gamma \) the distance between \( g \nu \) and \( \frac{1}{2}(\nu_1 + \nu_2) \) is at most \( \epsilon \). This is a compact \( \Gamma \)-invariant convex set, so by amenability of \( \Gamma \), if \( P_\epsilon \neq \emptyset \), the set \( P_{\epsilon, \Gamma} \subseteq P_\epsilon \) of measures that are fixed under \( \Gamma \) will be also a non-empty compact convex set. In this case, any extreme point of \( P_{\epsilon, \Gamma} \) would witness an ergodic \( \Gamma \)-invariant measure which is \( \epsilon \)-close to \( \frac{1}{2}(\nu_1 + \nu_2) \).

So in order to complete the proof, it remains to show that \( P_\epsilon \neq \emptyset \).

By the uniform specification property, there exists a finite set \( F \subset \Gamma \) such that for any \( x_1, x_2 \in X \) and any \( y \in \{0, 1\}^F \) there exists \( x \in X \) that \( \epsilon \)-interpolates \( x_1 \) and \( x_2 \) according to \( y \) in the following sense:

1. \( d(g(x), g(x_1)) \leq \epsilon \) whenever \( g(y)_h = 0 \) for all \( h \in F \).
2. \( d(g(x), g(x_2)) \leq \epsilon \) whenever \( g(y)_h = 1 \) for all \( h \in F \).
Let $X^*$ denote the space of closed subsets of $X$ (with the Fell topology, induced by the Hausdorff metric). Consider the map $\Phi_0 : X \times X \times \{0,1\}^\Gamma \to X^*$ defined by declaring for $x_1, x_2 \in X$ and $y \in \{0,1\}^\Gamma$ to be the set $\Phi_0(x_1, x_2, y) \in X^*$ of points $x \in X$ that $\epsilon$-interpolates $x_1$ and $x_2$ according to $y$ in the above sense. As explained, the uniform specification property implies that $\Phi_0(x_1, x_2, y)$ is non-empty for every $x_1, x_2 \in X$ and $y \in \{0,1\}^\Gamma$, and can be directly verified that $\Phi_0(x_1, x_2, y)$ is a closed subset of $X$. Furthermore, $\Phi_0 : X \times X \times \{0,1\}^\Gamma \to X^*$ is a clearly a Borel function. By Lemma 6.4 below, there exists a continuous function $\Psi : X^* \to X$ such that $\Psi(A) \in A$ for every non-empty $A \in X^* \phi(A) \in A$. Let $\Phi : X \times X \times \{0,1\}^\Gamma \to X$ be given by $\Phi = \Psi \circ \Phi_0$. Then $\Phi : X \times X \times \{0,1\}^\Gamma \to X$ is a Borel map (although not equivariant because $\Psi$ is not generally equivariant).

Let $\mu$ be the uniform Bernoulli measure on $\{0,1\}^\Gamma$ and let $\Gamma \curvearrowright (\{0,1\}^\Gamma, \mu)$ be the Bernoulli shift action. By amenability, for any $\epsilon > 0$ and any finite subset $F$ of $\Gamma$ there exists a measurable set $A \subset \{0,1\}^\Gamma$ such that $\mu(A) = \frac{1}{2} - \frac{1}{2}\epsilon$ and $\mu(\bigcap_{g \in F} g(A)) > \frac{1}{2} - \frac{3}{4}\epsilon$. See for example [21, Theorem 2.4].

It can be verified directly that the push-forward of $\nu_1 \times \nu_2 \times \mu$ via $\Phi$ is an element of $P_\epsilon$.

In the proof above we applied the following simple lemma:

**Lemma 6.4.** Let $X$ be a compact metrizable topological space, and let $X^*$ denote the space of closed subsets of $X$ (with the Fell topology, induced by the Hausdorff metric). Then there exists a “continuous selection map”, namely a map $\Psi : X^* \to X$ such that $\Psi(A) \in A$ for any non-empty $A \in X^*$.

**Proof.** Since any compact metrizable space is homeomorphic to a subset of the Hilbert cube $H := [0,1]^N$, and because any embedding $X \hookrightarrow H$ naturally induces an embedding $X^* \hookrightarrow H^*$, it suffices to prove the lemma for the case $X = H$. Let $\leq_H$ denote the lexicographical order on the Hilbert cube, defined by declaring for $x, y \in H$,

$$x \leq_H y \text{ if and only if } x = y \text{ or } \exists n \in \mathbb{N} \text{ s.t. } x_n < y_n \text{ and } x_k = y_k \text{ for all } k < n.$$ 

By a routine compactness argument any non-empty closed subset of $H$ admits a $\leq_H$-maximal element (this is essentially the Weierstrass extreme value theorem). Any map $\Psi : H^* \to H$ satisfying $\Psi(A) = \max_{\leq_H} A$ for every non-empty $A \in X^*$ is continuous, and satisfies the properties asserted in the statement of the lemma (by compactness of $H$ the empty set is an isolated point in $H^*$ so the value of $\Psi$ on the empty set cannot break continuity).

7. Further discussion and open questions

We conclude with a discussion of some further directions, questions and related problems.

7.1. Extension of random orders. A countable group $\Gamma$ has the IRO-extension property if every partial invariant random order on $\Gamma$ can be extended to a random invariant (total) order. As mentioned earlier, amenable groups have the IRO-extension property. In Section 3 we showed that $\text{SL}_3(\mathbb{Z})$ does not have the IRO-extension property, providing a first example for a countable group for which this property fails.

**Question 7.1.** Does there exist a non-amenable group with the IRO-extension property?

If the answer to Question 7.1 is negative, the IRO-extension property would be yet another characterization of amenability. If non-amenable groups with the IRO-extension property
exist, it could be of interest to understand if the IRO-extension property can be expressed in terms of other natural group properties.

7.2. Equivariant orderability and realization of probability preserving actions as IROs. In Section 5 we showed for any point in Ord(Γ) the stabilizer subgroup of Γ is always orderable. We also showed that a seemingly slightly stronger property is necessary and sufficient for a (non-atomic) ergodic probability preserving action to be realizable as an IRO.

Call an invariant random subgroup \( \mu \in \text{IRS}(\Gamma) \) **orderable** if it is supported on orderable subgroups, and **equivariantly orderable** if there exists a measurable equivariant function \( \pi : \text{Sub}(\Gamma) \to \text{p-Ord}(\Gamma) \) such that for almost every \( \Delta \in \text{Sub}(\Gamma) \) (with respect to \( \mu \)), the restriction of \( \pi(\Delta) \in \text{p-Ord}(\Gamma) \) to \( \Delta \) is an invariant total order.

A positive answer to the following question would yield a simplified characterization of probability preserving actions realizable as IRO’s.

**Question 7.2.** Is every orderable invariant random subgroup equivariantly orderable?

The argument given in the proof of Corollary 5.5 shows that any IRO of an orderable group \( \Gamma \) is equivariantly orderable. Thus, a positive solution to the question below would immediately imply a positive solution to Question 7.2

**Question 7.3.** Suppose \( \Gamma \) admits an ergodic IRS which is almost surely orderable and spanning in the sense that that \( \Gamma \) is the smallest normal subgroup which contains all the subgroups in the support of the IRS. Is \( \Gamma \) necessarily orderable?

Both ergodicity and the spanning assumptions are necessary here, as the following examples show:

**Example 7.4.** Let \( \Gamma \) be a finitely generated group which is not left orderable, such as \( \text{SL}_3(\mathbb{Z}) \). By finite generation we can pick an epimorphism \( \phi : F_n \to \Gamma \), where \( F_n \) is the free group on \( n \)-generators. Let \( \Delta = \{(x,x) \ | \ x \in F_n\} \triangleleft F_n \times F_n \) be the diagonal copy of \( F_n \) in the product, and \( N = \{(x,x) \ | \ x \in \ker(\phi)\} \). Now set \( G = (F_n \times F_n)/N \), and \( \tilde{G} = G \rtimes C_2 \) where the cyclic group \( C_2 \) acts via the obvious involution \( (x,y)N \mapsto (y,x)N \). Let \( F_1 = \{(x,e)N \ | \ x \in F_n\}, F_2 = \{(e,x)N \ | \ x \in F_n\} \) be the (injective) images of the two free factors in \( G \) and finally consider the IRS

\[
\mu = \delta_{F_1} + \delta_{F_2}.
\]

This is an IRS in both groups \( G < \tilde{G} \). Indeed \( F_1, F_2 < G \) so that the dirac measures \( \delta_{F_i} \) are \( G \)-invariant. In \( \tilde{G} \) these two are flipped by the involution so \( \mu \) is still invariant. \( \mu \) is supported on free groups, which are definitely left orderable, but the groups \( G, \tilde{G} \) themselves fail to be left orderable because they contain an isomorphic copy of \( \Gamma \cong \frac{\Delta}{N} \). These two examples just fall short from giving a counter example to Question 7.2. As an IRS on \( G \), \( \mu \) is spanning but fails to be ergodic; on \( \tilde{G} \), \( \mu \) is ergodic, but spans only \( G \).

7.3. Strong non-orderability. Say that a countable group \( \Gamma \) is **strongly non-orderable** if every \( \mu \in \text{IRO}(\Gamma) \) induces an essentially free probability preserving \( \Gamma \)-action. In particular, a strongly non-orderable group \( \Gamma \) does not admit left-orderable subgroups of finite index.

The Stuck Zimmer theorem [24] combined with the recent proof of Hurtado and Deroin that all higher rank irreducible lattices are non-orderable [6] gives rise to the following.
Corollary 7.5. Let $\Gamma < G$ be an irreducible lattice in a higher rank semisimple Lie group $G$ with property (T). Then $\Gamma$ is strongly non-orderable.

Proof. Assume that $\prec \in \text{IRO}(\Gamma)$ be an ergodic IRO which is not essentially free. The Stuck-Zimmer theorem implies that this IRO is supported on a finite orbit, and hence every order in $\text{Supp}(\mu)$ is fixed by a finite index subgroup of $\Gamma$, contradicting the main result of [6]. □

Finite groups are obviously strongly non-orderable, as is any normal subgroup of a strongly non-orderable group.

To the best of our knowledge, It is currently not known if a left orderable group can ever satisfy Kazhdan’s property (T). In view of the above, we formulate the following question:

Question 7.6. Is any group with Kazhdan’s property (T) strongly non-orderable?

References

1. Miklós Abért, Yair Glasner, and Bálint Virág, Kesten’s theorem for invariant random subgroups, Duke Math. J. 163 (2014), no. 3, 465–488. MR 3165420
2. Andrei Alpeev, Tom Meyerovitch, and Sieye Ryu, Predictability, topological entropy, and invariant random orders, Proc. Amer. Math. Soc. 149 (2021), no. 4, 1443–1457. MR 4242303
3. Lewis Bowen and Amos Nevo, Pointwise ergodic theorems beyond amenable groups, Ergodic Theory Dynam. Systems 33 (2013), no. 3, 777–820. MR 3062901
4. Raimundo Briceno, Kieffer-pinsker type formulas for gibbs measures on sofic groups, arXiv preprint arXiv:2108.06053 (2021).
5. Nishant Chandgotia and Tom Meyerovitch, Borel subsystems and ergodic universality for compact $\mathbb{Z}^d$-systems via specification and beyond, Proc. Lond. Math. Soc. (3) 123 (2021), no. 3, 231–312. MR 4311117
6. Bertrand Deroin and Sebastian Hurtado, Non left-orderability of lattices in higher rank semi-simple lie groups, arXiv preprint arXiv:2008.10687 (2020).
7. Bertrand Deroin, Andrés Navas, and Cristóbal Rivas, Groups, orders, and dynamics, arXiv preprint arXiv:1408.5805 (2014).
8. Tomasz Downarowicz, Piotr Oprocha, Mateusz Wieccek, and Guohua Zhang, Multiorders in amenable group actions, arXiv preprint arXiv:2108.03211 (2021).
9. Edward Formanek, Extending partial right orders on nilpotent groups, J. London Math. Soc. (2) 7 (1973), 131–134. MR 327605
10. E. Glasner and B. Weiss, Kazhdan’s property T and the geometry of the collection of invariant measures, Geom. Funct. Anal. 7 (1997), no. 5, 917–935. MR 1475550
11. Wen Huang, Lei Jin, and Xiangdong Ye, On extensions over semigroups and applications, Entropy 18 (2016), no. 6, Paper No. 230, 6. MR 3530053
12. Shizuo Kakutani, Random ergodic theorems and Markoff processes with a stable distribution, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, University of California Press, Berkeley and Los Angeles, Calif., 1951, pp. 247–261. MR 0044773
13. J. C. Kieffer, A generalized Shannon-McMillan theorem for the action of an amenable group on a probability space, Ann. Probability 3 (1975), no. 6, 1031–1037. MR 393422
14. J. Lindenstrauss, G. Olsen, and Y. Sternfeld, The Poulsen simplex, Ann. Inst. Fourier (Grenoble) 28 (1978), no. 1, vi, 91–114. MR 500918
15. Peter Linnell and Dave Witte Morris, Amenable groups with a locally invariant order are locally indicable, Groups Geom. Dyn. 8 (2014), no. 2, 467–478. MR 3231224
16. Dave Witte Morris, Some arithmetic groups that do not act on the circle, 2012.
17. Andrés Navas, On the dynamics of (left) orderable groups, Ann. Inst. Fourier (Grenoble) 60 (2010), no. 5, 1685–1740. MR 2766228
18. V. I. Oseledec, Markov chains, skew products and ergodic theorems for “general” dynamic systems, Teor. Veroyatnost. i Primenen. 10 (1965), 551–557. MR 0189123
19. B. S. Pickel’and A. M. Stepin, The property of the entropy equidistribution of commutative groups of metric automorphisms, Dokl. Akad. Nauk SSSR 198 (1971), 1021–1024. MR 0304615
20. A. H. Rhemtulla, *Right-ordered groups*, Canadian J. Math. **24** (1972), 891–895. MR 311538
21. Klaus Schmidt, *Amenability, Kazhdan’s property T, strong ergodicity and invariant means for ergodic group-actions*, Ergodic Theory Dynam. Systems **1** (1981), no. 2, 223–236. MR 661821
22. Adam S. Sikora, *Topology on the spaces of orderings of groups*, Bull. London Math. Soc. **36** (2004), no. 4, 519–526. MR 2069015
23. A. M. Stepin, *Equidistribution of the entropy of amenable groups of transformations*, Akad. Nauk Azerbaidzhan. SSR Dokl. **34** (1978), no. 9, 3–7. MR 522690
24. Garrett Stuck and Robert J. Zimmer, *Stabilizers for ergodic actions of higher rank semisimple groups*, Ann. of Math. (2) **139** (1994), no. 3, 723–747. MR 1283875
25. Dave Witte, *Arithmetic groups of higher Q-rank cannot act on 1-manifolds*, Proc. Amer. Math. Soc. **122** (1994), no. 2, 333–340. MR 1198459

Ben-Gurion University of the Negev. Departement of Mathematics. Be’er Sheva, 8410501, Israel.

Ben-Gurion University of the Negev. Departement of Mathematics. Be’er Sheva, 8410501, Israel.

Ben-Gurion University of the Negev. Departement of Mathematics. Be’er Sheva, 8410501, Israel.