A generating function associated with the alternating elements in the positive part of $U_q(\widehat{sl}_2)$

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1. Introduction

The quantized enveloping algebra $U_q(\widehat{sl}_2)$ has a subalgebra $U_q^+$, called the positive part [6, 18]. Both $U_q(\widehat{sl}_2)$ and $U_q^+$ appear in algebra [2, 4, 9, 10, 16], combinatorics [1, 11, 14, 15, 25, 26], mathematical physics [5, 8, 17], and representation theory [6, 12, 28].

In [20], M. Rosso introduced an embedding of the algebra $U_q^+$ into a $q$-shuffle algebra. In [7], I. Damiani obtained a Poincaré-Birkhoff-Witt (or PBW) basis for $U_q^+$. In her construction the PBW basis elements $\{E_{n+\delta+a_1}\}_{n\in\mathbb{N}}, \{E_{n+\delta}\}_{n\in\mathbb{N}}, \{E_{(n+1)\delta}\}_{n\in\mathbb{N}}$ are defined recursively. In [26], P. Terwilliger expressed the Damiani PBW basis elements in closed form, using the Rosso embedding of $U_q^+$.

In [25], Terwilliger used the Rosso embedding to obtain a type of element in $U_q^+$, said to be alternating. The alternating elements fall into four families, denoted by $\{W_{-n}\}_{n\in\mathbb{N}}, \{W_{n+1}\}_{n\in\mathbb{N}}, \{G_{n+1}\}_{n\in\mathbb{N}}, \{\hat{G}_{n+1}\}_{n\in\mathbb{N}}$. It was shown in [25] that the alternating elements $\{W_{-n}\}_{n\in\mathbb{N}}, \{W_{n+1}\}_{n\in\mathbb{N}}, \{G_{n+1}\}_{n\in\mathbb{N}}$ form a PBW basis for $U_q^+$; this PBW basis is called alternating. The alternating PBW basis was used in [2, 3, 21, 23, 24, 27].

In [25, Theorem 9.15], Terwilliger expressed $\{G_{n+1}\}_{n\in\mathbb{N}}$ in terms of the alternating PBW basis. His answer involved the elements $\{D_n\}_{n\in\mathbb{N}}$ with the following property: the generating function $D(t) = \sum_{n\in\mathbb{N}} D_n t^n$ is the multiplicative inverse of the generating function $G(t) = \sum_{n\in\mathbb{N}} \hat{G}_n t^n$ where $\hat{G}_0 = 1$. In [25, Section 11], Terwilliger used $\{D_n\}_{n\in\mathbb{N}}$ to describe how the Damiani PBW basis is related to the alternating PBW basis.
To motivate our results, we make some comments about $D(t)$. We mentioned that $D(t)$ is the multiplicative inverse of $\hat{G}(t)$. Using this relationship, the elements $\{D_n\}_{n \in \mathbb{N}}$ can be computed recursively from $\{\hat{G}_n\}_{n \in \mathbb{N}}$. Calculation of $D_n$ for $n \leq 3$ suggests that the elements $\{D_n\}_{n \in \mathbb{N}}$ admit a closed form. Our goal in this paper is to express $\{D_n\}_{n \in \mathbb{N}}$ in closed form. We will state our main result shortly.

First, we establish some conventions and notation.

In this paper, $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers, and $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is the set of integers. The letters $n, k, i, j, r, s, t$ always represent an integer. Let $\mathbb{F}$ denote a field. All algebras discussed are over $\mathbb{F}$, associative, and with a multiplicative identity. Let $q$ denote a nonzero scalar in $\mathbb{F}$ that is not a root of unity. For $n \in \mathbb{N}$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q.$$

We interpret $[0]_q! = 1$.

We will be looking at the positive part of $U_q(\widehat{sl}_2)$, denoted by $U_q^+$ [6, 18]. The algebra $U_q^+$ is defined by generators $A, B$ and the $q$-Serre relations

\begin{align*}
A^3B - \left[3\right]_q A^2BA + \left[3\right]_q ABA^2 - BA^3 &= 0, \\
B^3A - \left[3\right]_q B^2AB + \left[3\right]_q BAB^2 - AB^3 &= 0.
\end{align*}

Next we recall the Rosso embedding of $U_q^+$ into a $q$-shuffle algebra [20]. Let $x, y$ denote noncommuting indeterminates (called letters). Let $\mathbb{V}$ denote the free algebra generated by $x, y$. A product $v_1 v_2 \cdots v_n$ of letters is called a word, and $n$ is called the length of this word. The word of length 0 is called trivial and denoted by $1$. The words form a basis for the vector space $\mathbb{V}$, called the standard basis. The vector space $\mathbb{V}$ admits another algebra structure called the $q$-shuffle algebra. The $q$-shuffle product, denoted by $\star$, is defined recursively as follows:

- For $v \in \mathbb{V}$,

$$1 \star v = v \star 1 = v.$$

- For the letters $u, v$,

$$u \star v = uv + vuq^{(u,v)},$$

where

$$\langle x, x \rangle = \langle y, y \rangle = 2, \quad \langle x, y \rangle = \langle y, x \rangle = -2.$$

- For a letter $u$ and a nontrivial word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$u \star v = \sum_{i=0}^{n} v_1 \cdots v_i u v_{i+1} \cdots v_n q^{(u,v_1) + \cdots + (u,v_i)},$$

$$v \star u = \sum_{i=0}^{n} v_1 \cdots v_i u v_{i+1} \cdots v_n q^{(u,v_n) + \cdots + (u,v_{i+1})}.$$

- For nontrivial words $u = u_1 u_2 \cdots u_r$ and $v = v_1 v_2 \cdots v_s$ in $\mathbb{V}$,

$$u \star v = u_1 ( (u_2 \cdots u_r) \star v ) + v_1 (u \star (v_2 \cdots v_s)) q^{(v_1,u_1) + \cdots + (v_1,u_r)},$$

$$u \star v = (u \star (v_1 \cdots v_{s-1})) v_s + ( (u_1 \cdots u_{r-1}) \star v ) u_r q^{(u_r,v_1) + \cdots + (u_r,v_s)}.$$

Note that the $q$-shuffle product of two words of length $l_1, l_2$ is a linear combination of words of length $l_1 + l_2$. 

\[\]
Green showed in [13] that \( x, y \) satisfy the \( q \)-Serre relations in the \( q \)-shuffle algebra \( V \):
\[
x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0,
\]
\[
y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0.
\]
As a result there exists an algebra homomorphism \( \natural \) from \( U_q^+ \) to the \( q \)-shuffle algebra \( V \) that sends \( A \mapsto x, B \mapsto y \). The map \( \natural \) is injective by [20, Theorem 15]. Let \( U \) denote the subalgebra of the \( q \)-shuffle algebra \( V \) generated by \( x, y \). By construction, the image of \( \natural \) is \( U \).

We now mention some special words in \( V \) that will be useful later.

**Definition 1.1.** (See [25, Definition 5.2, Lemma 5.4].) We define \( G_0 = \tilde{G}_0 = 1 \).

For \( n \in \mathbb{N} \), define
\[
G_{n+1} = G_n yx, \quad \tilde{G}_{n+1} = \tilde{G}_n xy,
\]
\[
W_{-n} = \tilde{G}_n x, \quad W_{n+1} = y\tilde{G}_n.
\]

The words \( \{W_{-n}\}_{n \in \mathbb{N}}, \{W_{n+1}\}_{n \in \mathbb{N}}, \{G_n\}_{n \in \mathbb{N}}, \{\tilde{G}_n\}_{n \in \mathbb{N}} \) are called alternating.

**Example 1.2.** We have
\[
W_0 = x, \quad W_{-1} = xy, \quad W_{-2} = xyxy, \quad \ldots
\]
\[
W_1 = y, \quad W_2 = yxy, \quad W_3 = yxyxy, \quad \ldots
\]
\[
G_1 = xy, \quad G_2 = yxyx, \quad G_3 = yxyxyx, \quad \ldots
\]
\[
\tilde{G}_1 = xy, \quad \tilde{G}_2 = xyxy, \quad \tilde{G}_3 = xyxyxy, \quad \ldots
\]

By [25, Theorem 8.3], the alternating words are contained in \( U \).

It is shown in [25, Proposition 5.10] that with respect to \( \star \), \( \{W_{-n}\}_{n \in \mathbb{N}} \) mutually commute, \( \{W_{n+1}\}_{n \in \mathbb{N}} \) mutually commute, \( \{G_n\}_{n \in \mathbb{N}} \) mutually commute, and \( \{\tilde{G}_n\}_{n \in \mathbb{N}} \) mutually commute. Furthermore, by [25, Theorem 10.1] the alternating words \( \{W_{-n}\}_{n \in \mathbb{N}}, \{W_{n+1}\}_{n \in \mathbb{N}}, \{\tilde{G}_n\}_{n \in \mathbb{N}} \) form a PBW basis for \( U \).

In this paper, we focus on the alternating words \( \{\tilde{G}_n\}_{n \in \mathbb{N}} \). Consider their generating function
\[
\tilde{G}(t) = \sum_{n \in \mathbb{N}} \tilde{G}_n t^n.
\]

We will be discussing the multiplicative inverse of \( \tilde{G}(t) \) with respect to \( \star \). We now introduce this inverse.

**Definition 1.3.** (See [25, Definition 9.5].) We define the elements \( \{D_n\}_{n \in \mathbb{N}} \) of \( U \) in the following recursive way:
\[
D_0 = 1, \quad D_n = -\sum_{k=0}^{n-1} D_k \star \tilde{G}_{n-k} \quad (n \geq 1).
\]

Define the generating function
\[
D(t) = \sum_{n \in \mathbb{N}} D_n t^n.
\]

**Lemma 1.4.** (See [22, Lemma 4.1].) The generating function \( D(t) \) is the multiplicative inverse of \( \tilde{G}(t) \) with respect to \( \star \). In other words,
\[
\tilde{G}(t) \star D(t) = 1 = D(t) \star \tilde{G}(t).
\]
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Proof. The relation (2) can be checked routinely using (1).

For \( n \in \mathbb{N} \) we can calculate \( D_n \) recursively using (1).

**Example 1.5.** We list \( D_n \) for \( 0 \leq n \leq 3 \).

\[
\begin{align*}
D_0 &= 1, \\
D_1 &= -xy, \\
D_2 &= xyxy + [2]_q^2 xyyx, \\
D_3 &= -xyxyxy - [2]_q^4 xxyxxyy - [2]_q^4 xyxyxyy - [2]_q^4 xxyyxyy - [2]_q^4 xxyxyyy.
\end{align*}
\]

We are going to obtain a closed formula for \( D_n \).

To motivate the formula, let us examine Example 1.5. We can see that each \( D_n \) is a linear combination of words of length \( 2n \), and each coefficient is equal to \((-1)^n\) times a square. Furthermore, the words appearing in the linear combination have a certain type said to be Catalan. We now recall the definition of a Catalan word.

**Definition 1.6.** (See [26, Definition 1.3].) Define \( x = 1 \) and \( y = -1 \). A word \( a_1 \cdots a_k \) is Catalan whenever \( a_1 + \cdots + a_i \geq 0 \) for \( 1 \leq i \leq k-1 \) and \( a_1 + \cdots + a_k = 0 \). The length of a Catalan word is always even. For \( n \in \mathbb{N} \), let \( \text{Cat}_n \) denote the set of all Catalan words of length \( 2n \).

**Example 1.7.** We describe \( \text{Cat}_n \) for \( 0 \leq n \leq 3 \).

\[
\begin{align*}
\text{Cat}_0 &= \{1\}, \\
\text{Cat}_1 &= \{xy\}, \\
\text{Cat}_2 &= \{xyxy, xxyy\}, \\
\text{Cat}_3 &= \{xyxyxy, xxyyxy, xyxxyy, xxyxyy, xxxyyy\}.
\end{align*}
\]

We observe that for \( 0 \leq n \leq 3 \) each \( D_n \) is a linear combination of Catalan words of length \( 2n \). We now show that this observation is true for all \( n \in \mathbb{N} \).

**Proposition 1.8.** For \( n \in \mathbb{N} \), \( D_n \) is contained in the span of \( \text{Cat}_n \).

**Proof.** For \( n \in \mathbb{N} \), by Definition 1.1 we have that \( \tilde{G}_n = xyxy \cdots xy \) where the \( xy \) is repeated \( n \) times. The word \( \tilde{G}_n \) is Catalan by Definition 1.6. Note that the \( q \)-shuffle product of two Catalan words is a linear combination of Catalan words. The result follows by (1) and induction on \( n \).

**Definition 1.9.** For \( n \in \mathbb{N} \) and a word \( w \in \text{Cat}_n \), let \((-1)^n D(w)\) denote the coefficient of \( w \) in \( D_n \). In other words,

\[
D_n = (-1)^n \sum_{w \in \text{Cat}_n} D(w)w.
\] (3)

**Example 1.10.** In the table below, we list the Catalan words \( w \) of length \( \leq 6 \) and the corresponding \( D(w) \).

| \( w \) | \( D(w) \) |
|---|---|
| \( 1 \) | 1 |
| \( xy \) | 1 |
| \( xxy \) | [2]_q^2 |
| \( xyyy \) | 1 |
| \( xxyxy \) | [2]_q^2 |
| \( xyyyy \) | [2]_q^4 |
| \( xxyxyy \) | [2]_q^4 |
| \( xxxyyy \) | [2]_q^4 [3]_q^2 |

By (3), our goal of finding a closed formula for \( D_n \) reduces to finding a closed formula for \( D(w) \) where \( w \) is Catalan. The following is the main theorem of this paper.

**Theorem 1.11.** For \( n \in \mathbb{N} \) and a word \( w = a_1 \cdots a_{2n} \in \text{Cat}_n \), we have

\[
D(w) = \prod_{i=1}^{2n} [\bar{a}_1 + \cdots + \bar{a}_{i-1} + (\bar{a}_i + 1)/2]_q.
\] (4)
Moreover,

\[ D(w) = E(w)^2, \]

where

\[ E(w) = \prod_{1 \leq i \leq 2n \atop \bar{a}_i = x} [\bar{a}_1 + \cdots + \bar{a}_i]_q = \prod_{1 \leq i \leq 2n \atop \bar{a}_i = y} [\bar{a}_1 + \cdots + \bar{a}_i - 1]_q. \]  

**Remark 1.12.** There is a striking resemblance between (4) and [26, Definition 2.5]. While not explicitly used in our proofs, this resemblance did motivate our proof techniques and our interest in this entire topic.

## 2. The proof of Theorem 1.11

In this section, we prove Theorem 1.11.

**Definition 2.1.** For \( n \in \mathbb{N} \) and a word \( w = a_1 \cdots a_{2n} \in \text{Cat}_n \), we define

\[ D(w) = \prod_{i=1}^{2n} [\bar{a}_1 + \cdots + \bar{a}_i + (\bar{a}_i + 1)/2]_q, \]

\[ D_x(w) = \prod_{1 \leq i \leq 2n \atop \bar{a}_i = x} [\bar{a}_1 + \cdots + \bar{a}_i]_q, \]

\[ D_y(w) = \prod_{1 \leq i \leq 2n \atop \bar{a}_i = y} [\bar{a}_1 + \cdots + \bar{a}_i - 1]_q. \]

In order to prove Theorem 1.11, we establish the following for all Catalan words \( w \):

(i) \( D(w) = D(w) \);

(ii) \( D(w) = D_x(w)D_y(w) \);

(iii) \( D_x(w) = D_y(w) \).

Item (i) will be achieved in Theorem 2.23.

Item (ii) will be achieved in Lemma 2.2.

Item (iii) will be achieved in Lemma 2.8.

**Lemma 2.2.** For any Catalan word \( w \), we have

\[ D(w) = D_x(w)D_y(w). \]

**Proof.** Note that \((x + 1)/2 = 1\) and \((y + 1)/2 = 0\), so the result follows by Definition 2.1. \( \square \)

Next we will show item (iii). In order to do this, we now recall the concept of elevation sequences and profiles.

**Definition 2.3.** (See [26, Definition 2.6].) For \( n \in \mathbb{N} \) and a word \( w = a_1 \cdots a_n \), its elevation sequence is \((e_0, \ldots, e_n)\), where \( e_i = \bar{a}_0 + \cdots + \bar{a}_i \) for \( 0 \leq i \leq n \).

**Example 2.4.** In the table below, we list the Catalan words \( w \) of length \( \leq 6 \) and the corresponding elevation sequences.
\[ \begin{array}{|c|c|}
\hline
w & \text{elevation sequence of } w \\ \hline
1 & (0) \\ xy & (0, 1, 0) \\ xxy & (0, 1, 0, 1, 0) \\ xyy & (0, 1, 2, 1, 0) \\ xxyy & (0, 1, 0, 1, 0, 1, 0) \\ xxyxy & (0, 1, 2, 1, 0, 1, 0) \\ xxyyy & (0, 1, 2, 1, 0, 1, 0) \\ xxyxyy & (0, 1, 2, 1, 2, 1, 0) \\ xxyxyy & (0, 1, 2, 1, 2, 1, 0) \\ \hline
\end{array} \]

**Definition 2.5.** (See [26, Definition 2.8].) For \( n \in \mathbb{N} \) and a word \( w = a_1 \cdots a_n \), its **profile** is the subsequence of its elevation sequence consisting of the \( e_i \) that satisfy one of the following conditions:

- \( i = 0 \);
- \( i = n \);
- \( 1 \leq i \leq n - 1 \) and \( e_{i+1} - e_i \neq e_i - e_{i-1} \).

In other words, the profile of a word \( w \) is the subsequence of the elevation sequence of \( w \) consisting of the end points and turning points.

By a **Catalan profile** we mean the profile of a Catalan word.

**Example 2.6.** In the table below, we list the Catalan words \( w \) of length \( \leq 6 \) and the corresponding profiles.

\[ \begin{array}{|c|c|}
\hline
w & \text{profile of } w \\ \hline
1 & (0) \\ xy & (0, 1, 0) \\ xxy & (0, 1, 0, 1, 0) \\ xyy & (0, 1, 2, 0) \\ xxyy & (0, 1, 0, 1, 0, 1, 0) \\ xxyxy & (0, 2, 0, 1, 0) \\ xxyxy & (0, 1, 0, 2, 0) \\ xxyxy & (0, 2, 2, 1, 0) \\ xxyxy & (0, 3, 0) \\ \hline
\end{array} \]

**Lemma 2.7.** For a Catalan word \( w \) with profile \((l_0, h_1, l_1, \ldots, h_r, l_r)\), we have

\[
D_x(w) = \frac{[h_1]_q \cdots [h_r]_q}{[l_0]_q \cdots [l_r]_q},
\]

\[
D_y(w) = \frac{[h_1]_q \cdots [h_r]_q}{[l_0]_q \cdots [l_r]_q}.
\]

**Proof.** Follows from [26, Lemma 2.10] by direct computation.

**Lemma 2.8.** For any Catalan word \( w \), we have

\[ D_x(w) = D_y(w). \]

**Proof.** Follows from Lemma 2.7.

\[ \square \]
Lemma 2.9. For \( n \in \mathbb{N} \) and a word \( w \in \text{Cat}_n \) with profile \((l_0, h_1, l_1, \ldots, h_r, l_r)\), we have

\[
D(w) = D_x(w)^2 = D_y(w)^2 = \left( \frac{[h_1]_q^2 \cdots [h_r]_q^2}{[l_0]_q^2 \cdots [l_r]_q^2} \right)^2.
\]

Proof. Follows from Lemmas 2.2, 2.7, 2.8.

Motivated by Lemma 2.9, we make the following definition.

Definition 2.10. Given a Catalan profile \((l_0, h_1, l_1, \ldots, h_r, l_r)\), define

\[
D(l_0, h_1, l_1, \ldots, h_r, l_r) = \left( \frac{[h_1]_q^2 \cdots [h_r]_q^2}{[l_0]_q^2 \cdots [l_r]_q^2} \right)^2.
\]

Definition 2.11. For \( n \in \mathbb{N} \), we define

\[
D_n = (-1)^n \sum_{w \in \text{Cat}_n} D(w)w.
\]

We interpret \( D_0 = 1 \).

Next we will achieve a recurrence relation involving the \( D_n \). This will be accomplished in Proposition 2.17.

Lemma 2.12. For a Catalan profile \((l_0, h_1, l_1, \ldots, h_r, l_r)\) with \( r \geq 1 \),

\[
D(l_0, h_1, l_1, \ldots, h_r, l_r) = \sum_{j=\xi}^{r-1} D(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1} - 1, l_{j+1} - 1, \ldots, h_r - 1, l_r) \left( [h_{j+1}]_q^2 - [l_j]_q^2 \right),
\]

where \( \xi = \max\{j \mid 0 \leq j \leq r - 1, l_j = 0\} \).

Proof. To prove the above equation, consider the quotient of the right-hand side divided by the left-hand side. We will show that this quotient is equal to 1.

By Definition 2.10, the above quotient is equal to

\[
\sum_{j=\xi}^{r-1} \frac{[l_{j+1}]_q^2 \cdots [l_{r-1}]_q^2}{[h_{j+1}]_q^2 \cdots [h_r]_q^2} \left( [h_{j+1}]_q^2 - [l_j]_q^2 \right)
= \frac{1}{[h_{\xi+1}]_q^2 \cdots [h_r]_q^2} \sum_{j=\xi}^{r-1} [h_{j+1}]_q^2 [l_{j+1}]_q^2 \cdots [l_{r-1}]_q^2 \left( [h_{j+1}]_q^2 - [l_j]_q^2 \right)
= \frac{1}{[h_{\xi+1}]_q^2 \cdots [h_r]_q^2} \sum_{j=\xi}^{r-1} \left( [h_{j+1}]_q^2 \cdots [h_{\xi+1}]_q^2 [l_{j+1}]_q^2 \cdots [l_{r-1}]_q^2 - [h_{\xi+1}]_q^2 \cdots [h_r]_q^2 [l_j]_q^2 \cdots [l_{r-1}]_q^2 \right)
= \frac{1}{[h_{\xi+1}]_q^2 \cdots [h_r]_q^2} \left( [h_{\xi+1}]_q^2 \cdots [h_r]_q^2 [l_\xi]_q^2 \cdots [l_{r-1}]_q^2 \right)
= 1,
\]

where the last step follows from \( l_\xi = 0 \).

\[\square\]
Lemma 2.13. For any Catalan word \( w = a_1 \cdots a_m \), we have
\[
\frac{qx \ast w - q^{-1}w \ast x}{q - q^{-1}} = \sum_{i=0}^{m} a_1 \cdots a_i xa_{i+1} \cdots a_m [1 + 2a_1 + \cdots + 2a_i]_q.
\]

Proof. By the definition of the \( q \)-shuffle product, we have
\[
\frac{qx \ast w - q^{-1}w \ast x}{q - q^{-1}} = \sum_{i=0}^{m} a_1 \cdots a_i xa_{i+1} \cdots a_m q^1 + 2a_1 + \cdots + 2a_i + q^{-1+2a_{i+1} + \cdots + 2a_m} - q^{-1}.
\]
Observing that \((,,)\) is non-degenerate and symmetric. For any word \( w \) in \( V \) and any \( u \in V \), the scalar \((w,u)\) is the coefficient of \( w \) in \( u \).

For notation convenience, we bring in a bilinear form on \( V \).

Definition 2.14. (See [26, p. 6].) Let \((,,) : V \times V \to \mathbb{F}\) denote the bilinear form such that \((w,w) = 1\) for any word \( w \) in \( V \) and \((w,v) = 0\) for any distinct words \( w,v \) in \( V \).

Lemma 2.15. For any word \( v \) and any Catalan word \( w = a_1 \cdots a_m \), consider the scalar
\[
\left( \frac{(qx \ast w - q^{-1}w \ast x)y}{q - q^{-1}}, v \right) \tag{7}
\]
(i) If \( v \) is Catalan and of length \( m + 2 \), then the scalar (7) is equal to
\[
\sum_{i} [1 + 2a_1 + \cdots + 2a_i]_q,
\]
where the sum is over all \( i \) \( (1 \leq i \leq m) \) such that \( v = a_1 \cdots a_i xa_{i+1} \cdots a_m y \).
(ii) If \( v \) is not Catalan or is not of length \( m + 2 \), then the scalar (7) is equal to 0.

Proof. Follows from Lemma 2.13.

Lemma 2.16. For \( n \geq 1 \) and a word \( v \in \text{Cat}_n \), we have
\[
\mathcal{D}(v) = \sum_{w \in \text{Cat}_{n-1}} \mathcal{D}(w) \left( \frac{(qx \ast w - q^{-1}w \ast x)y}{q - q^{-1}}, v \right).
\]

Proof. By Lemma 2.15, it suffices to show that \( \mathcal{D}(v) \) is equal to
\[
\sum_{w,i} \mathcal{D}(w)[1 + 2a_1 + \cdots + 2a_i]_q, \tag{8}
\]
where the sum is over all ordered pairs \( (w,i) \) such that \( w = a_1 \cdots a_{2n-2} \in \text{Cat}_{n-1} \) and \( v = a_1 \cdots a_i xa_{i+1} \cdots a_{2n-2} y \).
Let \((l_0, h_1, l_1, \ldots, h_r, l_r)\) denote the profile of \(v\) and let \(\xi = \max\{j \mid 0 \leq j \leq r - 1, l_j = 0\}\).

To compute the sum (8), we study what kind of words \(w\) are being summed over and what is the coefficient for each corresponding \(D(w)\).

For any \(w\) being summed over in (8), its profile must be of the form
\[
(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1} - 1, l_{j+1} - 1, \ldots, l_{r-1} - 1, h_r - 1, l_r)
\]
for some \(j\) such that \(\xi \leq j \leq r - 1\). (If \(j < \xi\), then the profile of \(w\) contains \(l_\xi - 1 = -1\), which means \(w\) is not Catalan.)

For such \(w\), the coefficient of \(D(w)\) in (8) is
\[
\sum_{s=l_j}^{h_{j+1}-1} [1 + 2s]q,
\]
which is equal to
\[
[h_{j+1}]_q^2 - [l_j]_q^2
\]
by direct computation.

Therefore, by Lemma 2.12 we have
\[
\sum_{w,i} D(w)[1 + 2\bar{a}_1 + \cdots + 2\bar{a}_r]q = D(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1} - 1, l_{j+1} - 1, \ldots, l_{r-1} - 1, h_r - 1, l_r) \left( [h_{j+1}]_q^2 - [l_j]_q^2 \right)
\]
\[
= D(l_0, h_1, l_1, \ldots, h_r, l_r)
\]
\[
= D(v).
\]

\(\square\)

**Proposition 2.17.** For \(n \geq 1\),
\[
D_n = \frac{(q^{-1}D_{n-1} \star x - qx \star D_{n-1})y}{q - q^{-1}}.
\]

**Proof.** Given any word \(v\), we will show that its inner product with the right-hand side of (9) coincides with \((D_n, v)\).

If \(v\) does not have length \(2n\), then the two inner products are both 0.

If \(v\) is not Catalan, then \((D_n, v) = 0\) by Definition 2.11, and
\[
\left( \frac{(q^{-1}D_{n-1} \star x - qx \star D_{n-1})y}{q - q^{-1}}, v \right) = 0
\]
by Definition 2.11 and Lemma 2.15.

If \(v \in \text{Cat}_n\), then by Definition 2.11 and Lemma 2.16,
\[
\left( \frac{(q^{-1}D_{n-1} \star x - qx \star D_{n-1})y}{q - q^{-1}}, v \right) = (-1)^n \sum_{w \in \text{Cat}_{n-1}} D(w) \left( \frac{(qx \star w - q^{-1}w \star x)y}{q - q^{-1}}, v \right)
\]

Therefore, by Lemma 2.12 we have
\[
\sum_{w,i} D(w)[1 + 2\bar{a}_1 + \cdots + 2\bar{a}_r]q = D(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1} - 1, l_{j+1} - 1, \ldots, l_{r-1} - 1, h_r - 1, l_r) \left( [h_{j+1}]_q^2 - [l_j]_q^2 \right)
\]
\[
= D(l_0, h_1, l_1, \ldots, h_r, l_r)
\]
\[
= D(v).
\]

\(\square\)
\[ = (-1)^n D(v) \]
\[ = (D_n, v). \]

**Definition 2.18.** (See [25, Definition 9.11].) We define a generating function
\[ D(t) = \sum_{n \in \mathbb{N}} D_n t^n, \]
where \( D_n \) is from Definition 2.11.

Next we will show that \( D(t) = D(t) \). To do this, we will show that \( D(t) \) is the multiplicative inverse of \( \tilde{G}(t) \) with respect to \( \star \). This will be accomplished in Proposition 2.22.

**Lemma 2.19.** For \( k \in \mathbb{N} \), we have
\[ q \tilde{G}_k \star x = (q - q^{-1}) W_{-k} + q^{-1} x \star \tilde{G}_k. \]

**Proof.** Follows from the definition of \( \star \) by direct computation. \( \square \)

**Lemma 2.20.** For \( n \geq 1 \),
\[ D_n = - \sum_{k=1}^{n} \tilde{G}_k \star D_{n-k}. \quad (10) \]

**Proof.** We use induction on \( n \).

First assume that \( n = 1 \). Then (10) holds because
\[ D_0 = 1, \quad D_1 = -xy, \quad \tilde{G}_1 = xy. \]

Next assume that \( n \geq 2 \). By induction,
\[ D_{n-1} = - \sum_{k=1}^{n-1} \tilde{G}_k \star D_{n-1-k}. \quad (11) \]

In order to prove (10), it suffices to show
\[ \sum_{k=1}^{n-1} \tilde{G}_k \star D_{n-k} = -D_n - \tilde{G}_n. \quad (12) \]

For \( 1 \leq k \leq n - 1 \) we examine the \( k \)-summand in (12). We use the following notation: for a word \( w \) ending with the letter \( y \), the word \( wy^{-1} \) is obtained from \( w \) by removing the rightmost \( y \). Furthermore, for a linear combination \( A \) of words ending in \( y \), the element \( Ay^{-1} \) is obtained from \( A \) by removing the rightmost \( y \) of each word in the linear combination.

Note that \( \tilde{G}_k \) is a word ending in \( y \), and \( D_{n-k} \) is a linear combination of Catalan words which end in \( y \) by Definition 1.6, so
\[ \tilde{G}_k \star D_{n-k} = (\tilde{G}_k y^{-1} \star D_{n-k}) y + (\tilde{G}_k \star D_{n-k} y^{-1}) y. \quad (13) \]
We focus on the second term of the right-hand side of (13). By Proposition 2.17 and Lemma 2.19, we have
\[
\tilde{G}_k \star D_{n-k} y^{-1} \\
= -\frac{1}{q - q^{-1}} \tilde{G}_k \star (qx \star D_{n-k-1} - q^{-1} D_{n-k-1} \star x) \\
= -\frac{q}{q - q^{-1}} \tilde{G}_k \star x \star D_{n-k-1} + \frac{q^{-1}}{q - q^{-1}} \tilde{G}_k \star D_{n-k-1} \star x \\
= -W_{-k} \star D_{n-k-1} - \frac{q^{-1}}{q - q^{-1}} x \star \tilde{G}_k \star D_{n-k-1} + \frac{q^{-1}}{q - q^{-1}} \tilde{G}_k \star D_{n-k-1} \star x.
\]

By the above comment, and since \( \tilde{G}_k y^{-1} = W_{-k+1} \), we can write (13) as
\[
\tilde{G}_k \star D_{n-k} \\
= (W_{-k+1} \star D_{n-k}) y - (W_{-k} \star D_{n-k-1}) y \\
- \frac{q^{-1}}{q - q^{-1}} (x \star \tilde{G}_k \star D_{n-k-1}) y + \frac{q^{-1}}{q - q^{-1}} (\tilde{G}_k \star D_{n-k-1} \star x) y.
\]

We now sum the above equation over \( k \) from 1 to \( n - 1 \), using (11) and Proposition 2.17. We have
\[
\sum_{k=1}^{n-1} \tilde{G}_k \star D_{n-k} \\
= (W_0 \star D_{n-1}) y - (W_{-n+1} \star D_0) y + \frac{q^{-1}}{q - q^{-1}} (x \star D_{n-1}) y - \frac{q^{-1}}{q - q^{-1}} (D_{n-1} \star x) y \\
= (x \star D_{n-1}) y - \tilde{G}_n + \frac{q^{-1}}{q - q^{-1}} (x \star D_{n-1}) y - \frac{q^{-1}}{q - q^{-1}} (D_{n-1} \star x) y \\
= \frac{q}{q - q^{-1}} (x \star D_{n-1}) y - \frac{q^{-1}}{q - q^{-1}} (D_{n-1} \star x) y - \tilde{G}_n \\
= -D_n - \tilde{G}_n.
\]

We have verified (12), and (10) follows. \( \square \)

Definition 2.21. (See [26, p. 5].) Let \( \zeta : \mathbb{V} \rightarrow \mathbb{V} \) denote the \( \mathbb{F} \)-linear map such that
- \( \zeta (x) = y \),
- \( \zeta (y) = x \),
- For any word \( a_1 \cdots a_m \),
\[
\zeta (a_1 \cdots a_m) = \zeta (a_m) \cdots \zeta (a_1).
\]

By the above definition, \( \zeta \) is an antiautomorphism on the free algebra \( \mathbb{V} \). One can routinely check using the definition of \( \star \) that \( \zeta \) is also an antiautomorphism on the \( q \)-shuffle algebra \( \mathbb{V} \). Moreover, \( \zeta \) fixes \( G_n \) and \( D_n \) for all \( n \in \mathbb{N} \).

Proposition 2.22. We have
\[
\tilde{G}(t) \star D(t) = 1 = D(t) \star \tilde{G}(t).
\]

Proof. We have \( \tilde{G}_0 = 1 \) and \( D_0 = 1 \). By Lemma 2.20, for any \( n \geq 1 \) we have
\[
\sum_{k=0}^{n} \tilde{G}_k \star D_{n-k} = 0.
\]
By these comments,
\[ \tilde{G}(t) \star D(t) = 1. \] (14)

Applying \( \zeta \) to (14), we have
\[ D(t) \star \tilde{G}(t) = 1. \]

**Theorem 2.23.** The following hold.

(i) \( D(t) = D(t) \).
(ii) \( D_n = D_n \) for any \( n \in \mathbb{N} \).
(iii) \( D(w) = D(w) \) for any Catalan word \( w \).

**Proof.** Comparing Lemma 1.4 and Proposition 2.22, we obtain item (i). Item (ii) follows from item (i) by Definitions 1.3 and 2.18. Item (iii) follows from item (ii) by Definitions 1.9 and 2.11.

This finishes our proof of Theorem 1.11.

### 3. Some facts about \( \{D_n\}_{n \in \mathbb{N}} \)

In this section, we state some facts about \( \{D_n\}_{n \in \mathbb{N}} \) that we find attractive.

**Proposition 3.1.** (See [25, Lemma 9.7].) For \( n \geq 1 \),
- \( D_n \) is a polynomial in \( \tilde{G}_1, \ldots, \tilde{G}_n \) of degree \( n \), where each \( \tilde{G}_i \) is given the degree \( i \),
- \( \tilde{G}_n \) is a polynomial in \( D_1, \ldots, D_n \) of degree \( n \), where each \( D_i \) is given the degree \( i \).

**Proposition 3.2.** (See [25, Lemma 9.10].) For \( n, m \in \mathbb{N} \),
\[ D_n \star \tilde{G}_m = \tilde{G}_m \star D_n, \quad D_n \star D_m = D_m \star D_n. \]

**Proposition 3.3.** For \( n \geq 1 \),
\[ D_n = \frac{(q^{-1}D_{n-1} \star x - qx \star D_{n-1})y}{q - q^{-1}}. \] (15)

**Proof.** Follows from Proposition 2.17 and Theorem 2.23.

**Proposition 3.4.** For \( n \geq 1 \),
\[ D_n = \frac{x(q^{-1}y \star D_{n-1} - qD_{n-1} \star y)}{q - q^{-1}}. \]

**Proof.** Apply the antiautomorphism \( \zeta \) to each side of (15), and note that \( D_n \) is invariant under \( \zeta \).

Recall that for a linear combination \( A \) of words ending in \( y \), the element \( Ay^{-1} \) is obtained from \( A \) by removing the rightmost \( y \) of each word. We make a similar notation that for a linear combination \( B \) of words starting with \( x \), the element \( x^{-1}B \) is obtained from \( B \) by removing the leftmost \( x \) of each word.

**Proposition 3.5.** For \( n \geq 2 \),
\[ x^{-1}D_n y^{-1} + D_{n-1} = \frac{q^{-1}x^{-1}D_{n-1} \star x - q^2x \star x^{-1}D_{n-1}}{q - q^{-1}}. \] (16)
**Proof.** By the definition of the $q$-shuffle product, we have
\[ x \star D_{n-1} = xD_{n-1} + q^2 x(x \star x^{-1} D_{n-1}), \]
\[ D_{n-1} \star x = xD_{n-1} + x(x^{-1} D_{n-1} \star x). \]

The result follows from Proposition 3.3 and the two equations above. \( \square \)

**Proposition 3.6.** For $n \geq 2$,
\[ x^{-1} D_n y^{-1} + D_{n-1} = \frac{q^{-1} y \star D_{n-1} y^{-1} - q^3 D_{n-1} y^{-1} \star y}{q - q^{-1}}. \]

**Proof.** Apply the antiautomorphism $\zeta$ to each side of (16), and note that $D_n$ is invariant under $\zeta$. \( \square \)

Next we mention some PBW bases for $U_q^+$ that involve $\{D_{n+1}\}_{n \in \mathbb{N}}$. The readers may refer to [25, Definition 2.1] for a formal definition of a PBW basis.

**Proposition 3.7.** The elements $\{W_{-n}\}_{n \in \mathbb{N}},\{D_{n+1}\}_{n \in \mathbb{N}},\{W_{n+1}\}_{n \in \mathbb{N}}$ form a PBW basis for $U_q^+$ in any linear order that satisfies one of the following:

(i) $W_{-i} < D_{j+1} < W_{k+1}$ for $i, j, k \in \mathbb{N}$;
(ii) $W_{k+1} < D_{j+1} < W_{-i}$ for $i, j, k \in \mathbb{N}$;
(iii) $W_{k+1} < W_{-i} < D_{j+1}$ for $i, j, k \in \mathbb{N}$;
(iv) $W_{-i} < W_{k+1} < D_{j+1}$ for $i, j, k \in \mathbb{N}$;
(v) $D_{j+1} < W_{k+1} < W_{-i}$ for $i, j, k \in \mathbb{N}$;
(vi) $D_{j+1} < W_{-i} < W_{k+1}$ for $i, j, k \in \mathbb{N}$.

**Proof.** Follows from [25, Theorem 10.1] and Propositions 3.1, 3.2. \( \square \)

**Proposition 3.8.** The elements $\{E_{n\delta+a_0}\}_{n \in \mathbb{N}},\{D_{n+1}\}_{n \in \mathbb{N}},\{E_{n\delta+a_1}\}_{n \in \mathbb{N}}$ form a PBW basis for $U_q^+$ in the following linear order:
\[ E_{a_0} < E_{\delta+a_0} < E_{2\delta+a_0} < \cdots < D_1 < D_2 < D_3 < \cdots < E_{2\delta+a_1} < E_{\delta+a_1} < E_{a_1}. \]

**Proof.** Follows from [7, Section 5], [26, Theorem 1.7], [25, Proposition 11.9], and Proposition 3.2. \( \square \)

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