Exotic Quantum Double, Its Universal R-matrix
And Their Representations

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Abstract

The exotic quantum double and its universal R-matrix for quantum Yang-Baxter equation are constructed in terms of Drinfeld’s quantum double theory. As a new quasi-triangular Hopf algebra, it is much different from those standard quantum doubles that are the q-deformations for Lie algebras or Lie superalgebras. By studying its representation theory, many-parameter representations of the exotic quantum double are obtained with an explicit example associated with Lie algebra $A_2$. The multi-parameter R-matrices for the quantum Yang-Baxter equation can result from the universal R-matrix of this exotic quantum double and these representations.

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1. Introduction

In recent years, the quantum Yang-Baxter equation (QYBE) [1,2] has become a focus of the attention from both theoretical physicists and mathematicians. This is because the QYBE is a key to the complete integrability of many physical systems appearing in the quantum inverse scattering methods [3,4], the exactly-solvable models in statistical mechanics [5] and low-dimensional quantum field theory [6]. In solving the QYBE in a general way and classifying its solutions (R-matrices) algebraically, a remarkable mathematical structure—the quasi-triangular Hopf algebra (loosely called quantum group) are found in connection with the QYBE [7-10]. Among these developments, the Drinfeld’s quantum double [7] theory provides one with a general construction to systematically obtain solutions of the QYBE in terms of the quantum doubles (QDs), which usually are the ‘q-deformations’ of certain algebras, and their representations. The recent studies show that, not only the standard R-matrices [11-13], but also the non-standard ones [14-16], such as the R-matrices with non-additive spectral parameters [17-23], the colored R-matrices [24-28], can be obtained in the framework of Drinfeld’s QD theory, but for the latter the cyclic representations and other non-generic ones at roots of unity [30-39] and some paramterization of the quantum (universal enveloping) algebras [40-41] need to be considered. The purpose of the present paper is to search for the exotic quantum doubles, other than those ‘q-deformations’, so that the new universal R-matrix can be obtained for the QYBE based on Drinfeld’s quantum double theory.

To proceed our discussion conveniently, we need to outline some basic ideas in Drinfeld’s QD theory so that the notations used in this paper can be clarified. Suppose we are given two Hopf algebras A, B and a non-degenerate bilinear form < , >: $A \times B \rightarrow C$ (the complex field) satisfying the following conditions:

$$< a, b_1 b_2 > = < \Delta_A(a), b_1 \otimes b_2 >, a \in A, \quad b_1, b_2 \in B,$$

$$< a_1 a_2, b > = < a_2 \otimes a_1, \Delta_B(b) >, a_1, a_2 \in A, b \in A$$

$$< 1_A, b > = \epsilon_B(b), b \in B,$$

$$< a, 1_B > = \epsilon_A(a), a \in A$$

$$< S_A(a), S_B(b) > = < a, b >, a \in A, b \in B$$

where for $C = A, B, \Delta_C, \epsilon_C$ and $S_C$ are the coproduct, counit and antipode of C respectively; $1_C$ is the unit of C. Drinfeld’s QD theory (for a comprehensive reviews see the refs. [42,44]).
states the central results in the QD theory as follows:

**Theorem 1.** There exists a Hopf algebra $D$ satisfying the following conditions
1. $D$ contains $A$ and $B$ as Hopf subalgebras;
2. The mapping $A \times B \to D : a \otimes b \to ab$ is an isomorphism of vector space;
3. For any $a \in A, b \in B$, we have multiplication

\[ ba = \sum_{i,j} < a_i(1), S(b_j(1)) > < a_i(3), b_j(3) > a_i(2)b_j(2) \]  \hspace{1cm} (1.2)

where $c_i(k)(k = 1, 2, 3; c = a, b)$ are defined by

\[ \Delta^2(c) = (\text{id} \otimes \Delta)\Delta(c) = (\Delta \otimes \text{id})\Delta(c) = \sum_i c_i(1) \otimes c_i(2) \otimes c_i(3) \]

**Theorem 2** There exists an unique element

\[ \hat{R} = \sum_m a_m \otimes b_m \in A \times B \subset D \times D \]

obeying the “abstract” QYBE

\[ \hat{R}_{12}\hat{R}_{13}\hat{R}_{23} = \hat{R}_{23}\hat{R}_{13}\hat{R}_{12}, \]  \hspace{1cm} (1.3)

where $a_m$ and $b_m$ are the basis vectors of $A$ and $B$ respectively, and they are dual each other, i.e., $< a_m, b_n > = \delta_{m,n}$;

\[ \hat{R}_{12} = \sum_m a_m \otimes b_m \otimes 1, \hat{R}_{13} = \sum_m a_m \otimes 1 \otimes b_m, \hat{R}_{23} = \sum_m 1 \otimes a_m \otimes b_m \]

where $1$ is the unit of $D$.

Up to now, the QD’s built explicitly are only the quantum (universal enveloping) algebras and superalgebras and their parameterizations. They are the q-deformations of the universal algebras and possess a ‘standard’ quantum double structure that both the subalgebras $A$ and $B$ are non-commutative and non-cocommutative. This symmetric structure reflects the duality of $A$ and $B$. Notice that these standard quantum doubles approach the usual universal enveloping algebras (UEA) in the classical limit $q \to 1$. In this paper, we will construct so-called exotic quantum doubles (EQD) that are not those q-deformations and possess asymmetric dual structure that one of
the subalgebras A and B is commutative but non-cocommutative and another cocommutative but non-commutative. As new quasi-triangular Hopf algebras, these EQDs naturally enjoy the QYBE, but they have not the usual classical limit.

This paper is arranged as follows. In section 2, we take the sub-Borel subalgebra of the UEA of the classical Lie algebra as the Hopf subalgebra A with cocommutative coproduct in the QD construction and then built its quantum dual as a non-cocommutative but commutative Hopf subalgebra B. In section 3, we combine A and B to form the exotic quantum double and thereby obtain the new universal R-matrix for the QYBE. In section 4, we discuss an explicit example of EQDs, which is connected with the Lie algebra $A_2$ in details. In section 5, we study the representation theory of the EQD with the above example and construct a class of many-parameter representations to built the many-parameter R-matrices for the QYBE. Finally, in section 6, we give some remarks on the problems and the possible developments in the EQD.

2. Quantum Dual for Non-simple Lie Algebra

Let $\phi^+ : \{\alpha, \beta, \gamma, \ldots\}$ be the system of all positive roots with respect to a simple root system of a classical Lie algebra L. A Cartan elements $h \in H$(the Cartan subalgebra and all the positive root vectors $\{e_\alpha, | \alpha \in \phi^+\}$ generate an associative algebra A with the relations on the Cartan-Weyl basis

$$[h, e_\alpha] = \alpha(h)e_\alpha, \quad [e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}$$

where $\alpha \in H^*$ and the coefficients $N_{\alpha,\beta}$ enjoy the structure of the Lie algebra L. In fact, the algebra A is a subalgebra of the Borel subalgebra of the universal enveloping algebra (UEA) of the Lie algebra L. Defining the algebraic homomorphisms $\Delta : A \rightarrow A \otimes A, \epsilon : A \rightarrow C$ and the algebraic antihomomorphism $S : A \rightarrow A$ by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, S(x) = -x, \epsilon(x) = 0$$

where $x \in \{h, e_\alpha | \alpha \in \phi^+\}$, one gives the algebra a “trivial” (cocommutative) Hopf algebraic structure. It is a well-known fact in the theory of Hopf algebra since we can regarded the algebra A as an UEA of the non-simple Lie algebra with basis
\{h, e_{\alpha}, \alpha \in \phi^{+}\}. However, the Hopf algebraic dual (quantum dual) B of A is non-trivial (non-cocommutative) due to the duality of B to A. Now, we derive the structure of A in terms of this duality.

Because A is cocommutative, its dual is an Abelian algebra with commuting generators. So the associative algebraic structure is quite simple. To consider the Hopf algebraic structure, we set an order for the basis of A: If \( \alpha - \beta \) is a non-zero positive root, then we say \( \alpha \succ \beta \); the basis for A is written down to enjoy this order as:

\[
\{a(m, m_{\alpha}) = h^{m} \prod_{\alpha \in \phi^{+}} e_{\alpha}^{m_{\alpha}} = h^{m} \cdots e_{\beta}^{m_{\beta}} \cdots e_{\gamma}^{m_{\gamma}} \cdots e_{\delta}^{m_{\delta}} \cdots | \cdots \delta \succ \cdots \gamma \succ \cdots \beta, m_{\alpha} \in \mathbb{Z}^{+} = \{0, 1, 2, \ldots\}\}
\]

Suppose that the dual Hopf algebra B to A is generated by the dual generators \( t, f_{\alpha} \) to \( h, e_{\alpha} \) respectively. They are defined by the following pairs in terms of a bilinear form \( \langle , \rangle \):

\[
\langle h \cdot t \rangle = 1, \langle x \cdot t \rangle = 0, \langle e_{\alpha} \cdot f_{\alpha} \rangle = 1, \langle y \cdot f_{\alpha} \rangle = 0; \quad (2.3)
\]

where \( x \) and \( y \) are the basis elements of A other than \( h, e_{\alpha} \) respectively.

**Proposition 1.** For \( m_{\alpha}, n_{\alpha}, m \in \mathbb{Z}^{+}(\alpha \in \phi^{+}) \),

\[
\langle h^{m} \prod_{\alpha \in \phi^{+}} e_{\alpha}^{m_{\alpha}} , t^{l} \prod_{\alpha \in \phi^{+}} f_{\alpha}^{n_{\alpha}} \rangle = \delta_{m,l} m! \prod_{\alpha \in \phi^{+}} m_{\alpha}! \delta_{m_{\alpha},n_{\alpha}}; \quad (2.4)
\]

namely, the vectors

\[
b(m, m_{\alpha}) = \frac{t^{m}}{m!} \prod_{\alpha \in \phi^{+}} \frac{f_{\alpha}^{m_{\alpha}}}{m_{\alpha}!},
\]

form a dual basis for B to \( a(m, m_{\alpha}) \) respectively:

\[
\langle a(m, m_{\alpha}), b(n, n_{\alpha}) \rangle = \delta_{m,n} \prod_{\alpha \in \phi^{+}} \delta_{m_{\alpha},n_{\alpha}}
\]

**Proof.** Thanks the duality between A and B, we have

\[
\langle h^{l}, t^{m} \rangle = \langle \Delta(h^{l}), t^{m-1} \otimes t \rangle = \sum_{k=0}^{l} \frac{l!}{k!(l-k)!} \langle h^{l-k} \otimes h^{k}, t^{m-1} \otimes t \rangle = \cdots = l \delta_{l,m} \delta_{l,m}
\]

Similarly, for \( G = e_{\alpha}, F = f_{\alpha} \) respectively,

\[
\langle G^{m}, F^{n} \rangle = m! \delta_{m,n}
\]
Then,

\[ < h^m G^n, F^s > = \sum_{k=0}^{m} \sum_{r=0}^{n} \frac{m!n!}{(m-k)!k!(n-r)!r!} < h^{m-k} G^n \otimes h^k G^r, F \otimes F^{s-1} > \]

\[ = \sum_{k=0}^{m} \sum_{r=0}^{n} \frac{m!n!}{(m-k)!k!(n-r)!r!} \delta_{n-r,1} \delta_{m-k,0} < h^k G^r, F^{s-1} > \]

\[ = n < h^n G^{m-1}. F^{s-1} >= n! \delta_{n,s} \delta_{m,0}; \]

\[ < h^m G^n, t^r F^s > = \sum_{k=0}^{m} \sum_{r=0}^{n} \frac{m!n!}{(m-k)!k!(n-r)!r!} < h^{m-k} G^n \otimes h^k G^r, t^r \otimes F^{s} > \]

\[ = < h^m, t^l > < G^n, F^n > . \]

It follows from the above calculations that

\[ < a(m, m^\alpha), b(n, n^\alpha) >= < h^m, t^n > \prod_{\alpha \in \phi^+} < e_{\alpha}, f_{\alpha} >= m! \delta_{m,n} \prod_{\alpha \in \phi^+} m_{\alpha}! \delta_{m_{\alpha}, n_{\alpha}} \]

In this position, we can deduce the Hopf algrbraic structure, ie, \( \Delta = \Delta_B, \epsilon = \epsilon_B, S = S_B \) of the algebra B. Let us first consider \( \Delta(f_\gamma) \). Notice that the reduced linear form \( < , \Delta(f_\gamma) > \) is non-zero only on \( e_\gamma \otimes 1, h^n \otimes e_\gamma, h^n e_\alpha \otimes e_\beta (\beta > \alpha) \):

\[ < e_\gamma \otimes 1, \Delta(f_\gamma) > = < 1.e_\gamma, f_\gamma >= 1; \]

\[ < h^n \otimes e_\gamma, \Delta(f_\gamma) > = < e_\gamma h^n, f_\gamma >= < (h - \gamma(h))^n e_\gamma, f_\gamma >= (-\gamma(h))^n; \]

\[ < h^n e_\alpha \otimes e_\beta, f_\gamma >= < (h - \beta(h))^n e_\beta e_\alpha, f_\gamma > = (-\beta(h))^n < e_\alpha e_\beta - N_{\alpha,\beta} e_{\alpha,\beta}, f_\gamma >= -(\beta(h))^n N_{\alpha,\beta} \delta(\alpha + \beta, \delta). \]

Consequently,

\[ \Delta(f_\gamma) = f_\gamma \otimes 1 + \sum_{n=0}^{\infty} \frac{(-\gamma(h))^n t^n}{n!} \otimes f_\gamma \]

\[ - \sum_{\alpha, \beta \in \phi^+} N_{\alpha,\beta} \delta(\alpha + \beta, \gamma) \theta(\beta - \alpha) \sum_{n=0}^{\infty} \frac{(-\beta(h))^n t^n}{n!} f_\alpha \otimes f_\beta \]

where

\[ \delta(\alpha, \beta) = \begin{cases} 1, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha \neq \beta \end{cases} \]

\[ \theta(\alpha) = \begin{cases} 1, & \text{if } \alpha(\neq 0) \in \phi^+; \\ 0, & \text{if } \alpha \not\in \phi^+ \end{cases} \]
For $S(f_\gamma)$ and $\beta > \alpha$, the only non-zero pairs are

$$< h^n e_\alpha e_\beta, S(f_\gamma) > = < (-1)^n S(e_\beta e_\alpha h^n), S(f_\gamma) > = (-1)^n < e_\beta e_\alpha h^n, f_\gamma >$$

$$= (-1)^n < (h - \beta(h) - \alpha(h))^{n}(e_\alpha e_\beta - N_{\alpha, \beta} e_{\alpha + \beta}), f_\gamma >= -(\beta(h) + \alpha(h))^n N_{\alpha, \beta} \delta_{\alpha + \beta, \gamma};$$

$$< h^n e_\gamma, S(f_\gamma) >= -(-1)^n < S(e_\gamma h^n), S(f_\gamma) >= -(-1)^n < e_\gamma h^n, f_\gamma >$$

$$= -(-1)^n < (h - \gamma(h))^{n} e_\gamma, f_\gamma >= -(\gamma(h))^n$$

Consequently,

$$S(f_\gamma) = -\sum_{n=0}^{\infty} \frac{\gamma(h)^n t^n}{n!} f_\gamma$$

$$+ \sum_{\alpha, \beta \in \phi^+} N_{\alpha, \beta} \theta(\beta - \alpha) \delta(\alpha + \beta, \gamma) \sum_{n=0}^{\infty} \frac{(\beta(h) + \alpha(h))^n t^n}{n!} f_\alpha f_\beta$$

In the same way we derive $\Delta(h), S(h), \epsilon(f_\alpha)$ and so on. The results are listed as follows.

**Proposition 2.** The duality between $A$ and $B$ results in the the commutative associative algebraic structure and the non-cocomutative Hopf algebraic structure defined by

$$\Delta(t) = t \otimes 1 + 1 \otimes t$$

$$\Delta(f_\gamma) = f_\gamma \otimes 1 + e^{-\gamma(h)t} \otimes f_\gamma - \sum_{\alpha, \beta \in \phi^+} C(\alpha, \beta, \gamma) e^{-\beta(h)t} f_\alpha \otimes f_\beta,$$

$$S(f_\gamma) = -e^{\gamma(h)} (f_\gamma + \sum_{\alpha, \beta \in \phi^+} C(\alpha, \beta, \gamma) f_\alpha f_\beta), \quad (2.6)$$

$$S(h) = -e^{-\gamma(h)t} h, S(1) = 1; \epsilon(f_\gamma) = \epsilon(h) = 0, \epsilon(1) = 1.$$

where

$$C(\alpha, \beta, \gamma) = N_{\alpha, \beta} \theta(\beta - \alpha) \delta(\alpha + \beta, \gamma).$$

3. The Quantum Double and Its Universal R-Matrix

In this section we show how the algebra $A$ and its quantum dual $B$ can be combined to form a quasi-triangular Hopf algebra with the exotic structure. To define the multiplications between $A$ and $B$, we need to use the following formula

$$\Delta^2(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, x = h, t, e_\alpha, \alpha \in \phi^+$$

$$\Delta^2(f_\gamma) = f_\gamma \otimes 1 \otimes 1 + e^{-\gamma(h)t} \otimes f_\gamma \otimes 1 + e^{-\gamma(h)t} \otimes e^{-\gamma(h)t} \otimes f_\gamma$$
\[
\sum_{\alpha, \beta \in \phi^+} C(\alpha, \beta, \gamma)e^{-\beta(h)t} \otimes e^{-\beta(h)t} f_\alpha \otimes f_\beta - \sum_{\alpha, \beta \in \phi^+} C(\alpha, \beta, \gamma)e^{-\beta(h)t} f_\alpha \otimes f_\beta \otimes 1 \\
- \sum_{\alpha, \beta \in \phi^+} C(\alpha, \beta, \gamma)e^{-\beta(h)t} f_\alpha \otimes e^{-\beta(h)t} \otimes f_\beta - \\
\sum_{\alpha, \beta, \gamma \in \phi^+} C(\alpha, \beta, \gamma)C(\sigma, \delta, \beta)e^{-\delta(h)t} f_\alpha \otimes e^{-\delta(h)t} f_\sigma \otimes f_\delta,
\] (3.1)

Using the above equations and the definition (1.2), we calculate the commutators \([e_\alpha, f_\gamma], [h, f_\gamma], [t, e_\alpha]\) and so on:

\[
f_\gamma e_\gamma = e_\gamma, S(f_\gamma) > < 1, 1 > 1+ \\
< 1, S(e^{-\gamma(h)t}) > < 1, 1 > e_\gamma f_\gamma + < 1, 1 > S(e^{-\gamma(h)t}) > < e_\gamma f_\gamma, 1 > e^{-\gamma(h)t} \\
= -1 + e_\gamma f_\gamma + e^{-\gamma(h)t};
\]

\[
f_\gamma e_\eta = - \sum_{\alpha, \beta \in \phi^+} C(\alpha, \beta, \gamma) < e_\eta, S(e^{-\beta(h)t} f_\alpha) > < 1, 1 > f_\beta > e^{-\beta(h)t} f_\alpha \\
- \sum_{\alpha, \beta \in \phi^+} C(\alpha, \beta, \gamma) < 1, S(e^{-\gamma(h)t}) > < e_\eta, f_\beta > e^{-\beta(h)t} f_\alpha \\
= \sum_{\beta \in \phi^+} C(\gamma, \beta, \gamma) f_\beta - \sum_{\alpha \in \phi^+} C(\alpha, \eta, \gamma) e^{-\eta(h)t} f_\alpha + e_\eta f_\gamma;
\] (3.2)

\[
f_\gamma h = h, S(e^{-\gamma(h)t}) > < 1, 1 > f_\gamma + \\
< 1, S(e^{-\gamma(h)t}) > < 1, 1 > h f_\gamma + \gamma(h) f_\gamma + h f_\gamma; \\
t x = < 1, 1 > < 1, 1 > x t
\]

The above results are rewritten as follows

**Proposition 4.** The multiplication between \(A\) and \(B\) is defined by the following commutators

\[
[e_\alpha, f_\alpha] = 1 - e^{-\alpha(h)t}, \\
[h, f_\alpha] = -\alpha(h) f_\alpha, \\
[e_\alpha, f_\beta] = \sum_{\gamma \in \phi^+} C(\gamma, \alpha, \beta) e^{-\alpha(h)t} f_\gamma - \sum_{\gamma \in \phi^+} C(\alpha, \gamma, \beta) f_\gamma, \\
\alpha \neq \beta, \\
[t, x] = 0, x = h, e_\alpha
\] (3.3)
The above commutators combine the algebra $A$ with its quantum dual $B$ to form a non-cocommutative and non-commutative Hopf algebra $D(A)=D$ as the quantum double of $A$ (or $B$). As an associative algebra, it is generated by $h, t, e_\alpha, f_\alpha, (\alpha \in \phi^+)$ and the unit 1 obeying eqs. (2.1), and endowed with the Hopf algebraic structure by eqs. (2.2) and (2.6). Now, let us show that this Hopf algebra $D$ is also quasi-triangular. In fact, the construction of Drinfeld’s QD theory automatically perseveres the existence of the quasi-triangular structure. Intertwining $A$ and $B$, the universal R-matrix is a canonical element

$$\hat{R} = \sum_{m, m_\alpha = 0, \alpha \in \phi^+} \infty \ a(m, m_\alpha) \otimes b(m, m_\alpha) = e^{h \otimes t} \prod_{\alpha \in \phi^+} \exp(e_\alpha \otimes f_\alpha). \quad (3.4)$$

This element $\hat{R}(\in D \otimes D)$ endows the Hopf algebra $D$ with a quasi-triangular structure enjoyed by the following relations

$$\hat{R} \Delta(x) = \sigma \Delta(x) \hat{R},$$

$$(\Delta \otimes id) \hat{R} = \hat{R}_{13} \hat{R}_{23},$$

$$(id \otimes \Delta) \hat{R} = \hat{R}_{13} \hat{R}_{12}, \quad (3.5)$$

$$(\epsilon \otimes id) \hat{R} = 1 = (id \otimes \epsilon) \hat{R},$$

$$(S \otimes id) \hat{R} = \hat{R}^{-1} = (id \otimes S) \hat{R},$$

where $\sigma$ is such a permutation that $\sigma(x \otimes y) = y \otimes x, x, y \in D$. The eqs. (3.5) imply that the above constructed universal R-matrix satisfies the abstract QYBE. It is not too difficult to verify the above relations (3.5) by a straightforward calculation.

In the above discussion, we have constructed a new quantum "group" (quasi-triangular Hopf algebra) $D$ associated with an arbitrary classical Lie algebra in terms of Drinfeld’s QD theory. In comparison with the “standard” quantum ‘groups’ that are the q-deformations of UEA’s of classical Lie algebras and superalgebras, our quantum ‘group’ $D$ possesses some new features: 1. $D$ has not the usual classical limit since it is not a q-deformation of the QEA. 2. It has an exotic subalgebraic structure that the subalgebra $A$ is cocomutative but not commutative and the subalgebra $B$ commutative but not cocommutative. This asymmetric structure is quite different from the symmetric structure that both $A$ and $B$ are non-commutative and non-cocommutative. We will call $D$ exotic quantum double.
4. Example of The Exotic Quantum Double for $A_2$

In this section an explicit example of the exotic quantum double will be given in connection with the classical Lie algebra $A_2$. In this example, the subalgebra $A$ is taken to be an associative algebra generated by $h, a, b$ and the relations

\[
[h, a] = \mu a, \quad [h, b] = b, \quad (4.1a)
\]

\[
[a, [a, b]] = 0 = [b, [b, a]], \quad (4.1b)
\]

The generators $a$ and $b$ can be regarded as the root vectors with respect to the simple roots $\alpha_1$ and $\alpha_2$ respectively for $A_2$. The third positive root vector corresponding to $\alpha_1 + \alpha_2$ is just the commutator of $a$ and $b$, i.e.,

\[
c = [a, b], \quad (4.2)
\]

which satisfies

\[
[c, a] = 0 = [c, b],
\]

\[
[h, c] = (\mu + 1)c. \quad (4.3)
\]

The first equation in eq.(4.3) results from the Serre relation (4.1b). If we take $h_1$ and $h_2$ as the Cartan elements in the Chevalley basis for $A_2$ and

\[
[h_1, a] = 2a, \quad [h_1, b] = -b,
\]

\[
[h_2, a] = -a, \quad [h_2, b] = 2b, \quad (4.4)
\]

then,

\[
h = \frac{2\mu + 1}{3} h_1 + \frac{\mu + 2}{3} h_2. \quad (4.5)
\]

The cocomutative Hopf algebraic structure of $A$ is endowed with by

\[
\Delta(x) = x \otimes 1 + 1 \otimes x
\]

\[
S(x) = -x, \quad S(1) = 1; \quad \epsilon(x) = 0, \quad \epsilon(1) = 1
\]

Let $B$ be the quantum dual to $A$ and $t, d, f, g$ be its dual generators to $h, a, b, c$ respectively. According to the last sections, a straightforward calculation gives the Hopf algebraic structure of $B$:

\[
\Delta(d) = d \otimes 1 + e^{-\mu t} \otimes d,
\]

\[
\Delta(f) = f \otimes 1 + e^{-t} \otimes f,
\]

\[
\Delta(g) = g \otimes 1 + e^{-(\mu+1)t} \otimes g - e^{-t} d \otimes f,
\]

\[
\Delta(1) = 1 \otimes 1.
\]
\[ S(d) = -e^{\mu t}d, \]
\[ S(g) = -e^{(\mu + 1)t}(g + d \otimes f), \]
\[ S(f) = e^t f, \] \hspace{1cm} (4.7)

and the multiplication relation between A and B
\[ [h, d] = -\mu d, \]
\[ [h, f] = -f, \]
\[ [h, g] = -(\mu + 1)g, \]
\[ [a, d] = 1 - e^{-\mu t}, \]
\[ [b, f] = 1 - e^{-t}, \]
\[ [c, g] = 1 - e^{-(\mu + 1)t}, \] \hspace{1cm} (4.8)
\[ [a, g] = -f, \]
\[ [b, g] = e^{-t}d, \]
\[ [a, f] = 0 = [b, d], \]
\[ [c, d] = 0 = [c, f], \]
\[ [t, s] = 0, x = a, b, c, h. \]

The quantum double D(2) is generated by \(a, b, c, h, t, d, e, f\) with the relations (4.1a),(4.2) (4.3) and (4.8) as an associative algebra. Its quasi-triangular Hopf algebraic structure is endowed with by eqs.(4.7) and the universal R-matrix
\[ \hat{R} = e^{h \otimes t}e^{a \otimes d}e^{b \otimes f}e^{c \otimes g}, \] \hspace{1cm} (4.9)

5 The Representation Theory and Many-Parameter R-matrices

One purpose of building quantum double is to obtain the solutions of the QYBE in terms of its universal R-matrix and matrix representations. In order to find the solutions of QYBE associated with the exotic quantum double D, we should study its representation theory. In fact, for a given representation \(T[x]\) of D:
\[ T[x] : D \rightarrow End(V) \]
on the linear space V where x is a continuous parameter, we can construct a R-matrix
\[ R(x, y) = T[x] \otimes T[y](\hat{R}) \]
satisfying the QYBE

\[ R_{1,2}(x, y)R_{1,3}(x, z)R_{2,3}(y, z) = R_{2,3}(y, z)R_{1,3}(x, z)R_{1,2}(x, y), \]  (5.1)

Here, \( x, y \) and \( z \) appear as the color parameters [24] similar to the non-additive spectrum parameters in QYBE. This additivity for R-matrices was first found in ref.[15,16] for the chiral Potts model in statistical mechanics. Thus, it is necessary to study the representation theory and construct the many-parameter representations for the exotic quantum double \( D \). However, to write down an explicit representation of a general \( D \) is rather overlaborate. So we only discuss the typical example \( \text{D}(2) \) in this section, but the main ideas and method can be directly applied to the general case.

To simplify our discussion, we have to distinguish between the trivial and non-trivial \( D \)-modules.

**Definition 1.** The action of an operator on the representation space \( V \) is called to be trivial if its kernel is the whole space that it acts on.

**Definition 2.** A \( \text{D}(2) \)-module \( V \) is called to be trivial if at least one of the generators of \( D \) acts trivially on \( V \); otherwise, it is called non-trivial module.

Before studying the representation theory of \( \text{D}(2) \), we would like to give a remark on the above definitions. To study the trivial \( \text{D}(2) \)-module is much easier than that of a non-trivial one. In fact, the structure of a non-trivial \( \text{D}(2) \) module collapses into that of the module of a simpler algebra \( D' \). For example, if the action of \( t \) in \( \text{D}(2) \) is trivial, one only need to study the module of the associative algebra generated by \( h, a, b, c, d, f, g \) with non-zero commutation relations

\[
[a, b] = c, [a, g] = -f, [b, g] = d,
\]

\[
[h, a] = \mu a, [h, b] = b,
\]

For this reason, we will mainly study the non-trivial \( \text{D}(2) \)-model.

Having the above description, we are now in the position to prove a proposition as a central result for the representation theory of \( \text{D}(2) \).

**Proposition 5.** There does not exist a finite dimensional irreducible \( \text{D}(2) \)-module.

**Proof.** Suppose there exists a finite dimensional irreducible \( \text{D}(2) \)-model \( V \) and
$T : D \to End(V)$ is the corresponding finite dimensional irreducible representation. For simplicity we by $x$ denote $T(x)$ as follows for $x \in D(2)$. Since $t$ belongs to the center of $D$, $t$ must be a non-zero scalar in non-trivial finite dimensional irreducible representation according to the Schur lemma. Otherwise, if $t$ is zero, $V$ is trivial. Because $C$ is an algebraic closure, there must be an eigenvector $v$ such that

$$hv = \xi v, \xi \in C$$

Noticing the vectors $v, av, a^2v, ..., a^n v$... correspond to the distinct eigenvalues $\xi, \xi + \mu, \xi + 2\mu, ..., \xi + n\mu$... for $\mu \neq 0$, we come to the conclusion that there exists $r \in Z^+$ such that $v, av, a^2v, ..., a^{r-2}v, a^{r-1}v = u$ are linearly independent and $a^n v = 0$. Similarly, there are $s, q \in Z^+$ such that the non-zero $u, bu, b^2u, ..., b^{s-1}u = w$ are are linearly independent and $bw = b^sw = 0$; the non-zero vectors $w, cw, c^2w, ..., c^{q-1}w = z$ are linearly independent and $cz = c^qw = 0$. Then, we can prove that $a z = b z = c z = 0$ and thus the vector $z$ generates a $D(2)$-submodule

$$S = \text{Span}\{F(m, n, l) = d^m f^n g^l z \mid m, n, l \in Z^+\}$$

under the action of $D(2)$. Thanks to the irreducibility of $V$ and its finite dimension, we must have $S = V$ and conclude that there must exist $m', n', l'$ so that

$$dF(m' - 1, n, l) = 0,$$

$$fF(m, n' - 1, l) = 0,$$

$$gF(m, n, l' - 1) = 0,$$

that is to say, the dimension of $V$ is $m'n'l'$ of $S$. However, it follows from eq.(5.1) that

$$0 = adF(m' - 1, 0, 0) = ad^{m'} z$$

$$= [d^{m'} a + m'(1 - e^{-\mu t})] z = m'(1 - e^{-\mu t}) z,$$

that is, $m' = 0$. Similarly, $n' = l' = 0$. This means the $D(2)$-module is trivial.

According to the above proposition, for the study of non-trivial representation, we only need to focus on two cases, the indecomposable (reducible, but not completely reducible) representations and the infinite dimensional irreducible representations. Now, we only discuss the later. To construct an infinite dimensional irreducible representation explicitly, we define a Verma-like space

$$V(\eta, \pi) = \text{Span}\{ | M > = | m, n, l > = a^m b^n c^l \mid 0(\eta, \pi) > \mid m, n, l \in Z^+ \}$$
based on the vacuum-like state $|0(\eta, \pi)\rangle$:

$$d |0(\eta, \pi)\rangle = f |0(\eta, \pi)\rangle = g |0(\eta, \pi)\rangle = 0$$

$$h |0(\eta, \pi)\rangle = \eta |0(\eta, \pi)\rangle, t |0(\eta, \pi)\rangle = \pi |0(\eta, \pi)\rangle,$$

(5.3)

where $\eta, \pi \in C$. The existence of the vacuum-like state $|0(\eta, \pi)\rangle$ is easily proved by considering that $t$ and $h$ commute with each other and $(a, b, c), (d, f, g)$ and $(h, t)$ act as the ‘lifting’ operators, ‘lowering’ operator and the Cartan operators respectively for a classical Lie algebra.

**Proposition 6.** On the Verma-like space, the infinite dimensional representation $T^{[\eta, \pi]}$

$$h |M\rangle = [\eta + m\mu + (1 + \mu)t] |M\rangle,$$

$$a |M\rangle = |M + e^1\rangle,$$

$$b |M\rangle = |M + e^2\rangle - m |M - e^1 + e^3\rangle,$$

$$c |M\rangle = |M + e^3\rangle,$$

$$d |M\rangle = m(e^{-\mu\pi} - 1) |M - e^1\rangle,$$

$$f |M\rangle = n(e^{-\pi} - 1) |M - e^2\rangle,$$

$$g |M\rangle = l(e^{-(\mu+1)\pi} - 1) |M - e^3\rangle + mn(e^{-\pi} - 1) |M - e^1 - e^2\rangle,$$

(5.4)

is irreducible. Here $e^1 = (1, 0, 0), e^2 = (0, 1, 0), e^3 = (0, 0, 1)$ are the unit vectors in the lattice space $Z^3 : \{M = (m, n, l) | m, n, l \in Z^+\}$

**Proof.** Using the commutation relations of D(2), we can first prove by induction for $n \in Z^+$

$$da^n = a^n d + n(e^{-\mu t} - 1)a^{n-1},$$

$$fb^n = b^n f + n(e^{-t} - 1)b^{n-1},$$

$$g c^n = c^n g + n(e^{-(\mu+1)t} - 1)c^{n-1},$$

$$ga^n = a^n g + nf a^{n-1},$$

$$gb^n = b^n g - ne^{-t}db^{n-1},$$

$$ba^n = a^n b - nca^{n-1},$$

$$ab^n = b^n a + ncb^{n-1},$$

$$ha^n = a^n h + n\mu a^n,$$

(5.5)
\[ hb^n = b^n h + nb^n, \]
\[ he^n = c^n h + n(1 + \mu)b^n. \]

The eqs.(5.4) follows from eqs.(5.5) and (5.3) immediately. It is not difficult to verify that the eqs.(5.4) indeed define a representation of D(2). By considering that the indices \( m, n \) and \( l \) not only decrease but also increase by unit 1, it can be proved that this representation is irreducible if it is non-trivial.

Let us make an observation that there exist many parameter \( \mu, \pi \) and \( \eta \). Among them \( \mu \) and \( \eta \) are allowed by the quantum double structure and the representation theory respectively while \( \pi \) is due to the existence the central element \( t \). Since \( \eta \) and \( \pi \) can be used to distinguish the different representations, we can set \( x = (\eta, \pi) \) and obtain the colored R-matrices with two dimensional color parameters \( x \) where the parameter \( \mu \) is intrinsic and plays the similar role to that of the \( q \) in the standard quantum double -quantum algebras. In fact, for a rank-\( l \) classical Lie algebra, we can introduce \( l - 1 \) independent intrinsic parameters to the corresponding exotic quantum double since its Cartan subalgebra is \( l \) dimensional.

6. Discussions

To conclude this paper, we should give some remarks on our exotic quantum double and its relations to the known results, such as the Hopf algebraic structure for the function algebra on the formal group \([43,44]\), the extended Heisenberg-Weyl algebra (the boson algebra) as a quantum double \([45-47]\) and so on.

i.) From the construction of the exotic quantum double in this paper, we can see that a commutative (Abelian) algebra, eg., the subalgebra \( B \), can be endowed with a non-cocommutative Hopf algebraic structure and its quantum dual and quantum double can be deduced as non-commutative algebras. Such a process can be regarded as the inversion of the construction in this paper and maybe provide us a scheme of ‘quantization’ from commutative object to non-commutative one. An example of this ‘quantization’ was given \([45]\) recently. A simplest associative algebra is generated by two commuting generators \( X \) and \( H \). Its non-cocommutative Hopf algebraic structure is defined by

\[ \Delta(H) = H \otimes 1 + 1 \otimes H, \Delta(X) = X \otimes 1 + e^{-H} \otimes X, \]
Let $Y$ and $N$ be the dual generators to $X$ and $H$ respectively. Then the quantum dual has cocommutative Hopf algebraic structure the elements $X, Y, H$ and $N$ generate a quantum double $D(1)$ with the only non-zero commutation relations:

$$ [N, X] = X, [N, Y] = -Y, [X, Y] = 1 - e^{-H} $$

This quantum double $D(1)$ is just the special example that $A$ is the ‘Half’ UEA of $A_1$. There exists a homomorphism

$$ a \rightarrow X, a^+ \rightarrow Y, E \rightarrow 1 - e^{-H}, \hat{N} \rightarrow -N $$

from the boson algebra generated by the creation operator $a^+$, the annihilation operator $a$, the number operator $\hat{N}$ and the central operator $E$ to this quantum double where the only non-zero commutation relations for the boson algebra are

$$ [a, a^+] = E, [\hat{N}, a] = -a, [\hat{N}, a^+] = a^+ $$

This example shows the so-called ‘quatumization’ from commutative object to non-commutative one in which the quantum Yang-Baxter equation is enjoyed by the universal $R$-matrix

$$ \hat{R} = \exp(X \otimes Y)\exp(N \otimes H) $$

ii.) It has to be pointed out that there are some difficulties in the further developments of the exotic quantum double theory. When one take the subalgebra $B$ to be the whole UEA of a classical algebra, we hardly write down the dual basis explicitly and so the construction scheme of this paper can not work well. The similar problem also appears in the discussion in terms of the formal group. How to generalize the method and ideas of this paper to work on the case of the whole UEA other than a Borel subalgebra is the first open question we should mention. The second open question is how to find a finite dimensional representation for the exotic quantum double except the example for $A_1$ mentioned above. It is well-known that the finite dimensional $R$-matrices usually make sense in the quantum inverse scattering method and even in the exactly-solvable models in statistical mechanics. Thus, it is also expected that some new finite dimensional $R$-matrices can follow from the exotic quantum double through its universal $R$-matrix where the finite dimensional representations of the exotic quantum double must be used. However, although we can do it for the special case of $A_1$ by building the finite dimensional indecomposable representation of $D(1)$.
on certain quotient space on linear space D(1), we can not obtain a finite dimensional representation for other higher-rank exotic quantum double by the same method due to the existance of the multi-center in cetrain subalgebra. Therefore, there needs the futher works on finite dimensional representation.

\textbf{iii.}) In the formal group theory of Lie algbera [43], the bialgebra structure of the dual to the UEA of a classical Lie algebra can be given abstractly in terms of the formal group. It is not difficult to further define the antipode for this dual bialgebra. So, in this abstract way, the Hopf algebraic structure can be endowed with to this dual algebra. However, writing out the explicit Hopf algebraic structure, namely, the the explicit multiplication relations, coproduct, antipode and counit for the dual generators, completely depends on the explicit evolution of the Baker-Compbell-Hausdorff formula for classical Lie algebra. However, it is much difficult to do it even for the simple case e.g., SU(2). The study in this paper avoids this evaluation so that not only the dual Hopf algebraic structure is obtained, but also the corresponding quantum double - the exotic quantum double is built for the Borel subalgebra of the UEA of arbitrary classical Lie algebra by combining the two subalgebras dual to each other.

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