ON TATE CONJECTURE FOR THE SPECIAL FIBERS OF SOME UNITARY
SHIMURA VARIETIES

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ABSTRACT. Let $F$ be a totally real field in which a fixed prime $p$ is inert, and let $E$ be a CM extension of $F$ in which $p$ splits. We fix two positive integers $r, s \in \mathbb{N}$. We investigate the Tate conjecture on the special fiber of $G(U(r, s) \times U(s, r))$-Shimura variety. We construct cycles which we conjecture to generate the Tate classes and verify our conjecture in the case of $G(U(1, s) \times U(s, 1))$. We also discuss the general conjecture regarding special cycles on the special fibers of unitary Shimura varieties.

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INTRODUCTION

The study of the geometry of Shimura varieties lies at the heart of the Langlands program. Arithmetic information of Shimura varieties builds a bridge relating the world of automorphic representations and the world of Galois representations.

One of the interesting topics in this area is to understand the supersingular locus of the special fibers of Shimura varieties, or more generally, any interesting stratifications (e.g., Newton or Ekedahl-Oort stratification) of the special fibers of Shimura varieties. While most research results successfully obtained properties of local nature, e.g., dimension, smoothness, nonemptiness of strata, very little was known about the explicit global geometry of the stratification. The only exception that we are aware of is the work of Vollaard and Wedhorn [VW11] in which they showed that the supersingular locus of the special fiber of $GU(1, s)$-Shimura variety is a union of Deligne-Lusztig varieties.

In this paper, we however take a different approach, carrying forward a method introduced by the first author [He12, He11] and following the framework suggested by the second and the third authors [TX13+, TX14+]. The project started out to search for all interesting natural subvarieties of the special fiber of a given Shimura variety, which admit fibrations to (the special fibers of)

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1There has been some variants of this results, see [RTW13, HP13+]
other Shimura varieties. We will explain a general conjecture in this paper, which roughly predicts that all such subvarieties are governed by the Galois representations appearing on this Shimura variety. A special case of this conjecture is expected, on the one hand, to give a description of the superbasic locus of the Shimura variety, and one the other hand, to verify the Tate Conjecture for the special fiber of the Shimura variety (under certain generality hypothesis). As the project progresses, we were informed by X. Zhu that the union of these general conjectural subvarieties are supposed to be exactly certain Newton strata, and hence give those Newton strata an explicit description! This general framework should include the prior work \cite{VW11} as an example, (which describes the cycles in the corresponding case but did not prove the related Tate Conjecture). We believe that our conjecture will provide a completely new understanding of the special fibers of Shimura varieties.

We defer the detailed introduction to the first section, where the main Conjecture \ref{conj:main} is stated; we have to sacrifice the generality at various places for the accessibility of readers with various background. Remarks later hopefully will guide the readers towards a general version of the conjecture in terms of Langlands parameters. The rest of the paper is devoted to providing evidences for the special fiber of the Shimura variety (under certain generality hypothesis). As the project progresses, we were informed by X. Zhu that the union of these general conjectural subvarieties are supposed to be exactly certain Newton strata, and hence give those Newton strata an explicit description! This general framework should include the prior work \cite{VW11} as an example, (which describes the cycles in the corresponding case but did not prove the related Tate Conjecture). We believe that our conjecture will provide a completely new understanding of the special fibers of Shimura varieties.

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1. The conjecture on the special cycles

We will only discuss certain unitary Shimura varieties so that the description becomes explicit; one can easily reformulate the conjecture in other or general settings.

1.1. Notation. We fix a prime number $p$ throughout this paper. We fix an isomorphism $\iota_p : \mathbb{C} \cong \overline{\mathbb{Q}}_p$. Let $\mathbb{Q}^\ur_p$ be the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$.

Let $F$ be a totally real field of degree $f$ in which $p$ is inert. We label all real embeddings of $F$, or equivalently all $p$-adic embeddings of $F$ (into $\mathbb{Q}_p^\ur$) by $\tau_1, \ldots, \tau_f$ so that post-composition by the Frobenius map takes $\tau_i$ to $\tau_{i+1}$, with the convention that the subindices are taken modulo $f$. Let $E_0$ be an imaginary quadratic extension of $\mathbb{Q}$, in which $p$ splits. Put $E = E_0F$. Denote by $v$ and $\bar{v}$ the two $p$-adic places of $E_0$. Then $p$ splits into two primes $p$ and $\bar{p}$ in $E$, where $p$ is the $p$-adic place above $v$. Let $q_i$ denote the embedding $E \to E_p \cong F_p \cong \mathbb{Q}_p^\ur$ and $\bar{q}_i$ the analogous embedding which factors through $E_{\bar{p}}$ instead. Composing with $\iota_p^{-1}$, we regard $q_i$ and $\bar{q}_i$ as complex embeddings of $E$, and we put $\Sigma_{\infty, E} = \{q_1, \ldots, q_f, \bar{q}_1, \ldots, \bar{q}_f\}$.

1.2. Shimura Data. Let $D$ be a division algebra of $n^2$-dimensional over its center $E$, equipped with a positive involution $\ast$ which restricts to the complex conjugation $c$ on $E$. We assume that $D^{\opp} \cong D \otimes_{E, c} E$, and $D$ splits at $p$ so that we fix an isomorphism

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_n(E_p) \times M_n(E_{\bar{p}}) \cong M_n(\mathbb{Q}_p) \times M_n(\mathbb{Q}_{\bar{p}}),$$

where $\ast$ switches the two direct factors. We use $e$ to denote the element of $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ corresponding to the $(1, 1)$-elementary matrix in the first factor. Let $a_e = (a_i)_{1 \leq i \leq f}$ be a tuple of $f$ numbers with $a_i \in \{0, \ldots, n\}$. We choose $\beta_{a_e} \in (D^\times)^{\ast\ast} = \mathbb{Z}$ such that the following condition is satisfied:

\footnote{By a $(1, 1)$-elementary matrix, we mean an $n \times n$-matrix whose $(1, 1)$-entry is 1 and whose other entries are zero.}
Let $G_{a*}$ the algebraic group over $\mathbb{Q}$ such that $G_{a*}(R)$ for a $\mathbb{Q}$-algebra $R$ consists of elements $g \in (D_{opp} \otimes_{\mathbb{Q}} R)^{\times}$ with $g^{\ast} \beta_{a} g = c(g) \beta_{a}$ for some $c(g) \in R^{\times}$. If $G_{a*}^{1}$ denotes the kernel of the similitude character $c : G_{a*} \to \mathbb{G}_{m, \mathbb{Q}}$, then there exists an isomorphism

$$G_{a*}^{1}(\mathbb{R}) \cong \prod_{i=1}^{f} U(a_{i}, n - a_{i}),$$

where the $i$-th factor corresponds to the real embedding $\tau_{i} : F \hookrightarrow \mathbb{R}$.

Note that the assumption on $D$ at $p$ implies that

$$G_{a*}(\mathbb{Q}_{p}) \cong \mathbb{Q}_{p}^{\times} \times \text{GL}_{n}(\mathbb{Q}_{p}) \cong \mathbb{Q}_{p}^{\times} \times \text{GL}_{n}(\mathbb{Q}_{p, f}).$$

We put $V_{a*} = D$ and view it as a left $D$-module. Let $(\cdot, \cdot)_{a*} : V_{a*} \times V_{a*} \to \mathbb{Q}$ be the perfect alternating pairing given by

$$(x, y)_{a*} = (\text{Tr}_{F/\mathbb{Q}} \circ \text{Tr}_{D/F})(x \beta_{a} y^{\ast}), \quad \text{for } x, y \in V_{a*}.$$

Then $G_{a*}$ is identified with the similitude group associated to $(V_{a*}, (\cdot, \cdot)_{a*})$, i.e. for all $\mathbb{Q}$-algebra $R$, we have

$$G_{a*}(R) = \{ g \in \text{End}_{D \otimes_{\mathbb{Q}} R}(V_{a*} \otimes_{\mathbb{Q}} R) \mid (gx, gy)_{a*} = c(g)(x, y)_{a*} \text{ for some } c(g) \in R^{\times} \}. $$

Consider the homomorphism of $\mathbb{R}$-algebraic groups $h : \text{Res}_{\mathbb{R}}(\mathbb{G}_{m}) \to G_{a*, \mathbb{R}}$ given by

$$h(z) = \prod_{i=1}^{f} \text{diag}(z, \ldots, z, \bar{z}, \ldots, \bar{z}), \quad \text{for } z = x + \sqrt{-1}y.$$ 

Let $\mu_{h} : \mathbb{G}_{m, \mathbb{C}} \to G_{a*, \mathbb{C}}$ be the composite of $h_{\mathbb{C}}$ with the map $\mathbb{G}_{m, \mathbb{C}} \to \text{Res}_{\mathbb{C}}(\mathbb{G}_{m})_{\mathbb{C}} \cong \mathbb{C}^{\times} \times \mathbb{C}^{\times}$ given by $z \mapsto (z, 1)$. Here, the first copy of $\mathbb{C}^{\times}$ in $\text{Res}_{\mathbb{C}}(\mathbb{G}_{m})_{\mathbb{C}}$ is the one indexed by the identity element in $\text{Aut}_{\mathbb{C}}(\mathbb{C})$.

Let $E_{h}$ be the reflex field of $\mu_{h}$, i.e. the minimal subfield of $\mathbb{C}$ where the conjugacy class of $\mu_{h}$ is defined. It has the following explicit description. The group $\text{Aut}_{\mathbb{Q}}(\mathbb{C})$ acts naturally on $\Sigma_{\infty, E}$, and hence on the functions on $\Sigma_{\infty, E}$. Then $E_{h}$ is the subfield of $\mathbb{C}$ fixed by the stabilizer of the $\mathbb{Z}$-valued function $a$ on $\Sigma_{\infty, E}$ defined by $a(q_{i}) = a_{i}$ and $a(q_{i}) = n - a_{i}$. The isomorphism $\tau_{p}$ defines a $p$-adic place $v$ of $E_{h}$. By our hypothesis on $E$, the local field $E_{h,v}$ is an unramified extension of $\mathbb{Q}_{p}$ contained in $\mathbb{Q}_{p,f}$, the unique unramified extension over $\mathbb{Q}_{p}$ of degree $f$.

### 1.3. Unitary Shimura varieties of PEL-type.

Let $\mathcal{O}_{D}$ be a $\ast$-stable order of $D$ and $\Lambda_{a*}$ an $\mathcal{O}_{D}$-lattice of $V_{a*}$ such that $\langle \Lambda_{a*}, \Lambda_{a*} \rangle_{a*} \subseteq \mathbb{Z}$ and $\Lambda_{a*} \otimes_{\mathbb{Z}} \mathbb{Z}_{p}$ is auto-dual under the alternating pairing induced by $(\cdot, \cdot)_{a*}$. We put $K_{p} \cong \mathbb{Z}_{p}^{\times} \times \text{GL}_{n}(\mathbb{O}_{E_{p}}) \subseteq G_{a*}(\mathbb{Q}_{p})$, and fix an open compact subgroup $K_{p} \subseteq G_{a*}(\mathbb{A}^{\infty})$ such that $K = K_{p}K_{p}$ is neat, i.e. $G_{a*}(\mathbb{Q}) \cap gKg^{-1}$ for any $g \in G_{a*}(\mathbb{A}^{\infty})$.

Following [Ko92a], we have a unitary Shimura variety $\text{Sh}_{a*}$ defined over $\mathbb{Z}_{p,f}$

$\text{it represents the functor that takes a locally noetherian } \mathbb{Z}_{p,f}\text{-scheme } S \text{ to isomorphism classes of tuples } (\Lambda, \lambda, \eta), \text{ where}$

(1) $\Lambda$ is an $fn^{2}$-dimensional abelian variety over $S$ equipped with an action of $\mathcal{O}_{D}$ such that the induced action on $\text{Lie}(\Lambda/S)$ satisfies the Kottwitz determinant condition, that is, if we view the reduced relative de Rham homology $H_{1}^{\text{dR}}(\Lambda/S)^{\circ} := \oplus_{i=1}^{f} H_{1}^{\text{dR}}(\Lambda/S)$ and its quotient $\text{Lie}_{\Lambda/S}^{\circ} := \mathfrak{c} \cdot \text{Lie}(\Lambda/S)$ as a module over $\mathbb{F}_{p} \otimes_{\mathbb{Z}_{p}} \mathcal{O}_{S} \cong \bigoplus_{i=1}^{f} \mathcal{O}_{S}$, they, respectively, decompose into the direct sums of locally free $\mathcal{O}_{S}$-modules $H_{1}^{\text{dR}}(\Lambda/S)^{\circ}_{i}$ of rank $n$ and, their quotients, locally free $\mathcal{O}_{S}$-modules $\text{Lie}_{\Lambda/S,i}^{\circ}$ of rank $n - a_{i}$;
(2) $\lambda : A \to A^\vee$ is a prime-to-$p$ $O_D$-equivariant polarization such that the Rosati involution induces the involution $\ast$ on $O_D$;

(3) $\eta$ is a collection of, for each connected component $S_j$ of $S$ with a geometric point $\bar{s}_j$, $\pi_1(S_j, \bar{s}_j)$-invariant $K^p$-orbits of isomorphisms $\eta_j : \Lambda_{\alpha} \otimes \mathbb{Z} \cong T^{(p)}(A_{\bar{s}_j})$ such that the following diagram commutes for an isomorphism $\nu(\eta_j) \in \text{Hom}(\mathbb{Z}^{(p)}, \mathbb{Z}^{(p)}(1))$:

$$
\begin{array}{ccc}
\Lambda_{\alpha} \otimes \mathbb{Z} \cong T^{(p)} & \xrightarrow{\nu(\eta_j)} & \mathbb{Z}^{(p)}
\end{array}
$$

$$
\begin{array}{ccc}
\Lambda_{\alpha} \otimes \mathbb{Z} \cong T^{(p)} & \xrightarrow{(\cdot, \cdot)} & \mathbb{Z}^{(p)}
\end{array}
$$

where $\mathbb{Z}^{(p)} = \prod_{l \neq p} \mathbb{Z}_l$ and $T^{(p)}(A_{\bar{s}_j})$ denote the product of $l$-adic Tate modules of $A_{\bar{s}_j}$ for all $l \neq p$.

The Shimura variety $Sh_{\alpha}$ is proper and smooth over $\mathbb{Z}_{p}f$ of relative dimension $d(\alpha) := \sum a_i (n - a_i)$. Note that if $a_i \in \{0, n\}$ for all $i$, then $Sh_{\alpha}$ is of relative dimension zero.

We denote by $Sh_{\alpha}(\mathbb{C})$ the complex points of $Sh_{\alpha}$ via the embedding $\mathbb{Z}_{p}f \hookrightarrow \overline{\mathbb{Q}_p^{\ell}} \xrightarrow{\ell^{-1}} \mathbb{C}$. Let $K_{\infty} \subseteq G_{\alpha}(\mathbb{R})$ be the stabilizer of $h$ under the action of conjugation, and $X_{\infty}$ denote the $G_{\alpha}(\mathbb{R})$-conjugacy class of $h$. Then $K_{\infty}$ is a maximal compact modulo center subgroup of $G_{\alpha}(\mathbb{R})$. According to [Ko92a, page 400], the complex manifold $Sh_{\alpha}(\mathbb{C})$ is the disjoint union of $\#\ker^1(\mathbb{Q}, G_{\alpha})$ copies of

$$
G_{\alpha}(\mathbb{Q}) \backslash (G_{\alpha}(\mathbb{A}_{\infty}) \times X_{\infty}) / K \cong G_{\alpha}(\mathbb{Q}) \backslash (G_{\alpha}(\mathbb{A}_{\infty}) \times X_{\infty}) / K \times K_{\infty}.
$$

Here, if $n$ is even, then $\ker^1(\mathbb{Q}, G_{\alpha}) = (0)$, while if $n$ is odd then

$$
\ker^1(\mathbb{Q}, G_{\alpha}) = \ker \left( F^\times / \mathbb{Q}_E \rightarrow \mathbb{A}_F^\times / \mathbb{A}_E^\times \right).
$$

Note that $\ker^1(\mathbb{Q}, G_{\alpha})$ depends only on the CM extension $E/F$ and the parity of $n$.

1.4. $\ell$-adic cohomology. We fix a prime number $\ell \neq p$, and an isomorphism $\ell : \mathbb{C} \simeq \overline{\mathbb{Q}_l}$. Let $\xi$ be an algebraic representation of $G_{\alpha}$ over $\overline{\mathbb{Q}_l}$, and $\xi_{\mathbb{C}}$ be the base change via $\overline{\mathbb{Q}_l}$. The theory of automorphic sheaves allows us to attach to $\xi$ a lisse $\overline{\mathbb{Q}_l}$-sheaf $L_{\xi}$ over $Sh_{\alpha}$. If $\xi$ is the representation of $G_{\alpha}$ on the vector space $V_{\alpha}$ (Section 1.2), the corresponding $\ell$-adic local system is given by the rational $\ell$-adic Tate module (tensored with $\overline{\mathbb{Q}_l}$) of the universal abelian scheme over $Sh_{\alpha}$.

We assume that $\xi$ is irreducible. Let $\mathcal{H}_{\alpha} := Sh_{\alpha} \otimes \mathbb{Z}_{p, f}$ denote the special fiber of $Sh_{\alpha}$. Let $\mathcal{H}_{K} = \mathcal{H}(K, \overline{\mathbb{Q}_l})$ be the Hecke algebra of compactly supported $K$-bi-invariant $\overline{\mathbb{Q}_l}$-valued functions on $G_{\alpha}(\mathbb{A}_{\infty})$. The étale cohomology group $H^a_{et} (\mathcal{H}_{\alpha}, \mathcal{E}_{\xi})$ are equipped with a natural action of $\mathcal{H}_{K} \times \text{Gal}(\mathbb{F}_p/\mathbb{F}_{p^f})$. Since $Sh_{\alpha}$ is proper and smooth, there is no continuous spectrum and we have a canonical decomposition of $\mathcal{H}_{K} \times \text{Gal}(\mathbb{F}_p/\mathbb{F}_{p^f})$-modules:

$$
H^a_{et} (\mathcal{H}_{\alpha}, \mathcal{E}_{\xi}) = \bigoplus_{\pi \in \text{Irr}(G_{\alpha}(\mathbb{A}_{\infty}))} \text{t}_{\ell}(\pi^K) \otimes R_{\alpha, \ell}(\pi),
$$

where $\text{Irr}(G_{\alpha}(\mathbb{A}_{\infty}))$ denotes the set of irreducible admissible representations of $G_{\alpha}(\mathbb{A}_{\infty})$ with coefficients in $\mathbb{C}$, $\pi^K$ is $K$-invariant subspace of $\pi \in \text{Irr}(G_{\alpha}(\mathbb{A}_{\infty}))$, and $R_{\alpha, \ell}(\pi)$ is a certain $\ell$-adic representation of $\text{Gal}(\mathbb{F}_p/\mathbb{F}_{p^f})$ which we specify below.

**Hypothesis 1.5.** We consider an irreducible admissible representation $\pi$ of $G_{\alpha}(\mathbb{A}_{\infty})$ satisfying the following conditions:
(1) we have $\pi^K \neq 0$;
(2) there exists an irreducible representation $\pi_\infty$ of $G_{a_*}(\mathbb{R})$ such that $\pi \otimes \pi_\infty$ is automorphic, and that $\pi_\infty$ is cohomological in degree $d_{a_*}$ for $\xi$ in the sense that
\begin{equation}
H^{d_{a_*}}(\text{Lie}(G_{a_*}), K_{\infty}, \pi_\infty \otimes \xi_C) \neq 0 \tag{1.5.1}
\end{equation}
(3) for some rational prime $q$ which splits in $E_0$, there exists some place $x$ of $E$ above $q$ such that $D$ splits at $x$ and the representation $\pi_x$ is supercuspidal.

By [HT01 VI 2.7] and [Fa04 A.2.2, A.2.3], there are exactly $\prod_{i=1}^f \binom{n}{a_i}$ (resp. $\frac{1}{2} \binom{n}{n/2}$) irreducible representations $\pi_j$ of $G_{a_*}(\mathbb{R})$ satisfying condition (2) of Hypothesis 1.5, and they are all discrete series representations such that the cohomology group in (1.5.1) are all 1-dimensional (resp. 2-dimensional). Moreover, the automorphic multiplicity of $\pi \otimes \pi_j^{\infty}(\xi)$ is independent of $j$; so we denote it by $m_{a_*}^j(\pi)$ from now on.

Note that (1) of the above hypothesis implies that the $p$-component $\pi_p$ is unramified. We recall a more transparent description, due to Kottwitz [Ko92b], of the Galois action on $R_{a_*, \ell}(\pi)$. As $G_{a_*}(\mathbb{Q}_p) = \mathbb{Q}_p^* \times \text{GL}_n(E_p)$, we may write $\pi_p = \pi_{p,0} \otimes \pi_p$, where $\pi_{p,0}$ is a character of $\mathbb{Q}_p^*$ trivial on $\mathbb{Z}_p^\times$, and $\pi_p$ is an irreducible admissible representation of $\text{GL}_n(E_p)$ such that $\pi_p^{\text{GL}_n(\mathcal{O}_{E_p})} \neq 0$. Choose a square root $\sqrt{p}$ of $p$ in $\overline{\mathbb{Q}}$. Depending on this choice of $\sqrt{p}$, one has an (unramified) local Langlands parameter attached to $\pi_p$:

$$\varphi_{\pi_p} = (\varphi_{\pi_{p,0}}, \varphi_{\pi_p}) : W_{\mathbb{Q}_p} \to L(G_{a_*}(\mathbb{Q}_p)) \simeq \mathbb{C}^\times \times (\text{GL}_n(\mathbb{C})^{\mathbb{Z}/f\mathbb{Z}} \times \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)).$$

Here, $W_{\mathbb{Q}_p}$ denotes the Weil group of $\mathbb{Q}_p$, and $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ permutes cyclically the $f$ copies of $\text{GL}_n(\mathbb{C})$ though the quotient $\text{Gal}(\overline{\mathbb{Q}}_{p,f}/\mathbb{Q}_p) \simeq \mathbb{Z}/f\mathbb{Z}$. The image of $\varphi_{\pi_p}|_{W_{\mathbb{Q}_p,f}}$ lies in $(L(G_{a_*})^\circ : = \mathbb{C}^\times \times GL_n(\mathbb{C})^{\mathbb{Z}/f\mathbb{Z}}$. The cocharacter $\mu_h : GL_m(E_h) \to G_{a_*}(E_h)$ induces a character $\bar{\mu}$ of $(L(G_{a_*})^\circ$ over $E_h$. Let $\tau_{\mu_h}$ denote the algebraic representation of $(L(G_{a_*})^\circ$ with extreme weight $\bar{\mu}$. Denote by $\text{Frob}_{p,f}$ a geometric Frobenius element in $W_{\mathbb{Q}_p,f}$. Let $\overline{\mathbb{Q}}(\ell)^{1/2}$ denote the unramified representation of $W_{\mathbb{Q}_p,f}$ which sends $\text{Frob}_{p,f}$ to $(\sqrt{p})^{-f}$. Then $R_{a_*, \ell}(\pi)$ can be described in terms of $\varphi_{\pi_p}$ as follows.

**Theorem 1.6** ([Ko92b Theorem 1]. Under the above hypothesis and notation, we have an equality in the Grothendieck group of $W_{\mathbb{Q}_p}$-modules:

$$[R_{a_*, \ell}(\pi)] = \# \ker^1(\mathbb{Q}, G_{a_*}) m_{a_*}(\pi_\infty) \left[\ell(\tau_{\mu_h} \circ \varphi_{\pi_p}) \otimes \overline{\mathbb{Q}}(\ell)(-d_{a_*}^\bullet)\right].$$

In our case, one can make Kottwitz’s Theorem more transparent. Define an $\ell$-adic representation
\begin{equation}
\rho_{\varphi_p} = \ell(\varphi_{\pi_p}^{(1)}, \nu) \otimes \overline{\mathbb{Q}}(\ell)(1/n-1/2) : W_{\mathbb{Q}_p,f} \to \text{GL}_n(\overline{\mathbb{Q}}_{\ell}),
\end{equation}
where $\varphi_{\pi_p}^{(1)} : W_{\mathbb{Q}_p,f} \to \text{GL}_n(\mathbb{C})$ denotes the contragredient of the projection to the first (or any) copy of $\text{GL}_n(\mathbb{C})$. Both $\varphi_{\pi_p}$ and $\overline{\mathbb{Q}}(\ell)(1/2)$ depends on the choice of $\sqrt{p}$, but $\rho_{\varphi_p}$ does not. Explicitly,

\[\text{This automatically implies that } \pi_\infty \text{ has the same central character and infinitesimal character as the contragredient of } \xi_C.\]

\[\text{This representation is denoted by } | \cdot |_{\overline{\mathbb{Q}}_{\ell}}^{1/2} \text{ in } [Ko92b].\]

\[\text{The number } a(\pi_f) \text{ in loc. cit. can be computed using the formula in Lemma 4.2 of loc. cit. and the remarks after Hypothesis 1.5 above.}\]

\[\text{The reason why we normalize the Galois representation in this way is the following: if } \pi \otimes \pi_\infty \text{ is an automorphic representation of } G_{a_*}(\mathbb{A}), \text{ with } \pi_\infty \text{ cohomological in degree } d_{a_*} \text{ for } \xi, \text{ then there exists a conjugate self-dual base change } BC(\pi \otimes \pi_\infty) = (\Pi, \psi) \text{ of } \pi \otimes \pi_\infty \text{ to } GL_n(\mathbb{A}_E) \times \mathbb{A}_E^\times, \text{ and } \rho_{\varphi_p} \text{ is the restriction to } W_{E_p} \text{ of the representation of } \text{Gal}(\overline{\mathbb{Q}}/E) \text{ associated to } \Pi.\]
\(\rho_{\pi_p}(\text{Frob}_{p^f})\) is semi-simple with the characteristic polynomial given by \([\text{Gr}98\ (6.7)]\):

\[
(1.6.2) \quad X^n + \sum_{i=1}^{n} (-1)^{i}(\text{Np})^{(i-1)/2} a_p^{(i)} X^{n-i},
\]

where \(a_p^{(i)}\) is the eigenvalue on \(\pi_{p,1}^{\text{GL}_n(O_{E_p})}\) of the Hecke operator

\[
T_p^{(i)} = \text{GL}_n(O_{E_p}) \cdot \text{diag}(p, \ldots, p, 1, \ldots, 1) \cdot \text{GL}_n(O_{E_p}).
\]

An easy computation shows that \(r_{\mu_h} = \text{Std}^{-1}_{\pi_{p,1}} \otimes \otimes_{i=1}^{f} (\wedge^a \text{Std}^{i})\). Since the projection of \(\varphi_{p} |_{W_{Q, p^f}}\) to each copy of \(\text{GL}_n(C)\) is conjugate to each other, Theorem 1.6 is equivalent to

\[
(1.6.3) \quad [R_{a_{\ast}}(\pi)] = \# \ker^1(\mathbb{Q}, \rho_{a_{\ast}}) m_{a_{\ast}}(\pi) [\rho_{a_{\ast}}(\pi_p) \otimes \chi_{\pi_{p,0}}^{-1} \otimes \overline{\otimes}_\ell \left(\sum_{i} a_{a_j} a^{-1}\right)],
\]

where \(\rho_{a_{\ast}}(\pi_p) = r_{a_{\ast}} \circ \rho_{p}\) with \(r_{a_{\ast}} = \otimes_{i=1}^{f} \wedge^a \text{Std}\), and \(\chi_{\pi_{p,0}}\) denote the character of \(\text{Gal}(\overline{F}_p/\overline{F}_{p^f})\) sending \(\text{Frob}_{p^f}\) to \(\lambda(\pi_{p,0}(p^f))\).

1.7. **Tate Conjecture.** We recall first the Tate conjecture over finite fields. Let \(X\) be a projective smooth variety over a finite field \(\mathbb{F}_q\) of characteristic \(p\). For each prime \(\ell \neq p\) and integer \(r \leq \dim(X)\), we have a cycle class map

\[
\text{cl}_{X}^{\ell} : A^r(X) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_{\ell} \to H^r_{\text{et}}(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_{\ell})(r)^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)},
\]

where \(A^r(X)\) denotes the abelian group of codimension \(r\) algebraic cycles in \(X\) defined over \(\mathbb{F}_q\). Then the Tate conjecture predicts that this map is surjective. One has a geometric variant of Tate conjecture, which claims that the geometric cycle class map:

\[
\text{cl}_{X_{\overline{\mathbb{F}}_p}}^{\ell} : A^r(X_{\overline{\mathbb{F}}_p}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_{\ell} \to H^r_{\text{et}}(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_{\ell})(r)^{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{q^m})}
\]

is surjective. Here, the subscript “\(\text{fin}\)” means the subspace on which \(\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)\) acts through a finite quotient. Note that the surjectivity of \(\text{cl}_{X_{\overline{\mathbb{F}}_p}}^{\ell}\) implies that of \(\text{cl}_{X}^{\ell}\) by taking Galois invariants.

Consider the case \(X = \text{Sh}_{a_{\ast}}\) with \(d_{a_{\ast}}\) even. Let \(\pi\) be an irreducible admissible representation of \(G_{a_{\ast}}(\mathbb{A}_\infty)\) as in Theorem 1.6. By Theorem 1.6, the \(\pi\)-isotypic component of \(H_{d_{a_{\ast}}}(\text{Sh}_{a_{\ast}}\mathbb{F}_p, \overline{\mathbb{Q}}_{\ell})(d_{a_{\ast}})^{\text{fin}}\) is, up to Frobenius semi-simplification a\ isomorphic to \(\dim(\pi)^K \cdot \# \ker^1(\mathbb{Q}, \rho_{a_{\ast}}) \cdot m_{a_{\ast}}(\pi)\) copies of

\[
(1.7.1) \quad \left(\rho_{a_{\ast}}(\pi_p) \otimes \chi_{\pi_{p,0}}^{-1} \otimes \overline{\otimes}_\ell \left(\sum_{i} a_{a_j} a^{-1}\right)^{\text{fin}}\right).
\]

This implies that \(\chi_{\pi_{p,0}}(\text{Frob}_{p^f}) = \pi_{p,0}(p^f)\) is a root of unity. Hence, the dimension of \((1.7.1)\) equals to the sum of the dimensions of the Frobenius eigenspaces of \(\rho_{a_{\ast}}(\pi_p)\) with eigenvalues \((p^f)^{\sum_{i} a_{a_j} a^{-1}}\) for some root of unity \(\zeta\). In quite many examples, this space is known to be non-zero.

For instance, when \(f = 2, a_1 = r\) and \(a_2 = n - r\) for some \(1 \leq r \leq n - 1\), we have \(d(a_{\ast}) = 2r(n - r)\) and

\[
\rho_{a_{\ast}}(\pi_p) = (\wedge^r \rho_{\pi_p} \otimes \wedge^{n-r} \rho_{\pi_p}).
\]

If \(\rho_{\pi_p}(\text{Frob}_{p^f})\) has distinct eigenvalues \(\alpha_1, \ldots, \alpha_n\), then the eigenvalues of \(\text{Frob}_{p^f}\) on \(V_{\pi,a_{\ast}}\) are given by \(\alpha_{i_1} \cdots \alpha_{i_r} \cdots \alpha_{j_{n-r}}\), for distinct numbers \(i_1, \ldots, i_r\) and distinct numbers \(j_1, \ldots, j_{n-r}\). This product is exactly \((p^f)^{\sum_{i} a_{a_j} a^{-1}}(a^{(n)}_{a})\) (note that \(a^{(n)}_{a}\) is a root of unity) if the set \(\{i_1, \ldots, i_r\}\) and the set \(\{j_1, \ldots, j_{n-r}\}\) are the complement of each other as subsets of \(\{1, \ldots, n\}\). This condition is also necessary when the eigenvalues are sufficiently “generic” in the sense that \(\alpha_i/\alpha_j\) is not a root of

\footnote{Conjecturally, the Frobenius action on the étale \(\ell\)-adic cohomology groups of a projective smooth variety over a finite field is always semi-simple.}
unity for every pair $i \neq j$. In other words, the dimension of \( [1.7.1] \) is generically equal to \( \binom{n}{2} \). As Tate conjecture predicts, these cohomology classes should come from algebraic cycles. Our main conjecture addresses exactly this heuristic, predicting that those desired “generic” algebraic cycles are birationally equivalent to certain fiber bundles over the special fiber of some other Shimura varieties associated to an unitary group which is isomorphic to $G_{a*}$ at all finite places of $\mathbb{Q}$, but with different signatures at infinity. To make this precise, we need the following lemma. Since the proof is well-known to the expert, we leave it to Appendix A.

**Lemma 1.8.** For any tuple $b* = (b_i)_{1 \leq i \leq f}$ such that $b_i \in \{0, \ldots, n\}$ and $\sum_{i=1}^{f} b_i \equiv \sum_{i=1}^{f} a_i \pmod{2}$, there exists $\beta_{b*} \in (D^\times)^* = \{1\}$ such that if $G_{b*}$ denotes the corresponding algebraic group over $\mathbb{Q}$ defined in the similar way with $\beta_{a*}$ replaced by $\beta_{b*}$, then $G_{b*}(\mathbb{A}^\infty) \cong G_{a*}(\mathbb{A}^\infty)$ and

\[
G_{b*}^1(\mathbb{R}) \cong \prod_{i=1}^{f} U(b_i, n - b_i).
\]

Note that, for each tuple $b*$ as in Lemma [1.8] even though there is no canonical choice for $\beta_{b*} \in (D^\times)^*$, the group $G_{b*}$ is (up to isomorphism) independent of the choices of $\beta_{b*}$. In the sequel, we always fix a choice of $\beta_{b*}$ and as well as an isomorphism $G_{a*}(\mathbb{A}^\infty) \cong G_{b*}(\mathbb{A}^\infty)$.

This determines an isomorphism $\gamma_{a*,b*} : V_a \otimes_{\mathbb{Q}} \mathbb{A}^\infty \sim \rightarrow V_{b*} \otimes_{\mathbb{Q}} \mathbb{A}^\infty$ compatible with the alternating pairings on both sides. Recall that we have chosen a lattice $\Lambda_{a*} \subseteq V_a$ to define the moduli problem for $Sh_{a*}$. We put $\Lambda_{b*} := V_{b*} \cap \gamma_{a*,b*}(\Lambda_{a*} \otimes \hat{\mathbb{Z}})$. Then applying the construction of Subsection 1.3 to the lattice $\Lambda_{b*} \subseteq V_{b*}$ and the open compact subgroup $K^p \subseteq G_{a*}(\mathbb{A}^\infty) \cong G_{b*}(\mathbb{A}^\infty)$, we get a Shimura variety $Sh_{b*}$ over $\mathbb{Z}_{p^f}$ of level $K^p$ as well as its special fiber $Sh_{b*}$. Moreover, an algebraic representation $\xi$ of $G_{a*}$ over $\mathbb{Q}_{\ell}$ corresponds, via the fixed isomorphism $G_{a*}(\mathbb{A}^\infty) \cong G_{b*}(\mathbb{A}^\infty)$, to an algebraic representation of $G_{b*}$ over $\mathbb{Q}_{\ell}$. We use the same notation $\mathcal{L}_\xi$ to denote the étale sheaf on $Sh_{a*}$ and $Sh_{b*}$ defined by $\xi$.

**Conjecture 1.9.** Let $Sh_{a*}$ and $\mathcal{L}_\xi$ be as in Subsection 1.4. Let $\lambda$ be a dominant weight (with respect to a fixed maximal torus $T$ of $GL_n$) appearing in the representation $r_{a*} = \bigotimes_{i=1}^{f} \wedge^{a_i} \text{Std}$ of $GL_n$, with multiplicity $m_\lambda(a*).$ Write $\lambda$ as a sum of minuscule weights and therefore realize it as the highest weight of the representation $r_{b*} = \bigotimes_{i=1}^{f} \wedge^{b_i} \text{Std}$ of $GL_n$ for some set of numbers $b*$ in $\{0, \ldots, n\}$. Note that $\sum a_i = \sum b_i$ so that the Shimura variety $Sh_{b*}$ makes sense, and the étale sheaf $\mathcal{L}_\xi$ is also well defined on $Sh_{b*}$.

Then there exist varieties $Y_1, \ldots, Y_{m(\lambda(a*))}$ of dimension $\frac{d(a*) + d(b*)}{2}$ over $\mathbb{F}_{p^f}$, equipped with natural action of prime-to-$p$ Hecke correspondences, such that each $Y_j$ admits fits into diagram

\[
\begin{array}{ccc}
Y_j & \xrightarrow{pr_{a*}} & Y_j \\
\downarrow & & \downarrow \\
Sh_{a*} & \xrightarrow{pr_{b*}} & Sh_{b*} \\
\end{array}
\]

satisfying the following properties.

1. Each $b^{(j)}_*$ is a reordering of $b*$, and both $pr_{a*}$ and $pr_{b*(j)}$ are equivariant under prime-to-$p$ Hecke correspondences.

2. The morphism $pr_{a*}$ is a proper morphism and is birational onto the image; the morphism $pr_{b*(j)}$, up to some partial Frobenius morphism, is proper and generically smooth of dimension $\frac{d(a*) - d(b*)}{2}$ (note that $d_{a*} \equiv \bar{d}_{a*} \pmod{2}$ since $\sum a_i = \sum b_i$).

---

9Here, by minuscule weights, we meant weights of $GL_n$ for which the diagonal matrix $\text{Diag}(x_1, \ldots, x_n)$ is sent to $x_1 \ldots x_i$ for some $i \in \{0, \ldots, n\}$. In particular, we do not allow extra powers of determinants.
(3) The pullbacks via repectively \( \text{pr}_{a^*} \) and \( \text{pr}_{b^*}\) of the universal abelian variety on \( \text{Sh}_{a^*} \) and that on \( \text{Sh}_{b^*} \) to \( Y_j \) are quasi-isogenous, so that we have an isomorphism of the \( \ell \)-adic sheaves

\[
\text{pr}_{a^*}^* \mathcal{L}_\xi \cong \text{pr}_{b^*}^* \mathcal{L}_\xi.
\]

(4) Let \( \pi \) be an irreducible admissible representation of \( G_{a^*}(\mathbb{A}^\infty) \cong G_{b^*}(\mathbb{A}^\infty) \) satisfying Hypothesis 1.5 for signatures \( a^* \) and \( b^* \) (for all \( j \)) and such that the automorphic multiplicity \( m_{a^*}(\pi) = m_{b^*}(\pi) \)\(^{10}\) Conjugate the representation \( \rho_\pi \) so that \( \rho_\pi(\text{Frob}_{p^f}) \in T(Q_\ell) \); then it makes sense to evaluate a weight of \( T \) on \( \rho_\pi(\text{Frob}_{p^f}) \).

If the evaluations of \( \rho_\pi(\text{Frob}_{p^f}) \) at all weights of \( T \) are distinct, then, for any element \( w \) in the Weyl group of \( \text{GL}_n \), the natural homomorphism of \( \pi \)-isotypical components of the cohomology groups

\[
\bigoplus_{j=1}^{m_{\lambda}(a^*)} H^d_{\text{et}}(\text{Sh}_{a^*}, \overline{\mathbb{Q}}_p, \mathcal{L}_\xi)(d_{a^*})^{\text{Frob}_{p^f} = w\lambda} \rightarrow \bigoplus_{j=1}^{m_{\lambda}(a^*)} H^d_{\text{et}}(Y_j, \overline{\mathbb{Q}}_p, \mathcal{L}_\xi)(d_{a^*}^{b^*})^{\text{Frob}_{p^f} = w\lambda}
\]

is an isomorphism, where \( \text{pr}_{b^*} \), is induced by the trace map

\[
\text{Rpr}_{b^*}(\overline{\mathbb{Q}}_\ell) \rightarrow \overline{\mathbb{Q}}_\ell(-\frac{d_{a^*} - d_{b^*}}{2}),
\]

and the superscript \( \text{Frob}_{p^f} = w\lambda \) means to take the \( \text{Frob}_{p^f} \)-eigenspace with eigenvalue

\[
w\lambda \circ \rho_\pi(\text{Frob}_{p^f}) \cdot \chi_{F_{p^f}, (p^f)}(\sqrt{p})^{-f(n-1)\sum_i a_i}.
\]

In particular, when \( \xi \) is the trivial representation and the weight \( \lambda \) is a power of determinant (so automatically, \( \sum_i a_i \) is divisible by \( n \), and \( d_{a^*} \) is even), the cycles given by the images of \( Y_1, \ldots, Y_{m_{\lambda}(a^*)} \) parametrized by the discrete Shimura variety \( \text{Sh}_{a^*} \), generate the Tate classes of \( H^d_{\text{et}}(\text{Sh}_{a^*}, \overline{\mathbb{Q}}_p, (d_{a^*})^{[\pi]} \) when the evaluation of \( \rho_\pi(\text{Frob}_{p^f}) \) at all weights of \( T \) are distinct.

**Remark 1.10.** We immediately make a few remarks, starting with a couple of crucial ones and then moving to discussions on assumptions imposed in our setup and possible generalizations.

(1) A key feature of this Conjecture is that the codimension of the cycle map \( \text{pr}_{a^*} : Y_j \rightarrow \text{Sh}_{a^*} \) is the same as the fiber dimension of \( \text{pr}_{b^*} : Y_j \rightarrow \text{Sh}_{b^*} \).

(2) It seems that the fibers of \( \text{pr}_{b^*} : Y_j \rightarrow \text{Sh}_{b^*} \) are likely to be, up to partial Frobenius twist and birational transform, certain “iterated Deligne-Lusztig” varieties, that is, a tower of fibers, where the fiber at each step is certain Deligne-Lusztig varieties. (This phenomenon could be because we are essentially working with the group \( \text{GL}_n \).)

(3) Zhu pointed out to us that the union of the images of \( Y_1, \ldots, Y_{m_{\lambda}(a^*)} \) on \( \text{Sh}_{a^*} \) is expected to be exactly the closure of the Newton strata with slopes given by \( \lambda \). In fact, this is not that surprising because the generic Newton polygon of \( \text{Sh}_{b^*} \) is just given by \( \lambda \); so the universal abelian variety over the images of \( Y_j \) should have Newton polygon above \( \lambda \).

When \( \lambda \) is central, the Conjecture should read: irreducible components of the superbasic locus of the special fiber of a Shimura variety, generically, contribute to all Tate cycles in the cohomology. Implicitly, this means that the dimension of the superbasic locus is half of the dimension of the Shimura variety if and only if the Galois representations of the Shimura variety has generically non-trivial Tate classes.

\(^{10}\)It is expected that all \( m_{b^*}(\pi) \) and \( m_{a^*}(\pi) \) are 1.
Zhu further pointed out that the “correct way” to understand these varieties $Y_j$’s is that they are part of the Hecke correspondence at $p$ between the special fiber of two different Shimura varieties $Sh_{a\bullet}$ and $Sh_{b(j)}\bullet$. We hope that the Conjecture will bring interests into the study of such Hecke correspondences.

The assumption on the decomposition of the place $p$ in $E/\mathbb{Q}$ is just to simplify our presentation. We certainly expect the validity of analogous conjecture for the special fibers of general Shimura varieties of abelian type. Zhu pointed out to us that even if $p$ is ramified, we should expect Conjecture 1.9 continue to hold for (the special fiber of) the “splitting models” of Pappas and Rapoport \cite{PR05}. Some evidence of this has already appeared in the case of Hilbert modular varieties; see \cite{RX14+}.

The assumption that $D$ is a division algebra ensures that the Shimura variety will be compact and its cohomology does not have any endoscopy component. It would be certainly an interesting future question to study the case involving Eisenstein series and representations arising from endoscopy transfers.

One can certainly formulate the conjecture for more general Shimura varieties of PEL type (e.g. of type $C$), or more generally of abelian type (using the integral model of M. Kisin \cite{Ki12}), with one principle: the existence of special cycles are predicted by the Galois representation appearing on the Shimura variety; and with one caveat: the conjecture seems to be restricted to those $\lambda$ which are sum of minuscule weights (as required by Shimura varieties). It would certainly be an interesting future question to explore the non-minuscule weights case.

Conjecture 1.9(1)–(3) may have a degenerate situation: when $\sum_i a_i$ is not divisible by $n$, the representation $V_{a\bullet}$ does not contain a weight corresponding to an (integral) power of the determinant. In this case, the supersingular locus may still have a good description as, birationally, a union of fiber bundles over the special fiber of some other Shimura varieties for reductive groups which are not quasi-split at $p$. The first such example is given by Serre and Deuring \cite{Se96} in the case of modular curves, and more such examples are given in \cite{TX13+} and \cite{VW11}.

1.11. Known cases of Conjecture 1.9. Conjecture 1.9 is largely inspired by the work of second and the third authors \cite{TX13+} and \cite{TX14+}, where they proved the analogous conjecture for (the special fibers of) the Hilbert modular varieties (assuming that $p$ is inert in the totally real field).

Another strong evidence that motivates us to extend Conjecture 1.9 to the PEL case is the work of Vollaard and Wedhorn \cite{VW11}, where they considered certain stratification of the supersingular locus of the Shimura variety for $GU(1, s)$ for $s \in \mathbb{N}$. (They assumed $p$ to be inert in $E$ so the situation is slightly different.) They proved that the supersingular locus is the union of copies of some Deligne-Lusztig varieties; however, they were looking at a very different way to parametrize these Deligne-Lusztig varieties. The method of this paper should be applicable to their situation to verify the analogous Conjecture 1.9 (modulo a calculation of certain intersection numbers on the relevant Deligne-Lusztig variety). In fact, in their case, there will be only one collection of cycles, but the computation of the intersection matrix (only essentially one entry in this case) of them needs some hard-core Schubert calculus.

The aim of the rest of paper is to provide evidences for Conjecture 1.9 for some large rank groups. In particular, we will construct cycles in the case of the unitary group $G(U(r, s) \times U(s, r))$ for $s, r \in \mathbb{N}$ (Section 6); we expect these cycles to verify Conjecture 1.9 but we do not know how to compute the “intersection matrix” in general. Nonetheless, when $r = 1$, we are able to make the computation and prove Conjecture 1.9 (with trivial coefficients for the sake of simple presentation) in this case. (Section 3–5) We point out that our method should be applicable to quite many other examples, and even in general reduce Conjecture 1.9 to a question of combinatorial nature.
Unfortunately, it is not clear to us how one can get around these combinatorial difficulties. In the Hilbert case [TX14+], the combinatorial object is the so-called periodic semi-meander (for $GL_2$). The generalization of the usual (as opposed to periodic) semi-meander to other groups has been introduced; see [FKK14] for the corresponding references. The straightforward generalization to the periodic case does seem to agree with some of our computations with small groups. Nonetheless, the corresponding Gram determinant formula seems to be extremely difficult; even in the non-periodic case, we only know it for a special case; see [dF97].

We also mention that one can formulate a version of Conjecture 1.9 in the geometric Langlands setup. There has been some progress in this direction, which will appear in a forthcoming work of Zhu and the third author [XZ13+].

2. Toolbox

We first introduce the basic tools that we will use in this paper.

2.1. Notation. Recall that we have an isomorphism

$$\mathcal{O}_D \otimes_\mathbb{Z} \mathbb{Z}_p^f \cong \bigoplus_{i=1}^f (\mathcal{O}_D \otimes \mathcal{O}_{E,q_i} \mathbb{Z}_p^f \oplus \mathcal{O}_D \otimes \mathcal{O}_{E,q_i} \mathbb{Z}_p^f) \cong \bigoplus_{i=1}^f (M_n(\mathbb{Z}_p^f) \oplus M_n(\mathbb{Z}_p^f)).$$

Let $S$ be a locally noetherian $\mathbb{Z}_p^f$-scheme. An $\mathcal{O}_D \otimes_\mathbb{Z} \mathcal{O}_S$-module $M$ admits a canonical decomposition

$$M = \bigoplus_{i=1}^f (M_{q_i} \oplus M_{\bar{q}_i}),$$

where $M_{q_i}$ (resp. $M_{\bar{q}_i}$) is the direct summand of $M$ on which $\mathcal{O}_S$ acts via $q_i$ (resp. via $\bar{q}_i$). Then each $M_{q_i}$ has a natural action by $M_n(\mathcal{O}_S)$. Let $\epsilon$ denote the element of $M_n(\mathcal{O}_S)$ whose $(1,1)$-entry is 1 and other entries are 0. We put $M_i^\epsilon := \epsilon M_{q_i}$, and call it the reduced part of $M_{q_i}$.

Let $A$ be an $fn^2$-dimensional abelian variety over an $\mathbb{F}_p^f$-scheme $S$, equipped with an $\mathcal{O}_D$-action. The de Rham homology $H_1^{dR}(A/S)$ has a Hodge filtration

$$0 \to \omega_{A^{\vee}/S} \to H_1^{dR}(A/S) \to \text{Lie}_{A/S} \to 0,$$

compatible with the natural action of $\mathcal{O}_D \otimes_\mathbb{Z} \mathcal{O}_S$ on $H_1^{dR}(A/S)$. We call $H_1^{dR}(A/S)^\epsilon$ and $\omega_{A^{\vee}/S,i}^\circ$ respectively the reduced de Rham homology of $A/S$ and the reduced invariant 1-forms of $A^{\vee}/S$ (at $q_i$). In particular, the former is a locally free $\mathcal{O}_S$-module of rank $n$ and the latter is a subbundle of the former; when $A \to S$ satisfies the moduli problem in Subsection 1.3, $\omega_{A^{\vee}/S,i}^\circ$ is locally free of rank $a_q$.

The Frobenius map $A \to A^{(p)}$ induces a natural morphism

$$V : H_1^{dR}(A/S)^\circ \to H_1^{dR}(A/S)^{\circ, (p)}_{i-1},$$

where the index $i$ is considered as an element of $\mathbb{Z}/f\mathbb{Z}$, and the superscript “$(p)$” means the pull-back via the absolute Frobenius of $S$. The image of $V$ is exactly $\omega_{A^{\vee}/S,i}^{\circ, (p)}$. Similarly, the Verschiebung map $A^{(p)} \to A$ induces a natural morphism

$$F : H_1^{dR}(A/S)^{\circ, (p)}_{i-1} \to H_1^{dR}(A/S)^\circ_i.$$
The $\mathcal{O}_D$-action on $A$ induces a natural action of $\mathcal{O}_D$ on $\tilde{D}(A)$ that commutes with $F$ and $V$. Moreover, there is a canonical isomorphism $\tilde{D}(A)/p\tilde{D}(A) \simeq H^1_{dR}(A/k)$ compatible with all structures on both sides. For each $i \in \mathbb{Z}/f\mathbb{Z}$, we have the reduced part $\tilde{D}(A)_i = c\tilde{D}(A)_{qi}$. The Verschiebung and the Frobenius induce natural maps

$$V : \tilde{D}(A)_i^0 \to \tilde{D}(A)_{i-1}^0, \quad F : \tilde{D}(A)_i^0 \to \tilde{D}(A)_{i+1}^0.$$ 

Note that $\tilde{D}(A)_{qi} = (\tilde{D}(A)_i^0)^{\oplus n}$, and $\bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} \tilde{D}(A)_{qi}$ is the covariant Dieudonné module of the $p$-divisible group $A[p^{\infty}]$.

For any $fn^2$-dimensional abelian variety $A'$ over $k$ equipped with an $\mathcal{O}_D$-action, an $\mathcal{O}_D$-equivariant isogeny $A' \to A$ induces a morphism $\tilde{D}(A')_i^0 \to \tilde{D}(A)_i^0$ compatible with the action of $F$ and $V$. Conversely, we have the following

**Proposition 2.2.** Let $A$ be an abelian variety of dimension $fn^2$ over a characteristic $p$ prefect field $k$, equipped with an $\mathcal{O}_D$-action and an $\mathcal{O}_D$-compatible prime-to-$p$ polarization $\lambda$. Suppose given an integer $m \geq 1$ and a $W(k)$-submodule $\tilde{E}_i \subseteq \tilde{D}(A)_i^0$ for $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$(2.2.1) \quad p^m\tilde{D}(A)_i^0 \subseteq \tilde{E}_i, \quad F(\tilde{E}_i) \subseteq \tilde{E}_{i+1}, \quad \text{and} \quad V(\tilde{E}_i) \subseteq \tilde{E}_{i-1}.$$ 

Then there exists a unique abelian variety $A'$ over $k$ (depending on $m$) equipped with an $\mathcal{O}_D$-action, a prime-to-$p$ polarization $\lambda'$, and an $\mathcal{O}_D$-equivariant $p$-isogeny $\phi : A' \to A$ such that the natural inclusion $\tilde{E}_i \subseteq \tilde{D}(A')_i^0$ is naturally identified with the map $\phi_{*,i} : \tilde{D}(A')_i^0 \to \tilde{D}(A)_i^0$ induced by $\phi$ and such that $\phi^\vee \circ \lambda' \circ \phi = p^m\lambda'$. Moreover, we have

1. If $\dim \omega_{A'/k,i} = a_i$ and $\length_{W(k)}(\tilde{D}(A')_i^0/\tilde{E}_i) = \ell_i$ for $i \in \mathbb{Z}/f\mathbb{Z}$, then

$$\dim \omega_{A'/k,i} = a_i + \ell_i - \ell_{i+1}.$$ 

2. If $A$ is equipped with a prime-to-$p$ level structure $\eta$, then there exists a unique prime-to-$p$ level structure $\eta'$ on $A'$ such that $\eta = \phi \circ \eta'$.

**Proof.** By the Dieudonné theory, the Dieudonné submodules

$$\bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} (\tilde{E}_i/p^m\tilde{D}(A)_i^0)^{\oplus n} \subseteq \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} (\tilde{D}(A)_i^0/p^m\tilde{D}(A)_i^0)^{\oplus n}$$

corresponds to a closed subgroup scheme $H_p \subseteq A[p^m]$. The prime-to-$p$ polarization $\lambda$ induces a perfect pairing

$$\langle \bullet, \bullet \rangle : A[p^m] \times A[p^m] \to \mu_{p^m}.$$ 

Let $H_p^\perp = H_p^\perp \subseteq A[p^m]$ denote the orthogonal complement of $H_p$. Put $H_p = H_p \oplus H_p^\perp$. Let $\psi : A \to A'$ be the canonical quotient with kernel $H_p$, and $\phi : A' \to A$ be the quotient with kernel $\psi(A[p^m])$ so that $\psi \circ \phi = p^m\id_{A'}$ and $\phi \circ \psi = p^m\id_A$. By construction, $H_p \subseteq A[p^m]$ is a maximal totally isotropic subgroup. By [Mu73, §23, Theorem 2], there is a prime-to-$p$ polarization $\lambda'$ on $A'$ such that $p^m\lambda = \psi^\vee \circ \lambda' \circ \psi$. It follows also that $p^m\lambda' = \phi^\vee \circ \lambda \circ \phi$. The fact that $\phi_{*,i} : \tilde{D}(A')_i^0 \to \tilde{D}(A)_i^0$ is identified with the natural inclusion $\tilde{E}_i \subseteq \tilde{D}(A)_i^0$ follows from the construction. The existence and uniqueness of tame level structure is clear. The dimension of the differential forms can be computed as follows:

$$\dim \omega_{A'/k,i} = \dim \frac{V(\tilde{D}(A')_i^0)}{p\tilde{D}(A')_i^0} = \dim \frac{V(\tilde{E}_{i+1})}{p\tilde{E}_i}$$

$$= \dim \frac{V(\tilde{D}(A)_{i+1})}{p\tilde{D}(A)_i^0} - \length_{W(k)} \frac{V(\tilde{D}(A)_{i+1})}{V(\tilde{E}_{i+1})} + \length_{W(k)} \frac{p\tilde{D}(A)_i^0}{p\tilde{E}_i} = a_i - \ell_{i+1} + \ell_i.$$ 

$\square$
2.3. **Deformation theory.** We shall frequently use Grothendieck-Messing deformation theory to compare the tangent spaces of moduli spaces. We make this explicit in our setup.

Let \( \hat{R} \) be a noetherian \( \mathbb{F}_{p^2} \)-algebra and \( \hat{I} \subset \hat{R} \) an ideal such that \( \hat{I}^2 = 0 \). Put \( R = \hat{R}/\hat{I} \). Let \( \mathcal{C}_R \) denote the category of tuples \((A,\hat{\lambda},\hat{\eta})\), where \( A \) is an \( fn^2 \)-dimensional abelian variety over \( \hat{R} \) equipped with an \( O_D \)-action, \( \hat{\lambda} \) is a polarization on \( \hat{A} \) such that the Rosati involution induces the \(*\)-involution on \( O_D \), and \( \hat{\eta} \) is a level structure as in Subsection I.3(3). We define \( \mathcal{C}_R \) in the same way. For an object \((A,\lambda,\eta)\) in the category \( \mathcal{C}_R \), let \( H^1_{\text{cris}}(A/\hat{R}) \) be the evaluation of the first relative crystalline homology (i.e. dual crystal of the first crystalline cohomology) of \( A/R \) at the divided power thickening \( \hat{R} \rightarrow R \), and \( H^1_{\text{cris}}(A/R) \hat{i} := \tau H^1_{\text{cris}}(A/\hat{R}) \hat{i} \) be the i-th reduced part. We denote by \( \text{Def}(R,\hat{R}) \) the category of tuples \((A,\lambda,\eta,\hat{\omega}_i)_{i=1,...,f} \), where \((A,\lambda,\eta)\) is an object in \( \mathcal{C}_R \), and \( \hat{\omega}_i \subseteq H^1_{\text{cris}}(A/\hat{R}) \) for \( i \in \mathbb{Z}/f\mathbb{Z} \) is a sub-bundle that lifts \( \omega^\circ_{A/\hat{R},i} \subseteq H^1_{\text{dR}}(A/R) \). The following is a combination of Serre-Tate and Grothendieck-Messing deformation theory.

**Theorem 2.4** (Serre-Tate, Grothendieck-Messing). The functor \((\hat{A},\hat{\lambda},\hat{\eta}) \mapsto (A \otimes_R R,\lambda,\eta,\omega^\circ_{A/\hat{R},i})\), where \( \lambda \) and \( \eta \) are the natural induced polarization and level structure on \( \hat{A} \otimes_R R \), is an equivalence of categories between \( \mathcal{C}_R \) and \( \text{Def}(R,\hat{R}) \).

**Proof.** The main theorem of the crystalline deformation theory (cf. [Gr74], pp. 116–118), [MM90, Chap. II §1]) says that the category \( \mathcal{C}_R \) is equivalent to the category of objects \((A,\lambda,\eta)\) in \( \mathcal{C}_R \) together with a lift of \( \omega_{A/\hat{R}} \subseteq H^1_{\text{cris}}(A/R) \) to a subbundle \( \hat{\omega} \) of \( H^1_{\text{cris}}(A/\hat{R}) \), such that \( \hat{\omega} \) is stable under the induced \( O_D \)-action and is isotropic for the pairing on \( H^1_{\text{cris}}(A/R) \) induced by the polarization \( \lambda \). The additional information \( \hat{\omega} \) is clearly equivalent to the subbundle \( \hat{\omega}_i \subseteq H^1_{\text{cris}}(A/\hat{R}) \).

**Corollary 2.5.** If \( A_{\bullet} \) denote the universal abelian variety over \( \text{Sh}_{a_{\bullet}} \), then the tangent space \( T_{\text{Sh}_{a_{\bullet}}} \) of \( \text{Sh}_{a_{\bullet}} \) is

\[
\bigoplus_{i=1}^f \text{Lie}^\circ_{A_{\bullet,i}/\text{Sh}_{a_{\bullet}},i} \otimes \text{Lie}^\circ_{A_{\bullet}/\text{Sh}_{a_{\bullet}},i}.
\]

**Proof.** Let \( \hat{R} \) be a noetherian \( \mathbb{F}_{p^2} \)-algebra and \( \hat{I} \subset \hat{R} \) an ideal such that \( \hat{I}^2 = 0 \); put \( R = \hat{R}/\hat{I} \). By Theorem 2.4 to lift an \( R \)-point \((A,\lambda,\eta)\) of \( \text{Sh}_{a_{\bullet}} \) to an \( \hat{R} \)-point, it suffices to lift, for \( i = 1, \ldots, f \), the differentials \( \omega^\circ_{A/\hat{R},i} \subseteq H^1_{\text{cris}}(A/\hat{R}) \) to a subbundle \( \hat{\omega}_i \subseteq H^1_{\text{cris}}(A/\hat{R}) \). Such lifts form a torsor for the group

\[ \text{Hom}_R(\omega^\circ_{A/\hat{R},i},\text{Lie}^\circ_{A/\hat{R},i}) \otimes_R \hat{I}. \]

It follows from this

\[
T_{\text{Sh}_{a_{\bullet}}} \cong \bigoplus_{i=1}^f \text{Hom}(\omega^\circ_{A_{\bullet,i}/\text{Sh}_{a_{\bullet}},i},\text{Lie}^\circ_{A_{\bullet,i}/\text{Sh}_{a_{\bullet}},i}) \cong \bigoplus_{i=1}^f \text{Lie}^\circ_{A_{\bullet,i}/\text{Sh}_{a_{\bullet}},i} \otimes \text{Lie}^\circ_{A_{\bullet}/\text{Sh}_{a_{\bullet}},i}.
\]

Note that this proof also shows that \( \text{Sh}_{a_{\bullet}} \) is smooth. \( \square \)

2.6. **Notation in the real quadratic case.** For the rest of the paper, we assume \( f = 2 \) so that \( F \) is a real quadratic field in which \( p \) is inert. For non-negative integers \( r \leq s \) such that \( n = r + s \), we denote by \( G_{r,s} \) the algebraic group previously denoted by \( G_{a_{\bullet}} \) with \( a_1 = r \) and \( a_2 = s \); in particular, \( G_{r,s}(\mathbb{R}) = G(U(r, s) \times U(s, r)) \). If \( r', s' \) is another pairs of non-negative integers such that \( n = r' + s' \) \( r' \leq s' \), Lemma 1.8 gives an isomorphism \( G_{r,s}(\mathbb{A}^\infty) \cong G_{r',s'}(\mathbb{A}^\infty) \).

Let \( \text{Sh}_{r,s} \) be the Shimura variety over \( \mathbb{Z}/p^2 \) attached to \( G_{r,s} \) defined in Subsection 1.3 of some fixed sufficiently small prime-to-\( p \) level \( K^p \subseteq G_{r,s}(\mathbb{A}^\infty) \). Let \( \text{Sh}_{r,s} \) denote its special fiber over \( \mathbb{F}_{p^2} \). Let \( A = A_{r,s} \) denote the universal abelian variety over \( \text{Sh}_{r,s} \); it is a \( 2n^2 \)-dimensional abelian
variety, equipped with an action of $\mathcal{O}_D$ and a prime-to-$p$ polarization $\lambda_A$. Moreover, $\omega^{\sigma}_{\mathcal{A}^\vee/Sh_{r,s,1}}$ (resp. $\omega^{\sigma}_{\mathcal{A}^\vee/Sh_{r,s,2}}$) is a locally free module over $\text{Sh}_{r,s}$ of rank $r$ (resp. rank $s$).

3. **The Case of $G(U(1, n-1) \times U(n-1,1)$**

We will verify Conjecture 1.9 for $\text{Sh}_{1,n-1}$, namely the existence of some cycles $Y_j$ having morphisms to both $\text{Sh}_{0,n}$ and $\text{Sh}_{1,n-1}$ and generating Tate classes of $\text{Sh}_{1,n-1}$ under certain hypothesis on the Satake parameters. We always fix an isomorphism $G_{1,n-1}(\mathbb{A}^\infty) \cong G_{0,n}(\mathbb{A}^\infty)$, write $G(\mathbb{A}^\infty)$ for either group.

3.1. **Cycles on $\text{Sh}_{1,n-1}$**. For each integer $j$ with $1 \leq j \leq n$, we have a moduli space $Y_j$ over $\mathbb{F}_{p^2}$ that associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme $S$, the set of tuples $(A, \lambda, \eta, B, \lambda', \eta', \phi)$, where

- $(A, \lambda, \eta)$ is an $S$-point of $\text{Sh}_{1,n-1}$,
- $(B, \lambda', \eta')$ is an $S$-point of $\text{Sh}_{0,n}$, and
- $\phi : B \rightarrow A$ is an isogeny with kernel contained in $B[p]$

such that

- $p\lambda' = \phi^* \lambda \circ \phi$,
- $\phi \circ \eta' = \eta$, and
- the cokernel of the maps

$$\phi_{s,1} : H^1_{dR}(B/S)_1 \rightarrow H^1_{dR}(A/S)_1$$

and

$$\phi_{s,2} : H^1_{dR}(B/S)_2 \rightarrow H^1_{dR}(A/S)_2$$

are locally free $\mathcal{O}_S$-modules of rank $j - 1$ and $j$, respectively.

Note that there is a unique isogeny $\psi : A \rightarrow B$ such that $\psi \circ \phi = p \cdot \text{id}_B$ and $\phi \circ \psi = p \cdot \text{id}_A$. We have

$$\text{Ker}(\phi_{s,i}) = \text{Im}(\psi_{s,i}) \quad \text{and} \quad \text{Ker}(\phi_{s,i}) = \text{Im}(\psi_{s,i}),$$

where $\psi_{s,i}$ for $i = 1, 2$ is the induced morphism on the reduced de Rham homology in the evident sense. We have a diagram of morphisms:

$$\begin{array}{ccc}
\text{Sh}_{1,n-1} & \xrightarrow{\text{pr}_j} & Y_j \\
\text{pr}_j' & & \downarrow \\
\text{Sh}_{0,n} & & 
\end{array}$$

where $\text{pr}_j$ and $\text{pr}_j'$ send a tuple $(A, \lambda, \eta, B, \lambda', \eta', \phi)$ to $(A, \lambda, \eta)$ and $(B, \lambda', \eta')$, respectively. Letting $K^p$ vary, we see easily that both $\text{pr}_j$ and $\text{pr}_j'$ are equivariant under prime-to-$p$ Hecke actions given by the double cosets $K^p \setminus G(\mathbb{A}^{\infty,p}) / K^p$.

3.2. **Some Auxiliary Moduli Spaces**. The moduli problem for $Y_j$ is slightly complicated. We will introduce a more explicit moduli space $Y'_j$ below and then show they are isomorphic.

Consider the functor $Y'_j$ which associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme $S$, the set of tuples $(B, \lambda', \eta', H_1, H_2)$, where

- $(B, \lambda', \eta')$ is an $S$-valued point of $\text{Sh}_{0,n}$;
- $H_1 \subset H^1_{dR}(B/S)_1$ and $H_2 \subset H^1_{dR}(B/S)_2$ are local direct factors of rank $j$ and $j - 1$ respectively such that

$$(3.2.1) \quad V^{-1}(H^1_{dR}(p)) \subseteq H_1, \quad H_2 \subseteq F(H^1_{dR}(p)),$$

and the quotients $H_1/V^{-1}(H^1_{dR}(p))$ and $F(H^1_{dR}(p))/H_2$ are both locally free $\mathcal{O}_S$-modules (of rank one). Here, $F : H^1_{dR}(B/S)_1 \cong H^1_{dR}(B/S)_2$ and $V : H^1_{dR}(B/S)_1 \cong H^1_{dR}(B/S)_2$ are respectively the Frobenius and Verschiebung homomorphisms, which are actually isomorphisms because of the signature condition.
There is a natural projection \( \pi'_j : Y_j' \to \text{Sh}_{0,n} \) given by \( (B, \lambda', \eta', H_1, H_2) \mapsto (B, \lambda', \eta') \).

**Proposition 3.3.** The functor \( Y_j' \) is representable by a scheme \( Y_j' \) projective and smooth over \( \text{Sh}_{0,n} \) of dimension \( n - 1 \). Moreover, if \( (B, \lambda', \eta', H_1, H_2) \) denotes the universal object of \( Y_j' \), then the tangent bundle of \( Y_j' \) is

\[
T_{Y_j'} \cong ((H_1/V^{-1}(H_2^{(p)}))^s \otimes (H_1^{dR}(B/\text{Sh}_{0,n})_o/H_1)) \oplus (H_2 \otimes F(H_1^{(p)})/H_2).
\]

Here, for a coherent \( \mathcal{O}_{Y_j'} \)-module \( M \), we put \( M^* = \text{Hom}_{\mathcal{O}_{Y_j'}}(M, \mathcal{O}_{Y_j'}) \).

**Proof.** For each integer \( m \) with \( 0 \leq m \leq n \) and \( i = 1, 2 \), let \( \text{Gr}(H_1^{dR}(B/\text{Sh}_{0,n})_o^i, m) \) be the Grassmannian scheme over \( \text{Sh}_{0,n} \) that parametrizes subbundles of the universal Dieudonné module \( H_1^{dR}(B/\text{Sh}_{0,n})_o^i \) of rank \( m \). Then \( Y_j' \) is a closed subfunctor of the product of the Grassmannian schemes

\[
\text{Gr}(H_1^{dR}(B/\text{Sh}_{0,n})_o^1, j) \times \text{Gr}(H_1^{dR}(B/\text{Sh}_{0,n})_o^2, j - 1).
\]

The representability of \( Y_j' \) follows. Moreover, \( Y_j' \) is projective.

We show now that the structural map \( \pi'_j : Y_j' \to \text{Sh}_{0,n} \) is smooth of relative dimension \( n - 1 \). Let \( S_0 \to S \) be an immersion of \( \mathbb{F}_p \)-schemes with ideal \( I \) satisfying \( I^2 = 0 \). Suppose we are given a commutative diagram

\[
\begin{array}{ccc}
S_0 & \xrightarrow{g_0} & Y_j' \\
\downarrow{g} & & \downarrow{\pi'_j} \\
S & \xrightarrow{h} & \text{Sh}_{0,n}
\end{array}
\]

with solid arrows. We have to show that there is a morphism \( g \) as the dotted arrow that makes the whole diagram commute. Let \( B \) be the abelian scheme over \( S \) given by \( h \), and \( B_0 \) be the base change to \( S_0 \). The morphism \( g_0 \) gives rise to subbundles \( \overline{H}_1 \subset H_1^{dR}(B_0/S_0)_o^1 \) and \( \overline{H}_2 \subset H_1^{dR}(B_0/S_0)_o^2 \) with

\[
F(\overline{H}_1^{(p)}) \supset \overline{H}_2, \quad V_1(\overline{H}_2^{(p)}) \subset \overline{H}_1.
\]

Finding \( g \) is equivalent to finding subbundles \( H_i \subset H_1^{dR}(B/S)_o^i \) for \( i = 1, 2 \) which lift \( \overline{H}_j \)'s and satisfy (3.2.1). We note that \( F_s : \mathcal{O}_S \to \mathcal{O}_S \) factors through \( \mathcal{O}_{S_0} \). Hence \( V_1(H_2^{(p)}) \) and \( F(H_1^{(p)}) \) actually depend only on \( \overline{H}_1, \overline{H}_2 \), but not on the lifts \( H_1 \) and \( H_2 \). Therefore, the possible lifts \( H_2 \) form a torsor under the group

\[
\text{Hom}_{\mathcal{O}_{S_0}}(\overline{H}_2, F(\overline{H}_1^{(p)}/\overline{H}_2)) \otimes_{\mathcal{O}_{S_0}} I,
\]

and similarly the possible \( H_1 \)'s form a torsor under the group

\[
\text{Hom}_{\mathcal{O}_{S_0}}(\overline{H}_1/V^{-1}(\overline{H}_2^{(p)}), H_1^{dR}(B_0/S_0)_o/\overline{H}_1) \otimes_{\mathcal{O}_{S_0}} I.
\]

This proves that \( \pi'_j : Y_j' \to \text{Sh}_{0,n} \) is formally smooth, and hence smooth. To see that \( Y_j' \) has dimension \( n - 1 \), we take \( S_0 = y = \text{Spec}(k) \) with \( k \) a perfect field of characteristic \( p \) and \( S = \text{Spec}(k[e]/e^2) \). The arguments above show that the tangent space of the fiber \( Y_j' \) at \( y \) is

\[
T_{Y_j', y} \cong \text{Hom}_k(\overline{H}_2, F(\overline{H}_1^{(p)}/\overline{H}_2)) \oplus \text{Hom}_k(\overline{H}_1/V^{-1}(\overline{H}_2^{(p)}), H_1^{dR}(B_0/S_0)_o/\overline{H}_1),
\]

which has dimension \( j - 1 + (n - j) = n - 1 \). The description of the tangent bundle \( T_{Y_j'} \) is clear from the arguments above. \( \square \)
Remark 3.4. Let \((B, \lambda', \eta', H_1, H_2)\) be an \(S\)-point of \(Y'\).

(a) If \(j = n\), \(H_1\) has to be \(H_1^{\text{DR}}(B/S)\), and \(H_2\) is a hyperplane of \(H_1^{\text{DR}}(B/S)^2\). Condition \((3.2.1)\) is trivial. In this case, \(Y'_n\) is the projective space over \(\text{Sh}_{0,n}\) associated to \(H_1^{\text{DR}}(B/\text{Sh}_{0,n})^2\), where \(B\) is the universal abelian scheme over \(\text{Sh}_{0,n}\). So it is geometrically a union of \(\mathbb{P}^{n-1}_{\mathbb{F}_p}\).

(b) If \(j = 1\), then \(H_1\) is a line in \(H_1^{\text{DR}}(B/S)^1\) and \(H_2 = 0\). So \(Y_1'\) is the projective space over \(\text{Sh}_{0,n}\) associated to \(H_1^{\text{DR}}(B/\text{Sh}_{0,n})^1\).

(c) If \(j = 2\), \(H_2 \subseteq H_1^{\text{DR}}(B/S)^2\) is a line, and \(H_1 \subseteq H_1^{\text{DR}}(B/S)^1\) is sub-bundle of rank 2 such that \(F(H_1^{(p)})\) contains both \(H_2\) and \(F(V^{-1}(H_2^{(p)}))^{(p)}\). Therefore, if \(H_2 \neq F(V^{-1}(H_2^{(p)}))^{(p)}\), \(H_1\) is determined up to Frobenius pull-back. If \(H_2 \neq F(V^{-1}(H_2^{(p)}))^{(p)}\), then \(H_1\) might be any rank 2 sub-bundle containing \(V^{-1}(H_2^{(p)})\).

We fix a geometric point \(z = (B, \lambda', \eta') \in \text{Sh}_{0,n}(\mathbb{F}_p)\). It is possible to find good basis for \(H_1^{\text{DR}}(B/\mathbb{F}_p)^1\) and \(H_1^{\text{DR}}(B/\mathbb{F}_p)^2\) such that both \(F, V : H_1^{\text{DR}}(B/\mathbb{F}_p)^1 \to H_1^{\text{DR}}(B/\mathbb{F}_p)^2\) are given by the identity matrix. With these choices, we may identify the fiber \(Y'_{2,z} = \pi^{-1}(z)\) with a closed subvariety of

\[\text{Gr}(\mathbb{P}^n_{\mathbb{F}_p}, 2) \times \text{Gr}(\mathbb{P}^1_{\mathbb{F}_p}, 1).\]

Moreover, one may equip \(\text{Gr}(\mathbb{P}^n_{\mathbb{F}_p}, 1) \cong \mathbb{P}^{n-1}_{\mathbb{F}_p}\) with an \(\mathbb{F}_p^*\)-rational structure such that \(H_2 = F(V^{-1}(H_2^{(p)}))^{(p)}\) if and only if \([H_2] \in \mathbb{P}^{n-1}_{\mathbb{F}_p}\) is an \(\mathbb{F}_p^*\)-rational point. So \(Y'_{2,y}\) is isomorphic to a “Frobenius-twisted” blow-up of \(\mathbb{P}^{n-1}_{\mathbb{F}_p}\) at all of its \(\mathbb{F}_p^*\)-rational points. Here, “Frobenius-twisted” means that each irreducible component of the exceptional divisor has multiplicity \(p\). For instance, when \(n = 3\), each \(Y_{2,z}\) is isomorphic to the closed subscheme of \(\mathbb{P}^2_{\mathbb{F}_p} \times \mathbb{P}^2_{\mathbb{F}_p}\) defined by

\[\begin{align*}
   a_1b_1^p + a_2b_2^p + a_3b_3^p &= 0 \\
   a_1^pb_1 + a_2^pb_2 + a_3^pb_3 &= 0,
\end{align*}\]

where \((a_1 : a_2 : a_3)\) and \((b_1 : b_2 : b_3)\) are the homogeneous coordinates on the two copies of \(\mathbb{P}^2\).

Lemma 3.5. Let \((A, \lambda, \eta, B, \lambda', \eta', \phi)\) be an \(S\)-valued tuple of \(Y_j\). Then the image of \(\phi_{*,1}\) contains both \(\omega^\circ_{A^V/S_1}\) and \(F(H_1^{\text{DR}}(A/S)_1^{(p)})\), and the image of \(\phi_{*,2}\) is contained in both \(\omega^\circ_{A^V/S_2}\) and \(F(H_1^{\text{DR}}(A/S)_1^{(p)})\).

Proof. By the functoriality, \(\phi_{*,1}\) sends \(\omega^\circ_{B^V/S_2}\) to \(\omega^\circ_{A^V/S_2}\). Since \(\omega^\circ_{B^V/S_2} = H_1^{\text{DR}}(B/S)^2\) by the Kottwitz’s determinant condition, it follows that \(\text{Im}(\phi_{*,1})\) in contained in \(\omega^\circ_{A^V/S_2}\). Similar arguments by considering \(\psi_{*,1}\) shows that \(\omega^\circ_{A^V/S_1} \subseteq \text{Ker}(\psi_{*,1}) = \text{Im}(\phi_{*,1})\). The fact that \(\text{Im}(\phi_{*,2})\) is contained in \(F(H_1^{\text{DR}}(A/S)_1^{(p)})\) follows from the commutative diagram:

\[\begin{align*}
   H^{\text{dR}}(B/S)_1^{(p)} &\xrightarrow{\phi_{*,1}^{(p)}} H^{\text{dR}}(A/S)_1^{(p)} \\
   F &\cong F \\
   H^{\text{dR}}(B/S)_2 &\xrightarrow{\phi_{*,2}} H^{\text{dR}}(A/S)_2^{(p)},
\end{align*}\]

and the fact that the left vertical arrow is an isomorphism. The inclusion \(F(H_1^{\text{DR}}(A/S)_2^{(p)}) \subseteq \text{Im}(\phi_{*,1}) = \text{Ker}(\psi_{*,1})\) can be proved similarly using the functoriality of Verschiebung homomorphism. \(\square\)
3.6. Morphism from $Y_j$ to $Y'_j$. There is a natural morphism $\alpha : Y_j \to Y'_j$ for $1 \leq j \leq n$ defined as follows. For a tuple $(A, \lambda, \eta, B, \lambda', \eta', \phi)$ of $Y_j$ with values in an $\mathbb{F}_{p^2}$-scheme $S$, we define

$$H_1 := \phi_{*,1}(\omega_{A'/S,1}^{\circ}) \subseteq H_1^{dR}(B/S)_1^\circ,$$
and

$$H_2 := \psi_{*,2}(\omega_{A'/S,2}^{\circ}) \subseteq H_1^{dR}(B/S)_2^\circ.$$

Hence, there is a canonical isomorphism $\omega_{A'/S,2}/\text{Im}(\phi_{*,2}) \cong H_2$. From the commutative diagram (3.5.1), it is easy to see that $F(H_{1}^{(p)}) = \text{Ker}(\phi_{*,2}) = \text{Im}(\psi_{*,2})$, hence $H_2 \subseteq F(H_{1}^{(p)})$. Similarly, $V^{-1}(H_{2}^{(p)})$ is identified with $\text{Im}(\psi_{*,1}) = \text{Ker}(\phi_{*,1})$, hence $V^{-1}(H_{2}^{(p)}) \subseteq H_1$. Note that we have canonical isomorphisms:

$$\phi : (A, \lambda, \eta, B, \lambda', \eta', \phi) \mapsto (B, \lambda', \eta', H_1, H_2).$$

Proposition 3.7. The morphism $\alpha$ is an isomorphism.

Proof. Let $k$ be a perfect field containing $\mathbb{F}_{p^2}$. We first prove that $\alpha$ induces a bijection of points $\alpha : Y_j(k) \cong Y'_j(k)$. It suffices to show that there exists a morphism of sets $\beta : Y'_j(k) \to Y_j(k)$ inverse to $\alpha$. Let $y = (B, \lambda', \eta', H_1, H_2) \in Y'_j(k)$. We define $\beta(y) = (A, \lambda, \eta, B, \lambda', \eta', \phi)$ as follows.

Let $\tilde{\mathcal{E}}_i \subseteq \tilde{\mathcal{D}}(B)_i^{\circ}$ and $\tilde{\mathcal{E}}_2 \subseteq \tilde{\mathcal{D}}(B)_2^{\circ}$ be respectively the inverse image of $V^{-1}(H_{2}^{(p)}) \subseteq H_1^{dR}(B/k)_2^{\circ}$ and $F(H_{1}^{(p)}) \subseteq H_1^{dR}(B/k)_1^{\circ}$ via the natural reduction map $\tilde{\mathcal{D}}(B) \to H_1^{dR}(B/k)^{\circ} \cong \tilde{\mathcal{D}}(B)/\partial\tilde{\mathcal{D}}(B)^{\circ}$.

Applying Proposition 2.2 with $m = 1$, we get a triple $(A, \lambda, \eta)$ and an isogeny $\psi : A \to B$, where $A$ is an abelian variety over $k$ with an action of $\mathcal{O}_D$, $\lambda$ is a prime-to-$p$ polarization on $A$, and $\eta$ is a prime-to-$p$ level structure on $A$. By construction, $\psi$ is $\mathcal{O}_D$-equivariant such that $\psi^\vee \circ \lambda' \circ \psi = p\lambda$, and such that $\psi_{*,i} : \tilde{\mathcal{D}}(A)_i^{\circ} \to \tilde{\mathcal{D}}(B)_i^{\circ}$ is naturally identified with the inclusion $\tilde{\mathcal{E}}_i^{\circ} \subseteq \tilde{\mathcal{D}}(B)_i^{\circ}$ for $i = 1, 2$. An easy computation of Dieudonné modules shows that $\omega_{A'/k,i}^{\circ}$ has rank 1, and $\omega_{A'/k,2}$ has rank $n - 1$. Therefore, $(A, \lambda, \eta)$ is a point of $\text{Sh}_{1,n-1}$. Finally, we take $\phi : B \to A$ to be the unique isogeny such that $\phi \circ \psi = p \cdot \text{id}_A$ and $\psi \circ \phi = p \cdot \text{id}_B$. This finishes the construction of $\beta(y)$.

It is direct to check that $\beta$ is the set theoretic inverse to $\alpha : Y_j(k) \to Y'_j(k)$.

We show now that $\alpha$ induces an isomorphism on the tangent spaces at each closed point. Let $x = (A, \lambda, \eta, B, \lambda', \eta', \phi) \in Y_j(k)$ be a closed point. Consider the infinitesimal deformation over $k[e] = k[t]/t^2$. Note that $(B, \lambda', \eta')$ has a unique deformation $(\hat{B}, \hat{\lambda}', \hat{\eta}')$ to $k[e]$, namely the trivial deformation. By the Grothendieck-Messing deformation theory (cf. Theorem 2.4), giving a deformation $(\hat{A}, \hat{\lambda}, \hat{\eta})$ of $(A, \lambda, \eta)$ to $k[e]$ is equivalent to giving free $k[e]$-submodules $\hat{\omega}_{A'/k,i}^{\circ} \subseteq H_1^{\text{cris}}(A/k[e])^{\circ}$ for $i = 1, 2$ which lift $\omega_{A'/k,i}^{\circ}$. The isogeny $\phi$ and the polarization $\lambda$ deform to an isogeny $\hat{\phi} : \hat{B} \to \hat{A}$ and a polarization $\hat{\lambda} : \hat{A}^\vee \to \hat{A}$ (satisfying $p\hat{\lambda}' = \hat{\phi}^\vee \circ \hat{\lambda} \circ \hat{\phi}$), necessarily unique if they exist, if and only if

$$\hat{\omega}_{A'/k,i}^{\circ} \subseteq \phi_{*,2}(H_1^{dR}(A/k)_2^{\circ}) \otimes_k k[e]$$
and

$$\hat{\phi}_{*,1}(H_1^{dR}(B/k)_1^{\circ}) \otimes_k k[e]^{\vee} \subseteq (\hat{\omega}_{A'/k,1}^{\circ})^{\vee},$$
where the second inclusion comes from the consideration at the embedding $\hat{q}$ by taking duality using the polarization $\lambda$, and is equivalent to $\hat{\omega}_{A'/k,1}^{\circ} \subseteq \phi_{*,1}(H_1^{dR}(B/k)_2^{\circ}) \otimes_k k[e]$. As discussed before Proposition 3.7 we have $\text{Ker}(\phi_{*,1}) = V^{-1}(H_{2}^{(p)})$ and $F(H_{1}^{(p)}) = \text{Ker}(\phi_{*,2}) = \text{Im}(\psi_{*,2})$. Then giving such $\hat{\omega}_{A'/k,i}^{\circ}$ for $i = 1, 2$ is equivalent to giving liftings $\hat{H}_i$ to free $k[e]$-submodules $\hat{H}_1 \subseteq H_1^{dR}(B/k)_1^{\circ} \otimes_k k[e]$ for $i = 1, 2$ such that $\hat{H}_1 \supseteq V^{-1}(H_{2}^{(p)}) \otimes_k k[e]$ and $\hat{H}_2 \subseteq F(H_{1}^{(p)}) \otimes_k k[e]$. This is exactly the description of the tangent space of $Y'_j$ at $\alpha(x)$. This concludes the proof. \qed
In the sequel, we will always identify \( Y_j \) with \( Y'_j \). Before proceeding, we prove some results on the structure of \( \text{Sh}_{0,n}(\mathbb{F}_p) \).

3.8. Automorphisms of a point on \( \text{Sh}_{0,n} \). Let \( (V_{0,n} = D, \langle -, - \rangle_{0,n}) \) be the left \( D \)-module together with its alternating \( D \)-Hermitian pairing as in the definition of \( \text{Sh}_{0,n} \). Fix a point \( z = (B, \lambda, \eta) \in \text{Sh}_{0,n}(\mathbb{F}_p) \). Put \( C = \text{End}_{\mathcal{O}_D}(B)_{\mathbb{Q}} \), and denote by \( \dagger \) the Rosati involution on \( C \) induced by \( \lambda \). Let \( I \) be the algebraic group over \( \mathbb{Q} \) such that

\[
I(R) = \{ x \in C \otimes \mathbb{Q} R \mid xx^\dagger \in R^\times \}, \quad \text{for all } \mathbb{Q} \text{-algebras } R.
\]

**Proposition 3.9.** We have an isomorphism of \( \mathbb{Q} \)-algebraic groups: \( I \xrightarrow{\sim} G_{0,n} \).

**Proof.** We choose a finite field \( \mathbb{F}_{p^2} \) such that \( z \in \text{Sh}_{0,n}(\mathbb{F}_{p^2}) \) and all the elements of \( C \) are already defined over \( \mathbb{F}_{p^2} \). Since \( \text{Sh}_{0,n} \) is smooth (hence étale) over \( \mathbb{Z}_{p^2} \), there exists a point \( \tilde{z} = (\tilde{B}, \tilde{\lambda}, \tilde{\eta}) \in \text{Sh}_{0,n}(\mathbb{Z}_{p^2}) \) which reduces to \( z \) modulo \( p \). Put \( \tilde{B}_{\mathbb{Q}_p} = \tilde{B} \otimes \mathbb{Z}_{p^2} \mathbb{Q}_p \), and \( \tilde{B}_C \) the base change to \( \mathbb{C} \) via \( \iota^{-1}_p : \mathbb{Q}_p \xrightarrow{\sim} \mathbb{C} \). Then we have an injection of endomorphism rings

\[
\text{End}_{\mathcal{O}_D}(\tilde{B}_C)_{\mathbb{Q}} = \text{End}_{\mathcal{O}_D}(\tilde{B}_{\mathbb{Q}_p})_{\mathbb{Q}} \hookrightarrow \text{End}_{\mathcal{O}_D}(B)_{\mathbb{Q}} = C.
\]

Put \( H = H_1(\tilde{B}_C, \mathbb{Q}) \). It is a left \( D \)-module of rank 1 equipped with an alternating \( D \)-Hermitian pairing \( \langle -, - \rangle_\lambda \) induced by the polarization \( \lambda \). By results of Kottwitz \([\text{Ko92a} \ S8]\), for every place \( v \) of \( \mathbb{Q} \), the skew-Hermitian \( D_{\mathbb{Q}_v} \)-modules \( H_{\mathbb{Q}_v} \) and \( V_{0,n,\mathbb{Q}_v} \) are isomorphic. Then \( \text{End}_{\mathcal{O}_D}(\tilde{B}_C)_{\mathbb{Q}} \) consists of the elements of \( D^{\text{opp}} = \text{End}_D(H) \) that preserves the complex structure on \( H_{1,\mathbb{R}} \cong V_{0,n,\mathbb{R}} \) induced by \( h : \mathbb{C}^\times \to G_{0,n}(\mathbb{R}) \). But in our case \( h(i) \) is necessarily central (since \( G_{0,n} \) is compact). Thus \( \text{End}_{\mathcal{O}_D}(\tilde{B}_C)_{\mathbb{Q}} = D^{\text{opp}} \), and the Rosati involution on \( \text{End}_{\mathcal{O}_D}(\tilde{B})_C \) corresponds to the involution

\[
b \mapsto b^{\beta_0,n} = \beta_0,n b^* \beta_0^{-1}
\]

on \( D^{\text{opp}} \), where \( \beta_0,n \) is the element in the definition of \( \langle -, - \rangle_{0,n} \). Therefore, we have an inclusion of semi-simple \( \mathbb{Q} \)-algebras with involution: \( (D^{\text{opp}}, \beta_0,n) \subseteq (C, \dagger) \). If we can show that this is actually an equality, the proof of the Proposition will be finished. It suffices to show that \( \dim_{\mathbb{Q}} C \leq \dim_{\mathbb{Q}} D^{\text{opp}} \). We choose a prime number \( l \neq p \). Let \( V_l(B) \) be the \( \mathbb{Q}_l \)-Tate module. Then one has an injection

\[
C \otimes \mathbb{Q} Q_l \hookrightarrow \text{End}_{\mathcal{O}_{Q_l}}(V_l(B)) \cong D_{Q_l}^{\text{opp}}.
\]

In particular, we get

\[
\dim_{\mathbb{Q}} C \leq \dim_{\mathbb{Q}_l} D_{Q_l}^{\text{opp}} = \dim D^{\text{opp}}.
\]

**Remark 3.10.** With the notation of the proof above, if \( \pi_B \) denotes the Frobenius endomorphism of \( B \) relative to \( \mathbb{F}_{p^2} \), then Tate \([\text{Tat66}]\) shows that \( C \otimes \mathbb{Q}_l \) is exactly isomorphic to the centralizer of the image of \( \pi_B \) in \( \text{End}_{\mathcal{O}_{Q_l}}(V_l(B)) \). But we have seen in the proof that \( C \otimes \mathbb{Q}_l \xrightarrow{\sim} \text{End}_{\mathcal{O}_{Q_l}}(V_l(B)) \) is bijective. Hence, it follows that \( \pi_B \) is central in \( \text{End}_{\mathcal{O}_{Q_l}}(V_l(B)) \cong D_{Q_l}^{\text{opp}} \). Actually, using the fact that the Newton polygon of the \( p \)-divisible group of \( B \) has only slope 1/2, one can show that \( \pi_B = p^r \) if \( \mathbb{F}_{p^2} \) is sufficiently large.

In particular, there exists an isomorphism

\[
I(Q_p) \xrightarrow{\sim} G_{0,n}(Q_p) \cong Q_p^\times \times \text{GL}_n(Q_p^2)
\]

which is necessarily unique up to conjugation, because all automorphisms of \( \text{GL}_n(Q_p^2) \) are inner. This conjugacy class of isomorphisms can be explicitly described as follows. We put

\[
L_z = D(B)_1^{\circ, F^2=p} = \{ v \in D(B)_1^{\circ} : F^2(v) = pv \}.
\]
This is a free $\mathbb{Z}_p^2$-module of rank $n$, and we have \( \hat{D}(B)^{\perp} \simeq \mathbb{L}_z \otimes_{\mathbb{Z}_p^2} W(\overline{\mathbb{F}}_p) \). Put \( \mathbb{L}_z[1/p] = \mathbb{L}_z \otimes_{\mathbb{Z}_p^2} \mathbb{Q}_p \). An element \( x \in I(\mathbb{Q}_p) \) will induce an automorphism \( x_\mathbb{L} \in \text{Aut}_{\mathbb{Q}_p}(\mathbb{L}_z[1/p]) \simeq \text{GL}_n(\mathbb{Q}_p) \). Up to conjugation, the isomorphism \([3.10.1]\) is given as \( x \mapsto (xx^t, x_\mathbb{L}) \) for \( x \in I(\mathbb{Q}_p) \).

### 3.11. Isogenous classes.

Let \( \text{Isog}(z) \subseteq \text{Sh}_{0,n}(\overline{\mathbb{F}}_p) \) denote the subset of points \( z' = (B', \lambda', \eta') \) such that there exists an \( \mathcal{O}_p \)-equivariant quasi-isogeny \( \phi : B' \to B \) such that \( \phi^* \circ \lambda \circ \phi = c_0 \lambda' \) for some \( c_0 \in \mathbb{Q}_{>0} \); we denote such a quasi-isogeny by \( \phi : z' \to z \) for simplicity. We will give a group theoretic interpretation of \( \text{Isog}(z) \).

Put \( V^{(p)}(B) = T^{(p)}(B) \otimes_{\mathbb{Z}_p} \mathbb{A}^{\infty,p} \). Then the prime-to-\( p \) level structure \( \eta \) determines an isomorphism

\[
\eta : V_{0,n}^{(p)} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^{(p)}(B),
\]

unique up to right translation by elements of \( K^p \). For any \( z' = (B', \lambda', \eta') \in \text{Isog}(z) \) and a choice of \( \phi : z' \to z \) as above. The quasi-isogeny \( \phi \) induces an isomorphism \( \phi_* : V^{(p)}(B') \xrightarrow{\sim} V^{(p)}(B) \). Then there exists a \( g \in G_{0,n}(\mathbb{A}^{\infty,p}) \), unique up to right multiplication by elements of \( K^p \), such that the \( K^p \)-orbit of \( \phi_*^{-1} \circ \eta \circ g \) gives \( \eta' \). If one replaces \( \phi \) by \( \gamma \phi \) with \( \gamma \in I(\mathbb{Q}) \cong G_{0,n}(\mathbb{Q}) \), then \( g \) is replaced by \( \gamma g \).

Similarly, \( \phi \) induces an isomorphism \( \phi_* : \mathbb{L}_{\mathbb{L}'}[1/p] \xrightarrow{\sim} \mathbb{L}_z[1/p] \). Fix a \( \mathbb{Z}_p^2 \)-basis for \( \mathbb{L}_z \). Then there exists a \( g_\mathbb{L} \in \text{GL}_n(\mathbb{Q}_p^2) \) such that \( \phi_*(\mathbb{L}_{\mathbb{L}'}) = g_\mathbb{L}(\mathbb{L}_z) \), and the right coset \( \mathbb{g}_\mathbb{L} \mathbb{GL}_n(\mathbb{Z}_p^2) \) is independent of the choice of such a basis. We put \( g_p = (c_0, g_\mathbb{L}) \in \mathbb{Q}_p^\times \times \mathbb{GL}_n(\mathbb{Q}_p^2) \cong G_{0,n}(\mathbb{Q}_p) \). If one replaces \( \phi \) by \( \gamma \phi \) with \( \gamma \in I(\mathbb{Q}) \cong G_{0,n}(\mathbb{Q}) \), then \( (c_0, g_\mathbb{L}) \) is replaced by \( \gamma g_p = (\gamma \gamma^t c_0, \gamma g_\mathbb{L}) \).

Therefore, one obtains a well defined map

\[
\Theta_z : \text{Isog}(z) \to G_{0,n}(\mathbb{Q}) \setminus (G_{0,n}(\mathbb{A}^{\infty,p}) \times G_{0,n}(\mathbb{Q}_p^2)) / K
\]

with \( K = K^p \times K_p \), which attaches to \( z' \) the class of pairs \( (g, g_p) \).

**Proposition 3.12.** The map \( \Theta_z \) is bijective.

**Proof.** We show first the injectivity of \( \Theta_z \). Given \( z_1 = (B_1, \lambda_1, \eta_1), z_2 = (B_2, \lambda_2, \eta_2) \in \text{Isog}(z) \) that correspond to the same double coset, we have to prove that they are isomorphic. Choose quasi-isogenies \( \phi_1 : B_1 \to B \) and \( \phi_2 : B_2 \to B \), and let \( (g_1, g_{p,1}), (g_2, g_{p,2}) \in G_{0,n}(\mathbb{A}^{\infty,p}) \times G_{0,n}(\mathbb{Q}_p) \) denote the corresponding pairs. Then there exists \( \gamma \in G_{0,n}(\mathbb{Q}) \) and \( (k^p, k_p) \in K^p \times K_p \) such that \( \gamma(g_1, g_{p,1}) = (g_2 k^p, g_{p,2} k_p) \). Consider the quasi-isogeny

\[
\phi = \phi_2^{-1} \circ \gamma \circ \phi_1 : B_1 \to B_2.
\]

Then \( \phi \) induces isomorphisms \( T^{(p)}(B_1) \xrightarrow{\sim} T^{(p)}(B_2) \), and \( \mathbb{L}_{z_1} \xrightarrow{\sim} \mathbb{L}_{z_2} \). Since \( \hat{D}(B_1)^{\perp} \oplus \hat{D}(B_4)^{\perp} \simeq (\mathbb{L}_z \oplus F(\mathbb{L}_z)) \otimes_{\mathbb{Z}_p^2} W(\overline{\mathbb{F}}_p) \), it follows that \( \phi_* : \hat{D}(B_1) \xrightarrow{\sim} \hat{D}(B_2) \). Hence, \( \phi \) is an isomorphism of abelian varieties, and it is easily seen to be compatible with all structures. This shows the injectivity of \( \Theta_z \).

We now prove the surjectivity of \( \Theta_z \). Suppose that we are given \( (g, g_p) \in G_{0,n}(\mathbb{A}^{\infty,p}) \times G_{0,n}(\mathbb{Q}_p) \). We need to show that \( (g, g_p) \) comes from some \( z' \in \text{Isog}(z) \). Write \( g_p = (p^m u, g_\mathbb{L}) \in G_{0,n}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathbb{GL}_n(\mathbb{Q}_p^2) \) with \( u \in \mathbb{Z}_p^\times \). Up to multiplying \( (g, g_p) \) by an integer (viewed in \( \mathbb{Q}_p^\times \cong G_{0,n}(\mathbb{Q}) \)), we may assume that \( g(\Lambda_{0,n} \otimes \tilde{\mathbb{Z}}^{(p)}) \subseteq \Lambda_{0,n} \otimes \tilde{\mathbb{Z}}^{(p)} \), \( m \geq 0 \) and \( p^m \mathbb{L}_z \subseteq g_\mathbb{L}(\mathbb{L}_z) \subseteq \mathbb{L}_z \). Here, \( \Lambda_{0,n} \subseteq V_{0,n} \) denotes the \( \mathcal{O}_D \)-stable lattice used in the definition of \( \text{Sh}_{0,n} \). Then \( T_{z'}^{(p)} := \tilde{\eta} \circ g(\Lambda_{0,n} \otimes \tilde{\mathbb{Z}}^{(p)}) \) is an \( \mathcal{O}_D \)-stable sublattice of \( T^{(p)}(B) = \tilde{\eta} \Lambda_{0,n} \otimes \tilde{\mathbb{Z}}^{(p)} \). Choose an integer \( N \geq 1 \) coprime to \( p \) such that \( NT^{(p)}(B) \subseteq T_{z'}^{(p)} \) with quotient \( H^{(p)} \). It is a subgroup of \( B[N] \) stable under \( \mathcal{O}_D \).
Similarly, choose an integer \( m \geq 0 \) such that \( p^m \mathbb{L}_x \subseteq g_L(\mathbb{L}_x) \subseteq \mathbb{L}_x \). Put \( \tilde{\mathcal{E}}_1 = g_L(\mathbb{L}_x) \otimes \mathbb{Z}_{\mathfrak{p}} W(\mathbb{F}_p) \), \( \tilde{\mathcal{E}}_2 = F(\tilde{\mathcal{E}}_1) \). Then \( \tilde{\mathcal{E}}_1 \oplus \tilde{\mathcal{E}}_2 \) is a Dieudonné submodule of \( \tilde{\mathcal{D}}(B)^{\oplus} \), and the quotient

\[
\bigoplus_{i=1}^{2} (\tilde{\mathcal{E}}_i/p^m \tilde{\mathcal{D}}(B)^{\oplus})_i
\]
corresponds to a subgroup scheme \( H_p \) of \( B[p^m] \) stable under the action of \( \mathcal{O}_D \). Denote by \( H_p \subseteq B[p^m] \) the orthogonal complement of \( H_p \) under the perfect pairing \( B[p^m] \times B[p^m] \to \mu_{p^m} \). Put \( H_p = H_{p'} \) and \( H = H(p') \). Let \( \psi : B \to B' \) be the canonical isogeny with kernel \( H \), and \( \phi : B' \to B \) be the isogeny such that \( \phi \circ \psi = N p^m \text{Id}_B \). Then \( B' \) is an abelian scheme equipped with an induced \( \mathcal{O}_D \)-action. The induced map \( \phi_* : T^{(p)}(B') \to T^{(p)}(B) \) is naturally identified with the inclusion \( T^{(p)}_{\psi}(B') \subseteq T^{(p)}(B) \), and \( \phi_* : \tilde{\mathcal{D}}(B')^{\oplus}_i \to \tilde{\mathcal{D}}(B)^{\oplus}_i \) is identified with the inclusion \( \tilde{\mathcal{E}}_i \to \tilde{\mathcal{D}}(B)^{\oplus}_i \) for \( i = 1, 2 \). This implies in particular that \( \text{Lie}_{B',i} \) is of dimension \( n \) for \( i = 1, 2 \) and dimension \( 0 \) for \( i = 0 \). By \([\text{Mu74}, \S 23, \text{Theorem 2}]\), there is a prime-to-\( p \) polarization \( \lambda' \) on \( B' \) such that \( p^n \lambda' \) is equivalent to \( \phi \circ \lambda \circ \phi \). Finally, the orbit of \( \phi_*^{-1} \circ \eta \circ g \) under \( K_p \) well defines a \( K_p \)-level structure on \( B' \). Then \( \lambda' = (B', \lambda', \eta') \) well defines a point of \( \text{Isog}(z) \subseteq \text{Sh}_{0,n}(\mathbb{F}_p) \), and its image under \( \Theta_z \) is the class of \( (g, g_p) \). This finishes the proof of the subjectivity of \( \Theta_z \). \( \square \)

**Remark 3.13.** It follows from this Proposition and the description of \( \text{Sh}_{0,n}(\mathbb{C}) \) as in Subsection 1.3 that \( \text{Sh}_{0,n}(\mathbb{F}_p) \) consists of \( \# \ker^1(Q, G_{0,n}) \) isogenous classes of abelian varieties.

**Lemma 3.14.** Let \( N \) be a fixed non-negative integer. Up to replacing \( K_p \) by an open compact subgroup of itself, the Shimura variety \( \text{Sh}_{0,n} \) satisfies the following property: if \( (B, \lambda, \eta) \) is an \( \mathbb{F}_p \)-point of \( \text{Sh}_{0,n} \) and \( f : B \to B \) is an \( \mathcal{O}_D \)-quasi-isogeny such that

- \( p^N f \in \text{End}_{\mathcal{O}_D}(B) \),
- \( f' \circ \lambda \circ f = \lambda \),
- \( f \circ \eta = \eta \),

then \( f = \text{id} \).

**Proof.** It suffices to prove the Lemma for \( (B, \lambda, \eta) \) in a fixed isogenous class \( \text{Isog}(z) \) of \( \text{Sh}_{0,n}(\mathbb{F}_p) \).

We write \( G_{0,n}(\mathbb{A}^{\infty,p}) = \prod_{i \in I} G_{0,n}(Q_i) g_i K \) with \( K = K_p K_p \), where \( g_i \) runs through a finite set of representatives of the double coset

\[
G_{0,n}(Q) \backslash G_{0,n}(\mathbb{A}^{\infty,p}) / K.
\]

Let \( (B, \lambda, \eta) \) be a point of \( \text{Sh}_{0,n} \) corresponding to \( G_{0,n}(Q) g_i K \) for some \( i \in I \), and \( f \) be an \( \mathcal{O}_D \)-quasi-isogeny of \( B \) as in the statement. Then \( f \) is given by an element of \( G_{0,n}^1(Q) \). The condition that \( f \circ \eta = \eta \) is equivalent to that \( f \) maps to \( g_i K_p g_i^{-1} \subseteq G_{0,n}(\mathbb{A}^{\infty,p}) \). Moreover, \( p^N f \in \text{End}_{\mathcal{O}_D}(B) \) implies that the image of \( f \) in \( G_{0,n}(Q_p) \) belongs to \( \bigcap_{i,j} g_i K_p g_j^{-1} \), where \( \delta \) runs through the set

\[
\{(1, \text{diag}(p^{a_1}, p^{a_2}, \ldots, p^{a_n})) \in G_{0,n}(Q_p) \cong Q_p^\times \times \text{GL}_n(E_p) \mid n \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}.
\]

Write \( \prod_{i,j} g_i K_p g_j = \prod_{j \in J} h_j K_p \) for some finite set \( J \). Hence, it suffices to show that there exists an open compact subgroup \( K'' \subseteq K_p \) such that, for all \( g_i \), \( G_{0,n}^1(Q) \cap g_i (K'' \cdot h_j K_p) g_i^{-1} \) is equal to \( \{1\} \) if \( h_j K_p = K_p \), and it is empty otherwise. Since \( K \) is neat, we have \( G_{0,n}^1(Q) \cap g_i (K'' K_p) g_i^{-1} = \{1\} \) for any \( g_i \) and any \( K'' \subseteq K_p \). Note that this implies that \( G_{0,n}^1(Q) \cap g_i (K'' \cdot h_j K_p) g_i^{-1} \) contains at most one element (because if it contains both \( x \) and \( y \), then \( x^{-1} y \in G_{0,n}^1(Q) \cap g_i K_p g_i^{-1} = \{1\} \)). Let \( S \subseteq I \times J \) be the subset consisting of \((i,j)\) such that \( h_j K_p \neq K_p \) and \( G_{0,n}^1(Q) \cap g_i (K'' \cdot h_j K_p) g_i^{-1} \) indeed contains one element, say \( x_{i,j} \). Then \( x_{i,j} \neq 1 \) for all \((i,j) \in S \). Hence, one can choose a normal subgroup \( K'' \subseteq K_p \) so that \( x_{i,j} \neq g_i K'' g_i^{-1} \) for all \( i \). We claim that this choice of \( K'' \) will satisfy
the desired property. Indeed, if \( K^p = \prod b_i K^{p_i} \), then the double coset \( G_{0,n}(\mathbb{Q})\backslash G_{0,n}(\mathbb{A}^{\infty})/K^p K_p \) has a set of representatives of the form \( g_i b_i \). Then one has, for \( h_j K_p \neq K_p \),

\[
G_{0,n}(\mathbb{Q}) \cap g_i b_i (K^p h_j K_p) b_i^{-1} g_i^{-1} = G_{0,n}(\mathbb{Q}) \cap g_i (K^p h_j K_p) g_i^{-1} = \emptyset.
\]

Here, the second equality uses the fact that \( K^p \) is normal in \( K^\delta \). This finishes the proof.

We come back to the discussion on the cycles \( Y_j \subseteq \text{Sh}_{1,n-1} \) for \( 1 \leq j \leq n \). For a smooth variety \( X \) over \( \mathbb{F}_{p^2} \), we denote by \( T_X \) the tangent bundle of \( X \), and for a locally free \( \mathcal{O}_X \)-module \( M \), we put \( M^* = \text{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X) \).

**Proposition 3.15.** Let \((A, \lambda, \eta, B, \lambda', \eta', \phi^\text{univ})\) denote the universal object on \( Y_j \) for \( 1 \leq j \leq n \), and \( \mathcal{H}_i \subseteq H_1^{\text{dR}}(B/\text{Sh}_{0,n}) \) for \( i = 1, 2 \) be the universal sub-bundles on \( Y_j' \cong Y_j \).

1. The induced map \( T_{Y_j} \rightarrow \text{pr}_{1,*} T_{\text{Sh}_{1,n-1}} \) is universally injective, and we have canonical isomorphisms

\[
N_{Y_j}(\text{Sh}_{1,n-1}) := \text{pr}_{1,*} T_{\text{Sh}_{1,n-1}}/T_{Y_j} \\
\simeq (\mathcal{H}_1/V^{-1}(\mathcal{H}_2(p))) \otimes V^{-1}(\mathcal{H}_2(p)) \oplus (F(\mathcal{H}_1(p))/\mathcal{H}_2) \otimes (H_1^{\text{dR}}(B/\text{Sh}_{0,n+1}))_2/F(\mathcal{H}_1(p)) \simeq \Lambda_{A,1} \otimes \text{Coker}(\phi^\text{univ}) \oplus \Lambda_{A,2} \otimes \text{Im}(\phi^\text{univ}).
\]

2. Assume that \( K^p \) is sufficiently small so that the consequences of Lemma 3.3.4 hold for \( N = 1 \). For each fixed closed point \( z \in \text{Sh}_{0,n} \), the map \( \text{pr}_{j,z} := \text{pr}_j Y_{j,z} : Y_{j,z} \rightarrow \text{Sh}_{1,n-1} \) is a closed immersion, or equivalently, the morphism \( \text{pr}_{1,z} : Y_j \rightarrow \text{Sh}_{1,n-1} \times_{\text{Spec}(\mathbb{F}_{p^2})} \text{Sh}_{0,n} \) is a closed immersion.

3. The union of the images of \( \text{pr}_j \) for all \( 1 \leq j \leq n \) is the super-singular locus of \( \text{Sh}_{1,n-1} \), i.e. the reduced closed subscheme of \( \text{Sh}_{1,n-1} \) where all the slopes of the Newton polygon of the \( p \)-divisible group \( A[p^\infty] \) are \( 1/2 \).

**Proof.** (1) Let \( y = (A, \lambda, \eta, B, \lambda', \eta', \phi) \in Y_j(S) \) be a point of \( Y_j \) with values in an affine \( \mathbb{F}_{p^2} \)-scheme \( S \). Put \( \hat{S} = S \times_{\text{Spec}(\mathbb{F}_{p^2})} \text{Spec}(\mathbb{F}_{p^2}[t]/t^2) \). Then we have a natural bijection

\[
\mathcal{D}\text{ef}(y, \hat{S}) \cong \Gamma(S, y^* T_{Y_j}),
\]

where \( \mathcal{D}\text{ef}(y, \hat{S}) \) is the set of deformations of \( y \) to \( \hat{S} \). Similarly, we have \( \mathcal{D}\text{ef}(\pi_j \circ y, \hat{S}) \cong \Gamma(S, y^* \pi_{j,*} T_{\text{Sh}_{1,n-1}}) \). To prove the universal injectivity of \( T_{Y_j} \rightarrow \pi_{j,*} T_{\text{Sh}_{1,n-1}} \), it suffices to show that the natural map \( \mathcal{D}\text{ef}(y, \hat{S}) \rightarrow \mathcal{D}\text{ef}(\pi_j \circ y, \hat{S}) \) is injective. By crystalline deformation theory (Theorem 2.4), giving a point of \( \mathcal{D}\text{ef}(y, \hat{S}) \) is equivalent to giving \( \mathcal{O}_S \)-sub-bundles \( \mathcal{G}_{A,i} \subseteq H_1^{\text{cris}}(A/\hat{S})_i \) over \( \hat{S} \) such that

- \( \mathcal{G}_{A,i} \) lifts \( \omega_A^0/S_{i,1} \);
- \( \mathcal{G}_{A,i} \subseteq \text{Im}(\phi_{s,1}) \otimes \mathbb{F}_{p^2}[t]/t^2 \) and \( \text{Im}(\phi_{s,2}) \otimes \mathbb{F}_{p^2}[t]/t^2 \subseteq \omega_A^{0,2} \) are locally direct factors.

Hence, one sees easily that

\[
\mathcal{D}\text{ef}(y, \hat{S}) \cong \text{Hom}_{\mathcal{O}_{\hat{S}}}(\omega_{A/S_{i,1}}^0, \text{Im}(\phi_{s,1})/\omega_{A/S_{i,1}}^0) \oplus \text{Hom}_{\mathcal{O}_{\hat{S}}}(\omega_{A/S_{i,2}}^0/\text{Im}(\phi_{s,2}), H_1^{\text{dR}}(A/S)_{i,2}/\omega_{A/S_{i,2}}^0) \\
\cong \text{Lie}_{A/S_{i,1}}^0 \otimes (\text{Im}(\phi_{s,1})/\omega_{A/S_{i,1}}^0) \oplus (\omega_{A/S_{i,2}}^0/\text{Im}(\phi_{s,2}))^* \otimes \text{Lie}_{A/S_{i,2}}^0.
\]

Similarly, \( \mathcal{D}\text{ef}(y, \hat{S}) \) is given by the lifts of \( \omega_A^{0,i} \) for \( i = 1, 2 \) to \( \hat{S} \). These lifts are classified by \( \text{Hom}_{\mathcal{O}_{\hat{S}}}(\omega_{A/S_{i,1}}^0, H_1^{\text{dR}}(A/S)_{i,1}/\omega_{A/S_{i,1}}^0) \cong \text{Lie}_{A/S_{i,1}}^0 \otimes \text{Lie}_{A/S_{i,2}}^0 \). Hence, \( \mathcal{D}\text{ef}(y, \hat{S}) \) is canonically isomorphic to

\[
\text{Lie}_{A/S_{i,1}}^0 \otimes \text{Lie}_{A/S_{i,1}}^0 \oplus \text{Lie}_{A/S_{i,2}}^0 \otimes \text{Lie}_{A/S_{i,2}}^0.
\]
The natural map $\mathcal{D}(y, \hat{S}) \to \mathcal{D}(\pi_j \circ y, \hat{S})$ is induced by the natural maps

$$\text{Im}(\phi_{\ast, 1})/\omega_{A, 1}/S, 1 \twoheadrightarrow H_{1, \text{dr}}(A/S)_1^0/\omega_{A, 1}/S, 1 \cong \text{Lie}_{A, 1}/S, 1,$$

$$(\omega_{A, 1}/S, 2/\text{Im}(\phi_{\ast, 2}))^* \twoheadrightarrow \omega_{A, 2}/S, 2 \cong \text{Lie}_{A, 2}/S, 2.$$

It follows that $\mathcal{D}(y, \hat{S}) \to \mathcal{D}(\pi_j \circ y, \hat{S})$ is injective. To prove the formula for $N_{Y_j}(\text{Sh}_{1, n-1})$, we apply the arguments above to affine open subsets of $Y_j$. We see easily that

$$N_{Y_j}(\text{Sh}_{1, n-1}) \cong \text{Lie}_{A, 1}/S, 1 \otimes_{O_Y} \text{Coker}(\phi_{\ast, 1}^{\text{univ}}) \otimes \text{Lie}_{A, 2}/S, 2 \otimes_{O_Y} \text{Im}(\phi_{\ast, 2}^{\text{univ}})^*$$

$$\cong (\mathcal{H}_1/V^{-1}(\mathcal{H}_2^{(p)}))^* \otimes V^{-1}(\mathcal{H}_2^{(p)}) \oplus (F(\mathcal{H}_1^{(p)})/\mathcal{H}_2^{(p)})^* \otimes (H_{1, \text{dr}}(B/Y_j)_2^0/F(\mathcal{H}_1^{(p)})).$$

Here, the last step uses (3.61) and the isomorphism

$$\text{Im}(\phi_{\ast, 2}^{\text{univ}}) \cong H_{1, \text{dr}}(B/Y_j)_2^0/\text{Ker}(\phi_{\ast, 1}^{\text{univ}}) \cong H_{1, \text{dr}}(B/Y_j)_2^0/F(\mathcal{H}_1^{(p)}).$$

(2) By statement (1), $p_{r+1}$ induces an injection of tangent spaces at each closed points of $Y_{j, 2}$. To complete the proof, it suffices to prove that $\tau_{j, 2}$ induces injections on the closed points. Write $z = (B, \lambda', \eta') \in \text{Sh}_{0, n}(\mathbb{F}_p)$. Assume $y_1$ and $y_2$ are two closed points of $Y_{j, 2}$ with $\pi_j(y_1) = \pi_j(y_2) = (A, \lambda, \eta)$. Let $\phi_1, \phi_2 : B \to A$ be the isogenies given by $y_1$ and $y_2$. Then the quasi-isogeny $\phi_{1, 2} = \phi_2^{-1} \phi_1 \in \text{End}_Q(B)$ satisfies the conditions of Lemma 3.14 for $n = 1$. Hence, we get $\phi_{1, 2} = 1$, which is equivalent to $y_1 = y_2$. This proves that $\tau_{j, 2}$ is injective on closed points.

(3) Since all the points of $\text{Sh}_{0, n}(\mathbb{F}_p)$ are supersingular, it is clear that the image of each $p_{r+1}$ lies in the supersingular locus of $\text{Sh}_{1, n-1}$. Suppose now given a supersingular point $x = (A, \lambda, \eta) \in \text{Sh}_{1, n-1}(\mathbb{F}_p).

We have to show that there exists $(B, \lambda', \eta') \in \text{Sh}_{0, n}$ and an isogeny $\phi : B \to A$ such that $(A, \lambda, \eta, \lambda', \eta'; \phi)$ lies in $Y_j$ for some $1 \leq j \leq n$. Consider

$$L_Q = (\bar{D}(A)_{2}^{0}[1/p])^{F=x=p} = \{a \in \bar{D}(A)_{2}^{0}[1/p] \mid F^2(a) = pa\}.$$

Since $x$ is supersingular, $L_Q$ is a $\mathbb{Q}_p$-vector space of dimension $n$ by Dieudonné-Manin’s classification, and we have $L_Q \otimes_{\mathbb{Q}_p} W(\mathbb{F}_p)[1/p] = \bar{D}(A)_{2}^{0}[1/p]$. We put $\bar{E}_1 = (L_Q \cap \bar{D}(A)_{2}^{0}) \otimes_{\mathbb{Z}_p} W(\mathbb{F}_p)$, and $\bar{E}_2 = F(\bar{E}_1) \subseteq \bar{D}(A)_{2}^{0}$. Thus $\bar{E}_{\circ} = \bar{E}_1 \oplus \bar{E}_2$ is a Dieudonné submodule of $\bar{D}(A)_{\circ}$. Suppose that $\bar{E}_{\circ}$ contains $p\bar{D}(A)_{\circ}$ as a submodule. Then applying Lemma ?? with $m = 1$, we get an $O_D$-abelian variety $(B, \lambda', \eta')$ together with an $O_D$-isogeny $\phi : B \to A$ with $\phi^\circ \lambda' \circ \phi = p\lambda$. It is easy to see in this case that $(A, \lambda, \eta, B, \lambda', \eta', \phi)$ defines a point in $Y_j$ with $j = \dim_{\mathbb{F}_p}(\bar{D}(A)_{2}^{0}/\bar{E}_2)$. It then suffices to show that it is impossible to have $p\bar{D}(A)_{\circ} \not\subseteq \bar{E}_{\circ}$, or equivalently $\bar{D}(A)_{\circ} \not\subseteq \frac{1}{p}\bar{E}_{\circ}$. Suppose the contrary. The Kottwitz signature condition implies that both $F$ and $V : \bar{D}(A)_{2}^{0} \to \bar{D}(A)_{2}^{0}$ have cokernel isomorphic to $\mathbb{F}_p$. Thus the two injective morphisms

$$F : \bar{D}(A)_{2}^{0}/\bar{E}_{\circ} \to \bar{D}(A)_{2}^{0}/\bar{E}_{\circ}$$

have cokernel isomorphic to $\mathbb{F}_p$. It follows that for all $\alpha \geq 1$ there is only one pair

$$(3.15.1) \quad \text{F and V} : \left(\frac{1}{p^\alpha}\bar{E}_{\circ} \cap (\bar{D}(A)_{2}^{0} + \frac{1}{p}\bar{E}_{\circ})\right) / \frac{1}{p^\alpha}\bar{E}_{\circ} \to \left(\frac{1}{p^\alpha}\bar{E}_{\circ} \cap (\bar{D}(A)_{2}^{0} + \frac{1}{p}\bar{E}_{\circ})\right) / \frac{1}{p}\bar{E}_{\circ},$$

which is not an isomorphism. So there exists $\alpha \geq 1$ such that (3.15.1) are isomorphisms of non-zero $\mathbb{F}_p$-vector spaces (because there are at least two such pairs of morphisms for $\bar{D}(A)_{\circ} \not\subseteq \frac{1}{p}\bar{E}_{\circ}$). Multiplication by $p^\alpha$ gives isomorphisms:

$$\text{F and V} : \left(\frac{1}{p}\bar{E}_{\circ} \cap (p^\alpha\bar{D}(A)_{2}^{0} + \bar{E}_{\circ})\right) / \frac{1}{p}\bar{E}_{\circ} \to \left(\frac{1}{p}\bar{E}_{\circ} \cap (p^\alpha\bar{D}(A)_{2}^{0} + \bar{E}_{\circ})\right) / \frac{1}{p}\bar{E}_{\circ},$$

Now, on the one hand, Hilbert 90 implies that $\mathbb{L}' = \left(\frac{1}{p}\bar{E}_{\circ} \cap (p^\alpha\bar{D}(A)_{2}^{0} + \bar{E}_{\circ})\right)$ in fact generates the entire module. On the other hand, it is obvious that $\mathbb{L}' \subseteq \mathbb{L}'$ and $\mathbb{L}' \subseteq p^\alpha\bar{D}(A)_{2}^{0} + \bar{E}_{\circ} \subseteq \bar{D}(A)_{2}^{0}$. This means that $\mathbb{L}'$ is a finite $\mathbb{F}_p$-vector space, and hence $\mathbb{L}'$ is also $\mathbb{F}_p$-isomorphic to $\mathbb{L}'$.
This means that $\ell'$ and hence $L_\eta \cap \tilde{D}(A)^\circ$ generates the entire $(\frac{1}{2}E_1^\circ \cap (p^0\tilde{D}(A)^\circ + E_1^\circ))$. But this contradicts with the non-triviality of the vector spaces in $(3.15.1)$. Now the Proposition is proved.

\begin{proof}
We just prove the statement for $pr_1$, and the case of $pr_n$ is similar. Let $Z_1$ be the closed subscheme of $Sh_{1,n-1}$ defined by the vanishing of $V : \omega_{A^\vee,2} \to \omega_{n,1}^{(p)}$ (resp. $V : \omega_{A^\vee,1} \to \omega_{A^\vee,2}^{(p)}$).

To prove that $pr_1 : Y_1 \to Z_1$ is an isomorphism, it suffices to show that it induces a bijection between closed points and tangent spaces of $Y_1$ and $Z_1$. For any perfect field $k$ containing $\bar{F}_p$, one constructs a map $\theta : Z_1(k) \to Y_1(k)$ inverse to $pr_1 : Y_1(k) \to Z_1(k)$ as follows. Given $x = (A, \lambda, \eta) \in Z_1(k)$. Let $E_1^\circ = \tilde{D}(A)^\circ$ and $E_2^\circ \subseteq \tilde{D}(A)^\circ$ be the inverse image of $\omega_{A^\vee,k}[\eta] \subseteq \tilde{D}(A)^\circ$. Then the condition that $y \in Z_1$ implies that $E_1^\circ \equiv E_2^\circ$ is stable under $F$ and $V$. Applying Lemma ?? with $m = 1$, we get a tuple $(B, \lambda', \eta', \phi)$ such that $y = (A, \lambda, \eta, B, \lambda', \eta', \phi) \in Y_1(k)$. It is immediate to check that $x \mapsto y$ and $\pi_1$ is the set theoretic inverse of each other. It remains to show that $pr_1$ induces a bijection between $T_{Y_1,p}$ and $T_{Z_1,x}$. Proposition 3.15 already implies that we have an inclusion $T_{Y_1,k} \to T_{Z_1,x} \subseteq T_{Sh_{1,n-1},x}$; it suffices to check that $\dim T_{Z_1,x} = n - 1$. The tangent space $T_{Z_1,x}$ is the space of deformations $(\hat{A}, \hat{\lambda}, \hat{\eta})$ of $(A, \lambda, \eta)$ such that $V : \omega_{A^\vee,k}[\eta,1] \to \omega_{A^\vee/k[\eta],1}^{(p)}$ vanishes. This uniquely determines the lift $\hat{\omega}_{A^\vee,2} = \omega_{A^\vee/k[\eta,1]}^{(p)}$. So by the deformation theory (Theorem 2.4), the tangent space $T_{Z_1,x}$ is determined by the liftings $\hat{\omega}_{A^\vee,1} = \omega_{A^\vee/k[\eta],1}^{(p)}$; so it has dimension $n - 1$. This concludes the proof of the corollary.
\end{proof}

3.17. Geometric Jacquet-Langlands morphism. Let $\ell \not= p$ be a prime number. For $1 \leq i \leq n$, the diagram $(3.1.1)$ gives rise to a natural morphism

\begin{equation}
(3.1.1) \quad \mathcal{J}L_i : H^i_{et}(Sh_{0,n,\bar{F}_p}, \mathcal{O}_\ell) \to \frac{pr_i^*}{pr_i^*}H^0_{et}(Y_{i,z}, \mathcal{O}_\ell) \xrightarrow{\text{Gys}_{pt}} H^0_{et}(Y_{i,z}, \mathcal{O}_\ell) \xrightarrow{\text{Gys}_{pt}} H^2_{et}(n-1)(Sh_{1,n-1,\bar{F}_p}, \mathcal{O}_\ell)(n - 1),
\end{equation}

where the restriction of $\text{Gys}_{pt}$ to each $H^0_{et}(Y_{i,z}, \mathcal{O}_\ell)$ for $z \in Sh_{0,n}(\bar{F}_p)$ is the Gysin map associated to the closed immersion $Y_{i,z} \hookrightarrow Sh_{1,n-1,\bar{F}_p}$. It is clear that the image of $\mathcal{J}L_i$ is the subspace generated by the cycle classes of $[Y_{i,z}] \in A^{n-1}(Sh_{1,n-1,\bar{F}_p})$ with $z \in Sh_{0,n,\bar{F}_p}$. According to [He11], $\mathcal{J}L_i$ should be considered as a certain geometric realization of the Jacquet-Langlands correspondence between unitary groups. Putting all $\mathcal{J}L_i$ together, we get a morphism

\begin{equation}
(3.1.2) \quad \mathcal{J}L = \sum_i \mathcal{J}L_i : \bigoplus_{i=1}^n H^i_{et}(Sh_{0,n,\bar{F}_p}, \mathcal{O}_\ell) \to H^2_{et}(n-1)(Sh_{1,n-1,\bar{F}_p}, \mathcal{O}_\ell)(n - 1).
\end{equation}

Recall that we have fixed an isomorphism $G_{1,n-1}(A^\infty) \cong G_{0,n}(A^\infty)$, which we write uniformly as $G(A^\infty)$. Denote by $\mathcal{H}(K^p, \mathcal{O}_\ell) = \mathcal{O}_p[K^p \setminus G(A^\infty,p)/K^p]$ the prime-to-$p$ Hecke algebra. Then the morphism $(3.1.2)$ is $\mathcal{H}(K^p, \mathcal{O}_\ell)$-equivariant.
3.18. **Automorphic Representations.** Fix an isomorphism \( \iota_\ell : \mathbb{C} \to \overline{\mathbb{Q}}_\ell \). Let \( \mathcal{A}_K \) be the set of isomorphism classes of irreducible admissible representations \( \pi \) of \( G(A^\infty) \) (with \( \mathbb{C} \)-coefficients) satisfying Hypothesis 1.5 with \( a_\bullet = (1, n-1) \). For \( \pi \in \mathcal{A}_K \), we write \( \pi = \pi^p \otimes \pi_p \), where \( \pi^p \) (resp. \( \pi_p \)) is the prime-to-\( p \) part (resp. the \( p \)-component) of \( \pi \).

**Lemma 3.19.** Let \( \pi_1 \) and \( \pi_2 \) be two elements of \( \mathcal{A}_K \). If \( \pi_1^p \) and \( \pi_2^p \) are isomorphic, then we have \( \pi_{1,p} \cong \pi_{2,p} \).

**Proof.** Let \( \pi_{\infty,i} \) for \( i = 1, 2 \) be an admissible irreducible representation of \( G_{0,n}(\mathbb{R}) \) such that \( \iota_\ell(\pi_{\infty,i}) \otimes \pi_{\infty,i} \) is automorphic. Under condition (2) above, Harris and Labesse [HL04] show that each \( \iota_\ell(\pi_{\infty,i}) \otimes \pi_{\infty,i} \) base changes to an automorphic representation \( (\Pi_i, \psi_i) \) of \( \text{GL}_n(A_E) \times \mathbb{A}_E^\times \) such that

- \( \Pi_i \) is cuspidal and conjugate self-dual,
- for every unramified rational prime \( x \), the \( x \)-component of \( (\Pi_i, \psi_i) \) depends only on the \( x \)-component of \( \pi_{\infty,i} \), and
- if one writes \( \pi_{i,p} = \pi_{i,0} \otimes \pi_{i,p} \) as representation of \( G(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \text{GL}_n(\mathbb{E}_p) \), then \( \Pi_{i,p} = (\pi_{i,p} \otimes \pi_{i,p}^{-1}) \) as representation of \( \text{GL}_n(\mathbb{E}_p) \times \mathbb{Q}_p \mathbb{Q}_p^\times \mathbb{Q}_p^\times \). Here, \( \pi_{i,p}^{-1} \) denotes the complex conjugate of the contragredient of \( \pi_{i,p} \).

Then \( (\Pi_1, \psi_1) \) and \( (\Pi_2, \psi_2) \) are automorphic at almost all finite places. By the strong multiplicity one theorem for \( \text{GL}_n \), we get \( (\Pi_1, \psi_1) \cong (\Pi_2, \psi_2) \). By the description of \( (\Pi_{i,p}, \psi_{i,p}) \), it follows immediately that \( \pi_{1,p} \cong \pi_{2,p} \). \( \square \)

We fix an representation \( \pi \) in \( \mathcal{A}_K \). Lemma 3.19 implies that \( \pi \) is completely determined by its prime-to-\( p \) part. Thus taking the \( (\pi^p) \mathbb{K}^p \)-isotypical component of (3.17.2) is the same thing as taking the \( \pi^{K'} = (\pi^p)^{K'} \otimes (\pi_p)^{K'} \)-isotypical part. Let

\[
\mathcal{J}L_{\pi} : H^1(\text{Sh}_{0,n}, F_p, \mathbb{Q}_\ell)_{\pi} \to H^2(n-1)(\text{Sh}_{1,n-1}, F_p, \mathbb{Q}_\ell)_{\pi}(n-1)
\]

denote the morphism on the \( \pi^{K'} \)-isotypical components induced by \( \mathcal{J}L \). Recall that the image of \( \mathcal{J}L_{\pi} \) is included in \( H^2(n-1)(\text{Sh}_{1,n-1}, F_p, \mathbb{Q}_\ell)_{\pi}(n-1)^{\text{fin}} \), which is the maximal subspace of \( H^2(n-1)(\text{Sh}_{1,n-1}, F_p, \mathbb{Q}_\ell)_{\pi}(n-1) \) where the action of \( \text{Gal}(\overline{\mathbb{F}_p}/F_p) \) factors through a finite quotient. Then our main result claims that this inclusion is actually an equality under certain genericity condition conditions on \( \pi_p \). To make this precise, write \( \pi_p = \pi_{p,0} \otimes \pi_p \) as representation of \( G(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \text{GL}_n(\mathbb{E}_p) \). Let

\[
\rho_{\pi_p} : W_{\mathbb{Q}_p,2} \to \text{GL}_n(\mathbb{Q}_\ell)
\]

be the unramified representation of the Weil group of \( \mathbb{Q}_p^2 \) defined in (1.6.1). It induces a continuous \( \ell \)-adic representation of \( \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p^2) \), which we denote by the same notation. Then \( \rho_{\pi_p}(\text{Frob}_{p^2}) \) is semi-simple with characteristic polynomial (1.6.2). Then we have an explicit description of \( H^2(n-1)(\text{Sh}_{1,n-1}, F_p, \mathbb{Q}_\ell)(n-1) \) and \( H^0(\text{Sh}_{0,n}, F_p, \mathbb{Q}_\ell) \) in terms of \( \rho_{\pi_p} \) by (1.4.1) and (1.6.3). We denote by \( m_{\pi_p}(\pi) \) (resp. \( m_{0,n}(\pi) \)) the multiplicity \( m_{\alpha_\bullet}(\pi) \) appearing in (1.6.3) with \( a_\bullet = (1, n-1) \) (resp. with \( a_\bullet = (0,n) \)). By definition, if \( \pi_{\infty} \) denotes a discrete representation of \( G_{1,n-1}(\mathbb{R}) \) such that \( \iota_\ell(\pi) \otimes \pi_{\infty} \) is automorphic, then \( m_{\pi_p}(\pi) \) is the multiplicity of \( \iota_\ell(\pi) \otimes \pi_{\infty} \) in the space of automorphic representations of \( G_{1,n-1}(\mathbb{A}) \); similarly statements hold with \( G_{1,n-1} \) replaced by \( G_{0,n} \).

We can now state our main theorem.

**Theorem 3.20.** Fix a \( \pi \in \mathcal{A}_K \), and let \( \alpha_{p,1}, \ldots, \alpha_{p,n} \) denote the eigenvalues of \( \rho_{\pi_p}(\text{Frob}_{p^2}) \).

1. If \( \alpha_{p,1}, \ldots, \alpha_{p,n} \) are distinct, then the map \( \mathcal{J}L_{\pi} \) is injective.
Theorem 4.2. For integers $n, r$ with $0 \leq r \leq n$, let
\[
\binom{n}{r}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-r+1} - 1)}{(q - 1)(q^2 - 1) \cdots (q^r - 1)}
\]

The proof of this theorem will be given at the end of Section 5.

Remark 3.21. The equality $m_{1,n-1}(\pi) = m_{0,n}(\pi)$ should be a consequence of the conjectural descriptions of Arthur’s A-packets for unitary groups.

4. Fundamental intersection number

In this section, we will compute some intersection numbers on certain Deligne-Lusztig varieties. These numbers will play a key role in the computation in the next section of the intersection matrix of the cycles $Y_j$ on $\text{Sh}_{1,n-1}$.

Let $X$ be an algebraic variety of pure dimension $N$ over $\mathbb{F}_p$. For an integer $r \geq 0$, let $A^r(X)$ (resp. $A_r(X)$) denote the group of algebraic cycles on $X$ of codimension $r$ (resp. dimension $r$) modulo rational equivalence. If $Y \subseteq X$ is a sub-scheme equidimensional of codimension $r$, we denote by $[Y] \in A^r(X)$ the class of $Y$. We put $A^r(X) = \bigoplus_{i=0}^N A^i(X)$. For a zero-dimensional cycle $\eta \in A^N(X)$, we denote by
\[
\deg(\eta) = \int_X \eta
\]
the degree of $\eta$. Let $\mathcal{V}$ be a vector bundle over $X$. We denote by $c_i(\mathcal{V}) \in A^i(X)$ the $i$-th Chern class of $\mathcal{V}$ for $0 \leq r \leq N$, and put $c(\mathcal{V}) = \sum_{i=0}^N c_i(\mathcal{V})t^i$ in the free variable $t$.

4.1. A special Deligne-Lusztig variety. We fix an integer $n \geq 1$. For an integer $0 \leq k \leq n$, we denote by $\text{Gr}(n,k)$ the Grassmannian variety over $\mathbb{F}_p$ classifying $k$-dimensional subspaces of $\mathbb{F}_p^n$.

Given an integer $k$ with $1 \leq k \leq n$, let $Z^{(n)}_k$ be the sub-scheme of $\text{Gr}(n,k) \times \text{Gr}(n,k-1)$ whose $S$-valued points are the set of pairs $(L_1, L_2)$, where $L_1$ and $L_2$ are respectively subbundles of $\mathcal{O}_S^n$ of rank $k$ and $k-1$ satisfying $L_2 \subseteq L_1^{(p)}$ and $L_2^{(p)} \subseteq L_1$ (with locally free quotients). The same arguments as in Proposition 3.3 show that $Z^{(n)}_k$ is a smooth variety over $\mathbb{F}_p$ of dimension $n-1$. We denote the natural closed immersion by
\[
i_k : Z^{(n)}_k \hookrightarrow \text{Gr}(n,k) \times \text{Gr}(n,k-1).
\]

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the universal sub-bundles on $\text{Gr}(n,k) \times \text{Gr}(n,k-1)$ coming from the two factors, and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ the universal quotients, respectively. When there is no confusion, we still use $\mathcal{L}_i$ and $\mathcal{Q}_i$ for $i = 1, 2$ to denote their restrictions to $Z^{(n)}_k$. We put
\[
(4.1.1) \quad \mathcal{E}_k = (\mathcal{L}_1/\mathcal{L}_2^{(p)})^* \otimes \mathcal{L}_2^{(p)} \otimes (\mathcal{L}_1^{(p)}/\mathcal{L}_2) \otimes \mathcal{Q}_1^{(p)},
\]
which is a vector bundle of rank $n-1$ on $Z^{(n)}_k$. We have the top Chern class $c_{n-1}(\mathcal{E}_k) \in A^{n-1}(Z^{(n)}_k)$. We define the fundamental intersection number on $Z^{(n)}_k$ as
\[
(4.1.2) \quad N(n,k) := \int_{Z^{(n)}_k} c_{n-1}(\mathcal{E}_k).
\]

The main theorem we prove in this section is the following.

Theorem 4.2. For integers $n, r$ with $0 \leq r \leq n$, let
\[
\binom{n}{r}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-r+1} - 1)}{(q - 1)(q^2 - 1) \cdots (q^r - 1)}
\]
be the Gaussian binomial coefficients, and let \( d(n, k) = (2k-1)n-2k(k-1)-1 \) denote the dimension of \( \text{Gr}(n, k) \times \text{Gr}(n, k-1) \). Then, for \( 1 \leq k \leq n \), we have

\[
N(n, k) = (-1)^{n-1} \sum_{\delta=0}^{\min\{k-1, n-k\}} (n-2\delta)p^{d(n-2\delta, k-\delta)} \binom{n}{\delta}. 
\]

\[(4.2.1)\]

**Remark 4.3.** We point out that this theorem seems to be more than a technical result. It is at the heart of the understanding of these cycles we constructed. This is why we do not put the proof into the appendix.

**Proof.** We first claim that \( N(n, k) = N(n, n+1-k) \) for \( 1 \leq k \leq n \). Let \((L_1, L_2)\) be an \( S \)-valued point of \( \text{Gr}(n, k) \times \text{Gr}(n, k-1) \), and \( Q_i = O_{S}^n/L_i \) for \( i = 1, 2 \) be the corresponding quotient bundle. Then \((L_1, L_2) \mapsto (Q_2^*, Q_1^*)\) defines a duality isomorphism

\[\theta: \text{Gr}(n, k) \times \text{Gr}(n, k-1) \to \text{Gr}(n, n+1-k) \times \text{Gr}(n, n-k).\]

Since \( L_2^{(p)} \subseteq L_1 \) (resp. \( L_2 \subseteq L_1^{(p)} \)) is equivalent to \( Q_2^{*(p)} \subseteq Q_1^{*} \) (resp. to \( Q_1^{*(p)} \subseteq Q_2^{*} \)), \( \theta \) induces an isomorphism between \( Z_k^{(n)} \) and \( Z_{n+1-k}^{(n)} \). It is also direct to check that \( \mathcal{E}_k = \theta^*(\mathcal{E}_{n+1-k}) \). This verifies the claim. Now since the right hand side of \((4.2.1)\) is also invariant under replacing \( k \) by \( n+1-k \), it suffices to prove the Theorem when \( k \leq \frac{n+1}{2} \).

We reduce the proof of the theorem to an analogous situation where the twists are given on one of \( L_i \)'s. Let \( \tilde{Z}_k^{(n)} \) be the subscheme of \( \text{Gr}(n, k) \times \text{Gr}(n, k-1) \) whose \( S \)-valued points are the set of pairs \((\tilde{L}_1, \tilde{L}_2)\), where \( \tilde{L}_1 \) and \( \tilde{L}_2 \) are respectively subbundles of \( O_S^n \) of rank \( k \) and \( k-1 \) satisfying \( \tilde{L}_2 \subseteq \tilde{L}_1 \) and \( \tilde{L}_2^{(p)} \subseteq \tilde{L}_1 \). The relative Frobenius morphisms on the two Grassmanian factors induces two morphisms

\[
\begin{array}{ccc}
Z_k^{(n)} & \stackrel{\varphi}{\longrightarrow} & \tilde{Z}_k^{(n)} \\
(L_1, L_2) & \mapsto & (L_1^{(p)}, L_2) \\
(\tilde{L}_1, \tilde{L}_2) & \mapsto & (\tilde{L}_1, \tilde{L}_2^{(p)})
\end{array}
\]

such that the composition is the relative Frobenius on \( \tilde{Z}_k^{(n)} \). Using a simple deformation computation, we see that \( \varphi \) has degree \( p^{n-k} \) and \( \tilde{\varphi} \) has degree \( p^{k-1} \). Let \( \tilde{\mathcal{L}}_1 \) and \( \tilde{\mathcal{L}}_2 \) denote the universal sub-bundles on \( \text{Gr}(n, k) \times \text{Gr}(n, k-1) \) when restricted to \( \tilde{Z}_k^{(n)} \); let \( \tilde{\mathcal{Q}}_1 \) and \( \tilde{\mathcal{Q}}_2 \) denote the universal quotients, respectively. We put

\[
(4.3.1) \quad \tilde{\mathcal{E}}_k = (\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)})^* \otimes (\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2) \otimes (\tilde{\mathcal{Q}}_1^*, \tilde{\mathcal{Q}}_2^*),
\]

which is a vector bundle of rank \( n-1 \) on \( \tilde{Z}_k^{(n)} \).

Note that

\[\varphi^*(\tilde{\mathcal{E}}_k) = (\mathcal{L}_1^{(p)}/\mathcal{L}_2^{(p^2)})^* \otimes \mathcal{L}_2^{(p^2)} \otimes (\mathcal{L}_1^{(p)}/\mathcal{L}_2) \otimes \mathcal{Q}_2^{*(p)}.
\]

Comparing with \( \mathcal{E}_k \), we see that \( c_{n-1}(\varphi^*(\tilde{\mathcal{E}}_k)) = p^{k-1}c_{n-1}^*(\mathcal{E}_k) \), where the factor \( p^{k-1} \) comes from the Frobenius twist on the first factor. Thus, we have

\[
(4.3.2) \quad \int_{\tilde{Z}_k^{(n)}} c_{n-1}(\tilde{\mathcal{E}}_k) = (\text{deg } \varphi)^{-1} \int_{Z_k^{(n)}} c_{n-1}(\varphi^*(\tilde{\mathcal{E}}_k))
\]

\[= p^{k-n} \int_{\tilde{Z}_k^{(n)}} p^{k-1}c_{n-1}(\mathcal{E}_k) = p^{2k-n-1}N(n, k).
\]

Since \( d(n-\delta, k-\delta) + 2k - n - 1 = 2(k-\delta - 1)(n-k + \delta + 1) \), the Theorem is in fact equivalent to the following (for each fixed \( k \)).
Proposition 4.4. For $1 \leq k \leq \frac{n+1}{2}$, we have

$$\int_{\mathcal{Z}_1^{\langle n \rangle}} c_{n-1}(\tilde{\mathcal{E}}) = (-1)^{n-1} \sum_{\delta=0}^{k-1} (n-2\delta)p^{2(k-\delta)(n-k-\delta+1)} \binom{n}{\delta}.$$  

Remark 4.5. Before giving the proof of this proposition, we point out a variant of the construction of $\mathcal{Z}_k^{\langle n \rangle}$. Let $\tilde{\mathcal{Z}}_k^{\langle n \rangle}$ be the subscheme of $\text{Gr}(n,k) \times \text{Gr}(n,k-1)$ whose $S$-valued points are the set of pairs $(\tilde{L}_1', \tilde{L}_2')$ where $\tilde{L}_1'$ and $\tilde{L}_2'$ are respectively subbundles of $O_S^\otimes n$ of rank $k$ and $k-1$ satisfying $\tilde{L}_2' \subseteq \tilde{L}_1'$ and $\tilde{L}_2' \subseteq \tilde{L}_1'(p^2)$ (Note that the twist is on $\tilde{L}_1'$ as opposed to be on $\tilde{L}_2'$). This is again a certain partial-Frobenius twist of $\mathcal{Z}_k^{\langle n \rangle}$; it is smooth of dimension $n-1$. Define the universal subbundles and quotient-bundles $\tilde{\mathcal{E}}_1'$, $\tilde{\mathcal{L}}_2'$, $\tilde{Q}_1'$, $\tilde{Q}_2'$ similarly. We put

$$\tilde{\mathcal{E}}_k = (\tilde{\mathcal{E}}_1'/\tilde{\mathcal{L}}_2') \otimes \mathcal{L}_2'^* \oplus (\tilde{\mathcal{L}}_1'(p^2)/\tilde{\mathcal{E}}_2') \otimes (\tilde{\mathcal{Q}}_1'(p^2)).$$

Using the same argument as above, we see that, for every fixed $k$,

$$\int_{\mathcal{Z}_k^{\langle n \rangle}} c_{n-1}(\tilde{\mathcal{E}}) = p^{n+1-2k} N(n,k).$$

Note that the exponent is different from (4.3.2). So Proposition 4.4 for each fixed $k$ is equivalent to

$$\int_{\mathcal{Z}_k^{\langle n \rangle}} c_{n-1}(\tilde{\mathcal{E}}) = (-1)^{n-1} \sum_{\delta=0}^{k-1} (n-2\delta)p^{2(k-\delta)(n-k-\delta)} \binom{n}{\delta},$$

as $(k-\delta)(n-k-\delta) - (k-\delta-1)(n-k-\delta+1) = n - 2k + 1$.

Proof of Proposition 4.4. We first prove it in the case of $k = 1, 2$ and then we explain an induction process to deal with the general case.

When $k = 1$, $\tilde{\mathcal{Z}}_1^{\langle n \rangle}$ classifies a line-subbundle $\tilde{L}_1$ in $O_S^\otimes n$ with no additional condition (as $\tilde{L}_2$ is zero); so $\tilde{\mathcal{Z}}_1^{\langle n \rangle} \cong \mathbb{P}^{n-1}$ and $\tilde{\mathcal{L}}_1 = O_{\mathbb{P}^{n-1}}(-1)$. The vector bundle $\tilde{\mathcal{E}}_1$ equals to $\tilde{\mathcal{L}}_1 \otimes \tilde{\mathcal{Q}}_1^*$. It is straightforward to check that

$$c(\tilde{\mathcal{E}}_1) = (1 + c_1(O_{\mathbb{P}^{n-1}}(-1)))^n$$

and hence

$$\int_{\tilde{\mathcal{Z}}_1^{\langle n \rangle}} c_{n-1}(\tilde{\mathcal{E}}_1) = (-1)^{n-1} n;$$

the Proposition is proved in this case.

When $k = 2$, we consider a forgetful morphism

$$\psi: \tilde{\mathcal{Z}}_2^{\langle n \rangle} \longrightarrow \tilde{\mathcal{Z}}_1^{\langle n \rangle},$$

$$(\tilde{L}_1, \tilde{L}_2) \longmapsto (\tilde{L}_1).$$

This morphism is an isomorphism over the closed points $x \in \tilde{\mathcal{Z}}_1^{\langle n \rangle}(\mathbb{F}_p)$ for which $\tilde{L}_{2,x} \neq \tilde{L}_{2,x}^{(p^2)}$, because in this case, $\tilde{L}_{1,x}$ is forced to be $\tilde{L}_{2,x} + \tilde{L}_{2,x}^{(p^2)}$. On the other hand, for a closed point $x \in \tilde{\mathcal{Z}}_2^{\langle n \rangle}(\mathbb{F}_p)$ where $\tilde{L}_{2,x} = \tilde{L}_{2,x}^{(p^2)}$, i.e. for $x \in \tilde{\mathcal{Z}}_1^{\langle n \rangle}(\mathbb{F}_p, \tilde{L}_{2,x}^{(p^2)} \cong \mathbb{P}^{n-1}(\mathbb{F}_p^{p^2})), \psi^{-1}(x)$ is the space classifying a line $\tilde{L}_1$ in $\mathbb{P}_p^{\otimes n}/\tilde{L}_{2,x}$; so $\psi^{-1}(x) \cong \mathbb{P}^{n-2}$. A simple tangent space computation shows that $\psi$ is the blow-up morphism of $\tilde{\mathcal{Z}}_1^{\langle n \rangle} \cong \mathbb{P}^{n-1}$ at all of its $\mathbb{F}_p^{p^2}$-points. We use $E$ to denote the exceptional divisors, which is a disjoint union of $\mathbb{P}^{n-2}$ copies of $\mathbb{P}^{n-2}$.

Note that the vanishing of the morphism $\tilde{\mathcal{L}}_2 \rightarrow \tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)}$ defines the divisor $E$ (as we can see using deformation); so

$$O_{\tilde{\mathcal{Z}}_2^{\langle n \rangle}}(E) \cong \tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)} \otimes \tilde{\mathcal{L}}_2^{-1}.$$
Put $\eta = c_1(\tilde{L}_2) = \psi^*c_1(O_{\mathbb{P}^n-1}(-1))$ and $\xi = c_1(E)$. Then

$$c(\tilde{E}_2) = c((\tilde{L}_1/\tilde{L}_2^{(p^2)})^* \otimes \tilde{L}_2^{(p^2)}) \cdot c((\tilde{L}_1/\tilde{L}_2) \otimes \tilde{Q}_1)$$

(4.5.1)

$$= (1 - \xi + (p^2 - 1)\eta) \cdot (1 + \xi + p^2\eta)^n / (1 + \xi + (p^2 - 1)\eta),$$

where the computation of the second term comes from the following two exact sequences

$$0 \to (\tilde{L}_1/\tilde{L}_2) \otimes \tilde{Q}_1 \to (\tilde{L}_1/\tilde{L}_2)^{\oplus n} \to (\tilde{L}_1/\tilde{L}_2) \otimes \tilde{L}_1^* \to 0;$$

$$0 \to O_{\mathbb{P}^n(\mathbb{P})} \to (\tilde{L}_1/\tilde{L}_2) \otimes \tilde{Q}_1 \to (\tilde{L}_1/\tilde{L}_2) \otimes \tilde{L}_2^* \to 0.$$

Note that $\int_{\mathbb{P}^n(\mathbb{P})} \xi^j\eta^i = 0$ unless $(i, j) = (n - 1, 0)$ or $(0, n - 1)$, in which case we have

$$\int_{\mathbb{P}^n(\mathbb{P})} \eta^{n-1} = (-1)^{n-1} \quad \text{and} \quad \int_{\mathbb{P}^n(\mathbb{P})} \xi^{n-1} = (-1)^{n} \left(\frac{n}{1}\right).$$

Here, to prove the last formula, we used the fact that the restriction of $O_{\mathbb{P}^n(\mathbb{P})}(E)$ to each irreducible component $\mathbb{P}^{n-2}$ of $E$ is isomorphic to $O_{\mathbb{P}^n(\mathbb{P})}(-1)$. So it suffices to compute

- the $\xi^{n-1}$-coefficient of (4.5.1), which is the same as the $\xi^{n-1}$-coefficients of $(1 - \xi)(1 + \xi)^{n-1}$ and is equal to $2 - n$; and
- the $\eta^{n-1}$-coefficient of (4.5.1), which is the same as the $\eta^{n-1}$ coefficients of $(1 + (p^2 - 1)\eta)(1 + p^2\eta)^n / (1 + (p^2 - 1)\eta) = (1 + p^2\eta)^n$ and is equal to $np^{2(n-1)}$.

To sum up, we see that

$$\int_{\mathbb{P}^n(\mathbb{P})} c_{n-1}(\tilde{E}_2) = (-1)^{n-1}np^{2(n-1)} + (-1)^{n}(2 - n)\left(\frac{n}{1}\right).$$

which is exactly (4.4.1) for $k = 2$.

In general, we make an induction on $k$. Assume that the Proposition is proved for $k - 1 \geq 1$ and we now prove the Proposition for $k$ (assuming that $k \leq \frac{n+1}{2}$). By Remark 4.5, we get the similar intersection formula for $\tilde{E}_k$ on $\tilde{Z}_{k-1}$:

$$\int_{\tilde{Z}_{k-1}} c_{n-1}(\tilde{E}_k) = (-1)^{n-1}np^{2(n-1)} + (-1)^{n}(2 - n)\left(\frac{n}{1}\right).$$

(4.5.2)

We consider the moduli space $W$ over $\mathbb{P}^{p^2}$ whose S-points are tuples $(\tilde{L}_1, \tilde{L}_2 = \tilde{L}'_2, \tilde{L}'_3)$, where $\tilde{L}_1$, $\tilde{L}_2$, and $\tilde{L}'_3$ are respectively subbundles of $O_{\mathbb{P}^n}^{\otimes n}$ of rank $k$, $k - 1$, and $k - 2$ satisfying $\tilde{L}'_3 \subset \tilde{L}_2 \subset \tilde{L}_1$ and $\tilde{L}'_3 \subset \tilde{L}_2^{(p^2)} \subset \tilde{L}_1$. It is easy to use deformation theory to check that $W$ is a smooth variety of dimension $n - 1$. There are two natural morphisms

$$\psi_{12}: W \to \tilde{Z}_k \to \tilde{Z}_{k-1} \to (\tilde{L}_1, \tilde{L}_2 = \tilde{L}'_2, \tilde{L}'_3)$$

$$\psi_{23}: \tilde{Z}_k \to W \to (\tilde{L}_1, \tilde{L}_2) \to (\tilde{L}_1, \tilde{L}_2).$$

Let $E$ denote the subspace of $W$ whose closed points $x \in W(\mathbb{F}_p)$ are those such that $\tilde{L}_{2,x} = \tilde{L}'_{2,x}$, i.e. $\tilde{L}_{2,x}$ is an $\mathbb{F}_p$-rational subspace of $\mathbb{F}_p^{\otimes n}$ of dimension $k - 1$. It is clear that $E$ is a disjoint union of $(\frac{n}{k-1})^2$ copies (corresponding to the choices of $\tilde{L}_2$) of $\mathbb{P}^{k-2} \times \mathbb{P}^{n-k}$ (corresponding to the choice of $\tilde{L}'_3$ and $\tilde{L}_1$ respectively). It gives rise to a smooth divisor on $W$. 

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For a point $x \in (W \setminus E)(\mathbb{F}_p)$, we have $\tilde{L}_{2,x} \neq \tilde{L}_{2,x}^{(p^2)}$ and hence it uniquely determines both $\tilde{L}_{3,x}$ and $\tilde{L}_{1,x}$; so $\psi_{23}$ and $\psi_{13}$ are isomorphisms restricted to $W \setminus E$. On the other hand, when restricted to $E$, $\psi_{23}$ contracts each copy of $\mathbb{P}^{k-2} \times \mathbb{P}^{n-k}$ of $E$ into the first factor $\mathbb{P}^{k-2}$; whereas $\psi_{12}$ contracts each copy of $\mathbb{P}^{k-2} \times \mathbb{P}^{n-k}$ of $E$ into the second factor $\mathbb{P}^{n-k}$. It is clear from this (with a bit help of deformation argument) that $\psi_{23}$ is the blow-up of $Z_{k-1}^{(n)}$ along $\psi_{23}(E)$ and $\psi_{12}$ is the blow-up of $Z_{k}^{(n)}$ along $\psi_{12}(E)$; the divisor $E$ is the exceptional divisor for both blow-ups.

A simple deformation theory shows that the normal bundle of $E$ in $W$ when restricted to each component $\mathbb{P}^{k-2} \times \mathbb{P}^{n-k}$ is $\mathcal{O}_{\mathbb{P}^{k-2}}(-1) \otimes \mathcal{O}_{\mathbb{P}^{n-k}}(-1)$. Moreover, we can characterize $E$ as the zero locus of either one of the following natural homomorphisms

$$
\tilde{L}_2^{(p^2)}/\tilde{L}_3^{(p^2)} \rightarrow \tilde{L}_1/\tilde{L}_2, \quad \tilde{L}_2/\tilde{L}_3 \rightarrow \tilde{L}_1/\tilde{L}_2^{(p^2)}.
$$

So as a line bundle over $W$, we have

$$
\mathcal{O}_W(E) \cong (\tilde{L}_2^{(p^2)}/\tilde{L}_3^{(p^2)})^{-1} \otimes (\tilde{L}_1/\tilde{L}_2) \cong (\tilde{L}_2/\tilde{L}_3)^{-1} \otimes (\tilde{L}_1/\tilde{L}_2^{(p^2)}).
$$

We want to compare

$$
\int_{Z_{k-1}^{(n)}} c_{n-1}(\tilde{L}_2^{(p^2)}) = \int_W c_{n-1}(\psi_{23}(\tilde{L}_2^{(p^2)})) \quad \text{and} \quad \int_{Z_{k}^{(n)}} c_{n}(\tilde{L}_1/\tilde{L}_2^{(p^2)}) = \int_W c_{n-1}(\psi_{12}(\tilde{L}_1/\tilde{L}_2^{(p^2)})).
$$

We will show that they differ by $(2k - n - 2)(-1)^n \binom{n}{k-1} \mathbb{F}_p$, and this will conclude the proof of the Proposition by induction hypothesis (4.5.2). Indeed, we have

$$
(4.5.3) \quad c(\psi_{23}(\tilde{L}_2^{(p^2)})) = c((\tilde{L}_2/\tilde{L}_3)^{* (p^2)} \otimes \tilde{L}_3^{(p^2)}) \cdot c((\tilde{L}_2^{(p^2)}/\tilde{L}_3^{(p^2)}) \otimes \tilde{Q}_2^{* (p^2)}), \quad \text{and}
$$

$$
(4.5.4) \quad c(\psi_{12}(\tilde{L}_2)) = c((\tilde{L}_1/\tilde{L}_2^{(p^2)})^{* (p^2)} \otimes \tilde{L}_2^{(p^2)}) \cdot c((\tilde{L}_1/\tilde{L}_2^{(p^2)}) \otimes \tilde{Q}_1^{* (p^2)}),
$$

where $\tilde{Q}_1$ and $\tilde{Q}_2$ are the universal quotient vector bundles. Consider the following two exact sequences where the last two terms are identified:

$$
\begin{align*}
&0 \rightarrow (\tilde{L}_2/\tilde{L}_3)^{-1} \otimes \tilde{L}_3 \rightarrow (\tilde{L}_2/\tilde{L}_3)^{-1} \otimes \tilde{L}_2^{(p^2)} \rightarrow (\tilde{L}_2/\tilde{L}_3)^{-1} \otimes (\tilde{L}_2^{(p^2)}/\tilde{L}_3^{(p^2)}) \rightarrow 0 \\
&0 \rightarrow (\tilde{L}_1/\tilde{L}_2) \otimes \tilde{Q}_2^{* (p^2)} \rightarrow (\tilde{L}_1/\tilde{L}_2) \otimes \tilde{Q}_2^{* (p^2)} \rightarrow (\tilde{L}_2/\tilde{L}_3)^{-1} \otimes (\tilde{Q}_2^{* (p^2)}/\tilde{Q}_1^{* (p^2)}) \rightarrow 0
\end{align*}
$$

Here the right vertical isomorphism is given by

$$
(\tilde{L}_1/\tilde{L}_2) \otimes (\tilde{Q}_2^{* (p^2)}/\tilde{Q}_1^{* (p^2)}) \cong (\tilde{L}_1/\tilde{L}_2) \otimes (\tilde{L}_1^{(p^2)})^{-1}
$$

$$
\cong ((\tilde{L}_2^{(p^2)}/\tilde{L}_3^{(p^2)}) \otimes \mathcal{O}_W(E)) \otimes ((\tilde{L}_2/\tilde{L}_3) \otimes \mathcal{O}_W(E))^{-1} \cong (\tilde{L}_2/\tilde{L}_3)^{-1} \otimes (\tilde{L}_2^{(p^2)}/\tilde{L}_3^{(p^2)}).
$$

From these two exact sequences we see that

$$
c((\tilde{L}_2/\tilde{L}_3)^{-1} \otimes \tilde{L}_3^{(p^2)}) \cdot c(\mathcal{O}_W(E) \otimes (\tilde{L}_2^{(p^2)}/\tilde{L}_3) \otimes \tilde{Q}_2^{* (p^2)}) = c((\tilde{L}_1/\tilde{L}_2) \otimes \tilde{Q}_1^{* (p^2)}) \cdot c(\mathcal{O}_W(E) \otimes (\tilde{L}_2^{(p^2)})^{-1} \otimes \tilde{L}_2^{(p^2)}).
$$
Comparing this with \([4.5.3]\) and \([4.5.4]\), we get
\[
\begin{align*}
c_{n-1}(\psi_{12}^*(-\tilde{E}_k)) - c_{n-1}(\psi_{23}^*(-\tilde{E}_k)) = &
\left(c_{k-1}(\{L_1/\ell_2^{(n)}\}^{-1} \otimes \tilde{L}_2^{(p)}) - c_{k-1}(\mathcal{O}_W(E) \otimes (\tilde{L}_1/\ell_2^{(p)}))^{-1} \otimes \tilde{L}_2^{(p)})\right) \
& \cdot c_{n-k}(\tilde{L}_1/\ell_2 \otimes \tilde{Q}_1) \
& - c_{k-2}(\{L_2/\ell_3^{(n)}\}^{-1} \otimes \tilde{L}_3^{(p)}) \
& \cdot \left(c_{n-k+1}(\{L_2/\ell_3^{(n)}\}^{-1} \otimes \tilde{Q}_2^{(p)}) - c_{n-k+1}(\mathcal{O}_W(E) \otimes (\tilde{L}_2/\ell_3^{(p)}))^{-1} \otimes \tilde{Q}_2^{(p)})\right)
\end{align*}
\]
Recall that \(E\) is the exceptional divisor for the blow-up \(\psi_{23}\) centered at a disjoint union of \(\mathbb{P}^{k-2}\); so \(c_1(E)\) kills \(\psi_{23}^*(A(\tilde{Z}_{k-1}^{(n)}))\) for \(i \geq k - 1\). Similarly, \(c_1(E)\) kills \(\psi_{12}^*(A(\tilde{Z}_k^{(n)}))\) for \(i \geq n - k + 1\). As a result, we can rewrite the above complicated formula as
\[
\begin{align*}
c_{n-1}(\psi_{12}^*(-\tilde{E}_k)) - c_{n-1}(\psi_{23}^*(-\tilde{E}_k)) = &
\left(-c_1(E)^{k-2} \cdot c_{n-k}(\{L_1/\ell_2 \otimes \tilde{Q}_1^*) + c_{k-2}(\{L_2/\ell_3^{(n)}\}^{-1} \otimes \tilde{L}_3^{(p)}) \cdot c_1(E)^{n-k}\right) \
= &\left((-1)^{k-1}c_{n-k}(\{L_1/\ell_2 \otimes \tilde{Q}_1^*)|_{\psi_{12}(E)} + (-1)^{n-k}c_{k-2}(\{L_2/\ell_3^{(n)}\}^{-1} \otimes \tilde{L}_3^{(p)})|_{\psi_{23}(E)}\right)
\end{align*}
\]
For the first term, over each \(\mathbb{P}^{n-k}\) of \(\psi_{12}(E)\), it is to take the top Chern class of the canonical subbundle of rank \(n - k\) twisted by \(\mathcal{O}_{\mathbb{P}^{n-k}}(-1)\); it is \((-1)^{n-k}(n - k + 1)\) on each \(\mathbb{P}^{k-2}\). For the second term, over each \(\mathbb{P}^{k-2}\), it is again the canonical subbundle of rank \(k - 2\) twisted by \(\mathcal{O}_{\mathbb{P}^{k-2}}(-1)\); it is \((-1)^{k-2}(k - 1)\) on each \(\mathbb{P}^{k-2}\). To sum up, we have
\[
\begin{align*}
c_{n-1}(\psi_{12}^*(-\tilde{E}_k)) - c_{n-1}(\psi_{23}^*(-\tilde{E}_k)) = &
(-1)^{k-1}(-1)^{n-k}(n - k + 1)(\binom{n}{k-1})p^2 + (-1)^{n-k}(-1)^{k-2}(k - 1)(\binom{n}{k-1})p^2 \
= &\left((-1)^{n-k}(n - 2k + 2)(\binom{n}{k-1})p^2\right).
\end{align*}
\]
This concludes the proof of Proposition. \(\square\)

5. Intersection matrix of cycles on \(\text{Sh}_{1,n-1}\)

Throughout this section, we fix an integer \(n \geq 2\) and keep the notation of Section \(\S\). We will study the intersection theory of cycles \(Y_j\) for \(1 \leq j \leq n\) on \(\text{Sh}_{1,n-1}\) considered in Section \(\S\). For this, it does not hurt to assume the following.

**Hypothesis 5.1.** We assume that the tame level structure \(K^p\) is taken sufficiently small so that Proposition \(\S.5.12\) holds.

5.2. Hecke correspondence on \(\text{Sh}_{0,n}\). Recall that we have an isomorphism
\[
G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \text{GL}_n(\mathbb{E}_p) \cong \mathbb{Q}_p^\times \times \text{GL}_n(\mathbb{Q}_p^\times).
\]
Let \(K_p = \text{GL}_n(\mathcal{O}_{E_p})\) and \(K_p = \mathbb{Z}_p^\times \times K_p\). The Hecke algebra \(\mathbb{Z}[K_p \backslash \text{GL}_n(\mathbb{E}_p)/K_p]\) can be viewed as a sub-algebra of \(\mathbb{Z}[K_p \backslash \text{GL}_n(\mathbb{Q}_p)/K_p]\) with trivial factor at \(\mathbb{Q}_p^\times\)-component.

For \(\gamma \in \text{GL}_n(\mathbb{E}_p)\), the double coset \(T_p(\gamma) := K_p \gamma K_p\) defines a Hecke correspondence on \(\text{Sh}_{0,n}\). It induces a set theoretic Hecke correspondence
\[
T_p(\gamma): \text{Sh}_{0,n}(\overline{\mathbb{F}}_p) \to \mathcal{S}(\text{Sh}_{0,n}(\overline{\mathbb{F}}_p)),
\]
where \(\mathcal{S}(\text{Sh}_{0,n}(\overline{\mathbb{F}}_p))\) denotes the set of subsets of \(\text{Sh}_{0,n}(\overline{\mathbb{F}}_p)\). By Remark \(\S.5.13\), \(\text{Sh}_{0,n}(\overline{\mathbb{F}}_p)\) is a union of \# \(\ker^1(\mathbb{Q}, G_{0,n})\)-isogenous classes of abelian varieties. Fix a base point \(z_0 \in \text{Sh}_{0,n}(\overline{\mathbb{F}}_p)\). Then by Proposition \(\S.5.12\), there is a natural bijection
\[
\Theta_{z_0}: \text{Isog}(z_0) \xrightarrow{\sim} G_{0,n}(\mathbb{Q}) \backslash (G(\mathbb{A}_{\infty}^\times_p) \times G(\mathbb{Q}_p)) / K^p \times K_p.
\]
Write \(K_p \gamma K_p = \bigoplus_{i \in I} \gamma_i K_p\). If \(z \in \text{Isog}(z_0)\) corresponds to the class of \((g^p, g_p) \in G(\mathbb{A}_{\infty}^\times_p) \times G(\mathbb{Q}_p)\) with \(g_p = (g_p, \gamma_i, g_p)\), then \(T_p(\gamma)(z)\) consists of points in \(\text{Isog}(z_0)\) corresponding to the class of \((g^p, (g_p, \gamma_i, g_p))\) for all \(i \in I\).
By the definition of $\Theta_{z_0}$ in (3.11.1), $T_p(\gamma)$ has an alternative description as follows. Write $z = (A, \lambda, \eta)$, and let $\mathbb{L}_z$ denote the $\mathbb{Z}_p$-free module $\mathcal{D}(A)_{1,0,\mathfrak{F}^2=p}$. Then a point $z' = (B, \lambda', \eta') \in \text{Sh}_{0,n}(\overline{\mathbb{F}}_p)$ belongs to $T_p(\gamma)(z)$ if and only if there exists an $\mathcal{O}_{D'}$-equivariant $p$-quasi-isogeny $\phi : B' \to B$ (i.e. $p^m \phi$ is an isogeny of $p$-power order for some integer $m$) such that

1. $\phi' \circ \lambda \circ \phi = \lambda'$,
2. $\phi \circ \eta' = \eta$,
3. $\phi_* (\mathbb{L}_{z'})$ is a lattice of $\mathbb{L}_z[1/p] = \mathbb{L}_z \otimes_{\mathbb{Z}_p} \mathbb{Q}_p^2$ with the property: there exists a $\mathbb{Z}_p$-basis $(e_1, \ldots, e_n)$ for $\mathbb{L}_z$ such that $(\gamma e_1, \ldots, \gamma e_n)$ is a $\mathbb{Z}_p$-basis for $\phi_* (\mathbb{L}_{z'})$.

For given $z$, $z'$ and $\gamma$, such a $\phi$ is necessarily unique if it exists, by Lemma 3.14. Therefore, $T_p(\gamma)(z)$ is in natural bijection with the set of $\mathbb{Z}_p$-lattices $L' \subseteq \mathbb{L}_z[1/p]$ satisfying the property of (3) above.

For each integer $i$ with $0 \leq i \leq n$, we put

$$T_p(i) = T_p(\text{diag}(p, \ldots, p, 1, \ldots, 1))$$

with $i$ copies of $p$ on the main diagonal. By the discussion above, one has a natural bijection

$$T_p(i)(z) \sim \{ \mathbb{L}_{z'} \subseteq \mathbb{L}_z[1/p] \mid p\mathbb{L}_z \subseteq \mathbb{L}_{z'} \subseteq \mathbb{L}_z, \dim_{\mathbb{Z}_p} (\mathbb{L}_z/\mathbb{L}_{z'}) = i \}$$

for $z \in \text{Sh}_{0,n}(\overline{\mathbb{F}}_p)$. Note that $T_p(0) = \text{id}$, and we put usually $S_p := T_p(n)$. Then Satake isomorphism implies $\mathbb{Z}[K \setminus \text{GL}_n(E)]/K \approx \mathbb{Z}[T_p(1), \ldots, T_p(n-1), S_p, S_p^{-1}]$. More generally, for $0 \leq a \leq b \leq n$, we put

$$R_p^{(a,b)} = T_p(\text{diag}(p^2, \ldots, p^2, p, \ldots, p, 1, \ldots, 1))$$

with $a$ copies of $p^2$, $b-a$ copies of $p$, and $n-b$ copies of $1$. We note that $R_p^{(0,i)} = T_p(i)$, and $R_p^{(a,b)} S_p^{-1}$ is the Hecke operator $T_p(\text{diag}(p, \ldots, p, 1, \ldots, 1, p^{-1}, \ldots, p^{-1}))$ with $a$ copies of $p$, $b-a$ copies of $1$ and $n-b$ copies of $p^{-1}$. For the explicit relations between $R_p^{(a,b)}$ and $T_p^{(i)}$, see Proposition B.1.

5.3. Refined Gysin homomorphism. For an algebraic variety $X$ over $\overline{\mathbb{F}}_p$ of pure dimension $N$ and any integer $r \geq 0$, $A_r(X) = A^{N-r}(X)$ denotes the group of dimension $r$ (codimension $N - r$) cycles in $X$ modulo rational equivalence. Recall that the restriction of $\text{pr}_j : Y_j \to \text{Sh}_{1,n-1}$ to each $Y_{j,z}$ for $z \in \text{Sh}_{0,n}(\overline{\mathbb{F}}_p)$ and $1 \leq j \leq n$ is a regular closed immersion. There is a well defined Gysin homomorphism

$$\text{pr}_j^1 : A_{n-1}(\text{Sh}_{1,n-1,\overline{\mathbb{F}}_p}) \to A_0(Y_{j,z}),$$

whose composition with the natural projection $A_0(Y_{j,z}) \to A_0(Y_{j,z})$ is the refined Gysin map $(\text{pr}_j |_{Y_{j,z}})^!$ defined in [Fu98, 6.2] for regular immersions. Let $X \subseteq \text{Sh}_{1,n-1,\overline{\mathbb{F}}_p}$ be a closed sub-variety of dimension $n - 1$. Consider the Cartesian diagram

$$\begin{array}{ccc}
Y_{j,z} \times_{\text{Sh}_{1,n-1,\overline{\mathbb{F}}_p}} X & \xrightarrow{g_X} & X \\
\downarrow g_j & & \downarrow \\
Y_{j,z} & \xrightarrow{\text{pr}_j} & \text{Sh}_{1,n-1,\overline{\mathbb{F}}_p}.
\end{array}$$

Assume that the restriction of $g_X$ to each $Y_{j,z} \times_{\text{Sh}_{1,n-1}} X$ with $z \in \text{Sh}_{0,n}(\overline{\mathbb{F}}_p)$ is a regular closed immersion as well. Then $\text{pr}_j^1([X]) \in A_0(Y_{j,z})$ can be described as follows. Put $N_{Y_{j,z}}(\text{Sh}_{1,n-1}) = \pi_j^* (T_{\text{Sh}_{1,n-1}})/T_{Y_{j,z}}$, and similarly for $N_{Z \times \text{Sh}_{1,n-1}, Y_{j,z}}(Z)$. We define the excess vector bundle as

$$\mathcal{E}(Y_{j,z}, X) = g_j^* N_{Y_{j,z}}(\text{Sh}_{1,n-1})/N_{Y_{j,z} \times \text{Sh}_{1,n-1}} X.$$

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This is a vector bundle on $Y_{j,}\mathbb{F}_p \times_{\text{Sh}_{1,n-1}} X$, and let $r$ be its rank, which equals to the dimension of $Y_{j,}\mathbb{F}_p \times_{\text{Sh}_{1,n-1}} X$. Then the excess intersection formula [Pfl08, 6.3] shows that

$$\text{pr}_j^i([X]) = \sum_{z \in \text{Sh}_{0,n}(\mathbb{F}_p)} c_r(\mathcal{E}(Y_{j,z}, X)) \cdot [Y_{j,z} \times_{\text{Sh}_{1,n-1}} X],$$

where $c_r(\mathcal{E}_z)$ is the top Chern class of $\mathcal{E}(Y_{j,z}, X)$, and $[Y_{j,}\mathbb{F}_p \times_{\text{Sh}_{1,n-1}} X]$ is viewed as an element of $A_r(Y_{j,z})$.

**Proposition 5.4.** Let $i, j$ be integers with $1 \leq i \leq j \leq n$ and $z, z' \in \text{Sh}_{0,n}(\mathbb{F}_p)$.

1. The sub-varieties $Y_{i,z}$ and $Y_{j,z'}$ of $\text{Sh}_{1,n-1}$ have non-empty intersection if and only if there exists an integer $\delta$ with $0 \leq \delta \leq \min\{n - j, i - 1\}$ such that $z' \in R_p^{(j-i+\delta,n-\delta)} S_p^{-1}(z)$, or equivalently $z \in R_p^{(\delta,n-i-j-\delta)} S_p^{-1}(z')$, where $R_p^{(a,b)}$ and $S_p$ are the Hecke operators defined in Subsection 5.2.

2. If the condition in (1) is satisfied for some $\delta$, then $Y_{i,z} \times_{\text{Sh}_{1,n-1}} Y_{j,z'}$ is isomorphic to the variety $Z_i^{(n+i-j-2\delta)}$ defined in Subsection 4.1. Moreover, the excess vector bundles $\mathcal{E}(Y_{i,z}, Y_{j,z'})$ and $\mathcal{E}(Y_{j,z}, Y_{i,z})$ are both isomorphic to the vector bundle $(4.1.1)$ on $Z_i^{(n+i-j-2\delta)}$.

**Proof.** Let $(B_z, \lambda_z, \eta_z)$ and $(B_{z'}, \lambda_{z'}, \eta_{z'})$ be the polarized abelian varieties given by $z, z'$. Then $Y_{i,z} \times_{\text{Sh}_{1,n-1}} Y_{j,z'}$ is the moduli space of triples $(A, \lambda, \eta, \phi, \phi')$ where $\phi : B_z \to A$ and $\phi' : B_{z'} \to A$ are isogenies such that $(A, \lambda, \eta, B_z, \lambda_z, \eta_z, \phi)$ and $(A, \lambda, \eta, B_{z'}, \lambda_{z'}, \eta_{z'}, \phi')$ are points of $Y_{i,z}$ and $Y_{j,z'}$, respectively.

Assume first that $Y_{i,z} \times_{\text{Sh}_{1,n-1}} Y_{j,z'}$ is non-empty, and let $(A, \lambda, \eta, \phi, \phi')$ be an $\mathbb{F}_p$-valued point of it. Denote by $\omega^\circ_{A,v,k} \subseteq \tilde{D}(A)_k$ for $k = 1, 2$ the inverse image of $\omega^\circ_{A,v,k} \subseteq H^1_{\text{dR}}(A/\mathbb{F}_p) = \tilde{D}(A)_k$. We identify $\tilde{D}(B_z)_k$ and $\tilde{D}(B_{z'})_k$ with their images in $\tilde{D}(A)_k$ via $\phi_{z,*}, k$ and $\phi_{z',*}, k$. Then we have a diagram of inclusions of $W(\mathbb{F}_p)$-modules:

$$
\begin{array}{ccc}
\tilde{D}(B_z)_1^0 & \xrightarrow{j-i+\delta} & \tilde{D}(B_z)_2^0 \\
\mid & \downarrow \delta & \mid \\
\tilde{D}(B_{z'})_1^0 & \xleftarrow{j-i+\delta} & \tilde{D}(A)_1^0 \\
\end{array}
$$

By the definition of $Y_i$ and $Y_j$, we have

$$\text{dim}_{\mathbb{F}_p}(\tilde{D}(A)_1^0/\tilde{D}(B_z)_1^0) = \text{dim}_{\mathbb{F}_p}\text{Coker}(\phi_{z,*}) = i - 1,$$

and similarly, $\text{dim}_{\mathbb{F}_p}(\tilde{D}(B_{z'})_1^0/\tilde{D}(A)_1^0) = n - j$. Therefore, if we put

$$\delta = \text{dim}_{\mathbb{F}_p}(\tilde{D}(B_z)_1^0 + \tilde{D}(B_{z'})_1^0)/\tilde{D}(B_z)_1^0 = \text{dim}_{\mathbb{F}_p}(\tilde{D}(B_{z'})_1^0/(\tilde{D}(B_z)_1^0 \cap \tilde{D}(B_{z'})_1^0)),$$

we have $0 \leq \delta \leq \min\{i - 1, n - j\}$. Moreover, the quasi-isogeny $\phi_{z,z'} = \phi^{-1} \circ \phi' : B_{z'} \to B_z$ makes $B_{z'}$ an element of $\text{Isog}(z)$ in the notation of Subsection 3.11. We identify $\mathbb{L}_{z'}$ defined in (3.10.2) with a $\mathbb{Z}_{p^2}$-lattice of $\mathbb{L}_z[1/p]$ via $\phi_{z',*}, k$. Then

$$\text{dim}_{\mathbb{F}_p}(\mathbb{L}_z \cap \mathbb{L}_{z'})/p\mathbb{L}_z = \text{dim}_{\mathbb{F}_p}(\tilde{D}(B_z)_1^0 \cap \tilde{D}(B_{z'})_1^0)/p\tilde{D}(B_z)_1^0 = n + i - j - \delta.$$
Take a $\mathbb{Z}_p^2$-basis $(e_1, \ldots, e_n)$ of $\mathbb{L}_z$ such that the image of $(e_{j-i+\delta+1}, \ldots, e_n)$ in $\mathbb{L}_z/p\mathbb{L}_z$ form a basis of $(\mathbb{L}_z \cap \mathbb{L}_e)/p\mathbb{L}_z$ and such that $p^{-1}e_1, \ldots, p^{-1}e_{n-\delta+1}, p^{-1}e_n$ form a basis of $(\mathbb{L}_z + \mathbb{L}_e)/\mathbb{L}_z$. Then

\[(5.4.2) \quad (p e_1, \ldots, p e_{j-i+\delta}, e_{j-i+\delta+1}, \ldots, e_{n-\delta}, p^{-1} e_{n-\delta+1}, \ldots, p^{-1} e_n)\]

is a basis of $\mathbb{L}_e$, that is $z' \in R_p^{(j-i+\delta,n-\delta)}(z)$ according to the convention of Subsection 5.2.

Conversely, assume that there exists $\delta$ with $1 \leq \delta \leq \min\{i-1, n-j\}$ such that $z' \in R_p^{(j-i+\delta,n-\delta)}(z)$. We have to prove statement (2), then the non-emptyness of $Y_{i,z} \times_{Sh_{1,n-1}} Y_{j,z'}$ will follow automatically. Let $\phi_{z',z} : B_{z'} \to B_z$ be the unique quasi-isogeny which induces an identifies $\mathbb{L}_z$ with a $\mathbb{Z}_p^2$-lattice of $\mathbb{L}_z[1/p]$. By definition of $R_p^{(j-i+\delta,n-\delta)}(z)$, there exists a basis $(e_{\ell})_{1 \leq \ell \leq n}$ of $\mathbb{L}_z$ such that $\phi_{z',z}$ is a basis of $\mathbb{L}_{z'}$. One checks easily that $p(\mathbb{L}_z + \mathbb{L}_{z'}) \subseteq \mathbb{L}_z \cap \mathbb{L}_{z'}$. We put

\[M_k = \tilde{D}(B_z)^\circ_k \cap \tilde{D}(B_{z'})^\circ_k/p(\tilde{D}(B_z)^\circ_k + \tilde{D}(B_{z'})^\circ_k)\]

for $k = 1, 2$. Then one has

\[\dim_{\overline{\mathbb{F}}_p}(M_k) = \dim_{\mathbb{F}_p^2}(\mathbb{L}_z \cap \mathbb{L}_{z'})/p(\mathbb{L}_z + \mathbb{L}_{z'}) = n + i - j - 2\delta.\]

The Frobenius and Verschiebung on $\tilde{D}(B_z)$ induce two bijective Frobenius semi-linear maps $F : M_1 \to M_2$ and $V^{-1} : M_2 \to M_1$; we denote their linearizaiton by the same notation if no confusions arise. Let $Z_\delta(M_\bullet)$ be the moduli space which attaches to each $\overline{\mathbb{F}}_p$-scheme $S$ the set of pairs $(L_1, L_2)$, where $L_1 \subseteq M_1 \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_S$ and $L_2 \subseteq M_2 \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_S$ are sub-bundles of rank $i - \delta$ and $i + 1 - \delta$ respectively such that

\[L_2 \subseteq F(L_1^{(p)}), \quad V^{-1}(H_2^{(p)}) \subseteq L_1.\]

Note that there exists a basis $(\varepsilon_{k,1}, \ldots, \varepsilon_{k,n+i-j-2\delta})$ of $M_k$ for $k = 1, 2$ under which the matrices of $F$ and $V^{-1}$ are both identity. Indeed, by solving a system of equations of Artin-Schreier type, one can take a basis $(\varepsilon_{1,\ell})_{1 \leq \ell \leq n+i-j-2\delta}$ for $M_1$ such that

\[V^{-1}(F(\varepsilon_{1,\ell})) = \varepsilon_{1,\ell} \quad \text{for all} \quad 1 \leq \ell \leq n + i - j - 2\delta;\]

then we put $\varepsilon_{2,\ell} = F(\varepsilon_{1,\ell})$. Using these basis to identify both $M_1$ and $M_2$ with $\mathbb{F}_p^{n+i-j-2\delta}$, then it is clear that $Z_\delta(M_\bullet)$ is isomorphic to the variety $Z_1^{(n+i-j-2\delta)}$ considered in Subsection 4.1.

We have to establish an isomorphism between $Z_\delta(M_\bullet)$ and $Y_{i,z} \times_{Sh_{1,n-1}} Y_{j,z'}$. Let $(L_1, L_2)$ be a point of $Z_\delta(M_\bullet)$ with values in an $\overline{\mathbb{F}}_p$-scheme $S$. Note that there is a natural surjection

\[((\tilde{D}(B_z)^\circ_k \cap \tilde{D}(B_{z'})^\circ_k)/p\tilde{D}(B_z)^\circ) \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_S \to M_k \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_S;\]

we define $H_{z,k}$ for $k = 1, 2$ to be the inverse image of $L_k$ under this surjection. Then $H_{z,k}$ can be naturally viewed as a sub bundle of $\tilde{D}(B_z)^\circ_k \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_S$ of rank $i + 1 - k$, and we have $H_{z,2} \subseteq F(H_{z,1})$ and $V^{-1}(H_{z,2}^{(p)}) \subseteq H_{z,1}$ since $(L_1, L_2)$ verify similar properties. Therefore, $(L_1, L_2) \mapsto (B_z, S, \lambda_z, S, \eta_z, S, H_{z,1}, H_{z,2})$ gives rise to a well-defined map $\varphi_{i,z} : Z_\delta(M_\bullet) \to Y_{i,z}'$, where $(B_z, S, \lambda_z, S, \eta_z, S)$ is the base change of $(B_z, \lambda_z, \eta_z)$ to $S$. Similarly, we have a morphism $\varphi_{j,z'} : Z_\delta(M_\bullet) \to Y_{j,z'}'$ defined by $(L_1, L_2) \mapsto (B_{z'}, S, \lambda_{z'}, \eta_{z'}, H_{z',1}, H_{z',2})$, where $H_{z',k}$ is the inverse image of $L_k$ under the natural surjection:

\[((\tilde{D}(B_z)^\circ_k \cap \tilde{D}(B_{z'})^\circ_k)/p\tilde{D}(B_z)^\circ) \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_S \to M_k \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_S.\]

By Proposition 3.7 we get two maps

\[\varphi_{i,z} : Z_\delta(M_\bullet) \to Y_{i,z}, \quad \varphi_{j,z'} : Z_\delta(M_\bullet) \to Y_{j,z'}.\]

We claim that $\pi_i \circ \varphi_{i,z} = \pi_j \circ \varphi_{j,z}$, that is $(\varphi_{i,z}, \varphi_{j,z})$ defines a map

$\varphi : Z_\delta(M_\bullet) \to Y_{i,z} \times_{Sh_{1,n-1}} Y_{j,z'}$. 
Given an $S$-point $(L_1, L_2)$ of $Z_\delta(M_*)$, let $(A, \lambda, \eta, B_{z,S}, \lambda_z, \eta_z, \phi)$ and $(A', \lambda', \eta', B'_{z',S}, \lambda_{z'}, \phi'_{z'})$ be respectively the image of $(L_1, L_2)$ under $\varphi_{i,z}$ and $\varphi_{j,z'}$. To prove the claim, we have to show that there is an isomorphism $(A, \lambda, \eta) \cong (A', \lambda', \eta')$ as objects of $\text{Sh}_{1,n-1}$ with values in $S$. Consider the $p$-quasi-isogeny

$$f : A' \xrightarrow{\phi'} B'_{z',S} \xrightarrow{\phi_{z',z}} B_{z,S} \xrightarrow{\phi} A.$$  

It is easy to see that $f^\vee \circ \lambda \circ f = \lambda'$ and $f \circ \eta' = \eta$. We need to show that $f$ is an isomorphism of abelian varieties. It suffices to do this in the universal case, i.e., $S = Z_\delta(M_*)$; in particular, $S$ is integral. By Lemma 5.3 proved later, it suffices to check that, after base change from $S$ to an algebraically closed field $\kappa$ of characteristic $p$, $f$ is an isomorphism. To simplify the notation, we still use $B_z, B'_{z'}$ their base changes to $\kappa$. We identify $\hat{D}(B'_{z'})$ with $W(\kappa)$-lattices of $\hat{D}(B_2)[1/p]$ via the quasi-isogenies $\phi_{z',z} : B_{z'} \rightarrow B_z, \phi^{-1} : A \rightarrow B_z$ and $\phi_{z',z}^{-1} \circ \phi' : A' \rightarrow B_z$. Then by the construction of $A$ (cf. the proof of Proposition 3.7), $\hat{D}(A)$ and $\hat{\omega}_{A^\vee,1}$ fit into the diagram (5.4.1) such that there is a canonical isomorphism

$$(5.4.3) \quad L_1 \simeq \hat{\omega}_{A^\vee,1}/p(\hat{D}(B_2)_{i_1}^1 + \hat{D}(B_{z'}_{i_1}^1)) \subseteq (\hat{D}(B_2)_{1} \cap \hat{D}(B_{z'}_{1}^1))/p(\hat{D}(B_2)_{i_1}^1 + \hat{D}(B_{z'}_{1}^1)) = M_1;$$

similarly, we have

$$(5.4.4) \quad L_2 \simeq p\hat{\omega}_{A^\vee,2}/p(\hat{D}(B_2)_{i_2}^2 + \hat{D}(B_{z'}_{i_2}^2)) \subseteq (\hat{D}(B_2)_{2} \cap \hat{D}(B_{z'}_{2}^2))/p(\hat{D}(B_2)_{i_2}^2 + \hat{D}(B_{z'}_{2}^2)) = M_2.$$

It is easy to see that such relations determine $\hat{D}(A)$ uniquely from $(L_1, L_2)$. But the same arguments show that the same relations are satisfied with $A$ replaced by $A'$. Hence, we see that the quasi-isogeny $f$ induces an isomorphism between Dieudonné modules of $A$ and $A'$. As $f$ is a $p$-quasi-isogeny, this implies immediately that $f$ is an isomorphism of abelian varieties. This proves the claim.

It remains now to prove that $\varphi : Z_\delta(M_*) \isom Y_{i,z} \times_{\text{Sh}_{1,n-1}} Y'_{j,z'}$ is an isomorphism. It suffices to show that $\varphi$ induces bijections on closed points and tangent spaces. The arguments are quite similar to the proof of Proposition 3.7. Indeed, given a closed point $x = (A, \lambda, \eta, \phi, \phi')$ of $Y_{i,z} \times_{\text{Sh}_{1,n-1}} Y'_{j,z'}$, then one can construct a unique point $y = (L_1, L_2)$ of $Z_\delta(M_*)$ with $\varphi(y) = x$ by the relations (5.4.3) and (5.4.4). It follows immediately that $\varphi$ induces a bijection on closed points. Let $x$ and $y$ be as above. By the same arguments as in Proposition 3.3, the tangent space of $Z_\delta(M_*)$ at $y$ is given by

$$T_{Z_\delta(M_*)}(y) \simeq (L_1/V_1(L_2^{(p)}))^* \otimes (M_1/L_1) \oplus L_2 \otimes F(L_1^{(p)})/L_2.$$  

On the other hand, using Grothendieck-Messing deformation theory, one sees easily that the tangent space of $Y_{i,z} \times_{\text{Sh}_{1,n-1}} Y'_{j,z'}$ at $x$ is given by

$$T_{Y_{i,z} \times_{\text{Sh}_{1,n-1}} Y'_{j,z'}} : x = \text{Hom}_p(\omega_{A^\vee,1}, (\hat{D}(B_2)_{i_1} \cap \hat{D}(B_{z'}_{i_1}))/\hat{\omega}_{A^\vee,1})$$

$$\oplus \text{Hom}_p(\hat{\omega}_{A^\vee,2}/(\hat{D}(B_2)_{i_2} + \hat{D}(B_{z'}_{i_2})), \hat{D}(A)_{i_2}/\hat{\omega}_{A^\vee,2}).$$  

From (5.4.3) and (5.4.4), we see easily that

$$\omega_{A^\vee,1} \cong L_1/V_1(L_2^{(p)}), \quad \hat{D}(B_{z'}_{i_1}^1)/\hat{\omega}_{A^\vee,1} \cong M_1/L_1;$$

$$\hat{\omega}_{A^\vee,2}/(\hat{D}(B_2)_{i_2}^2 + \hat{D}(B_{z'}_{i_2}^2)) \cong L_2, \quad \hat{D}(A)_{i_2}/\hat{\omega}_{A^\vee,2} \cong F(L_1^{(p)})/L_2.$$  

Hence, it follows that $\varphi$ induces a bijection between $T_{Z_\delta(M_*)}(y)$ and $T_{Y_{i,z} \times_{\text{Sh}_{1,n-1}} Y'_{j,z'}}(x)$. This finishes the proof of Proposition 5.4.

Lemma 5.5. Let $S$ be an integral scheme in characteristic $p$, and $f : A' \rightarrow B$ be a $p$-quasi-isogeny of abelian schemes over $S$. Then $f$ is an genuine isogeny of abelian schemes if and only if so is after base change to all geometric points of $S$. 

□
Proof. The “only if” part is trivial. Assume now that \( f \) is a quasi-isogeny whose base change to every geometric point of \( S \) is a genuine isogeny. Choose an integer \( n \geq 1 \) such that \( f' = p^n f \) is an isogeny. Then \( f' \) is divisible by \( p^n \) after base change to every geometric point of \( S \). We have to show that \( f' \) is divisible by \( p^n \). By induction we may assume that \( n = 1 \). The statement is equivalent to the inclusion \( A[p] \subseteq \text{Ker}(f') \) of closed subgroup schemes locally free of finite type. Let \( S^\text{perf} \) denote the perfection of \( S \). Since \( S^\text{perf} \) is faithfully flat over \( S \), it suffices to check the inclusion after base change to \( S^\text{perf} \). Thus we may assume that \( S \) is perfect. By a result of Gabber (see also [La]), the crystalline Dieudonné functor for \( p \)-divisible groups over a perfect base is fully faithful. The isogeny \( f' \) is divisible by \( p \) if and only if its induced morphism on Dieudonné crystals is divisible by \( p \). But \( H^1_{\text{dR}}(A/S) \) is the reduction modulo \( p \) of the (covariant) Dieudonné crystal associated to \( A \), it suffices to prove that the induced morphism \( f'_*: \text{H}^1_{\text{dR}}(A/S) \to H^1_{\text{dR}}(B/S) \) is trivial. By assumption, this is the case after base change to every geometric point of \( S \). As \( S \) is reduced, thus \( f'_* \) vanishes on whole \( S \). \( \square \)

5.6. Applications to cohomology. Recall that we have a morphism \( \mathcal{J} \mathcal{L}_i (3.17.1) \) for each \( i = 1, \ldots, n \). We consider another map in the opposite direction:

\[
\nu_i : H^{2(n-1)}(\text{Sh}_{1,n-1,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell)(n-1) \xrightarrow{\text{pr}_i^*} H^{2(n-1)}(Y_{i,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} H^0(\text{Sh}_{0,n,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell),
\]

where the second isomorphism is induced by the trace map \( \text{Tr}_{\text{pr}_i^*} : R^{2(n-1)}(\gamma_i, \mathcal{L}_i) \to H^0(\text{Sh}_{0,n,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell) \).

For \( 1 \leq i, j \leq n \), we define

\[
m_{i,j} = \nu_j \circ \mathcal{J} \mathcal{L}_i : H^{0}(\text{Sh}_{0,n,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathcal{J} \mathcal{L}_i} H^{2(n-1)}(\text{Sh}_{1,n-1,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell)(n-1) \xrightarrow{\nu_j} H^0(\text{Sh}_{0,n,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell).
\]

Putting all the morphism \( \mathcal{J} \mathcal{L}_i \) and \( \nu_j \) together, we get a sequence of morphisms:

\[
(5.6.1) \quad \bigoplus_{i=1}^{n} H^{0}(\text{Sh}_{0,n,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathcal{J} \mathcal{L}_i} H^{2(n-1)}(\text{Sh}_{1,n-1,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell)(n-1) \xrightarrow{\nu_j} \bigoplus_{j=1}^{n} H^{0}(\text{Sh}_{0,n,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell).
\]

We see that the composed morphism above is given by the matrix \( M = (m_{i,j})_{1 \leq i, j \leq n} \), and we call it the intersection matrix of cycles \( Y_j \) on \( \text{Sh}_{1,n-1} \). All these morphisms are equivariant under the natural action of the Hecke algebra \( \mathcal{H}(K^p, \overline{\mathbb{Q}}_\ell) \). We will describe the intersection matrix in terms of the Hecke action of \( \overline{\mathbb{Q}}_\ell(K^p \backslash \text{GL}_n(E_p)/K_p) \) on \( H^0(\text{Sh}_{0,n,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell) \).

The group \( H^0(\text{Sh}_{0,n,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell) \) is the space of functions on \( \text{Sh}_{0,n}(\mathbb{F}_p) \) with values in \( \overline{\mathbb{Q}}_\ell \). For \( z \in \text{Sh}_{0,n}(\mathbb{F}_p) \), let \( e_z \) denote the characteristic function at \( z \). Then the image of \( z \) under \( K_p \gamma K_p \) for \( \gamma \in \text{GL}_n(E_p) \) is

\[
[K_p \gamma K_p]_*(e_z) = \sum_{z' \in T_p(\gamma)(z)} e_{z'},
\]

where \( T_p(\gamma)(z) \) means the set theoretic Hecke correspondence defined in Subsection 5.2. In the sequel, we will use the same notation \( T_p(\gamma) \) to denote the action of \( [K_p \gamma K_p] \) on \( H^0(\text{Sh}_{0,n,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell) \).

In particular, we have Hecke operators \( T_p^{(i)}(S_p, R_{p}^{(a,b)}, \ldots) \).

Proposition 5.7. For \( 1 \leq i \leq j \leq n \), we have

\[
m_{i,j} = \sum_{\delta=0}^{\min\{i-1,n-j\}} N(n+i-j-2\delta,i-\delta) R_p^{(j-i+\delta,n-\delta)} S_p^{-1},
\]

\[
m_{j,i} = \sum_{\delta=0}^{\min\{i-1,n-j\}} N(n+i-j-2\delta,i-\delta) R_p^{(\delta,n+i-j-\delta)} S_p^{-1},
\]

where \( N(n+i-j-2\delta,i-\delta) \) are the fundamental intersection numbers given by (4.2.1).

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Proof. We have a commutative diagram:

\[
\begin{array}{ccc}
A_{n-1}(Y_{i,p}) & \xrightarrow{pr_{i,*}} & A_{n-1}(Sh_{1,n-1,\mathbb{F}_p}) \\
\downarrow{cl} & & \downarrow{cl} \\
H^0(Y_{i,\mathbb{F}_p}, \mathbb{Q}_l) & \xrightarrow{Gys_{pr}_i} & H^{2(n-1)}(Sh_{1,n-1,\mathbb{F}_p}, \mathbb{Q}_l) (n-1) \xrightarrow{pr_{j}^*} H^{2(n-1)}(Y_{j,\mathbb{F}_p}, \mathbb{Q}_l).
\end{array}
\]

Here, the vertical arrows are cycles class maps, and \(pr_{i}^*\) is the refined Gysin map defined in (5.3.1). For \(z \in Sh_{0,n}(\mathbb{F}_p)\), the image of \(e_z\) under \(m_{i,j}\) is given by

\[
m_{i,j}(e_z) = Tr_{pr_{i}^*}(\pi^* Gys_{\pi_1}(Y_i, \ast)) = Tr_{pr_{i}^*}(cl(pr_{i}^* p_{i,j}[Y_i, z]))
\]

\[
= Tr_{pr_{i}^*}\left( \sum_{z' \in Sh_{0,n}(\mathbb{F}_p)} cl(c_r(E(Y_{j,z'}, Y_i, z))) \cdot cl(Y_{j,z'} \times Sh_{1,n-1} Y_i, z) \right)
\]

\[
= \sum_{z' \in Sh_{0,n}(\mathbb{F}_p)} \left( \int_{Y_{j,z'} \times Sh_{1,n-1} Y_i, z} c_r(E(Y_{j', z'}, Y_i, z)) \right) e_{z'},
\]

where we used (5.3.2) in the second step. By Proposition 5.4, \(e_{z'}\) has a non-zero contribution in the summation above if and only if there exists an integer \(\delta\) with \(0 \leq \delta \leq \min\{i-1, n-j\}\) such that \(z' \in R^{(j-i+\delta, n-\delta)} S^{1}_p(z)\); in that case, the coefficient of \(e_{z'}\) is given by \(N(n+i-j-2\delta, i-\delta)\) according to Theorem 4.2. The formula for \(m_{i,j}\) now follows immediately. The formula for \(m_{j,i}\) is proved in the same manner. \(\square\)

If we express \(m_{i,j}\) in terms of the elementary Hecke operators \(T^{(k)}_p\), we get the following

**Theorem 5.8.** Put \(d(n,k) = (2k-1)n - 2k(k-1) - 1\) for integers \(1 \leq k \leq n\). Then, for \(1 \leq i \leq j \leq n\), we have

\[
m_{i,j} = \sum_{\delta = 0}^{\min\{i-1, n-j\}} (-1)^{n+1+i-j} (n+i-j-2\delta)p^{d(n+i-j-2\delta, i-\delta)} T^{(j-i+\delta)}_p (n-\delta) S^1_p,
\]

\[
m_{j,i} = \sum_{\delta = 0}^{\min\{i-1, n-j\}} (-1)^{n+1+i-j} (n+i-j-2\delta)p^{d(n+i-j-2\delta, i-\delta)} T^{(j-i+\delta)}_p (n+i-j-\delta) S^1_p.
\]

**Proof.** We prove only the statement for \(m_{i,j}\), and that for \(m_{j,i}\) is similar. By Proposition 3.1 in Appendix A, the right hand side of the first formula above is

\[
\sum_{\delta = 0}^{\min\{i-1, n-j\}} (-1)^{n+1+i-j} (n+i-j-2\delta)p^{d(n+i-j-2\delta, i-\delta)} \left( \sum_{k=0}^{\delta} \binom{n+i-j-2\delta+2k}{k} R^{(j-i+\delta-k, n-\delta+k)}_p S^1_p \right)
\]

\[
= \sum_{r=0}^{\min\{i-1, n-j\}} (\star) R^{(j-i+r, n-r)}_p S^1_p.
\]

Here, we have put \(r = \delta - k\), and the expression in the bracket is

\[
\star = \sum_{k=0}^{\min\{i-1, n-j\}} (-1)^{n+1+i-j} (n+i-j-2r-2k)p^{d(n+i-j-2r-2k, i-r-k)} \binom{n+i-j-2r}{k} R^{(j-i+r-k, n-r-k)}_p S^1_p
\]

\[
= N(n+i-j-2r, i-r).
\]

The statement for \(m_{i,j}\) now follows from Proposition 5.7. \(\square\)
Example 5.9. We write down explicitly the intersection matrices when \( n \) is small.

(1) Consider first the case \( n = 2 \). This case is essentially the same as the Hilbert quadratic case studied in \([TX14+]\), and the intersection matrix writes as

\[
M = \begin{pmatrix}
-2p & T_p^{(1)} \\
T_p^{(1)}T_p^{(2)} & -2p
\end{pmatrix}.
\]

(2) When \( n = 3 \), Theorem 5.8 gives

\[
M = \begin{pmatrix}
3p^2 & -2pT_p^{(1)} & T_p^{(2)} \\
-2pT_p^{(2)} & 3p^4 + T_p^{(1)}T_p^{(2)} & -2pT_p^{(1)} \\
T_p^{(1)}S_p^{-1} & -2pT_p^{(2)} & 3p^2
\end{pmatrix}.
\]

(3) The intersection matrix for \( n = 4 \) writes:

\[
M = \begin{pmatrix}
-4p^3 & 3p^2T_p^{(1)} & -2pT_p^{(2)} & T_p^{(3)} \\
3p^2T_p^{(3)} & -4p^7 - 2pT_p^{(1)}T_p^{(3)}S_p^{-1} & 3p^4T_p^{(1)} + T_p^{(2)}T_p^{(3)}S_p^{-1} & -2pT_p^{(1)} \\
-2pT_p^{(2)}S_p^{-1} & 3p^4T_p^{(3)}S_p^{-1} + T_p^{(1)}T_p^{(2)}S_p^{-1} & -4p^7 - 2pT_p^{(1)}T_p^{(3)}S_p^{-1} & 3p^2T_p^{(1)} \\
T_p^{(1)}S_p^{-1} & -2pT_p^{(2)} & 3p^2T_p^{(3)}S_p^{-1} & -4p^3
\end{pmatrix}.
\]

5.10. Proof of Theorem 3.20. We prove first statement (1). Let \( \pi \in \mathcal{M}_{\mathcal{L}} \) as in the statement. Consider the \((\pi^p) K^p\)-isotypical direct factor of the \( \mathcal{M}(K^p, \overline{Q}_\ell)\)-equivariant sequence \([5.6.1]\). By Lemma 3.19 it is the same as taking \( \pi^p \)-isotypical components as \( \mathcal{M}(K, \overline{Q}_\ell)\)-modules on both sides:

\[
\bigoplus_{i=1}^n H^0(\mathcal{M}_{\mathcal{L}}) \to H^2(n-1)(\mathcal{M}_{\mathcal{L}}) \to H^2(n-1)(\mathcal{M}_{\mathcal{L}}) \to \bigoplus_{i=1}^n H^0(\pi).
\]

If \( a_p^{(i)} \) denotes the eigenvalues of \( T_p^{(i)} \) on \( \pi^p \) for each \( 1 \leq i \leq n \), then \( T_p^{(i)} \) acts as the scalar \( a_p^{(i)} \) on all the terms above. Therefore, \( \nu_{\pi} \circ \mathcal{J}_{\pi} \) is given by the matrix \( M_{\pi} \), which is obtained by replacing \( T_p^{(i)} \) by \( a_p^{(i)} \) in each entry of \( M \). By definition, the \( \alpha_{\pi,\pi'} \)'s are the roots of the Hecke polynomial:

\[
X^n + \sum_{i=1}^n (-1)^i p^{i(n-1)} a_p^{(i)} X^{n-i}.
\]

Then statement (1) follows easily from the following

Lemma 5.11. We have

\[
\det(M_{\pi}) = \pm p^{n(n^2-1)/3} \prod_{i<j} (\alpha_{\pi,i} - \alpha_{\pi,j})^2 \prod_{i=1}^n (\alpha_{\pi,i})^{n-1}.
\]

Here, \( \pm \) means that the formula is up to sign. In particular, \( \nu_{\pi} \circ \mathcal{J}_{\pi} \) is an isomorphism if the \( \alpha_{\pi,\pi'} \)'s are distinct.

Proof. Put \( \beta_i = \alpha_{\pi,i}/p^{n-1} \) for \( 1 \leq i \leq n \). For \( i = 1, \ldots, n \), let \( s_i \) be the \( i \)-th elementary symmetric polynomial in \( \beta_i \)'s. Then we have \( a_p^{(i)} = p^{i(n-1)} s_i \). It follows from Theorem 5.8 that the \((i, j)\)-entry of \( M_{\pi} \) with \( 1 \leq i \leq j \leq n \) is given by

\[
m_{i,j}(\pi) = s_n^{-1} \sum_{\delta=0}^{\min(i-1,n-j)} (-1)^{n+1+i-j} (n+i-j-2\delta) p^{d(n+i-j-2\delta,i-\delta)+(j-i+\delta)(n+i-j-\delta)+\delta(n-\delta)} s_{j-i+\delta} s_{n-\delta}.
\]
A key observation here is that the exponent index on \(p\) in each term above is independent of \(\delta\), and equals to \(e(i, j) := (n + 1)(i + j - 1) - (i^2 + j^2)\). Similar observation holds for \(i > j\). In summary, we get \(m_{i,j}(\pi) = s_{n}^{-1}p^{e(i,j)}m'_{i,j}(\pi)\) with

\[
m'_{i,j}(\pi) = \begin{cases} \sum_{\delta=0}^{\min\{i-1,n-j\}}(-1)^{n+1+i-j}(n+i-j-2\delta)s_{n-i+\delta}-s_{n-\delta}, & \text{if } i \leq j; \\ \sum_{\delta=0}^{\min\{j-1,n-i\}}(-1)^{n+1+j-i}(n+j-i-2\delta)s_{n+j-\delta}-s_{n+i-\delta}, & \text{if } i > j. \end{cases}
\]

For any \(n\)-permutation \(\sigma\), we have

\[
\sum_{i=1}^{n} e(i, \sigma(i)) = \frac{n(n^2 - 1)}{3}.
\]

Thus we get \(\det(M_{\pi}) = p^{\frac{n(n^2 - 1)}{3}} s_{n}^{-n} \det(m'_{i,j}(\pi))\). The rest of the computation is purely combinatorial, which is the case \(q = -1\) of Theorem \(\ref{thm:combinatorial}\) in Appendix B.

We prove now statement (2) of Theorem \(\ref{thm:main}\). Given statement (1), it suffices to prove that

\begin{equation}
\dim H^0(\text{Sh}_{0,n,\mathbb{F}_p}, \mathcal{Q}_\ell)_{\pi} \geq \dim H^2(\text{Sh}_{1,n-1,\mathbb{F}_p}, \mathcal{Q}_\ell)_{\pi}(n-1)^\text{fin}.
\end{equation}

Actually, by \(\text{(1.4.1)}\) and \(\text{(1.6.3)}\), we have

\[
H^0(\text{Sh}_{0,n,\mathbb{F}_p}, \mathcal{Q}_\ell)_{\pi} = \pi^K \otimes R_{(0,n),\ell}(\pi), \quad H^2(\text{Sh}_{1,n-1,\mathbb{F}_p}, \mathcal{Q}_\ell)_{\pi} = \pi^K \otimes R_{(1,n-1),\ell}(\pi).
\]

Write \(\pi_p = \pi_{p,0} \otimes \pi_p\) as a representation of \(G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \text{GL}_n(E_p)\). Let \(\chi_{\pi_p,0} : \text{Gal}(\mathbb{F}_p/\mathbb{F}_{p^2}) \to \mathbb{Q}_\ell^\times\) denote the character sending \(\text{Frob}_{p^2}\) to \(\pi_{p,0}(p^2)\), and let \(\rho_{\pi_p}\) as \(\text{(1.6.1)}\). Then, up to semi-simplification, we have

\[
[R_{(0,n),\ell}(\pi)] = m_{0,n}(\pi) \# \ker^1(\mathbb{Q}, G_{0,n}) \left[ \wedge^n \rho_{\pi_p} \otimes \chi_{\pi_p,0} \otimes \mathcal{Q}_\ell\left(\frac{n(n-1)}{2}\right) \right],
\]

\[
[R_{(1,n-1),\ell}(\pi)] = m_{1,n-1}(\pi) \# \ker^1(\mathbb{Q}, G_{1,n-1}) \left[ \rho_{\pi_p} \otimes \wedge^n \rho_{\pi_p} \otimes \chi_{\pi_p,0}^{-1} \otimes \mathcal{Q}_\ell\left(\frac{(n-1)(n-2)}{2}\right) \right].
\]

Note that

\[
\dim \left( \rho_{\pi_p} \otimes \wedge^n \rho_{\pi_p} \otimes \chi_{\pi_p,0}^{-1} \otimes \mathcal{Q}_\ell\left(\frac{(n-1)(n-2)}{2}\right) \right)^\text{fin} = \sum_{\zeta} \dim(\rho_{\pi_p} \otimes \wedge^n \rho_{\pi_p})_{\text{Frob}_{p^2}=p^{\alpha_{\pi_p,i}}},
\]

where the superscript “fin” means taking the subspace on which \(\text{Gal}(\mathbb{F}_p/\mathbb{F}_{p^2})\) acts through a finite quotient, and \(\zeta\) runs through all roots of unity. If \(\alpha_{\pi_p,i}/\alpha_{\pi_p,j}\) is not a root of unity, the right hand side above equals to the multiplicity of \(\prod_{i=1}^{n} \alpha_{\pi_p,i} = p^{\alpha_{\pi_p,i}}\) as eigenvalue of \((\rho_{\pi_p} \otimes \wedge^n \rho_{\pi_p})(\text{Frob}_{p^2})\), which is \(n\). Therefore, under these conditions on the \(\alpha_{\pi_p,i}\)'s, we have

\[
R_{1,n-1,\ell}(\pi)^\text{fin} \leq n,
\]

and equality holds if \(\text{Frob}_{p^2}\) is semi-simple on \(R_{1,n-1,\ell}(\pi)\). Now the inequality \(\text{(5.11.1)}\) follows immediately, since \(\# \ker^1(\mathbb{Q}, G_{1,n-1}) = \# \ker^1(\mathbb{Q}, G_{0,n})\) and \(m_{0,n}(\pi) = m_{1,n-1}(\pi)\) by assumption. This finishes the proof of Theorem \(\ref{thm:main}\).

\textbf{Remark 5.12}. We point out that the determinant of the intersection matrix computed by Theorem \(\ref{thm:combinatorial}\) holds when there is an auxiliary variable \(q\). It motivates us to ask, out of curiosity, whether there might be some quantum version of the construction of cycles, or even Conjecture \(\ref{conj:quantum}\) possibly for the geometric Langlands setup.

\section{Construction of Cycles in the Case of \(G(U(r, s) \times U(s, r))\)}

We keep the notation of Subsection \(\ref{subsec:notation}\). In this section, we will give the construction of certain cycles on Shimura varieties for \(G(U(r, s) \times U(s, r))\). We always assume that \(s \geq r\).
6.1. Description of the cycles in terms of Dieudonné modules. Let $\delta$ be a non-negative integer with $\delta \leq r$. We consider the case of Conjecture 1.9 when $a_1 = r$, $a_2 = s$, $b_1 = r - \delta$, and $b_2 = s + \delta$. Standard $GL_p$-representation computation shows that the weight multiplicity $m_\lambda(a_\bullet)$ in Conjecture 1.9 is $(s-r+2\delta)$. Conjecture 1.9 thus predicts the existence of $(s-r+2\delta)$ cycles $Y_j$ on $Sh_{r,s}$, each of dimension

$$rs + (r - \delta)(s + \delta) = 2rs - (s - r)\delta - \delta^2,$$

and each admits a rational map to $Sh_{r-\delta,s+\delta}$. The principal goal of this section is to construct these cycles, at least conjecturally. We start with the description in terms of the Dieudonné modules at closed points.

Consider the interval $[r - \delta, s + \delta]$; it contains $s - r + 2\delta$ unit segments with integer endpoints. We will parametrize the cycles on the Shimura variety by the $\delta$ elements subsets of these $s - r + 2\delta$ unit segments; there are exactly $(s-r+2\delta)$ such subsets. Let $j$ be one of them. Then we can write the union of all the segments in $j$ as

$$(6.1.1) \quad [j_{1,1}, j_{1,2}] \cup [j_{2,1}, j_{2,2}] \cup \cdots \cup [j_{\epsilon,1}, j_{\epsilon,2}]$$

such that all $j_{\alpha,i}$ are integers,

$$r - \delta \leq j_{1,1} < j_{1,2} < j_{2,1} < j_{2,2} < \cdots < j_{\epsilon,1} < j_{\epsilon,2} \leq s + \delta,$$

and we have $\sum_{\alpha=1}^\epsilon (j_{\alpha,2} - j_{\alpha,1}) = \delta$. For notation convenience, we put $j_{0,1} = j_{0,2} = 0$.

We define $Z_j$ to be the subset of $F_p$-points $z$ of $Sh_{r,s}$ such that the reduced Dieudonné modules $\tilde{D}(A_z)_{1}^p$ and $\tilde{D}(A_z)_{2}^p$ contain submodules $\tilde{E}_1$ and $\tilde{E}_2$ satisfying (2.2.1) for $m = \epsilon$, i.e.

$$p^i \tilde{D}(A_z)_{\epsilon}^p \subseteq \tilde{E}_{i}, \quad F(\tilde{E}_i) \subseteq \tilde{E}_{i+1}, \quad \text{and} \quad V(\tilde{E}_i) \subseteq \tilde{E}_{i-1},$$

and the following condition for $i = 1, 2$:

$$(6.1.2) \quad \tilde{D}(A_z)_{\epsilon}^p / \tilde{E}_i \simeq (W(F_p) / p^{i})^{\oplus j_{i,\epsilon}} \oplus (W(F_p) / p^{\epsilon-1})^{\oplus (j_{2,i} - j_{1,i})} \oplus \cdots \oplus (W(F_p) / p)^{\oplus (j_{\epsilon,i} - j_{\epsilon-1,i})}.$$}

We refer to the toy model discussed in Example 6.3 for the motivation of this condition. For a technical reason, we do not know if the set $Z_j$ is the set of $F_p$-points of a closed subscheme of $Sh_{r,s}$. But we shall show that a closely related subset of $Z_j$ is. See Remark 6.7.

Applying Proposition 2.2 with $m = \delta$, the submodules $\tilde{E}_1$ and $\tilde{E}_2$ give rise to a polarized abelian variety $(A'_z, \lambda'_z)$ over $z$ with an $O_D$-action and an $O_D$-equivariant isogeny $A'_z \to A_z$. Moreover, by (2.2.2), we have

$$\dim \omega_{A'_z} / F_p, 1 = \dim \omega_{A_z} / F_p, 1 + \sum_{\alpha=0}^{\epsilon-1} ((\epsilon - \alpha)(j_{\alpha+1,1} - j_{\alpha,1}) - (\epsilon - \alpha)(j_{\alpha+1,2} - j_{\alpha,2})) = r - \delta$$

and similarly $\dim \omega_{A'_z} / F_p, 2 = s + \delta$. So $A'_z$ satisfies the moduli problem for $Sh_{r-\delta,s+\delta}$; this suggests a geometric relationship between $Z_j$ and $Sh_{r-\delta,s+\delta}$ that we make precise in Definition 6.4.

We make an immediate remark that when $\delta = r$, the abelian variety $A_z$ coming from a point $z$ of $Z_j$ is isogenous to an abelian variety $A'_z$ that is a moduli object for the Shimura variety $Sh_{0,r,s}$. Thus both $A'_z$ and $A_z$ are supersingular. In other words, every $Z_j$ is contained in the supersingular locus of $Sh_{r,s}$. We certainly expect that the union of all $Z_j$ is exactly the supersingular locus, but we do not prove this general result in this paper. (See Proposition 3.15 [3] for a proof in the case when $r = 1$.)
6.2. **Towards a moduli interpretation.** We need to reinterpret the Dieudonné-theoretic condition defining $Z_j$ in a more geometric manner. For $\alpha = 0, \ldots, \epsilon$, we define submodules

$$\tilde{\mathcal{E}}_{\alpha,1} := \tilde{\mathcal{D}}(A_{\alpha})_{1} \cap \frac{1}{p^\alpha} \tilde{\xi}_1$$

and

$$\tilde{\mathcal{E}}_{\alpha,2} := \tilde{\mathcal{D}}(A_{\alpha})_{2} \cap \frac{1}{p^\alpha} \tilde{\xi}_2$$

of $\tilde{\mathcal{D}}(A_{\alpha})_{1}$ and $\tilde{\mathcal{D}}(A_{\alpha})_{2}$; they are easily seen to satisfy condition (2.2.1) with $m = \alpha$. Thus, Proposition 2.2 generates a polarized abelian variety $(A_{\alpha}, \lambda_{\alpha})$ with $\mathcal{O}_D$-action and an $\mathcal{O}_D$-equivariant isogeny $A_{\alpha} \to A_\epsilon$, where

$$(6.2.1) \quad r_\alpha := \dim \omega_{A_{\alpha}/\mathbb{F}_p,1}^\alpha = r - \sum_{\alpha' = 1}^{\alpha} (\tilde{j}_{\alpha',2} - \tilde{j}_{\alpha',1})$$

and

$$s_\alpha := \dim \omega_{A_{\alpha}/\mathbb{F}_p,2}^\alpha = n - \dim \omega_{A_{\alpha}/\mathbb{F}_p,1}^\alpha.$$

by the formula (2.2.2). In particular $r_0 = r$, $s_0 = s$, $r_\epsilon = r - \delta$, and $s_\epsilon = s + \delta$.

In fact, applying Proposition 2.2 to the sequence of inclusions

$$\tilde{\mathcal{E}}_i = \tilde{\mathcal{E}}_{i,1} \subset \tilde{\mathcal{E}}_{i-1,1} \subset \cdots \subset \tilde{\mathcal{E}}_{0,i} = \tilde{\mathcal{D}}(A_{\alpha})_{\alpha,i}^\circ,$$

we obtain a sequence of isogenies (each with $p$-torsion kernels):

$$(6.2.2) \quad A_{\alpha,i}' = A_\epsilon \phi_i A_{\epsilon-1} \phi_{i-1} \cdots \phi_1 A_0 = A_\alpha.$$

We have $\ker \phi_i \subseteq A_i[p]$, so that there also exists a unique isogeny $\psi_i : A_{i-1} \to A_i$ such that $\psi_i \phi_i$ and $\phi_i \psi_i$ are equal to multiplication by $p$.

For each $\alpha$, the cokernel of the induced map on cohomology

$$\phi_{\alpha,i} : H^1_{dR}(A_{\alpha}/\mathbb{F}_p, \mathbb{F}_p, 1) \to H^1_{dR}(A_{\alpha-1}/\mathbb{F}_p, \mathbb{F}_p, 1)$$

is canonically isomorphic to $\tilde{\mathcal{E}}_{\alpha-1,i}/\tilde{\mathcal{E}}_{\alpha,i}$ (resp. $\tilde{\mathcal{E}}_{\alpha,i}/\tilde{\mathcal{E}}_{\alpha-1,i}$), which as dimension $j_{\alpha,i}$ (resp. $n - j_{\alpha,i}$) over $\mathbb{F}_p$ by a straightforward computation using (6.1.2).

The upshot is that all these numeric information of the chain of isogenies (6.2.2) can be used to reconstruct $\tilde{\mathcal{E}}_i$ inside $\tilde{\mathcal{D}}(A_{\alpha})_{\alpha,i}^\circ$. This idea will be made precise after this important example.

**Example 6.3.** We give a good toy model for the isogenies of Dieudonné modules. This is the inspiration of the argument for the rest of this section. We start with the crystal $\tilde{\mathcal{D}}(A_{\alpha})_{\alpha,i}^\circ = \oplus_{i=1}^{n} W(\mathbb{F}_p) \mathbf{e}_j$ and $\tilde{\mathcal{D}}(A_{\alpha})_{\alpha,1}^\circ = \oplus_{j=1}^{n} W(\mathbb{F}_p) \mathbf{f}_j$. The maps $V_1 : \tilde{\mathcal{D}}(A_{\alpha})_{\alpha,1}^\circ \to \tilde{\mathcal{D}}(A_{\alpha})_{\alpha,2}^\circ$ and $V_2 : \tilde{\mathcal{D}}(A_{\alpha})_{\alpha,2}^\circ \to \tilde{\mathcal{D}}(A_{\alpha})_{\alpha,1}^\circ$, with respect to the given bases, are given by the diagonal matrices

$$\text{Diag}(1, \ldots, 1, p, \ldots, p), \quad \text{and} \quad \text{Diag}(1, \ldots, 1, p, \ldots, p).$$

Using the isogenies $\phi_{\alpha}$'s we may naturally identify $\tilde{\mathcal{D}}(A_{\alpha})_{\alpha,i}^\circ$ as lattices in $\tilde{\mathcal{D}}(A_{\alpha})_{\alpha,1}^\circ[\frac{1}{p}]$ with induced Frobenius and Verschiebung morphisms. For our toy model, we choose

$$\tilde{\mathcal{D}}(A_{\alpha})_{\alpha,1}^\circ = W(\mathbb{F}_p) \text{-span of} \quad \frac{1}{p^{\alpha-1}} \mathbf{e}_1, \ldots, \frac{1}{p^{\alpha-1}} \mathbf{e}_{j_{\alpha+1,1}}, \frac{1}{p^{\alpha-1}} \mathbf{e}_{j_{\alpha+1,1}+1}, \ldots, \frac{1}{p^{\alpha-1}} \mathbf{e}_{j_{\alpha+2,1}};$$

$$\tilde{\mathcal{D}}(A_{\alpha})_{\alpha,2}^\circ = W(\mathbb{F}_p) \text{-span of} \quad \frac{1}{p^{\alpha-2}} \mathbf{e}_{j_{\alpha+2,1}+1}, \ldots, \frac{1}{p^{\alpha-2}} \mathbf{e}_{j_{\alpha+2,1}+1}, \ldots, \frac{1}{p^{\alpha-2}} \mathbf{e}_{j_{\alpha+2,1}};$$

and

$$\tilde{\mathcal{D}}(A_{\alpha})_{\alpha,2}^\circ = W(\mathbb{F}_p) \text{-span of} \quad \frac{1}{p^{\alpha-2}} \mathbf{f}_{j_{\alpha+2,2}}, \frac{1}{p^{\alpha-2}} \mathbf{f}_{j_{\alpha+2,2}+1}, \ldots, \frac{1}{p^{\alpha-2}} \mathbf{f}_{j_{\alpha+2,2}};$$

In particular, the Verschiebung $V_1 : \tilde{\mathcal{D}}(A_\alpha)_{\alpha,1}^\circ \to \tilde{\mathcal{D}}(A_\alpha)_{\alpha,2}^\circ$ with respect to the basis above is given by

$$\text{Diag}(1, \ldots, 1, ???? \ldots ???, p, \ldots, p).$$
where ??? part is \( p \) if the place is in \([j_\alpha + 1, j_{\alpha+1}]\) for some \( \alpha \), and is 1 otherwise. Similarly, the Verscheibung \( V_2 : \mathcal{D}(A_0)^\circ_2 \to \mathcal{D}(A_0)^\circ_0 \) with respect to the basis above is given by

\[
\text{Diag}(1, \ldots, 1, ???, \ldots, ??, p, \ldots, p),
\]

where ??? part is 1 if the place is in \([j_\alpha + 1, j_{\alpha+1}]\) for some \( \alpha \), and is \( p \) otherwise.

**Definition 6.4.** Let \( j \) be as above. Define the numbers \( j_{\alpha,i} \) as in (6.1.1) and the numbers \( r_\alpha, s_\alpha \) as in (6.2.1). Let \( Y_j \) be the functor taking a locally noetherian \( \mathbb{F}_{p^2} \)-scheme \( S \) to the set of isomorphism classes of tuples

\[
(A_0, \ldots, A_\epsilon, \lambda_0, \ldots, \lambda_\epsilon, \eta_0, \ldots, \eta_\epsilon, \phi_1, \ldots, \phi_\epsilon, \psi_1, \ldots, \psi_\epsilon)
\]
such that:

1. For each \( \alpha \), \((A_\alpha, \lambda_\alpha, \eta_\alpha)\) is an \( S \)-point of \( \text{Sh}_{r_\alpha,s_\alpha} \).
2. For each \( \alpha \), \( \phi_\alpha \) is an \( \mathcal{O}_S \)-isogeny \( A_\alpha \to A_{\alpha-1} \), with kernel contained in \( A_\alpha[p] \), that is compatible with the polarizations in the sense that \( p\lambda_\alpha = \phi_\alpha^\circ \circ \lambda_{\alpha-1} \circ \phi_\alpha \) and with the tame level structures in the sense that \( \phi_\alpha \circ \eta_\alpha = \eta_{\alpha-1} \).
3. \( \psi_\alpha \) is the isogeny \( A_{\alpha-1} \to A_\alpha \) such that \( \phi_\alpha \circ \psi_\alpha = p \) and \( \psi_\alpha \circ \phi_\alpha = p \).
4. For each \( \alpha \) and \( i = 1, 2 \), the cokernel of the induced map \( \phi_{\alpha,s,i}^\text{dR} : H_1^\text{dR}(A_{\alpha-1}/S)_{i}^0 \to H_1^\text{dR}(A_\alpha/S)_{i}^0 \) is a locally free sheaf over \( S \) of rank \( j_{\alpha,i} \).
5. For each \( \alpha \) and \( i = 1, 2 \), the cokernel of the induced map \( \psi_{\alpha,s,i}^\text{dR} : H_1^\text{dR}(A_\alpha/S)_{i}^0 \to H_1^\text{dR}(A_{\alpha-1}/S)_{i}^0 \) is a locally free sheaf over \( S \) of rank \( n - j_{\alpha,i} \).
6. For each \( \alpha \), \( \text{Ker}(\phi_{\alpha,s,2}^\text{dR}) \) is contained in \( \omega_{A_\alpha/S,2}^\circ \).
7. For each \( \alpha \), the cokernel of \( \phi_{\alpha,s,1}^\text{dR} : \omega_{A_\alpha/S,1}^\circ \to \omega_{A_{\alpha-1}/S,1}^\circ \) is a locally free sheaf over \( S \) of rank \( r_{\alpha-1} - (r_{\alpha} - r_\epsilon) \).
8. For each \( \alpha \), the cokernel of \( \psi_{\alpha,s,1}^\text{dR} : \omega_{A_\alpha/S,1}^\circ \to \omega_{A_{\alpha-1}/S,1}^\circ \) is a locally free sheaf over \( S \) of rank \( r_{\alpha} - r_\epsilon \).

We point out some simple corollaries of these conditions which we shall use freely later:

1. By conditions (2) and (3), for each \( \alpha \) and \( i = 1, 2 \), we have \( \text{Im}(\psi_{\alpha,s,i}^\text{dR}) = \text{Ker}(\phi_{\alpha,s,i}^\text{dR}) \) and \( \text{Im}(\phi_{\alpha,s,i}^\text{dR}) = \text{Ker}(\psi_{\alpha,s,i}^\text{dR}) \).
2. Simple dimension count using conditions (1)(4)(5)(7)(8) implies that, for each \( \alpha \), the intersection \( \omega_{A_\alpha/S,1}^\circ \cap \text{Ker}(\phi_{\alpha,s,1}^\text{dR}) \) is locally free over \( S \) of rank \( r_\epsilon = r - \delta \); and the intersection \( \omega_{A_{\alpha-1}/S,1}^\circ \cap \text{Ker}(\psi_{\alpha,s,1}^\text{dR}) \) is locally free over \( S \) of rank \( r_{\alpha-1} - r_\epsilon \). So \( F(\text{Im}(\psi_{\alpha,s,1}^\text{dR})(p)) \) is a subbundle of rank \( j_{\alpha,1} - r_\epsilon \) and \( V^{-1}(\text{Im}(\phi_{\alpha,s,1}^\text{dR})(p)) \) is a subbundle of rank \( s_\alpha + r_\epsilon \).
3. We have a natural inclusion \( \psi_{\alpha,s,1}^\text{dR}(\omega_{A_{\alpha-1}/S,1}^\circ) \subseteq \omega_{A_\alpha/S,1}^\circ \cap \text{Ker}(\phi_{\alpha,s,1}^\text{dR}) \), which is an isomorphism of vector bundles of rank \( r_\epsilon \) as we shall show in the proof of Lemma 6.6 below.

**Remark 6.5.** Conditions (6)(7)(8) in Definition 6.4 did not appear in the definition \( Z_j \); but it is satisfied by the toy model in Example 6.3. They did not appear in moduli problem in Subsection 3.1 because they trivially hold in that case. The asymmetric conditions at the two places \( q_1 \) and \( q_2 \) is a reflection of choosing the degrees of \( \phi_\alpha \) to be related to \( j_{\alpha,i} \) as opposed to \( n - j_{\alpha-1,i} \). The purpose of keeping these two conditions in the moduli problem and carefully formulating them is to cut off unwanted components, so that the geometry of the moduli space \( Y_j \) is nicer (see Theorem 6.9). We think the picture is the following: \( Z_j \) is probably or at least heuristically the set of \( \mathbb{F}_{p^2} \)-points of a closed subscheme of \( \text{Sh}_{r_\epsilon,s} \). But this scheme has many irreducible components, which has overlaps with other \( Z_j \). Conditions (6)(7)(8) will help select one irreducible component that is "special" for \( j \). When taking the union of all images of \( Y_j \), we should still get the union of \( Z_j \).

\[\text{This is in fact a corollary of (2) and (4).} \]
Lemma 6.6. The moduli problem $Y_j$ is represented by a proper scheme $Y_j$ over $\mathbb{F}_{\rho^2}$, which we still denote by $Y_j$.

Proof. Standard arguments show that the moduli problem for conditions (1)–(6) is represented by a proper scheme over $\mathbb{F}_{\rho^2}$. To see the representability of $Y_j$, it suffices to show that conditions (7) and (8) are closed conditions. Clearly the following are closed conditions:

(7') For each $\alpha$, the cokernel of $\phi_{\alpha,1}^{dR} : \omega_{A_0}^* / S_1 \to \omega_{A_{0-1}}^* / S_1$ is a coherent sheaf over $S$ locally generated by $r_{\alpha-1} - (r_{\alpha} - r_{\epsilon})$ generators;

(8') For each $\alpha$, the cokernel of $\psi_{\alpha,1}^{dR} : \omega_{A_0}^* / S_1 \to \omega_{A_{0-1}}^* / S_1$ is a coherent sheaf over $S$ locally generated by $r_{\alpha} - r_{\epsilon}$ generators.

We will show that the apparently weaker conditions (7') and (8') imply conditions (7) and (8) (at the presence of conditions (1)–(6)); then the Lemma is clear. Recall that the inclusion in Definition 6.4(iii) holds unconditionally, so it makes sense to talk about the quotient. This quotient fits in the following exact sequence

$\begin{align*}
0 &\to \omega_{A_0}^* / S_1 \cap \text{Ker}(\phi_{\alpha,1}^{dR}) \to \omega_{A_0}^* / S_1 \to \omega_{A_{0-1}}^* / S_1 \to \omega_{A_{0-1}}^* / S_1 \to 0.
\end{align*}$

The lemma has now become a purely commutative algebra question. Let $0 \to B \to C \to D \to E \to 0$ denote the exact sequence above. For each point $x$ of $S$ with residue field $k(x)$, we have

$\text{Tor}_1^S(E, k(x)) \to (C/B) \otimes k(x) \to D \otimes k(x) \to E \otimes k(x) \to 0.$

We know from the moduli problem that $D$ is locally free of rank $r_{\alpha-1}$, and from conditions (7') and (8') that

$\dim E \otimes k(x) \leq r_{\alpha-1} - (r_{\alpha} - r_{\epsilon})$ and $\dim(C/B) \otimes k(x) \leq \dim C \otimes k(x) \leq r_{\alpha} - r_{\epsilon}.$

So this forces all inequalities above to be equalities and $\text{Tor}_1^S(E, k(x)) = 0$. By local criterion of flatness, $E$ is flat and hence locally free of rank $r_{\alpha-1} - (r_{\alpha} - r_{\epsilon})$. It then follows that $C/B$ is flat of locally free of rank $r_{\alpha} - r_{\epsilon}$. So $B \otimes k(x) = 0$ for any point $x$ on $S$; so $B = 0$. This proves the Lemma, and also the equality of the inclusion in Definition 6.4(iii).

The moduli space $Y_j$ admits morphisms to $\text{Sh}_{r,s}$ for all $0 \leq \alpha \leq \epsilon$, and that its image in $\text{Sh}_{r,s}$ is a closed subscheme whose $\mathbb{F}_{\rho^2}$-points are all contained in the set of points $Z_j$ considered in the previous section.

Remark 6.7. Conditions (6)(7)(8) in Definition 6.4 did not appear in the definition $Z_j$; but it is satisfied by the toy model in Example 6.3. They did not appear in moduli problem in Subsection 3.1 because they trivially hold in that case. The asymmetric conditions at the two places $q_1$ and $q_2$ is a reflection of choosing the degrees of $\phi_o$ to be related to $j_{o,i}$ as opposed to $n - j_{o-1,i}$. The purpose of keeping these two conditions in the moduli problem and carefully formulating them is to cut off unwanted components, so that the geometry of $Y_j$ is nicer (see Theorem 6.9). We think the picture is the following: $Z_j$ is probably or at least heuristically the set of $\mathbb{F}_{\rho^2}$-points of a closed subscheme of $\text{Sh}_{r,s}$. But this scheme has many irreducible components, which have overlaps with other $Z_j$’s. Conditions (6)(7)(8) will help select one irreducible component that is “special” for $j$. When taking the union of all the images of $Y_j$, we should still get the union of $Z_j$.

6.8. Geometry of $Y_j$. It is slightly difficult to understand the geometry of $Y_j$ directly; so we separate the discussion for each step $\phi_o : A_o \to A_{o-1}$. For each $\alpha = 1, \ldots, \epsilon$, we consider the moduli space $Y_{\alpha}$ that represents the functor taking a locally noetherian $\mathbb{F}_{\rho^2}$-scheme $S$ to the set of isomorphism classes of tuples

$(A_o, A_{o-1}, \lambda_o, \lambda_{o-1}, \eta_o, \eta_{o-1}, \phi_o, \psi_o)$
such that condition (1) of Definition 6.4 holds for \( \alpha \) and \( \alpha - 1 \), and conditions (2)–(8) of Definition 6.4 hold for \( \alpha \).

Clearly, \( Y_\alpha \) is a proper scheme of finite type over \( \mathbb{F}_p \), by (the proof of) Lemma 6.6. There are obvious forgetful functors from \( Y_\alpha \) to \( \text{Sh}_{\alpha,s_\alpha} \) and to \( \text{Sh}_{\alpha-1,s_{\alpha-1}} \). Moreover, the moduli spaces \( Y_\alpha \)'s and \( Y_j \) are related by the natural isomorphism

\[
Y_j \cong Y_1 \times_{\text{Sh}_{1,s_1}} Y_2 \times_{\text{Sh}_{2,s_2}} Y_3 \times_{\text{Sh}_{3,s_3}} \cdots \times_{\text{Sh}_{r_{\alpha-1},s_{\alpha-1}}} Y_\epsilon.
\]

**Theorem 6.9.** Each \( Y_\alpha \) is smooth of dimension \( r_\alpha s_\alpha + r_{\alpha-1}s_{\alpha-1} \).

**Proof.** (1) Let \( \hat{R} \) be a noetherian \( \mathbb{F}_p \)-algebra and \( \hat{I} \subset \hat{R} \) an ideal such that \( \hat{I}^2 = 0 \); put \( R = \hat{R}/\hat{I} \). We need to show that every \( R \)-point \((A_\alpha, A_{\alpha-1}, \lambda_\alpha, \lambda_{\alpha-1}, \eta_\alpha, \eta_{\alpha-1}, \phi_\alpha, \psi_\alpha)\) of \( Y_\alpha \) can be lifted to an \( \hat{R} \)-point (and we need to compute the corresponding tangent space). By Serre-Tate and Grothendieck-Messing deformation theory we recalled in Theorem 2.4 it is enough to lift, for \( i = 1, 2 \), the differentials \( \omega_{A^\alpha_{\hat{R},i}/R}^\alpha \subseteq H^1_{\text{cris}}(A_\alpha/R)^\circ_i \) to a subbundle \( \hat{\omega}_{\alpha,i} \subseteq H^1_{\text{cris}}(A_\alpha/\hat{R})^\circ_i \) such that

(a) \( \phi_{\alpha,1,*}^{\text{cris}}(\hat{\omega}_{\alpha,1}) \subseteq \hat{\omega}_{\alpha-1,1} \) and \( \psi_{\alpha,1,*}(\hat{\omega}_{\alpha-1,1}) \subseteq \hat{\omega}_{\alpha,1} \) (so that both \( \phi_\alpha \) and \( \psi_\alpha \) are lifted, which would automatically imply \( \text{Ker}(\phi_\alpha) \subseteq A_\alpha[p] \)),

(b) \( \hat{\omega}_{\alpha,2} \supseteq \text{Ker}(\phi_{\alpha,2}^{\text{cris}}) \),

and

(c) \( \hat{\omega}_{\alpha-1,1}/\phi_{\alpha,1,*}^{\text{cris}}(\hat{\omega}_{\alpha,1}) \) is flat over \( \hat{R} \) of rank \( r_\alpha - (r_\alpha - r_\epsilon) \), and \( \hat{\omega}_{\alpha,1}/\psi_{\alpha,1,*}^{\text{cris}}(\hat{\omega}_{\alpha-1,1}) \) is flat over \( \hat{R} \) of rank \( r_\alpha - r_\epsilon \).

We separate the discussion of lifts at \( q_1 \) and \( q_2 \), and show that the tangent space \( T_{Y_\alpha} \cong T_1 \oplus T_2 \) for the contribution \( T_1 \) and \( T_2 \) from the two places. We first look at \( q_2 \), as it is easier. The lift \( \hat{\omega}_{\alpha,2} \) is subject to condition (b); so its choices form a torsor for the group

\[
\text{Hom}_R \left( \omega_{A^\alpha_1/R_2}/\text{Ker}(\phi_{\alpha,2}^{\text{dr}}), \text{Lie}_{A_\alpha/R_2}^\circ \right) \otimes_R \hat{I}.
\]

Once the lift \( \hat{\omega}_{\alpha,2} \) is fixed, the choice of \( \hat{\omega}_{\alpha-1,2} \) is subject to condition (a) for \( i = 2 \). At the presence of (b), we always have

\[
\psi_{\alpha,2,*}(\hat{\omega}_{\alpha-1,2}) \subseteq \text{Im}(\psi_{\alpha,2,*}^{\text{cris}}) = \text{Ker}(\phi_{\alpha,2}^{\text{cris}}) \subseteq \hat{\omega}_{\alpha,2}.
\]

So it suffices to meet the condition \( \phi_{\alpha,2,*}^{\text{cris}}(\hat{\omega}_{\alpha,2}) \subseteq \hat{\omega}_{\alpha-1,2} \). So the choices of \( \hat{\omega}_{\alpha-1,2} \) form a torsor for the group

\[
\text{Hom}_R \left( \omega_{A^\alpha_{\hat{R},1}/R_2}/\phi_{\alpha,2,*}^{\text{cris}}(\omega_{A^\alpha_{\hat{R},2}}^\alpha), \text{Lie}_{A_{\alpha-1}/R_2}^\circ \right) \otimes_R \hat{I}.
\]

This implies that the contribution \( T_2 \) to the tangent space \( T_{Y_\alpha} \) at \( q_2 \) sits in a short exact sequence

\[
0 \to \text{Hom}(\omega_{A^\alpha_{\hat{R},2}/R_2}/\phi_{\alpha,2,*}^{\text{cris}}(\omega_{A^\alpha_{\hat{R},2}}^\alpha), \text{Lie}_{A_{\alpha-1}}^\circ) \to T_2 \to \text{Hom}(\omega_{A^\alpha_{\hat{R},2}/\text{Ker}(\phi_{\alpha,2}^{\text{dr}})}, \text{Lie}_{A_{\alpha}}^\circ) \to 0.
\]

In particular, \( T_2 \) is a locally free sheaf of rank

\[
(6.9.1) \quad (s_\alpha - (s_\alpha - j_\alpha,2))r_\alpha - 1 + (s_\alpha - j_\alpha,2)r_\alpha.
\]

We now look at the place \( q_1 \). We first lift

\[
\phi_{\alpha,1,*}^{\text{cris}}(\omega_{A^\alpha_{\hat{R},1}/R_1}^\alpha) \subseteq H^1_{\text{cris}}(A_{\alpha-1}/R_1)^\circ \to \text{a \ subbundle} \ \hat{H}_{\alpha-1} \subseteq \text{Im}(\phi_{\alpha,1,*}^{\text{cris}}) \subseteq H^1_{\text{cris}}(A_{\alpha-1}/\hat{R})^\circ
\]

of rank \( r_\alpha - r_\epsilon \). This subbundle \( \hat{H}_{\alpha-1} \) will be the image \( \phi_{\alpha,1,*}^{\text{cris}}(\hat{\omega}_{\alpha,1}) \); so that \( \hat{\omega}_{\alpha-1,1}/\phi_{\alpha,1,*}^{\text{cris}}(\hat{\omega}_{\alpha,1}) \) is locally free of rank \( r_\alpha - (r_\alpha - r_\epsilon) \) at the end. The choices of such lift such \( \hat{H}_{\alpha-1} \) form a torsor for the group

\[
\text{Hom}_R \left( \phi_{\alpha,1,*}^{\text{dr}}(\omega_{A^\alpha_{\hat{R},1}/R_1}), \text{Ker}(\psi_{\alpha,1,*}^{\text{cris}})/\phi_{\alpha,1,*}^{\text{cris}}(\omega_{A^\alpha_{\hat{R},1}/R_1}) \right) \otimes_R \hat{I}.
\]

Similarly, we lift

\[
\psi_{\alpha,1,*}^{\text{dr}}(\omega_{A^\alpha_{\hat{R},1}/R_1}^\circ) \subseteq H^1_{\text{cris}}(A_{\alpha}/R_1)^\circ \to \text{a \ subbundle} \ \hat{H}_{\alpha} \subseteq \text{Im}(\psi_{\alpha,1,*}^{\text{cris}}) \subseteq H^1_{\text{cris}}(A_{\alpha}/\hat{R})^\circ
\]
of rank \( r_\epsilon \). This subbundle \( \hat{H}_\alpha \) will be the image \( \psi_{\alpha,s_1}^{\text{cris}}(\hat{\omega}_{\alpha-1,1}) \); so that \( \hat{\omega}_{\alpha,1}/\psi_{\alpha,s_1}^{\text{cris}}(\hat{\omega}_{\alpha-1,1}) \) is locally free of rank \( r_\alpha - r_\epsilon \) at the end. The choices of such lift \( \hat{H}_\alpha \) form a torsor for the group

\[
\text{Hom}_R \left( \psi_{\alpha,s_1}^{\text{dR}}(\omega_{A_{\alpha-1}^V/R,1}), \text{Ker}(\phi_{\alpha,s_1}^{\text{dR}})/\psi_{\alpha,s_1}^{\text{dR}}(\omega_{A_{\alpha-1}^V/R,1}) \right) \otimes_R \hat{I}.
\]

Once the lifts \( \hat{H}_{\alpha-1} \) and \( \hat{H}_\alpha \) are fixed. We need to lift \( \omega_{A_{\alpha-1}^V/R,1} \) to a subbundle \( \hat{\omega}_{\alpha,1} \) of \( (\psi_{\alpha,s_1}^{\text{cris}})^{-1}(\hat{H}_{\alpha-1}) \) containing \( \hat{H}_\alpha \) of rank \( r_\alpha \). By Nakayama’s Lemma, the image under \( \phi_{\alpha,s_1}^{\text{cris}} \) of any such lift \( \hat{\omega}_{\alpha,1} \) is all of \( \hat{H}_\alpha \) because it is so modulo \( \hat{I} \). So there is no further restriction on such lift. All such lifts \( \hat{\omega}_{\alpha,1} \) form a torsor for the group

\[
\text{Hom}_R \left( \omega_{A_{\alpha-1}^V/R,1}/\psi_{\alpha,s_1}^{\text{dR}}(\omega_{A_{\alpha-1}^V/R,1}), (\text{Ker}(\phi_{\alpha,s_1}^{\text{dR}})^{-1} + \omega_{A_{\alpha-1}^V/R,1})/\omega_{A_{\alpha-1}^V/R,1} \right) \otimes_R \hat{I}.
\]

Similarly, we need to lift \( \omega_{A_{\alpha-1}^V/R,1} \) to a subbundle \( \hat{\omega}_{\alpha-1,1} \) of \( (\psi_{\alpha,s_1}^{\text{cris}})^{-1}(\hat{H}_\alpha) \) containing \( \hat{H}_{\alpha-1} \) of rank \( r_{\alpha-1} \). By Nakayama’s Lemma, the image under \( \psi_{\alpha,s_1}^{\text{cris}} \) of any such lift \( \hat{\omega}_{\alpha-1,1} \) is all of \( \hat{H}_{\alpha-1} \) because it is so modulo \( \hat{I} \). All such lifts \( \hat{\omega}_{\alpha-1,1} \) form a torsor for the group

\[
\text{Hom}_R \left( \omega_{A_{\alpha-1}^V/R,1}/\phi_{\alpha,s_1}^{\text{dR}}(\omega_{A_{\alpha-1}^V/R,1}), (\text{Ker}(\psi_{\alpha,s_1}^{\text{dR}})^{-1} + \omega_{A_{\alpha-1}^V/R,1})/\omega_{A_{\alpha-1}^V/R,1} \right) \otimes_R \hat{I}.
\]

To sum up, the contribution \( T_1 \) to the tangent space \( T_{Y_\alpha} \) at \( q_1 \) sits in a short exact sequence

\[
0 \to \text{Hom} \left( \frac{\omega_{A_{\alpha-1}^V/R,1}^{\psi_{\alpha,s_1}^{\text{dR}}}}{\phi_{\alpha,s_1}^{\text{dR}}(\omega_{A_{\alpha-1}^V/R,1})}, \frac{\ker(\phi_{\alpha,s_1}^{\text{dR}})/\omega_{A_{\alpha-1}^V/R,1}}{\phi_{\alpha,s_1}^{\text{dR}}(\omega_{A_{\alpha-1}^V/R,1})} \right) \oplus \text{Hom} \left( \frac{\omega_{A_{\alpha-1}^V/R,1}}{\phi_{\alpha,s_1}^{\text{dR}}(\omega_{A_{\alpha-1}^V/R,1})}, \frac{\ker(\psi_{\alpha,s_1}^{\text{dR}})/\omega_{A_{\alpha-1}^V/R,1}}{\psi_{\alpha,s_1}^{\text{dR}}(\omega_{A_{\alpha-1}^V/R,1})} \right) \to 0.
\]

This \( T_1 \) is a locally free sheaf over \( Y_\alpha \) of rank

\[
(6.9.2)
\]

\[
(r_\alpha - r_\epsilon)(n-j_{\alpha,1} - r_\alpha + r_\epsilon + r_\epsilon(j_{\alpha,1} - r_\epsilon) + (r_\alpha - r_\epsilon)(j_{\alpha,1} - r_\epsilon) + (r_{\alpha-1} - r_\alpha + r_\epsilon)(n-j_{\alpha,1} + r_\epsilon - r_{\alpha-1}).
\]

It is a straightforward but tedious task to check that the number \( (6.9.2) \) and the number \( (6.9.1) \) add up to \( s_\alpha r_\alpha + s_{\alpha-1} r_{\alpha-1} \). This concludes the proof of this Theorem.

**Definition 6.10.** A morphism \( f : X \to Y \) of varieties over \( \mathbb{F}_p \) is called a Frobenius factor if there is a morphism \( g : Y \to X \) such that \( g \circ f \) is some power of the Frobenius morphism on \( X \).

By [Hec12], Proposition 4.8], a proper morphism \( f : X \to Y \) of smooth varieties is a Frobenius factor if and only if it induces a bijection on closed points.

**Proposition 6.11.** The natural (proper) morphism \( \zeta_\alpha : Y_\alpha \to \text{Sh}_{r_{\alpha,s_\alpha}} \) is surjective. Moreover, \( Y_\alpha \) contains an open dense subscheme \( Y_\alpha^\circ \) such that the induced morphism \( \zeta_\alpha^\circ : Y_\alpha^\circ \to \text{Sh}_{r_{\alpha,s_\alpha}} \) can be factored as the composition \( Y_\alpha^\circ \xrightarrow{\theta_\alpha} Y_\alpha^* \xrightarrow{\zeta_\alpha^*} \text{Sh}_{r_{\alpha,s_\alpha}} \), where the first morphism is a Frobenius factor, and the second morphism is a smooth morphism of relative dimension \( r_{\alpha-1}s_{\alpha-1} - r_\alpha s_\alpha \).

**Proof.** In the moduli problem of \( Y_\alpha \), we use \( H_\alpha \) to denote \( \text{Im}(\psi_{\alpha,s_1}^{\text{dR},*}) = \text{Ker}(\phi_{\alpha,s_1}^{\text{dR},*}) \).

**Step I:** We first define the subscheme \( Y_\alpha^\circ \) of \( Y_\alpha \) to be the open subscheme where

(a) the intersection \( \omega_{A_{\alpha}^V/S_2}^{\psi_{\alpha,s_1}^{\text{dR}}} \cap V^{-1}(H_1^{(p)}) \) is a subbundle of (minimal) rank \( s_\alpha - (r_\alpha - r_\epsilon) \); and the intersection \( \omega_{A_{\alpha}^V/S_1}^{\psi_{\alpha,s_1}^{\text{dR}}} \cap V^{-1}(V^{-1}(H_1^{(p^2)})) \) is a subbundle of (minimal) rank \( r_\epsilon \);

(b) \( V(H_1^{(p)}) + F(H_1^{(p^2)}) \) is a subbundle of (maximal) rank \( j_{\alpha,1} \);

(c) \( F^{-1}(\omega_{A_{\alpha}^V/S_2}^{\psi_{\alpha,s_1}^{\text{dR}}}) \) has (minimal) rank \( s_\alpha \); and the intersection \( F^{-1}(\omega_{A_{\alpha}^V/S_2}^{\psi_{\alpha,s_1}^{\text{dR}}}) \cap V^{-1}(H_2^{(p^2)}) \) has (minimal) rank \( j_{\alpha,2} \).
We now explain that these are open conditions, and violating either condition will decrease the dimension of the tangent space as computed in Theorem 6.9, so the complement \( Y_\alpha \setminus Y_\alpha^0 \) is a closed subscheme of lower dimension, and hence \( Y_\alpha^0 \) is open and dense in \( Y_\alpha \).

(a) By Definition 6.4(ii), \( V^{-1}(H_1^{(p)}) \) is a subbundle of \( H_1^{dR}(A_\alpha/S)_1^0 \) of rank \( s_\alpha + r_\epsilon \), containing \( H_2 \). The intersection \( V^{-1}(H_1^{(p)}) \cap \omega_{A_\alpha/S,2}^0 \) has least rank \( (s_\alpha + r_\epsilon) + s_\alpha - n = s_\alpha + r_\epsilon - r_\alpha \). When the rank is higher, it would mean that the lift of \( \hat{\omega}_{\alpha,2} \) in the proof of Theorem 6.9 is subject to a further constraint.

When \( V^{-1}(H_1^{(p)}) \cap \omega_{A_\alpha/S,2}^0 \) is a subbundle of rank \( s_\alpha + r_\epsilon - r_\alpha \) (which is the case on an open subscheme of \( Y_\alpha \)), the second iterated preimage \( V^{-1}(V^{-1}(H_1^{(p^2)})) \) is a subbundle of \( H_1^{dR}(A_{\alpha/S})_1^0 \) of rank \( s_\alpha + r_\epsilon - r_\alpha + r_\alpha = s_\alpha + r_\epsilon \). It must contain \( H_1 \). So its intersection with \( H_1 + \omega_{A_{\alpha/S}}^0 \) has at least rank \( (s_\alpha + r_\epsilon) + (j_1 + r_\alpha - r_\epsilon) - n = j_1 \). Any bigger intersection will impose further constraint on the lifts \( \hat{H}_{\alpha-1} \) in the proof of Theorem 6.9. So over an open subscheme, the intersection of \( H_1 + \omega_{A_{\alpha/S}}^0 \) with \( V^{-1}(V^{-1}(H_1^{(p^2)})) \) is a subbundle of rank \( j_1 \)\(^{12}\).

Over this subscheme, the intersection \( \omega_{A_{\alpha/S}}^0 \cap V^{-1}(V^{-1}(H_1^{(p^2)})) \) has minimal rank \( r_\alpha + j_1 - (r_\alpha + j_1 - r_\epsilon) = r_\epsilon \). Any bigger rank would result in having extra constraint in the lifting of \( \hat{\omega}_1 \) in the proof of Theorem 6.9.

(b) Since the equality of Definition 6.4(iii) says that \( H_1 \cap \omega_{A_{\alpha/S}}^0 = \psi_{dR,1}(\omega_{A_{\alpha-1/S}}^0) \) (is a subbundle of rank \( r_\epsilon \)), (a) is equivalent to that

\[
\psi_{dR,1}(\omega_{A_{\alpha-1/S}}^0) + V^{-1}(F(H_1^{(p^2)}))
\]

is a subbundle of (maximal) rank \( j_{\alpha,1} + r_\alpha \). By Definition 6.4, \( F(H_1^{(p)}) \) is a subbundle of rank \( j_{\alpha,1} - r_\epsilon \), which is contained in \( H_2 \subseteq \omega_{A_{\alpha/S}}^0 \). So \( V^{-1}(F(H_1^{(p^2)})) \) is a subbundle of rank \( j_{\alpha,1} - r_\epsilon + r_\alpha \). So it suffices to show that, generically, \( \psi_{dR,1}(\omega_{A_{\alpha-1/S}}^0) \) and \( V^{-1}(F(H_1^{(p^2)})) \) do not intersect each other.

In view of the deformation argument of lifting \( \hat{H}_1 \) inside \( H_1 \) in the proof of Theorem 6.9, it suffices to show that \( H_1 \cap V^{-1}(F(H_1^{(p^2)})) \) has generically rank \( \leq j_{\alpha,1} - r_\epsilon \), which is equivalent to

\[(6.11.1) \quad \text{rank of } V(H_1) \cap F(H_1^{(p^2)}) \leq \text{rank of } V(H_1) - r_\epsilon.
\]

But \( V(H_1) = \psi_{dR,2}(\omega_{A_{\alpha-1/S}}^0) \) has maximal freedom within \( H_2 \) according to the deformation computation in the proof of Theorem 6.9, so \( F(H_1^{(p^2)}) \) being of corank \( r_\epsilon \) inside \( H_2 \) implies that \( (6.11.1) \) holds generically.

(c) can be proved in a similar way; so we leave it as an exercise for careful readers.

**Step II:** We define the moduli space \( Y_\alpha^\bullet \), which is to only remember \( H_1^\bullet = H_1^{(p)} \) and \( H_2^\bullet = H_2^{(p^2)} \).

More precisely, \( Y_\alpha^\bullet \) is the moduli space which represents the functor taking a locally noetherian \( \mathbb{F}_p \)-scheme \( S \) to the set of isomorphism classes of tuples

\[
(A_\alpha, \lambda_\alpha, \eta_\alpha, H_1^\bullet, H_2^\bullet)
\]

such that

1. \( (A_\alpha, \lambda_\alpha, \eta_\alpha) \) is an \( S \)-point of \( \text{Sh}_{r_\alpha, s_\alpha} \).
2. \( H_1^\bullet \) is a subbundle of \( H_1^{dR}(A_\alpha/S)_1^0 \) of rank \( j_{\alpha,1} \), and \( H_2^\bullet \) is a subbundle of \( H_1^{dR}(A_{\alpha/S})_2^0 \) of rank \( j_{\alpha,2} \).
3. We have \( F(H_1^{(p^2)}) \subseteq H_2^\bullet \), \( F(H_2^\bullet) \subseteq H_1^\bullet \), \( V(H_1^\bullet) \subseteq H_2^\bullet \), and \( V(H_2^\bullet) \subseteq H_1^\bullet \).

\(^{12}\)In fact, if \( \omega_{A_{\alpha/S}}^0 \cap V^{-1}(V^{-1}(H_1^{(p^2)})) \) is a subbundle of rank \( r_\epsilon \), it would force \( H_1 + \omega_{A_{\alpha/S}}^0 \cap V^{-1}(V^{-1}(H_1^{(p^2)})) \) to be a subbundle of rank

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(4) We have $H_2^s \subseteq \omega^{s,2}_{A_\alpha/S,2}$.

(5) The cokernel of $H_1^s \rightarrow \text{Lie}^{s,1}_{A_\alpha/S,1}$ is locally free of rank $s_\alpha + r_\epsilon - j_{\alpha,1}$.

(6) The cokernel of $H_1^p \rightarrow H_1^p(\alpha_{A_\alpha/S,1})/F(H_1^p(\alpha_{A_\alpha/S,1}))$ is locally free of rank $r_\alpha - r_\epsilon$.

(a') the intersection $\omega^{s,1}_{A_\alpha/S,1} \cap V^{-1}(H_1^s(\alpha_{A_\alpha/S,1}))$ is a subbundle of rank $s_\alpha - r_\alpha + r_\epsilon$; and $\omega^{s,2}_{A_\alpha/S,1} \cap V^{-1}(V^{-1}(H_1^s(\alpha_{A_\alpha/S,1})))$ is a subbundle of rank $r_\epsilon$.

(b') $V(H_1^s(\alpha_{A_\alpha/S,1}) + F(H_1^p(\alpha_{A_\alpha/S,1})))$ is a subbundle of rank $r_\alpha$.

(c') the intersection $F^{-1}(\omega^{s,2}_{A_\alpha/S,1}) \cap V^{-1}(H_1^s(\alpha_{A_\alpha/S,1}))$ is a subbundle of rank $r_\alpha$.

Standard arguments show that $Y_\alpha^*$ is a scheme separated of finite type over $\mathbb{F}_p$.

**Step III:** We show that there is a natural proper morphism $\theta_\alpha : Y_\alpha^* \rightarrow Y_\alpha^*$ given by

$$\theta_\alpha = (A_{\alpha,1}, \lambda_\alpha, \lambda_{\alpha-1}, \eta_\alpha, \eta_{\alpha - 1}, \phi_\alpha, \psi_\alpha) \mapsto (A_{\alpha,1}, \lambda_\alpha, \eta_\alpha, \text{Ker}(\phi_{\alpha,s,1}^{\text{dR}}(\alpha_{A_\alpha/S,1}), \text{Ker}(\phi_{\alpha,s,2}^{\text{dR}}(\alpha_{A_\alpha/S,1}))),$$

and $\theta_\alpha$ induces a bijection on the $\mathbb{F}_p$-points. (We shall show below in Step III that $Y_\alpha^*$ is smooth; then $\theta_\alpha$ would be a Frobenius factor by [He12, Proposition 4.8].) So the natural morphism $\zeta_\alpha : Y_\alpha \rightarrow \text{Sh}_{r_\alpha,s_\alpha}$ would factor as $Y_\alpha \xrightarrow{\theta_\alpha} Y_\alpha^* \xrightarrow{\zeta_\alpha} \text{Sh}_{r_\alpha,s_\alpha}$, where $\zeta_\alpha$ is given by forgetting $H_1^s$ and $H_1^p$.

We first check that $\theta_\alpha$ is well defined, by verifying the conditions in the moduli space of $Y_\alpha^*$. Conditions (1) and (2) are obvious. Condition (3) follows from the compatibility of the morphisms $\phi_\alpha$ and $\psi_\alpha$ with Frobenius and Verschiebung morphisms. Condition (4) follows from Definition 6.4.6.

For condition (5), we note that

$$\text{(6.11.2)} \quad H_1^p(\alpha_{A_\alpha/S,1})/(\text{Ker}(\phi_{\alpha,s,1}^{\text{dR}}) + \omega^{s,1}_{A_\alpha/S,1}) \xrightarrow{\phi_{\alpha,s,1}^{\text{dR}} \otimes 1} \phi_{\alpha,s,1}^{\text{dR}}(H_1^p(\alpha_{A_\alpha/S,1}))/\phi_{\alpha,s,1}^{\text{dR}}(\omega^{s,1}_{A_\alpha/S,1});$$

the latter is locally free of rank

$$n - j_{\alpha,1} - (r_\alpha - 1 - (r_\alpha - r_\epsilon)) = s_\alpha + r_\epsilon - j_{\alpha,1}$$

by Definition 6.4.7; so condition (5) holds. Condition (6) follows from the following sequence of isomorphisms:

$$\text{(6.11.3)} \quad \omega^{s,2}_{A_\alpha/S,2} \xrightarrow{\psi_{\alpha,s,2}^{\text{dR}}(\omega^{s,2}_{A_\alpha/S,2})} \frac{V(H_1^p(\alpha_{A_\alpha/S,1}))}{V(\psi_{\alpha,s,2}^{\text{dR}}(H_1^p(\alpha_{A_\alpha/S,1})))} \xrightarrow{\phi_{\alpha,s,1}^{\text{dR}}} \frac{H_1^p(\alpha_{A_\alpha/S,1})}{\text{Im}(\psi_{\alpha,s,2}^{\text{dR}})} + F(H_1^p(\alpha_{A_\alpha/S,1}))$$

where the first term is locally free of rank $r_\alpha - r_\epsilon$ by Definition 6.4.8. Conditions (a)(b)(c) clearly implies conditions (a')(b')(c'). The morphism $\theta_\alpha$ is proper because relative to $Y_\alpha$ is a moduli space for abelian varieties with only closed conditions.

By Proposition 2.2, the knowledge of $\text{Ker}(\phi_{\alpha,s,1}^{\text{dR}})$ and $\text{Ker}(\phi_{\alpha,s,2}^{\text{dR}})$ at each $\mathbb{F}_p$-point of $Y_\alpha$ is enough to recover $(A_{\alpha-1}, \lambda_{\alpha-1}, \eta_{\alpha-1})$ together with the natural isogenies $\phi$ and $\psi$. So the morphism $\theta_\alpha$ is injective on $\mathbb{F}_p$-points. Conversely, given a $\mathbb{F}_p$-point $(A_\alpha, \lambda_\alpha, \eta_\alpha, H_1, H_2)$, we shall find a preimage under $\theta_\alpha$. Indeed, conditions (3) allow us to invoke Proposition 2.2 to construct an abelian variety $(A_{\alpha-1}, \lambda_{\alpha-1}, \eta_{\alpha-1})$ together with the isogenies $\phi$ and $\psi$ satisfying Definition 6.4.1(4)-(6). Using isomorphisms 6.11.2 and 6.11.3 at the closed point, we deduce the equivalence of conditions (5)(6) above and Definition 6.4.7(8). Conditions (a)(b)(c) and (a')(b')(c') are clearly equivalent at an $\mathbb{F}_p$-point.

**Step IV:** We show that $Y_\alpha^*$ is smooth over $\text{Sh}_{r_\alpha,s_\alpha}$ of relative dimension $r_\alpha-1-s_\alpha-1-r_\alpha s_\alpha$, and hence is smooth itself. Then [He12, Proposition 4.8] would show that the morphism $\theta_\alpha$ is a Frobenius factor; this would conclude the proof of the Proposition.

For this, we take $\hat{R}$ a noetherian $\mathbb{F}_p$-algebra and $\hat{I}$ an ideal such that $\hat{I}^2 = 0$; put $R = \hat{R}/\hat{I}$. We need to show, for each $R$-point $y^* = (A_{\alpha}, \lambda_\alpha, \eta_\alpha, H_1^s, H_2^s)$ of $Y_\alpha^*$, if we are given a lift of $x = \zeta_\alpha(y^*)$ to an $\hat{R}$-point $\hat{x} = (A_\alpha, \hat{\lambda}_\alpha, \hat{\eta}_\alpha)$ of $\text{Sh}_{r_\alpha,s_\alpha}$, then there is an $\hat{R}$-point $\hat{y}^*$ of $Y_\alpha^*$ lifting $y^*$.
such that \( \zeta^* (\tilde{y}^*) = \hat{x} \). By Theorem 2.4, the lift \( \hat{x} \) is equivalent to lifts \( \hat{\omega}_i \) of \( \omega^{\circ\circ}_{i,H/R} \) as subbundles of \( H^{\text{cris}}_i(A_\alpha/R) \) for \( i = 1, 2 \). So it suffices to lift \( H_i^\bullet \) to a subbundle of \( H^{\text{cris}}_i(A_\alpha/R) \) such that the following conditions hold:

(i) \( \hat{H}_2^\bullet \subseteq \omega^{\circ\circ}_{A_\alpha/R} \); 
(ii) \( F(\hat{H}_1^\bullet (p^2)) \subseteq \hat{H}_2^\bullet \), (and \( F(H_1^\bullet) \subseteq H_1^\bullet \) is automatic at the presence of the condition above); 
(iii) \( \hat{H}_1^\bullet \subseteq V^{-1} (\hat{H}_2^\bullet) \) and \( \hat{H}_2^\bullet \subseteq V^{-1} (\hat{H}_1^\bullet (p^2)) \); 
(iv) the cokernel of \( \hat{H}_1^\bullet \to H_1^{\text{cris}}(A_\alpha/R) \) is a flat \( \hat{R} \)-module of rank \( s_\alpha + r_\epsilon - j_{\alpha,1} \) (so that \( \hat{H}_1^\bullet \cap \omega^{\circ\circ}_{A_\alpha,R} \) is a subbundle of rank \( r_\epsilon \)); 
(v) the cokernel of \( \hat{H}_2^\bullet \to H_1^{\text{cris}}(A_\alpha/R) \) is locally free of rank \( r_\alpha - r_\epsilon \); 
(vi) the intersection \( \hat{\omega}_2 \cap V^{-1} (\hat{H}_1^\bullet (p^2)) \) is a subbundle of rank \( s_\alpha + r_\epsilon - r_\alpha \); and the intersection \( \hat{\omega}_1 \cap V^{-1} (V^{-1} (\hat{H}_1^\bullet (p^2))) \) is a subbundle of rank \( r_\epsilon \); 
(vii) \( V(\hat{H}_1^\bullet \cap \omega^{\circ\circ}_{A_\alpha/R,1}) + F(\hat{H}_1^\bullet (p^2)) \) is a subbundle of rank \( j_{\alpha,1} \); 
(viii) the intersection \( F^{-1}(\omega^{\circ\circ}_{A_\alpha/R,1}) \cap V^{-1}(\hat{H}_2^\bullet) \) is a subbundle of rank \( j_{\alpha,2} \).

Conditions (iv)–(viii) are open conditions, just so that we have to avoid certain particular values of \( \hat{H}_1^\bullet \) and \( \hat{H}_2^\bullet \). We first determine \( \hat{H}_1^\bullet \cap \omega^{\circ\circ}_{A_\alpha/R,1} \). Note that condition (iii) implies that \( V(\hat{H}_1^\bullet) \subseteq V^{-1}(\hat{H}_1^\bullet (p^2)) \); so we must have 

\[
(6.11.4) \quad \hat{H}_1^\bullet \cap \omega^{\circ\circ}_{A_\alpha/R,1} \subseteq \omega^{\circ\circ}_{A_\alpha/R,1} \cap V^{-1}(V^{-1}(\hat{H}_1^\bullet (p^2))).
\]

But the both sides have the same rank \( r_\epsilon \) by condition (iv) and condition (a'). So they must be equal. Now, the lifting process can be summarized as first lifting \( \hat{H}_2^\bullet \) so that 

\[
V(\hat{H}_1^\bullet \cap \omega^{\circ\circ}_{A_\alpha/R,1}) + F(\hat{H}_1^\bullet (p^2)) \subseteq \hat{H}_2^\bullet \subseteq \omega^{\circ\circ}_{A_\alpha/R,2} \cap V^{-1}(\hat{H}_1^\bullet (p^2)).
\]

The quotient between the first inclusion has rank \( j_2 - j_1 \) by condition (b'), and the quotient between the second inclusion has rank \( s_\alpha + r_\epsilon - r_\alpha - j_{\alpha,2} \) by condition (a'). Once this \( \hat{H}_2^\bullet \) is fixed, \( \hat{H}_1^\bullet \) is determined by the inclusions:

\[
\hat{H}_1^\bullet \cap \omega^{\circ\circ}_{A_\alpha/R,1} \subseteq \hat{H}_1^\bullet \subseteq F^{-1}(\omega^{\circ\circ}_{A_\alpha/R,2}) \cap V^{-1}(\hat{H}_2^\bullet).
\]

Note that the first term has been determined to be equal to \( (6.11.4) \). The quotient for the first inclusion has rank \( j_1 - r_\epsilon \) and the quotient for the second inclusion has rank \( j_2 - j_1 \) by condition (c').

Summing up this discussion, we see that \( \zeta^\bullet \) is relatively smooth of relative dimension:

\[
(j_{\alpha,2} - j_{\alpha,1})(s_\alpha + r_\epsilon - r_\alpha - j_{\alpha,2}) - (j_{\alpha,1} - r_\epsilon)(j_{\alpha,2} - j_{\alpha,1}) = r_\alpha - s_\alpha - 1 - r_\alpha s_\alpha.
\]

This concludes the proof of the Proposition. \( \Box \)

**Remark 6.12.** Fixing a generic closed point of \( \text{Sh}_{r,s} \), one can see that the fiber of \( \zeta_\alpha \) is, up to a Frobenius factor, very close to certain Deligne-Lusztig variety. We expect that one can give \( Y_\alpha \) a stratification such that each stratum is smooth over the its image under \( \zeta_\alpha \), after modified by probably different Frobenius factors.

**Conjecture 6.13.** The varieties \( Y_j \) together with the natural morphisms to \( \text{Sh}_{r-s+\delta} \) and \( \text{Sh}_{r,s} \) satisfy the condition (3) of Conjecture 7.9. Moreover, the union of the images of \( Y_j \) in \( \text{Sh}_{r,s} \) is the closure of the locus where the Newton polygon of the universal abelian variety has slopes 0 and 1 each with multiplicity \( 2(r-\delta)n \), and slope \( \frac{1}{2} \) with multiplicity \( 2(n-2r+2\delta)n \).

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This Conjecture in the case of $r = \delta = 1$ was proved in Theorem 3.20.

### A. A calculation of Galois cohomology

In this appendix, we give the proof of Lemma 1.8. We keep the notation from Section 1.

**Proof of Lemma 1.8.** The proof is similar to [HT01, Lemma 1.7.1]. We will look for an $\alpha \in D^\times$ such that $\beta_\bullet = \alpha \beta_\bullet$ satisfies the conditions of the Lemma. Let $G_{a\bullet}^{\text{ad}}$ denote the adjoint group of $G_{a\bullet}$. Note that $G_{a\bullet}^{\text{ad}}$ is the Weil restriction to $\mathbb{Q}$ of an algebraic group $PG_{a\bullet}$ over $F$. The condition that $\beta_\bullet \in (D^\times)^{\ast = -1}$ is equivalent to the condition that $\beta_{a\bullet} \alpha^l \beta_{a\bullet}^{-1} = \alpha$. Thus $\alpha$ defines a class in $H^1(E/F, PG_{a\bullet})$. Conversely, every class in $H^1(E/F, PG_{a\bullet})$ arises in this way (See loc. cit. for details). Moreover, the group $G_{b\bullet}$ is the inner form of $G_{a\bullet}$ classified by

$$[\alpha] \in H^1(F, PG_{a\bullet}) \cong H^1(Q, G_{a\bullet}^{\text{ad}}).$$

Clozel shows [Cl91, Lemma 2.1] that if $n$ is odd then the natural restriction map

$$H^1(F, PG_{a\bullet}) \to \bigoplus_x H^1(F_x, PG_{a\bullet})$$

is surjective, where $x$ runs through all places of $F$. If $n$ is even he shows that there is an exact sequence

$$H^1(F, PG_{a\bullet}) \to \bigoplus_x H^1(F_x, PG_{a\bullet}) \xrightarrow{\phi = \sum_x \phi_x} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Here, $H^1(F_x, PG_{a\bullet})$ is the pointed set classifying the inner forms of $G_{a\bullet,F_x}$ over $F_x$, with the distinguished class given by $G_{a\bullet,F_x}^1$ itself. Moreover, if $x = \tau_i$ is an infinite place of $F$, then $H^1(F_x, PG_{a\bullet})$ is given by the classes of unitary groups $U(r_i, n - r_i)$ over $F_x \cong \mathbb{R}$ for some integer $0 \leq r_i \leq n$, and the map $\phi_x : H^1(F_x, PG_{a\bullet}) \to \mathbb{Z}/2\mathbb{Z}$ sends the class of $U(r_i, n - r_i)$ to $r_i - a_i \mod 2$. Therefore, regardless of the parity of $n$, there exists a class $\eta \in H^1(F, PG_F)$ whose image in $H^1(F_x, PG_{a\bullet})$ is

- the distinguished class if $x$ is a finite place,
- and equal to $b_i - a_i \mod 2$ if $x = \tau_i$ is an infinite place.

Such a class $\eta$ exists by the condition $\sum_{i=1}^f b_i \equiv \sum_{i=1}^f a_i \mod 2$. Then $\eta$ maps to zero in $H^1(E, PG_{a\bullet})$, because the natural map $H^1(E, PG_{a\bullet}) \to \bigoplus_x H^1(E_x, PG_{a\bullet})$ is injective. Thus $\eta$ is the image of some class $[\alpha] \in H^1(E, PG_{a\bullet})$ for some $\alpha \in D^\times$ with $\beta_{a\bullet} \alpha \beta_{a\bullet}^{-1} = \alpha$. This $\alpha$ satisfies the desired property by construction. \[\Box\]

### B. An explicit formula in the local spherical Hecke algebra for $\text{GL}_n$

In this appendix, let $F$ be a local field with ring of integers $\mathcal{O}$, $\varpi \in \mathcal{O}$ be a uniformizer, $\mathbb{F} = \mathcal{O} / \varpi \mathcal{O}$ and $q = \# \mathbb{F}$. Fix an integer $n \geq 1$. We consider the spherical Hecke algebra $\mathcal{H}_K = \mathbb{Z}[K \backslash \text{GL}_n(F)/K]$ with $K = \text{GL}_n(\mathcal{O})$. Here, the product of two double cosets $u = K x K$ and $v = K y K$ in $\mathcal{H}_K$ is defined as

$$(B.0.1) \quad u \cdot v = \sum_w m(u, v; w) \omega^{13},$$

We may also view elements of $\mathcal{H}_K$ as $\mathbb{Z}$-valued locally constant and compactly supported functions on $\text{GL}_n(F)$ which are bi-invariant under $K$, and define the product of $f, g \in \mathcal{H}_K$ as $(f \ast g)(x) = \int_{\text{GL}_n(F)} f(y) g(y^{-1}x) dy$, where $dy$ means the unique bi-invariant Haar measure on $\text{GL}_n(F)$ with $\int_K dy = 1$. For the equivalence between these two definitions, see [Gr98, p.4].
where the sum runs through all the double cosets \( w = KzK \) contained in \( KxKyK \), and
the coefficient \( m(u, v; w) \in \mathbb{Z} \) is determined as follows: If \( KxK = \bigsqcup_{i \in I} x_i K \) and \( KyK = \bigsqcup_{j \in J} y_j K \), then
\[
(B.0.2) \quad m(u, v; w) = \# \{ (i, j) \in I \times J \mid x_i y_j K = zK \text{ for a fixed element } z \text{ in } w \}.
\]
By the theory of elementary divisors, all double cosets \( KxK \) of the form
\[
T(a_1, \ldots, a_n) := K \text{ diag}(\varpi^{a_1}, \ldots, \varpi^{a_n}) K, \quad \text{for } a_i \in \mathbb{Z} \text{ with } a_1 \geq a_2 \geq \cdots \geq a_n.
\]
form a \( \mathbb{Z} \)-basis of \( \mathcal{H}_K \). We put
\[
R^{(r, s)} = T(1, 2, \ldots, r, 1, 2, \ldots, s, 1, 2, \ldots, n) \quad \text{for } 0 \leq r \leq s \leq n.
\]
We have thus \( R^{(0, s)} = T(s) \), and \( T(0) = K \).
Because of the lack of reference, we include a proof of the following elementary

**Proposition B.1.** For \( 1 \leq r \leq n \), let
\[
(B.1.1) \quad \binom{n}{r}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-r+1} - 1)}{(q - 1)(q^2 - 1) \cdots (q^r - 1)}
\]
be the Gaussian binomial coefficients, and put \( \binom{n}{0}_q = 1 \). Then for \( 0 \leq r \leq s \leq n \), we have
\[
T^{(r)} T^{(s)} = \sum_{i=0}^{\min\{r, n-s\}} \binom{s - r + 2i}{i}_q R^{(r-i, s+i)}.
\]

**Proof.** We fix a set of representatives \( \mathbb{F} \subseteq \mathcal{O} \) containing 0 of \( \mathbb{F} = \mathcal{O}/\varpi \mathcal{O} \). Then we have \( T^{(r)} = \bigsqcup_{x \in S(n, r)} xK \), where \( S(n, r) \) is the set of \( n \times n \) matrices \( x = (x_{i,j})_{1 \leq i,j \leq n} \) such that
- \( r \) of the diagonal entries equal to \( \varpi \) and the remaining \( n - r \) ones equal to 1;
- if \( i \neq j \), then \( x_{i,j} = 0 \) unless \( i > j \), \( x_{i,i} = 1 \) and \( x_{j,j} = \varpi \), in which case \( x_{i,j} \) can take any values in \( \mathbb{F} \).

For instance, the set \( S(3, 2) \) consists of matrices:
\[
\begin{pmatrix}
1 & 0 & 0 \\
\varpi & 0 & 0 \\
x_{2,1} & \varpi & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
\varpi & 0 & 0 \\
0 & 1 & 0 \\
x_{3,1} & 0 & \varpi
\end{pmatrix}
, \quad
\begin{pmatrix}
\varpi & 0 & 0 \\
0 & \varpi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
with \( x_{2,1}, x_{3,1}, x_{3,2} \in \mathbb{F} \). We have a similar decomposition \( T^{(s)} = \bigsqcup_{y \in S(n, s)} yK \). We write \( T^{(r)} T^{(s)} \) as a linear combination of \( T(a_1, \ldots, a_n) \) with \( a_i \in \mathbb{Z} \) and \( a_1 \geq \cdots \geq a_n \). By looking at the diagonal entries of \( xy \), we see easily that only \( R^{(r-i, s+i)} \) with \( 0 \leq i \leq \min\{r, n - s\} \) have non-zero coefficients, which we denote by \( C^{(r,s)}(n, i) \). To compute \( C^{(r,s)}(n, i) \), we have to count the pairs \((x, y) \in S(n, r) \times S(n, s)\) such that
\[
x y K = \text{ diag}(\varpi^2, \ldots, \varpi^2, \varpi, \ldots, \varpi, 1, \ldots, 1) K.
\]
Then \( x \) and \( y \) must be of the form
\[
x = \begin{pmatrix}
\varpi I_{r-i} & 0 & 0 \\
0 & A & 0 \\
0 & 0 & I_{n-s-i}
\end{pmatrix}, \quad
y = \begin{pmatrix}
\varpi I_{r-i} & 0 & 0 \\
0 & B & 0 \\
0 & 0 & I_{n-s-i}
\end{pmatrix},
\]
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where $I_k$ denotes the $k \times k$ identity matrix, and $A \in S(s - r + 2i, i)$, $B \in S(s - r + 2i, s - r + i)$ satisfying $AB \cdot GL_{s-r+2i}(O) = \varpi I_{s-r+2i}GL_{s-r+2i}(O)$. By [B.0.1], one obtains thus $C^{(r,s)}(n, i) = C^{(i,s-r+1)}(s-r+2i, i)$. Therefore, one is reduced to proving the following Lemma, which is a special case of our proposition.

Lemma B.2. Under the notation and hypothesis of Proposition above, assume moreover that $n = r + s$. Then the coefficient of $R_{(0,n)}$ in the product $T^{(r)}T^{(s)}$ is $(\binom{n}{r})_q$.

Proof. We make an induction on $n \geq 1$. The case $n = 1$ is trivial. We assume thus $n > 1$, and that the statement is true when $n$ is replaced by $n-1$. The case of $r = 0$ being trivial, we may assume that $r \geq 1$. We say a pair $(x, y) \in S(n, r) \times S(n, n-r)$ is admissible if $xyK = \varpi I_n K$. We have to show that the number of admissible pairs is equal to $(\binom{n}{r})_q$. Let $(x, y)$ be an admissible pair. Denote by $I$ (resp. by $J$) the set integers $1 \leq i \leq n$ such that $x_i, i = \varpi$ (resp. $y_i, i = \varpi$). Note that $(x, y)$ being admissible implies that $J = \{1, \ldots, n\} \setminus I$.

Assume first that $x_{1,1} = 1$. Then $x$ and $y$ must be of the form $x = \begin{pmatrix} 1 & 0 \\ * & A \end{pmatrix}$ and $y = \begin{pmatrix} \varpi & 0 \\ 0 & B \end{pmatrix}$ where $(A, B) \in S(n-1, r) \times S(n-1, n-1-r)$ admissible. Note that $xyK = \varpi I_n K$ always hold. We have $x_{i,1} = 0$ for $i \notin I$, and $x_{1,1}$ can take any values in $F$ for $i \in I$. Therefore, the number of admissible pairs $(x, y)$ with $x_{1,1} = 1$ is equal to $q^{\#I} = q^r$ times that of admissible $(A, B)$’s. The latter equals to $(\binom{n-1}{r})_q$ by induction hypothesis.

Consider now the case $x_{1,1} = \varpi$. One can write $x = \begin{pmatrix} \varpi & 0 \\ 0 & A \end{pmatrix}$, and $y = \begin{pmatrix} 1 & 0 \\ * & B \end{pmatrix}$ with $(A, B) \in S(n-1, r-1) \times S(n-1, n-r)$ admissible. Put $z = xy$. Then an easy computation shows that $z_{j,1} = y_{j,1}$ if $j \in J$, and $z_{j,1} = 0$ if $j \notin J$. Hence, $xyK = \varpi I_n K$ forces that $y_{j,1} = 0$ for all $j > 1$. Therefore, the number of admissible $(x, y)$ in this case equals to that of admissible $(A, B)$’s, which is $(\binom{n-1}{r-1})_q$ by induction hypothesis. The Lemma now follows immediately from the equality

$$(\binom{n}{r})_q = q^r \binom{n-1}{r}_q + \binom{n-1}{r-1}_q.$$

\[\square\]

C. A Determinant Formula

In this appendix, we prove the following elementary

Theorem C.1. Let $\alpha_1, \ldots, \alpha_n$ be $n$ indeterminates. For $i = 1, \ldots, n$, let $s_i$ denote the $i$-th elementary symmetric polynomial in $\alpha$’s, and $s_0 = 1$ by convention. Let $q$ be another indeterminate. We put $q_r = q^{r-1} + q^{r-3} + \cdots + q^{1-r}$. Consider the matrix $M_n(q) = (m_{i,j})$ given as follows:

$$m_{i,j} = \begin{cases} \sum_{\delta=0}^{\min\{i-1,n-j\}} q_{n+i-j-2\delta} s_{j-i+\delta} s_{n-\delta} & \text{if } i \leq j; \\ \sum_{\delta=0}^{\min\{j-1,n-i\}} q_{n+j-i-2\delta} s_{j-i+\delta} s_{n+j-i+\delta} & \text{if } i > j. \end{cases}$$

Then we have

$$\det(M_n(q)) = \prod_{i \neq j} (q \alpha_i - \frac{1}{q} \alpha_j).$$
Proof. Let \( N_n(q) \) be the resultant matrix of the polynomials \( f(x) = \prod_{i=1}^{n}(x + q^{-1}\alpha_i) \) and \( g(x) = \prod_{i=1}^{n}(x + q\alpha_i) \), that is \( N_n(q) \) is the \( 2n \times 2n \) matrix given by

\[
N_n(q) = \begin{pmatrix}
    s_0 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{1-n}s_{n-1} & q^{-n}s_n & 0 & \cdots & 0 \\
    0 & s_0 & q^{-1}s_1 & \cdots & q^{2-n}s_{n-2} & q^{1-n}s_{n-1} & q^{-n}s_n & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & s_0 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{-n}s_n \\
    s_0 & q_1s_1 & q_2s_2 & \cdots & q^{n-1}s_{n-1} & q^n s_n & 0 & \cdots & 0 \\
    0 & s_0 & q_1s_1 & \cdots & q^{n-2}s_{n-2} & q^{n-1}s_{n-1} & q^n s_n & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & s_0 & q_1s_1 & q_2s_2 & \cdots & q^n s_n \\
\end{pmatrix}.
\]

It is well known that \( \det(N_n(q)) = \prod_{i,j}((-q^{-1}\alpha_i + q\alpha_j)) \). Thus it suffices to show that \( \det(N_n(q)) = (q - q^{-1})^n \det(M_n(q)) \).

We first make the following row operations on \( N_n(q) \): subtract row \( i \) from row \( n + i \) for all \( i = 1, \ldots, n \). We obtain a matrix whose first column are all 0 expect the first entry being 1; moreover, one can take out a factor \((q - q^{-1})\) from row \( n + 1, \ldots, 2n \). Let \( N'_n(q) \) be the right lower \((2n-1) \times (2n-1)\) submatrix of the remaining matrix. Then we have

\[
N'_n(q) = \begin{pmatrix}
    s_0 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{1-n}s_{n-1} & q^{-n}s_n & 0 & \cdots & 0 \\
    0 & s_0 & q^{-1}s_1 & \cdots & q^{2-n}s_{n-2} & q^{1-n}s_{n-1} & q^{-n}s_n & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & s_0 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{-n}s_n \\
    q_1s_1 & q_2s_2 & q_3s_3 & \cdots & q^{n-1}s_{n-1} & q^n s_n & 0 & \cdots & 0 \\
    0 & q_1s_1 & q_2s_2 & \cdots & q^{n-2}s_{n-2} & q^{n-1}s_{n-1} & q^n s_n & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 0 & q_1s_1 & q_2s_2 & \cdots & q^n s_n \\
\end{pmatrix}
\]

with \( \det(N_n(q)) = (q - q^{-1})^n \det(N'_n(q)) \). Thus we are reduced to proving that \( \det(N'_n(q)) = \det(M_n(q)) \). Consider the \((2n-1) \times (2n-1)\) matrix \( R = \begin{pmatrix} I_{n-1} & 0 \\ C & D \end{pmatrix} \) with the lower \( n \times (2n-1) \) sub matrix given by

\[
(C \ D) = \begin{pmatrix}
    -q_1s_1 & -q_2s_2 & \cdots & -q_{n-1}s_{n-1} & 1 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{2-n}s_{n-2} & q^{1-n}s_{n-1} \\
    0 & -q_1s_1 & \cdots & -q_{n-2}s_{n-2} & 0 & 1 & q^{-1}s_1 & \cdots & q^{3-n}s_{n-3} & q^{2-n}s_{n-2} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & -q_1s_1 & 0 & 0 & 0 & \cdots & 1 & q^{-1}s_1 \\
    0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}.
\]

By a careful computation, one verifies without difficulty that \( RN'_n(q) = \begin{pmatrix} U & * \\ 0 & M_n(q) \end{pmatrix} \), where \( U \) is an \((n-1) \times (n-1)\)-upper triangular matrix with all diagonal entries equal to 1. Note that \( \det(R) = \det(D) = \det(U) = 1 \), it follows immediately that \( \det(N'_n(q)) = \det(M_n(q)) \). \( \square \)

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