VOLTERRA TYPE OPERATORS ON WEIGHTED DIRICHLET SPACES

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Abstract. The Carleson measures for weighted Dirichlet spaces had been characterized by Girela and Peláez, who also characterized the boundedness of Volterra type operators between weighted Dirichlet spaces. However, their characterizations for the boundedness are not complete. In this paper, we completely characterize the boundedness and compactness of Volterra type operators from the weighted Dirichlet spaces \( D^p_\alpha \) to \( D^q_\beta \) \((-1 < \alpha, \beta \) and \( 0 < p < q < \infty \)), which essentially complete their works. Furthermore, we investigate the order boundedness of Volterra type operators between weighted Dirichlet spaces.

1. Introduction

Let \( \mathbb{D} \) be the unit disk of a complex plane and \( H(\mathbb{D}) \) the space consisting of all the analytic functions on \( \mathbb{D} \). For \( 0 < p < \infty \), \(-1 < \alpha \), the weighted Bergman space \( A^p_\alpha \) on the unit disk \( \mathbb{D} \) consists of all the functions \( f \in H(\mathbb{D}) \) such that

\[
\|f\|_{A^p_\alpha} = \left( \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)\alpha dA(z) \right)^{1/p} < \infty,
\]

where \( dA(z) = \frac{1}{\pi} dx dy \) is the normalized Lebesgue area measure (see [8, 12, 29] for references). Then, the weighted Dirichlet space \( D^p_\alpha \) on \( \mathbb{D} \) consists of all the functions \( f \in H(\mathbb{D}) \) satisfying

\[
\|f\|_{D^p_\alpha} = \left( |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha dA(z) \right)^{1/p} < \infty.
\]

For any function \( g \in H(\mathbb{D}) \), the Volterra type operator \( T_g \) and its companion operator \( S_g \) are defined, respectively, by

\[
(T_g f)(z) = \int_0^z f(\omega)g'(\omega) d\omega \quad \text{and} \quad (S_g f)(z) = \int_0^z f'(\omega)g(\omega) d\omega
\]

for any \( f \in H(\mathbb{D}) \).

Let \( |I| \) denote the normalized Lebesgue length of \( I \), which is an interval of \( \partial \mathbb{D} \); then the Carleson square \( S(I) \) is defined as

\[
S(I) := \{ re^{i\theta} : e^{i\theta} \in I, 1 - |I| \leq r < 1 \}.
\]

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For any $s > 0$ and $\mu$ a positive Borel measure in $\mathbb{D}$, $\mu$ is an $s$-Carleson measure if there is a positive constant $C$ such that
\[
\mu(S(I)) \leq C|I|^s \quad \text{for all interval } I \subset \partial \mathbb{D}.
\]

For a space $X$ of analytic functions on $\mathbb{D}$, it is often useful to know the integrability properties of the functions $f \in X$. That is to determine for which positive Borel measure $\mu$ on $\mathbb{D}$ there is a continuous inclusion $X \subset L^p(\mu)$, or equivalently, by the closed graph theorem, there exists a positive constant $C$ such that for any $f \in X$,
\[
\|f\|_{L^q(\mu)} \leq C\|f\|_X.
\]

Duren [2] proved that the Hardy space $H^p \subset L^q(\mu)$, $0 < p \leq q < \infty$, if and only if $\mu$ is a $q/p$-Carleson measure, which extends the result obtained by Carleson [4] where the case $p = q$ was proven. For the weighted Bergman spaces, Luecking [13] proved that, for $0 < p \leq q < \infty$ and $-1 < \alpha$, $A^p_\alpha \subset L^q(\mu)$ if and only if $\mu$ is a $\frac{q(\alpha+2)}{p}$-Carleson measure.

For $0 < p < q < \infty$ and $-1 < \alpha$, Girela and Paláez [11] gave the characterizations of the measures $\mu$ for which $D^p_\alpha \subset L^q(\mu)$. In deed, the following theorem was proven:

**Theorem 1.** Suppose that $0 < p < q < \infty$, $-1 < \alpha$ and $\mu$ is a positive Borel measure in $\mathbb{D}$, then

1. If $p < \alpha + 2$, then $D^p_\alpha \subset L^q(\mu)$ if and only if $\mu$ is a $\frac{q(\alpha+2)-p}{p}$-Carleson measure;
2. If $p = \alpha + 2$, then $D^p_\alpha \subset L^q(\mu)$ if and only if there exists a positive constant $C$ such that for all interval $I \subset \partial \mathbb{D}$, it holds that $\mu(S(I)) \leq C \left(\log \frac{1}{|I|}\right)^{(1/p-1)q}$;
3. If $p > \alpha + 2$, then $D^p_\alpha \subset L^q(\mu)$ if and only if $\mu$ is a finite measure.

For the case of $p \geq q$, the corresponding characterizations were partly investigated in [10, 22, 26], where several questions were still open.

In section 2, we completely characterize the boundedness of Volterra type operators $T_g$ and $S_g$ from the weighted Dirichlet spaces $D^p_\alpha$ to $D^q_\beta$ ($-1 < \alpha, \beta$ and $0 < p < q < \infty$), which extend the works by Girela and Paláez in [11], where the original characterizations covered only the case $\alpha < p < \alpha + 2$. In section 3, we investigate the compactness of the Volterra type operators $T_g$ and $S_g$ from $D^p_\alpha$ to $D^q_\beta$ ($-1 < \alpha, \beta$ and $0 < p < q < \infty$). Finally, in section 4, we investigate the order boundedness of Volterra type operators between weighted Dirichlet spaces.

## 2. Boundedness of Volterra Type Operators

The Volterra type operator $T_g$ was introduced by Pommerenke [23] to study the exponentials of BMOA functions and in the meantime, he proved that $T_g$ acting on the Hardy-Hilbert space $H^2$ is bounded if and only if $g \in BMOA$. After his work, Aleman, Siskakis and Cima [1, 24] studied the boundedness and compactness of $T_g$ on the Hardy space $H^p$, where they showed that $T_g$ is bounded (compact) on $H^p$, $0 < p < \infty$, if and only if $g \in BMOA$ ($g \in VMOA$). For the related works, see [10]. Furthermore, Aleman and Siskakis [3] studied the boundedness and compactness of $T_g$ on the Bergman spaces while Galanopoulos et al. [10, 11] investigated the boundedness of $T_g$ and $S_g$ on the Dirichlet type spaces, and Xiao [27] studied $T_g$ and $S_g$ on $Q_p$ spaces.
Recently, Lin et al. [17, 18] characterized the boundedness of $T_g$ and $S_g$ acting on the derivative Hardy spaces $S^p(1 \leq p < \infty)$ and weighted Banach spaces with general weights. Mengestie [20] obtained a complete description of the boundedness and compactness of the invariant subspaces of the classical Volterra operator $T$ and the Littlewood-Paley formula, Constantin and Peláez [5] obtained the boundedness and compactness of the product of the Volterra type operators and composition operators on the weighted Fock spaces, and recently, he [21] studied the topological structure of the space of Volterra-type integral operators on the Fock spaces endowed with the operator norm. Furthermore, by applying the Carleson embedding theorem and the Littlewood-Paley formula, Constantin and Peláez [5] obtained the boundedness and compactness of $T_g$ on the weighted Fock spaces and investigated the invariant subspaces of the classical Volterra operator $T_z$ on such spaces.

The multiplication operator $M_g$ is defined by

$$(M_g f)(z) := g(z)f(z), \text{ for } f \in H(\mathbb{D}), z \in \mathbb{D}.$$  

The following relation holds:

$$(M_g f)(z) = f(0)g(0) + (T_g f)(z) + (S_g f)(z).$$

Then we characterize the boundedness of these operators.

**Theorem 2.** Let $-1 < \alpha, \beta, g \in H(\mathbb{D})$ and $0 < p < q < \infty$. Define $d \mu_{g,q,\beta}(z) := (1 - |z|^2)^\beta |g(z)|^q dA(z)$. Then the following statements hold:

1. If $p < \alpha + 2$, then $T_g : D^p_\alpha \to D^q_\beta$ is bounded if and only if $\mu_{g,q,\beta}(z)$ is a $g(\alpha+2-p)$-Carleson measure;

2. If $p = \alpha + 2$, then $T_g : D^p_\alpha \to D^q_\beta$ is bounded if and only if there exists a positive constant $C$ such that for all interval $I \subset \partial \mathbb{D}$, it holds that $\mu_{g,q,\beta}(S(I)) \leq C \left( \log \frac{1}{|I|} \right)^{(1/p-1)q}$;

3. If $p > \alpha + 2$, then $T_g : D^p_\alpha \to D^q_\beta$ is bounded if and only if $\mu_{g,q,\beta}$ is a finite measure, or equivalently, $g \in D^q_\beta$.

**Proof.** This follows directly from Theorem 1 and the closed graph theorem. □

**Theorem 3.** Let $-1 < \alpha, \beta, g \in H(\mathbb{D})$ and $0 < p < q < \infty$. Then $S_g : D^p_\alpha \to D^q_\beta$ is bounded if and only if $|g(z)| = O \left( (1 - |z|^2)^{\frac{2\alpha}{p} - \frac{2\beta}{q}} \right)$, as $|z| \to 1^-$.

**Proof.** First, suppose that $|g(z)| = O \left( (1 - |z|^2)^{\frac{2\alpha}{p} - \frac{2\beta}{q}} \right)$. Then for any $f \in D^p_\alpha$, it holds that

$$
\|S_g f\|_{D^q_\beta} = \left( \int_{\mathbb{D}} |f'(z)g(z)|^q (1 - |z|^2)^\beta dA(z) \right)^{1/q} \\
\leq C \left( \int_{\mathbb{D}} |f'(z)|^p |f'(z)|^{q-p} (1 - |z|^2)^{\frac{q(2\alpha+1)-2\beta}{p}} dA(z) \right)^{1/q} \\
\leq C ||f||_{D^p_\alpha}^{(q-p)/q} \left( \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha dA(z) \right)^{1/q} \\
\leq C ||f||_{D^p_\alpha}.
$$

Hence, $S_g : D^p_\alpha \to D^q_\beta$ is bounded.

Conversely, suppose that $S_g : D^p_\alpha \to D^q_\beta$ is bounded. Given $a \in \mathbb{D}$, define the function $f_a$ by

$$f_a(z) := \frac{(1 - |a|^2)^{(\alpha+2)/p}}{(1 - az)^{2(\alpha+2)/p - 1}}.$$
It is easy to prove that \( f_a \in D^p_\alpha \) and there exists a positive constant \( C \) such that for all \( a \in \mathbb{D}, \| f_a \|_{D^p_\alpha} \leq C \). Denoting \( \Delta(a, r) \) as the pseudo-hyperbolic disk with center \( a \) and radius \( r \), we have

\[
\left(1 - |a|^2\right)^{2+\beta-\frac{2\alpha+\beta}{p}}|g(a)|^q \leq C\left(1 - |a|^2\right)^{\beta-\frac{2\alpha+\beta}{p}} \int_{\Delta(a, r)} |g(\omega)|^q dA(\omega)
\]

\[
\leq C|a|^{-q} \int_{\Delta(a, r)} |(S_g f_a)'(\omega)|^q(1 - |\omega|^2)^\beta dA(\omega)
\]

\[
\leq C|a|^{-q} \|S_g f_a\|_{D^p_\alpha}^q
\]

\[
\leq C|a|^{-q}.
\]

Thus, \( |g(a)| = O\left(\left(1 - |a|^2\right)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}}\right) \), as \( |a| \to 1^- \).

As an immediate corollary, we obtain the known results originally proven by Zhao [28].

**Corollary 1.** Let \(-1 < \alpha, \beta, g \in H(\mathbb{D})\) and \(0 < p < q < \infty\). Then \( M_g : A^p_\alpha \to A^q_\beta \) is bounded if and only if \(|g(z)| = O\left(\left(1 - |z|^2\right)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}}\right)\), as \(|z| \to 1^-\).

**Proof.** This follows immediately from the fact that \( Dg = MgD \), where \( D \) is the differential operator.

**Theorem 4.** Let \(-1 < \alpha, \beta, g \in H(\mathbb{D})\) and \(0 < p < q < \infty\). Define \( d\mu_{g,a,\beta}(z) := (1 - |z|^2)^\beta|g'(z)|^q dA(z) \). Then the following statements hold:

1. If \( p < \alpha + 2 \), then \( M_g : D^p_\alpha \to D^q_\beta \) is bounded if and only if \( \mu_{g,a,\beta}(z) \) is a \( g_{(\alpha+2)-p}\)-Carleson measure and \(|g(z)| = O\left(\left(1 - |z|^2\right)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}}\right)\), as \(|z| \to 1^-\);

2. If \( p = \alpha + 2 \), then \( M_g : D^p_\alpha \to D^q_\beta \) is bounded if and only if \(|g(z)| = O\left(\left(1 - |z|^2\right)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}}\right)\) as \(|z| \to 1^-\) and there exists a positive constant \( C \) such that for all interval \( I \subset \partial\mathbb{D} \), it holds that \( \mu_{g,a,\beta}(S(I)) \leq C \left(\log \frac{1}{|I|}\right)^{(1/p-1)q} \);

3. If \( p > \alpha + 2 \), then \( M_g : D^p_\alpha \to D^q_\beta \) is bounded if and only if \(|g(z)| = O\left(\left(1 - |z|^2\right)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}}\right)\) as \(|z| \to 1^-\) and \( g \in D^q_\beta \).

**Proof.** Since \((Mgf)(z) = f(0)g(0) + (T_g f)(z) + (S_g f)(z)\), the sufficiency follows immediately from Theorem 2 and Theorem 3. It remains to prove the necessity. In this case, it is obvious that if we can prove that \(|g(z)| = O\left(\left(1 - |z|^2\right)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}}\right)\) as \(|z| \to 1^-\), then all the other statements follow immediately from Theorem 2 and Theorem 3 again.

Given \( a \in \mathbb{D} \), define the function \( F_a \) by

\[
F_a(z) := \frac{(1 - |a|^2)^{(\alpha+2)/p}}{(1 - \bar{a}z)^{2(\alpha+2)/p - 1}} - (1 - |a|^2)^{(p-\alpha-2)/p}.
\]

Then \( F_a(a) = 0 \), and the remainder of the proof is essentially similar to the converse part of the proof in Theorem 3. \( \square \)
3. Compactness of Volterra type operators

For any $s > 0$ and $\mu$ a positive Borel measure in $\mathbb{D}$, we say $\mu$ is a vanishing $s$-Carleson measure if

$$\mu(S(I)) = o(|I|^s) \quad \text{as } |I| \to 0.$$ 

**Theorem 5.** Suppose that $0 < p < q < \infty, -1 < \alpha$ and $\mu$ is a positive Borel measure in $\mathbb{D}$, then

1. If $p < \alpha + 2$, then $D_{\alpha}^p \subset L^q(\text{d}\mu)$ is compact if and only if $\mu$ is a vanishing $2(\alpha + 2 - p)$-Carleson measure;

2. If $p = \alpha + 2$, then $D_{\alpha}^p \subset L^q(\text{d}\mu)$ is compact if and only if $\mu(S(I)) = o\left(\left(\log \frac{1}{|I|}\right)^{(1/p-1)q}\right)$ as $|I| \to 0$;

3. If $p > \alpha + 2$, then $D_{\alpha}^p \subset L^q(\text{d}\mu)$ is compact if and only if $\mu$ is a finite measure.

**Proof.** (1) is known (see, for example, [15]).

For (2), we noticed that this condition is, in deed, a vanishing $((1 - 1/p)q, 0)$-logarithmic Carleson measure and the proof of it is basically similar to (ii) of Theorem 3.1 in [22].

Now for (3), since when $p > \alpha + 2$, it holds that $D_{\alpha}^p \subset H^\infty$, where $H^\infty$ is the space of all the bounded analytic functions on $\mathbb{D}$, then the compactness follows easily by the standard arguments. \hfill \Box

Then we characterize the compactness of these operators.

**Theorem 6.** Let $-1 < \alpha, \beta, g \in H(\mathbb{D})$ and $0 < p < q < \infty$. Define $d\mu_{g,q,\beta}(z) := (1 - |z|^2)^{\beta} |g'(z)|^q \text{d}A(z)$. Then the following statements hold:

1. If $p < \alpha + 2$, then $T_g : D_{\alpha}^p \to D_{\beta}^q$ is compact if and only if $\mu_{g,q,\beta}(z)$ is a vanishing $2(\alpha + 2 - p)$-Carleson measure;

2. If $p = \alpha + 2$, then $T_g : D_{\alpha}^p \to D_{\beta}^q$ is compact if and only if $\mu_{g,q,\beta}(S(I)) = o\left(\left(\log \frac{1}{|I|}\right)^{(1/p-1)q}\right)$ as $|I| \to 0$;

3. If $p > \alpha + 2$, then $T_g : D_{\alpha}^p \to D_{\beta}^q$ is compact if and only if $\mu_{g,q,\beta}$ is a finite measure, or equivalently, $g \in D_{\beta}^q$.

**Proof.** This follows directly from Theorem 5. \hfill \Box

**Theorem 7.** Let $-1 < \alpha, \beta, g \in H(\mathbb{D})$ and $0 < p < q < \infty$. Then $S_g : D_{\alpha}^p \to D_{\beta}^q$ is compact if and only if $|g(z)| = o\left(\left(1 - |z|^2\right)^{2\alpha/q - 2\beta/q}\right)$, as $|z| \to 1^-$.

**Proof.** First suppose that $|g(z)| = o\left(\left(1 - |z|^2\right)^{2\alpha/q - 2\beta/q}\right)$. Then, for any $\epsilon > 0$, there exists $r$ with $0 < r < 1$ such that $\frac{|g(z)|}{\left(1 - |z|^2\right)^{2\alpha/q - 2\beta/q}} < \epsilon$, whenever $|z| > r$.

Now, for any bounded sequence $\{f_n\}_{n=0}^\infty \subset D_{\alpha}^p$ such that $f_n$ converges to 0 locally
uniformly, it holds that
\[
\limsup_{n \to \infty} \|S_g f_n\|_{D^q_\beta} = \limsup_{n \to \infty} \left( \int_{\mathbb{D}} |f'_n(z)g(z)|^q(1 - |z|^2)^\beta dA(z) \right)^{1/q} \\
\leq \limsup_{n \to \infty} \left( \int_{\mathbb{D}} |f'_n(z)g(z)|^q(1 - |z|^2)^\beta dA(z) \right)^{1/q} \\
\leq \limsup_{n \to \infty} \frac{C^1}{q} \left( \int_{\mathbb{D}} |g|^q(1 - |z|^2)^\beta dA(z) \right)^{1/q} \\
\leq \limsup_{n \to \infty} \frac{C^1}{q} \|f_n\|_{D^q_\beta}^{(q-\alpha)/q} \left( \int_{\mathbb{D}} |f'_n(z)|^p(1 - |z|^2)^\alpha dA(z) \right)^{1/q} \\
\leq \limsup_{n \to \infty} \frac{C^1}{q} \|f_n\|_{D^q_\beta}^{(q-\alpha)/q} \\
\leq C e^{1/q}.
\]

Since \( \epsilon \) is arbitrary, it follows that \( S_g : D^p_\alpha \to D^q_\beta \) is compact.

Conversely, suppose that \( S_g : D^p_\alpha \to D^q_\beta \) is compact. Choose the functions \( f_a \) defined in the proof of Theorem 1, then the direct computation shows that \( f_a = 0 \). Thus, we have
\[
(1 - |a|^2)^{2+\beta - \frac{(2+\alpha)}{p}}|g(a)|^q \leq C(1 - |a|^2)^{\beta - \frac{(2+\alpha)}{p}} \int_{\Delta(a,r)} |g(\omega)|^q dA(\omega) \\
\leq C |a|^{-q} \int_{\Delta(a,r)} |(S_g f_a)(\omega)|^q(1 - |\omega|^2)^\beta dA(\omega) \\
\leq C |a|^{-q} \|S_g f_a\|_{D^q_\beta}^q \to 0 \quad \text{as} \quad |a| \to 1^-. 
\]

Thus, \( |g(a)| = o \left( (1 - |a|^2)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}} \right) \), as \( |a| \to 1^- \). \[ \square \]

As an immediate corollary, we obtain the known results originally proven by Čučišćević and Zhao [6].

**Corollary 2.** Let \(-1 < \alpha, \beta, g \in H(\mathbb{D})\) and \(0 < p < q < \infty\). Then \( M_g : A^p_\alpha \to A^q_\beta \) is compact if and only if \( |g(z)| = o \left( (1 - |z|^2)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}} \right) \), as \( |z| \to 1^- \).

**Theorem 8.** Let \(-1 < \alpha, \beta, g \in H(\mathbb{D})\) and \(0 < p < q < \infty\). Define \( d\mu_{g,q,\beta}(z) := (1 - |z|^2)^{\beta} |g(z)|^q dA(z) \). Then the following statements hold:

1. If \( p < \alpha + 2 \), then \( M_g : D^p_\alpha \to D^q_\beta \) is compact if and only if \( \mu_{g,q,\beta}(z) \) is a vanishing \( \frac{q(\alpha + 2 - p)}{p} \)-Carleson measure and \( |g(z)| = o \left( (1 - |z|^2)^{\frac{-2+\alpha}{p} - \frac{q+\beta}{q}} \right) \), as \( |z| \to 1^- \);

2. If \( p = \alpha + 2 \), then \( M_g : D^p_\alpha \to D^q_\beta \) is compact if and only if \( |g(z)| = o \left( (1 - |z|^2)^{\frac{-2+\alpha}{p} - \frac{q+\beta}{q}} \right) \) as \( |z| \to 1^- \) and \( \mu_{g,q,\beta}(S(I)) = o \left( \left( \log \frac{1}{|I|} \right)^{(1-p)q} \right) \) as \( |I| \to 0 \);

3. If \( p > \alpha + 2 \), then \( M_g : D^p_\alpha \to D^q_\beta \) is compact if and only if \( |g(z)| = o \left( (1 - |z|^2)^{\frac{-2+\alpha}{p} - \frac{q+\beta}{q}} \right) \) as \( |z| \to 1^- \) and \( g \in D^q_\beta \).
Proof. Since \((Mfg)(z) = f(0)g(0) + (Tgf)(z) + (Sgf)(z)\), the sufficiency follows immediately from Theorem 6 and Theorem 7. It remains to prove the necessary conditions and in this case, it is obvious that if we can prove that \(|g(z)| = a(1 - |z|^2)^{2+\alpha - \frac{2}{p}}|z| \rightarrow 1^−\), then all the other statements follow immediately from Theorem 6 and Theorem 7 again.

Given \(a \in \mathbb{D}\), define the function \(F_a\) by

\[
F_a(z) := \frac{(1 - |a|^2)^{(\alpha+2)/p}}{(1 - az)^{(2\alpha+2)/p-1}} - (1 - |a|^2)^{(p-\alpha-2)/p}. \]

Then \(F_a(a) = 0\), and the remainder of the proof is similar to that of Theorem 7. \(\square\)

4. ORDER BOUNDEDNESS OF VOLterra type operators

Let \(X\) be a Banach space of holomorphic functions defined on \(\mathbb{D}\), \(q > 0\), \((\Omega, A, \mu)\) a measure space and

\[
L^p(\Omega, A, \mu) := \{f \mid f : \Omega \rightarrow \mathbb{C} \text{ is measurable and } \int_{\Omega} |f|^p d\mu < \infty\}. \]

An operator \(T : X \rightarrow L^p(\Omega, A, \mu)\) is said to be order bounded if there exists \(g \in L^p(\Omega, A, \mu)\) such that for all \(f \in X\) with \(\|f\|_X \leq 1\), it holds that

\[
|T(f)(x)| \leq g(x), \quad \text{a.e. } [\mu]. \]

Order boundedness plays an important role in studying the properties of many concrete operators acting between Banach spaces like Hardy spaces, weighted Bergman spaces and so forth (see [13] [14] [23]). Recently, order boundedness of weighted composition operators between weighted Dirichlet spaces were studied in [14] [24]. In this section, we investigate the order boundedness of Volterra type operators between weighted Dirichlet spaces. Recall that in this case, an operator \(T : D^p_\alpha \rightarrow D^q_\beta\) is order bounded if and only if there exists \(g \in L^q(A_\beta)\) such that for all \(f \in D^p_\alpha\) with \(\|f\|_{D^p_\alpha} \leq 1\), it holds that

\[
|T(f)(z)| \leq g(z), \quad \text{a.e. } [A_\beta]. \]

Before proving the results, we first give some auxiliary lemmas.

**Lemma 1.** Let \(\alpha > -1\) and \(0 < p < \infty\). Denote \(\delta_z\) as the point evaluation functional on \(D^p_\alpha\), then

1. for \(p < \alpha + 2\), \(\|\delta_z\| \approx \frac{1}{|1 - |z|^2|^\alpha + 2 - p|/p}\),
2. for \(p = \alpha + 2\), \(\|\delta_z\| \approx \frac{1}{\log(|2 - |z|^2|^{1-p})/p}\),
3. for \(p > \alpha + 2\), \(\|\delta_z\| \approx 1\).

**Proof.** (1) and (2) follows from [14] Lemma 2.2 and Lemma 2.3 while (3) follows directly from the fact that \(D^p_\alpha \subset H^\infty\) for \(p > \alpha + 2\). \(\square\)

**Lemma 2.** Let \(\alpha > -1\) and \(0 < p < \infty\). Denote \(\delta'_z\) as the derivative point evaluation functional on \(D^p_\alpha\), then \(\|\delta'_z\| \approx \frac{1}{|1 - |z|^2|^\alpha + 2 - p|/p}\).

**Proof.** By definition, \(f \in D^p_\alpha\) if and only if \(f' \in A^p_\alpha\), thus the lemma follows from [12] Lemma 3.2. \(\square\)

Now we are ready to prove our results.
Theorem 9. Let $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ and $0 < p, q < \infty$. Then the following statements hold:

(1) If $p < \alpha + 2$, then $T_g : D^p_\alpha \rightarrow D^q_\beta$ is order bounded if and only if
\[ \int_D \frac{|g'(z)|^q}{(1-|z|^2)^{(\alpha+2-p)/p}} \, dA_\beta < \infty; \]

(2) If $p = \alpha + 2$, then $T_g : D^p_\alpha \rightarrow D^q_\beta$ is order bounded if and only if
\[ \int_D \frac{|g'(z)|^q}{\log(2/|z|)}^{q(1-p)/p} \, dA_\beta < \infty; \]

(3) If $p > \alpha + 2$, then $T_g : D^p_\alpha \rightarrow D^q_\beta$ is order bounded if and only if $g \in D^q_\beta$.

Proof. (1) Assume first that $T_g : D^p_\alpha \rightarrow D^q_\beta$ is order bounded. Then there exists $h \in L^q(A_\beta)$ such that for all $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \leq 1$, it holds that
\[ |f(z)g'(z)| \leq h(z), \quad \text{a.e. } [A_\beta]. \]

Hence, by Lemma 1 the inequality
\[ h(z) \geq |g'(z)||\delta_z| \geq \frac{|g'(z)|}{(1-|z|^2)^{(\alpha+2-p)/p}} \quad \text{holds a.e. } [A_\beta]. \]

Therefore, it holds that
\[ \int_D \frac{|g'(z)|^q}{(1-|z|^2)^{(\alpha+2-p)/p}} \, dA_\beta < \infty. \]

Conversely, suppose that
\[ \int_D \frac{|g'(z)|^q}{(1-|z|^2)^{(\alpha+2-p)/p}} \, dA_\beta < \infty. \]

Let
\[ h(z) = \frac{|g'(z)|}{(1-|z|^2)^{(\alpha+2-p)/p}}, \]

then by Lemma 1 for all $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \leq 1$,
\[ |f(z)g'(z)| \leq |g'(z)||\delta_z| \leq h(z), \quad \text{a.e. } [A_\beta]. \]

Accordingly, $T_g : D^p_\alpha \rightarrow D^q_\beta$ is order bounded.

The proof of (2) and (3) is almost similar to that of (1), thus we omit the details. \hfill \Box

By Theorem 2, Theorem 6 and Theorem 9 we obtain the following corollary.

Corollary 3. Let $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ and $\alpha + 2 < p < q < \infty$. Then the following statements are equivalent:

(1) $T_g : D^p_\alpha \rightarrow D^q_\beta$ is bounded;

(2) $T_g : D^p_\alpha \rightarrow D^q_\beta$ is compact;

(3) $T_g : D^p_\alpha \rightarrow D^q_\beta$ is order bounded;

(4) $g \in D^q_\beta$.

Theorem 10. Let $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ and $0 < p, q < \infty$. Then $S_g : D^p_\alpha \rightarrow D^q_\beta$ is order bounded if and only if
\[ \int_D \frac{|g(z)|^q}{(1-|z|^2)^{(\alpha+2-p)/p}} \, dA_\beta < \infty. \]

Proof. The proof is similar to that of Theorem 9 except that in this case, we resort to Lemma 2 instead of Lemma 1. \hfill \Box
Theorem 11. Let \(-1 < \alpha, \beta, g \in H(D)\) and \(0 < p, q < \infty\). Then the following statements hold:

1. If \(p < \alpha + 2\), then \(M_g : D^p_\alpha \to D^q_\beta\) is order bounded if and only if
   \[
   \int_D \frac{|g(z)|^q}{(1 - |z|^2)^{q(\alpha+2)/p}} dA_{\beta} < \infty;
   \]

2. If \(p = \alpha + 2\), then \(M_g : D^p_\alpha \to D^q_\beta\) is order bounded if and only if
   \[
   \int_D \frac{|g(z)|^q}{(1 - |z|^2)^{q(\alpha+2)/p}} dA_{\beta} + \int_D \frac{|g'(z)|^q}{(\log(2/|z|^2))^{q(1-p)/p}} dA_{\beta} < \infty;
   \]

3. If \(p > \alpha + 2\), then \(M_g : D^p_\alpha \to D^q_\beta\) is order bounded if and only if \(g \in D^q_\beta\) and
   \[
   \int_D \frac{|g(z)|^q}{(1 - |z|^2)^{q(\alpha+2)/p}} dA_{\beta} < \infty.
   \]

Proof. (1) Suppose that \(\int_D \frac{|g(z)|^q}{(1 - |z|^2)^{q(\alpha+2)/p}} dA_{\beta} < \infty\). Let \(f \in D^p_\alpha\) with \(\|f\|_{D^p_\alpha} \leq 1\), then by Lemma 1 and Lemma 2 we have
   \[
   |(f(z)g(z))'| \leq |f'(z)g(z)| + |f(z)g'(z)| \lesssim \frac{|g(z)|}{(1 - |z|^2)^{\alpha/p}} + \frac{|g'(z)|}{(1 - |z|^2)^{\alpha+2-p}/p}.
   \]

By taking
   \[
   h(z) = \frac{|g(z)|}{(1 - |z|^2)^{\alpha/2}} + \frac{|g'(z)|}{(1 - |z|^2)^{\alpha+2-p}/p},
   \]

then \(h \in L^q(A_\beta)\) since
   \[
   \int_D \frac{|g'(z)|}{(1 - |z|^2)^{\alpha+2-p}/p} dA_{\beta} \lesssim \int_D \frac{|g(z)|}{(1 - |z|^2)^{\alpha+2+p}/p} dA_{\beta} < \infty.
   \]

Accordingly, \(M_g : D^p_\alpha \to D^q_\beta\) is order bounded.

Conversely, assume that \(M_g : D^p_\alpha \to D^q_\beta\) is order bounded. Then there exists \(h \in L^q(A_\beta)\) such that for all \(f \in D^p_\alpha\) with \(\|f\|_{D^p_\alpha} \leq 1\), it holds that
   \[
   |(fg)'(z)| \leq h(z), \quad \text{a.e. } [A_\beta].
   \]

For any \(z \in D\), we consider the function
   \[
   f_z(\omega) = \frac{1 - |z|^2(\alpha+2)/p}{(1 - \bar{z}\omega)^{2(\alpha+2)/p-1}} - \frac{1 - |z|^2(\alpha+2)/p+1}{(1 - \bar{z}\omega)^{2(\alpha+2)/p+1}}, \quad \omega \in D.
   \]

An easy calculation shows that \(\|f_z\|_{D^p_\alpha} \lesssim 1\) and
   \[
   f'_z(\omega) = \bar{z} \left(\frac{2(\alpha+2) - p (1 - |z|^2)^{\alpha+2/p}}{1 - \bar{z}\omega)^{2(\alpha+2)/p}} - \frac{2(\alpha+2) (1 - |z|^2)^{\alpha+2/p+1}}{(1 - \bar{z}\omega)^{2(\alpha+2)/p+1}}\right), \quad \omega \in D.
   \]

Thus, we have \(f_z(z) = 0\) and \(f'_z(z) = \frac{\bar{z}g(z)}{(1 - |z|^2)^{\alpha+2/p}}\). Therefore,
   \[
   \frac{|\bar{z}g(z)|}{(1 - |z|^2)^{\alpha+2/p}} = |g'(z)f_z(z) + g(z)f'_z(z)| = |(gfg)'(z)| \lesssim h(z), \quad \text{a.e. } [A_\beta].
   \]

Hence, for \(|z| > 1/2\), it holds that
   \[
   \frac{|g(z)|}{(1 - |z|^2)^{\alpha+2/p}} \lesssim h(z), \quad \text{a.e. } [A_\beta].
   \]
For $|z| \leq 1/2$, it follows from the continuity of the function \( \frac{1}{(1 - |z|^2)^{(\alpha+2)/p}} \) that
\[
\frac{1}{(1 - |z|^2)^{(\alpha+2)/p}} \lesssim 1.
\]

Now, by taking the constant function 1 and the monomial \( z \) as the test function in \( D_p^\alpha \), we get that \( |g'(z)| \lesssim h(z) \) a.e. \( [A_\beta] \), and \( |g'(z)z + g(z)| \lesssim h(z) \) a.e. \( [A_\beta] \).

Thus, for $|z| \leq 1/2$, it also holds that
\[
|g(z)| \left(1 - |z|^2\right)^{\alpha+2} / p \lesssim h(z), \quad \text{a.e. } [A_\beta].
\]

In conclusion, for all $z \in \mathbb{D}$,
\[
\frac{|g(z)|}{(1 - |z|^2)^{(\alpha+2)/p}} \lesssim h(z), \quad \text{a.e. } [A_\beta],
\]
which implies that
\[
\int_\mathbb{D} \frac{|g(z)|^q}{(1 - |z|^2)^{(\alpha+2)/p}} dA_\beta < \infty.
\]

The proof of (2) and (3) are similar to that of (1) by some minor modifications. For example, in (2), we take the test function
\[
f_z(\omega) = \log\left(\frac{2 - \bar{z}\omega}{1 - |z|^2}\right) - \frac{\left(\log\left(\frac{2 - \bar{z}\omega}{1 - |z|^2}\right)\right)^2}{\log\left(\frac{2 - \bar{z}\omega}{1 - |z|^2}\right)^{1/p+1}}, \quad \omega \in \mathbb{D}.
\]

Thus the proof is complete. \(\Box\)

By Theorem 9, Theorem 10 and Theorem 11 we obtain the following corollary.

**Corollary 4.** Let $-1 < \alpha, \beta$, \( g \in H(\mathbb{D}) \) and $0 < p < \alpha + 2, 0 < q < \infty$. Then the following statements are equivalent:

1. $T_g : D_p^\alpha \rightarrow D_q^\beta$ is order bounded;
2. $S_g : D_p^\alpha \rightarrow D_q^\beta$ is order bounded;
3. $M_g : D_p^\alpha \rightarrow D_q^\beta$ is order bounded;
4. $\int_\mathbb{D} \frac{|g(z)|^q}{(1 - |z|^2)^{(\alpha+2)/p}} dA_\beta < \infty$, that is, \( g \in A_q^{\beta - 2(\alpha+2)/p} \).

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