Thermodynamics of a Photon Gas in Nonlinear Electrodynamics

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In this paper we analyze the thermodynamic properties of a photon gas under the influence of a background electromagnetic field in the context of any nonlinear electrodynamics. Neglecting the self-interaction of photons, we obtain a general expression for the grand canonical potential. Particularizing for the case when the background field is uniform, we determine the pressure and the energy density for the photon gas. Although the pressure and the energy density change when compared with the standard case, the relationship between them remains unaltered, namely $p = 3\rho$. Finally, we apply the developed formulation to the cases of Heisenberg-Euler and Born-Infeld nonlinear electrodynamics. For the Heisenberg-Euler case, we show that our formalism recover the results obtained with the 2-loop thermal effective action approach.

I. INTRODUCTION

Maxwell electrodynamics is one of the most successful theories in the history of physics. Classically, it is able to describe all the known electric and magnetic phenomena including the creation and propagation of electromagnetic wave. Its quantum version, QED, is the most successful QFT ever built and tested producing accurate results up to ten parts in a billion \cite{1}. Nevertheless, alternatives and extensions of Maxwell electrodynamics have been proposed since its creation in the second half of the nineteenth century. The motivations for proposing these extensions are quite diverse and include the problem of divergence for the classical Coulomb potential \cite{2,3}, experimental constrains for the photon mass \cite{5–8}, the classic study of vacuum polarization effects \cite{9,10}, modifications on electrodynamic in the context of branes \cite{12}, etc.

From the point of view of a gauge theory \cite{13}, Maxwell electrodynamics arises imposing four conditions to its Lagrangian: \(i\) the Lagrangian $L$ must be Lorentz invariant; \(ii\) $L$ should be gauge invariant for $U(1)$ symmetry group; \(iii\) $L$ must depend only on $A_\mu$, and its first derivative; \(iv\) the Lagrangian should contain only quadratic forms of $A_\mu$ and its first derivative. There are many examples of extensions of Maxwell electrodynamics which break, at least, one of the above four conditions. Violation of the first condition is usually done introducing tensor quantities that create preferred directions and boost-dependent effects \cite{14–17}. On the other hand, if it is introduced an usual term of mass for the photon - Proca electrodynamics \cite{18} - the second condition is necessarily violated. Moreover, if only the third condition is broken a Maxwell extension emerges known as Podolsky electrodynamics \cite{19–21}. Finally, if only the fourth condition is violated then a class of electrodynamics called nonlinear electrodynamics (NLED) arises \cite{22}. An example of NLED is Born-Infeld theory \cite{3,4}. This paper has NLED as background.

Models involving NLED appear in different branches of physics, but the most important situation in which they arise is in the context of vacuum polarization. Since the early thirties, we know that virtual electron loops induce a self-coupling of the electromagnetic field. In the energy scales below the electron mass and for a constant electromagnetic field this self-interaction could be represented by an effective field theory known as Heisenberg-Euler NLED. The one and two loops Heisenberg-Euler effective actions were calculated in \cite{9,23,24} and \cite{25,26} respectively. Besides vacuum polarization in QED, models of NLED emerge in other contexts such as string theories \cite{29,32} and in the description of radiation propagation inside specific materials \cite{33–36}.

Recently, in the context of deformed special relativity (DSR), some papers have studied thermodynamics consequences of modifications in the relativistic dispersion relation \cite{37–40}. These works are usually motivated by the possibility of Lorentz symmetry breakdown at Planck scales \cite{41,42}. Modified dispersion relations appear not only in DSR models but also in NLED scenarios. This interesting effect occurs in NLED when we study wave propagation in an electromagnetic background. Considering a background electromagnetic field it is possible to show that the interaction between a electromagnetic wave and this field produces a modified dispersion relation \cite{43–45}.

In the present work, we explore the thermodynamics properties of a photon gas under the influence of a background electromagnetic field in NLED scenarios. The paper is organized as follows. In section 2, we present the modified dispersion relations and perform the thermodynamic analysis for a photon gas in an arbitrary NLED background. In section 3, we apply the developed formu-
lation on Heisenberg-Euler NLED and compare the results with the effective field theory at finite temperature. Section 4 presents a comparison among the thermodynamic properties of the photon gas in the background of Maxwell and Born-Infeld electrodynamics. The final remarks and further perspectives are made in section 5. Finally, a discussion on the birefringence phenomenon is presented in Appendix A.

II. PHOTON GAS THERMODYNAMICS

A general NLED is described in terms of the Lagrangian $L(F, G)$ where $F = -\frac{1}{2} f^{\mu\nu} f_{\mu\nu} = \frac{1}{2} (E^2 - B^2)$ and $G = -\frac{1}{2} f^{\mu\nu} f_{\mu\nu} = \hat{E} \cdot \hat{B}$ are the contractions between the electromagnetic field tensor and its dual. The first step to obtain the modified dispersion relation is to introduce an energy scale $M$ and require that all other energy scales of the system (with exception of the field strength) are small when compared with $M$. This condition ensures that the electromagnetic field is slowly varying and restricts the radiation to be low frequency when compared with $M$. The next step is split the field strength $f^{\mu\nu}$ into a background field $F^{\mu\nu}$ and a plane wave $\phi^{\mu\nu}$. Finally, considering a linear approximation in $\phi^{\mu\nu}$ we obtain

\[
[\eta^{\mu\nu} + z_{\pm} (F, G) F^{\mu\alpha} F_{\alpha\nu}] k_{\mu} k_{\nu} = 0, \tag{1}
\]

where

\[
z_{\pm} (F, G) = \frac{-2F\sigma + L_F (L_{GG} + L_{FF}) \pm \sqrt{\delta}}{2[L_F^2 + 2L_F (L_{FG}G - L_{GG}F) - G^2\sigma]} \tag{2}
\]

with $L_F = \frac{\partial L}{\partial F}$, $L_G = \frac{\partial L}{\partial G}$,

\[
d = [L_F (L_{FF} - GGG) - 2F\sigma]^2 + 4[L_F L_{FG} - G\sigma]^2,
\]

and

\[
\sigma = L_{FF}L_{GG} - L_{FG}^2.
\]

This result was first obtained by the authors in\[44\].

The function $z_{\pm} (F, G)$ contains all the information of the NLED and the $\pm$ sign indicates the existence of two possible solutions. Besides, it is interesting to note that the linear approximation does not imply that the $F^{\mu\nu}$ is much stronger than the $\phi^{\mu\nu}$. Thus, the zero-field limit for the background field is well defined.

Using Minkowski’s signature, $\eta^{\mu\nu} = diag(+,-,-,-)$, and the 4-wave vector $k^\mu = (\omega, \mathbf{k})$,\[11\] can be rewritten to obtain the energy $\omega$ of a photon in terms of $z_{\pm} (F, G),$

\[
\omega_{\pm} = \frac{z_{\pm} S + \sqrt{z_{\pm}^2 S^2 - [1 + \hat{E}^2 z_{\pm}]^2 \{z_{\pm} R - \hat{k}^2\}}}{1 + \hat{E}^2 z_{\pm}} \tag{3}
\]

where

\[
S = (\hat{E} \times \hat{B}) \cdot \mathbf{k}, \quad R = (\mathbf{k} \times \hat{B})^2 - (\mathbf{k} \cdot \hat{E})^2.
\]

The expression\[3\] deserves some comments. First of all, there is not only one dispersion relation but two of them. It occurs because each transverse mode of wave propagation has a different dispersion relation, i.e. the energy $\omega$ depends on the photon polarization. This phenomenon is called birefringence and it is present in NLED models in general [46] (for details see appendix A). Moreover, these dispersion relations are coordinate dependent in general. The photon energy depends on the points of space-time because of its interaction with the background field. This is anticipated since the self-interaction processes of electromagnetic field are present in any nonlinear models.

Statistical Approach

With the general form of the photon dispersion relation in terms of NLED properties and the background field we now must resort to a statistical approach in order to obtain the thermodynamic properties of the photon gas. Since the bosonic nature of the photon is not affected by the generalization of the underlying dynamics, the Bose-Einstein statistics is still appropriate. There is an important point concerning the choice of reference frame. Although the phase volume $d^3x d^3p$ is Lorentz-invariant [47] the same can not be said about the dispersion relation [3]. Indeed, the dependence of $\omega_{\pm}$ on the background electromagnetic field makes it clearly dependent on the reference frame. In order to describe the thermal radiation, we must choose a reference frame comoving with the matter which produced the gas of photons. Thus the photons with the same energy are isotropically distributed in the space. It defines the reference frame in which the partition function must be calculated. Note that the same choice is done when we perform the usual calculation in Maxwell theory.\[3\]

In the grand canonical ensemble [45], the potential for the gas of photons is given by

\[
\Omega = -\frac{g}{V} \sum_j \ln \left(1 - e^{-\beta \omega_j}\right)
\]

1 This energy scale is specified in each NLED.

2 The units used in this paper are $c = h = 1$.

3 As it can be verified in the measurement of CMB, the effect of a different choice of reference frame distorts the black body radiation distribution.
where $\beta = \frac{1}{T}$, $V$ is the volume and $g$ takes account the internal degrees of freedom. In NLED each transverse mode corresponds to a different dispersion relation, so $g = 1$ and the expression above can be rewritten as

$$\Omega = \frac{-1}{(2\pi)^3 V} \left[ \int \ln (1 - e^{-\beta \omega^2}) d^3 x d^3 p + \int \ln (1 - e^{-\beta \omega^2}) d^3 x d^3 p \right],$$

(4)

It is clearly seen that in the cases of non-birefringent NLED, the grand canonical potential falls back to the known expression. The thermodynamical quantities are computed in the usual way i.e., the pressure $p$ and the energy density $\rho$ are given by

$$p = \frac{\Omega}{\beta} \quad \text{and} \quad \rho = -\left( \frac{\partial \Omega}{\partial \beta} \right).$$

(5)

Uniform electromagnetic field case

Although the equation (1) represents the general case, it is difficult to calculate the integrals mainly due to the spatial dependence of the electromagnetic field. Thus, for simplification purposes we consider a particular case where the electromagnetic field is uniform. Without loss of generality, it is possible to align the $x$ axis with the electric field and the magnetic field within the $xy$ plane. Then, equation (3) reduces to

$$\omega_\pm = \frac{y_\pm k_3 + \sqrt{a_\pm k_1^2 + b_\pm k_2^2 + c_\pm k_3^2 + e_\pm k_1 k_2}}{x_\pm},$$

where the subscript indicates the cartesian components and

$$a_\pm = (1 + E_0 z_\pm - B_0^2 z_\pm^2) x_\pm, \quad e_\pm = 2 x_\pm B_1 B_2 z_\pm,$$

$$b_\pm = (1 - B_0^2 z_\pm^2) x_\pm, \quad x_\pm = 1 + E_0^2 z_\pm,$$

$$c_\pm = x_\pm - x_\pm B_1^2 z_\pm - B_2^2 z_\pm, \quad y_\pm = E_1 B_2 z_\pm.$$

Note that all these factors are constants.

Due the similarity of the terms in the r.h.s. of (4), one can write

$$\Omega_\pm = -\frac{1}{(2\pi)^3} \int \ln (1 - e^{-\beta \omega_\pm}) d^3 k,$$

(6)

with

$$\Omega = \Omega_+ + \Omega_-.$$

Expanding the $\ln(...)$ in Taylor series and performing some variable substitutions the integral (6) is rewritten as

$$\Omega_\pm = A \sum_{n=1}^\infty \frac{1}{n} \left[ \int_0^\infty \int_0^{\pi} e^{-\frac{\beta \omega_\pm}{2}} \left( 1 + \frac{y_\pm}{\sqrt{x_\pm}} \cos \theta \right) r^2 \sin \theta dr d\theta \right],$$

where the constant $A$ is

$$A = \frac{1}{4\pi^2 \sqrt[4]{c_\pm (4a_\pm b_\pm - c_\pm^2)}}.$$

This integral converges under the requirements

$$x_\pm > 0 \quad \text{and} \quad |y_\pm| < \sqrt{c_\pm}.$$

(7)

The result is:

$$\Omega_\pm = \frac{\pi^2}{90} \frac{|c_\pm|}{(1 - z_\pm (B_1^2 + B_2^2))^2} \frac{1}{\beta^3}.$$  

(8)

The potential above was calculated on the black body rest frame. On the other hand, it is possible to choose a new frame which has a 4-velocity $u^\mu$ (in the black body rest frame $u^\mu = (1, 0, 0, 0)$). Thus, with $u^\mu$ we can define a new invariant

$$H = (u_\mu F^{\mu \alpha})(u_\nu F^{\nu \alpha}),$$

and rewrite (8) in terms of invariants $F$, $G$ and $H$:

$$\Omega_\pm = \frac{\pi^2}{90} \left[ 1 + 2z_\pm F - z_\pm^2 \frac{G^2}{2} \right] \frac{1}{\beta^3} \equiv K_\pm.$$

(9)

Because of the dependence of the constraints on $z_\pm$, they must be analyzed separately for each NLED. Two examples are given in the next sections.

Substituting the explicit form of $\Omega$ in (5) it is possible to obtain the pressure,

$$p (kT) = (K_+ + K_-) (kT)^4,$$

(10)

and the energy density

$$\rho (kT) = 3 (K_+ + K_-) (kT)^4.$$  

(11)

If we set $z_\pm = 0$ it follows $K_+ = K_- = \frac{\pi^2}{90}$ and the usual results are recovered.

An interesting result is that, despite of $p$ and $\rho$ being affected by the constant background field, the equation of state remains unaffected, i.e.

$$p = \frac{\rho}{3}.$$

This result is independent from a particular NLED.

Before the end of this section, it is interesting to reanalyze the approximation made at the beginning. To obtain the dispersion relation (1) it was supposed that the frequency associated with the radiation is smaller than the energy scale $M$. It implies that the approach developed is only valid when the average energy per photon $\varepsilon$ is much smaller than $M$. Thus, remembering that the photon numerical density $n$ is proportional to $(kT)^3$ we establish the following condition:

$$\varepsilon \sim \frac{\rho}{n} \sim (kT) \ll M.$$  

(12)
III. HEISENBERG-EULER EFFECTIVE LAGRANGIANS

In this section, we apply the above formulation to Heisenberg-Euler NLED. The 1-loop Heisenberg-Euler effective action [5] [10] is given by

\[ L^1 = F + \frac{e^2 ab}{8\pi^2} \int_0^\infty ds \frac{e^{-im^2 s}}{s} \times \left( \cot (\varepsilon s) \coth (\varepsilon b) - \frac{1}{ab} \left( \frac{1}{c^2} + 2 F \right) \right) \] (13)

where

\[ a^2 = \sqrt{F^2 + G^2 - F}, \quad b^2 = \sqrt{F^2 + G^2 + F}. \]

The constants \( e \) and \( m \) are respectively the charge and mass of the electron. It is worth noting that due to the fact \( G \) gains a minus sign after a parity transformation, the theory must only depend on \( G^2 \) to be parity invariant.

It is important to emphasize that the Heisenberg-Euler NLED is an effective description of QED in the low-energy regime. More precisely, the Lagrangian (13) is only valid for photons with energy lower than the electron mass. This characteristic scale sets \( M = m \), and thus our approach is valid only in the regime of \( kT \ll m \).

Expanding (13) in power series and integrating over \( s \) leads to

\[ L^1 = F + c_1 F^2 + c_2 G^2 + c_3 F^3 + c_4 FG^2 + ... \] (14)

with

\[ c_1 = \frac{8\alpha^2}{45m^4}, \quad c_2 = \frac{14\alpha^2}{45m^4}, \quad c_3 = \frac{2^8\pi\alpha^3}{315m^5}, \quad c_4 = \frac{2^5 \times 13\pi\alpha^3}{315m^8}, \]

where \( \alpha \) is the structure constant of QED.

Let us concentrate on a weak-field analysis expanding \( K_\pm \) to quadratic terms in the invariants:

\[ K_\pm \simeq \frac{\pi^2}{90} [1 + 2z_\pm (H - F)] \]
\[ + z_\pm^2 \left[ 3(H - 2F)^2 - G^2 + 4(H - 2F)F \right] + \mathcal{O}(3). \] (15)

By \( \mathcal{O}(3) \) we mean cubic combinations of invariants \( F, G \) and \( H \). The coefficient \( z_\pm \) should be linear in the invariants which means that

\[ z_+ = 2c_1 + 2(2c_2^2 - 4c_1c_2 + 3c_3)F + \mathcal{O}(2), \]
\[ z_- = 2c_2 + 2(2c_2^2 - 2c_1^2 + 2c_1c_2 + 2c_4)F + \mathcal{O}(2). \]

Besides, as we are concerned with first corrections to Maxwell electrodynamics all terms greater than \( \alpha^3 \) will be neglected. Thus, (15) is rewritten as

\[ K_+ \simeq \frac{\pi^2}{90} [1 + 4c_1 (H - F) + 12c_3 (H - F)F + \mathcal{O}(3)] \]
\[ K_- \simeq \frac{\pi^2}{90} [1 + 4c_2 (H - F) + 4c_4 (H - F)F + \mathcal{O}(3)] \]

and (11) leads to

\[ \rho(kT) = \frac{\pi^2}{15} (kT)^4 + \frac{44\alpha^2\pi^2}{675} \frac{(H - F)}{m^4} (kT)^4 \]
\[ + \frac{2^6 \times 37\alpha^3\pi^3}{3^3 \times 5^2 \times 7} \frac{F (H - F)}{m^8} (kT)^4 + \mathcal{O}(3). \] (16)

Note that \( (H - F) \) is always positive.

This result is in complete agreement with the calculation of QED effective action at finite temperature. In fact, it has been shown in [51] that in the thermal 1-loop QED effective Lagrangian at low-temperature expansion all the terms are damped by a factor \( e^{-\frac{kT}{m}} \). However, the thermal 2-loop effective action at low-temperature expansion in the weak-field limit produces a dominant term exactly as presented in (16).

We do not obtain terms proportional to \((kT)^6\) presented in the thermal 2-loop effective action because our approach is valid only in the regime of \( kT \ll m \). Moreover, the calculation at finite temperature shows that our model is an excellent approximation in the regime of \( \frac{kT}{m} < 0.05 \) where terms proportional to \((kT)^6\) are subdominants. It is worth mentioning that the consistency between the two approaches was not obvious a priori since we use the 1-loop Heisenberg-Euler effective action at zero temperature.

Let us move on and take into account the 2-loops Heisenberg-Euler effective action. According to [28], the effective Lagrangian is

\[ L^2 = F + \bar{c}_1 F^2 + \bar{c}_2 G^2 + \bar{c}_3 F^3 + \bar{c}_4 FG^2 + ... \]

with

\[ \bar{c}_1 = c_1 + \frac{\alpha^3}{\pi m^4} \frac{2^6}{3^4}, \quad \bar{c}_2 = c_2 + \frac{\alpha^3}{\pi m^4} \frac{263}{2 \times 3^4}, \]
\[ \bar{c}_3 = c_3 - \frac{\alpha^4}{m^8} \frac{2^3 \times 23 \times 53}{3^4 \times 5^2}, \quad \bar{c}_4 = c_4 - \frac{\alpha^4}{m^8} \frac{2^5 \times 541}{3^4 \times 5^2}. \]

Performing the same approximations as before we obtain

\[ \rho = \frac{\pi^2}{15} (kT)^4 + \left[ \frac{44\alpha^2\pi^2}{675} + \frac{391\alpha^3\pi^3}{5 \times 3^5} \right] \frac{(H - F)}{m^4} (kT)^4 \]
\[ + \frac{2^6 \times 37\alpha^3\pi^3}{3^3 \times 5^2 \times 7} \frac{F (H - F)}{m^8} (kT)^4 + \mathcal{O}(3), \] (17)

where the new term arises due the contribution of 2-loops effective action. As expected, this new term represents a small correction (about 1%) compared with \( \frac{44\alpha^2\pi^2}{675} \). However, for \( \frac{kT}{m} < 0.05 \) its contribution becomes dominant.
when compared to the last term in \(^{(17)}\). Restoring the (Gaussian) units we have

\[
\frac{F}{m^4} \rightarrow \frac{\hbar^3 c^3 F}{m^4 c^8} \rightarrow 7 \times 10^{-26} \left( \frac{E^2 - B^2}{2} \right).
\]

Thus,

\[
\frac{F}{m^4} < 0.05 \Rightarrow \left\{ \begin{array}{l}
B < 1.2 \times 10^{12} G \\
E < 3.6 \times 10^{14} V/cm.
\end{array} \right.
\]

As almost all electric and magnetic fields known respect the constraints above (exception for magnetars) the new term in \(^{(17)}\) is dominant over the last one.

Finally, it is important to emphasize that if there is consistency between our approach and the effective term in (17) is dominant over the last one.

The most well-known extension of Maxwell electrodynamics is the Born-Infeld electrodynamics which is produced by the following Lagrangian \(^{[3,4]}\):

\[
L_{BI} = b^2 \left[ 1 - \sqrt{1 - \frac{2F}{b^2} - \frac{G^2}{b^4}} \right],
\]

where \(b^2\) is an arbitrary real parameter with dimension of energy density. The Maxwell electrodynamics is recovered when \(b \rightarrow \infty\).

What makes the Born-Infeld NLED interesting are its special features. For example, \(L_{BI}\) has an upper limit value of the fields avoiding the problem of divergence of the electrostatic self-energy of a point charge. It is the only physical NLED which presents no birefringence \(^{[10]}\). Besides, a generalization of Born-Infeld model describes the electrodynamics on D-branes in the context of open string theories \(^{[29]}\).

Although Born-Infeld Lagrangian has no restrictions it possesses a characteristic energy scale associated with the \(b\) parameter. This characteristic scale sets \(M = \sqrt{b}\), and thus our approach is valid only in the regime of \(kT \ll \sqrt{b}\).

For the Born-Infeld Lagrangian, \(\delta = 0\) and \(^{(2)}\) is given by

\[
\zeta^{BI} = \zeta^{B1} = \frac{1}{2F - b^2}.
\]

In order to simplify the manipulation and for insight purposes we will analyze only when the uniform background field is purely magnetic or electric. In this case, we can write the energy densities for Born-Infeld model as

\[
\rho_{BI}(E_1, T) = \frac{\pi^2}{15} \left( \frac{1}{1 - \frac{E^2}{b^2}} \right) (kT)^4. \quad (18)
\]

and

\[
\rho_{BI}(B_1, T) = \frac{\pi^2}{15} \left( \frac{B_1^2}{b^2} + 1 \right) (kT)^4. \quad (19)
\]

It is important to look for the validity interval dictated by the constraints \(^{(17)}\). For the electrostatic case \(E_1 < b\), which is in agreement with the role that the constant \(b\) lays. For the magnetostatic case, the constraints are always satisfied, which means that \(B_1\) can assume any value. In order to compare the behaviors, a dimensionless comparison between the two \(\rho_{BI}\) and the standard \(\rho_{EM}\) is formed in figure \(^{[1]}\).

![Figure 1: Comparison between the energy densities for Born-Infeld and Maxwell electrodynamics.](image)

Figure \(^{[1]}\) shows that, for a given \(kT\), the two \(\rho_{BI}\) increase as the dimensionless field parameters increase and they are always greater than \(\rho_{EM}\). Physically, this phenomenon is due to the interaction of the photons with the background fields leading to an excitation of the gas i.e., the background fields transfer energy to the photon gas. It is worth noting that this energy transfer is more effective in the electrostatic case. Similar situation occurs with effective Heisenberg-Euler Lagrangians since the heat-bath engenders an extra excitation for the photon gas\(^{[6]}\).

V. FINAL REMARKS

In this paper the properties of a photon gas were studied in the context of a general NLED neglecting the photon self-interaction. As usual, the Bose-Einstein statistics was used and the calculation of the partition function

\(^{[6]}\) See the dominant correction in \(^{(17)}\).
was performed. The grand canonical potential was obtained considering an uniform background field, and for this case, we verified that although the energy density and pressure depend on the background field, the equation of state remains unaltered, namely $\rho = 3p$.

Applications of the results were done to Heisenberg-Euler and Born-Infeld nonlinear electrodynamics. Using the 1-loop Heisenberg-Euler effective action at zero temperature we were able to recover the results obtained in the thermal 2-loops approach [50]. Besides, including corrections at 2-loops (zero temperature) we predicted a new term which should be dominant in second order for the photon gas energy density. It would be interesting, in a future work, to verify if this new term can be derived from the thermal 3-loop Heisenberg-Euler effective action. For the Born-Infeld model, the interaction of the photons with the background field might produce an excitation of the gas.

The approach developed in this paper can be used to test the linearity of electrodynamics or set constraints for specific NLED. For example, one may look for deviations of Planck spectrum and/or modifications of energy density in situations of strong electromagnetic field (e.g. around magnetars). Although our approach applies to any NLED it does not take into account the self interaction of photons. This approximation limits the scope of possible applications. A solution is to choose a specific NLED. For example, one may look for deviations of Planck spectrum and/or modifications of energy density in situations of strong electromagnetic field (e.g. around magnetars). Although our approach applies to any NLED it does not take into account the self interaction of photons. This approximation limits the scope of possible applications. A solution is to choose a specific NLED and apply the procedure of quantization at finite temperature to obtain a complete description of the black body phenomenon [5]. In this situation, it would be possible, for example, to analyze the effects of a NLED in primordial cosmology.

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Appendix A: Birefringence Phenomenon

Following the procedure described in [44], the amplitude of electromagnetic field for a photon is given by

$$\varepsilon_{\mu\nu} = k_\mu \varepsilon_{\nu} - k_\nu \varepsilon_{\mu}$$

where

$$\varepsilon_{\mu} = \alpha a_\mu + \beta \bar{a}_\mu + \gamma k_\mu.$$ 

with $a^\mu \equiv F^{\mu\nu} k_\nu$ and $\bar{a}^\mu \equiv \bar{F}^{\mu\nu} k_\nu$. Using the relations $k_\mu = (\omega, -\vec{p})$, $a_\mu = (a_0, \vec{a})$ and $\bar{a}_\mu = (\bar{a}_0, \vec{\bar{a}})$, we can write the electric field $E_i = \varepsilon_{i\alpha}$ as

$$\bar{E} = \alpha \vec{v} + \beta \vec{a}$$

where $\vec{v} = -(a_0 \vec{p} + \omega \vec{a})$ and $\vec{a} = -(\bar{a}_0 \vec{\bar{p}} + \omega \vec{\bar{a}})$ are LI vectors. The equation [A1] states that the electric field for a photon is always located on the plane generated by $\vec{v}$ and $\vec{a}$.

On the other hand, the two dispersion relations $\omega_\pm$ are obtained imposing non-trivial solution to the system

$$\begin{pmatrix} M k^2 - L_{FF} a^2 & GL_{FG} k^2 - NL_{FG} \\ GL_{GG} k^2 - L_{FG} a^2 & M k^2 - NL_{GG} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with

$$M \equiv L_F + GL_F, \quad N \equiv a^2 + 2Fk^2, \quad k^2 \equiv k_\mu k^\mu \quad \text{and} \quad a^2 \equiv a_\mu a^\mu.$$ 

Each dispersion relations, $\omega_+$ and $\omega_-$, is associated with a respective vector $\vec{V}_+ = (\alpha_+, \beta_+)$ and $\vec{V}_- = (\alpha_-, \beta_-)$ which, in the language of eigenvalue problems, are eigenvectors. These two vectors define the LI set

$$\bar{E}_+ = \alpha_+ \vec{v} + \beta_+ \vec{a} \quad \text{and} \quad \bar{E}_- = \alpha_- \vec{v} + \beta_- \vec{a}.$$ 

Thus, any photon whose electric field $\bar{E}$ is in the direction $\bar{E}_+$ propagates with a dispersion relation $\omega_+$. Besides, the field $\bar{E}$ of any photon can be decomposed in

$$\bar{E} = \bar{a} \bar{E}_+ + \bar{\beta} \bar{E}_-,$$

producing the phenomenon of birefringence.

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