Abelian projection in $SU(N)$ gauge theories

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Abstract

The abelian projection in $SU(N)$ gauge theories is discussed in detail, as well as the construction of a disorder parameter to study dual superconductivity as a mechanism for color confinement. If the ideas of the large $N$ limit are correct, a universal $N$-independent behavior is expected for the suitable rescaled disorder parameter as a function of $\lambda = g^2 N$.

Key words: Confinement, monopoles, large-N

1 Introduction

There exists evidence from lattice simulations [1–3] that color confinement is produced by condensation of magnetic monopoles in the vacuum, i.e. dual superconductivity. Vacuum behaves as a dual superconductor in the confining phase, and goes to normal at the deconfining phase transition.

The evidence refers to $SU(2)$ and $SU(3)$ pure gauge theories. Preliminary data indicate that the same mechanism is at work in QCD with dynamical quarks [4], a fact which is in line with the ideas of $N_c \to \infty$ [5,6]. As $N_c \to \infty$ with $\lambda = g^2 N_c$ fixed the theory should preserve its structure; corrections $O(1/N_c)$ are expected to be small and under control. If this is true the mechanism of confinement should be the same at all values of $N_c$ and also in full QCD, quark loops being non-leading in the $1/N_c$ expansion.

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A direct check of these ideas can be done by exploring, by the same techniques used in Ref. [1,3], the symmetry of the confining vacuum in $SU(N)$ theories. The technique used there was to measure the vacuum expectation value $\langle \mu \rangle$ of an operator $\mu$, which creates a monopole as a function of the temperature, across the deconfining phase transition. A non zero $\langle \mu \rangle$ implies dual superconductivity. This is exactly what is found below the deconfining temperature $T_c$. Above $T_c$, $\langle \mu \rangle$ vanishes, and sectors with different magnetic charge are superselected.

Monopoles are defined by a procedure called abelian projection [7], which we summarize in the case of the $SU(2)$ gauge theory.

Let $\phi = \vec{\phi} \cdot \vec{\sigma}/2$ be any field in the adjoint representation, and $\hat{\phi} \equiv \vec{\phi}/|\vec{\phi}|$ its direction in color space, which is defined everywhere in a configuration except at zeros of $\vec{\phi}$.

A gauge invariant, color singlet field strength tensor can be defined:

$$F_{\mu \nu} = \hat{\phi} \cdot \vec{G}_{\mu \nu} - \frac{1}{g} \hat{\phi} \cdot (D_\mu \hat{\phi} \wedge D_\nu \hat{\phi}) \tag{1}$$

where

$$\vec{G}_{\mu \nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g \vec{A}_\mu \wedge \vec{A}_\nu$$

is the color field strength tensor, and

$$D_\mu \hat{\phi} = (\partial_\mu - g \vec{A}_\mu \wedge) \hat{\phi}$$

is the covariant derivative of $\hat{\phi}$. The two terms in Eq. (1) are separately color singlets and gauge invariant. The combination is chosen in such a way that bilinear terms in $\vec{A}_\mu \vec{A}_\nu$, and $\vec{A}_\mu \partial_\nu \hat{\phi}$ cancel. Indeed, by explicit computation:

$$F_{\mu \nu} = \partial_\mu (\hat{\phi} \cdot \vec{A}_\nu) - \partial_\nu (\hat{\phi} \cdot \vec{A}_\mu) - \frac{1}{g} \hat{\phi} \cdot (\partial_\mu \hat{\phi} \wedge \partial_\nu \hat{\phi}) \tag{2}$$

If we gauge transform to make $\hat{\phi} = \text{const}$, e.g. $\hat{\phi} = (0,0,1)$ then the second term in Eq. (2) vanishes and $F_{\mu \nu}$ becomes abelian

$$F_{\mu \nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 \tag{3}$$

This gauge transformation is known as abelian projection on $\hat{\phi}$, and is defined up to a residual $U(1)$, corresponding to rotations around $\hat{\phi}$.
Monopoles can be present in a noncompact formulation of the theory when \( \partial^\mu F^*_\mu \) can be different from zero [8]. A non zero magnetic current exists

\[
j_\nu = \partial^\mu F^*_\mu
\]

which is identically conserved

\[
\partial^\nu j_\nu = 0
\]

The corresponding \( U(1) \) symmetry can be either realized à la Wigner or Higgs broken. In the first case the Hilbert space is made of superselected sectors with definite magnetic charges. In the second case, under very general assumptions, the vacuum behaves as a dual superconductor.

The detection of dual superconductivity has been successfully performed for \( SU(2) \) and \( SU(3) \) gauge theories [1–3]. In this paper we want to analyze the abelian projection and the construction of the disorder parameter \( \mu \) for generic \( SU(N) \).

2 Abelian projection in \( SU(N) \) gauge theory

In analogy to the construction for \( SU(2) \), for an arbitrary operator \( \phi \) in the adjoint representation

\[
\phi = \sum \phi^a T^a
\]

we can define a field strength tensor \( F_{\mu\nu} \). Here \( T^a \ (a = 1 \ldots N^2 - 1) \) are the generators of \( SU(N) \) in the fundamental representation, with normalization

\[
\text{Tr} [T^a T^b] = \frac{1}{2} \delta^{ab}
\]

We write

\[
F_{\mu\nu} = \text{Tr} \{ \phi G_{\mu\nu} \} - \frac{i}{g} \text{Tr} \{ \phi [D_\mu \phi, D_\nu \phi] \}
\]

The normalization of \( \phi \) has been left indeterminate. A change in the normalization reflects in a change of the relative coefficients of the two terms in Eq. (6). The notation is the usual one.
\[ A_\mu = A_\mu^a T^a \]
\[ G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu] \]
\[ D_\mu \phi = \partial_\mu \phi - ig [A_\mu, \phi] \]

We want to investigate for what choice of \( \phi \), if any, bilinear terms in \( A_\mu A_\nu \), \( A_\mu \partial_\nu \phi \) cancel in Eq. (6). If this happens

\[ F_{\mu\nu} = c \left\{ \text{Tr} \{ \phi (\partial_\mu A_\nu - \partial_\nu A_\mu) \} - \frac{ig}{2} \text{Tr} \{ \phi [\partial_\mu \phi, \partial_\nu \phi] \} \right\} \quad (7) \]

and \( F_{\mu\nu} \) becomes abelian in the gauge where \( \phi \) is diagonal,

\[ F_{\mu\nu} = c \text{Tr} \{ \partial_\mu (\phi A_\nu) - \partial_\nu (\phi A_\mu) \} \quad (8) \]

The condition for the cancellation of bilinear terms \( A_\mu A_\nu \) in Eq. (6), for any \( A_\mu A_\nu \), reads

\[ X_2(A_\mu, \phi) \equiv \text{Tr} (\phi [A_\mu, A_\nu]) + \text{Tr} (\phi [[A_\mu, \phi], [A_\nu, \phi]]) = 0 \quad (9) \]

If \( \phi_0 \) is a solution of Eq. (9), then

\[ \phi = U(x) \phi_0 U^\dagger(x) \quad (10) \]

with arbitrary unitary matrix \( U(x) \) is also a solution. Indeed

\[ X_2(A_\mu(x), \phi) = X_2(U^\dagger(x)A_\mu(x)U(x), \phi_0) = 0 \]

since Eq. (9) holds for any choice of \( A_\mu \). In particular one can choose \( \phi_0 \) diagonal. The generic \( \phi_0 \), diagonal and constant, obeying Eq. (9) is (see appendix A)

\[ \Phi_0^q = \text{diag} \left( \frac{p}{N}, \frac{p}{N}, \ldots, \frac{p}{N}, \frac{q}{N}, \frac{q}{N}, \ldots, \frac{q}{N} \right) \quad (11) \]

where \( p + q = N \), and \( q = 1 \ldots N - 1 \). If the solution is continuous, it must be constant, since it cannot jump from one of the solutions in Eq. (11) to another. So the generic solution to our initial problem, which we call \( \Phi \), is given by Eq. (10), where \( \phi_0 \) can be any of the matrices \( \Phi_0^q \) of Eq. (11).

Let us now show that, if \( X_2 = 0 \), then the terms \( A_\mu \partial_\nu \Phi \) also vanish. Let us call \( X_1(A_\mu, \Phi) \) such terms, we have
\[ X_1(A, \Phi) = \text{Tr} \{ \Phi (\partial_\mu A_\nu - \partial_\nu A_\mu) \} + \text{Tr} \{ \Phi [[A_\mu, \Phi], \partial_\nu \Phi] \} \\
+ \text{Tr} \{ \Phi [\partial_\mu \Phi, [A_\nu, \Phi]] \} - \text{Tr} \{ \partial_\mu (\Phi A_\nu) - \partial_\nu (\Phi A_\mu) \} \\
= \text{Tr} \{ \Phi [[A_\mu, \Phi], \partial_\nu \Phi] \} + \text{Tr} \{ \Phi [\partial_\mu \Phi, [A_\nu, \Phi]] \} \\
- \text{Tr} \{ \partial_\mu \Phi A_\nu - \partial_\nu \Phi A_\mu \} \] (12)

Now [9]

\[ \partial_\mu \Phi = \partial_\mu (U \Phi U^\dagger) = [\Omega_\mu, \Phi] \quad \Omega_\mu = (\partial_\mu U) U^\dagger \]

and hence from Eq. (12)

\[ X_1(A, \Phi) = \text{Tr} \{ \Phi [[A_\mu, \Phi], [\Omega_\nu, \Phi]] + \Phi [[\Omega_\mu, \Phi], [A_\nu, \Phi]] - [\Omega_\mu, \Phi] A_\nu - [\Omega_\nu, \Phi] A_\mu \} \] (13)

which can also be written

\[ X_1(A, \Phi) = X_2(A + \Omega, \Phi) - X_2(\Omega, \Phi) - X_2(A, \Phi) \]

Since Eq. (9) is valid for arbitrary \( A_\mu \), one has:

\[ X_1(A, \Phi) = 0 \quad \text{if} \quad X_2(A, \Phi) = 0 \]

Then

\[ F_{\mu\nu} = \text{Tr} \{ \partial_\mu (\Phi A_\nu) - \partial_\nu (\Phi A_\mu) \} - \frac{i}{g} \text{Tr} \{ \Phi [\partial_\mu \Phi, \partial_\nu \Phi] \} \] (14)

In the abelian projected gauge, where \( \Phi = \Phi^0 \),

\[ F_{\mu\nu} = \text{Tr} \{ \partial_\mu (\Phi A_\nu) - \partial_\nu (\Phi A_\mu) \} \] (15)

On the other hand, the 't Hooft tensor for a pure gauge field \( \omega_\mu = -i (\partial_\mu U) U^\dagger \) is zero

\[ 0 = F_{\mu\nu}(\omega) = \text{Tr} \{ \partial_\mu (\Phi \omega_\nu) - \partial_\nu (\Phi \omega_\mu) \} - \frac{i}{g} \text{Tr} \{ \Phi [\partial_\mu \Phi, \partial_\nu \Phi] \} \] (16)

and finally, by use of Eq. (16), Eq. (14) can be rewritten

\[ F_{\mu\nu} = \text{Tr} \{ \partial_\mu (\Phi A_\nu) - \partial_\nu (\Phi A_\mu) \} - \text{Tr} \{ \partial_\mu (\Phi \omega_\nu) - \partial_\nu (\Phi \omega_\mu) \} \] (17)

showing that \( F_{\mu\nu} \) obeys Bianchi identities.
In conclusion an “abelian” field strength $F^q_{\mu\nu}$ can be defined for each of the fields $\Phi^q$ of Eq. (11), or any other field related to them by a gauge transformation. The normalization of the solution is fixed. The invariance group of $\Phi^q_0$ is

$$SU(q) \times SU(N - q) \times U(1) \quad (18)$$

and $\Phi^q(x)$ belongs to the coset of $\Phi^q_0$

$$\Phi^q(x) = U(x)\Phi^q_0U^\dagger(x) \quad (19)$$

$\Phi^q_0$ defines a symmetric subspace [10], in the sense that the full algebra of $SU(N)$ is the sum of the subalgebra $L_0$ of the little group of $\Phi^q_0$ plus the complement $L'$ to the full algebra, and

$$[L_0, L_0] \in L_0 \quad [L', L_0] \in L' \quad [L', L'] \in L_0 \quad (20)$$

The main property of symmetric spaces is that any element of the group $U$ can be uniquely split in the form

$$U = e^{iL'}e^{iL_0} \quad (21)$$

a property which will be used below.

Let us now discuss the abelian projection in the light of the above results. Following Ref. [7], let us consider a generic operator $X$, that transforms covariantly under gauge transformations. $X$ can be diagonalized by a gauge transformation:

$$X_D(x) = U_X(x)X(x)U_X^\dagger(x) \quad (22)$$

For each $\Phi^q_0$ in Eq. (11), a field transforming in the adjoint representation of $SU(N)$ can be defined:

$$\Phi^q(x) = U^\dagger_X(x)\Phi^q_0U_X(x) \quad (23)$$

These fields $\Phi^q$ can now be used to define $N - 1$ gauge-invariant field strength tensors:

$$F^q_{\mu\nu} = \text{Tr} \left\{ \Phi^q G_{\mu\nu} \right\} - \frac{i}{g} \text{Tr} \left\{ \Phi^q [\partial_\mu \Phi^q, \partial_\nu \Phi^q] \right\} \quad (24)$$
In the gauge where \( X \) is diagonal, \( \Phi^q(x) = \Phi^q_0 \) and, according to the above results, these tensors reduce to the abelian form:

\[
F^q_{\mu\nu} = \partial_\mu \text{Tr} \{ \Phi^q_0 A_\nu \} - \partial_\nu \text{Tr} \{ \Phi^q_0 A_\mu \}
\]  

(25)

The diagonal matrices \( \Phi^q_0 \) form a complete set of diagonal matrices, and hence

\[
X_D(x) = \sum_{q=1}^{N-1} c_q(x) \Phi^q_0
\]  

(26)

We denote by \( \alpha^a \) the diagonal matrices associated to the simple roots via:

\[
\alpha^a = \alpha^a_i H_i,
\]

If the generators of the Cartan subalgebra \( H_i \) are written in the standard basis:

\[
(H_m)_{ij} = \left( \sum_{k=1}^m \delta_{ik} \delta_{jk} - m \delta_{i,m+1} \delta_{j,m+1} \right) \frac{1}{\sqrt{2m(m+1)}}
\]

or

\[
H_m = \text{diag} (1,1,1 \ldots 1, -m, 0 \ldots 0) \frac{1}{\sqrt{2m(m+1)}}
\]

the matrices associated to the simple roots \( \alpha^a \) are:

\[
\alpha^1 = \alpha^1_i H_i = \frac{1}{2} \text{diag} (1, -1, 0, \ldots, 0)
\]

\[
\alpha^2 = \alpha^2_i H_i = \frac{1}{2} \text{diag} (0, 1, -1, 0, \ldots, 0)
\]

\[
\vdots
\]

\[
\alpha^a = \alpha^a_i H_i = \frac{1}{2} \text{diag} (0, 0, \ldots, 0, \underbrace{a,a+1}_{a+1}, -1, 0, \ldots, 0)
\]

(27)

It is then trivial to check that

\[
\text{Tr} \{ \Phi^a_0 \alpha^b \} = \frac{1}{2} \delta^{ab}
\]

(28)

and therefore

\[
\text{Tr} \{ X_D(x) \alpha^a \} = \frac{1}{2} c_a(x) = X^a_D - X^{a+1}_D
\]
For each point \( x \) such that \( c_a(x) = 0 \), two eigenvalues of \( X \) become degenerate:

\[
X_D^a(x) - X_D^{a+1} = 0,
\]

and the gauge transformation \( U_X \), Eq. (22), becomes singular. Such a singularity behaves as a magnetic charge with respect to the \( U(1) \) group of eq.(18) for \( q = a \).

Let \( A^D_\mu \) be the diagonal part of the gauge field in the abelian projected gauge [7]

\[
A^D_\mu = \text{diag} \left( a_\mu^1, \ldots, a_\mu^N \right)
\]

The diagonal matrix \( A^D_\mu \) can be expanded in the form:

\[
A^D_\mu = 2 \sum \tilde{a}_\mu^i \alpha^i
\]

The \( N - 1 \) abelian photons \( \tilde{a}_\mu^i \) coincide with the abelian fields defined via the abelian projection:

\[
\begin{align*}
\text{Tr} \left\{ \Phi_0^q A^D_\mu \right\} &= \tilde{a}_\mu^q \\
F_{\mu \nu}^q &= \partial_\nu \tilde{a}_\mu^q - \partial_\mu \tilde{a}_\nu^q
\end{align*}
\]

The singularities of the gauge transformation \( U_X \) at \( c_a(x) = 0 \) are magnetic charges with respect to the abelian field \( \tilde{a}_\mu^a \), a result that will be used to construct magnetically charged operators in the next Section.

### 3 Construction of the disorder parameter.

In the abelian projected representation, in which the operator \( \Phi \) is diagonal, the generic link \( U_\mu(x) \) can be cast in the form

\[
U_\mu(x) = V^{(i)}_\mu(x) D^{(i)}_\mu(x) \quad i = 1 \ldots N - 1
\]

where \( D^{(i)} = \exp(ic_\mu^i \alpha^i) \), \( \alpha^i \) being the diagonal matrix corresponding to the simple root \( \alpha^i \).

Indeed since the algebra of the little group of \( \phi^a, L_0 \), defines a symmetric space, \( U_\mu \) can be uniquely split as \( U_\mu = e^{iL_\mu'} e^{iL_0} \), as in Eq. (21). \( \alpha^i \) is an element of
$L_0$ and commutes with all the others, since $\alpha^i$ is invariant under $e^{iL_0}$

$$\text{Tr} \{ \Phi^a \alpha^b \} = \frac{1}{2} \delta^{ab} = \text{Tr} \left\{ e^{iL_0} \Phi^a e^{-iL_0} \alpha^b \right\} = \text{Tr} \left\{ \Phi^a e^{-iL_0} \alpha^b e^{iL_0} \right\}$$

(33)

$\alpha^i$ is the generator of the subgroup $U(1)$ of Eq. (18). Therefore

$$U_\mu = e^{iL_0'} e^{i\tilde{L}_0} e^{ie^i_{\mu} \alpha^i}$$

(34)

where $\tilde{L}_0$ is $L_0$ minus $\alpha^i$. The plaquette $\Pi_{\mu\nu}(x) \equiv U_\mu(x) U_\nu(x + \hat{\mu}) U_{\mu}^\dagger(x + \hat{\nu}) U_{\nu}^\dagger(x)$ can then be rewritten as the product of matrices of the form in Eq. (32)

$$\Pi_{\mu\nu}(x) = V_\mu(x) D_\mu(x) V_\nu(x + \hat{\mu}) D_{\mu}^\dagger(x + \hat{\nu}) V_{\mu}^\dagger(x + \hat{\nu}) D_{\nu}^\dagger(x)$$

$$= V_\mu(x) \tilde{V}_\nu(x + \hat{\mu}) \tilde{V}_{\mu}^\dagger(x + \hat{\nu}) \tilde{V}_{\nu}^\dagger(x) D_\mu(x) D_{\nu}(x + \hat{\mu}) D_{\mu}^\dagger(x + \hat{\nu}) D_{\nu}^\dagger(x)$$

$$= \tilde{\Pi}_{\mu\nu}(x) \Pi_{\mu\nu}^0(x)$$

(35)

An abelian $U(1)$ plaquette is thus defined. However, an alternative way of defining the abelian plaquette would be to operate the separation of $\Pi_{\mu\nu}$ directly as done for the single links. It is easy to see that the two definitions differ by terms $O(a^2)$. Indeed the second definition can be obtained by factorizing $e^{iL_0}$ from the product $V^i \tilde{V}$ in Eq. (35). The resulting $L_0$ would come from higher terms in the Baker Hausdorff formula, which are $O(a^2)$ and higher.

The lattice abelian projection is therefore intrinsically undefined by terms of order $O(a^2)$. A similar ambiguity comes out if the abelian field is defined by 2 × 2 or 2 × 1 Wilson loops, instead of the plaquette.

$U(1)$ monopoles are defined as in Ref. [11]. The angle $\theta_{\mu\nu}$ is defined by the equation

$$\Pi_{\mu\nu}^0 = e^{i\theta_{\mu\nu}}$$

(36)

The magnetic current defined as

$$j_\mu = \Delta_\nu \theta^*_{\mu\nu}$$

$$\theta^*_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} \theta_{\rho\tau}$$

(37)

$j_\mu$ identically vanishes by the Bianchi identity. However, if the angle $\theta_{\mu\nu}$ is defined modulo 2π:

$$\theta_{\mu\nu} = \tilde{\theta}_{\mu\nu} + 2\pi n_{\mu\nu}, -\pi \leq \theta_{\mu\nu} \leq \pi$$

(38)
then the magnetic current

\[ \tilde{j}_\mu = \Delta_\mu \tilde{\theta}_{\mu\nu}^* \]  

(39)
can be different from zero and is conserved, \( \Delta_\mu \tilde{j}_\mu = 0 \). The term proportional to \( n_{\mu\nu} \) counts the Dirac strings going through the plaquette, which are invisible.

A monopole at a fixed time exists in an elementary spatial cube such that one of the faces has \( n_{ij} = 1 \), the others \( n_{ij} = 0 \). The visible \( \tilde{\theta}_{\mu\nu}^* \) has then a flux of \( 2\pi \), which is balanced by the outgoing invisible string.

The operator \( \mu \) which adds a monopole at the site \( \vec{y} \) and time \( y^0 \) to a generic configuration, can be constructed as follows

\[ \mu = \exp \left\{ -\beta \sum (\tilde{S}_{0i}(y^0) - S_{0i}(y^0)) \right\} \]  

(40)

\( S_{0i} \) are the terms of the action involving space-time \((0, i)\) loops, with a space link at \( y^0 \) and the others at \( t \geq y^0 \). For example for the Wilson action

\[ S_{0i} = \sum_\vec{n} \Pi_{0i}(y^0, \vec{n}) \]

We will recall the construction for Wilson action: the generalization to actions containing loops other than the plaquette, e.g. improved actions, is straightforward.

In the favored abelian projection, according to Eq. (34)

\[ \Pi_{0i}(y^0, \vec{n}) = \tilde{\Pi}_{0i}(y^0, \vec{n}) \Pi_{0i}^0(y^0, \vec{n}) \]  

(41)

\( \tilde{S} \) is obtained from \( S \) by the following substitution in \( \Pi_{0i}(y^0, \vec{n}) \):

\[ U_i(y^0, \vec{n}) \rightarrow U_i(y^0, \vec{n}) e^{i\alpha \cdot b_i^n (\vec{n} - \vec{y})} = U_i'(y^0, \vec{n}) \]  

(42)

where \( b_i^\perp \) is the transverse component of the vector potential generating at \( \vec{n} \) a monopole sitting at \( \vec{y} \), \( \partial_i b_i^\perp = 0 \). Classical gauge ambiguities in \( b_i \) are contained in the longitudinal part, and do not affect the definition Eq. (42).

The plaquette \( \Pi_{0i}(y^0, \vec{n}) \) gets transformed to \( \Pi_{0i}'(y^0, \vec{n}) \)

\[ \Pi_{0i}'(y^0, \vec{n}) = U_i(y^0, \vec{n}) e^{i\alpha \cdot b_i^n (\vec{n} - \vec{y})} U_0(y^0, \vec{n} + \hat{i}) U_i^+(y^0 + 1, \vec{n}) U_0^+(y^0, \vec{n}) \]  

(43)
this can be viewed as a change of $U_i^\dagger(y_0 + 1, \vec{n})$:

$$U_i^\dagger(y_0 + 1, \vec{n}) \rightarrow U_i^\dagger(y_0, \vec{n} + \hat{i}) e^{i\alpha b^i_i(\vec{n} - \vec{y})} U_0(y_0, \vec{n} + \hat{i}) U_i^\dagger(y_0 + 1, \vec{n})$$

or

$$U_i(y_0 + 1, \vec{n}) \rightarrow U_i(y_0 + 1, \vec{n}) U_i^\dagger(y_0, \vec{n} + \hat{i}) e^{-i\alpha b^i_i(\vec{n} - \vec{y})} U_0(y_0, \vec{n} + \hat{i})$$

(44)

which is a multiplication of the link variable by an $SU(N)$ matrix and it can be reabsorbed in a change of variables. However $U_i(y_0 + 1, \vec{n})$ also appears in the plaquettes $\Pi_{ij}(y_0 + 1, \vec{n})$ and $\Pi_{0i}(y_0 + 1, \vec{n})$. Up to terms $O(a^2)$, the net effect will be that in $\Pi_{ij}(y_0 + 1, \vec{n})$,

$$\theta^0_{ij} \rightarrow \theta^0_{ij} + \Delta_i b_j - \Delta_j b_i$$

i.e. that a monopole has been added at $y_0 + 1$, and that a change like the one in Eq. (43) is produced at time $y_0 + 1$.

By successive iteration of this procedure one eventually come at a time $y'_0$ where an anti-monopole of type $a$ is situated, and then the procedure stops.

At $T = 0$ the correlator $\langle \bar{\mu}(\vec{x}, t) \mu(\vec{x}, 0) \rangle$ can be measured, and by cluster property

$$\langle \bar{\mu}(\vec{x}, t) \mu(\vec{x}, 0) \rangle \sim_{t \rightarrow \infty} |\langle \mu \rangle|^2 + ce^{-Mt}$$

whence $|\langle \mu \rangle|$ can be extracted.

At $T \neq 0$ $\langle \mu \rangle$ is measured directly, and $C^*$ boundary conditions in time are needed [1,3,4].

By the appropriate choice of $b^i_i(\vec{x} - \vec{y})$ in Eq. (35) a generic number of monopoles and anti-monopoles can be created at time $y_0$.

In numerical simulations it is convenient to measure

$$\rho = \frac{d}{d\beta} \log \langle \mu \rangle$$

which is much less noisy, and in terms of which

$$\langle \mu \rangle = \exp \left( \int_0^\beta \rho(x) dx \right)$$
The statement that $\langle \mu \rangle \neq 0$ for $T < T_c$ in the infinite volume limit corresponds to have $\rho$ volume independent and finite at large volumes. $\langle \mu \rangle = 0$, for $T > T_c$, is obtained if

$$\rho \rightarrow -kL + k' \quad (k > 0)$$

(45)

as the spatial size of the lattice, $L$, diverges. A behavior like Eq. (45) is numerically easy to test, and means that $\langle \mu \rangle$ is strictly zero in the thermodynamical limit. A direct measurement of $\langle \mu \rangle$ would only produce a value which is zero within (large) errors.

4 Conclusions

We have analyzed how to investigate dual superconductivity of the vacuum in the confined phase of $SU(N)$ gauge theories for arbitrary $N$, by deriving in detail the abelian projection, its symmetry properties, and the construction of a disorder parameter. Numerical simulations are in progress. We plan to measure $\langle \mu \rangle$, or better $\rho$, as a function of $\tilde{\beta} = \beta/N^2$ for different values of $N$. If the ideas about $1/N$ apply $\rho N^2$ should be a universal function of $\tilde{\beta}$, at sufficiently large $N$. Also the comparison of the different choices of $\langle \mu \rangle$ and of different abelian projections is of interest, on the way to understand the nature of dual excitations of QCD.

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5 Appendix

We want to show that the general diagonal, $x$ independent solution of Eq. (1) has the form of Eq. (9). Let us call $H_i \ (i = 1 \ldots N - 1)$ the independent generators belonging to the Cartan algebra of the group, $E_\alpha$ the generators belonging to the root $\alpha$.

The Lie algebra reads then

$$[H_i, H_j] = 0 \quad [H_i, E_\alpha] = \alpha_i E_\alpha$$

(46)

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \quad [E_\alpha, E_{-\alpha}] = \alpha_i H_i$$
In the fundamental representation

\[ \text{Tr}\{H_i H_j\} = \frac{1}{2} \delta_{ij} \quad \text{Tr}\{E_{\alpha}^{\dagger} E_{\alpha}\} = \frac{1}{2} \delta_{\alpha\beta} \]

\[ E_{\alpha}^{\dagger} = E_{-\alpha} \quad \text{Tr}\{H_i E_{\alpha}\} = 0 \]

In general for a diagonal \( \Phi \)

\[ \Phi = \sum c_i H_i \quad (47) \]

\[ A_\mu = A_i^\mu H_i + A_\mu^\alpha E_\alpha \quad (48) \]

We want to solve Eq. (9)

\[ X_2(A_\mu, \phi) \equiv \text{Tr} (\phi [A_\mu, A_\nu]) + \text{Tr} (\phi [[A_\mu, \phi], [A_\nu, \phi]]) = 0 \quad (49) \]

Using the multiplication law of the algebra, Eq. (46), and Eq. (48)

\[ [A_\mu, A_\nu] = A_\mu^\alpha A_\nu^\beta [E_\alpha, E_\beta] + A_\mu^\beta A_\nu^\alpha E_\alpha - A_\mu^\alpha A_\nu^\beta E_\alpha \quad (50) \]

The only contribution to \( X_2 \) comes from the first term of Eq. (49) and gives

\[ \frac{1}{2} c_i \alpha_i A_\mu^\alpha A_\nu^{-\alpha} \]

As for the second term of Eq. (42), since

\[ [\hat{\Phi}, A_\mu] = c_i A_\mu^\alpha [H_i, E_\alpha] = c_i \alpha_i A_\mu^\alpha E_\alpha \]

one has:

\[ [[\hat{\Phi}, A_\mu], [\hat{\Phi}, A_\nu]] = A_\mu^\alpha A_\nu^\beta c_i \alpha_i c_j \beta_j [E_\alpha, E_\beta] \]

and the second contribution to \( X_2 \) is

\[ -\frac{1}{2} A_\mu^\alpha A_\nu^{-\alpha} (c_i \alpha_i)^3 \]

In summary \( X_2 = 0 \) is equivalent to

\[ c_i \alpha_i = (c_i \alpha_i)^3 \quad \forall \alpha \quad (51) \]
By gauge transformations we can order the eigenvalues of $\Phi$ in decreasing order

$$\Phi = \text{diag}(\varphi_1 \ldots \varphi_n) \quad \varphi_1 \geq \varphi_2 \geq \ldots \varphi_n$$ (52)

It is easy to find $N-1$ independent solutions of Eq. (51), $\Phi^q = 2\mu^q$ where $\mu^q$ are the fundamental weights

$$\mu^q = \frac{1}{2} \begin{pmatrix} \frac{q}{N}, \frac{p}{N}, \ldots, \frac{p}{N}, -\frac{q}{N}, -\frac{q}{N}, \ldots, -\frac{q}{N} \end{pmatrix}$$ (53)

with $q = 1, 2, \ldots, N-1$.

Indeed for the simple roots

$$2\text{Tr} \{\mu^a \alpha^b\} = \delta^{ab}$$

A generic positive root $\alpha$ is the sum of simple roots, so that

$$2\text{Tr} \{\mu^a \alpha\} = 1 \text{ if } \alpha \text{ contains } \alpha^a$$

$$2\text{Tr} \{\mu^a \alpha\} = 0 \text{ if } \alpha \text{ does not contain } \alpha^a$$

The sign changes for negative roots. The $\mu$'s form a complete set for diagonal operators. The general solution will be of the form

$$\Phi = 2 \sum_A c^A \mu^A$$

$$2\text{Tr}(\hat{\Phi} \alpha^A) = c^A = 0, \pm 1$$

it is easy to see from Eq. (53) that

$$\Phi_1 - \Phi_2 = \frac{c_1}{N}$$

$$\Phi_2 - \Phi_3 = \frac{c_2}{N}$$

$$\ldots$$

Since the eigenvalues are ordered all the coefficients $c^A$ must be non negative. if two $c$'s were different from zero, $c^i, c^j$, then for the root $\alpha = \alpha^i + \alpha^{i+1} + \ldots + \alpha^j$ we would have for the scalar product

$$2\text{Tr} \{\Phi \alpha\} = 2$$
which is impossible. Therefore only one \( c \) can be different from zero and \( \Phi^a \) provides the generic solution of Eq. (9).

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