Lee-Yang edge singularity in the three-dimensional Gross-Neveu model at finite temperature

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Abstract

We discuss the relevance of the Lee-Yang edge singularity to the finite-temperature $Z_2$-symmetry restoration transition of the Gross-Neveu model in three dimensions. We present an explicit result for its large-$N$ free-energy density in terms of $\zeta(3)$ and the absolute maximum of Clausen’s function.

The Gross-Neveu model in $d = 3$ dimensions provides a remarkable example of a second order temperature driven phase transition in a theory which also exhibits dynamical symmetry breaking. The latter property is purely quantum field theoretical while the former one involves classical thermal fluctuations. We consider here the standard Lagrangian

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describing the Euclidean version of the Gross-Neveu model with $U(2N)$ symmetry [1–3]

$$\mathcal{L} = \bar{\psi}^a \partial \psi^a + \frac{G_0}{2N} (\bar{\psi}^a \psi^a)^2, \quad a = 1, 2, \ldots, N,$$

(1)

where $G_0$ is the coupling. The partition function for the theory can be written, after integrating out the fundamental four-component massless Dirac fermions $\bar{\psi}^a, \psi^a$, with the help of the auxiliary scalar field $\sigma(x)$ as [1–3]

$$Z_\sigma[G_0] = \int (D\sigma) e^{N \left[ 2 \text{Tr} \left[ \ln(-\partial^2 + \sigma^2) \right] - \frac{1}{2G_0} \int d^3x \sigma^2(x) \right]}.$$  

(2)

The model possesses a discrete $Z_2$ “chiral” symmetry as (1) is invariant under $\psi \rightarrow \gamma_5 \psi$. The usual $1/N$ expansion is generated if one expands as $\sigma(x) = \sigma_0 + O(1/\sqrt{N})$, provided that $\sigma_0$ satisfies the gap equation

$$\frac{\sigma_0}{G_0} = \int \frac{d^3p}{(2\pi)^3} \frac{4\sigma_0}{p^2 + \sigma_0^2}.$$  

(3)

One renormalizes (3) by introducing an UV cut-off $\Lambda$ as

$$\frac{1}{G_0} = \frac{1}{G_*} + \frac{1}{G_R} = 4 \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + \sigma_0^2},$$  

(4)

and obtains the renormalized coupling $1/G_R = -M/\pi$, where $M$ is the arbitrary mass scale introduced by renormalization. From (2) and (4) it is easy to calculate the leading-$N$ renormalized “effective action” $V_R(\sigma_0; G_R)$ defined as

$$\int d^3x V_R(\sigma_0; G_R) = -\ln \left[ \frac{Z_{\sigma_0}[G_R]}{Z_0[0]} \right],$$  

(5)

where the subtraction on the r.h.s. of (5) ensures a finite result. The result is [2]

$$V_R(\sigma_0, G_R) = \frac{N}{2\pi} \left( \frac{2}{3} |\sigma_0|^3 - M\sigma_0^2 \right).$$  

(6)

From (6) we can clearly separate three regimes:  

A) For $M < 0$ the minimum of (6) is always at the origin and the theory is in the $Z_2$-symmetric phase with $\sigma_0 = 0$.  

B) For $M > 0$ the minimum of (6) is at $\sigma_0 = M$, the theory is in the $Z_2$-broken phase and $M$ can be identified as the mass of the elementary fermionic fields.  

C) Finally, for $M = 0$,
the theory is at the critical point and it is a non-trivial three-dimensional conformal field theory (CFT).

Notice that the UV subtraction prescription in (5) has generated the $|\sigma_0|^3$ term in (6). This term manifests itself as the dominant contribution at the critical point $M = 0$. It is then conceivable that, to leading-$N$, the critical behavior of the model is somehow related to a $\phi^3$ theory. Of course, the true critical ground state is at $\sigma_0 = 0$ which would correspond to the zero-coupling critical point of the $\phi^3$ theory, or equivalently to a free-field theory. This is consistent with the well known mean field theory behavior, to leading-$N$, of all the critical quantities of the model [2].

On the other hand, it is well known [4, 5] that the IR limit of the theory with action

$$S = -\int \frac{1}{2} (\partial \phi)^2 + i(h - h_c)\phi + \frac{1}{3!}\lambda \phi^3 \right] d^d x,$$

(7)

dictates the critical behavior of an Ising model in a purely imaginary magnetic field $ih_c$ as the critical temperature is approached from above (from the symmetric phase). This critical point (Lee-Yang edge singularity [6]) corresponds to a non-unitary theory as it involves an imaginary coupling constant $\lambda$.

If the critical behavior of the Gross-Neveu model is in any way related to a $\phi^3$ theory, one would expect that the Lee-Yang singularity might become relevant as one approaches the critical point of the model from a suitable symmetric phase. To investigate such a possibility we introduce the ingredient of temperature $T$ by putting the model (2) in a slab geometry with one finite dimension of length $L = 1/T$. A crucial point is that the renormalization (4) is unaffected, since renormalizing the theory in the bulk suffices to remove the UV-divergences for finite temperature [1]. However, the gap equation (3) now becomes

$$\frac{\sigma_0}{G_0} = \frac{4\sigma_0}{L} \sum_{n=0}^{\infty} \int_{\Lambda} \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + \omega_n^2 + \sigma_0^2} + \frac{2\sigma_0^2}{\pi L} \int_{\Lambda} \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + \omega_n^2 + \sigma_0^2} - \frac{2\sigma_0}{\pi L} \ln \left(1 + e^{-L\sigma_0}\right).$$

(8)

Then, from (2) and (8) we can explicitly calculate the leading-$N$ renormalized “effective
action” \( V_R(\sigma_0, L; G_R) \) - now depending in addition on the “inverse” temperature \( L \) - as

\[
V_R(\sigma_0, L; G_R) = \frac{N}{2\pi L^3} \left[ \frac{2}{3} \sigma_0^3 L^3 - M \sigma_0^2 L^3 + 4Li_3 \left( -e^{-L\sigma_0} \right) - 4 \ln \left( e^{-L\sigma_0} \right) Li_2 \left( -e^{-L\sigma_0} \right) \right]
\]  

(9)

where \( Li_n(z) \) are the standard polylogarithms \([7]\). This “effective action” presents a remarkably explicit example of high-temperature symmetry restoration in a \textit{quantum} field theoretic system\(^3\) which is \textit{ordered} \((M > 0)\) at \( T = 0 \). The critical temperature is \( 1/L_c = T_c = M/2 \ln 2 \) \([2]\).

When \( M = 0 \) in (9), then for all \( T > 0 \) we approach the critical point from the symmetric phase. This is the regime where we would expect the appearance of the Lee-Yang singularity. This seems rather difficult to imagine as, despite the appearance in (9) of the cubic term \( \sigma_0^3 \) as a result of the UV subtraction prescription (5), the coefficient of this term is real. Nevertheless, one can show that (9) for \( M = 0 \) is in fact an \textit{even} function of \( \sigma_0 \). To see this we express (9) in terms of Nielsen’s generalized polylogarithms \( S_{n,p}(z) \) \([8]\) as follows

\[
V_R(\sigma_0, L; 0) = \frac{2N}{\pi L^3} \left[ S_{1,2}(z) + S_{1,2} \left( \frac{1}{z} \right) - \zeta(3) \right]
\]  

(10)

\[
S_{1,2}(z) = \frac{1}{2} \int_0^z \ln^2(1 - y) \, dy
\]  

(11)

\[
S_{1,2}(1) = 8 S_{1,2}(-1) = \zeta(3)
\]  

(12)

where we have set \( z = -e^{-L\sigma_0} \). From (10) we see that \( V_R(-\sigma_0, L; 0) = V_R(\sigma_0, L; 0) \). This remarkable property means that although the \( L \to \infty \) \((T \to 0)\) behavior of (9) looks like its is dominated by the cubic term (with real coefficient and ground state at \( \sigma_0 = 0 \)), fluctuations become important for all \( T > 0 \) and completely change the relevant underlying effective potential. To this end we point out that the step from (9) to (10) involves an all order resummation in \( \sigma_0 \), drive the theory towards another critical point.

\(^3\)Notice that although we are dealing with symmetry restoration in two dimension, the Mermin-Wanger-Coleman theorem is not violated as the relevant symmetry is discrete \((\mathbb{Z}_2 \text{ here})\).
From (10). Then, from (10) we conclude that away from \( \sigma_0 = 0 \) the critical theory is described by an effective Hamiltonian which is an even function of \( \sigma_0 \). If we view now \( \sigma_0 \) as a scalar order parameter and couple it to an external magnetic field, the critical behavior of such a system in the high temperature phase can be shown to correspond to a \( \phi^3 \) theory with purely imaginary coupling [4]. The critical point is determined by the non-zero solution of the gap equation (8) as

\[
\sigma_0 \left[ \sigma_0 + \frac{2}{L} \ln \left( 1 + e^{-\sigma_0 L} \right) \right] = 0 \Rightarrow \sigma_0 = \pm \frac{2\pi}{3L} , \tag{13}
\]

where we restricted \(-i\pi < L\sigma_0 < i\pi\) to avoid the cut of the logarithm. The fact that \( \sigma_0 \) is now purely imaginary, however, does not affect the reality properties of the effective potential and we obtain

\[
V_R(\pm \frac{2\pi}{3L}, L; 0) = \frac{N}{2\pi L^3} \left[ \frac{4}{3} \zeta(3) - \frac{8\pi}{3} \text{Cl}_2 \left( \frac{\pi}{3} \right) \right], \tag{14}
\]

where \( \text{Cl}_2(\theta) = \text{Im} \left[ Li_2(e^{i\theta}) \right] \) is Clausen’s function [7]. It is amusing to point out that \( \text{Cl}_2(\pi/3) \approx 1.014942\ldots \) is the absolute maximum of Clausen’s function which is a well-documented numerical constant.

Our result (14) corresponds to the leading-\( N \) free-energy density of the Lee-Yang edge singularity in \( d = 3 \). The parameter \( N \) should not be confused with the number of components of the underlying order parameter [12], but should be regarded as a suitable expansion parameter such that (14) is the leading approximation to the exact value of the free-energy density. Moreover, our result (14) corresponds to a new CFT in three-dimensions. Indeed, on general grounds [9,10] one expects that the free-energy density of a CFT placed in a slab geometry with one finite dimension of length \( L \) behaves as

\[
f_L - f_\infty = -\tilde{c} \Gamma(d/2)\zeta(d)/\pi^{d/2}L^d . \]

In \( d = 2 \) the parameter \( \tilde{c} \) is proportional to the central charge and the conformal anomaly [11]. However, corresponding results in \( d > 2 \) are still unknown. In \( d = 3 \) one easily obtains \( \tilde{c} = 3N \) for the case of \( N \) free massless four-component Dirac fermions [3]. The value of \( \tilde{c} \) for the Lee-Yang edge singularity which can be read-off from (14) is larger than \( 3N \), implying that the corresponding CFT is non-unitary. This is in accordance with the two-dimensional results [5].
In may cause some worry that we have connected the critical behavior of a unitary theory (Gross-Neveu) with a non-unitary one. Nevertheless, this is not a direct connection. Staying within the Gross-Neveu model and starting e.g. from the low temperature broken phase, we do not expect the appearance of the Lee-Yang critical behavior studied above. Namely, as we raise the temperature we simply expect that the $Z_2$ symmetry is restored at the critical temperature $T_c = M/2 \ln 2$ and then the system continues to be in the high-temperature symmetric temperature phase for all $T > T_c$. However, if we consider the Gross-Neveu model as a component of some enlarged theory, it is quite conceivable that the presence of other fields (e.g. gauge fields), or chemical potentials might account for a possible Lee-Yang critical behavior at $T > T_c$ as they could induce imaginary values for the minimum of the effective potential $\sigma_0$ [13]. Clearly, the enlarged system should still be described by a unitary theory. From this point of view, we expect our approach and results to be most suitable for discussing effects such as the recently studied symmetry nonrestoration [14], since the latter is related to an imaginary chemical potential. Our leading-$N$ calculations reproduce the well-known mean field theory results for the Lee-Yang edge singularity critical exponents. It would then be interesting to extend our results to next-to-leading order in $1/N$ for comparison with existing numerical calculations [15].

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