Noncommutative-Geometry Wormholes

Without Exotic Matter

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Abstract

A fundamental property of Morris-Thorne wormholes is the so-called flare-out condition that automatically results in a violation of the null energy condition (NEC) in classical general relativity. By contrast, in $f(R)$ modified gravity, the material threading the wormhole may actually satisfy the NEC, while, at the same time, the effective stress-energy tensor arising from the modified theory violates the NEC, thereby sustaining the wormhole. It is shown in this paper that noncommutative geometry, an offshoot of string theory, can be viewed as a special case of $f(R)$ gravity and can therefore sustain a wormhole without the use of exotic matter. It also provides a motivation for the choice of $f(R)$ in the modified gravitational theory.

Keywords: wormholes, noncommutative geometry, $f(R)$ modified gravity

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1 Introduction

Wormholes are tunnel-like structures in spacetime that link widely separated regions of our Universe or different universes altogether [1]. This spacetime geometry can be described by the metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

(1)
using units in which \( c = G = 1 \). In this line element, \( b = b(r) \) is called the *shape function* and \( \Phi = \Phi(r) \) is called the *redshift function*; the latter must be finite everywhere to avoid the presence of an event horizon. For the shape function we must have \( b(r_0) = r_0 \), where \( r = r_0 \) is the radius of the *throat* of the wormhole. Another important requirement is the *flare-out condition* at the throat: \( b'(r_0) < 1 \); also, \( b(r) < r \) near the throat. In classical general relativity, the flare-out condition can only be satisfied by violating the null energy condition (NEC):

\[
T_{\mu\nu}k^\mu k^\nu < 0
\]  

for all null vectors \( k^\mu \), where \( T_{\mu\nu} \) is the stress-energy tensor. In particular, for the outgoing null vector \((1, 1, 0, 0)\), the violation becomes

\[
T_{\mu\nu}k^\mu k^\nu = \rho + p_r < 0.
\]

Here \( T^t_t = -\rho \) is the energy density, \( T^r_r = p_r \) is the radial pressure, and \( T^{\theta}_\theta = T^{\phi}_\phi = p_t \) is the lateral pressure.

According to Lobo [2], these ideas can be extended to \( f(R) \) modified gravity by referring back to the Raychaudhury equation. In this theory, the Ricci scalar \( R \) in the Einstein-Hilbert action

\[
S_{\text{EH}} = \int \sqrt{-g} R \, d^4x
\]

is replaced by a nonlinear function \( f(R) \):

\[
S_{f(R)} = \int \sqrt{-g} f(R) \, d^4x.
\]

To extend these ideas, the stress-energy tensor \( T_{\mu\nu} \) has to be replaced by \( T_{\mu\nu}^{\text{eff}} \), the *effective* stress-energy tensor arising from the modified theory, leading to the Einstein field equations \( G_{\mu\nu} = \kappa^2 T_{\mu\nu}^{\text{eff}} \). The NEC now becomes

\[
T^{\text{eff}}_{\mu\nu}k^\mu k^\nu \geq 0.
\]

As a result, the violation of the (generalized) NEC becomes \( T^{\text{eff}}_{\mu\nu}k^\mu k^\nu < 0 \), which reduces to \( T_{\mu\nu}^{\text{eff}}k^\mu k^\nu < 0 \) in classical general relativity. According to Ref. [2], it now becomes possible in principle to allow the matter threading the wormhole to satisfy the NEC while retaining the violation of the generalized NEC, i.e., \( T^{\text{eff}}_{\mu\nu}k^\mu k^\nu < 0 \). So the necessary condition for maintaining a traversable wormhole has been met. According to Ref. [2], the higher-order curvature terms leading to the violation may be interpreted as a gravitational fluid that supports the wormhole.

The purpose of this paper is to show that noncommutative geometry, an offshoot of string theory, is not only an example of such a modified gravitational theory, it provides a motivation for the choice of the function \( f(R) \).
2 Noncommutative geometry

An important outcome of string theory is the realization that coordinates may become noncommutative operators on a $D$-brane $[3, 4]$. Noncommutativity replaces point-like objects by smeared objects $[5, 6, 7]$ with the aim of eliminating the divergences that normally occur in general relativity. Moreover, noncommutative geometry results in a fundamental discretization of spacetime due to the commutator $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix.

An effective way to model the smearing is to assume that the energy density of a static, spherically symmetric, and particle-like gravitational source has the form $[8, 9]$

$$\rho(r) = \frac{\mu\sqrt{\beta}}{\pi^2(r^2 + \beta)^2}. \quad (5)$$

Here the mass $\mu$ is diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty. Noncommutative geometry is an intrinsic property of spacetime and does not depend on any particular features such as curvature. Eq. (5) immediately yields the mass distribution

$$m_\beta(r) = \int_0^r 4\pi(r')^2 \rho(r')dr' = \frac{2M}{\pi} \left( \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r\sqrt{\beta}}{r^2 + \beta} \right), \quad (6)$$

where $M$ is now the total mass of the source.

3 Wormholes in modified gravity

In this section we adopt the point of view that noncommutative geometry is a modified gravity theory, but first we make the important observation that the Einstein field equations $G_{\mu\nu} = \kappa^2 T^\text{eff}_{\mu\nu}$ mentioned in Sec. 1 show that the noncommutative effects can be implemented by modifying only the stress-energy tensor, while leaving the Einstein tensor unchanged. As a result, the length scales can be macroscopic.

The next step is to show that our noncommutative-geometry background is a special case of $f(R)$ modified gravity. To that end, we need to list the gravitational field equations in the form used by Lobo and Oliveira [10]. Here we assume that $\Phi'(r) \equiv 0$; otherwise, according to Ref. [10], the analysis becomes intractable. (It is also assumed that for notational convenience, $\kappa = 1$ in the field equations.)

$$\rho(r) = F(r) \frac{b'(r)}{r^2}, \quad (7)$$

$$p_r(r) = -F(r) \frac{b(r)}{r^3} + F'(r) \frac{r b'(r) - b(r)}{2r^2} - F''(r) \left( 1 - \frac{b(r)}{r} \right), \quad (8)$$
and
\[ p_t(r) = -\frac{F'(r)}{r} \left( 1 - \frac{b(r)}{r} \right) + \frac{F(r)}{2r^3} \left[ b(r) - r b'(r) \right], \quad (9) \]
where \( F = \frac{df}{dR} \). If \( F(r) \equiv 1 \), then Eqs. (7) - (9) reduce to the usual field equations with \( \kappa = 1 \) and \( \Phi'(r) \equiv 0 \). So from \( \rho(r) = b'(r)/r^2 \) and Eq. (5), we obtain the shape function
\[ b(r) = \frac{M \sqrt{\beta}}{2\pi^2} \left( \frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r}{r^2 + \beta} - \frac{1}{\sqrt{\beta}} \tan^{-1} \frac{r_0}{\sqrt{\beta}} + \frac{r_0}{r_0^2 + \beta} \right) + r_0, \quad (10) \]
where \( M \) is now the mass of the wormhole, and from
\[ b'(r) = \frac{M \sqrt{\beta}}{\pi^2} \frac{r^2}{(r^2 + \beta)^2}, \quad (11) \]
we see that \( b'(r_0) < 1 \); the flare-out condition is thereby met.
According to Ref. [10], the Ricci scalar is
\[ R(r) = \frac{2b'(r)}{r^2}. \quad (12) \]

4 Avoiding exotic matter

Since we wish the matter threading the wormhole to obey the null energy condition, we require that \( \rho + p_r \geq 0 \); we also assume that \( \rho \geq 0 \). From Eqs. (7) and (8), we therefore need to satisfy the following conditions:
\[ \rho = \frac{Fb'}{r^2} \geq 0 \quad (13) \]
and
\[ \rho + p_r = \frac{(2F + rF')(b'r - b)}{2r^3} - F'' \left( 1 - \frac{b}{r} \right) \geq 0. \quad (14) \]
Using Eqs. (7) and (5), and replacing \( b'(r) \) by \( k \), we get
\[ F(r) = r \frac{\rho(r)}{2b'(r)} = \frac{1}{2b'(r)} \frac{\mu \sqrt{\beta}}{\pi^2} \frac{1}{(r^2 + \beta)^2}. \quad (15) \]
Eq. (12) now implies that
\[ r(R) = \sqrt{\frac{2b'}{R}}. \quad (16) \]
Substituting in Eq. (15) yields
\[ F(R) = \frac{\mu \sqrt{\beta}}{\pi^2} \frac{1}{R \left( \frac{2b'}{R} + \beta \right)^2}. \quad (17) \]
and

$$F'(R) = \frac{-2\sqrt{3}}{\pi^2} \frac{2b' + 2\beta R}{(2b' + \beta R^2)^3}. \quad (18)$$

Inequality (13) is evidently satisfied. Substituting in Inequality (14), we obtain (at or near the throat)

$$\rho + p_r = \frac{(2F + rF')(b'r - b)}{2r^3} = \frac{1}{r^3} \frac{\mu \sqrt{3}}{\pi^2} \frac{1}{(2b' + \beta R^2)^2} \left( 1 - \frac{r}{R} \frac{2b' + 2\beta R}{2b' + \beta R} \right) (b'r - b). \quad (19)$$

To show that \( \rho + p_r \mid_{r=r_0} > 0 \), recall from Eq. (11) that \( b'(r_0) \ll 1 \) since \( \beta \) is extremely small. So

$$\frac{r}{R} \mid_{r=r_0} = \frac{r_0^3}{2b'(r_0)} > 1, \quad (20)$$

as long as the throat size is not microscopic, i.e., \( r_0 > [2b'(r_0)]^{1/3} \). Since we also have \( (2b' + 2\beta R)(2b' + \beta R) > 1 \), it now follows that

$$\rho + p_r \mid_{r=r_0} > 0. \quad (21)$$

So the NEC is satisfied at the throat. Since \( F = \frac{df}{dR} \), Eq. (17) also yields

$$f(R) = \int_0^R \left( \frac{\mu \sqrt{3}}{\pi^2} \frac{1}{R'(2b' + \beta)} \right) dR' = \frac{\mu \sqrt{3}}{\pi^2} \frac{1}{R(2b' + \beta)} \ln \left( \frac{\beta R + 2b'}{\beta R + 2b'} \right) + C. \quad (22)$$

To check the violation of the generalized NEC, i.e., \( T^\mu_\nu k^\mu k^\nu < 0 \), we follow Lobo and Oliveira [10]:

$$\rho^\text{eff} + p^\text{eff}_r \mid_{r=r_0} = \frac{1}{F} \frac{r^b' - b}{r^3} + \frac{1}{F} \left( 1 - \frac{b}{r} \right) \left( F'' - F' \frac{b'r - b}{2r^2(1 - b/r)} \right) \mid_{r=r_0}$$

$$= \frac{1}{F} \frac{b'(r_0) - 1}{r_0^2} + \frac{1 - b'(r_0)}{2r_0} \frac{F'}{F}. \quad (23)$$

Since \( F'(R) < 0 \), it follows that

$$\rho^\text{eff} + p^\text{eff}_r \mid_{r=r_0} < 0. \quad (24)$$

So the generalized NEC is violated thanks to the stress-energy tensor \( T^\text{eff}_\mu_\nu \).

5 The high radial tension

Although not part of this study, noncommutative geometry plays another important role in wormhole physics. According to Ref. [1], for a moderately-sized
wormhole, the radial tension at the throat has the same magnitude as the pressure at the center of a massive neutron star. Attributing this outcome to exotic matter is rather problematical since exotic matter was introduced primarily to ensure the violation of the null energy condition.

It is shown in Ref. [11] that such an outcome can be accounted for by the noncommutative geometry background. Recalling that noncommutative geometry is an offshoot of string theory, this approach can be viewed as a foray into quantum gravity.

6 Conclusion

A fundamental geometric property of traversable wormholes is the flare-out condition $b'(r_0) < 1$. In classical general relativity, the flare-out condition can only be met by violating the NEC, $T_{\mu\nu} k^\mu k^\nu < 0$, for all null vectors $k^\mu$. In $f(R)$ modified gravity, the stress-energy tensor $T_{\mu\nu}$ is replaced by the effective stress-energy tensor $T_{\mu\nu}^{\text{eff}}$ arising from the modified theory. So it is possible in principle to have a violation of the generalized NEC, $T_{\mu\nu}^{\text{eff}} k^\mu k^\nu < 0$, while maintaining the NEC, $T_{\mu\nu} k^\mu k^\nu \geq 0$, for the material threading the wormhole. According to Ref. [2], the higher-order curvature terms leading to the violation may be interpreted as a gravitational fluid that supports the wormhole.

The purpose of this paper is to show that noncommutative geometry, an offshoot of string theory, can be viewed as a special case of $f(R)$ modified gravity, where $f(R)$ is given by Eq. (22). The result is a zero-tidal force traversable wormhole without exotic matter. The noncommutative-geometry background also provides a motivation for the choice of $f(R)$.

Sec. 5 reiterates another aspect on noncommutative geometry, the ability to account for the enormous radial tension in a Morris-Thorne wormhole, as shown in Ref. [11].

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