STUDIES OF THERMALLY UNSTABLE ACCRETION DISKS AROUND BLACK HOLES WITH ADAPTIVE PSEUDOSPECTRAL DOMAIN DECOMPOSITION. I. LIMIT-CYCLE BEHAVIOR IN THE CASE OF MODERATE VISCOSITY

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ABSTRACT

We present a numerical method for spatially 1.5-dimensional, time-dependent studies of accretion disks around black holes. The method originates from a combination of the standard pseudospectral and adaptive domain decomposition methods found in the literature, but with a number of improvements in both the numerical and physical senses. In particular, we introduce a new treatment for the connection at the interfaces of decomposed subdomains, construct an adaptive function for the mapping between the Chebyshev-Gauss-Lobatto collocation points and the physical collocation points in each subdomain, and modify the oversimplified one-dimensional basic equations of accretion flows to account for the effects of viscous stresses in both the azimuthal and radial directions. Our method is verified by reproducing the best results obtained previously by Szuszkiewicz & Miller on the limit-cycle behavior of thermally unstable accretion disks with moderate viscosity. A new finding is that, according to our computations, the Bernoulli function of the matter in such disks is always and everywhere negative, so that outflows are unlikely to originate from these disks. We are encouraged to study the more difficult case of thermally unstable accretion disks with strong viscosity in ongoing work.

Subject headings: accretion, accretion disks — black hole physics — hydrodynamics — instabilities

1. INTRODUCTION

The radiation pressure–supported inner regions of geometrically thin, optically thick Shakura-Sunyaev accretion disks (SSDs) around black holes (Shakura & Sunyaev 1973) are known to be thermally unstable (e.g., Kato et al. 1998, p. 155), but the occurrence of an instability does not necessarily mean that the disk will be disrupted after the characteristic growth time. A possible fate of the thermally unstable inner region of an SSD is so-called limit-cycle behavior, that is, nonlinear oscillation between two stable states. Similar to the case of dwarf novae, limit-cycle behavior was predicted from a local, steady state analysis (e.g., Kato et al. 1998, p. 176), that is, from the appearance of an $S$-shaped sequence of steady state solutions at a certain radius in the $\dot{M}-\Sigma$ (mass accretion rate vs. surface density) plane, with the lower and middle branches of the sequence corresponding to stable, gas pressure–supported SSD solutions and unstable, radiation pressure–supported SSD solutions, respectively, and the upper branch corresponding to the stable slim-disk solutions constructed by Abramowicz et al. (1988). This has been justified by a number of global, time-dependent numerical computations (Honma et al. 1991; Szuszkiewicz & Miller 1997, 1998, 2001; Teresi et al. 2004a, 2004b; Mayer & Pringle 2006). Unlike dwarf novae, however, only one astrophysical object, the Galactic microquasar GRS 1915+105, is known to exhibit the theoretically predicted limit-cycle luminosity variations (Nayakshin et al. 2000; Janiuk et al. 2002; Watarai & Mineshige 2003; Ohsuga 2006; Kawata et al. 2006).

Here we select the paper of Szuszkiewicz & Miller (2001, hereafter SM01) to represent existing theoretical work on the limit-cycle behavior of black hole accretion disks, for two reasons: First, SM01 adopted a diffusion-type prescription for viscosity; that is, the $r\phi$-component of the viscous stress tensor is expressed as

$$\tau_{r\phi} = \alpha H c_s \rho \frac{\partial \Omega}{\partial r},$$

where $\rho$ is the density, $\Omega$ is the angular velocity, $c_s$ is the sound speed, $H$ is the half-thickness of the disk, and $\alpha$ is a dimensionless constant parameter. All other relevant works have used the simple prescription

$$\tau_{r\phi} = -\alpha p,$$

where $p$ is the pressure and $\alpha$ is again a dimensionless constant, but with a different scaling [denoting $\alpha$ in eqs. (1) and (2) as $\alpha_1$ and $\alpha_2$, respectively, $\alpha_2 = (3\sqrt{6}/2)\alpha_1$]. It is known that direct integration of the differential equations describing transonic accretion disks with the diffusive form of viscosity is extremely difficult, while the $\alpha p$ viscosity prescription makes the task much easier (see the discussion in SM01). It should be noted, however, that equation (2) is only an approximation to equation (1) under a number of conditions (including assuming that the disk is stationary, geometrically thin, and in Newtonian Keplerian rotation and vertical hydrostatic equilibrium; e.g., Kato et al. 1998, p. 82). More seriously, as shown recently by Becker & Subramanian (2005), equation (1) is the only description proposed so far that is physically consistent close to the black hole event horizon, because of its diffusive nature, whereas equation (2), as well as some other viscosity prescriptions, implies an unphysical structure in the inner regions of black hole accretion disks. Our second reason is that SM01 carried out complete, very nice numerical computations; all the curves in their figures showing the evolution of disk structure are perfectly continuous and well resolved on the grid, whereas some fluctuations appear on the curves in the figures of other relevant works, which might make one worry whether some hidden numerical instabilities remained in the relevant code.

As evidenced by SM01, thermally unstable accretion disks undergo limit cycles when viscosity is moderate—that is, when the viscosity parameter $\alpha \approx 0.1$ (hereafter all numerical values of $\alpha$ are for $\alpha_2$ unless otherwise specified)—and the instability...
seems to be catastrophic when viscosity is weak, \( \alpha \approx 0.001 \). On the other hand, in the case of very strong viscosity, \( \alpha \approx 1 \), Chen et al. (1995) found that the S-shaped sequence of steady state solutions in the \( M-\Sigma \) plane does not form; instead, slim-disk solutions and optically thin advection-dominated accretion flow (ADAF) solutions (Narayan \\& Yi 1994; Abramowicz et al. 1995) combine into a single straight line. Accordingly, Takeuchi \\& Mineshige (1998) performed temporal evolution computations using the \( \alpha \) viscosity prescription with \( \alpha = 1 \) and proposed another possible fate of thermally unstable accretion disks: the very inner region of the disk could finally become an ADAF-like feature while the outer region retains the SSD state, forming a persistent two-phase structure. While this result is very interesting, since a phenomenological SSD+ADAF model has been quite successfully applied to black hole X-ray binaries and galactic nuclei (e.g., Narayan et al. 1998), SM01 stated that they could not make computations for \( \alpha = 1 \) because of difficulties keeping their code numerically stable, and they pointed out that it is worth considering whether the persistent ADAF feature obtained by Takeuchi \\& Mineshige (1998) would survive a change of the viscosity prescription to the diffusive form.

Our purpose here is to study thermally unstable accretion disks that are not disrupted by instabilities; that is, we wish to check whether the limit-cycle behavior is the only possible fate of these disks provided that viscosity is not too weak, or if a transition from the SSD state to the ADAF state is an alternative. As in SM01, we adopt the diffusive viscosity prescription of equation (1) and perform spatially 1.5-dimensional, time-dependent computations. But we choose a numerical method different from that of either SM01 or Takeuchi \\& Mineshige (1998), the adaptive pseudospectral domain decomposition method. With this technique, we hope to be able to perform computations for values of \( \alpha \) ranging from \( \sim 0.1 \) to \( \sim 1 \) and to obtain numerical results at the quality level of SM01. In this paper, we describe our numerical algorithm and techniques in detail and present computational results for \( \alpha = 0.1 \) as a test of our algorithm. We defer the results for larger values of \( \alpha \) to a subsequent paper.

2. NUMERICAL ALGORITHM

As the main purpose of this paper, in this section we present a numerical algorithm to solve a partial differential equation (or equations) of the general form

\[
\frac{\partial u(r, t)}{\partial t} = L(u(r, t)), \quad r \in [r_{\text{min}}, r_{\text{max}}],
\]

(3)

where \( u(r, t) \) is a physical quantity that is a function of the independent spatial variable \( r \) (e.g., the radius in a cylindrical coordinate system) and the time \( t \), and \( L \) is a partial differential operator on \( r \) and can be linear or nonlinear.

2.1. Scheme for Spatial Discretization

We first describe the standard Chebyshev pseudospectral method that is used to discretize the spatial differential operator \( L \). This method is explained in several textbooks (Gottlieb \\& Orszag 1983; Canuto et al. 1988; Boyd 2000; Peyret 2002). Recently, Chan et al. (2005, 2006) applied it to studies of astrophysical accretion flows and discussed its advantages.

Concretely, a series with a finite number of terms is used to approximate a physical quantity \( u(r) \) as

\[
u(r_k) = u(g(r_k)) = \sum_{n=0}^{N} \hat{u}_n T_n(r_k) = \sum_{n=0}^{N} \hat{u}_n \cos \left( \frac{nk\pi}{N} \right),
\]

(4)

where \( T_n(r_k) \) is the \( n \)-th order Chebyshev polynomial; \( r_k \) \( (k = 0, 1, 2, \ldots, N) \) represents the Chebyshev-Gauss-Lobatto collocation points and is defined as \( r_k \equiv \cos (k\pi/N) \), with \( N \) being the number of collocation points; \( r_0 = g(r_k) \) is the mapping from the Chebyshev-Gauss-Lobatto collocation points \( r_k \in [-1, 1] \) to the physical collocation points \( r_k \in [r_{\text{min}}, r_{\text{max}}] \) and is a strictly increasing function that satisfies both \( g(-1) = r_{\text{min}} \) and \( g(1) = r_{\text{max}} \); and the \( \hat{u}_n \) are the spectral coefficients and can be calculated from the physical values \( u(r_k) \) with a fast discrete cosine transform (FDCT; Press et al. 1992) \{conversely, if one has \( \hat{u}_n \), then \( u(r_k) \) can be obtained immediately by an inverse FDCT\}.

The radial derivative \( \partial u(r)/\partial r \) is also a function of \( r \) and in principle can similarly be approximated by a series that is obtained by using the chain rule:

\[
\frac{\partial u(r_k)}{\partial r} = \frac{1}{\partial g/\partial r} \frac{\partial u(g(r_k))}{\partial r} = \frac{1}{\partial g/\partial r} \sum_{n=0}^{N} \hat{u}_n T_n(r_k).
\]

(5)

The spectral coefficients \( \hat{u}_n \) can be calculated from \( \hat{u}_n \) with the three-term recursive relation

\[
\hat{u}_0 = 0,
\hat{u}_{n-1} = 2N\hat{u}_n,
\]

\[
c_n \hat{u}_n = \hat{u}_{n+2} + 2(n+1)\hat{u}_{n+1},
\]

(6)

where \( c_0 = 2 \) and \( c_n = 1 \) for \( n = 1, 2, \ldots, N \). Subsequently, \( \partial u(g(r_k))/\partial r \) is calculated from \( \hat{u}_n \) by an inverse FDCT and then substituted into equation (5) to obtain discrete spatial derivatives \( \partial u(r_k)/\partial r \).

To summarize, we define a discretized differential operator \( D \) for the continuous differential operator \( \partial u/\partial r \). This operator carries out the following work: (1) using an FDCT to calculate \( \hat{u}_n \) from \( u(r_k) \); (2) using the three-term recursive relation in equation (6) to obtain \( \hat{u}_n \) from \( \hat{u}_n \); and (3) using an inverse FDCT and equation (5) to obtain \( \partial u(r_k)/\partial r \). Finally, we use \( D \) to construct a discretized operator \( \hat{L} \) to approximate the operator \( L \) in equation (3). For example, if \( \hat{L} = \hat{L}(u_1 \partial u_1 + u_2 \partial u_2) \), where \( \partial \) denotes \( \partial /\partial r \), then \( \hat{L} \) can be constructed as \( \hat{L} = D(u_1 \partial u_1 + u_2 \partial u_2) \).

2.2. Scheme for Time Discretization

We adopt two schemes to perform the time integration; we use a third-order total variation diminishing (TVD) Runge-Kutta scheme (Shu \\& Osher 1988) to integrate the first two time steps, and then we change to a low CPU consumption scheme, the so-called third-order backward differentiation explicit scheme (Peyret 2002, pp. 130–133), to carry out the rest of the computations.

The third-order TVD Runge-Kutta scheme is expressed as

\[
\begin{align*}
u^{(1)} &= u^n + \frac{\Delta t}{2} \hat{L}(u^n), \\
u^{(2)} &= \frac{3}{4} u^n + \frac{1}{4} u^{(1)} + \frac{\Delta t}{2} \hat{L}(u^{(1)}), \\
u^{(n+1)} &= \frac{1}{4} u^n + \frac{3}{4} u^{(2)} + \frac{\Delta t}{2} \hat{L}(u^{(2)}),
\end{align*}
\]

(7)

where \( \Delta t \) is the time step, \( u^n \) and \( u^{(n+1)} \) are the values of the physical quantity \( u \) at the \( n \)-th and \( (n + 1) \)-st time levels, respectively, and \( u^{(1)} \) and \( u^{(2)} \) are two temporary variables.

The third-order backward differentiation explicit scheme can be written as

\[
\frac{1}{\Delta t} \sum_{j=0}^{3} a_j \nu^{n+1-j} = \sum_{j=0}^{2} b_j \hat{L}(u^{n-j}),
\]

(8)
where

\[
\begin{align*}
  a_0 &\equiv 1 + \frac{1}{1 + k_n} + \frac{1}{1 + k_n + k_{n-1}}, \\
  a_1 &\equiv -\frac{k_n(1 + k_n + k_{n-1})}{k_n(k_n + k_{n-1})}, \\
  a_2 &\equiv \frac{1 + k_n + k_{n-1}}{k_n(k_n + k_{n-1})(1 + k_n + k_{n-1})}, \\
  a_3 &\equiv -\frac{1 + k_n}{k_n(k_n + k_{n-1})(1 + k_n + k_{n-1})}, \\
  b_0 &\equiv \frac{(1 + k_n)(1 + k_n + k_{n-1})}{k_n(k_n + k_{n-1})}, \\
  b_1 &\equiv -\frac{1 + k_n + k_{n-1}}{k_n(k_n + k_{n-1})}, \\
  b_2 &\equiv \frac{1 + k_n}{k_n(k_n + k_{n-1})},
\end{align*}
\]

(9)

Here

\[
\begin{align*}
  k_n &\equiv \frac{r^n - r^{n-1}}{\Delta t}, \\
  k_{n-1} &\equiv \frac{r^{n-1} - r^{n-2}}{\Delta t},
\end{align*}
\]

(11)

with \(r^n, r^{n-1}\), and \(r^{n-2}\) being the times of the \(n\)th, \((n - 1)\)st, and \((n - 2)\)nd time levels, respectively.

Of these two time integration schemes, the former costs 3 times the latter’s CPU time per time step, but the latter is not able to start the integration by itself, whereas the former is. Therefore, we combine these two schemes in order to achieve a sufficiently high order of accuracy with minimal consumption of CPU time.

So far, we have fully discretized equation (3). In order to obtain a physically sound and numerically stable solution in a finite domain, it is in addition necessary to impose appropriate boundary conditions and to apply some filtering techniques to overcome the inevitable spurious nonlinear numerical instabilities that appear in the code. We leave the details of these to the Appendix.

### 2.3. Domain Decomposition

The numerical algorithm described in the previous two subsections and the Appendix has been a useful tool for solving partial differential equations and is essentially what was adopted by Chan et al. (2005). However, it turns out that, as we have experienced in our computations, the algorithm is insufficient for resolving so-called stiff problems. These are problems whose solution is characterized by two or more space scales and/or timescales of different orders of magnitude. In the spatial case, common examples in fluid mechanics are boundary-layer, shear-layer, viscous shock, interface, and flame-front problems. In all of these there exists a region (or regions) of small extent (with respect to the global characteristic length) in which the solution exhibits a very large variation (Peyret 2002, p. 298). When the Chebyshev pseudospectral method described in § 2.1 is applied to a stiff problem, the accuracy of the method can be significantly degraded and there may appear spurious oscillations that can lead to nonlinear numerical instabilities or spurious predictions of solution behavior (the so-called Gibbs phenomenon; Gottlieb & Shu 1997). The spectral filtering technique described in the Appendix is not able to completely remove these spurious oscillations, so the solution is still not well resolved, and sometimes the computation can even be destroyed by the growing spurious oscillations. A special method that has been developed to overcome these difficulties is domain decomposition (Bayliss et al. 1995; Peyret 2002). Here we mainly follow Bayliss et al. (1995) in using this method, but with a different technique to connect the decomposed subdomains.

The basic idea of domain decomposition is to divide a wide computational domain into a set of subdomains, so that each subdomain contains at most only one single region of rapid variation (i.e., with a stiff problem), and more grid points are collocated in this region by a special mapping function to enhance the resolution while the total consumption of CPU time is not substantially increased.

In each subdomain, the solution is obtained by taking into account some connection conditions at the interface between the two conjoint subdomains. In general, appropriate connection conditions are the continuity of both the solution and its spatial derivative normal to the interface (Bayliss et al. 1995; Peyret 2002). The continuity of the solution is satisfied naturally, but the continuity of its derivative cannot be achieved directly by the pseudospectral method, because of the use of FDCTs. To see this, let us divide the entire computational domain \(r \in [r_{\text{min}}, r_{\text{max}}]\) into \(M\) subdomains,

\[
S_i \equiv [r_{\text{min}}(i), r_{\text{max}}(i)], \quad i = 1, 2, \ldots, M,
\]

(12)

where \(r_{\text{min}} = r_{\text{min}}(1), r_{\text{max}} = r_{\text{max}}(1), r_{\text{min}}(2), r_{\text{max}} = r_{\text{min}}(3), \ldots, r_{\text{max}}(M) = r_{\text{max}}\) are the locations of the interfaces between the subdomains. Because an FDCT is used to calculate the numerical derivative in each subdomain \(S_i\), one obtains two values of the derivative at each interface. Let \(\partial u / \partial r\) and \(\partial^2 u / \partial r^2\) denote the left and right numerical derivatives of the physical quantity \(u\) at a certain interface, respectively. A seemingly rational choice to maintain the continuity of the derivative is to set the numerical derivative at the interface to be the mean of \(\partial u / \partial r\) and \(\partial^2 u / \partial r^2\), that is,

\[
\left. \frac{\partial u}{\partial r} \right|_{\text{interface}} = \frac{\partial u / \partial r + \partial^2 u / \partial r^2}{2}.
\]

(13)

Unfortunately, in practice this technique will often cause a numerical instability at the interfaces.

We find that the connection between two particular subdomains \(S_i\) and \(S_{i+1}\) can be made numerically stable if their discretizations satisfy an additional practical condition. Let \(r_{\text{int}}(i) = r_{\text{min}}(i+1)\) denote the location of the interface between \(S_i\) and \(S_{i+1}\), and let \(r_{N-1}^{(i)}\) and \(r_{N-1}^{(i+1)}\) \((r_{N-1}^{(i)} \in S_i, r_{N-1}^{(i+1)} \in S_{i+1})\) be the locations of the two nearest points to the interface; our computations show that if the condition

\[
|r_{\text{int}} - r_{N-1}^{(i)}| = |r_{\text{int}} - r_{N-1}^{(i+1)}|
\]

(14)

is satisfied, then the connection of derivatives represented by equation (13) will be numerically stable.

If stiff problems always appeared in a fixed spatial region, then the domain decomposition would be left unchanged. However, in general this is not the case. Instead, the location of the region in which a stiff problem appears changes with time (Bayliss et al. 1995). Therefore, the domain decomposition must be adjusted adaptively. To ensure that the connection condition (eq. [14]) is satisfied at the interfaces of newly divided subdomains, an adjustable mapping between the physical collocation points \(r_k\) in each new subdomain \(S_i\) and the Chebyshev-Gauss-Lobatto
collocation points \( \tilde{r}_k \) is needed. We adopt such a mapping in the form

\[
\rho^{(i)} = g \tilde{r}
\]

\[
= \rho_{\text{max}}^{(i)} + \frac{2}{\pi} \left( \rho_{\text{max}}^{(i)} - \rho_{\text{min}}^{(i)} \right) \arctan \left[ a \tan \left( \frac{\pi (\tilde{r} - 1)}{4} \right) \right],
\]

(15)

where

\[
\tilde{r} \in [-1, 1], \quad r^{(i)} \in S_i
\]

(see eq. [4]), in subdomain \( S_i \). This is a combination of two mapping functions,

\[
r^{(i)} = \frac{r_{\text{max}}^{(i)}}{2}(\tilde{r} + 1) - \frac{r_{\text{min}}^{(i)}}{2}(\tilde{r} - 1),
\]

(16)

\[
\tilde{r} = \frac{4}{\pi} \arctan \left[ a \tan \left( \frac{\pi (\tilde{r} - 1)}{4} \right) \right].
\]

(17)

Equation (16) is a trivial linear mapping (Chan et al. 2005), and equation (17) is the mapping proposed by Bayliss et al. (1995). The parameter \( a \) in equation (17) is adjustable, but this equation is only a mapping from \( \tilde{r} \in [-1, 1] \) to \( \tilde{r} \in [-1, 1] \). Therefore, we add equation (16) in order to make a complete mapping from \( \tilde{r} \in [-1, 1] \) to \( r^{(i)} \in S_i \). The combined mapping (eq. [15]) concentrates the discrete grid points toward \( r_{\text{max}}^{(i)} \) when \( a > 1 \) and toward \( r_{\text{min}}^{(i)} \) when \( a < 1 \), and it reduces to equation (16) when \( a = 1 \).

The adjustability of equation (15) is crucially important for achieving a numerically stable connection at the interfaces of subdomains. By substituting equation (15) into equation (14), we obtain

\[
d^{(i+1)} = \cot (\omega \arctan \{ d^{(i)} \tan \left[ \frac{\pi (\tilde{r}_i - 1)}{4} \right] \})
\]

(18)

with

\[
\omega \equiv \frac{(r_{\text{max}}^{(i)} - r_{\text{min}}^{(i)})}{(d^{(i+1)} - d^{(i)})}, \quad \tilde{r}_i \equiv \cos \left( \frac{\pi N}{N} \right), \quad \tilde{r}_r \equiv \cos \left( \frac{\pi (N - 1) N}{N} \right).
\]

where \( d^{(i)} \) and \( d^{(i+1)} \) are the mapping parameters for subdomains \( S_i \) and \( S_{i+1} \), respectively. Equation (18) can be used to determine the mapping parameters of every subdomain after specifying a decomposition of the computational domain \( S \) and the mapping parameter \( d^{(i)} \) of the innermost subdomain \( S_1 \) (=\( [r_{\text{min}}, r_{\text{max}}^{(i)}] \)). As a result, we obtain a particular collocation of discrete grid points within the whole computational domain \( [r_{\text{min}}, r_{\text{max}}] \). This collocation ensures a stable connection of the derivatives between any two conjoint subdomains (eq. [13]) and thus ensures a correct implementation of the pseudospectral method in each subdomain. The combination of the standard pseudospectral method and adaptive domain decomposition finally provides the name for our numerical algorithm as in the title of this paper.

3. LIMIT-CYCLE SOLUTIONS

We now verify our numerical algorithm by applying it to studies of thermally unstable black hole accretion disks with moderate viscosity and comparing our results with those of the representative work, SM01.

3.1. Basic Equations

We write the basic equations for viscous accretion flows around black holes in Eulerian rather than Lagrangian form as in SM01, because partial differential equations in the Eulerian description take the general form of equation (3), to which our numerical algorithm is suited. The basic equations to be solved are

\[
\frac{\partial \Sigma}{\partial t} = -v_x \frac{\partial \Sigma}{\partial r} - \Sigma \frac{\partial}{\partial r} (\rho \tau_v), \quad (19)
\]

\[
\frac{\partial v_x}{\partial t} = -v_x \frac{\partial v_x}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r^2 \frac{\partial \Sigma}{\partial r}), \quad (20)
\]

\[
\frac{\partial \tau_v}{\partial t} = -v_x \frac{\partial \tau_v}{\partial r} + \frac{\alpha}{r} \frac{\partial}{\partial r} \left( r^3 c_s^2 H \Sigma \frac{\partial \Omega}{\partial r} \right), \quad (21)
\]

\[
\frac{\partial H}{\partial t} = -v_x \frac{\partial H}{\partial r} + V_z, \quad (22)
\]

\[
\frac{\partial V_z}{\partial t} = -v_x \frac{\partial V_z}{\partial r} + 6 \frac{p}{\Sigma} - \frac{G m_{\text{BH}}}{r - r_g} \frac{H}{r}, \quad (23)
\]

\[
+ \frac{T}{12 - 10.5 \beta} \left\{ \alpha \Sigma c_s (r \partial \Omega / \partial r)^2 \right\} - F^-, \quad (24)
\]

Equations (19), (20), (21), and (24) represent conservation of mass, radial momentum, angular momentum, and energy, respectively. As in SM01, we adopt the diffusive form of viscosity (eq. [1]) in equations (21) and (24) and abandon the vertical hydrostatic equilibrium assumed in one-dimensional studies, instead introducing two new dynamical equations for the vertical acceleration (eq. [23]) and the evolution of the disk’s thickness (eq. [22]); thus, our studies can be regarded as 1.5-dimensional. In these equations \( v_x, l, l_k, V_z, M_{\text{BH}}, r_g, T, \beta, F^- \) are the radial velocity, specific angular momentum, Keplerian specific angular momentum, vertical velocity at the surface of the disk, black hole mass, gravitational radius \((\approx 2GM_{\text{BH}}/c^2)\), temperature, ratio of gas pressure to total pressure, and radiative flux per unit area away from the disk in the vertical direction, respectively. We use the “one zone” approximation for the vertically averaged disk, as in SM01, so that \( v_x, \Omega, l, l_k, p, c_s, \text{ and } T \) are all equatorial-plane quantities, while \( V_z \) and \( F^- \) are quantities at the disk’s surface. Additional definitions and relations among these quantities are

\[
\rho = \Sigma / H, \quad (25)
\]

\[
l_k = \sqrt{\frac{G m_{\text{BH}}^3}{(r - r_g)^2}}, \quad (26)
\]

\[
c_s = \sqrt{p / \rho}, \quad \Omega = l / r^2, \quad (27)
\]

\[
F^- = \frac{2 a T^4}{3 \pi k / 2 + \sqrt{3} + 1 / \tau_p}, \quad (28)
\]

\[
\rho = k p T + \rho_{\text{rad}}, \quad (29)
\]

\[
p_{\text{rad}} = F^- \left( \frac{r_g + 2}{2 \sqrt{3}} \right), \quad (30)
\]

\[
\tau_R = 0.34 \Sigma (1 + (6 \times 10^3 p T^{-3.5})), \quad (31)
\]

\[
\tau_p = 1.24 \times 10^{21} \Sigma q T^{-3.5}, \quad (32)
\]

\[
\beta = k p T / p, \quad (33)
\]
where equation (29) is a bridging formula that is valid for both optically thick and thin regimes, $p_{nad}$ is the radiation pressure, and $\tau_R$ and $\tau_P$ are the Rosseland and Planck mean optical depths.

### 3.2. Specific Techniques

As in SM01, in our code the inner edge of the grid is set at $r \approx 2.5r_g$, close enough to the central black hole so that the transonic point can be included in the solution, and the outer boundary is set at $r \approx 10^3r_g$, far enough away so that no perturbation from the inner regions can reach. A stationary transonic disk solution calculated with the $\alpha p$ viscosity prescription is used as the initial condition for the evolutionary computations. The $\alpha p$ viscosity prescription may seem inconsistent with evolutionary computations that adopt the diffusive viscosity prescription, but this does not matter. In fact, the initial condition affects only the first cycle of the obtained limit-cycle solutions, and all following cycles are nicely regular and repetitive. The time step $\Delta t$ is also adjusted in the same way as in SM01 to maintain numerical stability. We emphasize some techniques specific to our numerical algorithm below.

The solution to be obtained covers a wide range of the whole computational domain, and, in particular, thermal instability causes a violent variation of the solution (a stiff problem) in the inner region of the domain (inside $200r_g$). In such circumstances, the standard one-domain spectral method is certainly insufficient. Accordingly, we divide the whole computational domain into six subdomains and let each subdomain contain 65 grid points, so the total number of grid points is $65 \times 6 - 5 = 385$ (there are five overlapping points at the interfaces of the subdomains). At each time level, we apply the one-domain spectral method described in § 2.1 to each subdomain. In doing this, various techniques are used to remove or restrain spurious nonlinear oscillations and to properly treat the boundary conditions, in order to have a numerically stable solution in each subdomain, as well as a stable connection at each interface. Then we use the scheme described in § 2.2 to carry out the time integration over the time step $\Delta t$ to reach the next time level.

In general, spurious oscillations are caused by three factors: aliasing error, the Gibbs phenomenon, and the absence of viscous stress tensor components in the basic equations. The first of these is a numerical error specific to the spectral method when it is used to solve differential equations that contain nonlinear terms, and it can be resolved with the spectral filtering technique described in the Appendix (see Peyret [2002] for a detailed explanation and Chan et al. [2005] for a quite successful application of this technique).

However, the spectral filtering itself cannot resolve the Gibbs phenomenon characteristic of stiff problems, and the adaptive domain decomposition method described in § 2.3 becomes crucially important. In our computations, we set the spatial location where a large variation appears (e.g., the peak of the surface density $\Sigma$) as the interface of two particular subdomains and use the mapping equation (eq. [15]) along with the connection conditions (eqs. [13]–[14]) so that more grid points are concentrated on the two sides of the interface, to enhance the resolution, and a stable connection between the two subdomains is realized. As the location of large variations is not fixed and instead shifts during the time evolution, we follow this location, redivide the computational domain, and recalculate the mapping parameters for each new subdomain with equation (18). In practice, a stiff problem appears and its location shifts during the first $\sim 30$ s of each cycle, whose whole length is $\sim 700$ s. In these $30$ s we have to redivide the domain and reset the grid after every interval of $0.1$ s (or every $0.01$ s for the first few seconds), and the typical length of the time step $\Delta t$ is about $10^{-6}$ to $10^{-7}$ s. For the rest of the cycle (more than $600$ s), the stiff problem ceases, and then the grid reset is not needed and the time step can be much longer.

In addition to the above two factors in the numerical sense, there is a physical factor that can also cause spurious oscillations. The viscous stress tensor has nine spatial components, but in one-dimensional studies, usually only the $r\phi$-component is included in the basic equations, and the omission of the other components can result in numerical instabilities. In particular, Szuszkiewicz & Miller (1997) noted that if the tensor components $\tau_{rr}$ and $\tau_{\phi\phi}$ are neglected in the radial momentum equation, some instability would develop and cause termination of their computation because of the nondiffusive nature of the equation, and that the instability could be suppressed if a low level of numerical diffusion was added artificially into the equation. In our code we resolve the similar problem in a slightly different way. We add into the radial momentum equation (eq. [20]) two viscous forces $F_{rr}$ and $F_{\phi\phi}$ in the form given by equation (5) of Szuszkiewicz & Miller (1997) and accordingly add into the energy equation (eq. [24]) a heating term due to viscous friction in the radial direction as given by equation (15) of Szuszkiewicz & Miller (1997). We find through numerical experiments that when a very small viscosity in the radial direction is introduced, that is, with a radial viscosity parameter $\alpha_r \approx 0.05\alpha_0$, where $\alpha$ is the viscosity parameter in the $\alpha p$ viscosity prescription, the spurious oscillations due to the absence of viscous stress tensor components disappear and the solution stays nicely stable.

Solving partial differential equations in a finite domain usually requires one or both of the Dirichlet boundary condition, that is, changing the values of physical quantities at the boundary points to values supplied a priori, and the Neumann boundary condition, that is, adjusting the values of the derivatives of physical quantities at the boundary points to values supplied a priori. For spectral methods, Chan et al. (2005) introduced a new treatment, namely, the spatial filtering described in the Appendix, in order to ensure that the Dirichlet and Neumann boundary conditions can be imposed and numerical instabilities due to boundary conditions avoided. We note, however, that their spatial filtering treatment is applicable only for physical quantities whose boundary derivatives are equal, or approximately equal, to zero (e.g., the radial velocity $v_r$, the spatial derivative of which at the outer boundary is nearly zero). For quantities whose boundary derivatives are not negligible (e.g., the specific angular momentum $l$, whose spatial derivative at the outer boundary is not very small), the boundary treatment from finite-difference methods still works, that is, directly setting the values of those quantities at the boundary points to the specified boundary values.

SM01 supplied boundary conditions at both the inner and outer edges of the grid: at the inner edge $Dl/Dt = (\partial l/\partial r + v_r\partial l/\partial r)/l = 0$ and $\partial p/\partial r = 0$, and at the outer edge both $v_r$ and $l$ are constant in time with $l$ slightly smaller than the corresponding $f_k$. In our computations, we find that it is not necessary to supply a priori the two inner boundary conditions as in SM01. With the two outer boundary conditions as in SM01 and free inner boundary conditions instead, we are able to obtain numerically stable solutions of the basic equations, and these solutions automatically lead to an almost-zero viscous torque, that is, $Dl/Dt \approx 0$, and a nearly vanishing pressure gradient, $\partial p/\partial r \approx 0$, in the innermost zone. This proves that the inner boundary conditions assessed in SM01 are correct, but we think that our practice is probably more natural and more physical. Once the state of an accretion flow at the outer boundary is set, the structure and evolution of the flow will be controlled by the background gravitational field of the central black hole, and the flow should adjust
itself in order to be accreted successfully into the black hole. In particular, both the viscous torque and the pressure gradient of the flow must vanish on the black hole's horizon, and in the innermost zone a correct solution should asymptotically approach such a state; thus, in the computations there is no need to repeatedly apply inner boundary conditions.

### 3.3. Numerical Results

We have performed computations for a model accretion disk with black hole mass $M_{BH} = 10 M_\odot$, initial accretion rate $\dot{m} \equiv M_{\dot{m}} = 0.06 (\dot{M} = -2\pi r \Sigma \nu)$ is the accretion rate and $M_{\dot{m}}$ is the critical accretion rate, corresponding to the Eddington luminosity), and viscosity parameter $\alpha = 0.1$ (the equivalent $\alpha$ in the diffusive viscosity prescription is $\alpha = 0.1 \times [2/(3\sqrt{6})] \approx 0.0272$). It is known that the inner region of a stationary accretion disk with such physical parameters is supported by radiation pressure ($\Sigma < 0.4$) and is thermally unstable, and the disk is expected to exhibit limit-cycle behavior (Kato et al. 1998, pp. 155, 176). We have continued computations for several complete limit cycles, and a representative cycle is illustrated in Figures 1–6, which are for the time evolution of the radial distribution of the half-thickness of the disk $H$, temperature $T$, surface density $\Sigma$, effective optical depth $\tau_{\text{eff}} = \frac{\tau}{H/r_g}$, ratio of gas pressure to total pressure $\beta$, and accretion rate $\dot{m}$, respectively. Note that negative values of $\dot{m}$ signify an outflow in the radial direction, not in the vertical direction as the word “outflow” in the literature usually means.

The first panel of Figure 1 and the solid lines in Figures 2–6 show the disk just before the start of the cycle ($t = 0$ s). The disk is essentially in the SSD state; that is, it is geometrically thin ($H/r \ll 1$) and optically thick ($\tau_{\text{eff}} \gg 1$) everywhere, its temperature has a peak at $r \approx 6r_g$, its accretion rate is nearly constant in space, and its inner region (from $\sim 5r_g$ to $\sim 14r_g$) has $\beta < 0.4$ and is thermally unstable. Note that this configuration is not a stationary state and is obtained with the diffusive viscosity prescription, so it is very different from the initial condition at the beginning of the computation, which is a stationary solution with the $\alpha \nu$ viscosity prescription.
As the instability sets in \((t = 2 \text{ s}; \text{second panel of Fig. 1 and thin dashed lines in Figs. 2–6})\), in the unstable region \((r < 24r_g)\) the temperature rises rapidly, the disk expands in the vertical direction, and a very sharp spike appears in the surface density profile and, accordingly, in the optical depth and accretion rate profiles (exactly the stiff problem). The spikes move outward with time, forming an expansion wave, heating the inner material and pushing it into the black hole, and perturbing the outer material to depart from the SSD state. The expansion region is in fact essentially in the state of a slim disk, as it is geometrically thick \((H/r \lesssim 1)\), optically thick, very hot, and radiation pressure-supported \((\beta < 0.4)\); the front of the expansion wave forms the transition surface between the SSD and slim-disk states. At \(t = 12 \text{ s} \) (third panel of Fig. 1 and thin dot-dashed lines in Figs. 2–6), in the expansion region \(H\) and \(\dot{m}\) (negative, radial outflow) reach their maximum values, and the local \(\dot{m}\) (positive, inflow) exceeds 3, which is far above the initial value \(\dot{m} = 0.06\) and is even well above the critical value \(\dot{m} = 1\).

Once the wave front has moved beyond the unstable region \((r \lesssim 120r_g)\), the expansion starts to weaken, the temperature drops in the innermost part of the disk, and the material there deflates \((t = 23 \text{ s}; \text{fourth panel of Fig. 1 and thick dashed lines in Figs. 2–6})\). Subsequently, deflation spreads out through the disk, and the disk consists of three different regions: the inner part is geometrically thin, with the temperature and surface density being lower than their values at \(t = 0\) s; the middle part is what remains of the slim-disk state; and the outer part is still basically...
in the SSD state \( (t = 27 \text{ s}; \text{fifth panel of Fig. 1} \text{ and thick dot-dashed lines in Figs. 2} - 6) \).

The “outburst” part of the cycle ends when it has proceeded on the thermal timescale \( (t = 32 \text{ s}; \text{sixth panel of Fig. 1} \text{ and dotted lines in Figs. 2} - 6) \). What follows is a much slower process (on the viscous timescale) of refilling and reheating the inner part of the disk. Finally \( (t = 722 \text{ s}; \text{seventh panel of Fig. 1} \text{ and again the solid lines in Figs. 2} - 6) \), the disk returns to essentially the same state as at the beginning of the cycle. Then the thermal instability occurs again and a new cycle starts.

The bolometric luminosity of the disk, obtained by integrating the radiated flux per unit area, \( F^- \), over the disk at successive times, is shown in Figure 7 for three complete cycles. The luminosity exhibits a burst with a duration of about 20 s and a quiescent phase lasting for the remaining \( \sim 700 \text{ s} \) of the cycle. The amplitude of the variation is around 2 orders of magnitude, and during the outburst a super-Eddington luminosity is realized.

All these results obtained with our numerical method are similar to those of SM01, not only in that the limit-cycle behavior of thermally unstable accretion disks is confirmed, but also in the sense that the numerical solutions are of very good quality. In our computations we have been able to suppress all numerical instabilities and to remove all spurious oscillations, so that in our figures all the curves are perfectly continuous and smooth and all spikes are well resolved.

What is new, however, is that we have also computed the Bernoulli function (i.e., the specific total energy) of the accreted matter, which is expressed as

\[
B = \left[ 3(1 - \beta) + \frac{\beta}{\gamma - 1} \right] \frac{p}{\rho} + \frac{v^2 + V_z^2 + \Omega^2 r^2}{2} - \frac{GM_\text{BH}}{\sqrt{r^2 + H^2} - r_g} \tag{35}
\]

(cf. eq. [11.33] of Kato et al. 1998), where \( \gamma \) is the specific heat ratio, taken to be 5/3. Figure 8 shows the quantity \( B \) obtained in the whole computational domain, ranging from \( r \approx 2.5r_g \) to \( r \approx 10^3r_g \). It is clear that \( B \) has negative values over the entire spatial range (approaching zero for very large \( r \)) and for the whole duration of the cycle (in the figure the thick dot-dashed line, for \( t = 27 \text{ s} \), and the dotted line, for \( t = 32 \text{ s} \), coincide with the solid line for \( t = 0 \text{ s} \)). Note that in equation (35) the vertical kinetic energy \( \frac{1}{2}V_z^2 \) is included, and the gravitational energy is that for the surface of the disk. If the vertical kinetic energy is omitted and the gravitational energy is taken to be its equatorial-plane value as in one-dimensional models, then \( B \) will have even larger negative values. This result strongly supports the analytical demonstration by Abramowicz et al. (2000) that accretion flows with not-very-strong viscosity \( (\alpha \lesssim 0.1) \) have a negative Bernoulli function and implies that outflows are unlikely to originate from the thermally unstable accretion disks we consider here, because a positive \( B \) is a necessary, though not sufficient, condition for outflow production.

4. SUMMARY AND DISCUSSION

We have introduced a numerical method for studying thermally unstable accretion disks around black holes, which is essentially a combination of the standard one-domain pseudospectral method (Chan et al. 2005) and the adaptive domain decomposition method (Bayliss et al. 1995). As a test of our method, for the case of moderate viscosity we have reproduced the best numerical results obtained previously by SM01. Despite these similarities, we have made the following improvements over previous work in the numerical algorithm and concrete techniques, which have been proved effective in the practice of our computations:

1. In applying the domain decomposition method to resolve stiff problems, we developed a simple and useful connection technique to ensure numerically stable continuity for the derivative of a physical quantity across the interface of two conjoint subdomains, that is, equations (13) and (14), instead of the connection technique of Bayliss et al. (1995), which is seemingly complicated and was not explicitly explained.

2. We constructed a mapping function (eq. [15]) by adding a simple linear mapping (eq. [16]) to the adaptive mapping function (eq. [17]) proposed by Bayliss et al. (1995) so that the mapping between the Chebyshev-Gauss-Lobatto collocation points \( r_k \) and the physical collocation points \( r_k^\text{p} \), not just the mapping between two sets of collocation points \( r_k \) and \( r_k^\text{p} \), is complete; and the adjustability of equation (17) remains, which enables us to adaptively follow regions that incur a stiff problem and shift in space during the time evolution.

3. For the time integration, we use two complementary schemes, namely, the third-order TVD Runge-Kutta scheme and a third-order backward differentiation explicit scheme. The former is popular in one-domain spectral methods and is essentially what was used by Chan et al. (2005), and the latter can achieve the same accuracy and has the advantage of lower consumption of CPU time.

4. For the treatment of boundary conditions, we note that the spatial filtering technique developed by Chan et al. (2005) for spectral methods is useful but is by itself not sufficient, and the treatment traditionally used in finite-difference methods is still needed as a complement. We also find that once reasonable conditions are set at the outer boundary, our solutions behave in a physically consistent manner close to the black hole horizon, so that no inner boundary conditions are necessary as supplied by SM01.

5. We resolve the problem of spurious oscillations due to the absence of viscous stress tensor components in the basic equations.
in a way different from that of SM01. SM01 introduced an artificial viscosity term in the radial and vertical momentum equations. We instead improve the basic equations by including two viscous force terms, \( F_{rr} \) and \( F_{\theta\theta} \), in the radial momentum equation and a corresponding viscous heating term in the energy equation; all these terms were already mentioned by Szuszkiewicz & Miller in an earlier paper (Szuszkiewicz & Miller 1997). As for the vertical momentum equation, because of its crudeness in our 1.5-dimensional studies, we still adopt an artificial term, whose explicit form was kindly provided by Szuszkiewicz & Miller and is unpublished. We obtain solutions at the same quality level as SM01, but we think that our treatment is probably more physical in some sense. In particular, any modification in the momentum equation ought to require a corresponding modification in the energy equation, because otherwise the conservation of energy will not be correctly described.

Of these five improvements, we expect that the first two and the last one will be particularly helpful for subsequent studies of the strong-viscosity case \((\alpha \sim 1)\). In this case the viscous heating becomes extremely large, the “outburst” of the disk due to the thermal instability is predicted to be more violent, and the Gibbs phenomenon related to stiff problems can be even more serious than in the case of moderate viscosity studied in this paper. Our improved domain decomposition method is prepared to confront these difficulties. As for another nettlesome problem, that the absence of some viscous stress tensor components in 1- or 1.5-dimensional equations can also cause serious spurious oscillations, we think that although in the moderate-viscosity case they are equivalently effective as what were made by SM01, that is, introducing an artificial viscosity term in the momentum equations, our modifications for both the radial momentum and energy equations will display their advantages in the strong-viscosity case. In fact, the importance of the viscous forces \( F_{rr} \) and \( F_{\theta\theta} \) has long since been pointed out (e.g., Papaloizou & Stanley 1986). We think that the inclusion of a heating term in the energy equation in accordance with these two forces will be not only consistent in terms of physics, but hopefully also important in obtaining numerically stable solutions. With the preparations made in this paper, we next wish to achieve the goal of answering the question of the fate of thermally unstable black hole accretion disks with very large values of \( \alpha \): do these disks finally form stable and persistent SSD+ADAF configurations as suggested by Takeuchi & Mineshige (1998) or do they also undergo limit cycles, or something else? In view of the two facts that limit-cycle luminosity variations, even though having seemingly very reliable theoretical warranties, are not usually observed in black hole systems (GRS 1915+105 remains the only one known) and that outflows are already observed in many high-energy astrophysical systems that are believed to be powered by black hole accretion but are unlikely to originate from the thermally unstable accretion disks we study here, because of the negative Bernoulli function of the matter in these disks, it will definitely be interesting if some behavior other than the limit cycle for nonstationary black hole accretion disks or outflow formation from these disks can be demonstrated theoretically.

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APPENDIX

SPECTRAL FILTERING AND BOUNDARY CONDITIONS

When applied to solve nonlinear partial differential equations, a principal drawback of spectral methods is the aliasing error, which can cause spurious oscillations at high frequencies. Spectral filtering is a special technique developed to filter out the high-frequency modes in each time step to reduce this error. As in Chan et al. (2005; see also Gottlieb & Shu 1997; Peyret 2002), we use an exponential filter in spectral space:

\[
\sigma_\delta(n/N) \equiv \exp\left(-|\ln \epsilon||n/N|^{\delta}\right),
\]

where \( \epsilon \) is the machine accuracy and \( \delta \) is a parameter that can be determined from numerical trial and error. Then, instead of \( u(r_k) \) the filtered collocation values of the physical quantity \( u(r) \) at a collocation point \( r_k \) are given by

\[
\tilde{u}(r_k) = \sum_{n=0}^{N} \sigma_\delta\left(\frac{n}{N}\right) \tilde{u}_n \cos\left(\frac{nk\pi}{N}\right).
\]

As for the boundary conditions, the Dirichlet condition, \( u_{\text{boundary}} = u_0 \), the Neumann condition, \( (\partial u/\partial r)_{\text{boundary}} = u'_r \), or both are generally required. In order to avoid the appearance of discontinuous step functions at the boundaries, which would induce the Gibbs phenomenon, Chan et al. (2005) introduced another filtering technique, namely, the spatial filter. In our numerical algorithm, we either follow Chan et al. (2005) to impose the boundary conditions by using the spatial filter or we directly change the value of a certain physical quantity to have a given value at the boundary point, depending on whether the boundary derivative of the quantity is or is not very small. The spatial filter is monotonically decreasing,

\[
h(r) = \exp\left[-|\ln \epsilon||r - r_{\text{min}}|/r_{\text{max}} - r_{\text{min}}\right],
\]

where
for the outer boundary, and it is monotonically increasing,

\[ h(r) = \exp \left[ -\ln e \left( \frac{r_{\max} - r}{r_{\max} - r_{\min}} \right)^{\beta} \right], \]

(A4)

for the inner boundary. With this filter, the Dirichlet boundary condition can be imposed at each time step as

\[ u^n_k \rightarrow (u^n_k - u_0) h(r_k) + u_0, \]

(A5)

and the Neumann boundary conditions can be imposed as

\[ \left( \frac{\partial u}{\partial r} \right)_k^n \rightarrow \left( \frac{\partial u}{\partial r} \right)_k^n - u_0 \right) h(r_k) + u_0', \]

(A6)

where the superscript \( n \) and subscript \( k \) denote the relevant values at the \( n \)th time level and the collocation point \( r_k \), respectively.

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