A new conformal duality of spherically symmetric space–times

Hans - Jürgen Schmidt

1Institut für Mathematik, Universität Potsdam
PF 601553, D-14415 Potsdam, Germany

and

Institut für Theoretische Physik, Freie Universität Berlin,
Arnimallee 14, D-14195 Berlin, Germany

Abstract

A contribution linear in $r$ to the gravitational potential can be created by a suitable conformal duality transformation: the conformal factor is $1/(1 + r)^2$ and $r$ will be replaced by $r/(1 + r)$, where $r$ is the Schwarzschild radial coordinate. Thus, every spherically symmetric solution of conformal Weyl gravity is conformally related to an Einstein space. This result finally resolves a long controversy about this topic.

As a byproduct, we present an example of a spherically symmetric Einstein space which is a limit of a sequence of Schwarzschild–de Sitter space-times but which fails to be expressable in Schwarzschild coordinates. This example also resolves a long controversy.

*http://www.physik.fu-berlin.de/~hjschmi  e-mail: hjschmi@rz.uni-potsdam.de
Keyword(s): Alternative Theories of Gravity, Conformal invariance, spherical symmetry, Schwarzschild coordinates, Einstein spaces.
1 Introduction

From the Lagrangian

\[ L = C_{ijkl} C^{ijkl} \sqrt{-g} \]  

(1)

where \( C_{ijkl} \) is the conformally invariant Weyl tensor one gets the Bach tensor \[ B_{ij} = 2 C^{k}_{ij \; lk} + C_{ij \; lk} R_{lk} \]  

(2)

Recently, the solutions of the Bach equation \( B_{ij} = 0 \), i.e., the vacuum solutions of conformal Weyl gravity, enjoyed a renewed interest because in the static spherically symmetric case one gets a term linear in \( r \), (cf. [2] for a deduction and for the motivation):

\[ ds^2 = -A(r) dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2 \]  

(3)

with

\[ A(r) = 1 - 3\beta\gamma - \frac{(2 - 3\beta\gamma)r\beta}{r} + \gamma r - kr^2 \]  

(4)

Further discussion of this solution can be found in [3]. In [4], the viability of the term \( \gamma r \) is doubted, whereas in [2,3] just this part of the potential played the main role.\[\]

A solution of the Bach equation is called trivial if it is conformally related to an Einstein space, i.e. to a vacuum solution of the Einstein equation with

---

1 From dimensional analysis one can deduce the powers of \( r \) in three different ways as follows: a) In the Newtonian limit (i.e. \( \Delta \) is the flat-space Laplacian) one gets from the Einstein-Hilbert Lagrangian \( L_{EH} \) via \( \Delta \varphi = 0 \) the two spherically symmetric solutions \( \varphi = 1 \) and \( \varphi = 1/r \); and from \( L \) eq. (1) via \( \Delta \Delta \varphi = 0 \) one gets additionally \( \varphi = r \) and \( \varphi = r^2 \) (and, of course, all the linear combinations.) \( \varphi = 1 \) gives flat space, and \( \varphi = r^2 \) corresponds to the de Sitter space-time, so the essential terms are \( 1/r \) for \( L_{EH} \) and \( r \) for \( L \). b) \( L \) and \( L_{EH} \) differ by a factor \( < \text{length} >^2 \), so this should be the case for the potentials, too. c) Similarly one gets this as heuristic argument by calculating the Greens functions in momentum space.
arbitrary \( \Lambda \) [5]. So our question reads: Do non-trivial spherically symmetric solutions of the Bach equation exist? Up to now, contradicting answers have been given: Metric (3) with (4) is an Einstein space for \( \gamma = 0 \) only, so it seems to be a non-trivial solution for \( \gamma \neq 0 \), whereas in [5] (cf. also [6] for earlier references) it is stated that only trivial spherically symmetric solutions of the Bach equation exist.

It is the aim of the present paper to clarify this contradiction by introducing a new type of conformal duality within spherically symmetric space-times. The result will be that the value of \( \gamma \) in eq. (4) can be made vanish by a conformal transformation. Then the question whether this linear term is physically measurable or not depends on the question in which of these two conformal frames the non-conformal matter lives.

As a byproduct of this discussion we will present a new view to the question (see the different statements to this question in [9-12]) under which circumstances a spherically symmetric Einstein space can be expressed in Schwarzschild coordinates.

The paper is organized as follows: In sct. 2 we will deduce the new duality transformation, in sct. 3 we apply this transformation to the solution eq. (3,4), and in sct. 4 we look especially to those solutions where Schwarzschild coordinates do not apply.

\section{A new conformal duality transformation}

The general static spherically symmetric metric can be written

\[ ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2 \tag{5} \]

It will be a duality different from that one introduced in [7], cf. [8] for a review on conformal transformations between fourth-order theories of gravity.
where $d\Omega^2 = d\psi^2 + \sin^2 \psi d\phi^2$ is the metric of the standard 2-sphere. The functions $A$, $B$ and $C$ have to be positive.

The main simplification for solving the Bach equation for metric (5) was done in [2] as follows: The two possible gauge degrees of freedom (a redefinition of the radial coordinate $r$ and the conformal invariance of the Bach equation) can be used to get $r$ as Schwarzchild coordinate, i.e., $C(r) = r^2$, and $A(r) \cdot B(r) = 1$, i.e., one starts from the metric (3). The case when Schwarzchild coordinates do not apply will be discussed in sct. 4, here we concentrate on the following question: Do there exist conformal transformations of metric (3) which keep that metric form-invariant?

Of course, if $r$, $ds$, and $t$ are multiplied by the same non-vanishing constant $\alpha$, and the function $A$ will be redefined accordingly, then metric (3) remains form-invariant.

This conformal transformation with a constant conformal factor is called a homothetic transformation, and it will not be considered essential. Likewise, the transformation $r \rightarrow -r$ not changing the form of the metric (3) will not be considered essential.

Example: Let $A(r) = 1 - \frac{2m}{r}$, i.e., the Schwarzchild solution with mass parameter $m$. Let $\hat{r} = \alpha r$, $d\hat{s}^2 = \alpha^2 ds^2$, then $d\hat{s}^2$ represents the Schwarzchild solution with mass parameter $\hat{m} = \alpha m$.

One should expect that further conformal transformations do not exist because we already applied the conformal degree of freedom to reach the form (3) from the form (5). This expectation shall be tested in the following:

Let $b(r)$ be any non–constant function, and let the conformally transformed metric be $d\hat{s}^2 = b^2(r) ds^2$. With eq. (3) this reads

$$d\hat{s}^2 = -b^2(r)A(r)dt^2 + \frac{b^2(r)dr^2}{A(r)} + b^2(r)r^2d\Omega^2$$  \hspace{1cm} (6)

---

This applies also to negative values $\alpha$.  

---
Next, we have to assume that \( b(r) \cdot r \) is not a constant, and then we can introduce \( \tilde{r} = b(r) \cdot r \) as new Schwarzschild radial coordinate for metric (6). We get
\[
\frac{d\tilde{r}}{dr} = b(r) + r \frac{db}{dr} \tag{7}
\]
Form-invariance in the 00-component means that
\[
\tilde{A}(\tilde{r}) = b^2(r) A(r) \tag{8}
\]
and form-invariance in the 11-component implies
\[
\frac{b^2(r) dr^2}{A(r)} = \frac{d\tilde{r}^2}{\tilde{A}(\tilde{r})} \tag{9}
\]
Eqs. (8) and (9) together imply
\[
\frac{d\tilde{r}}{dr} = \pm b^2(r) \tag{10}
\]
If the lower sign appears we shall apply the transformation \( r \rightarrow -r \) to get the upper sign. So we get without loss of generality from eqs. (10) and (7)
\[
b^2(r) = b(r) + r \frac{db}{dr} \tag{11}
\]
The non-constant solutions of eq. (11) are
\[
b(r) = \frac{1}{1 + \alpha r} \tag{12}
\]
with a non-vanishing constant \( \alpha \). The assumption that \( b(r) \cdot r \) is not a constant is always fulfilled. We get
\[
\tilde{r} = \frac{r}{1 + \alpha r} \tag{13}
\]
which is valid for \( 1 + \alpha r \neq 0 \) and can be inverted to
\[
r = \frac{\tilde{r}}{1 + \tilde{\alpha} \tilde{r}} \tag{14}
\]
where \( \tilde{\alpha} = -\alpha \). Eqs. (13) and (14) are dual to each other: Exchange of tilted and untilted quantities changes the one of them to the other.
A likewise duality can be found for eq. (8) because of

\[ \tilde{b}(\tilde{r}) \cdot b(r) \equiv 1 \]  

(15)

and for eq. (12).

Factorizing out a suitable homothetic transformation we can restrict to the case \( \alpha = 1 \). Further, we restrict to the case that the denominator of eq. (13) is positive. Let us summarize this restricted case as follows:

Let \( A(r) \) be any positive function, let \( b(r) = 1/(1 + \alpha r) \) with \( \alpha = 1 \) and let

\[
\begin{align*}
    ds^2 &= -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2d\Omega^2 \\
\end{align*}
\]

Then the tilde-operator defined by \( \tilde{a} = -\alpha, \tilde{A}(\tilde{r}) = b^2(r)A(r), \)

\[ \tilde{r} = \frac{r}{1 + \alpha r} \]

and

\[
    d\tilde{s}^2 = b^2(r) \, ds^2
\]

represents a duality, i.e., the square of the tilde-operator is the identity operator.

3 Spherical symmetry and the Bach equation

Let us apply the duality from sect. 2 to the Schwarzschild–de Sitter solution, i.e., to metric (3) with

\[
A(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2
\]

(16)

That means, we have to insert eq. (16) into eqs. (6, 12, 14). Finally, we remove all the tildes, and we arrive at a metric which exactly coincides with eqs. (3,4): There is a one-to-one correspondence between the three parameters \( m, \Lambda, \alpha \) on the one hand, and \( \beta, \gamma, k \) on the other hand.
Here is the main result of the present paper: The Mannheim-Kazanas [2]-
solution given by eqs. (3,4) of the Bach equation is nothing but a conformally
transformed Schwarzschild-de Sitter metric; the 3-parameter set of solutions
(3,4) can be found by the conformal duality deduced in sect. 2.

It should be mentioned that the set of solutions of the full non-linear
field equation is really only 3-dimensional, and that this is in contrast to the
linearized equation which allows all linear combinations of 1, r, 1/r, and r^2,
i.e., a 4-dimensional set.

Up to now we had assumed that the metric is static and spherically sym-
metric. However, also the Bach equation allows to prove a Birkhoff-like the-
orem [6, 13]: Every spherically symmetric solution is conformally related to
a solution possessing a fourth isometry[^4] Another version of this result reads:
Every spherically symmetric solution is almost everywhere conformally re-
lated to an Einstein space. Furthermore, the necessary conformal factor can
always be chosen such that it maintains the spherical symmetry.

Why we need the restriction “almost everywhere” in the second ver-
sion? This applies to those points where the necessary conformal transfor-
mation becomes singular. Example: Take u = 1/r as new coordinate in the
Schwarzschild solution and apply an analytic conformal transformation such
that the metric can be analytically continued to negative values u via a reg-
ular point u = 0; by construction, this space-time solves the Bach equation,
but at u = 0 it fails to be conformally related to an Einstein space.

[^4]: This fourth isometry may be time-like or space-like, and we have a regular horizon
at surfaces where this character changes, so this is exactly analogous to the situation in
Einstein’s theory.
4 Applicability of Schwarzschild coordinates

To complete the discussion we want to give also those spherically symmetric solutions of the Bach equation which cannot be expressed in Schwarzschild coordinates. Before we do so, let us compare with the analogous situation in Einstein’s theory.

It has a long tradition to assume, see e.g. [9], that every static spherically symmetric line element can be expressed in Schwarzschild coordinates, i.e., that in metric (5), \( C(r) = r^2 \) can be achieved by a coordinate transformation. However, the topic is a little bit more involved.\(^5\)

Let us take a special example of the Schwarzschild-de Sitter metric: We insert \( m = l/3 > 0 \) and \( \Lambda = 1/l^2 \) into eqs. (3,16). For any positive constant \( \varepsilon \) we apply the following coordinate transformations

\[
\begin{align*}
  r &= l + \varepsilon x, \\
  t &= l^2 \tau / \varepsilon
\end{align*}
\]

and get

\[
ds^2 = -l^4 D d\tau^2 + \frac{dx^2}{D} + (l + \varepsilon x)^2 d\Omega^2
\]

with

\[
D = \frac{1}{\varepsilon^2} \left[ 1 - \frac{2l}{3(l + \varepsilon x)} - \frac{(l + \varepsilon x)^2}{3l^2} \right]
\]

\(^5\)In [10], sect. 23.2., page 595 one reads: “For a more rigorous proof that in any static spherical system Schwarzschild coordinates can be introduced, see Box 23.3.”. But that Box 23.3. at page 617 does not only give this proof, but also the necessary assumption: “ . . . such a transformation is possible, (i.e. nonsingular) only where \((\nabla r)^2 \neq 0\).” Later in the book (page 843) one can find the sentence: “The special case \((\nabla r)^2 = 0\) is treated in exercise 32.1.” and 3 pages later “We thank G.F.R. Ellis for pointing out the omission of the case \((\nabla r)^2 = 0\) in the preliminary version of this book.” Gaussian coordinates for metric (5), i.e., \( B \equiv 1 \), can always be chosen by a redefinition of \( r \), but Schwarzschild coordinates can be introduced only in regions where \( dC/dr \neq 0 \). On the other hand, the Schwarzschild radius comes out after one integration which has the result that usually, the order of the field equation will be reduced by one if expressed in Schwarzschild coordinates. This latter property is the very reason for the usefulness of them.
Developing this $D$ in a series in $\varepsilon$ it turns out that it is regular at $\varepsilon = 0$ and there its value reads $D = -x^2/l^2$. Therefore: Eq. (18) represents a one-parameter family of space-times analytic in the parameter $\varepsilon$, and for every $\varepsilon > 0$ it represents a spherically symmetric solution of the Einstein equation with $\Lambda = 1/l^2$. From continuity reasons, it represents a solution also for $\varepsilon = 0$. We get

$$ds^2 = l^2[-\frac{dx^2}{x^2} + x^2d\tau^2 + d\Omega^2]$$

(20)

which represents a spherically symmetric Einstein space which cannot be written in Schwarzschild coordinates. (It should be noted that in these coordinates, $x$ is timelike and $\tau$ is space-like.) The deduction of this solution presented here seems to be new. Nevertheless, it is already known, but usually it is not listed within the set of spherically symmetric Einstein spaces: In [11] it is listed in table 10.1. under the topic “$G_6$ with $\Lambda$-term”. In fact, metric (20) represents the cartesian product of two 2-spaces of constant and equal curvature, cf. [12]. Therefore, it is also a static metric and possesses a 6-dimensional isometry group.

A fortiori, metric (20) represents also a static spherically symmetric solution of the Bach equation, and this solution is not listed in refs. [2,3].

Further, let us mention that the cartesian product of two 2-spaces of constant curvature $P$ and $Q$ resp. represents an Einstein space iff $P = Q$, and it represents a solution of the Bach equation iff $P^2 = Q^2$. Thus, for $P = -Q \neq 0$ we get another static spherically symmetric solution of the Bach equation which cannot be expressed in Schwarzschild coordinates; however, it is conformally flat and therefore trivial, too.

Finally, we want to stress that the above consideration only dealt with vacuum solutions of conformal Weyl gravity; of course, the inclusion of non-conformal matter requests to fix one of the conformal frames, and it has to be discussed yet whether this shall be the Schwarzschild-de Sitter or in the
Mannheim-Kazanas frame.

The result of the present paper is that both solutions are conformally related, and that no further spherically symmetric solutions of the Bach equation exist.

Acknowledgement

Financial support from DFG is gratefully acknowledged. I thank the colleagues of Free University Berlin, where this work has been done, especially Prof. H. Kleinert, for valuable comments.

References

[1] R. Bach, Math. Zeitschr. 9 (1921) 110; H. Weyl, Sitzber. Preuss. Akad. d. Wiss. Berlin, Phys.-Math. Kl. (1918) 465.

[2] P. Mannheim, D. Kazanas, Gen. Relat. Grav. 26 (1994) 337; P. Mannheim, D. Kazanas, Phys. Rev. D 44 (1991) 417; P. Mannheim, Phys. Rev. D 58 (1998) 103511; N. Spyrou, D. Kazanas, E. Esteban, Class. Quant. Grav. 14 (1997) 2663.

[3] A. Edery, M. Paranjape, Phys. Rev. D 58 (1998) 024011; A. Edery, M. Paranjape, Gen. Relat. Grav. 31 (1999) in print.

[4] J. Demaret, L. Querella, C. Scheen, Class. Quant. Grav. 16 (1999) 749.

[5] H.-J. Schmidt, Ann. Phys. (Leipz.) 41 (1984) 435.

[6] R. Schimming, p. 39 in: M. Rainer, H.-J. Schmidt (Eds.) Current topics in mathematical cosmology, WSPC Singapore 1998.

[7] H.-J. Schmidt, gr-qc/9703002, Gen. Relat. Grav. 29 (1997) 859.
[8] V. Faraoni, E. Gunzig, P. Nardone: Conformal transformations in classical gravitational theories and in cosmology. gr-qc/9811047. Fund. Cosmic Physics, to appear 1999.

[9] M. v. Laue, Sitzber. Preuss. Akad. d. Wiss. Berlin, Phys.-Math. Kl. (1923) 27.

[10] C. Misner, K. Thorne, J. Wheeler: Gravitation, Freeman, San Francisco 1973.

[11] D. Kramer, H. Stephani, M. MacCallum, E. Herlt: Exact solutions of Einstein’s field equations, Verl. d. Wiss. Berlin 1980.

[12] M. Katanaev, T. Klösch, W. Kummer: Global properties of warped solutions in General Relativity, gr-qc/9807079.

[13] H.-J. Schmidt, Grav. and Cosmol. 3 (1997) 185; gr-qc/9709071.