Adding Negative Prices to Priced Timed Games

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Priced Timed Games

Timed Automaton with partition of states between 2 players + reachability objective + rates in locations + costs over transitions

Semantics in terms of infinite game with weights
Priced Timed Games

Timed Automaton with partition of states between 2 players + reachability objective + rates in locations + costs over transitions

Semantics in terms of infinite game with weights

Cost of a play:
- $\infty$ if not reached
- Otherwise, total payoff up to

$(\ell_1, 0)$
Priced Timed Games

Timed Automaton with partition of states between 2 players + reachability objective + rates in locations + costs over transitions

Semantics in terms of infinite game with weights

\[
(\ell_1, 0) \xrightarrow{0.4, \rightarrow} (\ell_4, 0.4)
\]
Priced Timed Games

Timed Automaton with partition of states between 2 players
+ reachability objective
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+ costs over transitions

Semantics in terms of infinite game with weights

Cost of a play:
\[ \begin{cases} + \infty & \text{if not reached} \\ \text{total payoff up to otherwise} \end{cases} \]
Priced Timed Games

Timed Automaton with partition of states between 2 players
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Semantics in terms of infinite game with weights

\[
(\ell_1, 0) \xrightarrow{0.4, \leftarrow} (\ell_4, 0.4) \xrightarrow{0.6, \rightarrow} (\ell_5, 0) \xrightarrow{1.5, \leftarrow} (\ell_4, 0) \xrightarrow{1.1, \rightarrow} (\ell_5, 0) \xrightarrow{2, \rightarrow} (\checkmark, 2)
\]
**Priced Timed Games**

Timed Automaton with partition of states between 2 players
+ reachability objective
+ rates in locations
+ costs over transitions

Semantics in terms of infinite game with weights

\[
\begin{align*}
(x < 1, x := 0, 0) & \quad x \leq 2, 0 \\
& \quad (\ell_2) \\
& \quad (\ell_3) \\
& \quad (x \leq 2) \\
& \quad x > 1, 1 \\
(\ell_1) & \quad x > 0 \\
& \quad x := 0, 0 \\
[x \leq 2] & \quad x \leq 1, 1 \\
& \quad (\ell_4) \\
& \quad (x \leq 2) \\
& \quad x \geq 1 \\
& \quad x := 0, 0 \\
& \quad (\ell_5) \\
& \quad (x \leq 2) \\
& \quad x \geq 1, 2 \\
(\ell_6) & \quad \checkmark
\end{align*}
\]

Cost of a play:
- \(\infty\) if not reached
- Total payoff up to otherwise

\[
\begin{align*}
\ell_1, 0 \quad \rightarrow & \quad 0.4, \rightarrow & \quad (\ell_4, 0.4) \quad \rightarrow & \quad (\ell_5, 0) \\
0.4 + 1 & \quad \rightarrow & \quad 0.6 & \quad \rightarrow & \quad 1.5 & \quad \rightarrow & \quad 1.1 & \quad \rightarrow & \quad 2 \\
-3 \times 0.6 & \quad \rightarrow & \quad +1.5 & \quad \rightarrow & \quad -3 \times 1.1 & \quad \rightarrow & \quad +2 \times 2 \rightarrow & \quad (\checkmark, 2) \\
& \quad = 3.8
\end{align*}
\]
Timed Automaton
with partition of states
+ reachability objective
  + rates in locations
  + costs over transitions

Semantics in terms of
infinite game with weights

Cost of a play:
{\[x < 1, x := 0,0\]}
{\[x > 1,1\]}
{\[x \geq 1\]}
{\[x := 0,0\]}

\[
\begin{align*}
(\ell_1, 0) \xrightarrow{0.4, \downarrow} (\ell_4, 0.4) \xrightarrow{0.6, \rightarrow} (\ell_5, 0) & \quad 0.4 + 1 \quad -3 \times 0.6 \quad +1.5 \quad -3 \times 1.1 \quad +2 \times 2 + 2 = 3.8 \\
(\ell_1, 0) \xrightarrow{0.2, \uparrow} (\ell_2, 0) \xrightarrow{0.9, \rightarrow} (\ell_3, 0.9) & \quad 0.2 \quad +0.9 \quad -0.2 \quad -0.9 \quad \ldots = +\infty (\checkmark \text{ not reached})
\end{align*}
\]
Priced Timed Games

Timed Automaton with partition of states between 2 players
+ reachability objective
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Semantics in terms of infinite game with weights

Cost of a play: \[
\begin{cases}
  +\infty & \text{if } \checkmark \text{ not reached} \\
  \text{total payoff up to } \checkmark & \text{otherwise}
\end{cases}
\]
Strategies and objectives

Strategy for each player: mapping of finite runs to a delay and an action
Strategies and objectives

Strategy for each player: mapping of finite runs to a delay and an action

Goal of player $\circ$: reach $\checkmark$ and minimize the cost
Goal of player $\Box$: avoid $\checkmark$ or, if not possible, maximize the cost
Strategies and objectives

Strategy for each player: mapping of finite runs to a delay and an action

Goal of player ○: reach ✓ and minimize the cost
Goal of player □: avoid ✓ or, if not possible, maximize the cost

Main object of interest:
\[ \overline{\text{Val}}(\ell, v) = \inf_{\sigma_\circ \in \text{Strat}_\circ} \sup_{\sigma_\square \in \text{Strat}_\square} \text{Wt}(\text{Play}((\ell, v), \sigma_\circ, \sigma_\square)) \in \mathbb{R} \cup \{-\infty, +\infty\} \]

What player ○ can guarantee as a payoff? and design good strategies
State of the art

$F \leq K$: $\exists$ a strategy in the PTG (priced timed game) for player $\Box$
reaching $\checkmark$ with a cost $\leq K$?
F_{\leq K} \triangleright: \exists \text{ a strategy in the PTG (priced timed game) for player } \bigcirc \text{ reaching } \checkmark \text{ with a cost } \leq K? 

- One-player case (*Priced timed automata*): optimal reachability problem is PSPACE-complete
  - Algorithm based on regions [Bouyer, Brihaye, Bruyère, and Raskin, 2007];
  - and hardness shown for timed automata with at least 2 clocks [Fearnley and Jurdziński, 2013, Haase, Ouaknine, and Worrell, 2012]
State of the art

\( F_{\leq K} \): \( \exists \) a strategy in the PTG (priced timed game) for player \( \bigcirc \) reaching \( \checkmark \) with a cost \( \leq K \)?

- One-player case (**Priced timed automata**): optimal reachability problem is PSPACE-complete
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- 2-player PTGs: **undecidable** [Bouyer, Brihaye, and Markey, 2006a], even with only non-negative costs and 3 clocks
State of the art

\( F \leq K \vdash: \exists \) a strategy in the PTG (priced timed game) for player \( \bigcirc \) reaching \( \checkmark \) with a cost \( \leq K \)?

- **One-player case (Priced timed automata):** optimal reachability problem is PSPACE-complete
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- **2-player PTGs:** undecidable [Bouyer, Brihaye, and Markey, 2006a], even with only non-negative costs and 3 clocks
- **PTGs with non-negative costs and strictly non-Zeno cost cycles:** exponential algorithm [Bouyer, Cassez, Fleury, and Larsen, 2004, Alur, Bernadsky, and Madhusudan, 2004]
State of the art

$F_{\leq K}^\top$: $\exists$ a strategy in the PTG (priced timed game) for player $\bigcirc$ reaching $\top$ with a cost $\leq K$?

- One-player case (Priced timed automata): optimal reachability problem is PSPACE-complete
  - Algorithm based on regions [Bouyer, Brihaye, Bruyère, and Raskin, 2007];
  - and hardness shown for timed automata with at least 2 clocks [Fearnley and Jurdziński, 2013, Haase, Ouaknine, and Worrell, 2012]

- 2-player PTGs: undecidable [Bouyer, Brihaye, and Markey, 2006a], even with only non-negative costs and 3 clocks

- PTGs with non-negative costs and strictly non-Zeno cost cycles: exponential algorithm [Bouyer, Cassez, Fleury, and Larsen, 2004, Alur, Bernadsky, and Madhusudan, 2004]

- One-clock PTGs with non-negative costs: exponential algorithm [Bouyer, Larsen, Markey, and Rasmussen, 2006b, Rutkowski, 2011, Hansen, Ibsen-Jensen, and Miltersen, 2013]
State of the art

\[ F_{\leq K}\checkmark \colon \exists \text{ a strategy in the PTG (-priced timed game) for player } \bigcirc \text{ reaching } \checkmark \text{ with a cost } \leq K? \]

- One-player case (**Priced timed automata**): optimal reachability problem is *PSPACE*-complete
  - Algorithm based on regions [Bouyer, Brihaye, Bruyère, and Raskin, 2007];
  - And hardness shown for timed automata with at least 2 clocks [Fearnley and Jurdziński, 2013, Haase, Ouaknine, and Worrell, 2012]
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This talk: **PTGs with negative costs**
Undecidability Results:
Constrained-Price Reachability

- Known: $F_{\leq K}$ undecidable for 3 or more clocks

Proof by reduction of 2-counter machines: $x_1 = \frac{1}{2c_1}$, $x_2 = \frac{1}{3c_2}$, $x_3$ for work

Theorem:

$F_{\leq K}$ undecidable for PTGs with 2 or more clocks
idem for $F_{\geq K}$, $F_{> K}$, $F_{= K}$, $F_{< K}$

New encoding: $x_1 = \frac{1}{5c_1 7c_2}$, $x_2$ for work

Simulation of “$\ell_k: \text{decrement } c_1; \text{goto } \ell_{k+1}$” for Reach($= 1$)

Diagram: State transitions and conditions for each state.
Other Undecidability Results

Theorem: Time-bounded Reachability

The following problem is undecidable for PTGs with 6 or more clocks:

Input: $K, T \in \mathbb{N}$

Question: $F \leq T \leq K$: $\exists$ strategy for $\bigcirc$ that reaches $\checkmark$

with cost $\leq K$ within time $T$?

Theorem: Repeated Reachability

The following problem is undecidable for PTGs with 3 or more clocks:

Input: $\eta \geq 0$

Question: $GF_{[-\eta,\eta]}$: $\exists$ strategy for $\bigcirc$ that visits $\checkmark$

infinitely often with a cost in $[-\eta, \eta]$?
Regain decidability?
More complex when negative costs

- Value $-\infty$: detection is as hard as mean-payoff. No hope for complexity better than $\text{NP} \cap \text{co-NP}$, or pseudo-polynomial.

- Memory complexity
  - Player $\bigcirc$ needs memory, even in the untimed setting: as seen in Axel’s talk
  - Player $\square$ may require infinite memory
One-clock Bi-Valued PTGs (1BPTGs)

Assumption: rates of locations \( \{p^-, p^+\} \) included in \( \{0, +d, -d\} \) (\( d \in \mathbb{N} \)) (no assumption on costs of transitions)

- Techniques of [Bouyer, Cassez, Fleury, and Larsen, 2004, Alur, Bernadsky, and Madhusudan, 2004] not applicable, e.g., because of Zeno costs cycles
- Exponential algorithms of [Bouyer, Larsen, Markey, and Rasmussen, 2006b, Rutkowski, 2011, Hansen, Ibsen-Jensen, and Miltersen, 2013] not working because of presence of negative costs
Intuition: it is sufficient for both players to play arbitrarily close to borders of regions, so that corner-point abstraction [Bouyer, Brinksma, and Larsen, 2008] can be adapted to this game setting...

Theorem:

- Computation of the value $\overline{\text{Val}}(\ell, v)$ of states of a 1BPTG in pseudo-polynomial time
- Synthesis of $\varepsilon$-optimal strategies for player $\bigcirc$ in pseudo-polynomial time

Theorem: Non-negative case

In case of a 1BPTG with only non-negative costs, all complexities drop down to polynomial.
Sketch of proof

1. Reduce the space of strategies in the 1BPTG
   ▶ restrict to uniform strategies w.r.t. timed behaviors

2. Build a finite priced game $\mathcal{G}$
   ▶ based on corner-point abstraction

3. Study $\mathcal{G}$
   ▶ thanks to the results presented in Axel’s talk

4. Lift results of $\mathcal{G}$ to the original 1BPTG

\[
\begin{align*}
x &< 1, x := 0, 0 \\
x &> 0, x := 0, 0 \\
x &\leq 1, x := 0, 0 \\
x &\geq 1, x := 0, 0 \\
x &\leq 1, x := 0, 0 \\
x &\geq 1, x := 0, 0 \\
x &\leq 1, x := 0, 0 \\
x &\geq 1, x := 0, 0 \\
\end{align*}
\]
Complete article published in the proceedings of CONCUR 2014\(^1\)

**Results**

- More undecidability results due to the presence of negative costs
- 1BPTGs are determined: \(\text{Val}(\ell, v) = \overline{\text{Val}}(\ell, v)\)
- Computation of the values, and synthesis of \(\varepsilon\)-optimal strategies for both players, in pseudo-polynomial time
- Strategy complexity: finite memory for \(\bigcirc\), infinite memory for \(\square\)
- In case of \(\geq 0\) prices, non-trivial class of 1-clock PTGs in PTIME
- Lifting of corner point abstraction to quantitative game setting

\(^1\)See also [http://arxiv.org/abs/1404.5894](http://arxiv.org/abs/1404.5894) for a complete version
Results

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- Implementation and test of this algorithm for real instances
- Decidability results for a bigger subset of PTGs with negative weights? careful since players may need to play far from boundaries in case of 2 clocks, or 1 clock and 3 distinct rates...

\(^1\text{See also } \text{http://arxiv.org/abs/1404.5894 for a complete version} \)
Summary and Future Work

\[ x ≤ 1 \]

\[ x = 1, x := 0 \]

\[ x = 1 \]

\[ 0 \xrightarrow{\text{x} ≤ 1} 1 \]

\[ 1 \xrightarrow{\text{x} = 1} -1 \]

\[ -1 \xrightarrow{\text{x} = 1} \]

\[ y = 0 \]

\[ x = 1 \]

\[ y = 1 \]

\[ 0 \xrightarrow{\text{x} ≤ 1, y := 0} 0 \]

\[ 0 \xrightarrow{\text{y} = 0} 1 \]

\[ 0 \xrightarrow{\text{x} = 1} 0 \]

\[ 0 \xrightarrow{\text{y} = 1} 1 \]

\[ 1 \xrightarrow{\text{x} = 1} 0 \]

\[ 1 \xrightarrow{\text{y} = 1} 0 \]
Thank you for your attention

Questions?
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