Collective excitations in Quantum Dot

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Abstract

We investigate different types of collective excitations in a quantum dot containing finite number of electrons at zero magnetic field. To estimate the excitation energies analytically we follow the energy weighted sum-rule approach. We consider the most general multipole excitation with angular momentum $l$, and the breathing mode excitation (monopole excitation) of a large quantum dot, for three different types of effective electron-electron interaction. These are the logarithmic interaction, the short range pseudopotential and the coulomb interaction. The ground state density of the many-body system is calculated within Thomas-Fermi approximation. The analytical results for the collective excitation energies and their dependence on the system size and other external parameters are discussed in detail.

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I. INTRODUCTION

With the advances in the nanofabrication technology, the study of the two dimensional electron gas (2 DEG) has become an interesting topic. Quantum dots are two-dimensional electrons confined within a finite area by applying various gate voltages. It is an example of a finite size quantum system where the number of particles can be varied from a few to few thousands. In recent years, these systems have received considerable attention as a tool to study the nature of electron-electron system in a finite size system. In particular one of the goals has been to determine the far-infrared response of the system and to understand the nature of collective excitations. Though the collective excitations have been extensively studied in other finite fermion systems like atomic nuclei [1,2] and metal clusters [3,4], in three dimensions, it still remains to be analysed in detail in two-dimensional quantum dots.

Recently, evidence has been found for a strong collective dipole mode and its splitting in the presence of the magnetic field in far infrared spectroscopy experiments [5,6]. There is likely a further evidence of a quadrupole excitation in a large quantum dot in the presence of magnetic field [6] in experiments done using far infrared spectroscopy. Recently collective spin and charge density excitations in a quantum dot containing 200 electrons have been observed [7]. Considerable amount of theoretical work has been done to study the nature of edge manetoplasmon excitations of quantum dots in strong magnetic fields [8–11]. The collective excitations have been studied within the classical hydrodynamical model of quantum dot [12,13]. A lot of effort has been made to understand the nature of the collective excitations in these nanostructures by using different theoretical tools.

In this paper we present detailed results on the nature of the collective excitations in quantum dot systems at zero magnetic field. We consider the multipole excitations and breathing modes of a quantum dot with finite number of electrons, and calculate analytically the excitation energies within sum rule approach. So far most of the microscopic calculations have been done numerically, and the results are therefore restricted by the particular choice of the device parameters. The main aim of this paper is to investigate the collective excitations in a dot containing large number of electrons analytically, to study the variation of the excitation energies with the device parameters and with the change of electron number. In a quasi zero dimensional nanostructure like quantum dot, the nature of electron-electron interaction is not very transparent. Other than the conventional coulomb interaction, we consider two other types of effective interactions. The advantage of obtaining an analytical expression is the variation of the collective modes with the interaction strength becomes clear. The ground state density profile is evaluated using the simple Thomas-Fermi (TF) approximation. The T-F approximation gives a good account of the bulk properties of the system when the number of particles, \(N\), is large [15]. The T-F density deviates from the full quantum mechanical density only in the region very close to the turning point and it neglects the asymptotic tail of the density. For the higher multipolarities the density fluctuations shift more towards the edge. But for estimating the low lying multipole excitations of a large quantum system, the T-F density is good enough and the edge correction of the density gives rise to a small correction over the leading order estimate. We estimate the edge correction for some important moments of density, and the corrections typically goes as, \(\sim c(l)/N^{3/4}\), where the term \(c(l)\) increases with increasing multipolarity \(l\). This estimate of the edge correction justifies the evaluation of different moments of a large quantum system by using...
T-F density. So that the low-lying collective excitations of a large dot can be estimated within the T-F approximation.

For the electron-electron interaction we have taken three different forms of the two-body interaction. These forms are chosen so as to facilitate analytical calculations of excitation energies. However, we also point out that in some limits these approximate the effective electron-electron interaction in a quantum dot. Since the effective interaction changes with the number of particles, it is useful to work with different approximate interactions.

This paper is organized as follows: In section II we discuss the ground state of the quantum dot in the simple TF approximation assuming various forms of the electron-electron interaction. In Sec.III we discuss the RPA dynamic polarisibility of quantum dot for different excitation operators, and the corresponding strength distributions of such excitations. Also we discuss how the collective excitation energy can be estimated using the moments of the corresponding strength distribution. We explicitly evaluate the multipole excitation energy for different models of quantum dot in Sec.IV. In Sec.V we derive the sum rules and excitation energy for the breathing mode of the dot, from simple scaling argument. Section VI contains a summary. In Appendix A and Appendix B, we outline the corrections that may arise from using T-F density for the ground state.

II. THOMAS-FERMI METHOD FOR GROUND STATE

The two dimensional Hamiltonian to describe the many electrons in a quantum dot may be written as

$$H = \sum_{i=1}^{N} \left[ \frac{p_{i}^2}{2m^*} + \frac{1}{2}m^*\omega^2 r_i^2 + \sum_{j>i=1}^{N} V(|\vec{r}_i - \vec{r}_j|) \right]. \quad (1)$$

Here we have assumed that the electrons are confined in a parabolic potential which is a good approximation to the device potential, $m^*$ is the effective mass of the electron in GaAs sample, which is 0.067 times the actual electron mass. We estimate the density profile of the many-body system by using T-F energy functional. It is well known that for the large number particles, T-F density profile agrees very well with the full quantum mechanical density, except very close to the turning point. Asymptotic tail of the density profile can be obtained by including the gradient corrections in the density functional, which includes the higher order $\hbar$ corrections in the T-F energy functional. In evaluating the energy weighted sum rules, T-F density gives the leading order contribution, and the small edge correction can give rise to the higher order $(1/N)$ corrections. In what follows, we assume that $N$ is sufficiently large so that these corrections may be neglected. The energy functional describing the ground state of this many body system is

$$E[\rho] = \int d^2r [\tau(r) + V_{sc}(r)\rho(r)], \quad (2)$$

where the self-consistent potential $V_{sc}$ is given by,

$$V_{sc} = \frac{1}{2}m^*\omega^2 r^2 + \frac{1}{2} \int d^2r' V(|\vec{r} - \vec{r}'|)\rho(r'). \quad (3)$$
Within the Thomas-Fermi approximation the kinetic-energy density is given by,
\[
\tau(r) = \frac{\hbar^2 \pi}{2m^* \rho^2(r)},
\]
for spin 1/2 electrons in an unpolarised dot. The self-consistent equation for determining the density is obtained by minimising the free energy
\[
F = (E - \mu N),
\]
where \(\mu\) is the chemical potential. Thus we have the following self-consistent equation for the density,
\[
\frac{\hbar^2 \pi \rho(r)}{m^*} + \frac{1}{2} m^* \omega^2 r^2 + \int d^2 r V(|\vec{r} - \vec{r}'|) \rho(r') = \mu.
\]
(4)
The chemical potential is determined by conserving the number of particles in the dot,
\[
\int d^2 r \rho(r) = N.
\]
(5)

In the following subsections we consider three different forms of the repulsive two body interaction and try to find out self-consistent density analytically. We first show that for logarithmic interaction and short-range interaction, the self-consistent density can be solved exactly. However, for coulomb interaction the above integral equation can not be solved exactly. Therefore we estimate the ground-state density profile by variational method.

A. Logarithmic interaction

The two dimensional Hamiltonian with logarithmic interaction is given by,
\[
H = \sum_{i=1}^{N} \left[ \frac{p_i^2}{2m^*} + \frac{1}{2} m^* \omega^2 r_i^2 \right] - g \sum_{i<j} \log \left( \frac{|\vec{r}_i - \vec{r}_j|}{a} \right).
\]
(6)
Here, \(a\) is an arbitrary parameter which determines the range of the two-body interaction, and \(g\) is the strength of the interaction. The advantage of taking logarithmic two-body interaction is that the Thomas-Fermi model is exactly solvable \cite{16,17}, and it can mimic the realistic two body interaction. This model may also be justified from the finite thickness effect of the two dimensional system \cite{17}. As shown by Zhang and DasSarma \cite{18}, the effective interaction for a quasi two-dimensional electron system in momentum space is given by,
\[
V(k) = \frac{e^2}{2\pi \epsilon k} \frac{1 + 9k/8b_z + 3k^2/8b_z^2}{(1 + k/b_z)^3}
\]
(7)
where, \(b_z\) has dimensions of inverse length. The effective potential given above behaves approximately as \(\log(r)\) over small distances and \(\frac{1}{r}\) for large distances. The strength of the logarithmic interaction can be written as \(g \approx \frac{3e^2 b_z}{8\epsilon}\).
Following Thomas-Fermi method the density distribution of the electron system can be determined from the following equation,

\[
\frac{\pi \hbar^2 \rho(r)}{m^*} + \frac{1}{2} m^* \omega^2 r^2 - g \int d^2 r' \rho(r') \ln \left| \frac{\vec{r} - \vec{r}'}{a} \right| = \mu. \tag{8}
\]

Now applying laplacian operator on both sides of the equation we obtain the following differential equation,

\[
\frac{\hbar^2 \pi}{m^*} \nabla^2 \rho(r) + 2 m^* \omega^2 - 2 \pi g \rho(r) = 0. \tag{9}
\]

Assuming the circular symmetry for the density of the ground state, the above equation can be written as,

\[
\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - 2 \tilde{g} y = 0, \tag{10}
\]

where, the dimensionless variable \( x = r/l_0 \) and \( y = \rho(r) - \frac{1}{\pi \tilde{g}^2} \). The harmonic oscillator length scale and the dimensionless coupling are defined as, \( l_0 = \sqrt{\frac{n}{m^* \omega}} \) and \( \tilde{g} = \frac{g}{\hbar \omega} \). The solution of the above differential equation is given in terms of modified Bessel functions,

\[
y(x) = A I_0(\sqrt{2 \tilde{g} x}) + B K_0(\sqrt{2 \tilde{g} x}), \tag{11}
\]

where \( A \) and \( B \) are constants to be fixed by the boundary conditions. In the limit \( x \to 0, I_0 \to 1 \), where as \( K_0 \to -\ln(x) \) \([13]\). The latter condition implies that the density at the centre is infinite which is unphysical since it would require infinite energy to push particles to the centre due to the Coulomb repulsion. Obviously minimisation of the ground state energy implies that \( B \) should be zero. The quantity \( A \) is then fixed by the boundary condition, \( \rho(r_0) = 0 \) at the turning point. We have therefore the full solution for the TF density,

\[
\rho(r) = \frac{1}{\pi \tilde{g}^2} \left[ 1 - \frac{1}{I_0(\sqrt{2 \tilde{g} x})} \right], \quad \text{for } x \leq x_0. \tag{12}
\]

The turning point \( r_0 \) can be determined from the normalisation condition,

\[
N = \int \rho(r) d^2 r = \frac{1}{\tilde{g}^2} \left[ \frac{\tilde{L}^2}{2} - \frac{\tilde{L} I_1(\tilde{L})}{I_0(\tilde{L})} \right]. \tag{13}
\]

here \( \tilde{L} = \sqrt{2 \tilde{g} x_0} \), and \( x = r/r_0 \). For \( \tilde{L} \gg 1 \) (which is equivalent to large \( N \) limit) the density becomes almost constant at the value \( \rho_0 = \frac{1}{\pi \tilde{g}^2} \), and the relation between \( \tilde{L} \) and the number of particles \( N \) becomes,

\[
2 \tilde{g} N = \tilde{L}^2 - 2 \tilde{L}. \tag{14}
\]

In the other limit \( \tilde{L} \ll 1 \) (which is equivalent to the weak coupling limit), the density becomes almost parabolic, and the dependence of \( \tilde{L} \) on \( N \) is given by,
\[ \tilde{g}^2 N = \frac{\tilde{L}^4}{16} (1 + \frac{1}{12} \tilde{L}^2) \]
\[ N = \frac{1}{4} x_0^4 (1 + \frac{1}{6} \tilde{g} x_0^2). \] (15)

In fig.1 we have compared the scaled density profile of the electrons with logarithmic interaction with the parabolic one, for two different values of the parameter \( \tilde{L} \). When \( \tilde{L} = 1 \), then the density becomes almost parabolic, but for \( \tilde{L} = 20 \), the density becomes very flat.

**B. Short range interaction**

We now consider the case when electrons in the dot are interacting by a very short range delta function interaction, \( V(r) = V_0 \delta (r) \). For a dilute Fermi gas the two body interactions can be approximated by delta function pseudopotential. The interaction strength \( V_0 \) is related to the s-wave scattering length \( a \) in the following way, \( V_0 = \frac{4 \pi \hbar^2 a}{m^*} \). Also it has been found that for 2-D electron gas in strong magnetic field the short range pseudo-potential is a very good approximation [20]. This system is also interesting because short range interaction among the fermions can mimic exclusion statistics [21].

The energy functional of the electrons with short range two body interaction can be written as,

\[ E[\rho] = \int d^2 r \left[ \frac{\hbar^2 \pi}{2 m^*} \rho^2(r) + \frac{1}{2} m^* \omega^2 r^2 \rho(r) + \frac{V_0}{2} \rho^2(r) \right]. \] (16)

By minimizing the energy functional with respect to the density, we obtain the following equation,

\[ \frac{\hbar^2 \pi}{m^*} \rho(r) + \frac{1}{2} m^* \omega^2 r^2 + V_0 \rho(r) = \mu, \] (17)

where \( \mu \) is the chemical potential of the system. The self consistent density can be written as,

\[ \rho(r) = \frac{1}{\pi l_0^2 \alpha_s} (\tilde{\mu} - \frac{1}{2} x^2), \] (18)

where, \( x \) is the dimensionless variable \( x = r/l_0 \) and other parameters are defined as, \( \alpha_s = 1 + \frac{\tilde{\nu}_0 m^*}{\pi \hbar^2} \) and \( \tilde{\mu} = \frac{\mu}{\hbar \omega} \). The turning point of the density distribution can be determined from the conservation of the total number of particles,

\[ N = \int d^2 r \rho(r) = \frac{x_0^4}{4 \alpha_s}. \] (19)

The results thus obtained with short range interaction are similar to the non-interacting but confined electrons. The only difference with the non-interacting electrons is the scaling factor \( 1/\alpha_s \) both in density and in the number of particles.
C. Coulomb interaction

Next we consider the case where the two-body electron electron interaction is $1/r$. For this form of the two-body interaction, T-F self consistency equation for density can not be solved analytically. To do the rest of the calculations analytically, we take a variational ansatz for the self consistent density profile and determine the parameters of the ansatz by minimising the energy functional. In zero magnetic field we can take the following ansatz for the density profile,

$$\rho(r) = \frac{1}{2\pi l_0^2 \alpha_c} (x_0^2 - x^2), \quad (20)$$

where, \(\alpha_c\) is the variational parameter, and \(x\) is the dimensionless variable \(r/l_0\). The density of the non interacting electron gas corresponds to the limit \(\alpha_c = 1\). The inclusion of the two body repulsion flattens the density profile, which can be thought of as the reduction of the frequency \(\omega_{eff}\) of the effective mean field potential. So the electron-electron interaction makes \(\alpha_c > 1\). In the case of short-range interaction the density profile is parabolic, and also for the logarithmic interaction density profile becomes parabolic in the strong confinement limit. We expect at zero magnetic field the ansatz can give good estimate of the ground state density in the strong confinement limit. We must, however, caution that this ansatz breaks down in the presence of a strong magnetic field. The ground state density here deviates from the parabolic form and is approximately \(\rho(r) \sim \sqrt{1 - (r/R)^2}\). For convenience we define \(\rho_0 = \frac{1}{2\pi l_0^2 \alpha_c}\). The turning point \(x_0\) can be calculated from the total number of electrons \(N\),

$$N = \int d^2 r \rho(r) = \frac{1}{2\pi l_0^2 \rho_0 x_0^4}. \quad (21)$$

Let us calculate the contributions to the energy coming from the different parts of the density functional as a function of the parameter \(\rho\). Contribution to the kinetic energy is given by,

$$E_{\tau} = \frac{\hbar^2 \pi}{2m^*} \int d^2 r \rho^2(r) = \frac{\hbar^2 \pi^2 \rho_0^2 l_0^4 x_0^6}{6m^*}. \quad (22)$$

The potential energy due to the external confinement is given by,

$$E_p = \frac{1}{2} m^* \omega_0^2 \int d^2 r \rho^2(r) = \frac{1}{12} m^* \omega_0^2 \rho_0 l_0^4 x_0^6. \quad (23)$$

The interaction energy within the Hartree approximation is given by,

$$E_H = \frac{1}{2} \int d^2 r d^2 r' \rho(r) V(|\vec{r} - \vec{r}'|) \rho(r), \quad (24)$$

where, \(V(|\vec{r} - \vec{r}'|) = \frac{e^2}{|\vec{r} - \vec{r}'|}\). In momentum space the above expression can be written as,

$$E_H = \frac{1}{2} \frac{e^2}{2\pi \epsilon} \int d^2 q \frac{1}{q} \rho(q) \rho(-q), \quad (25)$$

where,
\[ \rho(q) = \int \rho(r) e^{i \vec{k} \cdot \vec{r}} d^2 r = 4\pi \rho_0 \alpha^2 J_2(q \rho_0) \frac{q^2}{q^2}. \]  

(26)

After performing the integration we obtain the Hartree energy in the following form,

\[ E_H = \frac{\Gamma(\frac{1}{2})\Gamma(4) r_0^3}{2\Gamma(5/2)\Gamma(9/2)\Gamma(5/2)} \frac{e^2}{\epsilon} (\pi \rho_0^2 \rho_0)^2. \]  

(27)

In terms of the variational parameter \( \alpha_c \) and the number of particles \( N \), the total energy of the system is given by,

\[ E_0 = \frac{1}{3} \hbar \omega_0 N^{3/2} (\alpha_c^{1/2} + \frac{1}{\alpha_c^{1/2}}) + \frac{512}{315} \frac{\sqrt{2}}{\pi \alpha_c^{1/4} \epsilon_0} e^2 N^{7/4}. \]  

(28)

To get the minimum energy, the total energy is be minimized with respect to the variational parameter \( \alpha_c \) when the number of particles \( N \) is held fixed. The \( \alpha_c \) that one obtains at the minimum of the energy is therefore a function of \( N \), among other things. This is in contrast to the short-range interaction where \( \alpha_s \) is independent of \( N \).

Having thus determined the ground state energy and the density in the TF approximation, we now use these in evaluating the collective excitation energies for different types of collective excitations.

### III. SUM RULES AND COLLECTIVE EXCITATIONS

The excitation spectrum of any system is usually probed by applying external fields. The effect of the perturbing probe can be described through an interaction Hamiltonian,

\[ H_{\text{int}} = \lambda (F e^{i \omega t} + F^\dagger e^{-i \omega t}), \]  

(29)

where the quantity \( \lambda \) gives the strength of the oscillating field. Given an excitation operator \( F \), many useful quantities of the perturbed system can be calculated from so-called strength function,

\[ S_\pm = \sum_n | < n | F_\pm | 0 > |^2 \delta (E - E_n) , \]  

(30)

where, \( E_n \) and \( | n > \) are the excitation energy and excited state respectively, and \( F_+ = F, F_- = F^\dagger \). Various sum rules are derived through the moments of the strength function, defined as,

\[ m_k^\pm = \frac{1}{2} \int E^k (S_+(E) \pm S_-(E)) dE \]

\[ = \frac{1}{2} (| < 0 | F (\hat{H} - E_0)^k F^\dagger | 0 > \pm | < 0 | F^\dagger (\hat{H} - E_0)^k F | 0 > ). \]  

(31)

It is easy to see that, for a given \( k \), the moments may be expressed in terms of the commutators of the excitation operator \( F \) with the many body Hamiltonian \( H \). We give below some of the useful sum rules,
\[ m_0 = \frac{1}{2} < 0 | [F^\dagger, F] | 0 >, \] (32)
\[ m_1^+ = \frac{1}{2} < 0 | [F^\dagger, [H, F]] | 0 > \] (33)
\[ m_2^- = \frac{1}{2} < 0 | [J^\dagger, J] | 0 > \] (34)
\[ m_3^+ = \frac{1}{2} < 0 | [J^\dagger, [H, J]] | 0 > \quad ; \quad J = [H, F]. \] (35)

The choice of the excitation operator \( F \) is dictated by the physics that one wants to describe. In this paper we consider two types of collective excitations: the multipole excitation modes and the breathing modes in a the quantum dot. In two-dimensions multipole excitation operator can be written as \( F = \sum_i r_i^l e^{il\theta_i} = \sum_i z_i^l \), where \( z \) is the complex variable \( x + iy \). Angular part of any function in two dimensions can be expanded in terms of these operators. These multipole modes describe the angular momentum excitations of the system. The excitation operator for breathing mode is given by \( F = \sum_i r_i^2 \). The collective excitation energy can be estimated from the sum-rules of the corresponding excitation operators.

For a highly collective state the strength distribution becomes a sharply peaked function around the collective excitation energy. If the excitation operator \( F \) is hermitian, or if the Hamiltonian does not contain any time-reversal symmetry breaking term (as in the quantum dot without magnetic field), then we can take \( m_2^- = 0 \) and the low lying collective excitation energy is given by,

\[ E_c = \sqrt{m_3^+ / m_1^+}. \] (36)

Near the collective excitation energy, we can approximate the strength distribution by a delta function \( S_\pm (E) = \sigma_\pm \delta (E - E_c) \), then we can trivially show the above form of collective excitation. But in general the strength function has a finite width. Following ref. [9], one can then derive the above form of collective excitation energy for multipole excitations by using the variational principle. Given the \( N \) electron ground state \( | 0 > \), it is possible to find the collective excitation energy and the collective state \( | c > \), if one is able to find an operator \( O^\dagger \), which satisfies the following equation of motion,

\[ [\hat{H}, O^\dagger] = \hbar \omega_{\text{coll}} O^\dagger. \] (37)

The state \( O^\dagger | 0 > \) has excitation energy \( \hbar \omega_{\text{coll}} \). The excitation energy is then given by the following expression,

\[ \hbar \omega_{\text{coll}} = \frac{< 0 | [O, [\hat{H}, O^\dagger]] | 0 >}{< 0 | [O, O^\dagger] | 0 >}. \] (38)

We may now take the variational ansatz for \( O^\dagger \) as, \( O^\dagger = F + aJ \) with \( a \) as the variational parameter. Substituting this expression in the above equation, we obtain the excitation energy as,

\[ \hbar \omega_{\text{coll}} = \frac{m_1^+ + a^2 m_3^+}{2am_1^+}, \] (39)
where we have assumed \( m_2^- = 0 \), for the system in the absence of the magnetic field. By minimizing the energy with respect to the variational parameter \( a \), we obtain the collective excitation energy as \( \hbar \omega_{\text{coll}} = \sqrt{\frac{m_1}{m}} = E_c \).

### IV. MULTIPOLE EXCITATIONS IN QUANTUM DOT

The multipole excitations in a quantum dot lead to the change in total angular momentum. In particular, the dipole excitation in parabolic quantum dot has been well studied both theoretically and experimentally. The corresponding operator for the dipole excitation operator is given by \( F = x + iy \). One can generalise this and write for any multipole excitation operator \( F_l = z^l \). We use this form to calculate the relevant sum-rules for multipole excitations. The first moment of the strength distribution function may be written as,

\[
m_3^+(l) = \frac{1}{2} < 0 | [\sum z^i_l, \hat{H}, \sum z^i_l] | 0 > = \frac{\hbar^2 l^2}{m^*} \int d^2 r r^2 (l - 2) \rho^2(r), \tag{40}
\]

where \( \rho(r) \) is the ground state density distribution.

The calculation of \( m_3^+(l) \) is more complicated. To make the analysis transparent, we consider the various contributions to the third moment separately. The sum-rule \( m_3^+(l) \) may then be written in the following form,

\[
m_3^+(l) = m_3^+(T) + m_3^+(V) + m_3^+(ee), \tag{41}
\]

where the first term denotes the contribution from the kinetic energy operator in \( H \), the second term denotes the contribution from the confinement potential, and the last term is the contribution from the interaction part. We now calculate separately each one of these contributions.

The kinetic energy term in \( m_3^+(T) \) is given by,

\[
m_3^+(T) = 2\hbar^2 (l - 1) \frac{\hbar^2 l^2}{m^*} \int d^2 r r^2 (l - 2) (l \tau(r) + 2(l - 2) \lambda(r)), \tag{42}
\]

where, \( \tau(r) = < \frac{\vec{p}^2}{2m^*} > \), and \( \lambda(r) = < \sum_i \frac{\hbar^2 l^2}{2m^*} > = < \frac{(\vec{r} \times \vec{P})^2}{2m^*} > \). Here the angular brackets denotes the averages over the momentum distribution. Note that unlike the ground state where only the first term contributed, for the multipole excitations the centrifugal barrier, denoted by the second term, contributes to the sum rules. Using the Thomas-Fermi approximation this contribution is given by, \( \lambda(r) = \frac{1}{2} \tau(r) = \frac{\hbar^2}{4m^*} \rho^2(r) \). Substituting these expressions in the above equation we obtain,

\[
m_3^+(T) = \frac{2\hbar^4 \pi (l - 1)^2 \hbar^2 l^2}{m^*} \int d^2 r r^2 (l - 2) \rho^2(r). \tag{43}
\]

The contribution coming from the harmonic oscillator confinement in the sum-rule is given by,
where the central density $\rho$ distributions can be written as,

$$m_3(V) = lm^*\omega^2\hbar^2\frac{\hbar^2l^2}{m^*} \int d^2r r^{2(l-1)} \rho(r).$$

(44)

The last term $m_3(\text{ee})$ due to two body interaction can be written as,

$$m_3(\text{ee}) = \frac{1}{2}\hbar^2\frac{\hbar^2l^2}{m^*} \int d^2r V_H(r) r^{2(l-1)} \rho''(r) + (2l-1) \int d^2r V_H(r) r^{2l-3} \rho'(r)$$

$$+ \int d^2r \int d^2r' \rho'(r) r^{l-1} e^{-i\theta} V(|r - r'|) e^{i\theta'} r^{l-1} \rho'(r'),$$

(45)

where $V(r)$ is the two body interaction, $V_H = \int d^2r V(|r - r'|) \rho(r')$, and $\rho'(r) = \frac{\partial \rho(r)}{\partial r}$.

We thus have the general expressions for the sum rules where the only unknown is the ground state density $\rho(r)$. For specific interactions, the ground state density was calculated in Section II. We now use this to explicitly compute the sum rules. Note however, that the T-F density calculated in section II fails near the turning point, since the T-F density abruptly goes to zero. But for the lower multipolarities main contribution to the moments comes from the bulk region, and the diffusive edge region gives rise to small correction, which decreases with the increasing $N$. We discuss the corrections to this approximation in the appendices.

A. Logarithmic potential

For logarithmic potential the first moment $m_1^+$ is given by,

$$m_1^+ = m^*\hbar^2 l^2 \int d^2r r^{2(l-1)} \rho(r)$$

$$= m^*\hbar^2 l^2 \rho_c r_0^l 2\pi \int_0^1 dx x^{2l-1} [1 - \frac{I_0(x\tilde{L})}{I_0(L)}],$$

(46)

where the central density $\rho_c = \frac{1}{4\pi g r_0^2}$ and the dimensionless variable $x = r/r_0$, and $\tilde{L} = \sqrt{2g}x_0$.

In the case of the third moment $m_3^+$, kinetic energy and the confinement energy contributions can be written as,

$$m_3(T) + m_3(V) = \frac{\hbar^4 l^2}{m^*} [2(l - 1)^2 \frac{\hbar^4 \pi}{m^*} \int d^2r \rho^2(r) + l m^* \omega^2 \hbar^2 \int d^2r r^{2(l-1)} \rho(r)].$$

(47)

We now evaluate the remaining term $m_3(\text{ee})$. Integrating eq.(13) by parts we can write $m_3(\text{ee})$ as,

$$m_3(\text{ee}) = \frac{\hbar^4 l^2}{2m^*} [\int d^2r \nabla^2 V_H r^{2(l-1)} \rho(r) + 2(l - 1) \int d^2r \frac{\partial V_H}{\partial r} r^{2l-3} \rho(r) + I_3],$$

(48)

where, $V_H(r)$ is the hartree potential. For logarithmic interaction we can use the following relations,

$$\nabla^2 V_H(r) = -2\pi g \rho(r),$$

$$\frac{\partial V_H}{\partial r} = -m^* \omega^2 r - \frac{\hbar^2 \pi}{m^*} \frac{\partial \rho}{\partial r},$$

(49)

(50)
Therefore the term \( I_3 \) is given by,
\[
I_3 = \int d^2 r \int d^2 r' \rho(r) r^{l-1} e^{-i\theta} V(|\vec{r} - \vec{r}'|) e^{i\theta'} r'^{l-1} \rho'(r')
\]
\[= (2\pi)^2 \int d^2 q f(q) \int drr' J_l(qr) \rho'(r) \int drr' J_l(qr') \rho'(r'), \tag{51}\]
where \( f(q) \) is the two body potential in momentum space, and \( \rho'(r) = \frac{\partial \rho(r)}{\partial r} \). After doing the integration by parts and using some identities for Bessel functions we can rewrite the above expression in the following form,
\[
I_3 = \int d^2 q f(q) \int d^2 re^{i\vec{q}\cdot\vec{r}} e^{-i(l-1)\theta} r^{l-1} \rho(r) \int d^2 r' e^{-i\vec{q}\cdot\vec{r}'} e^{i(l-1)\theta'} r'^{l-1} \rho(r')
\]
\[= 2\pi g \int d^2 r r^{2(l-1)} \rho^2(r). \tag{52}\]
Using the above results we can write the multipole excitation energy in the following form,
\[
E_c^2 = \frac{m_3^+}{m_1^+} = \hbar^2 \omega^2 + \frac{3\hbar^4 \pi (l - 1)^2}{m^2} \int d^2 r r^{2(l-2)} \rho^2(r) \int d^2 r r^{2(l-1)} \rho(r). \tag{53}\]
The above equation can be written as,
\[
E_c^2 = \hbar^2 \omega^2 [1 + 6l(l - 1) \frac{f(\tilde{L})}{\tilde{L}^2}], \tag{54}\]
where,
\[
f(\tilde{L}) = \frac{(l - 1)}{l} \int_0^1 dx x^{2l-3} [1 - \frac{\tilde{L}(x\tilde{L})}{\tilde{L}(\tilde{L})}]^{2l-1} \int_0^1 dx x^{2l-1} [1 - \frac{\tilde{L}(x\tilde{L})}{\tilde{L}(\tilde{L})}]^{2l-1}. \tag{55}\]
In the large \( \tilde{L} \) limit, one can approximate the function \( [1 - \frac{\tilde{L}(x\tilde{L})}{\tilde{L}(\tilde{L})}] \) by a simple form \([1 - \exp(\tilde{L}(x - 1))]\), and the approximate value of \( f(\tilde{L}) \) is given by,
\[
f(\tilde{L}) = \frac{1 - 2F_1(2l - 2, 2l - 1; \tilde{L}) e^{-\tilde{L}} + F_1(2l - 2, 2l - 1; 2\tilde{L}) e^{-2\tilde{L}}}{1 - F_1(2l, 2l + 1; \tilde{L}) e^{-\tilde{L}}}, \tag{56}\]
where \( F_1 \) is the degenerate hypergeometric series \([19]\). In the large \( \tilde{L} \) limit asymptotic \( N \) dependence of the collective excitations are given by,
\[
E_c = \hbar \omega \sqrt{1 + \frac{3l(l - 1)}{\tilde{g}^2 N}}. \tag{57}\]
By substituting \( l = 1 \), for dipole excitation we can see that the excitation energy is \( \hbar \omega \), which obeys Kohn’s theorem \([23]\). In the other limit \( \tilde{L} \ll 1 \), (which is the weak coupling limit) the dispersion relation of the multipole modes are given by,
\[
E_c = \hbar \omega [3l - 2 - 6 \frac{(l - 1)}{l} \tilde{g} N^{1/2}]^{1/2}. \tag{58}\]
We can estimate the strength of the coupling from the relation \( \tilde{g} = \frac{2\hbar}{8\chi_{\omega}} \) given in sec.II. In GaAs sample the values of the parameters are \( b_z^{-1} = 58\AA \) and \( \epsilon = 12.6 \). For \( \hbar \omega = 5.4meV \) our estimate of the dimensionless coupling is \( \tilde{g} \approx 1.36 \). For this value of the coupling the variation of excitation energy with number of electrons for \( l = 1 - 4 \) are shown in figure 2.
B. Short range interaction

Now we calculate the sum-rules for the multipole excitations in the case of the short range interaction. For short-range interaction the density profile is given by,

\[ \rho(r) = \frac{1}{\pi l_0^2 \alpha_s} (\bar{\mu} - \frac{1}{2} x^2), \]  

(59)

where, \( \alpha_s = 1 + \frac{1}{\pi} \frac{m^*}{\hbar^2} \), and \( l_0^2 = \frac{\hbar}{m \omega} \). The first moment \( m_1^+ \) can be written as,

\[
m_1^+ = m^* \frac{\hbar^2 l^2}{m^* x^2 (l_0 - 1)} \int d^2 r r^2 \rho(r) \]

\[
= m^* \frac{\hbar^2 l^2}{m^* x^2 (l_0 - 1) x_0^2 (l + 1)}.
\]

(60)

Now let us calculate the third energy weighted moment \( m_3^+ \). Different contributions of this sum-rule are given below,

\[
m_3(T) = \frac{2 \hbar^4 \pi (l - 1)^2 \hbar^2 l^2}{m^* x^2 (l_0 - 1) x_0^2 (l + 1)} \int d^2 r r^2 (l - 1) \rho(r)
\]

\[
= \frac{\hbar^4 \pi (l - 1) \hbar^2 l^2}{m^* x^2 (l + 1) x_0^2 (l + 1)}.
\]

(61)

\[
m_3(V) = lm^* \hbar^2 \omega^2 \frac{\hbar^2 l^2}{m^* x^2 (l_0 - 1) x_0^2 (l + 1)} \int d^2 r r^2 \rho(r)
\]

\[
= lm^* \hbar^2 \omega^2 \frac{\hbar^2 l^2}{m^* x^2 (l + 1)}.
\]

(62)

\[
m_3(\text{ee}) = 0.
\]

(63)

From the above expressions collective excitation energy is given by,

\[
E_c^2 = \frac{m_3^+}{m_1^+} = l \hbar^2 \omega^2 + 2(l - 1) \frac{\hbar^2 \omega^2}{\alpha_s}.
\]

(64)

Once again for \( l = 1 \), the excitation energy is \( E_c = \hbar \omega \) consistent with Kohn’s theorem. For the value of the dimensionless parameter \( \alpha_s = 3.5 \) the dispersion relation of the modes \( l = 1 - 4 \) are shown in figure 3.

C. Coulomb interaction

For coulomb interaction we have taken a variational density of the following form,

\[ \rho(r) = \rho_0 (x_0^2 - x^2), \]  

(65)

where \( \rho_0 = \frac{1}{2 \pi l_0^2 \alpha_c} \) and \( x = r/l_0 \). Here \( \alpha_c \) is the variational parameter determined by minimizing the ground state energy as in Section IIC. This density is similar to that of short range interaction. The first moment therefore \( m_1^+ \) is given by,
The collective excitation energy for the multipole modes are given by,

\[
m_3(T) + m_3(V) = \frac{\hbar^2 l^2}{m^* l_0^2 l(l + 1)} \left[ 4(l - 1)\pi^2 \rho_0^2 [2(l + 1)] + \pi \hbar^2 \omega^2 \rho_0 m^* r_0^2 [2(l + 1)] + \frac{\pi \hbar^2 \omega^2 \rho_0 m^* r_0^2 [2(l + 1)]}{l_0^2 (l + 1)} \right].
\]  

(67)

We now calculate \( V_3(\rho) \) where

\[
m_3(\rho) = \frac{\hbar^4 l^2}{2 m^*} \int d^2 r \nabla^2 V_H r^{2(l - 1)} \rho(r) + 2(l - 1) \int d^2 r \frac{\partial V_H}{\partial r} r^{2l - 3} \rho(r)
\]

\[+ \int d^2 r \int d^2 r' r^{l - 1} \rho'(r) e^{-i\theta} \frac{1}{|\vec{r} - \bar{r}'|} e^{i\theta} r'^{l - 1} \rho'(r')].
\]

(68)

where \( \rho' = \frac{\partial \rho}{\partial r} \). The hartree potential \( V_H(r) \) is given by,

\[
V_H(r) = \frac{\epsilon^2}{\epsilon} \int d^2 r' \frac{1}{|\vec{r} - \bar{r}'|} \rho(r)
\]

\[= \frac{\epsilon^2}{\epsilon} \pi \rho_0 l_0^2 \left( \frac{r_0}{l_0} \right)^3 \frac{\Gamma(1/2)}{\Gamma(5/2)} F(1/2, -3/2, 1; \left( \frac{r}{r_0} \right)^2).
\]

(69)

After performing other integrals and some algebra we arrive at the final expression for \( m_3(\rho) \),

\[
m_3(\rho) = \frac{\hbar^4 l^2}{2 m^*} \frac{4\pi^2 \rho_0^2 2l}{\epsilon l_0} \left( \frac{r_0}{l_0} \right)^3 \frac{4\Gamma(l + 1/2)}{\pi \Gamma(l + 5/2)} - \frac{1}{l + 1} 3F_2(3/2, -1/2, l + 1; l + 2, 2; 1).
\]

(70)

The collective excitation energy for the multipole modes are given by,

\[
E_c^2(l) = [(l - 1)/(l - 1)] \frac{\hbar^2 \omega^2}{\alpha_c} + l\hbar^2 \omega^2
\]

\[+ l(l + 1) \frac{\epsilon^2}{\epsilon} \frac{\hbar \omega r_0}{\alpha_c l_0} \left[ \frac{4\Gamma(l + 1/2)}{\pi \Gamma(l + 5/2)} - \frac{1}{l + 1} 3F_2(3/2, -1/2, l + 1; l + 2, 2; 1) \right],
\]

\[
E_c(l) = \frac{\hbar \omega}{\alpha_c} [(l - 1)/\alpha_c + l(l + 1)] \left( \frac{\epsilon^2}{\epsilon} \frac{\hbar \omega}{\alpha_c} \right) \left( \frac{4N}{\alpha_c} \right)^{1/4} \times
\]

\[
\left[ \frac{4\Gamma(l + 1/2)}{\pi \Gamma(l + 5/2)} - \frac{1}{l + 1} 3F_2(3/2, -1/2, l + 1; l + 2, 2; 1) \right]^{1/2},
\]

(71)

where \( 3F_2(a_1, a_2, a_3; b_1, b_2; x) \) denotes the generalised hypergeometric function \([19]\). The validity of Kohn’s theorem is easily checked. For the dipole mode \( l = 1 \), we have

\[
3F_2(3/2, -1/2, 2; 3, 2; 1) = F(3/2, -1/2, 3; 1) = \frac{3! \Gamma(3/2) \Gamma(\frac{3}{2}) \Gamma(7/2)}{\Gamma(\frac{3}{2}) \Gamma(7/2)} \left[ 3F_2(1/2, 1/2, 3/2; 1) \right].
\]

Substituting this into the above equation we obtain the dipole energy, \( E_c(1) = \hbar \omega \).

In ref. [9] the multipole modes of a quantum dot with \( \hbar \omega = 5.6 meV \) containing 56 electrons has been calculated by using self consistent Hartree-Fock calculation. The multipole excitation energies for \( l = 2 - 4 \) are given by, 7.5, 9.0, 10.4 meV respectively. From our analytical expression we find the corresponding multipole excitation energies are 7.5, 8.9, 10.12 meV, which are in good agreement with the self-consistent numerical calculation. For \( \hbar \omega = 5.4 meV \) the multipole modes for \( l = 1 - 4 \) are plotted in the figure 4. Except in the case of dipole mode, the collective excitation energy decreases with increasing particle number.
V. BREATHING MODE EXCITATION

In this section we calculate the sum-rules and the excitation energy of the breathing mode in a quantum dot. This excitation is also known as the monopole excitation. In the breathing mode excitation average radius oscillates around its unperturbed value with a frequency of the excitation energy \[2\]. In a recent experiment on Bose-Einstein condensate this phenomena has been observed very clearly \[24\]. The monopole excitation operator is given by,

\[
F = \sum_i (x_i^2 + y_i^2).
\]

The first sum-rule \(m_1^+\) can be written as,

\[
m_1 = \frac{1}{2} \langle 0 | [F^+, [\hat{H}, F]] | 0 \rangle = \frac{2\hbar^2}{m^*} \langle 0 | \sum_i r_i^2 | 0 \rangle
\]

which is nothing but the mean-squared radius in the ground state.

The third moment \(m_3\) of the strength distribution can be calculated by using a simple scaling transformation of the ground state wave-function \[25\]. Consider the following transformation of the ground state wave function,

\[
|\phi_\eta > = e^{\tilde{\eta} J} |0 >,
\]

where \(J\) is an anti-harmitian operator \(J = [\hat{H}, F]\). Now \(J\) can be written as, \(J = -\frac{2\hbar^2}{m}(1 + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})\). If we take a new set of coordinates \(x' = \log(x)\) and \(y' = \log(y)\), then the operator \(J\) becomes \(-\frac{2\hbar^2}{m}(1 + \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'})\). Using these relations, it can be shown that the wave function and the coordinates undergo following scaling transformation,

\[
\phi_\eta(x, y) = e^{\tilde{\eta}} \phi_0(e^{\tilde{\eta}} x, e^{\tilde{\eta}} y),
\]

where, \(\tilde{\eta} = -\frac{2\hbar^2}{m^*} \eta\). The sum-rule \(m_3\) can be calculated from the scale transformed wave function \(\phi_\eta\), in the following way,

\[
m_3 = \frac{1}{2} \langle 0 | [J^+, [\hat{H}, J]] | 0 \rangle = \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \langle \eta | \hat{H} | \eta \rangle \bigg|_{\eta = 0}.
\]

Under this transformation density scales as,

\[
\rho_\eta(x, y) = e^{2\tilde{\eta}} \rho(e^{\tilde{\eta}} x, e^{\tilde{\eta}} y).
\]

We calculate the sum-rule \(m_3\) separately for different terms of the Hamiltonian. The sum-rule \(m_3\) can be written as,

\[
m_3 = m_3(T) + m_3(V) + m_3(ee).
\]
First let us calculate the contribution \( m_3(T) \) coming from the kinetic-energy density. Within the Thomas-Fermi approximation we have, \( T[\rho] \approx \int \rho^2(r) \). So the kinetic energy of the scaled density is given by,

\[
T[\rho_\eta] = e^{2\eta}T[\rho]. \tag{79}
\]

Using above relations \( m_3(T) \) can be evaluated in the following way,

\[
m_3(T) = \frac{1}{2} \frac{\partial^2 T[\rho_\eta]}{\partial \eta^2} |_{\eta=0} = 2\left(\frac{2h^2}{m^*}\right)^2 T[\rho]. \tag{80}
\]

Similarly, for quadratic confinement potential, the potential energy scales as \( V_\eta = e^{-2\tilde{\eta}}V \), and \( m_3(V) \) is given by,

\[
m_3(V) = 2\left(\frac{2h^2}{m^*}\right)^2 V. \tag{81}
\]

Finally we can calculate the term \( m_3(ee) \) due to the electron-electron interaction, in the following way,

\[
m_3(ee) = \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \left[ \frac{1}{2} \int d^2r \int d^2r' \rho(r)V(\frac{|\vec{r} - \vec{r}'|}{\tilde{\alpha}})\rho(r') \right] |_{\eta=0} = \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \left[ \log(\tilde{\alpha}) N^2 / 2 \right] = 0, \tag{82}
\]

where \( \tilde{\alpha} = e^{\tilde{\eta}} \). We now calculate the above quantities and monopole excitation energy for different types of interaction.

**A. Logarithmic interaction**

Using the density profile of the particles interacting by logarithmic potential, the first moment \( m_1 \) for monopole excitation can calculated exactly,

\[
m_1 = \frac{2h^2}{m^*} \int d^2rr^2\rho(r)
= \frac{\pi h^2 \rho_0 r_0^2}{m^*} \left[ 1 - \frac{4}{I_0(\tilde{L})} \frac{I_1(\tilde{L})}{\tilde{L}^2} \right], \tag{83}
\]

where the central density is \( \rho_0 = \frac{1}{\pi^{3/2}} \), and \( \tilde{L} = \sqrt{2g}x_0 \). The sum of the first two parts of the third moment \( m_3 \) is given by,

\[
m_3(T) + m_3(V) = 2\left(\frac{2h^2}{m^*}\right)^2 \left[ \frac{\pi^2 h^2}{m^*} + \frac{1}{2} m^* \omega_0^2 \right] \int d^2rr^2\rho(r). \tag{84}
\]

The third term \( m_3(ee) \) is given by,

\[
m_3(ee) = \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \left[ \frac{1}{2} \int d^2r \int d^2r' \rho(r)\log(\frac{|\vec{r} - \vec{r}'|}{\tilde{\alpha}a})\rho(r') \right] |_{\eta=0} = \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \left[ \log(\tilde{\alpha}) N^2 / 2 \right] = 0, \tag{85}
\]
\[ \tilde{\alpha} = e^{2\hbar^2 \eta / m^*}. \]
The monopole excitation energy is given by,

\[ E_m^2 = 2\hbar^2 \omega^2 \left[ 1 + \frac{4 f_m(\tilde{L})}{\tilde{L}^2} \right], \tag{86} \]

where \( f_m(\tilde{L}) \) is given by,

\[ f_m(\tilde{L}) = \frac{1}{2} \int_0^1 d^2 x [1 - \frac{I_0(x\tilde{L})}{I_0(\tilde{L})}] - \frac{1}{2} \int_0^1 d^2 x [1 - \frac{I_0(x\tilde{L})}{I_0(\tilde{L})}], \tag{87} \]

Using the approximate form for the density profile for \( \tilde{L} \gg 1 \), the function \( f_m(\tilde{L}) \) is given by,

\[ f_m(\tilde{L}) = 1 - \frac{2 F_1(2, 3; \tilde{L}) e^{-\tilde{L}} + F_1(2, 3; 2\tilde{L}) e^{-2\tilde{L}}}{1 - F_1(4, 5; \tilde{L}) e^{-\tilde{L}}}. \tag{88} \]

The asymptotic \( N \) dependence of the breathing mode is given by,

\[ E_m = \hbar \omega \sqrt{2 + \frac{4}{g^2 N}}. \tag{89} \]

In the weak coupling limit \( \tilde{L} \ll 1 \), the breathing mode excitation energy is given by,

\[ E_m = 2\hbar \omega \left[ 1 - \frac{3}{8 g} N^{1/2} \right]^{1/2}. \tag{90} \]

For \( g = 1.36 \), the breathing mode is shown in figure 2, and the excitation energy becomes almost constant at the value \( \sqrt{2\hbar \omega} \). This mode has higher excitation energy than the other multipole modes.

**B. Short range interaction**

The form of the short range pseudopotential is \( V = V_0 \delta(r) \), and the first moment \( m_1 \) for the breathing mode is given by,

\[ m_1 = \frac{2\hbar^2}{m^*} \int d^2 r r^2 \rho(r) = \frac{\hbar^2 l_0^2}{6 m^* \alpha x_0^6}. \tag{91} \]

where \( \alpha_s = 1 + \frac{V_0 m^*}{\hbar^2 k^*} \), and \( x = r/l_0 \). After doing some algebra the third moment \( m_3 \) can be written as,

\[ m_3 = m_3(T) + m_3(V) + m_3(ee) \]

\[ = 2 \left( \frac{2\hbar^2}{m^*} \right)^2 \left[ E_T + E_p + E_H \right] \]

\[ = 2 \left( \frac{2\hbar^2}{m^*} \right)^2 \frac{\hbar^2 \pi}{2 m^* (12\pi l_0^2 \alpha_s^2)} \frac{x_0^6}{12 \alpha_s} + \frac{1}{2} m^* \omega^2 \left( \frac{l_0^2 x_0^6}{12 \alpha_s} \right) + \frac{V_0}{2} \left( \frac{x_0^6}{12\pi l_0^2 \alpha_s^2} \right) \]

\[ = 2 \left( \frac{2\hbar^2}{m^*} \right)^2 \frac{m^* \omega^2 l_0^2 x_0^6}{12 \alpha_s}. \tag{92} \]
The excitation energy of the breathing mode is given by,

$$ E = \sqrt{\frac{m_3}{m_1}} = 2\hbar\omega. \quad (93) $$

Therefore for the short range interaction the excitation energy is independent of the interaction strength.

**VI. COULOMB INTERACTION**

Now let us consider the case where the electron-electron interaction is $1/r$ coulomb interaction. Variational density of the $N$ electron system is taken to be parabolic $\rho = \rho_0(x_0^2 - x^2)$. Where the central density is $\rho_0 = \frac{1}{2\pi^2a_c}$. The first sum-rule $m_1$ is given by,

$$ m_1 = \left(\frac{2\hbar^2}{m^*}\right) \int d^2rr^2\rho(r) $$

$$ = \frac{\pi\hbar^2\rho_0 l_0^2 x_0^6}{3m^*}. \quad (94) $$

Using the scale transformation properties of the density we obtain the second sum rule $m_3$ in the following form,

$$ m_3 = \frac{1}{2}\left(\frac{2\hbar^2}{m^*}\right)^2[4(T[\rho] + E_p) + E_H] $$

$$ = \frac{1}{2}\left(\frac{2\hbar^2}{m^*}\right)^2\frac{2\pi\rho_0 l_0^2 x_0^6}{3m^*} \left(\frac{\pi\hbar^2\rho_0}{m^*} + \frac{h\omega}{2}\right) + \frac{\Gamma(4)\Gamma(1/2)}{2(\Gamma(5/2))^2\Gamma(9/2)} \left(\frac{e^2}{\epsilon l_0}\right)^2\pi^2 l_0^4 \rho_0 x_0^6. \quad (95) $$

From the above expressions of the sum rules, the monopole excitation energy($\sqrt{m_3/m_1}$) can be written as,

$$ E^2 = 2\hbar^2\omega^2\left[\frac{1}{\alpha_c} + 1 + \frac{3\Gamma(4)\Gamma(1/2)}{4(\Gamma(5/2))^2\Gamma(9/2)} \left(\frac{e^2}{\epsilon l_0 h\omega}\right) x_0\right], $$

$$ E = \sqrt{2}\hbar\omega\left[1/\alpha_c + 1 + \frac{128\sqrt{2}}{105\pi} \left(\frac{e^2}{\epsilon l_0 h\omega}\right)^{3/4} N^{1/4}\right]^{1/2}. \quad (96) $$

In Fig.4 the $N$ dependence of the breathing mode is shown for $\hbar\omega = 5.4meV$ and it remains constant at the value $1.7\hbar\omega$.

**VII. SUMMARY**

In this paper we considered the linear response of quantum dot with finite number of electrons under the time dependent perturbation. We have considered different types of collective excitations of the dot at zero magnetic field. For a suitable excitation operator $F$, the imaginary part of the dynamic polarizability (or strength distribution) which is given by,
becomes a highly peaked function near the collective excitation energy. The collective excitation energy is well estimated by the energy weighted sum-rule approach, and is given by, \( E_{\text{coll}} = \sqrt{\frac{m_s}{m_1}} \). We have discussed in general both multipole excitations(also known as edge modes) and breathing mode excitations in a quantum dot. To calculate the collective excitation energies we have taken three different types of electron-electron interaction, which are, the logarithmic interaction, the short range pseudopotential and the coulomb interaction. Within the Thomas-Fermi approximation the ground state density has been calculated. This is a good approximation for large number of particles. For multipole excitations the angular momentum of the system changes, and the the excitation operator is defined as \( z^l \), which is the basis for the expansion of the angular part of any density fluctuation in 2-dimension. In quantum dot with large number of electrons and logarithmic two body interaction, multipole excitation energies form a band like structure around the energy \( \bar{\hbar} \omega \), and the energy spacing goes as, \( \sim \frac{1}{r_0^2} \). For short range interaction excitation energies are \( E = \hbar \omega \sqrt{l + 2(l - 1)/\alpha_s} \), which is independent of the number of electrons \( N \), but depends on the interaction parameter \( \alpha_s \). For coulomb interaction excitation energies take the following form,

\[
E^2 = \hbar^2 \omega^2 [l + (l - 1)/\alpha_c] + l(l + 1) (\frac{e^2}{\epsilon_0 \hbar \omega}) \frac{r_0}{\alpha_c l_0} \left[ \frac{4\Gamma(l + 1/2)}{\pi \Gamma(l + 5/2)} - \frac{1}{(l + 1)^3} \right] F_2 \left( \frac{3/2}{l + 2}; 1 \right) \]

(98)

For dipole mode all results are consistent with Kohn’s theorem, and the dipole excitation energy is \( \hbar \omega \).

For the breathing mode excitation, average value of the collective coordinate \( r^2 \) oscillates around its equilibrium value. For logarithmic interaction breathing mode excitation energy becomes \( \sqrt{2} \hbar \omega \) in large \( N \) limit. For short range interaction excitation energy is \( 2\hbar \omega \), which same as the excitation energy of the non-interacting system. For coulomb interaction the breathing mode remains almost constant at the value \( 1.7\hbar \omega \). As shown in the Appendices, the low lying excitations are well described by the T-F approximation for the ground state density in the large \( N \) limit. The corrections may also be calculated in order to correct for the inaccquercies that may arise for higher excitations.

Apart from the single particle excitations, collective excitations in large quantum dots and their dependence on the number of particles and external magnetic field has become very interesting topic. Detailed study of the collective excitations can give information about the electron-electron correlation in finite size quantum dot.

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APPENDIX A:

In this appendix we describe the corrections of the collective frequencies due to the edge corrections of the T-F density for a simple case. Since for the logarithmic two-body
interaction, the T-F density can be written in a closed form, we take this example. The density distribution for logarithmic interaction is given by,

$$\rho(r) = \frac{1}{\pi l_0^2 \tilde{g}} [1 - I_0(\sqrt{2\tilde{g}x}) I_0(\sqrt{2\tilde{g}x_0})],$$  \hspace{1cm} (A1)

where, $x = r/l_0$, and $r_0$ is the turning point. When $\tilde{L} = \sqrt{2\tilde{g}x_0} \gg 1$, then the density becomes almost flat. In this limit we can approximate the T-F density by a "Fermi-distribution" like density, which has a diffusive tail also and this density is given by,

$$\rho_a(r) = \rho_0 \frac{1}{1 + \exp \left(\frac{r-R}{a/R}\right)},$$  \hspace{1cm} (A2)

where, $\rho_0 = \frac{1}{\pi l_0^2 \tilde{g}}$, $R \approx r_0$, and $a \approx l_0/\sqrt{2\tilde{g}}$. The multipole excitation energies are given by,

$$E_c^2 = \hbar^2 \omega_0^2 + \frac{3\hbar^4 \pi (l-1)^2 Q_2(l)}{m^2 Q_1(l)},$$  \hspace{1cm} (A3)

where, $Q_2(l) = \int r^{2l-3} \rho_a(r)^2 dr$, and $Q_1(l) = \int r^{2l-1} \rho_a(r) dr$. Now we can expand $Q_1(l)$ and $Q_2(l)$ in a series of $a/R$, which is known as leptodermous expansion. Upto the leading order correction $Q_1$ and $Q_2$ are given by \cite{26},

$$Q_1(l) = \frac{\rho_0 R^{2l}}{2l} \left[1 + 2l(2l-1)\frac{\pi^2}{6} \left(\frac{a}{R}\right)^2\right],$$  \hspace{1cm} (A4)

$$Q_2(l) = \frac{\rho_0^2 R^{2(l-1)}}{2(l-1)} \left[1 - 2(l-1)\left(\frac{a}{R}\right)\right],$$  \hspace{1cm} (A5)

if we set $a = 0$ we recover the results given in section IV. Using the above results, upto the leading order in $a/R$, collective frequencies are given by,

$$E_c = \hbar \omega_0 \left[\sqrt{1 + \frac{3l(l-1)}{g^2 N}} + \frac{6l(l-1)^2}{(2g^2 N)^{3/2}}\right].$$  \hspace{1cm} (A6)

So the leading order correction goes as, $\sim (l/\sqrt{N})^3$, which is negligible for low lying excitations in a large system.

APPENDIX B:

In this appendix we estimate the correction of two important moments of the density, due to the diffusive edge of the density. These moments essentially control the excitation energies we are interested in. To include the asymptotic tail of the density, we write the total density in the following way,

$$\rho(r) = \rho_{TF}(r) + \delta \rho(r),$$  \hspace{1cm} (B1)

where $\rho_{TF}(r)$ is the Thomas-Fermi density, and $\delta \rho$ is the edge correction. Since the quantum mechanical density of the electrons in the harmonic oscillator potential asymptotically decays as $\sim e^{-r^2/l_0^2}$, we choose the following form of the edge correction,
\[ \delta \rho(r) = \delta \rho_0 e^{(r_0 - a)^2/l_0^2} e^{-r^2/l_0^2} \quad \text{for} \ r > r_0 - a, \]  

(B2)

where, \( a \) is the size of the region near the turning point \( r_0 \) where the T-F density deviates from the actual quantum mechanical density. In the large \( N \) limit \( a \ll r_0 \). At \( r = r_0 - a \) we match two densities, \( \rho_{TF}(r_0 - a) = \delta \rho_0 \). Now we consider the following moments,

\[
I_1(n) = \int r^n \rho(r) d^2r \\
I_2(n) = \int r^n \rho^2(r) d^2r. 
\]

(B3) (B4)

The corrections of the moments due the asymptotic tail of the density distribution may be written as,

\[
\delta I_1(n) \approx \delta \rho_0 \frac{n+2}{2} \frac{a}{l_0} \frac{1}{r_0^3} \int_{r_0}^{\infty} d^2r r^n e^{-r^2/l_0^2} \\
\delta I_2(n) \approx (\delta \rho_0)^2 \frac{2r_0^3}{l_0^2} \int_{r_0}^{\infty} d^2r r^n e^{-2r_0^2/l_0^2}. 
\]

(B5) (B6)

By using parabolic T-F density, we can calculate \( (I_1(n))_{TF} \) and \( (I_2(n))_{TF} \). The leading order fractional change of the moments due to the edge correction can be written as,

\[
\left| \frac{\delta I_1(n)}{(I_1(n))_{TF}} \right| \approx \frac{(n+2)(n+4)}{2} \frac{a}{l_0} \frac{1}{r_0^3} \\
= \frac{(n+2)(n+4)}{2} \frac{a}{l_0} \frac{1}{(4N)^{3/4}}, 
\]

(B7)

where \( \tilde{r}_0 = r_0/l_0 \), for noninteracting electrons \( \tilde{r}_0 = (4N)^{1/4} \). The correction of \( I_2(n) \) is given by,

\[
\left| \frac{\delta I_2(n)}{(I_2(n))_{TF}} \right| \approx \frac{n}{2} \frac{(n/2+2)(n/2+3)}{2} \frac{a}{l_0} \frac{1}{r_0^3} \left[ (4 \frac{a}{l_0} + \frac{1}{\tilde{r}_0}) \right] \\
\approx \frac{(n/2)^3}{2} \frac{a}{l_0} \frac{1}{(4N)^{3/4}} \left[ (4 \frac{a}{l_0} + \frac{1}{\tilde{r}_0}) \right]. 
\]

(B8)

Therefore the errors are at most of the order of \( n^3/N^{3/4} \). We may now estimate \( a/l_0 \) in the following way. The wave length \( \lambda \) of the particles near the fermi surface is of the order of \( \sim \frac{\hbar}{P_F(r)} \). The local fermi momentum is given by, \( P_F(r) = \sqrt{2m(\mu - V(r))} \). The semiclassical approximation breaks down when \( |\frac{\partial \lambda(r)}{\partial r}| \sim 1 \). From this condition we obtain \( a/l_0 \sim \frac{4N^{1/12}}{2r_0} = 1/(2(4N)^{1/12}) \). This estimate shows that the error can be reduced to even smaller value in the large \( N \) limit.

A more direct way of estimating corrections to the T-F limit is to compare the results with those obtained using the quantum mechanical density. This is rather hard in the case of interacting system, but easier in the non-interacting limit. The exact quantum mechanical density can be written as a sum over the squares of various orbital wavefunctions, \( \rho(r) = \sum_{n,l} |\psi_{n,l}(r)|^2 \), where the sum is restricted so that the number of orbitals included conserve number of particles \( N \). We calculated the excitation frequencies for two values of \( N \), by using quantum mechanical density, and compared them with frequencies obtained
from T-F density. For $N = 12$ and $l = 4$ excitation frequency T-F model deviates by only 10 percent, for the lower frequencies error is much less. For $N = 20$ and $l = 4$ collective mode the error due to semiclassical approximation reduces to 6 percent. This simple analytical calculation supports our large $N$ asymptotic analysis of the error due to the use of semiclassical density. If we consider the low-lying collective excitations in a large quantum system, then our calculation indicates that the error in the collective modes decreases asymptotically with the increasing number of particles.
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FIGURES

FIG. 1. Comparison of the density profile of electrons with logarithmic interaction with the parabolic density profile for two different values of $\tilde{L}$. Solid line shows the density of the electrons for $\tilde{L} = 1$, dashed line shows the density for $\tilde{L} = 20$, and dot-dashed line shows the parabolic density.

FIG. 2. Excitation energies ($\hbar \omega_c$) as a function of the number of particles (N), for logarithmic interaction with $\tilde{g} = 1.36$. Solid line shows the exact result and the dashed line shows asymptotic value of the excitation energy. $l$ values indicate the corresponding multipole modes and $n = 1$ level indicate the breathing mode.

FIG. 3. Excitation energies ($\hbar \omega_c$) as a function of the number of particles (N), for short range interaction with $\alpha_s = 3.5$. Multipole modes are denoted by corresponding $l$ values and $n = 1$ level indicate the breathing mode.

FIG. 4. Excitation energies ($\hbar \omega_c$) as a function of the number of particles (N), for coulomb interaction with $e^2/\epsilon_0 \hbar \omega = 1.45$. Multipole modes are denoted by corresponding $l$ values and $n = 1$ level indicate the breathing mode.
Fig. 1 (sinha)
Fig. 2
Fig. 4 (sinha)