THE Riemann-Roch Theorem
On Higher Dimensional
Complex Noncommutative Tori

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Abstract. We prove analogues of the Riemann-Roch Theorem and the Hodge Theorem for noncommutative tori (of any dimension) equipped with complex structures, and discuss implications for the question of how to distinguish “noncommutative abelian varieties” from “non-algebraic” noncommutative complex tori.

Introduction

There is by now a quite extensive literature on noncommutative differential geometry on noncommutative tori, and a much smaller literature on noncommutative complex analytic geometry for noncommutative complex tori, which is still in its infancy except for the case where the complex dimension is 1, in which case most of the obvious problems have been settled by work of Schwarz and Polishchuk [18, 16, 17, 19].

The purpose of this paper is to study noncommutative complex tori of arbitrary complex dimension, and in particular to try to find noncommutative analogues of the Riemann-Roch Theorem, the Hodge Theorem for the Dolbeault complex, and the Riemann characterization of abelian varieties within the class of all complex tori. Formulation of the main theorems is done in Section 1. In Section 2 we give the proofs of the Riemann-Roch Theorem and the Hodge Theorem. In Section 3 we examine various special cases and give noncommutative analogues (admittedly still imperfect) for the existence of non-algebraic complex tori in complex dimension > 1 and for the characterization of abelian varieties via Riemann forms.

1. General Setup

Let $\Theta$ be a skew-symmetric real matrix, say of size $d \times d$. The noncommutative torus $A_{\Theta}$ is the universal $C^*$-algebra with $d$ unitary generators $U_j$, $1 \leq j \leq d$.
subject to the basic commutation relation

\[ U_j U_k = e^{2\pi i \Theta_{jk}} U_k U_j. \]

This algebra carries a gauge action of \( \mathbb{T}^d \) via

\[ t \cdot (U_1^{n_1} \cdots U_d^{n_d}) = t_1^{n_1} \cdots t_d^{n_d} U_1^{n_1} \cdots U_d^{n_d}, \quad t = (t_1, \ldots, t_d) \in \mathbb{T}^d. \]

There are associated infinitesimal generators \( \delta_j \), which are *-derivations (really defined on the smooth subalgebra \( A_\Theta \), which we will introduce shortly), with \( \delta_j \) sending \( U_j \) to \( 2\pi i U_j \) and sending the other \( U_k, k \neq j \), to 0. Together, these define a tangent space or (commutative) Lie algebra \( \mathfrak{g} = \text{span}(\delta_1, \ldots, \delta_d) \). The algebra \( A_\Theta \) also carries a canonical tracial state \( \text{Tr} \) invariant under the gauge action, sending 1 to 1 and sending a monomial \( U_1^{n_1} \cdots U_d^{n_d} \) to 0 unless all of the \( n_j \) vanish. Because of the commutation relation, any element of \( A_\Theta \) has a canonical (formal) expansion in terms of the monomials \( U_1^{n_1} \cdots U_d^{n_d} \). This formal expansion may not converge in the \( C^* \)-norm, but there is a canonical smooth subalgebra \( A_\Theta \) consisting of elements of the form \( \sum_{n \in \mathbb{Z}^d} c_n U^n \) (here \( U^n \) is a shorthand for \( U_1^{n_1} \cdots U_d^{n_d} \)), for which the coefficients \( c_n \) are rapidly decreasing, i.e., form a sequence in the Schwartz space \( \mathcal{S}(\mathbb{Z}^d) \). The smooth subalgebra \( A_\Theta \) is exactly the algebra of \( C^\infty \) vectors for the gauge action of \( \mathbb{T}^d \), and plays the role of the algebra of \( C^\infty \) functions (with which it coincides if \( \Theta = 0 \)). Since we will want to use methods of differential geometry, we will almost always work with \( A_\Theta \) in place of its \( C^* \)-completion \( A_\Theta \).

**Definition 1.** Let \( A_\Theta \) and \( A_\Theta \) be as above, and assume that the “dimension” \( d = 2n \) is even. We will refer to \( \mathfrak{g} = \mathbb{R} \delta_1 + \cdots + \mathbb{R} \delta_{2n} \) as the tangent space. A complex structure on \( A_\Theta \) and \( A_\Theta \) is a choice of an endomorphism \( J \) of \( \mathfrak{g} \) satisfying \( J^2 = -1 \). It thus defines an isomorphism \( \mathfrak{g}_C \cong \mathfrak{g}^{\text{hol}} \oplus \mathfrak{g}^{\text{antihol}} \) as a direct sum of holomorphic and antiholomorphic tangent spaces, namely the \( \pm i \)-eigenspaces of \( J \). There is a similar splitting of the complexified cotangent space \( \mathfrak{g}_C^* \). The pair \((A_\Theta, J)\) will be called a noncommutative complex torus of complex dimension \( n \).

If \( J \) is as in Definition 1, then the standard \( n \)-torus \( \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n} \), together with the complex structure given by \( J \), can equivalently be viewed as \( \mathbb{C}^n/\Lambda \), with the standard complex structure on \( \mathbb{C}^n \) (given by multiplication by \( i \)), identified with the \( +i \)-eigenspace of \( J \), but with \( \Lambda \) now a “skewed” lattice in \( \mathbb{C}^n \) given by \( \Lambda = Q \cdot \mathbb{Z}^{2n} \), where \( Q \) is an \( n \times 2n \) matrix with values in \( \mathbb{C} \). Then a famous classical theorem, due basically to Riemann, though not rigorously proved until later in the 19th century,\(^1\) is

**Theorem 2** (Riemann Period Relations [4, Theorem 2.3]). The complex torus \( \mathbb{C}^n/\Lambda, \Lambda = Q \cdot \mathbb{Z}^{2n} \), is a complex abelian variety if and only if \( Q \) satisfies the Riemann conditions, i.e., after suitable change of basis, it can be put in the form \((\Omega, I_n)\) with \( \Omega \in M_n(\mathbb{C}) \) symmetric with positive-definite imaginary part.

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\(^1\) gives a detailed discussion of the history.
Note that the condition of Theorem 2 always holds in complex dimension 1, since any lattice in $\mathbb{C}$ can always be brought into the form $\mathbb{Z} + \tau \mathbb{Z}$, $\Re \tau > 0$, after multiplying by a nonzero complex number to send one of the basis vectors to 1. In higher dimensions, since the space of $J$’s can be identified with $\text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C})$, which has real dimension $2n^2$ or complex dimension $n^2$, whereas the Siegel upper half-space of symmetric matrices with positive-definite imaginary part can be identified with $\text{Sp}(2n, \mathbb{R})/U(n)$, which has real dimension $n^2 + n$ and complex dimension $\frac{1}{2}(n^2 + n)$ (strictly smaller than $n^2$ for $n > 1$), not every complex torus is an abelian variety.

There is an equivalent version of Theorem 2 which is sometimes more convenient:

**Theorem 3** (Riemann Period Relations, alternate version [15, Ch. I, §3], [14, Theorem 2.8], and [4, Theorem 3.1]). Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Then $X$ is a complex abelian variety if and only if it admits a Riemann form, i.e., there is a nondegenerate skew-symmetric bilinear form $E : \Lambda \times \Lambda \to \mathbb{Z}$ whose extension $E_R$ to a skew-symmetric real bilinear form on $\mathbb{C}^n$ satisfies $E_R(iv, iw) = E_R(v, w)$ and such that the associated hermitian form $H(v, w) = E_R(iv, w) + iE_R(v, w)$ is positive definite.

We will be interested in determining the extent to which something similar to Theorem 2 or Theorem 3 holds in the noncommutative world. The difficulty, of course, is that there is no accepted definition of a noncommutative abelian variety, or clarity about whether various possible definitions are equivalent or not. This problem was studied already in [13, 9, 10, 11], using different methods and slightly different definitions, but none of the approaches has yet completely solved the problem.

Given a complex structure $J$ on $A_{\Theta}$, one has a Hodge splitting of $\wedge g_{C}^*$ as a direct sum of pieces

$$\wedge^{p,q} g_{C}^* = \wedge^p (g_{C}^*)^{\text{hol}} \otimes \wedge^q (g_{C}^*)^{\text{antihol}}$$

and we define $\Omega^{p,q}(A_{\Theta}) = A_{\Theta} \otimes \wedge^{p,q} g_{C}^*$. The Dolbeault complex of $A_{\Theta}$ is

$$A_{\Theta} = \Omega^{0,0}(A_{\Theta}) \xrightarrow{\bar{\partial}} \Omega^{0,1}(A_{\Theta}) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{0,n}(A_{\Theta}),$$

where the $\bar{\partial}$ operator is defined by

$$\bar{\partial}(a) = \sum_j \bar{\partial}_j(a) \otimes d\bar{z}_j,$$

where $\{\bar{\partial}_j\}_{j=1, \ldots, n}$ is a basis for $g^{\text{antihol}}$ and $\{d\bar{z}_j\}_{j=1, \ldots, n}$ is the dual basis of $(g^{\text{antihol}})^*$. We extend this as usual to a map $\Omega^{p,q}(A_{\Theta}) \to \Omega^{p,q+1}(A_{\Theta})$. Note that this is a special case of the exterior differential calculus defined by Connes in [5], and we will use much of Connes’ theory in what follows.

Now we repeat a definition from [15]:
Definition 4. A vector bundle $E$ over $A$ will mean a finitely generated projective (right) module. (This is consistent with the usual notion of vector bundle because of the Serre-Swan Theorem.) A holomorphic vector bundle $E$ will mean such a bundle equipped with a holomorphic connection $\nabla$, meaning a map $E \to E \otimes (\mathfrak{g}^{\text{antihol}})^*$ satisfying the Leibniz rule

$$\nabla_{\bar{\partial}}(e \cdot a) = \nabla_{\bar{\partial}}(e) \cdot a + e \cdot \bar{\partial}(a).$$

Note that any vector bundle can be equipped with a holomorphic connection simply by writing $E = p(A \otimes \Theta)^r$ for some projection $p$, and then defining $\nabla = p(\bar{\partial})^r$. If $\nabla$ and $\nabla'$ are two holomorphic connections, then when one subtracts one from the other, the second term on the right in (1) cancels, and thus the space of holomorphic connections for a fixed projective module $E$ is a principal homogeneous space for $\text{End}_{A \otimes \Theta}(E) \otimes (\mathfrak{g}^{\text{antihol}})^*$. Note that $\text{End}_{A \otimes \Theta}(E)$ is a Morita equivalent noncommutative torus, which we can think of as acting on the left.

But most of the time we will be interested in flat holomorphic connections, that is, ones satisfying the flatness condition that one gets a Dolbeault complex

$$(2) \quad E = \Omega^{0,0}(E) \to \Omega^{0,1}(E) \to \cdots \to \Omega^{0,n}(E),$$

i.e., such that $(\nabla)^2 = 0$. The cohomology groups of (2) will be denoted $H^j(E)$. (We suppress mention of the choice of flat holomorphic connection, as this usually will be understood.) For example, when $E$ is a rank-one free module with the connection given by $\bar{\partial}$ itself, $H^j(E) \cong \bigwedge^j \mathbb{C}^n$ has dimension $\binom{n}{j}$, and $H^0(E)$ consists precisely of the constants $\mathbb{C}$.

Incidentally, the flatness condition $(\nabla)^2 = 0$ is a new and nontrivial condition when $n > 1$. As a result, in higher dimensions, it is not always immediately clear if a given projective module admits a flat holomorphic connection or not. In [20], Rieffel shows that if $\Theta$ is irrational, i.e., contains at least one irrational entry, then every projective module is “standard” and admits a standard connection with constant curvature, i.e., the curvature is just given by a(n explicit) linear functional on $\bigwedge^2 \mathfrak{g}$. If this linear functional is of Hodge type $(1,1)$, so it has no $(0,2)$-component, then the antiholomorphic part of the standard connection is a flat holomorphic connection. Explicit calculations in [10, 11] show that this may or may not be the case.

We need the flatness condition to define cohomology, since a noncommutative torus does not have “points” and thus we cannot use sheaf cohomology. However, $H^0(E) = \ker \nabla$, the space of “holomorphic sections,” is defined for any holomorphic bundle, whether or not the connection is flat.

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2This condition should be familiar from Kähler geometry.

3In fact, for generic $\Theta$, $A_{\Theta}$ is a simple algebra.
Definition 5. Let \((E, \nabla)\) be a holomorphic bundle in the sense of Definition 4. Its index or Euler characteristic \(\chi(E)\) is defined to be the index of the operator 
\[
\nabla + (\nabla)^*: \Omega^0,\text{even}(E) \to \Omega^0,\text{odd}(E),
\]
which is an elliptic operator in the sense of [3]. Note that this is well-defined whether or not \(\nabla\) is flat. The adjoint \((\nabla)^*\) here is defined using an appropriate Hilbert space completion of \(\Omega^{\bullet, \bullet}(E)\), which will be explained in detail in the proofs of the main theorems in Section 2. The Hilbert space inner product is not unique, but the non-uniqueness won’t affect the value of the index.

The following main theorems are in some sense “known” but we have not found an explicit reference for them, except in complex dimension 1, where they appear as [18, Proposition 2.5] and [16, Theorem 2.8].

**Theorem 6** (Riemann-Roch Theorem). Let \(J\) be a holomorphic structure on \(A_\Theta\), and let \(E\) be a holomorphic vector bundle over \((A_\Theta, J)\). Then 
\[
\chi(E) = \text{Ch}_{\text{top}}([E]).
\]

Here the right-hand side of the formula is defined as follows. The vector bundle \(E\) defines a class \([E] \in K_0(A_\Theta) \cong K_0(A_\Theta)\) (the equality of the two \(K\)-groups follows immediately from [3, Lemme 1]). This group is free abelian; in fact \(K_0(A_\Theta)\) is sent isomorphically under the Chern character to the exterior algebra \(\bigwedge^* H^1(T^{2n}) = \bigwedge^* \mathbb{Z}^{2n}\). (See [6] and [2].) The denominators in the Chern character cancel out for basically the same reasons as in [7, Proposition 4.3].) The notation \(\text{Ch}_{\text{top}}([E])\) means the component of the Chern character in the top-degree summand \(\bigwedge^{2n} \mathbb{Z}^{2n} \cong \mathbb{Z}\). In the case \(n = 1\) studied by Connes, Polishchuk, and Schwarz, this reduces to what would usually be called the Chern class \(c_1([E])\), or what Polishchuk and Schwarz call the degree.

**Theorem 7** (Hodge Theorem). If, in the situation of Theorem 6, the holomorphic connection on \(E\) is flat, so that the cohomology groups \(H^j(E)\) are defined, then these groups are finite-dimensional, and \(\chi(E) = \sum_{j=0}^n (-1)^j \dim H^j(E)\).

**Remark 8.** We would like to point one that Theorem 6 contains within it an extraordinary rigidity result. Namely, the Euler characteristic \(\chi(E)\) is independent of the complex structure \(J\) and of the holomorphic structure on \(E\), as well as of the noncommutativity parameter \(\Theta\) (within a family of holomorphic bundles for different noncommutative tori with varying \(\Theta\)).

However, the actual cohomology groups in Theorem 7 do indeed depend on this data, as it is easy to see from [16] that there are cases (with complex dimension \(n = 1\)) where one gets cancellation in this formula, whereas by [18, Proposition 2.5], for “standard” holomorphic bundles with nonzero \(c_1\) (i.e., nonzero degree), this does not happen, and only one of the two cohomology groups is nonzero. The paper [12] studies jumps in the cohomology groups \(H^j(E)\) (for \(E\) a rank-one free module) in more detail in the case \(n = 1\).
The rigidity of the index shows that noncommutative tori are indeed very special. Within the class of compact complex manifolds, one can find examples of diffeomorphic manifolds (with different complex structures) and different values of the Todd genus, which is the Euler characteristic (or index) of the structure sheaf $O$. This was first observed by Hirzebruch for almost-complex manifolds diffeomorphic to $\mathbb{CP}^3$ [8, §2.1] and improved by Kotschick [12], who gave examples that are actually diffeomorphic (but necessarily non-birational) complex projective varieties. Of course this cannot happen with complex tori since they always have vanishing Chern classes.

The next section, Section 2, contains the proofs of the main theorems, Theorem 6 and Theorem 7. The last section, Section 3, contains discussion of specific cases, as well as Theorems 13, 14, and 17, which are noncommutative replacements for the Riemann conditions, Theorems 2 and 3.

2. PROOFS OF THE RIEMANN-ROCH AND HODGE THEOREMS

In this section we give the proofs of the two main theorems, Theorem 6 and Theorem 7.

Proof of Theorem 6. We begin with the case where $E = (A_\Theta)^r$ is a free module over $A_\Theta$, in which case (as a Fréchet space) it can be identified with $S(T^{2n})^r$. If we take $\nabla$ to be just $(\bar{\partial})^r$, then (forgetting the noncommutative structure) $\nabla$ gives rise to the usual Dolbeault complex on the Schwartz space for a translation-invariant complex structure $J$ and flat holomorphic connection on $T^{2n}$. Choose a $J$-invariant Euclidean structure on $\mathbb{R}^{2n}$; this will give rise to a translation-invariant Kähler metric on $(T^{2n}, J)$. Thus the cohomology groups are $H^j(E) \cong (\bigwedge^j \mathbb{C}^n)^r$ (really one should use the conjugate space $(\bigwedge^j \mathbb{C}^n)^r$ for the reasons explained in [15, §1]) and the operator $D = \nabla + \nabla^*$ from even to odd forms is Fredholm, and $T = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$ is essentially self-adjoint (on $L^2(T^{2n})^r$ for our choice of Riemannian metric). By the usual Hodge Theorem, the index $\chi(E) = 0$.

If we keep $E$ a free module but change the holomorphic connection to $\nabla'$, the new connection differs from the old one by a multiplication operator of order 0, while $D$ is an elliptic differential operator of order 1. So the new operator $T' = \begin{pmatrix} 0 & D' \\ D'^* & 0 \end{pmatrix}$ differs from $T$ by a lower-order perturbation, and the index remains unchanged.

Next, suppose we are given a holomorphic connection $\nabla_E$ on $E$, as well as a complementary bundle $F$ with a holomorphic connection $\nabla_F$, so that the direct sum $E \oplus F$ is free. Then $\nabla = \nabla_E \oplus \nabla_F$ is a holomorphic connection for a free module, so by the cases we’ve already considered, $\ker(\nabla + \nabla^*)$ is finite-dimensional and the index of the operator from even to odd forms is 0. It follows that $D = \nabla_E + \nabla_F^*$ from even to odd forms is Fredholm, and $T = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$ is
essentially self-adjoint on the Hilbert space completion of $E$. In particular, the index $\chi(E)$ is well-defined, and independent of the choice of connection (by the independence result for free modules). We also obtain that $\chi(E) + \chi(F) = 0$.

In fact, we also obtain that $\chi(E)$ is independent of the complex structure $J$, since as we vary the $J$ continuously, we get a continuous family of Fredholm operators $T_J$, and thus the index cannot change.

With all of these facts at our disposal, we see that $\chi$ is a well-defined homomorphism $K_0(A_\Theta) \to \mathbb{Z}$. To prove the index theorem, it suffices to compute this map on a set of generators for $K_0$.

As noted in [10, Proof of Proposition 2.6], the theorem can now be reduced to the commutative case, i.e., to the Hirzebruch Riemann-Roch Theorem for complex manifolds. The reason is that Elliott [6] gives us a canonical isomorphism of $K_0(A_\Theta)$ with the free abelian group $K^0(T^{2n})$, that varies continuously (and thus remains constant) as $\Theta$ changes. First we compute the homomorphism $\chi$ in the commutative case. Choose a finite collection of vector bundles $E$ on $T^{2n}$ generating $K_0$. For each choice of $E$, we have (since the Todd polynomial of any complex torus is just the constant 1) by Hirzebruch Riemann-Roch

$$\chi(E) = \int_{T^{2n}} \text{Ch}(E) \cdot \text{Td}(T^{2n}) = \text{Ch}_{\text{top}}([E]).$$

So the theorem holds in the commutative case.

Now given the bundle $E$ over $T^{2n}$, the homotopy $t \mapsto t\Theta$, $t \in [0, 1]$ gives us a homotopy $E_t$ of vector bundles (so that $E_t$ is a vector bundle for $A_\Theta$ and the $K_0$ class of $E_t$ doesn’t change under the Elliott parametrization of $K_0$). In this way we again get a continuous family $T_t$ of Fredholm operators (on the same Hilbert space $L^2(E)$, the Hilbert space of $L^2$ sections of $E$ with respect to Lebesgue measure on the base and some hermitian metric on the bundle), and so the index doesn’t change. Thus the theorem holds in general. □

**Remark 9.** There is one little subtlety which we should note here, which is that in the commutative case, not every positive $K_0$-class is represented by a vector bundle. (It may only be represented by a formal difference of vector bundles. This was noted in [20, p. 320].) So if one starts with a holomorphic bundle for a noncommutative complex torus, after perturbing $\Theta$ to 0 it can happen that the bundle degenerates (i.e., no longer gives an actual bundle). This is not such a problem since one can take care to use $K_0$ generators that correspond to actual vector bundles even in the commutative case. (It suffices to jack up the rank by adding a free module in order to get into the stable range.)

**Proof of Theorem 7.** Suppose $(A_\Theta, J)$ is a noncommutative complex torus, and let $\nabla$ be a flat holomorphic connection on a smooth vector bundle $E$. We basically follow the method of proof of the classical Hodge Theorem, using the machinery developed in [11 Proposition 6.5] and [3]. The main technical issue is that $\nabla$ is *a priori* defined on $E$, which inherits a Fréchet space structure from $A_\Theta$, but
we will want to use analysis of operators on Hilbert spaces, so we need Hilbert space completions for the spaces $\Omega^{0,j}(E)$. For this we use $L^2(\text{Tr})$, the Hilbert space completion of $A_\Theta$ in the GNS representation of the tracial state $\text{Tr}$ (this will usually be a II$_1$ factor representation). In the commutative case $\Theta = 0$, $L^2(\text{Tr})$ simply becomes $L^2(T^{2n})$ with respect to normalized Lebesgue measure, so this is something familiar. Since $E$ is a direct summand in $(A_\Theta)^r$ for some $r$, we get a Hilbert space completion $H$ of $E$, namely the closure of $E$ in $L^2(\text{Tr})^r$, and Hilbert spaces $H_j$ completing $\Omega^{0,j}(E)$, with $H_0 = H$.

Next, we observe that $\nabla : \Omega^{0,j}(E) \to \Omega^{0,j+1}(E)$ is closable; this follows from the case of a free module with an arbitrary holomorphic connection, since $\bar{\partial}$ looks like a constant-coefficient first-order differential operator acting on the Schwartz space $S(T^{2n})$, and on a free module, $\nabla$ differs from $\bar{\partial}^r$ by a zero-order operator. We let $d_j : H_j \to H_{j+1}$ denote the closed extension of $\nabla$. Then

$$H_0 \xrightarrow{d_0} H_1 \xrightarrow{d_1} \cdots \xrightarrow{d_n} H^n$$

(note that the operators are closed but only densely defined) is a Hilbert complex in the sense of [3].

The operator $D = \nabla + \nabla^*$ (sending even forms to odd forms and vice versa) is elliptic, in the sense that if we combine it with the corresponding operator coming from a holomorphic connection (not necessarily flat) on a complementary vector bundle, we get an elliptic (vector-valued) first-order operator on $S(T^{2n})$. Thus it’s essentially self-adjoint on $\bigoplus H_j$. So all the theory of [3] §§2–3] immediately applies (see in particular Theorem 2.12 and Theorem 3.5), the Hilbert complex is Fredholm (with the same cohomology as the smooth complex), and the Hodge Theorem follows.

3. Examination of Some Specific Cases

3.1. The Product Case. The simplest kinds of complex higher-dimensional noncommutative tori are what one can call “products of noncommutative elliptic curves.” These are the noncommutative analogues of products of elliptic curves, which are very special even within the class of abelian varieties, let alone within all complex tori. For example, when $n = 2$, products of two elliptic curves vary within a moduli space of complex dimension 2, while abelian varieties of complex dimension 2 vary within a moduli space of complex dimension 3, and complex tori vary in a moduli space of complex dimension 4.

To be more precise, we make a definition.

**Definition 10.** A noncommutative complex torus $A_\Theta$ will be said to be of product type if the $2n \times 2n$ matrices $J$ and $\Theta$ both split as block direct sums of $2 \times 2$ matrices.
of the same form, i.e.,

$$\Theta = \begin{pmatrix} 0 & \theta_1 & 0 & \cdots & 0 \\ -\theta_1 & 0 & \theta_2 & \cdots & 0 \\ 0 & -\theta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \theta_n \\ -\theta_n & 0 & \cdots & \theta_n & 0 \end{pmatrix},$$

and similarly with $J$, except that the diagonal blocks in this case can be arbitrary $2 \times 2$ real matrices $J_j$ with trace 0 and determinant 1, $j = 1, \ldots, n$. Note that in the product-type case, $A_\Theta = A_{\theta_1} \otimes \cdots \otimes A_{\theta_n}$, where $A_\theta$ denotes the noncommutative 2-torus with parameter $\theta$, and the complex structure $J$ also splits as $J_1 \otimes \cdots \otimes J_n$.

For simplicity of notation, a complex noncommutative torus with complex dimension 1 will be called a noncommutative elliptic curve.

In the product-type case, we can carry over the results of [18], [16], [17], etc., to get a complete description of the holomorphic geometry of $(A_\Theta, J)$. As this should be regarded as the “trivial case” of higher-dimensional noncommutative tori, we do not go into exhaustive detail, but give one representative theorem. Recall that by [16], the holomorphic vector bundles over a noncommutative elliptic curve are completely classified. They are formed from iterated extensions of the standard holomorphic bundles considered in [18].

**Theorem 11.** Suppose $(A_\Theta, J) = (A_{\theta_1}, J_1) \otimes \cdots \otimes (A_{\theta_n}, J_n)$ is of product type, and let $E_j$ be a holomorphic vector bundle for the noncommutative elliptic curve $(A_{\theta_j}, J_j)$. Then $E = E_1 \otimes \cdots \otimes E_n$ is a holomorphic vector bundle for $(A_\Theta, J)$ with a flat holomorphic connection, and its cohomology groups are given by $H^\bullet(E) = \bigotimes_{j=1}^n H^\bullet(E_j)$. (The gradings add, and since $H^\bullet(E_j)$ is concentrated in degrees 0 and 1, $H^1(E)$ is the sum of the tensor products with $j$ factors in degree 1.)

**Proof.** It is clear that we get a flat holomorphic connection on $E$ by operating separately on the tensor factors. The cohomology calculation follows since the Dolbeault complex for $E$ splits as the tensor product of the complexes for the noncommutative elliptic curve factors.

3.2. The Splitting Case. A case which is not quite as simple as that of product type noncommutative complex tori, but where many of the same considerations apply, is the case where one has a “splitting.” This is the noncommutative analogue of the following classical situation. Suppose $\mathbb{C}^n/\Lambda$ is a complex torus, and suppose there is surjective $\mathbb{C}$-linear map $f: \mathbb{C}^n \to \mathbb{C}$ sending $\Lambda$ to a lattice in $\mathbb{C}$. That means the complex torus $\mathbb{C}^n/\Lambda$ has an elliptic curve $\mathbb{C}/f(\Lambda)$ as a quotient, and thus splits as an extension of this elliptic curve by the complex subtorus
(ker $f$)/$(\text{ker } f|_{\Lambda})$. When $n = 2$, we thus have an extension of one elliptic curve by another. The following is a fact about classical complex algebraic geometry.

**Remark 12.** Let $\mathbb{C}^2/\Lambda$ is a complex torus which is an extension of one elliptic curve by another. In other words, assume there exist $\tau, \tau'$ in the upper half-plane and complex linear maps $\alpha: \mathbb{C} \to \mathbb{C}^2$, $\beta: \mathbb{C}^2 \to \mathbb{C}$ such that

$$0 \to \mathbb{C} \xrightarrow{\alpha} \mathbb{C}^2 \xrightarrow{\beta} \mathbb{C} \to 0$$

and

$$0 \to (\mathbb{Z} + \tau'\mathbb{Z}) \xrightarrow{\alpha} \Lambda \xrightarrow{\beta} (\mathbb{Z} + \tau\mathbb{Z}) \to 0$$

are short exact. Then $\mathbb{C}^2/\Lambda$ need not be an abelian variety.

Indeed, the hypothesis of the proposition implies that after changing the basis for $\mathbb{C}^2$ by an element of $\text{GL}(2, \mathbb{C})$, we can assume $\Lambda$ is spanned by the vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} \tau' \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, e_4 = \begin{pmatrix} w \\ \tau \end{pmatrix},$$

for some $w \in \mathbb{C}$. If $w$ were 0 then clearly we’d have the product of two elliptic curves and this would be an abelian variety. In fact, an extension of abelian varieties within the category of abelian varieties has to split up to isogeny [14, Proposition 10.1]. But for sufficiently generic $w$, it is easy to see that this torus does not split; in fact, that $\mathbb{C}^2/\Lambda$ and the dual torus $(\mathbb{C}^2)^*/\Lambda^*$ are not isogenous, which implies that we do not have an abelian variety$^4$

In spite of Remark 12, we still get some useful information in the noncommutative splitting situation. Suppose $A_\Theta$ is a noncommutative torus with a holomorphic structure that “splits,” i.e., such that $J$ is of block diagonal form, say

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix},$$

with $J_1$ of size $2 \times 2$. Then we obtain an obvious holomorphic injection

$$\varphi: (A_\theta, J_1) \to (A_\Theta, J), \text{ where } \theta = \Theta_{12},$$

given by $u_j \mapsto U_j$, with $u_j$, $j = 1, 2$, the canonical generators of $A_\theta$, which respects smooth structures. The holomorphic geometry of the noncommutative elliptic curve $(A_\theta, J_1)$ is completely understood, thanks to [18], [16], and [17].

Let $E$ be a holomorphic vector bundle over $A_\theta$, $\theta = \Theta_{12}$ and $J_1$ as in (3). Then $\varphi_*(E) = E \otimes_{\varphi A_\theta} A_\Theta$ will be a holomorphic vector bundle over $(A_\Theta, J)$, and knowledge of the cohomology of $E$ should give some information about its cohomology.

To see what one should expect, consider the classical case where $X$ is a complex torus (of complex dimension $n$) with a (nontrivial) holomorphic group homomorphism to an elliptic curve $C$. We have a short exact sequence of complex tori

$$1 \to F \to X \xrightarrow{p} C \to 1,$$

and if $E$ is a holomorphic vector bundle over $C$, there is a Leray spectral sequence

$$H^k(C, R^\ell p_* p^* E) \Rightarrow H^{k+\ell}(X, p^* E).$$

$^4$Here $(\mathbb{C}^2)^*$ is the conjugate-linear dual of $\mathbb{C}^2$ and $\Lambda^* = \{ f \in (\mathbb{C}^2)^* : \text{Im } f(\lambda) \in \mathbb{Z} \ \forall \lambda \in \Lambda \}$ — see [13] I.2.
Here \( \mathcal{E} \) is the sheaf of germs of holomorphic sections of \( E \), whose cohomology is basically what we are calling \( H^*(E) \). By the projection formula, the \( E_2 \) term simplifies to give a Leray-Serre spectral sequence

\[
H^k(C, H^\ell(F, \mathcal{O}_F) \otimes \mathcal{E}) = H^k(C, \mathcal{E}) \otimes H^\ell(F, \mathcal{O}_F) \Rightarrow H^{k+\ell}(X, p^*\mathcal{E}).
\]

The equality on the left follows from the fact that since the fibration is topologically a product, there is no monodromy; i.e., \( \pi_1(C) \) acts trivially on the cohomology of the fiber. Furthermore, since \( C \) is a curve and \( \mathcal{O}_F \) has cohomology only up to dimension \( n-1 \), the spectral sequence is concentrated in the range \( 0 \leq k \leq 1, \ 0 \leq \ell \leq n-1 \). In particular there can’t be any differentials, and the edge homomorphism \( H^k(C, \mathcal{E}) \to H^k(X, p^*\mathcal{E}) \) is injective. So we get a lower bound on the size of \( H^0(X, p^*\mathcal{E}) \).

In a similar fashion, in the noncommutative context we have been talking about, we should get a lower bound on the dimension of \( H^0(\varphi_*(E) = E \otimes_{\varphi} \mathcal{A}_\Theta) \). To see this, let \( \nabla \) be a holomorphic connection on \( E \), and let \( \tilde{\partial}_1 \) on \( \mathcal{A}_\Theta \) correspond to \( \tilde{\partial} \) on \( A_\Theta \). We define \( \nabla' \) on \( \varphi_*(E) \) by \( \nabla_1' = \nabla \otimes 1 + 1 \otimes \tilde{\partial}_1, \nabla_j' = 1 \otimes \tilde{\partial}_j \) for \( j > 1 \). To check that this is consistent, observe that we have the relation \( ea \otimes b = e \otimes \varphi(a)b \) for \( e \in E, \ a \in \mathcal{A}_\Theta, \ b \in \mathcal{A}_\Theta \), but

\[
\nabla_1'(ea \otimes b) = \nabla(ea) \otimes b + ea \otimes \tilde{\partial}_1(b) \\
= (\nabla(e)a + e\tilde{\partial}(a)) \otimes b + ea \otimes \tilde{\partial}_1(b) \\
= \nabla(e) \otimes \varphi(a)b + e \otimes \varphi(\tilde{\partial}(a))b + e \otimes \varphi(a)\tilde{\partial}_1(b) \\
= (\nabla \otimes 1 + 1 \otimes \tilde{\partial}_1)(e \otimes \varphi(a)b) \\
= \nabla_1(e \otimes \varphi(a)b).
\]

And similarly, for \( j > 1 \), since \( \tilde{\partial}_j \) kills the image of \( \varphi \),

\[
\nabla_j'(ea \otimes b) = ea \otimes \tilde{\partial}_j(b) \\
= e \otimes \varphi(a)\tilde{\partial}_j(b) \\
= e \otimes \tilde{\partial}_j(\varphi(a)b) \\
= \nabla_j'(e \otimes \varphi(a)b).
\]

The Leibniz rule for \( \nabla' \) is clear since it holds for each \( \tilde{\partial}_j \). Finally, we check that the flatness condition holds. Suppose \( j, k > 1 \). Since \( \tilde{\partial}_j \) and \( \tilde{\partial}_k \) commute, so do
\[ \nabla_j' \] and \[ \nabla_k' \]. And 
\[
\nabla_j' \nabla_1 (e \otimes b) = \nabla_j' (\nabla (e) \otimes b + e \otimes \bar{\partial}_1 (b)) \\
= \nabla (e) \otimes \bar{\partial}_j (b) + e \otimes \bar{\partial}_j (\bar{\partial}_1 (b)) \\
= \nabla (e) \otimes \bar{\partial}_j (b) + e \otimes \bar{\partial}_1 (\bar{\partial}_j (b)) \\
= \nabla'_1 (e \otimes \bar{\partial}_j (b)) \\
= \nabla'_1 \nabla_j (e \otimes b).
\]
Thus \[ \nabla' \] is indeed a flat holomorphic connection on \( \varphi^*(E) \). Since it obviously annihilates \( e \otimes 1 \) for any \( e \in \ker \nabla \), we have proved the following:

**Theorem 13.** Suppose \( A_\Theta \) is a noncommutative torus with a holomorphic structure that “splits,” i.e., such that \( J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \), with \( J_1 \) of size \( 2 \times 2 \). Define the holomorphic injection
\[
\varphi: (A_\theta, J_1) \to (A_\Theta, J), \quad \text{where } \theta = \Theta_{12},
\]
by \( u_j \mapsto U_j \), with \( u_j, \ j = 1, 2 \), the canonical generators of \( A_\theta \). Then if \( E \) is a holomorphic bundle for \( (A_\theta, J_1) \) with holomorphic connection \( \nabla \), there is an induced flat holomorphic connection \( \nabla' \) on \( \varphi^*(E) = E \otimes_{\varphi} A_\Theta \), and \( \dim H^0(\varphi^*(E)) \geq \dim H^0(E) \).

### 3.3. The Non-Algebraic Case.

Finally, we prove a result which is an analogue of the fact that there exists a complex torus \( A_\Theta \) of complex dimension 2 with no non-constant meromorphic functions. (This phenomenon is analyzed in detail in [22].)

**Theorem 14.** There is a noncommutative complex torus of complex dimension 2 with the property that for any non-free standard projective module with constant curvature connection as in [20], the associated constant curvature holomorphic connection \( \nabla \) is non-flat.

**Remark 15.** We would conjecture, though it seems hard to figure out how to prove this, that one can even arrange for \( A_\Theta \) to admit no non-trivial “meromorphic functions.” Here a possible definition of meromorphic function is the one from [21, §6]: it is a formal quotient \( u^{-1} v \) where \( u \) is neither a left nor right zero divisor, and where \( u \) and \( v \) satisfy the Cauchy-Riemann equations
\[
\bar{\partial}_j (u) = f_j u, \quad \bar{\partial}_j (v) = f_j v.
\]

**Proof of Theorem 14.** We use [20, Theorem 4.5], which gives an explicit formula for the curvature of each standard vector bundle with its constant curvature connection.
connection. Each such bundle has a height $p \leq n$. When $n = 2$, there are three possibilities for $p$. The case $p = 0$ just corresponds to a free module with the trivial flat connection. So we only need to consider the cases $p = 1$ and $p = 2$. Let $\bar{\partial}_1, \bar{\partial}_2$ be a basis for $g^{\text{antihol}}$. The $(0, 2)$ component of the curvature, evaluated on $\bar{\partial}_1 \wedge \bar{\partial}_2$, is a non-zero constant multiplied by $\mu(\bar{\partial}_1 \wedge \bar{\partial}_2 \wedge \Theta)$ when $p = 2$, where $\mu$ is a generator of $\bigwedge^4(\mathbb{Z}^4)^*$, or is $\mu(\bar{\partial}_1 \wedge \bar{\partial}_2)$ when $p = 1$, where in this case $\mu$ is a decomposable element of $\bigwedge^2(\mathbb{Z}^4)^*$. Hence to get the desired conclusion we just need to choose $J$ so that $\mu(\bar{\partial}_1 \wedge \bar{\partial}_2) \neq 0$ for all $\mu = \alpha \wedge \beta$, for any linearly independent $\alpha$ and $\beta$ in the lattice $(\mathbb{Z}^4)^*$, and then choose $\Theta$ so that $\bar{\partial}_1 \wedge \bar{\partial}_2 \wedge \Theta \neq 0$. These conditions will be satisfied for “generic” $J$ and $\Theta$ (not lying in a countable union of proper subvarieties in the spaces of real $4 \times 4$ matrices with $J^2 = -1$ and $\Theta = -\Theta^t$).

Remark 16. It should be obvious that the same method of proof will work for any complex dimension $n > 2$ as well.

3.4. The Case of the Riemann Relations. The last case we consider will be the one where $J$ satisfies the Riemann conditions of Theorem 2. We will try to give a weak substitute for Theorem 3, showing that our noncommutative complex torus in this case admits a holomorphic bundle with “lots of holomorphic sections.”

Theorem 17. Let $(A_\Theta, J)$ be a noncommutative complex torus of complex dimension $n$, and assume that the underlying $(\mathbb{Z}^{2n}, J)$ admits a Riemann form $S$. Then there is a holomorphic bundle $E$ over $(A_\Theta, J)$ admitting a flat holomorphic connection, with $\dim H^0(E) > 1$.

Proof. We use some of the same argument that went into the proof of Theorem 3. Namely, we use [20, Theorem 4.5] to construct a standard module $E$ of height $p = 1$, but this time we arrange to have $\mu(\bar{\partial}_j \wedge \bar{\partial}_k) = 0$ for some $\mu = \alpha \wedge \beta$, $\alpha$ and $\beta$ in the lattice $\mathbb{Z}^{2n}$, and for all $j, k = 1, \ldots, n$. This corresponds to the construction of an ample line bundle with curvature of Hodge type $(1, 1)$, which is the essence of the proof in the classical case of Theorem 3.

A technical problem here is that we can’t just take pairing with $\mu$ to be given by the Riemann form $S$, because since $S$ is non-degenerate, it can’t be decomposable. (Any decomposable 2-form $\mu$ satisfies $\mu \wedge \mu = 0$, whereas $S \wedge S \wedge \cdots \wedge S \neq 0$.) However, we can write $S = S_1 + \cdots + S_n$ with $S_j$ decomposable for each $j$ and satisfying the same conditions as $S$ except that $S_j$ is the imaginary part of a positive semidefinite hermitian form $H_j$ with kernel of complex dimension $n - 1$. (To get the decomposition of $S$, merely choose a canonical basis of the lattice $\Lambda \cong \mathbb{Z}^{2n}$ as in [4, Definition 3.2.2], and let $S_j$ be equal to $S$ on the span of $\nu_j$ and $\nu_{j+n}$ and 0 on the span of the other basis vectors.)

Now take pairing with $\mu$ to be given by the alternating form $S_1$. In the Riemann form way of looking at things, we have embedded our lattice $\Lambda$ in $g^{\text{hol}}$, so that the complex structure is given by multiplication by $i$. The alternating form $S_1$ and
hermitian form $H_1$ factor through $\mathbb{C}\nu_1$ with lattice spanned by $\nu_1$ and $\nu_{n+1}$, and we are are reduced to the “splitting case” of Section 3.2. The conclusion now follows from the case of complex dimension 1 and Theorem 13.

**Remark 18.** Unfortunately, to get the analogue of an ample line bundle in the situation of Theorem 17, we really should take the tensor product of the holomorphic bundles corresponding to each of the $S_j$ (in the notation of the proof), but we don’t have a notion of tensor product of bundles in the general noncommutative case. The bundle constructed in the proof of Theorem 17 still has index 0.

But there is one case where we can get an analogue of an ample line bundle. If $\Theta$ splits as a block diagonal matrix with $2 \times 2$ blocks, in the sense that (in the notation of the proof of Theorem 17) $\nu_j$ and $\nu_{j+n}$ commute with $\nu_k$ and $\nu_{k+n}$ for $j \neq k$, then we get a holomorphic map from a tensor product of noncommutative elliptic curves into $(A_\Theta, J)$ as in the product case of Section 3.1, and the tensor product of the holomorphic bundles corresponding to the various $S_j$’s makes sense. So in this special case we get something close to the original theorem of Riemann.

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