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Some properties for Weyl’s projective Curvature Tensor of Generalized $W^h$-Birecurrent in Finsler Space

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Abstract

In this paper, we defined a Finsler space $F_n$ for which Weyl’s projective curvature tensor $W_{jkh}^i$ satisfies the generalized-birecurrence condition with respect to Cartan’s connection parameters $\Gamma_{jkh}^i$, given by the condition $W_{jkh \mid \mid lm}^i = \alpha_{lm}W_{jkh}^i + \beta_{lm} \left( \delta_k^i g_{jk} - \delta_j^i g_{jk} \right)$, where $l|m$ is h-covariant derivative of second order (Cartan’s second kind covariant differential operator) with respect to $x^l$ and $x^m$, successively, $\alpha_{lm}$ and $\beta_{lm}$ are non-null covariant vectors field and such space is called as a generalized $W^h$-birecurrent space and denoted briefly by $G W^h$-BRE$_n$. We have obtained the h-covariant derivative of the second order for Wely’s projective torsion tensor $W_{jkh}^i$, Wely’s projective deviation tensor $W_{jkh}^i$, and Wely’s projective curvature tensor $W_{jkh}^i$ and some tensors are birecurrent in our space. We have obtained the necessary and sufficient condition for Cartan’s third curvature tensor $R_{jkh}^i$, the associate curvature tensor $R_{jkh}^i$, and the associate torsion tensor $H_{kp,h}$ and the deviation tensor $H_{h}^i$ has been obtained in our space.

Key words: Finsler space, Generalized $W^h$- Birecurrent space, Weyl’s projective curvature tensor $W_{jkh}^i$, Cartan’s third curvature tensor $R_{jkh}^i$.

1. Introduction

On account of the different connections of Finsler space, the concept of the recurrent for different curvature tensors have been discussed by Matsumot [6], Pandey ([8], [9]), Dubey and Srivastava [4], Pandey and Misra [10], Pandey and Dwivedi [11], Verma [19], Dikshit [3], Qasem [14], Mishra and Lodhi [7], P.N. Pandey and Pal [12] and others. The generalized recurrent space was studied by De and Guha [2], Maralebhavi and Rathnamma [5], Pandey, Saxena and Goswani [13], Qasem and Al-Qashbari ([17],[18]), Qasem and Saleem [16] and others. Ahsan and Ali [1] who discussed a recurrent curvature tensor on some properties of W-curvature tensor of Weyl’s projective curvature tensor $W_{jkh}^i$ and others. Also, W-generalized birecurrent space studied by Qasem and Saleem [15] and others.

Let us consider an n-dimensional Finsler space equipped with the metric function $F$ satisfying the requisite conditions [19].

Let consider the components of the corresponding metric tensor $g_{ij}$, Cartan’s connection parameters $\Gamma_{jkh}^i$ and Berwald’s connection parameters $G_{jkh}^i$. These are symmetric in their lower indices.

The vectors $y_i$ and $y^i$ satisfy the following relations [19]:

\begin{align}
\text{a) } y_i = g_{ij} y^j, \quad \text{b) } y_i y^i = F^2 \quad \text{and} \quad \text{c) } \delta_j y^i = \delta^i_j.
\end{align}

The h-covariant derivative of second order for an arbitrary vector field with respect to $x^k$ and $x^l$, successively, we get

\begin{align}
X_{ijkl}^i = \delta_j \left( X_{ik}^i - \left( X_{ik}^j \right) \Gamma_{jkh}^i + \left( X_{ik}^j \right) \Gamma_{rkh}^i - \delta_r \left( X_{ik}^j \right) \Gamma_{rjs}^i \right) y^s.
\end{align}

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Taking skew-symmetric part with respect to the indices \( k \) and \( j \), we get the commutation formula for \( h \)-covariant differentiation as follows [19]:

\[
X^i_{kj} - X^i_{jk} = X^r K^l_{rkj} - \left( \partial_r X^l \right) K^s_{skj} y^s ,
\]

where

\[
K^l_{rkj} := \partial_l \Gamma^r_{kj} + \left( \partial_l \Gamma^r_{ij} \right) G^i_k + \Gamma^r_{mj} \Gamma^m_{kj} - j/k
\]

The tensor \( K^l_{rkj} \) as defined above is called Cartan’s fourth curvature tensor.

The metric tensor \( g_{ij} \) and the vector \( y^i \) are covariant constant with respect to the above process.

\[ g_{ij} = 0 \quad \text{and} \quad y^i_{jk} = 0 ~. \]

The process of \( h \)-covariant differentiation, with respect to \( x^k \), commute with partial differentiation with respect to \( y^j \) for arbitrary vector filed \( X^i \), according to [19]

\[
\partial_j \left( X^i_{ik} \right) - \partial_j X^i_{ij} = X^r \left( \partial_j \Gamma^r_{ik} \right) - \left( \partial_j X^l \right) P^i_{jk} ~.
\]

The quantities \( H^i_{kh} \) and \( H^i_{kh} \) form the components of tensors and they are called \( h \)-curvature tensor of Berwald (Berwald curvature tensor) and \( h(v) \)-torsion tensor, respectively and are defined as follow [19]:

\[
\begin{align*}
\text{a) } & H^i_{kh} := \partial_h G^i_k + G^i_k \Gamma^r_{hj} G^r_k - h/k \\
\text{and} & \\
\text{b) } & H^i_{kh} := \partial_h G^i_k + G^r_k \Gamma^i_{rhj} G^r_k - h/k
\end{align*}
\]

They are also related by [19]

\[
\begin{align*}
\text{a) } & H^i_{kh} \ y^j = H^i_{kh} , \quad \text{b) } H^i_{kh} = \partial_j H^i_{kh} \quad \text{and} \quad \text{c) } H^i_{jk} = \partial_j H^i_{kh} .
\end{align*}
\]

These tensors were constructed initially by means of the tensor \( H^i_k \), called the deviation tensor, given by

\[
\begin{align*}
\text{a) } & H^i_k := 2 \partial_h G^i_k - \partial_r G^i_h y^r + 2 G^r_k G^i_s - G^s_k G^i_h \\
\text{b) } & \partial_h G^i_k = G^i_k
\end{align*}
\]

In view of Euler’s theorem on homogeneous functions and by contracting the indices \( i \) and \( h \) in (1.8) and (1.9), we have the following:

\[
\begin{align*}
\text{a) } & H^i_{jk} y^j = -H^i_{kj} y^j = H^i_k \quad \text{and} \quad \text{b) } g_{ip} H^i_{jk} = H^i_{jk} .
\end{align*}
\]

The tensor \( W^i_{jk} \) is known as projective curvature tensor (generalized Wely’s projective curvature tensor ), the tensor \( W^i_{jk} \) is known as projective torsion tensor ( Wely’s torsion tensor ) and the tensor \( W^i_{jk} \) is known as projective deviation tensor ( Wely’s deviation tensor ) are defined by

\[
\begin{align*}
W^i_{jk} & = H^i_{jk} + \frac{2 \delta^i_j}{n+1} H_{[hk]} + \frac{2 y^i}{n+1} \partial_j H_{[hk]} \\
& + \frac{\delta^i_j}{n^2-1} \left( n H^j_{hj} + H_{hj} + y^r \partial_j H_{hr} \right) - \frac{\delta^i_j}{n^2-1} \left( n H^k_{jk} + H_{jk} + y^r \partial_j H_{kr} \right)
\end{align*}
\]

\[
\begin{align*}
W^i_{jk} & = H^i_{jk} + \frac{y^i}{n+1} H_{[jk]} + 2 \left( \frac{\delta^i_j}{n^2-1} \left( n H^k - y^r H^r \right) \right)
\end{align*}
\]

\[
\begin{align*}
W^i_{jk} & = H^i_{jk} - H \delta^i_j - \frac{1}{n+1} \left( \partial_r H^j_{[rk]} - \partial_j H \right) y^i
\end{align*}
\]

respectively.

The tensors \( W^i_{jk} \), \( W^i_{jk} \) and \( W^i_{jk} \) are satisfying the following identities [19]

\[
\begin{align*}
\text{a) } & W^i_{jk} y^j = W^i_{jk} \quad \text{and} \quad \text{b) } W^i_{jk} y^j = W^i_{jk} .
\end{align*}
\]

The projective curvature tensor \( W^i_{jk} \) is skew-symmetric in its indices \( k \) and \( h \).

Cartan’s third curvature tensor \( R^i_{jk} \) and the R-Ricci tensor \( R_{jk} \) are respectively given by [19]

\[
\begin{align*}
\text{a) } & W^i_{jk} y^j = W^i_{jk} \quad \text{and} \quad \text{b) } W^i_{jk} y^j = W^i_{jk} .
\end{align*}
\]

* The indices \( i, j, k, ... \) assume positive integral values from 1 to \( n \).
** \(-j/k \) means the subtraction from the former term by interchanging the indices \( k \) and \( j \).
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\[ R^i_{jkh} = \delta^h_{r} R^i_{jkr} + (\delta^h_{j} R^i_{kr}) G^i_k + C^i_{lm} \left( \delta^k_{l} R^i_{jmr} - G^i_m G^i_k \right) + \Gamma^i_{mk} \Gamma^r_{jm} - k/h \text{,} \]

b) \[ R^i_{jkh} y^j = H^i_{kh} = K^i_{kjh} y^j \text{,} \]

c) \[ g_{ij} R^i_{jkh} = R^i_{jkh} \text{ and } d) R^i_{jkh} y^j = H^i_k \text{.} \]

2. A Generalized \( W^h \)-Birecurrent Space

A Finsler space \( F^h \) for which Weyl’s projective curvature tensor \( W^i_{jkh} \) satisfies the recurrence property with respect to Cartan’s coefficient connection parameters \( \Gamma^i_{jk} \) which is characterized by the condition [16]

\[ W^i_{jkhil} = \lambda^i_{lm} W^i_{jkh} + \mu^i_{lm} (\delta^i_{l} g_{jk} - \delta^i_{k} g_{jh}) \text{,} \]

where \( l \) is \( h \)-covariant derivative of first order (Cartan’s second kind covariant differential operator) with respect to \( x^l \), the quantities \( \lambda^i_l \) and \( \mu^i_{lm} \) are non-null covariant vectors field. We shall call such space as a generalized \( W^h \)-recurrent space and we shall denote it briefly by \( G W^h \). \( F^h \).

Taking the \( h \)-covariant derivative for (2.1) with respect to \( x^m \) and using (1.5a), we get

\[ W^i_{jkhil} = \lambda^i_{lm} W^i_{jkh} + \mu^i_{lm} (\delta^i_{l} g_{jk} - \delta^i_{k} g_{jh}) \text{, where } g_{jkim} = 0 \text{.} \]

In view of (2.1), the above equation yields

\[ W^i_{jkhil} = \alpha^i_{lm} W^i_{jkh} + \beta^i_{lm} (\delta^i_{l} g_{jk} - \delta^i_{k} g_{jh}) \text{,} \]

where \( \alpha^i_{lm} = \lambda^i_{lm} + \lambda^i_{l} \lambda^i_{m} \) and \( \beta^i_{lm} = \lambda^i_{l} \lambda^i_{m} + \mu^i_{lm} \text{.} \)

Result 2.1. Every generalized \( W^h \)-recurrent space is generalized \( W^h \)-birecurrent space.

Transvecting the condition (2.2) by \( y^j \), using (1.5b), (1.14a) and (1.1a), we get

\[ W^i_{jkhil} = \alpha^i_{lm} W^i_{jkh} + \beta^i_{lm} (\delta^i_{l} y_k - \delta^i_{k} y_h) \text{.} \]

Transvecting (2.3) by \( y^k \), using (1.5b), (1.14b) and (1.1b), we get

\[ W^i_{jkhil} = \alpha^i_{lm} W^i_{jkh} + \beta^i_{lm} (\delta^i_{l} F^j - y_h y^i) \text{.} \]

Thus, we conclude

**Theorem 2.1.** In \( G W^h \)-BRF \( F^h \), the \( h \)-covariant derivative of the second order for Wely’s projective torsion tensor \( W^i_{jkh} \) and Wely’s projective deviation tensor \( W^i \) given by (2.3) and (2.4), respectively.

In view of the equation (1.3), we have

\[ W^i_{jkhil} = W^i_{jkh} K^i_{ljm} - W^i_{rkh} K^r_{ljm} - W^i_{rkh} K^r_{ljm} - W^i_{rkh} K^r_{ljm} - (\delta^i_{l} W^i_{jkh}) K^r_{kim} y^s \text{.} \]

Using the condition (2.2) and (1.15b) in (2.5), we get

\[ \alpha^i_{lm} W^i_{jkh} + \beta^i_{lm} (\delta^i_{l} g_{jk} - \delta^i_{k} g_{jh}) - \alpha^i_{ml} W^i_{jkh} - \beta^i_{ml} (\delta^i_{l} g_{jk} - \delta^i_{k} g_{jh}) = W^i_{jkh} K^i_{ljm} - W^i_{rkh} K^r_{ljm} - W^i_{rkh} K^r_{ljm} - W^i_{rkh} K^r_{ljm} - (\delta^i_{l} W^i_{jkh}) K^r_{kim} y^s \text{.} \]

If \( \alpha^i_{lm} \) and \( \beta^i_{lm} \) are skew-symmetric, then (2.6) can be written as

\[ \alpha^i_{lm} W^i_{jkh} + b^i_{lm} (\delta^i_{l} g_{jk} - \delta^i_{k} g_{jh}) = W^i_{jkh} K^i_{ljm} - W^i_{rkh} K^r_{ljm} - W^i_{rkh} K^r_{ljm} - (\delta^i_{l} W^i_{jkh}) K^r_{kim} y^s \text{.} \]

where \( \alpha^i_{lm} = 2 \alpha^i_{lm} \) and \( \beta^i_{lm} = 2 \beta^i_{lm} \text{.} \)

In view of the condition (2.2), the equation (2.7) is reduced to

\[ W^i_{jkhil} = W^i_{jkh} K^i_{ljm} - W^i_{rkh} K^r_{ljm} - W^i_{rkh} K^r_{ljm} - W^i_{rkh} K^r_{ljm} - (\delta^i_{l} W^i_{jkh}) K^r_{kim} y^s \text{.} \]

Transvecting (2.8) by \( y^j \), using (1.5b), (1.14a), (1.15b) and (1.1c), we get

\[ W^i_{jkhil} = W^i_{jkh} K^i_{ljm} - W^i_{rkh} H^i_{ljm} - W^i_{rkh} K^r_{ljm} - W^i_{rkh} K^r_{ljm} - (\delta^i_{l} W^i_{jkh} - W^i_{rkh}) H^i_{ljm} \text{.} \]
Transvecting (2.9) by $y^k$, using (1.5b), (1.14b) and (1.15b), we get
\[(2.10) \quad W^i_{\text{h}lm} = W^i_{\text{kh}lm} - W^i_{\text{rh}l}H^r_{\text{lm}} - W^i_{\text{lr}k}K^r_{\text{lm}} - \{ W^i_{\text{rkh}l} H^r_{\text{lm}} + ( \partial_r W^i_{\text{kh}l} - W^i_{\text{rk}h} ) H^r_{\text{lm}} \} y^k .\]

Thus, we conclude

\textbf{Theorem 2.2.} In $GW^h$-BRE$_n$, the $h$-covariant derivative of the second order for Weyl’s projective curvature tensor $W^i_{\text{jk}h}$, Weyl’s projective torsion tensor $W^i_{\text{kh}h}$ and Weyl’s projective deviation tensor $W^i_{\text{h}h}$ given by (2.8), (2.9) and (2.10), respectively, provided $\alpha_{im}$ and $\beta_{im}$ are skew-symmetric.

If $\alpha_{im}$ and $\beta_{im}$ are symmetric, then (2.6) can be written as
\[(2.11) \quad W^r_{\text{jk}h} K^i_{\text{rim}} = W^r_{kr} K^i_{\text{rlm}} + W^r_{jr} K^i_{\text{rklm}} + W^r_{jk} K^i_{\text{rklm}} + ( \partial_r W^i_{\text{jk}h}) H^r_{\text{lm}} .\]

Now, taking $h$- covariant derivative for (2.11), with respect to $x^h$ , we get
\[(2.12) \quad W^r_{\text{jk}h} K^i_{\text{rlm}} + W^r_{\text{kh}j} K^i_{\text{rlmin}} = \{ W^r_{\text{rkh}j} K^i_{\text{rlm}} + W^r_{\text{rj}h} K^i_{\text{rlm}} + W^r_{\text{jk}h} K^i_{\text{rlm}} + ( \partial_r W^i_{\text{jk}h}) H^r_{\text{lm}} \} y^k .\]

Again, taking $h$- covariant derivative for (2.12), with respect to $x^p$ , we get
\[(2.13) \quad \alpha_{np} W^i_{\text{jk}h} K^i_{\text{rlm}} + \beta_{np} ( \delta^i_h g_{jk} - \delta^i_k g_{jh} ) K^i_{\text{rlm}} + W^r_{\text{jk}h} K^i_{\text{rlmin}} + W^r_{\text{kh}j} K^i_{\text{rlmin}}\]
\[= \{ W^r_{\text{rkh}j} K^i_{\text{rlm}} + W^r_{\text{rj}h} K^i_{\text{rlm}} + W^r_{\text{jk}h} K^i_{\text{rlm}} + ( \partial_r W^i_{\text{jk}h}) H^r_{\text{lm}} \} y^k .\]

Using the condition (2.2) in the above equation, we get
\[(2.14) \quad \alpha_{np} W^i_{\text{jk}h} K^i_{\text{rlm}} + \beta_{np} ( \delta^i_h g_{jk} - \delta^i_k g_{jh} ) K^i_{\text{rlm}} + W^r_{\text{jk}h} K^i_{\text{rlmin}} + W^r_{\text{kh}j} K^i_{\text{rlmin}}\]
\[= \alpha_{np} \{ W^r_{\text{rkh}j} K^i_{\text{rlm}} + W^r_{\text{rj}h} K^i_{\text{rlm}} + W^r_{\text{jk}h} K^i_{\text{rlm}} + ( \partial_r W^i_{\text{jk}h}) H^r_{\text{lm}} \} y^k .\]

This shows that
\[(2.15) \quad \beta_{np} ( \delta^i_h g_{jk} - \delta^i_k g_{jh} ) K^i_{\text{rlm}} + W^r_{\text{jk}h} K^i_{\text{rlmin}} + W^r_{\text{kh}j} K^i_{\text{rlmin}} = 0 .\]

Transvecting (2.14) by $y^l$, using (1.5b), (1.15b), (1.14a),(1.1c) and (1.1a), we get
\[(2.16) \quad \{ W^r_{\text{rkh}h} H^i_{\text{lm}} + W^r_{\text{rj}h} K^i_{\text{rlm}} + W^r_{\text{kr}h} K^r_{\text{hlm}} + ( \partial_r W^i_{\text{kh}h} - W^i_{\text{rh}k} ) H^r_{\text{lm}} \} y^k\]
\[= \alpha_{np} \{ W^r_{\text{rkh}h} H^i_{\text{lm}} + W^r_{\text{rj}h} K^i_{\text{rlm}} + W^r_{\text{kr}h} K^r_{\text{hlm}} + ( \partial_r W^i_{\text{kh}h} - W^i_{\text{rh}k} ) H^r_{\text{lm}} \} y^k .\]

This shows that
\[(2.17) \quad \{ W^r_{\text{rkh}h} H^r_{\text{lm}} + W^r_{\text{rj}h} K^r_{\text{klm}} + W^r_{\text{kr}h} K^r_{\text{hlm}} + ( \partial_r W^i_{\text{kh}h} - W^i_{\text{rh}k} ) H^r_{\text{lm}} \} y^k\]
\[= \alpha_{np} \{ W^r_{\text{rkh}h} H^r_{\text{lm}} + W^r_{\text{rj}h} K^r_{\text{klm}} + W^r_{\text{kr}h} K^r_{\text{hlm}} + ( \partial_r W^i_{\text{kh}h} - W^i_{\text{rh}k} ) H^r_{\text{lm}} \} y^k .\]

if and only if
\[(2.18) \quad \beta_{np} ( \delta^i_h g_{yk} - \delta^i_k y_{gh} ) K^i_{\text{rlm}} + W^r_{\text{kh}p} K^i_{\text{rlmin}} + W^r_{\text{kh}h} K^i_{\text{rlmin}} = 0 .\]

Transvecting (2.16) by $y^k$, using (1.5b), (1.15b), (1.14b), (1.14a) and (1.1b), we get
\[(2.19) \quad \{ W^r_{\text{rkh}h} H^r_{\text{lm}} + W^r_{\text{rj}h} K^r_{\text{klm}} + \{ W^r_{\text{rkh}h} H^r_{\text{lm}} + ( \partial_r W^i_{\text{kh}h} - W^i_{\text{rh}k} ) H^r_{\text{lm}} \} y^k \} y^k\]
\[= \alpha_{np} \{ W^r_{\text{rkh}h} H^r_{\text{lm}} + W^r_{\text{rj}h} K^r_{\text{klm}} + \{ W^r_{\text{rkh}h} H^r_{\text{lm}} + ( \partial_r W^i_{\text{kh}h} - W^i_{\text{rh}k} ) H^r_{\text{lm}} \} y^k \} y^k .\]

This shows that

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\( (2.20) \quad \left[ W_{rh}^l H_{lm}^r + W_{r}^i K_{r}^lm + \{ W_{rkh}^l H_{lm}^r + (\hat{\alpha}_{r} W_{r}^l - W_{rkh}^l) H_{lm}^r \} y^k \right]_{tip} = \alpha_{np} \left[ W_{r}^l H_{lm}^r + W_{r}^i K_{r}^lm + \{ W_{rkh}^l H_{lm}^r + (\hat{\alpha}_{r} W_{r}^l - W_{rkh}^l) H_{lm}^r \} y^k \right] \\
\text{if and only if} \\
\( (2.21) \quad \beta_{np} (\delta_{h} f^2 \gamma_{h} y^{i}) K_{r}^lm + W_{r}^l K_{r}^lm \gamma_{h} + W_{r}^r \gamma_{h} + W_{r}^r K_{r}^lm \gamma_{h} = 0 \) .

Thus, we conclude

**Theorem 2.3.** In \( GW^{h} - BRF_n \), the tensors \( \{ W_{rkh}^l H_{lm}^r + W_{rh}^l K_{r}^lm + W_{kr}^l K_{r}^lm + (\hat{\alpha}_{r} W_{r}^l - W_{rkh}^l) H_{lm}^r \} \) \( \{ W_{rkh}^l H_{lm}^r + W_{rh}^l K_{r}^lm + W_{kr}^l K_{r}^lm + (\hat{\alpha}_{r} W_{r}^l - W_{rkh}^l) H_{lm}^r \} \) \( \{ W_{rkh}^l H_{lm}^r + W_{rh}^l K_{r}^lm + W_{kr}^l K_{r}^lm + (\hat{\alpha}_{r} W_{r}^l - W_{rkh}^l) H_{lm}^r \} \) \( \{ W_{rkh}^l H_{lm}^r + W_{rh}^l K_{r}^lm + W_{kr}^l K_{r}^lm + (\hat{\alpha}_{r} W_{r}^l - W_{rkh}^l) H_{lm}^r \} \) are behaves as birecurrent, if and only if (2.15), (2.18) and (2.21) hold, respectively ( provided that \( \alpha_{lm} \) and \( \beta_{lm} \) are symmetric).

3. The Necessary and Sufficient Condition

In this section, we shall obtain the necessary and sufficient condition for some tensors to be generalized birecurrent in \( GW^{h} - BRF_n \).

We know that Weyl’s projective curvature tensor \( W_{jkh}^l \) and Cartan’s third curvature tensor \( R_{jkh}^l \) are connected by the formula \( (1), (3) \)

\( (3.1) \quad W_{jkh}^l = R_{jkh}^l + \frac{1}{3} \left( \delta_{i}^k R_{jh} - g_{jk} R_{i}^h \right) . \)

Taking h-covariant derivative of (3.1), with respect to \( x^l \), we get

\( (3.2) \quad W_{jkh}^l = R_{jkh}^l + \frac{1}{3} \left( \delta_{i}^k R_{jh} - g_{jk} R_{i}^h \right) \)

Again, taking h- covariant derivative of (3.2), with respect to \( x^m \), we get

\( (3.3) \quad W_{jkh}^l = R_{jkh}^l + \frac{1}{3} \left( \delta_{i}^k R_{jh} - g_{jk} R_{i}^h \right) \)

Using the condition (2.2) in (3.3), we get

\( \alpha_{lm} W_{jkh}^i + \beta_{lm} (\delta_{i}^k g_{jk} - \delta_{i}^k g_{jh}) = R_{jkh}^l + \frac{1}{3} \left( \delta_{i}^k R_{jh} - g_{jk} R_{i}^h \right) \)

By using (3.1), the above equation can be written as

\( R_{jkh}^l = \alpha_{lm} R_{jkh}^l + \beta_{lm} (\delta_{i}^k g_{jk} - \delta_{i}^k g_{jh}) \)

\( + \frac{1}{3} \alpha_{lm} \left( \delta_{i}^k R_{jh} - g_{jk} R_{i}^h \right) - \frac{1}{3} \left( \delta_{i}^k R_{jh} - g_{jk} R_{i}^h \right) \)

This shows that

\( R_{jkh}^l = \alpha_{lm} R_{jkh}^l + \beta_{lm} (\delta_{i}^k g_{jk} - \delta_{i}^k g_{jh}) \)

if and only if

\( (\delta_{i}^k R_{jh} - g_{jk} R_{i}^h)_{lim} = \alpha_{lm} (\delta_{i}^k R_{jh} - g_{jk} R_{i}^h) \)

Thus, we conclude

**Theorem 3.1.** In \( GW^{h} - BRF_n \), Cartan’s third curvature tensor \( R_{jkh}^l \) is generalized birecurrent if and only if the tensor \( (\delta_{i}^k R_{jh} - g_{jk} R_{i}^h) \) is birecurrent.

Transvecting (3.4) by \( g_{ip} \), using (1.5a) and (1.15c), we get

\( R_{jpkh}^l = \alpha_{lm} R_{jpkh}^l + \beta_{lm} (g_{ph} g_{jk} - g_{pk} g_{jh}) \)

\( + \frac{1}{3} \alpha_{lm} \left( g_{pk} R_{jh} - g_{jk} R_{ph} \right)_{lim} \),

where \( g_{ip} R_{jh}^l = R_{ph} \), this shows that

\( R_{jpkh}^l = \alpha_{lm} R_{jpkh}^l + \beta_{lm} (g_{ph} g_{jk} - g_{pk} g_{jh}) \)

if and only if

\( (g_{pk} R_{jh} - g_{jk} R_{ph})_{lim} = \alpha_{lm} (g_{pk} R_{jh} - g_{jkh} R_{ph}) \)

Thus, we conclude

**Theorem 3.2.** In \( GW^{h} - BRF_n \), the associate curvature tensor \( R_{jpkh}^l \) is generalized birecurrent if and only if the tensor \( (g_{pk} R_{jh} - g_{jk} R_{ph}) \) is birecurrent.

Transvecting (3.4) by \( y^l \), using (1.5b), (1.15b), (1.1a) and (1.15d), we get

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\[
(H^i_{k|h})_{|lm} = \alpha_{tm}(H^i_k + \beta_{tm}\left( \delta^i_k y_k - \delta^i_k y_h \right) + \frac{1}{3} \alpha_{tm}\left( \delta^i_k H_k - y_k R^i_k \right) - \frac{1}{3} \left( \delta^i_k H_k - y_k R^i_h \right)_{|lm}.
\]

This shows that

\[
(H^i_{k|h})_{|lm} = \alpha_{tm}(H^i_k + \beta_{tm}\left( \delta^i_k y_k - \delta^i_k y_h \right) + \frac{1}{3} \alpha_{tm}\left( H^i_k y^i - F^2 R^i_k \right) - \frac{1}{3} \left( H^i_k y^i - F^2 R^i_h \right)_{|lm}.
\]

Thus, we conclude

**Theorem 3.3.** In $GW^h$-BFR, the $h$-covariant derivative of the second order for the $h(v)$-torsion tensor $H^i_{k|h}$ is given by (3.10) if and only if the tensor $\left( \delta^i_k H_k - y_k R^i_h \right)$ is birecurrent.

Also, transvecting (3.9) by $y^k$, using (1.5b), (1.10a) and (1.1b), we get

\[
(H^i_{h|k})_{|lm} = \alpha_{tm}(H^i_h + \beta_{tm}\left( \delta^i_h y_k - \delta^i_h y^i \right) + \frac{1}{3} \alpha_{tm}\left( H^i_h y^i - F^2 R^i_h \right) - \frac{1}{3} \left( H^i_h y^i - F^2 R^i_h \right)_{|lm}.
\]

This shows that

\[
(H^i_{h|k})_{|lm} = \alpha_{tm}(H^i_h + \beta_{tm}\left( \delta^i_h y_k - \delta^i_h y^i \right) + \frac{1}{3} \alpha_{tm}\left( H^i_h y^i - F^2 R^i_h \right) - \frac{1}{3} \left( H^i_h y^i - F^2 R^i_h \right)_{|lm}.
\]

Thus, we conclude

**Theorem 3.4.** In $GW^h$-BFR, the $h$-covariant derivative of the second order for the deviation tensor $H^i_{h}$ is given by (3.12) if and only if the tensor $\left( H^i_h y^i - F^2 R^i_h \right)$ is birecurrent.

Further, transvecting (3.9) by $g_{ip}$, using (1.5a) and (1.1b), we get

\[
(H^i_{p|h})_{|lm} = \alpha_{tm}(H^i_p + \beta_{tm}\left( g_{ph} y_k - g_{ph} y^i \right) + \frac{1}{3} \alpha_{tm}\left( g_{ph} H_h - y_k R_{ph} \right) - \frac{1}{3} \left( g_{ph} H_h - y_k R_{ph} \right)_{|lm}.
\]

where $g_{ip} R^i_h = R_{ph}$, this shows that

\[
(H^i_{p|h})_{|lm} = \alpha_{tm}(H^i_p + \beta_{tm}\left( g_{ph} y_k - g_{ph} y^i \right) + \frac{1}{3} \alpha_{tm}\left( g_{ph} H_h - y_k R_{ph} \right) - \frac{1}{3} \left( g_{ph} H_h - y_k R_{ph} \right)_{|lm}.
\]

Thus, we conclude

**Theorem 3.5.** In $GW^h$-BFR, the $h$-covariant derivative of the second order for the associate curvature tensor $H_{kp,h}$ is given by (3.14) if and only if the tensor $\left( g_{ph} H_h - y_k R_{ph} \right)$ is birecurrent.
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بعض الخصائص للموتر الإسقاطي لويلي في تعميم فضاء فنسلر المعاودة

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المملح

في هذه الورقة، عرفنا فضاء فنسلر في مفهوم كارتان في حالة:

\[ W^i_{jkl;m} = \alpha_{lm} W^i_{jkh} + \beta_{lm} (\delta^i_k g_{jk} - \delta^i_l g_{kj} ) , \quad W^i_{jkh} \neq 0 \]

حيث \( W^i_{jkl;m} \) هي مشتقة من الرتبة الثانية (مشتقة كارتان من النوع الثاني) بالنسبة إلى المسقطين الوضعيين \( x^i \) و \( x^m \) على التعاقب إذ أن \( \alpha_{lm} \) و \( \beta_{lm} \) هي حقول غير صفيرة لمتجهات متحدة. 

أوجدنا المشتقة من الرتبة الثانية \( GH^h-BRF_n \) للفضاء، وأطلقتنا عليه تعميم فضاء فنسلر لـ \( W^h_{jkl} \) - ثنائي المعاودة ورمزنا إليه بالرمز التالي و للمؤثر الإدواري لويلي \( W^i_{jkh} \) و كذلك للمؤثر الإسقاطي لويلي \( W^i_{jkh} \) في هذا الفضاء. كذلك، أوجدنا الشروط اللازم والكافي للمؤثر القوسي الثانى لـ \( R^i_{jkh} \) و موازقاته لتكون ثناية المعاودة المعممة، ثم، أوجدنا الشروط اللازم والكافي لإيجاد المشتقة \( H^i_{kp,h} \) للمؤثر الإدواري لـ \( H^i_{h} \) و كذلك للمؤثر الإدواري لـ \( H^i_{h} \) في هذا الفضاء.

الكلمات مفتاحية: فضاء فنسلر - تعميم فضاء فنسلر - ثنائي المعاودة - المؤثر الإسقاطي لويلي - للمؤثر القوسي الثالث لـ كارتان - للمؤثر الإدواري لـ \( R^i_{jkh} \) - تعميم فضاء فنسلر