Estimating coherence with respect to general quantum measurements

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The conventional coherence is defined with respect to a fixed orthonormal basis, i.e., to a von Neumann measurement. Recently, generalized quantum coherence with respect to general positive operator-valued measurements (POVMs) has been presented. Several well-defined coherence measures, such as the relative entropy of coherence $C_r$, the $l_1$ norm of coherence $C_{l_1}$ and the coherence $C_{T,\alpha}$ based on Tsallis relative entropy with respect to general POVMs have been obtained. In this work, we investigate the properties of $C_r$, $l_1$ and $C_{T,\alpha}$. We estimate the upper bounds of $C_{l_1}$; we show that the minimal error probability of the least square measurement state discrimination is given by $C_{T,1/2}$; we derive the uncertainty relations given by $C_r$, and calculate the average values of $C_r$, $C_{T,\alpha}$ and $C_{l_1}$ over random pure quantum states. All these results include the corresponding results of the conventional coherence as special cases.

I. INTRODUCTION

Coherence is a fundamental feature in quantum physics and a significant resource in quantum information processing. Since the rigorous framework [1] for quantifying coherence has been established, fruitful advances about coherence, both in theories and experiments, have been achieved, for reviews see e.g. [2, 3]. The conventional coherence is defined with respect to a fixed orthonormal basis. We know that an orthonormal basis $\{|j\rangle\}_{j=1}^d$ of a $d$-dimensional complex Hilbert space $H$ corresponds to a rank-one projective measurement (von Neumann measurement) with measurement operators $\{|j\rangle\langle j|\}_{j=1}^d$. Along this line, we may ask whether or not the notion of coherence can be generalized to general measurements. Recently, Bischof, Kampermann and Bruß generalized the conventional framework of coherence to the case of general positive operator-valued measurements (POVMs), by replacing the projective measurements with POVMs $[4, 5]$.

Let $S(H)$ be the set of all quantum states on $H$, $C(H)$ be the set of all channels on $H$ [6]. For a POVM $E = \{E_j\}_{j=1}^n$, $E_j \geq 0 \forall j$ and $\sum_{j=1}^n E_j = I_d$ with $I_d$ the identity on $H$, the set of POVM incoherent states is specified as $[4]$

$$I_p(H) = \{\rho \in S(H) | E_j \rho E_k = 0, \forall j \neq k\}.$$  \hspace{1cm} (1)

A channel $\phi \in C(H)$ is called a POVM incoherent channel $[5]$ if $\phi$ has a Kraus operator decomposition $\phi = \{K_l\}_l$ with $\sum_l K_l^\dagger K_l = I_d$ and there exists a block incoherent channel $\{K_l\}_l$ with respect to a canonical Naimark extension such that

$$K_l \rho K_l^\dagger \otimes |1\rangle\langle 1| = K_l^\dagger (\rho \otimes |1\rangle\langle 1|) K_l^\dagger, \ \forall \ l,$$  \hspace{1cm} (2)

where $\dagger$ stands for conjugate transpose. We call such a decomposition $\phi = \{K_l\}_l$ a POVM incoherent decomposition. We denote by $C_{\phi_1}(H)$ the set of all POVM incoherent channels.

A functional $C : S(H) \rightarrow R$ is called a POVM coherence measure with respect to the POVM $E = \{E_j\}_{j=1}^n$ if $C$ fulfills the following conditions $[5]$. 

(P1) Faithfulness: $C(\rho, E) \geq 0$, with equality if and only if $\rho \in I_p(H)$.

(P2) Monotonicity: $C(\phi_1(\rho, E), E) \leq C(\rho, E), \ \forall \phi_1 \in C_{\phi_1}(H)$.

(P3) Strong monotonicity: $\sum_j p_j C(\rho_j, E) \leq C(\rho, E), \ \forall \phi_1 = \{K_l\}_l$ is a POVM incoherent decomposition of $\phi_1$, $p_l = \text{tr}(\rho_l K_l^\dagger K_l)$, $\rho_l = K_l \rho K_l^\dagger / p_l$.

(P4) Convexity: $C(\sum_j p_j \rho_j, E) \leq \sum_j p_j C(\rho_j, E), \ \{\rho_j\}_j \subset S(H), \ \{p_j\}_j$ a probability distribution.

When the POVM $E$ is a rank-one projective measurement $E = \{|j\rangle\langle j|\}_{j=1}^d$, the definitions of POVM incoherent states and POVM incoherent channels, as well as the conditions (P1) to (P4) all reduce to the cases of the conventional coherence theory, for which various kinds of coherence measures have been proposed together with their operational interpretations and applications $[1, 7–20]$. However, less is known about the POVM coherence measures. Recently, several POVM coherence measures have been proposed, such as robustness of POVM coherence $C_{\text{rob}}(\rho, E) [5]$, $l_1$-norm of POVM coherence $C_{l_1}(\rho, E)$ $[5, 21]$, relative entropy of POVM coherence $C_r(\rho, E)$ $[4, 5]$, and POVM coherence $C_{T,\alpha}(\rho, E)$ based on Tsallis relative entropy $[21]$. In particular, the $C_r(\rho, E)$, $C_{l_1}(\rho, E)$ and $C_{T,\alpha}(\rho, E)$ allow for explicit expressions.
Furthermore, an alternative definition of POVM incoherent operation was introduced [22], and the quantifiers of POVM coherence based on max-relative entropy and coherent rank were studied [23],

\[ C_r(\rho, E) \] is defined by [4, 5],

\[ C_r(\rho, E) = \sum_j S(\sqrt{E_j} \rho \sqrt{E_j}) - S(\rho), \tag{3} \]

where \( S(M) = -\operatorname{tr}(M \log_2 M) \) is the entropy for positive semidefinite matrices \( M \).

\( l_1 \)-norm of POVM coherence \( C_{l_1}(\rho, E) \) [5, 21] is defined as

\[ C_{l_1}(\rho, E) = \sum_{j \neq k} ||\sqrt{E_j} \rho \sqrt{E_k}||_{tr}, \tag{4} \]

with \( ||M||_{tr} = \sqrt{\operatorname{tr}^2(M^*M)} \) the trace norm of matrix \( M \).

\( C_{T,\alpha}(\rho, E) \) is defined by [21]

\[ C_{T,\alpha}(\rho, E) = \frac{1}{\alpha - 1} \left( \sum_j \operatorname{tr}(\sqrt{E_j} \rho^\alpha \sqrt{E_j})^{1/\alpha} - 1 \right) \tag{5} \]

for \( \alpha \in (0, 1) \cup (1, 2) \). When \( E \) is a rank-one projective measurement, \( C_{T,\alpha}(\rho, E) \) returns to the standard coherence measure \( C_r(\rho, \{|j\rangle\langle j|\}_{j=1}^d) \) proposed in Ref. [14, 15]. A quantifier of standard coherence measure based on Tsallis entropy was proposed in Ref. [24], however it does not satisfy all conditions of BCP framework.

In this work, we investigate the properties and the estimations of \( C_r(\rho, E), C_{l_1}(\rho, E) \) and \( C_{T,\alpha}(\rho, E) \). We estimate the upper bounds and the averages for random pure quantum states, and explore their operational interpretations and the related uncertainty relations, which strengthen the necessity and applicability of the coherence with respect to POVMs. This paper is organized as follows. In section II, we discuss the upper bounds for \( C_{l_1}(\rho, E) \). In section III, we provide an operational interpretation for \( C_{T,1/2}(\rho, E) \). In section IV, we establish an uncertainty relation for \( C_r(\rho, E) \). In section V, we calculate the average values of \( C_{l_1}(\rho, E), C_r(\rho, E) \) and \( C_{T,\alpha}(\rho, E) \) for random pure states. Section VI is a brief summary.

II. UPPER BOUNDS FOR \( C_{l_1} \)

We first estimate the upper bounds of \( C_{l_1}(\rho, E) \). From the Theorem 4.29 in [25], we have that for any positive semidefinite matrices \( A, B \) and matrix \( X \),

\[ ||AXB||_{tr} \leq ||A^{p/2}||_{tr}^{1/p} ||X^{q/2}||_{tr}^{1/q} \tag{6} \]

for all positive numbers \( p > 1, q > 1 \) and \( 1/p + 1/q = 1 \). From (6) we obtain

\[ ||\sqrt{E_j} \rho \sqrt{E_k}||_{tr} \leq ||E_j^{p/2} \rho||_{tr}^{1/p} ||E_k^{q/2} \rho||_{tr}^{1/q}. \tag{7} \]

Summing over all \( j \neq k \) we get the following Theorem.

**Theorem 1.** For any POVM \( E = \{E_j\}_{j=1}^n = \{E_j\}_{j=1}^1 \) and quantum state \( \rho \) it holds that

\[ C_{l_1}(\rho, E) \leq \sum_{j \neq k} ||E_j^{p/2} \rho||_{tr}^{1/p} ||E_k^{q/2} \rho||_{tr}^{1/q}, \tag{8} \]

where \( p > 1, q > 1 \) and \( 1/p + 1/q = 1 \).

For \( j \neq k \), one sees that from (4), \( C_{l_1}(\rho, E) \) is dependent on both the state \( \rho \) and the commutativity of \( \{E_j, E_k\} \) and \( \rho \). However, the upper bound \( ||E_j^{p/2} \rho||_{tr}^{1/p} ||E_k^{q/2} \rho||_{tr}^{1/q} \) is a factorized form of \( E_j \) over \( \rho \) and \( E_k \) over \( \rho \). Theorem 1 sets a constraint on such factorization.

When \( p = q = 2 \), (8) becomes

\[ C_{l_1}(\rho, E) \leq \sum_{j \neq k} ||E_j^2 \rho||_{tr}^{1/2} ||E_k^2 \rho||_{tr}^{1/2} \]

\[ = (\sum_j ||E_j^2 \rho||_{tr}^{1/2})^2 - \sum_j ||E_j \rho||_{tr}. \tag{9} \]

When \( p \to 1 \) and \( q \to +\infty \), we have Theorem 2 below.

**Theorem 2.** For any POVM \( E = \{E_j\}_{j=1}^n = \{E_j\}_{j=1}^1 \) and quantum state \( \rho \), we have

\[ C_{l_1}(\rho, E) \leq 2 \sum_j (n - j) ||(E_j^{1/2} \rho)^\dagger||_{tr} \tag{10} \]

\[ \leq (n - 1) \sum_j ||(E_j^{1/2} \rho)||_{tr}, \tag{11} \]

where \( \{(E_j^{1/2} \rho)^\dagger\} \) stands for \( (E_j^{1/2} \rho) \leq (E_j^{1/2} \rho)^\dagger \leq \ldots \leq (E_n^{1/2} \rho) \).

**Proof.** Let \( E_k = \sum_{l_k} \lambda_{k,l_k}^t I(\lambda_{k,l_k}) \) be the eigendecomposition of \( E_k \), where \( I(\lambda_{k,l_k}) \) is the identity operator of the subspace spanned by all eigenvectors corresponding to the eigenvalues \( \lambda_{k,1} > \lambda_{k,2} > \ldots \) arranged in strictly decreasing order. Consequently, we get

\[ E_k^{q/2} = \sum_{l_k} (\lambda_{k,l_k}^t)^{q/2} I(\lambda_{k,l_k}), \]

\[ ||E_k^{q/2} \rho||_{tr} \leq \operatorname{tr} \left( \rho E_k^{q/2} \right) \leq \sqrt{\operatorname{tr} \left( \rho E_k^{q/2} \right) ||I_2 \rho||_{tr}} = ||E_k \rho||_{tr}^{q/2}, \]

\[ ||E_k^{q/2} \rho||_{tr} \leq \sqrt{\operatorname{tr} \left( \rho E_k^{q/2} \right)} \]

where

\[ \lambda(E_k) = \max \{ \lambda \in \sigma(E_k) | I(\lambda) \rho \neq 0 \}, \text{ when } E_k \neq 0, \]

\[ 0, \text{ when } E_k = 0, \]

\( \sigma(E_k) \) is the set of eigenvalues of \( E_k \).
When $p \to 1$ and $q \to +\infty$, we have $\|E_j^{p/2}\rho\|_\text{tr}^{1/p} \to \|E_j^{1/2}\rho\|_\text{tr}$. Therefore, from Theorem 1, $C_l(\rho, E) \leq \sum_{j \neq k} \min\{\|E_j^{1/2}\rho\|_{\text{tr}} \sqrt{\chi(E_k)}, \|E_k^{1/2}\rho\|_{\text{tr}} \sqrt{\chi(E_j)}\}$.

Since $E_k^{q/2} \leq I_d$, we get $\sqrt{\chi(E_k)} \leq 1$, and

$$C_l(\rho, E) \leq \sum_{j,k} \min\{\|E_j^{1/2}\rho\|_{\text{tr}}, \|E_k^{1/2}\rho\|_{\text{tr}}\}$$

$$= \sum_{j,k} \min\{\|E_j^{1/2}\rho\|_{\text{tr}}^2, \|E_k^{1/2}\rho\|_{\text{tr}}^2\}$$

$$= 2 \sum_{j < k} \min\{\|E_j^{1/2}\rho\|_{\text{tr}}^2, \|E_k^{1/2}\rho\|_{\text{tr}}^2\}$$

$$= 2 \sum_{j=1}^n \sum_{k=j+1}^n \|E_j^{1/2}\rho\|_{\text{tr}}^2$$

$$= 2 \sum_{j=1}^n (n-j)\|E_j^{1/2}\rho\|_{\text{tr}}^2, \quad (12)$$

which gives the inequality (10).

From the definition of $\{\|E_j^{1/2}\rho\|_{\text{tr}}^2\}$, we have

$$(n-j)\|E_j^{1/2}\rho\|_{\text{tr}}^2 \leq \sum_{i=1}^n \|E_i^{1/2}\rho\|_{\text{tr}}^2. \quad (13)$$

Adding $(n-j)\|E_j^{1/2}\rho\|_{\text{tr}}^2$ to both sides of (13) and summing over the index $j$, we obtain

$$2 \sum_{j=1}^n (n-j)\|E_j^{1/2}\rho\|_{\text{tr}}^2$$

$$\leq \sum_{j=1}^n \|E_j^{1/2}\rho\|_{\text{tr}}^2 + \sum_{j=1}^n (n-j)\|E_j^{1/2}\rho\|_{\text{tr}}^2. \quad (14)$$

Since the first term in (14),

$$\sum_{j=1}^n \|E_j^{1/2}\rho\|_{\text{tr}}^2 = \sum_{j=1}^n (j-1)\|E_j^{1/2}\rho\|_{\text{tr}}^2,$$

(14) leads to (11).

When $E$ is a rank-one projective measurement $E = \{|j\rangle\langle j|\}_{j=1}^d$, we have the following Corollary from the inequality (10) in Theorem 2.

**Corollary 1.** For rank-one projective measurement $E = \{|j\rangle\langle j|\}_{j=1}^d$, and any quantum state $\rho$, we have

$$C_l(\rho, \{|j\rangle\langle j|\}_{j=1}^d) \leq 2 \sum_{j} (d-j)\sqrt{\|\rho\|_j^2} \geq B_1, \quad (15)$$

where $\sqrt{\|\rho\|_j^2} \geq j$ stands for arranging $\sqrt{\|\rho\|_j^2}$ in nondecreasing order.

In Ref. [26] the authors presented upper bounds for $C_l(\rho, \{|j\rangle\langle j|\}_{j=1}^d)$, see Eqs. (2.7) in Ref. [26],

$$C_l(\rho, \{|j\rangle\langle j|\}_{j=1}^d) \leq \sum_{j} \sqrt{\|\rho\|_j^2} - 1 \equiv B_2, \quad (16)$$

$$C_l(\rho, \{|j\rangle\langle j|\}_{j=1}^d) \leq \sqrt{(d-1)(\text{tr}\rho - \sum_j \langle j|\rho|j\rangle^2)} \equiv B_3. \quad (17)$$

We compare the upper bound $B_1$ with the upper bounds $B_2$ and $B_3$ by detailed examples.

**Example 1.** Consider state

$$\rho = \frac{1}{2} \left( \begin{array}{cc} 1 - z & \frac{1}{2} + z \\ \frac{1}{2} + z & 1 + z \end{array} \right), \quad z \in [0, \frac{4}{5}], \quad (18)$$

under $d = 2$ orthonormal basis $\{|j\rangle\}_{j=1}^2$. We have

$$B_1 = \sqrt{\frac{1}{4} + (1 - z)^2},$$

$$B_2 = \sqrt{1 - z^2},$$

$$B_3 = \frac{1}{2}.$$  

We see that $B_1$ is tighter than $B_2$ for certain states. In particular, when $z = 0.5$ we have $B_2 > B_1 > B_3$; while when $z = 0.1$, $B_1 > B_2 > B_3$, see Fig. 1.

**Example 2.** Consider $\rho = |\psi\rangle\langle \psi|$, with

$$|\psi\rangle = x|1\rangle + 4x|2\rangle + \sqrt{1 - 17x^2}|3\rangle, \quad (19)$$

where $x \in [0, \frac{1}{\sqrt{17}}]$, $\{|j\rangle\}_{j=1}^3$ is an orthonormal basis. One has

$$B_1 = 12x,$$

$$B_2 = (5x + \sqrt{1 - 17x^2})^2 - 1,$$

$$B_3 = \sqrt{6(1 - 17x^4 - (1 - 17x^2)^2)}.$$  

One sees that when $x = 0.1$, $B_2 < B_1 < B_3$; while when $x = 0.21$, $B_2 < B_3 < B_1$, see Fig. 2.

Example 1 and Example 2 show that the upper bound $B_1$ is a new bound different from $B_2$ and $B_3$, and $B_1, B_2, B_3$ have no strict order of which is tighter than others.

### III. Operational Interpretation of $C_{T,1/2}$

In this section, we provide an operational interpretation for $C_{T,1/2}$. We first review the least square measurement (LSM). Quantum state discrimination is a fundamental problem in quantum physics, and plays a key role in quantum communication and quantum cryptography [27–31]. For an ensemble $\{\rho_j, \eta_j\}_{j=1}^n$ with $\{\rho_j\}_{j=1}^n$ quantum states and $\{\eta_j\}_{j=1}^n$ a probability distribution, two persons, Alice chooses one $\rho_j$ with probability $\eta_j$ and sends it to Bob, Bob performs a POVM $E = \{E_j\}_{j=1}^n$...
Concerning the relationship between $C_{T,1/2}$ and LSM, we have the following Theorem 3 and Theorem 4.

**Theorem 3.** Let $E = \{E_j\}_{j=1}^n$ be a POVM on the Hilbert space $H$ and $\rho$ a state in $H$. We have

$$C_{T,1/2}(\rho, E) = 2P^{\text{lsm}}(\{\rho_j, \eta_j\}_{j=1}^n),$$

where

$$\eta_j = \text{tr}(\rho E_j),$$

$$\rho_j = \eta_j^{-1/2} \sqrt{\rho} E_j \sqrt{\rho}.$$  

**Proof.** According to Eqs. (21), (22) and (26), we have

$$\rho_{\text{out}} = \sum_{j=1}^n \eta_j \rho_j = \rho,$$

$$M_j^{\text{lsm}} = \eta_j^{-\frac{1}{2}} \rho_{\text{out}} \rho_j \rho_{\text{out}}^{-\frac{1}{2}} = E_j.$$  

Consequently, we obtain

$$P^{\text{lsm}}(\{\rho_j, \eta_j\}_{j=1}^n) = 1 - \sum_j \eta_j \text{tr}(M_j^{\text{lsm}} \rho_j)$$

$$= 1 - \sum_j \text{tr}(E_j \sqrt{\rho} E_j \sqrt{\rho})$$

$$= 1 - \sum_j \text{tr}[(\sqrt{E_j \rho} \sqrt{E_j})^2]$$

$$= \frac{1}{2} C_{T,1/2}(\rho, E),$$

which completes the proof. ■

Theorem 3 shows that the minimal error probability of the least square measurement state discrimination is given by the POVM coherence $C_{T,1/2}(\rho, E)$ based on Tsallis relative entropy. If $\rho$ is POVM incoherent with respect to POVM $E = \{E_j\}_{j=1}^n$, i.e., $C_{T,1/2}(\rho, E) = 0$, then $p^{\text{lsm}}(\{\rho_j, \eta_j\}_{j=1}^n) = 0$, which means that $\{\rho_j, \eta_j\}_{j=1}^n$ can be perfectly discriminated by the least square measurement $E = \{E_j\}_{j=1}^n$.

From Theorem 3, we also see that for a given POVM $E = \{E_j\}_{j=1}^n$ and a state $\rho$, there exists an ensemble $\{\rho_j, \eta_j\}_{j=1}^n$ such that Eq. (24) holds. Conversely, we may ask for a given ensemble $\{\rho_j, \eta_j\}_{j=1}^n$, whether a POVM $E = \{E_j\}_{j=1}^n$ and a state $\rho$ exist so that Eq. (24) holds. Theorem 4 below shows that this is true, which can be verified similar to the proof of Theorem 3.

**Theorem 4.** For a given ensemble $\{\rho_j, \eta_j\}_{j=1}^n$ on the Hilbert space $H$, there exist a quantum state $\rho$ and a POVM $E = \{E_j\}_{j=1}^n$ such that

$$2P^{\text{lsm}}(\{\rho_j, \eta_j\}_{j=1}^n) = C_{T,1/2}(\rho, E),$$

where

$$\rho = \sum_{j=1}^n \eta_j \rho_j.$$
\[ E_j = \eta_j \rho^{-\frac{1}{2}} \rho_j \rho^{-\frac{1}{2}}. \] (30)

We remark that when the POVM \( E = \{E_j\}_{j=1}^n \) is a rank-one projective measurement \( E = \{|j\rangle\langle j|\}_{j=1}^n \), both the Theorem 3 and Theorem 4 reduce to the corresponding conclusions in the theory of conventional coherence [15].

IV. UNCERTAINTY RELATIONS GIVEN BY \( C_r \)

The entropic uncertainty relations play a central role in quantum cryptographic protocols [39]. In this section, we establish an entropic uncertainty relation given by the relative entropy of the POVM coherence \( C_r(\rho, E) \).

**Theorem 5.** Let \( E = \{E_j\}_{j=1}^n \) and \( F = \{F_k\}_{k=1}^m \) be two POVMs on the Hilbert space \( H \) and \( \rho \) a state in \( H \). We have

\[
C_r(\rho, E) + C_r(\rho, F) \geq 2\log_2 \frac{1}{c} - S(\rho),
\]

where \( c = \max_{jk} c_{jk}, c_{jk} = ||\sqrt{E_j}F_k||_2 \) denotes the operator norm (the maximal singular value).

**Proof.** Rewrite Eq. (3) as

\[
C_r(\rho, E) = \sum_{j=1}^n C_r(E_j) = \sum_{j=1}^n \text{tr}(E_j^2) - \text{tr}(E_j \rho)
\]

\[
C_r(\rho, F) = \sum_{k=1}^m C_r(F_k) = \sum_{k=1}^m \text{tr}(F_k^2) - \text{tr}(F_k \rho)
\]

By using the result [40] that

\[
S(\{\text{tr}(E_j)\}_{j=1}^n) + S(\{\text{tr}(F_k)\}_{k=1}^m) \geq 2\log_2 \frac{1}{c},
\]

we then can prove Theorem 5.

Theorem 5 can be improved when \( c \) is improved by \( c' \) as in [41, 42].

\[
c' = \min\{\text{max}_k ||\sum_j E_j F_k E_j||, \text{max}_j ||\sum_k F_k E_j F_k||\}
\]

Replacing \( c \) by \( c' \) in Eq. (31), it yields a stronger uncertainty relation [41, 42].

There is an operational interpretation of \( C_r(\rho, E) \) [5]: \( C_r(\rho, E) \) quantifies the private randomness generated by the POVM \( E \) on the state with respect to an eavesdropper holding optimal side information about the measured state. In this case Eq. (31) sets a tradeoff constraint between the POVMs \( E \) and \( F \). When POVMs \( E \) and \( F \) are rank-one projective measurements, Eq. (31) reduces the case of conventional coherence [43].

For any normalized pure state \( |\psi\rangle \), Theorem 5 gives rise to

\[
C_r(\psi, E) + C_r(\psi, F) \geq 2\log_2 \frac{1}{c'},
\]

where the lower bound is no longer state-dependent.

V. AVERAGES OF \( C_r, C_{T,\alpha} \) AND \( C_t \) OVER RANDOM PURE QUANTUM STATES

Random pure quantum states offer new insights into various phenomena in quantum physics and quantum information theory [44]. For the space of \( d \)-dimensional pure states, there exists a unique measure \( d\psi \) induced from the uniform Haar measure \( d\mu(U) \) on the unitary group \( U(d) \) [44]. This amounts to saying that any random pure state can be seen as a unitary matrix \( U \in U(d) \) performing on a fixed pure state \( \psi_0 \). In the following we always adopt this measure for random pure states [45–47]. The average value of the function \( g(\psi) \) over all pure states \( \psi \) then is defined as

\[
\int g(\psi) d\psi = \int g(U(\psi_0)) d\mu(U)
\]

The average of conventional coherence for pure quantum states has been extensively studied [45–47]. In [48] the authors presented the conventional coherence average over all orthonormal bases (rank-one projective measurements). In this section, we investigate the averages of \( C_r(\rho, E), C_{T,\alpha}(\rho, E) \) and \( C_t(\rho, E) \) with respect to POVM \( E \) for random pure states. We have Theorem 6 below.

**Theorem 6.** The averages of \( C_r(\psi, E), C_{T,\alpha}(\psi, E) \) and \( C_t(\psi, E) \) for random pure states \( \psi \) have the properties

\[
\int d\psi C_r(\psi, E) = -\frac{1}{d} \frac{1}{\ln 2}
\]

\[
\cdot \sum_{j=1}^n \sum_{k=1}^d (\Pi_{i \neq k} 1 - \lambda_j k) \sum_{m=2}^d \frac{1}{m} (\ln \lambda_j k - \sum_{m=2}^d \frac{1}{m}) (35)
\]

\[
\int d\psi C_{T,\alpha}(\psi, E) = \frac{1}{\alpha - 1} \sum_{j=1}^n B(E_j, \alpha - 1) - 1 \cdot (36)
\]

\[
\int d\psi C_t(\psi, E) \leq \sum_{j \neq k} B(E_j, \frac{p_{jk}}{2}) + B(E_k, \frac{q_{jk}}{2}) (37)
\]

where \( \{\lambda_j k\}_{j=1}^d \) are the eigenvalues of \( E_j \),

\[
B(E_j, \beta) = \frac{\Gamma(d)\Gamma(1 + \beta)}{\Gamma(d + \beta)} \sum_{k=1}^d (\Pi_{i \neq k} 1 - \lambda_j k) \lambda_j k^{d + \beta - 1}, \]

\[
\beta > 0, \Gamma(\cdot) \text{ is the Gamma function, and } p_{jk} > 1, q_{jk} > 1, 1/p_{jk} + 1/q_{jk} = 1 \text{ for any } j \neq k \text{. } B(E_j, \beta) \text{ is discussed in Appendix A of [49].}
\]

**Proof.** Let \( A \) be an operator on \( H \) with eigendecomposition,

\[
A = \sum_{j=1}^d \lambda_j |\varphi_j\rangle \langle \varphi_j|.
\] (39)
where \( \{ \lambda_j \}_{j=1}^d \) are the real eigenvalues and \( \{ | \varphi_j \rangle \}_{j=1}^d \) constitute an orthonormal basis of \( H \). Then one has [50]
\[
\int f(\langle \psi | A | \psi \rangle) d\psi = \Gamma(d) \sum_{j=1}^d \left( \frac{1}{\lambda_j - \lambda_k} \right) R_{d-1}[f](\lambda_j)
\]  
(40)
for any function \( f(\cdot) \), where \( R_{d-1}[f] \) is the Riemann-Liouville fractional integration defined by
\[
R_{\mu}[f](u) = \frac{1}{\Gamma(\mu)} \int_0^u f(w)(u-w)^{\mu-1} dw, \text{Re}\mu > 0. \quad (41)
\]
Note that [51]
\[
R_{\mu}[u^{\mu-1} \ln u](u) = \frac{\Gamma(\nu)}{\Gamma(\nu + \mu)} u^{\nu + \mu - 1}(\ln u + \Psi(\nu) - \Psi(\mu + \nu)), \quad (42)
\]
with \( \text{Re}\nu > 0, \text{Re}\mu > 0 \) and \( \Psi(\nu) = \frac{d}{d\nu} \ln \Gamma(\nu) \) the digamma function. Set \( \mu = d - 1 \) and \( \nu = 2 \). Using the properties \( \Gamma(d+1) = d! \) and \( \Psi(\mu + 1) = \Psi(\mu) + 1/\mu \), we get
\[
R_{d-1}[w \ln w](u) = \frac{1}{d!} u^d (\ln u - \sum_{m=2}^d \frac{1}{m}). \quad (43)
\]
Since
\[
\int d\psi C_\gamma(\psi, E) = -\sum_{j=1}^n \int d\psi \langle \psi | E_j | \psi \rangle \log_2 \langle \psi | E_j | \psi \rangle,
\]
we get (35) from (40) and (43).

Note that [51]
\[
R_{\mu}[u^{\mu-1}](u) = \frac{\Gamma(\nu)}{\Gamma(\nu + \mu)} u^{\nu + \mu - 1}, \quad (44)
\]
for \( \text{Re}\nu > 0 \) and \( \text{Re}\mu > 0 \). Set \( \mu = d - 1 \) and \( \nu - 1 = \beta \). We have
\[
R_{d-1}[w^\beta](u) = \frac{\Gamma(1 + \beta)}{\Gamma(d + \beta)} u^{d + \beta - 1}. \quad (45)
\]
By definition,
\[
\int d\psi C_{\gamma,\alpha}(\psi, E) = \frac{1}{\alpha - 1} \left( \sum_{j=1}^n \int d\psi \langle \psi | E_j | \psi \rangle \right)^{\frac{\alpha}{\alpha - 1} - 1}.
\]
Hence we obtain (36) from (40) and (45).

Moreover, from
\[
C_{\gamma_0}(\psi, E) = \sum_{j \neq k} \sqrt{\langle \psi | E_j | \psi \rangle \langle \psi | E_k | \psi \rangle},
\]
we have
\[
C_{\gamma_0}(\psi, E) \leq \sum_{j \neq k} \left( \frac{\langle \psi | E_j | \psi \rangle}{p_{jk}} + \frac{\langle \psi | E_k | \psi \rangle}{q_{jk}} \right), \quad (46)
\]
where we have used the Young inequality,
\[
ab \leq \frac{a^p}{p_{jk}} + \frac{b^q}{q_{jk}} \quad (47)
\]
for \( a \geq 0, b \geq 0, p_{jk} > 1, q_{jk} > 1 \) and \( \frac{1}{p_{jk}} + \frac{1}{q_{jk}} = 1 \). Taking into account (46), (40) and (45), one gets (37).

Note that if the eigenvalues \( \{ \lambda_{j,i} \}_{i=1}^d \) in (35) and (38) are degenerate, the problem can be still handled via the standard trick of taking the limits when the eigenvalues approach pairwise [50], see also [49] for (38). In particular, if \( p_{jk} = q_{jk} = 2 \) for any \( j \neq k \) in (46), we get
\[
C_{\gamma_0}(\psi, E) \leq n - 1. \quad (48)
\]
We remark that when the POVM \( E \) is a rank-one projective measurement \( E = \{|j\rangle \langle j|\}_{j=1}^d \), (35) gives rise to the corresponding result for conventional coherence [45],
\[
\int d\psi C_r(\psi, \{|j\rangle \langle j|\}_{j=1}^d) = \frac{1}{\ln 2} \sum_{m=2}^d \frac{1}{m}.
\]
To compare (36) with the one of conventional coherence, we consider the case of \( \alpha = 1/2 \). We see that
\[
\sum_{k=1}^d (\Pi_{l \neq k} \frac{1}{\lambda_{j,k} - \lambda_{j,l}}) \lambda_{j,l}^{d+1} \]
is a homogeneous symmetric polynomial in \( \{ \lambda_{j,k} \}_{k=1}^d \) of degree 2, which can be expressed in terms of elementary symmetric polynomials. Due to the homogeneity, there exist constants \( C_1 \) and \( C_2 \) such that
\[
\sum_{k=1}^d (\Pi_{l \neq k} \frac{1}{\lambda_{j,k} - \lambda_{j,l}}) \lambda_{j,l}^{d+1} = C_1 \left( \sum_{k=0}^d \lambda_{j,k} \right)^2 + C_2 \sum_{k<l} \lambda_{j,k} \lambda_{j,l}. \quad (49)
\]
Since
\[
\left( \sum_{k=0}^d \lambda_{j,k} \right)^2 = \sum_{k=0}^d \lambda_{j,k}^2 + 2 \sum_{k<l} \lambda_{j,k} \lambda_{j,l}, \quad (50)
\]
there further exist constants \( C_3 \) and \( C_4 \) such that
\[
\sum_{k=1}^d (\Pi_{l \neq k} \frac{1}{\lambda_{j,k} - \lambda_{j,l}}) \lambda_{j,l}^{d+1} = C_3 \left( \sum_{k=0}^d \lambda_{j,k} \right)^2 + C_4 \sum_{k=1}^d \lambda_{j,k}^2. \quad (51)
\]
Set \( \lambda_{j,1} = 1 \) and \( \{ \lambda_{j,k} = 0 \}_{k=2}^d \), we get \( C_3 + C_4 = 1 \). Let \( \lambda_{j,1} = 1, \lambda_{j,2} = 1/2 \) and \( \{ \lambda_{j,k} = 0 \}_{k=3}^d \), we have
\[ \frac{1}{2} C_3 + \frac{1}{4} C_4 = \frac{1}{2}. \] Consequently, we obtain \( C_3 = C_4 = \frac{1}{2} \) and
\[
\int d\psi C_{T,\frac{d}{2}}(\psi, E) = 2\{1 - \frac{1}{d+1} \sum_{j=1}^{n} (\text{tr} E_j)^2 + \text{tr}(E_j^2)\}. \quad (52)
\]

Hence, when the POVM \( E \) is a rank-one projective measurement \( E = \{ |j\rangle \langle j| \}_{j=1}^{d} \), (52) recovers the result in [47], \( \int d\psi C_{T,\frac{d}{2}}(\psi, \{ |j\rangle \langle j| \}_{j=1}^{d}) = 2(1 - 1)/(d + 1) \).

VI. SUMMARY

The conventional quantum coherence is defined with respect to an orthonormal basis, while the generalized quantum coherence studied is defined with respect to general POVM settings. We have investigated the properties of three well-defined POVM coherence measures, the \( C_r(\rho, E) \), \( C_{T,\alpha}(\rho, E) \) and \( C_{l_i}(\rho, E) \). We have provided the upper bounds of \( C_{l_i}(\rho, E) \), the operational interpretation for \( C_{T,1/2}(\rho, E) \), the uncertainty relations given by \( C_r(\rho, E) \), and calculated the averages of \( C_r(\rho, E) \), \( C_{T,\alpha}(\rho, E) \) and \( C_{l_i}(\rho, E) \) over random pure states. These results will strengthen the necessity of the concept of POVM coherence, and highlight the potential applications of these POVM coherence measures.

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