GEOMETRIC INVARIANTS OF 5/2-CUSPIDAL EDGES

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Abstract. We introduce two invariants called the secondary cuspidal curvature and the bias on 5/2-cuspidal edges, and investigate their basic properties. While the secondary cuspidal curvature is an analog of the cuspidal curvature of (ordinary) cuspidal edges, there are no invariants corresponding to the bias. We prove that the product (called the secondary product curvature) of the secondary cuspidal curvature and the limiting normal curvature is an intrinsic invariant. Using this intrinsity, we show that any real analytic 5/2-cuspidal edges with non-vanishing limiting normal curvature admits non-trivial isometric deformations, which provide the extrinsity of various invariants.

1. Introduction

The ordinary cusp or 3/2-cusp is a map-germ \((\mathbb{R},0) \rightarrow (\mathbb{R}^2,0)\) which is diffeomorphic (\(A\)-equivalent) to the map-germ \(t \mapsto (t^2, t^3)\) at the origin. It is known that the 3/2-cusp is the most frequently appearing singularity on plane curves. A cuspidal edge is a map-germ \((\mathbb{R}^2,0) \rightarrow (\mathbb{R}^3,0)\) which is \(A\)-equivalent to the map-germ \((u,v) \mapsto (u,v^2,v^3)\) at the origin (Figure 1 right), where two map-germs \(f,g : (\mathbb{R}^m,0) \rightarrow (\mathbb{R}^n,0)\) are \(A\)-equivalent if there exist diffeomorphisms \(\phi_s : (\mathbb{R}^m,0) \rightarrow (\mathbb{R}^m,0)\) and \(\phi_t : (\mathbb{R}^n,0) \rightarrow (\mathbb{R}^n,0)\) such that \(\phi_t \circ f \circ \phi_s = g\). A cuspidal edge is a kind of a direct product of a 3/2-cusp with an interval, and its differential geometric properties are well studied. In [26] (see also [21]), the cuspidal curvature for 3/2-cusps is defined. Roughly speaking, the cuspidal curvature measures a kind of wideness of 3/2-cusps. For cuspidal edges, the singular curvature and the limiting normal curvature are introduced in [19], and their geometric meanings are studied.

In this paper, we deal with the 5/2-cusp and 5/2-cuspidal edge. A 5/2-cusp (respectively, 5/2-cuspidal edge) is a map-germ \((\mathbb{R},0) \rightarrow (\mathbb{R}^2,0)\) (respectively, \((\mathbb{R}^2,0) \rightarrow (\mathbb{R}^3,0)\)) which is \(A\)-equivalent to the map-germ \(t \mapsto (t^2, t^5)\) (respectively, \((u,v) \mapsto (u,v^2,v^5)\)) at the origin (Figure 1 left). A 5/2-cusp is also called a rhamphoid cusp. Although 5/2-cuspidal edges do not generically appear but it is pointed out that they naturally appear in various differential geometric situations [4] [9] [18]. For 5/2-cusps, the cuspidal curvature vanishes. Hence, to measure the wideness of the 5/2-cusps, we need to consider a higher order invariant. In this paper, we define two curvatures on 5/2-cusps in addition to the invariants we mentioned in the above, which are the secondary cuspidal curvature and the bias of cusp. The secondary cuspidal curvature is an analogy of cuspidal curvature of 3/2-cusps, but as we will see in Section 2.4 there is no corresponding notion of bias of cusp for 3/2-cusp. Using this, we also define two curvatures for 5/2-cuspidal edges.
On the other hand, it is one of the fundamental problems in determining the intrinsity and extrinsity of invariants. It is proved that some basic invariants, such as the singular curvature and the product curvature, are intrinsic in [19, 14], and they have various applications. For example, the intrinsity of the product curvature is used to prove the existence of isometric deformations of real analytic cuspidal edges with non-vanishing limiting normal curvature in [15] and [8]. See [3, 7] for cross caps, and see [4] for other applications. In this paper, we determine whether the above invariants of 5/2-cuspidal edges are intrinsic or extrinsic, proving the existence of isometric deformations of real analytic 5/2-cuspidal edges with non-vanishing limiting normal curvature as in [15] and [8].

This paper is organized as follows. In Section 2, we define the secondary cuspidal curvature and the bias for 5/2-cusps, and study their geometric properties. In Section 3, we deal with 5/2-cuspidal edges and define two invariants on them. As an example, in Section 3.6, we calculate the invariants on the conjugate surfaces of spacelike Delaunay surfaces. In Section 4, we prove that the product (called the secondary product curvature) of the secondary cuspidal curvature and the limiting normal curvature is an intrinsic invariant. Using this intrinsity, we show the existence of isometric deformations of real analytic 5/2-cuspidal edges with non-vanishing limiting normal curvature, which yields the extrinsity of various invariants, see Table 2. Finally, in Section 5, we provide an intrinsic formulation of 5/2-cuspidal edges as a singular point of a positive semi-definite metric, called the Kossowski metric. Using an argument similar to that in Section 4, we prove the existence of isometric realizations of Kossowski metrics with intrinsic 5/2-cuspidal edges.

2. 5/2-cusps

In this section, we discuss the geometric properties of 5/2-cusps.

2.1. Invariants of 5/2-cusps. Let \( \gamma : (R, 0) \rightarrow (R^2, 0) \) be a map-germ, and \( \gamma'(0) = 0 \). We call \( \gamma \) is A-type if \( \gamma''(0) \neq 0 \). Let \( \gamma : (R, 0) \rightarrow (R^2, 0) \) be an A-type map-germ. The cuspidal curvature for \( \gamma \) at 0 is defined by

\[
\omega(\gamma, 0) = \frac{\det(\gamma''(0), \gamma'''(0))}{|\gamma''(0)|^{5/2}},
\]

which measures a kind of wideness of \( \gamma \) at 0 ([21]). We abbreviate \( \omega = \omega(\gamma) = \omega(\gamma, 0) \) if \( \gamma \) or 0 is clear. It is well known that an A-type map-germ \( \gamma \) is a 3/2-cusp if and only if \( \det(\gamma''(0), \gamma'''(0)) \neq 0 \), and hence \( \omega \neq 0 \).
Let $\gamma : (R, 0) \rightarrow (R^2, 0)$ be an $A$-type map-germ with $\det(\gamma''(0), \gamma'''(0)) = 0$. Then there exists $l \in R$ such that 

$$\gamma'''(0) = l\gamma''(0).$$

Then the secondary cuspidal curvature for $\gamma$ at 0 is defined by 

$$\omega_r(\gamma, 0) = \frac{\det \left( \gamma''(0), 3\gamma^{(5)}(0) - 10l\gamma^{(4)}(0) \right)}{|\gamma''(0)|^{3/2}}.$$

We abbreviate $\omega_r = \omega_r(\gamma, 0)$ as well. By a direct calculation, one can see that $\omega_r$ does not depend on the parameter of $\gamma$. On the other hand, a criterion for 5/2-cusp is known [13].

**Fact 2.1.** Let $\gamma : (R, 0) \rightarrow (R^2, 0)$ be a map-germ with $\gamma'(0) = 0$. Then $\gamma$ is a 5/2-cusp if and only if 

1. $\det(\gamma''(0), \gamma'''(0)) = 0,$
2. $3\det(\gamma''(0), \gamma^{(5)}(0))\gamma''(0) - 10\det(\gamma''(0), \gamma^{(4)}(0))\gamma'''(0) \neq (0, 0).$

We remark that by the condition [12] $\gamma''(0) \neq 0$. When $\gamma$ is $A$-type at 0, the conditions [1] and [2] are written as follows. By [1] there exists $l \in R$ such that $\gamma'''(0) = l\gamma''(0)$ and then, [2] is written as 

$$\det \left( \gamma''(0), 3\gamma^{(5)}(0) - 10l\gamma^{(4)}(0) \right) \neq 0.$$

Thus an $A$-type germ $\gamma$ is a 5/2-cusp if and only if $\omega = 0$ and $\omega_r \neq 0$.

Next we define the bias of cusp. Let $\gamma : (R, 0) \rightarrow (R^2, 0)$ be an $A$-type map-germ which is not a 3/2-cusp (i.e., $\omega = 0$). Then 

$$b(\gamma, 0) = \frac{\det(\gamma''(0), \gamma^{(4)}(0))}{|\gamma''(0)|^3}$$

does not depend on the parameter, and it is called the bias of cusp. We abbreviate $b = b(\gamma) = b(\gamma, 0)$ as well. Let $\gamma$ be an $A$-type germ. A line 

$$\left\{ u \lim_{t \to 0} \frac{\gamma'(t)}{|\gamma'(t)|} : u \in R \right\} = \left\{ u\gamma''(0) : u \in R \right\}$$

passing through $\gamma(0) = 0$ is called the tangent line of $\gamma$ at 0. We set two images of $\gamma$ as 

$$\gamma_+ = \gamma((0, \varepsilon)), \quad \gamma_- = \gamma((-\varepsilon, 0)),$$

for $\varepsilon > 0$. We see the following proposition.

**Proposition 2.2.** Let $\gamma$ be an $A$-type germ with $\omega = 0$. If $b \neq 0$ then for sufficiently small $\varepsilon$, the images $\gamma_+$ and $\gamma_-$ lie on the same side of the tangent line of $\gamma$. Moreover, if $\gamma$ is a 5/2-cusp and $b = 0$ then for sufficiently small $\varepsilon$, the images $\gamma_+$ and $\gamma_-$ lie on the both sides of the tangent line of $\gamma$.

**Proof.** By rotating $\gamma$ and by parameter change, we may assume that

$$\gamma = \left( \frac{t^2}{2}, \frac{t^4}{4!}\gamma_4 + \frac{t^5}{5!}\gamma_5(t) \right), \quad \gamma_4 \in R.$$

Then $b = \gamma_4$ and $\omega_r = 3\gamma_5(0)$. The claim of the proposition is obvious by these observations. \(\square\)

One can easily see that for 3/2-cusp, the images $\gamma_+$ and $\gamma_-$ always lie on the both sides of the tangent line of $\gamma$. Thus there is no similar notion of the bias of cusp for 3/2-cusps. If an $A$-type map-germ $\gamma$ with $\omega = 0$ satisfies $b(\gamma, 0) = 0$, then $\gamma$ is said to be balanced (see Figure 2).
Figure 2. The left figure shows a balanced $5/2$-cusp (i.e., $b = 0$), and the right one is non-balanced (i.e., $b \neq 0$). The dotted lines are the tangent line at each singular points. As we have shown in Proposition 2.2, the image of a balanced $5/2$-cusp extends over the two domains separated by the tangent line.

2.2. Behavior of the curvature function. Let $s_g$ be the arc-length function $s_g(t) = \int_0^t |\gamma'(t)| \, dt$ of a $A$-type germ $\gamma : (R, 0) \rightarrow (R^2, 0)$. It is shown that $(s(t) := ) \text{sgn}(t)\sqrt{|s_g(t)|}$ is $C^\infty$-differentiable and $s'(0) \neq 0$ ([23, Theorem 1.1]). Thus one can take $s(t)$ as a parameter, which is called the half-arclength parameter [23]. We have the following proposition.

**Proposition 2.3.** Let $\gamma : (R, 0) \rightarrow (R^2, 0)$ be a $5/2$-cusp, and $t$ a parameter. Let $\kappa$ be the curvature defined except $t = 0$. Then $\tilde{\kappa} = \text{sgn}(t)\kappa$ is a $C^\infty$ function, and

$$\tilde{\kappa}(0) = \frac{b}{3}, \quad \frac{d}{ds}\tilde{\kappa}(0) = \frac{\sqrt{2}}{24} \omega_r$$

holds, where $s$ is the half-arclength parameter.

**Proof.** We may assume that $\gamma$ is given by the form (2.1) without loss of generality. Then

$$\det(\gamma', \gamma'') = \frac{\gamma_4 t^3 + \gamma_5(0) t^4 + O(5)}{|\gamma'|^3} = \text{sgn}(t) \left( \frac{\gamma_4}{3} + \frac{\gamma_5(0)}{8} t + O(2) \right)$$

holds. Here, $O(n)$ stands for the terms whose degrees are greater than or equal to $n$. On the other hand, by $|\gamma'| = |t\sqrt{1 + t^4/36 + O(5)}| = |t + \gamma_4 t^4/72 + O(5)|$ and $s_g = t^2(1/2 + O(4))$, it holds that $s = t\sqrt{1/2 + O(4)}$ and $dt/ds = \sqrt{2}$ at 0. The proposition is obvious by the above calculations. \(\square\)

See [2] for another treatment of curvatures of curves with singularities.

2.3. Projection of space curves. Let $\Gamma : (R, 0) \rightarrow (R^3, 0)$ be a regular space curve, and let $t$ be an arc-length of $\Gamma$, and $e, n, b$ a Frenet frame. We set a plane curve $\gamma$ in the normal plane $(e(0))^\perp$ at 0:

$$\gamma(t) = \Gamma(t) - (\Gamma(t), e(0))\, e(0).$$

Remark that $\gamma'(0) = 0$. Then $\gamma$ at 0 is $A$-type if and only if $\kappa(0) \neq 0$, where $\kappa$ is the curvature of $\Gamma$. We assume that $\gamma$ is $A$-type (i.e., $\kappa(0) \neq 0$). Since

$$\omega(\gamma, 0) = \frac{\tau(0)}{\sqrt{\kappa(0)}}$$

holds.
γ at 0 is a 3/2-cusp if and only if τ(0) ≠ 0, where τ is the torsion of Γ. If γ is A-type but not a 3/2-cusp (i.e., κ(0) ≠ 0, τ(0) = 0), then
\[ b(γ, 0) = \frac{τ}{κ}(0), \quad ω_ν(γ, 0) = -\frac{κ^2τ' + 3κτ''}{κ^{5/2}}(0). \]
Thus under the assumption κ(0) ≠ 0, γ is a 3/2-cusp if and only if τ(0) ≠ 0, γ is not a 3/2-cusp and non-balanced if and only if τ(0) = 0, τ'(0) ≠ 0, and γ is a balanced 5/2-cusp if and only if τ(0) = τ'(0) = 0, τ''(0) ≠ 0.

3. INVARIANTS OF 5/2-CUSPIDAL EDGES

In this section, we discuss the geometric properties of 5/2-cuspidal edges.

3.1. Frontals. Let \( f : (R^2, 0) → (R^3, 0) \) be a map-germ. We call \( f \) a frontal if there exists a map \( ν : (R^2, 0) → S^2 \) satisfying \( (df(X), ν) = 0 \) for any \( X \in T_pR^2 \) and \( p \in (R^2, 0) \), where \( S^2 \) stands for the unit sphere in \( R^3 \). We call \( ν \) a unit normal vector field of \( f \). A frontal is called a front if \( (f, ν) \) is an immersion. Let \( f : (R^2, 0) → (R^3, 0) \) be a frontal, and \( ν \) a unit normal vector field of \( f \). We set
\[
λ = \det(f_u, f_v, ν)
\]
by taking a coordinate system \((u, v)\), and \( f_u = ∂f/∂u, f_v = ∂f/∂v \). We call \( λ \) a signed area density function. By the definition, \( S(f) = λ^{-1}(0) \), where \( S(f) \) is the set of singularities of \( f \). A singular point \( p \) of \( f \) is non-degenerate if \( df(ν) = 0 \). If \( p \) is a non-degenerate singular point, then \( S(f) \) near \( p \) is a regular curve. Let \( p \) be a singular point satisfying rank \( df_p = 1 \), then there exists a non-vanishing vector field \( η \) on a neighborhood \( U \) of \( p \) such that \( (η_0)_R = ker df_q \) for \( q ∈ S(f) \cap U \). We call \( η \) a null vector field. We note that the notion of non-degeneracy and null vector field are introduced in [11]. We remark that a non-degenerate singular point satisfies rank \( df_p = 1 \). A non-degenerate singular point \( p \) of \( f \) is called the first kind (respectively, second kind) if \( η_p \) is transverse to \( S(f) \) at \( p \) (respectively, \( η_p \) is tangent to \( S(f) \) at \( p \)). It is well-known that a singular point of the first kind on a front is cuspidal edge ([11] Proposition 1.3, see also [20] Corollary 2.5]).

3.2. Basic invariants for singular point of the first kind. In [19], the singular curvature and the limiting normal curvature are defined for cuspidal edges, namely singular points of the first kind of fronts. In [13] [14], the cuspidal curvature and the cusp-directional torsion are defined. These definitions are also valid for singular points of the first kind of frontals.

Let \( f : (R^2, 0) → (R^3, 0) \) be a frontal and \( ν \) a unit normal vector field. Let \( 0 \) be a singular point of the first kind. Taking a parameterization \( γ : (R, 0) → (R^2, 0) \) of \( S(f) \), the singular curvature \( κ_s \) and the limiting normal curvature \( κ_{ν} \) are defined by
\[
κ_s(t) = \text{sgn}(dλ(q)) \frac{det(γ', γ'', ν ∘ γ)}{|γ'|^3}(t), \quad κ_ν(t) = \frac{(γ'' ∘ γ)}{|γ'|^2}(t),
\]
respectively ([20]), where \( γ', γ'' \) are taken to be positively oriented. Let \( ξ \) be a vector field on \((R^2, 0)\) such that \( ξ_0 \) is tangent to \( S(f) \) at \( q ∈ S(f) \), and \( η \) be a null vector field. Then the cuspidal curvature \( κ_c \) and the cusp-directional torsion or the cuspidal torsion \( κ_d \) is defined by
\[
(3.2) \quad κ_c(t) = \frac{|ξ|^3/2det(ξ, η^2f, η^3f)|}{|ξ × η^2f|^{3/2}}|_{(u,v)=γ(t)},
\]
\[
(3.3) \quad κ_d(t) = \left( \frac{det(ξ, η^2f, ξη^2f)}{|ξ × η^2f|^2} - \frac{det(ξ, η^2f, ξ^2f)}{|ξ|^2|ξ × η^2f|^2} \right)|_{(u,v)=γ(t)},
\]
where \( \zeta_i f \) stands for the \( i \) times directional derivative of \( f \) by a vector field \( \zeta \). The invariant \( \kappa_c \) measures a kind of “wideness” of the singularity. Furthermore, it is shown that \( \kappa_{\Pi} = \kappa_c \kappa_v \) is an intrinsic invariant. See Section 4 for the definition of the intrinsic and the extrinsic of invariants. See [14] for detail. One can easily see that \( \kappa_c(0) \neq 0 \) if and only if \( f \) is a front. It is known that for two cuspidal edges, if \( \kappa_a, \kappa'_a, \kappa_v, \kappa'_v, \kappa_c, \kappa_t \) of them are coincide at 0, then 3-jets at 0 of them can be coincided for a coordinate system ([13] Theorem 6.1)], where \( t = d/dt \). In 3 [4], intrinsics and extrinsities of these invariants are investigated. See [17] for another approach to investigate cuspidal edges, and [24] for other applications of the above invariants (see also [25]).

3.3. Criterion and invariants for 5/2-cuspidal edges.

Let \( \xi, \eta \) be a coordinate system, \( S(f) = \{ v = 0 \} \), \( \eta = \partial_v \).

Proposition 3.1 ([6, Theorem 4.1]). The front germ \( f : (R^2, 0) \to (R^3, 0) \) is a 5/2-cuspidal edge if and only if

1. \( \eta \lambda(0) \neq 0 \),
2. \( \det(\xi f, \eta^2 f, \eta^3 f) = 0 \) on \( S(f) \),
3. \( \det(\xi f, \eta^2 f, 5\eta^3 f - 10l\eta^4 f)(0) \neq 0 \).

Here, \( \xi \) is a vector field on \( (R^2, 0) \) such that the restriction \( \xi|_{S(f)} \) is tangent to \( S(f) \), and \( \eta \) is a null vector field. Furthermore, \( \eta \) is a null vector field satisfying \( \langle \xi f, \eta^2 f \rangle(0) = \langle \xi f, \eta^3 f \rangle(0) = 0 \).

Then by [2], there exists \( l \in R \) such that \( \eta^3 f(0) = l\eta^2 f(0) \). This \( l \) is in the condition [3].

We remark that the condition [1] implies that 0 is a singular point of the first kind. As in [6, Lemma 4.2], the existence of vector field \( \tilde{\eta} \) for \( f \) is shown as follows: We take a coordinate system \( (u, v) \) satisfying

\[ S(f) = \{ v = 0 \} \quad \text{and} \quad \eta = \partial_v. \]

Then \( \eta \) is a null vector field, and since

\[ \eta^2 f = k_1 v^2 f_{uu} + 2k_1 vf_{uv} + k_1 f_u + f_{vv} \quad \text{and} \quad \eta^2 f(0) = k_1 f_u + f_{vv}, \]

\[ \langle \xi f, \eta^2 f \rangle(0,0) = k_1 \langle f_u, f_u \rangle(0,0) + \langle f_u, f_{vv} \rangle(0,0) = 0 \]

holds. Furthermore, let \( \xi, \eta \) satisfy \( \langle \xi f, \eta^2 f \rangle(0,0) = 0 \). Setting

\[ \tilde{\eta} = \frac{k_2 v^2}{2} \xi + \eta, \quad k_2 = -\frac{\langle f_u, f_{vv} \rangle}{\langle f_u, f_u \rangle}(0,0), \]

we see \( \langle \xi f, \tilde{\eta}^2 f \rangle(0,0) = 0 \) and

\[ \tilde{\eta}^3 f(0,0) = k_2 f_u(0,0) + f_{vv}(0,0). \]

Thus we have \( \tilde{\eta} \). Moreover, we see that the condition [2] implies that \( f(0) \) is not a front. This can be shown by the following lemma:

Lemma 3.2. The condition [2] is equivalent to \( \eta \nu = 0 \) on \( S(f) \).

Proof. First, we show that the condition [2] is equivalent to \( \det(\xi f, \nu, \eta \nu) = 0 \).

Since \( \langle \nu, \xi f \rangle = 0 \) and \( \langle \nu, \eta^2 f \rangle = -\langle \eta \nu, \eta f \rangle = 0 \), we see \( \nu \) is parallel to \( \xi f \times \eta^2 f \) on \( S(f) \). Thus

\[ (\xi f \times \eta f)^2 \det(\xi f, \nu, \eta \nu) = \frac{\det(\xi f, \eta^2 f, \eta(\xi f \times \eta^2 f))}{\det(\xi f, \eta^2 f, \eta f \times \eta^2 f + \xi f \times \eta^3 f)}. \]


Now $[\eta, \xi]|_f = \eta \xi - \xi \eta$ is parallel to $\xi$, and $\eta f = 0$ on $S(f)$. Since $\xi$ is tangent to $S(f)$, $\eta f = 0$ on $S(f)$. Hence $\eta \xi$ is parallel to $\xi$. Thus the left hand side of (3.7) is equal to
\begin{equation}
(3.8) \det(\xi, \eta^2 f, \eta^3 f)
\end{equation}
on $S(f)$. Since $\det(a, a \times b, a \times c) = |a|^2 \det(a, b, c)$ for vectors $a, b, c \in \mathbb{R}^3$, (3.8) is a non-zero multiple of $\det(\xi, \eta^2 f, \eta^3 f)$ on $S(f)$. Thus (3.8) is equivalent to $\det(\xi, \nu, \eta)$.

Then one can write $\nu = \alpha \xi + \beta \nu$. Then $\beta = \langle \nu, \xi \rangle = 0$. On the other hand, $[\xi, \eta] = [\eta, \xi] = \nu$. Since $\langle \nu, \xi \rangle (u, v) = \langle \nu, \eta \rangle (u, v) = 0$ for any $(u, v)$, it holds that $\langle \nu, \xi \rangle (u, v) + \langle \nu, \eta \xi \rangle (u, v) = 0$, $\langle \nu, \eta \xi \rangle (u, v) + \langle \nu, \xi \eta \xi \rangle (u, v) = 0$ for any $(u, v)$. Since $[\eta, \xi]$ is a vector field, $[\eta, \xi] = \nu$ is parallel to $\xi$ on $S(f)$. Thus $\langle \nu, [\eta, \xi] \rangle = 0$ on $S(f)$. Hence we have $\langle \xi, \eta^2 \rangle = \langle \eta, \xi^2 \rangle = \langle \nu, \eta \rangle = 0$ on $S(f)$. This completes the proof.

Lemma 3.3. Two real numbers $r_b$ and $r_c$ do not depend on the choice of $\xi$ and $\tilde{\eta}$.

Proof. We take a coordinate system $(u, v)$ satisfying that $S(f) = \{ v = 0 \}$, $f_v = 0$ on $S(f)$, and $\langle f_u, f_v \rangle = \langle f_u, f_{uvv} \rangle = 0$ on $S(f)$. In fact, by (3.5) and (3.6), we see $[\xi, \tilde{\eta}] = 0$. Thus we can find a coordinate system $(u, v)$ such that $\partial_u = \xi, \partial_v = \tilde{\eta}$. We set
\[ \xi = \alpha_1(u, v) \partial_u + \alpha_2(u, v) \partial_v, \quad \tilde{\eta} = \alpha_3(u, v) \partial_u + \alpha_4(u, v) \partial_v, \]
where $\alpha_i(u, v)$ $(i = 1, 2, 3, 4)$ is a smooth function such that $\alpha_1, \alpha_4 > 0, \alpha_3(u, 0) = 0$.

By the assumption (3.3), $(\alpha_3)_v = (\alpha_3)_{uv} = 0$ holds on the $u$-axis. By a straightforward calculation,
\[ \xi = \alpha_1 f_u, \quad \tilde{\eta} = \alpha_2 f_v, \quad \tilde{\eta} f = \alpha_4 f_{uvv} + f_u + f_{uvv}, \]
hold on the $u$-axis. Thus
\[ \frac{|f|_2^2 \det(\xi, \tilde{\eta}, \tilde{\eta})}{|\xi \times \tilde{\eta}|^2} = \frac{\det(\alpha_1 f_u, \alpha_2 f_v, \alpha_4 f_{uvv})}{|\alpha_1 f_u| |\alpha_2 f_v| |\alpha_4 f_{uvv}|} = \frac{\det(f_u, f_v, f_{uvv})}{|f_u||f_v|} \]
holds at 0, which shows the independence of $r_b$. By the above calculation, if $f_{uvv} = l f_v$, then $\tilde{\eta} f = (3(\alpha_4)_v + \alpha_4 l) \tilde{\eta}^2 f$. Moreover, we see
\[ \tilde{\eta} f = \alpha_4 (10(\alpha_4)_v f_{uvv} + \alpha_4 f_{uvv}). \]

Hence,
\[ \frac{|f|_2^5 |\xi \times \tilde{\eta}|^2}{|\xi \times \tilde{\eta}|^4} = \frac{\det(\xi, \tilde{\eta}^2 f, 3 \tilde{\eta}^2 f - 10 \tilde{\eta} f)}{|\xi \times \tilde{\eta}|^4} \]
we see

since it holds that

\[ f(t) = (t(0), \tilde{n}^2 f(\gamma(t)), \tilde{n}^3 f(\gamma(t))) \]

where \( \gamma(t) \) is a parameterization of \( S(f) \), and \( \tilde{n} \) is a null vector field satisfying

\[ \langle \xi f, \tilde{n}^2 f \rangle (\gamma(t)) = \langle \xi f, \tilde{n}^3 f \rangle (\gamma(t)) = 0. \]

Then we define \( r_b(t) \) and \( r_c(t) \) as

\[
\begin{align*}
\text{(3.9)} & \quad r_b(t) = \frac{|\xi f|^2 \det \left( \xi f, \tilde{n}^2 f, \tilde{n}^4 f \right)}{|\xi f \times \tilde{n}^2 f|^3} |_{(u,v)=\gamma(t)}, \\
\text{(3.10)} & \quad r_c(t) = \frac{|\xi f|^{5/2} \det \left( \xi f, \tilde{n}^2 f, 3\tilde{n}^2 f - 10 l \tilde{n}^4 f \right)}{|\xi f \times \tilde{n}^2 f|^{7/2}} |_{(u,v)=\gamma(t)},
\end{align*}
\]

respectively. The invariant \( r_b(t) \) is called the bias, and \( r_c(t) \) is called the secondary cuspidal curvature. We also define

\[ r_n(p) := \kappa_{\nu}(p) r_c(p) \]

for a singular point \( p \), which is called the secondary product curvature.

In [15], the bias \( r_b \) and the secondary cuspidal curvature \( r_c \) are used to investigate the cuspidal cross caps.

3.4. Geometric meanings. Here we study geometric meanings of \( r_b \) and \( r_c \). Let \( f : (R^2, 0) \to (R^3, 0) \) be a front and \( 0 \) a singular point of the first kind. Let \( \gamma(t) (\gamma(0) = 0) \) be a parameterization of \( S(f) \) and we set \( \tilde{\gamma}(t) := f(\gamma(t)) \). Since \( 0 \) is a singular point of the first kind, \( \tilde{\gamma}'(0) \neq 0 \). Thus we consider a slice of \( f \) by the normal plane \( (\tilde{\gamma}'(0)) \perp \) of \( \tilde{\gamma}'(0) \) passing through \( 0 \). It is defined by

\[ C = \{(u,v) ; (f(u,v), \tilde{\gamma}'(0)) = 0\}. \]

We take a coordinate system satisfying \( S(f) = \{ v = 0 \} \), \( \eta = \partial_v \) and \( \langle f_u, f_v \rangle = \langle f_{uu}, f_{uv} \rangle = 0 \) on \( S(f) \). Then we see that \( (f(u,v), \tilde{\gamma}'(0))u \neq 0 \) at \( 0 \). Thus we can take a parameterization of \( C \) by \( c(v) = (c_1(v), v) \). We set \( \hat{c} = f \circ c \). We remark that since \( \langle f(c(v)), \tilde{\gamma}'(0) \rangle = 0 \), it holds that \( c'_1(0) = 0 \). Furthermore, since \( f_{v}(u,0) = 0 \), it holds that \( f_{uv}(u,0) = f_{uvv}(u,0) = f_{uuv}(u,0) = 0 \). Then we have

\[ \hat{c}''(0) = f_{vvv}(0,0) + c''_1(0)f_u(0,0), \quad \hat{c}'''(0) = f_{vvv}(0,0) + c''_1(0)f_u(0,0). \]

Since \( \langle f(c(v)), \tilde{\gamma}'(0) \rangle = 0 \), it holds that \( c''_1(0) = -\langle f_u(0,0), f_{uvv}(0,0) \rangle = 0 \) and \( c''_1(0) = -\langle f_u(0,0), f_{uvv}(0,0) \rangle = 0 \). Furthermore, since

\[ \hat{c}^{(4)}(0) = f_{vvv}(0,0) + c^{(4)}_1 f_u(0,0), \quad \hat{c}^{(5)}(0) = f_{vvv}(0,0) + c^{(5)}_1 f_u(0,0), \]

we see

\[
\begin{align*}
\text{b}(\hat{c}, 0) = \frac{\det(f_u, f_{uv}, f_{vvv})}{|f_{uv}|^3}(0,0) = r_b, \\
\text{\omega}(\hat{c}, 0) = \frac{\det(f_u, f_{uv}, 3f_{vvv} - 10 f_{vvv})}{|f_{uv}|^{7/2}}(0,0) = r_c,
\end{align*}
\]

where the invariant \( b (\text{respectively, } \omega) \) is the bias of cusp (respectively, secondary cuspidal curvature) of \( \hat{c} \) as the plane curve in \( (\tilde{\gamma}'(0)) \perp \).
3.5. Normal form for 5/2-cuspidal edges. In [13], a normal form for cuspidal edges is given. We have the following.

**Proposition 3.4.** Let \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) be a 5/2-cuspidal edge. Then there exist a coordinate system \((u, v)\) on \((\mathbb{R}^2, 0)\) and an isometry \( \phi : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0) \) such that

\[
\Phi \circ f(u, v) = \left( u, \sum_{i=2}^5 a_i \frac{u^i}{i!} + \frac{v^2}{2} \sum_{i=2}^5 b_i \frac{v^i}{i!}, \sum_{i=1}^3 \frac{b_{12}}{i!} u^i v^i + \frac{b_{14}}{4!} u^4 v^4 + \sum_{i=4}^5 b_{2i} v^i \right) + h(u, v),
\]

where \( h(u, v) \) consists of the terms whose degrees are higher or equal to 6 of the form

\[
(0, u^6 h_1(u), u^5 h_2(u) + u^4 v^2 h_3(u) + u^2 v^4 h_4(u) + uv^5 h_5(u) + v^6 h_6(u, v)).
\]

Although this proposition can be shown by the same method of the proof of [13] Theorem 3.1, we give a proof in Appendix A for the completion. Under this normal form, the invariants defined above can be computed by

- \((\kappa_v(0), \kappa_v'(0), \kappa_v''(0), \kappa_v'''(0)) = \left( b_{20} - 2a_{2}b_{12}, b_{40} - 4a_{3}b_{12} - 2a_{2}b_{20} - 2a_{2}b_{22} - 3b_{22} - 4b_{22}b_{30}, b_{50} + 14a_{2}b_{12} - 7a_{2}b_{20} - 6a_{3}b_{12} - 6a_{3}b_{22} - 12b_{12}b_{20}b_{22} - 12b_{12}^2 b_{30} + 19b_{22}b_{30} + a_{2}(-6a_{3}b_{20} - 2b_{32} + 24b_{12} + 32b_{12}b_{12} \right),
- \((\kappa_s(0), \kappa_s'(0), \kappa_s''(0), \kappa_s'''(0)) = \left( a_{2} + 2b_{12}b_{20}, a_{4} - 4a_{2}b_{12} + b_{20} + 2b_{20}b_{22} + 4b_{12}b_{30} - 3a_{2}, a_{5} - a_{3}b_{20} + 19a_{3}, a_{6} - a_{3}b_{20} + 5b_{20} - 3a_{2}(4b_{12}b_{22} + 5b_{20}b_{30}) - 24b_{20}b_{12} + 2b_{12}(3b_{40} - 13b_{20}) + 6b_{22}b_{30} + 2b_{20}b_{32},
- \((\kappa_{s}(0), \kappa'_{s}(0), \kappa''_{s}(0)) = \left( 2b_{12} - 2b_{22} - a_{2}b_{20} - 2b_{32} + 4a_{2}b_{12} - a_{3}b_{20} - 2a_{3}b_{30} + 8b_{22}b_{12} \right),
- \((r_{b}(0), r'_{b}(0)) = \left( b_{20} - 12a_{2}b_{12},
- \((r_{c}(0) = 3b_{05}),
\]

and \( \kappa_{s} \equiv 0 \), where the prime means the differentiation with respect to the arc-length parameter of \( \tilde{\gamma} \). Looking at the boxed entries, we have the following proposition.

**Proposition 3.5.** Let \( f, g \) be germs of 5/2-cuspidal edges. If \( \kappa_{v}, \kappa_{v}', \kappa_{v}'', \kappa_{v}''', \kappa_{s}, \kappa_{s}', \kappa_{s}'', \kappa_{s}''', r_{b}, r'_{b}, r_{c} \) of them are coincide at 0, then there exist coordinate system \((u, v)\) and an isometry \( \Lambda \) of \( \mathbb{R}^3 \) such that

\[
j_{0}^{\Lambda} f(u, v) = j_{0}^{\Lambda}(A \circ g)(u, v),
\]

where \( j_{0}^{\Lambda} f(u, v) \) stands for the 5-jet of \( f \) with respect to \( (u, v) \) at 0.

Moreover, a parameterization of \( f(S(f)) \) as a space curve is given by \( f(u, 0) \). Since \( b_{04}, b_{14}, b_{05} \) do not appear in \( f(u, 0) \), they also do not appear in the curvature \( \kappa \) and the torsion \( \tau \) of \( f(u, 0) \). Thus we believe that the invariants \( r_{b}, r_{c} \) for 5/2-cuspidal edges are not mentioned before.

3.6. Invariants of 5/2-cuspidal edges on conjugate surfaces. We denote by \( \mathbb{R}^{3}_{1} \) the Lorentz-Minkowski 3-space with signature \((-+, +)\). A spacelike Delaunay surface with axis \( \ell \) is a surface in \( \mathbb{R}^{3}_{1} \) such that the first fundamental form (that is, the induced metric) is positive definite, it is of constant mean curvature (CMC, for short), and it is invariant under the action of the group of motions in \( \mathbb{R}^{3}_{1} \).
fixes each point of the line $\ell$. Such spacelike Delaunay surfaces are classified and they have conelike singularities (see [5], for details).

As in the case of CMC surfaces in $\mathbb{R}^3$, for a given (simply-connected) spacelike CMC surface in $\mathbb{R}^3$, there exists a spacelike CMC surface called the \emph{conjugate}. Any conjugate surface of a spacelike Delaunay surface is a spacelike helicoidal CMC surface$^4$ and it is shown in [6] that such spacelike helicoidal CMC surfaces have 5/2-cuspidal edges. We remark that spacelike zero-mean-curvature surfaces (i.e., maximal surface) never admit 5/2-cuspidal edges (cf. [6], see also [27]).

In this section, we compute the invariants $r_b$ and $r_c$ of 5/2-cuspidal edges on such spacelike helicoidal CMC surfaces, regarding them as surfaces in $\mathbb{R}^3$. More precisely, setting

$$\delta(u) = (u^2 + k + 1)^2 - 4k,$$

a non-totally-umbilical spacelike Delaunay surface with timelike axis is given by

$$f_{\text{Del}}(u, v) = \frac{1}{2H} \left( \int_0^u \sqrt{1 + \frac{4}{H^2}} \, d\tau, \, u \cos(2Hv), \, u \sin(2Hv) \right),$$

with a constant $k \in \mathbb{R}$ ($k \neq 1$), where $H$ is the mean curvature (see [6] for more details). Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a spacelike helicoidal CMC surface which is given as a conjugate surface of the Delaunay surface $f_{\text{Del}}$. Setting $\Delta(u) := \delta(u) - u^4$, such an $f$ can be written as follows: (cf. [6])

1. If $-1 < k < 1$ or $1 < k$, then $f$ is congruent to

$$f_T(u, v) = \left( \rho + \frac{1 - k}{2H(1 + k)} \phi, \rho \cos \phi, \rho \sin \phi \right),$$

where

$$\rho(u) := \sqrt{\delta(u)}, \quad \psi(u) := \int_0^u \sqrt{2(1 + k)} \, \tau^4 \, d\tau,$$

$$\phi(u, v) := \int_0^u \frac{2(1 + k)(1 - k)\tau^2}{\sqrt{\delta(u)}D(u)} \sqrt{\delta(u)}D(u) \, d\tau - \sqrt{\frac{1 + k}{2}} \nu.$$

2. If $k < -1$, then $f$ is congruent to

$$f_S(u, v) = \left( \rho \sinh \phi, \rho \cosh \phi, \psi + \frac{k - 1}{2H(1 + k)} \phi \right),$$

where

$$\rho(u) := -\sqrt{\delta(u)}, \quad \psi(v) := \int_0^u \sqrt{2(-k - 1)} \, \tau^4 \, d\tau,$$

$$\phi(u, v) := -\int_0^u \frac{2(-k - 1)(1 - k)\tau^2}{\sqrt{\delta(u)}D(u)} \sqrt{\delta(u)}D(u) \, d\tau - \sqrt{\frac{k - 1}{2}} \nu.$$

3. If $k = -1$, then $f$ is congruent to

$$f_L(u, v) = (\psi - \rho - \rho \phi^2, -2\rho \phi, \psi + \rho \phi^2) + H \left( \frac{\phi^3}{3} + \phi, \phi^2, \frac{\phi^3}{3} = \phi \right),$$

where $\rho(u) := u/2$,

$$\psi(u) := \int_0^u \sqrt{\tau^2 + 4 + \tau^2} \, d\tau, \quad \phi(u, v) := \int_0^u \frac{\sqrt{\tau^2 + 4 + \tau^2}}{2H\sqrt{\tau^2 + 4}} \, d\tau + v.$$
Figure 3. Spacelike helicoidal CMC surfaces ($H = 1/2$) having $5/2$-cuspidal edges in Lorentz-Minkowski 3-space $R^3_1$. These surfaces are conjugates of spacelike Delaunay surfaces of timelike axis. See [6] for more details.

Here, we consider the case of $f = f_T(u,v)$ given in (3.13). Similar computations can be applied in the cases of $f_S$ and $f_L$ given in (3.14) and (3.15), respectively. For simplification, we may assume that $H > 0$.

Since $(f_T)_u(0,v) = 0$, the singular set $S(f_T)$ is given by $S(f_T) = \{u = 0\}$ and $\eta = \partial_u$ gives a null vector field. Since the map $\nu : (R^2,0) \to S^2$ defined by

$$\nu = \frac{1}{\sqrt{2}\Delta \sqrt{\delta -(k+1)u^2}} \left( \sqrt{\delta \Delta}, -\sqrt{2(k+1)}u^3 \cos \phi - \sqrt{\delta}(k-1) \sin \phi, \right.$$

$$\left. -\sqrt{2(k+1)}u^3 \sin \phi + \sqrt{\delta}(k-1) \cos \phi \right)$$

is a unit normal vector field along $f_T$ (cf. Section 3.1), $f_T$ is a frontal. Then, we can check that $\eta\lambda(0,v) = -1/(2H^2\sqrt{k+1}) (\neq 0)$ holds, where $\lambda$ is the signed area density function (cf. (3.1)). Thus, we have that $f_T$ satisfies (1) in Proposition 3.1. Set $\xi(u,v) := \partial_v$ and

$$\tilde{\eta}(u,v) := \partial_u - \frac{2\text{sign}(k-1)}{(k-1)^2}u^2 \partial_v$$

(cf. (3.5), (3.6)). Then, we can check that $\langle \xi f_T(0,v), \tilde{\eta}^2 f_T(0,v) \rangle = 0$ and $\tilde{\eta}^3 f_T(0,v) = 0$. Hence, $f_T$ satisfies (2) in Proposition 3.1. Moreover, the constant $l$ is 0. Then, by a straightforward calculation, we have

$$\text{det} \left( \xi f_T, \tilde{\eta}^2 f_T, \tilde{\eta}^5 f \right)(0,v) = -\frac{24}{H^2|k-1|^3} (\neq 0).$$

Therefore, $f_T$ satisfies (3) in Proposition 3.1 and hence $f_T$ has $5/2$-cuspidal edges along $\gamma(v) = (0,v)$. The invariants are calculated as

$$r_c(0,v) = \frac{72H^{3/2}\sqrt{k+1}}{\sqrt{|k-1|}}, \quad r_\theta(0,v) = 0.$$

A helicoidal surface is a surface which is invariant under a non-trivial one-parameter subgroup of the isometry group of $R^3_1$. 

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1 A helicoidal surface is a surface which is invariant under a non-trivial one-parameter subgroup of the isometry group of $R^3_1$. 

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Similarly, in the case of \( k < -1 \), the invariants of \( f_S \) given in (3.14) are calculated as
\[
\begin{align*}
    r_c(0,v) &= \frac{72H^{3/2}\sqrt{-k-1}}{1-k \cosh \left( \frac{\sqrt{-k-1}v}{\sqrt{2}} \right)}, \\
    r_b(0,v) &= \frac{(1+k) \sinh \left( \frac{\sqrt{-k-1}v}{\sqrt{2}} \right)}{(1-k) \cosh \left( \frac{\sqrt{-k-1}v}{\sqrt{2}} \right)},
\end{align*}
\]
and in the case of \( k = -1 \), the invariants of \( f_L \) given in (3.15) are calculated as
\[
\begin{align*}
    r_c(0,v) &= \frac{72\sqrt{H}}{1+v^2}, \\
    r_b(0,v) &= \frac{6\sqrt{2}v}{H(1+v^2)^2}.
\end{align*}
\]

4. INTRINSITY AND EXTRINSITY OF INVARIANTS

Let \( f : (R^2, 0) \to (R^3, 0) \) be a map-germ. The induced metric or the first fundamental form of \( f \) is the metric on \((R^2, 0)\) defined by \( f^* \langle \cdot, \cdot \rangle \). A function \( I : (R^2, 0) \to R \), or \( I : S(f) \to R \) is an invariant if \( I \) does not depend on the choice of coordinate system on the source. An invariant \( I : (R^2, 0) \to R \), or \( I : S(f) \to R \) is intrinsic if it can be represented by a \( C^{\infty} \) function of \( E, F, G \) and their differential, where
\[
\begin{align*}
    E &= \langle f_u, f_u \rangle, \\
    F &= \langle f_u, f_v \rangle, \\
    G &= \langle f_v, f_v \rangle,
\end{align*}
\]
and \((u, v)\) is a coordinate defined in terms of the first fundamental form \( f^* \langle \cdot, \cdot \rangle \).

An invariant \( I : (R^2, 0) \to R \), or \( I : S(f) \to R \) is extrinsic if there exists a map \( \tilde{f} \) such that the first fundamental form of \( \tilde{f} \) is the same as \( f \), but \( I \) does not coincide. In [14][4], some of invariants of cuspidal edges are determined whether it is intrinsic or extrinsic (cf. [3] for invariants of cross caps). In this section, we show \( r_b \) is extrinsic.

4.1. Intrinsity criterion for 5/2-cuspidal edges. Let \( f : (R^2, 0) \to (R^3, 0) \) be a frontal-germ and 0 a non-degenerate singular point. If \( \kappa_v(0) \neq 0 \), then \( f \) is called non-\( \nu \)-flat at 0. Here, we shall show that the \( \mathcal{A} \)-equivalence class of 5/2-cuspidal edges can be determined intrinsically among non-\( \nu \)-flat frontal-germs.

Definition 4.1. Let \( f : (R^2, 0) \to (R^3, 0) \) be a frontal-germ such that 0 is a singular point of the first kind. A coordinate system \((u, v)\) around 0 is called adjusted at 0 if \( f_v(0,0) = 0 \). A coordinate system \((u, v)\) which is adjusted at 0 is called normally-adjusted at 0 if \((u, v)\) is compatible with the orientation of \((R^2, 0)\), \( E(0,0) = 1 \) and \( \lambda_v(0,0) = 1 \).

The existence of such a normally-adjusted coordinate system can be verified by the existence of normalized strongly adapted coordinate system [14] Definition 2.24, Proposition 2.25] (cf. [19] Lemma 3.2 and [14] Definition 3.7).

It was proved in [14] Corollary 3.14] that the Gaussian curvature \( K \) and the mean curvature \( H \) can be extended smoothly across 5/2-cuspidal edges. Then, we set
\[
H_q := H_v(0,0), \quad K_q := K_v(0,0),
\]
where \((u, v)\) is a coordinate system normally-adjusted at 0. We call \( K_q \) (respectively, \( H_q \)) the null-derivative Gaussian curvature (respectively, the null-derivative mean curvature) of 5/2-cuspidal edge at 0. We shall prove that the definition of null-derivative Gaussian and mean curvatures does not depend on a choice of normally-adjusted coordinate systems.

\textsuperscript{2}A coordinate system \((u, v)\) centered at \((0,0)\) is called normalized strongly adapted, if the singular set is given by the \( u \)-axis, \( \partial_v \) gives the null vector field along the \( u \)-axis, \( f_v(u,0) = 0 \), \( |f_u(u,0)| = |f_v(u,0)| = 1 \), \( (f_u(u,0), f_v(u,0)) = 0 \), and \( (f_u(u,v), f_v(u,v)) = 0 \) hold.
Lemma 4.2. If two coordinate systems \((u,v)\) and \((U,V)\) are normally-adjusted at 0, then
\[
U_u = 1, \quad U_v = 0, \quad V_v = 1
\]
holds at \((0,0)\). Moreover, the definition of null-derivative Gaussian and mean curvatures, \(K_\eta, H_\eta\), is independent of a choice of a coordinate system normally-adjusted at 0.

Proof. Since \(f_v = f_V = 0\) at \((0,0)\),
\[
f_v = U_v f_U + V_v f_V = U_v f_U
\]
yields \(U_v(0,0) = 0\). Since \((u,v) \mapsto (U,V)\) is orientation-preserving, \(J := U_u V_v - U_v V_u\) is positive-valued. In particular, \(J(0,0) = U_u(0,0)V_v(0,0) > 0\) holds. Setting \(\lambda := \det(f_u,f_v,\nu)\) and \(\Lambda := \det(f_U,f_V,\nu)\), we have \(\lambda = J \Lambda\). Then
\[
\lambda_v = J_v \Lambda + J \Lambda_v = J_v \Lambda + J (\Lambda U_u U_v + \Lambda V_v V_v)
\]
holds, and evaluating this at \((0,0)\), we have
\[
1 = U_u(0,0)V_v(0,0),
\]
which yields \(U_u(0,0) > 0\). Since \(J = U_u V_v > 0\) at \((0,0)\), \(V_v(0,0) > 0\) holds. Moreover, by \(1 = E = (f_u,f_u) = U_u^2 (f_U,f_U) = U_u^2(0,0) = 1\), we have \(U_u(0,0) = 1\). Substituting this into (4.3), \(V_v(0,0) = 1\) holds. Hence we have (4.2). Moreover, by \(\frac{\partial}{\partial v} = U_v \frac{\partial}{\partial U} + V_v \frac{\partial}{\partial V}\) holds at \((0,0)\). In particular, the definition of \(H_\eta, K_\eta\) as in (4.1) is independent of choice of a coordinate system normally-adjusted at 0. \(\Box\)

Since the Gaussian curvature \(K\) and the definition of normally-adjusted coordinate systems are intrinsic, the null-derivative Gaussian curvature \(K_\eta\) is an intrinsic invariant for 5/2-cuspidal edges. Now, we shall check the relationships among \(K\), \(H\), \(K_\eta\), \(H_\eta\) and other invariants.

Lemma 4.3. Let \(f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\) be a germ of 5/2-cuspidal edge. Then, the Gaussian curvature \(K\) and the mean curvature \(H\) of \(f\) satisfy
\[
K = \frac{1}{3} \kappa_v r_b - \kappa_t^2,
\]
\[
H = \frac{1}{2} \kappa_v + \frac{1}{6} r_b
\]
along the singular set, respectively. Moreover, the null-derivative Gaussian curvature \(K_\eta\) and the null-derivative mean curvature \(H_\eta\) of \(f\) satisfy
\[
K_\eta = \frac{1}{24} r_{\Pi},
\]
\[
H_\eta = \frac{1}{48} r_c
\]
along the singular set, respectively.

Proof. By Proposition 3.4, without loss of generality, we may assume that \(f\) is given by the form in (3.11). By a direct calculation, we have that
\[
\kappa_t(0) = 2b_{12}, \quad r_b(0) = b_{04}, \quad \kappa_v(0) = b_{20}, \quad r_c(0) = 3b_{05}, \quad r_{\Pi}(0) = 3b_{20}b_{05}
\]
and
\[
K = \frac{1}{3} b_{20}b_{04} - 4b_{12}^2, \quad H = \frac{1}{2} b_{20} + \frac{1}{6} b_{04}
\]
hold at \((0,0)\). Hence, (4.4) and (4.5) hold. On the other hand, since the coordinate system \((u,v)\) of \(f(u,v)\) given by the form in (3.11) is normally-adjusted at \((0,0)\), we have \(H_u = H_{\eta} \) and \(K_v = K_{\eta} \) at \((0,0)\). By a direct calculation, we have that

\[
H_u(0,0) = \frac{1}{16} b_{05}, \\
K_v(0,0) = \frac{1}{8} b_{20} b_{05},
\]

and hence, (4.6) and (4.7) hold.

\(\square\)

**Theorem 4.4.** For 5/2-cuspidal edges, the secondary product curvature \(r_\Pi \) is an intrinsic invariant.

**Proof.** By (4.4) in Lemma 4.3 and the fact that \(K_\eta \) is intrinsic, \(r_\Pi \) is intrinsic as well. \(\square\)

**Corollary 4.5.** Let \(f : (R^2, 0) \to (R^3, 0) \) be a frontal-germ such that 0 is a singular point of the first kind. If the singular point 0 is non-\(\nu\)-flat, \(f \) at 0 is a 5/2-cuspidal edge if and only if \(\kappa_\Pi = 0 \) along \(S(f)\) and \(r_\Pi(0) \neq 0 \).

**Proof.** By the definitions of \(\kappa_\nu \) given in (3.2), \(r_\nu \) given in (3.10) and the criterion (Proposition 3.1), \(f \) at 0 is a 5/2-cuspidal edge if and only if \(\kappa_\nu = 0 \) along \(S(f)\) and \(r_\nu(0) \neq 0 \). Therefore, imposing the non-\(\nu\)-flatness \(\kappa_\nu \neq 0 \), we have that 0 is a non-\(\nu\)-flat 5/2-cuspidal edge if and only if \(\kappa_\Pi = 0 \) along \(S(f)\) and \(r_\Pi(0) \neq 0 \). \(\square\)

Following Corollary 4.5 we give a definition of intrinsic 5/2-cuspidal edges for singular points of a certain metric, called Kossowski metric in Section 5 (cf. Definition 5.3).

### 4.2. Isometric deformations of 5/2-cuspidal edges

The following fact is a direct conclusion of \[8\] Theorem B:

**Fact 4.6** (\[8\]). Let \(f : (R^2, 0) \to (R^3, 0) \) be an analytic frontal-germ such that 0 is a singular point of the first kind, and \(\gamma : (R, 0) \to (R^2, 0) \) a singular curve. Assume that \(f \) has non-vanishing limiting normal curvature. Then, for given analytic functions germs \(\omega(t), \tau(t) \) at \(t = 0 \), there exists an analytic frontal-germ \(g = g_{\omega, \tau} \) such that

1. the first fundamental form of \(g_{\omega, \tau} \) coincides with that of \(f \),
2. the limiting normal curvature function of \(g_{\omega, \tau} \) along \(\gamma \) coincides with \(c_\omega(t) \) for a suitable choice of a unit normal vector field, and
3. \(\tau(t) \) gives the torsion function of \(\gamma_\nu(t) \), where \(\gamma_\nu(t) := g \circ \gamma(t) \).

The possibilities for congruence classes of such a \(g \) are at most two unless \(\tau \) vanishes identically. On the other hand, if \(\tau \) vanishes identically (i.e., \(\gamma_\nu \) is a planar curve), then the congruence class of \(g \) is uniquely determined.

Using Fact 4.6 we shall prove the following, which is an analogous result of \[15\] Theorem A and \[8\] Corollary D.

**Theorem 4.7** (Isometric deformation of 5/2-cuspidal edges). Let \(f : (R^2, 0) \to (R^3, 0) \) be a germ of analytic 5/2-cuspidal edge with non-vanishing limiting normal curvature, and \(\kappa_\nu(t) \) the singular curvature function along the singular curve \(\gamma(t) \). Take a germ of analytic regular space curve \(\sigma(t) \) such that its curvature function \(\kappa(t) \) satisfies

\[
\kappa > |\kappa_\nu|
\]

at \(0 \). Then, there exists a germ of analytic 5/2-cuspidal edge \(g_\sigma : (R^2, 0) \to (R^3, 0) \) with non-vanishing limiting normal curvature such that

1. the first fundamental form of \(g_\sigma \) coincides with that of \(f \),
2. the singular image \(g_\sigma \circ \gamma \) coincides with \(\sigma \).
The possibilities for congruence classes of such a \( g_\sigma \) are at most two unless \( \tau \) vanishes identically. On the other hand, if \( \tau \) vanishes identically (i.e., \( \sigma \) is a planar curve), then the congruence class of \( g_\sigma \) is uniquely determined.

Proof. Set \( \omega(t) \) as
\[
\omega(t) := \frac{1}{2} \log \left( \kappa(t)^2 - \kappa(t)^2 \right).
\]
Let \( \tau(t) \) be the torsion function of \( \sigma(t) \). By Fact 4.6 there exists an analytic frontal-germ \( g_\sigma := g_{\omega, \tau} : (R^2, 0) \to (R^3, 0) \) such that the items [1] [3] in Fact 4.6 hold. Thus, it suffices to show that \( g_\sigma \) has 5/2-cuspidal edge at 0. Since the first fundamental form of \( f \) coincides with that of \( g_\sigma \), the product curvature \( \kappa_1 \) and the secondary product curvature \( \tau_1 \) of \( f \) coincide with those of \( g_\sigma \), respectively. Therefore, by Corollary 4.5 we have that \( g_\sigma : (R^2, 0) \to (R^3, 0) \) has 5/2-cuspidal edge at 0.

In [8], the following is also proved.

**Fact 4.8 ([8 Corollary E]).** Let \( f_0, f_1 \) be two analytic frontal germs with non-degenerate singularities whose limiting normal curvature does not vanish. Suppose that they are mutually isometric. Then there exists an analytic 1-parameter family of frontal germs \( g_t \) \((0 \leq t \leq 1)\) satisfying the following properties:

1. \( g_0 = f_0 \) and \( g_1 = f_1 \),
2. \( g_t \) is isometric to \( g_0 \),
3. the limiting normal curvature of each \( g_t \) does not vanish.

Moreover, if both \( f_0 \) and \( f_1 \) are germs of cuspidal edges, swallowtails or cuspidal cross caps, then so are \( g_t \) for \( 0 \leq t \leq 1 \).

By this fact and Corollary 4.5 we also have the following result analogous to [8 Corollary E].

**Corollary 4.9.** Let \( f_0, f_1 \) be two analytic germs of 5/2-cuspidal edge whose limiting normal curvature does not vanish. Suppose that they are mutually isometric. Then there exists an analytic 1-parameter family of germs of 5/2-cuspidal edge \( g_t \) \((0 \leq t \leq 1)\) satisfying the following properties:

1. \( g_0 = f_0 \) and \( g_1 = f_1 \),
2. \( g_t \) is isometric to \( g_0 \),
3. the limiting normal curvature of each \( g_t \) does not vanish.

Proof. By Fact 4.8 there exists an analytic 1-parameter family of frontal germs \( g_t \) \((0 \leq t \leq 1)\) such that the items [1] [3] in Fact 4.8 hold. Since the limiting normal curvature of each \( g_t \) does not vanish and \( g_t \) is isometric to \( g_0 \) for each \( t \in [0, 1] \), Corollary 4.5 yields that \( g_t \) has 5/2-cuspidal edges. Hence, we have that the family \( \{g_t\}_{t \in [0, 1]} \) is the desired one.

### 4.3. Extrinicity of invariants

Let \( f : (R^2, 0) \to (R^3, 0) \) be a germ of non-\( \nu \)-flat 5/2-cuspidal edge and \( \gamma : (R, 0) \to (R^2, 0) \) a germ of a singular curve of \( f \). Let \( \tilde{\gamma} \) be the regular curve in \( R^3 \) given by \( \tilde{\gamma} := f \circ \gamma \), with arc-length parameter \( t \). Set \( \nu(t) := \nu(t) \) and \( \nu(t) := \nu(t) \). Then \( \{e, b, \nu\} \) is an orthonormal frame along \( \gamma \). Moreover we have
\[
\tilde{\gamma}'' = \kappa \nu + \kappa \nu', \quad b' = -\kappa e + \kappa \nu, \quad \nu' = -\kappa e - \kappa b.
\]
Let \( \kappa, \tau \) be the curvature and torsion functions of \( \tilde{\gamma} \), respectively. Substituting 4.8 into \( \kappa^2 \tau = \det(\tau', \tau'', \tau'''') \), we have the following.

**Lemma 4.10.** It holds that
\[
\kappa = \sqrt{\kappa^2 + \kappa''^2}, \quad \tau = \frac{\kappa' \kappa'' - \kappa \kappa'''}{\kappa^2} - \kappa'.
\]
In particular, if \( \kappa_s(t) = 0 \) along \( \gamma(t) \), then \( \kappa_t(t) = -\tau(t) \) holds.

As a corollary of Theorem 4.11 we prove the extrinsity of the limiting normal curvature \( \kappa_\nu \) (Corollary 4.11), the cuspidal torsion \( \kappa_t \) (Corollary 4.12), and the bias \( r_b \) (Corollary 4.13). We remark that the proof of Corollary 4.11 is analogous to that of [15] Corollary D).

**Corollary 4.11.** For 5/2-cuspidal edges, the limiting normal curvature \( \kappa_\nu \) is an extrinsic invariant.

**Proof.** Let us take a real-analytic germ of non-\( \nu \)-flat 5/2-cuspidal edge \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \). Denote by \( \kappa(t) \) and \( \tau(t) \) the curvature and torsion of \( \gamma(t) := f(\gamma(t)) \), respectively. By Fact 4.6 for a given analytic function \( \omega(t) \), there exists a non-\( \nu \)-flat real-analytic frontal-germ \( g_{\omega,\tau} \) such that \( g_{\omega,\tau} \) is isometric to \( f \), the limiting normal curvature function of \( g_{\omega,\tau} \) is \( c \omega(t) \), and \( \tau(t) \) gives the torsion function of \( g_{\omega,\tau}(\gamma(t)) \). Moreover, by Corollary 4.5, \( g_{\omega,\tau} \) is 5/2-cuspidal edge. Since we can choose \( \omega(t) \) arbitrary, the limiting normal curvature is extrinsic.

**Corollary 4.12.** For 5/2-cuspidal edges, the cuspidal torsion \( \kappa_t \) is an extrinsic invariant.

**Proof.** Let us take a real-analytic germ of non-\( \nu \)-flat 5/2-cuspidal edge \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) along the singular curve \( \gamma(t) \). Denote by \( \kappa(t) \) and \( \tau(t) \) the curvature and torsion of \( \gamma(t) := f(\gamma(t)) \), respectively. By Lemma 4.11 \( \tau(t) = -\kappa_t(t) \). Take an arbitrary analytic function \( \tilde{\tau}(t) \). Then, by the fundamental theorem of space curves, there exists an analytic regular space curve \( \sigma(t) \) in \( \mathbb{R}^3 \) whose curvature and torsion functions are given by \( \kappa(t) \) and \( \tilde{\tau}(t) \), respectively. Applying Theorem 4.1 to \( \sigma(t) \), there exists a real-analytic germ of non-\( \nu \)-flat 5/2-cuspidal edge \( g_\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) such that \( g_\sigma \) is isometric to \( f \) and \( \sigma \) gives the image of the singular set of \( g_\sigma \). Since \( \kappa_s \equiv 0 \) along the singular curve \( \gamma(t) \), Lemma 4.10 yields that the cuspidal torsion of \( g_\sigma \) is \( -\tilde{\tau}(t) \). Since we can choose \( \tilde{\tau}(t) \) arbitrary, the cuspidal torsion is extrinsic.

We remark that an analytic non-\( \nu \)-flat 5/2-cuspidal edge \( f \) satisfying \( \kappa_s \equiv 0 \) along \( S(f) \) exists. In fact, by rotating the plane curve \( (x(t), z(t)) := (1 + t^3, t^2) \) with respect to \( z \)-axis, we have such an example.

**Corollary 4.13.** For 5/2-cuspidal edges, the bias \( r_b \) is an extrinsic invariant.

**Proof.** Let us take a non-\( \nu \)-flat real-analytic 5/2-cuspidal edge satisfying \( \kappa_s \equiv 0 \) along the singular curve \( \gamma(t) \). Moreover, assuming \( \tau \equiv 0 \), then by Lemma 4.10 it holds that \( \kappa_t \equiv 0 \). Then, by Lemma 4.13

\[
K(\gamma(t)) = \frac{1}{3}r_b(t)\kappa_\nu(t).
\]

Let \( k \geq 0 \) be a non-negative real number. Since \( \kappa_\nu \neq 0 \), then by Theorem 4.12 there exists a family \( \{g^k\}_{k \geq 0} \) of real-analytic germs of 5/2-cuspidal edge such that, for each \( k \geq 0 \), \( g^k \) is non-\( \nu \)-flat, \( g^k \) has the same first fundamental form of \( f \), and the curvature function \( \kappa^k(t) \) of \( g^k(\gamma(t)) \) is given by \( \kappa^k(t) = \kappa(t) + k \), and the torsion is 0. Since \( f \) and \( g^k \) has the same first fundamental form for each \( k \geq 0 \), the singular curvature \( \kappa^k_s(t) \) of \( g^k \) vanishes identically along \( \gamma(t) \). Thus the limiting normal curvature of \( g \) is \( \kappa^k_\nu(t) = \kappa(t) + s(> 0) \). Thus the bias \( r^k_b \) for \( g^k \) is given by

\[
r^k_b(t) = 3\frac{K(\gamma(t))}{\kappa^k_\nu} = 3\frac{K(\gamma(t))}{\kappa(t) + s}.
\]

In particular, the bias is extrinsic. □
Remark 4.14. The secondary cuspidal curvature $r_c$ is also extrinsic, since $r_\Pi$ is intrinsic (Theorem 4.4), $\kappa_\nu$ is extrinsic (Corollary 4.11), and $r_c$ is written as $r_c = r_\Pi / \kappa_\nu$ when $\kappa_\nu \neq 0$. Moreover, the product $\kappa_\nu r_b$ is also extrinsic, since $\kappa_\nu r_b = 3(K + \kappa_t^2)$ holds by (4.4) and $\kappa_t$ is extrinsic (Corollary 4.12). On the other hand, by a proof similar to that of Corollary 4.12, we can prove that the cuspidal torsion $\kappa_t$ for cuspidal edges is also extrinsic.

4.4. Summary of intrinsity and extrinsity. We can summarize the intrinsity and extrinsity as follows. As seen in Section 2.1, the corresponding invariant of the bias of cusp $r_b$ does not exist for cuspidal edges.

| invariants | $\kappa_s$ | $\kappa_\nu$ | $\kappa_t$ | $\kappa_c$ | $\kappa_\Pi = \kappa_c \kappa_\nu$ |
|------------|------------|------------|------------|------------|----------------------------------|
| int/ext    | intrinsic  | extrinsic  | extrinsic  | extrinsic  | intrinsic                        |

Table 1. Intrinsity and extrinsity for cuspidal edges.

| invariants | $\kappa_s$ | $\kappa_\nu$ | $\kappa_t$ | $r_b$ | $r_c$ | $\kappa_\nu r_b$ | $\kappa_c r_b - 3\kappa_t^2$ | $r_\Pi = \kappa_\nu r_c$ |
|------------|------------|------------|------------|-------|------|------------------|------------------------|------------------|
| int/ext    | int        | ext        | ext        | ext   | ext  | int              | int                    | int               |

Table 2. Intrinsity (int) and extrinsity (ext) for 5/2-cuspidal edges. Here, we remark that the intrinsity of the invariant in the seventh slot can be verified by the identity $\kappa_\nu r_b - 3\kappa_t^2 = 3K$ (cf. (4.4)). With respect to the eighth slot, see Theorem 4.4.

5. ISOMETRIC REALIZATIONS OF INTRINSIC 5/2-CUSPIDAL EDGES

In this section, we deal with 5/2-cuspidal edge singularities without ambient spaces. We give a definition of intrinsic 5/2-cuspidal edges for singular points of Kossowski metrics, and prove the existence of their isometric realizations (Theorem 5.6) as in [15] and [8].

First, we briefly introduce the basic properties of Kossowski metrics. Further systematic treatments of Kossowski metrics are given in [4, 22, 8]. Let $ds^2$ be a germ of positive semi-definite metric on $(\mathbb{R}^2, 0)$. Assume that 0 is a singular point of $ds^2$, that is, $ds^2$ is not positive-definite at 0. Denote by $S(ds^2)$ the set of singular points. A non-zero tangent vector $v$ at 0 is called a null vector at 0 if $ds^2(v, x) = 0$ holds for every tangent vector $x$ at 0. A local coordinate neighborhood $(U, u, v)$ is called adjusted at 0 if $\partial v = \partial / \partial v$ gives a null vector at $(0, 0)$.

If $(U, u, v)$ is a local coordinate neighborhood adjusted at 0, then $F = G = 0$ holds at $(0, 0)$, where

$$ds^2 = E \, du^2 + 2F \, du \, dv + G \, dv^2.$$  

A singular point 0 is called admissible if there exists an local coordinate neighborhood $(U; u, v)$ adjusted at 0 such that $E_v = 2F_u, G_u = G_v = 0$ hold at $(0, 0)$.

Definition 5.1 (Kossowski metric). If each singular point is admissible, and there exists a smooth function $\lambda$ defined on a neighborhood $(U; u, v)$ of 0 such that

$$EG - F^2 = \lambda^2$$

on $U$, and $d\lambda \neq 0$ hold at $(0, 0)$, then $ds^2$ is called a (germ of) Kossowski metric, where $E, F, G$ are smooth functions on $U$ satisfying (5.1). Moreover, if we can choose $E, F, G,$ and $\lambda$ to be analytic functions, then the Kossowski metric is called analytic.
As shown in [4], the first fundamental form of a frontal-germ \( f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) whose singular points are all non-degenerate is a Kossowski metric.

Let \( ds^2 \) be a germ of Kossowski metric having a singular point at \( 0 \). By the condition \( d\lambda \neq 0 \) at \((0, 0)\), the implicit function theorem yields that there exists a regular curve \( \gamma(t) (|t| < \epsilon) \) in the uv-plane (called the singular curve) parameterizing \( S(ds^2) \). Then there exists a smooth non-zero vector field \( \eta \) such that \( \eta_0 \) gives a null vector for each \( q \in S(ds^2) \) near \((0, 0)\). We call \( \eta \) a null vector field.

**Definition 5.2.** If \( \eta \) is transversal to \( S(ds^2) \) at \( 0 \), the singular point \( 0 \) is called Type I (or an \( A_2 \) point).

For a Kossowski metric \( ds^2 \) induced from a frontal-germ \( f \), Type I singular points of \( ds^2 \) correspond to singular points of the first kind of \( f \).

According to [4], Proposition 2.25, for a Type I singular point, there exists a coordinate system \((U; u, v)\) centered at \( 0 \), such that

- the singular set \( S(ds^2) \) is given by the \( u \)-axis,
- \( \partial_u \) gives the null vector field,
- \( F = 0 \) on \( U \), and
- \( E(u, 0) = 1, E_v(u, 0) = G_v(u, 0) = 0, G_{vv}(u, 0) = 2 \)

hold, where \( E, F, G \) are smooth functions as in [5.1]. Such a coordinate system is called a normalized strongly adapted coordinate system. Since \( G_{vv}(u, 0) = 2 \) is equivalent to \( \lambda_v(u, 0) = \pm 1 \), by changing \( v \mapsto -v \) if necessary, we may assume that \( \lambda_v(u, 0) = 1 \). (Hence, in the case of Kossowski metrics induced from frontals in \( \mathbb{R}^3 \), the normalized strongly adapted coordinate systems are normally-adjusted, cf. Definition (5.1))

We shall review the definition of the product curvature for Type I singular points defined in [4]. Let \((U; u, v)\) be a normalized strongly adapted coordinate system centered at a Type I singular point \( 0 \). Denote by \( K \) the Gaussian curvature of \( ds^2 \) on \( U \setminus \{v = 0\} \). By [4] Proposition 2.27, \( vK(u, v) \) is a smooth function on \( U \). Then
\[
\tilde{\kappa}_\Pi := \lim_{v \to 0} vK(u, v)
\]
does not depend on a choice of a normalized strongly adapted coordinate system satisfying \( \lambda_v(0, 0) = 1 \), and is called the product curvature.

Now, assume that \( \tilde{\kappa}_\Pi \) vanishes along the \( u \)-axis. Then, \( K \) is a bounded smooth function on \( U \), and
\[
\tilde{K}_\eta := \lim_{u \to 0} K_v(u, v)
\]
does not depend on a choice of a normalized strongly adapted coordinate system satisfying \( \lambda_v(0, 0) = 1 \). We call \( \tilde{K}_\eta \) the secondary product curvature or the null-derivative Gaussian curvature.

**Definition 5.3.** Let \( ds^2 \) be a germ of Kossowski metric at a Type I singular point \( 0 \). If the product curvature \( \tilde{\kappa}_\Pi \) vanishes along \( S(ds^2) \), and the secondary product curvature \( \tilde{K}_\eta \) does not vanish at \( 0 \), then the singular point \( 0 \) is called an intrinsic 5/2-cuspidal edge.

The following lemma is a direct conclusion of Lemma 4.3 and Corollary 4.5

**Lemma 5.4.** Let \( f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be a non-\( \nu \)-flat frontal-germ having a singular point \( 0 \) of the first kind. Denote by \( ds^2 \) the first fundamental form of \( f \). Then, \( f \) at \( 0 \) is a 5/2-cuspidal edge if and only if \( 0 \) is an intrinsic 5/2-cuspidal edge (as a singular point of the Kossowski metric \( ds^2 \)).

We remark that the assumption of the non-\( \nu \)-flatness cannot be removed, since there exists a cuspidal edge with vanishing limiting normal curvature such that the corresponding singular points of \( ds^2 \) are intrinsic 5/2-cuspidal edges.
Kossowski [12] proved a realization theorem of Kossowski metrics, and a generalization and a refinement of Kossowski’s realization theorem is obtained in [8, Theorem B] (cf. [15]).

**Fact 5.5** (cf. [8, Theorem B]). Let $ds^2$ be a germ of analytic Kossowski metric on $(R^2, 0)$, and let $\gamma(t)$ ($|t| < \varepsilon$) be a singular curve passing through a singular point $0 = \gamma(0)$. Assume that $0$ is a Type 1 singular point of $ds^2$. Then, for given analytic function-germs $\omega(t)$, $\nu(t)$ at $t = 0$, there exists an analytic frontal-germ $f = f_{\omega, \nu} : (R^2, 0) \to (R^3, 0)$ satisfying the following properties:

1. $ds^2$ is the first fundamental form of $f$,
2. the limiting normal curvature function germ along the singular curve $\gamma$ coincides with $e^{\nu(t)}$ for a suitable choice of a unit normal vector field $\nu$,
3. $\tau(t)$ gives the torsion function germ of $\tilde{\gamma}(t) := f \circ \gamma(t)$.

The possibilities for the congruence classes of such an $f$ are at most two. Moreover, if $\tau$ vanishes identically (i.e. $\tilde{\gamma}$ is a planar curve), then the congruence class of $f$ is uniquely determined.

We remark that [8, Theorem B] is given under a more general assumption than that for Fact 5.5. Using this fact and an argument similar to that of Theorem 4.7, we have the following realization theorem of Kossowski metrics with intrinsic 5/2-cuspidal edges with prescribed singular images, which is an analogous result of [15, Theorem 12] and [8, Corollary D].

**Theorem 5.6.** Let $ds^2$ be a germ of analytic Kossowski metric on $(R^2, 0)$. Assume that $0$ is an intrinsic 5/2-cuspidal edge. Take a germ of analytic regular space curve $\sigma(t)$ such that its curvature function $\kappa(t)$ satisfies

$$\kappa > |\kappa_s|$$

at $0$, where $\kappa_s$ is the singular curvature of $ds^2$ along the singular curve $\gamma$. Then, there exists a germ of analytic 5/2-cuspidal edge $f_\sigma : (R^2, 0) \to (R^3, 0)$ with non-vanishing limiting normal curvature such that

1. the first fundamental form of $f_\sigma$ coincides with $ds^2$,
2. the singular image $f_\sigma \circ \gamma$ coincides with $\sigma$.

The possibilities for congruence classes of such an $f_\sigma$ are at most two unless $\tau$ vanishes identically. On the other hand, if $\tau$ vanishes identically (i.e., $\sigma$ is a planar curve), then the congruence class of $f_\sigma$ is uniquely determined.

**Appendix A. Proofs**

**A.1. Proof of Proposition 3.4.** We show the following proposition which is a normal form of a singular point of the first kind.

**Proposition A.1.** Let $f : (R^2, 0) \to (R^3, 0)$ be a frontal and $0$ a singular point of the first kind. Then there exist a coordinate system $(u, v)$ and an isometry of $R^3$ such that

$$A \circ f(u, v) = (u, a_2(u) + v^2/2, a_3(u) + v^2b_3(u, v))$$

for some functions $a_2, a_3, b_3$. If $0$ is a 5/2-cuspidal edge, $b_3$ has the form $b_3 = c_3(u) + v^2c_4(u) + v^3c_5(u, v)$ for some functions $c_3, c_4, c_5$.

**Proof.** Let $\nu$ be a unit normal vector field along $f$. Since $rank df_0 = 1$, by an isometry $A$ on $R^3$, we may assume $df_0(X) = (*, 0, 0)$ for any $X \in T_0R^2$ and $\nu(0, 0) = (0, 0, 1)$, where $*$ stands for a real number. Since $0$ is a singular point of the first kind, $S(f)$ is a regular curve in $(R^2, 0)$, and $\eta$ is transversal to $S(f)$. Thus there exists a coordinate system $(\bar{u}, \bar{v})$ satisfying $S(f) = \{\bar{v} = 0\}$ and $\eta = \bar{v}$. Since
\( f_0(0,0) = (a,0,0) \) \((a \neq 0)\), setting \( u = f_1(\bar{u}, \bar{v}), v = \bar{v} \), the coordinate system \((u,v)\) satisfies
\[
(A.1) \quad f(u,v) = (u, f_2(u,v), f_3(u,v)),
\]
where \( f_1(u,v) \) is the first component of \( f \). Since \( f_0(u,0) = 0 \), there exist functions \( a_2, a_3, b_2, b_3 \) such that \( f_i(u,v) = a_i(u) + v^2 b_i(u,v)/2 \) \((i = 2,3)\). Since 0 is non-degenerate, \( \lambda_c(0,0) \neq 0 \). Thus \( \det(f_u,f_{vv},\nu)(0,0) = b_2(0,0) \neq 0 \). Setting \( \bar{u} = u, \bar{v} = v\sqrt{|b_2(u,v)|} \), \( \text{(A.1)} \) is
\[
\begin{align*}
(\bar{u}, \bar{v}) &= (u, a_2(u) \pm \bar{v}^2/2, a_3(u) + \bar{v}^2 b_3(u, \bar{v})).
\end{align*}
\]
This shows the first assertion.

If 0 is a 5/2-cuspidal edge, then by Lemma, \( \det(f_u,f_{vv},f_{uvv})(u,0) = 0 \) holds. Thus we have the second assertion. \( \square \)

Proposition 3.4 is now obvious by Proposition \( \text{(A.1)} \).

A.2. Proof of Proposition 3.1

Proof of Proposition 3.1. To show Proposition 3.1, the independence of the condition on the choice of the vector field is essential. However, now it is obvious by Lemma 3.8. By Proposition \( \text{(A.1)} \) we may assume that \( f \) has the form \( f(u,v) = (u,v^2, v^6 c_5(u,v)) \). There exist functions \( c_6, c_7 \) such that \( c_5(u,v) = c_6(u,v^2) + vc_7(u,v^2) \). Thus we may assume that \( f \) has the form \( f(u,v) = (u,v^2, v^6 c_6(u,v^2)) \).

Vector fields \( \xi = \partial_u, \eta = \partial_v \) satisfy the condition of Proposition 3.1 with \( (3.4) \), and we see that \( l = 0 \). By condition \( (3.4) \) of Proposition 3.1, we see \( c_6(0,0) \neq 0 \). We set \( \Phi(X,Y,Z) = (X,Y,Z/c_6(X,Y)) \). Then \( \Phi \circ f = (u,v^2,v^5) \), which shows the assertion. \( \square \)

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