FRATTINI-INJECTIVITY AND MAXIMAL PRO-$p$ GALOIS GROUPS

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ABSTRACT. We call a pro-$p$ group $G$ Frattini-injective if distinct finitely generated subgroups of $G$ have distinct Frattinis. This paper is an initial effort toward a systematic study of Frattini-injective pro-$p$ groups (and several other related concepts). Most notably, we classify the $p$-adic analytic and the solvable Frattini-injective pro-$p$ groups, and we describe the lattice of normal abelian subgroups of a Frattini-injective pro-$p$ group.

We prove that every maximal pro-$p$ Galois group of a field that contains a primitive $p$th root of unity (and also contains $\sqrt{-1}$ if $p = 2$) is Frattini-injective. In addition, we show that many substantial results on maximal pro-$p$ Galois groups are in fact consequences of Frattini-injectivity. For instance, a $p$-adic analytic or solvable pro-$p$ group is Frattini-injective if and only if it can be realized as a maximal pro-$p$ Galois group of a field that contains a primitive $p$th root of unity (and also contains $\sqrt{-1}$ if $p = 2$); and every Frattini-injective pro-$p$ group contains a unique maximal abelian normal subgroup.

1. Introduction

Throughout, $p$ stands for a prime number. Given a pro-$p$ group $G$, we denote by $\Phi(G)$ the Frattini subgroup of $G$.

Definition 1.1. We say that a pro-$p$ group $G$ is Frattini-injective if the function $H \mapsto \Phi(H)$, from the set of finitely generated subgroups of $G$ into itself, is injective.

This paper is an initial effort toward a systematic study of Frattini-injective pro-$p$ groups (and several other related concepts). Straightforward examples of Frattini-injective pro-$p$ groups are provided by the free abelian pro-$p$ groups. In fact, they are the only Frattini-injective abelian pro-$p$ groups, since all Frattini-injective pro-$p$ groups are torsion-free or, better yet, they all have the unique extraction of roots property (see Corollary 2.3).

Every subgroup of a Frattini-injective pro-$p$ group is also Frattini-injective. Furthermore, if $U$ is an open subgroup of a Frattini-injective pro-$p$ group $G$, then $d(U) \geq d(G)$ (see Proposition 2.6). This is already an indication that Frattini-injectivity is a quite restrictive condition. Nonetheless, as we shall soon see, many significant pro-$p$ groups are indeed Frattini-injective.

Our first substantial result is

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**Theorem 1.2.** Let $G$ be a $p$-adic analytic pro-$p$ group of dimension $d \geq 1$. Then, $G$ is Frattini-injective if and only if it is isomorphic to one of the following groups:

1. the abelian group $\mathbb{Z}_p^d$;
2. the metabelian group $\langle x \rangle \rtimes \mathbb{Z}_p^{d-1}$, where $\langle x \rangle \cong \mathbb{Z}_p$ and $x$ acts on $\mathbb{Z}_p^{d-1}$ as scalar multiplication by $\lambda$, with $\lambda = 1 + p^s$ for some $s \geq 1$ if $p > 2$, and $\lambda = 1 + 2^s$ for some $s \geq 2$ if $p = 2$.

It turns out that the Frattini-injective solvable pro-$p$ groups are also quite scarce.

**Theorem 1.3.** Let $G$ be a solvable pro-$p$ group. Then, $G$ is Frattini-injective if and only if it is free abelian or isomorphic to a semidirect product $\langle x \rangle \rtimes A$, where $\langle x \rangle \cong \mathbb{Z}_p$, $A$ is a free abelian pro-$p$ group and $x$ acts on $A$ as scalar multiplication by $1 + p^s$, with $s \geq 1$ if $p$ is odd, and $s \geq 2$ if $p = 2$.

Next, we give a complete description of the lattice of abelian normal subgroups of a Frattini-injective pro-$p$ group.

**Theorem 1.4.** Let $G$ be a Frattini-injective pro-$p$ group. Then, $G$ has a unique maximal normal abelian subgroup $N$. Moreover, the following assertions hold:

(i) $N$ is isolated in $G$.
(ii) Every subgroup of $N$ is normal in $G$.
(iii) If $Z(G) \neq 1$, then $N = Z(G)$.
(iv) If $Z(G) = 1$ but $N \neq 1$, then $G \cong \mathbb{Z}_p \rtimes C_G(N)$ and $Z(C_G(N)) = N$.

There are two obvious ways of sharpening the Frattini-injectivity condition. Instead of confining to finitely generated subgroups, one can take all subgroups into consideration: We call a pro-$p$ group $G$ **strongly Frattini-injective** if the function $H \mapsto \Phi(H)$, from the set of all subgroups of $G$ into itself, is injective. Alternatively, we may require the map $H \mapsto \Phi(H)$ to be an embedding of posets: A pro-$p$ group $G$ is defined to be **strongly Frattini-resistant** (Frattini-resistant) if for all (finitely generated) subgroups $H$ and $K$ of $G$,

$$H \leq K \iff \Phi(H) \leq \Phi(K).$$

(To understand our reason for the choice of terms, see Section 4, where Frattini-resistance and another related concept, commutator-resistance, are introduced in a unified manner.)

Every (strongly) Frattini-resistant pro-$p$ group is (strongly) Frattini-injective. All solvable and all $p$-adic analytic Frattini-injective pro-$p$ groups are strongly Frattini-resistant. Additional examples of strongly Frattini-resistant groups are provided by the free pro-$p$ groups (Theorem 6.1).

If $G$ is a Demushkin group, then $G^{ab} \cong \mathbb{Z}_p^d$ or $G^{ab} \cong \mathbb{Z}/p^e\mathbb{Z} \times \mathbb{Z}_p^{d-1}$ for some $e \geq 1$; set $q(G) := p^e$ in the latter and $q(G) := 0$ in the former case.

**Theorem 1.5.** Let $G$ be a Demushkin pro-$p$ group. Then, the following assertions hold:
(i) If \( q(G) \neq p \), or \( q(G) = p \) and \( p \) is odd, then \( G \) is strongly Frattini-resistant.

(ii) If \( q(G) = 2 \) and \( d(G) > 2 \), then \( G \) is Frattini-injective, but not Frattini-resistant.

(iii) If \( q(G) = 2 \) and \( d(G) = 2 \), then \( G \) is not Frattini-injective.

The absolute Galois group of a field \( k \) is the profinite group \( G_k =: \text{Gal}(k_s/k) \), where \( k_s \) is a separable closure of \( k \). The maximal pro-\( p \) Galois group of \( k \), denoted by \( G_k(p) \), is the maximal pro-\( p \) quotient of \( G_k \). Equivalently, \( G_k(p) = \text{Gal}(k(p)/k) \), where \( k(p) \) is the compositum of all finite Galois \( p \)-extensions of \( k \) inside \( k_s \). De-
lineating absolute (maximal pro-\( p \)) Galois groups of fields within the category of profinite (pro-\( p \)) groups is one of the central problems of Galois theory.

**Theorem 1.6.** Let \( k \) be a field containing a primitive \( p \)th root of unity. If \( p = 2 \), in addition, assume that \( \sqrt{-1} \in k \). Then \( G_k(p) \) is strongly Frattini-resistant.

**Theorem 1.7.** For any field \( k \) and odd prime \( p \), every pro-\( p \) subgroup of the absolute Galois group \( G_k \) is strongly Frattini-resistant. Moreover, if \( \sqrt{-1} \in k \), then also every pro-2 subgroup of \( G_k \) is strongly Frattini-resistant.

In what follows, \( k \) is a field containing a primitive \( p \)th root of unity, and also \( \sqrt{-1} \in k \) if \( p = 2 \). In the last few decades, substantial progress has been made in the direction of finding necessary conditions for a pro-\( p \) group to be realiz-
able as the maximal pro-\( p \) Galois group of some field \( k \) (cf. for example, \([2, 3, 9, 10, 11, 12, 20, 21, 24, 25, 30, 38]\) and references therein). Most notably, it follows from the positive solution of the Bloch-Kato conjecture by Rost and Voevodsky (with a ‘patch’ of Weibel; cf. \([33, 38, 41]\)) that every maximal pro-\( p \) Galois group \( G_k(p) \) is quadratic, i.e., the cohomology algebra \( H^\bullet(G_k(p), \mathbb{F}_p) = \bigoplus_{n \geq 0} H^n(G_k(p), \mathbb{F}_p) \) is generated by elements of degree 1 and defined by homogeneous relations of degree 2 (see \([29]\)). Another restriction discovered recently concerns the external cohomological structure of \( G_k(p) \), more precisely, for every \( \varphi_1, \varphi_2, \varphi_3 \in H^1(G_k(p), \mathbb{F}_p) \) the triple Massey product \( \langle \varphi_1, \varphi_2, \varphi_3 \rangle \) is not essential (cf. \([12, 24, 25]\)).

In contrast to the above-mentioned properties of maximal pro-\( p \) Galois groups, Frattini-injectivity (and also Frattini-resistance) is a fairly elementary and quite palpable group theoretic condition; yet, it seems to be highly restrictive. For instance, within the classes of \( p \)-adic analytic and solvable pro-\( p \) groups, Frattini-injectivity completely characterizes maximal pro-\( p \) Galois groups.

**Corollary 1.8.** Let \( G \) be a solvable or \( p \)-adic analytic pro-\( p \) group. Then, \( G \) is Frattini-injective if and only if it is isomorphic to \( G_k(p) \) for some field \( k \) that contains a primitive \( p \)th root of unity, and also contains \( \sqrt{-1} \) if \( p = 2 \).

In particular, we recover a result due to Ware \([40]\) (when \( p \) is odd and \( k \) contains a primitive \( p^2 \)th root of unity; for \( p = 2 \) see \([17, 39]\)) and Quadrelli \([28]\) (for all \( k \); see also \([32]\)).
Corollary 1.9. Let \( G \) be a solvable or \( p \)-adic analytic pro-\( p \) group. Then \( G \) can be realized as a maximal pro-\( p \) Galois group of some field \( k \) that contains a primitive \( p \)th root of unity (and also \( \sqrt{-1} \in k \) if \( p = 2 \)) if and only if it is free abelian or isomorphic to a semidirect product \( \langle x \rangle \rtimes A \), where \( \langle x \rangle \cong \mathbb{Z}_p \), \( A \) is a free abelian pro-\( p \) group and \( x \) acts on \( A \) as scalar multiplication by \( 1 + p^s \), with \( s \geq 1 \) if \( p > 2 \), and \( s \geq 2 \) if \( p = 2 \).

As a corollary of Theorem 1.4, we obtain yet another well-known result on maximal pro-\( p \) Galois groups due to Engler and Nogueira [15] (for \( p = 2 \)) and Engler and Koenigsmann [14] (for \( p > 2 \)).

Corollary 1.10. Let \( k \) be a field containing a primitive \( p \)th root of unity (and \( \sqrt{-1} \in k \) if \( p = 2 \)). Then \( G_k(p) \) contains a unique maximal normal abelian subgroup.

More recently, 1-smooth cyclotomic pro-\( p \) pairs, a formal version of Hilbert 90 for pro-\( p \) groups (for a precise definition, see Section 7), have been investigated in an attempt to abstract essential features of maximal pro-\( p \) Galois groups (see [4], [13], [30], [31] and [32]).

Theorem 1.11. Let \( \mathcal{G} = (G, \theta) \) be a torsion-free 1-smooth cyclotomic pro-\( p \) pair. If \( p = 2 \), in addition, assume that \( \text{Im}(\theta) \leq 1 + 4\mathbb{Z}_2 \). Then \( G \) is strongly Frattini-resistant.

Consequently, many of the known properties of 1-smooth cyclotomic pro-\( p \) pairs can be obtained as consequences of Frattini-injectivity (see Section 7).

Outline of the paper: In Section 2, several elementary results on Frattini-injective pro-\( p \) groups are established. Theorem 1.2 is proved in Section 3. In Section 4, the concepts of Frattini-resistance and commutator-resistance are developed within the unifying framework of hierarchical triples. The proofs of Theorem 1.3 and Theorem 1.4 are given in Section 5. Section 6 is devoted to Free pro-\( p \) and Demushkin groups. In Section 7 Frattini-injectivity is investigated in the context of Galois theory. We close the paper with a brief section on another related concept, \( p \)-power resistance, and a section in which we formulate several problems that we hope will stimulate further research on Frattini-injective pro-\( p \) groups.

Notation: We take all group theoretic terms in the appropriate sense for topological groups; for instance, subgroups are assumed to be closed, homomorphisms are continuous, and generators are always understood to be topological generators. Let \( G \) be a pro-\( p \) group, \( H \) a subgroup of \( G \) and \( x, y \in G \). We use the following fairly standard notation: \( d(G) \) is the cardinality of a minimal generating set for \( G \); \( x^y = y^{-1}xy \) and \([x, y] = x^{-1}x^y \); the \( n \)th terms of the derived series and the lower central series of \( G \) are denoted by \( G^{(n)} \) and \( \gamma_n(G) \), respectively, with the exception of the commutator subgroup, which is always denoted by \([G, G] \); we write \( G^{ab} \) for
the abelianization of $G$; the center of $G$ is denoted by $Z(G)$; $N_G(H)$ and $C_G(H)$ are the normalizer and the centralizer of $H$ in $G$, respectively; $G^p$ is the subgroup of $G$ generated by $p$th powers of elements of $G$; the terms of the lower $p$-series are denoted by $P_i(G)$, so $P_1(G) = G$ and $P_{i+1}(G) = P_i(G)^p[G, P_i(G)]$ for $i \geq 1$.

2. Basic properties of Frattini-injective pro-$p$ groups

Frattini-injectivity is obviously a hereditary property, that is, every subgroup of a Frattini-injective pro-$p$ group is Frattini-injective. Furthermore, a Frattini-injective pro-$p$ group is necessarily torsion-free: if a pro-$p$ group $G$ has a non-trivial element of finite order, then it has an element, say $x$, of order $p$ and $\Phi(\langle x \rangle) = \{1_G\} = \Phi(\langle 1_G\rangle)$. Hence, the only Frattini-injective finite $p$-group is the trivial group, which henceforth will be tacitly disregarded.

Lemma 2.1. Let $G$ be a Frattini-injective pro-$p$ group, and let $H$ be a finitely generated subgroup of $G$. Then $N_G(H) = N_G(\Phi(H))$. In particular, $H \trianglelefteq G$ if and only if $\Phi(H) \trianglelefteq G$.

Proof. Since $\Phi(H)$ is a characteristic subgroup of $H$, it follows that $N_G(H) \leq N_G(\Phi(H))$. For the other inclusion, let $x \in G \setminus N_G(H)$; then $H \neq x^{-1}Hx$ and by Frattini-injectivity $\Phi(H) \neq \Phi(x^{-1}Hx) = x^{-1}\Phi(H)x$. Hence, $x \notin N_G(\Phi(H))$. \qed

A subgroup $H$ of a pro-$p$ group $G$ is said to be isolated (in $G$) if $x \in H$ whenever $x^p \in H$. (More generally, the condition implies that for every $\alpha \in \mathbb{Z}_p \setminus \{0\}$, if $x^\alpha \in H$, then $x \in H$.) A related concept (which will appear only later) is the isolator of a subgroup $H$ of a pro-$p$ group $G$; it is the smallest isolated subgroup of $G$ containing $H$.

Proposition 2.2. Every maximal abelian subgroup of a Frattini-injective pro-$p$ group is isolated.

Proof. Let $G$ be a Frattini-injective pro-$p$ group, and let $A$ be a maximal abelian subgroup of $G$. Consider an element $x \in G$ such that $x^p \in A$, and set $H := \langle x, A \rangle$. Then $x^p \in Z(H)$, and thus $\Phi(\langle x \rangle) = \langle x^p \rangle \trianglelefteq H$. It follows from Lemma 2.1 that $\langle x \rangle \trianglelefteq H$. This, in turn, implies that $[x, a] \in \langle x \rangle$ for every $a \in A$. Consequently,

$$1 = [x^p, a] = [x, a]x^{p-1}[x, a]x^{p-2} \cdots [x, a] = [x, a]^p.$$ 

Since all Frattini-injective pro-$p$ groups are torsion free, it follows that $[x, a] = 1$ for every $a \in A$. Hence, $H$ is abelian, and as $A$ is a maximal abelian subgroup of $G$, we get that $H = A$, i.e., $x \in A$. \qed

Corollary 2.3. Let $G$ be a Frattini-injective pro-$p$ group, $x, y \in G$, and $\alpha, \beta \in \mathbb{Z}_p \setminus \{0\}$. The following assertions hold:

(i) If $G$ is virtually abelian, then it is abelian.
(ii) All centralizers of elements of $G$ are isolated.
(iii) If $x^\alpha$ and $y^\beta$ commute, then $x$ and $y$ commute.
(iv) If $x^α = y^α$, then $x = y$, i.e., $G$ has the unique extraction of roots property.

Proof. (i) is an immediate consequence of Proposition 2.2. In order to prove (ii), suppose that $y^p \in C_G(x)$. Then $H := \langle x, y^p \rangle$ is an abelian group, and it is contained in some maximal abelian subgroup $A$ of $G$. By Proposition 2.2, $y \in A$, and hence $y \in C_G(x)$.

Suppose that $x^α$ and $y^β$ commute. Then $y^β \in C_G(x^α)$, and it follows from (ii) that $y \in C_G(x^α)$. By applying the same argument to the pair of commuting elements $y$ and $x^α$, we conclude that $x$ and $y$ commute, which proves (iii).

Now, (iv) is a consequence of (iii) and the fact that free abelian groups of rank two have the unique extraction of roots property. □

Remark 2.4. Note that all of the claims collected in Corollary 2.3 hold true for torsion-free (pro-$p$) groups all of whose maximal abelian subgroups are isolated. Furthermore, a pro-$p$ group $G$ has the unique extraction of roots property if and only if every maximal abelian subgroup of $G$ is isolated. Indeed, suppose that $G$ is a pro-$p$ group with the unique extraction of roots property. Then, $G$ is torsion free and distinct cyclic subgroups of $G$ have distinct Frattinis. As in the proof of Lemma 2.1, we deduce that for every cyclic subgroup $H$ of $G$, if $Φ(H) \leq G$, then $H \leq G$. Now observe that besides torsion-freeness, this is the only other property of Frattini-injective pro-$p$ groups used in the proof of Proposition 2.2.

Lemma 2.5. Let $G$ be a finitely generated pro-$p$ group, and let $M$ be a maximal subgroup of $G$. If $d(M) < d(G)$, then $d(G) = d(M) + 1$ and $Φ(G) = Φ(M)$.

Proof. Suppose that $d(M) < d(G)$, and let $x \in G \setminus M$. Since $G = \langle x, M \rangle$, we must have $d(G) = d(M) + 1$. Furthermore,

$$|G : Φ(M)| = |G : M||M : Φ(M)| = p^{1+d(M)} = p^{d(G)} = |G : Φ(G)|,$$

and as $Φ(M) ≤ Φ(G)$, it follows that $Φ(M) = Φ(G)$. □

Proposition 2.6. Let $G$ be a Frattini-injective pro-$p$ group. Then, every finitely generated subgroup $H$ of $G$ satisfies the following property: if $K$ is an open subgroup of $H$, then $d(K) ≥ d(H)$.

Proof. This follows directly from Lemma 2.5 by induction on the index of the open subgroup. □

3. Frattini-injective $p$-adic analytic pro-$p$ groups

A pro-$p$ group $G$ is said to be powerful if $p$ is odd and $[G,G] ≤ G^p$, or $p = 2$ and $[G,G] ≤ G^4$. A finitely generated powerful pro-$p$ group $G$ such that $|P_i(G) : P_{i+1}(G)| = |G : P_2(G)|$ for all $i ∈ \mathbb{N}$ is called uniform. By [7, Theorem 4.5], a powerful finitely generated pro-$p$ group is uniform if and only if it is torsion-free. Uniform pro-$p$ groups play a central role in the theory of $p$-adic analytic groups:
A topological group is $p$-adic analytic if and only if it contains an open uniform pro-$p$ subgroup (see [4, Theorems 8.1 and 8.18]).

In the seminal paper [23], Lazard defined saturable pro-$p$ groups. For our purposes, it is enough to know that every uniform pro-$p$ group is saturable. To every saturable pro-$p$ group $G$ one can associate a (saturable) $\mathbb{Z}_p$-Lie algebra $L_G$. Moreover, the assignment $G \mapsto L_G$ defines an equivalence between the category of saturable pro-$p$ groups and the category of saturable $\mathbb{Z}_p$-Lie algebras. One advantage of working with saturable pro-$p$ groups stems from the fact that every torsion-free $p$-adic analytic pro-$p$ group of dimension less than $p$ is saturable (see [16, Theorem E]), but in general not uniform.

A uniform pro-$p$ group is said to be hereditarily uniform if all of its open subgroups are also uniform.

**Proposition 3.1.** Every hereditarily uniform pro-$p$ group is Frattini-injective.

**Proof.** Let $G$ be a hereditarily uniform pro-$p$ group. Suppose that there are distinct subgroups $H$ and $K$ of $G$ such that $\Phi(H) = \Phi(K)$. Without loss of generality, we may assume that there is some $x \in H \setminus K$. Choose an open subgroup $U$ of $G$ such that $K \leq U$ and $x \notin U$. Then $x^p \in \Phi(H) = \Phi(K) \leq \Phi(U)$. Since $U$ is uniform, $\Phi(U) = U^p$ and $x^p = z^p$ for some $z \in U$ ([7, Lemma 3.4]). By unique extraction of roots in $G$ ([7, Lemma 4.10]), $x = z$, which yields a contradiction with $x \notin U$. □

The hereditarily uniform pro-$p$ groups were classified in [19]. It turns out that a uniform pro-$p$ group $G$ is hereditarily uniform if and only if it has a constant generating number on open subgroups, that is, $d(U) = d(G)$ for every open subgroup $U$ of $G$ (cf. [18] and [19, Corollary 1.12]).

**Proposition 3.2.** Let $G$ be a Frattini-injective $p$-adic analytic pro-$p$ group. Then, $G$ is virtually hereditarily uniform. More precisely, $G$ contains an open subgroup $U$ isomorphic to one of the following groups:

1. the abelian group $\mathbb{Z}_p^d$ with $d \geq 1$;
2. the metabelian group $\langle x \rangle \ltimes \mathbb{Z}_p^d$, where $d \geq 1$, $\langle x \rangle \cong \mathbb{Z}_p$, and $x$ acts on $\mathbb{Z}_p^d$ as scalar multiplication by $\lambda$, with $\lambda = 1 + p^s$ for some $s \geq 1$ if $p > 2$, and $\lambda = 1 + 2^s$ for some $s \geq 2$ if $p = 2$.

**Proof.** It follows from [7, Theorem 8.32] that there is an open subgroup $U$ of $G$ that is uniform. For every open subgroup $V$ of $U$, we have $d(V) \geq d(U)$ by Proposition 2.6, and also $d(U) \geq d(V)$ by [7, Theorem 3.8]. Hence, $U$ is a uniform pro-$p$ group with constant generating number on open subgroups. By [18, Corollary 2.4], $U$ is isomorphic to one of the groups listed in the proposition. In particular, $U$ is a hereditarily uniform pro-$p$ group (cf. [19, Corollary 1.12]). □

The rest of this section is devoted to eliminating the adverb “virtually” from Proposition 3.2. We begin with several lemmas.
Lemma 3.3. Let $p$ be an odd prime, and let $G = \langle x \rangle \rtimes N$, where $\langle x \rangle \cong \mathbb{Z}_p$ and $N \cong \mathbb{Z}_p^d$ for some $d \geq 1$, be a saturable pro-$p$ group. Suppose that $x^p$ acts on $N$ as scalar multiplication by $1 + p^s$ for some $s \geq 1$. Then, $s \geq 2$ and there is a unit $\alpha \in \mathbb{Z}_p^*$ such that $x^\alpha$ acts on $N$ as scalar multiplication by $1 + p^{s-1}$.

Proof. Note that the maximal subgroup $H := \langle x^p \rangle \rtimes N$ of $G$ is uniform, and therefore saturable. Consider the $\mathbb{Z}_p$-Lie algebras $L_G$ and $L_H$ associated to $G$ and $H$, respectively. Then, $L_H$ is a maximal subalgebra of $L_G$, and we can choose elements $y_1, y_2, \ldots, y_d$ from $N$ such that $\{x, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_d\}$ is a basis for $L_G$ and $\{px, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_d\}$ is a basis for $L_H$. Furthermore, we can assume that for each $1 \leq i \leq d$, we have $[\bar{y}_i, px]_{\text{Lie}} = p^s \bar{y}_i$. Hence, we must have $[\bar{y}_i, x]_{\text{Lie}} = p^{s-1} \bar{y}_i$. Moreover, $s - 1$ cannot be 0, since in that case $L_G$ would not be residually-nilpotent. Therefore, for some suitable unit $\alpha \in \mathbb{Z}_p^*$, the element $x^\alpha$ acts on $N$ as scalar multiplication by $1 + p^{s-1}$.  

Lemma 3.4. Let $G = \langle x \rangle \rtimes N$, where $\langle x \rangle \cong \mathbb{Z}_p$ and $N \cong \mathbb{Z}_p^d$ for some $d \geq 1$. Suppose that $x^p$ acts on $N$ as scalar multiplication by $1 + p^s$ for some $s \geq 1$ if $p > 2$, and as scalar multiplication by $1 + 2^s$ for some $s \geq 2$ if $p = 2$. If $p + 2 \leq \dim(G)$, then $d(G) > 2$.

Proof. Suppose that $p + 2 \leq \dim(G)$ and $d(G) = 2$. Then $G = \langle x, y \rangle$ for some $y \in N$. From $(x^p)^{-1}yx^p = y^{1+p^s}$, we get that the set $T = \{x^{-i}yx^i \mid 0 \leq i \leq p-1\}$ generates $N$. Hence, $d(N) \leq |T| \leq p < p + 1 \leq d = d(N)$, which yields a contradiction.

Lemma 3.5. Let $p$ be an odd prime, and let $G = \langle x \rangle \rtimes N$, where $\langle x \rangle \cong \mathbb{Z}_p$ and $N \cong \mathbb{Z}_p^{p-1}$, be a Frattini-injective pro-$p$ group. Suppose that $x^p$ acts on $N$ as scalar multiplication by $1 + p^s$ for some $s \geq 1$. Then $d(G) > 2$.

Proof. Suppose by way of contradiction that $d(G) = 2$. Then $G = \langle x, y \rangle$ for some $y \in N$. For $1 \leq i \leq p$, set $y_i := x^{-(i-1)}yx^{i-1}$. Since $(x^p)^{-1}yx^p = y^{1+p^s}$, the set $\{y_i \mid 1 \leq i \leq p\}$ generates $N$. Moreover, after (possibly) replacing $y$ by $x^{-(i-1)}yx^{i-1}$ for a suitable $1 \leq i \leq p$, we may assume that $\{y_1, y_2, \ldots, y_{p-1}\}$ is a basis for $N$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{p-1} \in \mathbb{Z}_p$ be such that $x^{-1}y_{p-1}x = y_1^{\alpha_1}y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-1}}$. Then

$$y_1^{1+p^s} = y_1^{p^s} = (y_{p-1})^{x^2} = (y_1^{\alpha_1}y_2^{\alpha_2} \cdots y_{p-2}^{\alpha_{p-2}}y_{p-1}^{\alpha_{p-1}})$$

$$= y_2^{\alpha_1}\cdot y_3^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}}(y_1^{\alpha_1}y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-1}})^{\alpha_{p-1}}$$

$$= y_1^{\alpha_1}\cdot y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}}y_3^{\alpha_2+\alpha_3} \cdots y_{p-2}^{\alpha_{p-3}+\alpha_{p-2}}y_{p-1}^{\alpha_{p-2}+\alpha_{p-1}}.$$

By comparing exponents, we get the following relations:
It readily follows that
\[ \alpha_i = (-1)^i \alpha_{p-1}^{p-i} \]
for \( 1 \leq i \leq p - 2 \) and \(-\alpha_{p-1}^{p} = 1 + p^s\). If \( s = 1 \), the equation \(-\alpha_{p-1}^{p} = 1 + p^s\) does not have a solution in \( \mathbb{Z}_p \). Hence, we may assume that \( s \geq 2 \), in which case, there is a unique \( \omega \in \mathbb{Z}_p \) such that
\[-w^p = 1 + p^s.\]
It follows that \( \alpha_i = (-1)^i \omega^{p-i} \) for \( 1 \leq i \leq p - 1 \).

Consider the open subgroup \( H := \langle x, y_1^p \rangle = \langle x \rangle \Phi(N) \) of \( G \). We have that \( y_1^{-p} y_2^p = [y_1^p, x] \in \Phi(H) \), and it follows by induction that
\[ y_{i+1}^{-p} y_{i+2} = y_i^{-p} y_{i+1}^p [y_i^{-p} y_{i+1}^p, x] \in \Phi(H) \]
for every \( 1 \leq i < p - 2 \).

Since \( \langle y_1^{-p} y_2^{p}, y_3^{-p} y_3^{p}, \ldots, y_{p-2}^{-p} y_{p-2}^{p}, y_1^{p} \rangle \) has index \( p \) in \( \Phi(N) \), it follows that
\[ \Phi(H) = \langle x, y_1^{-p} y_2^{p}, y_3^{-p} y_3^{p}, \ldots, y_{p-2}^{-p} y_{p-2}^{p}, y_1^{p} \rangle. \]

Next, consider the open subgroup \( K := \langle x, y_2^p \rangle \) of \( G \). Observe that \( K \neq H \).

For every \( \tilde{y} \in N \), we have \([\tilde{y}, x, y_2] = [\tilde{y}, x]\). Hence, as for the subgroup \( H \), we can deduce that \( y_i^{-p} y_{i+2}^p \in \Phi(K) \) for every \( 1 \leq i < p - 2 \). Using the identities
\[ y_i^\beta x = x y_i^\beta \]
\( (1 \leq i \leq p - 2) \) and \( y_{p-1}^\beta x = x (y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-1}})^\beta \), we obtain
\[
(xy_2)^p = x^2 y_3 y_3 \cdots y_3 y_2 = x^3 y_1 y_1 \cdots y_3 y_3 y_2 = \cdots = x^{p-2} y_{p-1} x y_{p-1} x (y_{p-1} \cdots y_3 y_2) \\
= x^{p-1} y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}} (y_{p-1}^{\alpha_{p-1}} x) y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}} (y_{p-1} \cdots y_3 y_2) \\
= x^{p-1} y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}} x (y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-1}})^{\alpha_{p-1}} y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}} (y_{p-1} \cdots y_3 y_2) \\
= x^p y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}} (y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-1}})^{1+\alpha_{p-1}} (y_{p-1} \cdots y_3 y_2) \\
= x^p y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}} (y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-1}})^{1+\alpha_{p-1}} y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}} (y_{p-1} \cdots y_3 y_2) \\
= x^p y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-2}} (y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{p-1}^{\alpha_{p-1}})^{1+\alpha_{p-1}} (y_{p-1} \cdots y_3 y_2). \\
\]
Moreover, it follows from \( (\text{II}) \) that
\[ (xy_2)^p = x^p y_1^{1+\alpha_1+p^s} y_2^{1+\alpha_2} y_3^{1+\alpha_3} \cdots y_{p-2}^{1+\alpha_{p-2}} y_{p-1}^{1+\alpha_{p-1}}. \]

Observe that \( \omega \equiv -1 \pmod{p} \), and thus \( \alpha_i \equiv -1 \pmod{p} \) for every \( 1 \leq i \leq p - 1 \). Let \( l_i \in \mathbb{Z}_p \) be such that \( 1 + \alpha_i = pl_i \) \( (1 \leq i \leq p - 1) \). Then
\[
(xy_2)^p (y_{p-2}^{1+\alpha_1+p^s} y_{p-3}^{1+\alpha_2+p^s} \cdots (y_{p-3}^{1+\alpha_2+p^s} y_{p-2}^{1+\alpha_2+p^s} + \cdots + l_2) (y_1^{1+\alpha_1+p^s} y_2^{1+\alpha_2+p^s} + \cdots + l_2)) \\
= x^p y_1^{(1+\alpha_1+p^s) + (1+\alpha_2) + \cdots + l_2} \\
= x^p y_1^{(1+\alpha_1+p^s) + (1+\alpha_2) + \cdots + (1+\alpha_{p-2}) + (1+\alpha_{p-1})}. \\
\]
where \( \gamma = (\alpha_1 + 1 + p^s) + (1 + \alpha_2) + (1 + \alpha_3) + \cdots + (1 + \alpha_{p-2}) + (1 + \alpha_{p-1}) \).
For $1 \leq m \leq \frac{p-1}{2}$, we have $1 + \alpha_{p-(2m-1)} = 1 + \omega^{2m-1} = (1 + \omega)u_m$, where $u_m = 1 - \omega + \omega^2 - \cdots - \omega^{2m-3} + \omega^{2m-1}$, and $1 + \alpha_{p-2m} = 1 - \omega^{2m} = (1 + \omega)v_m$, where $v_m = (1 - \omega)(1 + \omega^2 + \omega^4 + \cdots + \omega^{2(m-1)})$. From $(-1)^i \omega^{p-i} \equiv -1 \pmod{p}$, we get that

$$u_m \equiv 2m - 1 \pmod{p} \text{ and } v_m \equiv 2m \pmod{p}.$$ 

Hence,

$$\tilde{\gamma} := \frac{v_{p-1}}{u} + u_{p-1} + v_{p-3} + u_{p-3} + \cdots + v_1 + u_1$$

$$\equiv (p - 1) + (p - 2) + \cdots + 2 + 1 = \frac{(p - 1)p}{2} \equiv 0 \pmod{p}.$$ 

Since $\gamma = p^s + (1 + \omega)\tilde{\gamma}$ and $s \geq 2$, it follows that $p^2$ divides $\gamma$. Therefore, $x^p \in \Phi(K)$. Now it is easy to see that

$$\Phi(K) = \langle x^p, y_1^p y_2^p, y_2^{-p} y_3^p, \ldots, y_{p-2}^{-p} y_{p-1}^p, y_1^{p^2} \rangle = \Phi(H),$$

which yields a contradiction. Hence, we must have $d(G) > 2$. \hfill \Box

**Lemma 3.6.** Let $G = \langle x \rangle \times N$, where $\langle x \rangle \cong \mathbb{Z}_p$ and $N \cong \mathbb{Z}_p$, be a Frattini-injective pro-$p$ group. Suppose that $x^p$ acts on $N$ as scalar multiplication by $1 + p^s$ for some $s \geq 1$ if $p > 2$, and as scalar multiplication by $1 + 2^s$ for some $s \geq 2$ if $p = 2$. Then $d(G) > 2$.

**Proof.** Suppose that $d(G) = 2$. Then $G = \langle x, y \rangle$ for some $y \in N$, and $\{x^{-i}yx^i \mid 0 \leq i \leq p - 1\}$ is a basis for $N$. Set $y_1 := y$ and $y_{i+1} := [y_i, x]$ for $1 \leq i \leq p - 1$. Then $\{y_1, y_2, \ldots, y_p\}$ is also a basis for $N$.

It is easy to see that $y_2^p = y_2 y_3 \cdots y_{i+2} \cdots y_{k+2}$ for every $1 \leq k \leq p - 2$, and

$$y_2^{p-1} = \left( y_2^{p-2} y_3 \cdots y_{i+2} \cdots y_{k+2} \right)^x = y_2^{p-2} y_3^{(p-2)} \cdots y_{i+2}^{(p-2)} \cdots y_{k+2}^{(p-2)}$$

Moreover,

$$y_2^p = [y_2, x] = [y_1, y][y_1, x]^x \cdots [y_1, x]^{x^{p-2}} [y_1, x]^{x^{p-1}} = y_2 y_2^x \cdots y_2^{x^{p-2}} y_2^{x^{p-1}}.$$ 

By first expressing each $y_2^k$ ($1 \leq k \leq p - 1$) in terms of the basis $y_1, \ldots, y_p$ of $N$, and then applying the hockey-stick identity, $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$, to simplify exponents, we obtain $y_2^p = \gamma \gamma_{i+1} \cdots \gamma_{p-1} y_1^{x^p - 1} y_p^x$. Hence,

$$[y_p, x] = y_2^{x^p - (p)} y_2^{x^{p-1} - (p)} \cdots y_1^{x^{p-2} - (p)} y_p^{x^{p-1} - (p)},$$

Case 1: $s \geq 2$. Consider the open subgroup $H := \langle x, y_1^p, y_2 \rangle$ of $G$. It is not difficult to see that $d(H) = 3$. However, $K := \langle x, y_1^p \rangle$ is an open subgroup of $H$ with $d(K) < d(H)$, which contradicts Proposition 2.6.
Case 2: $s = 1$ (and thus $p > 2$). Consider the subgroups $H := \langle x, y \rangle$ and $K := \langle xy^{-1}, y \rangle$ of $G$. Observe that $H \neq K$. It is straightforward to see that

$$\Phi(N) = \langle [y_p, x], [[y_p, x], x], \ldots, [y_p, x] \rangle = \langle [y_p, xy^{-1}], \ldots, [y_p, xy^{-1}] \rangle.$$ 

Hence, $\Phi(N)$ is a subgroup of both $\Phi(H)$ and $\Phi(K)$. Moreover, it is not difficult to see that $\Phi(H) = \langle x^p, \Phi(N) \rangle$. Since $\gamma_3(\langle x, y \rangle^{-1}) \leq \Phi(N)$, it follows from the Hall-Petresco formula that

$$(xy^{-1})^p \equiv x^p \mod \Phi(N).$$

This implies that $\Phi(K) = \langle x^p, \Phi(N) \rangle = \Phi(H)$, a contradiction. \qed

**Lemma 3.7.** Let $G = \langle x \rangle \ltimes \langle y \rangle$, where $\langle x \rangle \cong \langle y \rangle \cong \mathbb{Z}_2$ and $x$ acts on $\langle y \rangle$ either as scalar multiplication by $-(1 + 2^s)$ for some $s \geq 2$ or by inversion. Then $G$ is not Frattini-injective.

**Proof.** Suppose first that $x$ acts on $\langle y \rangle$ as scalar multiplication by $-(1 + 2^s)$ for some $s \geq 2$. Consider the subgroups $H = \langle x, y^{2s-1} \rangle$ and $K = \langle xy, y^{2s-1} \rangle$ of $G$. Obviously, $H \neq K$. Moreover, $\Phi(H) = \langle x^2, y^{2s} \rangle$ and $\Phi(K) = \langle (xy)^2, y^{2s} \rangle$. Since $(xy)^2 = x^2y^{2s}$, it follows that $\Phi(H) = \Phi(K)$. Therefore, $G$ is not Frattini-injective.

Now suppose that $x$ acts on $\langle y \rangle$ by inversion. Then $(xy)^2 = x^2$, and thus $\Phi(\langle x \rangle) = \Phi(\langle xy \rangle)$. Hence, in this case also $G$ is not Frattini-injective. \qed

**Proof of Theorem 1.2.** One implication follows from Proposition 3.1. For the other implication, suppose that $G$ is Frattini-injective. By Proposition 3.2, $G$ contains an open hereditarily uniform subgroup $U$. If $U$ is abelian, then by Corollary 2.3 (i), $G \cong \mathbb{Z}_p^d$. Hence, we may assume that $U = \langle y \rangle \ltimes N$, where $\langle y \rangle \cong \mathbb{Z}_p$, $N \cong \mathbb{Z}_p^{d-1}$, and $y$ acts on $N$ as scalar multiplication by $\lambda = 1 + p^s$ for some $s \geq 1$ (or $s \geq 2$ if $p = 2$).

We proceed by induction on $|G : U|$. If $G = U$, there is nothing to prove; so, suppose that $|G : U| \geq p$. In fact, running along a subnormal series from $U$ to $G$, it suffices to consider the case $|G : U| = p$.

Let $x \in G \setminus U$; then $G = \langle x \rangle U$ and $x^p \in U$. Note that $N$ is the isolator of the commutator subgroup $[U, U]$ of $U$. Hence, $N$ is a characteristic subgroup of $U$. Since $U$ is normal in $G$, it follows that $N$ is also normal in $G$.

Consider the group $K = \langle x \rangle N$. We consider two separate cases: $x^p \in N$ and $x^p \notin N$.

**Case 1:** $x^p \in N$. Then, $K$ is a Frattini-injective pro-$p$ group that contains an abelian subgroup $N$ of index $p$. By Corollary 2.3 (i), $K \cong \mathbb{Z}_p^{d-1}$. From $\Phi((y^{-1}xy)) = \langle y^{-1}x^py \rangle = \langle (x^p)^\lambda \rangle = \Phi((x^\lambda))$, it follows that $y^{-1}xy \in \langle x \rangle$. Moreover, we have that $(x^\lambda)^p = (x^p)^\lambda = y^{-1}x^py = (y^{-1}xy)^p$. By Corollary 2.3 (iv), $y^{-1}xy = x^\lambda$. Therefore, $y$ acts on $K$ as scalar multiplication by $\lambda$ and $G = \langle y \rangle \ltimes K \cong U$ is of the required form.
Case 2: \( x^p \notin N \). Then, \( x^p = y^p w \) for some \( k \in \mathbb{N} \) and \( w \in N \).

Subcase 2.1: \( p \) is odd and \( k \geq 1 \). Since \( N \) is characteristic in \( U \), conjugation by \( x \) induces an action on \( U/N \cong \mathbb{Z}_p \). Moreover, as \( \text{Aut}(\mathbb{Z}_p) \cong C_{p-1} \times \mathbb{Z}_p \), this action must be trivial. Put \( z := x^{-1} y^{p^{k-1}} \); then \( z^p = (x^{-1} y^{p^{k-1}})^p \equiv x^{-p} y^{p^k} \equiv 1 \) (mod \( N \)). Hence, \( z^p \in N \), and after replacing \( x \) by \( z \), we return to Case 1.

Subcase 2.2: \( p = 2 \) and \( k \geq 1 \). Since \( \text{Aut}(\mathbb{Z}_2) \cong C_2 \times \mathbb{Z}_2 \), either \( y^x \equiv y \) (mod \( N \)) or \( y^x \equiv y^{-1} \) (mod \( N \)). We contend that the latter case does not occur. Indeed, suppose that \( y^x \equiv y^{-1} \) (mod \( N \)); thus \( y^x = y^{-1} n_0 \) for some \( n_0 \in N \). Then, for every \( n \in N \), we have

\[
(n^x)^{1+2^s} = (n^{1+2^s})^x = (n^y)^x = (n^x)y^x = (n^x)y^{-1}n_0 = (n^x)y^{-1} = (n^x)^{(1+2^s)}^{-1}.
\]

Hence, \((1 + 2^s)^2 = 1\), a contradiction.

Thus we must have \( [y, x] \in N \). Put \( z := x^{-1} y^{2^{k-1}} \); then \( z^2 = (x^{-1} y^{2^{k-1}})^2 \equiv x^{-2} y^{2^k} \equiv 1 \) (mod \( N \)). Therefore, \( z^2 \in N \), and after replacing \( x \) by \( z \), we return to Case 1.

Subcase 2.3: \( k = 0 \). Replacing \( y \) by \( yw \), we may assume that \( x^p = y \). We proceed by induction on \( d = \dim(G) \). Since \( N \) is normal in \( G \), we conclude that \( G = \langle x \rangle \rtimes N \). If \( p > d \), then \( p \) is odd and by [16, Theorem E], \( G \) is saturable. It follows from Lemma 3.3 that \( G \) is of the required form.

If \( p = d = 2 \), then by the classification of 2-adic analytic pro-2 groups ([16, Proposition 7.2]), either \( G \) is of the required form or \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), where the action is as scalar multiplication by \(-1 + 2 \ell \) for some \( t \geq 2 \) or by inversion. The latter case is excluded by Lemma 3.7.

Therefore, we may assume that \( d \geq \max\{p, 3\} \). Let \( \{z_1, z_2, \ldots, z_{d-1}\} \) be a basis for \( N \). For each \( 1 \leq i \leq d-1 \), set \( N_i := \langle x, z_i \rangle \). We claim that each \( N_i \) is of infinite index in \( G \). Indeed, if \( N_i \) is open in \( G \), then \( \dim(N_i) = \dim(G) = d \), and it follows from Lemma 3.4, Lemma 3.5 and Lemma 3.6 that \( d(N_i) > 2 \), a contradiction.

Hence, each \( N_i \) is Frattini-injective and of dimension \( \leq d - 1 \). By the induction hypothesis, \( N_i \) is of the required form (hereditarily uniform). It follows easily (by Lie theoretic methods, for example) that \( s \geq 2 \) and for each \( 1 \leq i \leq d - 1 \), there is \( \alpha_i \in \mathbb{Z}^N_p \) such that \( x^{-\alpha_i} z_i x^{\alpha_i} = z_i^{1+p^{s-1}} \). Since \( x^{-p} z_i x^p = z_i^{1+p^s} \) for each \( 1 \leq i \leq d - 1 \), it is not difficult to see that we must have \( \alpha_1 = \alpha_2 = \ldots = \alpha_{d-1} \). Therefore, \( G \) is of the required form. \( \square \)

We end this section with an example of a \( p \)-adic analytic pro-\( p \) group in which Frattini-injectivity fails in a rather extreme way. Let \( D_p \) be a central simple \( \mathbb{Q}_p \)-division algebra of index 2, \( \Delta_p \) the (unique) maximal \( \mathbb{Z}_p \)-order in \( D_p \) and \( \Psi \) the maximal ideal of \( \Delta_p \). Let \( SL_1(D_p) \) be the set of elements of reduced norm 1 in \( D_p \), and let \( G = SL_1(\Delta_p) := SL_1(D_p) \cap (1 + \Psi) \). Then, if \( p > 3 \), for every maximal subgroup \( M \) of \( G \) we have \( \Phi(M) = \Phi(G) \) (cf. [27, Lemma 2.26]).
4. Hierarchical triples

Definition 4.1. Let $G$, $K$ and $H$ be pro-$p$ groups with $H \leq K \leq G$. We say that $(G, K, H)$ is a hierarchical triple if for every $x \in G$, we have that $x \in K$ whenever $x^p \in H$. We call a pro-$p$ group $G$ Frattini-resistant if for every finitely generated subgroup $H$ of $G$, the triple $(G, H, \Phi(H))$ is hierarchical.

Recall that a pro-$p$ group $G$ is said to be strongly Frattini-injective if distinct subgroups of $G$ (not necessarily finitely generated) have distinct Frattinis. In the same vein, $G$ is defined to be strongly Frattini-resistant if $(G, H, \Phi(H))$ is a hierarchical triple for every subgroup $H$ of $G$.

Our first result shows that the definition of Frattini-resistance in terms of hierarchical triples coincides with the definition given in the introduction.

Proposition 4.2. A pro-$p$ group $G$ is Frattini-resistant if and only if for all finitely generated subgroups $H$ and $K$ of $G$,

$$\Phi(H) \leq \Phi(K) \implies H \leq K.$$  

In other words, a pro-$p$ group $G$ is Frattini-resistant if and only if the function $H \mapsto \Phi(H)$ is an embedding of the partially ordered set of finitely generated subgroups of $G$ into itself.

Proof. Suppose that $G$ is a Frattini-resistant pro-$p$ group, and let $H$ and $K$ be finitely generated subgroups of $G$ such that $\Phi(H) \leq \Phi(K)$. For every $x \in H$, we have that $x^p \in \Phi(K)$, and as $(G, K, \Phi(K))$ is a hierarchical triple, it follows that $x \in K$. Hence, $H \leq K$.

For the converse, suppose that $\Phi(H) \leq \Phi(K) \implies H \leq K$ for all finitely generated subgroups $H$ and $K$ of $G$. Let $L$ be a finitely generated subgroup of $G$. If $x^p \in \Phi(L)$ for some $x \in G$, then $\Phi(\langle x \rangle) = \langle x^p \rangle \leq \Phi(L)$, and thus $\langle x \rangle \leq L$, i.e., $x \in L$. It follows that $(G, L, \Phi(L))$ is a hierarchical triple, and therefore $G$ is Frattini-resistant.

Clearly, there is also a “strong” version of Proposition 4.2: A pro-$p$ group $G$ is strongly Frattini-resistant if and only if for all subgroups $H$ and $K$ of $G$, $\Phi(H) \leq \Phi(K) \implies H \leq K$.

Corollary 4.3. Every (strongly) Frattini-resistant pro-$p$ group is (strongly) Frattini-injective.

Proof. Let $G$ be a Frattini-resistant pro-$p$ group, and suppose that $\Phi(H) = \Phi(K)$ for some finitely generated subgroups $H$ and $K$ of $G$. By Proposition 4.2, $\Phi(H) \leq \Phi(K)$ implies $H \leq K$, and $\Phi(K) \leq \Phi(H)$ implies $K \leq H$. Therefore, $H = K$.

In like manner, the “strong” version of the corollary is a consequence of the “strong” version of Proposition 4.2.\[ \square \]

We develop next several results that could be useful when trying to prove that a given pro-$p$ group is (strongly) Frattini-resistant.
Proposition 4.4. Let $G$ be a pro-$p$ group, and suppose that $(G, U, \Phi(U))$ is a hierarchical triple for every open subgroup $U$ of $G$. Then, $G$ is strongly Frattini-resistant.

Proof. Let $H$ be a proper subgroup of $G$, and let $x \in G \setminus H$. Then, there exists an open subgroup $U$ of $G$ such that $H \leq U$ and $x \notin U$. By assumption, $(G, U, \Phi(U))$ is a hierarchical triple; hence, $x^p \notin \Phi(U)$. Since $\Phi(H) \leq \Phi(U)$, it follows that $x^p \notin \Phi(H)$. Therefore, $(G, H, \Phi(H))$ is a hierarchical triple. □

Corollary 4.5. A finitely generated Frattini-resistant pro-$p$ group is strongly Frattini-resistant.

Proof. This follows from Proposition 4.4 and the fact that every open subgroup of a finitely generated pro-$p$ group is also finitely generated. □

Recall that an epimorphism $\varphi : G \to H$ of pro-$p$ groups such that $\ker \varphi \leq \Phi(G)$ is called a Frattini-cover.

Proposition 4.6. Let $G$ be a pro-$p$ group. Suppose that for every open subgroup $U$ of $G$, there exists a Frattini-cover $\varphi : \langle x, U \rangle \to K$ onto a pro-$p$ group $K$ with the property that $(K, M, \Phi(M))$ is a hierarchical triple for every maximal subgroup $M$ of $K$. Then, $G$ is strongly Frattini-resistant.

Proof. Let $U$ be a proper open subgroup of $G$, and let $x \in G \setminus U$. Fix a Frattini-cover $\varphi : \langle x, U \rangle \to K$ onto a pro-$p$ group $K$ with the property that $(K, M, \Phi(M))$ is a hierarchical triple for every maximal subgroup $M$ of $K.$

Set $N := \Phi(\langle x, U \rangle)U$; then $N$ is a maximal subgroup of $\langle x, U \rangle$ which contains $U$, but does not contain $x$. Since $\varphi$ is a Frattini-cover, $M := \varphi(N)$ is a maximal subgroup of $K$ and $\varphi(x) \notin M$. Furthermore, $\varphi(x^p) = \varphi(x)^p \notin \Phi(M)$ (because $(K, M, \Phi(M))$ is a hierarchical triple), and as $\varphi(\Phi(N)) = \Phi(M)$, we get that $x^p \notin \Phi(N)$. Since $\Phi(U) \leq \Phi(N)$, it follows that $x^p \notin \Phi(U)$. Hence, $(G, U, \Phi(U))$ is a hierarchical triple. As $U$ was chosen to be an arbitrary (proper) open subgroup of $G$, it follows from Proposition 4.4 that $G$ is strongly Frattini-resistant. □

Remark 4.7. For a pro-$p$ group $G$ that is not finitely generated, the existence of appropriate Frattini-covers (as in Proposition 4.6) for all finitely generated subgroups of $G$, implies that $G$ is Frattini-resistant (although, not necessarily strongly Frattini-resistant).

Corollary 4.8. Let $G$ be a pro-$p$ group. Suppose that for every open subgroup $U$ of $G$ and for every maximal subgroup $M$ of $U$, the triple $(U, M, \Phi(M))$ is hierarchical. Then, $G$ is strongly Frattini-resistant.

Proof. For every open subgroup $U$ of $G$, the identity map $id_U : U \to U$ is a Frattini-cover satisfying the condition of Proposition 4.6. □
**Definition 4.9.** We define a pro-$p$ group $G$ to be **strongly commutator-resistant** (commutator-resistant) if $(H, \Phi(H), [H, H])$ is a hierarchical triple for every (finitely generated) subgroup $H$ of $G$.

**Proposition 4.10.** Every (strongly) commutator-resistant pro-$p$ group is (strongly) Frattini-resistant.

*Proof.* Let $G$ be a commutator-resistant pro-$p$ group, and let $H$ be a finitely generated subgroup of $G$. Consider an element $x \in G$ such that $x^p \in \Phi(H) = H^p[H, H]$. Then $x^p[H, H] = h^p[H, H]$ for some $h \in H$. Set $K := \langle x, H \rangle$; as $x, h \in K$ and $x^p[K, K] = h^p[K, K]$, it follows that $(x^{-1}h)^p \in [K, K]$. Since $K$ is finitely generated, $(K, \Phi(K), [K, K])$ is a hierarchical triple, and thus $x^{-1}h \in \Phi(K)$, or equivalently, $x\Phi(K) = h\Phi(K)$. Hence, we may replace $x$ by $h$ in a generating set for $K$.

It follows that $K = H$, and thus $x \in H$. This proves that $G$ is Frattini-resistant.

The “strong” version of the proposition can be proved in the same way. □

For an element $x$ of a pro-$p$ group $G$, we have that $x \in \Phi(G)$ if and only if $x[G, G] \in \Phi(G^{ab})$ (in other words, the abelianization homomorphism is a Frattini-cover). It follows that $(G, \Phi(G), [G, G])$ is a hierarchical triple if and only if every element of $G^{ab}$ of order $p$ is contained in $\Phi(G^{ab})$.

**Proposition 4.11.** Let $G$ be a pro-$p$ group. The following statements are equivalent:

(i) $(G, \Phi(G), [G, G])$ is a hierarchical triple.

(ii) Every element of order $p$ in $G^{ab}$ is contained in $\Phi(G^{ab})$.

(iii) Every element of order $p$ in $G^{ab}$ is a $p$th power.

*Proof.* This follows from the remarks made before the proposition and the fact that $\Phi(G^{ab})$ consists of the $p$th powers of elements of $G^{ab}$. □

As an immediate consequence of Proposition 4.11, we get the following

**Corollary 4.12.** Let $G$ be a pro-$p$ group. The following assertions hold:

(i) If $G$ is finitely generated, then $(G, \Phi(G), [G, G])$ is a hierarchical triple if and only if $G^{ab}$ does not contain a direct cyclic factor of order $p$.

(ii) If $G^{ab}$ is torsion-free, then $(G, \Phi(G), [G, G])$ is a hierarchical triple.

The following characterization of commutator-resistance is handy within the context of Galois theory.

**Proposition 4.13.** Let $G$ be a pro-$p$ group, and let $\pi : \mathbb{Z}_p/p^2\mathbb{Z}_p \rightarrow \mathbb{Z}_p/p\mathbb{Z}_p$ be the natural projection. Then, $(G, \Phi(G), [G, G])$ is a hierarchical triple if and only if for every homomorphism $\varphi : G \rightarrow \mathbb{Z}_p/p\mathbb{Z}_p$, there is a homomorphism $\psi : G \rightarrow \mathbb{Z}_p/p^2\mathbb{Z}_p$ such that $\pi \circ \psi = \varphi$.

*Proof.* Suppose that $(G, \Phi(G), [G, G])$ is a hierarchical triple, and let $\varphi : G \rightarrow \mathbb{Z}_p/p\mathbb{Z}_p$ be an epimorphism. Then $\varphi$ factors through an epimorphism $\tilde{\varphi} : G^{ab} \rightarrow \mathbb{Z}_p/p\mathbb{Z}_p$. 
Let \( x \in G^{\text{ab}} \setminus \ker \bar{\varphi} \). If \( px \in \Phi(\ker \varphi) \), then \( px = py \) for some \( y \in \ker \bar{\varphi} \), and \( p(x - y) = 0 \); this contradicts Proposition \ref{prop:4.14} (ii) since \( x - y \notin \Phi(G^{\text{ab}}) \). Hence, there exists a maximal subgroup \( M \) of \( \ker \bar{\varphi} \) that does not contain \( px \). It follows that \( G^{\text{ab}} / M \cong \mathbb{Z}_p / p^2 \mathbb{Z}_p \), and there is an obvious homomorphism \( \psi : G \to \mathbb{Z}_p / p^2 \mathbb{Z}_p \) such that \( \pi \circ \psi = \varphi \).

For the other direction, suppose that for every homomorphism \( \varphi : G \to \mathbb{Z}_p / p^2 \mathbb{Z}_p \), there exists a homomorphism \( \psi : G \to \mathbb{Z}_p / p^2 \mathbb{Z}_p \) such that \( \pi \circ \psi = \varphi \), and let \( x \in G \setminus \Phi(G) \). Choose a maximal subgroup \( M \) of \( G \) that does not contain \( x \), and consider the quotient homomorphism \( \varphi : G \to G / M \cong \mathbb{Z}_p / p \mathbb{Z}_p \) \( (\varphi(x) = 1 + p \mathbb{Z}_p) \). Let \( \psi : G \to \mathbb{Z}_p / p^2 \mathbb{Z}_p \) be a homomorphism such that \( \pi \circ \psi = \varphi \). Then \( \psi(x) \) generates \( \mathbb{Z}_p / p^2 \mathbb{Z}_p \), and thus \( \psi(x^p) = \psi(x)^p \neq 1 \). Since \([G,G] \leq \ker \psi\), it follows that \( x^p \notin [G,G] \). Therefore, \((G, \Phi(G), [G,G])\) is a hierarchical triple. \( \square \)

Considering \( \mathbb{Z}_p / p^2 \mathbb{Z}_p \) as a trivial \( G \)-module, the extension property of Proposition \ref{prop:4.13} comes down to saying that the natural projection \( \mathbb{Z}_p / p^2 \mathbb{Z}_p \to \mathbb{Z}_p / p \mathbb{Z}_p \) induces an epimorphism \( H^1(G, \mathbb{Z}_p / p^2 \mathbb{Z}_p) \to H^1(G, \mathbb{Z}_p / p \mathbb{Z}_p) \) of cohomology groups.

**Corollary 4.14.** Let \( G \) be a pro-\( p \) group, and suppose that \((U, \Phi(U), [U,U])\) is a hierarchical triple for every open subgroup \( U \) of \( G \). Then, \( G \) is strongly commutator-resistant. In particular, a finitely generated commutator-resistant pro-\( p \) group is strongly commutator-resistant.

**Proof.** By Proposition \ref{prop:4.13} the natural projection \( \pi : \mathbb{Z}_p / p^2 \mathbb{Z}_p \to \mathbb{Z}_p / p \mathbb{Z}_p \) induces an epimorphism \( \pi^* : H^1(U, \mathbb{Z}_p / p^2 \mathbb{Z}_p) \to H^1(U, \mathbb{Z}_p / p \mathbb{Z}_p) \) for every open subgroup \( U \) of \( G \) (where \( U \) is assumed to act trivially on \( \mathbb{Z}_p / p^2 \mathbb{Z}_p \)). It follows that \( \pi^* : H^1(H, \mathbb{Z}_p / p^2 \mathbb{Z}_p) \to H^1(H, \mathbb{Z}_p / p \mathbb{Z}_p) \) is an epimorphism for every subgroup \( H \) of \( G \) (cf. \cite{35} I.2.2, Proposition 8). Hence, by Proposition \ref{prop:4.13} \( G \) is strongly commutator resistant. \( \square \)

Before we turn to concrete classes of groups, we make one more useful observation.

**Proposition 4.15.** The properties Frattini-injective, Frattini-resistant and commutator-resistant (as well as their “strong” forms) are preserved under inverse limits.

**Proof.** Let \((G_i, \varphi_{i,j})_I\) be an inverse system of Frattini-injective pro-\( p \) groups with inverse limit \((G, \varphi_i)_{i \in I}\). Let \( H \) and \( K \) be finitely generated subgroups of \( G \) with \( \Phi(H) = \Phi(K) \). Then, for every \( i \in I \),

\[ \Phi(\varphi_i(H)) = \varphi_i(\Phi(H)) = \varphi_i(\Phi(K)) = \Phi(\varphi_i(K)). \]

Hence, \( \varphi_i(H) = \varphi_i(K) \) for all \( i \in I \), and thus \( H = K \).

It is equally easy to prove that all of the other properties are preserved under inverse limits. \( \square \)
5. FRATTINI-INJECTIVE SOLVABLE PRO-\(p\) GROUPS

Let \(G = \langle x \rangle \ltimes \mathbb{Z}_p^d\), where \(\langle x \rangle \cong \mathbb{Z}_p\) and \(x\) acts on \(\mathbb{Z}_p^d\) as scalar multiplication by \(1 + p^r\) with \(s \geq 1\) (\(s \geq 2\) if \(p = 2\)). It is easy to see that \(G^{ab} = \mathbb{Z}_p \ltimes (\mathbb{Z}_p/p^s\mathbb{Z}_p)^d\).

It follows from Corollary 4.12(i) that \(G\) is not commutator-resistant if \(s = 1\). On the other hand, it is not difficult to show that \(G\) is commutator-resistant for \(s \geq 2\). Moreover, a slight modification of the proof of Proposition 3.1 yields the following

**Proposition 5.1.** Every \(p\)-adic analytic Frattini-injective pro-\(p\) group is strongly Frattini-resistant.

It follows from Theorem 1.2 that all Frattini-injective \(p\)-adic analytic pro-\(p\) groups are metabelian. Conversely, we prove in this section that every solvable Frattini-injective pro-\(p\) group is metabelian and locally \(p\)-adic analytic.

**Lemma 5.2.** Let \(G = \langle x \rangle \ltimes A\), where \(\langle x \rangle \cong \mathbb{Z}_p\) and \(A\) is an abelian pro-\(p\) group, be a finitely generated Frattini-injective pro-\(p\) group. Then \(A\) is finitely generated.

**Proof.** Since \(G\) is finitely generated, \(A\) is finitely generated as a topological \(\langle x \rangle\)-module. The completed group algebra \(\mathbb{Z}_p[\langle x \rangle]\) can be identified with the formal power series algebra \(\mathbb{Z}_p[y]\) (by identifying \(x\) with \(1 + y\)); so, we may regard \(A\) as a right (topological) \(\mathbb{Z}_p[y]\)-module (cf. [32, Chapter 7]).

First suppose that \(A\) is a cyclic \(\mathbb{Z}_p[y]\)-module. Thus \(A \cong \mathbb{Z}_p[y]/I\) for some ideal \(I\) of \(\mathbb{Z}_p[y]\). We claim that \(I\) can not be the zero ideal. Indeed, identify \(A\) with \(\mathbb{Z}_p[y]\), and consider the subgroups \(H := \langle x, p^2y^0, py, y^2 \rangle\) and \(K := \langle x, p^2y^0, y^2 \rangle\) of \(G = \langle x \rangle \ltimes \mathbb{Z}_p[y]\) (where, in order to avoid confusion, we denote by \(y^0\) the identity element of \(\mathbb{Z}_p[y]\)). It is readily seen that \(H \neq K\) (in fact, \(K\) is a maximal subgroup of \(H\)), however,

\[
\Phi(H) = \Phi(K) = \langle x^p \rangle [p^2\mathbb{Z}_p[y] + (y^2)],
\]

which contradicts the assumption that \(G\) is Frattini-injective.

Hence, we may assume that \(I \neq (0)\). Since \(A\) is Frattini-injective, and thus torsion-free, there is an element \(a(y) = \sum_{n \geq 0} a_n y^n \in I\) that is not \(p\)-divisible in the abelian group \(\mathbb{Z}_p[y]\). Let \(m \geq 0\) be the smallest integer such that \(a_m \notin p\mathbb{Z}_p\); then \(a(y) \equiv b(y) \mod \Phi(\mathbb{Z}_p[y])\), where \(b(y) = \sum_{n \geq m} a_n y^n\). Since \(a_m\) is a unit in \(\mathbb{Z}_p\), there is \(c(y) \in \mathbb{Z}_p[y]\) such that \(b(y)c(y) = y^m\). Consequently, \(a(y)c(y) \equiv y^m \mod \Phi(\mathbb{Z}_p[y])\) and \((y^m) + \Phi(\mathbb{Z}_p[y]) \leq I + \Phi(\mathbb{Z}_p[y])\). From here it readily follows that \(A\) is finitely generated as a pro-\(p\) group.

Now suppose that \(A\) is generated by \(a_1, a_2, \ldots, a_d\) as \(\mathbb{Z}_p[y]\)-module. For each \(i = 1, \ldots, d\), let \(B_i\) be the submodule of \(A\) generated by \(a_i\); put \(H_i := \langle x, B_i \rangle = \langle x \rangle B_i\) and \(K_i := \langle x, B_i^n \rangle = \langle x \rangle B_i^n\). It follows from what has been already proved that \(B_i\) is a finitely generated free abelian pro-\(p\) group. Consequently, \(H_i\) is a \(p\)-adic analytic Frattini-injective pro-\(p\) group (since every extension of \(p\)-adic analytic pro-\(p\) groups is \(p\)-adic analytic). By Proposition 5.1, \(H_i\) is strongly Frattini-resistant.
For a given \( a \in A \), we have \([x, a] \in A\) and \([[x, a], a] = 1\). Hence, for every \( n \in \mathbb{N} \),
\[
[x, a^n] = [x, a][x, a^{n-1}][[x, a], a^{n-1}] = [x, a][x, a^{n-1}].
\]
It follows that \([x, a^n] = [x, a]^n\) for all \( n \in \mathbb{N} \). In particular, for each \( i = 1, \ldots, d \),
\[
[x, a_i]^p = [x, a_i^p] \in [K_i, K_i] \leq \Phi(K_i).
\]
Hence, \([x, a_i] \in K_i\) (because \( H_i \) is strongly Frattini-resistant). Since also \([x, a_i] \in B_i\), we get that
\[
[x, a_i] \in K_i \cap B_i = B_i^p \leq A^p.
\]
Note that \( A \) is generated (as a pro-\( p \) group) by \( x^{-\alpha}a_ix^\alpha \) (\( \alpha \in \mathbb{Z}_p \) and \( i = 1, \ldots, d \)). However, \([x, a_i] \in A^p = \Phi(A)\) implies that \( a_i\Phi(A) = x^{-1}a_ix\Phi(A) \). Therefore, \( a_1, \ldots, a_d \) suffice to generate \( A \).

**Lemma 5.3.** Let \( G \) be a non-abelian Frattini-injective metabelian pro-\( p \) group. Then \( G \cong \langle x \rangle \rtimes A \), where \( \langle x \rangle \cong \mathbb{Z}_p \), \( A \) is a free abelian pro-\( p \) group, and \( x \) acts on \( A \) as scalar multiplication by \( 1 + p^s \) with \( s \geq 1 \) if \( p \) is odd, and \( s \geq 2 \) if \( p = 2 \).

**Proof.** Let \( A \) be a maximal abelian subgroup of \( G \) containing \([G, G]\). Then \( A \) is a free abelian pro-\( p \) group (all torsion-free abelian pro-\( p \) groups are free abelian). Moreover, \( A \leq G \) (since \([G, G] \leq A\) and \( A \) is isolated in \( G \) (Proposition 2.2)).

Let \( x \in G \setminus A \), and let \( a_1, \ldots, a_d \in A \). Consider the subgroup \( H := \langle x, a_1, \ldots, a_d \rangle \) of \( G \). Let \( N \) be the normal subgroup of \( H \) generated (as a normal subgroup) by the elements \( a_1, \ldots, a_d \). Then \( N \leq A \), and hence \( N \) is abelian. Since \( A \) is isolated in \( G \), we also have \( \langle x \rangle \cap N = \{1\} \). Therefore, \( H = \langle x \rangle N \) is an internal semidirect product. By Lemma 5.2, \( N \) is a finitely generated free abelian pro-\( p \) group, and thus \( H \) is \( p \)-adic analytic.

By Theorem 1.2, either \( H \) is abelian or for some unit \( \alpha \) of \( \mathbb{Z}_p \), \( x^\alpha \) acts on \( N \) as scalar multiplication by \( 1 + p^s \) with \( s \geq 1 \) if \( p \) is odd, and \( s \geq 2 \) if \( p = 2 \). Since \( a_1, \ldots, a_d \) were chosen to be arbitrary elements of \( A \), it follows that \( x \) must act in the same way on all elements of \( A \). As \( A \) is a maximal abelian subgroup of \( G \), \( x \) can not commute with all elements of \( A \); so, \( x^\alpha \) (for some unit \( \alpha \)) acts on \( A \) as scalar multiplication by \( 1 + p^s \).

The group \( G/A \) is torsion-free since \( A \) is isolated in \( G \). We claim that \( G/A \cong \mathbb{Z}_p \). Suppose that this is not the case. Then, there exist \( x_1, x_2 \in G \) such that \( x_1A \) and \( x_2A \) generate in \( G/A \) a free abelian pro-\( p \) group of rank two. Fix \( a \in A \), \( a \neq 1 \), and consider the group \( L := \langle x_1, x_2, a \rangle \). Now, we know that \( x_1 \) and \( x_2 \) normalize the abelian group \( M := \langle [x_1, x_2], a \rangle \leq A \). Hence, \( M \leq L \) and \( L/M \cong \mathbb{Z}_p \times \mathbb{Z}_p \). This implies that \( L \) is \( p \)-adic analytic. It follows from Theorem 1.2 that all Frattini-injective \( p \)-adic analytic pro-\( p \) groups that have a quotient isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_p \) are abelian. However, \( L \) is not abelian since \( x_1 \) and \( x_2 \) do not commute with \( a \), a contradiction.
Let \( x \) be an element of \( G \) such that \( G/A = \langle xA \rangle \). Clearly, we may choose \( x \) in such a way that it acts on \( A \) as scalar multiplication by \( 1 + p^s \) (\( s \geq 1 \) if \( p \) is odd, and \( s \geq 2 \) if \( p = 2 \)). Therefore, \( G = \langle x \rangle \ltimes A \) is of the required form.

\[ \square \]

**Proof of Theorem 1.3.** Let \( G = \langle x \rangle \ltimes A \) be a semidirect product as in the statement of the theorem. It is easily seen that

\[ G = \lim_{\leftarrow} \langle x \rangle \ltimes A_i, \]

where \( \{ A_i \mid i \in I \} \) is the set of finitely generated direct factors of \( A \). Since all of the groups \( \langle x \rangle \ltimes A_i \) are Frattini-injective (Theorem 1.2), it follows from Proposition 4.15 that \( G \) is also Frattini-injective. (In fact, Proposition 5.1 and Proposition 4.15 imply that \( G \) is strongly Frattini-resistant.)

In light of Lemma 5.3, in order to prove the converse, it suffices to argue that there are no solvable Frattini-injective pro-\( p \) groups of derived length 3. Suppose to the contrary that \( G \) is such a group. By Lemma 5.3, \( [G, G] = \langle x \rangle \ltimes A \), where \( \langle x \rangle \cong \mathbb{Z}_p \), \( A \) is a free abelian pro-\( p \) group, and \( x \) acts on \( A \) as scalar multiplication by \( 1 + p^s \) with \( s \geq 1 \) if \( p \) is odd, and \( s \geq 2 \) if \( p = 2 \). Note that \( A \) is the isolator of \( G^{(2)} = A^{p^s} \) in \([G, G]\). Since \([G, G]\) is normal in \( G \) and \( G^{(2)} \) is characteristic in \([G, G]\), it follows that \( A \) is normal in \( G \).

There are elements \( y, z \in G \) such that \([y, z] \notin A \) (otherwise, we would have \([G, G] \leq A \) and \( G^{(2)} = 1 \), a contradiction); so, \([y, z]\) acts on \( A \) as scalar multiplication by some \( \lambda \neq 1 \). Clearly, \( \langle y, A \rangle \) and \( \langle z, A \rangle \) are metabelian groups, and it follows from Lemma 5.3 that \( y \) and \( z \) also act on \( A \) by scalar multiplication.

Fix \( a \in A, \ a \neq 1 \). Then the group \( \langle y, z \rangle \) acts on \( \langle a \rangle \cong \mathbb{Z}_p \), so we get a homomorphism \( \varphi : \langle y, z \rangle \rightarrow \text{Aut}(\mathbb{Z}_p) \). However, \([y, z] \notin \ker \varphi \), which is impossible since \( \text{Aut}(\mathbb{Z}_p) \) is abelian.

\[ \square \]

**Proof of Theorem 1.4.** It is readily seen (using Zorn’s lemma) that \( G \) contains a maximal normal abelian subgroup \( N \). The isolator of \( N \) is also a normal subgroup of \( G \), and it follows from Lemma 2.3 (iii) that it is abelian. Hence, \( N \) coincides with its isolator, and so it is isolated in \( G \).

Suppose that \( M \) is another maximal normal abelian subgroup of \( G \). Then, \( NM \) is a solvable Frattini-injective pro-\( p \) group. It follows from Theorem 1.3 that \( NM = \langle x \rangle \ltimes A \) (where the semidirect product is of the form described in the theorem). Without loss of generality, we may assume that the restriction to \( N \) of the projection homomorphism from \( NM \) onto \( \langle x \rangle \) is surjective. Hence, there is \( n \in N \) with \( n = xy \) for some \( y \in A \). For every \( a \in A, \ [a, n] = a^{p^s} \in N \); as \( N \) is isolated in \( G \), it follows that \( a \in N \). This implies that \( M \leq NM \leq N \), a contradiction. Hence, \( N \) is the unique maximal normal abelian subgroup of \( G \).
Let \( x \in G \). Then, the group \( \langle x, N \rangle \) is either abelian or metabelian with \( x \) acting on \( N \) by scalar multiplication. Furthermore, if \( Z(G) \neq 1 \), then \( Z(G) \leq N \), and \( x \) commutes with every element in \( N \). Now, it is clear that \((ii)\) and \((iii)\) hold.

Suppose that \( Z(G) = 1 \) but \( N \neq 1 \). Since \( N \) is normal in \( G \), the centralizer of \( N \) is also normal in \( G \). Moreover, as \( Z(C_G(N)) \) is characteristic in \( C_G(N) \), it follows that \( Z(C_G(N)) \) is a normal abelian subgroup of \( G \). Hence, \( Z(C_G(N)) \leq N \), and as the reverse inclusion is obvious, we get \( Z(C_G(N)) = N \).

It follows from Corollary 2.3 \((ii)\) that \( C_G(N) \) is isolated in \( G \) (because every intersection of isolated subgroups is also isolated). Hence, \( G/C_G(N) \) is torsion free. We need to prove that \( G/C_G(N) \) is pro-cyclic.

Fix \( n \in N \), \( n \neq 1 \). It follows from \((ii)\) that every non-trivial element of \( G/C_G(N) \) acts on \( \langle n \rangle \) by non-trivial scalar multiplication. Hence, \( G/C_G(N) \) embeds into \( \text{Aut}(\langle x \rangle) \). Since \( G/C_G(N) \) is torsion free, it follows that it is isomorphic to \( \mathbb{Z}_p \).

\[ \square \]

6. Free pro-\( p \) groups and Demushkin groups

Let \( F \) be a free pro-\( p \) group, and let \( H \) be a subgroup of \( F \). Then, \( H \) is also free pro-\( p \) and \( H^{ab} \) is a free abelian pro-\( p \) group. It follows from Corollary 4.12 \((ii)\) that \((H, \Phi(H), [H, H])\) is a hierarchical triple.

**Theorem 6.1.** Every free pro-\( p \) group is strongly commutator-resistant.

A pro-\( p \) group \( G \) is called a Demushkin group if it satisfies the following conditions:

\begin{enumerate}
    \item \( \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty \),
    \item \( \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1 \), and
    \item the cup-product \( H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p) \cong \mathbb{F}_p \) is a non-degenerate bilinear form.
\end{enumerate}

If \( k \) is a \( p \)-adic number field containing a primitive \( p \)th root of unity and \( k(p) \) is a maximal \( p \)-extension of \( k \), then \( \text{Gal}(k(p)/k) \) is a Demushkin group. Furthermore, the pro-\( p \) completion of any orientable surface group is also a Demushkin group. In fact, all Demushkin groups have many properties reminiscent of surface groups. For instance, every finite index subgroup \( U \) of a Demushkin group \( G \) is a Demushkin group with \( d(U) = |G : U|(d(G) - 2) + 2 \), and every subgroup of infinite index is free pro-\( p \). For a detailed exposition of the theory of Demushkin groups, see [35] or [26].

Let \( G \) be a Demushkin group. Since \( \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1 \), it follows that \( G \) is a one related pro-\( p \) group. Hence, there is an epimorphism \( \pi : F \to G \) where \( F \) is a free pro-\( p \) group of rank \( d := d(G) \) and \( \text{ker} \pi \) is generated as a closed normal subgroup by one element \( r \in \Phi(F) = F^p[F, F] \). It follows that either \( G^{ab} \cong \mathbb{Z}_p^d \) or \( G^{ab} \cong \mathbb{Z}/p^e \mathbb{Z} \times \mathbb{Z}_p^{d-1} \) for some \( e \geq 1 \); set \( q := p^e \) in the latter and \( q := 0 \) in the former case. Then, \( d \) and \( q \) are two invariants associated to \( G \). (When we wish to
emphasize the Demushkin group under consideration, we write $d(G)$ and $q(G)$ for the invariants of $G$.)

Demushkin groups were classified by Demushkin, Serre and Labute (\cite{Demushkin}, \cite{Serre}, \cite{Labute}, and \cite{Demushkin2}). We summarise the classification in the following

**Theorem 6.2.** Let $G$ be a Demushkin group with invariants $d$ and $q$. Then $G$ admits a presentation $G = \langle x_1, x_2, \ldots, x_d \mid r \rangle$, where

(i) if $q \neq 2$, then $d$ is even and

$$r = x_d^q [x_1, x_2] [x_3, x_4] \cdots [x_{d-1}, x_d];$$

(ii) if $q = 2$ and $d$ is even, then

$$r = x_1^{2+\alpha} [x_1, x_2] x_3^{2f} [x_3, x_4] \cdots [x_{d-1}, x_d]$$

for some $f = 2, 3, \ldots, \infty$ ($2f = 0$ when $f = \infty$) and $\alpha \in 4\mathbb{Z}_2$;

(iii) if $q = 2$ and $d$ is odd, then

$$r = x_1^2 x_2^{2f} [x_2, x_3] \cdots [x_{d-1}, x_d]$$

for some $f = 2, 3, \ldots, \infty$.

We first consider Demushkin groups with $q \neq 2$. (As is often the case, $q = 2$ takes more effort.)

**Theorem 6.3.** Let $G$ be a Demushkin pro-$p$ group. Then, the following assertions hold:

(i) If $q(G) \neq p$, then $G$ is strongly commutator-resistant.

(ii) If $q(G) = p$ and $p$ is odd, then $G$ is strongly Frattini-resistant, but not commutator-resistant.

**Proof.** (i) Suppose that $q(G) \neq p$, and let $H$ be a subgroup of $G$. If $|G : H| < \infty$, then $H$ is a Demushkin group with $q(H) \neq p$ (\cite{Demushkin} §3. Corollary), and it follows from Corollary 4.12 (i) that $(H, \Phi(H), [H, H])$ is a hierarchical triple; otherwise, $H$ is a free pro-$p$ group, and $(H, \Phi(H), [H, H])$ is a hierarchical triple by Theorem 6.1.

(ii) Suppose that $p$ is odd and $q(G) = p$. Since $G^{ab} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}_p^{d-1}$, it follows from Corollary 4.12 (i) that the triple $(G, \Phi(G), [G, G])$ is not hierarchical. Therefore, $G$ is not commutator-resistant.

Let $U$ be an open subgroup of $G$. Then, $U$ is a Demushkin group and there exist elements $x_1, \ldots, x_d \in U$ such that $U = \langle x_1, \ldots, x_d \mid x_1^q [x_1, x_2] [x_3, x_4] \cdots [x_{d-1}, x_d] \rangle$, where $q = q(U) = p^e$ for some $e \geq 1$. Consider the hereditarily uniform pro-$p$ group

$$K := \langle z_1, z_2, \ldots, z_d \mid [z_i, z_j] = 1 \text{ and } z_1^{-1} z_i z_1 = z_i^{1-q} \text{ for all } 2 \leq i, j \leq d \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p^{d-1}.$$ 

The assignment $x_1 \mapsto z_2$, $x_2 \mapsto z_1$ and $x_i \mapsto z_i$ for $3 \leq i \leq d$ defines a Frattini-cover $U \to K$. Since $K$ is Frattini-resistant (Proposition 5.1), it follows from Proposition 4.6 that $G$ is strongly Frattini-resistant. \hfill $\square$
Before we turn to the case $q = 2$, we prove two auxiliary results.

**Lemma 6.4.** Let $G$ be a pro-$p$ group that contains two distinct open subgroups with the same Frattini. Then, there are two distinct open subgroups $M$ and $N$ of $G$ such that $\Phi(M) = \Phi(N)$ and $M, N \leq \langle M, N \rangle$.

**Proof.** Let $X$ be the set of all pairs $(U, V)$ of distinct open subgroups of $G$ with $\Phi(U) = \Phi(V)$, and let

$$r := \min \{|G : U| - |G : V| | (U, V) \in X\}.$$ 

Set $X_r := \{(U, V) \in X | r = |G : U| - |G : V|\}$, and fix some $(U_0, V_0) \in X_r$. Among all pairs $(U, V) \in X_r$ such that $|G : U| = |G : U_0|$ and $|G : V| = |G : V_0|$, choose one, say $(M, N)$, that maximizes the index $|G : \langle U, V \rangle|$. We claim that both $M$ and $N$ are normal in $\langle M, N \rangle$. Suppose to the contrary that $M$ is not normal in $\langle M, N \rangle$. Then, there is $x \in N$ such that $M \neq M^x$. Moreover,

$$\Phi(M^x) = \Phi(M)^x = \Phi(N)^x = \Phi(N) = \Phi(M),$$

so $(M, M^x) \in X$. Since $|G : M| = |G : M^x|$, it follows that $r = 0$, and thus $|G : M| = |G : N|$. Furthermore, by the choice of $(M, N)$, we have that $|G : \langle M, M^x \rangle| \leq |G : \langle M, N \rangle|$. This, in turn, implies that $\langle M, M^x \rangle = \langle M, N \rangle$. However, this is only possible if $M = \langle M, N \rangle$, which yields a contradiction with $M \neq N$ and $|G : M| = |G : N|$.

\[ \square \]

**Lemma 6.5.** Let $G$ be a pro-$2$ group, and let $M$ and $N$ be normal open subgroups of $G$ such that $G = MN$ and $\Phi(M) = \Phi(N)$. Then

$$\log_2(|\Phi(G) : \Phi(M)|) \leq \log_2(|M : M \cap N|) \log_2(|N : M \cap N|).$$

**Proof.** For an arbitrary element $g = mn \ (m \in M, n \in N)$ of $G$, we have

$$g^2 = m^2 n^2 [m, n][[n, m], n].$$

Clearly, $m^2, n^2 \in \Phi(M)$, and as $[n, m] \in N$, we also get $[[n, m], n] \in \Phi(M)$. Therefore, $g^2 \in [M, N] \Phi(M) \leq \Phi(G)$. Since the squares of elements of $G$ generate $\Phi(G)$, it follows that $\Phi(G) = [M, N] \Phi(M)$.

From $\Phi(M) = \Phi(N) \leq M \cap N$, we deduce that $M/M \cap N$ and $N/M \cap N$ are elementary abelian $2$-groups. In addition, $\Phi(G)/\Phi(M)$ is also elementary abelian, and it is easily seen that

$$M/(M \cap N) \times N/(M \cap N) \to \Phi(G)/\Phi(M), (m(M \cap N), n(M \cap N)) \mapsto [m, n] \Phi(M),$$

is a well-defined bilinear map. Since $\Phi(G) = [M, N] \Phi(M)$, the induced linear transformation $M/(M \cap N) \otimes_{\mathbb{F}_2} N/(M \cap N) \to \Phi(G)/\Phi(M)$ is surjective. Hence,

$$\log_2(|\Phi(G) : \Phi(M)|) = \dim_{\mathbb{F}_2} \Phi(G)/\Phi(M) \leq \dim_{\mathbb{F}_2} M/(M \cap N) \otimes_{\mathbb{F}_2} N/(M \cap N)$$

$$= \dim_{\mathbb{F}_2} M/(M \cap N) \dim_{\mathbb{F}_2} N/(M \cap N) = \log_2(|M : M \cap N|) \log_2(|N : M \cap N|).$$

\[ \square \]
Theorem 6.6. Let $G$ be a Demushkin group with $q(G) = 2$. Then, the following assertions hold:

(i) If $d(G) = 2$, then $G$ is not Frattini-injective.
(ii) If $d(G) > 2$, then $G$ is Frattini-injective, but not Frattini-resistant.

Proof. (i) If $d(G) = 2$, then $G = \langle x_1, x_2 \mid x_1^{2+\alpha}[x_1, x_2] \rangle$ for some $\alpha \in 4\mathbb{Z}_2$. It is easily seen that $G$ is 2-adic analytic, however, it is not isomorphic to any of the pro-2 groups listed in Theorem 1.2 (for instance, it is obvious that $G$ is not powerful). Therefore, $G$ is not Frattini-injective.

(ii) Suppose that $d(G) > 2$, and assume that there are two distinct finitely generated subgroups $H$ and $K$ of $G$ such that $\Phi(H) = \Phi(K)$. Since a subgroup of $G$ is open if and only if it has open Frattini, $H$ and $K$ are either both open or they both have infinite index in $G$. Furthermore, in case that $H$ and $K$ are open, they necessarily have the same index in $G$ (since for $U$ open, $|G : \Phi(U)|$ is a strictly increasing function of $|G : U|$).

First suppose that $H$ and $K$ are open. By Lemma 6.5 we may further assume that $H$ and $K$ are both normal in $L := \langle H, K \rangle$. Moreover, it follows from Lemma 6.5 that

$$\log_2(|\Phi(L) : \Phi(H)|) \leq \log_2(|H : H \cap K|) \log_2(|K : H \cap K|).$$

Hence,

$$\log_2(|H : \Phi(H)|) = \log_2(|H : \Phi(L)|) + \log_2(|\Phi(L) : \Phi(H)|)$$

$$\leq d(L) + \log_2(|H : H \cap K|) \log_2(|K : H \cap K|).$$

On the other hand,

$$\log_2(|L : \Phi(H)|) = \log_2(|L : H|) + \log_2(|H : \Phi(H)|) = \log_2(|L : H|) + d(H).$$

Therefore,

$$\log_2(|L : H|) + d(H) \leq d(L) + \log_2(|H : H \cap K|) \log_2(|K : H \cap K|).$$

Since $L$ is a Demushkin group, we have $d(H) = |L : H|(d(L) - 2) + 2$; in addition, $|H : H \cap K| \leq |L : K| = |L : H|$ and $|K : H \cap K| \leq |L : H|$. Thus

$$\log_2(|L : H|) + |L : H|(d(L) - 2) + 2 \leq d(L) + \log_2(|L : H|)^2.$$
Now assume that $H$ and $K$ have infinite index in $G$. By Theorem A $(H, K)$ can not be free, so it must be open in $G$. Moreover, since $\Phi(H)$ has finite index in both $H$ and $K$, it follows from the Greenberg property of Demushkin groups (37, Theorem A) that $\Phi(H)$ also has finite index in $(H, K)$. This implies that $H$ and $K$ are open in $G$, a contradiction.

It remains to prove that $G$ is not Frattini-resistant. Suppose first that $d(G)$ is odd. Then $G = \langle x_1, \ldots, x_d \mid x_1^{2+\alpha}[x_1, x_2]x_2^{2f}[x_3, x_4] \cdots [x_d-1, x_d] \rangle$ for some $f = 2, 3, \ldots, \infty$. Let $H$ be the subgroup of $G$ generated by $x_2, x_3, \ldots, x_d$. Then $x_1 \notin H$, but $x_1^2 = [x_2, x_{d-1}] \cdots [x_3, x_2]x_2^{-2f} \in \Phi(H)$. Therefore, $G$ is not Frattini-resistant.

Now assume that $d(G)$ is even. Then

$$G = \langle x_1, \ldots, x_d \mid x_1^{2+\alpha}[x_1, x_2]x_2^{2f}[x_3, x_4] \cdots [x_d-1, x_d] \rangle$$

for some $f = 2, 3, \ldots, \infty$ and $\alpha \in 4\mathbb{Z}_2$. Let $M$ be a maximal subgroup of $G$ that contains the elements $x_2, x_3, \ldots, x_d$; then $x_1x_2 \notin M$. We may write the square of $x_1x_2$ as

$$(x_1x_2)^2 = x_1^2x_2^2[x_2, x_1][x_2, x_1] = (x_1^2x_2x_1^{-1})^2(x_1^2[x_1, x_2])[x_2, x_1]^2[x_2, x_1, x_2].$$

Clearly, $(x_1^2x_2x_1^{-1})^2 \in \Phi(M)$, and as $[x_2, x_1] \in M$, we also have $[x_2, x_1]^2 \in \Phi(M)$ and $[x_2, x_1, x_2] \in \Phi(M)$. Furthermore, $x_1^\alpha \in \Phi(M)$ (since $\alpha \in 4\mathbb{Z}_2$), and it follows from the relation of $G$ that

$$x_1^2[x_1, x_2] = x_1^{-\alpha}[x_1, x_{d-1}] \cdots [x_3, x_2]x_2^{-2f} \in \Phi(M).$$

Therefore, $(x_1x_2)^2 \in \Phi(M)$, which proves that $G$ is not Frattini-resistant.

\[\square\]

7. Maximal pro-$p$ Galois groups

Recall that we denote by $G_k = \text{Gal}(k_s/k)$ and $G_k(p) = \text{Gal}(k(p)/k)$ the absolute Galois group and the maximal pro-$p$ Galois group of a field $k$, respectively.

**Theorem 7.1.** Let $p$ be an odd prime, and let $k$ be a field that contains a primitive $p$th root of unity. Then $G_k(p)$ is a strongly Frattini-resistant pro-$p$ group.

**Proof.** By Corollary 4.8 it suffices to prove that for every subgroup $H$ of $G_k(p)$ and every maximal subgroup $M$ of $H$, the triple $(H, M, \Phi(M))$ is hierarchical. Let $F$, $K$ and $L$ be the fixed fields of $H$, $M$ and $\Phi(M)$, respectively. Then $K = F(\sqrt[p^a]{a})$ for some $a \in F^\times$ (since $F$ contains a primitive $p$th root of unity). Let $b \in k(p)$ be a root of the polynomial $X^{p^a} - \sqrt[p^a]{a}$; then $[K(b) : K]$ divides $p$, and hence $b \in L$.

Let $\sigma \in H \setminus M$. We claim that $\sigma^p$ does not fix $b$, and consequently $\sigma^p \notin \Phi(M)$.

Since the roots of the polynomial $X^{p^2} - a$ are $b^{p^i}_{p^2}$ (where $0 \leq i \leq p^2 - 1$, $\zeta_{p^2}$ is a primitive $p^2$th root of unity, we have $\sigma(b) = b^{p^i}_{p^2}$ for some $0 \leq s \leq p^2 - 1$. Furthermore, \( s \) is relatively prime to $p$, since $p \mid s$ would imply

$$\sigma(\sqrt[p^a]{a}) = \sigma(b^p) = \sigma(b)^p = [b^p_{p^2} = \sqrt[p^a]{a},$$
which yields a contradiction with $\sigma \notin M$.

Consider the action of $\sigma$ on the group $\mu_{p^2} = \langle \zeta_{p^2} \rangle$ of $p^2$th roots of unity. Since $\zeta_{p^2}^p \in k$, we have $\sigma(\zeta_{p^2}^p) = \zeta_{p^2}^p$, and thus $\sigma(\zeta_{p^2}) = \zeta_{p^2}^t$ for some $1 \leq t \leq p^2 - 1$ with $t \equiv 1 \pmod{p}$, i.e., $t = 1 + lp$ for some $0 \leq l \leq p - 1$. Moreover, by a simple calculation, we obtain

$$\sigma^p(b) = b \zeta_{p^2}^{s + ts + \ldots + t^{p-1}s}.$$

We need to prove that $p^2$ does not divide $s + ts + \ldots + t^{p-1}s = s(1 + t + \ldots + t^{p-1})$. Since $p$ does not divide $s$, it suffices to show that $p^2$ does not divide $1 + t + \ldots + t^{p-1}$. This is clearly the case if $t = 1$, so we may assume that $t = 1 + lp$ for some $1 \leq l \leq p - 1$. Then

$$1 + t + \ldots + t^{p-1} = \frac{t^p - 1}{t - 1} = \frac{\sum_{i=1}^{p-1} (p)(lp)^i}{lp} = p + p^2 \left[ \frac{p - 1}{2} + \sum_{i=3}^{p} \left( \frac{p}{i} \right) p^{i-3} i^{-1} \right].$$

Since $p$ is odd, it follows that $p^2$ does not divide $1 + t + \ldots + t^{p-1}$. \hfill $\square$

Clearly, the assumption that $p$ is an odd prime is essential in Theorem 7.1 (for instance, $\mathbb{C}/\mathbb{R}$ is a maximal 2-extension with Galois group cyclic of order two, which is obviously not Frattini-injective). However, the condition on the prime is used only in the last line of the proof (when $t \neq 1$). Hence, under the stronger assumption that $k$ contains a primitive $p^2$th root of unity, the theorem holds also for $p = 2$. In fact, in that case we can say more.

**Theorem 7.2.** Let $k$ be a field that contains a primitive $p^2$th root of unity. Then $G_k(p)$ is strongly commutator-resistant.

**Proof.** Let $H$ be a subgroup of $G_k(p)$ with fixed field $F$, and let $\varphi : H \to \mathbb{Z}_p/p\mathbb{Z}_p$ be an epimorphism. Denote by $K$ the fixed field of $\ker \varphi$. Then $K = F(\sqrt[p^2]{a})$ for some $a \in F^\times$, and $\varphi$ induces an isomorphism $\tilde{\varphi} : H/\ker \varphi \cong \text{Gal}(K/F) \to \mathbb{Z}_p/p\mathbb{Z}_p$. Since $F$ contains a $p^2$th root of unity, there is a field $L$ containing $K$ such that the extension $L/F$ is cyclic of degree $p^2$ (take $L := F(\sqrt[p^2]{a})$). By composing the restriction homomorphism $H \to \text{Gal}(L/F)$ with a suitable isomorphism from $\text{Gal}(L/F)$ to $\mathbb{Z}_p/p^2\mathbb{Z}_p$, we obtain an epimorphism $\psi : H \to \mathbb{Z}_p/p^2\mathbb{Z}_p$ such that $\pi \circ \tilde{\psi} = \varphi$, where $\pi : \mathbb{Z}_p/p^2\mathbb{Z}_p \to \mathbb{Z}_p/p\mathbb{Z}_p$ is the natural projection. It follows from Proposition 4.13 that $G_k(p)$ is strongly commutator resistant. \hfill $\square$

For a field $F$ containing a primitive $p$th root of unity, we denote by $F(\sqrt[p]{F^\times})$ the maximal $p$-Kummer extension of $F$. In terms of field extensions, the Frattini-resistance of $G_k(p)$ takes the following form.

**Corollary 7.3.** Let $k$ be a field that contains a primitive $p$th root of unity. If $p = 2$, in addition, assume that $\sqrt{-1} \in k$. Then, for all intermediate fields $F$ and $K$ of the extension $k(p)/k$,

$$F \subseteq K \iff F(\sqrt[p]{F^\times}) \subseteq K(\sqrt[p]{K^\times}).$$

Moreover, $F/k$ is Galois if and only if $F(\sqrt[p]{F^\times})/k$ is Galois.
Proof. Note that if $H$ is a subgroup of $G_k(p)$ with fixed field $F$, then $F(\sqrt[\varphi(F)]{F})$ is the fixed field of $\Phi(H)$. Hence, by Galois correspondence, the corollary follows from Theorem 7.1, Theorem 7.2 and Lemma 2.1.

As an immediate consequence of the torsion-freeness of Frattini-injective pro-$p$ groups, we obtain Becker’s restriction on the finite subgroups of $G_k(p)$ ([1]).

Corollary 7.4. Let $k$ be a field that contains a primitive $p$th root of unity.

(i) If $p$ is odd or $\sqrt{-1} \in k$, then $G_k(p)$ is torsion free and has the unique extraction of roots property. Furthermore, for every finitely generated subgroup $H$ of $G_k(p)$ and every open subgroup $U$ of $H$, we have $d(U) \geq d(H)$.

(ii) If $p = 2$, then every non-trivial finite subgroup of $G_k(p)$ is cyclic of order two.

Proof. (i) follows from the results of Section 2. For the proof of (ii), suppose that $H$ is a non-trivial finite subgroup of $G_k(2)$ with fixed field $F$. It follows from Theorem 7.2 that $k(2) = F(\sqrt{-1})$. Hence, $H$ is a cyclic group of order two. 

Before we turn to the more general context of 1-smooth cyclotomic pro-$p$ pairs, we give the proof of Theorem 1.7.

Proof of Theorem 1.7. First suppose that $p$ is an odd prime. Since every pro-$p$ subgroup of the absolute Galois group $G_k$ is contained in a $p$-Sylow subgroup, it suffices to prove that every $p$-Sylow subgroup $P$ of $G_k$ is strongly Frattini-resistant.

If $k$ is of characteristic $p$, then $P$ is a free pro-$p$ group by [26, Theorem 6.1.4]. Hence, $P$ is strongly Frattini-resistant by Theorem 6.1. If $k$ is of characteristic different than $p$, then the fixed field $F$ of $P$ contains a primitive $p$th root of unity and $P = G_F(p)$; in this case, the claim follows from Theorem 7.1. For $p = 2$, the result follows from Theorem 7.2. 

7.1. 1-smooth pro-$p$ groups. Following [10] and [13], we call a pair $\mathcal{G} = (G, \theta)$ consisting of a pro-$p$ group $G$ and a homomorphism $\theta : G \to 1+p\mathbb{Z}_p$ (where $1+p\mathbb{Z}_p$ is the group of 1-units of $\mathbb{Z}_p$) a cyclotomic pro-$p$ pair.

Given a cyclotomic pro-$p$ pair $\mathcal{G} = (G, \theta)$, let $Z_p(1)$ be the $G$-module with underlying abelian group $Z_p$ and $G$ action defined by $g \cdot v = \theta(g)v$ for all $g \in G$ and $v \in Z_p$. The pair $\mathcal{G}$ is said to be 1-smooth (or 1-cyclotomic; see [4], [13], [30], [31] and [32]), if for every open subgroup $U$ of $G$ and every $n \in \mathbb{N}$, the quotient map $Z_p(1)/p^nZ_p(1) \to Z_p(1)/pZ_p(1)$ (considered as a $U$-module homomorphism) induces an epimorphism

$$H^1(U, Z_p(1)/p^nZ_p(1)) \to H^1(U, Z_p(1)/pZ_p(1)).$$

Let $k$ be a field containing a primitive $p$th root of unity, and let $\mu_{p^\infty}$ be the group of all roots of unity in $k(p)$ of order a power of $p$. The cyclotomic pro-$p$ character $\theta_{k,p} : G_k(p) \to 1+p\mathbb{Z}_p$ is defined by $\sigma(\zeta) = \zeta^{\theta_{k,p}(\sigma)}$ for all $\sigma \in G_k(p)$ and
Let \( G \) be a pro-\( p \) group. Suppose that the abelian group \( \mathbb{Z}_p/p^2\mathbb{Z}_p \) can be endowed with a structure of a (topological) \( G \)-module in such a way that for every open subgroup \( U \) of \( G \), the quotient homomorphism \( \pi : \mathbb{Z}_p/p^2\mathbb{Z}_p \to \mathbb{Z}_p/p\mathbb{Z}_p \) induces an epimorphism

\[
\pi^* : H^1(U, \mathbb{Z}_p/p^2\mathbb{Z}_p) \to H^1(U, \mathbb{Z}_p/p\mathbb{Z}_p).
\]

Then, the following assertions hold:

(i) If \( \mathbb{Z}_p/p^2\mathbb{Z}_p \) is the trivial \( G \)-module, then \( G \) is strongly commutator-resistant.

(ii) Some maximal subgroup of \( G \) is strongly commutator-resistant.

(iii) If \( p \) is an odd prime, then \( G \) is strongly Frattini-resistant.

**Proof.** If \( G \) acts trivially on \( \mathbb{Z}_p/p^2\mathbb{Z}_p \), then every open subgroup of \( G \) satisfies the extension of homomorphisms property of Proposition 4.13. Hence, (i) follows from Corollary 4.14. In general (for an arbitrary action), we obtain a continuous homomorphism from \( G \) to \( \text{Aut}(\mathbb{Z}_p/p^2\mathbb{Z}_p) \) with kernel \( M \) such that \( |G : M| \leq p \). Now \( M \) acts trivially on \( \mathbb{Z}_p/p^2\mathbb{Z}_p \), and it follows from (i) that \( M \) is strongly commutator-resistant, whence (ii).

For the proof of (iii), let \( U \) be an open subgroup of \( G \), and let \( M \) be a maximal subgroup of \( U \). By Proposition 4.14, it suffices to prove that \((U, M, \Phi(M))\) is a hierarchical triple. Upon identifying \( U/M \) with \( \mathbb{Z}_p/p\mathbb{Z}_p \), we may consider the natural projection \( \varphi : U \to U/M \) as an element of \( H^1(U, \mathbb{Z}_p/p\mathbb{Z}_p) = \text{Hom}(U, \mathbb{Z}_p/p\mathbb{Z}_p) \).

The surjectivity of the homomorphism \( H^1(U, \mathbb{Z}_p/p^2\mathbb{Z}_p) \to H^1(U, \mathbb{Z}_p/p\mathbb{Z}_p) \) implies the existence of a derivation (1-cocycle) \( d : U \to \mathbb{Z}_p/p^2\mathbb{Z}_p \) such that \( \pi \circ d = \varphi \). For every \( x \in M \), we have \( \pi(d(x)) = \varphi(x) = 0 \); so, \( d(M) \leq p\mathbb{Z}_p/p^2\mathbb{Z}_p \). Since \( p\mathbb{Z}_p/p^2\mathbb{Z}_p \) is necessarily a trivial \( G \)-module, the restriction of \( d \) to \( M \) is a homomorphism. It follows that \( d(\Phi(M)) = 0 \).

Let \( x \in U \setminus M \). Then \( \varphi(x) \neq 0 \), and thus \( d(x) \notin p\mathbb{Z}_p/p^2\mathbb{Z}_p \). Now \( x \cdot d(x) = \alpha d(x) \) for some \( \alpha = 1 + lp \) with \( 0 \leq l \leq p - 1 \), and

\[
d(x^p) = (1 + \alpha + \ldots + \alpha^{p-1})d(x).
\]

If \( \alpha = 1 \), then \( d(x^p) = pd(x) \neq 0 \), and thus \( x^p \notin \Phi(M) \); otherwise

\[
1 + \alpha + \ldots + \alpha^{p-1} = \frac{\alpha^p - 1}{\alpha - 1} = \frac{\sum_{i=1}^{p-1} (lp)^i}{lp} = p + p^2 \left[ \frac{p-1}{2} l + \sum_{i=3}^{p} \left( \binom{p}{i} \right) p^{i-3}l^{i-1} \right].
\]

Since \( p \) is assumed to be an odd prime, it follows that \( p^2 \) does not divide \( \frac{\alpha^p - 1}{\alpha - 1} \). Therefore, \( d(x^p) \neq 0 \) and \( x^p \notin \Phi(M) \). \( \square \)

**Proof of Theorem 7.14.** For \( p \) odd, this follows from Lemma 7.13 (iii). If \( p = 2 \) and \( \text{Im}(\theta) \leq 1 + 4\mathbb{Z}_2 \), then \( G \) acts trivially on \( \mathbb{Z}_2(1)/4\mathbb{Z}_2(1) \) and by Lemma 7.13 \( G \) is commutator-resistant. \( \square \)
The following two corollaries, in particular, subsume Theorem 1.8 and Theorem 1.9. They were recently proved by Quadrelli [31], [32].

**Corollary 7.6.** Let $G$ be a $p$-adic analytic pro-$p$ group. Then, there exists a homomorphism $\theta : G \to 1 + p\mathbb{Z}_p$ (with $\text{Im}(\theta) \leq 1 + 4\mathbb{Z}_2$ if $p = 2$) such that $(G, \theta)$ is a $1$-smooth cyclotomic pro-$p$ pair if and only if $G$ is one of the groups listed in Theorem 1.2.

**Proof.** This follows from Theorem 1.11 and the well-known fact that the groups listed in Theorem 1.2 can be realized as maximal pro-$p$ Galois groups. \(\square\)

**Corollary 7.7.** Let $G$ be a solvable pro-$p$ group. Then, there exists a homomorphism $\theta : G \to 1 + p\mathbb{Z}_p$ (with $\text{Im}(\theta) \leq 1 + 4\mathbb{Z}_2$ if $p = 2$) such that $(G, \theta)$ is a $1$-smooth cyclotomic pro-$p$ pair if and only if $G$ is free abelian or it is a semidirect product $(x) \ltimes A$, where $(x) \cong \mathbb{Z}_p$, $A$ is a free abelian pro-$p$ group and $x$ acts on $A$ as scalar multiplication by $1 + p^s$ with $s \geq 1$ if $p$ is odd, and $s \geq 2$ if $p = 2$.

**Proof.** This follows from Theorem 1.11 and Theorem 1.3. \(\square\)

By Theorem 1.4, for a field $k$ containing a primitive $p$th root of unity (and also $\sqrt{-1} \in k$ if $p = 2$), $G_k(p)$ contains a unique normal abelian subgroup $N$. Moreover, it was proved in [30] that

$$N = \{h \in \ker \theta_{k,p} \mid ghg^{-1} = h^{\theta_{k,p}(g)} \text{ for all } g \in G_k(p)\},$$

where $\theta_{k,p}$ is the cyclotomic pro-$p$ character defined above.

Given a pro-$p$ group $G$ and a homomorphism $\theta : G \to 1 + p\mathbb{Z}_p$, let

$$Z_\theta(G) := \{h \in \ker \theta \mid ghg^{-1} = h^\theta(g) \text{ for all } g \in G\}.$$

Note that $Z_\theta(G)$ is a normal abelian subgroup of $G$.

**Proposition 7.8.** Let $G$ be a Frattini-injective pro-$p$ group containing a non-trivial normal abelian subgroup. The following assertions hold:

(i) There exists a unique homomorphism $\theta : G \to 1 + p\mathbb{Z}_p$ such that $Z_\theta(G)$ is the (unique) maximal abelian normal subgroup of $G$.

(ii) If $G$ is not pro-cyclic and for some $\psi : G \to 1 + p\mathbb{Z}_p$, $G = (G, \psi)$ is a 1-smooth cyclotomic pro-$p$ pair (with $\text{Im}(\psi) \leq 1 + 4\mathbb{Z}_2$ if $p = 2$), then $\psi = \theta$.

**Proof.** (i) The uniqueness part is obvious. Let $N$ be the unique maximal normal abelian subgroup of $G$, whose existence is guaranteed by Theorem 1.4. We define $\theta : G \to 1 + p\mathbb{Z}_p$ as follows: if $N = Z(G)$, then take $\theta$ to be the trivial homomorphism (i.e., $\theta(g) = 1$ for all $g \in G$); otherwise, by Theorem 1.4 (iv), $G = \langle x \rangle \ltimes C_G(N)$ for some suitable $x \in G$, and we let $\theta(C_G(N)) = 1$ and $xax^{-1} = a^{\theta(x)}$ for any (and hence every) $a \in N$. It readily follows from Theorem 1.4 that indeed $N = Z_\theta(G)$.

(ii) If $G = \langle x \rangle \ltimes A$ is metabelian (decomposed in a semidirect product as in Theorem 1.3), then $A$ is the isolator of $[G, G]$, and consequently, $\psi(A) = 1$. 

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Moreover, for every $n \in \mathbb{N}$, $a \in A$ and a derivation $d : G \to \mathbb{Z}_p(1)/p^n\mathbb{Z}_p(1)$, we have $\theta(x)d(a) = d(a^{\theta(x)}) = d(axa^{-1}) = d(a^{\psi(x)}) = \psi(x)d(a)$. Therefore, the surjectivity of the cohomology maps $H^1(U, \mathbb{Z}_p(1)/p^n\mathbb{Z}_p(1)) \to H^1(U, \mathbb{Z}_p(1)/p\mathbb{Z}_p(1))$ implies that $\psi = \theta$. By a similar argument, if $G$ is abelian (but not pro-cyclic), $\psi$ must be the trivial homomorphism.

In general, denoting by $N$ the unique maximal normal subgroup of $G$, for an arbitrary element $x \in G \setminus N$, the group $H := \langle x, N \rangle$ is metabelian and $(H, \psi_H)$ is a 1-smooth cyclotomic pro-$p$ pair. It follows from what has been already proved that $\psi(N) = 1$ and $\psi(x) = \theta(x)$. \qed

8. $p$-POWER-INJECTIVE PRO-$p$ GROUPS

**Definition 8.1.** We say that a pro-$p$ group $G$ is strongly $p$-power-injective ($p$-power-injective) if for all (finitely generated) subgroups $H$ and $K$ of $G$,

$$H^p = K^p \implies H = K.$$  

We call a pro-$p$ group $G$ strongly $p$-power-resistant ($p$-power-resistant) if for every (finitely generated) subgroup $H$ of $G$, the triple $(G, H, H^p)$ is hierarchical.

In our opinion, these are concepts deserving careful investigation. However, in this brief final section, we do little more than record several statements that follow from (or could be proved in a similar manner as) the main results of this paper.

**Proposition 8.2.** A pro-$p$ group $G$ is strongly $p$-power-resistant ($p$-power-resistant) if and only if for all (finitely generated) subgroups $H$ and $K$ of $G$,

$$H \leq K \iff H^p \leq K^p$$

Consequently, every (strongly) $p$-power-resistant pro-$p$ group is (strongly) $p$-power-injective.

**Proposition 8.3.** Let $G$ be a pro-$p$ group. If $(G, U, U^p)$ is a hierarchical triple for every open subgroup $U$ of $G$, then $G$ is strongly $p$-power-resistant. In particular, a finitely generated $p$-power-resistant pro-$p$ group is strongly $p$-power-resistant.

**Proposition 8.4.** A (strongly) Frattini-resistant pro-$p$ group is (strongly) $p$-power-resistant.

**Proof.** This follows from the fact that $G^p \leq \Phi(G)$ for every pro-$p$ group $G$. \qed

For a pro-2 group $G$, we have $\Phi(G) = G^2$. Hence, 2-power-injectivity (2-power-resistance) is the same as Frattini-injectivity (Frattini-resistance).

**Corollary 8.5.** (1) Every free pro-$p$ group is strongly $p$-power-resistant.

(2) Let $G$ be a Demushkin pro-$p$ group. Then, the following assertions hold:

(i) If $q(G) \neq p$, or $q(G) = p$ and $p$ is odd, then $G$ is strongly $p$-power-resistant.
(ii) If \( q(G) = 2 \) and \( d(G) > 2 \), then \( G \) is \( p \)-power-injective, but not \( p \)-power-resistant.

(iii) If \( q(G) = 2 \) and \( d(G) = 2 \), then \( G \) is not \( p \)-power-injective.

(3) Let \( k \) be a field that contains a primitive \( p \)-th root of unity. If \( p = 2 \), in addition, assume that \( \sqrt{-1} \in k \). Then \( G_k(p) \) is strongly \( p \)-power-resistant.

In contrast to Frattini-resistance, \( p \)-adic analytic \( p \)-power-resistant pro-\( p \) groups are ubiquitous.

**Proposition 8.6.** Every torsion free \( p \)-adic analytic pro-\( p \) group of dimension less than \( p \) is (strongly) \( p \)-power-resistant.

**Proof.** Let \( G \) be a torsion free \( p \)-adic analytic pro-\( p \) group of dimension less than \( p \). By [16, Theorem A], every closed subgroup of \( G \) is saturable. Thus to every subgroup \( H \) of \( G \) we can associate a saturable \( \mathbb{Z}_p \)-Lie algebra \( L_H \); moreover, \( L_{H^p} = pL_H \). Now let \( H \) and \( K \) be subgroups of \( G \) such that \( H^p \leq K^p \). Then

\[
H^p \leq K^p \implies pL_H \leq pL_K \implies L_H \leq L_K \implies H \leq K.
\]

Hence, by Proposition 8.2 \( G \) is a (strongly) \( p \)-power-resistant pro-\( p \) group. \( \square \)

**Corollary 8.7.** Suppose that \( p \geq 5 \). Then there are uncountably many pairwise non-commensurable \( p \)-power-resistant \( p \)-adic analytic pro-\( p \) groups.

**Proof.** This follows from [36, Theorem 1.1]. \( \square \)

#### 9. Final Remarks

In this final section, we formulate several problems that we hope will stimulate further research on Frattini-injective pro-\( p \) groups.

**Problem 1.** Is every finitely generated Frattini-injective pro-\( p \) group strongly Frattini-injective?

Theorem 6.6 provides examples of Frattini-injective pro-2 groups that are not Frattini-resistant. We do not know any such examples for \( p \) odd.

**Problem 2.** For \( p \) an odd prime, find examples of Frattini-injective pro-\( p \) groups that are not Frattini-resistant, or prove that such groups do not exist.

In [40], Ware proved that for \( p \) odd the maximal pro-\( p \) Galois group of a field containing a primitive \( p \)-th root of unity is either metabelian or it contains a non-abelian free pro-\( p \) subgroup.

**Problem 3.** Does every non-metabelian Frattini-injective (Frattini-resistant) pro-\( p \) group contain a non-abelian free pro-\( p \) subgroup?

Following [43], we call a pro-\( p \) group \( G \) absolutely torsion-free if \( H^{ab} \) is torsion-free for every subgroup \( H \) of \( G \). Let \( G = (G, \theta) \) be a 1-smooth cyclotomic pro-\( p \) pair (with \( \text{Im}(\theta) \leq 1 + 4\mathbb{Z}_2 \) if \( p = 2 \)), and suppose that \( G \) has a non-trivial center.
It follows from Proposition 7.8 that $\theta$ is the trivial homomorphism. Moreover, the proof of Proposition 4.13 can be adapted to show that $G$ is absolutely torsion-free.

**Problem 4.** Is every Frattini-injective (Frattini-resistant) pro-$p$ group with non-trivial center absolutely torsion-free?

In what follows assume that $p$ is odd.

**Problem 5.** Let $G$ be a strongly Frattini-resistant pro-$p$ group. Does there necessarily exist a homomorphism $\theta : G \to 1 + p\mathbb{Z}_p$ such that $(G, \theta)$ is a $1$-smooth cyclotomic pro-$p$ pair? (Note that if $G$ contains a non-trivial abelian normal subgroup, then Corollary 7.8 gives a description of the only possible candidate for $\theta$.)

**Problem 6.** Is every Frattini-resistant pro-$p$ group Bloch-Kato?

In [10], Efrat introduced the class $C$ of cyclotomic pro-$p$ pairs of elementary type. This class consists of all finitely generated cyclotomic pro-$p$ pairs which can be constructed from $\mathbb{Z}_p$ and Demushkin groups using free pro-$p$ products and certain semidirect products, known also as fibre products (cf. [10, 3]; see also [30, 7.5]). Given a field $k$ that contains a primitive $p$th root of unity, Efrat conjectured that if $G_k(p)$ is finitely generated, then the cyclotomic pro-$p$ pair $(G_k(p), \theta_k,p)$ is of elementary type; this is the so-called elementary type conjecture (cf. [8], [9] and [10]; see also [30, 7.5]).

It follows from [30, Theorem 1.4] and Theorem 1.11 that if $(G, \theta)$ is a cyclotomic pro-$p$ pair of elementary type, then $G$ is a strongly Frattini resistant pro-$p$ group.

**Problem 7.** Is there a finitely generated (strongly) Frattini resistant pro-$p$ group which is not of elementary type.

A negative answer to the above question would settle the elementary type conjecture. On the other hand, if there exist counter examples, then they will likely be pro-$p$ groups with exotic properties.

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