Primordial magnetic seed fields from extra dimensions

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Abstract

Dynamical extra dimensions break the conformal invariance of Maxwell’s equations in four dimensions. A higher dimensional background with \( n \) contracting extra dimensions and four expanding dimensions is matched to an effectively four dimensional standard radiation dominated universe. The resulting spectrum for the magnetic field is calculated taking into account also the momenta along the extra dimensions. Imposing constraints from observations an upper limit on the strength of magnetic seed fields is found. Depending on the number of extra dimensions, cosmologically interesting magnetic fields can be created.

1 Introduction

Magnetic fields are ubiquitous in the universe. Most galaxies, cluster of galaxies and even the Coma supercluster and radio galaxies at redshift \( z \simeq 2 \) have been found to be endowed with magnetic fields (for reviews see[1]-[3]).

The average field strength of the interstellar magnetic field in our Galaxy has been observed to be 3 – 4\( \mu \)G. Spiral galaxies in general seem to have magnetic fields with strength of the order of 10\( \mu \)G. The structure of these magnetic fields is determined by a large scale component with a coherence length of the order of the size of the visible disk and a small-scale component of tangled fields. There are a few spiral galaxies with exceptionally strong magnetic fields of the order of 50\( \mu \)G, which also have a very high star formation rate [4, 2].

Magnetic fields associated with elliptical galaxies have field strengths comparable to those observed in spiral galaxies. However, their structure seems to be quite distinct from that found for magnetic fields in spiral galaxies. The coherence length is much smaller than the corresponding galactic scales and the structure appears to be random.

In clusters of galaxies, magnetic fields of strength of the order of upto a few \( \mu \)G are found in the intracluster medium. The cluster center regions indicate strong magnetic fields with typical field strengths of the order of 10 – 30\( \mu \)G and in exceptional cases upto 70\( \mu \)G [5, 6]. The coherence length of the magnetic fields is of the order of the scale of the cluster galaxies.

There is also evidence for the existence of magnetic fields in structures on supercluster scale. The Coma-Abell 1367 supercluster is observed to have a magnetic field of strength 0.2 – 0.6\( \mu \)G.

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Finally, observations indicate the existence of magnetic fields in redshift $z \simeq 2$ radio galaxies.

There is no direct observational evidence of magnetic fields that are not associated with any collapsing or virialized structure. However, it is possible to put upper bounds on the strength of such cosmological magnetic fields from anisotropy measurements of the cosmic microwave background and from the abundances of light elements predicted by standard big bang nucleosynthesis.

To explain the widespread existence of large scale magnetic fields in the universe it is commonly assumed that a tiny magnetic seed field at the epoch of galaxy formation is amplified by a dynamo mechanism to its present strength of a few microgauss in our Galaxy. The dynamo amplifies an initial seed magnetic field exponentially. The amplification factor depends on the growth rate for the dominant mode of the dynamo and the amount of time during which the dynamo operates. In a flat universe with no cosmological constant the initial seed magnetic field needs to have at least a field strength of the order of $10^{-20}$ G to explain the current $\mu$G galactic field today. However, as it was pointed out in [10], this bound depends on the cosmological model. In a flat universe with non-vanishing cosmological constant the lower limit on the required initial magnetic field strength can be lowered significantly. For reasonable cosmological parameter the required strength of the initial seed magnetic field is of the order of $10^{-30}$ G.

There are different proposals for the origin of the magnetic seed field. A class of proposed models involves the creation of magnetic seed fields during an inflationary stage of the very early universe. In order to produce a magnetic seed field of significant strength the conformal invariance of Maxwell’s equations has to be broken, for example, by gravitational couplings of the photon.

The conformal invariance of Maxwell’s equations in four dimensions can also be broken if an embedding into a higher dimensional space-time with time-varying extra spatial dimensions is considered. In relation with the creation of seed magnetic fields this was first investigated in [13]. It is assumed that the $D$ dimensional space-time can be written as a direct product of a three dimensional space and an $n$ dimensional space. Vacuum space-times of this type are provided by the Kasner solutions, which in general admit two classes of solutions: either expanding three dimensions and collapsing extra dimensions or vice versa. The higher dimensional background with dynamical extra dimensions is matched to a standard four dimensional radiation dominated universe with static extra dimensions. In [13] it was found that magnetic fields of cosmologically interesting strength can be generated only in the case of contracting three dimensions and growing extra dimensions. The novel feature of the model under consideration here is that momenta along the extra dimensions are also taken into account. The final spectrum is obtained by integrating over these internal momenta. This leads to the generation of magnetic seed fields of cosmologically interesting strength in the case of expanding three dimensions and contracting extra dimensions. Imposing bounds from observations an upper bound on the strength of the magnetic field can be found.

Models with extra dimensions arise naturally in string/ M-theory which also led to the possibility of large extra dimensions [16]. In higher dimensional gravity the four dimensional Planck scale $M_4$ is no longer fundamental, instead the higher dimensional Planck scale $M_D$ becomes the fundamental scale. With the assumption that the $n$ extra dimensions have a characteristic size $R$, using Gauss’ law, the $D$-dimensional and the four-dimensional Planck masses $M_4$ and $M_D$, respectively, are related by [17]

$$M_4^2 = R^n M_D^{n+2}.$$ (1.1)
In the case of an expanding, external space and a contracting, internal space the exponents \( \lambda \) are of the form \( (2.5) \), where \( \partial_\eta \). The background space-time is assumed to be homogeneous and anisotropic with a line element,

\[
ds^2 = a^2(\eta) \left[ d\eta^2 - \delta_{ij} dx^i dx^j \right] - b^2(\eta) \delta_{AB} dy^A dy^B,
\]

where \( i, j = 1, ..., 3 \) and \( A, B = 4, ..., 3 + n, n \geq 1 \). \( a(\eta) \) and \( b(\eta) \) are the scale factor of the external, 3-dimensional space and the internal, \( n \)-dimensional space, respectively.

It is assumed that for \( \eta < -\eta_1 \) both scale-factors are functions of time. At \( \eta = -\eta_1 \) this is matched to a radiation dominated four dimensional flat universe with static extra dimensions, \( b(\eta) = \text{const}. \). The solutions are given by

\[
a(\eta) = a_1 \left( -\frac{\eta}{\eta_1} \right)^{\sigma}, \quad b(\eta) = b_1 \left( -\frac{\eta}{\eta_1} \right)^{\lambda}, \quad \text{for} \quad \eta < -\eta_1
\]

\[
a(\eta) = a_1 \left( \frac{\eta + 2\eta_1}{\eta_1} \right)^{\sigma}, \quad b(\eta) = b_1, \quad \text{for} \quad \eta \geq -\eta_1
\]

In the following we set \( a_1 = 1 = b_1 \).

For \( \eta < -\eta_1 \) the solution is given by the vacuum Kasner metric, which determines the exponents \( \sigma \) and \( \lambda \) as functions of the number of extra dimensions \( n \). These are related to the Kasner exponents \( \alpha_E \) and \( \alpha_I \), satisfying the Kasner conditions \( 3\alpha_E + n\alpha_I = 1 \) and \( 3\alpha_E^2 + n\alpha_I^2 = 1 \), by

\[
\sigma = \frac{\alpha_E}{1 - \alpha_E}, \quad \lambda = \frac{\alpha_I}{1 - \alpha_E}.
\]

(2.5)

In the case of an expanding, external space and a contracting, internal space the exponents \( \sigma \) and \( \lambda \) are of the form \( (2.6) \),

\[
\sigma = -\frac{1}{2} \left( \sqrt{\frac{3n}{n + 2}} - 1 \right), \quad \lambda = \sqrt{\frac{3}{n(n + 2)}}.
\]

(2.6)

Maxwell’s equations in \( D \) dimensions are given by \( \nabla_A F^{AB} = 0 \) with \( F_{AB} = \nabla_{[A} A_{B]}, \tilde{A}, \tilde{B} = 0, ..., n + 3 \). Here the interest is the electromagnetic field in the (3+1)-dimensional space-time. Thus it is assumed that \( A_i = A_i(x^i, y^B, \eta) \) and \( A_B = 0 \). Using the radiation gauge \( A_0 = 0, \nabla_i A^i = 0 \), Maxwell’s equations imply

\[
-\frac{1}{b^n} \partial_0 [b^n \partial_0 A_i] + \sum_{j=1}^{3} \partial_j \partial_j A_i + \left( \frac{a}{b} \right)^2 \sum_{B=4}^{3+n} \partial_B \partial_B A_i = 0.
\]

(2.7)

where \( \partial_0 \equiv \partial / \partial \eta \), \( \partial_i \equiv \partial / \partial x^i \) and \( \partial_B \equiv \partial / \partial y^B \).

Furthermore, the canonical field \( \Psi_i = b^2 A_i \) is introduced and the following expansion is used

\[
\Psi_i(\eta, x^i, y^A) = \int \frac{d^3 k d^n q}{(2\pi)^{3+n}} \sum_\alpha e^{in} \left[ a^\alpha_{l, \alpha} \Psi_l(\eta) e^{iA} + a^{\dagger}_{-l, \alpha} \Psi^*_l(\eta) e^{-iA} \right],
\]

(2.8)
where $l^\mu$ is a $(3+n)-$vector with components $l^i \equiv k^i$, $l^A \equiv q^A$. Moreover, $1 \cdot X = k \cdot x + q \cdot y$. $\alpha$ runs over the polarizations. In the background (2.3), $\eta < -\eta_1$, this results in the mode equation

$$\Psi_l'' + \left[ k^2 + \left( -\frac{\eta}{\eta_1} \right)^2 q^2 + \frac{1}{4 \eta^2} \right] \Psi_l = 0, \tag{2.9}$$

where $' \equiv \frac{\partial}{\partial \eta}$ and $N \equiv \frac{1}{4} (n\lambda - 1)^2 - \frac{1}{4}$. Furthermore, $\beta \equiv \sigma - \lambda$. $\beta < 0$ since only solutions with contracting extra dimensions will be discussed. $-1 \leq \beta < -1/(1 + \sqrt{3})$, where the lower boundary corresponds to $n = 1$ and the upper bound gives the value for large $n$.

Equation (2.9) can be solved in a closed form for one extra dimension $n = 1$. In general, for $n > 1$, to our knowledge, apart from the case $n = 6$, there are no solutions in closed form. However, it is possible to find approximate solutions.

- For $n = 1$ and $\eta < -\eta_1$ the equation for $\Psi_l$ (cf. equation (2.9)) reads

$$\Psi_l'' + \left[ k^2 + \left( -\frac{\eta}{\eta_1} \right)^2 q^2 + \frac{1}{4 \eta^2} \right] \Psi_l = 0, \tag{2.10}$$

which is solved by

$$\Psi_l = \sqrt{\pi} e^{\frac{\pi}{2} \eta \eta_1} (-k\eta)^\frac{1}{2} H^{(2)}_{\mu \eta_1} (-k\eta), \tag{2.11}$$

satisfying the Wronskian condition $\Psi_l^* \Psi_l - \Psi_l^* \Psi_l^* = i$ and $H^{(2)}_{\mu}(z)$ is the Hankel function of the second kind.

- For $n > 1$ and $\eta < -\eta_1$, in general approximate solutions can be found to the mode equation (2.9). In this case, there is a natural distinction into two cases [14, 15].

  i.) For $\left( -\frac{\eta}{\eta_1} \right)^2 q^2 < k^2$, or $\omega_q < \omega_k$ in terms of the physical frequencies $\omega_k = k/a(\eta)$ and $\omega_q = q/b(\eta)$, equation (2.9) becomes approximately,

$$\Psi_l'' + \left[ k^2 - \frac{N}{\eta^2} \right] \Psi_l = 0, \tag{2.12}$$

which is solved by

$$\Psi_l = \sqrt{\pi} \sqrt{-k\eta} H^{(2)}_{\mu} (-k\eta), \tag{2.13}$$

where $H^{(2)}_{\mu}$ is the Hankel function of the second kind and $\mu^2 \equiv \frac{1}{4} + N \Rightarrow \mu = \frac{1}{2} (n\lambda - 1)$. The mode functions satisfy the Wronskian condition.

ii.) For $\left( -\frac{\eta}{\eta_1} \right)^2 q^2 > k^2$, or $\omega_q > \omega_k$, equation (2.9) can be approximated by,

$$\Psi_l'' + \left( -\frac{\eta}{\eta_1} \right)^2 q^2 \Psi_l = 0, \tag{2.14}$$

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which is solved by

\[ \Psi_l = \frac{\sqrt{\pi}}{2} (-\kappa \eta)^{\frac{1}{2}} \mathcal{H}_{\mu}^{(2)} \left( (-q \eta) \kappa \left( -\frac{\eta}{\eta_1} \right)^\beta \right), \]  

(2.15)

where \( \kappa \equiv \frac{1}{\beta+1} \) and \( \mu = \frac{1}{2}(n\lambda - 1) \).

The case \( q = 0 \) is covered by the first case, \( \left( -\frac{\eta}{\eta_1} \right)^{2\beta} q^2 < k^2 \), thus the solutions are not written explicitly.

In the background (2.4), for \( \eta \geq -\eta_1 \), the mode equation is given by

\[ \Psi''_l + \left[ k^2 + \left( \frac{\eta + 2\eta_1}{\eta_1} \right)^2 q^2 \right] \Psi_l = 0. \]  

(2.16)

Introducing \( z \equiv \left( \frac{2q}{\eta_1} \right)^{\frac{1}{2}} (\eta + 2\eta_1) \) and \( \alpha \equiv -\frac{\eta k^2}{2q} \) this can be transformed into the equation for parabolic cylinder functions [10],

\[ \frac{d^2 \Psi_l}{dz^2} + \left[ \frac{z^2}{4} - \alpha \right] \Psi_l = 0, \]  

(2.17)

which is solved by

\[ \Psi_l = \frac{1}{\sqrt{2}} \left( \frac{\eta_1}{2q} \right)^{\frac{1}{4}} \left[ c_- E(\alpha, z) + c_+ E^*(\alpha, z) \right], \]  

(2.18)

where the Wronskian condition on the mode functions was applied and the normalization for the Bogoliubov coefficients \( |c_+|^2 - |c_-|^2 = 1 \) was used. Using the approximations ((19.24) of [20]) gives expressions for \( \Psi_l \) and \( \Psi'_l \) at \( \eta = -\eta_1 \).

i.) Namely, for \( \omega_q/\omega_k < 1 \), it is found that

\[ \Psi_l(-\eta_1) \sim \frac{1}{\sqrt{2k}} \left[ c_- e^{ik\eta_1 + i\frac{\pi}{4}} + c_+ e^{-ik\eta_1 - i\frac{\pi}{4}} \right] \]

\[ \Psi'_l(-\eta_1) \sim -\sqrt{\frac{k}{2}} \left[ c_- e^{i\eta_1 - i\frac{\pi}{4}} + c_+ e^{-i\eta_1 + i\frac{\pi}{4}} \right]. \]  

(2.19)

ii.) For \( \omega_q/\omega_k > 1 \) it follows that

\[ \Psi_l(-\eta_1) \sim \frac{1}{\sqrt{2q}} \left[ c_- e^{i\frac{q\eta_1}{2} + i\frac{\pi}{4}} + c_+ e^{-i\frac{q\eta_1}{2} - i\frac{\pi}{4}} \right] \]

\[ \Psi'_l(-\eta_1) \sim -\sqrt{\frac{q}{2}} \left[ c_- e^{i\frac{q\eta_1}{2} - i\frac{\pi}{4}} + c_+ e^{-i\frac{q\eta_1}{2} + i\frac{\pi}{4}} \right]. \]  

(2.20)
The total magnetic energy density is given by \[ 15 \]

\[ \rho = 2 \frac{R^n}{(2\pi)^{n+3}} \int \left[ \left( \frac{k}{a} \right)^2 + \left( \frac{q}{a} \right)^2 \right] \frac{1}{c_-} |c_-|^2 dV, \]  
(2.21)

where, assuming that the volume consists of two spheres, \[ dV = \frac{1}{a^3} \frac{2}{\Gamma(\frac{3}{2})} k^2 dk \wedge \frac{2}{\Gamma(\frac{3}{2})} q^{n-1} dq. \] At \( \eta = -\eta_1 \) the comoving wavenumbers \( k \) and \( q \) are equal to the physical momenta, since \( a_1 = 1 = b_1 \).

The spectral energy density \( \rho(\omega_k) = d\rho/d\log\omega_k \) is then given by

\[ \rho(\omega_k) = 16 \frac{R^n}{(2\pi)^{n+3}} \frac{\pi^{1+\frac{n}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} \omega_k^{4+n} \int dY [1 + Y^2]^{\frac{1}{2}} Y^{n-1} |c_-|^2, \]  
(2.22)

where \( Y \equiv \frac{\omega}{\omega_k} \), and \( \omega_k = \frac{k}{a}, \omega_q = \frac{q}{a} \).

During most of its history the universe had a very high conductivity, implying that a primordial magnetic field evolves while its flux is conserved. This makes the dimensionless ratio \( r \equiv \rho_B/\rho_\gamma \) approximately constant \[ 11 \], where \( \rho_B \) is the magnetic field energy density and \( \rho_\gamma \) is the energy density of the background radiation. Thus \( r \) is a good measure of the strength of a cosmological magnetic field. Furthermore, \( r = \Omega_{em}/\Omega_\gamma \), where \( \Omega = \rho/\rho_c \) with \( \rho_c \) the critical energy density, and \( \Omega_\gamma = (H_1/H)^2 (a_1/a)^3 \). Thus expressing the critical energy density in terms of the \( D \)-dimensional Planck mass \( M_D \), \( \rho_c = \frac{3}{\pi^2} R^n M_D^{n+2} H^2 \), leads to

\[ r(\omega_k) = \frac{16}{3} \frac{8\pi}{(2\pi)^{n+3}} \frac{\pi^{1+\frac{n}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} \omega_k a^{-n} \left( \frac{H_1}{M_D} \right)^{n+2} \left( \frac{\omega_k}{\omega_1} \right)^{4+n} \int_0^{Y_{max}} dY Y^{n-1} [1 + Y^2]^{\frac{1}{2}} |c_-|^2, \]  
(2.23)

where \( \omega_1 \equiv \frac{k_1}{a} \) and the maximal comoving wave number \( k_1 \sim H_1 \). Furthermore, an upper cut-off \( Y_{max} = \omega_{q_{max}}/\omega_k \) has been introduced. This is justified by the sudden transition approximation, which is used here, since at the transition time, \( \eta = -\eta_1 \), the metric is continuous but not its first derivative. This means that for modes with periods much larger than the duration of the transition phase, the transition phase can be treated as instantaneous. However, without an upper cut-off this type of approximation leads to an ultraviolet divergence \[ 21 \].

For \( q > 0 \) and \( n = 1 \), that is one extra dimension, continuously matching at \( \eta = -\eta_1 \) the solutions \[ 21 \] and \[ 21 \] on superhorizon scales \( k_\eta \ll 1, q_\eta \ll 1 \) leads to the following Bogoliubov coefficients for \( \omega_q/\omega_k < 1 \) and \( \omega_q/\omega_k > 1 \),

\[ c_- e^{ikn} \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{k_\eta} l} \left[ 1 + \frac{1}{2} \ln k_\eta - i k_\eta \ln k_\eta \right] e^{-i\frac{\pi}{4}} \]  
for \( Y < 1 \),  
(2.24)

\[ c_- e^{iqn} \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{q_\eta} l} \left[ 1 + \frac{1}{2} \ln q_\eta - i q_\eta \ln q_\eta \right] e^{-i\frac{\pi}{4}} \]  
for \( Y > 1 \).  
(2.25)

Neglecting subleading terms, it follows that the ratio \( r(\omega_k) \) is given by

\[ r(\omega_k) \sim \frac{1}{3\pi^3} \left( \frac{H_1}{M_4} \right)^3 \left( \frac{M_5}{M_4} \right)^{-3} \left( \frac{\omega_k}{\omega_1} \right)^3 \left( \frac{\omega_k}{\omega_1} \right)^2 \omega_{q_{max}}/\omega_1, \]  
(2.26)

where \( \omega_{q_{max}}(\eta) = \frac{\omega_{q_{max}}}{b} \) and it was assumed that \( \omega_{q_{max}} > \omega_k \).
The case \( Y < 1 \) includes the limit \( q = 0 \). Therefore together with \( \rho_{em}(\omega_k) = 2\pi^2|c_-|c_-(\omega_k)|^2 \) the following expression for the ratio of magnetic to background radiation energy density is obtained for \( q = 0, n = 1, \)

\[
r(\omega_k) \sim \frac{2}{3\pi^2} \left( \frac{H_1}{M_4} \right)^2 \left( \frac{\omega_k}{\omega_1} \right)^3 \left( \ln \frac{\omega_k}{\omega_1} \right)^2.
\]

(2.27)

For more than one extra dimension, \( n > 1 \), the solutions for \( \Psi_1 \) and \( \Psi_1' \) for \( \eta < -\eta_1 \) and \( \eta > -\eta_1 \) are matched at \( \eta = -\eta_1 \) for \( Y < 1 \) and \( Y > 1 \) for superhorizon modes, \( k\eta_1 \ll 1, q\eta_1 \ll 1 \). This leads to the following expressions for the Bogoliubov coefficient \( c_- \)

\[
e^{-ik\eta_1} \sim \frac{2^{\mu-k\frac{3}{2}}}{\sqrt{\pi}} \Gamma(\mu) \Gamma(\mu_1) \left( k\eta_1 \right)^{\frac{1}{2}+\mu} \left[ \left( \mu - \frac{1}{2} \right) \frac{1}{k\eta_1} + i \right] e^{-\frac{i\pi}{4}} \quad \text{for} \ Y < 1, \quad (2.28)
\]

\[
e^{-2\mu\frac{3}{2}} \sim \frac{2^{\mu-k\frac{3}{2}}}{\sqrt{\pi}} \Gamma(\mu) \Gamma(\mu_1) \left( k\eta_1 \right)^{\frac{1}{2}-\mu} \left[ \left( \mu - \frac{1}{2} \right) \frac{1}{q\eta_1} + i \right] e^{-\frac{i\pi}{4}} \quad \text{for} \ Y > 1. \quad (2.29)
\]

Using the expressions for \( |c_-| \) for \( Y < 1 \) and \( Y > 1 \), as provided by equations (2.28) and (2.29) for more than one extra dimension, \( n > 1 \), leads to the ratio of magnetic spectral energy density to background radiation density,

\[
r(\omega_k) \sim N a^{1+2\mu-k-n} \left( \frac{H_1}{M_D} \right)^{n+2} \left( \frac{\omega_{q_{\text{max}}}}{\omega_1} \right)^{n-2\mu\kappa} \left( \frac{\omega_k}{\omega_1} \right)^3.
\]

(2.30)

where

\[
N \equiv \frac{16}{3} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{2^{2\mu-3}}{\pi(n-2\mu\kappa)} \Gamma^2(\mu\kappa) \kappa^{1-2\mu\kappa} \left( \mu - \frac{1}{2} \right)^2
\]

where subleading terms have been omitted and \( \omega_{q_{\text{max}}} > \omega_k \) was assumed. The resulting spectrum is growing in frequency.

The expression for \( q = 0 \) can be derived using the expression for \( c_- \) for \( Y < 1 \) (cf. equation (2.28)). Together with \( \rho_{em} = 2\pi^2|c_-|^2 \) this implies for \( q = 0, n > 1 \)

\[
r(\omega_k) \sim \frac{n^\lambda - 2}{3\pi^2} \Gamma^2 \left( \frac{n\lambda - 1}{2} \right) (2 - n\lambda)^2 \left( \frac{H_1}{M_4} \right)^2 \left( \frac{\omega_k}{\omega_1} \right)^{4-n\lambda}.
\]

(2.31)

Furthermore, \( n\lambda = \sqrt{\frac{3}{n+2}} \). Since \( n\lambda < 4 \) the resulting spectrum for \( r(\omega_k) \) is increasing in frequency.

3 Constraining the model

The expressions for the ratio \( r(\omega_k) \) determining the ratio of the energy density of the magnetic field in comparison with the energy density of the background radiation contain several parameters apart from the physical frequencies \( \omega_k \) and \( \omega_{q_{\text{max}}} \). The free parameters are the Hubble parameter at the time of transition \( H_1 \), the \( D \)-dimensional Planck mass \( M_D \) and the number of extra dimensions \( n \).

There are several constraints from observations. \( r(\omega_k) \) has to be less than one for all frequencies in order not to overclose the universe. For \( r(\omega_k) \) increasing with frequency this implies \( r(\omega_k) < 1 \).
This is the case for the spectra given by equations (2.26) and (2.27) applicable for backgrounds with more than one extra dimension. In the case of one extra dimension the expressions for $r(\omega_k)$ (cf. equations (2.26) and (2.27)) have a maximum at some frequency $\omega_2$. Thus the constraint $r(\omega_2) < 1$ is imposed.

Newtonian gravity has been tested down to length scales of the order of 1 mm [18]. This implies the constraint $\frac{M_8}{M_4} \geq \left(1.616 \times 10^{-32}\right)^{n + \frac{1}{2}}$. Furthermore, with $T_1$ the temperature at the beginning of the radiation epoch, big bang nucleosynthesis requires that $T_1 > 10$ MeV. This imposes a bound on $H_1$ by using $H_1 M_4 = 1.66\omega^\frac{3}{2}(T_1)\left(\frac{T_1}{M_4}\right)^{2}$, where for $T_1 > 300$ GeV the number of effective degrees of freedom is given by $g_*(T_1) = 106.75$ [22], namely, $\log \frac{H_1}{M_4} > -40.94$.

The ratio $r$ calculated at the galactic scale $\omega_G^{-1}$ of order of 1 Mpc determines the strength of the primordial seed magnetic field at the time of galaxy formation. In the standard picture of a galactic magnetic dynamo operating since the time of galaxy formation, a seed magnetic field of at least $B_s \sim 10^{-20}$ G [9], corresponding to $r(\omega_G) > 10^{-37}$, is needed to explain the currently observed galactic magnetic field of a few $\mu$G [9]. However, taking into account a non-vanishing cosmological constant, it was shown in [10] that initial magnetic seed field strengths can be much below $10^{-20}$ G. Thus $r(\omega_G)$ can be as low as $10^{-57}$ and correspondingly the magnetic seed field $B_s \sim 10^{-30}$ G.

In the following, using the constraint $r(\omega_1) < 1$ or $r(\omega_2) < 1$, respectively, the constraint from the size of the extra dimension and from big bang nucleosynthesis an upper limit on the ratio $r(\omega_G)$ and thus the strength of the magnetic seed field strength at the time of galaxy formation is derived.

The strength of the seed field in terms of $r$ is given by $B_s \sim 3r_\frac{3}{2} \times 10^{-2}$ G [11].

In addition, the maximally amplified frequency calculated with respect to present day $\omega_1(\eta_0)$ is given by $\omega_1 \sim 6 \times 10^{11}$ Hz ($\frac{H_1}{M_4}$)$^{\frac{1}{2}}$ and the frequency corresponding to galactic scale, $\omega_G \sim 10^{-14}$Hz [13]. Furthermore, $r(\omega_G)$ is assumed to be of the form $r(\omega_G) = 10^{-m}$ where the exponent $m$ will be constrained by observational bounds. In the standard picture of the galactic dynamo, $m \leq 37$. In the following an upper bound on $-m$ will be found.

For one extra dimension, $n = 1$, the spectra (2.26) and (2.27) have a maximum at $\frac{\omega_G}{\omega_1} = e^{-\frac{2}{3}}$. Thus the constraint of the critical density is imposed by requiring $r(\omega_2) < 1$.

In the case where the momenta lying in the extra dimension are not taken into account, that is $q = 0$, $r(\omega_G) = 10^{-m}$ where $\omega_G = 10^{-14}$Hz implies,

$$-m = \log\frac{2}{3\pi^2} + \frac{1}{2}\log\frac{H_1}{M_4} + 3\log\frac{10^{-14}}{6 \times 10^{11}} + \log\left[\ln\frac{10^{-14}}{6 \times 10^{11}} - 1.1513\log\frac{H_1}{M_4}\right]^2. \quad (3.32)$$

Big bang nucleosynthesis requires $\log\frac{H_1}{M_4} > -40.94$ and the constraint $r(\omega_2) < 1$ implies $\log\frac{H_1}{M_4} < 1.2$. Evaluating $m$ at the upper limit $\log\frac{H_1}{M_4} = 1.2$ gives $r(\omega_G) < 10^{-74}$ corresponding to a magnetic seed field strength of $B_s < 10^{-39}$ G. Thus magnetic fields created in this setting are too weak in order to seed the galactic magnetic dynamo.

For $q > 0$ and $n = 1$ the various constraints mentioned above applied to the expression for $r(\omega_k)$ (cf. equation (2.26)) lead to the constraint on $m$

$$-m < \log\frac{9e^2}{4} + 3\log\frac{10^{-14}}{6 \times 10^{11}} - \frac{3}{2}\log\frac{H_1}{M_4} + \log\left[\ln\frac{10^{-14}}{6 \times 10^{11}} - 1.1513\log\frac{H_1}{M_4}\right]^2. \quad (3.33)$$

Evaluating $m$ at the lower bound $\log\frac{H_1}{M_4} = -40.94$ results in the bound $r(\omega_G) < 10^{-13}$ corresponding to a magnetic seed field strength of $B_s < 10^{-8}$ G. Thus in this case the lower bound on the magnetic seed field imposed by the galactic dynamo can be satisfied easily.
Assuming that \( T_1 \sim M_D \) results in an additional constraint on \( \log \frac{H_i}{M_4} \) by using the bound on the size of the extra dimensions. Namely, for any \( n \),

\[
\log \frac{H_i}{M_4} > \log 17.15 + \frac{2n}{n+2} \log(1.616 \times 10^{-32}). \tag{3.34}
\]

This gives a bound on \( \log \frac{H_i}{M_4} \) stronger than the one from big bang nucleosynthesis only up to three extra dimensions \( n \leq 3 \). In particular in the case at hand, for \( n = 1 \), it implies \( \log \frac{H_i}{M_4} > -19.96 \). Evaluating \( m \) at this value of \( \log \frac{H_i}{M_4} \) leads to \( r(\omega_G) < 10^{-43} \) and correspondingly the magnetic seed field strength \( B_s < 10^{-23} \) G. Thus in the case where \( T_1 \sim M_5 \) the created magnetic seed field satisfies the weaker bound of \( B_s > 10^{-30} \) G.

For more than one extra dimension \( n > 1 \) and \( q > 0 \) the constraint on \( r(\omega_k) \) (cf. equation (2.30)) at \( \omega_1 \) together with \( r(\omega_G) = 10^{-m} \) leads to

\[
-m < -\frac{3}{2} \log \frac{H_i}{M_4} + 3 \log \frac{10^{-14}}{6 \times 10^{11}}. \tag{3.35}
\]

Using the constraint from big bang nucleosynthesis \( \log \frac{H_i}{M_4} > -40.94 \) results in \( -m < -15.9 \) and thus \( r(\omega_G) < 10^{-16} \) and hence seed magnetic fields with strengths up to \( B_s < 10^{-10} \) G can be created. Assuming that the temperature at the beginning of the radiation epoch, \( T_1 \), is given by \( M_D \), that is \( T_1 \sim M_D \), changes the bound on \( m \) for two and three extra dimensions (cf. equation (8.34)). In this case, for \( n = 2 \) extra dimensions, \( -m < -31.5 \) implying \( r(\omega_G) < 10^{-32} \) and the magnetic field strength \( B_s < 10^{-18} \) G. For \( n = 3 \) extra dimensions, \( -m < -21.95 \) and hence \( r(\omega_G) < 10^{-22} \) and the magnetic field strength \( B_s < 10^{-13} \) G.

This is to be compared with the case where the internal momenta are not taken into account \( [13] \). Applying the constraints to equation (2.31) implies

\[
-m < \left(1 - \frac{n\lambda}{4}\right) \log \frac{2n\lambda-2}{3\pi^2} \Gamma^2 \left(\frac{n\lambda-1}{2}\right) (2-n\lambda)^2 + (4-n\lambda) \log \frac{10^{-14}}{6 \times 10^{11}}. \tag{3.36}
\]

In this case the bound on \( -m \) depends on the number of extra dimensions \( n \). This is related to the fact that the spectral index in the expression for \( r(\omega_k) \) (cf. equation (2.31)) is given by \( 4 - n\lambda \) and thus depends explicitly on the number of dimensions. In the case, where \( n > 1 \) and \( q > 0 \), the spectral index is 3, independent of the number of extra dimensions. In figure 11 the magnetic seed field strength \( B_s \) is plotted as a function of the number of extra dimensions \( n \) in the case \( n > 1 \), \( q = 0 \). As can be seen the resulting values for \( B_s \) are very small, much below even the weaker constraint, \( B_s > 10^{-30} \) G \( [10] \).

In the cases \( n = 1 \) and \( n > 1 \) for \( q > 0 \), \( \omega_{q_{\text{max}}} = q_{\text{max}}/b \) appears as a parameter in the expressions for \( r(\omega_k) \) (cf. equations (2.26) and (2.30)). Assuming that \( q_{\text{max}} \sim k_1 \) leads to \( \omega_{q_{\text{max}}}/\omega_1 \sim a \). Using this in \( r(\omega_2) < 1 \), for \( n = 1 \), and in \( r(\omega_1) < 1 \), for \( n > 1 \), leads in both cases to a constraint of the form

\[
\mathcal{N}_a a_0 \left(\frac{H_1}{M_4}\right)^{n+2} \left(\frac{M_D}{M_4}\right)^{-(n+2)} < 1, \tag{3.37}
\]

where \( \mathcal{N}_a = \frac{4}{27e^2\pi^3} \) for \( n = 1 \) and \( \mathcal{N}_a = \mathcal{N} \) for \( n > 1 \). If there are no additional constraints then equation (3.37) implies a lower bound on \( M_D/M_4 \), which has to be compared with the lower bound
provided by the size of the extra dimensions. However, if in addition $T_1 \sim M_D$ is imposed, then equation (3.37) together with $H_1 \sim 1.66g^2 \left(\frac{M_D}{M_4}\right)^2$ implies an upper bound on $M_D/M_4$, namely,

$$\log \frac{M_D}{M_4} < -\log \left(3 \times 10^{31} \mathcal{N}_s\right) - \frac{n + \frac{5}{2}}{n + 3} \log 1.66g^2, \quad (3.38)$$

where $a_0 \sim 3 \times 10^{31} \left(\frac{H_1}{M_4}\right)^\frac{1}{2}$ was used. This bound is always larger than the lower bound on $\log \frac{M_D}{M_4}$ provided by the size of the extra dimensions. Thus, the assumption $T_1 \sim M_D$ is consistent with the various constraints. Moreover, although this upper bound on $M_D/M_4$ leads to an upper bound on $H_1/M_4$, the maximal strength of the magnetic seed field is not changed, since for $n \geq 1, q > 0$ this was evaluated at the lower boundary of $\log \frac{H_1}{M_4}$.

4 Conclusions

The origin of magnetic fields on galactic and extragalactic scales is still an open problem. Different types of mechanisms have been proposed. In particular, in [13], the creation of magnetic fields due to dynamical extra dimensions was proposed. Along these lines, here, a model consisting of two phases has been investigated. A higher dimensional epoch with three expanding, external (spatial) dimensions and $n$ contracting, internal dimensions is matched to a standard radiation dominated phase with static extra dimensions. Taking the internal momenta into account the final expression for the ratio $r$ of magnetic field energy density to background radiation energy density is obtained by integrating over the internal modes. In doing so the sudden approximation requires the introduction of a maximal frequency in the internal momentum space. The resulting spectrum is constrained by imposing bounds from observations, such as, the constraint from critical energy density, the size of the extra dimensions and big bang nucleosynthesis.

For one extra dimension, $n = 1$, it was found that in the case where the momenta along the extra dimension are not taken into account, $q = 0$, only very weak magnetic seed fields are created,
$B_s < 10^{-39}$ G. However, in the case $q > 0$ magnetic seed fields as strong as $10^{-8}$ G can be obtained in general. Imposing the additional constraint $T_1 \sim M_5$ leads to magnetic seed fields $B_s < 10^{-23}$ G which satisfy the lower bound in a $\Lambda$ universe \cite{10}.

In models with more than one extra dimension, $n > 1$, strong magnetic seed fields can be created if the internal momenta are taken into account. In particular, not assuming that the temperature at the beginning of the radiation epoch is of the order of the $D$-dimensional Planck scale allows for the creation of seed magnetic fields with strengths of up to $10^{-10}$ G. For more than three extra dimensions, this also holds if $T_1 \sim M_D$ is assumed. With this assumption for two and three extra dimensions results in weaker magnetic seed fields, with maximal field strengths, $B_s < 10^{-18}$ G for two extra dimensions and $B_s < 10^{-13}$ G for three extra dimensions.

Therefore, in this particular model with extra dimensions, taking into account the momenta along the extra dimensions allows for the creation of strong magnetic fields.

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