Dimer coverings on the Sierpinski gasket with possible vacancies on the outmost vertices

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Abstract

We present the number of dimers $N_d(n)$ on the Sierpinski gasket $SG_d(n)$ at stage $n$ with dimension $d$ equal to two, three, four or five, where one of the outmost vertices is not covered when the number of vertices $v(n)$ is an odd number. The entropy of absorption of diatomic molecules per site, defined as $S_{SG_d} = \lim_{n \to \infty} \ln N_d(n)/v(n)$, is calculated to be $\ln(2)/3$ exactly for $SG_2(n)$. The numbers of dimers on the generalized Sierpinski gasket $SG_{d,b}(n)$ with $d = 2$ and $b = 3,4,5$ are also obtained exactly. Their entropies are equal to $\ln(6)/7$, $\ln(28)/12$, $\ln(200)/18$, respectively. The upper and lower bounds for the entropy are derived in terms of the results at a certain stage for $SG_d(n)$ with $d = 3,4,5$. As the difference between these bounds converges quickly to zero as the calculated stage increases, the numerical value of $S_{SG_d}$ with $d = 3,4,5$ can be evaluated with more than a hundred significant figures accurate.

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I. INTRODUCTION

The enumeration of close-packed dimers $N(G)$ on a graph $G$ was first considered by Fowler and Rushbrooke in enumerating the absorption of diatomic molecules on a surface [1]. The dimer coverings of a graph is a classical model in statistical physics and is called perfect matchings in mathematical literature. The dimer model on the square lattice was solved exactly by Kasteleyn [2] and Temperley and Fisher [3, 4]. The model is equivalent to various other statistical mechanical problems. For example, the zero-field partition function of Ising model on a planar lattice can be formulated as a dimer model on an associated planar lattice [5, 6]. It is also well known that there is a bijection between close-packed dimer coverings and spanning tree configurations on two related planar lattices [7]. A recent review on the enumeration of close-packed dimers on two-dimensional regular lattices is summarized in Ref. [8]. It is of interest to consider dimer coverings on self-similar fractal lattices which have scaling invariance rather than translational invariance. Fractals are geometric structures of noninteger Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [9, 10]. A well-known example of fractal is the Sierpinski gasket which has been extensively studied in several contexts [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. A dimer coverings will leave at least one vertex uncovered when the total number of vertices is an odd number, e.g., the rectangular lattice with both length and width odd [22, 23]. The vacancies that are not covered by any dimers can be considered as occupied by monomers. We allow such possible vacancies occur on the outmost vertices of the Sierpinski gasket. We shall derive rigorously the numbers of dimer coverings on the two-dimensional Sierpinski gasket and its generalization, and obtain upper and lower bounds for the entropy on the Sierpinski gasket with dimension equal to three, four or five.

II. PRELIMINARIES

We first recall some relevant definitions in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets $V$ and $E$ [24, 25]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in $G$. The degree or coordination number $k_i$ of a vertex $v_i \in V$ is the number of edges attached to it.
A $k$-regular graph is a graph with the property that each of its vertices has the same degree $k$. In general, one can associate a dimer (monomer) weight to each dimer (monomer) (see, for example [22]). For simplicity, all dimer (monomer) weights are set to one throughout this paper.

When the size of the graph increases as $v(G) \to \infty$, the number of dimer coverings $N(G)$ grows exponentially in $v(G)$. The entropy of absorption of diatomic molecules per site is given by

$$S_G = \lim_{v(G) \to \infty} \frac{\ln N(G)}{v(G)},$$

(2.1)

where $G$, when used as a subscript in this manner, implicitly refers to the thermodynamic limit. The dimer coverings considered here may not be close-packed dimers since there may be vacancies on the outmost vertices as mentioned above. Notice that we define the entropy per site rather than entropy per dimer. They differ by a factor of two in the thermodynamic limit regardless the presence of vacancies on the outmost vertices.

The construction of the two-dimensional Sierpinski gasket $SG_2(n)$ at stage $n$ is shown in Fig. 1. At stage $n = 0$, it is an equilateral triangle; while stage $n + 1$ is obtained by the juxtaposition of three $n$-stage structures. In general, the Sierpinski gaskets $SG_d$ can be built in any Euclidean dimension $d$ with fractal dimensionality $D = \ln(d+1)/\ln 2$ [12]. For the Sierpinski gasket $SG_d(n)$, the numbers of edges and vertices are given by

$$e(SG_d(n)) = \left(\frac{d+1}{2}\right)(d+1)^n = \frac{d}{2}(d+1)^{n+1},$$

(2.2)

$$v(SG_d(n)) = \frac{d+1}{2}[(d+1)^n + 1].$$

(2.3)

Except the $(d + 1)$ outmost vertices which have degree $d$, all other vertices of $SG_d(n)$ have degree $2d$. In the large $n$ limit, $SG_d$ is $2d$-regular.

![Fig. 1](image)

**FIG. 1:** The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG_2(n)$. 
The Sierpinski gasket can be generalized, denoted as $SG_{d,b}(n)$, by introducing the side length $b$ which is an integer larger or equal to two \[26\]. The generalized Sierpinski gasket at stage $n + 1$ is constructed with $b$ layers of stage $n$ hypertetrahedrons. The two-dimensional $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ are illustrated in Fig. 2, and those with $b = 4, 5$ at stage $n = 1$ in Fig. 3. The ordinary Sierpinski gasket $SG_d(n)$ corresponds to the $b = 2$ case, where the index $b$ is neglected for simplicity. The Hausdorff dimension for $SG_{d,b}$ is given by \[D = \frac{\ln(b+d-1)}{\ln b}\] \[26\]. For the two-dimensional Sierpinski gasket $SG_{2,b}(n)$ that will be considered here, the numbers of edges and vertices are given by

\[
e(SG_{2,b}(n)) = 3 \left[ \frac{b(b+1)}{2} \right]^n,
\]

\[
v(SG_{2,b}(n)) = \frac{b+4}{b+2} \left[ \frac{b(b+1)}{2} \right]^n + \frac{2(b+1)}{b+2}.
\]

Notice that $SG_{d,b}$ is not $k$-regular even in the thermodynamic limit.

**FIG. 2:** The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$.

**FIG. 3:** The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 4, 5$ at stage $n = 1$. 
III. THE NUMBER OF DIMER COVERINGS ON $SG_{2,b}(n)$ WITH $b = 2, 3, 4, 5$

In this section we derive rigorously the numbers of dimer coverings on the two-dimensional Sierpinski gasket $SG_2(n)$, equivalently $SG_{2,2}(n)$, and the generalized $SG_{2,b}(n)$ with $b = 3, 4, 5$. Let us start with the definitions of the quantities to be used. They are illustrated in Fig. 4, where only the outmost vertices of $SG_{2,b}(n)$ are shown.

**Definition III.1** Consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ at stage $n$. (i) Define $f_{2,b}(n)$ as the number of dimer coverings such that the three outmost vertices are vacant. (ii) Define $g_{2,b}(n)$ as the numbers of dimer coverings such that one certain outmost vertex, say the topmost vertex as illustrated in Fig. 4, is occupied by a dimer while the other two outmost vertices are vacant. (iii) Define $h_{2,b}(n)$ as the numbers of dimer coverings such that one certain outmost vertex, say the topmost vertex as illustrated in Fig. 4, is vacant while the other two outmost vertices are occupied by dimers. (iv) Define $t_{2,b}(n)$ as the number of dimer coverings such that all three outmost vertices are occupied by dimers.

![Fig. 4: Illustration for the configurations $f_{2,b}(n)$, $g_{2,b}(n)$, $h_{2,b}(n)$, and $t_{2,b}(n)$. Only the three outmost vertices are shown explicitly, where each open circle is vacant and each solid circle is occupied by a dimer.](image)

A. $SG_2(n)$

For the ordinary two-dimensional Sierpinski gasket, we use the notations $f_2(n)$, $g_2(n)$, $h_2(n)$, and $t_2(n)$ for simplicity. Because of rotational symmetry, there are three possible $g_2(n)$ and three possible $h_2(n)$ for non-negative integer $n$. The initial values at stage zero are $f_2(0) = 1$, $g_2(0) = 0$, $h_2(0) = 1$, $t_2(0) = 0$. The values at stage one are $f_2(1) = 0$, $g_2(1) = 2$, $h_2(1) = 0$, $t_2(1) = 2$. The value zero indicates that no such configurations are allowed. By Eq. (2.3), we have

$$v(SG_2(n)) = \frac{3}{2}(3^n + 1) = 3^n + 2 + \frac{1}{2}(3^n - 1)$$
where the Binomial expansion is used for $3^n = (2 + 1)^n$, such that the number of vertices for $SG_2(n)$ is odd for even $n$ and even for odd $n$. Therefore, $f(n)$, $h(n)$ are always zero for odd $n$ and $g(n)$, $t(n)$ are always zero for even $n$. Let us denote odd $n$ as $2m + 1$ and even $n$ as $2m$ with non-negative integer $m$ in the following discussion for $SG_2(n)$. These quantities satisfy simple recursion relations.

**Lemma III.1** For any odd $n = 2m + 1 > 0$,

$$f_2(2m + 2) = 2g_2^3(2m + 1) \ , \ (3.2)$$

$$h_2(2m + 2) = 2g_2^2(2m + 1)t_2(2m + 1) \ . \ (3.3)$$

For any even $n = 2m \geq 0$,

$$g_2(2m + 1) = 2f_2(2m)h_2^2(2m) \ , \ (3.4)$$

$$t_2(2m + 1) = 2h_2^3(2m) \ . \ (3.5)$$

**Proof** The Sierpinski gasket $SG_2(n + 1)$ is composed of three $SG_2(n)$ with three pairs of vertices identified. For each pair of identified vertices, either one of them is originally occupied by a dimer while the other one is vacant. The number $f_2(2m + 2)$ for $SG_2(2m + 2)$ consists of two configurations where all three of the $SG_2(2m + 1)$ are in the $g_2(2m + 1)$ status as illustrated in Fig. 5 such that Eq. (3.2) is verified.

![FIG. 5: Illustration for the expression of $f_2(2m + 2)$.

Similarly, $h_2(2m + 2)$ and $g_2(2m + 1)$, $t_2(2m + 1)$ can be obtained with appropriate configurations of its three constituting blocks as illustrated in Figs. 6, 7 and 8 to verify Eqs. (3.3), (3.4) and (3.5), respectively. □
It is elementary to solve $f_2(n)$, $g_2(n)$, $h_2(n)$, $t_2(n)$ in order to obtain the entropy for $SG_2$.

**Theorem III.2** For the two-dimensional Sierpinski gasket $SG_2(n)$ at stage $n = 2m$ or $n = 2m + 1$,

\[
\begin{align*}
    f_2(2m) &= h_2(2m) = 2^{\gamma_2(2m)} \\
    f_2(2m + 1) &= h_2(2m + 1) = 0
\end{align*}
\]

(3.6)

\[
\begin{align*}
    g_2(2m) &= t_2(2m) = 0 \\
    g_2(2m + 1) &= t_2(2m + 1) = 2^{\gamma_2(2m+1)}
\end{align*}
\]

(3.7)

where the exponent is

\[
\gamma_2(n) = \frac{1}{2}(3^n - 1) .
\]

(3.8)

Define the number of dimer coverings $N(SG_2(n))$ in Eq. (2.1) equal to $f_2(n = 2m)$ and equal to $g_2(n = 2m + 1)$ for even and odd $n$, respectively. With $v(SG_2(n)) = \frac{3}{2}(3^n + 1)$, the entropy is given by

\[
S_{SG_2} = \frac{1}{3} \ln 2 \simeq 0.23104906018...
\]

(3.9)
B. $SG_{2,3}(n)$

For the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 3$, we have

$$v(SG_{2,3}(n)) = \frac{7(6)^n + 8}{5} = 6^n + 2 + \frac{2}{5}(6^n - 1) = 6^n + 2 + 2 \sum_{j=1}^{n} \binom{n}{j} 5^{j-1}$$ (3.10)

by Eq. (2.5), such that the number of vertices is equal to three for $n = 0$ and becomes even for all positive integer $n$. Therefore, $f_{2,3}(n)$ and $h_{2,3}(n)$ are always zero for positive integer $n$, while the initial values remain $f_{2,3}(0) = 1$, $g_{2,3}(0) = 0$, $h_{2,3}(0) = 1$ and $t_{2,3}(0) = 0$. $g_{2,3}(n)$ and $t_{2,3}(n)$ satisfy recursion relations.

**Lemma III.3** For any positive integer $n$,

$$g_{2,3}(n + 1) = 6g_{2,3}^5(n)t_{2,3}(n) ,$$ (3.11)

$$t_{2,3}(n + 1) = 6g_{2,3}^4(n)t_{2,3}^2(n) ,$$ (3.12)

and for $n = 0$,

$$g_{2,3}(1) = 6f_{2,3}^2(0)h_{2,3}^4(0) = 6 ,$$ (3.13)

$$t_{2,3}(1) = 6f_{2,3}(0)h_{2,3}^5(0) = 6 .$$ (3.14)

**Proof** The Sierpinski gasket $SG_{2,3}(n + 1)$ is composed of six $SG_{2,3}(n)$ with six pairs of vertices identified and a set of three vertices identified. For each pair of identified vertices, either one of them is originally occupied by a dimer while the other one is vacant. For the set of three vertices, one of them is originally occupied by a dimer while the other two are vacant. The number $g_{2,3}(n + 1)$ for positive $n$ consists of six configurations where five of the $SG_{2,3}(n)$ are in the $g_{2,3}(n)$ status and one in the $t_{2,3}(n)$ status as illustrated in Fig. 9, such that Eq. (3.11) is verified.

![Diagram](image)

**FIG. 9:** Illustration for the expression of $g_{2,3}(n + 1)$ with positive $n$. The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.
Similarly, \( t_{2,3}(n + 1) \) with positive \( n \) for \( SG_{2,3}(n + 1) \) can be obtained with appropriate configurations of its six constituting \( SG_{2,3}(n) \) as illustrated in Fig. 10 to verify Eq. (3.12). Finally, \( g_{2,3}(1) \) and \( t_{2,3}(1) \) in Eqs. (3.13) and (3.14) are verified by Figs. 11 and 12 respectively. □

**FIG. 10:** Illustration for the expression of \( t_{2,3}(n + 1) \) with positive \( n \). The multiplication of three on the right-hand-side corresponds to the three possible orientations of \( SG_{2,3}(n + 1) \)

\[
\begin{align*}
\text{\( \triangledown \)} & = \text{\( \text{\( \triangledown \)} \times 3 + \text{\( \text{\( \triangledown \)} \times 3 \))} 
\end{align*}
\]

**FIG. 11:** Illustration for the expression of \( g_{2,3}(1) \). The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.

\[
\begin{align*}
\text{\( \triangledown \)} & = \text{\( \text{\( \triangledown \)} \times 2 + \text{\( \text{\( \triangledown \)} \times 2 \))} + \text{\( \text{\( \triangledown \)} \times 2 \))}
\end{align*}
\]

**FIG. 12:** Illustration for the expression of \( t_{2,3}(1) \). The multiplication of three on the right-hand-side corresponds to the three possible orientations of \( SG_{2,3}(n + 1) \)

\[
\begin{align*}
\text{\( \triangledown \)} & = \text{\( \text{\( \triangledown \)} \times 3 + \text{\( \text{\( \triangledown \)} \times 3 \))} 
\end{align*}
\]

It is elementary to solve \( g_{2,3}(n) \) and \( t_{2,3}(n) \) for positive \( n \) in order to obtain the entropy for \( SG_{2,3} \).

**Theorem III.4** For the generalized two-dimensional Sierpinski gasket \( SG_{2,3}(n) \) at stage \( n > 0 \),

\[
g_{2,3}(n) = t_{2,3}(n) = 6^{\gamma_{2,3}(n)},
\]
where the exponent is
\[ \gamma_{2,3}(n) = \frac{1}{5}(6^n - 1) . \]
(3.16)

Define the number of dimer coverings \( N(SG_{2,3}) \) in Eq. (2.1) equal to \( t_{2,3}(n) \). With
\[ v(SG_{2,3}) = \frac{(7(6)^n + 8)}{5} , \]
the entropy is given by
\[ S_{SG_{2,3}} = \frac{1}{7} \ln 6 \simeq 0.25596563846... \]
(3.17)

C. \( SG_{2,4}(n) \)

For the generalized two-dimensional Sierpinski gasket \( SG_{2,b}(n) \) with \( b = 4 \), we have
\[ v(SG_{2,4}) = \frac{4(10)^n + 5}{3} = 3 + \frac{4}{3}(10^n - 1) = 3 + \frac{4}{3} \sum_{j=1}^{n} \binom{n}{j} 9^j \]
(3.18)
by Eq. (2.5), such that the number of vertices is always odd for any \( n \). Therefore, \( g_{2,4}(n) \) and \( t_{2,4}(n) \) are zero for all \( n \), while the initial values remain \( f_{2,4}(0) = 1, g_{2,4}(0) = 0, h_{2,4}(0) = 1 \) and \( t_{2,4}(0) = 0 \). \( f_{2,4}(n) \) and \( h_{2,4}(n) \) satisfy recursion relations.

Lemma III.5 For any non-negative integer \( n \),
\[ f_{2,4}(n + 1) = 28f_{2,4}^4(n)h_{2,4}^5(n) , \]
(3.19)
\[ h_{2,4}(n + 1) = 28f_{2,4}^3(n)h_{2,4}^7(n) . \]
(3.20)

Proof The Sierpinski gasket \( SG_{2,4}(n + 1) \) is composed of ten \( SG_{2,4}(n) \) with nine pairs of vertices identified and three sets of three vertices identified. The number \( f_{2,4}(n + 1) \) for non-negative \( n \) consists of twenty eight configurations where four of the \( SG_{2,4}(n) \) are in the \( f_{2,4}(n) \) status and six in the \( h_{2,4}(n) \) status as illustrated in Fig. 13 such that Eq. (3.19) is verified.

Similarly, \( h_{2,4}(n+1) \) with non-negative \( n \) for \( SG_{2,4}(n+1) \) can be obtained with appropriate configurations of its ten constituting \( SG_{2,4}(n) \) as illustrated in Fig. 14 to verify Eq. (3.20).

It is elementary to solve \( f_{2,4}(n) \) and \( h_{2,4}(n) \) in order to obtain the entropy for \( SG_{2,4} \).
FIG. 13: Illustration for the expression of $f_{2,4}(n+1)$ with non-negative $n$. The multiplication of three on the right-hand-side corresponds to the three possible orientations of $SG_{2,4}(n+1)$.

**Theorem III.6** For the generalized two-dimensional Sierpinski gasket $SG_{2,4}(n)$ with non-negative integer $n$,

$$f_{2,4}(n) = h_{2,4}(n) = 28^{\gamma_{2,4}(n)},$$

where the exponent is

$$\gamma_{2,4}(n) = \frac{1}{9}(10^n - 1).$$

Define the number of dimer coverings $N(SG_{2,4}(n))$ in Eq. (2.1) equal to $f_{2,4}(n)$. With $v(SG_{2,4}(n)) = (4(10^n + 5))/3$, the entropy is given by

$$S_{SG_{2,4}} = \frac{1}{12} \ln 28 \simeq 0.27768370918...$$

**D. $SG_{2,5}(n)$**

For the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 5$, we have

$$v(SG_{2,5}(n)) = \frac{9(15)^n + 12}{7} = 15^n + 2 + \frac{2}{7}(15^n - 1) = 15^n + 2 + \frac{2}{7} \sum_{j=1}^{n} \binom{n}{j} 14^j$$

(3.24)
FIG. 14: Illustration for the expression of $h_{2,4}(n + 1)$ with non-negative $n$. The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.

by Eq. (2.5), such that the number of vertices is always odd for any $n$. Therefore, $g_{2,5}(n)$ and $t_{2,5}(n)$ are zero for all $n$, while the initial values remain $f_{2,5}(0) = 1$, $g_{2,5}(0) = 0$, $h_{2,5}(0) = 1$ and $t_{2,5}(0) = 0$. The figures of the recursion relations for $f_{2,5}(n)$ and $h_{2,5}(n)$ are too many to be shown here, and we state the following Lemma without proof.

**Lemma III.7** For any non-negative integer $n$,

\[ f_{2,5}(n + 1) = 200 f_{2,5}^6(n) h_{2,5}^9(n) , \]  

\[ h_{2,5}(n + 1) = 200 f_{2,5}^5(n) h_{2,5}^{10}(n) . \]

It is elementary to solve $f_{2,5}(n)$ and $h_{2,5}(n)$ in order to obtain the entropy for $SG_{2,5}$. 


**Theorem III.8** For the generalized two-dimensional Sierpinski gasket $SG_{2,5}(n)$ with non-negative integer $n$,

$$f_{2,5}(n) = h_{2,5}(n) = 200\gamma_{2,5}(n),$$

(3.27)

where the exponent is

$$\gamma_{2,5}(n) = \frac{1}{14}(15^n - 1).$$

(3.28)

Define the number of dimer coverings $N(SG_{2,5}(n))$ in Eq. (2.7) equal to $f_{2,5}(n)$. With $v(SG_{2,5}(n)) = (9(15)^n + 12)/7$, the entropy is given by

$$S_{SG_{2,5}} = \frac{1}{18}\ln 200 \simeq 0.29435096480...$$

(3.29)

As the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ for any $b$ is planar, it appears that the number of dimer coverings can be solved exactly. However, the number of configurations to be considered increases as $b$ increases and the recursion relations must be derived individually for each $b$. We have been unable to obtain a general expression of the number of dimer coverings on $SG_{2,b}(n)$ for arbitrary $b$.

**IV. THE NUMBER OF DIMER COVERINGS ON $SG_d(n)$ WITH $d = 3, 4, 5$**

In this section we present the number of dimer coverings on the Sierpinski gasket $SG_d(n)$ with $d = 3, 4, 5$ which is not planar. Instead of solving exactly the entropies for these Sierpinski gaskets, we obtain accurate upper and lower bounds for them.

**A. $SG_3(n)$**

For the three-dimensional Sierpinski gasket $SG_3(n)$, we use the following definitions.

**Definition IV.1** Consider the three-dimensional Sierpinski gasket $SG_3(n)$ at stage $n$. (i) Define $f_3(n)$ as the number of dimer coverings such that the four outmost vertices are vacant. (ii) Define $h_3(n)$ as the number of dimer coverings such that two certain outmost vertices are occupied by dimers and the other two outmost vertices are vacant. (iii) Define $s_3(n)$ as the number of dimer coverings such that all four outmost vertices are occupied by dimers.
As the number of vertices for $SG_3(n)$ is always even by Eq. (2.3), we do not have the dimer coverings such that one certain outmost vertices is occupied by a dimer and the other three outmost vertices are vacant, or one certain outmost vertices is vacant and the other three outmost vertices are occupied by dimers. The quantities $f_3(n)$, $h_3(n)$, and $s_3(n)$ are illustrated in Fig. 15, where only the outmost vertices are shown. There are $\binom{4}{2} = 6$ equivalent $h_3(n)$. The initial values at stage zero are $f_3(0) = 1$, $h_3(0) = 1$, $s_3(0) = 3$. These quantities satisfy recursion relations.

\begin{align}
  f_3(n+1) &= 8f_3(n)h_3^3(n) , \\
  h_3(n+1) &= 4f_3(n)h_3^2(n)s_3(n) + 4h_3^4(n) , \\
  s_3(n+1) &= 8h_3^3(n)s_3(n) .
\end{align}

Lemma IV.1  For any non-negative integer $n$,

\begin{align}
  f_3(n+1) &= 8f_3(n)h_3^3(n) , \\
  h_3(n+1) &= 4f_3(n)h_3^2(n)s_3(n) + 4h_3^4(n) , \\
  s_3(n+1) &= 8h_3^3(n)s_3(n) .
\end{align}

Proof  The Sierpinski gasket $SG_3(n+1)$ is composed of four $SG_3(n)$ with six pairs of vertices identified. For each pair of identified vertices, either one of them is originally occupied by a dimer while the other one is vacant. The number $f_3(n+1)$ for non-negative $n$ consists of eight configurations where one of the $SG_3(n)$ are in the $f_3(n)$ status and the other three are in the $h_3(n)$ status as illustrated in Fig. 16 such that Eq. (4.1) is verified.

Similarly, $h_3(n+1)$ and $s_3(n+1)$ for $SG_3(n+1)$ can be obtained with appropriate configurations of its four constituting $SG_3(n)$ as illustrated in Figs. 17 and 18 to verify Eqs. (4.2) and (4.3), respectively. □
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FIG. 16: Illustration for the expression of $f_3(n+1)$. The multiplication of four on the right-hand-side corresponds to the four possible orientations of $SG_3(n+1)$.

FIG. 17: Illustration for the expression of $h_3(n+1)$. The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.

The values of $f_3(n)$, $h_3(n)$, $s_3(n)$ for small $n$ can be evaluated recursively by Eqs. (4.1)-(4.3), but they grow exponentially, and do not have simple integer factorizations. To estimate the value of entropy for $SG_3$, we define the ratio

$$\alpha_3(n) = \frac{h_3(n)}{f_3(n)},$$

and its limit

$$\alpha_3 \equiv \lim_{n \to \infty} \alpha_3(n).$$

Lemma IV.2 Sequence $\{\alpha_3(n)\}_{n=1}^\infty$ decreases monotonically. The limit $\alpha_3$ is equal to $\sqrt{3}$.

FIG. 18: Illustration for the expression of $s_3(n+1)$. The multiplication of four on the right-hand-side corresponds to the four possible orientations of $SG_3(n+1)$. 

Proof  From Eqs. (4.1) and (4.3), the ratio \( s_3(n)/f_3(n) \) is invariant, that is equal to \( s_3(0)/f_3(0) = 3 \). Eq. (4.2) can be modified to be

\[
h_3(n + 1) = 12f_3^2(n)h_3^2(n) + 4h_3^4(n) .
\]  (4.6)

Although \( \alpha_3(0) = 1 \), it is clear that \( \alpha_3(n) \) is bounded below by \( \sqrt{3} \) for positive integer \( n \) because

\[
h_3^2(n + 1) - 3f_3^2(n + 1) = [12f_3^2(n)h_3^2(n) - 4h_3^4(n)]^2 \geq 0 .
\]  (4.7)

It follows that \( \alpha_3(n) \) decreases for positive \( n \) because

\[
\frac{h_3(n)}{f_3(n)} - \frac{h_3(n + 1)}{f_3(n + 1)} = \frac{h_3^2(n) - 3f_3^2(n)}{2f_3(n)h_3(n)} \geq 0 ,
\]  (4.8)

which implies that the limit \( \alpha_3 \) exists. From Eqs. (4.1) and (4.6), we have

\[
\frac{h_3(n + 1)}{f_3(n + 1)} = \frac{3}{2} \frac{f_3(n)}{h_3(n)} + \frac{1}{2} \frac{h_3(n)}{f_3(n)} .
\]  (4.9)

By taking the large \( n \) limit in Eq. (4.9), \( \alpha_3 \) is solved to be \( \sqrt{3} \). □

The general expressions for \( f_3(n) \) and \( h_3(n) \) in terms of quantities at stage \( m < n \) can be written as follows.

Lemma IV.3 For a non-negative integer \( m \) and any positive integer \( n > m \),

\[
f_3(n) = 2^{\frac{2(4)^{n-m+1}-5+3(-1)^{n-m}}{10}} f_3(m)^{\frac{2(4)^{n-m}+3(-1)^{n-m}}{5}} h_3(m)^{\frac{3(4)^{n-m}+3(-1)^{n-m}}{5}} \times \prod_{j=2}^{n-m} \left[ 3 + \alpha_3^2(n - j) \right]^{\frac{3(4)^{j-1}+3(-1)^{j-1}}{5}} ,
\]  (4.10)

\[
h_3(n) = 2^{\frac{4^{n-m+1}-5+3(-1)^{n-m}}{5}} f_3(m)^{\frac{2(4)^{n-m}+2(-1)^{n-m}}{5}} h_3(m)^{\frac{3(4)^{n-m}+2(-1)^{n-m}}{5}} \times \prod_{j=1}^{n-m} \left[ 3 + \alpha_3^2(n - j) \right]^{\frac{3(4)^{j-1}+2(-1)^{j-1}}{5}} .
\]  (4.11)

Here when \( n - m = 1 \), the product with lower limit two is defined to be one.
Proof. It is clear that Eqs. (4.10) and (4.11) are valid for \( n = m + 1 \) since \( f_3(m + 1) = 8f_3(m)h_3^2(m) \) and \( h_3(m + 1) = 4f_3^2(m)h_3^3(m)[3 + \alpha_3^2(m)] \) by Eqs. (4.1) and (4.6), respectively. Consider Eq. (4.10) holds for a certain positive integer \( n > m \), then

\[
f_3(n + 1) = 8f_3(n)h_3^2(n)
\]

\[
= 8 \times 2^{(4)^{n-m+1} + 3(-1)^{n-m}} \frac{f_3(m)}{5} \frac{h_3(m)}{5} \frac{3(4)^{n-m} - 3(-1)^{n-m}}{5} \\
\times \prod_{j=2}^{n-m} \left[ 3 + \alpha_3^2(n - j) \right] \frac{3(4)^{j-1} - 3(-1)^{j-1}}{5} \frac{2^{\frac{(4)^{n-m+1} + 3(-1)^{n-m}}{5}}}{5} \frac{f_3(m)}{5} \frac{6(4)^{n-m} - 6(-1)^{n-m}}{5} \\
\times h_3(m) \frac{9(4)^{n-m} + 6(-1)^{n-m}}{5} \prod_{j=1}^{n-m} \left[ 3 + \alpha_3^2(n - j) \right] ^{\frac{9(4)^{j-1} + 6(-1)^{j-1}}{5}} \\
= 2^{(4)^{n-m+1} + 3(-1)^{n-m}} \frac{f_3(m)}{5} \frac{8(4)^{n-m} - 3(-1)^{n-m}}{5} \frac{h_3(m)}{5} \frac{12(4)^{n-m} + 3(-1)^{n-m}}{5} \\
\times \prod_{j=2}^{n-m} \left[ 3 + \alpha_3^2(n - j) \right] \frac{12(4)^{j-1} + 3(-1)^{j-1}}{5} \left[ 3 + \alpha_3^2(n - 1) \right] ^3 \\
= 2^{(4)^{n-m+2} + 5(-1)^{n-m+1}} \frac{f_3(m)}{5} \frac{2^{(4)^{n-m+1} + 3(-1)^{n-m+1}}}{5} \frac{h_3(m)}{5} \frac{3(4)^{n-m+1} - 3(-1)^{n-m+1}}{5} \\
\times \prod_{j=2}^{n-m+1} \left[ 3 + \alpha_3^2(n + 1 - j) \right] \frac{3(4)^{j-1} - 3(-1)^{j-1}}{5} \tag{4.12}
\]

such that Eq. (4.10) is proved by induction. Eq. (4.11) can be established by the same procedure. \( \square \)

Let us state the following lemma without proof.

Lemma IV.4 If \( X(n + 1) = \frac{X(n)^2}{c} \) for non-negative integer \( n \) and \( X_0 \) is known, then

\[
X(n) = \frac{X(0)^{2^n}}{c^{2^n - 1}}. \tag{4.13}
\]

From above lemmas, we have the following bounds for the entropy.
**Lemma IV.5** The entropy for the number of dimer coverings on $SG_3(n)$ is bounded:

$$\frac{-\sqrt{3}\epsilon_3(m)^3}{720(4)^m} \leq S_{SG_3} - \left\{ \frac{2\ln f_3(m) + 3\ln h_3(m) + 5\ln 2 + \ln 3}{10(4)^m} + \frac{\sqrt{3}\epsilon_3(m)^3}{40(4)^m} \right\}$$

$$\leq \frac{\sqrt{3}\epsilon_3(m)^2}{40(4)^m [2\sqrt{3} - \epsilon_3(m)]}, \quad (4.14)$$

where $m$ is a positive integer and $\epsilon_3(n)$ is defined as $\alpha_3(n) - \sqrt{3}$.

**Proof** As $f_3(n)$ only differs from $s_3(n)$ by a factor of three, which is insignificant in the definition of the entropy, we will substitute $N(G) = f_3(n)$ in Eq. (2.1) for $SG_3$ so that

$$S_{SG_3} = \lim_{n \to \infty} \frac{\ln f_3(n)}{2(4^n + 1)}. \quad (4.15)$$

By Lemma [IV.3], we have

$$\ln f_3(n) = \frac{2(4)^{n-m} + 3(-1)^{n-m}}{5} \ln f_3(m) + \frac{3(4)^{n-m} - 3(-1)^{n-m}}{5} \ln h_3(m)$$

$$+ \frac{2(4)^{n-m+1} - 5 - 3(-1)^{n-m}}{10} \ln 2 + \Delta_3(n, m), \quad (4.16)$$

where

$$\Delta_3(n, m) = \sum_{j=2}^{n-m} \frac{3(4)^{j-1} - 3(-1)^{j-1}}{5} \ln [3 + \alpha_3^2(n - j)], \quad (4.17)$$

which is bounded as follows.

By Lemma [IV.2], we know $\epsilon_3(n)$ decreases monotonically to zero for positive integer $n$. $\epsilon_3(1) = \alpha_3(1) - \sqrt{3} = 2 - \sqrt{3}$. It is easy to find, by Eq. (4.19), that

$$\epsilon_3(n + 1) = \frac{\epsilon_3(n)^2}{2(\sqrt{3} + \epsilon_3(n))}. \quad (4.18)$$

Therefore, for any integer $n \geq m$ with $m$ fixed, we have

$$\epsilon_3(n + m) = \frac{\epsilon_3(m)^{2^n}}{(2\sqrt{3})^{2^n-1}} (1 + o(n)) \quad (4.19)$$

by Lemma [IV.4] and Eq. (4.18), where $o(n)$ is negative here and $o(n) \to 0$ as $n \to \infty$. Now $\Delta_3(n, m)$ in Eq. (4.17) can be rewritten,

$$\Delta_3(n, m) = \sum_{j=2}^{n-m} \frac{3(4)^{j-1} - 3(-1)^{j-1}}{5} \ln [6 + 2\sqrt{3}\epsilon_3(n - j) + \epsilon_3(n - j)^2]. \quad (4.20)$$
Since $\epsilon_3(n)$ is small for positive $n$, the logarithmic term can be written as
\[
\ln[6 + 2\sqrt{3}\epsilon_3(n - j) + \epsilon_3(n - j)^2] = \ln 6 + \frac{\sqrt{3}}{3}\epsilon_3(n - j)\left[1 - \frac{\xi_{n,j}\epsilon_3(n - j)^2}{18}\right],
\]
where $\xi_{n,j} \in (0, 1)$ so that
\[
\Delta_3(n, m) = \sum_{j=2}^{n-m} \frac{3(4)^{j-1} - 3(-1)^{j-1}}{5} \left\{ \ln 6 + \frac{\sqrt{3}\epsilon_3(n - j)}{3}\left[1 - \frac{\xi_{n,j}\epsilon_3(n - j)^2}{18}\right]\right\}
\]
\[
= \sum_{j=2}^{n-m} \frac{3\ln 6}{5} [(4)^{j-1} - (-1)^{j-1}]
\]
\[
+ \sum_{j=2}^{n-m} \frac{\sqrt{3}\epsilon_3(n - j)}{5} [(4)^{j-1} - (-1)^{j-1}]\left[1 - \frac{\xi_{n,j}\epsilon_3(n - j)^2}{18}\right].
\]
(4.22)

Because the $j = n - m$ term gives the largest contribution for $\Delta_3(n, m)$, it is easy to see that
\[
\frac{4^{n-m-1}\sqrt{3}\epsilon_3(m)(1 - \frac{\epsilon_3(m)^2}{18})(1 + o(n))}{5} \leq \Delta_3(n, m) - \frac{4^{n-m}\ln 6}{5}(1 + o(n)).
\]
(4.23)

On the other hand, since $\sum_{j=2}^{n-m} \epsilon_3(n - j)4^{j-1}[1 - \xi_{n,j}\epsilon_3(n - j)^2/18]$ is less than $\sum_{i=0}^{n-m-2} 4^{n-m-1}\epsilon_3(m + i)$, we have
\[
\Delta_3(n, m) - \frac{4^{n-m}\ln 6}{5}(1 + o(n)) \leq \frac{3(4)^{n-m}}{10}(1 + o(n)) \sum_{i=0}^{n-m-2} \left(\frac{\epsilon_3(m)}{2\sqrt{3}}\right)^{2^i}
\]
\[
\leq \frac{3(4)^{n-m}\epsilon_3(m)(1 + o(n))}{10[2\sqrt{3} - \epsilon_3(m)]},
\]
(4.24)

where we use Eq. (4.19) and the inequality
\[
\sum_{j=0}^{n-m-2} x^{2^j} = x + \sum_{j=1}^{n-m-2} x^{2^j} \leq x + \sum_{j=1}^{n-m-2} x^{2^j} \leq \frac{x + x^2}{1 - x^2} = \frac{x}{1 - x}
\]
(4.25)

for any $0 < x < 1$. The proof is completed by taking the infinite $n$ limit in Eq. (4.15).

The difference between the upper and lower bounds for $S_{SG3}$ quickly converges to zero as $m$ increases, and we have the following proposition.

**Proposition IV.1** The entropy for the number of dimer coverings on the three-dimensional Sierpinski gasket $SG3(n)$ in the large $n$ limit is $S_{SG3} = 0.42896389912...$. 

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The difference between the upper and lower bounds for $S_{SG3}$ quickly converges to zero as $m$ increases, and we have the following proposition.

**Proposition IV.1** The entropy for the number of dimer coverings on the three-dimensional Sierpinski gasket $SG3(n)$ in the large $n$ limit is $S_{SG3} = 0.42896389912...$.
By Eq. (4.19), we know
\[ \epsilon_3(7) \leq 2\sqrt{3}\left(\frac{2 - \sqrt{3}}{2\sqrt{3}}\right)^{2^6}, \] (4.26)
such that \( S_{SG_3} \) can be calculated with more than a hundred significant figures accurate when \( m \) is equal to seven in Eq. (4.14). It is too lengthy to be included here and is available from the authors on request.

**B. \( SG_4(n) \)**

For the four-dimensional Sierpinski gasket \( SG_4(n) \), we use the following definitions.

**Definition IV.2** Consider the four-dimensional Sierpinski gasket \( SG_4(n) \) at stage \( n \). (i) Define \( f_4(n) \) as the number of dimer coverings such that the five outmost vertices are vacant. (ii) Define \( h_4(n) \) as the number of dimer coverings such that two certain outmost vertices are occupied by dimers and the other three outmost vertices are vacant. (iii) Define \( s_4(n) \) as the number of dimer coverings such that one certain outmost vertex is vacant and the other four outmost vertices are occupied by dimers.

By Eq. (2.3), we have
\[ v(SG_4(n)) = \frac{5}{2}(5^n + 1) = 2(5)^n + 3 + \frac{1}{2}(5^n - 1) = 2(5)^n + 3 + \frac{1}{2} \sum_{j=1}^{n} \binom{n}{j} 4^j, \] (4.27)
such that the number of vertices for \( SG_4(n) \) is always odd. Therefore, we do not have the dimer coverings such that one certain outmost vertices is occupied by a dimer and the other four outmost vertices are vacant, or three certain outmost vertices are occupied by dimers and the other two outmost vertices are vacant, or all five outmost vertices are occupied by dimers. The quantities \( f_4(n), h_4(n), \) and \( s_4(n) \) are illustrated in Fig. 19, where only the outmost vertices are shown. There are \( \binom{5}{2} = 10 \) equivalent \( h_4(n) \) and \( \binom{5}{1} = 5 \) equivalent \( s_4(n) \). The initial values at stage zero are again \( f_4(0) = 1, h_4(0) = 1, s_4(0) = 3 \).

We write a computer program to obtain following recursion relations.
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**Lemma IV.6** For any non-negative integer $n$,

\begin{align*}
f_4(n + 1) &= 40f_4(n)h_4^3(n)s_4(n) + 24h_4^5(n) , \\
h_4(n + 1) &= 24f_4(n)h_4^2(n)s_4^2(n) + 40h_4^4(n)s_4(n) , \\
s_4(n + 1) &= 8f_4(n)h_4(n)s_4^4(n) + 56h_4^3(n)s_4^2(n) .
\end{align*}

(4.28)  (4.29)  (4.30)

The values of $f_4(n)$, $h_4(n)$, $s_4(n)$ for small $n$ can be evaluated recursively by Eqs. (4.28)-(4.30), but they grow exponentially, and do not have simple integer factorizations. To estimate the value of entropy for $SG_4$, we define the ratios

\[ \alpha_4(n) = \frac{h_4(n)}{f_4(n)} , \quad \beta_4(n) = \frac{s_4(n)}{h_4(n)} , \]

(4.31)

and their limits

\[ \alpha_4 \equiv \lim_{n \to \infty} \alpha_4(n) , \quad \beta_4 \equiv \lim_{n \to \infty} \beta_4(n) . \]

(4.32)

**Lemma IV.7** Sequence $\{\alpha_4(n)\}^\infty_{n=1}$ decreases monotonically while sequence $\{\beta_4(n)\}^\infty_{n=1}$ increases monotonically. The ratio $\beta_4(n)/\alpha_4(n)$ for positive $n$ increases monotonically to one.

**Proof** From Eqs. (4.28) to (4.30), we find

\[ h_4^2(n + 1) - f_4(n + 1)s_4(n + 1) = 256h_4^4(n)s_4(n)[h_4^2(n) - f_4(n)s_4(n)]^2 \geq 0 , \]

(4.33)

such that $\alpha_4(n) \geq \beta_4(n)$ for $n > 0$. It follows that for positive $n$,

\[ \frac{h_4(n)}{f_4(n)} - \frac{h_4(n + 1)}{f_4(n + 1)} = \frac{24h_4^2(n)[h_4^4(n) - f_4^2(n)s_4^2(n)]}{f_4(n)f_4(n + 1)} \geq 0 , \]

(4.34)

which shows that $\alpha_4(n)$ decreases, and

\[ \frac{s_4(n)}{h_4(n)} - \frac{s_4(n + 1)}{h_4(n + 1)} = \frac{16h_4^2(n)s_4^2(n)[f_4(n)s_4(n) - h_4^2(n)]}{h_4(n)h_4(n + 1)} \leq 0 , \]

(4.35)
which shows that $\beta_4(n)$ increases. From Eqs. (4.28) to (4.30), we have

\[
\frac{h_4(n+1)}{f_4(n+1)} = \frac{\beta_4(n)[3\beta_4(n) + 5\alpha_4(n)]}{5\beta_4(n) + 3\alpha_4(n)},
\]

and

\[
\frac{s_4(n+1)}{h_4(n+1)} = \frac{\beta_4(n)[\beta_4(n) + 7\alpha_4(n)]}{3\beta_4(n) + 5\alpha_4(n)},
\]

which leads to $\alpha_4 = \beta_4$ by taking the limit $n \to \infty$. The actual value of $\alpha_4$ and $\beta_4$ cannot be obtained by solving these equations. The numerical results give

\[
\alpha_4 = \beta_4 = 0.850772150002... 
\]

where more than a hundred significant figures can be evaluated when stage $n$ in Eq. (4.31) is equal to seven. □

The general expressions for $h_4(n)$ and $s_4(n)$ in terms of quantities at stage $m < n$ can be written as follows.

**Lemma IV.8** For a non-negative integer $m$ and any positive integer $n > m$,

\[
h_4(n) = 2^{\frac{3(n-m)-3}{4}} h_4(m) \frac{3^{n-m+1}}{4} s_4(m) \frac{5^{n-m-1}}{4} \\
\times \prod_{i=1}^{n-m} \left[ 5 + 3 \frac{\beta_4(n-i)}{\alpha_4(n-i)} \frac{3^{i-1+1}}{4} \right] \prod_{j=2}^{n-m} \left[ 7 + \frac{\beta_4(n-j)}{\alpha_4(n-j)} \frac{5^{j-1-1}}{4} \right],
\]

\[
s_4(n) = 2^{\frac{3(n-m)-3}{4}} h_4(m) \frac{3^{n-m-3}}{4} s_4(m) \frac{5^{n-m+3}}{4} \\
\times \prod_{i=2}^{n-m} \left[ 5 + 3 \frac{\beta_4(n-i)}{\alpha_4(n-i)} \frac{3^{i-1-3}}{4} \right] \prod_{j=1}^{n-m} \left[ 7 + \frac{\beta_4(n-j)}{\alpha_4(n-j)} \frac{5^{j-1+3}}{4} \right].
\]

Here when $n - m = 1$, the products with lower limit two are defined to be one.

**Proof** It is clear that Eqs. (4.39) and (4.40) are valid for $n = m + 1$ since $h_4(m+1) = 8h_4^2(m)s_4(m)[5 + 3\beta_4(m)/\alpha_4(m)]$ and $s_4(m+1) = 8h_4^2(m)s_4^2(m)[7 + \beta_4(m)/\alpha_4(m)]$ by Eqs.
Lemma IV.9 The entropy for the number of dimer coverings on \( SG_4(n) \) is bounded:

\[
-\frac{7\epsilon_4(m)^2}{640(5)^m[1 - \epsilon_4(m)/16]} \leq S_{SG_4} - \left\{ \frac{3\ln h_4(m) + \ln s_4(m) + 6\ln 2}{10(5)^m} - \frac{\epsilon_4(m)}{40(5)^m} \right\} \leq 0,
\]

where \( m \) is a positive integer and \( \epsilon_4(n) \) is defined as \( 1 - \beta_4(n)/\alpha_4(n) \).

Proof We substitute \( N(G) = s_4(n) \) in Eq. (2.1) for \( SG_4 \) so that

\[
S_{SG_4} = \lim_{n \to \infty} \frac{\ln s_4(n)}{5(5^n + 1)/2}.
\]
By Lemma IV.8. we have
\[ \ln s_4(n) = \frac{3(5)^{n-m} - 3}{4} \ln h_4(m) + \frac{5^{n-m} + 3}{4} \ln s_4(m) + \frac{3(5)^{n-m} - 3}{4} \ln 2 + \Delta_4(n, m) , \]
\[ (4.44) \]
where
\[ \Delta_4(n, m) = \sum_{i=2}^{n-m} \frac{3(5)^{i-1} - 3}{4} \left[ \ln 8 + \ln \left( 1 - \frac{3\epsilon_4(n-i)}{8} \right) \right] \\
+ \sum_{j=1}^{n-m} \frac{5^{j-1} + 3}{4} \ln \left[ 7 + \frac{\beta_4(n-j)}{\alpha_4(n-j)} \right] , \]
\[ (4.45) \]
which is bounded as follows.

By Lemma IV.7, we know \( \epsilon_4(n) \) decreases monotonically to zero for positive integer \( n \).
\( \epsilon_4(1) = 1 - \beta_4(1)/\alpha_4(1) = 4/49 \). It is easy to find, by Eqs. (4.36) and (4.37), that
\[ \epsilon_4(n+1) = \left[ \frac{2\epsilon_4(n)}{8 - 3\epsilon_4(n)} \right]^2 . \]
\[ (4.46) \]
Therefore, for any integer \( n \geq m \) with \( m \) fixed, we have
\[ \epsilon_4(n + m) = \frac{\epsilon_4(m) 2^n}{16^{2m-1} (1 + o(n))} . \]
\[ (4.47) \]
Now \( \Delta_4(n, m) \) in Eq. (4.45) can be rewritten,
\[ \Delta_4(n, m) = \sum_{i=2}^{n-m} \frac{3(5)^{i-1} - 1}{4} \left[ \ln 8 + \ln \left( 1 - \frac{3\epsilon_4(n-i)}{8} \right) \right] \\
+ \sum_{j=1}^{n-m} \frac{5^{j-1} + 3}{4} \ln \left[ 7 + \frac{\beta_4(n-j)}{\alpha_4(n-j)} \right] \\
= \sum_{i=2}^{n-m} \frac{3(5)^{i-1} - 1}{4} \left\{ \ln 8 - \frac{3\epsilon_4(n-i)}{8} \left[ 1 + \frac{3\xi_{n,i} \epsilon_4(n-i)}{8} \right] \right\} \\
+ \sum_{j=1}^{n-m} \frac{5^{j-1} + 3}{4} \left\{ \ln 8 - \frac{\epsilon_4(n-j)}{8} \left[ 1 + \frac{\xi_{n,j}' \epsilon_4(n-j)}{8} \right] \right\} , \]
\[ (4.48) \]
where \( \xi_{n,i}, \xi_{n,j}' \in (0, 1) \). It is easy to see that
\[ \Delta_4(n, m) - \frac{(3 \ln 2) 5^{n-m}}{4} (1 + o(n)) \leq - \frac{5^{n-m} \epsilon_4(m)}{16} (1 + o(n)) . \]
\[ (4.49) \]
By Eqs. (4.25) and (4.47), we have
\[ \Delta_4(n, m) - \frac{(3 \ln 2) 5^{n-m}}{4} (1 + o(n)) \]
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\[
\geq -5^{n-m}(1 + o(n)) \left(1 + \frac{3\epsilon_4(m)}{8}\right)^{n-m-1} \sum_{j=0}^{\infty} \left(\frac{\epsilon_4(m)}{16}\right)^{2^j}
\]

\[
\geq -\frac{\epsilon_4(m)5^{n-m}(1 + o(n))\left(1 + \frac{3\epsilon_4(m)}{8}\right)}{16 - \epsilon_4(m)}.
\] (4.50)

The proof is completed by taking the infinite \(n\) limit in Eq. (4.43). \(\square\)

The difference between the upper and lower bounds for \(S_{SG_4}\) quickly converges to zero as \(m\) increases, and we have the following proposition.

**Proposition IV.2** The entropy for the number of dimer coverings on the four-dimensional Sierpinski gasket \(SG_4(n)\) in the large \(n\) limit is \(S_{SG_4} = 0.56337479920\ldots\). The numerical value of \(S_{SG_4}\) can be calculated with more than a hundred significant figures accurate when \(m\) in Eq. (4.42) is equal to six. It is too lengthy to be included here and is available from the authors on request.

### C. \(SG_5(n)\)

For the five-dimensional Sierpinski gasket \(SG_5(n)\), we use the following definitions.

**Definition IV.3** Consider the five-dimensional Sierpinski gasket \(SG_5(n)\) at stage \(n\). (i) Define \(f_5(n)\) as the number of dimer coverings such that the six outmost vertices are vacant. (ii) Define \(g_5(n)\) as the number of dimer coverings such that one certain outmost vertex is occupied by a dimer and the other five outmost vertices are vacant. (iii) Define \(h_5(n)\) as the number of dimer coverings such that two certain outmost vertices are occupied by dimers and the other four outmost vertices are vacant. (iv) Define \(r_5(n)\) as the number of dimer coverings such that three certain outmost vertices are occupied by dimers and the other three outmost vertices are vacant. (v) Define \(s_5(n)\) as the number of dimer coverings such that two certain outmost vertices are vacant and the other four outmost vertices are occupied by dimers. (vi) Define \(t_5(n)\) as the number of dimer coverings such that one certain outmost vertex is vacant and the other five outmost vertices are occupied by dimers. (vii) Define \(u_5(n)\) as the number of dimer coverings such that all six outmost vertices are occupied by dimers.
The quantities $f_5(n)$, $g_5(n)$, $h_5(n)$, $r_5(n)$, $s_5(n)$, $t_5(n)$ and $u_5(n)$ are illustrated in Fig. 20, where only the outmost vertices are shown. The initial values are $f_5(0) = 1$, $g_5(0) = 0$, $h_5(0) = 1$, $r_5(0) = 0$, $s_5(0) = 3$, $t_5(0) = 0$, $u_5(0) = 15$. For the five-dimensional Sierpinski gasket $SG_5(n)$, the number of vertices is equal to six for $n = 0$ and odd for all positive integer $n$ by Eq. (2.3). Therefore, $f_5(n)$, $h_5(n)$, $s_5(n)$, $u_5(n)$ are always zero for positive integer $n$. There are $\binom{6}{1} = 6$ equivalent $g_5(n)$ and $t_5(n)$, and $\binom{6}{3} = 20$ equivalent $r_5(n)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig20}
\caption{Illustration for the dimer coverings $f_5(n)$, $g_5(n)$, $h_5(n)$, $r_5(n)$, $s_5(n)$, $t_5(n)$, $u_5(n)$. Only the six outmost vertices are shown explicitly, where each open circle is vacant and each solid circle is occupied by a dimer.}
\end{figure}

We write a computer program to obtain following recursion relations.

**Lemma IV.10** For any positive integer $n$,
\begin{align}
g_5(n+1) &= 40g_5^3(n)r_5(n)t_5^2(n) + 560g_5^2(n)r_5^3(n)t_5(n) + 424g_5(n)r_5^5(n) , \quad (4.51) \\
r_5(n+1) &= 4g_5^3(n)t_5^2(n) + 252g_5^2(n)r_5^2(n)t_5^2(n) + 636g_5(n)r_5^4(n)t_5(n) + 132r_5^6(n) , \quad (4.52) \\
t_5(n+1) &= 40g_5^2(n)r_5(n)t_5^3(n) + 560g_5(n)r_5^3(n)t_5^2(n) + 424r_5^5(n)t_5(n) , \quad (4.53)
\end{align}
and for $n = 0$,
\begin{align}
g_5(1) &= 280f_5(0)h_5^2(0)s_5^3(0) + 40f_5(0)h_5^3(0)s_5(0)u_5(0) + 680h_5^3(0)s_5^2(0) + 24h_5^5(0)u_5(0) \\
&= 15840 , \quad (4.54)
\end{align}
\begin{align}
r_5(1) &= 72f_5(0)h_5^2(0)s_5^2(0)u_5(0) + 120f_5(0)h_5(0)s_5^4(0) + 712h_5^3(0)s_5^3(0) + 120h_5^4(0)s_5(0)u_5(0)
\end{align}
\begin{equation}
= 44064 ,
\end{equation}

\begin{equation}
t_5(1) = 40f_5(0)h_5(0)s^3_5(0)u_5(0) + 280h^2_5(0)s^2_5(0)u_5(0) + 24f_5(0)s^5_5(0) + 680h^2_5(0)s^4_5(0)
\end{equation}

\begin{equation}
= 114912 .
\end{equation}

The values of \(g_4(n)\), \(r_4(n)\), \(t_4(n)\) for small positive \(n\) can be evaluated recursively by Eqs. (4.51)-(4.53), but they grow exponentially, and do not have simple integer factorizations.

To estimate the value of entropy for \(SG_5\), we define the ratio

\begin{equation}
\alpha_5(n) = \frac{r_5(n)}{g_5(n)} ,
\end{equation}

and its limit

\begin{equation}
\alpha_5 \equiv \lim_{n \to \infty} \alpha_5(n) .
\end{equation}

**Lemma IV.11** Sequence \(\{\alpha_5(n)\}_{n=1}^\infty\) decreases monotonically. The limit \(\alpha_5\) is equal to \(\sqrt{399/55}\).

**Proof** From Eqs. (4.51) and (4.53), the ratio \(t_5(n)/g_5(n)\) is invariant. Defined the ratio as \(c\), then

\begin{equation}
c = \frac{t_5(1)}{g_5(1)} = \frac{399}{55} .
\end{equation}

Eqs. (4.51) and (4.52) can be modified to be

\begin{equation}
g_5(n + 1) = 8g_5^6(n)r_5(n)P_5(n) ,
\end{equation}

\begin{equation}
r_5(n + 1) = 4g_5^6(n)Q_5(n) ,
\end{equation}

where

\begin{equation}
P_5(n) = 5c^2 + 70c^2\alpha_5^2(n) + 53\alpha_5^4(n) ,
\end{equation}

\begin{equation}
Q_5(n) = c^3 + 63c^2\alpha_5^2(n) + 159c\alpha_5^4(n) + 33\alpha_5^6(n) .
\end{equation}

It is clear that \(\alpha_5(n)\) is bounded below by \(\sqrt{c}\) because

\begin{equation}
r_5^2(n + 1) - g_5(n + 1)t_5(n + 1) = r_5^2(n + 1) - cg_5^2(n + 1)
\end{equation}
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\[ r_5(n) = 16[r_5^2(n) - cg_5^2(n)][c^4g_5^8(n) + 28c^3g_5^6(n)r_5^2(n) + 1542c^2g_5^4(n)r_5^4(n) + 1436cg_5^2(n)r_5^6(n) + 1089r_5^8(n)] \geq 0 \] (4.64)

by induction. It follows that \( \alpha_5(n) \) decreases for positive \( n \) because

\[
\frac{r_5(n)}{g_5(n)} = \frac{r_5(n + 1)}{g_5(n + 1)} = \frac{4}{g_5(n + 1)} \left\{ 19r_5^4(n)[r_5^2(n) - cg_5^2(n)] + 53r_5^2(n)[r_5^4(n) - c^2g_5^4(n)] + r_5^6(n) - c^3g_5^6(n) \right\} \geq 0 ,
\] (4.65)

which implies that the limit \( \alpha_5 \) exists. From Eqs. (4.60) and (4.61), we have

\[
\frac{r_5(n + 1)}{g_5(n + 1)} = \frac{c^3 + 63c^2\alpha_5^2(n) + 159c\alpha_5^4(n) + 33\alpha_5^6(n)}{2\alpha_5(n)[5c^2 + 70c\alpha_5^2(n) + 53\alpha_5^4(n)]} .
\] (4.66)

By taking the large \( n \) limit in Eq. (4.66) and the requirement that \( \alpha_5 \) must be real and positive, \( \alpha_5 \) is solved to be \( \sqrt{c} \). \( \square \)

The general expressions for \( g_5(n) \) and \( r_5(n) \) in terms of quantities at stage \( m < n \) can be written as follows.

**Lemma IV.12** For a non-negative integer \( m \) and any positive integer \( n > m \),

\[
g_5(n) = 2^{\frac{g(n)n-m-7+3(-1)^{n-m}}{4}} g_5(m) \prod_{i=1}^{n-m} P_5(n - i) \prod_{j=2}^{n-m} Q_5(n - j) \frac{g_5^6(n)g_5^6(n-1)}{r_5^6(n)g_5^6(n-1)} \] (4.67)

\[
r_5(n) = 2^{\frac{g(n)n-m-7+3(-1)^{n-m}}{4}} g_5(m) \prod_{i=2}^{n-m} P_5(n - i) \prod_{j=1}^{n-m} Q_5(n - j) \frac{g_5^6(n)g_5^6(n-1)}{r_5^6(n)g_5^6(n-1)} \] (4.68)

Here when \( n - m = 1 \), the products with lower limit two are defined to be one.
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Proof. It is clear that Eqs. (4.67) and (4.68) are valid for \( n = m + 1 \) since \( g_5(m + 1) = 8g_5(m)r_5(m)P_5(m) \) and \( r_5(m + 1) = 4g_5(m)Q_5(m) \) by Eqs. (4.60) and (4.61), respectively. Consider Eq. (4.67) holds for a certain positive integer \( n > m \), then

\[
g_5(n + 1) = 8g_5(n)r_5(n)P_5(n)
\]

\[
= 8 \times 2^{\frac{6n(n - m - 35 - 5(-1)^m)}{14}} g_5(m)^{\frac{6n - m + 1 + 5(-1)^{n - m}}{7}} r_5(m)^{\frac{6n - m - 5(-1)^{n - m}}{7}}
\]

\[
\times \prod_{i=1}^{n-m} P_5(n - i) \frac{5(-1)^{i} - 5(-1)^{i}}{7} \prod_{j=2}^{n-m} Q_5(n - j) \frac{5(-1)^{j - 1 + 5(-1)^{j}}}{7}
\]

\[
\times 2^{\frac{6n(n - m - 35 + 4(-1)^m)}{14}} g_5(m)^{\frac{6n - m + 1 - 6(-1)^{n - m}}{7}} r_5(m)^{\frac{6n - m + 6(-1)^{n - m}}{7}}
\]

\[
\times \prod_{i=1}^{n-m} P_5(n - i) \frac{6(-1)^{i} - 6(-1)^{i}}{7} \prod_{j=1}^{n-m} Q_5(n - j) \frac{6(-1)^{j - 1 - 6(-1)^{j}}}{7}
\]

\[
P_5(n)
\]

\[
= 2^{\frac{6n(n - m + 1 + 7(-1)^{n - m + 1})}{14}} g_5(m)^{\frac{6n - m + 2 + (-1)^{n - m + 1}}{7}} r_5(m)^{\frac{6n - m + 1 - (-1)^{n - m + 1}}{7}}
\]

\[
\times \prod_{i=1}^{n-m+1} P_5(n + 1 - i) \frac{6(-1)^{i} - 6(-1)^{i}}{7} \prod_{j=2}^{n-m+1} Q_5(n + 1 - j) \frac{6(-1)^{j - 1 - (-1)^{j}}}{7},
\]

such that Eq. (4.67) is proved by induction. Eq. (4.68) can be established by the same procedure. \( \square \)

From above lemmas, we have the following bounds for the entropy.

Lemma IV.13 The entropy for the number of dimer coverings on \( SG_5(n) \) is bounded:

\[
0 \leq S_{SG_5} \leq \left\{ \frac{2 \ln g_5(m)}{7(6)^m} + \frac{\ln r_5(m)}{21(6)^m} + \frac{14 \ln 2}{21(6)^m} + \frac{\ln c}{7(6)^m} + \frac{9 \epsilon_5(m)}{56 \sqrt{c}(6)^m} \right\}
\]

\[
\leq \frac{279 \epsilon_5(m)^2}{448c(6)^m \left[ 1 - \frac{\epsilon_5(m)}{8 \sqrt{c}} \right]},
\]

(4.70)

where \( m \) is a positive integer and \( \epsilon_5(n) \) is defined as \( \alpha_5(n) - \sqrt{c} \).
Proof  As \( u_5(n) \) is exactly zero for all positive \( n \), we will substitute \( N(G) = g_5(n) \) in Eq. (2.1) for \( SG_5 \) so that

\[
S_{SG_5} = \lim_{n \to \infty} \frac{\ln g_5(n)}{3(6^n + 1)}.
\]

(4.71)

By Lemma [IV.12] we have

\[
\ln g_5(n) = \frac{6^{n-m+1} + (-1)^{n-m}}{7} \ln g_5(m) + \frac{6^{n-m} - (-1)^{n-m}}{7} \ln r_5(m)
\]

\[
+ \frac{8(6)^{n-m} - 7(-1)^{n-m}}{14} \ln 2 + \Delta_5(n, m),
\]

(4.72)

where

\[
\Delta_5(n, m) = \sum_{i=1}^{n-m} \frac{6^i - (-1)^i}{7} \ln P_5(n - i) + \sum_{j=2}^{n-m} \frac{6^{j-1} + (-1)^j}{7} \ln Q_5(n - j),
\]

(4.73)

which is bounded as follows.

By Lemma [IV.11] we know \( \epsilon_5(n) \) decreases monotonically to zero for positive integer \( n \). \( \epsilon_5(1) = \alpha_5(1) - \sqrt{c} = 153/55 - \sqrt{399}/55 \). It is easy to find, by Eq. (4.66), that

\[
\epsilon_5(n + 1) = \frac{\epsilon_5(n)^2}{8\sqrt{c}}(1 + o(n)).
\]

(4.74)

Therefore, for any integer \( n \geq m \) with \( m \) fixed, we have

\[
\epsilon_5(m + n) = \frac{\epsilon_5(m)^{2^n}}{(8\sqrt{c})^{2^n - 1}}(1 + o(n)).
\]

(4.75)

From Eqs. (4.62) and (4.63), we have

\[
P_5(n + m) = 128c^2 + 352\sqrt{c} \epsilon_5(n + m)(1 + o(n)),
\]

(4.76)

and

\[
Q_5(n + m) = 256c^3 + 960c^2 \sqrt{c} \epsilon_5(n + m)(1 + o(n)),
\]

(4.77)

such that

\[
\ln P_5(n + m) = 7 \ln 2 + 2 \ln c + (1 + o(n)) \ln \left(1 + \frac{11\epsilon_5(n + m)}{4\sqrt{c}} \right),
\]

(4.78)

\[
\ln Q_5(n + m) = 8 \ln 2 + 3 \ln c + (1 + o(n)) \ln \left(1 + \frac{15\epsilon_5(n + m)}{4\sqrt{c}} \right).
\]

(4.79)

Now \( \Delta_5(n, m) \) in Eq. (4.73) can be rewritten,

\[
\Delta_5(n, m) = \sum_{i=1}^{n-m} \frac{(6)^i - (-1)^i}{7} [7 \ln 2 + 2 \ln c]
\]
\[ \sum_{i=1}^{n-m} \frac{(6)^i - (-1)^i}{7} (1 + o(n)) \ln (1 + \frac{11\epsilon_5(n - i)}{4\sqrt{c}}) \]
\[ + \sum_{j=2}^{n-m} \frac{(6)^{j-1} + (-1)^j}{7} [8 \ln 2 + 3 \ln c] \]
\[ + \sum_{j=2}^{n-m} \frac{(6)^{j-1} - (-1)^j}{7} (1 + o(n)) \ln (1 + \frac{15\epsilon_5(n - j)}{4\sqrt{c}}) \].

(4.80)

It is easy to see that
\[ \Delta_5(n, m) - \left[ \frac{10 \ln 2}{7} + \frac{3 \ln c}{7} \right] 6^{n-m} (1 + o(n)) \geq \frac{27\epsilon_5(m)}{56\sqrt{c}} 6^{n-m} (1 - o(n)) \].

(4.81)

By Eqs. (4.25) and (4.75), we have
\[ \Delta_5(n, m) - \left[ \frac{10 \ln 2}{7} + \frac{3 \ln c}{7} \right] 6^{n-m} (1 + o(n)) \]
\[ \leq \frac{27}{7} 6^{n-m} (1 + o(n)) \left( 1 + \frac{15\epsilon_5(m)}{4\sqrt{c}} \right) \sum_{j=0}^{n-m-1} \left( \frac{\epsilon_5(m)}{8\sqrt{c}} \right)^2 \]
\[ \leq \frac{27\epsilon_5(m) 6^{n-m} (1 + o(n)) \left( 1 + \frac{15\epsilon_5(m)}{4\sqrt{c}} \right)}{7[8\sqrt{c} - \epsilon_5(m)]}. \]

(4.82)

The proof is completed by taking the infinite \( n \) limit in Eq. (4.71). □

The difference between the upper and lower bounds for \( S_{SG_5} \) quickly converges to zero as \( m \) increases, and we have the following proposition.

**Proposition IV.3** The entropy for the number of dimer coverings on the five-dimensional Sierpinski gasket \( S_{SG_5}(n) \) in the large \( n \) limit is \( S_{SG_5} = 0.67042810305... \).

The numerical value of \( S_{SG_5} \) can be calculated with more than a hundred significant figures accurate when \( m \) in Eq. (4.70) is equal to six. It is too lengthy to be included here and is available from the authors on request.

We notice that the convergence of the upper and lower bounds of the entropy for dimer coverings on \( SG_d(n) \) is about the same for \( d = 3, 4, 5 \), similar to the results observed in [21] for the dimer-monomer model on \( SG_d(n) \).
V. SUMMARY

Compare the present results with those in Ref. [27], it is clear that the number of dimer coverings on the Sierpinski gasket SG \(_d(n)\) is less than that of dimer-monomers. The asymptotic growth constant \(z_{SG_d,b}\) for the dimer-monomer model defined as Eq. (2.1) of [27] corresponds to the entropy \(S_{SG_d,b}\) for the dimer coverings defined in Eq. (2.1). We summarize the values of \(S_{SG_d,b}\) and the ratio \(S_{SG_d,b}/z_{SG_d,b}\) in Table I. The value of \(S_{SG_d}\) increases as dimension \(d\) increases. Similarly for the generalized two-dimensional Sierpinski gasket, the exact value of \(S_{SG_{2,b}}\) increases slightly as \(b\) increases. For the cases studied, the ratio \(S_{SG_d}/z_{SG_d}\) also increases as dimension \(d\) increases, and \(S_{SG_{2,b}}/z_{SG_{2,b}}\) increases slightly as \(b\) increases.

It is interesting to compare entropy of dimer coverings on the Sierpinski gasket SG \(_d\) with that on the \(d\)-dimensional hypercubic lattice \(L_d\) which is also 2\(d\)-regular. The entropy of the square lattice was known to be \(G/\pi\) [4], where \(G\) is the Catalan number, for decades, while the entropy of the simple cubic lattice was estimated to be 0.44647 [28]. They are relatively larger than the entropies on \(SG_d\) with \(d = 2, 3\) presented here. The values of \(S_{L_d}\) and the ratio \(S_{SG_d}/S_{L_d}\) for \(d = 2, 3\) are given in Table I. It appears that as the \(d\) increases, the value \(S_{SG_d}\) approaches to the value \(S_{L_d}\) from below. As we have obtained the highly accurate value for the entropy on \(SG_d\) with \(d = 4, 5\), there is no numerical estimation for the entropy on \(L_d\) with \(d \geq 4\), to the best of our knowledge.

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TABLE I: Numerical values of $S_{SG, d,b}$, $S_L$, and the ratios $S_{SG, d,b}/z_{SG, d,b}$, $S_{SG, d}/S_L$. The last digits given are rounded off.

| $d$ | $b$ | $D$ | $S_{SG, d,b}$ | $S_{SG, d,b}/z_{SG, d,b}$ | $S_L$ | $S_{SG, d}/S_L$ |
|-----|-----|-----|---------------|----------------------------|------|----------------|
| 2   | 2   | $\frac{1}{5}$ ln 2 $\simeq$ 0.2310490602 | 0.3520510271 | $G/\pi \simeq 0.2915609040$ | 0.7924555624 |
| 2   | 3   | $\frac{1}{7}$ ln 6 $\simeq$ 0.2559656385 | 0.3811183712 | - | - |
| 2   | 4   | $\frac{1}{12}$ ln 28 $\simeq$ 0.2776837092 | 0.4054532859 | - | - |
| 2   | 5   | $\frac{1}{15}$ ln 200 $\simeq$ 0.2943509648 | - | - | - |
| 3   | 2   | 0.4289638991 | 0.5491430497 | 0.4465 | 0.9608 |
| 4   | 2   | 0.5633747992 | 0.6425502211 | - | - |
| 5   | 2   | 0.6704281031 | - | - | - |

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