Π à la node: disordered d-wave superconductors in two dimensions for the random masses

P. J. Hirschfeld and W.A. Atkinson

We review work on the problem of disorder in the 2D d-wave superconducting state, and show that the symmetries of the normal state and the disorder distribution are vital for understanding the low-energy behavior. Most previous theoretical results for the density of states (DOS) are reconciled by a combination of exact numerical solutions of the Bogoliubov-de Gennes equations and weak localization calculations, which suggest that a novel diffusive mode with momentum $(\pi, \pi)$ is responsible for a divergence of the DOS in the globally particle-hole symmetric case. We note briefly that the simple problem of a disordered tight-binding band of normal electrons displays some similar effects, which have been overlooked in the literature. Finally, in the physically realistic case of binary alloy disorder, no particle-hole symmetry, and an order parameter which is suppressed around each impurity site, a power law with nonuniversal exponent is predicted.

1. Introduction

Shortly after the early work of BCS, it was recognized that while non-magnetic impurities would not affect thermodynamic properties of a conventional superconductor, they might do so in hypothetical p-wave pairing systems. With the observation of a high density of low-energy excitations in superconducting heavy fermion systems, such states were considered as candidates for the ground state of compounds like UBe$_{13}$ and UPt$_3$.  

1. Physics Dept., Univ. Florida, PO Box 118440, Gainesville FL 32611
2. Physics Dept., Southern Illinois Univ., Carbondale, IL 62901-4401
Gorkov and Kalugin[3] and, independently, Ueda and Rice[4] pointed out that linear nodal regions of the order parameter on the 3D Fermi surface would lead to a finite residual density of states at zero energy, $\rho(0)$. This model was analogous to that used by Abrikosov and Gorkov[5] in their discussion of gapless superconductivity caused by magnetic scatterers, namely a self-consistent second-order perturbation theory in the impurity potential $U$ averaged over disorder. Pethick and Pines[6] noted that weak scattering models of this type were insufficient to describe transport properties measured on the heavy fermion systems, and proposed that large effective impurity scattering strengths, due to strong correlations, required a partial resummation of perturbation theory. Hirschfeld et al.[7] and Schmitt-Rink et al.[8] subsequently calculated the self-consistent $t$-matrix for impurities with phase shifts close to $\pi/2$, and found a pole in the residual density of states close to the Fermi level which broadened into a disorder-induced “plateau” at low-energies, yielding metallic behavior of the nodal quasiparticles at low temperatures in the superconducting state. Stamp[9] then investigated the local density of states around a single impurity within this model.

With the proposal of $d$-wave symmetry in the hole-doped cuprate superconductors in the early 1990s[10], the self-consistent $t$-matrix approximation (SCTMA) was used to provide a qualitative explanation of the effects of disorder on the $d$-wave state which provided important supporting evidence. For example, observations of a crossover of the London penetration depth from a $T$ to a $T^2$ behavior with a small concentration of Zn impurities were not initially understood to be a natural consequence of $d$ symmetry, because it was expected that in such experiments $T_c$ would be strongly suppressed in a $d$-wave system. In the unitarity limit, however, strong modifications of low-energy properties occur over an energy scale $\gamma$ (“impurity bandwidth”) which can be much larger than the normal state impurity scattering rate $\Gamma$ which controls $T_c$[11]. Analyses of transport properties also provided evidence for near-unitarity scattering, and helped rule out candidate extended $s$-wave states[12].

This relatively successful phenomenology was cast into doubt in a work by Nersesyan, Tsvelik and Wenger[13], who pointed out that the utility of the $t$-matrix approach was based on the ability to neglect multiple-impurity scattering processes (“crossed diagrams”), which are of higher order in $1/k_F\ell$ or the density of impurity sites $n_i$ in three dimensions. These diagrams have an additional log $E$ singularity in 2D coming from the line nodes of the $d_{x^2−y^2}$ gap, however, which leads to a breakdown of single-site perturbation theory and the need to resum at least a second class of crossed diagrams. This important paper led to a series of attempts to solve the problem nonperturbatively, following in most cases standard methods which had produced
Fig. 1. Self-energy after disorder averaging.

reliable results in the case of the disordered normal metal. By 2000 several such solutions had appeared in the literature, but if anything the situation was more confused, since the various papers had each claimed to calculate exactly the density of states of a disordered $d$-wave superconductor, but had arrived at dramatically different results. In particular, the claimed residual densities of states at zero energy were zero, finite, and infinite, respectively.

In this paper, we review efforts we began at that time to reconcile the various available theoretical results. We emphasize that our interest was primarily to understand real $d$-wave superconductors with strong nonmagnetic disorder, rather than classify models and search for universal physics. We also sought to understand which, if any, of the nonperturbative solutions was relevant to the “realistic” model and to experiments on cuprate superconductors. Most of this work, involving both numerical solutions of the Bogoliubov-de Gennes equations and analytical weak localization calculations, has been published previously. In these papers it was shown that most of the various nonperturbative solutions are correct, and that subtle differences in symmetries of the normal state band and disorder models led to the conflicting results. In particular, the divergent density of states found in “globally particle-hole symmetric models” (we use this language to describe models with normal state bands manifesting particle-hole symmetry on all energy scales, not just near the Fermi surface) was found to arise due to a qualitatively new diffusive mode with momentum $\left(\pi, \pi\right)$. It is interesting to note that this sensitivity to “details” of disorder does not exist in the theory of disorder in normal metals, but may be thought of as a
critical behavior of the $d$-wave state due to the linearly vanishing density of states.

The general principle that random systems of different symmetry should display qualitatively different low-energy behavior in the DOS has its origins in random matrix theory and was extended to $d$-wave superconductors by Senthil and Fisher and Bocquet et al. While this concept is very powerful, it has not always been possible to calculate the asymptotic low-energy DOS in a controlled way using field theoretical methods. This is particularly true for strong binary alloy disorder, which such methods have found difficult to treat. Very recently, however, some progress in this direction has been reported. Perhaps more importantly, there are no methods currently available other than numerical ones to treat off-diagonal (order parameter) disorder correlated with diagonal impurity locations. Since the $d$-wave order parameter is suppressed essentially to zero around strong scattering centers, this would appear to be an important effect, and we indeed find within BdG calculations that the case where order parameter response is self-consistently included in the numerical solutions to the Bogoliubov-de Gennes equations is qualitatively different from the case where the order parameter is assumed to be homogeneous. We argue that the physics of this correlated order response is relatively unexplored, and may lead to important changes in low energy transport, possibly even in three dimensions.

In the final part of this paper, we present some new results for a lattice model of a normal metal where global nesting symmetry is found to play a role similar to the $d$-wave case, and show that previous analyses have come to some erroneous conclusions.

2. Review of nonperturbative approaches

In their original papers, Nersesyan et al. considered models in which the scattering between nodes was artificially restricted, and mapped the problem onto a disordered chain which they then solved by bosonization, leading to predictions of vanishing density of states at the Fermi level, $\rho(E) \sim E^\alpha$, with $\alpha$ either a universal constant $1/7$ if quasiparticles were allowed to scatter across the Fermi surface, or a coupling constant-dependent result if only forward scattering within a node were allowed. Ziegler et al. then pointed out that a $d$-wave analog of the so-called Lloyd model in normal metals was exactly soluble for $\rho(E)$ over the whole frequency range, and yielded a constant density of states at $E \to 0$. To complete the proof, however, they were forced to assume a Lorenztian disorder distribution, as well as a rather strange normal state electronic dispersion,
ε_{ll}' = -t''(\cos 2k_x + \cos 2k_y)(3rd nearest neighbor hopping). Nersesyan et al.\cite{Nersesyan2002} criticized this work as nongeneric due to the use of the Lorentzian distribution, which has long-range tails with divergent moments, but Ziegler et al.\cite{Ziegler2003} were able to prove rigorous lower bounds to \(\rho(0)\) for other distributions. Shortly thereafter, Altland and Zirnbauer\cite{Altland1997} considered extending the Wigner-Dyson symmetry classification of disordered systems to superconductors (i.e. to systems with a Bogoliubov-de Gennes-type particle hole symmetry) and superconductor-normal interfaces. They argued that the number of possible symmetry classes (and, by implication, universal low-energy behavior) for disordered systems was restricted to the 10 Cartan Lie algebras, four of which apply to systems with BdG symmetry\cite{Altland1997}.

It may be worthwhile to review the general idea at this stage. The classification of disordered systems is based on the observation that some special symmetries obeyed by a random Hamiltonian for every realization of the disorder endow the statistical ensemble of Hamiltonians with the structure of a Lie algebra. The special symmetries are, in the context of tight-binding Hamiltonians describing the hopping of an electron in a random environment: time-reversal symmetry (TR), spin-rotation symmetry (SR), particle-hole symmetry, and sublattice symmetry. For a bipartite lattice, the sublattice or “chiral” symmetry alluded to corresponds to the statement that the Hamiltonian changes sign when all sites on sublattice A are multiplied by -1 (see below). Clearly an on-site potential breaks this symmetry, which is therefore very special. That these four symmetries are exhaustive from the point of view of group theory follows from the classification by Cartan of all possible finite-dimensional symmetric spaces. Combination of the scaling approach to localization and the Cartan classification of symmetric spaces then leads to the constructions of non-linear sigma models with the Cartan symmetric space as a target space to describe the physics of localization in the presence or absence of these special symmetries.

Senthil and Fisher\cite{Senthil1997} were the first to investigate how different symmetries affected the DOS of a disordered \(d\)-wave system. They first calculated the quantum corrections to the DOS and conductivity in the “metallic state” defined by the SCTMA “plateau”, an analogy originally suggested by Lee\cite{Lee1985}, and showed that the DOS was supressed as \(\delta \rho(E) \sim -\rho_0(v_\Delta/8v_F) \log \gamma/E\) for \(E \lesssim \gamma\), where \(v_F\) is the Fermi velocity, \(\rho_0\) is the plateau DOS, and \(v_\Delta\) is the gap velocity near the nodes, hinting at a vanishing DOS at \(E = 0\). In addition, they gave a heuristic argument for an effective 1D model whose low energy properties should correspond to the 4-node \(d\)-wave system, finding \(\delta \rho(E) \sim E\) below a strong localization scale \(E_1 \sim \gamma \exp -v_F/v_\Delta\) for a \(d\)-wave superconductor with TR- and SR-invariance (Cartan class CI), and \(\rho(E) \sim E^2\) if TR is broken (class C). Bocquet et al.\cite{Bocquet2000} mapped a more gen-
general problem of disordered Dirac fermions onto a supersymmetric field theory which could be solved by RG methods under certain conditions. They found as a special case that for disordered d-wave superconductors with both broken TR and SR symmetry (class D), the density of states diverges logarithmically as $E \to 0$. The class C and D results are irrelevant to the current work, in which we concern ourselves exclusively with singlet superconductors in zero external field with nonmagnetic disorder (generically class CI), but illustrated the importance of symmetry for the DOS and localization properties in the $d$-wave superconducting state.

All the above results were restricted to weak scattering, in the sense of a narrow distribution of site energies. Noting that in the cuprates there was considerable evidence for localized unitarity limit scattering centers, Pépin and Lee\textsuperscript{17} embarked upon a new approach to the disorder problem, considering formally the exact $t$-matrix for a given configuration of $N$ unitary scatterers in a half-filled tight-binding band. They predicted that the DOS was singular as $E \to 0$, and claimed that the leading such singularity was described by scattering $N$ times from a single impurity, leading to $\delta \rho(E) \sim 1/(E \log^2 \Delta/E)$ (an integrable divergence).

In Table 1, we collect the results of several nonperturbative calculations for disordered $d$-wave superconductors. For each reference, we indicate with a few words which model was considered and the Cartan symmetry class of the model in question. It will become clear from the discussion below that the symmetry class in each case was not always obvious to the authors of the work in question, and in some cases is still not settled.

3. Numerical results with homogeneous order parameter

Before reviewing numerical finite-size calculations, we remind the reader of some details of the $t$-matrix approach. We focus on the effect of point-like disorder, characterised by two parameters, the impurity concentration $n_i$ and the potential $U$. The latter is described by a $\delta$-function scattering potential, $\hat{U}(\mathbf{R} - \mathbf{R}_{\text{imp}}) = U_0 \delta(\mathbf{R} - \mathbf{R}_{\text{imp}}) \tau_3$, where the $\tau_i$ are the Pauli matrices in particle-hole space. The impurity $t$-matrix is given as usual by (in matrix notation) $\hat{T} = \hat{U} + \hat{U} \hat{G}_0 \hat{T}$, with $\hat{U}$ defined above. The result is

$$\hat{T}(\omega) \simeq \frac{g_0 - c\tau_3}{c^2 - g_0^2},$$

where $c = U_0^{-1} - g_3$, and $g_0$ and $g_3$ are the components of the momentum integrated Green function, $g_\alpha \equiv (1/2) \sum_k \text{Tr} \tau_\alpha \hat{G}(\mathbf{k}, E)$. For the single
impurity problem, Eq. (1) is exact, and $\hat{G} = \hat{G}^0$. The poles of this expression correspond to impurity resonances at position $\Omega_0$ with finite width due to coupling to the $d$-wave continuum, unless $\Omega_0 = 0$. From Eq. (1) it is clear that, since $g_0 \sim E$ up to log corrections, the resonance is not located exactly at the Fermi level for infinitely strong potential $U_0 = \infty$, unless $g_3 = 0$. This occurs only for a *globally* particle-hole symmetric electronic spectrum, e.g. a single tight-binding band at half-filling. The disorder-averaged self-energy is now defined in the limit of a density $n_i$ of independent impurities to be $\hat{\Sigma}(\mathbf{k}, E) \equiv n_i \hat{T}_{kk}(E)$, and determined self-consistently with the averaged $\hat{G}$ via the Dyson equation,

$$\hat{G}^{-1} - E - \xi_k \tau_3 - \Delta_k \tau_1 - \hat{\Sigma}(\mathbf{k}, E) \equiv \tilde{E} - \tilde{\xi}_k \tau_3 - \tilde{\Delta}_k \tau_1.$$  

In the SCTMA the impurity band is characterised by the scattering rate $\gamma = -\text{Im}\Sigma(E \rightarrow 0)$, that also approximately determines the energy at which $\rho(E)$ crosses over from linear ($|E| > \gamma$) to constant ($|E| < \gamma$). It is this constant behavior which Nersesyan et al. [13] have shown is not generally applicable in 2D.

The true low-energy behavior of the 2D system can be calculated within mean field theory by solving the BdG equations. In this paper we present results for a tight-binding lattice with $N = 1600$ sites and up to 50 disorder

| group      | class | model/method                                      | result $\rho(E \rightarrow 0)$ |
|------------|-------|--------------------------------------------------|-------------------------------|
| Nersesyan et al. | AIII | Dirac fermions w/ random gauge fields, supersymm./bosoniz., | $\sim |E|^{\alpha}$, $\alpha < 1$ dep. on # nodes, internode scattering |
| Ziegler et al. | AI  | Lloyd model                                       | $\sim \text{const.} + aE^2$ |
| Senthil-Fisher | CI  | $Sp(2n)$ nonlin. $\sigma$ model, $2+\epsilon$ expansion | $\sim |E|$ |
| Pépin-Lee     | C   | Leading sing. of $t$-matrix, $\frac{1}{2}$-filled unitarity limit | $\sim \frac{1}{(|E| \log^2 \Delta/|E|)}$ |
| Bocquet et al. | D   | Dirac fermions/NL$\sigma$ nonabelian bosoniz.       | $\sim \log |E|$ |
| Altland et al. | CI  | Dirac fermions/NL$\sigma$...                      | $\sim |E|^{\alpha}$ |
|              | AIII |                                                    | $\sim E^2$ const.            |
|              | C   |                                                    |                               |
|              | A   |                                                    |                               |
| Fabrizio et al., | AIII? | NL$\sigma$ model $\frac{1}{2}$-filled unitarity limit | $\sim e^{-A/\sqrt{\log|E|/|E|}}$ |

Table 1. Nonperturbative approaches to disordered $d$-wave DOS problem.
configurations. In matrix form, the mean-field Hamiltonian is

$$\mathcal{H} = \sum_{ij} \Phi_i^\dagger \begin{bmatrix} t_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -t_{ij}^* \end{bmatrix} \Phi_j$$

with $\Phi_i^\dagger = (c_i^\dagger, c_i)$. The subscripts $i$ and $j$ refer to site indices, and $t_{ij} = -t\delta_{i,j} + (U_i - \mu)\delta_{i,j}$ with $\delta_{i,j} = 1$ for nearest neighbour sites, and 0 otherwise. All energies in this work are measured in units of $t$, and the lattice constant is $a = 1$. The bond order-parameter is $\Delta_{ij} = -\frac{1}{2}\Delta_0[(-1)^{x_{ij}} - (-1)^{y_{ij}}]$ with $(x_{ij}, y_{ij})$ connecting sites $i$ and $j$, and where $\Delta_0$ is the homogeneous $d$-wave amplitude. The eigenstates of the finite-sized system are found using standard LAPACK routines to diagonalise Eq. 2, and the quasiparticle DOS defined by $ho(\omega) = N^{-1}\sum_n \delta(\omega - E_n)$ is then evaluated using a binning procedure. Spatial fluctuations arise naturally when one solves $\Delta_{ij}$ self-consistently in the presence of a disorder potential. For now, we simply hold $|\Delta_{ij}| = \Delta_0$ fixed. We focus primarily on the binary alloy distribution, where one assumes $U_i = U_0$ on $n_i$ of the sites, 0 elsewhere, and then averages over configurations. In this case one may compare directly with the SCTMA. In Figure 2a, we show that the SCTMA result agrees quite well with the exact result over most of the energy range. Very close to the Fermi level, the DOS appears to go to zero,

Fig. 2. a) DOS in binary alloy model near the unitary limit for $\Delta_0 = 2$. Main figure: $\mu = 0, n_i = 0.05$. Inset: $\mu = 0.2$ with $U_0^{-1} = 0.001$ (solid), 0.01 (dotted), 0.02 (dashed). Dashed-dotted line: SCTMA for $U_0^{-1} = 0.001$ b) Box disorder. Parameters: $\mu = 1.2, \Delta_0 = 2.0$. All energy units are in terms of the bare hopping $t$. 
except for the one case of very large \( U \) shown. This scale is actually determined in Figure 2a by the discreteness of the level spacing, consistent with the fact that the scale \( E_1 \) over which such suppressions are expected is exponentially small in the so-called “Dirac cone anisotropy” factor \( v_F/v_\Delta \approx 2t/\Delta_0 \). There are small deviations in the curves shown between SCTMA and the exact BdG solution, but they are confined to an energy \( E \lesssim \gamma \); we postpone discussion of these features for the moment. Note the divergence of the DOS at \( E \to 0 \) occurs only when the chemical potential is zero and the impurity scattering strength is infinite. In the language of the SCTMA, \( \mu = 0 \) and \( c = 0 \) are necessary conditions. Deviations from either of these symmetries splits the central peak and forces \( \rho(E) \to 0 \) at \( E = 0 \). The form of the divergence is difficult to deduce directly from the finite-size numerical results, although a claim has been made that the Pépin-Lee prediction provides a reasonable fit. It is worth noting that one can also split the peak as described with the addition of a next-nearest neighbor hopping to the Hamiltonian, thereby destroying the global particle-hole symmetry of the band even at \( \mu = 0 \). On the other hand, if one performs the same numerical “experiments” described above with the Ziegler et al. 3rd nearest neighbor dispersion, a constant DOS is always obtained, in agreement with Ref. 14.

Random site energy distributions, which we can also study with numerical methods by distributing \( U_i \) randomly (Gaussian, Lorentzian, “box” disorder), never give rise to the divergences seen in the binary alloy models. The density of states is again supressed to zero at sufficiently low energy, but the width of this supression is unobservably small for realistic Dirac cone anisotropies and disorder values (see Fig 2b). We have studied artificial cases with \( v_F/v_\Delta \approx 1 \), and in fact do find results consistent with a linear variation of \( \delta\rho(E) \) for strong disorder, but it is clear this result has no relevance for the real cuprates. It is furthermore obvious that the usual models with random site distributions mimic weak impurity scattering, as discussed in Ref. 14, and therefore miss these features of strong binary alloy disorder. An attempt has been made to treat unitarity scattering by conjecturing a duality symmetry connecting the perturbation expansions in the weak and strong scattering regimes. The prediction of this work is that a divergence should result whenever \( c = 0 \) for the single impurity problem. Note that this point of view implies that one should always be able to “fine-tune” the scattering potential to create a divergence in the disordered system, even away from the symmetric band case. While the idea is appealing on physical grounds, our numerical results do not support this conjecture (see Fig. 2a, and discussion in the next section). It appears that the ansatz of Ref. 21 is insensitive to nesting due to the neglect of crossing diagrams,
which spoil the duality at low energies.

4. Weak localization approach

\[ \text{T-matrix} \quad = \quad + \quad + \quad + \quad \ldots \]

\[ \text{Diffuson/ Cooperon} \quad = \quad + \quad + \]

\[ \text{Self-energy} \quad + \quad + \quad + \]

Fig. 3. Diagrams necessary for calculation of quantum correction to DOS. Dashed line is impurity interaction $\hat{U}$, and solid line is $d$-wave Nambu propagator $\hat{G}$.

The above numerical analysis is striking and does provide an explanation at a superficial level for the differences in the various nonperturbative approaches to the disordered $d$-wave problem. On the other hand, despite careful scaling calculations, any rigorous conclusions are necessarily limited by the finite size of the numerical experiments. In addition, the numerics do not provide any insight into the origin of the novel physics reflected in the divergence seen at half-filling and strong scattering, or other aspects of the results. For these reasons we have investigated analytically the leading quantum corrections to the density of states in the $d$-wave superconductor, following Senthil and Fisher in the weak scattering case, and extending the calculation to arbitrary scattering potential.

We first calculate the particle hole (diffuson $D_q$) and particle-particle (Cooperon $C_q$) propagators, keeping diagrams to formal leading order in $1/\sigma$, where $\sigma$ is the dimensionless conductance. These diagrams are shown
in Figure 3, along with the self-energy processes to the same order which determine the DOS. The usual diffusion mode associated with particle conservation is recovered at low energies,

\[ D_q(E, E') = C_q(E, E') = \frac{\gamma^2}{\pi \rho_0 D_0 q^2} \frac{1}{\rho(E - E')}, \tag{3} \]

where \( D_0 = \frac{v_F^2}{2\gamma} \log \frac{\hbar v_F}{\gamma} \) is the quasiparticle diffusion constant. The gapless mode in the Cooperon channel drives the leading singularity at low energies, yielding the Senthil-Fisher result for \( \delta \rho(E) \) given above, provided no further gapless diffusive modes exist. This is precisely the case if the Green’s function has, for a given impurity configuration, an additional symmetry relating its values on the two sublattices of the underlying crystal. We consider first the half-filled tight binding model on a square lattice with unitarity scattering case treated by Pépin-Lee, which displays the nesting (sublattice or “chiral”) symmetry \( \tau_2 G_k \tau_2 = G_{k+Q} \). Note again that the symmetry condition is only fulfilled if the potential is infinite, i.e. it effectively removes a site entirely from the lattice. It is easy to check that any second nearest neighbor hopping, model for the gap going beyond the simple nearest neighbor bond pairing model we have adopted, or any finite chemical potential will also break the symmetry. Any finite on-site potential \( U_0 \) does the same, regardless of whether or not the single-impurity resonance is fine-tuned to lie at the Fermi level, in contrast to the conjecture in Ref. 21.

![Fig. 4. Schematic dependence of low-energy DOS on symmetry class. SL and WL indicate strong and weak localization regimes, respectively.](image)

The extra symmetry induces a new massless diffusive mode in the Cooperon channel at \( \mathbf{q} = (\pi, \pi) \) leading to the divergence in \( \rho(E) \). How this
occurs may be seen by evaluating the DOS slightly away from the symmetric point,

$$\delta \rho(E) \simeq \frac{1}{2\pi D_0 \log \frac{v_F v_\Delta}{\gamma}} \left( -\log \frac{\gamma}{2E} + 5 \log \frac{\gamma}{\sqrt{4E^2 + \delta^2}} \right),$$

where $\delta$ is now the mass of the new $\pi$-mode. If the mass is finite, the Senthil-Fisher suppression of the DOS is recovered at energies below roughly $\delta/2$. On the other hand, if the mass vanishes, or if $\delta < E < \gamma$, the DOS will increase with decreasing energy due to the sign of the second factor in Eq. (4). How the mass varies with deviations from the symmetric point determines the intermediate-energy behavior of the DOS; for example, for infinite potential and a tight-binding band, the mass is given by $\delta \simeq \mu^2 / \log (v_F v_\Delta / \gamma)$. It is worth noting that if one tries to calculate this quantity in the nodal approximation, one can be easily misled; away from half-filling for infinite potential, its value is determined entirely by the Fermi surface curvature near the nodes. Finally, the Ziegler et al model can also be seen in this picture to be qualitatively different because the Hamiltonian contains no terms which mix the two sublattices; formally the Green’s function obeys the symmetry $\tau_3 \hat{G}_{\mathbf{k}} \tau_3 = \hat{G}_{\mathbf{k}+\mathbf{Q}}$. In this case, all singular contributions to $\rho(E)$ can be shown to cancel to leading order, consistent with a constant value at $E = 0$. Altland and Zirnbauer have pointed out that this model is equivalent in its localization properties to the symmetry class of a normal metal (class AI); it is nevertheless a true superconductor with a Meissner response. Since its electronic dispersion is very unphysical, this model, while soluble, has no predictive power for the cuprates. In Figure 4 we show the schematic behavior of the DOS for the various cases we have considered. Note that the functional form of the DOS at asymptotically low energies (below $E_1$) is not determined by this method, as indicated by the dashed portions of the curves.

Before leaving the subject of nesting symmetries in the homogeneous case, we ask what a symmetry analysis might have told us a priori about the low energy behavior of the DOS. Although the Hamiltonian of a bond-paired $d$-wave superconductor with half-filled band and unitarity limit scatterers obviously displays a Bogoliubov-de Gennes symmetry, it should not belong to the Cartan symmetry classes C,CI,D, DIII usually associated with superconductors. According to the usual definitions, only CI corresponds to the unbroken time-reversal and spin-rotation symmetry we consider here; the density of states for this class corresponds to the Fisher-Senthil result $\rho(E) \sim E$, as discussed above. The obvious conclusion is that the additional nesting symmetry combines with the BdG symmetry operation to move the system into a new Cartan class; this has been recently determined to be
the unitary chiral metal class AIII. However, it is puzzling that both the single-node model originally studied by Nersesyan et al. and the half-filled unitary scattering case both correspond to class AIII (see Table 1); the former model yields a power law DOS $\rho(E) \sim |E|^\alpha$, whereas the latter gives a divergence at small $E$.

The answer to this particular puzzle is that the Cartan classification is a purely algebraic (group theoretical) one which must sometimes be refined by allowing topological considerations, i.e., by deciding whether or not a physical model should be approximated in the context of the scaling theory of localization by a non-linear sigma model with the addition of possible topological terms. In this way models belonging to the same Cartan class can nevertheless display strikingly different localization properties. The most famous example thereof is the integer quantum Hall effect that naively belongs to class A but whose localization properties are described by a non-linear sigma model supplemented by a topological term. The model of Nersesyan et al. is also endowed with such a topological term, leading to the deviation from the naive AIII prediction. In addition, the low energy behavior depends on the strength of the disorder, and for sufficiently strong disorder gives a constant density of states.

Thus knowledge of the Cartan symmetry class is not a priori sufficient to fix the low-energy behavior; a model calculation is required. In the background is a more general set of questions raised in Refs. concerning whether a given disordered electron system “automatically” belongs to one of Cartan’s classes, or whether it requires some fine tuning of the disorder distribution. We do not address these issues here.

It is also obvious that there is a further discrepancy evident in Table 1, in that the same model is claimed to have two different asymptotic behaviors of the DOS as $E \to 0$ in Refs. and . This is why we have indicated the symmetry class of the model in both cases as AIII?. At present it is unclear if numerical approaches can distinguish between these two predictions, but one of them must be incorrect.

5. Inhomogeneous order parameter

Thus far we have considered numerical and analytical calculations with constant order parameter only. In self-consistent calculations with strong scatterers, $\Delta_{ij}$ vanishes (or nearly vanishes) along bonds connected to the impurity site, and rises to its asymptotic value over a few lattice constants. The naive expectation is that the pockets of normal metal which form around the impurities should enhance $\rho(0)$, but numerical studies show that the
superconducting gap tends to reopen relative to the SCTMA. In Figure 5, we exhibit this tendency for a realistic parameter set (i.e. no special nesting symmetries).

![Graph showing comparison of density of states]

Fig. 5. Comparison of density of states with homogeneous (A) and self-consistently determined (B) order parameter. Parameters: $n_i = 0.06$, $\mu = 1.2t$, $\Delta_0 = 0.8$.

We note that, close to unitarity (defined by $\Omega_0 = 0$), the homogeneous gap case is essentially constant at the Fermi level except for the single point at zero energy suppressed due primarily to finite size effects. By contrast, even close to unitarity the DOS exhibits a cusp-like behavior. Away from unitarity, the disorder-induced suppression occurs on a scale which we have argued scales with $\Omega_0$. Note this quantity is renormalized by the inhomogeneity in a nonuniversal way. The behavior of the low-energy DOS in these calculations is quite different from the non-self-consistently determined order parameter case. In Figure 6, we show that over a wide range at low energies, the DOS can be approximated by a power law, whose exponent is nonuniversal, in that it depends on the impurity potential, concentration, etc. Of
course, we can make no statements about the asymptotic low-energy form below some exponentially small strong localization scale. It is conceivable, even likely, that the lowest energy behavior is again given by the universal class CI (Senthil-Fisher) behavior. However, is at least interesting to ask whether off-diagonal (in particle-hole space) disorder correlated with bare (diagonal) disorder does not correspond to a different Cartan symmetry class, or whether it is amenable to this classification scheme at all. No field theoretical treatments of this problem are available to our knowledge. Again, we note that the asymptotic low energy behavior is, in any case, unlikely to be observed in experiment since the Dirac cone anisotropy $v_F/v_\Delta$ is always significantly greater than one.

![Graphs showing low-E power law DOS for self-consistently determined order parameter.](attachment:image.png)

Fig. 6. Low-E power law DOS for self-consistently determined order parameter. Left panel: solid line is fit to power law $\rho(E) \sim |E|^\alpha$ for $n_i = 0.04$ and $U_0 = 5$, close to unitarity limit $\Omega_0 = 0$. Right panel: power law exponent $\alpha$ and SCTMA $\rho(0)$ vs. impurity potential $U_0$ for $n_i = 0.02$.

The physical origin of the disorder-induced supression of the DOS relative to SCTMA is not completely clear. Despite the correlation with the single-impurity resonance energy, it appears to be a genuine many-impurity interference effect. We speculate that it may be formally related
to the physics of the Coulomb gap in interacting disordered metals, which can be captured by the Hartree-Fock approximation for the Coulomb interaction. Here, in analogous fashion, we treat the pairing interaction at the mean-field (BdG) level. Further work along these lines is in progress.

We conclude this section by noting that, while the generic case of the inhomogeneous order parameter is quite different from the homogeneous case, the nesting symmetries continue to play a role. For example, the $\rho(E)$ divergence at zero energy in the $\tau_2$ nesting case is preserved when one calculates the order parameter self-consistently, although its width is renormalized.

6. Half-filled normal metal

Since the nesting symmetries discussed in Section 4 occur also for normal metals in special cases, it is interesting to return to these simpler problems to see what one can learn. A 2D simple tight-binding band of electrons displays already in the pure case a weak logarithmic van Hove singularity at $E = 0$. Naively, one would expect generic disorder to mix the $k$-states near the Fermi level and cut off the weak nesting singularity, leading to finite DOS at half filling in the presence of any disorder. However, Nakhmedov et al. recently considered this problem in a weak localization framework, and came to the conclusion that the DOS was infinite. Such a conclusion is not without precedent in normal metals. Dyson studied 1D chains with chiral symmetry, i.e. systems where neither the pure Hamiltonian nor the disorder potential (e.g., random hopping) breaks the sublattice symmetry $\tau_2\hat{G}_k\tau_2 = \hat{G}_{k+Q}$. Gade and Wegner discussed a 2D example of a disordered nonlinear sigma model obeying the same symmetry. In these cases the DOS at zero energy was indeed shown to be infinite. However, in the more generic case studied by Nakhmedov et al., the on-site potential scattering explicitly breaks the sublattice symmetry; Gruzberg et al. therefore criticized the claim of infinite DOS, insisting the van Hove singularity should not affect the generic finite DOS one expects in a normal metal. They redid the weak localization calculation, obtaining a finite result.

It is of course easy to apply the numerical scheme developed for the $d$-wave superconductor to the half-filled tight-binding band. Our results are shown in Fig. 7. For finite impurity potential strength, the DOS exhibits no divergence at $E = 0$, supporting the claim of Gruzberg et al. On the other hand, for $U_0 \to \infty$, a narrow peak does appear. While with numerical methods we are unable to distinguish between a finite peak at zero energy and a true divergence, it is striking that this behavior appears to occur only when
the global sublattice symmetry $\tau_2 \hat{G}_k \tau_2 = \hat{G}_{k+\mathbf{Q}}$ applies ($\hat{G}$ is now diagonal) for a given disorder configuration. The infinite potential simply removes a site from the lattice and therefore does not break the sublattice symmetry. Formally, this system is in Cartan symmetry class BDI, whereas the finite potential case belongs to class AI. This distinction has been thoroughly discussed and understood in in disordered quantum wires\textsuperscript{42}, and on this basis we indeed anticipate that the numerical result in Figure 6 indicates a divergence for the case of infinite potential.\textsuperscript{43} Hints of the effects of this nesting symmetry were observed already in models with random hopping.\textsuperscript{44}

7. Conclusions

In this paper we have tried to review a variety of approaches to the problem of disorder in the $d$-wave state at low energies. While we succeeded, using a combination of numerical and analytical methods, in understanding the sensitivity of this state to different symmetries of the band and of the disorder realization, most of the issues involved were of a purely theoretical nature. This is due to several factors: first, the strong localization scale is exponentially small in the Dirac cone anisotropy (expected to be of order
10 in the cuprates), so within the calculations with homogeneous order parameter the SCTMA proves to be adequate, modulo some minor bumps at intermediate energies, if one works sufficiently far from half-filling. Secondly, the nesting symmetries considered do not apply in real systems, as they are always broken by finite impurity potential, nonzero chemical potential, and $t'$ hopping. Next, the systems in question are weakly three dimensional, and the mass in the $\pi$ mode will always saturate at a 3D crossover scale we estimate to be $t^2_{\perp}/\gamma$, where $t_{\perp}$ is the interlayer hopping. On this basis, we have suggested that if the physics of low-dimensional correlated scattering is ever to be observed experimentally, the best chances are in underdoped, intrinsically anisotropic materials where the interlayer coupling is weakest. There are some preliminary results that such a downturn in the DOS indeed occurs in highly underdoped LSCO but the situation is far from clear at the present writing. All other experimental results on systematically disordered cuprates of which we are aware (performed almost exclusively on systems near optimal doping) are consistent with a finite disorder-induced DOS.

The final effect we have discussed is the profound influence of order parameter inhomogeneity induced by and correlated with disorder in the $d$-wave superconductor. This phenomenon renders the problem considerably less amenable to analytical treatment, and certainly warrants further study. Weak localization effects are the primary cause of differences between the exact BdG calculations with homogeneous order parameter and the SCTMA, and it is clear that they will be irrelevant in a truly 3D system, where we know the crossed diagrams are negligible. On the other hand, we have no real evidence that the effect of order parameter inhomogeneity weakens significantly in higher dimensions, since there is no well-controlled “mean field theory of disorder” which includes it explicitly. It is interesting to ask whether these effects can strongly influence transport, since they fundamentally change the nature of scattering even at the single-impurity level, by adding a scattering channel with a completely different (Andreev) symmetry. Preliminary calculations of the conductivity in these models indicate that this is indeed the case.

Dedication. This paper is gratefully and affectionately dedicated by PJH to Peter Wölfle on the occasion of his 60th birthday.

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