MAXIMAL RATIO OF COEFFICIENTS OF DIVISORS
AND AN UPPER BOUND FOR HEIGHT FOR RATIONAL MAPS

CHONG GYU LEE

ABSTRACT. In this paper, we introduce the $D$-ratio of a rational map $f : \mathbb{P}^n \to \mathbb{P}^n$, defined over a number field $K$, whose indeterminacy locus is contained in a hyperplane $H$ on $\mathbb{P}^n$. The $D$-ratio $r(f)$ provides useful height inequalities on $\mathbb{P}^n(K) \setminus H$: there is a constant $C$, depending only on $f$, such that
\[
\frac{r(f)}{\deg f} h(f(P)) + C > h(P) \quad \text{for all } P \in \mathbb{P}^n(K) \setminus H.
\]
If the indeterminacy loci of $f_1$, $f_2$ are disjoint subsets in $H$, then there is a constant $C'$, depending only on $f_1$, $f_2$, such that
\[
\frac{1}{\deg f_1} h(f_1(P)) + \frac{1}{\deg f_2} h(f_2(P)) + C' > \left( 1 + \min_{l=1,2} \left( \frac{1}{r(f_l)} \right) \right) h(P) \quad \text{for all } P \in \mathbb{P}^n(K) \setminus H.
\]
Also, we provide some dynamical applications of those height inequalities.

1. Introduction

Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a rational map, defined over a number field $K$, and suppose its indeterminacy locus $I(f)$ is contained in a hyperplane $H$ on $\mathbb{P}^n$. In this article, we introduce the $D$-ratio $r(f)$ associated to $f$ and a resolution of indeterminacy. We use $r(f)$ to provide a relation between $h(P)$ and $h(f(P))$, where $h : \mathbb{P}^n(K) \to \mathbb{R}$ be the logarithmic absolute height function.

**Theorem A.** Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a rational map, defined over a number field $K$, such that $I(f)$ is contained in a hyperplane $H$ and let $r(f)$ be a $D$-ratio of $f$ obtained from a resolution of indeterminacy. Then, there is a constant $C$, depending only on $f$, such that
\[
\frac{r(f)}{\deg f} h(f(P)) + C > h(P) \quad \text{for all } P \in \mathbb{P}^n(K) \setminus H.
\]

**Theorem B.** Let $f_1, f_2 : \mathbb{P}^n \to \mathbb{P}^n$ be rational maps, defined over a number field $K$, such that the indeterminacy loci of $f_1, f_2$ are disjoint subsets of a hyperplane $H$ and let $r(f_1)$ be a $D$-ratio of $f_1$. Then, there is a constant $C$, depending only on $f_1, f_2$, such that
\[
\frac{1}{\deg f_1} h(f_1(P)) + \frac{1}{\deg f_2} h(f_2(P)) + C > \left( 1 + \min_{l=1,2} \left( \frac{1}{r(f_l)} \right) \right) h(P) \quad \text{for all } P \in \mathbb{P}^n(K) \setminus H.
\]

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We have Northcott’s theorem for endomorphisms: If \( \phi : \mathbb{P}^n \to \mathbb{P}^n \) is an endomorphism, then there are two nonnegative constants \( C_1, C_2 \), depending on \( \phi \), such that

\[
\frac{1}{\deg \phi} h(\phi(P)) + C_1 > h(P) > \frac{1}{\deg \phi} h(\phi(P)) - C_2 \quad \text{for all } P \in \mathbb{P}^n(\overline{K}).
\]

Northcott’s theorem is one of the essential theorems in arithmetic dynamics. For example, the Call-Silverman canonical height function [1] for an endomorphism on a projective space is well defined because of Northcott’s theorem.

In fact, Northcott’s theorem only holds for endomorphisms:

**Theorem C.** Let \( f : \mathbb{P}^n \to \mathbb{P}^n \) be a rational map defined over a number field \( K \). Suppose that \( f \) satisfies the following inequality for some nonempty Zariski open set \( U \) of \( \mathbb{P}^n \) and some constant \( C \):

\[
\frac{1}{\deg f} h(f(P)) + C > h(P) \quad \text{for all } P \in U(\overline{K}).
\]

Then, \( f \) is an endomorphism.

Note that \( h(P) \) is always bounded below by \( \frac{1}{\deg f} h(f(P)) - C_2 \) for any rational map \( f \) of degree \( d \) so that we only care about upper bound of \( h(P) \). (For details, see [23, Theorem B.2.5].)

Thus, the canonical height functions and other arithmetic-dynamical consequences may fail to exist in the presence of indeterminacy. However, \( D \)-ratios provide weaker height inequalities as stated in Theorem A and Theorem B. Those inequalities allow us to study arithmetic-dynamical properties of rational maps. For example, the Hénon map \( g : \mathbb{K}^n \to \mathbb{K}^n \) is a polynomial map of special dynamical interest. It is not an endomorphism on a projective space and hence Northcott’s theorem does not hold for \( g \). Nevertheless, it is well known that the Hénon map has good dynamical properties such as the boundedness of the set of periodic points. (For details, see [4, 15, 20].) Especially, a regular Hénon map \( g \), together with the inverse \( g^{-1} \), satisfies the hypothesis of Theorem B and hence \( g \) has a canonical height [10, 11, 14].

The main idea of the \( D \)-ratio is to generalize the case of endomorphisms. The degree of an endomorphism \( \phi : \mathbb{P}^n \to \mathbb{P}^n \) is the coefficient of \( H \) in \( \phi^*H = \deg \phi \cdot H \). In other words, we have

\[
\deg \phi = \sup \left\{ \delta \mid \frac{1}{\delta} \phi^*H - H \text{ is effective in } \text{Pic}(\mathbb{P}^n) \otimes \mathbb{R} \right\}.
\]

Then, the functorial property of the Weil height machine [23, Theorem B.3.2] gives the comparison of \( h(P) \) and \( h(\phi(P)) \):

\[
h_H(\phi^*(P)) = h_{\phi^*}(P) + O(1) = \deg \phi \cdot h_H(P) + O(1).
\]

Let \( f : \mathbb{P}^n \to \mathbb{P}^n \) be a rational map. Due to failure of the functoriality of the Weil height machine, we pass to a resolution of indeterminacy to work with a morphism: there exist a nonsingular projective variety \( V \) and a birational morphism \( \pi : V \to \mathbb{P}^n \) such that \( \tilde{f} = f \circ \pi \) extends to a morphism.

\[
\begin{align*}
\pi & : V \\
\mathbb{P}^n & \to \mathbb{P}^n \\
\phi & : \mathbb{P}^n \to \mathbb{P}^n \\
\end{align*}
\]

\[
\begin{align*}
\pi & \downarrow \quad \tilde{f} \downarrow \\
\mathbb{P}^n & \to \mathbb{P}^n \\
\phi & \downarrow \\
\mathbb{P}^n & \to \mathbb{P}^n \\
\end{align*}
\]
(In the case of endomorphisms, we may think $\pi$ to be the identity map on $\mathbb{P}^n$.) Now, we have two morphisms $\tilde{f}, \pi : V \to \mathbb{P}^n$. We can compare $\tilde{f}^*H$ and $\pi^*H$ in Pic($V$) and hence

$$h_H(\tilde{f}(P)) = h_{\tilde{f}^*H}(P) + O(1) \quad \text{and} \quad h_H(\pi(P)) = h_{\pi^*H}(P) + O(1).$$

Roughly, we define the $D$-ratio to be the constant $r(f)$ such that

$$\deg f \left( \frac{1}{r(f)} \right) = \sup \left\{ \delta \left| \frac{1}{\delta} \tilde{f}^*H - \pi^*H \text{ is $\mathbb{A}^n$-effective in Pic($V$) $\otimes$ $\mathbb{R}$} \right. \right\}.$$

Using properties of "$\mathbb{A}^n$-effective" divisors, we get Theorem A and Theorem B. Also, we conclude that

$$r(f) = 1 \quad \text{if and only if} \quad f \text{ is an endomorphism}. $$

Note that the definition of the $D$-ratio depends on the choice of a resolution of indeterminacy. However, we show that it depends only on the “strong factorization class” of the resolution. (See Lemma 4.4.) In particular, in dimension 2, the $D$-ratio depends only on $f$.

We provide two applications of Theorem A and Theorem B in arithmetic dynamics. For convenience, define

$$\text{Rat}^n(H) := \{ f : \mathbb{P}^n \to \mathbb{P}^n \mid I(f) \subset H \}.$$  

If $f \in \text{Rat}^n(H)$ is a rational map such that $r(f) < \deg f$, then Theorem A directly induces the following result:

**Theorem D.** Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial map, defined over a number field $K$, such that $r(f) < \deg f$. Then, the set of preperiodic points

$$\text{Preper}(f) := \left\{ P \in \mathbb{A}^n(K) \mid f^l(P) = f^m(P) \text{ for some } l \neq m \right\}$$

is a set of bounded height. Hence

$$\text{Preper}(f) \cap \mathbb{A}^n(K')$$

is a finite set for any number field $K'$.

The requirement $r(f) < \deg f$ in Theorem D is sharp; there are rational maps such that $r(f) = \deg f$ and $\text{Preper}(f)$ is not bounded. (See Example 5.3.) Still, we can find some information for such rational map $f$ if it has a good counterpart.

**Theorem E.** Let $S = \{ f_1, f_2 \}$ be a pair of polynomial maps, defined over a number field $K$, such that their indeterminacy loci are disjoint subsets of a hyperplane $H$, let $f(f_i)$ be a $D$-ratio of $f_i$ and let $\Phi_S$ be the monoid of rational maps generated by $S$:

$$\Phi_S := \{ f_{i_1} \circ \cdots \circ f_{i_m} \mid i_j = 1 \text{ or } 2, \ m \geq 0 \}.$$  

Define

$$\delta_S := \left( \frac{1}{1 + 1/r} \right) \left( \frac{1}{\deg f_1} + \frac{1}{\deg f_2} \right)$$

where $r = \max_{l=1,2}(r(f_l))$.

If $\delta_S < 1$, then

$$\text{Preper}(\Phi_S) := \bigcap_{f \in \Phi_S} \text{Preper}(f) \subset \mathbb{A}^n_K$$

is a set of bounded height.
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2. Preliminaries: Blowup and its Picard Group

In this section, we check the basic theory of the resolution of indeterminacy. For details, I refer [2, 7] to the reader. We will let $H$ be a fixed hyperplane of $\mathbb{P}^n$, let $\mathbb{A}^n = \mathbb{P}^n \setminus H$ and let $f$ be an element of $\text{Rat}^{n}(H)$ defined over a number field $K$ unless stated otherwise.

**Theorem 2.1** (Resolution of indeterminacy). Let $f : X \dashrightarrow Y$ be a rational map between proper varieties such that $X$ is nonsingular. Then there is a proper nonsingular variety $\tilde{X}$ with a birational morphism $\pi : \tilde{X} \to X$ such that $\phi = f \circ \pi : \tilde{X} \to Y$ extends to a morphism:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\phi} & Y \\
\pi \downarrow & & \downarrow \phi \\
X & \xrightarrow{f} & Y
\end{array}
$$

Using Hironaka’s Theorem (Theorem 2.4), we will observe the relation between the resolution of indeterminacy and the indeterminacy locus of $f$.

**Definition 2.2.** Let $\pi : \tilde{X} \to X$ be a birational morphism. Then, we say that a closed subscheme $I$ of $X$ is the center scheme of $\pi$ if the ideal sheaf $S$, corresponding to $I$, generates $\tilde{X}$:

$$
\tilde{X} = \text{Proj} \left( \bigoplus_{d \geq 0} S^d \right).
$$

**Definition 2.3.** Let $\pi : \tilde{X} \to X$ be a birational morphism. We say that $\pi$ is a monoidal transformation if its center scheme is a smooth irreducible subvariety of $X$. We say that $\tilde{X}$ is a successive blowup of $X$ if the corresponding birational map $\pi : \tilde{X} \to X$ is a composition of monoidal transformations.

**Theorem 2.4** (Hironaka). Let $f : X \dashrightarrow Y$ be a rational map between proper varieties such that $X$ is nonsingular. Then, there is a finite sequence of proper varieties $X_0, \ldots, X_r$ such that

1. $X_0 = X$.
2. $\rho_i : X_i \to X_{i-1}$ is a monoidal transformation.
3. If $T_i$ is the center scheme of $\rho_i$, then $\rho_1 \circ \cdots \circ \rho_{i-1}(T_i) \subset I(f)$ on $X$.
4. $f$ extends to a morphism $\tilde{f} : X_r \to Y$.
5. Consider the composition of all monoidal transformation $\rho : X_r \to X$. Then, the underlying subvariety of the center scheme $T$ of $\rho$, a subvariety made by the zero set of the ideal sheaf corresponding to $T$, is exactly $I(f)$.

**Proof.** See [8, Question (E) and Main Theorem II].

For notational convenience, we will define the following.
Definition 2.5. Let \( f : \mathbb{P}^n \to \mathbb{P}^n \) be a rational map. We say that a pair \((V, \pi)\) is a resolution of indeterminacy of \( f \) when \( V \) is a successive blowup of \( \mathbb{P}^n \) with a birational morphism \( \pi : V \to \mathbb{P}^n \) such that

\[
f \circ \pi : V \to \mathbb{P}^n
\]

extends to a morphism. And we call the morphism \( \phi := f \circ \pi \) a resolved morphism of \( f \).

In Section 3, we will find a basis of \( \text{Pic}(V) \) when \((V, \pi)\) is a resolution of indeterminacy of some rational map \( f \). Especially, we need a basis consisting of irreducible divisors. However, pullbacks of irreducible divisors may not be irreducible because of the exceptional part. So, we define the proper transformation, which preserves the irreducibility.

Definition 2.6. Let \( \pi : \tilde{X} \to X \) be a birational morphism with the center scheme \( \mathcal{I} \) and let \( D \) be an irreducible divisor on \( X \). We define the proper transformation of \( D \) by \( \pi \) to be

\[
\pi^# D = \pi^{-1}(D \cap U)
\]

where \( U = X \setminus Z(\mathcal{I}) \) and \( Z(\mathcal{I}) \) is the underlying subvariety made by the zero set of the ideal corresponding to \( \mathcal{I} \).

Proposition 2.7. Let \( V \) be a successive blowup of \( \mathbb{P}^n \) with a birational morphism \( \pi : V \to \mathbb{P}^n \); there are monoidal transformations \( \pi_i : V_i \to V_{i-1} \) such that \( V_r = V \) and \( V_0 = \mathbb{P}^n \). Let \( H \) be a hyperplane on \( \mathbb{P}^n \), let \( F_i \) be the exceptional divisor of the blowup \( \pi_i : V_i \to V_{i-1} \), let \( \rho_i = \pi_{i+1} \circ \cdots \circ \pi_r \) and let \( E_i = \rho_i^# F_i \). Then, \( \text{Pic}(V) \) is a free \( \mathbb{Z} \)-module with a basis

\[
\{H_V = \pi^# H, E_1, \cdots, E_r\}.
\]

Proof. [7, Exer.II.7.9] shows that

\[
\text{Pic}(\tilde{X}) \cong \text{Pic}(X) \oplus \mathbb{Z}
\]

if \( \pi : \tilde{X} \to X \) is a monoidal transformation. More precisely,

\[
\text{Pic}(\tilde{X}) = \{\pi^# D + nE \mid D \in \text{Pic}(X)\}
\]

where \( E \) is the exceptional divisor of \( \pi \) on \( \tilde{X} \). Apply it to each \( \pi_i \) and get the desired result. \( \square \)

3. \( \mathbb{A}^n \)-effective divisor

We roughly describe the \( D \)-ratio of \( f \) as follows: (Precise definition of the \( D \)-ratio will be given in Section 4.) \( r(f) \) is the constant such that

\[
\frac{\deg f}{r(f)} := \sup \left\{ \delta \left| \frac{1}{\delta} f^* H - \pi^* H \text{ is } \mathbb{A}^n \text{-effective in } \text{Pic}(V) \otimes \mathbb{R} \right. \right\}
\]

where \((V, \pi)\) is a resolution of indeterminacy, \( \tilde{f} \) is a resolved morphism and “\( \mathbb{A}^n \)-effective divisor” is the main topic of this section. Like the case of endomorphisms, we may use “effective” instead of new term “\( \mathbb{A}^n \)-effective.” However, it is hard to describe the effective cone of \( V \) even though \( \text{Pic}(V) \) is a free \( \mathbb{Z} \)-module. Moreover, we cannot control the base locus of all effective divisors. So, we will take the \( \mathbb{A}^n \)-effective cone \( \text{AE}(V) \) such that 1) \( \text{AE}(V) \) is a simple subset of the effective cone and 2) all element of \( \text{AE}(V) \) has the base locus outside of \( \mathbb{A}^n \). Remind that \( H \) is a fixed hyperplane of \( \mathbb{P}^n \), \( \mathbb{A}^n = \mathbb{P}^n \setminus H \) and \( f \) is an element of \( \text{Rat}^n(H) \) defined over a number field \( K \).
Definition 3.1. Let $V$ be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$ such that the underlying set of the center scheme of $\pi$ is a subset of $H$, let $H$ be a fixed hyperplane of $\mathbb{P}^n$ and let

$$\text{Pic}_R(V) = \mathbb{R}H_V \oplus \mathbb{R}E_1 \oplus \cdots \oplus \mathbb{R}E_r$$

with the basis $\{H_V, E_1, \cdots, E_s\}$ described in Proposition 2.7. We define the $\mathbb{A}^n$-effective cone of $V$ to be

$$\text{AE}(V) := \mathbb{R}^0H_V \oplus \mathbb{R}^0E_1 \oplus \cdots \oplus \mathbb{R}^0E_r$$

where $\mathbb{R}^0$ is the set of nonnegative real numbers. We say that an element $D \in \text{Pic}_R(V)$ is $\mathbb{A}^n$-effective if $D$ is contained in $\text{AE}(V)$ and denote it by

$$D \succ 0.$$ 

Moreover, on $\text{Pic}_R(V)$, we write

$$D_1 \succ D_2$$

if $D_1 - D_2$ is $\mathbb{A}^n$-effective.

Recall that $H$ is a fixed hyperplane. So, we have a fixed basis of $\text{Pic}(V)$. It implies the representation of an element in $\text{Pic}(V)$ is unique and hence the $\mathbb{A}^n$-effectiveness is well defined.

$\mathbb{A}^n$-effective divisors have some useful properties. To show them, we need the following lemma which is also important to define the $D$-ratio of a rational map later.

Lemma 3.2. Let $\pi : V \to \mathbb{P}^n$ and $\rho : W \to V$ be compositions of monoidal transformations such that the underlying sets of the center schemes of $\pi$ and $\pi \circ \rho$ are subsets of $H$, let $\{H_V, E_1, \cdots, E_r\}$ and $\{H_W, F_1, \cdots, F_s\}$ be bases of $\text{Pic}(V)$ and $\text{Pic}(W)$ respectively, described in Proposition 2.7 and let

$$\rho^*H_V = \rho^#H_V + \sum_{j=1}^s m_{0j}F_j \quad \text{and} \quad \rho^*E_i = \rho^#E_i + \sum_{j=1}^s m_{ij}F_j.$$ 

Then,

$$m_{ij} \geq 0 \quad \text{for all } i, j.$$ 

Furthermore,

$$\sum_{i=0}^r m_{ij} > 0 \quad \text{for all } j = 1, \cdots, s.$$ 

Proof. Fix $j \in \{1, \cdots, s\}$. Since the pullback of $E_i$ by $\rho$ is defined

$$\rho^*E_i = \rho^{-1}E_i$$

where $\rho^{-1}E_i$ is the scheme theoretic preimage. So, if $\rho(F_j) \subset E_i$, then $m_{ij} > 0$. Otherwise, $m_{ij} = 0$. Furthermore, since the underlying set $Z$ of the center of blowup of $W$ is in $H_V \cup \left( \bigcup_{i=1}^r E_i \right)$ by assumption, an irreducible subset $\rho(F_j)$ of $Z$ should be contained in one of irreducible components of $H_V \cup \left( \bigcup_{i=1}^r E_i \right)$, which is either $H_V$ or some $E_i$. Therefore, $m_{ij} > 0$ for at least one $i$. \qed
Proposition 3.3. Let $V$ be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$ and let $D, D_1, D_2, D_3 \in \text{Pic}_R(V)$.

(1) (Effectiveness) If $D$ is $\mathbb{A}^n$-effective, then $D$ is effective.
(2) (Boundedness) If $D$ is $\mathbb{A}^n$-effective, then $h_D(P)$ is bounded below on

$$\pi^{-1}(\mathbb{A}^n) = V \setminus \left(H_V \cup \left(\bigcup_{i=1}^r E_i\right)\right).$$

(3) (Transitivity) If $D_1 \succ D_2$ and $D_2 \succ D_3$, then $D_1 \succ D_3$.
(4) (Functoriality) If $\rho : W \to V$ is a monoidal transformation and $D_1 \succ D_2$, then $\rho^* D_1 \succ \rho^* D_2$.

Proof. (1) It is obvious since $\text{AE}(V)$ is a subset of the effective cone of $V$.

(2) Since $D$ is $\mathbb{A}^n$-effective, it is effective. By the positivity of the Weil height machine [23, Theorem B.3.2.(e)], we get that

$$h_D(P) > O(1) \quad \text{for all} \quad P \in V \setminus |D|$$

where $|D|$ is the base locus of $D$. The base locus of $D$, the intersection of all effective $(n-1)$-cycles linearly equivalent to $D$, is contained in any effective cycle linearly equivalent to $D$. By assumption, $D \sim p_0 H_V + \sum_{i=1}^r p_i E_i$ for some nonnegative integers $p_i$’s and hence $|D| \subseteq H_V \cup \left(\bigcup_{i=1}^r E_i\right)$. Therefore,

$$V \setminus \left(H_V \cup \left(\bigcup_{i=1}^r E_i\right)\right) \subseteq V \setminus |D|.$$ 

Furthermore, by assumption $I(f) \in H$ and Theorem [24]

$$\pi^{-1}(H) = H_V \cup \left(\bigcup_{i=1}^r E_i\right)$$

and hence

$$\pi^{-1}(\mathbb{P}^n \setminus H) = V \setminus \left(H_V \cup \left(\bigcup_{i=1}^r E_i\right)\right)$$

(3) If $D_1 \succ D_2$ and $D_2 \succ D_3$, then $D_1 - D_2$ and $D_2 - D_3$ are in $\text{AE}(V)$. Since $\text{AE}(V)$ is closed under addition by definition, $D_1 - D_3 = (D_1 - D_2) + (D_2 - D_3) \in \text{AE}(V)$.

(4) Let

$$\text{Pic}(V) = \mathbb{Z} H_V \oplus \mathbb{Z} E_1 \oplus \cdots \oplus \mathbb{Z} E_r,$$

let $W$ be a blowup of $V$ with a monoidal transformation $\rho : W \to V$. Then, $\text{Pic}(W)$ is still a free $\mathbb{Z}$-module:

$$\text{Pic}(W) = \mathbb{Z} H_V^\# \oplus \mathbb{Z} E_1^\# \oplus \cdots \oplus \mathbb{Z} E_r^\# \oplus \mathbb{Z} F$$

where $H_V^\# = \rho^* H_V$, $E_i^\# = \rho^* E_i$ and $F$ is the exceptional divisor of $W$ over $V$. Moreover,

$$\rho^* H_V = H_V^\# + m_0 F \quad \text{and} \quad \rho^* E_i = E_i^\# + m_i F$$

for some $m_i$, which are nonnegative integers by Lemma [3.2].
Therefore, for any $\mathbb{A}^n$-effective divisor $D = p_0 H_V + \sum_{i=1}^{r} p_i E_i \in \text{Pic}_R(V)$,

$$\rho^* D = p_0 (\rho^* H_V) + \sum_{i=1}^{r} p_i (\rho^* E_i) = p_0 H_V^\# + \sum_{i=1}^{r} p_i E_i^\# + \left( \sum_{i=0}^{r} p_i m_i \right) F$$

is $\mathbb{A}^n$-effective on $W$ because $p_i$'s and $m_i$'s are nonnegative integers. $\square$

4. Maximal ratio of coefficient of divisors

In this section, we introduce the main idea of this paper - the $D$-ratio. Since we fixed a hyperplane $H$ of $\mathbb{P}^n$, we have a fixed basis of $\text{Pic}(V)$ so that the representation of $D \in \text{Pic}(V)$ is unique. Hence, the maximal ratio of coefficients of $H_V$ and $E_i$'s in $\phi^* H$ and $\pi^* H$ is well defined.

**Definition 4.1.** Let $f \in \text{Rat}^n(H)$, let $(V, \pi)$ be a resolution of indeterminacy of $f$ and let $\phi$ be the resolved morphism of $f$ on $V$:

$$
\begin{array}{c}
V \\
\phi \downarrow \phi \\
\pi \downarrow \pi \\
\mathbb{P}^n \\
\end{array}
$$

Suppose that

$$\pi^* H = a_0 H_V + \sum_{i=1}^{r} a_i E_i \quad \text{and} \quad \phi^* H = b_0 H_V + \sum_{i=1}^{r} b_i E_i.$$  

If $b_i$ are nonzero for all $i$ satisfying $a_i \neq 0$, we define the $D$-ratio of $\phi$ to be

$$r(\phi) := \deg \phi \cdot \max_i \left( \frac{a_i}{b_i} \right).$$

If there is an $i$ satisfying $a_i \neq 0$ and $b_i = 0$, define

$$r(\phi) := \infty.$$  

**Remark 4.2.** Let $f \in \text{Rat}^n(H)$ and let $(V, \pi_V)$ be a resolution of indeterminacy of $f$. Then,

$$r(\phi) := \min \left\{ C \left| \frac{C}{\deg f} : \phi^* H - \pi^* H \succ 0 \right. \right\}.$$  

**Definition 4.3.** Let $f \in \text{Rat}^n(H)$ be a rational map defined over a number field $K$. Then, we define the $D$-ratio of $f$ on $V$,

$$r(f) = r(f, (V, \pi)) := r(\phi)$$

where $(V, \pi)$ is a resolution of indeterminacy of $f$ described in Theorem 2.4 and $\phi$ is the resolved morphism of $f$ on $V$.

In the rest of this paper, we will use $r(f)$ instead of $r(f, (V, \pi))$. The main application of the $D$-ratio is constructing height inequalities on $\mathbb{A}^n(K)$ and they do not care about the choice of $(V, \pi)$. Also, once we fix a resolution of indeterminacy $(V, \pi)$ of $f$, any successive blowup of $V$ will provides the same result.
Lemma 4.4. Let $(V, \pi_V)$ and $(W, \pi_W)$ be resolutions of indeterminacy of $f$ with resolved morphisms $\phi_V = f \circ \pi_V$ and $\phi_W = f \circ \pi_W$ respectively. Let $\tau = \pi_V^{-1} \circ \pi_W$.

$$
\begin{array}{c}
W \rightarrow \tau \rightarrow V \\
\pi_W \downarrow \phi_W \quad \phi_V \downarrow \pi_V
\end{array}
$$

$$
\begin{array}{c}
P^n \rightarrow P^n \xrightarrow{f} P^n
\end{array}
$$

Suppose that $\tau : W \rightarrow V$ allows strong factorization: there is a common blowup $U$ of $V$ and $W$ such that $\tau_V : U \rightarrow V$ and $\tau_W : U \rightarrow W$ are compositions of monoidal transformations.

$$
\begin{array}{c}
U \rightarrow \tau_V \rightarrow V \\
\pi_W \downarrow \phi_W \quad \phi_V \downarrow \pi_V
\end{array}
$$

$$
\begin{array}{c}
P^n \rightarrow P^n \xrightarrow{f} P^n
\end{array}
$$

Then,

$$r(\phi_V) = r(\phi_W).$$

Proof. Suppose

$$
\begin{align*}
\pi_V^* H &= a_0 H_V + \sum_{i=1}^r a_i E_i \\
\phi^*_V H &= b_0 H_V + \sum_{i=1}^r b_i E_i
\end{align*}
$$

First, consider the case that $W$ is a successive blowup of $V$. Suppose that $\rho : W \rightarrow V$ is a composition of monoidal transformations:

$$
\begin{array}{c}
P^n \xleftarrow{\rho} V \xrightarrow{\pi_V} W
\end{array}
$$

Since $\text{Pic}(V) = ZH_V \oplus ZE_1 \oplus \cdots \oplus ZE_r$, we get

$$
\text{Pic}(W) = ZH^\#_V \oplus ZE^\#_1 \oplus \cdots \oplus ZE^\#_r \oplus ZF_1 \oplus \cdots \oplus ZF_s
$$

where $H^\#_V = \rho^# H_V$, $E^\#_i = \rho^# E_i$ and $F_j$ are the irreducible components of the exceptional divisor of $W$ over $V$. Moreover, we may assume that

$$
\begin{align*}
\rho^* H_V &= H^\#_V + \sum_{j=1}^s m_{0,j} F_j \\
\rho^* E_i &= E^\#_i + \sum_{j=1}^s m_{i,j} F_j
\end{align*}
$$

for some integers $m_{i,j}$, which are nonnegative by Lemma 3.2. By assumption, $\phi_W = \phi_V \circ \rho$ and hence

$$
\pi^*_W H = \rho^* \pi^* H = \rho^* \left( a_0 H_V + \sum_{i=1}^r a_i E_i \right) = a_0 H^\#_V + \sum_{i=1}^r a_i E^\#_i + \sum_{j=1}^s \left( \sum_{i=0}^r a_i m_{i,j} \right) F_j
$$
and

\[ \phi_W^* H = \rho^* \phi_V^* H = \rho^* \left( b_0 H_V + \sum_{i=1}^r b_i E_i \right) = b_0 H_V^\# + \sum_{i=1}^r b_i E_i^\# + \sum_{j=1}^s \left( \sum_{i=0}^r b_i m_{i,j} \right) F_j. \]

If \( b_i = 0 \) and \( a_i \neq 0 \) for some \( i \), then \( r(\phi_V) = r(\phi_W) = \infty \) by definition. So, we may assume \( b_i > 0 \) for all \( i \). Then, by definition of the \( D \)-ratio, we get an inequality

\[ (B) \quad r(\phi_V) = \deg \phi_V \cdot \max_i \left( \frac{a_i}{b_i} \right) \geq \deg \phi_V \cdot \frac{a_i}{b_i} \quad \text{for all } i. \]

Because Lemma 3.2 and the fact \( b_i > 0 \), we have

\[ \sum_{i=0}^r b_i m_{i,j} \geq \sum_{i=0}^r m_{i,j} > 0 \quad \text{for all } j, \]

and hence all coefficient of \( \phi_W^* H \) is positive. Thus, we have

\[ r(\phi_W) = \deg \phi_W \cdot \max \left( \max_i \left( \frac{a_i}{b_i} \right), \max_j \left( \frac{\sum_{i=0}^r a_i m_{i,j}}{\sum_{i=0}^r b_i m_{i,j}} \right) \right). \]

Moreover, due to \( (B) \), we get

\[ \max_j \left( \frac{\sum_{i=0}^r a_i m_{i,j}}{\sum_{i=0}^r b_i m_{i,j}} \right) \leq \frac{\sum_{i=0}^r a_i m_{i,j} \deg \phi_V}{\sum_{i=0}^r b_i m_{i,j} \deg \phi_V} = \frac{r(\phi_V)}{\deg \phi_V} = \max_i \left( \frac{a_i}{b_i} \right). \]

Finally, \( \deg \phi_V = \deg \phi_W \) yields

\[ r(\phi_W) = \deg \phi_W \cdot \max \left( \max_i \left( \frac{a_i}{b_i} \right), \max_j \left( \frac{\sum_{i=0}^r a_i m_{i,j} \deg \phi_V}{\sum_{i=0}^r b_i m_{i,j} \deg \phi_V} \right) \right) = \deg \phi_V \cdot \max_i \left( \frac{a_i}{b_i} \right) = r(\phi_V). \]

Now let \((V, \pi_V)\) and \((W, \pi_W)\) be resolutions of indeterminacy of \( f \) allowing strong factorization: there is a common blowup \( U \) of \( V \) and \( W \) such that \( \tau_V : U \to V \) and \( \tau_W : U \to W \) are compositions of monoidal transformation. Then, \((U, \pi_U := \pi_V \circ \tau_V)\) is still a resolution of indeterminacy of \( f \):

\[
\begin{array}{c}
\pi_W \\
\phi_V \\
\pi_U \\
\end{array} \quad \begin{array}{c}
\phi_U \\
\tau_W \\
\tau_V \\
\end{array} \quad \begin{array}{c}
U \\
\leftarrow \quad W \quad \rightarrow V \\
\phi_V \\
\end{array}
\]

Then, the previous result says

\[ r(\phi_V) = r(\phi_U) = r(\phi_W). \]

\[ \square \]

**Proposition 4.5.** Let \( f, g \in \text{Rat}^n(H) \) be rational maps defined over a number field \( K \). Then,

1. \( r(f) = 1 \) if \( f \) is an endomorphism.
2. \( r(f) \geq 1 \).
(3) There is a resolution of indeterminacy \((U, \pi_U)\) of \(g \circ f\) such that the \(D\)-ratio of \(g \circ f\) on \(U\) satisfies the following inequality:
\[
\frac{r(f)}{\deg f} \cdot \frac{r(g)}{\deg g} \geq \frac{r(g \circ f)}{\deg(g \circ f)}.
\]

(4) If \(g\) is an endomorphism and \(f\) is a rational map on \(\mathbb{P}^n\), then \(r(g \circ f) = r(f)\).

Proof. (1) When \(f\) is an endomorphism, then \((\mathbb{P}^n, id)\) is a resolution of indeterminacy of \(f\). Thus, \(id^*H = H\) and \(f^*H = \deg f \cdot H\) and hence
\[
r(f) = \deg f \times \frac{1}{\deg f} = 1.
\]
If \((V, \pi)\) is an arbitrary resolution of indeterminacy of \(f\), then \(V\) is a successive blowup of \(\mathbb{P}^n\) so that
\[
r(f, (\mathbb{P}^n, id)) = r(f, (V, \pi))
\]
because of Lemma 4.3.

(2) Let \((V, \pi)\) be a resolution of indeterminacy of \(f\) with the resolved morphism \(\phi = f \circ \pi\). We may assume that the underlying set of the center of blowup is \(I(f)\) by Theorem 2.4. Suppose that
\[
\pi^*H = a_0H_V + \sum_{i=1}^{r} a_iE_i, \quad \phi^*H = b_0H_V + \sum_{i=1}^{r} b_iE_i.
\]
We can easily check that \(a_0 = 1\): because \(\pi(E_i) \subset I(f)\) and \(I(f)\) is a closed set of codimension at least 2, \(\pi^*E_i = 0\). Thus,
\[
\pi_*\pi^*H = \pi_*\left( a_0H_V + \sum_{i=1}^{r} a_iE_i \right) = \pi_*a_0H_V = a_0H.
\]
On the other hand, choose another hyperplane \(H'\) which satisfies \(I(f) \not\subset H'\). Then, since \(\pi\) is one-to-one outside of \(\pi^{-1}(I(f))\), we have
\[
\pi_*\pi^*H = \pi_*\pi^*H = H' = H.
\]
Therefore, \(\pi_*H_V = H\) and \(a_0 = 1\).

Now, let’s figure \(b_0\) out. We define the pull-back of \(H\) by \(\phi\) to be
\[
(C) \quad \phi^*H = \text{ord}_{H_V}(u \circ \phi) \cdot H_V + \sum_{i=1}^{r} \text{ord}_{E_i}(u \circ \phi) \cdot E_i
\]
where \(u\) is a uniformizer at \(H\). Apply \(\pi_*\) on \((C)\) and get
\[
\pi_*\phi^*H = \text{ord}_{H_V}(u \circ \phi) \cdot \pi_*H_V + \sum_{i=1}^{r} \text{ord}_{E_i}(u \circ \phi) \cdot \pi_*E_i = \text{ord}_{H_V}(u \circ \phi) \cdot H
\]
since \(\pi_*H_V = H\) and \(\pi_*E_i = 0\). Furthermore, because \(\phi = f \circ \pi\), \(f = [x_0^d, f_1, \ldots, f_n]\) and \(\pi\) is one-to-one on \(H_V\), we get
\[
\text{ord}_{H_V}(u \circ \phi) = \text{ord}_{H}(u \circ \phi \circ \pi^{-1}) = \text{ord}_{H}(u \circ f) = d.
\]
On the other hand, we have

\[ \pi_* \phi^* H = \pi_* \left( b_0 H_V + \sum_{i=1}^r b_i E_i \right) = b_0 \pi_* H_V + \sum_{i=1}^r b_i \pi_* E_i = b_0 H \]

and hence \( b_0 = d \).

Finally,

\[ r(f) = \deg f \cdot \max_i \left( \frac{a_i}{b_i} \right) \geq \deg f \cdot \frac{a_0}{b_0} = \deg f \cdot \frac{1}{\deg f} = 1. \]

(3) If \( r(f) = \infty \) or \( r(g) = \infty \), then it is clear. So, we may assume that \( r(f) \) and \( r(g) \) are finite. Let \((V, \pi_V)\) and \((W, \pi_W)\) be resolutions of indeterminacy of \( f \) and \( g \) obtained by Theorem 2.4 respectively and \( \phi = f \circ \pi_V, \psi = g \circ \pi_W \) are resolved morphisms defining the \( D \)-ratio of \( f \) and \( g \) respectively. Consider a rational map \( \phi' = \pi_W^{-1} \circ \phi : V \dashrightarrow W \) and find a resolution of indeterminacy \((U, \rho)\) of \( \phi' \) over \( V \):

\[
\begin{array}{ccc}
U & \xrightarrow{\rho} & \tilde{\phi} \\
\pi_U & & V \\
\pi_V & & W \\
\pi_W & & \mathbb{P}^n \\
& \phi \downarrow & \downarrow \psi \\
& f & \mathbb{P}^n \\
& \mathbb{P}^n & \xrightarrow{\psi} \mathbb{P}^n
\end{array}
\]

Note that \( U \) is a successive blowup of \( V \) and hence \( r(\phi) = r(\rho \circ \phi) \).

Let

\[ \alpha = \frac{r(\phi)}{\deg \phi} \quad \text{and} \quad \beta = \frac{r(\psi)}{\deg \psi}. \]

By the definition of \( r(\phi) \) and \( r(\psi) \), we have

\[ \pi_V^* H \prec \alpha \cdot \phi^* H \quad \text{and} \quad \pi_W^* H \prec \beta \cdot \psi^* H. \]

By the functoriality of the \( \mathbb{A}^n \)-effectiveness, we get

\[ (D) \quad \rho^* \pi_V^* H \prec \alpha \cdot \rho^* \phi^* H \quad \text{and} \quad \tilde{\phi}^* \pi_W^* H \prec \beta \cdot \tilde{\phi}^* \psi^* H. \]

Since the diagram commutes: \( \phi \circ \rho = \pi_W \circ \tilde{\phi} \), we get

\[ \rho^* \phi^* H = \tilde{\phi}^* \pi_W^* H. \]

So, we can connect inequalities in \( (D) \):

\[ \pi_U^* H = \rho^* \pi_V^* H < \alpha \cdot \rho^* \phi^* H = \alpha \cdot \tilde{\phi}^* \pi_W^* H < \alpha \beta \cdot \tilde{\phi}^* \psi^* H. \]

Therefore, \( \alpha \beta \) is a constant satisfying

\[ \alpha \beta \cdot (\psi \circ \tilde{\phi})^* H - \pi_V^* H = \alpha \beta \cdot \tilde{\phi}^* \psi^* H - \rho^* \pi_V^* H > 0, \]

where \( \psi \circ \tilde{\phi} \) is a resolved morphism of \( g \circ f \). It follows that \( \alpha \beta \geq \frac{r(\psi \circ \tilde{\phi})}{\deg \psi \circ \tilde{\phi}}. \)

(4) Let \((V, \pi)\) be a resolution of indeterminacy of \( f \) and suppose that

\[ \pi^* H = a_0 H_V + \sum_{i=0}^r a_i E_i, \quad \phi^* H = b_0 H_V + \sum_{i=0}^r b_i E_i. \]
Consider the following diagram:

Thus, we get

\[ \pi^* \text{id}_{P^n}^* H = \left( a_0 H_V + \sum_{i=1}^{r} a_i E_i \right), \quad \phi^* g^* H = \deg g \left( b_0 H_V + \sum_{i=1}^{r} b_i E_i \right) \]

and hence

\[ \frac{r(g \circ f)}{\deg(g \circ f)} = \max_i \left( \frac{a_i}{\deg g \cdot b_i} \right) = \frac{1}{\deg g} \max_i \left( \frac{a_i}{b_i} \right) = \frac{r(f)}{\deg f \deg g}. \]

Furthermore, \( \deg(g \circ f) = \deg f \cdot \deg g \) since \( g \) is an endomorphism. Therefore, we have the desired result:

\[ r(g \circ f) = r(f). \]

\[ \square \]

5. Upper bounds for height for rational map

In this section, we prove Theorem A and apply it to arithmetic dynamics. We start with Theorem C, which says that we can only expect a weaker height inequality than Northcott’s theorem.

**Theorem C.** Let \( f \in \text{Rat}^n(H) \) be a rational map defined over a number field \( K \). Suppose that \( f \) satisfies the following inequality for some nonempty Zariski open set \( U \) of \( P^n \) and some constant \( C \):

\[ \mathcal{E} \quad \frac{1}{\deg f} h(f(P)) + C > h(P) \quad \text{for all } P \in U(K) \setminus I(f). \]

Then, \( f \) is an endomorphism.

**Proof.** Suppose that there is a point \( Q \in I(f) \). Without loss of generality, we may assume that \( Q = [0, 0, \ldots, 0, 1] \). Let

\[ f(X) = [f_0(X), f_1(X), \ldots, f_n(X)] \]

where \( X = [X_0, \ldots, X_n] \), \( d = \deg f \) and \( f_i \) are homogeneous polynomials of degree \( d \). Then, we can claim that

\[ \deg_{X_n} f_i < d \]

for all \( i = 0, \ldots, n \); if there is an \( j \) such that \( \deg_{X_n} f_j = d \), then \( f_j(Q) \neq 0 \) and hence \( Q \) cannot be an indeterminacy point. Choose a point \( \alpha = [\alpha_0, \ldots, \alpha_{n-1}] \in \mathbb{P}^{n-1} \) and a projective line

\[ L_\alpha := \{ [x_0, \ldots, x_n] \mid \alpha_i x_j = \alpha_j x_i \text{ for all } 0 \leq i, j \leq n - 1 \} \cap H. \]

Precisely, \( L \) is the image of the closed embedding of \( \mathbb{P}^1 \):

\[ \iota_\alpha : \mathbb{P}^1 \to L \subset \mathbb{P}^n, \quad [Y_0, Y_1] \mapsto [\alpha_0 Y_0, \alpha_1 Y_1]. \]
Since $U$ is dense, so is $U' = U \setminus I(f)$. Therefore, there is an $\alpha$ such that $L_\alpha \cap U'$ is dense in $L_\alpha$. Moreover, $f$ is defined on $L_\alpha \cap U' = \{[\alpha_0, \cdots, \alpha_{n-1}, x_n]\}$ where $x_n = Y_1/Y_0$ and hence $f_i[\alpha_0, \cdots, \alpha_{n-1}, x_n]$ is one-variable polynomial of degree at most $\deg X_n f_i$ for all $i = 0, \cdots, n$. Thus, $f|_L$ is a morphism of degree $d' \leq \max_i \deg X_n f_i < d$ on $L$ and hence we have the following inequality:

$$h(P) > \frac{1}{d'} h(f(P)) - C' \quad \text{for all } P \in L(\overline{K}) \cap U',$$

which contradicts (E) $\square$.

**Theorem A.** Let $f \in \text{Rat}^n(H)$ be a rational map defined over a number field $K$ and let $r(f)$ be a $D$-ratio of $f$. Then, there is a constant $C$, depending only on $f$, such that

$$\frac{r(f)}{\deg f} h(f(P)) + C > h(P) \quad \text{for all } P \in \mathbb{A}^n(\overline{K}).$$

**Proof.** Let $(V, \pi)$ be a resolution of indeterminacy of $f$ with the resolved morphism $\phi = f \circ \pi_V$ and $r(f)$ be the $D$-ratio of $f$ on $V$. Suppose that

$$\pi^* H = a_0 H_V + \sum_{i=1}^r a_i E_i \quad \text{and} \quad \phi^* H = b_0 H_V + \sum_{i=1}^r b_i E_i$$

where $\{H_V, E_1, \cdots, E_r\}$ is the basis described in Proposition 2.7.

Let

$$E := \frac{r(f)}{\deg f} \phi^* H - \pi^* H.$$

By the definition of the $D$-ratio, $E$ is $\mathbb{A}^n$-effective. So, the height function corresponding to $E$,

$$h_E = \frac{r(f)}{\deg f} h(\phi(Q)) - h(\pi^*(Q))$$

is bounded below on $\pi^{-1}(\mathbb{A}^n)$ by Proposition 3.3 (1). Hence we have the following inequality:

$$\frac{r(f)}{\deg f} h(\phi(Q)) + C > h(\pi_V(Q)) \quad \text{for all } Q \in \pi^{-1}(\mathbb{A}^n).$$

Let $P \in \mathbb{A}^n(\overline{K})$ be an arbitrary point and take $Q_P = \pi^{-1}(P)$. Then $\phi(Q_P) = f(P)$ and $Q_P$ satisfies the above inequality. Therefore, we get the desired result:

$$\frac{r(f)}{\deg f} h(f(P)) + C > h(P) \quad \text{for all } P \in \mathbb{A}^n(\overline{K}).$$

$\square$

**Corollary 5.1.** Let $f \in \text{Rat}^n(H)$ be a rational map defined over a number field $K$. Then

$$r(f) = 1 \quad \text{if and only if} \quad f \text{ is an endomorphism.}$$

**Proof.** One direction is already done; see Proposition 3.3 (1). For the other direction, suppose that $r(f) = 1$. Then, by Theorem A, we get the inequality (E). So, $f$ is an endomorphism by Theorem C.

$\square$

We apply Theorem A to study dynamics of polynomial map. We can consider that a polynomial map $f : \mathbb{A}^n \to \mathbb{A}^n$ is an element $f \in \text{Rat}^n(H)$ such that $f(\mathbb{A}^n) \subset \mathbb{A}^n$. Thus, we can define $r(f)$ and apply Theorem A at all forward image $f^n(P) \in \mathbb{A}^n$. 
**Theorem D.** Let \( f : \mathbb{A}^n \to \mathbb{A}^n \) be a polynomial map, defined over a number field \( K \), such that \( r(f) < \deg f \). Then,

\[
\text{Preper}(f) = \left\{ P \in \mathbb{A}^n(K) \mid f^l(P) = f^m(P) \text{ for some } l, m \right\}
\]

is a set of bounded height and hence

\[
\text{Preper}(f) \cap \mathbb{A}^n(K')
\]

is finite for any number field \( K' \).

**Proof.** Let \( u = \frac{r(f)}{\deg f} < 1 \). Then, by Theorem A, we have

\[
(F) \quad u \cdot h(f(P)) > h(P) - C \quad \text{for all } P \in \mathbb{A}^n(K).
\]

Then, the iteration of \((F)\) provides

\[
u^l \cdot h(f^l(P)) > u^l \cdot h(f(f^{l-1}(P))) > u^{l-1} \left[ h(f^{l-1}(P)) - C \right] > u^{l-2} \left[ h(f^{l-2}(P)) - C \right] - u^{l-1}C > \ldots > h(P) - \left\{ 1 + u + \cdots + u^{l-1} \right\} C.
\]

Hence, we have

\[
\lim_{l \to \infty} u^l \cdot h(f^l(P)) > h(P) - \frac{C}{1 - u}.
\]

If \( P \) is a preperiodic point of \( f \), then the left hand side goes to zero so that

\[
\frac{C}{1 - u} > h(P) \quad \text{for all } P \in \mathbb{A}^n(K).
\]

\[\square\]

**Example 5.2.** Let

\[F(x, y) = (x^3 + y, x + y^2)\].

Then, after three blowing-ups along points (see Figure 1), we get a resolution of indeterminacy of \( F \). And, we have

\[\pi^*H = H_V + E_1 + 2E_2 + 3E_3, \quad \phi^*H = 3H_V + 2E_1 + 4E_2 + 6E_3.\]

Thus, \( r(F) = 3/2 < 3 \) and hence it has finitely many preperiodic points on \( \mathbb{A}^n(K) \).

**Example 5.3.** The condition \( r(f) < \deg f \) in Theorem D is sharp: let

\[F_0(x, y) = (x, y^2)\].

Then, after two blowing-ups along points (see Figure 2), we get a resolution of indeterminacy of \( F_0 \). And, we have

\[\pi^*H = H_V + E_1 + 2E_2, \quad \phi^*H = 2H_V + E_1 + 2E_2.\]
Thus, \( r(F_0) = 2 = \deg F_0 \). And, it has infinitely many integral fixed points \((n,0)\). Thus, \( \text{Preper}(F_0) \) is not bounded.

**Corollary 5.4.** Let \( f \in \text{Rat}^n(H) \) be a rational map, defined over a number field \( K \). If there is some number \( N \) satisfying \( r(f^N) < \deg(f^N) \), then

\[
\text{Preper}(f) = \left\{ P \in \mathbb{A}^n \mid f^l(P) = f^m(P) \text{ for some } l \neq m \right\}
\]

is a set of bounded height.

**Proof.** It is enough to show that \( \text{Preper}(f) = \text{Preper}(f^N) \).

One direction is clear;

\[
P \in \text{Preper}(f) \quad \Rightarrow \quad \mathcal{O}_f(P) \text{ is finite}
\]

\[
\Rightarrow \quad \mathcal{O}_{f^N}(P) \text{ is finite since } \mathcal{O}_{f^N}(P) \subset \mathcal{O}_f(P)
\]

\[
\Rightarrow \quad p \in \text{Preper}(f^N).
\]
So, we only have to show the other direction. Suppose that \( P \) is a preperiodic point of \( f \). Then,

\[
f^l(P) = f^m(P)
\]
holds for some natural numbers \( l < m \). Let \( g = f^{m-l} \). Then, we have

\[
g(f^l(P)) = f^l(P),
\]
which means \( f^l(P) \) is a fixed point of \( g \). So, we get

\[
g^N(f^l(P)) = f^l(P)
\]
and

\[
f^{kN-l}(g^N(f^l(P))) = f^{kN-l}(f^l(P)) = f^{kN}(P)
\]
for all positive integers \( k, N \). Since \( g = f^{m-l} \), we get

\[
f^{N(k+m-l)}(P) = f^{kN-l+N(m-l)+l}(P) = f^{kN-l}(g^N(f^l(P))) = (f^N)^k(P),
\]
which means that \( P \) is a preperiodic point of \( f^N \).

\[ \square \]

**Example 5.5.** Let

\[
G(x, y) = (y, x^2 + y).
\]
Then, after two blowing-ups (see Figure 3), the indeterminacy of \( G \) is resolved. And, we have

\[
\pi^*H = H_V + E_1 + 2E_2 \quad \text{and} \quad \psi^*H = 2H_v + E_1 + 2E_2
\]
so that \( r(G) = 2 \). But \( G^2(x, y) = (x^2 + y, x^2 + y^2 + y) \) which extends to a morphism. Thus, \( r(G^2) = 1 < 3 \). Therefore, it has finitely many preperiodic points on \( \mathbb{A}^n(K) \).

![Figure 3. Resolution of indeterminacy of \( G[X, Y, Z] = [YZ, X^2 + YZ, Z^2] \)](image)

**Example 5.6.** Let

\[
N(x, y, z) = (x + (x^2 - yz)z, y + 2(x^2 - yz)x + (x^2 - yz)^2z, z)
\]
be the Nagata map. Then, any points on the quadratic curve \( x^2 = yz \) is a fixed point of \( N(x, y, z) \). So, for any \( N \), \( r(N(x, y, z)) \geq \deg f^N_N \). Furthermore, the \( N \)-th iteration of \( N(x, y, z) \) is still a polynomial map of degree \( 5 \);

\[
f^N_N(x, y, z) = (x + N(x^2 - yz)z, y + 2N(x^2 - yz)x + N^2(x^2 - yz)^2z, z).
\]
Therefore, for any resolution of indeterminacy \((V_m, \pi_m)\) of any iteration of \(f_N^{\alpha}\), the resolved morphism \(\phi_N\) satisfies
\[
\deg f \cdot \deg f^{-1} - \deg f^{-1}(P) + C > h(P) \quad \text{for all } P \in \mathbb{P}^n(\mathbb{K}) \setminus H.
\]

**Remark 5.7.** Theorem D only considers preperiodic points of \(f\) in \(\mathbb{A}^n(\mathbb{K})\). Even though a rational map \(f\) satisfying \(\deg f < \deg f\) could have infinitely many periodic points on the hyperplane \(H\).

**Example 5.8.** Consider a rational map on Example 5.2:
\[
F(x, y, z) = [x^3 + yz^2, xz^2 + y^2z, z^3].
\]
It has infinitely many preperiodic points \(P = [a, b, 0] \) with \(a \neq 0\).

### 6. Jointly Regular Pairs

In this section, we will prove Theorem B, which is a generalization of results of Silverman [21], Kawaguchi [10] and the author [14] for jointly regular pairs: we say that \(S\) is a jointly regular pair if \(S = \{f_1, f_2\}\) is a set consisting of two rational maps whose indeterminacy loci are disjoint.

Silverman [21] proved a weaker result for a jointly regular family of polynomial maps: Let \(\{f_1, \cdots, f_k\}\) be a family of polynomial maps, defined over a number field \(K\). Suppose that the intersection of indeterminacy loci of \(f_i\)’s is empty. Then there is a constant \(C\) satisfying
\[
\sum_{i=1}^{k} \frac{1}{\deg f_i} h(f_i(P)) + C > h(P) \quad \text{for all } P \in \mathbb{P}^n(\mathbb{K}) \setminus H.
\]

Recently, Kawaguchi [11] and the author [14] independently proved Theorem B for regular polynomial automorphisms: we say that \(f\) is a regular polynomial automorphism if \(f: \mathbb{A}^n \to \mathbb{A}^n\) has the inverse map \(f^{-1}: \mathbb{A}^n \to \mathbb{A}^n\) and \(I(f) \cap I(f^{-1}) = \emptyset\). Then there is a constant \(C\) satisfying
\[
\frac{1}{\deg f} h(f(P)) + \frac{1}{\deg f^{-1}} h(f^{-1}(P)) + C > \left(1 + \frac{1}{\deg f \cdot \deg f^{-1}}\right) h(P) \quad \text{for all } P \in \mathbb{P}^n(\mathbb{K}).
\]

Note that \(r(f) = \deg f \cdot \deg f^{-1}\) if \(f\) is a regular polynomial automorphism. (For details, see [14].)

**Theorem B.** Let \(\{f_1, f_2\} \subset \text{Rat}^n(\mathbb{H})\) be a jointly regular pair of rational maps defined over a number field \(K\) and let \(r(f_i)\) be the \(D\)-ratio of \(f_i\). Then, there is a constant \(C\) satisfying
\[
\frac{1}{\deg f_1} h(f_1(P)) + \frac{1}{\deg f_2} h(f_2(P)) + C > \left(1 + \min_{l=1,2} \left(\frac{1}{r(f_l)}\right)\right) h(P) \quad \text{for all } P \in \mathbb{P}^n(\mathbb{K}) \setminus H.
\]

**Proof.** Let \((V_i, \pi_i)\) be resolutions of indeterminacy of \(f_i\) obtained by Theorem 2.1 and let \(\phi_i = f_i \circ \pi_i\) be the resolved morphisms of \(f_i\) respectively. Then, the underlying set of the center scheme of \(V_i\) are exactly \(I(f_i)\). So, we may assume that the centers of blowups \(V_1, V_2\) are disjoint. Suppose that
\[
\text{Pic}(V_i) = \mathbb{Z} \pi_i^* H \oplus \mathbb{Z}E_{l_1} \oplus \cdots \oplus \mathbb{Z}E_{l_r}.
\]
and
\[
\pi_i^* H = \pi_i^* H + \sum_{i=1}^{r_1} a_{iE_i} E_{l_i}, \quad \phi_i^* H = d_l \cdot \pi_i^* H + \sum_{i=1}^{r_2} b_{iE_i}.
\]
where \(d_l = \deg f_i\). Note that \(b_{l_0} = d_l\) from the proof of Proposition 4.5 (1).
Now, consider a blowup \( U \) of \( \mathbb{P}^n \) along union of centers of \( V_1 \) and \( V_2 \). Then, we have the following diagram:

\[
\begin{array}{c}
\mathbb{P}^n & \xrightarrow{\pi} & U & \xrightarrow{\rho_1} & V_1 \\
& & \phi_1 & & \phi_1 \\
& & \phi_2 & & \phi_2 \\
& & \pi_1 & & \pi_1 \\
& & \pi_2 & & \pi_2 \\
\end{array}
\]

Because \( I(f_1) \) and \( I(f_2) \) are disjoint, \( U \) is still a successive blowup of \( \mathbb{P}^n \). (For details, See Figure 4.) Moreover, we have

\[
\rho_1^* E_{1j} = \rho_1^# E_{1j} \quad \text{and} \quad \rho_2^* E_{2j} = \rho_2^# E_{2j}.
\]

Thus, let \( F_{ij} = \rho_i^# E_{ij} \) and get the following description of \( \text{Pic}(U) \):

\[
\text{Pic}(U) = \mathbb{Z} \pi_1^# H \oplus \mathbb{Z} F_{11} \oplus \ldots \oplus \mathbb{Z} F_{r_1,1} \oplus \mathbb{Z} F_{21} \oplus \ldots \oplus \mathbb{Z} F_{2r_2}
\]

\[
\rho_1^# \pi_1^# H = \rho_2^# \pi_2^# H = \pi^# H.
\]

Furthermore, Hironaka’s construction guarantees that

\[
\pi_1^{-1}(I(f_2)) \subset \pi_1^# H \quad \text{and} \quad \pi_2^{-1}(I(f_1)) \subset \pi_2^# H.
\]

\[\text{Figure 4. Exceptional divisors of } V_1, V_2 \text{ and } U\]
So, we have
\[ \rho_1^* \pi_2^* H = \pi_2^* H + \sum_{i=1}^{r_2} a_{2i} F_{2i}, \quad \text{and} \quad \rho_2^* \pi_1^* H = \pi_1^* H + \sum_{i=1}^{r_1} a_{1i} F_{1i}. \]

Apply \( \rho_l \) to \( \phi_l^* H \) and get
\[ \bar{\phi}_1^* H = \rho_1^* \left( d_1 \cdot \pi_1^* H \right) + \rho_1^* \left( \sum_{i=1}^{r_1} b_{li} E_{1i} \right) = d_1 \cdot \left( \pi_1^* H + \sum_{j=1}^{r_2} a_{2j} F_{2j} \right) + \sum_{i=1}^{r_1} b_{li} F_{1i}, \]
and
\[ \bar{\phi}_2^* H = \rho_2^* \left( d_2 \cdot \pi_2^* H \right) + \rho_2^* \left( \sum_{j=1}^{r_2} b_{2j} E_{2j} \right) = d_2 \cdot \left( \pi_2^* H + \sum_{i=1}^{r_1} a_{1i} F_{1i} \right) + \sum_{j=1}^{r_2} b_{2j} F_{2j}. \]
Therefore,
\[
\sum_{l=1}^{2} \frac{1}{d_l} \bar{\phi}_l^* H - \pi^* H = \sum_{l=1}^{2} \left[ \left( \pi_l^* H + \sum_{k \neq l} \sum_{j=1}^{r_k} a_{kj} F_{kj} \right) + \frac{1}{d_l} \sum_{i=1}^{r_l} b_{li} F_{li} \right] \\
- \left( \pi_l^* H + \sum_{k=1}^{2} \sum_{i=1}^{r_k} a_{li} F_{li} \right) \\
= \pi_l^* H + \sum_{l=1}^{2} \left( \frac{1}{d_l} \sum_{i=1}^{r_l} b_{li} F_{li} \right) \\
\geq \pi_l^* H + \sum_{l=1}^{2} \left( \frac{1}{d_l} \sum_{i=1}^{r_l} a_{li} F_{li} \right) \quad \text{\( \because r \left( \phi_l \right) \geq d_l a_{li} / b_{li} \)} \\
\geq \min \left( \frac{1}{r \left( \phi_l \right)} \right) \left( \pi_l^* H + \sum_{l=1}^{2} \sum_{i=1}^{r_l} a_{li} F_{li} \right) \\
= \min \left( \frac{1}{r \left( \phi_l \right)} \right) \pi^* H
\]
So, we get that the height function corresponding to an \( \mathbb{A}^n \) effective divisor
\[
\sum_{l=1}^{2} \frac{1}{d_l} \bar{\phi}_l^* H - \left( 1 + \min \left( \frac{1}{r \left( \phi_l \right)} \right) \right) \pi^* H
\]
is bounded below on \( \pi^{-1} (\mathbb{A}^n) \) and hence get the desired result. \( \square \)

**Corollary 6.1.** Let \( f : \mathbb{A}^n \to \mathbb{A}^n \) be a regular polynomial automorphism defined over a number field \( K \). Then, there is a constant \( C \) such that
\[
\frac{1}{\deg f} h(f(P)) + \frac{1}{\deg f^{-1}} h(f^{-1}(P)) + C > \left( 1 + \frac{1}{\deg f \cdot \deg f^{-1}} \right) h(P) \quad \text{for all} \quad P \in \mathbb{A}^n(\mathbb{K}).
\]

**Proof.** It is enough to show that \( r(f) = r(f^{-1}) = \deg f \cdot \deg f^{-1} \). \[14\] Lemma 3.5 shows that
\[
\pi_V^* H = H_V + d' E_V + M_V
\]
\[
\phi^*V = dH + E + I
\]
where \(d\) is the degree of \(\phi\), \(d'\) is the degree of \(\psi\) and \(d'I - M\) is an effective divisor. Furthermore, since support of \(d'I - M\) is not contained and doesn’t contain \(H\), it is actually \(\mathbb{A}^n\)-effective so that

\[
r(\phi) = d \times \max \left( \frac{1}{d'} d', \frac{M_i}{I_i} \right)
\]

where \(M_i \leq d'I_i\). Since the support of \(M_i, I_i\) does not contain \(H\) so that \(M_i < d'I_i\). Therefore,

\[
r(\phi) = d \times d'
\]

\[
\square
\]

Example 6.2. Let

\[
f(x, y) = (x^2 + y, y).
\]
We can check that \(r(f) = 1 \times \deg f = 2\). Therefore, with any \(g \in \text{Rat}^2(H)\) which makes jointly regular pair with \(f\) and \(r(g) < 2\), there is a constant \(C\) such that

\[
\frac{1}{2} h(f(P)) + \frac{1}{d} h(g(P)) + C > \left( 1 + \frac{1}{2} \right) h(P) \quad \text{for all } P \in \mathbb{A}^2(\mathbb{Q}).
\]

Remark 6.3. If a pair of rational maps is not jointly regular, then we may not have a similar inequality. For example, let

\[
f(x, y) = (x - y^d, y) \quad \text{and} \quad f^{-1}(x, y) = (x - y^d, y)
\]
where \(d\) is a natural number larger than 2. Then, for any \(x\),

\[
\frac{1}{d} h(f(x, 0)) + \frac{1}{d} h(f^{-1}(x, 0)) = \frac{2}{d} h((x, 0))
\]
so that it can’t bound \(h(x, 0) + C\) above for any constant \(C\).

If \(f_1 \in \text{Rat}^n(H)\) is a polynomial map such that \(r(f_1) \geq \deg f_1\), then we do not get information from Theorem A. However, if we can find a polynomial map \(f_2 \in \text{Rat}^n(H)\) such that \(\{f_1, f_2\}\) is a jointly regular pair, then we can apply Theorem B to get information for preperiodic points of a monoid generated by \(\{f_1, f_2\}\).

For each \(m \geq 0\), let \(W_m\) be the collection of ordered \(m\)-tuples chosen from \(\{1, 2\}\),

\[
W_m = \{(i_1, \ldots, i_m) \mid i_j \in \{1, 2\}\}
\]
and let

\[
W_* = \bigcup_{m \geq 0} W_m.
\]

Thus \(W_*\) is the collection of words of \(r\) symbols.

For any \(I = (i_1, \ldots, i_m) \in W_m\), let \(f_I\) denote the composition of corresponding polynomial maps in \(S\):

\[
f_I := f_{i_1} \circ \cdots \circ f_{i_m}.
\]
Definition 6.4. We denote the monoid of rational maps generated by \( S = \{ f_1, f_2 \} \) under composition by
\[
\Phi_S = \Phi := \{ \phi = f_I \mid I \in W_s \}.
\]
Let \( P \in \mathbb{A}^n \). The \( \Phi \)-orbit of \( P \) is defined to be
\[
\Phi(P) = \{ \phi(P) \mid \phi \in \Phi \}.
\]
The set of (strongly) \( \Phi \)-preperiodic points is the set
\[
\text{Preper}(\Phi_S) = \{ P \in \mathbb{A}^n \mid \Phi(P) \text{ is finite} \}.
\]

Theorem E. Let \( S = \{ f_1, f_2 \} \) be a jointly regular pair of polynomial maps, let \( f(f_I) \) be the D-ratio of \( f_I \) and let \( \Phi_S \) be the monoid of rational maps generated by \( S \). Define
\[
\delta_S := \left( \frac{1}{1 + 1/r} \right) \left( \frac{1}{\deg f_1} + \frac{1}{\deg f_2} \right)
\]
where \( r = \max_{l=1,2} (r(f_I)) \).
If \( \delta_S < 1 \), then
\[
\text{Preper}(\Phi_S) := \bigcap_{f \in \Phi_S} \text{Preper}(f) \subset \mathbb{A}^n_{\mathbb{Q}}
\]
is a set of bounded height.

Proof. By Theorem B, we have a constant \( C \) such that
\[
(G) \quad 0 \leq \left( \frac{1}{1 + 1/r} \right) \sum_{l=1}^{2} \frac{1}{d_l} h(f_I(Q)) - h(Q) + C \quad \text{for all } Q \in \mathbb{A}^n(\mathbb{Q}).
\]
Note that if \( r = \infty \), then \( \left( \frac{1}{1 + 1/r} \right) = 1 \), then it is done because of [21, Theorem 4]. Thus, we may assume that \( r \) is finite.

We define a map \( \mu : W_s \to \mathbb{Q} \) by the following rule:
\[
\mu_I = \mu_{(i_1, \ldots, i_m)} = \prod_{l=1}^{p_{I,l}} d_l^{i_l}
\]
where \( p_{I,l} = -|\{ t \mid i_t = l \}| \). Then, by definition of \( \delta_S \) and \( \mu_I \), the following is true:
\[
\delta_S^n = \left( \frac{r}{r+1} \right) \sum_{l=1}^{2} \frac{1}{d_l} \sum_{I \in W_m} \frac{1}{\deg f_{i_1} \cdots \deg f_{i_m}} = \left( \frac{r}{r+1} \right)^m \sum_{I \in W_m} \mu_I.
\]

Let \( P \in \mathbb{A}^n(\mathbb{Q}) \). Then, \((G)\) holds for \( f_I(P) \) for all \( I \in W_m \):
\[
0 \leq \left( \frac{r}{r+1} \right) \sum_{l=1}^{2} \frac{1}{d_l} h(f_I(f_I(P))) - h(f_I(P)) + C.
\]
Hence
\[
(H) \quad 0 \leq \sum_{m=0}^{M} \sum_{I \in W_m} \mu_I \left( \frac{r}{r+1} \right)^m \left[ \sum_{l=1}^{2} \frac{1}{d_l} h(f_I(f_I(P))) - \left( 1 + \frac{1}{r} \right) h(f_I(P)) + C \right].
\]
The main difficulty of the inequality is to figure out the constant term. From the definition of $\delta_S$, we have
\[
\sum_{m=0}^{M-1} \left( \frac{r}{r+1} \right)^m \sum_{I \in W_m} \mu_I = \sum_{m=1}^{M} \delta_S^m \leq \frac{1}{1 - \delta_S}.
\]

Now, do the telescoping sum and most terms in (H) will be canceled:
\[
\left( \sum_{m=0}^{M-1} \sum_{I \in W_m} \left( \frac{r}{r+1} \right)^m \mu_I \sum_{l=1}^{2} \frac{1}{d_l} h(f_l f_I(P)) \right) - \left( \sum_{m=1}^{M} \sum_{I \in W_m} \left( \frac{r}{r+1} \right)^{m-1} \mu_I h(f_I(P)) \right)
\]
\[
= \left( \sum_{m=0}^{M-1} \sum_{I \in W_m} \left( \frac{r}{r+1} \right)^m \mu_I \sum_{l=1}^{2} \frac{1}{d_l} h(f_l f_I(P)) \right) - \left( \sum_{m=0}^{M-1} \sum_{I \in W_m} \left( \frac{r}{r+1} \right)^m \mu_I h(f_I(P)) \right)
\]
\[
= 0.
\]

Therefore, the remaining terms in (H) are the first term when $m = M$ and the last term when $m = 0$. Thus, we get
\[
0 \leq \left[ \sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{K} \frac{1}{d_l} h(f_l f_I(P)) \right] - h(P) + \sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I C
\]
\[
\leq \left[ \sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{2} \frac{1}{d_l} h(f_l f_I(P)) \right] - h(P) + \frac{1}{1 - \delta_S} C.
\]

Let $P$ be a $\Phi$-periodic point and define the height of the images of $P$ by the monoid $\Phi$ to be
\[
h(\Phi(P)) = \sup_{R \in \Phi(P)} h(R).
\]

Since
\[
\sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{2} \frac{1}{d_l} = \left( \frac{r}{r+1} \right)^M \sum_{I \in W_{M+1}} \mu_I = (1 + \frac{1}{r}) \delta_S^{M+1},
\]
and
\[
h(\Phi(P)) \geq h(g(P)) \text{ for all } g \in \Phi,
\]
we get
\[
h(P) \leq \left[ \sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{2} \frac{1}{d_l} \right] h(\Phi(P)) + \frac{1}{1 - \delta_S} C
\]
\[
\leq \left( 1 + \frac{1}{r} \right) \delta_S^{M+1} h(\Phi(P)) + \frac{1}{1 - \delta_S} C.
\]

By assumption, $\delta_S < 1$ and $h(\Phi(P))$ is finite, so letting $M \to \infty$ shows that $h(P)$ is bounded by a constant that depends only on $S$. \qed
7. Relation with Height expansion coefficient

In this section, we will discuss the relation between the $D$-ratio and the height expansion coefficient of dominant rational maps and find a lower bound of $r(f)$.

Silverman defined the height expansion coefficient [22, Definition 1] as follows:

**Definition 7.1.** Let $\phi : W \dashrightarrow V$ be a rational map between quasiprojective varieties, all defined over $\mathbb{Q}$. Fix height functions $h_V$ and $h_W$ on $V$ and $W$ respectively, corresponding to ample divisors. The height expansion coefficient of $\phi$ (relative to chosen height function $h_V$ and $h_W$) is the quantity

$$\mu(\phi) = \sup_{\emptyset \neq U \subset W} \liminf_{P \in U, h(P) \to \infty} \frac{h_V(\phi(P))}{h_W(P)},$$

where the sup is over all nonempty Zariski dense open subsets of $W$.

It seems that there is no direct relation between $\mu(f)$ and $r(f)$. So, define a new quantity which will be a bridge between them.

**Definition 7.2.** Let $f \in \text{Rat}^n(H)$ be a rational map defined over a number field $K$. Then, we define the height expansion coefficient of $f$ on $\mathbb{A}^n$:

$$c(f) := \liminf_{P \in \mathbb{A}^n, h(P) \to \infty} \frac{h(f(P))}{h(P)}.$$

Now, we can find a relation between $c(f)$ and $r(f)$.

**Theorem 7.3.**

$$c(f)^{-1} \leq \frac{r(f)}{\deg f}.$$

**Proof.** We may assume that $P \in \mathbb{A}^n$ has sufficiently large $h(P)$. From Theorem A, we have

$$\frac{r(f)}{\deg f} h(f(P)) > h(P) + C \quad \text{for all } P \in \mathbb{A}^n$$

so that we may assume that $h(f(P))$ is sufficiently large, too. Hence,

$$\frac{r(f)}{\deg f} > \frac{h(P) + C}{h(f(P))}$$

for $P \in \mathbb{A}^n$ with sufficiently large $h(P)$. Therefore,

$$\frac{r(f)}{\deg f} \geq \limsup_{h(P) \to \infty} \frac{h(P) + C}{h(f(P))} = \frac{1}{c(f)}.$$

**Corollary 7.4.** Let $f \in \text{Rat}^n(H)$ be a rational map defined over a number field $K$. If there is a curve $C$ on $\mathbb{A}^n$ whose image under $f$ is a point, then $r(f) = \infty$.

**Proof.** It is enough to show $c(f)^{-1} = \infty$. Let $P_n$ be a sequence on the given curve $C$ whose height goes to infinity. Then,

$$c(f)^{-1} = \limsup_{h(P) \to \infty} \frac{h(P)}{h(f(P))} \geq \lim_{n \to \infty} \frac{h(P_n)}{h(f(P_n))} = \lim_{n \to \infty} \frac{h(P_n)}{h(Q)} = \infty$$
Furthermore, the relation between \( c(f) \) and \( \mu(f) \) is clear and hence we can build the following inequality.

**Proposition 7.5.** Let \( f \in \text{Rat}^n(H) \) be a rational map defined over a number field \( K \). Then

\[
\mu(f) \geq c(f) \geq \frac{\deg f}{r(f)}.
\]

**Proof.** It is clear since

\[
c(f) = \liminf_{P \in \mathbb{A}^n, h(P) \to \infty} \frac{h(f(P))}{h(P)}.
\]

is the case when \( U = \mathbb{A}^n \).

With Proposition 7.5 we can find an easier proof of Proposition 4.5 (1).

**Corollary 7.6.** Let \( f \in \text{Rat}^n(H) \) be a rational map defined over a number field \( K \). Then

\[
r(f) \geq 1.
\]

**Proof.** Remind that we have a lower bound of \( h(P) \): by [23, Theorem B.2.5], there exists a constant \( C \) such that

\[
\deg f \cdot h(P) + C > h(f(P)) \quad \text{for all } P \in \mathbb{P}^n(K) \setminus I(f).
\]

So, for any open set \( U \) of \( \mathbb{P}^n \), we have

\[
\liminf_{P \in U} \frac{h(f(P))}{h(P)} \leq \liminf_{P \in U, h(P) \to \infty} \frac{\deg f \cdot h(P) + C}{h(P)} = \deg f.
\]

and hence

\[
\mu(f) = \sup_U \liminf_{P \in U, h(P) \to \infty} \frac{h(f(P))}{h(P)} \leq \deg f.
\]

Therefore, by Proposition 7.5 we get

\[
\deg f \geq \mu(f) \geq \frac{\deg f}{r(f)}.
\]

**Example 7.7.** Let \( f[x, y, z] = [x^{km}, y^{(k-1)m}, z^m, z^{km}] \).

First, since \( f \equiv [1, 0, 0] \) on \( H \setminus [0, 1, 0] \),

\[
\liminf_{P \in U'} \frac{h(f(P))}{h(P)} = 0
\]

for all open set \( U' \not\subset \mathbb{A}^2 \). Thus,

\[
\mu(f) = \sup_{U: \text{open set } \not\emptyset} \liminf_{P \in U, h(P) \to \infty} \frac{h(f(P))}{h(P)} = \sup_{U \subset \mathbb{A}^2, \text{open}} \liminf_{P \in U} \frac{h(f(P))}{h(P)}.
\]

So, we may assume that \( U \subset \mathbb{A}^2 \).
Let \( T_\alpha = \{[x, y, z] \mid x = \alpha z\} \). Then, \( \bigcup_{\alpha \in \mu} U \) is Zariski dense in \( U \): there is an \( \alpha \) such that 
\[ U \cap T_\alpha \neq \emptyset. \]
Because \( U \cap T_\alpha \) is an open set of \( T_\alpha \), there is a sequence \( P_M = [\alpha, y_M, 1] \in U \cap T_\alpha \) such that 
\[ \lim_{M \to \infty} h(P_M) = \infty. \]
From the triangle inequality and the definition of the height, we get 
\[ h(y_M) \leq h(P_M) \leq h(y_M) + h(\alpha) = h(y_M) \]
and 
\[ h\left(y_M^{(k-1)m}\right) \leq h(f(P_M)) \leq \left[\alpha^d, y_M^{(k-1)m}, 1\right] \leq h\left(y_M^{(k-1)m}\right). \]
Moreover, if \( \lim_{M \to \infty} h(P_M) = \infty \), then 
\[ \lim_{M \to \infty} \frac{h(f(P_M))}{h(P_M)} = \lim_{M \to \infty} \frac{(k-1)m \cdot h(y_M)}{h(y_M)} = (k-1)m \]
and hence 
\[ \liminf_{P \in U} \frac{1}{h(P)} \frac{h(f(P))}{h(P)} \leq (k-1)m. \]

Next, figure out lower bound with \( r(f) \): we have \((V, \pi_V)\) a resolution of indeterminacy of \( f \) by \( k \) successive blowups and get a resolved morphism \( \phi_V \). Then, pull-backs of \( H \) by \( \pi_V \) and \( \phi_V \) are calculated as follows:
\[ \pi_V^*H = H_V + \sum_{i=1}^{k} iE_i \quad \text{and} \quad \phi_V^*H = kmH_V + \sum_{i=1}^{k} im(k-1)E_i. \]
Therefore, 
\[ r(f) = d \cdot \max_i \left( \frac{1}{d}, \frac{i}{im(k-1)} \right) = km \frac{1}{m(k-1)} = \frac{k}{k-1}. \]
and hence 
\[ (k-1)m = \mu(f) \geq c(f) \geq \frac{\deg f}{r(f)} = \frac{k-1}{k} = (k-1)m. \]

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