On quantization in light-cone variables compatible with wavelet transform

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Abstract

Canonical quantization of quantum field theory models is inherently related to the Lorentz invariant partition of classical fields into the positive and the negative frequency parts \( u(x) = u^+(x) + u^-(x) \), performed with the help of Fourier transform in Minkowski space. That is the commutation relations are being established between non localized solutions of field equations. At the same time the construction of divergence free physical theory requires the separation of the contributions of different space-time scales. In present paper, using the light-cone variables, we propose a quantization procedure which is compatible with separation of scales using continuous wavelet transform, as described in our previous paper [1].

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I. INTRODUCTION

The construction of quantum field theory models is inherently related to the Lorentz invariant partition of classical fields into the positive and the negative frequency parts

\[ u(x) = u^+(x) + u^-(x), \quad x \in \mathbb{R}^{1,3} \]

The quantization procedure itself is then based on the commutators of the energy-momentum tensor components \( T^{0\mu} \) with the positive and the negative frequency field operators \( u^{\pm}(k), k \in \mathbb{R}^3 \), with the field equations being used to eliminate the redundant component of the quantum fields.

The calculation of the \( n \)-point Green functions \( \langle u(x_1) \ldots u(x_n) \rangle \), the functional derivatives of the generating functional, is well known to suffer from loop divergences in both
UV and IR domains of momentum space. The way to rule out the divergences is to separate the contributions of different scales, which can be formally casted in the form [2]

\[ u(x) = \int u_a(x) \frac{da}{a}, \]

where the "scale component" \( u_a(x) \) is not yet well defined. The most known way to separate the scales is the renormalization group technique [3, 4] the less known is the wavelet transform in quantum field theory [1].

The consideration would be straightforward for Euclidean quantum field theory, where the projection of an arbitrary function \( u(x) \in L^2(\mathbb{R}^d) \) onto the scale \( a \) is given by the convolution

\[ u_a(b) := \int_{\mathbb{R}^d} \frac{1}{a^d} \tilde{g} \left( \frac{x - b}{a} \right) u(x) d^d x, \quad (1) \]

so that the function \( u(x) \) can be reconstructed from the set of its wavelet coefficients \( \{ u_a(b) \} \) by the inverse wavelet transform [5]

\[ u(x) = \frac{1}{C_g} \int_{\mathbb{R}_+ \times \mathbb{R}^d} \frac{1}{a^d} \tilde{g} \left( \frac{x - b}{a} \right) u_a(b) \frac{d^d b}{a} \quad (2) \]

The analyzing function \( g(x) \), satisfying rather loose admissibility condition \( \int \frac{|\tilde{g}(ak)|^2}{a} da = C_g < \infty \) is usually referred to as a basic wavelet.

The continuous wavelet transform (CWT) is a feasible alternative to the usual Fourier transform

\[ u(x) = \int e^{ikx} \tilde{u}(k) \frac{d^d k}{(2\pi)^d} \quad (3) \]

because in any measurement are not accessible exactly in a given point \( x \): to localize a particle in an interval \( \Delta x \) the measuring device requests a momentum transfer of order \( \Delta p \sim \hbar/\Delta x \). If \( \Delta x \) is too small the field \( u(x) \) at a fixed point \( x \) has no experimentally verifiable meaning. At the same time establishing of canonical commutation relations between the field operators is essentially based on Fourier transform (3). It is intuitively clear that the commutator \([u_{a_1}(b_1), u_{a_2}(b_2)]\) is a function of \( \frac{a_1}{a_2} \), which vanish if \( \log \frac{a_1}{a_2} \) is significantly different from zero. This fact is well known in radiophysics: if a field (system) is localized in a region of size \( a_1 \) centered at point \( b_1 \), it may be detected by other field with significant probability only in case when \( a_1 \) and \( a_2 \) have the same order. If the window width \( a_2 \) is too narrow or too wide in comparison to \( a_1 \) the probability of detection is low.

In the remainder of this paper we present the derivation of the canonical commutation relations between the field operators describing massive scalar field that depend on both position and resolution in \( \mathbb{R}^{1,3} \) Minkowski space.
II. CONTINUOUS WAVELET TRANSFORM

A. Basics of the continuous wavelet transform

Let $H$ be a Hilbert space of states for a quantum field $|\phi\rangle$. Let $G$ be a locally compact Lie group acting transitively on $H$, with $d\mu(\nu), \nu \in G$ being a left-invariant measure on $G$. Then, similarly to representation of a vector $|\phi\rangle$ in a Hilbert space of states $H$ as a linear combination of eigenvectors of momentum operator $|\phi\rangle = \int |p\rangle d\mu(p) \langle p| \phi\rangle$, any $|\phi\rangle \in H$ can be decomposed with respect to a representation $U(\nu)$ of $G$ in $H$ [6, 7]:

$$|\phi\rangle = \frac{1}{C_g} \int_G U(\nu)|g\rangle d\mu(\nu) \langle g| U^*(\nu)|\phi\rangle,$$  \hspace{1cm} (4)

where $|g\rangle \in H$ is referred to as an admissible vector, or basic wavelet, satisfying the admissibility condition

$$C_g = \frac{1}{\|g\|^2} \int_G |\langle g| U(\nu)|g\rangle|^2 d\mu(\nu) < \infty.$$  

The coefficients $\langle g| U^*(\nu)|\phi\rangle$ are referred to as wavelet coefficients.

If the group $G$ is abelian, the wavelet transform (4) with $G : x' = x + b'$ coincides with Fourier transform.

B. Euclidean space

The next to the abelian group is the group of the affine transformations of the Euclidean space $\mathbb{R}^d$

$$G : x' = aR(\theta)x + b,$$

$x, b \in \mathbb{R}^d, a \in \mathbb{R}_+, \theta \in SO(d), \hspace{1cm} (5)$

where $R(\theta)$ is the rotation matrix. We define unitary representation of the affine transform (5) with respect to the basic wavelet $g(x)$ as follows:

$$U(a, b, \theta)g(x) = \frac{1}{a^d}g\left(R^{-1}(\theta)\frac{x - b}{a}\right). \hspace{1cm} (6)$$

(We use $L^1$ norm [8, 9] instead of usual $L^2$ to keep the physical dimension of wavelet coefficients equal to the dimension of the original fields).

Thus the wavelet coefficients of the function $u(x) \in L^2(\mathbb{R}^d)$ with respect to the basic wavelet $g(x)$ in Euclidean space $\mathbb{R}^d$ can be written as

$$u_{a, \theta}(b) = \int_{\mathbb{R}^d} \frac{1}{a^d}g\left(R^{-1}(\theta)\frac{x - b}{a}\right) u(x) d^dx. \hspace{1cm} (7)$$

The wavelet coefficients (7) represent the result of the measurement of function $u(x)$ at the point $b$ at the scale $a$ with an aperture function $g$ rotated by the angle(s) $\theta$ [10].

The function $u(x)$ can be reconstructed from its wavelet coefficients (7) using the formula (4):
The normalization constant $C_g$ is readily evaluated using Fourier transform:

$$C_g = \int_0^\infty |\tilde{g}(aR^{-1}(\theta)k)|^2 \frac{da}{a} d\mu(\theta) = \int |\tilde{g}(k)|^2 \frac{d^d k}{|k|^d} < \infty.$$  

For isotropic wavelets

$$C_g = \int_0^\infty |\tilde{g}(ak)|^2 \frac{da}{a} = \int |\tilde{g}(k)|^2 \frac{d^d k}{S_d |k|^d},$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the area of unit sphere in $\mathbb{R}^d$.

It is helpful to rewrite continuous wavelet transform in Fourier form:

$$u(x) = \frac{1}{C_g} \int_0^\infty \frac{da}{a} \int \frac{d^d k}{(2\pi)^d} e^{ikx} \tilde{g}(ak) \tilde{u}_a(k),$$

$$\tilde{u}_a(k) = \frac{\tilde{g}(ak)}{\tilde{g}(a k_0)} \tilde{u}(k).$$

The wavelet function $\tilde{g}(ak)$ works as a band-pass filter, which injects a part of the energy carried by the k-mode of the function $u(x)$ into the "detector" of scale $a$, depending on how the product $|ak|$ is different from the unity.

Indeed, taking the plane wave $\phi(x) = (2\pi)^{-d/2}e^{i k_0 x}$ as an example of free particle with momentum $k_0$, so that $\hat{P}\phi(x) = k_0\phi(x)$, $\hat{P} = -i\partial_x$, we get

$$\tilde{\phi}(k) = \delta^d(k - k_0), \quad \tilde{\phi}_a(k) = \tilde{g}(ak)\delta^d(k - k_0)$$

and hence

$$\phi_a(b) = e^{ik_0 b} \tilde{g}(ak_0)$$  \hspace{1cm} (9)

The partial momentum per octave is

$$k_0 |\tilde{g}(ak_0)|^2,$$

so that the sum over all possible scales is $k_0$.

It is impossible however to do such separation in Minkowski space $\mathbb{R}^{1,3}$ in space-time coordinates $(t, x, y, z)$.

C. Minkowski space

To construct wavelet transform in Minkowski space it is convenient to turn from the space-time coordinates $x^\mu = (t, x, y, z)$ to the light-cone coordinates:

$$x^\mu = (x_+, x_-, y, z), \quad x_\pm = \frac{t \pm x}{\sqrt{2}}, \quad \mathbf{x}_\perp = (y, z).$$  \hspace{1cm} (10)

This is the so-called infinite momentum frame. The advantage of the coordinates (10) for the calculations in quantum field theory is significant simplification of the vacuum structure [11, 12]. The metrics in the light-cone coordinates becomes

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$  

The rotation matrix – the Lorentz boosts in $x$ direction and the rotations in $(y, z)$ plane
- has a block-diagonal form

\[ M(\eta, \phi) = \begin{pmatrix} e^\eta & 0 & 0 & 0 \\ 0 & e^{-\eta} & 0 & 0 \\ 0 & 0 & \cos \phi & \sin \phi \\ 0 & 0 & -\sin \phi & \cos \phi \end{pmatrix}, \]

so that \( M^{-1}(\eta, \phi) = M(-\eta, -\phi) \). Hyperbolic rotation in \((t, x)\) plane is determined by the hyperbolic rotation angle – the rapidity \( \eta \). The rotations in the transverse plane, not affected by the Lorentz contraction, are determined by the rotation angle \( \phi \).

The Poincare group can be extended by the scale transformations \( x' = ax \) to the affine group

\[ x' = aM(\eta, \phi)x + b, \]

with the representation written in the same form as that of wavelet transform in Euclidean space \( \mathbb{R}^d \), viz:

\[ U(a, b, \eta, \phi)u(x) = \frac{1}{a^4}u \left( M^{-1}(\eta, \phi) \frac{x - b}{a} \right), \]

defined in \( L^1 \) norm in accordance to \([1, 9]\).

So, we have straightforward generalization of the definition of wavelet coefficients of a function \( f(x) \in L^2(\mathbb{R}^{1,3}) \) with respect to the basic wavelet \( g[1, 13] \)

\[ W_{a,b,\eta,\phi}[f] = \int dx_+ dx_- d^2x_\perp \frac{1}{a^4}g \left( M^{-1}(\eta, \phi) \frac{x - b}{a} \right)f(x_+, x_-, x_\perp). \quad (11) \]

The difference from calculations in Euclidean space \( \mathbb{R}^4 \) is that the basic wavelet \( g(\cdot) \) cannot be defined globally on \( \mathbb{R}^{1,3} \). Instead, it should be defined in four separate domains impassible by Lorentz rotations:

\[ A_1 : k_+ > 0, k_- < 0; A_2 : k_+ < 0, k_- > 0; \]
\[ A_3 : k_+ > 0, k_- > 0; A_4 : k_+ < 0, k_- < 0, \]

where \( k \) is wave vector, \( k_\pm = \frac{\omega \pm k_x}{\sqrt{2}} \). Four separate wavelets should be defined in these four domains \([14, 15]\):

\[ g_i(x) = \int_{A_i} e^{ikx} \tilde{g}(k) \frac{d^4k}{(2\pi)^4}, \quad i = 1, 4. \quad (12) \]

We assume the following definition of the Fourier transform in light cone coordinates:

\[ f(x_+, x_-, x_\perp) = \int e^{ik_- x_+ + ik_+ x_- - ik_\perp x_\perp} \tilde{f}(k_-, k_+, k_\perp) \frac{dk_+ dk_- d^2k_\perp}{(2\pi)^4}. \]

Substituting the Fourier images into the definition (11) we get

\[ W_{a,b,\eta,\phi}^{i}[f; A_i] = \int e^{ik_- x_+ + ik_+ x_- - ik_\perp x_\perp} \tilde{f}(k_-, k_+, k_\perp) \frac{dk_+ dk_- d^2k_\perp}{(2\pi)^4}. \quad (13) \]

Similarly to the \( \mathbb{R}^d \) case, the reconstruction
formula is [13]:

\[
f(x) = \sum_{i=1}^{4} \frac{1}{C_{gi}} \int_{-\infty}^{\infty} d\eta \int_{0}^{2\pi} d\phi \int_{0}^{\infty} \frac{da}{a} \int_{M_{i}^{2}} \frac{db_{+}db_{-}d^{2}b_{\perp}}{a^{4}g_{i}} \left( M^{-1}(\eta) \frac{\xi - b}{a} \right) W_{i}^{\eta \phi}
\]

\[
= \sum_{i=1}^{4} \frac{1}{C_{gi}} \int_{-\infty}^{\infty} d\eta \int_{0}^{2\pi} d\phi \int_{0}^{\infty} \frac{da}{a} \int_{A_{i}} \frac{dk_{+}dk_{-}d^{2}k_{\perp}}{(2\pi)^{4}} e^{ik_{-}x_{+} + ik_{+}x_{-} - ik_{\perp}x_{\perp}} \times \nabla_{x}W_{i \eta \phi}(k) \tilde{g}(ak_{-}e^{\eta}, ak_{+}e^{-\eta}, aR^{-1}(\phi)k_{\perp})
\]

III. QUANTIZATION

Same as in standard quantum field theory, we wish to use the mass-shell delta function to get rid of redundant degrees of freedom [16]. Let us consider the massive scalar field in \( \mathbb{R}^{3} \) Minkowski space

\[
u(x) = \int e^{ikx}2\pi\delta(k^{2} - m^{2})\tilde{u}(k_{-}, k_{+}, k_{\perp}) \frac{d^{4}k}{(2\pi)^{4}}
\]

The Lorentz invariant scalar product and the invariant volume in \( k \)-space are

\[
k \cdot \nu = k_{0}x_{0} - k \cdot x = k_{-}x_{+} + k_{+}x_{-} - k_{\perp}x_{\perp}
\]

\[
\frac{d^{4}k}{(2\pi)^{4}} = \frac{dk_{0}dk_{x}dk_{y}dk_{z}}{(2\pi)^{4}} = \frac{dk_{-}dk_{+}d^{2}k_{\perp}}{(2\pi)^{4}}.
\]

For a massive scalar field because of the mass shell delta function \( \delta(2k_{+}k_{-} - k^{2} - m^{2}) \) only two domains \( A_{3} \) and \( A_{4} \) for which \( k_{+}k_{-} \) is positive will contribute to the decomposition of \( u(x) \). The integration over the \( k_{-} \) variable with the mass shell delta function gives

\[
k_{-} = \frac{k_{\perp}^{2} + m^{2}}{2k_{+}}
\]

After the substitution of integration variable \( k \rightarrow -k \) in integration over \( A_{4} \), the decomposition of \( u(x) \) takes the form

\[
u(x) = \int \left[ e^{ikx} \tilde{u} \left( \frac{k_{+}^{2} + m^{2}}{2k_{+}}, k_{+}, k_{\perp} \right) + e^{-ikx} \tilde{u} \left( -\frac{k_{+}^{2} + m^{2}}{2k_{+}}, -k_{+}, -k_{\perp} \right) \right] \times \theta(k_{+}) \frac{dk_{+}d^{2}k_{\perp}}{2k_{+}(2\pi)^{3}}
\]

\[
\equiv \int \left[ e^{ikx} \tilde{u}^{+}(k) + e^{-ikx} \tilde{u}^{-}(k) \right] \times \theta(k_{+}) \frac{dk_{+}d^{2}k_{\perp}}{2k_{+}(2\pi)^{3}}
\]

Both \( \tilde{u}^{+}(k) \) and \( \tilde{u}^{-}(k) \) are defined on a hemisphere in \( \mathbb{R}^{3} \) and can be decomposed into scale components by continuous wavelet transform in Euclidean space. The straightforward way to quantize the fields in the light-cone representation is to use the formal analogy between the decomposition (15) and the positive/negative frequency decomposition in usual coordinates \( (t, x) \) in the equal-
time quantization scheme
\[
\left\{ u(t, x), \frac{\partial L}{\partial \dot{u}(t, y)} \right\}_{t=0} = i\delta^3(x - y), \tag{16}
\]
where the curly brackets stand for the Poisson brackets substituted by commutator (anti-commutator) for Bose (Fermi) quantum fields.

Using the Lagrangian
\[
L = \frac{\partial u}{\partial x_+} \frac{\partial u}{\partial x_-} - \frac{1}{2} (\partial_\perp u)^2 - \frac{m^2}{2} u^2, \tag{17}
\]
we can infer that the \( x_+ = \frac{t + x}{\sqrt{2}} \) variable can be considered as "time" on the light-cone [17]. In analogy to common case the Poisson bracket can be then casted in the form
\[
\left\{ u(x_+ = 0, x_-), \frac{\partial u}{\partial y_-} \right\}_{y_+ = 0} = i\delta^3(x - y) \tag{18}
\]
Substituting decomposition (15) into the bracket (18) and changing the bracket to commutator one gets
\[
[\tilde{u}^-(k), \tilde{u}^+(q)] = 2k_+(2\pi)^3\delta^3(k - q). \tag{19}
\]
The latter equation is different from the standard commutation relation by changing the energy \( k_0 \) to the momentum \( k_+ \). The role of energy is played by \( k_- \) in the light-cone coordinates.

Substituting the inverse wavelet transform
\[
\tilde{u}^\pm(k_+, k_\perp) = \frac{1}{C_g} \int_0^\infty \tilde{g}(ak)\tilde{u}_a^\pm(k)\frac{da}{a}, \tag{20}
\]
where \( \tilde{u}_a^\pm(k) = \tilde{g}(ak)\tilde{u}_a^\pm(k), k \equiv (k_+, k_\perp) \), into the equality (19), and assuming an isotropic basic wavelet \( g(\cdot) \) for simplicity, we derive the commutation relations for the scale components
\[
[\tilde{u}_{a_1}^-(k), \tilde{u}_{a_2}^+(q)] = 16\pi^3C_g a_1\delta(a_1 - a_2)k_+\delta(k_+ - q_+)^2(k_\perp - q_\perp). \tag{21}
\]
The commutation relation (21) meets the general form of wavelet transform of the canonical commutation relations in Minkowski space, eq.(18) of [1] Introducing the vacuum state \( \Phi_p \) with the momentum \( p \) we get
\[
[u_{iai}^-(k), u_{ja\eta}^+(k')] = a\delta(a - a')\delta(\eta - \eta') \times \delta_{ij} C_{gi} [u^-(k), u^+(k')],
\]
declared on four Lorentz-invariant domains \( A_i, i = 1, 4 \). However, being defined on In the latter equations \( \tilde{u}^\pm(k) \) can be subjected to wavelet transform so that \( \tilde{u}^\pm(k) \) is
expressed by (20) with \( k \) having only 3 independent components. In this way we can construct the *multiscale* Fock space of states

\[
\Phi = \sum_{j,s} \int F_s^{(\cdots\cdot\cdot)}(a_1, k_1, \ldots, a_s, k_s) \tilde{u}^+_{j_1a_1}(k_1) \cdots \tilde{u}^+_{j_s a_s}(k_s) \frac{da_1 dk_1 + d^2 k_{1\perp}}{a_1 C_g 16 k_{1+} \pi^3} \cdots \frac{da_s dk_s + d^2 k_{s\perp}}{a_s C_g 16 k_{s+} \pi^3} \Phi_0,
\]

where \( k_i = (k_{i+}, k_{i\perp}) \) are three dimensional vectors, \( j \) denote all other indices of the quantum states, and \( \Phi_0 \) is a vacuum state \( u^-_i(x)\Phi_0 = 0 \).

**IV. CONCLUSIONS**

To be concluded, we have developed a quantization scheme suitable for applications in quantum theory of fields \( u_a(x) \), which explicitly depend on both position \( x \) and the scale (resolution) \( a \). It is not surprising, that such fields can form a prospective framework for analytic calculations in quantum chromodynamics, where most approved results are obtained either numerically lattice simulations [18], or analytically, with perturbation expansion being corrected by renormalization group methods [19]. In the latter case the obtained results, viz., process amplitudes, parton distribution functions, nucleon form factors, tacitly depend on some formal scale parameter \( \Lambda \), which is either cutoff momentum, or renormalization scale. From functional analysis point of view, this may suggest the use of space of functions which explicitly depend on both the position and the resolution. being operator-valued functions they certainly require commutation relations. The use of light-cone coordinates enables this construction. The massive scalar field quantization was choosen as a simple example. Perhaps the same technique can be used in general problems of quantum field theory, when wavelet transform is used to construct divergence free Green functions [2, 20, 21].

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