ORDER IDEALS AND A GENERALIZED KRULL HEIGHT THEOREM

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ABSTRACT. Let $N$ be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m})$. We give criteria for the height of the order ideal $N^*(x)$ of an element $x \in N$ to be bounded by the rank of $N$. The Generalized Principal Ideal Theorem of Bruns, Eisenbud and Evans says that this inequality always holds if $x \in \mathfrak{m}N$. We show that the inequality even holds if the hypothesis becomes true after first extending scalars to some local domain and then factoring out torsion. We give other conditions in terms of residual intersections and integral closures of modules.

We derive information about order ideals that leads to bounds on the heights of trace ideals of modules—even in circumstances where we do not have the expected bounds for the heights of the order ideals!

Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $N$ be a finitely generated $R$-module. For $x \in N$ we define the order ideal of $x$, written $N^*(x)$, to be the set of images of $x$ under homomorphisms $N \to R$.

The classical Krull height theorem (Krull [1928]) says that $r$ elements of $R$ either generate an ideal of height at most $r$, or the unit ideal. This may be interpreted by saying that if $N$ is free of rank $r$, then the order ideal of any element $x \in \mathfrak{m}N$ has height at most $r$. Eisenbud and Evans [1976] conjectured that the same statement would be true for any module $N$; they proved the conjecture for rings containing a field, and Bruns [1981] subsequently gave a general argument. This work leaves the question addressed in this paper:

**Under what circumstances does the order ideal of a minimal generator of $N$ have height at most the rank of $N$?**

We give criteria to settle this question in many cases, and use them in turn to prove related results bounding the heights of some residual intersections and trace ideals. An interesting feature is the need for studying Rees algebras and integral dependence of modules.

To get a feeling for the central question, consider the case where $R$ is a graded ring, $N$ is a graded module that represents a vector bundle $\mathcal{E}$ on $X = \text{Proj}(R)$, and $N$ is generated

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(perhaps only as a sheaf) by elements of degree 0. In this setting the order ideal of an element of degree 0 is just the vanishing locus of the corresponding section of the vector bundle; and there is a section which vanishes in codimension greater than \( r \) if and only if \( \mathcal{E} \) admits a sub-bundle isomorphic to \( \mathcal{O}_X \) if and only if the top Chern class \( c_r(\mathcal{E}) \) vanishes (for information on Chern classes see Fulton [1984, Chapter 3]).

For example, consider the cotangent bundle \( \Omega_{\mathbb{P}^{n-1}}^{1} \) of projective \( n-1 \)-space. It has no global sections, but its twist \( E = \Omega_{\mathbb{P}^{n-1}}^{1}(2) \) is a bundle of rank \( n-1 \) generated by its global sections. The corresponding module \( N \) over the polynomial ring \( R \) in \( n \) variables is the kernel of the map \( R^{n}(1) \to R^{2} \) sending the \( i \)th generator to the \( i \)th variable; it is generated in degree 0. The exact sequence
\[
0 \to \mathcal{E} \to \mathcal{O}^{n}(1) \to \mathcal{O}(2) \to 0
\]
and the fact that the “Chern polynomial”
\[
c_t(\mathcal{E}) = 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \cdots \in A^*(\mathbb{P}^{n-1}) = \mathbb{Z}[t]/(t^n)
\]
is multiplicative show that \( c_t(\mathcal{E}) \equiv (1 + t)^n/(1 + 2t) \mod(t^n) \). In particular \( c_{n-1}(\mathcal{E}) = \sum_{i=0}^{n-1} (-1)^i \binom{n}{n-1-i} 2^i \). A little arithmetic gives
\[
c_{n-1}(\mathcal{E}) = \begin{cases} 
0 & \text{if } n \text{ is even;} \\
1 & \text{if } n \text{ is odd.}
\end{cases}
\]
Thus we expect \( N \) to have an order ideal of height \( \geq n \) if and only if \( n \) is even. In Section 2 we present an algebraic analysis proving this result (and settling a number of related cases.)

A first suggestion of the role of integrality in the theory is illustrated by a question of Huneke and Koh: They asked whether the order ideals of elements in the integral closure of \( mN \) are always bounded by the rank of \( N \). We prove a still more general statement; the following is a special case of Theorem 3.1:

**Theorem.** Let \( R \) be an affine domain, let \( N \) be a finitely generated \( R \)-module of rank \( r \), and let \( x \in N \). Let \( R \to S \) be a homomorphism from \( R \) to a local domain \((S, \mathfrak{n})\) and write \( SN \) for \( N \otimes_R S \) modulo torsion. If the image of \( x \) in \( SN \) lies in \( \mathfrak{n}SN \), then the height of \( N^*(x) \) is at most the rank of \( N \).

In Section 2 we give systematic methods for constructing examples where \( N^*(x) \) has height greater than \( \operatorname{rank}(N) \). We illustrate our methods by constructing, among other things, a graded module \( N \) of rank 5 over a polynomial ring in 6 variables such that every homogeneous element of \( N \) has order ideal of height at most 5, but \( N \) contains inhomogeneous elements with order ideal of height 6!

To describe the more refined results of the paper we continue with the assumption that \( R \) is local. A basic construction of this paper (in a special case) is that of the **perpendicular module** of \( N \): If \( F \) is a free module of minimal rank with a surjection \( \pi : F \to N \) we define \( N^\perp = \operatorname{coker}(\pi^*) \). Set \( M = N^\perp \). We observe (Remark 2.2) that the order ideals of
generators of \( N \) correspond to the colon ideals of the form \( U :_R M = \{ f \in R \mid fN \subseteq U \} = \text{ann}(M/U) \) where \( M/U \) is a cyclic module. Thus the existence of order ideals of elements of \( N \) having extraordinary height is the same as the existence of submodules \( U \subseteq M \) as above with \( U :_R M \) of extraordinary height. A classic argument of McAdam [1983], as generalized in Section 2, shows that \( M \) is then integrally dependent on \( U \) (see Corollary 1.3), and thus \( \ell(M) < \mu(M) \). Under some circumstances we show that the condition \( \ell(M) = \mu(M) \) is actually necessary and sufficient for all \( x \in N \) to have order ideals of height \( \leq \text{rank}(N) \) (Proposition 3.6 and Proposition 3.9).

Using the related constructions, we are able to deduce information about the colon ideals from information on order ideals and vice versa. An example is the following special case of Proposition 4.3, which gives conditions under which all “residual intersections” of a module have the right height:

**Proposition.** Let \( R \) be a regular local ring containing a field and let \( M \) be a finitely generated torsion free \( R \)-module of rank \( e \). If \( s \) is an integer such that \( \text{Ext}^i_R(M,R) = 0 \) for \( 2 \leq i \leq s - 1 \) then, for every submodule \( U \subset M \) with \( \mu(U) - e + 1 \leq s \),

\[
\text{ht}(U :_R M) \leq \max\{0, \mu(U) - e + 1\}.
\]

In Section 4 we also investigate order ideals of modules of low rank. We show under mild hypotheses that if \( N \) has rank \( \leq 2 \), or \( N \) is a \( k \)-th syzygy module of rank \( k \), then all elements of \( N \) have order ideals of height \( \leq \text{rank}(N) \) (Proposition 4.1); under somewhat more stringent conditions we get a similar result for modules of rank \( 3 \) (Proposition 4.2).

In the last section we consider the relationship of trace ideals and order ideals. In the case of a module of rank \( 3 \) (or a \( k \)-th syzygy module of rank \( k + 1 \)) satisfying mild conditions there may well be order ideals that are too large; but we prove that the radical of such an order ideal must contain the whole trace ideal of \( N \), defined as the sum of all order ideals \( \text{tr}(N) = \{ f(x) \mid f \in N^* \text{ and } x \in N \} \) (Proposition 5.4). Finally we turn to the question of the possible heights of trace ideals. The surprising result is that we can give a stronger bound for the height of the trace ideal of \( N \) if the height of some order ideal exceeds \( \text{rank}(N) \) than in the contrary case (Theorem 5.5).

1. **Rees Algebras**

In this section we recall the general notion of Rees algebra of a module introduced in our [2000], and provide some results about integral dependence.

Let \( R \) be a Noetherian ring and let \( M \) be a finitely generated \( R \)-module. If \( Q \) is a prime ideal of \( R \) we write \( \mu_Q(M) \) for the minimal number of generators of \( M_Q \) over \( R_Q \). When \((R, \mathfrak{m})\) is a local ring we set \( \mu(M) = \mu_\mathfrak{m}(M) \). By \( -* \) we denote the functor \( \text{Hom}_R(-,R) \). We say that an \( R \)-linear map \( f : M \rightarrow F \) is a **versal map** from \( M \) to a free module if \( F \) is a free \( R \)-module and \( f^* \) is surjective. The latter condition means that every \( R \)-linear from \( M \) to a free \( R \)-module factors through \( f \). In our [2000, 0.1 and 1.3] we define the Rees
algebra of $M$ to be $\mathcal{R}(M) = \text{Sym}(M)/(\cap_g L_g)$ where the intersection is taken over all maps $g$ from $M$ to free $R$-modules, and $L_g$ denotes the kernel of $\text{Sym}(g)$. Equivalently, $\mathcal{R}(M)$ is isomorphic to the image of the map $\text{Sym}(f) : \text{Sym}(M) \to \text{Sym}(F)$, where $f : M \to F$ is any versal map to a free module. The Rees algebra of any finitely generated module exists and is unique up to canonical isomorphisms of graded $R$-algebras; in fact the construction is functorial. On a less trivial note we prove in our [2000, 1.4 and 1.5] that the above definition gives the usual notion of Rees algebras of ideals, and that over a $\mathbb{Z}$-torsion free ring $R$ any embedding $g : M \to G$ into a free module $G$ can be used to define the Rees algebra $\mathcal{R}(M)$ as the image of $\text{Sym}(g)$.

Let in addition $U \subset L$ be submodules of $M$, let $U', L'$ be the images of $U, L$ in $\mathcal{R}(M)$, and consider the $R$-subalgebras $R[U'] \subset R[L']$ of $\mathcal{R}(M)$. According to our [2000, 2.1] we say $L$ is integral over $U$ in $M$ if the ring extension $R[U'] \subset R[L']$ is integral; the largest such module $L$ (which exists and is unique) is called the integral closure of $U$ in $M$; finally, we say $M$ is integral over $U$ or $U$ is a reduction of $M$, if $M$ is integral over $U$ in $M$. In our [2000, 2.2] we record the following “valuative criterion of integrality”:

**Proposition 1.1.** Let $R$ be a Noetherian ring, let $M$ be a finitely generated $R$-module, let $U \subset L$ be submodules of $M$, and let $f : M \to F$ be a versal map from $M$ to a free $R$-module. The following are equivalent:

1. $L$ is integral over $U$ in $M$.
2. For every minimal prime $Q$ of $R$, the module $L'$ is integral over $U'$ in $M'$, where $'$ denotes images in $F/QF$.
3. For every map $M \to G$ to a free $R$-module and for every homomorphism $R \to S$ to a domain $S$, the module $L'$ is integral over $U'$ in $M'$, where $'$ denotes tensoring with $S$ and taking images in $S \otimes_R G$.
4. For every homomorphism $R \to V$ to a rank one discrete valuation ring $V$ whose kernel is a minimal prime of $R$, we have $U' = L'$, where $'$ denotes tensoring with $V$ and taking images in $V \otimes_R F$.
5. For every map $M \to G$ to a free $F$-module and every homomorphism $R \to V$ to a rank one discrete valuation ring $V$, we have $U' = L'$, where $'$ denotes tensoring with $V$ and taking images in $V \otimes_R G$.

Let $R$ be a Noetherian local ring with residue field $k$ and let $M$ be a finitely generated $R$-module. In our [2000, 2.3] we define the analytic spread $\ell(M)$ of $M$ to be the Krull dimension of $k \otimes_R \mathcal{R}(M)$. In case $k$ is infinite one has $\ell(M) = \{\mu(U) | U \text{ a reduction of } M\}$; furthermore $\ell(M) \leq \mu(M)$ and equality holds if and only if $M$ admits no proper reduction.

The next theorem is due to McAdam [1983, 4.1] in the case of ideals. Various cases with modules are treated in Rees [1987, 2.5], Kleiman-Thorup [1994, 10.7], Katz [1995, 2.4], and Simis-Ulrich-Vasconcelos [1999, 5.6]. Our proof is a reduction to the case treated by Rees. The result plays an important role in this paper.

**Theorem 1.2.** Let $R$ be a locally equidimensional universally catenary Noetherian ring, let $M$ be a finitely generated $R$-module and let $U$ be a submodule of $M$ generated by $t$ elements. If there exists a minimal prime $Q$ of $R$ such that $M/QM$ is not integral over
the image of $U$ in $M/QM$, then
\[ \text{ht}(U :_R M) \leq \max\{0, t + 1 - \mu_Q(M)\}. \]

**Proof.** We localize to assume that $R$ is local and equidimensional. Write $R' = R/Q, M' = M/QM$, and let $U'$ be the image of $U$ in $M'$. We have $\text{ht}(U :_R M) \leq \text{ht}(U' :_{R'} M')$, because $R'(U :_R M) \subset U' :_{R'} M'$ and $R$ is an equidimensional catenary local ring. Thus we may replace $R, U, M$ by $R', U', M'$ to assume that $R$ is a universally catenary local domain and $M$ is torsion free. We may assume that $U :_R M \neq 0$, and hence that rank $U = \text{rank } M$. With these assumptions the assertion was proved by Rees [1987, 2.5]. \(\square\)

**Corollary 1.3.** Let $R$ be a local equidimensional universally catenary Noetherian ring, let $M$ be a finitely generated $R$-module and let $U$ be a proper submodule of $M$ generated by $t$ elements. If $t(M) = \mu(M)$, then there exists a minimal prime $Q$ of $R$ such that $M/QM$ is not integral over the image of $U$ in $M/QM$. For any such $Q$,
\[ \text{ht}(U :_R M) \leq \max\{0, t + 1 - \mu_Q(M)\}. \]

**Proof.** We may assume that the residue field of $R$ is infinite. It follows from our hypothesis that $M$ is not integral over $U$. By Proposition 1.1 and the functoriality of the Rees algebra, there exists a minimal prime ideal $Q$ such that $M/QM$ is not integral over the image of $U$. The assertion now follows from Theorem 1.2. \(\square\)

### 2. Perpendicular Modules

Central to this paper is the following:

**Definition 2.1.** Let $R$ be a Noetherian ring and let $N$ be a finitely generated $R$-module with a choice of generators $x_1, \ldots, x_n$. Map a free $R$-module with basis $e_1, \ldots, e_n$ to $N$ by sending $e_i$ to $x_i$ and denote this map by $\pi$. We define $[x_1, \ldots, x_n] = \text{Coker}(\pi^*)$, and write $x_i^\perp$ for the image of $e_i^* \in \text{Coker}(\pi^*)$. When $R$ is local and the $x_1, \ldots, x_n$ are minimal generators, we set $N^\perp = [x_1, \ldots, x_n]^\perp$ and call it the **perpendicular module** to $N$ (indeed, this module only depends on $N$).

**Remark 2.2.** **Perpendicular modules and colons.** With notation as in 2.1, let $M = [x_1, \ldots, x_n]^\perp$ and let $U$ be the submodule of $M$ generated by $x_1^\perp, \ldots, x_{i-1}^\perp, x_{i+1}^\perp, \ldots, x_n^\perp$. We have
\[ N^*(x_i) = U :_R M. \]

The right hand side is clearly equal to the $i^{th}$ row ideal (the ideal generated by the elements of the $i^{th}$ row) of any matrix presenting $M$ with respect to the generating set $x_1^\perp, \ldots, x_n^\perp$. The $i^{th}$ row ideal equals $N^*(x_i)$ because the image of an element $f$ of $N^*$ under $\pi^*$ has $i^{th}$ component equal to $f(x_i)$.

An immediate consequence is that if $R$ is local and $N$ has no free summand, then $\mu(M) = n$. More generally, if $Q$ is any prime of $R$ we have $\mu_Q(M) = n - rf_Q(N)$, where $rf_Q(M)$ denotes the maximal rank of an $R_Q$-free direct summand of $M_Q$.

Combining the above remark with Theorem 1.2 we obtain the following consequence for order ideals.
Corollary 2.3. Let $R$ be a locally equidimensional universally catenary Noetherian ring, let $N$ be a finitely generated $R$-module, and let $Q$ be a minimal prime of $R$. Choose a generating set $x_1, \ldots, x_n$ of $N$, write $M = [x_1, \ldots, x_n]^{\perp}$, and let $U = Rx_1^{\perp} + \cdots + Rx_{n-1}^{\perp} \subset M$. If $M/QM$ is not integral over the image of $U$, then $\text{ht}(N^*(x_n)) \leq rf_Q(N)$.

Proof. Remark 2.2 shows that $N^*(x_n) = U :_R M$, and by Theorem 1.2, $\text{ht}(U :_R M) \leq n - \mu_Q(M)$. Finally, $n - \mu_Q(M) = rf_Q(N)$. □

The following version of the semicontinuity theorem for heights of ideals in a family will be useful in bounding the heights of order ideals:

Proposition 2.4. Let $(R, m)$ be an equidimensional universally catenary Noetherian local ring, let $Z = Z_1, \ldots, Z_n$ be indeterminates over $R$, and write $R' = R[Z]$.

1. If $\varphi$ is a matrix over $R$ with $n$ rows, then $\text{ht}((a \cdot \varphi)R) \leq \text{ht}((Z \cdot \varphi)R')$ for every vector $a$ of $n$ elements in $R$.
2. If $N$ is an $R$-module with generating set $x_1, \ldots, x_n$ and $y = \sum_{i=1}^{n} Z_i x_i \in N \otimes_R R'$, then $\text{ht}(N^*(x)) \leq \text{ht}((N \otimes_R R')^*(y))$ for every $x \in N$.

Proof. To see (1) write $S = R[Z, \{Z_i - a_i\}]$. The ring $R/(a \cdot \varphi)$ is obtained from $S/(Z \cdot \varphi)$ by factoring out the ideal generated by the $S$-regular sequence $Z_i - a_i, 1 \leq i \leq n$. Since $S$ is equidimensional and catenary, it follows that $\text{ht}((a \cdot \varphi)R) \leq \text{ht}((Z \cdot \varphi)S)$. Localizing further we deduce (1).

To prove (2) we apply (1) to a matrix $\varphi$ presenting $[x_1, \ldots, x_n]^{\perp}$ with respect to the generating set $x_1^{\perp}, \ldots, x_n^{\perp}$, and invoke Remark 2.2. □

Let $N$ be a finitely generated module over a Noetherian ring $R$. We say that $N$ satisfies $G_s$, where $s$ is a positive integer, if $N_Q$ is free of constant rank $r$ for every minimal prime $Q$ of $R$ and $\mu_Q(N) \leq \dim(R_Q) + r - 1$ for every prime $Q$ of $R$ with $1 \leq \dim(R_Q) \leq s - 1$. In case $G_s$ holds for every $s$, the module $N$ is said to satisfy $G_{\infty}$.

We say that $N$ has a rank and write $\text{rank}(N) = r$ if $N_Q$ is free of rank $r$ for every associated prime $Q$ of $R$. The module $N$ is said to be orientable if $N$ has rank $r$ and $(\Lambda^r N)^{**} \cong R$. This differs slightly from the definition given in Bruns [1987]. Our definition implies that $N_Q$ is the direct sum of a free $R_Q$-module and a torsion module for every prime $Q$ such that $\text{depth}(R_Q) \leq 1$. Thus if two modules in a short exact sequence are orientable, then so is the third as long as the right-hand module is torsion free locally in depth one.

Proposition 2.5. With notation as in 2.1, let $M = [x_1, \ldots, x_n]^{\perp}$. The module $N$ has a rank or is orientable if and only if $M$ has the same property.

Proof. We use the exact sequence

(*) $0 \rightarrow N^* \xrightarrow{\pi^*} (R^n)^* \rightarrow M \rightarrow 0$ 

from which it follows that $M$ has a rank if and only if $N^*$ has a rank. If $Q$ is an associated prime of $R$, then $N_Q$ is free if and only if $N_Q^*$ is free. (To see this, notice that if $f \in N_Q^*$ is a free generator, then the image of $f : N_Q \rightarrow R_Q$ is faithful, and hence $f$ is surjective.) Thus $N^*$ has a rank if and only if $N$ does.
Since $M$ is torsionfree, (*) shows that $M$ is orientable if and only if $N^*$ is. But $N^*$ is orientable if and only if $N$ is. □

Finally, we remark that every perpendicular module is torsionless (contained in a free module). Conversely, any finitely generated torsionless module is the perpendicular module of a finitely generated torsionless module with respect to some set of generators: If the torsionless module $N$ is generated by $x_1,\ldots,x_n$ then $N = [x_1^\perp,\ldots,x_n^\perp]$. (In general, $[x_1^\perp,\ldots,x_n^\perp]$ is the image of the natural map $N \to N^{**}$.)

3. Principal Ideal Theorems

We are now ready to prove our first main result. Recall that $rf_Q(N)$ denotes the maximal rank of a free summand of $N_Q$.

**Theorem 3.1. Generalized Height Theorem.** Let $R$ be a locally equidimensional universally catenary Noetherian ring, let $N$ be a finitely generated $R$-module, and let $x \in N$. Let $R \to S$ be a homomorphism of rings from $R$ to some Noetherian local domain $(S, n)$ and write $SN$ for $N \otimes_R S$ modulo $S$-torsion. If the image of $x$ in $SN$ lies in $nSN$, then

$$ht(N^*(x)) \leq \min_Q \{rf_Q(N)\}$$

where the minimum is taken over all minimal primes $Q$ of $R$ mapping to zero in $S$.

**Proof.** Replacing $S$ by a rank one discrete valuation ring $V$ containing $S$ and centered on $n$ we may assume that $S = V$. Notice that the order ideal of $x \otimes 1$ in the $V$-module $N \otimes_R V$ is a proper ideal.

Choose a generating set $x_1,\ldots,x_n = x$ of $N$ and a presentation $R^m \xrightarrow{\psi} R^n$ of $N$ with respect to this generating set. Consider the perpendicular module $M = \text{Im}(\psi^*) = [x_1,\ldots,x_n]^\perp \subset R^{m*}$ and its submodule $U = Rx_1^\perp + \cdots + Rx_n^\perp$. Let $Q$ be a minimal prime of $R$ mapping to zero in $S = V$. According to Corollary 2.3 it suffices to prove that $M/QM$ is not integral over the image of $U$.

In fact, writing $M' = \text{Im}(\text{Hom}_R(\psi,V)) = \text{Im}(\psi^* \otimes_R V) \subset R^{m*} \otimes_R V$, we have $M' = [x_1 \otimes 1,\ldots,x_n \otimes 1]^\perp$ and $M/QM$ maps to $M'$ with the image of $x_i^\perp$ being sent to $(x_i \otimes 1)^\perp$. Thus by the functoriality of Rees algebras it suffices to show that $M'$ is not integral over $U' = V(x_1 \otimes 1)^\perp + \cdots + V(x_{n-1} \otimes 1)^\perp$, or equivalently that $U' \neq M'$. (see for instance Rees [1987, 2.5] or Proposition 1.1). However according to Remark 2.2, $U' :_V M'$ is the order ideal of $x \otimes 1$ in $N \otimes_R V$. Since this ideal is proper we conclude that $U' \neq M'$. □

Theorem 3.1 says that whenever the height of the order ideal of $x$ exceeds the expected value, then the injection of $Rx$ into $N$ must be ‘valuatively split’, meaning that after passing to an arbitrary valuation, the induced map does split. Put differently, either the order ideal of $x$ has the expected height or else the height of the order ideal over any valuation ring becomes infinite. The set of elements having this property, but not splitting themselves, is a remarkable class, and exactly the class we wish to study.
Corollary 3.2. Let $R$ be a locally equidimensional universally catenary Noetherian ring and let $N$ be a finitely generated $R$-module. Let $x \in N$ and suppose that $\text{ht}(N^*(x)) > \max_Q \{rf_Q(N)\}$, where the maximum is taken over all minimal primes $Q$ of $R$.

1. If $N' \to N$ is an arbitrary epimorphism of $R$-modules and $x' \in N'$ is an element mapping to $x$, then for every integer $j$, $\text{Fitt}_{j+1}(N')$ is integral over $\text{Fitt}_j(N'/Rx')$.

2. If $I$ is an arbitrary integrally closed ideal in $R$, then the natural map $(Rx) \otimes_R (R/I) \to N \otimes_R (R/I)$ is injective.

Proof. To prove (1), we need to verify that for every homomorphism from $R$ to a discrete valuation ring $V$, $\text{Fitt}_{j+1}(N')V = \text{Fitt}_j(N'/Rx')V$, or equivalently, $\text{Fitt}_{j+1}(N' \otimes_R V) = \text{Fitt}_j(N' \otimes_R V/V(x' \otimes 1))$. However by Theorem 3.1, the image of $x$ generates a free $V$-summand of rank one in $VN$. Thus $x' \otimes 1$ generates a free $V$-summand of rank one in $N' \otimes_R V$, and the asserted equality of Fitting ideals is obvious.

To prove (2) we have to show that $IN \cap Rx \subseteq Ix$. Let $x_1, \ldots, x_n = x$ be a generating set of $N$. If $r \in R$ and $rx \in IN$, then $rx = \sum s_i x_i$ for some $s_i \in I$. Hence $s_1 x_1 + \cdots + s_{n-1} x_{n-1} + (s_n - r) x = 0$. Take $N'$ to be the $R$-module presented by the transpose of the vector $[s_1, \ldots, s_{n-1}, s_n - r]$, and let $x'$ be the last generator. Part (1) proves that $s_n - r$ is in the integral closure of the ideal $(s_1, \ldots, s_{n-1})$. Therefore $r \in I$, since $s_i \in I$ and $I$ is integrally closed. Hence $rx \in Ix$ as asserted.

Remark 3.3. In terms of matrices, Corollary 3.2(1) can be stated as follows: Let $\underline{x} = x_1, \ldots, x_n$ be a generating set of $N$ with $x_n = x$, let $\psi$ be a matrix with $n$ rows satisfying $\underline{x} \cdot \psi = 0$, and let $\psi'$ be the matrix obtained from $\psi$ by deleting the last row. Then for every integer $i$, $I_i(\psi)$ is integral over $I_i(\psi')$.

The second corollary answers in the affirmative a question asked by the second author and Jee Koh in the late 1980’s.

Corollary 3.4. Let $(R, \mathfrak{m})$ be an equidimensional universally catenary Noetherian local ring and let $N$ be a finitely generated $R$-module. If $x$ lies in $\mathfrak{m}N$, the integral closure of $\mathfrak{m}N$ in $N$, then $\text{ht}(N^*(x)) \leq \min \{\mu_Q(N)\}$, where the minimum is taken over all minimal primes $Q$ of $R$ so that $N_Q$ is free.

Proof. Let $Q$ be a minimal prime of $R$ so that $N_Q$ is free. Choose a local embedding $R/Q \hookrightarrow V$, where $(V, \mathfrak{n})$ is a rank one discrete valuation ring, and a versal map $f : N \to F$ from $N$ to a free module. As $x$ lies in the integral closure of $\mathfrak{m}N$ in $N$, Proposition 1.1 shows that $x' \in \mathfrak{m}N'$, where $'$ denotes tensoring with $V$ and taking images in $F \otimes_R V$. On the other hand since $N_Q$ is free, $f_Q$ is split injective and therefore $N' \cong VN$ as defined in Theorem 3.1. Thus the image of $x$ lies in $\mathfrak{m}VN \subseteq \mathfrak{n}VN$, and Theorem 3.1 immediately gives the conclusion.

Of course, Theorem 3.1 gives as an immediate corollary the theorem of Bruns, Eisenbud, and Evans in case the ring is equidimensional and catenary. The usual proofs reduce to this case, and from the paper of Bruns one obtains that if $(R, \mathfrak{m})$ is a local Noetherian ring, $N$ is a finitely generated $R$-module, and $x \in \mathfrak{m}N$, then

$$\text{ht}(N^*(x)) \leq \max_Q \{\mu_Q(N)\}$$
where the maximum ranges over the minimal primes $Q$ of $R$. However, we can observe that the proof reduces at once to the complete domain case and proves even a stronger result:

**Theorem 3.5.** Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $N$ be a finitely generated $R$-module. If $x \in \mathfrak{m}N$, then $\dim(R/N^*(x)) \geq \max_Q \{\dim(R/Q) - rf_Q(N)\}$ where the maximum is taken over all primes ideals in $R$. Thus,

$$\text{ht}(N^*(x)) \leq \min_Q \{\dim(R) - \dim(R/Q) + rf_Q(N)\}.$$

**Proof.** If $R$ is a complete local domain and $x \in \mathfrak{m}N$, then $\text{ht}(N^*(x)) \leq \text{rank}(N)$ by Theorem 3.1. Now the second assertion follows, and implies the first statement at once.

To prove the theorem for any Noetherian local ring, we only need to show the first inequality. We first reduce to the case where $R$ is complete by passing to the completion $\hat{R}$ of $R$. Let $Q$ be a prime in $R$ and choose a minimal prime $\hat{Q}$ over $Q\hat{R}$ such that $\dim(\hat{R}/\hat{Q}) = \dim(R/Q)$. Since the map $R_Q \to \hat{R}_Q$ is flat and local, it follows that $rf_Q(N) = rf_{\hat{Q}}(\hat{N})$. If the result holds for $\hat{R}$ and $\hat{N}$, we obtain

$$\dim(R/N^*(x)) = \dim(\hat{R}/\hat{N}^*(x)) \geq \dim(\hat{R}/\hat{Q}) - rf_{\hat{Q}}(\hat{N}) = \dim(R/Q) - rf_Q(N).$$

Henceforth we assume that $R$ is complete.

Choose an arbitrary prime ideal $Q$ in $R$ and let $J \subset R$ be the preimage of $(N/QN)^*(x)$, the order ideal of the image of $x$ in the $R/Q$-module $N/QN$. Clearly $N^*(x) \subset J$. Now $\dim(R/N^*(x)) \geq \dim R/J \geq \dim(R/Q) - rf_Q(N)$, where the last inequality follows from the complete domain case. \qed

Next we wish to find conditions on a module $N$ over a local ring which guarantee that $N^*(x)$ has the expected height for every $x \in N$. Obviously, such a module should not have any nontrivial free summands, which means that $\mu(N^\perp) = \mu(N)$. We can turn this necessary condition into a sufficient one if we replace $\mu(N^\perp)$ by $\ell(N^\perp)$:

**Proposition 3.6.** Let $R$ be an equidimensional universally catenary Noetherian local ring and let $N$ be a finitely generated $R$-module. If $\ell(N^\perp) = \mu(N)$, then for every $x \in N$,

$$\text{ht}(N^*(x)) \leq \max_Q \{rf_Q(N)\}.$$  

where the maximum is taken over all minimal primes $Q$ of $R$.

**Proof.** By Theorem 3.1 we may assume that $x$ can be extended to a minimal generating set $x_1, \ldots, x_n = x$ of $N$. Write $U = Rx_1^+ + \cdots + Rx_n^+ \subset M = [x_1, \ldots, x_n]^\perp \cong N^\perp$. As $\ell(M) = n$, $M$ cannot be integral over $U$. Hence by Proposition 1.1 and the functoriality of the Rees algebra, there exists a minimal prime $Q$ of $\hat{R}$ such that $M/QM$ is not integral over the image of $U$. The assertion now follows from Corollary 2.3. \qed

Here is a result bounding the analytic spread of a module from below. We will ultimately use it to give a partial converse of Proposition 3.6.
Proposition 3.7. Let $R$ be a Noetherian local ring with infinite residue field and let $M$ be a finitely generated torsion free $R$-module such that $M_Q$ is free of rank $e$ for every minimal prime $Q$ of $R$. Let $U$ be the submodule of $M$ generated by $t$ general linear combinations of a set of generators of $M$. If $\ell(M) \leq t$ and $M$ satisfies $G_{t-e+2}$, then $\text{ht}(U : M) \geq t - e + 2$.

Proposition 3.7 follows at once from the following more general version, which we phrase as a lower bound for the analytic spread of a module.

Proposition 3.7bis. Let $R$ be a Noetherian local ring with infinite residue field and let $M$ be a finitely generated torsionless $R$-module. Let $X \subseteq \text{Spec}(R)$ be the nonfree locus of $M$, and assume that for integers $e$ and $t$ we have $\mu_P(M) \leq \dim(R_P) + e - 1$ whenever $P \in X$ and $\dim(R_P) \leq t - e + 1$. Let $U$ be the submodule of $M$ generated by $t$ general linear combinations of a set of generators of $M$. If $\text{ht}(U :_R M) \leq t - e + 1$, then $\ell(M) \geq t + 1$.

Proof. Let $x_1, \ldots, x_t$ be the general elements of $M$ that generate $U$. Using basic element theory and induction on $i$, $0 \leq i \leq t$, one can show that $\mu_P(M/Rx_1 + \cdots + Rx_i) \leq \dim(R_P) + e - i - 1$ for every $P \in X$ with $i - e + 1 \leq \dim(R_P) \leq t - e + 1$. In particular $M_P = U_P$ for every $P \in X$ with $\dim(R_P) = t - e + 1$. Now suppose that $\ell(M) \leq t$. Then $U$ is a reduction of $M$ and hence $M_P = U_P$ for every prime $P \notin X$. Thus $\text{ht}(U :_R M) > t - e + 1$, which yields a contradiction. \qed

The following consequence of Proposition 3.7bis shows that under good circumstances the analytic spread is monotonic for inclusions:

Corollary 3.8. Let $R$ be an equidimensional universally catenary Noetherian local ring and let $M$ be a finitely generated $R$-module such that $M_Q$ is free of rank $e$ for every minimal prime $Q$ of $R$. Write $\ell = \ell(M)$ and assume that $M$ satisfies $G_{\ell - e + 1}$. If $M' \subseteq M$ is any submodule with $\text{codim}(M/M') \geq \ell - e + 1$, then $\ell(M') \geq \ell(M)$.

Proof. We may assume that the residue field of $R$ is infinite. Let $t = \ell - 1$, and let $U$ be the submodule generated by $t$ general linear combinations of a set of generators of $M'$. Since $t < \ell(M)$, the module $M$ is not integral over $U$. Thus by Proposition 1.2, there exists a minimal prime $Q$ of $R$ such that $\text{ht}(U :_R M) \leq \max\{0, t + 1 - \mu_Q(M)\} \leq t - e + 1$. Since $\text{codim}(M/M') \geq t - e + 2$ we have $\text{ht}(U :_R M') = \text{ht}(U :_R M) \leq t - e + 1$. Now Proposition 3.7bis gives $\ell(M') \geq t + 1 = \ell(M)$. \qed

Here is the promised partial converse of Proposition 3.6:

Proposition 3.9. Let $R$ be a Noetherian local ring with infinite residue field and let $N$ be a finitely generated $R$-module such that $N_Q$ is free of rank $r$ for every minimal prime $Q$ of $R$. Assume that $N^\perp$ satisfies $G_{\infty}$. If $\text{ht}(N^*(x)) \leq r$ for every $x \in N$, then $\ell(N^\perp) = \mu(N)$.

Proposition 3.9 is an immediate consequence of the following more general result.

Proposition 3.9bis. Let $R$ be a Noetherian local ring with infinite residue field and let $N$ be a finitely generated $R$-module. Let $X \subseteq \text{Spec}(R)$ be the nonfree locus of $N$, and assume that for an integer $r$ we have $\text{rfp}(N) \geq r + 1 - \dim(R_P)$ whenever $P \in X$. If $\text{ht}(N^*(x)) \leq r$ for every $x \in N$, then $\ell(N^\perp) = \mu(N)$. 
Proof. Set $M = N^\perp$. Write $n = \mu(N) \geq \mu(M)$ and let $U$ be a submodule of $M$ generated by $n - 1$ general linear combinations of generators of $M$. By Remark 2.2, there exists an element $x \in N$ such that $U :_{R} M = N^*(x)$. The latter ideal has height at most $r$ by assumption. Applying Proposition 3.7bis with $e = n - r$ and $t = n - 1$ we conclude that $\ell(M) \geq n$, hence $\ell(M) = n$. □

Combining Propositions 3.6 and 3.9bis one obtains the following. Assume that $R$ is an equidimensional universally catenary Noetherian local ring of dimension $d > 0$ with infinite residue field, and let $N$ be a finitely generated $R$-module that is free of constant $r$ locally on the punctured spectrum. For every $x \in N$ one has $\text{ht}(N^*(x)) \leq r < d$ if and only if $\ell(N^\perp) = \mu(N)$. To see this also notice that $\ell(N^\perp) \leq \dim(R) + \text{rank}(N^\perp) - 1 = \mu(N) + d - r - 1$.

Proposition 3.9 gives a systematic way of constructing modules of rank $r$ with elements whose order ideals have height exceeding $r$.

Example 3.10. Let $R$ be a Noetherian local ring with infinite residue field and let $I$ be an $R$-ideal of positive height. Suppose that $I$ satisfies $G_\infty$ and $\ell(I) \neq \mu(I)$. Let $N = I^\perp$ and notice that $I = N^\perp$. By Proposition 3.9, $N$ contains an element $x$ such that the height of $N^*(x)$ is strictly greater than the rank of $N$. Of course $N^*(x)$ is proper, since $N$ has no free summands.

If $I$ is the defining ideal of a monomial curve in $\mathbb{P}^3$ then the analytic spread of $I$ is at most 3 by Gimenez, Morales and Simis [1993]. Such an ideal satisfies $G_4$ by Herzog [1970]. If in addition $\mu(I) = 4$, then $I$ is $G_\infty$ and $\ell(I) \neq \mu(I)$. The module $N = I^\perp$ has rank 3, and — at least if the ground field is infinite — will have an order ideal of height 4.

To be explicit, let $I \subset R = k[z_1, \ldots, z_6]$ be the defining ideal of the monomial curve $t \mapsto (1, t^{\alpha - 1}, t^{\alpha + 1}, t^{2\alpha})$, for even numbers $\alpha > 0$. It is easy to check that this curve lies on the smooth quadric $ad - bc = 0$, and has divisor class $(\alpha - 1, \alpha + 1)$. Its ideal is thus minimally generated by 4 elements and all the conditions above are satisfied. (Actually the same is true for any curve in this divisor class, monomial or not.) The module $N$ may be explicitly described as the image of the right-hand map in the left exact sequence

$$
0 \to R \xrightarrow{\begin{pmatrix} ad - bc \\
ac + 1 - b^{\alpha - 1}d^2 \\
ac - b^\alpha d \\
b^{\alpha + 1} - a^2c^{\alpha - 1} \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} -b^\alpha & 0 & a & c \\
-ae^{\alpha - 1} & 0 & b & d \\
-b^{\alpha - 1}d & a & -c & 0 \\
-c^\alpha & b & -d & 0 \end{pmatrix}} R^4
$$

obtained by dualizing the first two steps of the minimal free resolution of $R/I$. From this we see at once that the third generator of $N$ has order ideal $(a, b, c, d)$, of height 4.

We now give an example of a graded module $N$ in which all homogeneous elements have order ideals of height at most rank $N$, although there are inhomogeneous elements whose order ideals have bigger height.

Example 3.11. Let $k$ be an infinite field and let $R = k[z_1, \ldots, z_6]$ be a polynomial ring,
graded with all the variables in degree 1. Set

\[ I = I_2 \left( \begin{array}{cccc} z_1 & z_2^2 & z_3^2 & 0 \\ 0 & z_4^2 & z_5^2 & z_6^2 \end{array} \right) \].

The perpendicular module \( N = I^\perp \) has rank 5. We claim first that all the order ideals of homogeneous elements of \( N \) have height at most 5. By Remark 2.2 and Corollary 2.3 it suffices to show that the ideal \( I \) has no reductions generated by 5 homogeneous elements.

Suppose on the contrary that \( J \) is a reduction of \( I \) generated by 5 homogeneous elements. The lowest degree part of \( I \) is generated by 3 analytically independent elements. The ideal \( J \) must contain all three since the ideal generated by the lowest degree elements of \( J \) is a reduction of the ideal generated by the lowest degree elements of \( I \). On the other hand, \( (z_1) \) contains the lowest degree part of \( I \). Let \( - \) denote images in \( R/(z_1) \). The ideal \( J \) is a reduction of \( I \). Thus \( I \) would have a reduction generated by 2 elements. However, this is impossible: \( I \) is generically a complete intersection of height 2, and not a complete intersection, so by Cowsik and Nori [1976] any reduction of \( I \) has at least 3 generators.

On the other hand, if we regrade the ring with degree \((z_1) = 2\), all the generators of \( I \) become homogeneous of the same degree. Since the generators of \( I \) satisfy the Plücker relation, \( I \) has a reduction \( J \) generated by 5 elements homogeneous in the new grading. As \( I \) is a complete intersection on the punctured spectrum, the ideal \( J : I \) has height 6, and thus \( N \) has an element whose order ideal has height 6 by Remark 2.2.

We finish this section with two classes of examples arising from the Koszul complex and the Buchsbaum-Rim complex, respectively.

Let \( R = k[z_1, \ldots, z_d] \) be a polynomial ring in \( d > 1 \) variables over a field \( k \), graded with the variables in degree 1, and write \( \Omega^i \) for the \( i \)th syzygy module of the maximal ideal \( m = (z_1, \ldots, z_d) \). The module \( \Omega^i \) has minimal free presentation \( \wedge^{i+2} R^d \rightarrow \wedge^{i+1} R^d \); in particular \( \Omega^d/m\Omega^d = \wedge^d k^d \). Using the self-duality of the Koszul complex, we see at once that \( (\Omega^i)^\perp = \Omega^{d-i-2} \) for \( 0 \leq i \leq d-2 \). Note that \( \Omega^{d-1} \) is a free module.

**Proposition 3.12.** If \( 2 \leq i \leq d-2 \), or if \( i = 1 \) and \( d \) is odd, then all elements of \( \Omega^i \) have order ideals of height at most the rank \( \binom{d-1}{i} \) of \( \Omega^i \).

**Proof.** We may suppose \( R \) is local. If \( 2 \leq i \leq d-3 \) there is nothing to prove, since \( \Omega^i \) has no free summand and its rank is greater than the the dimension of \( R \). If \( i = d-2 \) then \( (\Omega^i)^\perp = m \), which is generated by analytically independent elements. Similarly, if \( i = 1 \) and \( d \) is odd, then \( (\Omega^1)^\perp = \Omega^{d-3} \) is generated by analytically independent elements by Simis, Ulrich and Vasconcelos [1993, 3.1]. Now Proposition 3.6 yields the desired inequality in either case. \( \Box \)

We can prove a more precise result for homogeneous generators: We say that an element of \( \Omega^1 \) has rank \( t \) if its image in \( \Omega^1/m\Omega^1 = \wedge^2 k^d \) represents a linear transformation \( \wedge^d k^d \rightarrow k^d \) of rank \( t \). Since these linear transformations are alternating, the rank \( t \) is an even number. Any homogeneous minimal generator of rank \( 2s \) can be written as \( e_1 \wedge e_2 + \cdots + e_{2s-1} \wedge e_{2s} \), where the \( e_i \) are homogeneous minimal generators of \( R^d \).
**Proposition 3.13.** The height of the order ideal of a homogeneous generator $x$ of $\Omega_1$ is equal to the rank of $x$. In particular, if $d$ is even, there are elements of $\Omega_1$ with order ideals of height $d > \text{rank}(\Omega_1) = d - 1$.

**Proof.** Let $e_1, \ldots, e_d$ be homogeneous generators of $R^d$. If $d = 2s$ is even, then the generator $e_1 \wedge e_2 + \cdots + e_{2s-1} \wedge e_{2s}$ has rank $2s$, so the second statement follows from the first.

The module $\Omega_1$ is the image of the map $\wedge^2 R^d \to \wedge^1 R^d$ in the Koszul complex of $z_1, \ldots, z_n$. To prove the first statement, it suffices to consider the order ideal of the element $x$ that is the image of $e_1 \wedge e_2 + \cdots + e_{2s-1} \wedge e_{2s}$. The column corresponding to $e_{2i-1} \wedge e_{2i}$ has $\pm z_{2i}$ and $\pm z_{2i-1}$ in the $2i - 1$ and $2i$ places, respectively. Thus $x$ is mapped to an element of $R^d$ whose nonzero coordinates are $\pm z_1, \ldots, \pm z_{2s}$. Since the dual of the Koszul complex is exact, the components of the inclusion map $\Omega_1 \to R^d$ generate all the maps from $\Omega_1$ to $R$, and we see that the order ideal of $x$ is $(z_1, \ldots, z_{2s})$. $\square$

In contrast to Proposition 3.13 the next example shows that the kernel $N$ of a generic map $R^s \to R^t$ has only order ideals of height at most rank($N$) as long as $t > 1$.

**Proposition 3.14.** Let $(R, \mathfrak{m})$ be a local Gorenstein ring and let $t \leq s$ be integers. Let $\chi$ be a matrix with entries in $\mathfrak{m}$ and $\text{ht}(I(\chi)) = s - t + 1$. Set $N = \ker(\chi)$. Except in the case where $t = 1$ and $s$ is even, $\text{ht}(N^*(x)) \leq \text{rank}(N)$ for every $x \in N$.

**Proof.** We may assume that $s - t = \text{rank}(N) > 1$. Let $x_1, \ldots, x_n$ be a minimal generating set of $N$, let $S = R[Z_1, \ldots, Z_n]$ be a polynomial ring, and write $y = \sum_{i=1}^n Z_i x_i \in N \otimes_R S$. Further, let $I = (N \otimes_R S)^*(y)$ be the order ideal of $y$ and $J$ its unmixed part. By Proposition 2.4(2) it suffices to show that $\text{ht}(I_{mS}) \leq \text{rank}(N)$.

To this end write $M = N^\perp$. As in the proof of Remark 2.2 one sees that $\text{Sym}(M) \cong S/I$. Since $M$ satisfies $G_{\infty}$, one has $\dim(\text{Sym}(M)) = \dim(R) + \text{rank}(M)$ by Huneke and Rossi [1986, 2.6], which gives $\text{ht}(I) = \text{rank}(N) = s - t$. In this setting, Migliore, Nagel and Peterson show in [1999, 1.5b] that $I = J$ if $s - t$ is even, whereas $J/I \cong \text{Sym}_{(s-t-1)/2}(\text{coker}(\chi)) \otimes_R S$ if $s - t$ is odd. Thus for $s - t > 1$ odd and $t > 1$ one has $\mu_{mS}(J/I) > 1$, which yields $J_{mS} \neq S_{mS}$. Hence in either case

$$\text{ht}(I_{mS}) = \text{ht}(J_{mS}) = \text{ht}(J) = \text{ht}(I) = \text{rank}(N).$$

$\square$

## 4. Order Ideals of Low Rank Modules

It turns out that order ideals of elements in modules of low rank are particularly well-behaved. For example if $N$ is a module of rank 1, then (modulo torsion) $N$ is isomorphic to an ideal $I$ containing a nonzerodivisor. If $x \in I$ is a nonzerodivisor of $R$ then $I^*(x) = (x)_R$ is either the unit ideal or of grade 1. If on the other hand $x$ is a zerodivisor contained in an associated prime $Q$ of $R$, then $I^*(x)$ is also contained in $Q$ and thus has grade 0. The following propositions extend this kind of result to modules of rank 2 and 3 as well as $k^{th}$ syzygies of rank $k$ having finite projective dimension. The case of rank 2 modules over regular local rings had already been treated in Evans and Griffith [1982, p.377].
Proposition 4.1. Let $R$ be a Noetherian ring and let $N$ be a finitely generated $R$-module. Assume either

1. $N$ is orientable of rank 2; or
2. $R$ contains a field and $N$ is a $k$th syzygy of rank $k$ having finite projective dimension.

If $x \in N$ then $N^*(x) = R$ or $\text{grade}(N^*(x)) \leq \text{rank}(N)$.

Proof. We may assume that $R$ is local. Consider the exact sequence

$$0 \to Rx \to N \to X \to 0.$$ 

We suppose that $\text{grade}(N^*(x)) > \text{rank}(N)$. It follows that the annihilator of $x$ is 0, and we will show that $X$ is free. Hence $N^*(x) = R$ as required.

In case (1) we may assume that $N = N^*$. If we localize the above sequence at an arbitrary prime $Q$ with $\text{depth}(R_Q) \leq 2$, then the sequence splits and hence $X_Q$ is reflexive. If $\text{depth}(R_Q) \geq 3$ then $\text{depth}(X_Q) \geq 2$. It follows that $X$ is orientable and reflexive of rank one, hence free, completing the proof in this case.

Now suppose we are in case (2). After localizing at a prime $Q$ with $\text{depth}(R_Q) \leq k$, the above sequence splits. Since $N_Q$ is free by the Auslander-Buchsbaum formula, it follows that $X_Q$ is free as well for any such $Q$. If $Q$ is any prime with $\text{depth}(R_Q) > k$, then $\text{depth}(N_Q) \geq k$ and therefore $\text{depth}(X_Q) \geq k$. Thus by Hochster and Huneke [1990, 10.9], the module $X$ is a $k$th syzygy. It has rank $k - 1$ and finite projective dimension, so the version of the Evans-Griffith Syzygy Theorem due to Hochster and Huneke [1990, 10.8] and Evans and Griffith [1989, 2.4] implies that $X$ is free. 

The assumption of orientability in Proposition 4.1 is necessary: Let $k$ be a field, $R = k[Z_0, \ldots, Z_3]/(Z_0Z_3 - Z_1Z_2)$, and let $z_i$ denote the image of $Z_i$ in $R$. Let $M$ be the ideal $(z_0^2, z_0z_1, z_1^2)$. If $N = [z_0^2, z_0z_1, z_1^2]_+$ then $N$ is an $R$-module of rank 2. By Remark 2.2 the element $(z_0z_1)_+$ has order ideal $(z_0, \ldots, z_3)$, which has height 3.

Proposition 4.2. Let $R$ be a Gorenstein ring and let $N$ be an orientable $R$-module of rank 3 that satisfies $S_3$ and is free in codimension 2. If $x \in N$ then either $N^*(x) = R$ or $\text{ht}(N^*(x)) \leq 3$.

Proof. We may assume that $R$ is local, and we write $n = \mu(N)$, $e = n - 3$. By Proposition 4.1 we may assume that $N$ has no nontrivial free summand. We will prove that $\text{ht}(N^*(x)) \leq 3$.

Suppose the contrary and write $M = N^\perp$. By Theorem 3.1, $x$ can be extended to a minimal generating set of $N$. Using Remark 2.2, a row ideal in some minimal presentation matrix of $M$ then has height $> 3$. Let $u_1, \ldots, u_n$ be generic elements in $M \otimes_R R'$ defined over a local ring $R'$ that is obtained from $R$ by a purely transcendental residue field extension, and set $F = R'u_1 + \cdots + R'u_{e-1} \subset U = R'u_1 + \cdots + R'u_{n-1}$. The genericity of $u_1, \ldots, u_n$ implies that $\text{ht}(U:R'(M \otimes_R R')) > 3$ as shown in Proposition 2.4(1). To simplify notation we will write $R = R'$.

Because $N$ is $S_3$, the Acyclicity Lemma of Peskine and Szpiro [1972] shows that $\text{Ext}^1_R(N^*, R) = 0$. From the exact sequence (*) of Proposition 2.5 we see that $\text{Ext}^2_R(M, R) = 0$. 

Because $N$ has no free summands, $\mu(M) = n$ and hence $U \neq M$. The module $M$ is free locally in codimension 2, and by Proposition 2.5 it is orientable of rank $e$. Thus by Simis, Ulrich and Vasconcelos [1998, 3.2], $F \subset \text{the conclusion}$. Clearly $\text{Ext}_R^n(R/I, R) \cong \text{Ext}_R^n(I, R) \cong \text{Ext}_R^n(M, R) = 0$. Let $J$ be the image of $U$ in $I$. Notice that $\mu(J) \leq (n - 1) - (e - 1) = 3$. Furthermore $J \neq I$ and $\text{ht}(J : I) > 3$, because $I/J \cong M/U$.

If $\text{ht}(I) \geq 3$ then $J$ is a complete intersection and $\text{ht}(J : I) \leq 3$, and we are done. Otherwise $\text{ht}(I) = 2$. Then the condition $\text{Ext}_R^3(R/I, R) = 0$ implies that the factor ring of $R$ by any link of $I$ satisfies $S_2$; see Chardin, Eisenbud and Ulrich [1998, 4.4]. Since we are in a case where $I$ has height 2 and is a complete intersection in codimension 2, and $\mu(J) \leq 3$, we can apply Chardin, Eisenbud and Ulrich [1998, 3.4] to obtain $\text{ht}(J : I) \leq 3$ as required. □

In Proposition 4.2 the assumption of freeness in codimension 2 can be weakened to requiring that $\text{rf}_Q(N) \geq 2$ whenever $\dim(R_Q) = 2$. However, the $S_3$ condition is necessary, as can be seen from the monomial curves discussed in Example 3.10.

We can use Proposition 4.1 to prove that, under a vanishing hypothesis on some $\text{Ext}_R^i(M, R)$, the colon ideal $U : R M$ has at most the expected grade. In preparation, recall that if $M^*$ has a rank then $M$ does too.

**Proposition 4.3.** Let $R$ be a Noetherian local ring containing a field, let $M$ be a finitely generated torsion free $R$-module such that $M^*$ and $\text{Ext}_R^1(M, R)$ have finite projective dimension, and set $e = \text{rank}(M)$. If $s$ is an integer such that $\text{Ext}_R^i(M, R) = 0$ for $2 \leq i \leq s - 1$ then, for every submodule $U \subsetneq M$ with $\mu(U) - e + 1 \leq s$,

$$\text{grade}(U : R M) \leq \max\{0, \mu(U) - e + 1\}.$$  

**Proof.** We may assume that $\text{rank}(U) = e$ since otherwise $\text{grade}(U : R M) = 0$. Then $s \geq \mu(U) - e + 1 \geq 1$. Lowering $s$ if necessary, it suffices to prove that $\text{grade}(U : R M) \leq s$. If $s = 1$, then $U$ is free and thus $\text{grade}(U : R M) \leq 1 = s$. Therefore we may assume that $s \geq 2$. Suppose that $\text{grade}(U : R M) > s$. Choose a submodule $U \subsetneq V \subsetneq M$ such that $V/U$ is cyclic. Clearly $\text{grade}(U : R V) > s$ and $\text{grade}(V : R M) > s$. The latter inequality implies that $\text{Ext}_R^{s+1}(V, R) \cong \text{Ext}_R^{s+1}(M, R)$ for every $0 \leq i \leq s - 1$, forcing $V^*$ and $\text{Ext}_R^1(V, R)$ to have finite projective dimension and $\text{Ext}_R^{s+1}(V, R) = 0$ for $2 \leq i \leq s - 1$. We are free to replace $M$ by $V$ and from now on we assume that $M/U$ is cyclic.

Since $M/U$ is cyclic and $\mu(U) \leq s + e - 1$, there exists a generating set $u_1, \ldots, u_n$ of $M$ such that the first $n - 1$ elements generate $U$ and $n = s + e$. The module $M$ is torsionless since it is torsion free and has a rank. By the remark at the end of Section 2, there exists a finitely generated module $N$ with generators $x_1, \ldots, x_n$ so that $M = [x_1, \ldots, x_n]_{\perp}$ and $u_i = (x_i)_{\perp}$. By Remark 2.2, $U : R M = N^*(x_n)$. Using the sequence (*) of Proposition 2.5 we see that $N^{**}$ has finite projective dimension since $M^*$ and $\text{Ext}_R^1(M, R)$ have finite projective dimension, and that $N^{**}$ is an $s^{th}$ syzygy since $\text{Ext}_R^i(M, R) = 0$ for $2 \leq i \leq s - 1$. As $\text{rank}(N^{**}) = n - e = s$, Proposition 4.1(2) now proves that $\text{grade}(N^*(x_n)) \leq s$, giving the conclusion. □
Corollary 4.4. Let $R$ be a regular local ring containing a field and let $I$ be an ideal satisfying $\text{Ext}_R^i(R/I, R) = 0$ for $3 \leq i \leq s$. Then for every ideal $J \subseteq I$ with $\mu(J) \leq s$, $\text{ht}(J : I) \leq \mu(J)$. □

Note that Corollary 4.4 is interesting only if the height of $I$ is one or two, and reduces to the height two case. One should compare the results of Chardin, Eisenbud, and Ulrich [1998, 3.4 and 4.2], which yield the same conclusion for ideals of any height $g$ under the (incomparable) assumptions that $I$ satisfies $G_s$ and $\text{Ext}_R^i(R/I^{i-g}, R) = 0$ for $g+1 \leq i \leq s$. We did not expect a result that avoids reference to the powers of $I$!

Corollary 4.5. Let $R$ be a Noetherian local ring containing a field, let $M$ be a finitely generated torsion free $R$-module such that $M^*$ and $\text{Ext}_R^1(M, R)$ have finite projective dimension, and set $\text{rank}(M) = e$. If $s$ is an integer such that $R$ satisfies $S_{s+1}$, the module $M$ satisfies $G_{s+1}$, and $\text{Ext}_R^i(M, R) = 0$ for $2 \leq i \leq s-1$, then

$$\ell(M) \geq \min\{\mu(M), e + s\}.$$

Proof. We may assume that $R$ has an infinite residue field. Write $n = \mu(M)$ and $t = \min\{n,e+s\} - 1$. We may suppose that $t \geq 1$. Let $U$ be a submodule of $M$ generated by $t$ general linear combinations of generators of $M$. As $t \leq n - 1$ one has $U \neq M$ and therefore $\text{ht}(U :_RM) \leq t - e + 1$ by Proposition 4.3. But then Proposition 3.7bis implies that $\ell(M) \geq t + 1$. □

5. Heights of Trace Ideals

The surprising fact pursued in this section may be informally summarized by saying that if an order ideal of an element in a module $N$ is “bigger than it should be”, then the trace ideal of $N$ is not much larger than this order ideal.

Proposition 5.1. Let $R$ be a Noetherian ring, let $N = R y_1 + \cdots + R y_n$ be an $R$-module, and let $x, y$ be elements of $N$. Write $Y_i = N/RY_i$, and $Y = N/Ry$.

1. If $\text{ht}(N^*(x)/Y^*(x)) > 1$, then $N^*(y) \subset \sqrt{N^*(x)}$.

2. If $\text{ht}(N^*(x)/Y^*_i(x)) > 1$ for $1 \leq i \leq n$, then $\sqrt{\text{tr}(N)} = \sqrt{N^*(x)}$. In particular $\text{ht}(\text{tr}(N)) \leq \mu(N^*)$, unless $\text{tr}(N) = R$.

Proof. To prove (1) suppose there exists a prime ideal $Q$ of $R$ with $N^*(x) \subset Q$, but $N^*(y) \not\subset Q$. Replacing $R$ by $R_Q$ we may then assume that $N^*(x) \neq R$ and $N = R \oplus Y$. Writing $x = r + z$ with $r \in R, z \in Y$, we obtain $R \neq N^*(x) = Rr + N^*(y)$. Thus $\text{ht}(N^*(x)/N^*(y)) \leq 1$, which yields a contradiction. This proves (1).

Part (2) is an immediate consequence of (1) since $N^*(x) \subset \text{tr}(N) = N^*(y_1) + \cdots + N^*(y_n)$. □

Remark 5.2. The height assumptions in Proposition 5.1 are automatically satisfied if $R$ is locally equidimensional and catenary and if $\text{ht}(N^*(x)) > \text{bght}(Y^*(x)) + 1$ (for (1)) or $\text{ht}(N^*(x)) > \text{bght}(Y^*_i(x)) + 1$ (for (2)).

There is a corresponding statement for colon ideals. When combined with Remark 2.2 it could be used to deduce Proposition 5.1:
Proposition 5.3. Let $R$ be a Noetherian ring and let $M$ be a finitely generated $R$-module. Let $U = Ru_1 + \cdots + Ru_s \subset M$ be a submodule and write $U_i = Ru_1 + \cdots + Ru_i$. If $ht(U_i :_R M/U_{i-1} :_R M) > 1$ for $1 \leq i \leq s$, then $\sqrt{\text{Fitt}_j(M)} = \sqrt{\text{Fitt}_j(M/U)}$ for any $j \geq 0$. In particular $V :_R M \subset \sqrt{U :_R M}$ for every $s$-generated submodule $V \subset M$.

Proof. The asserted equality is equivalent to the statement that $u_1, \ldots, u_s$ form part of a minimal generating set of $M$ locally at each prime in the support of $M/U$. Thus it suffices to prove that for $1 \leq i \leq s$, the image of $u_i$ is a minimal generator of $M/U_{i-1}$ locally on the support of $M/U_i$. This reduces us to the case $s = 1$.

Suppose $u = u_1$ is not a minimal generator of $M$ locally at a prime $Q$ in the support of $M/U = M/U_1$. Replacing $R$ by $R_Q$ we may assume $(R, \mathfrak{m})$ is local, $M \neq 0$, and $u \in \mathfrak{m}M$. If $\varphi$ is a matrix with $n$ rows presenting $M$, then $I_n(\varphi) \neq R$ and we obtain a presentation matrix $\psi$ of $M/U$ by adding one column with entries in $\mathfrak{m}$. Thus by Bruns [1981, Corollary 1] (see also Eisenbud and Evans [1976, 2.1]), $ht(I_n(\psi)/I_n(\varphi)) \leq 1$, which gives $ht(\text{ann}(M/U)/\text{ann}(M)) \leq 1$, contrary to our assumption. □

In general one has the inclusion $\text{Fitt}_j(M/U) \subset \text{Fitt}_{j+s}(M)$ for any $s$-generated submodule $U$ of a finitely generated module $M$. One may ask which power of $\text{Fitt}_j(M)$ is contained in $\text{Fitt}_j(M/U)$ under the assumptions of Proposition 5.3.

The height assumption in Proposition 5.3 implies that $ht(U :_R M/0 :_R M) \geq 2s$. On the other hand a lower bound for $ht(U :_R M/0 :_R M)$ alone does not suffice to deduce the equality $\sqrt{\text{Fitt}_{j+s}(M)} = \sqrt{\text{Fitt}_j(M/U)}$. For instance, let $R = k[z_0, \ldots, z_n]$ be a polynomial ring over a field, $\varphi$ the 3 by $n$ matrix

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
z_0 & 0 & \cdots & 0 \\
z_1 & z_2 & \cdots & z_n
\end{pmatrix},
$$

which gives a map from $R^n$ to $R^3 = Re_1 \oplus Re_2 \oplus Re_3$, $M$ the cokernel of $\varphi$, and $U \subset M$ the submodule generated by the images of $e_1$ and $e_2$. In this case, one has $ht(U :_R M/0 :_R M) = n$, whereas $\text{Fitt}_2(M) = (z_0, \ldots, z_n) \not\subseteq \sqrt{\text{Fitt}_0(M/U)} = (z_1, \ldots, z_n)$.

Proposition 5.4. Let $R$ be a universally catenary Noetherian ring and let $N$ be a finitely generated $R$-module. Assume that one of the following conditions hold:

1. $R$ satisfies $S_3$ and $N$ is orientable of rank 3; or
2. $R$ is Gorenstein and $N$ is orientable of rank 4, satisfies $S_3$ and is free in codimension 2; or
3. $R$ satisfies $S_{k+1}$ and contains a field, and $N$ is a kth syzygy of rank $k + 1$ having finite projective dimension.

If $x \in N$ satisfies $ht(N^*(x)) > rank(N)$ then $\sqrt{\text{tr}(N)} = \sqrt{N^*(x)}$.

Proof. We may assume that $R$ is local, $N$ is torsion free and nonzero, and $N^*(x) \neq R$. If we are in case (3) with $k = 0$ then $N$ is orientable of rank 1, so $N^* \cong R$ and $N^*(x)$ has height 1, contradicting the hypothesis. In all other cases we may suppose that $R$ is $S_2$, hence equidimensional.
After passing to a purely transcendental extension of the residue field of $R$, we consider generic generators $y_1, \ldots, y_n$ of $N$ and set $Y_i = N/Ry_i$. By Proposition 5.1(2) and Remark 5.2, we only need to prove that $\text{bight}(Y_i^\ast(x)) \leq \text{rank}(N) - 1$ for $1 \leq i \leq n$. Write $y = y_i, Y = Y_i$, and notice that $Y^\ast(x) \subset N^\ast(x) \neq R$. By Proposition 2.4(2) $\text{ht}(N^\ast(y)) \geq \text{ht}(N^\ast(x)) > \text{rank}(N)$. Thus $Ry \cong R$ and assumptions (1), (2), and (3) pass from $N$ to $Y$, except that $\text{rank}(Y) = \text{rank}(N) - 1$. After localizing at minimal primes of $Y^\ast(x)$ we may apply Propositions 4.1 and 4.2 to conclude that $\text{bight}(Y^\ast(x)) \leq \text{rank}(Y) = \text{rank}(N) - 1$. 

We now consider bounds for the height of the trace ideal $\text{tr}(N) = N^\ast(N)$. Of course if $R$ is regular local and the height of the order ideal $N^\ast(x)$ is bounded by $\text{rank}(N)$ for every $x$ then $\text{ht}(\text{tr}(N)) \leq ri$ because the heights of ideals are subadditive in a regular local ring (Serre [1958, V.6.3]). In general, however, no such strong inequality is true. For the module $N$ of rank $r$ which is the image of the generic $m \times n$ matrix of rank $r$, all the heights of order ideals are bounded by $r$ but the height of the trace ideal is $r(m + n - r)$ (this is the height of the $1 \times 1$ minors of the generic matrix minus the height of the $r + 1) \times r + 1$ minors of that matrix—see Remark 2.2. Bruns [1981, Corollary 1] shows that this is a universal bound.

Curiously, if we assume that some element has an order ideal larger than the rank (which would seem to make the trace ideal larger) then we can get a bound which is asymptotically much sharper (as $(m + n)/r \to \infty$ it has order $(r - 2)m + (r - 3)n$ instead of $r(m + n)$). It is convenient at the same time to give a bound on $\text{ht}(N^\ast(U))$ for any $i$-generated submodule $U$.

**Theorem 5.5.** Let $R$ be an equidimensional universally catenary Noetherian local ring and let $N$ be an orientable $R$-module of rank $r$. Let $U \subset N$ be a submodule not containing any nontrivial free summand of $N$, and write $i = \mu(U), m = \mu(N^\ast)$. Then either $\text{ht}(N^\ast(x)) \leq r$ for every $x \in U$ or else

1. $\text{ht}(N^\ast(U)) \leq (r - 2)(m - r + 3) + (r - 3)i$, in case $R$ satisfies $S_3$;
2. $\text{ht}(N^\ast(U)) \leq (r - 3)(m - r + 3) + (r - 3)i$, in case $R$ is Gorenstein, and $N$ satisfies $S_3$ and is free in codimension 2;
3. $\text{ht}(N^\ast(U)) \leq (r - k)(m - r + 3) + (r - 3)i$, in case $R$ contains a field and satisfies $S_{k+1}$, and $N$ is a $k$th syzygy of finite projective dimension.

**Proof.** Assume that $\text{ht}(N^\ast(x)) > r$ for some $x \in U$. We will prove (1) and (2) by induction on $i$. Using Propositions 4.1 and 4.2 we see that $r \geq 3$ in (1), $r \geq 4$ in (2), and $r \geq k + 1$ in (3).

By Theorem 3.1 we can write $U = Rx_1 + \cdots + Rx_i$ with $\text{ht}(N^\ast(x_1)) > r$. If $i = 1$, then $\text{ht}(N^\ast(U)) = \text{ht}(N^\ast(x_1)) \leq m$, which yields the desired estimates in this case (as $r \geq 3$, or $r \geq 4$, or $r \geq k + 1$, respectively). Hence we may assume $i \geq 2$. Write $V = Rx_1 + \cdots + Rx_{i-1}$. If $N^\ast(x_i) \subset \sqrt{N^\ast(V)}$ we can replace $U$ by $V$ and apply the induction hypothesis (note that $x_1 \in V$ and $\text{ht}(N^\ast(x_1)) > r$). Otherwise, we may choose a prime $Q$ containing $N^\ast(V)$ such that $N^\ast(x_i) \not\subset Q$.

Set $X = N/Rx_i$, let $z_j$ be the image of $x_j$ in $X$ for $1 \leq j \leq i - 1$, and let $Z$ be the
image of $V$ in $X$. We estimate the height of $N^*(U)$ as follows:

$$
ht(N^*(U)) = ht(N^*(x_i) + N^*(V)) \leq m + ht(N^*(V)) \leq m + ht(N^*(V)_Q).
$$

Since $N^*(x_i) \not\subset Q$, $N_Q \cong R_Q \oplus X_Q$. We may write $x_j = r_j + z_j$ under this isomorphism. We deduce $N^*(x_j)_Q = (r_j, X^*(z_j))_Q$. It follows that

$$
N^*(V)_Q = (r_1, \ldots, r_{i-1})_Q + X^*(Z)_Q
$$
and $ht(X^*(z_1)_Q) \geq ht(N^*(x_1)_Q) - 1 > r - 1$. Furthermore, as $N^*(V)_Q \neq R_Q$ we may estimate the height of $N^*(V)_Q$; combining with the estimate above we obtain,

$$
ht(N^*(U)) \leq m + (i - 1) + ht(X^*(Z)_Q).
$$

We apply our induction hypothesis to $Z_Q \subset X_Q$ noting that $\mu(Z_Q) \leq i - 1$, $X_Q$ is orientable of rank $r - 1$, $\mu(X^*_Q) \leq m - 1$, and $ht(X^*_Q(z_1)) > r - 1$.

We formally set $k = 2$ for (1), $k = 3$ for (2), and $k = k$ for (3). By induction,

$$
ht(N^*(U)) \leq m + (i - 1) + ((r - 1) - k)((m - 1) - (r - 1) + 3) + ((r - 1) - 3)(i - 1)
$$
and the desired formulas follow. □

For example, let $I$ to be the ideal of the curve $t \mapsto (1, t, t^3, t^4)$ in $\mathbb{P}^3$ treated in Example 3.10. If $N = I^\perp$, then the inequality of Proposition 5.1(1) is sharp for every $i$ (here $r = 3, m = n = 4$). We do not have examples of rank $\geq 4$ where the inequality is sharp.
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