A SCHRODINGER EQUATION FOR MINI UNIVERSES

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ABSTRACT

We discuss how to fix the gauge in the canonical treatment of Lagrangians, with finite number of degrees of freedom, endowed with time reparametrization invariance. The motion can then be described by an effective Hamiltonian acting on the gauge shell canonical space. The system is then suited for quantization. We apply this treatment to the case of a Robertson–Walker metric interacting with zero modes of bosonic fields and write a Schrödinger equation for the on–shell wave function.

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1. Introduction.

Models of mini universes are characterized by a finite (small) number of gravitational and matter degrees of freedom. Now, a quantum theory for a mini universe may be relevant to the initial stages of the universe if an underlying field structure (strings, or any sort of quantum field treatment) is not yet predominant; then a quantum treatment of a few degrees of freedom could make sense [1,2]. The formulation of a consistent quantum theory for these models constitutes a very interesting problem.

The general covariance of General Relativity was at the basis of its very original formulation; and later, different approaches have investigated its canonical formulation. The introduction of the lapse function and shift vector, that are Lagrange multipliers, goes back to 1962 [3]. The investigations of the canonical structure of General Relativity have made large progress in the recent years together with the discussions about the definition and meaning of time (see [4-6] and the contributions to [7-9]). A general treatment of quantization of relativistic gauge systems is discussed in [10].

In mini universe models there is a residual invariance under reparametrization of time, similarly to what we find for particle problems in special relativity, and the Wheeler-DeWitt (WDW) equation [11,12] is the quantum expression of the correspondent constraint. Their canonical treatment requires analysis of the gauge invariance and reduction of the redundant degrees of freedom either by the BRST formalism or, alternatively, by reducing the phase space by gauge fixing.
The book by Henneaux and Teitelboim is a guideline to the canonical treatment of classical and quantum gauge systems [13].

In this paper our limited aim is to discuss their canonical structure and the quantum formulation of mini universe models. In our examples we will consider a Robertson-Walker (RW) universe, thus a single gravitational degree of freedom, its scale factor $a$.

As for matter, aiming to have consistently radiation with the same symmetry as the RW metric, we will introduce an SU(2) YM field and consider its zero mode [14-16]. Of course, the presence of a YM field is very welcome, as gauge fields constitute a fundamental ingredient of matter. This mini universe has the right properties of a radiation dominated universe: a separable WDW constraint, corresponding to the classical property of radiation density scaling as $a^{-4}$. A further component of matter that does not spoil this simplicity is a conformal scalar field, and we will introduce its zero mode for completeness.

The model under consideration does not contain, as it stands, interaction terms producing inflation; inflation must be introduced by hand in the Lagrangian in the form of a cosmological term and we will discuss briefly a form of gauge fixing for this case. We emphasize that the method we are presenting is not restricted to these cases; indeed, the next step will be to consider a minimally coupled scalar field. This is of great interest as it introduces a coupling to gravitation that induces inflation in a dynamical way. This model is more complex and will be discussed elsewhere.
The dynamics of the mini universe is expressed by a Lagrangian

\[ L = p_\mu \dot{q}_\mu - l C(q_\mu p_\mu) \quad \mu = 0, \ldots, N - 1 \quad (1.1) \]

which displays gauge invariance under reparametrization of time (here we reserve the word “gauge” to the case of time reparametrizations) and has close similarity to the dynamics of the free relativistic particle or of the relativistic harmonic oscillator. \( l \) is the Lagrange multiplier. In order to eliminate redundancy and reduce the canonical degrees of freedom the central point is to introduce a dynamical way of fixing the gauge in the form of a relationship between canonical variables and time:

\[ F(q, p, t) = 0. \quad (1.2) \]

The relationship (1.2), together with the constraint \( C = 0 \), must allow to eliminate one canonical degree of freedom as function of the remaining ones and of time; thus time is connected to the original set of canonical variables and the problem is reduced to gauge shell with \( 2(N - 1) \) canonical coordinates (analogously to the unitary gauge in field theories). Naturally the Poisson bracket \( \{F, C\} \) must not vanish even weakly (see for instance [13,17,18]), thus the set \( (F, C) \) is second class. The condition (1.2) must determine the Lagrange multiplier \( l(t) \) (essentially the lapse function: \( l = N/a \)) as a function of time and in general of canonical variables. In particular in some cases we will obtain the so-called cosmic time gauge \( N = 1 \), in others \( N = a \), the conformal time. The gauge fixing function \( F \) must be explicitly dependent upon time since the constraint is not linear in coordinates and momenta.
The reduction is implemented by a canonical transformation. For instance, if \( F \) is of the form \( F = p_0 - f(q_\mu, t) \) one can choose \( P_0 = F, Q_0 = q_0 \). Then the gauge shell is obtained by imposing \( P_0 = 0 \) and fixing \( Q_0 \) from the constraint \( C = 0 \). One canonical degree of freedom is thus eliminated. The motion is reduced to canonical form in \( N - 1 \) degrees of freedom with an effective Hamiltonian (in general time dependent) generating canonical motion on the gauge shell. Eq. (1.2) is the fundamental gauge fixing equation that states how time is connected to canonical variables. This implements a program of general relativity, namely time as defined in terms of physical variables.

The basic criterion for the choice of the gauge fixing condition must be the simplicity of the motion in the subspace of physical variables. Of course the difficulty resides in finding a gauge that simplifies the expression of the Hamiltonian on gauge shell (and must obey the requirements about gauge fixing that we know from field theories). The expression for \( l(t) \) is a consequence of that choice and thus not of primary interest in the procedure.

It is then natural to quantize the system in the reduced canonical space (gauge shell) by writing a time dependent Schrödinger equation, or, what has the same content, the Heisenberg equations. In a given gauge, time is connected to the canonical degree of freedom that has been eliminated by the constraint and gauge fixing. This is the way to decide which degree of freedom is better suited to play the role of time, whether the gravitation degree of freedom or a matter variable connected to the YM field (or to the conformal scalar). Of course, reduction of
the gravitational degree of freedom to the role of time in a Schrödinger equation is preferable, since its contribution in the constraint equation, for the simple cases under consideration, has opposite sign to that of the radiation degrees of freedom. However, if one introduces other sorts of matter, as for instance a scalar field, mixed terms will appear in the constraint and the gauging off of gravitation is more complicate. We shall examine this case in a next investigation.

The wave function introduced in this way has the usual properties of quantum mechanics, in particular a conserved current with positive definite probability density exists, while the WDW equation is a gauge constraint off-shell and the WDW wave function is non physical.

The method used here has no difficulty (except complication) to be implemented in the case of a set of first class constraints. Indeed one then introduces corresponding gauge fixing terms in order to form a set of second class constraints and then proceeds to the elimination of the redundant variables by the canonical transformation and establishes the effective Hamiltonian on the gauge shell.

2. Canonical gauge fixing.

As we have said, our aim is to discuss the canonical formulation of mini universe models in view of establishing the corresponding quantum theory. We use the ADM approach and in this paper we will treat the simple case of a RW metric. Systems of this kind are endowed with time reparametrization invariance. Other examples of systems with these characteristics are the relativistic particle
in Minkowsky space or the relativistic harmonic oscillator.

In our case the first-class Hamiltonian must be zero (as is the case when the canonical variables transform as scalars under reparametrization). In this case the extended Hamiltonian will contain only the constraint [13,17]:

$$H_E = l(t)C(q_\mu, p_\mu)$$  \hspace{1cm} (2.1)

where \( l(t) \) is the Lagrange multiplier that imposes the constraint \( C = 0 \).

As a warming up exercise let us consider the relativistic massive free particle. Its action is

$$S = \int dt \left( -p_0 \dot{x}_0 + p_i \dot{x}_i - l(t) C \right)$$  \hspace{1cm} (2.2)

where

$$C \equiv \frac{1}{2} \left( \bar{p}^2 + m^2 - p_0^2 \right) = 0$$  \hspace{1cm} (2.3)

is the constraint term. In order to show how the gauge fixing works, let us perform the following canonical transformation:

$$X_0 = x_0 - t p_0,$$  \hspace{1cm} (2.4a)

$$P_0 = p_0.$$  \hspace{1cm} (2.4b)

Let now take as gauge fixing condition

$$X_0 = 0.$$  \hspace{1cm} (2.5)

Of course the Poisson bracket \( \{X_0, C\} \) is not vanishing weakly (it is not a linear combination of \( X_0 \) and \( C \)). The use of the condition (2.5) in the canonical
equations of motion generated by (2.1) fixes the Lagrange multiplier as follows:

\[ l(t) = 1. \]  

(2.6)

Using the gauge fixing (2.5) and the constraint equation (2.3) the constrained action is

\[ S_c = \int dt \left( p_i \dot{x}_i - \frac{1}{2} P_0^2 \right) \]  

(2.7)

with \( P_0^2 \) given by (2.3) and (2.4). Thus, with this choice of gauge, in the sector \( p_i, x_i \) the motion is generated by the Hamiltonian

\[ H_{\text{eff}} = \frac{1}{2} P_0^2 = \frac{1}{2} (p^2 + m^2). \]  

(2.8)

Note that this Hamiltonian generates motion in the physical 6-dimensional space \( (x_i, p_i) \); this is equivalent to the motion in the sector \( p_i, x_i \) generated by the action (2.2) with the constraints (2.3) and (2.5). Let us note that the canonical transformation (2.4) must involve the time explicitly and that (2.5) can be thought of as the definition of time in terms of canonical variables. We remark that in this canonical gauge fixing the final goal is to obtain a simple expression for the effective Hamiltonian; the expression (2.6) of the Lagrange multiplier \( l(t) \) is a consequence of the gauge fixing (2.5) and of the equations of motion, and does not concern us too much. We will keep this in mind when dealing with mini universes in general relativity.

It is interesting to compare this result with a different gauge fixing condition. Let us choose as time the 0-th component of \( x_\mu \). To do that, eq. (2.4a) has to
be substituted by

\[ X_0 = x_0 - t. \]  

(2.4a')

Then by the same procedure one arrives to the square root Hamiltonian:

\[ H_{\text{eff}} = \sqrt{p^2 + m^2}. \]  

(2.9)

This simple example clarifies the method whose ultimate effectiveness resides in the possibility of finding a simple canonical transformation leading to a simple \( H_{\text{eff}} \).

After these preliminaries, let us go back to the general case (2.1) and define a canonical transformation of the form

\[ P_\mu = p_\mu - f_\mu(q_\nu; t), \]  

(2.10a)

\[ Q_\mu = q_\mu. \]  

(2.10b)

(the role of the \( p \)'s and \( q \)'s may be interchanged of course). From the Poisson identity \( \{ P_\mu, P_\nu \} = 0 \) follows, at least locally, that \( f_\mu = \partial/\partial q_\mu f(q_\nu, t) \). In the following we will denote partial differentiation by the symbol “\( | \)”. Then we have

\[ p_\mu \dot{q}_\mu = P_\mu \dot{Q}_\mu - f_\mu | t(Q_\mu; t) + \frac{d}{dt} f(Q_\mu; t). \]  

(2.11)

Now let us introduce the gauge fixing condition in the form

\[ P_0 = 0. \]  

(2.12)

Note that the Poisson bracket of the gauge fixing (2.12) and of the constraint, \( \{ P_0, C \} \), must not be weakly zero [13]. The function \( f \) must depend explicitly
on $t$ in order that the procedure works, namely to fix completely the Lagrange multiplier. Now the system has two constraints of second class, $P_0 = 0$ and $C = 0$. The variable $Q_0$ conjugate to $P_0$ is fixed by the constraint: $Q_0 = Q_c(Q_i, P_i; t)$, where $i = 1, \ldots, N - 1$ and $Q_c$ is obtained from

$$[C(Q_0, Q_i, P_0 = f|_0(Q_0, Q_i; t), P_i + f|_i)]_{Q_0 = Q_c} = 0. \quad (2.13)$$

The system is now on the “gauge shell”, the $2(N - 1)$-hypersurface in the $2N$-phase space $(Q_\mu, P_\mu)$ defined by

$$P_0 = 0, \quad Q_0 = Q_c(Q_i, P_i; t). \quad (2.14)$$

The effective Lagrangian in the $(Q_i, P_i)$ sector is obtained from (2.1) using (2.11) and (2.14). Neglecting a total derivative we have

$$L_{\text{eff}} = L[P_0 = 0, Q_0 = Q_c] = P_i \dot{Q}_i - H_{\text{eff}}(Q_i, P_i; t) \quad (2.15)$$

where the effective Hamiltonian on the gauge shell is

$$H_{\text{eff}}(Q_i, P_i; t) = f|_i(Q_c, Q_i; t). \quad (2.16)$$

The effective Hamiltonian is in general time dependent. One can check that the canonical equations

$$\dot{Q}_i = \frac{\partial H_{\text{eff}}}{\partial P_i}, \quad (2.17a)$$
$$\dot{P}_i = -\frac{\partial H_{\text{eff}}}{\partial Q_i} \quad (2.17b)$$

are equivalent to the original canonical system using (2.13) and (2.19) (see below).
Finally, for consistency with the gauge fixing condition (2.12) we require also

$$\dot{P}_0 = 0$$  \hspace{1cm} (2.18)

that determines the expression for the Lagrange multiplier:

$$l(t) = - \left[ \frac{\dot{f}_{|q_0}}{C|q_0 + \dot{f}_{|q_0 q_\mu} C|p_\mu \text{ s.t. } q_0=Q_c \text{ and } p_\mu=f_\mu} \right]$$  \hspace{1cm} (2.19)

In general the expression (2.19) can be complicated, however this does not necessarily concern us: the important fact is that the gauge fixing of the form (2.12) determines $l(t)$.

The problem of the motion is thus reduced to the task of determining the gauge function $f(q_\mu; t)$ so that $H_{\text{eff}}$ is as simple as possible. In the next section we will discuss the gauge fixing for RW universe in interaction with radiation.

In conclusion, let us remark that time is determined by the gauge fixing condition to be a function of the variables $q_\mu$ and $p_\mu$ from the condition $P_0 = 0$. As we shall see, often $t$ turns out to be a function only of a couple of canonical variables $p_0, q_0$, thus a single degree of freedom defines the time in that gauge. This is an interesting situation. A particular situation occurs when $t$ depends only on $q_0$ or $p_0$. This allows identification of time in that gauge with a canonical coordinate of immediate physical relevance, and of course we have in mind in the gravitational case the identification of $t$ with the scale factor $a$ of the RW universe. We shall see that, while this is in general possible, this is not the most advisable choice from the point of view of simplicity of $H_{\text{eff}}$. Indeed, the main criterion for the choice of the function $f$ is, as we have said, to obtain a simple
$H_{\text{eff}}$. With the identification of time with a degree of freedom $H_{\text{eff}}$ is in general time dependent.

In principle there is no great difficulty to generalize this method. After all, the four constraints in a general invariant theory are of first class and the procedure exposed here can be repeated. We stress, the problem is the determination of suitable canonical transformations (i.e. gauge fixing identities) so that the corresponding effective Hamiltonian is simple and possibly time independent.

3. Simple examples of gauge fixing.

We will apply the ideas exposed above to some simple cases of relevance to the discussion of mini universe models with RW or de Sitter spacetime coupled to zero modes of different fields. Let the constraint have the form

$$C \equiv \frac{1}{2} p_0^2 + V(q_0) - H_1(q_\mu, p_i) = 0 \quad (3.1)$$

We may identify $q_0$ as the gravitational degree of freedom, however this is not always needed.

Let us first assume that the constraint is separable and the 0-th degree of freedom has the form of a harmonic oscillator: this happens, for instance, when we deal with a conformal scalar field or with a RW closed metric. So we have

$$V(q_0) = \frac{1}{2} q_0^2, \quad (3.2a)$$

$$H_1 \equiv H_1(q_i, p_i). \quad (3.2b)$$
We choose the canonical transformation (2.10) as

\[ P_0 = p_0 + q_0 \cot t, \quad (3.3a) \]
\[ Q_0 = q_0 \quad (3.3b) \]

and the gauge fixing condition is of the form

\[ p_0 + q_0 \cot t = 0. \quad (3.4) \]

Gauge fixing conditions of this sort were introduced in [19] in the context of a gauge approach to systems of many relativistic particles. By the method exposed in section 2 one obtains a very interesting result for the effective Hamiltonian on the gauge shell:

\[ H_{\text{eff}} = H_1. \quad (3.5) \]

The simplicity of (3.5) shows the interest of the gauge fixing (3.4). \( l(t) \) is fixed:

\[ l(t) = -1. \quad (3.6) \]

Thus in this problem the natural choice of the time is arctg \( q_0/p_0 \).

One would intuitively like to identify \( q_0 \) as time. Let us see what happens if one assumes gauge fixing conditions of the form

\[ p_0 = \sqrt{2} t, \quad \text{or} \quad q_0 = -\sqrt{2} t. \quad (3.7) \]

Then respectively

\[ l(t) = -\frac{\sqrt{2}}{q_0}, \quad l(t) = -\frac{\sqrt{2}}{p_0}. \quad (3.8) \]
The effective Hamiltonian is time dependent:

\[ H_{\text{eff}} = 2\sqrt{H_1 - t^2}. \]  

(3.9)

The positive definiteness of the operator under square root implies that the support of \( q_0 \) or \( p_0 \) is restricted, in agreement with the general properties of the oscillatory motion in \( q_0, p_0 \). One sees the advantage of the gauge choice (3.4) since in that case \( H_{\text{eff}} \) is independent of time.

Let us end this part dedicated to the oscillator by noticing that of course a non compact oscillator requires hyperbolic functions in place of circular.

In mini universe models we shall encounter a quartic potential, for instance for the zero mode of the Yang–Mills field and of course in the case of a cosmological term. So let us discuss the gauge fixing in case \( V(q_0) \) of (3.1) has the form

\[ V(q_0) = kq_0^2 - \lambda q_0^4 \]

(3.10)

and (3.2b) holds. In this case it is convenient to choose the canonical transformation as

\[ P_0 = p_0 - \sqrt{2\lambda}(q_0^2 + g(t)), \]

(3.11a)

\[ Q_0 = q_0 \]

(3.11b)

where \( g(t) = t^2 - k/2\lambda \). The gauge fixing condition (2.12) is

\[ p_0 = \sqrt{2\lambda}(q_0^2 + g(t)) \]

(3.12)
which together with (2.18) allows to fix the Lagrange multiplier \( l(t) \):

\[
l(t) = -\frac{1}{\sqrt{2\lambda t q_0}}.
\]

We may now compute the effective Hamiltonian in the physical degrees of freedom:

\[
H_{\text{eff}} = \sqrt{2\lambda} \dot{g}(t)Q_c. \tag{3.14}
\]

In eq. (3.14) \( Q_c \) must be obtained from the constraint as in (2.13),

\[
Q_c^2 = \frac{1}{k + 2\lambda g(t)}(H_1 - \lambda g^2(t)) \tag{3.15}
\]

and thus

\[
H_{\text{eff}} = 2\sqrt{H_1 - \lambda(t^2 - k/2\lambda)^2} \tag{3.16}
\]

It is evident that the final form of \( H_{\text{eff}} \) is not simple and is time dependent. What is important is that time is essentially fixed by (3.12) as a function of \( p_0 \) and \( q_0 \).

Now let us give some hints about the case of a general potential in (3.1) (note that often this procedure is not the best to follow, as it happens, for instance, in the case just discussed).

Eq. (2.13) that defines \( Q_c(q_i, p_i; t) \) is now

\[
\left[ \frac{1}{2}(f_{|Q_0}(Q_0, Q_i; t))^2 + V(Q_0) - H_1(Q_0, Q_i, P_i + f_{|Q_i}) \right]_{Q_0 = Q_c} = 0. \tag{3.17}
\]

Remembering the form (2.16) of \( H_{\text{eff}} \) let us choose \( f \equiv f(Q_0; t) \); one needs to connect \( f_{|Q_0} \) to \( f_{|t} \) and we may proceed for instance as follows. Set

\[
\frac{1}{2}(f_{|q}(q; t))^2 + V(q) = g(t) F(q) \tag{3.18}
\]
where \( g(t) \) and \( F(q) \) arbitrary functions of \( t \) and of \( q \) respectively. Then

\[
f(Q_0; t) = \sqrt{2} \int_{Q_0}^{Q_c} dq \left[ H_1(q, Q_i, P_i) - V(q) \right]^{1/2} \quad (3.19)
\]

and the effective Hamiltonian has the expression

\[
H_{\text{eff}} = \frac{\dot{g}}{\sqrt{2}} \int_{Q_c}^{Q_c} dq \sqrt{F(q)} \left( g(t) - \frac{V(q)}{F(q)} \right)^{-1/2}. \quad (3.20)
\]

\( F(q) \) may be chosen so as to simplify the expression (3.19): for instance, if \( V(q_0) \) is of the form \( q_0^2 v(q_0) \) with \( v(q_0) \) a polynomial, then \( F = q^2 \) is a suitable choice.

Of course if the potential is particularly simple one may set

\[
F(q) = V(q). \quad (3.21)
\]

Then

\[
H_{\text{eff}} = \sqrt{2} \frac{d}{dt}(g - 1)^{1/2} \int_{Q_c}^{Q_c} dq \sqrt{V(q)}. \quad (3.22)
\]

Note that \( H_{\text{eff}} \) is in general time dependent also because of \( Q_c \). Finally, the Lagrange multiplier is

\[
l(t) = \left[ -\frac{f_{tq_0}^{Q_0}}{V'(q_0) + f_{tq_0}^{Q_0} + f_{tq_0}^{Q_0} - H_{1|q_0}(q_0, q_i, f_{tq_0}^{Q_0})} \right]_{q_0 = Q_c}. \quad (3.23)
\]

The previous results about the harmonic oscillator can be obtained from these general formulae choosing

\[
f(q_0; t) = -\frac{1}{2} \frac{q_0^2 \cotg t}{q_0}. \quad (3.24)
\]

In the end let us apply the method discussed above to a constraint of the form

\[
C = \frac{1}{2} p_0^2 - \frac{1}{2q_0^2} p_1^2 - q_0^4 V_1(q_1) + V_0(q_0). \quad (3.25)
\]
This case corresponds to the zero mode of a scalar field minimally coupled to gravity. Now $Q_c(q_i, p_i; t)$ is defined by

$$
\left[ \frac{1}{2} \left( f_{|q_0} \right)^2 - Q_0^4 V_1(Q_1) + V_0(Q_0) - (P_1 + f_{|Q_1}) \frac{1}{2} \frac{1}{Q_0^2} \right]_{Q_0 = Q_c} = 0. \quad (3.26)
$$

Choosing $f \equiv f(Q_0; t)$ and setting

$$
\frac{1}{2} \left( f_{q}(q; t) \right)^2 + V_0(q) \equiv q^4 g(t) \quad (3.27)
$$

we obtain the effective Hamiltonian

$$
H_{\text{eff}} = \frac{\dot{g}}{\sqrt{2}} \int^{Q_c} dq q^2 \left( g(t) - \frac{V_0(q)}{q^4} \right)^{-1/2} \quad (3.28)
$$

where now

$$
Q_c = \left( \frac{P_1^2}{2 (g(t) - V_1(Q_1))} \right)^{1/6}. \quad (3.29)
$$

In particular, if

$$
V_0(q_0) = q_0^4 \quad (3.30)
$$

(3.28) becomes

$$
H_{\text{eff}} = \pm \frac{1}{3} \frac{d}{dt} \sqrt{g - 1} \frac{P_1}{(g(t) - V_1(Q_1))^{1/2}}. \quad (3.31)
$$

Analogously to the cases discussed before, a suitable choice for $g(t)$ can simplify the effective Hamiltonian. Clearly, (3.31) depends explicitly on time.

Finally the Lagrange multiplier is

$$
l(t) = \left[ -\frac{q_0 f_{|q_0}}{2V_0 + q_0 V'_0 - 6q_0^4 V_1 + f_{|q_0} \left( f_{|q_0} + q_0 f_{|q_0 q_0} \right)} \right]_{q_0 = Q_c}. \quad (3.32)
$$

We will not give here a complete discussion of the gauge reduction of (3.25) because the coupling term $q_0^4 V_1(q_1)$ makes it difficult to obtain a simple effective
Hamiltonian. Let us discuss the simple case $V_1(q_1) = 0$ and $V_0(q_0) = q_0^2$. In this case it is convenient to choose

$$f(q_0; t) = \frac{1}{\sqrt{2} q_0^2} \sinh t$$

(3.33)

Using (3.26) we have that $Q_c(Q_i, P_i; t)$ is defined by

$$Q_c^2 = \pm \frac{1}{\sqrt{2}} \frac{P_1}{\cosh t}$$

(3.34)

and the effective Hamiltonian (2.16) can be written

$$H_{\text{eff}} = \pm \frac{1}{2} P_1$$

(3.35)

A surprising feature of (3.31) and (3.35) is that the effective Hamiltonian is linear in $P_1$. In the next section we will apply all these considerations to minisuperspace models.

4. Minisuperspace models.

Let us consider the action for gravity minimally coupled to SU(2) Yang-Mills (YM) field $A$ and a conformal scalar (CS) field $\varphi$. The reason for this choice is that for these fields the WDW equation decouples, a fact directly related to the property that the classical energy density scales as $a^{-4}$. The action is the sum of three terms:

$$I = I_{\text{GR}} + I_{\text{YM}} + I_{\text{CS}}.$$  

(4.1)

Here $I_{\text{GR}}$ represents the Einstein-Hilbert action of the gravitational field

$$I_{\text{GR}} = - \int_{\Omega} d^4x \sqrt{-g} (R + 2\Lambda) + 2 \int_{\Omega} d^3x \sqrt{h} \ K$$

(4.2a)
where $R$ is the scalar curvature tensor, $\Lambda$ the cosmological constant and $K$ the extrinsic curvature of the manifold $\Omega$. We use definitions as in Landau - Lifshitz and we have put $16\pi G \equiv L_p^2 \equiv M_p^{-2} = 1$ thus measuring all dimensional quantities in these units. The second and third term in (4.1), $I_{\text{YM}}$ and $I_{\text{CS}}$, represent respectively the YM and the CS field actions

\begin{align}
I_{\text{YM}} &= \frac{1}{2} \int F \wedge {}^* F, \\
I_{\text{CS}} &= \frac{1}{2} \int d^4x \sqrt{g} \left( \partial_{\mu} \varphi \partial^{\mu} \varphi + \frac{1}{6} R \varphi^2 \right) 
\end{align}

where $F = dA + A \wedge A$ is the field strength 2-form and we have set $= 1$ the gauge coupling constant.

We shall study spacetimes of topology $R \times H$, where $H$ is a homogeneous and isotropic three-surface. Hence we shall write

\[ ds^2 = N^2(t) dt^2 - a^2(t) \omega^p \otimes \omega^p \]

where $\omega^p$ are the 1-forms invariant under translations in space and $N(t)$ is the lapse function. The cosmic time corresponds to $N = 1$ and the conformal time to $N = a$. In the cases we are going to discuss, the $\omega^p$’s will satisfy the Maurer-Cartan structure equations:

\[ d\omega^p = \frac{k}{2} \epsilon_{pqr} \omega^q \wedge \omega^r \]

where $k = 0, 1$ and thus the line element (4.3) describes a flat or a closed Friedmann-Robertson-Walker universe. When $k = 1$, (4.3) has the $\text{SU}(2)_L \times \text{SU}(2)_R$. 

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SU(2)\(_R\) group of isometries. Using (4.3) and integrating over the spatial variables, the space density of the gravitational action (4.2a) can be written

\[ S_{GR} = 6 \int dt \left( -\frac{a\dot{a}^2}{N} + kNa - \frac{\Lambda}{3}Na^3 \right). \]  

(4.5)

Introducing the conjugate momentum

\[ p_a = -12 \frac{a\dot{a}}{N} \]  

(4.6)

(4.5) can be cast in the form

\[ S_{GR} = \int dt \left( p_a \dot{a} + \frac{N}{a} H_{GR} \right). \]  

(4.7)

Here \( H_{GR} \) is

\[ H_{GR} = \frac{1}{12} \left( \frac{1}{2}p_a^2 + V_a(a) \right) \]  

(4.8)

where

\[ V_a(a) = 72a^2 \left( k - \frac{\Lambda}{3}a^2 \right). \]  

(4.9)

Let us now introduce a YM field configuration with the same symmetry as the metric. We shall use a YM group SU(2) for simplicity and (in line with the metric) a form of the YM field with a single degree of freedom that was proposed in [14] (see also [16]); the case of a more general group has been investigated in [15]. The form of the field, written in the dreibein cotangent space, is

\[ A = \frac{i}{2} \xi(t)\sigma_p \omega^p. \]  

(4.10)

With the definition (4.10) \( A \) is evidently left- invariant; it is also right- invariant up to a gauge transformation [14,20]. \( \xi(t) \) is the single degree of freedom.
The field strength $F$ is:

$$F = \frac{i}{2} \sigma_\mu \dot{\xi} dt \wedge \omega^\mu + \frac{i}{4} \sigma_r \epsilon_{rpq} \xi (k - \xi) \omega^q \wedge \omega^r. \quad (4.11)$$

Thus the action of the YM field becomes

$$S_{YM} = \frac{3}{2} \int dt \left( \frac{a}{N} \dot{\xi}^2 - \frac{N}{a} \xi^2 (k - \xi)^2 \right) \quad (4.12)$$

and introducing the conjugate momentum

$$p_\xi = 3 \frac{a}{N} \dot{\xi}, \quad (4.13)$$

(4.12) can be cast in the form

$$S_{YM} = \int dt \left( p_\xi \dot{\xi} - \frac{N}{a} H_{YM} \right) \quad (4.14)$$

where $H_{YM}$ is

$$H_{YM} = \frac{1}{3} \left( \frac{1}{2} p_\xi^2 + V_\xi (\xi) \right) \quad (4.15)$$

and

$$V_\xi (\xi) = \frac{9}{2} \xi^2 (k - \xi)^2. \quad (4.16)$$

Now, let us discuss the CS field. Using (4.3) the CS field action becomes

$$S_{CS} = \frac{1}{2} \int dt \left( \frac{a}{N} (a \dot{\varphi} + \dot{a} \varphi)^2 - k N a \varphi^2 \right). \quad (4.17)$$

Defining the rescaled scalar field $\chi = \varphi a$, (4.17) can be cast in the form

$$S_{CS} = \frac{1}{2} \int dt \left( \frac{a}{N} \dot{\chi}^2 - k \frac{N}{a} \chi^2 \right). \quad (4.18)$$
Introducing as in the previous cases the conjugate momentum

\[ p_\chi = \frac{a}{N} \dot{\chi}, \quad (4.19) \]

(4.18) becomes

\[ S_{CS} = \int dt \left( p_\chi \dot{\chi} - \frac{N}{a} H_{CS} \right) \quad (4.20) \]

where

\[ H_{CS} = \frac{1}{2} (p_\chi^2 + k\chi^2). \quad (4.21) \]

In the conformal gauge, when \( k = 1 \), the Hamiltonian (4.21) describes a one-dimensional harmonic oscillator in the \( \chi \) variable.

The classical Friedmann - Einstein equation of motion is the constraint equation

\[ H_{YM} + H_{CS} = H_{GR}. \quad (4.22) \]

From the motion equations for the YM and the CS fields it is easy to obtain

\[ \left( \frac{a}{N} \dot{\xi} \right)^2 + \xi^2 (k - \xi)^2 = K_{YM}^2 \]
\[ \left( \frac{a}{N} \dot{\chi} \right)^2 + k\chi^2 = K_{CS}^2 \quad (4.23a, b) \]

where \( K_{YM} \) and \( K_{CS} \) are independent of \( t \); using (4.23), (4.22) becomes

\[ \frac{a^2 \dot{a}^2}{N^2} + \left( ka^2 - \frac{\Lambda}{3} a^4 \right) = \frac{1}{12} K^2 \quad (4.24) \]

where

\[ K^2 = 3K_{YM}^2 + K_{CS}^2. \quad (4.25) \]
In the cosmic gauge, $N = 1$, this reads

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\Lambda}{3} + \frac{1}{12} \frac{K^2}{a^4}. \quad (4.26)$$

We see that, as due to radiation, the energy density scales as $a^{-4}$. The meaning of $K^2$ is obvious:

$$K^2 = 2\rho(a = 1). \quad (4.27)$$

$K^2$ is thus the energy density of the radiation in a RW universe with scale factor $a$ of one Planck length.

Let us turn to gauge fixing and writing a Schrödinger equation. The complete action of our radiation filled mini universe is

$$S = \int dt \left( p_a \dot{a} + p_\xi \dot{\xi} + p_\chi \dot{\chi} - l(t) H_{\text{WDW}} \right) \quad (4.28)$$

where

$$H_{\text{WDW}} = H_{\text{YM}} + H_{\text{CS}} - H_{\text{GR}} \quad (4.29)$$

and

$$l(t) = \frac{N(t)}{a}. \quad (4.30)$$

Now we see that in this academic model we may choose the gauge fixing in essentially different ways. A very spontaneous and physically appealing choice consists in using the gravitational degree of freedom as connected to time. Let us first discuss the case $\Lambda = 0$, i.e. absence of the cosmological term, and a closed universe, $k = 1$. Then $H_{\text{GR}}$ is a harmonic oscillator and for a closed universe we use the gauge fixing identity of the form (3.4),

$$p_a = -12 a \cot t. \quad (4.31)$$
As a consequence

\[ l(t) = 1. \quad (4.32) \]

This is a conformal time gauge, \( N = a \).

With this choice of time, the Hamiltonian is time independent; the Schrödinger equation on the gauge shell takes the form

\[ i \frac{\partial}{\partial t} \Psi(\xi, \chi; t) = \left( H_{\text{CS}} + H_{\text{YM}} \right) \Psi(\xi, \chi; t). \quad (4.33) \]

Let us note that the classical Friedmann equation (4.24) for \( k = 1 \), in this gauge and with \( \Lambda = 0 \), reads

\[ \dot{a}^2 + a^2 = \frac{K^2}{12} \equiv a_{\text{M}}^2 \quad (4.34) \]

and a solution for the classical motion is

\[ a = a_{\text{M}} \sin t \quad (4.35) \]

From (4.6) we have

\[ p_a = -12\dot{a} \quad (4.36) \]

in agreement with the gauge condition (4.31). With the present definition of time, \( t = \arctg a/\dot{a} \), the region \( 0 \leq t \leq \pi/2 \) maps the expanding phase of the closed universe. Boundary conditions are of course expressed in the form

\[ \Psi(\chi, \xi; t_0) = f(\chi, \xi). \quad (4.37) \]

Eq. (4.33) expresses the evolution of the configuration in \( t \) and represents as usual in quantum mechanics the correlation amplitude for the different components of matter (\( \chi, \xi \) in the present case) in the universe.
Let us point out that a gauge choice identifying \( a \) with time,

\[
a = \frac{|t|}{\sqrt{6}}
\]  

leads to a time dependent Hamiltonian

\[
H_{\text{eff}} = 2\sqrt{H_{\text{YM}} + H_{\text{CS}}} - t^2.
\]  

The real problem comes with a cosmological term. In that case \( V_a \) has the form (4.9) and we then use the form (3.20) with \( g \rightarrow 12g \) for \( H_{\text{eff}} \) with the choice \( F(a) = a^2 \). Then

\[
H_{\text{eff}} = \sqrt{6\dot{g}} \int da \ a(g - 6k + 2\Lambda a^2)^{-1/2}
\]

\[
= \frac{1}{2\sqrt{2\lambda}} \frac{\dot{g}}{g^{1/2}} (g(g - 6k) + 2\Lambda(H_{\text{YM}} + H_{\text{CS}}))^{1/2}.
\]  

Let us now discuss an interesting different gauge fixing for the present, academic, case. We could very easily choose the time so as to be connected to the CS degree of freedom, since its Hamiltonian is a simple harmonic oscillator. This leads to ambiguities, already present in the classical discussions of the WDW equation, about self–adjointness of the quantum Hamiltonian \( H_{\text{GR}} \), and about the differential representation of the (now operator) \( p_a \). It is interesting to examine this case. Let us write the gauge fixing condition as

\[
p_\chi = \chi \ cot \ t.
\]  

Again this is a conformal time gauge, \( l(t) = 1 \). With this choice of time the Schrödinger equation takes the form

\[
i \frac{\partial}{\partial t} \Psi(a, \xi; t) = \left( H_{\text{YM}} - H_{\text{GR}} \right) \Psi(a, \xi; t).
\]  

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This identification of time connects the latter to a physically unclear entity as the CS field. However, this form is suited to discuss correlation between $a$ and $\xi$. We will see that we may draw some consequences in agreement with the correspondence principle. Since $t$ is physically not well defined, let us direct our attention to stationary states. With

$$\Psi(a, \xi; t) = e^{-iEt} \psi(a, \xi)$$

(4.43)

we obtain the stationary Schrödinger equation

$$\left( H_{YM} - H_{GR} \right) \psi(a, \xi) = E \psi(a, \xi).$$

(4.44)

This equation is similar to the WDW equation for pure gravity + YM (see [16] and also [21]), the difference being the presence of the term $E \psi$. The separability of the equation is the quantum form of the request that the density scales like $a^{-4}$. Let us write

$$\Psi(a, \xi) = \psi(a) \eta(\xi).$$

(4.45)

The equation for the YM wave function is

$$\frac{1}{3} \left( \frac{1}{2} p_\xi^2 + V_\xi(\xi) \right) \eta(\xi) = E_{YM} \eta(\xi).$$

(4.46)

As to boundary conditions, we remark that if we assume that the wave function tends to zero for large $|\xi|$, then eigenvalues are quantized.

Now let us turn to the gravitational degree of freedom again in the case of a closed universe. We have

$$\frac{1}{12} \left( \frac{1}{2} p_a^2 + V_a(a) \right) \psi_n(a) = E_n^g \psi_n(a).$$

(4.47)
The gravitational degree of freedom is a harmonic oscillator. It is then natural to set the boundary condition at \( a \to \infty \) by asking the square integrability of the wave function (for this suggestion see e.g. [16,22-24]). About the condition at \( a = 0 \), if we ask that \( \psi \to 0 \), then \( p_a = -i d/da \) is formally Hermitean and \( H_{GR} \) is self-adjoint. Let us accept for the moment these boundary conditions. Then the spectrum is given by the odd part of a harmonic oscillator (\( n_g \) odd)

\[
E_n^g = \frac{1}{2} + n_g
\]

and the wave functions are harmonic oscillator ones [16]. The eigenvalues of the two degrees of freedom are connected by

\[
E_n^g = E_{n^M}^Y - E.
\]

Let us now consider the correspondence principle with the classical gravitational motion for large oscillator quantum numbers \( n_g \) in the gauge \( l(t) = 1 \) [16]. We interpret of course, as we are led by the Schrödinger equation, \( |\psi|^2 \) as the probability density for the value \( a \) for the scale factor, to be compared through the correspondence principle to the classical probability density in the conformal time gauge distribution of the physical quantity \( a \) for an ensemble of trajectories. Now this is inversely proportional to the speed of \( a \) in that time gauge. Thus (probability not normalized)

\[
P_{cl}(t) = \frac{1}{da/dt}.
\]

From the classical equation of motion (4.34) we have in the conformal time gauge

\[
P_{cl}(t) = \frac{1}{\sqrt{a_M^2 - a^2}}.
\]
Now for the harmonic oscillator

\[ \Sigma_{av} |\psi_n(a)|^2 \rightarrow \frac{1}{\sqrt{a_M^2 - a^2}}. \quad (4.52) \]

So from the boundary conditions chosen it follows that the correspondence principle works properly in the conformal gauge \( N = a \) as noted in [16].

The extension of the discussion of the correspondence principle to the case of a universe with a cosmological term can be carried on along the lines shown in [16].

To conclude this section, let us explore briefly the case of gravity minimally coupled to a scalar field \( \phi \). Its action is

\[ I_{MCS} = \int d^4x \sqrt{g} \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \]

and the complete space density action in minisuperspace reduces to

\[ S = \int dt (p_a \dot{a} + p_\phi \dot{\phi} - l(t) (H_{MCS} - H_{GR}) \]

where

\[ H_{MCS} = \left( \frac{1}{2} \frac{p_\phi^2}{a^2} + a^4 V(\phi) \right). \]

Let us put, for simplicity, \( V(\phi) = 0 \) and consider a closed universe \( k = 1 \). Note that analogous results hold for the flat case \( k = 0 \). Using the linear gauge fixing

\[ p_a - 12a \sinh (t/\sqrt{3}) = 0, \]

we obtain the effective Hamiltonian

\[ H_{eff} = \pm p_\phi. \]
Note that we are essentially in the conformal gauge, in fact from (4.56) we have
\[ l = \frac{1}{2\sqrt{3} \cosh(t/\sqrt{3})}. \] (4.58)

Then the Schrödinger equation is
\[ \left( \frac{\partial}{\partial t} - \left(\pm\frac{\partial}{\partial \phi}\right) \right) \Psi = 0 \] (4.59)
which has the general solution
\[ \Psi = f(\phi \pm t) \] (4.60)

5. Conclusions.

The classical equations of motion for mini universes are endowed with a residual time reparametrization invariance. The ensuing constraint, upon quantization, becomes the WDW equation
\[ H_{WDW} \Psi_{WDW} = 0. \] (5.1)

Now this equation, fundamental as it is, contains well known ambiguities about which much has been written: absence of time, absence of conserved current, choice of boundary conditions, interpretation of the WDW wave function and normalization, and so on.

We have considered the classical Lagrangian for mini universes at its face value and have implemented the procedure of gauge fixing in the canonical scheme. Note that this procedure amounts to a definition of time in terms of
canonical coordinates. The choice of the gauge is in general a fine art and there is no a priori rule apart from the final simplicity of the effective Hamiltonian on gauge shell. This is actually the case also in gauge field theories.

Once the classical motion has been reduced to the gauge shell, the system can be quantized. Then one obtains the Schrödinger equation for a mini universe and the wave function has the usual properties of quantum mechanics: there is time, there is a conserved current and a positive density, thus the interpretation of the wave function is the usual one (we do not enter into the problem of the meaning of quantum mechanics as applied to the entire universe), and finally there is a $i \partial/\partial t$ in the equation.

Our treatment follows thus the lines of rational mechanics. In particular, we have not discussed here whether, in the case of a closed universe, time is bounded and about consequences.

Recently, possible ways to quantize the gravitational field by solving the contraints classically and then quantizing the reduced system have been discussed [25-29]. We believe this is the right direction.
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