Rotating cosmological cylindrical wormholes in GR and TEGR sourced by anisotropic fluids

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Given an anisotropic fluid source, we determine in closed forms, upon solving the field equations of general relativity (GR) and teleparallel gravity (TEGR) coupled to a cosmological constant, cylindrically symmetric four-dimensional cosmological rotating wormholes, satisfying all local energy conditions, and cosmological rotating solutions with two axes of symmetry at finite proper distance.

I. INTRODUCTION

In the teleparallel equivalent of general relativity (TEGR) \[1\] the vielbein vector fields \(e^a = e^a_\mu \partial_\mu \) are taken as fundamental variables instead of the metric \(g_{\mu \nu} \), related to each other by

\[
S_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu, \quad g^{\mu \nu} = \eta^{ab} e_a^\mu e_b^\nu, \quad e = \sqrt{|g|}, \tag{1}
\]

with \(\eta_{ab} = \text{diag}(+1, -1, -1, -1)\) being the metric of the 4-dimensional Minkowski spacetime and \(e \equiv |\det(e^a_\mu)| = \sqrt{|g|}\). In this work the tetrad indices, \(a, b, c \cdots, k, l\), and Greek coordinate indices run from 0 to 3.

The field equation in teleparallel gravity in the presence of matter fields take the form \[2\ 3\]

\[
I_\mu := -\delta^\mu_\nu \frac{f}{2} + 2\left[ e^{-1} e^a_\mu \partial_\rho \left( e e^a_\lambda S^\lambda_\nu \right) - T^\alpha_{\lambda \mu} S^\alpha_\nu \right] = -\kappa T_{(\text{mat})\mu}^\nu, \tag{2}
\]

where \(\kappa\) is the gravitational constant,

\[
f(T) = T + 2\Lambda, \tag{3}
\]

and \(T_{(\text{mat})\mu}^\nu\) is the matter stress-energy tensor (SET) which we assume to be that of an anisotropic fluid of the form

\[
T_{(\text{mat})\mu}^\nu = \rho u_\mu u^\nu + \sum_{i=1}^3 p_i e_i e^\nu, \tag{4}
\]

Here we have chosen \(e_0^\nu = u^\nu\) to be the four-velocity vector of the fluid. The remaining quantities used in \[2\], including the torsion \(T\), are defined by

\[
T^\alpha_{\mu \nu} = e^a_\nu \left( \partial_\mu e^b_\nu - \partial_\nu e^b_\mu \right),
\]

\[
K_{\lambda \mu \nu} = \frac{1}{2} \left( T_{\lambda \mu \nu} + T_{\lambda \nu \mu} - T_{\mu \lambda \nu} \right),
\]

\[
S^{\mu \nu} = \frac{1}{2} \left( K^{\mu \nu} - S^{\alpha \nu} T^\nu_\sigma + S^{\alpha \mu} T^\sigma_\nu \right),
\]

\[
T = T_{\lambda \mu \nu} S^{\lambda \mu \nu}. \tag{5}
\]

In the definition of \(T^\alpha_{\mu \nu}\) the connection \(\omega^a_{\mu \nu}\) has been taken identically zero as this is always possible in teleparallel gravity \[4\]. Some rotating cylinders in general relativity sourced by anisotropic fluids and \(\Lambda = 0\) have been determined in \[5\]. The purpose of this work is to consider the theories of general relativity (GR) and TEGR coupled to a cosmological constant and determine cylindrically symmetric rotating wormholes and other solutions sourced by anisotropic fluids \[6\]. In cylindrical coordinates \((x^0 = t, x^1 = r, x^2 = \phi, x^3 = z)\), we introduce the following non-diagonal vielbein to describe rotating solutions

\[
(e^a_\mu) = \begin{pmatrix} e^{\gamma(r)} & 0 & -e^{-\gamma(r)} & 0 \\ 0 & e^{\alpha(r)} & 0 & 0 \\ 0 & 0 & e^{\rho(r)} & 0 \\ 0 & 0 & 0 & e^{\phi(r)} \end{pmatrix}, \tag{6}
\]

resulting in the metric

\[
ds^2 = e^{2\gamma(r)} \left( dt - \Omega(r) e^{-2\gamma(r)} d\phi \right)^2 - e^{2\alpha(r)} dr^2 - e^{2\rho(r)} d\phi^2 - e^{2\phi(r)} dz^2. \tag{7}
\]

Using all that in \[5\] we obtain

\[
T = \frac{1}{2} e^{-2(\alpha + \beta + \gamma)} (\Omega' - 2\Omega \gamma')^2 + 2 e^{-2\Omega} (\beta' \gamma' + \beta' \mu' + \gamma' \mu'). \tag{8}
\]

\(S^{\mu \nu}\) may be given in a more compact form as:

\[
S^{\mu \nu} = \frac{1}{2} (T^{\mu \nu} + T^{\nu \mu} - T^{\mu \mu}) - \frac{1}{2} S^{\sigma \mu} T^\sigma_\nu + \frac{1}{2} S^{\sigma \nu} T^\sigma_\mu.
\]
The nonvanishing components of the SET are

\[ T^{(\text{mat})}_l = \rho, \quad T^{(\text{mat})}_r = -p_r, \quad T^{(\text{mat})}_\phi = -p_\phi, \quad T^{(\text{mat})}_z = -p_z, \quad T^{(\text{mat})}_\phi = -\Omega(\rho + p_\phi)e^{-2\gamma}, \] \] (9)

where we have set \( p_1 = p_r, p_2 = p_\phi, p_3 = p_z. \)

It has become customary to introduce the vortex \( \omega(r) \), which is the norm of the curl of the tetrad \( e_{\alpha}^{\mu} \) (see also \( 3, 7, 9 \)). This is related to \( \Omega(r) \) by

\[ \Omega(r) := 2e^{2\gamma(r)} \int_r^\infty e^{\beta(x) + \gamma(x)} \omega(x) dx, \] (10)

yielding

\[ T = 2\omega^2 + 2e^{-2\kappa}(\beta' \gamma' + \beta' \mu' + \gamma' \mu'). \] (11)

One may bring the field equations (2) to

\[ G_{\mu \nu} = -\kappa \tau_{\mu \nu}, \] (12)

where \( G_{\mu \nu} \) is the Einstein tensor and \( \tau_{\mu \nu} \) is the total SET including the cosmological constant and is defined by

\[ \tau_l^l = \rho - \frac{\Lambda}{\kappa}, \quad \tau_l^r = -\rho - \frac{\Lambda}{\kappa}, \]

\[ \tau_l^\phi = -p_\phi - \frac{\Lambda}{\kappa}, \quad \tau_l^z = -p_z - \frac{\Lambda}{\kappa}, \]

\[ \tau_l^\phi = -\Omega(\rho + p_\phi)e^{-2\gamma}. \] (13)

Since \( \nabla_\nu G_{\mu \nu} = 0 \), one must have

\[ \nabla_\nu \tau_{\mu \nu} = 0. \] (14)

The solutions we will derive in this work will satisfy the field equations of GR (12) and of TEGR (2). For TEGR, the solutions are constrained by (11).

Determining rotating and static solutions around an infinite axis is still a reviving topic. As we mentioned earlier, the purpose of this work is to construct rotating (as well as static) cylindrically symmetric solutions to the field equations of GR and TEGR sourced by anisotropic fluids coupled to a cosmological constant. In GR there is a set of rotating cylindrically symmetric perfect fluid solutions which may be appropriate as matched interiors. Among the known solutions in GR we find the rotating dust of Vishveshwara and Winicour (10), the perfect fluid sources with non-zero pressure of da Silva et al. (11), Davidson (12, 13) and Ivanov (14), and the family of Krasiński (15). The solutions that will be constructed in this work have share some physical and geometrical properties with the solutions known in the literature and have some other new properties.

In Sec. II we reduce the field equations. In Sec. III we restrict ourselves to anisotropic fluids with \( p_r = \omega_1 \rho, p_\phi = \omega_\phi \rho \) and \( p_z = \omega_2 \rho \), where the equation-of-state (EoS) parameters \( (\omega_r, \omega_\phi, \omega_z) \) are constants constrained by \(-1 \leq \omega_r \leq -1, -1 \leq \omega_\phi \leq 1\), and \(-1 \leq \omega_z \leq 1\). Section IV is devoted to the construction of rotating wormholes and Sec. V is devoted to the discussion of their physical and geometrical properties. In Sec. VI we provide cosmological rotating solutions with two axes of symmetry at finite proper distance. Our final conclusions are given in Sec. VII.

II. Reducing the Field Equations

Given that \( T^{(\text{mat})}_l = 0 \) and \( \tau_l^\phi = 0 \), the line \( l^\phi = 0 \) in (2), and the component \( l^\phi \) of Eq. (12), reduce upon using (9) and (13) to

\[ l^\phi = G_{\phi r} = -e^{2\gamma - \beta}[\omega(2\gamma' + \mu') + \omega'] = 0, \]

yielding

\[ \omega(r) = \omega_0 e^{-2\gamma - \mu}, \] (15)

where \( \omega_0 \) is a constant of integration. Using this in (10) we obtain

\[ \Omega(r) = 2\omega_0 e^{2\gamma(r)} \int_r^\infty e^{\beta(x) + \gamma(x)} - \mu(x) dx. \] (16)

The expression of \( G_{\phi r} \) is just half that of \( T \) (11), \( G_{\phi r} = \omega^2 + e^{-2\kappa}(\beta' \gamma' + \beta' \mu' + \gamma' \mu') = T/2, \) and this implies using (12) and (15) that \( p_r \) is given by

\[ p_r = \frac{T}{2\kappa} - \frac{\Lambda}{\kappa}. \] (17)

where the last term proportional to \( \Lambda \) is the radial pressure due to the cosmological constant and the first term, \( T/(2\kappa) \), is the radial pressure generated by a constant torsion. Thus, in TEGR, a ‘nonvanishing’ torsion generates a nonvanishing radial pressure \( T/(2\kappa) \) while in GR the relation (17) is written as \( G_{\phi r} = \kappa p_r + \Lambda \) and it just expresses the fact that the geometric entity \( G_{\phi r} \) is proportional to the sum of the pressures \( p_r \) and \( \Lambda/\kappa \).

Substituting (15) into (14) we obtain

\[ -4p_r - (\beta' + \gamma' + \mu') p_r - \gamma' \rho + \beta' p_\phi + \mu' p_z = 0. \] (18)

III. Anisotropic Fluids

We consider the simple case where \( p_r = \omega_1 \rho, p_\phi = \omega_\phi \rho \) and \( p_z = \omega_2 \rho \), with the EoS parameters

\[ R^a_{\beta \mu \nu} := \partial_\mu \Gamma^a_{\beta \nu} - \partial_\nu \Gamma^a_{\beta \mu} + \cdots \]
(ωr, ωϕ, ωz) being constants generally constrained by

\(-1 \leq \omega_r, \omega_\phi, \omega_z \leq 1, -1 \leq \omega_\phi \leq 1, -1 \leq \omega_z \leq 1 \).

(19)

The differential equation (18) becomes

\((\omega_r - \omega_\phi)\beta' + (1 + \omega_r)\gamma' + (\omega_r - \omega_z)\mu' + 4\omega_r\frac{\rho'}{\rho} = 0, \)

resulting in

\[\rho = \rho_0 \exp \left[\frac{(\omega_\phi - \omega_r)(1 + \omega_r)\gamma + (\omega_r - \omega_z)\mu}{4\omega_r}\right], \]

where \(\rho_0\) is a constant of integration. Using this in (17) we can evaluate the torsion from

\[T = 2(\kappa_\omega_1 + \Lambda). \]

(22)

Since the radial coordinate \(r\) can be changed at will by a coordinate transformation \(r \to \bar{r}\), from now on we fix the coordinate gauge to be

\[\alpha = \beta + \gamma + \mu. \]

(23)

In this gauge the independent field equations emanating from (12)

\[R_{\mu \nu} = -\kappa \left[\tau_{\mu \nu} - (1/2) \delta_{\mu \nu} \tau_c \right], \]

reduce to

\[2e^{-2\alpha} \gamma'' + 4\omega^2 = 2\Lambda + \kappa(1 + \omega_r + \omega_\phi + \omega_z)\rho, \]

(25)

\[2e^{-2\alpha} \mu'' = 2\Lambda + \kappa(-1 + \omega_r + \omega_\phi - \omega_z)\rho, \]

(26)

\[2e^{-2\alpha} \beta'' - 4\omega^2 = 2\Lambda + \kappa(-1 + \omega_r - \omega_\phi + \omega_z)\rho, \]

(27)

which are the equations \(R_{\mu \nu} = \cdots, R_{\phi \phi} = \cdots, \) respectively. The \(\phi\) equation (24) is a combination of (25) and (27) and the \(r\) equation (24) is a combination of (25), (26), (27) and \(G_{rr} = \omega^2 + e^{-2\alpha}(\beta'\gamma' + \beta'\mu' + \gamma'\mu')\) [recall that \(G_{rr} = T/2\)].

IV. COSMOLOGICAL ROTATING WORMHOLES

From now on, we assume \(\omega_r \neq 0\) to ensure that (21) remains valid. We first construct cosmological rotating cylindrical wormholes to GR. The counterpart solutions to TEGR are the same with the torsion given by (24).

We look for solutions with a constant shift function, that is, \(\gamma = 0, \mu = 0\). In this case \(\alpha = \beta\) (23) and Eq. (26) implies that \(\rho\) is a constant given by \(\rho_0 = 2\Lambda/\kappa(1 - \omega_r - \omega_\phi + \omega_z)\). Since \(\rho\) is constant, Eq. (21) implies

\[\omega_{\phi} = \omega_r, \]

(28)

and finally

\[\rho_0 = \frac{2\Lambda}{\kappa(1 - 2\omega_r + \omega_z)}. \]

(29)

Equations (15) and (25) imply that \(\omega^2\) is also a constant given by

\[\omega_0^2 = \frac{\Lambda(1 + \omega_z)}{1 - 2\omega_r + \omega_z} = \frac{\kappa(1 + \omega_z)}{2} \rho_0. \]

(30)

Using these values of \((\rho_0, \omega_0^2)\) in (27) we bring to the form (recall \(\alpha = \beta\))

\[e^{-2\alpha} \gamma'' = q^2 \equiv \frac{2\Lambda(1 - \omega_r + 2\omega_z)}{1 - 2\omega_r + \omega_z}. \]

(31)

This equations can be integrated in all three cases \(q^2 < 0, q^2 = 0\) and \(q^2 > 0\). In the case \(q^2 < 0\) we obtain cosmological rotating solutions with two axes of symmetry at finite proper distance (see Sec. [VI]). The case \(q^2 > 0\) (we may assume \(q > 0\)) yields cosmological rotating wormholes where the solution \(e^{2\alpha}\) is brought to the form

\[e^{2\alpha} = e^{2\beta} = \frac{c^2 \sec^2(cr)}{q^2}, \]

(32)

with \(c > 0\) being a constant of integration. As to \(\Omega\) is obtained from (16)

\[
\Omega(r) = 2\omega_0 \int_c^r e^{2\alpha(x)} dx = \frac{2c\omega_0}{q^2} \tan(cr),
\]

(33)

where we have dropped an additive constant of integration. The metric takes the form

\[
ds^2 = \left[dt - \frac{2c\omega_0 \tan(cr)}{q^2} d\phi\right]^2 - \frac{c^2 \sec^2(cr)}{q^2} dr^2 - \frac{c^2 \sec^2(cr)}{q^2} d\phi^2 - dz^2.
\]

(34)

We see that the spherical radius \(e^\beta = c \sec(cr)/q\) has a minimum value at \(r = 0\) and increases as \(|r|\) increases. On introducing the new radial coordinate \(u = c \tan(cr)/q\) and the new constant \(u_0^2 = c^2/q^2\), we bring the metric (34) to the manifestly wormhole form

\[
ds^2 = \left[dt - \frac{2c\omega_0 u}{q} d\phi\right]^2 - \frac{1}{q^2(u^2 + u_0^2)} du^2 - (u^2 + u_0^2) d\phi^2 - dz^2.
\]

(35)

A very similar solution describing a rotating wormhole, which is a solution to the Einstein–Maxwell equations, was determined in [16].

The TEGR wormhole solution is also given by (34) and (35) with \(T = 2\omega_0^2 > 0\) (22), which is a positive constant.

The metric component

\[g_{\phi\phi} = -\left(q^2 - 4\omega_0^2 u^2 + \frac{c^2}{q^2}\right), \]

(36)

is manifestly negative if \(q^2 - 4\omega_0^2 = 2\Lambda(1 + \omega_r)/(1 - 2\omega_r + \omega_z) \geq 0\) signaling the absence of closed timelike
curves (CTCs) \([17]\). For \(q^2 - 4\omega_0^2 < 0\), CTCs occur at large values of \(|u|\).

Let us see under which conditions the constraints \(q^2 > 0\), \(\omega_0^2 \geq 0\) and \(\rho_0 > 0\) are satisfied simultaneously. The gravitational constant \(\kappa\) being positive, we obtain the following constraints on the EoS parameters \((\omega_r, \omega_\phi, \omega_z)\) assuming they obey the general inequalities \([19]\). If \(\Lambda < 0\) we obtain

\[
\frac{1}{3} < \omega_r \leq 1 \quad \text{and} \quad \frac{\omega_r - 1}{2} < \omega_z < 2\omega_r - 1. \tag{37}
\]

If \(\Lambda > 0\) we obtain

\[
-1 \leq \omega_r \leq \frac{1}{3} \quad \text{and} \quad \frac{\omega_r - 1}{2} < \omega_z \leq 1, \quad \text{or} \tag{38}
\]

\[
\frac{1}{3} < \omega_r < 1 \quad \text{and} \quad 2\omega_r - 1 < \omega_z \leq 1. \tag{39}
\]

If, however, we want to avoid the presence of CTCs, we have to impose the fourth constraint \(q^2 - 4\omega_0^2 \geq 0\). This results in

\[
\Lambda > 0, \quad \omega_r = -1 \quad \text{and} \quad -1 < \omega_z \leq 1. \tag{40}
\]

This shows that, for a positive cosmological constant, it is always possible to have a rotating wormhole with a positive energy density and no CTCs. The anisotropic fluid is isotropic in a plane perpendicular to the axis of rotation with \(\omega_\phi = \omega_r = -1\).

V. PHYSICAL AND GEOMETRICAL PROPERTIES OF THE ROTATING WORMHOLE SOLUTION

Since the wormhole solutions (34) and (35) are not asymptotically flat, they are appropriate as matched interiors. Another reason why they are so is that the speed of rotation, \(\Omega = 2\omega_0 u/q \tag{33}\), increases linearly as one moves away from the axis of rotation at \(u = 0\). Consequently the linear speed of each fluid particle increases uncasingly as one moves away from the axis of rotation. Following the work done in \([16]\) one can match these wormholes to external rotating flat Minkowskian metrics in such a way that the radii of the cylindrical junction surfaces are chosen so that the interior wormhole region is exempt from CTCs.

Thus, the constraints (37)-(39), for \(\Lambda < 0\) and \(\Lambda > 0\), are fairly enough to obtain a physically acceptable wormhole solutions that certainly do not violate the weak energy condition \((\rho_0 \geq 0, \rho_0 + \rho_r \geq 0, \rho_0 + \rho_z \geq 0)\) and the dominant energy condition \((\rho_0 \geq 0, -\rho_0 \leq \rho_r \leq \rho_0, -\rho_0 \leq \rho_z \leq \rho_0)\). To satisfy the requirements of the strong energy condition \((\rho_0 \geq 0, \rho_0 + \rho_r \geq 0, \rho_0 + \rho_z \geq 0, \rho_0 + 2\rho_r + \rho_z \geq 0)\), the EoS parameters must obey the constraints (37) if \(\Lambda < 0\) and the following constraints if \(\Lambda > 0\):

\[
\omega_r = -1 \quad \text{and} \quad \omega_z = -1, \quad \text{or} \tag{41}
\]

\[
-1 < \omega_r < -\frac{1}{5} \quad \text{and} \quad -2\omega_r - 1 \leq \omega_z \leq 1, \quad \text{or} \tag{42}
\]

\[
-\frac{1}{5} \leq \omega_r \leq \frac{1}{3} \quad \text{and} \quad \omega_r - 1 < \omega_z \leq 1, \quad \text{or} \tag{43}
\]

\[
\frac{1}{3} < \omega_r < 1 \quad \text{and} \quad 2\omega_r - 1 < \omega_z \leq 1. \quad \text{or} \tag{44}
\]

Considered as interiors all the matter should be distributed inside cylindrical surfaces \(\Sigma_+\) and \(\Sigma_\omega\) of finite radii \(u_+ > u_0\) and \(u_- < -u_0\). Referring to \([3, 18]\), the effective mass \(m_+\) and angular momentum \(j_+\) per unit z-coordinate length of matter enclosed by \(\Sigma_+\) are defined by (there are similar definitions for \(m_-\) and \(j_-\) concerning the matter distribution enclosed by \(\Sigma_-\)):

\[
m_+ = 2\pi \int_0^{u_+} \tau_r^\mu \rho_\mu \sqrt{|g|} \, du, \tag{45}
\]

\[
j_+ = -2\pi \int_0^{u_+} \tau_\phi^\mu \rho_\mu \sqrt{|g|} \, du, \tag{46}
\]

where \(\tau_\mu^\nu\) is the total SET given in \([13]\) and \(g\) is the determinant of the metric. The vector \(n_\mu\) is the unit normal to the spacelike surface of integration, which is the hypersurface \(t = \text{const.}\) yielding \(n_\mu = \delta_\mu^t / \sqrt{g^{tt}}\) with \(g^{tt}\) being the component \(tt\) of the inverse metric. Using (35) along with \(g = -1/q^2\) we find

\[
m_+ = \frac{2\pi u_0}{q} (\rho_0 - \frac{\Lambda}{\kappa}) \int_{x_+}^{x_0} \frac{x^2 + 1}{\delta x^2 + 1} \, dx, \tag{47}
\]

\[
j_+ = \frac{4\pi u_0^2 \rho_0 (1 + \omega_r)}{q^2} \int_{x_+}^{x_0} \frac{x^2 + 1}{\delta x^2 + 1} \, dx, \tag{48}
\]

where \(p_\phi = p_r = \omega_r \rho_0\) has been used, and \(x \equiv u/u_0\), \(x_+ \equiv u_+ / u_0\) and \(\delta \equiv (q^2 - 4\omega_0^2) / q^2\). These integrals can be expressed in terms of the elliptic integral of the second kind and the log function for a wide range of parameters. It is obvious that \(j_+\) has the sign of \(\omega_0\) and is zero if the static case (no rotation) and in the extreme case \(\omega_r = -1\). The cosmological constant being small, we expect that in physically interesting situations to have \(\rho_0 > |\Lambda|/\kappa\) and thus \(m_+ > 0\).

The constants of integration \((\omega_0, u_0, q^2)\) and \(\rho_0 \propto \omega_0^2 \tag{30}\) are theoretically expressed in terms of \((m_+, j_+, x_+)\) once the integrals (47) and (48) are performed.

The curvature scalar, \(\mathcal{R}\), and the Kretschmann scalar, \(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\), assume the following expressions:

\[
\mathcal{R} = 2(\omega_0^2 - q^2), \tag{49}
\]

\[
R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = 4(q^4 - 6\omega_0^2 q^2 + 11\omega_0^4), \tag{50}
\]

which are finite constants.
VI. COSMOLOGICAL ROTATING SOLUTIONS WITH TWO AXES OF SYMMETRY AT FINITE PROPER DISTANCE

If the constant \( q^2 \) in (31) is negative we set \( Q^2 = -q^2 \) in (31) and the solution yields

\[
e^{2\alpha} = e^{2\beta} = \frac{c^2 \text{sech}^2(cr)}{Q^2}, \quad \Omega = \frac{2c\omega_0}{Q^2} \tanh(cr), \tag{51}
\]

where we have dropped an additive constant \( \Omega_0 \) in the expression of \( \Omega \). On setting \( u = c \tanh(cr)/Q \) and \( u_0^2 = c^2/Q^2 \), we bring the metric to the form

\[
ds^2 = \left[ dt - \frac{2c\omega_0 \tanh(cr)}{Q^2} \, d\phi \right]^2 - \frac{c^2 \text{sech}^2(cr)}{Q^2} \, d\phi^2 - dz^2,
\]

\[
ds^2 = \left( dt - \frac{2\omega_0 u}{Q} \, d\phi \right)^2 - \frac{1}{Q^2(u_0^2 - u^2)} \, du^2
- (u_0^2 - u^2) \, d\phi^2 - dz^2. \tag{52}
\]

Note that this solution shares with the wormhole solution (34)-(35) the same values of the physical quantities \((\rho_0, \omega_0^2)\), given in (29) and (30). The corresponding TEGR solution is also given by (52) and shares the same value of the torsion \( T = 2\omega_0^2 \) with the wormhole solution (34)-(35).

The circular radius \( e^\beta = \sqrt{u_0^2 - u^2} \) vanishes at \( u = \pm u_0 \) signaling the presence of two axes of symmetry. In the vicinity of these two axes the solution \( (52)-(53) \) has CTCs where \( g_{\phi\phi} = -(u_0^2 - u^2 - 4\omega_0^2 u^2/Q^2) \) becomes positive (the nonrotating solution, \( \omega_0 = 0 \), has no CTCs). The integral

\[
\int_{-u_0}^{u_0} \frac{1}{Q\sqrt{u_0^2 - u^2}} \, du = \frac{\pi}{Q'},
\]

being convergent, the two axes are at finite proper distance from each other and there is no spatial infinity. The absence of spatial infinity is also known for the nonrotating Melvin solution [17, 13, 20], however, for the latter the proper distance of the two axes of symmetry diverges (\( B \) is the magnetic field). Such nontrivial behaviors of the intrinsic geometry are familiar with static and rotating, cylindrically symmetric and/or axially symmetric, metrics and more examples are provided in [17].

VII. CONCLUSION

As we mentioned in the Introduction, the determination of rotating and static solutions around an infinite axis is still attracting much attention. In this work, we presented a first set of two cosmological (energy density constant) rotating solutions in GR and TEGR gravity sourced by anisotropic fluids (isotropic in a plane perpendicular to the axis of symmetry) and extended the existing list of solutions pertaining to GR.

We have shown that the EoS parameters obey a large set of values ensuring the satisfaction of all local energy conditions for the rotating wormholes. Such solutions can straightforwardly be matched to exterior rotating Mikowskian metrics.

The other cosmological rotating solution has two axes of symmetry at finite proper distance where one axis can be regularized upon appropriately fixing the value of the additive constant of integration in the expression of \( \Omega \) (which we have dropped) at the expense of rendering the time coordinate periodic.

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