Bottomonium Spectrum with Coupled-Channel Effects

Jia-Feng Liu$^1$ and Gui-Jun Ding$^{1,2}$

$^1$Department of Modern Physics,
University of Science and Technology of China, Hefei, Anhui 230026, China

$^2$Department of Physics, University of Wisconsin-Madison,
1150 University Avenue, Madison, WI 53706, USA

Abstract

We study the bottomonium spectrum in the nonrelativistic quark model with the coupled-channel effects. The mass shifts and valence $b\bar{b}$ component are evaluated to be rather large. We find that the hadronic loop effects can be partially absorbed into a reselection of the model parameters. No bottomonium state except $\Upsilon(5^1S_0)$ and $\Upsilon(5^3S_1)$ with mass around 10890 MeV is found in the quark models both with and without coupled-channel effects, so we suggest that $\Upsilon_b(10890)$ is an exotic state beyond the quark model, if it is confirmed to be a new resonance. The predictions for the $\chi_b(3P)$ masses are consistent with the ATLAS measurements. If some new bottomonium-like states are observed at LHCb or SuperB in the future, we can determine whether they are conventional bottomonium or exotic states by comparing their masses with the mass spectrum predicted in our work.

PACS numbers: 12.39.Jh, 12.40.Yx, 14.40.Pq, 14.40.Rt
I. INTRODUCTION

In past years, the spectroscopy of heavy flavor quarkonium has seen great progress, particularly the charmonium spectrum. Many charmonium-like states (such as $X(3872)$, $Y(4260)$ and so on) with remarkable and unexpected properties have been reported. These exotic states present great challenges to our understanding of the structure of heavy flavor quarkonium and quantum chromodynamics (QCD) at low energy, for a review, see Refs. [1–4]. On the other hand, many bottomonium states have been reported as well. In 2008, the spin-singlet pseudoscalar partner $\eta_b(1S)$ was found by the Babar Collaboration with mass $M = 9388.9^{+3.1}_{-2.3} \text{(stat)} \pm 2.7 \text{(syst)}$ MeV [5]. The $\Upsilon(3D_J)$ was discovered in 2010 in the $\pi^+\pi^-\Upsilon(1S)$ final state with mass $M = 10164.5 \pm 0.8 \text{(stat)} \pm 0.5 \text{(syst)}$ MeV [6]. In addition, the Babar Collaboration reported the P-wave spin-singlet $h_b(1P)$ via its radiative decay into $\gamma\eta_b(1S)$ with mass $M = 9902 \pm 4 \text{(stat)} \pm 1 \text{(syst)}$ MeV [7]. This state is confirmed by the Belle Collaboration [8], and its mass is measured to be $M = 9898.25 \pm 1.06 \text{(stat)}^{+1.03}_{-1.07} \text{(syst)}$ MeV. Meanwhile, the radial excitation state $h_b(2P)$ was also found by the Belle Collaboration with mass $M = 10259.76 \pm 0.64 \text{(stat)}^{+1.43}_{-1.03} \text{(syst)}$ MeV [8]. Recently the ATLAS Collaboration has reported the discovery of the $\chi_b(3P)$ state through reconstruction of the radiative decay modes of $\chi_b(3P) \rightarrow \Upsilon(nS,2S)\gamma$, and its mass barycenter is measured to be $10539 \pm 0.004 \text{(stat.)} \pm 0.008 \text{(syst.)}$ GeV [9]. In particular, the Belle Collaboration has observed an enhancement in the production process $e^+e^- \rightarrow \Upsilon(nS)\pi^+\pi^-(n = 1, 2, 3)$ [10]. The fit using a Breit-Wigner resonance shape yields a peak mass of $[10888.4^{+2.7}_{-2.6} \text{(stat)} \pm 1.2 \text{(syst)}]$ MeV and a width of $[30.7^{+8.3}_{-7.8} \text{(stat)} \pm 3.1 \text{(syst)}]$ MeV. In the following, we shall denote this state as $Y_b(10890)$. Moreover, the Babar Collaboration measured the $e^+e^- \rightarrow b\bar{b}$ cross section between 10.54 GeV and 11.20 GeV [11], the $\Upsilon(10860)$ and the $\Upsilon(11020)$ states, which are candidates of $\Upsilon(5S)$ and $\Upsilon(6S)$ respectively, were observed. Their masses and widths are fitted to be $M_{\Upsilon(10860)} = 10.876 \pm 0.002$ GeV, $\Gamma_{\Upsilon(10860)} = 43 \pm 4$ MeV, $M_{\Upsilon(11020)} = 10.996 \pm 0.002$ GeV and $\Gamma_{\Upsilon(11020)} = 37 \pm 3$ MeV, which are different from the previously measured values. In particular, two charged narrow structures at 10610 MeV and 10650 MeV in the $\pi^\pm \Upsilon(nS)(n = 1, 2, 3)$ and $\pi^\pm h_b(mP)(m = 1, 2)$ have been reported recently [12]. In summary, the current experimental data indicate that there may be exotic bottomonium-like structures similar to the charmonium sector. Furthermore, LHCb has begun to run [13], Belle will be updated to Belle II and a new SuperB factory will be built in Italy [14], we
expect that more heavy bottomonium states including the possible exotic extensions will be observed in the future.

Motivated by the above exciting experimental progress in $b\bar{b}$ states, we shall carry out a careful, detailed study of bottomonium spectroscopy in this work, notably the poorly understood higher-mass $b\bar{b}$ levels. Thus, we can determine whether future observed bottomonium-like states could be accommodated as canonical $b\bar{b}$ states by comparing their masses with the mass spectrum predicted in this work. It is well-known that simple potential models, which incorporate a color coulomb term at short distances, a linear scalar confining term at large distances, and a Gaussian-smeared one-gluon exchange spin-spin hyperfine interactions, have been frequently used to describe both the charmonium and bottomonium spectrums. Generally, the mixture between the quark model $b\bar{b}$ basis states and the two-meson continuum has been neglected in these models, which are called “quenched” quark models.

The effects of the “unquenched quark model” including virtual hadronic loops have been studied extensively in the framework of the coupled-channel method \cite{15-19}. The hadronic loop has turned out to be highly non-trivial, it can give rise to mass shifts to the bare hadron states and contribute continuum components to the physical hadron states. The possibility that loop effects may be responsible for the anomalously low masses of the new narrow charm-strange states $D^*_{s0}(2317)$ and $D_{s1}(2460)$ has been suggested by several groups \cite{20-23}. The hadronic loop in charmonium has been explored as well, and the mass shifts and continuum mixing due to loops of $D$, $D^*$, $D_s$ and $D^*_s$ meson pairs have been studied extensively \cite{24-31}. Both the mass shifts and the two-meson continuum components of the physical charmonium states were found to be rather large. In particular, a $J^{PC} = 1^{++}$ state with mass about 3872 MeV could possibly be generated dynamically.

Inspired by the large physical effects of hadronic loops in both the $D_{sJ}$ and charmonium states, we expect that the virtual hadronic loop should also play an important role in bottomonium spectroscopy. In this work, we shall study the bottomonium spectrum in detail, and take the hadronic loop effects into account. This paper is organized as follows. We present the framework of the coupled-channel analysis in section II. The non-relativistic potential model is outlined in section III. Section IV is devoted to the numerical results for the masses of the bottomonium states with and without hadronic loop effects, as well as phenomenological implications. We present our conclusions and discussion in section V.
II. FORMALISM OF COUPLED-CHANNEL ANALYSIS

![Coupled-channel diagram](image)

**FIG. 1:** Coupling of $b\bar{b}$ states to the $B\bar{B}$ mesons loop.

In bottomonium, the process $(b\bar{b}) \rightarrow (b\bar{n})(n\bar{b})$ via light quark pair $n\bar{n}$ creation would induce the hadronic loop shown in Fig. 1 where the initial bottomonium decays into intermediate virtual $B\bar{B}$ states and then reforms the original bottomonium state. Here $B(\bar{B})$ denotes a general $B(\bar{B})$ meson, it can be $B(\bar{B})$, $B_s(\bar{B}_s)$, $B^*(\bar{B}^*)$ or $B^*_s(\bar{B}^*_s)$, the same convention will be used henceforth without specification. Since the open-flavor decay couplings of bottomonium states to two-body $B\bar{B}$ final states are large, the resulting loop effects should be important. This kind of virtual hadronic loop is universal, but it is not usually included in quark potential models and is only partially present in the quenched lattice QCD. The coupled-channel model is an appropriate framework for analyzing these hadronic loop effects \[15 \text{-} 19\]. In the simplest version of the coupled-channel model \[25\], the full hadronic state is represented as

$$|\Psi\rangle = \left( \frac{\sum_{\alpha} c_\alpha |\psi_\alpha\rangle}{\sum_{BB} \chi_{BB}(\textbf{p}) |\phi_B\phi_{\bar{B}}\rangle} \right),$$

(1)

with the normalization condition $\sum_{\alpha} |c_\alpha|^2 + \sum_{BB} \int d^3p |\chi_{BB}(\textbf{p})|^2 = 1$. $|\psi_\alpha\rangle$ denotes the bare confined $bb$ states with the probability amplitude $c_\alpha$, $\phi_B(\phi_{\bar{B}})$ is the $b\bar{n}(b\bar{n})$ eigenstate describing the $B(\bar{B})$ meson, and $\chi_{BB}(\textbf{p})$ is the wavefunction in the two-meson channel $|\phi_B\phi_{\bar{B}}\rangle$. The wavefunction $|\Psi\rangle$ obeys the equation

$$\mathcal{H}|\Psi\rangle = M|\Psi\rangle,$$

(2)

where $H_0$ is the Hamiltonian for the valence $b\bar{b}$ system, with the eigenstates determined by $H_0|\psi_\alpha\rangle = M_\alpha|\psi_\alpha\rangle$. The Hamiltonian $H_{BB}$ acts between the constituents of $B$ and $\bar{B}$.
separately, where the interactions between $B$ and $\bar{B}$ are neglected. The continuum two-meson state $|\phi_B\phi_{\bar{B}}\rangle$ is the eigenstate of $H_{BB}$,

$$H_{BB}|\phi_B\phi_{\bar{B}}\rangle = (E_B + E_{\bar{B}})|\phi_B\phi_{\bar{B}}\rangle \simeq \left(m_B + m_{\bar{B}} + \frac{p^2}{2\mu_{BB}}\right)|\phi_{M_1}\phi_{M_2}\rangle \quad (3)$$

where $E_B = \sqrt{m_B^2 + p^2}$, $E_{\bar{B}} = \sqrt{m_{\bar{B}}^2 + p^2}$, $m_B$ and $m_{\bar{B}}$ are the masses of $B$ and $\bar{B}$ respectively, and $\mu_{BB} = \frac{m_B m_{\bar{B}}}{m_B + m_{\bar{B}}}$ is the reduced mass of the two-meson system. $H_I$ couples the bare state $|\psi_0\rangle$ with the two-body continuum $|\phi_B\phi_{\bar{B}}\rangle$. Let us consider one bare state $|\psi_0\rangle$, the matrix element of $H_I$ is of the following form:

$$\langle \phi_B\phi_{\bar{B}} | H_I | \psi_0 \rangle = h_0^B\bar{B}(p) \quad (4)$$

Substituting Eq.(3) and Eq.(4) into Eq.(2), we get the system of coupled equations for $c_0$ and $\chi_{BB}(p)$,

$$\begin{cases}
  c_0 M_0 + \sum_{BB} h_0^{BB}(p) \chi_{BB}(p) d^3 p = M c_0 \\
  (E_B + E_{\bar{B}}) \chi_{BB}(p) + c_0 h_0^{BB}(p) = M \chi_{BB}(p)
\end{cases} \quad (5)$$

This coupled-channel equation can be solved straightforwardly, and we finally obtain the master equation

$$M - M_0 + \sum_{BB} \Pi_{BB}(M) = 0. \quad (6)$$

Here $\Pi_{BB}(M)$ is the self-energy function for the hadronic loop induced by the intermediate states $B$ and $\bar{B}$, it is explicitly given by

$$\Pi_{BB}(M) = \int \frac{|h_0^{BB}(p)|^2}{E_B + E_{\bar{B}} - M - i\epsilon} d^3 p \quad (7)$$

Using the relation between the helicity amplitude $h_0^{BB}$ and the partial wave amplitude $M_{LS}$ [32], we have

$$\begin{align*}
\Pi_{BB}(M) &= \int_0^\infty dp \frac{p^2}{E_B + E_{\bar{B}} - M - i\epsilon} \int d\Omega_p |h_0^{BB}(p)|^2 \\
&= \int_0^\infty dp \frac{p^2}{E_B + E_{\bar{B}} - M - i\epsilon} \sum_{LS} |M_{LS}|^2 \\
&= \mathcal{P} \int_0^\infty dp \frac{p^2}{E_B + E_{\bar{B}} - M} \sum_{LS} |M_{LS}|^2 + i\pi \left(\frac{pE_B E_{\bar{B}}}{M} \sum_{LS} |M_{LS}|^2\right) |_{E_B + E_{\bar{B}} = M} \quad (8)
\end{align*}$$

Note that the imaginary part of the self-energy $\Pi_{BB}(M)$ arises only if the initial hadron mass is above the intermediate $BB$ threshold. Comparing with the two-body strong decay
width shown in Eq. (A.5), it is obvious that the imaginary part is exactly equal to half of the decay width, if the decay is not forbidden kinematically. For the bottomonium state above the threshold, its mass is determined by the real part of the master equation Eq. (6) [26, 27, 30]. The squared absolute value $|c_0|^2$ is proportional to the probability that the physical energy eigenstate is in the $b\bar{b}$ configuration, and the $b\bar{b}$ component is given by

$$P_{bb} = \frac{1}{\left(1 + \sum_{BB} \int \frac{|h_{BB}(p)|^2}{(E_B + E_{\bar{B}} - M)^2} d^3p\right)}$$ (9)

In this work the effects of virtual hadronic loops will be considered in the above framework, and we shall sum over the contributions of intermediate loops from two stable S-wave mesons $B\bar{B}$, $B\bar{B}^*$, $B^*\bar{B}$, $B^*\bar{B}^*$, $B_s\bar{B}_s$, $B_s\bar{B}_s^*$, $B_s^*\bar{B}_s$ and $B_s^*\bar{B}_s^*$, where antiparticles are indicated explicitly.

III. NON-RELATIVISTIC POTENTIAL MODEL

We use the standard non-relativistic potential model to describe the bare valence $b\bar{b}$ states. Its Hamiltonian is of the form

$$H_0 = \frac{\mathbf{p}^2}{m_b} + V_{\text{cou}}(r) + V_{\text{con}}(r) + C + V_{\text{sd}}(r)$$

$$V_{\text{cou}}(r) = -\frac{4}{3} \frac{\alpha_s}{r}, \quad V_{\text{con}}(r) = \sigma r,$$ (10)

where $m_b$ is the bottom quark mass, $V_{\text{cou}}(r)$ is the well-known color-Coulomb force, and $V_{\text{con}}(r)$ denotes the linear confinement potential. To restore the hyperfine and fine structure of the bottom spectrum, one needs to introduce the Fermi-Breit relativistic corrections term $V_{\text{sd}}(r)$, which includes spin-spin, spin-orbit and tensor force. This is explicitly given by

$$V_{\text{sd}}(r) = V_{SS}(r) + V_{LS}(r) + V_{T}(r),$$ (11)

where $V_{SS}(r)$, $V_{LS}(r)$ and $V_{T}(r)$ are spin-spin, spin-orbit and tensor operators respectively, and $V_{SS}$ is the contact hyperfine interaction,

$$V_{SS}(r) = \frac{2S_b \cdot S_{\bar{b}}}{3m_b^2} [\Delta(V_{\text{cou}}(r))] = \frac{32\pi \alpha_s}{9m_b^2} \bar{\delta}(r) \mathbf{S}_b \cdot \mathbf{S}_{\bar{b}},$$ (12)

where $\mathbf{S}_b$ is the spin of the bottom quark and $S_{\bar{b}}$ is the spin of the anti-bottom quark. The Gaussian smearing of the hyperfine interaction is introduced here,

$$\bar{\delta}(r) = \left(\frac{\kappa}{\sqrt{\pi}}\right)^3 e^{-\kappa^2 r^2}$$ (13)
The spin-orbit term is given by

$$V_{LS}(r) = \left[ 3 \frac{d}{dr} V_{\text{cou}}(r) - \frac{d}{dr} V_{\text{con}}(r) \right] \frac{\mathbf{L} \cdot \mathbf{S}}{2m_b^2 r} = \left( \frac{4\alpha_s}{r^3} - \frac{\sigma}{r} \right) \frac{\mathbf{L} \cdot \mathbf{S}}{2m_b^2},$$

(14)

where \( \mathbf{S} = \mathbf{S}_b + \mathbf{S}_{\bar{b}} \) is the total spin, and \( \mathbf{L} \) is the relative angular momentum between \( b \) and \( \bar{b} \). Finally, the tensor term is

$$V_T(r) = \frac{T_{b\bar{b}}}{m_b^3} \left[ \frac{1}{r} \frac{d}{dr} V_{\text{cou}}(r) - \frac{d^2}{dr^2} V_{\text{cou}}(r) \right] = \frac{4\alpha_s}{m_b^2 r^3} T_{b\bar{b}}$$

(15)

where \( T_{b\bar{b}} \) is the well-known tensor force operator,

$$T_{b\bar{b}} = 3 (\mathbf{S}_b \cdot \hat{\mathbf{r}})(\mathbf{S}_{\bar{b}} \cdot \hat{\mathbf{r}}) - \mathbf{S}_b \cdot \mathbf{S}_{\bar{b}}$$

(16)

The spin-dependent term \( V_{sd} \) is relativistically suppressed with respect to \( V_{\text{cou}} \) and \( V_{\text{con}} \), it is generally treated using leading order perturbation theory.

**IV. PREDICTIONS FOR THE BOTTOMONIUM SPECTRUM**

The interaction Hamiltonian \( H_I \), which couples the bare valence \( b\bar{b} \) with the two-body \( B\bar{B} \) continuum, is an essential element of this formalism. In this work, we shall use the well-established \( ^3P_0 \) model [33–36] to describe the mixing between the bare \( b\bar{b} \) state and the open bottom meson pair \( b\bar{q} \) and \( q\bar{b} \). The \( ^3P_0 \) model assumes that the Okubo, Zweig, and Iizuka (OZI) rule-allowed strong decay takes place via the creation of a quark-antiquark pair with \( J^{PC} = 0^{++} \) from the vacuum. The \( q\bar{q} \) pair production is described by the Hamiltonian,

$$H_I = g \sum_q \int d^3x \; \bar{\psi}_q(x) \; \psi_q(x)$$

(17)

where \( \psi_q \) is the Dirac quark field. Following the conventional calculating method in \( ^3P_0 \) model, one can then straightforwardly evaluate the valence-continuum coupling matrix element \( h_{BB}^0 \) in Eq. (4). Here we will use simple harmonic oscillator (SHO) wavefunctions for the involved mesons, with a universal oscillator parameter \( \beta \). The SHO wavefunction enables analytical calculation of the transition amplitudes, and it turns out to be a good approximation. In Appendix A, we present the analytical \( ^3P_0 \) amplitudes \( M_{LS} \) for two S-wave final state channels. Since the oscillator parameters of the initial and final states are different in these analytical expressions, for an initial state of higher radial excited bottomonium, we can obtain the required transition amplitudes by simply taking derivatives,
as is shown in Appendix B. Here we will take the typical values $\beta = 0.4$ GeV for numerical calculations, this value was frequently adopted in the literature \cite{36,39}, and it turned out to be a reasonable zeroth-order approximation in coupled-channel calculations as well \cite{27}.

The parameter $g$ has the form $g = 2m_q \gamma$, where $m_q$ is the constituent quark mass, and $\gamma$ is the effective strength of pair creation. While for the creation of strange quarks, the effective strength $\gamma_s = (m_q/m_s) \gamma$ is used, following Ref. \cite{25}, we shall take $\gamma = 0.322$. The masses of constituent quarks are chosen to be $m_u = m_d = 0.33$ GeV and $m_s = 0.55$ GeV as usual. In the following, we first present the predictions for the $b\bar{b}$ spectrum in the nonrelativistic potential model of section III, then we include the coupled-channel effects, and the phenomenological implications are discussed.

### A. Bottomonium spectrum in conventional non-relativistic potential model

As usual, we can determine the bare $b\bar{b}$ mass spectrum by solving the following Schrödinger equation numerically,

$$-rac{1}{m_b} \frac{d^2}{dr^2} u_{nl}(r) + [V(r) + \frac{l(l+1)}{m_b r^2}]u_{nl}(r) = E_{nl} u_{nl}(r)$$

where $E_{nl}$ is the energy eigenvalue, the potential $V(r)$ is the leading order one with $V(r) = V_{\text{con}}(r) + V_{\text{con}}(r) + C$. The wavefunction $\Psi_{nlm}$ of the system is closely related to $u_{nl}(r)$ by

$$\Psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi) \quad \text{and} \quad u_{nl}(r) = rR_{nl}(r)$$

The mass splitting within the multiplets is determined by the spin-dependent term $V_{sd}(r)$, which is taken to be a perturbation. As a result, we need to calculate the expectation value of $V_{sd}(r)$ between the leading order wavefunction $\Psi_{nlm}$, in order to get the bottomonium mass fine splitting. The only term which gives a nonvanishing contribution to $n^1S_0 - n^3S_1$ fine splitting is the spin-spin term. Since the expectation value of $S_b \cdot S_b$ in the $1^1S_0$ and $3^3S_1$ states is $-3/4$ and $1/4$ respectively, the bare masses of S-wave states are

$$M_0(n^1S_0) = E_{n0} - \frac{3}{4} \frac{32\pi\alpha_s}{9m_b^2} \langle \tilde{\delta}(r) \rangle$$

$$M_0(n^3S_1) = E_{n0} + \frac{1}{4} \frac{32\pi\alpha_s}{9m_b^2} \langle \tilde{\delta}(r) \rangle$$

where

$$\langle \tilde{\delta}(r) \rangle = \left(\frac{\kappa}{\sqrt{\pi}}\right)^3 \int_0^\infty e^{-\kappa^2 r^2} u_{n0}^2(r) dr$$
For the \( n^1P_1 - n^3P_J, n^1D_2 - n^3D_J, n^1F_3 - n^3F_J \) cases, the spin-orbital and tensor terms contribute as well. The corresponding bare masses are given by

\[
M_0(n^1P_1) = E_{n1} - \frac{32\pi\alpha_s}{4m_b^2}\langle\tilde{\delta}(r)\rangle \\
M_0(n^3P_J) = E_{n1} + \frac{1}{4m_b^2}\langle\tilde{\delta}(r)\rangle + \frac{1}{m_b^2}\left(A_J\alpha_s\langle\frac{1}{r}\rangle + B_J\sigma\langle\frac{1}{r}\rangle\right) \\
M_0(n^1D_2) = E_{n2} - \frac{32\pi\alpha_s}{4m_b^2}\langle\tilde{\delta}(r)\rangle \\
M_0(n^3D_J) = E_{n2} + \frac{1}{4m_b^2}\langle\tilde{\delta}(r)\rangle + \frac{1}{m_b^2}\left(C_J\alpha_s\langle\frac{1}{r}\rangle + D_J\sigma\langle\frac{1}{r}\rangle\right) \\
M_0(n^1F_3) = E_{n3} - \frac{32\pi\alpha_s}{4m_b^2}\langle\tilde{\delta}(r)\rangle \\
M_0(n^3F_J) = E_{n3} + \frac{1}{4m_b^2}\langle\tilde{\delta}(r)\rangle + \frac{1}{m_b^2}\left(E_J\alpha_s\langle\frac{1}{r}\rangle + F_J\sigma\langle\frac{1}{r}\rangle\right) \\
M_0(n^1G_4) = E_{n4} - \frac{32\pi\alpha_s}{4m_b^2}\langle\tilde{\delta}(r)\rangle \\
M_0(n^3G_J) = E_{n4} + \frac{1}{4m_b^2}\langle\tilde{\delta}(r)\rangle + \frac{1}{m_b^2}\left(G_J\alpha_s\langle\frac{1}{r}\rangle + H_J\sigma\langle\frac{1}{r}\rangle\right)
\]

Here \( \langle \ldots \rangle \) denotes the expectation value, which is defined in the same way as that in Eq.\((22)\). The coefficients \( A_J, B_J, C_J, D_J, E_J, F_J, G_J \) and \( H_J \) for each case are listed in Table II. Note that the tensor force \( T_{0b} \) could lead to the so-called S–D mixing. However, the mass splitting induced by the S–D mixing is a second-order perturbation effect of the hyperfine interactions \( V_{sd} \), thus its contribution is rather small and hence is neglected here. We find that the numerical value \( \langle \tilde{\delta}(r)\rangle \) is extremely small except for the S–wave states, since it would be proportional to the zero point value of the wavefunction, i.e., it is exactly zero if we don’t smear the hyperfine interaction. As a result, the following sum rules are satisfied quite well,

\[
M_0(n^1P_1) \simeq (5M_0(n^3P_2) + 3M_0(n^3P_0) + M_0(n^3P_1))/9 \\
M_0(n^1D_2) \simeq (7M_0(n^3D_3) + 5M_0(n^3D_2) + 3M_0(n^3D_1))/15 \\
M_0(n^1F_3) \simeq (9M_0(n^3F_4) + 7M_0(n^3F_3) + 5M_0(n^3F_2))/21 \\
M_0(n^1G_4) \simeq (11M_0(n^3G_5) + 9M_0(n^3G_4) + 7M_0(n^3G_3))/27
\]

For experimental input we use the masses of the 14 reasonably well-established \( b\bar{b} \) states, which are given in TableIII. Here we don’t include the \( \Upsilon(5S) \) and \( \Upsilon(6S) \) candidates \( \Upsilon(10860) \) and \( \Upsilon(11020) \), since their masses measured by the Babar Collaboration are different from
the Particle Data Group (PDG) averages [40]. The parameters that follow from fitting these masses are $\alpha_s=0.3840$, $\sigma=0.9155 \text{GeV/fm}$, $C = -0.7825 \text{ GeV}$, $m_b=5.19 \text{ GeV}$ and $\kappa=2.3 \text{ GeV}$. In the classical Godfrey-Isgur quark model [41], the effective strong coupling constant $\alpha_s$ at the bottomonium scale and the string tension $\sigma$ are determined to be about 0.25 and 0.18 $\text{GeV}^2$ respectively. Clearly the fitting value for $\sigma$ is approximately the same as that in [41], while $\alpha_s$ is found to be somewhat larger than that of [41]. Note that $\alpha_s$ is dissociated from the effective strong coupling constant determined by the hadronic width of the quarkonium in conventional potential models, it is only a purely phenomenological strength parameter of the short-range potential [19]. Given these values, we can predict the masses of the currently unknown $b\bar{b}$ states. The predicted spectrum is shown in Table III and we see that the mass sum rules in Eq. (31)-Eq.(34) are really satisfied rather well.

### B. Bottomonium spectrum with coupled-channel effects

Following the formalism presented in section II, we shall take the coupled-channel effects into account. By performing a fit to the 14 established experimental states given in Table III, the best values of the parameters are determined to be $\alpha_s=0.418$, $\sigma=0.818 \text{GeV/fm}$, $C = -0.62376 \text{ GeV}$, $m_b=5.18 \text{ GeV}$ and $\kappa=2.85 \text{ GeV}$. Note that the value of $\alpha_s$ here is larger than that in Ref. [41], while $\sigma$ is smaller than the fitting value of [41]. We can now straightforwardly evaluate the bare state masses, the mass shifts due to $B\bar{B}$ loops and the predictions for the bottomonium masses with coupled-channel effects included. The results are given in Table IV and Table V where the $b\bar{b}$ component is presented as well. It can be seen that the mass sum rules in Eq. (31)-Eq.(34) remain approximately intact. In order to clearly see the predictions for the bottomonium spectrum, we further plot the predicted masses in Fig. 2.
mentally measured data, and the dashed lines are predicted masses with coupled-channel effects included. Various $B\bar{B}$ thresholds are also shown.

From Table IV and Table VI we can see that the mass shifts are predicted to be close to each other, they are of order 100 MeV, although individual $B\bar{B}$ loops make different contributions to the mass shift of each state. Moreover, we see that the total $B\bar{B}$ components are rather large, they mostly scatter in the range of $0.1 \sim 0.25$. We conclude that the results for the hadronic loop contributions to the bottomonium states are consistent with the general loop theorem derived in Ref. [27]. We note that the mass shifts in the charmonium sector are predicted to be around 200 MeV [25] (a larger value of about 500 MeV was suggested in [27]), and the two-meson continuum components can be as large as 0.5. Therefore the hadronic loop effects in bottomonium are much smaller than those in the charmonium sector.

In order to compare the mass spectrum in the non-relativistic potential model, we show the predicted mass with coupled-channel effects in Table III as well. For the 16 observed states, both predictions agree well with the measured masses except $\Upsilon(4^3S_1)$, $\Upsilon(5^3S_1)$ and $\Upsilon(6^3S_1)$. We see that the masses of $\Upsilon(4^3S_1)$ and $\Upsilon(6^3S_1)$ are predicted to be rather close to the observation after the coupled-channel effects are included, but the mass of $\Upsilon(5^3S_1)$ is smaller than its measured value about 51 MeV. Whereas the non-relativistic potential model without hadronic loop predicts the $\Upsilon(5^3S_1)$ mass successfully, the departures of predictions from observation are 41.6 MeV and 57.4 MeV respectively for the $\Upsilon(4^3S_1)$ and $\Upsilon(6^3S_1)$.
states. The current experimental data indicates that coupled-channel effects can improve the agreement with observations, although it is not as satisfactory for the $\Upsilon(5^3S_1)$ state. But this is not the whole story, because the recently measured masses of $\Upsilon(5^3S_1)$ and $\Upsilon(6^3S_1)$ by the Babar Collaboration are different from the world averages. More precise measurements of the $\Upsilon(5^3S_1)$ and $\Upsilon(6^3S_1)$ states would be helpful in understanding the unquenched effects induced by hadronic loops in the bottomonium spectrum. Regarding the mass barycenter of the $\chi_b(3P)$ states, it is predicted to be 10523.3 MeV and 10537.9 MeV respectively for the included and not included coupled-channel effects. Both predictions are consistent with the ATLAS measurement, although the former is slightly smaller.

Inspecting the mass predictions shown in Table III, we can’t find a state with mass around 10890 MeV except $\Upsilon(5^1S_0)$ and $\Upsilon(5^3S_1)$, even if coupled-channel effects are considered. Therefore if $Y_b(10890)$ is confirmed to be a new resonance by future experiments, it should not be a canonical bottomonium state. Many interpretations have been proposed so far. A possible explanation is the existence of a tetraquark state $[bq][\bar{b}\bar{q}]$ [42, 43]. Another explanation is a $b\bar{b}$ counterpart to the $Y(4260)$ state, which may overlap with $\Upsilon(5S)$ [44]. Other interpretations such as final state interactions and so forth have been suggested as well [45]. In the same manner, if some new bottomonium-like state is observed by LHCb or SuperB in the future, we can determine whether the state could be accommodated as a conventional quark model state by comparing its mass with our predictions, and we can determine its assignment if it is.

Mixing between two bare $b\bar{b}$ states could arise through the hadron loop, provided that both bottomonium states could couple to the same intermediate $B\bar{B}$ state. Similar to the $S$–$D$ mixing, this kind of mixing would introduce corrections to the mass of the physical bottomonium. However, its contributions are of higher order in the valence-continuum coupling Hamiltonian $H_I$ with respect to the loop contributions discussed above. Moreover, as is stated by the loop theorem of Ref. [27], if the mass difference between various intermediate loop mesons is neglected, the mixing amplitude between two valence $b\bar{b}$ states vanishes unless both the orbital angular momentum and the spin of the two states are the same. Concrete numerical calculations show that this mixing effect is really quite small [27], and the same results are found in Ref. [25]. As a result, we have not considered this effect in the present work.

Finally, we note that although the bottomonium mass spectrums predicted in models with
and without coupled-channel effects are not drastically different from each other, i.e., part of
the hadronic loop effects can be absorbed into the redefinition of the model parameters, the
underlying physics is different. In the scenario with coupled-channel effects, there are sizable
$B\bar{B}$ components in the physical bottomonium states. As a result, the production and decay
of bottomonium would be different from those predicted in the quenched quark model. The
coupled-channel effects in hadronic transitions of bottomonium have been studied in Ref.
[46], and it was found that the inclusion of the coupled-channel effects can really improve
the theory of hadronic transitions. The mixture of $B\bar{B}$ continuum may also be important
in understanding some anomalous observations, since the same has turned out to be true
in the charmonium sector [16, 19, 47, 48]. This topic deserves much laborious and complex
work and is beyond the scope of the present work.

V. CONCLUSIONS AND DISCUSSIONS

Motivated by the recent experimental progress on the bottomonium spectrum, we spec-
ulate that some bottomonium-like states may be observed in the future similar to the char-
monium sector, particularly with the running of LHCb and SuperB. Because the coupled-
channel effects are very important in understanding the nature of the newly observed
charmonium-like states, such as $X(3872)$, the same is expected to be true in the bottomo-
nium sector. In this work, we investigate the hadronic loops effects in the bottomonium
spectrum. The coupling between the valence $b\bar{b}$ states and the two-meson $B\bar{B}$ continuum
is described in terms of $^3P_0$ model. The mass shifts and the $b\bar{b}$ component of the physical
bottomonium are calculated in detail. We find that the mass shifts for all states are similar
to each other, they are around 100 MeV, although the contributions of individual loops are
different for each state. The two-meson continuum $B\bar{B}$ components are found to be rather
large as well. The hadronic loop effects in bottomonium turn out to be smaller than the ones
in the charmonium sector. Moreover, we evaluate the mass spectrum in the conventional
constituent quark model, where coupled-channel effects are not taken into account. We find
that the hadronic loop effects can be partially absorbed into a reselection of the model pa-
rameters. Since the potential models both with and without coupled-channel effects don’t
predict a state with mass around 10890 MeV except for $\Upsilon(5\,^1S_0)$ and $\Upsilon(5\,^3S_1)$, we conclude
that $Y_b(10890)$ should be an exotic state beyond the quark model, if it is confirmed to be a
new resonance. Our prediction for the mass of $\chi_b(3P)$ states is consistent with the recent ATLAS measurement. Moreover, with the mass spectrum predicted in our work, we can determine whether a new bottomonium-like state observed in the future can be accommodated as a canonical bottomonium, and we can determine its assignment if it is.

Acknowledgments

We are grateful to Professor Mu-Lin Yan and Professor Dao-Neng Gao for stimulating discussions. Jia-Feng Liu would like to express special thanks to Hao-Ran Chang for beneficial suggestions and to Song-Bin Zhang for useful help with the numerical calculation. This work is supported by the National Natural Science Foundation of China under Grant No.10905053, Chinese Academy KJCX2-YW-N29 and the 973 project with Grant No. 2009CB825200. Jia-Feng Liu is supported in part by the National Natural Science Foundation of China under Grant No.10775124, No.11075149 and No.10975128.

[1] E. S. Swanson, Phys. Rept. 429, 243 (2006) [arXiv:hep-ph/0601110].
[2] E. Eichten, S. Godfrey, H. Mahlke and J. L. Rosner, Rev. Mod. Phys. 80, 1161 (2008) [arXiv:hep-ph/0701208].
[3] S. Godfrey and S. L. Olsen, Ann. Rev. Nucl. Part. Sci. 58, 51 (2008) [arXiv:0801.3867 [hep-ph]].
[4] N. Brambilla, S. Eidelman, B. K. Heltsley, R. Vogt, G. T. Bodwin, E. Eichten, A. D. Frawley and A. B. Meyer et al., Eur. Phys. J. C 71, 1534 (2011) [arXiv:1010.5827 [hep-ph]].
[5] B. Aubert et al. BABAR Collaboration, Phys. Rev. Lett. 101, 071801 (2008), arXiv:0807.1086.
[6] P. Sanchez et al. BABAR Collaboration, Phys. Rev. D. 82, 111102 (2010), arXiv:1004.0175.
[7] J. Lees et al. BABAR Collaboration, arXiv:1102.4565.
[8] I. Adachi et al. Belle Collaboration, arXiv:1103.3419.
[9] G. Aad et al. [ATLAS Collaboration], arXiv:1112.5154 [hep-ex].
[10] K. F. Chen et al. [Belle Collaboration], Phys. Rev. Lett. 100, 112001 (2008); I. Adachi et al. [Belle Collaboration], Phys. Rev. D 82, 091106 (2010), arXiv:0808.2445 [hep-ex].
[11] B. Aubert et al. [BABAR Collaboration], Phys. Rev. Lett. 102, 012001 (2009) arXiv:0809.4120 [hep-ex].
[12] I. Adachi et al. [Belle Collaboration] arXiv:1105.4583 [hep-ex].
[13] LHCb homepage, http://lhcb.web.cern.ch/lhcb
[14] B. O’Leary et al. [SuperB Collaboration], arXiv:1008.1541 [hep-ex].
[15] N. A. Tornqvist, Annals Phys. 123, 1 (1979); N. A. Tornqvist, Acta Phys. Polon. B 16, 503 (1985) [Erratum-ibid. B 16, 683 (1985)].
[16] S. Ono and N. A. Tornqvist, Z. Phys. C 23, 59 (1984); K. Heikkila, S. Ono and N. A. Tornqvist, Phys. Rev. D 29, 110 (1984) [Erratum-ibid. D 29, 2136 (1984)].
[17] N. A. Tornqvist, Z. Phys. C 68, 647 (1995) arXiv:hep-ph/9504372; N. A. Tornqvist and M. Roos, Phys. Rev. Lett. 76, 1575 (1996) arXiv:hep-ph/9511210.
[18] E. van Beveren, C. Dullemond and G. Rupp, Phys. Rev. D 21, 772 (1980) [Erratum-ibid. D 22, 787 (1980)]; E. van Beveren, G. Rupp, T. A. Rijken and C. Dullemond, Phys. Rev. D 27, 1527 (1983).
[19] E. Eichten, K. Gottfried, T. Kinoshita, J. B. Kogut, K. D. Lane and T. M. Yan, Phys. Rev. Lett. 34, 369 (1975) [Erratum-ibid. 36, 1276 (1976)]; E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane and T. M. Yan, Phys. Rev. Lett. 36, 500 (1976); E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane and T. M. Yan, Phys. Rev. D 17, 3090 (1978) [Erratum-ibid. D 21, 313 (1980)]; E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane and T. M. Yan, Phys. Rev. D 21, 203 (1980); E. J. Eichten, K. Lane, C. Quigg, Phys. Rev. D73, 014014 (2006) hep-ph/0511179.
[20] T. Barnes, F. E. Close and H. J. Lipkin, Phys. Rev. D 68, 054006 (2003) arXiv:hep-ph/0305025.
[21] E. van Beveren and G. Rupp, Phys. Rev. Lett. 91, 012003 (2003) arXiv:hep-ph/0305035.
[22] D. S. Hwang and D. W. Kim, Phys. Lett. B 601, 137 (2004) arXiv:hep-ph/0408154;
[23] Yu. A. Simonov and J. A. Tjon, Phys. Rev. D 70, 114013 (2004) arXiv:hep-ph/0409361.
[24] E. J. Eichten, K. Lane and C. Quigg, Phys. Rev. D 69, 094019 (2004) arXiv:hep-ph/0401210.
[25] Yu. S. Kalashnikova, Phys. Rev. D 72, 034010 (2005) arXiv:hep-ph/0506270.
[26] M. R. Pennington and D. J. Wilson, Phys. Rev. D 76, 077502 (2007) arXiv:0704.3384 [hep-ph].
[27] T. Barnes and E. S. Swanson, Phys. Rev. C 77, 055206 (2008) arXiv:0711.2080 [hep-ph].
[28] I. V. Danilkin and Yu. A. Simonov, Phys. Rev. D 81, 074027 (2010) arXiv:0907.1088 [hep-ph]; I. V. Danilkin and Yu. A. Simonov, Phys. Rev. Lett. 105, 102002 (2010) arXiv:1006.0211 [hep-ph].

[29] O. Zhang, C. Meng and H. Q. Zheng, Phys. Lett. B 680, 453 (2009) arXiv:0901.1553 [hep-ph].

[30] B. Q. Li, C. Meng and K. T. Chao, Phys. Rev. D 80, 014012 (2009) arXiv:0904.4068 [hep-ph].

[31] P. G. Ortega, J. Segovia, D. R. Entem and F. Fernandez, Phys. Rev. D 81, 054023 (2010) arXiv:1001.3948 [hep-ph].

[32] M. Jacob and G. C. Wick, Annals Phys. 7 (1959) 404 [Annals Phys. 281 (2000) 774].

[33] L. Micu, Nucl. Phys. B10, 521 (1969).

[34] A. Le Yaouanc, L. Oliver, O. Pene and J. C. Raynal, Phys. Rev. D 8 (1973) 2223; ibid., Phys. Rev. D 9, 1415 (1974); Phys. Rev. D 11, 680 (1975); Phys. Rev. D 11, 1272 (1975); Phys. Lett. B 71, 397 (1977).

[35] P. Geiger and E. S. Swanson, Phys. Rev. D 50, 6855 (1994) arXiv:hep-ph/9405238.

[36] E. S. Ackleh, T. Barnes and E. S. Swanson, Phys. Rev. D 54, 6811 (1996) arXiv:hep-ph/9604355.

[37] T. Barnes, F. E. Close, P. R. Page and E. S. Swanson, Phys. Rev. D 55, 4157 (1997) arXiv:hep-ph/9609339.

[38] T. Barnes, N. Black and P. R. Page, Phys. Rev. D 68, 054014 (2003) arXiv:nucl-th/0208072.

[39] J. F. Liu, G. J. Ding and M. L. Yan, Phys. Rev. D 82, 074026 (2010) arXiv:1008.0246.

[40] K. Nakamura et al. (Particle Data Group), J. Phys. G 37, 075021 (2010).

[41] S. Godfrey and N. Isgur, Phys. Rev. D 32, 189 (1985).

[42] M. Karliner and H. J. Lipkin, arXiv:0802.0649 [hep-ph].

[43] A. Ali, C. Hambrock and S. Mishima, Phys. Rev. Lett. 106, 092002 (2011) arXiv:1011.4856 [hep-ph].

[44] W. S. Hou, Phys. Rev. D 74, 017504 (2006) arXiv:hep-ph/0606016.

[45] C. Meng and K. T. Chao, Phys. Rev. D 78, 034022 (2008) arXiv:0805.0143 [hep-ph].

[46] H. -Y. Zhou and Y. -P. Kuang, Phys. Rev. D 44, 756 (1991).

[47] F. -K. Guo, C. Hanhart, G. Li, U. -G. Meissner and Q. Zhao, Phys. Rev. D 83, 034013 (2011) arXiv:1008.3632 [hep-ph].

[48] A. M. Badalian, V. D. Orlovsky, Y. A. Simonov and B. L. G. Bakker, arXiv:1202.4882 [hep-ph].
Appendix A: $^3P_0$ transition amplitudes

The derivation of the $^3P_0$ matrix elements has been discussed in detail in Ref. [36–39]. Starting from the pair production Hamiltonian $H_I$ given in Eq. (17), one can straightforwardly evaluate the $H_I$ matrix element $h_{fi}$ for the transition $A(b\bar{b}) \rightarrow B(q\bar{b}) + C(b\bar{q})$ in terms of overlap integrals in flavor, spin and spatial spaces,

$$h_{fi} = \langle BC|H_I|A\rangle_a + \langle BC|H_I|A\rangle_b$$

$$= I_{signature}(a)I_{flavor}(a)I_{spin+space}(a) + I_{signature}(b)I_{flavor}(b)I_{spin+space}(b) \quad (A.1)$$

Here $a$ and $b$ represent two decay diagrams in which the produced quark goes into meson $B$ and meson $C$ respectively. In the present work, only the first diagram is allowed. Therefore we have the flavor factor $I_{flavor}(a) = 1$ and $I_{flavor}(b) = 0$, as is listed in Table II. The spin-space part of $h_{fi}$ is explicitly given by

$$I_{spin+space}(a) = \int d^3k \Psi_{nALAM_LA}(k-P_B)\Psi^*_{nBLBM_LB}(k-rP_B)$$

$$\times \Psi^*_{nCLCM_LC}(k-rP_B)g \frac{m_3}{E_3}[\bar{u}_{ks}(v_{-ks})] \quad (A.2)$$

where $r = \frac{m_q}{m_q+m_b}$, and $m_q$ denotes created quark mass $m_u, m_d$ or $m_s$. $\Psi_{nALAM_LA}$ is the wavefunction of the initial meson $A$ in momentum space, and $\Psi_{nBLBM_LB}$ and $\Psi_{nCLCM_LC}$ are the wavefunctions of the final state mesons $B$ and $C$ respectively. Taking into account the phase space, we get the differential decay rate

$$\frac{d\Gamma_{A\rightarrow BC}}{d\Omega} = 2\pi PE_BE_CE_A |h_{fi}|^2 \quad (A.3)$$

where $P$ is the momentum of the final state mesons in the rest frame of meson $A$

$$P = \sqrt{[M_A^2-(M_B + M_C)^2][M_A^2-(M_B - M_C)^2]/(2M_A)}. \quad (A.4)$$

To compare with the experiments, we transform the amplitude $h_{fi}$ into the partial wave amplitude $M_{LS}$ by the recoupling calculation [32], then the decay width is

$$\Gamma(A \rightarrow B + C) = 2\pi \frac{PE_BE_CE_A}{M_A} \sum_{LS} |M_{LS}|^2. \quad (A.5)$$

Since we neglect mass splitting within the same isospin multiplet, to sum over all channels, one should multiply the mass shift due to a specific hadronic loop by the flavor factor $F$ which is listed in Table II.
TABLE II: Relevant flavor weight factors for bottomonium decay, where $|X\rangle = |b\bar{b}\rangle$.

| Generic Decay | Example | $I_{\text{flavor}}(a)$ | $I_{\text{flavor}}(b)$ | $\mathcal{F}$ |
|---------------|---------|------------------------|------------------------|---------------|
| $X \rightarrow BB$ | $X \rightarrow B^+ + B^-$ | 1 | 0 | 2 |
| $X \rightarrow B^*\bar{B}$ | $X \rightarrow B^{*+} + B^-$ | 1 | 0 | 4 |
| $X \rightarrow B^*\bar{B}$ | $X \rightarrow B^{*+} + B^{*-}$ | 1 | 0 | 2 |
| $X \rightarrow B_s\bar{B}_s$ | $X \rightarrow B^0_s + \bar{B}^0_s$ | 1 | 0 | 1 |
| $X \rightarrow B^*_s\bar{B}_s$ | $X \rightarrow B^{*0}_s + \bar{B}^{0}_s$ | 1 | 0 | 2 |
| $X \rightarrow B^*_s\bar{B}^*_s$ | $X \rightarrow B^{*0}_s + \bar{B}^{*0}_s$ | 1 | 0 | 1 |

We take all spatial wavefunctions to be simple harmonic oscillator forms with $\beta_A$ being the oscillator parameter of the initial meson $A$ and $\beta_B = \beta_C$ for the final state mesons $B$ and $C$. It turn out that the transition amplitude $\mathcal{M}_{LS}$ is proportional to an overall Gaussian factor, it can be expressed as

$$\mathcal{M}_{LS} = \frac{\gamma}{\sqrt{\pi}} e^{\frac{\nu^2 (r - 1)^2}{2 \beta_A^2 + \beta_B^2}} A_{LS}$$

where $r = \frac{m_q}{m_q + m_b}$, $m_q$ denotes created quark mass $m_u$, $m_d$ or $m_s$, and $P$ is the momentum of the final state mesons in the rest frame of meson $A$. The analytical expressions of the amplitudes for the decays into two stable $S$–wave final states are listed in the following. We note that our expressions are different from the results in Ref.[37]. The mass difference between the created quark and bottom quark and two oscillator parameters corresponding to initial and final states are considered in our expressions, and our results are more general than those in Ref.[37]. In the limit of $\beta_A = \beta_B = \beta$ and $r = 1/2$, the amplitudes presented below coincide with those of Ref.[37].

$$1S \rightarrow 1S + 1S$$

$$M_P = \frac{-8P \beta_A^{3/2} (2r \beta_A^2 + \beta_B^2)}{\sqrt{3} (2 \beta_A^2 + \beta_B^2)^{5/2}}$$

$$^3S_1$$

$$A_{10}(^3S_1 \rightarrow ^1S_0 + ^1S_0) = M_P \cdot ^1P_1$$

18
\[ A_{11}(^3S_1 \rightarrow ^3S_1 + ^1S_0) = -\sqrt{\frac{2}{3}} M_P \ 3P_1 \]  
\[ (A.9) \]

\[ A_{LS}(^3S_1 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 
\sqrt{\frac{4}{3}} M_P & 1P_1 \\
0 & 3P_1 \\
-\sqrt{\frac{20}{3}} M_P & 5P_1 \\
0 & 5F_1 
\end{cases} \]  
\[ (A.10) \]

\[ ^1S_0 \]

\[ A_{LS}(^1S_0 \rightarrow ^1S_0 + ^1S_0) = 0 \]  
\[ (A.11) \]

\[ A_{11}(^1S_0 \rightarrow ^3S_1 + ^1S_0) = -\sqrt{3} M_P \ 3P_0 \]  
\[ (A.12) \]

\[ A_{11}(^1S_0 \rightarrow ^3S_1 + ^3S_1) = \sqrt{6} M_P \ 3P_0 \]  
\[ (A.13) \]

\[ ^1P \rightarrow ^1S + ^1S \]

\[ M_S = -\frac{16\beta_A^{3/2}(2(2r^2\beta_A^2-\beta_B^2+r(\beta_B^2-2\beta_A^2))P^2+3\beta_B^2(2\beta_A^2+\beta_B^2))}{3\sqrt{\beta_B^2+\beta_A^2}(2\beta_A^2+\beta_B^2)^3} \]  
\[ (A.14) \]

\[ M_D = \frac{32P^2(r-1)\beta_A^{3/2}(2r^2\beta_A^2+\beta_B^2)}{\sqrt{15} \sqrt{\beta_B^2+\beta_A^2}(2\beta_A^2+\beta_B^2)^3} \]  
\[ (A.15) \]

\[ ^3P_2 \]

\[ A_{20}(^3P_2 \rightarrow ^1S_0 + ^1S_0) = M_D \]  
\[ (A.16) \]

\[ A_{21}(^3P_2 \rightarrow ^3S_1 + ^1S_0) = -\sqrt{\frac{3}{2}} M_D \]  
\[ (A.17) \]

\[ A_{LS}(^3P_2 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 
-\sqrt{\frac{5}{3}} M_S & ^5S_2 \\
\sqrt{\frac{3}{3}} M_D & ^1D_2 \\
-\sqrt{\frac{5}{3}} M_D & ^5D_2 
\end{cases} \]  
\[ (A.18) \]

\[ ^3P_1 \]

\[ A_{LS}(^3P_1 \rightarrow ^3S_1 + ^1S_0) = \begin{cases} 
M_S & ^3S_1 \\
-\sqrt{\frac{3}{2}} M_D & ^3D_1 
\end{cases} \]  
\[ (A.19) \]
\[ A_{LS}(^3P_0 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 0 & ^3S_1 \\ 0 & ^3D_1 \\ -\sqrt{5} M_D & ^5D_1 \end{cases} \] (A.20)

\[ A_{00}(^3P_0 \rightarrow ^1S_0 + ^1S_0) = \sqrt{\frac{3}{2}} M_S \ ^1S_0 \] (A.21)

\[ A_{LS}(^3P_0 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} \sqrt{\frac{1}{2}} M_S & ^1S_0 \\ -\sqrt{\frac{20}{3}} M_D & ^5D_0 \end{cases} \] (A.22)

\[ A_{LS}(^1P_1 \rightarrow ^3S_1 + ^1S_0) = \begin{cases} -\sqrt{\frac{2}{3}} M_S & ^3S_1 \\ -\sqrt{\frac{5}{3}} M_D & ^3D_1 \end{cases} \] (A.23)

\[ A_{LS}(^1P_1 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} M_S & ^3S_1 \\ \sqrt{\frac{10}{3}} M_D & ^3D_1 \\ 0 & ^5D_1 \end{cases} \] (A.24)

\[ 1D \rightarrow 1S + 1S \]

\[ M_P = -\frac{64 \sqrt{\frac{2}{3}} P(r - 1) \beta_A^{5/2} (2r^2 \beta_A^2 - \beta_B^2 + r(\beta_B^2 - 2\beta_A^2)) P^2 + 5 \beta_B^2 (2\beta_A^2 + \beta_B^2))}{5 \sqrt{\frac{2}{3}} + \frac{1}{35} \beta_B (2\beta_A^2 + \beta_B^2)^4} \] (A.25)

\[ M_F = -\frac{64 P^3 (r - 1)^2 \beta_A^{5/2} (2r \beta_A^2 + \beta_B^2)}{\sqrt{35} \sqrt{\frac{2}{3}} + \frac{1}{35} \beta_B (2\beta_A^2 + \beta_B^2)^4} \] (A.26)

\[ ^3D_3 \]

\[ A_{30}(^3D_3 \rightarrow ^1S_0 + ^1S_0) = M_F \ ^1F_3 \] (A.27)

\[ A_{31}(^3D_3 \rightarrow ^3S_1 + ^1S_0) = -\sqrt{\frac{4}{3}} M_F \ ^3F_3 \] (A.28)
\[ A_{LS}(^3D_3 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 
M_P & 5P_3 \\
\sqrt{\frac{7}{3}} M_F & 1F_3 \\
0 & 3F_3 \\
-\sqrt{\frac{5}{5}} M_F & 5F_3 \\
0 & 5H_3 
\end{cases} \] (A.29)

\[ A_{LS}(^3D_2 \rightarrow ^3S_1 + ^1S_0) = \begin{cases} 
-\sqrt{\frac{3}{8}} M_P & 3P_2 \\
-\sqrt{\frac{14}{15}} M_F & 3F_2 
\end{cases} \] (A.30)

\[ A_{LS}(^3D_2 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 
\frac{1}{2} M_P & 5P_2 \\
0 & 3F_2 \\
-\sqrt{\frac{66}{15}} M_F & 5F_2 
\end{cases} \] (A.31)

\[ A_{10}(^3D_1 \rightarrow ^1S_0 + ^1S_0) = -\sqrt{\frac{5}{12}} M_P & 1P_1 \] (A.32)

\[ A_{11}(^3D_1 \rightarrow ^3S_1 + ^1S_0) = -\sqrt{\frac{5}{24}} M_P & 3P_1 \] (A.33)

\[ A_{LS}(^3D_1 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 
-\sqrt{\frac{5}{6}} M_P & 3P_1 \\
0 & 3P_1 \\
\frac{1}{6} M_P & 5P_1 \\
-\sqrt{\frac{28}{5}} M_F & 5F_1 
\end{cases} \] (A.34)

\[ A_{LS}(^1D_2 \rightarrow ^3S_1 + ^1S_0) = \begin{cases} 
\frac{1}{2} M_P & 3P_2 \\
-\sqrt{\frac{7}{5}} M_F & 3F_2 
\end{cases} \] (A.35)

\[ A_{LS}(^1D_2 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 
\frac{1}{2} M_P & 3P_2 \\
0 & 5P_2 \\
\sqrt{\frac{14}{5}} M_F & 3F_2 \\
0 & 5F_2 
\end{cases} \] (A.36)
\[1F \rightarrow 1S + 1S\]

\[
M_D = -\frac{128 \sqrt{2} P^2 (r - 1)^2 \beta_A^{9/2} (2(2r^2 \beta_A^2 - \beta_B^2 + r(\beta_B^2 - 2\beta_A^2))) P^2 + 7\beta_B^2 (2\beta_A^2 + \beta_B^2))}{7(2\beta_A^2 + \beta_B^2)^{11/2}} \quad (A.37)
\]

\[
M_G = -\frac{256 \sqrt{2} P^4 (r - 1)^3 \beta_A^{9/2} (2r\beta_A^2 + \beta_B^2)}{3(2\beta_A^2 + \beta_B^2)^{11/2}} \quad (A.38)
\]

\[3F_4\]

\[
A_{40}(3F_4 \rightarrow 1^1 S_0 + 1^1 S_0) = M_G \quad 1^1 G_4 \quad (A.39)
\]

\[
A_{41}(3F_4 \rightarrow 3^1 S_1 + 1^1 S_0) = -\sqrt{\frac{5}{2}} M_G \quad 3^3 G_4 \quad (A.40)
\]

\[
A_{LS}(3F_4 \rightarrow 3^1 S_1 + 3^1 S_1) = \begin{cases} 
M_D & 5^5 D_4 \\
\sqrt{\frac{1}{3}} M_G & 1^1 G_4 \\
0 & 3^3 G_4 \\
-\sqrt{\frac{55}{32}} M_G & 5^5 G_4 \\
0 & 5^5 I_4
\end{cases} \quad (A.41)
\]

\[3F_3\]

\[
A_{LS}(3F_3 \rightarrow 3^1 S_1 + 1^1 S_0) = \begin{cases} 
-\sqrt{\frac{1}{3}} M_D & 3^3 D_3 \\
-\sqrt{\frac{27}{28}} M_G & 3^3 G_3 \\
0 & 3^3 D_3
\end{cases} \quad (A.42)
\]

\[
A_{LS}(3F_3 \rightarrow 3^1 S_1 + 3^1 S_1) = \begin{cases} 
\sqrt{\frac{1}{3}} M_D & 5^5 D_3 \\
0 & 3^3 G_3 \\
-\sqrt{\frac{35}{14}} M_G & 5^5 G_3
\end{cases} \quad (A.43)
\]

\[3F_2\]

\[
A_{20}(3F_2 \rightarrow 1^1 S_0 + 1^1 S_0) = -\sqrt{\frac{7}{20}} M_D \quad 1^1 D_2 \quad (A.44)
\]

\[
A_{21}(3F_2 \rightarrow 3^1 S_1 + 1^1 S_0) = -\sqrt{\frac{7}{30}} M_D \quad 3^3 D_2 \quad (A.45)
\]
\( A_{LS}(^3F_2 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 
0 & \text{ } ^5S_2 \\
-\sqrt{\frac{7}{60}} M_D & \text{ } ^1D_2 \\
0 & \text{ } ^3D_2 \\
\sqrt{\frac{1}{15}} M_D & \text{ } ^5D_2 \\
-\sqrt{\frac{26}{7}} M_G & \text{ } ^5G_2 
\end{cases} \) (A.46)

\( ^1F_3 \)

\( A_{LS}(^1F_3 \rightarrow ^3S_1 + ^1S_0) = \begin{cases} 
\frac{1}{2} M_D & \text{ } ^3D_3 \\
-\sqrt{\frac{7}{7}} M_G & \text{ } ^3G_3 \\
-\sqrt{\frac{1}{2}} M_D & \text{ } ^3D_3 \\
0 & \text{ } ^5D_3 \\
\sqrt{\frac{18}{7}} M_G & \text{ } ^3G_3 \\
0 & \text{ } ^5G_3 
\end{cases} \) (A.47)

\( A_{LS}(^1F_3 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 
\frac{1}{2} M_D & \text{ } ^3D_3 \\
-\sqrt{\frac{7}{7}} M_G & \text{ } ^3G_3 \\
-\sqrt{\frac{1}{2}} M_D & \text{ } ^3D_3 \\
0 & \text{ } ^5D_3 \\
\sqrt{\frac{18}{7}} M_G & \text{ } ^3G_3 \\
0 & \text{ } ^5G_3 
\end{cases} \) (A.48)

\( 1G \rightarrow 1S + 1S \)

\[ M_F = \frac{512 P^3 (r - 1) \beta_A^{11/2} (2(2r^2 \beta_A^2 - \beta_B^2) + r(\beta_B^2 - 2 \beta_A^2) - \beta_B^2 - 2 \beta_A^2)) P^2 + 9 \beta_B^2 (2 \beta_A^2 + \beta_B^2))}{21 \sqrt{15} \sqrt{7} \pi (2 \beta_A^2 + \beta_B^2)^{13/2}} \] (A.49)

\[ M_H = \frac{-512 P^5 (r - 1)^4 \beta_A^{11/2} (2r \beta_A^2 + \beta_B^2)}{3 \sqrt{231} \sqrt{7} \pi (2 \beta_A^2 + \beta_B^2)^{13/2}} \] (A.50)

\( ^3G_5 \)

\[ A_{50}(^3G_5 \rightarrow ^1S_0 + ^1S_0) = M_H \text{ } ^1H_5 \] (A.51)

\[ A_{51}(^3G_5 \rightarrow ^1S_0 + ^3S_1) = \sqrt{\frac{6}{5}} M_H \text{ } ^3H_5 \] (A.52)

\[ A_{LS}(^3G_5 \rightarrow ^3S_1 + ^3S_1) = \begin{cases} 
-\frac{2 \sqrt{7}}{3} M_F & \text{ } ^5F_5 \\
\frac{\sqrt{7}}{3} M_H & \text{ } ^1H_5 \\
0 & \text{ } ^3H_5 \\
-\frac{2 \sqrt{15}}{3} M_H & \text{ } ^5H_5 \\
0 & \text{ } ^5J_5 
\end{cases} \] (A.53)
\[ A_{LS}(^3G_4 \rightarrow ^1 S_0 + ^3 S_1) = \begin{cases} \frac{-\sqrt{35}}{6} M_F & ^3F_4 \\ \frac{2\sqrt{11}}{3\sqrt{3}} M_H & ^3H_4 \end{cases} \] (A.54)

\[ A_{LS}(^3G_4 \rightarrow ^3 S_1 + ^3 S_1) = \begin{cases} 0 & ^3F_4 \\ 0 & ^3H_4 \end{cases} \] (A.55)

\[ A_{LS}(^3G_3 \rightarrow ^1 S_0 + ^1 S_0) = M_F & ^1F_3 \] (A.56)

\[ A_{LS}(^3G_3 \rightarrow ^1 S_0 + ^3 S_1) = \begin{cases} \frac{1}{\sqrt{3}} M_F & ^3F_4 \\ 0 & ^3F_3 \end{cases} \] (A.57)

\[ A_{LS}(^3G_3 \rightarrow ^3 S_1 + ^3 S_1) = \begin{cases} \frac{1}{\sqrt{3}} M_F & ^3F_3 \\ 0 & ^3F_3 \\ -\frac{\sqrt{3}}{3\sqrt{2}} M_F & ^5F_3 \\ -2\frac{\sqrt{11}}{3} M_H & ^5H_3 \end{cases} \] (A.58)

\[ A_{LS}(^1G_4 \rightarrow ^1 S_0 + ^3 S_1) = \begin{cases} \frac{-\sqrt{7}}{3} M_F & ^3F_4 \\ \frac{\sqrt{11}}{3} M_H & ^3H_4 \end{cases} \] (A.59)
Appendix B: Recursion relations between wavefunctions

In Appendix A, we present the amplitudes for the ground bottomonium states decaying into two $S$–wave final states. If the initial state is the radial excited bottomonium, the corresponding amplitudes can certainly be derived in the same way. However, it is interesting to notice that the radially excited wavefunctions can be related to the lowest radial wavefunctions by differentiation,

$$\Psi_{1S}(p) = \frac{1}{\beta^{3/2} \pi^{3/4}} e^{-\frac{p^2}{2\beta}}$$  \hspace{1cm} (A.61)$$

$$\Psi_{2S}(p) = \frac{1}{\sqrt{6}\beta^2} (-3\beta^2 + 2p^2)\Psi_{1S}(p) = \frac{2}{\sqrt{6}}\beta \frac{\partial}{\partial \beta} \Psi_{1S}(p)$$  \hspace{1cm} (A.62)$$

$$\Psi_{3S}(p) = \frac{1}{2} \sqrt{\frac{15}{2}} \left( \frac{4p^4}{15\beta^4} - \frac{4p^2}{3\beta^2} + 1 \right)\Psi_{1S}(p) = \frac{1}{\sqrt{30}} \left( 3 + 2\beta \frac{\partial}{\partial \beta} + 2\beta^2 \frac{\partial^2}{\partial \beta^2} \right)\Psi_{1S}(p)$$  \hspace{1cm} (A.63)$$

$$\Psi_{4S}(p) = -\frac{1}{4} \sqrt{35} \left( \frac{-8p^6}{105\beta^6} + \frac{4p^4}{5\beta^4} - 2\frac{p^2}{\beta^2} + 1 \right)\Psi_{1S}(p)$$

$$= \frac{1}{3\sqrt{35}} \left( 15\beta \frac{\partial}{\partial \beta} + 6\beta^2 \frac{\partial^2}{\partial \beta^2} + 2\beta^3 \frac{\partial^3}{\partial \beta^3} \right)\Psi_{1S}(p)$$  \hspace{1cm} (A.64)$$

$$\Psi_{5S}(p) = \frac{3}{8} \sqrt{\frac{35}{2}} \left( \frac{16p^8}{945\beta^8} + \frac{-32p^6}{105\beta^6} + \frac{8p^4}{5\beta^4} + \frac{-8p^2}{3\beta^2} + 1 \right)\Psi_{1S}(p)$$

$$= \frac{1}{18\sqrt{70}} \left( 63 + 72\beta \frac{\partial}{\partial \beta} + 96\beta^2 \frac{\partial^2}{\partial \beta^2} + 24\beta^3 \frac{\partial^3}{\partial \beta^3} + 4\beta^4 \frac{\partial^4}{\partial \beta^4} + 2\beta^5 \frac{\partial^5}{\partial \beta^5} \right)\Psi_{1S}(p)$$  \hspace{1cm} (A.65)$$

$$\Psi_{6S}(p) = -\frac{3\sqrt{37}}{16} \left( \frac{-32p^{10}}{10395\beta^{10}} + \frac{16p^8}{189\beta^8} + \frac{-16p^6}{21\beta^6} + \frac{8p^4}{3\beta^4} + \frac{-10p^2}{3\beta^2} + 1 \right)\Psi_{1S}(p)$$

$$= \frac{1}{45\sqrt{77}} \left( \frac{675}{2} \beta \frac{\partial}{\partial \beta} + 240\beta^2 \frac{\partial^2}{\partial \beta^2} + 120\beta^3 \frac{\partial^3}{\partial \beta^3} + 20\beta^4 \frac{\partial^4}{\partial \beta^4} + 2\beta^5 \frac{\partial^5}{\partial \beta^5} \right)\Psi_{1S}(p)$$  \hspace{1cm} (A.66)$$
\[ \Psi_{2P}(p) = 4\sqrt{\frac{2!}{5! \pi^{1/4}}} \frac{1}{\beta^5} p(-5 + \frac{2p^2}{\beta^2}) Y_{1M_L}(\hat{p}) e^{-\frac{p^2}{2\beta^2}} = \sqrt{\frac{2}{5}} \frac{\partial}{\partial \beta} \Psi_{1P}(p) \] (A.67)

\[ \Psi_{3P}(p) = \frac{1}{2\sqrt{70}} (\frac{4p^4}{\beta^4} - \frac{28p^2}{\beta^2} + 35) \Psi_{1P}(p) = \frac{1}{\sqrt{70}} (5 + 2\beta \frac{\partial}{\partial \beta} + 2\beta^2 \frac{\partial^2}{\partial \beta^2}) \Psi_{1P}(p) \] (A.68)

\[ \Psi_{2D}(p) = 8\sqrt{\frac{3!}{7! \pi^{1/4}}} \frac{1}{\beta^7} p^2 (-7 + \frac{2p^2}{\beta^2}) Y_{2M_L}(\hat{p}) e^{-\frac{p^2}{2\beta^2}} = \sqrt{\frac{2}{7}} \frac{\partial}{\partial \beta} \Psi_{1D}(p) \] (A.69)

\[ \Psi_{3D}(p) = \frac{1}{6\sqrt{14}} \frac{4p^4}{\beta^4} - \frac{36p^2}{\beta^2} + 63) \Psi_{1D}(p) = \frac{1}{3\sqrt{14}} (7 + 2\beta \frac{\partial}{\partial \beta} + 2\beta^2 \frac{\partial^2}{\partial \beta^2}) \Psi_{1D}(p) \] (A.70)

\[ \Psi_{2F}(p) = \frac{1}{3\sqrt{2}} (-9 + \frac{2p^2}{\beta^2}) \Psi_{1F}(p) = \sqrt{\frac{2}{3}} \frac{\beta}{\beta} \Psi_{1F}(p) \] (A.71)

The differential operators depend only on \( \beta \), hence they can be pulled out of the integration in Eq.(A.2). Therefore the amplitudes for radially excited meson decays can be found by applying the above differential operators to the amplitudes listed in Appendix A. For example,

\[ M_{LS}(2^3S_1 \rightarrow^1 S_0 +^1 S_0) = \frac{2}{\sqrt{6}} \beta_A \frac{\partial}{\partial \beta_A} M_{LS}(^3S_1 \rightarrow^1 S_0 +^1 S_0) \] (A.72)

We have checked that the amplitudes obtained with this method are exactly the same as those that result from performing the overlap integral straightforwardly.
| states       | $M_{ex}$  | $M_{th}$ | $M_{np}$ | states       | $M_{ex}$  | $M_{th}$ | $M_{np}$ |
|--------------|-----------|----------|----------|--------------|-----------|----------|----------|
| $\eta_b(1^1S_0)$ | 9390.9    | 9391.8   | 9396.3   | $\Upsilon(2^3D_2)$ | —         | 10426.8  | 10442.1  |
| $\Upsilon(1^3S_1)$ | 9460.3    | 9460.3   | 9444.2   | $\Upsilon(2^3D_3)$ | —         | 10431.4  | 10447.0  |
| $h_b(1^1P_1)$    | 9899.9    | 9915.5   | 9907.9   | $h_b(3^1P_1)$     | —         | 10523.2  | 10537.2  |
| $\chi_{b0}(1^3P_0)$ | 9859.4    | 9875.3   | 9869.1   | $\chi_{b0}(3^3P_0)$ | —         | 10495.9  | 10508.1  |
| $\chi_{b1}(1^3P_1)$ | 9892.8    | 9906.8   | 9899.8   | $\chi_{b1}(3^3P_1)$ | —         | 10517.3  | 10531.1  |
| $\chi_{b2}(1^3P_2)$ | 9912.2    | 9929.6   | 9921.7   | $\chi_{b2}(3^3P_2)$ | —         | 10532.4  | 10547.9  |
| $\eta_b(2^1S_0)$    | —         | 10004.9  | 9994.1   | $\Upsilon(2^1F_3)$ | —         | 10566.4  | 10600.5  |
| $\Upsilon(2^3S_1)$ | 10023.3   | 10026.2  | 10010.2  | $\Upsilon(2^3F_2)$ | —         | 10560.9  | 10598.3  |
| $\Upsilon(1^1D_2)$ | —         | 10145.5  | 10154.1  | $\Upsilon(2^3F_3)$ | —         | 10566.1  | 10600.5  |
| $\Upsilon(1^3D_1)$ | —         | 10138.1  | 10146.6  | $\Upsilon(2^3F_4)$ | —         | 10567.9  | 10601.6  |
| $\Upsilon(1^3D_2)$ | 10164.5   | 10144.6  | 10153.2  | $\eta_b(4^1S_0)$ | —         | 10593.2  | 10612.3  |
| $\Upsilon(1^3D_3)$ | —         | 10149.3  | 10158.0  | $\Upsilon(4^3S_1)$ | 10579.4   | 10602.7  | 10621.0  |
| $h_b(2^1P_1)$     | 10259.8   | 10259.1  | 10257.7  | $\Upsilon(3^1D_2)$ | —         | 10655.7  | 10693.0  |
| $\chi_{b0}(2^3P_0)$ | 10232.5   | 10227.9  | 10225.6  | $\Upsilon(3^3D_1)$ | —         | 10650.9  | 10685.7  |
| $\chi_{b1}(2^3P_1)$ | 10255.5   | 10252.4  | 10251.0  | $\Upsilon(3^3D_2)$ | —         | 10654.5  | 10692.0  |
| $\chi_{b2}(2^3P_2)$ | 10268.7   | 10270.1  | 10269.3  | $\Upsilon(3^3D_3)$ | —         | 10662.8  | 10697.0  |
| $\Upsilon(1^1F_3)$ | —         | 10322.1  | 10343.1  | $\eta_b(5^1S_0)$ | —         | 10812.6  | 10852.3  |
| $\Upsilon(1^3F_2)$ | —         | 10319.5  | 10341.3  | $\Upsilon(5^3S_1)$ | 10865.0   | 10819.9  | 10859.6  |
| $\Upsilon(1^3F_3)$ | —         | 10322.0  | 10343.2  | $\eta_b(6^1S_0)$ | —         | 11008.0  | 11070.0  |
| $\Upsilon(1^3F_4)$ | —         | 10323.1  | 10344.0  | $\Upsilon(6^3S_1)$ | 11019.0   | 11022.6  | 11076.4  |
| $\eta_b(3^1S_0)$    | —         | 10337.9  | 10337.5  | $\Upsilon(1^1G_4)$ | —         | 10473.3  | 10505.4  |
| $\Upsilon(3^3S_1)$ | 10355.2   | 10351.9  | 10348.4  | $\Upsilon(1^3G_3)$ | —         | 10471.8  | 10505.8  |
| $\Upsilon(2^1D_2)$ | —         | 10427.98 | 10443.0  | $\Upsilon(1^3G_4)$ | —         | 10473.4  | 10505.8  |
| $\Upsilon(2^3D_1)$ | —         | 10420.4  | 10435.7  | $\Upsilon(1^3G_5)$ | —         | 10470.8  | 10504.7  |

TABLE III: The spectrum of the bottomonium states, where $M_{ex}$ is the PDG average for the measured mass \[40\]. $M_{np}$ denotes the prediction for the mass of the $b\bar{b}$ state in the conventional non-relativistic potential model presented in section III. $M_{th}$ is the theoretical prediction after the coupled-channel effects are taken into account. All the masses are in units of megaelectronvolts (MeV).
TABLE IV: Mass shifts (in MeV) and $b \bar{b}$ component $P_{bb}$ of bottomonium states due to individual $B \bar{B}$ loops. $M_{ex}$ is the PDG average for the measured mass, $M_0$ denotes the bare mass, and $M_{th}$ denotes the predicted masses with coupled-channel effects considered. $\delta M$ denotes the total mass shift. For simplicity, we have abbreviated the $B \bar{B}$ hadronic loop as “$B B$”, $B^{*} \bar{B}$ as “$B B^*$”, and so forth.
|      | $\Upsilon(2^3D_2)$ | $\Upsilon(2^3D_3)$ | $\Upsilon(3^1D_2)$ | $\Upsilon(3^3D_1)$ | $\Upsilon(3^3D_2)$ | $\Upsilon(3^3D_3)$ | $\Upsilon(1^3F_3)$ | $\Upsilon(1^3F_2)$ | $\Upsilon(1^3F_3)$ |
|------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $BB$ | 0                | 10.460           | 0                | 13.300           | 0                | 11.295           | 0                | 13.733           | 0                |
| $B^*B$ | 38.298       | 25.121           | 43.392           | 13.795           | 43.215           | 24.146           | 46.402           | 16.135           | 44.945           |
| $B^*B^*$ | 37.940       | 41.427           | 41.000           | 54.099           | 41.229           | 45.943           | 42.246           | 58.809           | 43.726           |
| $B_sB_s$ | 0            | 1.879            | 0                | 1.564            | 0                | 1.841            | 0                | 1.631            | 0                |
| $B_s^*B_s$ | 5.892       | 4.690            | 6.396            | 1.397            | 5.664            | 4.531            | 7.435            | 2.001            | 6.831            |
| $B_s^*B_s^*$ | 6.948       | 6.465            | 5.921            | 9.211            | 6.601            | 6.181            | 6.965            | 10.636           | 7.542            |
| $\delta M$ | 89.078     | 90.041           | 96.710           | 93.367           | 96.709           | 93.936           | 103.048          | 102.945          | 103.043          |
| $M_0$ | 10515.9        | 10521.5          | 10752.4          | 10744.3          | 10751.2          | 10756.7          | 10425.1          | 10422.5          | 10425.0          |
| $M_{th}$ | 10426.8       | 10431.4          | 10655.7          | 10650.9          | 10654.5          | 10662.8          | 10322.1          | 10319.5          | 10322.0          |
| $M_{ex}$ | —              | —                | —                | —                | —                | —                | —                | —                | —                |
| $P_{bb}$ | 0.836         | 0.835            | 0.922            | 0.645            | 0.763            | 0.588            | 0.829            | 0.827            | 0.829            |

TABLE V: The continuing of Table IV.