FIBONACCI EXPANSIONS

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ABSTRACT. Expansions in the Golden ratio base have been studied since a pioneering paper of Rényi more than sixty years ago. We introduce closely related expansions of a new type, based on the Fibonacci sequence, and we show that in some sense they behave better.

1. Introduction

Expansions of the form

\[ x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}, \quad (c_i) \in \{0,1\}^\mathbb{N} \]

in real bases \( q \in (1,2) \) have been first studied by Rényi [20]. Subsequently many works have been devoted to their connections to probability and ergodic theory, combinatorics, symbolic dynamics, measure theory, topology and number theory; see, e.g., [11, 19, 18, 21, 3, 4, 15, 2, 10] and their references. Following the discovery of surprising uniqueness phenomena by Erdős et al. [7], a rich theory of unique expansions has been uncovered [8, 5, 9, 16, 17, 6]. There are still many open problems, for example concerning the number of possible expansions of specific numbers in particular bases.

In the special case where \( q = \varphi := \frac{1+\sqrt{5}}{2} \approx 1.618 \) is the Golden ratio, Rényi proved that the average distribution of the digits 0 and 1 is not the same, and he computed their frequencies. In this paper we introduce the Fibonacci expansions

\[ x = \sum_{i=1}^{\infty} \frac{c_i}{F_i}, \quad (c_i) \in \{0,1\}^\mathbb{N}, \]

where the powers \( \varphi^i \) are replaced by the Fibonacci numbers:

\[ F_1 := 1, \quad F_2 := 1, \quad \text{and} \quad F_{i+2} := F_{i+1} + F_i, \quad i = 1, 2, \ldots. \]

They are closely related to the expansions in base \( \varphi \), because

\[ F_i = \frac{1}{\sqrt{5}} \left( \varphi^i + \frac{(-1)^{i+1}}{\varphi^i} \right) \]

for all \( i \) by Binet’s formula, whence \( F_i \) is the nearest integer to \( \varphi^i / \sqrt{5} \).

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The purpose of this work is to compare these two expansions, and to study more general Kakeya expansions of the form

\[ x = \sum_{i=1}^{\infty} c_i p_i, \quad (c_i) \in \{0, 1\}^\mathbb{N}, \]

where \((p_i)\) is a given Kakeya sequence, i.e., a sequence of positive numbers satisfying the conditions \(p_i \to 0\), and

\[ p_n \leq \sum_{i=n+1}^{\infty} p_i \quad \text{for all} \quad n. \]

We recall the following classical theorem:

**Theorem 1.1** (Kakeya [12, 13]). If \((p_i)\) is a Kakeya sequence, then a real number \(x\) has an expansion of the form (1.3) if and only if \(x \in [0, \sum_{i=1}^{\infty} p_i]\).

For example, \(\left(q^{-i}\right)\) is a Kakeya sequence for every \(q \in (1, 2]\), so that every \(x \in [0, \frac{1}{q-1}]\) has an expansion in base \(q\). A similar result holds for Fibonacci expansions. Setting

\[ S := \sum_{i=1}^{\infty} \frac{1}{F_i} \approx 3.360, \]

we have the following result:

**Theorem 1.2.** A real number \(x\) has an expansion in the Fibonacci base if and only if \(x \in [0, S]\).

Next we will investigate the number of expansions. It is clear that the expansions of \(0\) and \(\frac{1}{q-1}\) are unique in every base \(q \in (1, 2]\): \(c_i \equiv 0\) and \(c_i \equiv 1\), respectively. Otherwise, \(x\) may have several, even infinitely many expansions:

**Theorem 1.3** (Erdős et al. [8]). If \(q \in (1, \varphi)\), then every \(x \in (0, \frac{1}{q-1})\) has a continuum of expansions in base \(q\).

By a different proof, we will extend Theorem 1.3 to a class of Kakeya expansions:

**Theorem 1.4.** Let \((p_i)\) be a sequence of positive real numbers, satisfying the following conditions:

\[ p_i \to 0; \]

\[ p_n < \sum_{i=n+1}^{\infty} p_i \quad \text{for all} \quad n; \]

\[ p_{n-1} < \sum_{i=n+1}^{\infty} p_i \quad \text{for infinitely many} \quad n; \]

\[ p_n \leq 2p_{n+1} \quad \text{for all sufficiently large} \quad n. \]

Then every \(0 < x < S := \sum_{i=1}^{\infty} p_i\) has a continuum of expansions of the form (1.3).

The assumption \(q \in (1, \varphi)\) of Theorem 1.3 is sharp: if \(q = \varphi\), then for example \(1\) has only countably many expansions by a theorem of Erdős et al. [7]. Applying Theorem 1.3 we will prove that the Fibonacci expansions behave better:

**Theorem 1.5.** Every \(x \in (0, S)\) has a continuum of Fibonacci expansions.
For the reader's convenience we give a short proof of Theorem 1.1 in Section 2. Then Theorems 1.2, 1.3 and 1.5 are proved in Sections 3, 4 and 5, respectively. At the end of Section 4 we also deduce Theorem 1.3 from Theorem 1.4.

2. Proof of Theorem 1.1

First we prove Theorem 1.1. Given an arbitrary \( x \in (0, S] \), we define a function \( f : \mathbb{N} \to \{1, 2\} \) recursively as follows. Set \( s_1 := x \) and \( s_2 := S - x \). If \( n \geq 1 \) and \( f(1), \ldots, f(n - 1) \) have already been defined (no assumption if \( n = 1 \)), then we choose \( j \in \{1, 2\} \) such that

\[
p_n + \sum_{i < n, f(i) = j} p_i \leq s_j,
\]

and we define \( f(n) := j \). Such a \( j \) exists, for otherwise we would have

\[
2p_n + \sum_{i < n} p_i = 2 \sum_{j=1}^{2} \left( p_n + \sum_{i < n, f(i) = j} p_i \right) > s_1 + s_2 = S = \sum_{i=1}^{\infty} p_i,
\]

contradicting the Kakeya property (1.4).

The sets \( S_1 := \{ i \in \mathbb{N} : f(i) = 1 \} \) and \( S_2 := \{ i \in \mathbb{N} : f(i) = 2 \} \) form a partition \( \mathbb{N} \) such that

\[
\sum_{i \in S_j} p_i \leq s_j, \quad j = 1, 2.
\]

Both inequalities are in fact equalities because

\[
\sum_{i \in S_1} p_i + \sum_{i \in S_2} p_i = \sum_{i=1}^{\infty} p_i = S = s_1 + s_2.
\]

In particular,

\[
\sum_{i \in S_1} p_i = x.
\]

We need a lemma:

Lemma 3.1. We have \( F_i \to +\infty \). Furthermore,

\[
F_{n+1} \leq 2F_n \quad \text{and} \quad \frac{1}{F_n} < \sum_{i=n+1}^{\infty} \frac{1}{F_i}
\]

for all \( n \).

Proof. It follows from the definition that the Fibonacci sequence is a strictly increasing sequence of positive integers. Therefore \( F_i \to +\infty \).

Since \( F_2 < 2F_1 \) and \( F_3 = 2F_2 \), a trivial induction argument based on the identity \( F_{i+2} = F_{i+1} + F_i \) shows that \( F_{n+1} \leq 2F_n \) for all \( n \geq 1 \), and equality holds only if \( n = 2 \). From this, again by induction, we have \( F_{n+j} \leq 2^j F_n \) for all \( n \geq 1 \) and \( j \geq 1 \), and equality holds only if \( n = 2 \) and \( j = 1 \). Indeed, we have

\[
F_{n+j} \leq 2F_{n+j-1} \leq 4F_{n+j-2} \leq \cdots \leq 2^j F_n,
\]
and at least one the inequalities is strict unless \( n = 2 \) and \( j = 1 \). Therefore

\[
\sum_{j=1}^{\infty} \frac{1}{F_{n+j}} > \sum_{j=1}^{\infty} \frac{1}{2^{j}F_{n}} = \frac{1}{F_{n}}
\]

for every \( n \).

\[\square\]

**Proof of Theorem 1.2.** Since the Fibonacci numbers are positive, the formula \( p_{i} := \frac{1}{F_{i}} \) defines a Kakeya sequence by Lemma 3.1, and Theorem 1.2 follows from Theorem 1.1.

\[\square\]

We may place Theorem 1.2 into a broader framework, by considering expansions of the form

\[
(3.1) \quad x = \sum_{i=1}^{\infty} \frac{\varepsilon_{i}}{q^{i}(1 + \varepsilon_{i})}, \quad (\varepsilon_{i}) \in \{0, 1\}^{\mathbb{N}}
\]

with some given real numbers \( q \in (1, 2) \) and \( \varepsilon_{i} > -1 \). The following result shows that \((q^{-i})\), and even some perturbations of \((q^{-i})\) are Kakeya sequences.

**Proposition 3.2.** If \( 1 < q < 2 \) and

\[
(3.2) \quad 1 + \inf_{j} \varepsilon_{j} \geq q^{-1},
\]

then \( \left(\frac{1}{q^{i}(1+\varepsilon_{i})}\right) \) is a Kakeya sequence.

**Proof.** By our assumption we have \( q^{i}(1 + \varepsilon_{i}) > 0 \) for all \( i \), and \( 1 + \inf_{j} \varepsilon_{j} > 0 \). Hence

\[
q^{i}(1 + \varepsilon_{i}) \geq q^{i}(1 + \inf_{j} \varepsilon_{j}) \to +\infty,
\]

and therefore \( \frac{1}{q^{i}(1+\varepsilon_{i})} \to 0 \).

It remains to show for all \( n \geq 1 \) the inequalities

\[
\frac{1}{q^{n}(1 + \varepsilon_{n})} \leq \sum_{i=n+1}^{\infty} \frac{1}{q^{i}(1 + \varepsilon_{i})}.
\]

They follow from the relations

\[
\frac{1}{q^{n}(1 + \varepsilon_{n})} \leq \frac{1}{q^{n}(1 + \inf_{j} \varepsilon_{j})} \leq \sum_{i=n+1}^{\infty} \frac{1}{q^{i}(1 + \sup_{j} \varepsilon_{j})} \leq \sum_{i=n+1}^{\infty} \frac{1}{q^{i}(1 + \varepsilon_{i})},
\]

where the first and third inequalities are obvious, while the middle inequality is equivalent to

\[
\frac{1 + \sup_{j} \varepsilon_{j}}{1 + \inf_{j} \varepsilon_{j}} \leq \frac{1}{q^{-1}};
\]

i.e., to our assumption (3.2).

\[\square\]

**Example.** By Binet’s formula (1.2) the Fibonacci expansions are equivalent to the expansions (3.1) with \( q = \varphi \) and

\[
\varepsilon_{i} = \frac{(-1)^{i+1}}{\varphi^{2i}}
\]

In this case we have

\[
\inf_{j} \varepsilon_{j} = \varepsilon_{2} = -\frac{1}{\varphi^{2}} \quad \text{and} \quad \sup_{j} \varepsilon_{j} = \varepsilon_{1} = \frac{1}{\varphi^{2}}.
\]
Hence
\[
\frac{1 + \inf_j \varepsilon_j}{1 + \sup_j \varepsilon_j} = \frac{1 - \frac{1}{\varphi}}{1 + \frac{1}{\varphi}} = \frac{\varphi^4 - 1}{\varphi^2 (\varphi^2 + 1)} = \frac{\varphi^2 - 1}{\varphi^2} = \frac{\varphi^2 - 1}{\varphi + 1} = \varphi - 1,
\]
so that the condition of Proposition 3.2 is satisfied.

4. PROOF OF THEOREM 1.4

We need a new lemma. Let \((p_i)\) be as in Theorem 1.4. We say that \(p_n\) is a special element if the condition (1.8) is satisfied.

Lemma 4.1. There exists a \(K \geq 1\) such that if we remove from \((p_i)\) a special element \(p_k\) with \(k > K\), then the remaining sequence still satisfies the corresponding hypotheses of Theorem 1.4.

Proof. Fix an \(N \geq 1\) such that
\[ p_n \leq 2p_{n+1} \quad \text{for all } n \geq N, \]
and set
\[ \varepsilon := \min \left\{ -p_n + \sum_{i=n+1}^{\infty} p_i : n = 1, \ldots, N \right\}. \]
By assumption (1.7) we have \(\varepsilon > 0\). Since \(p_i \to 0\), there exists a \(K\) such that \(p_i < \varepsilon\) for all \(i \geq K\).

If we remove from \((p_i)\) a special element \(p_k\) with \(k \geq K\), then the conditions (1.6), (1.8) and (1.9) obviously remain valid. The condition (1.7) also remains valid for all \(n > k\) because the corresponding inequalities are unchanged, and it also remains valid for all \(n \leq N\) by the choice of \(K\). It remains to show that
\[ p_n + p_k < \sum_{i=n+1}^{\infty} p_i \quad \text{for all } N < n < k. \]
This is true for \(n = k - 1\) by (1.8) because \(p_k\) is a special element. Proceeding by induction, if (4.1) holds for some \(N < n < k\), then it also holds for \(n - 1\). Indeed, since \(p_{n-1} \leq 2p_n\) by our choice of \(N\), applying (4.1) we get
\[ p_{n-1} + p_k \leq 2p_n + p_k < p_n + \sum_{i=n+1}^{\infty} p_i = \sum_{i=n}^{\infty} p_i. \]

Proof of Theorem 1.4. Given \(0 < x < S\), using (1.6) we may apply repeatedly Lemma 4.1 to construct a sequence \(p_{i_1} > p_{i_2} > \cdots\) of special elements such that
\[ \sum_{j=1}^{\infty} p_{i_j} \leq \min \{x, S - x\}. \]
We may assume that \(i_{j+1} > i_j + 1\) for infinitely many indices \(j\); then after the removal of the elements \(p_{i_j}\) we still have an infinite sequence, that we denote by \((p'_i)\). Since after the removal of any finite number of elements \(p_{i_1}, \ldots, p_{i_m}\) the remaining sequence still satisfies the corresponding conditions (1.6) and (1.7) of Theorem 1.4, letting \(m \to \infty\) we conclude that \((p'_i)\) is a Kakeya sequence. Now we
may obtain a continuum of expansions of \( x \) as follows. Fix an arbitrary sequence 
\( (c_i) \subset \{0,1\} \). Then
\[
0 \leq \sum_{j=1}^{\infty} c_i p_{i_j} \leq \min\{x, S-x\}
\]
by (4.2), so that
\[
0 \leq x - \sum_{j=1}^{\infty} c_i p_{i_j} \leq x - \sum_{j=1}^{\infty} p_{i_j} = \sum_{i=1}^{\infty} p'_{i}.
\]
By Theorem 1.1 there exists a sequence \( (c'_i) \subset \{0,1\} \) such that
\[
\sum_{i=1}^{\infty} c'_i p'_{i_j} = x - \sum_{j=1}^{\infty} c_i p_{i_j}.
\]
Then
\[
\sum_{j=1}^{\infty} c_i p_{i_j} + \sum_{i=1}^{\infty} c'_i p'_{i_j}
\]
is an expansion of \( x \) in the original system \( (p_i) \) (the order of the positive terms is irrelevant), and different sequences \( (c_i) \) lead to different expansions. \( \square \)

In order to state a corollary of Theorem 1.4 we generalize the geometric sequences. Given a real number \( \rho > 0 \), a sequence \( (p_i) \) of positive numbers is called a \( \rho \)-sequence if
\[
p_{i+1} \geq \rho \cdot p_i \quad \text{for all} \ i.
\]
Corollary 4.2. If \( (p_i) \) is a \( \rho \)-sequence with \( \rho > \frac{1}{\varphi} \), and \( p_i \to 0 \), then every \( x \in (0, \sum_i p_i) \) has a continuum of expansions
\[
x = \sum_{i=1}^{\infty} c_i p_{i_j}, \quad (c_i) \in \{0,1\}^N.
\]
Proof. It suffices to check the conditions of Theorem 1.4. We have \( p_i \to 0 \) by assumption, and
\[
p_n \leq \frac{1}{\rho} p_{n+1} < \varphi p_{n+1} < 2p_{n+1}
\]
for all \( n \), so that the assumptions (1.6) and (1.9) are satisfied. Furthermore, since \( (p_i) \) is a \( \rho \)-sequence with \( \rho > \frac{1}{\varphi} \), and since \( \rho < 1 \) by our assumption \( p_i \to 0 \), the following relations hold for every \( n \geq 2 \):
\[
\sum_{i=n+1}^{\infty} p_i \geq p_{n-1} \sum_{i=2}^{\infty} \rho^i = \frac{\rho^2}{1-\rho} p_{n-1} > p_{n-1}.
\]
This proves (1.8) for all \( n \geq 2 \), and this implies (1.7). \( \square \)

Example. Corollary 4.2 reduces to Theorem 1.3 if \( p_i = q^{-i} \) with \( q \in (1, \varphi) \).
5. Proof of Theorem 1.5

For the proof of Theorem 1.5 we need some more properties of the Fibonacci sequence. We recall the identity

\[ F_i^2 = F_{i-1} F_{i+1} + (-1)^{i+1}, \quad i = 2, 3, \ldots. \]

It holds for \( i = 2 \) by a direct inspection: \( 1^2 = 1 \cdot 2 - 1 \). Proceeding by induction, if it holds for some \( i \geq 2 \), then it also holds for \( i + 1 \) because

\[ F_i^2 = F_{i-1} F_{i+1} + (-1)^{i+1} \]

\[ \implies F_i (F_i + F_{i+1}) = (F_{i-1} + F_i) F_{i+1} + (-1)^{i+1} \]

\[ \implies F_i F_{i+2} = F_{i+1}^2 + (-1)^{i+1} \]

\[ \implies F_{i+1}^2 = F_i F_{i+2} + (-1)^{i+2}. \]

Lemma 5.1. If \( k \) is a positive odd integer, then

\[ \frac{1}{F_k} < \sum_{i=k+2}^{\infty} \frac{1}{F_i}. \]

Proof. Fix any positive integer \( k \), and set

\[ \alpha := \frac{F_{k+1}}{F_k}, \quad \beta := \frac{F_{k+2}}{F_{k+1}}, \quad \gamma := \max\{\alpha, \beta\}. \]

Then \( F_{i+1} \leq \gamma F_i \) for \( i = k, k+1 \), and hence by induction for all \( i \geq k \). Since \( \alpha \neq \beta \) by (5.1), one of the equalities \( F_{k+1} \leq \gamma F_k \) and \( F_{k+2} \leq \gamma F_{k+1} \) is strict. Therefore

\[ \sum_{i=k+2}^{\infty} \frac{1}{F_i} \geq \frac{1}{F_{k+1}} \sum_{j=1}^{\infty} \frac{1}{\gamma^j} = \frac{1}{F_{k+1}} \cdot \frac{1}{\gamma - 1} = \frac{1}{F_k} \cdot \frac{1}{\alpha \cdot (\gamma - 1)}. \]

If \( k \) is odd, then this implies (5.2) because \( \beta > \alpha \) by (5.1), so that \( \gamma = \beta \), and hence

\[ \alpha \cdot (\gamma - 1) = \frac{F_{k+1}}{F_k} \cdot \left( \frac{F_{k+2}}{F_{k+1}} - 1 \right) = \frac{F_{k+2} - F_{k+1}}{F_k} = 1. \]

Proof of Theorem 1.5. The conditions (1.6), (1.7) et (1.9) of Theorem 1.4 are satisfied for \( p_i := 1/F_i \) by Lemma 3.1 and (1.8) is satisfied by Lemma 5.1.

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