Semiclassical Identification of Periodic Orbits in a Quantum Many-Body System

Maram Akila, Daniel Waltner, Boris Gutkin, Petr Braun and Thomas Guhr
Fakultät für Physik, Universität Duisburg-Essen, Lotharstraße 1, 47048 Duisburg, Germany

While a wealth of results has been obtained for chaos in single-particle quantum systems, much less is known about chaos in quantum many-body systems. We contribute to recent efforts to make a semiclassical analysis of such systems feasible, which is nontrivial due to the exponential proliferation of orbits with increasing particle number. Employing a recently discovered duality relation, we focus on the collective, coherent motion that together with the also present incoherent one typically leads to a mixture of regular and chaotic dynamics. We investigate a kicked spin chain as an example of a presently experimentally and theoretically much studied class of systems.

Introduction — The first step that later on led to the field of quantum chaos was arguably the introduction of Random Matrix Theory (RMT) in the early 50’s by Wigner to study statistical aspects of nuclei, for a review see Ref. [1–5]. Subsequently, RMT was applied to model the statistics of atomic and molecular spectra. Importantly, these systems are self-bound and interacting many-body systems. In the following decades the insight emerged that RMT also applies to a single quantum particle in a complicated potential. Numerical studies of billiards became popular to explore the connection between classical dynamics and quantum level statistics [4–7]. They led to the celebrated Bohigas-Giannoni-Schmit (BGS) conjecture stating that the level statistics of the quantum system should be described by RMT if the corresponding classical system is fully chaotic. Classical periodic orbits (POs) and the Gutzwiller trace formula made a detailed spectral analysis possible [1–2, 8, e.g. for the Hydrogen atom in a strong magnetic field [9–11], and also a heuristic understanding of the BGS conjecture [12]. Furthermore, in the early 80’s, the far-reaching connections between mesoscopic and quantum chaotic systems were uncovered [13–15]. In the early 90’s, deeper insights into the structure of POs led to much stronger arguments supporting the BGS conjecture [16].

The focus on single-particle systems which had started around 1980 let parts of the quantum chaos community almost forget that many-body systems were the objects of interest in the early days of quantum chaos. Only in recent years, new attempts to address many-body systems in the present context were put forward, in particular many-body localization [17–19] also observed in recent experiments [20, 21], spreading in self-bound many-body systems [22, 23], a semiclassical analysis of correlated many-particle paths in Bose Hubbard chains [24] and a trace formula connecting the energy levels to the classical many-body orbits [25, 26], to mention just a few. There are also attempts to study field theories semiclassically [27]. The first new aspect, specific to many body systems, is the existence of two large parameters — the number of particles \( N \) and the dimension of the Hilbert space controlled by the inverse of the effective Planck constant \( \hbar_{\text{eff}} \). Therefore, different semiclassical limits exist, see Ref. [28].

The second new feature is the complexity of many-body dynamics. In particular, many-body systems show collective motion, not present in single-particle systems. There are various definitions of collectivity. Here, we simply mean a coherent motion of all or of large groups of particles which can be identified in the classical phase space as well as in the quantum dynamics. Typically, a many-body system can exhibit incoherent, \textit{i.e.} non-collective, motion of its particles, coherent collective motion and forms of motion in between. Which dynamics emerges depends on the interaction, the excitation energy and the way how the system is probed. Importantly, collectivity has a strong impact on the level statistics as is known from numerous analyses of nuclear spectra. While incoherent particle motion leads to RMT statistics as in the famous example of the nuclear data ensemble [29–30], collective excitations often show Poisson statistics typical to integrable systems, as \textit{e.g.} in Ref. [31], see Ref. [3] for a review. In general, due to the mixed phase space, the BGS conjecture is not directly applicable to many-body systems.

Owing to the focus on the universal regime, the aforementioned studies of Bose-Hubbard chains [24, 25] did not take this collectivity into consideration, as only generic properties of chaotic dynamics were assumed. To illuminate the full complexity of the motion in many-body systems and the importance of collectivity from a semiclassical viewpoint, we consider a chain of \( N \) spins. This is a many-body generalization of the kicked top, often used as a model for single particle chaos [1]. We focus on the short time regime but consider arbitrary \( N \), where the collectivity plays a significant rôle. We have three main goals: First, we want to establish a new method for the semiclassical analysis of kicked many-body systems providing understanding of the quantum evolution in terms of classical many-particle orbits. The huge dimension of the Hilbert space might seem to raise an impenetrable barrier for the applicability of semiclassical tools such as the Gutzwiller trace formula [32]. We demonstrate that this problem can be circumvented by a suitable generalization of the recently discovered duality relation [28, 33] that maps properties of the time evol-
lution and of the enlargement of the system (by adding further particles) onto each other. Second, we wish to demonstrate the importance of collective motion. It may even dominate the quantum spectrum for large particle numbers. Third, we wish to provide a better understanding of spin chain dynamics as this class of systems is presently in the focus of theoretical [34–37] and experimental [38] research.

System Considered — We study a chain-like kicked quantum system of $N$ spins with nearest neighbor interaction [32] described by the Hamiltonian

$$\hat{H} = \hat{H}_I + \hat{H}_K \sum_{T=-\infty}^{\infty} \delta(t-T)$$

where the interaction part $\hat{H}_I$ and the kick part $\hat{H}_K$,

$$\hat{H}_I = \sum_{n=1}^{N} \frac{4J\sigma_{n+1}^{z}\sigma_{n}^{z}}{(j+1/2)^2}, \quad \hat{H}_K = \frac{2}{j+1/2}\sum_{n=1}^{N} \mathbf{b} \cdot \mathbf{s}_n ,$$

where $\mathbf{s}_n = (\sigma_n^x, \sigma_n^y, \sigma_n^z)$ are the operators for spin $n$ and quantum number $j$. Periodic boundary conditions, i.e. $\sigma_n^{x+1} = \sigma_1^x$, make the system translation invariant. Moreover, $J$ is the coupling constant and $\mathbf{b}$ a magnetic field, assumed without loss of generality to have the form $\mathbf{b} = (b^x, 0, b^z)$. We rescaled the terms in Eq. (2) by $j+1/2$ in order to keep them bounded for $j \to \infty$. The kicks act at discrete integer times $T$.

The one period evolution (Floquet) operator reads

$$\hat{U} = \hat{U}_I \hat{U}_K, \hat{U}_I = e^{-i(j+1/2)\hat{H}_I}, \hat{U}_K = e^{-i(j+1/2)\hat{H}_K},$$

where $(j+1/2)^{-1}$ takes on the rôle of the Planck constant $\hbar$. We find the corresponding classical system by replacing $\mathbf{s}_m \rightarrow \sqrt{j(j+1)} \mathbf{n}_m$ with a classical spin unit vector precessing on the Bloch sphere $\mathbf{n}_m$. The time evolution can therefore be interpreted as the action of two subsequent rotation matrices

$$\mathbf{n}_m(T+1) = R_{a}(4J\chi_m)R_{b}(2\mathbf{b})\mathbf{n}_m(T),$$

first around the magnetic field axis and then around the $z$ axis (Ising part) with angle $4J\chi_m, \chi_m = \sigma_m^{z-1} + \sigma_m^{z+1}$. The classical system can be cast in Hamiltonian form,

$$H(q, p) = \sum_{n=1}^{N} \left[ 4Jp_{n+1}p_n + \sum_{T=-\infty}^{\infty} \delta(t-T) \times 2 \left( b^x p_n + b^z \sqrt{1-p_n^2} \cos q_n \right) \right],$$

from which the canonical equations follow. The $N$-component vectors $p$ and $q$ are the conjugate momenta and positions of the $N$ (classical) spins, respectively. The vectors on the Bloch sphere are given by

$$\mathbf{n}_m = \left( \sqrt{1-p_m^2} \cos q_m, \sqrt{1-p_m^2} \sin q_m, p_m \right)$$
in terms of the canonical variables.

Periodic Orbit Expansion — In [33] we recently managed to express the trace of the propagator $\hat{U}$ to power $T$ for an interacting spin system in a Gutzwiller-type-of form valid in the limit $j \to \infty$

$$\text{Tr} \hat{U}^T \sim \sum_{s=T} \gamma_s e^{i(j+1/2)S_s} .$$

This is a sum over classical POs $\gamma$ of duration $T$ if they are well isolated. Here, $S_s = \int \mathbf{p} \cdot d\mathbf{q} - \int H(q, p) dt$ is the classical action and, for a sufficiently isolated orbit, $A_s = T_s^2 \gamma_s / \sqrt{\det(M_s - I)}$ with $T_s^2$ the primitive period of the orbit, $M_s$ the monodromy matrix and $G_s$ the Maslov phase of $\gamma$. The $2N$ eigenvalues $e^{\pm A_s}$ of $M_s$ determine the stability of the PO $\gamma$. For the Hamiltonian [39], most POs are neither fully stable nor unstable. The factor $A_s$ is finite if all $\lambda_i \neq 0$. If, however, $\lambda_i = 0$ for at least one marginal direction, then $A_s$ diverges. This happens, for instance, if the system undergoes a bifurcation, see [44].

The connection between the classical and the quantum system is revealed by taking the Fourier transform $\rho(S)$ of Eq. (7) in $j$. This is methodically similar to Ref. [10, 11] and was also used on the single particle kicked top by [44, 45]. We find

$$\rho(S) = \frac{1}{j_{\text{cut}}} \sum_{j=1}^{j_{\text{cut}}} e^{-i(j+1/2)S} \text{Tr} \hat{U}^T$$

which approximates the action spectrum as it features peaks of width approximately $\pi/j_{\text{cut}}$ whose positions are given by the actions modulo $2\pi$ of the POs with length $T$. The peak heights of isolated orbits are $A_s$, independent of $j_{\text{cut}}$.

Duality Relation — At this point, we have to overcome a severe problem. To resolve the peaks in $\rho(S)$ we need to compute $\text{Tr} \hat{U}^T$ for sufficiently large $j_{\text{cut}}$. But as its matrix dimension $(2j+1)^N \times (2j+1)^N$ grows exponentially with $N$, a direct calculation of the spectrum of $\hat{U}$ is impossible, e.g., even the propagator $\hat{U}^T$ for $N=19$ spins at $j=1$ has a matrix dimension of $10^9 \times 10^9$. Luckily, recently developed duality relations [28, 33] provide the solution and make possible, for the first time, a semiclassical analysis of genuine many-body orbits. The crucial ingredient is the exact identity

$$\text{Tr} \hat{U}^T = \text{Tr} \hat{U}^N.$$

The trace over the time-evolution operator $\hat{U}$ for $T$ time steps equals the trace over a “particle-number-evolution” operator $\hat{U}$ for $N$ particles. The form of the non-unitary dual operator $\hat{U}$ is similar to that of the time-evolution operator for a chain of $T$ spins. Its dimension $(2j+1)^T \times (2j+1)^T$.
(2j + 1)^T$ is governed by the time $T$ while the dimension $(2j + 1)^N \times (2j + 1)^N$ of the operator $\hat{U}$ grows with the particle number $N$. This duality allows us to calculate $\rho(S)$ for arbitrary $N$ as long as $T$ is sufficiently short. We generalize this duality approach, originally developed for $j = 1/2$ \cite{35}, to $j \gg 1$ to make it applicable in the present context \cite{46}. The form of $\hat{U}$ is given in the supplemental material.

**Incoherent versus Collective Motion** — Anticipating the results of the subsequent detailed analysis, we sketch the role of some collective orbits as opposed to the incoherent ones. Given translational invariance, a PO of the $M$-particle system generates a PO in the system with $kM$ particles for any integer $k$. We therefore introduce $N_{\gamma}^{(P)}$, analogous to $T_{\gamma}^{(P)}$, as the minimal number of particles required for a PO to close in the $N$-body system. Such orbits with relatively small $N_{\gamma}^{(P)} = N/k$ yield a form of collective dynamics as the spins $n$ and $n + N_{\gamma}^{(P)}$ move synchronously. The action of these orbits scales linearly with $N$, $S_\gamma = (N/N_{\gamma}^{(P)})S_{\gamma}^{P}$, where $S_{\gamma}^{P}$ is the action for $N_{\gamma}^{(P)}$ particles. Moreover, if, for example, an orbit for $N$ spins has a marginal direction, this implies a marginal direction in the corresponding PO for $kN$ spins. The number of periodic orbits grows strongly with $N$ which limits their resolution. However, collective orbits can stick out in such a remarkable way that they dominate the spectra.

**Details of the Analysis** — To demonstrate the power of our method we provide a numerical calculation of $|\rho(S)|$ for $T = 1$ and $N = 7$ spins in Fig. 1(a). In this case the number of POs is sufficiently low for a semiclassical analysis of individual peaks. The black peaks result from the Fourier transformation of $\text{Tr} \ U^T$, the positions of the colored peaks are given by the actions $S_{\gamma}$ of the POs derived from the Hamiltonian \cite{5}. Both, collective $N_{\gamma}^{(P)} = 1$ and non-collective $N_{\gamma}^{(P)} = 7$ types of POs are present. The former are highlighted by arrows in Fig. 1(a). The action spectrum $S_{\gamma}$ of POs is well reproduced, by the positions of the maxima of $|\rho(S)|$. The only exception is a peak marked by $\blacklozenge$ in Fig. 1(a), which is due to a complex predecessor of a nearly bifurcating orbit. The existence of such ghost orbits is known in the context of single-particle systems \cite{15}.

In Fig. 1(b) we depict $|\rho(S)|$, for $T = 1$, but now for $N = 19$ spins. Since the number of POs grows exponentially with $N$ it is not possible in this case to resolve all of them for computationally feasible values of $j_{\text{cut}}$. Nevertheless, it is possible to identify those that provide the most significant contribution to $\text{Tr} \ U^T$. Therefore we employ a simple filtering criterion by selecting only POs satisfying $A_{\gamma}^{-2} \propto |\det(M_{\gamma} - 1)| < 10^6$. As one can see, the actions of these orbits reproduce the positions of the most significant spikes of $|\rho(S)|$. The semiclassical reconstruction of $|\rho(S)|$ becomes additionally more challenging with growing $N$ due to the increasing number of nearly bifurcating POs. In contrast to an isolated PO where $|\rho(S)|$ does not scale with $j_{\text{cut}}$, we find for bifurcating orbits a nontrivial scaling $|\rho(S_{\gamma})| \sim (j_{\text{cut}})^{\alpha}$, where $\alpha$ depends on the bifurcation type. Importantly, for $T = 1$ the exponent $\alpha$ does not grow with $N$, see the lower dotted line in Fig. 2. For a single degree of freedom the effect of bifurcations was studied in \cite{41,47}. We found these results consistent with our numerics.

**FIG. 2:** Scaling exponent $\alpha$ of $|\rho(S_{\gamma})|$ as in Fig. 1(a) for $N = 7, 19$ spins. For $T = 1$, $\alpha$ depends on the number of spins. For $T = 1, 2$, $\alpha = 0.7,$ $j_{\text{cut}} = 4650$ and $b$ as in a). The positions of the classical POs as red lines for $N_{\gamma}^{(P)} = 1$, as blue lines for $N_{\gamma}^{(P)} = N$.
**Dominance of collectivity** — In both spectra in Fig. [1] collective orbits with $N_s^P < N$ play only a minor role. This appears to be a systematic effect for $T = 1$. In Fig. [2] we display $|\rho(S)|$ for $T = 2$. We find qualitatively similar pictures for larger $T$. The most striking features are the gigantic peaks, arising whenever the particle number is an integer multiple of four, $N = 4k$. They overshadow all other structures, see the panels for $N = 4, 8, 20, 100$. This is especially remarkable for $N = 100$ given the enormous number of POs. Furthermore, the scaling exponent $\alpha$ of the peak heights grows linearly with $N$, $\alpha(N) \sim \alpha_1 N$, where $\alpha_1 \approx 1/5$ in a wide parameter range. This structure is thus important for large $N$. Even $\alpha(4)$ is significantly larger than the corresponding values for bifurcations in single particle systems. This clearly indicates collective motion and, importantly, we can explain it by the emergence of certain classical structures in the phase space of the system as explained below. In the other panels of Fig. [3] where $N \neq 4k$, similar peaks are not seen. Due to a large number of POs we employ for $N = 6, 7$ the filtering condition $|\det(M_i - 1)| < 5 \times 10^5$. The collective orbit highlighted by the arrow, featuring $N_s^P = 1$, is the same for all panels and creates a discernible peak still for $N = 7$. The reason for this is that it is not sufficiently isolated but close to a pitchfork bifurcation.

**Semiclassical interpretation for the multiple-of-four collectivity** — The large peaks for $N = 4k$ do not result from isolated orbits, but from a four-dimensional manifold of POs. To explain this, we return to Eq. (4). We demand that the Ising rotation angle $4J \chi_m$ is identical for all spins, $\chi_m = \chi$, such that $R = R_b(4J\chi)R_b(2|b|)$ is a rotation by $\pi$ around the same axis for both time steps. This condition can be met if $N = 4k$. For $N = 4$ it imposes four restrictions (two per time step) on the angle of the Ising rotation of each pair of even and odd spins. After elementary calculations we obtain that the orientation of the spins must satisfy

$$\chi = n_{i-1}^z + n_{i+1}^z$$

$$\chi = (n_{i-1}^x + n_{i+1}^x) \cot \beta + (n_{i-1}^y + n_{i+1}^y) \frac{\cot |b|}{\sin \beta},$$

where $\beta = \arctan b^x/b^z$ is the angle between the magnetic field and the $z$ axis. The angle $\chi$ is calculated from

$$b^z \tan (2J\chi) = |b| \cot |b|. \quad (11)$$

The conditions (10) and (11) define a four-dimensional manifold. All points of it are POs with period $T = 2$ due to $R^2 = \mathbb{1}$ with the action $S_{\text{man}} = 2NJ\chi^2$. This manifold exists as repetition also for $N = 4k$. As $\chi$ is independent of $n$ for all spins their motion is highly correlated. They perform a collective solid body rotation keeping the angles between the spins constant over time. Fig. [3] shows a trajectory on the manifold after the action of each rotation. In the cases where Eq. (11) allows more than one possible solution for $\chi$, the system behavior is more complicated. Several manifolds with possible mixtures of the $\chi$, occur, and their number grows with $N$. As a result, in this regime $|\rho(S)|$ attains a more complicated profile with several maxima [16].

To quantify the impact of the PO manifold for fixed $j$ on the quantum spectrum it is instructive to study to what extent the phase the $\text{Tr} \hat{U}^T$ is determined by the action $S_{\text{man}}$ of the orbit leading to the largest peak in $|\rho(S)|$. Therefore we introduce

$$\Delta(j) = \text{Im \Log} \text{Tr} \hat{U}^T - (2j + 1)S_{\text{man}} \mod 2\pi. \quad (12)$$

As shown in Fig. [5] for $N \neq 4k$, $\Delta(j)$ is a wildly fluctuating function of $j$. However, for $T = 2$, $N = 4k$, $\Delta(j)$ is approximately constant. This implies that the phase of $\text{Tr} \hat{U}^T$ is strongly dominated by the single term provided by the PO manifold. This is due to the structure of the eigenvalues of the dual operator. For large $N$ the duality relationship [3] guarantees that $\text{Tr} \hat{U}^T$ is dominated by the eigenvalues of the dual operator $\hat{U}$ with largest magnitude. We find the remarkable property [46] that $\hat{U}$ possesses four eigenvalues with largest magnitude $a_l e^{i\varphi_l}$, $l = 1, \ldots, 4$ where $a_l = \text{const.} j^{\alpha_1}(1 + O(1/j))$ and

$$\varphi_l = (j + 1/2)S_{\text{man}}/N + \frac{\pi l}{2} + O(1/j) \mod 2\pi. \quad (13)$$

This is formally similar to a Bohr-Sommerfeld quantization rule for $S_{\text{man}}$. The existence of collective dynamics for $N = 4k$ finds here its correspondence in the fact that the contribution of the four eigenvalues $(a_l)^N e^{iN\varphi_l}$ to $\text{Tr} \hat{U}^T$ cancels due to the $l$-dependence of the phases in (13) except for $N = 4k$ where $N\varphi_l$ is independent of $l$.  

**Conclusions** — We carried out a semiclassical analysis of a (non-integrable) many-body quantum system. We studied a kicked spin chain as a representative of a class of systems which presently is in the focus of experimental and theoretical research. For the first time, we presented a unifying semiclassical approach to incoherent and to coherent, collective dynamics. The key tool was a recently discovered duality relation between the evolutions in time and particle number. It outmaneuvers the drastically increasing complexity of the problem with growing particle number. In the spin chain a certain type of collective motion strongly contributes to the spectra, whenever the particle number is an integer multiple of four. An experimental verification is likely to be feasible in view of the improving ability to control systems with larger numbers of spins.

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FIG. 3: For $T=2$, $|\rho(S)|$ as black curve with parameters $J=0.7$, $b^x=b^y=0.9$ and $j_{\text{cut}}=114$. The peaks for $N_{\gamma}(P)=N$ are blue, for $N_{\gamma}(P)=1$ red and yellow otherwise. For $N=4,8,20,100$, very large peaks, marked green, due to collective motion occur at actions $S_{\text{man}}$.

FIG. 4: Trajectory of a PO on the manifold, depicted after each rotation step for $J=0.7$ and $b^x=b^y=0.9$. Spins are ordered according to blue, yellow, green and red.

FIG. 5: Difference $\Delta(j)$ divided by $N$, for the manifolds with $N=4$ as orange circles and with $N=80$ as green squares, and for $N=3$ as blue diamonds.
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Supplemental material for “Semiclassical Identification of Periodic Orbits in a Quantum Many-Body System”

**Explicit Form of the Dual Operator** — In terms of the \((2j + 1)^T\) dimensional product basis,

\[
|\sigma\rangle = |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \cdots \otimes |\sigma_T\rangle
\]

(14)

with discrete single spin states \(\sigma_t \in \{-j, -j+1, \ldots, +j\}\), we can provide the explicit form of the dual operator \(\tilde{U} = \tilde{U}_I \tilde{U}_K\). The interaction part is a diagonal matrix with elements

\[
\langle \sigma | \tilde{U}_I | \sigma' \rangle = \delta_{\sigma, \sigma'} \prod_{t=1}^T \langle \sigma_t | \exp (2i \mathbf{b} \cdot \hat{\mathbf{s}}) | \sigma_{t+1} \rangle. \tag{15}
\]

The boundary conditions are periodic, i.e. \(\sigma_{T+1} = \sigma_1\). The kick part features a local structure

\[
\tilde{U}_K = \bigotimes_{t=1}^T \tilde{u}_K, \quad \langle \sigma | \tilde{u}_K | \sigma' \rangle = \exp \frac{4iJ\sigma \sigma'}{j + 1/2}. \tag{16}
\]

Although \(\tilde{u}_K\) is related to the interaction of \(\tilde{U}_I\) it is not diagonal.

An explicit example can be given in the integrable case \((\hat{b}^2 = 0)\) where the components of the dual operator are

\[
\tilde{U}_{nm} = \exp \left( i \frac{4JT}{j + 1/2} (n - j - 1)(m - j - 1) + 2iTb^2(n - j - 1) \right). \tag{17}
\]

The indices \(m, n\) run from 1 to \(2j + 1\) and time turns, in this case only, to a scaling of the system parameters.