Cosmological diagrammatic rules

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Abstract. A simple set of diagrammatic rules is formulated for perturbative evaluation of “in-in” correlators, as is needed in cosmology and other nonequilibrium problems. These rules are both intuitive, and efficient for calculational purposes.

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When calculating quantities relevant for cosmological evolution, one needs an efficient means to calculate operator expectation values, or more complicated expressions, in the “in-in,” or Schwinger-Keldysh, formalism.\textsuperscript{1} Various methods or rules have been derived to do this, in particular the closed time path formalism, and the rules outlined in [3] and [5–7]. This note formulates a refined set of such diagrammatic rules that seems both intuitive, and is efficient for calculational purposes. For example, this reduces the calculational complexity common in uses of the closed time path formalism.

As a simple and concrete example, take $\phi^3$ theory,

\begin{equation}
\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{g}{3!} \phi^3
\end{equation}

in a fixed background Robertson-Walker metric,

\begin{equation}
ds^2 = -dt^2 + a^2(t) ds_3^2,
\end{equation}

where $ds_3$ is the metric of a homogeneous space. It is useful to work with conformal time, $\eta$, instead of physical time, $t$, which is defined by

\begin{equation}
a(\eta)d\eta = dt.
\end{equation}

In the in-in formalism the expectation value of any operator $O$ (evaluated at time $\eta_0$, and up to vacuum normalization) is given by

\begin{equation}
\langle \Omega | O(\eta_0) | \Omega \rangle = \langle 0 | \bar{T} \left( e^{i \int_{\eta_0}^{\eta_0} d\eta H_I} \right) O(\eta_0) T \left( e^{-i \int_{\eta_0}^{\eta_0} d\eta H_I} \right) | 0 \rangle
\end{equation}

where $| \Omega \rangle$ is the vacuum of the interacting theory, $| 0 \rangle$ is the vacuum of the free theory, $T$ and $\bar{T}$ are time ordering and anti-ordering operators, respectively, and $H_I$ is the interaction hamiltonian in time $\eta$. The expectation value can be evaluated by expanding the exponential, and contracting fields, as in the usual Wick analysis of in-out amplitudes in flat-space field theory.

Concretely, consider the operator $O = \phi(x, \eta_0)\phi(x', \eta_0)$. The correction to the corresponding propagator to second order in $g$ follows from the second-order terms in the expansion of the exponentials. There are two kinds of terms. The first is

\begin{equation}
A(\eta_0, x, x') = \langle 0 | \phi(x, \eta_0)\phi(x', \eta_0) T \frac{1}{2!} \int dy \frac{-ig}{3!} \phi^3(y) \int dy' \frac{-ig}{3!} \phi^3(y') | 0 \rangle,
\end{equation}

with a corresponding term also from the left exponential. Here we use notation $y = (\eta, y)$, $dy = a^4 d^3y d\eta$, and the $\eta$ integrals range up to $\eta_0$. The second kind of term comes from the linear expansion in each of the exponentials:

\begin{equation}
\langle 0 | \bar{T} \int dy \frac{ig}{3!} \phi^3(y) \phi(x, \eta_0)\phi(x', \eta_0) T \int dy' \frac{ig}{3!} \phi^3(y') | 0 \rangle.
\end{equation}

Two kinds of propagator enter the corresponding expressions, the Wightman propagator,

\begin{equation}
W(x, x') = \langle 0 | \phi(x)\phi(x') | 0 \rangle,
\end{equation}

and the Feynman propagator,

\begin{equation}
G(x, x') = \langle 0 | T \phi(x)\phi(x') | 0 \rangle.
\end{equation}

\textsuperscript{1}For reviews, see for example [1–4].
Both expressions may be evaluated directly in terms of corresponding Wick contractions.

For (5), we find

$$A(\eta_0, x, x') = \frac{1}{2} \int dy(-ig) \int dy'(-ig) W(x, y) G(y, y')^2 W(x', y') \ .$$  \hspace{1cm} (9)

We could also directly find the term from expanding the left exponential, but observe that it simply gives the complex conjugate expression, with a combined contribution $2 \text{Re}A$.

Likewise, the contractions from (6) provide two separate terms. The first is

$$B(\eta_0, x, x') = \frac{1}{2} \int dy(i g) \int dy'(-ig) W(y, x) W(x', y') W^2(y, y') \ .$$  \hspace{1cm} (10)

The second is again the complex conjugate expression.

The expressions above are given by the following rules:

1. Draw a horizontal dotted line, corresponding to $\eta_0$, and place the external points of the correlator on this line.

2. At a given order, enumerate all placements of vertices either above or below the $\eta_0$ line, modulo reflections about this line. Then, draw all diagrams connecting these vertices with propagator lines, again modulo reflection.

3. Each propagator line crossing or ending on the dotted line gives a Wightman propagator, whose leftmost/rightmost time argument corresponds to the uppermost/lowermost vertex. Each propagator line below the dotted line gives a Feynman propagator, and each line above the dotted line gives the complex conjugate or time-reversed Feynman propagator.

4. Each vertex below the dotted line gives a $\mathcal{V} = -ig$ together with an integral, and each above gives an $\mathcal{V}^\dagger = ig$ and an integral.

5. Divide by the usual Feynman symmetry factors, where present.

6. Once the resulting diagrams are calculated, take twice their real part.

The corresponding diagrams for the amplitudes $A(\eta_0, x, x')$ and $B(\eta_0, x, x')$ are shown in figure 1. The $1/2$ in each corresponding expression is a symmetry factor.

If one works in a cosmological spacetime with a flat slicing, so that linear spatial momentum is conserved, these rules are easily reformulated working in momentum space (but retaining time $\eta$ as parameter). Specifically, in rule (3) one uses the momentum-space version of the propagator, and we replace

4. → 4.’ Vertices below/above the dotted line are accompanied by $V$ or $V^\dagger$, respectively; conserve momentum at each vertex and include an overall momentum-conserving delta function, integrate over all internal momenta, and integrate over the time coordinate of each vertex.

Clearly these rules have a trivial generalization to theories with more fields and more complicated vertices.
Examples. These rules appear to offer modest streamlining of existing calculations. For example, via these rules one immediately writes down (5) and (6), or the momentum-space expressions,

$$2\text{Re} A(\eta_0, k, k') = -g^2(2\pi)^3 \delta^3(k + k') \text{Re} \int a^4 \, d\eta a^4 \, d\eta' \quad \text{(11)}$$
$$\times \int dq W_k(\eta_0, \eta) G_q(\eta, \eta') G_{|k-q|}(\eta, \eta') W_k(\eta_0, \eta')$$
$$2\text{Re} B(\eta_0, k, k') = g^2(2\pi)^3 \delta^3(k + k') \text{Re} \int a^4 \, d\eta a^4 \, d\eta' \quad \text{(12)}$$
$$\times \int dq W_k(\eta, \eta_0) W_k(\eta, \eta') W_{|k-q|}(\eta, \eta') W_k(\eta_0, \eta')$$

where a useful shorthand for calculations is $dq = d^3q/(2\pi)^3$. In the special case of de Sitter space, we then use the Wightman propagator

$$\langle \phi_k(\eta) \phi_{k'}(\eta') \rangle = (2\pi)^3 \delta^3(k + k') W_k(\eta, \eta') \quad \text{(13)}$$

with

$$W_k(\eta, \eta') = U_k(\eta) U_k^*(\eta') \quad \text{(14)}$$

and

$$U_k(\eta) = \frac{H}{\sqrt{2k^3}} (1 + i\eta)e^{-i\eta} \quad \text{(15)}$$

and the corresponding Feynman propagator, with

$$G_k(\eta, \eta') = \theta(\eta - \eta') W_k(\eta, \eta') + \theta(\eta' - \eta) W_k(\eta', \eta) \quad \text{(16)}$$

We could also consider the same type of diagram, but with graviton-scalar interactions, e.g.

$$\mathcal{L}_3 = \frac{a}{2} \gamma_{ij} \partial_i \sigma \partial_j \sigma \quad \text{(17)}$$

and using the graviton propagator in transverse traceless gauge,

$$\langle \gamma_{ij}(k, \eta) \gamma_{kl}(k', \eta') \rangle = (2\pi)^3 \delta^3(k + k') 2\omega_{ij,kl}(k) W_k(\eta, \eta') \quad \text{(18)}$$
with polarization sum
\[ \omega_{ij,kl}(q) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} \]
\[ + \delta_{ij}\hat{q}_k\hat{q}_l + \delta_{kl}\hat{q}_i\hat{q}_j - \delta_{ik}\hat{q}_j\hat{q}_l - \delta_{il}\hat{q}_j\hat{q}_k - \delta_{jk}\hat{q}_i\hat{q}_l - \delta_{jl}\hat{q}_i\hat{q}_k + \hat{q}_i\hat{q}_j\hat{q}_k\hat{q}_l \]  

one immediately reproduces the two one-loop bubble diagrams given in eqs. (3.14) and (3.15)
of [8]. Likewise, with external gravitons, one has an immediate derivation of eq. (20) of [9]. In some cases, like the graviton bubble of [8], it is then possible to explicitly perform the $\eta$ integrals to find elementary expressions.

Another example is the tri-spectrum, calculated in [10]. There we have the six diagrams of figure 2 and figure 3, once reflection symmetry is accounted for. Then (taking the simpler case of an exchanged scalar, via the interaction (1)), we immediately write down the amplitude

$$\langle \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}\rangle = -g^2(2\pi)^3\delta^3 \left( \sum_i k_i \right) \cdot 2\text{Re} \left[ \int_{-\infty}^{\eta_0} a^4 d\eta \int_{-\infty}^{\eta_0} a^4 d\eta' \right] \times W_{k_1}(\eta, \eta)W_{k_2}(\eta, \eta)W_{k_3}(\eta, \eta')W_{k_4}(\eta, \eta')G_{|k_1+k_2|}(\eta, \eta')$$

$$-(2 \leftrightarrow 3) + (2 \leftrightarrow 4)$$

to be compared with (2.20) and subsequent formulas in [10].

Thus, we do find calculational streamlining, which we expect to be more significant for more complicated diagrams.

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