On the Orbits of Crossed Cubes

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Abstract

An orbit of \( G \) is a subset \( S \) of \( V(G) \) such that \( \phi(u) = v \) for any two vertices \( u, v \in S \), where \( \phi \) is an isomorphism of \( G \). The orbit number of a graph \( G \), denoted by \( \text{Orb}(G) \), is the number of orbits of \( G \). In [A Note on Path Embedding in Crossed Cubes with Faulty Vertices, Information Processing Letters 121 (2017) pp. 34–38], Chen et al. conjectured that \( \text{Orb}(CQ_n) = 2\lceil \frac{n}{2} \rceil - 2 \) for \( n \geq 3 \), where \( CQ_n \) denotes an \( n \)-dimensional crossed cube. In this paper, we settle the conjecture.

Keywords: Crossed cubes; Automorphism; Vertex-transitive; Orbits.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$ which are simply denoted by $V$ and $E$, respectively, when the context is clear. An automorphism of a graph $G = (V, E)$ is a mapping $\phi : V(G) \rightarrow V(G)$ such that there is an edge $uv \in E(G)$ if and only if $\phi(u)\phi(v)$ is also an edge in $E(G)$. A graph is vertex-transitive if, for any two vertices $u$ and $v$ of $G$, there is an automorphism $\phi$ such that $\phi(u) = v$. Clearly, every vertex-transitive graph is regular. However, not all regular graphs are vertex-transitive, e.g., crossed cubes [11,12] and the Frucht graph [19].

Definition 1.1. An orbit of $G$ is a subset $S$ of $V(G)$ such that $\phi(u) = v$ for any two vertices $u, v \in S$, where $\phi$ is an isomorphism of $G$. The orbit number of a graph $G$, denoted by $\text{Orb}(G)$, is the number of orbits in $G$.

By Definition 1.1, all vertex-transitive graphs $G$ are with $\text{Orb}(G) = 1$, e.g. hypercubes. In [11,12], Efe introduced the crossed cubes which will be defined in Section 2. Crossed cubes have several properties, e.g., smaller diameter and better embedding properties, which makes it compare favorably to the ordinary hypercubes [12,26,27]. The crossed cubes have been extensively studied [1–7,9–13,16–18,20–25,28–35]. In [21], Kulasinghe and Bettayeb showed that $\text{Orb}(\text{CQ}_n) > 1$ when $n \geq 5$, where CQ$_n$ is the $n$-dimensional crossed cube. In [5], Chen et al. showed that $\text{Orb}(\text{CQ}_5) = 2$ and conjectured that $\text{Orb}(\text{CQ}_n) = 2^{\lfloor \frac{n}{2} \rfloor -2}$ for $n \geq 3$. In this paper, we settle the conjecture.

The rest of this paper is organized as follows. In Section 2, we introduce the definition of crossed cubes. In Section 3, we show that $2^{\lfloor \frac{n}{2} \rfloor -2}$ is an upper bound of $\text{Orb}(\text{CQ}_n)$ for $n \geq 3$. In Section 4, we show that $2^{\lfloor \frac{n}{2} \rfloor -2}$ is also a lower bound of $\text{Orb}(\text{CQ}_n)$ for $n \geq 3$. Finally, Section 5 contains our concluding remarks.

2. Preliminaries

Definition 2.1. Two 2-bit binary strings $x_2x_1$ and $y_2y_1$ are pair related, denoted by $x_2x_1 \sim y_2y_1$, if and only if $(x_2x_1, y_2y_1) \in \{(00,00), (10,10), (01,11), (11,01)\}$. 

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The crossed cubes were introduced by Efe in [11,12]. The \( n \)-dimensional crossed cubes \( CQ_n \) contains \( 2^n \) vertices in which the degree of every vertex is \( n \). Every vertex of \( CQ_n \) is identified by a unique binary string, which is also called \textit{address}, of length \( n \). Let \( u = u_{n-1} \ldots u_0 \) be a vertex in \( V(CQ_n) \). A vertex \( u \) is an \textit{even vertex} (respectively, \textit{odd vertex}) if the value of \( u_{n-1} \ldots u_0 \) is even (respectively, odd). The \textit{negate} of \( u \) for \( 0 \leq i \leq n - 1 \) is denoted by \( \overline{u}_i \). For an index \( x \) with \( 0 \leq x \leq n - 1 \), we use \( P^x_v \) to denote the prefix \( u_{n-1} \ldots u_{x+1} \) while \( S^x_u \) denotes the suffix \( u_{x-2} \ldots u_0 \) (respectively, \( u_{x-1} \ldots u_0 \)) when \( x \) is odd (respectively, even). If \( u_i = v_i \) for all \( x + 1 \leq i \leq n - 1 \), then we use \( P^x_u = P^x_v \) to denote it. Furthermore, let \( S^x_u \sim S^x_v \) stand for \( u_2i+1u_{2i} \sim v_2i+1v_{2i} \) for all \( 0 \leq i \leq \left\lfloor \frac{x}{2} \right\rfloor - 1 \).

The edges of an \( n \)-dimensional crossed cube can be defined as follows.

**Definition 2.2.** Let \( u = u_{n-1} \ldots u_0 \) and \( v = v_{n-1} \ldots v_0 \) be two vertices in \( CQ_n \). There is an edge \( uv \) in \( E(CQ_n) \) if and only if there exists an index \( x \) with \( 0 \leq x \leq n - 1 \) such that the following conditions are satisfied

1. \( v_x = \overline{u}_x \)
2. \( v_{x-1} = u_{x-1} \) if \( x \) is odd,
3. \( P^x_v = P^x_u \), and
4. \( S^x_v \sim S^x_u \).

We say that vertex \( v \) is the \textit{kth-neighbor} of vertex \( u \) if \( u \) and \( v \) are adjacent along dimension \( k \). That is, \( v_k = \overline{u}_k \), \( v_{k-1} = u_{k-1} \) if \( k \) is odd, \( P^k_v = P^k_u \), and \( S^k_v \sim S^k_u \).

**Example 1.** Figure 1 depicts \( CQ_3 \) and \( CQ_4 \). For example, let \( u = 0011 \) and \( v = 0101 \). We can find that \( v_2 \sim \overline{u}_2 \), \( P^2_v = P^2_u = 0 \), and \( S^2_v = 01 \sim 11 = S^2_u \). Thus there is an edge \( uv \) in \( E(CQ_4) \).

As far as we know, the following theorem is the only result on the orbit number of crossed cubes.

**Theorem 2.3** ([21]). \( \text{Orb}(CQ_n) > 1 \) when \( n > 4 \).
3. The upper bound of $\text{Orb}(\text{CQ}_n)$

In this section, we show that $\text{Orb}(\text{CQ}_n) \leq 2\lceil \frac{n}{2} \rceil - 2$ for $n \geq 3$. In the rest of this paper, we use $f_i(u)$ to denote $u_{n-1} \ldots u_{i+1} \overline{u}_{i} u_{i-1} \ldots u_0$ for some $0 \leq i \leq n - 1$.

**Lemma 3.1.** For $n \geq 2$ and any odd $k$ with $1 \leq k < n$, the function $\phi(u) = f_k(u)$ for all vertices $u \in V(\text{CQ}_n)$ is an automorphism of $\text{CQ}_n$.

**Proof.** By the definition of automorphisms, we have to show that, for each edge $uv \in E(\text{CQ}_n)$, there is an edge $\phi(u)\phi(v) \in E(\text{CQ}_n)$. By Definition 2.2, we have that $P_x^u = P_x^v$, $v_x = \overline{u}_x$, and $S_x^u \sim S_x^v$ for some $0 \leq x \leq n - 1$. Moreover, $v_{x-1} = u_{x-1}$ when $x$ is odd. For simplicity, we only consider the case where $x$ is odd. The other case can be handled similarly. We distinguish the following three cases.

**Case 1.** $k < x$.

In this case, we have that $\phi(u) = P_x^u u_{x} u_{x-1} \ldots \overline{u}_k u_{k-1} S_k^u$ and $\phi(v) = P_x^v v_{x} v_{x-1} \ldots \overline{v}_k v_{k-1} S_k^v$.

Since only the $k$th bit is changed in $S_x^u$ and $S_x^v$, all we have to prove is that $\overline{u}_k u_{k-1} \sim \overline{v}_k v_{k-1}$.

Table 1 lists all possible cases of $\overline{u}_k u_{k-1}$ and $\overline{v}_k v_{k-1}$. It is easy to check that $\overline{u}_k u_{k-1} \sim \overline{v}_k v_{k-1}$ for any case. Thus there is an edge $\phi(u)\phi(v)$ in $E(\text{CQ}_n)$ and this case holds.

**Case 2.** $k = x$.

In this case, we have that $\phi(u) = P_x^u \overline{u}_x u_{x-1} S_x^u$ and $\phi(v) = P_x^v \overline{v}_x v_{x-1} S_x^v$. It is clear that $P_x^u = P_x^v$, $u_{x-1} = v_{x-1}$, and $S_x^u \sim S_x^v$. Note that $\overline{v}_x = \overline{u}_x = u_x$. By Definition 2.2, there is an
Table 1: All possible cases of $\pi_k u_{k-1}$ and $\pi_k v_{k-1}$

| $u_k u_{k-1}$ | $\overline{u}_k u_{k-1}$ | $v_k v_{k-1}$ | $\overline{v}_k v_{k-1}$ |
|---------------|-----------------------------|---------------|-----------------------------|
| 00            | 10                          | 00            | 10                          |
| 01            | 11                          | 11            | 01                          |
| 10            | 00                          | 10            | 00                          |
| 11            | 01                          | 01            | 11                          |

edge $\phi(u)\phi(v)$ in $E(CQ_n)$. Thus this case also holds.

**Case 3. $k > x$.**

In this case, only a bit in the same position of $P^x_u$ and $P^x_v$ is negated. Thus $P^x_{\phi(u)} = P^x_{\phi(v)}$ and all the other relations between $u_x$, $v_x$, $u_{x-1}$ and $v_{x-1}$, and $S^x_u$ and $S^x_v$ remain unchanged. By Definition 2.2, there is an edge $\phi(u)\phi(v)$ in $E(CQ_n)$. This completes the proof.

**Lemma 3.2.** For $n \geq 2$, the function $\phi(u) = f_{n-1}(u)$ for all vertices $u \in V(CQ_n)$ is an automorphism of $CQ_n$.

**Proof.** If $n$ is even, then $n - 1$ is an odd number. By Lemma 3.1, the function $\phi$ is an automorphism of $CQ_n$.

Now we consider the case where $n$ is odd. Assume that there is an edge $uv \in E(CQ_n)$ with $P^k_u = P^k_v$, $S^k_v \sim S^k_u$, and $v_k = \pi_k$ (and $v_{k-1} = u_{k-1}$ when $k$ is odd). If $k \neq n - 1$, then, by using a similar argument as in Lemma 3.1, we can prove that there is also an edge $\phi(u)\phi(v) \in E(CQ_n)$. It remains to consider the case where $k = n - 1$ and $n$ is odd. Accordingly, we have $\phi(u) = \pi_{n-1} S^{n-1}_u$ and $\phi(v) = \pi_{n-1} S^{n-1}_v$. It is easy to verify that $\phi(u)\phi(v)$ is also an edge in $E(CQ_n)$. This completes the proof.

**Lemma 3.3.** For $n \geq 2$, the function $\phi(u) = f_{n-2}(u)$ for all vertices $u \in V(CQ_n)$ is an automorphism of $CQ_n$.

**Proof.** If $n$ is odd, then $n - 2$ is an odd number. By Lemma 3.1, the function $\phi$ is an automorphism of $CQ_n$.

Now we consider the case where $n$ is even. Assume that there is an edge $uv \in E(CQ_n)$ with $P^k_u = P^k_v$, $S^k_v \sim S^k_u$, and $v_k = \pi_k$ (and $v_{k-1} = u_{k-1}$ when $k$ is odd). If $k \leq n - 2$, then, by using
similar argument as in Lemma 3.2, we can prove that there is also an edge \( \phi(u)\phi(v) \in E(CQ_n) \).

It remains to consider the case where \( k = n - 1 \). Accordingly, we have Table 2. It is easy to check from Table 2 that there is an edge \( \phi(u)\phi(v) \in E(CQ_n) \). This completes the proof. \( \square \)

| \( u_{n-1}u_{n-2} \) | \( v_{n-1}v_{n-2} \) | \( u_{n-1}u_{n-2} \) | \( v_{n-1}v_{n-2} \) |
|-----------------|-----------------|-----------------|-----------------|
| 00              | 10              | 01              | 11              |
| 01              | 11              | 00              | 10              |
| 10              | 00              | 11              | 01              |
| 11              | 01              | 10              | 00              |

**Lemma 3.4.** For odd \( n \geq 3 \) and \( u \in V(CQ_n) \), the following function \( \phi \) is an automorphism of \( CQ_n \):

\[
\phi(u) = \begin{cases} 
  f_{n-3}(u) & \text{if } u_{n-1} = 0 \\
  f_{n-3}(f_{n-2}(u)) & \text{otherwise.}
\end{cases}
\]

**Proof.** It is easy to show that the function \( \phi \) is a bijective function. It remains to prove that \( \phi \) is an automorphism of \( CQ_n \). Let \( uv \) be an edge in \( E(CQ_n) \) with \( P^k_u = P^k_v, u_k = v_k, \) and \( S^k_u = S^k_v \) for some \( 0 \leq k \leq n - 1 \). We claim that \( \phi(u)\phi(v) \) is also an edge in \( E(CQ_n) \). If \( k < n - 3 \), then it is clear that \( \phi(u)\phi(v) \) is also an edge in \( E(CQ_n) \). Thus we consider the cases where \( k = n - 1, n - 2, \) and \( n - 3 \). For the case where \( k = n - 1 \), if \( u_{n-1} = 0 \) (respectively, \( u_{n-1} = 1 \)), then \( v_{n-1} = 1 \) (respectively, \( v_{n-1} = 0 \)) and \( u_{n-2}u_{n-3} \sim v_{n-2}v_{n-3} \). After the mapping of \( \phi \), we can find that \( \phi(u)_{n-1} = 0, \phi(v)_{n-1} = 1 \) (respectively, \( \phi(u)_{n-1} = 1 \) and \( \phi(v)_{n-1} = 0 \)), and \( \phi(u)_{n-2}\phi(u)_{n-3} \sim \phi(v)_{n-2}\phi(v)_{n-3} \) (see the first, second, fifth, and sixth columns in Table 3). By using a similar argument, we can show that the claim holds for the other cases. This completes the proof. \( \square \)
Table 3: The leftmost three bits of $u$, $v$, $\phi(u)$, and $\phi(v)$

| $u$ | $v$ with $k = n - 1$ | $\phi(u)$ with $k = n - 1$ | $\phi(v)$ with $k = n - 1$ |
|-----|----------------------|-----------------------------|-----------------------------|
| 000 | 100                 | 001                         | 001                         |
| 001 | 111                 | 000                         | 000                         |
| 010 | 110                 | 011                         | 011                         |
| 011 | 101                 | 001                         | 010                         |
| 100 | 000                 | 110                         | 110                         |
| 101 | 011                 | 111                         | 010                         |
| 110 | 010                 | 111                         | 001                         |
| 111 | 001                 | 110                         | 100                         |

Lemma 3.5. For even $n \geq 4$ and $u \in V(CQ_n)$, the following function $\phi$ is an automorphism of $CQ_n$:

$$\phi(u) = \begin{cases} f_{n-4}(f_{n-3}(u)) & \text{if } u_{n-1}u_{n-2}u_{n-3}u_{n-4} \in \{0100,1000,0111,1011\} \\ f_{n-4}(u) & \text{otherwise.} \end{cases}$$

Proof. By using a similar argument as in Lemma 3.4, this lemma holds. \(\square\)

Lemma 3.6. Let $\phi$ be an automorphism defined in Lemmas 3.1-3.5. If $\phi(u) = v$, then $\phi(v) = u$.

Proof. It is clear that the lemma holds for the $\phi$ defined in Lemmas 3.1-3.3. For the case where $\phi$ is defined in Lemma 3.4 (respectively, Lemma 3.5), it is easy to check that $u = f_{n-2}(f_{n-2}(u))$ (respectively, $u = f_{n-3}(f_{n-3}(u))$) for $u$ with $u_{n-1} \neq 0$ (respectively, $u_{n-1}u_{n-2}u_{n-3}u_{n-4} \in \{0100,1000,0111,1011\}$). This further implies that the lemma holds for the $\phi$ defined in Lemmas 3.4 and 3.5. This completes the proof. \(\square\)

Corollary 3.7. Let $\phi$ be an automorphism defined in Lemmas 3.1-3.5. Every orbit contains exactly two vertices under the automorphism $\phi$.

Lemma 3.8. For $n \geq 2$ and $k$ even, the function $\phi(u) = f_k(u)$ for all vertices $u \in V(CQ_n)$ is an automorphism of $CQ_n$ only when $k$ is in $\{n - 2, n - 1\}$.

Proof. Note that if $k$ is even, then $k = n - 2$ when $n$ is even and $k = n - 1$ when $n$ is odd. If $k$ is in $\{n - 2, n - 1\}$, then, by using a similar argument as in Lemma 3.1, it is easy to show that negating the $k$th bit is an automorphism of $CQ_n$. It remains to show that it is impossible to find an automorphism of $CQ_n$ by negating the $k$th bit of the addresses of all vertices in $CQ_n$. 

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when \( k \) is even and \( k \notin \{n - 2, n - 1\} \). It is clear that there exists an edge \( uv \) in \( E(CQ_n) \) with \( k + 1 < x \) such that \( P^x_u = P^x_v \), \( v_k = \pi_x \), and \( S^x_v \sim S^x_u \) when \( k \notin \{n - 2, n - 1\} \). By examining all possible cases of \( u_{k+1}u_k \) and \( v_{k+1}v_k \) (see Table 4), the relation of \( u_{k+1}u_k \sim v_{k+1}v_k \) does not exist. Thus there is no edge between \( \phi(u) \) and \( \phi(v) \). This completes the proof.

Table 4: All possible cases of \( u_{k+1}u_k \) and \( v_{k+1}v_k \)

| \( u_{k+1}u_k \) | \( v_{k+1}u_k \) | \( v_{k+1}v_k \) | \( v_{k+1}v_k \) |
|-----------------|-----------------|-----------------|-----------------|
| 00              | 00              | 00              | 01              |
| 01              | 00              | 11              | 10              |
| 10              | 11              | 10              | 11              |
| 11              | 10              | 01              | 00              |

Lemma 3.9. For \( n \geq 3 \), \( \text{Orb}(CQ_n) \leq 2^{\lceil \frac{n}{2} \rceil} - 2 \).

Proof. By Corollary 3.7, every orbit contains exactly two vertices under an automorphism \( \phi \) defined in Lemmas 3.1-3.5. Note that, after applying two different automorphisms, we have that each orbit contains four distinct vertices. Since there are \( |\frac{n}{2}| + 2 \) different automorphisms defined in Lemmas 3.1-3.5, it follows that each orbit contains \( 2^{\lceil \frac{n}{2} \rceil + 2} \) distinct vertices. This further implies that \( \text{Orb}(CQ_n) \leq \frac{2^n}{2^{1 + \frac{2}{2}}} = 2^{\lceil \frac{n}{2} \rceil} - 2 \). This completes the proof.

Corollary 3.10. \( \text{Orb}(CQ_3) = \text{Orb}(CQ_4) = 1 \) and \( \text{Orb}(CQ_5) = \text{Orb}(CQ_6) = 2 \).

4. The lower bound of \( \text{Orb}(CQ_n) \)

Denote by \( N(v) = \{u \in V : uv \in E\} \) the open neighborhood of \( v \). A path (respectively, clique) in \( G \) of \( \ell \) vertices is denoted by \( P_\ell \) (respectively, \( K_\ell \)).

Definition 4.1. The \( P_4 \)-graph of a set \( S \subseteq V \) is a graph \( H \) with \( V(H) = S \) and there is an edge \( xy \) for \( x, y \in V(H) \) if and only if there is a \( P_4 \) from \( x \) to \( y \) in \( G \).

Example 2. Figure 2 depicts the \( P_4 \)-graphs of \( N(1) \) and \( N(4) \) in \( CQ_7 \). We only explain the construction of Figure 2(a). First we show that there is a \( P_4 \) in \( CQ_7 \) from vertex 0 to each vertex in \( N(1) \setminus \{0\} \). Since all vertices in \( N(1) \setminus \{0, 3\} \) have exactly three nonzero bits in their addresses, there is a \( P_4 \) in \( CQ_7 \) from vertex 0 to every vertex in \( N(1) \setminus \{0, 3\} \). It is easy to check
that the path 0, 1, 7, 3 is also a $P_4$ in $CQ_7$ from vertex 0 to vertex 3. Now we show that there is a $P_4$ in $CQ_7$ from vertex 3 to each vertex in $N(1) \setminus \{0, 3\}$. It is easy to find a $P_3$ from vertex 2 passing through vertices 6, 10, 18, 34, and 66 to vertices 7, 11, 19, 35, and 67, respectively. Since vertex 3 is adjacent to vertex 2. Thus there exists a $P_4$ from vertex 3 to each vertex in $N(1) \setminus \{0, 3\}$. The reason why there is no $P_4$ between any two vertices in $N(1) \setminus \{0, 3\}$ will be explained in Lemma 4.5.

Figure 2: The $P_4$-graphs of $N(1)$ and $N(4)$ in $CQ_7$.

**Lemma 4.2.** For $n \geq 5$, if $u$ is an even vertex in $CQ_n$, then the $P_4$-graph of $N(u)$ contains a $K_4$.

**Proof.** Let $u = u_n-1 \ldots u_0$ be an even vertex in $V(CQ_n)$. It is obviously that $u_0 = 0$. We can find that $w = u_n-1 \ldots u_1, x = u_n-1 \ldots \pi_10$, $y = u_n-1 \ldots \pi_2u_10$, and $z = u_n-1 \ldots \pi_3u_2u_10$ are four vertices in $N(u)$. It suffices to show that the induced subgraph of vertices $w, x, y, z$ in the $P_4$-graph of $N(u)$ is a $K_4$. It is easy to check that the following five paths are $P_4$ in $CQ_n$:

$$w = u_n-1 \ldots u_2u_11 \rightarrow u_n-1 \ldots \pi_2u_11 \rightarrow u_n-1 \ldots \pi_2u_10 \rightarrow u_n-1 \ldots u_2u_10 = x$$

$$w = u_n-1 \ldots u_2u_11 \rightarrow u_n-1 \ldots \pi_2u_11 \rightarrow u_n-1 \ldots \pi_2u_10 \rightarrow u_n-1 \ldots \pi_2u_10 = y$$

$$w = u_n-1 \ldots u_2u_11 \rightarrow u_n-1 \ldots \pi_3u_2u_11 \rightarrow u_n-1 \ldots \pi_3u_2u_10 \rightarrow u_n-1 \ldots \pi_3u_2u_10 = z$$

$$x = u_n-1 \ldots \pi_10 \rightarrow u_n-1 \ldots u_2u_11 \rightarrow u_n-1 \ldots \pi_2u_10 \rightarrow u_n-1 \ldots \pi_2u_10 = y,$$ and

$$x = u_n-1 \ldots \pi_10 \rightarrow u_n-1 \ldots u_2u_11 \rightarrow u_n-1 \ldots \pi_3u_2u_11 \rightarrow u_n-1 \ldots \pi_3u_2u_10 = z.$$ 

It remains to show that there is a $P_4$ from $y$ to $z$ in $CQ_n$. If $u_2 = 0$, then the path

$$y = u_n-1 \ldots \pi_30u_10 \rightarrow u_n-1 \ldots \pi_31u_10 \rightarrow u_n-1 \ldots \pi_4u_31u_10 \rightarrow u_n-1 \ldots u_4u_3u_2u_10 = z$$

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is a $P_4$ from $y$ to $z$; otherwise, the path
\[ y = u_{n-1} \ldots \pi_3 u_1 0 \to u_{n-1} \ldots \pi_4 u_3 1 u_1 0 \to u_{n-1} \ldots \pi_4 u_3 0 u_1 0 \to u_{n-1} \ldots u_4 \pi_2 u_1 0 = z \]
is a $P_4$ from $y$ to $z$. This completes the proof. \hfill \Box

**Lemma 4.3** ([21]). Let $v$ and $w$ be two vertices of $CQ_n$ such that $v_{2k+1}v_{2k} = w_{2k+1}w_{2k} = 01$ for some $k$. Then $v$ and $w$ cannot be adjacent along a dimension greater than or equal to $2k$.

**Lemma 4.4** ([21]). Let $v$ and $w$ be two vertices in $CQ_n$ such that $v_{2k+1}v_{2k} = 01$ and $w_{2k+1}w_{2k} = 10$ for some $k$. Then $v$ and $w$ cannot be adjacent.

**Lemma 4.5.** For $n \geq 5$, if $u$ is an odd vertex in $CQ_n$, then the $P_4$-graph of $N(u)$ contains no $K_3$.

**Proof.** If $u$ is an odd vertex, then there is exactly one even vertex in $N(u)$. It suffices to show that, in the $P_4$-graph of $N(u)$, there is no $K_3$ formed by any three odd vertices in $N(u)$. Let $x$, $y$, and $z$ be any three odd vertices in $N(u)$. Assume without loss of generality that $x = P_x^i \pi_i(u_{i-1}) S_x^i$, $y = P_y^j \pi_j(u_{j-1}) S_y^j$, and $z = P_z^k \pi_k(u_{k-1}) S_z^k$ with $i > j > k$, where $P_x^i = P_x^i, P_y^j = P_y^j, P_z^k = P_z^k, S_x^i \sim S_u^i, S_y^j \sim S_u^j, S_z^k \sim S_u^k$. Note that the bits in the parentheses will appear only when its corresponding index, i.e., $i, j, k$, is odd.

We will show that it is impossible that there is a $P_4$ from $x$ to $y$ and another $P_4$ from $x$ to $z$. We only consider the case where $u_1u_0 = 01$. The other case, i.e., $u_1u_0 = 11$, can be handled similarly.

If $u_1u_0 = 01$, then $x_1x_0 = y_1y_0 = z_1z_0 = 11$. Suppose to the contrary that there exit paths $P_4$ in $CQ_n$ from $x$ to $y$ and from $x$ to $z$. Let $x, v, w, y$ be the $P_4$ from $x$ to $y$. Note that the rightmost two bits of $v$ and $w$ must be in the set $\{01, 10\}$. By Lemmas 4.3 and 4.4, we have that $v_1v_0 = w_1w_0 = 10$. This implies that $P_x^0 = P_v^0$ and $P_y^0 = P_w^0$. Furthermore, vertex $v$ is adjacent to $w$ along dimension $i$. By using a similar argument, if there is a $P_4$ from $x$ to $z$, then it would be $x, v, s, z$ with $v_1v_0 = s_1s_0 = 10$ and $x_i = \pi_i = z_i$. Moreover, $v$ is adjacent to $s$ also along dimension $i$, a contradiction. Thus, in the $P_4$-graph of $N(u)$, there is no $K_3$ formed by any three odd vertices in $N(u)$ and the lemma follows. \hfill \Box
Lemma 4.6. If $\phi$ is an automorphism of $CQ_n$ for $n \geq 5$, then $\phi$ maps odd vertices to odd vertices and even vertices to even.

Proof. By Lemmas 4.2 and 4.5, the $P_4$-graphs of even and odd vertices are not isomorphism. Thus it is impossible to map an even vertex to an odd vertex by $\phi$ and the lemma follows. □

Lemma 4.7. Let $\phi$ be an automorphism of $CQ_n$ for $n \geq 5$. If $\phi(u) = v$ for $u, v \in V(CQ_n)$, then $\phi$ maps the $k$th neighbor of $u$ to the $k$th neighbor of $v$ for $k \in \{0, 1\}$.

Proof. Let $u = xu_1u_0$ and $v = yv_1v_0$, where $x = u_{n-1} \ldots u_2$ and $y = v_{n-1} \ldots v_2$. First, we consider the case where $u_1u_0 = 00$. That is, vertex $u$ is an even vertex. By Lemma 4.6, vertex $v$ is also an even vertex, namely, $v_0 = 0$. If $v_1 = 1$, then we can apply $f_1(x)$ for all $x \in V(CQ_n)$ after $\phi$ is applied. By Lemma 3.1, the new automorphism preserves the $k$th neighbor for $k = 1$ if and only if $\phi$ does. So, we can assume $v_1v_0 = 00$. Accordingly, we have $\phi(u) = \phi(x00) = y00 = v$.

To prove that $\phi$ preserves the $k$th neighbor for $k \in \{0, 1\}$, it suffices to show that $\phi(x01) = y01$, $\phi(x10) = y10$, and $\phi(x11) = y11$.

Now we show that $\phi(x01) = y01$. Since $x01$ is an odd vertex, by Lemma 4.6, vertex $\phi(x01)$ is an odd vertex, namely, its 0th bit is 1. Since there is an edge between $x00$ and $x01$, there is also an edge between $\phi(x00)$ and $\phi(x01)$. Recall that $\phi(x00) = y00$. Thus $\phi(x01)$ is adjacent to $y00$. This results in $\phi(x01) = y01$.

Next we prove that $\phi(x11) = y11$ and $\phi(x10) = y10$. By Lemma 4.6, vertex $\phi(x11)$ is an odd vertex, i.e., its 0th bit is 1. Furthermore, vertex $\phi(x11)$ is a neighbor of $\phi(x01) = y01$ since there is an edge between $x11$ and $x01$. Thus $\phi(x11) = z11$, where $z$ is a binary string of length $n - 2$. Since $\phi(x11) = z11$ is adjacent to $\phi(x10) = z10$ which is an even vertex and a neighbor of $y00$, it is impossible that $z \neq y$. For otherwise, $z10$ is not adjacent to $y00$, a contradiction. This yields $\phi(x11) = y11$ and $\phi(x10) = y10$. This establishes the proof of the lemma. □

Lemma 4.8. For $n \geq 5$, if $\phi$ is an automorphism of $CQ_{n+2}$, then $\hat{\phi}$ is an automorphism of $CQ_n$, where $\hat{\phi}(v) = \lfloor \frac{\phi(4v)}{4} \rfloor$ for $v \in V(CQ_n)$.

Proof. By Lemma 4.7, it is straightforward to check that $\hat{\phi}$ is a bijection. Moreover, if vertices
x and y are the kth neighbors to each other in CQn, then x00 and y00 are the (k+2)th neighbors in CQn+2. By Lemma 4.7 again, vertices φ(x00) and φ(y00) are higher (≥ 2) neighbors to each other in CQn+2. Consequently, vertices \( \frac{\phi(x00)}{4} \) and \( \frac{\phi(y00)}{4} \) are neighbors to each other in CQn. Therefore, function \( \hat{\phi} \) preserves adjacency.

\[ \square \]

**Theorem 4.9.** For \( n \geq 3 \), \( \text{Orb}(CQ_n) = 2^\left\lceil \frac{n}{2} \right\rceil - 2. \)

**Proof.** By Lemma 3.9, it follows that \( 2^\left\lceil \frac{n}{2} \right\rceil - 2 \) is an upper bound of \( \text{Orb}(CQ_n) \) when \( n \geq 3 \). It remains to show that \( 2^\left\lceil \frac{n}{2} \right\rceil - 2 \) is also a lower bound of \( \text{Orb}(CQ_n) \) when \( n \geq 3 \).

Let \( n = 2k + 3 \) (respectively, \( n = 2k + 4 \)) for \( k \geq 1 \) when \( n \) is odd (respectively, even). We claim that any automorphism \( \phi \) preserves even bits \( 2i \) for all \( 0 \leq i < k \). By Lemma 4.6, the claim holds when \( i = 0 \). By applying Lemma 4.8 \( i \) times for \( 0 \leq i < k \) and then by Lemma 4.6, we can find that bit \( 2i \) is preserved under automorphism \( \phi \). So vertices with different bits in any one of \( 2i \) for \( 0 \leq i < k \) are in different orbits. This further implies that \( \text{Orb}(CQ_n) \geq 2^\left\lceil \frac{n}{2} \right\rceil - 2. \)

By Lemma 3.9, this yields \( \text{Orb}(CQ_n) = 2^\left\lceil \frac{n}{2} \right\rceil - 2 \) for \( n \geq 3 \) and the theorem follows. \[ \square \]

5. Concluding remarks

In this paper, we derive the orbit number of crossed cubes. There are a lot of variants of hypercubes, e.g., folded cubes [14], twisted cubes [15], möbius cubes [8], etc. It is interesting to investigate the orbit number of those hypercube-like interconnection networks.

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