Abstract: Each of the descriptions of vertices, edges, and facets of the order and chain polytope of a finite partially ordered set are well known. In this paper, we give an explicit description of faces of 2-dimensional simplex of each polytope. These results mean a generalization in the case of 2-faces of the characterization known in the case of edges.

Keywords: order polytope; chain polytope; partially ordered set

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1. Introduction

The combinatorial structure of the order polytope $O(P)$ and the chain polytope $C(P)$ of a finite poset (partially ordered set) $P$ is explicitly discussed in [1]. Moreover, in [2], the problem when the order polytope $O(P)$ and the chain polytope $C(P)$ are unimodularly equivalent is solved. It is also proved that the number of edges of the order polytope $O(P)$ is equal to that of the chain polytope $C(P)$ in [3]. In the present paper we give an explicit description of faces of 2-dimensional simplex of $O(P)$ and $C(P)$ in terms of vertices. In other words, we show that triangles in 1-skeleton of $O(P)$ or $C(P)$ are in one-to-one correspondence with faces of 2-dimensional simplex of each polytope. These results are a direct generalizations of [4] (Lemma 4, Lemma 5).

2. Definition and Known Results

Let $P = \{x_1, \ldots, x_d\}$ be a finite poset. To each subset $W \subset P$, we associate $\rho(W) = \sum_{i \in W} e_i \in \mathbb{R}^d$, where $e_1, \ldots, e_d$ are the canonical unit coordinate vectors of $\mathbb{R}^d$. In particular $\rho(\emptyset)$ is the origin of $\mathbb{R}^d$.

A poset ideal of $P$ is a subset $I$ of $P$ such that, for all $x_i$ and $x_j$ with $x_i \in I$ and $x_j \leq x_i$, one has $x_j \in I$. An antichain of $P$ is a subset $A$ of $P$ such that $x_i$ and $x_j$ belonging to $A$ with $i \neq j$ are incomparable. The empty set $\emptyset$ is a poset ideal as well as an antichain of $P$. We say that $x_j$ covers $x_i$ if $x_i < x_j$ and $x_i < x_k < x_j$ for no $x_k \in P$. A chain $x_{j_1} < x_{j_2} < \cdots < x_{j_q}$ of $P$ is called saturated if $x_{j_q}$ covers $x_{j_{q-1}}$ for $1 < q \leq \ell$. A maximal chain is a saturated chain such that $x_{j_1}$ is a minimal element and $x_{j_\ell}$ is a maximal element of the poset. The rank of $P$ is $\gamma(C) - 1$, where $C$ is a chain with maximum length of $P$.

The order polytope of $P$ is the convex polytope $O(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $0 \leq a_i \leq 1$ for every $1 \leq i \leq d$ together with

$$a_i \geq a_j$$

if $x_i \leq x_j$ in $P$. 

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The chain polytope of $P$ is the convex polytope $\mathcal{C}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $a_i \geq 0$ for every $1 \leq i \leq d$ together with

$$a_i + a_{i+1} + \cdots + a_k \leq 1$$

for every maximal chain $x_1 < x_2 < \cdots < x_k$ of $P$.

One has $\dim(\mathcal{C}(P)) = \dim(\mathcal{O}(P)) = d$. The vertices of $\mathcal{O}(P)$ are those $\rho(I)$ for which $I$ is a poset ideal of $P$ (Lemma 1 (1) (Corollary 1.3)) and the vertices of $\mathcal{C}(P)$ is those $\rho(A)$ for which $A$ is an antichain of $P$ (Lemma 2.2). It then follows that the number of vertices of $\mathcal{O}(P)$ is equal to that of $\mathcal{C}(P)$. Moreover, the volume of $\mathcal{O}(P)$ and that of $\mathcal{C}(P)$ are equal to $e(P)/d!$, where $e(P)$ is the number of linear extensions of $P$ (Lemma 4.2). It also follows from Lemma 1 that the facets of $\mathcal{O}(P)$ are the following:

- $x_i = 0$, where $x_i \in P$ is maximal;
- $x_i = 1$, where $x_i \in P$ is minimal;
- $x_i = x_j$, where $x_j$ covers $x_i$,

and that the facets of $\mathcal{C}(P)$ are the following:

- $x_i = 0$, for all $x_i \in P$;
- $x_i + \cdots + x_k = 1$, where $x_1 < \cdots < x_k$ is a maximal chain of $P$.

In [4] a characterization of edges of $\mathcal{O}(P)$ and those of $\mathcal{C}(P)$ is obtained. Recall that a subposet $Q$ of finite poset $P$ is said to be connected in $P$ if, for each $x$ and $y$ belonging to $Q$, there exists a sequence $x = x_0, x_1, \ldots, x_s = y$ with each $x_i \in Q$ for which $x_{i-1}$ and $x_i$ are comparable in $P$ for each $1 \leq i \leq s$.

Lemma 1 ([4] (Lemma 4, Lemma 5)). Let $P$ be a finite poset.

1. Let $I$ and $J$ be poset ideals of $P$ with $I \neq J$. Then the convex hull of $\{\rho(I), \rho(J)\}$ forms an edge of $\mathcal{O}(P)$ if and only if $I \subset J$ and $J \setminus I$ is connected in $P$.
2. Let $A$ and $B$ be antichains of $P$ with $A \neq B$. Then the convex hull of $\{\rho(A), \rho(B)\}$ forms an edge of $\mathcal{C}(P)$ if and only if $(A \setminus B) \cup (B \setminus A)$ is connected in $P$.

3. Faces of 2-Dimensional Simplex

Using Lemma 1, we show the following description of faces of 2-dimensional simplex.

Theorem 1. Let $P$ be a finite poset. Let $I, J,$ and $K$ be pairwise distinct poset ideals of $P$. Then the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ forms a 2-face of $\mathcal{O}(P)$ if and only if $I \subset J \subset K$ and $K \setminus I$ is connected in $P$.

Proof. (“Only if”) If the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ forms a 2-face of $\mathcal{O}(P)$, then the convex hulls of $\{\rho(I), \rho(J)\}$, $\{\rho(J), \rho(K)\}$, and $\{\rho(I), \rho(K)\}$ form edges of $\mathcal{O}(P)$. It then follows from Lemma 1 that $I \subset J \subset K$ and $K \setminus I$ is connected in $P$.

(“If”) Suppose that the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ has dimension 1. Then there exists a line passing through the lattice points $\rho(I), \rho(J),$ and $\rho(K)$. Hence $\rho(I), \rho(J),$ and $\rho(K)$ cannot be vertices of $\mathcal{O}(P)$. Thus the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ has dimension 2.

Let $P = \{x_1, \ldots, x_d\}$. If there exists a maximal element $x_i$ of $P$ not belonging to $I \cup J \cup K$, then the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ lies in the facet $x_i = 0$. If there exists a minimal element $x_j$ of $P$ belonging to $I \cap J \cap K$, then the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ lies in the facet $x_j = 1$. Hence, working with induction on $d \geq 2$, we may assume that $I \cup J \cup K = P$ and $I \cap J \cap K = \emptyset$.

Suppose that $\emptyset = I \subset J \subset K = P$ and $K \setminus I = P$ is connected.

Case 1. $\sharp(J) = 1$. 

Let $J = \{x_i\}$ and $P' = P \setminus \{x_i\}$. Then $P'$ is a connected poset. Let $x_{i_1}, \ldots, x_{i_t}$ be the maximal elements of $P$ and $A_{i_j} = \{y \in P' \mid y < x_{i_j}\}$, where $1 \leq j \leq t$. Then we write

$$b_k = \begin{cases} 
\mathbb{I}(\{i_j \mid x_k \in A_{i_j}\}) & \text{if } k \notin \{i_1, \ldots, i_t, i_j\} \\
0 & \text{if } k = i_j \\
-\mathbb{I}(A_{i_j}) & \text{if } k \in \{i_1, \ldots, i_t\}
\end{cases}.$$

We then claim that the hyperplane $\mathcal{H}$ of $\mathbb{R}^d$ defined by the equation $h(x) = \sum_{k=1}^d b_k x_k = 0$ is a supporting hyperplane of $\mathcal{C}(P)$ and that $\mathcal{H} \cap \mathcal{C}(P)$ coincides with the convex hull of $\{\rho(\mathcal{C}), \rho(\mathcal{I}), \rho(P)\}$. Clearly $h(\rho(\mathcal{C})) = h(\rho(\mathcal{I})) = 0$ and $h(\rho(P)) = b_i = 0$. Let $I$ be a poset ideal of $P$ with $I \neq \emptyset$, $I \neq P$ and $I \neq J$. We have to prove that $h(\rho(I)) > 0$. To simplify the notation, suppose that $I \cap \{x_{i_1}, \ldots, x_{i_t}\} = \{x_{i_1}, \ldots, x_{i_u}\}$, where $0 \leq r < q$. If $r = 0$, then $h(\rho(I)) > 0$. Let $1 \leq r < q$, $I' = I \setminus \{x_i\}$, and $K = \bigcup_{j=1}^u (A_{i_j} \cup \{x_j\})$. Then $I'$ and $K$ are poset ideals of $P$ and $h(\rho(K)) = h(\rho(I')) = h(\rho(I))$. We claim $h(\rho(K)) > 0$. One has $h(\rho(K)) \geq 0$. Moreover, $h(\rho(K)) = 0$ if and only if no $z \in K$ belongs to $A_{i_{u+1}} \cup \cdots \cup A_{i_q}$. Now, since $P'$ is connected, it follows that there exists $z \in K$ with $z \in A_{i_{u+1}} \cup \cdots \cup A_{i_q}$. Hence $h(\rho(K)) > 0$. Thus $h(\rho(I)) > 0$.

**Case 2.** $\mathbb{I}(J) = d - 1$.

Let $P \setminus J = \{x_i\}$ and $P' = P \setminus \{x_i\}$. Then $P'$ is a connected poset. Thus we can show the existence of a supporting hyperplane of $\mathcal{C}(P)$ which contains the convex hull of $\{\rho(\mathcal{C}), \rho(\mathcal{I}), \rho(P)\}$ by the same argument in Case 1.

**Case 3.** $2 \leq \mathbb{I}(J) \leq d - 2$.

To simplify the notation, suppose that $J = \{x_1, \ldots, x_t\}$. Then $P \setminus J = \{x_{i+1}, \ldots, x_d\}$. Since $J$ and $P \setminus J$ are subposets of $P$, these posets are connected. Let $x_{i_1}, \ldots, x_{i_r}$ be the maximal elements of $J$ and $x_{i_{r+1}}, \ldots, x_{i_{r+t}}$ the maximal elements of $P \setminus J$. Then we write

$$A_{i_{s,j}} = \begin{cases} \{y \in J \mid y < x_{i_{s,j}}\} & \text{if } 1 \leq s \leq t \\
\{y \in P \setminus J \mid y < x_{i_{s,j}}\} & \text{if } q + 1 \leq s \leq r \end{cases}$$

and

$$b_k = \begin{cases} \mathbb{I}(\{i_{s,j} \mid x_k \in A_{i_{s,j}}\}) & \text{if } k \notin \{i_1, \ldots, i_{r+t}, i_{s,j}\} \\
-\mathbb{I}(A_{i_{s,j}}) & \text{if } k \in \{i_1, \ldots, i_{r+t}, i_{s,j}\}
\end{cases}.$$

We then claim that the hyperplane $\mathcal{H}$ of $\mathbb{R}^d$ defined by the equation $h(x) = \sum_{k=1}^d b_k x_k = 0$ is a supporting hyperplane of $\mathcal{C}(P)$ and $\mathcal{H} \cap \mathcal{C}(P)$ coincides with the convex hull of $\{\rho(\mathcal{C}), \rho(\mathcal{I}), \rho(P)\}$. Clearly $h(\rho(\mathcal{C})) = h(\rho(\mathcal{I})) = 0$, then $h(\rho(P)) = h(\rho(I)) + h(\rho(P \setminus J)) = 0$. Let $I$ be a poset ideal of $P$ with $I \neq \emptyset$, $I \neq P$ and $I \neq J$. What we must prove is $h(\rho(I)) > 0$.

If $I \subseteq J$, then $I$ is a poset ideal of $J$. To simplify the notation, suppose that $I \cap \{x_{i_1}, \ldots, x_{i_t}\} = \{x_{i_1}, \ldots, x_{i_s}\}$, where $0 \leq s < q$. If $s = 0$, then $h(\rho(I)) > 0$. Let $1 \leq s < q$, $K = \bigcup_{j=1}^s (A_{i_j} \cup \{x_j\})$. Then $K$ is a poset ideal of $I$ and $h(\rho(K)) \leq h(\rho(I))$. Thus we can show $h(\rho(K)) > 0$ by the same argument in Case 1 (Replace $r$ with $s$ and $P'$ with $J$).

If $J \subseteq I$, then $I \setminus J$ is a poset ideal of $P \setminus J$. To simplify the notation, suppose that $(I \setminus J) \cap \{x_{i_{r+1}}, \ldots, x_{i_{r+t}}\} = \{x_{i_{r+1}}, \ldots, x_{i_{r+t}}\}$, where $0 \leq t < r$. If $t = 0$, then $h(\rho(I)) = h(\rho(I)) + h(\rho(J)) = h(\rho(I \setminus J)) > 0$. Let $1 \leq t < r$, $K = \bigcup_{j=t+1}^{r+t} (A_{i_j} \cup \{x_j\})$. Then $K$ is a poset ideal of $P \setminus J$ and $h(\rho(K)) \leq h(\rho(I \setminus J)) = h(\rho(I)).$ Thus we can show $h(\rho(K)) > 0$ by the same argument in Case 1 (Replace $r$ with $q + t$, $q$ with $q + r$ and $P'$ with $P \setminus J$). Consequently, $h(\rho(I)) > 0$, as desired.

Let $A \triangle B$ denote the symmetric difference of the sets $A$ and $B$, that is $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

**Theorem 2.** Let $P$ be a finite poset. Let $A$, $B$, and $C$ be pairwise distinct antichains of $P$. Then the convex hull of $\{\rho(A), \rho(B), \rho(C)\}$ forms a 2-face of $\mathcal{C}(P)$ if and only if $A \triangle B$, $B \triangle C$ and $C \triangle A$ are connected in $P$. 
Proof. (“Only if”) If the convex hull of \( \{\rho(A), \rho(B), \rho(C)\} \) forms a 2-face of \( \mathcal{V}(P) \), then the convex hulls of \( \{\rho(A), \rho(B)\} \), \( \{\rho(B), \rho(C)\} \), and \( \{\rho(A), \rho(C)\} \) form edges of \( \mathcal{V}(P) \). It then follows from Lemma 1 
that \( A \triangle B, B \triangle C \) and \( C \triangle A \) are connected in \( P \).

(“If”) Suppose that the convex hull of \( \{\rho(A), \rho(B), \rho(C)\} \) has dimension 1. Then there exists a line passing through the lattice points \( \rho(A), \rho(B), \) and \( \rho(C) \). Hence \( \rho(A), \rho(B), \) and \( \rho(C) \) cannot be vertices of \( \mathcal{V}(P) \). Thus the convex hull of \( \{\rho(A), \rho(B), \rho(C)\} \) has dimension 2.

Let \( P = \{x_1, \ldots, x_d\} \). If \( A \cup B \cup C \neq P \) and \( x_i \notin A \cup B \cup C \), then the convex hull of \( \{\rho(A), \rho(B), \rho(C)\} \) lies in the facet \( x_i \), \( x_i = 0 \). Furthermore, if \( A \cup B \cup C = P \) and \( A \cap B \cap C \neq \emptyset \), then \( x_j \in A \cap B \cap C \) is isolated in \( P \) and \( x_j \) itself is a maximal chain of \( P \). Thus the convex hull of \( \{\rho(A), \rho(B), \rho(C)\} \) lies in the facet \( x_j = 1 \). Hence, working with induction on \( d \geq 2 \), we may assume that \( A \cup B \cup C = P \) and \( A \cap B \cap C = \emptyset \). As stated in the proof of [3] ([Theorem 2.1]), if \( A \triangle B \) is connected in \( P \), then \( A \) and \( B \) satisfy either (i) \( B \subset A \) or (ii) \( y < x \) whenever \( x \in A \) and \( y \in B \) are comparable. Hence, we consider the following three cases:

(a) If \( B \subset A \), then \( A \triangle B = A \setminus B \) is connected in \( P \), and thus \( \sharp(A \setminus B) = 1 \). Let \( A \setminus B = \{x_k\} \).

If \( C \cap A \neq \emptyset \), then \( C \cap A = \{x_k\} \), since \( A \cap B \cap C = C \cap B = \emptyset \). Namely \( x_k \) is isolated in \( P \).

Hence \( B \triangle C = B \cup C = A \cup B \cup C \) cannot be connected. Thus \( C \cap A = \emptyset \). In this case, we may assume \( z < x \) if \( x \in A \) and \( z \in C \) are comparable. Furthermore, \( P \) has rank 1.

(b) If \( B \subset A \) and \( B \cap A \neq \emptyset \), then we may assume \( y < x \) if \( x \in A \) and \( y \in B \) are comparable.

If \( C \subset B \) with \( C \cap A \cap B = \emptyset \), then as stated in (a), \( C \triangle A \) cannot be connected. Since \( C \nsubseteq B \), we may assume \( z < y \) if \( y \in B \) and \( z \in C \) are comparable. If \( C \cap B \neq \emptyset \), then \( C \cap A = \emptyset \) and \( P \) has rank 1 or 2. Similarly, if \( C \cap B = \emptyset \), then \( C \cap A = \emptyset \) and \( P \) has rank 2.

(c) Let \( B \subset A \) and \( B \cap A = \emptyset \). We may assume that if \( x \in A \) and \( y \in B \) are comparable, then \( y < x \). If \( C \subset B \), then we regard this case as equivalent to (a). Let \( C \subset B \). We may assume \( z < y \) if \( y \in B \) and \( z \in C \) are comparable. Moreover, if \( C \cap B \neq \emptyset \), then we regard this case as equivalent to (b).

If \( C \cap B = \emptyset \), then \( C \cap A = \emptyset \) and \( P \) has rank 2.

Consequently, there are five cases as regards antichains for \( \mathcal{V}(P) \).

Case 1. \( B \subset A, \ C \cap A = \emptyset \), and \( C \cap B = \emptyset \).

For each \( x_i \in B \) we write \( b_i \) for the number of elements \( z \in C \) with \( z < x_i \). For each \( x_i \in C \) we write \( c_j \) for the number of elements \( y \in B \) with \( x_j < y \). Let \( a_k = 0 \) for \( A \setminus B = \{x_k\} \). Clearly \( \sum_{x \in B} b_i = \sum_{x \in C} c_j = q \) where \( q \) is the number of pairs \( (y, z) \) with \( y \in B, z \in C \) and \( z < y \). Let \( h(x) = \sum_{x \in B} b_i x_i + \sum_{x \in C} c_j x_j + a_k x_k \) and let \( \mathcal{H} \) be the hyperplane of \( \mathbb{R}^d \) defined by \( h(x) = q \). Then \( h(\rho(A)) = h(\rho(B)) = h(\rho(C)) = q \). We claim that, for any antichain \( D \) of \( P \) with \( D \neq A, D \neq B, \) and \( D \neq C \), one has \( h(\rho(D)) < q \). Let \( D = B_1 \cup C_1 \) or \( D = \{x_k\} \cup C_1 \) with \( B_1 \subset B \) and \( C_1 \subset C \). Suppose \( D = B_1 \cup C_1 \). Since \( B \triangle C \) is connected and since \( D \) is an antichain of \( P \), it follows that \( \sum_{x \in B_1} b_i + \sum_{x \in C_1} c_j < q \). Thus \( h(\rho(D)) < q \). Suppose that \( D = \{x_k\} \cup C_1 \). It follows that \( \sum_{x \in C_1} c_j + a_k = \sum_{x \in C_1} c_j < \sum_{x \in C} c_j = q \). Thus \( h(\rho(D)) < q \).

Case 2. \( B \subset A, \ B \cap A \neq \emptyset, \ C \subset B, \ C \cap B \neq \emptyset, \ C \cap A = \emptyset \), and \( P \) has rank 1.

We define four numbers as follows:

\[
\alpha_i = \sharp\{y \in B \setminus A \mid y < x_i, x_i \in A \setminus B\};
\]

\[
\beta_j = \sharp\{x \in A \setminus B \mid x_j < x, x_j \in B \setminus A\};
\]

\[
\gamma_k = \sharp\{z \in C \setminus B \mid z < x_k, x_k \in B \setminus C\};
\]

\[
\delta_\ell = \sharp\{y \in B \setminus C \mid x_\ell < y, x_\ell \in C \setminus B\}.
\]
Since $P$ has rank 1, $B \subset A \cup C = P$. It follows that $A = (A \setminus B) \cup (B \setminus C)$, $C = (B \setminus A) \cup (C \setminus B)$. Then

$$
\sum_{x_i \in A} \alpha_i = \sum_{x_i \in A \setminus B} \alpha_i + \sum_{x_i \in B \setminus C} \alpha_i = q;
$$

$$
\sum_{x_i \in B \setminus A} \gamma_j + \sum_{x_i \in B \setminus C} \alpha_k = q;
$$

$$
\sum_{x_i \in C} \gamma_u = \sum_{x_i \in B \setminus A} \gamma_j + \sum_{x_i \in B \setminus C} \gamma_k + \sum_{x_i \in C \setminus B} \gamma_\ell = q,
$$

where $q_1$ is the number of pairs $(x, y)$ with $x \in A \setminus B, y \in B \setminus A$ and $y < x$, $q_2$ is the number of pairs $(y, z)$ with $y \in B \setminus C, z \in C \setminus B$ and $z < y$, and $q = q_1 + q_2$. Let

$$
h(x) = \sum_{x_i \in A} \alpha_i x_i + \sum_{x_i \in C} \gamma_u x_u
$$

and $H$ the hyperplane of $\mathbb{R}^d$ defined by $h(x) = q$. Then $h(\rho(A)) = h(\rho(B)) = h(\rho(C)) = q$. We claim that, for any antichain $D$ of $P$ with $D \neq A$, $D \neq B$ and $D \neq C$, one has $h(\rho(D)) < q$. Let $D = D_1 \cup D_2$ with $D_1$ is an antichain of $A \triangle B$ and $D_2$ is an antichain of $B \triangle C$. Since $A \triangle B, B \triangle C$ are connected, it follows that $h(\rho(D_1)) < q_1$ and $h(\rho(D_2)) < q_2$. Thus $h(\rho(D)) = h(\rho(D_1)) + h(\rho(D_2)) < q_1 + q_2 = q$.

**Case 3.** $B \subset A, B \cap A \neq \emptyset, C \subset B, C \cap B \neq \emptyset, C \cap A = \emptyset$, and $P$ has rank 2.

For each $x_i \in P$ we write $c(i)$ for the number of maximal chains, which contain $x_i$. Let $q$ be the number of maximal chains in $P$. Since each $x_i \in A$ is maximal element and each $x_k \in C$ is minimal element, $\sum_{x_i \in A} c(i) = \sum_{x_k \in C} c(k) = q$. Then

$$
\sum_{x_j \in B} c(j) = \sum_{x_j \in B \setminus A} c(s) + \sum_{x_j \in B \setminus C} c(t) + \sum_{x_j \in B \setminus (A \cup C)} c(u)
$$

$$
= \sum_{x_j \in B \setminus A} c(s) + \sum_{x_j \in B \setminus C} c(t) + \left( \sum_{x_j \in A \setminus B} c(v) - \sum_{x_j \in B \setminus C} c(t) \right)
$$

$$
= \sum_{x_j \in A} c(i) = q.
$$

Let $h(x) = \sum_{x_j \in P} c(i) x_i$ and $H$ the hyperplane of $\mathbb{R}^d$ defined by $h(x) = q$. Then $h(\rho(A)) = h(\rho(B)) = h(\rho(C)) = q$. We claim that, for any antichain $D$ of $P$ with $D \neq A, D \neq B$ and $D \neq C$, one has $h(\rho(D)) < q$. $D = A_1 \cup B_1 \cup C_1$ with $A_1 \subset A \setminus B, B_1 \subset B$, and $C_1 \subset C \setminus B$. Now, we define two subsets of $B$:

$$
B_2 = \{ x_j \in B \mid x_j < x_i, x_i \in A_1 \};
$$

$$
B_3 = \{ x_j \in B \mid x_k < x_j, x_k \in C_1 \}.
$$

Then $B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \emptyset$ and $B_1 \cup B_2 \cup B_3 \subset B_3$. Let $\sum_{x_j \in A} c(i) = q_{1}, \sum_{x_j \in B_1} c(j) = q_{2}, \sum_{x_j \in C_1} c(k) = q_{3}, \sum_{x_j \in B_2} c(i) = q_{1}', \sum_{x_j \in B_3} c(j) = q_{2}'$. Since $A \triangle B, B \triangle C$ are connected, it follows that $q_1 < q_1'$ and $q_3 < q_3'$. Hence

$$
h(\rho(D)) = \sum_{x_j \in A_1} c(i) + \sum_{x_j \in B_1} c(j) + \sum_{x_j \in C_1} c(k)
$$

$$
= q_1 + q_2 + q_3 < q_1' + q_2 + q_3'
$$

$$
= \sum_{x_j \in B_2} c(j) + \sum_{x_j \in B_1} c(j) + \sum_{x_j \in B_3} c(j) \leq \sum_{x_j \in B} c(j) = q.$$
Thus \( h(p(D)) < q \).

**Case 4.** \( B \notin A, B \cap A \neq \emptyset, C \cap B = \emptyset, \) and \( C \cap A = \emptyset \).

Since \( P \) has rank 2, we can show \( h(p(D)) < q \) by the same argument in Case 3 (Suppose \( C \cap B = \emptyset \)).

**Case 5.** \( B \notin A, B \cap A = \emptyset, C \cap B = \emptyset \) and \( C \cap A = \emptyset \).

Since \( P \) has rank 2, we can show \( h(p(D)) < q \) by the same argument in Case 3 (Suppose \( B \cap A = C \cap B = \emptyset \)).

In conclusion, each \( \mathcal{H} \) is a supporting hyperplane of \( \mathcal{C}(P) \) and \( \mathcal{H} \cap \mathcal{C}(P) \) coincides with the convex hull of \( \{p(A), p(B), p(C)\} \), as desired. \( \Box \)

**Corollary 1.** Triangles in 1-skeleton of \( \mathcal{O}(P) \) or \( \mathcal{C}(P) \) are in one-to-one correspondence with faces of 2-dimensional simplex of each polytope.

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