Approximate symmetries of Hamiltonians

Christopher T. Chubb\textsuperscript{1} and Steven T. Flammia\textsuperscript{1,2}

\textsuperscript{1}Centre for Engineered Quantum Systems, University of Sydney, Sydney, NSW, Australia.
\textsuperscript{2}Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA, USA.

May 5, 2017

Abstract
We explore the relationship between approximate symmetries of a gapped Hamiltonian and the structure of its ground space. We start by considering approximate symmetry operators, defined as unitary operators whose commutators with the Hamiltonian have norms that are sufficiently small. We show that when approximate symmetry operators can be restricted to the ground space while approximately preserving certain mutual commutation relations. We generalize the Stone-von Neumann theorem to matrices that approximately satisfy the canonical (Heisenberg-Weyl-type) commutation relations, and use this to show that approximate symmetry operators can certify the degeneracy of the ground space even though they only approximately form a group. Importantly, the notions of “approximate” and “small” are all independent of the dimension of the ambient Hilbert space, and depend only on the degeneracy in the ground space. Our analysis additionally holds for any gapped band of sufficiently small width in the excited spectrum of the Hamiltonian, and we discuss applications of these ideas to topological quantum phases of matter and topological quantum error correcting codes. Finally, in our analysis we also provide an exponential improvement upon bounds concerning the existence of shared approximate eigenvectors of approximately commuting operators under an added normality constraint, which may be of independent interest.

Contents

1 Introduction 1
  1.1 Results .......................................................... 3
2 Restriction to the ground space 5
3 Degeneracy lower bounds 8
  3.1 Stone-von Neumann Theorem .................................. 8
  3.2 One twisted pair ................................................. 8
  3.3 Two twisted pairs ................................................ 10
4 Minimum twisted commutation value 14
  4.1 Frobenius spectral bound ....................................... 15
  4.2 Higher norms and tightness ..................................... 17
5 Applications and open questions 17
  5.1 Local Hamiltonians .............................................. 18
  5.2 Topologically ordered systems ................................ 18
  5.3 Quantum codes .................................................. 19
References 20

A Approximate shared eigenvectors for approximately commuting matrices 22

B An algorithm for the certifiable degeneracy of a twisted pair 24

1 Introduction

Given a quantum system described by a Hamiltonian $H$, a symmetry is simply an operator that commutes with $H$. The symmetry can be block diagonalized with respect to the energy eigenspaces, and so the degeneracy within these blocks is constrained by the symmetry. In a system that possesses exact symmetries, a sufficiently
weak perturbation will preserve the number of states of any band gapped away from the rest of the spectrum, but the symmetries will generally become only approximate.

In this work we consider a natural converse to this: suppose we know that a system has some approximate symmetries and a gapped band, such as the ground space band. Can we “unperturb” the symmetries into exact symmetries within the given band? Can we also use the approximate group structure of the approximate symmetries to count the degeneracy within the band? We answer these questions in the affirmative, giving quantitative bounds on when such a procedure can be performed, and thus when such approximate symmetries can be used as certificates of ground space degeneracy.

A related area of mathematical research with a long and rich history is the relationship between the properties of approximately and exactly commuting matrices. An exemplary problem which dates back as far as the 1950s [2–6] is whether a pair of approximately commuting matrices lie near an exactly commuting pair, i.e. whether there exists a dimension independent \( \delta > 0 \) for each \( \epsilon > 0 \) such that for all \( H, S \) with \( \|H\|, \|S\| \leq 1 \),

\[
\|\tilde{H}, \tilde{S} : [\tilde{H}, \tilde{S}] = 0, \text{ where } \|H - \tilde{H}\|, \|S - \tilde{S}\| \leq \epsilon,
\]

where here and throughout the norm \( \| \cdot \| \) is the operator norm. Interpreting \( H \) as the Hamiltonian and \( S \) as a symmetry, this problem can be interpreted as whether approximate symmetries are necessarily near exact symmetries of a perturbed system. It has been shown that just such a theorem holds if all matrices are Hermitian [7–10]. A physical consequence of this is that a pair approximately commuting observables can be approximately simultaneously measured [10].

For unitary matrices the above is however known to be generally false [11]. This is due to a K-theoretic obstruction [12–14], though it is true if this obstruction vanishes [7, 8, 15], or under the assumption of a spectral gap [16]. Imposing a form of self-duality analogous to time-reversal symmetry the relevant K-theoretic obstruction reduces to the spin Chern number of a fermionic system [17], highlighting a link between the fields of topologically ordered quantum systems [18] and approximately commuting matrices.

Here we will consider Hamiltonians \( \hat{H} \) with multiple non-commuting approximate symmetries, and establish a connection to the ground space degeneracy. Ground space degeneracy is a property of a quantum system that plays a special role in several important applications, such as quantum coding theory and the study of phases of matter. Quantum codes, especially those encoded into ground spaces of local Hamiltonians, are the leading candidates for thermally stable quantum memories [19, 20]; in these models approximate symmetries constitute approximate logical operators and the ground space degeneracy corresponds to the code size. In the context of condensed matter systems, the link between symmetries and degeneracies plays an important role both in classical symmetry-breaking phases [21], and also in exotic quantum phases, such as those exhibited by topologically ordered models [18]. Unfortunately, determining the ground space degeneracy of (finite) systems is generally \#P-complete, even for gapped bands [22]. However, if we restrict to more structured examples such as 1D-local spin systems, then ground spaces can in fact be efficiently approximated [23, 24]. Our results show that when structure is present in the form of certain mutual commutation relations, one can obtain certifiable bounds on the degeneracy of a ground space by only knowing bounds on these relations. We go into more detail about these two applications in Section 5.

The form of non-commutation we will consider will involve \textit{twisted} commutation relations.

\begin{definition}[Twisted commutator] For \( \alpha \in [0, 1) \), the twisted commutator is defined as

\[
[X, Y]_\alpha := XY - e^{2\pi i \alpha} YX.
\]

We will refer to \( \alpha \) as the twisting parameter, and for some unitarily invariant norm \( \| \cdot \| \) we will refer to \( \| [\cdot, \cdot]_\alpha \| \) as the twisted commutation value. When considering a pair of operators in tandem such that each has a small twisted commutation value we will refer to it as a twisted pair.
\end{definition}

We note that the \( \alpha = 0 \) and \( \alpha = 1/2 \) cases correspond to the commutator and anti-commutator respectively.

Commuting operators exist in all dimensions, finite or infinite. Twisted commuting operators on finite-dimensional spaces, however, only exist in certain dimensions depending on the twisting parameter, e.g. no \( \alpha \neq 0 \) twisted commutator can non-trivially vanish in a one-dimensional space. For general operators, twisted commuting operators were studied in some detail in Ref. [25]. If we restrict to unitary operators however, the Stone-von Neumann Theorem\footnote{As usually stated, the Stone-von Neumann theorem is much more general than Theorem 1. We will only be concerned with twisted commutation in finite-dimensional spaces, and unconcerned with uniqueness, so this form will suffice for our purposes.} [26, 27] classifies the dimensions in which twisted commutation can occur.

\begin{theorem}[Finite-dimensional Stone-von Neumann theorem] Given \( \alpha = p/q \) with \( p, q \) coprime, then unitary operators \( X \) and \( Y \) which exactly twisted commute as

\[
[X, Y]_\alpha = 0
\]

only exist in dimensions which are multiples of \( q \).
\end{theorem}
In this paper we will generalize this connection into the regime of *approximately* twisted-commuting operators. Properties of both approximate commuting, and approximately twisted-commuting operators are reviewed in Ref. [28]. The rigidity of algebraic structures to small perturbations in the commutation relations that define them has been studied in several other settings, such as the soft torus [29,30] and approximate representations of groups [31–33].

Suppose we have a physical system with a self-adjoint Hamiltonian $H$, acting on a possibly infinite-dimensional Hilbert space. Let $\Pi$ be the orthogonal projector onto the finite-dimensional ground space, and $\bar{\Pi} := I - \Pi$. For simplicity, take the ground state energy of $H$ to be zero, such that $\Pi H = 0$. As well as this, we will assume that the excited states are gapped away from the ground space, such that they all have an energy at least $\Delta$, i.e. $H \geq \Delta \bar{\Pi}$. For such a system there exist two notions of symmetry we will discuss.

**Definition 2 (Symmetry).** We define a ground symmetry as an operator $U$ that commutes with the ground space projector $[U, \Pi] = 0$, and acts unitarily on the ground space $\Pi U \Pi = \Pi U \Pi = \Pi$. Moreover, we refer to a unitary as an $\epsilon$-approximate symmetry if it approximately commutes with the Hamiltonian with respect to a given unitarily invariant norm $\|[U,H]\| \leq \epsilon$.

Here we use $\|\cdot\|$ to denote any unitarily invariant norm.

The error thresholds we are going to consider will depend on the spectral gap $\Delta$ of the system in question. One way to improve the scaling with the gap would be to consider symmetries defined not by commutation with the Hamiltonian, but by commutation with functions of the Hamiltonian. For example we could consider commutation with an (unnormalized) Gibbs state $\|U,e^{-\beta H}\| \leq \epsilon$.

Such a symmetry can be seen to be an $\epsilon$-approximate symmetry of $H' = I - e^{-\beta H}$, which shares a ground space with $H$ and has a gap of $1 - e^{\beta \Delta}$. If we have some control over the temperature, such as in Monte Carlo simulations, then this gives a tradeoff we can use to improve the gap scaling. If for example we set $\beta = \ln(2)/\Delta$, then we get a fixed gap of $1/2$. A similar analysis could be performed with any function of $H$ which leaves the relevant band gapped.

1.1 Results

The main goal of this paper will be to establish a connection between twisted commuting symmetries and the ground space dimension, even when the relevant commutation relations are only approximate. A key feature of our bounds is that they can be expressed entirely in terms of the Hamiltonian, and do not require objects such as the ground space projector, which can often be prohibitively difficult to calculate, represent, or perform calculations with. Without access to the ground space projector, whether or not a unitary is a ground symmetry cannot be directly verified.

In Section 2 we will explore the relationship between approximate and ground symmetries, showing that an approximate symmetry is always near a ground symmetry. Extending this to the case of multiple symmetries, we will see that approximate symmetries can be restricted to the ground space with low distortion, implying the existence of unitaries on the ground space with certain twisted commutation relations. In showing these results, we will make repeated use of the following function and note some simple bounds on it,

$$f : [0,1] \to [0,1], \quad f(x) := 1 - \sqrt{1 - x}, \quad \frac{x}{2} \leq f(x) \leq x.$$  

Then our first main result is the following.

**Theorem 2 (Restriction to the ground space).** For two $\epsilon$-approximate symmetries $U$ and $V$ which approximately twisted commute $\|[U,V]\| \leq \delta$, then if $\xi := \epsilon/\Delta < 1$ there exists unitaries $u$ and $v$ acting on the ground space which also approximately twisted commute $\|[u,v]\| \leq \delta + 2\xi^2 + 4f(\xi^2)$.

Rather importantly, we note that the above theorem holds independent of the ground space dimension. This will allow us to use approximate symmetries alone as witnesses of ground space degeneracy, circumventing the need for direct access to the ground space, which is often inaccessible in non-exactly solvable models.
Note that for simplicity we will henceforth take the band in consideration to be an exactly degenerate ground space. We will see however that our proof will rely not on the bound $H \geq \Delta \Pi$, but on its relaxation $H^2 \geq \Delta^2 \Pi$, meaning that the band could be anywhere in the spectrum, so long as it is gapped on both sides by at least $\Delta$. Furthermore we can take $w := \|H\Pi\| \geq 0$ when our band has a potentially non-zero width. By considering the new Hamiltonian $H' := H - H\Pi$, we get that our restricted result holds for more general bands once the necessary changes have been made.

**Corollary 3** (Restriction to a general band). If there are two $\epsilon$-approximate symmetries $U$ and $V$ which approximately twisted commute

$$\| [U,V]_\alpha \| \leq \delta,$$

then if $\xi' := (\epsilon + w)/\Delta < 1$ there exists unitaries $u$ and $v$ acting on band of gap $\Delta$ and width $w$ which also approximately twisted commute

$$\| [u,v]_\alpha \| \leq \delta + 2\xi'^2 + 4f(\xi'^2).$$

Now that we have restricted our symmetries down to the ground space, by studying the relationship between dimensionality and approximate twisted commutation, we can hope to use these twisted symmetries as witnesses of ground space degeneracy. As above, we will henceforth adhere to the convention of upper case letters denoting operators which act on the system as a whole, and lower case operators which only act on the ground space.

In Section 3 we start by giving a proof of Theorem 1, and consider generalizing this argument to the case of approximately twisted commuting operators. We consider a twisted pair of unitaries, and construct states which can be used to lower bound the number of eigenvalues these operators possess. By doing so we will show that if these operators have a sufficiently small twisted commutation value in the operator norm, then a lower bound on their degeneracy can be inferred.

**Theorem 4.** If $u$ and $v$ are unitaries such that for some $d \in \mathbb{N}$

$$\| [u,v]_{1/d} \| < \frac{2}{d-1} \left[ 1 - \cos \frac{\pi}{d} \right],$$

then the dimension of each operator is at least $d$.

While we do not have a closed form bound on the twisted commutation value required to certify other dimensions ($d \neq 1/\alpha$), in Appendix B we discuss an algorithm to determine which degeneracies are certified by twisted pairs of given parameters. Using this we will plot the dimension that can be certified as a function of both the twisting parameter and the corresponding twisted commutation value.

In Appendix A we strengthen existing results on shared approximate eigenvectors for approximately commuting operators when a normality condition is introduced, exponentially improving the dimension dependence of the bounds relative to known results [34]. Using this, in Section 3.3 we consider extending this procedure to the case of two pairs of twisted commuting unitaries. Here we will once again construct a set of ground states, showing that for sufficient parameters that they are linearly independent. Using this we can obtain a similar dimensionality lower bound.

**Theorem 5.** If $u_1$, $u_2$, $v_1$ and $v_2$ are unitaries such that they satisfy the commutation relations

$$\| [u_1,u_2] \| \leq \gamma \quad \| [u_1,v_2] \| \leq \delta \quad \| [u_2,v_1] \| \leq \delta$$

and twisted commutation relations

$$\| [u_1,v_1]_{1/d_1} \| \leq \delta \quad \| [u_2,v_2]_{1/d_2} \| \leq \delta$$

with $d_1 \leq d_2$ and

$$\sqrt{d_1}d_2 + (d_1 + d_2)\delta < \frac{\sin^2(\pi/2d_1)}{(d_1d_2 - 1)^2},$$

then the dimension of each operator is at least $d_1d_2$.

In Section 4 we provide a more comprehensive analysis for the case of a single twisted pair. Leveraging results from spectral perturbation theory, we find an explicit closed form for the minimum twisted commutation value for a class of norms known as the $(p,k)$-Schatten-Ky Fan norms. These are defined as the $p$-norm of the largest $k$ singular values, or more formally as

$$\|X\|_{(p,k)} := \sup_A \left\{ (\text{Tr} |AX|^p)^{1/p} \left| A \right\| \leq 1, \text{ rank}(A) \leq k \right\}.$$  

For a $g$-dimensional operator, the special case $k = g$ reduces to the Schatten $p$-norm, and the case $p = 1$ reduces to the Ky Fan $k$-norm. In particular, the $p = \infty$ and $(p,k) = (2,g)$ special cases reduce to the operator and Frobenius norms respectively.
Theorem 6 (Minimum twisted commutation value). Suppose that $u$ and $v$ are $g$-dimensional unitaries, then for any $p \geq 2$ the twisted commutator is lower bounded

$$
\| [u, v] \|_{(p,k)} \geq 2k^{1/p} \sin \left( \pi \frac{|g\alpha| - g\alpha}{g} \right),
$$

where $\| \cdot \|_{(p,k)}$ is the $(p,k)$-Schatten-Ky Fan norm. Moreover this bound is tight, in that sense that there exist families of $g$-dimensional unitaries which saturate the above bounds and only depend on $|g\alpha|$, the nearest integer to $g\alpha$.

For a given twisted pair, all dimensions for which the twisted commutation value falls below this minimum can therefore be ruled out as valid dimensions. As this bound is not monotonic as a function of $g$, it not only provides a lower bound, but a full classification of which dimensions are disallowed.

After giving proofs of the main results outlined above in Sections 2, 3, and 4, we turn to broader discussion and applications of these ideas. Section 5 is devoted to discussion of future directions for this work that add the additional constraint that the Hamiltonian is local, and we discuss the relationship to the notions of topological order and topological quantum codes. In particular we show how recent numerical methods for studying quantum many-body systems [35] could leverage the bounds presented here to provide certificates of the topological degeneracy of certain quantum systems.

2 Restriction to the ground space

In this section we will make precise the notion that approximate symmetries can be utilized as proxies of ground symmetries. We first establish a relationship between approximate symmetries and the ground symmetries that they imply. Then we consider operators with approximate twisted commutation relations, and we show that these can also be restricted faithfully to the ground space with low distortion.

Constructing a ground symmetry from an approximate symmetry will come in two steps. First we will pinch the symmetry $U$ with respect to $\Pi$, giving an operator $P$ for which $[P,\Pi] = 0$. While this will render $P$ no longer unitary, we will see that its action upon the ground space will still be approximately unitary. Using this we will construct a nearby operator $\tilde{U}$ that retains commutation with the ground space projector, and acts unitarily on the ground space, thus constituting a ground symmetry.

We will start by showing that the off-diagonal blocks of an approximate symmetry are small, and then follow by showing that its action on the ground space is approximately unitary.

Lemma 2.1 (Small off-diagonal blocks). If $U$ is an $\epsilon$-approximate symmetry, then off-diagonal blocks of $U$ with respect to $\Pi$ have bounded norms, in particular $\| \Pi U \Pi + U \Pi \Pi \| \leq \epsilon/\Delta$. For Hamiltonians of the form $H = \Delta \Pi$, this inequality is tight.

Proof. We start by noting that $|A|^2 \geq |B|^2$ implies $|AX|^2 \geq |BX|^2$ for any $X$, where $|M| := \sqrt{MM^\dagger}$. Taking $X$ to be finite-rank, we have from Weyl’s inequalities [1] that the singular values of $AX$ majorize those of $BX$. Unitarily invariant norms$^2$ act as symmetric gauge functions on finite-rank operators [36–38], which implies from Refs. [36, Prop.IV.1, Thm.IV.2] that $\|AX\| \geq \|BX\|$—a similar argument for the adjoint also gives $\|XA\| \geq \|XB\|$. Because $H$ has a gapped band with projector $\Pi$, we have that $H^2 \geq \Delta^2 \Pi$. Using this, we can bound the off-diagonal blocks in terms of the commutator

$$
\Delta \| \Pi U \Pi + U \Pi \Pi \| = \| (\Pi \Delta + \Pi) (\Pi U \Pi + U \Pi \Pi) (\Delta \Pi + \Pi) \|
\leq \| (\Pi H + \Pi) (\Pi U \Pi + U \Pi \Pi) (H \Pi + \Pi) \|
= \| \Pi H U \Pi + U \Pi H \Pi \|,
$$

where the inequality follows from the aforementioned monotonicity property. Now using the unitary invariance of the norm (since $\Pi - \Pi$ is unitary), we find that

$$
\| \Pi H U \Pi + U \Pi H \Pi \| = \| \Pi H U \Pi - U \Pi H \Pi \|
= \| \Pi [H, U] \Pi + \Pi [H, U] \Pi \|
\leq \| [H, U] \| \leq \epsilon,
$$

where the second equality makes use of $H \Pi = 0$ and the first inequality is the pinching inequality.

With regard to tightness, if we take $H = \Delta \Pi$ then we can see $[H, U]$ has no on-diagonal blocks, and therefore $\Delta (\Pi U \Pi - U \Pi \Pi) = [H, U]$. Taking norms of both sides of this equation give $\Delta \| \Pi U \Pi + U \Pi \Pi \| = \| [H, U] \|$, meaning that $\| \Pi U \Pi + U \Pi \Pi \| \leq \epsilon/\Delta$ is tight.

$^2$Following Ref. [36,37] we adopt the normalization $\|A\| = \|A\|$ for all rank-1 operators $A$. 

5
Lemma 2.2 (Approximate unitarity on the ground space). For an \(\epsilon\)-approximate symmetry \(U\) with \(\xi := \epsilon/\Delta \leq 1\), the action on the ground space is approximately unitary

\[
\|\Pi - [\Pi U]_{\Pi}\| \leq f(\xi^2),
\]

where \(f(x) := 1 - \sqrt{1 - x}\). In the operator norm, this expression is tight.

**Proof.** First we can bound \(\|\Pi U\|_{\Pi}^2\) near \(\Pi\) by using the unitarity of \(U\) itself as

\[
\Pi - [\Pi U]_{\Pi}^2 = \Pi - [\Pi U^\dagger \Pi U]_{\Pi} = [\Pi U^\dagger \Pi U]_{\Pi} = [\Pi U^\dagger]_{\Pi}^2.
\]

Together with Lemma 2.1, the sub-multiplicativity of unitarily invariant norms on finite-rank operators let us conclude that \(\|\Pi - [\Pi U]_{\Pi}\| \leq \xi^2\). Next we need to use this bound on \(\|\Pi - [\Pi U]_{\Pi}\|^2\), and create a bound on \(\|\Pi - [\Pi U]_{\Pi}\|\).

Consider a function \(f(x) = \sum_{n=1}^{\infty} a_n x^n\), where \(a_n > 0\) and \(f(1) < \infty\). For any finite-rank operator \(0 \leq X \leq 1\), we can use the triangle inequality and submultiplicativity of \(\|\cdot\|\) to derive a Jensen-like inequality

\[
\|f(X)\| = \left\| \sum_{n=1}^{\infty} a_n X^n \right\| \leq \sum_{n=1}^{\infty} a_n \|X\|^n \leq \sum_{n=1}^{\infty} a_n \|X\|^n = f(\|X\|).
\]

If we let \(a_n = \Gamma(n-1/2)/2\sqrt{n!}\), then we get \(f(x) = 1 - \sqrt{1 - x}\) for \(x \in [0, 1]\). If we let \(X = \Pi - [\Pi U]_{\Pi}\), then applying the above gives

\[
\|\Pi - [\Pi U]_{\Pi}\| = \left\| f\left(\Pi - [\Pi U]_{\Pi}^2\right) \right\| \leq f\left(\|\Pi - [\Pi U]_{\Pi}^2\|\right) \leq f(\xi^2).
\]

We note that \(x/2 \leq f(x) \leq x\), which means that this bound improves upon the bound trivially given by the contractivity of \(\Pi U\),

\[
\|\Pi - [\Pi U]_{\Pi}\| \leq \|\Pi - [\Pi U]_{\Pi}^2\| \leq \xi^2.
\]

For the purposes of tightness, consider a two-dimensional Hilbert space, and a Hamiltonian \(H\) and unitary \(U\) given by

\[
H = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.
\]

In the operator norm \(\|U, H\| = \Delta \sin \phi\) and \(\|\Pi - [\Pi U]_{\Pi}\| = 1 - \cos \phi\), which saturates the above bound. 

Using these bounds we can now construct a ground symmetry \(\tilde{U}\) by pinching \(U\) with respect to \(\Pi\), and then restoring unitarity on the ground space.

Lemma 2.3 (Approximate symmetries are nearly ground symmetries). For an \(\epsilon\)-approximate symmetry \(U\) with \(\xi := \epsilon/\Delta \leq 1\), there exists a ground symmetry \(\tilde{U}\) which is close to \(U\)

\[
\|U - \tilde{U}\| \leq \xi + f(\xi^2),
\]

and closer still in the ground space

\[
\|\Pi (U - \tilde{U}) \Pi\| \leq f(\xi^2).
\]

The first inequality is tight to leading order in \(\xi\), and the second is tight in the operator norm.

**Proof.** We start by considering the polar of decompositon \(\Pi U/P\Pi = W [\Pi U]_{\Pi}\Pi\). As the ground space \(\text{im}(\Pi)\) is an invariant subspace of \(\Pi U\Pi\), we can take\(^3\) \(W\) to also leave the ground space invariant, \([W, \Pi] = 0\). Given this, we define our ground symmetry to be \(\tilde{U} := \Pi U \Pi + \Pi U \Pi\).

\(^3\)Such a \(W\) could be found by performing the polar decomposition restricted to the ground space, and padding the unitary out to act as the identity on the rest of the space.
We will now consider bounding the distance between $U$ and $\tilde{U}$ block-wise. The off-diagonal blocks are bounded by Lemma 2.1 as
\[
\left\| \Pi \left( \tilde{U} - U \right) \Pi + \Pi \left( U - \tilde{U} \right) \Pi + \Pi \left( \tilde{U} - U \right) \Pi \right\| \leq \xi.
\]
The bound on the ground space however follows from Lemma 2.2
\[
\left\| \Pi \left( \tilde{U} - U \right) \Pi \right\| = \left\| WH - W \Pi U \Pi \right\| = \left\| \Pi - \Pi U \Pi \right\| \leq f(\xi^2).
\]
Finally the fact that $U$ was unchanged on the excited space trivially implies
\[
\left\| \Pi \left( \tilde{U} - U \right) \Pi \right\| = 0.
\]
Putting everything together, this gives the desired bound
\[
\left\| \tilde{U} - U \right\| = \left\| \Pi \left( \tilde{U} - U \right) \Pi + \Pi \left( U - \tilde{U} \right) \Pi + \Pi \left( \tilde{U} - U \right) \Pi \right\|
\leq \left\| \Pi \left( \tilde{U} - U \right) \Pi + \Pi \left( U - \tilde{U} \right) \Pi + \Pi \left( \tilde{U} - U \right) \Pi \right\|
\leq \xi + f(\xi^2).
\]

As for tightness, Lemma 2.1 gives that $\left\| \Pi U \Pi + \Pi U \Pi \right\| = \left\| [H, U] / \Delta \right\|$ for Hamiltonians of the form $H = \Delta \Pi$. If we assume that $\left\| [U, H] \right\| = \epsilon$, then applying the pinching inequality gives
\[
\left\| U - \tilde{U} \right\| \geq \left\| \Pi \left( U - \tilde{U} \right) \Pi + \Pi \left( \tilde{U} - U \right) \Pi \right\| = \left\| \Pi U \Pi + \Pi U \Pi \right\| = \xi,
\]
which proves our bound on $U - \tilde{U}$ is tight to leading order in $\xi$. The tightness of the norm distance in the ground space follows directly from the tightness of Lemma 2.2.

We will now consider how the existence of nearby ground symmetries allows twisted commutation relations of approximate symmetries to be pulled down into the ground space.

**Theorem 2 (Restriction to the ground space).** For two $\epsilon$-approximate symmetries $U$ and $V$ which approximately twisted commute
\[
\left\| [U, V]_{\alpha} \right\| \leq \delta,
\]
then if $\xi := \epsilon / \Delta \leq 1$ there exists unitaries $u$ and $v$ acting on the ground space which also approximately twisted commute as
\[
\left\| [u, v]_{\alpha} \right\| \leq \delta + 2\xi^2 + 4f(\xi^2).
\]

**Proof.** Consider a $\tilde{U}$ and $\tilde{V}$ given by applying Lemma 2.3 to $U$ and $V$ respectively, such that
\[
\left\| \Pi U \Pi - \Pi \tilde{U} \Pi \right\|, \left\| \Pi V \Pi - \Pi \tilde{V} \Pi \right\| \leq f(\xi^2).
\]

Next we consider the twisted commutator of $U$ and $V$, and that of $\tilde{U}$ and $\tilde{V}$, both projected into the ground space. By expanding out the twisted commutators we have
\[
\Pi [U, V]_{\alpha} \Pi - \Pi [\tilde{U}, \tilde{V}]_{\alpha} \Pi = \left[ \Pi U \Pi, \Pi V \Pi \right]_{\alpha} - \left[ \Pi \tilde{U} \Pi, \Pi \tilde{V} \Pi \right]_{\alpha} + \Pi U \Pi \cdot \Pi V \Pi - e^{2\pi i \alpha} \Pi V \Pi \cdot \Pi U \Pi,
\]
\[
= \left( \Pi U \Pi - \Pi \tilde{U} \Pi \right) \cdot \Pi V \Pi - e^{2\pi i \alpha} \Pi V \Pi \cdot \left( \Pi U \Pi - \Pi \tilde{U} \Pi \right)
\]
\[
+ \Pi \tilde{U} \Pi \cdot \left( \Pi V \Pi - \Pi \tilde{V} \Pi \right) - e^{2\pi i \alpha} \left( \Pi V \Pi - \Pi \tilde{V} \Pi \right) \cdot \Pi U \Pi
\]
\[
+ \Pi U \Pi \cdot \Pi V \Pi - e^{2\pi i \alpha} \Pi V \Pi \cdot \Pi U \Pi.
\]

Using the triangle inequality, the contractivity of $\Pi U \Pi$ and $\Pi V \Pi$, and the bound on the off-diagonal blocks from Lemma 2.1, we can bound this as required:
\[
\left\| \Pi [U, V]_{\alpha} \Pi - \Pi [\tilde{U}, \tilde{V}]_{\alpha} \Pi \right\| \leq \left\| \left( \Pi U \Pi - \Pi \tilde{U} \Pi \right) \cdot \Pi V \Pi \right\|
+ \left\| \Pi V \Pi \cdot \left( \Pi U \Pi - \Pi \tilde{U} \Pi \right) \right\|
+ \left\| \Pi \tilde{U} \Pi \cdot \left( \Pi V \Pi - \Pi \tilde{V} \Pi \right) \right\|
+ \left\| \Pi V \Pi \cdot \left( \Pi V \Pi - \Pi \tilde{V} \Pi \right) \right\|
\leq 2 \left\| \Pi U \Pi - \Pi \tilde{U} \Pi \right\| + 2 \left\| \Pi V \Pi - \Pi \tilde{V} \Pi \right\|
+ \left\| \Pi U \Pi \cdot \Pi V \Pi \right\| + \left\| \Pi V \Pi \cdot \Pi U \Pi \right\|
\leq 4f(\xi^2) + 2\xi^2.
Next we let \( u \) and \( v \) be the restriction of \( \tilde{U} \) and \( \tilde{V} \) to the ground space respectively. As each are ground symmetries, \( u \) and \( v \) are both unitaries. If we consider the embedding of operators on the ground space back into the larger Hilbert space, then we can use the above to bound the twisted commutator of our ground space unitaries

\[
\| [u, v] \| \leq \| [U, V] \| + 2 \xi^2 + 4 f(\xi^2).
\]

Note that if we had a set of more than two unitaries, this additive growth in the twisted commutation value would hold equally for every pair separately.

3 Degeneracy lower bounds

In this section we show how twisted pairs of unitary operators can be used to give lower bounds on the degeneracy of the ground space. We start by considering an exact twisted pair and the Stone-von Neumann theorem. We will then show how this argument can be generalized to approximate twisted pairs, and how a lower bound on the degeneracy follows from an upper bound on the twisted commutator value. Finally we will see how this can also be extended to more general collections of twisted commuting operators through the example case of two twisted pairs that are approximately mutually commuting.

3.1 Stone-von Neumann Theorem

Consider a \( u \) and \( v \) which exactly twisted commute, so that \( uv = e^{2i\pi \alpha} vu \). Let \( (\lambda, |\psi\rangle) \) be an eigenpair of \( u \). Using the twisted commutation relation, we see that \( |\psi'\rangle := v|\psi\rangle \) forms a \( \lambda e^{2i\pi \alpha} \)-eigenvector. It follows that \( v \) forms an isomorphism between the \( \lambda \) and \( \lambda e^{2i\pi \alpha} \)-eigenspaces of \( u \), which allows us to conclude that their dimensions must be the same. Carrying this argument forward, we can see that any eigenspaces whose eigenvalues differ by any power of \( e^{2i\pi \alpha} \) must also be isomorphic.

Suppose we take \( \alpha \in \mathbb{Q} \), with \( \alpha = p/q \) with \( p, q \) coprime. As we can see in Fig. 1, a simple divisibility argument implies that the eigenspaces come in isomorphic multiples of \( q \), which therefore implies that the overall dimension of \( u \) and \( v \) is a multiple of \( q \) also.

We now generalize this connection between the twisted commutator and the spectrum of one of the operators to allow for only approximate twisted commutation.

3.2 One twisted pair

Let us first extend the above argument to the case of a single approximate twisted pair. For simplicity, we consider the case where \( \alpha = p/q \) with \( p = 1 \) and \( q = d \), so the corresponding phase in the twisted commutator
is $\eta := e^{2i\pi/d}$. This is not much of a restriction since if $p > 1$ we can replace $v$ with $v^p$ where $p$ is the modular multiplicative inverse of $p$ such that $pq = 1 \mod q$ and then apply the results of the $p = 1$ case. Under this substitution the twisted commutator will grow by at most a factor of $[q/2]$. However, in Appendix B we will show an alternative method that in fact works for arbitrary $\alpha \in \mathbb{R}$ and gives tighter bounds than this simple reduction. We also consider without loss of generality the case where $u$ has at least one $+1$ eigenvalue, which can always be achieved by redefining $u$ by multiplying by a complex unit phase factor.

Suppose we have two unitaries $u$ and $v$ such that

$$\| [u, v]_{1/d} \| = \| uv - \eta vu \| \leq \delta.$$ 

Our results will show that these operators must, for sufficiently small $\delta$, be at least $d$-dimensional. To do this we will explicitly show that $u$ has at least $d$ distinct eigenvalues.

Let $|\psi\rangle$ be a $+1$ eigenvector of $u$, i.e. $u|\psi\rangle = |\psi\rangle$. Consider the orbit of $|\psi\rangle$ under $v$, i.e. the states $|j\rangle := v^j|\psi\rangle$ for $j = -\left\lfloor \frac{d-1}{2} \right\rfloor, \ldots, \left\lceil \frac{d-1}{2} \right\rceil$. These vectors are precisely the vectors depicted in Figure 1. We first show that these are approximate eigenstates of $u$.

**Lemma 3.1** (Change in expectation value: One pair). The expectation value of $u$ with respect to $|j\rangle$ is approximately $\eta^j$, specifically 

$$|\langle j | u | j \rangle - \eta^j | \leq |j| \delta.$$  

**Proof.** This follows from the twisted commutator of $u$ and $v$ being small. By expanding the commutator and applying the triangle inequality we can see that $\| uv - \eta vu \| \leq \delta$ implies $\| v^{-j} uv - \eta^j u \| \leq |j| \delta$. From this we can see that the expectation value of $|j\rangle$ lies close to $\eta^j$:

$$|j| \delta \geq \| v^{-j} uv - \eta^j u \|$$

$$= \| v^{-j} uv^j - \eta^j u \|$$

$$\geq \langle \psi | v^{-j} uv^j - \eta^j u | \psi \rangle$$

$$\geq \langle \psi | v^{-j} uv^j | \psi \rangle - \eta^j \langle \psi | u | \psi \rangle$$

$$= \langle j | u | j \rangle - \eta^j.$$ 

So we can see that the $\{|j\rangle\}$ form a set of vectors with expectation values distributed approximately evenly around the unit circle, much like the states in the $\delta = 0$ case as seen in Fig. 1. To relate these states to the dimensions of $u$ and $v$, we will now show that there must exist an eigenvalue of $u$ near the expectation value of each state.

**Lemma 3.2** (Existence of eigenvalues). If there exists a state $|x\rangle$ such that

$$|\langle x | u | x \rangle - e^{i\theta} | \leq \zeta$$

then $u$ possesses a nearby eigenvalue $e^{i\phi}$ such that

$$|\phi - \theta | \leq \cos^{-1}(1 - \zeta).$$

**Proof.** The bound on the expectation value with respect to $u$ implies

$$\text{Re} \langle x | e^{-i\theta} u | x \rangle \geq 1 - \zeta.$$ 

As this expectation value is a convex combination of the eigenvalues of $u$, all of which lie on the unit circle, there must exists an eigenvalue of $e^{-i\theta} u$ with real value at least $1 - \zeta$ (see Fig. 2). This in turn implies that $u$ possesses an eigenvalue $e^{i\phi}$ such that

$$\text{Re} e^{i(\phi - \theta)} = \cos(\phi - \theta) \geq 1 - \zeta.$$ 

Combining the two above lemmas, we can place a lower bound on the number of distinct eigenvalues of $u$.

**Theorem 4.** If $u$ and $v$ are unitaries such that

$$\| [u, v]_{1/d} \| < \frac{2}{d-1} \left[ 1 - \cos \frac{\pi}{d} \right],$$

then the dimension of each operator is at least $d$. 

9
Lemma 3.2 gives that if there exists an expectation value in the blue region, there must exist an eigenvalue within the minor segment indicated by the dotted line.

Proof. From Lemma 3.1 we know that 

\[ |\langle j|u|j\rangle - e^{2\pi j/d}| \leq |j|\delta. \]

Applying Lemma 3.2 we therefore get that \( u \) must have a corresponding eigenvalue \( e^{i\phi_j} \) where

\[ |\phi_j - 2j\pi/d| \leq \cos^{-1}(1 - |j|\delta). \]

As such we can see that each eigenvalue is within some error of a \( d \)th root of unity.

Next we want to find a bound for \( \delta \) which ensures that these eigenvalues must be distinct, by bounding the regions in which these eigenvalues must exist away from each other. To do this we need \( |\phi_j - \phi_k| > 0 \) for all \( j \neq k \). Taking the worst case over \( j \neq k \):

\[
|\phi_j - \phi_k| = \frac{2\pi}{d} |j - k| + \left| \phi_j - \frac{2j\pi}{d} \right| - \left| \phi_k - \frac{2k\pi}{d} \right| \\
\geq \frac{2\pi}{d} |j - k| - \left| \phi_j - \frac{2j\pi}{d} \right| - \left| \phi_k - \frac{2k\pi}{d} \right| \\
\geq \frac{2\pi}{d} - \cos^{-1}\left(1 - \left| \frac{d - 1}{2} \right| \delta\right) - \cos^{-1}\left(1 - \left| \frac{d - 1}{2} \right| \delta\right).
\]

Here the last line follows from the fact that \( j \) and \( k \) cannot both saturate the worst-case distance of \( \lceil \frac{d - 1}{2} \rceil \). Therefore, the worst case can be chose without loss of generality to be \( j = \lfloor \frac{d - 1}{2} \rfloor \) and \( k = -\lfloor \frac{d - 1}{2} \rfloor \). Using the concavity of \( \cos^{-1}(z) \) over \( z \in [0,1] \), we can loosen this to

\[ |\phi_j - \phi_k| \geq \frac{2\pi}{d} - 2\cos^{-1}\left(1 - \frac{d - 1}{2}\delta\right). \]

Clearly this step is trivial for odd \( d \).

Thus we get that a sufficient condition for all of the eigenvalues to be distinct is that the right-hand side of this inequality is strictly positive, and therefore we have the equivalent condition

\[ \cos^{-1}\left(1 - \frac{d - 1}{2}\delta\right) < \frac{\pi}{d}. \]

Rearranging, we find the specified bound on \( \delta \) of

\[ \delta < \frac{2}{d - 1}\left[1 - \cos(\pi/d)\right]. \]

Above we have only considered the case \( d = 1/\alpha \), similar analysis could be performed for bounds required to certify dimensions \( d' \neq 1/\alpha \). In Appendix B we describe an algorithm for calculating which dimensions can be certified for an arbitrary pair of parameters \( \alpha \) and \( \delta \) — running this algorithm gives Fig. 3.

3.3 Two twisted pairs

Next we are going to argue that the above analysis can be extended to more general collections of twisted commuting symmetries. By way of example, we are going to consider the case of two twisted pairs

\[ \left\|\ket{u_1, v_1}_{1/d_1}\right\| \leq \delta, \quad \left\|\ket{u_2, v_2}_{1/d_2}\right\| \leq \delta, \]
each of which approximately commute

\[ \|[u_1, u_2]\| \leq \gamma, \quad \|[u_1, v_2]\| \leq \delta, \quad \|[u_2, v_1]\| \leq \delta. \]

The equivalent of Stone-von Neumann theorem laid down in Section 3.1 gives that for \( \gamma = \delta = 0 \), the dimension of such operators must be a multiple of \( d_1d_2 \). We are going to give bounds on \( \gamma \) and \( \delta \) below which we can prove the dimension to be at least \( d_1d_2 \).

Previously we bounded the dimension from below by bounding the number of distinct eigenvalues. This is possible because these eigenvalues imply the existence of an orthonormal set of associated eigenvectors. As \( u_1 \) and \( u_2 \) do not commute, they will not necessarily possess an orthonormal set of shared eigenvectors. Instead we will have to address these vectors more directly, constructing approximate shared eigenvectors and proving their linear independence. First we will see that the approximate commutation of \( u_1 \) and \( u_2 \) can be used to demonstrate the existence of such a vector.

The existence of approximate shared eigenvectors of approximately commuting matrices was first proven in generality by Bernstein in Ref. [34]. Whilst Bernstein considers potentially non-normal matrices, in our case both \( u_1 \) and \( u_2 \) are unitary. In Appendix A we leverage this additional structure to exponentially tighten the bounds on the approximate shared eigenvectors. One of the relevant bounds considered in Appendix A gives the following immediate corollary.

**Lemma 3.3** (Approximate eigenvector). There exists a vector \( |\psi\rangle \) such that, after multiplying \( u_1 \) and \( u_2 \) by appropriate phase factor, it is an approximate shared +1-eigenvector of both, namely that

\[ \|[u_1, \psi]\|, \|[u_2, \psi]\| \leq \sqrt{\gamma d_1d_2}/2. \]

**Proof.** Given an assumption that the dimension is at most \( d_1d_2 \), this is a direct application of Theorem A.1, which we consider in detail in Appendix A.

As in the case of a single pair, we will then consider the orbit of this vector under the action of products of \( v_1 \) and \( v_2 \). Let \( |i, j\rangle := v_1^iv_2^j|\psi\rangle \) for \( i = -\left\lfloor \frac{d_1-1}{2} \right\rfloor, \ldots, \left\lceil \frac{d_1-1}{2} \right\rceil \) and \( j = -\left\lfloor \frac{d_2-1}{2} \right\rfloor, \ldots, \left\lceil \frac{d_2-1}{2} \right\rceil \). For convenience once again let \( \eta_i := e^{2i\pi/d_i} \).
Lemma 3.4 (Change in expectation value: two pair). The states $|i, j\rangle$ are shared approximate eigenstates of $u_1$ and $u_2$. Specifically their approximate eigenvalues are the corresponding powers of $\eta_1$ and $\eta_2$

$$|\langle i, j|u_1|i, j\rangle - \eta_1^i|, |\langle i, j|u_2|i, j\rangle - \eta_2^j| \leq \sqrt{\gamma_{12}}/2 + (|i| + |j|)\delta.$$  

Proof. From Lemma 3.3 we have  

$$|\langle \psi|u_1|\psi\rangle - 1| \leq \sqrt{\gamma_{12}}/2.$$  

Applying an argument similar to that in Lemma 3.1 we can bound the change in eigenvalue under the action of $v_c$ as  

$$|\langle i, 0|u_1|i, 0\rangle - \eta_1^i| \langle \psi|u_1|\psi\rangle| \leq |i|\delta.$$  

Applying the same argument for $v_2$ gives  

$$|\langle i, j|u_1|i, j\rangle - \langle i, 0|u_1|i, 0\rangle| \leq \delta.$$  

The triangle inequality allows us to merge these three inequalities, giving the stated bound. A similar argument can be performed for $u_2$. \hfill \Box

In the single pair case, we used the expectation values to imply the existence of nearby eigenvalues. Due to the lack of a shared eigenbasis of $u_1$ and $u_2$, we cannot do the same in the two pair case.

The reason that a set of distinct eigenvalues lower bounds the dimension is that, for normal operators such as unitaries, the eigenvalues imply the existence of an orthonormal eigenbasis. Instead of proving the existence of such vectors indirectly through the eigenvalues, we could instead prove our vectors $\{|i, j\rangle\}$ to be linearly independent — this is the approach we will take.

To this end, we will start by showing two approximate eigenvectors of a unitary with inconsistent expectation values are approximately orthogonal.

Lemma 3.5 (Low overlap). If two normalized vectors $|x\rangle$ and $|y\rangle$ have expectation values with some unitary $w$ such that  

$$|\langle x|w|x\rangle - e^{i\theta_x}| \leq \zeta \quad \text{and} \quad |\langle y|w|y\rangle - e^{i\theta_y}| \leq \zeta$$  

then the two vectors have a bounded overlap  

$$|\langle x|y\rangle| \leq \sqrt{2\zeta}\csc \left(\frac{\theta_y - \theta_x}{4}\right).$$

Proof. Firstly, let $w' := e^{-i\theta_x}w$ and $\theta := \theta_y - \theta_x$. Next consider splitting the unit circle into three arcs $X$, $Y$, and $Z$. We let $X$ and $Y$ be centered on $\theta_x$ and $\theta_y$ respectively, and define them to be the largest possible regions such that they remain disjoint. We define $Z$ to be the remaining arc, as shown in Fig. 4. Note that by convexity any linear combination of eigenvectors whose eigenvalues lie in $X$ will have an expectation value in the segment subtended by $X$, and similar for $Y$.

Now split $|x\rangle$ into two components  

$$|x\rangle = \sqrt{1 - \lambda_x}|x_X\rangle + \sqrt{\lambda_x}|x_YZ\rangle,$$

where $|x_X\rangle$ is in the span of eigenvectors with values in $X$, and $|x_YZ\rangle$ similar for $Y$ and $Z$. By definition of $X$, we have  

$$\Re \langle x_YZ|w'|x_YZ\rangle \leq \cos(\theta/2) \leq \Re \langle x_X|w'|x_X\rangle \leq 1.$$
Next we use the bound on the expectation value.

\[
\zeta \geq |\langle x | w' | x \rangle - 1 |
\]

\[
\geq 1 - \text{Re} \langle x | w' | x \rangle
\]

\[
= 1 - (1 - \lambda_x) \text{Re} \langle x | w' | x \rangle - \lambda_x \text{Re} \langle xy | w' | xy \rangle
\]

\[
\geq 1 - (1 - \lambda_x) - \lambda_x \cos (\theta/2)
\]

\[
= 2\lambda_x \sin^2(\theta/4)
\]

Thus we conclude that \( \lambda_x \leq (\zeta/2) \csc^2(\theta/4). \) Similarly if we were to have decomposed \(|y\rangle\) into parts contained in \( Y \) and \( XZ \) as \(|y\rangle = \sqrt{1 - \lambda_y}|y\rangle + \sqrt{\lambda_y}|yz\rangle\) then \( \lambda_y \leq (\zeta/2) \csc^2(\theta/4). \)

Further decomposing

\[|xy| = \cos \varphi_x |x\rangle + \sin \varphi_x |y\rangle, \quad |yz| = \cos \varphi_y |y\rangle + \sin \varphi_y |z\rangle,\]

then the inner product has the form

\[
|\langle x | y \rangle | = \left| \sqrt{1 - \lambda_x} \sqrt{1 - \lambda_y} \cos \varphi_x \langle x | y \rangle + \sqrt{\lambda_x} \sqrt{1 - \lambda_y} \cos \varphi_x \langle y | y \rangle + \sqrt{\lambda_x} \sqrt{1 - \lambda_y} \sin \varphi_x \sin \varphi_y \langle x | z \rangle \right|
\]

\[
\leq \sqrt{1 - \lambda_x} \sqrt{1 - \lambda_y} \cos \varphi_x + \sqrt{\lambda_x} \sqrt{1 - \lambda_y} \sin \varphi_x \sin \varphi_y.
\]

Using the identity \(|A \cos \phi + B \sin \phi|^2 \leq |A|^2 + |B|^2\), we can maximize over \( \varphi_x \) to get

\[
|\langle x | y \rangle | \leq \sqrt{1 - \lambda_x} \sqrt{1 - \lambda_y} \cos \varphi_y + \sqrt{\lambda_x} \sqrt{1 - \lambda_y} \cos^2 \varphi_y.
\]

Using \( \cos \varphi_y \leq 1 \), we can simplify this bound to

\[
|\langle x | y \rangle | \leq \sqrt{\lambda_y} + \sqrt{\lambda_x}.
\]

Applying the \( \zeta \)-dependent bounds on the \( \lambda \) values, we get the stated bounds. \( \square \)

Now that we have a way of bounding the overlap between our vectors, we need to determine how low this overlap needs to be before linear independence can be ensured.

**Lemma 3.6 (Overlap threshold).** Take a set of normalized vectors \( S = \{ |v_i\rangle \} \) for \( 1 \leq i \leq n \). If the pairwise overlap between any two vectors is bounded \(|\langle v_i | v_j \rangle| < 1/(n - 1)\) for \( i \neq j \), then \( S \) is linearly independent.

**Proof.** Let \( G \) be the Gram matrix associated with \( S \). As each of the vectors is normalized \( G_{ii} = 1 \) for all \( i \).

As all of the non-diagonal entries are strictly modulus-bounded by \( 1/(n - 1) \), this matrix is strictly diagonally dominant, i.e.

\[|G_{ii}| > \sum_{j \neq i} |G_{ij}| \quad \text{for all } i.\]

From the Gershgorin circle theorem, such matrices are non-singular and full rank, allowing us to conclude that \( S \) is linearly independent.

Note that this analysis is tight, i.e. if \( |\langle v_i | v_j \rangle| = 1/(n - 1) \) for \( i \neq j \) then \( G \) is singular and \( \sum_i |v_i\rangle = 0 \). By considering the eigenvectors of such a Gram matrix, a set of vectors satisfying this can be backed out. \( \square \)

Given this bound, we can finally find the condition for our vectors to be linearly independent and therefore lower bound the dimension of the space in which they reside.

**Theorem 5.** If \( u_1, u_2, v_1 \) and \( v_2 \) are unitaries such that they satisfy the commutation relations

\[
||u_1, u_2|| \leq \gamma \quad ||u_1, v_2|| \leq \delta \quad ||u_2, v_1|| \leq \delta
\]

and twisted commutation relations

\[
||u_1, v_1||_{d_1} \leq \delta \quad ||u_2, v_2||_{d_2} \leq \delta
\]

with \( d_1 \leq d_2 \) and

\[
\sqrt{d_1} d_2 + (d_1 + d_2) \delta < \frac{\sin^2(\pi/2d_1)}{(d_1d_2 - 1)^2},
\]

then the dimension of each operator is at least \( d_1d_2 \).

13
Proof. From Lemma 3.4 we have that our vectors have expectation values bounded near powers of \( \eta_1 \) and \( \eta_2 \)

\[
\left| \langle i, j | u_1 | i, j \rangle - \eta_1 \right| , \left| \langle i, j | u_2 | i, j \rangle - \eta_2 \right| \leq \sqrt{3}d_1d_2/2 + (|i| + |j|) \delta.
\]

Take a pair of vectors \(|i, j\rangle\) and \(|i', j'\rangle\) such that \(i \neq i'\). Applying Lemma 3.5 with \(w = u_1\) we get that their overlap is bounded as

\[
|\langle i, j | i', j' \rangle|^2 \leq \sqrt{3}d_1d_2 + 2 \max \{|i| + |j|, |i'| + |j'|\} \delta \cdot \csc^2 \left( \frac{\pi (|i - i'|)}{2d_1} \right).
\]

Combining this with a similar argument for \(u_2\), and assuming \(d_1 \leq d_2\), we get that for \((i, j) \neq (i', j')\)

\[
|\langle i, j | i', j' \rangle|^2 \leq \left[ \sqrt{3}d_1d_2 + (d_1 + d_2)\delta \right] \csc^2 \left( \frac{\pi}{2d_1} \right)
\]

Thus we can see that

\[
\left[ \sqrt{3}d_1d_2 + (d_1 + d_2)\delta \right] \csc^2 \left( \frac{\pi}{2d_1} \right) < \frac{1}{(d_1d_2 - 1)^2}.
\]

implies \(|\langle i, j | i', j' \rangle| < 1/(d_1d_2 - 1)\) for all \((i, j) \neq (i', j')\). By Lemma 3.6 this means that the collection of vectors \(|\langle i, j \rangle\rangle\) are linearly independent, constructively proving the dimensionality of the operators in question to be at least \(d_1d_2\). Rearranging this gives the specified bound.

\[\Box\]

4 Minimum twisted commutation value

In the previous section we considered finding lower bounds on the dimensions of approximately twisting commuting operators. In the exact case, the Stone-von Neumann theorem (c.f. Theorem 1) tells us that unitaries \(x\) and \(y\) for which

\[ [x, y]_{1/d} = 0 \]

are not only at least \(d\)-dimensional, but are a multiple of \(d\)-dimensional. We might therefore hope for a more comprehensive understanding of twisted commutation that provides more information than simply a lower bound on the dimension. In this section we will consider the twisted commutator in the Schatten-Ky Fan norms \(\|\| := \|\|_{(p, k)}\) with \(p \geq 2\), and find the minimum possible twisted commutator value as a function of dimension.

Definition 3 (Minimum twisted commutator value). Let \(\Lambda_{(p, k)}^g\) be the minimum twisted commutator value, with respect to the Schatten-Ky Fan \((p, k)\)-norm, over all pairs of unitary matrices of dimension \(g\)

\[\Lambda_{(p, k)}^g := \min_{u, v \in U(g)} \|u, v\|_{(p, k)}\]

In this language, the Stone-von Neumann theorem gives that \(\Lambda_{(p, k)}^g = 0\) if and only if \(g \alpha \in \mathbb{Z}\). If we had an understanding of the values of \(\Lambda_{(p, k)}^g\) where \(g \alpha \notin \mathbb{Z}\), then we could use twisted commutation value as a way of certifying dimension. In particular, if one thinks of \(\alpha\) as fixed, and one knows the value \(\|u, v\|_{(p, k)}\) to be less than \(\Lambda_{(p, k)}^g\) for certain dimensions \(g\), then these certain dimensions are ruled out as possible dimensions of \(u\) and \(v\). In this section we will explicitly evaluate \(\Lambda_{(p, k)}^g\) for \(p \geq 2\).

To lower bound \(\Lambda_{(p, k)}^g\), we will utilize techniques from spectral perturbation theory to bound a related quantity known as the spectral distance. By considering a family of operators which twisted commute, we will furthermore show this bound to be tight.

Definition 4 (Spectral distance). The spectral \((p, k)\)-distance \(d_{(p, k)}(a, b)\) between two matrices \(a\) and \(b\) is the \((p, k)\)-norm of the vector containing the differences between eigenvalues of the two matrices, minimized over all possible orderings. If we let \(\lambda(x)\) denote the vector of eigenvalues of a \(g \times g\) matrix \(x\) then algebraically

\[d_{(p, k)}(a, b) := \min_{\sigma \in S_g} \|\sigma [\lambda(a)] - \lambda(b)\|_{(p, k)} = \min_{\sigma \in S_g} \left( \sum_{j=1}^k \left| \lambda_{\sigma(j)}(a) - \lambda_j(b) \right|^p \right)^{1/p}, \]

where the minimization is over all elements \(\sigma\) of the permutation group \(S_g\) on \(g\) symbols.
4.1 Frobenius spectral bound

Before attacking the spectral distance, we are first going to restrict ourselves to the case of the Frobenius norm ($p = 2, k = g$), where we shall denote the norm by $\| \cdot \|_F$, the corresponding spectral distance by $d_F(\cdot, \cdot)$, and the twisted commutator minimum by $\Lambda_{g,a}^{(F)}$. In this special case, the spectral distance between two normal matrices is bounded by their norm difference.

**Lemma 4.1** (Wielandt-Hoffman inequality [39]). For normal matrices $a$ and $b$, $d_F(a, b) \leq \|a - b\|_F$.

Once again let $\eta := e^{2 \pi i \alpha}$. Applying Wielandt-Hoffman to $\Lambda_{g,a}^{(F)}$ we see that the corresponding spectral distance provides a lower bound,

$$\Lambda_{g,a}^{(F)} = \min_{u,v \in U(g)} \| u^\dagger uv - \eta u \|_F \geq \min_{u,v \in U(g)} d_F(u^\dagger uv, \eta u) = \min_{u \in U(g)} d_F(u, \eta u).$$

Though $\| u^\dagger uv - \eta u \|_F$ depended on both $u$ and $v$, $d_F(u, \eta u)$ depends only on the spectrum of $u$, making for a much simpler optimization. This inequality will turn out to be tight for matrices minimizing the twisted commutator value.

Denote the eigenvalues of $u$ by $\{ e^{i \theta_j} \}$, then the spectral distance in question is given by

$$d^2_F(u, \eta u) := \min_{\sigma \in S_g} \sum_{j=1}^g \left| e^{i \sigma(j)} - e^{i (\theta_j + 2 \pi \alpha)} \right|^2 = \min_{\sigma \in S_g} \sum_{j=1}^g 4 \sin^2 \left( \frac{\theta_{\sigma(j)} - \theta_j - 2 \pi \alpha}{2} \right).$$

Define $f(\sigma; \theta_1, \ldots, \theta_g)$ to be the argument of the above optimization

$$f(\sigma; \theta_1, \ldots, \theta_g) := \sum_{j=1}^g 4 \sin^2 \left( \frac{\theta_{\sigma(j)} - \theta_j - 2 \pi \alpha}{2} \right)$$

(1) such that $d^2_F(u, \eta u) = \min_\sigma f(\sigma; \theta_1, \ldots, \theta_g)$. The optimization of $d^2_F(u, \eta u)$ can therefore be reduced to an optimization of $f(\sigma; \theta_1, \ldots, \theta_g)$.

We can now break the optimization of $f$ down into two parts. First we will show that for any assignment of permutation and angles, there exists a cyclic permutation, and adjusted angles, for which the value of $f$ is the same. This will allow us to consider a minimizing permutation which has only a single cycle without loss of generality. Secondly we shall see that, for such a cyclic permutation, the set of angles which minimize $f$ are those that are equally distributed around the unit circle. Given these, we will find an explicit minimum for $f$, and thus for $d_F(u, \eta u)$.

**Lemma 4.2** (Reduction to cyclic permutations). For a given multi-cycle permutation $\sigma$ and set of angles $\{ \theta_j \}$, there exists a cyclic permutation $\sigma'$ and set of adjusted angles $\{ \theta'_j \}$ such that

$$f(\sigma; \theta_1, \ldots, \theta_g) = f(\sigma'; \theta'_1, \ldots, \theta'_g).$$

**Proof.** Firstly, our indices can be reordered such that the cycles of $\sigma$ are contiguous, i.e. in cycle notation

$$\sigma = (1 \ldots k_1 - 1) (k_1 \ldots k_2 - 1) \ldots (k_n \ldots g),$$

for some $1 < k_1 \cdots < k_n \leq g$. (Note that the result is trivially true if $g = 1$, so we restrict to $g > 1$.) As $f$ only depends on the difference between angles whose indices are within the same cycle of $\sigma$, if we shift all the angles within the same cycle by the same amount, the value of $f$ will not change. For example if we take the change of angle

$$\theta'_j := \begin{cases} \theta_j - \theta_1 & 1 \leq j < k_1 \\ \theta_j - \theta_{k_1} & k_1 \leq j < k_2 \\ \vdots \\ \theta_j - \theta_{k_n} & k_n \leq j \leq g. \end{cases}$$

then $f(\sigma; \theta_1, \ldots, \theta_g) = f(\sigma'; \theta'_1, \ldots, \theta'_g)$. Notice that $\theta'_{k_1} = \theta'_{k_1} = \cdots = \theta'_{k_n} = 0$ by construction.

We now wish to merge the permutation $\sigma$ into a single cyclic permutation

$$\sigma' := (1 \ldots g).$$

To do this, the only entries of the permutation which need to be changed are those at the end of each cycle.

$$\begin{array}{lcl} \sigma(k_1 - 1) = 1 & \rightarrow & \sigma'(k_1 - 1) = k_1 \\
\sigma(k_2 - 1) = k_1 & \rightarrow & \sigma'(k_2 - 1) = k_2 \\
\vdots & \vdots & \\
\sigma(g) = k_n & \rightarrow & \sigma'(g) = 1. \end{array}$$
By definition of the adjusted angles however, the only indices that change are those for which the angles have already been made identical in the previous step, i.e. \( \theta_{\sigma(j)} = \theta'_{\sigma(j)} \) for all \( j \). As \( f \) only depends on \( \sigma \) through how it acts on the angles, this means that this doesn’t alter the value of \( f \), therefore \( f(\sigma; \theta'_1, \ldots, \theta'_g) = f(\sigma'; \theta'_1, \ldots, \theta'_g) \).

Now that we have addressed the nature of the optimal permutation, namely showing that it can be taken to be cyclic, we turn out attention to the optimal angles.

**Lemma 4.3.** For a given single-cycle permutation \( \sigma \), the sets of angles which optimize \( f \), as defined in Eq. (1), correspond to those evenly distributed around the unit circle, and the difference between adjacent angles \( \theta \) have double counted in \( f \). As Lemma 4.2 tells us that we can consider cyclic permutations without loss of generality, we can apply the

**Proof.** This result can be seen by recalling that the definition of \( \sigma \) by

\[
\sigma \leftrightarrow \theta \quad \text{for all } j, \quad \sigma(j) = \theta(j) + 2\pi m(j - 1)/g, \quad \text{where } \lfloor \cdot \rfloor \text{ denotes integer rounding. Moreover the corresponding minimal value of } f \text{ is }
\]

\[
\min \left\{ f(\sigma; \theta_1, \ldots, \theta_g) \right\} = 2\sqrt{g} \sin \left( \frac{\theta - \theta(j)}{g} \right).
\]

**Proof.** Denote both of the terms\(^4\) in \( f \) which depend non-trivially on \( \theta \) by \( f_j(\theta_j) \). Using the double angle formula and the auxiliary angle method, we can reduce the \( \theta_j \) dependence to a single sinusoidal term.

\[
f_j(\theta_j) = 4 \sin^2 \left( \frac{\theta - \theta(j) - 2\pi \alpha}{2} \right) + 4 \sin^2 \left( \frac{\theta(j) - \theta - 2\pi \alpha}{2} \right)
\]

\[
= 4 - 4 \cos \left( \frac{\theta(j) - \theta + 2\pi \alpha}{2} \right) \cos \left( \frac{\theta - \theta(j) + 2\pi \alpha}{2} \right).
\]

We can therefore see that the optimal \( \theta_j \), leaving all other angles fixed, satisfies

\[
\theta_j = \left( \theta(j) + \theta(j-1) \right)/2 \mod \pi.
\]

This implies that \( \theta(j) - \theta_j = \theta_j - \theta(j-1) \mod 2\pi \), i.e. \( \theta_j \) lies in at the ‘midpoint’ of its neighbors, as described by \( \sigma \). By inducting the above argument we find that \( \theta(j) - \theta_j = \theta(k) - \theta_k \mod 2\pi \) for all \( j, k \) meaning that all adjacent angles are equally spaced around the unit circle. This means that if we have \( g \) angles, and label our indices such that \( \sigma(j) = j + 1 \mod g \), then for some fixed integer \( m \), the optimal angles are of the form

\[
\theta_j = \theta_1 + 2\pi m(j - 1)/g. \quad (3)
\]

The only free parameter left now is \( m \), the spacing between adjacent points. Plugging these angles into the definition of \( f \) we find

\[
f(\sigma; \theta_1, \ldots, \theta_g) = 2\sqrt{g} \sin \left( \pi \left[ \frac{\theta - \theta_1}{g} \right] \right).
\]

This is in turn minimized for \( m = \left[ g/\theta_1 \right] \), giving the stated spacing and minima.

As this minimum of \( f \) is independent of the permutation \( \sigma \), we get an overall minimum for \( f \) for free.

**Corollary 7.** The minimum twisted commutator value (Definition 3) in the Frobenius norm \( \Lambda^{(F)}_{\gamma, \alpha} \) is lower bounded

\[
\Lambda^{(F)}_{\gamma, \alpha} \geq 2\sqrt{g} \sin \left( \pi \left[ \frac{g/\alpha}{g} \right] \right).
\]

**Proof.** This result can be seen by recalling that the definition of \( f \) in Eq. (1) gives that

\[
\min_{u \in U(\gamma)} d_{\gamma, \alpha}(u, \eta u) = \min_{\sigma, j} f(\sigma; \theta_1, \ldots, \theta_g).
\]

As Lemma 4.2 tells us that we can consider cyclic permutations without loss of generality, we can apply the minimum found in Lemma 4.3, giving

\[
\min_{u \in U(\gamma)} d_{\gamma, \alpha}(u, \eta u) = 2\sqrt{g} \sin \left( \pi \left[ \frac{g/\alpha}{g} \right] \right).
\]

Applying the Wielandt-Hoffman theorem (Lemma 4.1), we get that the above minimum spectral distance lower bounds the twisted commutator in the Frobenius norm, as required.

---

\(^4\)In saying there are two such terms we have assumed \( g \geq 3 \). If \( g = 1 \) the lemma is trivial (\( f \) is constant), and if \( g = 2 \) then we have double counted in \( f_j(\theta_j) \), but our analysis of its minimum remains valid.
4.2 Higher norms and tightness

With the above bound in hand, we now turn our attention to tightness. A canonical family of operators which exhibit twisted commutation is that of the generalized Pauli operators, also known as Sylvester’s clock and shift matrices

\[ C := \sum_j \omega^{j-1} |j\rangle \langle j|, \quad S := \sum_j |j \oplus 1\rangle \langle j| \]

where \( \omega = e^{2i\pi/g} \) is a primitive \( g \)-th root of unity, and \( \oplus \) denotes addition modulo \( g \). As \( S \) simply cyclically permutes the eigenbasis of \( C \), we can see that \( S^k C \) simply cyclically permutes the eigenbasis of \( C \), and therefore \( S^k S^* C S = \omega^k C \), or \( [C, S]_{1/g} = 0 \). By taking appropriate powers these operators can also yield pairs which twisted commute with a phase that is any power of \( \omega \), specifically we see \( [C, S^k]_{k/g} = 0 \). Suppose we take such a pair and evaluate the twisted commutator at an arbitrary phase \( \eta = e^{2i\pi \alpha} \). We then find,

\[ \| [C, S^k]_{\alpha} \|_F = \| CS^k - \eta S^k C \|_F = \| (1 - \omega^k \eta) CS^k \|_F = \sqrt{g} |1 - \omega^k \eta| = 2\sqrt{g} |\sin(\pi (\alpha + k/g))| . \]

If we now take \( k = -\lfloor g \alpha \rfloor \), then we saturate Corollary 7, proving tightness of the bound on \( \Lambda^{(F)}_{g,\alpha} \), allowing us to conclude

\[ \Lambda^{(F)}_{g,\alpha} = 2\sqrt{g} \sin \left( \pi \frac{\lfloor g \alpha \rfloor - g\alpha}{g} \right) . \]

For the above optimizations we restricted ourself to the \( p = 2 \) case of the Frobenius norm. The nature of the minimizers found allows us to pull this analysis up into minima for the \( p > 2 \) Schatten norms as well.

**Theorem 6** (Minimum twisted commutation value). Suppose that \( u \) and \( v \) are \( g \)-dimensional unitaries, then for any \( p \geq 2 \) the twisted commutator is lower bounded

\[ \| [u, v]_{\alpha} \|_{(p,k)} \geq 2k^{1/p} \sin \left( \pi \frac{\lfloor g \alpha \rfloor - g\alpha}{g} \right) , \]

where \( \| \cdot \|_{(p,k)} \) is the \( (p,k) \)-Schatten-Ky Fan norm. Moreover this bound is tight, in that sense that there exist families of \( g \)-dimensional unitaries which saturate the above bounds and only depend on \( \lfloor g\alpha \rfloor \), the nearest integer to \( g\alpha \).

**Proof.** By the equivalence of Schatten-Ky Fan norms, the minimum Frobenius norm will also provide a lower bound for other \( (p,k) \)-norms as well. Specifically for \( p \geq 2 \) we have

\[ \| M \|_{(p,k)} \geq k^{1/p} g^{-1/2} \| M \|_F . \]

Applying these to the definition of \( \Lambda^{(p,k)}_{g,\alpha} \), this bound gives that \( \Lambda^{(p,k)}_{g,\alpha} \geq k^{1/p} g^{-1/2} \Lambda^{(F)}_{g,\alpha} \) for \( p \geq 2 \). It turns out that this inequality is saturated by matrices \( M \) with flat spectra, i.e. those proportional to unitaries. It so happens that the clock and shift operators considered to demonstrate tightness have a twisted commutator with precisely this property, and therefore also saturate and demonstrate the tightness of the induced \( p \geq 2 \) bounds. We therefore conclude that

\[ \Lambda^{(p,k)}_{g,\alpha} = k^{1/p} g^{-1/2} \Lambda^{(F)}_{g,\alpha} = 2k^{1/p} \sin \left( \pi \frac{\lfloor g\alpha \rfloor - g\alpha}{g} \right) . \]
5.1 Local Hamiltonians

In this paper the only assumption we made about our Hamiltonian $H$ was the presence of a spectral gap. A natural additional structure to impose is that $H$ be a many-body Hamiltonian: decompose our Hilbert space into a tensor product of many smaller Hilbert spaces, and let our Hamiltonian take the form

$$H = \sum_k h_k$$

where each term $h_k$ acts non-trivially on a constant number of these tensor factor spaces. Additional to this we could also impose that the factors on which it acts are geometrically local as well. Under this special case it may be that either the bounds on degeneracy certification might be able to be improved, or we might be able to prove the existence of degeneracy witnesses with additional structure, e.g. such witnesses might act in a geometrically local fashion.

5.2 Topologically ordered systems

While the notions of approximate symmetry and degeneracy of a ground band are both robust to small perturbations, na"ively one can only consider perturbations of a strength no larger than the gap. For topologically ordered systems [18] however, we can afford much larger perturbations under certain locality assumptions.

Under the influence of local perturbations, the low-energy band structure, most notably the ground space degeneracy, is robust even if the overall strength of the perturbation is extensive [40]. Moreover, any symmetries which witnesses this degeneracy can be quasi-adiabatically continued [41] into approximate symmetries which witness the degeneracy of the ground band in the perturbed system. It is in this sense that the existence of degeneracy witnesses can be considered robust to even rather strong perturbations, at least for the ground band.

The family of abelian quantum double models possess symmetries supported on quasi-1D regions which satisfy twisted commutation relations related to the braid and fusion rules of the underlying anyons [42]. More general models such as non-abelian/twisted quantum doubles [42–44], and Levin-Wen string net models [45] are all believed to possess symmetries which satisfy more general commutation-like relations based on more general notions of commutation. One possible example is the twist product [46] which only commutes the two operators on part of the system, braiding them together.

$$\left(\sum_i A_i \otimes A'_i\right) \infty \left(\sum_j B_j \otimes B'_j\right) := \sum_{ij} A_i B_j \otimes B'_j A'_i.$$

An obvious extension of this work is to take various properties of these underlying systems implied by this commutation-like relations, and see if they too carry through into the regime of approximate relations.

In a recent paper, Bridgeman et. al. sought to classify the phases of 2D topologically ordered spin systems belonging to the same phase as abelian quantum doubles [35]. This was done by numerically optimizing twisted

Figure 5: The twisted commutator value minimum (in the operator norm, $p = \infty$) $\Lambda_{g,\alpha}^{(Op)}$. a) The dependence on the twisting parameter $\alpha$ for a few fixed dimensions $g$. The presence of roots at multiples of $1/g$ are those predicted by Theorem 1. b) Now fixing the twisting parameter $\alpha$, the dependence on the dimension $g$ is shown. Note that $g$ can only take integer values, indicated by the circles and pluses, with the continuous lines simply intended to guide the eye. The dotted black line indicates an $\alpha$-independent upper bound on $\Lambda_{g,\alpha}^{(Op)}$ given by applying the bound $|x - \lfloor x \rfloor| \leq 1/2$.
pairs of symmetries. This optimization was done over a tensor network \[47, 48\] ansatz of quasi-1D operators known as matrix product operators. For two operators \(L\) and \(R\), supported on intersecting quasi-1D regions, the cost function takes the form

\[
C(L, R; \alpha) \propto \epsilon_L^2 + \epsilon_R^2 + \delta^2
\]

where \(\epsilon_L := \|[L, H]\|_F\), \(\epsilon_R := \|[R, H]\|_F\), and \(\delta = \|[L, R]_\alpha\|_F\).

Minimizing \(C(L, R; \alpha)\) over \(L\) and \(R\) for a fixed \(\alpha\), they found that in the abelian quantum doubles the minimizers were unitary, and that both \(\epsilon_L\) and \(\epsilon_R\) vanish to within numerical accuracy, leaving only the twisted commutator value \(\delta\). By observing the values of \(\alpha\) for which the minimum cost is low, they hoped to classify the topological phases of the underlying Hamiltonian. By Theorem 2 we know that, at least to within numerical accuracy, the ribbon operators found restrict down to ground symmetries with the same twisted commutation relations. In Fig. 6 we compare, for the \(Z_5\) quantum double model, their numerically obtained values of this twisted commutator \(\delta_{\text{min}}\) with the minimal possible twisted commutator \(\Lambda_{5, \alpha}^{(F)}\), showing close agreement and lending support to the efficacy of this numerical method.

### 5.3 Quantum codes

One class of systems for which twisted commuting symmetries play a special role are quantum codes, in which they can be interpreted as logical operators \[19, 49, 50\]. For a quantum code encoding \(N\) codewords, the logical algebra must correspond to \(\text{Mat}_N(\mathbb{C})\), which necessarily contains a pair of operators \(X\) and \(Z\) such that \([X, Z]\) is 0; indeed the algebra generated by any two such operators \(X\) and \(Z\) is itself \(\text{Mat}_N(\mathbb{C})\).

While the existence of logical operators which \(\alpha = 1/N\) twisted commute can be ensured, we might only see and expect operators with twisted commutations characteristic of smaller ground spaces if we restrict the locality of these operators. Though the logical algebra is given by \(\text{Mat}_N(\mathbb{C})\), this space often naturally decomposes into a tensor product decomposition: the logical qudits. By geometrically restricting where on the system the operators can act, we can often restrict which factors the logical operators have nontrivial commutation relations with. This is the case for celebrated examples such as the toric code \[42\]. This can be seen above in Fig. 6, where the logical operators are restricted to string-like regions that are only sensitive to one \(\text{Mat}_{5}(\mathbb{C})\) factor of the larger \(\text{Mat}_{25}(\mathbb{C})\) logical algebra; one of the two 5-level qudits. In the same way that Ref. \[35\] sought to use the existence of twisted commuting symmetries to classify topological phases, how this existence varies with respect to the geometry imposed on these operators might provide a tool to probe what portion of the logical algebra is accessible on certain regions.

In the language of quantum codes, our results can be interpreted as bounds below which approximate logical operators imply the existence of a certain number of code words. A possible avenue for future work is whether there exists bounds below which not only can the number of codestates be bounded, but reliable encoding, decoding, and error correction can all be performed with these approximate logical operators. Understanding when information stored in such states is approximately preserved, as opposed to exactly preserved \[51\], could have interesting applications in approximate quantum error correction.
Acknowledgements

We thank Jacob Bridgeman for fruitful comments, and for computing the numerical data for Fig. 6 using the algorithm of Ref. [35]. For the use of American English in this paper, CTC apologises to the Commonwealth of Australia. This work was supported by the Australian Research Council via EQuS project number CE11001013 and STF was supported by an Australian Research Council Future Fellowship FT130101744.

References

[1] H. Weyl, “Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung),” *Mathematische Annalen* 71, 4, 441–469, (1911).

[2] P. Rosenthal, “Are Almost Commuting Matrices Near Commuting Matrices?,” *The American Mathematical Monthly* 76, 925, (1969).

[3] P. R. Halmos, “Some unsolved problems of unknown depth about operators on Hilbert space,” *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics* 76, 1, 67–76, (1976).

[4] S. Szarek, “On almost commuting Hermitian operators,” *Rocky Mountain Journal of Mathematics* 20, 583–591, (1990).

[5] K. R. Davidson, “Almost commuting hermitian matrices,” *Mathematica Scandinavica* 56, 222, (1985).

[6] I. D. Berg and K. R. Davidson, “Almost commuting matrices and a quantitative version of the Brown-Douglas-Fillmore theorem,” *Acta Mathematica* 166, 121–161, (1991).

[7] H. Lin, “Almost commuting selfadjoint matrices and applications,” in *Operator algebras and their applications*, 193–223, Fields Institute Communications, Vol. 13, (1997).

[8] I. Kachkovskiy and Y. Safarov, “Distance to normal elements in C*-algebras of real rank zero,” *Journal of the American Mathematical Society* 29, 61–80, arXiv:1403.2021, (2015).

[9] P. Friis and M. Rørdam, “Almost commuting self-adjoint matrices - a short proof of Huaxin Lin’s theorem,” *Journal für die reine und angewandte Mathematik* 1996, 479, 121–131, (1996).

[10] M. B. Hastings, “Making Almost Commuting Matrices Commute,” *Communications in Mathematical Physics* 291, 321–345, arXiv:0808.2474, (2009).

[11] D. Voiculescu, “Asymptotically commuting finite rank unitary operators without commuting approximants,” *Acta Scientiarum Mathematicarum* 45, 429–431, (1983).

[12] R. Exel and T. Loring, “Almost commuting unitary matrices,” in *Proceedings of the American Mathematical Society*, 106, 913–913, (1989).

[13] R. Exel and T. A. Loring, “Invariants of almost commuting unitaries,” *Journal of Functional Analysis* 95, 364–376, (1991).

[14] M.-D. Choi, “Almost Commuting Matrices Need not be Nearly Commuting,” *Proceedings of the American Mathematical Society* 102, 529, (1988).

[15] H. Lin, “Almost Commuting Unitaries and Classification of Purely Infinite Simple C*-Algebras,” *Journal of Functional Analysis* 24, 1–24, (1998).

[16] T. J. Osborne, “Almost commuting unitaries with spectral gap are near commuting unitaries,” arXiv:0809.0602, (2008).

[17] T. A. Loring and A. P. W. Sørensen, “Almost commuting orthogonal matrices,” *Journal of Mathematical Analysis and Applications* 420, 1051–1068, arXiv:1301.4575, (2014).

[18] X.-G. Wen, “Topological Order: From Long-Range Entangled Quantum Matter to a Unified Origin of Light and Electrons,” *ISRN Condensed Matter Physics* 2013, 1–20, arXiv:1210.1281, (2013).

[19] J. Preskill, *Quantum Error Correction*. Cambridge University Press, Cambridge, (2013).

[20] B. J. Brown, D. Loss, J. K. Pachos, C. N. Self, and J. R. Wootton, “Quantum memories at finite temperature,” *Reviews of Modern Physics* 88, 045005, arXiv:1411.6643, (2016).

[21] L. Landau, E. Lifshitz, V. Berestetskii, and L. Pitaevskii, *Course of Theoretical Physics*. Pergamon Press, (1951).

[22] B. Brown, S. T. Flammia, and N. Schuch, “Computational difficulty of computing the density of states,” *Physical Review Letters* 107, arXiv:1010.3060, (2011).

[23] C. T. Chubb and S. T. Flammia, “Computing the Degenerate Ground Space of Gapped Spin Chains in Polynomial Time,” 1–32, arXiv:1502.06907, (2015).

[24] Y. Huang, “A polynomial-time algorithm for approximating the ground state of 1D gapped Hamiltonians,” arXiv:1406.6355v3, (2014).

[25] J. Yang and H.-k. Du, “A note on commutativity up to a factor of bounded operators,” *Proceedings of the American Mathematical Society* 132, 1713–1721, (2004).
[26] B. C. Hall, Quantum Theory for Mathematicians, 267 of Graduate Texts in Mathematics. Springer New York, (2013).

[27] J. Rosenberg, “A selective history of the Stone-von Neumann theorem,” in Contemporary Mathematics, 331–353, (2004).

[28] M. Said, “Almost Commuting Elements in Non-Commutative Symmetric Operator Spaces,” (2014).

[29] S. Eilers and R. Exel, “Finite dimensional representations of the soft torus,” Proceedings of the American Mathematical Society 130, 727–732, arXiv:math/9810165, (1998).

[30] R. Exel, “The soft torus and applications to almost commuting matrices,” Pacific Journal of Mathematics 160, 207–217, (1993).

[31] L. Babai and K. Friedl, “Approximate representation theory of finite groups,” in Proceedings 32nd Annual Symposium of Foundations of Computer Science, 733–742, IEEE Comput. Soc. Press, (1991).

[32] J. C. Bridgeman, S. T. Flammia, and D. Poulin, “Detecting Topological Order with Ribbon Operators,” arXiv:1603.02275, (2016).

[33] R. Bhatia, Matrix Analysis, 169 of Graduate Texts in Mathematics. Springer New York, (1997).

[34] R. Schatten, Norm Ideals of Completely Continuous Operators. Springer Berlin, Heidelberg, (1960).

[35] J. Von Neuman, “Some matrix inequalities and metrization of matric spaces,” Tomsk Univ. Rev. 1, 286–299, (1937).

[36] A. J. Hoffman and H. W. Wielandt, “The variation of the spectrum of a normal matrix,” Duke Mathematical Journal 20, 37–39, (1953).

[37] M. Congedo, B. Afsari, A. Barachant, and M. Moakher, “Approximate Joint Diagonalization and Geometric Mean of Symmetric Positive Definite Matrices,” Public Library of Science 10, 4, (2015).

[38] D. Eynard, A. Kovnatsky, and M. M. Bronstein, “Almost-commuting matrices are almost jointly diagonalizable,” arXiv:1305.2135, (2013).

[39] F. Gygi, J.-L. Fattebert, and E. Schwegler, “Computation of Maximally Localized Wannier Functions using a simultaneous diagonalization algorithm,” Computer Physics Communications 155, 1–6, (2003).

[40] M. Congedo, B. Afsari, A. Barachant, and M. Moakher, “Approximate Joint Diagonalization and Geometric Mean of Symmetric Positive Definite Matrices,” Public Library of Science, 10, 4, (2015).

[41] D. Eynard, A. Kovnatsky, and M. M. Bronstein, “Laplacian colormaps: a framework for structure-preserving color transformations,” Computer Graphics Forum 33, 215–224, arXiv:1311.0119, (2014).
we have

Theorem 8 ([34]). Take $A$ and $B$ to be complex matrices of dimension $n \geq 2$. If $\|B\| \leq 1$, and for some $\delta > 0$ we have

$$\|(A, B)\| \leq \frac{\delta^n (1 - \delta)}{1 - \delta^{n-1}},$$

then for each eigenvalue $\lambda$ of $A$, there exists a $\mu$ and normalized $|x\rangle$ such that

$$\|A|x\rangle - \lambda|x\rangle\|, \|B|x\rangle - \mu|x\rangle\| \leq \delta.$$

Notice above the required bound on the commutator scales like $O(\delta^n)$ for small $\delta$. Below we will improve this dimension scaling by adding the additional assumption that one of the matrices is normal, allowing us to bring this down to a $O(\delta^2/n^2)$ dependence. First we will state the more general result, which only requires one of the matrices to be normal, followed by a more specialized result which applies when both matrices are normal.

The existence of an entire basis of shared approximate eigenvectors is closely related to approximate joint diagonalization, a problem that has been widely considered and has found application in fields such as quantum chemistry [53], machine learning [54] and image processing [55]. This literature is too vast to review in this appendix, but see Ref. [52] for a discussion of the relationship between approximately commuting matrices and joint diagonalization. Techniques similar to those used below have also been used in Ref. [56] to address the related problem of constructing nearby exactly commuting operators, in the case in which one matrix is Hermitian. Whilst this analysis gives better bounds than those presented below, it leverages a combinatorical construction [57] that explicitly uses the reality of the eigenvalues, and therefore cannot be directly applied to the case we will consider in which one matrix is normal, but not necessarily Hermitian.

Take $A$ and $B$ to be $n \times n$ matrices. Let $A$ be normal, with an eigenvalue decomposition $A = \sum_{i} \lambda_i |i\rangle\langle i|$. Next take $\lambda$ to be a specific eigenvalue of $A$. Let $I_0$ be the singleton set containing the index corresponding to $\lambda$, or all these indices if $\lambda$ is degenerate. Define $I_k$ to be all the indices whose eigenvalues are within some radius $r > 0$ in the complex plane (to be chosen later) of those in $I_{k-1}$, i.e.

$$I_k := \{i | \exists j \in I_{k-1} : |\lambda_i - \lambda_j| \leq r\}.$$

Clearly this sequence becomes fixed after at most $n$ terms, and so let $I := I_n$ be this fixed point. Intuitively $I$ can be thought of as the indices corresponding to eigenvalues which form a cluster around $\lambda$ where every eigenvalue in the cluster is linked to at least one other by a disk of radius $r$ in the complex plane.

By construction this set has two properties we require. First it is bounded away from any other index,

$$i \in I, j \notin I \implies |\lambda_i - \lambda_j| > r.$$

Second, because all of the eigenvalues corresponding to elements in $I$ have nearby neighbors in $I$, this means that the diameter of the disk containing all of the eigenvalues in $I$ has a diameter bounded by at most $nr$,

$$i \in I \implies |\lambda_i - \lambda| \leq nr.$$

Next let $V$ be the space spanned by the eigenvectors whose indices lies in $I$,

$$V := \text{Span} \{|i\rangle | i \in I\}.$$

Denoting the orthogonal complement of $V$ by $\bar{V}$, then we can decompose both $A$ and $B$ into blocks on $V \oplus \bar{V}$ as

$$A = \begin{pmatrix} AV & \cdot \\ \cdot & A_{\bar{V}} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} BVV & B_{V\bar{V}} \\ B_{V\bar{V}} & B_{\bar{V}\bar{V}} \end{pmatrix}.$$
Lemma A.1. If \(|A, B]\| \leq \epsilon$, with $A$ normal and decomposed as above, then $A_V$ is close to scalar, and the off-diagonal blocks of $B$ are bounded as

$$\|A_V - \lambda\bar{I}\| \leq n\epsilon \quad \text{and} \quad \|B_{VV}\| \leq n\epsilon/2.$$ 

Proof. Given that $A$ is normal, we can see that $A_V$ is approximately scalar due to the bound between eigenvalues in $I$:

$$\|A_V - \lambda\bar{I}\| = \max_{\epsilon \in \mathbb{I}} |\lambda_i - \lambda| \leq n\epsilon.$$

Next, using the fact that the operator norm dominates any component of a matrix, we can simply evaluate the relevant component of the commutator to bound elements of $B$:

$$\left|\langle i, [B, j]\rangle\right| \leq \frac{\epsilon}{|\lambda_i - \lambda_j|} < \frac{\epsilon}{r}.$$ 

This implies therefore that $\|B_{VV}\|_{\max} < \epsilon/r$, where $\|\cdot\|_{\max}$ denotes the elementwise max-norm. Using the fact that the operator norm exceeds the max-norm by at most the square root of the number of elements, we get

$$\|B_{VV}\| \leq \|B_{VV}\|_{\max} \times \sqrt{\dim V \times \dim \tilde{V}}.$$ 

Given that $\dim V + \dim \tilde{V} = n$, we have that $\dim V \times \dim \tilde{V} \leq n^2/4$, and so

$$\|B_{VV}\| < n\epsilon/2r.$$ 

Using these bounds, we can now put bounds on an approximate shared eigenvector. Imposing normality on both matrices, we can even impose the stricter requirement that both of the approximate eigenvalues are in fact exact eigenvalues.

Theorem A.1 (Shared approximate eigenvector). Suppose that $A$ and $B$ are $n \times n$ matrices, such that $A$ is normal and $\|A, B]\| \leq \epsilon$. For any $\lambda$ which is an eigenvalue of $A$, there exists a normalized $|u\rangle$ and $\mu$ such that

$$\|A|u\rangle - \lambda|u\rangle\|, \|B|u\rangle - \mu|u\rangle\| \leq n\sqrt{\epsilon/2}.$$ 

If $B$ is also normal, then for any $\lambda$ which is an eigenvalue of $A$, there exists a $\nu$ which is also an eigenvalue of $B$ and normalized $|w\rangle$ such that

$$\|A|w\rangle - \lambda|w\rangle\|, \|B|w\rangle - \nu|w\rangle\| \leq n\sqrt{\epsilon}.$$ 

Proof. Take $|u\rangle$ to be a right eigenvector of $B_{VV}$ (contained within $V$), of eigenvalue $\mu$. By Lemma A.1, this then gives that the relevant errors with respect to $A$ and $B$ behave as:

$$\|A|u\rangle - \lambda|u\rangle\| = \|A_V|u\rangle - \lambda|u\rangle\| \leq \|A_V\| \quad \|B|u\rangle - \mu|u\rangle\| = \|B_{VV}|u\rangle - \mu|u\rangle + B_{VV}|u\rangle\| \leq \|B_{VV}\|.$$ 

If we now let $r = \sqrt{\epsilon/2}$, we get the stated overall bound of $n\sqrt{\epsilon/2}$.

For the case of both matrices being normal, we can show that any approximate eigenvalue must lie near an exact eigenvalue. Taking $|w\rangle$ once again to be a right eigenvector of $B_{VV}$ with eigenvalue $\nu'$ (for a different value of $r$ to $|u\rangle$), we can see that

$$\|B|w\rangle - \nu'|w\rangle\| \leq n\epsilon/2r \quad \Rightarrow \quad \langle w|(B - \nu')\dagger(B - \nu)|w\rangle \leq n^2\epsilon^2/4r^2.$$ 

As $(B - \nu')\dagger(B - \nu')$ is positive semi-definite, the existence of such a $|w\rangle$ implies $(B - \nu')\dagger(B - \nu')$ possesses an eigenvalue at most $n^2\epsilon^2/4r^2$. By the normality of $B$, this implies in turn that $B$ contains an eigenvalue $\nu$ such that $|\nu - \nu'| \leq n\epsilon/2r$. Using this we can see that the error with respect to $B$ gains a factor of 2

$$\|B|w\rangle - \nu|w\rangle\| \leq \|B|w\rangle - \nu'|w\rangle\| + |\nu - \nu'| \leq n\epsilon/r.$$ 

Now taking $r = \sqrt{\epsilon}$, we find the stated bound of $n\sqrt{\epsilon}$.
An algorithm for the certifiable degeneracy of a twisted pair

In this appendix we sketch how, for a given pair of parameters $\alpha$ and $\delta$, we can calculate the minimum possible dimension of unitaries $u$ and $v$ such that $\|[u, v]\| \leq \delta$. Lemmas 3.1 and 3.2 give that for all $j \in \mathbb{Z}$, there exists an eigenvalue $e^{i\phi_j}$ of $u$ such that $|\phi_j - 2\pi \alpha j| \leq \cos^{-1}(1 - |j| \delta)$.

The question now is to find the minimum number of eigenvalues such that at least one lies in each of the above arcs. This is known as the transversal number, and can be efficiently calculated by a greedy algorithm [58]. We now sketch this algorithm for the example parameters $\alpha = 1/4$ and $\delta = 1/2$ (indicated by the turquoise dot in Fig. 3).

The first thing to note is that these arcs are trivial for $\delta |j| \geq 2$, in that they are the entire unit circle. For this reason we need only consider a finite number of arcs for $j = -\lfloor 2/\delta \rfloor, \ldots, \lfloor 2/\delta \rfloor$. In our case this corresponds $j = -3, \ldots, 3$. Below we have drawn these non-trivial arcs, omitting the trivial $j = 0$ arc.

Next we note that the $j = 0$ arc is simply a point, implying that $u$ must contain a $+1$ eigenvalue. Given this, any arc containing $+1$ can be thrown away (indicated in red below), allowing us to unfold our arcs on a circle into intervals on a line.

We then take the intervals to be sorted by end-point. Considering each interval in order, we place an eigenvalue at the end of each interval as necessary, indicated as a green line below. Note that any interval which already contains an included eigenvalue when we arrive at it can be ignored, indicated by the red interval below.

Including the already found eigenvalue at $+1$, this gives us the minimum number of points necessary to satisfy each arc. Applying the algorithm for a large number of points, we can plot the certified degeneracy as in Fig. 3.