We investigated incompressible and compressible magnetohydrodynamic (MHD) energy cascade rates in the solar wind at different heliocentric distances. We used in situ magnetic field and plasma observations provided by the Parker Solar Probe (PSP) mission and exact relations in fully developed turbulence. To estimate the compressible cascade rate, we applied two recent exact relations for compressible isothermal and polytropic MHD turbulence, respectively. Our observational results show a clear increase of the absolute value of the compressible and incompressible cascade rates as we get closer to the Sun. Moreover, we obtained an increase in both isothermal and polytropic cascade rates with respect to the incompressible case as compressibility increases in the plasma. Further discussion about the relation between the compressibility and the heliocentric distance is carried out. Furthermore, we compared both exact relations as compressibility increases in the solar wind and although we note a slightly trend to observe larger cascades using a polytropic closure, we obtained essentially the same cascade rate in the range of compressibility observed. Finally, we investigated the signed incompressible and compressible energy cascade rates and its connection with the real cascade rate.
I. INTRODUCTION

The solar wind is a well-studied supersonic plasma flow, from the Sun’s atmosphere to the edge of the heliosphere, characterized by a turbulent energy cascade rate (Tu and Marsch, 1995; Fraternale et al., 2019). The availability of in situ measurements from various orbiting spacecraft through different heliocentric distances allow a deep understanding of the essential role of turbulence from the large scales up to the kinetic scales in the solar wind plasma (Alexandrova et al., 2013; Bruno and Carbone, 2013; Kiyani, Osman, and Chapman, 2015; Chen, 2016). In this sense, the Parker Solar Probe (Fox et al., 2016) (PSP) mission has been exploring the inner heliosphere since 2018, approaching to the Sun with each orbit and enabling us to study the evolution of turbulence and to compare observations with theoretical predictions.

A prevailing challenge in the solar wind community is to enhance the models of the turbulent heating of the plasma, with particular focus in the near-Earth space (Sahraoui, Hadid, and Huang, 2020; Hadid et al., 2018) and the magnetosphere environments (Chen et al., 2020; Huang et al., 2020; Andrés et al., 2019, 2021; Andrés et al., 2022; Huang and Sahraoui, 2019; Kiyani, Osman, and Chapman, 2015; Alexandrova et al., 2013; Howes et al., 2012; Osman et al., 2011). Observations have shown that the solar wind proton temperature tends to decrease as a function of the radial distance from the Sun more gradually than the prediction of the adiabatic expansion model of the solar wind (Marsch et al., 1982; Vasquez et al., 2007; Pine et al., 2020). Although several scenarios have been proposed to explain these observations, the main candidate is certainly the local heating of the solar wind plasma via the turbulent cascade (Bruno and Carbone, 2013; Matthaeus and Velli, 2011; Marino and Sorriso-Valvo, 2023). Therefore, the energy at the largest scales in the solar wind will cascade within the inertial range until it reaches the ion scales where it is eventually transformed into thermal or kinetic energy of the plasma particles (see Sahraoui, Hadid, and Huang (2020); Kiyani, Osman, and Chapman (2015)).

Several theoretical efforts have been made in order to provide an estimation of the turbulent energy cascade rate at the solar wind. The first exact relation use the von-Kármán-Howarth dynamical equation (von Kármán and Howarth, 1938). This exact relation gives an expression for the energy dissipation or cascade rate $\varepsilon$ as a function of structure functions of the velocity turbulent field (e.g., Monin and Yaglom, 1975; Frisch, 1995). Galtier and Banerjee (2011) and Banerjee and Galtier (2014) generalized this exact relation to compressible isothermal and polytropic hydrodynamic (HD) turbulence, respectively. Galtier and Banerjee (2011) has found a new term on
the inertial range that behaves similarly as a source or a sink for the mean energy cascade rate. In particular, this new term is a product of increments and local mean values of the plasma variables, on the contrary to the typical structure functions present in the incompressible flows.

In the case of magnetized plasmas, the first exact relation was derived by Politano and Pouquet (1998a,b) for homogeneous, isotropic and incompressible magnetohydrodynamics (IMHD) turbulence. The authors recovered a scaling law for mixed third-order longitudinal structure and correlation functions. This exact law has been the subject of several numerical tests (e.g., Sorriso-Valvo et al., 2002; Mininni and Pouquet, 2009; Boldyrev, Mason, and Cattaneo, 2009; Wan et al., 2010; Verdini and Grappin, 2015); it has been used for the estimation of the incompressible cascade rate in space plasmas (Sorriso-Valvo et al., 2007; Sahraoui, 2008; Coburn et al., 2015; Simakov and Chacón, 2008; Masters et al., 2008) and the magnetic and kinetic Reynolds numbers (Weygand et al., 2007) in solar wind turbulence, and for the large-scale modeling of the solar wind (Matthaeus et al., 1999; MacBride, Smith, and Forman, 2008; MacBride, Forman, and Smith, 2005). Banerjee and Galtier (2013) derived an exact relation for some two-point correlation functions of the fields for isothermal compressible magnetohydrodynamic (CMHD) turbulence, written in terms of the flux or the source terms. On the other hand, Andrés and Sahraoui (2017) revisited the latter work and expressed the exact law as a function of a more adequate set of plasma variables, i.e., the plasma mass density, the plasma velocity field and the compressible Alfvén velocity. This theoretical work presented four different types of terms that are involved in the nonlinear cascade of energy in the inertial range: the hybrid terms (which can be written either as flux or source terms), the $\beta$-dependent terms and the well-known flux and source terms (see, Andrés and Banerjee (2019); Ferrand et al. (2020)). Recently, Simon and Sahraoui (2021) proposed a more general method that allows to derive the exact relation for any turbulent isentropic flow (i.e., constant entropy). The authors demonstrated that the well-studied MHD exact laws (incompressible and isothermal) and the new (polytropic) one can be obtained as specific cases of the general exact relation when the corresponding closure is specified. These formulations for IMHD and CMHD turbulence are used in the present study to estimate the non-linear transfer of energy in the solar wind.

A first attempt to include the compressibility in estimating the energy cascade rate using in situ observations was reported by Carbone et al. (2009) using a phenomenological model and Ulysses observations. The authors found a significant increase in the turbulent cascade rate with respect to the incompressible exact law. However, those results were based on a heuristic model, using
pseudo-energies (i.e., fluxes through the scale of the increments of the Elsässer fields), which are not conserved in CMHD theory (Marsch and Mangeney, 1987). In the same framework, Marino et al. (2008) showed that the compressible turbulent cascade seems to be able to supply the energy needed to account for the local heating of the non-adiabatic solar wind. Banerjee et al. (2016) and Hadid, Sahraoui, and Galtier (2017) have studied the impact of the compressible fluctuations in the energy cascade in the solar wind using a reduced form of the exact relation for CMHD turbulence and in situ observations from the Time History of Events and Macroscale Interactions during Substorms (THEMIS) (Auster et al., 2009) spacecraft. The authors found that the compressible fluctuations are shown to amplify by several order of magnitude the turbulent cascade rate with respect to the incompressible model. Recently, Bandyopadhyay et al. (2020) have computed the incompressible energy transfer rate between 35 and 55 solar radius using PSP observations during the first encounter. Andrés et al. (2021) extended previous observational studies computing the compressible energy transfer rate from $\sim 0.2$ au up to $\sim 1.7$ au, using PSP, THEMIS and Mars Atmosphere and Volatile EvolutioN (MAVEN) observations. The authors showed that, depending on the level of compressibility in the plasma, the different terms in the compressible exact relation were shown to have different impact in the total cascade rate (where the incompressible terms are included). Moreover, using more than 2 years of PSP observations, Andrés et al. (2022) studied the incompressible energy cascade rate using isotropic and anisotropic exact relations. The authors found a connection between the heliocentric distance, the local temperature of the plasma and the energy cascade rate, with a clear dominance of the perpendicular cascades over the parallel cascades as PSP approaches the Sun.

In the present paper, we use the PSP observations (magnetic field and plasma moments) at different heliocentric distances to compute the absolute value of the compressible and incompressible energy cascade rates. We investigated as well the signed incompressible and compressible energy cascade rates and its connection with the real cascade rate. Our goal is to discuss the impact of the plasma compressibility and heliocentric distance in the energy cascade in the solar wind. The paper is organized as follows: in section II we recall the theoretical CMHD set of equations and present briefly the main steps to derive the exact law for fully developed CMHD turbulence. In section III, we describe the data set composed of more than 3000 PSP events and the selection criteria used in the present study. In section IV, we present the main observational results. Finally, in section V we provide a summary and discussion of our main findings.
II. THEORETICAL MODELS AND EXACT RELATIONS

A. Compressible MHD model

In the present work, we used the 3D CMHD model given by the mass continuity equation, the momentum equation for the velocity field $u$, the induction equation for the magnetic field $B$ and the differential Gauss’s law (solenoidal condition $\nabla \cdot B = 0$). These equations can be written as (Marsch and Mangeney, 1987),

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho u),$$
$$\frac{\partial u}{\partial t} = -u \cdot \nabla u + u_A \cdot \nabla u_A - \frac{\nabla (P + P_M)}{\rho} - u_A (\nabla \cdot u_A) + f_k + d_k,$$
$$\frac{\partial u_A}{\partial t} = -u \cdot \nabla u_A + u_A \cdot \nabla u - \frac{u_A}{2} (\nabla \cdot u) + f_m + d_m,$$
$$0 = u_A \cdot \nabla \rho + 2\rho (\nabla \cdot u_A),$$

where $u$ is the velocity field (assuming a zeroth background flow speed) and $u_A = B/\sqrt{4\pi \rho}$ is the compressible Alfvén velocity with $B$ the total magnetic field and $\rho$ the mass density. Note that, the time dependence enters through $u$, $B$ and $\rho$. In addition, $P$ is the scalar isotropic pressure and $P_M = \rho u_A^2 / 2$ is the magnetic pressure. In the present work, we use two different equations of state which allow us to close the hierarchy of the fluid equation (so there is no need of an energy equation): the isothermal case $P = C_s^2 \rho$, where $C_s$ is the (constant) sound speed and the polytropic case $\gamma P = C_s \rho$, where $\gamma = 5/3$ is the ratio of the heat capacity at constant pressure to heat capacity at constant volume and $C_s$ is the (variable) sound speed. For a detailed discussion about the two equation of state see Section 2.1 in Simon and Sahraoui (2021). Finally, $f_{k,m}$ are the respectively mechanical and the curl of the electromotive large-scale forcing, and $d_{k,m}$ are respectively the small-scale kinetic and magnetic dissipation terms.

B. Exact relation in CMHD turbulence

In the CMHD model, the density total energy $E(x)$ and the density-weighted cross helicity $H(x)$ are given by,

$$E(x) = \frac{\rho}{2} (u \cdot u + u_A \cdot u_A) + \rho e,$$
$$H(x) = \rho (u \cdot u_A),$$
where $e$ is the internal energy. It is worth mentioning that a polytropic process is a thermodynamic process that obeys the relation: $PV^n = C$, where $P$ is the pressure, $V$ is a control volume, $n$ is the polytropic index, and $C$ is a constant. The polytropic process equation describes expansion and compression processes which include heat transfer. Some specific values of $\gamma$ correspond to particular cases like, $n = 0$ for an isobaric process and $n = \infty$ for an isochoric process. In addition, when the ideal gas law applies, $n = 1$ for an isothermal process and $n = \gamma$ for an isentropic process. Since these definitions are compatible with previous studies in the literature (see, Galtier and Banerjee, 2011; Banerjee and Galtier, 2013, 2014; Banerjee et al., 2016; Hadid, Sahraoui, and Galtier, 2017; Hadid et al., 2018; Andrés et al., 2019; Andrés et al., 2020; Simon and Sahraoui, 2021), we decided to use the names isothermal for a plasma that obey $P_{\text{iso}} \sim \rho$, and consequently $e_{\text{iso}} = C_{s0}^2 \ln(\rho/\rho_0)$, and polytropic for a plasma with $P_{\text{pol}} \sim \rho^n$ and $e_{\text{pol}} = (C_s^2 - C_{s0}^2)/(\gamma(\gamma - 1))$. In both cases, the sound speed $C_s$ (and its mean value $C_{s0}^2$) can be obtained from the perfect gas equation, $C_s^2 = \gamma k_B T_p/m_p$, where $T_p$ and $m_p$ are the proton temperature and mass, with $n = 1$ and $n = \gamma = 5/3$ for the isothermal and polytropic case, respectively (see, Galtier and Banerjee, 2011; Banerjee and Galtier, 2013). Finally, it is worth mentioning that any conclusion in the present paper is limited by these assumptions about the thermodynamics discussed here.

While the total energy is one of the ideal invariants, the density-weighted cross helicity is not. Nevertheless, both quantities are essential for the derivation of the exact law in CMHD turbulence (see, Andrés and Sahraoui, 2017; Simon and Sahraoui, 2021). In particular, we can define the two-point correlation function associated with the total energy, the helicity and the magnetic field by,

$$R_E(x, x') = \frac{\rho}{2}(u \cdot u' + u_A \cdot u_A') + \rho e', \quad (7)$$

$$R_H(x, x') = \frac{\rho}{2}(u \cdot u'_A + u_A \cdot u'), \quad (8)$$

$$R_B(x, x') = \frac{\rho}{2}(u_A \cdot u'_A) \quad (9)$$

where the prime denotes field evaluation at $x' = x + \ell$ ($\ell$ being the displacement vector). Using Eq. (1)-(4), the expressions (7)-(9) and following the usual assumptions for fully developed homogeneous turbulence (i.e., infinite kinetic and magnetic Reynolds numbers and a steady state with a balance between forcing and dissipation) (see, Banerjee and Kritsuk, 2018; Banerjee and Andrés, 2020), an exact relation for compressible MHD turbulence can be obtained in the inertial range as,

$$-2\varepsilon_C = \frac{1}{2} \nabla_{\ell} \cdot F_C + S_C + S_H + M_B, \quad (10)$$
where $\varepsilon_C$ is the total compressible energy cascade rate, $\mathbf{F}_C$ is the total compressible flux (defined below) and $S_C$, $S_H$ and $M_\beta$ are the so-called source, hybrid and $\beta$-dependent terms, respectively. For a detailed derivation and the explicit expressions of the total compressible energy cascade rate, see Andrés and Sahraoui (2017); Simon and Sahraoui (2021). It is worth mentioning that, the derivation of the exact law (10) does not require the assumption of isotropy (Andrés et al., 2022) and that it is independent of the dissipation mechanisms acting in the plasma (assuming that the dissipation acts only at the smallest scales in the system). In the present paper, in order to estimate the compressible energy cascade rate, we shall consider only the flux terms. The main reason is that the source, hybrid and $\beta$-dependent terms require computing the divergence of the plasma and the compressible Alfvén speeds, which can be done only using multi-spacecraft observations. Moreover, numerical results for supersonic and subsonic HD and MHD turbulence have shown that the source, hybrid and $\beta$-dependent terms are negligible with respect to the flux term in the inertial range (e.g., Kritsuk, Wagner, and Norman, 2013; Andres, Galtier, and Sahraoui, 2018; Andŕes et al., 2019; Ferrand et al., 2020).

The total compressible flux is a combination of two terms of different nature, a Yaglom-like term,

$$\mathbf{F}_{1C} = \langle [\delta(\rho \mathbf{u}) \cdot \delta \mathbf{u} + \delta(\rho \mathbf{u}_A) \cdot \delta \mathbf{u}_A] \delta \mathbf{u} - [\delta(\rho \mathbf{u}) \cdot \delta \mathbf{u}_A + \delta \mathbf{u} \cdot \delta(\rho \mathbf{u}_A)] \delta \mathbf{u}_A \rangle, \quad (11)$$

which is the compressible generalization of the incompressible term (Politano and Pouquet, 1998b,a), and a purely compressible flux term,

$$\mathbf{F}_{2C} = 2\langle \delta \rho \delta e \delta \mathbf{u} \rangle, \quad (12)$$

which is a contribution to the energy cascade rate due to the presence of density fluctuations in the plasma (Andrés and Sahraoui, 2017; Simon and Sahraoui, 2021). Here, we have introduced the usual increments, i.e., $\delta \alpha = \alpha' - \alpha$ (with $\alpha$ any scalar or vector function) and the angular bracket $\langle \cdot \rangle$ denotes an ensemble average. It is worth mentioning that the difference between the two compressible models is in the form of the specific internal energy $e$, therefore, the polytropic and isothermal cascade rate differences would arise only from the compressible flux term (12). Finally, assuming statistical isotropy, we can integrate the flux terms in (10) over a sphere of radius $\ell$ to obtain a scalar relation for isotropic turbulence. In compact form, (10) can be written as,

$$-\frac{4}{3} \varepsilon_C \ell = F_{1C} + F_{2C}, \quad (13)$$
where $F_{1C} + F_{2C} = (F_{1C} + F_{2C}) \cdot \hat{u}_{sw}$ is the flux term projected into the mean plasma flow velocity field $u_{sw}$. Note that, we called $\varepsilon_{1C} + \varepsilon_{2C} = (-3/4\ell)(F_{1C} + F_{2C})$.

In the incompressible limit $\rho \rightarrow \rho_0$, we recover the Politano and Pouquet law for fully developed incompressible MHD turbulence,

$$-\frac{4}{3} \varepsilon_I \ell = F_I,$$

where $F_i$ is the projection of $F_i = \rho_0\langle (\delta u^2) + (\delta u_A^2) \rangle \delta u - 2(\delta u \cdot \delta u_A) \delta u_A \rangle$ along the mean plasma flow velocity field (Politano and Pouquet, 1998a,b). Assuming the Taylor hypothesis, i.e., $\ell \approx u_0 \tau$, where $\tau$ is the time lag and $u_0$ is obtained by averaging over each time interval, all the quantities that are indexed by 0, Eqs. (13) and (14) can be expressed as a function of the time lags $\tau$.

### III. PSP OBSERVATIONS AND DATA SELECTION CRITERIA

In order to estimate the compressible energy cascade rate, we employed magnetic field measurements by the FIELDS flux gate magnetometer (MAG), along with proton density, velocity and temperature data from the Solar Probe Cup (SPC) of the SWEAP instrument suite (Bale et al., 2016; Fox et al., 2016; Kasper et al., 2016; Bale et al., 2019; Kasper et al., 2019; Case et al., 2020). Our analysis on the PSP observations involved a time interval between November 2, 2018 (Encounter 1), and December 30, 2020 (Encounter 6). This data set was divided into a series of samples of equal duration of 60 minutes. This particular time duration ensures having at least one correlation time of the turbulent fluctuations for each particular heliocentric distance (Parashar et al., 2020; Hadid, Sahraoui, and Galtier, 2017). To generate uniform time series, we re-sampled all the variables to 30 s time resolution. Thus, there are 10944 events using this data set. Regarding the presence of switchbacks (Bourouaine et al., 2020; de Wit et al., 2020; Hernández et al., 2021), the well-known rapid polarity reversals in magnetic fields, we did not take it into account in the present paper. To do this, we used the following criteria so as to discard the presence of switchbacks (keeping the intervals where the energy cascade rate is approximately constant).

In our data set, spurious spikes in the SPC moments, which are remnants of poor quality of fits, were removed. To accomplish this, consecutive filters were applied, as Figure 1 shows. First, for each event we applied a sharp mean value filter, which detect each spurious data using a threshold in the time series and replace it with the average between the previous and the next valid observation. When we say valid observation, we are referring to a real value of the data set, i.e.,
Figure 1. Schematic diagram of the filters used in order to remove spurious spikes from the data set.

values that have passed the first threshold that we applied. Although this filter removes most of the spurious data, in some cases leaves a fictitious large data value. Therefore, we applied the well-known Hampel filter in order to detect this fictitious outliers (Davies and Gather, 1993; Liu, Shah, and Jiang, 2004; Pearson et al., 2016). In few words, the Hampel filter uses a moving window implementation of predetermined size (in particular, we used a window of 8 points) to compute the local median \( m_i \) and the local standard deviation \( s_i \). Then, for each point \( x_i \),

\[
y_i = \begin{cases} 
  x_i & \text{if } |x_i - m_i| \leq n_\sigma s_i, \\
  m_i & \text{if } |x_i - m_i| > n_\sigma s_i,
\end{cases}
\]

if the absolute difference of the value of that point and the local median, \(|x_i - m_i|\) is above a threshold defined as \( n_\sigma \) times the local standard deviation, the value is replaced by the median. If not, the algorithm leaves the current point unchanged and proceeds to the next point. We applied this technique using a local window of 8 points and choosing a threshold value of \( n_\sigma = 3 \) (Bandyopadhyay et al., 2018). This choice of parameters help eliminate most of the undetected spikes and smooth the data set probably altered by the previous filter.

In addition, we considered only intervals that did not show large fluctuations of the energy
cascade rate over the large MHD scales, typically we retained events with std(ε_I)/mean(|ε_I|) < 0.85, where we have computed the standard deviation and the mean of the absolute value of the energy cascade over each event of 1 hour duration in the MHD scales. Finally, we analyzed the distribution of density fluctuations of the filtered data set. Figure 2 shows the occurrence rate for all the analyzed events for the numerical density, velocity, and Alfvén velocity field absolute values and their fluctuations, respectively. We can relate the distribution of density fluctuations with the compressibility ratio defined as √⟨ρ^2⟩ − ⟨ρ⟩^2/⟨ρ⟩. Since most of the events did not show high levels of compressibility rate, we only kept the events whose compressibility reached up to 30%. This leaves us a data set of 3298 events.

IV. OBSERVATIONAL RESULTS

In order to compute the compressible and incompressible energy cascade rate, we constructed correlation functions of the different turbulent fields at different time lags τ in the interval τ = [30, 3480]s. Therefore, once we had the energy cascade as a function of the time increments and in order to quantify statistically these increments, we took the average of the absolute values in the largest MHD scales, i.e., τ = [1000, 3000]s to obtain representative values of each event in the large scales (Hadid, Sahraoui, and Galtier, 2017; Andrés et al., 2020). It is worth to note that, the sign of the cascade rate can vary within the inertial range (Sorriso-Valvo et al., 2007) (showing a constant negative or positive sign for different time lags in the MHD range), which is related to the direct and inverse nature of the energy cascade rate (see, Coburn et al., 2015; Hadid, Sahraoui, and Galtier, 2017; Andrés et al., 2020). In particular, in the present paper, we focused on the study of the absolute value of the energy cascade in the MHD scales. In section IV D, we have also included an analysis of the limitations when using the absolute value of the cascade by making a comparison with the signed cascade rate. Finally, for the statistical convergence of the energy cascade rate, we analysed in detailed the correlation functions to assure its convergence, i.e. the constant value in the MHD scales). Further discussion is described in the Appendix VI.

A. The compressible and incompressible energy cascade rates

Figure 3 shows the absolute value of the total compressible energy cascade rate ⟨|ε_C|⟩ as a function of the absolute value of the incompressible cascade rate ⟨|ε_I|⟩ for (a) the isothermal (su-
We found a strong correlation between the cascade rate amplitude and the distance to the Sun. In particular, the closer to the Sun PSP is, the larger the energy cascade amplitude is. This is in agreement with previous work in the solar wind studying the incompressible energy cascade rate (e.g., Bandyopadhyay et al., 2020; Andrés et al., 2022). Likewise, Figure 4 shows the absolute value of the total compressible vs the incompressible energy cascade rate yet using the compressibility rate $\sqrt{\langle \rho^2 \rangle - \langle \rho \rangle^2} / \langle \rho \rangle$ (in percent) of each event as the colorbar. We observed an increase in the

Figure 2. The occurrence rate for the proton density, the proton and Alfvén velocity absolute mean values, in the first row, the corresponding fluctuations in the second row and the ratio between these magnitudes in the third row, respectively.
energy cascade rate as the level of compressibility grows, reaching up to 25%. Also, we note a clear correlation between the compressible and the incompressible cascade rates. In fact, given the weakly compressible nature of the solar wind, it is not surprising that these two energy cascades are similar and close to equality. Moreover, we observed (see Section IV B) that the first term of equation (13), i.e. the Yaglom-like term, dominates over the second one. As we mentioned above, this term tends to the incompressible expression in the limit of constant density. However, it is worth mentioning that, for the lowest values of the cascade rates in our data set (at \( \sim 0.8 \) au), there is a slight increase of the compressible cascade with respect to the incompressible one. It is worth noting that, as we are studying the total expression (13), we are not able to differentiate between the first and the second term which are the Yaglom-like term and the purely compressible one, respectively.

Figure 3. Total compressible energy cascade rate (absolute values) in the MHD scales as a function of the incompressible ones in the case of using (a) an isothermal (AS) and (b) polytropic (SS) model, respectively. The colobar corresponds to the heliocentric distance per event.

B. The isothermal and polytropic energy cascade components

Figure 5 and 6 show the absolute value of (a) the isothermal component \( \langle |\varepsilon_{2C}^{AS}| \rangle \) and (b) the polytropic component \( \langle |\varepsilon_{2C}^{SS}| \rangle \) (see Eq. 12) as a function of the Yaglom compressible generalization
Figure 4. Total compressible energy cascade rate (absolute values) in the MHD scales as a function of the incompressible ones in the case of using (a) an isothermal (AS) and (b) polytropic (SS) model, respectively. The colorbar corresponds to the level of compressibility (i.e., $\sqrt{\langle \rho^2 \rangle - \langle \rho \rangle^2} / \langle \rho \rangle$) per event.

component $\langle |\varepsilon_{1C}| \rangle$ (see Eq. 11), respectively. Again, the colorbar corresponds to the heliocentric distance and the compressibility percent, respectively. Although, in general, the dominant term in the MHD scales is $\langle |\varepsilon_{1C}| \rangle$, we noticed that even for the nearly incompressible solar wind (Compressibility $\leq \%10$), for some events the purely compressible term is relevant for an adequate estimation of the total cascade. We also observed that there are some events where the energy of the compressible term exceeds the Yaglom-like term. Moreover, the level of compressibility increases up to 20-25% in those events. Therefore, when we compared the compressible and the incompressible energy cascade in Figures 4 and 6, there is an impact of density fluctuation in the compressible components, even though most of the events are in a quasi incompressible state.

In the isothermal and polytropic components, we found the same behavior of the total compressible energy cascade with the heliocentric distance (as seen in Figure 3). Thus, we observe the fact that the energy cascade rates, both compressible and incompressible ones, are ordered by the heliocentric distance. Regarding the polytropic and the isothermal models, we noted that there is also a slight increase of the compressible term in the first case with respect to the second one. In addition, we note that there are larger cascade values when we use the polytropic model.
Figure 5. The compressible cascade rate component $\langle |\varepsilon_{2C}| \rangle$ as a function of the Yaglom-like term $\langle |\varepsilon_{1C}| \rangle$ in the MHD scales in the case of using (a) an isothermal (AS) and (b) polytropic (SS) model, respectively. The colorbar corresponds to the Heliocentric distance per event.

C. The effect of compressibility and heliocentric distance over the energy cascade rates

Figure 7 shows the bin-average absolute value of the energy cascade rate components as a function of (a) the compressibility and (b) the heliocentric distance. It is worth mentioning that we group events according to these two magnitudes and then, we segment and sort data values into bins. Finally, we took the average value of each bin. Moreover, in Fig. 7 (a) the colorbar corresponds to $\langle |r| \rangle$ and in Fig. 7 (b) the colorbar corresponds to the compressibility. In particular, we compared the behavior of the total expression of the compressible cascade, the compressible components, and the incompressible cascade. We reported the standard error as the error bars. Figure 7 summarizes the results seen in previous Figs. 3-6.

As we discussed previously (in Figure 5), we notice that as we approach to the Sun, the absolute value of the compressible energy cascade rate increases. Also, as the level of compressibility increases in the solar wind (see Figure 6), there is an important increase of the absolute value of the compressible energy cascade, approximately one order of magnitude. In the case of the incompressible cascade rate, the increment of the rate is less relevant compared to the compressible one. It is worth mentioning that, there is a clear relation between the compressibility and the
Figure 6. The compressible cascade rate component $\langle |\varepsilon_{2C}| \rangle$ as a function of the Yaglom-like term $\langle |\varepsilon_{1C}| \rangle$ in the MHD scales in the case of using (a) an isothermal (AS) and (b) polytropic model (SS), respectively. The colorbar corresponds to the level of compressibility per event.

distance to the Sun: as we travel away from the Sun, the larger the level of compressibility is. Interestingly, here we show that, on average, $\langle |\varepsilon_{SS}^{S}| \rangle > \langle |\varepsilon_{AS}^{S}| \rangle$ since 76% of the events satisfy this condition.

D. The sign of the incompressible and compressible energy cascade rates

In the previous sections, we have investigated the absolute value of the incompressible and compressible energy cascade rate. To investigate the connection between the absolute value of the cascade and the real cascade rate, we study the signed energy cascade rate in this subsection. Figure 8 and 9 show the signed (a-c) incompressible, (d-f) isothermal and (g-i) polytropic energy cascade rate distributions for different heliocentric distances $r$. On the one hand, in Figure 8, we compared the signed and the absolute value of the energy cascade. To do this, we have added in black dashed and dotted lines the mean value and the standard deviation of the signed energy cascade, respectively. Besides, we have included in a gray dashed line the mean of the absolute value of each distribution. In general, we observed that the mean of the absolute value of the cascade exceed the mean of the signed cascade. Therefore, it is worth mentioning that there is a
Figure 7. Average of the energy cascade rate for a given: (a) compressibility bin using a colorbar that corresponds to the heliocentric distance and (b) a heliocentric distance bin where the colorbar corresponds to the level of compressibility. The compressible isothermal and polytropic components ($\langle|\varepsilon_{AS2C}|\rangle$ and $\langle|\varepsilon_{SS2C}|\rangle$, resp.), the Yaglom generalization term ($\langle|\varepsilon_{1C}|\rangle$) and the incompressible cascade ($\langle|\varepsilon_I|\rangle$) are included.

limitation in the analysis of the absolute value of $\varepsilon$ since it is artificially increasing due to the use of the absolute value.

In Figure 9, we studied separately the amount of positive (red bars) and negative (blue bars) values of the energy cascade. For the sake of comparison, we have moved the negative values distribution to the positive axis using the absolute value of them. For each particular distribution, we added in red, blue and black dash lines the mean of the positive, negative and absolute values, respectively. We observed similar means for the positive and negative values for each distribution (similar results are obtained using the mean value of each distribution). Moreover, we found that the incompressible and compressible energy cascade rates have statistically the same amount of negative and positive values in our data set. On the other hand, the sign of the cascade is related to the direct and inverse cascade through the Taylor hypothesis (e.g., Sorriso-Valvo et al., 2007; Marino and Sorriso-Valvo, 2023). In particular, a positive (respectively negative) value of $\varepsilon$ imply a direct (inverse) energy cascade. Therefore, in Figure 9 we obtained a positive sign of the mean value of the incompressible and compressible (isothermal and polytropic) cascade which indicates a direct cascade at different heliocentric distances, despite having negative values of the energy cascade rate for individual hourly events.

Finally, we studied the relative sign between the compressible and incompressible cascade
Figure 10 shows the signed compressible cascade rate as a function of the signed incompressible one. We observed the same sign for both cascades in most of the events (\(\sim 97\%\)) for the averaged incompressible and compressible energy cascade rates. We concluded that for both isothermal and polytropic models, the compressible fluctuations do not affect the particular direction of the cascade rate.

Figure 8. Histograms of the total signed (a-c) \(\langle \varepsilon_I \rangle\), (d-f) \(\langle \varepsilon_{AS}^C \rangle\) and (g-i) \(\langle \varepsilon_{SS}^C \rangle\) distribution as a function of different heliocentric distances. Black dashed and dotted lines correspond to the mean and the standard deviation of the signed cascade, respectively. For the sake of comparison we have added in gray dashed line the mean of the absolute value of the energy cascade rate.
Figure 9. Histograms of the signed (a-c) $\langle \epsilon_I \rangle$, (d-f) $\langle \epsilon_{AS} \rangle$ and (g-i) $\langle \epsilon_{SS} \rangle$ distribution as a function of different heliocentric distances. Red and blue bars correspond to positive and negative values, respectively. Finally, red, blue and black dash lines correspond to the mean of the positive, negative and absolute values, respectively.

V. DISCUSSION AND CONCLUSIONS

In the present study, we estimated the absolute value of the energy cascade rate from both incompressible and compressible cases, using two different closures for the compressible model, i.e., the isothermal and the polytropic case. In particular, we computed the absolute value of the incompressible and compressible energy cascade rate in the solar wind at different heliocentric distances ($\sim 0.1 - 0.8$ au). Firstly, we found a clear correlation between the energy cascade rate
and the distance to the Sun (Figures 3 and 5). Our observational results show that, as we get closer to the Sun, the absolute value of the energy cascade rate increase. This observational result is compatible with recent studies (Adhikari et al., 2015; Bandyopadhyay et al., 2020; Adhikari et al., 2021; Andrés et al., 2021; Andrés et al., 2022). For instance, Andrés et al. (2022) showed a correlation between the incompressible energy cascade and the heliocentric distance in the inner heliosphere using isotropic and anisotropic exact relations. Also, Bandyopadhyay et al. (2020) have estimated the energy transfer rate from the first PSP perihelion using an incompressible exact relation and the von-Kármán decay law. The authors found that the energy cascade obtained near the perihelion is about 100 times higher than the average value at 1 au. Although we used a more complex theoretical model, as we include density fluctuations to model the expression for the energy cascade, our results show a similar increase of the absolute value of the incompressible and compressible energy cascade rate (up to two order of magnitude) as we approach to the Sun, probably due to the increase of the mean value of the amplitude of the fluctuations of plasma density, magnetic and velocity field.

Secondly, we observed an increase of the absolute value of the compressible energy cascade rate when the level of compressibility increases in the solar wind. We analyzed the competition between the Yaglom-like term (i.e., Eq. (11)) and the purely compressible term (i.e., Eq. (12)) in the total expression of the compressible cascade for both isothermal and polytropic models (see Figures 6 and 5). Despite the fact that the first term is dominant in most of the cases, we found that
the second term (purely compressible) still may play a relevant role for a proper estimation of the total compressible cascade in more compressible environments. Moreover, we obtained that, for the most compressible events (20%-25%), there is a slightly growth in the compressible cascade with respect to the incompressible one. Therefore, higher density fluctuations in the plasma lead to increasing $\varepsilon_C$ over $\varepsilon_I$ (Sorriso-Valvo et al., 2007; Marino et al., 2008; Andrés et al., 2019, 2021). Similarly, using THEMIS, MAVEN and PSP observations (at the first encounter), Andrés et al. (2021) reported moderate increases of the isothermal compressible cascade with respect to the incompressible one at different heliocentric distances. In our case of study, we expand these previous results including the polytropic model and much more extended data set. Our observational results showed more significant increments between the compressible and incompressible cascade, especially for the events with smaller cascade energy values. In addition, Hadid, Sahraoui, and Galtier (2017) showed that the energy cascade rate increases as compressibility increases in the plasma in the slow solar wind. Note that in the present work, almost all the events are in the slow solar wind (94%) so we confirmed this previous observational results.

Then, we reported the average of the absolute value of the energy cascade rate in order to study and compare the behavior between the compressibility levels and the heliocentric distance (in Figure 7). We related these two magnitudes and found out that the compressibility increases as we increase the distance to the Sun. Our results are in good agreement with Chen et al. (2020), who computed the magnetic compressibility coefficient $C_B = (\delta|B|/|\delta B|)^2$ and showed a clear decrease toward smaller heliocentric distances. They observed that the compressibility levels at PSP perihelion are an order of magnitude smaller at 1 au. However, their results are based on magnetic compressibility in the frequency domain unlike the present work where we computed the mean values in the real space. On the other hand, Adhikari et al. (2020) studied the frequency distribution of the solar wind compressibility between 0.17 au and 0.61 au and showed that density fluctuation distribution is concentrated mainly around ~ 0.15 au and decreases with the heliocentric distance. This result is compatible with the density fluctuation levels found by Andrés et al. (2021). Instead, we showed that the compressibility distribution reach its highest values approximately at 0.5 au and decreases as we get closer to the Sun.

In addition, we calculated the absolute value of the compressible energy cascade by using an isothermal and polytropic model based on previous theoretical works (Andrés and Sahraoui (2017); Simon and Sahraoui (2021)). From Figures 5, 6 and 7 mentioned above and, specially, Figure 7, we noticed that there is a clear trend of the polytropic cascade to be larger than the
isothermal one. In particular, Simon and Sahraoui (2021) observed the same behavior at MHD scales despite using few events unlike our present statistical work. However, it is worth mentioning that the two compressible models (isothermal and polytropic) give essentially the same cascade rate due to the fact that the contribution of the Yaglom-like term $\varepsilon_{1C}$ tends to be dominate over $\varepsilon_{2C}$ in the inertial MHD range in both cases.

We carried out an study of the sign of the energy cascade rate by analyzing the signed incompressible and compressible cascade distributions for different heliocentric distances. First, we compared the signed cascade with its absolute value distributions and as we expect there is a limitation in the analysis of the absolute value of the cascade rate since the absolute value over estimates the cascade values and these values could not be representative of the real cascade rate. Second, our observational results show statistically the same amount of positive and negative values in our data set. In fact, the mean value of these distributions are quite similar. Moreover, in all the distributions we found a positive sign for the mean value of signed energy cascade, which indicates a connection with a direct turbulent cascade process. The presence of positive and negative values of the cascade rate have been investigated previously in solar wind turbulence (Coburn et al., 2015; Stawarz et al., 2011, 2010). Using Advanced Composition Explorer (ACE) observations, Coburn et al. (2015) have found evidence for intermittent dynamics with a high level of variability relative to the mean cascade rates. In particular, the average cascade rate reported at 1 au is compatible to our results for large heliocentric distances. Finally, we included a brief study of the relative sign of the compressible and incompressible cascade rate values and we found that, in general, the compressible fluctuations do not affect the direction of the cascade rate. This is in good agreement with the previous observational results reported by Hadid, Sahraoui, and Galtier (2017). In particular, the authors showed that the correlation between the signed incompressible and reduced isothermal compressible cascades have the same sign for most of their cases at 1 au.

In summary, our observational results support the idea that the absolute value of the compressible energy cascade rate increases at small heliocentric distance and at large values of compressibility or density fluctuation levels. Moreover, the absolute value of the compressible energy cascade rate increases at small distances, while the fraction of the compressible cascade rate to the incompressible cascade rate seems to decrease as we move away from the Sun. This is a direct consequence from the fact that density fluctuations (or compressibility) decrease as we increase the heliocentric distance. Nevertheless, there are some aspects of this work that require improvement. First, we found some discrepancies about the intrinsic relation between the heliocentric
distances and the compressibility levels. Previous observational results (Andrés et al., 2021) have found that at smaller heliocentric distances, the compressibility and density fluctuations tends to be larger. Therefore, some other process may be acting to suppress/increase fluctuations as we get closer to the Sun. Second, in the present work we only focus in the MHD scales. Recent exact relations could be used to estimate the transfer of energy as we reach the sub ion scales (Andrés, Galtier, and Sahraoui, 2018; Simon and Sahraoui, 2021, 2022). In forthcoming works, further statistical investigation of these topics are going to be carried out using in situ data in more compressible environments (like the Earth’s magnetosheath).

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VI. APPENDIX: THE CONVERGENCE OF THE ENERGY CASCADE RATE

We studied the statistical convergence of the third-order moment technique and for some randomly chosen events, we used different time duration lengths to measure how constant the absolute value of the incompressible energy cascade rate actually is in the MHD scales. Figure 11 shows the average of the energy cascade rate in the MHD scales for each particular window (between 30 minutes and 12 hours) and we observed that, indeed, the cascade rate approaches a constant value compatible with the average at 1 hour.

In fact, the convergence of the energy cascade rate using third-order theory becomes relevant to provide a local measurement of the nonlinear terms. Finally, different tests of convergence of the incompressible (e.g., Wang et al., 2022; Coburn et al., 2015; Stawarz et al., 2009) and compressible (Hadid, Sahraoui, and Galtier, 2017) energy cascade rate have been applied to investigate the convergence of this technique.
Figure 11. (a) Absolute value of the incompressible energy cascade rate for one day of events of different time duration (width). (b) Average of the energy cascade for each event.

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