ON A CLASS OF SINGULARLY PERTURBED ELLIPTIC SYSTEMS WITH ASYMPTOTIC PHASE SEGREGATION

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Abstract. This work is devoted to study of a class of elliptic singular perturbed systems and their singular limit to a phase segregating system. We prove existence and uniqueness and study the asymptotic behaviour with convergence to a limiting problem as the interaction rate tends to infinity. The limiting problem is a free boundary problem such that at each point in the domain at least one of the components is zero which implies simultaneously all components can not coexist. We present a novel method, which provides an explicit solution of limiting problem for special choice of parameters. Moreover, we present some numerical simulations of the asymptotic problem.

Keywords: Singular perturbed system, segregation, free boundary problems, numerical approximation.

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1. Introduction and problem setting

In order to model strong interaction between multiple components with reaction and diffusion, different models have been proposed. Among these models the adjacent segregation models have been extensively studied from different point of views, to see about theoretical aspects we refer to [5, 6, 8, 12]. Most of the works are related to the case of two components, while [5] considers an extension to multiple components with strict segregation. Here we consider a different extension to multiple components that is still consist with the other models for the case of two components, the segregation behaviour is of different type for multiple ones however.

Let $Ω$ be bounded domain with $C^{1,α}$ smooth boundary. The model describes the steady state of $m$ species diffusing and interacting between all component in $Ω$. Let $u_i(x)$ denote the population density of the $i^{th}$ component. We study the following singular elliptic system introduced in [6], with unknowns $U^ε = (u_{1}^ε, \ldots, u_{m}^ε)$ which satisfy
\[
\begin{align*}
\Delta u_i^\varepsilon &= \frac{A_i(x)}{\varepsilon} F(u_1^\varepsilon, \ldots, u_m^\varepsilon) \quad \text{in } \Omega, \\
u_i^\varepsilon &\geq 0 \quad \text{in } \Omega, \\
u_i^\varepsilon &= \phi_i \quad \text{on } \partial \Omega,
\end{align*}
\] 
(1.1)

for \( i = 1, \ldots, m \). Here the function \( F : \mathbb{R}^m \to \mathbb{R} \) is given by

\[
F(u_1, \ldots, u_m) = \prod_{j=1}^m u_j^{\alpha_j},
\]

for an \( m \)-tuple \((\alpha_1, \ldots, \alpha_m)\) with \( \alpha_i \geq 1 \).

The main assumptions on boundary values and data are as below:

**Assumption 1.** The boundary data \( \phi_i \) are non-negative \( C^{1,\alpha} \) functions with following partial segregation property

\[
\prod_{i=1}^m \phi_i = 0 \quad \text{on } \partial \Omega.
\]

**Assumption 2.** The functions \( A_i(x) \) are smooth, positive and satisfy

\[
0 < A_i(x) \leq \sum_{j \neq i} A_j(x) \quad \text{in } \Omega.
\]

The system (1.1) and the limiting system for \( \varepsilon \downarrow 0 \) appear in theory of flames and are related to a model called Burke-Schumann approximation. The main assumption in Burke-Schumann model is that oxidizer and reactant mix on a thin sheet and the flame precisely occurs there. A way to justify the underlying assumption is to introduce a large parameter called Damköhler number, denoted by \( D_a \), which is the parameter measuring the intensity of the reaction (see [16]). Then, the a chemical reaction is described by

\[
\text{Oxidizer + Fuel} \rightarrow \text{Products}.
\]

Let \( Y_O \) and \( Y_F \), respectively, denote the mass fraction of the oxidizer and the fuel, then they satisfy the following system

\[
\begin{align*}
-\Delta Y_O + v(x) \cdot \nabla Y_O &= D_a Y_O Y_F \quad \text{in } \Omega, \\
-\Delta Y_F + v(x) \cdot \nabla Y_F &= D_a Y_O Y_F \quad \text{in } \Omega,
\end{align*}
\]

with given incompressible velocity field \( v \) and a Dirichlet boundary condition on \( \partial \Omega \).

In [6] a general Hölder estimate for a class of singular perturbed elliptic system (1.1) is shown. The authors applied this estimate to the well-known Burke-Schumann approximation in flame theory. Also they study the classical cases i,e., equidiffusional case with high activation energy approximation, non- equidiffusional case, and to nonlinear diffusion models. The limiting problems are nonlinear elliptic equations; they have Hölder or Lipschitz maximal global regularity.
We point out that L. Caffarelli and F. Lin in [5] studied the following system with different coupling term

\[
\begin{align*}
\Delta u_i^\varepsilon &= \frac{1}{\varepsilon} u_i^\varepsilon \sum_{j \neq i} u_j^\varepsilon(x) & \text{in } \Omega, \\
u_i^\varepsilon &\geq 0, & \text{in } \Omega, \\
u_i^\varepsilon(x) &= \phi_i(x) & \text{on } \partial \Omega, \\
i = 1, \ldots, m, 
\end{align*}
\]

where the boundary values satisfy

\[
\phi_i(x) \cdot \phi_j(x) = 0, \quad i \neq j \text{ on the boundary.}
\]

**Remark 1.1.** In system (1.1) choosing \( m = 2 \) and

\[
A_i(x) = 1, \quad \alpha_i = 1, \quad i = 1, 2,
\]

we get system (1.2) for \( m = 2 \) which has been studied extensively. Thus in (1.1) we are interested when \( m \geq 3 \).

To see different theoretical aspects of the system (1.2) we refer to [5, 12, 15] and references therein. In [5] the authors study the asymptotic limit; as \( \varepsilon \) tends to zero in system (1.2) and they show that limiting case yields to pairwise segregation. Furthermore, it is shown that away from a closed subset of the Hausdorff dimension less or equal \( n - 2 \) the free interfaces between various components are, in fact, \( C^{1,\alpha} \) smooth hyper surfaces.

For the numerical approximation of the system (1.2) we refer to [3, 4]. In [3] the authors propose a numerical scheme for a class of reaction-diffusion system with \( m \) densities having disjoint supports and are governed by a minimization problem. The proposed numerical scheme is applied for the spatial segregation limit of diffusive Lotka-Volterra models in presence of high competition and inhomogeneous Dirichlet boundary conditions. In [1] the proof of convergence of the finite difference scheme for a general class of the spatial segregation of reaction-diffusion, is given.

This work is devoted to analyse existence and uniqueness results for system (1.1), as well as a study of the qualitative properties of solutions to (1.1) as \( \varepsilon \) tends to zero. A particular novelty of the current work is to provide an explicit solution for an arbitrary number of components \( m \) when the parameter \( \varepsilon \) tends to zero in the following system

\[
\begin{align*}
\Delta u_i^\varepsilon &= \frac{A_i(x)}{\varepsilon} \prod_{j=1}^{m} u_j^\varepsilon(x) & \text{in } \Omega, \\
u_i^\varepsilon &\geq 0, & \text{in } \Omega, \\
u_i^\varepsilon(x) &= \phi_i(x) & \text{on } \partial \Omega, 
\end{align*}
\]

For the cases \( A_i(x) \) be same or are constants.
The outline of this paper is as follows: Section 2 consists the proof of existence and uniqueness of system (1.1). Section 3 deals with the limiting case as \( \varepsilon \) tends to zero. In Section 4 we give an explicit solution for limiting case together with a rate of convergence. Section 5 provides some numerical simulations of the singular limit.

2. Analysis of the Model for Fixed \( \varepsilon \)

In this section we prove existence and uniqueness of the solution of System (1.1) for fixed \( \varepsilon \). The proof is constructive and we implement it to obtain numerical approximation of (1.3)

Consider the following related time dependent parabolic system

\[
\begin{cases}
\frac{\partial u_i}{\partial t} - \Delta u_i = -\frac{A_i(x)}{\varepsilon} F(u_1, \ldots, u_m) & \text{in } \Omega \times (0, T) \\
u_i(\cdot, 0) = u_{i0} & \text{in } \Omega \\
u_i(x, t) = \phi_i(x) & \text{on } \partial \Omega \times [0, T),
\end{cases}
\]

where in (2.1) the initial values \( u_{i0}, i = 1, \ldots, m \) are non-negative and compatible with boundary data. Then by Theorem 2.1 in [10] we obtain

\[ u_i(x, t) \geq 0, \quad t > 0. \]

Also it is straight to show that as \( t \) tends to infinity

\[ u_i(x, t) \to u_i(x), \]

with \( u_i(x) \) being the solution of (1.1), see [7,11].

Let \( (u_1, \ldots, u_m) \) be a positive solution of the system (1.3) then

\[ u_i \leq M, \quad i = 1, \ldots, m, \]

where

\[ M = \max_{i=1,\ldots,m} \max_{x \in \partial \Omega} \phi_i(x). \]

We denote the harmonic extension of boundary data \( \phi_i \) with \( u_i^0 \). We multiply the following equation

\[ \Delta(u_i - u_i^0) = \frac{A_i(x)}{\varepsilon} \prod_{j=1}^m F(u_1, \ldots, u_m). \]

by \( (u_i - u_i^0)^+ \) where \( u^+(x) = \max(u(x), 0) \). Then integrating by parts gives

\[ -\int_{\Omega} |\nabla(u_i - u_i^0)^+|^2 dx = \int_{\Omega} \frac{A_i(x)}{\varepsilon} (u_i - u_i^0)^+ F(u_1, \ldots, u_m) dx. \]

Note that the integrand of right hand side is positive and

\[ u_i - u_i^0 = 0 \quad \text{on } \partial \Omega. \]

From here \( \int_{\Omega} |\nabla(u_i - u_i^0)^+|^2 dx = 0 \) which implies

\[ u_i \leq u_i^0 \quad \text{in } \Omega. \]
A standard maximum and nonnegativity principle for elliptic equations (cf. [14]) yields the following result. In sequel we use this result.

**Lemma 2.1.** Let \( u \in H^1(\Omega) \) be a weak solution of the system

\[
\begin{aligned}
\Delta u &= a u^\alpha(x) \quad \text{in } \Omega, \\
 u &= \phi \quad \text{on } \partial\Omega.
\end{aligned}
\]  

(2.2)

with \( a \) and \( \phi \) bounded and nonnegative, \( \alpha \geq 1 \) then

\[ 0 \leq u \leq M, \]

where

\[ M = \max_{x \in \partial\Omega} \phi_i(x). \]

In the next Theorem 2.2 we show the existence of nonnegative solutions to the original system. The main idea of the proof is to construct sub and super solution and decoupling the system in iterative way and to exploit the uniform \( L^\infty \) bounds, see also the proof in [15] for the proof of uniqueness of the solution for system (1.2).

**Theorem 2.2.** For each \( \varepsilon > 0 \), there exist a unique nonnegative solution

\[
(u_1^\varepsilon, \cdots, u_m^\varepsilon) \in H^1(\Omega)^m \cap L^\infty(\Omega)^m
\]

of the system (1.1).

**Proof.** Without loss of generality in the proof we set \( \alpha_i = 1 \) i.e.,

\[ F(u_1, \cdots, u_m) = \prod_{j=1}^m u_j. \]

To start, consider the harmonic extension \( u_0^\varepsilon \) given by

\[
\begin{aligned}
-\Delta u_0^\varepsilon &= 0 \quad \text{in } \Omega, \\
 u_0^\varepsilon &= \phi_i \quad \text{on } \partial\Omega.
\end{aligned}
\]  

(2.3)

Next, given \( u_i^k \) consider the solution of the following linear system

\[
\begin{aligned}
\Delta u_i^{k+1} &= A_i(x) u_i^1 u_i^2 \cdots u_i^k + \frac{1}{\varepsilon} u_i^{k+1} u_{i+1}^{k+1} \cdots u_m^{k+1} \\
 u_i^{k+1}(x) &= \phi_i(x)
\end{aligned}
\]  

(2.4)

Note that we can subsequently solve the equations for increasing \( i \) due to the triangular structure and always obtain a problem of the form considered in Lemma 2.1, hence the uniform bounds apply. We show that the following inequalities hold:

\[ u_i^0 \geq u_i^1 \geq u_i^2 \geq \cdots \geq u_i^{2k} \geq \cdots \geq u_i^{2k+1} \geq \cdots \geq u_i^3 \geq u_i^1, \quad \text{in } \Omega. \]

The first iteration for \( u_1 \) reads as

\[ \Delta u_1^1 = \frac{A_1(x)}{\varepsilon} u_1^1 u_2^0 \cdots u_m^0. \]
Note that since \( u^0 \geq 0 \), and boundary conditions \( \phi_i(x) \) are non negative then the weak maximum principle (see appendix) implies that \( u^1 \geq 0 \). The equation for \( u_2^1 \) in (2.4) is given by
\[
\Delta u_2^1 = \frac{A_2(x)}{2\varepsilon} ( u^1_0 u^1_2 u^0_3 \cdots u^0_m + u^1_1 u^1_2 u^0_3 \cdots u^0_m ) .
\]
Repeating the same argument, we obtain that \( u^1_2 \geq 0 \) and consequently \( u^1_i \geq 0 \) for \( i = 3, \cdots, m \).

Now we have
\[
\begin{cases}
  \Delta u^1_i \geq 0 & \text{in } \Omega, \\
  u^1_i(x) = u^0_i(x) = \phi_i(x) & \text{on } \partial \Omega.
\end{cases}
\]
Thus the comparison principle implies that \( u^1_i \leq u^0_i \). The same argument shows \( u^0_i \geq u^2_i \).

In the next step we verify the following inequalities hold
\( u^2_i \geq u^1_i \) for \( i = 1, \cdots, m \).

To do this, one verifies that inequality \( u^2_i \geq u^1_i \) holds then this fact can be used to prove inequality for \( i = 2, 3, \cdots, m \). Then the same arguments show that \( u^3_i \geq u^1_i \).

To proceed more with induction, assume that
\[
\begin{aligned}
  u^0_i &\geq u^2_i \geq \cdots \geq u^{2k}_i \geq u^{2k+1}_i \geq \cdots \geq u^3_i \geq u^1_i.
\end{aligned}
\] (2.6)
We show that
\[
 u^{2k+1}_i \leq u^{2k+2}_i .
\]
To show this, first we check for \( i = 1 \) and the same argument can be applied consequently. By (2.4) and the assumption in (2.6) we have
\[
\begin{cases}
  \Delta u^{2k+2}_1 = \frac{A_1(x)}{\varepsilon} u^1_1 u^{2k+2}_1 \prod_{j=2}^m u^{2k+1}_j \leq \frac{1}{\varepsilon} u^{2k+2}_1 \prod_{j=2}^m u^{2k}_j , \\
  \Delta u^{2k+1}_1 = \frac{A_1(x)}{\varepsilon} u^1_1 u^{2k+1}_1 \prod_{j=2}^m u^{2k}_j .
\end{cases}
\]
Note that \( u^{2k+1}_1 \) and \( u^{2k+2}_1 \) have the same boundary value so by the comparison principle
\[
 u^{2k+1}_1 \leq u^{2k+2}_1 .
\]
Now we proceed for \( i = 2, \cdots, m \). The same argument using the assumption \( u^{2k+1}_i \geq u^{2k-1}_i \) shows that
\[
 u^{2k+1}_i \leq u^{2k}_i .
\]
For the next step, we use the fact from previous step which states \( u^{2k+2}_i \leq u^{2k}_i \) to verify \( u^{2k+3}_i \geq u^{2k+1}_i \).
Now let $\bar{u}_i$ and $u_i$ be two families of functions such that
\[ u_i^{2k} \to \bar{u}_i \text{ uniformly in } \Omega, \]
\[ u_i^{2k+1} \to u_i \text{ uniformly in } \Omega. \]
Taking the limit in (2.4) yields for $i = 1, \cdots, m$ the followings hold
\[ \Delta \bar{u}_i = \frac{A_i(x)}{\varepsilon} (\bar{u}_1 \cdots \bar{u}_i \bar{u}_{i+1} \cdots \bar{u}_m + \bar{u}_1 \cdots \bar{u}_{i-1} \bar{u}_i \bar{u}_{i+1} \cdots \bar{u}_m) \text{ in } \Omega, \]
\[ \Delta u_i = \frac{A_i(x)}{\varepsilon} (u_1 \cdots u_i u_{i+1} \cdots u_m + u_1 \cdots u_{i-1} u_i u_{i+1} \cdots u_m) \text{ in } \Omega. \] (2.7)
The inequality $u_i^{2k} \geq u_i^{2k+1}$ implies that
\[ \bar{u}_i \geq u_i. \] (2.8)
We will show that in fact the equality holds. To do this, first consider the equations for the $m^{th}$
\[ \Delta \bar{u}_m = \frac{A_m(x)}{\varepsilon} \bar{u}_m (\bar{u}_1 \cdots \bar{u}_i \bar{u}_{i+1} \cdots \bar{u}_{m-1} + \bar{u}_1 \cdots \bar{u}_{i-1} \bar{u}_i \bar{u}_{i+1} \cdots \bar{u}_{m-1}) \text{ in } \Omega, \]
\[ \Delta u_m = \frac{A_m(x)}{\varepsilon} u_m (u_1 \cdots u_i u_{i+1} \cdots u_{m-1} + u_1 \cdots u_{i-1} u_i u_{i+1} \cdots u_{m-1}) \text{ in } \Omega, \]
\[ \bar{u}_m = u_m = \phi_m(x) \text{ on } \partial \Omega, \] (2.9)
which implies
\[ \bar{u}_m = u_m. \]
Now by checking the equation for $i = m - 1$ in (2.7) and using the previous fact $\bar{u}_m = u_m$, yields
\[ \bar{u}_{m-1} = u_{m-1}, \]
and argument is repeated backward which shows equality for every $i$.
To show uniqueness, assume there exists another positive solution $(w_1, \cdots, w_m)$ of system, then we show
\[ u_i = w_i, \quad i = 1, \cdots, m. \]
We will prove that the following equations hold:
\[ u_i^{2m+1} \leq w_i \leq u_i^{2m}, \quad \text{for } m \geq 0. \] (2.10)
To begin, we show that
\[ w_i \leq u_i^0. \] (2.11)
This is a consequence of the fact that $w_i$ satisfies
\[ \begin{cases} \Delta w_i \geq 0 & \text{in } \Omega, \\ w_i = u_i^0 & \text{on } \partial \Omega. \end{cases} \]
Next we compare $w_i$ with $u_1^i$ and we show $w_i \geq u_1^i$. As in existence part, first we check for $i = 1$ in inequality follows from (2.11) and
\[
\begin{cases}
\Delta w_1 = \frac{A_{1w}}{\epsilon} \prod_{j=2}^{m} w_j \quad \text{in } \Omega, \\
\Delta u_1^i = \frac{A_{1u}}{\epsilon} \prod_{j=2}^{m} u_j^0 \quad \text{in } \Omega.
\end{cases}
\]
Now we proceed by induction and we assume that the claim is true until $2k + 1$. This means that we have
\[u_i^{2k+1} \leq w_i \leq u_i^{2k}.
\]
Then we show
\[u_i^{2k+3} \leq w_i \leq u_i^{2k+2}.
\]
Again comparing the equations for $w_i$ and $u_i^{2k+2}$ and the using assumption $u_i^{2k+1} \leq w_i$ yields the following inequality
\[w_i \leq u_i^{2k+2}.
\]
The same reasoning for inequality $u_i^{2m+3} \leq w_i$ holds. Now taking limit in (2.10 ) shows that
\[w_i = u_i, \quad i = 1, \cdots, m.
\]

3. Limiting Problem

In this section we study properties of the solution for system (1.1) to provide estimates and compactness results to pass to the limit as $\epsilon$ tends to zero.

As we have seen in the last section, for each fixed $\epsilon$, the system (1.1) has a unique solution. Let $U^\epsilon = (u_1^\epsilon, \cdots, u_m^\epsilon)$ be the unique positive solution of system (1.1) for fixed $\epsilon$, then $u_i^\epsilon$ for $i = 1, \cdots, m$ satisfy the following differential inequalities:
\[-\Delta u_i^\epsilon \leq 0 \quad \text{in } \Omega.
\]
Also define $\widehat{u}_i^\epsilon$ as
\[\widehat{u}_i^\epsilon := u_i^\epsilon - \sum_{j \neq i} u_j^\epsilon,
\]
then considering the assumption 2, it is easy to verify that
\[-\Delta \widehat{u}_i^\epsilon \geq 0.
\]
Let $h_i$ and $H_i$ for $i = 1, \cdots m$ be harmonic with boundary value $\phi_i$ and $\widehat{\phi}_i$ respectively, where
\[\widehat{\phi}_i = \phi_i - \sum_{j \neq i} \phi_j,
\]
then we have
\[H_i \leq \widehat{u}_i^\epsilon \leq u_i^\epsilon \leq h_i.
\]
which implies
\[
\frac{\partial h_i}{\partial \nu} \leq \frac{\partial u^\varepsilon_i}{\partial \nu}.
\] (3.2)

In this part we show that the solution \( u^\varepsilon_i \) of system (1.1) has bound in \( W^{1,2}(\Omega) \) independently of \( \varepsilon \). To do this, we prove several lemmas.

**Lemma 3.1.** Assume \( x_0 \in \Omega \) and \( B_{2r}(x_0) \subset \Omega \). Let \( u \) satisfies the following
\[
\begin{cases}
\Delta u = f \geq 0 & \text{in } B_{2r}(x_0), \\
0 \leq u \leq M & \text{in } B_{2r}(x_0).
\end{cases}
\]

Then
\[
\int_{B_r(x_0)} f(x) \, dx \leq C_0 M r^{n-2},
\]
for some \( C_0 \) that only depends on dimension \( n \).

**Proof.** Without loss of generality, assume \( x_0 = 0 \). By Green’s formula for ball one has
\[
0 \leq u(0) = \int_{\partial B_{2r}(0)} u(x) \, dx - \int_{B_{2r}(0)} \left( \frac{\omega_n}{|x|^{n-2}} - \frac{\omega_n}{(2r)^{n-2}} \right) f(x) \, dx
\]
\[
\leq M - C_0 \int_{B_r} \frac{\omega_n}{|x|^{n-2}} f(x) \, dx \leq M - C_0 \frac{\omega_n}{r^{n-2}} \omega_n \int_{B_r(0)} f(x) \, dx.
\]
Next, rearranging terms proves the Lemma. \( \square \)

**Lemma 3.2.** Assume that \( u_i \) satisfies
\[
\begin{cases}
\Delta u_i = \frac{1}{\varepsilon} \prod_{j=1}^m u_j & \text{in } \Omega, \\
u_i \geq 0 & \text{in } \Omega, \\
u_i = \phi_i & \text{on } \partial \Omega.
\end{cases}
\] (3.3)

Then there exists a constant \( C_0 \) depends only on \( \Omega, n, r \) and \( \|\phi_i\|_{C^{1,\alpha}(\partial \Omega)} \) such that
\[
\int_{B_r(x_0) \cap \Omega} \frac{1}{\varepsilon} \prod_{j=1}^m u_j \, dx \leq C_0 r^{n-2}.
\]

**Proof.** The proof consider different cases.

1. If \( B_{2r}(x_0) \subset \Omega \) then it follows by previous Lemma 3.1.
2. If \( \exists k \) such that \( \phi_k = 0 \) on \( \partial \Omega \cap B_{2r}(x_0) \) then we may extend \( u_k \) to
\[
\overline{u}_k = \begin{dcases}
u_k & \text{in } \Omega, \\
nu_k & \text{in } \Omega^c.
\end{dcases}
\]
and apply the previous Lemma to \( \overline{u}_k \).
(3) If none of \( \phi_k \) vanishes on \( \partial \Omega \cap B_{2r}(x_0) \) then, since the product of boundary values is zero, there must be a \( \phi_i \) that vanishes at a point \( y_1 \in \partial \Omega \cap B_{2r}(x_0) \), we may assume that \( \phi_i(y_1) = 0 \). Also, since \( u_1 \geq 0 \) it follows that

\[
\frac{\partial u_1(y_1)}{\partial \nu} \leq 0.
\]

Now let \( h_1 \) solves

\[
\begin{align*}
\Delta h_1 &= 0 \quad \text{in } B_{4r}(x_0) \cap \Omega, \\
h_1 &= \phi_1 \quad \text{on } \partial \Omega \cap B_{4r}(x_0), \\
h_1 &= 0 \quad \text{on } \Omega \cap \partial B_{4r}(x_0).
\end{align*}
\]

(3.4)

Since \( \partial \Omega \) and \( \phi_1 \) are \( C^{1,\alpha} \) it follows that

\[
|\nabla h_1| \leq C_h, \quad \text{in } \Omega \cap B_{3r}(x_0).
\]

Now either \( w = u_1 - h_1 \) satisfies

\[
-C_h \leq \frac{\partial w}{\partial \nu} \leq C_*
\]

for some \( C_* \) (which we will decide) in which case we may apply the previous Lemma on \( w \) since

\[
\Delta w = \frac{1}{\varepsilon} \prod_{j=1}^m u_j + \frac{\partial w}{\partial \nu} H^{n-1} |\partial \Omega| \quad \text{in } B_{3r}(x_0),
\]

or there is a point \( y_2 \in \partial \Omega \cap B_{3r}(x_0) \) such that

\[
\frac{\partial w(y_2)}{\partial \nu} \geq C_*.
\]

Note that \( \phi_1(y_2) > 0 \) since otherwise,

\[
0 \geq \frac{\partial u_1(y_2)}{\partial \nu} = \frac{\partial w(y_2)}{\partial \nu} + \frac{\partial h_1(y_2)}{\partial \nu} \geq C_* - C_h > 0,
\]

provided \( C_* \) is large enough. Next, since \( \phi_1(y_2) > 0 \) there is another \( \phi_k \) say \( \phi_2 \), such that \( \phi_2(y_2) = 0 \). Let \( h_2 \) solves

\[
\begin{align*}
\Delta h_2 &= 0 \quad \text{in } B_{4r}(x_0) \cap \Omega, \\
h_2 &= \phi_2 \quad \text{on } \partial \Omega \cap B_{4r}(x_0), \\
h_2 &= 0 \quad \text{on } \Omega \cap \partial B_{4r}(x_0).
\end{align*}
\]

(3.5)

Then again \( |\nabla h_2| \leq C_h \) in \( B_{3r}(x_0) \) for some \( C_h \) depending only on the domain \( \Omega \) and \( \|\phi_2\|_{C^{1,\alpha}} \). Next let \( u_2 = w + h_2 + g \) in \( B_{4r}(x_0) \cap \Omega \) where

\[
\begin{align*}
\Delta g &= 0 \quad \text{in } B_{4r}(x_0) \cap \Omega, \\
g &= u_2 - w \quad \text{on } \partial \Omega \cap B_{4r}(x_0), \\
g &= 0 \quad \text{on } \partial B_{4r}(x_0) \cap \Omega.
\end{align*}
\]

(3.6)
Since \( g \) is bounded; \( |g| \leq 3M \) on \( \partial B_{4r}(x_0) \cap \Omega \), then it follows that
\[
|\nabla g| \leq C_g \quad \text{in} \quad B_{3r}(x_0) \cap \Omega,
\]
where \( C_g \) depends on the bound \( M, r \) and \( \Omega \). This leads in particular to
\[
0 \geq \frac{\partial u_2(y_2)}{\partial \nu} = \frac{\partial w(y_2)}{\partial \nu} + \frac{\partial h_1(y_2)}{\partial \nu} + \frac{\partial g(y_2)}{\partial \nu} \geq C_* - C_h - C_g \geq 0.
\]
This is a contradiction if \( C_* \) is large enough and this complete the proof.

\[\square\]

**Proposition 3.3.** Let \( u_1, \ldots, u_m \) be as in previous Lemma. Then there exists a constant \( C_0 \) (independent of \( \varepsilon \)) such that
\[
\|u_i\|_{W^{1,2}(\Omega)} \leq C_0.
\]

**Proof.** Cover \( \Omega \) by finitely say \( N \) balls \( B_r(x_k) \) and notice that
\[
\int_{\Omega} \frac{1}{\varepsilon} \prod_{j=1}^{m} u_j \leq \sum_{k=1}^{N} \int_{B_r(x_k)} \frac{1}{\varepsilon} \prod_{j=1}^{m} u_j \leq NC_0 r^{n-2}.
\]
Next let \( f = \frac{1}{\varepsilon} \prod_{j=1}^{m} u_j \) and define
\[
v = - \int_{\Omega} \frac{\omega_n}{|x - y|^{n-2}} f(y) \, dy.
\]
Then \( v \) satisfies
\[
\Delta v = f \chi_{\Omega} \quad \text{in} \quad \mathbb{R}^n,
\]
which implies
\[
\int_{B_R(0)} |\nabla v|^2 \, dx = \int_{B_R(0)} f(x)v(x) \, dx + \int_{\partial B_R(0)} v \frac{\partial v}{\partial \nu} \, ds \leq C. \tag{3.7}
\]
where \( R \) is chosen so large that \( \Omega \subset B_R(0) \). Now let \( u_i = H_i + v \) where
\[
\left\{ \begin{array}{l l}
\Delta H_i = 0 & \text{in} \quad \Omega \\
H_i = \phi_i - v & \text{on} \quad \partial \Omega.
\end{array} \right. \tag{3.8}
\]
Since \( \phi_i \in C^{1,\alpha} \) and \( v \in W^{1,2}(\Omega) \) by (3.7), then it follows that \( H_i \in W^{1,2}(\Omega) \) with bounds only depending on \( \|v\|_{W^{1,2}(\Omega)}, \|\phi_i\|_{C^{1,\alpha}(\Omega)} \) and (\( \Omega \)). In particular, \( \|u_i\|_{W^{1,2}(\Omega)} \) is bounded independent of \( \varepsilon \).

The above Lemma shows that up to a subsequence denoted with \( u_\varepsilon^i \) we get
\[
u_\varepsilon^i \rightharpoonup u_i \quad \text{in} \quad H^{1}_0(\Omega).
\]

The main result of this section is Theorem 3.4 which shows the asymptotic behaviour of system (1.3) as \( \varepsilon \) tends to zero.
**Theorem 3.4.** Let $U^\varepsilon = (u_1^\varepsilon, \cdots, u_m^\varepsilon)$ be a solution of the system at fixed $\varepsilon$. Let $\varepsilon$ tends to zero, then there exists $U \in H^1(\Omega)^m \cap L^\infty(\Omega)^m$ such that for all $i = 1, \cdots, m$:

1. $\Delta u_i \geq 0$ in the sense of distribution.
2. up to subsequences, $u_i^\varepsilon - u_i \to 0$ strongly in $H^1_0(\Omega)$.
3. $\prod_i u_i = 0$ a.e in $\Omega$.

**Proof.** Proposition (3.3) shows the existence of a weak limit $u_i$ such that, up to subsequences,

$$u_i^\varepsilon \rightharpoonup u_i \quad \text{in} \quad H^1_0.$$ 

The weak limit $u_i$ for $i = 1, \cdots, m$ satisfy the following differential inequalities

$$-\Delta u_i \leq 0, \quad -\Delta \hat{u}_i \geq 0 \quad \text{in} \quad \Omega,$$

since we can pass to the weak limit in the differential inequalities for $u_i^\varepsilon$ and $\hat{u}_i^\varepsilon$.

To show the strong convergence, we show that

$$\int_{\Omega} |\nabla u_i^\varepsilon|^2 \, dx \to \int_{\Omega} |\nabla u_i|^2 \, dx.$$ 

By weak lower semi continuously of Dirichlet norm just needs to show

$$\int_{\Omega} |\nabla u_i|^2 \, dx \geq \limsup \int_{\Omega} |\nabla u_i^\varepsilon|^2 \, dx.$$

We multiply the inequality $-\Delta u_i^\varepsilon \leq 0$ by $u_i^\varepsilon$ and integration by parts,

$$\int_{\Omega} |\nabla u_i^\varepsilon|^2 \, dx - \int_{\partial \Omega} u_i^\varepsilon \frac{\partial u_i^\varepsilon}{\partial n} \, ds \leq 0.$$ 

This implies

$$\int_{\partial \Omega} u_i \frac{\partial u_i}{\partial n} \, ds \geq \limsup \int_{\Omega} |\nabla u_i^\varepsilon|^2 \, dx. \quad (3.9)$$

Next we multiply the equation for $u_i^\varepsilon$ by $u_i$ to obtain

$$-\int_{\Omega} \nabla u_i^\varepsilon \cdot \nabla u_i \, dx + \int_{\partial \Omega} u_i \frac{\partial u_i^\varepsilon}{\partial n} \, ds = \int_{\Omega} u_i \prod_{j=1}^m u_j^\varepsilon \, dx.$$ 

Taking the limit as $\varepsilon_n$ tends to zero and considering the weak convergence of $u_i^\varepsilon$ and previous part to have

$$-\int_{\Omega} |\nabla u_i|^2 \, dx + \int_{\partial \Omega} u_i \frac{\partial u_i}{\partial n} \, ds = 0. \quad (3.10)$$

Form (3.9) and (3.10) the result holds.
(2) Fix a point \( x_0 \in \Omega \) and let the index \( i \) be such that
\[
u^\varepsilon_i(x_0) = \max_{1 \leq k \leq m} \nu^\varepsilon_k(x_0).
\]
Now assume that \( \nu^\varepsilon_i(x_0) = c > 0 \) then by Hölder continuity there is \( r \) such that
\[
|\nu^\varepsilon_i(x) - \nu^\varepsilon_i(x_0)| \leq \frac{c}{2}, \quad x \in B(x_0, r).
\]
Next we use the fact that the functions \( \nu^\varepsilon_i \) for \( i = 1, \cdots, m \) are subharmonic, using the mean value property for subharmonic functions (see the proof of theorem 2.1 in [13])
\[
\int_{\partial B(x_0, r)} |\nu^\varepsilon_i(x_0) - \nu^\varepsilon_i(y)| \, dy = \int_0^r \left( \int_{B(x_0, s)} \Delta \nu^\varepsilon_i \right) \frac{ds}{s^{n-1}} \geq r^2 \int_{B(x_0, r)} \Delta \nu^\varepsilon_i \, dx.
\] (3.11)
From here the following holds
\[
\int_{B(x_0, r)} \Delta \nu^\varepsilon_i \, dx = \int_{B(x_0, r)} \frac{\nu^\varepsilon_i}{\prod_{j \neq i} \nu^\varepsilon_j} \leq \frac{c}{2r^2}.
\] (3.12)
Note that in the ball \( B(x_0, r) \) we have \( \nu^\varepsilon_i \geq \frac{c}{2} \) so from (1.3) we obtain
\[
\int_{B(x_0, r)} \frac{1}{\varepsilon} \prod_{j \neq i} \nu^\varepsilon_j \, dx \leq \frac{1}{r^2}.
\] (3.13)
Next, in (3.13) let \( \varepsilon \) tend to zero which yields
\[
\prod_{j \neq i} \nu^\varepsilon_j(x) \rightarrow 0 \quad \text{in} \quad B(x_0, r).
\]

Let \( w_1 \) be the first eigenfunction of the Laplace operator in \( \Omega \), i.e.,
\[
\begin{align*}
-\Delta w_1 &= \lambda_1 w_1 \quad \text{in} \quad \Omega, \\
w_1 &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
The first eigenfunction does not change the sign and we may therefore take it to be positive and normalized it so that \( \|w_1\|_{L^\infty} = 1 \). Multiplying the equation
\[
\Delta \nu^\varepsilon_i = \frac{A_i}{\varepsilon} \prod_{i=1}^m \nu^\varepsilon_i,
\]
by \( w_1 \) and integrating over \( \Omega \) yields
\[
\int_{\Omega} w_1 \Delta \nu^\varepsilon_i \, dx = \int_{\Omega} \frac{A_i}{\varepsilon} \prod_{i=1}^m \nu^\varepsilon_i \, w_1 \, dx.
\]
Integration by parts and implementing that $w_1$ is zero on boundary, we obtain

$$
\int_{\Omega} \frac{A_i(x)}{\varepsilon} \prod_{i=1}^{m} u_i^\varepsilon(x) w_1 \, dx = \int_{\Omega} u_i^\varepsilon \Delta w_1 \, dx - \int_{\partial \Omega} w_i \frac{\partial w_1}{\partial n} \, ds
$$

$$
= \lambda_1 \int_{\Omega} u_i^\varepsilon \, dx - \int_{\partial \Omega} \phi_i \frac{\partial w_1}{\partial n} \, ds.
$$

Now from the bound on $u_i$ and the fact that normal derivative of the first eigenfunction on the boundary is bounded, we conclude

$$
\int_{\Omega} \frac{A_i(x)}{\varepsilon} \prod_{i=1}^{m} u_i^\varepsilon(x) w_1 \, dx \leq C.
$$

We know that for $i = 1, \cdots, m$ the solution $u_i^\varepsilon$ are Hölder continuous

$$
\|u_i^\varepsilon\|_{C^0} \leq C_i,
$$

where the constant $C_i$ is independent of $\varepsilon$. Note that, since

$$
w_1(x) > 0 \quad \text{in the interior of } \Omega,
$$

then the inequality in above yields that

$$
\int_{\Omega'} \Delta u_i^\varepsilon \leq C \quad \text{in compact subsets } \Omega' \subset \Omega, \quad (3.14)
$$

where the constant $C$ is independent of $\varepsilon$. For the rest, we show that for those points close enough to the boundary, $\Delta u_i^\varepsilon$ remains bounded. Let $C_i$ and $\beta_i$ denote the Hölder constant and Hölder exponent of $u_i^\varepsilon$. Choose the strip around boundary such that

$$
\text{dist}(x, \partial \Omega) \leq \left(\frac{\varepsilon}{C_i}\right)^{1/\beta_i} \quad \forall i = 1, \cdots, m. \quad (3.15)
$$

Let $y \in \partial \Omega$ be a point such that has minimum distance to $x$. Then by assumption on the boundary values, there is $k$ such that $u_k(y) = 0$ and

$$
\frac{|u_k^\varepsilon(x) - u_k^\varepsilon(y)|}{|x - y|^{\beta_k}} \leq C_k.
$$

The previous inequality and (3.15) imply that

$$
u_k^\varepsilon(x) \leq \varepsilon. \quad (3.16)
$$

Combining (3.14) and (3.16) yields that Laplace of $u_i$ is bounded.

Remark 3.1. The uniform bound of normal derivative of $u_i^\varepsilon$ yields estimates for limiting problem as follows. Integrate from

$$
\Delta u_i^\varepsilon = \frac{A_i(x)}{\varepsilon} \prod_{i=1}^{m} u_i^\varepsilon(x)
$$
to obtain
\[
\int_{\partial \Omega} \frac{\partial u^\varepsilon_i}{\partial n} \, ds = \int_{\Omega} \frac{A_i(x)}{\varepsilon} \prod_{i=1}^{m} u_i^\varepsilon(x) \, dx.
\]
From here we get
\[
\int_{\Omega} \frac{A_i(x)}{\varepsilon} \prod_{i=1}^{m} u_i^\varepsilon(x) \, dx \leq C
\]
this shows
\[
\int_{\Omega} A_i(x) \prod_{i=1}^{m} u_i^\varepsilon(x) \, dx \rightarrow 0 \quad \text{as } \varepsilon \text{ tends to zero.}
\]

**Definition 3.1.** Consider the non empty sets \( \Omega_i := \{ x \in \Omega : u_i(x) = 0 \} \). Then the free boundaries (interfaces) are define as
\[
\Gamma_{i,j} = \partial \Omega_i \cap \partial \Omega_j \cap \Omega.
\]
In the next Lemma we give the free boundary condition for the case \( A_i = 1 \).

**Lemma 3.5.** The following conditions holds on the free boundary \( \Gamma_{i,j} \).

1. \( \frac{\partial u_i}{\partial n}|_{\Omega_j} = -\frac{\partial u_i}{\partial n}|_{\Omega_i} \),
2. \( \frac{\partial u_k}{\partial n}|_{\Omega_j} - \frac{\partial u_k}{\partial n}|_{\Omega_i} = \frac{\partial u_k}{\partial n}|_{\Omega_j} \quad k \neq i, j \).

**Proof.** Let \( x_0 \) be a free boundary point in \( \Gamma_{i,j} \). Note that
\[
\Delta(u_k - u_j) = 0 \quad \text{in } B_r(x_0) \setminus \Gamma_{i,j}.
\]
In the sense of distribution we have
\[
\Delta(u_k - u_j) = \frac{\partial(u_k - u_j)}{\partial n} H^{n-1}|_{\Gamma_{i,j}} \quad \text{in } B_r,
\]
Splitting \( B_r = (B_r \cap \Omega_i) \cup (B_r \cap \Omega_j) \) and considering the fact that in \( \Omega_j \) we have \( u_j = 0 \) the second relation is proved.

\( \square \)

**Remark 3.2.** : In [15] the uniqueness of the limiting solution of system (1.2) for arbitrary number of components, is shown. Consider the metric space \( \sum \) defined by
\[
\sum = \{(u_1, u_2, \cdots, u_m) \in \mathbb{R}^m : u_i \geq 0, u_i \cdot u_j = 0 \quad \text{for } i \neq j\}.
\]
In [15] (see Theorem 1.6 ) it is shown that the limiting solution \((u_1, \cdots, u_m)\) of (1.2) is a harmonic map into the space \( \sum \). By definition the harmonic map is the critical point of the following energy functional
\[
\int_{\Omega} \sum_{i=1}^{m} \frac{1}{2} |\nabla u_i|^2 dx,
\]
among all nonnegative segregated states $u_i \cdot u_j = 0$, a.e. with the same boundary conditions.

Also in [2] an alternative proof of uniqueness for limiting case for system (1.2) is given which is more direct and based on properties of limiting solutions. Although some properties of limiting solution for systems (1.3) and (1.2) are similar, the proof of uniqueness for system (1.3) in the case $\varepsilon$ tends to zero remains challenging problem.

Define the energy associated to $m$ densities defined by

$$E(U) = \int_\Omega \sum_i |\nabla u_i(x)|^2 dx.$$ 

Now consider the following problem

$$\min E(U),$$

over the closed but non-convex set

$$S = \left\{ (u_1, \cdots, u_m) : u_i \in H^1(\Omega), u_i \geq 0, \prod_{i=1}^m u_i(x) = 0, u_i|_{\partial \Omega} = \phi_i \right\}.$$ 

Existence of a minimizer is direct. The following variation

$$v_i = (1 + \varepsilon \varphi_i) u_i, \quad i = 1, \cdots, m,$$

with $\varphi_i \in C^\infty_c(\Omega)$ yields the following

$$u_i \geq 0, \quad u_i \cdot \Delta u_i = 0, \quad \prod_{j=1}^m u_j = 0.$$ 

This implies that each $u_i$ is harmonic in its support which does not hold for our limiting solution. In fact, for system (1.3) in Theorem 3.2 we show that

$$\prod_{i=1}^m u_i = 0 \quad \text{and} \quad \Delta u_i \text{is bounded.}$$

Figure 1 also shows $u_1$ is not smooth in its support $\Delta u_1$ are Dirac measures on interfaces.

### 4. Explicit solutions in the limiting case

In this section we give an explicit solution and the rate of convergence for the limiting solution of the following system

$$\begin{cases}
\Delta u_i^\varepsilon = \frac{A_i(x)}{\varepsilon} \prod_{j=1}^m (u_j^\varepsilon)^{a_j}(x) & \text{in } \Omega, \\
u_i = \phi_i & \text{on } \partial \Omega,
\end{cases} \quad (4.1)$$

for the cases that $A_i(x)$ are the same or $A_i = C_i$, constants.
4.1. Construction of Solutions. It is easy to check that for every $\varepsilon$
\[
\Delta (u_1^\varepsilon - u_{i+1}^\varepsilon) = 0, \quad i = 1, \cdots, m - 1
\]
which remains true as $\varepsilon$ tends to zero. First of all define
\[
w_i = u_1 - u_{i+1}, \quad i = 1, \cdots, m - 1,
\]
(4.2)
then $w_i$ is the harmonic extension of the Dirichlet value $\phi_1 - \phi_{i+1}$. This means that $w_i$ for $i = 1, \cdots, m - 1$ is the solution of
\[
\begin{cases}
\Delta w_i = 0 & \text{in } \Omega, \\
w_i = \phi_1 - \phi_{i+1} & \text{on } \partial \Omega.
\end{cases}
\]
(4.3)
Note that the nonnegativity of the $u_i$ is equivalent to $u_1 \geq w_i$. Thus, an obvious candidate solution is given by
\[
u_1(x) = \max \left( \max_{i=1, \ldots, m-1} w_i(x), 0 \right)
\]
and
\[
u_i = u_1 - w_i, \quad i = 2, \ldots, m.
\]
(4.4)
(4.5)
Obviously, by this construction we have $u_i \geq 0$ and moreover
\[u_1(x)u_2(x) \cdots u_m(x) = 0, \quad \text{for all } x \in \Omega.
\]
To see the latter, let $x$ be fixed and $j$ such that $w_j(x) \geq w_i(x)$ for all $i$. Then
\[u_j(x) = u_1(x) - w_j(x) = 0.
\]
We finally need to verify $\Delta u_i \geq 0$. For $u_1$ this follows from the fact that maximum of harmonic function is subharmonic then for the rest of $u_i$ it follows from (4.4) and (4.5).

Remark 4.1. Let $v_i$ be defined as below:
\[v_i = u_2 - u_i,
\]
then set
\[u_2 = \max \left\{ \max_{i=1, \ldots, m-1} v_i(x), 0 \right\}.
\]
From this we can recover other components by
\[u_i = v_i - u_2, \quad i = 1, 3, \cdots, m.
\]
One can check this choice gives the same solutions as in (4.4) and (4.5), for the case $m = 3$ is straightforward.
4.2. **Convergence Rate.** We now turn our attention to a rate of convergence of the solutions as $\varepsilon \to 0$. Note that

$$w_i = u_i^\varepsilon - u_i^\varepsilon, \quad i = 1, \ldots, m$$

is harmonic with Dirichlet data $\phi_i - \phi_i$, hence coincides with the one in the previous section, in particular independent of $\varepsilon$.

We thus have

$$\Delta u_i^\varepsilon = \frac{1}{\varepsilon} \prod_{i=1}^{m} u_i^\varepsilon \prod_{i=1}^{m} (u_i^\varepsilon - w_i)$$

Now we have $0 \leq u_i^\varepsilon - w_i$ and $u_i^\varepsilon - w_i \geq u_i^\varepsilon - u_1$, hence

$$\varepsilon \Delta u_1^\varepsilon \geq |u_1^\varepsilon - u_1|^m,$$

respectively

$$\varepsilon \int_{\Omega} |\nabla (u_1^\varepsilon - u_1)|^2 \, dx + \int_{\Omega} |u_1^\varepsilon - u_1|^{m+1} \, dx \leq -\varepsilon \int_{\Omega} \nabla (u_1^\varepsilon - u_1) \cdot \nabla u_1 \, dx.$$

Applying Young’s inequality on the right-hand side we deduce

$$\|u_1^\varepsilon - u_1\|_{L^{m+1}(\Omega)} \leq C \varepsilon^{1/(m+1)}.$$  

5. **Numerical Study of the Limiting Problem**

This section provides some examples of numerical approximations to the limiting problem of the following system.

$$\begin{cases} 
\Delta u_i = \frac{1}{\varepsilon} \prod_{j} u_j \quad \text{in } \Omega, \\
u_i = \phi_i \quad \text{on } \partial \Omega.
\end{cases}$$

In our examples we implemented directly mimicking the fixed point technique in the existence proof of Theorem 2.2 with value of $\varepsilon$ and the method in Section 4 which demonstrate those give basically the same as epsilon goes to zero

**Example 5.1.** Let $\Omega = B_1, m = 3$. The boundary values $\phi_i$ for $i = 1, 2, 3$ are defined by

$$\phi_1(1, \Theta) = \begin{cases} 
|\sin(\frac{2}{3}\Theta)| & 0 \leq \Theta \leq \frac{4\pi}{3}, \\
0 & \text{elsewhere},
\end{cases} \quad \phi_2(1, \Theta) = \begin{cases} 
|\sin(\frac{3}{2}\Theta)| & \frac{2\pi}{3} \leq \Theta \leq 2\pi, \\
0 & \text{elsewhere}.
\end{cases}$$

$$\phi_3(1, \Theta) = \begin{cases} 
|\sin(\frac{3}{2}\Theta)| & \frac{4\pi}{3} \leq \Theta \leq 2\pi + \frac{2\pi}{3}, \\
0 & \text{elsewhere}.
\end{cases}$$

Here the boundary conditions satisfy

$$\phi_1 \cdot \phi_2 \cdot \phi_3 = 0.$$

The surface of $u_1$ is depicted in Figure 1. Also one can check the jump in gradient of $u_1$ along $\Gamma_{2,3}$ which has shown in part 2 of Lemma (3.5). In Figure 2,
Example 5.2. Let $\Omega = [-1, 1] \times [-1, 1]$ and $m = 4$. The boundary values $\phi_i$, $(i=1,2,3,4)$ are given as follows:

- $\phi_1 = \begin{cases} 1 - x^2 & x \in [-1, 1] \& y = 1, \\ 0 & \text{elsewhere}. \end{cases}$
- $\phi_2 = \begin{cases} 2(1 - y^2) & y \in [-1, 1] \& x = 1, \\ 0 & \text{elsewhere}. \end{cases}$
- $\phi_3 = \begin{cases} 3(1 - x^2) & x \in [-1, 1] \& y = -1, \\ 0 & \text{elsewhere}. \end{cases}$
- $\phi_4 = \begin{cases} 4(1 - y^2) & y \in [-1, 1] \& x = -1, \\ 0 & \text{elsewhere}. \end{cases}$

We implemented the iterative scheme given by Lemma 2.2 with $\varepsilon = 10^{-8}$ and method given by (4.4) and (4.5). The obtained solutions are same and the surface of $u_1$ is given in (6). The interfaces are shown in Figure ??.

In Figure (5 we draw the Laplace of $u_1$ on the interfaces. We know Laplace of $u_1$ is Dirac along interfaces so we scaled $\Delta u_1$ by multiplying by mesh size.

Example 5.3. Next, we change boundary values as below.

- $\phi_1 = \begin{cases} 1 - x^2 & x \in [-1, 1] \& y = 1, \\ 1 - y^2 & y \in [-1, 1] \& x = 1, \\ 0 & \text{elsewhere}. \end{cases}$
- $\phi_2 = \begin{cases} 2(1 - y^2) & y \in [-1, 1] \& x = -1, \\ 2(1 - x^2) & x \in [-1, 1] \& y = 1, \\ 0 & \text{elsewhere}. \end{cases}$
- $\phi_3 = \begin{cases} 3(1 - x^2) & x \in [-1, 1] \& y = 1, \\ 3(1 - y^2) & y \in [-1, 1] \& x = 1, \\ 0 & \text{elsewhere}. \end{cases}$
- $\phi_4 = \begin{cases} 4(1 - x^2) & -1 \leq x \leq 1 \& y = -1, \\ 4(1 - y^2) & y \in [-1, 1] \& x = -1, \\ 0 & \text{elsewhere}. \end{cases}$

The following picture shows the interfaces
Figure 2. Surface of $u_1 + u_2 + u_3$.

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Figure 3. surface of $u_1$.

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Figure 4. Free boundary and supports of the components.

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Figure 5. Laplace of $u_1$ as measure(scaled) on the interfaces. The mesh size is $\triangle x = \triangle y = 10^{-3}$. 
Figure 6. Free boundaries.