A Positive-Definite Scalar Product for Free Proca Particle

Vít Jakubský
Ústav jaderné fyziky AV ČR
250 68 Řež
Czech Republic
e-mail: jakub@ujf.cas.cz

Jaroslav Smejkal
Ústav technické a experimentální fyziky ČVUT
Horská 3a/22
128 00 Praha 2 - Nové Město. Czech Republic
e-mail: Jaroslav.Smejkal@utef.cvut.cz

Abstract
We implement recent results of pseudo-Hermitian quantum mechanics to description of relativistic massive particle with spin-one. We derive a one-parameter family of Lorentz invariant positive-definite scalar products on the space of solutions of Proca equation.

1 Introduction
As the non-relativistic quantum theory established to be a consistent physical theory, a natural task of intertwining the quantum mechanics with relativistic theory raised. The first attempts to match these two theories has led to a relativistic quantum kinematics governed by Klein-Gordon, Dirac and Proca equations [1]. A manifestly covariant equation describing a free massive vector boson

\[ \partial_{\mu} F^{\mu \nu} + m^2 A^{\nu} = 0, \quad F^{\mu \nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}. \]

has been derived by Proca [1]. Similarly to its spin-zero counterpart, Klein-Gordon equation, it contains a second time derivative. This has a deep consequences resulting in the well-known puzzle of probabilistic interpretation of the associated bosonic theories.

In the forties of the previous century, the formalism of the relativistic equations has been unified formally (let us mention works of Sakata and Taketani [3], Foldy [4] and Feschbach and Villars [5] in this context). Similarly to Dirac equation, evolution equation was rewritten as \( i \partial_t \Psi = H_{ST} \Psi \). The sequential lost of its manifest covariance was balanced by

\(^{1}\)Hierarchy of covariant equations of higher spin has been found, see [2]. For our present purposes, we will consider system with spin \( s \leq 1 \).
improved description of spin-one particles and anti-particles as the wave function consists of six components. On the contrary to Dirac equation, Hamiltonian represented by $6 \times 6$ matrix

$$H_{ST} = (\tau_3 + i\tau_2) \frac{P^2}{2m} + \tau_3 m - i\tau_2 \frac{(SP)^2}{m},$$

ceases to be Hermitian. Indeed, it rather fulfills the relation

$$H_{ST}^\dagger \tau_3 = \tau_3 H_{ST}.$$  

As the Hamiltonian $H_{ST}$ is Hermitian with respect to an inner product $(\Psi_1, \Psi_2)_{\tau_3} = \int \Psi_1^\dagger \tau_3 \Psi_2 d^3x$, we call it pseudo-Hermitian [6]. Although a reasonable contribution of the multi-component formalism, it did not resolve the puzzle of the consistent probabilistic interpretation for boson particles. Indefiniteness of the inner product implies existence of wave functions with negative quadrat of the norm. Consequently, continuity equation with indefinite density of probability had to be reinterpreted as a current conservation law, see [5].

Recently, several people imagined that the use of the bosonic Hamiltonians $H_{ST}$ in relativistic quantum mechanics may be seen as one of the most characteristic applications of the so called $PT$-symmetric version of Quantum Mechanics as proposed by Bender and Boettcher [7]. Thanks to a concentrated effort and debate (cf. its small sample in [8] - [25]) it became clear that even the operators $O$ with the “weakened” Hermiticity property

$$O^\dagger = \Theta O \Theta^{-1} \neq O, \quad \Theta = \Theta^\dagger > 0,$$  

(to be called quasi-Hermiticity) may represent, in full accord with the standard postulates and probabilistic interpretation of Quantum Mechanics, observable quantities [10].

The constructive approach of the pseudo-Hermitian quantum mechanics was applied to Klein-Gordon equation [26, 27] successfully. In the following article, we intend to perform derivation of the positive metric operator $\Theta$ for the free massive particle with spin one.

The description of our spin-one results will be organized as follows. Firstly we shall summarize the formulation of a system described by Proca equation. We refresh standard solution of the evolution equation. We then fix an inertial frame in a way discussed, originally, by Taketani, Sakata and Tamm [3]. This will enable us to outline the construction of the massive vector boson Hamiltonian (i.e., of the generator of the time evolution) which appears manifestly non-Hermitian. With the purpose of achieving an appropriate physical interpretation for the system rewritten in the Schrödingerian form we then summarize the application of a few key ideas of the $PT$-symmetric Quantum Mechanics to our problem.

Finally, we re-emphasize that the correct probabilistic interpretation of the Proca’s equation must be based on the introduction and specification of a “correct” metric operator $\Theta \neq I$ in the “physical” Hilbert space of states. We discuss Lorentz invariance of the associated scalar product.

---

2The matrices $S_j$ constitute three dimensional representation of rotational group, see (9), $\tau_j$ are Pauli matrices.
2 Proca particle

2.1 Covariant formulation

The equation (1) can be rewritten in a more transparent manner without lost of its manifest covariance. Its differentiation and consequent summation reveal that the individual components of the field have to fulfill Klein-Gordon equation and are binded by an additional constraint

\[(m^2 - \Box) A_\mu = 0, \quad \partial_{\nu} A^{\nu} = 0,\]

As could be expected in spin-one system, only three components of the field are linearly independent.

The free Klein-Gordon equation is solved by the wave functions

\[A^\mu_\pm(x) = N_\pm(p) a^\mu_\pm(p) e^{-i p x}, \quad p_0 = \pm \omega, \quad \omega = |\sqrt{p^2 + m^2}|\]

while the additional condition acquires the form

\[p a^{\pm}_\pm(p) = 0.\]

The three linearly independent four-vectors \(a^{\pm}_\pm(p)\) may be chosen in the form

\[a^{\pm,1}_\pm(p) = (0, a^{\pm,1}_\pm(p)^T, \quad a^{\pm,2}_\pm(p) = (0, a^{\pm,2}_\pm(p)^T \quad a^{\pm,3}_\pm(p) = \left(\pm \frac{|p|}{\omega}, \frac{p}{|p|}\right)^T\]

where the space-like components (i.e., polarization vectors) in the first two terms are perpendicular to \(p\) while the third one is parallel to \(p\). For the sake of definiteness we may fix, say,

\[a^{\pm,1}_\pm(p) = \left(\frac{0, -p_3, p_2}{\sqrt{p_3^2 + p_2^2}}\right), \quad a^{\pm,2}_\pm(p) = \left(\frac{(p_2^2 + p_3^2, -p_1 p_2, -p_1 p_3)}{\sqrt{(p_3^2 + p_2^2)^2 + p_1^2 p_2^2 + p_1^2 p_3^2}}\right)\]

It is convenient to distinguish the three independent solutions of (3) by their spin properties. In terms of the three-dimensional representation of the rotation group

\[S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\]

we may define the operator of helicity which in momentum representation reads

\[\hat{h}(p) = \frac{p S}{|p|} = n S = i \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}.\]

We compose another set of solutions of (3) by a linear combination of the vectors \(a_j\)

\[u^{\pm,1}_\pm(p) = \frac{a^{\pm,1}_\pm(p) + i a^{\pm,2}_\pm(p)}{\sqrt{2}}, \quad u^{\pm,-1}_\pm(p) = \frac{a^{\pm,1}_\pm(p) - i a^{\pm,2}_\pm(p)}{\sqrt{2}}\]

\[u^{\pm,0}_\pm(p) = a^{\pm,3}_\pm(p).\]
It can be verified that the spatial components of the four-vectors \( u_{\pm,h} \) are eigenstates of the helicity operator \( \hat{h} \):

\[
\hat{h} u_{\pm,h}(p) = h u_{\pm,h}(p), \quad h = 1, -1, 0.
\]  

Finally, combining (5), (7) and (11) we get

\[
A_{\pm,h}^\mu(x,t) = \mathcal{N}_{\pm,h} u^\mu_{\pm,h}(p) e^{i p x \mp i \omega t},
\]

i.e. an explicit form of wave functions with sharp value of energy and helicity in x-representation.

### 2.2 Hamiltonian and its wave functions

As was mentioned in introduction, Feschbach-Villars formulation of integer-spin relativistic equations is closely related to the conception of pseudo-Hermitian operators. In this section, we refresh Schrödinger-like reformulation of the equations (1). Let us introduce a special denotation of components of antisymmetric tensor \( F^{\mu \nu} \) that is in a formal accordance with the one used in electromagnetism

\[
F_{ij} = -\varepsilon_{ijm} B_m, \quad F_{0j} = -E_j, \quad i,j = 1, 2, 3.
\]

In a fixed inertial frame, the equations (1) then acquire the form

\[
B = \text{rot} A, \quad A_0 = -m^{-2} \text{div} E, \quad \frac{\partial A}{\partial t} = -E - \text{grad} A_0
\]

\[
\frac{\partial E}{\partial t} = m^2 A - \text{rot} B.
\]  

An elimination of \( B \) and \( A_0 \) gives

\[
\frac{\partial A}{\partial t} = -E + \frac{\nabla^2 E}{2m^2} + \frac{\dot{q}}{2m^2} E, \quad \frac{\partial E}{\partial t} = m^2 A - \frac{\nabla^2 A}{2} + \frac{\dot{q}}{2} A,
\]

where we abbreviated

\[
\dot{q} = 2 \text{grad} \text{div} - \nabla^2.
\]

Now, it is easy to arrive at the Schrödinger-like matrix equation

\[
i \frac{\partial \Psi}{\partial t} = H \Psi,
\]

where we defined

\[
\Psi^T = (m A, i E)^T.
\]  

The equation (16) prescribes the time-evolution of our system in its first-quantized form. In the absence of any external forces the quantum Hamiltonian \( H \) of the vector boson is a matrix containing differential operators as its elements

\[
H = \begin{pmatrix}
0 & -m + \frac{\text{grad} \text{div}}{m} \\
-m + \frac{\nabla^2}{m} & -m + \frac{\text{grad} \text{div}}{m}
\end{pmatrix}.
\]

\[3\]The Hamiltonian \( H \) can be transformed into a rather standard expression for vector boson Hamiltonian [2] wide-spread in the literature (see cf. [3], [6]). The relation is

\[H_{ST} = AH A^{-1}, \quad A = i \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\]
In coordinate representation this manifestly non-Hermitian $6 \times 6$–dimensional matrix has the transparent operator structure,

$$H = \frac{1}{m} \begin{pmatrix} 0 & 0 & 0 & \partial_1^2 - m^2 & \partial_1 \partial_2 & \partial_1 \partial_3 \\ 0 & 0 & 0 & \partial_2 \partial_1 & \partial_2^2 - m^2 & \partial_2 \partial_3 \\ 0 & 0 & 0 & \partial_3 \partial_1 & \partial_3 \partial_2 & \partial_3^2 - m^2 \\ -\partial_1^2 - \omega^2 & -\partial_1 \partial_2 & -\partial_1 \partial_3 & 0 & 0 & 0 \\ -\partial_2 \partial_1 & -\partial_2^2 - \omega^2 & -\partial_2 \partial_3 & 0 & 0 & 0 \\ -\partial_3 \partial_1 & -\partial_3 \partial_2 & -\partial_3^2 - \omega^2 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

This is a non-Hermitian, $\mathcal P$–pseudo-Hermitian operator generating the free motion,

$$H^\dagger = \mathcal P H \mathcal P^{-1} \neq H, \quad \mathcal P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{20}$$

where $I$ is $3 \times 3$ unit matrix.

Quite often, the system is considered confined to a finite box of side length $L$. For the particular periodic boundary conditions imposed at its walls one then arrives at the “quantized” energies on the mass shell,

$$p_0^\pm, n = \pm \omega^\pm_n = \pm \sqrt{p_n^2 + m^2}, \quad p_n = \frac{2\pi}{L} n$$

with $n$ being a vector of integer components. Vice versa, the continuity of the energies can be restored in the $L \to \infty$ limit whenever necessary. Considering finite $L$ until section 4, let us omit the indexes $n$.

To distinguish spin characteristics of wave functions, we introduce another observable which commutes with $H$ in momentum representation

$$\Lambda_{i,j}(p) = \Lambda_{i+3,j+3}(p) = -\frac{i|p|\varepsilon_{ijk}p_k}{p^2} = \frac{Sp}{|p|},$$

$$\Lambda_{i,j+3}(p) = \Lambda_{i+3,j}(p) = 0, \quad i,j = 1,2,3. \tag{21}$$

This is a block diagonal operator which examines the spin projection into momentum direction, i.e. it measures the helicity of our vector boson. It is easy to verify that the latter operator is also pseudo-Hermitian with respect to $\mathcal P$,

$$\Lambda^\dagger = \mathcal P \Lambda \mathcal P^{-1}.$$

The six independent eigenvectors of both $(18)$ and $(21)$ may be constructed most easily in terms of the above-introduced functions $(11)$ in momentum representation,

$$\Psi_{\pm,h}(p) = \begin{pmatrix} m u_{\pm,h}(p) \\ \mp \omega u_{\pm,h}(p) + p u_{\pm,h}^0(p) \end{pmatrix}, \quad h = 1,-1,0. \tag{22}$$

We arrive at the complete right-action Schrödinger equation for the fixed-helicity states,

$$H(p)\Psi_{\pm,h}(p) = \pm \omega \Psi_{\pm,h}(p), \quad \Lambda(p)\Psi_{\pm,h}(p) = h \Psi_{\pm,h}(p), \quad .$$

Employing pseudo-Hermiticity of the operators, we can write the relations for their adjoint ones immediately,

$$H^\dagger(p)\mathcal P \Psi_{\pm,h}(p) = \pm \omega \mathcal P \Psi_{\pm,h}(p), \quad \Lambda^\dagger(p)\mathcal P \Psi_{\pm,h}(p) = h \mathcal P \Psi_{\pm,h}(p), \quad h = \pm 1, 0. \tag{23}$$
It is not difficult to show that the wave functions $\Psi_{\pm,h}(p)$ fulfill the relation

$$
\langle \Psi_{\epsilon,h}(p), \mathcal{P}\Psi_{\epsilon',h'}(p') \rangle = \delta_{\epsilon\epsilon'} \delta_{h h'}
$$

In a way described thoroughly in ref. [26], the set of vectors $\{\Psi_{\pm,h}(p), \mathcal{P}\Psi_{\pm,h}(p)\}$ forms a biorthonormal basis of our system in the momentum representation.

### 3 Construction of physical metric operator

Let us, for a while, neglect the particular structure of the operators $H$ and $\Lambda$ and consider them just as non-Hermitian operators admitting spectral decomposition in terms of biorthonormal basis [28] $\{|n,h\rangle, |n,h\rangle\rangle\rangle |\langle\langle n,h|\langle m,j\rangle = \delta_{mn}\delta_{hj}\}^\dagger$

$$
H = \sum_n E_n|n,h\rangle \langle n,h| , \quad \Lambda = \sum_n h|n,h\rangle \langle n,h| \tag{24}
$$

The most general operator $\mathcal{P}$ with respect to which $H$ and $\Lambda$ are pseudo-hermitian [20] ($H^\dagger = \mathcal{P}^{-1} H \mathcal{P}$, $\Lambda^\dagger = \mathcal{P}^{-1} \Lambda \mathcal{P}$) is

$$
\mathcal{P} = \sum_{n,h} |n,h\rangle \rangle \pi_{n,h} |\langle\langle n,h| , \quad \pi_{n,h} \in (-\infty, \infty). \tag{25}
$$

Clearly, family of the operators [26] contains positive-definite members. These are just the wanted candidates for a physical metric operator. They are of the form

$$
\Theta = \sum_{n,h} |n,h\rangle \rangle \theta(n,h) |\langle\langle n,h| , \quad \theta(n,h) > 0. \tag{26}
$$

In the language of ref. [10] the (quasi-)Hermiticity of $H$ and $\Lambda$ can be recovered by the following redefinition of the scalar product,

$$
\langle \psi|\phi\rangle_{\text{phys}} \equiv \langle \psi|\Theta|\phi\rangle.
$$

It holds $\langle \psi|H\phi\rangle_{\text{phys}} = \langle H\psi|\phi\rangle_{\text{phys}}$, $\langle \psi|\Lambda\phi\rangle_{\text{phys}} = \langle \Lambda\psi|\phi\rangle_{\text{phys}}$. Further examination may proceed in framework of theory of Hermitian operators where Hilbert space is generated by rather non-trivial metric $\Theta$ and is spanned by $\{|n,h\rangle\}$.

Refreshing the particular properties of $H$, $\Lambda$ and $\mathcal{P}$ again, we can rewrite [26] for our system immediately. Indeed, picking up [20], [22] and the vectors in [23] we get

$$
\Theta_L(p) = \sum_h \left[ \theta(\omega, h) \mathcal{P}\Psi_{+,h}(p) (\mathcal{P}\Psi_{+,h}(p))^\dagger + \theta(-\omega, h) \mathcal{P}\Psi_{-,h}(p) (\mathcal{P}\Psi_{-,h}(p))^\dagger \right],
$$

$$
\theta(\pm, h) > 0, \quad h = \pm 1, 0. \tag{27}
$$

Performing quite straightforward calculations, we arrive to the following expression

$$
\Theta_L(p) = \begin{pmatrix}
\Theta(p)_{11} & \Theta(p)_{12} \\
\Theta(p)_{12} & \Theta(p)_{22}
\end{pmatrix}. \tag{28}
$$
Considering the explicit form of the involved quantities (7), (8), (11) and (22), it could be expected that the resulting matrix-operators $\Theta_{ij}$ will be rather complicated. However, these revealed to be of an unexpectedly simple structure

\[
\Theta(p)_{11} = s_{+,1}J^T(p) + s_{+,0}K(p) + \frac{m^2}{\omega^2} s_{+,0}K(p) \\
\Theta(p)_{12} = m \omega \left[ s_{+,1}J^T(p) + s_{+,0}K(p) \right] \\
\Theta(p)_{22} = \frac{m^2}{\omega^2} \left[ s_{+,1}J^T(p) + s_{+,0}K(p) \right],
\]

\[
s_{\pm,h} = \pm \theta(\omega, h) + \theta(-\omega, h),
\]

in terms of $3 \times 3$ Hermitian matrices $J$ and $K$ with elements

\[
J(p)_{ij} = \frac{p^2 \delta_{ij} - p_i p_j + \imath \epsilon_{ijk} p_k p_j}{8m^2 p^2}, \quad K(p)_{ij} = \frac{p_i p_j}{4m^2 p^2}, \quad i < j, \quad j = 1, 2, 3.
\]

Moreover, the matrices $J$ and $K$ can be expressed as polynomials of the helicity operator

\[
J = \frac{\hat{h}^2 - \hat{h}}{8m^2}, \quad K = \frac{I - \hat{h}^2}{4m^2}.
\]

Then, the metric components (29) reveal to be functions of energy and helicity operators. The expression (29) represents the candidate for proper metric operator.

However, it suffers from a strong ambiguity represented by the six unknown functions $\theta(\pm \omega, h)$. In the following section, we will impose an additional physically motivated requirement which will reduce the unwanted degrees of freedom.

## 4 Lorentz invariance

Let us send the length $L$, determining the volume in which the particle is confined, in infinity. Then the problem appears how to describe the system from various inertial frames of reference. To resolve the puzzle, it is crucial to find an appropriate representation of Poincare group on a space spanned by wavefunctions.

It is an immediate physical requirement that the scalar product should be independent on the reference frame. Writing this condition explicitly, we have

\[
(\Psi'_1, \Theta \Psi'_2) = (\Psi_1, \Theta \Psi_2),
\]

where $\Psi'_j$ are transformed wavefunctions. In standard case where $\Theta = 1$, this condition is ensured by unitary representation of Poincare group on the Hilbert space of states.

Let us consider infinitesimal transformations of the wavefunctions

\[
\Psi'(x, t) = (1 + \imath \epsilon M)\Psi(x, t), \quad \epsilon \sim 0
\]

where $M$ is a generator of Poincare algebra in an appropriate representation. Then the invariance of scalar product leads to the operator equation

\[
M^\dagger \Theta = \Theta M.
\]
It can be expected that this will reduce the free parameters in the metric operator additionally.

To handle with the relation (34), we need to find an explicit form of generators of Poincare group. As the transformation properties of wavefunctions (13) are known since Wigner [29], it is possible to derive the transformation law for (17) directly. Instead of performing this straightforward but rather lengthy computation, it is sufficient to refer the literature. In 1958, Foldy encountered the problem for relativistic systems with various spins [4] (see also [30]). Let us follow the existing procedure by introducing a transformation

\[ B = (8 \omega)^{-1/2} \begin{pmatrix} \omega + m + \frac{q}{\omega + m} & -\omega - m + \frac{q}{\omega + m} \\ -\omega - m - \frac{q}{\omega + m} & \omega + m + \frac{q}{\omega + m} \end{pmatrix}, \chi = B \Psi, \quad (35) \]

where \( \Psi \) is six component wave function (17). The transformation brings the Hamiltonian \( H \) in (18) into the diagonal form

\[ H_F = BHB^{-1} = \omega \beta, \quad (36) \]

while the operator \( \Lambda \) remains unchanged under this transformation. Foldy found that the generators of infinitesimal transformations of Poincare group are

\[ H_F = \beta \omega, \quad P = I p, \quad K = \frac{\beta}{2} (x \omega + \omega x) - \beta \frac{S \times p}{m + \omega} - I t p, \quad J = I (x \times p + S), \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad (37) \]

These constitute a direct sum of two irreducible representations of Poincare algebra\(^5\). They are hermitian with respect to the scalar product

\[ (\chi_1, \chi_2) = \int \chi_1^\dagger \chi_2 d^3x. \quad (38) \]

The representation (37) is called Foldy’s canonical representation and appears in many works considering relativistic equations of free particles [31], [32].

As we require \( B \) to be a unitary mapping the new metric operator reads \( \tilde{\Theta} = (B^{-1})^\dagger \Theta B^{-1} \) and is block diagonal\(^6\)

\[ \tilde{\Theta} = \begin{pmatrix} \tilde{\Theta}_+ & 0 \\ 0 & \tilde{\Theta}_- \end{pmatrix}. \]

The diagonal sub-matrices-operators can be written in a compact form

\[ \tilde{\Theta}_\pm = \frac{1}{4m^2 \omega} \left[ \theta(\pm \omega, 1) + \theta(\pm \omega, -1) - \frac{2 \omega^2}{m^2} \theta(\pm \omega, 0) \right] \hat{h}^2 \]

\(^4\)It has been found by Case [6] as a generalization of Foldy-Wouthuysen transformation to spin one.

\(^5\)For explicit form of discrete symmetries see [4].

\(^6\)The initial indefinite metric \( P \) appearing in (20) revealed to be proportional to a proper space inversion

\[ (B^{-1})^\dagger PB^{-1} = \frac{1}{m} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \]

discrete transformation within the Poincare group (see [3]).
\[
+ (\theta(\pm \omega, 1) - \theta(\pm \omega, -1)) \hbar + \frac{2 \omega^2}{m^2} \theta(\pm \omega, 0) I \right].
\] (39)

Now, let us return to the Lorentz invariance condition (34). The generators (37) are hermitian with respect to (38). Thus, we can rewrite the condition as

\[ M \tilde{\Theta} = \tilde{\Theta} M \implies [M, \tilde{\Theta}] = 0. \]

It is a well known result of representation theory that the operator commuting with all generators of an irreducible representation is proportional to identity on the representation space. In our case, we deal with reducible representation of the Poincare algebra. Consequently, not the metric operator \( \tilde{\Theta} \) but its particular components \( \Theta_{\pm} \) have to be multiples of the identity operator. Considering their explicit forms (39), we arrive to the set of relations for unknown functions \( \theta(\pm \omega, j) \)

\[
\begin{align*}
\theta(\pm \omega, 1) + \theta(\pm \omega, -1) &= \frac{2 \omega^2}{m^2} \theta(\pm \omega, 0) \\
\theta(\pm \omega, 1) &= \theta(\pm \omega, -1) \\
\omega \theta(\pm \omega, 0) &\sim 1.
\end{align*}
\] (40)

Their solution restrict ambiguity of the metric operator drastically since we obtain

\[
\theta(\pm \omega, 0) = \frac{\alpha_+ m}{\omega}, \quad \theta(\pm \omega, j) = \frac{\alpha_+ \omega}{m}, \quad j = \pm 1,
\] (41)

where \( \alpha_{\pm} \) are dimensionless positive real numbers.

The evolution equation in Foldy’s canonical form is \( i \partial_t \chi = H_f \chi \). It is homogenous, i.e. its two solutions which differ by multiplication constant are physically equivalent. Thus, one of the parameters \( \alpha_{\pm} \) can be immersed into a wave function. Concluding, the most general form of the positive definite, Lorentz invariant metric operator for free spin-one particle in Foldy’s canonical representation is \(^7\)

\[
\tilde{\Theta} = \frac{1}{m^3} \begin{pmatrix} I & 0 \\ 0 & \gamma I \end{pmatrix}, \quad \gamma = \frac{\alpha_-}{\alpha_+}.
\] (42)

Finally, let us leave the canonical formalism and return to the space of solutions (13) of Proca equation (1). We can state the following

**Theorem:** Positive-definite and Lorentz invariant scalar products on the space of solutions of Proca equation (1) form a one-parameter set. Explicitly, the scalar product of two solutions of Proca equation \( A_j = (A^0_j, A_j)^T, \ j = 1, 2 \) is

\[
\langle A_1, A_2 \rangle \equiv -\frac{i \alpha}{2m} \left[ \langle A_1, \dot{A}_2 + \text{grad} A^0_2 \rangle - \langle \dot{A}_1 + \text{grad} A^0_1, A_2 \rangle \right]
\]

\(^7\)It can be transformed, up to a multiplicative constant, into a diagonal matrix.

\[
(D^{-1})^T \tilde{\Theta} D^{-1} = \frac{1}{m^3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix}.
\]

The remaining free parameter is eliminated from the metric. However, probabilistic interpretation remains non-unique as the ambiguity moves to the definition of wavefunction.
\[
+ \left( A_1, \left[ \frac{m}{\omega} I + \frac{(pS)^2}{2m^2\omega} \right] A_2 \right) + \left( \dot{A}_1 + \text{grad} A_0^0, \left[ \frac{\dot{\omega}}{2m^3 I} - \frac{(pS)^2}{2m^3\omega} \right] \left( \dot{A}_2 + \text{grad} A_0^0 \right) \right)
\] (43)

where \( \alpha \) is in \((-1,1)\), \( (.,.) \) denotes standard scalar product and \( \dot{\omega} = \int d\omega |p| \langle p \rangle \).

**Proof:**

Let us recall the definition of \( \Psi = (mA_i, -i\partial_t A - i \text{grad} A^0)^T \) and get the reverse transformation of \( \tilde{\Theta} = \Theta \equiv \left( \Theta_1, \Theta_2 \right) \). Inserting these quantities into

\[
((A_1, A_2)) \equiv (\Psi_1, \Theta_2).
\]

and denoting \( \alpha \equiv \alpha_+ - \alpha_- \), we arrive to the expression (13).

## 5 Discussion

In the present article, we considered a free massive vector boson. After a short introduction into the its standard quantum description, we re-derived its description within multicomponent Hamiltonian formalism. Consequently, we summarized main concepts and results concerning pseudo-Hermitian diagonalizable operators with discrete spectra and applied them to the quantum system. As the dynamics of free massive spin-one particle can be rewritten via apparently non-Hermitian operators (18), we aimed to apply framework of pseudo-Hermitian quantum mechanics to this quantum relativistic system. Particularly, we constructed a positive definite operator that plays the role of the positive definite metric on the space of solutions of evolution equation (16).

Its construction proceeded in two steps. First, we considered the particle to be confined in a box with finite length \( L \) of side. Then, the resulting positive-definite metric operator acquired the form (29). It revealed to be a function of energy and helicity operators. It had six degrees of freedom represented by unknown functions \( \theta(\pm j, j) \), \( j = \pm 1, 0 \).

Subsequently, we made a transition to the open space by sending \( L \) to infinity. We discussed Lorentz invariance of the metric operator (29). This physically motivated requirement reduced the ambiguity of the metric. We found, that the Lorentz invariant positive-definite scalar products (12) create a one parameter family. We found an explicit form of scalar product for solutions of Proca equation (43) which is in a close analogy to its spin-zero counterpart derived by Mostafazadeh (26). In this way, we contributed to the probabilistic interpretation of the Proca wave functions.

At the very end, let us still make a few remarks. Attempts to construct a proper probabilistic interpretation within the relativistic quantum mechanics are not new. To our best knowledge, the most explicit and the oldest work in this direction was done by Mathews (32). In comparison with the approach of the present article and (26), he enquired the problem from the very opposite side. He started the construction with a known representation of Poincare algebra for free particles. He sought an operator which would
generate Lorentz invariant scalar product. Although not required from the beginning, the resulting operator was positive-definite. Pseudo-Hermitian quantum mechanics enables construction of the metric operator in relativistic quantum systems where the generators of Poincare group are not given explicitly. In this sense, our approach can be seen as a more general.

Acknowledgement

Participation of VJ was partially supported by the project LC06002.

References

[1] Klein O 1926 Z. Physik 37 895;
Fock V 1926 Z. Physik 38 242 and 39 226;
Gordon W 1926 Z. Physik 40 117;
Dirac P A M 1928 Proc. Roy. Soc. (London) A117 610;
Proca A 1936 J. Physique Rad. 7 347;
Dirac P A M 1942 Proc. Roy. Soc. London A 180 1;
Pauli W 1943 Rev. Mod. Phys. 15 175

[2] Lifshitz E M, Pitaevskii L P, Berestetskii V B, Quantum Electrodynamics, Oxford 1996.

[3] Taketani M and Sakata S 1940 Proc. Phys.-Math. Soc. (Japan) 22 757;
Tamm I 1940 Compt. rend. acad. sci (U.R.S.S) 29 551

[4] Foldy L L 1956 Phys.Rev. 102 568

[5] Feshbach H and Villars F 1958 Rev. Mod. Phys. 30 24

[6] Case K M 1954 Phys. Rev. 95 1323

[7] Bender C M and Boettcher B 1998 Phys. Rev. Lett. 80 4243

[8] Hatano N and Nelson D R 1996 Phys. Rev. Lett. 77 570;
Hatano N and Nelson D R 1997 Phys. Rev. B 56 8651;
Bender C M and Milton K A 1997 Phys. Rev. D 55 R3255;
Bender C M, Boettcher S and Meisinger P N 1999 J. Math. Phys. 40 2201;
Bender C M, Brody D C and Jones H F 2002 Phys. Rev. Lett. 89 0270401;
G. Scolariici, L. Solombrino, J. Math. Phys. 44 (2003) 4450;
Bender C M and Jones H F 2004 Phys.Lett. A 328 102.
Bender C M 2004 Czechosl. J. Phys. 54 1027;
Jones H F 2004 Czechosl. J. Phys. 54 1107;
Mostafazadeh A 2004 Czechosl. J. Phys. 54 1125
[9] Caliceti E, Graffi S and Maioli M 1980 Commun. Math. Phys. 75 51;
    Caliceti E and Graffi S 2004 J. Phys. A 37 2239;
    Fernández F M, Guardiola R, Ros J and Znojil M 1998 J. Phys. A: Math. Gen. 31
    10105;
    Bíla H 2004 Czech. J. Phys. 54 1049

[10] Scholtz F G, Geyer H B and Hahne F J W 1992 Ann. Phys. (NY) 213 74

[11] Buslaev V and Grecchi V 1993 J. Phys. A: Math. Gen. 26 5541;
    Alvarez G 1995 J. Phys. A: Math. Gen. 27 4589

[12] Dorey P, Dunning C and Tateo R 2001 J. Phys. A: Math. Gen. 34 5679;
    Shin K C 2001 J. Math. Phys. 42 2513

[13] Znojil M 1999 Phys. Lett. A 259 220

[14] Andrianov A A, Cannata F, Dedonder J-P and Ioffe M V 1999 Int. J. Mod. Phys. A
    14 2675;
    Lérai G and Znojil M 2000 J. Phys. A: Math. Gen. 33 7165;
    Znojil M 2001 LANL arXiv math-ph/0104012 reprinted 2004 Rendiconti del Circ.
    Mat. di Palermo, Ser. II, Suppl. 72 211;
    Bagchi B, Quesne C and Znojil M 2001 Mod. Phys. Lett. A 16 2047;
    Bagchi B, Quesne C and Roychoudhury R 2005 J. Phys. A 38 L647

[15] Heiss W D and Harney H L 2001 Eur. Phys. J. D 17 149;
    Heiss W D 2005 Czech. J. Phys. 55 1107;
    Günther U and Stefani F 2005 Czech. J. Phys. 55 1099

[16] Langer H and Tretter C 2004 Czech. J. Phys. 54 1113

[17] Znojil M 2001 Phys. Lett. A. 285 7;
    Jakubský V and Znojil M 2004 Czech. J. Phys. 54 1101

[18] Ramirez A and Mielnik B 2003 Rev. Mex. Fis. 49S2 130

[19] Albeverio S, Fei S-M and Kurasov P 2002 Lett. Math. Phys. 59 227;
    Znojil M and Jakubský V 2005 J. Phys. A: Math. Gen. 38 5041;
    Jakubský V and Znojil M (2005) J. Phys. 55 1113;
    Fei S-M 2005 Czech. J. Phys. 55 1085

[20] Mostafazadeh A 2002 J. Math. Phys. 43 205, 2814, 3944 and 6343

[21] Bagchi B, Mallik S and Quesne C 2002 Mod. Phys. Lett. A 17 1651;
    Mostafazadeh A 2003 LANL arXiv quant-ph/0310164
    Kleefeld F 2003 AIP conf. proc. 660 325;
[22] Mostafazadeh A and Batal A 2004 J. Phys. A: Math. Gen. 37 11645
[23] Japaridze, G. S. 2002 J.Phys. A: Math. Gen. 35 1709;
    Geyer H B, Scholtz F G and Snyman I 2004 Czech. J. Phys. 54 1069;
    Geyer H B and Snyman I 2005 Czech. J. Phys. 55 1091
    Heiss W D 2005 Czech. J. Phys. 55 1107
    Kretschmer R and Szymanowski L 2004 Phys. Lett. A 325 112
[24] F. G. Scholz, H. B. Geyer, Phys. Lett. B in print
[25] Mostafazadeh A 2005 J. Phys. A: Math. Gen. 38 6557
[26] Mostafazadeh A 2003 Class. Quant.Grav. 20 155 and arXiv quant-ph/0307059, unpublished;
    Mostafazadeh A and Zamani F 2003 arXiv quant-ph/0312078, unpublished;
    Mostafazadeh A 2004 Ann. Phys. (NY) 309 1;
    Mostafazadeh A 2004 Czechosl. J. Phys. 54 93;
    Mostafazadeh A 2003 J. Math.Phys. 44 974
[27] Znojil M 2004 Czechosl. J. Phys. 54 151;
    Znojil M 2005 Czechosl. J. Phys. 55 1187
[28] Znojil M 2006 J. Phys. A: Math. Gen. 39 441
[29] Wigner E P 1939 Ann. Math. 40 149
[30] Fushchich W I and Nikitin A G 1994 Symmetries of Equations of Quantum Mechanics
    (New York: Allerton Press)
[31] Fronsdal 1958 Phys. Rev. 113 1367; Weaver D I, Hammer C I, Good R H, 1964 Phys. Rev. 135 B241
[32] Mathews P M 1966 Phys. Rev. 143 985