The many mathematical faces of Mermin’s proof of the Kochen-Specker theorem

Leon Loveridge\textsuperscript{a,*} and Raouf Dridi\textsuperscript{b,†}

\textsuperscript{a}. Department of Computer Science, University of Oxford, Wolfson Building, Parks Rd, Oxford, UK OX1 3QD
\textsuperscript{b}. Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada

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Abstract

Mermin’s simple “pentagram” proof of the Kochen-Specker theorem is examined from various perspectives. We emphasise the many mathematical structures intimately related to Kochen-Specker proofs, ranging through functional analysis, sheaf theory and topos theory, Coxeter groups and algebraic geometry. Some novel results are presented along the way.

1 Introduction

The assortment of results collectively referred to as the Kochen-Specker theorem, discovered by Bell \cite{1} and Kochen and Specker \cite{2} in the 1960s, still occupies a prominent position in the foundations of quantum mechanics. The physical and philosophical content of these theorems expound essential differences between classical and quantum mechanics, \textit{grosso modo} informing us that the mathematical structure of quantum theory is incompatible with a realist interpretation of the theory in the spirit of classical statistical mechanics.

There now exists a vast literature on many aspects of the Kochen-Specker theorem. On the mathematical side this includes “small”/efficient proofs (e.g., \cite{3,4} for proofs in $\mathbb{R}^3$, the record there for the smallest number of rays being held by Conway and Kochen—see \cite{5}), general graph theoretic treatments \cite{6-8}, generalisations to unsharp observables (e.g., \cite{9-11}), preparation and transformation contextuality \cite{9}, operator algebras \cite{12}, (co-)sheaf and (co-)presheaf \cite{13-18} approaches and geometry and cohomology \cite{19,20}, “logical” Bell inequalities \cite{21}, topos theory \cite{17,22,23}, and non-contextual inequalities \cite{6,24} akin in style to the Bell inequality \cite{25,26}, whose violation detects contextuality, to name but a few. Contextuality has also recently been pinpointed as a resource for quantum computation \cite{27,29}.

In this paper we use Mermin’s simple “pentagram” proof \cite{30} of the Kochen-Specker theorem in Hilbert space dimension 8, along with a corresponding state-dependent version, to emphasise the many mathematical structures intimately related to Kochen-Specker proofs. This functions as an introduction to the subject of Kochen-Specker proofs as well as a survey of the many mathematical viewpoints that now exist on the Kochen-Specker theorem.

*leon.loveridge@cs.ox.ac.uk
†dridi.raouf@gmail.com
After providing some background and motivating the problem that Bell and Kochen and Specker addressed, we give Mermin’s state-independent proof based on the non-existence of a valuation for 10-qubit observables, as well as the corresponding state-dependent version. This gives rise to colouring proofs, based on the impossibility of assigning values to projections/rays arising from the spectral resolution of the given observables. We mention another general form of Kochen-Specker scenario called “all-versus-nothing” arguments of which the Mermin system is an example. We also present a result due to Clifton showing that Mermin’s proof may be adapted to the case of position-momentum contextuality for three degrees of freedom. An algebraic perspective due to Döring is then presented, before a discussion of Coxeter groups, $E_8$ in particular, which is seen to naturally arise from the Mermin system.

Three sheaf-theoretic formulations are then given, based on the work of Isham and collaborators and focusing on the so-called spectral presheaf (e.g., [22]), followed by a covariant approach of Heunen, Landsman, and Spitters, and finally an operational approach initiated by Abramsky and Brandenburger. A novel algebraic geometric framework is then provided, followed by a short section providing some conclusions and avenues for further investigation.

2 Background on the Kochen-Specker Theorem

A basic question in the foundations of quantum mechanics is whether the probabilistic structure of the orthodox formalism is a reflection of a fundamental property of the quantum world, or comes about as an effective description arising from the theorist’s lack of detail about the inner workings of quantum phenomena.

If it is the latter, it is compelling to believe that quantum mechanical observables should have values prior to, and independently of, measurement. Although these values cannot, in general, be realised by a quantum state, they should, in principle, be realised by a “microstate”. In Newtonian physics, an observable quantity is represented by a (Borel/continuous/smooth) function $f : T^*\mathcal{M} \to \mathbb{R}$ on the phase space $T^*\mathcal{M}$. The value $\nu_a(f)$ of $f$ in a state $s \in T^*\mathcal{M}$ is given by $\nu_a(f) = f(s)$. Therefore, the possible values of an observable $f$ are the spectral values of $f$ in the sense of $C^*$-algebras. Moreover, given a function $h : \mathbb{R} \to \mathbb{R}$, the observable $h(f) := h \circ f : T^*\mathcal{M} \to \mathbb{R}$ has value $\nu_a(h \circ f) = h(f(s))$ in the state $s \in T^*\mathcal{M}$.

It is then natural to look for a “hidden state” underlying the quantum description that behaves analogously to a state in classical physics, and specifically that values quantum observables in an appropriate way. With $\mathcal{H}$ a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ the collection of bounded linear operators (to be viewed as either a von Neumann or $C^*$-algebra), the hidden state would then be given as a valuation obeying analogous conditions to the classical case, namely $\nu : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ such that for any self-adjoint element $a \in \mathcal{L}(\mathcal{H})$ and function $f : \mathbb{R} \to \mathbb{R}$,

1. $\nu(a) \in \text{spec}(a)$ (SPEC),
2. $\nu(f(a)) = f(\nu(a))$ (FUNC),

where condition (2) (FUNC) is understood in the sense of the appropriate functional calculus. An immediate consequence of FUNC is that $\nu$ is linear and multiplicative on commuting operators. Alternatively, one can say that $\nu$ is a multiplicative linear functional on $C^*(a)$ (with the natural extension to complex combinations), and is therefore a character of $C^*(a)$.
In a finite dimensional Hilbert space, the collection of all such characters are in one-to-one correspondence with the set of non-zero eigenvalues of \( a \) and, therefore, characters are generic as valuations.

We also note that \( \nu \) can be extended naturally to the whole of \( \mathcal{L}(\mathcal{H}) \) and FUNC to complex-valued functions of the appropriate type.

### 3 Mermin’s Pentagram: Combinatoric Proofs

#### 3.1 Colouring and parity proofs

Mermin provided a simple construction for ruling out a valuation \( \nu \) on a collection of ten self-adjoint operators in \( \mathbb{C}^8 \), arranged in five contexts, given here as collections of commuting observables. With \( \sigma_{x,z} \in M_2(\mathbb{C}) \) denoting the Pauli operators and writing \( X_1 := \sigma_x \otimes 1 \otimes 1 \), etc., applying FUNC (and for the final equality, also SPEC) yields

\[
\begin{align*}
\nu(X_1X_2X_3) &= \nu(X_1)\nu(X_2)\nu(X_3), \\
\nu(X_1Z_2Z_3) &= \nu(X_1)\nu(Z_2)\nu(Z_3), \\
\nu(Z_1X_2Z_3) &= \nu(Z_1)\nu(X_2)\nu(Z_3), \\
\nu(Z_1Z_2X_3) &= \nu(Z_1)\nu(Z_2)\nu(X_3), \\
\nu(X_1X_2X_3)\nu(X_1Z_2Z_3)\nu(Z_1X_2Z_3)\nu(Z_1Z_2X_3) &= -1; \\
\end{align*}
\]

for the final equation each observable appearing as an argument in \( \nu \) commutes with all others inside a \( \nu(\cdot) \), and we exploit the anticommutativity of \( X_i \) and \( Z_i \). On the other hand, taking products of the left-hand sides and right-hand sides of (1)–(4) gives

\[
\nu(X_1X_2X_3)\nu(X_1Z_2Z_3)\nu(Z_1X_2Z_3)\nu(Z_1Z_2X_3) = 1
\]

by SPEC, and therefore there exists no context-independent valuation \( \nu \) compatible with the above constraints, yielding a combinatoric obstruction to a hidden state acting as a valuation. This contradiction constitutes a so-called parity proof of the Kochen-Specker theorem.

Colouring proofs can also be extracted from the Mermin system in more than one way; Kerhaghan and Peres [32], Waegell and Aravind [33], and Toh [34] have each given an example. Kernaghan and Peres note that each context generates 8 mutually orthogonal vectors, given as the simultaneous eigenvectors of the (commuting) observables appearing in the context. Following Waegell and Aravind [33], we may form the rank-one projections \( P_{[v_i]} \equiv P_i \), and given that for the first context, for example, \( \sum_{i=1}^{8} P_i = 1 \), we have \( \sum_{i=1}^{8} \nu(P_i) = 1 \). Therefore, precisely one \( P_i \) must have value \( \nu(P_i) = 1 \), with all others having a value of zero. This amounts to colouring precisely one vector/ray out of a complete orthogonal set green, for instance, and all of the others red. The assumption of non-contextuality manifests as the requirement that the given colour must be independent of the context with which the projection is associated (since the contexts have non-zero intersections). It is found that such a colouring is impossible, yielding another combinatoric proof.

To conclude this subsection, we also mention state-dependent versions of the Kochen-Specker theorem based on the Mermin system. Equation (5) is consistent with \( \nu(X_1X_2X_3) = \)
1, \( \nu(X_1 Z_2 Z_3) = -1 \), \( \nu(Z_1 X_2 Z_3) = -1 \), and \( \nu(Z_1 Z_2 X_3) = -1 \), yielding the equations

\[
\begin{align*}
\nu(X_1) \nu(X_2) \nu(X_3) &= 1, \\
\nu(X_1) \nu(Z_2) \nu(Z_3) &= -1, \\
\nu(Z_1) \nu(X_2) \nu(Z_3) &= -1, \\
\nu(Z_1) \nu(Z_2) \nu(X_3) &= -1.
\end{align*}
\]

Again, by taking products of the left-hand side and the right-hand side, a contradiction arises. This is clearly weaker than the contradiction implied by equations (11)–(15). The state-
dependence comes from observing that the numbers on the right-hand side are eigenvalues of the products \( X_1 X_2 X_3 \), \( X_1 Z_2 Z_3 \), etc., in the Greenberger-Horne-Zeilinger (GHZ) state

\[
\Psi = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)
\]

Thus the contradiction may be understood as the impossibility of the given observables having values compatible with the quantum predictions in a specific state, in this case the GHZ state \( |\Psi\rangle \).

The Hasse diagram illustrating the state-dependent Mermin poset \( P \) is as follows:

\[
\begin{align*}
\{X_1, X_2, X_3\} & \quad \{X_1, Z_2, Z_3\} & \quad \{Z_1, X_2, Z_3\} & \quad \{Z_1, Z_2, X_3\} \\
X_1 & \quad X_2 & \quad X_3 & \quad Z_3 & \quad Z_2 & \quad Z_1
\end{align*}
\]

\[\text{(11)}\]

3.2 “All-versus-nothing” arguments

Mermin [30] dubbed the contradiction obtained from the GHZ state (i.e., (7)–(11)) an “all-
versus-nothing” (AvN) argument. Similar strong forms of contextuality (to be discussed
further in Section 6.3) have appeared in the literature, and have been generalised in [20],

where a general form of AvN argument is provided based on \( mod\ -\ n \) linear equations and is

independent of the quantum formalism.

A rich source of examples of quantum AvN arguments is provided in the setting of sta-
biliser quantum mechanics. \( P_n \) denotes the \( n \)-qubit Pauli group, i.e., products of \( P_1 = \langle jX, jY, jZ, j\rangle \), with \( j \in \{\pm 1, \pm i\} \). Then, an AvN argument, in direct analogy with the

GHZ case already considered, is a system of local equations of parity type which have no

global solution.

The main theorem on this subject in [20] shows that any “AvN triple” gives rise to an

AvN argument.

**Definition 3.1** An AvN triple \( \langle e, f, g \rangle \) in \( P_n \) is a triple \( \langle e, f, g \rangle \) of elements of \( P_n \) with global

phases \( +1 \) which pairwise commute, and which satisfy the following conditions:

1. For each \( i \in \{1, \ldots, n\} \) at least two of \( e_i, f_i, g_i \) are equal.

2. The number of \( i \) such that \( e_i = g_i = f_i \), all distinct from \( \mathbb{1} \), is odd.

Then,
Theorem 3.2. Let $S$ be the subgroup of $\mathcal{P}_n$ generated by an AvN triple, and $V_S$ the subspace stabilised by $S$. For every state $\Psi$ in $V_S$, the empirical model (a precise definition is given in (L.3)) realised by $\Psi$ under the Pauli measurements admits an all-versus-nothing argument.

Here, “empirical model” refers to the probability table arising from joint distributions defined by $\Psi$ and the observables appearing in each context.

3.3 Position-Momentum contextuality for three spacetime degrees of freedom

Clifton [31] has observed that the Mermin proof encapsulated in equations (11)–(15) may be adapted to provide a proof of contextuality based on the position and momentum of a spinless particle in three spacetime dimensions. He considers the Weyl form of the canonical commutation relation (CCR) in $\mathcal{H} \equiv L^2(\mathbb{R}^3)$. With $Q$ and $P$ the usual position and momentum operators acting in (some dense domain of) $\mathcal{H}$ (i.e., $Q = (Q_1, Q_2, Q_3)$, etc.), $p, q \in \mathbb{R}^3$, and writing $U(p) \equiv e^{ipQ}$ and $V(q) \equiv e^{iqP}$, the CCR takes the form

$$U(p)V(q) = e^{-ipq}V(q)U(p).$$

The component operators $U(p_i) \equiv e^{ip_iQ_i}$ and $V_i = e^{iq_iP_i}$ satisfy the same form of CCR as (12) for each $i$; furthermore,

$$[U(p_i), V(q_i)] = 0 \text{ whenever } p_iq_i = 2ni\pi,$$

and, with $\llbracket , \rrbracket_+$ denoting the anticommutator,

$$[U(p_i), V(q_i)]_+ = 0 \text{ whenever } p_iq_i = (2n+1)i\pi.$$

Then, as Clifton observed, one may consider once more a valuation $\nu$, defined on self-adjoint operators in $\mathcal{L}(\mathcal{H})$ and naturally extended to all bounded operators in the obvious way.

The commutativity and anticommutativity displayed by equations (13) and (14) suggest that one may find suitable values of $p_i, q_i$ to reproduce the Mermin proof, and this is precisely what is done in [31], where Clifton chooses $p_iq_i = (2n+1)i\pi$ for all $i$. The Mermin contradiction stemming from equations (11)–(15) relies on the fact that the operators in question are involutions, and therefore that $\nu(X_i^2) = \nu(Z_i^2) = 1$. Such a property is, however, lost for $U(p_i)$ and $V(q_i)$. Nevertheless, with $U(-q_i)$, the inverse of $U(q_i)$, etc., the equations

$$\nu(U(-p_1)U(-p_2)U(-p_3)) = \nu(U(-p_1))\nu(U(-p_2))\nu(U(-p_3)),$$

$$\nu(V(q_1)V(q_2)V(p_3)) = \nu(V(q_1))\nu(V(q_2))\nu(U(p_3)), $$

$$\nu(V(-q_1)U(p_2)V(-q_3)) = \nu(V(-q_1))\nu(U(p_2))\nu(V(-q_3)),$$

$$\nu(U(p_1)V(-q_2)V(-q_3)) = \nu(U(p_1))\nu(V(-q_2))\nu(V(-q_3)),$$

hold by FUNC. Precisely akin to the Mermin proof, taking products of the right-hand sides and left-hand sides therefore gives

$$\nu(U(-p_1)U(-p_2)U(-p_3))\nu(V(q_1)V(q_2)V(p_3))\nu(V(-q_1)U(-p_2)V(-q_3)) = 1,$$

and, in direct analogy to Mermin, noticing that each operator appearing as an argument of $\nu$ commutes with all others appearing in $\nu$, the anticommutativity yields

$$\nu(U(-p_1)U(-p_2)U(-p_3))\nu(V(q_1)V(q_2)V(p_3))\nu(V(-q_1)U(-p_2)V(-q_3))$$

$$\nu(U(p_1)V(-q_2)V(-q_3)) = -1,$$

giving the required contradiction.
4 An Operator Algebraic Perspective: Döring’s Theorem

The most general form of the Kochen-Specker theorem is due to Döring [12] and concerns valuations on a general von Neumann algebra $\mathcal{R}$ (viewed as a von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$).

**Theorem 4.1 (Döring)**

Let $\mathcal{R}$ be a von Neumann algebra, with self-adjoint part $\mathcal{R}_{sa}$, without type $I_1$ or $I_2$ summand. Then there is no $\nu : \mathcal{R}_{sa} \to \mathbb{C}$ satisfying $\text{SPEC}$ and $\text{FUNC}$.

Döring actually provides a number of different Kochen-Specker-type theorems in [12] which arise from two Gleason-type theorems. The simplest, corresponding closely in its conclusion to the original Kochen-Specker result (in which $\mathcal{R} = M_3(\mathbb{R})$), and of which the situation arising from the Mermin system is an example, is a direct consequence of Gleason’s theorem [36], which we recall states that completely additive measures on the projection lattice $\mathcal{P}(\mathcal{H})$ are given by normal states and thus by density operators via the usual trace formula. The lack of a dispersion-free (i.e., $\{0,1\}$-valued) state on the projection lattice is then enough to rule out a valuation $\nu$. The form of Kochen-Specker theorem thus arising is valid on type $I$ factors.

Remarkably, as Döring demonstrates, any non-abelian von Neumann algebra without summands of type $I_1$ and $I_2$ has enough structure to display a Kochen-Specker contradiction. We refer to [12] for a proof which relies on a generalised version of Gleason’s theorem (see [37–40]; given also in [41]) applicable to general von Neumann algebras (i.e., not necessarily factors, and certainly not of type $I$). Roughly speaking, this version of Gleason’s theorem shows that additive measures extend to states provided that the given algebras have no type $I_2$ summand.

It is immediately obvious that the von Neumann algebra generated by the state-independent Mermin proof falls within the remit of Döring’ theorem. Indeed, it is of the simplest form: consider the ten observables $A_i$ appearing in Mermin’s proof, and form $\mathcal{A} := \{A_i\}''$ (prime denoting commutant). Of course, $\mathcal{A} = \{X_1, X_2, X_3, Z_1, Z_2, Z_3\}''$, and it is readily verified that $\mathcal{A} = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \cong M_8(\mathbb{C})$. Clearly, $\mathcal{A}$ is a type $I_8$ factor and and hence there is no valuation on $\mathcal{A}_{sa}$.

5 Geometry: Root Systems, Weyl Chambers and the Emergence of $E_8$

Mermin’s proof connects to the Coxeter group $E_8$ in two simple steps: one collects together the eigenstates defined by the contexts, and then a process of “completion” is effected as described below. The final set turns out to be the root system of $E_8$, which can be verified using Coxeter diagrams. Additionally, one can exploit the simple transitive action of $E_8$ on its set of chambers and get a description of Mermin’s proof in terms of galleries.

We briefly review the mathematics involved and refer the reader to [42,43] for details. A Coxeter matrix indexed by $S$, with $S$ a finite set, is a function $m_{ij} : S \times S \to \{1, 2, \cdots \} \cup \{+\infty\}$ satisfying $m_{ii} = 1$ and $m_{ij} = m_{ji} > 1$ for $i \neq j$. Associated to a Coxeter matrix is a Coxeter diagram. Its set of nodes is $S$ and the two nodes $i$ and $j$ are connected if $m_{ij} \geq 3$: we assign 0, 1, 2, or 3 edges between $i$ and $j$ when $m_{ij}$ is 2, 3, 4, or 6, respectively. The Coxeter group $W = W_S$ associated to a Coxeter matrix $m$ is the group with generators $s_i, i \in S$ and relations: (1) $s_i^2 = 1$ and (2) $(s_is_j)^{m_{ij}} = 1$ (the braid relations). The pair $(W, S)$ is called
a Coxeter system. The canonical geometric realisation of Coxeter groups uses root systems. There, the elements of $S$ are represented by real reflections. A root system $\Phi \subset \mathbb{R}^n$ (with $n = \text{card}(S)$) is a finite set of vectors (roots) satisfying the two simple properties:

1. For all $\alpha \in \Phi$: $\lambda \alpha \in \Phi$ iff $\lambda = \pm 1$,
2. For all $\alpha, \beta \in \Phi$:

$$s_\alpha(\beta) := \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \alpha \in \Phi.$$ 

The pairing is the standard inner product on $\mathbb{R}^n$. The transformation $s_\alpha$ is the reflection with respect to the hyperplane (wall) $\{x : (x, \alpha) = 0\}$; it is an involution. Reflections $s_\alpha$ for $\alpha \in \Phi$ generate the group $W$ (more precisely, generate the Tits realisation of $W$ as a reflection group). By property (2), it is a symmetry group of $\Phi$ and $W$ is a subgroup of the group $\subset O(\mathbb{R}^n)$ of all orthogonal transformations on $\mathbb{R}^n$. Hence, $W(\Phi)$ preserves angles and lengths.

We return now to Mermin’s proof and explain the emergence of the group $E_8$. Each context in the proof yields a basis for the Euclidean space $\mathbb{R}^8$. We get 40 vectors in total, corresponding to the five bases. By reflections, this set of vectors is completed to 240 vectors $\alpha_i$ with $\alpha_i + 120 = -\alpha_i$ for all $1 \leq i \leq 120$. Coxeter’s diagram computation yields the following diagram, which uniquely identifies the Coxeter group $E_8$ (a fact about finite irreducible Coxeter groups).

![Coxeter diagram of $E_8$.](image)

A Weyl chamber of a root system $\Phi$ is the geometrical realisation of the generating set $S$; it is the connected region of $\mathbb{R}^n$ bounded by the walls of the different generating reflections in $S$. A gallery is a sequence of chambers $S_i$ with each successive pair sharing a face of codimension one. For instance, if $w = s_1 \cdots s_n$, then the sequence $S, s_1S, s_1s_2S, \cdots, s_1s_2 \cdots s_nS$ is a gallery. The Coxeter group $W$ acts transitively on its set of chambers; this will help us to translate Mermin’s proof from the level of roots to that of chambers. First, we fix a reference chamber $S_0$. Suppose $S_0$ is the chamber given by the diagram of Figure 1. The reflection $s_1$ (with respect to the first root) is then represented by the chamber

![Reference chamber](image)

The second reflection $s_2$ is represented by

![Second reflection](image)

etc. The fact that $m_{1,2} = 2$ means that two galleries $S_0, S_1, s_2S_1$ and $S_0, S_2, s_1S_2$ intersect
6 (Co-)presheaves and (Co-)sheaves

The Kochen-Specker theorem can be elegantly phrased in terms of functors from a category to the category of sets. In this section we review three such formulations: the spectral presheaf $\Sigma$ of Isham et al., where the (state-independent) Kochen-Specker theorem concerns the (non-)existence of points of $\Sigma$; the covariant approach of Heunen et al., and the operational approach of Abramsky et al.

6.1 Isham et al.

The base category under consideration in the work of Isham, Butterfield, and Hamilton (e.g., [15]) is $\mathcal{V}(\mathcal{A})$, with $\mathcal{A}$ a $C^*$-algebra, $\mathcal{V}(\mathcal{A})$ denoting the partially ordered set (poset) of unital abelian $C^*$-subalgebras, and elements of $\mathcal{V}(\mathcal{A})$ now referred to as contexts. The morphisms in $\mathcal{V}(\mathcal{A})$ are inclusion relations; $U \rightarrow V$ if and only if $U \subseteq V$ as abelian algebras (we will also use $\hookrightarrow$ for inclusion arrows).

A presheaf of particular relevance to the Kochen-Specker theorem is the spectral presheaf $\Sigma : \mathcal{V}(\mathcal{A}) \rightarrow \text{Set}$. For each $V \in \mathcal{V}(\mathcal{A})$,

- $\Sigma(V) := \{\lambda_V : \lambda_V$ is a character of $V\}$ equipped with the topology of pointwise convergence, i.e., $\Sigma(V)$ is the Gelfand spectrum of $V$ (in the case of the Mermin proof, the topology is discrete);
- on inclusions $U \hookrightarrow V$,
- $\Sigma(U \subseteq V) : \Sigma(V) \rightarrow \Sigma(U)$; $\lambda_V \mapsto \lambda_V|_U \equiv \lambda_U$, where the restriction is understood in the sense of function restriction.

The state-independent Kochen-Specker theorem concerns the existence of a global section (or point) $\tau : 1 \rightarrow \Sigma$ (1 is the terminal element). If such a section were to exist, for each $V$, $\tau_V : \{\ast\} \rightarrow \Sigma(V)$, and for each $U \hookrightarrow V$, $\tau_V(\ast)(\lambda_V) = \lambda_V|_U \equiv \lambda_U$. In other words, we would have a valuation for each context giving rise to a global valuation.

The Mermin proof can be phrased in these terms when restricted to the poset $\mathcal{P}$. Let $\tau$ be such a putative global section. Then, since $\tau_V(\ast)$ is a character of each abelian algebra $V_i$,

$$\tau_1(\ast)(X_1 X_2 X_3) \tau_1(\ast)(X_1) \tau_1(\ast)(X_2) \tau_1(\ast)(X_3) = 1,$$
$$\tau_2(\ast)(X_1 Z_2 Z_3) \tau_2(\ast)(X_1) \tau_2(\ast)(Z_2) \tau_2(\ast)(Z_3) = 1,$$
$$\tau_3(\ast)(Z_1 X_2 Z_3) \tau_3(\ast)(Z_1) \tau_3(\ast)(X_2) \tau_3(\ast)(Z_3) = 1,$$
$$\tau_4(\ast)(Z_1 Z_2 X_3) \tau_4(\ast)(Z_1) \tau_4(\ast)(Z_2) \tau_4(\ast)(X_3) = 1,$$
$$\tau_5(\ast)(X_1 X_2 X_3) \tau_5(\ast)(X_1 Z_2 Z_3) \tau_5(\ast)(Z_1 X_2 Z_3) \tau_5(\ast)(Z_1 Z_2 X_3) = -1.$$
However, the choice of characters on each algebra is constrained by the requirement that \( \tau_1(*) \langle X_1 \rangle = \tau_2(*) \langle X_1 \rangle, \tau_5(*) \langle X_1X_2X_3 \rangle = \tau_1(*) \langle X_1X_2X_3 \rangle \), etc. After making these replacements in equations (21)–(25), one again meets with a contradiction.

Given a unit vector \( \varphi \in \mathbb{C}^8 \) one can define a subpresheaf \( w^\varphi \) of \( \Sigma \) in which state-dependent proofs may be considered. Indeed, as demonstrated in [23], the GHZ state \( |\Psi\rangle \) gives rise to the presheaf \( w^\Psi \) which lacks a global section.

A primary motivation for Isham et al. for constructing \( \Sigma \) and \( w^\varphi \) as they do is that they may be viewed as objects in the topos \( \text{Set}^{\mathcal{V}(A)} \), i.e., the topos of all set-valued presheaves on \( \mathcal{V}(A) \). A topos has a multivalued “truth object” provided by the subobject classifier; Isham and Döring [44] then argue for a connection between the quantum probabilities given by the Born rule and the truth values in \( \text{Set}^{\mathcal{V}(A)} \). Furthermore, they argue, the relaxation of a Boolean truth system to a more general, many-valued intuitionistic one paves the way for a “neo-realist” interpretation of quantum theory.

6.2 Heunen-Landsman-Spitters

Heunen, Landsman, and Spitters [17] have provided another topos-theoretic view on the Kochen-Specker theorem. With \( \mathcal{V}(A) \) denoting the poset of unital, abelian \( C^* \)-subalgebras of some fixed \( C^* \)-algebra \( A \), they construct the topos \( \mathcal{T}(A) \)—the topos of all covariant set-valued functors on \( \mathcal{V}(A) \).

The point is to internalise the (in general) non-abelian algebra \( A \), yielding \( \mathfrak{A} \) internal to \( \mathcal{T}(A) \) (in this section we follow the convention of the authors of [17] in underlining internal entities) which is abelian, and is referred to as the Bohrification of \( A \).

\( \mathfrak{A} \) is the “tautological functor”, defined on objects by

\[
\mathfrak{A}(C) = C,
\]

and on any arrow \( C \subseteq D, \mathfrak{A}(C) \hookrightarrow \mathfrak{A}(D) \). The \( C^* \)-algebra \( A \) is a (not necessarily commutative) \( C^* \)-algebra in the topos \( \text{Set} \), whereas, as proved in [17], \( \mathfrak{A} \) is a commutative \( C^* \)-algebra in the “universe of discourse” provided by \( \mathcal{T}(A) \).

The covariant “version” of the spectral presheaf \( \Sigma \) from Section 6.1, which will be denoted \( \Sigma \), gives the internal Gelfand spectrum \( \Sigma(A) \) of the internal abelian algebra \( \mathfrak{A} \). \( \Sigma(A) \) is an internal locale; the points of a general locale \( X \) in a topos are the frame maps \( \mathcal{O}(X) \to \Omega \), with \( \Omega \) being the subobject classifier of the topos in question. As shown in [17], the Kochen-Specker theorem now reads:

**Theorem 6.1 (Heunen-Landsman-Spitters)**

Let \( \mathcal{H} \) be a Hilbert space for which \( \dim \mathcal{H} \geq 3 \) and let \( A = \mathcal{L}(\mathcal{H}) \). Then \( \Sigma(A) \) has no points.

The proof is roughly as follows (see [17], Theorem 6, and the accompanying proof): internally, a point \( \rho : * \to \Sigma \) of the locale \( \Sigma \) can be combined with a self-adjoint operator \( a \in \mathfrak{A}_{sa} \) through its Gelfand transform \( \hat{a} : \Sigma \to \mathbb{R} \) to give a point \( \hat{a} \circ \rho : * \to \mathbb{R} \), the latter denoting the locale of Dedekind reals, again viewed internally. This defines an internal multiplicative linear map (natural transformation) \( \nu_\rho : \mathfrak{A}_{sa} \to \text{Pt}(\mathbb{R}) \) with components \( \nu_\rho(V) : \mathfrak{A}_{sa}(V) \to \text{Pt}(\mathbb{R})(V) \), which coincides externally with \( C_{sa} \to \mathbb{R} \), and is precisely the valuation that is ruled out by the Kochen-Specker theorem [2].

With regard to the Mermin system, replacing \( \mathcal{V}(A) \) by \( \mathcal{P} \) generated by the five contexts involved, we may view the (non-commutative) \( C^* \)-algebra \( A_P \) again as an internal commutative \( C^* \)-algebra \( \mathfrak{A}_P \) in the topos \( \text{Sets}^P \equiv \mathcal{T}(A_P) \) with associated internal locale \( \Sigma(A_P) \). That
this locale has no points follows from considering valuations $v_p(V)$ for the abelian algebras of each context, yielding again the insoluble set of equations (21)–(25), with $v_p(V)$ coinciding with $\tau(s)_V$.

6.3 Abramsky et al.

Abramsky and Brandenburger [18] have presented an operational, sheaf-theoretic description of contextuality, and Abramsky, with Mansfield and Barbosa, have developed cohomological techniques [19, 20] for identifying its presence. Their approach is independent of the Hilbert space formalism, thereby doing away with SPEC and FUNC and instead focussing directly on the (im-)possibility of a global section for a compatible family of no-signalling distributions.

Following Abramsky and Brandenburger, we refer to a finite collection of observables $X$, understood as a topological space with the discrete topology and distinguished open sets (the contexts here) $\{C_i\} \equiv \mathcal{M}; \bigcup C_i = X$ as a measurement cover. Each $C_i$ is a maximal set of observables which may be measured jointly. The outcomes of a measurement of any observable $A \in X$ are given by the set $O$ and, therefore, under the assumption that all observables have the same outcome sets, for a measurement of $n$ observables in $X$, outcomes are in $O \times O \times \ldots \times O \equiv O^n$. The event sheaf $\mathcal{E}$ is defined by $\mathcal{E}(U) := O^U \equiv \{f : U \to O\}$ for $U \subset X$, i.e., the collection of assignments for observables in $U$. Elements of $\mathcal{E}^U$, or sections above $U$, are therefore viewed as possible assignments of observables in $U$. An empirical model $e$ specifies joint probability distributions $e(C)$ over assignments in $\mathcal{E}(C)$, where $C \in \mathcal{M}$, to be compatible in the sense of the no-signalling principle (or, equivalently, the sheaf condition: see [18]). The support presheaf $S_e$ of $e$ specifies a subpresheaf of $\mathcal{E}$ defined by $S_e(U) = \{s \in \mathcal{E}(U) : s \in \text{supp } e_U\}$.

Abramsky and Brandenburger [18] identify three levels of contextuality/non-locality arranged in a proper hierarchy—probabilistic non-locality, possibilistic non-locality, and strong contextuality—in which the state-dependent Mermin system, and indeed all AvN arguments, occupies the strongest level of strong contextuality. The next level down corresponds to the situation where outcome probabilities can be neglected and all the matters is what is possible, i.e., on the level of supports; the Hardy model occupies this level and is not strongly contextual. In turn, the Bell model is probabilistically non-local but not possibilistically non-local, thus showing that each level in the hierarchy can be realised by quantum mechanics. It should be noted, however, that quantum mechanics is not a necessary requirement—for example, Podolosky-Rosen boxes are also strongly contextual [18].

**Definition 6.2** The model $e$ is called strongly contextual if $S_e(X) = \emptyset$.

As shown in [18], the GHZ models (for at least three parties) are strongly contextual; We also note that all AvN arguments are strongly contextual. To exhibit strong contextuality in our familiar setting we turn our attention to the Mermin state-dependent proof, with the following table:

|      | + | + | + | + | + | + | + | - | - | - | - | - | - |
|------|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_1$ = $X_1, X_2, X_3$ | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| $C_2$ = $X_1, Y_2, Y_3$ | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| $C_3$ = $Y_1, X_2, Y_3$ | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| $C_4$ = $Y_1, Y_2, X_3$ | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
There are four “potential” global sections compatible with the assignments for \( C_2, C_3, \) and \( C_4 \), namely

\[
\begin{align*}
    s_1 : (X_1, X_2, X_3, Y_1, Y_2, Y_3) &\mapsto (+ + + +), \\
    s_2 : (X_1, X_2, X_3, Y_1, Y_2, Y_3) &\mapsto (+ - + +), \\
    s_3 : (X_1, X_2, X_3, Y_1, Y_2, Y_3) &\mapsto (- + + +), \\
    s_4 : (X_1, X_2, X_3, Y_1, Y_2, Y_3) &\mapsto (- - - -).
\end{align*}
\]

As can be seen from the table, none of these are compatible with the constraints imposed by \( C_1 \), thus yielding the required contradiction.

To conclude this section, we briefly remark that Abramsky et al. [19, 20] have observed cohomological witnesses for contextuality, including strong contextuality, by imposing a ring structure on the outcome spaces. These tools are strong enough to identify the presence of an AvN argument, for instance. In general, their cohomological construction yields sufficient conditions under which a Kochen-Specker contradiction occurs; unfortunately, these are sometimes not necessary [45].

### 7 Algebraic Geometry

It was observed in subsection 6.1 that the topology on \( \Sigma(V) \) is discrete in the Mermin example. If one views the discrete topology as the Zariski topology, then one can phrase the contradiction in the framework of algebraic geometry. For that, let \( A \) be a commutative ring with identity. It will be, for us, of the form

\[
\mathbb{Z}_2[i, j, \cdots, k]/p := \{ \text{polynomials in } i, j, \cdots, k \mod p \},
\]

i.e., a coordinate ring of an affine variety \( V(p) \subset \mathbb{Z}_2^n \). The ideal \( p \), in our context, represents the algebraic constraints involved in Mermin’s type of proofs. The discreteness of the situation allows the replacement of the Gelfand spectrum by the prime spectrum of an algebraic variety to be defined next. Specifically, the spectrum \( \Sigma(A) \) of the commutative algebra \( A \) is replaced by the prime spectrum of a ring \( A \) associated to \( A \).

**Definition 7.1 (Prime spectrum of a ring)** The prime spectrum of a ring \( A \) is the set \( \text{PSpec } A \) of all prime ideals in \( A \).

In our case of coordinate rings, \( \text{PSpec } A \) has two types of elements: maximal ideals and non-maximal (but prime) ideals. The first type corresponds to points in the affine space \( \mathbb{Z}^n \), whereas prime ideals which are not maximal are irreducible subvarieties of our variety. Each element \( f \in A \) defines a function on the prime spectrum of \( A \). Let \( p \) be an element of \( \text{PSpec } A \). We denote by \( \kappa(p) \) the quotient field \( A/p \), called the residue field of \( \text{PSpec } A \) at \( p \). We can define the value of \( f \) at \( p \) to be the image of \( f \) via the ring homomorphism \( v_p : A \rightarrow \kappa(p) \). If \( A \) is a coordinate ring of an affine variety over an algebraically closed field \( K \) and \( p \) is the maximal ideal corresponding to a point of the variety, then \( \kappa(p) = K \) and \( f(p) \) is just the valuation of \( f \) at this point.

The goal of the remainder of this section is to explain contextuality of Mermin’s system in the language of algebraic geometry we have just reviewed. Notation is as defined in the previous section. We “identify” each set \( S_e(C) \), where \( C \in \mathcal{P} \) and \( \mathcal{P} \) is the poset generated by \( \mathcal{M} \), as an affine variety, and local sections in \( S_e(C) \) as maximal (ring) ideals in \( \text{PSpec } A_C \).
This yields for each $C$ a coordinate ring $A_C$. The set of rings $A_C$, which are partially ordered by inclusion, define the poset $\text{CoordRings}(\mathcal{P})$. We can define the functor

$$
\mathcal{F} : \text{CoordRings}(\mathcal{P}) \rightarrow \text{Sets}
$$

$$
A \mapsto \{ v_p, \ p \in \text{PSpec} A \}.
$$

(31)

For Mermin-type Kochen-Specker proofs, the functor $\mathcal{F}$ has no point. We explain this in the Mermin example, which should be enough to convince the reader of the validity of the latter statement. For the first context, we have the coordinate ring $A_1 := \mathbb{Z}_2 [s_1, s_2, s_3] / (s_1 s_2 s_3 - 1)$ where, for instance, the local section $(s_1, s_2, s_3) \mapsto (-1, -1, 1)$ of $S_v$ at the first context defines a maximal ideal $p = (s_1 + 1, s_2 + 1, s_3 - 1) \in \text{PSpec} A_1$. For the second context, we have the coordinate ring $A_2 := \mathbb{Z}_2 [s_1, s_2, s_3] / (s_1 s_2 s_3 + 1)$, and similarly for the two other maximal contexts. For the context $\{x_1\}$, the coordinate ring is $A_{1,2} := \mathbb{Z}_2 [s_1]$, with two proper maximal ideals $(s_1 - 1)$ and $(s_1 + 1)$; i.e., the corresponding affine variety is the whole line $\mathbb{Z}_2$.

For each ring $A_i$ and each maximal ideal $p \in \text{PSpec} A_i$, the local valuation $v_p : A_i \rightarrow \kappa(p) = A_i/p$ takes $x \in A_i$ and returns $v_p(x) = x \mod p$. In the case of $A_1$ and $p = (s_1 + 1, s_2 + 1, s_3 - 1)$, we have

$$
v_p(x_1(s_1, s_2, s_3)) = v_p(s_1) = s_1 \mod p = -1,
$$

$$
v_p(x_2(s_1, s_2, s_3)) = v_p(s_2) = s_2 \mod p = -1,
$$

$$
v_p(x_3(s_1, s_2, s_3)) = v_p(s_3) = s_3 \mod p = 1.
$$

(32)

The functor $\mathcal{F}$ has no points for GHZ3: for any maximal ideal $p_1 \in A_1$, one has

$$
v_{p_1}(s_1 s_2 s_3) = v_{p_1}(s_1) v_{p_1}(s_2) v_{p_1}(s_3) = 1,
$$

(33)

which is nothing but the equation (1). For the three other maximal rings, we have similar equations valid for all maximal ideals $p_i \in \text{Spec} A_i$. The system is inconsistent and the functor $\mathcal{F}$ is pointless.

8 Discussion and Outlook

As has been presented, there is a remarkable variety and mathematical richness surrounding Kochen-Specker proofs, much of which is exhibited by Mermin’s simple system. We saw that the collection of observables given in that example, along with a given state where appropriate, yield naturally combinatoric contradictions in the form of insoluble equations, impossible colourings, and so on. Mermin’s system, as Clifton showed, could also be adapted to the case of position and momentum observables.

Döring provided a non-combinatorial proof of the Kochen-Specker theorem in the language of von Neumann algebras, thereby providing a genuine generalisation. We then provided some geometric insight on the Mermin proof through Coxeter groups and their root systems, and observed that $E_8$ arises naturally in this context. The presheaf perspectives presented then provide another way of phrasing Kochen-Specker-type contradictions—three such approaches were given. Finally, an algebraic geometric framework was outlined. It was observed that in the finite scenario, the Gelfand spectrum of an abelian algebra can be replaced by the prime spectrum of an algebraic variety, and the statement that the spectral presheaf of Isham and
Butterfield has no global section “translates” into the statement that the prime spectrum functor has no global section. This algebraic geometric approach seems to warrant further exploration.

We only briefly touched upon the role played by cohomology in the Kochen-Specker theorem. We believe this to be an area for much further work, particularly when phrased in the language of toposes [46]. Let $\mathcal{E}$ be the topos $\text{Sets}^{\text{op}}$ and $\mathcal{AB}(\mathcal{E})$ the category of abelian group objects, i.e., presheaves from the poset $\mathcal{P}$ into the category $\mathcal{AB}$ of abelian groups. The Giraud axiom for generators implies that the abelian category $\mathcal{AB}(\mathcal{E})$ has enough injectives. The global sections functor $\Gamma : \mathcal{E} \to \text{Sets}$ induces a functor (again denoted) $\Gamma : \mathcal{AB}(\mathcal{E}) \to \mathcal{AB}$, which is left exact and preserves injectives. For any abelian group object $A \in \mathcal{E}$, the cohomology groups $H^n(\mathcal{E}, A)$ are defined as the right derived functors of $\Gamma$. Now, the functors $\Sigma$ and $S_e$ are in $\mathcal{AB}(\mathcal{E})$ in the case of GHZ3; i.e., they are functors of vector spaces. Mermin’s proof translates into the functors $H^n$, for both $\Sigma$ and $S_e$, being all equal to the zero functor.

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