The Expansion in Width for Domain Walls in Nematic Liquid Crystals in External Magnetic Field *

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March 24, 2022

Abstract

The improved expansion in width is applied to curved domain walls in uniaxial nematic liquid crystals in external magnetic field. In the present paper we concentrate on the case of equal elastic constants. We obtain approximate form of the director field up to second order in magnetic coherence length.

PACS numbers: 61.30.Jf, 11.27.+d, 02.30.Mv
Preprint TPJU-26/98

*Paper supported in part by KBN grant 2P03B 095 13.
1 Introduction

Liquid crystals are probably the best materials for experimental and theoretical studies of topological defects. Variety of defects, relatively simple experiments in which one can observe them, and soundness of theoretical models of dynamics of relevant order parameters make liquid crystals unique in this respect. Literature on topological defects in liquid crystals is enormous, therefore we do not attempt to review it here. Let us only point the books [1], [2], [3] in which one can find lucid introductions to the topic as well as collections of references.

Our paper is devoted to dynamics of domain walls in uniaxial nematic liquid crystals in an external magnetic field. Static, planar domain walls were discussed for the first time in [5]. We would like to approximately calculate director field of a curved domain wall. We use a method, called the improved expansion in width, whose general theoretical formulation has been given in [5, 6]. Appropriately adapted expansion in width can also be applied to disclination lines [7].

The expansion in width is based on the idea that transverse profiles of the curved domain wall and of a planar one differ from each other by small corrections which are due to curvature of the domain wall. We calculate these corrections perturbatively. Formally, we expand the director field in a parameter which gives the width of the domain wall, that is the magnetic coherence length $\xi_m$ in the case at hand, but actually terms in the expansion involve dimensionless ratios $\xi_m/R_i$, where $R_i$ are (local) curvature radius of the domain wall. Therefore, our expansion is expected to provide a good approximation when curvature radius of the domain wall are much larger than the magnetic coherence length. For planar domain walls the perturbative solution reduces to just one term which coincides with a well-known exact solution. As we shall see below, the improved expansion in width is not quite straightforward – there are consistency conditions and rather special coordinate system is used – but that should be regarded as a reflection of nontriviality of evolution of curved domain walls. Actually, several first terms in the expansion can be calculated without any difficulty, and the whole approach looks quite promising.

In the present paper we consider the simplest and rather elegant case of equal elastic constants. In order to take into account differences of values of the elastic constants for real liquid crystals one can use, for example, the
following two strategies: perturbative expansion with respect to deviations
of the elastic constants from their mean value, or the expansion in width
generalized to the unequal constants case. In the former approach, the equal
constant approximate solution obtained in the present paper can be used as
the starting point for calculating corrections. The case of unequal elastic
constants we will discuss in a subsequent paper.

The plan of our paper is as follows. We begin with general description of
domain walls in uniaxial nematic liquid crystals in Section 2. Next, in Section
3, we introduce a special coordinate system comoving with the domain wall.
Section 4 contains description of the improved expansion in width. In Section
5 we discuss consecutive terms in the expansion up to the second order in
$\xi_m$. Several remarks related to our work are collected in Section 5.

2 Domain walls in nematic liquid crystals

In this Section we recall basic facts about domain walls in uniaxial nematic
liquid crystals [1], [2]. We fix our notation and sketch background for the
calculations presented in next two Sections.

We shall parametrize the director field $\vec{n}(\vec{x}, t)$ by two angles $\Theta(\vec{x}, t)$,
$\Phi(\vec{x}, t)$:

$$\vec{n} = \begin{pmatrix} \sin \Theta \cos \Phi \\ \sin \Theta \sin \Phi \\ \cos \Theta \end{pmatrix}. \quad (1)$$

In this way we get rid of the constraint $\vec{n}^2 = 1$.

We assume that the splay, twist and bend elastic constants are equal
($K_{11} = K_{22} = K_{33} = K$). In this case Frank—Oseen—Zöcher elastic free
energy density can be written in the form

$$\mathcal{F}_e = \frac{K}{2} (\partial_\alpha \Theta \partial_\alpha \Theta + \sin^2 \Theta \partial_\alpha \Phi \partial_\alpha \Phi). \quad (2)$$

Our notation is as follows: $\alpha = 1, 2, 3$; $\partial_\alpha = \partial / \partial x^\alpha$; $x^\alpha$ are Cartesian
coordinates in the usual 3-dimensional space $R^3$; $\vec{x} = (x^\alpha)$. In formula (2)
we have abandoned a surface term which is irrelevant for our considerations.

In order to have stable domain walls it is necessary to apply an external
magnetic field $\vec{H}_0$ [1], [2]. We assume that $\vec{H}_0$ is constant in space and time.
Without any loss in generality we may take
\[ \vec{H}_0 = \begin{pmatrix} 0 \\ 0 \\ H_0 \end{pmatrix}. \]

Then the magnetic field contribution to free energy density of the nematic is given by the following formula
\[ \mathcal{F}_m = -\frac{1}{2} \chi_a H_0^2 \cos^2 \Theta. \] (3)

Here \( \chi_a \) is the anisotropy of the magnetic susceptibility. It can be either positive or negative. For concreteness, we shall assume that \( \chi_a > 0 \). Our calculations can easily be repeated if \( \chi_a < 0 \). The ground state of the nematic is double degenerate: \( \Theta = 0 \) and \( \Theta = \pi \) give minimal total free energy density \( \mathcal{F} = \mathcal{F}_e + \mathcal{F}_m \). It is due to this degeneracy that stable domain walls can exist.

Dynamics of the director field is mathematically described by the equation
\[ \gamma_1 \frac{\partial \vec{n}}{\partial t} + \frac{\delta F}{\delta \vec{n}} = 0, \] (4)

where
\[ F = \int d^3x \mathcal{F}. \]

\( \gamma_1 \) is the rotational viscosity of the liquid crystal, and \( \frac{\delta}{\delta \vec{n}} \) denotes the variational derivative with respect to \( \vec{n} \). Equation (4) is equivalent to the following equations for the \( \Theta \) and \( \Phi \) angles
\[ \gamma_1 \frac{\partial \Theta}{\partial t} = K \Delta \Theta - \frac{K}{2} \sin(2\Theta) \partial_\alpha \Phi \partial_\alpha \Phi - \frac{1}{2} \chi_a H_0^2 \sin(2\Theta), \] (5)

\[ \gamma_1 \sin^2 \Theta \frac{\partial \Phi}{\partial t} = K \partial_\alpha (\sin^2 \Theta \partial_\alpha \Phi), \] (6)

where \( \Delta = \partial_\alpha \partial_\alpha \).

The domain walls arise when the director field is parallel to the magnetic field \( \vec{H}_0 \) in one part of the space and anti-parallel to it in another. In between there is a layer – the domain wall – across which \( \vec{n} \) smoothly changes its orientation from parallel to \( \vec{H}_0 \) to the opposite one, that is \( \Theta \) varies from 0 to \( \pi \) or vice versa. The angle \( \Phi \) does not play important role. The Ansatz
\[ \Phi = \Phi_0 \] (7)
with constant $\Phi_0$ trivially solves Eq.(6). Then, Eq.(5) is the only equation we have to solve. In the following we assume the Ansatz (7), hereby restricting the class of domain walls we consider. It is clear from formula (2) that domain walls with varying $\Phi$ have higher elastic free energy than the walls with constant $\Phi$.

Let us recall the static planar domain wall [1, 2]. We assume that it is parallel to the $x^1 = 0$ plane. Then

$$\Theta = \Theta_0(x^1), \quad \Phi_0 = \text{const},$$

(8)

where

$$\Theta_0 \big|_{x^1 \to -\infty} = 0, \quad \Theta_0 \big|_{x^1 \to +\infty} = \pi.$$  

(9)

One could also consider an "anti-domain wall" obtained by interchanging 0 and $\pi$ on the r.h.s. of boundary conditions (9). Equation (5) is now reduced to the following equation

$$K\Theta_0'' = \frac{1}{2} \chi_a H_0^2 \sin(2\Theta_0),$$

(10)

where $'$ denotes $d/dx^1$. This equation is well-known in soliton theory as the sine-Gordon equation, see e.g., [3]. It is convenient to introduce the magnetic coherence length $\xi_m$,

$$\xi_m = \left(\frac{K}{\chi_a H_0^2}\right)^{1/2}.$$  

(11)

The functions

$$\Theta_0(x^1) = 2 \arctan(\exp \frac{x^1 - x^1_0}{\xi_m})$$

(12)

with arbitrary constant $x^1_0$ obey Eq.(10) as well as the boundary conditions (9). The planar domain walls are homogeneous in the $x^1 = 0$ plane. Their transverse profile is parametrized by $x^1$. Width of the wall is approximately equal to $\xi_m$, in the sense that for $|x^1 - x^1_0| \gg \xi_m$ values of $\Theta_0$ differ from 0 or $\pi$ by exponentially small terms.

The planar domain wall solution of Eqs.(5), (6) contains two arbitrary constants: $\Phi_0$ and $x^1_0$. The arbitrariness of $\Phi_0$ is due to the assumption that the elastic constants are equal. Then the free energy density $F$ is invariant with respect to $\Phi \to \Phi + \text{const}$. If the elastic constants are not equal this invariance is lost, and in the case of planar domain walls $\Phi_0$ can take
only discrete values $n\pi/2, n = 0, 1, 2, 3$. The constant $x_0^1$ appears because of invariance of Eqs.(5), (6) with respect to the translations $x^1 \to x^1 + \text{const.}$

Notice that $\Theta_0(x_0^1) = \pi/2$. Hence at $x^1 = x_0^1$ the director $\vec{n}$ is perpendicular to $\vec{H}_0$. In fact, the boundary conditions (9) imply that for any domain wall there is a surface on which $\vec{n}\vec{H}_0 = 0$. Such surface is called the core of the domain wall. The magnetic free energy density $F_m$ has a maximum on the core.

The planar domain wall (12) plays very important role in our approach. In a sense, it is taken as the zeroth order approximation to curved domain walls. The trick consists in using a special coordinate system comoving with the curved domain wall. Such a coordinate system encodes shape and motion of the domain wall regarded as a surface in the space. Internal dynamics of the domain wall, like details of orientation of the director inside the domain wall, is then calculated perturbatively in the comoving reference frame with the function (12) taken as the leading term.

3 The comoving coordinates

The first step in our construction of the perturbative solution consists in introducing the coordinates comoving with the domain wall. Two coordinates $(\sigma^1, \sigma^2)$ parametrize the domain wall regarded as a surface in the $R^3$ space, and one coordinate, let say $\xi$, parametrizes the direction perpendicular to the domain wall. For convenience of the reader we quote main definitions and formulas below [6].

We introduce a smooth, closed or infinite surface $S$ in the usual $R^3$ space. It is supposed to lie close to the domain wall. Its shape mimics the shape of the domain wall. In particular we may assume that $S$ coincides with the core at certain time $t_0$. Points of $S$ are given by $\vec{X}(\sigma^i, t)$, where $\sigma^i$ ($i = 1, 2$) are two intrinsic coordinates on $S$, and $t$ denotes the time. We allow for motion of $S$ in the space. The vectors $\vec{X}_k, \ k = 1, 2$, are tangent to $S$ at the point $\vec{X}(\sigma^i, t)$ [4]. They are linearly independent, but not necessarily orthogonal to each other. At each point of $S$ we also introduce a unit vector $\vec{p}(\sigma^i, t)$ perpendicular to $S$, that is

$$\vec{p}\vec{X}_k = 0, \quad \vec{p}^2 = 1.$$\footnote{We use the notation $f_{,k} \equiv \partial f/\partial \sigma^k$.}
The triad \( (\vec{X}, k, \vec{p}) \) forms a local basis at the point \( \vec{X} \) of \( S \). Geometrically, \( S \) is characterized by the induced metric tensor on \( S \)

\[ g_{ik} = \vec{X}_i \vec{X}_k, \]

and by the extrinsic curvature coefficients of \( S \)

\[ K_{il} = \vec{p} \vec{X}_{il}, \]

where \( i, k, l = 1, 2 \). They appear in Gauss-Weingarten formulas

\[ \vec{X}_{ij} = K_{ij} \vec{p} + \Gamma_{ij}^l \vec{X}_l, \quad \vec{p}_i = -g^{il} K_{lj} \vec{X}_j. \] (13)

The matrix \( (g^{ik}) \) is by definition the inverse of the matrix \( (g_{kl}) \), i.e. \( g^{ik} g_{kl} = \delta_i^l \), and \( \Gamma_{ik}^l \) are Christoffel symbols constructed from the metric tensor \( g_{ik} \). Two eigenvalues \( k_1, k_2 \) of the matrix \( (K_i^j) \), where \( K_i^j = g^{il} K_{lj} \), are called extrinsic curvatures of \( S \) at the point \( \vec{X} \). The main curvature radii are defined as \( R_i = 1/k_i \).

The comoving coordinates \( (\sigma^1, \sigma^2, \xi) \) are introduced by the following formula

\[ \vec{x} = \vec{X}(\sigma^i, t) + \xi \vec{p}(\sigma^i, t). \] (14)

\( \xi \) is the coordinate in the direction perpendicular to the surface \( S \). In the comoving coordinates this surface has very simple equation: \( \xi = 0 \). We will use the compact notation: \( (\sigma^1, \sigma^2, \xi) = (\sigma^\alpha) \), where \( \alpha=1, 2, 3 \) and \( \sigma^3 = \xi \). The coordinates \( (\sigma^\alpha) \) are just a special case of curvilinear coordinates in the space \( R^3 \). In these coordinates the metric tensor \( (G_{\alpha \beta}) \) in \( R^3 \) has the following components:

\[ G_{33} = 1, \quad G_{3k} = G_{k3} = 0, \quad G_{ik} = N_i^l g_{lr} N_k^r, \]

where

\[ N_i^l = \delta_i^l - \xi K_i^l, \]

\( i, k, l, r = 1, 2 \). Simple calculations give

\[ \sqrt{G} = \sqrt{g} N, \]

where \( G = \det(G_{\alpha \beta}) \), \( g = \det(g_{ik}) \) and \( N = \det(N_{ik}) \). For \( N \) we obtain the following formula

\[ N = 1 - \xi K_i^i + \frac{1}{2} \xi^2 (K_i^j K^i_j - K_i^j K^j_i). \]
Components $G^{\alpha\beta}$ of the inverse metric tensor in $R^3$ have the form

$$G^{33} = 1, \ G^{3k} = G^{k3} = 0, \ G^{ik} = (N^{-1})_r^i g^l r (N^{-1})_l^k,$$

where

$$(N^{-1})_r^i = \frac{1}{N} \left( (1 - \xi K^l_r) \delta^i_r + \xi K^i_r \right).$$

We see that dependence on the transverse coordinate $\xi$ is explicit, while $\sigma^1, \sigma^2$ appear through the tensors $g_{ik}, K^l_r$ which characterize the surface $S$.

The comoving coordinates $(\sigma^\alpha)$ have in general certain finite region of validity. In particular, the range of $\xi$ is given by the smallest positive $\xi_0(\sigma^i, t)$ for which $G = 0$. It is clear that such $\xi_0$ increases with decreasing extrinsic curvature coefficients $K^l_i$, reaching infinity for the planar domain wall, for which $K^l_i = 0$. We assume that the surface $S$ (hence also the domain wall) is not curved too much. Then, that region is large enough, so that outside it there are only exponentially small tails of the domain wall which give negligible contributions to physical characteristics of the domain wall.

The comoving coordinates are utilised to write Eq.(5) in a form suitable for calculating the curvature corrections. Let us start from the Laplacian $\Delta \Theta$. In the new coordinates it has the form

$$\Delta \Theta = \frac{1}{\sqrt{G}} \frac{\partial}{\partial \sigma^\alpha} \left( \sqrt{G} G^{\alpha\beta} \frac{\partial \Theta}{\partial \sigma^\beta} \right).$$

The time derivative on the l.h.s. of Eq.(5) is taken under the condition that all $x^\alpha$ are constant. It is convenient to use time derivative taken at constant $\sigma^\alpha$. The two derivatives are related by the formula

$$\frac{\partial}{\partial t} |_{x^\alpha} = \frac{\partial}{\partial t} |_{\sigma^\alpha} + \frac{\partial \sigma^\beta}{\partial t} |_{x^\alpha} \frac{\partial}{\partial \sigma^\beta},$$

where

$$\frac{\partial \xi}{\partial t} |_{x^\alpha} = -\vec{p} \cdot \vec{X}, \ \frac{\partial \sigma^i}{\partial t} |_{x^\alpha} = -(N^{-1})_k^i g^{kr} \vec{X}_r (\dot{\vec{X}} + \xi \dot{\vec{p}}),$$

the dots stand for $\partial/\partial t |_{\sigma^\alpha}$. Let us also introduce the dimensionless coordinate

$$s = \xi/\xi_m.$$
Now we can write equation (5) transformed to the comoving coordinates $(\sigma^i, s)$ (with the Ansatz (7) taken into account):

\[
\begin{align*}
\gamma_1 \xi^2_m \left( \frac{\partial \Theta}{\partial t} \bigg|_{\sigma^a} - \frac{1}{\xi_m} \vec{p} \cdot \dot{\vec{X}} \frac{\partial \Theta}{\partial s} - (N^{-1})^i_k g^{kr} \dot{X}_r (\dot{X} + \xi_m s \dot{p}) \frac{\partial \Theta}{\partial \sigma^i} \right) \\
= \frac{\partial^2 \Theta}{\partial s^2} - \frac{1}{2} \sin(2\Theta) + \frac{1}{N} \frac{\partial N}{\partial s} \frac{\partial \Theta}{\partial s} + \xi_m^2 \frac{1}{\sqrt{g} N} \frac{\partial}{\partial \sigma^j} \left( G^{jk} \sqrt{g} N \frac{\partial \Theta}{\partial \sigma^k} \right),
\end{align*}
\]

where $\gamma_1$ is the cosmological constant.

Equation (15) is the starting point for construction of the expansion in width.

4 The improved expansion in width

We seek domain wall solutions of Eq.(15) in the form of expansion with respect to $\xi_m$, that is

\[
\Theta = \Theta_0 + \xi_m \Theta_1 + \xi_m^2 \Theta_2 + \ldots.
\]

Inserting formula (16) in Eq.(15) and keeping only terms of the lowest order ($\sim \xi_m^0$) we obtain the following equation

\[
\frac{\partial^2 \Theta_0}{\partial s^2} = \frac{1}{2} \sin(2\Theta_0),
\]

which essentially coincides with Eq.(10) after the rescaling $x^1 = \xi_m s$. Its solutions

\[
\Theta_{s_0}(s) = 2 \arctan(\exp(s - s_0)),
\]

essentially have the same form as the planar domain walls (12), but now $s$ gives the distance from the surface $S$. This surface will be determined later.

In the remaining part of the paper we shall consider curvature corrections to the simplest solution

\[
\Theta_0(s) = 2 \arctan(\exp s).
\]

Because already $\Theta_0$ interpolates between the ground state solutions 0, $\pi$, the corrections $\Theta_k$, $k \geq 1$, should vanish in the limits $s \to \pm \infty$.

Equations for the corrections $\Theta_k$, $k \geq 1$, are obtained by expanding both sides of Eq.(15) and equating terms proportional to $\xi_m^k$. These equations can be written in the form

\[
\hat{L} \Theta_k = f_k,
\]
with the operator $\hat{L}$

$$\hat{L} = \frac{\partial^2}{\partial s^2} - \cos(2\Theta_0) = \frac{\partial^2}{\partial s^2} + \frac{2}{\cosh^2 s} - 1. \quad (20)$$

The last equality in (20) can be obtained, e.g., from Eq.(17): inserting $\Theta_0$ given by formula (18) on the l.h.s. of Eq.(17) we find that $\sin(2\Theta_0) = -2\sinh s/\cosh^2 s$, and $\cos(2\Theta_0) = 1 - 2/\cosh^2 s$. The expressions $f_k$ on the r.h.s. of Eqs.(19) depend on the lower order contributions $\Theta_l$, $l < k$. Straightforward calculations give

$$f_1 = \partial_s \Theta_0 (K^r_r - \frac{\gamma_1}{K} \dot{p} \dot{X}), \quad (21)$$

$$f_2 = -\sin(2\Theta_0) \Theta_1^2 + s \partial_s \Theta_0 K^i_j \dot{K}_i^j + \partial_s \Theta_1 (K^r_r - \frac{\gamma_1}{K} \dot{p} \dot{X}), \quad (22)$$

$$f_3 = \frac{2\gamma_1}{K} (\partial_t \Theta_1 - g^{kr} \dot{X}_r \dot{X} \partial_k \Theta_1) - 2 \sin(2\Theta_0) \Theta_1 \Theta_2 - \frac{2}{3} \cos(2\Theta_0) \Theta_2^3 + s \partial_s \Theta_1 K^i_j \dot{K}_i^j + \frac{1}{2} s^2 \partial_s \Theta_0 K^r_r \left((K^r_r)^2 - 3 K^i_j K_i^j\right) - \frac{1}{\sqrt{g}} \partial_j (\sqrt{g} g^{jk} \partial_k \Theta_1) + \partial_s \Theta_2 (K^r_r - \frac{\gamma_1}{K} \dot{p} \dot{X}), \quad (23)$$

and

$$f_4 = \frac{2\gamma_1}{K} (\partial_t \Theta_2 - s g^{jk} \dot{p} X_k \partial_j \Theta_1) - \frac{2\gamma_1}{K} g^{jk} \dot{X}_k (\partial_j \Theta_2 + s K^i_j \partial_i \Theta_1) - \sin(2\Theta_0) (\Theta_2^2 + 2 \Theta_1 \Theta_3 - \frac{1}{3} \Theta_4) + 2 \cos(2\Theta_0) \Theta_2^3 + s \partial_s \Theta_2 K^i_j \dot{K}_i^j + s^3 \partial_s \Theta_0 \left((K^r_r)^4 + \frac{1}{2} (K^r_r K^r_r)^2 - 2 (K^r_r)^2 K^i_j K_i^j\right) - \frac{s^2}{2} \partial_s \Theta_1 K^r_r \left((K^r_r)^2 - 3 K^i_j K_i^j\right) - \frac{1}{\sqrt{g}} \partial_j (\sqrt{g} g^{jk} \partial_k \Theta_2) - \frac{2s}{\sqrt{g}} \partial_j (\sqrt{g} g^{jk} \partial_k \Theta_1) + s g^{jk} (\partial_j K^r_r) \partial_k \Theta_1 + \partial_s \Theta_3 (K^r_r - \frac{\gamma_1}{K} \dot{p} \dot{X}), \quad (24)$$

where $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial \sigma^i$. We have taken into account the fact that $\Theta_0$ does not depend on $\sigma^i$.

Notice that all Eqs.(19) for $\Theta_k$ are linear. The only nonlinear equation in our perturbative scheme is the zeroth order equation (17).

It is very important to observe that operator $\hat{L}$ has a zero-mode, that is a function $\psi_0(s)$ which quickly vanishes in the limits $s \to \pm\infty$, and which obeys the equation

$$\hat{L} \psi_0 = 0.$$
Inserting $\Theta_{s_0}(s)$ in Eq.(17), differentiating that equation with respect to $s_0$ and putting $s_0 = 0$ we obtain as an identity that $\hat{L}\psi_0 = 0$ where

$$\psi_0(s) = \frac{1}{\cosh s}. \quad (25)$$

The presence of this zero-mode is related to the invariance of Eq.(17) with respect to translations in $s$, therefore it is often called the translational zero-mode. Let us multiply both sides of Eqs.(19) by $\psi_0(s)$ and integrate over $s$. Integration by parts gives

$$\int_{-\infty}^{\infty} ds \psi_0 \hat{L} \Theta_k = \int_{-\infty}^{\infty} ds \Theta_k \hat{L} \psi_0 = 0.$$

Hence, we obtain the consistency (or integrability) conditions

$$\int_{-\infty}^{\infty} ds \psi_0(s)f_k(s) = 0, \quad (26)$$

where $f_k$ are given by formulas of the type (21) - (24). We shall see in the next Section that these conditions play very important role in determining the curved domain wall solutions.

Using standard methods one can obtain the following formulas for vanishing in the limits $s \to \pm \infty$ solutions $\Theta_k$ of Eqs.(19):

$$\Theta_k = G[f_k] + C_k(\sigma^i, t)\psi_0(s), \quad (27)$$

where

$$G[f_k] = -\psi_0(s) \int_0^s dx \psi_1(x)f_k(x) + \psi_1(s) \int_{-\infty}^s dx \psi_0(x)f_k(x). \quad (28)$$

Here $\psi_0(s)$ is the zero-mode (25) and

$$\psi_1(s) = \frac{1}{2}(\sinh s + \frac{s}{\cosh s}) \quad (29)$$

is the other solution of the homogeneous equation

$$\hat{L}\psi = 0.$$

The second term on the r.h.s. of formula (27) obeys the homogeneous equation $\hat{L}\Theta_k = 0$. It vanishes when $s \to \pm \infty$. 

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The solutions (27) contain as yet arbitrary functions \( C_k(\sigma^i, t) \). Also \( \vec{X}(\sigma^i, t) \) giving the comoving surface \( S \) has not been specified. It turns out that conditions (26) are so restrictive that they essentially fix those functions. The extrinsic curvature coefficients \( K^i_l \) and the metric \( g_{ik} \) will follow from \( \vec{X}(\sigma^i, t) \).

One can worry that \( G[f_k], k \geq 1 \), given by formula (28) do not vanish when \( s \to \pm \infty \) because the second term on the r.h.s. of formula (28) is proportional to \( \psi_1 \), which exponentially increases in the limits \( s \to \pm \infty \). However, the integrals

\[
\int_{-\infty}^{s} dx \psi_0 f_k
\]

vanish in that limit due to the consistency conditions (26). Moreover, qualitative analysis of Eq.(15) shows that \( f_k \sim (\text{polynomial in } s) \times \exp(-|s|) \) for large \( |s| \), hence those integrals behave like \( (\text{polynomial in } s) \times \exp(-2|s|) \) for large \( |s| \) ensuring that all \( G[f_k] \) exponentially vanish when \( |s| \to \infty \).

5 The approximate domain wall solutions

In this Section we discuss the approximate solutions obtained with the help of the perturbative scheme we have just described. We present formulas for \( \Theta_1 \) and \( \Theta_2 \), an equation for \( \vec{X}(\sigma^i, t) \) which determines motion of the surface \( S \), as well as equations for the functions \( C_1, C_2 \).

The zeroth order solution is already known, see formula (18). This allows us to discuss the consistency condition with \( k = 1 \). Substituting \( f_1 \) from formula (21) and noticing that

\[
\partial_s \Theta_0 = \frac{1}{\cosh s} = \psi_0(s)
\]

we find that that condition is equivalent to

\[
\frac{\gamma_1}{K} p^r \vec{X} = K^r_r.
\]

This condition is in fact the equation for \( \vec{X} \). It is of the same type as Allen-Cahn equation [10], but in our approach it governs motion of the auxiliary surface \( S \).
Let us now turn to the perturbative corrections. After taking into account Allen-Cahn equation (30) we have \( f_1 = 0 \). Therefore, the total first order contribution has the form

\[
\Theta_1 = \frac{C_1(\sigma^i, t)}{\cosh s}.
\]  

(31)

The second order contribution \( \Theta_2 \) is calculated from formula (28) with \( f_2 \) given by formula (22). Using the results (30), (31) we obtain the following formula

\[
\Theta_2 = \psi_2(s)C^2_1(\sigma^i, t) + \psi_3(s)K^i_jK^j_i + \frac{C_2(\sigma^i, t)}{\cosh s},
\]

(32)

where

\[
\psi_2(s) = -\frac{\sinh s}{2\cosh^2 s},
\]

\[
\psi_3(s) = \frac{1}{2}s\cosh s - \frac{s}{2\cosh s} - \psi_1(s)\ln(2\cosh s)
\]

\[
+ \frac{s^2\sinh s}{4\cosh^2 s} - \frac{1}{4\cosh s}\int^s_0 dx x^2\frac{x}{\cosh^2 x}.
\]

The integral in \( \psi_3(s) \) can easily be evaluated numerically. Due to the consistency conditions, the functions \( C_1, C_2 \) in formulas (31), (32) are not arbitrary, see below.

The consistency condition (26) with \( k = 2 \) does not give any restrictions — it can be reduced to the identity \( 0 = 0 \). More interesting is the next condition, that is the one with \( k = 3 \). Inserting formula (23) for \( f_3 \) and calculating necessary integrals over \( s \) we find that it can be written in the form of the following inhomogeneous equation for \( C_1(\sigma^i, t) \)

\[
\frac{\gamma_1}{K}(\partial_k C_1 - g^{kr}\tilde{X}_r\tilde{X}_\partial_k C_1 - \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k C_1) - K^i_jK^j_i C_1
\]

\[= \frac{\alpha^2}{24}K^r_i \left((K^i_i)^2 - 3K^i_jK^j_i \right).
\]

(33)

We have also used Allen-Cahn equation (30). Equation (33) determines \( C_1 \) provided that we fix initial data for it. Similarly, the consistency condition coming from the fourth order (\( k = 4 \)) is equivalent to the following homogeneous equation for \( C_2 \)

\[
\frac{\gamma_1}{K}(\partial_k C_2 - g^{kr}\tilde{X}_r\tilde{X}_\partial_k C_2)
\]

\[= \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k C_2) - K^i_jK^j_i C_2 = 0.
\]

(34)
The formulas (16), (18), (31) and (32) give a whole family of domain walls. To obtain one concrete domain wall solution we have to choose initial position of the auxiliary surface $S$. Its positions at later times are determined from Allen-Cahn equation (30). We also have to fix initial values of the functions $C_1, C_2$, and to find the corresponding solutions of Eqs.(33), (34). Notice that we are not allowed to choose the initial profile of the domain wall because the dependence on the transverse coordinate $s$ is explicitly given. It is known from formulas (18), (31) and (32). Any choice of the initial data gives an approximate domain wall solution. Of course such a choice should not lead to large perturbative corrections at least in certain finite time interval. Therefore one should require that at the initial time $\xi_m C_1 \ll 1, \xi_m^2 C_2 \ll 1, \xi_m K'_i \ll 1$. The domain wall is located close to the surface $S$ because for large $|s|$ the perturbative contributions vanish and the leading term $2 \arctan(e^s)$ is close to one of the vacuum values $0, \pi$.

Let us remark that Eqs.(30), (33) and (34) imply that a planar domain wall ($K'_i = 0$) can not move, in contradistinction with relativistic domain walls for which uniform, inertial motions are possible.

In the presented approach we describe evolution of the domain wall in terms of the surface $S$ and of the functions $C_1, C_2$. These functions can be regarded as fields defined on $S$. In some cases Eqs. (30), (33), (34) can be solved analytically, one can also use numerical methods. Anyway, these equations are much simpler than the initial Eq.(5).

The presented formalism is invariant with respect to changes of coordinates $\sigma^1, \sigma^2$ on $S$. In particular, in a vicinity of any point $\bar{X}$ of $S$ we can choose the coordinates in such a way that $g_{ik} = \delta_{ik}$ at $\bar{X}$. In these coordinates Eq.(30) has the form

$$\frac{\gamma_1 v}{K} = \frac{1}{R_1} + \frac{1}{R_2},$$

where $v$ is the velocity in the direction $\bar{p}$ perpendicular to $S$ at the point $\bar{X}$, and $R_1, R_2$ are the main curvature radii of $S$ at that point.

As an example, let us consider cylindrical and spherical domain walls. If $S$ is a straight cylinder of radius $R$ then $R_1 = \infty$, $R_2 = -R(t)$, $v = \dot{R}$ and Eq.(35) gives

$$R(t) = \sqrt{R_0^2 - \frac{2K}{\gamma_1}(t - t_0)},$$

where $R_0$ is the initial radius. The origin of the Cartesian coordinate frame
is located on the symmetry axis of the cylinder $S$ (which is the $z$-axis), $\vec{p}$ is the outward normal to $S$, and $s = (\sqrt{r^2 - z^2} - R(t))/\xi_m$, where $r$ is the radial coordinate in $R^3$. As $\sigma^1, \sigma^2$ we take the usual cylindrical coordinates $z, \phi$. Equations (33), (34) reduce to

$$\frac{\gamma_1}{K} \partial_t C_1 - \left( \partial_z^2 C_1 + \frac{1}{R^2} \partial_\phi^2 C_1 \right) - \frac{1}{R^2} C_1 = \frac{\pi^2}{12} \frac{1}{R^3}, \quad (37)$$

$$\frac{\gamma_1}{K} \partial_t C_2 - \left( \partial_z^2 C_2 + \frac{1}{R^2} \partial_\phi^2 C_2 \right) - \frac{1}{R^2} C_2 = 0. \quad (38)$$

If $C_1, C_2$ at the initial time $t_0$ have just constant values $C_1(0), C_2(0)$ on the cylinder, then

$$C_1(t) = \frac{\pi^2}{12R(t)} \ln(R_0/R(t)) + \frac{R_0}{R(t)} C_1(0), \quad C_2(t) = \frac{R_0}{R(t)} C_2(0). \quad (39)$$

General solutions of Eqs.(37), (38) can be found by splitting $C_1, C_2$ into Fourier modes, but we shall not present them here.

The case of spherical domain wall is quite similar. Now $S$ is a sphere of radius $R$ and $R_1 = R_2 = -R, v = \dot{R}$. Equation (35) gives

$$R(t) = \sqrt{R_0^2 - \frac{4K}{\gamma_1}(t-t_0)}, \quad (40)$$

Now the origin is located at the center of the sphere, $s = (r - R(t))/\xi_m$, and $\vec{p}$ is the outward normal to $S$. As $\sigma^k$ we take the usual spherical coordinates. Then, Eqs.(33), (34) can be written in the form

$$\frac{\gamma_1}{K} \partial_t C_1 - \frac{1}{R^2} \left( \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta C_1) + \frac{1}{\sin^2 \theta} \partial_\phi^2 C_1 \right) - \frac{2}{R^2} C_1 = \frac{\pi^2}{6} \frac{1}{R^3}, \quad (41)$$

and

$$\frac{\gamma_1}{K} \partial_t C_2 - \frac{1}{R^2} \left( \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta C_2) + \frac{1}{\sin^2 \theta} \partial_\phi^2 C_2 \right) - \frac{2}{R^2} C_2 = 0. \quad (42)$$

General solution of these equations can be obtained by expanding $C_1, C_2$ into spherical harmonics. In the particular case when $C_1, C_2$ are constant on the sphere $S$ the solutions $C_k(t)$ have the same form (39) as in the previous case except that now $R(t)$ is given by formula (40).
In the both cases our approximate formulas are expected to be meaningful as long as $R(t)/\xi_m \gg 1$.

Because we know the transverse profile of the domain wall, we can express the total free energy $F$ by geometric characteristics of the domain wall. One should insert our approximate solution for $\Theta$ in formulas (2) and (3) for $F_e$ and $F_m$, respectively, and to perform integration over $s$. The volume element $d^3x$ is taken in the form

$$d^3x = \xi_m \sqrt{G} d^2\sigma ds.$$  

For simplicity, let us consider curved domain walls for which $C_1 = 0 = C_2$

at the time $t_0$. Straightforward calculation gives

$$F = -\frac{K}{2} \frac{V}{\xi_m^2} + \frac{2K}{\xi_m} |S|$$

$$-\frac{\pi^2}{6} K \xi_m \int d^2\sigma \sqrt{g} \left( \frac{1}{R_1^2} + \frac{1}{R_2^2} - \frac{1}{R_1 R_2} \right) + \text{terms of the order } \xi_m^3, \quad (43)$$

where $|S|$ denotes the area of the surface $S$, and $V$ is the total volume of the liquid crystalline sample. The first term on the r.h.s. of this formula is just a bulk term which appears because the smallest value of the magnetic free energy density has been chosen to be equal to $-K/(2\xi_m^2)$. The proper domain wall contribution starts from the second term. This term gives the main contribution of the domain wall to $F$. One can think about the corresponding constant free energy $2K/\xi_m$ per unit area. The third term on the r.h.s. of formula (43) represents the first perturbative correction. It is of the order $(\xi_m/R_i)^2$ when compared with the main term, and within the region of validity of our perturbative scheme it is small. One can easily show that this term is negative or zero. Hence, it slightly diminishes the total free energy. In this sense, the domain walls have negative rigidity — bending them without stretching (i.e., with $|S|$ kept constant) diminishes the free energy.

6 **Remarks**

We would like to add several remarks about the expansion in width and the approximate domain wall solutions it yields.
1. In the presented approach dynamics of the curved domain wall in the three dimensional space is described in terms of the comoving surface $S$ and of the functions $C_k, k \leq 1$, defined on $S$. The profile of the domain wall has been explicitly expressed by these functions, by the transverse coordinate $\xi$, and by the geometric characteristics of $S$. The surface $S$ and the functions $C_k$ obey equations (30), (33), (34) which do not contain $\xi$. In particular cases these equations can be solved analytically, and in general one can look for numerical solutions. Such numerical analysis is much simpler than it would be in the case of the initial equation (5) for the angle $\Theta$, precisely because one independent variable has been eliminated.

2. We have used $\xi_m$ as a formal expansion parameter. This may seem unsatisfactory because it is a dimensionful quantity, hence it is hard to say whether its value is small or large. What really matters is smallness of the corrections $\xi_m \Theta_1, \xi_m^2 \Theta_2$. This is the case if $\xi_m C_1 \ll 1, \xi_m^2 C_2 \ll 1$ and $\xi_m K_i^j \ll 1$, as it follows from formulas (31) and (32).

3. Notice that an assumption that $S$ coincides with the core for all times in general would not be compatible with the expansion in width. If we assume that $C_1 = 0 = C_2$ at certain initial time $t_0$, Eq.(33) implies that $C_1 \neq 0$ at later times (unless the r.h.s. of it happens to vanish). Then, it follows from formulas (16), (18) and (31) that $\Theta \neq \pi/2$ at $s = 0$, that is on $S$.

4. In the present work we have neglected effects which could come from perturbations of the exponentially small tails of the domain wall. For example, consider a domain wall in the form of infinite straight cylinder flattened from two opposite sides. Its front and rear flat sides have zero curvatures, and according to Eq.(35) they do not move. In our approximations the domain wall shrinks from the sides where the mean curvature $1/R_1 + 1/R_2$ does not vanish. Now, in reality the front and rear parts interact with each other. This interaction is exponentially small only if the two flat parts are far from each other. We have neglected it altogether assuming the $2 \arctan(e^s)$ asymptotics at large $s$. In this sense, our approximate solution takes into account only the effects of curvature.

5. Finally, let us mention that the dynamics of domain walls in nematic liquid crystals can also be investigated with the help of another approximation scheme called the polynomial approximation. In the first paper [11] it has been applied to a cylindrical domain wall, and in the second one to a planar soliton. Comparing the two approaches, the polynomial approximation is much cruder than the expansion in width. It also contains more arbitrariness,
e. g., in choosing concrete form of boundary conditions at $|s| \to \infty$. On the other hand, that method is much simpler. It can be useful for rough estimates.

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