INEQUALITIES BETWEEN DIRICHLET AND NEUMANN EIGENVALUES OF THE POLYHARMONIC OPERATORS

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Abstract. We prove that \( \mu_{k+m}^m < \lambda_k^m \), where \( \mu_{k+m}^m \) (\( \lambda_k^m \)) are the eigenvalues of \((-\Delta)^m\) on \( \Omega \subset \mathbb{R}^d \), \( d \geq 2 \), with Neumann (Dirichlet) boundary conditions.

1. Introduction and main result

Let \( m \in \mathbb{N} \). Let \( \Omega \) be a bounded domain (i.e., a bounded connected open set) in \( \mathbb{R}^d \), \( d \geq 2 \), such that the embedding \( H^m(\Omega) \subset L^2(\Omega) \) is compact. We denote here by \( H^m(\Omega) \) the standard Sobolev space of (complex valued) functions in \( L^2(\Omega) \) with weak derivatives up to order \( m \) in \( L^2(\Omega) \), endowed with the standard scalar product

\[
\langle u, v \rangle_{H^m(\Omega)} = \int_{\Omega} D^m u \cdot D^m v + u\bar{v} \, dx,
\]

and induced norm

\[
\| u \|_{H^m(\Omega)} = \left( \int_{\Omega} |D^m u|^2 + |u|^2 \, dx \right)^{1/2},
\]

where

\[
D^m u \cdot D^m v = \sum_{j_1, \ldots, j_m = 1}^d \left( \frac{\partial^m_{j_1 \ldots j_m} u}{\partial x_{j_1} \cdots \partial x_{j_m}} \right) \left( \frac{\partial^m_{j_1 \ldots j_m} v}{\partial x_{j_1} \cdots \partial x_{j_m}} \right).
\]

Here and in what follows \( \partial^m_{j_1 \ldots j_m} \) denotes \( \frac{\partial^m}{\partial x_{j_1} \cdots \partial x_{j_m}} \).

We consider the following variational problem:

\[
\int_{\Omega} D^m u \cdot D^m \phi \, dx = \Lambda \int_{\Omega} u \phi \, dx, \quad \forall \phi \in H(\Omega),
\]

in the unknowns \( u \in H(\Omega) \) and \( \Lambda \in \mathbb{R} \). Here \( H(\Omega) \) denotes a subspace of \( H^m(\Omega) \) containing \( H^m_0(\Omega) \), the closure in \( H^m(\Omega) \) of \( C_0^\infty(\Omega) \). From the hypotheses on \( H(\Omega) \) and the fact that \( H^m(\Omega) \subset L^2(\Omega) \) is compact, it follows that problem (1.4) admits an increasing sequence of non-negative eigenvalues of finite multiplicity diverging to \( +\infty \).

We will denote by \( \{ \lambda_k^m \}_{k \in \mathbb{N}} \) the eigenvalues of (1.4) when \( H(\Omega) = H^m_0(\Omega) \). Here we agree to repeat the eigenvalues according to their multiplicity. We will denote
by \( \{u_m^i\}_{k \in \mathbb{N}} \) the corresponding eigenfunctions, normalized by \( \int_{\Omega} u_m^i u_m^j dx = \delta_{ij} \) for all \( i, j \in \mathbb{N} \). It is easy to check that \( \lambda_m^0 > 0 \) for all \( m \in \mathbb{N} \).

We will denote by \( \{\mu_k^m\}_{k \in \mathbb{N}} \) the eigenvalues of (1.4) when \( H(\Omega) = H^m(\Omega) \), where we agree to repeat the eigenvalues according to their multiplicity. We will denote by \( \{v_k^m\}_{k \in \mathbb{N}} \) the corresponding eigenfunctions, normalized by \( \int_{\Omega} v_m^i v_m^j dx = \delta_{ij} \) for all \( i, j \in \mathbb{N} \). It is easy to check that \( \mu_k^m = \cdots = \mu_{n(d,m)}^m = 0 \) and \( \mu_{n(d,m)+1}^m > 0 \) for all \( m \in \mathbb{N} \), where \( n(d,m) \) denotes the dimension of the space of polynomials of degree at most \( m - 1 \) in \( \mathbb{R}^d \).

We will call \( \{\lambda_k^m\}_{k \in \mathbb{N}} \) the Dirichlet spectrum of the polyharmonic operator \((-\Delta)^m\) and \( \{\mu_k^m\}_{k \in \mathbb{N}} \) the Neumann spectrum of the polyharmonic operator \((-\Delta)^m\). In fact problem (1.4) is the weak formulation of

\[
\begin{cases}
(-\Delta)^m u = \Lambda u, & \text{in } \Omega, \\
u = \frac{\partial u}{\partial v} = \cdots = \frac{\partial^{m-1} u}{\partial v^{m-1}} = 0, & \text{on } \partial \Omega
\end{cases}
\]

when \( H(\Omega) = H_0^m(\Omega) \), and of

\[
\begin{cases}
(-\Delta)^m u = \Lambda u, & \text{in } \Omega, \\
N_i u = \cdots = N_m u = 0, & \text{on } \partial \Omega
\end{cases}
\]

when \( H(\Omega) = H^m(\Omega) \). Here \( N_i u, i = 1, \ldots, m, \) are uniquely defined complementing boundary operators of degree at most \( 2m - 1 \) which correspond to the classical Neumann boundary conditions for the polyharmonic operator. In particular, if \( m = 1 \) then \( N_1 u = \frac{\partial u}{\partial v} \); if \( m = 2 \) then \( N_1 u = \frac{\partial^2 u}{\partial v^2} \) and \( N_2 u = \text{div}_0(D^2 u) - \frac{\partial \Delta u}{\partial v} \).

It is in general a quite involved task to write the explicit form of the Neumann boundary conditions for \( m \geq 3 \).

The eigenvalues \( \lambda_k^m \) admit the following variational characterization

\[
\lambda_k^m = \min_{\dim(W)=k} \max_{W \subset H_0^m(\Omega), w \neq 0} \frac{\int_{\Omega} |D^m w|^2 dx}{\int_{\Omega} |w|^2 dx},
\]

and the eigenvalues \( \mu_k^m \) admit the following variational characterization

\[
\mu_k^m = \min_{W \subset H^m(\Omega), w \neq 0} \max_{\dim(W)=k} \frac{\int_{\Omega} |D^m w|^2 dx}{\int_{\Omega} |w|^2 dx}.
\]

From (1.7) and (1.8) it follows immediately that \( \mu_k^m \leq \lambda_k^m \) for all \( m, k \in \mathbb{N} \).

General inequalities between the eigenvalues of the Dirichlet and Neumann Laplacian on \( \Omega \subset \mathbb{R}^d, d \geq 2 \), have been widely studied in the past. The study of such inequalities was initiated by Payne [6] who proved that \( \mu_{k+2}^1 < \lambda_k^1 \) for a bounded smooth convex domain in \( \mathbb{R}^2 \). He also conjectured in [7] that in general \( \mu_{k+1}^1 \leq \lambda_k^1 \) should hold (and even more, under additional assumptions like convexity). Generalization to higher dimension of Payne’s result were achieved independently by Aviles [8] and Levine-Weinberger [9]. In particular, Aviles proved that \( \mu_{k+1}^1 < \lambda_k^1 \) for smooth bounded domains in \( \mathbb{R}^d \) with boundary of non-negative mean curvature. Levine and Weinberger established a family of inequalities of the form \( \mu_{k+r}^1 < \lambda_k^1 \), \( 1 \leq r \leq d \), under a series of assumptions on the sign of the principal curvatures of the boundary. In particular, if the domain is convex, \( r = d \) (extending the result of Payne), while if the mean curvature is non-negative, \( r = 1 \) (recovering Aviles’ result). A definitive answer to Payne’s conjecture was given by Friedlander [10] who proved \( \mu_{k+1}^1 < \lambda_k^1 \) for a general smooth bounded domain in \( \mathbb{R}^d \). An alternative
proof (which does not require smoothness of the boundary) has been proposed in \cite{2}.

We will prove an analogue of Friedlander’s result for the polyharmonic operator $(-\Delta)^m$ namely,

**Theorem 1.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, such that the embedding $H^m(\Omega) \subset L^2(\Omega)$ is compact. Then

$$
\mu_{k+m}^m < \lambda_k^m
$$

for all $m, k \in \mathbb{N}$.

**Remark 1.2.** We remark that when $d = 1$ it is easy to verify that $\mu_{k+m}^m = \lambda_k^m$ for all $m, k \in \mathbb{N}$. Let $\Omega = (0, 1)$. The general solution of $(-1)^m u^{(2m)}(x) = \lambda u(x)$ with $\lambda \in \mathbb{R}$, $\lambda > 0$, is given by $u(x) = \sum_{j=0}^{2m-1} \alpha_j \exp \left( \frac{\pi i (j+1)}{2m} \lambda \# x \right)$ for some $\alpha_0, \ldots, \alpha_{2m-1} \in \mathbb{C}$. Dirichlet boundary conditions read $u(0) = u(1) = u'(0) = u'(1) = \cdots = u^{(m-1)}(0) = u^{(m-1)}(1) = 0$, while Neumann boundary conditions read $u^{(m)}(0) = u^{(m)}(1) = \cdots = u^{(2m-1)}(0) = u^{(2m-1)}(1) = 0$. In both cases, boundary conditions produce a homogeneous system of $2m$ equations in $2m$ unknowns $\alpha_0, \ldots, \alpha_{2m-1}$ which has a solution if and only if the determinant of the associated matrix is zero. Let us indicate by $\det M_D(\lambda)$ $\det M_N(\lambda)$ the determinants of the systems produced by the Dirichlet and Neumann boundary conditions, respectively. All the positive Dirichlet (Neumann) eigenvalues are exactly the solutions of $\det M_D(\lambda) = 0$ ($\det M_N(\lambda) = 0$). It is straightforward to see that the two equations $\det M_D(\lambda) = 0$ and $\det M_N(\lambda) = 0$ have the same positive solutions. Moreover, by standard Sturm-Liouville theory, all the positive eigenvalues of both Dirichlet and Neumann problems have multiplicity one. The first Dirichlet eigenvalue $\lambda_1^m$ is positive. On the contrary, 0 is an eigenvalue of the Neumann problem of multiplicity $m$, in fact the functions $1, x, \ldots, x^{m-1}$ form a basis of the corresponding eigenspace. We conclude then that $\mu_{k+m}^m = \lambda_k^m$ for all $m, k \in \mathbb{N}$.

### 2. Proof of Theorem 1.1

We will prove now Theorem 1.1. We prove first the following lemma.

**Lemma 2.1.** Let $\{c_j\}_{j=1}^n \subset \mathbb{C}$ be a set of $n$ distinct complex numbers and let $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$, $\omega \neq 0$, be fixed. Then $\{e^{c_j \omega \cdot x}\}_{j=1}^n \subset C^\infty(\mathbb{R}^d, \mathbb{C})$ is a linearly independent set of functions.

**Proof.** Assume that there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with $\alpha_j \neq 0$ for some $j$ such that $\sum_{j=1}^n \alpha_j e^{c_j \omega \cdot x} = 0$. We can assume without loss of generality that $\omega_1 \neq 0$. Let us differentiate $n - 1$ times the equality $\sum_{j=1}^n \alpha_j e^{c_j \omega \cdot x} = 0$ with respect to $x_1$. We find that $\sum_{j=1}^n \alpha_j (c_{j,1})^k e^{c_{j,1} \omega \cdot x} = 0$ for all $k = 0, \ldots, n - 1$ and $x \in \mathbb{R}^d$. In particular, for $x = 0$ we have $\sum_{j=1}^n \alpha_j (c_{j,1})^k = 0$ for all $k = 0, \ldots, n - 1$. This is a system of $n$ equations in $n$ unknowns $\alpha_1, \ldots, \alpha_n$. The associated matrix is a square Vandermonde matrix with non-zero determinant since all the $c_{j,1}$ are distinct by hypothesis. Then necessarily $\alpha_1 = \cdots = \alpha_n = 0$, which contradicts the fact that $\alpha_j \neq 0$ for some $j$. This concludes the proof.

We are now ready to prove Theorem 1.1.
Proof of Theorem 7.1 Let $U$ denote the subspace of $H^m_0(\Omega)$ generated by the first $k$ Dirichlet eigenfunctions $u_1^m, \ldots, u_k^m$. By hypothesis, $\dim(U) = k$. Let us also denote by $\{\xi_j\}_{j=1}^m$ the set of the $m$-th roots of the unity, namely $\xi_j = e^{2\pi i j}$ for $j = 0, \ldots, m - 1$. Let $\omega \in \mathbb{R}^d$ be a non-zero vector and let $V_\omega$ be the space generated by $\{e^{i\xi_j \omega \cdot x}|\Omega\}_{j=1}^m$.

From Lemma 2.1 it follows that $\dim(V_\omega) = m$ (we can assume without loss of generality that $0 \in \Omega$).

We observe now that $H^m_0(\Omega) \cap V_\omega = \{0\}$. In fact we can proceed as in the proof of Lemma 2.1 and assume that there exist $\alpha_1, \ldots, \alpha_m$ with $\alpha_j \neq 0$ for some $j$ such that $v = \sum_{j=1}^m \alpha_j e^{i\xi_j \omega \cdot x} \in H^m_0(\Omega)$. Assume $\omega \neq 0$ and differentiate $m - 1$ times the function $v$ with respect to $x_1$. We find that $\sum_{j=1}^m \alpha_j (i\xi_j \omega_1) e^{i\xi_j \omega \cdot x} = 0$ for all $x \in \partial \Omega$ and $k = 0, \ldots, m - 1$. Again, this is not possible unless $\alpha_1 = \cdots = \alpha_m = 0$.

This proves that the sum $U + V_\omega$ is a direct sum for all choices of $\omega \neq 0$, hence $\dim(U + V_\omega) = k + m$. It is easy to verify that $V_\omega \cap V_z = \{0\}$ for all non-zero $\omega, z \in \mathbb{R}^d$, $\omega \neq z$ (see also Lemma 2.1). Let $N \subset H^m(\Omega)$ be the space generated by all the Neumann eigenfunctions corresponding to the eigenvalue $\lambda_k^m$. If $\lambda_k^m$ is not a Neumann eigenvalue, we set $N = \{0\}$. Since all the eigenvalues have finite multiplicity, $\dim(N) < \infty$.

We note that $U \cap N = \{0\}$. This is trivial if $\lambda_k^m$ is not a Neumann eigenvalue since $N = \{0\}$. Otherwise, assume that there exists $u \neq 0 \in U \cap N$. Hence

$$\int_\Omega D^m u \cdot D^m \phi dx = \lambda_k^m \int_\Omega u \phi dx, \quad \forall \phi \in H^m(\Omega).$$

Let us denote by $\tilde{u}$ the extension by 0 of $u$ to $\mathbb{R}^d$ which belongs to $H^m(\mathbb{R}^d)$, since $u \in U \subset H^m_0(\Omega)$. Then

$$\int_\Omega D^m \tilde{u} \cdot D^m \phi dx = \lambda_k^m \int_\Omega \tilde{u} \phi dx, \quad \forall \phi \in H^m(\mathbb{R}^d).$$

This implies that $\tilde{u}$ is analytic in $\mathbb{R}^n$ and satisfies $(-\Delta)^m \tilde{u} = \lambda_k^m \tilde{u}$, and therefore $\tilde{u} = 0$ in $\mathbb{R}^d$. In particular $u = 0$, a contradiction.

We conclude then that there exists $\omega \in \mathbb{R}^d$ with $|\omega|^{2m} = \lambda_k^m$ such that $(U + V_\omega) \cap N = \{0\}$. In fact, we have that $U \cap N = \{0\}$, $V_\omega \cap V_z = \{0\}$ for all $z \neq \omega$ and $U \cap V_\omega = \{0\}$ for all $\omega$. Since we can choose any $\omega \in \mathbb{R}^d$ with $|\omega|^{2m} = \lambda_k^m$ and the space of such $\omega$ has infinite dimension (if $d \geq 2$), then we necessarily find a $\omega$ such that $(U + V_\omega) \cap N = \{0\}$ (the space $N$ has finite dimension).

We set then $V := V_\omega$. Now let $W := U + V$. Clearly $W \subset H^m(\Omega)$ with $\dim(W) = k + m$. From (1.8) it follows immediately that

$$\mu_{k+m}^m \leq \max_{w \neq 0} \frac{\int_\Omega |D^m w|^2 dx}{\int_\Omega |w|^2 dx}. \quad (2.1)$$

We will prove now that

$$\int_\Omega |D^m w|^2 dx \leq \lambda_k^m \int_\Omega |w|^2 dx \quad (2.2)$$
for all \( w \in W \). Any \( w \in W \) is of the form \( w = u + v \) with \( u \in U \) and \( v \in V \). We have

\[
\int_\Omega |D^m w|^2 \, dx = \int_\Omega \left( |D^m u|^2 + |D^m v|^2 + 2 \text{Re} \left( D^m u \cdot D^m v \right) \right) \, dx \nln \int_\Omega \left( |D^m u|^2 + |D^m v|^2 + 2 \text{Re} \left( u (-\Delta)^m v \right) \right) \, dx.
\]

The second equality is a consequence of the fact that \( u \in H^m_0(\Omega) \) and of \( m \) integrations by parts. From the definition of \( U \) it follows that

\[
\int_\Omega |D^m u|^2 \, dx \leq \lambda_k^m \int_\Omega |u|^2 \, dx
\]

for all \( u \in U \). Assume to know that

\[
|D^m v|^2 = \lambda_k^m |v|^2
\]

and

\[
(-\Delta)^m v = \lambda_k^m v
\]

for all \( v \in V \). Then from (2.3), (2.4), (2.5) and (2.6) it follows

\[
\int_\Omega |D^m w|^2 \, dx \leq \lambda_k^m \int_\Omega \left( |u|^2 + |v|^2 + 2 \text{Re} \left( uv \right) \right) \, dx = \lambda_k^m \int_\Omega |u + v|^2 \, dx = \lambda_k^m \int_\Omega |w|^2 \, dx,
\]

which implies \( \mu_{k+m}^m \leq \lambda_k^m \) by (2.1). Then, in order to prove \( \mu_{k+m}^m \leq \lambda_k^m \) we have to prove (2.7) and (2.9). Let \( v \in V \), i.e., \( v = \sum_{j=1}^m \alpha_j e^{i \xi_j \omega_x} \) for some \( \alpha_1, ..., \alpha_m \in \mathbb{C} \). We have

\[
-\Delta v = \sum_{j=1}^m \alpha_j \xi_j^2 |\omega|^2 e^{i \xi_j \omega_x},
\]

which immediately implies

\[
(-\Delta)^m v = \sum_{j=1}^m \alpha_j \xi_j^{2m} |\omega|^{2m} e^{i \xi_j \omega_x} = \lambda_k^m \sum_{j=1}^m \alpha_j e^{i \xi_j \omega_x} = \lambda_k^m v,
\]

and hence the validity of (2.9). Moreover,

\[
|D^m v|^2 = \sum_{j_1, \ldots, j_m=1}^d \left| \partial_{j_1 \ldots j_m} v \right|^2 = \sum_{j_1, \ldots, j_m=1}^m \left| \sum_{j=1}^m \alpha_j \xi_j^m \omega_{j_1} \ldots \omega_{j_m} e^{i \xi_j \omega_x} \right|^2 = \sum_{k,l=1}^m \alpha_k \alpha_l \xi_k \omega_{j_1} \omega_{j_2} \ldots \omega_{j_m} e^{i \xi_k \omega_x} - e^{-i \xi_l \omega_x} = \sum_{k,l=1}^m \alpha_k \alpha_l \xi_k \omega_{j_1} \omega_{j_2} \ldots \omega_{j_m} e^{-i \xi_k \omega_x} - e^{i \xi_l \omega_x} \sum_{j_1, \ldots, j_m=1}^d \omega_{j_1}^2 \ldots \omega_{j_m}^2 = |\omega|^{2m} \sum_{k,l=1}^m \alpha_k \alpha_l \xi_k \omega_{j_1} \omega_{j_2} \ldots \omega_{j_m} e^{-i \xi_k \omega_x} = \lambda_k^m \sum_{j=1}^m \alpha_j e^{i \xi_j \omega_x} = \lambda_k^m |v|^2,
\]
therefore (2.3) holds.

We have then proved that \( \mu_{k+m} \leq \lambda_k^m \) for all \( m, k \in \mathbb{N} \). We observe that the inequality is actually strict. In fact, assume that \( \mu_{k+m} = \lambda_k^m \) is a Neumann eigenvalue. Then equality holds in (2.3) and this implies that there exists a non-zero \( u \in W \cap N \), but this is not possible since \( W \cap N = \{0\} \) by construction. This concludes the proof.

We conclude this note with a few remarks.

**Remark 2.2.** We remark that the proof of the strict inequality in Theorem 1.1 does not work in dimension \( d = 1 \) since we have only two choices of \( \omega \in \mathbb{R} \) with \( |\omega|^{2m} = \lambda_k^m \), and not necessarily \( (U + V) \cap N = \{0\} \) (in dimension \( d \geq 2 \) we can choose any vector \( \omega \) of the sphere of radius \( (\lambda_k^m)^{1/2} \), and hence we can always find a vector \( \omega \) such that \( (U + V) \cap N = \{0\} \)). For example, let us consider the case of the Laplacian in one dimension. We can choose the space \( V \) in the proof of Theorem 1.1 either to be the space generated by \( e^{i(\lambda_k^m)x} \) or the space generated by \( e^{-i(\lambda_k^m)x} \). It is easy to see that there exists \( v \in (U + V) \cap N \), \( v \neq 0 \). In fact, any element of \( U + V \) is of the form \( \sum_{j=1}^m \alpha_j \sin((\lambda_j^m)^{1/2}x) + \beta e^{i(\lambda_k^m)^{1/2}x} \) (we have chosen here the space \( V \) generated by \( e^{i(\lambda_k^m)^{1/2}x} \)), for some \( \alpha_1, ..., \alpha_k, \beta \in \mathbb{C} \). Choose now \( \alpha_j = 0 \) for \( j = 1, ..., k-1 \) and \( \alpha_k = -i\beta \). Then \( v = \cos((\lambda_k^m)^{1/2}x) \) which is clearly a non-zero function in \( N \).

**Remark 2.3.** We recall that the eigenvalue 0 of the operator \( (-\Delta)^m \) with Neumann boundary conditions has multiplicity \( n(d, m) \), the dimension of the space of polynomials in \( \mathbb{R}^d \) of degree at most \( m - 1 \). We have \( n(d, m) > m \) for all \( m \in \mathbb{N} \) and \( d \geq 2 \) (while \( n(1, m) = m \) for all \( m \)). Hence it is natural to conjecture that actually \( \mu_{n(d, m) + k} \leq \lambda_k^m \) for all \( m, k \in \mathbb{N} \) (or even the strict inequality). If we want to repeat the proof of Theorem 1.1 in order to obtain this last inequality, we should find for all \( \lambda_k^m \) a \( n(d, m) \)-dimensional subspace \( V \) of \( H^m(\Omega) \) of functions such that, for all \( v \in V \)

i) \( (-\Delta)^m v = \lambda_k^m v \) (in the \( L^2(\Omega) \) sense);
ii) \( |D^m v|^2 = \lambda_k^m |v|^2 \) (in the \( L^2(\Omega) \) sense);
iii) \( H^m_0(\Omega) \cap V = \{0\} \).

If we want the strict inequality, we shall additionally require that \( V + U \) is linearly independent with \( N \) (\( U, N \) are as in the proof of Theorem 1.1).

We note that such a space \( V \) cannot be generated by \( n(d, m) \) functions of the form \( e^{i\omega_j \cdot x} \) with \( \omega_j \in \mathbb{C}^d \) for all \( j = 1, ..., n(d, m) \), since in general we cannot find more that \( m \) distinct \( \omega_j \in \mathbb{C}^d \) such that \( v = \sum_{j=1}^{n(d, m)} e^{i\omega_j \cdot x} \) satisfy i), ii) and iii) (one can easily verify such claim for \( m = 1, d = 2 \) and \( n(2, 2) = 3 \)). We also note that actually less than ii) is needed to conclude as in Theorem 1.1 it is sufficient that \( \int_\Omega |D^m v|^2 dx \leq \lambda_k^m \int_\Omega |v|^2 dx \) for all \( v \in V \), however this seems to be false in general if \( V \) is generated by more than \( m \) functions of the form \( e^{i\omega_j \cdot x} \).

**Remark 2.4.** One may think to prove inequalities among Dirichlet and Neumann eigenvalues of the polyharmonic operators by proving inequalities among \( \lambda_k^1 \) and \( \lambda_k^m \) and inequalities among \( \mu_k^1 \) and \( \mu_k^m \) and combining them with Friedlander’s result [3].

Inequalities between Dirichlet eigenvalues of the Laplacian and of the polyharmonic operators can be easily obtained via min-max principle (1.7) and suitable
integrations by parts. In particular, for any \( u \in H^m_0(\Omega) \), \( m \geq 1 \) we have

\[
\int_\Omega |D^m u|^2 \, dx \leq \left( \int_\Omega |D^{m+1} u|^2 \, dx \right)^{1/2} \left( \int_\Omega |D^{m-1} u|^2 \, dx \right)^{1/2}.
\]

(2.11)

In order to prove formula (2.11) we recall the following identity which holds for all \( u \in H^m_0(\Omega) \) and which is obtained integrating by parts \( \int_\Omega |D^m u|^2 \, dx \) (see e.g., [3]):

\[
\int_\Omega |D^m u|^2 \, dx = \begin{cases} 
\int_\Omega |\Delta \bar{u}|^2 \, dx, & \text{if } m \text{ is even,} \\
\int_\Omega |\nabla \Delta \bar{u}|^2 \, dx, & \text{if } m \text{ is odd.}
\end{cases}
\]

Assume now that \( u \in H^{m+1}_0(\Omega) \) (actually \( u \in H^{m+1}(\Omega) \cap H^m_0(\Omega) \) is sufficient). Let \( m \) be even. Then

\[
\int_\Omega |D^m u|^2 \, dx = \int_\Omega |\Delta \bar{u}|^2 \, dx = - \int_\Omega |\nabla \Delta \bar{u}|^2 \, dx \\
\leq \left( \int_\Omega |\nabla \Delta \bar{u}|^2 \, dx \right)^{1/2} \left( \int_\Omega |\nabla \Delta \bar{u}|^2 \, dx \right)^{1/2} \\
= \left( \int_\Omega |D^{m+1} u|^2 \, dx \right)^{1/2} \left( \int_\Omega |D^{m-1} u|^2 \, dx \right)^{1/2},
\]

where we have used Hölder’s inequality. This proves (2.11) in the case that \( m \) is even. The case \( m \) odd is treated similarly. Inequality (2.11) is then proved.

In particular, when \( m = 2 \) we deduce from (2.11) that

\[
\left( \frac{\int_\Omega |\Delta u|^2 \, dx}{\int_\Omega u^2 \, dx} \right)^{1/2} \leq \left( \frac{\int_\Omega |D^2 u|^2 \, dx}{\int_\Omega u^2 \, dx} \right)^{1/2}.
\]

By induction on \( m \) it follows that for any \( u \in H^{m+1}_0(\Omega) \),

\[
\left( \frac{\int_\Omega |D^m u|^2 \, dx}{\int_\Omega u^2 \, dx} \right)^{1/m} \leq \left( \frac{\int_\Omega |D^{m+1} u|^2 \, dx}{\int_\Omega u^2 \, dx} \right)^{1/(m+1)}.
\]

This implies, by (1.7), that \((\lambda^m_k)^{1/m}\) is an increasing function of \( m \) for all \( k \in \mathbb{N} \), hence \((\lambda^m_k)^m \leq \lambda^m_k\) for all \( m, k \in \mathbb{N} \) (the inequality is actually strict). Hence, from [3] it follows that \((\mu^m_k)^m < \lambda^m_k\) for all \( m, k \in \mathbb{N} \).

One wish to obtain now a reverse inequality between Neumann eigenvalues of the Laplacian and the polyharmonic operator in order to close the chain. However such inequalities are unavailable, and actually it is not clear whether a sort of monotonicity holds in the Neumann case. The inequality \((\mu^m_k) \leq (\mu^1_k)^m\) is trivially true only if \( d = 1 \) and follows by testing eigenfunctions of the Neumann Laplacian into (1.3). On the contrary, already for \( m = 2 \) and \( d \geq 2 \) it is not at all understood if such an inequality always holds.

We prove here, as an example, that the inequality \( \mu^2_k \leq (\mu^1_k)^2 \) holds for convex domains in \( \mathbb{R}^d, d \geq 2 \). In fact, for any \( u \in H^2(\Omega) \) real-valued we have

\[
\int_\Omega |D^2 u|^2 \, dx = \int_\Omega (\Delta u)^2 \, dx + \frac{1}{2} \int_{\partial \Omega} \partial_u (|\nabla u|^2) \, d\sigma - \int_{\partial \Omega} \Delta u \partial_u u \, d\sigma.
\]

(2.12)

Here \( \nu \) denotes the outer unit normal to \( \partial \Omega \) and \( d\sigma \) the measure element of \( \partial \Omega \). Equality (2.12) follows from the pointwise identity \( |D^2 u|^2 = \frac{1}{2} \Delta (|\nabla u|^2) - \Delta u \cdot \nabla u \)
which holds for smooth real-valued functions $u$. Now, we note that

$$
(2.13) \quad \frac{1}{2} \partial_{\nu}(|\nabla u|^2)_{|\partial \Omega} = \nabla (\partial_{\nu} u) \cdot \nabla u_{|\partial \Omega} - D_\nu \cdot \nabla u = \nabla_{\partial \Omega} (\partial_{\nu} u) \cdot \nabla u_{|\partial \Omega} + \partial_{\nu} (\partial_{\nu} u) \nu \cdot \nabla u_{|\partial \Omega} - II (\nabla_{\partial \Omega} u, \nabla_{\partial \Omega} u).
$$

Here $\nabla_{\partial \Omega} u_{|\partial \Omega}$ denotes the tangential component of the gradient of $u$ on the boundary, and $II(\cdot, \cdot)$ the second fundamental form on $\partial \Omega$ (in fact $II = D_\nu$).

Assume now that $u \in H^2(\Omega)$ is such that $\partial_{\nu} u = 0$ on $\partial \Omega$ (in the sense of $L^2(\partial \Omega)$) and that $II \geq 0$ in the sense of quadratic forms. Then $\nabla u_{|\partial \Omega} = \nabla_{\partial \Omega} u$ (the gradient of $u$ restricted on the boundary belongs to the tangent space to the boundary). This fact combined with (2.12) and (2.13) implies that for such $u$ and $\Omega$

$$
\int_{\Omega} |D^2 u|^2 dx \leq \int_{\Omega} (\Delta u)^2 dx.
$$

Moreover, if $\Omega$ is a convex domain, then all eigenfunctions of the Neumann Laplacian belong to $H^2(\Omega)$ by standard elliptic regularity, and their normal derivatives vanish at the boundary (in $L^2(\partial \Omega)$). Hence, when $m = 2$, taking into (1.3) as $k$-dimensional subspace of $H^2(\Omega)$ of test functions the space generated by the first $k$ eigenfunctions of the Neumann Laplacian, we immediately obtain $\mu^2_k \leq (\mu^1_k)^2$ for all $k \in \mathbb{N}$.

However, also in the convex case, we cannot conclude more than $\mu^2_{k+1} < \lambda^2_k$, which is a weaker result with respect to $\mu^2_{k+2} < \lambda^2_k$ proved in Theorem 1.1.

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