ON CONVERGENCE SETS OF POWER SERIES WITH HOLOMORPHIC COEFFICIENTS

BASMA AL-SHUTNAWI, HUA LIU, AND DAOWEI MA

Abstract. We consider convergence sets of formal power series of the form $f(z, t) = \sum_{n=0}^{\infty} f_n(z) t^n$, where $f_n(z)$ are holomorphic functions on a domain $\Omega$ in $\mathbb{C}$. A subset $E$ of $\Omega$ is said to be a convergence set in $\Omega$ if there is a series $f(z, t)$ such that $E$ is exactly the set of points $z$ for which $f(z, t)$ converges as a power series in a single variable $t$ in some neighborhood of the origin. A $\sigma$-convex set is defined to be the union of a countable collection of polynomially convex compact subsets. We prove that a subset of $\mathbb{C}$ is a convergence set if and only if it is $\sigma$-convex.

1. Introduction

The purpose of this article is to describe the convergence sets of formal power series with holomorphic coefficients. The study of convergence sets comes from generalizations of Hartogs Theorem (see [2, 4, 8, 11]). Our approach is motivated by recent work [3, 4, 9] concerning formal power series $F(z, t) = \sum_{n=0}^{\infty} P_n(z) t^n$, whose coefficients are polynomials of one or more complex variables. In some of these studies the authors focus on power series of the form $F(z, t) = f(tz_1, \ldots, tz_N) = \sum_{n=0}^{\infty} P_n(z) t^n$, where $P_n(z)$ are homogeneous polynomials of degree $n$ for $n \in \mathbb{N}$. We say that $E \subset \mathbb{C}^N$ is the convergence set of $F$ if for every $z \in E$ there exists some $r_z > 0$ such that $F(z, t)$ converges for $t < r_z$ while for each $z \in \mathbb{C}^N \setminus E$ the radius of convergence of $F(z, t)$ equals 0. Note that it is always assumed that $\deg P_n \leq n$ in these investigations.

Some related problems from operator calculus and renormalization of quantum field theory (see, e.g., [6]) are concerned with the formal series of the form

$$F(T, t) = \sum_{n=0}^{\infty} K_n(T) t^n,$$

where $K_n(T)$ belong to the $C^*$ algebra generated by some operator $T$. It is necessary to discuss the convergence set of $F(z, t)$ for both spectrum analysis of $F(T, t)$ and perturbation theory. Here $K_n(T)$ are holomorphic functions on some neighborhood of the compact set $K$, the spectra of $T$. It is desirable to find the necessary and sufficient conditions for a set $E$ to be the convergence set of some $F(z, t) = \sum_{n=0}^{\infty} f_n(z) t^n$. This article answers the question completely when $N = 1$.

2000 Mathematics Subject Classification. Primary: 32A05, 30C85.

Key words and phrases. formal power series, analytic functions, convergence sets.
2. Convergence Sets

Let \( \Omega \) be an open subset in the complex space \( \mathbb{C}^N \). Denote by \( \mathcal{O}(\Omega) \) the set of holomorphic functions on \( \Omega \).

We consider the power series of the form

\[
f(z, t) = \sum_{n=0}^{\infty} f_n(z)t^n, \quad z \in \Omega,
\]

where \( f_n(z) \in \mathcal{O}(\Omega) \) and \( t \) is a complex variable. We denote by \( \mathcal{O}(\Omega)[[t]] \) the collection of the series of form (1).

**Definition 2.1.** Let \( f(z, t) \in \mathcal{O}(\Omega)[[t]] \). We define the *convergence set of \( f \) in \( \Omega \) by*

\[
\text{Conv}_\Omega(f) = \{ z \in \Omega : f(z, t) \text{ converges in some neighborhood of 0} \},
\]

or equivalently,

\[
\text{Conv}_\Omega(f) = \{ z \in \Omega : |f_n(z)| \leq r_z^n \text{ for some } r_z > 0 \text{ and every } n \in \mathbb{N} \}.
\]

**Definition 2.2.** A subset \( E \subset \Omega \) is said to be a *convergence set in \( \Omega \) if there exists an \( f \in \mathcal{O}(\Omega)[[t]] \) such that \( E = \text{Conv}_\Omega(f) \). A convergence set in \( \mathbb{C}^N \) is also simply called as *convergence set*.

**Proposition 2.3.** Let \( K \) be a polynomially convex compact set in \( \mathbb{C}^N \). Then \( K \) is a convergence set in \( \mathbb{C}^N \).

**Proof.** Let \( m \) be any positive integer and \( y \in \mathbb{C}^N \setminus K \). Since \( K \) is a polynomially convex, there exists a polynomial \( P_y(z) \) such that \( |P_y(y)| > m \) and \( |P_y(z)| \leq 1 \) for \( z \in K \).

Set \( U_y = \{ x \in \mathbb{C}^N : |P_y(x)| > m \} \). The open cover \( U_y, y \in \mathbb{C}^N \setminus K \), of the set \( \mathbb{C}^N \setminus K \) contains a countable subcover \( U_{yk}, k = 1, 2, \ldots \). Now denote by \( P_{mk}(z) = P_{yk}(z) \). For each \( m \) we get a sequence \( \{ P_{mk} \}_{k=1}^{\infty} \). Since the set \( \{ P_{mk} \} \) is countable we can arrange it as a sequence \( \{ h_j(z) \}_{j=1}^{\infty} \). Set

\[
f(z, t) = \sum_{j=1}^{\infty} h_j^2(z)t^j.
\]

Suppose that \( z \in K \). Then for each \( j \), \( |h_j(z)| \leq 1 \). Hence \( z \in \text{Conv}_{\mathbb{C}^N}(f) \). Consequently, \( K \subset \text{Conv}_{\mathbb{C}^N}(f) \).

Now suppose that \( z \in \mathbb{C}^N \setminus K \). Then for each \( m \in \mathbb{N} \) there is a \( k \in \mathbb{N} \) such that \( |P_{mk}(z)| \geq m \). It follows that the sequence \( \{ |h_j(z)| \} \) is unbounded. So the formal power series \( f(z, t) \) is divergent at \( z \). Consequently, \( \text{Conv}_{\mathbb{C}^N}(f) \subset K \). That is, \( K \) is a convergence set in \( \mathbb{C}^N \).

The following proposition provides a necessary condition for a set to be a convergence set.

**Proposition 2.4.** Let \( E \) be a convergence set in \( \Omega \subset \mathbb{C}^N \). Then \( E \) is an \( F_\sigma \) set.

**Proof.** Suppose that \( E = \text{Conv}_\Omega(f) \), where

\[
f(z, t) = f_0(z) + f_1(z)t + \cdots + f_n(z)t^n + \cdots \in \mathcal{O}(\Omega)[[t]].
\]
For $j \in \mathbb{N}$ denote by $K_j = \{ z \in \Omega : \text{dist}(z, \partial \Omega) \geq \frac{1}{j} \}$ and $|z| < j$. Then $\Omega = \bigcup_{j=1}^{\infty} K_j$ and each $K_j$ is contained in the interior of $K_{j+1}$. We now prove that

$$E = \bigcup_{j=1}^{\infty} \bigcap_{n=0}^{\infty} \{ z \in K_j : |f_n(z)| \leq j^n \}.$$  

For $z \in E$, suppose that $z \in K_L$ for some integer $L$. By the definition of convergence set, there exists a positive integer $J$ such that

$$|f_n(z)| < J^n, \quad n \in \mathbb{N}.$$  

Let $m = \max\{L, J\}$. Then $z \in \bigcap_{n=0}^{\infty} \{ z \in K_m : |f_n(z)| \leq m^n \}$. On the other hand, assume that $z \in \bigcup_{j=1}^{\infty} \bigcup_{l=1}^{\infty} \{ z \in K_j \setminus B_l : |f_n(z)| \leq j^n \}$. Then there exist a positive integer $j$ such that $z \in \bigcap_{n=0}^{\infty} \{ z \in K_j \setminus B_l : |f_n(z)| \leq j^n \}$. So $|f_n(z)| \leq j^n$ for all $n$, i.e., $z \in E$.

It is clear that $\bigcap_{n=0}^{\infty} \{ z \in K_j \setminus B_l : |f_n(z)| \leq j^n \}$ are closed. By (2), $E$ is an $F_\sigma$ set. \qed

The converse of Theorem 2.4 is not true, for which we will give a counterexample in the next section.

We now discuss the intersection of several convergence sets.

**Proposition 2.5.** Let $E_1, \ldots, E_k$ be convergence sets in $\Omega$. Then the intersection $E := \cap_{j=1}^{k} E_j$ is also a convergence set in $\Omega$.

**Proof.** It suffices to prove that the intersection of two convergence sets in $\Omega$ is a convergence set in $\Omega$. Suppose that we have two formal power series in $\mathcal{O}(\Omega)([t])$

$$f(z, t) = f_0(z) + f_1(z)t + \cdots + f_n(z)t^n + \cdots,$$

and

$$g(z, t) = g_0(z) + g_1(z)t + \cdots + g_n(z)t^n + \cdots$$

and the corresponding convergence sets $A = \text{Conv}_{\Omega}(f)$ and $B = \text{Conv}_{\Omega}(g)$, respectively. Define by

$$F(z, t) = F_0(z) + F_1(z)t + \cdots + f(z, t^2) + tg(z, t^2)$$

$$= f_0(z) + g_0(z)t + f_1(z)t^2 + g_1(z)t^3 + \cdots.$$  

Then

$$F_n(z) = \begin{cases} f_2(z), & \text{if } n \text{ is even;} \\ g_{n+1}(z), & \text{if } n \text{ is odd.} \end{cases}$$  

For $z \in A \cap B$ suppose that $|f_n(z)| < r_A^n$ and $|g_n(z)| < r_B^n$ for some $r_A, r_B > 0$ and every $n \in \mathbb{N}$. Then $|F_n(z)| < (\max\{r_A, r_B\})^n$. So $A \cap B \subset \text{Conv}_{\Omega}(F)$.

On the other hand, for $z \in \text{Conv}_{\Omega}(F)$ suppose that $|F_n(z)| < r^n$. Then both $|f_n(z)|$ and $|g_n(z)|$ are less than $(r^2)^n$, i.e., $\text{Conv}_{\Omega}(F) \subset A \cap B$. \qed

In the rest of the article we only consider the case $N = 1$. We discuss some properties for the convergence sets in the complex plane. For any countable set $E$ in $\mathbb{C}$ we in the following theorem construct a formal series whose convergence set in $\mathbb{C}$ is exactly $E$. For the convenience of the proof, denote by $i.e.$,

$$N_r(S) = \{ z \in \mathbb{C} : |z - p| < r \text{ for some } p \in S \}$$
for $S \subset \mathbb{C}$ and $r > 0$.

**Theorem 2.6.** Let $S = \{z_1, z_2, \ldots\}$ be a countable infinite subset of $\mathbb{C}$. Define an $F \in \mathbb{C}[z][[t]]$ by

$$F(z, t) = \sum_{n=0}^{\infty} C_n \left[ \prod_{j=1}^{n} (z - z_j) \right] t^n,$$

where $C_n = (n/\gamma_n)^n$, and

$$\gamma_n = \min\left( \frac{1}{2^{n+1}} \min_{1 \leq i < j \leq n+1} |z_i - z_j|, 1/n \right).$$

Then $\text{Conv}_C(F) = S$.

**Proof.** Note that $\gamma_n$ is positive since $z_i$ are pairwise distinct. Let $L_j = \{z_1, \ldots, z_{i+1}\}$ for $j \in \mathbb{N}$. We now prove that

$$\bigcap_{j=k}^{\infty} N_{\gamma_j}(L_j) = L_k, \ k \in \mathbb{N}. \quad (4)$$

We only need to prove $\bigcap_{j=k}^{\infty} N_{\gamma_j}(L_j) \subset L_k$, which would follow from the following statement:

$$\bigcap_{s=k}^{\infty} N_{\gamma_s}(L_s) \subset N_{\gamma_j}(L_k), \text{ for } j \geq k. \quad (5)$$

We prove (5) by induction on $j$. It is obvious for $j = k$ since $N_{\gamma_k}(L_k) = U_k$. Suppose the statement is true for $j = k$. Let $z \in \bigcap_{s=k}^{\infty} N_{\gamma_s}(L_s)$ and by (4), $z \notin \bigcup_{\ell=k+1}^{N+1} N_{\gamma_N}(z_\ell)$. Since $\gamma_{N+1} \leq \gamma_N$ we see that $\bigcup_{\ell=k+1}^{N+1} N_{\gamma_{N+1}}(z_\ell) \subset \bigcup_{\ell=k+1}^{N+1} N_{\gamma_N}(z_\ell)$, hence

$$z \notin \bigcup_{\ell=k+1}^{N+1} N_{\gamma_{N+1}}(z_\ell). \quad (6)$$

By the induction hypothesis $z \in N_{\gamma_N}(L_k)$ and by (4), $z \notin \bigcup_{\ell=k+1}^{N+1} N_{\gamma_N}(z_\ell)$. Since $\gamma_{N+1} \leq \gamma_N$ we see that $\bigcup_{\ell=k+1}^{N+1} N_{\gamma_{N+1}}(z_\ell) \subset \bigcup_{\ell=k+1}^{N+1} N_{\gamma_N}(z_\ell)$, hence

$$z \notin \bigcup_{\ell=k+1}^{N+1} N_{\gamma_{N+1}}(z_\ell). \quad (7)$$

On the other hand, we know that

$$z \in N_{\gamma_{N+1}}(L_{N+1}) = \left( \bigcup_{\ell=k+1}^{N+1} N_{\gamma_{N+1}}(z_\ell) \right) \cup \left( N_{\gamma_{N+1}}(L_k) \right). \quad (8)$$

By (7) and (8), $z \in N_{\gamma_{N+1}}(L_k)$. This completes the induction step, and therefore the statement is proved.

Now let $P_0(z) = 1$ and, for $n \geq 1$,

$$P_n(z) = C_n \prod_{k=1}^{n} (z - z_k). \quad (9)$$
Then $F(z, t) = \sum_{n=0}^{\infty} P_n(z)t^n$. For $n \geq k$, we have $P_n(z_k) = 0$. It follows that $S \subset \text{Conv}_C(F)$.

Now suppose that $z \not\in S$. By (1), $z \not\in \bigcap_{j=k}^{\infty} U_j$ for each $k$. It follows that there is a strictly increasing sequence $j_k$ of positive integers such that $z \not\in U_{j_k}$ for $k = 1, 2, \ldots$. Then

$$\text{(10)} \quad |P_{j_k}(z)| \geq (\frac{j_k}{\gamma_{j_k}})^{j_k} \prod_{i=1}^{j_k} (\gamma_i) \geq (j_k)^{j_k},$$

since $\gamma_i \geq \gamma_{j_k}$ for $i \leq j_k$. This implies that $z \not\in \text{Conv}_C(f)$. Therefore $S = \text{Conv}_C(f)$. ∎

Example 2.7. By Theorem 2.6 the set $\mathbb{Q}$ of rational numbers is a convergence set in $\mathbb{C}$. But by Theorem 2.4 the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is not so, since it is not an $F_\sigma$ set.

Theorem 2.8. Let $\Omega \subset \mathbb{C}$, let $S$ be a countable dense subset of $\Omega$, let $\{C_n\}$ be a sequence of positive numbers, and let $A = \{a_1, a_2, \ldots, a_k\}$ be any finite subset of $\Omega$. Then there exists an enumeration $\{z_1, z_2, \ldots\}$ of $S$, such that $A \subset \text{Conv}_\Omega(F)$, where $F$ is defined by

$$F(z, t) = \sum_{n=0}^{\infty} C_n \left[ \prod_{j=1}^{n} (z - z_j) \right] t^n.$$

Proof. Suppose that the diameter of $A$ is $d$. Let $S = \{s_1, s_2, \ldots\}$. We choose distinct points $z_1, z_2, \ldots$, from $S$ such that the following are satisfied:

1. $|z_{l(k+1)+i} - a_i| < d$, and $C_{l(k+1)+i+p}|z_{l(k+1)+i} - a_i| < \frac{d}{(l+2)!}$, for $l = 0, 1, 2, \ldots, i = 1, 2, \ldots, k, p = 0, 1, \ldots, k$;
2. for $l = 1, 2, \ldots, z_{l(k+1)} = s_{\tau(l)}$, where $\tau(l) = \min\{p \in \mathbb{N} : |s_p - a_i| < ld, s_p \in S \setminus \{z_1, \ldots, z_{l(k+1)-1}\}\}$.

We now show that

$$|C_n(a_i - z_1) \cdots (a_i - z_n)| < (2d)^n, \quad n \geq k + 1, \quad i = 1, \ldots, k.$$

Fix $n \geq k + 1$ and $1 \leq i \leq k$. Choose $l \geq 0$ and $0 \leq p \leq k$ such that $n = l(k+1) + i + p$. Since

$$|a_i - z_{m(k+1)+j}| \leq |a_i - a_j| + |a_j - z_{m(k+1)+j}| < d + d = 2d,$$

for $m = 1, 2, \ldots$ and $1 \leq j \leq k$, and since

$$|a_i - z_{m(k+1)}| < (m+1)d < (m+1)(2d),$$

for $m = 1, 2, \ldots$, we have

$$\text{(11)} \quad |C_n(a_i - z_1) \cdots (a_i - z_n)|$$

$$= |(a_i - z_1) \cdots (a_i - z_{l(k+1)+i-1})| \cdot |C_{l(k+1)+i+p}(a_i - z_{l(k+1)+i})| \cdot |a_i - z_{l(k+1)+i+1}| \cdots |a_i - z_{l(k+1)+i+p}|$$

$$\leq (l + 2)! (2d)^{l(k+1)+i-1} \frac{d}{(l+2)!} (2d)^p$$

$$= (2d)^{n-1} (l + 2)! \frac{d}{(l+2)!} < (2d)^n.$$
In the above formula, the factor $(l + 2)!$ comes from those factors $(a_i - z_{a(k+1)})$ for $\alpha = 0, 1, \cdots, l + 1$. By (III) we know that the convergence radius of $F(z,t)$ is $> 1/(2d)$ for all $a_i \in A$ as the power series of $t$, which completes the proof. 

3. $\sigma$-convex-sets and Convergence Sets

Definition 3.1. Let $E$ be a compact subset of $\Omega$. The holomorphic hull of $E$ in $\Omega$ is defined by

$$\hat{E}_\Omega = \{ z \in \Omega : |h(z)| \leq \max_{\zeta \in E} |h(\zeta)| \text{ for all } h \in \mathcal{O}(\Omega) \}.$$ 

If $E = \hat{E}_\Omega$ we say that $E$ is holomorphically convex in $\Omega$.

Proposition 3.2. [5] Theorem 1.3.4] A compact subset $E$ of $\Omega \subset \mathbb{C}$ is holomorphically convex in $\Omega$ if and only if none of the bounded connected components of its complement is contained in $\Omega$.

By Proposition 3.2 it is easy to prove the following proposition.

Proposition 3.3. Let $E$ be a compact subset of $\Omega$. Then $\hat{E}_\Omega = E \cup (\bigcup C_\alpha)$, where $\{C_\alpha\}$ is the set of bounded connected components of $\mathbb{C} \setminus E$ that are contained in $\Omega$.

Lemma 3.4. Let $K$ be a holomorphically convex compact set in $\Omega \subset \mathbb{C}$. Then $\mathbb{C} \setminus K$ has only finite components.

Proof. Suppose that $\mathbb{C} \setminus K$ has an infinite number of bounded components. We may denote by $C_\alpha, \alpha = 1, 2, \cdots$, those components since $\mathbb{C} \setminus K$ is an open set. Again denote $d = \max_{w \in K} |w|$. It is obvious that $\{ z \in \mathbb{C} : |z| > d \}$ must be in the unbounded component of $\mathbb{C} \setminus K$. So $C_\alpha \cap \{ z \in \mathbb{C} : |z| > d \} = \emptyset, \forall \alpha \in 1, 2, \cdots$. We get that $U = \bigcup_{\alpha = 1}^{\infty} C_\alpha \subset \{ z \in \mathbb{C} : |z| \leq d \}$. That is, $U$ is bounded.

By Proposition 3.2 for any $j \in \mathbb{N}$, $C_j$ can not be contained in $\Omega$. Then we may choose $w_j \in C_j \setminus \Omega$. Let $w$ be a limited point of the bounded set $\{ w_j \}_{j=1}^\infty$. Without the confusion, we may assume that $w_j \to w$ when $j \to \infty$. Denote by

$$d_j = 2\sup\{ r > 0 : \text{ There exists } a \in C_j \text{ such that } D(a, r) \subset C_j \}$$

the inner diameter of $C_j$.

By the above argument we get $\sum_{j=1}^{\infty} \frac{1}{2} \pi d_j^2 \leq \text{area}(U) \leq \pi d^2$. So we obtain that $d_j$ tends to 0. Now by (12), we have

$$\text{dist}(w_j, K) \leq d_j$$

because $D(w_j, r) \subset C_j$ for all $r \leq \text{dist}(w_j, K) = \text{dist}(w_j, \partial C_j)$. Thus we obtain that $\text{dist}(w, K) = \lim_{j \to \infty} \text{dist}(w_j, K) = 0$, i.e., $w \in K \subset \Omega$. It is impossible because $\Omega$ contains only inner points while $w$ is a limited point of points outer of $\Omega$. Thus $\mathbb{C} \setminus K$ has only finite bounded components. 

Definition 3.5. A subset $K$ of $\Omega \subset \mathbb{C}$ is said to be a $\sigma$-holomorphically-convex set in $\Omega$ if it is the union of a countable collection of holomorphically convex compact subsets of $\Omega$. Moreover, $K$ is a $\sigma$-convex-set if it is the union of a countable collection of polynomial convex compact sets.

Lemma 3.6. Every holomorphically convex compact $K$ in $\Omega$ is a $\sigma$-convex-set.
Proof. By Lemma 3.4, $\mathbb{C} \setminus K$ has only finite components. We first suppose that it has only one bounded one, named as $C$. Let $a \in C$ and $0 < r < R$ such that $K \subset \mathbb{R}_{rR} = \{z \in \mathbb{C} : r < |z - a| < R\}$. For $1 < j \in \mathbb{N}$, set
\[ E_j = \{z = \rho e^{i\theta} + a : r < \rho < R; 0 \leq \theta \leq \left(1 - \frac{1}{j}\right)2\pi\} \]
and $K_j = K \cap E_j$. Then $K = \cup_{j=2}^{\infty} K_j$. We need to prove that $K_j$ is polynomially convex for all $j > 1$.

Let $w \notin K_j$ for some $j \in \mathbb{N}$. If $w$ is not in $K$, it should be in $C$ or the unbounded component of $\mathbb{C} \setminus K$. And in either case, $w$ can be connected with $\infty$ by a path not intersecting with $K_j$. If $w$ is not in the simply connected set $E_j$ it is still path-connected with $\infty$. So $\mathbb{C} \setminus K_j$ has no bounded component, that is, $K_j$ is polynomially convex by Proposition 3.2 and the remark after Definition 3.1.

For the case that $K$ is $m$-connected the proof is still valid when we take place of the fan family $\mathbb{R}_{rR} = \{z \in \mathbb{C} : r < |z - a| < R\} \setminus E_j$ by a curved fan family, each of which cut $K$ into a polynomially convex set. $\square$

The following proposition is obvious.

Proposition 3.7. Let $\{K_j\}_{j=1}^{\infty}$ be a sequence of $\sigma$-holomorphically-convex sets in $\Omega$. Then $\cup_{j=1}^{\infty} K_j$ is $\sigma$-holomorphically-convex.

By Lemma 3.6 and the above proposition we have the following corollary.

Corollary 3.8. Let $E$ be a $\sigma$-holomorphically-convex set in $\Omega$. Then it is also $\sigma$-convex.

So we only say the $\sigma$-convex set and not the $\sigma$-holomorphically-convex set in the rest of this paper.

Here are two easy examples of $\sigma$-convex sets. Let $\Delta(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.

Example 3.9. Every open set in the complex plane is a $\sigma$-convex set. Suppose $E \subset \mathbb{C}$ is open, then
\begin{equation}
E = \bigcup_{j=1}^{\infty} \Delta(w_j, \frac{r_{w_j}}{2}),
\end{equation}
where $\{w_j\}_{i=1}^{\infty}$ is a dense countable subset of $E$ and $r_{w_j} = \text{dist}(w_j, \partial E)$.

Example 3.10. The unit circle $\Gamma$ is a $\sigma$-convex set since $\Gamma = \cup_{j=1}^{\infty} \{e^{i\theta} : 0 \leq \theta \leq \left(1 - \frac{1}{j}\right)2\pi\}$.

The following is a counterexample.

Example 3.11. (Sierpinski triangle): Let $T$ be the equilateral triangle with vertices $A, B, C$ and sides of length 1. And let $D, E, F$ be the mid-points of sides $AB, BC, AC$ respectively. Denote 4 equilateral triangles by
\[ T_1 = \Delta_{ADF}, T_2 = \Delta_{DBE}, T_3 = \Delta_{ECF}, T_4 = \Delta_{DEF}, \]
among which $T_4$ is only the inverted triangle. Forgot $T_4$ and for $i_1 = 1, 2$ or 3 we still obtain $T_{i_1} = T_{i_11} \cup T_{i_12} \cup T_{i_13} \cup T_{i_14}$. Continue the process, for every $T_{i_1...i_k}$ ($i_l = 1, 2$
or 3, \(1 \leq l \leq k\), we obtain the decomposition: 
\[T_{i_1 \ldots i_k} = T_{i_1 \ldots i_{k-1}} \cup T_{i_1 \ldots i_{k-2}} \cup T_{i_1 \ldots i_{k-3}} \cup T_{i_1 \ldots i_{k-4}}\]

where \(T_{i_1 \ldots i_{k-4}}\) is inverted one. It is obvious that the side length of \(T_{i_1 \ldots i_k}\) is \(\frac{1}{2^{l-2}}\).

The Sierpinski triangle \(S\) is defined by
\[
S = \bigcap_{i=1}^{\infty} \bigcup_{i=1,2} T_{i_1 \ldots i_k} = T \setminus \bigcup_{i=1,2} \bigcup_{1 \leq i \leq l \leq k} T_{i_1 \ldots i_k}^o,
\]
where \(T_{i_1 \ldots i_k}^o\) means the inner part of \(T_{i_1 \ldots i_k}\).

Let \(S\) be contained in one open set \(\Omega\) in \(\mathbb{C}\). The Sierpinski triangle is not a \(\sigma\)-convex set. Otherwise, suppose that in \(\Omega\) there exists a sequence of holomorphically convex compact sets \(K_j, j = 1, 2, \ldots\), such that \(S = \bigcup_{j=1}^{\infty} K_j\). But as the closed set of the complete metric space \(\Omega\), the Sierpinski triangle is itself a complete metric space, and hence a set of second category. By Bair category theorem there is at least one \(K_j\) containing an induced non-empty open set \(V\) in \(S\). So there exists an open subset \(U \subset \Omega\), such that \(V = U \cap S \subset K_j\). Let \(v \in V \subset U\). For every \(k \in \mathbb{N}\) satisfying \(2^{-k} < \text{dist}(v, \partial U)\), by (14), \(v\) belongs to some one \(T_{i_1 \ldots i_k}\). Then \(T_{i_1 \ldots i_k} \subset U\). So \(T_{i_1 \ldots i_k}^o\) is also in \(U\). Since \(\partial T_{i_1 \ldots i_k}^o \subset S\) we get \(\partial T_{i_1 \ldots i_k}^o \subset V \subset K_j\). Then \(T_{i_1 \ldots i_k}^o\) is one of the complement of \(K_j\), it is clear that \(T_{i_1 \ldots i_k}^o\) is also in \(K_j\), which contradicts with Lemma 3.4.

**Proposition 3.12.** Let \(K_1, K_2\) be compact sets in \(\mathbb{C}\) with \(K_1 \cap K_2 = \emptyset\). Then \((K_1 \cup K_2)^\wedge = \widehat{K_1} \cup \widehat{K_2}\).

**Proof.** It suffices to show that \((K_1 \cup K_2)^\wedge \subset \widehat{K_1} \cup \widehat{K_2}\). Let \(U\) be a bounded connected component of \(\mathbb{C} \setminus (K_1 \cup K_2)\), and put \(Q = \partial \overline{U}\). Then 
\[Q \subset \partial \overline{U} \subset \partial U \subset \partial (\mathbb{C} \setminus (K_1 \cup K_2)) = \partial K_1 \cup K_2 \subset \overline{K_1 \cup K_2} = K_1 \cup K_2.\]

Since \(Q\) is connected, it follows that \(Q \subset K_1\) or \(K_2\), and hence \(U \subset \overline{Q} \subset \widehat{K_1} \cup \widehat{K_2}\). Therefore, 
\((K_1 \cup K_2)^\wedge \subset \widehat{K_1} \cup \widehat{K_2}\).

**Theorem 3.13.** Let \(E\) be a \(\sigma\)-convex set. Then there exist polynomially convex compact sets \(E_n, n = 1, 2, \ldots\) in \(\Omega\), such that \(E_n \subset E_{n+1}\) for \(n \geq 1\) and \(E = \bigcup_{n=1}^{\infty} E_n\).

**Proof.** By the definition of \(\sigma\)-convex-set, \(E\) can be written as 
\[
E = \bigcup_{j=1}^{\infty} K_j,
\]
where \(K_j\) is the polynomially convex compact set for each \(j\). For \(r > 0\) we denote by \(N_r(A)\) the \(r\)-neighborhood of \(A\). For a positive integer \(n\) set
\[
F_{n1} = L_{n1} = K_1.
\]

And step by step define
\[
L_{nj} = K_j \setminus N_{1/n} \left( \bigcup_{i=1}^{j-1} K_i \right), \quad F_{nj} = \widehat{L}_{nj}, \quad \text{for } 2 \leq j \leq n.
\]

Again denote by
\[
E_n = \bigcup_{j=1}^{n} F_{nj}.
\]
By Proposition 3.12 we obtain

\[ E_n = \bigcup_{j=1}^{n} F_{nj} = \bigcup_{j=1}^{n} \hat{L}_{nj} = \bigcup_{j=1}^{n} L_{nj}, \]

i.e., the set \( E_n \) is the polynomial hull of \( \bigcup_{j=1}^{n} L_{nj} \). Since \( L_{nj} \subset L_{(n+1)j} \) for each \( n \in \mathbb{N} \) and \( 1 \leq j \leq n \) we have \( E_n \subset E_{n+1} \).

We now prove that

\[ E = \bigcup_{n=1}^{\infty} E_n. \]

Let \( z \in E \). Assume that \( j \) is the first positive integer such that \( z \in K_j \), i.e., \( z \) in \( K_j \) but not in \( K_i \), \( i < j \). Then for \( n > 1/\text{dist}(z, \cup_{i=1}^{j-1} K_i) \) we have \( z \in K_j \setminus N_{\frac{1}{n}}(\cup_{i=1}^{j-1} K_i) = L_{nj} \).

And so \( E \subset \bigcup_{n=1}^{\infty} E_n \). On the other hand, let \( z \in \bigcup_{n=1}^{\infty} E_n \). Then there exist \( n \) such that \( z \in E_n = \bigcup_{j=1}^{n} F_{nj} \subset \bigcup_{j=1}^{n} \hat{K}_j = \bigcup_{j=1}^{n} K_j \subset E \). Therefore \( E = \bigcup_{n=1}^{\infty} E_n \). \( \square \)

We denote by \( d(E, F) \) the Euclidean distance between subsets \( E \) and \( F \).

**Lemma 3.14.** Let \( E, \{K_n\}, \{E_n\} \) be as in Theorem 3.13. Let \( U_n = N_{\frac{1}{m}}(E_n) \). Then for every positive integer \( m \), we have

\[ \bigcap_{j=m}^{\infty} U_j \subset E. \]

**Proof.** Otherwise, we suppose that there is an \( m > 0 \) and \( z \in \mathbb{C} \) such that

\[ z \in \left( \bigcap_{j=m}^{\infty} U_j \right) \setminus E. \]

We first claim that

\[ z \in N_{\frac{1}{m}} \left( \bigcup_{j=1}^{m} F_{nj} \right), \text{ for } n = m, m+1, \ldots \]

We prove \( \text{(21)} \) by induction on \( n \). \( \text{(21)} \) obviously holds for \( n = m \) by \( E_n = \bigcup_{j=1}^{m} F_{nj} \) and \( \text{(20)} \). If there exists a positive integer \( N \) so that \( \text{(21)} \) is true for \( n = N - 1 \) but not \( n = N \). Put

\[ Q = \bigcup_{j=m+1}^{N} F_{nj}, \quad R = Q \cap \left( \bigcup_{j=1}^{m} F_{nj} \right), \quad S = \left( \bigcup_{j=1}^{m} F_{nj} \right) \setminus R. \]

Suppose that \( z \in R \). Then there exists \( i \) and \( k \) with \( 1 \leq i \leq m < k \leq N \) such that \( z \in F_{N_i} \cap F_{N_k} \subset K_i \cap F_{N_k} \). Hence, \( z \) is not in \( L_{N_k} \). Since \( z \in F_{N_k} = \hat{L}_{N_k} \), we have that \( z \) belongs to one of the bounded components of the complement of \( L_{N_k} \). But \( N_{\frac{1}{N}}(z) \subset N_{\frac{1}{N}}(K_i) \) is contained in the complement of \( L_{N_k} \). So \( N_{\frac{1}{N}}(z) \) is just in this bounded component. Then \( N_{\frac{1}{N}}(z) \) is contained in the polynomially convex of \( L_{N_k} \), i.e., \( N_{\frac{1}{N}}(z) \subset \hat{L}_{N_k} = F_{N_k} \). So we obtain

\[ N_{\frac{1}{N}}(R) \subset Q. \]
Fix $i \leq m$. Let $z$ be in $K_i$ but not in $F_{N_k}$ for $m \leq k \leq N$. Then $z$ must be in the unbounded component of the complement of $L_{N_k}$ for $m \leq k \leq N$. Since $N_{1\over N}(z) \subset N_{1\over N}(K_i)$ does not intersect with $L_{N_k}$, we have that $N_{1\over N}(z)$ is just in that unbounded component. That is $N_{1\over N}(z) \cap \widetilde{L}_{N_k} = \emptyset$ for each $m \leq k \leq N$. So we have

\begin{equation}
(24) \quad d(S, Q) \geq {1 \over N}.
\end{equation}

Since (21) is assumed to hold for $N - 1$, and $N_{1\over 3N - 3} \left( \bigcup_{j=1}^{m} F_{N-1,j} \right)$ is the subset of $N_{1\over 3N - 3} \left( S \cup R \right)$, we see that

\begin{equation}
(25) \quad z \in N_{1\over 3N - 3} \left( S \cup R \right).
\end{equation}

But by the assumption that $z \notin E$, (23) and

\begin{equation}
N_{1\over 3N - 3}(R) \subset N_{1\over 3N - 3}(R) \subset Q \subset E
\end{equation}

we obtain

\begin{equation}
(26) \quad z \notin N_{1\over 3N - 3}(R) \subset N_{1\over 3N - 3}(R).
\end{equation}

It follows that

\begin{equation}
(27) \quad z \in N_{1\over 3N - 3}(S).
\end{equation}

By (24) we obtain

\begin{equation}
\begin{aligned}
d(z, Q) &\geq d(Q, S) - d(z, S) \\
&\geq {1 \over N} - {1 \over 3N - 3} \geq {1 \over 3N}.
\end{aligned}
\end{equation}

Thus

\begin{equation}
(28) \quad z \notin N_{1\over 3N}(Q).
\end{equation}

Combining (26) with (28) contradicts with $z \in U_N = N_{1\over 3N}(Q \cup S)$. Then we complete the induction step for (21). At last, by $\bigcup_{j=1}^{m} F_{nj} \subset \bigcup_{j=1}^{m} K_j$, we see that

\begin{equation}
(29) \quad d(z, \bigcup_{j=1}^{m} K_j) < {1 \over 3n}, \ \forall n \geq m.
\end{equation}

Therefore, $z \in \bigcup_{j=1}^{m} K_j \subset E$, contradicting with (20). The proof is complete. \hfill $\Box$

**Theorem 3.15.** Let $E$ be a convergence set in $\Omega$. Then $E$ is a $\sigma$-convex set. Moreover, there exists an ascending sequence $\{E_j\}$ of holomorphically convex compact sets in $\Omega$ such that $E = \bigcup_{j=1}^{\infty} E_j$.

**Proof.** Let $E$ be the convergence set $\text{Conv}_\Omega(f)$ for

\begin{equation}
f(z, t) = f_0(z) + f_1(z)t + \cdots + f_n(z)t^n + \cdots,
\end{equation}

where $f_n(z) \in \mathcal{O}(\Omega)$. Denote by

\begin{equation}
E_{jn} = \{ z \in \Omega : \text{dist}(z, \partial \Omega) \geq 1/j, |z| \leq j, |f_n(z)| \leq j^n \}, \ \forall j, n \in \mathbb{N},
\end{equation}

\begin{equation}
E_j = \bigcup_{n=1}^{\infty} E_{jn}.
\end{equation}

Then $E_j \subset E_{j+1}$, and $E = \bigcup_{j=1}^{\infty} E_j$ is a $\sigma$-convex set. The proof is complete. \hfill $\Box$
where \( \text{dist}(z, \partial \Omega) \) is the distance between \( z \) and the boundary of \( \Omega \). Every \( E_{jn} \) is obviously compact. Let

\[
E_j = \bigcap_{n=1}^{\infty} E_{jn}, \; j \in \mathbb{N}.
\]

Then we have

\[
E_j \subset E_{j+1}, \; \text{for } j \geq 1.
\]

By the proof of Theorem 2.4, \( E = \bigcup_{j=1}^{\infty} E_j \). It follows from the definition of holomorphically convex sets that each \( E_{jn} \) is holomorphically convex in \( \Omega \). It is also a direct consequence of the definition that the intersection of a family of holomorphically convex sets in \( \Omega \) is holomorphically convex. Therefore each \( E_j \) is a holomorphically convex compact set in \( \Omega \). The proof of Theorem 3.15 is complete.

For the domain \( \Omega \) and a positive integer \( m \) denote by

\[
\Omega_m = \{ z : z \in \Omega, \text{dist}(z, \partial \Omega) \geq 1/m, |z| \leq m \}.
\]

**Lemma 3.16.** Let \( K \) be a polynomially convex compact subset in \( \Omega \), \( U \subset \Omega \) an open set containing \( K \), and \( m \) a positive integer. Then there exist a finite number of polynomials, \( P_{m1}(z), \ldots, P_{m\ell}(z) \), such that

\[
|P_{mj}(z)| \leq 1, \; j = 1, \ldots, \ell, \; \text{for all } z \in K,
\]

and

\[
\max_j \{|P_{m1}(z)|, \ldots, |P_{m\ell}(z)|\} \geq m, \; \text{for all } z \in \Omega_m \setminus U.
\]

**Proof.** Due to the polynomial convexity of \( K \), for each \( z_0 \in \Omega_m \setminus U \) there exists a polynomial \( Q(z) \) such that

\[
|Q(z)| \leq 1, \; \text{for } z \in K, \; \text{and } |Q(z_0)| \geq m + 1.
\]

Then there is some neighborhood \( V(z_0) \) of \( z_0 \) such that \( |Q(z)| \geq m \) for each \( z \in V(z_0) \). Since \( \Omega_m \setminus U \) is compact there are a finite number of such open sets \( V(z_1), \ldots, V(z_\ell) \) covering \( \Omega_m \setminus U \). The corresponding polynomials are denoted by \( P_{mj}(z), \; \forall j = 1, \ldots, \ell \). Then \( P_{mj}(z), \forall j = 1, \ldots, \ell, \) satisfy (34) and (35). \( \square \)

Now we prove the main theorem of this paper.

**Theorem 3.17.** \( E \subset \Omega \) is a convergence set in \( \Omega \) if and only if it is \( \sigma \)-convex.

**Proof.** By Corollary 3.8 and Theorem 3.15 we only need to prove 'if'.

Let \( E \) be a \( \sigma \)-convex set. By Theorem 3.13 there exist polynomially convex compact sets \( E_n \) such that \( E = \bigcup_{n=1}^{\infty} E_n \), and \( E_n \subset E_{n+1} \) for \( n \geq 1 \). Let \( U_n \) be the neighborhood of \( E_n \) in Theorem 3.14 then we have

\[
\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} U_n \subset E.
\]
And by $E_k \subset E_{k+1}$ and $E_k \subset U_n$ for $n \geq k$ we obtain

$$E = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} U_n.$$ 

Now for each $k \in \mathbb{N}$, let $P_{k_1}, \ldots, P_{k_{n_k}}$ be the polynomials for $\Omega_{n} = \Omega_{k}, K = E_k$ and $U = U_k$ by Lemma 3.16. Then we have

$$|P_{k_s}(z)| \leq 1, \text{ for } z \in E_k, 1 \leq s \leq n_k$$

and for every $z \in \Omega \setminus U_k$ there exists $j, 1 \leq j \leq n_k$, such that

$$|P_{kj}(z)| \geq k.$$

Enumerate the countable set of polynomials $\{\{P_{kj}\}_{j=1}^{n_k}\}_{k=1}^{\infty}$ by $\{h_{\ell}\}_{\ell=1}^{\infty}$ such that $h_1 = P_{11}, h_2 = P_{12}, \ldots, h_{n_1} = P_{1n_1}, h_{n_1+1} = P_{21}, \ldots$. Define

$$f_0(z) = 1$$

and

$$f_{\ell}(z) = h_{\ell}^\ell(z), \forall \ell \in \mathbb{N}^+.$$

For any $z \in E$ there exist $k$ such that $z \in E_n$ for $n \geq k$. Then for $\ell > n_1 + \cdots + n_k$ we have $|f_{\ell}(z)| \leq |h_{\ell}(z)| \leq 1$. It implies that $z \in \text{Conv}_{\Omega}(f)$. Thus $E \subset \text{Conv}_{\Omega}(f)$.

For any $z \in \Omega \setminus E = \Omega \setminus (\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n = \bigcap_{n=1}^{\infty} (\Omega \setminus U_n)$, there exist infinite positive integer $m$ such that $z \in \Omega_m$. Then for any $k > m$ we have $z \in \Omega_k$. By Lemma 3.16 and taking $\Omega_m = \Omega_k, K = E_k, U = U_k$, we have a polynomial $P_{kj}(z)$ such that $|P_{kj}(z)| > k$. Let $\ell = n_1 + \cdots + n_{k-1} + j$. We obtain

$$|f_{\ell}(z)| > k^\ell \geq m^\ell.$$

To summarize, for each $z \notin E$, and for each positive integer $m$, there exist infinite positive integers $\ell$ such that $|f_{\ell}(z)| > m^\ell$. Consequently, it implies that $z \notin \text{Conv}_{\Omega}(f)$. Hence $\text{Conv}_{\Omega}(f) \subset E$. Therefore $E = \text{Conv}_{\Omega}(f)$. \qed

It is difficult to directly deal with the union of a countable collection of convergence sets in $\Omega$. But by Theorem 3.17, it is equivalent to the case of $\sigma$-convex sets, while the latter is easily done by definition.

**Corollary 3.18.** The union of a countable collection of convergence sets in $\Omega$ is a convergence set in $\Omega$.

**Remark.** In this paper we don’t confine the degree of the coefficient polynomials in the definition of the convergence set. However it is mentioned in the first section that $\text{Conv}(f)$ is a polar set if the degree of coefficients $P_n(z)$ are assumed not great than $n$. While it is surprising that it is not essential for the degree of the coefficient polynomial when it is great than $n$. For $\varepsilon > 0$, let the $n$-th term coefficients of $f(z, t)$ is polynomial with degree not great than $n^{1+\varepsilon}$. We call $\text{Conv}(f)$ as the $\varepsilon$-convergence-set. Then the $E \subset \mathbb{C}$ is a convergence set if only if it is an $\varepsilon$-convergence-set. In fact, let $E = \text{Conv}(f)$, where $f(z, t) = \sum_{j=0}^{\infty} P_n(z) t^n$. Denote by $d_n$ the degree of $P_n, n = 0, 1, 2, \ldots$. We take $m_n$ be a sequence of increasing integers satisfy that $m_n > (\frac{d_{m_n-1}}{m_n-1})^2$. Now let $F(z, t) = \sum_{j=0}^{\infty} (P_n(z))^{m_n} t^{nm_n}$. Then $E = \text{Conv}(f) = \text{Conv}(F)$, and $E = \text{Conv}(F)$ be an $\varepsilon$-convergence-set.
Acknowledgment. We thank Buma Fridman for helpful discussions. Part of the third named author’s work was done while visiting Tsinghua University Yau Mathematical Sciences Center during his sabbatical leave in spring 2014. He is grateful for the Center’s hospitality and financial support.

References

[1] S.S. Abhyankar, T.T. Moh, A reduction theorem for divergent power series, J. Reine Angew. Math., 241(1970), 27—33.
[2] A.F. Beardon, Iteration of rational functions, 3rd ed. Springer, New York, 1991.
[3] B.L. Fridman, D. Ma, Osgood-Hartogs type properties of power series and smooth functions, Pacific J. Math., 251(2011), 67—79.
[4] B.L. Fridman, D. Ma, T.S. Neelon, On convergence sets of divergent power series, Annales Polonici Mathematici, 106(2012), 193–198.
[5] L. Hormander, An Introduction to Complex Analysis in Several Variables, 3nd ed, North-Holland, 1990.
[6] O. Knill, Renormalization of random Jacobi operators, Comm. Math. Phys., 164(1994), 1, 195—215.
[7] P. Lelong, On a problem of M.A. Zorn, Proc. Amer. Math. Soc., 2(1951), 11—19.
[8] N. Levenberg, R.E. Molzon, Convergence sets of a formal power series, Math. Z., 197(1988), 411—420.
[9] D. Ma, T.S. Neelon, On convergence sets of formal power series, Complex Analysis and its Synergies, (2015) 1:4, DOI 10.1186/s40627-015-0004-4.
[10] A. Sathaye, Convergence sets of divergent power series, J. Reine Angew. Math., 283(1976), 86—98. Kokyoroku in Math. 14, Tokyo, 1982.
[11] K. Spallek, P. Tworzewski, T. Winiarski, Osgood-Hartogs-theorem of mixed type, Math. Ann, 288(1990), 75—88.