A THEOREM FOR DISTINCT ZEROS OF L-FUNCTIONS

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Abstract

In this paper, we establish a simple criterion for two $L$-functions $L_1$ and $L_2$ satisfying a functional equation (and some natural assumptions) to have infinitely many distinct zeros. Some related questions have already been answered in the particular case of Automorphic forms using so-called Converse Theorems (see [4]). Deeper results can also be stated for elements of the Selberg class (see [1]). However, we shall give here a general answer that do not use any advanced topics in analytic number theory. Therefore, this paper should be accessible to anyone who has some basic notions in measure-theory and advanced complex analysis.

Throughout this paper, we distinguish two kinds of Dirichlet series, as it is usually done. A meromorphic function $f$ on a domain $\Omega \subseteq \mathbb{C}$ is represented by a General Dirichlet Series (GDS) if $f$ can be written in the form

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

where $\lambda_n$ is a sequence of real and non-negative numbers strictly increasing to infinity) when $\text{Re}(s)$ is large enough. In the same framework, $f$ is represented by an Ordinary Dirichlet Series (ODS) if $f$ can be written in the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

when $\text{Re}(s)$ is large enough. In the event that the GDS is absolutely convergent (when $\text{Re}(s)$ is large), we shall say that this is an absolutely convergent GDS. In this case, we note $\sigma_a(f)$ the abscissa of absolute convergence of the series (this number depends only on $f$ and not on the form of the Dirichlet series as we will see, Corollary [3]). We recall that for ODS, the finite abscissa of convergence is equivalent to the finite abscissa of absolute convergence via the inequality $\sigma_c(f) \leq \sigma_a(f) \leq \sigma_c(f) + 1$ (where $\sigma_c(f)$ is the abscissa of convergence of the ODS).

The purpose of this paper is to provide a relationship between two Dirichlet series which may be extended to a meromorphic function of finite growth on the whole complex plane, satisfying a functional equation and whose sets of zeros (counted with multiplicity) differ from one another only by a finite number of elements. More precisely, we shall prove
Theorem 1. Let $\gamma$ be a meromorphic function on $\mathbb{C}$, $k$ a real. Suppose that $L_1$ and $L_2$ are two meromorphic functions on $\mathbb{C}$ such that $L_i$ (for $i = 1$ and 2) satisfies the following conditions

(i) There exists a meromorphic function $L_i^*$ on $\mathbb{C}$ and a positive real $N_i$ such that $L_i(k - s) = N_i^s \gamma(s)L_i^*(s) \quad (\forall s \in \mathbb{C})$,

(ii) there exists a polynomial $P_i$ such that $P_iL_i$ is holomorphic and of finite order in $\mathbb{C}$.

Assume further that $L_1/L_2$ has only finitely many poles and that $L_1/L_2$ and $L_1^*/L_2^*$ are represented by ODS.

Then there exist an integer $N$ and complex numbers $(a_d)_{d | N}$ such that $L_1(s) = \left( \sum_{u | N} a_u u^{-s} \right) L_2(s) \quad (\forall s \in \mathbb{C})$.

The proof will infer that $N = N_1/N_2$ when $N_1/N_2$ is actually a positive integer. The latter result can be seen as a generalization of a famous result of Hamburger (see [3]). It has the following well-known corollaries (we left the proof for the reader).

Corollary 1. Let $\chi$ and $\varphi$ be two distinct primitive Dirichlet characters. Then $L(\chi, s)/L(\varphi, s)$ has infinitely many poles.

Corollary 2. Let $f$ and $g$ be two linearly independent Hecke forms of integer weight $k$ for $\Gamma_0(N)$. Then $L(f, s)/L(g, s)$ has infinitely many poles.

I FUNCTIONS REPRESENTED BY GDS

We recall that for a measure $\mu$ on the measurable space ($[0, \infty), B[0, \infty]$) (as usual, $B[0, \infty]$ denotes the Borel sets of the topological space $[0, \infty)$), the Laplace transform $L\mu$ of $\mu$ is defined by:

$L\mu(s) := \int_0^\infty e^{-st}\mu(dt)$.

When the image measure $|\mu|$ has finite mass, $\mu$ will be referred to as absolutely convergent: in this case, the Laplace transform of $\mu$ converges for $\Re(s) \geq 0$.

The Laplace transform will be useful for us through the following example: let $\nu$ be the measure on ($[0, \infty), B[0, \infty]$) defined by

$\nu = \sum_{n=1}^{\infty} a_n \delta_{\lambda_n}$.

It is straightforward to see that the corresponding GDS is the Laplace transform of $\nu$ (remark that the measure is absolutely convergent if and only if $\sigma_\nu(L\nu)$ is non-positive). Therefore, via a uniqueness theorem for Laplace transform, we should be able to derive an equivalent result for GDS.

Theorem 2. Let $\mu_1$ and $\mu_2$ be two absolutely convergent measures on ($[0, \infty), B[0, \infty]$) such that $L\mu_1(s) = L\mu_2(s)$ for all $s \geq 0$. Then $\mu_1 = \mu_2$.

Note that the assumption: $L\mu_1(s) = L\mu_2(s)$ for all $s \geq 0$ can be weaken in: $L\mu_1(s) = L\mu_2(s)$ for all $s \geq \sigma$ (for a $\sigma \in \mathbb{R}$), if we assume that the measure $(t \mapsto e^{-\sigma t})\mu_i$ is absolutely convergent (for $i = 1$ and 2).
Proof: Setting $\mu := \mu_1 - \mu_2$ (also absolutely convergent), the assumption is equivalent to
\[
\int_0^\infty e^{-st}\mu(dt) = 0 \quad (\forall s \geq 0).
\]
By the Stone-Weierstrass theorem, for any continuous function $f$ on $[0, 1]$, we can find a sequence $(P_n)$ of polynomials such that
\[
\|f - P_n\|_\infty := \sup_{t \in [0, 1]} |f(t) - P_n(t)| \xrightarrow{n \to \infty} 0.
\]
Hence,
\[
\left|\int_0^\infty f(e^{-t})\mu(dt)\right| = \left|\int_0^\infty f(e^{-t}) - P_n(e^{-t})\mu(dt)\right| \leq |\mu([0, \infty))||f - P_n||_\infty
\]
and since the right-hand side is arbitrary small (for large values of $n$), we conclude that
\[
\int_0^\infty f(e^{-t})\mu(dt) = 0
\]
for any continuous function $f$ on $[0, 1]$. Therefore, we fix $a < b$ in $(0, \infty)$ and we define $f_\varepsilon$ (for $\varepsilon > 0$) by
\[
f_\varepsilon(x) := \begin{cases} 
\frac{1}{\varepsilon}(x - e^{-a} + \varepsilon) & e^{-a} - \varepsilon \leq x \leq e^{-a} \\
1 & e^{-a} \leq x \leq e^{-b} \\
\frac{1}{\varepsilon}(x - e^{-b} - \varepsilon) & e^{-b} \leq x \leq e^{-b} + \varepsilon \\
0 & \text{otherwise.}
\end{cases}
\]
If $\varepsilon$ is small enough, $f_\varepsilon$ is continuous and defined on $[0, 1]$. Thus,
\[
|\mu([a, b])| = \left|\int_0^\infty 1_{[a, b]}(t)\mu(dt)\right| \leq \left|\int_0^\infty f_\varepsilon(e^{-t})\mu(dt)\right| + 2|\mu([0, \infty))|\varepsilon = 2|\mu([0, \infty))|\varepsilon
\]
and finally $\mu([a, b]) = 0$ for all $b > a > 0$ (the case $a = 0$ can be done similarly). To conclude, we define
\[
C := \{[a, b]|a, b \in [0, \infty)\} \subset B[0, \infty).
\]
$C$ is a $\pi$-system such that $\sigma(C) = B[0, \infty)$ and on which $\mu$ and the trivial measure coincide. Finally, knowing that $\mu([0, \infty)) = 0$, a well-known result from measure-theory allows us to conclude that $\mu$ is the trivial measure. Therefore $\mu_1 = \mu_2$.

This theorem has many consequences in the theory of GDS. It says in particular that a very few functions are represented by an absolutely convergent GDS.

**Corollary 3 (Uniqueness Theorem for GDS).** Let $f : [a, \infty) \to C$ for a certain $a < \infty$. Then $f$ can be represented by at most one absolutely convergent GDS.

Proof: Suppose that we can write (for $s$ large enough)
\[
f(s) = \sum_{n=1}^\infty a_ne^{-\lambda_ns} = \sum_{n=1}^\infty b_ne^{-\mu_ns}
\]
where the two series are absolutely converging for a certain $\sigma \in \mathbb{R}$. Without loss of generality, we may and shall assume that $\sigma = 0$ (taking $a'_n := a_ne^{-\sigma\lambda_n}$ and $b'_n := b_ne^{-\sigma\mu_n}$). We introduce
\[
\nu_1 := \sum_{n=1}^\infty a_n\delta_{\lambda_n}, \quad \nu_2 := \sum_{n=1}^\infty b_n\delta_{\mu_n}.
\]
Those two measures are absolutely convergent (since $\sigma = 0$) and $\mathcal{L}\nu_1(s) = \mathcal{L}\nu_2(s)$ for all $s \geq 0$. The previous theorem implies that $\nu_1 = \nu_2$. Therefore, by measuring the singletons $(\{\lambda_n\})$, $(\{\mu_n\})$ under this equality, one obtains that the two series are the same.
Corollary 4. A non-constant rational fraction is not represented by an absolutely convergent GDS.

Proof: We assume the opposite, i.e. there exists $F$ a non-constant rational function represented by an absolutely convergent GDS. For $s \geq \sigma_a(F)$,

$$F(s) = \frac{a_n s^n + \cdots + a_0}{b_m s^m + \cdots + b_0} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.$$  

From the partial fraction principle, we know that we can write $F$ as

$$F(s) = \sum_{i \in \{1, \ldots, p\}} F_i(s) = P(s) + \sum_{i \in \{1, \ldots, p\}} F_i(s)$$

where $P$ is a polynomial and where \( (z_i), (c_{i,j}) \) and positive integers \( (n_i) \). Now, by the absolute convergence of the GDS, \( \lim_{s \to \infty} F(s) \) is finite, we denote it $\ell$. $P$ must be constant equal to $\ell$ since \( F_i(s) \) tends to zero at infinity.

Let $f_i$ (for $i \in \{1, \ldots, p\}$) be the function

$$f_i(t) := \sum_{r=1}^{n_i} c_{i,r} e^{z_i t}e^{-r-1t}$$

(whose Laplace transform is $F_i$) and let $\lambda$ be the Lebesgue measure on $\mathbb{R}$. In the same framework, Theorem 2 implies that the two measures

$$\ell \delta_0 + \sum_{i=1}^{p} f_i \lambda \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \delta_{\lambda_n}$$

are the same. We easily see that $f_i$ (for $i \in \{1, \ldots, p\}$) must be zero. A contraction since $F$ is non-constant.

II THE SPACE $\mathcal{O}_N$

For convenience, we introduce a definition : a meromorphic function $f$ is of finite order in a set $\Omega \subseteq \mathbb{C}$ if a $\rho > 0$ exists such that $f(s) \ll e^{\rho |s|^p}$ in $\Omega$. It extends the usual definition of the finite order for entire functions (taking $\Omega = \mathbb{C}$).

Definition. For $N$ a positive real, let $\mathcal{O}_N$ (resp. $\mathcal{G}_N$) be the space of functions $f$ that satisfy :

1. $f$ is meromorphic on $\mathbb{C}$ and represented by an ODS (resp. absolutely convergent GDS),
2. $\forall s \in \mathbb{C}, \ f(-s) = N^s g(s)$ where $g$ is represented by an ODS (resp. absolutely convergent GDS),
3. $M > 0$ exists such that $f$ is of finite order in the set

$$\Omega_M(f) := \{ -1 - \sigma_a(g) \leq \text{Re}(s) \leq \sigma_a(f) + 1 \} \cap \{|\text{Im}(s)| \geq M\}.$$  

It is clear that if $f$ belongs to $\mathcal{G}_N$ (resp. $\mathcal{O}_N$) then $g$ belongs to $\mathcal{G}_N$ (resp. $\mathcal{O}_N$). The goal of this section is to understand the structure of $\mathcal{O}_N$.

Lemma 1. $f$ belongs to $\mathcal{G}_1$ if and only if $f$ is constant. In particular $\mathcal{G}_1 = \mathcal{O}_1 = \text{Vect}\{1\}$. 

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We define the reverse inclusion. We take \( f \).

Clearly, the constant function equal to zero belongs to \( \mathcal{O} \).

Proof: We split the complex plane into the following sets

\[
\Omega_1 := \Omega_M(f) \\
\Omega_2 := \{ \text{Re}(s) \geq \sigma_a(f) + 1 \} \\
\Omega_3 := \{ \text{Re}(s) \leq -1 - \sigma_a(g) \} \\
\Omega_4 := \{-1 - \sigma_a(g) \leq \text{Re}(s) \leq \sigma_a(f) + 1 \} \cap \{ |\text{Im}(s)| \leq M \}.
\]

By the fact that \( f(s) = g(s) \) and the general theory of Dirichlet series, we know that \( f \) is bounded on \( \Omega_2 \) and \( \Omega_3 \). In particular, \( f \) is bounded on the lines \( \{ \text{Re}(s) = \sigma_a(f) + 1 \} \) and \( \{ \text{Re}(s) = -1 - \sigma_a(g) \} \).

We can apply the Phragmèn-Lindelöf principle to \( \Omega_1 \) to conclude that \( f \) is bounded on this set. Thus, since \( \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 = \mathbb{C} \), the only poles of \( f \) are in \( \Omega_4 \) which is bounded. By the isolated zeros principle, \( f \) has only finitely many poles. Let \( \rho_1, \rho_2, \ldots, \rho_m \) be those poles counted with multiplicity. To sum up, we can find \( A \) and \( B \) two positive constants such that

\[
\prod_{i=1}^{m} (s - \rho_i)f(s) \leq A + B|s|^m \quad (\forall s \in \mathbb{C}).
\]

The general form of Liouville’s theorem then implies that the left hand side is a polynomial in \( s \). Hence, \( f \) is a rational fraction in \( s \). By Corollary 3 \( f \) must be constant.

Lemma 2. If \( N > 0 \) is not an integer, \( \mathcal{O}_N = \{0\} \).

Proof: Clearly, the constant function equal to zero belongs to \( \mathcal{O}_N \). We now establish the reverse inclusion. We take \( f \in \mathcal{O}_N \) and write \( \lfloor N \rfloor \neq N \) the entire part of \( N \).

We define

\[
\tilde{f} : s \mapsto N^{s/2} \left( f(s) - \sum_{u=1}^{\lfloor N \rfloor} \frac{a_u}{u^s} \right) = \sum_{n=1}^{\infty} a_{\lfloor N \rfloor + n} e^{-s(\log(N) - \log(\lfloor N \rfloor + n))}.
\]

An easy computation and the above formula show that \( \tilde{f} \) belongs to \( \mathcal{G}_1 \) and as a consequence is constant by Lemma 1 since \( \log(\lfloor N \rfloor + 1) > \log(\sqrt{N}) \) (because \( N > 1 \)) and the absolute convergence of the series (for \( \text{Re}(s) \) large), \( f(s) \) tends to zero as \( s \) tends to infinity: this constant must be zero. It gives

\[
f(s) = \sum_{u=1}^{\lfloor N \rfloor} a_u \frac{u^s}{u^s} \quad (\text{for } \text{Re}(s) \text{ large}).
\]

Since \( N^{-s}f(-s) = g(s) \) (\( \forall s \in \mathbb{C} \)), we have

\[
g(s) = \sum_{u=1}^{\lfloor N \rfloor} a_u \left( \frac{u}{N} \right)^s \quad (\text{for } \text{Re}(s) \text{ large})
\]

and thus everywhere on \( \mathbb{C} \) by analytic continuation. But, by assumption, \( g \) is represented by an ODS. From Corollary 3 since the right-hand side is absolutely convergent (as a finite sum), we obtain that this right-hand side must be an ODS. A contradiction since \( N \) is not an integer.

The complete structure of \( \mathcal{O}_N \) is given by the following theorem:

Theorem 3. For all \( N > 0 \), the space \( \mathcal{O}_N \) is a finite-dimensional \( \mathbb{C} \)-vector space and when \( N \) is an integer,

\[
\mathcal{O}_N = \text{Vect}_{\lfloor i \rfloor N} \left\{ s \mapsto \frac{1}{u^s} \right\}.
\]

In particular, \( \dim(\mathcal{O}_N) = d(N) \) (where \( d(N) \) denotes the number of positive divisors of \( N \) if \( N \) is an integer and 0 otherwise).
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Proof: Without loss of generality, we assume that $N \geq 1$ is an integer (the other case is given by the previous lemma). It follows directly from the definition that $O_N$ is a $C$-vector space. Since $u^s = \left( \frac{u^s}{N} \right)^N$, for $u$ a positive divisor of $N$, we have

$$s \mapsto \frac{1}{u^s} \in O_N$$

and thus,

$$\text{Vect}_{u|N} \left\{ s \mapsto \frac{1}{u^s} \right\} \subseteq O_N \subseteq G_N$$

Similarly, we also have

$$\text{Vect}_{u \in \{1, \ldots, N\}} \left\{ s \mapsto \frac{1}{u^s} \right\} \subseteq G_N$$

We turn on to the other inclusion. Let $f(s) = \sum_{n \geq 1} a_n n^{-s} \in O_N \subseteq G_N$ and $\tilde{f}$ be the function defined as

$$\tilde{f}(s) : s \mapsto N^{s/2} \left( f(s) - \sum_{n = 1}^{N} \frac{a_n}{u^s} \right) = N^{s/2} \sum_{n > N} \frac{a_n}{u^s}$$

which is in $G_N$ thanks to the vector-space structure. By construction, $\tilde{f}$ admits a GDS of the form

$$\tilde{f}(s) = \sum_{n = 1}^{\infty} b_n e^{-s \mu_n} = \sum_{n = 1}^{\infty} a_{N+n} e^{-s \left( \log(\sqrt{N}) - \log(N+n) \right)}$$

where $(\mu_n)$ is a strictly increasing sequence of positive numbers (as in the proof of the previous lemma). Consequently, $s \mapsto \tilde{f}(s)$ lies in $G_1$. By Lemma 1, $\tilde{f}$ is constant on $C$; this constant must be zero by construction, since $f$ has no constant term. We have:

$$f(s) = \sum_{u = 1}^{N} \frac{a_u}{u^s}$$

Finally $f$ must belong to $O_N$, so we should add the condition $u|N$ in the index of the sum (this can be seen by invoking Corollary 3 again). Thus

$$\text{Vect}_{u|N} \left\{ s \mapsto \frac{1}{u^s} \right\} \supseteq O_N$$

which proves the result.

III PROOF OF THEOREM 1

It remains to prove a lemma.

Lemma 3. Let $f$ and $g$ be two entire functions of finite order such that $f/g$ is entire. Then $f/g$ is of finite order.

Proof: Since $f$ and $g$ are of finite order, they both have finite genus (see 2 for a general study of the order, the genus and Hadamard’s factorization theorem). Thus, they both admit a Hadamard product:

$$f(s) = s^n e^{P(s)} \prod_{\rho \in \mathcal{Z}(f)} \left( 1 - \frac{s}{\rho} \right) \exp \left( \frac{s}{\rho} + \frac{s^2}{2\rho^2} + \ldots + \frac{s^h}{h!\rho^h} \right)$$

$$g(s) = s^m e^{Q(s)} \prod_{\rho \in \mathcal{Z}(g)} \left( 1 - \frac{s}{\rho} \right) \exp \left( \frac{s}{\rho} + \frac{s^2}{2\rho^2} + \ldots + \frac{s^r}{r!\rho^r} \right)$$
where we use the notation $\mathcal{Z}(f)$ for the multiset of zeros of $f$. Here, $P$ and $Q$ are two polynomials, and $h$ (resp. $r$) is the genus of $f$ (resp. $g$). By assumption $\mathcal{Z}(g) \subset \mathcal{Z}(f)$ (where the inclusion for multisets means that all elements of $\mathcal{Z}(g)$ are in $\mathcal{Z}(f)$ with a lower multiplicity). Consequently, the ratio of the two products over zeros of $f$ and $g$ can be simplified in

$$\exp \left( \sum_{i=r+1}^{h} \frac{s^i}{i} \left( \sum_{\rho \in \mathcal{Z}(g)} \frac{1}{\rho^i} \right) \right) \prod_{\rho \in \mathcal{Z}(f) \setminus \mathcal{Z}(g)} \left( 1 - \frac{s}{\rho} \right) \exp \left( \frac{s + s^2 + \ldots + s^h}{h} \right)$$

if $h \geq r$, and in

$$\exp \left( - \sum_{i=h+1}^{r} \frac{s^i}{i} \left( \sum_{\rho \in \mathcal{Z}(g)} \frac{1}{\rho^i} \right) \right) \prod_{\rho \in \mathcal{Z}(f) \setminus \mathcal{Z}(g)} \left( 1 - \frac{s}{\rho} \right) \exp \left( \frac{s + s^2 + \ldots + s^h}{h} \right)$$

if $r \geq h$. From the definition of the genus, the following two series

$$\sum_{\rho \in \mathcal{Z}(g)} \frac{1}{\rho^i} \leq \sum_{\rho \in \mathcal{Z}(f)} \frac{1}{\rho^i}$$

are absolutely convergent for $i \in \{r, \ldots, h\}$. Hence, in those two cases, the right-hand side is entire and of finite order and this completes the proof.

We shall now reveal the proof of the main result.

**Proof of Theorem**

We suppose that

$$L_1(k-s) = N_1^* \gamma(s)L_1^*(s) \quad (\forall s \in \mathbb{C}),$$

$$L_2(k-s) = N_2^* \gamma(s)L_2^*(s) \quad (\forall s \in \mathbb{C})$$

and we define

$$\rho(s) := \frac{L_1(s + \frac{k}{2})}{L_2(s + \frac{k}{2})} \quad \text{and} \quad \rho^*(s) := \left( \frac{N_1}{N_2} \right)^{k/2} \frac{L_1^*(s + \frac{k}{2})}{L_2^*(s + \frac{k}{2})}.$$ 

We will show that $\rho$ belongs to $\mathcal{O}_{N_1/N_2}$.

1. $\rho$ is a meromorphic function on $\mathbb{C}$ (as the quotient of two meromorphic functions) represented by an absolutely convergent ODS.

2. $\forall s \in \mathbb{C}, \rho(-s) = (N_1/N_2)^s \rho^*(s)$ where $\rho^*$ is also represented by an absolutely convergent ODS.

3. We introduce $f_1(s) := (P_1L_1)(s + k/2)$ (resp. $f_2(s) := (P_2L_2)(s + k/2)$). By assumption, $f_1/f_2$ has finitely many poles: let $\mathcal{Z}$ be the finite multiset of those poles. From Lemma 3 (with $f(s) = f_1(s)$ and $g(s) = f_2(s)/\prod_{\rho \in \mathcal{Z}} (s - \rho)$), we know that

$$\prod_{\rho \in \mathcal{Z}} (s - \rho) \frac{f_1(s)}{f_2(s)}$$

is entire and of finite order in $\mathbb{C}$. Thus, $\rho$ is the product of a rational fraction and an entire function since

$$\rho(s) = \left( \frac{P_2(s + k/2)}{P_1(s + k/2)\prod_{\rho \in \mathcal{Z}} (s - \rho)} \right) \left( \prod_{\rho \in \mathcal{Z}} (s - \rho) \frac{f_1(s)}{f_2(s)} \right).$$

Taking $M$ to be strictly larger than all the imaginary parts of the poles of the rational fraction, we obtain the third condition that defines $\mathcal{O}_{N_1/N_2}$.
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Hence $\rho \in \mathcal{O}_{N_1/N_2}$. If $N_1/N_2$ is not an integer, $\rho$ is the constant function equal to zero and the result is clear. If $N : = N_1/N_2$ is an integer, there exist complex numbers $(a_d)$ such that

$$\rho(s) = \sum_{u \mid N} \frac{a_u}{u^s} \quad (\forall s \in \mathbb{C})$$

(by Theorem 3) and this is the result.

Note that the proof infers $N = N_1/N_2$.

References

[1] A. Perelli E. Bombieri. Distinct zeros of $L$-functions. Acta Arithmetica, 83(3):271–281, 1998.

[2] P. Garret. Weierstrass and Hadamard products. Complex analysis course on Weierstrass and Hadamard products, pages 1–8, 2015.

[3] H. Hamburger. Über die riemannsche funktionalgleichung der $\zeta$-funktion. Mathematische Zeitschrift, 11(3-4):224–245, 1921.

[4] R. Raghunathan. A comparison of zeros of $L$-functions. Mathematical Research Letters, 6:155–167, 1999.