Harmonic Oscillator, Coherent States, and Feynman Path Integral†‡

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Abstract

The Feynman path integral for the generalized harmonic oscillator is reviewed, and it is shown that the path integral can be used to find a complete set of wave functions for the oscillator. Harmonic oscillators with different (time-dependent) parameters can be related through unitary transformations. The existence of generalized coherent states for a simple harmonic oscillator can then be interpreted as the result of a (formal) invariance under a unitary transformation which relates the same harmonic oscillator. In the path integral formalism, the invariance is reflected in that the kernels do not depend on the choice of classical solutions.

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I Introduction

Quantum mechanics can be seen as a one-dimensional field theory. As in classical field theory, such as electrodynamics, the Green function method is a convenient tool for the field theory. It is therefore very tempting to try to find Green functions for quantum mechanical systems. One of the traditional approaches to the Green function may be in finding the complete set of modes for the Hamiltonian (or the Schrödinger operator) of the system. Feynman provides a formal solution to the Green function problem: He showed that the path integral gives the kernel and Green function [1, 2]. Among the examples where path integrals have been carried out exactly, the generalized harmonic oscillator (GHO) is a special one, as Feynman and Hibbs have pointed out [2]. In this contribution, the quantum mechanics of the GHO will be discussed based on the author’s works in recent years [3, 4, 5]. It will be shown that the Feynman path integral carried out in terms of classical solutions can be used to give a complete set of wave functions for the oscillator.

In the GHO system, there exist generalized coherent (coherent and squeezed) states which are described by classical solutions: The center of probability distribution of a coherent state moves along a classical solution, and the motion of the width of the distribution for a squeezed state is associated with the classical solutions, as was first found in a simple harmonic oscillator (SHO) system [3, 7]. In recent years, it has been shown that the harmonic oscillators of different (time-dependent) parameters can be related through a unitary transformation [4, 8]. The existence of the generalized coherent states for a SHO can then be interpreted as a result of the (formal) invariance under a unitary transformation which relates the same SHO. The path integral formalism provides a very intuitive reason for the fact that the quan-

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*This is the definition for the generalized coherent state adopted in this contribution. Of course, there are many other definitions which must be connected in some way [6].
tum mechanics of a GHO is described by the classical solutions of the system, and the invariance is reflected in that the path integral does not depend on the choice of classical solutions.

In the next section, the Feynman path integral for a quantum mechanical system will be briefly reviewed. In Sec. III, the path integral will be applied to a GHO. In Sec. IV, the unitary relations between the different GHOs of different (time-dependent) parameters will be discussed, and it will be argued that the existence of generalized coherent states for a SHO is a result of the invariance under a unitary transformation. We will also show that the invariant (action variable) found by Lewis [9] is, in the quantum theory, nothing but the transformed Hamiltonian from that of a SHO system through the unitary transformation, which shows why the invariant should be an invariant. The final section will be devoted to discussions.

II The Feynman Path Integral: A Brief Review

Let’s consider a system with the Hamiltonian, $H_N(x_1, x_2, \cdots, x_N, p_1, p_2, \cdots, p_N)$, which gives the Schrödinger equation,

$$O_N \Psi = 0,$$

with the Schrödinger operator

$$O_N(t, x_1, x_2, \cdots, x_N) = -i\hbar \frac{\partial}{\partial t} + H_N. \tag{2}$$

The Green function problem is to find the solution of the equation:

$$O_N(t_b, x_{1b}, x_{2b}, \cdots, x_{Nb})G_N(b; a) = \delta(t_b - t_a) \prod_{i=1}^{N} \delta(x_{ib} - x_{ia}). \tag{3}$$

The kernel may be a convenient step toward the Green function, and for a non-interacting or an isolated system it may be enough for the analysis of
the given system. The kernel, which is a familiar object in mathematics, is the solution of the equation:

\[ O_N(t_b, x_{1b}, x_{2b}, \cdots, x_{Nb})K_N(b; a) = 0, \]  

(4)

with the initial condition,

\[ K_N(b, a) \to \prod_{i=1}^{N} \delta(x_{ib} - x_{ia}) \quad \text{as} \quad t_b \to t_a + 0. \]  

(5)

Feynman provides a formal solution for the kernel: He showed that the kernel can be obtained through the path integral

\[ K(b, a) = \int_{x_{1a}}^{x_{1b}} \cdots \int_{x_{N_a}}^{x_{Nb}} \exp\left[ i \frac{1}{\hbar} \int_{t_a}^{t_b} L(t, x_1, \cdots, x_N) \right] Dx_1 \cdots Dx_N. \]  

(6)

In his book with Hibbs [2], he then asserted that the Green function can be written as

\[ G(b, a) = \begin{cases} K(b, a) & \text{if } t_b > t_a, \\ 0 & \text{if } t_b < t_a. \end{cases} \]  

(7)

Assuming a complete set of wave functions \( \{\psi_{n_1, n_2, \cdots, n_N}\} \) for the \( O_N \), an alternative expression for the kernel satisfying Eqs. (4,5) is given as

\[ K(b, a) = \sum_{n_1, \cdots, n_N} \psi_{n_1, n_2, \cdots, n_N}(t_b, x_{1b}, \cdots, x_{Nb})\psi^*_{n_1, n_2, \cdots, n_N}(t_a, x_{1a}, \cdots, x_{Na}). \]  

(8)

If the exact kernel can be found, the expression in Eq. (8) may be used to find a complete set. If we do not require \( K(b, a) = 0 \) for \( t_b < t_a \), in this expression, there is a relation:

\[ K^*(b, a) = K(a, b). \]  

(9)

Though the kernel has been studied for various models through the path integral or the mode summation method, the list of the cases where the exact kernel is known is still limited [10, 11].
III A Generalized Harmonic Oscillator and the Path Integral

As shown by Feynman and Hibbs [2], the path integral for a general quadratic system can be carried out, and the kernel is written in terms of the classical action up to a term which purely depends on the initial time \( t_a \) and the final time \( t_b \) of the path integral. A Lagrangian of an \( N \)-dimensional quadratic system may be written as

\[
L = \sum_{i=1}^{N} \left( \frac{1}{2} M(t) \dot{x}_i^2 - \frac{1}{2} M(t) w^2(t) x_i^2 + F_i(t) x_i ight) + \frac{d}{dt}(M(t)a(t)x_i^2) + \frac{d}{dt}(b_i(t)x_i) + f(t),
\]

(10)

while, for \( N = 1 \), it gives a general quadratic system in one dimension. This Lagrangian clearly shows that the classical equation of motion of the quadratic system is same to that of a GHO of time-dependent mass and frequency with an external force:

\[
\frac{d}{dt}(M \dot{x}_i) + M(t) w^2(t) x_i = F_i(t),
\]

(11)

since the terms written as a total derivative with respect to time do not affect the classical equation of motion. Indeed, the quantum mechanics of the GHO has been a subject of intensive study over a long period, mainly relying upon the invariant (See Sec. IV).

In addition to the observation by Feynman and Hibbs, the fact that the kernel should satisfy the Schrödinger equation and the initial condition can be used to determine the kernel uniquely for the GHO [3]. If the two linearly independent homogeneous solutions of Eq. (11) are denoted as \( u(t), v(t) \) with a particular solution \( x_{ip}(t) \), then one can find that the kernel is written as

\[
K(b, a)
\]
in Eq. (8), a complete set of wave functions 
\{\psi_{n_1, n_2, \cdots, n_N}(t, x_1, x_2, \cdots, x_N)\}
are defined as
\[
\psi_{n_1, n_2, \cdots, n_N}(t, x_1, x_2, \cdots, x_N) = \left(\frac{\Omega}{\pi \hbar \rho^2(t)}\right)^{N/4} \exp\left[\frac{\mathbf{r}^2}{2\hbar} - \frac{\mathbf{r}_p^2}{2\hbar} + \frac{\mathbf{f}(t)}{\hbar}\right] \times \prod_{i=1}^{N} \frac{1}{\sqrt{2^{n_i} n_i!}} \left[\frac{\rho(t)}{\rho(t)}\right]^{n_i} \mathcal{H}_{n_i}(\sqrt{\frac{\Omega x_i - x_i p(t)}{\hbar}}),
\]
with
\[
\rho(t) = \sqrt{u^2(t) + v^2(t)},
\]
as in the one-dimensional case [3, 5, 12].

The Hamiltonian corresponding to the Lagrangian in Eq. (10) is written, in terms of the operators $\vec{r}$ and $\vec{p}$, as

$$H = \frac{\vec{p}^2}{2M(t)} - a(t)[\vec{p} \cdot \vec{r} + \vec{r} \cdot \vec{p}] + \frac{1}{2} M(t)c(t)\vec{r}^2$$

$$- \frac{\vec{b}(t)}{M(t)} \cdot \vec{p} + \vec{d}(t) \cdot \vec{r} + \left( \frac{\vec{b}^2(t)}{2M(t)} - f(t) \right),$$

(17)

where

$$c(t) = w^2 + 4a^2 - 2\ddot{a} - 2\frac{M}{M}a, \quad \vec{d}(t) = 2a\vec{b} - \vec{b}(t) - \vec{F}(t).$$

(18)

Though the terms proportional to $a(t)$ and $\vec{b}(t)$ make the Hamiltonian look rather complicated, from the wave function in (15), it can be found that such terms can be removed from the system through a simple unitary transformation [3]. For the one-dimensional case, the quantum description of a general quadratic system can therefore be found, through a unitary transformation, from that of a GHO system. From now on we only consider the case of $N = 1$ [13]. We also take $a(t), b(t), f(t)$ to vanish, since the general case can be recovered through a simple unitary transformation.

IV Generalized Coherent States from the Unitary Invariance

An historical account of the generalized coherent states of the simple harmonic oscillator (SHO) system has been well summarized by Klauder and Skagerstam [3] and by Nieto [7]. In recent years, it has been shown that harmonic oscillators of different (time-dependent) parameters can be related through a unitary transformation [3, 4]. In this section, it will be shown that the existence of generalized coherent states in a SHO system can then be
interpreted as a result of the (formal) invariance under a unitary transformation which relates the same SHO \[4\].

The generalized harmonic oscillator (GHO) of time-dependent mass and frequency with an external force may be described by the Hamiltonian:

\[
H_F = \frac{p^2}{2M(t)} + \frac{1}{2} M(t) w^2(t) x^2 - F(t)x = H - F(t)x. \tag{19}
\]

The Hamiltonian for the SHO system of unit mass and unit frequency may be given by

\[
H_s = \frac{p^2}{2} + \frac{x^2}{2}. \tag{20}
\]

We define the three different Shrödinger operators

\[
O_F(t) = -i\hbar \frac{\partial}{\partial t} + H, \quad O(t) = -i\hbar \frac{\partial}{\partial t} + H, \quad O_s(\tau) = -i\hbar \frac{\partial}{\partial \tau} + H_s, \tag{21}
\]

which correspond to the Hamiltonian \(H_F\), \(H\), and \(H_s\), respectively.

With \(x_p\) being a particular solution of the classical equation of motion for the system described by \(H_F\), as in the previous section, we define \(\xi\) through the relation

\[
\dot{\xi}(t) = \frac{1}{2} \left[ M w^2 x^2_p - M \dot{x}_p^2 \right].
\]

A unitary operator defined by

\[
U_F = e^{i\xi} \exp\left[ i \frac{\hbar}{4} M \dot{x}_p x \right] \exp\left[ -i \frac{\hbar}{4} x_p p \right] \tag{22}
\]

then gives the relation

\[
U_F O(t) U_F^\dagger = O_F(t) \tag{23}
\]

connecting the systems described by \(H\) and \(H_F\). If \(\tau\), the time of the SHO system, and \(t\), the time of the system described by \(H\), are related by

\[
d\tau = \frac{\Omega}{M(t) \rho^2(t)} dt, \tag{24}
\]

then the unitary operator given by

\[
U_S = \exp\left[ i \frac{\hbar}{4} M(t) \dot{\rho}(t) x^2 \right] \exp\left[ -i \frac{\hbar}{4} \frac{\rho^2(t)}{\Omega} (xp + px) \right] \tag{25}
\]
also connects the system described by $H_s$ and $H$, as

$$U_S O_s(\tau) U_S^\dagger \mid_{\tau=\tau(t)} = \frac{M \rho^2}{\Omega} O = \left( \frac{dt}{d\tau} \right) O. \quad (26)$$

The two unitary relations given in Eqs. (23,26) and the unitary transformation mentioned in the previous section thus relate the SHO system of unit mass and unit frequency to a general quadratic system, provided the solutions of classical equation of motion of the quadratic systems are known. For simplicity, we only consider the relations between the systems described by $H_F$ and $H_s$ in detail.

If $\phi_s(x)$ is an eigenstate of Hamiltonian $H_s$ satisfying $H_s = E\phi_s(x)$, then from the unitary relations, a wave function for the system $H_F$, satisfying $O_F(t)\psi = 0$, is given by

$$\psi(t,x) = e^{-iE\tau/\hbar} U_F U_S \phi_s(x) \mid_{\tau=\tau(t)} = \left( \frac{\Omega}{\rho^2} \right)^{1/4} \left( \frac{u(t) - iv(t)}{\rho(t)} \right)^{E/\hbar} e^{i\xi} \exp\left[ \frac{i}{\hbar} M \dot{x}_p x \right] \times \exp\left[ \frac{i}{2\hbar} M \frac{\dot{\rho}}{\rho} (x - x_p)^2 \right] \phi_s(\sqrt{\Omega/\rho^2} (x - x_p)). \quad (28)$$

From the well-known eigenfunctions of the SHO system, then a complete set for the system described by $H_F$ is therefore given as $\{\psi_n(t,x) | n = 0, 1, 2 \cdots \}$, where

$$\psi_n(t,x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{\Omega}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{\rho}} \left( \frac{u - iv}{\rho} \right)^{n+1/2} \exp\left[ \frac{i}{\hbar} (M \dot{x}_p x + \xi) \right] \times \exp\left[ \frac{(x - x_p)^2}{2\hbar} (-\frac{\Omega}{\rho^2} + iM \frac{\dot{\rho}}{\rho}) \right] H_n(\sqrt{\frac{\Omega x - x_p}{\rho}}). \quad (29)$$

For the case where $M(t) = 1 = w(t)$ and $F(t) = 0$, the unitary relations in Eqs. (23,26) become the desired relations between the same system. As one example, if we choose the classical solutions as $u(t) = \cos t$, $v(t) = \sin t$ and $x_p(t) = 0$, then $U_S = U_F = 1$. However, there are choices which give non-trivial $U_S$ and/or $U_F$; for example, the choice $u(t) = \cos t$, $v(t) = \sin t$, and
$x_p(t) = \cos t$ gives $U_S = 1$ with a non-trivial $U_F$, so that $\psi_n(t, x)$ describes a coherent state whose probability distribution moves along with $x_p(t)$ \((= \cos t)\). On the other hand, if we choose $u(t) = \cos t$, $v(t) = C \sin t$ \((C \neq 1)\), with $x_p(t) = 0$, then $U_F = 1$ but $U_S$ gives a squeezed state whose probability distribution pulsates periodically with the period 1. In this way, it is clear that the presence of generalized coherent states in SHO systems is a result of the invariance of the Schrödinger operator under the unitary transformation [Due to the relation in Eq. (26), this invariance is up to a time-scaling given in Eq. (24) and a multiplication of a purely time-dependent term].

An operator $I$ which has long been of interest for the GHO can be obtained from $H_s$ as

$$I = U_F U_S H_s U^\dagger_S U^\dagger_F. \quad (30)$$

By comparing the Schrödinger operators, one can find that

$$H_F = \left(\frac{d\tau}{dt}\right) I + U_F U_S (-i\hbar \frac{\partial}{\partial t}) U^\dagger_S U^\dagger_F, \quad (31)$$

The expression of $H_F$ given in Eq. (31) can be used to show that $I$ is an invariant of the system of $H_F$ satisfying

$$i\hbar \frac{\partial}{\partial t} I = [H_F, I]. \quad (32)$$

The explicit expression of $I$ is given as \[5\]

$$I = \frac{1}{2\Omega} \frac{\Omega^2}{p^2} (x - x_p)^2 + \left\{ M\dot{\rho}(x - x_p) - \rho(p - M\dot{x}_p) \right\}^2. \quad (33)$$

If we simply take the operators $x$ and $p$ as the canonical variables of classical physics, one can find that $I$ is the action variable satisfying

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H]_{PB} = 0. \quad (34)$$

\(^1\text{Note that, even for } F(t) = 0, \text{ a non-trivial } x_p \text{ can be chosen \[4\].}\)
For $x_p = 0$, the expression of $I$ has first been found by Lewis [9] through the asymptotic theory of Kruskal. As far as the quantum theory of GHO is concerned, the unitary transformation method gives a simple way to find the invariant and clearly shows the reason why the invariant is in fact an invariant.

V Discussions

The Feynman path integral for the GHO has been reviewed, to show that the path integral can be used to find a complete set of wave functions for the oscillator. Harmonic oscillators with different parameters have been shown to be related through unitary transformations, and the existence of generalized coherent states for a SHO has been interpreted as a result of the (formal) invariance under a unitary transformation which relates the same SHO. It has been shown that the unitary transformation applied to the Hamiltonian of a SHO gives an invariant of the time-dependent system; this feature will also be true for any time-dependent system which is related to a time-independent system through a unitary transformation. This invariant becomes an exact action variable if the operators are considered as classical variables, while the harmonic oscillator is also interesting in that it provides a fruitful ground to study the relations between classical and quantum physics [14, 5].

For the harmonic oscillator with an additional inverse-square potential, it has been known that the path integral can be carried out exactly [15]. For this case, it has been shown that a unitary transformation method could be used to find a complete set from a simple case [4]. The unitary method can also be applied for the $N$-body harmonic oscillators interacting through inverse-square potential (Calogero-Sutherland Model) [16], while the kernel has not been known for this case. Indeed, it would be very interesting, if we could evaluate the kernel for an interacting many-body system.
There could be different representations for the same model, as the representation of a GHO system is determined by the choice of classical solutions. However, I believe that the kernel will be unique for a given model. A simple anyon model may be realized as a non-interacting system of two-body harmonic oscillators in two dimension obeying fractional statistics. Though this model with fractional statistics is described by the same Hamiltonian of the non-interacting harmonic oscillators with usual statistics, the kernels for the two systems are different, which shows that the statistics determine the model in this case \cite{17, 18}.

Last, but not least, we have to add that the solutions presented in this contribution are still formal, in that the explicit wave functions and kernel will be given once the classical solutions are found. Some general features of the classical solutions have been discussed in Ref. \cite{19}. In general, the classical problems are not trivial at all and amount to solving the one-dimensional, time-independent Schrödinger equation with an arbitrary potential.

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