Article
Stochastic Approximate Algorithms for Uncertain Constrained K-Means Problem

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Abstract: The k-means problem has been paid much attention for many applications. In this paper, we define the uncertain constrained k-means problem and propose a (1 + ε)-approximate algorithm for the problem. First, a general mathematical model of the uncertain constrained k-means problem is proposed. Second, the random sampling properties of the uncertain constrained k-means problem are studied. This paper mainly studies the gap between the center of random sampling and the real center, which should be controlled within a given range with a large probability, so as to obtain the important sampling properties to solve this kind of problem. Finally, using mathematical induction, we assume that the first j – 1 cluster centers are obtained, so we only need to solve the j-th center. The algorithm has the elapsed time O((|P| + d)/(kn)) and outputs a collection of size O((|P| + d)/(kn)) of candidate sets including approximation centers.

Keywords: stochastic approximate algorithms; uncertain constrained k-means; approximation centers

1. Introduction

The k-means problem has received much attention in the past several decades. The k-means problems consists of partitioning a set P of points in d-dimensional space \( \mathbb{R}^d \) into k subsets \( P_1, \ldots, P_k \) such that \( \sum_{i=1}^{k} \sum_{p \in P_i} ||p - c_i||^2 \) is minimized, where \( c_i \) is the center of \( P_i \), and \( ||p - q|| \) is the distance between two points of \( p \) and \( q \). The k-means problem is one of the classical NP-hard problems, and has been paid much attention in the literature [1–3].

For many applications, each cluster of the point set may satisfy some additional constraints, such as chromatic clustering [4], r-capacity clustering [5], r-gather clustering [6], fault tolerant clustering [7], uncertain data clustering [8], semi-supervised clustering [9], and l-diversity clustering [10]. The constrained clustering problems was studied by Ding and Xu, who presented the first unified framework in [11]. Given a point set \( P \subseteq \mathbb{R}^d \) and a positive integer \( k \), a list of constraints \( L \), the constrained k-means problem is to partition \( P \) into \( k \) clusters \( P = \{ P_1, \ldots, P_k \} \), such that all constraints in \( L \) are satisfied and \( \sum_{p \in P} \sum_{\lambda \in L} ||\lambda - c(P)\||^2 \) is minimized, where \( c(P) = 1/|P| \sum_{x \in P} x \) denotes the centroid of \( P \).

In recent years, particular research has been focused on the constrained k-means problem. Ding and Xu [11] showed the first polynomial time approximation scheme with running time \( O(2^{poly(k/\epsilon)}(\log n)^{3} n) \) for the constrained k-means problem, and obtained a collection of size \( O(2^{poly(k/\epsilon)}(\log n)^{k+1}) \) of candidate approximate centers. The existing fastest approximation schemes for the constrained k-means problem takes \( O(2^{O(k/\epsilon)} n) \) time [12,13], which was first shown by Bhattacharya, Jaiswai, and Kumar [12]. Their algorithm gives a collection of size \( O(2^{O(k/\epsilon)}) \) of candidate approximate centers. In this paper, we propose the uncertain constrained k-means problem, which supposes that all
points are random variables with probabilistic distributions. We present a stochastic approximate algorithm for the uncertain constrained k-means problem. The uncertain constrained k-means problem can be regarded as a generalization of the constrained k-means problem. We prove the random sampling properties of the uncertain constrained k-means problem, which are fundamental for our proposed algorithm. By applying random sampling and mathematical induction, we propose a stochastic approximate algorithm with lower complexity for the uncertain constrained k-means problem.

This paper is organized as follows. Some basic notations are given in Section 2. Section 3 provides an overview of the new algorithm for the uncertain constrained k-means problem. In Section 4, we discuss the detailed algorithm for the uncertain constrained k-means problem. In Section 5, we investigate the correctness, success probability, and running time analysis of the algorithm. Section 6 concludes this paper and gives possible directions for future research.

2. Preliminaries

**Definition 1** (Uncertain constrained k-means problem). Given a random variable set $X \subseteq \mathbb{R}^d$, the probability density function $f_X(s)$ for every random variable $X \in X$, a list of constraints $L$, and a positive integer $k$, the uncertain constrained k-means problem is to partition $X$ into $k$ clusters $\mathcal{X} = \{\mathcal{X}_1, \ldots, \mathcal{X}_k\}$, such that all constraints in $L$ are satisfied and $\sum_{X \in X} \sum_{X \in X} \int_{\mathbb{R}^d} ||s - c(\mathcal{X}_i)||^2 f_X(s) ds$ is minimized, where $c(\mathcal{X}_i) = \frac{1}{|\mathcal{X}_i|} \sum_{X \in \mathcal{X}_i} \int_{\mathbb{R}^d} s f_X(s) ds$ denotes the centroid of $\mathcal{X}_i$.

**Definition 2** ([13]). Let $X$ be a set of random variables in $\mathbb{R}^d$, $f_X(s)$ be probability density function for every random variable $X \in X$, and $q \in \mathbb{R}^d$ and $P$ be a set of points in $\mathbb{R}^d$, $p \in P$.

- Define $f_2(q, X) = \sum_{X \in X} \int_{\mathbb{R}^d} ||s - q||^2 f_X(s) ds$.
- Define $c(X) = \frac{1}{|X|} \sum_{X \in X} \int_{\mathbb{R}^d} s f_X(s) ds$.
- Define $\text{dist}(X, P) = \min_{p \in P} \int_{\mathbb{R}^d} ||s - p|| f_X(s) ds$.

**Definition 3** ([13]). Let $X$ be a set of random variables in $\mathbb{R}^d$, $f_X(s)$ be the probability density function for every random variable $X \in X$, and $X_1, \ldots, X_k$ be a partition of $X$.

- Define $m_j = c(\mathcal{X}_j)$.
- Define $\beta_j = \frac{|\mathcal{X}_j|}{|X|}$.
- Define $\sigma_j = \sqrt{\frac{f_2(m_j, X_j)}{|\mathcal{X}_j|}}$.
- Define $\text{OPT}_k(X) = \sum_{j=1}^k \sum_{X \in \mathcal{X}_j} \int_{\mathbb{R}^d} ||s - c(\mathcal{X}_j)||^2 f_X(s) ds = \sum_{j=1}^k f_2(m_j, \mathcal{X}_j)$.
- Define $\sigma_{\text{opt}} = \sqrt{\frac{\text{OPT}_k(X)}{|X|}} = \sqrt{\sum_{j=1}^k \beta_j \sigma_j^2}$.

**Lemma 1.** For any point $x \in \mathbb{R}^d$ and a random variable set $X \subseteq \mathbb{R}^d$, $f_2(x, X) = f_2(c(X), X) + |X||c(X) - x|^2$. 
Proof. Let \( f_X(s) \) be the probability density function for every random variable \( X \in \mathcal{X} \).

\[
f_2(x, \mathcal{X}) = \sum_{x \in \mathcal{X}} \int_{\mathbb{R}^d} ||s - x||^2 f_X(s) ds
\]

(1)

\[
e = \sum_{x \in \mathcal{X}} \int_{\mathbb{R}^d} ||s - c(\mathcal{X}) + c(\mathcal{X}) - x||^2 f_X(s) ds
\]

(2)

\[
= \sum_{x \in \mathcal{X}} \int_{\mathbb{R}^d} ||s - c(\mathcal{X})||^2 f_X(s) ds + \sum_{x \in \mathcal{X}} \int_{\mathbb{R}^d} ||c(\mathcal{X}) - x||^2 f_X(s) ds
\]

(3)

\[
= f_2(c(\mathcal{X}), \mathcal{X}) + ||c(\mathcal{X}) - x||^2 \sum_{x \in \mathcal{X}} \int_{\mathbb{R}^d} f_X(s) ds
\]

(4)

\[
= f_2(c(\mathcal{X}), \mathcal{X}) + |\mathcal{X}||c(\mathcal{X}) - x|^2.
\]

(5)

The (3) equality follows from the fact that \( \sum_{x \in \mathcal{X}} \int_{\mathbb{R}^d} (s - c(\mathcal{X})) f_X(s) ds = 0. \)

Lemma 2. Let \( \mathcal{X} \) be a set of random variables in \( \mathbb{R}^d \) and \( f_X(s) \) be the probability density function for every random variable \( X \in \mathcal{X} \). Assume that \( \mathcal{Y} \) is a set of random variables obtained by sampling random variables from \( \mathcal{X} \) uniformly and independently. For \( \forall \delta > 0 \), we have:

\[
Pr(||c(\mathcal{Y}) - c(\mathcal{X})||^2 > \frac{1}{\delta |\mathcal{Y}|} \sigma^2) < \delta,
\]

(6)

where \( \sigma^2 = \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \int_{\mathbb{R}^d} ||s - c(\mathcal{X})||^2 f_X(s) ds. \)

Proof. First, observe that

\[
E(c(\mathcal{Y})) = c(\mathcal{X}), \quad E(||c(\mathcal{Y}) - c(\mathcal{X})||^2) = \frac{1}{|\mathcal{Y}|} \sigma^2
\]

(7)

where \( \sigma^2 = \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \int_{\mathbb{R}^d} ||s - c(\mathcal{X})||^2 f_X(s) ds. \) Then apply the Markov inequality to obtain the following.

\[
Pr(||c(\mathcal{Y}) - c(\mathcal{X})||^2 > \frac{1}{\delta |\mathcal{Y}|} \sigma^2) < \delta.
\]

(8)

Lemma 3. Let \( \mathcal{Q} \) be a set of random variables in \( \mathbb{R}^d \), \( f_X(s) \) be the probability density function for every random variable \( X \in \mathcal{Q} \), and \( \mathcal{Q}_1 \) be an arbitrary subset of \( \mathcal{Q} \) with \( |\mathcal{Q}| \) random variables for some \( 0 < \alpha \leq 1 \). Then \( ||c(\mathcal{Q}) - c(\mathcal{Q}_1)|| \leq \sqrt{\frac{1-\alpha}{\alpha}} \sigma \), where \( \sigma^2 = \frac{1}{|\mathcal{Q}|} \sum_{x \in \mathcal{Q}} \int_{\mathbb{R}^d} ||s - c(\mathcal{Q})||^2 f_X(s) ds. \)

Proof. Let \( \mathcal{Q}_2 = \mathcal{Q} \setminus \mathcal{Q}_1 \). By Lemma 1, we have the following two equalities.

\[
f_2(c(\mathcal{Q}), \mathcal{Q}_1) = f_2(c(\mathcal{Q}_1), \mathcal{Q}_1) + |\mathcal{Q}_1||c(\mathcal{Q}_1) - c(\mathcal{Q})|^2,
\]

(9)

\[
f_2(c(\mathcal{Q}), \mathcal{Q}_2) = f_2(c(\mathcal{Q}_2), \mathcal{Q}_2) + |\mathcal{Q}_2||c(\mathcal{Q}_2) - c(\mathcal{Q})|^2.
\]

(10)

Let \( L = ||c(\mathcal{Q}_1) - c(\mathcal{Q}_2)|| \). By the definition of the mean point, we have:

\[
c(\mathcal{Q}) = \frac{1}{|\mathcal{Q}|} \sum_{x \in \mathcal{Q}} \int_{\mathbb{R}^d} sf_X(s) ds = \frac{1}{|\mathcal{Q}|} (|\mathcal{Q}_1|c(\mathcal{Q}_1) + |\mathcal{Q}_2|c(\mathcal{Q}_2)).
\]

(11)

Thus, the three points \( \{c(\mathcal{Q}), c(\mathcal{Q}_1), c(\mathcal{Q}_2)\} \) are collinear, while \( ||c(\mathcal{Q}_1) - c(\mathcal{Q})|| = (1 - \alpha)L \) and \( ||c(\mathcal{Q}_2) - c(\mathcal{Q})|| = \alpha L \). Meanwhile, by the definition of \( \sigma \), we have \( \sigma^2 = \)
\[
\frac{1}{|Q|} \sum_{x \in Q} f_x(s)ds + \sum_{x \in Q_2} \int_{\mathbb{R}^d} \|s - c(Q)\|^2 f_x(s)ds.
\]
Combining Equality (9) and Equality (10), we have:

\[
\sigma^2 \geq \frac{1}{|Q|} \left( \|Q_1\| |c(Q_1) - c(Q)|^2 + \|Q_2\| |c(Q_2) - c(Q)|^2 \right) 
= a((1 - \alpha)L)^2 + (1 - \alpha)(\alpha L)^2 
= a(1 - \alpha)L^2.
\]

(12)

Thus, we have \( L \leq \frac{\sigma}{\sqrt{a(1 - \alpha)}} \), which means that \( |c(Q) - c(Q_1)| = (1 - \alpha)L \leq \sqrt{\frac{1}{1 - \alpha}} \sigma \).

Lemma 4 ([12]). For any \( x, y, z \in \mathbb{R}^d \), then \( \|x - z\|^2 \leq 2\|x - y\|^2 + 2\|y - z\|^2 \).

3. Overview of Our Method

In this section, we first introduce the main idea of our methodology to solve the uncertain constrained \( k \)-means problem.

Considering the optimal partition \( \mathbb{X} = \{X_1, \ldots, X_k\} | |X_i| \geq \ldots \geq |X_k| \) of \( \mathcal{X} \), since \( |X_i|/|\mathcal{X}| \geq 1/k \), if we could sample a set \( S \) of size \( O(k/\epsilon) \) from \( \mathcal{X} \) uniformly and independently, then at least \( O(1/\epsilon) \) random variables in \( S \) are from \( X_i \) with a certain probability. All subsets of \( S \) of size \( O(1/\epsilon) \) could be enumerated to discover the approximate center of \( X_i \).

We assume that \( C_{j-1} = \{c_1, \ldots, c_{j-1}\} \) is the set including approximate centers of the \( X_1, \ldots, X_j \). Let \( B_j = \{x \in \mathcal{X} | \text{dist}(X, C_{j-1}) = \min_{c \in C_{j-1}} \int_{\mathbb{R}^d} \|s - c\|^2 f_X(s)ds \leq r_j\} \), where \( r_j = \sqrt{\frac{2\sigma}{4n k^2}} \sigma_{\text{opt}} \). The set \( X_j \) is divided into two parts: \( X_{j}^{\text{out}} \) and \( X_{j}^{\text{in}} \), where \( X_{j}^{\text{out}} = X_j \setminus B_j \) and \( X_{j}^{\text{in}} = X_j \cap B_j \). For each random variable \( X \), let \( \tilde{X} \) be the nearest point (particular random variable) in \( C_{j-1} \) to \( X \). Let \( \tilde{X}_{j}^{\text{in}} = \{ \tilde{x} | x \in X_{j}^{\text{in}} \} \), and \( \tilde{X}_{j} = \tilde{X}_{j}^{\text{in}} \cup X_{j}^{\text{out}} \).

If most of the random variables of \( X_j \) are in \( X_{j}^{\text{in}} \), our idea is to use the center of \( \tilde{X}_{j}^{\text{in}} \) to approximate the center of \( X_j \). The center of \( \tilde{X}_{j}^{\text{in}} \) is found based on \( C_{j-1} \). If most of the random variables of \( X_j \) are in \( X_{j}^{\text{out}} \), our ideal is to replace the center of \( X_j \) with the center of \( \tilde{X}_{j} \). For seeking out the approximate center of \( \tilde{X}_{j} \), we should find out a subset \( S' \) by uniformly sampling from \( \tilde{X}_{j} \). However, the set \( X_{j}^{\text{out}} \) is unknown. We need to find the set \( S' \cap X_{j}^{\text{out}} \). We apply a branching strategy to find a set \( Q \) such that \( \mathcal{X} \setminus B_j \subseteq Q \), and \( |Q| < 2|\mathcal{X} \setminus B_j| \). Then, a random variables set \( S \) is obtained by sampling random variables from \( \mathcal{X} \) independently and uniformly. And the set \( \mathcal{X} \setminus B_j \subseteq Q \) can be replaced by a subset \( \mathcal{X}^* \) of \( \mathcal{X} \) in \( X_{j}^{\text{out}} \). Based on \( \mathcal{X} \) and \( \tilde{X}_{j}^{\text{in}} \), the approximation center of \( X_j \) could be obtained. Therefore, the algorithm presented in this paper outputs a collection of size \( O\left((\frac{1991k}{\epsilon^2})^{8k/\epsilon \cdot n} \right) \) of candidate sets containing approximation centers, and has the running time \( O\left((\frac{1991k}{\epsilon^2})^{8k/\epsilon \cdot n} \right) \).

4. Our Algorithm cMeans

Given an instance \((\mathcal{X}, k, L)\) of the uncertain constrained \( k \)-means problem, \( \mathbb{X} = \{X_1, \ldots, X_k\} \) denotes an optimal partition of \((\mathcal{X}, k, L)\). There exist six parameters \((\epsilon, Q, g, k, C, U)\) in our cMeans, where \( \epsilon \in [0, 1] \) is the approximate factor, \( Q \) is the input random variable set, \( g \) is the number of centers, \( k \) is the number of the clusters, \( C \) is the set of approximate cluster centers, and \( U \) is a collection of candidate sets including the approximate center. Let \( M = \frac{g}{\epsilon}, N = \frac{2938k}{\epsilon^3} \), where \( M \) is the size of subsets of the sampling set and \( N \) is...
the size of the sampling set. Without loss of generality, assume that values of $M$ and $N$ are integers.

We use the branching strategy to seek out the approximate centers of clusters in $X$. There exist two branches in our algorithm \texttt{cMeans}, which can be seen in Figure 1. On one branch, a size $N$ set $S_1$ is obtained by sampling from $Q$ uniformly and independently; $S_2$ is constructed by $S_1$ and $M$ copies of each point in $C$. Moreover, we consider each subset $S'$ of size $M$ of $S_2$, and the centroid $c$ of $S'$ is solved to represent the approximate center of $X_{k-g+1}$, and our algorithm \texttt{cMeans}(\epsilon, Q, g - 1, k, C \cup \{c\}, U) is used to obtain the remaining $g - 1$ cluster centers.

On the other branch, for each random variable $X \in Q$, we calculate the distance between $X$ and $C$ first. $H$ denotes the set of all distances of random variables in $X$ to $C$, where $H$ is a multi-set. We should obtain the median value $m$ for all values in $H$, which is the $\lfloor |H|/2 \rfloor$-th element if all of the values in $H$ are sorted. In the second branch, $Q$ is divided into two parts, $Q'$ and $Q''$, based on $m$ such that for $\forall X' \in Q'$, $X'' \in Q''$, $\text{dist}(X', C) \leq \text{dist}(X'', C)$, where $|Q'| = \lceil |Q|/2 \rceil$, $|Q''| = \lfloor |Q|/2 \rfloor$. Subroutine \texttt{cMeans}(\epsilon, Q'', g, k, C, U) is used to obtain the remaining $g$ cluster centers. Therefore, we present the specific algorithm for seeking out a collection of candidate sets in the Algorithm 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Flow chart of our algorithm \texttt{cMeans}.}
\end{figure}
### Algorithm 1: \texttt{cMeans}(\epsilon, Q, g, k, C, U)

**Input:** \((\epsilon, Q, g, k, C, U)\)

**Output:** a collection of candidate sets

1. \(M = \frac{6}{\epsilon}, N = \frac{79380k}{\epsilon^3}, S_1 = S_2 = H = \emptyset\);
2. if \(g = 0\) then
   3. add \(C\) to the collection \(U\);
3. sample a set \(S_1\) of size \(N\) from \(Q\) independently and uniformly;
4. if \(C = \emptyset\) then
   5. \(S_2 = S_1\);
5. else
   6. \(S_2 = S_1 \cup \{M\text{ copies of each point in } C\}\);
7. for each subset \(S'\) of size \(M\) of \(S_2\) do
   8. compute the centroid \(c\) of \(S'\);
   9. \(\text{cMeans}(\epsilon, Q, g - 1, k, C \cup \{c\}, U)\);
10. end
11. for each random variable \(X \in Q\) do
12. compute \(\text{dist}(X, C)\), and add \(\text{dist}(X, C)\) to \(H\);
13. obtain the median value \(m\) of all values in \(H\), which is the \([\frac{|H|}{2}]\)th element if all the values in \(H\) are sorted;
14. divide \(Q\) into \(Q'\) and \(Q''\) by \(m\) such that for \(\forall X' \in Q', X'' \in Q''\),
   15. \(\text{dist}(X', C) \leq \text{dist}(X'', C)\), where \(|Q'| = \lceil \frac{|Q|}{2} \rceil, |Q''| = \lfloor \frac{|Q|}{2} \rfloor\);
16. if \(|Q''| \geq 1\) then
   17. \(\text{cMeans}(\epsilon, Q'', g, k, C, U)\);
18. end
19. end

### 5. Analysis of Our Algorithm \texttt{cMeans}

We investigate the success probability, correctness, and time complexity analysis of the algorithm \texttt{cMeans} in this section.

**Lemma 5.** There exists a candidate set, with a probability of at least \(1/12k\), including the approximate center \(C_k = \{c_1, \ldots, c_k\}\) in \(U\) satisfying \([m_j - c_j] \leq \frac{9}{m} e\sigma^2 + \frac{1}{100\epsilon} c_\text{opt}\) (1 ≤ \(j\) ≤ \(k\)).

The following Lemmas from 6 to 16 are used to prove Lemma 5. We prove Lemma 5 via induction on \(j\). For \(j = 1\), we can obtain \(\beta_1 \geq 1/k\) easily, and prove the success probability first.

**Lemma 6.** In the process of finding \(c_1\) in our algorithm \texttt{cMeans}, by sampling a set of \(79,380k/\epsilon^3\) random variables from \(X\) independently and uniformly, denoted by \(S_1\), the probability that at least \(6/\epsilon\) random variables in \(S_2\) are from \(X_1\) is at least \(1/2\).

**Proof.** In our algorithm \texttt{cMeans}, we assume that \(S_1 = S_1, \ldots, S_N\), where \(N = 79,380k/\epsilon^3\). Let \(x_1', \ldots, x_N'\) be the corresponding random variables of elements in \(S_1\). If \(S_i \in X_1\), then
$x'_i = 1$. Otherwise $x'_i = 0$. It is known easily that $Pr[S_i \in \mathcal{X}_1] \geq \frac{1}{k}$. Let $x = \sum_{i=1}^N x'_i \nu = \sum_{i=1}^N E(x'_i)$. We obtain that $u \geq 79,380k/e^3$. Then,

$$Pr[x > \frac{6}{e}] = 1 - Pr[x \leq \frac{6}{e}]$$

$$= 1 - Pr[x \leq \frac{6e^2 \cdot 79,380}{79,380} e^3]$$

$$\geq 1 - Pr[x \leq \frac{e^2}{13,230}]$$

$$\geq 1 - e^{-\frac{(1 - \frac{e^2}{13,230})^2 79,380}{2}}$$

$$\geq 1 - e^{-\frac{(1 - \frac{1}{5})^2 79,380}{2}}$$

$$\geq 1 - e^{-\frac{(1 - \frac{1}{5})^2 79,380}{2}}$$

$$\geq \frac{1}{2}.$$

\[ \square \]

From Lemma 6, an $S^*$ with size $6/e$ of $S_2$ can be obtained, and the probability that all points in $S^*$ are from $\mathcal{X}_1$ is at least $1/2$. Let $c_i$ denote the centroid of $S^*$, and $\delta = 5/6$. For $|S^*| = 6/e$, by Lemma 2, we conclude that $||m_1 - c_1||^2 \leq \frac{1}{6}ec^2$ holds with a probability of at least $1/6$. Then, the probability that a subset $S^*$ of size $6/e$ of $S_2$ can be found such that $||m_1 - c_1||^2 \leq \frac{1}{6}ec^2$ holds is at least $1/12$. Therefore, we conclude that Lemma 5 holds for $j = 1$.

Moreover, we assume that for $j \leq j_0 (1 \leq j_0)$, Lemma 5 holds with a probability of at least $1/12$. Considering the case $j = j_0 + 1$, we prove Lemma 5 by the following two cases: (1) $|\mathcal{X}^{out}_j| \leq \frac{e}{4\Theta} \beta_j n$; (2) $|\mathcal{X}^{out}_j| > \frac{e}{4\Theta} \beta_j n$.

5.1. Analysis for Case 1: $|\mathcal{X}^{out}_j| \leq \frac{e}{4\Theta} \beta_j n$

Since $|\mathcal{X}^{out}_j| \leq \frac{e}{4\Theta} \beta_j n$, most of the random variables of $\mathcal{X}_j$ are in $B_j$. Our idea is to replace the center of $\mathcal{X}_j$ with the center of $\mathcal{X}^{in}_j$ in $\tilde{X}_j$. Thus, we need to find the approximate center $c_j$ of $\mathcal{X}^{in}_j$ and the bound distance $||m_j - c_j||$. We divide the distance $||m_j - c_j||$ into the following three parts: $||m_j - m^{in}_j||$, $||m^{in}_j - \tilde{m}^{in}_j||$, and $||\tilde{m}^{in}_j - c_j||$. We first study the distance between $m_j$ and $m^{in}_j$.

**Lemma 7.** $||m_j - m^{in}_j|| \leq \sqrt{\frac{e}{4\Theta^2}}$.  

**Proof.** Since $|\mathcal{X}_j| = \beta_j n$ and $|\mathcal{X}^{out}_j| \leq \frac{e}{4\Theta} \beta_j n$, the proportion of $\mathcal{X}^{in}_j$ in $\mathcal{X}_j$ is at least $1 - \frac{e}{4\Theta}$. By Lemma 3, $||m_j - m^{in}_j|| \leq \sqrt{\frac{e/4\Theta}{1-e/4\Theta}} \leq \sqrt{\frac{e}{4\Theta}}$. \[ \square \]

**Lemma 8.** $||m^{in}_j - \tilde{m}^{in}_j|| \leq r_j$. 

[Contextual notes for citation and further reading]
Proof. Since \( m_j^{in} = \frac{1}{|X_j^{in}|} \sum_{X \in X_j^{in}} \int_{\mathbb{R}^d} s f_X(s) ds \), and \( \tilde{m}_j^{in} = \frac{1}{|X_j^{in}|} \sum_{X \in X_j^{in}} \tilde{X} \), we can obtain the following:

\[
||m_j^{in} - \tilde{m}_j^{in}|| = || \frac{1}{|X_j^{in}|} \sum_{X \in X_j^{in}} \int_{\mathbb{R}^d} s f_X(s) ds - \frac{1}{|X_j^{in}|} \sum_{X \in X_j^{in}} \tilde{X} || \\
= \frac{1}{|X_j^{in}|} || \sum_{X \in X_j^{in}} \int_{\mathbb{R}^d} (s - \tilde{X}) f_X(s) ds || \\
\leq \frac{1}{|X_j^{in}|} \sum_{X \in X_j^{in}} || s - \tilde{X} || f_X(s) ds \\
\leq \frac{1}{|X_j^{in}|} \sum_{X \in X_j^{in}} r_j \\
= r_j.
\]

\[ q \]

**Lemma 9.** \( f_2(\tilde{m}_j^{in}, \tilde{X}_j^{in}) \leq 2|X_j^{in}|^2 r_j^2 + 2f_2(m_j, X_j^{in}) - |X_j^{in}||m_j - \tilde{m}_j^{in}|^2 \).

**Proof.** Since \( |\tilde{X}_j^{in}| = |X_j^{in}| \), by 1, we have \( f_2(m_j, \tilde{X}_j^{in}) = f_2(\tilde{m}_j^{in}, \tilde{X}_j^{in}) + |X_j^{in}||\tilde{m}_j^{in} - m_j| \). Then,

\[
f_2(\tilde{m}_j^{in}, \tilde{X}_j^{in}) = f_2(m_j, \tilde{X}_j^{in}) - |X_j^{in}||\tilde{m}_j^{in} - m_j|^2 \\
= \sum_{X \in X_j^{in}} || \tilde{X} - m_j ||^2 - |X_j^{in}||\tilde{m}_j^{in} - m_j|^2 \\
= \sum_{X \in X_j^{in}} \int_{\mathbb{R}^d} || \tilde{X} - m_j ||^2 f_X(s) ds - |X_j^{in}||\tilde{m}_j^{in} - m_j|^2 \\
= \sum_{X \in X_j^{in}} \int_{\mathbb{R}^d} || \tilde{X} - s + s - m_j ||^2 f_X(s) ds - |X_j^{in}||\tilde{m}_j^{in} - m_j|^2 \\
\leq \sum_{X \in X_j^{in}} \int_{\mathbb{R}^d} (2|| \tilde{X} - s ||^2 + 2|s - m_j|^2) f_X(s) ds - |X_j^{in}||\tilde{m}_j^{in} - m_j|^2 \\
\leq 2|X_j^{in}|^2 r_j^2 + 2 \sum_{X \in X_j^{in}} \int_{\mathbb{R}^d} || s - m_j ||^2 f_X(s) ds - |X_j^{in}||\tilde{m}_j^{in} - m_j|^2 \\
= 2|X_j^{in}|^2 r_j^2 + 2f_2(m_j, X_j^{in}) - |X_j^{in}||m_j - \tilde{m}_j^{in}|^2.
\]

**Lemma 10.** In the process of finding \( c_j \) in our algorithm cMeans, for the set \( S_2 \) in step 5, a subset \( S^* \) of size \( 6/e \) of \( S_2 \) can be obtained such that all random variables in \( S^* \) are from \( \tilde{X}_j^{in} \). Let \( c_j \) be the centroid of \( S^* \). Then, the inequality \( ||\tilde{m}_j^{in} - c_j||^2 \leq \frac{2}{9} r_j^2 + \frac{4}{525} r_j^2 - \frac{1}{2} e ||m_j - \tilde{m}_j^{in}||^2 \) holds with a probability of at least \( 1/6 \).

**Proof.** For each point \( p \in C_{j-1} \), \( 6/e \) copies of \( p \) are added to \( S_2 \) in step 9 in our algorithm cMeans. Thus, a subset \( S^* \) of size \( 6/e \) of \( S_2 \) can be obtained such that all random variables
Lemma 12.  

In $S^*$ are from $\tilde{X}_j^{in}$. Let $\delta = 5/6$. Since $|S^*| = 6/\varepsilon$, by Lemma 2, $||\tilde{m}_j^{in} - c_j||^2 \leq \frac{\varepsilon f_2(m_j, X_j^{in})}{|X_j^{in}|^2}$ holds with a probability of at least $1/6$. Assume that $||\tilde{m}_j^{in} - c_j||^2 \leq \frac{\varepsilon f_2(m_j, X_j^{in})}{|X_j^{in}|^2}$.

Then,

$$||\tilde{m}_j^{in} - c_j||^2 \leq \frac{\varepsilon f_2(m_j, X_j^{in})}{|X_j^{in}|}$$

(34)

$$\leq \frac{1}{5} \cdot \frac{2|X_j^{in}|^2 + 2f_2(m_j, X_j^{in}) - |X_j^{in}|||m_j - \tilde{m}_j^{in}||^2}{|X_j^{in}|}$$

(35)

$$= \frac{2}{5} \varepsilon r_j^2 + \frac{2}{5} |X_j^{in}| - \frac{1}{5} |m_j - \tilde{m}_j^{in}|^2$$

(36)

$$\leq \frac{2}{5} \varepsilon r_j^2 + \frac{2}{5} |X_j^{in}| - \frac{1}{5} |m_j - \tilde{m}_j^{in}|^2$$

(37)

$$\leq \frac{2}{5} \varepsilon r_j^2 + \frac{2}{5} \beta_j n \sigma_j^2 \left( \frac{1 - \varepsilon}{49 \beta_j n} \right) - \frac{1}{5} |m_j - \tilde{m}_j^{in}|^2$$

(38)

$$\leq \frac{2}{5} \varepsilon r_j^2 + \frac{49}{120} \beta_j n \sigma_j^2$$

(39)

□

Lemma 11. If $c_j$ satisfies $||\tilde{m}_j^{in} - c_j||^2 \leq \frac{\varepsilon}{5} r_j^2 + \frac{49}{120} \beta_j n \sigma_j^2$, then $||m_j - c_j||^2 \leq \frac{\varepsilon}{10} r_j^2 + \frac{1}{10} \beta_j n \sigma_j^2 r_j$.  

Proof. Assume that $c_j$ satisfies $||\tilde{m}_j^{in} - c_j||^2 \leq \frac{\varepsilon}{5} r_j^2 + \frac{49}{120} \beta_j n \sigma_j^2$. Then,

$$||m_j - c_j||^2 = ||m_j - \tilde{m}_j^{in} + \tilde{m}_j^{in} - c_j||^2$$

(40)

$$\leq 2||m_j - \tilde{m}_j^{in}||^2 + 2||\tilde{m}_j^{in} - c_j||^2$$

(41)

$$\leq (2 - \frac{2}{5} \varepsilon)||m_j - \tilde{m}_j^{in}||^2 + \frac{4}{5} \varepsilon r_j^2 + \frac{49}{60} \beta_j n \sigma_j^2$$

(42)

$$\leq (2 - \frac{2}{5} \varepsilon)||m_j - m_j^{in} + m_j^{in} - \tilde{m}_j^{in}||^2 + \frac{4}{5} \varepsilon r_j^2 + \frac{49}{60} \beta_j n \sigma_j^2$$

(43)

$$\leq (2 - \frac{2}{5} \varepsilon)(2||m_j - m_j^{in}||^2 + 2||m_j^{in} - \tilde{m}_j^{in}||^2) + \frac{4}{5} \varepsilon r_j^2 + \frac{49}{60} \beta_j n \sigma_j^2$$

(44)

$$\leq (2 - \frac{2}{5} \varepsilon)(\frac{1}{24} \varepsilon r_j^2 + 2r_j^2) + \frac{4}{5} \varepsilon r_j^2 + \frac{49}{60} \beta_j n \sigma_j^2$$

(45)

$$\leq \frac{9}{10} \varepsilon r_j^2 + 4r_j^2$$

(46)

$$= \frac{9}{10} \varepsilon r_j^2 + \frac{1}{10} \beta_j n \sigma_j^2 r_j$$

(47)

□

5.2. Analysis for Case 2: $|X_j^{out}| > \frac{\varepsilon}{10} |X_j|$  

Let $\tilde{X}_j = \tilde{X}_j^{in} \cup X_j^{out}$, and $\tilde{m}_j$ denote the centroid of $\tilde{X}_j$. Our idea is to replace the center of $X_j$ with the center of $\tilde{X}_j$. But it is difficult to seek out the center of $\tilde{X}_j$. Thus, we try to find an approximate center $c_j$ of $\tilde{X}_j$.  

Lemma 12. $\frac{|X_j^{out}|}{|X_j|} \geq \frac{n^2}{|X_j|}$.
Proof.

\[
|\chi_{\text{out}}^j| \geq |X_j| \sum_{i=1}^{j-1} |X_i \setminus B_j| + |X_j^\text{out}| + \sum_{i=j+1}^k |X_i| \tag{48}
\]

\[
\geq |X_j^\text{out}| \sum_{i=1}^{j-1} f_2(c_i, X_i) + |X_j^\text{out}| + \sum_{i=j+1}^k |X_i| \tag{49}
\]

\[
\geq \left( \frac{1}{1+e} \right) n_{\text{opt}} r_j + |X_j^\text{out}| + \sum_{i=j+1}^k |X_i| \tag{50}
\]

\[
|\chi_{\text{out}}^j| \geq \left( \frac{1}{1+e} \right) n_{\text{opt}} r_j + |X_j^\text{out}| + \sum_{i=j+1}^k |X_i| \tag{51}
\]

\[
|\chi_{\text{out}}^j| \geq \frac{|X_j^\text{out}|}{40(1+e)k\beta_j n} + |X_j^\text{out}| + (k-j)\beta_j n + \frac{e^2}{40}\beta_j n \tag{52}
\]

\[
|\chi_{\text{out}}^j| \geq \frac{|X_j^\text{out}|}{40(1+e)k\beta_j n} + \frac{|X_j^\text{out}|}{40}\beta_j n + (k-j)\beta_j n \tag{53}
\]

\[
|\chi_{\text{out}}^j| \geq \frac{(80k+k)49 + (e-49j)e}{e^2} \geq \frac{49}{3969k} \tag{54}
\]

Lemma 13. \(|m_j - \tilde{m}_j| \leq r_j\).

Proof.

\[
|m_j - \tilde{m}_j| = \left| \frac{1}{|X_j|} \sum_{X \in \chi_j} \int_{\mathbb{R}^d} s f_X(s) ds - \frac{1}{|X_j|} \left( \sum_{X \in \chi_j^{\text{in}}} \tilde{X} + \sum_{X \in \chi_j^{\text{out}}} \int_{\mathbb{R}^d} s f_X(s) ds \right) \right| \tag{55}
\]

\[
= \frac{1}{|X_j|} \left| \sum_{X \in \chi_j^{\text{in}}} \int_{\mathbb{R}^d} (s - \tilde{X}) f_X(s) ds \right| \tag{56}
\]

\[
= \frac{1}{|X_j|} \sum_{X \in \chi_j^{\text{in}}} \int_{\mathbb{R}^d} ||s - \tilde{X}|| f_X(s) ds \tag{57}
\]

\[
\leq \frac{1}{|X_j|} \sum_{X \in \chi_j^{\text{in}}} r_j \tag{58}
\]

\[
= \frac{|\chi_j^{\text{in}}|}{|X_j|} r_j \tag{59}
\]

\[
\leq r_j \tag{60}
\]

Lemma 14. \(f_2(\tilde{m}_j, \tilde{X}_j) \leq 2 f_2(m_j, X_j) + 4\beta_j n r_j^2\).
Proof. 
\[
f_2(\bar{m}_j, \bar{y}_j) = \sum_{x \in X^\infty_{y}} ||\bar{X} - \bar{m}_j||^2 + \sum_{x \in X^\infty_{y}} \int_{x} ||s - \bar{m}_j||^2 f_X(s)ds
\]
\[
= \sum_{x \in X^\infty_{y}} \int_{x} ||\bar{X} - \bar{m}_j||^2 f_X(s)ds + \sum_{x \in X^\infty_{y}} \int_{x} ||s - \bar{m}_j||^2 f_X(s)ds
\]
\[
\leq \sum_{x \in X^\infty_{y}} \int_{x} (2||\bar{X} - s||^2 + 2||s - \bar{m}_j||^2) f_X(s)ds + \sum_{x \in X^\infty_{y}} \int_{x} ||s - \bar{m}_j||^2 f_X(s)ds
\]
\[
\leq 2 \sum_{x \in X^\infty_{y}} \int_{x} ||\bar{X} - s||^2 f_X(s)ds + 2 \sum_{x \in X^\infty_{y}} \int_{x} ||s - \bar{m}_j||^2 f_X(s)ds
\]
\[
\leq 2|X^\infty_{y}|r_7^2 + 2f_2(\bar{m}_j, \bar{y}_j)
\]
\[
= 2|X^\infty_{y}|r_7^2 + 2f_2(m_j, X_j) + 2|X_j|||m_j - \bar{m}_j||^2
\]
\[
\leq 2f_2(m_j, X_j) + 4\beta_j \delta r_j^2
\]

Lemma 15. In the process of finding \(c_j\) in our algorithm cMeans, we assume that \(Q\) satisfies \(X \setminus B_j \subseteq Q\) and \(|Q| < 2||X \setminus B_j||\). For the set \(S_2\) in step 5, a subset \(S^*\) of size \(6/e\) of \(S_2\) can be obtained such that all random variables in \(S^*\) are from \(\bar{X}_i\) with a probability of 1/2. Let \(c_j\) denotes the centroid of \(S^*\). Then, the inequality \(||\bar{m}_j - c_j||^2 \leq \frac{4}{3} \epsilon r_j^2 + \frac{2}{3} \sigma \epsilon \delta^2\) holds with a probability of at least 1/6.

Proof. In our algorithm cMeans, we assume that \(S_1 = S_1, \ldots, S_N\), where \(N = 79380k/e^3\). Let \(x'_1, \ldots, x'_N\) be the corresponding random variables of elements in \(S_1\). If \(x_i \in X^\infty_{y}\), obtain \(x'_i = 1\), or else \(x'_i = 0\). It is known easily that \(Pr[x_i \in X^\infty_{y}] \geq \frac{e^2}{7938k}\) by Lemma 12. Let \(x = \sum_{i=1}^N x'_i, u = \sum_{i=1}^N E(x'_i)\). We obtain that \(u \geq 10/e\), and
\[
Pr[x > \frac{6}{e}] = 1 - Pr[x \leq \frac{6}{e}]
\]
\[
\geq 1 - Pr[x \leq \frac{3}{5} u]
\]
\[
\geq 1 - e^{-\frac{1}{2} \frac{3}{5} u}
\]
\[
\geq 1 - e^{-\frac{1}{2} \frac{3}{5} \frac{3}{2} \frac{5}{2}}
\]
\[
\geq 1 - e^{-\frac{5}{2}}
\]
\[
\geq \frac{1}{2}.
\]
Then, the probability that at least \(6/e\) random variables in \(S_1\) are from \(X^\infty_{y}\) is at least 1/2. Since \(S_2 = S_1 \cup \{6/e\} \) copies of each point in \(C\), a subset \(S^*\) of size \(6/e\) of \(S_2\) can be obtained, and the probability that all random variables in \(S^*\) are from \(\bar{X}_i\) is at least 1/2. Let \(c_j\) denote the centroid of \(S^*\) and \(\delta = 5/6\). For \(|S^*| = 6/e\) and \(|\text{widetilde}{d}.X_j| = |X_j|\),
by Lemma 2, $||\tilde{m}_j - c_j||^2 \leq \frac{\beta}{5} \frac{f_2(\tilde{m}_j, \tilde{X}_j)}{|X_j|} = \frac{\beta}{5} \frac{f_2(\tilde{m}_j, \tilde{X}_j)}{|X_j|}$ holds with a probability of at least 1/6.

Assume that $||\tilde{m}_j - c_j||^2 \leq \frac{\beta}{5} \frac{f_2(\tilde{m}_j, \tilde{X}_j)}{|X_j|}$. Then,

$$||\tilde{m}_j - c_j||^2 \leq \frac{\beta}{5} \frac{f_2(\tilde{m}_j, \tilde{X}_j)}{|X_j|} \leq \frac{2f_2(m_j, X_j) + 4\beta n r^2}{5 |X_j|} \leq \frac{4\epsilon^2_j + 2\epsilon^2_{opt}}{5}.$$ (76)

\[\square\]

**Lemma 16.** If $c_j$ satisfies $||\tilde{m}_j - c_j||^2 \leq \frac{\beta}{5} \frac{f_2(\tilde{m}_j, \tilde{X}_j)}{|X_j|}$, then $||m_j - c_j||^2 \leq \frac{9\epsilon^2_j}{10} + \frac{1}{10\beta k}\epsilon^2_{opt}$.  

**Proof.** Assume that $c_j$ satisfies $||\tilde{m}_j - c_j||^2 \leq \frac{\beta}{5} \frac{f_2(\tilde{m}_j, \tilde{X}_j)}{|X_j|}$, then,

$$||m_j - c_j||^2 = ||m_j - \tilde{m}_j + \tilde{m}_j - c_j||^2$$ (77)

$$\leq 2||m_j - \tilde{m}_j||^2 + 2||\tilde{m}_j - c_j||^2$$ (78)

$$\leq 2r_j^2 + \frac{8\epsilon^2_j}{5} + \frac{4\epsilon^2_{opt}}{5}$$ (79)

$$= \frac{4\epsilon^2_j}{5} + (2 + \frac{8\epsilon}{5})r_j^2$$ (80)

$$\leq \frac{9\epsilon^2_j}{10} + \frac{1}{10\beta k}\epsilon^2_{opt}.$$ (81)

\[\square\]

**Lemma 17.** Given an instance $(X, k, L)$ of the uncertain constrained k-means problem, where the size of $X$ is $n$, for $\forall e \in \{0, 1\}, k \geq 2$, we assume that by using our algorithm cMeans(e, X, k, C, U) (C and U are initialized as empty sets), a collection U of candidate sets including approximate centers is obtained. If there exists a set $C_k = \{c_1, \ldots, c_k\}$ in U satisfying that $||m_j - c_j||^2 \leq \frac{9\epsilon^2_j}{10} + \frac{1}{10\beta k}\epsilon^2_{opt}$ (1 \leq j \leq k), then $C_k$ is a $(1 + \epsilon)$-approximation for the uncertain constrained k-means problem.

**Proof.** Assume that $C_k = c_1, \ldots, c_k$ is a set in U satisfying that $||m_j - c_j||^2 \leq \frac{9\epsilon^2_j}{10} + \frac{1}{10\beta k}\epsilon^2_{opt}$ (1 \leq j \leq k). Then,

$$\sum_{j=1}^{k} f_2(c_j, X_j) = \sum_{j=1}^{k} (f_2(m_j, X_j) + |X_j|||m_j - c_j||^2)$$ (82)

$$\leq \sum_{j=1}^{k} (f_2(m_j, X_j) + \beta n (\frac{9\epsilon^2_j}{10} + \frac{1}{10\beta k}\epsilon^2_{opt}))$$ (83)

$$\leq \sum_{j=1}^{k} (f_2(m_j, X_j) + \frac{9\epsilon n}{10} \sum_{j=1}^{k} \beta_j \epsilon_j^2 + \frac{1}{10\beta k}\epsilon^2_{opt})$$ (84)

$$\leq \sum_{j=1}^{k} (f_2(m_j, X_j) + \frac{9\epsilon n}{10} \epsilon^2_{opt} + \frac{1}{10\beta k}\epsilon^2_{opt})$$ (85)

$$= (1 + \epsilon) \cdot OPT_k(P).$$ (86)

\[\square\]

5.3. Time Complexity Analysis

We analyze the time complexity for our algorithm cMeans in this section.

**Lemma 18.** The time complexity of our algorithm cMeans is $O(4^k (13\sqrt{2}k \epsilon)^{\frac{1}{2}} k \epsilon^{-1} \frac{1}{n} d^4)$.
Given an instance
The above formula can be simplified as

In our algorithm cMeans, steps 5–9 have a run time of $O(k/e^3)$, step 11 have a run time of $O(d/e)$, and steps 13–16 have a run time of $O(knd)$. Let $T(n, g)$ denote the time complexity of algorithm cMeans, where $g$ is the number of center centers, and $n$ is the size of $Q$.

If $g = 0$, $T(n, 0) = O(1)$. When $n = 1$, $T(1, g) = a(T(1, g - 1) + O(d/e)) + O(k/e^3)$.

Because $a > k/e^3$, $T(1, g) = a(T(1, g - 1) + O(d/e)) \leq a^g \cdot T(1, 0) + g \cdot a^g \cdot O(d/e) = O(g \cdot a^g \cdot d/e)$. Therefore, $T(1, g) \leq O(4^g (13231e^2k)^{6g/e})^{1/2}d$, where $e = 2.7183$.

For $\forall n \geq 2$ and $g \geq 1$, the recurrence of $T(n, g)$ could be obtained as follows:

$$T(n, g) = a \cdot T(n, g - 1) + T(\lceil n/2 \rceil, g) + a \cdot O(\frac{d}{e}) + O(\frac{k}{e^3}) + O(knd).$$

Because $a > k/e^3$, two constants $b_1$ and $b_2$ with $b_1 \geq 1$ and $b_2 \geq 1$ could be obtained to arrive at the following recurrence.

$$T(n, g) \leq a \cdot T(n, g - 1) + T(\lceil n/2 \rceil, g) + a \cdot b_1 \cdot \frac{d}{e} + b_2 \cdot knd.$$

Now we claim that $T(n, g) \leq b_1 \cdot b_2 \cdot \frac{1}{e} \cdot a^g \cdot 2^g \cdot nd - b_1 \cdot \frac{d}{e} \geq a(b_1 \cdot b_2 \cdot \frac{1}{e} \cdot a^g - 1) \cdot 2^{g(s-1)} \cdot nd - b_1 \cdot \frac{d}{e}$

$$+ b_1 \cdot b_2 \cdot \frac{1}{e} \cdot a^g \cdot 2^{gs} \cdot \lceil n/2 \rceil d - b_1 \cdot \frac{d}{e} + a \cdot b_1 \cdot \frac{d}{e} + b_2 \cdot knd.$$

The above formula can be simplified as $\frac{1}{e^2} \cdot b_1 \cdot a^g 2^g \geq k$, which holds for $\forall g \geq 1$. For $a = (\frac{13231e^2k}{e^3})^{6g/e}$, $T(n, k) = O(4^k (13231e^2k)^{6k/e})^{1/2}nd$. \[\Box\]

Thus, we can obtain the following Theorem 2.

**Theorem 2.** Given an instance $(X, k, L)$ of the uncertain constrained k-means problem, where the size of $X$ is $n$, for $\forall \epsilon \in (0, 1), k \geq 2$, by using our algorithm cMeans$(e, X, k, C, U)$, a collection $U$ of candidate sets including approximate centers can be obtained with a probability of at least $1/12^2$ such that $U$ includes at least one candidate set including approximate centers that is a $(1 + \epsilon)$-approximation for the uncertain constrained k-means problem, and the time complexity of our algorithm cMeans is $O(4^k (13231e^2k)^{6k/e})^{1/2}nd$.

6. Conclusions

In this paper, we defined the uncertain constrained k-means problem first, and then presented a stochastic approximate algorithm for the problem in detail. We proposed a general mathematical model of the uncertain constrained k-means problem, and studied the random sampling properties, which are very important to deal with the uncertain constrained k-means problem. By applying a random sampling technique, we obtained a $(1 + \epsilon)$-approximate algorithm for the problem. Then, we investigated the success probability, correctness and time complexity analysis of our algorithm cMeans, whose running time is $O(4^k (13231e^2k)^{6k/e})^{1/2}nd$. However, there also exists a big gap between the current algorithms for the uncertain constrained k-means problem and the practical algorithms for the problem, which has been mentioned in [13] similarly.
We will try to explore a much more practical algorithm for the uncertain constrained $k$-means problem in future. It is known that the 2-means problem is the smallest version of the $k$-means problem, and remains NP-hard. The approximation schemes for the 2-means problem can be generalized to solve the $k$-means problem. Due to the particularity of the uncertain constrained 2-means problem, we will study approximation schemes for the uncertain constrained $k$-means problem through approximation schemes of the uncertain constrained 2-means problem. Additionally, we will apply the proposed algorithm to some practical problems in the future.

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