

G-GAMES WITH COALITIONS

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Abstract. This paper models games where the strategies are nodes of a graph G (we denote them as G-games) and in presence of coalition structures. The cases of one-shot and repeated games are presented. In the latter situation, coalitions are assumed to move from a strategy to another one under the constraint that they are adjacent in the graph. We introduce novel concepts of pure and mixed equilibria which are comparable with classical Nash and Berge equilibria. A Folk Theorem for G-games of repeated type is presented. Moreover, equilibria are proven to be described through suitably defined Markov Chains, hence leading to a constrained Monte Carlo Markov Chain procedure.

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1. Introduction

We present evolutionary game models where the strategies have a graph structure. The players may pass from a strategy to an adjacent one in the graph or, alternatively, they can hold their positions. The target of each player is the maximization of her/his payoff, and players are allowed to form coalitions.

Taking the strategies of the game as the nodes of a graph has an intuitive motivation. In fact, we have in mind that the change of a decision is usually a stepwise process over a continuum of alternatives, where the decider modifies her/his status by selecting a status which is close to the previous one. Think about a game where the strategies of the single players are the positions on a lattice, and at any time they have to decide a new position in the neighborhood of the previous one. Such situations commonly appear in the real world. This is the case of physical restrictions in the actions, with players constrained to move from a position (or strategy) to an adjacent one. An example might be the location strategies with time constraints: each player selects a geographical location which is achievable from her/his current position in no more than 10 minutes walking. It is clear that the selected position will be close (adjacent, in some sense) to the previous one.

Some relevant contributions in game theory discuss the cases of interacting players, who are then viewed as nodes of a graph (see e.g. [33] and the recent contribution [41] with references therein). This setting is associated to the strategic behavior of the agents, whose connections of local types might create global behaviors.

Less attention has been paid to the case in which strategies are linked together. In this respect, it is important to give credit to some remarkable examples in the literature of games with strategies exhibiting a graph structure.

A graph-based structure of the strategies is presented in the context of game colouring graphs, which is a well-known problem in game theory. In such a model, two players alternatively select and colour a node of the graph under some adjacency constraints and with opposite targets (for the details, see e.g. [12]).

Under different conditions, we mention also the game of cops and robbers (see e.g. [3] and the monograph [12]). In this case, the players (cops and robbers) move on
a graph at each step to a node which is adjacent to the previous position, with the intuitive target to escape (robber) and to catch the robber by occupying her/his same node (cops).

The trapping games are also relevant (see [8]). Here a couple of players is considered, and they are assumed to select alternatively strategies along the arcs of a graph. The selection of each vertex twice is forbidden, so that one of the players loses the game when she/he cannot move (the player is trapped).

We feel that such games are close to us for one important aspect: the players move on strategies which are adjacent nodes of a graph. However, we are more general than the mentioned models, in that we abandon the binary situation win-lose and the alternation of the players in moving, consider more than two players and admit the presence of coalitions of players. Moreover, we present here a game theoretical modeling since we do not fix payoffs or a specific target for the players but, rather, we develop a general framework to be adapted to a plethora of special cases.

As mentioned above, in the proposed setting we consider a partition of the set of the players in coalitions — and in so doing, we fix a coalition structure of the game (see e.g. [6, 35]).

We introduce and study the equilibria of the game. At this aim, we move from the breakthrough work of Nash [34] — which relies to the context of non-cooperative games — and consider the Berge extension of the classical Nash equilibria to coalitions of players (see [10]). Specifically, we extend such a concept by including the presence of a graph structure for the strategies. In particular, we provide a definition of equilibria on the basis of the comparison of the payoffs of adjacent strategies, hence extending the concept of the standard Berge equilibria. Indeed, Berge equilibria are subcases of our setting, in that they are included in our definition when the graph of the strategies is complete.

Our framework is then adapted to the repeated games (see e.g. [5, 7, 9, 18, 29]). More specifically, we consider a class of such games where the players move on adjacent strategies in the graph from a time to the next one. The time horizon $T$ of the repeated game can be finite or infinite. The final payoff of each coalition is assumed to be derived by the sum of the payoffs realized at each step of the repeated game. In the situation of infinitely played repeated games — sometimes called superfames — scholars introduce often a discount factor $\delta \in (0, 1)$ for having the convergence of the payoffs series (see e.g. [1, 2, 23] and references therein). The discount factor is not a mere mathematical device: it has also the intuitive economic meaning of under evaluating the future payoffs of the repeated games.

In the present paper, we avoid the introduction of the discount factor by removing by definition the convergence problems of the aggregated payoffs. Specifically, in accord to classical game theory literature, we take the total payoff of the supergame as the liminf of the mean of the payoffs of the one-shot games (see e.g. [23]). In so doing, we are in line with the literature dealing with infinitely played repeated games without discount factor for the payoffs, where the convergence problems in the formulation of the series of the total payoff are removed by definition (see e.g. [17]).

We prove that it is sufficient that the dynamics on the graph is driven by Markov chains, and this leads also to add results to the field of Monte Carlo Markov Chain (MCMC). For a wide perspective on MCMC, see [4, 13], while a review of the endless applications of MCMC should include relevant contributions like [24, 25].

In the contexts of the Markov chains, [16] studies the equilibria of a repeated game in dependence of the memory of the players. The authors consider that the action or strategy at any time depends only on the public knowledge related to the
previous $K$ stages of the game (memory $K$). Hence the players construct equilibria collected in $\Gamma_K$. The space of equilibria $\Gamma_K$ is analyzed and compared with $\Gamma_\infty$, i.e. the equilibria constructed using all the past history (unbounded memory).

In [28] a one-period game with unitary memory is studied ($K = 1$) and a Folk Theorem for bargaining games is presented.

For what concerns the memory of the repeated games, in our context we prove that the connectedness of $G$ leads to $\Gamma_K = \Gamma_\infty$, for each $K \geq 1$. Moreover, we are able to derive a new version of the Folk Theorem tailored to our specific context (see also versions of Folk Theorems in [7, 9, 15, 19, 22, 31]). In particular, we deal with the case $T = \infty$. Differently with the standard case of infinitely repeated games without discounting – where Folk Theorem states simply that any equilibrium of the one-shot game is also an equilibrium for the repeated game (see the seminal contribution of [36]) – we here introduce a natural constraint to let such equilibria be consistent with the graph $G$ of the strategies.

In illustrating how the paper flows, some details on the treated topics are also provided. In particular, Section 2 contains the preliminary and notations which serve for formalizing the game models we deal with. We denote such game models as $G$-games, to point the attention on the graph of the strategies $G$, and assume that $G$ is a finite graph. Moreover, such a section is devoted to the definition of the (pure) $C$-equilibria for games with coalition structure $C$. The connection between $G$-games and standard games is also discussed.

Section 3 is a technical one. It discusses the definition of (mixed) $C$-equilibria in a static context. By construction, such equilibria are independent from the graph $G$ and depend only on the coalition structure $C$. Theorem 1 is an existence result for mixed $C$-equilibria for $G$-games, that is an arrangement with our notation of [10, 34].

Section 4 is divided into two subsections. In the first one, we focus on MCMC problems when Markov chains are linked to graphs. Indeed, the presence of an adjacency constraint over the strategies leads to a constrained version of MCMC, in the sense that we will admit only nonnull transition probabilities between two adjacent states (strategies). In this context, we are able to say whenever, given a distribution $\mu$, it is possible to construct a homogenenous or a nonhomogeneous Markov chain having $\mu$ as empirical distribution (see Theorems 2 and 3). A definition of a specific class of graphs – the $C$-decomposable graphs – is introduced in Definition 4 on the basis of the strong products of graphs (see [37]). This definition will be used to introduce the repeated $G$-games.

In particular, $C$-decomposability leads to the possibility that each coalition of players might select a strategy only on the basis of the knowledge of the past history of the game, without the need of assuming communications among coalitions. Moreover, in this case, we do a specific construction of the MCMC that is computationally less heavy than in the general case (see Theorem 4).

The second subsection represents the conclusion of the arguments developed in the previous parts of the paper. In fact, we introduce therein a dynamical setting by providing the definition of repeated $G$-games (see the general Definition 5 and the more specific Definition 6). Here we are able to observe that the (pure) $C$-equilibria might turn out to be meaningful when a game is played $T = 2$ times or when $T$ is unknown – where unknown should be intended in the sense that the coalitions cannot do any prediction about the end of the game.

Differently, the (mixed) $C$-equilibria can be used to prove a version of the Folk Theorem in the context of repeated $G$-games with $T = \infty$ and under some conditions on the information. In particular, Definition 7 formalizes $C$-equilibria for
the $G$-repeated games when $T = \infty$. Then, Theorem 8 states that the multidimensional Markov chain introduced in the previous subsection can be viewed as a $C$-equilibrium for the $G$-repeated games with $T = \infty$, minimal information – i.e., coalitions have knowledge only of their previously selected strategies – and in both cases of initial strategies assigned by an external referee or selected by players themselves. As a corollary of Theorem 8 we have the above-mentioned new version of the Folk Theorem, which guarantees the existence of a $C$-equilibrium for the $G$-repeated games in the considered framework.

Last section provides some conclusive remarks and traces lines for future research.

2. Definition of a $G$-game

Consider a set of $n$ players $V = \{1, \ldots, n\}$. The $j$-th player has a set of $k_j$ (pure) strategies collected in $S_j = \{s_j^1, \ldots, s_j^{k_j}\}$. We denote the product space of the strategies as

$$S = \prod_{j \in V} S_j,$$

and an element $s \in S$ is a profile of strategies. For $j \in V$, $\pi_j : S \rightarrow \mathbb{R}$ denotes the payoff function of the $j$-th player. Hence, for $s = (s_1, \ldots, s_n) \in S$, $\pi_j(s)$ is the payoff of the $j$-th player when the players use the profile of strategies $s$. The vector of payoffs is $\pi = (\pi_1, \ldots, \pi_n)$.

For convenience we introduce a notation for strategies substitution. For a given set $C \subset V$ and $s, t \in S$ we define the profile of strategies $[s, t; C] \in S$, by setting, for its $j$-th component,

$$[s, t; C]_j = \begin{cases} t_j & \text{if } j \in C; \\ s_j & \text{if } j \notin C. \end{cases}$$

A set $C \subset V$ of players is called coalition. We are interested in the coalitions which form a partition $C = \{C_1, \ldots, C_r\}$ of $V$ and we will call this partition a coalition structure. We consider games which present a coalition structure.

The payoff of a coalition $C$ is a function $\Pi_C : S \rightarrow \mathbb{R}$. Sometimes, it is natural to consider $\Pi_C$ as the sum of the payoffs of the single players belonging to $C$, i.e. $\Pi_C(s) = \sum_{i \in C} \pi_i(s)$, for $s \in S$.

For a given coalition $C \subset V$ we write the set of the pure strategies of $C$ as

$$S_C = \prod_{\ell \in C} S_\ell.$$

Let us consider a $G$-game with coalition structure $C = \{C_1, \ldots, C_r\}$. Let $s_{C_\ell} \in S_{C_\ell}$ for $\ell = 1, \ldots, r$. The profile of strategies can be written by $s = (s_{C_1}, \ldots, s_{C_r}) \in S$. In this formalism the strategy of the $j$-th player can be obtained as the projection on $S_j$ of $s_{C_\ell}$ where $\ell \in \{1, 2, \ldots, r\}$ is the unique element having $j \in C_\ell$.

The vector of the payoffs of the coalitions $C$ is denoted by $\Pi = (\Pi_{C_1}, \ldots, \Pi_{C_r})$.

Hereafter, we consider that the elements of $S$ are nodes of a graph $G = (S, E)$. Some notations for graphs are now presented. Given two graphs $G = (S, E)$ and $G' = (S', E')$ we say that $G'$ is a subgraph of $G$ if $S' \subset S$ and $E' \subset E$, and we write $G' \subset G$. Moreover, the subgraph $G' \subset G$ is said to be an induced subgraph of $G$ if $s, t \in S'$ and $\{s, t\} \in E$ imply $\{s, t\} \in E'$. In this case we write $G' = G[S']$.

The profiles of strategies $s, t \in S$ are declared adjacent in $G$ if $\{s, t\} \in E$ or $s = t$. We define a $G$-game with coalition $C$ as the quadruple $(V, C, \Pi, G)$, where $\Pi$ are the payoffs of the coalitions $C$.

A key concept of the present study is the equilibrium of the $G$-game in presence of coalitions.
**Definition 1.** Given a graph $G = (S, E)$ and a $G$-game $(V, C, \Pi, G)$ where $C$ is a partition of $V$, we say that $s \in S$ is a pure $C$-equilibrium for the $G$-game if

$$\Pi_C(s) \geq \Pi_C([s, s; C]),$$

for any $C \in C$ and any $s \in S$ such that $\{s, \bar{s}\} \in E$.

Sometimes we refer to pure $C$-equilibria simply as $C$-equilibria.

Notice that the concept of $C$-equilibrium in Definition 1 coincides with the classical Berge equilibrium (see [10]) when the considered graph $G$ is complete. Indeed, Berge equilibrium is a Nash one for coalitions, without any structure of the set of strategies. Some easy remarks follow.

**Remark 1.** If all the nodes of the graph $G$ are isolated, then any $s \in S$ is a $C$-equilibrium.

If the graph $G$ is complete and $C = \{\{1\}, \ldots, \{n\}\}$ then all the couples of elements of $S$ are formed by adjacent strategies and the $G$-game becomes the standard game $(V, S, \pi)$. In this case, $s \in S$ is a $C$-equilibrium for the $G$-game $(V, C, \Pi, G)$ if and only if it is a pure Nash equilibrium for $(V, S, \pi)$. If the graph $G$ is complete and $C = \{V\}$ with

$$\Pi_V(s) = \sum_{i \in V} \pi_i(s), \text{ for } s \in S,$

Then any $C$-equilibrium for the $G$-game is a Pareto optimal solution for the game $(V, S, \pi)$. The finiteness of $S$ guarantees that each $G$-game admits a $C$-equilibrium when $C = \{V\}$ and the total payoff of $V$ is the sum of the individual payoffs.

Thus, our definition of $C$-equilibrium for a $G$-game is not only more general than Berge equilibrium, but it is also a generalization of the Nash equilibrium and Pareto optimal solution for a game.

For a $G$-game with a coalition structure $C$, we collect the $C$-equilibria in the set $E_C$. The set $E_C$ contains the set of the pure Berge equilibrium but it can be empty.

### 3. Mixed $C$-equilibria

In this section we introduce and analyse the mixed equilibria for a $G$-game. The context is static, in the sense that the game is played only one time. Therefore, it will be clear that all the mixed $C$-equilibrium, here presented, will not depend on the choice of the graph $G$. However, the relevance of the graph will be clear on the section dealing with the repeated $G$-games. This part is analogous to Berge (for coalitions) or Nash mixed equilibria (see [10, 54]); it is presented only for ease of reading the paper. We notice that in [10, 34] the pure equilibria are also mixed equilibria. As we will see in our presentation this is not longer true.

A mixed strategy for a coalition $C \in \mathcal{C}$ is a distribution on $S_C$, and we denote it by $\Lambda_C = (\lambda_C(s) : s \in S_C)$. In presence of mixed strategies, the payoff of a coalition $C$ will be a random variable. Reasonably, we will also consider that the single coalitions act independently one each other because they are not communicating. Thus, the choice of the individual mixed strategies for all the coalitions $C_1, \ldots, C_r$ fixes also a product distribution on $S$ for the game. Hence, the expected payoff for coalition $C \in \mathcal{C}$ is

$$E_{\Lambda_{C_1} \times \ldots \times \Lambda_{C_r}}(\Pi_C) = \sum_{s_{C_1} \in S_{C_1}} \cdots \sum_{s_{C_r} \in S_{C_r}} \Pi_C(s_{C_1}, \ldots, s_{C_r}) \prod_{\ell=1}^r \lambda_{C_{\ell}}(s_{C_{\ell}}),$$

where $E_{\Lambda_{C_1} \times \ldots \times \Lambda_{C_r}}$ is the expected value with respect to the product distribution $\Lambda_{C_1} \times \ldots \times \Lambda_{C_r}$ and $\Lambda_{C_{\ell}}$, for $\ell = 1, \ldots, r$, is the distribution selected by the coalition $C_{\ell}$.
Definition 2. Let us consider a $G$-game $(V, C, \Pi, G)$ with coalition structure $C = \{ C_1, \ldots, C_r \}$. A mixed $C$-equilibrium for the $G$-game is a product distribution $\Lambda = \prod_{t=1}^{\infty} \Lambda_{C_t}$, where
\[
E_{\Lambda_{C_1} \times \cdots \times \Lambda_{C_r}}(\Pi_{C_t}) \geq E_{\Lambda_{C_1} \times \cdots \times \Lambda_{C_{t-1}} \times \Lambda_{C_{t+1}} \times \cdots \times \Lambda_{C_r}}(\Pi_{C_t}),
\]
for any $\ell \in \{ 1, \ldots, r \}$ and for any distribution $\Lambda_{C_t}$ on the space $S_{C_t}$.

We collect all the mixed $C$-equilibria in the set $\mathcal{M}_C$.

As already announced above, Definition 2 does not depend on the presence of the graph $G$, and can be provided for a generic game whose strategies are not nodes of a graph. However, the reference to $G$-games will be useful in the next Section.

Theorem 1. For a given $G$-game $(V, C, \Pi, G)$ there exists a mixed $C$-equilibrium.

Proof. Let us see the coalition $C \in C$ as a single player having space of strategies given by $S_C$. The payoff of the player identified with the coalition $C$ is $\Pi_C$. Therefore we are dealing with a standard game as defined by Nash in [34]. By applying Theorem 1 of [34] one obtains the existence of a product distribution $\bar{\Lambda} = \prod_{t=1}^{\infty} \bar{\Lambda}_{C_t}$ which satisfies (3).

For a given $G$-game $(V, C, \Pi, G)$, let us define
\[
\hat{\mathcal{M}}_C = \{(s_{C_1}, \ldots, s_{C_r}) \in S : \delta_{s_{C_1}} \times \cdots \times \delta_{s_{C_r}} \in \mathcal{M}_C \}.
\]
We notice that $\mathcal{E}_C \cap \hat{\mathcal{M}}_C$ is the set of the pure Berge equilibria. Furthermore, in general, we have that $\mathcal{E}_C \not\subset \hat{\mathcal{M}}_C$ and $\hat{\mathcal{M}}_C \not\subset \mathcal{E}_C$.

4. MONTE CARLO MARKOV CHAIN ON GRAPHS AND REPEATED $G$-GAMES

This section deals with a constrained MCMC procedure in presence of graphs and its application to the repeated $G$-games, which will be defined below. The MCMC part can be treated separately and it has a relevance and an interest in itself.

4.1. MCMC on graphs. We deal with a MCMC problem. In particular, we construct some Markov chains which are linked to the graph $G$.

Definition 3. We say that a stochastic process $X = (X(t) : t \in \mathbb{N})$ on $S$ is consistent with the graph $G = (S, E)$ if, for each $t \in \mathbb{N}$, $X(t)$ and $X(t+1)$ are adjacent in $G$ with probability one.

We notice that if a process $X = (X(t) : t \in \mathbb{N})$ is consistent with a graph $G$ then it is also consistent with any graph $G' \supset G$.

Given a finite graph $G = (S, E)$ and a distribution $\mu = (\mu(s) : s \in S)$, we will provide an answer to the following question:

**Q** Is it possible to construct a (not necessarily homogeneous) Markov chain $X = (X(t) : t \in \mathbb{N})$ which is consistent with $G$ and such that its empirical distribution converges almost surely to $\mu$ as $t$ goes to infinity?

Thus, we want to construct a Markov chain $X = (X(t) : t \in \mathbb{N})$ with the following properties: $X$ is consistent with the graph $G$ and
\[
\lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} 1_{\{X(m) = s\}} = \mu(s), \quad s \in S \quad a.s.. \tag{4}
\]

We provide an answer to question **Q** in all possible situations and we show that when $G$ is connect it is possible to construct such a Markov chain. Moreover, in constructing such a Markov chain, we are in the framework of the MCMC theory, even if here the Markov chain is constrained to have transitions only between adjacent states of $G$. 

All the possible situations, along with the related answers to question Q, can be distinguished in four cases:

(i) If the distribution $\mu$ is concentrated on a unique $\bar{s} \in S$, i.e. $\mu = \delta_{\bar{s}}$, then one can construct the constant Markov chain $X = (X(t) : t \in \mathbb{N})$ such that $X(t) = \bar{s}$, for each $t$. By Definition 3 and the concept of adjacent states, one has that $X$ is consistent with $G$ and (4) is trivially satisfied.

(ii) If $G[supp(\mu)]$ is not connected but $supp(\mu)$ is contained in a connected component of $G$, then one can construct a nonhomogeneous Markov chain which is consistent with $G$ and fulfilling condition (4) (see Theorem 2 below).

(iii) If $G[supp(\mu)]$ is not connected and $supp(\mu)$ is not contained in a unique connected component of $G$, then it does not exist a stochastic process which is consistent with $G$ and fulfilling (4) (see Theorem 3 part b. below).

(iv) If $G[supp(\mu)]$ is connected, then one can construct a homogeneous Markov chain consistent with $G$ which satisfies (4) (see Theorem 3 part a. below).

We now deal with item (ii).

Notice that, in this case, there exists a connected component of $S$, say $\hat{S}$, such that $supp(\mu) \subset \hat{S}$ and $supp(\mu) \neq \hat{S}$. Without loss of generality and to avoid the introduction of further notation, we assume that $G$ is connected and we identify $\hat{S}$ with $S$.

For a given distribution $\mu = (\mu(s) : s \in S)$, let us define, in case (ii), the non-empty set

$$A_k = \left\{ s \in S : \mu(s) < \frac{1}{k} \right\}, \quad k \geq 1$$

and let the distribution $\eta_k = (\eta_k(s) : s \in S)$ be

$$\eta_k(s) = \frac{1}{|A_k|} 1_{(s \in A_k)}, \quad s \in S,$$

i.e. $\eta_k$ is the uniform distribution on $A_k$. We also define the distribution $\mu_k = (\mu_k(s) : s \in S)$ as

$$\mu_k = \frac{1}{k} \eta_k + \frac{k-1}{k} \mu. \quad (5)$$

Notice that

$$||\mu_k - \mu||_{TV} = \frac{1}{k} ||\eta_k - \mu||_{TV} \leq \frac{1}{k},$$

where $|| \cdot ||_{TV}$ is the total variation norm (see e.g. [32]).

Let $N$ denote the cardinality of $S$. Since $supp(\mu)$ is not connected in $G$, then it contains at least two points. Since $supp(\mu) \subset \hat{S}$ and $\hat{S}$ is connected, then $N \geq 3$. By construction, for $k$ large enough and since $N \geq 3$, one has that

$$\mu_k(s) \geq \frac{1}{(N-1)k}, \quad s \in S. \quad (6)$$

Let us label the elements of $S = \{s^1, \ldots, s^N\}$ such that

$$\mu(s^1) \geq \mu(s^2) \geq \cdots \geq \mu(s^N).$$

Let us take an integer $k$ such that

$$k > \frac{1}{\min\{\mu(s) > 0 : s \in S\}}.$$

According to definition (5), with the previous selection of $k$, one has

$$\mu_k(s^1) \geq \mu_k(s^2) \geq \cdots \geq \mu_k(s^N) > 0. \quad (7)$$
We now construct the transition matrix $P^{(\mu_k, G)} = (p_{l, r} : l, r = 1, \ldots, N)$ related to the distribution $\mu_k$ and to the graph $G = (S, E)$. For each $l, m = 1, \ldots, N,$

$$p_{l, m} = \begin{cases} p_i, & \text{if } l < m \text{ and } \{s^l, s^m\} \in E; \\ \frac{\mu_k(s^m)}{\mu_k(s^l)} p_l, & \text{if } l > m \text{ and } \{s^l, s^m\} \in E; \\ p_l, & \text{if } l = m; \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

where

$$p_l = 1 - \sum_{m': m' > l} 1_{\{(s^l, s^{m'}) \in E\}} + \sum_{m': m' < l} \frac{\mu_k(s^{m'})}{\mu_k(s^l)} 1_{\{(s^l, s^{m'}) \in E\}}$$

and

$$p = \min_{l=1,\ldots,N} \frac{1}{2 \left( \sum_{m': m' > l} 1_{\{(s^l, s^{m'}) \in E\}} + \sum_{m': m' < l} \frac{\mu_k(s^{m'})}{\mu_k(s^l)} 1_{\{(s^l, s^{m'}) \in E\}} \right)} \tag{9}$$

Notice that by definition $p \leq \frac{1}{2}$. In fact, since $G$ is connected, there exists at least an edge $\{s^l, s^m\} \in E$, with $m > 1$; thus the denominator of (9) is at least equal to 2, when $l = 1$. The transition matrix $P^{(\mu_k, G)}$ is well defined, since $G$ is connected.

Formula (8) assures that the couple $(\mu_k, P^{(\mu_k, G)})$ is reversible. Moreover, $P^{(\mu_k, G)}$ is irreducible, since $G$ is connected, thus $\mu_k$ is the unique invariant distribution of $P^{(\mu_k, G)}$. One can also see that $P^{(\mu_k, G)}$ is aperiodic since, by construction, $p_l \geq \frac{1}{k}$ for $l = 1, \ldots, N$.

We introduce the ergodic coefficient of Dobrushin (see [20] and [13] p. 235), which is defined as

$$\delta(P) = 1 - \inf_{i, j = 1, \ldots, N} \sum_{h=1}^{N} p_{i, h} \wedge p_{j, h} \tag{10}$$

where $P = (p_{i, j} : i, j = 1, \ldots, N)$ is a stochastic matrix.

**Lemma 1.** Given the transition matrix $P^{(\mu_k, G)}$ on $S$ constructed above, with $N = |S| \geq 3$, the Dobrushin’s ergodic coefficient can be bounded from above as follows

$$\delta((P^{(\mu_k, G)})^{N-1}) \leq 1 - \left( \frac{c_N}{k} \right)^{N-1}$$

where $c_N = \frac{1}{2(N-1)^2}$ and $k$ is large enough.

**Proof.** For $k$ large enough, condition $N \geq 3$, and inequalities (6) and (7) provide

$$1 \leq \frac{\mu_k(s^m)}{\mu_k(s^l)} \leq k(N-1), \text{ for } l > m. \tag{11}$$

Thus, by (11) one obtains $p \geq \frac{c_N}{k}$, for $k$ large enough. Then one has that, if $p_{l, m} \neq 0,$

$$p_{l, m} \geq \frac{c_N}{k}. \tag{12}$$

For $k$ large enough, since the graph $G$ is connected and $p_l \geq \frac{1}{k}$ for each $l = 1, \ldots, N$, then (12) gives that

$$p_{l, m}^{(N-1)} \geq \left( \frac{c_N}{k} \right)^{N-1}, \quad l, m = 1, \ldots, N,$$

where $p_{l, m}^{(N-1)}$ is the transition probability from $s^l$ to $s^m$ in $(N-1)$ steps.

Then, by definition of the ergodic coefficient of Dobrushin in (10), one has the thesis. \hfill $\square$
For a given distribution over $S$, namely $\lambda = (\lambda(s) : s \in S)$, we construct a non-homogeneous Markov chain $X = (X(t) : t \in \mathbb{N})$ with $\lambda$ as initial distribution. The transition matrix of the Markov chain $X$ at time $t \in \mathbb{N}$ will be denoted by $P(t) = (p_{i,j}(t) : i,j = 1,\ldots,N)$.

Let us consider an increasing sequence of times $(t_\ell : \ell \in \mathbb{N})$, and pose

$$P(t) = \sum_{k=1}^{\infty} P(\mu_k, G) 1_{\{t \in [t_k, t_{k+1})\}}.$$ \hfill (13)

**Theorem 2.** Given a connected graph $G = (S, E)$ and a distribution $\mu = (\mu(s) : s \in S)$, any Markov chain $X = (X(t) : t \in \mathbb{N})$ constructed above with sequence $(t_\ell : \ell \in \mathbb{N})$ and $t_\ell = \ell^2 N$, for $\ell \in \mathbb{N}$, satisfies $\mathbb{P}$. To prove the result, we first check that

$$\lim_{\ell \to \infty} \frac{1}{t_\ell} \sum_{m=0}^{t_\ell-1} 1_{\{X(m)=s\}} = \mu(s), \quad s \in S \quad a.s. \quad \text{\hfill (14)}$$

By definition of $(t_\ell : \ell \in \mathbb{N})$ in the hypotheses, one has

$$\lim_{\ell \to \infty} \frac{t_{\ell+1} - t_\ell}{t_\ell} = 0,$$

and then (14) is equivalent to (14).

For $\varepsilon > 0$ and $s \in S$ let us define the sequence of events $(B_\ell(\varepsilon, s) : \ell \in \mathbb{N})$ as

$$B_\ell(\varepsilon, s) = \left\{ \left| \mu(s) - \frac{1}{t_\ell} \sum_{m=t_\ell}^{t_\ell+1-1} 1_{\{X(m)=s\}} \right| < \varepsilon \right\}. \hfill (15)$$

To obtain (14) it is enough that, for each $\varepsilon > 0$ and $s \in S$ one has

$$\mathbb{P} \left( \liminf_{\ell \to \infty} B_\ell(\varepsilon, s) \right) = 1.$$  

Now, take the auxiliary independent random variables $(Y(t) : t \in \mathbb{N})$ with values on $S$ such that $Y(i)$ has distribution $\mu_k$ if $i \in [t_k, t_{k+1})$ (see (5) for the definition of $\mu_k$).

Notice that for each initial distribution $\vartheta$ on $S$, Lemma [1] and Dobrushin’s Theorem (see (13)) give that

$$\|\vartheta P(t_\ell) - \mu_\ell\|_{TV} \leq \delta(P(t_\ell)^{N-1}) \left( \frac{\varepsilon N}{\ell} \right)^{\frac{1}{2}} \frac{\ell^{2N}}{N} \leq \exp \left( -c_N^2 \frac{\ell N}{N-1} \right), \hfill (16)$$

for $\ell$ large enough.

Let $\epsilon_N = c_N^{-1}$. Given $i \geq 0$ and $k \geq 1$, by the maximal coupling (see (22) and inequality (16) one can couple $X(t_\ell + k \ell^2 N + i)$ with $Y(t_\ell + k \ell^2 N + i)$ so that

$$\mathbb{P}(X(t_\ell + k \ell^2 N + i) \neq Y(t_\ell + k \ell^2 N + i)) \leq \exp \left( -\epsilon_N \frac{\ell N + 1}{N-1} \right), \hfill (17)$$

for $\ell$ large enough.

Let us define the sequence of events $(A_{\ell,i} : \ell \in \mathbb{N}, i \in [0, \ell^2 N])$ by

$$A_{\ell,i} = \left\{ X(t_\ell + a \ell^2 N + i) = Y(t_\ell + a \ell^2 N + i) : a \geq 1 \text{ and } t_\ell + k \ell^2 N + i \leq t_{\ell+1} - 1 \right\}, \hfill (18)$$

for each $\ell \in \mathbb{N}$ and $i \in [0, \ell^2 N)$.

For $\ell$ large enough and by subadditivity, one has

$$\mathbb{P}(A_{\ell,i}) \geq 1 - (\ell + 1)^{5N} \exp \left( -\epsilon_N \frac{\ell N + 1}{N-1} \right).$$
We also set $\hat{A}_\ell = \bigcap_{k=0}^{\ell N-1} A_{\ell k}$. Then, for $\ell$ large enough,

$$
P(\{X(t) = Y(t) : t \in [t_\ell + \ell^2 N, t_{\ell + 1})\}) = P(\hat{A}_\ell) \geq 1 - (\ell + 1)^7 N \exp \left(-\hat{c}_N \frac{\ell^N + 1}{N - 1}\right).$$

(19)

By (19) and the first Borel-Cantelli lemma, it follows that $P(\liminf_{\ell \to \infty} \hat{A}_\ell) = 1$.

Now, for $\varepsilon > 0$ and $s \in S$, let us define the sequence of events $(\hat{B}_\ell(\varepsilon, s) : \ell \in \mathbb{N})$

$$
\hat{B}_\ell(\varepsilon, s) = \left\{ \left| \mu(s) - \frac{1}{t_{\ell + 1} - t_\ell} \sum_{m=t_\ell}^{t_{\ell + 1} - 1} 1_{\{Y(m) = s\}} \right| < \frac{\varepsilon}{2} \right\}.
$$

(20)

A straightforward calculation gives that

$$
\liminf_{\ell \to \infty} (\hat{B}_\ell(\varepsilon, s) \cap \hat{A}_\ell) \subset \liminf_{\ell \to \infty} B_\ell(\varepsilon, s).
$$

Therefore to end the proof it is enough to show that $P(\liminf_{\ell \to \infty} \hat{B}_\ell(\varepsilon, s)) = 1$. Such a result is a consequence of the large deviation bounds for i.i.d. Bernoulli random variables and the first Borel-Cantelli lemma. This concludes the proof. □

Remark 2. The definition of $(t_\ell : \ell \in \mathbb{N})$ provided in Theorem 3 represents only one of the possible choices. In this respect, it is interesting to note that the proof of Theorem 3 can be adapted to other sequences $(t_\ell : \ell \in \mathbb{N})$. For example, one can take $t_{\ell + 1} - t_\ell \geq c \ell^5 N^{-1}$, with $c > 0$. In this case, for any $\ell \in \mathbb{N}$, there exists $I_\ell \in \mathbb{N}$ and an increasing sequence

$$
t^{(0)}_\ell, t^{(1)}_\ell, \ldots, t^{(I_\ell)}_\ell
$$

such that $t_\ell = t^{(0)}_\ell$, $t^{(I_\ell)}_\ell = t_{\ell + 1}$ and the following property holds

$$
\lim_{\ell \to \infty} \sup_{I \in \{0, 1, \ldots, I_\ell - 1\}} \frac{t^{(I+1)}_\ell - t^{(I)}_\ell}{t^{(I+1)}_\ell - t^{(I)}_\ell} = 0; \quad \lim_{\ell \to \infty} \frac{t^{(0)}_\ell - t^{(I_\ell - 1)}_\ell}{t^{(I_\ell - 1)}_\ell} = 0.
$$

By reproducing the arguments of the proof of Theorem 3 for the new sequences $t^{(0)}_\ell, t^{(1)}_\ell, \ldots, t^{(I_\ell)}_\ell$, one obtains that a new Markov chain defined with this new sequence of times satisfies (3).

Next example shows that the convergence of the distribution $\mu_k$ to the distribution $\mu$ should not be taken too fast and $t_{\ell + 1} - t_\ell$ should be not taken too small in order to have (4).

Example 1. Let us consider a graph $G = (S, E)$ with $S = \{s^1, s^2, s^3, s^4\}$ and $E = \{\{s^1, s^3\}, \{s^1, s^4\}, \{s^2, s^4\}\}$.

Let us take the distribution $\mu = (\mu(s) : s \in S)$ having $\mu(s^1) = \mu(s^2) = \frac{1}{2}$, and define $t_\ell = \ell$, for each $\ell \in \mathbb{N}$, and the sequence of distributions $(\hat{\mu}_\ell : \ell \in \mathbb{N})$ where $\hat{\mu}_\ell = \nu_{2^\ell}$. We take a non-homogeneous Markov chain $X = (X(t) : t \in \mathbb{N})$ with transition matrix $P(\ell) = (p_{m,n}(\ell) : m, n = 1, 2, 3, 4)$, at time $\ell$, given by

$$
P(\ell) = P(\hat{\mu}_\ell, G), \quad \ell \in \mathbb{N}.
$$

In particular, $\|\hat{\mu}_\ell - \mu\|_{TV} \leq \frac{1}{2^\ell}$.

A straightforward computation gives that at time $\ell$

$$
p = \frac{1}{2^{\ell + 1}}, \quad (21)
$$

accordingly to the definition of $p$ given in (9). Thus, (21) gives that $p_{1,1}(\ell) = 1 - \frac{1}{2^\ell}$ at time $\ell$. Therefore, the Borel-Cantelli’s Lemma guarantees that

$$
|\{(\ell \in \mathbb{N} : X(\ell) = s^1, X(\ell + 1) \neq s^1)\}| < \infty \quad a.s.,
$$
Furthermore, $X \rightarrow R$ excessively high is fast but, if one tries to have an simulated annealing (see [30]). In both cases the hope is that the rate of convergence and therefore local minima (case of simulated annealing) or not convergence to the distribution $Q$ (case of our framework). In this case, the response to question Q might be wrong, even if the Markov chain is consistent with the graph $G$.

Next result provides an answer to $G$ even if the Markov chain is consistent with the graph $G$.

The following two sentences hold true:

**Theorem 3.** The following two sentences hold true:

a. if $G[\text{supp}(\mu)]$ is connected, then each homogeneous Markov chain $X = (X(t) : t \in \mathbb{N})$ with state space $\text{supp}(\mu)$ having transition matrix equal to $P(\mu,G[\text{supp}(\mu)])$ defined in (8) satisfies (1). Furthermore, $X$ is consistent with $G$;

b. if $G[\text{supp}(\mu)]$ is not connected, then each homogeneous Markov chain consistent with $G$ does not satisfies (4).

**Proof.** We prove a.. Since $G[\text{supp}(\mu)]$ is connected, then the transition matrix $P(\mu,G[\text{supp}(\mu)])$ is well defined. Moreover, $\mu$ is the unique invariant distribution of $P(\mu,G[\text{supp}(\mu)])$ because $P(\mu,G[\text{supp}(\mu)])$ is irreducible. Now, by applying the ergodic theorem, one has (1). The consistence of $X$ with $G$ follows from the fact that, for $l \neq m$, $p_{il,m} > 0$ implies $\{s',s''\} \in E$.

We prove b. by contradiction. Assume that (4) holds true for a Markov chain $(X(t) : t \in \mathbb{N})$. Then for each $s \in \text{supp}(\mu)$ one has

$$\mathbb{P}(\{\lim_{t \to \infty} X(t) = s, \text{ i.o.}\}) = 1.$$  \hspace{1cm} (22)

Let us consider $s', s'' \in \text{supp}(\mu)$ which belong to two different connected components of $G[\text{supp}(\mu)]$. By (22), it follows that $\mathbb{P}(T < \infty) = 1$ where

$$T = \inf\{t \in \mathbb{N} : X(t) \in \{s', s''\}\}.$$  \hspace{1cm} (23)

Without loss of generality one can assume that $\mathbb{P}(X(T) = s') > 0$. Then, by the consistence of $X$ with the graph $G$, one has that

$$\mathbb{P}(\{t \in \mathbb{N} : X(t) = s''\} = \emptyset|X(T) = s') = 1.$$  \hspace{1cm} (24)

Therefore

$$\mathbb{P}(\{X(t) = s'', \text{ i.o.}\}) < 1,$$

and this contradicts (22). \hspace{1cm} \Box

**Remark 3.** We notice that, by Theorem 2 a., it is possible, for any $\varepsilon > 0$, to select a homogeneous Markov chain $X = (X(t) : t \in \mathbb{N})$ having transition matrix equal to $P(\mu,k,G)$ (see (12) and (13)), with $k \geq \lceil \frac{1}{\varepsilon} \rceil$, satisfying

$$\lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} 1_{\{X(m) = s\}} = \mu(s), \hspace{1cm} s \in S \hspace{1cm} a.s.$$  \hspace{1cm} (25)

Furthermore, $X$ is consistent with $G$.

Some consequences of Theorems 2 and 3 arise. Let us consider a function $f : S \rightarrow \mathbb{R}$

Under condition of Theorem 2 or of Theorem 3 a. one obtains

$$\lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} f(X(m)) = \mathbb{E}_{\mu}(f), \hspace{1cm} a.s.$$  \hspace{1cm} (26)
where $E_\mu$ is the expected value with respect to the distribution $\mu$, i.e.

$$E_\mu(f) = \sum_{s \in S} f(s) \mu(s).$$

If (23) holds true, then

$$\lim_{t \to \infty} \left| \frac{1}{t} \sum_{m=0}^{t-1} f(X(m)) - E_\mu(f) \right| \leq \varepsilon \max_{s \in S} |f(s)|, \quad a.s.. \quad (25)$$

We now need the definition of product of graphs. The usefulness of such a definition will be clear in the next section on the repeated games. Thus, in the light of the subsequent definitions and results, we use the same notation employed in the formalization of the games.

**Definition 4.** Given a finite set $V$, a partition $\mathcal{C} = \{C_1, \ldots, C_r\}$ of $V$ and a connected finite graph $G = (S, E)$ where $S = \bigcup_{j \in V} S_j$, we say that $G$ is $\mathcal{C}$-decomposable if $G = G_1 \otimes G_2 \otimes \ldots \otimes G_r$, where $G_1 = (S_{C_1}, E_1), \ldots, G_r = (S_{C_r}, E_r)$ and $\otimes$ is the strong product for graphs introduced by [37], i.e. given $(s_{C_1}, \ldots, s_{C_r}), (\bar{s}_{C_1}, \ldots, \bar{s}_{C_r}) \in S$ they are adjacent with respect to $G$ if and only if for any $\ell = 1, \ldots, r$ the vertices $s_{C_\ell}, \bar{s}_{C_\ell} \in S_{C_\ell}$ are adjacent with respect to $G_\ell$. We say that $G = (G_1, \ldots, G_r)$ is the $\mathcal{C}$-decomposition of $G$.

We point out that, for a given coalition structure $\mathcal{C}$, if the graph $G$ has a $\mathcal{C}$-decomposition $G' = (G'_1, \ldots, G'_r)$ then it is unique.

We also notice that a complete graph is trivially $\mathcal{C}$-decomposable, for each partition $\mathcal{C} = \{C_1, \ldots, C_r\}$ of $V$. In this case, the graphs $G_1, \ldots, G_r$ of $G$ are complete.

We consider $\mathcal{C} = \{C_1, \ldots, C_r\}$ a partition of $V$, a $\mathcal{C}$-decomposable graph $G = (S, E)$ with $\mathcal{C}$-decomposition given by $G = (G_1, \ldots, G_r)$, where $G_h = (S_{C_h}, E_h)$, for each $h = 1, \ldots, r$. Let us take a product distribution $\mu = \prod_{h=1}^{r} \mu_{C_h}$, where $\mu_{C_h}$ is a distribution on the space $S_{C_h}$.

In order to proceed, we construct $r$ independent Markov chains $X_{C_1} = (X_{C_1}(t) : t \in \mathbb{N}), \ldots, X_{C_r} = (X_{C_r}(t) : t \in \mathbb{N})$ such that the $h$-th Markov chain $X_{C_h}$ has state space $S_{C_h}$ and an arbitrary initial distribution $\lambda_{C_h} = (\lambda_{C_h}(s_{C_h}) : s_{C_h} \in S_{C_h})$, for each $h = 1, \ldots, r$.

Moreover, by replacing $S$ with $S_{C_h}$ and $\mu$ with $\mu_{C_h}$, we replicate the construction provided before Theorem 2. In so doing, we take $k \in \mathbb{N}$ to define the distribution $\mu_{C_h,k} = (\mu_{C_h,k}(s_{C_h}) : s_{C_h} \in S_{C_h})$.

Now, take a sequence of increasing times $(t^{(C_h)}_{\ell} : \ell \in \mathbb{N})$, such that

$$\min_{h=1, \ldots, r} t^{(C_h)}_{\ell+1} - t^{(C_h)}_{\ell} \geq c \delta^{\bar{N}-1}, \quad (26)$$

with $c$ a positive constant.

The transition matrices of $X_{C_h}$ are $(P_{C_h}(t) : t \in \mathbb{N})$ as in (13):

$$P_{C_h}(t) = \sum_{k=1}^{\infty} P(\mu_{C_h,k}, G_h) 1_{\{t \in [t^{(C_h)}_{\ell}, t^{(C_h)}_{\ell+1}) \}}. \quad (27)$$

We introduce the Markov chain $X = (X(t) = (X_{C_1}(t), \ldots, X_{C_r}(t)) \in S : t \in \mathbb{N})$.

Next result is similar to Theorem 2 but it is based on the $r$ independent Markov chains constructed above. In the context of MCMC, this framework provides a remarkable simplification of the computational complexity, in that dealing with $r$ independent Markov chains with state spaces $S_{C_1}, \ldots, S_{C_r}$ is more affordable than only one Markov chain with state space given by $S = S_{C_1} \times \ldots \times S_{C_r}$. Furthermore, as we will see and as preannounced above, such a context will be of theoretical usefulness in defining the repeated $G$-games.
Let us consider a finite set $V$ and a partition $C = \{C_1, \ldots, C_r\}$ of $V$. Let $S = \prod_{C \in \mathcal{C}} S_C$, and a $C$-decomposable connected graph $G = (S, E)$, with $C$-decomposition $G = (G_1, \ldots, G_r)$.

Let us take a product distribution $\mu = \prod_{h=1}^r \mu_{C_h}$ and consider the $r$ independent Markov chains $X_{C_1} = (X_{C_1}(t) : t \in \mathbb{N})$, $\ldots$, $X_{C_r} = (X_{C_r}(t) : t \in \mathbb{N})$ constructed above, and the Markov chain $X = (X, t)$. Then

$$\lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} 1\{X(m) = s\} = \lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} \prod_{h=1}^r 1\{X_{C_h}(m) = s_{C_h}\} = \prod_{h=1}^r \mu_{C_h}(s_{C_h}) = \mu(s),$$

for each $s = (s_{C_1}, \ldots, s_{C_r}) \in S$.

**Proof.** By (28) follows that

$$\lim_{t \to \infty} \left[ \mathbb{P}\left( X \in \bigcup_{h=1}^r \bigcup_{m=1}^\infty \left( \left[ t \ell_C^h + \frac{m}{t} \ell_C^h \right], \left[ t \ell_C^h + \frac{m+1}{t} \ell_C^h \right) \right) \right] = 0.$$

In fact, for each $h = 1, \ldots, r$,

$$\lim_{t \to \infty} \left[ \mathbb{P}\left( X \in \bigcup_{m=1}^\infty \left( \left[ t \ell_C^h + \frac{m}{t} \ell_C^h \right], \left[ t \ell_C^h + \frac{m+1}{t} \ell_C^h \right) \right) \right] = 0,$$

since

$$\lim_{t \to \infty} \frac{\ell_{C_h}^2}{c_{C_h}^2 N} \leq \lim_{t \to \infty} \frac{\ell_{C_h}^2}{c_{C_h}^2 N} = 0.$$

Thus, the times in $\bigcup_{h=1}^r \bigcup_{m=1}^\infty \left( \left[ t \ell_C^h + \frac{m}{t} \ell_C^h \right], \left[ t \ell_C^h + \frac{m+1}{t} \ell_C^h \right) \right)$ can be neglected in the procedure of checking (28), i.e.

$$\lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} 1\{X(m) = s\} = \lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} 1\{X(m) = s\} \cdot 1\{m \notin \bigcup_{h=1}^r \bigcup_{m=0}^\infty \left( \left[ t \ell_C^h + \frac{m}{t} \ell_C^h \right], \left[ t \ell_C^h + \frac{m+1}{t} \ell_C^h \right) \right) \}.$$

and also

$$\lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} 1\{X(m) = s\} = \lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} \left[ 1\{X(m) = s\} + 1\{m \notin \bigcup_{h=1}^r \bigcup_{m=0}^\infty \left( \left[ t \ell_C^h + \frac{m}{t} \ell_C^h \right], \left[ t \ell_C^h + \frac{m+1}{t} \ell_C^h \right) \right) \} \right].$$

Let us define the set of times $A = \bigcup_{h=1}^r \bigcup_{m=0}^\infty \left( \left[ t \ell_C^h + \frac{m}{t} \ell_C^h \right], \left[ t \ell_C^h + \frac{m+1}{t} \ell_C^h \right) \right)$. Now we introduce the independent random variables $Y_{C_h}(t) : t \in \mathbb{N}, h = 1, \ldots, r$. The random variables $Y_{C_h}(t) : t \in \mathbb{N}$, with label $h$, take value on $S_C$. Moreover, if $t \in \left( \left[ t \ell_C^h + \frac{m}{t} \ell_C^h \right], \left[ t \ell_C^h + \frac{m+1}{t} \ell_C^h \right) \right)$ then $Y_{C_h}(t)$ has distribution $\mu_{C_h}$. We now adapt formula (17) to the Markov chain $X_{C_h}$. If $\bar{h} \notin A$ then for each $h = 1, \ldots, r$ there exists $\bar{h}_h$ such that $\bar{h}_h$ belong to $[t \ell_{C_h}^h, t_{h+1} \ell_{C_h}^h)$. In this case formula (17) becomes

$$\mathbb{P}(X_{C_h}(\bar{h}) = Y_{C_h}(\bar{h})) \geq 1 - \exp \left( -\hat{c}_N \left[ \frac{\bar{h}^{N+1}}{N-1} \right] \right),$$

where we recall that $\hat{c}_N = \left[ \frac{c_{N-1}^2}{2 N^2+1} \right]$. Hence, for any $\bar{h} \notin A$ one has that there exist $\bar{h}_1, \ldots, \bar{h}_r \in \mathbb{N}$ such that $\bar{h} \in \bigcap_{h=1}^r \left( \left[ t \ell_{C_h}^h + \frac{m}{t} \ell_{C_h}^h \right], \left[ t \ell_{C_h}^h + \frac{m+1}{t} \ell_{C_h}^h \right) \right)$. Therefore, using the independence of the random variables $Y$’s and the independence of the Markov chains $X$’s, one has

$$\mathbb{P}(X_{C}(\bar{h}), \ldots, X_{C}(\bar{h})) = (Y_{C}(\bar{h}), \ldots, Y_{C}(\bar{h})) \geq 1 - \sum_{h=1}^r \exp \left( -\hat{c}_N \left[ \frac{\bar{h}^{N+1}}{N-1} \right] \right).$$
For \( \bar{t} \in \bigcap_{h=1}^{r} [1^{(G_h)} + \frac{\ell}{h} + 2N, 1^{(G_h)} + \frac{\ell}{h} + 1) \), the distribution of \((Y_{C_1}(\bar{t}), \ldots, Y_{C_r}(\bar{t}))\) coincides with \(\prod_{h=1}^{r} \mu_{C_h, \bar{\ell}_h} \). Thus, we have

\[
\left\| \mu - \prod_{h=1}^{r} \mu_{C_h, \bar{\ell}_h} \right\|_{TV} \leq \sum_{h=1}^{r} \frac{1}{\ell_h} \tag{31}
\]

Notice that any \( \bar{\ell}_h \) increases to infinity when \( \bar{t} \) goes to infinity. Therefore, the left-hand side of (31) goes to zero as \( \bar{t} \) goes to infinity. Inequalities (30) and (31) give an upper bound for the distance in total variation between the law of \( X(\bar{t}) \) and the distribution \( \mu \).

Now, by following the arguments in the proof of Theorem 2, we obtain equation (28).

In the same setting of the previous theorem, consider a graph \( G = (\mathcal{S}, E) \) and a connected graph \( G' = (\mathcal{S}, E') \) such that \( E' \subset E \) and \( G' \) is \( C \)-decomposable. Theorem 4 can be applied to the graph \( G' \) and the Markov chain \( X \). In any case \( X \) is also consistent with the graph \( G \), which is connected by construction having more edges than \( G' \). Thus, in some sense, Theorem 4 can be applied also to the supergraph \( G \) of \( G' \).

4.2. Repeated \( G \)-games. We now give the general definition of a repeated game in presence of a coalition structure \( \mathcal{C} = \{C_1, \ldots, C_r\} \), and then we present our specific setting.

Definition 5. A repeated game is a game played \( T \) times, with \( \bar{T} \in \mathbb{N} \cup \{\infty\} \), by a set of players \( V \) with a coalition structure \( \mathcal{C} = \{C_1, \ldots, C_r\} \), where the coalition \( C \in \mathcal{C} \) has set of strategies \( \mathcal{S}_C \). Each coalition \( C \in \mathcal{C} \) has an initial strategy \( s_C(0) \) at time \( t = 0 \). Furthermore, coalition \( C \) selects at time \( 1 \leq t < \bar{T} \) a strategy \( s_C(t) \in \mathcal{S}_C \), where such a selection can depend only on the available information on the previous history of the game. The payoff function of \( C \in \mathcal{C} \) at any time is given by \( \Pi_C : \mathcal{S} \to \mathbb{R} \). The payoff of the \( T \) times repeated game for the coalition \( C \in \mathcal{C} \) is

\[
\Pi_C^{(T)} = \begin{cases} \frac{1}{T} \sum_{m=0}^{T-1} \Pi_C(s_{C_1}(m), \ldots, s_{C_r}(m)), & \text{if } T < \infty; \\ \liminf_{T \to \infty} \frac{1}{T} \sum_{m=0}^{T-1} \Pi_C(s_{C_1}(m), \ldots, s_{C_r}(m)), & \text{if } T = \infty. \end{cases}
\]

In the proposed context of \( G \)-games, each coalition \( C \) can select at time \( t + 1 \) only a strategy (or action) \( s_C(t + 1) \) that is adjacent to the strategy \( s_C(t) \) selected at time \( t \). Such a requirement cannot be satisfied for a general graph, because it would imply the construction of a strategy \( s_C(t + 1) \) by knowing the decisions that are doing the other coalitions. To convince the reader of this problem, we present a simple example in the setting of two players.

Example 2. Consider \( V = \{1, 2\} \), \( \mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \), with \( \mathcal{S}_1 \equiv \mathcal{S}_2 \equiv \{s_1, s_2\} \). The graph is \( G = (\mathcal{S}, E) \), where

\[ E = \{(s_1, s_1), (s_2, s_1), (s_1, s_1), (s_1, s_2), (s_2, s_2), (s_2, s_1), (s_2, s_2)\}. \]

Notice that \( (s_1, s_1) \) and \( (s_2, s_2) \) are not adjacent, and therefore it is impossible to have that \( s(t) = (s_1, s_1) \) and \( s(t + 1) = (s_2, s_2) \). In any case, both players have the opportunity to move from \( s_1 \) to \( s_2 \) if the other player decides to remain in \( s_1 \).

Thus, the selection of the strategy by a player should not depend only on the past, but also on the current choices of the other player. However, this situation is not considered in Definition 5.
For the reasons expressed above and explained through Example we will define the repeated $G$-games when $G$ can be written as product of graphs according to Definition.

We are ready to present the definition of repeated $G$-games.

**Definition 6.** Consider a connected graph $G = (S, E)$, a set of players $V$ and a coalition structure $\mathcal{C} = \{C_1, \ldots, C_r\}$. Assume that $\mathcal{G} = (G_1, \ldots, G_r)$ is the $\mathcal{C}$-decomposition of $G$. A repeated $G$-game is a repeated game such that, for $t \in \mathbb{N}$, any coalition $C \in \mathcal{C}$ can select at time $t + 1$ only strategies $s_C(t + 1)$’s in $S_C$ which are adjacent to the strategy $s_C(t) \in S_C$ selected at time $t$.

Let us focus on the information. We will present two extreme cases.

(I) All the coalitions are aware about the previous history of the repeated game. This is the maximum available level of information, called maximal information.

(Im) Any coalition has knowledge of time $t$ and of its previously selected strategies before $t$, without any information on the choices of the others coalitions. We call this case minimal information.

We also assume that the vector of the initial strategies of the coalitions at time zero are of two types.

(P0) The initial strategies are decided by the players themselves.

(R0) There is a referee of the game who assigns the initial strategies.

In both cases when the information is minimal the coalitions are not aware about the initial strategies of the others.

We notice that it is often not possible or really hard to load the entire past history of the game into memory. Therefore, we will focus mainly on Markovian processes, where the only datum needed is the current time $t$ and the strategy selected at time $t - 1$.

We now present an immediate result showing the relevance of the pure $\mathcal{C}$-equilibria, given in Definition for $T = 2$.

**Theorem 5.** Consider a repeated $G$-game with coalition structure $\mathcal{C} = \{C_1, \ldots, C_r\}$ where $T = 2$, information is of type (Im) and initial strategies are of type (R0). If the vector of initial strategies $\bar{s} = (\bar{s}_{C_1}, \ldots, \bar{s}_{C_r}) \in E_{\mathcal{C}}$, then it is a Berge equilibrium for $\mathcal{C}$ at time $t = 1$ where the space of strategies available to any coalition $C \in \mathcal{C}$ is restricted to the strategies adjacent to $\bar{s}_C$.

We also believe that $\mathcal{C}$-equilibria, defined in Section become relevant when the coalitions are not aware about the terminal time of the game. This is the case of a final stage of the game not under the control of the single coalitions. For instance, each coalition has the opportunity to abandon the game and such an abandonment would determine the end of the game itself. In a different context, the game might end at the occurrence of an event whose distribution is not known. In all such situations one can reasonably guess that coalitions do not consider the consequences on the long-term of their strategies and they play the game by implicitly assuming that each round is the last one. Thus, it is reasonable to believe that if the coalitions achieve at time $t$ a $\mathcal{C}$-equilibrium, then they will play such strategies at each time $t > \bar{t}$. Indeed, such a way to play leads to a stage-wise maximization of their payoffs.

We are ready to give the definition of equilibrium for the repeated $G$-games. According to the specific framework we will focus on, we restrict our attention to the case of minimal information and $T = \infty$. 

Definition 7. Consider a connected graph $G = (S, E)$ and a repeated $G$-game with coalition structure $C = \{C_1, \ldots, C_r\}$. Assume that $G = (G_1, \ldots, G_r)$ is the $C$-decomposition of $G$. Assume $T = \infty$, minimal information and initial strategies of type $(P_0)$ or $(R_0)$. Consider $r$ adapted and independent stochastic processes $X_{C_1} = (X_{C_1}(t) : t \in \mathbb{N}), \ldots, X_{C_r} = (X_{C_r}(t) : t \in \mathbb{N})$ taking values in $S_{C_1}, \ldots, S_{C_r}$ and consistent with $G_1, \ldots, G_r$, respectively. We say that $X = (X_{C_1}, \ldots, X_{C_r})$ is a $C$-equilibrium for the repeated $G$-game when

\[
\mathbb{E}(\liminf_{t \to \infty} \frac{1}{t} \sum_{j=0}^{t-1} \Pi_{C_j}(X_{C_1}(m), \ldots, X_{C_r}(m))) \geq \mathbb{E}(\liminf_{t \to \infty} \frac{1}{t} \sum_{j=0}^{t-1} \Pi_{C_j}(X_{C_1}(m), \ldots, X_{C_r}(m), X_{C_{r+1}}(m), \ldots, X_{C_r}(m))),
\]

for each $\ell = 1, \ldots, r$ and for any adapted stochastic process $\tilde{X}_{C_\ell}$ which is consistent with $G_\ell$ and independent from $X_{C_1}, \ldots, X_{C_{r-1}}, X_{C_{r+1}}, \ldots, X_{C_r}$.

The previous definition is meaningful only in the case of minimal information because we are requiring the independence of the strategy processes. We now illustrate the connection between the equilibria in $\mathcal{M}_C$ and the $C$-equilibria for the repeated $G$-games defined in Definition 7. Specifically, we will present a version of the Folk Theorem for our framework. To proceed, we need a preliminary general result. We state it directly in the language of the $G$-games, for the sake of notation.

Theorem 6. Consider a connected graph $G = (S, E)$ and a repeated $G$-game with coalition structure $C = \{C_1, \ldots, C_r\}$. Assume $T = \infty$ and that $G = (G_1, \ldots, G_r)$ is the $C$-decomposition of $G$. Consider $\Lambda_{C_1} \times \ldots \times \Lambda_{C_r} \in \mathcal{M}_C$ and $r$ independent Markov chains $X_{C_1} = (X_{C_1}(t) : t \in \mathbb{N}), \ldots, X_{C_r} = (X_{C_r}(t) : t \in \mathbb{N})$ with state space $S_{C_1}, \ldots, S_{C_r}$ and consistent with $G_1, \ldots, G_r$, respectively, such that

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} \prod_{h=1}^r 1_{\{X_{C_h}(m) = s_{C_h}\}} = \prod_{h=1}^r \Lambda_{C_h}(s_{C_h}), \quad \text{a.s.,}
\]

for each $s_{C_h} \in S_{C_h}$.

Then, almost surely,

\[
\Pi_{C_\ell}^{(\infty)} = \mathbb{E}_{\Lambda_{C_1} \times \ldots \times \Lambda_{C_r}}(\Pi_{C_\ell}) = \lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} \Pi_{C_\ell}(X_{C_1}(m), \ldots, X_{C_r}(m)) \geq \mathbb{E}(\liminf_{t \to \infty} \frac{1}{t} \sum_{j=0}^{t-1} \Pi_{C_j}(X_{C_1}(m), \ldots, X_{C_r}(m), X_{C_{r+1}}(m), \ldots, X_{C_r}(m))),
\]

for each $\ell = 1, \ldots, r$ and stochastic process $\tilde{X}_{C_\ell}$ consistent with $G_\ell$ and independent from the Markov chains $X_{C_1}, \ldots, X_{C_{r-1}}, X_{C_{r+1}}, \ldots, X_{C_r}$.

Proof. By formula (34) we deduce the first two equalities in (34), so we have to prove only the inequality in (34).

Let us take $\ell = 1, \ldots, r$. Consider a stochastic process $\tilde{X}_{C_\ell}$ consistent with $G_\ell$ and independent from the Markov chains $X_{C_1}, \ldots, X_{C_{r-1}}, X_{C_{r+1}}, \ldots, X_{C_r}$. From the finiteness of $S$ one has the tightness of the distributions on $S$. Therefore, there exists a sequence $(t_n : n \in \mathbb{N})$ and a distribution $\tilde{\Lambda}_{C_\ell}$ on $S_{C_\ell}$ such that

\[
\lim_{n \to \infty} \frac{1}{t_n} \sum_{m=0}^{t_n-1} 1_{\{\tilde{X}_{C_\ell}(m) = s_{C_\ell}\}} = \tilde{\Lambda}_{C_\ell}(s_{C_\ell}), \quad \text{a.s.,}
\]
for each \( s_{C_r} \in \mathcal{S}_{C_r} \).

The independence assumption of \( \tilde{X}_{C_r} \) from \( X_{C_1}, \ldots, X_{C_{r-1}}, X_{C_{r+1}}, \ldots, X_{C_r} \) and (33) give

\[
\mathbb{E}( \lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} \mathbf{1}_{\{(X_{C_1}(m), \ldots, X_{C_{r-1}}(m), \tilde{X}_{C_r}(m), X_{C_{r+1}}(m), \ldots, X_{C_r}(m)) = s\}} ) = \\
\hat{\Lambda}_{C_r}(s_{C_r}) \left[ \prod_{k=1, \ldots, r, k \neq \ell} \Lambda_{C_k}(s_{C_k}) \right],
\]

(36)
for any \( s = (s_{C_1}, \ldots, s_{C_r}) \in \mathcal{S} \). Thus, (36) leads to

\[
\mathbb{E}( \lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} \Pi_{C_1}(X_{C_1}(m), \ldots, X_{C_{r-1}}(m), \tilde{X}_{C_r}(m), X_{C_{r+1}}(m), \ldots, X_{C_r}(m)) ) = \\
= \mathbb{E}_{\hat{\Lambda}_{C_1} \times \ldots \times \hat{\Lambda}_{C_{r-1}} \times \hat{\Lambda}_{C_r}} \left( \Pi_{C_r} \right).
\]

(37)
By the assumption that \( \Lambda_{C_1} \times \ldots \times \Lambda_{C_r} \in \mathcal{M}_C \), one has that

\[
\mathbb{E}_{\Lambda_{C_1} \times \ldots \times \Lambda_{C_{r-1}} \times \Lambda_{C_r}} \left( \Pi_{C_r} \right) \leq \mathbb{E}_{\hat{\Lambda}_{C_1} \times \ldots \times \hat{\Lambda}_{C_r}} \left( \Pi_{C_r} \right).
\]

The thesis comes from

\[
\liminf_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} \Pi_{C_1}(X_{C_1}(m), \ldots, X_{C_{r-1}}(m), \tilde{X}_{C_r}(m), X_{C_{r+1}}(m), \ldots, X_{C_r}(m)) \leq \\
\leq \lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} \Pi_{C_1}(X_{C_1}(m), \ldots, X_{C_{r-1}}(m), \tilde{X}_{C_r}(m), X_{C_{r+1}}(m), \ldots, X_{C_r}(m)).
\]

\( \square \)

The previous theorem says that the considered Markov chains form a C-equilibrium for the repeated G-game when \( T = \infty \), initial strategies of types (\( P_0 \)) or (\( R_0 \)) and minimal information (see Definition 7). From Theorem 4 we know that such Markov chains exist and we have constructed them. We point out that Equation (33) represents a global condition on the empirical distribution related to all the coalitions. Differently, Theorem 5 contains a local condition which actually implies (33). In fact, in Theorem 5 we have constructed \( r \) independent Markov chains such that if

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{m=0}^{t-1} \mathbf{1}_{\{X_{C_1}(m) = s_{C_1}\}} = \Lambda_{C_1}(s_{C_1}), \quad \text{a.s.,}
\]

for each \( \ell = 1, \ldots, r \), then one obtains also (33). This is relevant in game theory at least in the case of minimal information. In fact, the independence assumption of the considered Markov chains is always valid under the condition of minimal information. In fact, in such cases, any coalition does not know the strategies selected at the previous steps by the other coalitions playing the game.

By Theorems 4 and Theorem 5 we obtain the following.

**Corollary 1.** Consider a connected graph \( G = (\mathcal{S}, E) \) and a repeated G-game with coalition structure \( \mathcal{C} = \{C_1, \ldots, C_r\} \). Assume \( T = \infty \), initial strategies of types (\( P_0 \)) or (\( R_0 \)) and information of type (\( I_m \)). Assume also that \( \mathcal{G} = \{G_1, \ldots, G_r\} \) is the C-decomposition of \( G \).

For any \( \Lambda_{C_1} \times \ldots \times \Lambda_{C_r} \in \mathcal{M}_C \), there exist \( r \) independent Markov chains \( X_{C_1} = (X_{C_1}(t) : t \in \mathbb{N}), X_{C_2} = (X_{C_2}(t) : t \in \mathbb{N}) \) with state space \( \mathcal{S}_{C_1}, \ldots, \mathcal{S}_{C_r} \) and consistent with \( G_1, \ldots, G_r \), respectively, such that \( X = (X_{C_1}, \ldots, X_{C_r}) \) is a C-equilibrium for the repeated G-game.
Corollary 1 is a version of the Folk Theorem in our framework, which guarantees the existence of $\mathcal{C}$-equilibria for the repeated game. It is also important that the $\mathcal{C}$-equilibria analysed are Markovian that means that the single coalition has to memorize only the current time and the strategy played in the previous stage of the game.

As a by-product, Corollary 1 can be also related to the memory $K$ of the single coalitions – i.e., the knowledge by a coalition $C \in \mathcal{C}$ of the strategies played at the previous $K$ stages of the repeated $G$-games by $C$. In particular, it states that the $\mathcal{C}$-equilibria are equilibria for any repeated $G$-games where any coalition has at least memory $K = 1$.

5. Conclusions

This paper has introduced and analyzed the $G$-games, i.e. games whose strategies are nodes of a graph $G$. This class of games presents interesting features either under a theoretical as well as under a practical point of view.

Indeed, several real-world situations can be modeled through game models where the players are physically constrained to move sequentially from a strategy to an adjacent one. The introduction of a graph of the strategies serves to capture such constraints. We specifically deal with $G$-games with a coalition structure. In so doing, we admit the presence of interactions among the players. Notice that the presented framework does not exclude the case of absence of constraints – one can take $G$ complete – and the possibility of noncooperative games – by taking coalitions formed by single players.

A detailed exploration of several aspects is carried out. In particular, we define the equilibria of the coalitions of pure and mixed type and present a version of the Folk Theorem for the class of infinitely played $G$-repeated games with connected $G$. The definition of $\mathcal{C}$-equilibria represents an extension of the (pure) Berge and Nash equilibria. However, we do not enter here the challenging problem of equilibria selection (see the seminal work [27] and e.g. [11, 14, 21, 26, 29, 40]), leaving this topic for future research.

It is important to note that Theorem 4 can be seen as an universality result. In fact, assume that multiple equilibria are attained and a selection criterion identifies one of them as the valid one for all the coalitions. Then, such a valid equilibrium can be achieved under the requirement that $G$ is connected.

We also point out that coalitions are fixed in the developed game model. The theme of coalition structure generation – i.e., the problem of partitioning the players, according to a specific criterion (see e.g. [38, 39]) – is beyond the material presented in this paper. Also such a challenging research theme is left for future development of the study of the $G$-games.

References

[1] D. Abreu. On the theory of infinitely repeated games with discounting. *Econometrica*, 56(2):383–396, 1988.
[2] D. Abreu, D. Pearce, and E. Stacchetti. Toward a theory of discounted repeated games with imperfect monitoring. *Econometrica*, 58(5):1041–1063, 1990.
[3] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Appl. Math.*, 8(1):1–11, 1984.
[4] C. Andrieu, N. De Freitas, A. Doucet, and M. I. Jordan. An introduction to MCMC for machine learning. *Machine learning*, 50(1-2):5–43, 2003.
[5] W. B. Arthur, Y. M. Ermoliev, and Y. M. Kaniovski. Path-dependent processes and the emergence of macro-structure. *European journal of operational research*, 30(3):294–303, 1987.
[6] R. J. Aumann and J. H. Dréze. Cooperative games with coalition structures. *Internat. J. Game Theory*, 3:217–237, 1974.
[7] R. J. Aumann and L. S. Shapley. Long-term competition—a game-theoretic analysis. In Essays in game theory (Stony Brook, NY, 1992), pages 1–15. Springer, New York, 1994.
[8] R. Basu, A. E. Holroyd, J. B. Martin, and J. Wåstlund. Trapping games on random boards. *Ann. Appl. Probab.*, 26(6):3727–3753, 2016.
[9] J.-P. Benoist and V. Krishna. Finitely repeated games. *Econometrica*, 53(4):905–922, 1985.
[10] C. Berge. *Théorie générale des jeux à n personnes*. Mémoire Sci. Math., no. 138. Gauthier-Villars, Paris, 1957.
[11] K. Binmore, L. Samuelson, and P. Young. Equilibrium selection in bargaining models. *Games Econ. Behav.*, 45(2):296–328, 2003. Special issue in honor of Robert W. Rosenthal.
[12] A. Bonato and R. J. Nowakowski. The game of cops and robbers on graphs, volume 61 of Student Mathematical Library. American Mathematical Society, Providence, RI, 2011.
[13] P. Brémaud. *Markov chains*, volume 31 of Texts in Applied Mathematics. Springer-Verlag, New York, 1999. Gibbs fields, Monte Carlo simulation, and queues.
[14] H. Carlsson and E. van Damme. Global games and equilibrium selection. *Econometrica*, 61(5):989–1018, 1993.
[15] B. Chen and S. Takahashi. A folk theorem for repeated games with unequal discounting. *Games Econ. Behav.*, 76(2):571–581, 2012.
[16] H. L. Cole and N. R. Kocherlakota. Finite memory and imperfect monitoring. *Games Econ. Behav.*, 53(1):59–72, 2005.
[17] M. W. Cripps and J. P. Thomas. Reputations and commitment in two-person repeated games without discounting. *Econometrica*, 63(6):1401–1419, 1995.
[18] P. A. David. Path dependence, its critics and the quest for historical economics. *Evolution and path dependence in economic ideas: Past and present*, 15:40, 2001.
[19] J. Deb, J. González-Díaz, and J. Renault. Uniform folk theorems in repeated anonymous random matching games. *Games Econ. Behav.*, 108:1–23, 2016.
[20] R. Dobrushin. Central limit theorem for non-stationary Markov chains. I. *Teor. Veroyatnost. i Primenen.*, 1:72–89, 1956.
[21] J. Duffy and J. Ochs. Equilibrium selection in static and dynamic entry games. *Games Econ. Behav.*, 76(1):97–116, 2012.
[22] D. Fudenberg and E. Maskin. The folk theorem in repeated games with discounting or with incomplete information. *Econometrica*, 54(3):533–554, 1986.
[23] D. Fudenberg and E. Maskin. Nash and perfect equilibria of discounted repeated games. *J. Econ. Theory*, 51(1):194–206, 1990.
[24] W. R. Gilks, S. Richardson, and D. J. Spiegelhalter, editors. *Markov chain Monte Carlo in practice*. Interdisciplinary Statistics. Chapman & Hall, London, 1996.
[25] P. J. Green. Reversible jump Markov chain Monte Carlo computation and Bayesian model determination. *Biometrika*, 82(4):711–732, 1995.
[26] J. C. Harsanyi. A new theory of equilibrium selection for games with complete information. *Games Econ. Behav.*, 8(1):91–122, 1995. Nobel Symposium on Game Theory (Björkborn, 1993).
[27] J. C. Harsanyi and R. Selten. *A general theory of equilibrium selection in games*. MIT Press, Cambridge, MA, 1988. With a foreword by Robert Aumann.
[28] P. J-J. Herings, A. Meshalkin, and A. Predtetchinski. A one-period memory folk theorem for multilateral bargaining games. *Games Econ. Behav.*, 103:185–198, 2012.
[29] M. Kandori, G. J. Mailath, and R. Rob. Learning, mutation, and long run equilibria in games. *Econometrica*, 61(1):29–56, 1993.
[30] S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi. Optimization by simulated annealing. *Science*, 220(4598):671–680, 1983.
[31] M. Laclau. A folk theorem for repeated games played on a network. *Games Econ. Behav.*, 76(2):711–737, 2012.
[32] T. Lindvall. *Lectures on the coupling method*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1992. A Wiley-Interscience Publication.
[33] R. B. Myerson. Graphs and cooperation in games. *Math. Oper. Res.*, 2(3):225–229, 1977.
[34] J. Nash. Non-cooperative games. *Ann. of Math. (2)*, 54:286–295, 1951.
[35] G. Owen. *Game theory*. Academic Press, Inc., San Diego, CA, third edition, 1995.
[36] A. Rubinstein. Equilibrium in supergames with the overtaking criterion. *J. Econ. Theory*, 21(1):1–9, 1979.
[37] G. Sabidussi. Graph multiplication. *Math. Z.*, 72:446–457, 1959/1960.
[38] T. Sandholm, K. Larson, M. Andersson, O. Shehory, and F. Tohmé. Coalition structure generation with worst case guarantees. *Artificial Intelligence*, 111(1-2):209–238, 1999.
[39] T. W. Sandholm and V. R. Lesser. Coalitions among computationally bounded agents. *Artificial Intelligence*, 94(1-2):99–137, 1997.
[40] T. C. Schelling. The strategy of conflict. Harvard university press, 1980.

[41] G. Szabó and G. Fáth. Evolutionary games on graphs. Phys. Rep., 446(4-6):97–216, 2007.

[42] X. Zhu. The game coloring number of planar graphs. J. Combin. Theory Ser. B, 75(2):245–258, 1999.

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