Some examples of different descriptions of energy-momentum
density in the context of Bianchi IX cosmological model

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Abstract

Based on the Bianchi type IX metric, we calculate the energy and momentum density components of the gravitational field for the five different definitions of energy-momentum, namely, Tolman, Papapetrou, Landau-Lifshitz, Møller and Weinberg. The energy densities of Møller and Weinberg become zero for the spacetime under consideration. In the other prescriptions, i.e., Tolman, Papapetrou and Landau-Lifshitz complexes, we find different non-vanishing energy-momentum densities for the given spacetime, supporting the well-known argument in General Relativity that the different definitions may lead to different distributions even in the same spacetime background.

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I. INTRODUCTION

The interpretation of energy-momentum as a significant conserved quantity is one of the most interesting and stimulating problems in the theory of Einstein General Relativity (GR). There have been numerous efforts to acquire a well-defined illustration for the energy and momentum localization in the literature. Unfortunately, there is still no prevalent accepted interpretation of energy and momentum distributions in GR [1].

The energy-momentum conservation in GR can be written as

\[ \nabla_\mu T^\mu_\nu = 0, \quad (\mu, \nu = 0, 1, 2, 3), \quad (1) \]

where \( T^\mu_\nu \) indicates the symmetric energy-momentum tensor including the matter and all non-gravitational fields. In 1915, Einstein [2] acquired an expression for the energy-momentum complexes comprised of the contribution from gravitational field energy by introducing the energy-momentum \( t^\mu_\nu \) which is not a tensor and is called the gravitational field pseudotensor. The energy-momentum complex satisfies the local conservation laws, i.e.

\[ T^\mu_\nu,\mu \equiv \frac{\partial}{\partial x^\mu} \left( \sqrt{-g}(T^\mu_\nu + t^\mu_\nu) \right) = 0, \quad (2) \]

where the energy-momentum tensor \( T^\mu_\nu \) is replaced by the energy-momentum complex \( T^\mu_\nu \) which is a combination of the tensor \( T^\mu_\nu \) plus the pseudotensor \( t^\mu_\nu \) but in the ordinary form of conservation laws. So, we have

\[ T^\mu_\nu = \theta^\mu_\nu,\lambda, \quad (3) \]

where \( \theta^\mu_\nu,\lambda \) are denoted as the superpotentials and are not uniquely determined.

With an appropriate choice of a coordinates system, the pseudotensor \( t^\mu_\nu \) can be identified such to disappear at a special point. Schrodinger demonstrated that the pseudotensor can vanish outside the schwarzschild radius utilizing an appropriate choice of coordinates. There have been many efforts in order to attain a more fitting quantity to illustrate the distribution of energy-momentum on account of matter, non-gravitational fields and gravitational field pseudotensor. Einstein supported the expression of pseudotensor to portray the gravitational field and explained that this energy-momentum pseudo-complex prepares reasonable expressions for the complete energy-momentum of the closed systems. Many authors have prescribed different explanations for the energy-momentum complex [2–9]. These explanations can only give significant outcomes if the computations are carried out in cartesian...
coordinates. In 1982, Penrose \cite{10} initiated the proposal of quasi-local energy to find the energy-momentum of a curved spacetime by using any coordinate system. Many theoretical physicists \cite{11} considered an assortment of different proposals of the quasi-local energy to study different models of the universe. Very general results for the most general nonstatic spherically symmetric metric is known by Virbhadra \cite{12}. However these proposals of the energy-momentum complexes could not lead to some unique definition of energy in GR because each of these quasi-local expressions have their own issues.

In this paper, we present some well-known energy-momentum densities based on Bianchi IX cosmological model to study the problem of localization of the energy and momentum in GR. The spatially homogeneous and anisotropic Bianchi models play a significant role in modern cosmology. However, on account of intricate essence of the field equations, there are minor works on anisotropic models (see, e.g., \cite{13} and also \cite{14}), particularly on the Bianchi IX cosmological model \cite{15} (the so-called Mixmaster universe \cite{16}). Several generalizations to the Mixmaster universe have also been considered in detail, by some authors (for a comprehensive review, see \cite{17}). The Bianchi universes provide useful paradigms to investigate the nonlinear behavior of the Einstein equations, due to the time-dependency of the gravitational fields (see, for example, \cite{18} and references therein). Studying on the anisotropic models was first considered after finding the anisotropic behavior of the microwave background radiation. In other words, no decisive confirmation is perceived that the early universe had essentially the same properties on the early era. Hence, it is credibly applicable to study the universes established upon the different Bianchi type metrics. A general study of the dynamical properties of anisotropic Bianchi universes in the context of GR is presented by Perez \cite{19}. In this setup, the Bianchi type VIII and IX universes are dynamically equivalent. In 2005, the authors in Ref. \cite{20} applied the canonical quantum theory of gravity (Quantum Geometrodynamics) to the homogeneous Bianchi type IX cosmological model, and accordingly they developed the framework for the quantum theory of homogeneous cosmology. In 2009, Bakas et al \cite{21} considered spatially homogeneous (but generally non-isotropic) cosmologies in the Hořava-Lifshitz gravity and compared them to those of GR using Hamiltonian methods. They exhibited that the Mixmaster dynamics is completely dominated by the quadratic Cotton tensor potential term for very small volume of the universe, by focusing on the closed-space cosmological model (Bianchi type IX). Recently, Damour and Spindel \cite{22} have studied the minisuperspace quantization of spatially
homogeneous (Bianchi) cosmological universes sourced by a Dirac spinor field. They presented the exact quantum solution of the Bianchi type II system and discussed the main qualitative features of the quantum dynamics of the (classically chaotic) Bianchi type IX system. Barrow and Yamamoto [23], in a recent paper, have investigated the stability of the Einstein static universe as a non-LRS Bianchi type IX solution of the Einstein equations in the presence of both non-tilted and tilted fluids. They have found that the static universe is unstable due to homogeneous perturbations of Bianchi type IX to the future and the past.

Moreover, the comprehension of type IX solutions, due to its anisotropy, with their oscillatory treatment in the direction of the initial singularity is prevalently related to one of the keys towards a more clear comprehension of singularities in GR and thus could be an interesting candidate to test the quantum theory [24]. We therefore focus on Bianchi IX cosmological model which demonstrates a specifically rich dynamical structure. The Bianchi type IX universe is defined by the line element:

$$ds^2 = -dt^2 + S^2(t)dx^2 + R^2(t)\left[dy^2 + \sin^2 y\,dz^2\right] - S^2(t)\cos y\left[2dx - \cos y\,dz\right]dz,$$

(4)

where the functions $S$ and $R$ are function in $t$ and determined from the field equations. We will use the above line element to acquire the energy and momentum densities in the next section.

The paper is organized in the following. In Section II by applying the energy-momentum definitions of Tolman, Papapetrou, Landau-Lifshitz, Møller and Weinberg, we calculate the energy-momentum densities of the universe based on Bianchi type IX metric, respectively. Conclusion is presented in Section III.

II. ENERGY-MOMENTUM COMPLEXES: SOME EXAMPLES

The energy-momentum in Tolman’s prescription [3] has the form [28]

$$\Upsilon_\mu^\nu = \frac{1}{8\pi}U_\mu^{\nu,\lambda},$$

(5)

where the Tolman’s superpotential $U_\mu^{\nu,\lambda}$ is defined by

$$U_\mu^{\nu,\lambda} = \sqrt{-g}\left(-g^{\kappa\nu}V_{\mu,\kappa,\lambda} + \frac{1}{2}g_\mu^\kappa g^{\kappa\beta}V_{\nu,\beta,\lambda}\right),$$

(6)
with
\[ V^\alpha_{\beta\gamma} = -\Gamma^\alpha_{\beta\gamma} + \frac{1}{2}g^\alpha_{\beta\gamma} - \Gamma^\delta_{\beta\gamma} + \frac{1}{2}g^\alpha_{\gamma\delta}\Gamma^\gamma_{\delta\beta}. \] (7)

The locally conserved energy-momentum complex of Tolman includes contributions from the matter plus all gravitational and non-gravitational fields \[29\]. The Tolman energy and momentum complex satisfies the local conservation laws as follows,
\[ \Upsilon^\nu_{\mu,\nu} = 0, \] (8)

where \( \Upsilon^0_\mu \) is a combination of the energy-momentum tensor including the matter and all non-gravitational fields plus the gravitational field pseudotensor. Therefore we can describe the quantity \( \Upsilon^0_0 \) as representing the energy density of the whole physical system including gravitation, and describe the quantity \( \Upsilon^0_i \) as representing the components of the total momentum density. In order to calculate the energy and momentum density components for the Bianchi type IX metric, we need to compute the essential non-zero components of \( U^{0\lambda}_\mu \) which give
\[
\begin{align*}
U^{02}_0 &= -S(t) \cos y, \\
U^{01}_1 &= \frac{1}{2}R(t) \sin y \left( \dot{S}(t)R(t) - 2S(t)\dot{R}(t) \right), \\
U^{00}_2 &= -\frac{1}{2}S(t)R^2(t) \cos y, \\
U^{02}_2 &= U^{03}_3 = -\frac{1}{2}R^2(t) \dot{S}(t) \sin y, \\
U^{01}_3 &= -R(t) \sin y \cos y \left( \dot{S}(t)R(t) - S(t)\dot{R}(t) \right),
\end{align*}
\] (9)

where overdot abbreviates \( \partial/\partial t \). Substituting these values for Eq. (5), we obtain the components of energy and momentum density in the prescription of Tolman as follows
\[
\Upsilon^0_0 = \frac{S(t)}{8\pi},
\] (10)
\[
\Upsilon^0_1 = \Upsilon^0_3 = 0, \quad \Upsilon^0_2 = -\frac{R(t) \cos y}{8\pi} \left( \dot{S}(t)R(t) + S(t)\dot{R}(t) \right).
\] (11)

The energy and momentum in the prescription of Papapetrou \[4\] takes the form
\[
\Omega^{\mu\nu} = \frac{1}{16\pi} N^{\mu\nu\lambda\kappa}_{\lambda\kappa},
\] (12)

where
\[
N^{\mu\nu\lambda\kappa} = \sqrt{-g} \left( g^{\mu\nu} \eta^{\lambda\kappa} - g^{\mu\lambda} \eta^{\nu\kappa} + g^{\lambda\kappa} \eta^{\mu\nu} - g^{\nu\kappa} \eta^{\mu\lambda} \right). \] (13)

The Papapetrou’s superpotential \( N^{\mu\nu\lambda\kappa} \) is symmetric on its first pair of indices with \( \eta^{\mu\nu} \) that is the Minkowski metric. The Papapetrou energy-momentum complex obeys the local
conservation laws,
\[ \Omega^\mu\nu,\nu = 0, \]  
(14)
where \(\Omega^{00}\) and \(\Omega^{i0}\) represent the energy and momentum density components respectively. In this prescription, the essential non-zero components of \(N^{\mu0\lambda\kappa}\) corresponding to the metric (11) yield the following expressions:

\[
\begin{align*}
N^{0000} &= 2S(t)R^2(t) \sin y, \\
N^{0011} &= \frac{R^2(t)}{S(t)} \left( S^2(t) - 1 \right) \sin y - S(t) \cos y \cot y, \\
N^{1010} &= -\frac{R^2(t)}{S(t)} \left( S^2(t) + 1 \right) \sin y - S(t) \cos y \cot y, \\
N^{3010} &= N^{1030} = N^{0031} = N^{0013} = -S(t) \cot y, \\
N^{0022} &= S(t) \left( R^2(t) - 1 \right) \sin y, \\
N^{2020} &= -S(t) \left( R^2(t) + 1 \right) \sin y, \\
N^{3030} &= -\frac{S(t)}{\sin y} \left( R^2(t) \sin^2 y + 1 \right), \\
N^{0033} &= \frac{S(t)}{\sin y} \left( R^2(t) \sin^2 y - 1 \right).
\end{align*}
\]  
(15)
Replacing the above expressions in Eq. (12), one can obtain the energy and momentum densities in Papapetrou’s prescription. So, we have

\[
\begin{align*}
\Omega^{00} &= \frac{\pi \sin y}{16} \left[ 8R(t)\dot{R}(t)\dot{S}(t) + 2R^2(t)\ddot{S}(t) + S(t) \left( 4\dot{R}^2(t) + 4R(t)\dddot{R}(t) - R^2(t) + 1 \right) \right], \\
\Omega^{20} = \Omega^{30} = 0, \quad \Omega^{10} &= -\frac{\pi \cos y}{16} \left( \dot{S}(t) + R^2(t)\dot{S}(t) + 2S(t)R(t)\dot{R}(t) \right). \\
\end{align*}
\]  
(16)

The energy and momentum in the prescription of Landau and Lifshitz (5) is given by

\[
L^{\mu\nu} = \frac{1}{16\pi} S^{\mu\nu\lambda\kappa},_{\lambda\kappa}, \quad \Omega^{00} = \frac{\pi \sin y}{16} \left[ 8R(t)\dot{R}(t)\dot{S}(t) + 2R^2(t)\ddot{S}(t) + S(t) \left( 4\dot{R}^2(t) + 4R(t)\dddot{R}(t) - R^2(t) + 1 \right) \right], \\
\Omega^{20} = \Omega^{30} = 0, \quad \Omega^{10} &= -\frac{\pi \cos y}{16} \left( \dot{S}(t) + R^2(t)\dot{S}(t) + 2S(t)R(t)\dot{R}(t) \right).
\]  
(17)

The Landau-Lifshitz’s energy-momentum complex \(L^{\mu\nu}\) confirms the local conservation laws

\[
L^{\mu\nu},_\nu = 0. \quad (20)
\]
following relations:

\[ S^{0011} = -S^{0101} = -R^2(t) \left( R^2(t) \sin^2 y + S^2(t) \cos^2 y \right), \]
\[ S^{0022} = -S^{0202} = \sin^2 y S^{0033} = -\sin^2 y S^{0303} = -S^2(t) R^2(t) \sin^2 y, \]
\[ S^{0031} = S^{0013} = -S^{0103} = -S^{0301} = -S^2(t) R^2(t) \cos y. \]

Substituting these components for (18), one can obtain the energy and momentum densities which give the following relations:

\[ L^{00} = 2 S^2(t) R^2(t) \left( \sin^2 y - \cos^2 y \right), \]
\[ L^{01} = L^{03} = 0, \quad L^{02} = 2 \sin(2y) \left( R^2(t) S(t) \dot{S}(t) + R(t) \dot{R}(t) S^2(t) \right). \]

The energy and momentum in the prescription of Møller [8] is given by

\[ M_\mu^\nu = \frac{1}{8\pi} \chi^\mu_\nu - \chi^\nu_\mu, \]

where the antisymmetric superpotential \( \chi^\mu_\nu \) has the form

\[ \chi^\mu_\nu = -\chi^\nu_\mu = \sqrt{-g} \left( g_{\nu\sigma,\mu} - g_{\nu\mu,\sigma} \right) g^{\mu k} g^{\nu \sigma}, \]

where \( g \) is the determinant of the metric \( g_{\mu\nu} \). It can be simply shown that the Møller’s energy-momentum complex satisfies the local conservation laws

\[ M_\nu^{\mu, \mu} = 0. \]

\( M_0^0 \) is the energy density and \( M_i^0 \) are the momentum density components. For the line element given by Eq. (4) the required non-vanishing components of \( \chi^{03}_\nu \) are

\[ \chi^{01}_1 = -2 R^2(t) \dot{S}(t) \sin y, \]
\[ \chi^{02}_2 = \chi^{03}_3 = -2 S(t) R(t) \dot{R}(t) \sin y, \]
\[ \chi^{01}_3 = -R(t) \left( \dot{S}(t) R(t) - S(t) \dot{R}(t) \right) \sin(2y). \]

Entering the above components in Eq. (24), we can find the energy and momentum densities as follows

\[ M_0^0 = 0, \]
\[ M_1^0 = M_3^0 = 0, \quad M_2^0 = \frac{-S(t) R(t) \dot{R}(t) \cos y}{4\pi}. \]

The energy density is zero in Møller’s prescription, as can be seen from Eq. (28).
The Weinberg’s energy-momentum complex is expressed by the equation

\[ W^{\mu\nu} = \frac{1}{16\pi} \Delta^{\mu\nu\lambda}_{\lambda}, \]  

where Weinberg’s superpotential \( \Delta^{\mu\nu\lambda} \) is antisymmetric on its first pair of indices which defines as

\[ \Delta^{\mu\nu\lambda} = \partial^\mu h^{\kappa\nu\lambda}_\kappa - \partial^\nu h^{\kappa\mu\lambda}_\kappa - \partial^\lambda h^{\kappa\mu\nu}_\kappa + \partial^\nu h^{\mu\lambda}_\kappa - \partial^\mu h^{\nu\lambda}_\kappa, \]

where \( \partial_\mu \equiv \partial/\partial x^\mu, \partial^\mu \equiv \partial/\partial x_\mu \) and \( h^{\mu\nu}_\lambda \) shows the symmetric tensor defined as \( h^{\mu\nu}_\lambda = g^{\mu\nu} - \eta^{\mu\nu} \). The energy-momentum in Weinberg’s prescription satisfies the local conservation laws

\[ W^{\mu\nu},_\nu = 0. \] 

\( W^{00} \) and \( W^{10} \) are the energy and momentum density components respectively. The following non-zero components of \( \Delta^{\mu0\lambda} \) are required to find energy-momentum densities in this prescription

\[ \begin{align*}
\Delta^{101} &= \frac{2}{(R^2(t)S^3(t)\sin^2 y)} \left( R^3(t)S^3(t)\dot{R}(t) \left( (S^2(t) \cos^2 y + 2) \sin^4 y ight) + R^5(t)S^2(t)\dot{S}(t) \left( (S^2(t) + 2) \sin^4 y \cos^2 y + 4S^5(t)R^2(t)\dot{R}(t) \sin^2 y \cos^2 y ight) + R^7(t)\dot{S}(t) \left( (S^2(t) + 1) \sin^6 y + R^3(t)S^4(t)\dot{S}(t) \sin^2 y \cos^2 y \cos^2 y + 1 \right) + R^2(t)S^7(t)\dot{R}(t) \sin^2 y \cos^2 y \cos^2 y + 1 \right) + S^7(t)\dot{R}(t) \cos^2 y \left( 3 \cos^2 y + \sin^2 y + 1 \right), \\
\Delta^{103} &= \frac{\sin(2y)}{(R(t)S(t)\sin y)} \left( 2R^2(t)S^3(t)\dot{R}(t) \sin^2 y + S^5(t)\dot{R}(t) \left( 3 \cos^2 y + \sin^2 y + 1 \right) + R^5(t)\dot{S}(t) \left( (S^2(t) + 1) \sin^4 y + R^3(t)S^2(t)\dot{S}(t) \sin^2 y \cos^2 y + 1 \right) + R^2(t)S^5(t)\dot{R}(t) \sin^2 y \left( \cos^2 y + 1 \right) \right), \\
\Delta^{200} &= -\frac{4\cot y}{R^2(t)\sin^2 y}, \\
\Delta^{202} &= \frac{2}{R^2(t)S^3(t)\sin^2 y} \left( \dot{R}(t)S^3(t) \cos^2 y + \dot{S}(t)R^3(t) \sin^2 y + \dot{R}(t)R^3(t) \sin^2 y + \dot{R}(t)S^3(t) \dot{R}(t) \sin^2 y \right), \\
\Delta^{301} &= \frac{\cos y}{(S(t)\sin y)} \left( R^2(t) \sin^2 y + S^2(t) \cos^2 y \right) \Delta^{202}, \\
\Delta^{303} &= \frac{1+\cos^2 y}{\sin^2 y} \Delta^{202}.
\end{align*} \]

We find the components of energy and momentum density distribution in the prescription of Weinberg as follows

\[ W^{00} = 0, \] 
\[ W^{10} = W^{30} = 0, \] 
\[ W^{20} = \frac{\pi \cot y \dot{R}(t) \left( 2R^2(t) - 1 \right)}{2R^2(t) \sin^2 y}. \]

As can be seen from Eq. (34), the energy density is zero in Weinberg’s prescription.
III. CONCLUSION

In conclusion, we have demonstrated some examples of the different descriptions of the energy-momentum density in the context of Bianchi IX cosmological model. We have found that the energy-momentum complexes of Møller and Weinberg provide the zero energy density in the gravitational background under consideration. Also, it is possible to vanish both Møller’s and Weinberg’s momentum density components at a specific spacetime point, e.g., \( y = \pi/2 \). However, this vanishing by no means holds in general. So, these results may be supported by this statement which tells us that different energy and momentum complexes can give the same results for the same gravitational background [25].

In the remaining prescriptions, i.e., Tolman, Papapetrou and Landau-Lifshitz complexes, we have acquired different non-zero energy and momentum densities which sustain this statement that different energy and momentum complexes could yield different energy and momentum distributions for a given gravitational background [26]. In fact, in all the prescriptions for the spacetime under consideration, each of the different expressions might indicate a physically and geometrically consequence connected to the boundary conditions. Furthermore, according to the equivalence principle, the appearance of pseudotensors as noncovariant objects is the origin of these discrepancies which reflects the fact that the gravitational field cannot be perceived at a point. Therefore, according to Ref. [27], the main outcome of this paper points out that energy cannot be localized in this type of time-dependent gravitational background of the spacetime.

Acknowledgments

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[28] Throughout this paper, Latin indices (i, j, ...) represent the spatial coordinate values while Greek indices (\mu, \nu, ...) represent the spacetime labels. We set the fundamental constants equal to unity; \( G = c = 1 \).

[29] Note that the elements of the energy-momentum complex containing matter plus all gravitational and non-gravitational fields are the same in all other prescriptions as follows in this paper.