The rational points on certain Abelian varieties over function fields

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Abstract

In this paper, we consider Abelian varieties over function fields that arise as twists of Abelian varieties by cyclic covers of irreducible quasi-projective varieties. Then, in terms of Prym varieties associated to the cyclic covers, we prove a structure theorem on their Mordell-Weil group. Our results give an explicit method for construction of elliptic curves, hyper- and super-elliptic Jacobians that have large ranks over function fields of certain varieties.

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1 Introduction and main results

Let $A$ be an Abelian variety defined over an arbitrary global field $k$. By the Mordell-Weil Theorem, the set $A(k)$ of $k$-rational points on $A$ is a finitely generated abelian group \cite{12}. In other words, one has $A(k) \cong A(k)_{\text{tors}} \oplus \mathbb{Z}^r$ where $A(k)_{\text{tors}}$ is a finite subgroup of $A(k)$ that is called the torsion subgroup; and $r$ is a non-negative number $r$ which is called the (Mordell-Weil) rank of $A$ over $k$ and is denoted by $\text{rk}(A(k))$. It is a mysterious quantity associated to an Abelian variety. Finding Abelian varieties with large ranks is one of the most challenging problems in Arithmetic and Diophantine geometry.

For example, when $A$ is an elliptic curve defined over $k = \mathbb{Q}$ a folklore conjecture asserts that the rank of an elliptic curve over $\mathbb{Q}$ can be arbitrary large \cite{12}. This conjecture should now be regarded as being in serious doubt by the results of J. Park and et. al., in \cite{10}. In short, they predict that the
number of elliptic curves over \( \mathbb{Q} \) with rank \( \geq 21 \) is finite. However, in 2006, Elkies showed the existence of an elliptic curve over \( \mathbb{Q} \) with 28 independent generators. In the case of elliptic curves over quadratic number fields \( k = \mathbb{Q}(\sqrt{d}) \) the largest known rank is 30 with \( d = -3 \), which has been found by F. Najman. To see the equation of these curves and more information on the high rank elliptic curves over rational numbers and quadratic number fields, we refer the reader to [3].

In contrast, for each prime \( p \), there are known explicit elliptic curves over \( k = \mathbb{F}_p(t) \) with arbitrary large rank \([13, 14]\). In the case \( k = \mathbb{C}(t) \), it has been proved that for a very general elliptic curve \( E \) over \( k \) with height \( d \geq 3 \) and every finite rational extension \( k' \) of \( k \) the Mordell-Weil group \( E(k') \) is a trivial group, see [15].

In this paper, we generalize the main result of Hazama in [5] to an arbitrary cyclic \( s \)-covers of irreducible quasi-projective varieties for any integer \( s \geq 2 \). We fix a global field \( k \) of characteristic \( \geq 0 \) not dividing \( s \), so that it contains an \( s \)-th root of unity, which is denoted by \( \zeta \). Let us denote by \( \mathcal{A}[s](k) \) the subgroup of \( k \)-rational \( s \)-division points of \( \mathcal{A} \). We assume that there exists an order \( s \) automorphism \( \sigma \in \text{Aut}(\mathcal{A}) \). Given the irreducible quasi-projective varieties \( \mathcal{V} \) and \( \mathcal{V}' \) with function fields \( K \) and \( K' \), respectively, we denote by \( \text{Prym}_{\mathcal{V}'/\mathcal{V}} \) the Prym variety associated to the cyclic \( s \)-cover \( \pi : \mathcal{V}' \to \mathcal{V} \), all defined over \( k \). See Section 2 for the definition and some properties of \( \text{Prym}_{\mathcal{V}'/\mathcal{V}} \).

**Theorem 1.1** Notation being as above, we assume that there exists a \( k \)-rational point \( v'_0 \in \mathcal{V}'(k) \). Then we have an isomorphism of Abelian groups:

\[
\mathcal{A}_a(K) \cong \text{Hom}_k(\text{Prym}_{\mathcal{V}'/\mathcal{V}}, \mathcal{A}) \oplus \mathcal{A}[s](k).
\]

Moreover, if \( \text{Prym}_{\mathcal{V}'/\mathcal{V}} \) is \( k \)-isogenous to \( \mathcal{A}^n \times \mathcal{B} \) for some positive integer \( n \) and and an Abelian variety \( \mathcal{B} \) over \( k \) with \( \dim(\mathcal{B}) = 0 \) or \( \dim(\mathcal{B}) > \dim(\mathcal{A}) \) and no irreducible components \( k \)-isogenous to \( \mathcal{A} \), then \( \text{rk}(\mathcal{A}_a(K)) \geq n \cdot \text{rk}(\text{End}_k(\mathcal{A})) \).

As an application of this theorem, for given integers \( 2 \leq s \leq r \leq n \), we consider the cyclic \( s \)-cover \( \pi : C_n \to \mathcal{V}_n \) where \( C_n \) is the product of \( n \) copies of the curve \( C_{s,f} \) given by the affine equation \( y^s = f(x) \) with \( f(x) \in k[x] \) of degree \( r \), and \( \mathcal{V}_n \) is the quotient of \( C_n \) by a certain cyclic subgroup of
order \(s\) of \(\text{Aut}(C_n)\), see Section 5. Let \(C_{s,f}^\xi\) be the twist of \(C_{s,f}\) by the cyclic extension \(L|K\), where \(K = k(V_n)\) and \(L = k(C_n)\). Denote by \(J(C_{s,f})\) the Jacobian variety of \(C_{s,f}\) and let \(J(C_{s,f})[s](k)\) to be its subgroup of \(k\)-rational \(s\)-division points. We have the following result.

**Theorem 1.2** With the above notations and assuming that there exists \(k\)-rational point \(c \in C_{s,f}(k)\), we have

\[
J(C_{s,f}^\xi)(K) \cong \left(\text{End}_k(J(C_{s,f}))\right)^n \oplus J(C_{s,f})[s](k),
\]

an isomorphism of Abelian groups and hence,

\[
\text{rk}(J(C_{s,f}^\xi)(K)) \geq n \cdot \text{rk}(\text{End}_k(J(C_{s,f}))).
\]

The structure of this paper is as follows. In section 2, we investigate some of the properties of the Prym varieties associated to the cyclic covers of quasi-projective varieties. In section 3, we recall the main result of Hazama from [5] that we are going to extend in this paper. Then, we prove Theorems 1.1 and 1.2 in sections 4 and 5, respectively.

## 2 The Prym variety associated to the cyclic covers

The notion of Prym variety was introduced by Mumford in [9] and has been extensively studied for double covers of curves in [1]. It has been generalized for double covers of irreducible quasi-projective varieties in [5]. Here, we generalize this notion to the case of cyclic \(s\)-covers of varieties.

**Definition 2.1** For an integer \(s \geq 2\), the Prym variety of the cyclic \(s\)-cover \(\pi : V' \rightarrow V\) of irreducible quasi-projective varieties over \(k\) is defined by the quotient Abelian variety

\[
Prym_{V'/V} := \frac{Alb(V')}{\text{Im}(id + \tilde{\gamma} + \cdots + \tilde{\gamma}^{s-1})},
\]

where \(Alb(V')\) is the Albanese variety and \(\tilde{\gamma}\) is the automorphism of \(Alb(V')\) induced by an order \(s\) automorphism \(\gamma \in \text{Aut}(V')\) defined over \(k\).

We note that if both of the varieties \(V\) and \(V'\) are curves, then this definition is compatible with the one given in [5], by the following lemma.

**Lemma 2.2** Given an integer \(s \geq 2\), let \(\pi : V' \rightarrow V\) be a cyclic \(s\)-cover of irreducible quasi-projective varieties, both as well as \(\pi\) defined over \(k\).
Suppose that \( \gamma \in \operatorname{Aut}(\mathcal{V}') \) is an automorphism of order \( s \) defined over \( k \). Denote by \( \tilde{\gamma} \) the automorphism of the Albanese variety \( \operatorname{Alb}(\mathcal{V}') \) induced by \( \gamma \). Then there is a \( k \)-isogeny of Abelian varieties,

\[
\text{Prym}_{\mathcal{V}'/\mathcal{V}} \sim_k \ker(id + \tilde{\gamma} + \cdots + \tilde{\gamma}^{s-1} : \operatorname{Alb}(\mathcal{V}') \to \operatorname{Alb}(\mathcal{V}'))^\circ,
\]

where \((*)^\circ\) means the connected component of its origin.

**Proof.** Let \( \mathcal{A} = \operatorname{Alb}(\mathcal{V}') \) and denote its dimension by \( m \). Given \( \gamma \in \operatorname{Aut}(\mathcal{A}) \) of order \( s \). Define \( m_1 := \dim \ker(id - \gamma)^\circ \), and \( m_2 := \dim \ker(id + \gamma + \cdots + \gamma^{s-1})^\circ \). Then, \( m = m_1 + m_2 \) by considering the induced action on the tangent space of \( \mathcal{A} \) at the origin. We have \( \gamma(P) = P \) for each point \( P \) belonging to the intersection of \( \ker(id - \gamma)^\circ \) and \( \ker(id + \gamma + \cdots + \gamma^{s-1})^\circ \), so \( 0 = (id + \gamma + \cdots + \gamma^{s-1})(P) = sP \), which implies that \( \ker(id - \gamma)^\circ \cap \ker(id + \gamma + \cdots + \gamma^{s-1})^\circ \subseteq \mathcal{A}[s] \). Thus \( \mathcal{A} \) is \( k \)-isogenous to their product, i.e.,

\[
\mathcal{A} \sim_k \ker(id - \gamma)^\circ \times \ker(id + \gamma + \cdots + \gamma^{s-1})^\circ.
\]

Moreover, we note that \( \operatorname{Im}(id + \gamma + \cdots + \gamma^{s-1}) \subseteq \ker(id - \gamma)^\circ \) and

\[
m - m_2 = \dim \operatorname{Im}(id + \gamma + \cdots + \gamma^{s-1}) = \dim \ker(id - \gamma)^\circ = m_1.
\]

Therefore, \( \operatorname{Im}(id + \gamma + \cdots + \gamma^{s-1}) = \ker(id - \gamma)^\circ \) that gives the desired result.

Here, we describe a general method of construction of new \( s \)-cover using the given ones, which we will use in the proof of Theorems 1.1 and 1.2.

Let \( \pi_i: \mathcal{V}'_i \to \mathcal{V}_i \) for \( i = 1, 2 \) be \( s \)-covers of irreducible quasi-projective varieties, all defined over \( k \). Assume there exist \( k \)-rational simple points \( v'_i \in \mathcal{V}'_i \). Denote by \( G_i \) the cyclic Galois group of the corresponding function field extensions. Then, the covering \( \pi_1 \times \pi_2: \mathcal{V}'_1 \times \mathcal{V}'_2 \to \mathcal{V}_1 \times \mathcal{V}_2 \) has Galois group \( G_1 \times G_2 \cong \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/s\mathbb{Z} \). Suppose that \( \mathcal{W} \) is its intermediate cover \( \mathcal{V}'_1 \times \mathcal{V}'_2/G \), where \( G \) is the group generated by \( \gamma = (\gamma_1, \gamma_2) \in \operatorname{Aut}(\mathcal{V}'_1 \times \mathcal{V}'_2) \).

Let \( \tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) \) be the order \( s \) automorphism in \( \operatorname{Aut}(\operatorname{Alb}(\mathcal{V}'_1) \times \operatorname{Alb}(\mathcal{V}'_2)) \) corresponding to \( \gamma \), where \( \tilde{\gamma}_i \) is an automorphism of \( \operatorname{Alb}(\mathcal{V}'_i) \) induced by \( \gamma_i \in \operatorname{Aut}(\mathcal{V}'_i) \) of order \( s \geq 2 \) for \( i = 1, 2 \). Then there exists a \( k \)-rational isomorphism

\[
\phi := \operatorname{Alb}(\mathcal{V}'_1) \times \operatorname{Alb}(\mathcal{V}'_2) \to \operatorname{Alb}(\mathcal{V}'_1 \times \mathcal{V}'_2),
\]

given by \( \phi = \tilde{\phi}_1 + \tilde{\phi}_2 \), where \( \tilde{\phi}_1 : \operatorname{Alb}(\mathcal{V}'_1) \to \operatorname{Alb}(\mathcal{V}'_1) \times \operatorname{Alb}(\mathcal{V}'_2) \) is induced by the inclusion map \( \phi_1 : \mathcal{V}'_1 \to \mathcal{V}'_1 \times \mathcal{V}'_2 \) defined by \( \phi_1(v) = (v, v_2) \) and \( \phi_2(v) = (v_1, v) \). By this isomorphism, we have \( \ker(\mu) \sim_k \ker(\mu_1) \times \ker(\mu_2) \), where \( \mu := id + \tilde{\gamma} + \cdots + \tilde{\gamma}^{s-1} \) and \( \mu_i := id + \tilde{\gamma}_i + \cdots + \tilde{\gamma}_i^{s-1} \) for \( i = 1, 2 \). This implies that \( \ker(\mu)^\circ \sim_k \ker(\mu_1)^\circ \times \ker(\mu_2)^\circ \). Therefore, applying Lemma 2.2 and putting everything together, we conclude the following result.
Proposition 2.3 As a $k$-rational isogeny of Abelian varieties, we have

$$Prym_{V'_1 \times V'_2 / W} \sim_k Prym_{V'_1 / V_1} \times Prym_{V'_2 / V_2}.$$ 

3 The result of Hazama

In [4, 5], Hazama gave an explicit method of construction of Abelian varieties that have large rank over function fields using the twist theory [2, 4]. In [16], Wang extended the result of [4] to cyclic covers of the projective line with prime degrees. Inspired by Hazama’s result, in [17], Yamagishi reduces the problem of constructing elliptic curves of rank $n \geq 1$ with generators to the problem of finding rational points on a certain varieties. By providing a parametrization for the rational points on those varieties, she gets all of the elliptic curves of rank $1 \leq n \leq 7$ defined over a field of characteristic different from two.

Here, we briefly recall the main result of Hazama from [5]. Let $A$ be an Abelian variety over $k$ with characteristic different from two. Suppose that $\pi : V' \to V$ is a double cover with Prym variety $Prym(V'/V)$, of irreducible quasi-projective varieties $V$ and $V'$ defined over $k$. Let $\mathcal{K}$ and $\mathcal{K}'$ be the function field of $V$ and $V'$, respectively, and $G$ the Galois group of the extension $\mathcal{K}'|\mathcal{K}$. Denote by $A_a$ the twist of $A$ by the 1-cocycle $a = (a_u) \in Z^1(G, Aut(A))$ defined by $a_{id} = id$ and $a_\iota = -id$.

**Theorem 3.1** With the above notations, assume that there exist a $k$-rational simple point $v'_0 \in V'$. Then we have an isomorphism of Abelian groups:

$$A_a(\mathcal{K}) \cong Hom_k(Prym(V'/V), A) \oplus A[2](k).$$

Moreover, if $Prym_{V'/V}$ is $k$-isogenous with $E^n \times B$ for some positive integer $n$, where $E$ is an elliptic curve over $k$ and $B$ is an Abelian variety with no simple component $k$-isogenous to $E$, then $rk(E_b(\mathcal{K})) = n \cdot rk(End_k(E))$.

We refer the reader to 2.2 and 2.3 in [5], for the proof of the above theorem.

4 Proof of Theorem 1.1

Suppose that the natural map $i_{V'} : V' \to Alb(V')$ sends $v'_0$ to the origin of $Alb(V')$ so that $i_{V'}$ is defined over $k$. Then, using Theorem 4 of chapter II in [7], we have $A(\mathcal{K}') = \{ k$-rational maps $V' \to A \} \cong Hom_k(Alb(V'), A) \oplus$
\[ A(k), \text{ where } P \in A(K') \text{ corresponds to the pair } (\lambda, Q) \in \text{Hom}_k(\text{Alb}(V'), A) \oplus A(k) \text{ such that } P(v') = \lambda(i_{V'}(v')) + Q \text{ for each } v' \in V'. \] This implies that the action of \( \gamma^j \in G \) is given by \( \gamma^j(\lambda, Q) = (\lambda \circ \gamma^j, Q) \) for \( j = 0, \ldots, s - 1 \), where \( \gamma \) is the automorphism of the Albanese variety \( \text{Alb}(V') \) induced by \( \gamma \in \text{Aut}(V') \). Since \( \gamma^s = \text{id} \) and hence \( \tilde{\gamma}^s = \text{id} \), so

\[ A_n(K) \cong \{ P \in A(K') : b_{ij} \cdot \gamma^j(P) = P, \; \forall \gamma^j \in G \}, \]

by applying the proposition 1.1 in [4]. This implies that \( (\lambda, Q) \in A_n(K) \) if and only if \( \gamma^j(\lambda, Q) = (\lambda \circ \gamma^j, Q) = (\lambda \circ \tilde{\gamma}^s - j, Q) = \gamma^{s-j}(\lambda, Q) \). Thus, \( (\lambda, Q) \in A_n(K) \) if and only if \( \lambda \) annihilates \( \text{Im}(\text{id} + \tilde{\gamma} + \cdots + \tilde{\gamma}^{s-1}) \) and \( Q \in A[s](k) \). Therefore,

\[ A_n(K) \cong \text{Hom}_k(\text{Prym}_{V'/V}, A) \oplus A[s](k). \]

Furthermore, if we assume that \( \text{Prym}_{V'/V} \) is \( k \)-isogenous with \( A^n \times B \) for some positive integer \( n \), where \( A \) and \( B \) are Abelian varieties defined over \( k \) such that \( \dim(B) = 0 \) or \( \dim(B) > \dim(A) \) and none of irreducible components of \( B \) is \( k \)-isogenous to \( A \), then

\[ A_n(K) \cong \text{Hom}_k(A^n \times B, A) \oplus A[s](k) \cong \text{Hom}_k(A^n, A) \oplus \text{Hom}_k(B, A) \oplus A[s](k) \cong (\text{End}_k(A))^n \oplus \text{Hom}_k(B, A) \oplus A[s](k). \]

Therefore, as \( \mathbb{Z} \)-modules, we have \( \text{rk}(A_n(K)) \geq n \cdot \text{rk}(\text{End}_k(A)) \).

### 5 The proof of the theorem 1.2
Given the integers \( 2 \leq s \leq r \leq n \), fix a polynomial \( f(x) \in k[x] \) of degree \( r \). Consider the curve \( C_{s,f} : y^s = f(x) \) with a rational point \( c \in C_{s,f}(k) \). It admits an order \( s \) automorphism \( \iota : (x, y) \mapsto (x, \zeta \cdot y) \). For each \( 1 \leq i \leq n \), let \( C^{(i)}_{s,f} \) a copy of \( C_{s,f} \) defined by the affine equation \( y^s_i = f(x_i) \) and denote by \( \iota_i \) the corresponding automorphism for each of these curves. Define \( C_n := \prod_{i=1}^n C^{(i)}_{s,f} \) which can be expressed by the simultaneous equations \( y^s_i = f(x_i) \) for \( i = 1, \ldots, n \). Let \( G = \langle \gamma \rangle \) to be the order \( s \) cyclic subgroup of \( \text{Aut}(C_n) \), where \( \gamma := (\iota_1, \ldots, \iota_n) \), and define \( V_n := C_n/G \). If we assume that \( L \) is the function field of \( C_n \), i.e., \( L = k(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \), where \( x_1, x_2, \ldots, x_n \) are independent transcendentals variables and each \( y_i \) defines a degree \( s \) extension by the equation \( y^s_i - f(x_i) = 0 \), then \( K = k(V_n) \) the
function field of $V_n$ is equal to the set of all invariant elements of $L$ by the action of $G$, i.e., $K = L^G = k(x_1, \ldots, x_n, y_1^{s-1}y_2, \ldots, y_1^{s-1}y_{n-1})$. Since 
\[(y_1^{s-1}y_{i+1})^s = f(x_1)^{s-1}f(x_{i+1})\]
holds for $i = 1, \ldots, n - 1$, so by defining $z_i := y_1^{s-1}y_{i+1}$ the variety $V_m$ can be expressed by $z_i^s = f(x_1)^{s-1}f(x_{i+1})$, for $i = 1, \ldots, n - 1$. Hence, $L|K$ is a cyclic extension of degree $s$ determined by $y_1^s = f(x_1)$, i.e.,
\[L = K(y_1) = k(x_1, \ldots, x_n, z_1, \ldots, z_{n-1})(y_1)\].

Define $C_{s,f}$ to be the twist of $C_{s,f}$ by the extension $L|K$ and let $J(C_{s,f})$ to denote its Jacobian variety. In a similar way as Corollary 3.1 in [4], one can see that $C_{s,f}$ is defined by the affine equation $f(x_1)z^s = f(x)$. It is also easy to check that $C_{s,f}$ contains the following $K$-rational points:
\[P_i := (x_1, 1) \text{ and } P_i := (x_{i+1}, z_i/f(x_1)) \text{ for } (1 \leq i \leq n - 1). \quad (5.1)\]

**Remark 5.1** The construction of the varieties $C_n$ and $V_n$ generalizes that given by Yamagishi in [17], which is used to find elliptic curves of high rank that have a given set of algebraic numbers as $x$-coordinates of the generators of their Mordell-Weil group.

By the fact that the Albanese and Jacobian varieties of curves coincide and applying Lemma 2.2 to $V' = C_{s,f}^{(i)} = C_{s,f}$ and $V = \mathbb{P}^1$, we have
\[
P_{\text{Prym}}(C_{s,f}^{(i)}/\mathbb{P}^1) = \frac{J(C_{s,f}^{(i)})}{\text{Im}(id + \bar{i} + \cdots + i^{s-1})} \sim_k \ker(id + \bar{i} + \cdots + i^{s-1}).
\]

Since $0 = id - \bar{i}^s = (id - \bar{i})(id + \bar{i} + \cdots + i^{s-1})$ and $id \neq \bar{i}$, we have
\[0 = id + \bar{i} + \cdots + i^{s-1} \in \text{End}(J(C_{s,f}^{(i)})) = \text{End}(J(C_{s,f})),\]
which implies that $P_{\text{Prym}}(C_{s,f}^{(i)}/\mathbb{P}^1) = J(C_{s,f}^{(i)})$ for each $i = 1, \ldots, n$. By applying Proposition 2.3 one can get an $k$-isogeny of Abelian varieties
\[
P_{\text{Prym}}(C_{s,f}/V_n) \sim_k \prod_{i=1}^n P_{\text{Prym}}(C_{s,f}^{(i)}/\mathbb{P}^1) = J(C_{s,f})^n. \quad (5.2)
\]

Let us denote by $Q_i$ the image of $P_i$ $(i = 1, \ldots, n)$ given by (5.1) under the canonical embedding of $C_{s,f}^{(i)}$ into $J(C_{s,f})$. Define $a = (a_n) \in Z^1(G, \text{Aut}(J(C_{s,f})))$ by $a_{id} = id$ and $a_{\gamma^j} = \bar{i}^j$ where $\gamma^j \in G$ and $\bar{i} : J(C_{s,f}) \to \mathbb{P}^1$.
\( J(C_{s,f}) \) is the automorphism induced by \( \iota : C_{s,f} \rightarrow C_{s,f} \). Denote by \( J(C_{s,f})_a \) the twist of \( J(C_{s,f}) \) with the 1-cocycle \( a \). Then, \( J(C_{s,f})_a = J(C_{s,f}^\xi) \) by the lemma on page 172 in [4]. Applying the theorem [4,1] for \( V' = C_n, V = V_n, \) and \( A = J(C_{s,f}) \), we have

\[
J(C_{s,f}^\xi)(K) \cong \text{Hom}_k(\text{Prym}_{C_n/V_n}(J(C_{s,f})), J(C_{s,f})[s](k))
\]

\[
\cong \text{Hom}_k(J(C_{s,f})^n, J(C_{s,f}))[s](k)
\]

\[
\cong (\text{End}_k(J(C_{s,f})))^n \oplus J(C_{s,f})[s](k).
\]

Thus, as \( \mathbb{Z} \)-modules, we have \( \text{rk}(J(C_{s,f}^\xi)(K)) \geq n \cdot \text{rk}(\text{End}_k(J(C_{s,f}))) \). Tracing back the above isomorphisms shows that the points \( Q_1, \cdots, Q_n \) belong to the set of independent generators of \( J(C_{s,f}^\xi)(K) \).

References

[1] Beauville, A.: Variétés de Prym et Jacobiennes intermediaires, Ann. scient. Éc. Norm. Sup.”, 10, (1977), 309-391.

[2] Borel, A., and Serre, J. -P.: Théorèmes de finitude en cohomologie galoisienne, Comment. Math. Helv., 39, (1964), 111-164.

[3] Dujella, A.: High rank elliptic curves with prescribed torsion, http://www.maths.hr/~duje/tors.htm, (2017)

[4] Hazama, F.: On the Mordell-Weil group of certain abelian varieties defined over function fields, J. Number Theory, 37, (1991), 168-172.

[5] Hazama, F.: Rational points on certain abelian varieties over function fields, J. Number Theory, 50, (1995), 278-285.

[6] Hindry, H., and Silverman, J. H.: Diophantine geometry: An introduction, Graduate Text in Mathematics, Vol. 201, Springer-Verlag, New york (2001).

[7] Lang, S.: Abelian Varieties, Springer-Verlag, New York/Berlin (1983).

[8] Lange, H., and Ortega, A.: Prym varieties of cyclic coverings, Geom. Dedicata., 150, (2011), 391-403.

[9] Mumford, D.: Prym varieties (I), Contributions to analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, (1974), 325-350.
[10] PARK J., AND ET. AL.: A heuristic for boundedness of ranks of elliptic curves, https://arxiv.org/abs/1602.01431, (2016)

[11] SALAMI, S.: On some different related problems in Diophantine geometry, Ph.D thesis, Universidade Federal do Rio de Janeiro, Brazil, (2017).

[12] SILVERMAN, J. H.: The Arithmetic of Elliptic Curves, second edition, Graduate Text in Mathematics, Vol. 106, Springer-Verlag, New York (2009).

[13] TATE, J. T. AND SHAFAREVICH, I. R.: The rank of elliptic curves, Akad. Nauk SSSR, 175, (1967), 770-773.

[14] ULMER, D.: Elliptic Curves with Large rank over function fields, Annals Math., 155, No. 1 (2002), 295-315.

[15] ULMER, D.: Rational curves on elliptic surfaces, J. Algebraic Geom., 26, (2017), 357-377.

[16] WANG, W. B.: On the Twist of Abelian Varieties Defined by the Galois Extension of Prime Degree, Journal of Algebra, 163 (3), (1994), 813 - 818.

[17] YAMAGISHI, H.: A unified method of construction of elliptic curves with high Mordell-Weil rank, Pacific J. Math, 191, (1999), 507-524.