Symmetry reduction of loop quantum gravity

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Abstract
The relation between standard loop quantum cosmology (LQC) and full loop quantum gravity (LQG) fails already at the first nontrivial step: the configuration space of LQC cannot be embedded into the configuration space of full LQG due to a topological obstruction. We investigate this obstruction in detail, because many topological obstructions are the source of physical effects. For this, we derive the topology of a large class of subspaces of the LQG configuration space. This allows us to find the extension of the standard LQC configuration space that admits an embedding in agreement with Fleischhack (arXiv:1010.0449v1 [math-ph]). We then construct the embedding for flat FRW LQC and find the reassuring result that it coincides asymptotically with standard LQC.

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1. Introduction

The construction of a UV-complete theory of quantum gravity is one of the main open problems in theoretical physics, which sparked research programs in a variety of directions. A particularly fruitful program is loop quantum gravity (LQG) [2–4], as it represents a mathematically well-developed quantum field theoretic framework. An important source of insight for the development of LQG is loop quantum cosmology (LQC) which is a symmetry-reduced quantum mechanical model that exhibits many features of full LQG. The expectation that LQC captures essential features of LQG is often motivated by the following argument: on the one hand, one expects that quantum effects are most important near a singularity of classical general relativity (GR); on the other hand, one expects in light of the BKL conjecture [5–7] that the dynamics near a singularity of GR is well approximated by the dynamics of decoupled homogeneous cosmologies, so one may expect that LQC provides essential insight for the dynamics of full LQG in situations where the quantum effects are expected to become most significant.
Symmetry-reduced models of LQG, in particular LQC, are obtained as ‘loop quantizations’ of symmetry reductions of the classical theory underlying LQG. This ‘classical reduction then quantization’ procedure however weakens the link between the reduced and full theory, because important features of the full quantum field theory may be overlooked by going through a classical symmetry reduction and requantization. Ideally, one would construct a symmetry-reduced model directly from the full theory. An obstacle to such a construction of LQC out of LQG was exposed in [8], where it was shown that the embedding of the spectra of the algebras of configuration operators of standard LQC into LQG is not continuous. However, it turns out that the embedding into piecewise linear LQG is continuous [9]. Recently, in [1] a mathematical proof has been presented, which demonstrates how the configuration space of LQC has to be extended under the requirement of smooth embeddability of the LQC configuration space into the configuration space of full LQG.

A phenomenological investigation of this issue has not been performed either in the LQC framework or in the spin foam cosmology (SFC) framework (see e.g. [10, 11]). It can be easily understood why SFC are insensitive: standard spin foams are not a path integral version of LQG. They are from the point of view of canonical LQG a restriction of full LQG to piecewise linear LQG with low-valent vertices and unknotted edges. However, standard LQC is embeddable into full piecewise linear LQG. The embeddability issue is thus avoided at the very start of the program. This is the technical reason why the investigation in this paper is independent of the significant advances in SFC.

Canonical LQC is on the other hand sensitive to the issue. It seems that the standard LQC avoided the issue so far because only the topological obstruction to embeddability is known and the workaround in terms of piecewise linear LQG exists (we will comment on this in section 5.2); however, exploring possible deviations from standard LQC should be worthwhile. It is the purpose of this paper to perform the first step in this direction. For this, it is first necessary to investigate the topological origin of the obstruction to embedding standard LQC into LQG in detail and then to provide an explicit construction of an extension of standard LQC that permits an embedding. The investigation of this mathematical problem is physically motivated by the fact that configuration space topology is the source of interesting effects, e.g., in solid-state physics or in Euclidean QFT. Moreover, one of the most celebrated results of full LQG is the kinematic discreteness of the area operator. This can be understood as a direct consequence of the topology of the configuration space. We can in light of these precedences not a priori exclude that an embeddable version of LQC admits a very different phenomenology than standard LQC. The investigation of this phenomenology is however a vast subject and is thus beyond the scope of this paper. The main results of this paper are as follows.

(1) The induced topology on one-dimensional affine subspaces of the configuration space of full LQG is the spectrum of the algebra of continuous asymptotically almost periodic functions. This topology is finer than the Bohr compactification of the group \((\mathbb{R}, +)\), which is used to model the compactness of the configuration space of full LQG in standard LQC.

(2) We construct an embeddable version of flat FRW LQC, which differs from standard LQC by the introduction of configuration operators that vanish at infinity. However, we find

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3 Piecewise linear LQC [9] is an interesting modification of standard LQG, where arbitrary graphs are replaced by piecewise linear graphs but where edges can be knotted and vertices can possess arbitrarily high valence. The kinematic theory differs significantly from standard LQG, while the diffeomorphism-invariant theory can be shown to be equivalent to diffeomorphism-invariant standard LQG if a particular extension of the diffeomorphism group is chosen. However, the precise construction of the path groupoid and extension of the diffeomorphism group may have observable consequences and it is thus not for a theorist but for experiment to decide which construction if any is realized in nature.
the reassuring result that standard LQC and embeddable LQC coincide asymptotically. Standard LQC thus captures the universal holonomy modifications correctly.

(3) We do not see any obstruction to generalizing our construction of an embeddable flat FRW LQC to other models such as e.g. [12].

The paper is structured as follows: we first perform necessary asymptotic analysis in sections 2–4, which can be skipped by a physically interested reader. The content of the individual sections is as follows.

In section 2, we re-examine the issue of non-embeddability of the standard LQC configuration space into the configuration space of full LQG. As found in [8], the obstruction to embeddability of standard (flat FRW, i.e. isotropic Bianchi I) LQC is the violation of almost periodic dependence of general LQG spin-network functions when evaluated on the minisuperspace variables used in LQC. As it turns out [1] the solution is to extend the standard configuration space of LQC [13] to cylindrical functions, which are supported on arbitrary edges, not only on edges which are straight with respect to the metric of a chosen cosmological background. From the mathematical side, the problem of constructing this extension is equivalent to solving the differential equation for the parallel transport (in this context also referred to as holonomy) of the Ashtekar connection along arbitrary edges. As usual on the flat FRW minisuperspace, the Ashtekar connection can be parametrized by a real parameter $c$. As a warm-up, we apply the first obvious approach. That is, we apply slight perturbations to straight edges. This leads to a series expansion of the general solution into powers of a perturbation parameter $\epsilon$, where by construction the zeroth order is the standard LQC construction. Demanding the perturbation contributions to remain small for arbitrary large values of $c$ turns out to require $\epsilon \sim c^{-1}$. Hence, the perturbation approach already hints to a solution where contributions of perturbations lead to corrections as inverse powers of $c$.

In section 3, we use the Liouville–Green Ansatz [14] to determine the dependence of an arbitrary spin-network function on the minisuperspace variables. For this, the general solution to the holonomy ODE in terms of a series expansion in inverse powers of $c$ is constructed to arbitrary finite order in $c^{-1}$. We provide an explicit finite upper bound for the error of truncating the series at arbitrary finite order.

In section 4, the limit $c \to \infty$ of the constructed solution is analyzed. It is shown to coincide with the standard LQC construction. We show that the desired extension of the LQC configuration space consists of the standard part of functions almost periodic in the parameter $c$ plus functions which vanish for $c = 0$ and $c \to \infty$ as observed in [1].

In section 5, we use this result, to construct an explicit embedding of isotropic Bianchi I LQC into LQG and discuss its relation with standard LQC.

In section 6, we conclude with remarks on the physical interpretation of this result. We conclude our presentation in section 6.1 with an outlook on future work.

To complete our presentation, appendices A and B contain parts of the explicit computations. Additionally an alternative derivation for the geometric interpretation of the parameter $c$ in terms of the scalar curvature is presented, which uses a recently developed coordinate-free description [15] for the Ashtekar variables.

2. Setup

In this section, we give a brief introduction to the symmetric setup used for the construction of standard LQC [13]. In appendix C, a more detailed coordinate-free treatment due to [15] is provided.
2.1. Holonomy ODE for the homogeneous isotropic cosmological model

Assume a trivial principal $SU(2)$ fiber bundle and choose a coordinate chart covering a subset of $\Sigma$ containing $e$. Then pullback of the real Ashtekar–Barbero connection as an $su(2)$-valued one-form to $\Sigma$ can locally be written as $A(x) = A^I_i(x) \, dx^i \otimes \tau_I$, where $\{\tau_I\}_{I=1,2,3}$ denotes a basis of $su(2)$. The embedded edge $e$ can be written as a map $e : \mathbb{R} \supset [0, T] \ni t \mapsto (e(t) \in \Sigma)$.

The parallel transport of $A$ along the edge $e$ is referred to as its holonomy and is defined by the following ODE (we follow the conventions of [8]):

$$\frac{d}{dt} h(e(t)) = -A(e(t)) h(e(t)) \quad \text{with initial condition } h(e(0)) = I_{SU(2)} \quad (2.1)$$

and $A(e(t)) = A^I_i(e(t)) \tau_i \otimes \tau_I$. In what follows we will denote derivatives with respect to $t$ by dots (e.g. $\dot{e}(t) := (\tau_i \partial_{e^i})(t) = \frac{d}{dt} e(t)$). Also we frequently suppress the dependence on $t$ to shorten our notation. Hence, we write for (2.1)

$$\dot{h} = -A(\dot{e}) h.$$

In this paper, we are particularly concerned with homogeneous cosmology. This is obtained through a simple transitive action of a three-dimensional Lie-group $G$ on $\Sigma$, which allows the identification of $\Sigma$ with $G$. Using this identification, an invariant basis $\{x_a\}_{a=1,2,3}$ in $T \Sigma$ can be described by left-/right-invariant vector fields on $G$. If one follows an integral curve $I$ of one of these vector fields on $G$ respectively $\Sigma$, then the components of the metric tensor are constant along that curve:

$$\langle x_a, x_b \rangle|_I = g_{ab}|_I = \text{const.}$$

Given the invariant basis $\{x_a\}_{a=1,2,3}$, the dual invariant basis of $T^* \Sigma$, $\{x^b\}_{b=1,2,3}$, can be obtained from its definition $x^b(x_a) = \delta^b_a$. For Bianchi I, we use $G = \mathbb{R}^3$. We can furthermore impose isotropy by enlarging the isometry group to $E(3)$. Using a general result on symmetric connections [16], one obtains a parametrization of isotropic Bianchi I connections by $A = c \cdot \delta^b_a dx^a \otimes \tau_b$. Using $\dot{e} = e^b(t) \partial_{x^b}$, we find $A(e) = c \cdot \delta^b_a \dot{e}^a \otimes \tau_b$. Using the defining representation of $SU(2)$, the holonomy $h$ can be written in matrix form

$$h = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{with } |a|^2 + |b|^2 = 1.$$

Taking the usual basis of $su(2)$,

$$\tau_1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_3 = i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and using the shorthands

$$\dot{e}^1(t) := \dot{x}, \quad \dot{e}^2(t) := \dot{y}, \quad m = \dot{x} - i \dot{y}, \quad \dot{e}^3(t) := \dot{z}, \quad n = \dot{z},$$

we can write (2.1) as

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = -ic \begin{pmatrix} n & m \\ -\bar{m} & -\bar{n} \end{pmatrix} \begin{pmatrix} a \\ -\bar{b} \end{pmatrix} \quad (2.2)$$

with initial conditions $a(0) = 1, b(0) = 0$. From that we obtain two first-order ODEs

$$\dot{a} = ic(na - mb) \quad \dot{b} = ic(nb + ma). \quad (2.3)$$
This can be transformed into a second-order ODE for \( a \):

\[
\ddot{a} = Na + M\dot{a} \quad \text{with} \quad M := \frac{\dot{m}}{m} \quad \text{and} \quad N := \imath(c\dot{m} - M n + \imath c(m\ddot{m} + n^2)),
\]

where the initial condition for \( a \) now reads as

\[
a(0) = 1 \quad \dot{a}(0) = \imath c n(0).
\]

Using the transformation

\[
d = \frac{\dot{a}}{\sqrt{m}} \quad \text{where} \quad m \neq 0 \quad \text{(2.6)}
\]

we rewrite (2.4) to

\[
d = \left[ \frac{1}{2}M^2 - \frac{1}{2}M + N \right] \dot{d} \quad \text{(2.7)}
\]

Equation (2.7) is a linear ODE of second order. Its solution \( d(t) \) consists of a linear combination of two fundamental solutions \( d^{(+)}(t) \) and \( d^{(-)}(t) \), that is,

\[
d = d^{(+)} + d^{(-)}.
\]

With this, the solution to (2.4) can be written as

\[
a = \sqrt{m}(A^{(+)}(t)d^{(+)} + A^{(-)}d^{(-)}),
\]

where \( A^{(+)} \) and \( A^{(-)} \) are constants to be chosen such that the initial conditions (2.5) are satisfied.

2.2. Non-embeddability of configuration spaces

In [8], it was shown that the configuration space of LQC (given by \( \mathbb{R}_{\text{Bohr}} \), i.e. the Bohr compactification of the real line) cannot be continuously embedded into the configuration space of LQG (the space \( \mathcal{A} \) of generalized connections). It was found that in order to retain continuity, it is necessary that solutions (2.8) depend almost periodically on the parameter \( c \) for arbitrary edges, not only edges which are straight with respect to the symmetric background. However, it was shown explicitly that generically this is not the case.

Spiral arcs. In [8], an exact solution to ODE (2.5) was found for spiral edges. This is the most general case in which the ODE has constant coefficients, because in the spiral case, \( M(t) = M(0) = M_0 \) and \( n(t) = n(0) = n_0 \) are constants. Anticipating the notation of section 3, ODE (3.3) reads

\[
d = \kappa^2 \Lambda^2 d \quad \text{with} \quad \Lambda^2 = \Lambda^2(\kappa) := 1 + \frac{\alpha_0}{\kappa} + \frac{\beta_0}{\kappa^2},
\]

where \( \kappa := \imath c, \alpha_0 := -M_0 n_0 \) and \( \beta_0 := \frac{1}{2}M_0^2 \) and we have used the arc-length parametrization \( |m|^2 + n^2 = 1 \) (see figure 1 for illustration). It has two fundamental solutions \( \dot{d}(t)(\kappa, t) = e^{\kappa\Lambda t} \) and we can obtain a solution to the holonomy ODE (2.5) and to the original ODE system (2.3) by analogy with section 3.2 by imposing the initial conditions \( (a(0), 0) = 1, \dot{a}(\kappa, 0) = \kappa n_0, b(\kappa, 0) = 0, b(\kappa, 0) = \kappa n(0) \). The solution\(^4\) is given by

\[
a(\kappa, t) = 2 \sqrt{m(t)m(0)} \sum_{\ell = \pm 1} \left\{ \left[ 1 + \ell \right] \frac{2\kappa n_0 - M_0}{2\kappa \Lambda} e^{\ell \kappa \Lambda t} \right\}
\]

\[
b(\kappa, t) = \sqrt{m(t)m(0)} \sum_{\ell = \pm 1} \left\{ \ell \frac{1}{2\Lambda} e^{\ell \kappa \Lambda t} \right\}.
\]

\(^4\) We denote it by German-type letters.
Figure 1. Spiral arc as described by our setup. Recall that \( m = \dot{x} - i \dot{y} \) and \( n = \dot{z} \). We have \( R^2 = x^2 + y^2 = \frac{\mu^2}{\lambda} \) and the step height per period is given by \( D = \frac{2\pi}{\lambda} \). The limit \( \lambda \to 0 \) corresponds to a line in the \((x,y)\)-plane with the direction \((\Im(\mu), -\Re(\mu))\). The limit \( \lambda \to \infty \) corresponds to a line in the \( z \)-direction. In the case of a planar circle \((e = 0)\), we have \( R^2 = \lambda^{-2} \).

To obtain an explicit parametrization for solutions (2.10), we set \( m(t) = \mu \cdot e^{i\lambda t} \) (w.l.o.g. \( \mu \in \mathbb{C}, \lambda \in \mathbb{R} \)); hence \( M_0 = i\lambda \) and \( m(0) = \mu \). Moreover, let \( n_0 := v \) with \( v \in \mathbb{R} \),\(|\mu|^2 + v^2 = 1\) and \( \Delta := \sqrt{c^2/\lambda^2 + c(c - \nu\lambda)} \).

Then (2.10) reads
\[
\begin{align*}
a(k, t) &= \frac{1}{\Delta} \left\{ \cos(\Delta t) + \frac{i}{2\Delta} (2\nu c - \lambda) \sin(\Delta t) \right\} \\
\tilde{b}(k, t) &= \frac{1}{\lambda} \frac{\sqrt{c^2/\lambda^2 + c(c - \nu\lambda)}}{\Delta} \sin(\Delta t).
\end{align*}
\]

Obviously (2.11) exhibits a non-almost dependence of \( a(k, t), b(k, t) \) on \( c \) unless the underlying curve is a line (in this case, \( \lambda = 0 \) and \( v = 0 \)). However, one observes from (2.11) that in the limit \( c \to \infty \) periodicity is asymptotically restored. This observation is the starting point for looking at the properties of the holonomies along arbitrary edges. The first obvious Ansatz is to look at the effect of small perturbations of straight edges.

**Perturbation of straight edges.** The detailed computation can be found in appendix B. Here we only quote the result. The general setup is given as follows: let \( e_0(t) = (e_0^1(t), e_0^2(t), e_0^3(t)) = (x_0, y_0, z_0) \). Now assume that the edge \( y_0 \) is deformed into another edge \( \gamma \) such that \( \gamma(t) = (x_0 + \epsilon \tilde{x}, y_0 + \epsilon \tilde{y}, z_0 + \epsilon \tilde{z}) \), where \( \tilde{x} = \tilde{x}(t), \tilde{y} = \tilde{y}(t), \tilde{z} = \tilde{z}(t) \) and \( \epsilon = \text{const} \) is a small deformation parameter. For \( e_0 \) being a straight line and under some simplifying assumptions described in appendix B.5, one obtains a solution to (2.8) in terms of a formal power series in the perturbation parameter \( \epsilon \). However, one finds from the computation in appendix B that in order to ensure ‘small’ contributions from the perturbation even for large \( c \), one has to require that \( \epsilon \sim c^{-1} \). One then obtains a solution to (2.8) which looks as follows (see appendix B.5 for computational details):

\[
a(k, t) = \sum_{\sigma = \pm} \epsilon^{\sigma \pi} \{ 1 + O(c^{-1}) \}. \tag{2.12}
\]

From (2.12) it becomes obvious that the property of asymptotic almost periodicity carried by solutions (2.10) is not a special property of considering spiral arcs. Rather it is a generic property valid for general perturbations of straight edges. Hence, we are referred to look for a general solution of (2.10) for large \( c \) in terms of an inverse power series of \( c \). Fortunately,
there is a well-developed mathematical framework available (see e.g. [14]), which uses the so-called Liouville–Green Ansatz. This is employed in the following section.

3. Solution by a Liouville–Green Ansatz

3.1. Liouville–Green Ansatz

In this section, we compute an expression for the holonomy along arbitrary edges. It is based on the presentation in [14]. In principle, this approach is viable for general differential equations of the form

\[ \tilde{d}(\kappa, t) = \kappa^{2n} f(\kappa, t) a(\kappa, t), \]

where \( n \in \mathbb{N}, \kappa \in \mathbb{C} \) is a complex parameter and \( f(\kappa, t) \) is an analytic function of \( t \in \mathbb{C} \) with the uniform asymptotic expansion

\[ f(\kappa, t) = f_0(t) + \frac{f_1(t)}{\kappa} + \frac{f_2(t)}{\kappa^2} + \cdots \text{ for } |\kappa| \to \infty. \]

Using (2.4) we can rewrite equation (2.7) as

\[ \tilde{d}(\kappa, t) = \kappa^2 \left( 1 + \frac{\alpha(t)}{\kappa} + \frac{\beta(t)}{\kappa^2} \right) d(\kappa, t) \quad \text{with} \quad 1 = f_0(t) = \overline{m}(t)m(t) + n(t)^2 \]

\[ \alpha(t) := f_1(t) = \dot{n}(t) - M(t)n(t) \]

\[ \beta(t) := f_2(t) = \frac{1}{2} M^2(t) - \frac{1}{2} \dot{M}(t) \]

\[ f_k(t) = 0 \quad \text{if } k \geq 2. \]

and \( t \in \mathbb{R}, \kappa = ic \) and we have chosen arc-length parametrization in order to set \( \overline{m}(t)m(t) + n(t)^2 = 1 \). For shortness of the notation, we will from now on suppress the dependence on \( \kappa, t \) where possible.

Now the strategy is to write (3.3) as

\[ \tilde{d} - \kappa^2 d = (\kappa \alpha + \beta) d \]

and to regard the rhs as an inhomogeneity. Then two fundamental solutions \( d^{(\pm)}(\kappa, t) \) to (3.4) can be obtained from the Liouville–Green Ansatz

\[ d^{(\ell)}(\kappa, t) = e^{\ell \kappa t} X^{(\ell)}(t) \left( \sum_{r=0}^{\infty} \frac{A^{(\ell)}_r(t)}{\kappa^r} \right), \]

where \( \ell = \pm 1 \) and \( A^{(\ell)}_0(t) = \text{const} = 1 \). Plugging this Ansatz into (3.4), we obtain

\[ \tilde{d}^{(\ell)} = e^{\ell \kappa t} X^{(\ell)} \left\{ \kappa^2 X^{(\ell)} + \kappa \left( X^{(\ell)} A_1^{(\ell)} + 2 \ell \dot{X}^{(\ell)} \right) + \left( X^{(\ell)} A_1^{(\ell)} + 2 \ell \dot{X}^{(\ell)} A_1^{(\ell)} + X^{(\ell)} \dot{A}_1^{(\ell)} + X^{(\ell)} \ddot{X}^{(\ell)} \right) \right\} \]

\[ + \sum_{r=1}^{\infty} \kappa^{-r} \left( X^{(\ell)} A_{r+2}^{(\ell)} + 2 \ell \dot{X}^{(\ell)} A_{r+1}^{(\ell)} + X^{(\ell)} A_r^{(\ell)} + 2 \ell \dot{X}^{(\ell)} \dot{A}_r^{(\ell)} + 2 \dot{X}^{(\ell)} \dot{A}_{r+1}^{(\ell)} + 2 \dot{X}^{(\ell)} A_r^{(\ell)} + X^{(\ell)} \ddot{A}_r^{(\ell)} \right) \]

and

\[ (\kappa^2 + \kappa \alpha + \beta) d^{(\ell)} = e^{\ell \kappa t} X^{(\ell)} \left\{ \kappa^2 + \alpha \left( A_1^{(\ell)} + \alpha \right) + \beta + A_2^{(\ell)} \right\} \]

\[ + \sum_{r=1}^{\infty} \kappa^{-r} \left( A_{r+2}^{(\ell)} + \alpha A_{r+1}^{(\ell)} + \beta A_r^{(\ell)} \right) \].
Now we compare coefficients for every order of $\kappa$:

\[ \mathcal{O}(\kappa^2): \quad 0 = 0 \]
\[ \mathcal{O}(\kappa^1): \quad 2\ell\dot{X}^{(t)} = \alpha X^{(t)} \]
\[ \mathcal{O}(\kappa^0): \quad \ddot{X}^{(t)} + 2\ell\beta A_r^{(t)}X^{(t)} = \beta X^{(t)} \quad (3.6) \]
\[ \mathcal{O}(\kappa^{-r}): \quad \ddot{\tilde{X}}^{(t)} + 2\ell\dot{A}_{r+1}^{(t)}X^{(t)} + 2\tilde{X}^{(t)}\tilde{A}_r^{(t)} + \dot{X}^{(t)}A_r^{(t)} = \beta X^{(t)}A_r^{(t)}. \]

These equations can be integrated. From $\mathcal{O}(\kappa^1)$ we find\(^5\)

\[ X^{(t)}(t) = X_0^{(t)}e^{\frac{\ell}{2}\int_0^t \alpha(s)\,ds} \quad \text{with } X_0^{(t)} \text{ an integration constant,} \quad (3.7) \]

yielding the following identities:

\[ \frac{\dot{X}^{(t)}}{X^{(t)}} = \frac{\ell}{2\alpha} \quad \text{and} \quad \frac{\ddot{X}^{(t)}}{X^{(t)}} = \frac{\ell}{2\dot{\alpha}} + \frac{1}{4}\alpha^2. \quad (3.8) \]

We can use (3.8) in order to rewrite the $\mathcal{O}(\kappa^0)$ and $\mathcal{O}(\kappa^{-r})$ expressions of (3.6) to obtain

\[ \dot{A}_1^{(t)} = \frac{\ell}{2} \left( \beta - \frac{\ell}{2}\dot{\alpha} - \frac{1}{4}\alpha^2 \right) \quad (3.9) \]
\[ \dot{A}_{r+1}^{(t)} = \frac{\ell}{2} \left( \beta - \frac{\ell}{2}\dot{\alpha} - \frac{1}{4}\alpha^2 \right) A_r^{(t)} - \frac{\ell}{2}\alpha \dot{A}_r^{(t)} - \frac{\ell}{2}\dot{A}_r^{(t)} \quad (3.10) \]

which can be integrated for $r \geq 0$ to

\[ A_1^{(t)}(t) = C_1^{(t)} + \frac{\ell}{2} \int_0^t \left( \beta(s) - \frac{\ell}{2}\dot{\alpha}(s) - \frac{1}{4}\alpha^2(s) \right)\,ds \quad (3.11) \]
\[ A_{r+1}^{(t)}(t) = C_{r+1}^{(t)} - \frac{\ell}{2}\alpha(t)A_r^{(t)}(t) - \frac{\ell}{2}\int_0^t \left( \beta(s) + \frac{\ell}{2}\dot{\alpha}(s) - \frac{1}{4}\alpha^2(s) \right)A_r^{(t)}(s)\,ds, \quad (3.12) \]

where $C_1^{(t)}$ and $C_{r+1}^{(t)}$ are integration constants. Here we have used the fact that $\ell^2 = 1$. Note the different signs in front of the $\dot{\alpha}$ term in the integrals on the right-hand sides of (3.11) and (3.12), which result from a partial integration of the $\alpha \dot{A}_r^{(t)}$-term in (3.10).

The integration constants $C_1^{(t)}$, $C_{r+1}^{(t)}$ and $X_0^{(t)}$ can be fixed by imposing the initial conditions (2.5) and using Ansatz (2.8).

### 3.2. Power-series solution to the holonomy ODE for general edges

We can now construct a formal solution to the original ODE (2.5) and to the original ODE system (2.3). To summarize, we have

\[ a(\kappa, t) = \sqrt{m(t)}(A_{(+)} d^{(+)}(\kappa, t) + A_{(-)} d^{(-)}(\kappa, t)) \quad (13.13) \]
\[ \dot{a}(\kappa, t) = \frac{\ell}{2} M(t) a(\kappa, t) + \sqrt{m(t)}(A_{(+)} \dot{d}^{(+))(\kappa, t)} + A_{(-)} \dot{d}^{(-)}(\kappa, t)), \quad (13.14) \]

with initial conditions

\[ a(\kappa, 0) = 1, \quad \dot{a}(\kappa, 0) = \kappa n(0) \quad (13.15) \]

and respectively

\[ b(\kappa, t) = \sqrt{m(t)}(B_{(+)} d^{(+)}(\kappa, t) + B_{(-)} d^{(-)}(\kappa, t)) \quad (13.16) \]

\(^5\) Here and in what follows we will always integrate from 0 to $t$ and insert the appropriate integration constants when necessary.

\(^8\)
with initial conditions
\[ b(\kappa, 0) = 0, \quad \dot{b}(\kappa, 0) = \kappa m(0) \]  \tag{3.17}
and \( d^{(\pm)}(\kappa, t) \) given by (3.5).

**Fixing \( X_0^{(\ell)} \).** Without loss of generality, we can set the integration constant \( X_0^{(\ell)} \) in (3.7) to \( X_0^{(\ell)} = 1 \), because as a constant it can be absorbed into \( A_{(+)}, A_{(-)} \) respectively \( B_{(+)}, B_{(-)} \) in (3.13) and (3.16). Therefore, we have
\[ d^{(\ell)}(\kappa, t) = e^{\ell(\kappa + \frac{1}{2} \int z^2 (t) \, dt)} \left( \sum_{\ell=0}^{\infty} \frac{A^{(\ell)}(t)}{\kappa^\ell} \right). \]  \tag{3.18}

### 3.2.2. Solution for \( b(\kappa, t) \)

We have
\[ b(\kappa, 0) = 0 = \sqrt{m(0)} \left( B_{(+)\lambda} \sum_{n=0}^{\infty} \frac{A_n^{(\ell)}(0)}{\kappa^n} + B_{(-)\lambda} \sum_{n=0}^{\infty} \frac{A_n^{(-\ell)}(0)}{\kappa^n} \right). \]  \tag{3.19}

Now we compare coefficients for every order of \( \kappa \):
\[ \mathcal{O}(\kappa^0) : A_0^{(\ell)} = 1, \text{ hence } 0 = B_{(+)\lambda} + B_{(-)\lambda} \quad \Rightarrow \quad B_{(+)\lambda} = -B_{(-)\lambda} \]
\[ \mathcal{O}(\kappa^{-n})|_{n>0} : \text{ with } B_{(+)\lambda} = -B_{(-)\lambda} \text{ we have } 0 = A_n^{(+)\lambda}(0) - A_n^{(-)\lambda}(0) \quad \Rightarrow \quad A_n^{(+)\lambda}(0) = A_n^{(-)\lambda}(0). \]  \tag{3.20}

Additionally, we have
\[ \dot{b}(\kappa, 0) = \kappa m(0) = \frac{M(0)}{2} \sqrt{m(0)} \left( B_{(+)\lambda} \dot{d}^{(+)\ell}(\kappa, 0) + B_{(-)\lambda} \dot{d}^{(-)\ell}(\kappa, 0) \right) \]
\[ \overset{(3.17)}{=} \frac{M(0)}{2} \sqrt{m(0)} B_{(+)} \left( \dot{d}^{(+)\ell}(\kappa, 0) - \dot{d}^{(-)\ell}(\kappa, 0) \right) \]
\[ \overset{(3.20)}{=} \sqrt{m(0)} B_{(+)} \left\{ 2\kappa + \alpha(0) \right\} \sum_{n=0}^{\infty} \frac{A_n^{(\ell)}(0)}{\kappa^n} + \sum_{n=0}^{\infty} \frac{\dot{A}_n^{(+)\ell}(0)}{\kappa^n} - \sum_{n=0}^{\infty} \frac{\dot{A}_n^{(-)\ell}(0)}{\kappa^n} \right\}. \]  \tag{3.21}

The comparison of coefficients gives
\[ \mathcal{O}(\kappa^1) : A_0^{(\ell)} = 1, \text{ hence } m(0) = 2 \sqrt{m(0)} B_{(+)\lambda} \quad \Rightarrow \quad B_{(+)\lambda} = -B_{(-)\lambda} = \frac{\sqrt{m(0)}}{2} \]
\[ \mathcal{O}(\kappa^0) : 0 = \alpha(0) + 2A_0^{(+)\ell}(0) \quad \Rightarrow \quad A_0^{(+)\ell}(0) = A_0^{(-)\ell}(0) = -\frac{\alpha(0)}{2} \]
\[ \mathcal{O}(\kappa^{-n})|_{n>0} : A_n^{(+)\ell} = A_n^{(-)\ell} \quad \Rightarrow \quad A_{n+1}^{(\ell)}(0) = -\frac{\alpha(0)}{2} A_n^{(\ell)}(0) - \frac{1}{2} A_n^{(+)\ell}(0) + \frac{1}{2} A_n^{(-)\ell}(0). \]  \tag{3.22}

If we compare this to (3.11) and (3.10), we obtain
\[ C_1^{(+)} = C_1^{(-)} = -\frac{\alpha(0)}{2} \quad \text{ and } \quad C_{n+1}^{(+)} = \frac{1}{2} A_n^{(-)\ell}(0), \quad C_{n+1}^{(-)} = -\frac{1}{2} A_n^{(+)\ell}(0). \]  \tag{3.23}

### 3.2.2. Solution for \( a(\kappa, t) \)

We have
\[ a(\kappa, 0) = 1 = \sqrt{m(0)} \left( A_{(+)\lambda} \sum_{n=0}^{\infty} \frac{A_n^{(+)\ell}(0)}{\kappa^n} + A_{(-)\lambda} \sum_{n=0}^{\infty} \frac{A_n^{(-)\ell}(0)}{\kappa^n} \right). \]  \tag{3.24}
Now we compare coefficients for every order of $\kappa$,

\[ O(\kappa^0) : A_0^{(0)} = 1, \text{ hence } 1 = \sqrt{m(0)}(A_{(+)} + A_{(-)}) \sim A_{(-)} - [m(0)]^{-\frac{1}{2}} - A_{(+)} \]

\[ O(\kappa^{-n})_{n>0} : 0 = A_{(+)}A_n^{(+)}(0) + A_{(-)}A_n^{(-)}(0) \sim A_n^{(-)}(0) = -\frac{A_{(-)}}{A_{(+)}}A_n^{(+)}(0). \]

(3.25)

Additionally, we have

\[
\begin{align*}
\dot{a}(\kappa, 0) &= \kappa m(0) = \frac{M(0)}{2} a(\kappa, 0) + \sqrt{m(0)} \left( A_{(+)}\ddot{d}^{(+)}(\kappa, 0) + A_{(-)}\ddot{d}^{(-)}(\kappa, 0) \right) \\
&= \frac{M(0)}{2} + \sqrt{m(0)} \left\{ A_{(+)} \left[ \kappa + \frac{\alpha(0)}{2} \right] \sum_{n=0}^{\infty} A_n^{(+)}(0) \frac{1}{k_n} + \sum_{n=0}^{\infty} \dot{A}_n^{(+)}(0) \right\} \\
&\quad + A_{(-)} \left[ - \left( \kappa + \frac{\alpha(0)}{2} \right) \sum_{n=0}^{\infty} A_n^{(-)}(0) \frac{1}{k_n} + \sum_{n=0}^{\infty} \dot{A}_n^{(-)}(0) \right].
\end{align*}
\]

(3.26)

The comparison of coefficients gives $A_0^{(t)} = 1$

\[ O(\kappa^1) : n(0) = \sqrt{m(0)}(A_{(+)} - A_{(-)}) \sim A_{(+)} = \frac{m(0)}{2\sqrt{m(0)}} \]

\[ O(\kappa^0) : 0 = \frac{M(0)}{2} a(\kappa, 0) + \sqrt{m(0)} \left[ A_{(+)} \left( A_{(+)}^{(0)}(0) + \frac{n(0)}{2} \right) - A_{(-)} \left( A_{(-)}^{(0)}(0) + \frac{n(0)}{2} \right) \right] \\
\sim \frac{1}{2} A_n^{(+)}(0) = -\frac{\alpha(0)n(0) + M(0)}{\alpha(0)n(0) + M(0)} \]

\[ O(\kappa^{-n})_{n>0} : 0 = A_{(+)} \left[ A_n^{(+)}(0) + \frac{n(0)}{2} A_n^{(0)}(0) + A_n^{(+)}(0) \right] - A_{(-)} \left[ A_n^{(-)}(0) + \frac{n(0)}{2} A_n^{(-)}(0) - A_n^{(-)}(0) \right] \\
\sim A_n^{(-)}(0) = -\frac{n(0)}{2} A_n^{(-)}(0) - \frac{1}{2} A_n^{(-)}(0) - \frac{1}{2} \dot{A}_n^{(-)}(0) \]

\[ A_n^{(+)}(0) = -\frac{n(0)}{2} A_n^{(+)}(0) + \frac{1}{2} A_n^{(+)}(0) + \frac{1}{2} \dot{A}_n^{(+)}(0) \]

(3.27)

where we have just multiplied the second to last row by $-\frac{A_{(-)}}{A_{(+)}}$ in order to arrive at the last row.

If we compare this to (3.11) and (3.10) for $t = 0$, we obtain

\[ C_{n+1}^{(+)} = -\frac{\alpha(0)n(0) + M(0)}{2n(0) + 1} \text{ and } C_{n+1}^{(-)} = -\frac{1}{2} \dot{A}_n^{(-)}(0) = \frac{1}{2} \dot{A}_n^{(-)}(0) \]

(3.28)

3.2.3. Final solution. We have thus constructed a formal solution to the ODE system (2.3) with initial conditions (2.5):

\[ a(\kappa, t) = \sqrt{m(t)}(A_{(+)}d^{(+)}(\kappa, t) + A_{(-)}d^{(-)}(\kappa, t)) \]

\[ b(\kappa, t) = \sqrt{m(t)}(B_{(+)}d^{(+)}(\kappa, t) + B_{(-)}d^{(-)}(\kappa, t)). \]

(3.29)

Using the arc-length parametrization $s(t) m(t) + n(t)^2 = 1$ and

\[ \alpha(t) := \dot{n}(t) - M(t)n(t), \quad \beta(t) := \frac{1}{2} M^2(t) - \frac{1}{2} M(t), \]

we then have for $\ell = \pm 1$

\[ d^{(\ell)}(\kappa, t) = e^{\ell(\kappa + \frac{1}{2} \int_0^t \alpha(s) ds)} \left( \sum_{n=0}^{\infty} \frac{A_n^{(\ell)}(t)}{k_n^\ell} \right), \]

(3.30)
with
\[ A_{0}^{(t)}(t) = 1 \]
\[ A_{1}^{(t)}(t) = C_{1}^{(t)} + \frac{\ell}{2} \int_{0}^{t} \left( \beta(s) - \frac{\ell}{2} \alpha(s) - \frac{1}{4} \alpha^{2}(s) \right) A_{r}^{(t)}(s) \, ds \]
\[ A_{r+1}^{(t)}(t) = C_{n+1}^{(t)} - \frac{1}{2} \alpha(t) A_{r}^{(t)}(t) - \frac{\ell}{2} A_{r}^{(t)}(t) + \frac{\ell}{2} \int_{0}^{t} \left( \beta(s) + \frac{\ell}{2} \alpha(s) - \frac{1}{4} \alpha^{2}(s) \right) A_{r}^{(t)}(s) \, ds, \]
where the last line is valid for \( r \geq 1 \). For \( a(\kappa, t) \), we have the integration constants
\[ C_{1}^{(t)} = \frac{\ell}{2} \cdot n(0) + \frac{\ell}{\sqrt{m(0)}} \]
\[ C_{1}^{(t)} = -\frac{\alpha(0) n(0) + M(0)}{2(n(0) + \ell)} \]
\[ C_{r+1}^{(t)} = \frac{\ell}{2} \cdot n(0) + \frac{\ell}{2} \cdot \beta(t) A_{r}^{(t)}(0) \]
and for \( b(\kappa, t) \), we have the integration constants
\[ B_{1}^{(t)} = \frac{\sqrt{m(0)}}{2} \]
\[ C_{1}^{(t)} = \frac{\ell}{2} \cdot n(0) \]
\[ C_{r+1}^{(t)} = \frac{\ell}{2} \cdot \beta(t) A_{r}^{(t)}(0). \]

Given these solutions in terms of a formal power series, it remains to discuss the finiteness properties of this series. This issue will be discussed in the following section.

4. Asymptotics and finiteness of the solution

4.1. Asymptotic behavior of SU(2) holonomies

Using the Liouville–Green method and Horn asymptotics (see e.g. [14]), it is straightforward to find the asymptotic behavior of the holonomy differential equation. We present the general procedure here and refer to appendix A for details.

Recall that the general second-order linear ordinary differential equation \( \ddot{a}(t) = A(t) \dot{a}(t) + B(t) a(t) \) is transformed into the standard form \( \ddot{d} = \left( B(t) + \frac{1}{2} A^{2}(t) - \frac{1}{2} A(t) \right) d(t) \)
using \( a(t) = d(t) \exp \left( \frac{\ell}{2} \int_{0}^{t} ds A(s) \right) \). This transforms the holonomy differential equation (in arc-length parametrization for \( r \)) into
\[ \ddot{d}(t) = (\kappa^{2} + \kappa \alpha(t) + \beta(t)) d(t), \]
where we introduced \( \kappa := \imath c \). The Liouville–Green Ansatz for the solution is
\[ d^{(\pm)}(t) = \exp \left( \pm \kappa t + \frac{1}{2} \int_{0}^{t} ds \alpha(s) \right) \left( 1 + \sum_{n=1}^{\infty} \frac{A_{n}^{(\pm)}(t)}{\kappa^{n}} \right), \]
which suggests that \( d^{(\pm)}(t) \sim C^{(\pm)} e^{\mp t} \) as \( |\kappa| \rightarrow \infty \) with the prefactor \( C^{(\pm)} \). To prove this behavior, one needs convergence of this Ansatz which is rarely given. In fact the Liouville–Green Ansatz for a generic holonomy diverges. However, any truncated Ansatz \( \sum_{k=1}^{n} \frac{A_{k}^{(\pm)}(t)}{\kappa^{k}} \) is well behaved (if the \( n \)th derivatives of \( \alpha, \beta \) are bounded) and hence we consider the truncation error
\[ \epsilon_{n}^{(\pm)}(t) = d^{(\pm)}(t) - \sum_{k=1}^{n} \frac{A_{k}^{(\pm)}(t)}{\kappa^{k}}. \]
The asymptotic behavior can be obtained from bounding the truncation error using the following steps:

1. insert the truncation error into the ODE to obtain an equation for \( \epsilon^{(\pm)}_{n}(t) \). This ODE can be solved using variation of constants in Horn’s asymptotic Ansatz \( Z^{k} = (\kappa + \frac{1}{2} \alpha)^{-1} \exp \left( \pm (\kappa t + \frac{1}{2} \int ds \alpha) \right) \);
(2) the key of Horn’s Ansatz is that it solves the ODE to all positive powers of $\kappa$; hence, the kernel of the integral equation obtained form the variation of constants does not contain any positive powers of $\kappa$ and thus yields finite bounds on the solution when $|\kappa| \to \infty$;

(3) finiteness of the coefficients of the truncated Liouville–Green series together with the bound on the truncation error yields the anticipated asymptotic behavior of $d^{(\pm)}(t) \to C^{(\pm)} e^{\pm \kappa t}$ as $|\kappa| \to \infty$ and thus also for $a(t) = d(t) \exp(\frac{1}{2} \int_0^t \text{d}s A(s))$, since $A$ is independent of $\kappa$;

(4) the asymptotic behavior of the matrix elements of the holonomy is then determined by the asymptotic behavior of the linear combination of $d^{(\pm)}(t)$ that satisfies the initial condition for the specific matrix element, which yields the anticipated asymptotic periodicity of the holonomy.

A significant simplification to estimate of the truncation error can be obtained by using methods developed in [17]. There it is assumed that the edges along which the holonomy is integrated are holomorphic in a finite radius in the complex plane around the part of the real $t$-axes that is integrated, because this yields a very simple bound on the Liouville–Green coefficients $d^{(\pm)}(t)$. This method is applicable if one is able to satisfy initial conditions s.t. all integration constants $C^{(r)}(t)$ in the Liouville–Green recursion (3.31) vanish. Unfortunately this requirement is in general not compatible with the initial conditions (2.5). A solution to (2.3) fulfilling the initial condition (2.5) does in general not have vanishing $C^{(r)}$. This is only possible in very special cases.

Let us now discuss this result: by construction, series (3.30) provides a formal solution to ODE (3.4). However, it is not obvious whether this series is finite or whether it converges.

**Finiteness.** As explicitly shown in appendix A, for analytic edges series (3.30) is finite: at every finite inverse order of $\kappa$, the rest term

$$\epsilon^{(t)}_{n>0}(\kappa, t) := d^{(t)}(\kappa, t) - e^{\frac{\lambda_t}{2}} \frac{e^{\alpha t}}{\kappa} \sum_{k=0}^{n-1} \frac{A_k^{(t)}(t)}{\kappa^k}$$

obeys the explicit bound

$$|\epsilon^{(t)}_{n}(\kappa, t)| \leq \frac{\exp \left( \int_0^t |\alpha(s)| \text{d}s \right)}{\lambda(\kappa, t)} \left( \int_0^t |\Delta^{(t)}_{n}(\kappa, \tau)| \text{d}\tau \right) \times \exp \left( \frac{1}{2} \int_0^t |\alpha(s)| \text{d}s \right) \int_0^t |\chi(\kappa, s)| \text{d}s \right) \leq 2 \Phi(t)^2 \frac{\Phi(\tau) |\Delta^{(t)}_{n}(\kappa, \tau)| }{\lambda(\kappa, t) |\chi^{(n-1)}(\kappa)|} \exp \left( \frac{\Phi(t) \int_0^t |\chi(\kappa, s)| \text{d}s }{\lambda(\kappa, t)} \right),$$

where $\lambda(\kappa, t) := \inf_{0 \leq \tau < t} \left| \kappa + \frac{1}{2} \alpha(t) \right|$ and $|\Delta^{(t)}_{n}(\kappa, \tau)| \leq |\kappa|^{n-1} \exp \left( \int_0^\tau |\alpha(s)| \text{d}s \right) |\Delta^{(t)}_{n}(\kappa, \tau)|$. Here we have set $X^{(t)}(t) = e^{\frac{\lambda_t}{2}} \frac{e^{\alpha t}}{\kappa} \sum_{k=0}^{n-1} \frac{A_k^{(t)}(t)}{\kappa^k}$ and the functions $\alpha(t), \beta(t)$ are given as in (3.3). In the last line, we have introduced the shorthand $\Phi(t) := \exp \left( \frac{1}{2} \int_0^t |\alpha(s)| \text{d}s \right)$. Hence for fixed $t$, we find that $|\epsilon^{(t)}_{n}(\kappa, t)| \sim O(\kappa^{-n})$.

At the same time, given the finiteness of the interval $[0, t] \subset \mathbb{R}$ and assuming the analyticity properties of the functions $\alpha(t), \beta(t)$ as in appendix A.2, it is obvious from (3.31) that for

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6 In this case, one can apply Cauchy’s integral formula in order to replace derivatives of $\alpha(t), \beta(t)$ in (3.31) by integral expressions.

7 See the paragraph on convergence at the end of this section.
every $r < \infty$, it holds that $|A^{(t)}_{r+1}(t)|$ is bounded from above, because it is a finite combination of lower order terms.

**Convergence.** In appendix A.1, a general estimate given for $|A^{(t)}_{r+1}(t)|$ is derived which only depends on $r$. There the results of [17] are applied. These results only hold under quite restrictive assumptions on the recursion relation (3.31); in particular, it requires the integration constants $C^a_{n}\beta(t)$ to be identically zero. Moreover, the functions $\alpha(t)$ and $\beta(t)$ are required to be holomorphic in a small strip around the $\Re$-axis in the complex plane, in order to use Cauchy’s integral formula to replace derivatives with respect to $t$ by contour integrals. Unfortunately, these requirements are too restrictive to be applied for $|A^{(t)}_{r+1}(t)|$ for general curves $\gamma$. Interestingly, the requirement $C^a_{n}\beta(t) = 0$ $\forall n > 1$ is fulfilled for spiral arcs, for which however the general explicit solution is known [8].

4.2. Asymptotic behavior of holonomy matrix elements for symmetric connections

In summary, the last subsection has shown the following.

**Lemma 1.** Given the analyticity condition (A.28), (A.30) and quasi-arc-length parametrization

\[\begin{align*}
a(\kappa, t) & \to \sqrt{m(t)/m(0)} \left\{ \cosh \left( \kappa t + \frac{1}{2} \int_0^t \alpha(s) \, ds \right) - \sinh \left( \kappa t + \frac{1}{2} \int_0^t \alpha(s) \, ds \right) \right\} \quad (4.5)
b(\kappa, t) & \to -i \sqrt{m(t)/m(0)} \sinh \left( \kappa t + \frac{1}{2} \int_0^t \alpha(s) \, ds \right)
\end{align*}\]

in the limit $|c| \to \infty$.

Note that this gives the asymptotics only. In order to rederive the property that $|a(\kappa, t)|^2 + |b(\kappa, t)|^2 = 1$, one has to start from the general solutions of section 3.2.3, compute $|a(\kappa, t)|^2$ and $|b(\kappa, t)|^2$ and then take the limit.

An edge $e$ is parametrized by $0 \leq t \leq T$, which allows us to define the inverse edge $e^{-1}(t) := e(T - t)$. Lemma 1 thus implies that there exist $t_2(e) > 0$ and $t_2(e^{-1}) > 0$, where asymptotics (4.5) holds. We can thus restrict our attention to the compact interval $t_2(e)/2 \leq t \leq T - t_2(e^{-1})/2$. For any point $t$ in this interval, we are able to consider $e^+(\lambda) := e(t + \lambda)$ and $e^-(\lambda) := e(t - \lambda)$, and lemma 1 states that there are $t_2^+, t_2^- > 0$ such that asymptotics (4.5) holds for $0 \leq t \leq t_2^+$ on $e^+_e$ and analogously for $e^-_e$, implying that any point $t$ in the interval $t_2(e)/2 \leq t \leq T - t_2(e^{-1})/2$ has an open neighborhood where the asymptotics holds. Hence, by compactness of the interval $t_2(e)/2 \leq t \leq T - t_2(e^{-1})/2$, one can find a finite open covering of this interval where the asymptotics holds for each element of the covering. Thus, the asymptotics of the holonomy is given by a finite number of matrix products the matrix elements of each having asymptotics (4.5), which implies that the holonomy of the entire edge has exponential asymptotics in $\kappa$.

Now we introduce the following.

**Definition 1.** A complex-valued function is called asymptotically almost periodic, iff it can be written as the sum of a continuous almost periodic function and a continuous function that vanishes at infinity and at zero.

**Lemma 2.** The continuous asymptotically almost periodic functions form an algebra over $\mathbb{C}$, and each element splits uniquely into the almost periodic functions plus functions vanishing at infinity.

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8 There are mixing terms which contribute then.
Proof. Let \( f_i \) be asymptotically almost periodic, so there exist continuous almost periodic functions \( a_i \) and continuous functions \( b_i \) vanishing at infinity, such that \( f_i = a_i + b_i \).

Algebra: \( af \) is asymptotically almost periodic, since for \( \alpha \in \mathbb{C} \), \( aa \) is almost periodic and \( ab \) vanishes at infinity. \( f_1 + f_2 \) is asymptotically almost periodic since \( a_1 + a_2 \) is almost periodic and \( b_1 + b_2 \) vanishes at infinity. \( f_1 f_2 \) is asymptotically almost periodic, since \( a_1 a_2 \) is almost periodic and \( a_1 b_2 + a_2 b_1 + b_1 b_2 \) vanishes at infinity.

Uniqueness: Assume that \( f \) is asymptotically almost periodic and there exist continuous \( a_1 \neq a_2 \) almost periodic and continuous \( b_1 \neq b_2 \) s.t. \( f = a_1 + b_1 = a_2 + b_2 \), so \( a_1 - a_2 = b_2 - b_1 \), which implies that \( a_1 - a_2 \) vanishes at infinity; hence \( a_1 = a_2 \). Thus, \( f \mapsto a_1 + b_1 \) is unique.

If \( c = 0 \), the original ODE system (2.3) has constant solutions which are almost periodic. \( \square \)

Let us recall that a gauge-variant spin-network function \( T_\gamma \) is a finite collection of edges \( \gamma = (e_1, \ldots, e_n) \) with a matrix element of an irreducible representation \( \rho \) of the gauge group \( (SU(2) \text{ in the present case}) \) associated with each edge, such that \( T_\gamma (A) = \prod_{i=1}^n \rho^{j_i}(b_i(A))_{m,n} \).

Using that a matrix element of an irreducible representation is a polynomial function of the matrix elements of the fundamental representation, we find using lemma 2.

**Theorem 4.1.** Let \( cA_\gamma \) denote the symmetric connection as described in section 2.1. Given a spin-network function \( T_\gamma \) s.t. each edge \( e_i \in \gamma \) can be written \( e_i = e_{i,1} \circ \ldots \circ e_{i,k} \) such that each \( e_{i,j} \) satisfies the analyticity condition (A.28); then \( T_\gamma (cA_\gamma) \) is an asymptotically almost periodic function of \( c \).

This leads us to the idea to group the set of all spin-network functions \( T_\gamma \) into equivalence classes, where \( T_\gamma \sim T_\gamma' \) are called equivalent if \( T_{\gamma'} (A)|_{cA_\gamma} = T_{\gamma'} (A)|_{cA_\gamma} \), that is, if they coincide when evaluated on a symmetric connection \( cA_\gamma \). In section 5.3, this observation will be used in order to construct an embedding of the extended configuration space of LQC (given by the set \( A \text{AAP} \) of asymptotically almost periodic functions) into \( CYl(\overline{\mathbb{A}}) \), the configuration observable algebra of full LQG.

4.3. Generalization to the full configuration space

Remarkably, the previous construction generalizes straightforwardly to any\(^9\) bounded reference connection \( A_\gamma \): let us consider a one-dimensional linear subspace of \( \overline{\mathbb{A}} \), the space of generalized connections. Let this subspace contain connections of the form \( cA_\gamma \), where \( A_\gamma \) is a bounded \( su(2) \)-valued reference one-form, i.e. its components \( (A_\gamma)_{ji}^k \) are bounded in a trivialization \( A_\gamma = (A_\gamma)_{ij} dx^i \otimes e_j \). Using the notation

\[
\begin{align*}
  n(t) &= (A_\gamma)_{ij}^2 (e(t)) \xi^i(t) \\
  m(t) &= \left( (A_\gamma)_{ij}^1 (e(t)) - i (A_\gamma)_{ij}^2 (e(t)) \right) \xi^i(t),
\end{align*}
\]

we impose the ‘quasi arc-length’ parametrization condition \( n^2(t) + |m(t)|^2 = 1 \), such that the holonomy ODE (2.1) becomes

\[
\dot{w}(t) = M(t) \dot{w}(t) + (ic(n(t) - M(t)n(t)) - c^2)w(t),
\]

where \( M(t) = \frac{\dot{m}(t)}{m(t)} \) and \( w = a \) or \( b \) with the initial condition \( a(0) = 1 \), \( b(0) = 0 \). We thus have a tool to investigate the compactification of the configuration space of LQG in a general ‘radial’\(^10\) through the induced Gel’fand topology on the radial coordinate \( c \). To

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\(^9\) Not necessarily symmetric.

\(^10\) We understand radial in the sense of a fictitious Banach space.
rephrase this: every gauge-variant spin-network function $T_r$ exhibits an asymptotically almost periodic dependence on $e$ if the connection $A$ is parametrized\footnote{For a general connection $A \in \mathcal{A}$ with bounded components $A^k(x) < \infty \ \forall x \in \Sigma$ and $\forall a,i = 1,2,3$, in a chosen trivialization, choose e.g. $c := \sup_{x,a,i} |A^k(x)|$ and introduce $(A^k_a(x)) := e^{-1} A^k_a(x)$.} as $cA_x$.

5. Quantum symmetry reduction

To construct a quantum embedding, we will follow the construction outlined in [18]. This construction allows one to canonically construct a reduced quantum theory from a full quantum theory such that the matrix elements of the two theories are guaranteed to match up. In the simplest case, one considers a Schrödinger representation of a quantum observable algebra $\mathcal{A}$ represented on $\mathcal{H} = L^2(\mathbb{R}, dm)$ and an embedding $i : \mathcal{X}_o \to \mathbb{R}$ of a reduced configuration space $\mathcal{X}_o$ (consisting of symmetric configurations) into the full compact quantum configuration space $\mathcal{X}$. If one finds a linear map $r : C(\mathcal{X}_o) \to C(\mathbb{R})$ such that $i^* r = id_{C(\mathcal{X}_o)}$, then one can construct ket–bra operators for the reduced theory and an induced representation using the pair $(i, r)$. The operators in the reduced theory turn out to have a canonical interpretation in terms of operators in the full theory.

We will later perform this construction for flat FRW cosmology, where it is straightforward to work out $i$ as the evaluation at symmetric connections. It thus suffices to construct the map $r$ in order to obtain an explicit quantum symmetry reduction.

5.1. Symmetric quantum connections

A classical connection $\omega$ (on a trivial bundle) is determined by the connection one-form $A = \sigma^* \omega$ obtained as its pullback under a section $\sigma$. A connection one-form then defines a homeomorphism from the groupoid $\mathcal{P}$ of piecewise analytic paths to the gauge group $\mathcal{G}$ by

$$A : \mathcal{P} \to G, \quad e \mapsto h_e(A),$$

where $h_e$ denotes the holonomy along the path $e$. A quantum connection is thus conveniently defined as an element of $\text{Hom}(\mathcal{P}, \mathcal{G})$, which imposes no further restriction on the homeomorphism. This is in contrast to a classical connection where differentiability allows us to reconstruct the connection components unambiguously from a much smaller subgroupoid $\mathcal{P}_0$, which is local and can be constructed using in each chart $(U^d, [\phi^d]_{i=1}^d)$ as follows.

The base space is $\mathcal{P}_0 = U$. At each point $x_0 \in U^d$, we consider the $d$ linearly independent coordinate directions $d\phi^a$ and the $d$ corresponding families of integral curves $e^a_{\nu,i} : (0, t) \to U^d$ which we consider as arc-length parametrized in the Euclidean metric on $\mathbb{R}^d$. The elements of the groupoid $\mathcal{P}_0$ then consist of all finite concatenations of these integral curves modulo zero paths and the source map is $s(e^a_{\nu,i}) = x_0$ and range map is $r(e^a_{\nu,i}) = e^a_{\nu,i}(t)$. The components of a classical connection are then by construction completely determined in terms of the restriction to $\mathcal{P}_0$ through the holonomy differential equation

$$A^a_\nu(x_0) r^t = \partial_t h_{e^a_{\nu,i}}(A) \big|_{t=0}.$$

Let us now consider the subspace $\mathcal{S}$ of classical connections that are invariant under the action of a symmetry group $\mathbb{H}$. We can use equation (5.2) to define a classical symmetric connection (tautologically) as follows.

**Definition 2** (Classical symmetry). A morphism from $A : \mathcal{P} \to \mathcal{G}$ is called classically $\mathbb{H}$-symmetric iff there exists a diffeomorphism $\phi$ and gauge transformation $g$ and an element $B \in \mathcal{S}$ s.t. $A(e) = g^{-1}(s(e)) h_{\phi(e)}(B) g(r(e))$ for all $e \in \mathcal{P}_o$.\footnote{For a general connection $A \in \mathcal{A}$ with bounded components $A^k(x) < \infty \ \forall x \in \Sigma$ and $\forall a,i = 1,2,3$, in a chosen trivialization, choose e.g. $c := \sup_{x,a,i} |A^k(x)|$ and introduce $(A^k_a(x)) := e^{-1} A^k_a(x)$.}
Since $\mathcal{P}_o$ is a subgroupoid of $\mathcal{P}$, we can immediately apply this definition to quantum connections. However, using this definition does not seem to capture a useful notion of symmetry for quantum connections as can be seen from a simple example: let $B \in S$ and $\overline{B} \notin S$ and define the homeomorphism as the extension by groupoid composition of a generating set that contains a generating set of $\mathcal{P}_o$ and generators that cannot be decomposed into elements of $\mathcal{P}_o$:

$$A : e \mapsto \begin{cases} h_e(B) & e \in \mathcal{P}_o \\ h_e(\overline{B}) & e \in \overline{\mathcal{P}_o} \end{cases}$$ (5.3)

where $\overline{\mathcal{P}_o}$ denotes all elements of $\mathcal{P}$ which do not admit nontrivial decomposition that contains an element of $\mathcal{P}_o$. This quantum connection does not appear symmetric for almost all paths, except for the ones in $\mathcal{P}_o$. We are thus led to use a stronger definition of symmetry for quantum connections.

**Definition 3** (Quantum symmetry). A morphism $A : \mathcal{P} \to G$ is called quantum $\mathbb{H}$-symmetric iff there exist a diffeomorphism $\phi$ and gauge transformation $g$ and an element $B \in S$ s.t.

$$A(e) = g^{-1} s(e) h_{\phi(e)}(B) g(r(e))$$ for all $e \in \mathcal{P}$.

These two definitions of symmetry applied to the morphism defined by the holonomy of a classical connection coincide due to the invertability of (5.2), but weed out strange quantum connections as the one defined through equation (5.3).

Note that usual LQC is motivated by a model of LQG defined on $\mathcal{P}_o$, so the difference between these two definitions does not occur there and one usually uses definition 2.

### 5.2. Alternative notion of symmetry

We will subsequently use the notion of quantum symmetry in definition 3, because it is closest to the standard interpretation of LQC and diffeomorphism covariant. However, ideally one would like a manifestly diffeomorphism-invariant notion of symmetry. To obtain such a notion, we switch for simplicity to metric canonical variables $(g_{ab}, \pi^{ab})$ and consider the metric configuration variable. Our aim is to find a manifestly diffeomorphism-invariant description of flat homogeneous metrics.

This can be done by observing that locally (i.e. up to nonlocal integration constants) the diffeomorphism class of a metric is given by three scalars, which can be chosen to be $\phi_1(x) = R(x)$, $\phi_2(x) = R_{ab}(x)R^{ab}(x)$ and $\phi_3(x) = \frac{\det(R(x))}{\det(g(x))}$. A flat homogeneous metric satisfies

$$\phi_i(x) = 0.$$ (5.4)

This condition is manifestly diffeomorphism invariant, because the functions $x \mapsto \phi_i(x)$ are scalar. This allows us to compare the flat homogeneous sector of full LQG with the flat homogeneous sector of piecewise linear LQG, because the two theories coincide at the diffeomorphism-invariant level. One would naively expect that quantizing the analog of equation (5.4) in Ashtekar–Barbero variables in both standard LQG and piecewise linear LQG would lead to the same result, because the two theories coincide at the diffeomorphism-invariant level if the suitable extension of the diffeomorphism group is chosen on both sides. But the issue is more subtle than that, because to be able to use this argument one needs the quantization of equation (5.4) to be unique\(^{12}\). This is by no means guaranteed and we have to expect to encounter quantization ambiguities even in this very sophisticated program and we

\(^{12}\) One also needs to find a quantization of equation (5.4) that transforms covariantly under the particular extensions of the diffeomorphism group.
would ultimately have to resort to experimental predictions to determine whether any of the quantizations describes nature.

Similarly, whether nature is described by full LQG or piecewise linear LQG or neither has to be decided by experiment. It is thus worthwhile to explore possible differences between the two theories.

5.3. An explicit quantum symmetry reduction

The configuration algebra of standard LQC is the closed span of exponential functions of the connection parameter $c$, so the exponential functions are a dense (using the $\| \cdot \|_\infty$-norm) subset of the configuration operators. The assumed relation $r$ between the exponential functions $e^{ict}$ in standard LQC with full LQG configuration operators is given by the observation that holonomy matrix $h_t$ elements along straight edges of the length $t$ exhibit this exponential dependence on $c$, so $r(e^{ict}) = h_t$. This relation extends straightforwardly to the span of the exponential functions by linearity. To extend it to the closed span however, one needs to assume a bound $\| r(f) \|_{\text{LQG}} \leq \| f \|_\infty$ to ensure convergence.

In this first attempt, we set difficulties concerning issues of convergence and completion aside for now, but caution that one might be forced to amend the definition of $r$ to extend it to the closure. We will denote by $C_c(\mathbb{R})$ functions of compact support and by $C_0(\mathbb{R})$ functions which vanish at infinity.

The construction strategy for the quantum embedding follows [18]: let $A(c)$ be the embedding of the symmetry-reduced configuration space, parametrized by $c$ into the space of all connections, so $A$ can be used to pull back any configuration observable (in particular cylindrical functions) $T(A)$ to a function $T(A(c))$ on the symmetry-reduced configuration space, which defines equivalence classes of configuration observables in the full theory whose pullback to the symmetry-reduced configuration space coincides. Thus, a symmetry reduction is encoded in these equivalence classes. To relate the symmetry-reduced model to the full theory, one constructs an embedding of the symmetry-reduced configuration observables $f(c)$. For this, we have to find a representative $r(f)$ in the full theory for each equivalence class, such that $r(f) |_{A(c)} = f(c)$.

The difference between our present treatment and standard LQC is that our treatment forces us not to restrict ourselves to continuous almost periodic functions $f_c \in AP(\mathbb{R})$, but to continuous asymptotically almost periodic functions $f \in AAP(\mathbb{R})$. Given a continuous asymptotically almost periodic function $f \in AAP(\mathbb{R})$, we can (by definition) find a pair $f_c \in AP(\mathbb{R})$, $f_0 \in C_0(\mathbb{R})$ s.t. $f = f_c + f_0$. Lemma 2 provides that the pair $f_c$, $f_0$ is unique. In order to construct an explicit embedding of $AAP(\mathbb{R})$ into $Cyl(A)$, the configuration space of full LQG, we now construct a generating set of functions, which allows us to describe every $f \in AAP(\mathbb{R})$ without referring to the explicit series (3.30). For this, we start with the definition of some functions (these are illustrated in figure 2).

For each $a \leq b$ let us define $\chi_{a,b} \in C_0(\mathbb{R})$ through

$$\chi_{a,b}(x) := \begin{cases} 0 & : x < a - 1 \\ x - a + 1 & : a - 1 \leq x < a \\ 1 & : a \leq x < b \\ b - x + 1 & : b \leq x < b + 1 \\ 0 & : x \geq b + 1 \end{cases} \quad (5.5)$$

13 It is in particular not obvious whether the completion of the embedding in the full LQG-norm lies within the Hilbert space of full LQG.
For each \(a \leq b\) and \(f \in C_c(\mathbb{R})\) with \(\text{supp}(f) \subset [a, b]\), we introduce the continuous \((b-a+2)\)-periodic function \(f_{a,b} \in AP(\mathbb{R})\) defined through its values at the defining interval,

\[
\begin{align*}
f_{a,b}(x) := \begin{cases} 
0 & : a - 1 < x \leq a \\
f(x) & : a < x \leq b \\
0 & : b < x < b + 1 
\end{cases},
\end{align*}
\]

and extended to all of \(\mathbb{R}\) by \((b-a+2)\) periodicity\(^{14}\). For a specific positive function \(g(x)\) and each \(A < a - 1, a < b, b + 1 < B\), we define the continuous \((B - A)\) periodic function \(G_{a,b}^A \in AP(\mathbb{R})\) defined on the initial period \(A \leq x < B\) as

\[
\begin{align*}
G_{a,b}^A(x) := \begin{cases} 
0 & : A \leq x < a - 1 \\
\frac{\int g(x) \chi_a(x)}{\int g(x)} & : a - 1 \leq x < b + 1 \\
0 & : b + 1 \leq x < B 
\end{cases},
\end{align*}
\]

and extended to all of \(\mathbb{R}\) by \((B - A)\) periodicity. Let us furthermore assume \(g \in C_c(\mathbb{R})\) s.t. \(\forall x : g(x) > 0\) and \(\exists g_o > 0 : g(x) \leq \frac{g_o}{|x|}\) for \(|x| > x_o\).

**Lemma 3.** For every \(f \in AAP(\mathbb{R})\), there is a Cauchy sequence of sums of exponential functions plus \(g\) times sums of exponential functions converging to \(f\) in \(\|\cdot\|_\infty\).

**Proof.** We use that there is a unique split \(f = f_a + f_o\) with \(f_a \in AP(\mathbb{R})\) and \(f_o \in C_c(\mathbb{R})\). Using the density of the exponential functions \(AP(\mathbb{R})\), we find a Cauchy sequence of finite sums of exponential functions converging to \(f_o\).

For \(f_a\) we use that for every \(\epsilon > 0\), there exists \(a, b \in \mathbb{R}\) and \(\tilde{f} \in C_c(\mathbb{R})\) with \(\text{supp}(\tilde{f}) \subset [a, b]\) s.t. \(\|f - \tilde{f}\|_\infty \leq \epsilon\). Using \(\tilde{f} = \chi_{a,b} f_{a,b}\), we find that for every \(\epsilon > 0\), there exist \(a, b\) and \(\tilde{f}_{a,b} \in AP(\mathbb{R})\) s.t. \(\|f - \chi_{a,b} f_{a,b}\| \leq \epsilon\). Moreover, using \(g(x) > 0\) and \(g(x) \leq \frac{g_o}{|x|}\) for \(x > x_o\), we find for \(\epsilon > 0\) that there exist \(A, B \in \mathbb{R}\) s.t. \(\|g G_{a,b}^A - \chi_{a,b}\|_\infty \leq \epsilon\).

Hence, from

\[
\|f_o - g G_{a,b}^A \tilde{f}_{a,b}\|_\infty = \|f_o - (g G_{a,b}^A - \chi_{a,b}) \tilde{f}_{a,b}\| \leq \epsilon (1 + \|\tilde{f}_{a,b}\|_\infty)
\]

and \(\|f_o\|_\infty \leq \|f_o\|_\infty + \epsilon\) and the density of the exponential functions in \(AP(\mathbb{R})\), we find that for every \(\epsilon' > 0\) there exists a sum of exponentials \(h\) yielding \(\|f_o - gh\|_\infty \leq \epsilon'\). \(\square\)

**An explicit embedding.** Now we use this result in order to construct an explicit embedding of \(AAP(\mathbb{R})\) into \(Cyl(A)\) (the configuration space of full LQG). For this, we use the spiral-arc

\(^{14}\)That is, we just successively concatenate copies of the defining interval along the real axis in order to obtain a periodic function.
solution of section 2.2 and the defining representation of $SU(2)$. First we construct a function $g(c) > 0$ with $g(c) \leq \frac{\alpha_{\text{cyl}}}{c^2}$ for $c > c_0$. Recall solution (2.11)
\[
a(\kappa, t) = e^{\frac{1}{2}i\kappa t} \left( \cos(\Delta t) + \frac{1}{2\Delta}(2\kappa c - \lambda) \sin(\Delta t) \right),
\]
where $\Delta = \left[ \frac{e^2}{4} + c(c - \nu \lambda) \right]^{1/2}$ and $\lambda, \nu \in \mathbb{R}, \mu \in \mathbb{C}, \nu^2 + |\mu|^2 = 1$. Now denote the explicit dependence on the parameters $\mu, \nu, \lambda, t$ as subscripts $a_{\mu, \nu, \lambda, t}$ (because $a$ is independent of $\mu$) and compute
\[
B_{\mu, t}(c) = |a_{0, 0, t}(c) - e^{\pm i\delta}a_{0, \lambda, t}(c)|^2
\]
\[
= \frac{\lambda^2}{4c^2 + \lambda^2} \sin^2 \left( \sqrt{c^2 + \frac{\lambda^2}{4}t} \right) + \left( \cos(ct) - \cos \left( \sqrt{c^2 + \frac{\lambda^2}{4}t} \right) \right)^2
\]
\[
(5.8)
\]
as well as
\[
A_{t}(c) = a_{1, 0, t}(c) = e^{i\pi t}.
\]
We verify for $t > 0$ and $\lambda > 0$ that $B_{\mu, t}(c) = O(c^{-2})$ for $c \to \infty$, by inspecting both summands separately: First, $\frac{\lambda^2}{c^2 + \lambda^2} \sin^2 \left( \sqrt{c^2 + \frac{\lambda^2}{4}t} \right) = O(c^{-2})$, because the modulus of the sine is bounded by 1 and the quotient is $O(c^{-2})$ as $c \to \infty$. Second, $\left( \cos(ct) - \cos \left( \sqrt{c^2 + \frac{\lambda^2}{4}t} \right) \right)^2 = O(c^{-2})$ because expanding the square root inside the argument of the second cosine around 1 cancels to first order the first cosine and the remainder is of $O(c^{-1})$ which is squared to yield $O(c^{-2})$. Moreover, for $\lambda = 2$ and $t = 1$, we verify $B_{2, 1}(c) > 0$ as follows: both summands are squares and thus positive semidefinite. The first summand vanishes for $c = \frac{1}{2}k\pi - \frac{\pi}{2}$ for integer $k$, while inserting these values for $c$ into the argument of the sine is incompatible with $\frac{\pi}{2}$ for the integer $n$. Hence, $B_{2, 1}(c)$ can be used for the construction in the proof of lemma 3.

To construct cylindrical functions in LQG whose restriction to homogeneous isotropic connections reduces to functions (5.9) and (5.8) of the homogeneous isotropic connection parameter $c$, we realize $a$ as the holonomy matrix element $h_{11}$ and $\delta$ as the holonomy matrix element $h_{22}$ in the fundamental representation. Moreover, we fix three general points $\xi, \nu, \zeta$, which we denote as superscripts e.g. $h_{\nu}^\xi$, and specify spiral parameters $\nu, \lambda, t$, which we denote as arguments $h_{\nu}^{\lambda, t}(\nu, \lambda, t)$ and define the embedding map $r$ as the linear extension of
\[
r(1) = 1
\]
\[
r(e^{i\nu}) = h_{11}^\nu(1, 0, t)
\]
\[
r(B_{2, 1}(c)e^{i\nu}) = h_{11}^\nu(1, 0, t)\left( h_{11}^\nu(0, 0, 1) - e^{-i\nu}h_{11}^\nu(0, 2, 1) \right)\left( h_{22}^\nu(0, 0, 1) - e^i\nu h_{22}^\nu(0, 2, 1) \right),
\]
\[
(5.12)
\]
where $t \neq 0$ in (5.11) and the 1 in (5.10) is understood as the unit function.

*An explicit measure for the symmetry-reduced sector.* Let us now consider the image of $r$ within the LQG inner product: observing that $r(e^{i\nu})$ is (upon normalization) a gauge-variant spin-network function on a different graph for different $t$, we conclude
\[
\frac{1}{|r(e^{i\nu})|_{\text{LQG}}|r(e^{i\nu})|_{\text{LQG}}} (r(e^{i\nu}), r(e^{i\nu}))_{\text{LQG}} = \delta_{t, t}
\]
\[
(5.13)
\]
where we used $|T|_{\text{LQG}} = \sqrt{(T, T)_{\text{LQG}}}$ and the Kronecker delta $\delta_{t, t}$. Moreover, observing that $r(B_{2, 1}(c)e^{i\nu})$ is a linear combination of gauge-variant spin-network functions on a graph always different from $r(e^{i\nu})$ and for $t \neq 0$ on a different graph than $r(B_{2, 1}(c)e^{i\nu})$, we find
\[
\frac{1}{|r(B_{2, 1}(c)e^{i\nu})|_{\text{LQG}}|r(e^{i\nu})|_{\text{LQG}}} (r(B_{2, 1}(c)e^{i\nu}), r(e^{i\nu}))_{\text{LQG}} = 0
\]
\[
(5.14)
\]
\[
\frac{1}{|r(B_{2,1}(c) e^{ie c})|_{\text{LQG}}} \langle r(B_{2,1}(c) e^{ie c}), r(B_{2,1}(c) e^{ie c}) \rangle_{\text{LQG}} = \delta_{1,1}^K. \tag{5.15}
\]

We have thus established that orthogonality of our generating set and can return to the question of completion: for this purpose, let us introduce the induced norm \( f \) in the span of the generating set:

\[
\|f\|_{\text{ind}} := |r(f)|_{\text{LQG}}. \tag{5.16}
\]

which we can freely use to construct a completion of the span of the generating set. However, we have to caution that we were not able to prove that the completions in \( \|\cdot\|_{\infty} \) are contained in the completion using \( \|\cdot\|_{\text{ind}} \). Hence, we cannot exclude the possibility that the embedding map \( r \) needs to be amended.

5.4. Asymptotic analysis and the BKL picture

Let us briefly discuss some physical consequences of the construction described in the previous two subsections.

The BKL-conjecture [5–7] suggests that the generic evolution of GR near a singularity is such that spatial derivatives become negligible compared to time derivatives, i.e. each point in space evolves separately as a homogeneous cosmology, so the understanding of cosmological models seems to hint toward an understanding of the evolution near a generic singularity. Since large time derivatives of the metric imply large values of the Ashtekar connection in a fixed trivialization and a fixed chart, the first approach for the investigation of the behavior of LQG observables near a generic singularity is to investigate the asymptotic dependence of observables and in particular spin-network functions on homogeneous connections. Thus, our result, stating that the dependence of a spin-network function \( T(A) \) on the homogeneous scaling parameter \( c \) of a connection \( A = cA_\star \) becomes asymptotically almost periodic as \( c \to \infty \), suggests that the generic behavior of spin-network functions near a singularity is well approximated by an almost periodic dependence on the scaling parameter \( c \). This scenario has been investigated in the context of LQC [19–22].

However, in light of the non-embeddability of standard LQC into LQG, one expects modifications to the standard LQC dynamics, since it is expected that quantum corrections will become important not at infinite momentum scale, but rather at momentum scale of order 1 in Planck units; one expects from dimensional analysis of \( T(cA_\star) = \psi(c)(1 + \mathcal{O}(1/c)) \), where \( \psi \) is almost periodic, that the \( \mathcal{O}(1/c) \) corrections are not necessarily negligible when approaching the Planck scale and become negligible only when going far beyond the Planck scale. It is thus possible that the use of definition 3 could lead to observable deviations from results derived using definition 2. However, the existence of quantum connections of the type defined in equation (5.3) suggests to use definition 3.

Note that the difference between the two definitions is not resolved when using piecewise linear LQG, because \( \tilde{P}_\rho \) in equation (5.3) is still larger than \( P_\rho \).

Let us conclude this discussion with the remark that in order to derive quantitative differences between embeddable and non-embeddable LQC, one needs to choose a particular effective Hamilton constraint. This goes far beyond the scope of this paper, but a naive estimate of the difference can be made as follows: the simplest version LQC modifies the classical connection parameter \( c \) to \( \frac{1}{\beta} \sin(\beta c) \) and then quantizes using the modified variables as fundamental building blocks. In light of the present discussion, one should however rather use a modification of the kind

\[
c \to \frac{1}{\beta} \left( \sin(\beta c) + \beta \mathcal{O}\left(\frac{1}{\beta c}\right) \right) \tag{5.17}
\]
to take the asymptotic behavior of full LQG holonomies into account. The leading order is thus standard LQC and the content of the paper can be summarized as the reassuring statement: ‘Standard LQC coincides asymptotically with embeddable LQC and thus captures the universal holonomy modification.’

6. Conclusions and outlook

Loop quantum cosmology (LQC) is used as a tool for the study of conceptual issues in full loop quantum gravity (LQG), because it is a simple toy model that exhibits many features of the full theory. Moreover, since on the one hand it is expected that quantum effects of gravity are most significant in the vicinity of GR singularities and on the other hand since the BKL conjecture [5–7] states that the dynamics of GR in the vicinity of a generic singularity is well approximated by decoupled homogeneous cosmologies, one expects that LQC provides insight into the behavior of LQG where its effects are expected to be the most important.

However, to be able to draw such conclusions, one needs to understand the relation between full LQG and the symmetry-reduced model. The nature of this relation is not as obvious as one might think, because standard LQC is obtained as a ‘loop quantization’ of a classical symmetry reduction of GR and not as a symmetry reduction of LQG. Moreover, one finds that the structures of the two theories do not match up: in particular, one can show that the embedding of the configuration space of LQC into the configuration space of LQG is not continuous [8]. Since LQG is a rather restrictive field theoretic framework that does not admit arbitrary changes, we are led to the investigation of symmetry-reduced models of LQG in this paper. In other words, we assume standard LQG rather than piecewise linear LQG as the underlying full theory.

It turns out that the problem of non-embeddability of standard LQC into full LQG stems from the restriction to piecewise linear curves in the ‘loop quantization’ of homogeneous cosmology, which in turn implies that the holonomy matrix elements, which are the building blocks of the configuration operators in the ‘loop quantization’ program, are linear combinations of exponential functions of the symmetric connection parameter. This motivated our first Ansatz to start with standard LQC, but to allow for small perturbations of the linear curves used in the construction. To do this, we formally solved the holonomy differential equation as a power series in the perturbation parameter, such that the first term in the series is precisely the unperturbed linear curve. This provides an intuitive understanding of the relation of standard LQC and its embeddable extension. Moreover, it turns out that the formal solution cannot converge for large values of the symmetric connection parameter unless the perturbation parameter scales with the inverse of the symmetric connection parameter.

This inverse scaling motivated our second Ansatz, the asymptotic expansion of the solution to the holonomy differential equation as an inverse power series in the symmetric connection parameter. The resulting Liouville–Green series is in general divergent, but one can show that the truncated Liouville–Green series captures the asymptotic behavior correctly. It follows that the solutions to the holonomy differential equation can be characterized as a sum of a continuous periodic function and a continuous function that vanishes at infinity, implying that the algebra generated by polynomials thereof can be characterized as a continuous asymptotically almost periodic function. It turns out that this result generalizes from symmetric connections to arbitrary one-parameter families of the connections of form $A = cA_\alpha$, where spin-network functions can be shown to be asymptotically almost periodic functions of the parameter $c$.

In the following, we provide a definition of a symmetric quantum connection that does not tacitly assume classical smoothness properties of a quantum connection. The asymptotic
almost periodic dependence of configuration operators on the symmetric connection parameter shows that one has to include at least one nonlinear curve in the construction of an embeddable symmetry-reduced model of LQG and also allowed us to give an explicit construction of an embeddable symmetry-reduced model of full LQG, which contains standard LQC and a ‘minimal’ extension. This extension consists, in accordance with [1], of configuration observables that vanish both for large and small values of the symmetric connection parameter.

The most likely physical interpretation of our result is that the extension does not change the generic physical behavior in the limit of the infinite symmetric connection parameter that was established in LQC, because it consists of functions that vanish for large connection parameters. Thus, elements of the extension are not expected to affect the behavior of the model in the limit of the infinite connection parameter. Specific physical predictions are however likely to change, because quantum gravity effects are expected to play an important role not at the infinite connection parameter but at values of order 1 (or even less) in Planck units, where the configuration operators introduced through the extension do not vanish. In particular, if we are using the usual configuration representation, there are wavefunctions in the extension that vanish for the large connection parameter, or physically speaking near the cosmological singularity, which are to be expected to yield very different physical behavior than almost periodic wavefunctions.

6.1. Outlook

The results presented in this paper shed new light on the conceptual question: What symmetry means at the level of the kinematical LQG-Hilbert space \( \mathcal{H}_0 \) and how symmetric states can be constructed thereon? The novel answer we get from our analysis is that every specifically chosen symmetric connection \( c_{A^*} \) induces a decomposition of \( \mathcal{H}_0 \) into equivalence classes of cylindrical functions, where two cylindrical functions are called equivalent, if their restriction to \( c_{A^*} \) coincides. Hence, we can take the opposite point of view and define a particular symmetry as an analogous decomposition of \( \mathcal{H}_0 \) into equivalence classes of cylindrical functions.

Starting from this observation, there are several future directions for investigation.

An important task is to examine the mathematical properties of the symmetry-equivalence classes in more detail. The ultimate goal here is to work out the general construction of a symmetric Hilbert space \( \mathcal{H}_{\text{sym}} \) by taking the quotient of \( \mathcal{H}_0 \) and the equivalence relation\(^{15}\). It will be important to generalize the construction of section 5.3 and to see how a measure on \( \mathcal{H}_{\text{sym}} \) can be constructed and to understand whether the symmetry-equivalence classes preserve the inductive structure of \( \mathcal{H}_0 \).

Secondly, given the explicit embedding of the extended LQC configuration space into \( \mathcal{H}_0 \) as described in section 5.3, we have already proved that the \( \text{AAP}(\mathbb{R}) \) functions form an algebra. However, one needs to work out the action of flux operators of full LQG on a particular embedding of \( \text{AAP}(\mathbb{R}) \) in \( \mathcal{A} \). Does this action preserve the symmetric sector and moreover the symmetry-equivalence classes in \( \mathcal{H}_0 \)? If it does, the action of geometric operators and ultimately the constraint operators on the symmetry-equivalence classes has to be analyzed in detail. These questions can be investigated very naturally using the results of this paper, because they suggest to define symmetric operators as operators on the LQG-Hilbert space as follows: recall that we have an equivalence relation \( \sim \) of cylindrical functions through the pullback under \( i : c \mapsto c_{A^*} \), such that any cylindrical function can be written as the sum of

\(^{15}\) This generalizes to very general symmetries (e.g. discrete symmetries) and the questions raised here are applicable in these cases as well.
a fixed representative \( f \) in the equivalence class plus a function \( f^\circ \) whose pullback under \( i \) vanishes. We then call an operator \( O \) symmetric if it satisfies
\[
\langle \text{Cyl}_1, O\text{Cyl}_2 \rangle = \langle \text{Cyl}_1 + f^\circ_1, O(\text{Cyl}_2 + f^\circ_2) \rangle,
\]
for all cylindrical functions \( \text{Cyl}_1 \) and cylindrical functions \( f^\circ_1 \) whose pullback under \( i \) vanishes. Note that this relation is by construction linear, meaning that if \( O_1 \) and \( O_2 \) are symmetric, then \( O_1 + \alpha O_2 \) for \( \alpha \in \mathbb{C} \) is symmetric as is \( O_1 O_2 \).

A third very important goal is to analyze the effect the extension of the configuration space of LQC from \( \mathbb{R}^{\text{Boh}} \) to the spectrum \( \Delta_1(\text{AAP}(\mathbb{R})) \) has. Certainly the first goal is to construct a measure on the extended Hilbert space. Besides starting from the extended configuration space itself, there is the possibility of starting from our explicit embedding and its usage of the kinematical inner product on \( \mathcal{H}_0 \) using the Ashtekar–Lewandowski measure. As described above, one might be able to define a measure on \( \mathcal{H}_0 \) and to use it for the extended version of LQC. Then one is able to work out the consequences for the physical predictions of LQC.

Lastly, there are some purely mathematical questions: from the viewpoint of differential geometry, it will be interesting to see how our procedure (described in section 4.3) for computing the holonomy of a connection in the asymptotic regime of a large rescaling can be generalized to connections on arbitrary bundles. Moreover, it is necessary to work out its relation to the curvature of the connection as discussed in appendix C.3 in more detail. From a complex analysis viewpoint, one is led to ask when solutions to the holonomy equations are analytic in \( c \). This could be investigated e.g. through extending the constructions of section 5.3 to complex \( c \) and applying results of [17]. From a functional analytic viewpoint, one needs to address the convergence properties of the embedding map, e.g. through proving bounds that imply convergence.

Acknowledgments

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Appendix A. Boundedness of the formal solution

By construction, series (3.30) provides a formal solution to ODE (3.4). However, it is not clear whether this series converges. We thus truncate it and investigate the rest term
\[
\epsilon_{n>0}(\kappa, t) := d(\kappa, t) - d(\kappa, t) + \sum_{k=0}^{n-1} A_k(t),
\]
separately. Insertion into the holonomy ODE yields
\[
\epsilon_{n}(\kappa, t) = (\kappa^2 + \kappa \alpha + \beta) \epsilon_{n}(\kappa, t) + \frac{d}{d t} \epsilon_{n}(\kappa, t) \left( \beta A_{n-1} - \left( \frac{X}{x} A_{n-1} + 2 \frac{X}{x} A_{n-1} + \frac{X}{x} A_{n-1} \right) \right),
\]

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\]
where $\Delta_n(k, t) = \varphi(\epsilon(t))X_n(t) - \frac{2\Delta(t)}{\epsilon(t)}$. To investigate the asymptotic behavior of $\epsilon_n(k, t)$, we construct a fundamental system with Wronskian $\varphi$ that solves equation (A.2) to all positive powers in $k$.

$$Z_{(\sigma)}(k, t) = \frac{e^{\sigma(kt + \frac{1}{2} \int_0^t a(\tau) d\tau)}}{\sqrt{k + \frac{1}{2} \sigma(t)}},$$

(A.3)

with $\sigma = \pm 1$, which satisfies

$$\dot{Z}_{(\sigma)} = (k^2 + \kappa \alpha + \gamma)Z_{(\sigma)},$$

(A.4)

where $\gamma(k, t) = \frac{2a(t)}{4(k + \frac{1}{2} a(t))} - \frac{\dot{a}(t)}{4(k + \frac{1}{2} a(t))}$. Variation of the constants of

$$\epsilon_n(k, t) = \mu_n^{(t)}Z_{(+)} + \nu_n^{(t)}Z_{(-)}$$

yields

$$\dot{\mu}_n^{(t)}Z_{(+)} + \dot{\nu}_n^{(t)}Z_{(-)} = (\beta - \gamma)\epsilon_n^{(t)} + \Delta_n^{(t)},$$

(A.5)

yielding for $\epsilon_n^{(t)}(k, t)$

$$\epsilon_n^{(t)}(k, t) = \epsilon_n^{(t)}(k, 0) + \int_0^t K(t, \tau, k)\chi(k, \tau)\epsilon_n^{(t)}(k, \tau) + \Delta_n^{(t)}(\tau) d\tau,$$

(A.6)

where $K(t, \tau, k) = \frac{1}{2}(Z_{(+)}(k, t)Z_{(-)}(k, \tau) - Z_{(-)}(k, t)Z_{(+)}(k, \tau))$ and $\chi(k, \tau) = \beta(\tau) - \gamma(k, \tau)$. Imposing the initial condition $\epsilon_n^{(t)}(k, 0) = 0$, $\epsilon_n^{(t)}(k, 0) = 0$ on the integral equation, we can construct a series solution given by $\epsilon_n^{(t)} = \sum_{m=0}^{\infty} (\epsilon_{n,m}^{(t)} - \epsilon_{n,m}^{(t)})$, where the summands are determined by

$$\epsilon_{n,0}^{(t)}(k, t) = 0,$$

(A.7)

$$\epsilon_{n,1}^{(t)}(k, t) = \int_0^t d\tau K(t, \tau, k)\Delta_n^{(t)}(k, \tau),$$

$$\epsilon_{n,m+1}^{(t)}(k, t) - \epsilon_{n,m}^{(t)}(k, t) = \int_0^t d\tau K(t, \tau, k)\chi(k, \tau)(\epsilon_{n,m}^{(t)}(k, \tau) - \epsilon_{n,m+1}^{(t)}(k, \tau)).$$

To find a bound on this series, we impose

$$\begin{align*}
|K(t, \tau, k)| &\leq p(k, t) q(\tau) \\
K_n(k) &:= \sup_{0 \leq \tau \leq l}[|q(\tau)|] \\
K_n(k) &:= \sup_{0 \leq \tau \leq l}[|p(k, \tau)| q(\tau)] \\
\phi_n^{(t)}(k, t) &:= \int_0^t |\Delta_n^{(t)}(k, \tau)| d\tau \\
\psi(k, t) &:= \int_0^t |\chi(k, \tau)| d\tau,
\end{align*}$$

(A.9)

where $p, q \geq 0$, such that we can show

$$\begin{align*}
\frac{|\epsilon_{n,m+1}^{(t)}(k, t) - \epsilon_{n,m}^{(t)}(k, t)|}{p(k, t)} &\leq K \phi_n^{(t)}(k, t) \frac{K_n^m \psi_m(k, t)}{m!} \\
\frac{|\epsilon_{n,1}^{(t)}(k, t) - \epsilon_{n,0}^{(t)}(k, t)|}{p(k, t)} &\leq K \phi_n^{(t)}(k, t) \frac{K_n \psi(k, t)}{2} \\
\frac{|\epsilon_{n,1}^{(t)}(k, t) - \epsilon_{n,0}^{(t)}(k, t)|}{p(k, t)} &\leq K \phi_n^{(t)}(k, t) \frac{K_n \psi(k, t)}{2}.
\end{align*}$$

(A.10)

by complete induction. Induction beginning:

$$\begin{align*}
|\epsilon_{n,1}^{(t)}(k, t) - \epsilon_{n,0}^{(t)}(k, t)| &\leq |||\epsilon_{n,0}^{(t)}(t)|| \leq \int_0^t d\tau |K(t, \tau, k)| |\Delta_n^{(t)}(k, \tau)| \\
&\leq p(k, t) \int_0^t d\tau q(\tau) |\Delta_n^{(t)}(k, \tau)| \\
&\leq p(k, t) K \int_0^t d\tau |\Delta_n^{(t)}(k, \tau)| \\
&\leq p(k, t) K \phi_n^{(t)}(k, t).
\end{align*}$$

(A.11)
Induction step:

\[
\left| e_{n,m+1}^{(\ell)}(\kappa, t) - e_{n,m}^{(\ell)}(\kappa, t) \right| = \frac{1}{p(\kappa, t)} \left| \int_0^\tau d\tau K(t, \tau, \kappa) \chi(\kappa, \tau) \left( e_{n,m}^{(\ell)}(\kappa, \tau) - e_{n,m-1}^{(\ell)}(\kappa, \tau) \right) \right|
\]

\[
\leq \frac{1}{p(\kappa, t)} \int_0^\tau d\tau q(\tau) \left| \chi(\kappa, \tau) \right| \left| e_{n,m}^{(\ell)}(\kappa, \tau) - e_{n,m-1}^{(\ell)}(\kappa, \tau) \right|
\]

\[
\leq \int_0^\tau d\tau q(\tau) \left| \chi(\kappa, \tau) \right| K \phi_n^{(\ell)}(\kappa, \tau) \frac{K^{m-1} \psi^{m-1}(\kappa, \tau)}{(m-1)!} p(\kappa, \tau)
\]

\[
\leq K K_o^{(\ell)}(\kappa) \int_0^\tau d\tau \left| \chi(\kappa, \tau) \right| \phi_n^{(\ell)}(\kappa, \tau) \frac{\psi^{m-1}(\kappa, \tau)}{(m-1)!}
\]

\[
= K K_o^{(\ell)}(\kappa) \phi_n^{(\ell)}(\kappa, t) \frac{\psi^m(\kappa, t)}{m!}.
\]

(A.12)

where we used the fact that by construction \( \phi_n^{(\ell)}(\kappa, t) \) and \( \psi(\kappa, t) \) are non-decreasing in the interval \([0, \tau]\). The series is then bounded by

\[
\left| e_n^{(\ell)}(\kappa, t) \right| = \left| \sum_{m=0}^{\infty} \left( e_{n,m+1}^{(\ell)}(\kappa, t) - e_{n,m}^{(\ell)}(\kappa, t) \right) \right|
\]

\[
\leq p(\kappa, t) K \phi_n^{(\ell)}(\kappa, t) \sum_{m=0}^{\infty} \left( K_o^{(\ell)}(\kappa) \psi(\kappa, t) \right)^m \frac{1}{m!}
\]

\[
= p(\kappa, t) K \phi_n^{(\ell)}(\kappa, t) e^{K_o^{(\ell)}(\kappa) \psi(\kappa, t)}.
\]

(A.13)

Using \( \lambda(\kappa, t) := \inf_{0 \leq \tau \leq t} \{ |\kappa + \frac{1}{2} \alpha(t)| \} \), which is \( O(\kappa) \) as \( |\kappa| \to \infty \), we can bound \( K(t, \tau, \kappa) \) by

\[
K(t, \tau, \kappa) = \frac{1}{2} \left| Z_{(+)}(\kappa, t) Z_{(-)}(\kappa, \tau) - Z_{(-)}(\kappa, t) Z_{(+)}(\kappa, \tau) \right|
\]

\[
= \frac{1}{2} \left| e^{(t-\tau) + \frac{1}{2} \int_0^\tau \alpha(s) \, ds} - e^{-k(t-\tau) - \frac{1}{2} \int_0^\tau \alpha(s) \, ds} \right|
\]

\[
\leq \frac{1}{2 \lambda(\kappa, t)} \left( |e^{(t-\tau) + \frac{1}{2} \int_0^\tau \alpha(s) \, ds}| + |e^{-k(t-\tau) - \frac{1}{2} \int_0^\tau \alpha(s) \, ds}| \right)
\]

\[
= \frac{1}{2 \lambda(\kappa, t)} \left( |e^{\frac{1}{2} \int_0^\tau \alpha(s) \, ds}| + |e^{-k \int_0^\tau \alpha(s) \, ds}| \right)
\]

\[
= \frac{1}{\lambda(\kappa, t)} \left( |e^{\frac{1}{2} \int_0^\tau \alpha(s) \, ds}| + |e^{-k \int_0^\tau \alpha(s) \, ds}| \right)
\]

\[
= \frac{\lambda(\kappa, t)}{\lambda(\kappa, t)} \cdot \phi^{-\frac{1}{2} \int_0^\tau \alpha(s) \, ds}.
\]

(A.14)
Here we have used that $\kappa = \imath\epsilon (\epsilon \in \mathbb{R})$ is purely imaginary. We can thus choose

\[
p(\kappa, t) = \frac{\exp \left( \frac{1}{2} \int_0^t |\alpha(s)| \, ds \right)}{\lambda(\kappa, t)}
\]

\[
q(\tau) = \exp \left( - \frac{1}{2} \int_0^\tau |\alpha(s)| \, ds \right).
\]

(A.15)

which allows us to set

\[
\mathcal{K} = \sup_{0 \leq \tau \leq t} |q(\tau)| = \sup_{0 \leq \tau \leq t} \left| \exp \left( - \frac{1}{2} \int_0^\tau |\alpha(s)| \, ds \right) \right|
\]

\[
\mathcal{K}_n(\kappa) = \sup_{0 \leq \tau \leq t} |p(\kappa, \tau) q(\tau)| = \frac{\exp \left( \frac{1}{2} \int_0^t |\alpha(s)| \, ds \right)}{\lambda(\kappa, t)}.
\]

We thus established the bound

\[
|\epsilon_n^{(\tau)}(\kappa, t)| \leq \frac{\exp \left( \frac{1}{2} \int_0^t |\alpha(s)| \, ds \right)}{\lambda(\kappa, t)} \left( \int_0^\tau |\Delta_n^{(\tau)}(\kappa, \tau)| \, d\tau \right)
\]

\[
\times \exp \left( \exp \left( \frac{1}{2} \int_0^\tau |\alpha(s)| \, ds \right) \int_0^\tau |\chi(s)| \, ds \right) \left( \frac{1}{\lambda(\kappa, t)} \right).
\]

(A.17)

which ensures that $\epsilon_n^{(\tau)}(\kappa, t) \to 0$ as $|\kappa| \to \infty$ if $\alpha(t)$, $\Delta_n^{(\tau)}(\kappa, t)$, $\chi(\kappa, t)$ are bounded.

### A.1. Estimate of the error term

Let us assume that $\psi$, $\alpha$ are analytic in a domain $D(d)$ that contains all complex numbers $z$ for which $\min_{z \in I} |z - x| \leq d$, where $I$ is the integration path for the Liouville–Green expansion and furthermore assume that there are constants $k_1, k_2, A$ s.t. for all $x \in D(d)$, we have the bounds

\[
|\psi(x)| \leq k_1, \quad \int_0^t \! dx |\psi(x)| \leq k_2 |\alpha(x)| \leq A,
\]

(A.18)

so we can define $k = \max\{k_1, \frac{k_2}{2}, \frac{4}{3}\}$. The bounds yield $|\psi^{(m)}(t)| \leq k_1^\frac{m!}{2^m}$ and $|\alpha^{(m)}(x)| \leq A^\frac{m!}{2^m}$ through Cauchy’s integral formula.

**Lemma 4.** Given the above bounds, the Liouville–Green coefficients satisfy for $t$ within the integration path and $n \geq 2$: $|d_n^{(m)}(t)| \leq C_n \frac{k_1^\frac{m!}{2^m} + k_2^\frac{m!}{2^m}}{(n-2)!}$ with $C_n = (1 + 2kd^2)^\frac{(1+2kd^2)_-(n-2)!}{(n-2)!}$.

**Proof.** By complete induction following [17]. Preparation:

\[
|d_1(t)| = \frac{1}{2} \left| \int_0^t \! du |\psi(u)| \right| \leq \frac{1}{2} k_2
\]

\[
|d_1^{(m)}| = \frac{1}{2} |\psi^{(m-1)}(t)| \leq \frac{1}{2} k_1 \frac{(m-1)!}{2^{m-1}}.
\]

(A.19)

Induction beginning:

\[
|d_2(t)| \leq \frac{1}{2} \left| \int_0^t \! du |d_1(u)||\psi(u)| + \frac{1}{4} |\psi(t)| + \frac{1}{2} \left| \int_0^t \! du |\alpha(u)||d_1^{(1)}(u)| \right|
\]

\[
\leq \frac{1}{4} \left( k_2^2 + k_1 + Ak_2 \right) \leq \frac{k}{4} (1 + 2kd^2).
\]

(A.20)
Furthermore denoting the digamma function by \(\psi\) and using \(\sum_{k=1}^{m} \frac{1}{k} = 2 \gamma + \psi(m)\) as well as \(\frac{1}{m} + 2 \gamma + \psi(m) \leq 1\) for all natural numbers \(m\), we find

\[
|d_2^{(m)}(t)| \leq \frac{1}{2} \left| \frac{d^m}{dt^m} (d_1(t))^2 \right| + \frac{1}{4} \left| \psi^{(m)}(t) \right| + \frac{1}{2} \left| \frac{d^{m-1}}{dt^{m-1}} (\alpha(t)d_1(t)) \right|
\]

\[
\leq \frac{1}{4} k \frac{m!}{d^m} + \frac{1}{2} \left( 2 |d_1^{(m)}(t)||d_1(t)| + \sum_{k=1}^{m-1} \binom{m}{k} |d_1^{(m-k)}(t)||d_1^{(k)}(t)| \right)
\]

\[
+ \frac{1}{2} \sum_{l=0}^{m-1} \left( m - 1 \right) \frac{A(m - l - 1)! k l!}{d^{m-l-1} d^l}
\]

\[
\leq \frac{1}{4} k \frac{m!}{d^m} \left( 1 + \frac{d^2}{m} \right) \left( 1 + 2(\gamma + \psi(m)) + kd^2 \right)
\]

\[
\leq k \frac{m!}{d^m} \left( 1 + 2kd^2 \right).
\]  \(\text{(A.21)}\)

Induction step:

\[
|d_{n+1}(t)| \leq \frac{1}{2} \left( \int_0^t du |\psi(u)| |d_n(u)| + |d_n^{(1)}(t)| \right) + \left| \int_0^t du \alpha(u) d_n^{(1)}(u) \right|
\]

\[
\leq \frac{1}{2} \left( k C_n (n - 2)! + k C_n (n - 1)! + A k \frac{C_n (n - 2)!}{2} \right)
\]

\[
\leq k \frac{C_n (n - 1)!}{2^{n-1}} \left( 1 + 2k \frac{d^2}{n - 1} \right).
\]  \(\text{(A.22)}\)

Moreover, using

\[
\frac{1}{2} \left| \frac{d^n}{dt^n} (\alpha(t)d_1(t)) \right| \leq \frac{1}{2} \sum_{l=0}^{m-1} \left( m - 1 \right) \frac{\alpha^{(m-l-1)}(t)||d_n^{(l+1)}(t)||}{d^{m-l-1} d^l}
\]

\[
\leq \frac{k}{2^{n+1}} C_n (m + n - 1)! \frac{d^{m+n-1}}{d^{m-1}} \frac{n}{n - 1}
\]

\[
\leq \frac{k}{2^{n+1}} C_n (m + n - 1)! \frac{d^{m+n-1}}{d^{m-1}} \frac{n}{n - 1}
\]  \(\text{(A.23)}\)

as well as

\[
\frac{1}{2} \left( \left| \frac{d^{m-1}}{dt^{m-1}} \psi(t)d_n(t) \right| + |d_n^{(m+1)}(t)| \right) \leq \frac{k}{2^{n+1}} \frac{(m + n - 1)!}{d^{m+n-1}} C_n \left( 1 + 2k \frac{d^2}{n - 1} \right).
\]  \(\text{(A.24)}\)

we find

\[
|d_n^{(m)}(t)| \leq \frac{k}{2^{n+1}} \frac{(m + n - 1)!}{d^{m+n-1}} C_n \left( 1 + 2k \frac{d^2}{n - 1} \right).
\]  \(\text{(A.25)}\)

If we assume these analyticity conditions and that the initial conditions are such that all integration constants \(C_n^{(i)}\) in the Liouville–Green recursion relations \((3.31)\) vanish, we can use equation \((A.17)\) to provide a bound on the truncation error that is independent of the Liouville–Green coefficient. This is rather useful, because the explicit form, particularly of higher order coefficients, is very complicated. We then obtain

\[
|\Delta_n^{(i)}(\kappa, \ell)| \leq \frac{2}{|\kappa|^n} e^\ell \frac{d^{(i)}(\alpha(s))}{d^n(t)}
\]

\[
\leq \frac{2}{|\kappa|^n} e^\ell \frac{d^{(i)}(\alpha(s))}{C_n k (n - 1)!} \frac{1}{2^{n-1} d^{n-1}}.
\]  \(\text{(A.26)}\)
Insertion into the truncation error yields the bound
\[ |\epsilon(t)\{\kappa, t\}| \leq e^{\int_{a}^{b}\frac{ds}{\lambda\{\kappa, t\}}} \exp\left[\int_{a}^{b}\frac{ds}{\lambda\{\kappa, t\}}\right] 2 |\epsilon|^{n} \exp\left[\int_{a}^{b}\frac{ds}{\lambda\{\kappa, t\}}\right] C_{n}\left(\frac{n-1}{2}\right) |t|. \] (A.27)

A.2. Real analytic edges

We assume that the components of the connection and of the edge are real analytic in the chosen trivialization. For \( A \neq 0 \), we perform a position-independent gauge transformation \( \tau \mapsto g_{a}^{-1}\tau g_{a} \), such that \( m(0) \neq 0 \). Then there are \( t_{o}, \epsilon > 0 \) s.t. \( \inf_{\omega \geq t \geq t_{o}} |m(s)| \geq \epsilon \).

Specifically, we assume that there exists \( t_{i} > 0 \) such that for \( 0 \leq \tau \leq t_{i} \),
\[
\begin{align*}
\beta_{R}(\tau) &= \text{Re}\left(\frac{1}{4}M^{2}(\tau) - \frac{1}{4}M(\tau)\right) \\
\beta_{I}(\tau) &= \text{Im}\left(\frac{1}{4}M^{2}(\tau) - \frac{1}{4}M(\tau)\right) \\
\alpha_{R}(\tau) &= \text{Re}(\hat{n}(\tau) - M(\tau)n(\tau)) \\
\alpha_{I}(\tau) &= \text{Im}(\hat{n}(\tau) - M(\tau)n(\tau))
\end{align*}
\] (A.28)

are real analytic for \( 0 \leq \tau \leq t_{i} \). Then for each \( i \in \mathbb{N} \), there exist
\[
\begin{align*}
\alpha_{R}^{i}, \beta_{R}^{i}, t^{R}_{a,i} &> 0 \text{ s.t. } \alpha_{R}^{i} > \sup_{0 \leq \tau \leq t^{R}_{a,i}} \left| \frac{\partial \alpha_{R}(\tau)}{\partial \tau} \right| \\
\alpha_{I}^{i}, \beta_{I}^{i}, t^{I}_{a,i} &> 0 \text{ s.t. } \alpha_{I}^{i} > \sup_{0 \leq \tau \leq t^{I}_{a,i}} \left| \frac{\partial \alpha_{I}(\tau)}{\partial \tau} \right|
\end{align*}
\] (A.29)

which lets us define \( \alpha_{i} := \max\{\alpha_{R}^{i}, \alpha_{I}^{i}\}, \beta_{i} := \max\{\beta_{R}^{i}, \beta_{I}^{i}\} \) and
\[
t_{i} := \min\{t^{R}_{a,i}, t^{I}_{a,i}, t^{R}_{a,i}, t^{I}_{a,i}, t^{R}_{a,i}, t^{I}_{a,i}, t^{R}_{a,i}, t^{I}_{a,i}, t_{0}, t_{1}\}. \] (A.30)

We thus have the bounds
\[
|\alpha(\tau)| \leq \alpha_{o} \text{ for } t < t_{o} \\
|\beta(\tau)| \leq \beta_{o} \text{ for } t < t_{o}
\] (A.31)

which implies\(^{16}\)
\[
|\chi(\kappa, t)| = |\beta(\kappa, t) - \gamma(\kappa, t)| \leq |\beta(t)| + |\gamma(\kappa, t)| \\
\leq \beta_{o} + \frac{\alpha^{2}(t)}{4} + \frac{3\alpha^{2}(t)}{16(\kappa + \frac{\alpha(t)}{2})^{2}} - \frac{\alpha(t)}{4(\kappa + \frac{\alpha(t)}{2})} \text{ for } t < t_{o} \\
\leq \beta_{o} + \frac{1}{4} \alpha_{o}^{2} + \frac{3}{16} \lambda^{2}(\kappa, t) + \frac{5 \alpha_{o}}{4 \lambda(\kappa, t)} \text{ for } t < t_{2}, \] (A.32)

\[
|\Delta^{I}(\kappa, t)| = 2 e^{\frac{t}{2} \int_{a}^{b} d\tau \alpha(\tau)} \Delta^{I}(\kappa, t) \\
\overset{(3.10)}{=} 2 e^{\frac{t}{2} \int_{a}^{b} d\tau \alpha(\tau)} \left| \alpha(t) - \frac{1}{4} \alpha^{2}(t) - \frac{\alpha(t)}{2} \right| \\
\leq 2 e^{\frac{t}{2} \int_{a}^{b} d\tau |\alpha(\tau)|} \left| \beta(t) - \frac{1}{4} \alpha^{2}(t) + \frac{\alpha(t)}{2} \right| \\
\leq 2 \left( \beta_{o} + \frac{1}{4} \alpha_{o}^{2} + \frac{1}{2} \alpha_{1} \right) \text{ for } t < t_{1}. \] (A.33)

\(^{16}\)Again, \( \lambda(\kappa, t) := \inf_{0 \leq \tau \leq |\kappa + \frac{1}{4} \alpha(\tau)|} \).
We thus have a bound on the truncation error

\[ |\varepsilon^{(l)}_1(\kappa, t) - \Delta^{(l)}_1(\kappa, \tau)| \leq 2 \alpha_0 \left( 2 \alpha_0 + 2 \alpha_1 \right) \left( e^{\frac{1}{2} \alpha t} - 1 \right) \]  

for \( t < t_1 \).  

(A.34)

We thus have a bound on the truncation error

\[ |\varepsilon^{(l)}_1(\kappa, t)| \leq e^{\alpha t} \left( e^{\frac{1}{2} \alpha t} - 1 \right) \frac{4 \beta_0 + 2 \alpha_1}{2 \alpha_0 \lambda(\kappa, t)} \left[ e^{\frac{1}{2} \alpha t} \cdot \left( 4 \beta_0 \lambda^2(\kappa, t) + 4 \alpha_0^2 \lambda^2(\kappa, t) + 3 \alpha_1^2 + 4 \alpha_2 \lambda(\kappa, t) \right) \right] \]  

valid for \( t < t_2 \) which, using that by construction \( \lambda = \mathcal{O}(|c|) \) for \( c \to \infty \equiv |\kappa| \to \infty \), establishes

\[ |\varepsilon^{(l)}_1(\kappa, t)| \to 0 \quad \text{for} \quad |c| \to \infty. \]  

(A.36)

Appendix B. Perturbation approach

Restricting to a particular type of edge refers to a background. Even worse it prevents LQC to be continuously embedded into LQG. For this, one would need the solution to (2.7) for the isotropic case but for arbitrary edges.

In this section, we now construct the general solution to (2.7) in terms of a perturbation around a known explicit solution of section 2.2. For this, we use methods of [14], similar to the Liouville–Green approximation described there. However, instead of giving an approximate solution we will compute the exact solution in terms of a formal power series in the perturbation.

B.1. General perturbation

Suppose a solution to (2.7) is given for a special edge \( e_0(t) = (e_0^1(t), e_0^2(t), e_0^3(t)) = (x_0, y_0, z_0) \). Now assume that the edge \( e_0 \) is deformed into another edge \( e \) such that \( e(t) = (x_0 + \varepsilon \tilde{x}, y_0 + \varepsilon \tilde{y}, z_0 + \varepsilon \tilde{z}) \), where \( \tilde{x} = \tilde{x}(t), \tilde{y} = \tilde{y}(t), \tilde{z} = \tilde{z}(t) \), and \( \varepsilon = \text{const} \) is a deformation parameter. Then we can write the components of the tangent vector \( \dot{e} \) as

\[ \begin{align*}
\dot{x} &= \dot{x}_0 + \varepsilon \dot{\tilde{x}}, \\
\dot{y} &= \dot{y}_0 + \varepsilon \dot{\tilde{y}}, \\
\dot{z} &= \dot{z}_0 + \varepsilon \dot{\tilde{z}}.
\end{align*} \]

This implies

\[ \begin{align*}
m &= (\dot{x}_0 - i \dot{y}_0) + \varepsilon (\dot{\tilde{x}} - i \dot{\tilde{y}}), \\
n &= \dot{z}_0 + \varepsilon \dot{\tilde{z}_0}.
\end{align*} \]  

(B.1)

Moreover, we introduce

\[ M_0 := \frac{m_0}{m_0} \quad \text{and} \quad \tilde{M} := \frac{\tilde{m}}{m_0} \quad \text{with} \quad \tilde{M} = \frac{\tilde{m}}{m_0} - \tilde{M}_0 \]  

\[ \tilde{M} = \frac{\tilde{m}}{m_0} - 2M_0(\dot{\tilde{M}} + \tilde{M}_0) \]
in order to expand
\[
M = \frac{\dot{m}}{m} = \frac{\dot{m}_0 + \hat{\varepsilon} \tilde{m}}{m_0 + \hat{\varepsilon} \tilde{m}} = \frac{1}{m_0} \left( \dot{m}_0 + \hat{\varepsilon} \tilde{m} \right) \frac{1}{1 - (-\varepsilon M)}
\]
\[
= (M_0 + \varepsilon (\tilde{M} + \tilde{M} M_0)) \sum_{k=0}^{\infty} (-1)^k \varepsilon^k \tilde{M}^k
\]
\[
= \sum_{k=0}^{\infty} (-1)^k \varepsilon^k [\tilde{M}_0 \tilde{M}^k + \varepsilon^{k+1} (\tilde{M} \tilde{M}^k + M_0 \tilde{M}^{k+1})]
\]
\[
= M_0 + \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \left[ \tilde{M}_0 \tilde{M}^k - \tilde{M} \tilde{M}^{k-1} - M_0 \tilde{M}^k \right]
\]
\[
= M_0 - \frac{\tilde{M}}{M} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \tilde{M}^k,
\]
where we need to assume \(|\varepsilon \tilde{M}| < 1\) from the first to second line in order to ensure that the geometric series converges. Note that this assumption is reparametrization independent: if we introduce a new edge parameter \(T = T(t)\), then for an arbitrary function \(f(t)\), it holds that \(\dot{f}(t) = \frac{dt}{dt} f(T(t)) = \frac{dT}{dt} \dot{T}(t)\); hence, the \(\tilde{T}\)-terms cancel by construction of \(\tilde{M}\). As a consequence, it is straightforward to write down
\[
M^2 = M_0^2 - 2 M_0 \frac{\tilde{M}}{M} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \tilde{M}^k + \left( \frac{\tilde{M}}{M} \right)^2 \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} (-1)^{k+r} \varepsilon^{k+r} \tilde{M}^{k+r}
\]
\[
= M_0^2 - 2 M_0 \frac{\tilde{M}}{M} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \tilde{M}^k + \left( \frac{\tilde{M}}{M} \right)^2 \sum_{r=2}^{\infty} (r-1) (-1)^{r-1} \varepsilon^{r-1} \tilde{M}^{r-1}
\]
\[
= M_0^2 + (2 M_0 \tilde{M}) \varepsilon + \left( \frac{\tilde{M}}{M} \right) \sum_{r=2}^{\infty} (-1)^{r-1} \varepsilon^{r-1} \tilde{M}^{r-1} \left( (r-1) \tilde{M} - 2 M_0 \right)
\]

and
\[
\dot{M} = \dot{M}_0 - \frac{\ddot{M} \tilde{M} - \dot{\tilde{M}}^2}{M^2} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \tilde{M}^k - \frac{\ddot{M}}{M} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k k \tilde{M}^{k-1} \tilde{M}
\]
\[
= \dot{M}_0 - \left[ \frac{\ddot{M}}{M} - \left( \frac{\dot{\tilde{M}}}{M} \right)^2 \right] \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \tilde{M}^k - \left[ \frac{\dot{\tilde{M}}}{M} \right] \sum_{k=1}^{\infty} \left( (-1)^k \varepsilon^k k \tilde{M}^k \right)
\]
\[
= \dot{M}_0 + \ddot{\tilde{M}} \varepsilon - \frac{\dot{\tilde{M}}^2}{M} \sum_{k=2}^{\infty} (-1)^k \varepsilon^k \tilde{M}^k \{ (k-1) \tilde{M}^2 + \frac{\dot{\tilde{M}}^2}{M} \}
\]
Finally, we use the expansion
\[
|m|^2 + n^2 = \left( \dot{x}_0 + \hat{\varepsilon} \dot{\tilde{x}} \right)^2 + \left( \dot{y}_0 + \hat{\varepsilon} \dot{\tilde{y}} \right)^2 + \left( \dot{z}_0 + \hat{\varepsilon} \dot{\tilde{z}} \right)^2
\]
\[
= \dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2 + 2(\dot{x}_0 \dot{\tilde{x}} + \dot{y}_0 \dot{\tilde{y}} + \dot{z}_0 \dot{\tilde{z}}) + (\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2) \hat{\varepsilon}^2
\]
\[
=: |\dot{m}_0|^2 + n_0^2 + 2 Y \varepsilon + (|\dot{\tilde{m}}|^2 + n_0^2) \hat{\varepsilon}^2,
\]
in order to express
\[
N = \i \varepsilon (\dot{n} - \dot{n} \tilde{m} + \i \varepsilon (m \tilde{m} + n^2))
\]
\[
= -\varepsilon^2 (|m_0|^2 + n_0^2 + 2 Y \varepsilon + (|\tilde{m}_0|^2 + n_0^2) \hat{\varepsilon}^2) + \i \varepsilon (\dot{n}_0 + \hat{\varepsilon} \tilde{n}) - \i \varepsilon (n_0 + \hat{\varepsilon} \tilde{n}) M
\]
\[= -e^2 (\ldots) + \mathcal{O}(n_0 + e\tilde{n}) - \mathcal{O}(n_0 + e\tilde{n}) \left\{ M_0 - \frac{\dot{M}}{M} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \tilde{M}^k \right\} \]

\[= -e^2 (\ldots) + \mathcal{O}(n_0 + e\tilde{n}) \left\{ M_0 - \frac{\dot{M}}{M} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \tilde{M}^k \left\{ n_0 \varepsilon^k + \tilde{n} \varepsilon^{k+1} \right\} \right\} \]

\[= -e^2 (\ldots) + \mathcal{O}(n_0 + e\tilde{n}) - \mathcal{O}(n_0 + e\tilde{n}) \left\{ -n_0 \dot{M} \varepsilon + \frac{\dot{M}}{M} \left( n_0 - \frac{\tilde{n}}{M} \right) \sum_{k=2}^{\infty} (-1)^k \tilde{M}^k \varepsilon^k \right\} \]

So we are finally able to expand the bracket term in (2.7) into orders of \( \varepsilon \):

\[\left\{ \frac{1}{4} M^2 - \frac{1}{2} M + N \right\} = \frac{1}{4} M_0^2 - \frac{1}{2} M_0 - e^2 (|m_0|^2 + n_0^2) + \mathcal{O}(n_0 + M_0 n_0) \]

\[+ \varepsilon \left\{ \frac{1}{2} (M_0 \dot{M} - \tilde{M}) - 2e^2 \hat{Y} + \mathcal{O}(n_0 \dot{M} - n_0 \tilde{M}) \right\} \]

\[+ \varepsilon^2 \left\{ \frac{1}{4} (\tilde{M})^2 - 2M_0 \tilde{M} \right\} + \frac{1}{2} ((\tilde{M})^2 + \tilde{M} \tilde{M}) - e^2 (|\tilde{m}|^2 - \tilde{n}) + \mathcal{O}(n_0 \dot{M} - \tilde{n}) \}

\[+ \sum_{k=3}^{\infty} (-1)^k \tilde{M}^k \varepsilon^k \left\{ \frac{1}{4} \tilde{M} \tilde{M}^{-1} ((k-1) \tilde{M} \tilde{M}^{-1} - 2M_0) + \frac{1}{2} \tilde{M}^2 ((k-1) \tilde{M}^2 + \tilde{M} \tilde{M}) \right\} \]

\[+ \varepsilon^2 \tilde{M} \tilde{M}^{-1} \left( n_0 - \tilde{n} \tilde{M}^{-1} \right) \right\}, \quad (B.2) \]

where we have assumed \( |\varepsilon \tilde{M}| < 1 \) and \( Y := \tilde{x}_0 \tilde{x}^2 + \tilde{y}_0 \tilde{y}^2 + \tilde{z}_0 \tilde{z}^2 \).

### B.2. Simplifications

As it stands (B.2) holds in full generality. However, in order to solve (2.7) with the perturbative Ansatz, we require \( M_0 \) and \( n_0 \) to be constants. Hence, \( M_0 = 0 = n_0 \). Moreover, we may assume that we perturb \( \epsilon_0(t) \) only in the perpendicular direction, that is, for the edge tangents \( e_0, \tilde{e} \), it holds that \( (e_0, \tilde{e}) = (e_0, e_0) = |e_0|^2 \). Then it follows that \( Y := \tilde{x}_0 \tilde{x}^2 + \tilde{y}_0 \tilde{y}^2 + \tilde{z}_0 \tilde{z}^2 = 0 \).

If we leave the beginning point of the edge fixed under perturbation, we have \( \epsilon_0(t = 0) = \tilde{e}(t = 0) \). Moreover, without loss of generality we can choose \( \epsilon_0(t) \) such that \( \epsilon_0(t = 0) = \tilde{e}(t = 0) \).

Finally we are free to choose a convenient parametrization, for example, arc-length parametrization of \( \epsilon_0(t) \), such that \( |m_0|^2 + n_0^2 = 1 \). We will refrain from the latter and write using \( \kappa := 1 \epsilon \):

\[\left\{ \frac{1}{4} M^2 - \frac{1}{2} M + N \right\} = \frac{1}{4} M_0^2 - \frac{1}{2} M_0 - \kappa^2 (|m_0|^2 + n_0^2) - \kappa M_0 n_0 \]

\[+ \varepsilon \left\{ \frac{1}{2} (M_0 \dot{M} - \tilde{M}) + \kappa (\tilde{n} - n_0 \tilde{M}) \right\} \]

\[+ \varepsilon^2 \left\{ \frac{1}{4} (\tilde{M})^2 - 2M_0 \tilde{M} \right\} + \frac{1}{2} ((\tilde{M})^2 + \tilde{M} \tilde{M}) - \kappa^2 (|\tilde{m}|^2 - \tilde{n} + \kappa \tilde{M} (n_0 \tilde{M} - \tilde{n}) \}

\[+ \sum_{k=3}^{\infty} (-1)^k \tilde{M}^k \varepsilon^k \left\{ \frac{1}{4} \tilde{M} \tilde{M}^{-1} ((k-1) \tilde{M} \tilde{M}^{-1} - 2M_0) \right\} \]
\[ + \frac{1}{2} \tilde{\mathcal{M}}^{-2} (k - 1)(\tilde{\mathcal{M}}^2 + \tilde{\mathcal{M}}) + \kappa \tilde{\mathcal{M}}^{-1} (m_0 - \tilde{\mathcal{M}}^{-1}) \]

\[ =: \mathcal{K}^2(\kappa) + \sum_{k=1}^{\infty} \tilde{e}^k f_k(\kappa, t), \] (B.3)

where we have introduced the obvious shorthand in the last line. In particular,

\[ \mathcal{K}^2(\kappa) := \frac{1}{2} M_0^2 + \kappa^2 (|m_0|^2 + n_0^2) - \kappa M_0 n_0. \] (B.4)

### B.3. Solution by the Liouville–Green method

With expansion (B.3), we can write (2.7) as

\[ \tilde{d} = \left\{ \mathcal{K}^2(\kappa) + \sum_{k=1}^{\infty} \tilde{e}^k f_k(\kappa, t) \right\} d. \] (B.5)

Now we make a Liouville–Green Ansatz, similar to section 3. That is, we set

\[ d(\kappa, t) = \sum_{\sigma = \pm} \sigma \mathcal{K} \tilde{d}^{(\sigma)}(\kappa, t) \]

\[ \dot{d}(\kappa, t) = \sum_{\sigma = \pm} \sigma \mathcal{K} \tilde{d}_n(\sigma)(\kappa, t) + \tilde{d}^{(\sigma)}(\kappa, t) \cdot \tilde{e}^n \]

\[ \ddot{d}(\kappa, t) = \sum_{\sigma = \pm} \sigma \mathcal{K} \tilde{d}_n(\sigma)(\kappa, t) + 2\sigma \mathcal{K} \tilde{d}_n(\sigma)(\kappa, t) + \tilde{d}^{(\sigma)}(\kappa, t) \cdot \tilde{e}^n. \] (B.6)

In the last line we have used the fact that \( \sigma^2 = 1 \). As we will see at the end of this computation, the choice of initial conditions

\[ d^{(\sigma)}(\kappa, 0) = 1 \quad \text{and} \quad \dot{d}^{(\sigma)}(\kappa, 0) = 0, \] (B.7)

which have to hold for arbitrary \( \epsilon \), will be convenient. Now we plug Ansatz (B.6) into (B.5).

For clarity we will again suppress the dependence on \( t, \kappa \) and simply write e.g. \( d^{(\sigma)}_n \) instead of \( d^{(\sigma)}_n(\kappa, t) \). This gives

\[ \sum_{\sigma = \pm} \mathcal{K} e^{\kappa t} \sum_{n=0}^{\infty} \left\{ \mathcal{K}^2 \tilde{d}_n^{(\sigma)} + 2\sigma \mathcal{K} \tilde{d}_n^{(\sigma)} + \mathcal{K} \tilde{d}_n^{(\sigma)} - \mathcal{K} \tilde{d}_n^{(\sigma)} - 2\mathcal{K} \tilde{d}_n^{(\sigma)} + \kappa \tilde{d}_n^{(\sigma)} \sum_{k=1}^{\infty} \tilde{e}^k \right\} e^n = 0. \]

This has to hold at any point of \( \epsilon(t) \), that is, for arbitrary values of \( t \). Hence, it must hold that

\[ \sum_{n=0}^{\infty} \left\{ 2\sigma \mathcal{K} \tilde{d}_n^{(\sigma)} + \tilde{d}_n^{(\sigma)} - \tilde{d}_n^{(\sigma)} \sum_{k=1}^{\infty} \tilde{e}^k \right\} e^n = \sum_{n=1}^{\infty} \epsilon^n \left\{ \tilde{d}_n^{(\sigma)} + 2\sigma \mathcal{K} \tilde{d}_n^{(\sigma)} - \sum_{k=1}^{n} d_n^{(\sigma)} \tilde{e}_k + \tilde{d}_0^{(\sigma)} + 2\sigma \mathcal{K} \tilde{d}_0^{(\sigma)} = 0. \]

This has to hold for arbitrary values of \( \epsilon \), hence separately in every order of \( \epsilon \):

\[ \mathcal{O}(\epsilon^0) \tilde{d}_0^{(\sigma)} + 2\sigma \mathcal{K} \tilde{d}_0^{(\sigma)} = 0 \] (B.8)

\[ \mathcal{O}(\epsilon^{n=0}) \tilde{d}_n^{(\sigma)} + 2\sigma \mathcal{K} \tilde{d}_n^{(\sigma)} = \gamma_n^{(\sigma)} \] with \( \gamma_n^{(\sigma)} := \sum_{k=1}^{n} d_n^{(\sigma)} \tilde{e}_k. \) (B.9)
B.3.1. Solution to $O(\varepsilon^0)$. This is a homogeneous linear ODE of second order with constant coefficients. Its solution is given by
\[ d_0^{(\sigma)} = A_0^{(\sigma)} + B_0^{(\sigma)} e^{-2\kappa K t} \]
with integration constants $A_0^{(\sigma)}, B_0^{(\sigma)}$.

B.3.2. Solution to $O(\varepsilon^{n>0})$. This second-order ODE is inhomogeneous but still linear with constant coefficients. Its general solution $d_n^{(\sigma)}$ can be obtained as a linear combination
\[ d_n^{(\sigma)} = d_n^{(\sigma)_{\text{HOM}}} + d_n^{(\sigma)_{\text{SP}}} \]
of the general solution $d_n^{(\sigma)_{\text{HOM}}}$ to the homogeneous equation plus a special solution $d_n^{(\sigma)_{\text{SP}}}$ to the inhomogeneous equation. The homogeneous part is equivalent to the $O(\varepsilon^0)$ case and given by
\[ d_n^{(\sigma)_{\text{HOM}}} = A_n^{(\sigma)_{\text{HOM}}} + B_n^{(\sigma)_{\text{HOM}}} e^{-2\kappa K t} \]
with the integration constants $A_n^{(\sigma)_{\text{HOM}}}$ and $B_n^{(\sigma)_{\text{HOM}}}$. The special solution $d_n^{(\sigma)_{\text{SP}}}$ can be obtained from $d_n^{(\sigma)_{\text{HOM}}}$ by the method of variation of constants, that is, we make the Ansatz
\[ d_n^{(\sigma)_{\text{SP}}} (\kappa, t) = A_n^{(\sigma)_{\text{SP}}} (\kappa, t) + B_n^{(\sigma)_{\text{SP}}} (\kappa, t) e^{-2\kappa K t}. \]

With the usual requirement $\int A_n^{(\sigma)_{\text{SP}}} (\kappa, t) + B_n^{(\sigma)_{\text{SP}}} (\kappa, t) e^{-2\kappa K t} = 0$, this gives
\[ \dot{d}_n^{(\sigma)_{\text{SP}}} (\kappa, t) = -2\kappa K B_n^{(\sigma)_{\text{SP}}} (\kappa, t) e^{-2\kappa K t} \]
\[ \ddot{d}_n^{(\sigma)_{\text{SP}}} (\kappa, t) = (-2\kappa K B_n^{(\sigma)_{\text{SP}}} (\kappa, t) + 4\kappa^2 B_n^{(\sigma)_{\text{SP}}} (\kappa, t)) e^{-2\kappa K t}. \]

If we plug this Ansatz into (B.9), we obtain (recall that $\sigma^2 = 1$)
\[ \boxed{ B_n^{(\sigma)} (\kappa, t) = -\frac{\sigma}{2K} \int_0^t e^{2\kappa K (s-t)} \lambda_n^{(\sigma)} (\kappa, s) \, ds + B_n^{(\sigma)_{\text{SP}}} } \]
\[ A_n^{(\sigma)} (\kappa, t) = \frac{\sigma}{2K} \int_0^t \lambda_n^{(\sigma)} (\kappa, s) \, ds + A_n^{(\sigma)_{\text{SP}}} , \]
where $A_n^{(\sigma)_{\text{SP}}}$ and $B_n^{(\sigma)_{\text{SP}}}$ are again integration constants. Therefore, the special solution $d_n^{(\sigma)_{\text{SP}}} (\kappa, t)$ is given by
\[ d_n^{(\sigma)_{\text{SP}}} (\kappa, t) = A_n^{(\sigma)_{\text{SP}}} + B_n^{(\sigma)_{\text{SP}}} + \frac{\sigma}{2K} \int_0^t (1 - e^{2\kappa K (s-t)}) \lambda_n^{(\sigma)} (\kappa, s) \, ds. \]

Using (B.10), the final solution $d_n^{(\sigma)}$ of (B.9) is given by
\[ d_n^{(\sigma)} = D_n^{(\sigma)_{\text{SP}}} + B_n^{(\sigma)_{\text{HOM}}} e^{-2\kappa K t} + \frac{\sigma}{2K} \int_0^t (1 - e^{2\kappa K (s-t)}) \lambda_n^{(\sigma)} (\kappa, s) \, ds, \]
where we have introduced the overall integration constant $D_n^{(\sigma)_{\text{SP}}} := A_n^{(\sigma)_{\text{HOM}}} + A_n^{(\sigma)_{\text{SP}}} + B_n^{(\sigma)_{\text{SP}}}$. 

B.3.3. Implementation of initial conditions. Now we implement the initial conditions (B.7) into (B.11):
\[ d^{(\sigma)} (\kappa, 0) = 1 = \sum_{n=0}^{\infty} d_n^{(\sigma)} (\kappa, 0) \cdot \varepsilon^n = A_0^{(\sigma)} + B_0^{(\sigma)} + \sum_{n=1}^{\infty} (D_n^{(\sigma)_{\text{SP}}} + B_n^{(\sigma)_{\text{HOM}}}) \cdot \varepsilon^n \]
\[ \dot{d}^{(\sigma)} (\kappa, 0) = 0 = \sum_{n=0}^{\infty} d_n^{(\sigma)} (\kappa, 0) \cdot \varepsilon^n = -2\kappa K \left( B_0^{(\sigma)} + \sum_{n=1}^{\infty} B_n^{(\sigma)_{\text{HOM}}} \cdot \varepsilon^n \right). \]
As these conditions have to hold for arbitrary $\varepsilon$, it follows that
\begin{equation}
A_0^{(s)} = 1 \quad B_0^{(s)} = 0 \quad \text{and} \quad B_{n,\text{HOM}}^{(s)} = 0 = D_{n,\text{SP}}^{(s)}.
\end{equation}
Therefore,
\begin{equation}
d^{(s)}(\kappa, t) = e^{\sigma K t} \left\{ 1 + \frac{\sigma}{2 K} \sum_{n=1}^{\infty} \int_0^t (1 - e^{2\sigma K (s-t)}) \lambda_n^{(s)}(\kappa, s) \, ds \right\}, \tag{B.12}
\end{equation}
where $\lambda_n^{(s)}(\kappa, t) := \sum_{k=1}^{n} \lambda_n^{(s)}(\kappa, t) f_k(\kappa, t)$. Moreover, by construction $d_0^{(s)}(\kappa, t) = 1$ and $f_k(\kappa, t)$ is given according to (B.3) and $K$ is defined in (B.4).

### B.4. Solution

Now we are set up to construct the solution to (2.4) for an arbitrary edge $e(t)$ using Ansatz (2.8) in terms of a perturbation about an edge $e_0(t)$. According to appendix B.2, we demand
\begin{equation}
e_0(0) = e(0) \quad \text{and} \quad \dot{e}_0(0) = \dot{e}(0).
\end{equation}
This implies $m(0) = m_0(0)$. Imposing the initial conditions (2.5), we obtain
\begin{align*}
a(\kappa, 0) &= 1 = A_{(+)}d^{(+)}(\kappa, 0) + A_{(-)}d^{(-)}(\kappa, 0) = \sqrt{m_0(0)}(A_{(+)} + A_{(-)}) \\
\dot{a}(0) &= \kappa n_0(0) = A_{(+)}d^{(+)}(\kappa, 0) + A_{(-)}d^{(-)}(\kappa, 0) = K(A_{(+)} - A_{(-)}).
\end{align*}
Hence, $A_{(s)} = \frac{1}{2} \left[ \frac{1}{\sqrt{m_0(0)}} + \sigma \frac{n_0(0)}{K(\kappa)} \right]$ and we obtain the final solution
\begin{equation}
a(\kappa, t) = \sqrt{m(t)} \sum_{\sigma = \pm} A_{(\sigma)} e^{\sigma K t} \left\{ 1 + \frac{\sigma}{2 K} \sum_{n=1}^{\infty} e^{n \sigma} \int_0^t (1 - e^{2\sigma K (s-t)}) \lambda_n^{(s)}(\kappa, s) \, ds \right\}, \tag{B.13}
\end{equation}
with $\lambda_n^{(s)}(\kappa, s)$ and $K = K(\kappa)$ given below (B.3) and $m$ given in (B.1).

### B.5. Perturbation about a line

Certainly the functions $f_k(\kappa, t)$ of (B.3), which are needed in order to explicitly compute (B.13), are still quite complicated. Also the definition (B.4) of $K$ involves a square root taken from a complex number. To simplify this situation, we can construct a solution of (2.7) for a general edge $e(t)$ as follows. Given $e(t)$ we construct the solution to (2.7) as a perturbation of a line $e_0(t)$, for which again
\begin{equation}
e_0(0) = e(0) \quad \text{and} \quad \dot{e}_0(0) = \dot{e}(0) \tag{B.14}
\end{equation}
holds. Without loss of generality, we can choose the maximal simplification for $e_0$ being a line. That is,
\begin{equation}
e_0(t) = (x_0(t), 0, 0) \quad e(t) = (x(t), \varepsilon \tilde{y}(t), \varepsilon \tilde{z}(t)). \tag{B.15}
\end{equation}
Moreover, we choose arc-length parametrization of the line, that is, we demand
\begin{equation}1 = \tilde{x}_0 = m(0) = m_0(0). \tag{B.16}
\end{equation}
Then we have
\begin{align*}m_0 = \dot{x}_0 = 1, \quad M_0 = M_0 = m_0, \quad n_0 = n_0 = \dot{n}_0, \quad K^2 = \kappa^2 = -c^2 \quad \tilde{m} = \tilde{y}, \quad \tilde{M} = \tilde{m} = \tilde{n}, \quad \tilde{n} = \tilde{z}. \tag{B.17}
\end{align*}
Under these assumptions, expression (B.3) can be simplified to
\begin{align*}
\left\{ \frac{1}{4} \dot{M}^2 - \frac{1}{2} \dot{M} + N \right\} &= \kappa^2 + \varepsilon \varepsilon \tilde{n} + \varepsilon^2 \left\{ \frac{3}{4} (\tilde{M})^2 + \frac{1}{2} \tilde{M} \dot{M} + \kappa^2 (\tilde{m}^2 + \tilde{n}^2) - \kappa \tilde{M} \tilde{n} \right\} \\
&+ \sum_{k=3}^{\infty} \varepsilon^k (-1)^k \tilde{M}^{k-2} \left\{ \frac{3}{4} (k-1)(\tilde{M})^2 + \frac{1}{2} \tilde{M} \dot{M} - \kappa \tilde{M} \tilde{n} \right\}. \tag{B.18}
\end{align*}
and expression (B.13) reads

\[ a(\kappa, t) = \sum_{\sigma = \pm} \Theta^{\sigma}(\kappa, t) \left\{ 1 + \frac{\sigma}{2k} \sum_{n=1}^{\infty} \rho^n \int_0^t (1 - e^{2\sigma x(s-\kappa)}) \lambda_{\sigma}^{\sigma}(\kappa, s) \, ds \right\}. \]  

(B.19)

Note that if we set \( \varepsilon = c^{-1} \), then for \( c \to \infty \) solution (B.19) takes the form

\[ a(\kappa, t) = \sum_{\sigma = \pm} \Theta^{\sigma}(1 + \mathcal{O}(c^{-1})). \]  

(B.20)

In fact, this property holds for arbitrary curves in the limit \( c \to \infty \). The proof is given in section 4.

**Appendix C. Geometric interpretation of the parameter \( c \)**

**C.1. The Ashtekar connection revisited**

For completeness, we will briefly sketch some of the insights obtained in [15], where a coordinate-free treatment of the Ashtekar connection is developed. Note that unlike in the rest of this paper, we denote vectors by the symbol '\( \cdot \)'.

We are working with a \((3+1)\)-decomposition of spacetime \((\mathcal{M}, g) \cong \mathbb{R} \times \Sigma\). The Cauchy surfaces \( \Sigma \) are orientable. Hence, at every \( m \in \mathcal{M} \) we can decompose the tangent space \( T_m \mathcal{M} = T_m \mathcal{M}^1 \oplus T_m \mathcal{M}^2 \) into two orthogonal subspaces: the component \( T_m \mathcal{M}^1 \cong T_m \Sigma \) tangent to \( \Sigma \) and the normal component \( T_m \mathcal{M}^2 \cong N_m \), where \( n \) is the surface surface normal unit vector and \( N \in \mathbb{R} \). With the signature \((s, +, +, +)\) of \( \mathcal{M} \) \((s = \pm 1)\), we have \( g(n, n) = s \) and \( g(n, X) = 0, g(X, X) > 0 \) for every \( X \in T_m \Sigma \). To shorten notation, we use the isomorphism between the local tangent spaces \( T_m \Sigma \sim \mathbb{R}^3 \) and the Lie algebra \( \mathfrak{so}(3) \) induced by the equivalence of the defining representation of \( \mathfrak{so}(3) \) on \( \mathbb{R}^3 \) and the adjoint representation of \( \mathfrak{so}(3) \) on \( \mathfrak{so}(3) \). Let \( \{e_l\}_{l=1,2,3} \) be an orthonormal basis-frame of \( T_m \Sigma \), e.g. the Cartesian standard basis of \( \mathbb{R}^3 \). Moreover, let \( \{\tau_l\}_{l=1,2,3} \) be a basis of \( \mathfrak{so}(3) \sim \mathfrak{su}(2) \), orthonormal with respect to the Cartan–Killing–metric thereon, e.g. \( (\tau_l)_{jk} = -\epsilon_{ljk} \). Now we choose a fixed identification \( e_l \leftrightarrow \tau_l \) and can write any element \( X \in T_m \Sigma \) equivalently as \( X = \sum X_l e_l \leftrightarrow \sum X_l \tau_l \), where \( X_l = g(X, e_l) \) denotes the expansion of \( X \) into \( \{e_l\}_{l=1,2,3} \). Consequently, we will write capital Latin indices to denote both \( \mathfrak{so}(3) \) indices and indices with respect to the orthonormal frame \( \{e_l\} \). Then we can decompose the covariant derivative \( \nabla^M \) on \( \mathcal{M} \) coming from the Lévi-Cività connection as

\[ \nabla^M_{e_k} e_l = (\nabla^M_{e_k} e_l)^{\perp} + (\nabla^M_{e_k} e_l)^{\parallel} \]

\[ = \sum_j g(\nabla^\Sigma_{e_k} e_l, e_j) e_j + sg(\nabla^M_{e_k} e_l, n)n \]

\[ = \sum_j g(\nabla^\Sigma_{e_k} e_l, e_j) e_j + sg(K(e_k, e_l), n)n, \]  

(C.1)

where in the last line we denote by \( \nabla^\Sigma \) the covariant derivative coming from the Lévi-Cività connection on \( \Sigma \), which is for any \( e_l, e_j \in T_m \Sigma \) precisely given by the tangential component of \( \nabla^M \). Now we introduce the shorthands

\[ k_{kl} = k(e_k, e_l) = g(K(e_k, e_l), n) := g(\nabla^M_{e_k} e_l, n) = -g(e_l, \nabla^M_{e_k} n), \]

(C.2)
where \( k_{KI} \) are the components of extrinsic curvature and \( W_n : T_m \Sigma \to T_n \Sigma \) is called the Weingarten map\(^7\). Note the symmetry of \( k_{KI} = \frac{1}{2}(k_{KI} + k_{IK}) \) by construction, the torsion \( T^M \) of the Lévi-Civita-connection \( \bar{\nabla}^M \) vanishes and we have

\[
T^M(e_K, e_I) = 0 = \nabla^M_{e_K} e_I - \nabla^M_{e_I} e_K - [e_K, e_I]
\]

and therefore \( k_{KI} = k_{IK} \) if the vectors \( e_K, e_I \) commute, that is, \([e_K, e_I] = 0\). For the Ashtekar connection, we have for any two smooth vector fields \( X, Y \in \Gamma(T\Sigma) \)

\[
\nabla_Y^A X = \nabla_X^A Y + \beta W_n(X) \cdot Y,
\]

where \( \beta \) is the Barbero–Immirzi parameter and we have introduced the product

\[
X \cdot Y = \sum_{I,J,K} X_I Y_J e_I e_J = \sum_{I,J,K} X_I Y_J e_{IJK} e_K.
\]

where we have expanded \( X = \sum_I g(X, e_I) e_I =: \sum_I X_I e_I \). We can write this for the orthonormal basis \( \{e_I\}_{I=1,2,3} \) as

\[
\nabla_{e_I}^A e_J = \nabla_{e_I}^e e_J - \beta \sum_L k_{IL} e_L \cdot e_J
\]

\[
= \nabla_{e_I}^e e_J - \beta \sum_{L,M} k_{IL} e_{LM} e_M
\]

or in components using \( g(e_I, e_J) = \delta_{IJ} \)

\[
A_{IJK} := (A_I)_{JK} = g(\nabla_{e_I}^e e_J, e_K) = \Gamma_{IJK} - \beta \sum_L k_{IL} e_{LJK}.
\]

Here \( \Gamma_{KLM} \) correspond to the usual Christoffel symbols\(^18\).

\section*{C.2. Curvature of the Ashtekar connection}

Our previous considerations imply for the curvature of the Ashtekar connection

\[
R^A(e_I, e_J) e_K = \nabla_{e_I}^A \nabla_{e_J}^A e_K - \nabla_{e_J}^A \nabla_{e_I}^A e_K - \nabla_{[e_I, e_J]}^A e_K
\]

\[
= \sum_M \left\{ (e_I (A_{JKM}) - e_J (A_{IKM}) + \sum_L (A_{JKL}A_{ILM} - A_{IKL}A_{JLM} - C_{ILM}A_{JK}) \right\} e_M
\]

\[
= R(e_I, e_J) e_K + \beta \sum_{L,M} \left\{ (e_I (k_{IL}) - e_J (k_{IL})) e_{KLM} + \sum_N \left( \Gamma_{ILN} - \Gamma_{JLN} \right) k_{LMN} e_{KMN} \right\} e_M
\]

\[
+ \beta^2 \sum_M \left( (k_{IK} k_{JM} - k_{IM} k_{JK}) e_M \right) \tag{C.7}
\]

where \( C_{ILM} = \Gamma_{ILN} - \Gamma_{JLN} = g([e_I, e_J], e_K) \) from the torsion freeness of \( \bar{\nabla}^E \). This gives for the scalar curvature of \( \bar{\nabla}^A \)

\[
R^A := \sum_{I,J} g(R^A(e_I, e_J) e_J, e_I)
\]

\[
= \sum_{I,J} \left\{ (e_I (A_{IJI}) - e_J (A_{IJI}) + \sum_L (A_{IJL}A_{ILI} - A_{IHL}A_{JLI} - C_{IHL}A_{ILI}) \right\}
\]

\[
= R + \beta^2 \sum_{I,J} \left\{ (k_{IK} k_{IJ} - k_{IJK}) \right\}
\]

\[
= R - \beta^2 \beta \left\{ (tr(k))^2 - tr(k^2) \right\}. \tag{C.8}
\]

\(^7\)By construction, \( W_n \) has only components in \( T_m \Sigma \). We have \( g(n, n) = s = \text{const} \) and hence \( (\bar{\nabla}^A_M)(n, n) = 0 = e_K(n) + 2g(\bar{\nabla}^A_M n, n) \), where we use the symmetry of \( g \) and the fact that \( \bar{\nabla}^A \) is metric.

\(^8\)Its components can be obtained for an arbitrary basis from the well-known Koszul formula [23]: \( \Gamma_{IJK} = g(\nabla_{e_I}^e e_J, e_K) = \frac{1}{4} \left( e_I g(e_J, e_K) - e_K g(e_I, e_J) + e_J g(e_K, e_I) - g(e_I, [e_J, e_K]) + g(e_K, [e_I, e_J]) + g(e_J, [e_K, e_I]) \right) \).
where the term proportional to $\beta$ vanishes, because it is antisymmetric in $I, J$. This can be compared to the decomposition of the four-dimensional curvature scalar under a $(3+1)$-decomposition $\mathcal{M} \cong \Sigma \times \mathbb{R}$ of the manifold [15]:

$$R^M = R^X + s((\text{tr}(k))^2 - \text{tr}(k^2)) + 2s \text{div}(- \text{tr}(k)n + \nabla_n^M n)$$

$$= R^X + (s + \beta^2)((\text{tr}(k))^2 - \text{tr}(k^2)) + 2s \text{div}(- \text{tr}(k)n + \nabla_n^M n).$$ (C.9)

Here $\text{div}(X) = \sum_i g(\nabla_{e_i}^M X_e) + s g(\nabla_n^M X, n)$ denotes the divergence of a vector field $X \in \Gamma(T\mathcal{M})$. Start with the Hamilton constraint given in [2], p 124, as

$$H = - \frac{s}{\sqrt{\det q}} (K^i_j K^j_i - K^i_i K^j_j) E_i^a E_j^b - \sqrt{\det q} R,$$ (C.10)

where $i, j, l = 1, 2, 3$ denote $su(2)$-indices and $a, b$ denote indices in $T\Sigma$; moreover, the densitized triads $E_i^a$ are given in terms of the non-densitized inverse triads $e_i^a$ as $E_i^a = \sqrt{\det q} e_i^a$. This can be rewritten as

$$H = - \frac{s}{\sqrt{\det q}} (\beta^2 - (\text{tr}(K))^2 - (\text{tr}(k))^2) - R.$$ (C.11)

Using (C.8), this can be written as the well-known form

$$- \frac{\beta^2}{\sqrt{\det q}} H = sR^X + (\beta^2 - s)R.$$ (C.12)

This finishes the short digression on the coordinate-free description of the Ashtekar variables. We will use this formalism in order to reobtain the geometric meaning of the parameter $c$ in the case of the homogeneous isotropic Bianchi I universe.

C.3. Interpretation of the symmetric connection for spatially homogeneous isotropic models

In what follows we will examine the properties of the Ashtekar connection on Friedmann–Robertson–Walker spacetimes, whose metric tensor can be written on $\mathcal{M}_I \equiv I \times \Sigma (I \subset \mathbb{R},$ open) as

$$g = s \, dt \otimes dt + a(t)^2 g_\Sigma,$$ (C.13)

where $(\Sigma, g_\Sigma)$ is a connected Riemannian manifold with the constant sectional curvature $K_\Sigma = \kappa$ and $g_\Sigma$ is independent of $t$. At every $m \in \mathcal{M}_I$ we decompose $T_m \mathcal{M}_I$ into the tangent space $T_m \Sigma$ and an orthogonal direction spanned by the surface normal $n = \frac{\partial}{\partial t}$. Assuming $[n, X] = 0 \ \forall X \in T_m \Sigma$, we have

$$g(W_n(X), Y) \overset{(C.2)}{=} g(\nabla^{\mathcal{M}}_n n, Y)$$

footnote 16 \[ \overset{\text{footnote 16}}{=} \frac{1}{2} \frac{\partial}{\partial t} (a(t)^2 g_\Sigma(X, Y)) \]

$$= a(t) \dot{a}(t) g_\Sigma(X, Y) \rightarrow W_n = \frac{\dot{a}(t)}{a(t)} \text{Id}_{T\Sigma}.$$ (C.14)

Hence, (C.3) reads

$$\nabla^X_Y = \nabla^\Sigma_Y + \beta \frac{\dot{a}}{a} X \bullet Y$$ (C.15)

or in components as in (C.5)

$$A_{IJK} := (A_I)_{JK} = g(\nabla^I e_J, e_K) = \Gamma_{JK} - \beta \sum_L k_{IL} \epsilon_{LJK}$$

$$= \frac{1}{2} (C_{IK} + C_{JK} - C_{IK}) - \beta \frac{\dot{a}}{a} \sum_L s_{LJK}.$$ (C.16)

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where we have used the product structure $e_I \cdot e_J = \sum_{IJK} e_{IJK} e_K$. Moreover, $C_{IJK}$ is $g(e_I, e_J, e_K)$, and the result for $\Gamma_{IJK}$ follows directly from footnote 18. In the case of Bianchi I, we have $\Gamma_{IJK} = 0$ and $g(\partial_m, \partial_n) = a^2 \delta_{mn}$ hence the coordinate basis $\{\partial_n\}$ can be related to an orthonormal basis $\{e_I\}$ of $T \Sigma$ by $\partial_n = \sum m e_m e_I = a \delta_{mn} e_I$. Hence, $A_{IJK} = -\beta \frac{2}{3} \sum_I \delta_I e_I e_{IJK} = (A_I^M e_M)_{JK} = -A_{I}^M e_{MJK}$. Therefore, $A_I^M = \beta \frac{2}{3} \delta_I^M$ and $A^M = \sum_I e_I A_I^M = \beta a \frac{2}{3} \sum_I \delta_I \delta_I^M = \beta \delta_{M}$. Therefore, we obtain the usual [13] identification

$$c = \beta \dot{a}.$$  

(C.17)

This yields for (C.7) in an orthonormal frame $\{e_I\}$ on $T \Sigma$

$$R^I(e_I, e_J) e_K = R^I_{JK} = (\dot{a}/a)^2 \sum_I e_I e_J e_K$$  

(C.18)

and we consistently obtain for Bianchi I with $R^I = 0$ (C.8)

$$R^I = -6\beta \left(\frac{\dot{a}}{a}\right)^2$$

where we have used the fact that $k_{IL} = -\frac{2}{3} \delta_{IL}$ and dim $\Sigma = 3$. Evaluating (C.9) for Bianchi I, we find in agreement with [24], p 97,

$$R^M = -6\beta \left(\frac{\dot{a}}{a} + \frac{\ddot{a}}{a}\right).$$  

(C.20)

This shows that in a situation where $\ddot{a} \ll \dot{a}$, one can understand the limit $c \to -\infty$ as a blow-up of scalar 4-curvature on $M$, that is, a situation close to a singularity. The fact that in the limit $c \to -\infty$ we reobtain the LQC framework indicates an affirmation of the BKL picture, as discussed in section 5.4.

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