Polyharmonic Maass forms for PSL(2, \mathbb{Z})

Jeffrey C. Lagarias¹ · Robert C. Rhoades²

Dedicated to the memory of Marvin Knopp.

Received: 1 April 2014 / Accepted: 30 July 2015 / Published online: 7 March 2016
© Springer Science+Business Media New York 2016

Abstract We discuss the space of polyharmonic Maass forms of even integer weight on PSL(2, \mathbb{Z}) \backslash \mathbb{H}. We explain the role of the real-analytic Eisenstein series \( E_k(z, s) \) and the differential operator \( \frac{\partial}{\partial s} \) in this theory.

Keywords Modular forms · Polyharmonic · Harmonic · Maass forms

Mathematics Subject Classification 11F55 · 11F37 · 11F12

1 Introduction

Classical Maass forms of even integer weight \( k \in 2\mathbb{Z} \) for PSL(2, \mathbb{Z}) are smooth functions on \( \mathbb{H} := \{ z = x + iy \in \mathbb{C} : \text{Im}(z) > 0 \} \) such that

(1) (Modular invariance condition) \( f(z) = f \bigg|_k \gamma(z) \) for each \( z \in \mathbb{H} \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}) \) where the \textit{slash operator} of weight \( k \in \mathbb{Z} \) is defined by

---

Research of the first author was partially supported by NSF Grants DMS-1101373 and DMS-1401224. Research of the second author was partially supported by an NSF Mathematical Sciences Postdoctoral Fellowship.

Robert C. Rhoades
rob.rhoades@gmail.com

Jeffrey C. Lagarias
lagarias@umich.edu

¹ Department of Mathematics, The University of Michigan, Ann Arbor, MI 48109-1043, USA
² Center for Communications Research, Princeton, NJ 08540, USA
\[ g \big|_{k} \gamma(z) = (cz + d)^{-k} g \left( \frac{az + b}{cz + d} \right). \]

That is, \( f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z) \) holds for \( \gamma \in \text{PSL}(2, \mathbb{Z}) \).

(2) (Laplacian eigenfunction condition) \( f \) satisfies \( (\Delta_k - \lambda) f = 0 \) for some \( \lambda \in \mathbb{C} \), where
\[
\Delta_k := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]
is the weight \( k \) hyperbolic Laplacian. (We follow the convention of Maass [24] for the sign of the Laplacian. Many authors call \(-\Delta_k\) the hyperbolic Laplacian, for example [15].)

(3) (Moderate growth condition) There exists an \( \alpha \in \mathbb{R} \) such that \( f(x + iy) = O(y^\alpha) \) as \( y \to \infty \), uniformly in \( x \in \mathbb{R} \).

Such forms have played an important role in number theory and automorphic forms, see for instance, Sect. 1.9 of [12]. More generally, we may consider forms of integer weight \( k \in \mathbb{Z} \), but nothing is gained since all odd integer weight forms on \( \text{PSL}(2, \mathbb{Z}) \) must vanish identically, due to the weight \( k \) modular invariance condition applied to \( \gamma = \pm \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \).

In this paper we study the situation where the eigenvalue \( \lambda = 0 \), which is the case corresponding to holomorphic modular forms, where we relax the Laplacian eigenfunction condition to require only that the forms satisfy
\[
(\Delta_k)^m f(z) = 0 \tag{1.1}
\]
for some non-negative integer \( m \). We call such \( f(z) \) polyharmonic Maass forms, in parallel with the literature on polyharmonic functions for the Euclidean Laplacian, on which there is an extensive literature, starting around 1900 (see Almansi [1], Aronszajn et al. [3], Render [28]). The integer parameter \( m \) in (1.1) might be termed the (harmonic) order in parallel with the literature on polyharmonic functions. However, we will use the term harmonic depth because the term “order” is used in conflicting ways in the literature.\(^1\) Moreover, we allow the harmonic depth to take half-integer values, as follows. We assign to any non-zero holomorphic modular form \( f(z) \) the harmonic depth \( \frac{1}{2} \), because it is annihilated by the \( \partial \)-bar operator \( \overline{\partial} \) which is “half” of the harmonic Laplacian \( \Delta_k = \left( y^2 \frac{\partial}{\partial z} + 2iky \right) \frac{\partial}{\overline{\partial} z} \). In addition, to any weight \( k \) Maass form \( f(z) \) such that \( (\Delta_k)^m f(z) = g(z) \) with \( g(z) \) a non-zero holomorphic modular form, we assign harmonic depth \( m + \frac{1}{2} \).

A sequel paper [2] will treat functions satisfying the more general equation
\[
(\Delta_k - \lambda)^m f(z) = 0 \tag{1.2}
\]
\(^1\) The partial differential equations literature assigns order \( 2m \) to \( (\Delta_k)^m \) when treated in terms of the differential operators \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \). A second conflict is that \( p \)-harmonic Maass form is used by Bruggeman [9] with a different meaning, to refer to a function annihilated by the \( p \)-Laplacian, in which \( p \) corresponds to the weight parameter \( k \) in our notation.
Polyharmonic Maass forms for $\text{PSL}(2, \mathbb{Z})$

for some non-negative integer $m$, where $\lambda \in \mathbb{C}$. We call functions satisfying (1.2) \textit{shifted polyharmonic functions} with \textit{eigenvalue shift} $\lambda$. If they transform as modular forms of weight $k$, we call them \textit{shifted polyharmonic depth $m$ Maass forms}. We let $V_k^m(\lambda)$ denote the vector space of all such weight $k$ forms with eigenvalue $\lambda$ on $\text{PSL}(2, \mathbb{Z})$ that have moderate growth at the cusp. For $\lambda \neq 0$ we refer to the minimal $m$ annihilating such a function as its \textit{(shifted) harmonic depth}; it is always an integer since holomorphic forms do not occur. From this more general perspective the case $\lambda = 0$ is exceptional in allowing forms having half-integer depth.

In this paper we determine the spaces $V_k^m(0)$ for all even integer weights $k$ for the full modular group $\text{PSL}(2, \mathbb{Z})$, showing that it is finite dimensional and explicitly exhibiting a full set of basis elements. The finite dimensionality of these spaces has entirely to do with the moderate growth condition; if this is relaxed, then the resulting space of solutions can be infinite dimensional. Our main observation is that the new members of the vector spaces in the harmonic depth $m$ case for $m \geq 3/2$ involve derivatives in the $s$-variable of non-holomorphic Eisenstein series $E_k(z, s)$, evaluated at $s = 0$.

1.1 Polyharmonic Maass forms

Forms satisfying the first three conditions of Sect. 1 with the Laplacian eigenfunction condition relaxed to $(\Delta_k)^m f = 0$ are called weight $k$ $m$-harmonic Maass forms for $\text{PSL}(2, \mathbb{Z})$. Let $V_k^m(0)$ denote the vector space of such functions, allowing $m \in \frac{1}{2}\mathbb{Z}$ with $m \geq 1/2$. It is known that this space is finite dimensional, without explicitly determining the dimension, see [5, Theorem 8.5]. The object of this paper is to give an explicit construction of a basis of $V_k^m(0)$. Moreover, we explain the role of the non-holomorphic Eisenstein series in this construction.

The non-holomorphic Eisenstein series $E_0(z, s)$ of weight 0 for $\text{PSL}(2, \mathbb{Z})$, is given by

$$E_0(z, s) := \frac{1}{2} \left( \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{y^s}{|mz + n|^{2s}} \right) = \frac{1}{2} \xi(2s) \left( \sum_{(c,d) \in \mathbb{Z}^2 : (c,d) = 1} \frac{y^s}{|cz + d|^{2s}} \right)$$

(1.3)

with $z = x + iy \in \mathbb{H}$ and $\Re(s) > 1$. The completed weight 0 Eisenstein series is

$$\hat{E}_0(z, s) := \pi^{-s} \Gamma(s)E_0(z, s).$$

For each $z \in \mathbb{H}$ the completed series is known to analytically continue to a meromorphic function of $s$ that satisfies the functional equation

\[\sum_{z \in \mathbb{H}} \hat{E}_0(z, s) f(z) = 0.\]
\( \hat{E}_0(z, s) = \hat{E}_0(z, 1 - s) \).

The singularities of \( \hat{E}_0(z, s) \) are simple poles at \( s = 0 \) and \( s = 1 \). For our purposes it is better to work with the \textit{doubly completed non-holomorphic Eisenstein series}

\[
\hat{E}_0(z, s) = s(s - 1)\pi^{-s} \Gamma(s)E_0(z, s)
\]  

(1.4)

which removes the two poles and maintains the symmetry between \( s \) and \( 1 - s \). See Theorem 3.6 for details. We define the Taylor coefficients with respect to \( s \) by \( F_n(z) \), namely

\[
\hat{E}_0(z, s) = \sum_{n=0}^{\infty} F_n(z)s^n.
\]  

(1.5)

**Theorem 1.1** For integer \( m \geq 1 \) the vector space \( V_0^m(0) \) of weight 0 \( m \)-harmonic Maass forms has \( V_0^m(0) = V_0^{m-1/2}(0) \) and is \( m \)-dimensional. Moreover, the set \( \{F_0(z), \ldots, F_{m-1}(z)\} \) is a basis for \( V_0^m(0) \).

We call this set the \textit{Taylor basis}. For example, the space \( V_0^{1/2}(0) \) is one dimensional and is spanned by the constant functions.

Automorphic forms, including Maass forms, are related to forms of different weights via differential operators called the (Maass) weight raising and lowering operators. See, for instance, Sect. 2.1 of [12]. In the theory of harmonic Maass forms there is a closely related operator which moves harmonic forms of weight \( k \) to forms of weight \( 2 - k \). This operator was introduced by Bruinier and Funke [10] and is given by

\[
\xi_k := 2iy^k \frac{\partial}{\partial \bar{z}}.
\]  

(1.6)

In other words, if \( f : \mathbb{H} \to \mathbb{C} \) is a smooth function, then \( \xi_k(f) = 2iy^k \frac{\partial}{\partial \bar{z}} f \), using the Wirtinger derivative \( \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}) \).

Hence, in the context of studying weight 0 harmonic forms it is natural to also introduce and study weight 2 forms. We define the completed real-analytic weight 2 Eisenstein series for \( \text{Re}(s) > 1 \) by

\[
\hat{E}_2(z, s) = \pi^{-(s+1)} \Gamma(s + 1)E_2(z, s)
\]

\[
= \pi^{-(s+1)} \Gamma(s + 1) \left( \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{y^s}{(mz + n)^2|mz+n|^{2s}} \right),
\]  

(1.7)

and the \textit{doubly completed real-analytic weight 2 Eisenstein series} by

\[
\hat{E}_2(z, s) := (s + 1)s\pi^{-(s+1)} \Gamma(s + 1)|\zeta(2s + 2)\left( \frac{1}{2} \sum_{(c,d) \neq (c,d) = 1} \frac{y^s}{(cz+d)^2|cz+d|^{2s}} \right).
\]  

(1.8)
Parallel to the case of $\hat{E}_0(z, s)$ this series has an analytic continuation to $s \in \mathbb{C}$ (see Theorem 3.6). Define its Taylor coefficients by

$$\hat{E}_2(z, s) =: \sum_{n=0}^{\infty} G_n(z) s^n. \quad (1.9)$$

In analogy with Theorem 1.1 we have the following.

**Theorem 1.2** For integer $m \geq 0$, the vector space $V_2^m(0)$ of weight 2, $m$-harmonic Maass forms has $V_2^m(0) = V_2^{m+1/2}(0)$ and is $m$-dimensional. Moreover, $G_0(z) \equiv 0$ and the set $\{G_1(z), \ldots, G_m(z)\}$ is a basis for $V_2^m(0)$.

Again we call this a Taylor basis of $V_2^m(0)$. For example, the space $V_{1/2}^1(0) = \{0\}$.

To explore the relationship between these weight 0 and weight 2 forms it is convenient to introduce symmetrized versions of the Taylor coefficients $\{F_j\}$ and $\{G_j\}$. Define the symmetrized Taylor basis functions

$$\tilde{F}_n(z) := (-1)^n \left( F_n(z) + \sum_{\ell=1}^{n} \binom{n + \ell}{n} F_{n-\ell}(z) \right) \quad (1.10)$$

$$\tilde{G}_n(z) := G_n(z) + \sum_{\ell=1}^{n} (-1)^\ell \binom{n + \ell}{n} G_{n-\ell}(z). \quad (1.11)$$

These functions are obtained by a triangular change of basis from the Taylor bases given above.

**Theorem 1.3** For the symmetrical basis functions (1.10) and (1.11), we have

1. $\Delta_0 \tilde{F}_n(z) = \tilde{F}_{n-1}(z)$ and $\xi_0 \tilde{F}_n(z) = \tilde{G}_n(z)$
2. $\Delta_2 G_n(z) = G_{n-1}(z)$ and $\xi_2 G_n(z) = F_{n-1}(z)$.

This theorem can be summarized by the picture in Fig. 1.

One may view this picture as a set of two “towers” interconnected by a set of “ramps” which are given by the operators $\xi_k$ and $\xi_{2-k}$. The “ramp” structure incorporates well-known factorizations of the unshifted Laplacians

$$\Delta_k = \xi_{2-k} \xi_k, \quad (1.12)$$

which themselves move down the towers. The bottom elements of the towers of symmetrized Taylor bases are well known. We have

$$\tilde{F}_0(z) = \frac{1}{2} \quad (1.13)$$

$$\tilde{F}_1(z) = \frac{1}{2} \gamma - \frac{1}{2} \log(4\pi e) - \log \left( \sqrt{y} |\Delta(z)|^{1/4} \right), \quad (1.14)$$
where $\gamma$ is Euler’s constant. The second of these formulas is essentially equivalent to the “$s = 0$” version of Kronecker’s first limit formula for the Eisenstein series $E_0(z, s)$. In addition we have

$$\widetilde{G}_0(z) = 0$$
$$\widetilde{G}_1(z) = \frac{\pi}{6} - \frac{1}{2y} - 4\pi \left( \sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi inz} \right)$$

which is related to the Fourier series expansion for $E_2(z, s)$.

Complementing this “tower” structure given by the Laplacians $\Delta_k$ is the operator $\frac{\partial}{\partial s}$, which allows movement up the “tower.” More precisely, the Taylor coefficients are given by $F_n(z) = \frac{1}{n!} \frac{\partial^n}{\partial s^n} E(z, s) \mid_{s=0}$. To move up the “tower” one level at a time, set

\begin{align*}
\text{harmonic depth} & \quad \text{weight 0} & \quad \text{weight 2} & \quad \text{harmonic depth} \\
\vdots & \quad \xi_2 & \quad \Delta_2 & \quad \vdots \\
5/2 & \quad \widetilde{F}_2(z) & \quad \Delta_0 & \quad \widetilde{G}_2(z) & \quad 2 \\
\xi_0 & \quad \Delta_0 & \quad \widetilde{G}_1(z) & \quad 1 \\
3/2 & \quad \widetilde{F}_1(z) & \quad \Delta_0 & \quad \widetilde{G}_1(z) & \quad 1 \\
\xi_2 & \quad \Delta_0 & \quad \widetilde{G}_0(z) & \equiv 0 & \quad 0 \\
1/2 & \quad \widetilde{F}_0(z) & \quad \Delta_0 & \quad \widetilde{G}_0(z) & \equiv 0
\end{align*}

Fig. 1 Tower and ramp structure for weights 0 and 2
\[
\widehat{E}_0^{(n)}(z, s) := \frac{1}{n!} \left( \frac{\partial}{\partial s} \right)^n \widehat{E}_0(z, s).
\]

Then we have \(F_n(z) = \widehat{E}_0^{(n)}(z, 0)\) and
\[
F_{n+1}(z) = \frac{1}{(n+1)} \left. \frac{\partial}{\partial s} \widehat{E}_0^{(n)}(z, s) \right|_{s=0}.
\]

Hence, there is a sort of “switching” between the differential operator \(\frac{d}{ds}\) and the Laplacian operator \(\Delta_k\) moving down the tower.

There are a number of references in the literature where the differentiation with respect to \(s\) has appeared in the context of Eisenstein series and automorphic forms. For instance, the works of Kudla–Rapoport–Yang [20–22,33,34], Duke–Imamoglu–Toth [13,14], and Duke and Li [16, p.2] contain such results. W. Duke pointed out that in 1961 Siegel [30, Chap. 3, p. 83] noted a recursion relating the Laplacian acting on Taylor coefficients of weight 0 non-holomorphic Eisenstein series, which Proposition 8.3 extends. The book of Bruggeman [8] makes a general study of families of automorphic forms \(F(z, s)\) in which \(s \in \mathbb{C}\) is the family parameter.\(^2\)

While harmonic Maass forms have played a significant role in the emerging theory of Ramanujan’s mock theta functions (see [27], for example), there are few examples of \(m\)-harmonic Maass forms for \(m > 1\). One exception is the appearance of a 2-harmonic form in the work of Bringmann–Diamantis–Raum [6]. Their work is related to non-critical modular \(L\)-values.

1.2 Shifted and arbitrary integer weight polyharmonic Maass forms

In Sect. 2 we state corresponding results for polyharmonic Maass forms (i.e., \(\lambda = 0\)) for all even integer weights \(k\) and all harmonic depths \(m \geq 1/2\). This case is complicated by the presence of holomorphic cusp forms, which appear at harmonic depth 1/2. We determine the dimension for each \((k, m)\). An important feature is Proposition 6.6, which implies that any non-zero holomorphic cusp form is not the image under the operator \(\xi_{2-k}\) of any harmonic depth \(m = 3/2\) polyharmonic Maass form, hence not the image under \(\Delta_k\) of a harmonic depth 2 Maass form. This result shows that polyharmonic Maass forms are of value in understanding classical modular forms. Namely, for \(k \geq 4\) it provides a new characterization of holomorphic Eisenstein series viewed inside the vector space \(M_k\) of holomorphic modular forms: For \(k \geq 4\) the one-dimensional space of holomorphic Eisenstein series comprises the range of the Laplacian \(\Delta_k\) acting on the space \(V^2_k(0)\) of depth 2 polyharmonic Maass forms having moderate growth at the cusp.

As already mentioned, a sequel paper [2] will treat shifted polyharmonic Maass forms satisfying \((\Delta_k - \lambda)^m f(z) = 0\) for a fixed \(\lambda \neq 0\). The case of general \(\lambda\) includes Maass cusp forms, and via non-holomorphic Eisenstein series also has a connection with the Riemann zeta zeros, observed by Zagier [35]. The harmonic depth of such

\(^2\) Bruggeman [8, Sect. 1.2.4] uses a family variable \(s\) that equals \(s_{\text{usual}} - \frac{1}{2}\) with \(s = s_{\text{usual}}\) used here.
f is always an integer m. The analogs of Figs. 1 and 2 of this paper for \( \lambda \neq 0 \) have “towers” with actions of Laplacians \( \Delta_k \) and \( \Delta_{2-k} \); however, the “ramps” are replaced by rungs of a “ladder” at each fixed depth m, with the depth preserved by the action of the operators \( \xi_k \) and \( \xi_{2-k} \).

1.3 Roadmap

Section 2 states further main results for polyharmonic Maass forms of arbitrary even integer weight other than 0 or 2. Section 3 contains known results on classical holomorphic modular forms (Sect. 3.1), and basic facts on non-holomorphic Eisenstein series, including their Fourier expansions and functional equations (Sect. 3.2). Section 4 formulates results on the Fourier expansions of polyharmonic Maass forms, some based on [2]. Section 5 presents results about the Bruinier-Funke non-holomorphic differential operator \( \xi_k \), showing that it preserves spaces of functions of moderate growth at the cusp. Section 6 establishes for \( \lambda = 0 \) that holomorphic cusp forms are not the image under \( \Delta_k \) of any bi-harmonic Maass form. Section 7 computes the action of \( \xi_k \) on non-holomorphic Eisenstein series. Section 8 gives recursions for the Taylor series coefficients of \( \hat{E}_k(z; s_0) \) in the \( s \)-variable, which are functions of \( z \). It shows these coefficients are polyharmonic Maass forms and determine recursion relations they satisfy under the action of \( \Delta_k \) and \( \xi_k \). Section 9 constructs modified bases of \( V^m_k(0) \) which satisfies simpler recursion relations with respect to \( \Delta_k \) and \( \xi_k \). Section 10 presents proofs of the main theorems of Sect. 1, namely Theorems 1.1, 1.2, and 1.3. Section 11 then completes the proofs of the main results of Sect. 2.

1.4 Notation

We define the completed Riemann zeta function by

\[
\hat{\zeta}(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s).
\]

For a fixed \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we write \( \gamma \cdot z = \frac{az + b}{cz + d} \).

2 General even integer weight polyharmonic Maass forms

Section 1 presented results dealing with weight 0 and weight 2 polyharmonic Maass forms. This section describes the analogous picture for arbitrary even integer weights. The scenario for arbitrary weight is complicated by the presence of cuspidal holomorphic modular forms.

**Definition 2.1** A holomorphic modular form for \( \text{SL}_2(\mathbb{Z}) \) of weight \( k \in \mathbb{Z} \) is a holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) satisfying \( f |_k \gamma(z) = f(z) \), i.e.

\[
(cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right) = f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})
\]
and \( f(z) = O(y^A) \) as \( y \to \infty \) for some \( A \). Moreover, if \( f(z) = O(y^{-k/2}) \) as \( y \to \infty \) \( f \) is called a cusp form.

The set of all holomorphic modular forms is a vector space, denoted \( M_k \). Moreover, let \( S_k \) denote the space of all such cusp forms. With our convention on half-integer harmonic depth we have \( M_k = V_{1/2}^k \).

For \( z \in \mathbb{H} \), an even integer \( k \), and \( \text{Re}(s) > 1 \) define for \( \text{Re}(s) > 1 - k \) the (non-holomorphic) Eisenstein series

\[
E_k(z, s) := \frac{1}{2} \left( \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \text{SL}_2(\mathbb{Z})} \psi_s \left| \gamma(z) \right| \right)
\]

\[
= \frac{1}{2} \zeta(2s) \left( \sum_{(c,d) \in \mathbb{Z}^2 : (c,d) = 1} \frac{y^s}{(cz + d)^k |cz + d|^{2s}} \right), \tag{2.1}
\]

where \( \psi_s(z) = \text{Im}(z)^s \). For \( k \geq 4 \) an even integer the value \( s = 0 \) gives the unnormalized holomorphic Eisenstein series

\[
E_k(z, 0) := \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(mz + n)^k}. \tag{2.2}
\]

The Eisenstein series \( E_k(z, s) \) meromorphically continues in the \( s \)-variable with a functional equation given in terms of the completed non-holomorphic Eisenstein series:

\[
\hat{E}_k(z, s) := \pi^{-(s + \frac{k}{2})} \Gamma(s + \frac{k}{2} + \frac{|k|}{2}) E_k(z, s). \tag{2.3}
\]

The functional equation is

\[
\hat{E}_k(z, s) = \hat{E}_k(z, 1 - k - s),
\]

which has critical line \( \text{Re}(s) = \frac{1-k}{2} \). The function \( \hat{E}_k(z, s) \) is an entire function of \( s \) for \( k \neq 0 \) and for \( k = 0 \) it has simple poles at \( s = 0, 1 \), see Theorem 3.6.

Define the doubly completed (non-holomorphic) Eisenstein series

\[
\hat{\hat{E}}_k(z, s) := (s + \frac{k}{2})(s + \frac{k}{2} - 1) \hat{E}_k(z, s) \tag{2.4}
\]
which is an entire function of \( s \). By convention we denote the Taylor coefficients at
\( s = 0 \) for \( k \in 2\mathbb{Z} \) by
\[
\tilde{E}_k(z, s) = \begin{cases} 
\sum_{n=0}^{\infty} F_{n,k}(z)s^n & \text{for weights } k \leq 0, \\
\sum_{n=0}^{\infty} G_{n,k}(z)s^n & \text{for weights } k \geq 2.
\end{cases} \tag{2.5}
\]

For weights 0 and 2 we have \( F_{n,0}(z) = F_n(z) \) and \( G_{n,2}(z) = G_n(z) \) as given in Sect. 1. The cases \( k = 0, 2 \) are distinguished by the property that the factor \((s + \frac{k}{2})(s + \frac{k}{2} - 1)\) in \( \tilde{E}_k(z, s) \) has a zero at \( s = 0 \), leading to different behavior than the general case; this is one reason we have treated them separately. The “ramp” and “tower” structure is slightly altered in the general case, as pictured below.

The following two theorems contain results for general even integer weights \( k \neq 0, 2 \) which parallel Theorems 1.1, 1.2, and 1.3.

**Theorem 2.2** Let \( m \geq 1 \) be an integer, and \( k \in 2\mathbb{Z} \) an even integer, with \( k \neq 0 \) or 2.

1. For an even integer \( k \leq -2 \), \( V_k^m(0) \) is \( m \)-dimensional. Moreover, \( \{F_{0,k}(z), \ldots, F_{m-1,k}(z)\} \) is a basis for \( V_k^m(0) \). In this case \( V_k^{1/2}(0) = \{0\} \) and \( \dim(V_k^m(0)) = \dim(V_k^{m+\frac{1}{2}}(0)) \) for all \( m \geq 0 \).
2. For an even integer weight \( k \geq 4 \),
\[
V_k^m(0) = E_k^m + S_k
\]
where \( E_k^m \) is an \( m \)-dimensional subspace spanned by \( \{G_{0,k}(z), \ldots, G_{m-1,k}(z)\} \). Moreover, \( S_k \) consists of cusp forms and has dimension \( \max([k/12] - 1, 0) \) if \( k \equiv 2 \) (mod 12) and \( [k/12] \) if \( k \neq 2 \) (mod 12) and \([x]\) is the largest integer less than or equal to \( x \). In this case \( V_k^{1/2}(0) = M_k \) contains the holomorphic Eisenstein series \( G_{0,k}(z) \) and cusp forms \( S_k \), and \( \dim(V_k^m(0)) = \dim(V_k^{m+\frac{1}{2}}(0)) \) for all \( m \geq 1 \).

The next result describes the “ramp” and “tower” structure for \( k \geq 4 \), with corresponding weight \( 2 - k \leq -2 \), using modified basis functions.

**Theorem 2.3** Let \( m \geq 1 \) be an integer and \( k \in 2\mathbb{Z} \) an even integer, with \( k \geq 4 \). Then

1. There are modified basis functions \( \tilde{G}_{n,k}(z) \) and \( \tilde{F}_{n,2-k}(z) \) such that the following are true.
   (a) (“Ramp” relations)
   \[
   \xi_k \tilde{G}_{n,k}(z) = (k-1)\tilde{F}_{n-1,k}(z) \quad \text{and} \quad \xi_{2-k} \tilde{F}_{n,2-k}(z) = \tilde{G}_{n,2-k}(z).
   \]
   (b) (“Tower” relations)
   \[
   \Delta_k \tilde{G}_{n,k}(z) = (k-1)\tilde{G}_{n-1,k}(z) \quad \text{and} \quad \Delta_{2-k} \tilde{F}_{n,2-k}(z) = (k-1)\tilde{F}_{n-1,2-k}(z).
   \]
Fig. 2 Tower and ramp structure for weights $k$ and $2 - k$ for $k \geq 4$

(2) One choice of modified functions takes the form

$$
\tilde{G}_{n,k}(z) = G_{n,k}(z) + \sum_{\ell=1}^{n} \frac{1}{(k-1)^\ell} \left( \frac{n+\ell}{n} \right) G_{n-\ell,k}(z)
$$

$$
\tilde{F}_{n,2-k}(z) = (-1)^n \left( F_{n,k}(z) + (-1)^n \sum_{\ell=1}^{n} (-1)^\ell \frac{1}{(k-1)^\ell} \left( \frac{n+\ell}{n} \right) F_{n-\ell,k}(z) \right).
$$

The modified basis functions are not unique; their form is classified in Theorem 9.1. Figure 2 shows the “tower” and “ramp” structure for positive weights $k \geq 4$, paired with its dual weight $2 - k \leq -2$.

For even weights $k \geq 4$ the holomorphic Eisenstein series $\tilde{G}_{0,k}(z) = c_k E_k(z)$ occurs at harmonic depth $1/2$. This picture is reversed from the case of weight $k = 2$, paired with dual weight 0. These results are proved in Sect. 11.

3 Background

This section recalls well-known results on holomorphic modular forms for $SL(2, \mathbb{Z})$ and presents results on Maass’s Eisenstein series as discussed in Maass [24], and parallel properties for the non-holomorphic Eisenstein series $E_k(z, s)$ considered here.
# 3.1 Classical holomorphic modular forms for SL$_2(\mathbb{Z})$

This section collects some well-known results concerned with holomorphic modular forms for SL($2, \mathbb{Z}$). We reference Zagier’s article [36]; however, there are many good references for these results.

**Theorem 3.1** (Corollary p. 15 of [36]) For $k < 0$ and for $k$ odd, $\dim(M_k) = 0$. For $k \geq 0$ even

$$\dim(M_k) = \begin{cases} \left\lceil \frac{k}{12} \right\rceil + 1 & k \not\equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor & k \equiv 2 \pmod{12} \end{cases}$$

where $[x]$ is the largest integer less than or equal to $x$.

A source of holomorphic modular forms is the Eisenstein series. For $k \in 2\mathbb{Z}$ with $k \geq 4$ the (normalized) holomorphic Eisenstein series is

$$E_k(z) = \frac{1}{2} \sum_{c,d \in \mathbb{Z}, (c,d)=1} \frac{1}{(cz+d)^k}.$$  (3.1)

Clearly $E_k(z) = 1 + o(1) \neq 0$ as $y \to \infty$.

**Proposition 3.2** (Proposition 4, p. 16 of [36]) The ring of holomorphic modular forms $M_* = \bigoplus_{k \in 2\mathbb{Z}} M_k$ for PSL($2, \mathbb{Z}$) is freely generated under multiplication by $E_4(z)$ and $E_6(z)$. Moreover, $M_k = E_k + S_k$ with $S_k$ being the space of cusp forms and $\dim(E_k) = 1$ for $k = 0$ and $k \geq 4$ and is zero otherwise.

To parallel computations in later sections we record the Fourier series of $E_k(z)$.

**Proposition 3.3** (Proposition 5, p. 16 of [36]) The Fourier expansion of the Eisenstein series $E_k(z)$ for an even integer $k \geq 4$ is given by

$$E_k(z) = \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \left( -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \right),$$

where $\sigma_{\ell}(n) = \sum_{d|n, d > 0} d^\ell$, $\zeta$ is the Riemann zeta function, and $B_k$ is the $k$th Bernoulli number defined by the generating function $\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x-1}$.

Finally we recall the notion of a weakly holomorphic modular form.

**Definition 3.4** A weakly holomorphic modular form $f : \mathbb{H} \to \mathbb{C}$ on SL($2, \mathbb{Z}$) of weight $k$ is a holomorphic function satisfying $f |_k \gamma(z) = f(z)$ for all $\gamma \in \text{SL}(2, \mathbb{Z})$ such that $f$ has at most linear exponential growth as $y \to \infty$ (i.e., $f(iy) = O(e^{Ay})$ for some $A$ as $y \to \infty$). We let $M_k^!$ denote the vector space of weakly holomorphic modular forms of weight $k \in 2\mathbb{Z}$ for PSL($2, \mathbb{Z}$).
Such functions may have poles at the cusp, and a basic example of a weakly
holomorphic modular form of weight 0 having such a pole is the modular invari-
ant \( j(z) = 1728 \frac{E_4(z)^3}{E_4(z)^3 - E_6(z)} \). The vector space \( M'_k \) is infinite dimensional for all \( k \geq 0 \).

### 3.2 Properties of non-holomorphic Eisenstein series

Recall from (2.1) that for \( k \in 2\mathbb{Z} \) the non-holomorphic Eisenstein series \( E_k(z, s) \) is

\[
E_k(z, s) := \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{y^s}{|mz + n|^{2s}(mz + n)^k}.
\]  \hspace{1cm} (3.2)

This series converges absolutely for \( \text{Re}(s) > 1 - k \). The resulting function transforms
under elements of \( \text{PSL}(2, \mathbb{Z}) \) as

\[
E_k \left( \frac{az + b}{cz + d}, s \right) = (cz + d)^k E_k(z, s) \quad \text{when} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})
\]

For even weights \( k \geq 4 \) at \( s = 0 \) this function specializes to an unnormalized version \(^3\)
of the holomorphic Eisenstein series \( E_k(z) \), given in Sect. 3.1, with

\[
E_k(z, 0) = \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \frac{1}{(mz + n)^k} = \frac{1}{2} \frac{\Gamma(k)}{\zeta(k)} E_k(z).
\]

Using results on Maass’s non-holomorphic Eisenstein series \( G(z, \bar{z}; \alpha, \beta) \), as
treated in [24, Chap. 4], one may obtain for the completed non-holomorphic Eisenstein
series \( \hat{E}_k(z, s) = \pi^{-\frac{1 + k}{2}} \Gamma(s + \frac{k}{2} + |\frac{k}{2}|) E_k(z, s) \) the following result.

**Proposition 3.5** (Fourier expansion of \( \hat{E}(z, s) \)) For \( k \in 2\mathbb{Z} \), the completed non-
holomorphic Eisenstein series \( \hat{E}(z, s) \) has the Fourier expansion

\[
\hat{E}_k(z, s) = C_0(y, s) + (-1)^{\frac{1}{2}} (\sqrt{2} \pi)^{2s+k} \pi^{-s-\frac{k}{2}} \Gamma \left( s + \frac{k}{2} + |\frac{k}{2}| \right) \times \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\sigma_{2s+k-1}(n)}{\Gamma \left( s + \frac{k}{2} (1 + \text{sgn}(n)) \right)} \left( 2\pi |n| y \right)^{-\frac{k}{2}} W_{1-\text{sgn}(n)k,s+\frac{k}{2}} (4\pi |n| y)^{2\pi i n x}
\]

\(^3\) With our scaling \( E_k(z, 0) = \frac{1}{2} G_k(z) \), where \( G_k(z) \) is the usual unnormalized holomorphic Eisenstein
series in Serre [29].
in which \( W_{\kappa,\mu}(z) \) denotes the Whittaker \( W \)-function and the constant term is

\[
C_0(y, s) = \left( \frac{\Gamma \left( s + \frac{k}{2} + \frac{|k|}{2} \right)}{\Gamma \left( s + \frac{k}{2} \right)} \right) \wzeta(2s + k) y^s + (-1)^\frac{k}{2} \frac{\Gamma \left( s + \frac{k}{2} \right) \Gamma \left( s + \frac{k}{2} + \frac{|k|}{2} \right)}{\Gamma(s + k) \Gamma(s)} \wzeta(2s - k) y^{1-s-k}.
\]

and \( \sigma_s(n) = \sum_{d|n, d > 0} d^s \).

Proof This result is given in [2], translating results in [24, Chap. IV]. The Whittaker \( W \)-function is described in Whittaker and Watson [32, Sect.16.12], and in [2]. \( \square \)

We now collect various analytic properties of the Eisenstein series.

**Theorem 3.6** (Properties of \( E_k(z, s) \)) Let \( k \in 2\mathbb{Z} \).

(1) (Analytic Continuation) For fixed \( z \in \mathbb{H} \), the completed weight \( k \) Eisenstein series

\[
\hat{E}_k(z, s) := \pi^{-s+k} \Gamma \left( s + \frac{k}{2} \right) E_k(z, s)
\]

analytically continues to the \( s \)-plane as a meromorphic function. For \( k = 0 \) its has two singularities, which are simple poles at \( s = 0 \) and \( s = 1 \) with residues \( -\frac{1}{2} \) and \( \frac{1}{2} \), respectively. For \( k \neq 0 \) it is an entire function.

(2) (Functional Equation) For fixed \( z \in \mathbb{H} \), the completed weight \( k \) Eisenstein series satisfies the functional equation

\[
\hat{E}_k(z, s) = \hat{E}_k(z, 1 - k - s). \tag{3.3}
\]

The doubly completed series

\[
\hat{\hat{E}}_k(z, s) := \left( s + \frac{k}{2} \right) \left( s + \frac{k}{2} - 1 \right) \hat{E}_k(z, s)
\]

is an entire function of \( s \) for all \( k \in 2\mathbb{Z} \) and satisfies the same functional equation

\[
\hat{\hat{E}}_k(z, s) = \hat{\hat{E}}_k(z, 1 - k - s). \tag{3.4}
\]

The center line of these functional equations is \( \text{Re}(s) = \frac{1-k}{2} \).

(3) \( (\Delta_k\text{-Eigenfunction} ) \) \( E_k(z, s) \) is a (generalized) eigenfunction of the non-Euclidean Laplacian operator \( \Delta_k \) with eigenvalue \( \lambda = s(s + k - 1) \). That is, for all \( s \in \mathbb{C} \),

\[
\Delta_k E_k(z, s) = s(s + k - 1) E_k(z, s). \tag{3.5}
\]

This eigenfunction property holds for the completed functions \( \hat{E}_k(z, s) \) and \( \hat{\hat{E}}_k(z, s) \).
Proof This result is given in [2], deduced from \( G(z, \bar{z}; \alpha, \beta) \) in [24, Chap. IV]. The case \( k = 0 \) is treated in Sect. 4.4 of Kubota [19]. For (1) compare Miyake [25, Corollary 7.2.11].

We note an interesting consequence: we may obtain the doubly completed Eisenstein series from the singly completed one by applying a differential operator, a shifted Laplacian. This supplies an analytic construction of the “double completion.”

**Corollary 3.7** For \( k \in 2\mathbb{Z} \) the doubly completed non-holomorphic Eisenstein series \( \hat{E}(z, s) \) is obtainable from the completed Eisenstein series \( \hat{E}(z, s) \) by

\[
\left( \Delta_k + \frac{k(k-2)}{4} \right) \hat{E}_k(z, s) = \hat{\hat{E}}_k(z, s).
\]

**Proof** By Theorem 3.6 (3) we have

\[
\left( \Delta_k + \frac{k(k-2)}{4} \right) \hat{E}_k(z, s) = \left( s + k + 1 \right) \hat{E}(z, s) = \hat{\hat{E}}(z, s).
\]

\( \Box \)

4 Polyharmonic Fourier series

We formulate results on the form of Fourier series of (shifted) polyharmonic functions, some taken from [2]. We give results Fourier series for general \( \lambda = s(s+k-1) \), although this paper considers only \( \lambda = 0 \), because our results require taking derivatives with respect to \( s \).

4.1 Harmonic Fourier coefficients

The following result gives the allowable functional form of the Fourier coefficients for periodic functions in the hyperbolic plane that satisfy \( (\Delta_k - \lambda)h_n(z)e^{2\pi inx} = 0 \) and have moderate growth at the cusp, meaning \( O(y^c) \) for some finite \( c \) as \( y \to \infty \).

This result is due to Maass [23, Hilfssatz 6] and involves Whittaker \( W \)-functions, as treated in [32, Sect. 16.12].

**Theorem 4.1** (Harmonic Fourier Coefficients of Moderate Growth) Let \( k \in 2\mathbb{Z} \) and suppose that \( f_n(z) = h_n(y)e^{2\pi inx} \) is a shifted harmonic function for \( \Delta_k \) on \( \mathbb{H} \) with eigenvalue \( \lambda \in \mathbb{C} \), i.e., it satisfies

\[
(\Delta_k - \lambda)f_n(z) = 0 \quad \text{for all} \quad z = x + iy \in \mathbb{H}.
\]

Write \( \lambda = s_0(s_0 + k - 1) \) for some \( s_0 \in \mathbb{C} \) (there are generally two choices for \( s_0 \)). Suppose also that \( f_n(z) \) has at most polynomial growth in \( y \) at the cusp. Then the complete set of such functions \( h_n(y) \) is given in the following list.
(1) Suppose \( n \neq 0 \), and let \( \epsilon = \text{sgn}(n) \in \{ \pm 1 \} \). Then

\[
h_n(y) = y^{-\frac{k}{2}} \left( a_0(W_{\frac{k}{2}, s_0 + \frac{k-1}{2}}(4\pi |n|y)) \right)
\]

for some constant \( a_0 \in \mathbb{C} \), with \( W_{\kappa, \mu}(z) \) denoting a W-Whittaker function.

(2a) Suppose \( n = 0 \) with \( s_0 \neq \frac{1-k}{2} \) (equivalently, with \( \lambda \neq -(\frac{1-k}{2})^2 \)). Then

\[
h_n(y) = a_0^+ y^{s_0} + a_0^- y^{1-k-s_0}
\]

for some constants \( a_0^+, a_0^- \in \mathbb{C} \).

(2b) Suppose \( n = 0 \) with \( s_0 = \frac{1-k}{2} \) (equivalently, with \( \lambda = -(\frac{1-k}{2})^2 \)). Then

\[
h_n(y) = \sum_{j=0}^{\infty} \frac{\partial^j}{\partial s^j}(y^s)|_{s=s_0} = \sum_{j=0}^{\infty} a_j (\log y)^j y^{k-\frac{1}{2}}
\]

for some constants \( a_0, a_1 \in \mathbb{C} \).

Proof For \( n \neq 0 \) this follows from Maass [24, Lemma 6, Chap. 4, p. 181]. For \( n = 0 \) it is a simple calculation. \( \Box \)

4.2 Polyharmonic Fourier coefficients

We state a result proved in [2] which specifies the allowable form of individual Fourier coefficients \( h_n(z) \) which satisfy \( (\Delta_k - \lambda)^m h_n(z)e^{2\pi i nx} = 0 \) and have moderate growth at the cusp. This result is based on the fact that polyharmonic functions in \( z \) in the Fourier coefficients are obtainable by repeated partial derivatives \( \frac{\partial}{\partial s} \) of the eigenfunctions with respect to the eigenvalue parameter \( s \). This fact holds because although the operator \( \frac{\partial}{\partial s} \) commutes with the Laplacian \( \Delta_k \), it does not commute with \( \Delta_k - s(s+k-1)I \) but instead satisfies the commutator identity

\[
\left[ \Delta_k - s(s+k-1)I, \frac{\partial}{\partial s} \right] = (1-k-2s)I.
\]

of Heisenberg type.

Theorem 4.2 (Polyharmonic Fourier Coefficients) Let \( k \in 2\mathbb{Z} \) and suppose that \( f_n(z) = h_n(y)e^{2\pi i nx} \) is a shifted polyharmonic function for \( \Delta_k \) on \( \mathbb{H} \) with eigenvalue \( \lambda \in \mathbb{C} \), i.e., it satisfies

\[
(\Delta_k - \lambda)^m f_n(z) = 0 \text{ for all } z = x + iy \in \mathbb{H}.
\]

Write \( \lambda = s_0(s_0 + k - 1) \) for some \( s_0 \in \mathbb{C} \) (there are generally two choices for \( s_0 \)). Suppose also that \( f_n(z) \) has at most polynomial growth in \( y \) at the cusp. Then the complete set of such functions \( h_n(y) \) is given in the following list.
(1) Suppose \( n \neq 0 \), and let \( \epsilon = \text{sgn}(n) \in \{\pm 1\} \). Then

\[
h_n(y) = y^{-\frac{k}{2}} \left( \sum_{j=0}^{m-1} a_j \frac{\partial^j}{\partial s^j} \left( W_{n \epsilon k, s+\frac{k-1}{2}} \right) \right)_{s=s_0} (4\pi |n| y)\]

for some constants \( a_j \in \mathbb{C} \), with \( W_{\kappa, \mu}(z) \) denoting the \( W \)-Whittaker function.

(2a) Suppose \( n = 0 \) with \( s_0 \neq \frac{1-k}{2} \) (equivalently, with \( \lambda \neq -(\frac{1-k}{2})^2 \)). Then

\[
h_n(y) = \sum_{j=0}^{m-1} a_j^+ \left( \frac{\partial^j}{\partial s^j} \left( y^s \right) \right)_{s=s_0} + \sum_{j=0}^{m-1} a_j^- \left( \frac{\partial^j}{\partial s^j} \left( y^{1-k-s} \right) \right)_{s=s_0}
\]

\[
= \sum_{j=0}^{m-1} a_j^+ (\log y)^j y^{s_0} + \sum_{j=0}^{m-1} a_j^- (-\log y)^j y^{1-k-s_0}
\]

for some constants \( a_j^+, a_j^- \in \mathbb{C} \).

(2b) Suppose \( n = 0 \) with \( s_0 = \frac{1-k}{2} \) (equivalently, with \( \lambda = -(\frac{1-k}{2})^2 \)). Then

\[
h_n(y) = \sum_{j=0}^{2m-1} a_j \frac{\partial^j}{\partial s^j} \left( y^s \right)_{s=s_0} = \sum_{j=0}^{2m-1} a_j (\log y)^j y^{\frac{k-1}{2}}.
\]

for some constants \( a_j \in \mathbb{C} \).

Remark (1) For each \( \lambda \in \mathbb{C} \) there are two choices of \( s \) except \( \lambda = -(\frac{1-k}{2})^2 \), where there is a unique choice. These choices of \( s \) correspond to the variable switch \((\kappa, \mu) \) to \((\kappa, -\mu) \) which corresponds to \( s \mapsto 1 - (k+s) \). Either one of the choices leads to a basis of the same vector space of functions.

(2) If no growth conditions are imposed on \( f_n(z) \) at the cusp then, when \( n \neq 0 \), additional terms are allowed in the Fourier expansion, which involve a suitable linearly independent Whittaker function, call it \( \tilde{M}_{\epsilon k, s+\frac{k-1}{2}}(y) \), and its derivatives with respect to \( s \), see [2] for further discussion. The Whittaker \( W \)-functions \( W_{\kappa, \mu}(y) \) have modulus going to 0 exponentially fast as \( y \to \infty \), while the independent functions \( \tilde{M}_{\kappa, \mu}(y) \) have modulus growing exponentially fast in \( y \) as the real variable \( y \to \infty \).

4.3 Polyharmonic Fourier series expansions

Theorem 4.2 implies a Fourier expansion formula valid for all \( m \)-harmonic Maass forms with shifted eigenvalue \( \lambda \). We introduce a new notation for these functions. For \( n \neq 0 \) set \( \epsilon = \frac{n}{|n|} \in \{\pm 1\} \), and for each \( m \geq 0 \) set

\[
u_{\epsilon, k, [m]}(y; s_0) := y^{-\frac{k}{2}} \frac{\partial^m}{\partial s^m} \left( W_{\kappa, s+\frac{k-1}{2}} \right) (4\pi |n| y)_{s=s_0}.
\]
For $n = 0$ with $s_0 \neq \frac{1-k}{2}$, for each $m \geq 0$ set

$$u^{[m],+}_{k,0}(y; s_0) := \frac{\partial^m}{\partial y^m} y^s|_{s=s_0} = (\log y)^m y^{s_0}$$

$$u^{[m],-}_{k,0}(y; s_0) := \frac{\partial^m}{\partial y^m} y^{1-k-s}|_{s=s_0} = (-1)^m (\log y)^m y^{1-k-s_0}.$$

For $n = 0$ with $s_0 = \frac{1-k}{2}$ set

$$u^{[m],+}_{k,0}\left(z; \frac{1-k}{2}\right) = (\log y)^{2m-2} y^{\frac{1-k}{2}}, \quad \text{and} \quad u^{[m],-}_{k,0}\left(z; \frac{1-k}{2}\right) = (\log y)^{2m-1} y^{\frac{1-k}{2}}.$$

Then we have the following result.

**Theorem 4.3** (Fourier expansion in $V^m_k(\lambda)$) Let $f(z) \in V^m_k(\lambda)$ for some $k \in 2\mathbb{Z}$. Let $m \geq 1$, and fix an $s \in \mathbb{C}$ with $\lambda = s(s+k-1)$. Then the Fourier expansion of $f(z)$ exists and has the form

$$f(z) = \sum_{j=0}^{m-1} \left( c^{+}_{0,j} u^{[j],+}_{k,0}(y) + c^{-}_{0,j} u^{[j],-}_{k,0}(y) \right)$$

$$+ \sum_{\epsilon \in \{\pm\}} \left( \sum_{n=1}^{\infty} \sum_{j=0}^{m-1} c^{-}_{\epsilon,j,n} u^{[j],-}_{\epsilon,k,n}(y) e^{2\pi i (\epsilon n) x} \right),$$

in which $c^{\pm}_{\epsilon,j}$ and $c^{-}_{\epsilon,j,n}$ are constants. This Fourier expansion converges absolutely and uniformly to $f(z)$ on compact subsets of $\mathbb{H}$.

**Proof** A weight $k$ modular form on $\text{PSL}(2, \mathbb{Z})$ has $f(z) = f(z+1)$, using $\gamma = \left( \begin{array}{ll} 1 & 1 \\ 0 & 1 \end{array} \right)$, whence it has a Fourier expansion of the form $f(z) \sim \sum_{n \in \mathbb{Z}} h_n(y) e^{2\pi i n x}$ with coefficient functions

$$h_n(y) = \int_0^1 f(z) e^{-2\pi i n x} \, dx.$$

Because $f(z)$ has moderate growth of order $O(y^K)$ as $y \to \infty$, for some fixed finite $K$, we conclude that each $h_n(y)$ separately has moderate growth of the same order. Additionally, the shifted polyharmonic condition $(\Delta_k - \lambda)^m f(z) = 0$ implies that each of its Fourier coefficients separately satisfy

$$(\Delta_k - \lambda)^m \left( h_n(y) e^{2\pi i n x} \right) = 0.$$

This fact holds by separation of variables, since the application of $(\Delta_k - \lambda)$ to any function $f_1(y) e^{2\pi i n x}$ yields another function $f_2(y) e^{2\pi i n x}$, whence $(\Delta_k - \lambda)^m \left( h_n(y) e^{2\pi i n x} \right) = \hat{h}_n(y) e^{2\pi i n x}$, whence
\[
0 = (\Delta_k - \lambda)^m (f(x + iy)) = \sum_{n \in \mathbb{Z}} (\Delta_k - \lambda)^m \left( \tilde{h}_n(y)e^{2\pi inx} \right) = \sum_{n \in \mathbb{Z}} \tilde{h}_n(y)e^{2\pi inx}.
\]

The uniqueness of Fourier series expansions of a real-analytic function then implies that all \( \tilde{h}_n(y) = 0. \) Finally Theorem 4.2 applies to each \( f_n(z) \) separately to give an expansion of the given form.

Finally the absolute and uniform convergence on compact subsets follows from the known real-analyticity of such \( f(z) \). \( \square \)

### 4.4 1-Harmonic Fourier expansions

We give a version of the Fourier expansion for the case \( m = 1 \) and eigenvalue \( \lambda = 0 \), involving incomplete Gamma functions, which is a special case of a Fourier expansion for 1-harmonic Maass forms that appears in the literature. We relate it to the Fourier expansion version given in terms of Whittaker functions in Theorem 4.3.

**Lemma 4.4** For any \( k \in 2\mathbb{Z}, \) if \( f(z) \in V^1_k(0), \) then it has a Fourier expansion of form

\[
f(z) = \sum_{n=1}^{\infty} b_{-n} \Gamma(1 - k, 4\pi |n|y)e^{-2\pi iny} + (b_0 y^{-1 - k} + a_0) + \sum_{n=1}^{\infty} a_n e^{2\pi iny} \tag{4.2}
\]

in which \( \Gamma(\kappa, y) = \int_y^{\infty} t^{\kappa-1} e^{-t} dt \) denotes the incomplete Gamma function.

**Proof** Since \( f(z) \) has moderate growth the Fourier expansion of Theorem 4.3 applies, taking \( m = 1. \) Thus we have

\[
f(z) = y^{-\frac{k}{2}} \left( \sum_{n=1}^{\infty} \tilde{b}_{-n} W_{\frac{k}{2}, \frac{1-k}{2}} \left( 4\pi |n|y \right)e^{-2\pi inx} + (\tilde{b}_0 y^{1-k} + \tilde{a}_0) + \sum_{n=1}^{\infty} \tilde{a}_n W_{\frac{k}{2}, \frac{1-k}{2}} \left( 4\pi |n|y \right)e^{2\pi inx} \right),
\]

for certain complex constants \( \tilde{a}_n, \tilde{b}_{-n} \) for all \( n \geq 0. \)

We assert that for \( n \leq -1, \)

\[
(4\pi |n|y)^{-\frac{k}{2}} W_{\frac{k}{2}, \frac{1-k}{2}} \left( 4\pi |n|y \right) = \Gamma(1 - k, 4\pi |n|y)e^{-2\pi n y} = \Gamma(1 - k, 4\pi |n|y)e^{2\pi |n|y}. \tag{4.3}
\]

To prove the assertion we use the identity \([26, (13.18.5)]\)

\[
z^{\frac{1}{2}-\mu} e^{z^2} \Gamma(2\mu, z) = W_{\mu-\frac{1}{2}, \mu}(z),
\]

in which we take \( \mu = \frac{1-k}{2} \) and \( z = 4\pi |n|y \) to obtain

\[
W_{\frac{k}{2}, \frac{1-k}{2}} \left( 4\pi |n|y \right) = (4\pi |n|y^k \frac{1}{2} e^{2\pi |n|y} \Gamma(1 - k, 4\pi |n|y),
\]

\( \square \) Springer
from which (4.3) follows.

We next assert that for \( n \geq 1 \) we have

\[
(4\pi ny)^{-\frac{k}{2}} W_{\frac{k}{2} - \frac{1}{4}}(4\pi ny) = e^{-2\pi ny}.
\]

(4.4)

To show this we use the identity [26, (13.4.31)]

\[ W_{\kappa, \mu}(z) = W_{\kappa, -\mu}(z). \]

and the identity [26, (13.18.17)] valid for integer \( n \geq 0 \) and integer \( \alpha \) that

\[
W_{\frac{\alpha + 1}{2} + n, \frac{\alpha}{2}}(z) = (-1)^n n! e^{-\frac{z}{2}} z^{\frac{\alpha + 1}{2}} L_n^{(\alpha)}(z),
\]

with \( L_n^{(\alpha)}(z) \) being a (modified) Laguerre polynomial of degree \( n \). We choose \( n = 0 \) to obtain

\[
W_{\frac{\alpha + 1}{2}, -\frac{\alpha}{2}}(z) = W_{\frac{\alpha + 1}{2}, \frac{\alpha}{2}}(z) = e^{-\frac{z}{2}} z^{\frac{\alpha + 1}{2}},
\]

since \( L_0^{(\alpha)}(z) = 1 \). We take \( \alpha = k - 1 \) and \( z = 4\pi ny \) to obtain (4.4).

Combining the two assertions, the coefficients of the new Fourier expansion (4.2) are related to the old by \( a_0 = \tilde{a}_0, b_0 = \tilde{b}_0 \), and for all \( n \geq 1 \),

\[
a_n = (4\pi n)^{\frac{k}{2}} \tilde{a}_n \quad \text{and} \quad b_{-n} = (4\pi |n|)^{\frac{k}{2}} \tilde{b}_{-n}. \tag{4.5}
\]

\[ \square \]

Remark A more general form of this formula for weak Maass forms appears in Bruinier and Funke [10], see Sect. 6.1. (Compare also the discussion in the introduction of [15].) Bruinier and Funke note in their equations (3.2a) and (3.2b) and the sentence before those equations that any \( f(z) \in M_k^! \) has a Fourier expansion of the form

\[
f(z) = b_0 y^{1-k} + \sum_{n \geq 0} a_n e^{2\pi i nz} + \sum_{n < 0} b_n \Gamma(1 - k, 4\pi |n| y) e^{2\pi i nz}. \tag{4.6}
\]

The condition of at most polynomial growth made above imposes the strengthened restrictions on the ranges of summation in the two parts.

5 The \( \xi \)-operator

5.1 Properties of the differential operators \( \xi_k \)

Bruinier and Funke [10, Proposition 3.2] introduce the operator

\[
\xi_k(f)(z) = 2iy^k \frac{\partial}{\partial z} f(z),
\]

\[ \square \]
which is related to the Maass raising and lowering operators.

We state basic lemmas concerning the differential operators $\xi_k$, mentioned in the introduction to [15].

**Lemma 5.1** Let $f : \mathbb{H} \to \mathbb{C}$ be a $C^\infty$-function of two real variables $(x, y)$, and set $z = x + iy$. Then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ and for any integer $k$,

$$\xi_k ((cz + d)^{-k} f(\gamma \cdot z)) = (cz + d)^{k-2} (\xi_k f)(\gamma \cdot z).$$

**Proof** This is a calculation, see for example [2]. $\square$

Lemma 5.1 implies that if $f(z)$ is a weight $k$ (holomorphic or non-holomorphic) modular form for a discrete subgroup $\Gamma$ of $\text{PSL}(2, \mathbb{Z})$, with no growth conditions imposed on any cusp, then $\xi_k f$ is a weight $2 - k$ modular form for $\Gamma$, again imposing no growth conditions at any cusp.

**Lemma 5.2** The operator $\Delta_k = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ factorizes as

$$\Delta_k = \xi_{2-k} \xi_k.$$

Thus $\Delta_{2-k} = \xi_k \xi_{2-k}$.

**Proof** This is an easy calculation. $\square$

### 5.2 Action of $\xi_k$ on Fourier coefficient functions

We recall a result on the $\xi_k$-operator action on the Fourier coefficient functions in Theorem 4.1. Note that the operator $\xi_k$ maps weight $k$ to weight $2 - k$, maps Fourier coefficient $n$ to Fourier coefficient $-n$, and maps the holomorphic parameter $s$ to the anti-holomorphic parameter $-\bar{s}$.

**Lemma 5.3** (1) Let $n \leq -1$. Then

$$\xi_k \left( y^{-\frac{k}{2}} W_{\frac{k}{2}+s+k-\frac{1}{2}} (4\pi |n|y) e^{2\pi inx} \right) = -y^{-(\frac{2-k}{2})} W_{2-k, -\bar{s}+\frac{1}{2}} (4\pi |n|y) e^{-2\pi inx}.$$

(2) Let $n \geq 1$. Then

$$\xi_k \left( y^{-\frac{k}{2}} W_{\frac{k}{2}+s+k-\frac{1}{2}} (4\pi ny) e^{2\pi inx} \right)$$

$$= \bar{s}(1 - k - \bar{s}) y^{-(\frac{2-k}{2})} W_{\frac{2-k}{2}, -\bar{s}+\frac{1}{2}} (4\pi |n|y) e^{-2\pi inx}.$$

(3) There holds

$$\xi_k (y^s) = \bar{s} y^{1+k+\bar{s}} \text{ and } \xi_k (y^{1-(s+k)}) = (1 - k - \bar{s}) y^{-\bar{s}}.$$

**Proof** These results follow by calculations given in [2]. $\square$
5.3 $\xi_k$ preserves moderate growth

**Lemma 5.4** The action of the $\xi$ operator on shifted polyharmonic vector spaces preserves the moderate growth property for all $\lambda$. That is,

$$\xi_k(V_k^m(0)) \subseteq V_{2-k}^m(0).$$

**Proof** Now $f(z) \in V_k^m(0)$ has $(\Delta_k)^m f(z) = 0$. Applying Lemma 5.2 we have

$$0 = \xi_k(\Delta_{2-k})^m f(z) = \xi_k(\xi_{2-k}\xi_k)^m f(z) = (\xi_k^2)^m(\xi_k f(z)) = (\Delta_{2-k})^m(\xi_k f(z)).$$

It remains to show that $\xi_k f(z)$ has moderate growth. We expand the polyharmonic function $f(z)$ in Fourier series, using Theorem 4.3, noting that the Fourier series coefficients for $n \neq 0$ only involve $s$-derivatives of $W$-Whittaker functions. We apply $\xi_k$ term by term to the resulting Fourier series. By Lemma 5.3, we have for $n \leq -1$ that

$$\xi_k \left( y^{-\frac{k}{2}} W_{\frac{k}{2},s+\frac{k-1}{2}}(4\pi |n| y)e^{2\pi inx} \right) = -y^{-(\frac{k}{2}-1)} W_{\frac{k}{2},-\bar{s}+\frac{1}{2}}(4\pi |n| y)e^{-2\pi inx},$$

while for $n \geq 1$,

$$\xi_k \left( y^{-\frac{k}{2}} W_{\frac{k}{2},s+\frac{1}{2}}(4\pi ny)e^{2\pi inx} \right) = \bar{s}(1 - k - \bar{s}) y^{-(\frac{k}{2}-1)} W_{-(\frac{k}{2}),-\bar{s}+\frac{1}{2}}(4\pi |n| y)e^{-2\pi inx}.$$

Differentiating repeatedly with respect to $s$, and noting that

$$\frac{\partial}{\partial s} \xi_k = \xi_k \frac{\partial}{\partial \bar{s}} \text{ and } \frac{\partial}{\partial \bar{s}} \xi_k = \xi_k \frac{\partial}{\partial s},$$

we obtain, for $n \leq -1$,

$$\xi_k \left( \frac{\partial^j}{\partial s^j} \left( y^{-\frac{k}{2}} W_{\frac{k}{2},s+\frac{k-1}{2}}(4\pi |n| y)e^{2\pi inx} \right) \right) = -y^{-(\frac{k}{2})} \frac{\partial^j}{\partial \bar{s}^j} W_{\frac{k}{2},-\bar{s}+\frac{1}{2}}(4\pi |n| y)e^{-2\pi inx}.$$

For the case $n \geq 1$ the repeated $s$-derivatives give many more terms, but they all involve polynomials in $\bar{s}$ times $\frac{\partial^k}{\partial \bar{s}^k} W_{\frac{k}{2},-\bar{s}+\frac{1}{2}}(4\pi |n| y)$ with $0 \leq k \leq j$. All the resulting $s$-derivatives of the Whittaker $W$-functions, have rapid decay at the cusp (uniformly for a fixed value $s = s_0$, in our case $s = 0$). This may be shown by differentiating an integral representation of the Whittaker $W$-function with respect to the $s$-parameter, see [2] for details. The resulting Fourier series expansion has moderate growth at the cusp, certifying that $\xi_k f(z) \in V_{2-k}^m(0)$. \qed

**Remark** (1) This argument is the special case $\lambda = 0$ of a result proved in [2].
There exist weight $k$ real-analytic modular forms $f(z)$ not having moderate growth at the cusp with the property that $\xi_k f(z)$ has moderate growth at the cusp. The simplest examples are members of $M^1_k \setminus M_k$ which have linear exponential growth at the cusp but are annihilated by $\xi_k$ since they are holomorphic functions.

6 Polyharmonic non-liftability of holomorphic cusp forms

6.1 Weak Maass forms

We consider the liftability problem in the context of larger spaces of weak Maass forms in which lifts do exist. Such spaces were originally introduced in a more general context in Bruinier and Funke [10], see also Bruggeman [9].

**Definition 6.1** For $k \in 2\mathbb{Z}$ the space $H_k(0)$ of weight $k$ Harmonic weak Maass forms is those $f: \mathbb{H} \rightarrow \mathbb{C}$, such that

1. For all $\gamma \in \text{PSL}(2, \mathbb{Z})$, $f(\gamma z) = (cz + d)^k f(z)$.
2. $\Delta_k f(z) = 0$
3. (Linear exponential growth at the cusp) There is a positive constant $A = A(f) < \infty$ such that $|f(z)| \leq e^{Ay}$ for all $y \geq y_0$.

The vector space $H_k = H_k(0)$ is infinite dimensional. By definition the holomorphic functions in the space $H_k$ comprise the space $M^1_k$ of weakly holomorphic weight $k$ modular forms, cf. Definition 3.4.

**Proposition 6.2** (Bruinier and Funke [10, Prop. 3.2]) For $\lambda = 0$ and all $k \in 2\mathbb{Z}$ the map $f(z) \mapsto \xi_k f(z)$ defines a conjugate-linear mapping

$$\xi_k : H_k \rightarrow M^1_{2-k}.$$ 

Its kernel is $M^1_k$.

This mapping was shown to be surjective by Bruggeman [9, Theorem 1.1]. The members of $H_k$ have a Fourier expansion of shape

$$f(z) = \sum_{n \ll \infty} c^-_f(n) \Gamma(1 - k, 4\pi |n|y)q^n + \sum_{n \gg -\infty} c^+_f(n)q^n,$$

with $q = e^{2\pi iz}$ and $z = x + iy$. Here $n \gg -\infty$ means that $n$ is bounded below, $n \ll \infty$ means $n$ is bounded above. We call

$$f^+(z) := \sum_{n \gg -\infty} c^+_f(n)q^n$$

the holomorphic part of $f(z)$, and

$$f^-(z) = \sum_{n \ll \infty} c^-_f(n) \Gamma(k - 1, 4\pi |n|y)q^n$$
the non-holomorphic part of \( f(z) \). We also call the finite sum

\[
P(f)(z) := \sum_{n \leq 0} c_f^+(n)q^n
\]

the principal part of \( f(z) \) (or of \( f^+(z) \)).

The inclusion \( V_k^1(0) \subset H_k \) immediately follows by considering the Fourier series of members of \( V_k^1(0) \) given in Lemma 4.4.

**Definition 6.3** For \( k \in 2\mathbb{Z} \) the space \( H_k^+ \) of weight \( k \) harmonic weak Maass forms of eigenvalue \( \lambda = 0 \) is those \( f \in H_k^+ \), whose Fourier expansion has non-holomorphic part of form

\[
f^-(z) = \sum_{n < 0} c_f^-(n) \Gamma(k - 1, 4\pi |n| y)q^n,
\]

That is, \( f^-(z) \) has rapid decay as \( y \to \infty \).

The subspace \( H_k^+ \) has an alternate characterization as the set of all \( f(z) \in H_k \) such that \( \xi_k f(z) \in S_k \) is a holomorphic cusp form.

**Proposition 6.4** (Bruinier and Funke [10, Theorem 1.1]) For \( k \in 2\mathbb{Z} \) the space \( H_k^+ \) contains \( M_k^1 \) and there is an exact sequence (regarded as \( \mathbb{R} \)-vector spaces)

\[
0 \to M_k^1 \to H_k^+ \to S_{2-k} \to 0,
\]

in which the third map is \( \xi_k \) (which is conjugate linear). Moreover, there is a well-defined bilinear form \( \{ \cdot, \cdot \} \) defined for \( f(z) \in H_k^+, g(z) \in S_{2-k} \) by

\[
\{ g, f \} := (g, \xi_k(f))_{2-k},
\]

in which \( (-, -)_{2-k} \) denotes the Petersson inner product on modular forms of weight \( 2 - k \) (at least one a cusp form). This bilinear form gives a non-degenerate pairing of \( S_{2-k} \) with \( H_k^+ / M_k^1 \).

The sequence above is not an exact sequence of \( \mathbb{C} \)-vector spaces because the map \( \xi_k \) is conjugate linear, compare [10, Corollary 3.8].

**6.2 Regularized Petersson inner product**

We recall that for weight \( k \) holomorphic modular forms the Petersson inner product is defined on \( M_k \times S_k \) by

\[
(f, g)_k := \int_{\mathcal{F}} f(z)\overline{g(z)}y^k \frac{dx dy}{y^2},
\]

The terminology is used in Bruinier, Ono and Rhoades [11], who call this space \( H_k \), omitting the \( + \), see [11, Remark 6].
Polyharmonic Maass forms for PSL(2, ℤ)

in which \( \mathcal{F} = \{ z : |z| \geq 1, 0 \leq x \leq 1 \} \) is the standard fundamental domain for PSL(2, ℤ).

A (regularized) inner product generalizing the Petersson inner product was introduced in Borcherds [4]. Let \( \mathcal{F}_T \) denote the standard fundamental domain for PSL(2, ℤ) cut off at height \( T \),

\[ \mathcal{F}_T = \{ z : |z| \geq 1 \text{ with } 0 \leq x \leq 1, y \leq T \}. \]

Let \( g \in M_k \) and let \( h \in M_k^! \) a weight \( k \) weakly holomorphic modular form of weight \( k \) on SL(2, ℤ). Then for modular forms of weight \( k \), set

\[ (g, h)^{\text{reg}} := \lim_{T \to \infty} \int_{\mathcal{F}_T} g(z) \overline{h(z)} y^k \frac{dx dy}{y^2}. \]

Remark If \( h \) is a cusp form, then \((g, h)^{\text{reg}}\) is the usual Petersson inner product.

**Lemma 6.5** Let \( f(z) \in H^+_{2-k} \) have \( \xi_{2-k} f(z) = g(z) \), where \( g \in S_k \) is a non-zero weight \( k \) holomorphic cusp form. Then

\[ f(z) = \sum_{n \gg -\infty} c^+_f(n) q^n + \sum_{n < 0} c^-_f(n) \Gamma(k - 1, 4\pi |n| y) q^n, \]

with \( q = e^{2\pi i z} \), and

\[ g(z) = \xi_{2-k} f(z) = \sum_{n=1}^{\infty} c^-_f(-n)(4\pi n)^{k-1} q^n. \] (6.1)

Moreover, if \( h(z) \in M_k \) with \( h(z) = \sum_{n=0}^{\infty} c_h(n)^+ q^n \), then the Petersson inner product

\[ (h, \xi_{2-k} f)_k = \sum_{n=0}^{\infty} c^+_h(n) c^+_f(-n). \] (6.2)

In particular, for \( h = g \), then

\[ (g, g)_k = (g, \xi_{2-k} f)_k = -\sum_{n=1}^{\infty} (4\pi n)^{k-1} c^-_f(-n) c^-_f(-n). \] (6.3)

**Proof** The Fourier series (6.1) follows from \( \xi_k \) applied term by term to the Fourier series of \( f(z) \), using Lemma 4.4 and Lemma 5.3(1). (See also the introduction to [11].) The conversion (4.5) is applied twice, as

\[ b_{-n}(4\pi |n|)^{\frac{k}{2}} = \tilde{b}_{-n} \quad \longrightarrow_{\xi_{2-k}} \quad -\tilde{b}_{-n} = \tilde{a}_{n} = a_{n}(4\pi n)^{-\frac{k}{2}}, \]

yielding \( a_{n} = -\tilde{b}_{-n}(4\pi n)^{k-1} \). The Petersson inner product formula (6.2) is a special case of Bruinier-Funke [10, Prop. 3.4]. □
6.3 Non-liftability of holomorphic cusp forms

We show that there are no preimages under \( \xi_{2-k} \) of holomorphic cusp forms for \( \text{PSL}(2, \mathbb{Z}) \) that have polynomial growth at the cusp.

**Proposition 6.6** Let \( g \in S_k \) be a weight \( k \) holomorphic cusp form for \( \text{SL}(2, \mathbb{Z}) \).

1. Let \( M \) be a non-holomorphic weight \( k \) modular form for \( \text{SL}(2, \mathbb{Z}) \) satisfying
   \[
   \Delta_k M(z) = g(z).
   \]
   Then \( M \) cannot have polynomial growth in \( y \) as \( y \to \infty \).

2. There is no weight \( 2-k \) modular form \( f(z) \in V^1_{2-k}(0) \) such that
   \[
   \xi_{2-k} f(z) = g(z).
   \]

**Proof** (1) Assume for a contradiction that such an \( M(z) \) exists, in which case \( M(z) \in V^2_k(0) \). Lemma 5.4 now shows that \( f(z) := \xi_k M(z) \in V^2_{2-k}(0) \). We now assert \( f(z) \in V^1_{2-k}(0) \). This holds since \( f(z) \) has moderate growth and
   \[
   \xi_{2-k} f(z) = \xi_{2-k} (\xi_k M(z)) = \Delta_k (M(z)) = g(z),
   \]
   which yields
   \[
   \Delta_{2-k} (f)(z) = \xi_k (\xi_{2-k} f(z)) = \xi_k g(z) = 0,
   \]
   because \( g(z) \) is a holomorphic form. Thus (1) will follow if we prove (2).

(2) We have \( f(z) \in V^1_{2-k}(0) \), and we \( f(z) \in H_{2-k} \) since \( V^1_{2-k}(0) \subseteq H_{2-k} \). Since \( \xi_{2-k} f \in S_k \), we have \( f(z) \in H_{2-k}^+ \). According to Proposition 6.4 there is a unique class of lifts \([ f ] \in H^+_{2-k}/M_{2-k}^1 \) whose image is \( g(z) \). We need to prove that no member \( f(z) \) of this class has moderate growth at the cusp. Since our given \( f(z) \in V^1_{2-k}(0) \), by Lemma 4.4 it has a Fourier expansion of the form:
   \[
   f(z) = \sum_{n=1}^{\infty} b_{-n} \Gamma(k-1, 4\pi |n| y) e^{-2\pi i nz} + (b_0 y^{1-k} + a_0) + \sum_{n=1}^{\infty} a_n e^{2\pi i nz}.
   \]

By Lemma 6.5 we have
   \[
   g(z) = \xi_{2-k} f(z) = (k-1) b_0 - \sum_{n=1}^{\infty} b_{-n} (4\pi n)^{k-1} e^{2\pi i nz}.
   \]

We see in addition that \( b_0 = 0 \), since \( g(z) \) is a cusp form. We next compute the Petersson inner product \( (g, g)_k \) using Lemma 6.5 to be
   \[
   (g, g)_k = (g, \xi_{2-k} f)_k = - \sum_{n \geq 1} b_{-n} (4\pi n)^{k-1} a_{-n} = 0.
   \]
Here we used \( c^+_f(-n) = a_{-n} = 0 \) for all \( n \geq 1 \), because the Fourier terms \( c^+_f(-n)q^{-n} \) are not of moderate growth. This is a contradiction because the Petersson inner product \( (g, g)_k > 0 \) since \( g(z) \neq 0 \). We conclude such an \( f(z) \) cannot exist, which proves (2).

\[ \square \]

Remark (1) One may explicitly construct harmonic weak Maass forms that do map to holomorphic cusp forms under \( \xi_{2-k} \) using Maass-Poincaré series, see Bringmann and Ono [7], Bruinier et al. [11], and the discussion in Kent [18, Sect. 1.3.2].

(2) It is also interesting to check how the non-holomorphic Eisenstein series \( f(z) = \hat{E}_{2-k}(z, 0) \) evade the argument in the proof above. To allow Eisenstein series we must use a regularized inner product. A non-zero value can occur due to the presence of the \( n = 0 \) term in the inner product \((g, g)_{reg}\).

7 Action of \( \xi_k \) on non-holomorphic Eisenstein series

For later use we compute the action of the \( \xi_k \)-operator on the non-holomorphic Eisenstein series.

**Proposition 7.1** Let \( k \in 2\mathbb{Z} \). Then

\[
\xi_k \hat{E}_k(z, s) = \begin{cases} 
\hat{E}_{2-k}(z, -\bar{s}) & \text{if } k \leq 0, \\
\bar{s}(\bar{s} + k - 1) \hat{E}_{2-k}(z, -\bar{s}) & \text{if } k \geq 2.
\end{cases}
\]

In addition

\[
\xi_k \hat{E}_k(z, s) = \begin{cases} 
\hat{E}_{2-k}(z, -\bar{s}) & \text{if } k \leq 0, \\
\bar{s}(\bar{s} + k - 1) \hat{E}_{2-k}(z, -\bar{s}) & \text{if } k \geq 2.
\end{cases}
\]

We obtain this result by first studying the action of \( \xi_k \) on the uncompleted Eisenstein series \( E_k(z, s) \).

**Lemma 7.2** For \( k \in 2\mathbb{Z} \) there holds

\[
\xi_k E_k(z, s) = \bar{s}E_{2-k}(z, \bar{s} + k - 1).
\]

**Proof** Taking \( \psi_s(z) = y^s \), we have \( \psi_s(\gamma z) = \frac{y^s}{|cz + d|^s} \). Using Lemma 5.1 and taking \( \Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \), we have for \( \text{Re}(s) > 2 \) that
\[ \xi_k E_k(z, s) = \frac{1}{2} \sum_{\gamma = \left( \begin{array}{cc} * & * \\ c & d \end{array} \right) \in \Gamma_\infty \setminus \text{SL}(2, \mathbb{Z})} \xi_k (cz + d)^{-k} \psi_s(\gamma z) \]

\[ = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}(2, \mathbb{Z})} (cz + d)^{k-2} (\xi_k \psi_s)(\gamma z) \]

\[ = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}(2, \mathbb{Z})} (cz + d)^{k-2} \psi_{\tau - 1 + k}(\gamma z) \]

using the identity \( 2iy^k \frac{\partial}{\partial z} y^s = \bar{s}y^{\bar{s} - 1 + k} \). The identity holds on compact subsets of \( \text{Re}(s) > 2 \) and of \( z \in \mathbb{H} \) because the sums on both sides converge absolutely and uniformly on these domains. It now follows for all \( s \in \mathbb{C} \) and all \( z \in \mathbb{H} \) by analytic continuation, away from poles. \( \square \)

One can easily deduce from Lemma 7.2 a direct proof of Theorem 3.6(3). Namely since \( \Delta_k = \xi_{2-k} \circ \xi_k \), we obtain

\[ \Delta_k E_k(z, s) = \xi_{2-k} \hat{E}_{2-k}(z, \bar{s} - 1 + k) = s(s + k - 1) E_k(z, s). \]

**Proof of Proposition 7.1** Lemma 7.2 gives

\[ \xi_k \hat{E}_k(z, s) = \frac{s}{\bar{s}} \frac{\Gamma \left( s + \frac{k}{2} + \frac{|k|}{2} \right)}{\Gamma \left( \frac{s + k}{2} + \frac{|k|}{2} \right)} \hat{E}_{2-k}(z, \bar{s} - 1 + k) \]

\[ = \frac{s}{\bar{s}} \frac{\Gamma \left( s + \frac{k}{2} + \frac{|k|}{2} \right)}{\Gamma \left( \frac{s + k}{2} + \frac{|k|}{2} \right)} \hat{E}_{2-k}(z, -\bar{s}), \]

where the functional equation of \( \hat{E}_k(z, s) \) was used to get the rightmost equality. For \( k \geq 2 \), the \( \Gamma \)-quotient is \( \frac{\Gamma(s + k)}{\Gamma(s + k - 1)} = (s + k - 1) \). For \( k \leq 0 \), the \( \Gamma \)-quotient is \( \frac{\Gamma(s)}{\Gamma(s + 1)} = \frac{1}{s} \). The result for \( \hat{E}_k(z, s) \) follows.

The result for \( \hat{E}(z, s) = (s + \frac{k}{2})(s + \frac{k}{2} - 1) \hat{E}_k(z, s) \) follows on noting that

\[ \left( s + \frac{k}{2} \right) \left( s + \frac{k}{2} - 1 \right) = \left( -\bar{s} + \frac{2 - k}{2} \right) \left( -\bar{s} + \frac{2 - k}{2} - 1 \right). \]

\( \square \)

**8 Taylor series of non-holomorphic Eisenstein series**

We show that the Taylor series coefficients of the doubly completed Eisenstein series \( \hat{E}_k(z, s) \) in the \( s \)-variable at \( s = 0 \) define polyharmonic Maass forms of eigenvalue \( \frac{123}{123} \).
\( \lambda = s(s + k - 1) \). We consider the doubly completed Eisenstein series \( \hat{E}_k(z, s) \) rather than the singly completed \( \tilde{E}_k(z, s) \) because it is an entire function of \( s \) for all \( k \). The case \( \lambda = 0 \) has special features compared to the general \( \lambda \) case, which we treat in [2].

We will use rescaled Taylor coefficients of \( E_k(z, 0) \), matching the notation used in Sect. 2, namely

\[
\hat{E}_k(z, s) = \begin{cases} 
\sum_{n=0}^{\infty} F_{n,k}(z)s^n & \text{for weights } k \leq 0, \\
\sum_{n=0}^{\infty} G_{n,k}(z)s^n & \text{for weights } k \geq 2.
\end{cases}
\]

(8.1)

In consequence the derivatives with respect to \( s \) exhibit

\[
\frac{\partial^n}{\partial s^n} \hat{E}_k(z, s) \big|_{s=0} = \begin{cases} 
n!F_{n,k}(z) & \text{for weights } k \leq 0, \\
n!G_{n,k}(z) & \text{for weights } k \geq 2.
\end{cases}
\]

(8.2)

### 8.1 Initial Taylor coefficients at \( s = 0 \) doubly completed weight 0 and 2 Eisenstein series

We use the abbreviated notation \( F_n(z) = F_{n,0}(z) \) and \( G_n(z) = G_{n,2}(z) \).

**Lemma 8.1** The Taylor series of the doubly completed Eisenstein series \( \hat{E}_0(z, s) = \sum_{n=0}^{\infty} F_n(z)s^n \) at \( s_0 = 0 \) has

\[
F_0(z) = \hat{E}_0(z, 0) = \frac{1}{2}.
\]

In addition

\[
F_1(z) = \frac{\partial}{\partial s} \hat{E}_0(z, s) \big|_{s=0} = -\frac{1}{2} \gamma + \log(4\pi) + \log \left( \sqrt{y} |\Delta(z)|^{\frac{1}{2}} \right).
\]

**Proof** The first assertion is derived from the Fourier expansion in Proposition 3.5, using \( \hat{E}_0(z, s) = s(s-1)\hat{E}_0(z, s) \), after observing that for \( k = 0 \) all the Fourier terms with \( n \neq 0 \) vanish, and the constant term becomes

\[
s(s-1)C_0(y, s) \big|_{s=0} = s(s-1)\hat{\zeta}(2s)y^s \big|_{s=0} + s(s-1)\hat{\zeta}(2-2s)y^{1-s} \big|_{s=0} = \frac{1}{2},
\]

coming from the simple pole of \( \hat{\zeta}(2s) \) at \( s = 0 \) having residue \(-\frac{1}{2}\).

The second assertion is deduced from Kronecker’s first limit formula [30, Chap. 1, Theorem 1] which states\(^5\)

\[
2E_0(z, s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(\sqrt{y} |\eta(z)|^2)) + O(s-1),
\]

\(^5\) Siegel’s definition of the Eisenstein series has an extra factor of 2 compared to (1.3).
where \( \gamma \) is Euler’s constant, and the Dedekind eta function satisfies \(|\eta(z)| = |\Delta(z)|^{1/24}\). Using the functional equation for \( \hat{E}_0(z, s) = \hat{E}_0(z, 1-s) \) we obtain

\[
E_0(z, s) = \gamma(s) E_0(z, 1-s)
\]

with

\[
\gamma(s) := \frac{\pi^{-(1-s)} \Gamma(1-s)}{\pi^{-s} \Gamma(s)} = s \pi^{2s-1} \frac{\Gamma(1-s)}{\Gamma(1+s)}.
\]

Computing Laurent expansions at \( s = 0 \) of \( \gamma(s) \) and \( E_0(z, 1-s) \) yields analogous Kronecker’s limit formula to be derived at \( s = 0 \). We obtain

\[
E_0(z, s) = -\frac{1}{2} + \left( -\log(2\pi) - \log \sqrt{|\Delta(z)|^{|1/2|}} \right) s + O(s^2).
\]

It was given in Stark [31], see also Katayama [17, (3.2.8)]. Another Laurent series calculation using \( \hat{E}_0(z, s) = s(s-1) \hat{E}_0(z, s) \) yields

\[
\hat{E}_0(z, s) = \frac{1}{2} + \left( -\frac{1}{2}(\gamma + 1) + \frac{1}{2} \log(4\pi) + \log(\sqrt{|\Delta(z)|^{|1/2|}}) \right) s + O(s^2).
\]

\[
\text{Lemma 8.2} \quad \text{The Taylor series of the doubly completed Eisenstein series } \hat{E}_2(z, s) = \sum_{n=0}^{\infty} G_n(z)s^n \text{ at } s_0 = 0 \text{ has } G_0(z) \equiv 0
\]

and

\[
G_1(z) = \frac{\partial}{\partial s} \hat{E}_2(z, s)|_{s=0} = \frac{\pi}{6} - \frac{1}{2y} - 4\pi \left( \sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi inz} \right)
\]

where \( z = x + iy \in \mathbb{H} \). This function is holomorphic in \( z \) except for the term \( \frac{1}{2y} \) in the constant term of its Fourier expansion.

\[
\text{Proof} \quad \text{We start from the Fourier series expansion of } \hat{E}_2(z, s) \text{ given in Proposition 3.5. It gives}
\]

\[
\hat{E}_2(z, 0) = \left( \hat{\xi}(2) - \frac{1}{2y} \right) - 4\pi \left( \sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi inz} \right),
\]

where we use \( W_{1, \frac{z}{2}}(y) = y e^{-\frac{1}{2}y} \) [26, (13.18.2)] to get \( \frac{1}{2\pi iny} W_{1, \frac{z}{2}}(4\pi ny)e^{2\pi inx} = e^{2\pi inz} \), and we also use \( \lim_{s \to 0} (-s) \hat{\xi}(-2s)y^s = -\frac{1}{2} \). It is analytic at \( s = 0 \) so that \( \hat{E}_2(z, s) \) has \( G_0(z) = \hat{E}_2(z, 0) \equiv 0 \). Furthermore we have \( G_1(z) = \frac{\partial}{\partial s} \hat{E}_2(z, s)|_{s=0} = \hat{E}_2(z, 0), \) which with \( \hat{\xi}(2) = \frac{\pi}{6} \) gives the result. \( \square \)
8.2 Recursions for $\Delta_k$ and $\xi_k$ action on Taylor coefficients of Eisenstein series

We establish recursion relations under $\Delta_k$ and $\xi_k$ relating the Taylor coefficients $F_k(z)$ and $G_k(z)$.

**Proposition 8.3** With the Taylor coefficients of $E_k(z, s)$ defined in (8.1) and setting $F_n^{-, k}(z) = (-1)^n F_{n, k}(z)$, we have

1. For $k \geq 2$ an even integer,
   \[
   \Delta_k G_{n,k}(z) = (k - 1)G_{n-1,k}(z) + G_{n-2,k}(z),
   \]
   where we define $G_{1,k}(z) = G_{2,k}(z) \equiv 0$.
2. For $k \leq 0$ an even integer,
   \[
   \Delta_k F_{n,k}^{-}(z) = (1-k)F_{n-1,k}^{-}(z) + F_{n-2,k}^{-}(z),
   \]
   where we define $F_{-1,k}^{-}(z) = F_{-2,k}^{-}(z) \equiv 0$.
3. For $k \geq 2$ an even integer,
   \[
   \xi_k G_{n,k}(z) = (k-1)F_{n-1,2-k}^{-}(z) + F_{n-2,2-k}^{-}(z).
   \]
4. For $k \leq 0$ an even integer,
   \[
   \xi_k F_{n,k}^{-}(z) = G_{n,2-k}(z).
   \]

**Proof** (1) Let $k \geq 2$. Then

\[
\Delta_k(\hat{E}_k(z, s)) = \Delta_k\left(\sum_{n=0}^{\infty} G_{n,k}(z) s^n\right) = \sum_{n=0}^{\infty} \Delta_k G_{n,k}(z) s^n.
\]

By Theorem 3.6 (3),

\[
\Delta_k(\hat{E}_k(z, s)) = s(s + k - 1)\hat{E}_k(z, s)
\]

\[
= \sum_{n=0}^{\infty} G_{n,k} s^{n+2} + (k-1) \sum_{n=0}^{\infty} G_{n,k} s^{n+1}
\]

\[
= (k-1)G_{0,k}(z)s + \sum_{n=2}^{\infty} ((k-1)G_{n-1,k}(z) + G_{n-2,k}(z))s^n
\]

Term by term comparison establishes $\Delta_k G_{n,k}(z) = (k-1)G_{n-1,k}(z) + G_{n-2,k}(z)$ for all $n \geq 0$, using the convention that $G_{1,k}(z) = G_{2,k}(z) \equiv 0$.

(2) Let $k \geq 0$. An argument similar to (1) yields $\Delta_k F_{n,k}^{-}(z) = (k-1)F_{n-1,k}^{-}(z) + F_{n-2,k}^{-}(z)$ for all $n \geq 0$, using the convention that $F_{-1,k}^{-}(z) = F_{-2,k}^{-}(z) \equiv 0$. Substituting $F_{n,k}^{-}(z) = (-1)^n F_{n,k}(z)$ yields (8.4).
(3) We argue similarly to (1) using Proposition 7.1. Suppose \( k \geq 2 \). Then

\[
\xi_k(\hat{E}_k(z, s)) = \xi_k \left( \sum_{n=0}^{\infty} G_{n,k}(z) s^n \right) = \sum_{n=0}^{\infty} \xi_k G_{n,k}(z)(\bar{s}^n).
\]

By Proposition 7.1,

\[
\xi_k(\hat{E}_k(z, s)) = \bar{s}(\bar{s} + k - 1) \hat{E}_{2-k}(z, -\bar{s}) = \bar{s}(\bar{s} + k - 1) \left( \sum_{n=0}^{\infty} F_{n,2-k}(z)(\bar{s})^n \right)
= (k - 1)F_{0,2-k}(z)\bar{s} + \sum_{n=2}^{\infty} ((k - 1)F_{n-1,2-k}(z) + F_{n-2,2-k}(z))(\bar{s})^n.
\]

Term by term comparison in \( \bar{s}^n \) yields, \( \xi_k G_{n,k}(z) = (k - 1)F_{n-1,2-k}(z) + F_{n-2,2-k}(z) \) for all \( n \geq 0 \).

(4) Let \( k \leq 0 \), so

\[
\xi_k(\hat{E}_k(z, s)) = \sum_{n=0}^{\infty} \xi_k F_{n,k}(z)(\bar{s}^n) = \sum_{n=0}^{\infty} F_{n,k}(z)(-\bar{s}^n).
\]

By Proposition 7.1,

\[
\xi_k(\hat{E}_k(z, s)) = \hat{E}_{2-k}(z, -\bar{s}) = \bar{s}(\bar{s} + k - 1) \sum_{n=0}^{\infty} G_{n,2-k}(z)(-\bar{s})^n.
\]

Term by term comparison in \( -\bar{s}^n \) yields \( \xi_k F_{n,k}^{-}(z) = G_{n,k}(z) \) for all \( n \geq 0 \). \( \square \)

### 8.3 Taylor coefficients of Eisenstein series at \( s = 0 \) are polyharmonic Maass forms

Now we show that these Taylor coefficients are polyharmonic Maass forms.

**Theorem 8.4** With the Taylor coefficients of \( E_k(z, s) \) defined in (8.1) we have

1. For all \( n \geq 0 \) the functions \( F_{n,k}(z) \) (for weights \( k \leq 0 \)) and \( G_{n,k}(z) \) (for weights \( k \geq 2 \)) are polyharmonic Maass forms of depth at most \( n + 1 \), i.e., they belong to \( V_k^{n+1}(0) \).
2. For weights \( k \leq 0 \) and \( n \geq 0 \) the function \( F_{n,k}(z) \in V_k^{n+1}(0) \setminus V_k^n(0) \). For weights \( k \geq 4 \) and \( n \geq 0 \) the function \( G_{n,k}(z) \in V_k^{n+1}(0) \setminus V_k^n(0) \). For weight \( k = 2 \) and \( n \geq 1 \) the function \( G_{n,2}(z) \in V_2^n(0) \setminus V_2^{n-1}(0) \).

**Proof** (1) The \((n + 1)\)-harmonicity of both \( F_{n,k}(z) \) (\( k \leq 0 \)) and \( G_{n,k}(z) \) (\( k \geq 2 \)) follows by induction on \( n \) using the recursions (1) and (2) of Proposition 8.3. These recursions establish the base cases \( \Delta_k F_{0,k}(z) = 0 \), \( \Delta_k G_{0,k}(z) = 0 \) and \( \Delta_k F_{1,k}(z) = 

\( \copyright \) Springer
(k − 1)F_{0,k}(z), \ \Delta_k G_{1,k}(z) = (k − 1)G_{0,k}(z). The induction step for n ≥ 2 uses the two-step recursions (8.3) and (8.4).

The functions \( F_{n,k}(z) \) (resp. \( G_{n,k}(z) \)) transform as weight \( k \) modular forms as a property inherited from \( \hat{\mathcal{E}}_k(z, s) \). To show membership of these functions in \( V_k^{n+1}(0) \) it remains to show these functions have moderate growth at the cusp. We indicate details for \( F_{n,k}(z) \), the argument for \( G_{n,k}(z) \) being similar. We use the \( s \)-derivative relation

\[
F_{m,k}(z) = \frac{1}{m!} \left. \frac{\partial^m}{\partial s^m} \hat{\mathcal{E}}(z, s) \right|_{s=0}.
\]

We apply it term by term to the Fourier series for \( \hat{\mathcal{E}}(z, s) \) and then set \( s = 0 \) to obtain the Fourier series for \( F_{m,k}(z) \). Applying \( \frac{\partial}{\partial s} \) repeatedly to the Fourier coefficients of \( \hat{\mathcal{E}}(z, s) \) reveals that these coefficients involve polynomials in \( s \) of degree at most \( 2m \) times terms of the form \( y^{\frac{k}{2}} \frac{\partial^j}{\partial s^j} W_{\frac{k}{2}+s+\frac{k+1}{2}}(4\pi |n|y)e^{2\pi inx} \) times for \( 0 \leq j \leq m-1 \) given in Theorem 4.2. When we set \( s = 0 \), it follows that the individual nonconstant Fourier coefficients \( (n \neq 0) \) each have rapid decay at the cusp, while the constant term has at most polynomial growth in \( y \) as \( y \to \infty \), bounded by \( y^{k+\epsilon} \). Finally uniform decay estimates in \( n \) of the \( W \)-Whittaker function family and a bounded number \( s \)-derivatives of them imply that the full Fourier series of \( F_{m,k}(z) \) has moderate growth at the cusp, compare [2].

(2) For \( k \neq 2 \) prove the result on membership in \( V_k^{n+1}(0) \setminus V_k^n(0) \) by induction on \( n \geq 0 \). By convention \( V_k^0(0) = \{0\} \), and the base case assures \( F_{0,k}(z) \neq 0 \) for weight \( k \geq 0 \) and \( G_{0,k}(z) \neq 0 \) for weight \( k \geq 4 \). The non-vanishing of \( F_{0,k}(z) \) for weights \( k \geq -2 \) and of \( G_{0,k}(z) \) for weight \( k \geq 4 \) are equivalent to non-vanishing of \( \hat{\mathcal{E}}_k(z, 0) \), which follows from non-vanishing of the constant term in the Fourier series expansion of \( \hat{\mathcal{E}}(z, 0) \) in Proposition 3.5, and the fact that \( (s + \frac{k}{2})(1 - (s + \frac{k}{2})x) \neq 0 \). The non-vanishing for weight \( k = 0 \) is given by Lemma 8.1. For the step \( n = 1 \) we have for \( k \leq 0 \) that \( \Delta_k F_{1,k}(z) = (k - 1)F_{0,k}(z) \in V_k^1(0) \setminus V_k^0(0) \), which certifies \( F_{1,k}(z) \in V_k^2(0) \setminus V_k^1(0) \). Similarly for \( k \geq 4 \) we have \( \Delta_k G_{1,k}(z) = (k - 1)G_{0,k}(z) \in V_k^1(0) \setminus V_k^0(0) \), which certifies \( G_{1,k}(z) \in V_k^2(0) \setminus V_k^1(0) \). The induction step with \( n \geq 2 \) is completed for \( k \leq 0 \) using the recursion (8.4) using \( (k - 1)F_{n-1,k}(z) \in V_k^n(0) \setminus V_k^{n-1}(0) \) and \( F_{n-2,k}(z) \in V_k^{n-1}(0) \) yielding \((k - 1)F_{n-1,k}(z) + F_{n-2,k}(z) \in V_k^n(0) \setminus V_k^{n-1}(0) \), certifying \( F_{n,k}(z) \in V_k^{n+1}(0) \setminus V_k^n(0) \). The argument for \( k \geq 4 \) is similar.

Weight \( k = 2 \) is exceptional, because we have \( G_{0,2}(z) \equiv 0 \) by Lemma 8.2. In this case Lemma 8.2 gives \( G_{1,2}(z) \neq 0 \), and we prove \( G_{n,k}(z) \in V_k^n(0) \setminus V_k^{n-1}(0) \) for \( n \geq 1 \) by a similar induction, using \( G_{1,2}(z) \neq 0 \) as the base case. ☐

Theorem 8.4 motivates the following definition.

**Definition 8.5** The **polyharmonic Eisenstein space** \( E^n_k(0) \) is the vector space of weight \( k \) polyharmonic Maass forms of harmonic depth at most \( m \) that are spanned by the Taylor series coefficients of the doubly completed Eisenstein series \( \hat{\mathcal{E}}_k(z, s) \).

**Corollary 8.6** For each \( k \in 2\mathbb{Z} \) and each \( m \geq 1 \) the polyharmonic Eisenstein space \( E_k^m(0) \) has dimension \( m \) and is contained in \( V_k^m(0) \). For \( k \leq 0 \) a basis is \( \{F_{j,k}(z) : \).
0 \leq j \leq m - 1\}. For k = 2 a basis is \{G_{j,2}(z) : 1 \leq j \leq m\}. For k \geq 4 a basis is \{G_{j,k}(z); 0 \leq j \leq m - 1\}.

**Proof** Theorem 8.4 identifies the given Taylor coefficient functions as being in the space \(E_k^m(0)\) and being linearly independent. At the same time it identifies each Taylor coefficient functions as being linearly independent of the full set of lower numbered ones. \qed

### 9 Modified Taylor coefficient basis

We obtain modified basis functions of the polyharmonic Eisenstein space \(E_k^m(0)\) that simplifies the action of \(\Delta_k\) and \(\xi_k\), via a triangular change of basis of the Taylor coefficients of \(\hat{E}_k(z, s)\).

**Theorem 9.1** (Modified polyharmonic Eisenstein space basis) Let \(k \in 2\mathbb{Z}\), with \(k \geq 2\). Let \(\{c_{n,k,\ell} : n \geq 0; 1 \leq \ell \leq n + 1\}\) be a set of (complex) constants and define for \(n \geq 0\) the set of functions

\[
\tilde{G}_{n,k}(z) := G_{n,k}(z) + \sum_{\ell=1}^{n} (-1)^{\ell} c_{n,k,\ell} G_{n-\ell,k}(z),
\]

(9.1)

\[
\tilde{F}_{n,2-k}(z) := (-1)^{n} (F_{n,k}(z) + \sum_{\ell=1}^{n} c_{n,k,\ell} F_{n-\ell,2-k}(z)),
\]

(9.2)

in which \(G_{n,k}(z)\) are the Taylor coefficients of \(\hat{E}_k(z, s)\) and \(F_{n,k}(z)\) are the Taylor coefficients of \(\hat{E}_{2-k}(z, s)\). Then for all sets of constants that obey for all \(n \geq 1\) the connection equations:

\[
c_{n,k,\ell} = \frac{1}{k-1} c_{n,k,\ell-1} + c_{n-1,k,\ell}, \quad \text{for } 0 \leq \ell \leq n,
\]

(9.3)

in which we set \(c_{n,k,0} = 1\) and \(c_{n,k,-1} = 0\), and which involve the free constants \(c_{n,k,n+1}\) (for \(n \geq 0\)) these functions will simultaneously satisfy:

1. ("Ramp" relations)

\[
\xi_k \tilde{G}_{n,k}(z) = (k - 1) \tilde{F}_{n-1,2-k}(z) \quad \text{for } n \geq 1,
\]

(9.4)

\[
\xi_{2-k} \tilde{F}_{n,2-k}(z) = \tilde{G}_{n,k}(z) \quad \text{for } n \geq 1.
\]

(9.5)

2. ("Tower" relations)

\[
\Delta_k \tilde{G}_{n,2-k}(z) = (k - 1) \tilde{G}_{n-1,k}(z) \quad \text{for } n \geq 1,
\]

(9.6)

\[
\Delta_{2-k} \tilde{F}_{n,2-k}(z) = (k - 1) \tilde{F}_{n-1,2-k}(z) \quad \text{for } n \geq 1.
\]

(9.7)

**Remark** The connection equations have infinitely many solutions, since may specify arbitrarily the values of \(c_{n,k,n+1}\) for all \(n \geq 0\).
Proof Suppose \( k \geq 2 \). It suffices to verify the “ramp” relations (1) hold, since the “tower relations” (2) then follow using \( \Delta_k = \xi_k \xi_2 \) and \( k - \xi_2 = \xi_{2-k} \xi_k \). Set \( c_{n,k,0} = 1 \) and \( c_{n,k,-1} = 0 \) for all \( n \geq 0 \). By Proposition 8.3 (3) we obtain

\[
\xi_k \tilde{G}_{n,k}(z) = \xi_k \left( \sum_{\ell=0}^{n} (-1)^{\ell} c_{n,k,\ell} G_{n-\ell,k}(z) \right)
\]

\[
= \sum_{\ell=0}^{n} (-1)^{\ell} c_{n,k,\ell} \left( (k - 1)(-1)^{n-\ell-1} F_{n-\ell-1,2-k}(z) + (-1)^{n-\ell-2} F_{n-\ell-2,2-k}(z) \right)
\]

\[
= (-1)^{n-1} \sum_{\ell=0}^{n} c_{n,k,\ell} \left( (k - 1) F_{n-\ell-1,2-k}(z) - F_{n-\ell-2,2-k}(z) \right)
\]

\[
= (-1)^{n-1} (k - 1) \left( \sum_{\ell=0}^{n-1} F_{n-1-\ell,2-k}(z) (c_{n,k,\ell}) - \frac{1}{k - 1} c_{n,k,\ell-1} \right).
\]

The connection equations, after complex conjugation, state

\[
\bar{c}_{n,k,\ell} = \frac{1}{k - 1} c_{n,k,\ell-1} + c_{n-1,k,\ell}.
\]

Substituting these in the last equation yields

\[
\xi_k \tilde{G}_{n,k}(z) = (-1)^{n-1} (k - 1) \sum_{\ell=0}^{n-1} F_{n-1-\ell,2-k}(z) \bar{c}_{n-1,k,\ell} = (k - 1) \tilde{F}_{n-1,2-k}(z),
\]

which verifies one “ramp” relation.

Using next Proposition 8.3 (4) we obtain

\[
\xi_k \tilde{F}_{n,2-k}(z)
\]

\[
= \xi_{2-k} \left( (-1)^{n} \sum_{\ell=0}^{n} c_{n,k,\ell} F_{n-\ell,2-k}(z) \right) = (-1)^{n} \sum_{\ell=0}^{n} c_{n,k,\ell} \xi_{2-k}(F_{n-\ell,2-k}(z))
\]

\[
= (-1)^{n} \sum_{\ell=0}^{n} c_{n,k,\ell} (-1)^{n-\ell} G_{n-\ell,k}(z) = \sum_{\ell=0}^{n} c_{n,k,\ell} (-1)^{\ell} G_{n-\ell,k}(z) = \tilde{G}_{n,k}(z),
\]

which verifies the other “ramp” relation. \( \square \)

Remark There is freedom of choice in choosing the boundary values \( c_{n,k,n+1} \) for \( n \geq 0 \). We call the choice \( c_{n,k,n+1} = 0 \) for all \( n \) the zero boundary conditions. Table 1 presents value for the case of weight \( k = 2 \). The main diagonal in this table is Catalan numbers \( c_{n,2,n} = \frac{1}{n+1} \binom{2n}{n} \). One gets another elegant solution with the choice \( c_{n,k,n+1} = \binom{2n+1}{n+1} \) which corresponds to \( c_{n,k,n} = \binom{2n}{n} \); we call these binomial...
Table 1 Values of $c_{n,2,\ell}$, with 0 boundary conditions

| $n$ | $\ell$ |
|-----|--------|
| 0   | 1 0    |
| 1   | 1 1 0  |
| 2   | 1 2 2 0|
| 3   | 1 3 5 5 0|
| 4   | 1 4 9 14 14 0|
| 5   | 1 5 14 28 42 42 0|
| 6   | 1 6 20 48 90 132 132 0|
| 7   | 1 7 27 75 165 297 429 429 0|

Table 2 Values of $c_{n,2,\ell}$, with $\binom{2n+1}{n+1}$ boundary conditions

| $n$ | $\ell$ |
|-----|--------|
| 0   | 1 1    |
| 1   | 1 2 3  |
| 2   | 1 3 6 10|
| 3   | 1 4 10 20 35|
| 4   | 1 5 15 35 70 126|
| 5   | 1 6 21 56 126 252 462|
| 6   | 1 7 28 84 210 462 924 1716|
| 7   | 1 8 36 120 330 792 1716 3432 6435|

boundary conditions. In the binomial case one obtains $c_{n,k,\ell} = \frac{1}{(k-1)!} \binom{n+\ell}{n}$, as we will check when proving Theorem 2.2. Table 2 below gives values for weight $k = 2$.

10 Proofs of Theorems 1.1, 1.2, and 1.3

In this section we prove the three main theorems of Sect. 1

Proposition 10.1 For weight $k = 0$ the space $V^1_0$ is one dimensional and consists of constants. The spaces satisfy for all $n \geq 0$ the equality

$$V^{n+\frac{1}{2}}_0 = V^{n+1}_0.$$  

For integer $n \geq 1$ the space $V^n_0$ is at most $n$ dimensional.

Proof We consider the case $k = 0$. The space $V^{1/2}_0 = M_0$ is the space of weight 0 holomorphic modular forms, which has dimension 1 and is spanned by the constant functions (Theorem 3.1). We next check that the space $V^1_0$ is also generated by the constant functions, so that $V^1_0 = V^{1/2}_0$. First, we observe that every $f(z) \in V^1_0$ has a Fourier expansion of the form

\[ f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i nz}. \]
\[ f(z) = \sum_{n=1}^{\infty} b_{-n} e^{-2\pi inz} + (b_0 y + a_0) + \sum_{n=1}^{\infty} a_n e^{2\pi inz} \]

for some sequences of complex numbers \( a_n \) and \( b_n \). This fact follows starting from the Fourier series given in Lemma 4.4, observing that \( \Gamma(1, y) = \int_y^\infty e^{-t} dt = e^{-y} \) whence

\[
\Gamma(1, 4\pi |n|y) e^{2\pi inz} = \Gamma(1, 4\pi |n|y) e^{-2\pi ny} e^{2\pi inx} = e^{2\pi ny} e^{2\pi inx} = e^{2\pi inz} \text{ for } n \leq -1.
\]

Applying \( \xi_0 \) to this form, by Lemma 5.4 we obtain \( \xi_0 f(z) = V_2^1 \), and computation of \( \xi_0 \) on the Fourier expansion above yields

\[
\xi_0 f(z) = b_0 + \left( \sum_{n=1}^{\infty} b_{-n} (4\pi n) e^{2\pi inz} \right).
\]

This Fourier expansion certifies that \( \xi_0(f(z)) \) is a weight 2 holomorphic modular form for \( SL(2, \mathbb{Z}) \). There are no non-zero holomorphic modular forms by a classical result (Theorem 3.1), whence \( \xi_0 f(z) = 0 \) and \( \xi_0(V_0^1) = \{0\} \). It follows that \( f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz} \) is a holomorphic modular form, so \( f(z) \in V_0^{1/2} \) and \( V_0^{1/2} = V_0^1 \).

We next show for all \( n \geq 1 \) the equality of vector spaces

\[
V_0^{n+1/2} = V_0^{n+1}.
\]

Clearly \( V_0^{n+1/2} \subset V_0^{n+1} \). Suppose for a contradiction that the inclusion is strict, so there exists \( f(z) \in V_0^{n+1} \) and \( f(z) \notin V_0^{n+1/2} \). Then \( g(z) := \Delta^n(f)(z) \in V_0^1 \) but \( g(z) \notin V_0^{1/2}(0) \), which contradicts \( V_0^{1/2} = V_0^1 \). The equality follows.

We now show \( \dim(V_0^{n+1}) \leq \dim(V_0^n) + 1 \) by induction on \( n \geq 0 \). The base case \( n = 0 \) is established. We prove the induction step by contraction. Let \( \dim(V_0^n) = m \) and suppose to the contrary that \( \dim(V_0^{n+1}) \geq m + 2 \). Since the constants are in \( V_0^1(0) \), we may choose \( m + 2 \) linearly independent functions \( f_0(z), ..., f_{m+1}(z) \) in \( V_0^{n+1}(0) \) with \( f_0(z) = 1 \) being constant. Now \( \Delta_0(f_i)(z) : 1 \leq i \leq m+1 \) all lie in \( V_0^n \) so must be linearly dependent, say \( \sum_{i=1}^{m+1} \alpha_i \Delta_0(f_i) = 0 \). Thus \( g(z) := \sum_{i=1}^{m+1} \alpha_i \Delta_0(f_i) \) satisfies \( \Delta_0(g(z)) = \sum_{i=1}^{m+1} \alpha_i \Delta_0(f_i) = 0 \) whence \( g(z) \in V_0^1(0) \). This forces \( g(z) = \alpha_0 \) to be constant, which gives a nontrivial linear relation \( \sum_{i=1}^{m+1} \alpha_i f_i(z) = \alpha_0 f_0(z) \) contradicting our assumption of linear independence. The induction step is proved. \( \square \)

**Proposition 10.2** For weight \( k = 2 \) the space \( V_2^{1/2} \) is trivial, and the space \( V_2^1 \) is one dimensional, spanned by \( E_2(z, 0) \). The spaces satisfy for all \( n \geq 0 \) the equality

\[
V_2^n = V_2^{n+1/2}.
\]

For integer \( n \geq 0 \) the space \( V_2^n \) is at most \( n \) dimensional.
Proof. It is a classical result that the space $V^{1/2}_2 = M_2$ of holomorphic weight 2 forms is trivial. (Theorem 3.1). The space $V^1_2$ has dimension at least 1 since $\Delta_2(\hat{E}_2(z, 0)) = 0$ by Theorem 3.6(3). Furthermore $\xi_2(\hat{E}_2(z, 0)) = \frac{1}{2}$ is a constant function.  

Now $\xi_2(V^1_2) \subset V^0_0$ by Lemma 5.4, and $V^1_0$ consists of constant functions. If $\xi_2(V^1_2)$ were of dimension 2 or greater, then its image under $\xi_2$ would have a linear dependence, which we could use to produce a non-zero function $f(z) \in V^1_2$ with $g(z) = \xi_2 f(z) = 0$. By Lemma 4.4 every $f(z) \in V^1_2$ has a Fourier expansion of the form

$$f(z) = \sum_{n=1}^{\infty} b_{-n} \Gamma(-1, 4\pi |n|y) e^{-2\pi i n z} + \left(\frac{b_0}{y} + a_0\right) + \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$ 

for some sequences of complex numbers $a_n$ and $b_n$. A calculation using Lemma 4.4 and Lemma 5.3(1) shows that $g(z) = \xi_2 f(z)$ has Fourier expansion

$$g(z) = \xi_2 f(z) = -\sum_{n=1}^{\infty} \frac{b_{-n}}{4\pi n} e^{2\pi i n z} = 0.$$ 

We conclude that all $b_n = 0$, which says that $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ is a weight 2 holomorphic modular form, so $f(z) = 0$. Thus dim$(V^1_1) = 1$, and it is generated by any non-zero scalar multiple of $\hat{E}_2(z, 0)$.

We next show for all $n \geq 1$ the equality of vector spaces

$$V^n_2 = V^{n+1/2}_2.$$ 

Clearly $V^n_2 \subset V^{n+1/2}_2$. Suppose for a contradiction that the inclusion is strict, so there exists $f(z) \in V^{n+1/2}_2$ and $f(z) \notin V^n_2$. Then $g(z) := \Delta^n(f)(z) \in V^{1/2}_2$ is non-zero, otherwise it is in $V^n_2$, which contradicts the fact that $V^{1/2}_2$ is $\{0\}$. The equality follows.

Finally one may show by induction on $n$ that dim$(V^{n+1}_2) \leq$ dim$(V^n_2) + 1$ holds for all $n \geq 0$ exactly as in Proposition 10.1. \qed

Proof of Theorems 1.1 and 1.2 An upper bound dim$(V^n_k) \leq n$ for $k = 0$ follows from Proposition 10.1, which shows also $V^n_0 = V^{n-1}_0$. An upper bound dim$(V^n_k) \leq n$ for $k = 2$ follows from Proposition 10.2, which shows $V^n_2 = V^{n+1/2}_2$. It remains to show a matching lower bound dim$(V^n_k) \geq n$ for $k = 0, 2$ and to give a basis. The Eisenstein space $E^n_k(0) \subset V^n_k$ for all $k \in 2\mathbb{Z}$, so the required lower bound follows from the assertion dim$(E^n_k(0)) = n$, given in Corollary 8.6, which shows $V^n_k = E^n_k(0)$.

Corollary 8.6 now supplies the asserted bases for $V^n_k(0)$ for $k = 0, 2$. \qed

\footnote{The constant can be found using Proposition 7.1 as $s \to 0$ in $\xi_2(\hat{E}_2(z, s)) = (\zeta + 1)(\pi \hat{E}_0(z, -\zeta))$, using the simple pole of $\hat{E}_0(z, s)$ at $s = 0$ with residue $-\frac{1}{2}$.}
Proof of Theorem 1.3  We apply Theorem 9.1 with \( k = 2 \). We need only to prove that the particular choice

\[
c_{n,2,\ell} = \binom{n+\ell}{\ell}
\]

satisfies the hypotheses of that result. We have \( c_{n,2,0} = \binom{n}{0} = 1 \) and \( c_{n,2,-1} = \binom{n-1}{-1} = 0 \). Since \( k = 2 \) the connection equation (9.3) asserts \( c_{n,2,\ell} = c_{n,2,\ell-1} + c_{n-1,2,\ell} \) which is exactly the binomial coefficient recursion

\[
\binom{n+\ell}{\ell} = \binom{n-1}{\ell-1} + \binom{n-1}{\ell}.
\]

The result follows. In this case the other boundary condition states \( c_{n,2,n+1} = \binom{2n+1}{n+1} \).

\[\square\]

11 Proof of Theorems 2.2 and 2.3

We now treat the arbitrary even integer weight case, excluding \( k = 0 \) or 2. The proof of Theorem 2.2 is similar to that of Theorems 1.1, 1.2, and 1.3.

Proof of Theorem 2.2  In following proof we let the weight parameter \( k \geq 4 \), and the case of weights \( \leq -2 \) are represented as \( 2 - k \).

We first establish assertion (2) for harmonic depth \( m = 1 \). Suppose we are given a weight \( k \geq 4 \) form \( f(z) \in V_1^k(0) \). It has a Fourier expansion of the form

\[
f(z) = \sum_{n=1}^{\infty} b_{-n} \Gamma(1-k, 4\pi |n| y) e^{-2\pi i nz} + (b_0 y^{1-k} + a_0) + \sum_{n=1}^{\infty} a_n e^{2\pi i nz},
\]

see Lemma 4.4. Now \( \xi_k f(z) \in V_{2-k}^1(0) \) by Lemma 5.4, and a calculation as in Lemma 6.6 we have

\[
\xi_k f(z) = (1-k) b_0 - \sum_{n=1}^{\infty} \overline{b_{-n}(4\pi n)^{k-1} e^{2\pi i nz}}.
\]

Thus \( \xi_k f(z) \) is a holomorphic modular form, and by Theorem 3.1 it is identically 0. It follows that its Fourier series has \( b_n = 0 \) for all \( n \geq 0 \), and we conclude that \( f(z) \) is itself holomorphic. Therefore \( V_1^k(0) = V_{2-k}^1(0) = M_k \). Now by Proposition 3.2 \( M_k = E_k^1(0) + S_k \), where \( E_k^1(0) \) contains the weight holomorphic \( k \) Eisenstein series, which by (2.2) which is one dimensional. Moreover, \( S_k \) is the space of weight \( k \) cusp forms. This establishes the assertion (2) for \( m = 1 \), i.e., for \( V_k^1(0) \) for \( k \geq 4 \).
Next we establish the assertion (1) for \( V_{2-k}^1(0) \) for harmonic depth \( m = 1 \). Let \( f(z) \in V_{2-k}^1(0) \). Then \( \xi_{2-k} f(z) \in V_k^1(0) = E_k^1(0) + S_k \), whence we have

\[
\xi_{2-k} f(z) = c_E E_k(z, 0) + g(z)
\]

where \( g(z) \in S_k \) is a cusp form. Proposition 7.1 shows that for weights \( 2 - k \leq 0 \) that the function \( F_{0,2-k}(z) = \hat{E}_{2-k}(z, 0) \) has

\[
\xi_{2-k} F_{0,2-k}(z) = \xi_{2-k} \hat{E}_{2-k}(z, 0) = \hat{E}_k(z, 0) = \frac{k}{2} \left( \frac{k}{2} - 1 \right) \pi^{-\frac{k}{2}} \Gamma(k) E_k(z, 0).
\]

Now \( \frac{k}{2} \left( \frac{k}{2} - 1 \right) \pi^{-\frac{k}{2}} \Gamma(k) \neq 0 \) since we assumed \( k \geq 4 \). Thus there is a constant \( C \) such that \( h(z) = f(z) - CF_{0,2-k}(z) \) has

\[
\xi_{2-k} (h(z)) = \xi_{2-k} (f(z) - CF_{0,2-k}(z)) = g(z).
\]

Now \( F_{0,2-k} \in V_{2-k}^1(0) \) by Proposition 8.3(5). so \( h(z) := f(z) - CF_{0,2-k}(z) \in V_{2-k}^1(0) \). Since \( h(z) \) is a preimage of a cusp form under \( \xi_{2-k} \), Proposition 6.6 implies that \( g(z) = 0 \). Therefore, \( f(z) \) is a multiple of \( F_{0,2-k}(z) \), and we have established that \( V_{2-k}^1(0) \) is one dimensional, spanned by \( F_{0,2-k}(z) \). We know that \( V_{2-k}^1(0) = M_k \) is \( M_k = \{ 0 \} \), because there are no holomorphic modular forms of negative weight, hence \( V_{2-k}^1(0) \neq V_{2-k}^1(0) \). Finally if \( f(z) \in V_{2-k}^{3/2}(0) \), then \( \Delta_{2-k}(f)(z) \in V_{2-k}^{1/2}(0) = \{ 0 \} \), so \( f(z) \in V_{2-k}^1(0) \) and \( V_{2-k}^1(0) = V_{2-k}^{3/2}(0) \). This establishes assertion (1) for \( m = 1 \) for weights \( 2 - k \leq -2 \).

We have proved assertions (1) and (2) for \( m = 1 \) and complete the proof by induction on \( m \geq 1 \). For assertion (1), the induction hypothesis says that \( V_{2-k}^m(0) \) is \( m \)-dimensional. Now Lemma 7.2 shows that \( V_{2-k}^{m+1}(0) \) is at least \( m + 1 \) dimensional, since \( F_{m,k}(z) \in V_{2-k}^{m+1}(0) \setminus V_{2-k}^m(0) \). Suppose that there is a form \( f(z) \in V_{2-k}^{m+1}(0) \) linearly independent from the set \( \{ F_{0,2-k}(z), \ldots, F_{m-2-k}(z) \} \). Then \( \Delta_{2-k} f(z), \Delta_{2-k} F_{0,k}(z), \ldots, \Delta_{2-k} F_{m,k}(z) \in V_{2-k}^m(0) \) are linearly dependent, so there exist \( C, c_0, c_1, \ldots, c_m \) so that

\[
\Delta_{2-k} \left( Cf(z) + c_0 F_{0,k}(z) + \cdots + c_m F_{m,k}(z) \right) = 0.
\]

It now follows that \( Cf(z) + c_0 F_{0,k}(z) + \cdots + c_m F_{m,k}(z) = DF_{0,k}(z) \) holds for some constant \( D \), contracting linear independence. Thus \( V_{2-k}^{m+1}(0) \) has dimension \( m + 1 \).

It is easy to deduce that \( V_{2-k}^{m+1}(0) = V_{2-k}^{m+\frac{3}{2}}(0) \) by applying \( (\Delta_k)^m \) to a given \( f(z) \in V_{2-k}^{m+\frac{3}{2}}(0) \).

The proof of the induction step for assertion (2) is similar. We must also use the result of Proposition 6.6 which shows there are no 1-harmonic lifts of cusp forms. \( \square \)
Proof of Theorem 2.3  This result follows from Theorem 9.1. It remains only to show for \( k \geq 4 \) that

\[
c_{n,k,\ell} = \frac{1}{(k-1)^\ell} \binom{n + \ell}{\ell}
\]

satisfies the connection equations of that result. It is easy to check that \( c_{n,k,0} = 1 \) and \( c_{n,k,-1} = 0 \). The connection equations (9.3) become the identity

\[
\frac{1}{(k-1)^\ell} \binom{n + \ell}{\ell} = \frac{1}{k-1} \left( \frac{1}{(k-1)^{\ell-1}} \binom{n + \ell - 1}{\ell - 1} \right) + \frac{1}{(k-1)^\ell} \binom{n + \ell - 1}{\ell}.
\]

\[\square\]

Acknowledgments  The authors thank N. Andersen, W. Duke and the reviewer for helpful comments. The first author thanks the Clay Foundation for support as a Clay Senior Fellow at ICERM, an NSF-supported institute, where some work on this paper was done.

References

1. Almansi, E.: Sull’ integrazione dell’equazione differenziale \( \Delta^2u = 0 \). Ann. Mat. Ser. III(11), 1–59 (1899)
2. Andersen, N., Lagarias, J. C., Rhoades, R. C.: Shifted polyharmonic Maass forms for \( \text{PSL}(2, \mathbb{Z}) \) (in preparation)
3. Aronszajn, N., Crease, T.M., Lipkin, L.J.: Polyharmonic Functions. Oxford University Press, Oxford (1983)
4. Borchers, R.E.: The Gross-Kohnen-Zagier theorem in higher dimensions. Duke Math. J. 97, 219–233 (1999)
5. Borel, A.: Automorphic Forms on \( \text{SL}_2(R) \). Cambridge University Press, Cambridge (1997)
6. Bringmann, K., Diamantis, N., Raum, M.: Mock period functions, sesquiharmonic Maass forms, and non-critical values of \( L \)-functions. Adv. Math. 233, 115–134 (2013)
7. Bringmann, K., Ono, K.: Lifting cusp forms to Maass forms with an application to partitions. Proc. Natl. Acad. USA 104, 3725–3731 (2007)
8. Bruggeman, R.: Families of Automorphic Forms. Birkhäuser, Basel (1994)
9. Bruggeman, R.: Harmonic lifts of modular forms. Ramanujan J. 33(1), 55–82 (2014)
10. Bruinier, J., Funke, J.: On two geometric theta lifts. Duke Math. J. 125, 45–90 (2004)
11. Bruinier, J.H., Ono, K., Rhoades, R.: Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues. Math. Ann. 342(3), 673–693 (2008) (Erratum 345(1), 31 (2009))
12. Bump, D.: Automorphic Forms and Representations. Cambridge University Press, Cambridge (1997)
13. Duke, W., Imamoglu, Ö., Tóth, Á.: Cycle integrals of the \( J \)-function and mock modular forms. Ann. Math. 173(2), 947–981 (2011)
14. Duke, W., Imamoglu, Ö., Tóth, Á.: Real quadratic analogous of traces of singular moduli. IMRN 13, 3082–3094 (2011)
15. Duke, W., Imamoglu, Ö., Tóth, Á.: Regularized inner products of modular functions. Ramanujan J. doi:10.1007/s11139-013-9544-5
16. Duke, W., Li, Y.: Harmonic Maass forms of weight one. Duke Math. J. 164(1), 39–113 (2015)
17. Katayama, K.: A generalization of gamma functions and Kronecker’s limit formulas. J. Number Theory 130, 1642–1674 (2010)
18. Kent, Z.: \( p \)-adic analysis and mock modular forms. Ph.D. Thesis, University of Hawaii (2010)
19. Kubota, T.: Elementary Theory of Eisenstein Series. Halstead Press, New York (1973)
20. Kudla, S.: Special cycles and derivatives of Eisenstein series. Heegner Points and Rankin L-series. Mathematical Sciences Research Institute Publications, vol. 49, pp. 243–270. Cambridge University Press, Cambridge (2004)
21. Kudla, S., Rapoport, M., Yang, T.-H.: Derivatives of Eisenstein series and Faltings heights. Compos. Math. **140**, 887–951 (2004)
22. Kudla, S., Yang, T.: Eisenstein series for $SL(2)$. Sci. China Math. **53**(9), 2275–2316 (2010)
23. Maass, H.: Die Differentialgleichungen in der Theorie der elliptischen Modulfunktionen. Math. Ann. **125**, 235–263 (1953)
24. Maass, H.: Lectures on Modular Functions of One Complex Variable. Tata Institute of Fundamental Research, Bombay (1964) (revised 1983)
25. Miyake, T.: Modular Forms. Translated and corrected by Y. Maeda. Springer, New York (1989, reprinted 2006)
26. NIST Digital Library of Mathematical Functions. http://www.dlmf.nist.gov/, Release 1.0.7 of 2014-03-21
27. Ono, K.: Unearthing the visions of a master: harmonic Maass forms and number theory. In: Proceedings of the 2008 Harvard-MIT Current Developments in Mathematics Conference, pp. 347–454. Somerville (2009)
28. Render, H.: Real Bargmann spaces, Fischer decompositions, and sets of uniqueness for polyharmonic functions. Duke Math. J. **142**(2), 313–352 (2008)
29. Serre, J.-P.: A Course in Arithmetic. Springer, New York (1973)
30. Siegel, C.L.: Advanced analytic number theory. Fundamental Research Studies in Mathematics, vol. 9, 2nd edn. Tata Institute of Fundamental Research, Mumbai (1980)
31. Stark, H.M.: Class fields and modular forms of weight one. Lecture Notes in Mathematics, vol. 601, pp. 277–287. Springer, Berlin (1977)
32. Whittaker, E.M., Watson, G.N.: A Course of Modern Analysis, 4th edn. Cambridge University Press, Cambridge (1927) (reissued 1996)
33. Yang, T.-H.: Faltings heights and the derivative of Zagier’s Eisenstein series. Hegner Points and Rankin L-series. Mathematical Sciences Research Institute Publications, vol. 49, pp. 271–284. Cambridge University Press, Cambridge (2004)
34. Yang, T.-H.: The second term of an Eisenstein series. In: Second International Congress of Chinese Mathematicians. New Studies in Advanced Mathematics, vol. 4, pp. 233–248. International Press, Somerville (2004)
35. Zagier, D.: Eisenstein series and the Riemann zeta function. In: Automorphic Forms, Representation Theory and Arithmetic (Bombay, 1979). Studies in Mathematics, vol. 10, pp. 275–301. Tata Institute of Fundamental Research, Bombay (1981)
36. Zagier, D.: Elliptic modular forms and their applications. In: The 1-2-3 of Modular Forms: Lectures at a Summer School in Nordfjordeid, pp. 1–103. Universitext, Springer, Berlin (2008)