A regularity criterion for the Navier-Stokes equations via two entries of the velocity Hessian tensor

Zujin Zhang\textsuperscript{1,*}

Department of Mathematics, Sun Yat-sen University
Guangzhou 510275, Guangdong, P.R. China

Abstract

We consider the Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^3$, and provide a sufficient condition to ensure the smoothness of the solution. It involves only two entries of the velocity Hessian tensor.

Keywords: Incompressible Navier-Stokes equations, regularity criterion, global regularity, weak solutions, strong solutions

2010 MSC: 35Q30, 35B45, 76D03

1. Introduction

This paper is concerned with the global regularity of solutions of the three-dimensional Navier-Stokes equation (NSE):

\begin{align}
\frac{\partial}{\partial t}u - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= 0, \quad \text{in } \mathbb{R}^3 \times (0,T), \\
\nabla \cdot u &= 0, \quad \text{in } \mathbb{R}^3 \times (0,T), \\
\nabla \cdot u &= 0, \quad \text{on } \mathbb{R}^3 \times \{t = 0\} ,
\end{align}

where $T > 0$ is a given time, $u = (u_1, u_2, u_3)$ is the velocity field, $p$ is a scalar pressure, and $u_0$ is the initial velocity field satisfying $\nabla \cdot u_0 = 0$ in the sense of distributions.

The global existence of a weak solution $u$ to (1) with initial data of finite energy is well-known since the work of Leary [19], see also Hopf...
However, the issue of uniqueness and regularity of \( u \) was left open, and is still unsolved up to date. Pioneered by Serrin [25, 26] and Prodi [24], there have been a lot of literatures devoted to finding sufficient conditions to ensure the smoothness of \( u \). These conditions involve either

- the velocity \( u \), see [10, 11, 13, 25, 26, 27, 28], which states
  \[
  u \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1, \quad 3 \leq \beta \leq \infty;
  \]

- or several components of the velocity \( u \), the velocity gradient \( \nabla u \), the vorticity \( \omega = \text{curl} \ u \), or the pressure gradient \( \nabla p \), see [1, 2, 3, 4, 6, 5, 8, 9, 10, 12, 16, 17, 18, 20, 21, 22, 23, 32, 33, 34, 35, 36, 37, 39, 43, 44, 47, 48], and references cited therein.

We remark that many of the regularity criteria established in the above cited papers have been extended to the following three dimensional magneto-hydrodynamic equations (MHD):

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - (b \cdot \nabla) b - \Delta u + \nabla p &= 0, \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\
\partial_t u + (u \cdot \nabla) b - (b \cdot \nabla) u - \Delta b &= 0, \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\
\nabla \cdot u &= 0, \quad \nabla \cdot b = 0, \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\
\nabla \cdot u &= 0, \quad \nabla \cdot b = 0, \quad \text{on } \mathbb{R}^3 \times \{t = 0\}.
\end{aligned}
\]

Here \( u = (u_1, u_2, u_3) \) is the velocity field, \( b = (b_1, b_2, b_3) \) is the magnetic field, \( u_0, b_0 \) are the corresponding initial data, and \( p \) is a scalar pressure. The interested readers are referred to [14, 30, 31, 38, 40, 46] and references cited therein.

Motivated by [7] and [42], we consider in this paper the regularity criterion involving \( \partial_1 \partial_3 u_3 \) and \( \partial_2 \partial_3 u_3 \) only. Before stating the precise result, let us recall the weak formulation of (1).

**Definition 1.** Let \( u_0 \in L^2(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \), and \( T > 0 \). A measurable \( \mathbb{R}^3 \)-valued vector \( u \) is said to be a weak solution of (1) if the following conditions hold:

1. \( u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \);
2. \( u \) solves (1) in the sense of distributions; and
3. the energy inequality, that is,

\[ \|u(t)\|_2^2 + \nu \int_{t_0}^{t} \|\nabla u(s)\|_2^2 \, ds \leq \|u(t_0)\|_2^2, \tag{4} \]

for almost every \( t_0 \) (including \( t_0 = 0 \)) and every \( t \geq t_0 \).

Our regularity criterion now reads:

**Theorem 2.** Let \( u_0 \in H^1(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \), and \( T > 0 \). Suppose \( u \) is the corresponding solution on \([0, T]\) of \( (1) \). If

\[ \partial_1 \partial_3 u_3, \, \partial_2 \partial_3 u_3 \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2 + \frac{1}{\beta}; \quad 1 < \beta \leq \infty, \tag{5} \]

then \( u \) is smooth on \((0, T)\).

Before proving this theorem in Section 2, we collect here some notations used throughout this paper and make some remarks on our result.

The usual Lebesgue spaces \( L^q(\mathbb{R}^3) \) \((1 \leq q \leq \infty)\) is endowed with the norm \( \|\cdot\|_q \). For a Banach space \((X, \|\cdot\|)\), we do not distinguish it with its vector analogues \( X^3 \), thus the norm in \( X^3 \) is still denoted by \( \|\cdot\| \); however, all vector- and tensor-valued functions are printed boldfaced. We also denote by

\[ \partial_i \varphi = \frac{\partial \varphi}{\partial x_i}, \quad \partial^2_{ij} \varphi = \partial_i \partial_j \varphi = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \quad 1 \leq i \leq 3 \]

the first- and second-order derivatives of a function \( \varphi \); by

\[ \nabla_h \varphi = (\partial_1, \partial_2) \varphi, \quad \Delta_h \varphi = (\partial^2_{11} + \partial^2_{22}) \varphi \]

the horizontal gradient, horizontal Laplacian of \( \varphi \).

**Remark 3.** Noticing that

\[ \lim_{\beta \to 1} \left( 2 + \frac{1}{\beta} \right) = 3, \]

we almost establish a Serrin-type regularity criterion via \( \partial_1 \partial_3 u_3 \) and \( \partial_2 \partial_3 u_3 \) only.
Remark 4. Due to the orthogonal transformation invariance of NSE (1), we easily extends our regularity criterion as

\[ \partial_i \partial_k u_k, \partial_j \partial_k u_k \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2 + \frac{1}{\beta}, \quad 1 < \beta \leq \infty, \]

where \( \{i, j, k\} = \{1, 2, 3\}. \)

Remark 5. Our result seems to be more involved than that in [5] in the following sense. In [5], only conditions on 1 component is needed among the total 9 components of the velocity gradient tensor

\[ \partial \otimes u = \nabla u = [\partial_i u_j], \]

the ratio is 1/9. And our result requires regularity of 2 entries among the total 27 entries of the velocity Hessian tensor

\[ \partial^2 \otimes u = \nabla^2 u = [\partial^2_{ij} u_k], \]

the ratio being 2/27, which is less than 1/9.

Remark 6. In classical and numerical analysis, we do not only rely on the graph of the differential to study properties of a function, but also utilize the graph of the Hessian to do so. In this point of view, our result is a complement to that in [5, 42].

Remark 7. Our proof in Section 2 is different from that in [5, 42], due to the our assumptions on the Hessian. We shall first bound \( \|\nabla_3 u\|_2 \), then estimate \( \|\nabla u\|_2 \). In fact, tracking the proof of that in [5, 42] we will obtain a not-so-good result.

2. Proof of the main result

In this section, we shall prove Theorem 2. First, let us recall and prove some technical lemmas.

The first one being a component-reducing technique due to Kukavica and Ziane [17].

Lemma 8. Assume \( u = (u_1, u_2, u_3) \in C^\infty_c(\mathbb{R}^3) \) is divergence free. Then

\[ \sum_{i,j=1}^2 \int_{\mathbb{R}^3} u_i \partial_i u_j \Delta u_j dx = \frac{1}{2} \sum_{i,j=1}^2 \int_{\mathbb{R}^3} \partial_i u_j \partial_i u_j \partial_3 u_3 dx \]

\[ -\int_{\mathbb{R}^3} \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 dx + \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2 u_1 \partial_3 u_3 dx. \quad (6) \]
And the next two lemmas are variants of multiplicative Sobolev inequalities in $\mathbb{R}^3$, similar in spirit to that in [5].

**Lemma 9.** Let $1 < r \leq 3$. Assume $f$, $g$, $h \in C_c^\infty(\mathbb{R}^3)$. Then there exists a constant $C > 0$ such that

$$
\int_{\mathbb{R}^3} f \ g \ h \ dx \leq C \|f\|^{1/r}_{\mathbb{R}^3} \|\partial_3 f\|^{1/r}_{\mathbb{R}^3} \|g\|^{1/r}_{\mathbb{R}^3} \|\partial_1 g\|^{1/r}_{\mathbb{R}^3} \|\partial_2 g\|^{1/r}_{\mathbb{R}^3} \|h\|^{1/r}_{\mathbb{R}^3} \|\partial_1 h\|^{1/r}_{\mathbb{R}^3} \|\partial_2 h\|^{1/r}_{\mathbb{R}^3}.
$$

**Proof.**

$$
\int_{\mathbb{R}^3} f \ g \ h \ dx_1 dx_2 dx_3
\leq \int_{\mathbb{R}^3} \max_{x_3} |f| \left( \int_{\mathbb{R}^3} |g|^{2} \ dx_3 \right)^{1/2} \left( \int_{\mathbb{R}^3} |h|^{2} \ dx_3 \right)^{1/2} \ dx_1 dx_2
\leq \left[ \int_{\mathbb{R}^3} \left( \max_{x_3} |f| \right)^{r} \ dx_1 dx_2 \right]^{1/r} \left[ \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |g|^{2} \ dx_3 \right)^{\frac{r}{r+1}} \ dx_1 dx_2 \right]^{\frac{r+1}{r}}
\cdot \left[ \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |h|^{2} \ dx_3 \right)^{\frac{r}{r+1}} \ dx_1 dx_2 \right]^{\frac{r+1}{r}}
\leq C \left[ \int_{\mathbb{R}^3} |f|^{r-1} |\partial_3 f| \ dx_1 \right]^{1/r} \left[ \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |g|^{\frac{2r}{r+1}} \ dx_1 dx_2 \right)^{\frac{r+1}{r}} \ dx_3 \right]^{1/2}
\cdot \left[ \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |h|^{\frac{2r}{r+1}} \ dx_1 dx_2 \right)^{\frac{r+1}{r}} \ dx_3 \right]^{1/2}
\leq C \|f\|^{1/r}_{\mathbb{R}^3} \|\partial_3 f\|^{1/r}_{\mathbb{R}^3} \|g\|^{1/r}_{\mathbb{R}^3} \|\partial_1 g\|^{1/r}_{\mathbb{R}^3} \|\partial_2 g\|^{1/r}_{\mathbb{R}^3} \|h\|^{1/r}_{\mathbb{R}^3} \|\partial_1 h\|^{1/r}_{\mathbb{R}^3} \|\partial_2 h\|^{1/r}_{\mathbb{R}^3}.
$$

The same argument also yields

**Lemma 10.** Let $1 < r \leq 3$. Assume $f$, $g$, $h \in C_c^\infty(\mathbb{R}^3)$. Then there exists a constant $C > 0$ such that

$$
\int_{\mathbb{R}^3} f \ g \ h \ dx \leq C \|f\|^{1/r}_{\mathbb{R}^3} \|\partial_1 f\|^{1/r}_{\mathbb{R}^3} \|g\|^{1/r}_{\mathbb{R}^3} \|\partial_2 g\|^{1/r}_{\mathbb{R}^3} \|\partial_3 g\|^{1/r}_{\mathbb{R}^3} \|h\|^{1/r}_{\mathbb{R}^3} \|\partial_1 h\|^{1/r}_{\mathbb{R}^3} \|\partial_2 h\|^{1/r}_{\mathbb{R}^3} \|\partial_3 h\|^{1/r}_{\mathbb{R}^3}.
$$
Proof of Theorem 2.

Step 1. Preliminary reduction.

For any \( \varepsilon \in (0, T) \), due to the fact that \( \nabla u \in L^2(0, T; L^2(R^3)) \), we may find a \( \delta \in (0, \varepsilon) \), such that \( \nabla u(\delta) \in L^2(R^3) \). Take this \( u(\delta) \) as initial data, there exists an \( \hat{u} \in C([\delta, \Gamma^*], H^1(R^3)) \cap L^2(0, \Gamma^*; H^2(R^3)), \) where \([\delta, \Gamma^*]\) is the life span of the unique strong solution, see [29]. Moreover, \( \hat{u} \in C^\infty(R^3 \times (\delta, \Gamma^*)) \).

According to the uniqueness result, \( \hat{u} = u \) on \([\delta, \Gamma^*]\). If \( \Gamma^* \geq T \), we have already that \( u \in C^\infty(R^3 \times (0, T)) \), due to the arbitrariness of \( \varepsilon \in (0, T) \). In case \( \Gamma^* < T \), our strategy is to show that \( \|\nabla u(t)\|_2 \) remains bounded independently of \( t \nearrow \Gamma^* \). The standard continuation argument then yields that \([\delta, \Gamma^*]\) could not be the maximal interval of existence of \( \hat{u} \), and consequently \( \Gamma^* \geq T \). This concludes the proof.

For this purpose, let us choose a \( \tau \) sufficiently close to \( \Gamma^* \) such that

\[
\int_{\tau}^{\Gamma^*} \left[ \| (\partial_1, \partial_2) \cdot \partial_3 u_3(s) \|_{L^2}^2 + \| \nabla u(s) \|_{L^2}^2 \right] ds < \tilde{\varepsilon},
\]

where \( \tilde{\varepsilon} > 0 \) is small and will be chosen later on.

We shall first in Step 2 establish the bounds of \( \|\nabla_h u(t)\|_2 \) for \( t \in [\tau, \Gamma^*) \).

The estimation of \( \|\nabla_h u(t)\|_2 \) is derived in Step 3.

Step 2. \( \|\nabla_h u(t)\|_2 \) estimates.

Taking the inner product of \( (1)_1 \) with \(-\Delta u\), we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla_h u(t)\|_2^2 + \nu \|\nabla_h u(t)\|_2^2 = \int_{R^3} (u \cdot \nabla) u \cdot \Delta u \, dx
\]

\[
= \sum_{i,j=1}^2 \int_{R^3} u_i \partial_i \Delta u_j \, dx + \sum_{j=1}^2 \int_{R^3} u_3 \partial_3 u_j \, dx + \sum_{i=1}^3 \int_{R^3} u_i \partial_i u_3 \, dx
\]

\[
\equiv I_1 + I_2 + I_3.
\]

Invoking Lemma 8, \( I_1 \) can be rewritten as

\[
I_1 = \frac{1}{2} \sum_{i,j=1}^2 \int_{R^3} \partial_i u_j \partial_i u_j \partial_3 u_3 \, dx - \int_{R^3} \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 \, dx + \int_{R^3} \partial_1 u_2 \partial_2 u_1 \partial_3 u_3 \, dx.
\]
For $I_2$, $I_3$, it follows by integrating by parts and noticing the divergence free condition $\nabla u = 0$, that
\[
I_2 = \sum_{j=1}^{2} \int_{\mathbb{R}^3} u_3 \partial_3 u_j \Delta u_j \, dx
\]
\[
= - \sum_{j,k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j \, dx + \frac{1}{2} \sum_{j,k=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_3 \left( \partial_k u_j \right)^2 \, dx, 
\]  
(10)

\[
I_3 = \sum_{i=1}^{3} \int_{\mathbb{R}^3} u_i \partial_3 u_3 \Delta u_3 \, dx
\]
\[
= \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_j u_i \partial_3 u_3 \partial_j u_3 \, dx. 
\]  
(11)

Gathering (9), (10), (11) together, (8) becomes
\[
\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h u\|_{L^2}^2 \leq C \int_{\mathbb{R}^3} |\nabla_h u_3| \cdot |\nabla u| \cdot |\nabla_h u| \, dx + C \int_{\mathbb{R}^3} |\partial_3 u_3| \cdot |\nabla_h u|^2 \, dx
\]
\[
= J_1 + J_2. 
\]  
(12)

We now apply Lemmas 9 and 10 with $\beta = \frac{2}{d-1}$ to bound $J_1$, $J_2$ respectively as
\[
J_1 = C \int_{\mathbb{R}^3} |\nabla_h u_3| \cdot |\nabla u| \cdot |\nabla_h u| \, dx
\]
\[
\leq C \|\nabla_h u_3\|_{L^2}^{2(d-1)} \|\nabla_\beta \partial_3 u_3\|_{L^2}^{\frac{d}{2}} \|\nabla u\|_{L^2}^{2(d-1)} \|\nabla_\beta \nabla_h u\|_{L^2}^{\frac{d}{2}}, 
\]  
(13)

\[
J_2 = C \int_{\mathbb{R}^3} |\partial_3 u_3| \cdot |\nabla_h u|^2 \, dx
\]
\[
\leq C \|\partial_3 u_3\|_{L^2}^{2(d-1)} \|\partial_1 \partial_3 u_3\|_{L^2}^{\frac{d}{2}} \|\nabla_\beta \nabla h u\|_{L^2}^{2(d-1)} \|\nabla_\beta \nabla_h u\|_{L^2}^{\frac{d}{2}}
\]
\[
\leq C \|\nabla_h u\|_{L^2}^{2(d-1)} \|\partial_1 \partial_3 u_3\|_{L^2}^{\frac{d}{2}} \|\nabla_\beta \nabla_h u\|_{L^2}^{\frac{d}{2}}, 
\]  
(14)

where in the last inequality, we use the divergence free condition $\nabla \cdot u = 0$.

Substituting (13), (14) into (12), we obtain by Young inequality that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h u\|_{L^2}^2 \leq C \|\nabla_h u\|_{L^2}^{2(d-1)} \|\nabla u\|_{L^2}^{2(d-1)} \|\partial_1 \partial_3 u_3\|_{L^2}^{\frac{d}{2}} \|\nabla_\beta \nabla_h u\|_{L^2}^{\frac{d}{2}}
\]
\[ \leq C \| \nabla_h u \|_2^2 \| \nabla u \|_2^2 \| (\partial_1, \partial_2) \partial_3 u_3 \|_{\frac{d}{\beta}}^\frac{d}{\beta} + \nu \| \nabla_h u \|_2^2 \]

i.e.
\[ \frac{d}{dt} \| \nabla_h u \|_2^2 + \nu \| \nabla_h u \|_2^2 \leq C \| \nabla_h u \|_2^2 \| (\partial_1, \partial_2) \partial_3 u_3 \|_{\frac{d}{\beta}}^\frac{d}{\beta} . \quad (15) \]

Integrating this inequality over \([\tau, t]\), for any \(t \in [\tau, T^*]\), we get
\[ \| \nabla_h u(t) \|_2^2 + \nu \int_\tau^t \| \nabla_h u(s) \|_2^2 \, ds \leq \| \nabla_h u(\tau) \|_2^2 + C \int_\tau^t \| (\partial_1, \partial_2) \partial_3 u_3(s) \|_{\frac{d}{\beta}}^\frac{d}{\beta} \| \nabla_h u(s) \|_2^2 \| \nabla u(s) \|_2 \, ds. \quad (16) \]

Young inequality then implies
\[
\sup_{\tau \leq t \leq T^*} \| \nabla_h u(t) \|_2^2 \leq \| \nabla_h u(\tau) \|_2^2 + C \sup_{\tau \leq t \leq T^*} \| \nabla_h u(t) \|_2^2 \cdot \int_\tau^{T^*} \left[ \| (\partial_1, \partial_2) \partial_3 u_3(s) \|_{\frac{d}{\beta}}^\frac{d}{\beta} + \| \nabla u(s) \|_2 \right] \, ds. 
\]

Hence if \(\tilde{\varepsilon}\) in (7) is choose so that
\[ C \tilde{\varepsilon} < \frac{1}{2}, \]
we see
\[ \sup_{\tau \leq t \leq T^*} \| \nabla_h u(t) \|_2^2 \leq 2 \| \nabla_h u(\tau) \|_2^2. \quad (17) \]

**Step 3.** \(\| \nabla u(t) \|_2\) estimates.

Taking the inner product of (1) with \(-\Delta u\) in \(L^2(\mathbb{R}^3)\), we see
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|_2^2 + \nu \| \Delta u \|_2^2 = \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u \, dx \\
= \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta_h u \, dx + \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \partial_3^2 u \, dx \\
= K_1 + K_2. \quad (18) 
\]
The term $K_1$ can be dominated similarly as in Step 2, and we find
\[
K_1 = \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \Delta_h \mathbf{u} \, dx \\
\leq C \| (\partial_1, \partial_2) \partial_3 u_3 \|^\beta_{\beta+1} \| \nabla_h \mathbf{u} \|_2 \| \nabla \mathbf{u} \|_2 + \frac{\nu}{4} \| \Delta \mathbf{u} \|_2^2. \tag{19}
\]
Meanwhile for $K_2$, we have, by integrating by parts and noticing the divergence free condition $\nabla \cdot \mathbf{u} = 0$, that
\[
K_2 = \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \partial_{33} \mathbf{u} \, dx \\
= \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_i \partial_3 u_j \partial_3 u_j \, dx \\
= \sum_{i=1}^{2} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_i \partial_3 u_j \partial_3 u_j \, dx + \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j \, dx \\
= \sum_{i=1}^{2} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_i \partial_3 u_j \partial_3 u_j \, dx - \sum_{j=1}^{3} (\partial_1 u_1 + \partial_2 u_2) \partial_3 u_j \partial_3 u_j \, dx.
\]

Applying Hölder inequality, interpolation inequality, Sobolev inequality and Young inequality yields
\[
K_2 \leq C \int_{\mathbb{R}^3} |\nabla_h \mathbf{u}| \cdot |\nabla \mathbf{u}|^2 \, dx \\
\leq C \| \nabla_h \mathbf{u} \|_2 \| \nabla \mathbf{u} \|_4^2 \\
\leq C \| \nabla_h \mathbf{u} \|_2 \| \nabla \mathbf{u} \|_2^{1/2} \| \Delta \mathbf{u} \|_2^{3/2} \\
\leq C \| \nabla_h \mathbf{u} \|_2 \| \nabla \mathbf{u} \|_2^2 + \frac{\nu}{4} \| \Delta \mathbf{u} \|_2^2. \tag{20}
\]
Now, replacing (19), (20) into (18), we get
\[
\frac{d}{dt} \| \nabla \mathbf{u} \|_2^2 + \nu \| \Delta \mathbf{u} \|_2^2 \leq C \| (\partial_1, \partial_2) \partial_3 u_3 \|^\beta_{\beta+1} \| \nabla_h \mathbf{u} \|_2^2 \| \nabla \mathbf{u} \|_2 + C \| \nabla_h \mathbf{u} \|_2 \| \nabla \mathbf{u} \|_2^2.
\]
Thanks to (17), we obtain further that
\[
\frac{d}{dt} \| \nabla \mathbf{u} \|_2^2 + \nu \| \Delta \mathbf{u} \|_2^2 \leq C \| \nabla_h \mathbf{u} \|_2^2 \left[ 1 + \| (\partial_1, \partial_2) \partial_3 u_3 \|^\beta_{\beta+1} \| \nabla \mathbf{u} \|_2 \right] \\
+ 4C \| \nabla_h \mathbf{u}(\tau) \|_4^2 \| \nabla \mathbf{u} \|_2, \quad \text{on } [\tau, \Gamma^{*}).
\]
Invoking Gronwall inequality then implies $\| \nabla \mathbf{u}(t) \|_2, t \in [\tau, \Gamma^{*})$ is uniformly bounded, as desired. This completes the proof of Theorem 2. \qed
References

[1] H. Beirão da Veiga, A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$, Chinese Ann. Math. Ser. B 16 (1995), 407–412.

[2] H. Beirão da Veiga, L.C. Berselli, On the regularizing effect of the vorticity direction in incompressible viscous flows, Differential Integral Equations 15 (2002), 345–356.

[3] L.C. Berselli, G.P. Galdi, Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations, Proc. Amer. Math. Soc. 130 (2002), 3585–3595.

[4] C.S. Cao, Sufficient conditions for the regularity to the 3D Navier-Stokes equations, Discrete Contin. Dyn. Syst. 26 (2010), 1141–1151.

[5] C.S. Cao, E.S. Titi, Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor, arXiv: 1005.4463 [math. AP] 25 May 2010.

[6] C.S. Cao, E.S. Titi, Regularity criteria for the three-dimensional Navier-Stokes equations, Indiana Univ. Math. J. 57 (2008), 2643–2661.

[7] C.S. Cao, J.H. Wu, Two new regularity criteria for the 3D MHD equations, J. Differential Equations 248 (2010), 2263–2274.

[8] D. Chae, J. Lee, Regularity criterion in terms of pressure for the Navier-Stokes equations, Nonlinear Anal. TMA 46 (2001), 727–735.

[9] P. Constantin, C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier-Stokes equations, Indiana Univ. Math. J. 42 (1993), 775–789.

[10] L. Escauriaza, G. Seregin, V. Šverák, Backward uniqueness for parabolic equations, Arch. Ration. Mech. Anal. 169 (2003), 147–157.

[11] E.B. Fabes, B.F. Jones, N.M. Riviére, The initial value problem for the Navier-Stokes equations with data in $L^p$, Arch. Rational Mech. Anal. 45 (1972), 222–240.
[12] J.S. Fan, S. Jiang, G.X. Ni, On regularity criteria for the n-dimensional Navier-Stokes equations in terms of the pressure, J. Differential Equations 244 (2008), 2963–2979.

[13] Y. Giga, Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations 62 (1986), 186–212.

[14] C. He, Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, J. Differential Equations 213 (2005), 235–254.

[15] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachr. 4 (1951), 213-231.

[16] J.M. Kim, On regularity criteria of the Navier-Stokes equations in bounded domains, J. Math. Phys. 51 (2010), 053102.

[17] I. Kukavica, M. Ziane, Navier-Stokes equations with regularity in one direction. J. Math. Phys. 48 (2007), 065203.

[18] I. Kukavica, M. Ziane, One component regularity for the Navier-Stokes equations. Nonlinearity 19 (2006), 453–469.

[19] J. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace. Acta Math. 63 (1934), 193–248.

[20] J. Neustupa, A. Novotný, P. Penel, An interior regularity of a weak solution to the Navier-Stokes equations in dependence on one component of velocity, Topics in Mathematical Fluid Mechanics, Quaderni di Matematica, Dept. Math., Seconda University, Napoli, Caserta, Vol. 10, pp. 163–183 (2002); see also A remark to interior regularity of a suitable weak solution to the Navier-Stokes equations, CIM Preprint No. 25, 1999.

[21] J. Neustupa, P. Penel, Anisotropic and geometric criteria for interior regularity of weak solutions to the 3D Navier-Stokes equations, in Mathematical Fluid Mechanics (Recent Results and Open Problems), Advances in Mathematical Fluid Mechanics, edited by J. Neustupa, and P. Penel (Birkhäuser, Basel-Boston-Berlin, 2001), pp. 239–267.
[22] P. Penel, M. Pokorný, *On anisotropic regularity criteria for the solutions to 3D Navier-Stokes equations*, J. Math. Fluid Mech. doi:10.1007/s00021-010-0038-6.

[23] P. Penel, M. Pokorný, *Some new regularity criteria for the Navier-CStokes equations containing the gradient of velocity*, Appl. Math. **49** (2004), 483–493.

[24] G. Prodi, *Un teorema di unicità per le equazioni di Navier-Stokes*, Ann. Mat. Pura Appl. **48** (1959), 173–182.

[25] J. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **9** (1962), 187–191.

[26] J. Serrin, *The initial value problems for the Navier-Stokes equations*, in Nonlinear Problems, edited by R. E. Langer (University of Wisconsin Press, Madison, WI, (1963).

[27] H. Sohr, *Zur Regularitätstheorie der instationären Gleichungen von Navier-Stokes*, (German) [On the regularity theory of the nonstationary Navier-Stokes equations] Math. Z. **184** (1983), 359–375.

[28] H. Sohr, W. von Wahl, *On the regularity of the pressure of weak solutions of Navier-Stokes equations*, Arch. Math. (Basel) **46** (1986), 428–439.

[29] R. Temam, *Navier-Stokes equations, Theory and numerical analysis*, North-Holland, 1977.

[30] J.H. Wu, *Generalized MHD equations*, J. Differential Equations **195** (2003), 284–312.

[31] J.H. Wu, *Regularity criteria for the generalized MHD equations*, Comm. Partial Differential Equations **33** (2008), 285–306.

[32] Y. Zhou, *A new regularity criterion for the Navier-Stokes equations in terms of the direction of vorticity*, Monatsh. Math. **144** (2005), 251–257.

[33] Y. Zhou, *A new regularity criterion for the Navier-Stokes equations in terms of the gradient of one velocity component*. Methods Appl. Anal. **9** (2002), 563–578.
[34] Y. Zhou, *A new regularity criterion for weak solutions to the Navier-Stokes equations*, J. Math. Pures Appl. **84** (2005), 1496–1514.

[35] Y. Zhou, *Direction of vorticity and a new regularity criterion for the Navier-Stokes equations*. ANZIAM J. **46** (2005), 309–316.

[36] Y. Zhou, *On a regularity criterion in terms of the gradient of pressure for the Navier-Stokes equations in $\mathbb{R}^n$*, Z. Angew. Math. Phys. **57** (2006), 384–392.

[37] Y. Zhou, *On regularity criteria in terms of pressure for the Navier-Stokes equations in $\mathbb{R}^3$*, Proc. Amer. Math. Soc. **134** (2006), 149–156.

[38] Y. Zhou, *Regularity criteria for the 3-D MHD equations in terms of the pressure*, Internat. J. Non-Linear Mech. **41** (2006), 1174–1180.

[39] Y. Zhou, *Regularity criteria in terms of pressure for the 3-D Navier-Stokes equations in a generic domain*, Math. Ann. **328** (2004), 173–192.

[40] Y. Zhou, *Remarks on regularities for the 3D MHD equations*, Discrete Contin. Dyn. Syst. **12** (2005), 881–886.

[41] Y. Zhou, M. Pokorný, *On a regularity criterion for the Navier-Stokes equations involving gradient of one velocity component*, J. Math. Phys. **50** (2009), 123514.

[42] Y. Zhou, M. Pokorný, *On the regularity to the solutions of the Navier-Stokes equations via one velocity component*, Nonlinearity **23** (2010), 1097–1107.

[43] X.C. Zhang, *A regularity criterion for the solutions of 3D Navier-Stokes equations*, J. Math. Anal. Appl. **346** (2008), 336–339.

[44] Z.F. Zhang, Q.L. Chen, *Regularity criterion via two components of vorticity on weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$*, J. Differential Equations **216** (2005), 470–481.

[45] Z.J. Zhang, *A Serrin-type regularity criterion for the Navier-Stokes equations via one velocity component*, to appear.

[46] Z.J. Zhang, *Remarks on the regularity criteria for generalized MHD equations*, J. Math. Anal. Appl. **375** (2011), 799–802.
[47] Z.J. Zhang, M. Lu, L.D. Ni, Some Serrin-type regularity criteria for weak solutions to the Navier-Stokes equations, submitted.

[48] Z.J. Zhang, Two new regularity criteria for the 3D Navier-Stokes equations via two entries of the velocity gradient tensor, submitted.