On Characterizations of Metric Regularity of Multi-valued Maps *

M. Ivanov and N. Zlateva

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Dedicated to Professor Alexander D. Ioffe

Abstract

We provide a new proof along the lines of the recent book of A. Ioffe of a 1990’s result of H. Frankowska showing that metric regularity of a multi-valued map can be characterized by regularity of its contingent variation – a notion extending contingent derivative.

Keywords: surjectivity, metric regularity, multi-valued map.

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1 Introduction

Metric regularity, as well as, the equivalent to it linear openness and pseudo-Lipschitz property of the inverse, are very important concepts in Variational Analysis. They have been intensively studied as it can be seen in a number of recent monographs, e.g. [1, 8, 3, 7] and the references therein. A very rich and instructive survey on metric regularity is the book of A. Ioffe [6].

It may be noted in Chapter V of [6] that the modulus of regularity of a multi-valued map between Banach spaces is estimated in terms of the

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tangential cones to its graph. The estimates are precise, but they are not characteristic. This is because in infinite dimensions a map may well be regular and the tangential cones to its graph be insufficiently informative, for details see [5].

In [4] H. Frankowska introduced the notion of contingent variation of a multi-valued map which extends Bouligand tangential cone. This notion can precisely characterize metric regularity.

Let \((X, d)\) and \((Y, d)\) be metric spaces and let

\[ F : X \rightrightarrows Y \]

be a multi-valued map. If \(V \subset Y\) the restriction \(F^V\) is defined by

\[ F^V(x) := F(x) \cap V, \quad \forall x \in X, \]

see [6, p.54]. The properties related to the so restricted map are called restricted.

For example, the multi-valued map \(F : X \rightrightarrows Y\) is called restrictedly Milyutin regular on \((U, V)\), where \(U \subset X\) and \(V \subset Y\), if there exists a number \(r > 0\) such that

\[ B(v, rt) \cap V \subset F(B(x, t)) \]

whenever \((x, v) \in \text{Gr } F \cap (U \times V)\) and \(B(x, t) \subset U\), where \(B(x, t)\) is the closed ball with center \(x\) and radius \(t\): \(B(x, t) := \{u \in X : d(u, x) \leq t\}\), and \(\text{Gr } F = \{(x, v) : v \in F(x)\}\).

The supremum of all such \(r\) is called modulus of surjection, denoted by

\[ \text{sur}_m F^V(U|V) \]

By convention, \(\text{sur}_m F^V(U|V) = 0\) means that \(F\) is not restrictedly Milyutin regular on \((U, V)\).

This notion taken from [6] is explained in great detail in Section 2 below.

In the literature, e.g. [6, Section 5.2], there are various estimates of \(\text{sur}_m F^V(U|V)\) and related moduli in terms of derivative-like objects. Unlike the so called co-derivative criterion, see [6, Section 5.2.3], most of the primal estimates are not characteristic in general. Here we re-establish one primal criterion which complements [6, Section 5.2.2] and is, moreover, characteristic. It is essentially done by H. Frankowska in [4], see also [5]. There a new derivative-like object is defined as follows.
Let \((X,d)\) be a metric space, \((Y,\|\cdot\|)\) be a Banach space, \(F : X \rightrightarrows Y\) be a multi-valued map. For \((x,y) \in \text{Gr} F\) the \textit{contingent variation} of \(F\) at \((x,y)\) is the closed set

\[
F^{(1)}(x,y) := \limsup_{t \to 0^+} \frac{F(B(x,t)) - y}{t},
\]

where \(\limsup\) stands for the Kuratowski limit superior of sets.

Equivalently, \(v \in F^{(1)}(x,y)\) exactly when there exist a sequence of reals \(t_n \downarrow 0\) and a sequence \((x_n, y_n) \in \text{Gr} F\) such that \(d(x, x_n) \leq t_n\) and

\[
\left\| v - \frac{y_n - y}{t_n} \right\| \to 0, \text{ when } n \to \infty.
\]

This notion extends the so-called contingent, or graphical, derivative usually denoted by \(DF(x,y)\), e.g. [6, pp.163, 202].

Our main result can now be stated. As usual, \(B_Y\) denotes the closed unit ball of the Banach space \((Y,\|\cdot\|)\).

**Theorem 1.** Let \((X,d)\) be a metric space and \((Y,\|\cdot\|)\) be a Banach space, let \(U \subset X\) and \(V \subset Y\) be non-empty open sets. Let \(F : X \rightrightarrows Y\) be a multi-valued map with complete graph. \(F\) is restrictedly Milytin regular on \((U, V)\) with \(\text{sur}_m F^V (U | V) \geq r > 0\) if and only if

\[
F^{(1)}(x,v) \supset r B_Y \text{ for all } (x,v) \in \text{Gr} F \cap (U \times V). \tag{1}
\]

This result is essentially established by H. Frankowska in [4, Theorem 6.1 and Corollary 6.2]. However, there it is presented as a characterization of local modulus of regularity in terms of the local variant of the condition (1). Here we render the characterization global. The technique in [4] is different, but it again depends on Ekeland Variational Principle.

The rest of the article is organized as follows. In Section 2 we provide for reader’s convenience the relevant material from [6]. We also present in another form the first criterion for Milyutin regularity from [6]. In Section 3 we prove Theorem 1.
2 Milyutin regularity

Let \((X, d)\) and \((Y, d)\) be metric spaces. Let \(U \subset X\) and \(V \subset Y\), let \(F : X \rightrightarrows Y\) be a multi-valued map and let \(\gamma(\cdot)\) be extended real-valued function on \(X\) assuming positive values (possibly infinite) on \(U\).

**Definition 2.** *(linear openness, [6, Definition 2.21])* \(F\) is said to be \(\gamma\)-open at linear rate on \((U, V)\) if there is an \(r > 0\) such that

\[
B(F(x), rt) \cap V \subset F(B(x, t)),
\]

if \(x \in U\) and \(t < \gamma(x)\), i.e.

\[
B(v, rt) \cap V \subset F(B(x, t)),
\]

whenever \((x, v) \in \text{Gr} F\), \(x \in U\) and \(t < \gamma(x)\).

Denote by \(\text{sur}_\gamma F(U|V)\) the upper bound of all such \(r > 0\) and call it *modulus of \(\gamma\)-surjection of \(F\) on \((U, V)\)*. If no such \(r\) exists, set \(\text{sur}_\gamma F(U|V) = 0\).

**Definition 3.** *(metric regularity, [6, Definition 2.22])* \(F\) is said to be \(\gamma\)-metrically regular on \((U, V)\) if there is \(\kappa > 0\) such that

\[
d(x, F^{-1}(y)) \leq \kappa d(y, F(x)),
\]

provided \(x \in U\), \(y \in V\) and \(\kappa d(y, F(x)) < \gamma(x)\).

Denote by \(\text{reg}_\gamma F(U|V)\) the lower bound of all such \(\kappa > 0\) and call it *modulus of \(\gamma\)-metric regularity of \(F\) on \((U, V)\)*. If no such \(\kappa\) exists, set \(\text{reg}_\gamma F(U|V) = \infty\).

**Theorem 4.** *(equivalence theorem, [6, Theorem 2.25])* The following are equivalent for any metric spaces \(X, Y\), any \(F : X \rightrightarrows Y\), any \(U \subset X\), \(V \subset Y\) and any extended real-valued function \(\gamma(\cdot)\) which is positive on \(U\):

a) \(F\) is \(\gamma\)-open at linear rate on \((U, V)\);

b) \(F\) is \(\gamma\)-metrically regular on \((U, V)\).

Moreover (under the convention \(0.\infty = 1\)),

\[
\text{sur}_\gamma F(U|V).\text{reg}_\gamma F(U|V) = 1.
\]

**Definition 5.** *(regularity, [6, Definition 2.26])* We say that \(F : X \rightrightarrows Y\) is \(\gamma\)-regular on \((U, V)\) if the equivalent properties of Theorem 4 are satisfied.
Definition 6. (Miluytin regularity, [6, Definition 2.28]) Set
\[ m_U(x) := d(x, X \setminus U). \]
We shall say that \( F \) is Miluytin regular on \((U, V)\) if it is \( \gamma \)-regular on \((U, V)\) with \( \gamma(x) = m_U(x) \).

We will need also Ekeland Variational Principle (see [9, p.45]): Let \((M, d)\) be a complete metric space, and \(f: M \rightarrow \mathbb{R} \cup \{+\infty\}\) be a proper, lower semicontinuous and bounded from below function. Assume that \( f(\overline{x}) \leq \inf f + \lambda \varepsilon \) for some \( \overline{x} \in M \) and \( \lambda \varepsilon > 0 \). Then there is \( \overline{y} \in M \) such that
(i) \( f(\overline{y}) \leq f(\overline{x}) - \lambda d(\overline{x}, \overline{y}) \);
(ii) \( d(\overline{x}, \overline{y}) \leq \varepsilon \);
(iii) \( f(x) + \lambda d(x, \overline{y}) \geq f(\overline{y}) \), for all \( x \in M \).

The following characterization of Miluytin regularity is very similar in form (in fact equivalent) to the so called first criterion for Miluytin regularity, see [6, Theorem 2.47]. It is also similar to [2, Proposition 2.2], but there it is stated in local form. We present here a proof for reader’s convenience.

Following [6, p.35] for \( \xi > 0 \) we denote by \( d_\xi \) the product metric
\[ d_\xi((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), \xi d(y_1, y_2)\}, \tag{2} \]
where \( x_i \in X, y_i \in Y, i = 1, 2, \) and \((X, d)\) and \((Y, d)\) are metric spaces.

Theorem 7. Let \((X, d), (Y, d)\) be metric spaces. Let \( F: X \rightrightarrows Y \) be a multi-valued map with complete graph. Let \( U \subset X \) and \( V \subset Y \). Then
\[ \text{sur}_m F(U|V) = \sup\{r \geq 0 : \exists \xi > 0 \text{ such that} \]
\[ \forall (x, v) \in \text{Gr} F, x \in U, y \in V \text{ satisfying } 0 < d(y, v) < rm_U(x) \]
\[ \exists (u, w) \in \text{Gr} F \text{ such that } d(y, w) < d(y, v) - rd_\xi((x, v), (u, w))\}. \tag{3} \]

Proof. Let us denote by \( s_1 \) the left hand side of the above equation, i.e. \( s_1 := \text{sur}_m F(U|V) \). In other words,
\[ s_1 = \sup\{r \geq 0 : B(v, rt) \cap V \subset F(B(x, t)), \forall (x, v) \in \text{Gr} F, x \in U, t < m_U(x)\}. \]

Denote by \( s_2 \) the right hand side of the equation.
We need to show that \( s_1 = s_2 \).
First, we will show that $s_1 \leq s_2$.
If $s_1 = 0$ we have nothing to prove.

Let $s_1 > 0$. Take $0 < r < r' < s_1$. Let $x \in U$, $v \in F(x)$ be fixed. Let $y \in V$ be such that $0 < d(y, v) < rm_U(x)$. In particular $0 < d(y, v) < r'm_U(x)$.

Set $t := \frac{d(y, v)}{r'}$. Then $t < m_U(x)$. By $r' < s_1 = \text{sur}_m F(U|V)$ and by the definition of $\text{sur}_m F(U|V)$ it holds that $y \in B(v, r't) \cap V \subset F(B(x, t))$, i.e. $y \in F(B(x, t))$. So, there exists $u \in B(x, t)$ such that $y \in F(u)$.

Fix $\xi$ such that $0 < \xi r' < 1$. Then
\[
d_\xi((x, v), (u, y)) = \max\{d(x, u), \xi d(v, y)\} \leq \max\{t, \xi r't\} = t \max\{1, \xi r'\} = t,
\]
so
\[
r'd_\xi((x, v), (u, y)) \leq r't = d(y, v).
\]
Observe that $d_\xi((x, v), (u, y)) > 0$ since $d(v, y) > 0$. The latter and $r' > r$ yield
\[
rd_\xi((x, v), (u, y)) < r't < d(y, v),
\]
or
\[
0 < d(y, v) - rd_\xi((x, v), (u, y)).
\]
Since $0 = d(y, y)$ we get that
\[
d(y, y) < d(y, v) - rd_\xi((x, v), (u, y))
\]
and (3) holds with $w = y$ as $(u, y) \in \text{Gr} F$.

This means that $r \leq s_2$. Finally, $s_1 \leq s_2$.

Second, we will prove that $s_2 \leq s_1$.
If $s_2 = 0$ we have nothing to prove.

Let now $s_2 > 0$. Let $0 < r < s_2$. Let us fix $x_0 \in U$, $v_0 \in F(x_0)$ and $0 < t < m_U(x_0)$.

Fix $y \in V$ such that $d(y, v_0) \leq rt$, i.e. $y \in B(v_0, rt) \cap V$. Let $M := \text{Gr} F$, and let $\xi > 0$ correspond to $r$ in the definition of $s_2$. It is clear that $(M, d_\xi)$ is a complete metric space.

Consider the function $f : M \to \mathbb{R}$ defined as $f(u, w) := d(w, y)$.

Then $f \geq 0$ and it is continuous on $M$. Since $f(x_0, v_0) = d(v_0, y) \leq rt$, by Ekeland Variational Principle there exists $(x_1, v_1) \in M$ such that
\[
\begin{align*}
(i) & \quad f(x_1, v_1) \leq f(x_0, v_0) - rd_\xi((x_1, v_1), (x_0, v_0)); \\
(ii) & \quad d_\xi((x_1, v_1), (x_0, v_0)) \leq t;
\end{align*}
\]
(iii) \( f(u, w) + rd_\xi((u, w), (x_1, v_1)) \geq f(x_1, v_1) \), for all \((u, w) \in M\). 

Or, equivalently

(i) \( d(v_1, y) \leq d(v_0, y) - rd_\xi((x_1, v_1), (x_0, v_0)) \leq rt - rd_\xi((x_1, v_1), (x_0, v_0)) \);

(ii) \( d(x_1, x_0) \leq t \) \quad \xi d(v_1, v_0) \leq t;

(iii) \( d(w, y) + rd_\xi((u, w), (x_1, v_1)) \geq d(v_1, y) \), for all \((u, w) \in M\).

Set \( p := d(v_1, y) \).

Assume that \( p > 0 \). Take \( t' \) such that \( t < t' < m_U(x_0) \). For \( x \in B\left(x_1, \frac{p}{r} + t' - t\right) \) we have that

\[
\begin{align*}
    d(x, x_0) &\leq d(x, x_1) + d(x_1, x_0) \\
    &\leq \frac{p}{r} + t' - t + d(x_1, x_0) \\
    \text{(using (i))} &\leq \frac{rt - rd(x_1, x_0)}{r} + t' - t + d(x_1, x_0) \\
    &= t - d(x_1, x_0) + t' - t + d(x_1, x_0) \\
    &= t'.
\end{align*}
\]

Hence \( B\left(x_1, \frac{p}{r} + t' - t\right) \subset B(x_0, t') \subset U \). Then \( \frac{p}{r} + t' - t \leq m_U(x_1) \), and \( \frac{p}{r} < m_U(x_1) \) because \( t' - t > 0 \). Hence, \( 0 < d(v_1, y) < rm_U(x_1) \). But now (iii) contradicts (iii).

Therefore, \( p = 0 \) and then \( y = v_1 \in F(x_1) \). Since by (ii) \( x_1 \in B(x_0, t) \), we have \( y \in F(B(x_0, t)) \cap V \).

Since \( x_0 \in U \), \( v_0 \in F(x_0) \), \( y \in B(v_0, rt) \cap V \) and \( 0 < t < m_U(x_0) \) were arbitrary, this means that \( r \leq s_1 \). Since \( 0 < r < s_2 \) was arbitrary, \( s_2 \leq s_1 \), and the proof is completed.

In the definitions of regularity properties it is not required that \( F(x) \subset V \). Such requirements can be included in the definitions as follows.

**Definition 8. (restricted regularity, [6, Definition 2.35])** Set \( F^V(x) := F(x) \cap V \). We define restricted \( \gamma \)-openness at linear rate and restricted \( \gamma \)-metric regularity on \((U, V)\) by replacing \( F \) by \( F^V \).

The equivalence Theorem 4 also holds for the restricted versions of the properties. The case is the same with Theorem 7, where the proof needs only small adjustments when working with \( F^V \) instead of \( F \).
3 Proof of the main result

The proof of our main result relies on the following Lemma.

Lemma 9. Let \((X, d)\) be a metric space and \((Y, \| \cdot \|)\) be a Banach space, let \(U \subset X\) and \(V \subset Y\) be non-empty sets and let

\[
F : X \rightharpoonup Y
\]

be a multi-valued map.

If for some \(r > 0\) it holds that

\[
F(1)(x, v) \supset r B_Y \quad \text{for all } (x, v) \in \text{Gr} F \cap (U \times V),
\]

then for any \(0 < r' < r\) and any \(\xi \in (r^{-1}, (r')^{-1})\) it holds that for any \(x \in U\) and any \(v \in F^V(x)\) and \(y \in V \setminus \{v\}\) there is \((u, w) \in \text{Gr} F\) such that

\[
\| y - w \| < \| y - v \| - r' d_\xi((x, v), (u, w)).
\]

Proof. Let \(r' \in (0, r)\) be fixed.

Fix \(\xi > 0\) such that \((r')^{-1} > \xi > r^{-1}\).

Take \((x, v)\) such that \((x, v) \in \text{Gr} F \cap (U \times V)\).

Fix \(y \in V\) such that \(0 < \| y - v \|\).

Set \(\bar{v} := r \frac{y - v}{\| y - v \|}\). Obviously \(\| \bar{v} \| = r\). By assumption, \(F(1)(x, v) \ni \bar{v}\).

By definition of the contingent variation there exist \(t_n \downarrow 0, u_n \in X\) as well as \(w_n \in Y\) and \(z_n \in Y\) such that \(w_n \in F(u_n), d(x, u_n) \leq t_n, \| z_n \| \to 0\) and

\[
v + t_n \bar{v} = w_n + t_n z_n.
\] (4)

Note first that for \(n\) large enough

\[
\xi \| w_n - v \| > t_n \geq d(x, u_n) \Rightarrow d_\xi((x, v), (u_n, w_n)) = \xi \| w_n - v \|. \tag{5}
\]

Indeed, \(\| w_n - v \| = t_n \| v - z_n \| \geq t_n (r - \| z_n \|)\) and, since \(\xi (r - \| z_n \|) \to \xi r > 1\) as \(n \to \infty\), we have \(\xi \| w_n - v \| > t_n\) for \(n\) large enough.

From (4) we have

\[
y - w_n = y - v - t_n \bar{v} + t_n z_n. \tag{6}
\]

Since

\[
y - v - t_n \bar{v} = (1 - t_n r \| y - v \|^{-1}) (y - v),
\]

...
and since $1 - t_n r \|y - v\|^{-1} > 0$ for $n$ large enough, we have for such $n$ that

$$\|y - v - t_n \bar{v}\| = (1 - t_n r \|y - v\|^{-1}) \|y - v\| = \|y - v\| - t_n r.$$  

Combining the latter with (6) we get for $n$ large enough

$$\|y - w_n\| = \|y - v - t_n \bar{v} + t_n z_n\|$$

$$\leq \|y - v - t_n \bar{v}\| + t_n \|z_n\|$$

$$= \|y - v\| - t_n (r - \|z_n\|).$$ (7)

On the other hand, (4) can be rewritten as $w_n - v = t_n \bar{v} - t_n z_n$, hence

$$\|w_n - v\| = t_n \|\bar{v} - z_n\| \leq t_n (r + \|z_n\|),$$

and using this estimate we obtain that

$$\liminf_{n \to \infty} \frac{t_n (r - \|z_n\|)}{r' \xi \|v - w_n\|} \geq \liminf_{n \to \infty} \frac{t_n (r - \|z_n\|)}{r' \xi t_n (r + \|z_n\|)} = \frac{1}{r' \xi} > 1.$$  

From this and (7) we have that for large $n$

$$\|y - w_n\| < \|y - v\| - r' \xi \|v - w_n\|.$$ 

Using (5) we finally obtain that for all $n$ large enough

$$\|y - w_n\| < \|y - v\| - r' d_\xi ((x, v), (u_n, w_n))$$

and the claim follows. \qed

Proving our main result is now straightforward.

**Theorem 1.** Let $(X, d)$ be a metric space and $(Y, \| \cdot \|)$ be a Banach space, let $U \subset X$ and $V \subset Y$ be non-empty open sets. Let $F : X \rightrightarrows Y$

be a multi-valued map with complete graph.

$F$ is restrictedly Milytin regular on $(U, V)$ with $\text{sur}_m F^V(U|V) \geq r > 0$ if and only if

$$F^{(1)}(x, v) \supset rB_Y \text{ for all } (x, v) \in \text{Gr } F \cap (U \times V).$$
Proof. Let

\[ F^{(1)}(x, v) \supset rB_Y \text{ for all } (x, v) \in \text{Gr } F \cap (U \times V). \]

From Lemma 9 and Theorem 7 it follows that \( \text{sur}_m F^V(U|V) \geq r \).

Conversely, let \( \text{sur}_m F^V(U|V) \geq r > 0 \). This means that

\[ B(v, rt) \cap V \subset F(B(x, t)) \]

whenever \((x, v) \in \text{Gr } F, x \in U, v \in V\) and \( t < m_U(x) \).

Take arbitrary \((x, v) \in \text{Gr } F^V, x \in U\) and note that \( m_U(x) > 0 \) because \( U \) is open. Take positive \( t \) such that \( t < m_U(x) \).

For any \( y \in rB_Y \) it holds that \( v + ty \in B(v, rt) \). Moreover, \( v + ty \in V \) will be true for small \( t \) because \( V \) is open. Then, by assumption, \( v + ty \in F(B(x, t)) \), so \( y \in \frac{F(B(x, t)) - v}{t} \) which means that \( y \in F^{(1)}(x, v) \). Hence, \( F^{(1)}(x, v) \supset rB_Y \).

\[ \square \]

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