THE FIELD OF VALUES BOUND ON IDEAL GMRES
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Abstract. A widely known result of Howard Elman, and its improvements due to Gerhard Starke, Michael Eiermann and Oliver Ernst, gives a bound on the (worst-case) GMRES residual norm using quantities related to the field of values of the given matrix and of its inverse. In this note we give a simple and direct proof that these bounds also hold for the ideal GMRES approximation. Our work was motivated by a question of Otto Strnad, a student at the Charles University in Prague.

1. Bounds on the GMRES residual norms. Consider a linear algebraic system $Ax = b$ with a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. The GMRES method of Saad and Schultz [10] is an iterative method that constructs approximations $x_k$, $k = 1, 2, \ldots$, such that
\[ \| r_k \| = \| b - Ax_k \| = \min_{p \in \pi_k} \| p(A)r_0 \|. \] (1.1)

Here $\| v \| \equiv \langle v, v \rangle^{1/2}$ denotes the Euclidean norm, $\pi_k$ denotes the set of polynomials $p$ of degree at most $k$ and with $p(0) = 1$, and $r_0 \equiv b - Ax_0$, for a given initial approximation $x_0$.

Let $M = \frac{1}{2}(A + A^T)$ denote the symmetric part of $A$. Assuming that $M$ is positive definite, Elman derived in his PhD thesis of 1982 [3, Theorem 5.4 and 5.9] the following bound on the $k$th relative GMRES residual norm (stated in [3] for the GCR method):
\[ \frac{\| r_k \|}{\| r_0 \|} \leq \left( 1 - \frac{\lambda_{\min}(M)^2}{\lambda_{\max}(A^T A)} \right)^{k/2}; \] (1.2)

see also the subsequent paper [2, Theorem 3.3]. Denote by $\mathcal{F}(A)$ the field of values of $A$, and by $\nu(\mathcal{F}(A))$ the distance of $\mathcal{F}(A)$ from the origin,
\[ \nu(\mathcal{F}(A)) \equiv \min_{z \in \mathcal{F}(A)} |z|. \]

In his Habilitation thesis of 1994 [11, Section 2.2] and in his subsequent paper [12, Theorem 3.2], Starke proved that the $k$th relative GMRES residual norm for a matrix $A$ with positive definite symmetric part is bounded by
\[ \frac{\| r_k \|}{\| r_0 \|} \leq \left( 1 - \nu(\mathcal{F}(A))\nu(\mathcal{F}(A^{-1})) \right)^{k/2}. \] (1.3)

Note that if $M$ is positive definite, then $\nu(\mathcal{F}(A)) = \lambda_{\min}(M)$, and
\[ \frac{\lambda_{\min}(M)}{\| A \|^2} \leq \min_{w \in \mathbb{C}^n \setminus \{0\}} \left| \frac{\langle Aw, w \rangle}{\langle w, w \rangle} \frac{\langle w, w \rangle}{\langle Aw, Aw \rangle} \right| = \min_{v \in \mathbb{C}^n \setminus \{0\}} \left| \frac{\langle A^{-1}v, v \rangle}{\langle v, v \rangle} \right| = \nu(\mathcal{F}(A^{-1})). \]

Hence, as pointed out by Starke in [11, 12], the bound (1.3) improves Elman’s bound (1.2). In [1 Corollary 6.2], Eiermann and Ernst proved that (1.3) holds for any nonsingular matrix $A$, i.e. they proved this bound without the assumption on the symmetric part of $A$.

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2. Worst-case and ideal GMRES. For each given $A$, $b$ and $x_0$, the corresponding $k$th relative GMRES residual norm is bounded by the $k$th worst-case GMRES residual norm, which in turn is bounded by the $k$th ideal GMRES approximation (introduced in [6]),

$$\frac{\|r_k\|}{\|r_0\|} \leq \max_{v \neq 0} \min_{p \in \pi_k} \frac{\|p(A)v\|}{\|v\|} \leq \min_{p \in \pi_k} \max_{v \neq 0} \frac{\|p(A)v\|}{\|v\|} = \min_{p \in \pi_k} \|p(A)\|. \quad (2.1)$$

Note that the right hand sides in the bounds (1.2) and (1.3) both do not depend on $r_0$. Hence both right hand sides represent upper bounds on worst-case GMRES, i.e.

$$\max_{v \neq 0} \min_{p \in \pi_k} \frac{\|p(A)v\|}{\|v\|} \leq \left( 1 - \frac{\lambda_{\min}(M)^2}{\lambda_{\max}(A^TA)} \right)^{k/2}, \text{ if } M = \frac{1}{2}(A + A^T) \text{ is positive definite, worst-case GMRES} \quad (2.2)$$

and

$$\max_{v \neq 0} \min_{p \in \pi_k} \frac{\|p(A)v\|}{\|v\|} \leq \left( 1 - \nu(F(A))\nu(F(A^{-1})) \right)^{k/2}, \text{ if } A \text{ is nonsingular. worst-case GMRES} \quad (2.3)$$

It has been shown by examples in [4, 13], that there exist matrices $A$ and iteration steps $k$ for which the value of the $k$th ideal GMRES approximation is larger than the value of the $k$th worst-case GMRES residual norm, i.e. the second inequality in (2.1) can be strict. The example in [13] even shows that the ratio of worst-case and ideal GMRES can be arbitrarily small. Therefore a natural question is whether the right hand sides of (2.2) and (2.3) also represent upper bounds on ideal GMRES, i.e. whether

$$\min_{p \in \pi_k} \|p(A)\| \leq \left( 1 - \frac{\lambda_{\min}(M)^2}{\lambda_{\max}(A^TA)} \right)^{k/2}, \text{ if } M = \frac{1}{2}(A + A^T) \text{ is positive definite, ideal GMRES} \quad (2.4)$$

and

$$\min_{p \in \pi_k} \|p(A)\| \leq \left( 1 - \nu(F(A))\nu(F(A^{-1})) \right)^{k/2}, \text{ if } A \text{ is nonsingular. ideal GMRES} \quad (2.5)$$

The two bounds (2.4) and (2.5) are stated in our paper [9, p. 168], and also in the book [8, p. 296], but no proof is given there. The other publications in this context mentioned above (namely [1, 2, 3, 11, 12]) do not mention ideal GMRES, as they deal with (worst-case) GMRES only. However, a closer inspection of the statement of (1.2) in [2, equation (3.3)] reveals that this statement actually contains the stronger result (2.4). The bound (2.4) on ideal GMRES is not stated in any of these works, and we are unaware of a simple, direct proof of this bound in the previous literature. The following section gives such a proof.
3. Proof of the ideal GMRES bound. In this section we consider the general complex setting, i.e. $A \in \mathbb{C}^{n \times n}$, $b \in \mathbb{C}^n$, and $\langle x, y \rangle = y^H x$ where $H$ denotes Hermitian transposed. Similarly, we will allow the polynomials from the set $\pi_k$ to have complex coefficients in general.

Consider a given unit norm vector $v$ and the problem

$$\min_{\alpha \in \mathbb{C}} \| v - \alpha A v \|^2.$$  

It is easy to show that the minimum is attained for

$$\alpha_* = \frac{\langle v, A v \rangle}{\langle A v, A v \rangle},$$

and that

$$\| v - \alpha_* A v \|^2 = 1 - \frac{\langle v, A v \rangle}{\langle A v, A v \rangle} = 1 - \frac{\langle A^{-1} w, w \rangle}{\langle v, v \rangle} \frac{\langle A v, v \rangle}{\langle v, v \rangle},$$

where $w \equiv A v$.

Next recall that the ideal and worst-case GMRES approximations are equal in the step $k = 1$; see Joubert [7, Theorem 1] or Greenbaum and Gurvits [5, Theorem 2.5]. Using this fact and $\alpha_*$ from above we see that

$$\min_{p \in \pi_k} \| p(A) \| \leq \min_{\alpha \in \mathbb{C}} \| (I - \alpha A)^k \| \leq \min_{\alpha \in \mathbb{C}} \| I - \alpha A \|^k$$

$$= \min_{\| v \| = 1} \max_{\alpha \in \mathbb{C}} \| v - \alpha A v \|^k$$

$$= \max_{\| v \| = 1} \left( \min_{\alpha \in \mathbb{C}} \| v - \alpha A v \|^2 \right)^{k/2}$$

$$= \max_{\| v \| = 1} \left( 1 - \frac{\langle v, A v \rangle}{\langle A v, A v \rangle} \right)^{k/2}$$

$$\leq \left( 1 - \min_{w \in \mathbb{C}^n} \frac{\langle A^{-1} w, w \rangle}{\langle w, w \rangle} \right)^{k/2} \min_{w \in \mathbb{C}^n} \frac{\langle A v, v \rangle}{\langle v, v \rangle}$$

$$= (1 - \nu(A) \nu(A^{-1}))^{k/2}.$$

Moreover, if the Hermitian part $M = \frac{1}{2}(A + A^H)$ is positive definite, we can bound $\nu(A)$ and $\nu(A^{-1})$ from below by

$$\lambda_{\min}(M) \leq \nu(A), \quad \frac{\lambda_{\min}(M)^2}{\| A \|^2} \leq \nu(A^{-1}).$$

Consequently, the following theorem has been shown.

**Theorem 3.1.** If $A \in \mathbb{C}^{n \times n}$ is a nonsingular matrix, then

$$\min_{p \in \pi_k} \| p(A) \| \leq \left( 1 - \nu(A) \nu(A^{-1}) \right)^{k/2}, \quad (3.1)$$

where $\mathcal{F}(A)$ denotes the field of values of $A$, and $\nu(A)$ is the distance of $\mathcal{F}(A)$ from the origin. Moreover, if $M = \frac{1}{2}(A + A^H)$ is positive definite, then

$$\min_{p \in \pi_k} \| p(A) \| \leq \left( 1 - \frac{\lambda_{\min}(M)^2}{\lambda_{\max}(A^H A)} \right)^{k/2}. \quad (3.2)$$
Note that the derivation of (3.1) is based on replacing the optimal polynomial of degree \(k\) from the \(k\)th ideal GMRES approximation by the polynomial \((1 - \alpha z)^k\). Since the latter has only one \(k\)-fold root in the complex plane, the bound (3.1) cannot be expected to be sharp in general.

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