ASYMMETRIC RAMSEY PROPERTIES OF RANDOM GRAPHS FOR CLIQUES AND CYCLES

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Abstract. We say that $G \to (F, H)$ if, in every edge colouring $c : E(G) \to \{1, 2\}$, we can find either a 1-coloured copy of $F$ or a 2-coloured copy of $H$. The well-known Kohayakawa–Kreuter conjecture states that the threshold for the property $G(n, p) \to (F, H)$ is equal to $n^{-1/m_2(F,H)}$, where $m_2(F,H)$ is given by

$$m_2(F,H) := \max \left\{ \frac{e(J)}{v(J) - 2 + 1/m_2(H)} : J \subseteq F, e(J) \geq 1 \right\}.$$

In this paper, we show the 0-statement of the Kohayakawa–Kreuter conjecture for every pair of cycles and cliques.

1. Introduction

We say that a graph $G$ is a Ramsey graph for the pair of graphs $(F, H)$ if, in every edge colouring $c : E(G) \to \{1, 2\}$, we can find either a 1-coloured copy of $F$ or a 2-coloured copy of $H$. We write $G \to (F, H)$ if $G$ is Ramsey for $(F, H)$, and $G \not\rightarrow (F, H)$ otherwise. It follows from Ramsey’s Theorem [11] that, for each pair of graphs $(F, H)$, there exists a graph $G$ such that $G \to (F, H)$.

The study of whether or not the binomial random graph $G(n, p)$ is Ramsey for a symmetric pair of graphs was initiated by Frankl and Rödl [2], and Luczak, Ruciński, and Voigt [8]. They showed that the probability threshold for having $G(n, p) \to (K_3, K_3)$ is of order $n^{-1/2}$. In 1995, Rödl and Ruciński [12][13] determined the probability threshold for $G(n, p) \to (F, F)$ for almost all non-empty graphs $F$. They showed that, if $F$ has a component which is not a star or a path of length three, then the threshold is of order $n^{-1/m_2(F)}$, where

$$m_2(F) := \max \left\{ \frac{e(J) - 1}{v(J) - 2} : J \subseteq F, v(J) \geq 3 \right\}.$$

The parameter $m_2(F)$ is called the $m_2$-density of the graph $F$. Here, $v(J)$ and $e(J)$ denote the size of the vertex set and of the edge set of the graph $J$, respectively. The remaining cases were addressed subsequently by Friedgut and Krivelevich [3].

A natural generalisation of this problem is to determine a threshold function $p(F, H)$ for the property $G(n, p) \to (F, H)$, for any asymmetric pair of graphs $(F, H)$. This problem was posed in 1997 by Kohayakawa and Kreuter [5], who proved that $p(C_\ell, C_k) = \Theta(n^{1-\ell/(\ell-1)k})$.
for any pair of cycles \((C_\ell, C_k)\) with \(k \geq \ell \geq 3\). In the same paper, they conjectured that 
\[ p(F, H) = \Theta(n^{-1/m_2(F,H)}) ,\]
where
\[ m_2(F, H) := \max \left\{ \frac{e(J)}{v(J) - 2 + 1/m_2(H)} : J \subseteq F, e(J) \geq 1 \right\} , \]
for any pair of graphs such that \(m_2(F) \geq m_2(H) \geq 1\). Since the Kohayakawa–Kreuter conjecture was posed, there have been many attempts to solve it (see, for example, [4, 6, 9]). In a recent breakthrough, Mousset, Nenadov and Samotij [10] showed that 
\[ p(F, H) = O(n^{-1/m_2(F,H)}) ,\]
whenever 
\[ m_2(F) \geq m_2(H) \geq 1, \]
the so-called 1-statement. In contrast, much less is known about the 0-statement, that is, the statement that 
\[ p(F, H) = \Omega(n^{-1/m_2(F,H)}) ,\]
whenever 
\[ m_2(F) \geq m_2(H) \geq 1, \]
one possible reason for that is that the 0-statement seems to depend on the structural behaviour of Ramsey graphs.

As far as we know, the 0-statement is only proved for two types of pairs of graphs. Kohayakawa and Kreuter [5] established the 0-statement for all pairs of cycles while Marciniszyn, Skokan, Spöhel and Steger [9] addressed all pairs of cliques.

In this paper, we show that the 0-statement holds for any pair of cliques and cycles. This is the first 0-statement result for different types of graphs.

**Theorem 1.1.** For all \(\ell, r \geq 4\) there exists \(c > 0\) such that, if 
\[ p = p(n) \leq c n^{-1/m_2(K_r, C_\ell)} ,\]
then
\[ \lim_{n \to \infty} \mathbb{P}[G(n, p) \rightarrow (K_r, C_\ell)] = 0. \]

Combining Theorem 1.1 with the results of [5, 9] and [10], we establish the Kohayakawa–Kreuter conjecture for any pair of cycles and cliques with at least 3 vertices. We remark that we do need the assumption \(\ell, r \geq 4\) in our proof, so our result does not imply the earlier results involving \(K_3\).

The main tool behind the proof of Theorem 1.1 is a structural characterisation of Ramsey graphs for the pair \((K_r, C_\ell)\) via a ‘container type’ argument (see Theorem 2.1), which is a rephrasing of the idea used in previous works. Roughly speaking, we find a family \(\mathcal{I}\) of graphs with the following properties: (a) \(|\mathcal{I}|\) is small; (b) for every graph \(G\) with \(G \rightarrow (K_r, C_\ell)\) there exists \(I \in \mathcal{I}\) such that \(I \subseteq G\); and (c) for each \(I \in \mathcal{I}\), either \(I\) is small and dense or very structured. We provide the details in Section 2.

The rest of the paper is organised as follows. In Section 2, we prove Theorem 1.1 in Section 3, we provide the main technical lemmas of this paper; in Section 4, we prove some structural lemmas about Ramsey graphs; in Section 5, we describe the algorithms used to prove our main technical theorem (see Theorem 2.1); finally, in Section 6, we do a careful analysis of these algorithms. In the appendix, we provide some simple calculations involving \(m_2\)-densities, for completeness.

### 2. The main technical result

In this section, we present the main technical result of this paper and deduce Theorem 1.1 from it. In order to state this result, we need a little notation. For a graph \(G\), define \(\lambda(G)\)
by
\[ \lambda(G) = v(G) - \frac{e(G)}{m_2(K_r, C_\ell)}. \]
For any positive real numbers \( M, \varepsilon \) and any positive integer \( n \), define
\[ J_1(\varepsilon) = \{ G : \lambda(G) \leq -\varepsilon \} \quad \text{and} \quad J_2(M, n) = \{ G : \lambda(G) \leq M \text{ and } e(G) \geq \log n \}, \]
where the logarithm is in base 2. Finally, for any natural numbers \( r, \ell \) and \( n \), let
\[ R_n(K_r, C_\ell) = \{ G : v(G) = n \text{ and } G \to (K_r, C_\ell) \}. \]
When \( r \) and \( \ell \) are clear from context, we write \( R_n \) for \( R_n(K_r, C_\ell) \). In addition, we set
\[ R(K_r, C_\ell) = \bigcup_{n \in \mathbb{N}} R_n(K_r, C_\ell). \]

The connection between \( \lambda, J_1(\varepsilon), J_2(M, n) \) and \( R_n \) is contextualised in the next theorem.

**Theorem 2.1.** For any integers \( r, \ell \geq 4 \), there exist positive constants \( M = M(r, \ell) \) and
\( \varepsilon = \varepsilon(r, \ell) \) such that the following holds. For every \( n \in \mathbb{N} \), there exists a function \( f : R_n(K_r, C_\ell) \to J_1(\varepsilon) \cup J_2(M, n) \) such that \( f(G) \subseteq G \) for all \( G \in R_n \) and
\[ |f(R_n)| \leq (\log n)^M. \]

In the language of hypergraph containers [1, 14], Theorem 2.1 provides a relatively small collection \( f(R_n) \) of fingerprints. Additionally to \( |f(R_n)| \) being small, each graph \( f(G) \) either has a very small value of \( \lambda \) (negative, and bounded away from 0), or a fairly small (though possibly positive) value of \( \lambda \) and is very large. To obtain such a collection and the function \( f \) in Theorem 2.1 we employ an algorithm adapted from [7].

Theorem 2.1 is easily deduced from Theorem 2.1. The proof of Theorem 2.1 is given in the next four sections.

**Proof of Theorem 2.1.** Given \( r, \ell \geq 4 \), let \( M \) and \( \varepsilon \) be positive constants given by Theorem 2.1 and set \( c = 2^{-2M} \). For each \( n \in \mathbb{N} \), let \( p = p(n) \leq cn^{-1/m_2(K_r, C_\ell)} \). Let \( f \) be the function given by Theorem 2.1 let \( \Gamma \sim G(n, p) \) and suppose that \( \Gamma \in R_n \). Then, \( f(\Gamma) \subseteq \Gamma \) and \( f(\Gamma) \in J_1(\varepsilon) \cup J_2(M, n) \). Let \( I_1 := f(R_n) \cap J_1(\varepsilon) \) and \( I_2 := f(R_n) \cap J_2(M, n) \). Thus,
\[ \mathbb{P}(\Gamma \to (K_r, C_\ell)) \leq \mathbb{P}(F \subseteq \Gamma \text{ for some } F \in I_1 \cup I_2). \]
Since \( \lambda(F) \leq -\varepsilon \) for each \( F \in I_1 \) and \( c \leq 1 \), we have
\[ \mathbb{P}(F \subseteq \Gamma) \leq n^{v(F)} p^{e(F)} \leq c^{e(F)} n^{\lambda(F)} \leq n^{-\varepsilon} \]
for every \( F \in I_1 \). Similarly, we have
\[ \mathbb{P}(F \subseteq \Gamma) \leq c^{e(F)} n^{\lambda(F)} \leq n^{-M} \]
for every \( F \in I_2 \), as \( \lambda(F) \leq M \) and \( e(F) \geq \log n \) for each \( F \in I_2 \), and by our choice of \( c \). Applying the union bound to (2) and using (3) and (4), we obtain that
\[ \mathbb{P}(G(n, p) \to (K_r, C_\ell)) \leq (\log n)^M \cdot (n^{-\varepsilon} + n^{-M}), \]
since \(|\mathcal{I}_1 \cup \mathcal{I}_2| \leq (\log n)^M\). As the expression on the right hand side tends to 0 as \(n \to \infty\), this implies the theorem.

\[ \square \]

3. Proof of Theorem 2.1

In this section, we state the main technical lemmas of this paper, and deduce Theorem 2.1 from them. We also introduce some notation that we use during the proof.

Let \(r, \ell\) be positive integers. We follow the approach by [5] and [7] and bring our problem into the hypergraph setting. Given a graph \(G = (V, E)\), let \(\mathcal{G}_{r,\ell}(G)\) be the hypergraph on the edge set of \(G\) whose hyperedges correspond to the copies of \(K_r\) and \(C_\ell\) in \(G\). We suppress \(G\), \(r\) and \(\ell\) from the notation whenever they are clear from context. Define

\[
\mathcal{E}'_1(G) = \{ E(F) : F \cong K_r, F \subseteq G \} \quad \text{and} \quad \mathcal{E}'_2(G) = \{ E(F) : F \cong C_\ell, F \subseteq G \}.
\]

(5)

Analogously, if \(\mathcal{H} \subseteq \mathcal{G}_{r,\ell}(G)\), then we set \(\mathcal{E}_1(\mathcal{H}) := \mathcal{E}_1(G) \cap E(\mathcal{H})\) and \(\mathcal{E}_2(\mathcal{H}) := \mathcal{E}_2(G) \cap E(\mathcal{H})\).

The reason why we deal with \(\mathcal{G}_{r,\ell}(G)\) instead of \(G\) is as follows. In order to build our fingerprints algorithmically, we would like to deal with a subgraph \(H\) of \(G\) which has the two following properties: (1) every edge \(e \in E(H)\) is contained in an \(\ell\)-cycle; (2) for every copy \(C\) of \(C_\ell\) in \(H\) every \(e \in C\), there exists a copy \(K\) of \(K_r\) such that \(E(K) \cap E(C) = \{e\}\). These properties would allow us to build the fingerprint of \(G\) algorithmically by attaching either a copy of \(K_r\) or a copy of \(C_\ell\) to the current graph at each step. However, such graph \(H\) might not exist. Even if \(H\) is minimal (with respect to subgraph containment) for \(H \to (K_r, C_\ell)\), we can only deduce that for every \(e \in E(H)\) there exist \(K \cong K_r\) and \(C \cong C_\ell\) in \(H\) such that \(E(K) \cap E(C) = \{e\}\). But this does not directly imply that property (2) holds. We can overcome this problem by considering subhypergraphs of \(\mathcal{G}_{r,\ell}(G)\) which are \(*\)-critical. This property was first considered in [7].

Definition 3.1 (\(*\)-critical). Let \(\mathcal{E}_1\) and \(\mathcal{E}_2\) be two families of sets on a vertex set. We say that a hypergraph \(\mathcal{H} = \mathcal{E}_1 \cup \mathcal{E}_2\) is \(*\)-critical with respect to \((\mathcal{E}_1, \mathcal{E}_2)\) if the following two properties hold. For each \(e \in V(\mathcal{H})\), there exists a hyperedge \(F \in \mathcal{E}_2\) such that \(e \in F\); and for each \(F \in \mathcal{E}_2\) and each \(e \in F\), there exists a hyperedge \(E \in \mathcal{E}_1\) such that \(E \cap F = \{e\}\). When \(\mathcal{E}_1\) and \(\mathcal{E}_2\) are clear from context, we say that the hypergraph \(\mathcal{H}\) is \(*\)-critical.

Let \(\text{Crit}_{r,\ell}(G)\) be the set of all \(*\)-critical subhypergraphs of \(\mathcal{G}_{r,\ell}(G)\). The next lemma shows that if \(G \to (K_r, C_\ell)\), then there are subhypergraphs of \(\mathcal{G}_{r,\ell}(G)\) which are \(*\)-critical. This is a direct consequence of Lemma 4.1 and we prove it in Section 4.

Lemma 3.2. Let \(r, \ell \geq 4\) be integers. If \(G \to (K_r, C_\ell)\), then \(\text{Crit}_{r,\ell}(G) \neq \emptyset\).

Given a hypergraph \(\mathcal{H} \subseteq \mathcal{G}_{r,\ell}(G)\), we define the underlying graph of \(\mathcal{H}\), denoted by \(G(\mathcal{H})\), to be the subgraph of \(G\) whose edge set is \(\bigcup_{E \in E(\mathcal{H})} E\). The following lemma is central to our proof. We prove it in Section 4.

Lemma 3.3. Let \(r, \ell \geq 4\) be integers. There exists \(\varepsilon = \varepsilon(r, \ell) > 0\) such that the following holds. If \(\mathcal{H} \in \text{Crit}_{r,\ell}(H)\), then \(\lambda(G(\mathcal{H})) \leq -\varepsilon\).
In order to find the function \( f : \mathcal{R}_n(K_r, C_\ell) \to \mathcal{J}_1(\varepsilon) \cup \mathcal{J}_2(M, n) \) in Theorem 2.4, we define an algorithm \textsc{Hypertree} in Section 5. For each \( G \in \mathcal{R}_n(K_r, C_\ell) \), this algorithm takes some hypergraph \( \mathcal{H} \in \text{Crit}_{r,\ell}(G) \) as input and creates a pair \((\mathcal{H}_T, D_T)\) as output, where \( \mathcal{H}_T \subseteq \mathcal{H} \) and \( D_T \subseteq \mathbb{N} \). The fingerprint \( f(G) \) will be \( G(\mathcal{H}_T) \), and the auxiliary set \( D_T \) will help us to count all the possible outputs of \textsc{Hypertree} and to ensure that \( G(\mathcal{H}_T) \) belongs to \( \mathcal{J}_1(\varepsilon) \cup \mathcal{J}_2(M, n) \). For the detailed description of \textsc{Hypertree}, we refer the reader to Section 5. Let \textsc{Hypertree}(\mathcal{H}) denote the execution of \textsc{Hypertree} on input \( \mathcal{H} \). Its basic properties are given by the next lemma.

**Lemma 3.4.** Let \( n, r, \ell \geq 4 \) be integers and \( G \) be a graph on \( n \) vertices. For any hypergraph \( \mathcal{H} \in \text{Crit}_{r,\ell}(G) \), \textsc{Hypertree}(\mathcal{H}) generates a sequence of hypergraphs \( \mathcal{H}_0 \subseteq \ldots \subseteq \mathcal{H}_T \subseteq \mathcal{H} \) and sets \( D_0 \subseteq \ldots \subseteq D_T \) for which the following holds.

(a) \( \mathcal{H}_0 = \{ E \} \), for some \( E \in \mathcal{E}_1(\mathcal{H}) \); that is, the underlying graph of \( \mathcal{H}_0 \) is a copy of \( K_r \) in \( G \);

(b) \( v(\mathcal{H}_0) < v(\mathcal{H}_1) < \ldots < v(\mathcal{H}_T) \);

(c) \( T \) is the smallest integer such that \( \lambda(G_T) \leq -\varepsilon \) or \( T \geq \log n \), where \( G_T = G(\mathcal{H}_T) \) and \( \varepsilon \) is the constant given by Lemma 3.3;

(d) \textsc{Hypertree}(\mathcal{H}) returns the pair \((\mathcal{H}_T, D_T)\).

Our next lemma is the most important property of the \textsc{Hypertree} algorithm (for a proof, see Section 6). We shall use it together with Lemma 3.4 to deduce that the underlying graph given by the output of \textsc{Hypertree}(\mathcal{H}) belongs to \( \mathcal{J}_1(\varepsilon) \cup \mathcal{J}_2(M, v(G)) \) whenever \( \mathcal{H} \) is a hypergraph in \( \text{Crit}_{r,\ell}(G) \), where \( M = M(r, \ell) > 0 \). As the underlying graph of the output hypergraph is a subgraph of \( G \), this will establish the existence of a function \( f : \mathcal{R}_n(K_r, C_\ell) \to \mathcal{J}_1(\varepsilon) \cup \mathcal{J}_2(M, n) \) such that \( f(G) \subseteq G \).

**Lemma 3.5.** For all integers \( r, \ell \geq 4 \), there exists \( \delta = \delta(r, \ell) > 0 \) such that the following holds. For any graph \( G \) and any hypergraph \( \mathcal{H} \in \text{Crit}_{r,\ell}(G) \), the sequence \((\mathcal{H}_i, D_i)_{i=0}^T\) generated by \textsc{Hypertree}(\mathcal{H}) satisfies

1. \( \lambda(G_i) = \lambda(G_{i-1}) \) for all \( i \notin D_T \), and
2. \( \lambda(G_i) \leq \lambda(G_{i-1}) - \delta \) for all \( i \in D_T \),

where \( G_i = G(\mathcal{H}_i) \) for each \( i \in \{0, \ldots, T\} \).

We say that the \( i \)-th step of \textsc{Hypertree}(\mathcal{H}) is degenerate if \( i \in D_T \), and non-degenerate otherwise. The stopping conditions on \( \lambda \) (see Lemma 3.4(c)) combined with Lemma 3.5 imply that \textsc{Hypertree}(\mathcal{H}) must have a bounded number of degenerate steps. This will be essential to show that the set of all possible outputs given by \textsc{Hypertree} is at most polylogarithmic in \( n \) when applied over \( \bigcup_{v(G)=n} \text{Crit}_{r,\ell}(G) \). To be more precise, for each \( n, r, \ell \geq 4 \), consider the family of non-isomorphic graphs

\[
\text{Out}_{r,\ell}(n) = \bigcup_{G : v(G)=n} \{ G(\mathcal{H}_T) : \mathcal{H} \in \text{Crit}_{r,\ell}(G) \},
\]
where $T = T(\mathcal{H})$ and $\mathcal{H}_T$ are the stopping time and the output given by $\text{Hypertree}(\mathcal{H})$, respectively. The next lemma bounds the size of $\text{Out}_{r,\ell}(n)$.

**Lemma 3.6.** For all $r, \ell \geq 4$, there exists $C > 0$ such that $|\text{Out}_{r,\ell}(n)| \leq (\log n)^C$, for all $n \in \mathbb{N}$.

We prove this lemma in Section 6. Now, we are ready to prove Theorem 2.1 assuming all the lemmas stated in this section.

**Proof of Theorem 2.1** Fix $n, r, \ell \geq 4$. For each $G \in \mathcal{R}_n(K_r, C_\ell)$, let $\mathcal{H}(G)$ be a $*$-critical hypergraph in $\text{Crit}_{r,\ell}(G)$. By Lemma 3.2, such a hypergraph must exist. Let $(\mathcal{H}_i(G))_{i=0}^T$ be the sequence of hypergraphs generated by $\text{Hypertree}(\mathcal{H}(G))$. By Lemma 3.3, the last hypergraph of this sequence, namely $\mathcal{H}_T(G)$, is the hypergraph output by $\text{Hypertree}(\mathcal{H}(G))$.

Define

$$f : \mathcal{R}_n \to \text{Out}_{r,\ell}(n)$$

$$G \mapsto G(\mathcal{H}_T(G)).$$

As $\mathcal{H}_T(G) \subseteq \mathcal{H}(G)$ by Lemma 3.4 and $G(\mathcal{H}(G)) \subseteq G$ by construction, we have $f(G) \subseteq G$ for each $G \in \mathcal{R}_n$. Moreover, by Lemma 3.5(c), we have $\lambda(f(G)) \leq -\varepsilon$ or $T \geq \log n$. In the first case, $f(G)$ belongs to the set of graphs $J_1(\varepsilon) = \{H : \lambda(H) \leq -\varepsilon\}$.

In the second case, we claim that $f(G) \in J_2(C, n)$, where $C = \lambda(K_r)$. To see that, first note that the sequence $(v(H_i(G)))_{i=0}^T$ is strictly increasing by Lemma 3.5(b). For simplicity, let $G_i = G(H_i(G))$. Since $e(G_i) = v(H_i(G))$ for every $i \in \{0, \ldots, T\}$, we have

$$e(f(G)) = e(\mathcal{H}_T(G)) \geq T \geq \log n. \tag{6}$$

Moreover, by Lemma 3.5, we have $\lambda(G_i) \leq \lambda(G_{i-1})$ for each $i \in \{0, \ldots, T\}$, where $G_0 \cong K_r$ by Lemma 3.4(a). In particular,

$$\lambda(f(G)) = \lambda(G_T) \leq \lambda(G_0) = \lambda(K_r). \tag{7}$$

Together, (6) and (7) imply that $f(G) \in J_2(C, n)$, where $C = \lambda(K_r)$. This proves our claim.

Now, it only remains to show that $|f(\mathcal{R}_n)| \leq (\log n)^{C_0}$, for some constant $C_0 > 0$. But, this follows directly from Lemma 3.6. We finish the proof by setting $M = \max\{C_0, C\}$. □

### 4. The structural lemmas

In this section, we obtain some key structural information about Ramsey hypergraphs and prove Lemmas 3.2 and 3.3.

Given two families of sets $\mathcal{E}_1$ and $\mathcal{E}_2$ on a vertex set and a hypergraph $\mathcal{H}$, define

$$\mathcal{E}_1(\mathcal{H}) = \mathcal{E}_1 \cap E(\mathcal{H}) \quad \text{and} \quad \mathcal{E}_2(\mathcal{H}) = \mathcal{E}_2 \cap E(\mathcal{H}).$$

We refer to the hyperedges of $\mathcal{E}_1(\mathcal{H})$ and $\mathcal{E}_1(\mathcal{H})$ as, respectively, hyperedges of type 1 and 2. We say that $\mathcal{H}$ is Ramsey for $(\mathcal{E}_1, \mathcal{E}_2)$, and we write $\mathcal{H} \to (\mathcal{E}_1, \mathcal{E}_2)$, if the following holds.
For every 2-colouring $c : V(H) \to \{1, 2\}$, there exists a hyperedge $E \in \mathcal{E}_i(H)$ such that
\[ c(E) = \{i\}, \]
for some $i \in \{1, 2\}$. Conversely, we write $H \to (\mathcal{E}_1, \mathcal{E}_2)$ if $H \to (\mathcal{E}_1, \mathcal{E}_2)$ is not satisfied. Clearly, if $H \subseteq F$ and $H \to (\mathcal{E}_1, \mathcal{E}_2)$, then $F \to (\mathcal{E}_1, \mathcal{E}_2)$. Therefore, we may concentrate on the minimal hypergraphs $H$ that satisfy $H \to (\mathcal{E}_1, \mathcal{E}_2)$.

We call a hypergraph $H$ Ramsey minimal with respect to $(\mathcal{E}_1, \mathcal{E}_2)$ if $H \to (\mathcal{E}_1, \mathcal{E}_2)$, yet the removal of any hypervertex or hyperedge from $H$ destroys this property. Minimal Ramsey hypergraphs have the $\star$-critical property, as the next lemma shows. Its proof follows the same steps as the proof of [7] Claim 1. A particular case of it can be also found in [5] Section 3.

**Lemma 4.1.** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two disjoint families of sets on a vertex set. If a hypergraph $H$ is Ramsey minimal with respect to $(\mathcal{E}_1, \mathcal{E}_2)$, then the following holds. For each $i \in \{1, 2\}$, each hyperedge $E \in \mathcal{E}_i$, and each hypervertex $e \in E$, there exists a hyperedge $F \in \mathcal{E}_3-i$ such that $E \cap F = \{e\}$. In particular, $H$ is $\star$-critical.

**Proof.** Fix any hyperedge $E \in \mathcal{E}_i(H)$, for some $i \in \{1, 2\}$, and any hypervertex $e \in E$. Let $H \setminus E$ be the hypergraph with vertex set $V(H)$ and hyperedge set $E(H) \setminus \{E\}$. Consider any colouring $c : V(H) \to \{1, 2\}$ for which there is no hyperedge of type $j$ in $H \setminus E$ coloured $j$ under $c$, for all $j \in \{1, 2\}$. This colouring exists because $H$ is Ramsey minimal. As $H \to (\mathcal{E}_1, \mathcal{E}_2)$, all the hypervertices in $E$ must be coloured $i$ under $c$. Moreover, $E$ is the only monochromatic hyperedge under $c$ which has the same colour as its number type. Now, let $c' : V(H) \to \{1, 2\}$ be the colouring such that $c'(f) = c(f) \iff f \neq e$ (recall that $e \in E$). As $H \to (\mathcal{E}_1, \mathcal{E}_2)$ and $E$ is not monochromatic of colour $i$ under $c'$, there must exist a hyperedge $F \in \mathcal{E}_3-i(H)$ such that $c'_E(F) = 3 - i$ and $E \cap F = \{e\}$, as required. \hfill $\Box$

Now we are ready to prove Lemma 3.2.

**Proof of Lemma 3.2.** If $G \to (K_r, C_\ell)$, then $G_{r, \ell}(G) \to (\mathcal{E}_1, \mathcal{E}_2)$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are defined in (3). Let $H$ be an arbitrary Ramsey minimal subhypergraph of $G_{r, \ell}(G)$. Then $H$ is $\star$-critical, by Lemma 4.1 and therefore $H \in \text{Crit}_{r, \ell}(G)$, as required. \hfill $\Box$

We now turn to the proof of Lemma 3.3. In order to prove it, we require some structural information about underlying graphs of $\star$-critical hypergraphs. This structural information is obtained in Lemma 4.3 and, before stating it, it is worth to point out the following observation.

**Observation 4.2.** Let $r, \ell \geq 3$. For any graph $H$ and any hypergraph $H \in \text{Crit}_{r, \ell}(H)$, we have $d(v) \geq r$ for all $v \in V(G(H))$.

In fact, if $H$ is $\star$-critical, then for any $e \in E(G(H))$ there exists $K \in \mathcal{E}_1(H)$ and $C \in \mathcal{E}_2(H)$ such that $C \cap K = \{e\}$. As $K$ and $C$ are copies of $K_r$ and $C_\ell$ contained in $G(H)$, respectively, we can easily infer that $d(v) \geq r$ for all $v \in V(G(H))$.

Now, we need to set some notations. For each graph $G$, define
\[ A = A(G) = \{v \in V(G) : d(v) = r\} \quad \text{and} \quad B = B(G) = \{v \in V(G) : d(v) > r\}. \quad (8) \]
By Observation 4.2, $V(G)$ can be partitioned into $V(G) = A \cup B$ whenever $G$ is the underlying graph of a $\ast$-critical hypergraph. Below, we use $N(v)$ to denote the neighbourhood of a vertex $v$ in $G$ and, for each $S \subseteq V(G)$, we write $d_S(v) = |N(v) \cap S|$. Our structural lemma is as follows.

**Lemma 4.3.** Let $r \geq 3$ and $\ell \geq 4$ be integers, and $H$ be a graph. For any hypergraph $H \in \text{Crit}_{r,\ell}(H)$, we have that

1. $A$ is an independent set in $G(H)$ and
2. $d_B(v) \geq r - 2$, for all $v \in V(G(H))$.

**Proof.** First, let us prove item (1). Suppose for a contradiction that there are two adjacent vertices $u, v \in V(G(H))$ such that $d(u) = d(v) = r$. As $H$ is $\ast$-critical, there exists an $r$-clique $R_1 \in \mathcal{E}_1(H)$ and an $\ell$-cycle $C_1 \in \mathcal{E}_2(H)$ such that $E(R_1) \cap E(C_1) = \{u, v\}$. Now, let $u_c$ be the neighbour of $u$ in $V(C_1) \setminus \{u, v\}$ and fix any vertex $w \in V(R_1) \setminus \{u, v\}$. Then, we have the following claim.

**Claim 4.4.** There exists an $\ell$-cycle $C_2 \in \mathcal{E}_2(H)$ such that either $\{u_c, u, w\} \subseteq V(C_2)$ or $\{v, u, w\} \subseteq V(C_2)$.

**Proof of the Claim.** As $H$ is $\ast$-critical, there exists an $r$-clique $R_2$ and an $\ell$-cycle $C_2$ such that $E(R_2) \cap E(C_2) = \{u, w\}$. If $R_2 = R_1$, then $C_2$ must contain $u_c$, as $d(u) = r$. This settles the first part of the claim. If $R_2 \neq R_1$, then $R_2$ must contain $u_c$, again because $d(u) = r$. Moreover, as $u_c \in V(R_2)$ and $\ell \geq 4$, we cannot have $v \in R_2$. Together, these imply that $V(R_2) = \{u_c\} \cup V(R_1) \setminus \{v\}$. As $v$ is the only vertex in $V(R_1)$ not contained in $R_2$, it follows that $v \in C_2$. This settles the second part.

Let $C_2$ be the cycle given by the claim above. If $\{u_c, u, w\} \subseteq V(C_2)$, then there exists an $r$-clique $R_2 \in \mathcal{E}_1(H)$ such that $E(R_2) \cap E(C_2) = \{u_c, u\}$. As $V(R_2) \subseteq N(u) \cup \{u\} \setminus \{w\}$, $R_2$ has no choice but to contain $v$. In particular, this implies that $v$ is a neighbour of $u_c$, which gives us a contradiction, as $\ell \geq 4$. If $\{v, u, w\} \subseteq V(C_2)$, then there exists an $r$-clique $R_2 \in \mathcal{E}_1(H)$ such that $E(R_2) \cap E(C_2) = \{u, v\}$. As $V(R_2) \subseteq N(u) \cup \{u\} \setminus \{w\}$, $R_2$ has no choice but to contain $u_c$. In particular, this implies again that $v$ is a neighbour of $u_c$, which gives us a contradiction. This proves item (1).

To show item (2), we consider two cases: (i) either $d_B(v) = d(v)$, or (ii) $d(v) > d_B(v)$. In the first case, Observation 4.2 gives us $d_B(v) = d(v) \geq r$. In the second case, there is a vertex $u \in N(v) \cap A$ and, since $H$ is $\ast$-critical, there is also an $r$-clique $R$ in $G(H)$ such that $\{u, v\} \in E(R)$. As $A$ is an independent set, we must have $V(R) \setminus \{u\} \subseteq B$, which implies that $v$ has at least $r - 2$ neighbours in $B$.

Let $m(G) = e(G)/v(G)$ be the edge density of $G$. Now, we are ready to prove Lemma 3.3.

**Proof of Lemma 3.3.** To simplify the notation, set $G = G(H)$. Observe that $\lambda(G) \leq -\varepsilon$ if and only if $m_2(K_r, C_\ell)^{-1} - m(G)^{-1} \geq \varepsilon e(G)^{-1}$. The last inequality holds for some $\varepsilon = \ldots$
\( \varepsilon(r, \ell) > 0 \) if there exists \( \delta = \delta(r, \ell) > 0 \) such that \( m(G) > m_2(K_r, C_\ell) + \delta \). Thus, we can reduce our problem to finding such \( \delta \). In order to do so, we shall first bound \( e(G) \).

By Observation 4.2 the set \( V(G) \) can be partitioned into \( V(G) = A \cup B \), where \( A = A(G) \) and \( B = B(G) \) were defined in (8). Thus, we can write

\[
2e(G) = \sum_{v \in A} d(v) + \sum_{v \in B} d(v).
\]

As \( d(v) = r \) for all \( v \in A \), we have \( \sum_{v \in A} d(v) = |A| \). Now, to bound the sum \( S = \sum_{v \in B} d(v) \), observe that this sum counts twice each edge inside the set \( B \) and counts once each edge across \( A \) and \( B \). By Lemma 4.3(1), we have \( e(A, B) = r|A| \), as \( A \) is an independent set and \( d(v) = r \) for each \( v \in A \). By Lemma 4.3(2), \( d_B(v) \geq r - 2 \) for each \( v \in B \). Together, these imply that

\[
S \geq r|A| + (r - 2)|B|.
\]

Furthermore,

\[
S \geq (r + 1)|B|,
\]

as \( d(v) \geq r + 1 \) for each \( v \in B \). Therefore,

\[
2e(G) \geq r|A| + \max \left\{ r|A| + (r - 2)|B|, (r + 1)|B| \right\}.
\]

As \( v(G) = |A| + |B| \), we have

\[
2m(G) \geq \max \left\{ \frac{2r|A| + (r - 2)|B|}{|A| + |B|}, \frac{r|A| + (r + 1)|B|}{|A| + |B|} \right\}
= r - 2 + \max \left\{ (r + 2)x, 3 - x \right\},
\]

where \( x = |A|/v(G) \). The last expression attains its minimum value when \( x = 3/(r + 3) \), and hence

\[
m(G) \geq \frac{r + 1}{2} - \frac{3}{2(r + 3)}.
\]

A straightforward calculation shows that \( m_2(K_r, C_\ell) = \frac{(\ell)(\ell - 1)}{(r - 1)(\ell - 1) - 1} \) (see Lemma 6.1). From this expression, we can see that \( \ell \mapsto m_2(K_r, C_\ell) \) is decreasing. Thus, in order to conclude our proof, it suffices to show that

\[
\frac{r + 1}{2} - \frac{3}{2(r + 3)} > m_2(K_r, C_4).
\]

Using again the expression we have for \( m_2(K_r, C_\ell) \), an easy calculation shows that the last inequality holds for every \( r \geq 4 \). This completes the proof of the lemma. \( \square \)
5. The Algorithms

In this section, we formally describe the algorithm Hypertree and its subroutine Flower, and prove Lemma 5.3. Let \( n, r, \ell \geq 4 \) be fixed integers throughout this section.

First, let us recall some notation from Section 3. Given any graph \( G \), \( \text{Crit}_{r,\ell}(G) \) denotes the set of all \( \ast \)-critical subhypergraphs of \( \mathcal{G}_{r,\ell}(G) \), the hypergraph whose hyperedges correspond to the copies of \( K_r \) and \( C_\ell \) on \( G \). For any \( \mathcal{H} \subseteq \mathcal{G}_{r,\ell}(G) \), we denote \( \mathcal{E}_1(\mathcal{H}) = E(\mathcal{H}) \cap \mathcal{E}_1(\mathcal{G}) \) and \( \mathcal{E}_2(\mathcal{H}) = E(\mathcal{H}) \cap \mathcal{E}_2(\mathcal{G}) \), where

\[
\mathcal{E}_1(\mathcal{G}) = \{ E(F) : F \cong K_r, F \subseteq G \} \quad \text{and} \quad \mathcal{E}_2(\mathcal{G}) = \{ E(F) : F \cong C_\ell, F \subseteq G \}.
\]

The underlying graph of \( \mathcal{H} \) is denoted by \( G(\mathcal{H}) \).

We find it instructive to first provide an informal overview of Hypertree. This algorithm takes a hypergraph \( \mathcal{H} \in \text{Crit}_{r,\ell}(G) \) as input, for some graph \( G \) on \( n \) vertices, builds a sequence \((\mathcal{H}_i)_{i=0}^T \) of subhypergraphs of \( \mathcal{H} \) and outputs \( \mathcal{H}_T \). The algorithm seeks to find a subhypergraph \( F \subseteq \mathcal{H} \) for which the following holds. The graph \( F = G(\mathcal{F}) \), which is a subgraph of \( G \), satisfies (1) \( \lambda(F) \leq -\varepsilon \) or (2) \( \lambda(F) \leq M \) and \( e(F) \geq \log n \), for some positive constants \( \varepsilon = \varepsilon(r, \ell) \) and \( M = M(r, \ell) \).

In the first step, the algorithm picks a hyperedge \( E_0 \in \mathcal{E}_1(\mathcal{H}) \) and sets \( \mathcal{H}_0 = \{ E_0 \} \). In the 1-th step, it attaches a hyperedge \( E_i \in \mathcal{E}_1(\mathcal{H}) \) to the current hypergraph \( \mathcal{H}_{i-1} \) to build \( \mathcal{H}_i \). It is required that such a copy intersects \( G(\mathcal{H}_{i-1}) \) in at least two vertices, but is not a subgraph of \( G(\mathcal{H}_{i-1}) \). If no such hyperedge exists, then the algorithm runs a subroutine which we call Flower. This algorithm, when called within Hypertree, returns (1) a hyperedge \( C \in \mathcal{E}_2(\mathcal{H}) \) which intersects \( \mathcal{H}_{i-1} \) in at least one hypervertex and it is not contained in \( \mathcal{H}_{i-1} \); and (2) a collection of hyperedges in \( \mathcal{E}_1(\mathcal{H}) \) so that each intersect \( C \) in exactly one hypervertex. The output of Flower is attached to \( \mathcal{H}_{i-1} \) to build \( \mathcal{H}_i \). We defer the exact description of Flower until after the description of Hypertree.

The algorithm Hypertree also uses a canonical labelling function to guarantee that the number of non-isomorphic underlying graphs given by output hypergraphs is not very large (cf. Lemma 3.16). Define \( \sigma_0 : E(K_r) \to \{ 1, 2, \ldots, \binom{n}{r} \} \) to be a fixed labelling of the edges of \( K_r \). For each sequence of underlying graphs \((G(\mathcal{H}_0), \ldots, G(\mathcal{H}_t))\) generated by Hypertree at step \( t \), we define its associated sequence of labellings \((\sigma_i)_{i=0}^t \) in the following recursive way. Given \( \sigma_{i-1} : E(G(\mathcal{H}_{i-1})) \to \mathbb{N} \), define \( \sigma_i : E(G(\mathcal{H}_i)) \to \mathbb{N} \) to be a function which satisfies the following properties: (1) \( \sigma_i(e) = \sigma_{i-1}(e) \) for every \( e \in E(G(\mathcal{H}_{i-1})) \); (2) \( \sigma_i \) labels the edges in \( E(G(\mathcal{H}_i)) \setminus E(G(\mathcal{H}_{i-1})) \) with the next natural numbers larger than \( e(G(\mathcal{H}_{i-1})) \); and (3) outside \( E(G(\mathcal{H}_{i-1})) \), \( \sigma_i \) is determined by the unlabelled graph generated by the edges in \( E(G(\mathcal{H}_i)) \setminus E(G(\mathcal{H}_{i-1})) \) and by the set of vertices in \( G(\mathcal{H}_{i-1}) \) that meet these edges. We say that \( \sigma_i \) is the canonical extension of \( \sigma_{i-1} \). For each \( i \), note that \( \sigma_i \) is also an ordering of \( V(\mathcal{H}_i) \).
A hyperedge $E$ always corresponds to a set of edges of some underlying graph, and so we denote by $V(E)$ the set $\cup_{e \in E} e$. Next, is the formal description of the HYPERTREE algorithm. Recall that $\varepsilon = \varepsilon(r, \ell)$ is the small positive constant given by Lemma 3.3.

**Algorithm 1: HYPERTREE**

**Input:** A hypergraph $H \in \text{Crit}_{r, \ell}(G)$, for some graph $G$ on $n$ vertices

**Output:** A pair $(H_T, D_T)$, where $H_T \subseteq H$ and $D_T \subseteq \mathbb{N}$

/* Initialise: */

1. $i = 1$, $D_0 = \emptyset$, $H_0 = \{E_0\}$ for some $E_0 \in \mathcal{E}_1(H)$

   Let $\sigma_0 : E(V(H_0)) \to \mathbb{N}$ be equal to the canonical labelling of $K_r$

2. while $\lambda(G(H_0)) > -\varepsilon$ and $i < \log n$ do

3.   if there exists $E \in \mathcal{E}_1(H)$ such that $|V(E) \cap V(G(H_{i-1}))| \geq 2$ and $E \not\subseteq V(G(H_{i-1}))$

4.     then

5.       set $H_i = H_{i-1} \cup \{E\}$ and $D_i = D_{i-1} \cup \{i\}$

6.     end else let $H_F$ be the output of $\text{FLOWER}(H_{i-1}, H, \sigma_{i-1})$

7.       set $H_i = H_{i-1} \cup H_F$

8.       if $|V(G(H_i)) \setminus V(G(H_{i-1}))| = (r - 1)(\ell - 1) - 1$ then set $D_i = D_{i-1}$

9.       else set $D_i = D_{i-1} \cup \{i\}$

10.      end

11.     Let $\sigma_i : V(H_i) \to \mathbb{N}$ be the canonical extension of $\sigma_{i-1}$ to $H_i$. $i \mapsto i + 1$

12. return $(H_i, D_i)$

Let us now turn to the subroutine $\text{FLOWER}$. The input of $\text{FLOWER}$ is a triple $(H_i, H, \sigma_i)$, where $H$ is a $\star$-critical subhypergraph of $\mathcal{G}_{r, \ell}(G)$, for some graph $G$, $H_i \subseteq H$ and $\sigma_i : V(H_i) \to \mathbb{N}$ is an ordering of the vertices of $H_i$. When called within HYPERTREE, the output is a subhypergraph of $H$ called flower. For a hyperedge $C$ of type 2 and hyperedges $P_1, \ldots, P_t$ of type 1, we call the hypergraph $H_F = \{C, P_1, \ldots, P_t\}$ a flower if $t < \ell$ and $|C \cap P_s| = 1$ for all $1 \leq s \leq t$. The hyperedges $P_1, \ldots, P_t$ are called petals. Observe that, in the ‘graph world’, a flower corresponds to a copy of $C_\ell$ and $t$ copies of $K_r$ that intersect $C$ in exactly one edge (and possibly more vertices).

We next introduce some non-standard notation that we need in the algorithm description to ensure that the total number of non-isomorphic hypergraphs produced by HYPERTREE is not too big. For each $i \in \mathbb{N}$, let $\mathcal{C}(i, H_i)$ be the set of all $\star$-critical hypergraphs which generate the hypergraph $H_i$ in the $i$-th step of HYPERTREE and which enter $\text{FLOWER}$ at
step $i+1$. Now, define

$$\overline{H}_i = \bigcup_{H \in \mathcal{C}(i, H_i)} \{E \in H : E \subseteq V(H_i)\}.$$ 

Observe that $G(H_i) = G(\overline{H}_i)$. Now we are ready to state the formal description of FLOWER.

Algorithm 2: FLOWER

Input: A triple $(H_{i-1}, H, \sigma_{i-1})$, where $H$ is a hypergraph in $\text{Crit}_{r, \ell}(G)$, for some graph $G$, $H_{i-1} \subseteq H$, and $\sigma_{i-1}$ is an ordering of $V(H_{i-1})$

Output: A flower $H_F = \{C\} \cup \{P_e : e \in C \setminus V(H_{i-1})\}$, where $C \in \mathcal{E}_2(H)$,

\[
P_e \in \mathcal{E}_1(H) \text{ for all } e \in C \setminus V(H_{i-1})
\]

/* Find a seed: */

1 Let $e_0 \in E(G(H_{i-1}))$ be the smallest edge under the labelling $\sigma_{i-1}$ for which $e_0 \notin C$

2 let $C \in \mathcal{E}_2(H)$ be a hyperedge containing $e_0$ such that $C \not\subseteq V(H_{i-1})$

\[
\text{for every } e \in C \setminus V(H_{i-1}) \text{ do }
\]

3 \quad let $P_e \in \mathcal{E}_1(H)$ be such that $C \cap P_e = \{e\}$

end

return $\{C\} \cup \{P_e : e \in C \setminus V(H_{i-1})\}$

In general, the algorithm FLOWER may return an error, since an edge $e_0$ as in line 1 or a cycle $C$ as in line 2 may not exist. However, such concerns are void when FLOWER is called within HYPERTREE, as the next lemma shows.

Lemma 5.1. Let $G$ be a graph on $n$ vertices and $H \in \text{Crit}_{r, \ell}(G)$. Suppose that the algorithm FLOWER is called in the $i$-th step of HYPERTREE($H$). Then, the edge $e_0$ in line 1 the cycle $C$ in line 2 and the clique $P_e$ in line 3 of FLOWER$(H_{i-1}, H, \sigma_{i-1})$ exist. In particular, the algorithm runs without errors and finishes in finite time. Moreover, $e_0$ is uniquely determined by $H_{i-1}$ and $\sigma_{i-1}$.

Proof. First, let us show the existence of $e_0$. We claim that $H_{i-1}$ cannot be $*$-critical. Otherwise, $\lambda(G(H_{i-1})) \leq -\varepsilon$, by Lemma 3.31 and this would imply that HYPERTREE($H$) has not entered the while loop in the $i$-th step. In particular, FLOWER would not be called in the $i$-th step of HYPERTREE($H$). Now, observe that for every $C \in \mathcal{E}_2(H_{i-1})$ and every $e \in C$, there exists $K \in \mathcal{E}_1(H_{i-1})$ such that $K \cap C = \{e\}$. In fact, if $C \in \mathcal{E}_2(H_{i-1})$, then there exists a hypergraph $F \in \mathcal{C}(i-1, H_{i-1})$ such that $C \in \mathcal{E}_2(F)$. As $F$ is $*$-critical, there must also exist $K \in \mathcal{E}_1(F)$ such that $K \cap C = \{e\}$. But, because HYPERTREE($F$) has entered the else-statement at step $i$, we have $K \subseteq V(H_{i-1})$, and hence $K \in \mathcal{E}_1(H_{i-1})$. It follows that the only reason for which $H_{i-1}$ is not $*$-critical is because there exists an edge $e \in E(G(H_{i-1}))$ for which there is no hyperedge in $\mathcal{E}_2(H_{i-1})$ containing $e$. Among these edges, we take $e_0$ to be the one with the smallest labelling under $\sigma_{i-1}$. By construction, $e_0$ only depends on $H_{i-1}$ and $\sigma_{i-1}$. As $H$ is $*$-critical, the existence of the cycle $C$ as in line 2 and the petals $P_e$ as in line 3 is straightforward. 

Lemma 5.2. Under the same assumptions as in Lemma 5.1, FLOWER(\(\mathcal{H}_{i-1}, \mathcal{H}, \sigma_{i-1}\)) output a flower \(\mathcal{H}_F = \{C\} \cup \{P_e : e \in C \setminus V(\mathcal{H}_{i-1})\}\) which satisfies the following properties:

(\(F_1\)) \(C \in \mathcal{E}_i(\mathcal{H})\) but \(C \not\subseteq V(\mathcal{H}_{i-1})\).
(\(F_2\)) \(P_e \in \mathcal{E}_1(\mathcal{H})\) and \(C \cap P_e = \{e\}\) for every \(e \in C \setminus V(\mathcal{H}_{i-1})\), and
(\(F_3\)) \(|V(P_e) \cap V(G(\mathcal{H}_{i-1}))| \leq 1 \leq |C \cap E(G(\mathcal{H}_{i-1}))|\) for all \(e \in C \setminus V(\mathcal{H}_{i-1})\).

Proof. The properties \((F_1)\) and \((F_2)\) are immediate from the algorithm description and Lemma 5.1 As FLOWER was called in the \(i\)-th step of HYPERTREE(\(\mathcal{H}\)), line 3 of HYPERTREE was not executed. This means that for each petal \(P_e\) we have \(|V(P_e) \cap V(G(\mathcal{H}_{i-1}))| \leq 1\), which proves the first inequality in \((F_3)\). The second inequality follows from the existence of \(e_0\) in line 1 as \(e_0 \in C \cap E(G(\mathcal{H}_{i-1}))\). \(\square\)

With Lemmas 5.1 and 5.2 at our hands, we now deduce Lemma 3.4.

Proof of Lemma 3.4. In its initialisation, HYPERTREE(\(\mathcal{H}\)) algorithm sets \(D_0 = \emptyset\) and \(\mathcal{H}_0 = \{E_0\}\), for some \(E_0 \in \mathcal{E}_1(\mathcal{H})\). This already establishes Part \((a)\) Now, for each step \(i\) of the while loop, HYPERTREE(\(\mathcal{H}\)) executes one of the following actions. Either it sets \(\mathcal{H}_i = \mathcal{H}_{i-1} \cup \{E\}\) for some \(E \in \mathcal{E}_1(\mathcal{H})\) (Case 1, cf. line 4), or it sets \(\mathcal{H}_i = \mathcal{H}_{i-1} \cup \mathcal{H}_F\), where \(\mathcal{H}_F\) is the output of the algorithm FLOWER(\(\mathcal{H}_{i-1}, \mathcal{H}, \sigma_{i-1}\)) (Case 2, cf. line 6). We remark that one of them must be executed because \(\mathcal{H}_{i-1} \not\subseteq \text{Crit}_{r,\ell}(G)\), as \(\lambda(G(\mathcal{H}_{i-1})) > -\varepsilon\) (see Lemma 3.3).

In either case, \(\mathcal{H}_{i-1} \subseteq \mathcal{H}_i \subseteq \mathcal{H}\) (cf. Lemma 5.2 for the second case). Similarly, it is easy to see from lines 4, 7 and 8 that \(D_{i-1} \subseteq D_i \subseteq \mathbb{N}\).

Let \(T\) be the number of iterations of the while loop in line 2 of HYPERTREE(\(\mathcal{H}\)). By the while loop condition and the increase of \(i\) by one in every iteration (see line 10), HYPERTREE(\(\mathcal{H}\)) must stop in at most \(\log n\) iterations. Moreover, the while loop guarantees that \(T\) is the smallest integer such that \(\lambda(G_T) \leq -\varepsilon\) or \(T \geq \log n\), where \(G_T = G(\mathcal{H}_T)\) and \(\varepsilon\) is the constant given by Lemma 3.3. This establishes Part \((c)\).

Since HYPERTREE(\(\mathcal{H}\)) should return \((\mathcal{H}_T, D_T)\) (see line 11), we also establish Part \((d)\). Now, it remains to show Part \((b)\). Let \(i\) be any step of the while loop in HYPERTREE(\(\mathcal{H}\)). In Case 1, the hyperedge \(E\) satisfies \(E \not\subseteq V(\mathcal{H}_{i-1})\) (cf. line 4). In Case 2, Lemma 5.2 implies that the output \(\mathcal{H}_F = \{C, P_1, \ldots, P_t\}\) given by FLOWER(\(\mathcal{H}_{i-1}, \mathcal{H}, \sigma_{i-1}\)) satisfies \(C \not\subseteq V(\mathcal{H}_{i-1})\). In both cases, \(v(\mathcal{H}_{i-1}) < v(\mathcal{H}_i)\). \(\square\)

6. The algorithm analysis

In this section, we prove Lemmas 3.5 and 3.6 and hence complete the proof of Theorem 2.1. The former states that in each step of the algorithm, \(\lambda(G_i)\) either decreases by an additive constant in a degenerate step, or that its value remains the same. The latter then states that the number of non-isomorphic structures that the algorithm can generate is not too big.

Recall the description of HYPERTREE. Throughout this section, let \(G_i\) denote the graph \(G(\mathcal{H}_i)\), where \(\mathcal{H}_i\) is the hypergraph generated in \(i\)-th step of HYPERTREE. For all \(1 \leq i \leq T\), the graph \(G_i\) is obtained from \(G_{i-1}\) by adding either an \(r\)-clique or an underlying graph of
a flower \( \{C, P_1, \ldots, P_t\} \) to it, depending whether HYPERTREE executes the if-clause in lines 3–4 or the else-clause in lines 5–8. In the latter case, we will analyse the change in \( \lambda \) by adding first \( C \), and then one petal (copy of \( K_r \)) at a time. Thus, it makes sense to pin down the effect of adding a copy of \( K_r \) to an arbitrary graph \( F \) first.

For two graphs \( F_1 \) and \( F_2 \), we denote by \( F_1 \cap F_2 \) the subgraph with vertex set \( V(F_1) \cap V(F_2) \) and edge set \( E(F_1) \cap E(F_2) \). The graph \( F_1 \cup F_2 \) is defined analogously. For any graph \( F \), recall that \( \lambda(F) = v(F) - e(F)/m_2(K_r, C_\ell) \). Then, we can write

\[
\lambda(F_1 \cup F_2) - \lambda(F_1) = v(F_1 \cup F_2) - v(F_1) - \frac{e(F_1 \cup F_2) - e(F_1)}{m_2(K_r, C_\ell)}
= v(F_2) - v(F_1 \cap F_2) - \frac{e(F_2) - e(F_1 \cap F_2)}{m_2(K_r, C_\ell)}. \tag{9}
\]

Now, define

\[
\beta_{r, \ell}(J) = r - v(J) - \frac{\binom{r}{2} - e(J)}{m_2(K_r, C_\ell)}. \tag{10}
\]

By \( 9 \), we have

\[
\lambda(F_1 \cup F_2) - \lambda(F_1) = \beta_{r, \ell}(J) \tag{11}
\]
in the case when \( F_2 \cong K_r \) and \( J = F_1 \cap F_2 \). Before stating the lemma which encompasses how \( \beta_{r, \ell}(J) \) behaves for various subgraphs \( J \subseteq K_r \), we need the following lemma which provides closed formulas for \( m_2(C_\ell) \), \( m_2(K_r) \) and \( m_2(K_r, C_\ell) \). We prove it in the appendix.

**Lemma 6.1.** Let \( r, \ell \geq 4 \) be integers. Then,

\[
m_2(C_\ell) = \frac{\ell - 1}{\ell - 2}, \quad m_2(K_r) = \frac{r + 1}{2}, \quad \text{and} \quad m_2(K_r, C_\ell) = \frac{\binom{r}{2}}{r - 2 + (\ell - 2)/(\ell - 1)}.
\]

In particular, \( r/2 < m_2(K_r, C_\ell) < m_2(K_r) \).

In our next lemma, we obtain upper bounds for \( \beta_{r, \ell}(J) \) for every subgraph \( J \not\subseteq K_r \) with at least two vertices.

**Lemma 6.2.** Let \( r, \ell \geq 4 \) be integers. Let \( J \not\subseteq K_r \) such that \( v(J) \geq 2 \). Then,

(a) \( \beta_{r, \ell}(J) < 0 \),

(b) \( \beta_{r, \ell}(K_2) = 1/m_2(K_r, C_\ell) - (\ell - 2)/(\ell - 1) > -1 \),

(c) \( \beta_{r, \ell}(J) \leq \beta_{r, \ell}(K_2) \) if \( d(v) = 1 \) for some \( v \in V(J) \). The equality holds if and only if \( J \cong K_2 \).

**Proof.** First, let us prove part (a). When \( J \not\subseteq K_r \) has \( r \) vertices, we can easily see from (10) that \( \beta_{r, \ell}(J) < 0 \). Thus, let us assume that \( 2 \leq v(J) < r \). Observe that \( \beta_{r, \ell}(J) < 0 \) if the following inequalities are satisfied:

\[
m_2(K_r, C_\ell) < m_2(K_r) \leq \frac{\binom{r}{2} - e(J)}{r - v(J)}. \tag{12}
\]
It remains to show that both inequalities in (12) are true. The first inequality follows from Lemma 6.1. As \( m_2(K_r) = \left( \binom{r}{2} - 1 \right) / (r - 2) \) by Lemma 6.1, the last inequality in (12) is equivalent to

\[
\frac{\binom{r}{2} - 1}{r - 2} \leq \frac{\binom{r}{2} - e(J)}{r - v(J)}.
\]

(13)

If \( v(J) = 2 \), then \( e(J) \leq 1 \) and hence (13) holds. If \( 3 \leq v(J) < r \), then the last inequality can be rearranged to

\[
\frac{e(J) - 1}{v(J) - 2} \leq \frac{\binom{r}{2} - 1}{r - 2}.
\]

As \( e(J) \leq \binom{v(J)}{2} \) and \( (\binom{r}{2} - 1) / (r - 2) = (r + 1) / 2 \), the last inequality holds whenever \( v(J) \leq r \). This establishes part (a).

To show part (c), first note that \( d(v) = 1 \) for some \( v \in V(J) \) if and only if \( K_2 \subseteq J \subseteq K_{r-1} \cdot K_2 \), where \( K_{r-1} \cdot K_2 \) denotes the graph obtained from \( K_{r-1} \) by adding a pendant edge. When \( J \cong K_2 \), the equality in (c) holds trivially. Thus, let us assume that \( v(J) \geq 3 \) and \( J \subseteq K_{r-1} \cdot K_2 \). In this case, the inequality \( \beta_{r,t}(J) < \beta_{r,t}(K_2) \) is equivalent to

\[
\frac{e(J) - 1}{v(J) - 2} < m_2(K_r, C_\ell).
\]

(14)

But, for any \( J \subseteq K_{r-1} \cdot K_2 \) such that \( v(J) \geq 3 \), we have

\[
\frac{e(J) - 1}{v(J) - 2} \leq m_2(K_{r-1} \cdot K_2) = \max \left\{ m_2(K_{r-1}), \frac{e(K_{r-1} \cdot K_2) - 1}{v(K_{r-1} \cdot K_2) - 2} \right\} = \frac{r}{2},
\]

by definition of \( m_2(\cdot) \) and the identity \( m_2(K_{r-1}) = r/2 \) (see Lemma 6.1). As \( m_2(K_r, C_\ell) > r/2 \) by Lemma 6.1, this finishes the proof of part (c).

For part (b), the identity \( \beta_{r,t}(K_2) = 1/m_2(K_r, C_\ell) - (\ell - 2)/(\ell - 1) \) follows readily from the definition of \( \beta_{r,t} \) in (10) and the identity for \( m_2(K_r, C_\ell) \) in Lemma 6.1. Finally, \( m_2(K_r, C_\ell) > 0 \) and \( (\ell - 2)/(\ell - 1) < 1 \) imply that \( \beta_{r,t}(K_2) > -1 \).

□

Now we are ready to prove Lemma 3.5.

Proof of Lemma 3.5. Suppose that \( \text{HYPERTREE}(\mathcal{H}) \) executes the if-statement in lines 3–4 in the \( i \)-th iteration of its while loop. Then, \( i \in D_i \) and hence \( i \in D_r \), by Lemma 3.4. Moreover, \( G_i = G_{i-1} \cup K \) for some \( K \cong K_r \) such that \( |V(G_{i-1}) \cap V(K)| \geq 2 \) and \( K \not\subseteq G_{i-1} \).

Observe that graph \( J = G_{i-1} \cap K \) satisfies the assumptions of Lemma 6.2 and hence, by (11), \( \lambda(G_i) - \lambda(G_{i-1}) = \beta_{r,t}(J) < 0 \).

Now, suppose that \( \text{HYPERTREE}(\mathcal{H}) \) executes the else-statement in lines 5–8 in the \( i \)-th iteration of its while loop. Let \( \mathcal{H}_F = \{ C \} \cup \{ P_e : e \in C \setminus E(G_{i-1}) \} \) be the flower returned by \( \text{FLOWER}(\mathcal{H}_{i-1}, \mathcal{H}, \sigma_{i-1}) \). Recall all the properties of \( \mathcal{H}_F \) given by Lemma 5.2. In order to bound the difference \( \lambda(G_i) - \lambda(G_{i-1}) \), we first analyse the increment \( \lambda(G_{i-1} \cup C) - \lambda(G_{i-1}) \).
Let $J_0$ be the graph $G_{i-1} \cap C$. By (9), we have

$$\lambda(G_{i-1} \cup C) - \lambda(G_{i-1}) = \ell - v(J_0) - \frac{\ell - e(J_0)}{m_2(K_r, C_\ell)} \leq \left( \frac{\ell - 2}{\ell - 1} - \frac{1}{m_2(K_r, C_\ell)} \right) \cdot |E(C) \setminus E(G_{i-1})|,$$  

(15)

where in the inequality we use that $v(J_0) \geq 2$ and $v(J_0) > e(J_0)$, as $K_2 \subseteq J_0 \subseteq C_\ell$ (see Lemma 5.2). Note that equality holds in (15) if and only if $J_0 \cong K_2$.

By Lemma 6.2(a), the contribution of each petal of $\mathcal{H}_F$ to $\lambda$ is negative. But, the contribution of $C$ to $\lambda$, which is bounded by (15), may be positive (and large). However, as we shall show, the contribution of $C$ to $\lambda$ is smaller or equal than the absolute value of the sum of all the contributions of each petal of $\mathcal{H}_F$ to $\lambda$. In order to prove this, we recursively find a subsequence of petals $(P_j)_{j=1}^t$ in $\mathcal{H}_F$ such that the intersection graph $P_j \cap (G_{i-1} \cup C \cup P_1 \cup \ldots \cup P_{j-1})$ has potentially many isolated vertices. These isolated vertices allow us to gain a sufficiently negative contribution to $\lambda$ from each petal in the sequence, and hence ‘beat’ the contribution given by the cycle in (15). This sequence of petals does not necessarily contain all the petals of $\mathcal{H}_F$, but this is not a problem. By Lemma 6.2(a) all the petals in $\mathcal{H}_F$ give a negative contribution to $\lambda$, and hence we may discard some petals from the analysis (and adding them later will not increase the value of $\lambda$).

We define this sequence of petals iteratively. Let $\{u_0, \ldots, u_{t-1}\}$ be a cyclic ordering of the vertices of $C$ such that $u_0u_{t-1}$ is an edge of $G_{i-1} \cap C$, which must exist by property (F3) of Lemma 5.2. By convention, define $u_{-1} = u_{t-1}$. Now, define $A_0 = E(C) \setminus E(G_{i-1})$ and construct a nested sequence of sets $(A_s)_{s \geq 0}$ in the following recursive way. For each $s \in \mathbb{N}$, let

$$m_s = \min \{m : u_mu_{m+1} \in A_{s-1}\}$$

and let $P_s = P_{u_m}u_{m+1}$ be the unique petal in $\mathcal{H}_F$ which covers the edge $u_mu_{m+1}$, meaning that $P_s \cap C = \{u_mu_{m+1}\}$. Then, set

$$A_s = A_{s-1} \setminus \{u_mu_{m+1} : u_mu_{m+1} \in V(P_s)\}.$$

Let $t$ be the smallest integer such that $A_t = \emptyset$, and note that $t \leq |A_0| = |E(C) \setminus E(G_{i-1})|$. For simplicity, denote $G_{i-1}^{(0)} = G_{i-1} \cup C$ and, more generally, for each $1 \leq s \leq t$, let

$$G_{i-1}^{(s)} = G_{i-1} \cup C \cup P_1 \cup \ldots \cup P_s.$$

Now, observe that

$$\lambda(G_i) - \lambda(G_{i-1}) \leq \lambda(G_{i-1}^{(0)}) - \lambda(G_{i-1}) + \sum_{s=1}^t \lambda(G_{i-1}^{(s)}) - \lambda(G_{i-1}^{(s-1)}).$$

(16)

Indeed, Lemma 6.2(a) together with (11) imply that we can discard the petals in $\{P_e : e \in E(C) \setminus E(G_{i-1})\}$ which do not belong to the chosen sequence $P_1, \ldots, P_t$. Moreover, equality
holds if and only if $P_1, \ldots, P_t$ are all the petals in the flower $\mathcal{H}_F$. To bound each increment in (16), we next analyse the structure of the graph $J_s := G_i^{(s-1)} \cap P_s$. Define
\[ I_s = \{ u_{m+1} \in V(J_s) \setminus \{ u_{m+1} \} : u_m u_{m+1} \in A_{s-1} \}. \]

**Claim 6.3.** The degree of $u_{m+1}$ in $J_s$ is 1 and $I_s$ is a set of isolated vertices in $J_s$.

**Proof.** Let $u_m$ be any vertex in $I_s \cup \{ u_{m+1} \}$ and $w$ be any vertex in $J_s$. We affirm that $u_m$ is adjacent to $w$ inside the graph $J_s$ if and only if $\{ u_m w \} = C \cap P_s$. Indeed, we cannot have $\{ u_m w \} \in E(G_i^{(s-1)})$, otherwise $P_s$ would be an $r$-clique which intersects $G_i^{(s-1)}$ in at least 2 vertices, contradicting Lemma 5.2 (F9). If $w \not= u_{m+1}$, we also cannot have $\{ u_m w \} \in E(P_1 \cup \cdots \cup P_{s-1})$, otherwise $u_{m-1} u_m \notin A_{s-1}$, and hence $u_m \notin I_s \cup \{ u_{m+1} \}$. As $u_m u_{m+1}$ is the only edge in $P_s \cap C$, it follows that $u_{m+1}$ is the only neighbour of $u_{m+1}$ in $J_s$, and that $u_m$ is isolated in $J_s$ for any $u_m \in I_s$.

Let $\tilde{J}_s$ be the subgraph of $J_s$ induced by the vertex set $V(J_s) \setminus I_s$. By the previous claim, we have $E(\tilde{J}_s) = E(J_s)$, which implies that $\beta_{r, \ell}(J_s) = \beta_{r, \ell}(\tilde{J}_s) - |I_s|$ (see (10)). By (11), we obtain
\[
\lambda(G_i^{(s)}) - \lambda(G_i^{(s-1)}) = \beta_{r, \ell}(J_s) = \beta_{r, \ell}(\tilde{J}_s) - |I_s| \leq \beta_{r, \ell}(K_2) - |I_s| \quad (17)
\]
for every $1 \leq s \leq t$. In the last inequality we use Lemma 6.2 (c) as $d(u_{m+1}) = 1$ by Claim 6.3. Moreover, by Lemma 6.2 (c) equality holds if and only if $\tilde{J}_s \cong K_2$. When $\tilde{J}_s \cong K_2$, note that we also have $|I_s| = 0$, as the only vertex $u_{m+1} \in V(J_s)$ such that $u_m u_{m+1} \in A_{s-1}$ is $u_{m+1} = u_{m+1}$.

Combining (15), (16) and (17), we have
\[
\lambda(G_i) - \lambda(G_i-1) \leq \left( \frac{\ell - 2}{\ell - 1} - \frac{1}{m_2(K_r, C_\ell)} \right) \cdot |E(C) \setminus E(G_i-1)| + t \beta_{r, \ell}(K_2) - \sum_{s=1}^{t} |I_s|. \quad (18)
\]
From the definitions of $A_s$ and $I_s$, it is easy to see that $\sum_s (|I_s|+1) = |A_0| = |E(C) \setminus E(G_i-1)|$. And, by Lemma 6.2 (b) we have $\beta_{r, \ell}(K_2) = m_2(K_r, C_\ell)^{-1} - (\ell - 2)/(\ell - 1)$. Then, (18) is equivalent to
\[
\lambda(G_i) - \lambda(G_i-1) \leq (\beta_{r, \ell}(K_2) + 1) \cdot (t - |A_0|). \quad (19)
\]
By Lemma 6.2, $\beta_{r, \ell}(K_2) > -1$ and, as we have $t \leq |A_0|$, it follows that
\[
(\beta_{r, \ell}(K_2) + 1) \cdot (t - |A_0|) \leq 0. \quad (20)
\]
Clearly, equality in (20) holds if and only if $t = |A_0|$. We conclude that $\lambda(G_i) - \lambda(G_i-1) \leq 0$ in the case when we add the flower $\{ C \} \cup \{ P_e : e \in C \setminus E(G_i-1) \}$.

Observe that $\lambda(G_i) - \lambda(G_i-1) = 0$ if and only if we have equalities in (15) - (20). This means that we must have $C \cap G_i-1 \cong K_2$ (and hence $|A_0| = \ell - 1$), $t = |A_0|$ and
\[ P_s \cap (G_i-1 \cup C \cup P_1 \cup \cdots \cup P_{s-1}) \cong K_2, \]
for each $1 \leq s \leq t$. As $e \in E(P_s \cap C)$, we infer that none of the $\ell - 1$ petals intersect outside the cycle $C$ and that the only petals sharing a vertex are consecutive petals, which share exactly one vertex. This happens if and only if $|V(G_i) \setminus V(G_{i-1})| = (r - 1)(\ell - 1) - 1$, whence $i$ is not added to $D_i$ (cf. line 7 of HYPERTREE), and then $i \notin D_T$. This proves (1).

The existence of $\delta = \delta(r, \ell)$ for (2) readily follows by noting that there are only $C = C(r, \ell)$ non-isomorphic configurations of such flowers and cliques (and how they intersect with $G_{i-1}$). This finishes the proof of the lemma.

It remains to prove Lemma 3.6 which bounds the number of non-isomorphic underlying graphs of hypergraphs that HYPERTREE may output. In principle, $|\text{Out}_{r,\ell}(n)|$ could be very large, but this is avoided with the help of the canonical labelling function (recall its definition given before the description of HYPERTREE). For each $t = 1, \ldots, \lceil \log n \rceil$, these vertex labellings assist the construction of $G(\mathcal{H}_t)$ from $G(\mathcal{H}_{t-1})$, and for all but a constant number of steps, we will see that this construction is unique. That is, it does not depend on the input, it depends only on $G(\mathcal{H}_{t-1})$ and $\sigma_{t-1}$. This is the reason why we get at most a polylogarithmic bound on the number of outputs.

In order to bound $|\text{Out}_{r,\ell}(n)|$, we first bound how many pairs $(G(\mathcal{H}_t), \sigma_t)$ HYPERTREE can produce in step $t$, for all $t = 1, \ldots, \lceil \log n \rceil$. To do so, we need to recall and define some notation. For a $*$-critical hypergraph $\mathcal{H}$, let $T(\mathcal{H})$ be the stopping time of HYPERTREE$(\mathcal{H})$. For any $t \geq 0$ and any $*$-critical hypergraph $\mathcal{H}$ such that $T(\mathcal{H}) \geq t$, let $\mathcal{H}_t(\mathcal{H})$ be the hypergraph obtained in step $t$ of HYPERTREE$(\mathcal{H})$. Let $D_t(\mathcal{H})$ be the accompanying set and $\sigma_t(\mathcal{H}) : V(\mathcal{H}_t) \to \mathbb{N}$ be the canonical vertex labelling of $\mathcal{H}_t$, and hence of $E(G(\mathcal{H}_t))$. For each $n \in \mathbb{N}$, define $\text{Crit}_{r,\ell}(n) = \bigcup_{t=0}^{\lceil \log n \rceil} \text{Crit}_{r,\ell}^{(2)}(G)$. Finally, for each $t, n \in \mathbb{N}$ and each set $D \subseteq \{1, \ldots, t\}$, define

$$G(t, D, n) = \bigcup \{ (G(\mathcal{H}_t), \sigma_t) : \mathcal{H}_t = \mathcal{H}_t(\mathcal{H}), \sigma_t = \sigma_t(\mathcal{H}) \},$$

where the union is over all $\mathcal{H} \in \text{Crit}_{r,\ell}(n)$ such that $D_t(\mathcal{H}) = D$ and $T(\mathcal{H}) \geq t$.

Our next lemma gives an upper bound on the size of $G(t, D, n)$.

**Lemma 6.4.** For all $r, \ell \geq 4$ there exists $C > 0$ such that $|G(t, D, n)| \leq t^C |D|$, for all $t, n \in \mathbb{N}$ and $D \in \{1, \ldots, t\}$.

**Proof.** To simplify notation, set $g(t, D, n) := |G(t, D, n)|$. First, note that $G(0, \emptyset, n)$ contains only one pair, and hence $g(0, \emptyset, n) = 1$. In fact, for every $*$-critical hypergraph $\mathcal{H}$, $\mathcal{H}_0(\mathcal{H})$ consists of a single hyperedge of type 1 (cf. Lemma 3.4(a)) and $\sigma_0$ is a fixed labelling of $V(\mathcal{H}_0)$.

Now, we claim that for each $t \geq 1$ and each $D \subseteq \{1, \ldots, t\}$, we have

$$g(t, D, n) \leq \begin{cases} g(t-1, D, n) & \text{if } t \notin D; \\ g(t-1, D \setminus \{t\}, n) \cdot (4tt^2)^{t^2} & \text{if } t \in D. \end{cases} \quad (21)$$

First, assume that $t \notin D$ and let $(G(\mathcal{H}_t), \sigma_t) \in G(t, D, n)$. Let $\mathcal{H}$ be any hypergraph in $\text{Crit}_{r,\ell}(n)$ such that $\mathcal{H}_t(\mathcal{H}) = \mathcal{H}_t$, $\sigma_t(\mathcal{H}) = \sigma_t$ and $D_t(\mathcal{H}) = D$. Note that $t \notin D$ implies that
\(D_t(\mathcal{H}) = D_{t-1}(\mathcal{H})\). And this happens if and only if \(\mathcal{H}_t = \mathcal{H}_{t-1} \cup \mathcal{H}_F\), for some flower \(\mathcal{H}_F\) such that

\[
|V(G(\mathcal{H}_t)) \setminus V(G(\mathcal{H}_{t-1}))| = (r-1)(\ell-1) - 1,
\]

see line 7 of HYPTREE. For (22) to hold, observe that \(G(\mathcal{H}_F)\) must intersect \(G(\mathcal{H}_{t-1})\) in exactly one edge, which is given by line 1 of FLOWER(\(\mathcal{H}_{t-1}, \mathcal{H}, \sigma_{t-1}\)). Once we have this edge, called \(e_0\), we can see that FLOWER generates only one type of flower \(\mathcal{H}_F\) such that (22) holds and \(G(\mathcal{H}_F) \cap G(\mathcal{H}_{t-1})\) is equal to \(e_0\). Moreover, by Lemma 5.1, \(e_0\) only depends on \(G(\mathcal{H}_{t-1})\) and the canonical labelling \(\sigma_{t-1}\). Therefore, for any other hypergraph \(\mathcal{H}'\) such that \(G(\mathcal{H}_{t-1}(\mathcal{H}')) = G(\mathcal{H}_{t-1}(\mathcal{H}))\), \(\sigma_{t-1}(\mathcal{H}') = \sigma_{t-1}\) and \(t \notin D_t(\mathcal{H}')\), the flower \(\mathcal{H}'_F\) given by FLOWER(\(\mathcal{H}_{t-1}(\mathcal{H}'), \mathcal{H}', \sigma_{t-1}\)) also satisfies \(V(\mathcal{H}') \cap V(\mathcal{H}_{t-1}) = \{e_0\}\) for the same \(e_0\). As \(\sigma_t\) is the canonical labelling extending \(\sigma_{t-1}\) to \(\mathcal{H}_t\), which is uniquely determined by \(G(\mathcal{H}_{t-1})\) and the unlabelled graph \(G(\mathcal{H}_F)\), we also have \(\sigma_i(\mathcal{H}') = \sigma_t\). This implies that there is an injection \(\mathcal{G}(t, D, n) \rightarrow \mathcal{G}(t-1, D, n)\) mapping \((\mathcal{H}_t, \sigma_t)\) to \((\mathcal{H}_{t-1}, \sigma_{t-1})\), and hence \(g(t, D, n) \leq g(t-1, D, n)\). This proves the first inequality in (21).

To show the second inequality, note that in step \(t\) of HYPTREE(\(\mathcal{H}\)) one of the following holds: (1) The algorithm has stopped; (2) \(\mathcal{H}_t = \mathcal{H}_{t-1} \cup \{E\}\), for some \(E \in \mathcal{E}_1(\mathcal{H})\); or (3) \(\mathcal{H}_t = \mathcal{H}_{t-1} \cup \{\mathcal{H}_F\}\), for some flower \(\mathcal{H}_F \subseteq \mathcal{H}\). Let \(H = G(E)\) or \(H = G(\mathcal{H}_F)\) be the underlying graph of the hyperedges that are added. Note that \(v(H) \leq \ell r^2\) and, as the number of vertices in \(G(\mathcal{H}_t)\) is at most \(\ell r^2\), there are at most \((\ell r^2)^2 \cdot 2\ell r^2\) ways to choose the subgraph \(H \cap G(\mathcal{H}_{t-1})\). Once this subgraph is fixed, there are at most \(2\ell r^2\) ways to choose the edges of \(H \setminus G(\mathcal{H}_{t-1})\) in \(G(\mathcal{H}_t)\). This implies that the graph \(G(\mathcal{H}_t)\) may be obtained from \(G(\mathcal{H}_{t-1})\) by at most \((4t\ell r^2)^2\ell r^2\) ways, and hence \(g(t, D) \leq g(t-1, D) \cdot (4t\ell r^2)^2\ell r^2\). As \(g(0, \emptyset, n) = 1\), we establish our lemma with \(C = (\ell r^2)^4\) by iterating the inequalities in (21).

Now, we are ready to prove Lemma 3.6

Proof of Lemma 3.6. We first claim that there exists a constant \(C_1 = C_1(r, \ell) > 0\) such that \(|D_T(\mathcal{H})| \leq C_1\) for all \(\mathcal{H} \in \text{Crit}_{r, \ell}(n)\). Recall that \(T = T(\mathcal{H})\) denotes the stopping time of HYPTREE(\(\mathcal{H}\)). Fix any hypergraph \(\mathcal{H} \in \text{Crit}_{r, \ell}(n)\) and let \(G_i = G(\mathcal{H}_i)\) for \(i = 0, \ldots, T\).

By Lemma 3.5, we have \(\lambda(G_i) \leq \lambda(G_{i-1}) - \delta\) if \(i \in D_T(\mathcal{H})\), and \(\lambda(G_i) = \lambda(G_{i-1})\) if \(i \notin D_T(\mathcal{H})\), where \(\delta = \delta(r, \ell) > 0\). As \(\lambda(G_0) = \lambda(K_r)\) (by Lemma 3.4(a)) and \(\lambda(G_{T(\mathcal{H}) - 1}) > -\varepsilon\) (by Lemma 3.4(c)), it follows that \(|D_T(\mathcal{H})| \leq 1 + (\lambda(K_r) - \varepsilon)/\delta\). As \(\varepsilon\) only depends on \(r\) and \(\ell\), this proves our claim.

By Lemma 3.4(c), the stopping time \(T\) is bounded from above by \(\log n\). Since \(|D_T| \leq C_1\), the size of \(\text{Out}_{r, \ell}(n)\) is bounded by the size of

\[
\bigcup_{t \leq \log n} \bigcup_{|D| \leq C_1} \mathcal{G}(t, D, n),
\]
where $G(t, D, n)$ was defined just above Lemma 6.4 Using the bound on $|G(t, D, n)|$ given by Lemma 6.4, we conclude that

$$|\text{Out}_{r, \ell}(n)| \leq \sum_{t=1}^{[\log n]} \sum_{D \subseteq [t] : |D| \leq C_1} t^{M_1 |D|} \leq (\log n)^{C_0},$$

for some $C_0 = C_0(r, \ell) > 0$. □

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Appendix A. Proof of Lemma 6.1

Note that every subgraph $J \subseteq C_\ell$ is a forest, and so we have $e(J) \leq v(J) - 1$. Thus, for every $J \subseteq C_\ell$ with $v(J) \geq 3$ this implies that $(e(J) - 1)/(v(J) - 2) \leq 1$. On the other hand,

$$\frac{e(C_\ell) - 1}{v(C_\ell) - 2} = \frac{\ell - 1}{\ell - 2} > 1,$$

which implies $m_2(C_\ell) = (\ell - 1)/(\ell - 2)$. Now, let us analyse subgraphs of $K_r$. For each $J \subseteq K_r$, we have $e(J) \leq \binom{v(J)}{2}$. Thus,

$$\frac{e(J) - 1}{v(J) - 2} \leq \frac{\binom{v(J)}{2} - 1}{v(J) - 2} = \frac{v(J) + 1}{2},$$

for each $J \subseteq K_r$ such that $v(J) \geq 3$. It follows that $m_2(K_r) = (r + 1)/2$. Next, for each $\ell \geq 3$, consider the function $f_\ell : \mathbb{N} \to \mathbb{Q}$ defined by

$$f_\ell(t) = \frac{\binom{t}{2}}{t - 2 + m_2(C_\ell)^{-1}}.$$

It is not hard to check that $(f_\ell(t))_{t \geq 3}$ is monotone increasing (for every given $\ell$). Since $m_2(C_\ell) = (\ell - 1)/(\ell - 2)$, we have

$$m_2(K_r, C_\ell) = f_\ell(r) = \frac{\binom{r}{2}}{r - 2 + (\ell - 2)/(\ell - 1)}. \quad (23)$$

It follows readily from this identity that $m_2(K_r, C_\ell)$ is strictly decreasing in $\ell$, and thus,

$$m_2(K_r, C_\ell) \leq m_2(K_r, C_3) = \frac{r(r - 1)}{2r - 3} < \frac{r + 1}{2} = m_2(K_r) \quad (24)$$

for every $r \geq 4$. Finally, the identity in (23) implies that

$$m_2(K_r, C_\ell) = \frac{\binom{r}{2}(\ell - 1)}{(r - 1)(\ell - 1) - 1} = \frac{r}{2} \cdot \frac{1}{1 - \frac{1}{(r-1)(\ell-1)}} > \frac{r}{2}.$$

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