Test-Particle Motion in the Nonsymmetric Gravitational Theory

J. Légaré and J. W. Moffat
Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7
(May 30, 1995)

We present a derivation of the equation of motion for a test-particle in the framework of the non-symmetric gravitational theory. Three possible couplings of the test-particle to the non-symmetric gravitational field are explored. The equation of motion is found to be similar in form to the standard geodesic equation of general relativity, but with an extra antisymmetric force term present. The equation of motion is studied for the case of a static, spherically symmetric source, where the extra force term is found to take the form of a Yukawa force.

I. INTRODUCTION

We consider here the problem of test-particle motion in the weak-field limit of the nonsymmetric gravitational theory (NGT) (see [1] for a description of the structure of the NGT). The problem is approached from the point of view of Lagrangian mechanics: a scalar Lagrangian that couples particle quantities to the components of the gravitational field is postulated, and the Euler-Lagrange equation is used to determine the equation of motion.

In general relativity (GR), the scalar Lagrangian of choice describing the coupling of particle quantities to the gravitational field is well-known:

\[ L_{GR} = \left( g_{\mu \nu} \dot{\xi}^\mu \dot{\xi}^\nu \right)^{1/2}. \]

However, the metric in GR is a symmetric tensor \( g_{\mu \nu} = g(\mu \nu) \). In the NGT, this is no longer the case, as the metric picks up a skew component. A problem therefore arises: if we describe the coupling of the test-particle to the NGT gravitational field simply by \( L_{GR} \), particle quantities will not couple to the skew components of the metric. This leads to a more serious problem: if particles do not couple to the skew components of the metric, how did the skew components come to be?

The program therefore consists in finding a Lagrangian which describes the coupling of the particle quantities to both the symmetric (GR) components of the gravitational field, as well as to the nonsymmetric (NGT) elements of this field. We present three possible couplings herein. In formulating these Lagrangians, we have restricted ourselves to couplings that are linear in the velocities.

II. TEST-PARTICLE MOTION

As in GR, spacetime in the NGT is described by a four dimensional manifold \( M \) and a metric \( g_{\mu \nu} \). However unlike GR, the NGT field equations not only determine the symmetric components \( g_{\mu \nu} = s_{\mu \nu} \) of the metric, but also the antisymmetric components \( g_{[\mu \nu]} = a_{\mu \nu} \). Let \( s_{\mu \nu} \) be defined by \( s_{\mu \nu} s_{\nu \beta} = \delta^\mu_\beta \). Note that \( s_{\mu \nu} \neq g(\mu \nu) \), and similarly, \( a_{\mu \nu} \neq g([\mu \nu]) \). The inverse of the full metric is defined by \( g^{\alpha \mu} g_{\alpha \nu} = g^{\mu \nu} g_{\nu \alpha} = \delta^\mu_\alpha \).

The equation of particle motion is to be derived from a scalar Lagrangian \( L \). This Lagrangian will be decomposed into two pieces: \( L = L_{GR} + L_{NGT} \), where

\[ L_{GR} = (s_{\mu \nu} \dot{\xi}^\mu \dot{\xi}^\nu)^{1/2}. \]

Here, \( \sigma \) is an affine parameter, while \( \xi^\mu(\sigma) \) and \( \dot{\xi}^\mu(\sigma) = d\xi^\mu/d\sigma \) are the particle’s path and four-velocity (with respect to the affine parameter \( \sigma \)), respectively. We will take \( L_{NGT} \) to have the form \( L_{NGT} = (1/2) \lambda A_\mu \dot{\xi}^\mu \), where \( A_\mu \) is a covector independent of the particle velocity \( \dot{\xi}^\mu \), and \( \lambda \) is a coupling constant. The Euler-Lagrange equation of particle mechanics

\[ \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{\xi}^\alpha} - \frac{\partial L}{\partial \xi^\alpha} = 0 \]

allows us to conclude that the contribution of \( L_{NGT} \) to the equation of motion will be of the form

\[ \frac{d}{d\sigma} \frac{\partial L_{NGT}}{\partial \dot{\xi}^\alpha} - \frac{\partial L_{NGT}}{\partial \xi^\alpha} = \frac{\lambda}{2} \left( \frac{dA_\alpha}{d\sigma} - \dot{\xi}^\mu \frac{\partial A_\mu}{\partial \xi^\alpha} \right) = \lambda \dot{\xi}^\mu \frac{\partial A_\mu}{\partial \xi^\alpha} = -\lambda f_{[\alpha \mu]} \xi^\mu, \]
We use the notation $T$ of motion. Inserting (1) and (4) into (2) and using (3) gives

$$L_1 = \frac{1}{2} C_1 \lambda F^\eta \dot{\xi}_\eta = \frac{1}{2} C_1 \lambda \epsilon^{\mu\nu\lambda\eta} F_{\mu\nu\lambda} s_{\eta\alpha} \dot{\xi}^\alpha$$  \hspace{1cm} (4a)

$$L_2 = \frac{1}{2} C_2 \lambda g[^{123} \tau \nu] F_{[\mu\nu\lambda]} \dot{\xi}^\lambda$$  \hspace{1cm} (4b)

$$L_3 = \frac{C_3 \lambda}{2} \frac{g[^{123} \nu \mu]}{\sqrt{-g}} s_{\lambda\mu} \dot{\xi}^\nu.$$  \hspace{1cm} (4c)

The field-strength tensor $F_{[\mu\nu\lambda]}$ is defined by

$$F_{[\mu\nu\lambda]} = \partial[\lambda g_{\mu\nu}] = \frac{1}{2} (\partial_{\lambda} a_{\mu\nu} + \partial_{\mu} a_{\nu\lambda} + \partial_{\nu} a_{\lambda\mu}).$$  \hspace{1cm} (5)

The constant $\lambda$ couples the test-particle to the NGT skew field, and has dimensions of a length. The constants $C_i$ ($i = 1, 2, 3$) are dimensionless constants measuring the relative strengths of the three interactions. The symbol $\epsilon^{\mu\nu\lambda\eta}$ is the fully antisymmetric Levi-Civita tensor density, defined by

$$\epsilon^{\mu\nu\lambda\eta} = \begin{cases} +1 & \text{if } \mu\nu\lambda\eta \text{ is an even permutation of 1230}, \\ -1 & \text{if } \mu\nu\lambda\eta \text{ is an odd permutation of 1230}, \\ 0 & \text{otherwise.} \end{cases}$$

We use the notation $T_{\mu\nu} = \sqrt{-g} T_{\mu\nu}$, for some tensor $T_{\mu\nu}$. All indices are lowered with $s_{\mu\nu}$ and raised with $s^{\mu\nu}$.

We are now in a position to use the covariant Euler-Lagrange equation of particle mechanics to derive the equation of motion. Inserting (1) and (2) into (3) using (3) gives

$$0 = \frac{d\dot{\xi}^\beta}{d\sigma} + s^{\beta\alpha} \left( \dot{\xi}^\nu \frac{ds_{\alpha\nu}}{d\sigma} - \frac{1}{2} s_{\mu\nu,\alpha} \dot{\xi}^\mu \dot{\xi}^\nu \right) - \kappa \lambda s^{\beta\alpha} f_{[\alpha\mu]} \dot{\xi}^\mu,$$  \hspace{1cm} (6)

where we have contracted with $s^{\beta\alpha}$ and where $\kappa^2 = s_{\mu\nu} \dot{\xi}^\mu \dot{\xi}^\nu$. We have assumed that $\kappa$ is a constant; this assumption will be justified in the next section. $f_{[\alpha\mu]}$ is given by

$$f_{[\alpha\mu]} = C_1 \partial_{[\alpha} \left( \frac{\epsilon^{\rho\sigma\nu\lambda]}{\sqrt{-g}} F_{\sigma\nu\lambda\rho]} s_{\mu\eta} \right) + C_2 \partial_{[\alpha} \left( g[^{\rho\nu}] F_{[\eta\nu\mu]} \right) + C_3 \partial_{[\alpha} \left( \frac{g[^{\rho\nu}]}{\sqrt{-g}} s_{\mu\rho\eta} \right).$$  \hspace{1cm} (7)

If we define the symmetric Christoffel symbols by

$$\left\{ \beta \atop \mu\nu \right\} = \frac{1}{2} s^{\beta\alpha \gamma} (s_{\alpha\nu,\mu} + s_{\mu\alpha,\nu} - s_{\mu\nu,\alpha}),$$  \hspace{1cm} (8)

we can rewrite (3) as

$$\frac{d^2 \dot{\xi}^\beta}{d\sigma^2} + \left\{ \beta \atop \mu\nu \right\} \frac{d\dot{\xi}^\mu}{d\sigma} \frac{d\dot{\xi}^\nu}{d\sigma} = \kappa \lambda s^{\beta\alpha} f_{[\alpha\mu]} \frac{d\dot{\xi}^\mu}{d\sigma}.$$  \hspace{1cm} (9)

### III. Conservation Laws

Contracting (3) with $s_{\alpha\beta} X^\alpha$ where $X^\alpha$ is some vector, gives

$$\frac{d}{d\sigma} (s_{\alpha\beta} X^\alpha \dot{\xi}^\beta) = \frac{d}{d\sigma} (X_B \dot{\xi}^\beta) = \frac{1}{2} \dot{\xi}^\nu X^\alpha \partial_\alpha s_{\mu\nu} + s_{\alpha\mu} \dot{\xi}^\nu \partial_\nu X^\alpha + \kappa \lambda f_{[\alpha\mu]} \dot{\xi}^\mu X^\alpha.$$  \hspace{1cm}

However, the Lie derivative of the symmetric part of the metric is given by

$$\mathcal{L}_X [s]_{\mu\nu} = X^\alpha \partial_\alpha s_{\mu\nu} + s_{\alpha\mu} \partial_\nu X^\alpha + s_{\alpha\nu} \partial_\mu X^\alpha.$$
Therefore,
\[ \frac{d(X_\beta \dot{\xi}^\beta)}{d\sigma} = \frac{1}{2} \xi^{\mu} \xi^{\nu} \mathcal{L}_{[s]} \mu\nu + \kappa \lambda f_{[\alpha\mu]} \dot{\xi}^\mu X^\alpha. \] (10)

If in (10) we take \( X^\alpha \) to be the components of a Killing vector, we have that
\[ \frac{d(X_\beta \dot{\xi}^\beta)}{d\sigma} = \kappa \lambda f_{[\alpha\mu]} \dot{\xi}^\mu X^\alpha. \]

If the right-hand side of this equation vanishes, we have the result familiar from GR that \( X_\beta \dot{\xi}^\beta \) is a constant of the motion. For instance, it will be shown below that the only non-vanishing component of \( f_{[\alpha\mu]} \) for a static, spherically-symmetric system is \( f_{[rt]} \). Since \( X = \partial / \partial \phi \) is a Killing vector of a spherically-symmetric system, we find that \( s_{\phi\phi} \dot{\xi}^{\phi} = r^2 \dot{\phi} \sin \theta \) is a constant of the motion, corresponding to the conservation of angular momentum.

This is not a generic feature: in a static, spherically-symmetric system, \( X = \partial / \partial t \) is also a Killing vector, but since \( f_{[rt]} \neq 0 \), then \( s_{\phi\phi} \dot{\xi}^{\phi} = \gamma t \) is not a conserved quantity. The nature of \( \gamma t \) in a static, spherically-symmetric system will be explored further in the next section, where it will be shown that \( \gamma t \) is one part of a conserved quantity.

Setting \( X^\alpha = \dot{\xi}^\alpha \) in (10) leads to
\[ \frac{d}{d\sigma} (s_{\alpha\beta} \dot{\xi}^\alpha \dot{\xi}^\beta) = \frac{1}{2} \xi^{\mu} \xi^{\nu} \mathcal{L}_{[s]} \mu\nu + \kappa \lambda f_{[\alpha\mu]} \dot{\xi}^\mu \dot{\xi}^\alpha. \]

Now,
\[ \mathcal{L}_{[s]} \mu\nu \dot{\xi}^\mu \dot{\xi}^\nu = \dot{\xi}^\mu \dot{\xi}^\nu \frac{ds_{\mu\nu}}{d\sigma} + 2s_{\mu\nu} \dot{\xi}^\mu \frac{d\dot{\xi}^\mu}{d\sigma} = 2\kappa \lambda f_{[\mu\nu]} \dot{\xi}^\mu \dot{\xi}^\nu \equiv 0. \]

Evidently, the last term in (10) vanishes identically, leaving
\[ \frac{d}{d\sigma} (s_{\alpha\beta} \dot{\xi}^\alpha \dot{\xi}^\beta) = \frac{d\kappa^2}{d\sigma} = 0. \] (11)

This justifies our assumption that \( \kappa \) is a constant. In fact, \( \kappa \) is a first-integral of the motion, corresponding physically to the constancy of the mass of the test-particle.

In GR, massive particles are usually assumed to have \( \kappa^2 = s_{\mu\nu} \dot{\xi}^\mu \dot{\xi}^\nu \neq 0 \). Indeed, it is generally assumed that \( \kappa^2 = 1 \), which defines the affine parameter as the proper time: \( d\tau^2 = \kappa^2 d\sigma^2 \). Then, the equation of motion becomes
\[ \frac{d^2 \xi^\beta}{d\tau^2} + \left\{ \frac{\beta}{(\mu\nu)} \right\} \frac{d\xi^\mu}{d\tau} \frac{d\xi^\nu}{d\tau} = \lambda s^{\beta\alpha} f_{[\alpha\mu]} \frac{d\xi^\mu}{d\tau}. \] (12)

On the other hand, massless particles (such as photons) are usually taken to have \( \kappa^2 = 0 \). Since the mass of the particle does not enter into (10), we might be tempted to postulate that this is the correct equation of motion for a massless particle. However, for such a massless particle, the right hand side of (10) vanishes, since \( \kappa^2 = 0 \). The equation of motion for a massless particle in the NGT is therefore written:
\[ \frac{d^2 \xi^\beta}{d\sigma^2} + \left\{ \frac{\beta}{(\mu\nu)} \right\} \frac{d\xi^\mu}{d\sigma} \frac{d\xi^\nu}{d\sigma} = 0. \] (13)

This is, of course, the geodesic equation of GR. We therefore conclude that massless test-particles in this model do not couple directly to the antisymmetric components of the NGT gravitational field, but rather follow geodesics of the symmetric background \( s_{\mu\nu} \). This is equivalent to the statement that massless particles have no structure.

**IV. MASSIVE TEST-PARTICLE MOTION IN A STATIC SPHERICALLY SYMMETRIC FIELD**

We now consider the motion of a test-particle in the gravitational field of a static, spherically symmetric source. We will restrict ourselves to the case of massive test-particles. The coordinates are \( \xi^1 = r, \xi^2 = \theta, \xi^3 = \phi \) and \( \xi^0 = t \). For the case of “non-magnetic” NGT considered here, the spherically-symmetric metric takes the form
\[ g_{\mu\nu} = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & f \sin \theta \\ 0 & 0 & -f \sin \theta & -\beta \sin^2 \theta \end{bmatrix}. \] (14a)

Here \( \alpha, \beta, \gamma, \) and \( f \) are functions of \( \xi^1 = r \) only. The inverse metric is

\[ g^{\mu\nu} = \begin{bmatrix} 1 / \gamma & 0 & 0 & 0 \\ 0 & -1 / \alpha & 0 & 0 \\ 0 & 0 & -\beta / (\beta^2 + f^2) & f \csc \theta / (\beta^2 + f^2) \\ 0 & 0 & -f \csc \theta / (\beta^2 + f^2) & -\beta \csc^2 \theta / (\beta^2 + f^2) \end{bmatrix}. \] (14b)

Since \( s_{\mu\nu} \) is a diagonal matrix, it is a simple matter to find its inverse

\[ s^{\mu\nu} = \begin{bmatrix} 1 / \gamma & 0 & 0 & 0 \\ 0 & -1 / \alpha & 0 & 0 \\ 0 & 0 & -1 / \beta & 0 \\ 0 & 0 & 0 & -1 / \beta \sin^2 \theta \end{bmatrix}. \] (14c)

This satisfies \( s^{\mu\nu} s_{\mu\alpha} = \delta^\alpha_\nu \). The function \( \beta \) is assumed to be \( \beta = r^2 \). In cases where \( M \ll r \), the metric components \( \gamma \) and \( \alpha \) take on the Schwarzschild form:

\[ \gamma = \frac{1}{\alpha} = 1 - \frac{2M}{r}, \] (15)

so that \( \alpha \gamma = 1 \). As we limit ourselves to a study of test-particle motion in weak-fields, we will assume that it is always true that \( \alpha \) and \( \gamma \) take on the Schwarzschild form.

Note that the only independent, non-zero component of \( a_{\mu\nu} \) is \( a_{23} = f \sin \theta \). In the case of “magnetic” NGT, there is also a contribution from \( a_{10} = w \), where \( w \) is a function of \( r \) only. It can be shown that when expanding about a spherically symmetric GR background, the only solution for magnetic NGT that yields asymptotically flat space is \( w = 0 \).

From the definition of the NGT Christoffel symbols, it is found that

\[ \left\{ \begin{array}{l} r \\ \phi \end{array} \right\}_{(rr)} = \frac{\alpha'}{2\alpha} = -\frac{M}{r^2} \left( \frac{1}{1 - 2M/r} \right) \] (16a)

\[ \left\{ \begin{array}{l} r \\ \phi \end{array} \right\}_{(\phi\phi)} = \sin^2 \theta \left\{ \begin{array}{l} r \\ \phi \end{array} \right\}_{(\theta\theta)} = -\frac{r \sin^2 \theta}{\alpha} = -r \sin^2 \theta \left( 1 - \frac{2M}{r} \right) \] (16b)

\[ \left\{ \begin{array}{l} r \\ (\phi) \end{array} \right\}_{(rr)} = \frac{\gamma'}{2\gamma} = \frac{M}{r^2} \left( 1 - \frac{2M}{r} \right) \] (16c)

\[ \left\{ \theta \\ (r\phi) \right\}_{(rr)} = \frac{1}{r} \] (16d)

\[ \left\{ \theta \\ (\phi\phi) \right\}_{(\phi\phi)} = -\sin \theta \cos \theta \] (16e)

\[ \left\{ \phi \\ (r\phi) \right\}_{(\phi\phi)} = \frac{\cos \theta}{\sin \theta} \] (16f)

\[ \left\{ t \\ (rt) \right\}_{(rt)} = \frac{\gamma'}{2\gamma} = \frac{M}{r^2} \left( 1 - \frac{2M}{r} \right) \] (16g)

A prime denotes differentiation with respect to \( r \). We have assumed that \( M \ll r \), and hence limited the expressions for the Christoffel symbols to their Schwarzschild form.

The determinant of the metric is given by

\[ g = \det(g_{\mu\nu}) = -(r^4 + f^2) \sin^2 \theta. \] (17)

In the static, spherically symmetric field, the skew field-strength tensor \( F_{[\mu\nu\lambda]} \) has only one independent, non-zero component,

\[ F_{[\theta\phi r]} = \frac{1}{3} \partial_r a_{\theta\phi} = \frac{1}{3} f' \sin \theta. \]
On the other hand, $g^{(\nu \nu)}$ vanishes:

$$g^{(\theta \phi \phi)} = \frac{\partial}{\partial \phi} \left( \frac{f}{\sqrt{r^4 + f^2}} \right) \equiv 0.$$ 

From (7), it follows that the skew force $f_{(\alpha \sigma)}$ also has one independent component:

$$f_{(rt)} = \frac{d}{dr} \left( \frac{C_1 \gamma f'}{\sqrt{r^4 + f^2}} \right).$$

For convenience, we will set $C_1 = 1$ and $C_2 = C_3 = 0$, as these terms will not contribute to the motion for the case of static, spherical symmetry.

In section III, it was found that $J = r^2 \dot{\phi} \sin \theta$ was a constant of the motion, corresponding to the conservation of angular momentum per unit rest mass. Using (16) and (17) in (12) yields the equations of motion for a test-particle in the field of a static, spherically symmetric source:

$$0 = \frac{d^2 r}{d \tau^2} + \frac{\alpha'}{2 \alpha} \left( \frac{dr}{d \tau} \right)^2 - \frac{r \sin^2 \theta}{\alpha} \left( \frac{d \phi}{d \tau} \right)^2 - \frac{r}{\alpha} \left( \frac{d \theta}{d \tau} \right)^2 + \frac{\gamma'}{2 \alpha} \left( \frac{dt}{d \tau} \right)^2 \tag{18a}$$

$$0 = \frac{d^2 \theta}{d \tau^2} + \frac{2 \dot{\theta} dr}{r d \tau} - \sin \theta \cos \theta \left( \frac{d \phi}{d \tau} \right)^2 \tag{18b}$$

$$0 = \frac{d^2 t}{d \tau^2} + \frac{\gamma'}{\gamma} \frac{dt}{d \tau} + \frac{\lambda}{\gamma} \frac{dr}{d \tau} \left( \frac{\gamma f'}{\sqrt{r^4 + f^2}} \right). \tag{18c}$$

The $\phi$ component is omitted, as it is already integrated.

We can satisfy (18b) identically by letting $\theta(\tau_0) = \pi/2$ and $\dot{\theta}(\tau_0) = 0$ for some proper time $\tau_0$ (see [3], p. 71). Orbits therefore lie in a plane and by choosing $\theta(\tau_0) = \pi/2$, we have fixed that plane. It follows that $J = r^2 \dot{\phi}$.

We can write (18a) as

$$\frac{1}{\gamma} \frac{d}{d \tau} \left( \frac{dt}{d \tau} \right) + \frac{\lambda}{\gamma} \frac{d}{d \tau} \left( \frac{\gamma f'}{\sqrt{r^4 + f^2}} \right) = 0.$$ \hspace{1cm} (19)

Since $\lambda$ is a constant, we conclude from this that

$$E = \frac{\gamma dt}{d \tau} + \frac{\lambda \gamma f'}{\sqrt{r^4 + f^2}} \tag{19}$$

is a constant of the motion. $E$ represents the energy at infinity per unit rest mass. We see that, as was suggested earlier, although $\gamma \dot{t}$ is not a constant of the motion, it does form one part of a conserved quantity.

Using these results, we can rewrite (18a) as

$$0 = \frac{d^2 r}{d \tau^2} + \frac{\alpha'}{2 \alpha} \left( \frac{dr}{d \tau} \right)^2 - \frac{J^2}{r^4 \alpha} + \frac{\lambda}{\alpha \gamma} \left( E - \frac{\lambda \gamma f'}{\sqrt{r^4 + f^2}} \right) \frac{d}{dr} \left( \frac{\gamma f'}{\sqrt{r^4 + f^2}} \right) \tag{20}$$

\hspace{1cm} $$+ \frac{\gamma'}{2 \alpha \gamma^2} \left( E - \frac{\lambda \gamma f'}{\sqrt{r^4 + f^2}} \right)^2.$$

In order to extract a useful result from this, we will make certain simplifying assumptions. There are two regimes to consider, corresponding to $\mu r \gg 1$ (large $r$) and $\mu r \ll 1$ (small $r$), respectively. In both cases, we will assume $M \ll r$ and $M \ll 1/\mu$.

We treat first the case of large $r$. In this case, we are interested in the corrections to the Newtonian gravitational force acting on the particle. We therefore rewrite (20) in terms of $r(t)$:
\[
\frac{d^2 r}{dt^2} + \frac{\alpha'}{2\alpha} \left( \frac{dr}{dt} \right)^2 - \frac{J_N^2}{r^3} + \frac{\gamma'}{2\alpha} - \frac{\lambda\gamma}{\alpha} \left( E - \frac{\lambda\gamma f'}{\sqrt{r^4 + f^2}} \right)^{-1} \frac{d}{dr} \left( \frac{\gamma f'}{\sqrt{r^4 + f^2}} \right).
\]

Here, \(mJ_N \equiv mr^2d\phi/dt\) is the Newtonian value of the angular momentum. We recognize the left-hand side as the usual GR contributions to the equation of motion.

For large \(r\), it can be shown \cite{foot1} that
\[
f \approx \frac{sM^2}{3} \frac{e^{-\mu r}(1 + \mu r)}{(\mu r)^{\mu M}},
\]
where \(s\) is an arbitrary constant and \(M\) is the Schwarzschild mass of the source. The constant \(1/\mu\) is a fundamental constant in the NGT, and represents the range of the skew components \(a_{\mu\nu}\). Assuming \(\mu M \ll 1\), we can write \((\mu r)^{\mu M} \sim 1\), leaving
\[
f \approx \frac{sM^2}{3} e^{-\mu r}(1 + \mu r). \tag{21}
\]

To our order of approximation,
\[
\frac{\lambda\gamma}{\alpha} \left( E - \frac{\lambda\gamma f'}{\sqrt{r^4 + f^2}} \right)^{-1} \frac{d}{dr} \left( \frac{\gamma f'}{\sqrt{r^4 + f^2}} \right) \approx E\lambda \frac{d}{dr} \left( \frac{\gamma f'}{\sqrt{r^4 + f^2}} \right) \approx \frac{E\lambda sM^2}{3} \frac{e^{-\mu r}(1 + \mu r)}{r^2}.
\]

Therefore, the radial equation of motion may be written
\[
\frac{d^2 r}{dt^2} - \frac{J_N^2}{r^3} = - \frac{M}{r^2} - \frac{E\lambda sM^2}{3} \frac{e^{-\mu r}(1 + \mu r)}{r^2}, \tag{22}
\]
where we have assumed that the particle is moving slowly, so that \(dr/dt \ll 1\). For a spin \(1^+\) particle exchange, as is obtained in the linear approximation of the NGT \cite{foot2}, \(\lambda < 0\), yielding a repulsive Yukawa force in \cite{foot2}.

At the other end of the spectrum, we have the case where \(r\) can be taken as small. More precisely, this is the region where \(\mu r\) and \(M/r\) are both small. We are more interested here in the orbit of the particle. It is interesting to take a different approach: taking \(\theta = \pi/2\) and \(\theta = 0\), the normalization condition \(s_{\mu\nu} \xi^\mu \xi^\nu = 1\) can be written
\[
1 = \gamma \left( \frac{dt}{d\tau} \right)^2 - \alpha \left( \frac{dr}{d\tau} \right)^2 - r^2 \left( \frac{d\phi}{d\tau} \right)^2 = \gamma \left( \frac{dt}{d\tau} \right)^2 - \alpha J^2 \left[ \left( \frac{du}{d\phi} \right)^2 + \frac{1}{r^2}\alpha \right],
\]
where \(J = r^2d\phi/d\tau\), \(u = 1/r\), and where we have written \(r = r(\phi)\). Using \cite{foot3}, this becomes
\[
\left( \frac{du}{d\phi} \right)^2 = \frac{1}{\alpha\gamma J^2} \left( E - \frac{\lambda\gamma f'}{\sqrt{r^4 + f^2}} \right)^2 - \frac{1}{r^2}\alpha - \frac{1}{\alpha J^2}.
\]

Differentiating this with respect to \(\phi\) gives the equation describing the orbit of the particle:
\[
\frac{d^2 u}{d\phi^2} = -\frac{r^2}{2} \frac{d}{dr} \left[ \frac{1}{\alpha\gamma J^2} \left( E - \frac{\lambda\gamma f'}{\sqrt{r^4 + f^2}} \right)^2 - \frac{1}{\alpha J^2} - \frac{1}{r^2}\alpha \right]. \tag{23}
\]

Thus far, the treatment has been entirely general. We can specialize to the case where \(\mu r\) and \(M/r\) are small. It can then be shown \cite{foot2} that
\[
f \approx \frac{sM^2}{3} \left( 1 - \frac{1}{2}(\mu r)^2 + \frac{2M}{r} \right).
\]

Typically, the term \(2M/r\) in the parenthesis will dominate the term \((\mu r)^2\). However, for interest sake we will keep both terms. Taking the Schwarzschild forms \(\gamma = 1/\alpha = 1 - 2M/r\), we arrive at the orbit equation:
\[
\frac{d^2 u}{d\phi^2} + u = \frac{M}{J^2} \left[ 1 + \frac{E\lambda sM^2}{3} \right] 3Mu^2 + \frac{8\lambda sM^3u^3}{3J^2}. \tag{24}
\]

This equation is similar to the orbit equation from GR (see \cite{foot4}, p. 186), the only differences being the last term on the right-hand side and the factor multiplying the first term on the right-hand side. The former term represents the lowest-order NGT correction to the GR result.
V. CONCLUSIONS

In [5], it is argued that an analysis of particle motion in the strong-field regime from the point of view of the matter response equation (see [6]) is inconclusive due to the uncertain nature of the stress tensor, $T^{\mu\nu}$. A similar judgement is passed on an analysis of the type performed in [7], since “If singularities are tolerated in general relativity, then the completeness of this theory is even more thoroughly shattered.” As NGT is claimed to be a candidate for a non-singular theory of gravitation, an analysis of particle motion by means of point-like singularities is even more questionable. However, these types of analyses can be performed in the weak-field regime.

Far be it for us to discuss the validity of such criticism, we have preferred to avoid the problem of solving the motion of particles in strong gravitational fields by restraining ourselves to an analysis of test-particle motion strictly in the weak-field regions of the static, spherically-symmetric solution of the field equations. The particular choice of the Lagrangian $L_{GR}$ leading to the familiar left-hand sides of (12) and (13) is guided by the matter response equation, which in NGT may be written (see [8] for a derivation of the matter response equation):

$$T^{(\beta \rho)}_{\rho \beta} + s^{\beta \lambda} g_{\mu \lambda} T^{[\rho \nu]}_{[\mu \nu]} + \left\{ \beta_{\mu \nu} \right\} T^{\mu \nu} = 0,$$

where

$$\left\{ \beta_{\mu \nu} \right\} = \frac{1}{2} s^{\beta \alpha} \left( g_{\alpha \nu, \mu} + g_{\mu \alpha, \nu} - g_{\mu \nu, \alpha} \right).$$

The resemblance of this object to the Christoffel symbol defined in (8) is obvious. The matter response equation is a result of the full NGT field equations, valid in both strong- and weak-field regimes. Should the “correct” stress tensor $T^{\mu \nu}$ be known, it is reasonable to conclude that the predicted test-particle equation of motion would contain the Christoffel connection. For instance, we might expect that in the case of massless particles this would lead to (13).

However, things are not so simple: according to Wheeler [5], the arguments of the previous paragraph are conclusive only in the weak-field regions of spacetime. Taking the weak-field limit to be the deciding factor, we find the number of candidates for the title of “equation of test-particle motion” multiplies quickly. For instance, the left-hand sides of (12) and (13) can be written $\nabla_{\xi} [\xi]^\beta_{\mu \nu}$, where $\nabla_{\mu}$ is the connection defined with respect to the Christoffel symbols. Although we cannot obtain the resulting equations of motion from an action principle, we could have just as easily postulated that the aforementioned connection be defined with respect to some other set of connection coefficients. There are two obvious possibilities in the NGT: $\Gamma^\beta_{\mu \nu}$ and $W^\beta_{\mu \nu}$. The latter is unconstrained, while the former is constrained to have a vanishing torsion vector: $\Gamma^\beta_{[\mu \nu]} = 0$. In the weak-field limit, both of these connections reduce to the Christoffel symbols, and therefore lead to the same equation of motion. These possibilities will be the subject of a future study.

Keeping the previous arguments in mind, we have brought forth three possible candidates to couple the test-particle quantities to the skew field of NGT. It is found that, in this scheme, massless particles would not directly couple to the skew field.

Of the three proposed candidates, only one is found to generate any interaction at all in a static, spherically-symmetric field. The correction to the Newtonian gravitational force acting on the particle is worked out in the weak-field regime and is described by a repulsive Yukawa force. The correction to the orbit of a particle is also calculated; the lowest-order NGT correction is a $1/r^3$ term.

ACKNOWLEDGMENTS

We would like to thank M. Clayton, N.J. Cornish, and T. Demopoulos for their help and for many stimulating discussions. The form of the coupling [12] was suggested by M. Clayton. This work was supported by the Natural Sciences and Engineering Research Council of Canada. We would like to thank the University of West Indies, Cave Hill Campus, for their hospitality during the course of this work. J. Légaré would like to thank the Government of Ontario for their support of this work.

[1] J. W. Moffat, University of Toronto Preprint UTPT-94-28, to be published in J. Math. Phys.
[2] N. J. Cornish, University of Toronto Preprint UTPT-94-37, 1994.
[3] A. Papapetrou, Lectures on General Relativity (D. Reidel Publishing Company, Dordrecht, 1974).
[4] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (John Wiley & Sons, New York, 1972).
[5] J. A. Wheeler, Revs. Modern Phys. 33, 63 (1961).
[6] A. Papapetrou, Proc. Phys. Soc. (London) A64, 57 (1951).
[7] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. 39, 65 (1938).
[8] J. Légare and J. W. Moffat, University of Toronto Preprint UTPT-94-36, to be published in Gen. Rel. Grav.