Compact Hermitian Young Projection Operators

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Abstract: In this paper, we describe a compact and practical algorithm to construct Hermitian Young projection operators for irreducible representations of the special unitary group SU($N$), and discuss why ordinary Young projection operators are unsuitable for physics applications. The proof of this construction algorithm uses the iterative method described by Keppeler and Sjödahl in [1]. We further show that Hermitian Young projection operators share desirable properties with Young tableaux, namely a nested hierarchy when "adding a particle". We end by exhibiting the enormous advantage of the Hermitian Young projection operators constructed in this paper over those given by Keppeler and Sjödahl.

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1 Introduction & outline

1.1 Historical overview

More than a hundred years ago, the representation theory of compact, semi-simple Lie groups, in particular also of \( \text{SU}(N) \), was a hot topic of research. Most known to physicists is the work done by Clebsch and Gordan, where the product representations of \( \text{SU}(N) \) can be classified using the Clebsch-Gordan coefficients \([2–4]\). This is the textbook method for \( N = 2 \) to find the irreducible representations of spin of an \( m \)-particle configuration, giving an explicit change of basis. While this method is perfectly adequate also for \( N \neq 2 \), it requires one to choose the parameter \( N \) at the start of the calculation. Thus, this approach is of little use to us, as we mainly strive to apply representation theory in a context of QCD, where it is often essential for \( N \) (representing \( N_c \), the number of colors in this case) to be a parameter to be varied at the end of the calculation to get a better understanding of underlying structures \([5–7]\).

Shortly after the research by Clebsch and Gordan was conducted, Élie Cartan introduced another method of finding the irreducible representations of Lie groups via finding certain subalgebras of the associated Lie algebras \([8]\) since known as Cartan subalgebras. This method is based on finding the highest weights corresponding to the irreducible representations, and then constructing all basis states within it. This process was used by Gell-Mann in 1961 \([9, 10]\) when he introduced the eight-fold way (here \( N \) represents \( N_f \) the number of flavors) to order hadrons into flavor multiplets such as baryon octet and decuplet featuring prominently corresponding to the irreducible representations, and then constructing all basis states within it. This immediately induces a product representation of \( \text{SU}(N) \), a representation on \( V^\otimes m \), if one uses this basis of \( V \) to induce a basis on \( V^\otimes m \) so that a general element \( v \in V^\otimes m \) takes the form \( v = v^{i_1 \ldots i_m} e_{(i_1)} \otimes \cdots \otimes e_{(i_m)} \):

\[
U \circ v = U \circ v^{i_1 \ldots i_m} e_{(i_1)} \otimes \cdots \otimes e_{(i_m)} := U^i_{j_1} \cdots U^i_{j_m} v^{j_1 \ldots j_m} e_{(i_1)} \otimes \cdots \otimes e_{(i_m)} \tag{1}
\]

\(1\)This is an elusive piece of knowledge, Fulton \([13, \text{chapter 8.2, lemma 4}]\), for example, provides the basis for finding highest weight vectors directly from tableaux without fixing \( N \).
Since all the factors in $V^\otimes m$ are identical, the notion of permuting the factors is a natural one and leads to a linear map on $V^\otimes m$ according to

$$\rho \circ v = \rho \circ v^{i_1 \ldots i_m} e_{i_1} \otimes \cdots \otimes e_{i_m} := v^{\rho(i_1) \ldots \rho(i_m)} e_{(i_1)} \otimes \cdots \otimes e_{(i_m)}$$

(2)

where $\rho$ is an element of $S_m$, the group of permutations of $m$ objects. From the definitions (1) and (2) one immediately infers that the product representation commutes with all permutations on any $v \in V^\otimes m$:

$$U \circ \rho \circ v = \rho \circ U \circ v .$$

(3)

In other words, any such permutation $\rho$ is an invariant of $\text{SU}(N)$ (or in fact any Lie group $G$ acting on $V$):

$$U \circ \rho \circ U^{-1} = \rho .$$

(4)

It can further be shown that these permutations in fact span the space of all linear invariants of $\text{SU}(N)$ over $V^\otimes m$ [16]. The permutations are thus referred to as the primitive invariants of $\text{SU}(N)$ over $V^\otimes m$. The full space of linear invariants is then given by

$$\text{API}(\text{SU}(N), V^\otimes m) := \left\{ \sum_{\sigma \in S_m} \alpha_{\sigma} \sigma \mid \alpha_{\sigma} \in \mathbb{R}, \sigma \in S_m \right\} \subset \text{Lin}(V^\otimes m) .$$

(5)

Note we are exclusively focusing on $\text{API}(\text{SU}(N), V^\otimes m)$ and make no efforts to directly discuss the invariants on $V^\otimes m \otimes V^{\ast \otimes m'}$. For $\text{SU}(N)$ these are implicitly included due to the presence of $\epsilon^{i_1 \ldots i_N}$ as a second invariant besides $\delta_{ij}$ – the construction of explicit algorithms tailored to expose this structure are beyond the scope of this paper. For a more comprehensive introduction to invariant theory, readers are referred to [2, 15–17].

Like all projection operators onto irreducible multiplets of $\text{SU}(N)$ in $V^\otimes m$, the Young projection operators are elements of $\text{API}(\text{SU}(N), V^\otimes m)$ and thus have a natural expansion in terms of permutations via (5) and the associated birdtrack visualizations. The Young projection operators satisfy the following three properties:

1. The complete set of Young projection operators for $\text{SU}(N)$ over $V^\otimes m$ sum up to the identity element of $V^\otimes m$,

2. Young projection operators are mutually orthogonal, i.e. $Y \cdot Y' = 0$ if $Y \neq Y'$.

3. Young projection operators are idempotent, that is they satisfy $Y \cdot Y = Y$.

Thus, the Young projection operators split the space into mutually orthogonal subspaces, which can be shown to be irreducible, [2]. This, together with equation (3) then implies that the Young projection operators classify all irreducible representations of $\text{SU}(N)$, [2, 3, 16]. The projection operators are both compact and can be constructed keeping $N$ as a parameter, both desirable properties for the practitioner, but are afflicted by one deficiency: Young projection operators on $V^\otimes m$ are not Hermitian once $m \geq 3$ (see section 1.2).

With Young’s contributions, the representation theory of compact, semi-simple Lie groups was considered a fully understood and complete theory from approximately 1950 onward, even though many misconceptions, in particular about the full extent of the theory remained, in particular among casual practitioners.

In the 1970’s Penrose devised a graphical method of dealing with primitive invariants of Lie groups including Young projection operators, [18, 19], which was subsequently applied in a collaboration with MacCallum, [20]. This graphical method, now dubbed the birdtrack formalism, was modernized and further developed by Cvitanović, [16], in recent years. The immense benefit of the birdtrack formalism is that it makes the actions

$^2$Permuting the basis vectors instead involves $\rho^{-1}$: $v^{\rho(i_1) \ldots \rho(i_m)} e_{(i_1)} \otimes \cdots \otimes e_{(i_m)} = v^{\rho^{-1}(1) \ldots (i_m)} e_{(\rho^{-1}(1))} \otimes \cdots \otimes e_{(\rho^{-1}(i_m))}$
of the operators visually accessible and thus more intuitive. For illustration, we give as an example the permutations of $S_3$ written both in their cycle notation (see [2] for a textbook introduction) as well as birdtracks:

$$\xymatrix{ & \ar[r] & \ar[l] & \ar[l] \cr \text{id} & & & \cr (12) & & & \cr (13) & & & \cr (23) & & & \cr (123) & & & \cr (132) & & & \cr}$$

(6)

The action of each of the above permutations on a tensor product $v_1 \otimes v_2 \otimes v_3$ is clear, for example

$$(123) (v_1 \otimes v_2 \otimes v_3) = v_3 \otimes v_1 \otimes v_2.$$ 

In the birdtrack formalism, this equation is written as

$$\xymatrix{v_1 \ar[r] & v_3 \ar[l] \cr v_2 \ar[l] \cr v_3 \ar[l] \cr v_2 \ar[l] \cr v_1 \ar[l] \cr v_3 \ar[l] \cr v_2 \ar[l] \cr v_3 \ar[l] \cr} = \xymatrix{v_3 \ar[r] & v_2 \ar[l] \cr v_1 \ar[l] \cr v_3 \ar[l] \cr v_2 \ar[l] \cr v_1 \ar[l] \cr v_3 \ar[l] \cr v_2 \ar[l] \cr v_3 \ar[l] \cr} ,$$

where each term in the product $v_1 \otimes v_2 \otimes v_3$ (written as a tower $\xymatrix{v_1 \ar[r] & v_3 \ar[l] \cr v_2 \ar[l] \cr v_3 \ar[l] \cr v_2 \ar[l] \cr v_1 \ar[l] \cr v_3 \ar[l] \cr v_2 \ar[l] \cr v_3 \ar[l] \cr}$) can be thought of as being moved along the lines of $\xymatrix{v_1 \ar[r] & v_3 \ar[l] \cr v_2 \ar[l] \cr v_3 \ar[l] \cr v_2 \ar[l] \cr v_1 \ar[l] \cr v_3 \ar[l] \cr v_2 \ar[l] \cr v_3 \ar[l] \cr}$. Birdtracks are thus naturally read from right to left as is also indicated by the arrows on the legs.\footnote{The direction of arrows thus encodes whether the leg is acting on $V$ or its dual $V^*$, c.f. section 3.1.}

The representation theory of SU($N$) found a short-lived revival in 2014, when Keppeler and Sjödahl contrived a construction algorithm for Hermitian projection operators (based on the idempotents already found by Young), [1]. This paper arose out of a need for Hermitian operators in a physics context. In their paper, the birdtrack formalism was used to devise a recursive construction algorithm for Hermitian projection operators. However, even though this algorithm does produce Hermitian operators satisfying all the properties 1–3 of Young projection operators while keeping $N$ as a parameter, the expressions due to Keppeler and Sjödahl soon become extremely long and thus computationally expensive and impractical.

In this paper, we give a considerably more efficient and thus more practical construction algorithm for Hermitian Young projection operators yielding compact expressions. We further show that, unlike Young projection operators, Hermitean Young projection operators mimic the tableau hierarchy of Young tableaux – a fact that has bee overlooked by KS. The remainder of this present section 1 gives a detailed outline of this paper and lists all goals that will be achieved along the way.

### 1.2 Where non Hermitian Young projection operators fail to deliver

Among practitioners, many misconceptions still exist with regards to Young projection operators. The probably most generic one stems from the presentation of Young tableaux and Young projection operators in the literature: It is usually explained in parallel that

1. Young tableaux follow a progressive hierarchy, in the sense that tableaux consisting of $n$ boxes can be obtained from Young tableaux of $n−1$ boxes merely by adding the box $\xymatrix{1}$ in the appropriate place.
For example, the tableaux
\[
\begin{array}{c}
1 & 2 \\
\otimes 3 \\
1 & 2 & 3
\end{array}
\quad \text{and} \quad \begin{array}{c}
1 & 2 \\
\otimes 3 \\
1 & 2 & 3
\end{array}
\]

and also
\[
\begin{array}{c}
1 & 2 \\
\otimes 3 \\
1 & 2 & 3
\end{array}
\quad \text{and} \quad \begin{array}{c}
1 & 2 \\
\otimes 3 \\
1 & 2 & 3
\end{array}
\]

(7)

Since this is a key concept, we will need some notation and nomenclature to refer to it. In general, for a particular Young tableau with \( n - 1 \) boxes \( \Theta \), we will denote the set of all Young tableaux that can be obtained from \( \Theta \) by adding the box \( \begin{array}{c} n \end{array} \) by
\[
\left\{ \Theta \otimes \begin{array}{c} n \end{array} \right\};
\]
this set will also be referred to as the child-set of \( \Theta \).

2. This is complemented by the fact that the Young projection operators span the full space, that is
\[
\sum_{\Theta \in Y_n} Y_\Theta = \text{id}_n,
\]
where \( Y_n \) is understood to be the set of all Young tableaux consisting of \( n \) boxes (for a fixed \( n \)), and \( \text{id}_n \) is the identity operator on the space \( V^\otimes n \). Equation (9) is also known as the completeness relation of Young projection operators. In particular
\[
Y_{\begin{array}{c} 1 \\ \otimes 2 \end{array}} + Y_{\begin{array}{c} 1 \\ \otimes 3 \end{array}} = \text{id}_2
\]
and
\[
Y_{\begin{array}{c} 1 \\ \otimes 2 \\ \otimes 3 \end{array}} + Y_{\begin{array}{c} 1 \\ \otimes 3 \\ \otimes 2 \end{array}} + Y_{\begin{array}{c} 2 \\ \otimes 1 \\ \otimes 3 \end{array}} = \text{id}_3.
\]

The completeness relations offer decompositions of unity in both cases.

The hierarchy relation (7) of Young tableaux and the completeness relation (9) of Young projection operators then might lead the unwary reader to (incorrectly) infer that the tableau hierarchy (7) automatically implies that this decomposition of unity is in fact nested, i.e. that the child tableaux correspond to projection operators that furnish decompositions of their parent projectors so that the identities
\[
Y_{\begin{array}{c} 1 \\ \otimes 2 \end{array}} + Y_{\begin{array}{c} 1 \\ \otimes 3 \end{array}} \nRightarrow Y_{\begin{array}{c} 1 \\ \otimes 2 \\ \otimes 3 \end{array}},
\]
\[
Y_{\begin{array}{c} 1 \\ \otimes 2 \end{array}} + Y_{\begin{array}{c} 1 \\ \otimes 3 \end{array}} \nRightarrow Y_{\begin{array}{c} 2 \\ \otimes 1 \\ \otimes 3 \end{array}}.
\]

(12)

would hold. Both of these “equations” can easily be shown to be false by direct calculation. The authors have not found any literature that clearly states that relation (7) holds for Young tableaux only, and does not have a counterpart in terms of Young projection operators.

In physics applications, (9) is often not sufficient, and we require a counterpart of (7) for a suitable set of projection operators, thus repairing the failure of “equation” (12). The desired analogue exists, it is given by the Hermitian Young projection operators introduced by Keppeler and Sjödahl (KS) [1], although
This practically crucial observation was not mentioned by KS in their paper. In the present paper, we will explicitly demonstrate that the tableau hierarchy (7) can be transferred to the KS-operators in the desired manner: Using $P_\Theta$ to denote the Hermitian Young projection operator corresponding to the tableau $\Theta$, it turns out that the decompositions in our example are indeed nested, so that
\begin{equation}
P_{\Theta_1} + P_{\Theta_2} = P_{\Theta_1 + \Theta_2}
\end{equation}
hold, and that this generalizes to all Hermitian projectors corresponding to Young tableaux. Thus, the first goal of this paper will be to show that this pattern holds in general:

**Goal 1** We are interested in a nested decomposition of projection operators in analogy to the hierarchy relation of Young tableaux discussed in eqns. (7) and (8) to operators, thus generalizing eq. (13) to
\begin{equation}
\sum_{\Phi \in \{\Theta \otimes n\}} P_\Phi = P_\Theta.
\end{equation}

1. We will first show that equation (14) does not hold if the $P_i$ are the (non-Hermitian) Young projection operators; this will be done in section 3.2.

2. In section 3.3.2 we will find our intuition restored when we show that equation (14) does hold for Hermitian operators. At the end of this section, we will discuss that a more general version of (14) holds,
\begin{equation}
\sum_{\Phi \in \{\Theta \otimes m \otimes \cdots \otimes n\}} P_\Phi = P_\Theta;
\end{equation}
that is, the summation property of Hermitian Young projection operators holds over various generations of tableaux, c.f. eq. (70).

### 1.3 Shorter is better

Having motivated the necessity of Hermitian Young projection operators, we will now shift our focus to their application. In particular, the authors of this paper are foremost interested in applications in a QCD context, such as is laid out in [21]. With this objective in mind, the Hermitian Young projection operators conceived by KS, [1], soon lose all practical usefulness as the number of factors in $V \otimes m$ grows: the expressions become too long and thus computationally expensive; a quality, that is explained in section 4.3.

An array of practical tools [22] particularly suited for the birdtrack formalism [16], in which the Hermitian Young projection operators by KS were constructed, allows to devise a new construction principle for Hermitian Young projection operators, which we could not resist to dub MOLD-construction. As elements in the algebra of invariants, the MOLD-operators are identical to the KS-operators, however their expressions in terms of symmetrizers and antisymmetrizers as well as the number of steps used in the construction is shorter, often dramatically so. We gain access to all the desired properties of the KS-operators at a much lower computational cost: their idempotency, their mutual orthogonality, their completeness relation\(^5\), and also the hierarchy relation (14). A clear comparison between the MOLD- and the KS-constructions and the resulting expressions for the Hermitian Young projection operators can be found in section 4.3. This leads to the second goal of this paper:

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\(^4\)This will be accomplished by using a shortened version of the KS-operators; a construction principle for these shortened operators is given in section 3.3.1.

\(^5\)All of these properties of the KS-operators are described in Theorem 3 and in [1].
Goal 2 We will provide a construction principle for Hermitian Young projection operators that produces compact, and thus practically useful expression for these operators, section 4. An explicit comparison of the algorithms is given in 4.3.

2 Young tableaux, birdtracks, notations and conventions

Before we set out to achieve Goals 1 and 2, we will provide a short sketch of birdtracks as they relate to Young tableaux in section 2.1, mainly to prepare for section 2.2 where we establish the notation used in this paper. For a more comprehensive introduction to the birdtrack formalism, refer to [16].

2.1 Birdtracks & Young tableaux

Our aim in this section is to establish a link between Young tableaux [2] and birdtracks [16, 18–20], as it is our ultimate goal is to use the tools presented in this paper in a QCD context where SU(N) with \( N = N_c = 3 \) is the gauge group of the theory [21] in a manner that allows us to keep \( N \) as a parameter in order to have direct access to additional structure, not least the large \( N_c \) limit.

As mentioned earlier, one way to generate the projection operators corresponding to the irreducible representations of SU(N) without being forced to choose a numerical value for \( N \) at the outset is via the method of Young projection operators, which can be constructed from Young tableaux, see for example [2, 3, 13, 23] and other standard textbooks.

We therefore begin with a short memory-refresher on Young tableaux, our main source for this will be [2]. A Young tableau is defined to be an arrangement of \( m \) boxes which are left-aligned and top-aligned, and each box is filled with a unique integer between 1 and \( m \) such that the numbers increase from left to right in each row and from top to bottom in each column\(^6\). For example, among the two conglomerations of boxes

\[
\Theta = \begin{array}{c}
1 & 3 & 6 \\
2 & 5 & 7 \\
4
\end{array}
\quad \text{and} \quad
\tilde{\Theta} = \begin{array}{c}
3 & 4 & 1 \\
2 & 6 & 7 \\
5
\end{array}
\]

\( \Theta \) is a Young tableau but \( \tilde{\Theta} \) is not since \( \tilde{\Theta} \) is neither top aligned nor are the numbers increasing within each column and row. The study of Young tableaux is the topic of several books, e.g. [13], and is thus too vast a topic to fully explore here.

Throughout this paper, \( \mathcal{Y}_n \) will denote the set of all Young tableaux consisting of \( n \) boxes. For example,

\[
\mathcal{Y}_3 := \left\{ \begin{array}{c}
1 & 2 & 3 \\
1 & 2 \\
1 & 3 \\
1
\end{array} \right\} \cup \left\{ \begin{array}{c}
1 & 2 \otimes 3 \\
1 & 2 \otimes 3
\end{array} \right\}
\]

We will denote a particular Young tableau by an upper case Greek letter, usually \( \Theta \) or \( \Phi \).

To establish the connection with birdtrack notation, let us consider a symmetrizer over elements 1 and 2, \( S_{12} \), corresponding to a Young tableau

\[
\begin{array}{c}
1 & 2
\end{array}
\]

\( ^6 \)In some references, the presently described tableau may also be referred to as a standard Young tableau [13, 23].
as symmetrizers always correspond to rows of Young tableaux [2]. We know that this symmetrizer $S_{12}$ is given by $\frac{1}{2} (\text{id} + (12))$, where id is the identity and (12) denotes the transposition that swaps elements 1 and 2. Graphically, we would denote this linear combination as [16]

$$S_{12} = \frac{1}{2} \begin{array}{c}
\begin{array}{c}
\text{---}
\text{---}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{---}
\text{---}
\end{array}
\end{array}.$$

This operator is read from right to left\(^7\), as it is viewed to act as a linear map from the space $V \otimes V$ into itself. In this paper, permutations and linear combinations thereof will always be interpreted as elements of $\text{Lin}(V^\otimes n)$, where $\text{Lin}(V^\otimes n)$ denotes the space of linear maps over $V^\otimes n$. In particular, we will denote the sub-space of $\text{Lin}(V^\otimes n)$ that is spanned by the primitive invariants of $\text{SU}(N)$ by $\text{API}(\text{SU}(N), V^\otimes n)$.

Following [16], we will denote a symmetrizer over an index-set $N$, $S_N$, by an empty (white) box over the index lines in $N$. Thus, the symmetrizer $S_{12}$ is denoted by $\begin{array}{c}
\begin{array}{c}
\text{---}
\text{---}
\end{array}
\end{array}$, corresponding to the Young tableau $\begin{array}{c}
\begin{array}{c}
1
2
\end{array}
\end{array}$. Similarly, an antisymmetrizer over an index-set $M$, $A_M$, is denoted by a filled (black) box over the appropriate index lines. For example,

$$A_{12} = \begin{array}{c}
\begin{array}{c}
\text{---}
\text{---}
\end{array}
\end{array},$$

since antisymmetrizers correspond to columns of Young tableaux, [2]. For any Young tableau $\Theta$, one can form an idempotent, the so-called Young projection operator corresponding to $\Theta$ [2, 13, 14, 16, 23]: Let $S_{R_i}$ denote the symmetrizer corresponding to the $i^{th}$ row of the tableau $\Theta$, and let $S_{\Theta}$ denotes the set (or product, it does not matter since the symmetrizers $S_{R_i}$ are disjoint by the definition of a Young tableau) of the symmetrizers $S_{R_i}$,

$$S_{\Theta} = S_{R_1} \cdots S_{R_k}.$$

Similarly, let

$$A_{\Theta} = A_{C_1} \cdots A_{C_l},$$

where $A_{C_j}$ corresponds to the $j^{th}$ column of $\Theta$. Then, the object

$$Y_\Theta := \alpha_\Theta \cdot S_{\Theta}A_{\Theta}$$

is an idempotent, where $\alpha_\Theta$ is a combinatorial factor involving the hook length of the tableau $\Theta$ [13, 23]. $Y_\Theta$ is called the Young projection operator corresponding to $\Theta$. Besides being idempotent\(^8\)

$$Y_\Theta \cdot Y_\Theta = Y_\Theta,$$

Young projection operators are also mutually orthogonal: If $\Theta$ and $\Phi$ are two Young tableaux consisting of the same number of boxes, then

$$Y_\Theta \cdot Y_\Phi = \delta_{\Theta \Phi} Y_\Theta.$$  \hspace{1cm} (17b)

Furthermore, Young projection operators satisfy a completeness relation, that is, the Young projection operators corresponding to the tableaux in $\mathcal{Y}_n$ sum up to the identity operator on the space $V^\otimes n$,

$$\sum_{\Theta \in \mathcal{Y}_n} Y_\Theta = 1_n.$$  \hspace{1cm} (17c)

These three operators allow the Young projection operators associated to the Young tableaux in $\mathcal{Y}_n$ to fully classify the irreducible representations of $\text{SU}(N)$ over $V^\otimes n$ [2, 15–17].

\(^7\)This is no longer strictly true for birdtracks representing primitive invariants of $\text{SU}(N)$ over a mixed algebra $V^\otimes m \otimes (V^*)^\otimes n$ which includes dual vector spaces. A more informative discussion on this is out of the scope of this paper; readers are referred to [16].

\(^8\)This property is surprisingly hard to prove without the simplification rules paraphrased in section 2.3 [22].

8
It should be noted that, since all symmetrizers in $S_\Theta$ (resp. antisymmetrizers in $A_\Theta$) are disjoint, each index line enters at most one symmetrizer (resp. antisymmetrizer) in birdtrack notation. Thus, one may draw all symmetrizers (resp. antisymmetrizers) underneath each other.

As an example, we construct the birdtrack Young projection operator corresponding to the following Young tableau,

$$\Theta = \begin{array}{c}
1 & 3 & 4 \\
2 & 5 \\
\end{array}.$$  \hspace{1cm} (18)

$Y_\Theta$ is given by

$$Y_\Theta = 2 \cdot S_{134}S_{25}A_{12}A_{35},$$

where the constant 2 is the combinatorial factor that ensures the idempotency of $Y_\Theta$. In birdtrack notation, the Young projection operator $Y_\Theta$ becomes

$$Y_\Theta = 2$$

where we were able to draw the two symmetrizers in $S_\Theta$ and the antisymmetrizers in $A_\Theta$ underneath each other, as claimed.

Following [2], we define the set of horizontal permutations of a Young tableau $\Theta$, $h_\Theta$, to be set of all permutations that do not mix entries across rows in $\Theta$. Similarly, the set of vertical permutations $v_\Theta$ is the set of all permutations that do not mix entries across columns of $\Theta$. For example, for the tableau $\Theta$ as defined in (18),

$$h_\Theta = \{\text{id}, (13), (14), (34), (134), (143), (25)\} \quad \text{and} \quad v_\Theta = \{\text{id}, (12), (35)\}$$

In particular, the set of symmetrizers and antisymmetrizers corresponding to a Young tableau $\Theta$ then obeys, [2]

$$h_\Theta S_\Theta = S_\Theta h_\Theta = S_\Theta \quad \text{and} \quad v_\Theta A_\Theta = A_\Theta v_\Theta = \text{sign}(v_\Theta) A_\Theta$$

for all $h_\Theta \in h_\Theta$ and for all $v_\Theta \in v_\Theta$, where $\text{sign}(v_\Theta)$ is the signature of the permutation $v_\Theta$.

### 2.2 Notation & conventions

In the literature, there is a great multitude of (sometimes conflicting) conventions and notations regarding birdtracks, Young symmetrizers and other quantities used in this paper. We will devote this section to laying down the conventions that will be used here.

#### 2.2.1 Structural relationships between Young tableaux of different sizes

Throughout this paper, $Y_\Theta$ shall denote the normalized Young projection operator corresponding to a Young tableau $\Theta$, and $P_\Theta$ will refer to the normalized Hermitian Young projection operator corresponding to $\Theta$. Furthermore, for any operator $O$ consisting of symmetrizers and anti-symmetrizers, the symbol $\bar{O}$ will refer to a product of symmetrizers and antisymmetrizers without any additional scalar factors. For example,

$$Y_\Theta := \begin{array}{c}
\frac{4}{3} \\
\end{array} \begin{array}{c}
\Rightarrow \alpha_\Theta \\
\Rightarrow Y_\Theta \\
\end{array}$$
Thus, for a birdtrack operator $O$, comprised solely of symmetrizers and antisymmetrizers, $\bar{O}$ denotes the graphical part alone,

$$O := \omega \bar{O},$$

(20)

where $\omega$ is some scalar. The benefit of this notation is that the barred operator stays unchanged under multiplication with a non-zero scalar $\lambda$,

$$\lambda \cdot O \neq O \quad \text{but} \quad \lambda \cdot \bar{O} = \bar{O}.$$

It should be noted that $\bar{Y}_\Theta$ and $\bar{P}_\Theta$ are only quasi-idempotent, while $Y_\Theta$ and $P_\Theta$ are idempotent. We will denote the normalization constants of $Y_\Theta$ and $P_\Theta$ by $\alpha_\Theta$ and $\beta_\Theta$ respectively, such that

$$Y_\Theta := \alpha_\Theta \bar{Y}_\Theta \quad \text{and} \quad P_\Theta := \beta_\Theta \bar{P}_\Theta;$$

(21)

where the normalization constant $\beta_\Theta$ is given together with the appropriate construction principles for $P_\Theta$ (we encounter three different versions, one each for the original KS construction in Theorem 3, the simplified KS construction in Theorem 1, and the MOLD version in Theorem 5).

It is a well-known fact that Young tableaux in $\mathcal{Y}_n$ can be built from Young tableaux in $\mathcal{Y}_{n-1}$ by adding the box $[n]$ at an appropriate place as to not destroy the properties of Young tableaux; such places are referred to as outer corners, [23]. In this way, the Young tableau $\Theta = \begin{array}{c|c|c} 1 & 2 & 3 \\ \hline 4 & 5 & \end{array}$ generates the subset $\{ \Theta \otimes [5] \}$ of $\mathcal{Y}_5$,

$$\Theta = \begin{array}{c|c|c} 1 & 2 & 3 \\ \hline 4 & 5 & \end{array} \in \mathcal{Y}_4$$

(22)

This operation is not a map in the mathematical sense as it does not yield a unique result. The reverse operation, taking away the box with the highest entry, is a map; let us denote this map by $\pi$. $\pi$ can then repeatedly be applied to the resulting tableau,

$$\begin{array}{c|c|c} 1 & 2 & 3 \\ \hline 4 & 5 & \end{array} \xrightarrow{\pi} \begin{array}{c|c|c} 1 & 2 & 3 \\ \hline 4 & \_ & \_ \end{array} \xrightarrow{\pi} \begin{array}{c|c|c} 1 & 2 & 3 \\ \hline \_ & 4 & \_ \end{array} \xrightarrow{\pi} \begin{array}{c|c|c} 1 & 2 & 3 \\ \hline \_ & \_ & 4 \end{array}.$$

**Definition 1 (parent map and ancestor tableaux)** Let $\Theta \in \mathcal{Y}_n$ be a Young tableau. We define its parent tableau $\Theta^{(1)} \in \mathcal{Y}_{n-1}$ to be the tableau obtained from $\Theta$ by removing the box $[n]$ of $\Theta$. Furthermore, we will define a parent map $\pi$ from $\mathcal{Y}_n$ to $\mathcal{Y}_{n-1}$, for a particular $n$,

$$\pi : \mathcal{Y}_n \to \mathcal{Y}_{n-1},$$

(23a)

which acts on $\Theta$ by removing the box $[n]$ from $\Theta$,

$$\pi : \Theta \mapsto \Theta^{(1)}, \quad \text{for } \Theta \in \mathcal{Y}_n.$$
In general, we define the successive action of the parent map \( \pi \) by
\[
Y_n \xrightarrow{\pi} Y_{n-1} \xrightarrow{\pi} Y_{n-2} \xrightarrow{\pi} \ldots \xrightarrow{\pi} Y_{n-m},
\]
and denote it by \( \pi^m \),
\[
\pi^m : Y_n \rightarrow Y_{n-m}, \quad \pi^m := Y_n \xrightarrow{\pi} Y_{n-1} \xrightarrow{\pi} Y_{n-2} \xrightarrow{\pi} \ldots \xrightarrow{\pi} Y_{n-m}.
\]
We will further denote the tableau obtained from \( \Theta \) by applying the map \( \pi \) \( m \) times, \( \pi^m(\Theta) \), by \( \Theta^{(m)} \), and refer to it as the ancestor tableau of \( \Theta \) \( m \) generations back. Applying the map \( \pi^m \) to a Young tableau \( \Theta \) then yields the unique tableau \( \Theta^{(m)} \).

\[ (24a) \]

2.2.2 Embeddings and images of linear operators

Any operator \( O \in \mathrm{Lin}(V^\otimes n) \) can be embedded into \( \mathrm{Lin}(V^\otimes m) \) for \( m > n \) in several ways, simply by letting the embedding act as the identity on \((m-n)\) of the factors; how to select these factors is a matter of what one plans to achieve. The most useful convention for our purposes is to let \( O \) act on the first \( n \) factors and operate with the identity on the remaining \((m-n)\) factors. We will call this the canonical embedding.

On the level of birdtracks, this amounts to letting the index lines of \( O \) coincide with the top \( n \) index lines of \( \mathrm{Lin}(V^\otimes m) \), and the bottom \((m-n)\) lines of the embedded operator constitute the identity birdtrack of size \((m-n)\). For example, the operator \( \bar{Y}^1_2^3 \) is canonically embedded into \( \mathrm{Lin}(V^\otimes 5) \) as
\[
\leftarrow \rightarrow.
\]

Furthermore, we will use the same symbol \( O \) for the operator as well for its embedded counterpart. Thus, \( \bar{Y}^1_2^3 \) shall denote both the operator on the left as well as on the right hand side of the embedding (25).

Lastly, if a Hermitian projection operator \( A \) projects onto a subspace completely contained in the image of a projection operator \( B \), then we denote this as \( A \subset B \), transferring the familiar notation of sets to the associated projection operators. In particular, \( A \subset B \) if and only if
\[
A \cdot B = B \cdot A = A
\]
for the following reason: If the subspaces obtained by the consecutive application of the operators \( A \) and \( B \) in any order is the same as that obtained by merely applying \( A \), then not only need the subspaces onto which \( A \) and \( B \) project overlap (as otherwise \( A \cdot B = B \cdot A = 0 \)), but the subspace corresponding to \( A \) must be completely contained in the subspace of \( B \) - otherwise the last equality of (26) would not hold. Hermiticity is crucial for these statements - they thus do not apply to most Young projection operators on \( V^\otimes m \) if \( m \geq 3 \).

A familiar example for this situation is the relation between symmetrizers of different length: a symmetrizer \( S_N \) can be absorbed into a symmetrizer \( S_{N'} \), as long as the index set \( N \) is a subset of \( N' \), and the same statement holds for antisymmetrizer, [16]. For example,
\[
\left\[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{array} \right\] = \left\[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{array} \right\} = \left\[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{array} \right\].
\]

Thus, by the above notation, \( S_{N'} \subset S_N \), if \( N \subset N' \). Or, as in our example,
\[
\left\[ \begin{array}{cccc} 1 & 2 \\ 2 & 1 \end{array} \right\} \subset \left\[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{array} \right\].
\]
In particular, it follows immediately from the definition of the ancestor tableau (Definition 1, eq. (24c)) that

\[ S_{\Theta(k)} S_{\Theta} = S_{\Theta} S_{\Theta(k)} \quad \text{and} \quad A_{\Theta(k)} A_{\Theta} = A_{\Theta} A_{\Theta(k)} \]  

(27)

for every ancestor tableau \( \Theta(k) \) of \( \Theta \).

### 2.3 Cancellation rules

One of the suspected reasons why the birdtrack formalism has not yet gained as much popularity as it ought is because there exist virtually no practical rules which allow easy manipulation of birdtrack operators in the literature. In [22], we establish various rules designed to easily manipulate birdtrack operators comprised of symmetrizers and anti-symmetrizers. Since all operators considered in this paper are of this form, the simplification rules of [22] are immediately applicable to this paper; in particular, none of the proofs of the construction algorithms in this paper would have been possible without these rules. Thus, we choose to summarize the most important results of [22] here. For the proofs of these rules, readers are referred to [22].

The simplification rules of [22] fall into two classes:

1. **Cancellation rules** (Theorems 1 and 2, section 2.3): these rules are to cancel large chunks of birdtrack operators, thus making them shorter (often significantly so) and more practical to use. These rules are used in several places throughout this paper, in particular in the proof of the shortened KS-operators (Corollary 1) and the proof of the construction of MOLD-operators (Theorem 5). Since the cancellation rules are used multiple times throughout this paper, we provide these rules in this present section.

2. **Propagation rules** (Theorem 6, section A.1.1): these rules allow one to commute (sets of) symmetrizers through (sets of) antisymmetrizers and vice versa. These rules come in handy when trying to expose the implicit Hermiticity of a birdtrack operator. In this paper, we use these rules in the proof of Theorem 4, which is why we defer the re-statement of the propagation rules to appendix A.1.

**Theorem 1 (cancellation of wedged ancestor-operators)** Consider two Young tableaux \( \Theta \) and \( \Phi \) such that they have a common ancestor tableau \( \Gamma \). Let \( Y_{\Theta} \), \( Y_{\Phi} \) and \( Y_{\Gamma} \) be their respective Young projection operators, all embedded in an algebra that is able to contain all three. Then

\[ Y_{\Theta} Y_{\Gamma} Y_{\Phi} = Y_{\Theta} Y_{\Phi}. \]  

(28)

For example, consider the Young tableaux

\[ \Theta = \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & \end{array} \quad \text{and} \quad \Phi = \begin{array}{ccc}
1 & 2 & 4 \\
3 & \end{array}, \]

which have the common ancestor

\[ \Gamma = \begin{array}{cc}
1 & 2 \\
3 & \end{array} \]

Then, the product \( Y_{\Theta} Y_{\Gamma} Y_{\Phi} \) is given by

\[ Y_{\Theta} Y_{\Gamma} Y_{\Phi} = 4 \]
where $\alpha_\Theta \Theta = 4$. According to Theorem 1, we are allowed to cancel the operator $Y_\Phi$, hence reducing the above product to

$$Y_\Theta Y_\Phi = 3 \cdot \overline{Y}_\Theta Y_\Phi,$$

where $\alpha_\Theta \Theta = 3$.

A more general cancellation-Theorem is:

**Theorem 2** (cancellation of parts of the operator) Let $\Theta \in Y_n$ be a Young tableau and $M \in \text{API}(\text{SU}(N), V^\otimes m)$ be an algebra element. Then, there exists a (possibly vanishing) constant $\lambda$ such that

$$O := S_\Theta A_\Theta = \lambda \cdot Y_\Theta. \tag{29}$$

If furthermore the operator $O$ is non-zero, then $\lambda \neq 0$. One instance in which $O$ is guaranteed to be non-zero is if $M$ is of the form

$$M = A_{\Phi_1} S_{\Phi_2} A_{\Phi_3} S_{\Phi_4} \cdots A_{\Phi_{k-1}} S_{\Phi_k}, \tag{30}$$

where $A_{\Phi_i} \supset A_\Theta$ and $S_{\Phi_j} \supset S_\Theta$ for every $i \in \{1, 3, \ldots k-1\}$ and for every $j \in \{2, 4, \ldots k\}$.

As an example, consider the operator

$$O := \overline{S}_{125, 344} \cdot \{A_{13}\} \cdot \{S_{12}, S_{34}\} \cdot \{A_{13}, A_{24}\}.$$

This operator meets all conditions of the above Theorem 2: the sets $\{S_{125}, S_{344}\}$ and $\{A_{13}, A_{24}\}$ together constitute the birdtrack of a Young projection operator $\overline{Y}_\Theta$ corresponding to the tableau

$$\Theta := \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4
\end{array}.$$

The set $\{A_{13}\}$ corresponds to the ancestor tableau $\Theta_{(2)}$, and the set $\{S_{12}, S_{34}\}$ corresponds to the ancestor tableau $\Theta_{(1)}$ and can thus be absorbed into $A_\Theta$ and $S_\Theta$ respectively, c.f. eq. (27). Hence $O$ can be written as

$$O = S_\Theta A_{\Theta_{(2)}} S_{\Theta_{(1)}} A_\Theta.$$

Then, according to the above Cancellation-Theorem 2, we may cancel the wedged ancestor sets $A_{\Theta_{(2)}}$ and $S_{\Theta_{(1)}}$ at the cost of a non-zero constant $\tilde{\lambda}$. In particular, we find that

$$O = \tilde{\lambda} \cdot \overline{Y}_\Theta,$$

which is proportional to $Y_\Theta$.

### 3 Hermitian Young projection operators

Throughout this paper, we will be working with linear maps over linear spaces, in particular with maps in $\text{API}(\text{SU}(N), V^\otimes m) \subset \text{Lin}(V^\otimes m)$. All the familiar tools from linear algebra (as can be found in [24] and other standard textbooks) apply but will likely look unfamiliar when employed in the language of birdtracks. We thus devote this section to translate the most important tools for this paper into the language of birdtracks.
3.1 Hermitian conjugation of linear maps in birdtrack notation

We begin by recalling the definition of Hermitian conjugation for linear maps. Let $U$ and $W$ be linear spaces, and let $\langle \cdot, \cdot \rangle_U : U \rightarrow \mathbb{F}$, where $\mathbb{F}$ is a field usually taken to be $\mathbb{C}$ or $\mathbb{R}$, denote the scalar product defined on $U$, and similarly for $W$. Furthermore, let $P : U \rightarrow W$ be an operator. The scalar products then furnish a definition of the Hermitian conjugate of $P$ (denoted by $P^\dagger : W \rightarrow U$) in the standard way:

$$\langle w, Pu \rangle_W = \langle P^\dagger w, u \rangle_U$$  \hspace{1cm} (31)

for any $u \in U$ and $w \in W$, [24]. In our case, $u$ and $w$ will be elements of $V^\otimes m$, both $u$ and $w$ appear as tensors with $m$ upper indices $u_{i_1...i_m}$ and $w_{i_1...i_m}$.

Eq. (31) is equivalent to requiring the following diagram to commute,

\[ \begin{array}{ccc}
V^\otimes m \otimes V^\otimes m & \xrightarrow{1_w \otimes 1_U} & V^\otimes m \otimes V^\otimes m \\
\downarrow U \otimes 1_W & & \downarrow \langle \cdot, \cdot \rangle \\
V^\otimes m \otimes V^\otimes m & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{C}
\end{array} \] \hspace{1cm} (32)

The scalar product between these maps is then defined in the usual way, $\langle u, w \rangle = u^\dagger w \in \mathbb{C}$. The map $u^\dagger$ is an element of the dual space $(V^*)^\otimes m$ and thus needs to be equipped with lower indices $u_{j_1...j_m}$.

Complex conjugation $*$ is necessary, since the vector space $V$ may be complex. Hence, we have that

$$\langle u, w \rangle = u^\dagger w = u_{i_1...i_m} w^{i_1...i_m} \in \mathbb{C}.$$  \hspace{1cm} (33)

In the above, all indices of $u^\dagger$ and $w$ were contracted so that the outcome of the scalar product lies in a field, in our case $\mathbb{C}$. Graphically, let us represent a tensor with $j$ lower indices and $i$ upper indices by a box which has $j$ legs exiting on the right and $i$ legs exiting on the left,

\[ T^{a_1a_2...a_i}_{b_1b_2...b_j} \rightarrow \begin{array}{c} a_1 \ \ \ \ \ \ \ \ \ b_1 \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ ...
\end{array} \]

\[ T_{a_1a_2...a_i}^{b_1b_2...b_j} \rightarrow \begin{array}{c} a_1 \ \ \ \ \ \ \ \ \ b_1 \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ ...
\end{array} \]

Therefore, the tensors $u_{i_1...i_m}$ and $w^{i_1...i_m}$ will have $m$ legs exiting on the right and left respectively,

$$u_{i_1...i_m} \rightarrow \begin{array}{c} a_1 \ \ \ \ \ \ \ \ \ b_1 \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ ...
\end{array} \quad \text{and} \quad w^{i_1...i_m} \rightarrow \begin{array}{c} a_1 \ \ \ \ \ \ \ \ \ b_1 \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ ...
\end{array} \]$$

from now on, we will suppress the index labels of the birdtracks corresponding to the tensors in question. The scalar product $\langle u, w \rangle$ is diagrammatically represented as

$$\langle u, w \rangle \hspace{1cm} (34)$$

---

10 At this point we recall that the basis vectors $e$ of $V^\otimes m$ are denoted with lower indices; therefore $u$ and $w$ act on the $e'$s as linear maps.

11 Since basis vectors $\omega$ of $(V^*)^\otimes m$ have upper indices.
where the contraction of indices is indicated via the connection of corresponding index lines in the birdtrack. In (34), we see that the birdtrack corresponding to \( \langle u, w \rangle \) does not have any index lines exiting on either the right or the left, indicating that it is indeed a scalar. We will now consider a scalar product \( \langle u, Pw \rangle \), where \( P : \text{Lin}(V^\otimes m) \to \text{Lin}(V^\otimes m) \) is an operator, to find its Hermitian dual \( P^\dagger \). \( P \) must thus have \( m \) lower and \( m \) upper indices,

\[
P^{i_1 \ldots i_m}_{j_1 \ldots j_m} \rightarrow \begin{array}{c}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array}
\]

where the \( j \)-indices act on an element of \( \text{Lin}(V^\otimes m) \) via index contraction. The scalar product \( \langle u, Pw \rangle \) will then be given by

\[
\langle u, Pw \rangle = (u^{i_1 \ldots i_m})^* \cdot P^{i_1 \ldots i_m}_{j_1 \ldots j_m} \cdot w^{j_1 \ldots j_m} \rightarrow u \begin{array}{cc}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array} \quad w \begin{array}{c}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array} = u \begin{array}{cc}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array} \quad w \begin{array}{c}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array}.
\]

(35)

The adjoint of \( P \), \( P^\dagger \) is defined to be the object such that relation (31) holds. Thus, \( P^\dagger \) acts on the dual space, \( P^\dagger : \text{Lin}(V^*^\otimes m) \to \text{Lin}(V^*^\otimes m) \), which again means that it has \( m \) upper and \( m \) lower indices, but now the \( j \) indices act on the element of \( \text{Lin}(V^*^\otimes m) \),

\[
P^\dagger = (P^{i_1 \ldots i_m}_{j_1 \ldots j_m})^* = P_{j_1 \ldots j_m}^{i_1 \ldots i_m}.
\]

(36)

It should be noted that once again, the raising and lowering of indices induces a complex conjugation of the tensor components, as we have already seen for \( u \) in (33).

\[
\langle P^\dagger u, w \rangle = (u^{j_1 \ldots j_m})^* (P^{i_1 \ldots i_m}_{j_1 \ldots j_m})^* \cdot u^{i_1 \ldots i_m} \rightarrow u \begin{array}{cc}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array} \quad P^\dagger \begin{array}{cc}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array} \quad w \begin{array}{c}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array} = u \begin{array}{cc}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array} \quad P^\dagger \begin{array}{cc}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array} \quad w \begin{array}{c}
p \\
\hline
i_1 \quad \cdots \quad i_m \quad j_1 \quad \cdots \quad j_m
\end{array}.
\]

in direct correspondence with equation (35). For birdtrack operators, the Hermitean conjugate can thus be graphically formed by reflecting the birdtrack about its vertical axis and reversing the arrows, for example

\[
\text{reflect} \rightarrow \text{rev. arr} \rightarrow \text{i.e.} \quad \Rightarrow \quad \text{reflect} \rightarrow \text{rev. arr} \rightarrow \text{i.e.} \quad \Rightarrow.
\]

\[12\]The projection operators considered in this paper are real and thus remain unaffected by complex conjugation. This is no longer true for group elements or representations.
The mirroring of birdtracks under Hermitian conjugation immediately implies the unitarity of the primitive invariants (and thus that we are dealing with a unitary representation of $S_m$ on $V^\otimes m$): the inverse permutation of any primitive invariant $\rho \in S_m$ is obtained by traversing the lines of the birdtrack corresponding to $\rho$ in the opposite direction. \[ \begin{align*} \begin{array}{c|c} \hline \rowcolor{Gray} & \begin{array}{c} \text{The inverse permutation} \\ \text{of any primitive invariant} \\ \text{is obtained by traversing} \\ \text{the lines of the} \\ \text{birdtrack} \\
 \end{array} \\
 \hline \begin{array}{c} \text{any } \rho \in S_m \\
 \end{array} & \begin{array}{c} \rho^{-1} = \rho^\dagger \\
 \end{array} \\
 \hline \end{array} \end{align*} \]

However, since “traversing the lines in the opposite direction” clearly corresponds to flipping the birdtrack about its vertical axis and reversing the direction of the arrows, we have that \[ \rho^{-1} = \rho^\dagger \quad \forall \rho \in S_m ; \quad (40) \]
the primitive invariants are unitary.

These obvious Hermiticity properties of the primitive invariants make it easy to judge Hermiticity of an operator once it is expanded in this basis set. This is no longer the case in other representations: While any mirror symmetric birdtrack represents a Hermitian operator, the converse is not true in all representations. Despite a lack of apparent mirror symmetry, the product birdtrack

\[ \begin{align*} \begin{array}{c|c} \hline \rowcolor{Gray} & \begin{array}{c} \text{is Hermitian, as can be shown by either} \\ \text{using the simplification rules of Theorem 6} \\
\text{(app. A.1.1) which} \\
\text{allow us to recast (41) in an explicitly} \\
\text{mirror symmetric form, or by expanding it} \\
\text{fully in terms of primitive invariants.} \\
\end{array} \\
\hline \begin{array}{c} \text{In this paper, we will always consider birdtrack operators with lines} \\
\text{directed from right to left (as is indicated} \\
\text{by the arrows on the legs). To reduce clutter, we will from now on} \\
\text{suppress the arrows and (for example) simply write} \\
\end{array} \\
\hline \end{array} \end{align*} \]

3.2 Why equation (12) and its generalization cannot hold

In section 1.2 equation (12), we claimed that

\[ \begin{align*} Y_{123} + Y_{123} & \neq Y_{123} . \quad (42) \end{align*} \]

We have now acquired the necessary tools to show why the two sides fail to match. Assuming equality in (42) and adopting birdtrack notation this relation takes the form

\[ \begin{align*} \begin{array}{c|c} \hline \rowcolor{Gray} & \begin{array}{c} \text{and would imply that} \\
\end{array} \\
\hline \begin{array}{c} 4 \begin{array}{c} \text{is Hermitian, but the left hand side is not:} \\
\end{array} \\
\end{array} \\
\hline \end{array} \end{align*} \]

From section 3.1, the right hand side of (43) is Hermitian, but the left hand side is not:

\[ \begin{align*} Y_{123} \neq Y_{123}^\dagger \end{align*} \]
(as is evident from the expansions in (38), specifically the last terms shown). Thus we have arrived at a contradiction. In a similar way, it can be falsely concluded that $Y_{\Phi}$ is Hermitian. In fact, by assuming that
\begin{equation}
\sum_{\Phi \in \Theta \otimes \Theta} Y_{\Phi} = Y_{\Theta} \quad \text{for } \Theta \in \mathcal{Y}_{n-1}
\end{equation}
holds for Young projection operators $Y_{\Phi}$, it is possible to show that all Young projection operators are Hermitian. In the argument for an arbitrary $n$, an additional step which was not present in the example for $n = 3$ is required. This step however is present in the proof for $n = 4$; we therefore go through this case explicitly. Let us assume that equation (44) holds for $n = 3$,
\begin{align*}
Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} + Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}
\end{array}} &= Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}},
\end{align*}
and also for $n = 4$,
\begin{align*}
Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} &= Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} + Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}
\end{array}} + Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 4 \\
2 & 3
\end{array}
\end{array}} \quad (45) \\
Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} &= Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} + Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}
\end{array}} + Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 4 \\
2 & 3
\end{array}
\end{array}} \quad (46) \\
Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} &= Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} + Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}
\end{array}} + Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 4 \\
2 & 3
\end{array}
\end{array}} \quad (47) \\
Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} &= Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} + Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}
\end{array}} + Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 4 \\
2 & 3
\end{array}
\end{array}} \quad (48)
\end{align*}
Equations (45) and (48) tell us that the operators $Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}}$ and $Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 4 \\
2 & 3
\end{array}
\end{array}}$ are Hermitian\textsuperscript{11}. To show that the remaining operators are Hermitian, we notice that a similarity transformation with an element $\rho$ of $S_4$ of the form $Y_{\Theta} \mapsto \rho Y_{\Theta} \rho^\dagger$ does not change the Hermiticity of the operator $Y_{\Theta}$. Thus, for example the operator $(34) \cdot Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} \cdot (34)$, where (34) $\in S_4$ is a transposition, is still Hermitian. It is now easy to check (via direct calculation) that\textsuperscript{14}
\begin{align*}
(34) \cdot Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} \cdot (34) &= Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}},
\end{align*}
where $(243)^\dagger = (234)$, and similarly
\begin{align*}
(34) \cdot Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}} \cdot (34) &= Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}},
\end{align*}
We therefore conclude that also the Young projection operators $Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}}$, $Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}
\end{array}}$, $Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 4 \\
2 & 3
\end{array}
\end{array}}$ and $Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 5 \\
2 & 4
\end{array}
\end{array}}$ are Hermitian. The remaining two operators $Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{array}}$ and $Y_{\begin{array}{cc}
\begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}
\end{array}}$ can thus be written as linear combinations of Hermitian operators,
\textsuperscript{11}This can be concluded in the same way that we previously found that $Y_{\Phi}$ is Hermitian.
\textsuperscript{14}In fact, [2] defines the Young projection operator of a tableau $\Theta$ that can be obtained from $\Phi$ by reordering the entries of $\Phi$ according to a permutation $\rho$ as $Y_{\Theta} := \rho^\dagger Y_{\Phi} \rho$.
using equations (46) and (47),
\[
Y_1 = Y_1 - \left( Y_2 + Y_3 \right),
\]
\[
Y_2 = Y_2 - \left( Y_3 + Y_4 \right),
\]
leading us to conclude that they are Hermitian as well. Thus, we have (falsely!) found that all Young projection operators corresponding to Young tableaux in \( Y_4 \) are Hermitian.

Proceeding in a similar fashion, it is possible to show that all Young projection operators are Hermitian, which is clearly false. Therefore, we must conclude that assumption (44) does not hold for Young projection operators, and have thus achieved part 1 of Goal 1 of this paper.

Since the obstacle to summability is the lack of Hermiticity of the Young operators, the above discussion provides a strong hint that (44) might hold for a Hermitian version of the Young projection operators, as was already claimed in the Introduction, section 1.2. In section 3.3.2, we show explicitly that this is true, completing part 2 of Goal 1. In order to be able to do so, we first need to describe how to obtain Hermitian Young projection operators. This will be the subject of the following section.

3.3 KS Construction principle for Hermitian Young projection operators

A construction principle for Hermitian Young projection operators has recently been found by Keppeler and Sjödahl [1]. We will now paraphrase this construction principle, see Theorem 3, as it forms a basis for proving that equation (13) and its generalizations indeed hold, section 3.3.2. We will further use this Theorem as a starting point for a new construction principle, which leads much more compact expression for Hermitian Young projection operators, section 4. We will give Keppeler and Sjödahl’s Theorem 3 without proof; a formal proof can be found in [1].

**Theorem 3 (KS Hermitian Young projectors)** Let \( \Theta \in Y_n \) be a Young tableau. If \( n \leq 2 \), then the Hermitian Young projection operator \( P_\Theta \) corresponding to the tableau \( \Theta \) is given by
\[
P_\Theta := Y_\Theta.
\]
This provides a termination criterion for an iterative process that obtains \( P_\Theta \) from \( P_\Theta^{(1)} \) via\(^{15}\)
\[
P_\Theta := P_\Theta^{(1)} Y_\Theta P_\Theta^{(1)},
\]
once \( n > 2 \). In (50) \( P_\Theta^{(1)} \) is understood to be canonically embedded in the algebra \( V^{\otimes n} \). Thus, \( P_\Theta \) is recursively obtained from the full chain of its Hermitian ancestor operators \( P_\Theta^{(m)} \).

The above operators satisfy the same properties as the Young projection operators (c.f. (17)):

\[
* \text{Idempotency:} \quad P_\Theta \cdot P_\Theta = P_\Theta
\]
\[
* \text{Orthogonality:} \quad P_\Theta \cdot P_\Phi = \delta_\Theta\Phi P_\Theta
\]
\[
* \text{Completeness:} \quad \sum_{\Theta \in Y_n} P_\Theta = 1_n
\]

\(^{15}\)In [1], eq. (50) is given as \( P_\Theta = P_\Theta^{(1)} \otimes Y_\Theta \otimes P_\Theta^{(1)} \); however, since the \( P_i \) and \( Y_j \) are understood to be linear maps on the space \( V^{\otimes n} \), this equation is merely a product of linear maps. The authors therefore deem the tensor-product-notation introduced by KS unnecessarily complicated, and denote this product of linear maps as shown above.
As an example, consider the Young tableau
\[
\Theta = \begin{array}{c}
1 & 2 & 1 \\
3 & 5
\end{array}
\]
with ancestor tableaux
\[
\Theta_{(1)} = \begin{array}{c}
1 & 2 & 1 \\
3
\end{array}, \quad \Theta_{(2)} = \begin{array}{c}
1 & 2 \\
3
\end{array} \quad \text{and} \quad \Theta_{(3)} = \begin{array}{c}
1 & 2
\end{array}^{16}
\]
When constructing the Hermitian Young projection operator \( P_\Theta \) according to the KS-Theorem 3, we first have to find \( P_{\Theta_{(3)}} \), \( P_{\Theta_{(2)}} \) and \( P_{\Theta_{(1)}} \). According to the Theorem, \( P_{\Theta_{(3)}} = Y_{\Theta_{(3)}} \), since \( \Theta_{(3)} \in \mathcal{Y}_2 \). Then, following the iterative procedure of the KS-Theorem, \( P_{\Theta_{(2)}} \) and \( P_{\Theta_{(1)}} \) are given by
\[
P_{\Theta_{(2)}} = P_{\Theta_{(3)}} Y_{\Theta_{(2)}} P_{\Theta_{(3)}} = Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}},
\]
and
\[
P_{\Theta_{(1)}} = P_{\Theta_{(2)}} Y_{\Theta_{(1)}} P_{\Theta_{(2)}} = Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}} Y_{\Theta_{(1)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}}.
\]
Lastly, the desired operator \( P_\Theta \) is
\[
P_\Theta = P_{\Theta_{(1)}} Y_{\Theta} P_{\Theta_{(1)}} = Y_{\Theta_{(3)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}} Y_{\Theta_{(1)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}} Y_{\Theta_{(1)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}}.
\]
As a birdtrack, the above operator can be written as
\[
P_\Theta = \frac{128}{9} Y_{\Theta_{(1)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}},
\]
where
\[
\frac{128}{9} = \left( \alpha_{\Theta_{(3)}} \right)^8 \left( \alpha_{\Theta_{(2)}} \right)^4 \left( \alpha_{\Theta_{(1)}} \right)^2 \alpha_\Theta
\]
is the appropriate normalization constant arising from the KS-algorithm.
Let us emphasize that KS have proven that this or any other operator constructed with their algorithm is Hermitian. The operator (54) is however not symmetric under a flip about its vertical axis, and thus Hermiticity is not visually obvious. An additional advantage of the construction algorithm described in the following section 4 is that it will necessarily yield mirror-symmetric operators, making their Hermiticity immediately visible.

3.3.1 Beyond the KS-construction

The results regarding Hermitian Young projection operators presented up until now are all taken from [1]. We will now move beyond the established results and show that

1. the KS-operators can be simplified to yield more compact expressions (Corollary 1)

\[\text{16We do not have to consider the ancestor } \Theta_{(4)}, \text{ since } \Theta_{(3)} \in \mathcal{Y}_2 \text{ and thus terminates the recursion (50).}\]
2. the KS-operators obey equation (14),

\[ \sum_{\Phi \in \{\Theta \otimes n\}} P_\Phi = P_\Theta ; \]

this will be shown in section 3.3.2.

In [22], we found several simplification rules for birdtrack operators, some of which are summarized in section 2.3. In particular, Theorem 1 can be used to shorten the above operator (54) to

\[ P_\Theta = Y_\Theta(3) Y_\Theta(2) Y_\Theta(1) Y_\Theta Y_\Theta(1) Y_\Theta(2) Y_\Theta(3) \]

(55)

\[ = 8 \cdot Y_\Theta(3) Y_\Theta(2) Y_\Theta(1) Y_\Theta Y_\Theta(1) Y_\Theta(2) Y_\Theta(3) \]

(56)

where \((\alpha_\Theta(3) \alpha_\Theta(2) \alpha_\Theta(1))^2 \alpha_\Theta = 8\). The above expression for \(P_\Theta\) is clearly considerably shorter than the expression given in (54). In fact, Theorem 1 allows us to systematically shorten the KS-projection operators, exposing a new, much simpler general form:

\[ \text{Corollary 1 (strictly ordered Hermitian Young projectors)} \]

Let \(\Theta \in \mathcal{Y}_n\) be a Young tableau. Then, the corresponding Hermitian Young projection operator \(P_\Theta\) is given by

\[ P_\Theta = Y_{\Theta(n-2)} Y_{\Theta(n-3)} Y_{\Theta(2)} Y_{\Theta(1)} Y_{\Theta(2)} Y_{\Theta(1)} Y_{\Theta(2)} \cdots Y_{\Theta(n-4)} Y_{\Theta(n-3)} Y_{\Theta(n-2)} . \]

(57)

This result simply follows from a repeated application of Theorem 1, where we notice that \(\Theta_{(n-2)} \in \mathcal{Y}_2\) necessarily.

Even though this simplification is already quite substantial, it is by no means the simplest form achievable. We will present a new construction principle in section 4, creating even more compact and thus easier usable Hermitian Young projection operators. The proof of this construction will however make use of the KS-Theorem 3, see app. A.

### 3.3.2 Spanning subspaces with Hermitian operators

In section 1.2, we claimed that

\[ P_\Xi + P_\Xi = P_\Xi \]

holds if the operators \(P_\Xi\) are Hermitian. We will now set about proving the more general version of this equation: We will show that

\[ \sum_{\Phi \in \{\Theta \otimes n\}} P_\Phi = P_\Theta \]

(58)

holds for every \(\Theta \in \mathcal{Y}_{n-1}\) if the \(P_\Xi\) are the Hermitian operators introduced previously. We begin by showing that a projection operator \(P_\Theta\) projects onto a subspace of the image of an operator \(P_{\Theta_{(m)}}\), where \(\Theta_{(m)}\) is an ancestor tableau of \(\Theta\). In particular, this will mean that the image of an operator \(P_\Theta\) is a subset of the image of its parent operator \(P_{\Theta_{(1)}}\).

\[ \text{Lemma 1 (Subspaces corresponding to Hermitian Young projection operators are nested)} \]

Let \(\Theta \in \mathcal{Y}_n\) be a Young tableau and let \(\Theta_{(m)}\) be its ancestor tableau, with \(m < n\). Furthermore, let \(P_\Theta\) and \(P_{\Theta_{(m)}}\) be the Hermitian Young projection operators corresponding to these tableaux. Then, the image of \(P_\Theta\) lies entirely in the image of \(P_{\Theta_{(m)}}\),

\[ P_\Theta P_{\Theta_{(m)}} = P_\Theta = P_{\Theta_{(m)}} P_\Theta . \]

(59)
We wish to draw attention how the following proof of Lemma 1 makes use of some of the simplification rules given in section 2.3, as this will be mirrored in the proofs of the main Theorems given in appendix A.

Proof of Lemma 1: To prove the inclusion of the subspaces, it suffices to show that the product of the operators satisfies eq. (59) (c.f. eq. (26)). What this relation implies is that if we first act the product \( P_\theta P_\Theta \) (or equivalently \( P_\theta P_\Theta(m) \)) on an object \( x \), we obtain the same outcome as if we only act \( P_\theta \) on \( x \). Hence, \( P_\theta \) must correspond to a smaller subspace than \( P_\Theta(m) \), and this subspace must completely be contained in the subspace corresponding to \( P_\Theta(m) \). From the shortened KS construction, Corollary 1, the Hermitian Young projection operators \( P_\theta \) and \( P_\Theta(m) \) are given by

\[
P_\theta = Y_{\Theta_1} Y_{\Theta_2} \cdots Y_{\Theta_{m+1}} Y_{\Theta_1} Y_{\Theta_2} \cdots Y_{\Theta_{n-2}} Y_{\Theta_{n-1}}\]
\[
P_\Theta(m) = Y_{\Theta_1} Y_{\Theta_2} \cdots Y_{\Theta_{m+1}} Y_{\Theta_1} Y_{\Theta_2} \cdots Y_{\Theta_{n-2}} Y_{\Theta_{n-1}}.
\]

When forming the product \( P_\theta P_\Theta(m) \), we see a lot of cancellation of wedged ancestor operators due to Theorem 1,

\[
P_\theta \cdot P_\Theta(m) = Y_{\Theta_1} Y_{\Theta_2} \cdots Y_{\Theta_{m+1}} Y_{\Theta_1} Y_{\Theta_2} \cdots Y_{\Theta_{n-2}} Y_{\Theta_{n-1}} = Y_{\Theta_{n-1}} Y_{\Theta_{n-2}} \cdots Y_{\Theta_1} Y_{\Theta_2} \cdots Y_{\Theta_{n-2}} Y_{\Theta_{n-1}}.
\]

The above can easily be identified to be the operator \( P_\Theta \), yielding the first equality \( P_\theta P_\Theta(m) = P_\Theta \). The second equality can similarly be shown, leading to the desired result.

We note that Lemma 1 does not hold for non-Hermitian Young projection operators, as we will show shortly. First however, let us show that equation (58) holds: Recall the completeness relation of Hermitian Young projection operators, eq. (53),

\[
\sum_{\Theta \in Y_{n-1}} P_\Theta = \text{id}_{n-1},
\]

where \( \text{id}_k \) is the identity operator on the space \( V^\otimes k \). Equation (60) can be canonically embedded into the space \( V^\otimes n \) as was discussed in section 2.2.2. In order to make the embedding of the operator \( P_\Theta \) explicit, we will – for this section only – make the identity operator on the last factor explicitly visible in the birdtrack spirit and denote the embedded operator by the symbol \( P_\Theta.17 \) The embedded equation (60) thus is

\[
\sum_{\Theta \in Y_{n-1}} P_\Theta = \text{id}_n.
\]

Even though (61) is a decomposition of unity, a finer decomposition of \( \text{id}_n \) (also using only orthogonal objects) is obtained with Hermitian Young projection operators corresponding to Young tableaux in \( Y_n \),

\[
\sum_{\Phi \in Y_n} P_\Phi = \text{id}_n.
\]

\[17\text{In birdtrack notation, the canonically embedded operator } P_\Theta \text{ will be } P_\Theta \text{ with an extra index line on the bottom, making the notation } P_\Theta \text{ intuitive.}
Since clearly $\mathcal{Y}_n$ is the union of all the sets \{\Theta \otimes \square\}, for all $\Theta \in \mathcal{Y}_{n-1}$, the sum (62) can be split into

\[
\sum_{\Phi \in \mathcal{Y}_n} P_{\Phi} = \sum_{\Theta \in \mathcal{Y}_{n-1}} \left( \sum_{\Psi \in \{\Theta \otimes \square\}} P_{\Psi} \right) = \text{id}_n. \quad (63)
\]

Since both (61) and (63) are a decomposition of $\text{id}_n$, they must be equal to each other, yielding

\[
\sum_{\Theta \in \mathcal{Y}_{n-1}} P_{\Theta} = \sum_{\Theta \in \mathcal{Y}_{n-1}} \left( \sum_{\Psi \in \{\Theta \otimes \square\}} P_{\Psi} \right). \quad (64)
\]

Let us now multiply the above equation with a particular operator $P_{\Theta'}$ on $V^\otimes n$, where $\Theta'$ is a particular tableau in $\mathcal{Y}_{n-1}$. Due to the orthogonality property (eq. (52), Theorem 3) and the inclusion property (eq. (59), Lemma 1) of Hermitian Young projectors, it follows that

\[
\sum_{\Theta \in \mathcal{Y}_{n-1}} \delta_{\Theta \Theta'} P_{\Theta} = \sum_{\Theta \in \mathcal{Y}_{n-1}} \left( \delta_{\Theta \Theta'} \sum_{\Psi \in \{\Theta \otimes \square\}} P_{\Psi} \right) = P_{\Theta'},
\]

yielding the desired equation (58). The reason why this proof breaks down for Young projection operators $Y_{\Theta}$, is that the last step does not hold for all Young projection operators,

\[
\left[ \sum_{\Theta \in \mathcal{Y}_{n-1}} \left( \sum_{\Psi \in \{\Theta \otimes \square\}} Y_{\Psi} \right) \right] \cdot Y_{\Theta'} \neq \sum_{\Theta \in \mathcal{Y}_{n-1}} \left( \delta_{\Theta \Theta'} \sum_{\Psi \in \{\Theta \otimes \square\}} Y_{\Psi} \right),
\]

and similarly for left multiplication of $Y_{\Theta'}$. This is due to the fact that not all Young projection operators obey the inclusion properties of Lemma 1: While there exist some Young projection operators $Y_{\Theta}$ for which either $Y_{\Phi} Y_{\Phi(m)} = Y_{\Phi}$ or $Y_{\Phi(m)} Y_{\Phi} = Y_{\Phi}$ holds\(^\text{18}\), both conditions cannot hold for a particular non-Hermitian operator $Y_{\Phi}$, as is shown in the following Lemma 2.

**Lemma 2 (non-Hermitian Young operators don’t commute with their ancestor operators)** Let $\Theta$ be a Young tableau and $\Theta_{(m)}$ be its ancestor $m$ generations back, where $m$ is a strictly positive integer, $m > 0$. If $Y_{\Theta}$ is not Hermitian, $Y_{\Theta}$ and $Y_{\Theta_{(m)}}$ do not commute,

\[
[Y_{\Theta}, Y_{\Theta_{(m)}}] \neq 0. \quad (66)
\]

**Proof of Lemma 2:** We present a proof by contradiction: Suppose there exists a non-Hermitian Young projection operator $Y_{\Theta}$ which commutes with its ancestor operator $Y_{\Theta_{(m)}}$,

\[
Y_{\Theta} Y_{\Theta_{(m)}} = Y_{\Theta_{(m)}} Y_{\Theta}. \quad (67)
\]

If we multiply equation (67) with the operator $Y_{\Theta}$ on the right, and use Theorem 1 to simplify the LHS of the resulting equation, we obtain

\[
\frac{Y_{\Theta} Y_{\Theta_{(m)}} Y_{\Theta}}{=Y_{\Theta} Y_{\Theta}} = \frac{Y_{\Theta_{(m)} Y_{\Theta}}}{=Y_{\Theta}} \Rightarrow Y_{\Theta} = Y_{\Theta_{(m)}} Y_{\Theta}
\]

\[
\iff \alpha_{\Theta} \cdot S_{\Theta} A_{\Theta} = \alpha_{\Theta_{(m)}} \cdot S_{\Theta_{(m)}} A_{\Theta_{(m)}} S_{\Theta} A_{\Theta}. \quad (68)
\]

\(^\text{18}\)For most Young projection operators, neither of these conditions are true, as can be seen on the example $Y_{\Theta} = Y_{\square}$ and $Y_{\Theta_{(m)}} = Y_{\square \otimes \square}$.
If \( h_\Theta \) denotes the set of all horizontal permutations corresponding to \( \Theta \) as introduced in section 2.1, then the LHS of equation (68) satisfies

\[
h_\Theta (\alpha_\Theta \cdot S_\Theta A_\Theta) = (\alpha_\Theta \cdot S_\Theta A_\Theta) \quad \text{for every } h_\Theta \in h_\Theta.
\]

Thus, the RHS of equation (68) must also satisfy

\[
h_\Theta (\alpha_{\Theta(m)} \cdot S_{\Theta(m)} A_{\Theta(m)} S_{\Theta} A_\Theta) = (\alpha_{\Theta(m)} \cdot S_{\Theta(m)} A_{\Theta(m)} S_{\Theta} A_\Theta) \quad \text{for every } h_\Theta \in h_\Theta.
\]

This then implies that the symmetrizers \( S_{\Theta(m)} \) and \( S_\Theta \) must be equal, as otherwise equation (69) cannot hold, [2]. Following similar steps, one can conclude that \( A_{\Theta(m)} \) and \( A_\Theta \) must be equal. Thus, assumption (67) yields

\[
Y_{\Theta(m)} = Y_\Theta,
\]

which is a contradiction for any \( m > 0 \). Thus, we conclude that

\[
[Y_\Theta, Y_{\Theta(m)}] \neq 0
\]

for every Young projection operator \( Y_\Theta \) and all of its ancestor operators \( Y_{\Theta(m)} \) with \( m > 0 \).

Let us now go beyond result (58), as it can be generalized even further: Since the Hermitian operators sum up to their Hermitian parent operators (eq.(65)), and these in turn sum to their Hermitian parent operators, the summation property necessarily holds over multiple generations. This statement also follows straight from Lemma 1, which states that the image of a Hermitian Young projection operator \( P_\Theta \) is contained in the image of its Hermitian ancestor operator \( P_{\Theta(m)} \), where \( m \) can be any positive integer. Therefore, if \( Y_{\Theta,n} := \{ \Theta \otimes m \otimes \cdots \otimes n \} \) is the subset of \( Y_n \) containing all tableaux that have \( \Theta \in Y_{m-1} \) as their ancestor, then

\[
\sum_{\Phi \in Y_{\Theta,n}} P_\Phi = P_\Theta,
\]

confirming eq. (15). For example, if \( \Theta = \begin{array}{c|c|c} 1 & 2 & 3 \end{array} \), then \( P_\Theta \) can be written as a sum of the following Hermitian Young projection operators corresponding to tableaux in \( Y_5 \),

\[
\sum_{\Phi \in Y_{\Theta,5}} P_\Phi = P_{\Theta(m)}.
\]

This concludes the first Goal of this paper.

Figure 1 shows all Hermitian Young projectors corresponding to Young tableaux in \( Y_n \) up to and including \( n = 4 \). The arrows in this figure indicate which operators sum to which ancestor operators. The summation property of Hermitian Young projectors was not mentioned in [16, fig. 9.1] where a virtually identical figure can be found.
Figure 1: Hierarchy of Young tableaux and the associated nested Hermitian Young projector decompositions (in the sense of embeddings into API (SU(N), V⊗4)): Projection operators that are contained in a gray box correspond to equivalent irreducible representations of SU(N), as their corresponding Young tableaux have the same shape [2, 16]. The dimension of the irreducible representation corresponding to a (set of) operators(s) is given on the left [16, fig. 9.1]. The arrows indicate which operators sum to which ancestor – this summation property of Hermitian Young projection operators was not observed by [16].
4 An algorithm to construct compact expressions of Hermitian Young projection operators

We will now come to Goal 2 of this paper and provide a construction principle that allows us to directly arrive at compact expressions for Hermitian Young projection operators (see Theorem 5 below). This construction yields much shorter expressions than the previously encountered KS algorithm (Theorem 3), or even the shortened version of Theorem 1, as is exemplified in Fig. 2.

4.1 Lexically ordered Young tableaux

It turns out that the ordering of the numbers in the Young tableau plays a vital role in our algorithm. Thus, we will first establish what we mean by the lexical order of a Young tableau. To do so, we will introduce the column- and row-word corresponding to a tableau.

Definition 2 (column- and row-words & lexical ordering) Let $\Theta \in \mathcal{Y}_n$ be a Young tableau. We define the column-word of $\Theta$, $C_\Theta$, to be the column vector whose entries are the entries of $\Theta$ as read column-wise from left to right. Similarly, the row-word of $\Theta$, $R_\Theta$, is defined to be the row vector whose entries are those of $\Theta$ read row-wise from top to bottom.

We will call a tableau $\Theta$ lexically ordered, if either $C_\Theta$ or $R_\Theta$ or both are in lexical order. In particular, we say that $\Theta$ is column-ordered (resp. row-ordered), if $C_\Theta$ (resp. $R_\Theta$) is in lexical order.

For example, the tableau

$$\Phi := \begin{array}{ccc}
1 & 5 & 7 & 9 \\
2 & 6 & 8 \\
3 \\
4
\end{array}$$

has a column-word

$$C_\Phi = (1, 2, 3, 4, 5, 6, 7, 8, 9)^t$$

and a row-word

$$R_\Phi = (1, 5, 7, 9, 2, 6, 8, 3, 4).$$

From this, we see that $\Phi$ in (72) is lexically ordered. In particular, it is column-ordered (but not row-ordered).

In Theorem 4 we will describe a construction principle for the Hermitian Young projection operators corresponding to lexically ordered tableaux. This will form a starting point for the general construction principle of the Hermitian Young projection operators given in section 4.2. It is clear that Keppeler and Sjödahl had noticed that the projectors associated ordered tableaux are special: In the appendix of [1] they discuss two examples of Hermitian Young projection operators (which happen to correspond to lexically ordered tableaux) constructed according to the KS-Theorem, and argue that these operators can be simplified quite drastically. The procedure leads eventually to the same expressions that emerge directly from Theorem 4. However, Keppeler and Sjödahl do not establish the connection to the lexical order of the Young tableau and do not even hint at a general construction principle.

It should be noted that Definition 2 of the row-word is different to the definition given in the standard literature such as [13, 23]: there, the row word is read from bottom to top rather than from top to bottom. However, for the purposes of this paper, Definition 2 is more useful than the standard definition.
Theorem 4 (lexical Hermitian Young projectors)  Let $\Theta \in Y_n$ be row-ordered. Then, the corresponding Hermitian Young projection operator $P_\Theta$ is given by

$$P_\Theta = \alpha_\Theta \cdot \bar{Y}_\Theta \hat{Y}_\Theta. \tag{73a}$$

On the other hand, if $\Theta \in Y_n$ is a column-ordered tableau, then the corresponding Hermitian Young projection operator $P_\Theta$ is given by

$$P_\Theta = \alpha_\Theta \cdot \bar{Y}_\Theta \hat{Y}_\Theta. \tag{73b}$$

The proof of this Theorem is deferred to Appendix A.1. It is directly evident from eqns. (73a) and (73b) that $P_\Theta$ is Hermitian in both cases. Since Hermitian conjugation in birdtrack notation amounts to reflection about a vertical axis, the formulae also guarantee that Hermiticity is directly visible as a reflection symmetry of the associated birdtrack diagrams.

As an example, consider the Young tableau

$$\Theta = \begin{array}{c|c|c} 1 & 2 \\ \hline 3 \end{array}$$

which has a lexically ordered row-word $\mathcal{R}_\Theta = (1, 2, 3)$. The associated Hermitian Young projection operator $P_\Theta$ according to the Lexical-Theorem 4 is given by

$$P_\Theta = \begin{array}{c|c|c} 4 & 3 \\ \hline \alpha_\Theta \end{array} \bar{Y}_\Theta \hat{Y}_\Theta.$$ 

The Hermiticity of this operator is prominently visible in its mirror symmetry.

4.2 Young tableaux with partial lexical order

We will now give a construction principle for compact expressions of Hermitian Young projection operators corresponding to general, not necessarily lexically ordered, tableaux. The goal is to use what partial order there is to a diagram to simplify an optimized iterative procedure. As a first step we need to be able to quantify how “un-ordered” a Young tableau is; we define a Measure Of Lexical Disorder.

Definition 3 (measure of lexical disorder (MOLD))  Let $\Theta \in Y_n$ be a Young tableau. We define its Measure Of Lexical Disorder (MOLD) to be the smallest natural number $M(\Theta) \in \mathbb{N}$ such that

$$\Theta(M(\Theta)) = \pi^{M(\Theta)}(\Theta) \tag{74}$$

is a lexically ordered tableau. (Recall from Definition 1 that $\pi^{M(\Theta)}$ refers to $M(\Theta)$ consecutive applications of the parent map $\pi$ to the tableau $\Theta$.)

We note that the MOLD of a Young tableau is a well-defined quantity, since one will always eventually arrive at a lexically ordered tableau, as, for example, all tableaux in $Y_3$ are lexically ordered. This then implies that the MOLD of a tableau $\Theta \in Y_n$ has an upper bound,

$$M(\Theta) \leq n - 3, \tag{75}$$

making it a well-defined quantity. As an example, consider the tableau

$$\Phi := \begin{array}{c|c|c} 1 & 2 & 4 \\ \hline 3 & 5 \end{array}.$$
The MOLD of the above tableau is $M(\Theta) = 2$, since two applications of the parent map generate a lexically ordered tableau, but just one application of $\pi$ on $\Phi$ would not be sufficient.

$$
\begin{array}{c}
1 & 2 & 4 \\
3 & 5 \\
\end{array}
\xrightarrow{\pi} 
\begin{array}{c}
1 & 2 & 4 \\
3 & \Theta \\
\end{array}
\xrightarrow{\pi} 
\begin{array}{c}
1 & 2 \\
3 & \Theta \\
\end{array}
$$

We will now give the main Theorem of this section, the construction principle of Hermitian Young projection operators corresponding to Young tableaux $\Theta$, using the MOLD of the latter. To do so, we distinguish four cases; the reason why they have to be dealt with separately is given in the analysis following the Theorem, section 4.2.1.

**Theorem 5 (MOLD operators)** Consider a Young tableau $\Theta \in \mathcal{Y}_n$ with MOLD $M(\Theta) = m$. Furthermore, suppose that $\Theta_m$ has a lexically ordered row-word. Then, the Hermitian Young projection operator corresponding to $\Theta$, $P_\Theta$, is, for $m$ even,

$$
P_\Theta = \beta_\Theta \cdot S_{\Theta(m)} \cdot A_{\Theta(m-1)} \cdot S_{\Theta(m-2)} \cdots \cdot S_{\Theta(2)} \cdot A_{\Theta(1)} \cdot Y_{\Theta} Y_\Theta^\dagger A_{\Theta(1)} \cdot S_{\Theta(2)} \cdots \cdot S_{\Theta(m-2)} \cdot A_{\Theta(m-1)} \cdot S_{\Theta(m)}; \quad (76a)
$$

and, for $m$ odd,

$$
P_\Theta = \beta_\Theta \cdot S_{\Theta(m)} \cdot A_{\Theta(m-1)} \cdot S_{\Theta(m-2)} \cdots \cdot A_{\Theta(2)} \cdot S_{\Theta(1)} \cdot Y_{\Theta} Y_\Theta^\dagger A_{\Theta(1)} \cdot S_{\Theta(2)} \cdots \cdot S_{\Theta(m-2)} \cdot A_{\Theta(m-1)} \cdot S_{\Theta(m)} \cdot A_{\Theta(m)}; \quad (76b)
$$

Similarly, if $\Theta_m$ has a lexically ordered column-word, $P_\Theta$ is given by, for $m$ even,

$$
P_\Theta = \beta_\Theta \cdot A_{\Theta(m)} \cdot S_{\Theta(m-1)} \cdot A_{\Theta(m-2)} \cdots \cdot A_{\Theta(2)} \cdot S_{\Theta(1)} \cdot Y_{\Theta} Y_\Theta^\dagger A_{\Theta(1)} \cdot S_{\Theta(2)} \cdots \cdot S_{\Theta(m-2)} \cdot A_{\Theta(m-1)} \cdot A_{\Theta(m)}; \quad (76c)
$$

and, for $m$ odd,

$$
P_\Theta = \beta_\Theta \cdot A_{\Theta(m)} \cdot S_{\Theta(m-1)} \cdot A_{\Theta(m-2)} \cdots \cdot S_{\Theta(2)} \cdot A_{\Theta(1)} \cdot Y_{\Theta} Y_\Theta^\dagger A_{\Theta(1)} \cdot S_{\Theta(2)} \cdots \cdot A_{\Theta(m-2)} \cdot S_{\Theta(m-1)} \cdot A_{\Theta(m)} \cdot A_{\Theta(m)}; \quad (76d)
$$

In the above, all symmetrizers and antisymmetrizers are understood to be canonically embedded into the algebra $V^\otimes n$; $\beta_\Theta$ is a non-zero constant chosen such that $P_\Theta$ is idempotent.

The formal proof of this Theorem can be found in Appendix A.2. A comparative example of a Hermitian Young projection operator constructed using MOLD and KS is given in section 4.3, Fig 2. In Theorem 5, it should be noted that we have not provided a form of the constant $\beta_\Theta$. This normalization-constant however can easily be found for specific operators by direct calculation since the MOLD-operators are very much suited for automated calculations on a computer, as is described in section 4.3. We would like to draw the reader’s attention to the fact that the symmetrizers and antisymmetrizers in all four expressions of Theorem 5 strictly alternate, including those inside the Young projectors.

As an example, consider the Young tableau

$$
\Theta := \begin{array}{c}
1 & 2 & 4 \\
3 & 5 \\
\end{array}
$$

This tableau has MOLD 2 (i.e. even MOLD), and $\Theta_2$ has a lexically ordered row-word. Thus, we have to construct the Hermitian Young projection operator $P_\Theta$ corresponding to $\Theta$ according to equation (76a). $P_\Theta$
is therefore given by

\[ P_{\Theta} = \beta_{\Theta} \cdot S_{\Theta(1)} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(2)} S_{\Theta(2)} \]

\[ = \beta_{\Theta} \cdot S_{\Theta(1)} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(2)} S_{\Theta(2)} \]

A direct calculation in Mathematica reveals that \( \beta_{\Theta} = 4 \) for \( P_{\Theta} \) to be idempotent.

### 4.2.1 A Short Analysis of the MOLD-Theorem 5

We now pause for a moment to look at the four cases presented in Theorem 5 in more detail and emphasize their differences. We hope to convey an intuitive feel as to why the corresponding operators are constructed the way they are.

First, let us look at the first two operators (76a) and (76b). Both these operators \( P_{\Theta} \) have a symmetrizer on the outside, namely \( S_{\Theta(m)} \), opposed to the operators (76c) and (76d) which have an antisymmetrizer on the outside. This stems from the iterative construction of Hermitian Young projection operators given by the KS-Theorem 3: By the Lexical-Theorem 4 we know that \( P_{\Theta(m)} \) is given by

\[ P_{\Theta(m)} = \alpha_{\Theta} \cdot \bar{Y}_{\Theta(m)} \bar{\bar{Y}}_{\Theta(m)} = \alpha_{\Theta} \cdot S_{\Theta(m)} A_{\Theta(m)} S_{\Theta(m)}, \]  

(77)

since \( P_{\Theta(m)} \) is assumed to correspond to a row-ordered Young tableau. When we thus construct \( P_{\Theta} \) recursively according to KS, [1], we find that

\[ P_{\Theta} \propto P_{\Theta(m)} ... Y_{\Theta} ... P_{\Theta(m)} \propto S_{\Theta(m)} A_{\Theta(m)} S_{\Theta(m)} ... Y_{\Theta} ... S_{\Theta(m)} A_{\Theta(m)} S_{\Theta(m)}. \]  

(78)

Thus, we expect there to be symmetrizers on the outside of the operators \( P_{\Theta} \) in expressions (76a) and (76b). Following a similar logic, we expect there to be antisymmetrizers on the outside of operators (76c) and (76d).

Lastly, we discuss the importance of the distinction between even and odd \( m \) in the above MOLD-Theorem 5. In the construction of all \( P_{\Theta} \) in the Theorem, we find that they consist of products of alternating symmetrizers and antisymmetrizers to more and more recent generations of \( \Theta \) as we move further to the center of \( P_{\Theta} \). If the operator \( P_{\Theta} \) thus starts with \( S_{(m)} \) on the outside, as it does in equations (76a) and (76b), and the product has alternating sets of symmetrizers and antisymmetrizers each going up one generation, then the parity of \( m \) will decide whether the set corresponding to the tableau \( \Theta_{(1)} \) in the product \( P_{\Theta} \) is a set of symmetrizers or antisymmetrizers. Thus, the central three sets of symmetrizers and antisymmetrizers in the product \( P_{\Theta} \) will then either be

\[ A_{\Theta} S_{\Theta} A_{\Theta} = \bar{Y}_{\Theta} \bar{Y}_{\Theta} \]  

or

\[ S_{\Theta} A_{\Theta} S_{\Theta} = \bar{Y}_{\Theta} \bar{Y}_{\Theta} \]  

(79)

dependent on the nature of the sets corresponding to \( \Theta_{(1)} \), but keeping the alternating structure of symmetrizers and antisymmetrizers.

The fact that the central sets of \( P_{\Theta} \) in all four equations of the above Theorem 5 are either product of (79) opposed to simply \( Y_{\Theta} \) or \( \bar{Y}_{\Theta} \) can be attributed to the fact the \( P_{\Theta} \) is Hermitian and we would like its Hermiticity to be visually explicit.
4.3 The advantage of using MOLD

The practical advantages of our construction opposed to the KS-Theorem 3 are striking. To illustrate this we return to the same example used in [22] to demonstrate the effectiveness of the simplification rules derived there. The Young tableau

\[
\Phi := \begin{array}{cccc}
1 & 2 & 4 & 7 \\
3 & 6 & 5 & 8 \\
9 & \\
\end{array}
\] (80)

leads to an expression with 127 symmetrizer- and antisymmetrizer-sets which reduce to an object with only 13 such sets after applying the cancellation and propagation rules of [22].

Both construction principles (KS and MOLD) are iterative in the sense that they both require knowledge about the ancestor tableaux of a tableau \( \Theta \in \mathcal{Y}_n \). For the construction of KS as it was originally described in [1], one needs all ancestor operators of \( \Theta \) up until \( \Theta_{(n-2)} \), while the MOLD construction merely uses the ancestor tableaux up until \( \Theta_{M(\Theta)} \), which is at most \( \Theta_{(n-3)} \) according to (75). This one tableau difference does not seem excessive at first glance, but one should keep in mind that the difference is at least one tableau, often more. However the bulk of the computing power used to generate \( P^K_S \) comes from the fact that, in addition to the ancestor tableaux of \( \Theta \), one further requires information about the explicit form of the ancestor Hermitian Young projectors \( P_\Theta \) all the way up to \( P_{\Theta(2)} \), which have to be calculated separately. The MOLD-construction merely uses the Young sets of symmetrizers or antisymmetrizers (S and A respectively) of the ancestor tableaux of \( \Theta \) up to \( \Theta_{M(\Theta)} \), which can be immediately read off the tableaux and thus needs minimal computing power.

Using the MOLD-Theorem 5, one arrives at the shorter version directly, after a considerably shorter recursive path and without the need for additional simplifications. One bypasses a long repetitive list of steps altogether! The 127/13 length ratio between \( P^K_S \Phi \) and \( P^M_O \Phi \)20 is strikingly apparent in Fig. 2. This makes the MOLD algorithm a lot more practical to work with analytically. The algorithm really comes into its own when used in symbolic algebra programs: the MOLD construction allows us to efficiently create projection operators for considerably larger Young tableaux than the iterative KS equivalent. In particular, for the example in Figure 2, the fact that the MOLD algorithm simply avoids a long series of steps makes it over 18600 times faster than its KS equivalent: It generates its result in approximately 0.0038 seconds, while KS takes approximately 71 seconds (not even taking into account the cost of the simplification steps to arrive at the final result) on a modern laptop.

\[
\beta^K_S \cdot \hat{P}^K_S = P = \beta^M_O \cdot \hat{P}^M_O
\]

but \( \beta^K_S \neq \beta^M_O \) in general.

---

Figure 2: For a size comparison, this figure shows \( P^K_S \Phi \) (top) and \( P^M_O \Phi \) (bottom) for the tableau \( \Phi \) as defined in (80) using two different constructions. The top operator was constructed using the KS-algorithm, while the bottom operator was constructed using MOLD. Both operators and the associated graphics were generated in Mathematica.
Unlike $p^{KS}_{\Theta}$, $p^{MOLD}_{\Theta}$ is obviously and visibly Hermitian by construction.\(^{21}\) Given that birdtracks are meant to be a tool that makes dealing with these operators visually clear, this is a clear advantage.

We have given a construction principle for compact expressions of Hermitian Young projection operators, the MOLD-operators, in section 4.2, and we have now seen that the MOLD-operators are indeed more useful for practical calculations. We have thus achieved Goal 2 of this paper.

5 Conclusion & outlook

The representation theory of SU($N$) is an old theory with many successful applications in physics. Yet some of the tools remain awkward and only applicable in specific situations, like the general theory of angular momentum or the construction of Young projection operators that lack Hermiticity. Newer tools like the birdtrack formalism remain only partially connected with these time honored results. We have a very specific interest in applications to QCD in the JIMWLK context, in jet physics, in energy loss and generalized parton distributions, so we have aimed at creating a set of tools that we know will aid in these applications and, in the process have pointed out where the existing tools fall short of our needs.

1. We have found that projection operators built on Young tableaux are uniquely suited to calculations that keep $N$ as a parameter.

To simply list the irreducible multiplets in $V \otimes \omega^m$, Young’s procedure forms descendant tableaux of those representing the irreducibles contained in $V \otimes (\omega^{m-1})$, portraying an iterative procedure of “adding a particle” in each step.

To parallel this in terms of projection operators and the associated subspaces (i.e. to implement eq. (70) which represents the general case of the summation relations collected in Fig. 1) –as one needs to do to actually perform calculations in physics applications– we have established that one needs Hermitian versions of these Young projection operators as those constructed earlier by Keppeler and Sjödahl [1]. This sets the backdrop for the remaining developments.

2. Having motivated the necessity for Hermitian Young projection operators, we are faced with the fact that the KS algorithm quickly becomes unwieldy – the iterative procedures are computationally expensive and produce long expressions (Fig. 2). We have earlier presented simplification rules [22] to distill these down to more compact expressions, but that does not alleviate the computational cost.

To address this issue, we have provided a new algorithm based on the Measure Of Lexical Disorder (MOLD) of a tableau in sec. 4.2 that drastically reduces the calculational footprint of the procedure compared to the KS method. This algorithm almost completely incorporates the simplifications of [22] at vastly reduced calculational cost (only isolated cases of MOLD-operators still allow for further simplification with the tools presented in [22]). All the algorithms are eminently suited for implementation in symbolic algebra programs: all our explorations and examples have been generated in Mathematica.

In particular, the operators shown in Fig. 2 were generated in Mathematica: the operator on the top was constructed using the KS-Theorem, while the operator on the bottom resulted from the MOLD-construction. The automated calculation was significantly improved with the MOLD-algorithm, as the MOLD-operator in Fig. 2 was obtained approximately 18600 times faster than the KS-equivalent on a modern laptop – an improvement of 4 orders of magnitude.

\(^{21}\) $p^{KS}_{\Theta}$ do not exhibit their Hermiticity directly since the center-piece of the KS operator is the Young projection operator $\bar{Y}_{\Theta}$, which is inherently non-Hermitian. We need to rely on the proof given in [1] to be assured of their Hermiticity.
Our own list of applications for the tools and insights presented in this paper are QCD centric: Global singlet state projections of Wilson-line operators that appear in a myriad of applications due to factorization of hard and soft contributions help analyzing the physics content in all of them; this will be explored further in [21]. We hope that our presentation is suitable to unify perspectives provided by the various approaches to representation theory of SU(N) and that the results prove useful beyond these immediate applications.

Acknowledgements: H.W. is supported by South Africa’s National Research Foundation under CPRR grant nr 90509. J.A-Z. was supported (in sequence) by the postgraduate funding office of the University of Cape Town (2014), the National Research Foundation (2015) and the Science Faculty PhD Fellowship of the University of Cape Town (2016).

A Proofs

This appendix provides the proofs of the Theorems given in section 4.

A.1 Proof of Theorem 4 “lexical Hermitian Young projectors”

The proof of Theorem 4 makes use of propagation rules of birdtrack operators [22]. We thus summarize the applicable rules of [22] in section A.1.1, before giving the proof of the Lexical-Theorem 4 in section A.1.2.

A.1.1 Propagation rules

We first require the definition of a new quantity, an amputated tableau:

**Definition 4 (amputated (Young) tableaux)** Let \( \Theta \) be a (Young) tableau and let \( \mathcal{C} \) denote a particular column in \( \Theta \). We construct the row-amputated tableau of \( \Theta \) according to \( \mathcal{C} \), \( \mathcal{S}_r[\mathcal{C}] \), by removing all rows of \( \Theta \) which do not overlap with \( \mathcal{C} \).

Similarly, if \( \mathcal{R} \) is a particular row in \( \Theta \), we construct the column-amputated tableau of \( \Theta \) according to \( \mathcal{R} \), \( \mathcal{S}_c[\mathcal{R}] \), by removing all columns that do not overlap with \( \mathcal{R} \).

For example, for the following tableau \( \Theta \), the row amputated tableau \( \mathcal{S}_r \) according to column \((3, 4, 7)^t\) is

\[
\begin{array}{ccc}
1 & 3 & 5 & 9 \\
2 & 4 & 8 & 10 \\
6 & 7 & 15 \\
11 \\
12
\end{array}
\mapsto \begin{array}{ccc}
1 & 3 & 5 & 9 \\
2 & 4 & 8 & 10 \\
6 & 7 & 13 \\
\end{array}
\]

(81)

the rows (11) and (12) were deleted, since they did not overlap with the shaded column \((3, 4, 7)^t\). Another example for column amputation is shown in the step from eq. (87) to (88). The idea of amputated tableaux is necessary to describe the following simplification rule for birdtrack operators:

**Theorem 6 (propagation of (anti-) symmetrizers)** Let \( \Theta \) be a Young tableau and \( O \) be a birdtrack operator of the form

\[
O = S_\Theta A_\Theta S_{\Theta \setminus \mathcal{R}}.
\]

(82)
in which the symmetrizer set \( S_{\Theta \setminus R} \) arises from \( S_\Theta \) by removing precisely one symmetrizer \( S_R \). By definition \( S_R \) corresponds to a row \( R \) in \( \Theta \) such that \( S_\Theta = S_{\Theta \setminus R} S_R = S_R S_{\Theta \setminus R} \).

If the column-amputated tableau of \( \Theta \) according to the row \( R \), \( \Theta_c [R] \), is \textit{rectangular}, then the symmetrizer \( S_R \) may be propagated through the set \( A_\Theta \) from the left to the right, yielding

\[
O = S_\Theta A_\Theta S_{\Theta \setminus R} = S_{\Theta \setminus R} A_\Theta S_\Theta,
\]

which implies that \( O \) is Hermitian. We may think of this procedure as moving the missing symmetrizer \( S_R \) through the intervening antisymmetrizer set \( A_\Theta \).

Noting that \( S_\Theta = S_\Theta S_R = S_RS_\Theta \) we immediately augment this statement to

\[
S_\Theta A_\Theta S_{\Theta \setminus R} = S_{\Theta \setminus R} A_\Theta S_\Theta = S_\Theta A_\Theta S_\Theta.
\]

(84)

If the roles of symmetrizers and antisymmetrizers are exchanged, we need to verify that the row-amputated tableau \( \Theta_r [C] \) with respect to a column \( C \) is rectangular to guarantee that

\[
A_\Theta S_\Theta A_{\Theta \setminus C} = A_{\Theta \setminus C} S_\Theta A_\Theta = A_\Theta S_\Theta A_\Theta.
\]

(85)

This amounts to moving the missing antisymmetrizer \( A_C \) through the intervening symmetrizer set \( S_\Theta \).

As an example, consider the operator \( Q \) given by

\[
Q := \begin{bmatrix}
S_\Theta & A_\Theta & S_{\Theta \setminus R}
\end{bmatrix},
\]

(86)

To check if the amputated tableau is rectangular we first need to reconstruct \( \Theta \) with rows corresponding to \( S_\Theta \) and columns corresponding to \( A_\Theta \). Evidently,

\[
\Theta = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & \\
6 & 7
\end{bmatrix}.
\]

(87)

In \( \Theta \), we have marked the row \((6, 7)\) corresponding to the symmetrizer \( S_{67} \), which we would like to propagate through to the right. Thus, in accordance with Theorem 6, we form the column-amputated tableau of \( \Theta \) according to the row \((6, 7)\),

\[
\Theta_c [(6, 7)] = \begin{bmatrix}
1 & 2 \\
4 & 5 \\
6 & 7
\end{bmatrix},
\]

(88)

and see that it is indeed a rectangular tableau. Thus, we may propagate the symmetrizer \( S_{67} \) from the left to the right,

\[
Q = \begin{bmatrix}
S_\Theta & A_\Theta & S_{\Theta \setminus R}
\end{bmatrix} = \begin{bmatrix}
S_\Theta & A_\Theta & S_{\Theta \setminus R}
\end{bmatrix} = \begin{bmatrix}
S_\Theta & A_\Theta & S_{\Theta \setminus R}
\end{bmatrix}.
\]
A.1.2 Proof of Theorem 4:

We will present a proof by induction: First, we proof the Base Step for the projection operators of \( SU(N) \) over \( V^{\otimes 3} \) (i.e. with 3 legs) since this is the smallest instant for which the KS-algorithm produces a new operator. Thereafter, we will consider a general projection operator corresponding to a Young tableau \( \Theta \in \mathcal{Y}_{m+1} \) with a lexically ordered column-word (the proof for operators corresponding to row-ordered Young tableaux is very similar and thus left as an exercise to the reader). We will assume that Theorem 4 is true for the Hermitian operator corresponding to its parent tableau \( P_{\Theta(1)} \), where \( \Theta(1) \in \mathcal{Y}_m \); this is the Induction Hypothesis. Then, we show that the projection operators obtained from the KS-Theorem reduce to the expression given in the Lexical-Theorem 4,

\[
P_{\Theta} = P_{\Theta(1)} Y_{\Theta} = \alpha_{\Theta} \cdot Y_{\Theta}^{\dagger} Y_{\Theta}, \tag{89}
\]

concluding the proof.

The Base Step: For the projection operators of \( SU(N) \) over \( V^{\otimes 1} \) or \( V^{\otimes 2} \) (i.e. with 1 or 2 legs), the proof of (89) is trivial since all Young projection operators are automatically Hermitian; thus, \( \bar{Y}_{\Theta}^{\dagger} \) = \( \bar{Y}_{\Theta} \), and (89) reduces to

\[
\alpha_{\Theta} \cdot Y_{\Theta}^{\dagger} Y_{\Theta} = \alpha_{\Theta} \cdot \bar{Y}_{\Theta} \cdot \bar{Y}_{\Theta} = \bar{Y}_{\Theta}. \tag{90}
\]

Since all Young projection operators \( Y_{\Theta} \) with \( \Theta \in \mathcal{Y}_{1,2} \) have normalization constant 1 (as can easily be checked by looking at all three of them explicitly), \( Y_{\Theta} = \bar{Y}_{\Theta} \) holds for these operators. Thus, the Lexical-Theorem 4 returns the original, already Hermitian operators as does the original KS-algorithm. The first nontrivial differences occur for \( n = 3 \). We use this as the base step. Here, we have the following Young projection operators corresponding to their respective Young tableaux,

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\end{array} & \quad 4 \quad 3 \quad 4 \quad 3
\end{align*}
\tag{91a}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\end{array} & \quad 1 \quad 3 \quad 1 \quad 2 \quad 1 \quad 2 \quad 3
\end{align*}
\tag{92}
\]

In (91), the first and last operator are already Hermitian and have normalization constant 1. Therefore, the Lexical-Theorem will return these operators unchanged, c.f. eq. (90).

The second and third tableaux in (91) are lexically column-ordered and row-ordered respectively. Thus, we must construct their Hermitian Young projection operators according to prescriptions (73b) and (73a) respectively.

Table 1 shows that the construction of the Hermitian projection operators for (91) obtained from the Lexical-Theorem 4 is equivalent that of the KS-Theorem 3, [1], thus concluding the base step of this proof:
KS-Theorem 3, [1] (Multiplying Hermitian parent on either side) | Lexical-Theorem 4 (Multiplying by Hermitian conjugate on appropriate side)

Table 1: This table contrasts the construction of Hermitian Young projection operators according to the KS-Theorem 3 (left), with that according to the Lexical-Theorem 4 (right). Despite visible algorithmic differences, the results are identical.

The Induction Step: Let $\Theta \in \mathcal{Y}_{m+1}$ be a tableau with a lexically ordered column-word, and let $\Theta^{(1)} \in \mathcal{Y}_m$ be its parent tableau. Clearly, the column-word of $\Theta^{(1)}$ is also in lexical order. We will assume that the Lexical-Theorem 4 holds for the Hermitian Young projection operator $P_{\Theta^{(1)}}$, i.e. that

$$P_{\Theta^{(1)}} = \alpha_{\Theta^{(1)}} \cdot \bar{Y}^\dagger_{\Theta^{(1)}} \bar{Y}_{\Theta^{(1)}},$$

and we will refer to this condition as the Induction Hypothesis. Thus, according to this induction hypothesis, $P_{\Theta^{(1)}}$ can be written in terms of sets of symmetrizers and antisymmetrizers corresponding to the tableau $\Theta^{(1)}$ as

$$P_{\Theta^{(1)}} = \alpha_{\Theta^{(1)}} \cdot \bar{Y}^\dagger_{\Theta^{(1)}} \bar{Y}_{\Theta^{(1)}} \cdot \bar{P}_{\Theta^{(1)}},$$

where we used the fact that $S_{\Theta^{(1)}} S_{\Theta^{(1)}} = S_{\Theta^{(1)}}$. We now construct $\bar{P}_\Theta$ from $\bar{P}_{\Theta^{(1)}}$ using the KS-Theorem 3; we have

$$\bar{P}_\Theta = \bar{P}_{\Theta^{(1)}} \cdot \bar{Y}^\dagger_{\Theta^{(1)}} \bar{Y}_{\Theta^{(1)}},$$

In the above, we have chosen to ignore the proportionality constant for now, as carrying it with us would draw attention away from the important steps of the proof. Once we have shown that $\bar{P}_\Theta = \bar{Y}^\dagger_{\Theta^{(1)}} \bar{Y}_{\Theta^{(1)}}$, we will show that the proportionality constant $\alpha_\Theta$ given in (93) is indeed the one we require for $P_\Theta$ to be idempotent.

Since $\Theta^{(1)}$ is the parent tableau of $\Theta$, the images of all symmetrizers and antisymmetrizers in $Y_\Theta$ (and thus $P_\Theta$) are contained in the images of the symmetrizers and antisymmetrizers in $Y_{\Theta^{(1)}}$, respectively.\(^{22}\)

$$S_\Theta \subset S_{\Theta^{(1)}} \quad \text{and} \quad A_\Theta \subset A_{\Theta^{(1)}},$$

and hence

$$S_{\Theta^{(1)}} S_\Theta = S_\Theta = S_\Theta S_{\Theta^{(1)}} \quad \text{and} \quad A_{\Theta^{(1)}} A_\Theta = A_\Theta = A_\Theta A_{\Theta^{(1)}},$$

c.f. eq. (27). Therefore, we are able to factor $S_{\Theta^{(1)}}$ out of $S_\Theta$ in (94) to obtain

$$\bar{P}_\Theta = \bar{P}_{\Theta^{(1)}} \cdot \bar{Y}^\dagger_{\Theta^{(1)}} \bar{Y}_{\Theta^{(1)}}$$

\(^{22}\)We use the notation introduced in section 2.2.2.
Since \( Y_{\Theta(1)}^\dagger = \alpha_{\Theta(1)} Y_{\Theta(1)}^\dagger \) is a projection operator, it follows that \( \bar{Y}_{\Theta(1)}^\dagger \bar{Y}_{\Theta(1)}^\dagger \propto \bar{Y}_{\Theta(1)}^\dagger \). Hence, the above reduces to

\[
\bar{P}_\Theta = \alpha_{\Theta(1)} S_{\Theta(1)} A_{\Theta} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(1)} = A_{\Theta} S_{\Theta} S_{\Theta(1)} A_{\Theta(1)} A_{\Theta(1)} ,
\]

where we used eq. (95) to reabsorb \( S_{\Theta(1)} \) into \( S_{\Theta} \) and \( A_{\Theta(1)} \) into \( A_{\Theta} \). Thus

\[
\bar{P}_\Theta = A_{\Theta} S_{\Theta} A_{\Theta(1)} .
\] (97)

To complete the proof, we have to distinguish two cases: The case where \( \boxed{\text{boxed}} \) lies in the first row of \( \Theta \), and the case where it is positioned in any \textit{but} the first row.

1. Suppose \( \boxed{\text{boxed}} \) lies in the first row of \( \Theta \). Since this is the box containing the highest value in the tableau \( \Theta \), there is no box positioned below it (otherwise \( \Theta \) would not be a Young tableau). Thus, the leg \( m+1 \) is not contained in any antisymmetrizer (of length \( > 1 \)), yielding the sets \( A_{\Theta(1)} \) and \( A_{\Theta} \) identical,

\[
\bar{P}_\Theta = A_{\Theta(1)} S_{\Theta} A_{\Theta(1)} A_{\Theta(1)} = A_{\Theta} S_{\Theta} S_{\Theta(1)} A_{\Theta(1)} A_{\Theta(1)} .
\]

We now apply the Cancellation-Theorem 2 to the part of \( P_\Theta \) in the red box to obtain

\[
\bar{P}_\Theta = A_{\Theta} S_{\Theta} A_{\Theta} \] (98)
as required.

2. Suppose now that \( \boxed{\text{boxed}} \) is situated in any \textit{but} the first row of \( \Theta \). In this case the leg \( m+1 \) does enter an antisymmetrizer (of length \( > 1 \)), thus \( A_{\Theta(1)} \neq A_{\Theta} \) - a new strategy is needed. To understand the obstacles, let us once again look at the operator \( P_\Theta \) as described by equation (97),

\[
\bar{P}_\Theta = A_{\Theta(1)} S_{\Theta} A_{\Theta(1)} A_{\Theta(1)} .
\] (99)

\textbf{Describing the strategy:} In (99) we have suggestively marked a part of \( P_\Theta \) in colour: if we were allowed to \textit{exchange} the sets \( A_{\Theta(1)} \) and \( A_{\Theta} \), replacing \( \bar{P}_\Theta \) by

\[
A_{\Theta} S_{\Theta} A_{\Theta(1)} A_{\Theta(1)} ,
\] (100)

we would be able to factor the symmetrizer \( S_{\Theta(1)} \) out of \( S_{\Theta} \) by relation (95) and use the fact that \( Y_{\Theta(1)} = \alpha_{\Theta(1)} Y_{\Theta(1)} \) is a projection operator to obtain

\[
S_{\Theta(1)} \rightarrow S_{\Theta(1)} S_{\Theta(1)} = A_{\Theta} S_{\Theta} S_{\Theta(1)} A_{\Theta(1)} A_{\Theta(1)} .
\] (101)

\text{Re-absorbing} \( S_{\Theta(1)} \) into \( S_{\Theta} \) yields

\[
S_{\Theta} S_{\Theta(1)} \rightarrow S_{\Theta} \] (102)
From there a similar argument as is needed to justify the missing step from (99) to (100) can be used to show that

\[ A_\Theta S_\Theta A_{\Theta(1)} = A_\Theta S_\Theta A_\Theta , \quad (103) \]

yielding the desired form of \( \bar{P}_\Theta \). The main obstacle in achieving this result thus lies in the justification of the exchange of antisymmetrizers in the step from (99) to (100).

The full argument: We will accomplish this exchange of \( A_{\Theta(1)} \) and \( A_\Theta \) within the marked region of (99) in the following way: Consider the Young tableaux \( \Theta_{(1)} \) and \( \Theta \) as depicted in Figure 3:

\[ \Theta_{(1)} = \begin{array}{llll}
\text{m} & \text{m+1} \\
\text{m+2} & \text{m+3}
\end{array} \quad \text{and} \quad \Theta = \begin{array}{llll}
\text{m} & \text{m+1} \\
\text{m+2} & \text{m+3}
\end{array} \]

Figure 3: This figure gives a schematic depiction of the Young tableaux \( \Theta_{(1)} \) and \( \Theta \). The boxes that are common in the two tableaux have been marked in colour. The box with entry \( m+1 \) has to lie in the bottom-most position of the last column of \( \Theta \), as otherwise the column-word of \( \Theta \), \( C_\Theta \), would not be in lexical order, contradictory to our initial assumption. The requirement that \( C_\Theta \) is lexically ordered therefore also uniquely determines the position of the box \( m \), as is indicated in this figure.

Since, by assumption, \( m+1 \) does not lie in the first row of \( \Theta \), the leg \( m+1 \) is contained in an antisymmetrizer (of length \( >1 \)) in \( A_\Theta \), as was already mentioned previously. Let us denote this antisymmetrizer by \( A^{m+1}_\Theta \in A_\Theta \). Furthermore, let \( A^m_{\Theta(1)} \) be the corresponding antisymmetrizer of the tableau \( \Theta_{(1)} \); in other words, \( A^m_{\Theta(1)} \) is the antisymmetrizer \( A^{m+1}_\Theta \) with the leg \( m+1 \) removed. Hence \( A^m_{\Theta(1)} \supset A^{m+1}_\Theta \), using the notation introduced in section 2.2.2.

Since \( \Theta_{(1)} \) is the parent tableau of \( \Theta \), all columns but the last will be identical in the two tableaux, see Figure 3. Thus, the antisymmetrizers corresponding to any but the last row will be contained in both sets \( A_{\Theta(1)} \) and \( A_\Theta \), which in particular implies that

\[ A_\Theta = A_{\Theta(1)} A^{m+1}_\Theta \quad (104) \]

since \( A^m_{\Theta(1)} \supset A^{m+1}_\Theta \). Thus, if we were able to commute the antisymmetrizer \( A^{m+1}_\Theta \) through the set \( S_\Theta \) from the right to the left (and then absorb \( A^m_{\Theta(1)} \) into \( A^{m+1}_\Theta \)), we could cast \( P_\Theta \) into the desired form (103). In fact, this is exactly what we will do: According to Theorem 6, the antisymmetrizer \( A^{m+1}_\Theta \) can be propagated through the set \( S_\Theta \) if the row-amputated Young tableau \( \mathcal{S}_r \) according to the last column of \( \Theta \) is rectangular. Thus, let us form this amputated tableau,

\[ \mathcal{S}_r = \begin{array}{llll}
\text{m} & \text{m+1} \\
\text{m+2} & \text{m+3}
\end{array} \]

This tableau is indeed rectangular\(^{23}\), allowing us to propagate the antisymmetrizer \( A^{m+1}_\Theta \) from the right to the left,

\[ \bar{P}_\Theta = A_\Theta S_\Theta A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(1)} . \]

\(^{23}\)It is important to note that the above amputated tableau would not be rectangular if \( \Theta \) were not lexically ordered, as then \( m+1 \) could be situated in a column other than the last one. Thus, for non-lexically ordered tableaux, the proof breaks down at this point.
Having demonstrated, that \( A_{\Theta(1)} \) and \( A_{\Theta} \) may be swapped, it is possible to simplify \( \bar{P}_\Theta \) as shown in (101)–(102),

\[
\begin{align*}
\bar{P}_\Theta &= A_{\Theta} S_{\Theta} A_{\Theta(1)} = A_{\Theta} S_{\Theta(1)} A_{\Theta(1)} = \overbrace{A_{\Theta} S_{\Theta(1)} A_{\Theta}}^{= Y_{\Theta(1)}}.
\end{align*}
\]

We once again use Theorem 6 to obtain the desired form of \( \bar{P}_\Theta \),

\[
\bar{P}_\Theta = A_{\Theta} S_{\Theta} A_{\Theta(1)} \xrightarrow{\text{Thm. 6}} A_{\Theta} S_{\Theta} A_{\Theta}.
\]

It remains to show that the normalization constant given in (89) is the right one: that is, we will show that \( P_\Theta = \alpha_\Theta \bar{P}_\Theta \), where \( \bar{P}_\Theta = \overline{Y}_\Theta Y_\Theta = A_{\Theta} S_{\Theta} A_{\Theta} \) (as was found in (98) and (105)), is indeed a projection operator. We will establish this by simply squaring \( P_\Theta \) and requiring that it is idempotent:

\[
P_\Theta P_\Theta = \alpha_\Theta^2 \cdot (A_{\Theta} S_{\Theta} A_{\Theta}) (A_{\Theta} S_{\Theta} A_{\Theta}) = \alpha_\Theta^2 \cdot A_{\Theta} S_{\Theta} A_{\Theta} = \overline{Y}_\Theta Y_\Theta,
\]

where we have used the fact that \( A_{\Theta} A_{\Theta} = A_{\Theta} \). By the idempotency of Young projection operators \( Y_\Theta \), it follows that \( \overline{Y}_\Theta Y_\Theta = 1/\alpha_\Theta Y_\Theta \), simplifying (106) as

\[
P_\Theta P_\Theta = \alpha_\Theta^2 \cdot A_{\Theta} S_{\Theta} A_{\Theta} = \alpha_\Theta \cdot A_{\Theta} S_{\Theta} A_{\Theta} = P_\Theta;
\]

this concludes the proof of this Theorem 4.

A.2 Proof of Theorem 5 “partially lexical Hermitian Young projectors” (or MOLD-Theorem)

We now present a proof of the MOLD-Theorem 5 by induction, using the Lexical-Theorem 4 as a base step:

Consider a Young tableau \( \Theta \) with MOLD \( M(\Theta) \) such that \( \Theta_{(M(\Theta))} \) has a lexically ordered row-word; the proof for \( \Theta_{(M(\Theta))} \) having lexically ordered column-word is very similar and thus left as an exercise to the reader. We will provide a Proof by Induction on the MOLD of \( \Theta, M(\Theta) \). Furthermore, we will for now ignore the proportionality constant \( \beta_\Theta \) and concentrate on the birdtrack \( \bar{P}_\Theta \) only. From the steps in the following proof, it will become evident that \( \beta_\Theta \neq 0 \) and \( \beta_\Theta < \infty \) (as is explicitly discussed at the appropriate places within the proof), ensuring that \( P_\Theta := \beta_\Theta \bar{P}_\Theta \) is a non-trivial (i.e. non-zero) projection operator.

The Base Step: Suppose that \( M(\Theta) = 0 \). In that case, \( \Theta \) itself has a lexically ordered row-word. Then, by the MOLD-Theorem, \( \bar{P}_\Theta \) must be of the following form

\[
\bar{P}_\Theta = S_{\Theta} A_{\Theta} S_{\Theta};
\]

this agrees with the result we obtained from the Lexical-Theorem 4 for which we have already given a full proof in the Appendix A.1. Also, the normalization constant \( \beta_\Theta = \alpha_\Theta \neq 0 \), as required by the MOLD-Theorem. Thus, the base step of the induction is fulfilled.
The Induction Step: Let us now consider a Young tableau $\Theta$, such that the MOLD-Theorem holds for its parent tableau $\Theta^{(1)}$. Further, assume that $M(\Theta^{(1)}) = m$, for some positive integer $m$ with $\Theta^{(m)}$ being row-ordered; thus, we have that $M(\Theta) = m + 1$. We can now have one of two situations, either $m$ is even, or $m$ is odd. First, suppose that $m$ is even. Then, according to the MOLD-Theorem, the birdtrack of the projection operator $P_{\Theta^{(1)}}$ is given by (c.f. eq. (76a))

$$P_{\Theta^{(1)}} = C \left\{ S_{\Theta^{(1)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} \right\} C^\dagger,$$

where we defined $C$ to be

$$C := S_{\Theta^{(m+1)}} A_{\Theta^{(m)}} S_{\Theta^{(m-1)}} \cdots S_{\Theta^{(3)}} A_{\Theta^{(2)}}.$$

We will now construct the birdtrack $\bar{P}_{\Theta}$ according to the KS-Theorem 3 [1]; this yields

$$\bar{P}_{\Theta} = \bar{P}_{\Theta^{(1)}} \mathcal{Y}_\Theta \bar{P}_{\Theta^{(1)}}$$

$$= C \left\{ S_{\Theta^{(1)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} \cdots A_{\Theta^{(m)}} S_{\Theta^{(m-1)}} S_{\Theta^{(m+1)}} A_{\Theta^{(m+1)}} \cdots S_{\Theta^{(1)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} \right\} C^\dagger,$$  \(108\)

where we absorbed $S_{\Theta^{(m+1)}}$ into $S_{\Theta}$. We notice that the parts of $\bar{P}_{\Theta}$ denoted by $C$ are already in the form that we want them to be. We thus focus our attention on the part of $\bar{P}_{\Theta}$ inside the gray box. If we can show that the parts within this gray box can be written as

$$\bar{P}_{\Theta} = C \left\{ S_{\Theta^{(1)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} \cdots A_{\Theta^{(m)}} S_{\Theta^{(m-1)}} S_{\Theta^{(m+1)}} A_{\Theta^{(m+1)}} \cdots S_{\Theta^{(1)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} \right\} C^\dagger,$$  \(110\)

then we have completed the proof. We will accomplish this goal in two steps:

1. We will use the Cancellation-Theorem 2 (see section 2.3) to cancel the wedged ancestor sets of (anti-) symmetrizers in the gray box and thus reduce $\bar{P}_{\Theta}$ to

$$\bar{P}_{\Theta} = C \left\{ S_{\Theta^{(1)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} \right\} C^\dagger.$$  \(109\)

2. We then make use of the Hermiticity of $P_{\Theta}$ to show that

$$\bar{P}_{\Theta} = C \left\{ S_{\Theta^{(1)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} S_{\Theta^{(1)}} \right\} C^\dagger.$$  \(110\)

Let us start the two-step-process:

1. The first step is easily accomplished: We factor a set $S_{\Theta^{(1)}}$ out of $S_{\Theta}$ and a set $A_{\Theta^{(1)}}$ out of $A_{\Theta}$,

$$\bar{P}_{\Theta} = C \left\{ S_{\Theta^{(1)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} \cdots A_{\Theta^{(m+2)}} S_{\Theta^{(m+1)}} A_{\Theta^{(m+1)}} \cdots S_{\Theta^{(1)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} \right\} C^\dagger.$$  \(111\)
We now encounter sets of symmetrizers and antisymmetrizers corresponding to ancestor tableaux $\Theta_{(k)}$ with $1 < k \leq m$ wedged between sets belonging to the tableau $\Theta_{(1)}$. Thus, we may use Theorem 2 to simplify the operator $P_\Theta$,

$$P_\Theta = C \begin{array}{c}
S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} \cdots A_{\Theta_{(m+2)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}}^\dagger
\end{array}$$

$$= C \begin{array}{c}
S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}}^\dagger
\end{array} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}}$$

(112)

Absorbing $S_{\Theta_{(1)}}$ into $\Theta_\Theta$, using the fact that $\tilde{Y}_\Theta$ is quasi-idempotent ($\tilde{Y}_\Theta \cdot \tilde{Y}_\Theta \propto \tilde{Y}_\Theta$), and finally absorbing $A_{\Theta_{(1)}}$ into $A_{\Theta}$ yields the desired form (109) for $\tilde{P}_\Theta$,

$$\tilde{P}_\Theta = C \begin{array}{c}
S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}}^\dagger
\end{array} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}}$$

$$= C \begin{array}{c}
S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}}^\dagger
\end{array} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}}$$

(113)

concluding this step of the proof.

2. For the second step of the proof, we first notice that the operator obtained in the previous step, operator (113), is Hermitian; this is due to the fact that $\tilde{P}_\Theta$ as given in (108) was constructed according to the iterative method described in the KS-Theorem 3, [1]. In particular, this implies that $\tilde{P}_\Theta = \tilde{P}_\Theta^\dagger$,

$$\tilde{P}_\Theta = C \begin{array}{c}
S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}}^\dagger
\end{array} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}} \alpha A_{\Theta_{(1)}} S_{\Theta_{(1)}}$$

(114)

When we gave a proof of the Lexical-Theorem 4 in Appendix A.1, we were able to prove that $A_{\Theta_{(1)}}$ can be extended to become $A_{\Theta}$ by using techniques described in Appendix A.1.1. Now however, we are no longer able to use these techniques, as most amputated tableaux $\Theta'$ or $\Theta''$ would not be rectangular (as can be easily verified by an example). We therefore need a different strategy to arrive at the desired form for $P_\Theta$.

In addition to $P_\Theta$ as given in (113), let us define the operator $\bar{O}$ by

$$\bar{O} := C \begin{array}{c}
S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}} A_{\Theta_{(1)}} S_{\Theta_{(1)}}^\dagger
\end{array}$$

(115)

clearly, this operator is Hermitian by construction due to its symmetry. We seek to show that $P_\Theta = \bar{O}$ in order to conclude the second step of this proof. This will be accomplished by showing that

$$\bar{O} \subset \tilde{P}_\Theta \quad \text{and} \quad \tilde{P}_\Theta \subset \bar{O},$$

(116)

where we use the notation introduced in section 2.2.2. These inclusions will then lead us to conclude that the subspaces onto which $O$ and $\tilde{P}_\Theta$ project are equal, rendering the two operators equal, $O = \tilde{P}_\Theta$. 39
Let us start by proving the two inclusions (116): As discussed in section 2.2.2, the first inclusion holds if and only if $\bar{O} \cdot \bar{P}_\Theta = P_\Theta = P_\Theta \cdot \bar{O}$, c.f. equation (26). We thus need to examine the product of $\bar{O}$ and $\bar{P}_\Theta$. We consider

$$\bar{O} \cdot \bar{P}_\Theta = C S_{\Theta(1)} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(1)} C^\dagger \cdot C S_{\Theta(2)} A_{\Theta(2)} S_{\Theta(2)} A_{\Theta(2)} C^\dagger .$$

Similar to what was done in part 1, we use Theorem 2 to simplify the central part of the product $\bar{O} \cdot \bar{P}_\Theta$

$$\bar{O} \cdot \bar{P}_\Theta = C S_{\Theta(1)} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(1)} C^\dagger \cdot C S_{\Theta(2)} A_{\Theta(2)} S_{\Theta(2)} A_{\Theta(2)} S_{\Theta(1)} C^\dagger ,$$

yielding

$$\bar{O} \cdot \bar{P}_\Theta = C S_{\Theta(1)} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(1)} \underbrace{S_{\Theta(2)} A_{\Theta(2)}}_{=Y_{\Theta}} S_{\Theta(1)} C^\dagger = \bar{O} .$$

Hence, we found that $\bar{O} \cdot \bar{P}_\Theta = \bar{O}$. Recalling that both operators $\bar{O}$ and $\bar{P}_\Theta$ are Hermitian, it follows that

$$\bar{O} = \bar{O} = (\bar{O} \cdot \bar{P}_\Theta)^\dagger = \bar{P}_\Theta^\dagger \cdot \bar{O}^\dagger = \bar{P}_\Theta \cdot \bar{O}.$$

Thus, we have shown that both equalities, $\bar{O} \cdot \bar{P}_\Theta = \bar{O}$ and $\bar{P}_\Theta \cdot \bar{O} = \bar{O}$, hold, implying the first inclusion $\bar{O} \subset \bar{P}_\Theta$.

To prove the second inclusion in (116), we need to consider the product $\bar{P}_\Theta \cdot \bar{O}$,

$$\bar{P}_\Theta \cdot \bar{O} = \underbrace{C S_{\Theta(1)} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(1)} \underbrace{S_{\Theta(2)} A_{\Theta(2)}}_{=Y_{\Theta}} S_{\Theta(1)} C^\dagger}_{(117)}.$$

Once again, we may use Theorem 2 to simplify this product as

$$\bar{P}_\Theta \cdot \bar{O} = \underbrace{\underbrace{C S_{\Theta(1)} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(1)}}_{=Y_{\Theta}} \underbrace{S_{\Theta(2)} A_{\Theta(2)} S_{\Theta(1)}}}_{(117)} C^\dagger.$$

We recognize the right hand side of the above equation (117) to be the operator $\bar{P}_\Theta$. We thus found that $\bar{P}_\Theta \cdot \bar{O} = \bar{P}_\Theta$. Once again, we make use of the Hermiticity of the operators $\bar{O}$ and $\bar{P}_\Theta$ to see that

$$\bar{P}_\Theta = \bar{P}_\Theta^\dagger = (\bar{P}_\Theta \cdot \bar{O})^\dagger = \bar{O} \cdot \bar{P}_\Theta^\dagger = \bar{O} \cdot \bar{P}_\Theta ,$$

yielding the desired inclusion $\bar{P}_\Theta \subset \bar{O}$. We have thus managed to prove both inclusions in (116), forcing us to conclude that the two operators $\bar{O}$ and $\bar{P}_\Theta$ are equal, $\bar{O} = \bar{P}_\Theta$, yielding

$$\bar{P}_\Theta = \bar{O} = \underbrace{C S_{\Theta(1)} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(1)} S_{\Theta(2)}}_{(118)} C^\dagger ,$$

as desired by eq. (113).

Suppose now that $m$ is odd. The proof for odd $m$ will also be conducted in two steps, just as for even $m$. We will only give an outline of this proof, as the steps are very similar to those for even $m$.

By the induction hypothesis, the projection operator $P_{\Theta(1)}$ is of the form

$$P_{\Theta(1)} = \underbrace{S_{\Theta(m+1)} A_{\Theta(m)} \ldots S_{\Theta(2)} A_{\Theta(1)} S_{\Theta(1)} A_{\Theta(1)}}_{=Y_{\Theta(2)}} \underbrace{S_{\Theta(2)} \ldots A_{\Theta(m)} S_{\Theta(m+1)}}_{=Y_{\Theta(1)}} .$$

40
Constructing the birdtrack of the Hermitian Young projection operator $P_\Theta$, $\bar{P}_\Theta$, according to the KS-Theorem 3 \[1\] gives

$$P_\Theta = P_{\Theta(1)} Y_\Theta P_{\Theta(1)}$$

$$= C_\Theta \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array}$$

where $C_\Theta := S_{\Theta(m+1)} \cdots S_{\Theta(2)}$. We again use Theorem 2 to simplify the operator (119),

$$\bar{P}_\Theta = S_{\Theta(m+1)} \cdots S_{\Theta(2)} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array}.$$

We then define an operator $\bar{O}$ by

$$\bar{O} := S_{\Theta(m+1)} \cdots S_{\Theta(2)} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array}.$$

Using Theorem 2 as well as the fact that both $\bar{P}_\Theta$ and $\bar{O}$ are Hermitian by construction, we may show the inclusions $P_\Theta \subset \bar{O}$ and $\bar{O} \subset P_\Theta$, to conclude that $\bar{P}_\Theta = \bar{O}$, yielding the desired result

$$\bar{P}_\Theta = S_{\Theta(m+1)} \cdots S_{\Theta(2)} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array} \begin{array}{cccc} \Theta(1) & \Theta(2) \cdots & \Theta(m+1) \end{array}.$$

The proof of equations (76c) and (76d) in the MOLD-Theorem follows the same steps as the proof of equations (76a) and (76b) given above and is thus left as an exercise to the reader.

Lastly, we notice that the idempotency of $P_\Theta$ in each of the cases (76) can be verified by using the Cancellation-Theorem 2: For example if $P_\Theta$ is constructed according to (76a), it follows that

$$P_\Theta \cdot P_\Theta = \beta_\Theta^2 \cdot S_{\Theta(m)} \cdots \Theta(2) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m)$$

$$= \beta_\Theta^2 \cdot S_{\Theta(m)} \cdots \Theta(2) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m) \cdots \Theta(m),$$

where $\lambda$ is a non-zero constant, since all the cancelled sets can be absorbed into $S_\Theta$ and $A_\Theta$ respectively (c.f. Theorem 2). Thus, defining

$$\beta_\Theta := \frac{1}{\lambda} < \infty$$

ensures that $P_\Theta$ is indeed a projection operator.

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