SUM RULES FOR HIGHER-TWIST PARTON DISTRIBUTIONS

M. Burkardt
New Mexico State University, Las Cruces,
NM 88003, USA
E-mail: burkardt@nmsu.edu

In deep-inelastic scattering experiments, there is a general connection between subtractions in dispersion relations, violations of sum-rules and $\delta$-functions in parton distribution functions. It is explained why one might expect a small violation in sum rules for the twist-3 distribution functions $g_T(x)$ and $h_L(x)$ when the sum-rules are applied to $x \neq 0$ data only. The non-perturbative predictions are studied in the context of a one-loop model.

1 Introduction

In the theoretical analysis of deep-inelastic scattering (DIS), one usually applies the operator product expansion (OPE) for $Q^2 \to \infty$ to the Compton amplitude—a procedure which implicitly involves analytic continuation of the Compton amplitude to the regime where $Q^2 > 2M\nu$. Formally this step is accomplished by invoking dispersion relations. However, at least in principle, it may happen that there appear subtractions in these dispersion relations which then manifest themselves as a violation of naive sum-rules.

In this note, we study the issue of subtractions in the context of the higher-twist parton distributions $h_L(x)$ and $g_T(x)$, which are defined as correlation functions in a light-like direction

\[ S \cdot n h_L(x) = \frac{1}{2M} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS | \bar{\psi}(0) \sigma^+ \gamma_5 \psi(\lambda n) | PS \rangle \]
\[ S_\perp g_T(x) = \frac{1}{2p^+} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS | \bar{\psi}(0) \gamma_\perp \gamma_5 \psi(\lambda n) | PS \rangle, \]

(1)

where $p^± = \frac{1}{\sqrt{2}} (p^0 \pm p^3)$ denotes the usual light-cone coordinates and the light-like vector $n^\mu$ projects out $n \cdot A = A^+$ for all 4-vectors $A^\mu$.

Like the more familiar polarized twist-two distributions $g_1(x)$ and $h_1(x)$, these twist-three distributions are important physical quantities which summarize low energy properties of the nucleon—in high-energy scattering processes $g_T(x)$ appears as a $1/Q$ correction in DIS, while $h_L(x)$ can be measured for example in the nucleon-nucleon polarized Drell-Yan process.

\[ ^a \text{Naive, because they are derived assuming that there are no subtractions.} \]
Upon integrating over $x$ in Eq. (1), comparing with similar definitions for $g_1$ and $h_1$, and using Lorentz invariance one can thus derive the sum-rules

$$\int_{-1}^{1} dx h_L(x) = \int_{-1}^{1} dx h_1(x) \quad (2)$$

$$\int_{-1}^{1} dx g_T(x) = \int_{-1}^{1} dx g_1(x) \quad (3)$$

Eq. (3) is also known as the Burkhardt-Cottingham (BC) sum-rule. For the parton distribution functions, defined as light-cone correlations (1), these sum-rules (2,3) are a direct consequence of Lorentz invariance, and therefore hardly anybody would question their validity (assuming the integrals converge). However, the issue here is whether these sum rules are also valid when applied to experimental data! Using the operator product expansion in the Bjorken limit, one can show that for $x \neq 0$, the parton distribution functions (defined as a light-cone correlations) agree with the corresponding experimentally measured structure functions. In DIS experiments, the point $x = 0$ must always be excluded since it corresponds to $Q^2 = 0$. In practice, DIS experiments can only measure

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} dx [g_T(x) + g_T(-x)] \quad (4)$$

and similarly for the other distributions. Therefore, what really is being tested when one tests the above sum-rules is neither Lorentz invariance nor the OPE but whether or not Eqs. (2,3) receive a nonzero contribution from the point $x = 0$, i.e. whether or not $h_L(x)$ and $g_T(x)$ [defined as in Eq. (1)] contain $\delta(x)$-type singularities at the origin.

The paper is organized as follows. In the next section, we will explain the general connections between subtractions in dispersion relations, violations of sum-rules and $\delta$-functions in parton distribution functions. The main question is whether a situation as described above does actually occur or whether it is only of academic interest. In order to understand the implications for QCD, we will use the moment relations derived from the OPE to derive relations between the small $x$ behaviour of polarized twist-2 distributions ($g_1(x)$ and $h_1(x)$) and the coefficient of the $\delta$-function of polarized twist-3 distributions [$h_L(x)$ and $g_1(x)$]. Finally, these predictions are illustrated using a one-loop model.

### 2 Real and Imaginary parts of the Compton Amplitude in DIS

In this section, general connections between subtraction constants, violation of sum-rules and $\delta(x)$-terms in parton distributions will be discussed. For this
purpose, we will denote $G(x, Q^2)$ and $T(x, Q^2)$ the real and imaginary part respectively of some generic forward Compton amplitude, where $x = \frac{Q^2}{2m\nu}$.

The Compton amplitude is an analytic function of $x$, except along a cut from $0 < x < 1$ (the cut appears along the physical region for DIS!). Therefore, it should be possible to relate the real and imaginary part using a dispersion relation

$$T(x', Q^2) = \frac{1}{\pi} \int_0^1 dx \frac{x'}{x'^2 - x^2} G(x, Q^2) + p \left( \frac{1}{x'}, Q^2 \right),$$

where $p(z, Q^2)$ is a polynomial in $z$ and where $|x'| > 1$. In the language of dispersion theory, the ‘polynomial subtraction’ is related to ‘$J=0$ fixed poles’.

In the theoretical analysis of DIS one usually applies the operator product expansion (OPE) for $|x'| > 1$, yielding

$$T(x', Q^2) = \sum_{n=0,2,..}^{1} \frac{1}{x'^{n+1}} a_n$$

where the $a_n$ can be expressed as matrix elements of the form

$$a_n = \langle P, S | \bar{\psi} \Gamma D^{n-1} \psi | P, S \rangle,$$

which appear in the OPE of $T J_\mu(\xi) J_\nu(0)$. Here, $\Gamma$ is some Dirac matrix which depends on the particular structure function (e.g. polarized or unpolarized DIS) and $D^n$ denotes an $n-th$ order covariant derivative. In order to keep the discussion as general as possible, $\Gamma$ will not be specified here any further.

In the most simple case, i.e. when there is no subtraction in Eq. (5) and $p(z) \equiv 0$, the $a_n$ can be expressed through the moments of $G$ by expanding the geometric series in Eq. (5), yielding

$$a_n = \frac{1}{\pi} \int_0^1 dx x^{n-1} G(x, Q^2).$$

However, in the following we want to investigate what happens if $p(z) \neq 0$. As a specific example, let us assume that $p \left( \frac{1}{x'} \right) = \frac{c}{x'}$, where $c$ is some (possibly $Q^2$ dependent) constant. Such a scenario has several important consequences:

- First of all, the simple relation between the $a_n$ and $\int_0^1 dx x^{n-1} G(x, Q^2)$ (5) is spoiled for the lowest moment and instead one finds

$$a_n = \begin{cases} \frac{1}{\pi} \int_0^1 dx x^{n-1} G(x, Q^2) & n = 3, 5, .. \\ \frac{1}{\pi} \int_0^1 dx x^{n-1} G(x, Q^2) + c & n = 1 \end{cases}$$
As a consequence, the naive sum-rule $a_1 = \frac{1}{\pi} \int_0^1 dx G(x, Q^2)$ is of course violated if $c \neq 0$

Suppose that one defines a parton distribution function $g(x)$ through the moments, i.e. by requiring that

$$a_n = \frac{1}{\pi} M_n[g] \equiv \frac{1}{\pi} \int_{-1}^1 dxx^{n-1}g(x, Q^2) \quad n \geq 0$$

then $g(x, Q^2)$ differs from the experimentally measured $G(x, Q^2)$ by a $\delta$ function at the origin, i.e.

$$g(x, Q^2) + g(-x, Q^2) = G(x, Q^2) + c\pi \delta(x).$$  

This result should be intuitively clear since the above subtraction affects only the lowest moment, which means that there must be a $\delta$-function present.

To summarize the above discussion, we emphasize that there is no problem with the OPE if a sum-rule derived for parton distributions fails when applied to experimental data. All it means is that the corresponding dispersion relation has a subtraction and the parton distribution function (defined as a generalized function through the light-cone moments) has a $\delta$-function at the origin, which is not present in the experimental DIS data. Therefore, an experiment which measures $\lim_{\varepsilon \to 0} \int_\varepsilon^1 dx G(x, Q^2)$ would miss the $\delta$ function and hence a “violation” of the sum-rule would be observed.

There exist a number of toy models where such $\delta(x)$ terms have been observed in parton distribution for models in 2 and 4 dimensions. Because of lack of space, the reader is referred to these references and we focus in the following on the most important question, namely whether such $\delta$ functions occur in QCD.

### 3 OPE, Moments and $\delta$-Functions in QCD

In this section we start from the relation among the moments of $h_L(x), h_1(x)$ and $g_1(x)$ (note that $M_n[\tilde{h}_L] = 0$ for $n \leq 2$).

$$M_n[h_L] = \begin{cases} M_n[h_1] & n = 1 \\ \frac{2}{n+1} M_n[h_1] + M_n[\tilde{h}_L] + \frac{m_n}{M_{n+1}} M_{n-1}[g_1] & n \geq 2 \end{cases}$$  

(12)
where $\tilde{h}_L$ is the interaction-dependent twist-three part of $h_L$. Upon inverting the moment relation, one finds
\begin{align}
 h_L(x, \mu^2) &= 2x \int_x^1 dy \frac{h_1(y, \mu^2)}{y^2} + \tilde{h}_L(x, \mu^2) + \frac{m_q}{M} \left[ \frac{g_1(x, \mu^2)}{x} - 2x \int_x^1 dy \frac{g_1(y, \mu^2)}{y^2} \right] \\
 h_L(x, \mu^2) &= -2x \int_{-1}^x dy \frac{h_1(y, \mu^2)}{y^2} + \tilde{h}_L(x, \mu^2) + m_q M \left[ \frac{g_1(x, \mu^2)}{x} + 2x \int_{-1}^x dy \frac{g_1(y, \mu^2)}{y^2} \right]
\end{align}
(13) (14)

for $x > 0$ and $x < 0$ respectively. Now we multiply Eq. (13) by $x^\beta$, integrate from 0 to 1 and let $\beta \to 0$, yielding
\begin{align}
 \int_{0+}^1 dx h_L(x, \mu^2) &= \int_{0+}^1 dx \left( h_1(x, \mu^2) + h^3_L(x, \mu^2) \right) + \frac{m_q}{2M} \lim_{\beta \to 0} \beta \int_{0}^1 dx x^{\beta-1} g_1(y, \mu^2), \\
\text{while multiplying Eq. (14) by } |x|^\beta \text{ and integration from } -1 \text{ to } 0 \text{ yields}
\int_{-1}^0 dx h_L(x, \mu^2) &= \int_{-1}^0 dx \left( h_1(x, \mu^2) + h^3_L(x, \mu^2) \right) - \frac{m_q}{2M} \lim_{\beta \to 0} \beta \int_{-1}^0 dx |x|^{\beta-1} g_1(y, \mu^2).
\end{align}
(15) (16)

Adding Eqs. (16) and (15) and
\begin{itemize}
  \item assuming that there is no $\delta(x)$ in $h_1$, i.e. assuming that
  \begin{equation}
  \int_{-1}^0 dx h_1(x) + \int_{0+}^1 dx h_1(x) = \int_{-1}^1 dx h_1(x)
  \end{equation}
  (17)
  \item assuming that there is no $\delta(x)$ in $h^3_L$ either and using that (from OPE), the lowest moment of $h^3_L$ vanishes identically i.e. assuming that
  \begin{equation}
  \int_{-1}^0 dx h^3_L(x) + \int_{0+}^1 dx h^3_L(x) = \int_{-1}^1 dx h^3_L(x) = 0
  \end{equation}
  (18)
  \item using that
  \begin{equation}
  \lim_{\beta \to 0} \beta \int_{0}^1 dx x^{\beta-1} g_1(y, \mu^2) = g_1(0+, \mu^2)
  \end{equation}
  (19)
\end{itemize}

one thus finds
\begin{align}
\int_{0+}^1 dx \left[ h^3_L(x, \mu^2) - \tilde{h}^3_L(x, \mu^2) \right] &= \int_{-1}^0 dx h_L(x) + \int_{0+}^1 dx h_L(x) \\
&= \int_{-1}^1 dx h_1(x, \mu^2) + \frac{m_q}{2M} \left[ g_1(0+) - g_1(0-) \right].
\end{align}
(20)
Since the OPE also tells us that the first moments of \( h_L \) and \( h_1 \) ought to be the same (if 0 is included in the integration), i.e. \( \int_{-1}^{1} dx h_L(x, \mu^2) = \int_{-1}^{1} dx h_1(x, \mu^2) \), we thus conclude

\[
h_L(x, \mu^2) = h_L^{reg}(x, \mu^2) - \frac{m_q}{2M} [g_1(0+, \mu^2) - g_1(0-, \mu^2)] \delta(x). \tag{21}
\]

A similar analysis applied to \( g_T(x) \) yields

\[
g_T(x) = \int_{-x}^{x} dy g_1(y) + \frac{m}{M} \left[ \frac{h_1(x)}{x} - \int_{-y}^{y} dy h_1(y) y^2 \right] + \tilde{g}_T(x) - \int_{-x}^{x} dy \tilde{g}_T(x) y \tag{22}
\]

And hence under similar assumptions as above

\[
g_T(x, \mu^2) = g_T^{reg}(x, \mu^2) - \frac{m_q}{M} [h_1(0+, \mu^2) - h_1(0-, \mu^2)] \delta(x) \tag{23}
\]

To summarize these results, although the OPE ensures that the integrals of \( h_L \) and \( g_T \) are the same as those of \( h_1 \) and \( g_1 \) respectively, this statement is strictly true only if the origin is included in the integration. By analytic continuation of the moments, we find that the behavior at the origin of \( h_L \) and \( g_T \) might be singular enough and the above statement about equality of the lowest moments seems to be violated if the origin is excluded from the integrals.

4 A one-Loop Model for \( h_L(x) \)

In order to illustrate the results from the previous section in a concrete example, we consider \( h_L(x) \) for a massive quark in a one-loop model.

In such a model one finds for example

\[
h_L(x) \propto \bar{u}(P, S) \int \frac{d^2k}{(2\pi)^3} \frac{\gamma^\mu i}{k^\mu - m + i\varepsilon} \gamma^5 i \gamma^\nu u(P, S) D_{\mu\nu}(P - k) \tag{24}
\]

where \( k^+ = XP^+ \) and \( D_{\mu\nu}(P - k) \) is the gluon propagator. Here we have suppressed wave function renormalization terms \([x \delta(x - 1)]\) because they are not relevant for the behaviour near \( x = 0 \).

In Eq. (24) the \( \delta(x) \) terms can arise from terms with \( k^- \) in numerator, because for those terms integrals of the form

\[
\int dk^- \frac{k^-}{(k^2 - m^2 + i\varepsilon)^2 (p - k)^2 + i\varepsilon} \tag{25}
\]

\( ^{b}\text{In the previous section, we found that the } \delta(x) \text{ term is explicitly proportional to the quark mass quark!} \)
diverge linearly when $k^+ = 0$. In order to see this, we rewrite

$$k^- = P^- - \frac{(\vec{P}_\perp - \vec{k}_\perp)^2}{2(P^+ - k^+)} - \frac{(P - k)^2}{2(P^+ - k^+)}$$ (26)

and note that the first two terms on the r.h.s. of Eq. (26) give “regular” expression when inserted into Eq. (25). However, the third term on the r.h.s. of Eq. (26) cancels one of the denominators in Eq. (25), yielding

$$\int \frac{dk^-}{2\pi} \frac{1}{[k^2 - m^2 + i\varepsilon]^2} = \frac{i}{2} \frac{\delta(k^+)}{k_\perp^2 + m^2}. \quad (27)$$

A detailed analysis\(^3\) shows that in $h_L$, the $\delta(k^+)$-terms survive, yielding a nonzero $\delta(x)$ term that is proportional to $g_1(0+)$,\(^1\) which confirms Eq. (21). A similar 1-loop analysis also shows that for $g_\perp$, the $\delta(k^+)$ term (at 1 loop) is multiplied by $k^+ \Rightarrow \text{no } \delta(x)$. This is consistent with the result that $h^{1\text{-loop}}_L(0) = 0$ at one loop.

5 Summary and Discussion

At least in principle, one cannot exclude subtractions in the dispersion relation between real & imaginary part of Compton amplitude. Whenever such a subtraction is present, this also implies a $\delta(x)$-term in the corresponding parton distribution (defined as light-cone correlation).

A decomposition of $h_L$ and $g_T$ into twist-2 and twist-3 pieces suggests that $\lim_{\beta \to 0} \int_0^1 dx g_\perp(x) \neq \int_0^1 dx g_\perp(x)$ and therefore $g_\perp(x) = g_\perp(x)^{\text{reg}} + c\delta(x)$. A similar result is derived for $h_L$.

The nonzero coefficient of $\delta(x)$ in $h_L(x)$ was confirmed by explicit one-loop calculations for $h_L(x)$ as well as $g_1(0+) - g_1(0-)$. At one loop, no $\delta(x)$ term was found in $g_\perp(x)$. This is consistent with the fact that $h^{1\text{-loop}}_L(0) = 0$.

The prediction of these $\delta(x)$ terms from the moment relations [Eqs. (21), (23), together with their one loop verification, are the main result of this note.

Although the one-loop calculation yields $h_1(0) = 0$, this result changes at next to leading order (NLO)\(^8\), i.e. $h^{NLO}_1(0) \neq 0$. Therefore, using Eq. (23), we expect that the BC sum-rule\(^4\) is also violated, but only at NLO.

In summary, both the $h_L$ sum rule as well as the $g_T$ are expected to be violated if the point $x = 0$ is not included in the integrals. For $h_L$ the violation appears already at leading order, while the violation for $g_T$ does not appear until NLO. Both for $h_L$ and $g_T$, the violation is proportional to $\frac{m_q}{M}$, i.e.

\(^c\)At one loop, $g_1(x) = 0$ for $x < 0$ and thus $g_1(0-) = 0$.
$q = u, d$ the effect is expected to be very small. However, the contribution from strange quarks might be a significant.

Recently, it has also been suggested that $\delta(x)$ terms might be present in the twist-two distribution $g_1(x)$. Although there is not enough space here to discuss the results of Ref. 9 in detail, it should be emphasized that the general features that have been discussed in this note also apply in that case.

One of the main differences between DIS and deeply virtual Compton Scattering (DVCS) is that DIS measures only the imaginary part of the forward Compton amplitude, while DVCS should allow measurements of the full (real & imaginary part) Compton amplitude — including non-forward matrix elements thereof. The virtue of DVCS in this context is twofold. First of all, since also the real part of the Compton amplitude is being measured, one could directly test whether or not a subtraction appears in the dispersion relation. Furthermore, any $\delta$-functions on forward parton distribution functions get smeared out in non-forward distributions, i.e. the measurement of the non-forward Compton amplitude acts as some kind of regularization. Because of these features, studying near forward parton distributions using DVCS may also be of help in clarifying the issues discussed in this paper.

Acknowledgements: It is a pleasure to thank X, Ji, Y. Koike and L. Mankiewicz for helpful discussions. This work was supported in part by a grant from DOE (DE-FG03-95ER40965), by a fellowship from JSPS and in part by TJNAF.

References

1. R.L. Jaffe and X. Ji, Nucl. Phys. B 375, 527 (1992).
2. D. Adams et al. (SMC), Phys. Lett. B 336, 125 (1994); K. Abe et al. (E143), Phys. Rev. Lett. 76, 587 (1996).
3. M. Burkardt, Phys. Rev. D 52, 3841 (1995).
4. H. Burkhardt and W.N. Cottingham, Ann. Phys. 56, 453 (1970).
5. M. Burkardt, Nucl. Phys. A 373, 613 (1992); Phys. Lett. B 268, 419 (1991);
6. Y. Koike and K. Tanaka, Prog. Theor. Phys. Suppl. 120, 247 (1995).
7. R.D. Tangerman and P.J. Mulders, hep-ph/9408303.
8. A. Hayashigaki, Y. Kanazawa and Y. Koike, Phys. Rev. D 56, 7350 (1997); W. Vogelsang, Phys. Rev. D 57, 1886 (1998).
9. S.D. Bass, Eur. Phys. J.A 5, 17 (1999).

\[\text{The phase information can be obtained from interference with the Bethe-Heitler process.}\]