Concerning the Navier-Stokes problem

Alexander G. Ramm

Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA.; ramm@math.ksu.edu

Received: 16 June 2020; Accepted: 27 August 2020; Published: 31 August 2020.

Abstract: The problem discussed is the Navier-Stokes problem (NSP) in $\mathbb{R}^3$. Uniqueness of its solution is proved in a suitable space $X$. No smallness assumptions are used in the proof. Existence of the solution in $X$ is proved for $t \in [0, T]$, where $T > 0$ is sufficiently small. Existence of the solution in $X$ is proved for $t \in [0, \infty)$ if some a priori estimate of the solution holds.

Keywords: Navier-Stokes equations, uniqueness of the solutions.

MSC: 35Q30, 76D05.

1. Introduction

There is a large literature on the Navier-Stokes problem (NSP) in $\mathbb{R}^3$ (see [1], Chapter 5) and references therein. The global existence and uniqueness of a solution in $\mathbb{R}^3$ was not proved. The goal of this paper is to prove uniqueness of the solution to NSP in a suitable functional space. No smallness assumptions are used in our proof.

The NS problem in $\mathbb{R}^3$ consists of solving the equations

$$
\nu' + (v, \nabla)v = -\nabla p + \nu \Delta v + f, \quad x \in \mathbb{R}^3, \ t \geq 0, \quad \nabla \cdot v = 0, \quad v(x, 0) = v_0(x). \tag{1}
$$

Vector-functions $v = v(x, t), f = f(x, t)$ and the scalar function $p = p(x, t)$ decay as $|x| \to \infty$ uniformly with respect to $t \in \mathbb{R}_+ := [0, \infty)$, $v' := v_t, \nu = \text{const} > 0$ is the viscosity coefficient, the velocity $v$ and the pressure $p$ are unknown, $v_0$ and $f$ are known, $\nabla \cdot v_0 = 0$. Equations (1) describe viscous incompressible fluid with density $\rho = 1$.

We use the integral equation for $v$:

$$
v(x, t) = F - \int_0^t ds \int_{\mathbb{R}^3} G(x - y, t - s)(v, \nabla)v dy. \tag{2}
$$

Equation (2) is equivalent to (1), see [2]. Formula for the tensor $G$ is derived in [2], see also [1], p.41. The term $F = F(x, t)$ depends only on the data $f$ and $v_0$ (see equation (18) in [2] or formula (5.42) in [1]):

$$
F := \int_{\mathbb{R}^3} g(x - y)v_0(y)dy + \int_0^t ds \int_{\mathbb{R}^3} G(x - y, t - s)f(y, s)dy. \tag{3}
$$

We assume throughout that $f$ and $v_0$ are such that $F$ is bounded in all the norms we use. Let $X$ be the Banach space of continuous functions with respect to $t$ with the norm

$$
\|v\| := \int_{\mathbb{R}_+} |\hat{v}(\xi, t)| (1 + |\xi|) d\xi, \tag{4}
$$

where $t > 0$, and $\hat{v} := (2\pi)^{-3} \int_{\mathbb{R}^3} v(x, t)e^{-i\xi \cdot x} dx$. Taking the Fourier transform of (2) yields

$$
\hat{v} = \hat{F} - \int_0^t ds \hat{G} \ast i\xi \hat{v} := B(\hat{v}), \tag{5}
$$
where $\ast$ denotes the convolution in $\mathbb{R}^3$ and for brevity we omitted the tensorial indices: instead of $G_{mp\bar{\nu}_j}(i\xi_j)\bar{\nu}_p$, where one sums up over the repeated indices, we wrote $G(\xi, t-s)\bar{\nu} \ast (i\xi\bar{\nu})$. From formula (5.9) in [1] it follows that

$$
|G| \leq ce^{-v\xi^2(t-s)}.
$$

(6)

By $c > 0$ we denote various constants independent of $t$ and $\xi$. Let $S(\mathbb{R}^3 \times \mathbb{R}_+)$ and $S(\mathbb{R}^3)$ be the L.Schwartz spaces. Our results are:

**Theorem 1.** Assume that $f$ and $v_0$ are in $S(\mathbb{R}^3 \times \mathbb{R}_+)$ and $S(\mathbb{R}^3)$ respectively. Then there is at most one solution to NSP in $X$.

**Theorem 2.** The solution to NSP in $X$ exists for $t \in [0, T]$ if $T > 0$ is sufficiently small.

**Theorem 3.** The solution $v(x, t)$ to NSP in $X$ exists for all $t \geq 0$ if an a priori estimate $\sup_{t \geq 0} \|\bar{\nu}(\xi, t)\| < c_a$ holds, where $c_a > 0$ is a constant depending only on the data.

2. Proofs

**Proof of Theorem 1.** Let $\bar{\nu}$ and $\bar{\omega}$ belong to $X$ and solve equation (5). Denote $z := \bar{\nu} - \bar{\omega}$. Then (5) implies

$$
z = -\int_0^t dsG(z \ast i\xi\bar{\nu} + \bar{\omega} \ast i\xi z).
$$

(7)

Let $\|z(\xi, t)\| := u(t)$ and $\int_{\mathbb{R}^3} := \int$. From (7) and (6) one gets

$$
u(t) \leq c \int_0^t ds \int d\xi e^{-v\xi^2(t-s)}(1 + |\xi|) \left[ |z(\xi - \zeta, s)||\xi||\bar{\nu}(\xi, s)| d\xi + \int |\bar{\omega}(\xi - \zeta, s)||\xi||z(\zeta, s)| d\xi \right].
$$

(8)

Let $\eta := \xi - \zeta$. One has:

$$
\int d\xi |\bar{\nu}| |\bar{\omega}(1 + |\xi|)|z(\zeta - \xi, s)| e^{-v\xi^2(t-s)} \leq \|\bar{\nu}(s)\| \max_{\xi \in \mathbb{R}^3} \left\{ e^{-v(1+|\eta|)\xi^2(t-s)} \frac{1 + |\eta + \zeta|}{1 + |\eta|} \right\}.
$$

(9)

Furthermore,

$$
\max_{\xi \in \mathbb{R}^3} \left\{ e^{-v(1+|\eta|)\xi^2(t-s)} \frac{1 + |\eta + \zeta|}{1 + |\eta|} \right\} = (1 + |\eta|)^{-1} \max_{p \in \mathbb{R}_+} \left\{ (1 + p) e^{-v p^2(t-s)} \right\} \leq 1 + \frac{c_v}{(t-s)^{1/2}},
$$

(10)

where $c_v = cv^{-0.5}$. Indeed, if $h(r) = (1 + r)e^{-v(t-s)^2}$, then $\max_{r \geq 0} h(r) = h(R) \leq 1 + \frac{c_v}{(t-s)^{1/2}}$, where $R = 1 + \left(1 + \frac{1}{2v(t-s)}\right)^{1/2}$ and $h'(R) = 0$.

A similar estimate holds for the second integral in (8):

$$
\int d\xi |\bar{\omega}(1 + |\xi|)|z(\zeta, s)| \int d\xi e^{-v\xi^2(t-s)}(1 + |\xi|)|\bar{\omega}(\zeta - \xi, s)| \leq u(s) \max_{\xi \in \mathbb{R}^3} \int dp |\bar{\omega}(p, s)|(1 + |p + \zeta|) e^{-v\xi^2(t-s)}.
$$

(11)

The right side of (11) is $u(s)f$, where

$$
f = \int dp |\bar{\omega}(p, s)|(1 + p) \max_{\xi \in \mathbb{R}^3} \left( \frac{1 + |p + \zeta|}{1 + p} e^{-v\xi^2(t-s)} \right) \leq \|\bar{\omega}\| \left(1 + \frac{c_v}{(t-s)^{1/2}} \right).
$$

(12)

From (7)–(12) one gets

$$
u(t) \leq C(t) \int_0^t \left(1 + \frac{c_v}{(t-s)^{1/2}} \right) u(s) ds,
$$

(13)

where $C(t) > 0$ is a continuous function and $u(t) > 0$. Note that $C(t)$ is a continuous function of $t$ for all $t \geq 0$ because we assume that the solutions $\bar{\nu}$ and $\bar{\omega}$ belong to $X$ and $C(t)$ is the sum of the norms of the two
elements of $X$. The Volterra inequality (13) has only the trivial solution $u(t) = 0$, as follows from Lemma 1, proved below. Theorem 1 is proved. □

**Lemma 1.** Inequality (13) has only the trivial non-negative solution $u(t) = 0$.

**Proof of Lemma 1.** Denote $\frac{d(t)}{c(t)} = q(t)$. Then

$$q(t) \leq \int_0^t \left( 1 + \frac{c_v}{(t-s)^{1/2}} \right) C(s) q(s) ds := \int_0^t K(t,s) q(s) ds. \quad (14)$$

The kernel $K(t,s) > 0$ is weakly singular. Any solution $q \geq 0$ to (14) satisfies the estimate $0 \leq q \leq Q$, where $Q \geq 0$ solves the Volterra equation

$$Q(t) = \int_0^t K(t,s) Q(s) ds. \quad (15)$$

This equation has only the trivial solution $Q = 0$. Lemma 1 is proved. □

**Proof of Theorem 2.** From (5) after multiplying by $1 + |ξ|$, integrating over $\mathbb{R}^3$ and using calculations similar to the ones in equation (12), one gets

$$u(t) \leq b(t) + c \int_0^t \left( 1 + \frac{c_v}{(t-s)^{1/2}} \right) u^2(s) ds := A(u), \quad (16)$$

where $b(t) := \int |\hat{F}(ξ, t)| (1 + |ξ|) dξ$ and $u(t) := \|\hat{a}(ξ, t)\|$. For sufficiently small $T$ equation $U = AU$ is uniquely solvable by iterations according to the contraction mapping principle. If $\sup_{t \in [0, T]} b(t) \leq c_0$ and $T$ is sufficiently small, then a ball $\sup_{t \in [0, T]} u(t) \leq c_1$, $c_1 > c_0$, is mapped by the operator $A$ into itself and $A$ is a contraction mapping. The operator $A$ maps positive functions into positive functions. Thus, $u(t) \leq U(t)$. Theorem 2 is proved. □

**Proof of Theorem 3.** Under the assumption of Theorem 3 inequality (16) implies:

$$u(t) \leq b(t) + c c_v \int_0^t \left( 1 + \frac{c_v}{(t-s)^{1/2}} \right) u(s) ds := A_1(u), \quad (17)$$

The corresponding equation $U = A_1 U$ is a linear Volterra integral equation. It has a unique solution defined for all $t \geq 0$, and $0 \leq u(t) \leq U(t)$. Theorem 3 is proved. □

**Remark 1.** The following a priori estimates for solutions to NSP hold:

$$\|v\|_{L^2(\mathbb{R}^3)} \leq c, \quad \int_0^t \|\nabla v(x,s)\|_{L^2(\mathbb{R}^3)}^2 ds \leq c, \quad (18)$$

and

$$\sup_{t \in [0, T]} |\hat{a}(ξ, t)| \leq c + c T^{1/2}, \quad \sup_{t \geq 0, ξ \in \mathbb{R}^3} |ξ| |\hat{a}(ξ, t)| < c. \quad (19)$$

**Proof of (18).** First estimate (18) is well known. It remains to prove the second estimate (18). For this, multiply (1) by $v$ and integrate over $\mathbb{R}^3$ to get (see [1]):

$$0.5 \frac{d}{dt} \int v^2 dx + v \int |
abla v|^2 dx = \int fv dx.$$ \hspace{1cm}

Integrating over $t$ one gets:

$$0.5 \int v^2 dx + \int_0^t ds \int |
abla v(x,s)|^2 dx \leq 0.5 \int v^2 dx + \int_0^t ds \int fv dx.$$
One has \( \int_0^t ds \int f v dx \leq \int_0^t ds \left( \int |f(x,s)|^2 \right)^{1/2} \left( \int |v(x,s)|^2 \right)^{1/2} \leq c. \) Indeed, it is assumed that \( f \) decays fast, so \( \sup_{t \geq 0} \int_0^t ds (\int |f|^2 dx)^{1/2} \leq c. \) Using this and estimates (18) we get \( \int_0^t ds \int |f| v dx \leq c. \) Thus, the second estimate (18) is proved.

Proof of estimate (19). From Equation (5) one gets:

\[
|\tilde{\theta}| \leq |\tilde{F}| + c \int_0^1 e^{-\nu \xi^2 (t-s)} |\tilde{\theta}| \ast (|\xi| |\tilde{\theta}|) ds := |\tilde{F}| + I.
\]

One has \( \sup_{t \geq 0} |\tilde{F}| \leq c \) under the assumptions of Theorem 1. By the Cauchy inequality, the first estimate (18) and Parseval’s equality one gets \( |\tilde{\theta}| \ast (|\xi| |\tilde{\theta}|) \leq ||\tilde{\theta}||_{L^2(\mathbb{R}^3)} |||\xi| \tilde{\theta}||_{L^2(\mathbb{R}^3)}. \) Thus, using the Cauchy inequality, and the second estimate (18), one gets

\[
I \leq c \int_0^1 e^{-\nu \xi^2 (t-s)} |||\xi| \tilde{\theta}||_{L^2(\mathbb{R}^3)} ds \leq c t^{1/2} \left[ \int_0^t |||\xi| \tilde{\theta}||_{L^2(\mathbb{R}^3)}^2 \right]^{1/2} \leq c t^{1/2}.
\]

From (20) and (21) estimate (19) follows.

The second estimate (19) is proved in [1], p. 50, inequality (5.39), under the assumption of Theorem 1.

Conflicts of Interest: “The author declares no conflict of interest.”

References

[1] Ramm, A. G. (2019). Symmetry Problems. The Navier-Stokes Problem, Morgan & Claypool Publishers, San Rafael, CA.

[2] Ramm, A. G. (2017). Global existence, uniqueness and estimates of the solution to the Navier-Stokes equations, Applied Mathematics Letters, 74, 154-160.

© 2020 by the author; licensee PSRP, Lahore, Pakistan. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (http://creativecommons.org/licenses/by/4.0/).