Limits of the uni-directional pulse propagation approximation

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I apply the method of characteristics to both bi-directional and uni-directional pulse propagation in dispersionless media containing nonlinearity of arbitrary order. The differing analytic predictions for the shocking distance quantify the effects of the uni-directional approximation used in many pulse propagation models. Results from numerical simulations support the theoretical predictions, and reveal the nature of the coupling between forward and backward waves.

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I. INTRODUCTION

Most approaches to optical pulse propagation rely on an approximation where the fields only propagate forwards. Even the recently derived extensions of typical propagation methods used in nonlinear optics (e.g. [1, 2]) assume a complete decoupling between oppositely propagating fields to optimise the calculation. Moreover, those based directly on Maxwell’s equations (e.g. [3, 4, 5]) or the second order wave equation (e.g. [6, 7, 8]), are often simplified to work in the forward-only limit, where backward propagating fields are set to zero. This is despite directional decompositions of Maxwell’s equations (e.g. [3, 9]) indicating that nonlinearity inevitably couples the forward and backward waves together – and even creates a backward field if one is not present. Usually we assume that a forward wave will not generate a significant backward wave via the nonlinearity because the backward component is very poorly phase matched. In contrast, deliberately trying to phase match the backward wave was suggested in the 1960’s [10] and some progress has been made in \( \chi^{(2)} \) materials in recent years (e.g. [11, 12, 13, 14]); however such attempts are not the focus of this paper.

Here I use the phenomenon of carrier wave shock[ing as a tool to probe the fundamental limits of a uni-directional model under the influence of the intrinsically bi-directional nonlinear coupling. Carrier wave shocks are discontinuities in the electric (and magnetic field) profiles, and develop as the nonlinear effects distort the waveform as it propagates. Assuming an initially uni-directional field, I compare analytic predictions for the shocking distance from uni- and bi-directional theories. These are based on the Method of Characteristics (MOC) [16], and show a clear difference between the predicted shocking distances in the two cases. The (exact) bi-directional theory is based on the second order wave equation (as in [15, 17]), whereas the (approximate) uni-directional theory is derived from the \( G^\pm \) directional fields [3] description. I support the theoretical results with pseudospectral spatial domain (PSSD) simulations [1] for both second order (\( \chi^{(2)} \)) and third order (\( \chi^{(3)} \)) nonlinearities.

The results in this paper act as a bound on the validity of one-dimensional optical propagation models using a uni-directional approximation. Since linear dispersion and finite nonlinear response times will typically diminish any generation of a backward wave, it is clear that any model which can be assumed uni-directional on the basis of this paper will be more so in practise. These results do not tell us whether the uni-directional approximation would be more or less robust for models incorporating transverse effects such as self-focussing, but they at least establishes a point of reference, valid for beams with a weak spatial variation. Note also that I consider only propagation within bulk media, since surfaces or interfaces can cause reflections, and clearly require a bi-directional model.

Section II briefly describes the MOC, and presents a prediction for the shocking distance in the bi-directional case due to simple nonlinearities of arbitrary order; section III follows with analogous calculations using an explicitly uni-directional wave equation. Next, section IV considers the analytic and numerical results, section V considers the role of the backward field, and then section VI presents some conclusions.

II. BI-DIRECTIONAL MODEL

Analytic formulae for the shocking distance in materials with instantaneous response have been calculated for both third-order nonlinearities (by Rosen [15]) and a simple second order case (by Radnor [18]). Both calculations used the MOC to predict the formation of a value discontinuity in the field at certain points within the electric field profile of a pulse or wave. Here I generalise the treatment to allow for an instantaneous perturbative nonlinearity of arbitrary order, following the calculation of Kinsler et al. [17]. Note that the calculation below

\( \chi^{(2)} \)

\( \chi^{(3)} \)
The usual second order wave equation for \( E \) in spectral domain, the distortion corresponds to a build-up of the coefficients, the displacement field is using the MOC can be found. Ignoring the tensor nature rather than the most general form, because then solutions significant quantities of higher order harmonic content.

In this paper, I consider nonlinearities of the simplest, rather than the most general form, because then solutions using the MOC can be found. Ignoring the tensor nature of the coefficients, the displacement field is

\[
D = \epsilon_0 \left( E + \chi^{(1)} E + \chi^{(m)} E^m \right).
\]  

(1)

The usual second order wave equation for \( E \) in this case is

\[
\frac{c^2}{n_0^2} \frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial z^2} + \chi^{(m)} \frac{\partial^2 E^m}{\partial t^2},
\]

(2)

where \( \epsilon_r = 1 + \chi^{(1)} = n_0^2 \) is the (relative) dielectric constant and \( n_0 \) the linear refractive index.

The MOC treats each point on the waveform \( E(t) \) separately, and considers how it will move as the wave propagates. The line traced out by the movement of any one of these points is a characteristic. The equation associated with eqn. (2) that governs the characteristic lines of \( E \) is

\[
\frac{\partial E}{\partial t} + v_m(E) \frac{\partial E}{\partial z} = 0.
\]

(3)

with the velocity \( v_m(E) \) being that for a point on the wave with field strength \( E \), which is given by

\[
v_m(E) = \frac{c}{n_0} \left[ 1 + m \chi^{(m)} E^{m-1}/n_0^2 \right]^{-1/2}.
\]

(4)

In this picture, the nonlinearity manifests itself by giving different (fixed) velocities to characteristics of different \( E \). Thus, as the wave profile \( E(t) \) propagates forward in space \( z \), the wave profile becomes distorted by temporal compressions or expansions. A shock occurs if a region is compressed to the point where two characteristics intersect. For a \( \chi^{(3)} \) nonlinearity, characteristics with higher \( E^2 \) move more slowly, dragging the peaks toward later times; a shock will first occur on the profile where \( E^2 \) is changing most rapidly in time. Waves approaching the point of shocking are shown in fig. 1.

![FIG. 1: The progressive distortion of an initially sinusoidal wave profile as the shocking distance is approached for (a) \( \chi^{(3)} \) and (b) \( \chi^{(2)} \) nonlinearities in a dispersionless medium. In the spectral domain, the distortion corresponds to a build-up of significant quantities of higher order harmonic content.](image1)

Using eqn. (4) along with the construction shown in fig. 2, we can derive a simple formula for the distance to shocking. The figure shows two characteristics AC and BC, originating from points A and B, and converging towards a shock at C after a distance of \( L \). We have drawn the case where the speed associated with AC (represented by its gradient) is lower than that of BC. From the geometry of the figure, it is easy to show that

\[
\frac{dv}{dt} = \frac{v}{t} = \frac{v^2}{L},
\]

(5)

where \( t, v, \) and \( L = vt \) are respectively time, velocity, and distance. Differentiating the velocity leads to

\[
\frac{dv_m}{dt} = -\frac{mc}{2 \left( 1 + m \chi^{(m)} E^{m-1}/n_0^2 \right)^{3/2}} \frac{\partial (E^{m-1})}{\partial t},
\]

(6)

and combining eqns. (5) and (7) yields

\[
L_m = \frac{v^2_m}{\frac{dv_m}{dt}} = 2c n_0 \sqrt{1 + m \chi^{(3)} E^{m-1}/n_0^2}.
\]

(8)

Thus \( L_m \) depends on \( E(t) \), and so will vary across the pulse profile. As the profile propagates and evolves, so will the predictions for \( L_m \), nevertheless, the point at which shocking occurs relative to the origin remains fixed. For a given profile, a shock will occur first at the point...
where the $L_m$ reaches its minimum value. We can therefore define the shocking distance, for any arbitrary waveform $E(t)$, as

$$S_m = \frac{2cn_0}{m} \min\left[\frac{C_m/\chi^{(m)}}{(-\partial E^{-1}/\partial t)}\right], \quad (9)$$

where

$$C_m = \sqrt{1 + m\chi^{(m)}}E^{-1}/n_0^2. \quad (10)$$

We see that the shocking distance is inversely proportional to the nonlinear strength $\chi^{(m)}$ (as would be expected), and that the other important quantity is a derivative of powers of the field, i.e. $(\partial E^{-1}/\partial t)$. Note that for $\chi^{(m)} > 0$, the nonlinear correction factor $C_m$ will always be greater than 1 for odd-order nonlinearities, but may be less than 1 for even-order nonlinearities. In the limit of weak nonlinearity, $m\chi^{(m)}E^{-1}/n_0^2 \ll 1$,

$$S_m \approx \frac{2cn_0}{m} \min\left[\frac{1/\chi^{(m)}}{(-\partial E^{-1}/\partial t)}\right]. \quad (11)$$

### III. UNI-DIRECTIONAL MODEL

Using the directional field approach of Kinsler et al. \cite{3} we can write a pair of coupled first order wave equations for directional $G^\pm$ fields under the influence of an $m$th order nonlinearity without dispersion. In the present case of a dispersionless medium, these are defined by using $G^\pm = \sqrt{\epsilon}E_x \pm \sqrt{\mu_0}H_y$, where $\epsilon = \mu_0(1 + \chi^{(1)})$. This combination of scaled fields provides us with a $G^+$ field whose Poynting vector points forward along the $z$ axis, and a $G^-$ field whose Poynting vector points backward. To get a uni-directional model, we simply set the backward propagating $G^-$ field to zero, leaving us with a single first order wave equation for the forward propagating $G^+$ field. Importantly, we do not require the use of an exponential carrier function to impart the directionality (see e.g. \cite{14, 20}), as this is achieved by the construction of the $G^\pm$ fields themselves. As in section \[11\] the calculation below can easily be generalised to include a sum of nonlinear terms of different order, if so desired.

After defining $E^\pm = G^\pm/2\sqrt{\epsilon}$ \cite{21}, the coupled bi-directional wave equations are

$$\frac{\partial E_\pm}{\partial z} + \frac{n_0}{c} \frac{\partial E_\pm}{\partial t} = \pm \frac{\chi^{(m)}}{2c} \frac{\partial (E_+ + E_-)^m}{\partial t}. \quad (12)$$

Setting $E_- = 0$ gives us a single, uncoupled, unidirectional wave equation which we can rewrite as

$$\frac{\partial E_+}{\partial t} + v_{m+}(E_+) \frac{\partial E_+}{\partial z} = 0, \quad (13)$$

with velocity

$$u_m(E_+) = \frac{c}{n_0} \left[1 + m\chi^{(m)}E_+^{-1}/2n_0^2\right]. \quad (14)$$

Comparing eqn. \[13\] to eqn. \[3\] we see they have the same form; so that this wave equation also describes the motion of its characteristics. Note also that eqn. \[14\] is equivalent to a first-order expansion of the square root term in eqn. \[1\]. We can again use the method of characteristics. Differentiating the velocity $u_m$ gives

$$\frac{du_m}{dt} = \frac{n_0}{c} mc\chi^{(m)}/(1 + m\chi^{(m)}E_+^{-1}/2n_0^2)^2 \frac{\partial (E_+^{-1})}{\partial t}. \quad (15)$$

and combining eqns. \[5\] and \[15\] yields the shocking distance

$$S_{m+} = \frac{2cn_0}{m} \min\left[\frac{1/\chi^{(m)}}{(-\partial E^{-1}/\partial t)}\right]. \quad (16)$$

This is the same as the weak nonlinearity limit of the bi-directional prediction in eqn. \[9\], i.e. it lacks the correction factor $C_m$ defined in eqn. \[10\].

It is important to note that eqn. \[10\] was found using an explicitly uni-directional formalism, in which the only approximation was of uni-directional propagation with no coupling to the backward wave. Thus it allows an unambiguous comparison of uni-directional and bi-directional propagation.

### IV. DISCUSSION

We can see by comparing the shocking distances predicted by the (exact) bi-directional theory in eqn. \[9\] and the (approximate) uni-directional theory in eqn. \[10\] that both theoretical predictions have the same two dominant trends: they are inversely dependent on the nonlinear coefficient $\chi^{(m)}$, and on the gradient of the $(m-1)$th power of the field.

The difference between the bi- and uni-directional theories lies in the nonlinear correction factor $C_m$, which only becomes significant for extremely strong nonlinearities. For small nonlinearities, and correspondingly long shocking distances, the poorly phase matched backward component does not build up, so the forward field (and hence shocking distances) are barely affected. In fused silica near the damage threshold, effective nonlinearities of order $\chi^{(3)}E^2 \simeq 0.06$ can be achieved. In the absence of dispersion, this amount of nonlinearity would lead to shocking distances of less than three wavelengths (e.g. $S_3 \sim 2\mu$m for $\lambda = 800$nm), a regime in which $C_3 \simeq 1.04$.

For stronger nonlinearities, significant conversion can take place before the field has propagated even one wavelength, let alone the several needed for the averaging effect caused by poor phase matching. The growth of a significant backward component then affects the propagation; indeed it can even be very strongly altered if $m\chi^{(m)}E_+^{-1}/2n_0^2 \sim 1$. However, such extreme nonlinearities are not achievable in realistic systems, since they require field intensities far beyond the damage thresholds of standard nonlinear materials. Nevertheless, the point of this paper is to examine the limits of the uni-directional approximation: adding more realistic material models would test only those models, not the uni-directional approximation.
To support the theory, I have also done simulations of the bi-directional and uni-directional cases using the PSSD technique. These are (a) straightforward simulations of Maxwell’s equations, which are naturally bi-directional; (b) bi-directional simulations of Maxwell’s equations, using the $G^\pm$ directional fields; and (c) uni-directional simulations, using a $G^+$ only forward propagating model. I treat nonlinear orders $m = 2, 3$ only and propagate two-cycles of a (sinusoidal) CW field sampled by 512 or 2048 points, giving time resolutions of $dt = 0.0128\,\text{fs}$ and $0.0032\,\text{fs}$. For the $E, H$ simulations I use a Yee-style staggered grid and the PSSD method; for the $G^\pm$ or $G^+$ simulations I use a leapfrog method, which, being analogous to the staggered grid, ensures that the numerical performance of the simulations is comparable. Numerical “shocks” are detected by using the local discontinuity detection (LDD) technique.

The shocking distances vary slightly from the comparable theory for two reasons. First, the LDD technique is not a perfect predictor of shocks. Second, the staggered grids used by the simulations make it very hard to perfectly match the two initial fields values needed, especially in the case of strong nonlinearity.

Figs. 3 and 4 exhibit different trends for the scaled shocking distances $\chi^{(3)} S_3$ and $\chi^{(2)} S_2$; a difference due to the nonlinear correction $C_m$, which contains a field dependent part $F_m = m \chi^{(m)} E^{m-1}/n_0^2$. In the even-order case $F_m$ can take either sign; whereas for odd-order it can only have the sign of $\chi^{(m)}$. Thus for sufficiently strong nonlinearities, $C_m$ could become zero, implying a zero shocking distance. This can happen for odd-order nonlinearities only if $\chi^{(m)}$ is negative, but will always be possible in the even-order case. This exotic “instant shock” behaviour is a mathematical prediction rather than a physical one, since the effective nonlinearities required are far in excess of those attainable in experiment, and realistic nonlinearities do not have an instantaneous response. Nevertheless, it is instructive to examine the reason for this surprising effect, as it demonstrates how backward waves can influence the forward ones.

We see this instant shock regime in fig. 4, where the trend for $\chi^{(2)} S_2$ is downward, with a sudden dip toward zero when the peak field $E_0$ gives $F_2 = -1$, and remaining at zero for still stronger nonlinearities. This causes numerical difficulties, leading to relatively poor agreement between theory and simulation in this region on fig. 4, although this improves as the temporal resolution is increased.

On fig. 5 we see the forward and backward field components $G^\pm$ in the case of a positive $\chi^{(3)}$, where $\chi^{(3)} S_3$
increases with nonlinearity. Both are steepest near $\sim 1\text{fs}$, where $G^-$ has a gradient of the opposite sign to that of $G^+$, so the gradient of $E \sim G^+ + G^-$ is reduced. Consequently the backward $G^-$ wave has the effect of increasing the shocking distance; although the effect would reverse for a negative $\chi^{(3)}$. Note that both these components are strongly coupled to each other.

In contrast, in the $\chi^{(2)}$ case on fig. 6, the equivalent region (near 1.3fs) shows that the gradients of $G^+$ and $G^-$ can have the same sign, enhancing the gradient of $E$. Closer inspection shows that the gradient of $G^-$ abruptly switches sign where $G^+$ has a point of inflection, so the $G^-$ field mitigates shocking on one side, and enhances it on the other. The enhancement then acts like a feedback process, where the backward wave steepens the gradient, which in turn enhances the backward wave, and so on. Thus not only does the backward wave decrease the shocking distance, but for sufficiently strong nonlinearities the feedback can cause a shock in an infinitesimally short distance – just as predicted by eqn. (10).

Note that since the effect of the $\chi^{(2)}$ nonlinearity depends on the sign of the field, the backward field components will tend to increase the shocking distances in some regions of the field profile, but reduce them in others. However, since we always look for the minimum shocking distance, this will always be reduced, whatever the sign of $\chi^{(2)}$.

VI. CONCLUSIONS

I have demonstrated the fundamental limits of the widely used uni-directional propagation approximation. This was done by comparing analytic results for the shocking distance obtained from both an exact bi-directional model, and an approximate uni-directional model. These theoretical results were for simple nonlinearities of arbitrary order, based on the method of characteristics, and are supported by numerical simulations of both models. The exact bi-directional results were based on Maxwell’s equations using $E \& H$. The approximate uni-directional results relied on the construction of the $G^\pm$ directional fields [3], which enabled the forward-only approximation to be made without introducing any additional assumptions.

The results show that the condition $|m\chi^{(m)}E^{m-1}/n_0^m| \ll 1$ must hold for the uni-directional approximation to be true; even when no backward field is initially present. This condition is usually easily satisfied in nonlinear optical materials – even with fields strong enough to approach the damage threshold. If this condition does not hold, then significant non phase-matched forward-backward coupling can occur, affecting the propagation accordingly: in extreme cases causing behaviour like the “instant shock” discussed in section V. Such features, demonstrated in a comparison between two models which only differ by a uni-directional approximation made in one, provide an important indication of the limitations of a uni-directional description.

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[1] T. Brabec, F. Krausz, “Nonlinear optical pulse propagation in the single-cycle regime”, Phys. Rev. Lett. 78, 3282-3285 (1997).
[2] P. Kinsler, G.H.C. New, “Few cycle pulse propagation”, Phys. Rev. A 67, 023813 (2003).
[3] P. Kinsler, S.B.P. Radnor, G.H.C. New, “Theory of di-
rectional pulse propagation”, Phys. Rev. A. 72, 063807 (2005).
[4] M. Kolesik, J.V. Moloney, M. Mlejnek, “Unidirectional optical pulse propagation equation”, Phys. Rev. Lett. 89, 283902 (2002).
[5] J.C.A. Tyrrell, P. Kinsler, G.H.C. New, “Pseudospectral Spatial-Domain: A new method for nonlinear pulse propagation in the few-cycle regime with arbitrary dispersion”, J. Mod. Opt. 52, 973-986 (2005).
[6] K.J. Blow, D. Wood, “Theoretical description of transient stimulated Raman scattering in optical fibers”, IEEE J. Quant. Electron. 25, 2665-2673 (1989).
[7] A. Ferrando, M. Zacaretés, P.F. de Córdoba, D. Binosi, A. Montero, “Forward-backward equations for nonlinear propagation in axially invariant optical systems”, Phys. Rev. E 71, 016601 (2005).
[8] G. Genty, P. Kinsler, B. Kibler, J.M. Dudley, “Nonlinear envelope equation modeling of sub-cycle dynamics and harmonic generation in nonlinear waveguides”, Opt. Express 15, 5382-5387 (2007).
[9] P. Kinsler, “Theory of directional pulse propagation: detailed calculations”, arXiv:physics/0611216.
[10] S.E. Harris, “Proposed backward wave oscillation in the infrared”, Appl. Phys. Lett. 9, 114-116 (1966).
[11] J.U. Kang, Y.J. Ding, W.K. Burns, J.S. Melinger, “Backward second-harmonic generation in periodically poled bulk LiNbO3”, Opt. Lett. 22, 862-864 (1997).
[12] Y.J. Ding, J.B. Khurgin, “Backward optical parametric oscillators and amplifiers”, IEEE J. Quant. Electron. 32, 1574-1582 (1996).
[13] Y.J. Ding, J.U. Kang, J.B. Khurgin, “Theory of backward second-harmonic and third-harmonic generation using laser pulses in quasi-phase-matched second-order nonlinear medium”, IEEE J. Quant. Electron. 34, 966-974 (1998).
[14] J.Z. Sanborn, C. Hellings, T.D. Donnelly, “Breakdown of the slowly-varying-amplitude approximation: generation of backward-traveling, second-harmonic light”, J. Opt. Soc. Am. B20, 152-157 (2003).
[15] G. Rosen, “Electromagnetic shocks and the self-annihilation of intense linearly polarized radiation in an ideal dielectric material”, Phys. Rev. 139, A539-A543 (1965).
[16] G. B. Whitham, “Lectures on Wave propagation” (Wiley, New York, 1979).
[17] P. Kinsler, S.B.P. Radnor, J.C.A. Tyrrell, G.H.C. New, “Optical carrier wave shocking: Detection and dispersion”, Phys. Rev. E 75, 066603 (2007).
[18] S.B.P. Radnor (Department of Physics, Imperial College London, London, personal communication, 2006).
[19] S.B.P. Radnor, L.E. Chipperfield, P. Kinsler, G.H.C. New, “Carrier wave self-steepening and application to high harmonic generation”, submitted to Phys. Rev. A (2007).
[20] L.W. Casperson, “Field-equation approximations and amplification in high-gain lasers: Numerical results”, Phys. Rev. A 44, 3291-3304 (1991).
[21] P. Kinsler, “Pulse propagation methods in nonlinear optics”, arXiv:0707.0982.
[22] K.S. Yee, “Numerical solution of initial boundary value problems involving Maxwell’s equations in isotropic media”, IEEE Trans. Antennas Propagat. AP-14, 302-307 (1966).

Appendix: Time propagated MOC

1. Bi-directional form

Here we take the expression in eqn. [3], and swap the roles of $t$ and $z$. The associated equation governing the characteristic lines of $E(z)$ is

$$w_m(E)\frac{\partial E}{\partial t} + \frac{\partial E}{\partial z} = 0. \quad (17)$$

with the (inverse) velocity $w_m(E)$ given by

$$w_m(E) = \frac{n_0}{c} \left[ 1 + m\chi(m)E^{-1}/n_0^2 \right]^{1/2}. \quad (18)$$

Using eqn. [13] along with as time-space swapped version of the construction shown in fig. 2 we can derive a simple formula for the time to shocking. From the geometry, it is easy to show that

$$\frac{dw}{dz} = \frac{w}{z} = \frac{w^2}{T} \quad (19)$$

where $z$, $w$, and $T = wz$ are respectively distance, inverse velocity, and time. Differentiating the inverse velocity leads to

$$\frac{dw_m}{dz} = \frac{n_0}{c} \left[ 1 + m\chi(m)E^{-1}/n_0^2 \right]^{1/2} \frac{dE^{-1}}{dz} \quad (20)$$

and combining eqns. [19] and [22] yields

$$T_m = w_m\frac{dw_m}{dz} \quad (23)$$

$$= \frac{w_m^3}{m\chi(m)} \left( \frac{dE^{-1}}{dz} \right)^{-1}. \quad (24)$$

For a given profile, a shock will occur first at the point where $T$ reaches its minimum value. We can therefore define the shocking time $T_m$, for any arbitrary waveform $E(z)$, as

$$T_m = \frac{2n_0^3}{mc} \min C_m/E^{m-1}/dz \quad (25)$$

where

$$C_m = \sqrt{1 + m\chi(m)E^{-1}/n_0^2} \quad (26)$$
\[ C_m^3 \simeq 1 + \frac{3}{2} m \chi^{(m)} E^{m-1}/n_0^2. \]  

(27)

In the spatially propagated form (eqn. [19]), the correction term is \( C_m \), not the \( C_{m0} \), seen here.

Note that the size of the first order correction term is independent of the nonlinear strength, and depends only on the properties of the field profile:

\[
\delta T_m = \frac{2n_0^3}{mc\chi^{(m)}} \times \frac{3m \chi^{(m)} E^{m-1}}{2n_0^2} \left( \frac{dE^{-1}}{dz} \right)^{-1} 
\]

(28)

\[
= \frac{6n_0^3m \chi^{(m)} E^{m-1}}{2n_0^2mc\chi^{(m)}} \times \left( \frac{dE^{-1}}{dz} \right)^{-1} 
\]

(29)

\[
= \frac{3n_0 E^{m-1}}{c} \times \left( \frac{dE^{-1}}{dz} \right)^{-1}. 
\]

(30)

2. Uni-directional form

Here we take the expression in eqn. [19], and swap the roles of \( t \) and \( z \). The associated equation governing the characteristic lines of \( E(z) \) is

\[
w_{m+}(E) \frac{\partial E}{\partial t} + \frac{\partial E}{\partial z} = 0. 
\]

(31)

with the (inverse) velocity \( w_{m+}(E) \) given by

\[
w_{m+}(E) = \frac{n_0}{c} \left[ 1 + m \chi^{(m)} E^{m-1}/2n_0^2 \right]. 
\]

(32)

Using eqn. [19] along with as time-space swapped version of the construction shown in fig. 2, we can derive a simple formula for the time to shocking. From the geometry, it is easy to show that

\[
dw \over dz = \frac{w}{T} \]

(33)

where \( z, w, \) and \( T = wz \) are respectively distance, inverse velocity, and time. Differentiating the inverse velocity leads to

\[
dw_{m+} \over dz = \frac{n_0}{c} \frac{m \chi^{(m)} E^{m-1}/2n_0^2}{dE^{-1}/dz} \]

(34)

\[
= \frac{n_0 m \chi^{(m)} E^{m-1}}{c} \frac{2n_0^2}{dz} 
\]

(35)

\[
= \frac{m \chi^{(m)} E^{m-1}}{2n_0 c^2} \frac{dz}{dw_{m+}}. 
\]

(36)

and combining eqns. [19] and [22] yields

\[
T_{m+} = w_{m+}^2 \frac{dw_{m+}}{dz} 
\]

(37)

\[
= w_{m+}^2 \frac{2n_0 c}{m \chi^{(m)}} \left( \frac{dE^{-1}}{dz} \right)^{-1} 
\]

(38)

\[
= \frac{n_0^2}{c^2} \left( 1 + m \chi^{(m)} E^{m-1}/2n_0^2 \right)^2 \frac{2n_0 c}{m \chi^{(m)}} \left( \frac{dE^{-1}}{dz} \right)^{-1} 
\]

(39)

For a given profile, a shock will occur first at the point where \( T \) reaches its minimum value. We can therefore define the shocking time \( T_{m+} \), for any arbitrary waveform \( E(z) \), as

\[
T_{m+} = \frac{2n_0^3}{mc \chi^{(m)}} D_m^{2}/\chi^{(m)} dE^{-1}/dz, 
\]

(40)

where

\[
D_m = 1 + m \chi^{(m)} E^{m-1}/2n_0^2. 
\]

(41)

\[
D_m^2 \simeq 1 + m \chi^{(m)} E^{m-1}/n_0^2. 
\]

(42)

In the spatially propagated form (eqn. [19]), the \( D_m \)-like term is simply 1.

Note that the size of the first order correction term is independent of the nonlinear strength, and depends only on the properties of the field profile:

\[
\delta T_{m+} = \frac{2n_0^3}{mc \chi^{(m)}} \times \frac{m \chi^{(m)} E^{m-1}}{n_0^2} \left( \frac{dE^{-1}}{dz} \right)^{-1} 
\]

(43)

\[
= \frac{2n_0^3m \chi^{(m)} E^{m-1}}{n_0^2mc \chi^{(m)}} \times \left( \frac{dE^{-1}}{dz} \right)^{-1} 
\]

(44)

\[
= \frac{2n_0 E^{m-1}}{c} \times \left( \frac{dE^{-1}}{dz} \right)^{-1}. 
\]

(45)

3. Comparison: Bi- vs Uni-directional

The comparison is not as neat as for the spatially propagated case, but nevertheless is very similar. For the small nonlinearity limit (i.e. \( m \chi^{(m)} E^{m-1}/n_0^2 \ll 1 \), the difference is just the factor of 1 + \( m \chi^{(m)} E^{m-1}/2n_0^2 \) that appears for the spatially propagated results in the same limit.

Note that the shocking time \( T \) is not just related to the shocking distance \( L \) by a simple factor of \( c \), since in the spatially propagated picture, the peak-dragging effect corresponds to a time offset which needs to be added to the propagation time \( L/c \).