Isochrone spacetimes

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Abstract
We introduce the relativistic version of the well-known Henon’s isochrone spherical models: static spherically symmetrical spacetimes in which all bounded trajectories are isochrone in Henon’s sense, i.e., their radial periods do not depend on their angular momenta. Analogously to the Newtonian case, these “isochrone spacetimes” have as particular cases the so-called Bertrand spacetimes, in which all bounded trajectories are periodic. We propose a procedure to generate isochrone spacetimes by means of an algebraic equation, present explicitly several families of these spacetimes, and discuss briefly their main properties. We identify, in particular, the family whose Newtonian limit corresponds to the Henon’s isochrone potentials and that could be considered as the relativistic extension of the original Henon’s proposal for the study of globular clusters. Nevertheless, isochrone spacetimes generically violate the weak energy condition and may exhibit naked singularities, challenging their physical interpretation in the context of General Relativity.

Keywords Static spherically symmetrical spacetimes · Geodesic motion · Isochrone orbits

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1 Introduction

The so-called isochrone spherical models were introduced by Michel Hénon in the fifties [1–3] in the study of the dynamics of globular clusters, see [4] for a brief review on the subject. Globular clusters are dense, roughly spherically symmetric, distributions of stars whose dynamics is usually described through an averaged gravitational potential, leading naturally to the study of general central potentials. Henon’s isochrone models are intimately related to the classical Bertrand’s theorem. Strictly speaking, an isochrone model in Henon’s sense is a Newtonian spherically symmetric gravitational field for which the radial periods of bounded orbits do not depend on their angular momenta. The Newtonian and the harmonic oscillator potentials, which according to Bertrand’s theorem are the only central potentials for which all bounded trajectories are periodic (closed), are also isochrone in Henon’s sense. In fact, Bertrand’s theorem can be considered as a refinement of the concept of isochrone potential, the Newtonian and the harmonic oscillator potentials are the only isochrone potential with closed orbits, see Section 4.4 of [5].

In principle, non-relativistic gravitational configurations as the isochrone spherical models might be effectively attained in globular clusters as a result of a dynamical mechanism called resonant relaxation, as suggested by Hénon in his original works [1–3], see also [6, 7] for more modern approaches to this subject. For further recent developments on the dynamics of isochrone potentials, see [5, 8–10]. The family of Henon’s spherical isochrone potentials includes, besides the two cases of Bertrand’s theorem, three other central potentials, namely the so-called Hénon potential

\[ V_{He}(r) = -\frac{k}{b + \sqrt{b^2 + r^2}}, \]  

(1)

and, respectively, the bounded and hollowed potentials

\[ V_{bo}(r) = \frac{k}{b + \sqrt{b^2 - r^2}}, \]  

(2)

\[ V_{ho}(r) = -\frac{k}{r^2} \sqrt{r^2 - b^2}, \]  

(3)

where \( b \) and \( k \) are positive constants. It is clear that the potentials (2) and (3) are not defined for all \( r \) and that the Newtonian potential arises from the limit \( b \to 0 \) of \( V_{He} \) or \( V_{ho} \). The isochrone potentials (1) and (3) are asymptotically Newtonian for large \( r \), while (1) and (2) are effectively harmonic near the center \( r = 0 \).

Some years ago, Perlick [11] introduced the notion of a Bertrand spacetime, namely a spherically symmetric and static spacetime in which any bounded trajectory of test bodies is periodic, clearly extending the classical Bertrand’s theorem to the realm of General Relativity. From Bertrand’s theorem, we have also that the azimuthal angle
for the Newtonian and harmonic potential cases is given by $\Theta = \frac{\pi}{\beta}$ with, respectively, $\beta = 1$ and $\beta = 2$. We remind that the azimuthal angle corresponds to the angular variation between the closest and farthest points to the center of a bounded trajectory. Rather surprisingly, Perlick showed that it is always possible to construct a static spherically symmetrical spacetime in which all test body bounded trajectories are periodic for any rational value of the parameter $\beta$. We know that any spherically symmetric and static spacetime can be cast in a spherical Schwarzschild coordinate system where its metric takes the form

$$g_{ab}dx^a dx^b = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\Omega^2,$$  \hspace{1cm} (4)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ denotes the usual metric on the unit sphere $S^2$. There are two types of Bertrand spacetimes. For the first type, we have

$$\frac{1}{f(r)} = G + \sqrt{\kappa + r^{-2}},$$  \hspace{1cm} (5)

$$h(r) = \beta^2 (1 + \kappa r^2),$$  \hspace{1cm} (6)

whereas the second case corresponds to

$$\frac{1}{f(r)} = G \mp \frac{r^2}{1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - Kr^4}},$$  \hspace{1cm} (7)

$$h(r) = \frac{\beta^2 ((1 - Dr^2)^2 - Kr^4)}{2 \left(1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - Kr^4}\right)},$$  \hspace{1cm} (8)

with arbitrary constants $G$, $\kappa$, $D$, and $K$. For an enlightening geometrical interpretation of the Bertrand spacetimes and their relation with the standard harmonic and Newtonian potentials on curved 3-dimensional manifolds, see [12]. For the Bertrand spacetimes of the first type, the Newtonian limit is obtained by setting $\kappa = 0$ (locally flat condition) and $\beta = 1$ (absence of conical singularity at the origin), and clearly corresponds to the $r^{-1}$ interaction. On the other hand, for the second type of spacetimes, the locally flat condition and the absence of conical singularity requires, respectively, $D = K = 0$ and $\beta = 2$, and we have eventually the harmonic potential case.

In the present paper, we introduce the relativistic version of the Henon’s isochrone models, which we denominate “isochrone spacetimes”. They correspond to the static spherically symmetric spacetimes (4) in which all bounded timelike trajectories are isochrone in Henon’s sense, i.e., their radial periods do not depend on their angular momenta. As in the non-relativistic Newtonian case [10], the isochrony condition for a static spherically symmetric spacetime is equivalent to demand that the relativistic radial action be additively separable in the two pertinent constants of motion, namely the energy and the angular momentum. However, as we will see, in clear contrast with the Newtonian case, such condition is not a very stringent restriction in the relativistic domain. Besides the rather trivial extension of the Bertrand spacetimes for the case of
real $\beta$, there are many other families of isochrone spacetimes. We propose a procedure to generate such new spacetimes by means of an algebraic equation and present explicitly some families of solutions. In particular, we present a family whose Newtonian limit corresponds to Henon’s isochrone potentials (1), (2) and (3). Some of these spacetimes are asymptotically flat and others have regular centers in an analogous way of their Newtonian counterparts and, hence, they could be useful as a relativistic extension of the original Henon’s proposal for the study of globular clusters. However, isochrone spacetimes generically violate the weak energy condition and may exhibit naked singularities and, consequently, their physical interpretation in General Relativity is rather challenging. In the next section, we will introduce the isochrone spacetimes and present a procedure to generate them by exploring an algebraic equation. We will present explicitly three large families of these spacetimes and discuss briefly some of their main properties. The third and last section is devoted to some concluding remarks, including the issue of the violation of the weak energy conditions and some brief comments about the causal structure of isochrone spacetimes.

2 The isochrone spacetimes

The pertinent Lagrangian for the motion of test bodies in a static spherically symmetrical spacetime with metric (4) reads

$$\mathcal{L} = - f \dot{t}^2 + \frac{\dot{r}^2}{h} + r^2 \dot{\phi}^2,$$

with the dot denoting the usual derivation with respect to the proper time of the timelike geodesic, where we have, without loss of generality due the spherical symmetry, restricted the motion to the equatorial plane. The geodesic equation derived from (9) admits 3 constants of motion, which are

$$f \dot{t} = 1, \quad \ell = r^2 \dot{\phi}, \quad E = - \frac{1}{f} + \frac{\dot{r}^2}{h} + \frac{\ell^2}{r^2},$$

with $\ell$ and $E$ interpreted, respectively, as the trajectory angular momentum and energy. This is a simple one-dimensional motion problem and the orbit radial period can be determined from the conserved quantities by exploring elementary methods. We have

$$T(E, \ell) = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{\sqrt{h(r) \sqrt{E - U_\ell(r)}}},$$

where $r_{\text{min}}$ and $r_{\text{max}}$ are the usual return points of the effective potential

$$U_\ell(r) = - \frac{1}{f(r)} + \frac{\ell^2}{r^2},$$

which is expected to have a local minimum at $r_0$ corresponding to the circular orbits. We also assume $U''_\ell(r_0) > 0$, i.e., the dynamical stability of the circular orbit. Moreover,
all functions here are assumed to be sufficiently smooth. The other important quantity for our purposes in this work is the azimuthal angle, which is given by

$$\Theta(E, \ell) = \ell \int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2 \sqrt{h(r)} \sqrt{E - U(r)}}$$  \hspace{1cm} (13)$$

and corresponds to the angular variation between the closest and farthest points to the center of a bounded trajectory. Notice that the apsidal angle, another common quantity used in this kind of analysis, is defined as the angular variation during one radial period and, hence, is twice the azimuthal angle. Although the effective potential (12) is indeed the relativistic counterpart of the sum of the Newtonian gravitational potential and the centrifugal barrier term, the presence of the factor $h(r)$ in (11) and (13) spoils any other useful analogy here. From (4), we see that $h(r) = 1$ basically reduces the problem to the Newtonian one, since in this case we can interpret $V(r) = -1/f(r)$ as a Newtonian potential in a spatially flat spacetime. Hence, the genuine relativistic problem demands a non-constant $h(r)$, which implies in a spatially curved spacetime, preventing the direct use of any Newtonian result in the present case. In particular, there is no place here for the introduction of the so-called Henon variables and, consequently, the identification of the parabolic properties of the isochrone potentials, see [5]. For further details on the interpretation of the effective potential (12) on spatially curved manifolds, see [12].

The isochrony condition $\frac{\partial T}{\partial \ell} = 0$ is equivalent to demand $\frac{\partial \Theta}{\partial E} = 0$ since we have $T = \frac{1}{2} \frac{\partial \mathcal{A}_r}{\partial E}$ and $\Theta = -\frac{1}{2} \frac{\partial \mathcal{A}_r}{\partial E}$, with $\mathcal{A}_r$ standing for the so-called radial action of the problem

$$\mathcal{A}_r(E, \ell) = 2 \int_{r_{\min}}^{r_{\max}} \frac{\sqrt{E - U(r)} dr}{\sqrt{h(r)}}.$$  \hspace{1cm} (14)$$

Notice that, as in the Newtonian case [10], the isochrony condition for static spherically symmetrical spacetimes is fully equivalent to demand that the relativistic radial action be additively separable, i.e., $\mathcal{A}_r(E, \ell) = \mathcal{B}(E) + \mathcal{C}(\ell)$. In other words, we have $\frac{\partial T}{\partial \ell} = 0$, and consequently $\frac{\partial \Theta}{\partial E} = 0$, if and only if the radial action is additively separable in the constants $E$ and $\ell$. Unfortunately, however, due to the generic presence of spatial curvature (non constant $h(r)$), the relativistic isochrony condition is not enough to single out relativistic isochrone potentials as it happens in the Newtonian case.

For the sake of notation simplicity, we will drop hereafter the $\ell$ index denoting dependence in $U(r)$. It is convenient now to follow [11] and introduce a different integration variable $R$ such that

$$\frac{dR}{R^2} = \frac{dr}{r^2 \sqrt{h}},$$  \hspace{1cm} (15)$$

in terms of which the azimuthal angle (13) reads

$$\Theta(E) = \ell \int_{R_{\min}}^{R_{\max}} \frac{dR}{R^2 \sqrt{E - U(R)}},$$  \hspace{1cm} (16)$$
Fig. 1  Aspect of a generic effective potential $U(R)$ near its local minimum at $R = R_0$. The potential is inverted in each of the two branches $R_-(U) \leq R_0$ and $R_+(U) \geq R_0$. It is clear that $R_+ - R_-$ must be a function of $U$. However, in the vicinity of $R_0$, $U$ is well described by a parabola since we assume $U''_0 = U''(R_0) > 0$ and, consequently, $R_+ - R_- = \frac{2\sqrt{2}}{\sqrt{U_0}} \sqrt{U - U_0}$ near $R_0$, which is the main motivation for the proposed expression (23)

where the effective potential is now given by

$$U(R) = \ell^2 v(R) - w(R),$$

(17)

with

$$v(R) = \frac{1}{r^2(R)}, \quad w(R) = \frac{1}{f(r(R))},$$

(18)

and

$$h(r) = \left(\frac{rv'R^2}{2}\right)^2 = \left(\frac{R^2}{r^2R'}\right)^2,$$

(19)

where the prime $'$ denotes the derivative of the function for its considered variable. Despite Eq. (16) in the new radial variable $R$ is formally identical to the equivalent Newtonian one, the effective potential (17) is quite different. In particular, the term corresponding to the centrifugal barrier now is an arbitrary function and not a fixed $R^{-2}$ term as we would have in the Newtonian case. It is clear that the problem in General Relativity is much less restricted and that, in principle, many other solutions are indeed possible, exactly in the same way of the Bertrand spacetime problem. We stress that the spacetime functions $v(R)$ and $w(R)$ are independent of the trajectory initial conditions and, hence, cannot depend on $\ell$ nor $E$. Also, the condition $R'(r) > 0$ tacitly assumed in (15) and its direct consequence $v'(R) < 0$ from (18) will be important to select the correct solutions in the subsequent analysis. In terms of the new radial variable $R$, the metric (4) is given by

$$g_{ab}dx^a dx^b = -\frac{dt^2}{w(R)} + \frac{dR^2}{(R^2v(R))^2} + \frac{d\Omega^2}{v(R)}.$$  

(20)
Let us now decompose the motion range \([R_{\text{min}}, R_{\text{max}}]\) into the branches \(R_- \leq R_0 = R(r_0)\) and \(R_+ \geq R_0\), where \(U(R)\) in inverted in each of these branches, see Fig. 1 and [11]. We can write (16) as

\[
\Theta = \ell \int_{U_0}^{E} \frac{1}{\sqrt{E - U}} \frac{d}{dU} \left( \frac{1}{R_-} - \frac{1}{R_+} \right) dU, \tag{21}
\]

where \(U_0 = U(R_0)\). The isochrony condition corresponds to require an azimuthal angle independent of \(E\), i.e., \(\Theta(E, \ell) = \frac{\pi}{\beta \ell}\). Eq. (21) can be inverted by using the Abel equation [11], leading in this case to

\[
\frac{1}{R_-} - \frac{1}{R_+} = \frac{2}{\ell \beta \ell} \sqrt{U - U_0}. \tag{22}
\]

We follow now the same approach [13] we have used recently for deriving the classical isochrone potentials (1), (2), and (3). By inspecting the analytical properties of the potential \(U(R)\) near its minimal at \(R = R_0\), see Fig. 1, we can write

\[
R_+ - R_- = \frac{\sqrt{U - U_0}}{F(U)}, \tag{23}
\]

where \(F(U)\) is an arbitrary function such that \(F(U_0) = \frac{\sqrt{U''_0}}{2\sqrt{2}}\), with \(U''_0 = U''(R_0)\). Notice that since \(F(U)\) is assumed to be arbitrary, there is indeed no loss of generality in choosing the form (23) for the function corresponding to \(R_+(U) - R_-(U)\). Nevertheless, this choice is a very convenient one since, from (22) and (23), we will have

\[
\sqrt{U - U_0} = F(U) R_+ - \frac{A}{R_+} = - \left( F(U) R_- - \frac{A}{R_-} \right), \tag{24}
\]

implying finally

\[
U - U_0 = \left( F(U) R - \frac{A}{R} \right)^2, \tag{25}
\]

valid for both branches, with

\[
A = \frac{\ell \beta \ell}{2}. \tag{26}
\]

From (22) and (23) we have also the useful relation

\[
\ell^2 \beta^2 \ell = \frac{R_0^4 U''_0}{2}. \tag{27}
\]

In summary, an isochrone effective potential \(U(R)\) must obey the algebraic relation (25) for a function \(F(U)\) such that (23) holds. Hence, we indeed have a procedure to generate isochrone effective potentials: once a function \(F(U)\) is given, we can formally solve (25) for \(U\) and obtain the corresponding isocrone effective potential. However, of course, we cannot always get explicit solutions with the required properties for
this equation. Fortunately, we can do it for some algebraic choices for \( F(U) \). Let us consider now these explicit relevant examples.

### 2.1 First case: constant \( F(U) \)

The simplest choice is a constant \( F(U) = \alpha \). From (25), we have in this case simply

\[
U(R) = BR^2 + \frac{C}{R^2} + D, \tag{28}
\]

with \( B = \alpha^2, C = A^2 = \ell^2 \beta_\ell^2 / 4, \) and \( U_0 = \alpha \beta_\ell - D \). The parameter \( \alpha \) is completely arbitrary and could depend, in principle, also on \( \ell \) and, hence, the decomposition of \( U(R) \) in “centrifugal” and “central potential” parts as (17) can involve the first two terms of (28) rather arbitrarily. The most general solution corresponds to set

\[
v(R) = B_1 R^2 + \frac{C_1}{R^2} + D_1, \tag{29}
\]

and

\[
w(R) = B_2 R^2 + \frac{C_2}{R^2} + D_2. \tag{30}
\]

Of course, we are exploring an underlying linear structure of the problem, we are considering \( v(R) \) and \( w(R) \) as generic linear combinations of the three terms of (28) such that \( B = B_1 \ell^2 + B_2, C = C_1 \ell^2 + C_2, \) and \( D = D_1 \ell^2 + D_2 \). This will be valid for all cases considered here. Once \( v(R) \) is given, we can recast the original spherical coordinates (4) by using (18) and (29). Assuming \( K = 4B_1 C_1 \neq 0, \) we will have

\[
R^2 = \frac{1 - D_1 r^2 \mp \sqrt{(1 - D_1 r^2)^2 - Kr^4}}{2B_1 r^2}, \tag{31}
\]

leading to

\[
h = \frac{2C_1 ((1 - D_1 r^2)^2 - Kr^4)}{1 - D_1 r^2 \pm \sqrt{(1 - D_1 r^2)^2 - Kr^4}}. \tag{32}
\]

Notice that (8) and (32) are identical, up to the definition of the constant \( C_1 \). For the determination of \( f(r) \), we will explore the linear structure of the problem and consider

\[
w(R) = w(R) - \xi v(R) + \xi/r^2, \]

where (18) was invoked, with \( \xi = C_2/C_1 \). Hence, without loss of generality, one can consider instead the function (30) its equivalent form

\[
w(R) = B'_2 R^2 + D'_2 + \frac{\xi}{r^2}, \tag{33}
\]

with \( B'_2 = B_2 - \xi B_1 \neq 0 \) and \( D'_2 = D_2 - \xi D_1 \), leading finally to

\[
\frac{1}{f} = G \mp \frac{\rho r^2}{1 - D_1 r^2 \pm \sqrt{(1 - D_1 r^2)^2 - Kr^4}} + \frac{\xi}{r^2}, \tag{34}
\]
where \( G = D'_2 \) and \( \rho = 2B'_2C_1 \). The extra \( r^{-2} \) term in (33) and (34) can be also understood as a manifestation of the well-known gauge invariance of the problem: if 
\( U(R) \) is an isochrone potential, then \( \dot{U}(R) = U(R) + \Delta + \Lambda v(R) \), for any constants \( \Delta \) and \( \Lambda \), will be also isochrone, but with the parameters \( E \) and \( \ell \) redefined accordingly. The spacetime functions (32) and (34) define our first class of isochrone spacetimes, for which we have

\[
\beta^2 \ell = 4C_1 \left( 1 + \frac{\xi}{\ell^2} \right). \tag{35}
\]

The Newtonian limit for this family is obtained by setting \( D_1 = K = 0 \) and \( C_1 = 1 \), and a simple inspection of (34) reveals that it corresponds to the harmonic potential plus an extra \( r^{-2} \) interaction term. The Bertrand spacetimes of the second type arise by considering the case \( \xi = 0 \) and a rational \( \sqrt{C_1} \). The extra \( r^{-2} \) term is associated with a spacetime singularity at \( r = 0 \) as one can see by noticing that \( f \sim r^2 \) for \( r \to 0 \) and \( \xi \neq 0 \), and that the scalar curvature of the metric (4) in this case reads

\[
\mathcal{R} \sim \frac{2 - 3 (r h' + 2h)}{r^2} \tag{36}
\]

for \( r \to 0 \), diverging for the two possibilities of (32). The presence of this singularity for \( \xi \neq 0 \) is generic for all families of isochrone spacetimes discussed here.

Let us now inspect the particular cases with \( K = 0 \). First, of course, one cannot have both \( B_1 = 0 \) and \( C_1 = 0 \) simultaneously, since it would correspond to no centrifugal barrier at all, which is incompatible with (18). For the case \( B_1 = 0 \) and \( C_1 \neq 0 \), the expressions (32) and (34) are still valid in the limit \( B_1 \to 0 \), choosing the pertinent signs. The case \( B_1 \neq 0 \) and \( C_1 = 0 \) is also possible but somehow distinct, since it does not have a Newtonian limit. It corresponds to a second family of isochrone spacetimes for which

\[
h = \frac{(1 - D_1 r^2)^3}{B_1 r^4} \tag{37}
\]

and

\[
\frac{1}{f} = G + \frac{\rho r^2}{1 - D_1 r^2} + \frac{\xi}{r^2}. \tag{38}
\]

For this family, we have

\[
\beta^2 \ell = \frac{4\rho}{B_1 \ell^2}. \tag{39}
\]

Since \( \rho \neq 0 \), otherwise \( U(R) \) would not have a local minimum, there is no limit of constant \( \beta_\ell \) for these spacetimes and, consequently, they do not have a Bertrand limit. The clear possibility of having \( h(r_\ast) = 0 \), with \( r_\ast > 0 \), for \( D_1 > 0 \) in (37) (and also in (32)) and its implication for the causal structure of the underlying spacetime deserve some comments. We will address these points in the last section. Finally, notice the condition \( B'_2 \neq 0 \) assumed in (30) is necessary by a rather subtle reason. If \( B'_2 = 0 \), the effective potential has only the centrifugal barrier term, and the requirement of a local minimum \( U(R) \) will be equivalent of a local minimum of \( v(R) \), but due to (19), we will have \( h(r_0) = 0 \). We will also return to this point in the last section.
2.2 Second case: linear $F(U)$

The second simplest choice for our problem is, of course, the linear $F(U)$ case. Since we can always add a constant to $U$, one can consider without loss of generality $F(U) = \alpha U$. We will proceed along the same lines of the preceding case and solve (25). We get

$$U(R) = \frac{B\sqrt{p + \epsilon R^2}}{R^2} + \frac{C}{R^2} + D,$$  \hspace{1cm} (40)

where $U_0 = -\epsilon \alpha^2 B^2 - D$, with $\epsilon$ assuming two possible values $\epsilon = \pm 1$. The other parameters in the potential (40) are such that

$$p = \frac{4A\alpha + 1}{4\alpha^4 B^2},$$ \hspace{1cm} (41)

$$C = \frac{2A\alpha + 1}{2\alpha^2},$$ \hspace{1cm} (42)

and it is clear that one requires $p > 0$ for $\epsilon = -1$. The signs of $B$ and $C$ must also be conveniently chosen to guarantee a local minimum for the effective potential $U(R)$. One can advance from (40) that the flat space limit $(R = r)$ of $U(R)$ will comprehend all Henon’s isochrone potentials (1), (2) and (3), but occasionally with some extra $r^{-2}$ terms. As in the preceding case, let us first consider the linear combinations

$$v(R) = B_1 \frac{\sqrt{p + \epsilon R^2}}{R^2} + \frac{C_1}{R^2} + D_1,$$ \hspace{1cm} (43)

and

$$w(R) = B'_2 \frac{\sqrt{p + \epsilon R^2}}{R^2} + D'_2 + \frac{\xi}{r^2},$$ \hspace{1cm} (44)

with $C_1 \neq 0$ and $B'_2 \neq 0$. Reintroducing the original spherical coordinates, we have

$$R^2 = \epsilon C_1 \left( \frac{r + \mu \Phi(r)}{\Xi(r) - \epsilon vr} - \mu \right),$$ \hspace{1cm} (45)

where $\mu = p/C_1$, $v = B_1/2\sqrt{C_1}$,

$$\Phi(r) = \epsilon \left( \frac{1}{r} - D_1 r \right),$$ \hspace{1cm} (46)

and

$$\Xi^2(r) = \epsilon + \kappa r^2 + \mu \Phi^2(r),$$ \hspace{1cm} (47)

with $\kappa = v^2 - \epsilon D_1$, leading to

$$h = \frac{\epsilon C_1 r \left( r + \mu \Phi(r) + 2\epsilon v\mu \left( \Xi(r) - \epsilon vr \right) \right) \Xi^2(r)}{(r + \mu \Phi(r))^2}$$ \hspace{1cm} (48)
and
\[
\frac{1}{f} = G + \frac{\epsilon \rho (r + \mu \Phi(r)) (\Xi(r) - \epsilon vr)}{(r + \mu \Phi(r))^2 - \mu (\Xi(r) - \epsilon vr)^2} + \frac{\xi}{r^2}, \tag{49}
\]

The general expression for the azimuthal angle of the metric with the functions (48) and (49) is rather complicated. For \( \mu \neq 0 \), we have from (41) and (42)

\[
\alpha^2 = \frac{1}{2C_1 \xi_\ell} \left(1 - \sqrt{1 - \xi_\ell}\right), \tag{50}
\]

with

\[
\xi_\ell = \mu \left(2\nu + \frac{\rho}{\ell^2 + \xi}\right)^2, \tag{51}
\]

leading to

\[
\beta_\ell^2 = 2C_1 \left(1 + \frac{\xi}{\ell^2}\right) \left(\frac{1}{\xi_\ell} - 1\right) \left(1 - \sqrt{1 - \xi_\ell}\right). \tag{52}
\]

The case \( \mu = 0 \) follows straightforwardly from the \( \mu \to 0 \) limit of (52).

The Newtonian limit of this family of isochrone spacetimes corresponds to set \( \kappa = \nu = 0 \) (locally flat condition) and \( C_1 = 0 \) (no conic singularity at the origin), and it corresponds to the classic Henon’s potentials (1), (2), and (3), with an extra \( r^{-2} \) interaction. Furthermore, the parameter \( \xi \) can be properly chosen to reproduce (1), (2), and (3) exactly, eliminating the singularity at the origin. In this sense, this family of isochrone spacetimes can be considered the relativistic version of the Henon’s original potentials. The Bertrand limit corresponds to the case \( \mu = \xi = 0 \) and a rational \( \sqrt{C_1} \), and coincide with the Perlick first family defined by (5) and (6). The family defined by the functions (48) and (49) has some asymptotically flat spacetimes, they correspond to the cases with \( \Xi \) constant for large \( r \), which demands \( \epsilon = 1 \) and \( \mu D_1^2 + \kappa = \mu D_1^2 - D_1 + \nu^2 = 0 \), and in contrast with the Bertrand family defined by (5) and (6), these asymptotically flat isochrone spacetimes can admit curved spatial sections.

As in the preceding case, we are left with the \( C_1 = 0 \) case, which does not have a Newtonian limit. We have for this case

\[
h = \frac{B_1 r \left(\sqrt{B_1^2 r^2 + 4p\Phi^2} - \epsilon B_1 r\right) \left(B_1^2 r^2 + 4p\Phi^2\right)}{8p\Phi^2}, \tag{53}
\]

\[
\frac{1}{f} = G + \frac{\rho \left(\sqrt{B_1^2 r^2 + 4p\Phi^2} - \epsilon B_1 r\right)}{2pr} + \frac{\xi}{r^2}, \tag{54}
\]

and

\[
\beta_\ell^2 = \frac{C_2}{\ell^2} \left(\frac{1}{\xi_\ell} - 1\right) \left(1 - \sqrt{1 - \xi_\ell}\right), \tag{55}
\]

with \( \tilde{\xi}_\ell = \rho (\ell^2 + \xi)/\rho^2 \). This family does not admit asymptotically flat spacetimes or Bertrand limit.
2.3 Third case: quadratic $F(U)$

Our last explicit case is also the next natural one: the quadratic $F(U)$. Again, thanks to the gauge invariance of the problem, we can consider without loss of generality $F(U) = \alpha(U^2 + \gamma)$. Equation (25) in this case will be an irreducible fourth-order polynomial, but it admits the simple solution

$$U(R) = B\sqrt{1\over R + p} + C \over R + D,$$  \hspace{0.5cm} (56)

where $\alpha^2 = {1\over 4B^2C}$, $\gamma = -B^2p$, $U_0 = -B^2Cp - D$, and

$$A = {C^3 \over 2B}.$$  \hspace{0.5cm} (57)

The corresponding spacetimes are not asymptotically flat and have no Newtonian limit. We have for $C_1 \neq 0$:

$$h = {C_1^2 r^2 \Omega^2 \over 4 (B_1 + \Omega)^2},$$  \hspace{0.5cm} (58)

$$1 \over f = G + \rho \sqrt{\Omega^2 + B_1 (B_1 + 2\Omega)} + \xi \over r^2, \hspace{0.5cm} (59)$$

with

$$\Omega^2 = B_1^2 + 4C_1 \left(C_1 p - D + {1\over r^2}\right),$$  \hspace{0.5cm} (60)

and

$$\beta^2 \ell = {\sqrt{C_1} (\ell^2 + \xi) \over \ell^2 \left(2\rho + {B_1 \over C_1} (\ell^2 + \xi)\right)}. \hspace{0.5cm} (61)$$

For case with $C_1 = 0$, we have

$$h = \left({B_1^2 r^3 \over 4 (1 - D_1 r^2)}\right)^2, \hspace{0.5cm} (62)$$

$$1 \over f = G + \rho \left( (1 - D_1 r^2)^2 - B_1^2 p \right) \over B_1^2 r^4 + \xi \over r^2, \hspace{0.5cm} (63)$$

and

$$\beta^2 \ell = {\rho^{3\over 2} \over 2B_1 (\ell^2 + \xi)}. \hspace{0.5cm} (64)$$

These families do not admit asymptotically flat spacetimes and have no Bertrand limit.
3 Final remarks

We have introduced the notion of isochrone spacetimes and presented a procedure to generate them by means of the algebraic Eq. (25). We have obtained explicit expressions for the families of spacetimes corresponding to the cases with constant, linear, and quadratic $F(U)$, but many others solutions are indeed possible. We remind that we are looking for solutions of (25) allowing for, at least, a three-dimensional vector space that will give origin to the two functions $v(R)$ and $w(R)$ of the effective potential (17). With the help of Maple, we could find explicit solutions for (25) with the required properties for $F(U) = \alpha \sqrt{U} + \gamma$, $F(U) = \alpha U^{-1} + \gamma$, and $F(U) = \frac{\alpha U}{U + \gamma}$, among others, but the resulting expressions are too cumbersome to be useful in our context. For cases with $F(U)$ involving any rational expressions with non-linear polynomials, the problem of solving (25) reduces to find the roots of higher order irreducible polynomials in $U$, and in some cases the solutions are indeed compatible with our requirements, despite their typical intricate expressions. We do not expect any compatible solution for non-algebraic functions $F(U)$, but we could not prove that they indeed do not exist. Nevertheless, the constant and linear $F(U)$ cases are particularly relevant here because they have the Bertrand spacetimes [11] as special limits, see Fig. 2. Moreover, the linear case has as Newtonian limit the Henon’s isochrone potentials (1), (2) and (3), and, hence, such family of isochrone spacetimes could be useful as a relativistic extension of the original Henon’s proposal for the study of globular clusters. They might also generalize some recent models for galactic dark matter [14] based in the Bertrand spacetimes of the first type. It is worth stressing that only the constant and linear $F(U)$ cases have isochrone Newtonian limits and, hence, they are probably the most relevant isochrone spacetimes.

The causal structure of isochrone spacetimes certainly deserves a deeper investigation, but some issues are already clear. One can see from all explicit families of spacetime of the previous section that one can have, for some choice of parameters, $h(r_*) = 0$ with $r_* > 0$. In such a case, from the metric (4), we see that $r = r_*$ is a null sphere and, hence, a candidate to be an event horizon provided no spacetime singularity is present at $r = r_*$. Incidentally, this is the reason why one cannot have

![Fig. 2](#)

Fig. 2 Schematic representation of the physically most relevant isochrone spacetimes. Left: the constant $F(U)$ family of Sect. 2.1. It has the Bertrand spacetimes of the second type (7) and (8) as its limit with closed orbits, and both have the harmonic potential as their Newtonian limit. Right: The linear $F(U)$ family of Sect. 2.2. Its particular cases with closed orbits are the Bertrand spacetimes of the first type (5) and (6), and its respective Newtonian limit corresponds to the Henón isochrone potentials (1), (2) and (3). The Newtonian potential arises as the Newtonian limit of the Bertrand spacetimes of the first type and as the closed orbits case of the isochrone potentials.
$B_2' = 0$ in the discussions of the last section, since in such a case we would have $r_0 = r_*$, a circular null orbit, and our analysis is focused from the beginning only on timelike geodesics. From (19), we see that $r_*$ corresponds to a critical point of $v(R)$, and from (20) we see that no evident spacetime singularity is expected at the critical points of $v(R)$, provided $w(R)$ be regular there. Hence, despite that $R$ defined in (15) is a tortoise-like coordinate, there should be a possibility of extending the isochrone spacetime across $r_*$, giving origin to a black hole spacetime in the case of absence of singularities at $r = r_*$. The possibility of having an isochrone black hole is certainly instigating from a theoretical point of view. As an illustration, let us consider the case with $C_1 = 0$ for a constant $F(U)$, which corresponds to the functions (37) and (38), with the parameters $G = \xi = 0$ and $\rho^{\frac{1}{2}} = B_1^{\frac{1}{2}} = D_1^{\frac{1}{2}} = r_*^{-1}$ for the sake of simplicity,

$$g_{ab} dx^a dx^b = -\frac{r_*^2 - r^2}{r^2} dt^2 + \frac{r_*^2 r^4}{(r_*^2 - r^2)^3} dr^2 + r^2 d\Omega^2. \quad (65)$$

Both the scalar curvature and the Kretschmann invariant suggest that the null suface $r = r_*$ is not singular, and hence the metric (65) could be extended naturally for $r > r_*$. Of course, this metric is not static in this exterior region, and hence our discussion on isochrone orbits does not apply there. On the other hand, we can also see form the curvature invariants that there is a naked spacetime singularity in the interior region at $r = 0$.

We finishing noticing that the isochrone spacetimes generically violates the weak energy condition, as one can see from the Einstein tensor evaluated for the metric (4). We have for the temporal component

$$G_{00} = \frac{f(r) \left( 1 - \frac{d}{dr} h(r) \right)}{r^2}, \quad (66)$$

and it is clear that one can have $G_{00} < 0$ for sufficiently large $r$ for all families we have considered in this paper. This raises a pertinent question about the physical interpretation of the isochrone spacetimes, since in the context of General Relativity they would require exotic matter. Nevertheless, in the context of modified gravity, the energy conditions can be considerably different from those ones of General Relativity [15], and perhaps some of these theories could have isochrone spacetimes as physically viable solutions.

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Data availability This paper has no associated data.

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