The problem of dispersion-free probabilities in Gleason-type theorems for a two-dimensional Hilbert space

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Abstract

As it is known, Gleason’s theorem is not applicable for a two-dimensional Hilbert space since in this situation the set of Gleason’s axioms is not strong enough to imply Born’s rule thus leaving room for a dispersion-free probability measure i.e., one that has only the values 0 and 1. To strengthen Gleason’s axioms one must add at least one more assumption.

But, as it is shown in the presented paper, alternatively, one can alter one of the implicit assumptions lying in the foundation of Gleason’s theorem. Particularly, the assumption that the closed subspaces of the Hilbert space form an orthomodular lattice can be replaced by the assumption that the closed subspaces belonging to the incommutable projection operators are the elements of the different Boolean lattices that do not meet each other. Because of this change, no assumption additional to Gleason’s axioms is necessary to imply Born’s rule for two-dimensional quantum systems.

Keywords: Quantum mechanics; closed subspaces; lattice structures; Burnside’s Theorem; probability measures

1 Introduction

Consider the set \( \mathcal{L}(\Sigma) \) of the closed subspaces of the two-dimensional Hilbert space \( \mathcal{H} = \mathbb{C}^2 \), namely,

\[
\mathcal{L}(\Sigma) = \left\{ \text{ran}(\hat{P}^{(Q)}_1), \text{ran}(\hat{P}^{(Q)}_2) \right\}_{Q=0}^3,
\]

(1)

where \( \text{ran}(\hat{P}^{(Q)}_n) \) are the column spaces of the qubit projection operators \( \hat{P}^{(Q)}_n \) defined by the formula

\[
\hat{P}^{(Q)}_n = \frac{1}{2} \begin{pmatrix}
1 + (-1)^n(\delta_{Q0} - \delta_{Q3}) & (-1)^n(-\delta_{Q1} + i\delta_{Q2}) \\
(-1)^n(-\delta_{Q1} - i\delta_{Q2}) & 1 + (-1)^n(\delta_{Q0} + \delta_{Q3})
\end{pmatrix},
\]

(2)

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in which $\delta_{ab}$ is the Kronecker delta and $Q = \{0, 1, 2, 3\}$.

Let $\leq$ be a binary relation indicating that, for certain pairs of $\text{ran}(\hat{P}^{(Q)}_n)$ in the set $\mathcal{L}(\Sigma)$, one of the $\text{ran}(\hat{P}^{(Q)}_n)$ precedes the other in the ordering. Then, the set $\mathcal{L}(\Sigma)$ together with the binary relation $\leq$ constitute the orthomodular lattice $(\mathcal{L}(\Sigma), \leq)$.

Due to the one-one correspondence between the closed subspaces and the projection operators, the set $\mathcal{L}(\Sigma)$ corresponds to the collection $\Sigma$ of all the projection operators, commutable and incommutable alike, on the Hilbert space $\mathbb{C}^2$, namely,

$$\Sigma = \left\{ \{\hat{P}^{(Q)}_1, \hat{P}^{(Q)}_2\}_{Q=0}^3 \right\} .$$

Consequently, the lattice $(\mathcal{L}(\Sigma), \leq)$ corresponds to the lattice $(\Sigma, \leq)$, in which every pair of elements $a, b \in \Sigma$ possesses a meet denoted by $a \land b$ and a join denoted by $a \lor b$.

Let a continuous function

$$\text{Pr}: \Sigma \rightarrow [0, 1]$$

be a probability measure in a pure state $|\Psi\rangle$. The value of this function at $\hat{P}^{(Q)}_n$, i.e., $\text{Pr}[\hat{P}^{(Q)}_n]$, denotes the probability that the proposition corresponding to projection operator $\hat{P}^{(Q)}_n$ will be verified if the system is prepared in the state $|\Psi\rangle$.

Assume that in any state of the system one must have

$$\text{Pr}[\hat{P}^{(0)}_1] = \text{Pr}[^0] = 0 , \text{Pr}[\hat{P}^{(0)}_2] = \text{Pr}[^1] = 1 ,$$

where $^0$ and $^1$ stand for the zero and identity operators respectively.

In line with this, for each set of the mutually orthogonal projectors $\{\hat{P}^{(Q)}_1, \hat{P}^{(Q)}_2\}$ one must further have

$$\text{Pr}[\hat{P}^{(Q)}_1 \land \hat{P}^{(Q)}_2] = 0 ,$$

$$\text{Pr}[\hat{P}^{(Q)}_1 \lor \hat{P}^{(Q)}_2] = \text{Pr}[\hat{P}^{(Q)}_1 + \hat{P}^{(Q)}_2] = \text{Pr}[\hat{P}^{(Q)}_1] + \text{Pr}[\hat{P}^{(Q)}_2] = 1 .$$

It is trivially to see that under such assumptions, the lattice $(\Sigma, \leq)$ can admit a dispersion-free probability measure, i.e., one that has only the values 0 and 1; for example, the following measure:

$$\text{Pr}[\hat{P}^{(1)}_1] = \text{Pr}[\hat{P}_{x+}] = 1 , \text{Pr}[\hat{P}^{(1)}_2] = \text{Pr}[\hat{P}_{x-}] = 0 ,$$
\[
\begin{align*}
\text{Pr}[\hat{P}_1^{(2)}] &= \text{Pr}[\hat{P}_{y+}] = 1, \quad \text{Pr}[\hat{P}_2^{(2)}] = \text{Pr}[\hat{P}_{y-}] = 0, \\
\text{Pr}[\hat{P}_1^{(3)}] &= \text{Pr}[\hat{P}_{z+}] = 1, \quad \text{Pr}[\hat{P}_2^{(3)}] = \text{Pr}[\hat{P}_{z-}] = 0.
\end{align*}
\] (9) (10)

The problem with the said measure, however, is that it allows a joint probability for the non-commuting projection operators, e.g., \(\text{Pr}[\hat{P}_{z+} \land \hat{P}_{z+}] = \text{Pr}[\hat{P}_{z+}] \cdot \text{Pr}[\hat{P}_{z+}] = 1\), which contradicts the basic principle of quantum mechanics ruling out a direct and precise joint verification of those operators.

To overcome this problem, one may replace the projection operators \(\hat{P}_n^{(Q)}\) by \textit{effects} \(\hat{E}_i\), that is, self-adjoint operators bounded between \(\hat{0}\) and \(\hat{1}\), namely,

\[
\hat{0} \leq \hat{E}_i \leq \hat{1} \implies 0 \leq \langle \Psi | \hat{E}_i | \Psi \rangle \leq 1,
\] (11)

and demand

\[
\text{Pr}[\hat{E}_1 + \hat{E}_2 + \ldots] = \text{Pr}[\hat{E}_1] + \text{Pr}[\hat{E}_2] + \ldots
\] (12)

in case of

\[
\hat{E}_1 + \hat{E}_2 + \ldots \leq 1,
\] (13)

where the mutual commutativity of the effects \(\hat{E}_1, \hat{E}_2, \ldots\) is not required \[1\]. Then, the fact that \(\text{Pr}[\hat{E}_1 \land \hat{E}_2] \neq 1\) will preclude the joint verification of incompatible properties associated with \(\hat{E}_i\).

Yet, the replacement of the qubit projection operators \(\hat{P}_n^{(Q)}\) by the effects \(\hat{E}_i\) (corresponding to elements of a positive-operator-valued measure, POVM, \[2\]) brings about a new problem, namely, the problem of the interpretation of the effects \(\hat{E}_i\). To be sure, unlike the projection operators \(\hat{P}_n^{(Q)}\) whose number is limited (by the dimension of the Hilbert space associated with the system), the number of the effects \(\hat{E}_i\) is unlimited. More importantly, one always obtains the same result when performing two consecutive verifications of the proposition corresponding to the projection operator \(\hat{P}_n^{(Q)}\), while this need not be true for the properties associated with the effects \(\hat{E}_i\) \[3\].

Another way to avoid the dispersion-free probability measure \(\text{Pr}[\hat{E}_1, \hat{E}_2, \ldots]\) is to show that the said measure is \textit{illogical}, i.e., not justifiable from rationality principles \[4\].

However, this approach is not conceptually neutral. That is, the pertinence of the rationality principles to quantum mechanics strongly depends on the interpretation of the state vector \(|\Psi\rangle\). Thus, in the Bayesian approach to quantum mechanics \[5\] \[6\] \[7\] \[8\] \[9\], probabilities – and therefore the state vector \(|\Psi\rangle\) – represent an agent’s degrees of belief (which can be rational or not rational), rather than objective properties of physical systems (as it is assumed in accordance with an ontic
interpretation of $|\Psi\rangle$). In view of that, the idea of excluding the dispersion-free probability measure from consideration based on the rationality principles can be rationalized only within QBism, i.e., the Bayesian interpretation of quantum mechanics.

On the other hand, to avoid the measure $[8]$, one can alter the implicit assumption making this measure possible, namely, the assumption that the closed subspaces $\text{ran}(\hat{P}(Q))$ form the orthomodular lattice $(\mathcal{L}(\Sigma), \leq)$.

The presented paper demonstrates that replacing this assumption by a different one, according to which the closed subspaces of $\mathbb{C}^2$ belonging to the incommutable projection operators are the elements of the different invariant-subspace lattices that do not meet each other, resolves the problem of the dispersion-free probability measure.

2 Probability measures of the closed subspaces of $\mathcal{H}$

Recall that the column space (a.k.a. range) of the projection operator $\hat{P}$ – the closed (under addition and multiplication) subspace belonging to $\hat{P}$ – is the subset of the vectors $|\Phi\rangle$ of the Hilbert space $\mathcal{H}$ that are in the image of $\hat{P}$, namely,

$$\text{ran}(\hat{P}) = \left\{ |\Phi\rangle \in \mathcal{H} : \hat{P}|\Phi\rangle = |\Phi\rangle \right\}.$$  

(14)

Let us define the elementary event $E$ as the following outcome

$$E \equiv \left\{ |\Psi\rangle : |\Psi\rangle \in \text{ran}(\hat{P}) \right\}.$$  

(15)

This event occurs if the vector $|\Psi\rangle$ associated with the state of the system resides in the closed subspace $\text{ran}(\hat{P})$ and, as a result, the proposition corresponding to the projection operator $\hat{P}$ is verified (i.e., has the value of the truth).

Along these lines, the sample space $\Omega$ can be defined in the following way:

$$\Omega \equiv \left\{ |\Psi\rangle : |\Psi\rangle \in \mathcal{H} \land |\Psi\rangle \in \text{ran}(\hat{1}) \right\} = \mathcal{H},$$  

(16)

where $\text{ran}(\hat{1}) = \mathcal{H}$. Since all vectors associated with physically meaningful states of the system must differ from zero, no $|\Psi\rangle$ resides in $\text{ran}(\hat{0}) = \{0\}$. Thus, the event

$$\left\{ |\Psi\rangle : |\Psi\rangle \neq 0 \land |\Psi\rangle \in \text{ran}(\hat{0}) \right\} = \emptyset$$  

(17)

must be impossible.

Assume that the probability of the event $E$ denoted by $\Pr[E]$ is a non-negative real number, namely,
\[ \Pr[E] \geq 0 \quad . \] (18)

Also, assume that

\[ \Pr[H] = 1 \quad , \] (19)

\[ \Pr[\varnothing] = 0 \quad . \] (20)

As the column spaces \( \text{ran}(\hat{P}) \) and \( \text{ran}(\hat{P}') \) of different orthogonal projection operators \( \hat{P} \) and \( \hat{P}' \) are orthogonal to each other, i.e.,

\[ \text{ran}(\hat{P}) \wedge \text{ran}(\hat{P}') = \{ 0 \} \quad , \] (21)

the events \( E \) and \( E' \equiv \{ |\Psi\rangle : |\Psi\rangle \in \text{ran}(\hat{P}') \} \) are mutually exclusive (i.e., disjoint):

\[ E \wedge E' = \left\{ |\Psi\rangle : |\Psi\rangle \neq 0 \wedge |\Psi\rangle \in \text{ran}(\hat{P}) \wedge \text{ran}(\hat{P}') \right\} = \varnothing \quad . \] (22)

For this reason, the probability of the conjunction \( E \) and \( E' \), i.e., \( \Pr[E \wedge E'] \), must be zero. Then, in accordance with the addition law of probability, the probability of the disjunction of these events, i.e., \( \Pr[E \vee E'] \), must be the sum of the probabilities \( \Pr[E] \) and \( \Pr[E'] \).

On the other hand, the disjunction of \( E \) and \( E' \) is

\[ E \vee E' = \left\{ |\Psi\rangle : |\Psi\rangle \neq 0 \wedge |\Psi\rangle \in \text{ran}(\hat{P}) \vee \text{ran}(\hat{P}') \right\} \quad , \] (23)

where

\[ \text{ran}(\hat{P}) \vee \text{ran}(\hat{P}') = \text{ran}(\hat{P}) + \text{ran}(\hat{P}') \quad , \] (24)

Therefore,

\[ E \vee E' = \left\{ |\Psi\rangle : |\Psi\rangle \neq 0 \wedge |\Psi\rangle \in \text{ran}(\hat{P}) \right\} + \left\{ |\Psi\rangle : |\Psi\rangle \neq 0 \wedge |\Psi\rangle \in \text{ran}(\hat{P}') \right\} \quad , \] (25)

which implies

\[ \Pr[E + E'] = \Pr[E] + \Pr[E'] \quad . \] (26)
3 Lattice structures on the set of the closed subspaces of $\mathcal{H}$

Let us denote by $\Sigma = \{ \Sigma^{(Q)} \}_{Q} = \{ \{ \hat{P}^{(Q)} \}_{n=1}^{N} \}_{Q}$ the collection of all the projection operators, commutable and incommutable alike, on the finite Hilbert space $\mathcal{H}$ (whose dimension is greater than 1). This collection spans the Hilbert space $\mathcal{H}$ and so $\Sigma$ is equal to $A(\mathcal{H})$, the algebra of all linear transformations on $\mathcal{H}$.

Let us introduce $L(\Sigma)$ – i.e., the set of all the closed subspaces belonging to $\Sigma$. Theoretically, there are two options regarding ordering of the elements in the set $L(\Sigma)$: The partial order on $L(\Sigma)$ is defined by the meet on $L(\Sigma)$ or the meet on $L(\Sigma)$ is defined by the partial order on $L(\Sigma)$ [10].

Consider the first option: Assume that the partial order on $L(\Sigma)$ is defined by the meet operation, namely,

$$L(\Sigma) = \bigwedge_{Q} \left( \bigwedge_{n=1}^{N} \text{ran}(\hat{R}^{(Q)}_{n}) \right) = \bigwedge_{Q} \left( \bigcap_{n=1}^{N} \text{ran}(\hat{R}^{(Q)}_{n}) \right) = \ldots \text{ran}(\hat{R}^{(Q)}_{n}) \cap \text{ran}(\hat{R}^{(Q')}_{m}) \ldots , \quad (27)$$

where $\text{ran}(\hat{R}^{(Q)}_{n})$ and $\text{ran}(\hat{R}^{(Q')}_{m})$ are the members of the sets of the closed subspaces invariant under $\hat{P}^{(Q)}_{n}$ and $\hat{P}^{(Q')}_{m}$, explicitly,

$$\text{ran}(\hat{R}^{(Q)}_{n}) \in \left\{ \text{ran}(0), \text{ran}(\hat{P}^{(Q)}_{n}), \text{ran}(\hat{1} - \hat{P}^{(Q)}_{n}), \text{ran}(\hat{1}) \right\} , \quad (28)$$

$$\text{ran}(\hat{R}^{(Q')}_{m}) \in \left\{ \text{ran}(0), \text{ran}(\hat{P}^{(Q')}_{m}), \text{ran}(\hat{1} - \hat{P}^{(Q')}_{m}), \text{ran}(\hat{1}) \right\} . \quad (29)$$

Then, the set $L(\Sigma)$ must be irreducible in accordance with Burnside’s theorem on incommutable algebras [11] [12] [13] [14], that is, it must contain no nontrivial closed subspace:

$$\Sigma = A(\mathcal{H}) \implies L(\Sigma) = \left\{ \{0\}, \mathcal{H} \right\} . \quad (30)$$

In this case, the closed subspaces $\text{ran}(\hat{P}^{(Q)}_{n})$ and $\text{ran}(\hat{P}^{(Q')}_{m})$ belonging to the incommutable projection operators $\hat{P}^{(Q)}_{n}$ and $\hat{P}^{(Q')}_{m}$ cannot be the elements of $L(\Sigma)$, in other words, they do not meet each other. In symbols:

$$\text{ran}(\hat{P}^{(Q)}_{n}), \text{ran}(\hat{P}^{(Q')}_{m}) \not\in L(\Sigma) \implies \text{ran}(\hat{P}^{(Q)}_{n}) \And \text{ran}(\hat{P}^{(Q')}_{m}) , \quad (31)$$

where the cancelation of $\And$ indicates that the meet operation cannot be defined for the subspaces $\text{ran}(\hat{P}^{(Q)}_{n})$ and $\text{ran}(\hat{P}^{(Q')}_{m})$.

Consider the second option: Assume the binary relation $\leq$ over the set $L(\Sigma)$ before anything else. Then, by definition, each two-element subset of $L(\Sigma)$ has a meet, e.g.,
\[
\left\{ \text{ran}(\hat{P}_n^{(Q)}), \text{ran}(\hat{P}_{m}^{(Q')}) \right\} \subseteq \mathcal{L}(\Sigma) \implies \text{ran}(\hat{P}_n^{(Q)}) \land \text{ran}(\hat{P}_{m}^{(Q')}) \in \mathcal{L}(\Sigma) .
\]

As it has just been demonstrated, there are two opposite assumptions about the lattice structure on the set \(\mathcal{L}(\Sigma)\), namely, the invariant-subspace lattice \((\mathcal{L}(\Sigma), \land)\) and the orthomodular lattice \((\mathcal{L}(\Sigma), \leq)\), represented by the formulas (31) and (32) in that order.

### 4 Compatibility of the verifications

According to the definition of the maximal context \(\Sigma^{(Q)} = \{\hat{P}_n^{(Q)}\}_{n=1}^{N}\), all the projection operators \(\hat{P}_n^{(Q)} \in \Sigma^{(Q)}\) are orthogonal to each other and resolve to the identity operator \(\hat{1}\), i.e.,

\[
\sum_{n=1}^{N} \hat{P}_n^{(Q)} = \hat{1} .
\]

In view of that, it holds

\[
\sum_{n=1}^{N} \text{ran}(\hat{P}_n^{(Q)}) = \text{ran}(\hat{1}) = \mathcal{H} .
\]

This leads to

\[
\Pr \left[ \sum_{n=1}^{N} E_n \right] = \sum_{n=1}^{N} \Pr[E_n] = 1 .
\]

Suppose that the system is prepared in a pure state \(|\Psi_k^{(Q)}\rangle \neq 0\) lying in the column space of the projection operator \(\hat{P}_k^{(Q)} \in \Sigma^{(Q)}\). Consider the event \(E_k\):

\[
E_k \equiv \left\{ |\Psi_k^{(Q)}\rangle \right\} .
\]

Because \(\text{ran}(\hat{P}_k^{(Q)}) \land \text{ran}(\hat{P}_{n \neq k}^{(Q)}) = \{0\}\) and \(|\Psi_k^{(Q)}\rangle \neq 0\), the event where the vector \(|\Psi_k^{(Q)}\rangle\) resides in both \(\text{ran}(\hat{P}_k^{(Q)})\) and \(\text{ran}(\hat{P}_{n \neq k}^{(Q)})\) is impossible. In symbols,

\[
E_{n \neq k} \equiv \left\{ |\Psi_k^{(Q)}\rangle : |\Psi_k^{(Q)}\rangle \in \text{ran}(\hat{P}_k^{(Q)}) \land \text{ran}(\hat{P}_{n \neq k}^{(Q)}) \right\} = \emptyset .
\]

This entails \(\Pr[E_{n \neq k}] = 0\) and so, in accordance with (35), \(\Pr[E_k] = 1\). Subsequently, if the system is prepared (found) in the state residing in the column space of any projection operator from the maximal context \(\hat{P}_n^{(Q)} \in \Sigma^{(Q)}\), the probability of each event \(E_n\) will be dispersion-free, i.e., will have only the values 0 and 1.
Next, consider the events \( E'_m \) relating to the verification of the propositions corresponding to the projection operators \( \hat{P}^{(Q')} \) from another maximal context \( \Sigma^{(Q')} \) that can occur in the state \( |\Psi_k^{(Q')}\rangle \).

The events \( \{E'_m\}_{m=1}^N \) are mutually exclusive as
\[
E'_l \land E'_{m \neq l} = \left\{ |\Psi^{(Q)}_k\rangle : |\Psi^{(Q)}_k\rangle \neq 0 \land |\Psi^{(Q)}_k\rangle \in \text{ran}(\hat{P}^{(Q')}_l) \land \text{ran}(\hat{P}^{(Q')}_{m\neq l}) \right\} = \emptyset .
\] (38)

On the other hand, under the assumption of the invariant-subspace lattice \((\mathcal{L}(\Sigma), \land)\), the events \( \{E'_m\}_{m=1}^N \) are objectively indeterminate since \( \text{ran}(\hat{P}^{(Q')}_k) \land \text{ran}(\hat{P}^{(Q')}_{m \neq l}) \) cannot be defined. In symbols,
\[
\{E'_m\}_{m=1}^N \equiv \left\{ |\Psi^{(Q)}_k\rangle : |\Psi^{(Q)}_k\rangle \in \text{ran}(\hat{P}^{(Q')}_k) \land \text{ran}(\hat{P}^{(Q')}_{m \neq l}) \right\} = \emptyset .
\] (39)

So, when one has no other information than that exactly \( N \) mutually exclusive events \( E'_m \) can occur in the state \( |\Psi^{(Q)}_k\rangle \), one is justified in assigning each the probability \( \text{Pr}[E'_m] = \frac{1}{N} \).

This means that under the assumption of the invariant-subspace lattice \((\mathcal{L}(\Sigma), \land)\), the closed subspaces of the finite Hilbert space \( \mathcal{H} \) with \( \text{dim}(\mathcal{H}) > 1 \) do not admit probabilities having only the values 0 and 1.

5 Concluding remarks

In keeping with this assumption, the probability of the verification for a quantum system of dimension 2 can be presented as
\[
\text{Pr}[\hat{P}_{x\pm} \land \hat{P}_{z\pm}] = \text{Pr} \left[ \left\{ |\Psi_{x\pm}\rangle : |\Psi_{x\pm}\rangle \in \text{ran}(\hat{P}_{x\pm}) \land \text{ran}(\hat{P}_{z\pm}) \right\} \right] = \frac{1}{2} ,
\] (40)

which implies Born’s rule
\[
|\langle \Psi_{x\pm} |\Psi_{z\pm}\rangle|^2 = \frac{1}{2} .
\] (41)

As it is known, Gleason’s theorem \[15\] is not applicable for the two-dimensional Hilbert space \( \mathbb{C}^2 \) since in that case the set of Gleason’s axioms \([5]-[7]\) is not strong enough to imply Born’s rule thus leaving room for the dispersion-free probability measure \([8]-[10]\). So, to strengthen Gleason’s axioms one must add at least one more assumption \[16\].

But, as it has been shown in the presented paper, alternatively, one can change the implicit assumption lying in the foundation of Gleason’s theorem. Namely, the assumption that the closed subspaces of \( \mathbb{C}^2 \) form the orthomodular lattice \((\mathcal{L}(\Sigma), \leq)\), where
\[
\mathcal{L}(\Sigma) = \left\{ \{0\}, \text{ran}(\hat{P}_{x+}), \text{ran}(\hat{P}_{x-}), \text{ran}(\hat{P}_{y+}), \text{ran}(\hat{P}_{y-}), \text{ran}(\hat{P}_{z+}), \text{ran}(\hat{P}_{z-}), \mathbb{C}^2 \right\} ,
\] (42)
can be replaced by the assumption that the closed subspaces \( \text{ran}(\hat{P}_{x\pm}), \text{ran}(\hat{P}_{y\pm}) \) and \( \text{ran}(\hat{P}_{z\pm}) \) belonging to the incommutable projection operators are the elements of the different invariant-subspace lattices \( (\mathcal{L}(\Sigma^{(x)}), \wedge), (\mathcal{L}(\Sigma^{(y)}), \wedge) \) and \( (\mathcal{L}(\Sigma^{(z)}), \wedge) \), where

\[
\mathcal{L}(\Sigma^{(x)}) = \left\{ \{0\}, \text{ran}(\hat{P}_{x+}), \text{ran}(\hat{P}_{x-}), \mathbb{C}^2 \right\},
\]

\[
\mathcal{L}(\Sigma^{(y)}) = \left\{ \{0\}, \text{ran}(\hat{P}_{y+}), \text{ran}(\hat{P}_{y-}), \mathbb{C}^2 \right\},
\]

\[
\mathcal{L}(\Sigma^{(z)}) = \left\{ \{0\}, \text{ran}(\hat{P}_{z+}), \text{ran}(\hat{P}_{z-}), \mathbb{C}^2 \right\},
\]

that do not meet each other as \( \mathcal{L}(\Sigma^{(x)}) \cap \mathcal{L}(\Sigma^{(y)}) \cap \mathcal{L}(\Sigma^{(z)}) = \{ \{0\}, \mathbb{C}^2 \} \).

As a result, no assumption additional to Gleason’s axioms is necessary to imply Born’s rule (41).

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