Symmetry Finder for neutrino oscillation in matter: Interplay between $\nu$SM and non-unitarity

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Abstract: Symmetry Finder (SF), a systematic method for finding reparametrization symmetry in neutrino oscillation in matter, has been successfully applied to the Denton et al. (DMP) and the atmospheric-resonance perturbation theories to uncover, respectively, the eight 1-2 and sixteen 1-3 state exchange symmetries. Here we apply SF to the DMP theory extended to DMP-UV by including unitarity violation (UV), a possible low-energy manifestation of physics beyond the $\nu$SM. Implementation of UV into SF yields the additional two very different constraints, which nonetheless allow remarkably consistent solutions, the eight DMP-UV symmetries. Treatment of one of the constraints, the genuine non-unitary part, leads to the key identity which entails the UV $\alpha$ parameter transformation only by rephasing, which innovates the invariance proof of the Hamiltonian. It is shown that the reparametrization symmetry as a whole distinguishes between the $\nu$SM and UV sectors, suggesting its utility for diagnostics of the extended neutrino theory. An interesting picture, the $\nu$SM - UV inter-sector communications through the phases emerges.
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1 Introduction

Symmetry plays a fundamental role in constructing gauge theories for particle interactions. In much wider contexts, symmetry offers a powerful method for understanding systems in quantum mechanics and quantum field theories [1]. There exists a large variety of implications and utilities of symmetry. Even outside gauge and/or space-time symmetries, the approximate symmetry in the hadron spectrum, for example, brought us the valuable insights, revealing some of the key features of the strong force [2–4].

In neutrino physics, there have been discussions on symmetries mostly from the various phenomenological points of view. The large mixing angles $\theta_{23}$ and $\theta_{12}$ triggered interests in symmetry between the first and the second octants [5–7], which may possibly be elaborated into the discrete symmetries in flavor physics models [8]. An approximate or exact symmetries of the oscillation probability in systems within the $\nu$SM [9–11], the neutrino-mass-embedded Standard Model, or beyond [12], provided understanding of the parameter degeneracy [9, 10, 13–15]. Discussions on the related but different aspects are, e.g., in refs. [16] and [17].

The reparametrization symmetries discussed in the two previous papers [18, 19] and a concurrently progressing work [20], as well as the one in this paper, do not fall into the same categories as mentioned above, apart from possible tie to refs. [5–7]. By “symmetry”, we mean this particular type throughout this paper. Invariance under the reparametrization implies that there is another way of parametrizing the equivalent solution of the theory. This special characteristic of the symmetry may trigger the several questions. One of the most important would be: What is the meaning of existence of the symmetry? We all agree that eventually we must be able to answer this question, but at present we are not ready to do the job, unfortunately.

One of the major obstacles to answer the question is that we know too little about the symmetry. At this stage, therefore, we think it worthwhile to take an “experimentalists’
approach” to the symmetry. That is, we try to dig out all the symmetries, assuming that they exist in all the reasonable, i.e., well-defined and consistent, frameworks of neutrino oscillations.

But, how? To make an “experimental search” for the symmetries we need a systematic way of hunting the symmetry. Now we call the readers’ attention to “Symmetry Finder” (SF) which serves for this purpose as introduced and fully examined in refs. [18–20]. For the preceding treatment in vacuum, see ref. [21]. The list of the symmetries that are identified by using the SF method in the perturbative frameworks of neutrino oscillation in matter within the $\nu$SM includes:

- The eight symmetries of the 1-2 state exchange type are uncovered in refs. [18, 20] in the Denton et al. (DMP) [22] and the solar-resonance perturbation (SRP) [23] theories, respectively, with the very similar properties to each other.

- The sixteen symmetries of the 1-3 state exchange type are found [19] in the helio-perturbation theory [24].

In this paper we try to enlarge our field of “experimental search” of the symmetries to the neutrino theory with non-unitary flavor mixing matrix [29–33]. Non-unitarity, for short, is one of the promising ways to discuss physics beyond the $\nu$SM. See, e.g., refs. [34–44] for an incomplete list of the additional references on non-unitarity. Even more generic framework of adding “non-standard interactions” (NSI) [25] is vigorously investigated as a possible description of physics beyond the $\nu$SM. See e.g. refs. [45–48] for reviews of NSI, and refs. [49–52] for constraints on NSI.

As a concrete setting for our symmetry search we use the extension of the DMP to include unitarity violation (UV), the DMP-UV perturbation theory formulated in ref. [53]. The DMP [22] is one of the “globally valid” frameworks, roughly speaking, the one whose region of validity covers both the solar and the atmospheric resonances. See ref. [54] for an alternative. Or, in other words, it covers the most of the terrestrial neutrino measurements [20]. Then, what we should do is to extend the SF framework to include the effect of non-unitarity, as we will do below, and carry out the similar SF analysis as done for the $\nu$SM DMP [18]. It would entail an extended, or un-extended, DMP-UV symmetries.

But, we have encountered something unexpected. In fact, there was an indication for it in our previous work: The oscillation probability derived to first order in the DMP-UV expansion [53] obeys Symmetry IA- and IB-DMP in the nomenclature defined in ref. [18], without involving any UV sector variables’ transformation. Nonetheless, we hesitated to mention about the feature there because we did not understand it. See the first three columns in Table 1 for the classification scheme and $\nu$SM variables’ transformations. In the SF framework, as will be explained in section 5, the 1-2 space rotation is involved and

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\[ \text{It is a version of the atmospheric-resonance perturbation theory with the title “helio-perturbation”, a shorthand for the “helio-terrestrial-ratio perturbation” with the expansion parameter } \Delta m_{21}^2 / \Delta m_{31}^2. \] In this article, the term “resonance” should be understood in a less strict sense than usual, see refs. [25–28].

\[ \text{We are aware that in physics literatures UV usually means “ultraviolet”. But, in this paper UV is used as an abbreviation for “unitarity violation” or “unitarity violating”.} \]
the symmetry transformation must act on the UV sector variables as well. It casts doubt on the above mentioned property of Symmetry IA- and IB-DMP, leaving it as a mystery.

Now let us go to Table 1 which summarizes the results of our investigation in this paper. The fourth column tabulates the transformation property of the UV sector variables $\tilde{\alpha}_{\beta\gamma}$, where the Greek indices are for the flavor $e, \mu, \tau$. For the definition of the $\tilde{\alpha}$ parameters, see section 2. According to the SF analysis results, no UV $\tilde{\alpha}$ transformation is involved in Symmetry IA- and IB-DMP, confirming the above mentioned feature. But, in the remaining Symmetry $X$, $X = $ IIA, IIB, · · ·, IVB, the $\tilde{\alpha}$ parameters do transform in conjunction with the $\nu$SM variables’ transformations.

Then, what is wrong in our naive argument above? We will resolve the apparent puzzle in our SF treatment in sections 6 and 7. What is more important is the above-mentioned feature that the SM theory, as a whole, recognizes and distinguishes the Symmetry X, $X = $ IIA, IIB, · · ·, confirming the above mentioned feature. But, in the remaining SM and UV parts. This possibility will be discussed further in this paper.

**Table 1:** All the reparametrization symmetries of the 1-2 state exchange type in the DMP-UV perturbation theory are summarized by giving the transformation properties of the $\nu$SM and UV $\tilde{\alpha}$ variables. In the text we refer them as “Symmetry IA-DMP-UV”, etc.

| Symmetry | Vacuum parameter transf. | Matter parameter transf. | UV parameter transf. |
|----------|--------------------------|--------------------------|---------------------|
| IA       | none                     | $\lambda_1 \leftrightarrow \lambda_2$, $c_\psi \rightarrow \mp s_\psi$, $s_\psi \rightarrow \mp c_\psi$ | none |
| IB       | $\theta_{12} \rightarrow -\theta_{12}$, $\delta \rightarrow \delta + \pi$. | $\lambda_1 \leftrightarrow \lambda_2$, $c_\psi \rightarrow \mp s_\psi$, $s_\psi \rightarrow \mp c_\psi$ | none |
| IIA      | $\theta_{23} \rightarrow -\theta_{23}$, $\theta_{12} \rightarrow -\theta_{12}$. | $\lambda_1 \leftrightarrow \lambda_2$, $c_\psi \rightarrow \pm s_\psi$, $s_\psi \rightarrow \pm c_\psi$ | $\tilde{\alpha}_{\mu e} \rightarrow -\tilde{\alpha}_{\mu e}$, $\tilde{\alpha}_{\tau\mu} \rightarrow -\tilde{\alpha}_{\tau\mu}$ |
| IIB      | $\theta_{23} \rightarrow -\theta_{23}$, $\delta \rightarrow \delta + \pi$. | $\lambda_1 \leftrightarrow \lambda_2$, $c_\psi \rightarrow \mp s_\psi$, $s_\psi \rightarrow \mp c_\psi$ | same as IIA |
| IIIA     | $\theta_{13} \rightarrow -\theta_{13}$, $\theta_{12} \rightarrow -\theta_{12}$. | $\lambda_1 \leftrightarrow \lambda_2$, $\phi \rightarrow -\phi \quad c_\psi \rightarrow \pm s_\psi$, $s_\psi \rightarrow \pm c_\psi$ | $\tilde{\alpha}_{\mu e} \rightarrow -\tilde{\alpha}_{\mu e}$, $\tilde{\alpha}_{\tau e} \rightarrow -\tilde{\alpha}_{\tau e}$ |
| IIIB     | $\theta_{13} \rightarrow -\theta_{13}$, $\delta \rightarrow \delta + \pi$. | $\lambda_1 \leftrightarrow \lambda_2$, $\phi \rightarrow -\phi \quad c_\psi \rightarrow \mp s_\psi$, $s_\psi \rightarrow \mp c_\psi$ | same as IIIA |
| IVA      | $\theta_{23} \rightarrow -\theta_{23}$, $\theta_{13} \rightarrow -\theta_{13}$. | $\lambda_1 \leftrightarrow \lambda_2$, $\phi \rightarrow -\phi \quad c_\psi \rightarrow \mp s_\psi$, $s_\psi \rightarrow \mp c_\psi$ | $\tilde{\alpha}_{\tau e} \rightarrow -\tilde{\alpha}_{\tau e}$, $\tilde{\alpha}_{\tau\mu} \rightarrow -\tilde{\alpha}_{\tau\mu}$ |
| IVB      | $\theta_{23} \rightarrow -\theta_{23}$, $\theta_{13} \rightarrow -\theta_{13}$, $\theta_{12} \rightarrow -\theta_{12}$, $\delta \rightarrow \delta + \pi$. | $\lambda_1 \leftrightarrow \lambda_2$, $\phi \rightarrow -\phi \quad c_\psi \rightarrow \pm s_\psi$, $s_\psi \rightarrow \pm c_\psi$ | same as IVA |

To summarize Introduction, we examine in this paper the reparametrization symmetry of the 1-2 state exchange type using the SF method in the DMP-UV perturbation theory. This is the first systematic investigation of such symmetry in theory of neutrino evolution with non-unitary flavor mixing matrix in matter. We, therefore, believe that the topic is sufficiently interesting to investigate in its own right. Moreover, we expect that the lessons we will learn in this study give us an intriguing possibility of using the symmetry
to diagnose a low-energy theory treatment of new physics beyond the $\nu$SM.

The organization of this paper contains a compact review of the DMP-UV perturbation theory in section 3, and its treatment by the $V$ matrix method in section 4. The SF framework for the DMP-UV perturbation theory is introduced in section 5, and its zeroth-order solutions are given in Table 2. The UV parts of the first-order SF equations are fully analyzed in sections 6 and 7. The obtained DMP-UV symmetries are summarized in Table 1, which is given above. An all-order proof of the DMP-UV symmetries is given in section 8. The summary and outlook in section 9 conclude this paper.

### 2 Three active-neutrino system with non-unitary flavor mixing matrix

This section is to define the system of three active neutrinos propagating under the influence of non-unitary flavor mixing matrix and the matter potential. In our formalism the three active neutrino evolution in matter in the presence of non-unitary flavor mixing is based on the Schrödinger equation in the vacuum mass eigenstate basis [31, 33], the “check basis”,

$$
i \frac{d}{dx} \tilde{\nu} = \frac{1}{2E} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix} + N^\dagger \begin{pmatrix} a - b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -b \end{pmatrix} N \begin{pmatrix} \tilde{\nu} \equiv \tilde{H}\tilde{\nu} \end{pmatrix}$$

(2.1)

In eq. (2.1), which also defines the check-basis Hamiltonian $\tilde{H}$, $N$ denotes the $3 \times 3$ non-unitary flavor mixing matrix which relates the flavor neutrino states to the vacuum mass eigenstates as

$$\nu_\alpha = N_{\alpha i} \tilde{\nu}_i.$$  

(2.2)

In eq. (2.2) and hereafter, the subscript Greek indices $\alpha$, $\beta$, or $\gamma$ run over $e, \mu, \tau$, and the Latin indices $i, j$ run over the mass eigenstate indices 1, 2, and 3. $E$ is neutrino energy and $\Delta m_{ji}^2 = m_j^2 - m_i^2$. The usual phase redefinition of neutrino wave function is done to leave only the mass squared differences. Notice, however, that doing this phase redefinition or not (see eq. (8.1)) does not affect our symmetry discussion in this paper.

The functions $a(x)$ and $b(x)$ in eq. (2.1) denote the Wolfenstein matter potentials [25] due to charged current (CC) and neutral current (NC) reactions, respectively,

$$a(x) = 2\sqrt{2}G_F N_e E \approx 1.52 \times 10^{-4} \left( \frac{Y_e \rho}{g \text{ cm}^{-3}} \right) \left( \frac{E}{\text{GeV}} \right) \text{ eV}^2,$$

$$b(x) = \sqrt{2}G_F N_n E = \frac{1}{2} \left( \frac{N_n}{N_e} \right) a,$$

(2.3)

where $G_F$ is the Fermi constant. $N_e$ and $N_n$ are the electron and neutron number densities

$^4$If one determines the coupling constant from the experimentally measured rate of muon decay, one obtains $G_\mu$, which is related to the tree-level Fermi constant $G_F$ as $G_\mu = \sqrt{(NN^\dagger)_{ee}(NN^\dagger)_{\mu\mu}}G_F$ [29]. Given the existing bound, see e.g., ref. [30], the difference between $G_\mu$ and $G_F$ is small. Even if this is adopted one can easily show that it does not affect our symmetry discussion. $NN^\dagger$ transforms under Symmetry X as $NN^\dagger \rightarrow \text{Rep}(X) NN^\dagger \text{Rep}(X)^\dagger$, where $\text{Rep}(X)$ is defined in (6.4). Then, $(NN^\dagger)_{\beta\beta}$ is invariant under all the DMP-UV symmetries. Therefore, the difference between using $G_\mu$ or $G_F$ is just a multiplicative constant which remains unchanged under the symmetry transformations.
in matter. \( \rho \) and \( Y_e \) denote, respectively, the matter density and number of electrons per nucleon in matter. These quantities except for \( G_F \) are, in principle, position dependent.

We use so called the \( \alpha \) parametrization \[30\] for the non-unitary flavor mixing matrix \( N \),

\[
N = (1 - \alpha) U = \left( \begin{array}{ccc}
\alpha_{ee} & 0 & 0 \\
\alpha_{\mu e} & \alpha_{\mu\mu} & 0 \\
\alpha_{\tau e} & \alpha_{\tau\mu} & \alpha_{\tau\tau}
\end{array} \right) U, \tag{2.4}
\]

where \( U \equiv U_{\text{MNS}} \) denotes the usual unitary \( \nu \text{SM} \) mixing matrix \[55\]. The \( \alpha \) parametrization originates in refs. \[4, 56\]. In fact, the definitions of \( U \) and \( \alpha \) matrices are convention dependent \[57\]. In eq. (2.4) we have used the Particle Data Group (PDG) convention \[58\].

For convenience of our discussion, following ref. \[18\], we use the solar (SOL) convention for the \( U \) and \( \alpha \) matrices. They are defined by the phase redefinition of those of the PDG convention:

\[
\begin{align*}
U_{\text{SOL}} &= U_{\text{PDG}} \equiv \begin{bmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{bmatrix} \\
\alpha_{\text{SOL}} &= \alpha_{\text{PDG}} \equiv \begin{bmatrix} \tilde{\alpha}_{ee} & 0 & 0 \\ \tilde{\alpha}_{\mu e} & \tilde{\alpha}_{\mu\mu} & 0 \\ \tilde{\alpha}_{\tau e} & \tilde{\alpha}_{\tau\mu} & \tilde{\alpha}_{\tau\tau} \end{bmatrix}, \tag{2.5}
\end{align*}
\]

with the obvious notations \( s_{ij} \equiv \sin \theta_{ij} \) etc., \( \delta \) for the lepton analogue of the quark CP violating KM phase \[59\], and \( \tilde{\alpha}_{\beta\gamma} \) for the SOL convention \( \alpha \) parameters which we use.\(^5\)

The second line in eq. (2.5) defines the rotation matrices in the 2-3, 1-3, and 1-2 spaces in order.

### 3 DMP-UV perturbation theory in a nutshell

To present our formulation of the DMP-UV perturbation theory, an extension of DMP to include non-unitarity,\(^6\) we rely on the basic formulation given in our previous paper \[53\], which will be referred for details. On the other hand, we have to go beyond the treatment of ref. \[53\] for the following three reasons: (1) While we remain in the first-order treatment in most part of this paper, we have to keep the second-order UV term in the Hamiltonian

\(^5\)The other useful form of the \( U \) matrix is the one with the atmospheric (ATM) convention used e.g., in refs. \[22, 24, 57\] in which the phase factor \( e^{\pm i\delta} \) is attached to \( s_{23} \) as opposed to \( s_{12} \) in the SOL convention.

\(^6\)The SF method is applied first to the DMP theory \[22\] partly because the first-order expressions of the oscillation probabilities are already very accurate, see e.g., ref. \[60\]. Later in this paper and in ref. \[20\] the SF analyses for the symmetries are performed in the DMP-UV and the helio-UV perturbation theories, respectively. Despite the parallelism at high level between the SF treatments in these two UV extended theories we try to make our presentation in this paper self-contained. It is because of the marked difference between the structure of symmetries possessed by these two theories, the eight 1-2 exchange symmetries in DMP-UV as in Table 1, and the sixteen 1-3 exchange symmetries in the helio-UV perturbation theories \[20\].
when we attempt to give an all-order proof of the symmetries in section 8. (2) In the SF framework we need so called the \( V \) matrix method [61], which will be formulated for our theory in section 4. (3) We take the eigenvalue-renormalized basis for the eigenstate in matter, which is slightly different from the one in ref. [53].

The DMP-UV perturbation theory has two kind of the expansion parameters, \( \epsilon \) and the UV \( \alpha \) parameters. \( \epsilon \) is defined as

\[
\epsilon \equiv \frac{\Delta m_{31}^2}{\Delta m_{31}^2}, \quad \Delta m_{31}^2 \equiv \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2, \quad (3.1)
\]

where \( \Delta m_{31}^2 \) is the “renormalized” atmospheric \( \Delta m^2 \) used in ref. [24]. The same quantity is known as the effective \( \Delta m_{ee}^2 \) in the \( \nu_e \to \nu_e \) channel in vacuum [62]. While we prefer usage of \( \Delta m_{31}^2 \) in the context of the present paper, the question of which symbol should be appropriate to use here is under debate [24]. The authors of ref. [63] make the choice alternative to ours. The other expansion parameters are the \( \alpha_{\beta\gamma} \) parameters (2.5) which represent the UV effects.

We start from the tilde-basis Hamiltonian

\[
\tilde{H} \equiv (U_{13}U_{12})H(U_{13}U_{12})^\dagger = \tilde{H}_{\nu SM} + \tilde{H}^{(1)}_{UV} + \tilde{H}^{(2)}_{UV}, \quad (3.2)
\]

where

\[
\tilde{H}_{\nu SM} = \frac{\Delta m_{31}^2}{2E} \left\{ \begin{bmatrix} a(x) / \Delta m_{31}^2 + s_{13}^2 + e s_{12}^2 & 0 & c_{13}s_{13} \\ 0 & \epsilon_{12}^2 & 0 \\ c_{13}s_{13} & 0 & \epsilon_{13}^2 + s_{12}^2 \end{bmatrix} + c_{12}s_{12} \begin{bmatrix} 0 & c_{13}e^{i\delta} & 0 \\ c_{13}e^{-i\delta} & 0 & -s_{13}e^{-i\delta} \\ 0 & -s_{13}e^{i\delta} & 0 \end{bmatrix} \right\}. \quad (3.3)
\]

The UV part has the first and second order terms in the \( \tilde{\alpha} \) parameters

\[
\tilde{H}^{(1)}_{UV} = \frac{b}{2E} U_{23}^\dagger A (U_{23}) U_{23},
\]

\[
\tilde{H}^{(2)}_{UV} = -\frac{b}{2E} U_{23}^\dagger A^{(2)} U_{23}, \quad (3.4)
\]

where

\[
A \equiv \begin{bmatrix} 2\tilde{\alpha}_{ee} \left( 1 - \frac{a(x)}{b(x)} \right) & \tilde{\alpha}_{\mu e}^* & \tilde{\alpha}_{\tau e}^* \\ \tilde{\alpha}_{\mu e} & 2\tilde{\alpha}_{\mu \mu} & \tilde{\alpha}_{\tau \mu} \\ \tilde{\alpha}_{\tau e} & \tilde{\alpha}_{\tau \mu} & 2\tilde{\alpha}_{\tau \tau} \end{bmatrix},
\]

\[
A^{(2)} \equiv \begin{bmatrix} \tilde{\alpha}_{ee}^2 \left( 1 - \frac{a(x)}{b(x)} \right) + |\tilde{\alpha}_{\mu e}|^2 + |\tilde{\alpha}_{e \tau}|^2 & \tilde{\alpha}_{\mu e}^* \tilde{\alpha}_{\mu \mu} + \tilde{\alpha}_{\tau e}^* \tilde{\alpha}_{\tau \mu} & \tilde{\alpha}_{\tau e}^* \tilde{\alpha}_{\tau \tau} \\ \tilde{\alpha}_{\mu e}^* \tilde{\alpha}_{\mu \mu} + \tilde{\alpha}_{e \tau}^* \tilde{\alpha}_{\tau \mu} & \tilde{\alpha}_{\mu \mu}^2 + |\tilde{\alpha}_{\tau \mu}|^2 & \tilde{\alpha}_{\tau \mu}^* \tilde{\alpha}_{\tau \tau} \\ \tilde{\alpha}_{e \tau}^* \tilde{\alpha}_{\tau \tau} & \tilde{\alpha}_{\tau \mu}^* \tilde{\alpha}_{\tau \tau} & \tilde{\alpha}_{\tau \tau}^2 \end{bmatrix}. \quad (3.5)
\]

In \( \tilde{H}_{\nu SM} \) in eq. (3.3), the rephasing to remove the NC potential is understood [53]. For a consistent nomenclature \( A \) must carry the superscript as \( A^{(1)} \), but for simplicity of the expressions we omit it throughout this paper. In what follows we keep omitting the superscript \( (1) \) for many of the quantities in first order in the \( \tilde{\alpha} \) parameters, because our treatment will be free from the second order terms apart from section 8.
3.1 The renormalized eigenvalue (bar) basis

We use the two successive rotations in the 1-3 and 1-2 spaces with the angles $\phi$ and $\psi$, $\theta_{13}$ and $\theta_{12}$ in matter, respectively, to diagonalize $H_{\nu SM}$, see ref. [22]. The notation “matter-dressed $\theta_{13}$ ($\theta_{12}$)” for $\phi$ ($\psi$) may also be used. The diagonalized basis is denoted as the “bar basis” with the Hamiltonian

$$H = U_{12}^\dagger(\psi, \delta)U_{13}^\dagger(\phi)H_{13}(\phi)U_{12}(\psi, \delta) = H_{\nu SM}^{(0)} + H_{\nu SM}^{(1)} + H_{\nu UV} + H_{\nu UV}^{(2)}.$$  

The notations are such that $U_{12}(\psi, \delta)$, for example, implies $U_{12}(\theta_{12}, \delta)$ with replacement of $\theta_{12}$ by $\psi$. The $\nu SM$ part of $H$ takes the form in the SOL convention as

$$H_{\nu SM}^{(0)} + H_{\nu SM}^{(1)} = \frac{1}{2E} \begin{bmatrix} \lambda_1^{\nu SM} & 0 & 0 \\ 0 & \lambda_2^{\nu SM} & 0 \\ 0 & 0 & \lambda_3^{\nu SM} \end{bmatrix} + \epsilon c_{12}s_{12}\sin(\phi - \theta_{13})\frac{\Delta m_{ren}^2}{2E} \begin{bmatrix} 0 & 0 & -s_\psi \\ 0 & 0 & c_\psi e^{-i\delta} \\ -s_\psi & c_\psi e^{i\delta} & 0 \end{bmatrix},$$

where the first and second terms in eq. (3.7) defines $H_{\nu SM}^{(0)}$ and $H_{\nu SM}^{(1)}$, respectively. $\lambda_i^{\nu SM}/2E$ denote the zeroth-order eigenvalues of the $\nu SM$ Hamiltonian given in ref. [22]. The UV part of the bar-basis Hamiltonian reads

$$H_{UV}^{(1)} = \frac{b}{2E} G, \quad H_{UV}^{(2)} = -\frac{b}{2E} G^{(2)}.$$  

where the $G$ matrices are the 2-3, 1-3 and 1-2 rotated $A$ matrices in eq. (3.5):

$$G = U_{12}(\psi, \delta)^\dagger U_{13}^\dagger(\phi)U_{23}(\theta_{23})AU_{13}(\phi)U_{12}(\psi, \delta),$$

$$G^{(2)} = U_{12}(\psi, \delta)^\dagger U_{13}^\dagger(\phi)U_{23}(\theta_{23})A^{(2)}U_{23}(\theta_{23})U_{13}(\phi)U_{12}(\psi, \delta).$$

For simplicity in the discussion of reparametrization symmetry we take the renormalized bar basis in which the diagonal elements of the $G$ matrices are absorbed into the eigenvalues of $H^{(0)}$:

$$\lambda_i = \lambda_i^{\nu SM} + bG_{ii} - bG_{ii}^{(2)} \quad (i = 1, 2, 3).$$  

The off-diagonal parts of the $G$ and $G^{(2)}$ matrices are left in $H_{UV}^{(1)}$ and $H_{UV}^{(2)}$, and is treated as perturbation. Notice that since the matter-dressed mixing angles are defined through the process of diagonalizing the $\nu SM$ part, $H_{\nu SM}^{(0)} + H_{\nu SM}^{(1)}$, the shifted eigenvalues do not affect them.

From now on until section 8, we concentrate on the first order term in the Hamiltonian and parametrize the $G$ matrix as

$$G = DHD^\dagger = \begin{bmatrix} H_{11} & e^{i\delta}H_{12} & H_{13} \\ e^{-i\delta}H_{21} & H_{22} & e^{i\delta}H_{23} \\ H_{31} & e^{i\delta}H_{32} & H_{33} \end{bmatrix},$$

where $D \equiv \text{diag}(e^{i\delta}, 1, e^{i\delta})$, with the diagonal elements untouched, $G_{ii} = H_{ii}$. The explicit expressions of $H_{ij}$ are given in Appendix A. We have introduced the $H$ matrix because
of the two reasons: (1) Overall $e^{\pm i\delta}$ factors in the elements $G_{12}$ and $G_{23}$ are taken out to prevent proliferation of hidden $\delta$ in the expressions of the probability. (2) It makes the $\delta$ dependences of the $\nu$SM and UV parts of the SF equation more coherent.

Then, by using the renormalized eigenvalues, the bar-basis Hamiltonian is given to first order in the DMP-UV expansion as

$$\hat{H} = \frac{1}{2E} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} + \epsilon c_{12}s_{12} \sin(\phi - \theta_{13}) \frac{\Delta m_{\text{ren}}^2}{2E} \begin{bmatrix} 0 & 0 & -s_{\psi} \\ 0 & 0 & c_{\psi}e^{-i\delta} \\ -s_{\psi} c_{\psi}e^{i\delta} & 0 \end{bmatrix} + \frac{b}{2E} \begin{bmatrix} 0 & e^{i\delta}H_{12} & H_{13} \\ e^{-i\delta}H_{21} & 0 & e^{-i\delta}H_{23} \\ H_{31} & e^{i\delta}H_{32} & 0 \end{bmatrix}. \quad (3.12)$$

Hereafter we denote the first term of eq. (3.12) as $\hat{H}^{(0)}$, and the second and third as $\hat{H}^{(1)}$ which are treated as perturbation. After identifying the unperturbed and perturbed Hamiltonian, there is a standard route to compute the $S$ matrix and the oscillation probability. This task is carried out explicitly in the $\nu_{\mu} \to \nu_e$ channel in ref. [53] to first order in the DMP-UV expansion. As in ref. [53] we use the uniform matter density approximation until section 8.

4 $V$ matrix method

Symmetry Finder (SF) [18–20] is formulated by using the $V$ matrix formalism [61], and for this reason we construct it for the DMP-UV perturbation theory. If we have the expression of the flavor eigenstate in terms of the mass eigenstate basis in matter as

$$\nu = V \bar{\nu}, \quad \text{(4.1)}$$

the oscillation probability can readily be calculated as

$$P(\nu_{\beta} \to \nu_{\alpha}) = \delta_{\alpha\beta} - 4 \sum_{j>i} \text{Re}[V_{\alpha i}V_{\beta j}^*V_{\alpha j}^*V_{\beta i}] \sin^2 \left( \frac{(\lambda_j - \lambda_i)x}{4E} \right) - 2 \sum_{j>i} \text{Im}[V_{\alpha i}V_{\beta j}^*V_{\alpha j}^*V_{\beta i}] \sin \left( \frac{(\lambda_j - \lambda_i)x}{2E} \right). \quad (4.2)$$

The $V$ matrix method has been utilized for this purpose, e.g., in refs. [22, 24]. The explicit computation of the $V$ matrix elements is carried out for the SRP theory in ref. [20].

4.1 $V$ matrix at the zeroth and first orders

However, since we deal with the theory with non-unitary mixing matrix a proper care is needed to calculate the $V$ matrix. We recall that the flavor eigenstate $\nu_\alpha$ is related with the vacuum mass eigenstate (denoted as the check basis) as $\nu_\alpha = N_\alpha \bar{\nu}_i = \{(1 - \bar{\alpha})U\}_{\alpha i} \bar{\nu}_i$, see eqs. (2.2) and (2.4). From eqs. (3.2) and (3.6), the relation between the bar and the check bases are given by $\bar{H} = U_{12}^\dagger(\psi, \delta)U_{13}^\dagger(\phi)U_{13}U_{12}^\dagger HU_{13}^\dagger U_{13}^\dagger(\phi)U_{12}(\psi, \delta)$. Here, $U_{13}$ and $U_{12}$
without arguments implies the rotation matrices in vacuum, see eq. (2.5). Therefore, the relation between the check-basis and bar-basis states is
\[
\bar{\nu} = U_{12}^\dagger U_{13}^\dagger U_{13}(\phi)U_{12}(\psi, \delta)\bar{\nu}.
\] (4.3)

Then, the flavor state is connected to the bar-basis state as
\[
\nu = (1 - \tilde{\alpha})U\bar{\nu} = (1 - \tilde{\alpha})U_{23}(\theta_{23})U_{13}(\phi)U_{12}(\psi, \delta)\bar{\nu},
\] (4.4)
which gives the unperturbed part in the \(V\) matrix. Namely, the zeroth-order and the first order “genuine UV part” of the \(V\) matrix are given, respectively, as
\[
V^{(0)} = U_{23}(\theta_{23})U_{13}(\phi)U_{12}(\psi, \delta),
\]
\[
V^{(1)}_{\text{UV}} = -\tilde{\alpha}U_{23}(\theta_{23})U_{13}(\phi)U_{12}(\psi, \delta) = -\tilde{\alpha}V^{(0)}.
\] (4.5)

When the perturbation \(\tilde{H}^{(1)}\) is switched on, the other type of the first-order term in the \(V\) matrix, the “unitary evolution” (EV) part \(V^{(1)}_{\text{EV}}\), is generated. See ref. [57] for more about these concepts, the “genuine UV” and “unitary evolution” parts, which will be briefly recollected in section 4.3. To compute the EV part of the first-order correction to the \(V\) matrix we take out the prefactor \((1 - \tilde{\alpha})\) because the effect of this \(\tilde{\alpha}\) produces the second order effect. Then, the method for obtaining the first-order correction for the \(V\) matrix is identical to the one used in computing the first-order correction to the wave function in quantum mechanics. When we write \(\bar{\nu}_i = \bar{\nu}_i^{(0)} + \bar{\nu}_i^{(1)}\), the first-order correction can be calculated as
\[
\bar{\nu}_i^{(1)} = \sum_{j \neq i} \frac{2E\tilde{H}^{(1)}_{ij}}{\lambda_i - \lambda_j} \bar{\nu}_j^{(0)} = W_{ij}\bar{\nu}_j^{(0)},
\] (4.6)
where we have defined the \(W\) matrix. Since \(\tilde{H}\) in eq. (3.12) has the two first order terms, one from the \(\nu\)SM and the other due to EV (UV-induced but unitary evolution) effect, the \(W\) matrix can be written in the form of addition of these terms, \(W = W_{\nu\text{SM}} + W_{\text{EV}}\), and their explicit forms are given in eq. (4.10). A short note for clarification of eq. (4.6) is in Appendix B.

Then, the energy eigenstate calculated to first order can be written, using the \(W\) matrix defined in eq. (4.6), as
\[
\bar{\nu} = \bar{\nu}^{(0)} + \bar{\nu}^{(1)} = (1 + W)\bar{\nu}^{(0)}
= (1 + W_{\nu\text{SM}} + W_{\text{EV}})[U_{23}(\theta_{23})U_{13}(\phi)U_{12}(\psi, \delta)]^\dagger \nu.
\] (4.7)
where we have used the zeroth order relation \(\nu = V^{(0)}\bar{\nu}^{(0)} = [U_{23}(\theta_{23})U_{13}(\phi)U_{12}(\psi, \delta)]\bar{\nu}^{(0)}\). Inverting this relation and adding the contribution from the genuine unitary part in eq. (4.5), we obtain the expression of the flavor state by the mass eigenstate in matter to first order as
\[
\nu = \left[-\tilde{\alpha}V^{(0)} + V^{(0)}(1 + W_{\nu\text{SM}} + W_{\text{EV}})^\dagger\right]\bar{\nu} \equiv V\bar{\nu},
\] (4.8)
where we have used the expression of the zeroth-order \(V\) matrix in eq. (4.5).
4.2 *V* matrix to first order: Summary

For the convenience of formulating the SF equation we rewrite the expression of the flavor state (4.8) with use of the *V* matrix in the following form:

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau 
\end{pmatrix} = U_{23}(\theta_{23})U_{13}(\phi)U_{12}(\psi, \delta) \left\{ 1 + W_{\nu SM}^{(1)} + W_{EV}^{(1)} - Z_{UV}^{(1)} \right\}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 
\end{pmatrix},
\]

where

\[
W_{\nu SM}^{(1)}(\theta_{23}, \phi, \psi; \lambda_1, \lambda_2) \equiv W_{\nu SM} U_{\nu SM}^{\dagger} = e c_{12} s_{12} \sin(\phi - \theta_{13})
\begin{pmatrix}
0 & 0 & -s_\psi \frac{\Delta m^2_{32}}{\lambda_3 - \lambda_2} \\
0 & 0 & c_\psi e^{-i\delta} \frac{\Delta m^2_{32}}{\lambda_3 - \lambda_2} \\
-s_\psi \frac{\Delta m^2_{32}}{\lambda_3 - \lambda_1} & -c_\psi e^{i\delta} \frac{\Delta m^2_{21}}{\lambda_3 - \lambda_1} & 0
\end{pmatrix},
\]

\[
W_{EV}^{(1)}(\theta_{23}, \phi, \psi; \lambda_1, \lambda_2) \equiv W_{EV}^{\dagger} =
\begin{pmatrix}
0 & -e^{-i\delta} H_{21} \frac{b}{\lambda_2 - \lambda_1} & H_{13} \frac{b}{\lambda_3 - \lambda_1} \\
e^{i\delta} H_{12} \frac{b}{\lambda_2 - \lambda_1} & 0 & -H_{23} e^{-i\delta} \frac{b}{\lambda_3 - \lambda_2} \\
-H_{31} \frac{b}{\lambda_3 - \lambda_1} & -H_{32} e^{i\delta} \frac{b}{\lambda_3 - \lambda_2} & 0
\end{pmatrix}.
\]

That is, we label the mass eigenstate basis, the bar-basis state, \(\nu_i\) \((i = 1, 2, 3)\) to make the state label more explicit in our discussion of the reparametrization symmetry which involves the state exchange. \(W_{\nu SM}\) and \(W_{EV}\) are calculated by using eq. (4.6). Noticing that the \(H\) matrix is hermitian.

In eq. (4.9), we give \(V^{(0)}\) in eq. (4.8) the explicit form as in eq. (4.5), and move it to the front position in the right-hand side of eq. (4.9). For this purpose we have introduced \(Z_{UV}^{(1)}\) with the definition

\[
\tilde{\alpha} V^{(0)} = V^{(0)} Z_{UV}^{(1)},
\]

which can readily be solved for \(Z_{UV}^{(1)}\) as

\[
Z_{UV}^{(1)}(\theta_{23}, \phi, \psi; \tilde{\alpha}_{\beta \gamma}) = (V^{(0)})^{\dagger} \tilde{\alpha} V^{(0)}.
\]

4.3 Computing the oscillation probability using the *V* matrix method

Having obtained the \(V\) matrix as in eq. (4.8), it is straightforward to compute the oscillation probability by using the formula (4.2). We restrict ourselves into the zeroth- and first-order terms of the oscillation probability. In the UV extensions of the perturbative formulations of neutrino oscillation in matter using the \(S\) matrix method [53, 57, 64], the probability can be written to first order in the DMP-UV expansion as

\[
P(\nu_\beta \to \nu_\alpha) = P(\nu_\beta \to \nu_\alpha)_{\nu SM}^{(0)} + P(\nu_\beta \to \nu_\alpha)_{\nu SM}^{(1)} + P(\nu_\beta \to \nu_\alpha)_{EV}^{(1)} + P(\nu_\beta \to \nu_\alpha)_{UV}^{(1)},
\]

where the first two terms denote the standard \(\nu SM\) contributions. \(P(\nu_\beta \to \nu_\alpha)_{EV}^{(1)}\) is the contribution from EV part, the UV effect-driven but describing the unitary evolution effect. \(P(\nu_\beta \to \nu_\alpha)_{UV}^{(1)}\) is the genuine non-unitary contribution which violates unitarity at the \(S\) matrix and the probability levels [57].
The first two terms, \( P(\nu_\beta \to \nu_\alpha)^{(0)}_{\nu SM} \) and \( P(\nu_\beta \to \nu_\alpha)^{(1)}_{\nu SM} \), are fully calculated in ref. [22]. See also arXiv v3 of ref. [65] for the less abstract expressions in all the relevant channels. Therefore, we just concentrate into the UV related parts, \( P(\nu_\beta \to \nu_\alpha)^{(1)}_{\nu SM} \) and \( P(\nu_\beta \to \nu_\alpha)^{(1)}_{\nu SM} \), in this paper. In our \( V \) matrix formulation, \( P(\nu_\beta \to \nu_\alpha)^{(1)}_{\nu SM} \) comes from \( V_{UV} \) in eq. (4.5), and \( P(\nu_\beta \to \nu_\alpha)^{(1)}_{\nu SM} \) from \( V(0)^{EV}_{W} \) part of eq. (4.8).

The computed results of \( P(\nu_\mu \to \nu_e)^{(1)}_{EV} \) and \( P(\nu_\mu \to \nu_e)^{(1)}_{UV} \) are given in Appendices C.1 and C.2, respectively. One can readily see that \( P(\nu_\mu \to \nu_e)^{(1)}_{UV} \) is identical with the corresponding formula in ref. [53], and that \( P(\nu_\mu \to \nu_e)^{(1)}_{EV} \) agrees with the one in ref. [53], apart from that our expression in Appendix C misses the \( (b_x)/2E \) term, eq. (4.9) in ref. [53]. But, there is no problem because the \( (b x)/2E \) term shows up if we expand the renormalized eigenvalues (3.10) by \( b_{Gi} = b_{Hi} \), as will be shown in eq. (D.5) in Appendix D.

5 Symmetry Finder for the DMP-UV perturbation theory

The underlying principle of Symmetry Finder (SF) is very simple. Suppose that the flavor basis state (i.e., wave function) \( \nu \) allows the two different expressions by the mass eigenstate \( \bar{\nu} \) in vacuum or in matter [18, 21]:

\[
\nu = U(\theta_{23}, \theta_{13}, \theta_{12}, \delta)\bar{\nu} = U(\theta'_{23}, \theta'_{13}, \theta'_{12}, \delta')\bar{\nu}',
\]

(5.1)

where the quantities with “prime” imply the transformed ones, and \( \nu' \) may involve eigenstate exchanges and/or rephasing of the wave functions. If it is in matter the mixing angles and the CP phase can be elevated to the matter-dressed variables. As eq. (5.1) represents the unique flavor state by the two different sets of the physical parameters, it implies a symmetry. This is nothing but the key statement of SF [18–21].

5.1 Symmetry Finder (SF) equation

We embody SF in eq. (5.1) and the associated machinery, the SF equation, in the DMP-UV perturbation theory. We define another state physically equivalent to the one in eq. (4.9):

\[
F \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = FU_{23}(\theta_{23})U_{13}(\phi)U_{12}(\psi, \delta)R^tR \left\{ 1 + W^{(1)}_{\nu SM} + W^{(1)}_{EV} - Z^{(1)}_{UV} \right\} R^tR \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix},
\]

(5.2)

with use of the flavor state rephasing matrix \( F \) and the generalized 1-2 state exchange matrix \( R \) parametrized as\(^7\)

\[
F = \begin{bmatrix} e^{i\tau} & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & -e^{i(\delta + \alpha)} & 0 \\ e^{-i(\delta + \beta)} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

(5.3)

respectively, where \( \tau, \sigma, \alpha, \) and \( \beta \) denote the arbitrary phases. Since the rephasing of the states does not change its physical content, the states defined by eqs. (4.9) and (5.2) are physically equivalent to each other.

\(^7\)In ref. [18], we have used the notation \( G \) for the \( R \) matrix in eq. (5.3). We use here the symbol \( R \) not to confuse it to the \( G \) matrix in eq. (3.9).
Now, we introduce the SF equation, the DMP version of eq. (5.1). If the flavor state (5.2) can be written in a form of the same state but with the transformed parameters, it implies existence of symmetry. The concrete form of the statement reads:

\[
\begin{bmatrix}
e^{i\tau} & 0 & 0 \\
0 & e^{i\sigma} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} e^{i\sigma} \\
0 & -s_{23} e^{-i\sigma} & c_{23}
\end{bmatrix}
\begin{bmatrix}
c_{13} & 0 & s_{13} e^{i\tau} \\
0 & 1 & 0 \\
-s_{13} e^{-i\tau} & 0 & c_{13}
\end{bmatrix}
\times \left(1 + W_{\nu SM}^{(1)}(\Phi; \nu_1, \lambda_2) + W_{EV}^{(1)}(\Phi, \tilde{\alpha}; \lambda_1, \lambda_2) - Z_{UV}^{(1)}(\Phi, \tilde{\alpha})\right)
\]

That is, if we find a solution of the SF equation (5.4) we identify a reparametrization symmetry in the DMP-UV theory. In eq. (5.4), \(\Phi\) denotes the collective representation of all the parameters involved, \(\theta_{23}, \phi, \psi, \delta, \theta_{12}, \) and \(\theta_{13}\), and \(\Phi'\) their transformed ones. \(\tilde{\alpha}'\) is for the transformed \(\tilde{\alpha}\), the collective notation for the \(\tilde{\alpha}\) parameters, and \(\delta' = \delta + \xi\). The SF equation is valid to first order in the DMP-UV perturbation theory.

Clarifying remarks on the structure of the SF equation and the relationship between the DMP and DMP-UV theories are in order: (1) Due to the perturbative formulation of the SF equation it can be decomposed into the zeroth and the first order parts, which will be denoted as the first and second conditions, respectively. The former contains only the \(\nu SM\) variables. (2) The three first-order entities in the second condition, \(W_{\nu SM}^{(1)}(\Phi'; \lambda_1, \lambda_2)\), \(W_{EV}^{(1)}(\Phi', \tilde{\alpha}'; \lambda_1, \lambda_2)\), and \(Z_{UV}^{(1)}(\Phi, \tilde{\alpha})\), do not affect to each other due to the differences in their variable dependence. One can confirm this property by the explicit treatment of the second condition on the UV parts in sections 5.4 and 5.5, in addition to the \(\nu SM\) part which is worked out in ref. [18]. Therefore, the second condition decomposes into the three independent equations. See eq. (5.8).

### 5.2 The first condition

By eliminating all the first-order terms in the SF equation (5.4), we obtain the first condition. We look for the solution under the ansatz \(s_{23} e^{i\sigma} = s_{23}'\) and \(s_\phi e^{i\tau} = s_\phi'\). Apparently we have no other choice within the present SF formalism. The ansatz implies that the possible values of \(\tau\) and \(\sigma\) are restricted to integer multiples of \(\pi\).

Then the first condition takes the form

\[
FU_{12}(\psi, \delta) R^\dagger = U_{12}(\psi', \delta + \xi),
\]

\[\text{(5.5)}\]
where we recall that
\[
U_{12}(\psi, \delta) = \begin{bmatrix}
c_\psi & s_\psi e^{i\delta} & 0 \\
-s_\psi e^{-i(\alpha - \tau)} & c_\psi & 0 \\
0 & 0 & 1
\end{bmatrix}.
\] (5.6)

It is shown [18] that the DMP first condition, which is identical to ours, can be reduced to
\[
c_\psi' = -s_\psi e^{-i(\alpha - \tau)} = -s_\psi e^{i(\beta + \sigma)},
\]
\[
s_\psi' = c_\psi e^{i(\beta + \tau - \xi)} = c_\psi e^{-i(\alpha - \sigma - \xi)}.
\] (5.7)

We note that under the above restriction of \(\tau\) and \(\sigma\) being integer multiples of \(\pi\), eq. (5.7) implies that all the rest of the phase parameters, \(\xi, \alpha, \) and \(\beta\), must also be integer multiples of \(\pi\). All the solutions of the first condition are obtained in ref. [18], and they are summarized in Table 2. The universal nature of the first three columns of Table 2 over the DMP, SRP, and the helio-perturbation theories is pointed out in ref. [20].

**Table 2:** All the solutions of the first condition (5.7) for Symmetry X where \(X = IA, IB, \ldots, IVB\), and the rephasing matrix Ref(X) (6.4) are tabulated. The labels “upper” and “lower” imply the upper and lower sign in the corresponding rows in Table 1.

| Symmetry | \(\tau, \sigma, \xi\) | \(\alpha, \beta\) | Rep(X) |
|----------|-----------------|-----------------|--------|
| IA       | \(\tau = \sigma = 0, \xi = 0\) | \(\alpha = \beta = 0\) (upper) | diag(1,1,1) |
|          |                 | \(\alpha = \pi, \beta = -\pi\) (lower) |        |
| IB       | \(\tau = \sigma = 0, \xi = \pi\) | \(\alpha = \pi, \beta = -\pi\) (upper) | same as IA |
|          |                 | \(\alpha = \beta = 0\) (lower) |        |
| IIA      | \(\tau = 0, \sigma = -\pi, \xi = 0\) | \(\alpha = \pi, \beta = 0\) (upper) | diag(1,-1,1) |
|          |                 | \(\alpha = 0, \beta = \pi\) (lower) |        |
| IIB      | \(\tau = 0, \sigma = -\pi, \xi = \pi\) | \(\alpha = 0, \beta = \pi\) (upper) | same as IIA |
|          |                 | \(\alpha = \pi, \beta = 0\) (lower) |        |
| IIIA     | \(\tau = \pi, \sigma = 0, \xi = 0\) | \(\alpha = 0, \beta = \pi\) (upper) | diag(-1,1,1) |
|          |                 | \(\alpha = \pi, \beta = 0\) (lower) |        |
| IIIB     | \(\tau = \pi, \sigma = 0, \xi = \pi\) | \(\alpha = 0, \beta = \pi\) (upper) | same as IIIA |
|          |                 | \(\alpha = \pi, \beta = 0\) (lower) |        |
| IVA      | \(\tau = \sigma = \pi, \xi = 0\) | \(\alpha = \pi, \beta = -\pi\) (upper) | diag(-1,-1,1) |
|          |                 | \(\alpha = \beta = 0\) (lower) |        |
| IVB      | \(\tau = \sigma = \pi, \xi = \pi\) | \(\alpha = \pi, \beta = -\pi\) (upper) | same as IVA |
|          |                 | \(\alpha = \beta = 0\) (lower) |        |

5.3 The second condition

The first-order terms in the SF equation (5.4) constitute the second condition which can be decomposed into the \(\nu\)SM, EV, and the UV parts,
\[
R\mathcal{W}^{(1)}_{\nu\text{SM}}(\theta_{23}, \delta, \phi, \psi; \lambda_1, \lambda_2)R^\dagger = \mathcal{W}^{(1)}_{\nu\text{SM}}(\theta'_{23}, \delta + \xi, \phi', \psi'; \lambda_2, \lambda_1),
\]
\[
R\mathcal{W}^{(1)}_{\text{EV}}(\theta_{23}, \delta, \phi, \psi, \tilde{\alpha}_{\beta\gamma}; \lambda_1, \lambda_2)R^\dagger = \mathcal{W}^{(1)}_{\text{EV}}(\theta'_{23}, \delta + \xi, \phi', \psi', \tilde{\alpha}'_{\beta\gamma}; \lambda_2, \lambda_1),
\]
\[
R\mathcal{Z}^{(1)}_{\text{UV}}(\theta_{23}, \delta, \phi, \psi, \tilde{\alpha}_{\beta\gamma})R^\dagger = \mathcal{Z}^{(1)}_{\text{UV}}(\theta'_{23}, \delta + \xi, \phi', \psi', \tilde{\alpha}'_{\beta\gamma}).
\] (5.8)
The decomposability of the second condition implies, together with the common first condition in the both DMP and DMP-UV theories, that the symmetries of the DMP-UV theory cannot be larger than the eight symmetries of the νSM DMP. The question is whether all of them survive in the UV extension.

The second condition works as follows: The νSM part, the first line in eq. (5.8), determines the transformation property of ψ etc., as fully worked out in ref. [18]. It produces the eight DMP symmetries, IA, IB, · · ·, IVB, where the type A (B) means that no δ is involved (δ is involved) in the symmetry transformations. See Table 1. With the knowledge of the eight νSM symmetries we examine the second conditions on \( \mathcal{W}_{EV}^{(1)} \) and \( \mathcal{Z}_{UV}^{(1)} \) to know if the consistent solutions exist. In the rest of this section, we reduct the EV and UV second conditions a little further to make them ready to solve. It will be followed by the solutions of the \( \mathcal{Z}_{UV}^{(1)} \) and \( \mathcal{W}_{EV}^{(1)} \) equations in the next two sections 6 and 7, respectively.

5.4 The second condition on the unitary evolution part

The second condition (5.8) with the explicit form of \( \mathcal{W}_{EV}^{(1)} \) in eq. (4.10) can be written, using the \( H \) matrix defined in eq. (3.11), as

\[
\begin{bmatrix}
0 & e^{i(\delta + \alpha + \beta)} H_{12} \frac{b}{\lambda_2 - \lambda_1} & -e^{i\alpha} H_{21} \frac{b}{\lambda_2 - \lambda_1} & -e^{i(\delta + \beta)} H_{23} \frac{b}{\lambda_3 - \lambda_1} \\
-e^{-i(\delta + \alpha + \beta)} H_{21} \frac{b}{\lambda_2 - \lambda_1} & 0 & e^{-i(\delta + \beta)} H_{13} \frac{b}{\lambda_3 - \lambda_1} \\
e^{-i\alpha} H_{32} \frac{b}{\lambda_3 - \lambda_2} & -e^{i(\delta + \beta)} H_{31} \frac{b}{\lambda_3 - \lambda_1} & 0 & e^{-i(\delta + \xi)} H_{13}^{\prime} \frac{b}{\lambda_3 - \lambda_1} \\
e^{-i(\delta + \xi)} H_{13}^{\prime} \frac{b}{\lambda_3 - \lambda_2} & e^{-i(\delta + \xi)} H_{31}^{\prime} \frac{b}{\lambda_3 - \lambda_1} & e^{i(\delta + \xi)} H_{32}^{\prime} \frac{b}{\lambda_3 - \lambda_1} & 0
\end{bmatrix}
\]

We notice that it can be written in the condensed form as

\[
H_{12}^{\prime} = -e^{i(\alpha + \beta - \xi)} H_{21},
H_{23}^{\prime} = e^{-i(\beta - \xi)} H_{13},
H_{13}^{\prime} = -e^{i\alpha} H_{23}.
\]

The condition on \( H_{ij} \) can be obtained from that on \( H_{ij} \) using the hermitisity \( H_{ij} = H_{ij}^\dagger \).

5.5 The second condition on the genuine non-unitary part

Using eq. (4.12) the second condition with \( \mathcal{Z}_{UV}^{(1)} \) in eq. (5.8) takes the form

\[
R \left[ V^{(0)}(\theta_{23}, \phi, \psi, \delta) \right]^\dagger \tilde{\alpha} V^{(0)}(\theta_{23}, \phi, \psi, \delta) R = \left[ V^{(0)}(\theta_{23}', \phi', \psi', \delta + \xi) \right]^\dagger \tilde{\alpha}' V^{(0)}(\theta_{23}', \phi', \psi', \delta + \xi).
\]

where \( V^{(0)}(\theta_{23}, \phi, \psi, \delta) \) is defined in eq. (4.5). Then, the transformed \( \tilde{\alpha}' \) can be obtained in a closed form as

\[
\tilde{\alpha}' = V^{(0)}(\theta_{23}', \phi', \psi', \delta + \xi) R \left[ V^{(0)}(\theta_{23}, \phi, \psi, \delta) \right]^\dagger \tilde{\alpha} V^{(0)}(\theta_{23}, \phi, \psi, \delta) R^\dagger \left[ V^{(0)}(\theta_{23}', \phi', \psi', \delta + \xi) \right]^\dagger.
\]
Notice the vastly different features of the two second conditions, the one for $Z^{(1)}_{UV}$ in eq. (5.12) and the other for $W^{(1)}_{EV}$ in eq. (5.10). It makes consistency between them highly nontrivial.

6 Solution of the SF equation: Genuine non-unitary part

As we have already pointed out, possibility of UV extension of the $\nu$SM DMP symmetries depends on whether or not the second conditions on $W^{(1)}_{EV}$ and $Z^{(1)}_{UV}$ produce independently the consistent solutions for the transformation properties of the $\tilde{\alpha}$ parameters. Therefore, we first examine the second condition (5.12) on $Z^{(1)}_{UV}$, the genuine non-unitary part, because it will reveal an interesting feature for the SF formalism itself. But in the first place, of course, it will tell us how $\tilde{\alpha}$ transform under the DMP-UV symmetries.

If we use the simplified notation $[V'RV^\dagger] \equiv V^{(0)}(\theta'_{23}, \phi', \psi', \delta + \xi)R[V^{(0)}(\theta_{23}, \phi, \psi, \delta)]^\dagger$, eq. (5.12) can be written as $\alpha' = [V'RV^\dagger] \alpha [V'RV^\dagger]^\dagger$. Therefore, we calculate $[V'RV^\dagger]$ first. To calculate it in a transparent way, we define $C[12]$, 

$$C[12] \equiv \begin{bmatrix} c_\phi' & s_\phi' e^{i(\delta + \xi)} & 0 \\ -s_\phi' e^{-i(\delta + \xi)} & c_\phi' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -e^{i(\delta + \alpha)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi e^{i\delta} & 0 \\ s_\psi e^{-i\delta} & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(6.1)

such that the key ingredient $[V'RV^\dagger]$ can be written as

$$[V'RV^\dagger] = V^{(0)}(\theta'_{23}, \phi', \psi', \delta + \xi)R[V^{(0)}(\theta_{23}, \phi, \psi, \delta)]^\dagger$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23}' & s_{23}' \\ 0 & -s_{23}' & c_{23}' \end{bmatrix} \begin{bmatrix} c_\phi' & 0 & s_\phi' \\ 0 & 1 & 0 \\ -s_\phi' & 0 & c_\phi' \end{bmatrix} C[12] \begin{bmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23} & c_{23} \end{bmatrix}.$$ 

(6.2)

We will see immediately below that $C[12]$ and $[V'RV^\dagger]$ in eq. (6.2) are equal to each other in a quite interesting manner. See eq. (6.3).

6.1 Useful Identities for the rephasing matrix

We calculate $C[12]$ and $[V'RV^\dagger]$ by inserting the solutions of the SF equation tabulated in Table 1, and the phase parameters $\alpha, \beta$, etc. given in Table 2. Since each solution for Symmetry X has the upper and lower $\pm$ signs there are total sixteen cases. The computed results we have obtained for them are the ones totally unexpected to us. Despite the profound dependences on the $\nu$SM parameters in $C[12]$ and $[V'RV^\dagger]$, the computed results are the constants, which depend only on the symmetry type, Symmetry X with $X = I, II, III, IV$.

$$C[12] = V^{(0)}(\theta'_{23}, \phi', \psi', \delta + \xi)R[V^{(0)}(\theta_{23}, \phi, \psi, \delta)]^\dagger = \text{Rep}(X).$$ 

(6.3)

---

8We generically quote symmetry as “Symmetry X” when X applies to all the DMP-UV symmetries. When we quote X for a particular property in a more specific way, such as e.g., “X = II, III, and IV”, it means that the property holds for the both XA and XB. For our repeated use of the phrase “O transforms under the transformations of Symmetry X”, we simply say “O transforms under Symmetry X” to avoid cumbersome repetition of the words.
Rep(X) is the rephasing matrix which is introduced to characterize the transformation property of the flavor basis Hamiltonian as $H \rightarrow \text{Rep}(X)H\text{Rep}(X)\dagger$ under Symmetry X [18]. It is the key concept in the proof of the DMP symmetries as the Hamiltonian symmetries. Rep(X) is given by $\text{Rep}(I) = \text{diag}(1,1,1)$, and
\begin{align*}
\text{Rep}(II) &= \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \\
\text{Rep}(III) &= \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \\
\text{Rep}(IV) &= \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\end{align*}

In Appendix E, a sketchy proof of the identity eq. (6.3) is given for Symmetry IV. The computations of $C[12]$ and $[V'RV]\dagger$ for the rest of symmetries $X=I, II, III$, are left for the interested readers.

We observe that the identity has several interesting consequences. They include: (1) a simple transformation property of $\tilde{\alpha}$ as discussed below, (2) an innovation in the Hamiltonian proof of the symmetries as will be shown in section 8, and (3) a possible speculation on its topological nature. We will also give a conjecture on possible further enlargement of symmetry in section 9.

### 6.2 Solution of the second condition: Genuine non-unitary part

The solution of the second condition (5.12) can readily be obtained by using the second identity in eq. (6.3):

$$\tilde{\alpha}' = \text{Rep}(X) \tilde{\alpha} \text{Rep}(X)\dagger,$$

which implies that $\tilde{\alpha}' = \tilde{\alpha}$ for Symmetry $X = I$, and for $X = II, III, and IV$, in order

$$\tilde{\alpha}' = \begin{bmatrix}
\tilde{\alpha}_{ee} & 0 & 0 \\
-\tilde{\alpha}_{me} & \tilde{\alpha}_{\mu\mu} & 0 \\
\tilde{\alpha}_{\tau e} & -\tilde{\alpha}_{\tau\mu} & \tilde{\alpha}_{\tau\tau}
\end{bmatrix}, \\
\begin{bmatrix}
\tilde{\alpha}_{ee} & 0 & 0 \\
-\tilde{\alpha}_{me} & \tilde{\alpha}_{\mu\mu} & 0 \\
-\tilde{\alpha}_{\tau e} & \tilde{\alpha}_{\tau\mu} & \tilde{\alpha}_{\tau\tau}
\end{bmatrix}, \\
\begin{bmatrix}
\tilde{\alpha}_{ee} & 0 & 0 \\
\tilde{\alpha}_{me} & \tilde{\alpha}_{\mu\mu} & 0 \\
-\tilde{\alpha}_{\tau e} & -\tilde{\alpha}_{\tau\mu} & \tilde{\alpha}_{\tau\tau}
\end{bmatrix}. \quad (6.6)

The resulting transformation properties of the $\tilde{\alpha}$ parameters and Rep(X) are summarized in Table 1 and Table 2, respectively.

The obtained solution for Symmetry I implies that no $\tilde{\alpha}$ parameters' transformation is involved, which indicates that Symmetry I- and IB-DMP-UV involve only the $\nu_{SM}$ transformation. But, to make a careful statement we need to go back to the unrenormalized eigenvalue basis. This point will be discussed in section 7.4.

### 7 Solution of the SF equation: Unitary evolution part

We analyze the the second condition eq. (5.10) for the unitary evolution part to determine the transformation property of the $H_{ij}$ parameters. After verifying the consistency with the $\tilde{\alpha}$ parameter transformations (6.6), we examine the invariance of the oscillation probability under the $H_{ij}$ parameters.

---

9Rep(II) = diag (-1,1,-1) in ref. [18], but this is equivalent to diag (1,-1,1) as in eq. (6.4). Similarly, Rep(IV) can be written as diag (1,1,-1). The overall sign of Rep(X) does not affect the physical observables.
7.1 Solution of the second condition: Unitary evolution part

The solutions of the first condition depend not only on the symmetry types denoted generically as $X A$ and $X B$, but also the upper and lower signs of the phase parameters $\alpha$, $\beta$, etc., as summarized in Table 2. Using the phase parameters, one can show that the second condition (5.10) implies that $H_{ij}$ transform under Symmetry $X$ as

- Symmetry $IA$, $IIB$: $H'_{12} = -H_{21}$, $H'_{23} = \pm H_{13}$, $H'_{13} = \mp H_{23}$,
- Symmetry $IB$, $IA$: $H'_{12} = H_{21}$, $H'_{23} = \mp H_{13}$, $H'_{13} = \pm H_{23}$,
- Symmetry $IIA$, $IVB$: $H'_{12} = H_{21}$, $H'_{23} = \pm H_{13}$, $H'_{13} = \mp H_{23}$,
- Symmetry $IIIB$, $IVA$: $H'_{12} = -H_{21}$, $H'_{23} = \mp H_{13}$, $H'_{13} = \pm H_{23}$. (7.1)

where $\pm$ (or $\mp$) sign refers to the upper and lower signs in Table 2 and Table 1, which are synchronized between them. Notice that the transformation property of $H_{ji}$ can be obtained by using the hermiticity $H_{ji} = (H_{ij})^\ast$. The pairings may look curious because the pair $IA$ and $IIB$, and also $IB$ and $IIA$, differ in the property that the former (latter) does not (does) contain the transformation of the $\tilde{\alpha}$ parameters. But, we reassure these pairings in the next section 7.2. It appears that the pairing of the two symmetries is dictated by the transformation property of $\psi$, which is the same inside the pair.

7.2 Consistency between the $H_{ij}$ and the $\tilde{\alpha}$ parameter transformations

Now, we are left with the consistency check between the solutions of the second conditions derived from the genuine unitary part (6.6) and the unitary evolution part (7.1). The way we carry it out is to use the transformation properties of the $\nu$SM variables and the $\tilde{\alpha}$ parameters in eq. (6.6), the both of whose are in Table 1, to obtain the $H_{ij}$ transformation properties and see if they agree with the ones in eq. (7.1).

Then, we immediately encounter the problem for $H_{ii}$ ($i = 1, 2, 3$), the diagonal elements. Their transformation properties are absent in eq. (7.1). It is because they are absorbed into the eigenvalues, see eq. (3.10). Note that $G_{ii} = H_{ii}$. In fact, $H_{ii}$ calculated with the above recipe transform, under all Symmetry $X = IA, IB, \cdots, IVB$, as

\[ H_{11} \leftrightarrow H_{22}, \quad H_{33} \text{ is invariant}. \] (7.2)

i.e., $H_{11}$-$H_{22}$ exchange, and $H_{33}$ is invariant. It must be the case because we are dealing with the 1-2 state exchange symmetry $\lambda_1 \leftrightarrow \lambda_2$ for which the both transformations $\lambda_{1}\text{SM} \leftrightarrow \lambda_{2}\text{SM}$ and $H_{11} \leftrightarrow H_{22}$ must occur simultaneously under all Symmetry $X$. Therefore, the consistency is met for the diagonal $H_{ii}$.

Now we discuss the off-diagonal $H_{ij}$ ($i \neq j$). For concreteness we give explicit treatments of Symmetry I and II. In Appendix A the $H_{ij}$ elements are given by using the $K_{ij}$ elements, whose latter depend only on $\theta_{23}$, $\phi$ and the $\tilde{\alpha}$ parameters. Since none of them is involved in the transformations of Symmetry IA and IB, all the $K_{ij}$ elements are invariant under these symmetries. Then, the $H_{ij}$ elements transform under IA and IB only through the $\nu$SM parameter transformations in Table 1 as

\[ H_{12} \rightarrow -H_{21}, \quad H_{13} \rightarrow \mp H_{23}, \quad H_{23} \rightarrow \pm H_{13}, \quad \text{(IA)}, \]
\[ H_{12} \rightarrow H_{21}, \quad H_{13} \rightarrow \pm H_{23}, \quad H_{23} \rightarrow \mp H_{13}, \quad \text{(IB)}, \] (7.3)
which resolves the apparent puzzle mentioned in section 1. Notice that they reproduce the relevant lines in eq. (7.1). It means that no \( \tilde{\alpha} \) parameter transformation is involved in Symmetry IA- and IB-DMP-UV, and only the \( \nu\text{SM} \) parameter transformations suffice. Thus, the SF treatment reproduces the somewhat puzzling result we have mentioned in section 1.

Under Symmetry IIA and IIB, the two \( K_{ij} \) elements flip sign

\[
K_{12} \rightarrow -K_{12}, \quad K_{23} \rightarrow -K_{23},
\]

and all the other \( K_{ij} \) elements are invariant. Then, one can show that the resulting \( H_{ij} \) element transformations are as

\[
H_{12} \rightarrow H_{21}, \quad H_{13} \rightarrow \pm H_{23}, \quad H_{23} \rightarrow \pm H_{13} \quad \text{(IIA)},
\]

\[
H_{12} \rightarrow -H_{21}, \quad H_{13} \rightarrow \mp H_{23}, \quad H_{23} \rightarrow \pm H_{13} \quad \text{(IIB)}.\]

Therefore, the pairings between IA-IIB, and IB-IIA in the first and second lines of eq. (7.1), which we have referred as “curious” are reproduced.

Similarly one can work out the \( H_{ij} \) transformation properties for Symmetry III and IV to confirm eq. (7.1). Therefore, the \( \tilde{\alpha} \) parameter transformation from the second condition on the genuine non-unitary part \( Z^{(1)}_{UV} \) is perfectly consistent with the \( H_{ij} \) transformation property derived from that of the unitary evolution part \( W^{(1)}_{EV} \).

### 7.3 Invariance of the oscillation probability

The oscillation probability \( P(\nu_\mu \rightarrow \nu_e)^{(1)}_{EV} \) given in Appendix C.1 is written in terms of the \( \nu\text{SM} \) and \( H_{ij} \) parameters without any naked \( \tilde{\alpha} \) parameters. Therefore, showing the invariance under Symmetry X can be carried out straightforwardly for all the eight symmetries with the transformation properties of these parameters given in Table 1 and eq. (7.1). This exercise for invariance proof, simple but slightly lengthy, is left for the interested readers.

On the other hand, the probability \( P(\nu_\mu \rightarrow \nu_e)^{(1)}_{UV} \) in Appendix C.2 consists of the \( \nu\text{SM} \) and the naked \( \tilde{\alpha} \) parameters. We can use the transformation properties of these variables summarized in Table 1 to prove the invariance under the all Symmetry X.

In this paper we do not discuss the oscillation channels other than \( \nu_\mu \rightarrow \nu_e \) explicitly, because we will prove the Hamiltonian invariance in section 8 which automatically applies to all the oscillation channels.

### 7.4 Distinguishing between the \( \nu\text{SM} \) and the UV sectors

Here is a clarifying note on our claim that the symmetry can distinguish between the \( \nu\text{SM} \) and the UV sectors of the theory. To make a clearer statement on this point, let us momentarily take the original framework in ref. [53] in which we restrict to the unrenormalized treatment of the eigenvalues \( \lambda_i = \lambda_i^{\nu\text{SM}} \) by removing the \( H_{ii} \) term in eq. (3.10). Now there exists the \( (bx)/2E \) term in \( P(\nu_\mu \rightarrow \nu_e)^{(1)}_{EV} \) in eq. (D.5). Then, there is no mixed up between the \( \nu\text{SM} \) and the UV variables.

Suppose then that the theory is (second-) quantized. Our results on Symmetry IA and IB imply that the IA or IB symmetry generators act on the \( \nu\text{SM} \) operators, but not on
the UV operators. Whereas the symmetry generators of the remaining symmetries, X=II, III, IV, do act on both the νSM and UV operators. In this sense, the symmetry of the state exchange type, Symmetry X-DMP-UV as a whole, recognizes the both νSM and UV sectors of the theory, and can distinguish between them.  

7.5 The DMP-UV symmetry: Summary

Here, we give our summary of all the eight DMP-UV symmetries, denoted as Symmetry X-DMP-UV where X = IA, IB, ···, IVB. In Table 1 we give the transformation properties of the νSM and the UV \( \tilde{\alpha} \) parameters under Symmetry X. In Table 2, we tabulate the solutions of the first condition (5.7) and the rephasing matrix Rep(X) given in eq. (6.4) for all Symmetry X-DMP-UV. Up until this section our results are valid to first order in the DMP-UV expansion. We attempt at an all-order proof of all Symmetry X in the next section 8.

8 DMP-UV symmetry as a Hamiltonian symmetry

In this section we show that all the DMP-UV symmetries summarized in Table 1 are the symmetries of the flavor basis Hamiltonian \( H_{\text{flavor}} \). In unitary case in vacuum \( H_{\text{flavor}} = U \hat{H} U^\dagger \), where \( \hat{H} \) is the vacuum mass eigenstate basis Hamiltonian, and \( U \) the νSM flavor mixing matrix, see eq. (2.5). In non-unitary case in matter, since the flavor basis ν is related to the mass eigenstate basis \( \hat{\nu} \) as \( \nu = N \hat{\nu} \), \( H_{\text{flavor}} = N \hat{H} N^\dagger \), where the check basis Hamiltonian \( \hat{H} \) is defined in eq. (2.1). We denote \( H_{\text{flavor}} \) constructed in this way as \( H_{\text{VM}} \).

We can construct \( H_{\text{flavor}} \) in an alternative way. To formulate the DMP-UV perturbation theory we diagonalized \( \hat{H} \) to obtain the bar basis Hamiltonian which is presented in eq. (3.12) to first order. Then, we can transform back to the check basis, and then transform to the flavor basis using \( \nu = N \hat{\nu} \). The thereby obtained \( H_{\text{flavor}} \) is denoted as \( H_{\text{Diag}} \). The subscripts in \( H_{\text{VM}} \) and \( H_{\text{Diag}} \) imply “vacuum-matter” and “diagonalized”, respectively. Of course they are equal to each other, \( H_{\text{VM}} = H_{\text{Diag}} \).

8.1 Transformation property of \( H_{\text{VM}} \)

Using \( N = (1 - \tilde{\alpha}) U \) and \( NN^\dagger = (1 - \tilde{\alpha})(1 - \tilde{\alpha})^\dagger \), 2E times \( H_{\text{VM}} = N \hat{H} N^\dagger \) can be written as

\[
2E H_{\text{VM}} = (1 - \tilde{\alpha}) U(\Xi) \begin{bmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{bmatrix} U(\Xi)^\dagger + (1 - \tilde{\alpha})^\dagger \cdot \begin{bmatrix} a - b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -b \end{bmatrix} \cdot (1 - \tilde{\alpha})^\dagger \cdot (1 - \tilde{\alpha})^\dagger,
\]

\( (8.1) \)

\(^{10}\)A related question is whether or not Symmetry IA and IB are special apart from the point discussed here, and if yes in what sense. In the exact treatment of neutrino oscillation in uniform density matter [66] it appears that IA and IB are the only known reparametrization symmetries as noted in refs. [18, 21]. We do not know the reason why the symmetry is less numerous in this theory.

\(^{11}\)Usage of \( H_{\text{VM}} \) and \( H_{\text{Diag}} \) instead of our previous notations \( H_{\text{LHS}} \) and \( H_{\text{RHS}} \), respectively, is to unify our notation with ref. [20].
where we have used a collective notation $\Xi$ for all the vacuum parameters involved. Here we have used a slightly different phase-redefined basis from the one in eq. (2.1) to make the vacuum Hamiltonian $\propto \text{diag}(m_1^2, m_2^2, m_3^2)$ making it more symmetric. But it does not affect our symmetry discussion.

We have shown in ref. [18] that the vacuum term transforms under Symmetry $X$ as

$$
\begin{align*}
U(\Xi) 
\begin{bmatrix}
  m_1^2 & 0 & 0 \\
  0 & m_2^2 & 0 \\
  0 & 0 & m_3^2 
\end{bmatrix}
U(\Xi)^\dagger & \rightarrow \text{Rep}(X) 
\begin{bmatrix}
  m_1^2 & 0 & 0 \\
  0 & m_2^2 & 0 \\
  0 & 0 & m_3^2 
\end{bmatrix}
U(\Xi)^\dagger 
\text{Rep}(X)^\dagger.
\end{align*}
$$

(8.2)

where $\text{Rep}(X)$ is defined in eq. (6.4). Using the transformation property of the $\tilde{\alpha}$ parameters in eq. (6.5), $\tilde{\alpha}' = \text{Rep}(X) \tilde{\alpha} \text{Rep}(X)^\dagger$, the matter term in eq. (8.1) which originates from the $\nu\text{SM}$ and the UV sectors of the theory obeys the same transformation property as in the vacuum term. Then, the whole $H_{\text{VM}}$ transforms under Symmetry $X$ as

$$
H_{\text{VM}} \rightarrow \text{Rep}(X)H_{\text{VM}} \text{Rep}(X)^\dagger,
$$

(8.3)

which means that $H_{\text{VM}}$ is invariant under Symmetry $X$ up to the rephasing factor $\text{Rep}(X)$. By being the diagonal matrix with the elements $e^{\pm i\pi}$, $\text{Rep}(X)$ does not affect physical observables as it can be absorbed into the neutrino wave functions.

We note that our success in demonstrating the transformation property of $H_{\text{VM}}$ in a transparent way as above heavily owes the $\alpha$ parametrization [30] of the non-unitary matrix, eq. (2.4). It entails the neat $\tilde{\alpha}$ transformation (6.5), and enabled us to have the simple and revealing form of $H_{\text{VM}}$ in eq. (8.1).

Therefore, the invariance property (8.3) ultimately comes from the fact that the transformation property of the $\tilde{\alpha}$ parameters is determined by the identical rephasing matrix $\text{Rep}(X)$ that governs the transformation of the $\nu\text{SM}$ part of the Hamiltonian. Despite that it must be the case for proving the Hamiltonian invariance, it is remarkable to see that it indeed occurs, being enforced by the genuine UV part of the SF equation (5.12). It indicates an intriguing interplay between the $\nu\text{SM}$ and UV sectors in the theory. In passing, we note that we do not use the property that the matter density is uniform to obtain the invariance proof, the feature which prevails in the proof of invariance of $H_{\text{Diag}}$ in section 8.2.

### 8.2 Transformation property of $H_{\text{Diag}}$: New method

In this section we discuss $H_{\text{Diag}}$ to show that it is invariant under Symmetry $X$-DMP-UV with the same rephasing matrix as needed for $H_{\text{VM}}$. By $H_{\text{Diag}}$ we mean the flavor-basis Hamiltonian written in terms of the diagonalized variables. Using the expression of the flavor-basis state by the bar-basis state $\nu_{\alpha} = [(1 - \tilde{\alpha})U_{23}U_{13}(\phi)U_{12}(\psi, \delta)]_{\alpha j} \bar{\nu}_{j}$ given in eq. (4.4), $H_{\text{Diag}}$ is given by

$$
H_{\text{Diag}} = (1 - \tilde{\alpha})U_{23}(\theta_{23})U_{13}(\phi)U_{12}(\psi, \delta)\tilde{H}U_{12}^\dagger(\psi, \delta)U_{13}^\dagger(\phi)U_{23}^\dagger(\theta_{23})(1 - \tilde{\alpha})^\dagger.
$$

(8.4)

The bar-basis Hamiltonian $\tilde{H}$ is given in eq. (3.12), which ignores the second order term eq. (3.8) with the $G^{(2)}$ matrix in eq. (3.9). In this section we proceed with the bar-basis Hamiltonian (3.12) without the second order term to prove invariance of $H_{\text{Diag}}$ under
Symmetry X. In the next section 8.3 we will present a simple argument to show that our proof of invariance prevails even after we include the second order effect.

Since we use an entirely new method to prove the invariance $H_{\text{Diag}}$, we include the $\nu$SM part as well, though its invariance has been fully discussed in ref. [18]. From the identity (6.3) one obtains

$$V^{(0)}(\theta_2', \phi', \delta') = \text{Rep}(X)V^{(0)}(\theta_{23}, \phi, \psi, \delta)R^\dagger.$$  \hspace{1cm} (8.5)

Then, $H_{\text{Diag}}$ in eq. (8.4) with use of eq. (4.5) transforms under Symmetry X as

$$H_{\text{Diag}} = (1 - \tilde{\alpha})V^{(0)}(\theta_{23}, \phi, \psi, \delta)\bar{H}(\theta_{23}, \phi, \psi, \delta; \bar{\alpha}_{B\gamma}, \lambda_i) \left[V^{(0)}(\theta_{23}, \phi, \psi, \delta)\right]^\dagger (1 - \tilde{\alpha})^\dagger$$

$$\rightarrow_{\text{Symmetry X}} (1 - \tilde{\alpha'})V^{(0)}(\theta'_{23}, \phi', \psi', \delta')\bar{H}(\theta'_{23}, \phi', \psi', \delta'; \bar{\alpha}_{B\gamma}', \lambda_i') \left[V^{(0)}(\theta'_{23}, \phi', \psi', \delta')\right]^\dagger (1 - \tilde{\alpha'})^\dagger$$

$$= \text{Rep}(X)(1 - \tilde{\alpha})V^{(0)}(\theta_{23}, \phi, \psi, \delta)R^\dagger \bar{H}(\theta'_{23}, \phi', \psi', \delta'; \bar{\alpha}_{B\gamma}', \lambda_i')R \left[V^{(0)}(\theta_{23}, \phi, \psi, \delta)\right]^\dagger (1 - \tilde{\alpha})^\dagger \text{Rep}(X)^\dagger.$$  \hspace{1cm} (8.6)

Note that $R$ is the “untransformed” matrix. What is remarkable is that one can show by using the $H_{ij}$ transformation property in eq. (7.1) that

$$R^\dagger \bar{H}(\theta'_{23}, \phi', \psi', \delta'; \bar{\alpha}_{B\gamma}', \lambda_i')R = \bar{H}(\theta_{23}, \phi, \psi, \delta; \bar{\alpha}_{B\gamma}, \lambda_i)$$  \hspace{1cm} (8.7)

for all Symmetry X-DMP-UV symmetries where X= IA, IB, ···, IVB. The proof must be done for all the Symmetry X with the upper and lower signs of the solutions of the first condition.\footnote{A careful reader might have detected an extreme similarity with the treatment in ref. [20] for the helio-UV perturbation theory. But, we remark that despite the similarity at the equation level, the calculation needed for proof of eq. (8.7) differs in its figure and in volume.} The fact that eq. (8.7) holds implies that $H_{\text{Diag}}$ transforms under Symmetry X as

$$H_{\text{Diag}} \rightarrow \text{Rep}(X)H_{\text{Diag}}\text{Rep}(X)^\dagger.$$  \hspace{1cm} (8.8)

That is, $H_{\text{Diag}}$ is invariant apart from the rephasing factors $\text{Rep}(X)$ and $\text{Rep}(X)^\dagger$. Notice again that $\text{Rep}(X)$ is solely rooted in the $\nu$SM, see eq. (6.3), but also governs the UV part of the theory.

### 8.3 Including the second-order UV effect

Now let us include the second-order UV effect into our proof of invariance by turning on $\bar{H}_{\text{UV}}^{(2)} = -(b/2E)G^{(2)}$ in eq. (3.8). Recapitulating our nomenclature, once the $A^{(2)}$ matrix is given as in eq. (3.5), one can define the $G^{(2)}$ matrix as in eq. (3.9). Then, we can similarly define $H^{(2)}$ matrix as $G^{(2)} = DH^{(2)}D^\dagger$ as in eq. (3.11). If one wants to obtain the explicit forms of the $H^{(2)}_{ij}$ matrix elements, one can follow eq. (A.1) by replacing $A$ by $A^{(2)}$ in Appendix A.

We first show that the transformation properties of the $A^{(2)}$ matrix is identical with that of the $A$ matrix, the both defined in eq. (3.5). Notice that only the $\tilde{\alpha}$ parameters
transform in them. Using the $\tilde{\alpha}$ parameter transformations (6.6) and with the explicit expressions of the $A$ and $A^{(2)}$ matrices it is a straightforward to show that the transformation properties of the $A$ and $A^{(2)}$ matrices under Symmetry $X$ are the same. It means that the transformation properties of the $G^{(2)}$ matrix under Symmetry $X$, where $X=IA, IB, \cdots, IVB$, is the same as that of $G$ matrix, because $G = [V^{(0)}(\theta_{23}, \phi, \psi, \delta)]^\dagger AV^{(0)}(\theta_{23}, \phi, \psi, \delta)$ and $G^{(2)} = [V^{(0)}(\theta_{23}, \phi, \psi, \delta)]^\dagger A^{(2)} V^{(0)}(\theta_{23}, \phi, \psi, \delta)$, see eq. (3.9).

Our last step is to show that the key equation (8.7) for proof of invariance of the Hamiltonian is satisfied after the second order effect $-(b/2E)G^{(2)}$ is included. To prove eq. (8.7) we have used the $H_{ij}$ transformations in eq. (7.1). But, one can readily show that the above constructed $H_{ij}^{(2)}$ matrix elements transform under Symmetry $X$ in the same way as $H_{ij}$. Since inclusion of the second order UV term merely changes $H_{ij}$ to $H_{ij} - H_{ij}^{(2)}$ in eq. (8.7), and their transformation properties are the same, the invariance proof given in section 8.2 remains valid with inclusion of the second order UV effect.

To summarize, we have shown in this section that the flavor basis Hamiltonian $H_{\text{flavor}}$ (the both $H_{\nu\nu}$ and $H_{\text{flavor}}$) transforms as $H_{\text{flavor}} \rightarrow \text{Rep}(X) H_{\text{flavor}} \text{Rep}(X)^\dagger$ under Symmetry $X$. This establishes the property of Symmetry $X$ as the Hamiltonian symmetry which holds in all-orders in the DMP-UV perturbation theory in all the oscillation channels.

9 Summary and outlook

We have discussed the reparametrization symmetry of the 1-2 state exchange type in the DMP-UV perturbation theory, an extended version of the DMP to include non-unitarity with use of the $\alpha$ parametrization. This is the third successful application of Symmetry Finder (SF) method after the cases of $\nu$SM DMP [18] and the helio-perturbation [19] theories. We have found that all of the eight $\nu$SM DMP symmetries have UV (unitarity violation) extension, which entails Symmetry X-DMP-UV where $X=IA, IB, \cdots, IVB$. The symmetries are summarized in Table 1.

By showing the invariance of the flavor basis Hamiltonian under the transformation of Symmetry $X$ in Table 1, up to rephasing, we have assured that all the symmetries hold in all orders in the DMP-UV perturbation theory. They constitute the reparametrization symmetries uncovered for the first time in neutrino theory with non-unitarity. Later the other UV-extended symmetries, Symmetry X-helioP-UV where $X=IA, IB, \cdots, IVB$, duplicated with or without “f” ($s_{12}$ sign flip) labels joined the list [20]. Their presence in the UV extended theory strengthens our assumption that the reparametrization symmetry exists in any well-defined theory of neutrino evolution.

We recall that the effects of UV manifest in the two forms called the genuine UV (abbreviated as “UV”) part, and the unitary evolution (abbreviated as “EV”) part in the $V$ matrix. See ref. [57] for the concepts and section 4.3 for a brief account. With use of the $\alpha$ parametrization for non-unitary mixing matrix [30], the former “UV” and the latter “EV” parts produce the two independent constraints on the UV sector:

\[ \text{Rep}(\Pi)_{\text{helioP}} = \text{Rep}(IV), \text{ and } \text{Rep}(IV)_{\text{helioP}} = \text{Rep}(II). \]
The “UV” constraint, thanks to the key identity (6.3), results in the transformation property \( \tilde{\alpha} \rightarrow \text{Rep}(X) \tilde{\alpha} \text{Rep}(X) \dagger \), where \( \text{Rep}(X) \) denotes the rephasing matrix which is defined in eq. (6.4), the quantity introduced for the \( \nu \)SM DMP symmetry [18].

The “EV” constraint produces the transformation property eq. (7.1) of the \( H_{ij} \) elements defined in eq. (3.11), see Appendix A for the explicit expressions of \( H_{ij} \).

Despite the completely independent nature of the “UV” and “EV” constraints, the deduced \( \tilde{\alpha} \) and \( H_{ij} \) transformation properties are perfectly consistent with each other. This remarkable consistency testifies that the SF method works for the neutrino theory with non-unitarity.

Thus, with the added new symmetries in the DMP-UV theory, the experimentalists’ approach to the reparametrization symmetry in neutrino oscillation is in progress. Yet, great many questions are still left apart from the raison d’ être question raised in section 1, and the correctness of assumption of the “universal existence of symmetry”. For the latter, a counter example, if found, would stimulate the progress in our understanding of the symmetry. Generically, the typical “frequently asked” questions would be, for example:

- What is a possible utility of the symmetry, apart from the cross checking the formulas?
- Take the theory in which the symmetry exists, for example, the \( \nu \)SM DMP. How big is the symmetry in this theory?

While the both of them are not the easy questions to answer, let us try to address them.

### 9.1 Diagnostics of a low-energy theory of new physics

Possible capability of the symmetry for distinguishing the \( \nu \)SM and UV sectors of the theory is the starting point of our investigation of the symmetry in the DMP-UV theory. Why does such diagnostics capability important? Suppose that a UV-causing new physics at high energies has an effective description at low energy. Further suppose that the experiments start to detect something looks like “UV” effects. Obviously we want to make a guess from the low energy data the figure of the origin of the UV effects. To extract the answer of such question, we think, it would be of great help if we understand in which manner the UV effect is implemented into the low energy effective theory.

From our symmetry viewpoint we ask the question: How does the symmetry transformation affect the UV variables? Does the UV variables’ transformation have any visible correlations with the \( \nu \)SM variables’ ones? Or, possibly more concretely: Is it natural to expect that the UV \( \alpha \) parameters undergo a similar transformation when the \( \nu \)SM variables transform by “discrete rotation”? These are the question to investigate in which way the UV new physics is buried into the low energy theory.

Our result indicates that response of the UV \( \tilde{\alpha} \) parameters to the \( \nu \)SM variable transformations is not what we naively expected as above. We have found that the UV \( \tilde{\alpha} \) parameters transform as \( \tilde{\alpha} \rightarrow \text{Rep}(X) \tilde{\alpha} \text{Rep}(X) \dagger \), where \( \text{Rep}(X) \) is the constant diagonal matrix with the phase sensitive elements \( e^{\pm i\pi} \). That is, as far as the transformation

\[ \tilde{\alpha} \rightarrow \text{Rep}(X) \tilde{\alpha} \text{Rep}(X) \dagger, \]
property under the symmetry is concerned, the UV sector variables \( \tilde{\alpha} \) communicate with the \( \nu \) SM variables only via the phase sensitive quantity \( \text{Rep}(X) \). Unfortunately, the real meaning of this result eludes us at this moment.

### 9.2 The key identity and its possible topological nature

Probably, the most important observation made in this paper is the identity (6.3), which can be written with abbreviations as \( V^{(0)}(\Phi') R [V^{(0)}(\Phi)]^\dagger = \text{Rep}(X) \).\(^{15}\) It is often called as the “key identity” in this paper because (1) the above simple transformation property of the \( \tilde{\alpha} \) parameters is its direct consequence, (2) it renovates the Hamiltonian proof of the symmetry, and (3) it naturally leads to a conjecture for larger reparametrization symmetries as described in section 9.4 below.

However, the nature of the identity is not understood and is highly intriguing. While the left-hand side has full of the \( \nu \) SM variables dependences, the right-hand side is the constant diagonal matrix with the elements \( \pm 1 = e^{\pm i\pi} \). Such a coherent behavior of the sixteen (eight symmetries duplicated by the upper and lower signs) quantities is not thinkable without a particular reason. The only possibility which we are aware, to the best of our knowledge, is somehow \( V^{(0)}(\Phi') R [V^{(0)}(\Phi)]^\dagger \) has a topological nature. While its mathematical proof eludes us, since it is so natural, we suspect that a solid argument for backing up the topological origin of the identity could exist. Since the author presented an argument in favor of this possibility in analogy with the \( U(1) \) charge quantization around a vortex in ref. \[20\], we do not repeat it here.

### 9.3 The \( \nu \) SM-UV inter-sector communications through the phases

Here, we remark that we have another evidence for the \( \nu \) SM-UV inter-sector communications through the phases. It is the unsuppressed lepton KM phase \( \delta - \alpha \) parameters’ phase correlations investigated in detail in refs. \[53, 57, 64\]. It is entirely natural to expect existence of order-unity correlations between the \( \nu \) SM and UV phases from the viewpoint of unitarity polygon, a generalization of the unitarity triangle, in larger unitary theory \[32, 33, 57\]. A possible connection between the two phase-related features of the \( \nu \) SM-UV communications is intriguing and is awaiting further investigation.

Notice that the interactions between the \( \nu \) SM and UV variables always occur through the first-order term of the Hamiltonian, the last term in eq. (3.12). Certainly it opens the way of communication between the two sectors in the way which manifests at low energies, as originally dictated by the high energy theory but brought to low scales when the high-energy sector is integrated out \[29\]. Nonetheless, it is the first-order suppressed effect, and the physical picture behind it is not transparent even if one solves exactly the system with non-unitarity, as done in ref. \[33\]. Here, we take a different approach to the picture of inter-sector communications using the symmetry probe. Whether we have succeeded to dig out the crucial feature of such communications, or it is just one aspect of the many-face “polyhedron” remains to be seen.

---

\(^{15}\)\( V^{(0)}(\Phi) = U_{23}(\theta_{23}) U_{13}(\phi) U_{12}(\psi, \delta) \) with \( \Phi \) (\( \Phi' \)) being the collective notation for the arguments before (after) the transformation. \( R \) denotes the generalized 1-2 exchange matrix. See section 9.2.
This shows our current status of “diagnostics” understanding of the theory with the UV effect by using the reparametrization symmetry. Notwithstanding the speculated topological nature of the identity is true or not, the $\nu$SM-UV inter-sector communication through the phases is a natural and real outcome from our study of reparametrization symmetry in this paper.

9.4 How big is the symmetry?

How big is the reparametrization symmetry for a given theory? To our current understanding, the answer depends on which theories we talk about, and relatedly on which types of the state exchange examined. For an alternative attempt aiming at finding larger symmetries, see ref. [63]. Among our reparametrization symmetries à la SF, there exist the eight 1-2 exchange symmetries in the DMP [18], DMP-UV theories (this paper), and in the SRP (solar-resonance perturbation) theory [20]. As far as the sixteen 1-3 exchange symmetries are concerned they exist, to date, only in the helio- and helio-UV perturbation theories [20]. With the current technology we cannot make arbitrary choice of the state exchange type in a given theory.

Within the symmetries identified by the SF framework, they all possess the structure $X = I, II, III, IV$, apart from the doublings due to the types A or B, or possibly with or without “f” ($s_{12}$ sign flip) indices. We suspect that the symmetry structure $X = I, II$, III, IV, exhausts the candidate list for symmetries for the following reasoning: We have to have the rephasing matrix Rep($X$) for the Hamiltonian proof of Symmetry $X$. Rep($X$) must be a diagonal matrix because otherwise it alters the physical observables by changing the flavor labels. As far as the real diagonal matrices with the elements $\pm 1$ are concerned, our Rep($X$) with $X = I, II, III, IV$ in Table 2 constitute all the possible choices. The remaining possibility is the case of complex diagonal matrix Rep($X$), whose existence, however, eludes us at this moment.

We want to call the readers’ attention on a new possibility that might be suggested from our result. The identity eq. (6.3) and the transformation property of $H_{\text{diag}}$ in eq. (8.6) with the equality (8.7) needed for invariance proof, do not refer at least formally, which states are exchanged. Notice that they are all the key elements for the Hamiltonian proof of the DMP and DMP-UV symmetries. Therefore, we suspect that even more generic state exchange symmetry exists which might be extended to $S_3$, for example, for the three-neutrino system. If it is the case, the formulation would become a more abstract one because the task is not transparent, at best, given the difficulty of formulating 1-3 exchange symmetry with an explicit parametrization of the $U$ matrix [19]. At the same time the key identity eq. (6.3) must be elevated to match to this generalization with the suitably generalized $R$ matrix. The topological nature of the identity might show up more naturally in this extended setting. The author believes that this possibility is worth to explore.

Acknowledgments

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\section{H matrix elements in first order: Summary}

In ref. \cite{53}, we have defined the $G$ matrix, the same one as in eq. (3.9), in a two-step form

\begin{align}
G &= U_{12}(\psi, \delta) K U_{12}(\psi, \delta), \\
K &= U_{13}^\dagger(\phi) U_{23}^\dagger(\theta_{23}) A U_{23}(\theta_{23}) U_{13}(\phi), \tag{A.1}
\end{align}

where $A$ is defined in eq. (3.5). We follow the same style in this paper, except that we use the $H$ matrix elements to display the $G$ matrix elements, $G = DHD^\dagger$, $D = \text{diag}(e^{i\delta}, 1, e^{i\delta})$ as defined in eq. (3.11).

The explicit expressions of the $H$ matrix elements through the $K$ matrix elements are given by

\begin{align*}
H_{11} &= c_\psi^2 K_{11} + s_\psi^2 K_{22} - c_\psi s_\psi \left( e^{i\delta} K_{12} + e^{-i\delta} K_{12} \right), \\
H_{12} &= \left[ c_\psi s_\psi (K_{11} - K_{22}) + \left( c_\psi^2 e^{-i\delta} K_{12} - s_\psi^2 e^{i\delta} K_{21} \right) \right], \\
H_{13} &= (c_\psi K_{13} - s_\psi e^{i\delta} K_{23}), \\
H_{21} &= \left[ c_\psi s_\psi (K_{11} - K_{22}) + \left( c_\psi^2 e^{i\delta} K_{21} - s_\psi^2 e^{-i\delta} K_{12} \right) \right], \\
H_{22} &= s_\psi^2 K_{11} + c_\psi^2 K_{22} + c_\psi s_\psi \left( e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right), \\
H_{23} &= (s_\psi K_{13} + c_\psi e^{i\delta} K_{23}), \\
H_{31} &= (c_\psi K_{31} - s_\psi e^{-i\delta} K_{32}), \\
H_{32} &= (s_\psi K_{31} + c_\psi e^{-i\delta} K_{32}), \\
H_{33} &= K_{33}. \tag{A.2}
\end{align*}

The $K$ matrix elements are given by

\begin{align*}
K_{11} &= 2c_\phi^2 \tilde{\alpha}_{ee} \left( 1 - \frac{a}{b} \right) + 2s_\phi^2 \left[ s_{23}^2 \tilde{\alpha}_{\mu\mu} + c_{23}^2 \tilde{\alpha}_{\tau\tau} + c_{23}s_{23} \text{Re}(\tilde{\alpha}_{\tau\mu}) \right], \\
&\quad - 2c_\phi s_\phi \text{Re}(s_{23}^2 \tilde{\alpha}_{\mu\mu} + c_{23} \tilde{\alpha}_{\tau\tau}), \\
K_{12} &= c_\phi \left( c_{23} \tilde{\alpha}_{\mu e}^* - s_{23} \tilde{\alpha}_{\tau e}^* \right) - s_\phi \left[ 2c_{23}s_{23}(\tilde{\alpha}_{\mu\mu} - \tilde{\alpha}_{\tau\tau}) + c_{23}^2 \tilde{\alpha}_{\tau\mu} - s_{23}^2 \tilde{\alpha}_{\tau\mu}^* \right] = (K_{21})^*, \\
K_{13} &= 2c_\phi s_\phi \left[ \tilde{\alpha}_{ee} \left( 1 - \frac{a}{b} \right) - (s_{23}^2 \tilde{\alpha}_{\mu\mu} + c_{23} \tilde{\alpha}_{\tau\tau}) \right] \\
&\quad + c_\phi^2 \left( s_{23} \tilde{\alpha}_{\mu e}^* + c_{23} \tilde{\alpha}_{\tau e}^* \right) - s_\phi^2 \left( s_{23} \tilde{\alpha}_{\mu e} + c_{23} \tilde{\alpha}_{\tau e} \right) - 2c_{23}s_{23} c_\phi s_\phi \text{Re}(\tilde{\alpha}_{\tau\mu}) = (K_{31})^*, \\
K_{22} &= 2 \left[ c_{23}^2 \tilde{\alpha}_{\mu\mu} + s_{23}^2 \tilde{\alpha}_{\tau\tau} - c_{23}s_{23} \text{Re}(\tilde{\alpha}_{\tau\mu}) \right], \\
K_{23} &= s_\phi \left( c_{23} \tilde{\alpha}_{\mu e} - s_{23} \tilde{\alpha}_{\tau e} \right) + c_\phi \left[ 2c_{23}s_{23}(\tilde{\alpha}_{\mu\mu} - \tilde{\alpha}_{\tau\tau}) + c_{23}^2 \tilde{\alpha}_{\tau\mu} - s_{23}^2 \tilde{\alpha}_{\tau\mu}^* \right] = (K_{32})^*, \\
K_{33} &= 2s_\phi^2 \tilde{\alpha}_{ee} \left( 1 - \frac{a}{b} \right) + 2c_\phi^2 \left[ s_{23}^2 \tilde{\alpha}_{\mu\mu} + c_{23}^2 \tilde{\alpha}_{\tau\tau} + c_{23}s_{23} \text{Re}(\tilde{\alpha}_{\tau\mu}) \right] \\
&\quad + 2c_\phi s_\phi \text{Re}(s_{23}^2 \tilde{\alpha}_{\mu\mu} + c_{23} \tilde{\alpha}_{\tau\tau}). \tag{A.3}
\end{align*}

Notice that the $K$ matrix elements are free from the $\nu$SM vacuum mixing angles apart from $\theta_{23}$ and contain no $\delta$ in the SOL convention. $K_{ji} = K_{ij}^*$ and $K_{ii}$ are real.
### B First order correction in the wave function in quantum mechanics

In ref. [24], the numerator of eq. (4.6) is written as $\bar{H}^{(1)}_{ji}$, not $(\bar{H}^{(1)}_{ji})^*$ which is equivalent to eq. (4.6). The complex conjugate can be disregarded in ref. [24] because the $\bar{H}^{(1)}$ elements are real due to the usage of the ATM convention of the $U$ matrix. But, in our case and in general it must be kept. Though this point must be obvious from eq. (26) in ref. [61], we want to write a pedagogical note here not to leave an ambiguity on this point.

The non-degenerate stationary state perturbation theory formulated for the ket state reads

$$\lvert \nu_i \rangle^{(1)} = \lvert \nu_i \rangle^{(0)} + \sum_{j \neq i} \frac{(H_1)_{ij}}{E_i^{(0)} - E_j^{(0)}} \lvert \nu_j \rangle^{(0)},$$  

(B.1)

with $E_i^{(0)}$ being the zeroth-order eigenvalues and $H_1$ perturbed Hamiltonian. In terms of bra state it can be written as

$$\langle \nu_i \rangle^{(1)} = \langle \nu_i \rangle^{(0)} + \sum_{j \neq i} \langle \nu_j \rangle^{(0)} \left( H_1^\dagger \right)_{ji} \frac{1}{E_i^{(0)} - E_j^{(0)}}$$

$$= \langle \nu_i \rangle^{(0)} + \sum_{j \neq i} \langle \nu_j \rangle^{(0)} \frac{(H_1)_{ij}}{E_i^{(0)} - E_j^{(0)}}.$$  

(B.2)

Since the wave function we have used in this paper corresponds to the bra state, we obtain eq. (4.6). In fact, if the complex conjugation is missed, we obtain a wrong expression of $W_{\nu_{e}^{SM}}$ in eq. (4.10) which is different from the correct one by $\delta \rightarrow -\delta$. Notice that our expression in eq. (4.10) is consistent with the one in ref. [18], which is obtained by the rephasing from the ATM convention result.

### C The probabilities $P(\nu_\mu \rightarrow \nu_\nu)^{(1)}_{\text{EV}}$ and $P(\nu_\mu \rightarrow \nu_\nu)^{(1)}_{\text{UV}}$

#### C.1 Unitary evolution part $P(\nu_\mu \rightarrow \nu_\nu)^{(1)}_{\text{EV}}$

The unitary evolution part of the oscillation probability $P(\nu_\mu \rightarrow \nu_\nu)^{(1)}_{\text{EV}}$ is given in a form separated to the T-even and T-odd parts: $P(\nu_\mu \rightarrow \nu_\nu)^{(1)}_{\text{EV}} = P(\nu_\mu \rightarrow \nu_\nu)^{(1)}_{\text{EV, T-even}} + P(\nu_\mu \rightarrow \nu_\nu)^{(1)}_{\text{EV, T-odd}}.$
\[ P(\nu_\mu \rightarrow \nu_e)_{\text{EV}}^{(1)} \mid T \text{-even} \]
\[
= \left[ 2c_\phi^2 \left( (c_{23}^2 - s_{23}^2 s_\phi^2) \sin 4\psi + \sin 2\theta_{23}s_\phi \cos \delta \cos 4\psi \right) \text{Re} \left( H_{12} \right) - 2\sin 2\theta_{23}c_\phi^3 s_\phi \cos 2\psi \sin \text{Im} \left( H_{12} \right) \right] \times \left( \frac{b}{\lambda_2 - \lambda_1} \right)^2 \frac{\sin^2 \left( \lambda_2 - \lambda_1 \right)x}{4E} \\
+ [-4 \left\{ c_\phi s_\phi c_\phi^2 s_\psi (c_{23}^2 + s_{23}^2 \cos 2\phi) \text{Re} \left( H_{23} \right) + c_{23} s_{23} c_\phi c_\psi \cos \delta \left( s_\phi^2 c_\psi^2 + s_\psi^2 \cos 2\phi \right) \text{Re} \left( H_{23} \right) \right] (-1 + c_\phi c_\psi) \text{Im} \left( H_{23} \right) \right\} \left( \frac{b}{\lambda_3 - \lambda_2} \right)^2 \frac{\sin^2 \left( \lambda_3 - \lambda_2 \right)x}{4E} \\
+ 4 \left\{ c_\phi s_\phi c_\phi s_\psi \left[ c_{23}^2 c_\phi^2 - s_{23}^2 \cos 2\phi (1 + s_\psi^2) \right] - c_{23} s_{23} c_\phi s_\psi \cos \delta \left( s_\phi^2 (1 + s_\psi^2) - s_\psi^2 \cos 2\phi \right) \text{Re} \left( H_{23} \right) \right] \left( \frac{b}{\lambda_3 - \lambda_1} \right)^2 \frac{\sin^2 \left( \lambda_3 - \lambda_1 \right)x}{4E} \\
+ 4 \left\{ c_\phi s_\phi c_\phi s_\psi \left[ c_{23}^2 s_\psi^2 - s_{23}^2 \cos 2\phi (1 + c_\psi^2) \right] + c_{23} s_{23} c_\phi s_\psi \cos \delta \left[ s_\phi^2 (1 + c_\psi^2) - c_\psi^2 \cos 2\phi \right] \text{Re} \left( H_{23} \right) \right] \left( \frac{b}{\lambda_3 - \lambda_2} \right)^2 \frac{\sin^2 \left( \lambda_3 - \lambda_2 \right)x}{4E} \\
+ 4 \left\{ - (c_{23}^2 + s_{23}^2 \cos 2\phi) c_\phi s_\phi c_\psi s_\psi^2 + c_{23} s_{23} c_\phi s_\psi \cos \delta \left( s_\phi^2 s_\psi^2 + \cos 2\phi c_\psi^2 \right) \text{Re} \left( H_{13} \right) \right] \left( \frac{b}{\lambda_3 - \lambda_1} \right)^2 \frac{\sin^2 \left( \lambda_3 - \lambda_1 \right)x}{4E} \\
+ 4 \left\{ (c_{23}^2 + s_{23}^2 \cos 2\phi) c_\phi s_\phi c_\psi s_\psi + c_{23} s_{23} c_\phi c_\psi \cos \delta \left( s_\phi^2 c_\psi^2 + s_\psi^2 \cos 2\phi \right) \text{Re} \left( H_{23} \right) \right] \left( \frac{b}{\lambda_3 - \lambda_2} \right)^2 \frac{\sin^2 \left( \lambda_3 - \lambda_2 \right)x}{4E} \right]. \quad (C.1)
\[ P(\nu_\mu \to \nu_e)^{(1)}_{\text{T-odd}} = 8 \left[ -c_{23}s_{23}c_{\phi}^2 s_\phi \left( \cos \delta \text{Im} (H_{12}) + \cos 2\psi \sin \delta \text{Re} (H_{12}) \right) \left( \frac{b}{\lambda_2 - \lambda_1} \right) 
\right. \\
+ c_{\phi} c_\psi \left\{ - \cos 2\theta_{23} s_\theta c_\psi s_\phi + c_{23}s_{23} \cos \delta \left( s_\psi^2 - s_\phi^2 c_\psi^2 \right) \right\} \text{Im} (H_{23}) \\
+ \left. c_{23}s_{23} \sin \delta \left( s_\phi^2 - c_\phi^2 s_\psi^2 \right) \text{Re} (H_{23}) \right\} \left( \frac{b}{\lambda_3 - \lambda_2} \right) \\
+ c_{\phi} s_\psi \left\{ \cos 2\theta_{23} s_\theta c_\psi s_\phi + c_{23}s_{23} \cos \delta \left( c_\psi^2 - s_\phi^2 s_\psi^2 \right) \right\} \text{Im} (H_{13}) \\
+ c_{23}s_{23} \sin \delta \left( s_\phi^2 - c_\phi^2 c_\psi^2 \right) \text{Re} (H_{13}) \left( \frac{b}{\lambda_3 - \lambda_1} \right) \\
\times \sin \left( \frac{\lambda_2 - \lambda_1}{4E} \right) \sin \left( \frac{\lambda_3 - \lambda_1}{4E} \right) \sin \left( \frac{\lambda_3 - \lambda_2}{4E} \right). \tag{C.2} \]

C.2 Genuine non-unitary part \( P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}} \)

The genuine non-unitary part of the oscillation probability reads

\[ P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}} = -2 (\tilde{\alpha}_{ee} + \tilde{\alpha}_{\mu\mu}) \\
\times \left[ s_{23}^2 \sin^2 2\phi \left\{ s_\phi^2 \sin^2 \left( \frac{\lambda_3 - \lambda_2}{4E} \right) + \frac{1}{4E} \right\} + c_\phi^2 \sin^2 2\psi \left( c_{23}^2 - s_{23}^2 s_\phi^2 \right) \sin^2 \left( \frac{\lambda_2 - \lambda_1}{4E} \right) \\
+ 4J_{mr} \cos \delta \left\{ -\sin^2 \left( \frac{\lambda_3 - \lambda_2}{4E} \right) + \sin^2 \left( \frac{\lambda_3 - \lambda_1}{4E} \right) + \cos 2\psi \sin^2 \left( \frac{\lambda_2 - \lambda_1}{4E} \right) \right\} \\
+ 8J_{mr} \sin \delta \sin \left( \frac{\lambda_2 - \lambda_1}{4E} \right) \sin \left( \frac{\lambda_3 - \lambda_1}{4E} \right) \sin \left( \frac{\lambda_3 - \lambda_2}{4E} \right) \right\} \left[ s_\phi^2 \sin \left( \frac{\lambda_3 - \lambda_2}{2E} \right) - \sin \left( \frac{\lambda_3 - \lambda_1}{2E} \right) \right] - c_\phi^2 \sin \left( \frac{\lambda_2 - \lambda_1}{2E} \right) \\
+ 2c_{23}c_\phi \sin 2\psi \text{Re} \left( \tilde{\alpha}_{\mu\mu} e^{i\delta} \right) \left\{ s_\phi^2 \sin^2 \left( \frac{\lambda_3 - \lambda_2}{4E} \right) - \sin^2 \left( \frac{\lambda_3 - \lambda_1}{4E} \right) \right\} + c_\phi^2 \cos 2\psi \sin^2 \left( \frac{\lambda_2 - \lambda_1}{4E} \right) \\
+ c_{23}c_\phi \sin 2\psi \text{Im} \left( \tilde{\alpha}_{\mu\mu} e^{i\delta} \right) \left\{ s_\phi^2 \sin \left( \frac{\lambda_3 - \lambda_2}{2E} \right) - \sin \left( \frac{\lambda_3 - \lambda_1}{2E} \right) \right\} - c_\phi^2 \sin \left( \frac{\lambda_2 - \lambda_1}{2E} \right) \\
+ s_{23} \sin 2\phi \left[ \cos \delta \text{Re} \left( \tilde{\alpha}_{\mu\mu} e^{i\delta} \right) + \sin \delta \text{Im} \left( \tilde{\alpha}_{\mu\mu} e^{i\delta} \right) \right] \right\} \left[ s_\phi^2 \sin \left( \frac{\lambda_3 - \lambda_2}{4E} \right) + c_\phi^2 \sin \left( \frac{\lambda_3 - \lambda_1}{4E} \right) \right] - c_\phi^2 \sin^2 2\psi \sin^2 \left( \frac{\lambda_2 - \lambda_1}{4E} \right) \\
+ s_{23} \sin 2\phi \left[ \sin \delta \text{Re} \left( \tilde{\alpha}_{\mu\mu} e^{i\delta} \right) + \cos \delta \text{Im} \left( \tilde{\alpha}_{\mu\mu} e^{i\delta} \right) \right] \left\{ s_\phi^2 \sin \left( \frac{\lambda_3 - \lambda_2}{2E} \right) + c_\phi^2 \sin \left( \frac{\lambda_3 - \lambda_1}{2E} \right) \right\}, \tag{C.3} \]

where \( J_{mr} \equiv c_{23}s_{23}c_\phi^2 s_\phi c_\psi s_\phi \) is the Jarlskog factor \([67]\) in matter.

D (bx)/2E term in \( P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}} \)

The expressions of the probabilities \( P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}} \) and \( P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}} \) given in Appendix C are based on the renormalized treatment of the eigenvalues \( \lambda_i = \lambda_i^{\text{SM}} + bH_a \) in eq. (3.10). To compare the present expression of \( P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}} \) to the one from the un-renormalized treatment of ref. [53], we calculate the first-order corrections that are produced when we expand the eigenvalues \( \lambda_i \) in \( P(\nu_\mu \to \nu_e)^{(0)}_{\nu \text{SM}} \) given for example in ref. [65]. Notice that this
is the only source of the \((bx)/2E\) term in first order in perturbation. In the other parts of the theory the eigenvalue expansion merely induces the second order terms.

We briefly describe the computation for it with use of the \(S\) matrix method because it is easier. We expand the eigenvalues to first order

\[
e^{-i\frac{\lambda_i}{2E}x} = e^{-ih_{ix}x} + \left(-i\frac{bx}{2E}\right) H_{ii} e^{-ih_{ix}x}, \tag{D.1}\]

where we have defined

\[
h_i \equiv \frac{\lambda_i^{SM}}{2E} \quad (i = 1, 2, 3).
\]

Then, the flavor basis \(S\) matrix element can be expanded to first order as

\[
S_{\epsilon\mu}^{(0)}(\lambda_i) = S_{\epsilon\mu}^{(0)}(\lambda_i^{SM}) + S_{\epsilon\mu}^{(1)}(H_{ii}), \tag{D.3}\]

where

\[
S_{\epsilon\mu}^{(0)}(\lambda_i^{SM}) = c_{23}c_\phi e^{i\delta}c_\psi s_\psi \left( e^{-ih_2x} - e^{-ih_1x} \right) + s_{23}c_\phi s_\psi \left[ e^{-ih_3x} - \left( c_\psi^2 e^{-ih_1x} + s_\psi^2 e^{-ih_2x} \right) \right],
\]

\[
S_{\epsilon\mu}^{(1)}(H_{ii}) = \left(-i\frac{bx}{2E}\right) \left\{ c_{23}c_\phi e^{i\delta}c_\psi s_\psi \left( H_{22} e^{-ih_2x} - H_{11} e^{-ih_1x} \right) \right.
\]

\[
+ \left. s_{23}c_\phi s_\psi \left( H_{33} e^{-ih_3x} - c_\psi^2 H_{11} e^{-ih_1x} - s_\psi^2 H_{22} e^{-ih_2x} \right) \right\}. \tag{D.4}\]

The interference between \(S_{\epsilon\mu}^{(0)}(\lambda_i^{SM})\) and \(S_{\epsilon\mu}^{(1)}(H_{ii})\) produces the required first-order correction, \((bx)/2E\) term in \(P(\nu_\mu \to \nu_\epsilon)_{\text{EV}}^{(1)}\), as

\[
2\text{Re} \left[ \left\{ S_{\epsilon\mu}^{(0)}(\lambda_i^{SM}) \right\}^* S_{\epsilon\mu}^{(1)}(H_{ii}) \right]
\]

\[
= 2 \left(\frac{bx}{2E}\right) \left\{ (H_{22} - H_{11}) \right.
\]

\[
\times \left[ 2J_{mr} \sin \delta \sin^2 \left( \frac{h_2 - h_1}{2} \right) \right. \left. + \left( c_\psi^2 c_\phi^2 s_\psi^2 c_{23} - s_{23}^2 c_\phi^2 \right) \right.
\]

\[
+ \left. (H_{33} - H_{22}) \right. \left[ 2J_{mr} \sin \delta \sin^2 \left( \frac{h_3 - h_2}{2} \right) \right. \left. + \left( s_{23}^2 c_\phi^2 s_\psi^2 - J_{mr} \cos \delta \right) \right.
\]

\[
+ \left. (H_{33} - H_{11}) \right. \left[ -2J_{mr} \sin \delta \sin^2 \left( \frac{h_3 - h_1}{2} \right) \right.
\]

\[
+ \left. \left( s_{23}^2 c_\phi^2 s_\psi^2 + J_{mr} \cos \delta \right) \sin(h_3 - h_1)x \right\}. \tag{D.5}\]

Using the expression of \(H_{ii}\) in terms of the \(K_{ij}\) variables as given in Appendix A, it can be shown that eq. (D.5) precisely reproduces eq. (4.9) in ref. [53].

### E An explicit proof of the identity eq. (6.3) for Symmetry IV

It is easy to prove the first equality in eq. (6.3), \(C[12] = \text{Rep}(X)\), by explicit calculation of \(C[12]\) for all the eight DMP symmetries with use of the values of the parameters \(\alpha, \beta\) etc.
for each symmetry, which have the upper and lower signs as given in Table 2. To show how the second equality holds, we examine the cases of symmetries IVA and IVB and compute \([V'RV^\dagger]\). Inserting \(C[12]=\text{Rep}(IV)\) in eq. (6.2) we obtain

\[
V^{(0)}(\theta'_{23}, \phi', \psi', \delta')R \left[ V^{(0)}(\theta_{23}, \phi, \psi, \delta) \right]^\dagger \\
= \begin{bmatrix} 1 & 0 & 0 & \ c'_\phi & 0 & s'_\phi \\ 0 & c'_{23} & s'_{23} & 0 & 1 & 0 \\ 0 & -s'_{23} & c'_{23} & -s'_\phi & 0 & c'_\phi \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & \ c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & s_\phi & 0 & c_\phi & 0 & s_{23} & c_{23} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

(E.1)

thanks to the properties that

\[
\begin{bmatrix} c'_\phi & 0 & s'_\phi \\ 0 & 1 & 0 \\ -s'_\phi & 0 & c'_\phi \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c'_{23} & s'_{23} \\ 0 & -s'_{23} & c'_{23} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

(E.2)

Notice that \(s'_\phi = -s_\phi\) and \(s'_{23} = -s_{23}\) in Symmetry IVA and IVB. What happens is that when \(\text{Rep}(IV)\) move to the left to get out to the front, it remedies the transformed parameters into the un-transformed parameters through passage, which occurs for both \(s_\phi\) and \(s_{23}\) in Symmetry IV. Similarly, one can prove the second equality in eq. (6.3) for all the remaining symmetries \(X=I, II, \text{and } III\). For Symmetry I, \(\text{Rep}(I) = 1\), and no transformation on \(s_{23}\) and \(s_\phi\) is needed. For Symmetry II (III) the “sign remedy” occurs only for \(s_{23} (s_\phi)\).

References

[1] S. Coleman, “Aspects of Symmetry: Selected Erice Lectures,”
https://doi.org/10.1017/CBO9780511565045

[2] M. Ikeda, S. Ogawa and Y. Ohnuki, “A Possible Symmetry in Sakata’s Model for Bosons-Baryons System,” Prog. Theor. Phys. 22 (1959), 715-724 doi:10.1143/PTP.22.715

[3] M. Gell-Mann, “Symmetries of baryons and mesons,” Phys. Rev. 125 (1962), 1067-1084 doi:10.1103/PhysRev.125.1067

[4] S. Okubo, “Note on unitary symmetry in strong interactions,” Prog. Theor. Phys. 27 (1962), 949-966 doi:10.1143/PTP.27.949

[5] G. L. Fogli, E. Lisi, D. Montanino and G. Scioscia, “Three flavor atmospheric neutrino anomaly,” Phys. Rev. D 55 (1997), 4385-4404 doi:10.1103/PhysRevD.55.4385 [arXiv:hep-ph/9607251 [hep-ph]].

[6] G. L. Fogli, E. Lisi and A. Palazzo, “Quasi energy independent solar neutrino transitions,” Phys. Rev. D 65 (2002), 073019 doi:10.1103/PhysRevD.65.073019 [arXiv:hep-ph/0105080 [hep-ph]].

[7] A. de Gouvea, A. Friedland and H. Murayama, “The Dark side of the solar neutrino parameter space,” Phys. Lett. B 490 (2000), 125-130 doi:10.1016/S0370-2693(00)00989-8 [arXiv:hep-ph/0002064 [hep-ph]].
[8] G. Altarelli and F. Feruglio, “Discrete Flavor Symmetries and Models of Neutrino Mixing,” Rev. Mod. Phys. 82 (2010), 2701-2729 doi:10.1103/RevModPhys.82.2701 [arXiv:1002.0211 [hep-ph]].

[9] G. L. Fogli and E. Lisi, “Tests of three flavor mixing in long baseline neutrino oscillation experiments,” Phys. Rev. D 54 (1996), 3667-3670 doi:10.1103/PhysRevD.54.3667 [arXiv:hep-ph/9604415 [hep-ph]].

[10] H. Minakata and H. Nunokawa, “Exploring neutrino mixing with low-energy superbeams,” JHEP 10 (2001), 001 doi:10.1088/1126-6708/2001/10/001 [arXiv:hep-ph/0108085 [hep-ph]].

[11] H. Minakata and S. Uchinami, “Parameter Degeneracy in Neutrino Oscillation – Solution Network and Structural Overview –,” JHEP 04 (2010), 111 doi:10.1007/JHEP04(2010)111 [arXiv:1001.4219 [hep-ph]].

[12] P. Coloma and T. Schwetz, “Generalized mass ordering degeneracy in neutrino oscillation experiments,” Phys. Rev. D 94 (2016) no.5, 055005 [erratum: Phys. Rev. D 95 (2017) no.7, 079903] doi:10.1103/PhysRevD.94.055005 [arXiv:1604.05772 [hep-ph]].

[13] J. Burguet-Castell, M. B. Gavela, J. J. Gomez-Cadenas, P. Hernandez and O. Mena, “On the Measurement of leptonic CP violation,” Nucl. Phys. B 608 (2001), 301-318 doi:10.1016/S0550-3213(01)00248-6 [arXiv:hep-ph/0103258 [hep-ph]].

[14] V. Barger, D. Marfatia and K. Whisnant, “Breaking eight fold degeneracies in neutrino CP violation, mixing, and mass hierarchy,” Phys. Rev. D 65 (2002), 073023 doi:10.1103/PhysRevD.65.073023 [arXiv:hep-ph/0112119 [hep-ph]].

[15] H. Minakata, H. Nunokawa and S. J. Parke, “Parameter Degeneracies in Neutrino Oscillation Measurement of Leptonic CP and T Violation,” Phys. Rev. D 66 (2002), 093012 doi:10.1103/PhysRevD.66.093012 [arXiv:hep-ph/0208163 [hep-ph]].

[16] A. de Gouvea and J. Jenkins, “The Physical Range of Majorana Neutrino Mixing Parameters,” Phys. Rev. D 78 (2008), 053003 doi:10.1103/PhysRevD.78.053003 [arXiv:0804.3627 [hep-ph]].

[17] S. Zhou, “Symmetric formulation of neutrino oscillations in matter and its intrinsic connection to renormalization-group equations,” J. Phys. G 44 (2017) no.4, 044006 doi:10.1088/1361-6471/aa5f7d [arXiv:1612.03537 [hep-ph]].

[18] H. Minakata, “Symmetry finder: A method for hunting symmetry in neutrino oscillation,” Phys. Rev. D 104 (2021) no.7, 075024 doi:10.1103/PhysRevD.104.075024 [arXiv:2106.11472 [hep-ph]].

[19] H. Minakata, “Symmetry Finder applied to the 1–3 mass eigenstate exchange symmetry,” Eur. Phys. J. C 81 (2021) no.11, 1021 doi:10.1140/epjc/s10052-021-09810-5 [arXiv:2107.12086 [hep-ph]].

[20] H. Minakata, “Symmetry in neutrino oscillation in matter: New picture and the $\nu$SM - non-unitarity interplay,” [arXiv:2210.09453 [hep-ph]].

[21] S. Parke, “Theoretical Aspects of the Quantum Neutrino,” doi:10.1142/9789811207402_0008 [arXiv:1801.09643 [hep-ph]].

[22] P. B. Denton, H. Minakata and S. J. Parke, “Compact Perturbative Expressions For Neutrino Oscillations in Matter,” JHEP 06 (2016), 051 doi:10.1007/JHEP06(2016)051 [arXiv:1604.08167 [hep-ph]].
[23] I. Martinez-Soler and H. Minakata, “Perturbing Neutrino Oscillations Around the Solar Resonance,” PTEP 2019 (2019) no.7, 073B07 doi:10.1093/ptep/ptz067 [arXiv:1904.07853 [hep-ph]].

[24] H. Minakata and S. J. Parke, “Simple and Compact Expressions for Neutrino Oscillation Probabilities in Matter,” JHEP 01 (2016), 180 doi:10.1007/JHEP01(2016)180 [arXiv:1505.01826 [hep-ph]].

[25] L. Wolfenstein, “Neutrino Oscillations in Matter,” Phys. Rev. D 17 (1978), 2369-2374 doi:10.1103/PhysRevD.17.2369

[26] S. P. Mikheyev and A. Y. Smirnov, “Resonance Amplification of Oscillations in Matter and Spectroscopy of Solar Neutrinos,” Sov. J. Nucl. Phys. 42 (1985), 913-917

[27] V. D. Barger, K. Whisnant, S. Pakvasa and R. J. N. Phillips, “Matter Effects on Three-Neutrino Oscillations,” Phys. Rev. D 22 (1980), 2718 doi:10.1103/PhysRevD.22.2718

[28] A. Y. Smirnov, “Solar neutrinos: Oscillations or No-oscillations?,” [arXiv:1609.02386 [hep-ph]].

[29] S. Antusch, C. Biggio, E. Fernandez-Martinez, M. B. Gavela and J. Lopez-Pavon, “Unitarity of the Leptonic Mixing Matrix,” JHEP 10 (2006), 084 doi:10.1088/1126-6708/2006/10/084 [arXiv:hep-ph/0607020 [hep-ph]].

[30] F. J. Escrihuela, D. V. Forero, O. G. Miranda, M. Tortola and J. W. F. Valle, “On the description of nonunitary neutrino mixing,” Phys. Rev. D 92 (2015) no.5, 053009 [erratum: Phys. Rev. D 93 (2016) no.11, 119905] doi:10.1103/PhysRevD.92.053009 [arXiv:1503.08879 [hep-ph]].

[31] M. Blennow, P. Coloma, E. Fernandez-Martinez, J. Hernandez-Garcia and J. Lopez-Pavon, “Non-Unitarity, sterile neutrinos, and Non-Standard neutrino Interactions,” JHEP 04 (2017), 153 doi:10.1007/JHEP04(2017)153 [arXiv:1609.08637 [hep-ph]].

[32] C. S. Fong, H. Minakata and H. Nunokawa, “A framework for testing leptonic unitarity by neutrino oscillation experiments,” JHEP 02 (2017), 114 doi:10.1007/JHEP02(2017)114 [arXiv:1609.08623 [hep-ph]].

[33] C. S. Fong, H. Minakata and H. Nunokawa, “Non-unitary evolution of neutrinos in matter and the leptonic unitarity test,” JHEP 02 (2019), 015 doi:10.1007/JHEP02(2019)015 [arXiv:1712.02798 [hep-ph]].

[34] E. Fernandez-Martinez, M. B. Gavela, J. Lopez-Pavon and O. Yasuda, “CP-violation from non-unitary leptonic mixing,” Phys. Lett. B 649 (2007), 427-435 doi:10.1016/j.physletb.2007.03.069 [arXiv:hep-ph/0703098 [hep-ph]].

[35] S. Goswami and T. Ota, “Testing non-unitarity of neutrino mixing matrices at neutrino factories,” Phys. Rev. D 78 (2008), 033012 doi:10.1103/PhysRevD.78.033012 [arXiv:0802.1434 [hep-ph]].

[36] S. Antusch, M. Blennow, E. Fernandez-Martinez and J. Lopez-Pavon, “Probing non-unitary mixing and CP-violation at a Neutrino Factory,” Phys. Rev. D 80 (2009), 033002 doi:10.1103/PhysRevD.80.033002 [arXiv:0903.3986 [hep-ph]].

[37] S. Antusch, S. Blanchet, M. Blennow and E. Fernandez-Martinez, “Non-unitary Leptonic Mixing and Leptogenesis,” JHEP 01 (2010), 017 doi:10.1007/JHEP01(2010)017 [arXiv:0910.5957 [hep-ph]].
[38] S. Antusch and O. Fischer, “Non-unitarity of the leptonic mixing matrix: Present bounds and future sensitivities,” JHEP 10 (2014), 094 doi:10.1007/JHEP10(2014)094 [arXiv:1407.6607 [hep-ph]].

[39] S. F. Ge, P. Pasquini, M. Tortola and J. W. F. Valle, “Measuring the leptonic CP phase in neutrino oscillations with non-unitary mixing,” Phys. Rev. D 95 (2017) no.3, 033005 doi:10.1103/PhysRevD.95.033005 [arXiv:1605.01670 [hep-ph]].

[40] E. Fernandez-Martinez, J. Hernandez-Garcia and J. Lopez-Pavon, “Global constraints on heavy neutrino mixing,” JHEP 08 (2016), 033 doi:10.1007/JHEP08(2016)033 [arXiv:1605.08774 [hep-ph]].

[41] D. Dutta and P. Ghoshal, “Probing CP violation with T2K, NO$
u$A and DUNE in the presence of non-unitarity,” JHEP 09 (2016), 110 doi:10.1007/JHEP09(2016)110 [arXiv:1607.02500 [hep-ph]].

[42] S. Parke and M. Ross-Lonergan, “Unitarity and the three flavor neutrino mixing matrix,” Phys. Rev. D 93 (2016) no.11, 113009 doi:10.1103/PhysRevD.93.113009 [arXiv:1508.05095 [hep-ph]].

[43] S. A. R. Ellis, K. J. Kelly and S. W. Li, “Current and Future Neutrino Oscillation Constraints on Leptonic Unitarity,” JHEP 12 (2020), 068 doi:10.1007/JHEP12(2020)068 [arXiv:2008.01088 [hep-ph]].

[44] P. Coloma, J. López-Pavón, S. Rosauro-Alcaraz and S. Urrea, “New physics from oscillations at the DUNE near detector, and the role of systematic uncertainties,” JHEP 08 (2021), 065 doi:10.1007/JHEP08(2021)065 [arXiv:2105.11466 [hep-ph]].

[45] T. Ohlsson, “Status of non-standard neutrino interactions,” Rept. Prog. Phys. 76 (2013), 044201 doi:10.1088/0034-4885/76/4/044201 [arXiv:1209.2710 [hep-ph]].

[46] O. G. Miranda and H. Nunokawa, “Non standard neutrino interactions: current status and future prospects,” New J. Phys. 17 (2015) no.9, 095002 doi:10.1088/1367-2630/17/9/095002 [arXiv:1505.06254 [hep-ph]].

[47] Y. Farzan and M. Tortola, “Neutrino oscillations and Non-Standard Interactions,” Front. in Phys. 6 (2018), 10 doi:10.3389/fphy.2018.00010 [arXiv:1710.09360 [hep-ph]].

[48] P. S. Bhupal Dev, K. S. Babu, P. B. Denton, P. A. N. Machado, C. A. Argüelles, J. L. Barrow, S. S. Chatterjee, M. C. Chen, A. de Gouvêa and B. Dutta, et al. “Neutrino Non-Standard Interactions: A Status Report,” SciPost Phys. Proc. 2 (2019), 001 doi:10.21468/SciPostPhysProc.2.001 [arXiv:1907.00991 [hep-ph]].

[49] S. Davidson, C. Pena-Garay, N. Rius and A. Santamaria, “Present and future bounds on nonstandard neutrino interactions,” JHEP 03 (2003), 011 doi:10.1088/1126-6708/2003/03/011 [arXiv:hep-ph/0302093 [hep-ph]].

[50] S. Antusch, J. P. Baumann and E. Fernandez-Martinez, “Non-Standard Neutrino Interactions with Matter from Physics Beyond the Standard Model,” Nucl. Phys. B 810 (2009), 369-388 doi:10.1016/j.nuclphysb.2008.11.018 [arXiv:0807.1003 [hep-ph]].

[51] C. Biggio, M. Blennow and E. Fernandez-Martinez, “General bounds on non-standard neutrino interactions,” JHEP 08 (2009), 090 doi:10.1088/1126-6708/2009/08/090 [arXiv:0907.0097 [hep-ph]].

[52] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler and J. Salvado, “Updated
constraints on non-standard interactions from global analysis of oscillation data," JHEP 08 (2018), 180 doi:10.1007/JHEP08(2018)180 [arXiv:1805.04530 [hep-ph]].

[53] H. Minakata, “Toward diagnosing neutrino non-unitarity through CP phase correlations,” PTEP 2022 (2022) no.6, 063B03 doi:10.1093/ptep/ptac078 [arXiv:2112.06178 [hep-ph]].

[54] S. K. Agarwalla, Y. Kao and T. Takeuchi, “Analytical approximation of the neutrino oscillation matter effects at large $\theta_{13}$,” JHEP 04 (2014), 047 doi:10.1007/JHEP04(2014)047 [arXiv:1302.6773 [hep-ph]].

[55] Z. Maki, M. Nakagawa and S. Salata, “Remarks on the unified model of elementary particles,” Prog. Theor. Phys. 28 (1962), 870-880 doi:10.1143/PTP.28.870

[56] J. Schechter and J. W. F. Valle, “Neutrino Masses in SU(2) x U(1) Theories,” Phys. Rev. D 22 (1980), 2227 doi:10.1103/PhysRevD.22.2227

[57] I. Martinez-Soler and H. Minakata, “Standard versus Non-Standard CP Phases in Neutrino Oscillation in Matter with Non-Unitarity,” PTEP 2020 (2020) no.6, 063B01 doi:10.1093/ptep/ptaa062 [arXiv:1806.10152 [hep-ph]].

[58] P. A. Zyla et al. [Particle Data Group], “Review of Particle Physics,” PTEP 2020 (2020) no.8, 083C01 doi:10.1093/ptep/ptaa104

[59] M. Kobayashi and T. Maskawa, “CP Violation in the Renormalizable Theory of Weak Interaction,” Prog. Theor. Phys. 49 (1973), 652-657 doi:10.1143/PTP.49.652

[60] G. Barenboim, P. B. Denton, S. J. Parke and C. A. Ternes, “Neutrino Oscillation Probabilities through the Looking Glass,” Phys. Lett. B 791 (2019) 351 doi:10.1016/j.physletb.2019.03.002 [arXiv:1902.00517 [hep-ph]].

[61] H. Minakata and H. Nunokawa, “CP violation versus matter effect in long baseline neutrino oscillation experiments,” Phys. Rev. D 57 (1998), 4403-4417 doi:10.1103/PhysRevD.57.4403 [arXiv:hep-ph/9705208 [hep-ph]].

[62] H. Nunokawa, S. J. Parke and R. Zukanovich Funchal, “Another possible way to determine the neutrino mass hierarchy,” Phys. Rev. D 72 (2005), 013009 doi:10.1103/PhysRevD.72.013009 [arXiv:hep-ph/0503283 [hep-ph]].

[63] P. B. Denton and S. J. Parke, “Parameter symmetries of neutrino oscillations in vacuum, matter, and approximation schemes,” Phys. Rev. D 105 (2022) no.1, 013002 doi:10.1103/PhysRevD.105.013002 [arXiv:2106.12436 [hep-ph]].

[64] I. Martinez-Soler and H. Minakata, “Physics of parameter correlations around the solar-scale enhancement in neutrino theory with unitarity violation,” PTEP 2020 (2020) no.11, 113B01 doi:10.1093/ptep/ptaa112 [arXiv:1908.04855 [hep-ph]].

[65] H. Minakata, “Neutrino amplitude decomposition in matter,” Phys. Rev. D 103 (2021) no.5, 053004 doi:10.1103/PhysRevD.103.053004 [arXiv:2011.08415 [hep-ph]].

[66] H. W. Zaglauer and K. H. Schwarzer, “The Mixing Angles in Matter for Three Generations of Neutrinos and the Msw Mechanism,” Z. Phys. C 40 (1988), 273 doi:10.1007/BF01555889

[67] C. Jarlskog, “Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Violation,” Phys. Rev. Lett. 55 (1985), 1039 doi:10.1103/PhysRevLett.55.1039