Towards quantum simulation of spin systems using continuous variable quantum devices

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We study Bosonic representation of spin Ising model with the application of simulating two level systems using continuous variable quantum processors. We decompose the time evolution of spin systems into a sequence of continuous variable logical gates and analyze their structure. We provide an estimation of quantum circuit scaling with the size of the spin lattice system. Furthermore, we discuss the possibility of using a Gaussian Boson sampling device to estimate the ground state energy of Ising Hamiltonian. The result has potential application in developing hybrid classical-quantum algorithms such as continuous variable version of variational quantum eigensolver.

I. INTRODUCTION

It is known that simulating a quantum system is a computationally hard problem\textsuperscript{[1–5, 7–11]}. The difficulty stems from the exponential explosion in the computational resources needed for the simulation, leading to strong limitations on the size of quantum systems that can be simulated using modern supercomputers.\textsuperscript{[1]} As observed by Feynman in 1982, a quantum simulator can resolve the problem of exponential explosion in the required computational resources and perform the simulation in polynomial time\textsuperscript{[12, 13]}. Over the past 40 years, there have been extensive research and tremendous efforts to build quantum devices as well as quantum algorithms that can realize quantum simulations\textsuperscript{[1–5, 7–11]}.

In a typical classical simulation, one begins with the Hamiltonian that describes the energy of the physical system and solves the dynamical equations to obtain the time evolution of the system of interest. That information is used to compute the ground state energy or other relevant properties of the system. In a quantum simulation, however, the Hamiltonian of the physical system is first mapped into the degrees of freedom of another quantum system that we can fully control and manipulate to extract information\textsuperscript{[1, 2, 5]}. If the quantum information is transformed and stored faithfully, the quantum simulator emulates the dynamic of the physical system of interest. By applying specific quantum operations and measurements on the quantum processor, the time evolution can be studied and the expectation value of the Hamiltonian for a given state can be computed\textsuperscript{[1–5, 7–11]}.

One of the most important problems in quantum simulation is that of simulating spin systems. In particular, Ising spin systems have been widely used to understand phase transition in magnetic systems and to study lattice gas and spin glass systems\textsuperscript{[12–14]}. It has been shown that the classical simulation of an Ising model is an NP-complete problem\textsuperscript{[15]} and many NP-complete problems can be mapped to Ising model\textsuperscript{[16]}. In particular, the Hamiltonian of a molecular system can be mapped to an Ising model\textsuperscript{[17]}. Hence, solving the Ising model on a quantum simulator paves the way to solving many complex problems of scientific and industrial interests.

Continuous variable (CV) quantum computing is a branch of quantum information processing where information is stored in quantum observables with continuous spectrum—such as positions and momenta operators. In the conventional qubit model, information is stored in discrete levels of a quantum system such as spins or two levels system of an atom or a superconducting system\textsuperscript{[18, 19]}. In analogy to qubit, the building block of a CV quantum processor are photonic modes whose states are described in an infinite dimensional Hilbert space. Quantum information is encoded in the amplitude and phase of electromagnetic waves or light. The associated CV quantum gates are implemented by passive optical elements such as phase shifters, beam splitters, interferometers or active interaction with non-linear crystals.\textsuperscript{[20, 21]} CV quantum computing has been shown to be equivalent to qubit based quantum computing\textsuperscript{[22]}

In this work, we discuss the simulation of Ising spin model on a continuous variable (CV) quantum processor (for simulations using qubit based quantum processors and quantum annealers see\textsuperscript{[23–30]} and references therein.). Through Jordan-Schwinger transform, we first map Edward-Anderson and Heisenberg Hamiltonian for spin systems to Bosonic degrees of freedom used in CV quantum computing. This results in quadratic and quartic interaction terms in the CV quantum modes. We analyze these terms and study CV gate decomposition of the resulting Hamiltonian in terms of the universal CV gate set. This will enable us to provide a resource es-


II. BOSONIC HAMILTONIAN FOR SPIN SYSTEM

A. Spin Hamiltonian

Spin systems are usually defined on a $d$-dimensional lattice with $N$ number of sites each being occupied by spin $S_i$. There are two main components to the energy of the spin system: (i) due to the interaction between spins with their nearest neighbors and (ii) due to interaction of the individual spins with an external magnetic field. Thus, In the presence of an external magnetic field, $\vec{B} = B_0 \hat{z}$, the energy of such a system is given by the following Hamiltonian:

$$\hat{H}_{\text{Ising}} = - \sum_{k,l=1,k<l}^N J_{kl} \hat{s}_k \hat{s}_l - B_0 \sum_{k=1}^N \hat{s}_k.$$  \hfill (2.1)

where $J_{kl}$ are the coupling constants between spins at the $k^\text{th}$ and the $l^\text{th}$ sites and $\hat{s}_i$ denotes the spin operator (with eigenvalues $s_i = \{+1, -1\}$) at $i^\text{th}$ site. The first term of the Hamiltonian represent the internal spin-spin interaction and the second term represents the energy due interaction of an individual spin with the external magnetic field, taken to be homogeneous across the lattice. Ising model is one of the most widely studied spin system [12, 38, 39], in which each spins is given by a two state system, given by $\{+1, -1\}$. In a magnetic system, the binary state represents the two configurations, namely aligned $|\uparrow\rangle$ and anti-aligned $|\downarrow\rangle$ states with an external magnetic field.

The Ising model was formulated by Wilhelm Lenz and Ernst Ising to describe the behavior of anti-ferromagnetism in a spin lattice [12, 38, 39]. This simple and yet rich model has become a very popular model for representing distinct physical systems due to various interpretation of “spin”. For instance, one interpretation of “spin” can be the presence ($s = +1$) or the absence ($s = -1$) of a molecule in a certain cell of a “lattice gas”, and so, the model can be used for studying the critical behavior of a fluid system [12-14]. Since the first application of Ising model to magnetic and molecular systems, it has been used to describe different physical systems in various fields ranging from physics, chemistry, biology to artificial intelligence. Finding the ground state of Ising model is an NP hard problem [15] that leads to mapping of many NP hard problems to the Ising model [16]. This puts Ising model as one of the most important cornerstones of quantum simulation. A successful quantum simulation of the Ising model would lead the way to solving many interesting problems, hence potentially showcasing quantum advantage over the state-of-the-art classical simulation methods.

The Ising model (or spin systems in general) can be considered as a special class of Heisenberg models which, in addition to the spin-spin interaction in $z$-direction ($J^z$) considered in the first term of Eq. (2.1), also has transverse interactions in $x$ and $y$ directions. The Hamiltonian for such a system can be written as:

$$\hat{H} = \hat{H}_{\text{trans}} + \hat{H}_{\text{Ising}},$$ \hfill (2.2)

where

$$\hat{H}_{\text{trans}} = - \sum_{k,l=1,k<l}^N \left( J_{kl}^x \hat{s}_k^x \hat{s}_l^x + J_{kl}^y \hat{s}_k^y \hat{s}_l^y + J_{kl}^z \hat{s}_k^z \hat{s}_l^z \right).$$ \hfill (2.3)

Therefore the Heisenberg Hamiltonian takes the following form:

$$\hat{H} = - \sum_{k,l=1,k<l}^N \left( J_{kl}^x \hat{s}_k^x \hat{s}_l^x + J_{kl}^y \hat{s}_k^y \hat{s}_l^y + J_{kl}^z \hat{s}_k^z \hat{s}_l^z \right) - B_0 \sum_{k=1}^N \hat{s}_k.$$ \hfill (2.4)

In this model, $J_{kl}^\alpha$ with $\alpha = \{x,y,z\}$ respectively are $x$, $y$ and $z$ components of the coupling constants between spins at the $k^\text{th}$ and $l^\text{th}$ site. The above model assumes that the external field is homogeneous across the lattice and the spin-spin interaction is isotropic. In the following we will continue to consider binary states for the spin with eigenvalues $s_k^\alpha = \{+1, -1\}$ (corresponding to states $\{\uparrow\}, \{\downarrow\}$) and analyze the Heisenberg Hamiltonian $\hat{H}$ in Eq. (2.4), of which Ising model is a special case, owing to the isotropy of the spin-spin interactions. In the next subsection we will transform the spin degrees of freedom to Bosonic degrees of freedom using Jordan-Schwinger transformation [40, 41].

B. Bosonic representation of spin Hamiltonian

In this paper we are interested in studying the spin systems using a CV quantum computing device whose physical realization is based on Bosonic degrees of freedom. Therefore, in order to implement the spin Hamiltonian of Eq. (2.4), we first need to transform the spin degrees of freedom $s_j$ to Bosonic degrees of freedom suitable for a CV quantum device. We will consider the standard Jordan-Schwinger transformation of the angular momentum operators to carry out the transformation from spin states $s_j$ to the desired Bosonic operators.

The paper is organized as follows. Section II gives a brief introduction to the Ising model under consideration and presents mapping of the Ising Hamiltonian to CV degrees of freedom. In section III we discuss various terms in the CV Hamiltonian, their decomposition in to universe CV quantum gates and a resource count to simulate the Ising model. In section IV we discuss estimation of quartic terms in Hamiltonian involving four mode optical correlation using a Gaussian Boson sampling device. We summarize our result in section V.
which in our case will be based on quantum photonics. We will first transform the spin degrees of freedom to photonic creation and annihilation operators which are then written in terms of the continuous degrees of freedom represented by the quantum optical modes.

Let us consider two uncoupled harmonic oscillators described pairs of creation (\(a_i^\dagger\)) and annihilation \(a_i\) operators where the index \(i\) represents the two types of oscillators with \(|\uparrow\rangle\) and \(|\downarrow\rangle\). The associated number operators are then defined as follows \([41]\):

\[
\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i, \quad \hat{n}_j = \hat{a}_j^\dagger \hat{a}_j.
\]  

(2.5)

Here, we have assumed the usual commutation relations for \(\hat{a}_i^\dagger\) and \(\hat{a}_i\):

\[
[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{n}_i, \hat{a}_j^\dagger] = \delta_{ij} \hat{a}_j, \quad [\hat{n}_i, \hat{a}_j] = -\delta_{ij} \hat{a}_i.
\]  

(2.6)

The Jordan-Schwinger transformation is a way to write representations of SU(N) with Bosons \([40, 41]\). For SU(2) it is defined by:

\[
\hat{S}_\alpha = \frac{1}{2} \sum_{i,j} \hat{a}_i^\dagger (\hat{\sigma}_\alpha)_{ij} \hat{a}_j
\]  

(2.7)

where \(\sigma_\alpha\) for \(\alpha = x, y, z\) are the Pauli matrices and \(\hat{a}_i^\dagger, \hat{a}_i\) are creation and annihilation operators for two coupled harmonic oscillators. With the choice of spin basis \(|\uparrow\rangle| = |0\rangle\) and \(|\downarrow\rangle| = |1\rangle\), we can write mix operators defined as follows:

\[
\hat{S}_+ = \hat{S}_x + i \hat{S}_y = \hat{a}_z^\dagger \hat{a}_1,
\]

\[
\hat{S}_- = \hat{S}_x - i \hat{S}_y = \hat{a}_z \hat{a}_1^\dagger,
\]

\[
\hat{S}_z = \frac{1}{2} (\hat{n}_1 - \hat{n}_1).
\]  

(2.8)

Note that the \(\hat{S}_+\) and \(\hat{S}_-\) operators act like spin flip operators, i.e. \(\hat{S}_+\) annihilates a spin of type \(|\downarrow\rangle\) and creates a spin of type \(|\uparrow\rangle\). Similarly, \(\hat{S}_-\) annihilates a spin of type \(|\uparrow\rangle\) and creates a spin of type \(|\downarrow\rangle\). The operator \(\hat{S}_z\), however simply counts the difference between the number of spins of two types. All these operators leave the sum of total spins i.e. \(\hat{n}_1 + \hat{n}_1\) unchanged. Furthermore, we can define:

\[
\hat{S}_x = \frac{1}{2} (\hat{S}_+ + \hat{S}_-), \quad \hat{S}_y = -i \frac{1}{2} (\hat{S}_+ - \hat{S}_-).
\]  

(2.9)

It is straightforward to show that the newly defined operators \(\hat{S}_i\) follow the following commutation relations:

\[
[\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad [\hat{S}_+, \hat{S}_-] = 2 \hat{S}_z.
\]  

(2.10)

The anti-commutation relation between \(\hat{S}_+\) and \(\hat{S}_-\) becomes:

\[
\{\hat{S}_+, \hat{S}_-\} = \hat{n}_1 + \hat{n}_1 + 2 \hat{n}_1 \hat{n}_1.
\]  

(2.11)

In the model considered here, each lattice site is occupied by a maximum of one spin-half particle which can be in one of the two states: \(|\uparrow\rangle| = |n_1 = 1, n_1 = 0\rangle\) or \(|\downarrow\rangle| = |n_1 = 0, n_1 = 1\rangle\). Therefore, at a given lattice site, \(\hat{n}_1 + \hat{n}_1 = 1\) and \(\hat{n}_1 \hat{n}_1 = 0\).

### TABLE I: The coefficients \(C\) and \(F\) of quartic terms in Eq. (2.16).

| \(a_k\), \(b_k\), \(c_k\), \(d_k\) | \(C\) | \(F\) |
|---------------------------------|-----|-----|
| \(X_k\), \(X_k\), \(X_k\), \(X_k\) | +1 | +1 |
| \(P_k\), \(P_k\), \(P_k\), \(P_k\) | +1 | +1 |
| \(X_k\), \(X_k\), \(P_k\), \(P_k\) | +1 | -1 |
| \(P_k\), \(P_k\), \(X_k\), \(P_k\) | +1 | -1 |
| \(X_k\), \(P_k\), \(P_k\), \(P_k\) | -1 | +1 |

This is a ‘holonomic’ constraint which yields the following anti-commutation relation for the spin system with spin-half particles:

\[
\{\hat{S}_+, \hat{S}_-\} = 1.
\]  

(2.12)

This plays a key role in the algebra of the operator and puts \(\hat{S}_+\) and \(\hat{S}_-\) on the similar footings of the ladder operators associated to a single harmonic oscillator. Furthermore, The holonomic constraint, \(\hat{n}_1 + \hat{n}_1 = 1\), projects the infinite dimension harmonic oscillator space to a finite dimensional spin space, while making the Jordan-Schwinger transformation exact on each site.\(^2\)

Using the Jordan-Schwinger mapping i.e. writing \((\hat{s}_x, \hat{s}_y, \hat{s}_z)\) in terms of \((\hat{S}_x, \hat{S}_y, \hat{S}_z)\), \(\hat{S}_+\) and \(\hat{S}_-\), Eq. (2.4) can be written as:

\[
\hat{H} = \sum_{k,l=1,k<l}^N \left(-\frac{J_{kl}^x + J_{kl}^y}{4} \hat{S}_x^k \hat{S}_y^l + \frac{J_{kl}^x - J_{kl}^y}{4} \hat{S}_y^k \hat{S}_x^l + h.c.\right)
\]

Now, using the mapping from \(S_+\) and \(S_\pm\) to creation and annihilation operators given in Eq. (2.3), the Hamiltonian above can be written as:

\[
\hat{H} = \sum_{k,l=1,k<l}^N \left(-\frac{J_{kl}^x + J_{kl}^y}{4} \hat{a}_x^k \hat{a}_x^l + h.c.\right)
\]

\[
-\frac{J_{kl}^x - J_{kl}^y}{4} \hat{a}_y^k \hat{a}_y^l + h.c.\right)
\]

\[
-\frac{J_{kl}^y \hat{a}_y^k \hat{a}_y^l - J_{kl}^x \hat{a}_y^k \hat{a}_y^l - J_{kl}^x \hat{a}_y^k \hat{a}_y^l + J_{kl}^y \hat{a}_y^k \hat{a}_y^l + \hat{n}_1 \hat{n}_1\right)
\]

\[
-\frac{B_0}{2} \sum_{k=1}^N \left(\hat{n}_1^k - \hat{n}_1^k\right)
\]  

(2.14)

\(^2\) Note that at the operator level, the ‘holonomic’ constraint may not be satisfied in general, in which case the anti-commutation relation of Eq. (2.12) can be satisfied only under an approximation.
where ‘h.c.’ refers to Hermitian conjugate.

Eq. (2.14) is the desired Hamiltonian transformed from spin representation to the Bosonic creation and annihilation operators, which are discrete Bosonic degrees of freedom. Recall that, in this paper, we are interested in implementing the Ising model using CV quantum computer. This requires us to write creation and annihilation operators in terms of continuous degrees of freedom which in this case are the quadrature operators, \( q \in \{ \hat{X}, \hat{P} \} \):

\[
\hat{a} = \hat{X} + i \hat{P}, \\
\hat{a}^\dagger = \hat{X} - i \hat{P}, \\
\hat{n} = \hat{X}^2 + \hat{P}^2 - \frac{n}{2},
\]  

(2.15)

where the standard commutation relation, \([\hat{X}, \hat{P}] = i/2\), is used. Using Eq. (2.15), we can write the Hamiltonian in terms of the continuous variables \( \hat{X} \) and \( \hat{P} \), which we collectively denote as \( q \) for conciseness as follows:

\[
\hat{H} = \sum_{k,l=1}^{N} \left( -J_{kl}^{(1)} + J_{kl}^{(2)} \right) C_{k l k' l' z} (\hat{q}_{k} \hat{q}_{l} \hat{q}_{l'} \hat{q}_{k'}) \\
- \sum_{k,l=1}^{N} \left( -J_{kl}^{(1)} + J_{kl}^{(2)} \right) F_{k l k' l' z} (\hat{q}_{k} \hat{q}_{l} \hat{q}_{l'} \hat{q}_{k'}) \\
- \sum_{k,l=1}^{N} J_{kl}^{(2)} (\hat{q}_{k}^{2} \hat{q}_{l}^{2} + \hat{q}_{l}^{2} \hat{q}_{k}^{2} - \hat{q}_{k}^{2} \hat{q}_{l}^{2} - \hat{q}_{l}^{2} \hat{q}_{k}^{2}) \\
- \frac{B_0}{2} \sum_{k=1}^{N} \hat{q}_{k}^{2}. 
\]  

(2.16)

In the equation above we have used the following convention to denote the indices \((k, l)\) and the spin orientation \((\uparrow, \downarrow)\): \( k_1 := k \uparrow, k_2 := k \downarrow, l_1 := l \uparrow, l_2 := l \downarrow \). When transforming from creation/annihilation operators to \( \hat{X} \) and \( \hat{P} \) variable, we find that \( \hat{q}_{k} \hat{q}_{l} \hat{q}_{l'} \hat{q}_{k'} \) is non-zero only for 8 combinations of \( \hat{X} \) and \( \hat{P} \) variable. These non-zero terms are shown in Table-I, for which \( C_{kl} = \frac{1}{2} (J_{kl}^{(1)} + J_{kl}^{(2)}) \) and \( F_{kl} = \frac{1}{2} (J_{kl}^{(1)} - J_{kl}^{(2)}) \) and \( C \) and \( F \) are given explicitly in Table-II. Similarly to Eq. (2.16) we have used the notation: \( k_1 := k \uparrow, k_2 := k \downarrow, l_1 := l \uparrow, l_2 := l \downarrow \).

### III. CV GATE DECOMPOSITION

Typically the total Hamiltonian is expressed in terms of summation of various terms. In our case, the Hamiltonian in Eq. (2.16) is composed of two quartic terms with coefficients \( C \) and \( F \), product of quadratic terms and one quadratic term. To simulate the time evolution operator, \( e^{i \hat{H}} \), where \( \hat{H} = \sum_{j} \hat{H}_j \) is a sum of operators with \( \hat{H}_j \) denoting the quartic and quadratic terms, one can use the Lie-Trotter product formula \([42, 43]\). Trotterization approximates the time evolution operator as a series of \( K \) time steps of size \( t/K \), during which, the non-commutativity of the Hamiltonian terms, \( \hat{H}_j \), is neglected. Precisely,

\[
e^{\frac{it}{K} \sum_{j} \hat{H}_j} \approx \left( \prod_{j=1}^{N} e^{\frac{it}{K} \hat{H}_j} \right) K + O \left( \frac{N^2 t^2 \Gamma^2}{K} \right) 
\]  

(3.1)

where \( \Gamma = \max_j ||\hat{H}_j|| \) is the largest Hamiltonian norm. The last term accounts for the error in this approximation and it vanishes as \( K \to \infty \). However, infinite product terms would be equivalent to an infinitely long circuit depth which is not feasible in practice. Therefore, one needs to find a trade-off so that \( K \) is large enough to achieve a reasonable accuracy, and at the same time, small enough to avoid exploiting the number of gates in the quantum circuit.

Using Eq. (3.1), the time evolution operator of \( \hat{H} \) in Eq. (2.16) is approximated as

\[
e^{i \hat{H} t} \approx \left[ \prod_{k,l=1}^{K} e^{i \frac{t}{K} (\hat{q}_{k} \hat{q}_{l} \hat{q}_{l'} \hat{q}_{k'})} \right] K \\
\times e^{i \frac{t}{K} (\hat{q}_{k}^{2} \hat{q}_{l}^{2} + \hat{q}_{l}^{2} \hat{q}_{k}^{2} - \hat{q}_{k}^{2} \hat{q}_{l}^{2} - \hat{q}_{l}^{2} \hat{q}_{k}^{2})} \\
\times e^{i \frac{B_0}{2K} \hat{q}_{k}^{2}},
\]

where the coefficients \( g_{kl} := \frac{1}{4} (C_{kl} + J_{kl}^{(1)} + F_{kl} - J_{kl}^{(2)}) \) and \( C \) and \( F \) are given explicitly in Table-I. Similarly to Eq. (3.1) we have used the notation: \( k_1 := k \uparrow, k_2 := k \downarrow, l_1 := l \uparrow, l_2 := l \downarrow \).

#### A. Universal optical gates

In order to optically implement the above time evolution operator, one needs to decompose each term as a sequence of universal optical gates, defined as follows \([18, 44]\):

\[
R := e^{i \alpha \hat{X}}, \\
G := e^{i \alpha \hat{X}^{2}}, \\
V := e^{i \alpha \hat{X}^{3}}, \\
\mathcal{F} := e^{i \frac{\alpha}{2} (\hat{X}^{2} + \hat{P}^{2})}, \\
C_z := e^{i \alpha \hat{X} \hat{P}}.
\]  

(3.3)

The above set consists of single mode gates namely rotation gate \((R)\), Gaussian gate \((G)\), cubic phase gate \((V)\) and Fourier transform gate \((\mathcal{F})\) and two modes gate control phase gate \((C_z)\). As discussed later in this section, we need two additional gates for a complete decomposition of the time evolution operator:

1. quartic gate, \( Q_k(\alpha) = e^{i \alpha \hat{X}^{4}_k} \)

2. the translating or the shift gate, \( T_k(A) = e^{i A \hat{P} \hat{X}} \), where \( A \) is the shift parameter.

Let us show how to implement \( Q \) and \( T \) using the universal gate set. For an arbitrary operator function \( f(\hat{X}) \), it is straightforward to show that \( T_k(A) f(\hat{X}_k) T_k(-A) = f(\hat{X}_k + A) \) and \( \mathcal{F} f(\hat{X}) \mathcal{F}^{-1} = f(\hat{P}) \). Furthermore, using the identity relation

\[
x_k^4 = (x_k^2 - x_k^2 - 2 x_k x_k^2, 
\]  

(3.4)
the quartic gate \( Q(\alpha) \) can be decomposed to:

\[
Q_2(\alpha) = e^{i \alpha X_2^4} = e^{i \hat{P}_1 X_2^2 e^{i \alpha X_2^2} e^{-i \alpha \hat{P}_1 X_2^2}} = T_1(\hat{X}_2^2) G_1(\alpha) T_1^\dagger(\hat{X}_2^2) G_1(-\alpha) T_1(-\alpha X_2^2) T_1^\dagger.
\]

Similarly to the quartic gate \( Q \), the shift gate \( T \) can also be decomposed in terms for the universal gate set. As discussed in Ref. [45]:

\[
e^{i \alpha \hat{x}_2} = \mathcal{F} C_z \mathcal{F}^\dagger \quad (3.6)
\]

\[
e^{i \alpha \hat{x}_2^2} = e^{i \hat{P}_1 e^{i \alpha \hat{x}_2} e^{-i \alpha \hat{P}_1} e^{i \alpha \hat{x}_2} e^{i \hat{P}_1}} = e^{i \alpha \hat{x}_2} e^{-i \alpha \hat{P}_1} e^{i \alpha \hat{x}_2} e^{i \hat{P}_1} X_2^2 \quad (3.7)
\]

\[
V'_l C_z(-\alpha) V_l^\dagger C_z(-2\alpha) V_l^\dagger \quad (3.8)
\]

where \( V' = \mathcal{F} V \mathcal{F}^\dagger \). In principle, one can find a general decomposition relation for \( e^{i \alpha \hat{x}_2^4} \). However, up to the second order is enough for our purposes.

Note that for implementing every quartic operator, \( Q \), three second-order shift gate are required, and, for each of them, five cubic phase gates (\( V \)) and three phase control gates (\( C_z \)) are required. Thus, in total, each \( Q \) costs us 15 cubic phase gates (\( V \)) and 9 control phase gates (\( C_z \)). Now that we have understanding of the decomposition of quartic gate and shift gate, we elaborate on the decomposition of time evolution operator for the Hamiltonian in Eq. \( 2.16 \).

### B. Decomposing the spin Hamiltonian

In the following, we consider individual term of the Hamiltonian in Eq. \( 2.16 \) and discuss their decomposition in terms of continuous variable gates.

- **Four mode quartic terms**: Let consider the first and second term in the Hamiltonian which are of the form \( \hat{q}_{k_1} \hat{q}_{k_2} \hat{q}_{l_1} \hat{q}_{l_2} \). These terms are quartic in order and contain products of four modes. In order to implement this in terms of the gates of the form \( X^4 \) we can use the following identity:

\[
\hat{q}_{k_1} \hat{q}_{k_2} \hat{q}_{l_1} \hat{q}_{l_2} = \frac{1}{192} \left[ (\hat{q}_{k_1} + \hat{q}_{k_2} + \hat{q}_{l_1} + \hat{q}_{l_2})^4 \right.
\]

Note that each of these quartic gates is in the form of summation (with + or - signs) of \( \hat{q}_{k_1}, \hat{q}_{k_2}, \hat{q}_{l_1}, \) and \( \hat{q}_{l_2} \), e.g. the first term is \( (\hat{q}_{k_1} + \hat{q}_{k_2} + \hat{q}_{l_1} + \hat{q}_{l_2})^4 \). These terms can be obtained by starting with a quartic gate corresponding to \( \hat{q}_{k_1}^4 \) followed by shifting it three times as follows:

\[
e^{i \alpha (\hat{q}_{k_1} + \hat{q}_{k_2} + \hat{q}_{l_1} + \hat{q}_{l_2})^4} = T_1(\hat{q}_{k_2}) T_1(\hat{q}_{k_3}) T_1(\hat{q}_{k_4}) e^{i \alpha \hat{q}_{k_1}^4} T_1(\hat{q}_{l_2}) T_1(\hat{q}_{l_3}) T_1(\hat{q}_{l_4}). \quad (3.10)
\]

Using similar expressions we can obtain the quartic gate implementation of the each term on the right hand side of Eq. \( 3.9 \). Hence, in order to implement each of the quartic terms on the right hand side of Eq. \( 3.10 \), we need 6 first order shift gate, in addition to the three second order shift gates. This amounts to a total of \( 8 \times 6 = 48 \) first order and \( 8 \times 3 = 24 \) second order shift gate.

As an alternative to Eq. \( 3.9 \), we can expand the product of four modes using the following identity:

\[
\hat{q}_{k_1} \hat{q}_{k_2} \hat{q}_{l_1} \hat{q}_{l_2} = \frac{1}{24} \left[ (\hat{q}_{k_1} + \hat{q}_{k_2} + \hat{q}_{l_1} + \hat{q}_{l_2})^4 \right.
\]

\[
- \sum_{l \neq m \neq n} (\hat{q}_{l_1} + \hat{q}_{m} + \hat{q}_{n})^4
\]

\[
+ \sum_{l \neq m} (\hat{q}_{l_1} + \hat{q}_{m})^4 - \sum_{l} \hat{q}_{l_1}^4 \right]. \quad (3.11)
\]

Here, \( l, m, n \in \{k_1, k_2, l_1, l_2\} \). For this implementation, \( (1 \times 6 + 4 \times 4 \times 6 + 2 \times 4) = 38 \) first-order shift gates and \( 3 \times (1 + 4 + 6 + 4) = 45 \) second-order shift gates. Therefore, by choosing the first identity, we need much fewer (approximately half as many) second order shift gates while the number of first order shift gates are comparable for the two identities. Therefore, the first identity in Eq. \( 3.9 \) requires fewer cubic phase gates, \( V \) and control phase gates, \( C_z \).

Note that \( \hat{q} \) in Eq. \( 2.16 \) can be either position or momentum operator. We assume that all momentum operators can be converted to the position operator using a Fourier transform gate, i.e. \( f(\hat{P}) = \mathcal{F} f(\hat{X}) \mathcal{F}^\dagger \). Therefore, by considering the first identity relation in Eq. \( 3.9 \), the decomposition of four-mode position operator is
\[ e^{i\alpha (\hat{x}_\nu \hat{\sigma}_\mu \hat{x}_\nu)} = \prod_{\tilde{s}} S_{\tilde{s}} \left[ T_i(\hat{X}_\nu) T_i(\hat{X}_\mu) T_j(\hat{X}_j) e^{-i\frac{\alpha}{2\pi} \hat{x}_\nu^2} T_j^\dagger(\hat{X}_j) T_i^\dagger(\hat{X}_\mu) T_i^\dagger(\hat{X}_\nu) \right] \]

(3.12)

where the \( S \) operators determine the sign of the elements in \( (X_j, X_\mu, X_\nu) \) by spanning \( \tilde{s} \in \{ (+, +, +), (-, -, -), (+, -, -), (-, +, +), (-, +, -), (+, -, +), (-, -, +), (-, -, +), (+, +, -), (-, +, +) \} \). Hence, four modes terms in the Hamiltonian can be written in terms of quartic operators. The quartic operators can further be implemented in terms of cubic phase gates and control phase gates as shown in Eq. (3.8).

- **Two mode quartic terms**: The third term in Eq. (2.16) is also quartic, however, it appears as a product of two quadratic terms corresponding to two different modes, i.e., \( q_i q_j \). Thus, we use the identity relation

\[ q_i^2 q_j^2 = \frac{1}{12} \{ (\hat{q}_i + \hat{q}_j)^4 + (\hat{q}_i - \hat{q}_j)^4 - 2 \hat{q}_i^2 - 2 \hat{q}_j^2 \}, \]  

(3.13)

which leads to the following decomposition in terms of quartic gates:

\[ e^{i\alpha (\hat{x}_m^2 \hat{x}_n^2)} = T_j(\hat{X}_j) e^{i\frac{\alpha}{2} \hat{x}_j^2} T_j^\dagger(\hat{X}_j) \times T_i(\hat{X}_m) e^{i\frac{\alpha}{2} \hat{x}_m^2} T_i^\dagger(\hat{X}_m) \times e^{-2i\alpha \hat{x}_j^2} \times e^{-2i\alpha \hat{x}_m^2}. \]  

(3.14)

Therefore, for each of the terms we need two quartic operators, four shift gates and two quadratic operators. The quartic operators can be implemented in terms of cubic phase gates and control phase gates as shown in Eq. (3.8). The quadratic operators can be easily implement using Gaussian gates. Like before, for those terms that \( \hat{q} = \hat{P} \) operator, we need to first apply a Fourier transform operator and turn it to an \( \hat{X} \) operator and then use the above decomposition.

- **Single mode quadratic terms**: The last term of the Hamiltonian, i.e., \( q^2 \), is due to interaction with external magnetic field and it can be simply implemented by using a Gaussian gate, i.e., \( G(\alpha) = e^{i\alpha \hat{x}^2} \).

Using the decomposition discussed above, all the terms of the Hamiltonian in Eq. (2.16) can be implemented using universal CV gates.

### IV. HYBRID QUANTUM SIMULATION

Computation of the expectation value of the Hamiltonian using the CV gate decomposition discussed in the previous section requires non-linear gates and a fault tolerant CV quantum computer which is not expected to be available in the near future. However, it has been shown that the noisy intermediate-scale quantum devices (NISQ) can potentially be used to implement hybrid quantum algorithm while the full fledged quantum computer is under development [31-37]. In this section, we discuss a hybrid method to estimate the expectation value of the spin Hamiltonian discussed here using an optical continuous variable NISQ device.

Recall that, in order to implement a CV version of variational quantum eigensolver algorithm, we need to estimate the expectation value of each terms of the Hamiltonian, i.e. \( \langle H \rangle = \sum_{j=1}^{N} \langle H_j \rangle \). The quadratic terms can be computed using photon counting measurement of each mode: terms with \( \langle q_i^2 \rangle \) and \( \langle q_i^2 q_j^2 \rangle \) are given by photon number measurement \( \langle \hat{n}_i \rangle \) and two detector correlation measurement \( \langle \hat{n}_i \hat{n}_j \rangle \). For the quartic terms containing four modes, \( \hat{q}_1 \hat{q}_2 \hat{q}_3 \hat{q}_4 \), which would require non-linear gates and a fault tolerant quantum computer, we present the following proposal where we use the so-called Gaussian Boson sampling, a continuous variable NISQ device.

#### A. Using Gaussian Boson sampling (GBS)

The Gaussian Boson sampling device measures the system in the photon number basis (e.g. Bosonic Fock states of four modes are, \( |n_k n_k n_l n_l\rangle \)) and samples from the following distribution

\[ \Pr(\tilde{m}) = \frac{1}{\sqrt{\tilde{m}!}} \det \sigma_0 \text{Haf}(A_S) \]

(4.1)

where \( \tilde{m} = m_1 m_2 \ldots m_M \) is a string of photon numbers detected in \( M \) modes and \( \sigma_0 = \sigma_A + \frac{1}{2M} \) with \( \sigma_A \) being the covariance matrix of the physical modes of the GBS device [46,47]. The \( A_S \) is a submatrix of a matrix \( A \) where \( S \) is determined by the detected photon string, \( \tilde{m} \). The matrix \( A \) is related to the covariance matrix of the physical modes via

\[ A = \chi_{2M} [I_{2M} - (\sigma_A + \frac{1}{2M})^{-1}] \]

(4.2)

Let \( \tilde{q} = (\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_m) \) be a random vector that follows a normal distribution \( \tilde{q} \sim N(\mu, \Sigma) \), where \( \mu \) is the mean and \( \Sigma = \{ \langle \hat{q}_i \hat{q}_j \rangle \} \) is the covariance matrix. For the case of zero mean, \( \mu = 0 \), with even number of modes, \( m \), the mean value of the products is [48],

\[ \langle \hat{q}_1 \hat{q}_2 \ldots \hat{q}_m \rangle = \text{Haf}(\Sigma). \]

(4.3)

The above theorem relates the expectation value of each individual term of the transverse Hamiltonian to the Hafnian of the covariance matrix.
If we use a direct mapping from the modes of interests, \( \hat{q}_i \) to the physical modes of the GBS device (i.e., if we choose \( \Sigma = \sigma_A \)), the GBS device can obtain an estimation of \( \text{Haf}(A) \) after post selection on \( P(1, 1, \ldots, 1) \) where one photon is detected in all modes. However, the quantity of our interest is \( \text{Haf}(\Sigma) \) not \( \text{Haf}(A) \). Since the relation between \( A \) and the covariance of the physical system in Eq. (4.2) is not linear, we expect that extracting \( \text{Haf}(\Sigma) \) from \( \text{Haf}(A) \) is very non-trivial. Therefore, the direct mapping where \( \sigma_A = \Sigma \) is not suitable.

One can consider engineering the GBS circuit such that \( A = \Sigma \). However, in general, such a matrix \( A \) does not relate to valid covariance matrix to represent a physical system.

Let \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \) be a \( 2M \times 2M \) block diagonal symmetric matrix. According to [49], one of the requirements that \( A \) must satisfy in order to map to a valid covariance matrix of a Gaussian state is \( [A_{11}, A_{12}] = 0 \) and \( A_{12} \geq 0 \). On the other hand, a covariance matrix \( \Sigma \) with \( 2M \times 2M \) dimension and is given by

\[
\Sigma = \begin{pmatrix} C & D \\ D^T & C \end{pmatrix},
\]

\[
C_{ij} = \frac{1}{2} \langle \hat{b}_i \hat{b}_j \rangle,
\]

\[
D_{ij} = \langle \hat{b}_i \hat{b}_j \rangle
\]

where \( \hat{b}_i^\dagger, \hat{b}_i \) are Bosonic creation-annihilation operators. In general, \( [C, D] \neq 0 \), and therefore, if we consider \( A = \Sigma \), a mapping between the modes of interest and the physical modes of the GBS device may not exist for an arbitrary \( \Sigma \). However, it was shown in Ref. [49] that by doubling the number of optical modes and considering \( A = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix} \) a valid covariance matrix \( \sigma_A \) exists. Thus, the quantity of our interest can be measured by

\[
\langle q_1 q_2 q_3 q_4 \rangle = \text{Haf}(\Sigma) = \sqrt{\text{Haf}(A)} \quad \text{(4.5)}
\]

We highlight that obtaining the above results has a few requirements which must be considered for experimental implementation. First, the initial state needs to be Gaussian with zero mean which can be easily implemented using a displacement operator. Second, it is required to do a post-selection on the measurement outcome with specific photon pattern \( m = (1, 1, \ldots, 1) \). Therefore, a single photon detector is required which is not easy to implement and currently an active field of research. Moreover, in order to implement a spin lattice with \( N \) sites, we require \( 4N \) optical modes, a factor of two due to Bosonization and another factor of two for estimating the Hafnian.

V. CONCLUSION

In this paper, we studied gate decomposition of Ising system in the continuous variable (CV) paradigm of quantum computing. We used the Jordan-Wigner transformation to transform the Hamiltonian of transverse Ising model into a Bosonic Hamiltonian. The result was then used to decompose the time evolution operator of Ising model in terms of universal gate sets in continuous variable quantum computing. Multi-mode terms were analyzed in details to obtain a compact expression for gate components. In addition to application in quantum circuit model, we discussed how the current NISQ devices such as Gaussian Boson sampling can be used to estimate ground states of Ising-type Hamiltonian. The result can potentially be used for hybrid classical-quantum algorithms such as variational quantum eigensolver.

Ising spin systems are among the most studied problems in quantum simulation. In addition to their wide use in understanding phase transition in magnetic systems, it has been shown that many NP hard problems can be mapped to an Ising model [50]. Therefore, solving an Ising model on a quantum computer opens up new avenues to simulating NP-hard problems on quantum computer.

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