Traversable Wormholes with Exponential Shape Function in Modified Gravity and in General Relativity: A Comparative Study

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Abstract

We propose a novel shape function, on which the metric that models traversable wormholes is dependent. With this shape function, the energy conditions, equation of state and anisotropy parameter are analyzed in $f(R)$ gravity, $f(R, T)$ gravity and general relativity, where $R$ is the scalar curvature and $T$ is the trace of stress energy tensor. Furthermore, the consequences obtained with respect to these theories are compared. In addition, the existence of wormhole geometries is investigated.

Keywords: Traversable Wormhole; Exponential shape function; $f(R, T)$ gravity; $f(R)$ gravity

1 Literature survey

Wormhole solutions of Einstein’s field equations in general relativity do not satisfy the classical energy conditions. These connect two universes or two remote parts of the same universe. These were first studied by Flamm \[1\] in a simplest form. After that, Einstein & Rosen \[2\] first introduced a mathematical model of wormhole representing the connection of two asymptotically flat spaces through a bridge which is known as Einstein-Rosen bridge. It was observed that an exotic matter must be present to produce antigravity for stability of Einstein-Rosen bridge. Otherwise due to gravity, the throat will collapse into a singularity and hence the passage through the wormhole will stop. Morris & Thorne \[3\] made a significant contribution in the study of wormholes, discovered traversable wormholes and began an active area of research. Some examples of traversable wormholes are discussed in \[4\]-\[6\]. In literature, various studies have been done to investigate the stability of wormholes and the matter passing through them \[7\]-\[11\]. The models of multiverses have also been constructed using the concept of wormholes \[12\]-\[15\]. Lemos et al. \[16\] reviewed traversable wormholes, analyzed them due to the effect of cosmological constant and explored their various properties. Lobo \[17\] studied physical properties of wormhole solutions. Böhmer et al. \[18\] obtained various wormhole solutions using a linear relationship between energy density and pressure and explored phantom wormhole geometries. Cataldo & Meza \[19\] explored wormhole structures filled with matter components of two types. Cataldo et al. \[20\] studied static spherically symmetric wormholes sustained by matter sources with isotropic pressure and showed their non-existence in the presence of zero-tidal-force. Wang and Meng \[21\] studied wormholes in the context
of bulk viscosity. They considered three classes of viscous models and obtained various wormhole solutions. Moradpour [22] carried out a study of traversable wormholes in general relativity and Lyra geometry. He found a possibility of a Lyra displacement vector field so that energy conditions are satisfied in a Lyra manifold. Tsukamoto and Kokubu [23] investigated linear stability of thin shell wormholes filled with a barotropic fluid. Barros and Lobo [24] studied wormhole structures using three form fields and obtained various solutions. They found the validation of weak and null energy conditions in the presence of three-form fields.

Several experimental data have declared the current cosmic acceleration of the universe [25–28]. Einstein’s theory of general relativity was found unable to explain this acceleration. Consequently, various modified theories have been introduced in literature. The $f(R)$ and $f(R, T)$ theories are well known theories among them. Starobinsky [29] introduced first $f(R)$ model taking $f(R) = R + \alpha R^2$ representing inflationary scenario. Caroll et al. [30] made some correction in gravitational action by adding the term $R^{-n}$, where $n > 0$, and explained cosmic acceleration. Many viable cosmological models in modified $f(R)$ gravity have been discussed in [31–40]. Harko et al. [41] generalized the gravitational action $f(R)$ by including the stress energy tensor term and developed a new theory called $f(R, T)$ theory of gravity. Houndjo [42] obtained the unification of both deceleration and acceleration phases without neglecting of matter in the context of $f(R, T)$ gravity. Baffou et al. [43] studied the cosmological evolution of deceleration and equation of state parameters using $f(R, T)$ gravity. Various cosmological models have been studied in $f(R, T)$ theory of gravity [44–54]. For recent reviews on not only dark energy problem but also modified gravity theories, see, e.g. [55–63].

Modified theories have also been used for the exploration of wormhole geometry. Hochberg et al. [64] solved semi classical field equations representing wormholes. Nojiri et al. [65] used effective equation method and determined the possibility of the induction of wormholes in early time. Furey and Bendictis [66] considered gravitational action with non-linear powers of Ricci scalar and explored the existence of static wormhole. Dotti et al. [67] obtained wormhole solutions in higher dimensional gravity. Lobo and Oliveira [68] studied traversable wormholes in $f(R)$ gravity. They determined the factors that are responsible for the dissatisfaction of the null energy condition and support the wormhole structures. They obtained various solutions by taking various equations of state and considering some particular shape functions with constant redshift function. Saiedi and Esfahani [69], using constant shape and redshift functions, obtained wormhole solutions in the background of $f(R)$ gravity and investigated null and weak energy conditions. Bouhmadi-Lopez et al. [70] studied spherically symmetric wormholes and investigated stability regions. Duplessis and Easson [71] obtained exotic traversable wormhole and black hole solutions in scale-free $R^2$ gravity which do not require violation of null energy condition. Najafi et al. [72] considered an extra space-like dimension, examined its effect on scale factor, shape function and energy density and explored traversable wormhole in the framework of FRW model. Rahaman et al. [73] studied wormhole solutions for various shape functions. Bahamonde et al. [74] studied wormholes in $f(R)$ gravity. They constructed a dynamical wormhole and found it asymptotically approaching towards the FLRW universe. Moraes et al. [75] obtained analytical general solutions for static wormholes in $f(R, T)$ gravity. Zubair et al. [76] investigated energy conditions and wormhole solutions taking three types of fluids in $f(R, T)$ gravity. Moreas and Sahoo [77] studied static wormholes in $f(R, T)$ theory of gravity and presented some models of wormholes using different assumptions for the matter content. Amir et al. [78] carried out a study of the shadow of charged wormholes in the context of Einstein-Maxwell-dilaton theory. Kuhffittig [79] explored wormholes in $f(R)$ gravity. He considered various shape functions, derived corresponding $f(R)$ functions and found wormhole
solutions. He also considered a special form of function $f(R)$ and obtained wormhole solutions. Recently, Godani and Samanta [80] investigated energy conditions for traversable wormhole in $f(R)$ gravity with shape functions of two types and found the presence of phantom fluid in wormhole structure.

The motivation of this paper is to develop a new shape function to study the wormhole solutions in different theories of gravitation. Therefore, in this paper, a new shape function is defined and wormhole solutions are explored in $f(R)$ gravity, $f(R, T)$ gravity and GR. The section-wise description is as follows: In Section 2, the wormhole structure is discussed. In Sections 3 & 4, brief reviews of $f(R)$ and $f(R, T)$ gravities, respectively, are presented. In Section 5, the field equations are solved and energy condition terms are computed with respect to both modified theories. In Section 6, the findings are discussed and finally, in Section 7, the work is concluded.

2 Wormhole structure

A static and spherically symmetric wormhole structure is defined by the metric

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2 \theta d\Phi^2),$$

where the functions $b(r)$ and $e^{2\Phi(r)}$ are called as shape and redshift functions respectively. The radial coordinate $r$ varies from $r_0$ to $\infty$, where $r_0$ is called the radius of the throat. The angles $\theta$ and $\phi$ vary from 0 to $\pi$ and 0 to $2\pi$ respectively. To avoid the presence of horizons and singularities, the redshift function should be finite and non-zero. The shape function should satisfy the following properties: (i) $b(r)/r < 1$ for $r > r_0$, (ii) $b(r_0) = r_0$ at $r = r_0$, (iii) $b(r)/r \to 0$ as $r \to \infty$, (iv) $b(r)/r < b'(r)/r^2 > 0$ for $r > r_0$ and (v) $b'(r) - 1 \leq 0$ at $r = r_0$. The condition (i) is necessary for the radial metric component to be negative. The shape function possesses minimum value equal to $r_0$ given by condition (ii). To obtain asymptotically flat space time as $r \to \infty$, the condition (iii) is required. Conditions (iv) and (v) are known as flaring out condition which are required to obtain traversable wormholes.

In this paper, we defined new shape function $b(r)$ as follows:

$$b(r) = \frac{r}{\exp(r - r_0)}. \quad (2)$$

This shape function satisfies all the conditions discussed above (Fig. (1)). Now, we consider an equilateral slice $\theta = \pi/2$ for the embedding of a 2-dimensional surface in a 3-dimensional Euclidean space. For constant time $t$, the metric takes the form

$$ds^2 = \frac{1}{1 - b(r)/r}dr^2 + r^2 d\phi^2. \quad (3)$$

In cylindrical coordinates, the metric for the 3-dimensional Euclidean space is given by

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2. \quad (4)$$

For the embedding of an axially symmetric embedded surface, $z$ will be dependent on $r$ only. Therefore,

$$ds^2 = (1 + \left(\frac{dz}{dr}\right)^2)dr^2 + r^2 d\phi^2. \quad (5)$$
From Equations 3 and 5 the expression for the embedding surface is obtained as

$$\frac{dz}{dr} = \pm \left( \frac{r}{b(r)} - 1 \right)^{-1/2}. \quad (6)$$

At $r = r_0$, this embedding surface is ill defined i.e. $\frac{dz}{dr} = \infty$ at $r = r_0$. Now, the proper radial distance is defined as

$$l(r) = \pm \int_{r_0}^{r} \left( \frac{r - b(r)}{r} \right)^{-\frac{1}{2}} \, dr \quad (7)$$

Using shape function (2),

$$l(r) = \pm \int_{r_0}^{r} \left( 1 - \exp(r_0 - r) \right)^{-\frac{1}{2}} \, dr. \quad (8)$$

For this proper radial distance $l(r)$ to be well behaved everywhere, it should be finite everywhere in space time. At a far distance from the throat of wormhole, the space becomes asymptotically flat, so $\frac{dz}{dr} \to 0$ as $l(r) \to \pm \infty$.

Let a traveler starts journey radially from a point in the lower universe to a point in the upper universe. Then the acceleration experienced by the traveler should not be greater than $g = 9.8$ m/sec$^2$. In [3][4], the acceleration experienced by a traveler traveling radially is defined as

$$a = \pm \sqrt{1 - \frac{b(r)}{r} \exp^{-\phi(r)} (\gamma \exp \phi(r))'c^2}$$

$$= \pm \sqrt{1 - \exp(r_0 - r) \exp^{-\phi(r)} (\gamma \exp \phi(r))'c^2}, \quad (9)$$

where $\gamma = (1 - v^2/c^2)^{-1}$; $v$ is the radial velocity of traveler. Thus, $|a| \leq g$. 

Figure 1: Plot for various conditions satisfied by the shape function $b(r)$
As obtained in \[3,4\], for a traveler having the size of the body equal to \(\epsilon\), the expressions for radial and tidal constraints, respectively, are given as

\[
\left| (1 - \frac{b}{r})(-\phi'' - \phi'^2 + \frac{b'r - b}{2r(r-b)}\phi')c^2 \right| |\epsilon| \leq g
\] (10)

and

\[
\left| \frac{\gamma^2 c^2}{2r^2} \left[ \frac{v^2}{c^2} (b' - \frac{b}{r}) + 2(r-b)\phi' \right] \right| |\epsilon| \leq g.
\] (11)

Expressions (10) and (11) provide constraints for the shape function and the velocity of the traveler to cross the wormhole.

For the zero-tidal-force Schwarzschild-like wormholes, the metric takes the form

\[
ds^2 = -dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2 \theta d\Phi^2).
\] (12)

For metric (12), the lateral tidal constraint gives

\[
| r \exp(r_0 - r) \frac{v^2 c^2}{2r^2} |,
\]
however the radial tidal constraint vanishes everywhere. For the non-relativistic motion, \(v << c\) which implies that \(\gamma \approx 1\). So, at \(r = r_0\), \(v \leq \sqrt{0.8r}\).

### 3 Field equations of \(f(R)\) gravity

The gravitational action for Einstein’s theory of general relativity is defined as

\[
S_G = \frac{1}{16\pi} \int [R + L_m] \sqrt{-g} d^4x,
\] (13)

where \(R\) is the curvature scalar, \(L_m\) is the matter Lagrangian density and \(g\) is the determinant of the metric \(g_{\mu\nu}\).

It was found inadequate to describe the present accelerating phase of our universe. Consequently, the action (13) was generalized by replacing \(R\) with an arbitrary function \(f(R)\). It is defined as

\[
S_G = \frac{1}{16\pi} \int [f(R) + L_m] \sqrt{-g} d^4x.
\] (14)

Varying Eq. (14) with respect to \(g_{\mu\nu}\), the field equations are given as

\[
F(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + \square F(R)g_{\mu\nu} = T_{\mu\nu},
\] (15)

where \(R_{\mu\nu}\) is Ricci tensor and \(F(R) = \frac{df}{dR}\). For the matter source of the wormhole, the energy momentum tensor is defined as

\[
T_{\mu\nu} = \frac{\partial L_m}{\partial g^{\mu\nu}},
\]

\[
= (\rho + p_t)u_\mu u_\nu - p_t g_{\mu\nu} + (p_r - p_t)X_\mu X_\nu,
\] (16)
where $\rho$ denotes the energy density, $p_t$ and $p_r$ denote tangential and radial pressures respectively and $u_\mu$ & $X_\mu$ stand for the four velocity and radial vectors respectively such that

$$u^\mu u_\mu = -1 \text{ and } X^\mu X_\mu = 1.$$  

(17)

The Einstein’s field equations for the metric $\mathbb{1}$ in $f(R)$ gravity come out to be:

$$\rho = F(r) \frac{b'(r)}{r^2}$$

(18)

$$p_r = -F(R) \frac{b(r)}{r^3} + F'(r) \frac{rb'(r) - b(r)}{2r^2} - F''(r) (1 - \frac{b(r)}{r^3})$$

(19)

$$p_t = -\frac{F'(r)}{r} (1 - \frac{b(r)}{r^3}) + \frac{F(r)}{2r^3} (b(r) - rb'(r)),$$

(20)

where $R$ is defined as $R(r) = \frac{2b'(r)}{r^2}$ and prime upon a function means the differentiation of that function with respect to radial coordinate $r$.

4 Field equations of $f(R, T)$ gravity

Harko et al. [41] modified Einstein’s general relativity by replacing $R$ with an arbitrary function $f(R, T)$ of $R$ and $T$, where $T$ is the trace of stress energy tensor, and defined the gravitational action as

$$S_G = \frac{1}{16\pi} \int [f(R, T) + L_m] \sqrt{-g} d^4 x$$

(21)

Varying action (21) with respect to the metric, field equations are

$$f(R, T) R_{\mu\nu} - \frac{1}{2} f(R, T) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R(R, T) = 8\pi T_{\mu\nu} - f_T(R, T) \theta_{\mu\nu},$$

(22)

where $\nabla^\mu$ denote the covariant derivative, $\Box = -\nabla^\mu \nabla_\mu$, $\theta_{\mu\nu} = -2T_{\mu\nu} + g^{\mu\sigma} L_m - 2g^{\gamma\sigma} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\gamma\sigma}}$, and $f_R = \frac{\delta f(R, T)}{\delta R}$ and $f_T = \frac{\delta f(R, T)}{\delta T}$.

Then

$$\theta_{\mu\nu} = -2T_{\mu\nu} - pg^{\mu\nu}.$$  

(23)

For $f(R, T) = R + 2f(T)$, the gravitational field equations from Eq.(22) are obtained as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} - 2f'(T) T_{\mu\nu} - 2f'(T) \theta_{\mu\nu} + f(T) g_{\mu\nu}$$

(24)

where $'$ stands for the derivative of $f(T)$ with respect to $T$. Using Eq.(23) in Eq.(24),

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} + 2f'(T) T_{\mu\nu} + [2pf'(T) + f(T)] g_{\mu\nu}.$$  

(25)

The field equations for the wormhole metric $\mathbb{1}$ are come out to be

$$\frac{b'}{r^2} = (8\pi + \lambda) \rho - \lambda (p_r + 2p_t)$$

(26)
\[-\frac{b}{r^3} = \lambda \rho + (8\pi + 3\lambda)p_r + 2\lambda p_t\]  \hfill (27)

\[\frac{b - b'}{2r^3} = \lambda \rho + \lambda p_r + (8\pi + 4\lambda)p_t\]  \hfill (28)

5 Wormhole solutions

Tsujikawa [81] introduced a viable cosmological \(f(R)\) model with the function \(f(R)\) defined as

\[f(R) = R - \mu R_c \tanh \frac{R}{R_c},\]  \hfill (29)

where \(\mu\) and \(R_c\) are positive constants. This model satisfies both cosmological and local gravity constraints. For \(R >> R_c\), \(f(R) \cong R - \mu R_c(1 - \exp(-2\frac{R}{R_c}))\). At the de-Sitter point,

\[\mu = \frac{x_1 \cosh^2(x_1)}{2 \sinh(x_1) \cosh(x_1) - x_1},\]  \hfill (30)

where \(x_1 = R_1/R_c\). For the stability of the model at de-Sitter point, \(x_1 > 0.920\) and \(\mu > 0.905\). For \(R > 0\), \(f_{RR} > 0\) and for \(\mu < 1\), \(f_R > 0\). Hence, for \(0.905 < \mu < 1\), \(f_R > 0\) and \(f_{RR} > 0\) for all positive values of \(R\).

Harko et al. [41] proposed an \(f(R, T)\) model with the function \(f(R, T) = R + 2\lambda T\). In literature, various cosmological models are explored using this model.

In this section, the solutions of wormhole metric (1) and energy condition terms are derived with respect to above \(f(R)\) and \(f(R, T)\) models which are as follows:

5.1 Case 1: For \(f(R)\) model

The wormhole solutions are

\[\rho = \frac{(r - 1)e^{r_0 - r}}{r^2} \left( m \text{sech}^2 \left( \frac{2(r - 1)e^{r_0 - r}}{cr^2} \right) - 1 \right) \]  \hfill (31)

\[p_r = \frac{e^{-r} \text{sech}^4 \left( \frac{2(r - 1)e^{r_0 - r}}{cr^2} \right)}{8c^2 r^2} \left( 4 \left( c^2 (m - 1)e^{r_0} + 4mr^2 (e^r - e^{r_0}) \right) \cosh \left( \frac{4(r - 1)e^{r_0 - r}}{cr^2} \right) \right. \]  
\[\left. + 4c^2 me^{r_0} - c^2 e^{r_0} \cosh \left( \frac{8(r - 1)e^{r_0 - r}}{cr^2} \right) - 3c^2 e^{r_0} + 4cmr^2 e^{r_0} \sinh \left( \frac{4(r - 1)e^{r_0 - r}}{cr^2} \right) \right) \]  
\[+ 32m^2 e^{r_0} - 32me^{r} r^2 \]  \hfill (32)

\[p_t = \frac{e^{-r} \left( ce^{r_0} - m \left( ce^{r_0} - 4 \left( e^r - e^{r_0} \right) \tanh \left( \frac{2(r - 1)e^{r_0 - r}}{cr^2} \right) \right) \text{sech}^2 \left( \frac{2(r - 1)e^{r_0 - r}}{cr^2} \right) \right)}{2cr} \]  \hfill (33)
From Equations (31), (32) and (33),

\[
\begin{align*}
\rho + p_r &= \frac{(r - 1)e^{r_0 - r}}{r^2} \left( \text{msech}^2 \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right) - 1 \right) + \frac{e^{-r} \text{sech}^4 \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right)}{8c^2 r^2} \\
&\times \left( 4 \left( c^2 (m-1)e^{r_0} + 4mr^2 (e^r - e^{r_0}) \right) \cosh \left( \frac{4(r - 1)e^{r_0-r}}{cr^2} \right) + 4c^2 me^{r_0} \\
&- c^2 e^{r_0} \cosh \left( \frac{8(r - 1)e^{r_0-r}}{cr^2} \right) - 3c^2 e^{r_0} + 4cmr^2 e^{r_0} \sinh \left( \frac{4(r - 1)e^{r_0-r}}{cr^2} \right) \\
&+ 32mr^2 e^{r_0} - 32me^r r^2 \right) \\
\rho + p_t &= \frac{(r - 1)e^{r_0 - r}}{r^2} \left( \text{msech}^2 \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right) - 1 \right) \\
&+ \frac{e^{-r} \left( ce^{r_0} - m (ce^{r_0} - 4 (e^r - e^{r_0}) \tanh \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right) \right) \text{sech}^2 \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right)}{2cr} \\
\rho - |p_r| &= \frac{(r - 1)e^{r_0 - r}}{r^2} \left( \text{msech}^2 \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right) - 1 \right) - \frac{e^{-r} \text{sech}^4 \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right)}{8c^2 r^2} \\
&\times \left( 4 \left( c^2 (m-1)e^{r_0} + 4mr^2 (e^r - e^{r_0}) \right) \cosh \left( \frac{4(r - 1)e^{r_0-r}}{cr^2} \right) + 4c^2 me^{r_0} \\
&- c^2 e^{r_0} \cosh \left( \frac{8(r - 1)e^{r_0-r}}{cr^2} \right) - 3c^2 e^{r_0} + 4cmr^2 e^{r_0} \sinh \left( \frac{4(r - 1)e^{r_0-r}}{cr^2} \right) \\
&+ 32mr^2 e^{r_0} - 32me^r r^2 \right) \\
\rho - |p_t| &= \frac{(r - 1)e^{r_0 - r}}{r^2} \left( \text{msech}^2 \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right) - 1 \right) \\
&- \frac{e^{-r} \left( ce^{r_0} - m (ce^{r_0} - 4 (e^r - e^{r_0}) \tanh \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right) \right) \text{sech}^2 \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right)}{2cr} \\
pt - pr &= \frac{e^{-r} \text{sech}^4 \left( \frac{2(r-1)e^{r_0-r}}{cr^2} \right)}{8c^2 r^2} \left( 4 \left( c^2 (m-1)e^{r_0} + 4mr^2 (e^r - e^{r_0}) \right) \cosh \left( \frac{4(r - 1)e^{r_0-r}}{cr^2} \right) \right)
\end{align*}
\]

\[35\]

\[36\]

\[37\]

5.2 Case 2: For \(f(R, T)\) model

In \(f(R, T)\) gravity, for every choice of function \(f(R, T)\) several theoretical models are obtained. From Equations [20, 27, 28] and [2], we can have
Figure 2: Plots for Density, NEC, DEC, $\triangle$ & $\omega$
ρ = \frac{-(r - 1)e^{ro - r}}{2(\lambda + 4\pi)r^2} \quad (38)

p_r = \frac{e^{ro - r}}{(2\lambda + 8\pi)r^2} \quad (39)

p_t = \frac{4\lambda r + 16\pi r}{4(\lambda r + 16\pi r)} \quad (40)

From Equations (38), (39) and (40),

\rho + p_r = \frac{(-r + 2)e^{ro - r}}{2(\lambda + 4\pi)r^2} \quad (41)

\rho + p_t = -\frac{e^{ro - r}}{2(\lambda + 4\pi)r^2} + \frac{e^{ro - r}}{4\lambda r + 16\pi r} \quad (42)

\rho - |p_r| = -\frac{e^{ro - r}}{2(\lambda + 4\pi)r^2} - \frac{(r - 1)e^{ro - r}}{(2\lambda + 8\pi)r^2} \quad (43)

\rho - |p_t| = -\frac{e^{ro - r}}{2(\lambda + 4\pi)r^2} - \frac{(r - 1)e^{ro - r}}{4\lambda r + 16\pi r} \quad (44)

If \triangle < 0, then the geometry is said to be attractive; if \triangle > 0, then the geometry is said to be repulsive and if \triangle = 0, then the geometry has an isotropic pressure.

6 Results and discussion

The alternate theories have a significant role in the exploration of wormhole geometries. The metric of wormhole depends mainly on two functions: shape function and redshift function. For simplicity, the redshift function is taken constant. However, the shape function that contributes for the shape of the wormhole is newly defined in terms of exponential function as \( b(r) = \exp(r - r_0) \). Using this shape function, wormhole structure is investigated by determining various energy conditions, anisotropy parameter and equation of state parameter in the context of \( f(R) \), \( f(R, T) \) and general relativity theories.

From the field equations and energy condition terms derived in \( f(R) \) gravity, in Section 3, the following results are drawn: The value of \( \mu > 0.905 \) does not affect the results. The energy density is obtained to be a positive function decreasing to zero with the increment of radial coordinate \( r \) from 0 to 1. While for \( r > 1 \), it is found to be negative, which is unacceptable (Figs. 2(a)). The null energy condition in terms of pressure is defined as \( \rho + p_r \geq 0 \) and \( \rho + p_t \geq 0 \). The first null energy condition term \( \rho + p_r \) is found to be undefined for \( r \in (0, 44.61) \), negative for \( r \in [44.61, 58.75) \cup [63.72, \infty) \) (Figs. 2(b), 2(d)) and positive for \( r \in (58.75, 63.72) \) (Fig. 2(c)). The second null energy condition term i.e. \( \rho + p_t \) is negative for \( r \in [2.01, 59.25] \cup \{1\} \) and positive for rest values of \( r \) (Figs. 2(e), 2(f)). Thus, the null energy condition is not dissatisfied for every \( r > r_0 \). Hence the null energy condition is violated near the throat of the wormholes, which supports the existence of the wormhole solutions. The dominated energy condition is defined as \( \rho - |p_r| \geq 0, \rho - |p_t| \geq 0 \) and \( \rho \geq 0 \). The first dominated energy condition term i.e. \( \rho - |p_r| \) is undefined for \( r \in (0, 44.71) \) and less than zero for \( r \geq 44.71 \) (Fig. 2(g)). The second dominated
energy condition term $\rho - |p_t|$ is obtained to be positive for $0 < r \leq 0.67$ and negative for $r > 0.67$ (Figs. 2(h), 2(i)). Subsequently, the dominate energy condition is also violated near the throat of the wormhole, which supports the existence of the wormhole solutions. The anisotropy parameter is defined as $\Delta = p_t - p_r$. The anisotropy parameter is undefined for $0 < r < 44.71$, hence there is no idea about the nature of the geometry (i.e., whether the geometry is repulsive or attractive) of the wormhole within the range $0 < r < 44.71$. The anisotropy parameter is obtained to be positive for $r \in [44.71, 58.84) \cup (63.81, \infty)$, which indicates the nature of the geometry is repulsive. Finally, the anisotropy parameter is found to be negative for $r \in [58.84, 63.81]$, which indicates the nature of the geometry of the wormhole is attractive. This shows that the geometry is neither attractive nor repulsive for all values of $r$. It possesses repulsive or attractive natures in different intervals. Please see the figure (Figs. 2(j), 2(K)) for more detail observations about the anisotropy parameter. The equation of state parameter in terms of radial pressure is defined as $w = \frac{p_r}{\rho}$. It is obtained to be undefined for $r \in (0, 44.71)$, positive for $r \in [44.71, 58.83)$ and negative for $r \in [58.83, \infty)$ (Figs. 2(l), 2(m)). Hence, we may conclude that for $r \in (0, 44.71)$, we do not have any knowledge about the matter, which is contained in the wormhole. For $r \in [44.71, 58.83)$, we conclude that the wormhole may contain only ordinary type of matter. For $r \in [58.83, \infty)$, we may conclude that the wormhole may contain some abnormal type of matter, which is may be called as dark energy. This shows that the wormholes are not filled with same type of matter for all values of $r$.

Now, from the field equations and energy condition terms derived for $f(R, T)$ gravity in Section 3, the following analysis is done: the energy density $\rho$ is positive and decreases to zero as $r$ increases from 0 to 1 and $\lambda$ increases from $-4\pi$ to $\infty$ or $r$ increases from 1 to $\infty$ and $\lambda$ decreases from $-4\pi$ to $-\infty$ (Figs. 3(a), 3(b)). So, we can get positive energy density for $0 < r < 1$, if $\lambda > -4\pi$, unless energy density is negative for the same range of $r$. Similarly, we can have positive energy.
density for \( r > 1 \), if \( \lambda < -4\pi \), unless energy density is negative for the same range of \( r \). Hence, it is observed that the entire wormhole does not contain either positive energy density or negative energy density. Therefore, we may conclude that if the energy density is negative near throat, then outside throat it is positive or vice versa. So, to observe negative or positive energy density near the throat of the wormhole, we will have to restrict \( \lambda < -4\pi \) or \( \lambda > -4\pi \) respectively. Similarly, to observe negative or positive energy density outside the throat of the wormhole, we will have to restrict \( \lambda > -4\pi \) or \( \lambda < -4\pi \) respectively. From the equations (39) and (40) it is observed that the radial pressure \( (p_r) \) and the tangential pressure \( (p_t) \) are negative for \( \lambda < -4\pi \) and positive for \( \lambda > -4\pi \). Both terms \( \rho + p_r \) and \( \rho + p_t \) are obtained to be negative and increasing to zero as \( r \) increases from 0 to 2 and \( \lambda \) decreases from \(-4\pi\) to \(-\infty\) or \( r \) increases from 2 to \( \infty \) and \( \lambda \) increases from \(-4\pi\) to \( \infty \) (Figs. 3(c)-3(f)). If we consider \( \lambda < -4\pi \), then the null energy condition \( \rho + p_r \) and \( \rho + p_t \) will be negative near the throat of the wormhole, i.e. the null energy condition is violated near the throat of the wormhole, however, the null energy condition is satisfied outside the throat of the wormhole for \( \lambda < -4\pi \). Both dominated energy condition terms \( \rho - |p_r| \geq 0 \) and \( \rho - |p_t| \geq 0 \) are found to be negatively increasing functions tending towards zero as \( r \) increases from 0 to 1 and \( \lambda \) decreases from \(-4\pi\) to \(-\infty\) or \( r \) increases from 1 to \( \infty \) and \( \lambda \) increases from \(-4\pi\) to \( \infty \) (Figs. 3(g)-3(j)). Similarly, the dominated energy condition is violated near the throat for \( \lambda < -4\pi \), however, the dominated energy condition is satisfied outside the throat of the wormhole for \( \lambda < -4\pi \). Thus, it is observed that all the energy conditions are violated near the throat of the wormhole, however, these are satisfied outside the throat of the wormhole for \( \lambda < -4\pi \). The anisotropy parameter \( \Delta \) is negative i.e. the geometry is attractive, if \( r \) increases from 0 to 2 and \( \lambda \) increases from \(-4\pi\) to \( \infty \) or \( r \) increases from 2 to \( \infty \) and \( \lambda \) decreases from \(-4\pi\) to \( -\infty \) (Figs. 3(k), 3(l)). The anisotropy parameter \( \Delta \) is positive i.e. the geometry is repulsive if \( r \) increases from 0 to 2 and \( \lambda \) decreases from \(-4\pi\) to \( -\infty \) or \( r \) increases from 2 to \( \infty \) and \( \lambda \) increases from \(-4\pi\) to \( \infty \) (Figs. 3(m), 3(n)). It is observed that the geometry of the wormhole near the throat is repulsive and outside the throat is attractive, if we consider \( \lambda < -4\pi \). Similarly, it is observed that the geometry of the wormhole near the throat is attractive and outside the throat is repulsive, if we consider \( \lambda > -4\pi \). Hence, logically we can consider either \( \lambda > -4\pi \) or \( \lambda < -4\pi \), because at a time two range can not be possible. Hence, precisely we may restrict that the \( \lambda < -4\pi \) is acceptable for the existence of wormhole solutions in \( f(R, T) \) gravity. Hence, we conclude that \( \lambda \) play as an important role for the existence of wormhole solutions in \( f(R, T) \) gravity. The equation of state parameter \( \omega \) is found to be dependent on \( r \) only not on \( \lambda \). It is positive for \( r \) lying between 0 and 1 and it is negative for \( r \geq 1 \). For \( r = 1 \), \( \omega = -1 \) and for \( r > 1 \), \( -1 < \omega < 0 \) (Figs. 3(o), 3(p)).

Taking \( \lambda = 0 \) in Equations (38)-(44), the model reduces to GR. For this particular case, it is observed that for \( 0 < r < 1 \), the energy density \( \rho > 0 \), for \( r = 1 \), \( \rho = 0 \) and for \( r > 1 \), \( \rho < 0 \). Since the observations suggest positive density, therefore its acceptable value lies near the throat in the range \( 0 < r < 1 \). Both null energy condition terms \( \rho + p_t \) and \( \rho + p_r \) are positive near the throat, however, these are becoming negative outside the throat of the wormhole. So, it is observed that the null energy condition is satisfied near throat, however, it is violated outside throat. This result is not compatible with the existence of the wormhole solutions. As per the existence of wormhole solutions the null energy conditions must be violated near the throat. However, in this work we obtained the null energy condition is satisfied near the throat and this may be the cause of newly defined exponential shape function \( b(r) = \frac{r}{\exp(r-r_0)} \). Hence, we may conclude that the exponential shape function may not be support for the existence of wormhole solutions in general relativity, however, it supports in \( f(R) \) and \( f(R, T) \) gravity.

13
7 Conclusions

Tsujikawa [81] proposed a viable cosmological $f(R)$ model that fulfills the conditions of stability and local gravity. Harko et al. [41] introduced $f(R, T)$ theory of gravity as an alternative for the description of dark energy. In the present work, the frameworks of these modified gravities are used for the exploration of wormhole structures. First, a newly shape function is defined in terms of the exponential function. Then this shape function is used in field equations of both $f(R)$ and $f(R, T)$ gravities and hence the energy density, null and dominated energy condition terms, equation of state parameter and anisotropy parameter are calculated. In case of $f(R)$, it is found that for some range of radial coordinate null and dominated energy conditions validates and for some of its range these conditions violates. In this case, the geometry also does not possesses the same type of nature. Therefore, we may conclude that the wormhole may contain some abnormal type of matter, which is may be called as dark energy. This shows that the wormholes are not filled with same type of matter for all values of $r$.

However, in case of $f(R, T)$ gravity, we observed, all energy conditions are violated near the throat of the wormhole, while all energy conditions are satisfied outside the throat of the wormhole for $\lambda < -4\pi$. It is also observed that the geometry of the wormhole near the throat is repulsive and outside the throat is attractive for $\lambda < -4\pi$. Eventually, we conclude that $\lambda$ play as an important role for the existence of wormhole solutions in $f(R, T)$ gravity.

For $\lambda = 0$, it reduces to wormhole solutions in general relativity. From the results and discussion section it is observed that the null energy condition is satisfied near the throat of the wormhole in general theory of relativity and this may be the cause of newly defined exponential shape function $b(r) = \frac{r}{\exp(r - r_0)}$. Hence, we may conclude that the exponential shape function may not be support for the existence of wormhole solutions in general relativity, however, it supports in $f(R)$ and $f(R, T)$ gravity. Thus, for the model undertaken, the $f(R, T)$ theory of gravity is found to be a most suitable choice to describe the existence of exotic matter near the throat of the wormhole with newly defined exponential shape function $b(r) = \frac{r}{\exp(r - r_0)}$.

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