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SHARP HARDY-LERAY AND RELLICH-LERAY INEQUALITIES FOR CURL-FREE VECTOR FIELDS

NAOKI HAMAMOTO AND FUTOSHI TAKAHASHI

Abstract. In this paper, we prove Hardy-Leray and Rellich-Leray inequalities for curl-free vector fields with sharp constants. This complements the former work by Costin-Maz’ya [2] on the sharp Hardy-Leray inequality for axisymmetric divergence-free vector fields.

1. Introduction

In this paper, we concern the classical functional inequalities called Hardy-Leray and Rellich-Leray inequalities for smooth vector fields and study how the best constants will change according to the pointwise constraints on their differentials.

Let \( N \in \mathbb{N} \) be an integer with \( N \geq 2 \) and put \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \). In the following, \( C_c^\infty (\mathbb{R}^N)^N \) denotes the set of smooth vector fields

\[ u = (u_1, u_2, \ldots, u_N) : \mathbb{R}^N \ni x \mapsto u(x) \in \mathbb{R}^N \]

having compact supports on \( \mathbb{R}^N \). Let \( \gamma \neq 1 - N/2 \). Then it is well known that

\[
\left( \gamma + \frac{N}{2} - 1 \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} |x|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 |x|^{2\gamma} dx
\]

holds for any vector field \( u \in C_c^\infty (\mathbb{R}^N)^N \) with \( u(0) = 0 \) if \( \gamma < 1 - N/2 \). This is a higher dimensional extension of the 1-dimensional inequality by G. H. Hardy, see [8], also [12], and was first proved by J. Leray [10] in 1933 when the weight \( \gamma = 0 \), see also the book by Ladyzhenskaya [9]. It is also known that the constant \( \left( \gamma + \frac{N}{2} - 1 \right)^2 \) is sharp and never attained. In [2], Costin and Maz’ya proved that if the smooth vector fields are axisymmetric and subject to the divergence-free constraint \( \text{div} u = 0 \), then the constant \( \left( \gamma + \frac{N}{2} - 1 \right)^2 \) in (1) can be improved and replaced by a larger one. More precisely, they proved the following:

**Theorem A.** (Costin-Maz’ya [2]) Let \( N \geq 3 \). Let \( \gamma \neq 1 - N/2 \) be a real number and \( u \in C_c^\infty (\mathbb{R}^N)^N \) be an axisymmetric divergence-free vector field. Assume that \( u(0) = 0 \) if \( \gamma < 1 - N/2 \). Then

\[
C_{N,\gamma} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} |x|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 |x|^{2\gamma} dx
\]

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holds with the optimal constant \( C_{N, \gamma} \) given by

\[
C_{N, \gamma} = \begin{cases} 
(\gamma + \frac{N}{2} - 1)^2 \frac{2N + 1 + (\gamma - \frac{N}{2})^2}{N - 1 + (\gamma - \frac{N}{2})^2} & (N \geq 4, \gamma \geq 1), \\
(\gamma + \frac{N}{2} - 1)^2 + 2 + \min_{\kappa \geq 0} \left( \kappa + \frac{4(N-1)(\gamma-1)}{\kappa + N - 1 + (\gamma - \frac{N}{2})^2} \right) & (N = 3, \gamma > 1), \\
(\gamma + \frac{1}{2})^2 + 2 & (N = 2, \gamma > 1).
\end{cases}
\]

Note that the expression of the best constant \( C_{N, \gamma} \) is slightly different from that in [2] when \( N \geq 4 \), but a careful checking the proof in [2] leads to the above formula in Theorem A. Choosing \( \gamma = 0 \) in Theorem A, we see that the best constant in (1) is actually improved for axisymmetric divergence-free vector fields in the sense that

\[
C_{N,0} = \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx
\]

holds with the optimal constant \( C_{N,0} \) given by

\[
C_{N,0} = \left( \frac{N}{2} - 1 \right)^2 \frac{N^2 + 4N + 4}{N + 1 + N} > \left( \frac{N-2}{2} \right)^2.
\]

In 2-dimensional case, the result in [2] reads as follows:

**Theorem B.** (Costin-Maz’ya [2]) Let \( \gamma \neq 0 \) be a real number and \( u \in C_c^\infty(\mathbb{R}^2) \) be a divergence-free vector field. We assume that \( u(0) = 0 \) if \( \gamma < 0 \). Then

\[
C_{2, \gamma} = \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^2} |\nabla u|^2 \, dx
\]

holds with the optimal constant \( C_{2, \gamma} \) given by

\[
C_{2, \gamma} = \begin{cases} 
\frac{3 + (\gamma - 1)^2}{1 + (\gamma - 1)^2} & \text{for } |\gamma + 1| < \sqrt{3}, \\
\gamma^2 + 1 & \text{otherwise}.
\end{cases}
\]

When \( N = 2 \), the divergence-free field \( u \) in Theorem B need not be axisymmetric. Furthermore, if we consider \( u^+ = (-u_2, u_1) \) for \( u = (u_1, u_2) \) in Theorem B, then the condition \( \text{div} u = 0 \) is replaced by \( \text{curl} u^+ = 0 \) and also \( |\nabla u|^2 = |\nabla u^+|^2 \). Thus the above inequality in Theorem B holds also for curl-free vector fields with the same constant.

Motivated by this observation, our aim in this paper is to generalize Costin-Maz’ya’s result for curl-free vector fields when \( N = 2 \) to higher-dimensional cases. In addition, we also consider the Rellich type inequality involving the higher-order derivative, \( \Delta u \), for curl-free vector fields. We refer to [5] for the Rellich-Leray inequality for divergence-free vector fields. See also [6], [7] for other improvements of [2].

Now, main results of this paper are as follows:

**Theorem 1.** Let \( \gamma \neq 1 - N/2 \) be a real number and let \( u \in C_c^\infty(\mathbb{R}^N) \) be a curl-free vector field. We assume that \( u(0) = 0 \) if \( \gamma < 1 - N/2 \). Then

\[
H_{N, \gamma} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx
\]
Let \( \phi \) corresponds to the higher-dimensional analogue of Theorem B in the sense that \( C \) Beltrami operator on the sphere \( \nabla \) that free fields \( u \) unchanged even if we additionally assume the axisymmetry condition on the curl-free vector fields.

Remark 5. \( \phi \) holds with the optimal constant \( \frac{C}{\|\mathbf{u}\|^2} \).

Corollary 4. \( \phi \) is needed. Theorem 1 corresponds to the higher-dimensional analogue of Theorem B in the sense that \( C_{2;\gamma} = H_{2;\gamma} \).

For curl-free vector fields \( \mathbf{u} \), Poincaré’s lemma implies that there exists a smooth scalar potential \( \phi \) such that \( \mathbf{u} = \nabla \phi \). Thus by using the potential function \( \phi \), Theorem 1 is equivalent to the following corollary.

**Corollary 2.** Let \( \gamma \neq 1 - N/2 \) be a real number and let \( \phi \in C^\infty_c(\mathbb{R}^N) \). We assume that \( \nabla \phi(0) = 0 \) if \( \gamma < 1 - N/2 \). Then

\[
H_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\nabla \phi|^2}{|x|^2} |x|^{2\gamma} \, dx \leq \int_{\mathbb{R}^N} |D^2 \phi|_2^2 |x|^{2\gamma} \, dx
\]

with the optimal constant \( H_{N,\gamma} \) given in (3). Here \( D^2 \phi(x) = \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} (x) \right)_{1 \leq i, j \leq N} \) denotes the Hessian matrix of \( \phi \).

By similar arguments, we prove the following Rellich-Leray inequality for curl-free vector fields.

**Theorem 3.** Let \( \gamma \neq 2 - N/2 \) be a real number and let \( \mathbf{u} \in C^\infty_c(\mathbb{R}^N)^N \) be a curl-free vector field. We assume that \( \int_{\mathbb{R}^N} |x|^{2\gamma-4} |\mathbf{u}|^2 \, dx < \infty \). Then

\[
R_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|x|^4} |x|^{2\gamma} \, dx \leq \int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |x|^{2\gamma} \, dx
\]

with the optimal constant \( R_{N,\gamma} \) given by

\[
R_{N,\gamma} = \min_{\nu \in \mathbb{R} \cup \{0\}} \left( \frac{1 - \frac{N}{2} - \gamma}{3 - \frac{N}{2} - \gamma} \right)^2 + \alpha_\nu \left( \frac{2 - \gamma}{2 - \gamma} - \alpha_\nu \right)^2
\]

for \( \gamma \neq 3 - N/2 \), where we put \( \alpha_s = s(s + N - 2), \quad s \in \mathbb{R} \), and

\[
R_{N,3-N/2} = \begin{cases} 
4(N-2)^2 & \text{for } N = 2, 3, 4, \\
(N+3)(N-1) & \text{for } N \geq 5.
\end{cases}
\]

**Corollary 4.** Let \( \gamma \neq 2 - N/2 \) be a real number and let \( \phi \in C^\infty_c(\mathbb{R}^N) \) be a potential function such that \( \int_{\mathbb{R}^N} |x|^{2\gamma-4} |\nabla \phi|^2 \, dx < \infty \). Then

\[
R_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\nabla \phi|^2}{|x|^4} |x|^{2\gamma} \, dx \leq \int_{\mathbb{R}^N} |\nabla \Delta \phi|^2 |x|^{2\gamma} \, dx
\]

holds with the optimal constant \( R_{N,\gamma} \) as in (5) and (6).

**Remark 5.** The best constants \( H_{N,\gamma} \) and \( R_{N,\gamma} \) in Theorem 1 and 3 are respectively unchanged even if we additionally assume the axisymmetry condition on the curl-free fields \( \mathbf{u} \). Indeed, \( \psi_\nu(\sigma) = P_\nu(-\cos \theta_1) \), where \( P_\nu \) is a Legendre polynomial of \( \nu \)-th order (see Appendix in [5]), is the axisymmetric eigenfunction of the Laplace-Beltrami operator on the sphere \( S^{N-1} \) associated with the eigenvalue \( \alpha_\nu = \nu(\nu +...
proved the following: Let the simple formula a ball on test vector fields for the proof of the sharpness of the constants. Since the test vector fields introduced in [2] may not have compact supports, we will use different we prove Theorem 3 and the sharpness of the constants (5) and (6). Since the test

Concerning Corollary 2 which is equivalent to Theorem 1, we should remark that the similar results already exist by [13], [3]; see also [4] Chapter 6.5. More precisely, improving the work by Tertikas and Zographopoulos [13], Ghoussoub and Moradifam ([3]: Appendix B) proved the following: Let $C_c^\infty(B_R)$ denote the set of smooth functions having compact supports on a ball $B_R \subset \mathbb{R}^N$ with radius $R$. Define

$$A_{N,\gamma}(R) = \inf \left\{ \frac{\int_{B_R} |\Delta \phi|^2 |x|^{2\gamma} \, dx}{\int_{B_R} |\nabla \phi|^2 |x|^{2\gamma} \, dx} : \phi \in C_c^\infty(B_R) \right\}.$$ 

Assume $\gamma \geq 1 - N/2$. Then $A_{N,\gamma}(R)$ is independent of $R$, and is equal to

$$A_{N,\gamma} = \min_{\nu \in \mathbb{N} \cup \{0\}} \left\{ \left( \frac{(N-4+2\gamma)(N-2\gamma)}{4} + \alpha_\nu \right)^2 \right\},$$

where $\alpha_\nu = \nu(N + \nu - 2)$ ($\nu \in \mathbb{N} \cup \{0\}$) is the $\nu$-th eigenvalue of the Laplace-Beltrami operator on the unit sphere $S^{N-1}$ in $\mathbb{R}^N$. Note that by the simple formula

$$|D^2 \phi|^2 = \sum_{i,j=1}^N \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)^2 = \text{div} \left( \frac{1}{2} \nabla |\nabla \phi|^2 - (\Delta \phi) \nabla \phi \right) + (\Delta \phi)^2,$$

for $\phi \in C_c^\infty(B_R)$, we have $\int_{B_R} |D^2 \phi|^2 \, dx = \int_{B_R} |\Delta \phi|^2 \, dx$ which implies $H_{N,0} = A_{N,0}$. However, in weighted cases, it holds $\int_{B_R} |D^2 \phi|^2 |x|^{2\gamma} \, dx = \int_{B_R} |\Delta \phi|^2 |x|^{2\gamma} \, dx$, and in general we have $H_{N,\gamma} \neq A_{N,\gamma}$. Also the inequality in Corollary 4 seems new.

The organization of this paper is as follows: In §2, we recall the method by Costin-Maz’ya in [2] and derive the equivalent curl-free condition in polar coordinates. In §3, we prove Theorem 1 and the sharpness of the constant (3). In §4, we prove Theorem 3 and the sharpness of the constants (5) and (6). Since the test vector fields introduced in [2] may not have compact supports, we will use different test vector fields for the proof of the sharpness of the constants.

### 2. Preparation: Costin-Maz’ya’s setting

In this section, we recall the method of Costin-Maz’ya [2] and derive the polar coordinate representation of the curl-free condition.

**Spherical polar coordinate.** We introduce the spherical polar coordinates

$$(\rho, \theta_1, \theta_2, \cdots, \theta_{N-2}, \theta_{N-1}) \in (0, \infty) \times (0, \pi)^{N-2} \times [0, 2\pi)$$
whose relation to the standard Euclidean coordinates \( \mathbf{x} = (x_1, \cdots, x_N) \in \mathbb{R}^N \) is given by
\[
\mathbf{x} = \rho(\cos \theta_1, \pi_1 \cos \theta_2, \pi_2 \cos \theta_3, \cdots, \pi_{k-1} \cos \theta_k, \cdots, \pi_{N-2} \cos \theta_{N-1}, \pi_{N-1}),
\]
hereafter we use the notation
\[
\pi_0 = 1, \quad \pi_k = \prod_{j=1}^{k} \sin \theta_j, \quad (k = 1, 2, \cdots, N - 1)
\]
for simplicity. Also we use the notation
\[
\partial_\rho = \frac{\partial}{\partial \rho}, \quad \partial_\theta_k = \frac{\partial}{\partial \theta_k}, \quad (k = 1, 2, \cdots, N - 1)
\]
for the partial derivatives, and
\[
dx = \prod_{k=1}^{N} dx_k = dx_1 dx_2 \cdots dx_N, \quad d\sigma = \prod_{k=1}^{N-1} (\sin \theta_k)^{N-k-1} d\theta_k
\]
for the volume elements on \( \mathbb{R}^N \) and \( S^{N-1} \).

The orthonormal basis vector fields \( e_\rho, e_{\theta_1}, e_{\theta_2}, \cdots, e_{\theta_{N-1}} \) along the polar coordinates are given by
\[
(7) \quad \begin{cases} 
    e_\rho = \frac{\partial \mathbf{x}}{\partial \mathbf{\rho}} = (\cos \theta_1, \pi_1 \cos \theta_2, \pi_2 \cos \theta_3, \cdots, \pi_{N-2} \cos \theta_{N-1}, \pi_{N-1}) , \\
    e_{\theta_k} = \frac{\partial \mathbf{x}}{\partial \theta_k} = \frac{1}{\pi_k} \partial_\theta_k e_\rho, \quad (k = 1, 2, \cdots, N - 1)
\end{cases}
\]
that are clearly independent of the radius \( \rho \). Note that we can rewrite them as
\[
e_\rho = (\cos \theta_1, \pi_1 \cos \theta_2, \pi_2 \cos \theta_3, \cdots, \pi_{k-1} \cos \theta_k, \pi_k \varphi_k),
\]
\[
e_{\theta_k} = \left(0, 0, \cdots, 0, -\sin \theta_k, \cos \theta_k \varphi_k\right),
\]
where
\[
\varphi_k = \left(\cos \theta_{k+1}, \frac{\pi_{k+1}}{\pi_k} \cos \theta_{k+2}, \frac{\pi_{k+2}}{\pi_k} \cos \theta_{k+3}, \cdots, \frac{\pi_{N-2}}{\pi_k} \cos \theta_{N-1}, \frac{\pi_{N-1}}{\pi_k}\right) \in S^{N-k-1}
\]
is a \((N-k)\)-vector, which depends only on \( \theta_{k+1}, \cdots, \theta_{N-1} \). From these expressions, we can easily check the orthonormality of \( e_\rho, e_{\theta_1}, e_{\theta_2}, \cdots, e_{\theta_{N-1}} \).

For any smooth vector field \( \mathbf{u} = (u_1, u_2, \cdots, u_N) : \mathbb{R}^N \to \mathbb{R}^N \), its polar components \( u_\rho, u_{\theta_1}, u_{\theta_2}, \cdots, u_{\theta_{N-1}} \) as \( \mathbb{R} \)-valued smooth functions are defined by
\[
\mathbf{u} = u_\rho e_\rho + \sum_{k=1}^{N-1} u_{\theta_k} e_{\theta_k}.
\]
The second term of the right-hand side is denoted by
\[
\mathbf{u}_{\sigma} = \sum_{k=1}^{N-1} u_{\theta_k} e_{\theta_k}
\]
and we call this the spherical component of \( \mathbf{u} \). Thus we have the polar decomposition of \( \mathbf{u} \):
\[
(8) \quad \mathbf{u} = u_\rho e_\rho + \mathbf{u}_{\sigma}
\]
which gives the decomposition of \( u \) into the radial and the spherical parts. Also by using the chain rules together with (7), we have

\[
\partial_\rho = e_\rho \cdot \nabla, \quad \text{and} \quad \frac{1}{\rho} \partial_{\theta_k} = \pi_{k-1} e_{\theta_k} \cdot \nabla, \quad (k = 1, \ldots, N - 1),
\]

which give the polar decomposition of the gradient operator \( \nabla \):

\[
\nabla = e_\rho \partial_\rho + \frac{1}{\rho} \nabla_\sigma,
\]

where

\[
\nabla_\sigma = \sum_{k=1}^{N-1} \frac{e_{\theta_k}}{\pi_{k-1}} \partial_{\theta_k}
\]
is the gradient operator on \( S^{N-1} \).

Moreover, it is well-known that the polar representation of the Laplace operator \( \Delta = \sum_{k=1}^{N} \partial^2 \partial x_k^2 \) is given by

\[
\Delta = \frac{1}{\rho^{N-1}} \partial_\rho \left( \rho^{N-1} \partial_\rho \right) + \frac{1}{\rho^2} \Delta_\sigma,
\]

where

\[
\Delta_\sigma = \sum_{k=1}^{N-1} \frac{(\sin \theta_k)^{k+1-N}}{\pi_{k-1}} \partial_{\theta_k} \left( (\sin \theta_k)^{N-k-1} \partial_{\theta_k} \right) = \sum_{k=1}^{N-1} \frac{1}{\pi_{k-1}} D_{\theta_k} \partial_{\theta_k}
\]
is the Laplace-Beltrami operator on \( S^{N-1} \) and for every \( k = 1, \ldots, N - 1 \)

\[
D_{\theta_k} = \partial_{\theta_k} + (N - k - 1) \cot \theta_k
\]
is the adjoint operator of \( -\partial_{\theta_k} \) in \( L^2(d\sigma, S^{N-1}) \) : it holds that

\[
-\int_{S^{N-1}} f (\partial_{\theta_k} g) d\sigma = \int_{S^{N-1}} (D_{\theta_k} f) g d\sigma
\]
for any smooth functions \( f, g \) on \( S^{N-1} \).

We also introduce the deformed radial coordinate \( t \in \mathbb{R} \) by the Emden transformation

\[
t = \log \rho.
\]
Note that (13) leads to the transformation law of the differential operators \( \rho \partial_\rho = \partial_t \).

By this transformation, it is easy to check that the polar decomposition of \( \nabla \) , \( \Delta \) in (9) , (11) are also given by

\[
\rho \nabla = e_\rho \partial_\rho + \nabla_\sigma,
\]

\[
\rho^2 \Delta = \partial_t^2 + (N - 2) \partial_t + \Delta_\sigma.
\]

For the later use, we prove the following lemma.

**Lemma 7.** Let \( \nabla_\sigma \) and \( \Delta_\sigma \) are given by (10) and (12) respectively. Then for any \( f \in C^\infty(S^{N-1}) \), \( \sigma = e_\rho \in \mathbb{S}^{N-1} \) and \( \alpha \in \mathbb{C} \), there holds that

\[
\Delta_\sigma (e_\rho f) - e_\rho \Delta_\sigma f = (2 \nabla_\sigma - (N - 1) e_\rho) f,
\]

\[
\Delta_\sigma \nabla_\sigma f - \nabla_\sigma \Delta_\sigma f = ((N - 3) \nabla_\sigma - 2 e_\rho \Delta_\sigma) f,
\]

\[
\Delta_\sigma \left( f e_\rho + \alpha \nabla_\sigma f \right) = e_\rho \left( (1 - 2\alpha) \Delta_\sigma f - (N - 1) f \right) + (2 + (N - 3)\alpha) \nabla_\sigma f + \alpha \nabla_\sigma \Delta_\sigma f.
\]
Proof. Take any $f \in C^\infty(S^{N-1})$. We identify $f$ with the function $\tilde{f} \in C^\infty(\mathbb{R}^N \setminus \{0\})$ defined by $\tilde{f}(x) = f(|x|)$. Since $\tilde{f} = \tilde{f}$ does not depend on the radius $\rho$, we have $\nabla_\sigma f = \rho \nabla f$ by (9) and $\Delta_\sigma f = \rho^2 \Delta f$ by (11). Thus we compute

$$\Delta_\sigma(e_\rho f) - e_\rho \Delta f = \rho^2 \Delta \left( \frac{x f}{\rho} \right) - \frac{x}{\rho} \rho^2 \Delta f$$

$$= \rho^2 \left( \frac{\Delta(x f)}{\rho} + 2 \left( (\nabla \rho^{-1}) \cdot \nabla \right) (x f) + (\Delta \rho^{-1})x f \right) - \rho x \Delta f$$

$$= 2 \rho (\nabla f \cdot \nabla) x - 2 (\nabla \rho \cdot \nabla) (x f) + \rho^3 (\Delta \rho^{-1}) e_\rho f$$

$$= 2 \rho \nabla f - 2 \partial_\rho (\rho e_\rho f) - (N - 3) e_\rho f$$

$$= (2 \nabla - (N - 1) e_\rho) f,$$

here we have used $\nabla \rho \cdot \nabla = \partial_\rho$ and $\Delta \rho^{-1} = -(N - 3) \rho^{-3}$. This proves (16). Similarly, also noting the commutativity $\Delta \nabla = \nabla \Delta$ and using $\Delta \rho = (N - 1) \rho^{-1}$, we have

$$(\Delta_\sigma \nabla_\sigma - \nabla_\sigma \Delta_\sigma) f = \rho^2 \Delta \nabla_\sigma f - \rho \nabla \Delta_\sigma f$$

$$= \rho^2 \Delta (\rho \nabla f) - \rho \nabla (\rho^2 \Delta f)$$

$$= \rho^2 ((\Delta \rho) \nabla f + 2 (\nabla \rho \cdot \nabla) \nabla f) - \rho (\nabla \rho^2) \Delta f$$

$$= (N - 1) \rho \nabla f + 2 \rho^2 \partial_\rho (\rho^{-1}) \nabla_\sigma f - 2 \rho^2 e_\rho \Delta f$$

$$= (N - 3) \nabla_\sigma f - 2 e_\rho \Delta_\sigma f.$$}

This proves (17). Finally, by (16) and (17), we see

$$\Delta_\sigma(e_\rho f + \alpha \nabla_\sigma f) = \Delta_\sigma(e_\rho f) + \alpha \Delta_\sigma \nabla_\sigma f$$

$$= (e_\rho \Delta_\sigma + 2 \nabla_\sigma - (N - 1) e_\rho) f + \alpha (\nabla_\sigma \Delta_\sigma + (N - 3) \nabla_\sigma - 2 e_\rho \Delta_\sigma) f$$

$$= e_\rho ((1 - 2 \alpha) \Delta_\sigma f - (N - 1) f) + (2 + (N - 3) \alpha) \nabla_\sigma f + \alpha \nabla_\sigma \Delta_\sigma f.$$

This proves (18).

Representing the curl-free condition in polar coordinates. In the following, let “$\cdot$” denote the standard inner product in $\mathbb{R}^N$, “$\wedge$” the wedge product for differential forms and “$d$” the exterior derivative operator. For a vector field $\mathbf{a} = (a_1, a_2, \ldots, a_N) : \mathbb{R}^N \to \mathbb{R}^N$, we define the vector valued 1-form $\mathbf{da} = (da_1, da_2, \ldots, da_N)$. If $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ is a vector field, then $\mathbf{u} \cdot \mathbf{da}$ denotes the 1-form $\sum_{k=1}^N u_k da_k$. Now, for any $C^\infty$ vector field $\mathbf{u} : \mathbb{R}^N \to \mathbb{R}^N$ with variable $\mathbf{x} = (x_1, \ldots, x_N)$, we define its curl as the differential 2-form

$$\text{curl } \mathbf{u} = d(\mathbf{u} \cdot d\mathbf{r}).$$

This can be expressed in terms of the standard Euclidean coordinates, according to the usual manipulations for the exterior derivative $d$ and the wedge product $\wedge$:

$$d(\mathbf{u} \cdot d\mathbf{r}) = \sum_{k=1}^N du_k \wedge dx_k = \sum_{k=1}^N \sum_{j=1}^N \frac{\partial u_k}{\partial x_j} dx_j \wedge dx_k = \sum_{j<k} \left( \frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right) dx_j \wedge dx_k.$$

As well as the standard representation, we want to find a representation of $d(\mathbf{u} \cdot d\mathbf{r})$ in terms of the polar coordinates $(\rho, \theta_1, \ldots, \theta_{N-1})$. For this purpose, first we
differentiate the unit vector field \( e_\rho \) given by (7) and expand it in the spherical-coordinate basis:

\[
de_\rho = \sum_{k=1}^{N-1} \frac{\partial e_\rho}{\partial \theta_k} d\theta_k = \sum_{k=1}^{N-1} e_{\theta_k} \pi_{k-1} d\theta_k .
\]

Then, taking the inner product with the vector field \( u = u_\rho e_\rho + \sum_{k=1}^{N-1} u_{\theta_k} e_{\theta_k} \) and also taking its exterior derivative, we see that

\[
u \cdot de_\rho = \sum_{k=1}^{N-1} u_{\theta_k} \pi_{k-1} d\theta_k ,
\]

\[
d(u \cdot de_\rho) = dp \wedge \sum_{k=1}^{N-1} (\partial_{\theta_k} u_{\theta_k}) \pi_{k-1} d\theta_k + \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \partial_{\theta_j} (\pi_{k-1} u_{\theta_k}) d\theta_j \wedge d\theta_k .
\]

Also we have

\[
u \cdot dx = \nu \cdot (dp e_\rho + \rho de_\rho) = \nu dp + \rho \nu \cdot de_\rho .
\]

From these relations, we obtain the polar representation of the curl of \( \nu \):

\[
d(u \cdot dx) = d(u_\rho dp + \rho \nu \cdot de_\rho)
\]

\[
= du_\rho \wedge dp + dp \wedge (u \cdot de_\rho) + \rho d(u \cdot de_\rho)
\]

\[
= dp \wedge \left( -du_\rho + \sum_{k} u_{\theta_k} \pi_{k-1} d\theta_k + \sum_{k} \rho \partial_{\theta_k} u_{\theta_k} \pi_{k-1} d\theta_k \right)
\]

\[
+ \rho \sum_{j} \sum_{k} \partial_{\theta_j} (\pi_{k-1} u_{\theta_k}) d\theta_j \wedge d\theta_k
\]

\[
= dp \wedge \left( \sum_{k} \left( \pi_{k-1} \partial_{\theta_k} (\rho u_{\theta_k}) - \partial_{\theta_k} u_\rho \right) \right) d\theta_k
\]

\[
+ \rho \sum_{j<k} \left( \partial_{\theta_j} (\pi_{k-1} u_{\theta_k}) - \partial_{\theta_k} (\pi_{j-1} u_{\theta_j}) \right) d\theta_j \wedge d\theta_k .
\]

Therefore, the curl-free condition \( d(\nu \cdot x) = 0 \) for the vector field \( \nu \) is represented by

\[
\begin{align*}
\partial_{\theta_j} (\rho \pi_{k-1} u_{\theta_k}) &= \partial_{\theta_k} u_\rho, \\
\partial_{\theta_j} (\pi_{k-1} u_{\theta_k}) &= \partial_{\theta_k} (\pi_{j-1} u_{\theta_j}) \quad \left( j, k = 1, 2, \ldots, N-1 \right).
\end{align*}
\]

We claim that the second relation in (19) is actually a consequence of the first. Indeed, by integrating the first equation in (19) on any interval \((0, r] \subset \mathbb{R}\) with respect to the measure \( dp \), we have \( r \pi_{k-1} u_{\theta_k} = \partial_{\theta_k} \int_{0}^{r} u_\rho dp \) for every \( k \). Thus the function \( \phi \in C^\infty(\mathbb{R}^N \setminus \{0\}) \) defined by \( \phi(x) = \frac{1}{|x|} \int_{0}^{|x|} \rho(x/|x|) dp \) satisfies \( \pi_{k-1} u_{\theta_k} = \partial_{\theta_k} \phi \) for all \( k \). Then the second relation in (19) is equivalent to \( \partial_{\theta_j} \partial_{\theta_k} \phi = \partial_{\theta_k} \partial_{\theta_j} \phi \), which holds trivially. This proves the claim.

Consequently we have proved that a vector field \( \nu \in C^\infty(\mathbb{R}^N)^N \) is curl-free if and only if

\[
\partial_{\theta_k} (\rho u_{\theta_k}) = \frac{1}{\pi_{k-1}} \partial_{\theta_k} u_\rho , \quad k = 1, \ldots, N-1 .
\]

That is, using the same vector notation as in (8) and (9), we have

\[
\partial_{\theta_k} (\rho u_\sigma) = \nabla_\sigma u_\rho \quad (\rho, \sigma) \in \mathbb{R}_+ \times \mathbb{S}^{N-1} .
\]
In what follows we also call (20) the curl-free condition for \( u \).

**Brezis-Vazquez, Maz’ya transformation.** Let \( \varepsilon \neq 1 \) be a real number. As in [1], [11], we introduce a new vector field \( v \) by the formula

\[
v(x) = \rho^{1-\varepsilon} u(x).
\]

Then the curl-free condition (20) is transformed into

\[
\nabla_{\sigma}(\rho^{\varepsilon-1} v_\rho) = \partial_\rho(\rho^\varepsilon v_\sigma) ,
\]

that is,

\[
(\varepsilon + \rho \partial_\rho)v_\sigma = \nabla_\sigma v_\rho .
\]

**Fourier transformation in radial direction.** In the following, let us use the abbreviation \( v(t, \sigma) = v(\rho \sigma) \) for a vector field \( v(x) = v(\rho \sigma) \), where \( t = \log \rho \) is the Emden transformation given in (13). As in [2], we apply the one-dimensional Fourier transformation

\[
v(t, \sigma) \mapsto \hat{v}(\lambda, \sigma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda t} v(t, \sigma) dt
\]

with respect to the variable \( t \). By the transformation law between the derivative operators

\[
\rho \partial_\rho = \partial_t \mapsto \partial_\lambda = i \lambda ,
\]

the curl-free condition (22) is changed into the equation

\[
(\varepsilon + i \lambda)\hat{v}_\sigma = \nabla_\sigma \hat{v}_\rho ,
\]

that is,

\[
\hat{v}_\sigma = \frac{1}{\varepsilon + i \lambda} \nabla_\sigma f \quad \text{where} \quad f = \hat{v}_\rho .
\]

Thus we see that \( \hat{v}_\sigma \) is expressed by the spherical gradient of some function \( f = \hat{v}_\rho \).

In this sense, we may consider \( f \) as a kind of scalar potential of \( \hat{v}_\sigma \), corresponding to the fact that the curl-free vector field \( u \) has a scalar potential.

Now we have proved the following proposition:

**Proposition 8.** Let \( \varepsilon \neq 1 \) and let \( u \) be a smooth vector field on \( \mathbb{R}^N \). Then \( u \) is curl-free if and only if its Brezis-Vázquez, Maz’ya transformation \( v(t, \sigma) = e^{\varepsilon(1-\varepsilon)} u(e^{\lambda} \sigma) \) satisfies

\[
(\varepsilon + \partial_\lambda)v_\sigma = \nabla_\sigma v_\rho .
\]

In particular, if \( u \) is curl-free and has a compact support on \( \mathbb{R}^N \), then the Fourier transformation of \( v \) satisfies

\[
\hat{v}(\lambda, \sigma) = f e_\rho + \frac{1}{\varepsilon + i \lambda} \nabla_\sigma f
\]

for some complex-valued scalar function \( f = f(\lambda, \sigma) \in C^\infty(\mathbb{R} \times \mathbb{S}^{N-1}) \).
We list up some formulae for \( \hat{\mathbf{v}} \) and its differentials. The square length of \( \hat{\mathbf{v}} \) is

\[
|\hat{\mathbf{v}}|^2 = |f|^2 + \frac{1}{\varepsilon^2 + \lambda^2} |\nabla_\sigma f|^2.
\]

By using Lemma 7, we also see that

\[
-\Delta_\sigma \hat{v} = e_\rho \left( (N - 1)f + \left( \frac{2}{\varepsilon + i\lambda} - 1 \right) \Delta_\sigma f \right) - \left( \frac{N - 3}{\varepsilon + i\lambda} + 2 \right) \nabla_\sigma f - \frac{1}{\varepsilon + i\lambda} \nabla_\sigma \Delta_\sigma f.
\]

Then integrating \( |\hat{\mathbf{v}}|^2 \), \(-\hat{\mathbf{v}} \cdot \Delta_\sigma \hat{\mathbf{v}}\) and \( |\Delta_\sigma \hat{\mathbf{v}}|^2 \) over \( S^{N-1} \), we find that

\[
\begin{align*}
\int_{S^{N-1}} |\hat{\mathbf{v}}|^2 \, d\sigma &= \int_{S^{N-1}} \mathcal{F} \left( 1 + \frac{1}{\varepsilon^2 + \lambda^2} (-\Delta_\sigma) \right) \, d\sigma, \\
\int_{S^{N-1}} |\nabla_\sigma \hat{\mathbf{v}}|^2 \, d\sigma &= \int_{S^{N-1}} \mathcal{F} \left( N - 1 + \left( 1 + \frac{3 - 4\varepsilon - N}{\varepsilon^2 + \lambda^2} \right) (-\Delta_\sigma) + \frac{1}{\varepsilon^2 + \lambda^2} (-\Delta_\sigma)^2 \right) \, d\sigma, \\
\int_{S^{N-1}} |\Delta_\sigma \hat{\mathbf{v}}|^2 \, d\sigma &= \int_{S^{N-1}} \mathcal{F} \left( (N - 1)^2 + \left( 2N + 2 + \frac{(N - 3)^2 - 8\varepsilon}{\varepsilon^2 + \lambda^2} \right) (-\Delta_\sigma) \right. \\
&\quad \left. + \left( 1 + \frac{10 - 8\varepsilon - 2N}{\varepsilon^2 + \lambda^2} \right) (-\Delta_\sigma)^2 + \frac{1}{\varepsilon^2 + \lambda^2} (-\Delta_\sigma)^3 \right) \, d\sigma.
\end{align*}
\]

Thus, we have proved the following lemma.

**Lemma 9.** Let \( \hat{\mathbf{v}} = f e_\rho + \frac{1}{\varepsilon + i\lambda} \nabla_\sigma f \) as in (24). Then

\[
\begin{align*}
\int_{S^{N-1}} |\hat{\mathbf{v}}|^2 \, d\sigma &= \int_{S^{N-1}} \mathcal{F}(\lambda, -\Delta_\sigma) \, d\sigma, \\
\int_{S^{N-1}} |\nabla_\sigma \hat{\mathbf{v}}|^2 \, d\sigma &= \int_{S^{N-1}} \mathcal{F}(\lambda, -\Delta_\sigma) \, d\sigma, \\
\int_{S^{N-1}} |\Delta_\sigma \hat{\mathbf{v}}|^2 \, d\sigma &= \int_{S^{N-1}} \mathcal{F}(\lambda, -\Delta_\sigma) \, d\sigma
\end{align*}
\]

where the three polynomials \( \alpha \mapsto P_k(\lambda, \alpha) \) \((k = 1, 2, 3)\) are given by

\[
\begin{align*}
P_1(\lambda, \alpha) &= 1 + \frac{1}{\varepsilon^2 + \lambda^2} \alpha, \\
P_2(\lambda, \alpha) &= N - 1 + \left( 1 + \frac{3 - 4\varepsilon - N}{\varepsilon^2 + \lambda^2} \right) \alpha + \frac{1}{\varepsilon^2 + \lambda^2} \alpha^2, \\
P_3(\lambda, \alpha) &= (N - 1)^2 + \left( 2N + 2 + \frac{(N - 3)^2 - 8\varepsilon}{\varepsilon^2 + \lambda^2} \right) \alpha \\
&\quad + \left( 1 + \frac{10 - 8\varepsilon - 2N}{\varepsilon^2 + \lambda^2} \right) \alpha^2 + \frac{1}{\varepsilon^2 + \lambda^2} \alpha^3.
\end{align*}
\]

### 3. Proof of Theorem 1

Let \( \gamma \neq 1 - N/2 \) be a real number and put \( \varepsilon = 2 - N/2 - \gamma \neq 1 \). If the right-hand side of (2) diverges, there is nothing to prove. When the right-hand side of (2) is finite, the smoothness of \( \mathbf{u} \) implies the existence of an integer \( m > \varepsilon - 2 \) such that \( \nabla \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^m) \) as \( |\mathbf{x}| \to +0 \). If \( \varepsilon < 1 \), then the vector field \( \mathbf{v}(\mathbf{x}) \) in (21) is Hölder continuous at \( \mathbf{x} = 0 \) and satisfies \( \mathbf{v}(0) = 0 \). When \( \varepsilon > 1 \), again
the assumption \( u(0) = 0 \) implies \( u(x) = O(|x|^{m+1}) \) and \( v(x) = O(|x|^{m+2-\varepsilon}) \), thus the same properties hold for \( u \). Also since
\[
\nabla u = \nabla (\rho^2 v) = \rho^{-2} ((\varepsilon - 1) x \otimes v + \rho \nabla v)
\]
by (14), and since \( \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \partial x \cdot v \, d\sigma \, dt \) vanishes, we calculate
(25)
\[
\int_{\mathbb{R}^N} |x|^{2\gamma} |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} |x|^{4-2\varepsilon-N} |\nabla u|^2 \, dx
\]
\[
= \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}} |e_\rho \otimes (\varepsilon - 1 + \partial t) x + \nabla x \cdot v|^2 \, dt \, d\sigma
\]
\[
= \int_{\mathbb{S}^{N-1} \times \mathbb{R}} ((\varepsilon - 1)^2 |v|^2 + |\partial x \cdot v|^2 + |\nabla x \cdot v|^2) \, d\sigma \, dt
\]
\[
= \int_{\mathbb{S}^{N-1} \times \mathbb{R}} (\varepsilon - 1)^2 |\partial x|^2 + |\nabla x|^2 \partial x \cdot \, d\sigma \, dt
\]
and
(26)
\[
\int_{\mathbb{R}^N} |x|^{2\gamma-2} |u|^2 \, dx = \int_{\mathbb{R}^N} |x|^{2-2\varepsilon-N} |u|^2 \, dx
\]
\[
= \int_{\mathbb{S}^{N-1}} \int_{0}^{\infty} \frac{|v|^2 \, d\sigma}{\rho} = \int_{\mathbb{R}^N} |v|^2 \, dt \, d\sigma
\]
\[
= \int_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial x|^2 \, d\sigma \, dt = \int_{\mathbb{R} \times \mathbb{S}^{N-1}} \mathcal{T} (\lambda - \Delta \lambda) \, f \, d\sigma \, dt
\]
by Lemma 9. Therefore, by (25) and (26), the optimal constant in (2) can be expressed as
(27)
\[
H_{N,\gamma} = \inf_{u \neq 0, c_{0,0} = 0} \frac{\int_{\mathbb{R}^N} |x|^{2\gamma} |\nabla u|^2 \, dx}{\int_{\mathbb{R}^N} |x|^{2\gamma-2} |u|^2 \, dx} = \inf_{f \neq 0} \frac{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} \mathcal{T} Q_1 (\lambda - \Delta \lambda) \, f \, d\sigma \, dt}{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} \mathcal{T} P_1 (\lambda - \Delta \lambda) \, f \, d\sigma \, dt},
\]
where \( Q_1 (\lambda, \alpha) = ((\varepsilon - 1)^2 + \lambda^2) P_1 (\lambda, \alpha) + P_2 (\lambda, \alpha) \).

**Calculation of a lower bound.** In the same manner as Costin-Maz’ya [2], we expand \( f \) in \( L^2 (\mathbb{S}^{N-1}) \) by eigenfunctions \( \{ \psi_\nu \}_{\nu \in \{0\} \cup \mathbb{N}} \) of \( -\Delta \lambda \) as
(29)
\[
f (\lambda, \sigma) = \sum_{\nu = 0}^{\infty} f_\nu (\lambda) \psi_\nu (\sigma), \quad \begin{cases}
-\Delta \lambda \psi_\nu = \alpha_\nu \psi_\nu, \\
\alpha_\nu = \nu (\nu + N - 2), & (\nu = 0, 1, 2, \ldots)
\end{cases}
\]
Then we find that (27) is estimated from below by
\[
H_{N,\gamma} \geq \inf_{f \neq 0} \frac{\sum_{\nu \in \mathbb{N} \cup \{0\}} \int_{\mathbb{R}} Q_1 (\lambda, \alpha_\nu) |f_\nu (\lambda)|^2 \, d\lambda}{\sum_{\nu \in \mathbb{N} \cup \{0\}} \int_{\mathbb{R}} P_1 (\lambda, \alpha_\nu) |f_\nu (\lambda)|^2 \, d\lambda} \geq \inf_{\lambda \in \mathbb{R} \setminus \{0\}} \inf_{\nu \in \{0\} \cup \mathbb{N}} \frac{Q_1 (\lambda, \alpha_\nu)}{P_1 (\lambda, \alpha_\nu)},
\]
where $P_1$, $Q_1$ are the same as in Lemma 9, (28) and where in the last inequality we have used Lemma 10 in Appendix, applied to $X = \{(\nu, \lambda) \in (\mathbb{N} \cup \{0\}) \times \mathbb{R}\}$, $\mu = \left( \sum_{\nu \in \mathbb{N} \cup \{0\}} \delta_\nu \right) \times d\lambda$ and $g(\nu, \lambda) = |f_\nu(\lambda)|^2$. Therefore, we have

$$H_{N, \tau} \geq \inf_{\kappa > 0} \inf_{\nu \in \mathbb{N} \cup \{0\}} F(\kappa, \alpha_\nu) \quad (30)$$

with $F(\kappa, \cdot)$ defined by

$$F(\kappa, \alpha) = \frac{Q_1(\sqrt{2}\kappa, \alpha)}{P_1(\sqrt{2}\kappa, \alpha)} = (\varepsilon - 1)^2 + N - 1 + \kappa + \alpha - 2\alpha \frac{2\varepsilon + N - 2}{\varepsilon^2 + \kappa + \alpha} \quad (31)$$

for $\kappa > 0$ and $\alpha \geq 0$. Here we also define $F(0, \alpha)$ by

$$F(0, \alpha) = \lim_{\kappa \downarrow 0} \frac{Q_1(\lambda, \alpha)}{P_1(\lambda, \alpha)} = \lim_{\kappa \downarrow +0} F(\kappa, \alpha) \quad (32)$$

$$= \begin{cases} 
(\varepsilon - 1)^2 + N - 1 + \alpha - 2\alpha \frac{2\varepsilon + N - 2}{\varepsilon^2 + \alpha} & \text{for } \alpha > 0 \\
(\varepsilon - 1)^2 + N - 1 & \text{for } \alpha = 0 
\end{cases}.$$

In this setting, we calculate the right-hand side of (30). In the case $\varepsilon < 1 - N/2$, by differentiating (31) directly with respect to $\alpha$, we see that

$$\frac{\partial}{\partial \alpha} F(\kappa, \alpha) = 1 - 2(2\varepsilon + N - 2) \frac{\varepsilon^2 + \kappa}{(\varepsilon^2 + \kappa + \alpha)^2} > 0.$$

Thus $0 \leq \alpha \mapsto F(\kappa, \alpha)$ is monotone increasing for each $\kappa > 0$, and

$$F(\kappa, \alpha) \geq F(\kappa, 0) = (\varepsilon - 1)^2 + N - 1 + \kappa > F(0, 0) = F(0, 0) = F(0, 0),$$

that implies

$$\inf_{\kappa > 0} \inf_{\nu \in \mathbb{N} \cup \{0\}} F(\kappa, \alpha_\nu) = F(0, \alpha_0) \quad \text{when } \varepsilon < 1 - N/2.$$

In the case $\varepsilon \geq 1 - N/2$, by (31) we see that $F(\kappa, \alpha)$ is increasing in $\kappa > 0$ for each $\alpha \geq 0$. Thus $F(\kappa, \alpha) \geq F(0, \alpha)$ and

$$\inf_{\kappa > 0} \inf_{\nu \in \mathbb{N} \cup \{0\}} F(\kappa, \alpha_\nu) = \inf_{\nu \in \mathbb{N} \cup \{0\}} F(0, \alpha_\nu).$$

To evaluate the right-hand side, we compute

$$\frac{\partial}{\partial \alpha} F(0, \alpha) = 1 - 2(2\varepsilon + N - 2) \frac{\varepsilon^2}{(\varepsilon^2 + \alpha)^2} = \frac{\varepsilon^4 - 4\varepsilon^3 + 2(\varepsilon - (N - 2))\varepsilon^2 + \alpha^2}{(\varepsilon^2 + \alpha)^2} \geq \frac{\varepsilon^2(\varepsilon + 2)^2 + \alpha^2}{(\varepsilon^2 + \alpha)^2} > 0 \quad \text{if } \alpha \geq N.$$

Thus we have $F(0, \alpha) > F(0, N)$ for any $\alpha \geq N$, which implies $F(0, \alpha_\nu) \geq F(0, \alpha_2) = F(0, 2N)$ for all $\nu \geq 2$. This in turn implies

$$\inf_{\nu \in \mathbb{N} \cup \{0\}} F(0, \alpha_\nu) = \min_{\nu \in \{0, 1, 2\}} F(0, \alpha_\nu).$$

Moreover, by computing

$$F(0, \alpha_2) - F(0, \alpha_1) = F(0, 2N) - F(0, N - 1)$$

$$= \frac{(N + 1)\varepsilon^2((\varepsilon - 2)^2 + N - 1) + 2N(N - 1)}{(\varepsilon^2 + N - 1)(\varepsilon^2 + 2N)} > 0,$$
we see that
\[ \inf_{\nu \in \{0, 1, 2\}} F(0, \alpha_\nu) = \min_{\nu \in \{0, 1\}} F(0, \alpha_\nu). \]

Therefore, by calculating
\[
F(0, \alpha_1) - F(0, \alpha_0) = F(0, N - 1) - F(0, 0) = (N - 1) \frac{(\varepsilon - 2)^2 - (N + 1)}{\varepsilon^2 + N - 1},
\]
it turns out that
\[
\inf_{\kappa > 0} \inf_{\nu \in \mathbb{N} \cup \{0\}} F(\kappa, \alpha_\nu) = \min_{\nu \in \{0, 1\}} F(0, \alpha_\nu)
\]
(33)
\[
= \begin{cases} 
F(0, \alpha_1) & \text{for } (\varepsilon - 2)^2 \leq N + 1, \\
F(0, \alpha_0) & \text{for } (\varepsilon - 2)^2 > N + 1
\end{cases}
\]
when \( \varepsilon \geq 1 - N/2 \). The expression (33) holds true even for \( \varepsilon < 1 - N/2 \) since \( \varepsilon < 1 - N/2 \) implies \( (\varepsilon - 2)^2 > N + 1 \).

Inserting this result into (30), we have
\[
H_{N, \gamma} \geq \min_{\nu \in \{0, 1\}} F(0, \alpha_\nu)
\]
\[
= \begin{cases} 
F(0, \alpha_1) = (\varepsilon - 1)^2 \frac{\varepsilon^2 + 3(N - 1)}{\varepsilon^2 + N - 1} & \text{for } |\varepsilon - 2| \leq \sqrt{N + 1}, \\
F(0, \alpha_0) = (\varepsilon - 1)^2 + N - 1 & \text{otherwise.}
\end{cases}
\]

Returning to \( \varepsilon = 2 - N/2 - \gamma \), we arrive at the desired infimum value in Theorem 1.

**Optimality for** \( H_{N, \gamma} \). In this subsection, we prove that the former lower bound of \( H_{N, \gamma} \) is indeed realized as an equality:
\[
H_{N, \gamma} = \min_{\nu \in \{0, 1\}} F(0, \alpha_\nu) = \min_{\nu \in \{0, 1\}} \lim_{|\lambda| \searrow 0} \frac{Q_1(\lambda, \alpha_\nu)}{P_1(\lambda, \alpha_\nu)}.
\]

For that purpose, let \( \nu_0 \in \{0, 1\} \) be such that
\[
\min_{\nu \in \{0, 1\}} F(0, \alpha_\nu) = F(0, \alpha_{\nu_0}).
\]

By the argument in the last subsection, it is enough to prove that there exists a sequence of curl-free vector fields \( \{u_n\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^N)^N \) such that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |x|^{2\gamma - 2} |\nabla u_n|^2 dx = \lim_{|\lambda| \searrow 0} \frac{Q_1(\lambda, \alpha_{\nu_0})}{P_1(\lambda, \alpha_{\nu_0})}.
\]

For the construction of \( \{u_n\}_{n \in \mathbb{N}} \), take any nonnegative \( h \in C_c^\infty(\mathbb{R}) \), \( h \neq 0 \) and put \( h_n(t) = h(t/n) \) for \( n \in \mathbb{N} \). Set
\[
v_n(\rho, \sigma) = e_\rho \left( ch_n(t) + h'_n(t) \right) \psi_{\nu_0}(\sigma) + h_n(t) \nabla_\sigma \psi_{\nu_0}(\sigma)
\]
where \( \rho = e^t \) and \( \psi_{\nu_0} \) denotes an eigenfunction of \( -\Delta_\sigma \) associated with the eigenvalue \( \alpha_{\nu_0} = \nu_0(t_0 + N - 2) \). Then it is clear that \( v_n \) satisfies (23). Define
\[
u_n(\rho, \sigma) = \rho^{-\gamma} v_n(\rho, \sigma)
\]
for \( \gamma = 2 - N/2 - \gamma \). Then \( \{u_n\}_{n \in \mathbb{N}} \) is a sequence of curl-free vector fields having compact supports on \( \mathbb{R}^N \setminus \{0\} \). Put
\[
f_n(\lambda, \sigma) = (v_n)_\rho(\lambda, \sigma) = (\varepsilon + i\lambda) \hat{h}_n(\lambda) \psi_{\nu_0}(\sigma)
\]
and compute the Hardy-Leray quotient for \( u_n \) by using (25) and (26). We see that
\[
\frac{\int_{\mathbb{R}^N} |x|^2 |\nabla u_n|^2 \, dx}{\int_{\mathbb{R}^N} |x|^{2\gamma-2} |u_n|^2 \, dx} = \frac{\int_{\mathbb{R}^N} Q_1(\lambda, -\Delta \sigma) f_n \, dx \, d\lambda}{\int_{\mathbb{R}^N} Q_1(\lambda, -\Delta \sigma) f_n \, dx \, d\lambda} = \frac{\int_{\mathbb{R}^2} (\epsilon^2 + \lambda^2) P_1(\lambda, \alpha_{\nu_0}) |\tilde{h}_n(\lambda)|^2 \, d\lambda}{\int_{\mathbb{R}^2} P_1(\lambda, \alpha_{\nu_0}) |\tilde{h}_n(\lambda)|^2 \, d\lambda} = \frac{\int_{\mathbb{R}^2} Q_0(\lambda, \alpha_{\nu_0}) |\tilde{h}_n(\lambda)|^2 \, d\lambda}{\int_{\mathbb{R}^2} P_0(\lambda, \alpha_{\nu_0}) |\tilde{h}_n(\lambda)|^2 \, d\lambda},
\]
here
\[
P_{01}(\lambda, \alpha) = (\epsilon^2 + \lambda^2) P_1(\lambda, \alpha) = \epsilon^2 + \alpha + \lambda^2, \quad Q_{01}(\lambda, \alpha) = (\epsilon^2 + \lambda^2) Q_1(\lambda, \alpha)
\]
are polynomials in \( \lambda \). Note that \( \tilde{h}_n(\lambda) = \tilde{h}(t/n)(\lambda) = n \tilde{h}(n\lambda) \). Thus if \( \epsilon^2 + \alpha_{\nu_0} \neq 0 \), then we have
\[
\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 |x|^{2\gamma} \, dx}{\int_{\mathbb{R}^N} |u_n|^2 |x|^{2\gamma-2} \, dx} = \frac{\int_{\mathbb{R}^2} Q_0(\lambda, \alpha_{\nu_0}) |\tilde{h}(n\lambda)|^2 \, d\lambda}{\int_{\mathbb{R}^2} P_0(\lambda, \alpha_{\nu_0}) |\tilde{h}(n\lambda)|^2 \, d\lambda} \to \frac{Q_{01}(\lambda, 0)}{P_{01}(\lambda, 0)} = \frac{Q_1(\lambda, \alpha_{\nu_0})}{P_1(\lambda, \alpha_{\nu_0})}
\]
as \( n \to \infty \). In the case \( \epsilon = 0 = \alpha_{\nu_0} \), by using
\[
P_{01}(\lambda, 0) = \lambda^2, \quad Q_{01}(\lambda, 0) = N \lambda^2 + \lambda^4,
\]
we can check that
\[
\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 |x|^{2\gamma} \, dx}{\int_{\mathbb{R}^N} |u_n|^2 |x|^{2\gamma-2} \, dx} = \frac{\int_{\mathbb{R}^2} Q_{01}(\lambda, 0) |\tilde{h}(n\lambda)|^2 \, d\lambda}{\int_{\mathbb{R}^2} P_{01}(\lambda, 0) |\tilde{h}(n\lambda)|^2 \, d\lambda} \to N = \lim_{|\lambda| \to +0} \frac{Q_{1}(\lambda, 0)}{P_{1}(\lambda, 0)}
\]
as \( n \to \infty \). Thus we have proved (34) which shows the optimality of \( H_{N, \gamma} \) in the class of curl-free vector fields in \( C_0^\infty(\mathbb{R}^N)^N \).

\[\square\]

4. Proof of Theorem 3

Let \( \gamma \neq 2 - N/2 \) be a real number and put \( \epsilon = 3 - N/2 - \gamma \neq 1 \). Under the transformation \( v = \rho^{1-\epsilon} u \) in (21), the gradient vector field is transformed as
\[
\nabla v = \nabla (\rho^{1-\epsilon} u) = (1 - \epsilon) \rho^{-\epsilon} e_\rho \otimes u + \rho^{1-\epsilon} \nabla u,
\]
which leads to
\[
|\rho \nabla v|^2 = (1 - \epsilon)^2 |\rho^{1-\epsilon} u|^2 + 2(1 - \epsilon) \rho^{2-2\epsilon} u \cdot \rho \partial_\rho u + \rho^{2-2\epsilon} |\rho \nabla u|^2.
\]
On the other hand, the assumption \( \int_{\mathbb{R}^N} |x|^{2-2\epsilon - N} |u|^2 \, dx < \infty \) and the smoothness of \( u \) imply that
\[
u(x) = O \left(|x|^m\right), \quad \nabla u(x) = O \left(|x|^{m-1}\right) \quad \text{as} \quad |x| \downarrow 0
\]
for some integer \( m > \epsilon - 1 \) if \( \epsilon > 1 \). Therefore, we see that \( v \) must satisfy
\[
|v(0)| = \lim_{\rho \to 0} |\rho \nabla v| = 0
\]
by (38) when $\varepsilon > 1$.

Next, we see the $\Delta u$ is written in terms of $v$ as follows:

$$\Delta u = (2\varepsilon + N - 4)\partial_t v + \partial^2 v + \Delta v,$$

here we have used (15) and $\Delta v = 2\varepsilon - 1$. Note that $\int |v| dx = \int |v| dx$ and $\int \partial_t v \cdot v dx = 0$ and $\int \partial^2 v \cdot v dx = -\int \Delta v dx$ by (39). Thus by using (40), Lemma 9, and noting $(2\varepsilon + N - 4)^2 - 2\varepsilon - 1 = (N - 2)^2 + 2\varepsilon - 1$, we find that the both integrals of the Rellich-Leray inequality (4) are written as

$$\int |x|^{2\gamma} |\Delta u|^2 dx = \int |x|^{2\gamma - 2N} |\Delta u|^2 dx$$

$$= \int_{\mathbb{R}^N} \int_0^\infty |\alpha_{\varepsilon - 1} v + (2\varepsilon + N - 4)\partial_t v + \partial^2 v + \Delta v|^2 d\rho$$

$$= \int_{\mathbb{R}^N} \int_0^\infty \left( (\alpha_{\varepsilon - 1} v + (2\varepsilon + N - 4)\partial_t v + \partial^2 v + \Delta v)^2 \right) dt$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( (\alpha_{\varepsilon - 1} v + (2\varepsilon + N - 4)\partial_t v + \partial^2 v + \Delta v)^2 \right) d\lambda d\sigma$$

and

$$\int |x|^{2\gamma - 4} |u|^2 dx = \int |x|^{2\gamma - 2N} |u|^2 dx$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( (\alpha_{\varepsilon - 1} v + (2\varepsilon + N - 4)\partial_t v + \partial^2 v + \Delta v)^2 \right) d\lambda d\sigma.$$

Therefore, by (41) and (42), the optimal constant in (4) can be expressed as

$$R_{N, \gamma} = \inf_{u \neq 0, u \text{inf} u = 0} \frac{\int |x|^{2\gamma} |\Delta u|^2 dx}{\int |x|^{2\gamma - 2N} |u|^2 dx} = \inf_{f \neq 0} \frac{\int \mathcal{J}_2 (\lambda, -\Delta) f d\lambda d\sigma}{\int \mathcal{J}_1 (\lambda, -\Delta) f d\lambda d\sigma}$$

with the polynomial $Q_2 (\lambda, \alpha)$ given by

$$Q_2 (\lambda, \alpha) = (\alpha_{\varepsilon - 1} + ((N - 2)^2 + 2\varepsilon - 1) \lambda^2 + \lambda^4) P_1 (\lambda, -\Delta)$$

and

$$+ 2 (\lambda^2 - \varepsilon - 1) P_2 (\lambda, -\Delta) + P_3 (\lambda, -\Delta).$$

Calculation of a lower bound. As in (29), we expand $f$ in terms of eigenfunctions of $-\Delta$. Then by (43), (44), and Lemma 10, we find

$$R_{N, \gamma} = \inf_{f \neq 0} \frac{Q_2 (\lambda, \alpha) f d\lambda d\sigma}{P_1 (\lambda, -\Delta) f d\lambda d\sigma} = \inf_{f \neq 0} \frac{F (\kappa, \alpha) f d\lambda d\sigma}{\inf_{f \neq 0} F (\kappa, \alpha) f d\lambda d\sigma},$$
where for $\kappa > 0$ and $\alpha \geq 0$, $F(\kappa, \alpha)$ is defined as

$$F(\kappa, \alpha) = \frac{Q_2(\sqrt{\kappa}, \alpha)}{P_1(\sqrt{\kappa}, \alpha)}$$

$$= \alpha_{\varepsilon-1}^2 + ((N - 2)^2 + 2\alpha_{\varepsilon-1})\kappa + \kappa^2 + \frac{2(\kappa - \alpha_{\varepsilon-1})P_2(\sqrt{\kappa}, \alpha) + P_3(\sqrt{\kappa}, \alpha)}{P_1(\sqrt{\kappa}, \alpha)}.$$

By directly calculating further, we can check that

$$F(\kappa, \alpha) = \kappa^2 + \frac{4\alpha(1 - \varepsilon)(N + 2\varepsilon - 2)^2\kappa}{(\varepsilon^2 + \alpha)(\kappa + \varepsilon^2 + \alpha)}$$

$$+ \left( \frac{N^2}{2} + 2 \left( \frac{\varepsilon + \frac{N - 4}{2}}{\varepsilon^2 + \alpha} \right)^2 + 2\alpha \right) \kappa + \frac{(\varepsilon - \alpha)^2}{\varepsilon^2 + \alpha} (\alpha_{\varepsilon} - \alpha)^2$$

for $\varepsilon = 3 - N/2 - \gamma \neq 0$, and

$$F(\kappa, \alpha) = \kappa^2 + \frac{4(N - 2)^2\kappa}{\kappa + \alpha} + ((N - 2)^2 + 4 + 2\alpha) \kappa + (4 + \alpha)\alpha$$

for $\varepsilon = 0$. We also define $F(0, \alpha)$ as

$$F(0, \alpha) = \lim_{\alpha\to+0} \frac{Q_2(\lambda, \alpha)}{P_1(\lambda, \alpha)} = \lim_{\alpha\to+0} F(\kappa, \alpha)$$

$$= \begin{cases} 
(\varepsilon - 1)^2 + \alpha (\alpha_{\varepsilon} - \alpha)^2, & \text{for } \varepsilon \neq 0, \alpha \geq 0, \\
(4 + \alpha)\alpha, & \text{for } \varepsilon = 0, \alpha > 0, \\
4(N - 2)^2, & \text{for } \varepsilon = 0, \alpha = 0.
\end{cases}$$

In these settings, from now on we evaluate the expression

$$\inf_{\nu \in \mathbb{N}\cup\{0\}} \inf_{\kappa > 0} F(\kappa, \alpha_{\nu}).$$

If $\varepsilon < 1$, it is clear that the map $0 < \kappa \mapsto F(\kappa, \alpha)$ is increasing for any fixed $\alpha \geq 0$. Also, if $\varepsilon > 1$, estimating $\partial_\kappa F(\kappa, \alpha)$ from below by

$$\frac{\partial F(\kappa, \alpha)}{\partial \kappa} = 2\kappa \left( \frac{-4\alpha(\varepsilon - 1)(N + 2\varepsilon - 2)^2}{(\kappa + \varepsilon^2 + \alpha)^2} + \frac{N^2}{2} + 2 \left( \frac{\varepsilon + \frac{N - 4}{2}}{\varepsilon^2 + \alpha} \right)^2 + 2\alpha \right)$$

$$\geq -\frac{4\alpha(\varepsilon - 1)(N + 2\varepsilon - 2)^2}{(\varepsilon^2 + \alpha)^2} + \frac{N^2}{2} + 2 \left( \frac{\varepsilon + \frac{N - 4}{2}}{\varepsilon^2 + \alpha} \right)^2 + 2\alpha$$

$$\geq -\frac{\varepsilon - 1}{\varepsilon^2}(N + 2\varepsilon - 2)^2 + \frac{N^2}{2} + 2 \left( \frac{\varepsilon + \frac{N - 4}{2}}{\varepsilon^2 + \alpha} \right)^2 + 2\alpha$$

$$\geq -\frac{1}{4}(N + 2\varepsilon - 2)^2 + \frac{N^2}{2} + 2 \left( \frac{\varepsilon + \frac{N - 4}{2}}{\varepsilon^2 + \alpha} \right)^2 + 2\alpha$$

$$= \left( \frac{\varepsilon + \frac{N - 4}{2}}{\varepsilon^2 + \alpha} \right)^2 + \frac{N^2 - 4}{2} + 2\alpha \geq 0,$$

we see again that $F(\kappa, \alpha)$ is increasing with respect to $\kappa > 0$ for any $\alpha \geq 0$. Therefore we have

$$\inf_{\kappa > 0} F(\kappa, \alpha) = F(0, \alpha)$$

for all $\varepsilon \neq 1$, which implies

$$\inf_{\nu \in \mathbb{N}\cup\{0\}} \inf_{\kappa > 0} F(\kappa, \alpha_{\nu}) = \inf_{\nu \in \mathbb{N}\cup\{0\}} F(0, \alpha_{\nu}).$$
Moreover, we can check that
\[ \frac{\partial F(0, \alpha)}{\partial \alpha} \geq 0, \quad \alpha \geq \max \{\alpha_1, \alpha_2\}, \]
see Lemma 11. This implies that \( \inf_{\nu \in \mathbb{N} \cup \{0\}} F(0, \alpha_{\nu}) \) is attained. Therefore, we have the desired estimate:

\[ R_{N, \gamma} \geq \min_{\nu \in \mathbb{N} \cup \{0\}} F(0, \alpha_{\nu}) \quad \text{with} \quad F(0, \alpha_{\nu}) = \lim_{|\lambda| \to 0} \frac{Q_2(\lambda, \alpha_{\nu})}{P_1(\lambda, \alpha_{\nu})}. \]

Furthermore, we see that \( \min_{\nu \in \mathbb{N} \cup \{0\}} F(0, \alpha_{\nu}) \) is given by
\[ \min_{\nu \in \mathbb{N} \cup \{0\}} F(0, \alpha_{\nu}) = \min_{\nu \in \mathbb{N} \cup \{0\}} \frac{(\varepsilon - 2)^2 + \alpha_{\nu}}{\varepsilon^2 + \alpha_{\nu}} (\alpha_{\varepsilon} - \alpha_{\nu})^2 \]
for \( \varepsilon = 3 - N/2 - \gamma \neq 0 \), and
\[ \min_{\nu \in \mathbb{N} \cup \{0\}} F(0, \alpha_{\nu}) = \min \left\{ 4(N - 2)^2, (4 + \alpha_1)\alpha_1 \right\} \]
\[ = \begin{cases} 4(N - 2)^2 = F(0, \alpha_0) & \text{for } N = 2, 3, 4, \\ (N + 3)(N - 1) = F(0, \alpha_1) & \text{for } N \geq 5 \end{cases} \]
for \( \varepsilon = 3 - N/2 - \gamma = 0 \). This gives the lower bound of \( R_{N, \gamma} \). In the next subsection we will show that the above inequality is indeed the equality.

**Optimality for \( R_{N, \gamma} \).** To show that the inequality (46) is indeed the equality, let \( \nu_0 \in \mathbb{N} \cup \{0\} \) be such that \( F(0, \alpha_{\nu_0}) = \min_{\nu \in \mathbb{N} \cup \{0\}} F(0, \alpha_{\nu}) \) is satisfied. We use the sequence of curl-free vector fields \( \{u_n\}_{n \in \mathbb{N}} \) in (36) again with (35), however for \( \varepsilon = 3 - N/2 - \gamma \). Then, as in the proof of Theorem 1, we obtain the following expression:

\[ \frac{\int_{\mathbb{R}^N} |x|^{2\gamma} |\Delta u_n|^2 \, dx}{\int_{\mathbb{R}^N} |x|^{2\gamma - 4} |u_n|^2 \, dx} = \frac{\int_{\mathbb{R}^{N-1} \times \mathbb{R}} T_n Q_2(\lambda, -\Delta_{\sigma}) f_n \, d\sigma d\lambda}{\int_{\mathbb{R}^{N-1} \times \mathbb{R}} T_n P_1(\lambda, -\Delta_{\sigma}) f_n \, d\sigma d\lambda} = \frac{\int_{\mathbb{R}} (\varepsilon^2 + \lambda^2) Q_2(\lambda, \alpha_{\nu_0}) |\tilde{h}_n(\lambda)|^2 \, d\lambda}{\int_{\mathbb{R}} (\varepsilon^2 + \lambda^2) P_1(\lambda, \alpha_{\nu_0}) |\tilde{h}_n(\lambda)|^2 \, d\lambda} = \frac{\int_{\mathbb{R}} Q_{02}(\lambda, \alpha_{\nu_0}) |\tilde{h}_n(\lambda)|^2 \, d\lambda}{\int_{\mathbb{R}} P_{01}(\lambda, \alpha_{\nu_0}) |\tilde{h}_n(\lambda)|^2 \, d\lambda}, \]

where \( P_{01}(\lambda, \alpha) \) is the same as in (37) and
\[ Q_{02}(\lambda, \alpha) = (\varepsilon^2 + \lambda^2)^2 Q_2(\lambda, \alpha) \]
is a polynomial in \( \lambda \). When \( \varepsilon = 0 \) and \( \alpha_{\nu_0} = 0 \), by using the facts
\[ Q_{02}(\lambda, 0) = 4(N - 2)^2 \lambda^2 + (N^2 - 4N + 8) \lambda^4 + \lambda^6 \]
and \( P_{01}(\lambda, 0) = \lambda^2 \), we prove that
\[ \lim_{n \to \infty} \frac{\int_{\mathbb{R}^N} |x|^{2\gamma} |\Delta u_n|^2 \, dx}{\int_{\mathbb{R}^N} |x|^{2\gamma - 4} |u_n|^2 \, dx} = 4(N - 2)^2 = F(0, 0). \]
Thus as in the proof of Theorem 1, we can show that
\[
\lim_{n \to \infty} \frac{\int_{\mathbb{R}^N} |x|^{2\gamma} |\Delta u_n|^2 \, dx}{\int_{\mathbb{R}^N} |x|^{2\gamma - 4} |u_n|^2 \, dx} = F(0, \alpha_0)
\]
for all cases \(\varepsilon^2 + \alpha_{\varepsilon_0} \neq 0\) and \(\varepsilon^2 + \alpha_{\varepsilon_0} = 0\). This leads to the optimality of \(R_{N, \gamma}\). \(\square\)

5. Appendix.

In this appendix, we prove technical lemmas.

**Lemma 10.** Let \((X, \mathcal{M}, \mu)\) be a measure space and let \(\xi, \eta : X \to \mathbb{R}\) be a \(\mu\)-measurable function such that \(\xi \neq 0 \mu\)-a.e. Suppose \(g : X \to \mathbb{R}\) is a \(\mu\)-measurable function satisfying, \(g \equiv 0 \mu\)-a.e., \(0 < \int_X \xi \, d\mu < \infty\), and \(\int_X |\eta| \, d\mu < \infty\). Then we have
\[
\int_X \eta g \, d\mu \geq \text{ess inf}_{x \in X} \frac{\eta(x)}{\xi(x)} \int_X g \, d\mu.
\]

**Proof.** Let \(I = \text{ess inf}_{x \in X} \frac{\eta(x)}{\xi(x)}\). Then \(\frac{\eta}{\xi} \geq I \mu\)-a.e. Multiply the both sides by \(\eta \equiv 0 \mu\)-a.e., \(\eta g \geq I \xi g \mu\)-a.e. By integrating over \(X\), we obtain
\[
\int_X \eta g \, d\mu \geq I \int_X \xi g \, d\mu
\]
which leads the result. \(\square\)

**Lemma 11.** Let \(F(0, \alpha)\) be given by (45). Then we have
\[
\frac{\partial F(0, \alpha)}{\partial \alpha} \geq 0 \quad \text{for} \quad \alpha \geq \max\{\alpha_1, \alpha_2\}.
\]

**Proof.** Recall \(\alpha_1 = N - 1\) and \(\alpha_2 = \varepsilon (\varepsilon + N - 2)\). It is enough to show the lemma when \(\varepsilon \neq 0\) and \(F(0, \alpha) = \frac{(\varepsilon - 2\varepsilon^2 + \alpha)(\alpha - \alpha)^2}{\varepsilon + \alpha}\). A direct computation shows that
\[
\frac{\partial F(0, \alpha)}{\partial \alpha} = \frac{2(\alpha - \alpha)}{(\alpha + \varepsilon^2)^2} f_\varepsilon(\alpha), \quad \text{where}
\]
\[
f_\varepsilon(\alpha) = \alpha^2 + 2(\varepsilon^2 - \varepsilon + 1)\alpha + \varepsilon^2(\varepsilon - 1)^2 + 2\alpha(1 - \varepsilon).
\]
Since \(\varepsilon^2 - \varepsilon + 1 > 0\) for any \(\varepsilon \in \mathbb{R}\), we see that \(f_\varepsilon\) is strictly increasing for \(\alpha \geq 0\).

Thus if we show (i) \(f_\varepsilon(\alpha_2) \geq 0\) if \(\alpha_2 \geq \alpha_1\), and (ii) \(f_\varepsilon(\alpha_1) \geq 0\) if \(\alpha_1 \geq \alpha_2\), then \(f_\varepsilon(\alpha) \geq 0\) for any \(\alpha \geq \max\{\alpha_1, \alpha_2\}\), which concludes the lemma.

To prove (i), we observe that \(f_\varepsilon(\alpha_2) = (\alpha_2 + \varepsilon^2)(\alpha_2 + (\varepsilon - 2)^2)\). Thus if \(\alpha_2 \geq \alpha_1 = N - 1 > 0\), clearly we have \(f_\varepsilon(\alpha_2) > 0\).

To prove (ii), we observe that \(f_\varepsilon(\alpha_1) = f_\varepsilon(N - 1) = \varepsilon^4 - 6\varepsilon^3 + 8\varepsilon^2 - 2\varepsilon + N^2 - 1\).

We need to prove this quartic function is nonnegative for \(\varepsilon \in \mathbb{R}\) such that \(\alpha_1 \geq \alpha_2\), i.e., \(-(N - 1) \leq \varepsilon \leq 1\). However, this is an elementary fact. \(\square\)

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