Optimal transport of two ions under slow spring-constant drifts

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Received 1 October 2014, revised 4 February 2015
Accepted for publication 26 February 2015
Published 23 June 2015

Abstract

We investigate the effect of slow spring-constant drifts of the trap used to shuttle two ions of different mass. We design transport protocols to suppress or mitigate the final excitation energy by applying invariant-based inverse engineering, perturbation theory, and a harmonic dynamical normal-mode approximation. A simple, explicit trigonometric protocol for the trap trajectory is found to be robust with respect to the spring-constant drifts.

Keywords: trapped ions, shortcuts to adiabaticity, quantum information

1. Introduction

A possible scalable architecture for quantum information processing relies on shuttling small numbers of trapped ions among storing and processing sites in multi-electrode configurations \cite{1-5}. Transport of two ions of different species is particularly relevant as one of them may be used for cooling and the other one to encode the qubit \cite{6}. Diabatic transport of two equal ions has been recently realized \cite{7,8}. It was recognized \cite{7} that different masses would require special consideration since all modes may be excited by the transport. On the theory side, equal-mass two-ion transport has been studied in \cite{9} to design fast protocols without final excitation by invariant-based inverse engineering, whereas the design of fast transport protocols of two ions with different mass was tackled in \cite{10} using a harmonic approximation in normal mode coordinates that is accurate up to very short transport times, of the order of a few oscillations of the ions.

The transport protocols are subjected to noise and perturbations. In current experiments, the errors in the spring constant due to slow drifts of imperfect calibration are likely to dominate others. This means that the spring constant for each run of the experiment stays constant, but it may change from run to run, differing from the ideal value used to set the protocol. The effect of these errors was studied in \cite{11} for single ion-transport. For two ions of equal mass, the normal mode coordinates become proportional to center-of-mass (CM) and relative coordinates and are exactly decoupled. In that case, only the CM can be excited by the motion of a harmonic trap \cite{9}, so that the results and techniques in \cite{11} (valid for one ion or a decoupled CM motion) are directly applicable. For unequal masses though, this decoupling of coordinates does not hold so that a different approach is needed.

In this paper we investigate the effect of spring-constant perturbations on the transport of two ions of different mass within a harmonic approximation in dynamically defined normal mode coordinates \cite{10}, and apply invariant-based inverse engineering combined with perturbation theory in the relative error parameter to design transport protocols that suppress or mitigate the final excitation energy. In sections 2 and 3 we briefly introduce the invariant-based inverse engineering method and the dynamical normal modes; in section 4 we design protocols that suppress effectively the excitation energy up to very small shuttling times, of interest for current quantum information processing applications.

2. Invariant-based engineering method

In this section, we provide a brief review of invariant-based engineering for shuttling one ion \cite{12}. As the Hamiltonian is
quadratic, the structure and properties of dynamical invariants and propagators are known [13, 14] and may be used to design the trap motion. The harmonic transport of one ion is described by the effective 1D Hamiltonian

\[
\hat{H}_0(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(\hat{q} - Q_0(t))^2,
\]

(1)

where \(\hat{q}\) and \(\hat{p}\) are the position and momentum operators, \(\omega/(2\pi)\) is the frequency of the trap, and \(Q_0(t)\) the position of its moving center. The corresponding quadratic-in-momentum Lewis–Riesenfeld invariant [15–17] is given (up to an arbitrary multiplicative constant) by [12]

\[
\hat{I}(t) = \frac{1}{2m}\left[\hat{p} - m\alpha(t)\right]^2 + \frac{1}{2}m\omega^2\left[\hat{q} - \alpha(t)\right]^2,
\]

(2)

where the dot represents a time derivative, and the function \(\alpha(t)\) must satisfy the auxiliary equation

\[
\dot{\alpha} + \omega^2(\alpha - Q_0) = 0,
\]

(3)

so that the invariant condition holds

\[
\frac{d\hat{I}(t)}{dt} = \frac{\hat{\theta}(t)}{\hbar} + \frac{1}{i\hbar}\left[\hat{I}(t), \hat{H}_0(t)\right] = 0.
\]

(4)

The expectation value of \(\hat{I}(t)\) remains constant for solutions of the time-dependent Schrödinger equation \(i\hbar \partial_t \Psi(q, t) = \hat{H}_0(t)\Psi(q, t)\). The solutions can be expressed in terms of independent ‘transport modes’ \(\Psi(q, t) = \sum_n c_n\Phi(q, t; n)\), where \(\Phi(q, t; n) = e^{i\theta(n)}\phi(q, t; n)\), \(n = 0, 1, \ldots\) is the mode index; \(c_n\) are time-independent coefficients; \(\phi(q, t; n)\) are the orthogonal eigenvectors of the invariant \(\hat{I}(t)\) satisfying \(\dot{\phi}(q, t; n) = \lambda(n)\phi(q, t; n)\), with real time-independent eigenvalues \(\lambda(n)\). Finally, the Lewis–Riesenfeld phase is

\[
\theta(t; n) = \frac{1}{\hbar} \int_0^t \left(\phi(t'; n) | i\hbar \frac{\partial}{\partial t'} - \hat{H}_0(t') \right)\phi(t'; n)\,dt'.
\]

(5)

For the harmonic trap [12],

\[
\phi(q, t; n) = \exp\left(-\frac{m\omega^2}{\hbar^2} q^2\right) \phi^{(0)}(q - \alpha(t); n),
\]

(6)

where \(\phi^{(0)}(q; n)\) are the eigenstates of equation (1) for \(Q_0(t) = 0\). Note that in harmonic transport \(\alpha(t)\) is the center of the transport modes which obeys the classical Newton equation (3).

To transport the ion between 0 and \(d\) in a time \(T\), the trajectory \(Q_0\) should satisfy

\[
Q_0(0) = 0, \quad Q_0(T) = d.
\]

(7)

The inverse engineering strategy is to design the invariant first, via \(\alpha(t)\), and then get \(Q_0(t)\) from the Newton equation (3). To guarantee the commutativity of \(\hat{I}(t)\) and \(\hat{H}_0(t)\) at initial time \(t = 0\) and final time \(t = T\) (which implies the shuttling from initial to final trap eigenstates without final excitation), and the continuity of trap motion, the designed \(\alpha(t)\) should satisfy the boundary conditions [12]

\[
\alpha(0) = 0, \quad \alpha(T) = d,
\]

\[
\dot{\alpha}(0) = 0, \quad \dot{\alpha}(T) = 0,
\]

\[
\ddot{\alpha}(0) = 0, \quad \ddot{\alpha}(T) = 0.
\]

(8)

The first line of conditions in equation (8) sets the states at the desired locations. The second one leaves them at rest. The third line is not necessary to achieve commutativity, but it assures the continuity of the trap motion. If the second derivatives do not vanish the trap will not be centered at 0 and \(d\) according to equation (3). This means that instantaneous trap displacements would be required at the boundary times, for example from 0 to \(\dot{\alpha}(0)/\omega^2\) at time zero. Approaching that ideal jump in practice might not be easy.

In the next section we shall show how to extend these ideas to two ions of different mass. Notice that an alternative method described in [12], the compensating force approach, may formally be applied to ion chains to avoid excitation. This however requires applying different forces to ions of different mass, whereas in the available technology in linear Paul traps the forces are proportional to the charge [10].

### 3. Dynamical normal modes for two ions in a moving trap

The Hamiltonian for 1D two-ion transport can be written as

\[
\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}m_1\omega_1^2(\hat{q}_1 - Q_0)^2 + \frac{1}{2}m_2\omega_2^2(\hat{q}_2 - Q_0)^2 + \frac{C_c}{\hat{q}_1 - \hat{q}_2},
\]

(9)

where \(\hat{q}_1\), \(\hat{q}_2\), \(\hat{p}_1\) and \(\hat{p}_2\) are the position and momentum operators of the two ions (we assume the ion 1 to be always on the right of ion 2 due to their strong repulsion), \(C_c = e^2/\varepsilon\varepsilon_0\) is the Coulomb constant \((\varepsilon_0\) the vacuum permittivity), \(m_1\) and \(m_2\) are the masses of the two ions, and \(\omega_1\) and \(\omega_2\) are the (angular) frequencies of the ions when they move independently in the trap. They are related to the spring constant \(u_0\) by \(u_0 = m_1\omega_1^2 = m_2\omega_2^2\). For equal masses, the Hamiltonian can be separated using CM and relative coordinates, see the appendix A. Here we focus on different masses, \(m_1 \neq m_2\), so the separability does not hold. An alternative description is given by the dynamical, mass-weighted normal-mode coordinates for the moving trap [10]. In operator form

\[
\hat{q}_\pm = a_\pm \sqrt{\hbar m} \left(\hat{q}_1 - Q_0 - \frac{l}{2}\right) + b_\pm \sqrt{\mu m} \left(\hat{q}_2 - Q_0 + \frac{l}{2}\right),
\]

(10)

with conjugate momenta

\[
\hat{P}_\pm = \frac{1}{\sqrt{\mu}} \left(a_\pm \hat{P}_1 + b_\pm \sqrt{\mu} \hat{P}_2\right).
\]

(11)
Let us now consider a modified spring-constant $\lambda(1 + \lambda)$, where the relative error $\lambda$ with respect to the ideal value $u_0$ remains constant during the transport time. This implies that, for each ion, the squared frequencies are $\omega_{\mu}^2(1 + \lambda)/(2\pi)$. The Hamiltonian in the laboratory frame will take the form

$$
\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \frac{1}{2} m_1\omega_1^2(1 + \lambda)(\hat{q}_1 - Q_0)^2 + \frac{1}{2} m_2\omega_2^2(1 + \lambda)(\hat{q}_2 - Q_0)^2 + \frac{C\varepsilon}{\hat{q}_1 - \hat{q}_2}.
$$

We define new coordinates as

$$
\hat{q}_+^{\prime} = a_+\sqrt{m}(\hat{q}_1 - Q_0 - \frac{l'}{2}) + b_+\sqrt{m}(\hat{q}_2 - Q_0 + \frac{l'}{2}),
$$

$$
\hat{q}_-^\prime = a_-\sqrt{m}(\hat{q}_1 - Q_0 + \frac{l'}{2}) + b_-\sqrt{m}(\hat{q}_2 - Q_0 - \frac{l'}{2}),
$$

where the equilibrium distance is now $l' = l(1 + \lambda)^{-1}$, whereas the expressions for normal-mode momenta are not affected by the error. Within the harmonic approximation, the Hamiltonian for normal coordinates becomes

$$
\hat{H}_N \approx \frac{1}{2} (\hat{p}_1 - R_0) + \frac{1}{2} (\hat{p}_2 - R_0)^2 + \frac{1}{2} Q_1^2(1 + \lambda)\hat{q}_1^2 + \frac{1}{2} Q_2^2(1 + \lambda)\hat{q}_2^2.
$$

Now we use a transformation that shifts the momenta to the trap frame

$$
\hat{U}_t = e^{-\frac{i}{\hbar}(P_0\hat{A}_t - \frac{1}{2} P_0 \hat{A}_t^2)},
$$

with $[\hat{P}_+, \hat{q}_+] = [\hat{P}_+, \hat{q}_-] = [\hat{P}_-, \hat{q}_+] = [\hat{P}_-, \hat{q}_-]$. The corresponding Hamiltonian $\hat{H}_N = \hat{U}_t \hat{H}_N \hat{U}_t^\dagger - i\hbar \hat{U}_t \partial_t \hat{U}_t^\dagger$ for the transformed wave function $|\psi\rangle = \hat{U}_t |\psi\rangle$ is (neglecting the terms that depend only on time)

$$
\hat{H}_N' = \hat{H}_+ + \hat{H}_-,
$$

where

$$
\hat{H}_+ = \frac{\hat{p}_+^2}{2} + \frac{\Omega_+^2}{2}(\hat{q}_+ - q_0^\prime)^2,
$$

$$
\hat{H}_- = \frac{\hat{p}_-^2}{2} + \frac{\Omega_-^2}{2}(\hat{q}_- - q_0^\prime)^2,
$$

with

$$
\Omega_+^2 = \Omega_+^2(1 + \lambda),
$$

$$
q_0^\prime = -\frac{\Omega_+}{\Omega_+^2} = q_0(1 + \lambda)^{-1},
$$

$$
q_0^\prime = -\frac{\Omega_-}{\Omega_+^2}.
$$

To have common initial and final states for the dynamics driven by the Hamiltonians (19) and (20), and agreement between the Hamiltonians at these boundary times, $R_0$ should
satisfy the boundary conditions
\[ R_{0z}(0) = R_{0z}(T) = 0, \]
\[ \dot{R}_{0z}(0) = \dot{R}_{0z}(T) = 0, \]
which implies (from equation (15))
\[ \dot{Q}_0(0) = \ddot{Q}_0(T) = 0, \]
\[ \dot{Q}_0(0) = \dddot{Q}_0(T) = 0. \]

The unperturbed ‘trajectories’ \( \alpha_z \) play the role of \( \alpha_\epsilon \) in each mode; note that they are ‘trajectories’ in a normal-mode coordinate space. They satisfy
\[ \dddot{\alpha}_z + \Omega_z^2 (\alpha_z - q_{0z}) = 0, \]
as well as the boundary conditions to make the excitation energy for the unperturbed spring constant zero at \( T \),
\[ \alpha_z(0) = 0, \quad \alpha_z(T) = 0, \]
\[ \dot{\alpha}_z(0) = 0, \quad \dot{\alpha}_z(T) = 0, \]
\[ \ddot{\alpha}_z(0) = 0, \quad \ddot{\alpha}_z(T) = 0, \]
compare them to the ones in equation (8).

The perturbed trajectories, denoted as \( F_z(t) \), satisfy instead
\[ \dddot{F}_z(t) + \Omega_z^2 [F_z(t) - q_{0z}] = 0. \]

Both \( \alpha_z \) and \( F_z \) may be found by explicit integral expressions. With or without interaction the functions and their derivatives vanish at \( t = 0 \),
\[ \alpha_z(t) = \Omega_z \int_0^t dt' q_{0z}(t') \sin [\Omega_z (t - t')] , \]
\[ F_z(t) = \Omega_z^2 \int_0^t dt' q_{0z}(t') \sin [\Omega_z (t - t')] . \]

The boundary conditions for \( F_z(0) \) and \( \dot{F}_z(0) \) may be inferred from the physically motivated assumption that the initial state is the ground state of the Hamiltonian (19) irrespective of the \( \lambda \) value.

Defining the correction \( f_z(t) \) by \( F_z(t) = \alpha_z(t) + f_z(t) \), we have
\[ \dddot{f}_z(t) + \Omega_z^2 f_z(t) = \lambda \dddot{\alpha}_z - B_z \Omega_z^2, \]
where
\[ B_z = q_{0z} \left[ 1 - (1 + \lambda)^{-1} \right], \]
which can be solved formally as
\[ f_z(t) = \frac{1}{\Omega_z} \int_0^t \left[ \lambda \dddot{\alpha}_z(t') - B_z \Omega_z^2 \right] dt' \times \sin [\Omega_z (t - t')] dt. \]

The energy can be calculated exactly within the harmonic approximation and it takes at final time \( T \) the form
\[ \{ \dot{H}_z(T) \} = \{ \phi_z(T; n) [\dot{H}_z(T) \phi_z(T; n)] \}
\[ = \left( n + \frac{1}{2} \right) \beta \Omega_z^2 + E_z(T), \]
where \( \phi_z(q_z; T; n) = \exp \left[ \frac{\Omega_z(0) q_z}{\beta} \right] \) \( \phi(0)[q_z - F_z(T); n] \) are the eigenstates of the invariants \( \tilde{E}_z \) corresponding to the Hamiltonians \( \dot{H}_z(21) \), and the final excitation energy for each mode is
\[ E_z(T) = \frac{1}{2} \left\{ \int_0^T \left[ \lambda \dddot{\alpha}_z(t) - B_z(t) \Omega_z^2 \right] \sin (\Omega_z^2 t) dt \right\}^2 + \frac{1}{2} \left\{ \int_0^T \left[ \lambda \dddot{\alpha}_z(t) - B_z(t) \Omega_z^2 \right] \cos (\Omega_z^2 t) dt \right\}^2 . \]

The total excitation energy is \( E_z(T) = E_z+(T) + E_z-(T) \).

In order to eliminate the excitation energy, the designed protocol should satisfy
\[ \int_0^T \left[ \lambda \dddot{\alpha}_z(t) - B_z(t) \Omega_z^2 \right] \cos (\Omega_z^2 t) dt = 0, \]
\[ \int_0^T \left[ \lambda \dddot{\alpha}_z(t) - B_z(t) \Omega_z^2 \right] \sin (\Omega_z^2 t) dt = 0. \]

Since \( \lambda \) is generally not known or drifts from run to run of the experiment, we simplify the condition. We use the Taylor expansion \((1 + \lambda)^{-1} = 1 - \lambda + \cdots \) and keep in the first order of \( \lambda \), so \( B_z \approx \lambda q_{0z} \). We also approximate \( \sin (\Omega_z^2 t) \approx \sin (\Omega_z t) \), \( \cos (\Omega_z^2 t) \approx \cos (\Omega_z t) \), so the dominant order of the condition (37), using equation (26) and dropping constants, is
\[ \int_0^T \dddot{\alpha}_z(t) \cos (\Omega_z t) dt = 0, \]
\[ \int_0^T \dddot{\alpha}_z(t) \sin (\Omega_z t) dt = 0. \]

As \( Q_0(t) \) depends on \( \alpha_\epsilon \) and \( \alpha_z \) via equations (22) and (15), and it should be a unique function, common to both modes, we may assume an ansatz for \( Q_0(t) \) with free parameters, and then get \( \alpha_\epsilon \) from equation (26) to satisfy the conditions (27)–(29). We solve equation (26) using the condition (27). Since equation (25) implies that the condition (29) of \( \ddot{\alpha}_z \) is also satisfied, we should thus design \( Q_0 \) to satisfy the conditions (7), (24), and (25); and moreover \( \alpha_\epsilon \) should satisfy equations (28), and the integrals in equation (38) have to be nullified. In the following, we compare the performance of different protocols for \( Q_0(t) \).

One possible ansatz is the polynomial (see figure 1)
\[ Q_0(t) = \sum_{j=0}^{13} \beta_j j^j, \]
where the coefficients are found from the above 14 conditions: (7), (24), (25), (28) and (38). The final excitation versus \( \lambda \) is shown in figure 2 for a couple of final times.
A simpler option is the trigonometric ansatz, $Q_0(t) = d \left[ \beta_0 + \sum_{j=1}^{4} \beta_j \cos \left( \frac{2\beta_j t}{T} \right) \right]$, that satisfies the conditions (7), (24), (25) and (28) with just five parameters,

$$Q_0(t) = d \left[ \frac{1}{2} + \left( -\frac{9}{16} + 2\beta_3 + 5\beta_4 \right) \cos \left( \frac{\pi t}{T} \right) 
+ \frac{1}{16} \left( 1 - 48\beta_3 - 96\beta_4 \right) \cos \left( \frac{3\pi t}{T} \right) 
+ \beta_3 \cos \left( \frac{5\pi t}{T} \right) + \beta_4 \cos \left( \frac{7\pi t}{T} \right) \right]. \quad (40)$$

where $\beta_3$ and $\beta_4$ can be given explicitly as functions of $T$,

$$\beta_3 = -\frac{49 \left( T^2 \Omega_c^2 - 25 \pi^2 \right) \left( T^2 \Omega_c^2 - 25 \pi^2 \right)}{2048 T^4 \Omega_c^4 \Omega^2},$$

$$\beta_4 = \frac{5 \left( T^2 \Omega_c^2 - 49 \pi^2 \right) \left( T^2 \Omega_c^2 - 49 \pi^2 \right)}{2048 T^4 \Omega_c^4 \Omega^2}. \quad (41)$$

The behavior of this cosine protocol is quite remarkable. In particular, even though the exact vanishing of the integrals in (38) is not imposed, they are indeed negligible for times larger than approximately eight periods (of particle 1). (The integrals are doable explicitly and the result is given in the appendix B.) The reason is that the mode trajectories $\alpha_2$ vary slowly with respect to the faster oscillation of $\sin (\Omega_2 t)$ or $\cos (\Omega_2 t)$, see figure 3. Of course this cancellation will not hold for very short process times $T$ of the order of a few oscillations, but in that short-time regime the harmonic approximation breaks down anyway. As shown in figure 2, the cosine protocol (40) is as stable as the polynomial protocol (39), even more stable for the longer final time. The range of validity of the harmonic approximation is examined in figure 4 using classical dynamics. This classical approximation is enough to detect significant deviations from the harmonic behavior and much less demanding computationally than a full quantum calculation. The figure shows the final excitation of two classically moving ions for $\lambda = 0$, with respect to the equilibrium energy, using both protocols and the exact Hamiltonian (9). The protocols would be excitation-free for $\lambda = 0$ in the harmonic approximation, so the excitation at short times is due to anharmonicities and mode-coupling. The polynomial ansatz is slightly more robust with respect to them, holding negligible excitation up to nine oscillation periods, versus ten oscillation periods for the cosines. The trigonometric ansatz provides in summary an excellent, simple way to eliminate the spring constant error for the transport of two ions, as it is given by explicit time-dependent coefficients.

Finally, let us compare the results of the cosine protocol with the protocols derived in [11] for one particle. As stated earlier, the CM is coupled to the relative motion unless the masses are equal. For equal masses only the CM is relevant to design the trap trajectory. For unequal masses, we may try to engineer a $Q_{0}(t)$ approximately neglecting the coupling, in other words, considering a single uncoupled (CM) particle with Hamiltonian $\hat{H}_{CM}^2 = \hat{\Omega}_{CM}^2 (\hat{Q} - \hat{Q}_0)^2$, where $\hat{\Omega}$ and $\hat{P}$ are conjugate CM position and momentum operators, $M = m_1 + m_2$, and $\hat{\Omega}_{CM}^2 = \frac{2\omega_{CM}^2}{M}$. Specifically we may design $\hat{Q}_{0}(t)$ as in [11], Section IV, to make it robust versus the spring-constant errors we are interested in here. With that new $\hat{Q}_{0}(t)$ we compute the excitation using equation (37). It is indeed much larger than the excitation of the cosine protocol as shown in figure 5. We conclude that for unequal masses the two-mode approach is clearly superior to a simplistic approach based on a single uncoupled CM coordinate.

![Figure 1](image1.png)

**Figure 1.** Trap trajectories versus time for the protocols in equation (39) (red dashed line) and equation (40) (blue solid line) for $T = 10.5 T_0$, $\omega_1 = 2\pi \times 2$ MHz, $\omega_0 = 2\pi \times \omega_1$, $d = 370 \mu m$, and $n = 0$. The ions (1 and 2) are $^9\text{Be}^+$ and $^{24}\text{Mg}^+$. (See in [11], Section IV, to make it robust versus the spring-constant errors we are interested in here. With that new $\hat{Q}_{0}(t)$ we compute the excitation using equation (37). It is indeed much larger than the excitation of the cosine protocol as shown in figure 5. We conclude that for unequal masses the two-mode approach is clearly superior to a simplistic approach based on a single uncoupled CM coordinate.)

![Figure 2](image2.png)

**Figure 2.** Excitation suppression. Excitation energy $E_s$ versus the error for $T = 7.5 T_0$ (a) and $T = 10.5 T_0$ (b), using the protocol in equation (39) (red dashed line) and equation (40) (blue solid line). Other parameters are the same as in figure 1.
5. Conclusion

We have found trap trajectories to transport without final excitation two ions of different mass. The trajectories are designed to be robust with respect to errors in the spring constant. To achieve that goal we have combined invariant-based inverse engineering and a harmonic approximation in dynamically defined normal-mode coordinates. Shortcuts to adiabaticity with enhanced robustness have been designed as well for discrete systems [18–20], and some of the results and techniques may be applied to ion transport. In particular the robustness can be improved systematically if necessary as in [19], by nullifying integrals associated with higher orders in the relative error parameter. The design of trap-transport functions robust with respect to random, noisy perturbations of the spring constant requires a different treatment and will be tackled elsewhere.

Acknowledgments

We are grateful to Margarita and Vladimir Man’ko for their inspiring research throughout the years. We dedicate this article to them as a little tribute to their studies on dynamical invariants. This work was supported by the Grants No. IT472-10, No. FIS2009-12773-C02-01, No. UFI 11/55, No. 11474193, No. 61176118, No. 13PJ1403000, No. 2013310811003, and the Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning. MP acknowledges a fellowship by UPV/EHU.

Appendix A. Equal masses

In this appendix we discuss the equal mass limit and explain the connection between the results for CM and relative coordinates in [11] and the dynamical normal-mode approach followed in this paper.
CM-relative coordinates: in the CM-relative coordinates, the Hamiltonian in (9) for \( m_1 = m_2 = m \), can be written as

\[
\hat{H} = \frac{\hat{P}_c^2}{2M} + \frac{1}{2}M\omega_{cm}^2 \left( \hat{Q} - Q_0 \right)^2 + \frac{1}{2}m_r\omega_r^2 \hat{r}_c^2 + \frac{1}{2}m_c\omega_c^2 \hat{r}_c^2 + \frac{C_r}{\hat{r}},
\]

where

\[
\hat{Q} = \frac{\hat{q}_1 + \hat{q}_2}{2}, \quad \hat{r} = \hat{q}_1 - \hat{q}_2,
\]

\[
\hat{P} = \hat{p}_1 + \hat{p}_2, \quad \hat{P}_c = \frac{\hat{p}_1 - \hat{p}_2}{2}, \quad M = 2m, \quad m_r = \frac{m_c}{2}, \quad \omega_{cm} = \omega_r = \omega_c.
\]  

The two coordinates are uncoupled, and only the motion of the center of mass depends on the trajectory \( Q_0(t) \) of the trap. Thus, the design of shortcuts without final excitation reduces to an effective one-particle transport problem.

Normal-modes: for equal masses \( \mu = 1 \), so equations (12), (14) and (15) become

\[
a_+ = \frac{1}{\sqrt{2}}, \quad b_+ = -\frac{1}{\sqrt{2}}, \quad a_- = \frac{1}{\sqrt{2}}, \quad b_- = \frac{1}{\sqrt{2}}, \quad \Omega_0^2 = 3\omega_r^2, \quad \Omega_0^2 = \omega_c^2, \quad P_{0+} = 0, \quad P_{0-} = \sqrt{2m}Q_0.
\]

The new coordinates and momenta are

\[
\hat{q}_+ = \frac{1}{\sqrt{2}}(\hat{q}_1 - \hat{q}_2 - l), \quad \hat{\dot{q}}_+ = \frac{1}{\sqrt{2}}(\hat{q}_1 + \hat{q}_2 - 2Q_0), \quad \hat{p}_+ = \frac{1}{\sqrt{2}m}(\hat{p}_1 - \hat{p}_2), \quad \hat{\dot{p}}_+ = \frac{1}{\sqrt{2}m}(\hat{p}_1 + \hat{p}_2).
\]

Then the Hamiltonian in normal-coordinate form (13) takes the form

\[
\hat{H}_N = \frac{1}{2}\hat{P}_c^2 + \frac{1}{2}m\Omega_0^2 \hat{q}_+^2 + \sum_{n=1}^\infty \left( \frac{nC_n}{m} \right)^2 (-1)^n \hat{q}_+^n,
\]

If we apply to the wave function that evolves with \( \hat{H}_N \) the unitary transformation

\[
\hat{U}_0 = \int d\Omega dr dq_+ dq_- |Q, r \rangle \langle q_+, q_-|, \quad (A.8)
\]

the new Hamiltonian will be \( \hat{H}_{NC} = \hat{U}_0 \hat{H}_N \hat{U}_0^\dagger + i\hbar \frac{\partial}{\partial t} \hat{U}_0^\dagger \). For the first part we substitute the definitions (A.7) in the Hamiltonian (A.6),

\[
\hat{H}_I = \frac{\hat{P}_c^2}{2M} + \frac{1}{2}m\omega_r^2 \hat{r}_c^2 + \frac{C_r}{\hat{r}}, \quad \hat{P}_c = \frac{1}{\sqrt{2}m}M\omega_{cm}^2 \left( \hat{Q} - Q_0 \right)^2 - Q_0 \hat{P}.
\]

For the second part we calculate

\[
\frac{\partial \hat{U}_0}{\partial t} = \int d\Omega dr dq_+ dq_- |Q, r \rangle \frac{\partial}{\partial t} \langle q_+| q_- |, \quad (A.10)
\]

where \( \langle Q, r| q_+| q_- \rangle = \delta [Q - Q(q_+, q_-)]\delta [r - r(q_+, q_-)] = \delta_0 \delta_r \) and

\[
\frac{\partial \langle q_+| q_- \rangle}{\partial t} = \frac{\partial \langle q_+| q_0 \rangle}{\partial t} + \delta_0 \delta_r \frac{\partial \langle q_0| q_- \rangle}{\partial t} = -Q_0 \delta_0 \delta_r \delta_0 Q_0.
\]

The second part is thus

\[
\hat{H}_2 = i\hbar \frac{\partial \hat{U}_0}{\partial t} \hat{U}_0^\dagger = \hat{Q}_0^\dagger \hat{P},
\]

so the Hamiltonian \( \hat{H}_{NC} = \hat{H}_I + \hat{H}_2 \) coincides with the one in equation (A.1).

Appendix B. Integrals in the cosine protocol

Using the protocol (40), we get

\[
\Omega_0^2 \int_0^T \alpha_z(t) \cos(\Omega_\perp t) \, dt + \Omega_0^2 \int_0^T \alpha_z(t) \sin(\Omega_\perp t) \, dt = \frac{11025 \pi^8 - 12916 \pi^6 T^2 \Omega_0^2}{2D_0} \times \left[ 1 + \cos(\Omega_z T) - \sin(\Omega_z T) \right],
\]

where \( D_0 = (11025 \pi^8 - 12916 \pi^6 T^2 \Omega_0^2 + 1974 \pi^8 T^4 \Omega_0^4 - 84 \pi^6 T^6 \Omega_0^6 + T^8 \Omega_0^8) \Omega_0^2 \). One of the integrals is zero when \( \Omega_z T = 2k\pi + \frac{\pi}{2} \) or \( 2k\pi + \pi \), with \( k = 0, 1, 2, \ldots \).

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