Nonlinear devices, such as transistors, enable contemporary computing technologies. We theoretically investigate nonlinear effects, bearing a high fundamental scientific and technical relevance, in magnonics with emphasis on superconductor-ferromagnet hybrids. Accounting for finite magnon chemical potential, we theoretically demonstrate magnonic spin-Joule heating, the spin analogue of conventional electronic Joule heating. Besides suggesting a key contribution to magnonic heat transport in a broad range of devices, it provides insights into the thermal physics of non-conserved bosonic excitations. Considering a spin-split superconductor self-consistently, we demonstrate its interface with a ferromagnetic insulator to harbor large tunability of spin and thermal conductances. We further demonstrate hysteretic rectification effect in this hybrid, where the hysteresis results from the superconducting state bistability.

Introduction.—The spin carriers in magnetic insulators - magnons - constitute a fertile platform for science and technology due to their nonconserved bosonic nature as well as solid-state host that admits strong interactions. Since they transport information without a movement of electrons, and Joule heating due to charge current flow, they are also touted as low-dissipation alternatives to electrons as information carriers [1–10]. While magnon chemical potential vanishes in equilibrium due to their nonconserved nature, it becomes nonzero in nonequilibrium situations due to their almost conserved nature at short time scales. This underlies a vast range of phenomena such as Bose-Einstein condensation [11–13] and spin transport [14–17], where the importance of nonzero chemical potential in these phenomena has recently been, and continues to be, recognized [14, 18]. Overall, a broad range of magnonic devices have already been demonstrated [17, 19–26].

From the perspective of both devices and exciting physics, nonlinearities are highly desired, whether for unconventional computing paradigms [27] or Maxwell’s daemon-like switches [28, 29]. In this context, the high potential of synergy between magnonics and superconductors has just begun to be realized [30–33]. The latter admit strong nonlinear effects since a new and small energy scale - the superconducting gap - determines properties such as quasiparticle density of states [34–37]. Recent years have seen an upsurge of activity in this context with several exciting phenomena discussed, experimentally [33, 38–46] as well as theoretically [35–37, 47–53]. However, the possibility of transport influencing, and even destroying, the superconducting state was not considered. This consideration is the pinnacle of nonlinearity offered by superconductor and has been exploited successfully in various charge transport based devices [54–56].

Here, we theoretically investigate spin and heat transport in hybrids consisting of a superconductor (S) interfaced with a ferromagnetic insulator (FI). Our focus is on exploiting the strong nonlinearities available in this hybrid for new device concepts and fundamental physics. Hence, we evaluate the superconducting state self-consistently. Our first key finding is that magnonic spin current flow results in a spin-Joule heating given by $I_m^2R_m$, where $I_m$ is the magnon current and $R_m$ is the spin resistance. This finding is not specific to the hybrids considered and bears relevance for magnon chemical potential driven spin transport in general. In bulk magnets, the spin-Joule heating power per unit volume becomes $j_m^2\rho_m$. Here, $j_m$ is the magnon current density and $\rho_m$ the spin resistivity, is defined via $j_m = -\nabla\mu_m/\rho_m$, where $\mu_m$ is the magnon chemical potential [14]. Our second set of findings demonstrates a control over the transport coefficients, such as spin and thermal conductances, in the S-FI interface via spin-splitting in the S layer. Further, via self-consistent calculations, we demonstrate a hysteretic rectification effect in the spin current vs. spin chemical potential difference. This is rooted in the dependence of the superconducting state on the spin chemical potential, as the latter results in depairing of spin-singlet Cooper pairs [30]. Besides the interesting directionality, such an “I-V” characteristic could enable devices with built-in memory and threshold behavior, beneficial in some unconventional computing architectures [27].

Spin-Joule heating.—Considering an interface between an FI and S [Fig. 1 (a)], the heat generated per unit time in FI and S is given by $\dot{Q}_{FI} = \dot{E}_{FI} - \mu_sN_m$ and $\dot{Q}_S = \dot{E}_S - \mu_sN_s$. Here, $\dot{E}_{FI}$ ($\dot{E}_S$) is the rate of energy change in FI (S), $\mu_m$ is the magnon chemical potential in FI, $\mu_s$ the spin accumulation in S, $N_m$ is the rate...
of magnon number change in FI, and \( \dot{N}_s \) is the rate of electron-hole pair change in S [13]. On account of energy and spin conservation, we obtain: \( \dot{E}_S = -\dot{E}_{FI} \equiv \dot{E} \) and \( \dot{N}_s = -\dot{N}_m \equiv \dot{N} = I_s/\hbar \), with \( I_s \) the spin current across the interface. Here, the inclusion of \( \mu_{m,s} \) terms in the definition of heat [57] is necessary when considering effects, such as heat generation, up to second order in \( \mu_{m,s} \). The physical justification of this contribution, which is well-known for electrons [58, 59], is as follows. A particle added to an ensemble at an energy below the chemical potential needs to absorb energy from other particles in order to respect the statistical distribution enforced by a nonzero chemical potential [58]. Thus, this causes a cooling of the ensemble as a whole.

The average heat flow from FI to S is obtained as \( \dot{Q} = (\dot{Q}_S - \dot{Q}_{FI})/2 = E - \mu I_s/\hbar \). Before we evaluate the spin and heat flow across the interface below, we pause to examine the total heat generation \( \Delta Q \equiv \dot{Q}_S + \dot{Q}_{FI} \) in our system. Considering temperature to remain uniform, the heat generation due to chemical potential driven spin transport is simplified to:

\[
\Delta \dot{Q} = I_m \Delta \mu = I_m^2 \rho_m R_m,
\]

where \( I_m = I_s/\hbar \), \( \Delta \mu \equiv \mu_m - \mu_s \), and the linear response relation \( I_m = \Delta \mu/\rho_m \) will be derived below. This is the spin analogue of Joule heating expression for charge current flow across an interface. A generalization of this result to an interface between two FIs with different magnon chemical potentials leads to a similar expression for the heating. In the continuum limit, this leads to magnonic spin-Joule heating power per unit volume in the FI bulk:

\[
P = j_m^2 \rho_m,
\]

where \( j_m = -\nabla \mu_m/\rho_m \) is the magnon current density and \( \rho_m \), the magnon spin resistivity.

Equation (2) is a general result with broad consequences for magnonic spin transport in different materials, hybrids, and regimes. The charge conservation allows for relating electronic Joule heating directly to work done by the external battery [59]. Such a straightforward identification does not appear possible for magnonic spin transport. Nevertheless, the spin-Joule heating is also derived from the work done by external sources that maintain nonzero chemical potentials in the system. Due to the high sensitivity of superconductor-based thermometers [60], the FI/S hybrids investigated below offer a suitable platform for an experimental measurement of the spin-Joule heating.

**Spin and heat currents in linear response.**—We consider the system shown in Fig. 1(a). A superconductor is placed in contact with a ferromagnetic insulator, and a spin splitting field \( h \) is introduced to the former. This system may be influenced by a spin chemical potential on the superconductor side, a nonequilibrium magnon chemical potential on the ferromagnetic insulator side, or a temperature gradient across the system, all of which may result in the flow of heat and spin currents due to exchange interactions between electrons and magnons at the interface. We show that the spin splitting field leads to a significant asymmetry in the transport properties of the heterostructure, with respect to the orientation of \( h \). The main physics behind this effect illustrated in Fig. 1(b). Spin can be transmitted between the two materials when, e.g., a spin down quasiparticle in the superconductor experiences a spin flip upon reflection at the interface, accompanied by the creation of a magnon in the ferromagnetic insulator, or vice versa. At low temperatures, this process is suppressed due to the presence of the superconducting gap. However, the size of this gap can be tuned by \( h \). In a spin split superconductor, the density of states for the two spin species is shifted relative to each other by a value of \( 2h \) [30, 35, 37]. This means that the effective gap that must be overcome by a spin-down quasiparticle undergoing a spin flip is increased if the spin splitting field is parallel to the magnetization in the ferromagnetic insulator (\( h > 0 \)), or reduced if it is antiparallel (\( h < 0 \)). Hence, the latter configuration is more amenable to the generation of transport currents.

We study the interface between the superconductor and the ferromagnetic insulator using a tunneling Hamiltonian approach [13, 15, 47, 48, 61],

\[
H = H_{SC} + H_{FI} + H_{\text{int}},
\]

\[\text{(3)}\]
where \( H_{SC} \) describes a Zeeman split superconductor,

\[
H_{SC} = \sum_{k\nu} \xi_k c_{k\nu}^\dagger c_{k\nu} - \sum_k \left[ \Delta c_{k\uparrow}^\dagger c_{-k\downarrow} + \Delta^* c_{-k\downarrow}^\dagger c_{k\uparrow} \right] - \sum_{k\nu,s,s'} \hbar \sigma_{ss'} \nu_k c_{k\nu}^\dagger c_{k\nu},
\]

with \( \xi_k = \hbar^2 k^2/2m - \mu \), with chemical potential \( \mu \), and \( \hbar \) is the exchange field, assumed to be directed along the \( z \) axis. The Hamiltonian of the ferromagnetic insulator is given within the Holstein-Primakoff approximation as \[62\]

\[
H_{FI} = \sum_k \hbar \omega_k a_{k\uparrow}^\dagger a_{k\uparrow},
\]

with magnon operators \( a_k \) and \( a_k^\dagger \). We assume a quadratic dispersion of the form \( \hbar \omega_k = \Delta_m + J_m k^2 \).

The two materials may communicate by the exchange of spin, in which a magnon on the ferromagnet side is either absorbed or created by a quasiparticle spin flip on the superconductor side. This process is captured by \[48\]

\[
H_{int} = \sum_{kk'} \left[ W_{kk'} s_k a_k^\dagger a_{k'}^\dagger + W_{kk'}^* s_k^\dagger a_{k'} a_k \right],
\]

with \( s_k^\dagger = s_k^\dagger \pm s_k^\dagger \), and \( s_k^\dagger = \frac{1}{2} \sum_{q,s,s'} \sigma_{ss'} c_{k+q,s'} \).

The transport properties of this system are most conveniently studied on the ferromagnetic insulator side, where the tunneling spin current becomes \( I_s = i \langle [N_m, H] \rangle \), with \( N_m = \sum_k a_k^\dagger a_k \). This gives,

\[
I_s = i \sum_{kk'} \left( W_{kk'} s_k a_{k'}^\dagger a_k - W_{kk'}^* s_k^\dagger a_{k'} a_k \right).
\]

In a similar way, the average heat current is \( \dot{Q} = i \langle [H_{FI}, H] \rangle / \hbar - \tilde{\mu} I_s / \hbar \), giving

\[
\dot{Q} = \frac{i}{\hbar} \sum_{kk'} (\hbar \omega_{k'} - \tilde{\mu}) \left( W_{kk'} s_k a_{k'}^\dagger a_k - W_{kk'}^* s_k^\dagger a_{k'} a_k \right).
\]

These quantities may be calculated by using Green function techniques \[63\], resulting in

\[
I_s = -2 |W|^2 V_{FI} \hbar \int d\omega \nu_m(\hbar \omega) \chi_s(\hbar \omega) \times [n_{FI}(\hbar \omega - \mu_m) - n_{SC}(\hbar \omega - \mu_s)], \tag{9}
\]

\[
\dot{Q} = -2 |W|^2 V_{FI} \hbar \int d\omega (\hbar \omega - \tilde{\mu}) \nu_m(\hbar \omega) \chi_s(\hbar \omega) \times [n_{FI}(\hbar \omega - \mu_m) - n_{SC}(\hbar \omega - \mu_s)], \tag{10}
\]

under the approximation of \( W_{kk'} \approx W \). Here, \( V_{FI} \) is the volume of the FI, \( n_j(\varepsilon) = \left[ e^{\varepsilon/k_BT} - 1 \right]^{-1} \) is the Bose-Einstein distribution function in material \( j \), \( \nu_m(\varepsilon) = \sqrt{\varepsilon - \Delta_m / 4\pi^2 J_s^{3/2}} \) is the magnon density of states and

\[
\chi_s(\hbar \omega) = -\pi \hbar^2 V_{SC}^2 \int d\varepsilon F_{\Delta} \nu(\varepsilon + h\omega - h) \times [f(\varepsilon + h\omega - \mu_s/2) - f(\varepsilon + \mu_s/2)] \tag{11}
\]

is the transverse spin susceptibility of the superconductor, with \( \nu(\varepsilon) = \Re \left[ \varepsilon / \sqrt{\varepsilon^2 - \Delta^2} \right] \) the superconducting density of states, \( F_{\Delta} = 1 + |\Delta|^2 / (\varepsilon + h) (\varepsilon + h\omega - h) \) the coherence factor, and \( f(\varepsilon) = \left[ e^{\varepsilon/k_BT} + 1 \right]^{-1} \) the Fermi-Dirac distribution function.

For small temperature differences across the tunnel junction, \( \Delta T = T_{FI} - T_{SC} \), or a small difference in chemical potentials, \( \Delta \mu = \mu_m - \mu_s \), Eqs. (9) and (10) may be linearized, to obtain

\[
\left( I_s / \dot{Q} \right) \approx \left( \frac{G \ h \alpha}{\alpha \ kT} \right) \left( \frac{\Delta \mu}{\Delta T / T} \right), \tag{12}
\]

with \( T = (T_{FI} + T_{SC}) / 2 \). This defines the spin conductivity \( G \), the spin-dependent Seebeck coefficient \( \alpha \), and
FIG. 3. The spin and heat current as a function of the chemical potential difference, $\Delta \mu$, exhibiting a switching effect in which both $I_s$ and $Q$ exhibit a jump in magnitude on the order of 100. Here, an exchange field of $h = 0.7\Delta_0$ is applied, which is close to the critical field at which superconductivity is destroyed. We have set $T_{SC} = 0.1T_c$, $T_{FI} = 0.5T_c$, and $\mu_m = 0$. The red (blue) curve describes a situation in which superconductivity is regained (destroyed). These two curves are different due to the hysteresis caused by the bistability of the superconducting gap $\Delta$ as a function of the effective spin splitting $h_{\text{eff}} = h - \mu_s/2$, as shown in the inset. Here, $I_0 = |W|^2V_{F\text{I}}h\chi_0\Delta_0/2\pi^2J_s^{2/3}$, with $\chi_0 = \pi h^2V_{\text{SC}}^2\nu_0^3\Delta_0$, and $Q_0 = I_0\Delta_0/h$.

The heat conductivity $\kappa$ as
\[
G = -2|W|^2V_{F\text{I}}h \int d\omega \frac{\chi_0(h\omega)\nu_m(h\omega)}{4k_BT \sinh^2 \frac{h\omega - \mu}{2k_BT}}, \quad (13)
\]
\[
\alpha = -2|W|^2V_{F\text{I}} \int d\omega \frac{(h\omega - \mu)^2 \chi_0(h\omega)\nu_m(h\omega)}{4k_BT^2 \sinh^2 \frac{h\omega - \mu}{2k_BT}}, \quad (14)
\]
\[
\kappa = -2|W|^2V_{F\text{I}} \int d\omega \frac{(h\omega - \mu)^2 \chi_0(h\omega)\nu_m(h\omega)}{2k_BT^2}, \quad (15)
\]

For concreteness, we set the critical temperature of the superconductor to $T_c = 1\, \text{K}$, corresponding to a superconducting gap of $\Delta_0 = 1.76k_BT_c \approx 150\, \text{meV}$. This is consistent with the superconducting properties of aluminum [64]. For the ferromagnetic insulator, a typical value the magnon gap is $\Delta_m/k_B \approx 1\, \text{K}$ [14], and so for simplicity, we set $\Delta_m = \Delta_0$.

Control over the interfacial conductances and rectification. To investigate the effect of the spin splitting field $h$ on the transport coefficients given in Eqs. (13)–(15), we define the quantity $P_x = (x_\uparrow - x_\downarrow)/(x_\uparrow + x_\downarrow)$, which we refer to as the polarization of $x$, for $x \in \{G,\alpha,\kappa\}$, where $\uparrow$ ($\downarrow$) indicates $h > 0$ ($h < 0$). The result is shown in Fig. 2, for $|h| \in [0.0.6]$ for a range of temperatures $T$. It is seen that for the lowest temperature considered, $T/T_c = 0.3$, there is a significant polarization. It is negative, indicating that larger currents are to be expected for $h < 0$, consistent with the physical picture presented in Fig. 1(b). We also see that the effect diminishes as the temperature of the heterostructure approaches $T_c$, at which point the superconductor transitions to a normal metal, with no modulation of the density of states, and thus no polarization.

The gap in the density of states, which is present in the superconducting state, but not in the normal metal state, has the potential for an interesting application. In the following, we set the temperature of the superconductor to $T_{SC} = 0.1T_c$, and in the FI to $T_{FI} = 0.5T_c$. Hence, a temperature gradient is maintained across the interface, and both spin and heat currents are flowing between the two materials. On the other hand, the magnitude of these currents is largely reduced compared to a normal metal due to the superconducting gap. Next, we set the spin splitting field to $h = 0.7\Delta_0$. This is close to the critical field at which the superconductor transitions to the normal state, but the size of the gap still remains close to the maximal value $\Delta_0$, as evidenced by a self-consistent determination of $\Delta(h)$. We note that the superconducting gap responds to a spin chemical potential as $[65] \Delta(h) \rightarrow \Delta(h_{\text{eff}})$, with $h_{\text{eff}} = h - \mu_s/2$. Hence, if $\mu_s < 0$, the two contributions will add up, which has the potential of bringing the superconductor into the normal-state regime at some critical field $h_{\text{eff}}$. In the opposite case, $\mu_s$ partially cancels $h$, and the normal-state system returns to the superconducting state at $h_{\text{eff}}$. We note that due to the hysteresis caused by the bistability of $\Delta(h_{\text{eff}})$ in the transition region [66–68], $h_{\text{eff}}^+$ is generally not equal to $h_{\text{eff}}^-$, as indicated in the inset of Fig. 3. In any case, the point is that $\mu_s$ can cause superconductivity to either be destroyed or regained, which produces an abrupt change in the size of the currents. To illustrate this effect, we plot $I_s$ and $Q$, as given by Eqs. (9) and (10), in Fig. 3(a) and (b), respectively, for $\Delta/\Delta_0 \in [-0.6, 0.0]$, keeping $\mu_m = 0$ fixed. We find that both $I_s$ and $Q$ feature jumps in magnitude on the order of 100 when a transition takes place, for this parameter set.

Summary. Exploiting the weaker energy scale of superconducting gap, we have demonstrated a broad range of nonlinear effects in the context of magnonic spin transport in superconductor-ferromagnetic insulator hybrids. The predicted control over interfacial conductances and hysteretic rectification I-V characteristics open avenues for integrating magnonic devices into unconventional computing architectures, for example. Our theoretical demonstration of magnonic spin-Joule heating provides valuable insights into the wide range of studies and de-
periences involving chemical potential-driven spin transport.

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Supplemental Material

The Hamiltonian considered in this work is given as

\[ H = H_{SC} + H_{FI} + H_{int}, \tag{S1} \]

where

\[ H_{SC} = \sum_{k,s} \xi_k c_{k,s}^\dagger c_{k,s} - \sum_k \left[ \Delta c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \Delta^* c_{k\downarrow} c_{-k\uparrow} \right] - \hbar \sum_{k,\alpha\beta} \sigma\bar{\alpha}\bar{\beta} c_{k\alpha}^\dagger c_{k\beta}, \tag{S2} \]

\[ H_{FI} = \sum_k \hbar \omega_k a_k^\dagger a_k, \tag{S3} \]

represent the initial, unperturbed system, and

\[ H_{int} = \sum_{k,k'} \left[ W_{kk'} s_k^\dagger a_{k'} + W_{kk'}^* s_k a_{k'}^\dagger \right] \tag{S4} \]

describes their interaction, assumed to be turned on at time \( t = 0 \). Here, \( s_k = \hbar \sum_q c_{k+q,\downarrow}^\dagger c_{k,\downarrow} \) describes a spin flip processes, and \( a_k \) is a magnon annihilation operator. As described in the main text, the spin and heat currents are respectively given as

\[ I_s(t) = i \sum_{kk'} \left\langle \left[ W_{kk'}(t) s_k^\dagger(t) a_{k'}(t) - W_{kk'}^*(t) s_k(t) a_{k'}^\dagger(t) \right] \right\rangle, \tag{S5} \]

\[ \dot{Q}(t) = i \frac{\hbar}{\beta} \sum_{kk'} \left( \hbar \omega_{k'} - \mu \right) \left\langle \left[ W_{kk'}(t) s_k^\dagger(t) a_{k'}(t) - W_{kk'}^*(t) s_k(t) a_{k'}^\dagger(t) \right] \right\rangle, \tag{S6} \]

where the spin and magnon chemical potential has been introduced via the transformation \( c_{k\sigma} \rightarrow c_{k\sigma} e^{-i\mu_\tau t/\hbar} \) and \( a_k \rightarrow a_k e^{-i\mu_s t/\hbar} \), such that \( W_{kk'}(t) = W_{kk'} e^{-i(\mu_\tau - \mu_s) t/\hbar} \), where we have set \( \mu_\tau = -\mu_s = -\mu_s/2 \). We note further that Eqs. (S5) and (S6) are given in the interaction representation, so that the time evolution of \( s_k \) and \( a_k \) are governed by \( H_0 = H_{SC} + H_{FI} \). To proceed, we define Green functions in the two materials as

\[ \chi_{kk'}(t,t') = -i \left\langle T s_k(t) s_{k'}^\dagger(t') \right\rangle, \tag{S7} \]

\[ D_{kk'}(t,t') = -i \left\langle T a_k(t) a_{k'}^\dagger(t') \right\rangle, \tag{S8} \]

where \( T \) is the time ordering operator, as well as a mixed Green function of the form

\[ G_{kk'}(t,t') = -i \left\langle T a_k(t) s_{k'}(t') \right\rangle. \tag{S9} \]
In terms of Eq. (S9), Eqs. (S5) and (S6) become

\begin{align}
I_s(t) &= -2 \sum_{kk'} \Re \left[ W_{kk'}(t) G^<_k(t,t) \right], \\
\dot{Q}(t) &= -\frac{2}{\hbar} \sum_{kk'} \left( \hbar \omega_{kk'} - \bar{\mu} \right) \Re \left[ W_{kk'}(t) G^<_k(t,t) \right],
\end{align}

with \( G^<_k(t,t') = -i \left\langle \hat{s}^+_k(t') a_k(t) \right\rangle \). To find this lesser Green function, we first determine Eq. (S9) on the Keldysh contour,

\[ G_{kk'}(\tau_1, \tau_2) = -i \left\langle T_K a_k(\tau_1) \hat{s}^+_k(\tau_2) \right\rangle = -i \left\langle T_K a_k(\tau_1) \hat{s}^+_k(\tau_2) \exp \left[ -\frac{i}{\hbar} \int_C d\tau \ H_{\text{int}}(\tau) \right] \right\rangle_0, \]

where \( T_K \) implies time ordering along the Keldysh contour \( C \), and the subscript 0 indicates averaging over the unperturbed system. To lowest order in the perturbation one finds

\[ G_{kk'}(\tau_1, \tau_2) \approx \frac{-(i)^2}{\hbar} \sum_{qq'} \int_C d\tau \left[ W_{qq'}(\tau) \left\langle T_K a_k(\tau_1) \hat{s}^+_k(\tau_2) \hat{s}^+_q(\tau) a_q(\tau) \right\rangle_0 + W_{qq'}(\tau) \left\langle T_K a_k(\tau_1) \hat{s}^+_k(\tau_2) \hat{s}^+_q(\tau) a_q(\tau) \right\rangle_0 \right]. \]

The first term is immediately seen to vanish, as it only contains contractions between magnon annihilation operators. From the second term we get

\[ G_{kk'}(\tau_1, \tau_2) = \frac{1}{\hbar} \int_C d\tau \ W^*_{kk'}(\tau) D_{kk}(\tau_1, \tau) \chi_{kk'}(\tau, \tau_2), \]

by a straightforward application of Wick’s theorem. From the Langreth rules, we thus find the lesser Green function to be

\( G^<_k(t,t) = \frac{1}{\hbar} \int dt W^*_{kk'}(t) \left[ D^R_{kk}(t,t) \chi^<_k(t,t) + D^<_{kk}(t,t) \chi^A_{kk'}(t,t) \right], \)

where \( D^R_{kk}(t,t') = -i\theta(t-t') \left\langle a_k(t), a_k^{\dagger}(t') \right\rangle_0 \), \( D^<_{kk}(t,t') = +i\theta(t'-t) \left\langle a_k(t), a_k^{\dagger}(t') \right\rangle_0 \) are the retarded and advanced magnon Green functions, respectively. Similar definitions apply to \( \chi^R \) and \( \chi^A \). Exploiting time translation invariance (implying that we consider times long after the perturbation has been switched on), we find

\[ G^<_k(t,t) = \hbar W^*_{kk'} e^{i(\mu_m - \mu_s)t/\hbar} \int \frac{d\omega}{2\pi} \left[ D^R_{kk'}(\omega - \frac{\mu_m}{\hbar}) \chi^<_k(\omega - \frac{\mu_s}{\hbar}) + D^<_{kk'}(\omega - \frac{\mu_m}{\hbar}) \chi^A_k(\omega - \frac{\mu_s}{\hbar}) \right]. \]

Inserting into Eqs. (S10) and (S11) thus gives

\begin{align}
I_s &= 4\hbar \sum_{kk'} \int \frac{d\omega}{2\pi} |W_{kk'}|^2 \Im D^R_{kk'}(\omega - \frac{\mu_m}{\hbar}) \Im \chi^R_{kk'}(\omega + \frac{\mu_s}{\hbar}) \left[ n_{\text{FI}}(\omega - \frac{\mu_m}{\hbar}) - n_{\text{SC}}(\omega - \frac{\mu_s}{\hbar}) \right], \\
\dot{Q} &= 4 \sum_{kk'} \int \frac{d\omega}{2\pi} (\hbar \omega_{kk'} - \bar{\mu}) |W_{kk'}|^2 \Im D^R_{kk'}(\omega - \frac{\mu_m}{\hbar}) \Im \chi^R_{kk'}(\omega + \frac{\mu_s}{\hbar}) \left[ n_{\text{FI}}(\omega - \frac{\mu_m}{\hbar}) - n_{\text{SC}}(\omega - \frac{\mu_s}{\hbar}) \right],
\end{align}

where we have used the identities \( M^A(\omega) = (M^R(\omega))^* \) and \( M^<(\omega) = 2i \Im M^R(\omega) n_j(\omega) \), with \( n_j \) the Bose-Einstein distribution function in material \( j \). In the following, we approximate \( W_{kk'} \to W \). The retarded, noninteracting magnon Green function, albeit with a chemical potential \( \mu_m \), is given as

\[ D^R_{kk}(\omega) = \frac{1}{\hbar \omega - \hbar \omega_k + \mu_m + i\delta}, \]

where \( \delta \) is an infinitesimal quantity. Hence, \( \Im D^R_{kk}(\omega - \mu_m/\hbar) = -\pi \delta(\hbar \omega - \hbar \omega_k) \). Furthermore, we let \( \sum_{k'} \to V_{\text{FI}} \int d(\hbar \omega_k) \nu_m(\hbar \omega_k) \), where \( V_{\text{FI}} \) is the volume of the ferromagnetic insulator and \( \nu_m(\varepsilon) = \sqrt{\varepsilon - \Delta_m^2/4\pi^2 J_s^3/2} \) is the magnon chemical potential, so that we get

\begin{align}
I_s &= -2 |W|^2 V_{\text{FI}} \sum_k \int d\omega \nu_m(\hbar \omega) \Im \chi^R_{kk}(\omega + \frac{\mu_s}{\hbar}) \left[ n_{\text{FI}}(\omega - \frac{\mu_m}{\hbar}) - n_{\text{SC}}(\omega + \frac{\mu_s}{\hbar}) \right], \\
\dot{Q} &= -2 |W|^2 V_{\text{FI}} \sum_k \int d\omega (\hbar \omega - \bar{\mu}) \nu_m(\hbar \omega) \Im \chi^R_{kk}(\omega + \frac{\mu_s}{\hbar}) \left[ n_{\text{FI}}(\hbar \omega - \mu_m) - n_{\text{SC}}(\hbar \omega - \mu_s) \right],
\end{align}
Next, we require an expression for $\chi_{kk'}^{R}(\omega)$. We begin from Eq. (S7), which is given in terms of the electron creation and annihilation operators in imaginary time as

$$\chi_{kk'}(\tau_1, \tau_2) = -\hbar^2 \sum_{qq'} \left\langle T c_{q}^{\dagger}(\tau_1) c_{q+k, \downarrow}(\tau_1) c_{q'+k', \downarrow}(\tau_2) c_{q'}(\tau_2) \right\rangle. \quad (S20)$$

As we are considering a superconductor, it is convenient to introduce the Nambu basis, $\psi_k = \begin{pmatrix} c_{k\uparrow} & c_{k\downarrow} & c_{-k\uparrow} & c_{-k\downarrow} \end{pmatrix}^T$, from which the superconducting Green function may be found as

$$C(k, k'; \tau_1, \tau_2) = -i \left\langle \psi_k(\tau_1) \psi_{k'}^{\dagger}(\tau_2) \right\rangle \left( \begin{array}{cccc} -\hbar^2 \langle T c_{k\uparrow}(\tau_1) c_{k\uparrow}^{\dagger}(\tau_2) \rangle & 0 & 0 & \langle T c_{k\downarrow}(\tau_1) c_{-k\downarrow}^{\dagger}(\tau_2) \rangle \\ 0 & -\hbar^2 \langle T c_{k\downarrow}(\tau_1) c_{k\downarrow}^{\dagger}(\tau_2) \rangle & \langle T c_{k\uparrow}(\tau_1) c_{-k\uparrow}^{\dagger}(\tau_2) \rangle & 0 \\ 0 & \langle T c_{-k\uparrow}(\tau_1) c_{k\uparrow}^{\dagger}(\tau_2) \rangle & -\hbar^2 \langle T c_{-k\downarrow}(\tau_1) c_{k\downarrow}^{\dagger}(\tau_2) \rangle & 0 \\ -\hbar^2 \langle T c_{-k\downarrow}(\tau_1) c_{-k\downarrow}^{\dagger}(\tau_2) \rangle & 0 & 0 & \langle T c_{-k\uparrow}(\tau_1) c_{-k\uparrow}^{\dagger}(\tau_2) \rangle \end{array} \right) \right. \left( \begin{array}{c} G_{\uparrow\uparrow}(k, k'; \tau_1, \tau_2) \\ G_{\uparrow\downarrow}(k, k'; \tau_1, \tau_2) \\ G_{\downarrow\uparrow}(k, k'; \tau_1, \tau_2) \\ G_{\downarrow\downarrow}(k, k'; \tau_1, \tau_2) \end{array} \right) \right) \quad (S21)$$

To find find Eq. (S20) we employ Wick’s theorem, making sure to include all non-zero contractions that contain correlators appearing in Eq. (S21). The result is

$$\chi_{kk'}(\tau) = -\hbar^2 \delta_{kk'} \sum_q G_{\uparrow\uparrow}(q; -\tau) G_{\downarrow\downarrow}(q + k; \tau) - F_{\uparrow\uparrow}(q, -\tau) F_{\downarrow\downarrow}(q + k; \tau) \quad (S22)$$

where we have used the identity $F_{kk'}(q, \tau) = -F_{kk'}(q, -\tau)$, as well as defined $\tau = \tau_1 - \tau_2$, and $G_{ss'}(q) \equiv G_{ss'}^1(q, q)$. The superconductor Green function is given in terms of the Matsubara frequencies as $C(k, i\omega_n) = 1/(i\omega_n - H_{SC})$, from which we find

$$C(k, i\omega_n) = \left( \begin{array}{cccc} \frac{i\omega_n + \xi_k + \hbar - \frac{1}{2} \mu_s}{i\omega_n + h \frac{1}{2} \mu_s} & 0 & 0 & \frac{i\omega_n + \xi_k + \hbar + \frac{1}{2} \mu_s}{i\omega_n + h \frac{1}{2} \mu_s} \\ 0 & \frac{i\omega_n + \xi_k + \hbar - \frac{1}{2} \mu_s}{i\omega_n + h \frac{1}{2} \mu_s} & -\Delta^* & 0 \\ 0 & \frac{i\omega_n + \xi_k + \hbar + \frac{1}{2} \mu_s}{i\omega_n + h \frac{1}{2} \mu_s} & \frac{i\omega_n + \xi_k + \hbar - \frac{1}{2} \mu_s}{i\omega_n + h \frac{1}{2} \mu_s} & -\Delta \\ \frac{i\omega_n - \xi_k + \hbar - \frac{1}{2} \mu_s}{i\omega_n + h \frac{1}{2} \mu_s} & 0 & 0 & \frac{i\omega_n - \xi_k + \hbar + \frac{1}{2} \mu_s}{i\omega_n + h \frac{1}{2} \mu_s} \end{array} \right). \quad (S23)$$

From this, the gap may be self-consistently determined from the relation

$$\Delta = -\frac{1}{\beta} \sum_{k, i\omega_n} F_{\uparrow\uparrow}(k, i\omega_n). \quad (S24)$$

In frequency space, Eq. (S22) takes the form

$$\chi_{kk'}(i\omega_n) = -\frac{\hbar^2}{\beta} \delta_{kk'} \sum_{q, i\omega_m} G_{\uparrow\uparrow}(q, i\omega_m + i\omega_n) G_{\downarrow\downarrow}(q + k, i\omega_m) - F_{\uparrow\uparrow}(q, i\omega_m + i\omega_n) F_{\downarrow\downarrow}(q + k, i\omega_m)$$

$$- F_{\uparrow\uparrow}(q, i\omega_m + i\omega_n) F_{\downarrow\downarrow}(q + k, i\omega_m) + G_{\uparrow\uparrow}(q - k, i\omega_m + i\omega_n) G_{\downarrow\downarrow}(q, i\omega_m),$$

and after inserting Eq. (S23) we get

$$\chi_{kk'}(i\omega_n) = -\frac{\hbar^2}{\beta} \delta_{kk'} \sum_{q, i\omega_m} \sum_{\lambda, \lambda' = \pm 1} \frac{1}{i\omega_m + i\omega_n + h - \frac{1}{2} \mu_s - \lambda \epsilon_{q} i\omega_m - h + \frac{1}{2} \mu_s - \lambda \epsilon_{q+k}} \left[ \frac{1}{2} + \frac{\xi_{q} \xi_{q+k}}{2 \lambda \epsilon_{q} \epsilon_{q+k}} \right].$$
with \( E_q = \sqrt{\xi_q^2 + |\Delta|^2} \). Performing the Matsubara sum thus gives

\[
\chi_{kk'}^R (h \omega) = \hbar^2 \delta_{kk'} \sum_q \sum_{\lambda, \lambda' = \pm 1} \left[ \frac{1}{2} + \frac{\xi_q \xi_{q+k} + \Delta^2}{2 \lambda' E_q E_{q+k}} \right] \frac{f(\lambda E_{q+k} + h - \frac{1}{2} \mu_s) - f(\lambda E_q - h + \frac{1}{2} \mu_s)}{h \omega + i \delta - \lambda' E_{q+k} + \lambda E_q - 2h + \mu_s},
\]

(S25)

after analytic continuation \((i \omega_n \rightarrow h \omega + i \delta)\), where \( f(E) \) is the Fermi-Dirac distribution function. We seek the quantity \( \sum_k \Im \chi_{kk'}^R (\omega) \), which becomes

\[
\sum_k \Im \chi_{kk'}^R (\omega) = -\pi \hbar^2 \sum_{kk'} \sum_{\lambda, \lambda' = \pm 1} \left[ \frac{1}{2} + \frac{\xi_k \xi_{k'} + \Delta^2}{2 \lambda' E_k E_{k'}} \right] \left( f(\lambda' E_{k'} + h - \frac{1}{2} \mu_s) - f(\lambda E_k - h + \frac{1}{2} \mu_s) \right) \times \delta(h \omega - \lambda' E_{k'} + \lambda E_k - 2h + \mu_s).
\]

(S26)

Next we approximate \( \sum_k \simeq V_{SC} \nu_0 \int d\xi_k \), where \( V_{SC} \) is the volume of the superconductor, and \( \nu_0 \) is the density of states at the Fermi level. In addition, it’s convenient to change integration variable from \( \xi_k \) to \( E_k \equiv E \), so that \( d\xi = \nu(E) dE \), where \( \nu(E) = \Re \left[ |E| / \sqrt{E^2 - |\Delta|^2} \right] \) is the normalized superconducting density of states, with integration limits \( E \in [0, \infty] \). All in all, this gives

\[
\sum_k \Im \chi_{kk'}^R (\omega) = -\pi \hbar^2 V_{SC}^2 \nu_0^2 \int_0^\infty dE \nu(E) \nu(\lambda E + h \omega - 2h + \mu_s) \times \left[ \frac{1}{2} + \frac{\lambda' \Delta^2}{2 \nu(E) \nu(\lambda E + h \omega - 2h + \mu_s) + 2 \lambda' \nu(E) \nu(\lambda E + h \omega - 2h + \mu_s)} \right] \times \left( f(\lambda E + h \omega - h + \frac{1}{2} \mu_s) - f(\lambda E - h + \frac{1}{2} \mu_s) \right).
\]

(S27)

Note that we may set \( \nu(E) = \nu(\lambda E) \), and hence we find, after summing over \( \lambda \) and \( \lambda' \),

\[
\sum_k \Im \chi_{kk'}^R (\omega) = -\pi \hbar^2 V_{SC}^2 \nu_0^2 \int_0^\infty dE \left[ 1 + \frac{\Delta^2}{E(E + h \omega - 2h + \mu_s)} \right] \nu(E) \nu(\lambda E + h \omega - 2h + \mu_s) \times \left( f(E + h \omega - h + \frac{1}{2} \mu_s) - f(E - h + \frac{1}{2} \mu_s) \right).
\]

(S28)

Finally, we define \( \chi_s(h \omega) = \sum_k \chi_{kk}^R (\omega - \frac{\mu_s}{\hbar}) \), which gives

\[
\chi_s(h \omega) = -\pi \hbar^2 V_{SC}^2 \nu_0^2 \int_{-\infty}^\infty dE F_\Delta \nu(E + h \omega) \nu(E + h \omega - h) \left( f(E + h \omega - \frac{1}{2} \mu_s) - f(E + \frac{1}{2} \mu_s) \right),
\]

where \( F_\Delta = 1 + |\Delta|^2/(E + h)(E + h \omega - h) \). The final expression for the spin and heat currents is found by inserting into Eqs. (S18) and (S19). One gets

\[
I_s = -2|W|^2 V_{FI} \hbar \int d\omega \nu_m(h \omega) \chi_s(h \omega) \left[ n_{FI} \left( \omega - \frac{\mu_m}{\hbar} \right) - n_{SC} \left( \omega - \frac{\mu_s}{\hbar} \right) \right],
\]

(S29)

\[
\dot{Q} = -2|W|^2 V_{FI} \int d\omega (h \omega - \mu_m) \nu_m(h \omega) \chi_s(h \omega) \left[ n_{FI} \left( \omega - \frac{\mu_m}{\hbar} \right) - n_{SC} \left( \omega - \frac{\mu_s}{\hbar} \right) \right],
\]

(S30)

which correspond to Eqs. 9 and 10 in the main article.