Conservative Contextual Linear Bandits

Abbas Kazerouni, Mohammad Ghavamzadeh, Yasin Abbasi-Yadkori, and Benjamin Van Roy

1Stanford University, abbask@stanford.edu
2Adobe Research, ghavamza@adobe.com
3Adobe Research, abbasiya@adobe.com
4Stanford University, bvr@stanford.edu

Abstract

Safety is a desirable property that can immensely increase the applicability of learning algorithms in real-world decision-making problems. It is much easier for a company to deploy an algorithm that is safe, i.e., guaranteed to perform at least as well as a baseline. In this paper, we study the issue of safety in contextual linear bandits that have application in many different fields including personalized ad recommendation in online marketing. We formulate a notion of safety for this class of algorithms. We develop a safe contextual linear bandit algorithm, called conservative linear UCB (CLUCB), that simultaneously minimizes its regret and satisfies the safety constraint, i.e., maintains its performance above a fixed percentage of the performance of a baseline strategy, uniformly over time. We prove an upper-bound on the regret of CLUCB and show that it can be decomposed into two terms: 1) an upper-bound for the regret of the standard linear UCB algorithm that grows with the time horizon and 2) a constant (does not grow with the time horizon) term that accounts for the loss of being conservative in order to satisfy the safety constraint. We empirically show that our algorithm is safe and validate our theoretical analysis.

1 Introduction

Many problems in science and engineering can be formulated as decision-making problems under uncertainty. Although many learning algorithms have been developed to find a good policy/strategy for these problems, most of them do not provide any guarantee that their resulting policy will perform well, when it is deployed. This is a major obstacle in using learning algorithms in many different fields, such as online marketing, health sciences, finance, and robotics. Therefore, developing learning algorithms with safety guarantees can
immensely increase the applicability of learning in solving decision problems. A policy generated by a learning algorithm is considered to be safe, if it is guaranteed to perform at least as well as a baseline. The baseline can be either a baseline value or the performance of a baseline strategy. It is important to note that since the policy is learned from data, and data is often random, the generated policy is a random variable, and thus, the safety guarantees are in high probability.

Safety can be studied in both offline and online scenarios. In the offline case, the algorithm learns the policy from a batch of data, usually generated by the current strategy or recent strategies of the company, and the question is whether the learned policy will perform as well as the current strategy or no worse than a baseline value, when it is deployed. This scenario has been recently studied heavily in both model-based (e.g., [7]) and model-free (e.g., [3, 13, 14, 11, 10, 6]) settings. In the model-based approach, we first use the batch of data and build a simulator that mimics the behavior of the dynamical system under study (hospital’s ER, financial market, robot), and then use this simulator to generate data and learn the policy. The main challenge here is to have guarantees on the performance of the learned policy, given the error in the simulator. This line of research is closely related to the area of robust learning and control. In the model-free approach, we learn the policy directly from the batch of data, without building a simulator. This line of research is related to off-policy evaluation and control. While the model-free approach is more suitable for problems in which we have access to a large batch of data, such as in online marketing, the model-based approach works better in problems in which data is harder to collect, but instead, we have good knowledge about the underlying dynamical system that allows us to build an accurate simulator.

In the online scenario, the algorithm learns a policy while interacting with the real system. Although (reasonable) online algorithms will eventually learn a good or an optimal policy, there is no guarantee for their performance along the way (the performance of their intermediate policies), especially at the very beginning, when they perform a large amount of exploration. Thus, in order to guarantee safety in online algorithms, it is important to control their exploration and make it more conservative. Consider a manager that allows our learning algorithm runs together with her company’s current strategy (baseline policy), as long as it is safe, i.e., the loss incurred by letting a portion of the traffic handled by our algorithm (instead of by the baseline policy) does not exceed a certain threshold. Although we are confident that our algorithm will eventually perform at least as well as the baseline strategy, it should be able to remain alive (not terminated by the manager) long enough for this to happen. Therefore, we should make it more conservative (less exploratory) in a way not to violate the manager’s safety constraint. This setting has been studied in the multi-armed bandit (MAB) [15]. [15] considered the baseline policy as a fixed arm in MAB, formulated safety using a constraint defined based on the performance of the baseline policy (mean of the baseline arm), and modified the UCB algorithm [2] to satisfy this constraint.

In this paper, we study the notion of safety in contextual linear bandits, a setting that has application in many different fields including online personalized ad recommendation\footnote{Other definitions of safety have been studied in contextual linear bandits (e.g., Example 2 in [9]).}.
We first formulate safety in this setting, as a constraint that must hold \textit{uniformly in time}, in Section \ref{section2}. Our goal is to design learning algorithms that minimize regret under the constraint that at any given time, their expected sum of rewards should be above a fixed percentage of the expected sum of rewards of the baseline policy. This fixed percentage depends on the amount of risk that the manager is willing to take. Then in Section \ref{section3}, we propose an algorithm, called \textit{conservative linear UCB} (CLUCB), that satisfies the safety constraint. At each round, CLUCB plays the action suggested by the standard linear UCB (LUCB) algorithm (e.g., \cite{5,8,14,9}), only if it satisfies the safety constraint for the worst choice of the parameter in the confidence set, and plays the action suggested by the baseline policy, otherwise. We also prove an upper-bound for the regret of CLUCB, which can be decomposed into two terms. The first term is an upper-bound on the regret of LUCB that grows at the rate $\sqrt{T \log(T)}$. The second term is constant (does not grow with the horizon $T$) and accounts for the loss of being conservative in order to satisfy the safety constraint. This improves over the regret bound derived in \cite{15} for the MAB setting, where the regret of being conservative grows with time. In Section \ref{section4} we show how CLUCB can be extended to the case that the reward of the baseline policy is unknown without a change in its rate of regret. Finally in Section \ref{section5} we report experimental results that show CLUCB behaves as expected in practice and validate our theoretical analysis.

\section{Problem Formulation}

In this section, we first review the standard linear bandit setting and then introduce the conservative linear bandit formulation considered in this paper.

\subsection{Linear Bandit}

In the linear bandit setting, at any time $t$, the agent is given a set of (possibly) infinitely many actions/options $A_t$, where each action $a \in A_t$ is associated with a feature vector $\phi^t_a \in \mathbb{R}^d$. At each round $t$, the agent should select an action $a_t \in A_t$. Upon selecting $a_t$, the agent observes a random reward $y_t$ generated as

$$y_t = \langle \theta^*, \phi^t_{a_t} \rangle + \eta_t,$$

where $\theta^* \in \mathbb{R}^d$ is the unknown reward parameter, $\langle \theta^*, \phi^t_{a_t} \rangle = r^t_{a_t}$ is the expected reward of action $a_t$ at time $t$, i.e., $r^t_{a_t} = \mathbb{E}[y_t]$, and $\eta_t$ is a random noise such that

\begin{assumption}
Each element $\eta_t$ of the noise sequence $\{\eta_t\}_{t=1}^\infty$ is conditionally $\sigma$-sub-Gaussian, i.e.,

$$\forall \zeta \in \mathbb{R}, \quad \mathbb{E}\left[e^{\zeta \eta_t} \mid a_{1:t}, \eta_{1:t-1}\right] \leq \exp \left(\frac{\zeta^2 \sigma^2}{2}\right).$$

The sub-Gaussian assumption automatically implies that $\mathbb{E}[\eta_t \mid a_{1:t}, \eta_{1:t-1}] = 0$ and $\text{Var}[\eta_t \mid a_{1:t}, \eta_{1:t-1}] \leq \sigma^2$.
\end{assumption}


Note that the above formulation contains time-varying action sets and time-dependent feature vectors for each action, and thus, includes the linear contextual bandit setting. In linear contextual bandit, if we denote by $x_t$, the state of the system at time $t$, the time-dependent feature vector $\phi_a^t$ for action $a$ will be equal to $\phi(x_t, a)$, the feature vector of state-action pair $(x_t, a)$.

We also make the following standard assumption on the unknown parameter $\theta^*$ and feature vectors:

**Assumption 2.** There exist constants $B, D \geq 0$ such that $\|\theta^*\|_2 \leq B$, $\|\phi_a^t\|_2 \leq D$, and $\langle \theta^*, \phi_a^t \rangle \in [0, 1]$, for all $t$ and all $a \in A_t$.

We define $B = \{\theta \in \mathbb{R}^d : \|\theta\|_2 \leq B\}$ and $F = \{\phi \in \mathbb{R}^d : \|\phi\|_2 \leq D, \langle \theta^*, \phi \rangle \in [0, 1]\}$ to be the parameter space and feature space, respectively.

Obviously, if the agent knows $\theta^*$, she will choose the optimal action $a^*_t = \arg \max_{a \in A_t} \langle \theta^*, \phi_a^t \rangle$ at each round $t$. Since $\theta^*$ is unknown, the agent’s goal is to maximize her cumulative expected rewards after $T$ rounds, i.e., $\sum_{t=1}^T \langle \theta^*, \phi_a^t \rangle$, or equivalently, to minimize its (pseudo)-regret, i.e.,

$$R_T = \sum_{t=1}^T \langle \theta^*, \phi_a^t \rangle - \sum_{t=1}^T \langle \theta^*, \phi_a^t \rangle, \tag{2}$$

which is the difference between the cumulative expected rewards of the optimal and agent’s strategies.

### 2.2 Conservative Linear Bandit

The conservative linear bandit setting is exactly the same as the linear bandit, except that there exists a baseline policy $\pi_b$ (the company’s strategy) that at each round $t$, selects action $b_t \in A_t$ and incurs the expected reward $r_{b_t}^t = \langle \theta^*, \phi_{b_t}^t \rangle$. We assume that the expected rewards of the actions taken by the baseline policy, $r_{b_t}^t$, are known. This is often a reasonable assumption, since we usually have access to a large amount of data generated by the baseline policy, as it is our company’s strategy, and thus, have a good estimate of its performance (see Remark 1 at the end of this Section). We relax this assumption in Section 4 and extend our proposed algorithm to the case that the reward function of the baseline policy is not known in advance.

Another difference between the conservative and standard linear bandit settings is the **performance constraint**, which is defined as follows:

**Definition 1** (Performance Constraint). At each round $t$, the difference between the performances of the baseline and the agent’s policies should remain below a pre-defined fraction $\alpha \in (0, 1)$ of the baseline performance. This constraint may be written formally as

$$\sum_{i=1}^t r_{b_t}^i - \sum_{i=1}^t r_{a_t}^i \leq \alpha \sum_{i=1}^t r_{b_t}^i, \quad \forall t \in \{1, \ldots, T\},$$

4
or equivalently as
\[
\sum_{i=1}^{t} r_{a_i}^i \geq (1 - \alpha) \sum_{i=1}^{t} r_{b_i}^i, \quad \forall t \in \{1, \ldots, T\}.
\]  

The parameter \( \alpha \in (0, 1) \) controls how conservative the agent should be. Small values of \( \alpha \) show that only small losses are tolerated, and thus, the agent should be overly conservative, whereas large values of \( \alpha \) indicate that the manager is willing to take risk, and thus, the agent can explore more and be less conservative. Here given the value of \( \alpha \), the goal of the agent is to select her actions in a way to both minimize her regret (2) and satisfy the performance constraint (3). In the next section, we propose a linear bandit algorithm to achieve this goal.

Remark 1. As mentioned above, it is often reasonable to assume that we have access to a good estimate of the baseline reward function. If in addition to this estimate, we have access to the data generated by the baseline policy that are used to compute this estimate, we can use them in our algorithm. The reason we do not use the data generated by the actions suggested by the baseline policy in constructing the confidence sets in our algorithm is mainly to keep the analysis simple. However, when we deal with the more general case of unknown baseline reward in Section 4, we construct the confidence sets using all available data, including those generated by the baseline policy. It is also important to note that having a good estimate of the baseline reward function does not necessarily mean that we know the unknown parameter \( \theta^* \), since the data used for this estimate has only been generated by the baseline policy, and thus, may only provide a good estimate of \( \theta^* \) in a limited subspace.

3 A Conservative Linear Bandit Algorithm

In this section, we propose a linear bandit algorithm, called conservative linear upper confidence bound (CLUCB), that is based on the optimism in the face of uncertainty principle, and given the value of \( \alpha \), both minimizes the regret (2) and satisfies the performance constraint (3). Algorithm 1 contains the pseudocode of CLUCB. At each round \( t \), CLUCB uses the previous observations and builds a confidence set \( C_t \) that with high probability contains the unknown parameter \( \theta^* \). It then selects the optimistic action

\[
a'_t \in \arg\max_{a \in \mathcal{A}_t} \max_{\theta \in C_t} \langle \theta, \phi^i_a \rangle,
\]

which has the best performance among all the actions available in \( \mathcal{A}_t \), within the confidence set \( C_t \). In order to make sure that constraint (3) is satisfied, the algorithm plays the optimistic action \( a'_t \), only if it satisfies the constraint for the worst choice of the parameter \( \theta \in C_t \). To make this more precise, let \( S_{t-1} \) be the set of rounds \( i \) before round \( t \) at which CLUCB has played the optimistic action, i.e., \( a_i = a'_i \). In other words, \( S_{t-1} = \{1, 2, \ldots, t-1\} - S_{t-1} \) is the set of rounds \( j \) before round \( t \) at which CLUCB has followed the baseline policy, i.e., \( a_j = b_j \).
Algorithm 1 Pseudocode of CLUCB

**Input:** $\alpha, \mathcal{B}, \mathcal{F}$

**Initialize:** $S_0 = \emptyset$, $z_0 = 0 \in \mathbb{R}^d$, and $\mathcal{C}_1 = \mathcal{B}$

for $t = 1, 2, 3, \cdots$ do

Find $(a'_t, \tilde{\theta}_t) \in \arg \max_{(a, \theta) \in \mathcal{A}_t \times \mathcal{C}_t} \langle \theta, \phi_{a_t}^t \rangle$

Compute $L_t = \min_{\theta \in \mathcal{C}_t} \langle \theta, z_{t-1} + \phi_{a_t}^t \rangle$

if $L_t + \sum_{i \in S_{t-1}} r_i \geq (1 - \alpha) \sum_{i=1}^t r_i$ then

Play $a_t = a'_t$ and observe reward $y_t$ defined by (1)

Set $z_t = z_{t-1} + \phi_{a_t}^t$, $S_t = S_{t-1} \cup t$, $S_{c_t} = S_{c_{t-1}}$

Given $a_t$ and $y_t$, construct the confidence set $\mathcal{C}_{t+1}$ according to (5)

else

Play $a_t = b_t$ and observe reward $y_t$ defined by (1)

Set $z_t = z_{t-1}$, $S_t = S_{t-1}$, $S_{c_t} = S_{c_{t-1}} \cup t$, $\mathcal{C}_{t+1} = \mathcal{C}_t$

end if

end for

In order to guarantee that it does not violate constraint (3), at each round $t$, CLUCB plays the optimistic action, i.e., $a_t = a'_t$, only if

$$\min_{\theta \in \mathcal{C}_t} \left[ \sum_{i \in S_{t-1}} r_i + \sum_{i \in S_{t-1}} \phi_{a_t}^i + \langle \theta, \phi_{a_t}^t \rangle \right] \geq (1 - \alpha) \sum_{i=1}^t r_i,$$

and plays the conservative action, i.e., $a_t = b_t$, otherwise. In the next section, we will describe how CLUCB constructs and updates the confidence sets $\mathcal{C}_t$.

### 3.1 Construction of Confidence Sets

CLUCB starts by the most general confidence set $\mathcal{C}_1 = \mathcal{B}$ and updates its confidence set only when it plays an optimistic action. This is mainly for simplification and is based on the idea that since the reward function of the baseline policy is known ahead of time, playing a baseline action does not provide any new information about the unknown parameter $\theta^*$. However, this can be easily changed to update the confidence set after each action. This is in fact what we do in the algorithm proposed in Section 4. We follow the approach of [1] to build confidence sets for the unknown parameter $\theta^*$. Let $S_t = \{i_1, \ldots, i_{m_t}\}$ be the set of rounds up to and including round $t$ at which CLUCB has played the optimistic action. Note that we have defined $m_t = |S_t|$. For a fixed value of $\lambda > 0$, let

$$\hat{\theta}_t = \left( \Phi_t \Phi_t^T + \lambda I \right)^{-1} \Phi_t Y_t,$$

be the regularized least square estimate of $\theta$ at round $t$, where $\Phi_t = [\phi_{a_{i_1}}^{i_1}, \ldots, \phi_{a_{i_{m_t}}}^{i_{m_t}}]$ and $Y_t = [y_{i_1}, \ldots, y_{i_{m_t}}]^T$. For a fixed confidence parameter $\delta \in (0, 1)$, we construct the confidence
set for the next round $t + 1$ as

$$C_{t+1} = \left\{ \theta \in \mathbb{R}^d : \|\theta - \hat{\theta}_t\|_{V_t} \leq \beta_{t+1} \right\}, \quad (5)$$

where $\beta_{t+1} = \sigma \sqrt{d \log \left( \frac{1+ (m_t+1)D^2 / \lambda}{\delta} \right) + \sqrt{\lambda B} }$, $V_t = \lambda I + \Phi_t \Phi_t^\top$, and the weighted norm is defined as $\|x\|_V = \sqrt{x^\top V x}$ for any $x \in \mathbb{R}^d$ and any positive definite $V \in \mathbb{R}^{d \times d}$. Note that similar to the linear UCB algorithm (LUCB) in [1], the sub-Gaussian parameter $\sigma$ and the regularization parameter $\lambda$ that appear in the definitions of $\beta_{t+1}$ and $V_t$ should also be given to the CLUCB algorithm as input.

The following proposition (Theorem 2 in [1]) shows that the confidence sets constructed as in (5) contain the true parameter $\theta^*$ with high probability.

**Proposition 1.** For any $\delta > 0$ and the confidence set $C_t$ defined by (5), we have

$$P \left[ \theta^* \in C_t, \ \forall t \in \mathbb{N} \right] \geq 1 - \delta.$$ 

Proposition 1 indicates that at each round $t$, the CLUCB algorithm satisfies the performance constraint (3) with probability at least $1 - \delta$. This is because at each round $t$, CLUCB ensures that (3) holds for all $\theta \in C_t$ and $P[\theta^* \in C_t] \geq 1 - \delta$.

### 3.2 Regret Analysis of CLUCB

In this section, we prove a regret bound for the proposed CLUCB algorithm. Let $\Delta^t_{b_t} = r^t_{a_t} - r^t_{b_t}$ be the baseline gap at round $t$, i.e., the difference between the expected rewards of the optimal and baseline actions at round $t$. This quantity shows how sub-optimal the action suggested by the baseline policy is at round $t$. We make the following assumption on the performance of the baseline policy $\pi_b$.

**Assumption 3.** There exist $0 \leq \Delta_l \leq \Delta_h$ and $0 < r_l < r_h$ such that, at each round $t$,

$$\Delta_l \leq \Delta^t_{b_t} \leq \Delta_h \quad \text{and} \quad r_l \leq r^t_{b_t} \leq r_h. \quad (6)$$

An obvious candidate for both $\Delta_h$ and $r_h$ is 1, as all the mean rewards are confined in $[0, 1]$. The reward lower-bound $r_l$ ensures that the baseline policy maintains a minimum level of performance at each round. Finally, $\Delta_l = 0$ is a reasonable candidate for the lower-bound of the baseline gap.

The following proposition shows that the regret of CLUCB can be decomposed into the regret of a linear UCB (LUCB) algorithm (e.g., [1]) and a regret caused by being conservative in order to satisfy the performance constraint (3).

**Proposition 2.** The regret of CLUCB can be decomposed into two terms as follows:

$$R_T(\text{CLUCB}) \leq R_{S_T}(\text{LUCB}) + n_T \Delta_h, \quad (7)$$

where $R_{S_T}(\text{LUCB})$ is the cumulative (pseudo)-regret of LUCB at rounds $t \in S_T$ and $n_T = |S^*_T| = T - |S_T| = T - m_T$ is the number of rounds (in $T$ rounds) at which CLUCB has played the conservative action.
Proof. From the definition of regret (2), we have

\[
R_T(\text{CLUCB}) = \sum_{t=1}^{T} r_{a_t^*}^t - \sum_{t=1}^{T} r_{a_t}^t \\
= \sum_{t \in S_T} (r_{a_t^*}^t - r_{a_t}^t) + \sum_{t \in S_T^c} (r_{a_t^*}^t - r_{b_t}^t) \\
= \sum_{t \in S_T} (r_{a_t^*}^t - r_{a_t}^t) + \sum_{t \in S_T^c} \Delta_t \\
\leq \sum_{t \in S_T} (r_{a_t^*}^t - r_{a_t}^t) + n_T \Delta_h.
\] (8)

The result follows from the fact that for \( t \in S_T \), CLUCB plays the exact same actions as LUCB, and thus, the first term in (8) represents LUCB’s regret for these rounds. \( \square \)

The regret bound of LUCB for the confidence set defined by (5) can be derived from the results of [1]. Let \( \mathcal{E} \) be the event that \( \theta^* \in \mathcal{C}, \forall t \in \mathbb{N} \), which according to Proposition holds with probability at least \( 1 - \delta \). The following proposition provides a bound for \( R_{S_T}(\text{LUCB}) \).

Since this proposition is a direct application of Theorem 3 in [1], we omit its proof here.

**Proposition 3.** On event \( \mathcal{E} \), for any \( T \in \mathbb{N} \), we have

\[
R_{S_T}(\text{LUCB}) \leq 4 \sqrt{m_T d \log \left( \lambda + \frac{m_T D}{d} \right)} \\
\times \left[ B \sqrt{\lambda} + \sigma \sqrt{2 \log \left( \frac{1}{\delta} \right)} + d \log \left( 1 + \frac{m_T D}{\lambda d} \right) \right] \\
= O \left( d \log \left( \frac{D}{\lambda \delta T} \right) \sqrt{T} \right). \tag{9}
\]

Now in order to bound the regret of CLUCB, we only need to find an upper bound on \( n_T \), i.e., the number of times CLUCB deviates from LUCB and selects the action suggested by the baseline policy. We start this part of the proof with the following lemma.

**Lemma 4.** For given \( k \in \mathbb{N}, \lambda > 0 \) and any sequence \( X_1, X_2, \cdots, X_k \) in \( \mathbb{R}^d \) such that \( \forall i: \|X_i\|_2 \leq D \), let \( V_0 = \lambda I \) and \( V_i = \lambda I + \sum_{j=1}^{i} X_j X_j^\top \) for \( 1 \leq i \leq k \). Then, we have

\[
\sum_{i=1}^{k} \min \left( 1, \|X_i\|_{V_{i-1}}^2 \right) \leq 2d \log \left( 1 + \frac{kD^2}{\lambda d} \right). \tag{10}
\]

Lemma 4 is a direct application of Lemma 11 in [1] and we omit its proof here. The following theorem provides a bound on the number of rounds at which CLUCB acts conservatively and follows the baseline policy \( \pi_b \).
Theorem 5. Assume that $\lambda \leq D^2$. On event $\mathcal{E}$, for any horizon $T \in \mathbb{N}$, we have

$$n_T \leq 1 + 114d^2 \left( \frac{B\sqrt{\lambda} + \sigma}{\Delta_l + \alpha r_l} \right)^2 \left[ \log \left( \frac{64d(B\sqrt{\lambda} + \sigma)D}{\sqrt{\lambda} \delta (\Delta_l + \alpha r_l)} \right) \frac{\Delta}{\Delta_l + \alpha r_l} \right].$$

Proof. Let $\tau$ be the last round at which CLUCB plays conservatively (action suggested by the baseline policy), i.e., $\tau = \max \{1 \leq t \leq T \mid a_t = b_t\}$. From Algorithm 1 at round $\tau$, we may write

$$\min_{\theta \in C_{t^*}} \left\langle \theta, \phi_{a_t^*}^t + \sum_{t \in S_{t-1}} \phi_{a_t}^t \right\rangle + \sum_{t \in S_{\tau-1}} r_{bt}^t < (1 - \alpha) \sum_{t=1}^{\tau} r_{bt}^t,$$

or equivalently,

$$\alpha \sum_{t=1}^{\tau} r_{bt}^t < \sum_{t \in S_{\tau}} \left[ r_{bt}^t - \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle + r_{bt}^t - \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle \right] + \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle - \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle + \sum_{t \in S_{t-1}} \phi_{a_t}^t).$$

We may rewrite (11) as

$$\alpha \sum_{t=1}^{\tau} r_{bt}^t < \sum_{t \in S_{t-1}} \left[ r_{bt}^t - \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle + r_{bt}^t - \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle \right]$$

$$+ \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle - \min_{\theta \in C_{t}} \left\langle \theta, \phi_{a_t^*}^t \right\rangle + \sum_{t \in S_{t-1}} \left\langle \theta, \phi_{a_t}^t \right\rangle$$

$$= \sum_{t \in S_{t-1}} \left[ r_{bt}^t - \max_{\theta \in C_{t}} \left\langle \theta, \phi_{a_t^*}^t \right\rangle + \max_{\theta \in C_{t}} \left\langle \theta, \phi_{a_t}^t \right\rangle \right]$$

$$+ \left[ r_{bt}^t - \max_{\theta \in C_{t}} \left\langle \theta, \phi_{a_t^*}^t \right\rangle + \max_{\theta \in C_{t}} \left\langle \theta, \phi_{a_t}^t \right\rangle \right]$$

$$+ \max_{\theta \in C_{t}} \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle + \sum_{t \in S_{t-1}} \phi_{a_t}^t).$$

Note that for each $t \in S_{t-1}$, we have

$$r_{bt}^t - \max_{\theta \in C_{t}} \left\langle \theta, \phi_{a_t^*}^t \right\rangle \leq r_{bt}^t - \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle = -\Delta_{bt},$$

and similarly, for the round $\tau$, we have

$$r_{bt}^t - \max_{\theta \in C_{t}} \left\langle \theta, \phi_{a_t^*}^t \right\rangle \leq r_{bt}^t - \left\langle \theta^*, \phi_{a_t^*}^t \right\rangle = -\Delta_{bt}.$$

Using inequalities (12) to (14), we may rewrite (11) as
\[
\alpha \sum_{t=1}^{\tau} r_{bt} < \sum_{t \in S_{\tau-1}} \left[ -\Delta_{bt}^t + \max_{\theta \in \mathcal{C}_t} \langle \theta - \theta^*, \phi_{at}^t \rangle \right. \\
- \Delta_{bt}^t + \max_{\theta \in \mathcal{C}_t} \langle \theta - \theta^*, \phi_{at}^t \rangle \\
+ \max_{\theta \in \mathcal{C}_t} \left\langle \theta^* - \theta, \phi_{at}^t + \sum_{t \in S_{\tau-1}} \phi_{at}^t \right\rangle \\
\leq -(m_{\tau-1} + 1) \Delta_t + \max_{\theta \in \mathcal{C}_t} \langle \theta - \theta^*, \phi_{at}^t \rangle \\
+ \sum_{t \in S_{\tau-1}} \max_{\theta \in \mathcal{C}_t} \langle \theta - \theta^*, \phi_{at}^t \rangle \\
+ \max_{\theta \in \mathcal{C}_t} \left\langle \theta^* - \theta, \phi_{at}^t + \sum_{t \in S_{\tau-1}} \phi_{at}^t \right\rangle
\]

(15)

On the other hand, it follows from (15) and the fact that all rewards are in \([0, 1]\) that
\[
\alpha \sum_{t=1}^{\tau} r_{bt} \leq -(m_{\tau-1} + 1) \Delta_t + 4(m_{\tau-1} + 1).
\]

(17)

Combining (16) and (17), we may write
\[
\alpha \sum_{t=1}^{\tau} r_{bt} \leq -(m_{\tau-1} + 1) \Delta_t + 4m_{\tau-1} + 4 \beta_{\tau} \left[ \min \left( \| \phi_{at}^\tau \|_{V^{-1}}^2, 1 \right) \right]
\]

(18)

In order to write the next equation more compactly, let us define \( \Gamma \) as
\[
\Gamma = \left[ \min \left( \| \phi_{at}^\tau \|_{V^{-1}}^2, 1 \right) + \sum_{t \in S_{\tau-1}} \min \left( \| \phi_{at}^t \|_{V^{-1}}^2, 1 \right) \right].
\]

From Cauchy-Schwarz inequality and Lemma 4, we have
\[
\alpha \sum_{t=1}^{\tau} r_{b_t}^t \leq -(m_{\tau-1} + 1)\Delta_t + 4\beta_\tau \sqrt{(m_{\tau-1} + 1) \times \Gamma} \\
\leq -(m_{\tau-1} + 1)\Delta_t + 4\beta_\tau \sqrt{2(m_{\tau-1} + 1)d \log \left(1 + \frac{m_{\tau}D^2}{\lambda d}\right)} \\
= -(m_{\tau-1} + 1)\Delta_t \\
+ 8\sqrt{(m_{\tau-1} + 1)d \log \left(1 + \frac{(m_{\tau-1} + 1)D^2}{\lambda d}\right)} \times \left[\sqrt{\lambda B} + \sigma \sqrt{d \log \left(\frac{\lambda + (m_{\tau-1} + 1)D^2}{\lambda \delta}\right)}\right] \\
\leq -(m_{\tau-1} + 1)\Delta_t \\
+ 8d(B\sqrt{\lambda} + \sigma) \log \left(\frac{2(m_{\tau-1} + 1)D^2}{\lambda \delta}\right) \sqrt{(m_{\tau-1} + 1)},
\]

where the last inequality follows from the fact that \(\lambda \leq D^2\). On the other hand, since \(\forall t: r_{b_t}^t \geq r_t \) and \(\tau = n_{\tau-1} + m_{\tau-1} + 1\), we may write

\[
\alpha r_l n_{\tau-1} \leq -(m_{\tau-1} + 1)(\Delta_t + \alpha r_l) \\
+ 8d(B\sqrt{\lambda} + \sigma) \log \left(\frac{2(m_{\tau-1} + 1)D^2}{\lambda \delta}\right) \sqrt{(m_{\tau-1} + 1)}.
\]

The RHS of (19) is only positive for a finite range of \(m_{\tau}\), and thus, has a finite upper-bound. For \(m = (m_{\tau-1} + 1), c_1 = 8d(B\sqrt{\lambda} + \sigma), c_2 = \frac{2D^2}{\lambda \delta}\) and \(c_3 = (\Delta_t + \alpha r_l)\), Lemma 8 reported in Appendix A provides the following upper-bound on the RHS (and thus for the LHS) of (19):

\[
\alpha r_l n_{\tau-1} \leq 114d^2 \frac{(B\sqrt{\lambda} + \sigma)^2}{\Delta_t + \alpha r_l} \left[\log \left(\frac{64d(B\sqrt{\lambda} + \sigma)D}{\sqrt{\lambda \delta} (\Delta_t + \alpha r_l)}\right)\right]^2.
\]

The result follows from \(n_T = n_{\tau} = n_{\tau-1} + 1\).

We now have all the necessary ingredients to derive a regret bound on the performance of the CLUCB algorithm. We report the regret bound of CLUCB in Theorem 6, whose proof is a direct consequence of the results of Propositions 2 and 3, and Theorem 5.

**Theorem 6.** With probability at least \(1 - \delta\), the CLUCB algorithm satisfies the performance constraint (3) for all \(t \in \mathbb{N}\), and has the following regret bound:

\[
R_T(\text{CLUCB}) = O \left(d \log \left(\frac{DT}{\lambda \delta}\right) \sqrt{T} + \frac{K \Delta_h}{\alpha r_l (\alpha r_l + \Delta_l)}\right),
\]

where \(K\) is a constant that depends only on the parameters of the problem as

\[
K = 1 + 114d^2 \frac{(B\sqrt{\lambda} + \sigma)^2}{\Delta_t + \alpha r_l} \left[\log \left(\frac{64d(B\sqrt{\lambda} + \sigma)D}{\sqrt{\lambda \delta} (\Delta_t + \alpha r_l)}\right)\right]^2.
\]
Remark 2. The first term in the regret bound (20) is the regret of LUCB, which grows at the rate $\sqrt{T \log(T)}$. The second term accounts for the loss incurred by being conservative in order to satisfy the performance constraint (3). Our results indicate that this loss does not grow with time (since CLUCB will be conservative only in a finite number of times). This improves over the regret bound derived in [15] for the MAB setting, where the regret of being conservative grows with time. Furthermore, the regret bound of Theorem 6 clearly indicates that CLUCB’s regret is larger for smaller values of $\alpha$. This perfectly matches the intuition that the agent must be more conservative, and thus, suffers higher regret for smaller values of $\alpha$. Theorem 6 also indicates that CLUCB’s regret is smaller for smaller values of $\Delta_h$, because when the baseline policy $\pi_b$ is close to optimal, the algorithm does not lose much by being conservative.

4 Unknown Baseline Reward

In this section, we consider the case where the expected rewards of the actions taken by the baseline policy, $r_{i,t}$, are unknown at the beginning. We show how the CLUCB algorithm presented in Section 3 should be changed to handle this case, and present a new algorithm, called CLUCB2. We prove a regret bound for CLUCB2, which is at the same rate as that for CLUCB. This shows that the lack of knowledge about the reward function of the baseline policy does not hurt our algorithm in terms of the rate of the regret.

Algorithm 2 contains the pseudocode of CLUCB2. The main difference with CLUCB is in the condition that should be checked at each round $t$ to see whether we should play the optimistic action $a_t'$ or the conservative action $b_t$. This condition should be selected in a way that CLUCB2 satisfies the constraint (3). We may rewrite (3) as

$$\sum_{i \in S_{t-1}} r_i a_i + \alpha \sum_{i \in S_{t-1}} r_i b_i \geq (1 - \alpha) (\phi_{b_t}^t + \sum_{i \in S_{t-1}} r_i b_i).$$

(21)

If we lower-bound the LHS and upper-bound the RHS of (21), we obtain

$$\min_{\theta \in C_t} \langle \theta, \sum_{i \in S_{t-1}} \phi_{a_i} r_i a_i + \phi_{a_t} r_t a_t \rangle + \alpha \min_{\theta \in C_t} \langle \theta, \sum_{i \in S_{t-1}} \phi_{b_i} \rangle \geq (1 - \alpha) \max_{\theta \in C_t} \langle \theta, \sum_{i \in S_{t-1}} \phi_{b_i} \rangle.$$

(22)

Since each confidence set $C_t$ is built in a way to contain the true parameter $\theta^*$ with high probability, it is easy to see that (21) is satisfied whenever (22) is true.

CLUCB2 uses both optimistic and conservative actions, and their corresponding rewards in building its confidence sets. Specifically for any $t$, we let $\Phi_t = [\phi_{a_1}^t, \phi_{a_2}^t, \ldots, \phi_{a_t}^t]$, $Y_t = [y_1, y_2, \ldots, y_t]^\top$, $V_t = \lambda I + \Phi_t^T \Phi_t$, and define the least-square estimate after round $t$ as

$$\hat{\theta}_t = (\Phi_t^T \Phi_t + \lambda I)^{-1} \Phi_t^T Y_t.$$

(23)
Algorithm 2 CLUCB2

Input: $\alpha, r, B, F$
Initialize: $n \leftarrow 0$, $z \leftarrow 0$, $w \leftarrow 0$, $v \leftarrow 0$ and $C_1 \leftarrow B$
for $t = 1, 2, 3, \cdots$ do
    Let $b_t$ be the action suggested by $\pi_b$ at round $t$
    Find $(a'_t, \tilde{\theta}) = \operatorname{arg \, max}_{(a, \theta) \in A_t \times C_t} \langle \theta, \phi_{a'_t} \rangle$
    Find $R_t = \max_{\theta \in C_t} \langle \theta, v + \phi_{a'_t} \rangle$ and
    $L_t = \min_{\theta \in C_t} \langle \theta, z + \phi_{b_t} \rangle + \alpha \max_{\theta \in C_t} \{ \min_{\theta \in C_t} \langle \theta, w \rangle, nr \}$
    if $L_t \geq (1 - \alpha)R_t$ then
        Play $a_t = a'_t$ and observe $y_t$ defined by (1)
        Set $z \leftarrow z + \phi_{a'_t}$ and $v \leftarrow v + \phi_{b_t}$
    else
        Play $a_t = b_t$ and observe $y_t$ defined by (1)
        Set $w = w + \phi_{b_t}$ and $n \leftarrow n + 1$
    end if
Given $a_t$ and $y_t$, construct the confidence set $C_{t+1}$ according to (24)
end for

Given $V_t$ and $\hat{\theta}_t$, the confidence set for round $t + 1$ is constructed as
$$C_{t+1} = \left\{ \theta \in C_t : \| \theta - \hat{\theta}_t \|_{V_t} \leq \beta_{t+1} \right\},$$
where $C_1 = B$ and $\beta_t = \sigma \sqrt{d \log \left( \frac{1+td^2}{\delta} \right)} + B\sqrt{\lambda}$. Similar to Proposition 1, we can easily prove that the confidence sets built by (24) contain the true parameter $\theta^*$ with high probability, i.e., $\mathbb{P}[\theta^* \in C_t, \forall t \in \mathbb{N}] \geq 1 - \delta$.

Remark 3. Note that unlike the CLUCB algorithm, here we build nested confidence sets, i.e., $\cdots \subseteq C_{t+1} \subseteq C_t \subseteq C_{t-1} \subseteq \cdots$, which is necessary for the proof of the algorithm. Potentially, this can increase the computational complexity of CLUCB2, but from a practical point of view, the confidence sets become nested automatically after sufficient data has been observed. Therefore, the nested constraint in building the confidence sets can be relaxed at sufficiently large rounds.

The following theorem guarantees that CLUCB2 satisfies the safety constraint (3) with high probability, while its regret has the same rate as that of CLUCB and is worse than that of LUCB only up to an additive constant.

Theorem 7. With probability at least $1 - \delta$, the CLUCB2 algorithm satisfies the performance constraint (3) for all $t \in \mathbb{N}$, and has the following regret bound
$$R_T(\text{CLUCB2}) = O \left( d \log \left( \frac{DT}{\lambda \delta} \right) \sqrt{T} + \frac{K \Delta_\delta}{\alpha^2 r_l^2} \right),$$ (25)
where $K$ is a constant that depends only on the parameters of the problem as

$$K = 256d^2 (B\sqrt{\lambda} + \sigma)^2 \left[ \log \left( \frac{10d(B\sqrt{\lambda} + \sigma)\sqrt{D}}{\alpha r(\lambda \delta)^{1/4}} \right) \right]^2 + 1.$$ 

We report the proof of Theorem 7 in Appendix B. The proof follows the same steps as that of Theorem 6 with additional non-trivial technicalities that have been highlighted there.

5 Simulation Results

In this section, we provide simulation results to illustrate the performance of the proposed CLUCB algorithm. We considered a time independent action set of 100 arms each having a time independent feature vector living in the $\mathbb{R}^4$ space. These feature vectors and the parameter $\theta^*$ are randomly drawn from $\mathcal{N}(0, I_4)$ such that the mean reward associated to each arm is positive. The observation noise at each time step is also generated independently from $\mathcal{N}(0, 1)$, and the mean reward of the baseline policy at any time is taken to be the reward associated to the 10'th best action. We have taken $\lambda = 1$, $\delta = 0.001$ and the results are averaged over 1,000 realizations.

In Figure 1, we plot the per-step regret (i.e., $\frac{R_t}{T}$) of LUCB and CLUCB for different values of $\alpha$ over a horizon $T = 40,000$. Figure [4] shows that the per-step regret of CLUCB remains
Figure 2: Percentage of the rounds, in the first 1,000 rounds, at which the safety constraint is violated by LUCB and CLUCB for different values of $\alpha$.

constant at the beginning (the conservative phase), because during this phase, CLUCB follows the baseline policy to make sure that the performance constraint (3) is satisfied. As expected, the length of the conservative phase decreases as $\alpha$ is increased, since the performance constraint is relaxed for larger values of $\alpha$, and hence, CLUCB starts playing optimistic actions more quickly. After this initial conservative phase, CLUCB has learned enough about the optimal action and its performance starts converging to that of LUCB. On the other hand, Figure 1 shows that the per-step regret of CLUCB at the first few periods remains much lower than that of LUCB. This is because LUCB plays agnostic to the safety constraint, and thus, may select very poor actions in its initial exploration phase. In regard to this, Figure 2 plots the percentage of the rounds, in the first 1,000 rounds, at which the safety constraint (3) is violated by LUCB and CLUCB for different values of $\alpha$. According to this figure, CLUCB always satisfies the performance constraint for all the values of $\alpha$, while LUCB fails in a significant number of rounds, specially for small values of $\alpha$ (i.e., tight constraint).

To better see the effect of the safety constraint on the regret of the algorithms, Figure 3 plots the per-step regret achieved by CLUCB at round $t = 40,000$ for different values of $\alpha$, as well as that for LCUB. As expected from our analysis and is shown in Figure 1, the performance of CLUCB converges to that of LUCB after an initial conservative phase. Figure 3 confirms that such convergence happens more quickly for larger values of $\alpha$, where the safety constraint is relaxed.
Figure 3: Per-step regret of LUCB and CLUCB for different values of $\alpha$, at round $t = 40,000$.

6 Conclusion

In this paper, we studied the concept of safety in contextual linear bandits to address the challenges that arise in implementing such algorithms in practical situations such as ad recommendation systems. Most of the existing linear bandit algorithms, such as LUCB [1], suffer from a large regret at their initial exploratory rounds. This unsafe behavior is not acceptable in many practical situations, where having a reasonable performance at any time is necessary for a learning algorithm to be considered reliable and to remain in production.

To guarantee safe learning, we formulated a conservative linear bandit problem, where the performance of the learning algorithm (measured in terms of its cumulative rewards) at any time is constrained to be at least as good as a fraction of the performance of a baseline policy. We proposed a conservative version of the LUCB algorithm, called CLUCB, to solve this constraint problem, and showed that it satisfies the safety constraint with high probability, while achieving a regret bound equivalent to that of LUCB up to an additive time-independent constant. We designed two versions of CLUCB that can be used depending on whether the reward function of the baseline policy is known or unknown, and showed that in each case, CLUCB acts conservatively (i.e., plays the action suggested by the baseline policy) only at a finite number of rounds, which depends on how suboptimal the baseline policy is. We reported simulation results that support our analysis and show the performance of the proposed CLUCB algorithm.
References

[1] Y. Abbasi-Yadkori, D. Pál, and C. Szepesvári. Improved algorithms for linear stochastic bandits. In Advances in Neural Information Processing Systems, pages 2312–2320, 2011.

[2] P. Auer, N. Cesa-Bianchi, and P. Fischer. Finite-time analysis of the multiarmed bandit problem. Machine Learning Journal, 47:235–256, 2002.

[3] L. Bottou, J. Peters, J. Quinonero-Candela, D. Charles, D. Chickering, E. Portugaly, D. Ray, P. Simard, and E. Snelson. Counterfactual reasoning and learning systems: The example of computational advertising. Journal of Machine Learning Research, 14:3207–3260, 2013.

[4] W. Chu, L. Li, L. Reyzin, and R. Schapire. Contextual bandits with linear payoff functions. In Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics, pages 208–214, 2011.

[5] V. Dani, T. Hayes, and S. Kakade. Stochastic linear optimization under bandit feedback. In COLT, pages 355–366, 2008.

[6] N. Jiang and L. Li. Doubly robust off-policy value evaluation for reinforcement learning. In Proceedings of the Thirty-Third International Conference on Machine Learning, 2016.

[7] M. Petrik, M. Ghavamzadeh, and Y. Chow. Safe policy improvement by minimizing robust baseline regret. In Advances in Neural Information Processing Systems, 2016.

[8] P. Rusmevichientong and J. Tsitsiklis. Linearly parameterized bandits. Mathematics of Operations Research, 35(2):395–411, 2010.

[9] D. Russo and B. Van Roy. Learning to optimize via posterior sampling. Mathematics of Operations Research, 39(4):1221–1243, 2014.

[10] A. Swaminathan and T. Joachims. Batch learning from logged bandit feedback through counterfactual risk minimization. Journal of Machine Learning Research, 16:1731–1755, 2015.

[11] A. Swaminathan and T. Joachims. Counterfactual risk minimization: Learning from logged bandit feedback. In Proceedings of The 32nd International Conference on Machine Learning, 2015.

[12] G. Theocharous, P. Thomas, and M. Ghavamzadeh. Building personal ad recommendation systems for life-time value optimization with guarantees. In Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, pages 1806–1812, 2015.

[13] P. Thomas, G. Theocharous, and M. Ghavamzadeh. High confidence off-policy evaluation. In Proceedings of the Twenty-Ninth Conference on Artificial Intelligence, 2015.
[14] P. Thomas, G. Theocharous, and M. Ghavamzadeh. High confidence policy improvement. In Proceedings of the Thirty-Second International Conference on Machine Learning, pages 2380–2388, 2015.

[15] Y. Wu, R. Shariff, T. Lattimore, and C. Szepesvári. Conservative bandits. In Proceedings of The 33rd International Conference on Machine Learning, pages 1254–1262, 2016.
A Technical Details of the Proof of Theorems 5 and 7

In the proof of Theorems 5 and 7, we use the following lemma to bound the RHS of (16) and (43).

Lemma 8. For any \( m \geq 2 \) and \( c_1, c_2, c_3 > 0 \), the following holds

\[
-c_3m + c_1\sqrt{m}\log(c_2m) \leq \frac{16c_1^2}{9c_3} \left( \log \left( \frac{2c_1\sqrt{c_2e}}{c_3} \right) \right)^2.
\]  
(26)

Proof. Define the LHS of (26) as a function \( g(m) \), \( m \geq 2 \), i.e.,

\[
g(m) = -c_3m + c_1\sqrt{m}\log(c_2m).
\]

Firstly, note that we have

\[
g'(m) = -c_3 + \frac{c_1(2 + \log(c_2m))}{2\sqrt{m}} \quad \text{and} \quad g''(m) = \frac{-c_1\log(c_2m)}{4m\sqrt{m}}.
\]

This implies that since \( c_2 > 1 \), \( g \) is a differentiable concave function over its domain \([2, \infty)\), and thus, we can find \( m^* \), the global maximum of function \( g \). The first order condition implies that \( g'(m^*) = 0 \), which gives us

\[
2 + \log(c_2m^*) = \frac{2c_1}{c_3} \sqrt{m^*}.
\]  
(27)

Plugging this into the definition of \( g \), we obtain

\[
g^* = \max_{m \geq 2} g(m) = g(m^*) = c_3m^* - 2c_1\sqrt{m^*}.
\]

Now, we use the change of variable \( x = \frac{c_1}{2c_2} \sqrt{m^*} \), which by (A) gives us

\[
g^* = \frac{4c_1^2}{c_3} (x^2 - x).
\]  
(28)

On the other hand, (27) becomes

\[
2 + \log \left( \frac{4c_1^2}{c_3^2} \right) + 2\log(x) = 4x.
\]

Taking exp from both sides gives us

\[
\frac{e^{4x}}{x^2} = \frac{4c_1^2c_2e^2}{c_3^2}.
\]

Now, since \( x^2 \leq e^x \), for all \( x \geq 0 \), it follows from (A) that

\[
\frac{4c_1^2c_2e^2}{c_3^2} = \frac{e^{4x}}{x^2} \geq \frac{e^{4x}}{e^x} = e^{3x},
\]

19
which indicates that

\[ x \leq \frac{1}{3} \log \left( \frac{4c_1^2 c_2 e^2}{c_3} \right). \]

Plugging this into (28) gives us

\[ g^* \leq \frac{4c_1^2}{c_3} x^2 \leq \frac{4c_1^2}{9c_3} \left[ \log \left( \frac{4c_1^2 c_2 e^2}{c_3} \right) \right]^2 = \frac{16c_1^2}{9c_3} \left[ \log \left( \frac{2c_1 \sqrt{c_2 e}}{c_3} \right) \right]^2. \]

The statement follows from the fact that \( g(m) \leq g^* \), for any \( m \geq 2 \).

\[ \Box \]

B Proof of Theorem 7

Proof. Suppose the confidence sets do not fail, which is true with probability at least \( 1 - \delta \). Then, CLUCB2 satisfies constraints are all satisfied, since it ensures that those constraints are satisfied by the worst parameter in the confidence set at any time.

Similar to Proposition 2, we can decompose the regret of CLUCB2 as

\[ R_T(\text{CLUCB2}) = \sum_{t \in S_T} \left( r_{a_t} - r_{a_t}^* \right) + \sum_{t \in S_T} \left( r_{a_t}^* - r_{b_t} \right) \]

\[ \leq \sum_{t \in S_T} \left( r_{a_t}^* - r_{a_t} \right) + n_T \Delta_h, \]

where \( n_T = |S_T^c| \) is the number of times CLUCB2 follows the baseline policy in \( T \) time steps. Now note that for \( t \in S_T \), CLUCB2 is following the action suggested by LUCB and hence,

\[ \sum_{t \in S_T} \left( r_{a_t}^* - r_{a_t} \right) \leq R_{ST}(\text{LUCB}) \leq R_T(\text{LUCB}), \] (29)

where \( R_{ST}(\text{LUCB}) \) denotes the regret of LUCB played at time steps \( t \in S_T \) which is upper bounded by the regret of LUCB played at all \( T \) time steps. On the other hand, by Proposition 3, we have the following regret bound for LCUB:

\[ R_T(\text{LUCB}) = O \left( d \log \left( \frac{D}{\lambda \delta T} \right) \sqrt{T} \right). \]

Thus, it follows that

\[ R_T(\text{CLUCB2}) = O \left( d \log \left( \frac{D}{\lambda \delta T} \right) \sqrt{T} + n_T \Delta_h \right), \] (30)

Note that according to [24], the confidence set \( C_t \) which CLUCB2 uses to find the optimistic action at round \( t \) is built based not only on the observations made by previously played optimistic actions, but also by the observations made when the baseline policy has been followed at rounds before \( t \). Therefore, the confidence set \( C_t \) used by CLUCB2 at round
$t$ would be tighter than what LUCB would have had if it was applied only to rounds $i \in S_t$. Hence, the first inequality in (29) still holds.

Given (30), we only need to show that CLUCB2 follows the baseline policy only at a finite number of rounds. Let $\tau$ be the last round that CLUCB2 follows the baseline policy (plays conservatively), i.e., $\tau = \max \{1 \leq t \leq T \mid a_t = b_t\}$. At time $\tau$, let

$$L_\tau = \min_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}} \phi_{a_i}^i + \phi_{a_i}^r \right> + \alpha \max \left( \min_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}} \phi_{b_i}^i \right>, n_{\tau-1} r_l \right),$$

which satisfies $L_\tau \geq \min_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}} \phi_{a_i}^i + \phi_{a_i}^r \right> + \alpha n_{\tau-1} r_l$, and

$$R_\tau = \max_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}\cup \{\tau\}} \phi_{b_i}^i \right>.$$

From Algorithm 2 at time $\tau$, we have $L_\tau < (1 - \alpha)R_\tau$ which with some simple algebra translates to

$$\alpha n_{\tau-1} r_l \leq \max_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}\cup \{\tau\}} \phi_{b_i}^i \right> - \min_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}} \phi_{a_i}^i + \phi_{a_i}^r \right> - \alpha \max_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}\cup \{\tau\}} \phi_{b_i}^i \right>.$$  \hspace{1cm} (31)

The rest of the proof is devoted to use (31) and prove a time independent upper bound on $n_{\tau-1}$. Unlike in the proof of Theorem 5, we rely on the nested property of the confidence sets built in (24) in this proof.

If the confidence sets do not fail (i.e., $\theta^* \in C_\tau$), then since $\forall \ i \leq \tau : C_\tau \subseteq C_i$, we have

$$\max_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}\cup \{\tau\}} \phi_{b_i}^i \right> \leq \sum_{i \in S_{\tau-1}\cup \{\tau\}} \max_{\theta \in C_i} \left< \theta, \phi_{b_i}^i \right> \leq \sum_{i \in S_{\tau-1}\cup \{\tau\}} \max_{\theta \in C_i} \left< \theta, \phi_{b_i}^i \right>.$$  \hspace{1cm} (32)

First, it follows from (32) and the fact that $\langle \theta^*, \phi_{b_i}^i \rangle \geq r_l$ that

$$-\alpha \max_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}\cup \{\tau\}} \phi_{b_i}^i \right> \leq -\alpha (m_{\tau-1} + 1) r_l.$$  \hspace{1cm} (33)

On the other hand, by the definition of optimistic action $a_i^*$ at round $i$ it follows that $\max_{\theta \in C_i} \left< \theta, \phi_{b_i}^i \right> \leq \max_{i \in A_i} \max_{\theta \in C_i} \left< \theta, \phi_{a_i}^i \right> = \max_{\theta \in C_i} \left< \theta, \phi_{a_i}^i \right>$. Then, from (32) it follows that

$$\max_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}\cup \{\tau\}} \phi_{b_i}^i \right> \leq \sum_{i \in S_{\tau-1}} \max_{\theta \in C_i} \left< \theta, \phi_{a_i}^i \right> + \max_{\theta \in C_\tau} \left< \theta, \phi_{a_i}^r \right>.$$  \hspace{1cm} (34)

Furthermore, since $\forall \ i \leq \tau : C_\tau \subseteq C_i$, we have

$$\min_{\theta \in C_\tau} \left< \theta, \sum_{i \in S_{\tau-1}} \phi_{a_i}^i + \phi_{a_i}^r \right> \geq \sum_{i \in S_{\tau-1}} \min_{\theta \in C_\tau} \left< \theta, \phi_{a_i}^i \right> + \min_{\theta \in C_\tau} \left< \theta, \phi_{a_i}^r \right> \geq \sum_{i \in S_{\tau-1}} \min_{\theta \in C_i} \left< \theta, \phi_{a_i}^i \right> + \min_{\theta \in C_\tau} \left< \theta, \phi_{a_i}^r \right>.$$  \hspace{1cm} (35)
Combining (33), (34), (35) with (31) gives
\[
\alpha n_{r-1} r_i \leq -\alpha (m_{r-1} + 1) r_i + \sum_{i \in S_{r-1}} \left[ \max_{\theta \in C_i} \langle \theta, \phi^i_{a_i} \rangle - \min_{\theta \in C_i} \langle \theta, \phi^i_{a_i} \rangle \right] + \left[ \max_{\theta \in C_r} \langle \theta, \phi^r_{a_r} \rangle - \min_{\theta \in C_r} \langle \theta, \phi^r_{a_r} \rangle \right].
\]  
(36)

Now, note that for any \( i \), the reward of playing any action is in \([0, 1]\) and hence, \[
\max_{\theta \in C_i} \langle \theta, \phi^i_{a_i} \rangle - \min_{\theta \in C_i} \langle \theta, \phi^i_{a_i} \rangle \leq 1. \]
On the other hand, since the confidence sets do not fail, we have
\[
\max_{\theta \in C_i} \langle \theta, \phi^i_{a_i} \rangle - \min_{\theta \in C_i} \langle \theta, \phi^i_{a_i} \rangle = 2 \max_{\theta \in C_i} \langle \theta - \hat{\theta}_{i-1}, \phi^i_{a_i} \rangle \leq 2 \left\| \theta - \hat{\theta}_{i-1} \right\|_{V_{i-1}} \left\| \phi^i_{a_i} \right\|_{V_{i-1}} \leq 2 \beta_i \left\| \phi^i_{a_i} \right\|_{V_{i-1}}.
\]
Hence since \( \beta_i \)'s are non-decreasing and all larger than 1, it follows that
\[
\left[ \max_{\theta \in C_i} \langle \theta, \phi^i_{a_i} \rangle - \min_{\theta \in C_i} \langle \theta, \phi^i_{a_i} \rangle \right] \leq 2 \beta_r \min \left(1, \left\| \phi^i_{a_i} \right\|_{V_{i-1}} \right). 
\]  
(37)

Similarly, we can show that
\[
\left[ \max_{\theta \in C_r} \langle \theta, \phi^r_{a_r} \rangle - \min_{\theta \in C_r} \langle \theta, \phi^r_{a_r} \rangle \right] \leq 2 \beta_r \min \left(1, \left\| \phi^r_{a_r} \right\|_{V_{r-1}} \right). 
\]  
(38)

Substituting (37) and (38) in (36) gives
\[
\alpha n_{r-1} r_i \leq -\alpha (m_{r-1} + 1) r_i + 2 \beta_r \left[ \sum_{i \in S_{r-1}} \min \left(1, \left\| \phi^i_{a_i} \right\|_{V_{i-1}} \right) + \min \left(1, \left\| \phi^r_{a_r} \right\|_{V_{r-1}} \right) \right]. 
\]  
(39)

Bounding the RHS of (39) gives rise to another key difference between this proof and that of Theorem 5. Note that \( V_i = \lambda_i + \sum_{j \in S_i} \phi^j_{b_j} (\phi^j_{a_j})^T + \sum_{j \in S_i} \phi^j_{b_j} (\phi^j_{b_j})^T \) is built not only based on the actions played at non-conservative rounds but also based on the actions played at conservative times (\( j \in S_i^c \)), and hence Lemma 4 cannot be directly used to bound the RHS of (39). Instead, we define for any \( i \): \( \tilde{V}_i = \lambda_i + \sum_{j \in S_i} \phi^j_{b_j} (\phi^j_{a_j})^T \) which satisfies \( V_i = \tilde{V}_i + \sum_{j \in S_i^c} \phi^j_{b_j} (\phi^j_{b_j})^T \). Therefore it follows that
\[
\left\| \phi^i_{a_i} \right\|_{V_{i-1}} \leq \left\| \phi^i_{a_i} \right\|_{\tilde{V}_{i-1}}, \quad \left\| \phi^r_{a_r} \right\|_{V_{r-1}} \leq \left\| \phi^r_{a_r} \right\|_{\tilde{V}_{r-1}},
\]
and hence, from (39) it follows that
\[
\alpha n_{r-1} r_i \leq -\alpha (m_{r-1} + 1) r_i + 2 \beta_r \left[ \sum_{i \in S_{r-1}} \min \left(1, \left\| \phi^i_{a_i} \right\|_{\tilde{V}_{i-1}} \right) + \min \left(1, \left\| \phi^r_{a_r} \right\|_{\tilde{V}_{r-1}} \right) \right]. 
\]  
(40)
Now, similar to the proof of Theorem 5, we define

$$\Gamma = \left[ \sum_{i \in S_{r-1}} \min \left( 1, \| \phi_{a_i} \|^2_{V_{r-1}} \right) + \min \left( 1, \| \phi_{a_i}^T \|^2_{V_{r-1}} \right) \right],$$

which by Lemma 4 satisfies

$$\Gamma \leq 2d \log \left( 1 + \frac{(m_{r-1} + 1) D^2}{\lambda d} \right).$$

(41)

On the other hand, for $n_{r-1} \geq 3^2$ we have

$$\beta_{r} = \sigma \sqrt{d \log \left( \frac{1 + (m_{r-1} + 1 + n_{r-1})}{\delta} \right) D^2/\lambda + B \sqrt{\lambda}} \leq (\sigma + B \sqrt{\lambda}) \sqrt{d \log \left( \frac{1 + (m_{r-1} + 1) n_{r-1} D^2/\lambda}{\delta} \right)},$$

(42)

where we used $\tau = m_{r-1} + n_{r-1} + 1$.

Using (41) and (42), and an application of Cauchy-Schwarz inequality on (40) gives

$$\alpha n_{r-1} r_l \leq -\alpha (m_{r-1} + 1) r_l + 2 \beta_{r} \sqrt{(m_{r-1} + 1) \Gamma}$$

$$\leq -\alpha (m_{r-1} + 1) r_l + 2 \beta_{r} \sqrt{2d (m_{r-1} + 1) \log \left( 1 + \frac{(m_{r-1} + 1) D^2}{\lambda d} \right)}$$

$$= -\alpha (m_{r-1} + 1) r_l + 3d (B \sqrt{\lambda} + \sigma) \log \left( \frac{2n_{r-1} D^2}{\lambda \delta} (m_{r-1} + 1) \right) \sqrt{(m_{r-1} + 1)}. \quad (43)$$

Note that in contrary to the proof of Theorem 5 where only $m_{r-1}$ appeared on the RHS of (19), here both $n_{r-1}$ and $m_{r-1}$ both appear on the RHS. To bound $n_{r-1}$, we first provide an upper bound on the RHS of (43) in terms of $n_{r-1}$. For $m = (m_{r-1} + 1)$, $c_1 = 3d (B \sqrt{\lambda} + \sigma)$, $c_2 = \frac{2n_{r-1} D^2}{\lambda \delta}$ and $c_3 = \alpha r_l$, Lemma 8 provides the following upper bound on the RHS (and hence the LHS) of (43):

$$\alpha n_{r-1} r_l \leq 16 d^2 \left( \frac{B \sqrt{\lambda} + \sigma}{\alpha r_l} \right)^2 \left[ \log \left( \frac{24d (B \sqrt{\lambda} + \sigma) D \sqrt{n_{r-1}}}{\sqrt{\lambda \delta \alpha r_l}} \right) \right]^2,$$

which is equivalent to

$$\sqrt{n_{r-1}} \leq 4d \left( \frac{B \sqrt{\lambda} + \sigma}{\alpha r_l} \right) \log \left( \frac{24d (B \sqrt{\lambda} + \sigma) D \sqrt{n_{r-1}}}{\sqrt{\lambda \delta \alpha r_l}} \right). \quad (44)$$

Now, note that the LHS of (43) grows linearly with $\sqrt{n_{r-1}}$ while the RHS grows logarithmically. Thus, such an inequality holds only for a finite number of $n_{r-1}$’s. Lemma 9 applied

\footnote{If this condition does not hold, then it results in the simple bound $n_{r-1} \leq 2$.}
with \( x = \sqrt{n_{\tau-1}} \), \( c_1 = 4d\frac{B\sqrt{\lambda + \sigma}}{\alpha r_1} \) and \( c_2 = \frac{24d(B\sqrt{\lambda + \sigma})D}{\sqrt{\lambda \delta} \alpha r_1} \), gives the following upper bound on \( n_{\tau-1} \):

\[
n_{\tau-1} \leq 256 \frac{d^2(B\sqrt{\lambda + \sigma})^2}{\alpha^2 r_1^2} \left[ \log \left( \frac{10d(B\sqrt{\lambda + \sigma})\sqrt{D}}{\alpha r_1(\lambda \delta)^{1/4}} \right) \right]^2.
\]

Therefore, CLUCB2 follows the baseline policy only at

\[
n_T = n_r = n_{\tau-1} + 1 \leq 256 \frac{d^2(B\sqrt{\lambda + \sigma})^2}{\alpha^2 r_1^2} \left[ \log \left( \frac{10d(B\sqrt{\lambda + \sigma})\sqrt{D}}{\alpha r_1(\lambda \delta)^{1/4}} \right) \right]^2 + 1
\]

rounds, and hence according to (30), achieves a regret bound of

\[
R_T(CLUCB2) = O \left( d \log \left( \frac{D}{\lambda \delta} T \right) \sqrt{T} + K \frac{\Delta_h}{\alpha^2 r_1^2} \right),
\]

where

\[
K = 256d^2(B\sqrt{\lambda + \sigma})^2 \left[ \log \left( \frac{10d(B\sqrt{\lambda + \sigma})\sqrt{D}}{\alpha r_1(\lambda \delta)^{1/4}} \right) \right]^2 + 1.
\]

C Technical Detail Used in the Proof of Theorem 7

We used the following Lemma in the proof of Theorem 7.

Lemma 9. Let \( c_1 \) and \( c_2 \) be two positive constants such that \( \log(c_1 c_2) \geq 1 \). Then, any \( x > 0 \) satisfying \( x \leq c_1 \log(c_2 x) \) also satisfies \( x \leq 2c_1 \log(c_1 c_2) \).

Proof. Assume that \( x \leq c_1 \log(c_2 x) \) holds. Define \( a = \frac{1}{c_1 c_2} \) and change the variable to \( z = c_2 x \). Then, we have

\[
a z \leq \log(z).
\]

Let \( q(z) = az \) and \( l(z) = \log(z) \) and define \( z^* = 2/a \log(1/a) \). First, since \( \log(t^2) \leq t \) for any \( t > 0 \), we have

\[
\frac{1}{a} \geq \log \left( \frac{1}{a^2} \right) \Rightarrow \log \frac{1}{a} \geq \log \left( 2 \log \frac{1}{a} \right) \Rightarrow 2 \log \frac{1}{a} \geq \log \left( \frac{2}{a} \log \frac{1}{a} \right).
\]

By the definition of \( z^*, q \) and \( l \), the last inequality is equivalent to

\[
q(z^*) \geq l(z^*).
\] (45)

Furthermore, since \( \log(1/a) \geq 1 \), then for any \( z \geq z^* \):

\[
q'(z) = a \geq \frac{a}{2 \log(1/a)} = l'(z^*) \geq l'(z).
\] (46)

Thus, it follows from (45) and (46) that \( q(z) \geq l(z) \) for all \( z \geq z^* \). Thus, \( az \leq \log(z) \) is possible only for \( z \leq z^* \). Replacing the definition of \( a, z \) and \( z^* \), we deduce that \( x \leq c_1 \log(c_1 c_2) \) is possible only if \( x \leq 2c_1 \log(c_1 c_2) \). 

\[\square\]