LOCAL OPERATIONS AND EVENTUALLY OPEN ACTIONS

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Abstract. We study continuous actions of Polish groups on Polish spaces. We develop Scott analysis introduced by Hjorth for studying orbit equivalence relations. We define eventually open actions and prove that this property characterizes the actions endowed with a complete system of hereditarily countable invariant structures.

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0. Introduction

In this paper we study continuous actions of Polish groups on Polish spaces (say $G$ on $X$) by means of generalized Scott invariants introduced by Hjorth in [4]. Modifying the generalized Hjorth-Scott analysis we approach the orbit equivalence relation in a fashion which exploits descriptive set theoretical viewpoint slightly more intensively.

The basic tool of Hjorth's work are hereditarily countable structures $\phi_\alpha(x, U, V)$, $U \subseteq_{\text{open}} X$, $V \subseteq_{\text{open}} G$, corresponding to Scott characteristics ([4], Chapter 6.2). We rather concentrate on associated sets $B_\alpha(x, U, V) = \{y \in X : \phi_\alpha(y, U, V) = \phi_\alpha(x, U, V)\}$ (we call them $\alpha$-pieces) and their presentations with use of some operation of local saturation. This direction can be considered as a generalization of the notion of canonical partitions introduced by Becker in [2]. Following this way we are able to supplement Hjorth's work with a couple of new statements concerning the sets $B_\alpha(x, U, V)$. In particular in Theorem 12 we present a canonical form for $B_\alpha(x, U, V)$. This immediately implies that the sets $B_\alpha(x, U, V)$ are Borel and moreover this describes their Borel complexity.

The original motivation for this result is connected with the problems of coding of $G$-orbits in admissible sets. In order to extend the results of [7] and [6] to the general case of Polish $G$-spaces, Theorem 12 looks very helpful. This stuff will be considered in a separate paper.

In this paper we first concentrate on refining topologies by extending the initial basis by families of sets of the form $B_\beta(x, U, V)$. Applying Theorem 12 we show that the original topology enriched upon
some natural families of these sets generates on $X$ finer topologies endowed with the same Borel structure as the initial one so that each $B_\alpha(x, X, G)$ with the corresponding subspace topology becomes a Polish $G$-space. This generalizes a similar theorem proved by Hjorth in [4] in the case when $\alpha$ is $\gamma^*(x)$, the generalized Scott rank of $x$.

We then study when the maps $G \to Gx$ defined by $g \to gx$ are open (for all $x \in X$) with respect to appropriate topologies mentioned above. We prove that this property is equivalent to the property that the generalized Scott analysis applied to the orbit equivalence relation leads to a complete system of invariants. Moreover we can restate this as a very simple condition which we call \textit{eventual openness of the action} (this is the content of Theorem 33).

It is worth noting that it is proved in [4] that orbit equivalence relations equipped with a complete system of generalized Scott invariants are classifiable by countable models, i.e. they are Borel reducible to the isomorphism relation on the space $\text{Mod}(\mathcal{L})$ of all countable structures of some countable language $\mathcal{L}$.

The operation of \textit{local saturation} and its basic properties are presented in Section 1. Section 2 is devoted to our approach to the generalized Hjorth-Scott analysis. Eventually open actions are studied in Section 3. Along with the local counterpart of saturation we apply there a local version of Vaught transforms. This may be interesting in itself.

1. PRELIMINARIES

In the first part of this section we recall standard notation and facts concerning Polish group actions. In the second one we define \textit{local saturation} - a new operation arising in this context. This operation is of particular importance for this paper.

1.1. Notation. A \textit{Polish space} (group) is a separable, completely metrizable topological space (group). If a Polish group $G$ continuously acts on a Polish space $X$, then we say that $X$ is a \textit{Polish $G$-space}. We say that a subset of $X$ is \textit{invariant} if it is $G$-invariant. All basic facts concerning Polish $G$-spaces can be found in [3], [4] and [5].

Let $G$ be a Polish group, $\mathcal{N}$ be a countable basis of $G$ and $G_0 = \{g_i : i \in \omega\}$ be a countable dense subgroup of $G$. Let $\mathcal{V} \subseteq \mathcal{N}$, $\mathcal{V} = \{V_m : m \in \omega\}$, be a countable basis of open neighborhoods of the unity of $G$. We shall assume that $V = V^{-1}$ and $V^g \in \mathcal{V}$, whenever $V \in \mathcal{V}$ and $g \in G_0$. Besides $\mathcal{V}$ we shall use the symbol $\hat{\mathcal{V}}$ to denote the set of all (not only basic) symmetric neighbourhoods of the unity $1_G$.

Let $\langle X, \tau \rangle$ be a Polish $G$-space and $\mathcal{U} = \{U_n : n \in \omega\}$ be a countable basis of $X$. We assume that for every $U \in \mathcal{U}$ and $g \in G_0$ we have $gU \in \mathcal{U}$.

Since we shall use Vaught transforms, recall the corresponding definitions. The Vaught $*$-transform of a set $B \subseteq X$ with respect to an open $H \subseteq G$ is the set $B^*H = \{x \in X : \{g \in H : gx \in B\}$
is comeager in \( H \)}, the Vaught \( \Delta \)-transform of \( B \) is the set \( B^\Delta H = \{ x \in X : \{ g \in H : gx \in B \} \) is not meager in \( H \}. It is known that for any \( x \in X \) and \( g \in G \), \( gx \in B^{+H} \Leftrightarrow x \in B^{+Hg} \) and \( gx \in B^\Delta H \Leftrightarrow x \in B^{\Delta Hg} \).

It is worth noting that for any open \( B \subseteq X \) and any open \( K < G \) we have \( B^\Delta K = KB \), where \( KB = \{ gx : g \in K, x \in B \} \). Indeed, by continuity of the action for any \( x \in KB \) and \( g \in K \) with \( gx \in B \) there are open neighbourhoods \( K_1 \subseteq K \) and \( B_1 \subseteq KB \) of \( g \) and \( x \) respectively so that \( K_1B_1 \subseteq B \); thus \( x \in B^\Delta K \). Moreover for every countable ordinal \( \alpha \) if \( B \in \Sigma^0_\alpha(X) \), then \( B^\Delta H \in \Sigma^0_\alpha(X) \) and if \( B \in \Pi^0_\alpha(X) \), then \( B^{+H} \in \Pi^0_\alpha(X) \).

Other basic properties of Vaught transforms can be found in [3] and [5].

### 1.2. Local saturation

When \( G \) admits a basis of open subgroups at its unity \( 1_G \), then every Polish \( G \)-space admits a basis consisting of the sets which are invariant with respect to some open basic subgroup of \( G \). Since such a \( G \) is isomorphic to a closed permutation group (see [3] for details), we may easily generalize Scott method described in 6.1 of [4] to analyse orbit equivalence relations arising in these situations.

To handle with difficulties of the general case we introduce a local variant of the operation of saturation which generalizes the concept of a local orbit introduced in [4].

**Definition 1.** Let \( U \subseteq X \) be open\(^1\) and \( V \in \mathcal{V} \). For every \( A \subseteq X \) we define inductively an increasing sequence \( (V_U[n]A)_{n \in \omega} \) of subsets of \( X \) as follows:

\[
\begin{align*}
V_U^0 A &= A \cap U, \\
V_U^{n+1} A &= V(V_U^n A) \cap U.
\end{align*}
\]

The set \( V_U A = \bigcup_{n \in \omega} V_U[n] A \) is called the local \( V_U \)-saturation of \( A \). For every \( x \in X \) we shall write \( V_U x \) instead of \( V_U \{ x \} \) and call this set the local \( V_U \)-orbit of \( x \).

We see that \( V_U A \) is contained in \( U \) and contains \( A \cup U \). It is nonempty if and only if \( A \cap U \neq \emptyset \), in particular \( V_U x \neq \emptyset \) if and only if \( x \in U \). \( V_U A \) is a union of all local \( V_U \)-orbits of elements of \( A \).

There is a natural correspondence between local orbits and suitably defined subsets of \( G \) (which already appeared in [4]). Let us introduce the corresponding definition and formulate basic facts.

**Definition 2.** Let \( U \subseteq X \) be open, \( A \subseteq U \) and \( V \in \mathcal{V} \). For every \( x \in U \) we define an increasing sequence of subsets of \( G \) as follows.

\[
\begin{align*}
(V)_U^0(0) &= \{ 1_G \}, \\
(V)_U^n(n+1) &= \{ gh : ghx \in U, h \in (V)_U^n(n), g \in V \}.
\end{align*}
\]

\(^1\)Actually we do not need to demand that \( U \) is open, the definition makes sense for any \( U \).
Then we put \( \langle V \rangle_U^x = \bigcup_{n \in \omega} \langle V \rangle_U^x(n) \).

If \( x \not\in U \), we put \( \langle V \rangle_U^x = \emptyset \).

Finally we define \( \langle V \rangle_U^A = \bigcap \{ {\langle V \rangle_U^x : x \in A} \} \).

By this definition we see that \( V_U x = \langle V \rangle_U^x \). The following simple lemma collects the very basic properties of introduced sets.

**Lemma 3.** Let \( U \subseteq X \) be open, \( V \in \hat{V} \) and \( x \in X \). Then

1. \( \langle V \rangle_U^x(n + 1) = \{ gh : h \in \langle V \rangle_U^x(1), g \in \langle V \rangle_U^{hx}(n) \} \), for every \( n \in \omega \).
2. If \( n, k \geq 0 \) and \( h \in \langle V \rangle_U^x(k) \), then \( \langle V \rangle_U^{hx}(n) h \subseteq \langle V \rangle_U^x(n + k) \) and \( \langle V \rangle_U^x(n) h^{-1} \subseteq \langle V \rangle_U^{hx}(n + k) \). Consequently, \( \langle V \rangle_U^{hx} h = \langle V \rangle_U^x \).
3. For every \( h \in \langle V \rangle_U^x(1) \), \( n \in \omega \), \( f \in \langle V \rangle_U^{hx}(n) \) there are open \( U_x \subseteq U \), \( W_h \subseteq \langle V \rangle_U^x(1) \) and \( W_f \subseteq \langle V \rangle_U^{hx}(n) \) such that \( x \in U_x \), \( h \in W_h \), \( f \in W_f \) and \( f' h' \in \langle V \rangle_U^{hx}(n + 1) \), for every \( y \in U_x \), \( h' \in W_h \) and \( f' \in W_f \).

Consequently \( \langle V \rangle_U^x(n), n \in \omega \), and \( \langle V \rangle_U^x \) are open.

**Proof.** (1), (2) are easy consequences of the definition, then (3) follows by induction from continuity of the action. \( \Box \)

**Remark.** If \( x \in U \), then the equality \( \langle V \rangle_U^{hx} h = \langle V \rangle_U^x \) is not true unless \( h \in \langle V \rangle_U^x \). Applying point (2) of the lemma above we can easily check that for any \( h, h' \in G \), the sets \( \langle V \rangle_U^{hx} h, \langle V \rangle_U^{hx} \) are either equal or disjoint. Thus the family \( \{ \langle V \rangle_U^{hx} h : h \in G, hx \in U \} \) is a partition of the set \( \{ h \in G : hx \in U \} \) into open sets. \( \Box \)

By classical results orbits of elements under continuous (Borel) actions are Borel sets. If we slightly modify the proof, we see that this remains true for local orbits.

**Corollary 4.** Every local orbit under a continuous action is a Borel set.

**Proof.** Let \( U \subseteq X \) be open, \( V \in \hat{V} \) and \( x \in X \). Put \( W = \{ g \in G : gx \in V_U x \} \). We have \( W x = V_U x \) and \( W = \langle V \rangle_U^x G_x \), where \( G_x \) is the stabilizer of \( x \). By Lemma 3 we see that \( W \) is an open subset of \( G \). Let \( T_x \) be a Borel transversal of \( G/G_x \). Then \( W \cap T_x \) is also Borel and the function \( g \to gx \) is a bijection from \( W \cap T_x \) onto \( V_U x \). Hence the latter has to be Borel. \( \Box \)

The other simple properties of the operation of local saturation are collected below.

2In this form, i.e. as classes of the appropriate equivalence relation, the sets we are discussing appear in [3].
Lemma 5. Let $U, U' \subseteq X$ be open and $V, V' \in \hat{V}$. For every $A, B, A_1, A_2, \ldots \subseteq X$, $x, y \in U$ and $f \in G$ the following statements hold.

1. If $A \subseteq B$, $U' \subseteq U$ and $V' \subseteq V$ then $V' \cap U A \subseteq V U B$.
2. $V_U(\bigcup_{n} A_n) = \bigcup_{n} V_U A_n$.
3. $V_U(V_U A) = V_U A$.
4. $V_U x = V_U y$ if and only if $x \in V_U y$.
5. If $x \in U'$ and $V' \subseteq V$, then $V_U x = U \cap \bigcup\{(g V' \cap V') x \cap g U' : g \in G_0 \cap \langle V \rangle_U^x\}$.
6. $f(V_U A) = (V f)^U (f A)$.

Proof. (1) and (2) are immediate.

(3) By (1), (2) we have $V_U(V_U A) = V_U(\bigcup_{n} V_U^{[n]} A) = \bigcup_{n} V_U(V_U^{[n]} A)$. Let $n \in \omega$ be arbitrary. It follows from the definition that $V_U(V_U^{[n]} A) = \bigcup_{i \geq n} V_U^{[i]} A$. Since the family $\{V_U^{[n]} A\}_{n \in \omega}$ is increasing, we see that $V_U A = \bigcup_{i \geq n} V_U^{[i]} A$ which completes the proof.

(4) ($\Rightarrow$) is obvious.

(5) ($\Leftarrow$) immediately follows by Lemma 3(2).

(5) To prove $\subseteq$ consider an arbitrary $h \in \langle V \rangle_U^x$. By Lemma 3(3) we can find an open set $W \subseteq \langle V \rangle_U^x$ such that $h \in W$ and $W x \subseteq U$. We may additionally demand that $W^{-1} W \subseteq V'$ (since $V'$ is symmetric, then also $W W^{-1} \subseteq V'$). Thus the equality $g \in G_0 \cap W$ we have $h \in g V' \cap V' g$ and $h x \in g U'$.

For the converse inclusion observe that $V' g x \cap U \subseteq V_U x$ whenever $g x \in V_U x$.

(6) follows from the fact that for every $n \in \omega$ we have $f(V_U^{[n]} A) = (V f)^U_{[n]} (f A)$. The latter can be obtained by an easy inductive argument. \[\square\]

The new concept of saturation entails a new concept of invariantness - local invariantness.

Definition 6. Let $U \subseteq X$ be open, $V \in \hat{V}$ and $A \subseteq X$. We say that $A$ is locally $V_U$-invariant if $V_U A = A \cap U$.

Remark. It follows that $A$ is locally $V_U$-invariant if and only if $V_U x \subseteq A$, for every $x \in A$. Observe also that $A$ is locally $V_U$-invariant whenever $V(A \cap U) \cap U = A \cap U$. Indeed, the equality $\neg \neg V(A \cap U) \cap U = A \cap U$ implies by induction that for every $n \in \omega$ we have $V_U^{[n]} A = A \cap U$ and thus $V_U A = A \cap U$. On the other hand we have $A \cap U \subseteq V(A \cap U) \cap U \subseteq V_U A$. Thus the equality $V_U A = A \cap U$ implies that $V(A \cap U) \cap U = A \cap U$.

Obviously every $A$ such that $A \supseteq U$ or $U \cap A = \emptyset$ is locally $V_U$-invariant. Moreover it can be justified by easy straightforward arguments that the family of all locally $V_U$-invariant subsets of $X$
forms a complete Boolean algebra. By Lemma 5 we see that for every $A$, $V_U A$ is a $V_U$-invariant set containing $A \cap U$. If $A$ is open then $V_U A$ is open by Definition 1.

2. Sets arising in Polish group actions

This section can be considered both as systematization and some improvement of the material contained in Section 6.2 of [4]. It is divided into two subsections. In the first one we modify the generalized Scott analysis developed by Hjorth. The basic tool of Hjorth’s work are hereditarily countable structures $\phi_\alpha(x, U_n, V_m)$ corresponding to Scott characteristics. We suggest slightly different approach and concentrate on associated sets $B_\alpha(x, U, V) = \{y \in X : \phi_\alpha(y, U, V) = \phi_\alpha(x, U, V)\}$.

We characterize the sets $B_\alpha(x, U, V)$ with use of the operation of local saturation and study them slightly further in order to present this material in a complete form.

This direction can be considered as a generalization of the notion of canonical partitions (see [2]). Following this way we are able to supplement Hjorth’s work with a couple of new statements concerning the sets $B_\alpha(x, U, V)$. The main result of this part is Theorem 12 which makes possible to express the sets $B_\alpha(x, U, V)$ in a canonical form. This possibility is of fundamental importance for our study. In particular it enables us to prove that the sets $B_\alpha(x, U, V)$ are Borel and describe their Borel complexity.

The second subsection is devoted to refining topologies by extending the initial basis by families of $\beta$-pieces, $\beta < \alpha$. Applying Theorem 12 we show that the original basis enriched upon the family $\{B_\beta(x', U_n, V_m) : x' \in V_U x \cap U_n, n, m \in \omega, \beta < \alpha\}$ generates on $X$ a finer topology endowed with the same Borel structure as the initial one so that $B_\alpha(x, X, G)$ with the corresponding subspace topology becomes a Polish $G$-space. It is worth noting that a similar theorem is proved by Hjorth in [4] in the case when $\alpha$ is $\gamma(x)$, the generalized Scott rank of $x$. Thus our theorem can be considered as a generalization of it.

2.1. Borel partitions. As we have already mentioned $\alpha$-invariants $\phi_\alpha(x, U, V)$ were introduced by Hjorth as a counterpart of $\alpha$-invariants studied by Scott. From now on we fix a countable basis $U = \{U_n : n \in \omega\}$ of $X$ and a countable basis $V = \{V_n : n \in \omega\}$ of open symmetric neighbourhoods of $1_G$.

**Definition 7. (Hjorth)** For every $U \in U$, $x \in U$ and $V \in V$ we define a set $\phi_\alpha(x, U, V)$ by simultaneous induction on the ordinal $\alpha$:

\[
\phi_1(x, U, V) = \{U : U \in U, V_U x \cap U \neq \emptyset\},
\]

\[
\phi_{\alpha+1}(x, U, V) = \{\phi_\alpha(x', U_n, V_m), n, m \} : x' \in V_U x, U_n \subseteq U, V_m \subseteq V\},
\]

\[
\phi_\lambda(x, U, V) = \{\phi_\alpha(x, U, V), \alpha : \alpha < \lambda\} \text{ for } \lambda \text{ limit .}
\]
Every $\phi_\alpha(x, U, V)$, $1 \leq \alpha < \omega_1$, defines the set $B_\alpha(x, U, V) = \{y \in U : \phi_\alpha(y, U, V) = \phi_\alpha(x, U, V)\}$. We call the sets of this form $\alpha$-pieces. Additionally we treat every basic open $U$ as a 0-piece. In the lemma below we put together the properties of $\alpha$-pieces that can be found in [4].

**Lemma 8.** (Hjort) Let $V \in V$, $U \in U$, $x \in U$ and $\alpha$ be an ordinal, $\alpha > 0$. Then the following statements are true.

1. $B_\alpha(x, U, V)$ is locally $V_U$-invariant, $V_U x \subseteq B_\alpha(x, U, V) \subseteq U$ and $B_\alpha(x, U, V) = B_\alpha(z, U, V)$ for every $z \in B_\alpha(x, U, V)$.
2. For any $z \in U$ the sets $B_\alpha(x, U, V)$, $B_\alpha(z, U, V)$ are either equal or disjoint.
3. If $x \in U_n \subseteq U$, $V_m \subseteq V$ and $\beta \leq \alpha$, then $B_\alpha(x, U_n, V_m) \subseteq B_\beta(x, U, V)$.
4. $hB_\alpha(x, U, V) = B_\alpha(hx, hU, V^h)$, for all $h \in G_0$.

**Remark.** While discussing $\alpha$-pieces we may omit conditions $U_n \subseteq U$ and $V_n \subseteq V$ in the formula defining $\phi_{\alpha+1}(x, U, V)$ and let $U_n, V_n$ vary over all elements of $U$ and $V$ respectively. This is because the set $B_{\alpha+1}(x, U, V)$ coincides with the set

$$\{y : \{\langle \phi_\alpha(y', U_n, V_m), n, m \rangle : y' \in V_U y \cap U_n\} = \{\langle \phi_\alpha(x', U_n, V_m), n, m \rangle : x' \in V_U x \cap U_n\}\}.$$ 

To see this note that the latter set is obviously included in $B_{\alpha+1}(x, U, V)$.

To get the converse inclusion we proceed as follows. Consider any $y$ which belongs to $B_{\alpha+1}(x, U, V)$ and any triple $(x', U_n, V_m)$ with $x' \in V_U x \cap U_n$. Take any $U_1 \subseteq U_n \cap U$ containing $x'$ and any $V_j \subseteq V_m \cap V$. According to the assumption on $y$ we may find $y' \in V_U y$ such that $\phi_\alpha(x', U_1, V_j) = \phi_\alpha(y', U_1, V_j)$, i.e. $y' \in B_\alpha(x', U_1, V_j)$. By Lemma [3] the latter implies $y' \in B_\alpha(x', U_n, V_m)$, i.e. $\phi_\alpha(x', U_n, V_m) = \phi_\alpha(y', U_n, V_m)$. This proves that the set $\{\langle \phi_\alpha(x', U_n, V_m), n, m \rangle : x' \in V_U x \cap U_n\}$ is contained in the set $\{\langle \phi_\alpha(y', U_n, V_m), n, m \rangle : y' \in V_U y \cap U_n\}$. The symmetric argument shows that

$$\{\langle \phi_\alpha(y', U_n, V_m), n, m \rangle : y' \in V_U y \cap U_n\} \subseteq \{\langle \phi_\alpha(x', U_n, V_m), n, m \rangle : x' \in V_U x \cap U_n\}.$$ 

By Proposition 2.C.2 of the paper of Becker [2] there exists a unique partition of $X$, $X = \bigcup \{Y_t : t \in T\}$, into invariant $G_\delta$ sets $Y_t$ such that every $G$-orbit of $Y_t$ is dense in $Y_t$. To construct this partition we define for any $t \in 2^\mathbb{N}$ the set

$$Y_t = (\bigcap \{GA_j : t(j) = 1\}) \cap (\bigcap \{X \setminus GA_j : t(j) = 0\})$$

and take $T = \{t \in 2^\mathbb{N} : Y_t \neq \emptyset\}$.

Observe that the family $\{B_1(x, X, G) : x \in X\}$ is just the canonical partition defined by Becker. Moreover for every ordinal $0 < \alpha < \omega_1$ the family $\{B_\alpha(x, X, G) : x \in X\}$ is a partition of $X$.
approximating the original orbit partition. Below we will see that every such a partition also can be
obtained in a canonical way mimicking the construction of the canonical partition.

Proposition 9. For every \( U \in \mathcal{U} \), \( x \in U \) and \( V \in \mathcal{V} \) the following equalities hold:

\[
B_1(x, U, V) = \bigcap_n \{ V_U U_n : V_U x \cap U_n \neq \emptyset \} \cap \bigcap_n \{ X \setminus V_U U_n : V_U x \cap U_n = \emptyset \},
\]

\[
B_{\alpha+1}(x, U, V) = \bigcap_{n,m} \{ V_U B_\alpha(y, U_n, V_m) : y \in U_n, V_U x \cap B_\alpha(y, U_n, V_m) \neq \emptyset \} \cap \bigcap_{n,m} \{ X \setminus V_U B_\alpha(y, U_n, V_m) : y \in U_n, V_U x \cap B_\alpha(y, U_n, V_m) = \emptyset \},
\]

\[
B_\lambda(x, U, V) = \bigcap \{ B_\alpha(x, U, V) : \alpha < \lambda \}, \text{ for } \lambda \text{ limit }.
\]

Proof. The first and the last equalities are obvious. We have to prove the second one.

(\( \subseteq \)) Take any \( z \in B_{\alpha+1}(x, U, V) \) and a triple \((y, U_n, V_m)\) such that \( y \in U_n \).

If \( V_U x \cap B_\alpha(y, U_n, V_m) \neq \emptyset \), then for some \( x' \in V_U x \) we have \( x' \in B_\alpha(y, U_n, V_m) \), i.e. \( \phi_\alpha(x', U_n, V_m) = \phi_\alpha(y, U_n, V_m) \). By the assumption on \( z \) there is \( z' \in V_U z \) such that \( \phi_\alpha(z', U_n, V_m) = \phi_\alpha(y, U_n, V_m) \), i.e. \( z' \in B_\alpha(y, U_n, V_m) \).

On the other hand if \( V_U x \cap B_\alpha(y, U_n, V_m) = \emptyset \), then for every \( x' \in V_U x \) we have \( x' \notin B_\alpha(y, U_n, V_m) \), i.e. \( \phi_\alpha(x', U_n, V_m) \neq \phi_\alpha(y, U_n, V_m) \). This implies \( \phi_\alpha(y, U_n, V_m), n, m \notin \phi_{\alpha+1}(x, U, V) \). Suppose towards contradiction that \( z \in V_U B_\alpha(y, U_n, V_m) \). Then there is \( z' \in V_U z \) such that \( z' \in B_\alpha(y, U_n, V_m) \), i.e. \( \phi_\alpha(z', U_n, V_m) = \phi_\alpha(y, U_n, V_m) \). Hence we get

\[
\{ \phi_\alpha(y, U_n, V_m), n, m \} \notin \phi_{\alpha+1}(z, U, V).
\]

Therefore we see that \( \phi_{\alpha+1}(z, U, V) \neq \phi_{\alpha+1}(x, U, V) \), thus \( z \notin B_{\alpha+1}(x, U, V) \). This contradicts our assumptions.

(\( \supseteq \)) Suppose that \( z \notin B_{\alpha+1}(x, U, V) \). Then there is a pair \((n, m)\) such that

\[
\{ \phi_\alpha(x', U_n, V_m) : x' \in V_U x \cap U_n \} \neq \{ \phi_\alpha(z', U_n, V_m) : z' \in V_U z \cap U_n \}.
\]

Therefore one of the following cases holds:

1° There is \( x' \in V_U x \) such that \( V_U z \cap B_\alpha(x', U_n, V_m) = \emptyset \);

2° There is \( z' \in V_U z \) such that \( V_U x \cap B_\alpha(z', U_n, V_m) = \emptyset \).

Either case implies

\[
z \notin \bigcap \{ V_U B_\alpha(y, U_n, V_m) : y \in U_n \subseteq U, V_m \subseteq V, V_U x \cap B_\alpha(y, U_n, V_m) \neq \emptyset \}
\]

\[
\cap \{ X \setminus V_U B_\alpha(y, U_n, V_m) : y \in U_n \subseteq U, V_m \subseteq V, V_U x \cap B_\alpha(y, U_n, V_m) = \emptyset \}. \square
\]

From now on we shall use Proposition 9 as a definition of an \( \alpha \)-piece.
Corollary 10. Hence we may formulate the following assertion.

\[ B_\alpha(x, U, V) = \bigcap_{n,m} \{ V_U B_\beta(y, U_n, V_m) : y \in U_n, V_U x \cap B_\beta(y, U_n, V_m) \neq \emptyset \} \cap \bigcap_{n,m} \{ X \setminus V_U B_\beta(y, U_n, V_m) : y \in U_n, V_U x \cap B_\alpha(y, U_n, V_m) = \emptyset \} . \]

Hence we may formulate the following assertion.

Corollary 10. Let \( x, y \in U \). Then \( B_\alpha(x, U, V) = B_\alpha(y, U, V) \) if and only if the local orbits \( V_U x \) and \( V_U y \) intersect exactly the same \( \beta \)-pieces, for every \( 0 \leq \beta < \alpha \).

In the next lemma we formulate another important property of \( \alpha \)-pieces: every element of \( V_U B_\alpha(x, U_n, V_m) \) can be surrounded by some \( \alpha \)-piece entirely contained in \( V_U B_\alpha(x, U_n, V_m) \). This property will be frequently applied in the rest of the paper.

Lemma 11. Let \( V \in \mathcal{V} \), \( U \in \mathcal{U} \) and \( x \in U_n \cap U \). Then for every \( m \in \omega \), ordinal \( \alpha > 0 \) and \( y \in V_U B_\alpha(x, U_n, V_m) \) there are \( U_i \subseteq U_n \) and \( h \in G_0 \) such that \( h \in \langle V \rangle^U \) (i.e. \( h \in \langle V \rangle^U \) for every \( t \in U_i \), \( y \in hU_i \) and \( B_\alpha(y, hU_i, (V_m)^h) \subseteq V_U B_\alpha(x, U_n, V_m) \)).

Proof. If \( y \in V_U B_\alpha(x, U_n, V_m) \), then the local orbit \( V_U y \) intersects \( B_\alpha(x, U_n, V_m) \). Thus the intersection \( \langle V \rangle^U \cap \{ g \in G : gy \in B_\alpha(x, U_n, V_m) \} \) is nonempty. By Lemma 3(3) we see that \( \langle V \rangle^U \) is open. The set \( \{ g \in G : gy \in B_\alpha(x, U_n, V_m) \} \) is open either. Indeed if \( gy \in B_\alpha(x, U_n, V_m) \), then by Lemma \( \alpha \) the local orbit \( (V_m)^g \) is entirely contained in \( B_\alpha(x, U_n, V_m) \) and so \( \langle V \rangle^U \cap \{ g \in G : gy \in B_\alpha(x, U_n, V_m) \} \) is open, i.e. the set \( \{ g \in G : gy \in B_\alpha(x, U_n, V_m) \} \) is open. Thus we are done since the set \( \langle V \rangle^U \cap \{ g \in G : gy \in B_\alpha(x, U_n, V_m) \} \) is open by Lemma 3(3).

To sum up \( \langle V \rangle^U \cap \{ g \in G : gy \in B_\alpha(x, U_n, V_m) \} \) is a nonempty open set. Therefore it contains an element of \( G_0 \), i.e. we can find \( g \in G_0 \cap \langle V \rangle^U \) such that \( gy \in B_\alpha(x, U_n, V_m) \). Put \( y' = gy \) and \( h = g^{-1} \). We have \( h \in G_0 \) and \( hy' = y \). By Lemma 3(2) we see that \( h \in \langle V \rangle^U \). According to Lemma 3(3) there is a basic open \( U_i \subseteq U_n \) containing \( y' \) such that \( h \in \langle V \rangle^U \). Since \( B_\alpha(y', U_i, V_m) \subseteq U_i \) and \( B_\alpha(y', U_i, V_m) \subseteq B_\alpha(y', U_n, V_m) = B_\alpha(x, U_n, V_m) \), then

\[ B_\alpha(hy', hU_i, (V_m)^h) = hB_\alpha(y', U_i, V_m) \subseteq V_U B_\alpha(x, U_n, V_m) . \]

\( \square \)

Now we are ready to formulate the main result of this part. Despite its technical character this theorem shed a new light on the nature of \( \alpha \)-pieces. In particular it enables us to prove that \( \alpha \)-pieces are Borel sets.
Theorem 12. Let $U \in \mathcal{U}$, $V \in \mathcal{V}$ and $x \in U$. Then for every ordinal $\alpha > 0$ the following equality is true

$$B_{\alpha+1}(x, U, V) = \bigcap_{n,m} \left( \bigcup \{ B_\alpha(gx, hU_i, (V_m)^h) : U_i \subseteq U_n, h \in G_0 \cap (V_U)^U_g, g \in G_0 \cap (V)^U_g, gx \in hU_i \} \right) \cap \bigcap_{n,m} \left( (X \setminus U_n) \cup \bigcup \{ B_\alpha(gx, U_n, V_m) : g \in G_0 \cap (V)^U_g, gx \in U_n \} \right).$$

Proof. We will apply Proposition 8. Consider arbitrary $U_n$, $V_m$ and $y \in U_n$ such that $B_\alpha(y, U_n, V_m) \cap V_U x \neq \emptyset$. The set $V_U B_\alpha(y, U_n, V_m)$ can be presented as a union of $\alpha$-pieces in the way described in Lemma 11. If we throw aside the elements that can be surrounded by some $\alpha$-piece disjoint from $V_U x$, then we may limit ourselves to $\alpha$-pieces containing elements of the form $gx$, where $g \in G_0 \cap (V)^U_g$. As a result we obtain the following formula.

$$V_U B_\alpha(y, U_n, V_m) \cap \bigcap_{i,j} \left( (X \setminus V_U) B_\alpha(z, U_i, V_j) : V_U x \cap B_\alpha(z, U_i, V_j) = \emptyset \right)$$

$$= \bigcup \{ B_\alpha(gx, hU_i, (V_m)^h) : U_i \subseteq U_n, h \in G_0 \cap (V_U)^U_g, g \in G_0 \cap (V)^U_g, gx \in hU_i \} \cap \bigcap_{i,j} \left( (X \setminus V_U) B_\alpha(z, U_i, V_j) : V_U x \cap B_\alpha(z, U_i, V_j) = \emptyset \right).$$

Next consider the intersection

$$\bigcap_{n,m} \{ X \setminus V_U B_\alpha(y, U_n, V_m) : y \in U_n, V_U x \cap B_\alpha(y, U_n, V_m) = \emptyset \}.$$

By Lemma 11 it is a complement of a union

$$\bigcup \{ B_\alpha(y, U_n, V_m) : y \in U_n, V_U x \cap B_\alpha(y, U_n, V_m) = \emptyset \}$$

To complete the proof observe that by Lemma 8.2 for any $n, m$ and $y \in U_n$ the following equivalence is true:

$$B_\alpha(y, U_n, V_m) \cap V_U x = \emptyset \text{ if and only if } B_\alpha(y, U_n, V_m) \subseteq U_n \setminus \bigcup \{ B_\alpha(gx, U_n, V_m) : g \in G_0 \cap (V)^U_g, gx \in U_n \}.$$ 

$\square$

Involving Theorem 12 in an inductive argument we can prove the following statement.

Corollary 13. Let $x, y \in X$, $V \in \mathcal{V}$, $U \in \mathcal{U}$, $\alpha \geq 1$ be an ordinal and

$$\rho(\alpha) = \begin{cases} 
2n & \text{if } \alpha = n, \text{ where } n < \omega \\
\beta + 2n & \text{if } \alpha = \beta + n, \text{ where } n < \omega \text{ and } \beta \text{ is limit.}
\end{cases}$$

Then we have $B_\alpha(x, U, V) \in \Pi^\rho_{\rho(\alpha)}(X)$. 
Finally it follows from Lemma 8(2) and Theorem 12 that for every ordinal \( \alpha \geq 1 \) the family of \( \{ B_\alpha(x, X, G) : x \in X \} \) forms a partition of the space \( X \) into invariant Borel sets which Borel rank is bounded by a countable ordinal.

**Corollary 14.** For every \( V \in \mathcal{V} \) and \( U \in \mathcal{U} \) the family \( \{ B_\alpha(x, U, V) : x \in U \} \) is a partition of \( U \) into locally \( V_\mathcal{U} \)-invariant \( \Pi^0_{\rho(\alpha)} \)-sets. In particular the family \( \{ B_\alpha(x, X, G) : x \in X \} \) is a partition of the whole space \( X \) into \( G \)-invariant \( \Pi^0_{\rho(\alpha)} \)-sets.

In the end of this section we shall prove another property of \( \alpha \)-pieces. Lemma 3(4) states that for any \( \alpha \)-piece \( B_\alpha(x, U, V) \) and any \( h \in G_0 \), the set \( hB_\alpha(x, U, V) \) is an \( \alpha \)-piece defined with respect to basic open \( hU, V^h \). Since \( \alpha \)-pieces are defined only with respect to basic open \( U, V \), the above is not true in the general case of any \( h \in G \). Instead we can prove a related property which can be viewed as a generalization of Lemma 5(3).

**Lemma 15.** Let \( \alpha > 0 \) be an ordinal, \( U, U' \in \mathcal{U}, V, V' \in \mathcal{V}, x \in U \) and \( h \in G \). If \( hx \in U' \subseteq hU \) and \( V' \subseteq V^h \), then for every ordinal \( \beta \leq \alpha \) we have

\[
B_\alpha(hx, U', V') \subseteq hB_\beta(x, U, V).
\]

**Proof.** First observe that according to Lemma 5(3) we have to consider only the case \( \alpha = \beta \). We proceed by induction on \( \alpha \) applying Proposition 9.

Assume \( \alpha = 1 \). Suppose towards contradiction that \( B_1(hx, U', V') \not\subseteq hB_1(x, U, V) \). Then there is \( y \in B_1(hx, U', V') \) which does not belong to \( hB_1(x, U, V) \). Hence by the assumption that \( U' \subseteq hU \) we have \( h^{-1}y \in U \) and \( h^{-1}y \not\in B_1(x, U, V) \). So according to Lemma 8(2) we see that \( B_1(x, U, V) \cap B_1(h^{-1}y, U, V) = \emptyset \). Then there is a basic open set \( U_n \) meeting one of the local orbits \( V_U x \), \( V_U(h^{-1}y) \) and disjoint from the other. W.l.o.g. we may assume that \( V_U x \cap U_n \neq \emptyset \) while \( V_U(h^{-1}y) \cap U_n = \emptyset \). This implies \( V_U x \subseteq V_U U_n \) and \( V_U(h^{-1}y) \cap V_U U_n = \emptyset \). Using Lemma 5(6) we get \( (V^h)_hU x \subset (V^h)_hU U_n \) while \( (V^h)_hU y \cap (V^h)_hU U_n = \emptyset \). Then by Lemma 5(1) we have \( (V')_U h x \subset (V^h)_hU U_n \) and \( (V')_U y \cap (V^h)_hU U_n = \emptyset \). Since according to Lemma 5(3) the set \( (V^h)_hU U_n \) is open, the latter conditions imply that it contains a basic open set meeting \( (V')_U h x \) and disjoint from \( (V')_U y \). This contradicts the assumption that \( y \in B_1(x, U', V') \).

For the successor step assume that for every \( n, m, k, l \in \omega \) and \( h \in G \) satisfying the conditions \( hx \in U_n \subseteq hU_k \) and \( V_m \subseteq V_l^h \) we have \( B_\alpha(hx, U_n, V_m) \subseteq hB_\alpha(x, U_k, V_l) \). Suppose towards contradiction that \( B_{\alpha+1}(hx, U', V') \not\subseteq hB_{\alpha+1}(x, U, V) \). Then there is \( y \in B_{\alpha+1}(hx, U', V') \) such that \( y \not\in hB_{\alpha+1}(x, U, V) \). Then \( h^{-1}y \not\in B_{\alpha+1}(x, U, V) \) and so \( B_{\alpha+1}(x, U, V) \) is disjoint from \( B_{\alpha+1}(h^{-1}y, U, V) \). By Proposition 9 there is an \( \alpha \)-piece \( B_\alpha(z, U_k, V_l) \) meeting exactly one of the local orbits \( V_U x \) and \( V_U(h^{-1}y) \). W.l.o.g. assume that \( B_\alpha(z, U_k, V_l) \) meets \( V_U x \). We have
$V_U x \subseteq V_U B_\alpha(z, U_k, V_l)$ and $V_U (h^{-1} y) \cap V_U B_\alpha(z, U_k, V_l) = \emptyset$. Since $x \in V_U B_\alpha(z, U_k, V_l)$, then according to Lemma 11 there are $i, j \in \omega$ such that $B_\alpha(x, U_i, V_j) \subseteq V_U B_\alpha(z, U_k, V_l)$. Hence $B_\alpha(x, U_i, V_j)$ is disjoint from $V_U (h^{-1} y)$ either, and so we may assume that $x = z$, $i = k$ and $j = l$.

Now take some $U_n, V_m$ so that $hx \in U_n \subseteq hU_k$ and $V_m \subseteq V^h$. We claim that the local orbit $(V')_U y$ is disjoint from $B_\alpha(hx, U_n, V_m)$. Otherwise the local orbit $(V^h)_h y$ is not disjoint from $B_\alpha(hx, U_n, V_m)$ either. By the inductive assumption we have $B_\alpha(hx, U_n, V_m) \subseteq hB_\alpha(x, U_k, V_l)$. Hence $(V^h)_h y$ meets $hB_\alpha(x, U_k, V_l)$ which by Lemma 6(6) implies that $V_U (h^{-1} y)$ meets $B_\alpha(x, U_k, V_l)$. This contradiction shows that the claim that $(V')_U y \cap B_\alpha(hx, U_n, V_m) = \emptyset$ is true.

Hence we can apply Proposition 9 to conclude that $y \notin B_{\alpha + 1}(hx, U', V')$. This contradicts our assumption and completes the successor step. The limit step is immediate. 

Using the lemma above we can prove the following assertion. It will be applied in the proof of Theorem 17.

**Corollary 16.** For every $U \in \mathcal{U}$, $V \in \mathcal{V}$, $x \in U$, $h \in G$ and ordinal $\alpha > 0$ the set $hB_\alpha(x, U, V)$ is a union of appropriate $\alpha$-pieces.

**Proof.** Take any $y \in hB_\alpha(x, U, V)$. According to Lemma 8(2) we may assume that $y = hx$. For some $n, m \in \omega$ we have $hx \in U_n \subseteq hU$ and $(V_m) \subseteq V^h$. Then by the lemma above we get $B_\alpha(hx, U_n, V_m) \subseteq hB_\alpha(x, U, V)$. □

The results above starting from Proposition 9 till Theorem 12 and its consequences concerning borelness of the sets $B_\alpha(x, U, V)$ is our contribution in study of $\alpha$-pieces.

### 2.2. Finer topologies.

Since every piece of the canonical partition is a $G_\delta$-subset of $X$, it is a Polish space with the topology inherited from the original Polish topology on $X$. This fact is generalized by Hjorth (see 4) who proved that for every $x \in X$ the set $B_{\gamma^2(x) + 2}(x, X, G)$ is a Polish $G$-space with respect to the topology generated by the family $\{B_{\gamma^2(x)}(x', U_n, V_m) : x' \in Gx \cap U_n, n, m \in \omega\}$.

We improve this result and show that for every ordinal $\alpha > 0$ every $\alpha$-piece of the form $B_\alpha(x, X, G)$ is a Polish $G$-space with respect to the topology generated by ‘earlier’ $\beta$-pieces and this topology generates the same Borel structure as the original topology. Our proof is based on Theorem 12 and the theorem on Borel families by Sami.

From now on we shall use the following notation for every ordinal $\beta$:

$$
B^x_0 = \mathcal{U},
$$

$$
B^x_\beta = \{B_\gamma(x', U, V) : U \in \mathcal{U}, x' \in U \cap G_\alpha x, V \in \mathcal{V}\} \text{ for } \beta > 0,
$$

$$
B^z_{< \beta} = \bigcup\{B^x_\gamma : \gamma < \beta\}.
$$
Theorem 17. Let $x \in X$ and $0 < \alpha < \omega_1$ be an ordinal. The set $B_\alpha(x, X, G)$ with the (relative) topology generated by the family $B^\leq_\alpha$ as basic open sets is a Polish $G$-space with the same Borel structure as the original topology.

Additionally for every $U \in \mathcal{U}$, $V \in \mathcal{V}$ with $x \in U$ the set $B_\alpha(x, U, V)$ is a Polish space with respect to this topology.

Proof. As we have already mentioned $B_1(x, X, G)$ is a $G_\delta$ subset of $\langle X, \tau \rangle$, thus it is a Polish space with respect to the (relative) topology generated by $U$. Therefore below we will deal only with $\alpha > 1$. We shall use the following result by Sami (see [8], Lemma 4.2).

Let $\langle X, t \rangle$ be a topological space and $0 \leq \zeta < \omega_1$. Let $F$ be a Borel family of rank $\zeta$, i.e. a family of subsets of $X$ which can be decomposed into subfamilies of two types $F = \bigcup \{P_\xi : 0 \leq \xi < \zeta\} \cup \{S_\xi : 0 \leq \xi < \zeta\}$ satisfying the following conditions:

1. $S_0$ consists of open sets,
2. $P_\zeta = \{X \setminus A : A \in S_\zeta\}$, for $0 \leq \xi < \zeta$,
3. every element of $S_\xi$ is a union of a countable subfamily of $\bigcup \{P_\eta : 0 \leq \eta < \xi\}$, for $0 \leq \xi < \zeta$.

If $\langle X, t \rangle$ is a Polish space then the topology generated by a family of intersections of finite subsets of the union $t \cup F$ is also Polish.

We start with some preliminary work. For every $0 \leq \xi < \omega_1$ we define the sets $S_\xi$ and $P_\xi$. First we put:

$S_0 = \{V_U A : A, U \in \mathcal{U}, V \in \mathcal{V}\}$,
$P_0 = \{X \setminus D : D \in S_0\}$,

$S_1 = P_0$,
$P_1 = \{X \setminus D : D \in S_1\} = S_0$,

$S_2 = \{(\bigcup \{X \setminus V_U A : A \in \mathcal{U}, x' \in V_U A\} : U \in \mathcal{U}, V \in \mathcal{V}, x' \in G_0 x)\} \cup \{(\bigcup \{V_U A : A \in \mathcal{U}, x' \notin V_U A\} : U \in \mathcal{U}, V \in \mathcal{V}, x' \in G_0 x)\}$,
$P_2 = \{X \setminus D : D \in S_2\}$.

Observe that $\bigcup \{(S_i \cup P_i) : 0 \leq i < 2\}$ is a Borel family of rank 3, where

$S_2 = \{(X \setminus B_1(x', U, V)) : U \in \mathcal{U}, x' \in U, V \in \mathcal{V}, x' \in G_0 x\}$,
$P_2 = B^*_1$. 

We proceed similarly at each successor stage. Every successor ordinal has one of the forms: \( \xi + 2n + 1 \) or \( \xi + 2n + 2 \), where \( n \) is a natural number and \( \xi = 0 \) or \( \xi \) is a limit ordinal. We define:

\[
S_{\xi+2n+1} = \left\{ \bigcup \{ B_{\xi+n}(gx', hU_i, \langle V_m \rangle^h) : U_i \subseteq U_n, h \in G_0 \cap \langle V \rangle_{U_i}^h, g \in G_0 \cap \langle V \rangle_{U_i}^{x'}, \right. \\
gx' \in hU_i \big\} : n, m \in \omega, U \in U, V \in V, x' \in G_0x \bigcup \big\{ (X \setminus U_n) \cup \bigcup \{ B_{\xi+n}(gx', U_n, V_m) : g \in G_0 \cap \langle V \rangle_{U_i}^x, gx' \in U_n \big\} : n, m \in \omega, U \in U, V \in V, x' \in G_0x \big\},
\]

\[
P_{\xi+2n+1} = \{ X \setminus D : D \in S_{\xi+2n+1} \},
\]

\[
S_{\xi+2n+2} = \{ (X \setminus B) : B \in B_{\xi+2n+1}^x \},
\]

\[
P_{\xi+2n+2} = B_{\xi+n+1}^x.
\]

Finally, for every limit \( \xi < \omega \) we put

\[
S_\xi = \{ X \setminus B : B \in B_\xi^x \},
\]

\[
P_\xi = B_\xi.
\]

We claim that for every \( 1 \leq \zeta < \omega \) the family \( \bigcup \{ (S_\xi \cup P_\xi) : \xi < \zeta \} \) is a Borel family of rank \( \zeta \). It is clear that such a family satisfies conditions 1-2 of the Sami’s theorem. We have to check that it also satisfies condition 3. We apply an inductive argument. It is obvious for \( \zeta = 1, 2 \). The case of a limit \( \zeta \) immediately follows from the inductive assumption.

For the successor step take an arbitrary \( \zeta \leq 1 \) and suppose that the family \( \bigcup \{ (S_\xi \cup P_\xi) : 0 \leq \xi < \zeta \} \) satisfies condition 3. We have to check that the family \( \bigcup \{ (S_\xi \cup P_\xi) : 0 \leq \xi < \zeta + 1 \} \) also satisfies condition 3. Since

\[
\bigcup \{ (S_\xi \cup P_\xi) : 0 \leq \xi < \zeta + 1 \} = \bigcup \{ (S_\xi \cup P_\xi) : 0 \leq \xi < \zeta \} \cup S_\zeta \cup P_\zeta,
\]

it suffices to prove that every element from \( S_\zeta \) is a union of elements of the set \( \bigcup \{ P_\xi : 0 \leq \xi < \zeta \} \). Consider two cases.

1° \( \zeta \) is a successor ordinal. Then there are unique ordinals \( \gamma \) and \( n \) such that \( n \) is a natural number, \( \gamma \) equals 0 or is a limit ordinal and \( \zeta \) has one of the following form: \( \gamma + 2n + 1 \) or \( \gamma + 2n + 2 \). In the first case the desired property follows by the definition. For the second case consider any element \( D = X \setminus B_{\gamma+n+1}(x', U, V) \) from \( S_{\gamma+2n+2} \). Applying Theorem 12 we have

\[
D = \bigcup \{ U \setminus \bigcup \{ B_{\alpha}(gx', hU_i, \langle V_m \rangle^h) : U_i \subseteq U_n, h \in G_0 \cap \langle V \rangle_{U_i}^h, g \in G_0 \cap \langle V \rangle_{U_i}^{x'}, \right. \\
gx' \in hU_i \big\} : n, m \in \omega, U \in U, V \in V, x' \in G_0x \bigcup \big\{ (X \setminus U_n) \cup \bigcup \{ B_{\alpha}(gx', U_n, V_m) : g \in G_0 \cap \langle V \rangle_{U_i}^x, gx' \in U_n \big\} \big\}
\]

Hence we see that \( D \) is a union of elements from \( P_{\gamma+2n+1} \).
2° $\zeta$ is limit. Then by Proposition 3 we have

\[ P_\zeta = \{ B : B \in B^\zeta \} = \{ \bigcap_{\xi < \zeta} B_\xi(gx, U, V) : g \in G_0, U \in U, V \in V \}, \]

\[ S_\zeta = \{ X \setminus B : B \in B^\zeta \} = \{ \bigcup_{\xi < \zeta} (X \setminus \bigcap_{\xi < \zeta} B_\xi(gx, U, V)) : g \in G_0, U \in U, V \in V \}. \]

To show that every element of $S_\zeta$ is a union of elements from $\bigcup\{ P_\xi : \xi < \zeta \}$ we shall use the following property.

**Claim 1.** If $\xi$ equals 0 or is a limit ordinal and $n$ is a natural number then $B^{\xi + n}_\zeta \subseteq S^{\xi + 2n + 1}_\zeta$.

**Proof of Claim 1.** Consider an arbitrary element $B^{\xi + n}_\zeta(x', U, V)$ from $B^{\xi + n}_\zeta$. If we substitute $U_n = U$ and $V_m = V$ in the first formula defining elements of $S^{\xi + 2n + 1}_\zeta$, then we get the set

\[ D = \bigcup\{ B^{\xi + n}_\zeta(gx', hU, (V)^h) : U_i \subseteq U \ h \in G_0 \cap \langle V \rangle_{V}^{U_i}, g \in G_0 \cap \langle V \rangle_{\zeta}, gx' \in hU_i \}. \]

The condition $(*)$ $U_i \subseteq U \ \& \ h \in G_0 \cap \langle V \rangle_{V}^{U_i} \ \& \ g \in G_0 \cap \langle V \rangle_{\zeta} \ \& \ gx' \in hU_i$ and Lemma 5 (1)-(3) imply that $D \subseteq B^{\xi + n}_\zeta(x', U, V)$. On the other hand since $h = g = 1_G$ and $U_i = U$ also satisfy condition $(*)$, thus we get $D \supseteq B^{\xi + n}_\zeta(x', U, V)$.

Applying this property and the assumption that $\zeta$ is limit we see that $B^{\xi + \zeta}_\zeta \subseteq \bigcup_{\xi < \zeta} S_\zeta$. Hence $\{ X \setminus B : B \in B^{\xi + \zeta}_\zeta \} \subseteq \bigcup_{\xi < \zeta} P_\xi$. Therefore every element of $S_\zeta$ is a countable union of elements of the set $\bigcup_{\xi < \zeta} P_\xi$.

This completes Case 2°.

Now let $\gamma$ and $k$ be the unique ordinals such that $\alpha = \gamma + k$, $k$ is a natural number and $\gamma$ equals 0 or is a limit ordinal. Define

\[ \hat{\alpha} = \begin{cases} \gamma + 2(k - 1) + 1 & \text{if } k > 0 \\ \alpha & \text{if } k = 0. \end{cases} \]

Put $\mathcal{F}_\alpha = \bigcup\{ P_\xi \cup S_\xi : 0 \leq \xi < \hat{\alpha} \}$. We have proved that $\mathcal{F}_\alpha$ is a Borel family of rank $\hat{\alpha}$ and $U \subseteq \mathcal{F}_\alpha$. Hence it is a subbase of the Polish topology finer then the initial topology generated by $U$. Since $B^{\xi + \zeta}_\zeta \subseteq \mathcal{F}_\alpha$, then $B_\alpha(x, X, G)$ is a $G_\delta$ set with respect to this topology. Thus every $B_\alpha(x, X, G)$ is also a Polish space with the inherited topology. We now show that the family $\{ B_\alpha(x, X, G) \cap B : B \in B^{\xi + \zeta}_\zeta \}$ is a basis of the topology. So we have to prove that every set of the form $B_\alpha(x, X, G) \cap D$, where $D \in \mathcal{F}_\alpha$ is a union of elements from $\{ B_\alpha(x, X, G) \cap B : B \in B^{\xi + \zeta}_\zeta \}$.

This is an immediate consequence of the following claim.
Claim 2. Let $\zeta < \beta < \alpha$. Then for every $x' \in G_0 x$, $U \in U$, $V \in V$ and $n, m \in \omega$ the sets below are unions of elements from the family $\{B_\alpha(x, X, G) \cap B : B \in B_2^\alpha\}$:

\[
B_\alpha(x, X, G) \cap B_\zeta(x', U_n, V_m),
\]

\[
B_\alpha(x, X, G) \setminus \bigcup \{B_{\zeta+n}(gx', hU_i, (V_m)^h) : U_i \subseteq U_n, h \in G_0 \cap \langle V \rangle_{U_i}^U, g \in G_0 \cap \langle V \rangle_{U_i}^{x'}, gx' \in hU_i \},
\]

\[
B_\alpha(x, X, G) \cap (U_n \setminus \bigcup \{B_\zeta(hx', U_n, V_m) : h \in G_0 \cap \langle V \rangle_{U}^{x'} \}).
\]

Proof of Claim 2. Take any $y \in B_\alpha(x, X, G) \cap B_\zeta(x', U_n, V_m)$. By Lemma $[8](1)$, (3) we see that $B_\alpha(x, X, G) \cap B_\beta(y, U_n, V_m) \subseteq B_\alpha(x, X, G) \cap B_\zeta(x', U_n, V_m)$. Since $y \in B_\alpha(x, X, G)$, then we may apply Corollary $10$ to find some $x'' \in G_0 x$ such that $B_\beta(x'', U_n, V_m) = B_\beta(y, U_n, V_m)$.

To settle the second part of this claim consider any $y \in B_\alpha(x, X, G)$ which does not belong to the union

\[
\bigcup \{B_{\zeta+n}(gx', hU_i, (V_m)^h) : U_i \subseteq U_n, h \in G_0 \cap \langle V \rangle_{U_i}^U, g \in G_0 \cap \langle V \rangle_{U_i}^{x'}, gx' \in hU_i \}.
\]

Applying the argument from the proof of Theorem $12$ we see that

\[
y \in B_\alpha(x, X, G) \setminus \bigcup \{B_\zeta(x', U_n, V_m) \}.
\]

By Lemma $[8](3)$ and Proposition $9$ we have $B_\beta(y, U, V) \subseteq X \setminus \bigcup \{B_\zeta(x', U_n, V_m) \}$. We now find $x'' \in G_0 x$ such that $B_\beta(x'', U_n, V_m) = B_\beta(y, U_n, V_m)$ and finish the proof.

Similarly we prove the last part of the claim.

We now see that $\{B_\alpha(x, X, G) \cap B : B \in B_2^\alpha\}$ generates on $B_\alpha(x, X, G)$ the (relative) topology defined by $F_\alpha$. Since every $\alpha$-piece $B_\alpha(x, U, V)$ is a $G_\delta$ subset of $B_\alpha(x, X, G)$ we see that the additional statement of this theorem is true either.

Now it suffices to show that the action $a : G \times B_\alpha(x, X, G) \to B_\alpha(x, X, G)$ is continuous with respect to each coordinate. Take an arbitrary basic open set $B = B_\beta(x', U, V) \cap B_\alpha(x, X, G)$, where $\beta < \alpha$, $U \in U$, $V \in V$ and $x' \in G_0 x \cap U$. To prove continuity with respect to the first coordinate fix some $y \in B_\alpha(x, X, G)$ and consider the set $\{h \in G : hy \in B\}$. If $h$ is an element of this set, i.e. $hy \in B$, then according to Lemma $[8](1)$ $V_U^h h \subseteq B$. Hence $\langle V \rangle_{U}^{hy}$ is an open neighbourhood of $h$ contained in $\{h \in G : hy \in B\}$.

To prove continuity with respect to the second coordinate fix some $h \in G$ and consider the set

\[
\{y \in B_\alpha(x, X, G) : hy \in B\} = h^{-1}B_\beta(x', U, V) \cap B_\alpha(x, X, G).
\]
By Corollary\textsuperscript{10} the set $h^{-1}B_\beta(x',U,V)$ is a union of $\beta$-pieces. Hence $h^{-1}B_\beta(x',U,V) \cap B_\alpha(x,X,G)$ is a union of $\beta$-pieces meeting $V_Ux$, thus it is open with respect to $t^*_\alpha$. This proves continuity with respect to the second coordinate. $\square$

From now on let $t^*_\alpha$ denote the Polish topology on $B_\alpha(x,X,G)$ described above. Observe that in the case when $\alpha$ is a successor ordinal and $\alpha = \beta + 1$, the topology $t^*_\alpha$ is also (relatively) generated by a smaller basis, namely $B^*_{\beta}$. It follows directly from Claim 2.

Since $t^*_\alpha$ is finer then the original (relative) topology on $B_\alpha(x,X,G)$ all operations and sets introduced so far can be considered with respect to $t^*_\alpha$. We shall use the superscript $t^*_\alpha$ to stress that a given object is constructed in the $G$-space $B_\alpha(x,X,G)$ with respect to the topology $t^*_\alpha$. Let us illustrate this idea.

**Example.** Take arbitrary $x \in X$ and an ordinal $\alpha > 0$. Consider $B_\alpha(x,X,G)$ as a $G$-space with respect to the topology $t^*_\alpha$. Fix some enumeration $\{D_\alpha : n \in \omega\}$ of its basis $U^*_\alpha = \{B_\alpha(x,X,G) \cap B : B \in B^*_{<\alpha}\}$. Then for every $\beta > 0$, $D \in U^*_\alpha$ and $V \in V$, we define the $\beta$-piece $B^*_{\beta}(y,D,V)$ with respect to $t^*_\alpha$ using the scheme from Proposition\textsuperscript{9} in the following way:

\[
B^*_{1}(z,D,V) = \bigcap_{n} \{V_D D_n : V_D z \cap D_n \neq \emptyset \} \cap \bigcap_{n} \{B_\alpha(x,X,G) \setminus V_D D_n : V_D z \cap D_n = \emptyset \}, \\
B^*_{\beta+1}(z,D,V) = \bigcap_{n,m} \{V_D B^*_{\beta}(y,D_n,V_m) : y \in D_n, V_D z \cap B^*_{\beta}(y,D_n,V_m) \neq \emptyset \} \cap \bigcap_{n,m} \{B_\alpha(x,X,G) \setminus V_D B^*_{\beta}(y,D_n,V_m) : y \in D_n, V_D z \cap B^*_{\beta}(y,D_n,V_m) = \emptyset \}, \\
B^*_{\lambda}(z,D,V) = \bigcap \{B^*_{\beta}(z,D,V) : \beta < \lambda\}, \text{ for } \lambda \text{ limit}. 
\]

There is a natural relationship between $\alpha$-pieces constructed with respect to the subsequent topologies.

**Proposition 18.** Let $V \in V$, $U \in U$, $x \in X$ and $x' \in Gx \cap U$. Let $\gamma, \alpha$ be ordinals such that $1 \leq \gamma < \alpha < \omega_1$. Then for every $y \in B_\alpha(x,X,G) \cap B_\gamma(x',U,V)$ and $\beta < \omega_1$ the following equality holds.

\[
B_{\alpha+\beta}(y,U,V) = \begin{cases} 
B^*_{\beta+1}(y,B_\alpha(x,X,G) \cap B_\gamma(x',U,V),V) & \text{if } \beta \text{ is finite} \\
B^*_{\beta}(y,B_\alpha(x,X,G) \cap B_\gamma(x',U,V),V) & \text{if } \beta \text{ is infinite.}
\end{cases}
\]

**Proof.** We shall give only a sketch of the proof.

By Lemma\textsuperscript{31}), (3) we see that $B_{\alpha+\beta}(y,U,V) \subseteq B_{\alpha}(x,X,G) \cap B_\gamma(x',U,V)$. Then by Corollary\textsuperscript{10} we conclude that the set $B_{\alpha+\beta}(y,U,V)$ consists of all elements $z \in B_{\alpha}(x,X,G) \cap B_\gamma(x',U,V)$ such that the local orbits $V_Uz$ and $V_uy$ intersect the same sets from $B_{<\alpha+\beta}$.

On the other hand since $B_{\alpha}(x,X,G) \cap B_\gamma(x',U,V)$ is locally $V_U$-invariant, then $V_{(B_\alpha(x,X,G) \cap B_\gamma(x',U,V))} z = V_Uz$ whenever $z \in B_{\alpha}(x,X,G) \cap B_\gamma(x',U,V)$. Applying Corollary\textsuperscript{10} to the $G$-space $\langle B_\alpha(x,X,G), t^*_\alpha \rangle$
we conclude that the set $B^{t_x}_{\beta+1}(y, B_{\alpha}(x, X, G) \cap B_{\gamma}(x', U, V), V)$ consists of all elements $z \in B_{\alpha}(x, X, G) \cap B_{\gamma}(x', U, V)$ such that the local orbits $V_U z$ and $V_U y$ intersect the same sets from $B^{t_x}_{\leq \beta+1}$.

Now the required property follows by induction on $\beta$ with use of Lemma 8.3 both for the original topology and $t_{\alpha}^x$. □

This proposition is not involved into main results of the paper. For completeness we just describe some application of it. We start with Hjorth’s generalization the notion of a Scott rank. In [4] Hjorth proves that to every $x \in X$ we can assign a cardinal invariant which can be treated as a counterpart of a Scott rank. The definition is based on the following lemma.

**Lemma 19. (Hjorth)** For every $x \in X$ there is some $\gamma < \omega_1$ such that for all $U \in U$, $V \in V$ and $x', x'' \in Gx$ we have

$$(\exists \alpha < \omega_1)(B_{\alpha}(x', U, V) \neq B_{\alpha}(x'', U, V)) \Rightarrow (B_{\gamma}(x', U, V) \neq B_{\gamma}(x'', U, V)).$$

For every $x \in X$ we denote by $\gamma^*(x)$ the least ordinal $\gamma$ satisfying the statement of Lemma 19.

It is proved in [4] that in the case $G = S_\infty$ we have $B_{\gamma^*(x)+2}(x, X, G) = Gx$, for every $x \in X$. It remains true for any closed permutation group but fails in the general case of an arbitrary Polish group $G$. Hjorth proves the following weaker statement.

**Theorem 20. (Hjorth)** For every $x \in X$, $U \in U$, $V \in V$ and $x' \geq \gamma^*(x) + 2$ we have

$$B_{\gamma^*(x)+2}(x', U, V) = B_{\alpha}(x, U, V).$$

In particular $B_{\gamma^*(x)+2}(x, X, G) = B_{\alpha}(x, X, G)$.

The following assertion is a direct consequence of Proposition 18.

**Corollary 21.** For every ordinal $1 \leq \alpha < \omega_1$ we have

$$B^{t_x}_{\alpha^*+(\gamma^*)^2}(x, X, G) = B^{t_x}_{\alpha^*}(x, X, G).$$

3. **Eventually open actions**

The generalized Scott analysis is an important tool in studying orbit equivalence relations. The result of Hjorth from [4] which we mentioned in Introduction can be expressed in terms of $\alpha$-pieces as follows:

If the system of generalized Scott invariants for a $G$-space $X$ is complete, i.e. for every $x \in X$ the piece $B_{\gamma^*(x)+2}(x, X, G)$ coincides with $Gx$, then the orbit equivalence relation induced by the $G$-action is classifiable by countable models.

In this section we study a property of continuous actions of $G$ on $X$ which, as we will see later, is equivalent to the equality $B_{\gamma^*(x)+2}(x, X, G) = Gx$ for all $x \in X$. 
Definition 22. We say that the action $G$ on $X$ is eventually open if for every $x \in X$ and $V \in \mathcal{V}$ there are $n, m \in \omega$ such that $x \in U_n$ and $(V_m)_{V_n}x \subseteq Vx$.

Observe that every Polish group has an eventually open action, e.g. the action by left multiplication on itself. Moreover if a Polish group admits a basis of open subgroups at its unity then all their continuous actions are eventually open. We now show that the converse is not true. The proof of this assertion is based on the idea described in [3].

Proposition 23. All continuous actions of $\mathbb{R}$ on Polish spaces are eventually open.

Proof. Let $X$ be a Polish $\mathbb{R}$-space, $x \in X$ and $I \subseteq \mathbb{R}$ be an open, symmetric interval. Since $S_x$, the stabilizer of $x$, is a closed subgroup of $\mathbb{R}$, then either $S_x = \mathbb{R}$ or $S_x$ is nowhere dense. In the first case we have $\mathbb{R}x = \{x\}$ and so $J_U x \subseteq Ix$ for every open $J$ and $U$. In the second case $(-\infty, 0) \setminus S_x \neq \emptyset$ and $(0, \infty) \setminus S_x \neq \emptyset$. Hence there is $U$ containing $x$ such that both sets $(-\infty, 0) \setminus \overline{U}$ and $(0, \infty) \setminus \overline{U}$ are nonempty. Then by continuity of the action we can choose $J = (-a, a) \subseteq I$ and $k, l \in \mathbb{N}$ such that $(ka, (k+2)a)x \cap \overline{U} = \emptyset$ and $(-(l+2)a, -la)x \cap \overline{U} = \emptyset$. Then we see that $J_U x \subseteq [-la, ka]x$.

We have $\{b \in \mathbb{R} : bx \in Ix\} = IS_x$, $IS_x$ is open and so $[-la, ka] \setminus IS_x$ is a compact subset of $\mathbb{R}$. Since $[-la, ka]x \cap Ix = ([-la, ka] \setminus IS_x)x$, then $[-la, ka]x \cap Ix$ is a compact subset of $X$ not containing $x$. Hence there is a basic open $U_n$ containing $x$ such that $U_n \subseteq U \setminus ([la, ka]x \setminus Ix)$. Since $J_{U_n} x \subseteq J_U x \cap U_n \subseteq [-la, ka]x \cap U_n$, then we finally conclude $J_{U_n} x \subseteq Ix$. □

Thus for Polish group actions eventual openness is a property weaker than being induced by a group admitting a basis of open subgroups at its unity. Nevertheless as we shall see below eventual openness is equivalent to completeness of the system of generalized Scott invariants.

The rest of the section is divided into two parts. The first one is devoted to local counterparts of Vaught transforms $\Delta$ and $*$ which will be applied in the second part in the proof of the result announced above. Moreover we will see later that eventually open actions are exactly those for which the map $G \to Gx$: $g \to gx$ is open with respect to $t^*_x$, for every $x \in X$. This property motivates the name. At this place note that using Theorem 20 and Lemma 3) we have the following fact.

Proposition 24. Let $G$ be a Polish group, $X$ be a Polish $G$-space and $x \in X$. The following statements are equivalent:

(i) There is $\alpha \geq 1$ such that the map $G \to Gx$: $g \to gx$ is open with respect to $t^*_x$.

(ii) The map $G \to Gx$: $g \to gx$ is open with respect to $t^*_x$.

3.1. Local Vaught transforms $\Delta_U V$ and $*_U V$. In the introductory section we saw that in some special cases local $V_U$-saturation turns Borel sets into Borel sets. In general if $A$ is a Borel set,
then $V_U A$ is analytic (recall that $VA$ is analytic whenever $A$ is Borel). Then the question of some $V_U$-local counterparts of Vaught transforms arises.

**Definition 25.** Let $V \in \bar{V}$ and $U \subseteq X$ be open. For every Borel set $A \subseteq X$ we define:

$$
A^{\Delta V(V, 1)} = (A \cap U)^\Delta V \cap U,
A^{\Delta V(V, n+1)} = (A^{\Delta V(V, n)})^\Delta V \cap U,
A^{\Delta V} = \bigcup_n A^{\Delta V(V, n)},
$$

$$
A^{*V(1)} = (\langle A \cap U \rangle \cup (X \setminus U))^{*V} \cap U,
A^{*V(n+1)} = (A^{*V(V, n)}) \cup (X \setminus U)^{*V} \cap U,
A^{*V} = \bigcap_n A^{*V(V, n)}.
$$

**Remark.** It is clear that if we substitute in the above definition $A$ by $A \cap U$, then we obtain exactly the same sets, especially $A^{\Delta V} = (A \cap U)^{\Delta V}$ and $A^{*V} = (A \cap U)^{*V}$. Therefore we can limit ourselves to subsets of $U$, while discussing properties of $V_U$-local Vaught transforms.

It is natural to ask if the sequences $(A^{\Delta V(V, n)})$ and $(A^{*V(V, n)})$ are monotone. The positive answer to this question is one of the consequences of the following statements.

**Lemma 26.** Let $V \in \bar{V}$, $U \subseteq X$ be open, $x \in U$ and $A \subseteq X$ be a Borel set. Then for every natural number $n > 0$ the following conditions are satisfied:

1. $x \in A^{\Delta V(V, n)}$ iff $x \in A^{\Delta(V)_{\bar{V}}(n)}$.
2. $x \in A^{*V(V, n)}$ iff $x \in A^{*(V)_{\bar{V}}(n)}$.

**Proof.** (1) We proceed by induction. Since $\{g \in V : gx \in A \cap U\} \subseteq \langle V \rangle_{\bar{V}}^x(1)$ and $\langle V \rangle_{\bar{V}}^x(1)$ is open, then $\{g \in V : gx \in A \cap U\}$ is nonmeager in $V$ if and only if it is nonmeager in $\langle V \rangle_{\bar{V}}^x(1)$. This settles the case $n = 1$.

Now assume that for some $n > 0$, every Borel set $B \subseteq X$ and every $y \in U$ we have $y \in B^{\Delta V(V, n)}$ iff $y \in B^{\Delta(V)_{\bar{V}}(n)}$.

If $x \in A^{\Delta V(V, n+1)}$, then according to Definition 25 $x \in U$ and there is $h \in V$ such that such that $hx \in A^{\Delta V(V, n)}$. This implies $h \in \langle V \rangle_{\bar{V}}^x(1)$ and by the inductive assumption $hx \in A^{\Delta(V)_{\bar{V}}^x(n)}$. Then there is a non-meager set $K \subseteq \langle V \rangle_{\bar{V}}^x(n)$ such that $f hx \in A$ whenever $f \in K$. Hence $K hx \subseteq A$, $K h$ is a non-meager subset of $\langle V \rangle_{\bar{V}}^{hx}(n) h$ and $\langle V \rangle_{\bar{V}}^{hx}(n) h \subseteq_{\text{open}} \langle V \rangle_{\bar{V}}^x(n+1)$. Therefore $x \in A^{\Delta(V)_{\bar{V}}(n+1)}$, which completes the forward direction.

To prove the converse suppose that $x \in A^{\Delta(V)_{\bar{V}}(n+1)}$. This means that the set $K = \{f \in G : fx \in A\}$ is nonmeager in $\langle V \rangle_{\bar{V}}^x(n + 1)$. Then the set $\hat{K} = \{(g, h) : gh \in K\}$ is nonmeager in the set $\{(g, h) : h \in \langle V \rangle_{\bar{V}}^x(1), g \in \langle V \rangle_{\bar{V}}^x(n), gh \in \langle V \rangle_{\bar{V}}^x(n + 1)\}$. By Lemma 3 we see that the latter set is an open subset of $\langle V \rangle_{\bar{V}}^x(n) \times \langle V \rangle_{\bar{V}}^x(1)$. Hence $\hat{K}$ is nonmeager in the product $\langle V \rangle_{\bar{V}}^x(n) \times \langle V \rangle_{\bar{V}}^x(1).$ On
the other hand borelness of $A$ implies borelness of the set $\hat{K}$. Thus we can apply Kuratowski-Ulam
Theorem to see that there is a non-meager set $F \subseteq \langle V \rangle_\mu^\gamma(1)$ such that for every $h \in F$ the set
$\{g \in G : ghx \in A\}$ is non-meager in $\langle V \rangle_\mu^{hx}(n)$. Hence
$$x \in (A^\Delta(V,(n+1)))^\Delta \cap U = A^\Delta(V,(n+1)).$$
This completes the proof of (1). We can prove (2) in a similar way. □

**Corollary 27.** Let $V \in \hat{V}$ and $U \subseteq X$ be open. For every Borel set $A \subseteq X$ the sequence $(A^\Delta(V,(n)))$
is increasing while the sequence $(A^\ast(V,(n)))$ is decreasing.

**Proof.** This is an easy consequence of the lemma above, Definition 2 and standard properties of
the original topological Vaught transforms. □

Let $x \in U$. Intuitively $x$ is an element of $A^\Delta(V)$ if $A$ contains some "big" part of its local $V_U$-orbit.
Similarly $x$ belongs to $A^\ast(V)$ if "almost whole" $V_Ux$ is contained in $A$. The following statement is a
precise formulation of this idea. It also follows directly from Definition 20 and Lemma 20.

**Corollary 28.** Let $V \in \hat{V}$, $U \subseteq X$ be open and $A \subseteq X$ be a Borel set. Then for every $x \in U$ the
following conditions are satisfied:

1. $x \in A^\Delta(V)$ iff $x \in A^\Delta(V,0)$.
2. $x \in A^\ast(V)$ iff $x \in A^\ast(V,0)$.

Now it is clear that the basic properties of $V_U$-local Vaught transforms are similar to the properties
of the original (topological) Vaught transforms.

**Lemma 29.** Let $V, V' \in \hat{V}$, $U \subseteq X$ be open and $x \in U$. Let $A, A_0, A_1, \ldots, A_n, \ldots$ be Borel subsets
of $X$. Then the following statements hold.

1. If $V' \subseteq V$, then $A^\Delta(V') \subseteq A^\Delta(V)$ and $A^\ast(V') \supseteq A^\ast(V)$.
2. $(U \setminus A)^\Delta(V) = U \setminus A^\Delta(V)$.
3. $(\bigcup_n A_n)^\Delta(V) = \bigcup_n (A_n^\Delta(V))$ and $(\bigcap_n A_n)^\ast(V) = \bigcap_n (A_n^\ast(V))$.
4. $x \in A^\Delta(V) \iff x \in \bigcup \{A^\Delta(V',g) \cap U : g \in G_0 \cap \langle V \rangle^x_U, V \subseteq V, V' \subseteq \langle V \rangle^x_U\}$;
   
   $x \in A^\ast(V) \iff x \in \bigcap \{A^\Delta(V',g) \cap U : g \in G_0 \cap \langle V \rangle^x_U, V \subseteq V, V' \subseteq \langle V \rangle^x_U\}.$
5. For every countable ordinal $\alpha > 0$ we have:
   
   (i) If $A \in \Sigma^0_\alpha$, then $A^\Delta(V) \in \Sigma^0_\alpha$ and $A^\ast(V) \in \Pi^0_\alpha$.
   
   (ii) If $A \in \Pi^0_\alpha$, then $A^\ast(V) \in \Pi^0_\alpha$ and $A^\Delta(V) \in \Sigma^0_\alpha$.

**Proof.** (1) and (3) are immediate consequences of the analogous properties of the original Vaught transforms.
(2) By Corollary 28(1) \( x \in (U \setminus A)^{\Delta V} \) if and only if \( x \in (U \setminus A)^{\Delta (V)_U} \). By standard properties of the original Vaught transforms, the latter is equivalent to \( x \in U \setminus A^* (V)_U \). Then by Corollary 28(2) we get \( x \in U \setminus A^* V \).

(4) Applying Corollary 28 together with the properties of the original Vaught transforms we get the equivalence

\[
(*) \quad x \in A^{\Delta V} \iff x \in \bigcup \{ (A^W) \cap U : W \subseteq \langle V \rangle_U \text{ open } \}.
\]

Since the group operation is continuous, for every basic open \( W \subseteq \langle V \rangle_U \) we can find \( V'' \subseteq V \) and \( g \in G_0 \cap \langle V \rangle_U \) such that \( V''g \subseteq W \). Then we have \( A^W \subseteq A^* (V''g) \). This completes the proof of the first equivalence. We can prove the second one in a similar way.

(5) We may easily derive it from the definition using induction and the analogous properties of the original Vaught transforms. \( \square \)

We close the discussion of \( V_U \)-local Vaught transforms with the following important property.

**Lemma 30.** For every open \( U \subseteq X \), Borel set \( A \subseteq X \) and \( V \in \hat{V} \), the following statements hold.

1. \( A^* V \subseteq A^{\Delta V} \subseteq V_U A \).
2. \( A^{\Delta V} \) and \( A^* V \) are locally \( V_U \)-invariant.
3. If \( A \) is locally \( V_U \)-invariant, then \( A^{\Delta V} = A^* V = A \cap U \).

**Proof.** (1) is immediate by Corollary 28.

(2) Accordingly to the remarks following the definition of local \( V_U \)-invariantness, we have to prove that if \( x \in A^{\Delta V} \) (\( x \in A^* V \)) then \( V_U x \subseteq A^{\Delta V} \) (resp. \( V_U x \subseteq A^* V \)).

Take an arbitrary \( x \in A^{\Delta V} \) (\( x \in A^* V \)). Then by Corollary 28 the set \( K = \{ g \in \langle V \rangle_U : gx \in A \} \) is nonmeager (comeager) in \( \langle V \rangle_U \). Thus for any \( h \in \langle V \rangle_U \) the set \( Kh^{-1} \) is nonmeager (comeager) in \( \langle V \rangle_U h_x \). By Corollary 28 again, the latter yields \( hx \in A^{\Delta V} \) (resp. \( hx \in A^* V \)). Thus we are done since \( V_U x = \langle V \rangle_U hx \).

(3) is immediate by Corollary 28 and Lemma 30(1). \( \square \)

By point (3) of the lemma above if \( A \) is locally \( V_U \)-invariant, then \( V_U A = A^{\Delta V} \). We may generalize the property as follows.

**Proposition 31.** Let \( V \in \hat{V}, U \subseteq X \) be open and \( A \subseteq U \) be a Borel set. If there are open \( U' \subseteq U \) and \( V' \in \hat{V} \) such that \( A \subseteq U' \) and \( A \) is locally \( V_U' \)-invariant, then \( V_U A = A^{\Delta V} \).

**Proof.** We have \( \supseteq \) by point (1) of the previous lemma.

To prove \( \subseteq \) take any element \( hx \) where \( x \in A \) and \( h \in \langle V \rangle_U \). Since \( A \) is locally \( V_U' \)-invariant thus \( \langle V' \rangle_U h^{-1} h_x \) is locally \( V_U' \)-invariant. Then also \( \langle V' \rangle_U h^{-1} h_x \subseteq A \). By Lemma 3 \( \langle V \rangle_U h_x = \langle V \rangle_U h^{-1} \). Thus we see that \( \{ f \in \langle V \rangle_U : fhx \in A \} \) contains an open set \( \langle V' \rangle_U h^{-1} \) which proves \( hx \in A^{\Delta V} \). \( \square \)
This lemma together with Lemma 29(5) yield the following assertion.

**Corollary 32.** Let $V \in \hat{V}, U \subseteq X$ be open, $A \subseteq U$ and $A \in \Pi^0_\alpha(X)$ for some ordinal $\alpha$. If there are open $U' \subseteq U$ and $V' \in \hat{V}$ such that $A \subseteq U'$ and $A$ is locally $V'_{U'}$-invariant, then $V_U A \in \Sigma^0_{\alpha+1}(X)$.

### 3.2. Eventual openness and generalized Scott analysis.

**Theorem 33.** Let $G$ be a Polish group and $X$ be a Polish $G$-space. The following statements are equivalent:

1. The $G$-action on $X$ is eventually open.
2. For every $V \in \mathcal{V}, U \in \mathcal{U}, x \in X$, ordinal $\alpha \geq 1$ and every locally $V_U$-invariant Borel set $A \in \Sigma^0_\alpha(X) \cup \Pi^0_\alpha(X)$ if $x \in A$, then $B_\alpha(x, U, V) \subseteq A$.
3. For every $x \in X$ we have $Gx = B_{\gamma, (x)+2}(x, X, G)$.
4. For every $x \in X$ the map $G \to Gx$: $g \to gx$ is open with respect to $\iota_{\gamma, (x)+2}$.

**Proof.** (1)$\Rightarrow$(2) We apply an inductive argument.

Consider the case $\alpha = 1$. If $A$ is a locally $V_U$-invariant open set containing $x$, then the local orbit $V_U x$ meets some basic open set $U_n \subseteq A$. Then $V_U U_n \subseteq A$ and so

$$A \supseteq \bigcap \{V_U U_i : U_i \cap V_U x \neq \emptyset\} \supseteq B_1(x, U, V).$$

If $A$ is a locally $V_U$-invariant closed set containing $x$, then we have

$$A \supseteq \bigcap \{X \setminus V_U U_n : A \cap U_n = \emptyset\} \supseteq \bigcap \{X \setminus V_U U_n : V_U x \cap U_n = \emptyset\} \supseteq B_1(x, \sigma).$$

To go through the successor step assume that for every $x' \in Gx, U_n, V_m$ and every locally $(V_m)_{U_n}$-invariant $A \in \Sigma^0_\alpha \cup \Pi^0_\alpha(X)$ we have $B_\alpha(x', U_n, V_m) \subseteq A$, whenever $x' \in A$. Then consider an arbitrary $V_U$-invariant set $A \in \Sigma^0_{\alpha+1} \cup \Pi^0_{\alpha+1}(X)$ such that $x \in A$.

1° If $A \in \Sigma^0_{\alpha+1}(X)$ then $A$ can be presented as a union $A = \bigcup A_i$ such that $\{A_i : i < \omega\} \subseteq \Pi^0_\alpha(X)$. Since $A$ is locally $V_U$-invariant, then according to Lemma 30(3) and Lemma 29(3) we have $A = A_{\Delta^0 V} = \bigcup_i A_i^{\Delta^0 V}$. There is $i < \omega$ such that $x \in A_i^{\Delta^0 V}$. Then by Lemma 29(4), there are $g \in G_0 \cap \langle V_i \rangle_{U_i}$ and $V' \subseteq V$ such that $V'g \subseteq \langle V_i \rangle_{U_i}$ and $x \in A_i^{\star(V'g)}$. This means that the set $\{h \in G : hx \in A_i\}$ is comeager in $V'g$ or equivalently the set $T = \{h \in G : hx \in A_i\}$ is comeager in $V'$. Let $G_{gx}$ be the stabilizer of $gx$. Since $T = TG_{gx}$ and $V'G_{gx} = \{h \in G : hx \in V'g\}$ thus $T$ is comeager in $\{h \in G : hx \in V'g\}$. The $G$-action is eventually open, so there are $U_n \subseteq U$ containing $gx$ and $V_m \subseteq V'$ such that $(V_m)_{U_n}gx \subseteq V'g$. Then $(V_m)_{U_n}^{\star(gx)} \subseteq \{h \in G : hx \in V'g\}$, and so $T$ is comeager in $(V_m)_{U_n}^{\star(gx)}$. By Corollary 28 the latter implies $gx \in A_i^{\star(U_n)V_m}$. Using Lemmas 29(5) and 30(3) we see that $A_i^{\star(U_n)V_m}$ is a locally $(V_m)_{U_n}$-invariant $\Pi^0_\alpha$-set. Thus we can apply the
inductive assumption to get $B_\alpha(gx, U_n, V_m) \subseteq A^{U_n V_m}_i$. On the other hand Lemmas 30(1) and 29(1) together with the assumption that $A$ is locally $V_U$-invariant imply

$$A_i^{U_n V_m} \subseteq A_i^{\Delta U_n V_m} \subseteq (V_m)_{U_n} A \subseteq A.$$

Therefore $V_U B_\alpha(gx, U_n, V_m) \subseteq A$ and so $B_{\alpha+1}(x, U, V) \subseteq A$.

2° If $A$ is a locally $V_U$-invariant $\Pi^0_{\alpha+1}$ set then $X \setminus A$ is a locally $V_U$-invariant $\Sigma^0_{\alpha+1}$ set. Suppose that $B_{\alpha+1}(x, U, V) \not\subseteq A$. Then there is $y \in B_{\alpha+1}(x, U, V)$ such that $y \in X \setminus A$. By 1° this implies $B_{\alpha+1}(y, U, V) \subseteq X \setminus A$. This by Lemma 30(1) contradicts the assumption that $x \in A$.

(2) $\Rightarrow$ (3) Since $Gx$ is an invariant Borel set, then there is an ordinal $\alpha$ such that $B_\alpha(x, X, G) \subseteq Gx$. Hence by Theorem 20 we have $B_{\gamma^*(x)+2}(x, X, G) \subseteq Gx$.

(3) $\Rightarrow$ (4) By Effros theorem on $G_\delta$-orbits.

(4) $\Rightarrow$ (1) Take an arbitrary $V \in V$. The set $Vx$ is $t_{\gamma^*(x)+2}^x$-open, so there are $n, m \in \omega$ such that $B_{\gamma^*(x)+1}(x, U_n, V_m) \cap Gx \subseteq Vx$. Hence $(V_m)_{U_n} x \subseteq Vx$. □

If $X$ is a Polish $G$-space under an eventually open action, then by the theorem of Hjorth (see Lemma 6.30. [4]) the orbit equivalence relations induced on $X$ is classifiable by countable models. Thus for every ordinal $\alpha \geq 1$ the orbit equivalence relation induced on the Polish $G$-space $\langle B_\alpha(x, X, G), t_\alpha^x \rangle$ is also classifiable by countable models. By another theorem of Hjorth (see Corollary 3.19. [4]) this in turn implies that the $G$-action on $B_\alpha(x, X, G)$ is not turbulent with respect to $t_\alpha^x$. In particular the action of $G$ on $B_{\gamma^*(x)+2}(x, X, G)$ is not turbulent with respect to $t_{\gamma^*(x)+2}^x$.

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