Dilaton transformation under abelian and non-abelian T-duality in the path-integral approach

J. De Jaegher\textsuperscript{a}, J. Raeymaekers\textsuperscript{a}, A. Sevrin\textsuperscript{b} and W. Troost\textsuperscript{a}\textsuperscript{*}

\textsuperscript{a}Instituut voor theoretische fysica, Katholieke Universiteit Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium
\textsuperscript{b}Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

ABSTRACT

We present a convenient method for deriving the transformation of the dilaton under T-duality in the path-integral approach. Subtleties arising in performing the integral over the gauge fields are carefully analysed using Pauli-Villars regularization, thereby clarifying existing ambiguities in the literature. The formalism can not only be applied to the abelian case, but, and this for the first time, to the non-abelian case as well. Furthermore, by choosing a particular gauge, we directly obtain the target-space covariant expression for the dual geometry in the abelian case. Finally it is shown that the conditions for gauging non-abelian isometries are weaker than those generally found in the literature.

\textsuperscript{*}Jeanne.DeJaegher@fys.kuleuven.ac.be, Joris.Raeymaekers@fys.kuleuven.ac.be, asevrin@tena4.vub.ac.be, Walter.Troost@fys.kuleuven.ac.be
1 Introduction

This paper is devoted to the clarification of several technical aspects of T-duality in bosonic string models in the path-integral approach (for a review, see [1], [2]).

The paper consists of three parts, in the first of which we elucidate the derivation of the dilaton transformation under (abelian) T-duality. The setup is the usual one: one gauges an isometry of a bosonic sigma model and introduces a Lagrange multiplier to constrain the corresponding field strength to vanish. Integrating over two different sets of variables in the path integral yields the original and the dual model, related by the so-called Buscher rules. The transformation of the metric and torsion potential comes from the classical contribution of the integration over the gauge fields. The dilaton transformation however is more subtle since it is a ‘quantum effect’ coming from a functional determinant which should be carefully regularized. A first attempt to account for the dilaton transformation from the path-integral point of view was made by Buscher in [3]. However, repeating this calculation, one finds that a term is missing from the result for the functional determinant. This term cannot be absorbed in a shift of the dilaton or the other background fields. We resolve this apparent catastrophe by carefully retracing the steps which lead from the gauged model back to the original one: one encounters then another functional determinant which in [3] was taken to be field independent, while in [4, 5] it was realized that this determinant should be regularized in a field dependent way so as to cancel the unwanted term. The presentation in [4, 5] was however restricted to a special class of sigma models, characterized by a static metric and zero torsion, while here it is obtained for a generic background.

For this purpose, we develop an unambiguous way to deal with the regularization of these functional determinants through the use of Pauli-Villars (PV) regularization: divergent Feynman diagrams are regularized by introducing extra massive fields called PV fields. It has the advantage that these fields enter as extra terms in the Lagrangian and are fixed once and for all. As a consequence, the dependence of the path integral measure on parameters such as the conformal factor of the 2d metric, can be read off directly from the mass terms of the PV fields [6]. Specifically, we use PV fields to regularize the gauged sigma-model. We then proceed to compute both original and dual partition functions, and we obtain the usual transformation of the dilaton field. We argue that the dilaton transformation is a regularization-independent effect not affected by ambiguities in the definition of the path integral measure for the gauged model.

In the second part of the paper we tackle the covariant derivation. We introduce a way to fix the gauge in a universal way and obtain the dual model expressed in arbitrary coordinates. These covariant T-duality rules facilitate the discussion of global issues like scaling and singularities of the dual model. Though the coordinate-independent form of the Buscher rules was previously derived in the Hamiltonian formulation [7], in which T-duality is seen as a canonical transformation, quantum effects are more easily calculable in our approach.

One can also consider target spaces which have several non-commuting isometries. The dualization procedure can be generalized to these backgrounds and is called non-abelian T-
duality [8], [9]. The dilaton transformation for this case has not yet been derived from first principles in the literature. In the third part of the paper we present such a derivation following the same method as in the abelian case. The starting point for this derivation is the action with gauged non-commuting isometries. The gauging of non-abelian isometries presented in [10] has the disadvantage of not being applicable to the abelian case. We have found that it is possible to relax the conditions on target space given in [10] such that the gauging of non-commuting isometries includes this case.

2 Abelian T-duality

In this section we will demonstrate in detail how to obtain, through a regularised computation, the T-dual partition functions for closed bosonic strings, including the quantum corrections.

A standard method to perform T-duality is the following: one gauges an isometry of a bosonic sigma-model and introduces a Lagrange multiplier. Integrating over the Lagrange multiplier yields back the original theory, while integrating over the gauge fields brings up the T-dual theory. Both describe equivalent string theories, where the strings move in different background fields which are related by the Buscher rules. These include a quantum effect, a shift of the dilaton field. The necessity for this shift is deduced from the observation that, when the original background fields solve the first order \( \beta \)-functions, the classical dual background fields may not (for a recent discussion, see e.g. [11]). In this section, we provide a fully regularized synthetic computation of this effect.

Let us start from the usual action for a closed bosonic string in a non-trivial background\(^1\) \((G_{\mu\nu}, B_{\mu\nu}, \Phi)\) with vanishing 1-loop \( \beta \)-functions. Suppose the action is invariant under the global transformation

\[
\delta X^0 = \epsilon.
\]

and that none of the background fields depends upon this coordinate. We can gauge this isometry by introducing a gauge field \( A_a \) and a Lagrange multiplier \( \chi \) which constrains the corresponding field strength to vanish. The starting point for obtaining two dual actions is the following partition function:

\[
Z[G_{\mu\nu}, B_{\mu\nu}, \Phi, X^M] = \int \frac{[dA_a][dX^0][d\chi]}{\text{gauge volume}} e^{-S_{gauged}}.
\]  

(2.1)

The \( X^M \) are treated as classical background fields throughout.

The gauged action reads

\[
S_{gauged} = \frac{1}{4\pi\alpha'} \int d^2 \sigma \sqrt{g} \left\{ \frac{1}{\sqrt{g}} B_{MN} \partial_a X^M \partial_b X^N + G_{00} \right\} \partial_a X^0 \partial_b X^0 + A_a \partial_a X^0 + A_b \partial_b X^0 + 2 \left( g^{ab} B_{0M} \right) \partial_a X^0 + A_a \partial_b X^M
\]

\footnote{We denote local world-sheet coordinates by \( \sigma^a \) and local target space coordinates by \( X^\mu = (X^0, X^M) \).}
The original string model is obtained by integrating over the Lagrange multiplier $\chi$, so that the gauge field becomes pure gauge, and gauge fixing it to 0. This leaves the $X^0$ integral. An alternative way, which makes clearer the procedure to obtain the dual theory, and opens the way to treat functional determinants less cavalierly, is to parametrize the gauge field $A_a$ in terms of two scalars $\tilde{\alpha}, \tilde{\beta}$ (Hodge decomposition)

$$A_a = \partial_a \tilde{\alpha} + \frac{\epsilon_{ac}}{\sqrt{g}} \partial_c \tilde{\beta}. \quad (2.3)$$

The gauged isometry of the action translates into the fact that it depends on the combination $X^0 + \tilde{\alpha}$, so that gauge fixing is performed simply by fixing either $X^0$ or $\tilde{\alpha}$ to some convenient value. The action for the Lagrange multiplier $\chi$ and the remaining gauge field degree of freedom $\tilde{\beta}$ contains a background independent d’Alembertian, so that integrating out these fields gives, at first sight, just a constant — but we will see that this naive expectation is not valid if we adopt the most straightforward regularization that respects 2-dimensional diffeomorphism invariance.

The dual model is obtained most simply by shifting the $A_a$-field after completing the square, while fixing the gauge on $X^0$. We find it most convenient to choose the coordinate equal to the Lagrange multiplier

$$X^0 = \chi, \quad (2.4)$$

instead of putting it to zero (as is often done): this shows more explicitly how the Lagrange multiplier $\chi$ becomes a coordinate of the dual target space\footnote{In the next section, the coordinate free formulation of this gauge choice will be shown to lead to the covariant form of the dual geometry.}. The resulting action is

$$S_{gauged} = \frac{1}{4\pi\alpha'} \int d^2\sqrt{g} \left\{ G_{00} g^{ab} A_a A_b + (g^{ab} \tilde{G}_{MN} + i \frac{\epsilon^{ab}}{\sqrt{g}} \tilde{B}_{MN}) \partial_a X^M \partial_b X^N + \alpha' R \Phi + 2(g^{ab} \tilde{G}_{0M} + i \frac{\epsilon^{ab}}{\sqrt{g}} \tilde{B}_{0M}) \partial_a \chi \partial_b X^M + g^{ab} \tilde{G}_{00} \partial_a \chi \partial_b \chi \right\}, \quad (2.5)$$

which, apart from the quadratic gauge field term, is recognised to be a bosonic sigma-model (with $\chi$ interpreted as the zeroth coordinate) in the background $(\tilde{G}_{\mu\nu}, \tilde{B}_{\mu\nu}, \Phi)$:

$$\tilde{G}_{00} = \frac{1}{G_{00}},$$

$$\tilde{G}_{0M} = \frac{B_{0M}}{G_{00}},$$

$$\tilde{G}_{MN} = G_{MN} - \frac{G_{0M} G_{0N} - B_{0M} B_{0N}}{G_{00}}.$$

$$\tilde{B}_{0M} = \frac{G_{0M}}{G_{00}},$$

$$\tilde{B}_{MN} = B_{MN} - \frac{G_{0M} B_{0N} - B_{0M} G_{0N}}{G_{00}}. \quad (2.6)$$
Thus one obtains the classical Buscher rules for the relation between the metrics. To compute the quantum corrections, we introduce a regularization.

### 2.1 Regularization

Since all functional integrals we want to perform are Gaussian, regularization is rather straightforward. We adopt a method dating back to Pauli and Vilars. It can be viewed as adding extra fields (PV-fields) to the theory that have very large masses. In loops, they cancel the divergences of the original fields. Physical results are obtained by letting the masses tend to infinity, possibly after adding further terms to the action to make the limit finite. Consistency on comparison of the original and the dual model will be achieved only if we specify the same PV-action to provide the regularization in both computations.

The following recipe constructs a PV action that is guaranteed to regularize all one-loop diagrams (with external $\phi$-lines) \[6\]:

- take the second derivative $\frac{\partial^2 S(\phi)}{\partial (\phi(x)) \partial (\phi(x')}$ of the action $S(\phi)$ for the ordinary physical fields $\phi$;
- sandwich this matrix with PV fields $\Phi$, one for every $\phi$; thus, PV fields have the same kinetic term as the field they regularize.
- choose a mass term $M^2 \Phi T \Phi$, with $T$ a (non-degenerate) matrix that may depend on the fields $\phi$.
- add these constructs to the action; include a minus sign for diagrams with a closed PV-loop. \[3\].
- if necessary for the regularization of momentum integrals, add several sets of PV-fields with identical actions but values $M^2_j$ for the masses, weighing the loops with factors $c_j$: effectively, integration over PV fields is defined through
  $\int [d\Phi_j] e^{-\int d^2 \sigma \Phi_j A \Phi_j} = (det A)^{\frac{c_j}{2}}$. \[(2.7)\]

One imposes the regularization conditions

$$\sum_j c_j = 1,$$

$$\sum_j c_j M^2_j = 0.$$ \[(2.8)\]

Implementing this for the action \[(2.2)\] with the change of variables \[(2.3)\], where three PV fields are introduced corresponding to the fields $\{X_0 + \tilde{\alpha}, \tilde{\beta}, \chi\}$, leads to an awkward non-diagonal action. A considerable simplification occurs if we decompose the gauge field $A_a$ in a slightly different way, viz.

$$A_a = \partial_a \alpha + \frac{\epsilon_a^c}{\sqrt{g} G_{00}} \partial_c \beta.$$ \[(2.9)\]

\[\text{4}\]If one wishes, the minus sign prescription may be dispensed with at the expense of introducing three fields, two fermionic and one bosonic, in stead of the one "bosonic" PV-field. Since for the present purpose this would not change anything, we stick to the simpler representation.
That such a decomposition is always possible locally can be argued as follows: one first defines a symmetric, positive bilinear inner product on the space of \( n \)-forms by

\[
(\omega, \eta) = \int G_{00} \omega \wedge \star \eta.
\]

(2.10)

This defines the hermitean conjugate \( d^\dagger \) of the operator \( d \). A generalised Hodge decomposition theorem then states that locally every form \( \omega \) can be decomposed into an exact and a coexact part. This decomposition is orthogonal with respect to the inner product (2.10). Applying this to the one-form \( A \) we get the decomposition (2.9). The parametrization (2.9) has the advantage that the \( \alpha, \beta \) cross-terms vanish so that the kinetic energy operator for \( \alpha \) and \( \beta \) is diagonal.

We now introduce the (sets of) Pauli-Villars fields \( Y_0^i, Y_1^i, Y_2^i \) to regularize integrations over \( \chi, \alpha \) and \( \beta \) respectively. These are the only integrations that need to be performed for comparison of the dual versions. We choose the mass terms to respect worldsheet reparametrization invariance, which however entails breaking two dimensional conformal invariance.

From (2.7) and (2.9) we find

\[
S_{PV}[Y_0^i] = \frac{1}{4\pi\alpha'} \sum_i \int d^2\sigma \sqrt{g} \frac{1}{G_{00}} (g^{ab} \partial_a Y_0^i \partial_b Y_0^i + M_i^2 (Y_0^i)^2),
\]

\[
S_{PV}[Y_1^i] = \frac{1}{4\pi\alpha'} \sum_i \int d^2\sigma \sqrt{g} G_{00} (g^{ab} \partial_a Y_1^i \partial_b Y_1^i + M_i^2 (Y_1^i)^2),
\]

\[
S_{PV}[Y_2^i] = \frac{1}{4\pi\alpha'} \sum_i \int d^2\sigma \sqrt{g} \frac{1}{G_{00}} (g^{ab} \partial_a Y_2^i \partial_b Y_2^i + M_i^2 (Y_2^i)^2).
\]

(2.11)

The path-integral measure is now explicitly defined by

\[
\left[ \frac{dX_0}{dA_a} | d\chi \right] \text{gauge volume} = \prod_\sigma \left( d(X_0(\sigma) + \alpha(\sigma)) \prod_j d(Y_1^j(\sigma)) \right)

\left( d(\beta(\sigma)) \prod_k d(Y_2^k(\sigma)) \right)

\left( d(\chi(\sigma)) \prod_i d(Y_0^i(\sigma)) \right).
\]

(2.12)

The path integral measures in (2.12) are (line by line) invariant under world-sheet reparametrizations and Weyl rescalings: because of the ”statistics” of the PV fields (compare (2.7)) the Jacobian for transformation of a field cancels with the transformation of the corresponding PV-field (see for example [6]). Hence, with this choice of mass terms, the partition function is invariant under reparametrizations of the world-sheet, and a possible breaking of Weyl invariance comes exclusively from the mass terms.

The world sheet reparametrization invariance permits us to (locally) choose a conformal gauge:

\[
g_{ab} = \rho \delta_{ab}.
\]

One might consider including a (formal) Jacobian factor for the change of variables from \( A_a \) to \( \alpha, \beta \). This would amount to no more than a change of normalization of the partition function for the gauged sigma model and drops out of the comparison of the original to the dual partition function. We simply adopt (2.12) as our definition.
Loop effects can be accessed via the dependence of the functional integrals on this conformal factor.

2.2 The quantum contributions

2.2.1 In the original formulation of the model

The original formulation of the sigma model is obtained by integrating over the gauge field and the Lagrange multiplier $\chi$, leaving the $X^0$ integral. Starting from the gauged action (2.1), we set $\alpha = 0$. This gauge choice implies that the $X^0$ integral is regularized by the set of Pauli-Villars fields $Y^i_1$. The $\chi$ and $\beta$ integrations are carried out, together with the integrals over $Y^i_0$ and $Y^i_2$ that regularize them. The resulting path integral contains first of all the bosonic string sigma-model action in the original background ($G_{\mu\nu}$, $B_{\mu\nu}$, $\Phi$), with isometry coordinate $X^0$, the remaining part is the regularized integral over $\beta$ and $\chi$:

$$Z = \int [dX^0] \prod_i [dY^i_1] \exp \left( -S[G_{\mu\nu}, B_{\mu\nu}, \Phi, X^\mu] - S_{PV}[Y^i_1] \right) e^{-W}$$

with

$$e^{-W} = \int [d\beta][d\chi] \prod_j [dY^j_0][dY^j_2] \exp \left( -S_{PV}[Y^j_0] - S_{PV}[Y^j_2] \right)$$

$$- \frac{1}{4\pi\alpha'} \int d^2\sigma \left\{ \frac{1}{G_{00}} \delta^{ab} \left( \partial_a \beta \partial_b \beta + 2i\partial_a \chi \partial_b \beta \right) + 2(\delta^{ab} G_{0M} + i\varepsilon^{ab} B_{0M})\partial_b X^M \frac{\varepsilon^c_a}{G_{00}} \partial_c \beta \right\}.$$  (2.14)

The kinetic energy for $\beta$ and $\chi$ can be diagonalised by completing the square in $\beta$ and defining a shifted variable $\beta' = \beta + i\chi$:

$$e^{-W} = \int [d\beta'][d\chi] \prod_j [dY^j_0][dY^j_2] \exp \left( -S_{PV}[Y^j_0] - S_{PV}[Y^j_2] \right)$$

$$- \frac{1}{4\pi\alpha'} \int d^2\sigma \left\{ \frac{1}{G_{00}} \delta^{ab} \left( \partial_a \beta' \partial_b \beta' + \partial_a \chi \partial_b \chi \right) - 2(\beta' - i\chi)e^c_a \partial_c [\left( \delta^{ab} \frac{G_{0M}}{G_{00}} + i\varepsilon^{ab} \frac{B_{0M}}{G_{00}} \right) \partial_b X^M] \right\}.$$  (2.15)

Integrating over $\beta'$, $\chi$ and their corresponding PV fields, we obtain

$$W[\rho, G_{00}] = -\sum_i \frac{c_i}{2} \text{Tr} \ln \left( \frac{\mathcal{O} + \rho M^2}{\mathcal{O}} \right),$$

with

$$\mathcal{O} = \begin{pmatrix} \delta^{ab}(-\partial_a \partial_b - \Lambda_a \partial_b) & 0 \\ 0 & \delta^{ab}(-\partial_a \partial_b - \Lambda_a \partial_b) \end{pmatrix},$$

and

$$\Lambda_a = \partial_a \ln G_{00}.$$  (2.18)
The variation of $W$ with respect to the conformal factor is calculated in appendix A by using the heat-kernel expansion \[12\]. The result can be integrated with respect to $\rho$ to give (see A.7)

$$W = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \left[ \frac{1}{12} R \Box^{-1} R + \frac{1}{4} \Box^{-1} R \Lambda^2 + \frac{1}{2} R \ln G_{00} \right] + \text{Weyl inv. terms.} \quad (2.19)$$

The first term can be seen as an extra contribution to the Liouville action coming from the gauge field and the Lagrange multiplier. Note the dependence of (2.19) on $G_{00}$, that, with this regularization, comes in through the functional integrals leading to the original sigma model. This dependence may be missed if one does these integrals naively, and in particular the term quadratic in $\Lambda$ is absent in \[3\]. The term with $\ln G_{00}$ will give rise to the dilaton shift; the other terms can not be absorbed in the dilaton, but will eventually drop out.

The computation via the conformal anomaly gives no control over possible Weyl invariant terms \[5\]. However, these are restricted to be of the form $2c_1 \int d^2\sigma \sqrt{g} g^{ab} \Lambda_a \Lambda_b$, with an undetermined coefficient $c_1$ from each operator $\delta^{ab} (-\partial_a \partial_b - \Lambda_a \partial_b)$.

Summarizing,

$$Z = \exp \left( -\frac{1}{4\pi} \int d^2\sigma \sqrt{g} \left[ \frac{1}{12} R \Box^{-1} R + \frac{1}{4} \Box^{-1} R \Lambda^2 + \frac{1}{2} R \ln G_{00} + 2c_1 \Lambda^2 \right] \right) \int [dX^0] \prod_i [dY^i_1] \exp \left( -S[G_{\mu\nu}, B_{\mu\nu}, \Phi, X^\mu] - S_{PV}[Y^i_1] \right). \quad (2.20)$$

2.2.2 In the dual formulation

The dual model is obtained after integration over $\alpha$ and $\beta$. The resulting partition function reads

$$Z = \int [d\chi] \prod_i [dY^i_0] \exp \left( -S[\tilde{G}_{\mu\nu}, \tilde{B}_{\mu\nu}, \Phi, X^\mu] - S_{PV}[Y^i_0] \right) e^{-\tilde{W}}, \quad (2.21)$$

with

$$\tilde{W}[\rho, G_{00}] = -\sum_j \frac{c_j}{2} \text{Tr} \ln \left( \frac{\tilde{O} + \rho M^2}{\tilde{O}} \right), \quad (2.22)$$

where

$$\tilde{O} = \begin{pmatrix} \delta^{ab}(\partial_a \partial_b - \Lambda_a \partial_b) & 0 \\ 0 & \delta^{ab}(\partial_a \partial_b + \Lambda_a \partial_b) \end{pmatrix}.$$ 

The variation of $\tilde{W}$ with respect to the conformal factor is calculated in appendix A, again using the heat-kernel expansion. The result can be integrated with respect to $\rho$ to give (A.8)

$$\tilde{W} = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \left[ \frac{1}{12} R \Box^{-1} R + \frac{1}{4} \Box^{-1} R \Lambda^2 \right] + \text{Weyl inv. terms.} \quad (2.23)$$

A comparison with Buscher’s computation \[3\] done in the appendix, shows that the $\Lambda^2$ was not present in his results although it should have been (A.9). Since the Weyl invariant term is symmetric under $\Lambda \to -\Lambda$, its contribution is again $2c_1 \int d^2\sigma \sqrt{g} g^{ab} \Lambda_a \Lambda_b$. The resulting
partition function at the dual side consists of the integrated anomaly and the classical dual action:

\[
Z = \exp \left( -\frac{1}{4\pi} \int d^2\sigma \sqrt{g} \left[ \frac{1}{12} R \square^{-1} R + \frac{1}{4} \square^{-1} R \Lambda^2 + 2c_1 \Lambda^2 \right] \right) \cdot \int [d\chi] \prod_i [dY^i_0] \exp \left( - S[\tilde{G}_{\mu\nu}, \tilde{B}_{\mu\nu}, \Phi, \chi, X^M] - S_{PV}[Y^i_0] \right).
\] (2.24)

2.2.3 The dilaton shift

Comparing the results in (2.20) and (2.24), the Liouville action and the terms quadratic in \( \Lambda \) are identical on both sides. This is just as well, since they could not be absorbed in any of the background fields, and would therefore constitute a breaking of duality at the quantum level. The difference that remains resides in a term that can be absorbed in a shift of the dilaton field on either side:

\[
\tilde{\Phi} = \Phi - \frac{1}{2} \ln G_{00}.
\] (2.25)

Summarizing, the identity

\[
\int [dX^0] \prod_i [dY^i_1] \exp(-S[G_{\mu\nu}, B_{\mu\nu}, \Phi, X^\mu] - S_{PV}[Y^i_1])
\]

\[
= \int [d\chi] \prod_i [dY^i_0] \exp(-S[\tilde{G}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{\Phi}, \chi, X^M] - S_{PV}[Y^i_0])
\] (2.26)

shows the equivalence of the bosonic strings propagating in dual backgrounds, both having a shift in the zeroth coordinate as global isometry.

It is important to note that this equivalence is valid to all orders, on condition that the integration over the isometry coordinate is regularized as indicated by the Pauli-Villars actions present in (2.23), spelled out explicitly in (2.11). If one prefers to perform the regularization differently, one may have to compensate for this with an additional finite counterterm. In the proof of this equivalence, we have kept the other string coordinates \( X^M \) fixed. If one also considers quantum fluctuations of these fields, they may give further quantum contributions to (for example) beta functions, but, provided one regularizes them in identical ways on both sides, this is irrelevant for the T-duality rules (2.23) and (2.6) as computed here. However, if one adopts a regularization of the \( X^M \) coordinates on the original side that is different from the dual side, counterterms have to be introduced in the form of further corrections to the T-duality rules as in [4], [13].

3 Target-space covariant abelian T-duality rules

We now proceed with the derivation of the Buscher rules in a target space covariant way. After the construction of the action where one or several commuting global isometries are gauged, we
concentrate on the gauge fixing that is needed for dualizing the model at the quantum level and present the Buscher rules in a covariant form. While the classical rules were already obtained in [7] using canonical transformations, the quantum treatment was not done up till now as no suitable gauge fixing was available.

3.1 The gauging

Consider again the action for a bosonic string in non-zero (D)-dimensional background fields

\[
S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left\{ (g^{ab}G_{\mu\nu} + i\frac{\epsilon^{ab}}{\sqrt{g}} B_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)} \Phi \right\}.
\]  

(3.1)

Following [10], the transformation \(\delta X^\mu = \varepsilon k^\mu\) is a symmetry of \(S\) if

- \(k^\mu\) is a Killing vector:

\[
L_k G_{\mu\nu} = k^\rho G_{\mu\nu,\rho} + k^\rho G_{\rho\nu} + k^\rho G_{\mu\rho} = 0.
\]  

(3.2)

- The Lie derivative of \(B_{\mu\nu}\) is a total derivative:

\[
L_k B \equiv i_k dB + d_i k B = d(v + i_k B),
\]  

(3.3)

where \(dv = i_k dB\)

- The dilaton is invariant:

\[
L_k \Phi = k^\mu \partial_\mu \Phi = 0.
\]  

(3.4)

Promoting the isometry to a gauge symmetry, \(\delta X^\mu = \varepsilon(\sigma) k^\mu\), and applying the Noether procedure results in an invariant action,

\[
S_{gauged} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left\{ g^{ab}G_{\mu\nu}(\partial_a X^\mu + A_a k^\mu)(\partial_b X^\nu + A_b k^\nu) 
+ i\epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - 2i\epsilon^{ab} A_a v_\nu \partial_b X^\nu 
+ \alpha' R^{(2)} \Phi \right\},
\]  

(3.5)

where we introduced a gauge field \(A_a\) transforming as

\[
\delta A_a = -\partial_\mu \varepsilon.
\]  

(3.6)

Invariance is easily checked once it is realized that \(v_\mu \partial_a X^\mu\) can be chosen to be invariant. Indeed, under a gauge transformation \(v_\mu \partial_a X^\mu\) transforms as a total derivative,

\[
\delta (v_\mu \partial_a X^\mu) = \partial_\mu (\varepsilon k^\mu v_\mu).
\]  

(3.7)
However, it is clear from the definition of $v$ that it is only defined up to an exact one-form. In this way we have an additional shift symmetry,

$$\delta_{shift} \left( v_\mu \partial_\alpha X^\mu \right) = \partial_\alpha h,$$

with $h$ arbitrary. For a given $\varepsilon$ we can make $v_\mu \partial_\alpha X^\mu$ invariant by making a compensating shift transformation with parameter $h = -\varepsilon k^\mu v_\mu$.

The starting point for the duality transformation is this gauged action supplemented with a Lagrange multiplier term

$$S_{lm} = \frac{i}{2\pi\alpha'} \int d^2\sigma \epsilon^{ab} A_a \partial_b \chi,$$

so that the partition function reads:

$$Z[G_{\mu\nu}, B_{\mu\nu}, \Phi, X^\mu, g^{ab}] = \int [dA_a][d\chi][d\eta] e^{-S_{gauged} - S_{lm}}.$$  \hspace{1cm} (3.10)

The role of the $\eta$ variable will be explained in the next paragraph. As in the rest of the paper, we always assume a trivial world sheet topology. Using the methods developed in [14] and [15], our analysis could be generalized to non-trivial topologies as well.

### 3.2 The gauge fixing for the dual theory

The key ingredient of our analysis is the choice of a local gauge, meaning that the gauge is fixed using fields transforming without derivatives on the transformation parameter. As a consequence, the gauge fixing determinant will be a trivial normalization factor of the path integral having no influence on the computation of the other non-trivial determinants.

The Killing vector $k^\mu$ generates the abelian isometry. One can always define a 1-form $\omega$ that satisfies

$$\omega_\mu k^\mu = 1.$$  \hspace{1cm} (3.11)

One can take $\omega$ to be exact and write it as $d$ of a coordinate in the direction of the isometry [16]:

$$\omega_\mu dX^\mu = d\eta.$$  \hspace{1cm} (3.12)

Our gauge choice is then

$$\eta = \chi.$$  \hspace{1cm} (3.13)

such that the Lagrange multiplier becomes the new isometry coordinate of the dual model. This condition can always be reached taking the transformation parameter to be

$$\varepsilon = -\eta + \chi.$$  \hspace{1cm} (3.14)

Integration over the Lagrange multiplier will then cancel the infinite volume of the gauge group. Note that this gauge choice is simply the covariant counterpart of the “natural” gauge choice of our first derivation (2.4).
3.3 The covariant Buscher rules

The calculation is now exactly the same as in adapted coordinates. The result is a set of covariant abelian T-duality rules:

\[ \tilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{k_\mu k_\nu - (v_\mu - \omega_\mu)(v_\nu - \omega_\nu)}{k^2}, \]

\[ \tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{k_\mu (v_\nu - \omega_\nu) - k_\nu (v_\mu - \omega_\mu)}{k^2}, \]

\[ \tilde{\Phi} = \Phi - \frac{1}{2} \ln k^2. \]  

This discussion is trivially generalised to backgrounds with several commuting isometries.

4 Dilaton transformation for non-abelian duality.

In this section we present a systematic derivation of the dilaton transformation for non-abelian duality \[8\][9], i.e. T-duality with respect to a non-commuting group of isometries of the background. The derivation will proceed along the same lines as in the abelian case.

4.1 Conditions for non-abelian symmetries

We start with a bosonic string sigma model on a background which we now suppose to have \( n \) independent isometry vectors \( k_\mu^i, i = 1 \ldots n \) that satisfy the algebra

\[ [k_i, k_j] = f_{ij}^k k_k, \]  

(4.1)

where the \( f_{ij}^k \) are the structure constants of the isometry group \(^6\). The action is invariant under the global transformations \( \delta X^\mu = \epsilon^i k^\mu_i \) if the following conditions are satisfied \[10\]:

- \( k^\mu_i \) are Killing vectors:
  \[ \mathcal{L}_i G = 0. \]  

(4.2)

- The Lie derivative of \( B = \frac{1}{2} B_{\mu\nu} dX^\mu \wedge dX^\nu \) is a total derivative:
  \[ \mathcal{L}_i B = d(v_i + i_i B), \]  

(4.3)

or

\[ i_i dB = dv_i. \]  

(4.4)

- The dilaton is invariant:
  \[ \mathcal{L}_i \Phi = 0. \]  

(4.5)

\(^6\)Anomaly considerations require the isometry group to have traceless generators in the adjoint representation \(^[10]\).
We used the notation $\mathcal{L}_i \equiv L_{k_i}$ and $i_i \equiv i_{k_i}$ to denote the Lie derivative and index contraction respectively. As before, the one-forms $v_i$ are only defined up to a closed form. By taking a Lie derivative of (4.4) one also finds that

$$\mathcal{L}_i v_j = f^k_{ij} v_k + w_{ij},$$

(4.6)

with $w_{ij}$ a set of unspecified closed one-forms. We also define, for later use, the set of functions $c_{ij}$ by

$$c_{ij} \equiv i_j v_i.$$

### 4.2 Gauging and further conditions

We promote the isometries to a gauge symmetry, $\delta X^\mu = \epsilon^i(\sigma) k_i^\mu$. It is important to realize that gauging non-abelian isometries imposes further conditions on the background [10]. These conditions are usually arrived at as follows: the freedom to shift the $v_i$ by a closed form is used to put the forms $w_{ij}$ to zero. This is not always possible, the integrability conditions to achieve this are derived in [10]. The Noether procedure then leads to one further condition: $c_{(ij)} = 0$. However, in the previous section we showed that for abelian isometries no conditions on $w_{ij}$ or $c_{ij}$ were required to gauge the model. This suggests that the gauging conditions usually adopted are too restrictive and can be relaxed to include the models with abelian isometries. In order to arrive at these more general conditions we refrain from fixing the shift freedom on the $v_i$ and include such shifts as compensating gauge transformations of the $v_i$, just as we did previously in the abelian models. Under a gauge transformation, $v_{j\mu} \partial_\alpha X^\mu$ transforms as

$$\delta (v_{j\mu} \partial_\alpha X^\mu) = \epsilon^i \left( f^k_{ij} v_{k\mu} + w_{ij\mu} \right) \partial_\alpha X^\mu + c_{ji} \partial_\alpha \epsilon^i.$$

(4.7)

As $v_j$ is only defined modulo an exact one-form, we get an additional shift symmetry

$$\delta_{\text{shift}} (v_{j\mu} \partial_\alpha X^\mu) = \partial_\alpha h_j,$$

(4.8)

with the functions $h_j$ completely arbitrary. A Noether procedure as in [10] determines the functions $h_j$. While doing this one finds an additional requirement: the closed one-forms $w_{ij}$ should be exact as well,

$$w_{ij} = d\lambda_{ij}.$$  

(4.9)

In fact this is only a condition on the antisymmetric part of $w_{ij}$, as one has

$$w_{(ij)} = dc_{(ij)} \quad \text{or} \quad \lambda_{(ij)} = c_{(ij)},$$

(4.10)

which can be seen by combining the variation in eq. (4.7) with the alternative expression

$$\delta (v_{j\mu} \partial_\alpha X^\mu) = \partial_\alpha (\epsilon^i c_{ji}) - 3\epsilon^i k^\rho_{ki} k^\nu_{kj} \partial_\mu B_{\nu\rho} \partial_\alpha X^\mu.$$  

(4.11)

In this way we find for given $\epsilon^i$ the following compensating shift transformation,

$$h_j = -\epsilon^i \lambda_{ij} = -\epsilon^i c_{(ij)} - \epsilon^i \lambda_{[ij]},$$

(4.12)
As a consequence the combined gauge and compensating shift transformation of \( v_{j\mu} \partial_a X^\mu \) becomes

\[
\delta (v_{j\mu} \partial_a X^\mu) = \epsilon^i f^k_{ij} v_{k\mu} \partial_a X^\mu - \partial_a \epsilon^i \left( \lambda_{[ij]} + c_{[ij]} \right),
\]

(4.13)

and \( c_{ij} \) transforms homogeneously,

\[
\delta c_{ij} = \epsilon^k \left( f^l_{kij} c_{lj} + f^l_{jk} c_{il} \right).
\]

(4.14)

The gauged action becomes

\[
S_{\text{gauged}} = \frac{1}{4 \pi \alpha'} \int d^2 \sigma \left\{ \sqrt{g} g^{ab} G_{\mu \nu} (\partial_a X^\mu - A^i_a k^{i\mu})(\partial_b X^\nu - A^i_b k^{i\nu}) \\
+ i \epsilon^{ab} B_{\mu \nu} \partial_a X^\mu \partial_b X^\nu + 2 i \epsilon^{ab} A^i_a v_{ib} \partial_b X^\nu \\
- i \epsilon^{ab} (c_{[ij]} + \lambda_{[ij]}) A^i_a A^i_b + \alpha' R^{(2)} \Phi \right\},
\]

(4.15)

which is invariant under the gauge transformations

\[
\delta X^\mu = \epsilon^i k^i, \\
\delta A^i_a = \partial_a \epsilon^i + f^l_{jk} A^j_b \epsilon^k,
\]

(4.16)

provided \( \lambda_{[ij]} \) transforms homogeneously as well,

\[
\delta \lambda_{[ij]} = \epsilon^k f^l_{kij} \lambda_{[ij]} + \epsilon^k f^l_{kji} \lambda_{[ij]}.
\]

(4.17)

This implies further conditions on the background. The transformation of \( \lambda_{[ij]} \) can be decomposed as

\[
\delta \lambda_{[ij]} = \epsilon^k L_k \lambda_{[ij]} + \delta_{\text{shift}} \lambda_{[ij]}.
\]

(4.18)

The compensating shift transformation is immediately obtained from eqs. (4.6) and (4.12) which give

\[
\delta_{\text{shift}} \lambda_{ij} = -\epsilon^k \left( L_i \lambda_{kj} - f^l_{ij} \lambda_{kl} \right).
\]

(4.19)

Combining eqs. (4.17), (4.18) and (4.19) yields the extra condition

\[
L_k \lambda_{[ij]} + L_{[j} \lambda_{k|i]} = -f^l_{ij} \lambda_{kl} + f^l_{kij} \lambda_{[ij]} - f^l_{kji} \lambda_{[ij]}.
\]

(4.20)

This condition generalises the one found in [10] and incorporates the abelian models for which it is trivially satisfied. One can also check that it is invariant under the shifts \( v_i \to v_i + df_i \) for arbitrary functions \( f_i \). The conditions obtained in [10] form a special solution to our conditions. Indeed, in [10], one required that the one-forms \( v_i \) could be chosen in such a way that \( w_{ij} = 0 \), implying that \( c_{(ij)} \) is a constant which had to vanish. In this case \( \lambda_{ij} \) can be chosen to be zero and eq. (4.20) is trivially satisfied.

We proceed by adding the Lagrange multiplier term to eq. (4.15),

\[
S_{lm} = \frac{i}{4 \pi \alpha'} \int d^2 \sigma \epsilon^{ab} F_{ab} \chi,
\]

(4.21)
where the $\chi_i$ transform as

$$\delta \chi_i = -f^i_{jk}\chi_j \epsilon^k.$$  

Again the starting point for the duality transformation is the partition function for the gauged model

$$Z[G_{\mu\nu}, B_{\mu\nu}, \Phi, X^M] = \int [dA^i_a] [dX^i] [d\chi^i] B[f_i] \det F e^{-S_{gauged} - S_{lm}},$$ \hspace{1cm} (4.22)

with $X^\mu = (X^i, X^M)$ and the $X^i$ are a subset of coordinates parametrizing the orbits of the isometry group, $B$ is a suitably chosen functional of some gauge-fixing functions $f^i$ and $\det F$ is the corresponding Fadeev-Popov determinant.

### 4.3 The classical T-dual theory

As in the abelian case, the dual model is obtained by integrating out the gauge fields and making a gauge choice which gives the Lagrange multipliers the meaning of functions on target space. The same procedure will be followed here.

The gauged action can be written as follows:

$$S_{gauged} + S_{lm} = S + \frac{1}{4\pi \alpha'} \int d^2\sigma \sqrt{g}[A^i_a f^{ab} A^j_b + h^a_i A^i_a],$$ \hspace{1cm} (4.23)

with

$$f^{ab}_{ij} = g^{ab} G_{ij} + \frac{i}{\sqrt{g}} \epsilon^{ab} D_{ij},$$

$$h^a_i = -2g^{ab} k^{ib} + 2 \frac{i}{\sqrt{g}} \epsilon^{ab} (v^{ib} - \partial_b \chi^i),$$

$$G_{ij} = k^i_{i\mu} k^j_{j\nu},$$

$$D_{ij} = -(c_{[ij]} + \lambda_{[ij]}) - f^{ik} \chi^k.$$ \hspace{1cm} (4.24)

After completing the square in the gauge fields $A^i$ the partition function reads

$$Z = \int [dA^i_a] [dX^i] [d\chi] B[f_i] \det F$$

$$\exp -\left(S + \frac{1}{4\pi \alpha'} \int d^2\sigma \sqrt{g} h^a_i (f^{-1})^{ij}_{ab} h^b_j\right)$$

$$\exp -\left(\frac{1}{4\pi \alpha'} \int d^2\sigma \sqrt{g} f^{ab}_{ij} A^i_a A^j_b\right)$$ \hspace{1cm} (4.25)

We still have to specify the gauge-fixing functions $f_i$. In contrast to the abelian case, specifying a general gauge choice that works for all gauged sigma-models is still an open problem. In the following we will consider, in analogy with the abelian case, local gauge choices of the form

$$\chi_i = \eta_i(X^\mu),$$ \hspace{1cm} (4.26)
where the $\eta_i$ are unspecified functions on target space. This means that $f_i = \chi_i - \eta_i$. Such a choice (locally) fixes the gauge if the Fadeev-Popov determinant induced by it

$$\det \mathcal{F} = \det[\mathcal{L}_j \eta_i + f^k_{ij} \eta_k],$$

(4.27)
doesn’t vanish. This determinant, being independent of the 2-dimensional metric, will not contribute to the dilaton transformation and will be ignored in the following.

A more explicit form for the classical dual action

$$S[\tilde{G}_{\mu\nu}, \tilde{B}_{\mu\nu}, \Phi] = S - \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{f} h^a_i (f^{-1})^{ij}_{ab} h^b_j,$$

(4.28)
can be found by inverting the matrix $f$:

$$(f^{-1})^{ij}_{ab} = g_{ab} (\mathcal{G} - \mathcal{D} \mathcal{G}^{-1} \mathcal{D})^{-1} ij - \frac{i \epsilon_{ab}}{\sqrt{g}} [\mathcal{G}^{-1} \mathcal{D} (\mathcal{G} - \mathcal{D} \mathcal{G}^{-1} \mathcal{D})^{-1}]^{ij}.\quad (4.29)$$

After some algebra we obtain the following form for the dual metric $\tilde{G}_{\mu\nu}$ and torsion $\tilde{B}_{\mu\nu}$:

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - [k_{ij} k_{\mu\nu} - (v_{\mu\nu} - \partial_{\mu} \eta_i)(v_{\nu\mu} - \partial_{\nu} \eta_j)] (\mathcal{G} - \mathcal{D} \mathcal{G}^{-1} \mathcal{D})^{-1} ij$$

$$+ 2k_{ij} (v_{\mu\nu} - \partial_{\mu} \eta_j)[\mathcal{G}^{-1} \mathcal{D} (\mathcal{G} - \mathcal{D} \mathcal{G}^{-1} \mathcal{D})^{-1}]^{ij},$$

$$\tilde{B}_{\mu\nu} = B_{\mu\nu} + [k_{ij} k_{\mu\nu} - (v_{\mu\nu} - \partial_{\mu} \eta_i)(v_{\nu\mu} - \partial_{\nu} \eta_j)] [\mathcal{G}^{-1} \mathcal{D} (\mathcal{G} - \mathcal{D} \mathcal{G}^{-1} \mathcal{D})^{-1}]^{ij}$$

$$- 2k_{ij} (v_{\mu\nu} - \partial_{\nu} \eta_j)(\mathcal{G} - \mathcal{D} \mathcal{G}^{-1} \mathcal{D})^{-1} ij.\quad (4.30)$$

### 4.4 Regularization

We still have to incorporate the contribution from the integral over the gauge fields

$$\int [dA^i_a] \exp \left( \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{f} f^{ab}_{ij} A^i_a A^j_b \right).\quad (4.31)$$

A suitable choice of path-integral measure for the gauge fields will reduce this calculation to the one performed in the abelian case. The $A^i_a$ can be arranged in a column vector

$$\vec{A} = \left( \begin{array}{c} A^i_1 \\ A^i_2 \end{array} \right), \quad i, j = 1 \ldots n,$$

so that (4.31) now reads

$$\int [dA^i_a] \exp \left( \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \vec{A}^T f \vec{A} \right).\quad (4.32)$$

The matrix $f$ should now be seen as a $2n \times 2n$ matrix

$$f = \left( \begin{array}{cc} \mathcal{G} & i\mathcal{D} \\ -i\mathcal{D} & \mathcal{G} \end{array} \right).$$

The eigenvalues of matrix $f$ are (at least) twofold degenerate. Indeed, making use of the relation

$$\det \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \det AD \det (1 - D^{-1} CA^{-1} B),$$
one easily shows that the characteristic polynomial \( \det(f - \lambda) \) is a complete square:

\[
\begin{align*}
\det(f - \lambda) &= \det(G - \lambda)^2 \det(1 - ((G - \lambda)^{-1}D)^2) \\
&= \det(G - \lambda + D) \det(G - \lambda - D) \\
&= \det(G - \lambda + D) \det(G - \lambda + D)^T \\
&= (\det(G + D - \lambda))^2.
\end{align*}
\]

This means that there exists a matrix \( R \) such that

\[
RfR^{-1} = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_1, \cdots, \lambda_n),
\]

where \( \lambda_i \) are the eigenvalues of the matrix \( G + D \). Defining

\[
\tilde{A} = R\tilde{A}
\]

we can, as in (2.9), locally define scalars \( \alpha^i, \beta^i \) such that

\[
\tilde{A}^i_a = \partial_a \alpha^i + \frac{e^b}{\sqrt{g_{\lambda_i}}} \partial_b \beta^i
\]

and the path integral measure is defined as

\[
[dX^i][dA^i_a][d\chi] = \prod_{\sigma} \left( d(\alpha^i(\sigma))d(\beta^i(\sigma))d(\chi(\sigma))d(X^i(\sigma)) \right)
\]

The calculation now proceeds precisely as in the abelian case. We introduce sets of PV fields \( Y_{r_0}^i, Y_{r_1}^i, Y_{r_2}^i \) with kinetic terms

\[
\begin{align*}
S_{PV}[Y_{r_0}^i] &= \frac{1}{4\pi\alpha'} \sum_r \int d^2\sigma \sqrt{g} \frac{1}{\lambda_i} (g^{ab} \partial_a Y_{r_0}^i \partial_b Y_{r_0}^i + M_{r_0}^2(Y_{r_0}^i)^2), \\
S_{PV}[Y_{r_1}^i] &= \frac{1}{4\pi\alpha'} \sum_r \int d^2\sigma \sqrt{g} \lambda_i (g^{ab} \partial_a Y_{r_1}^i \partial_b Y_{r_1}^i + M_{r_1}^2(Y_{r_1}^i)^2), \\
S_{PV}[Y_{r_2}^i] &= \frac{1}{4\pi\alpha'} \sum_r \int d^2\sigma \sqrt{g} \frac{1}{\lambda_i} (g^{ab} \partial_a Y_{r_2}^i \partial_b Y_{r_2}^i + M_{r_2}^2(Y_{r_2}^i)^2),
\end{align*}
\]

and the path integral measure is defined as

\[
[dX^i][dA^i_a][d\chi] = \prod_{\sigma} \left( d(\alpha^i(\sigma))d(\beta^i(\sigma))d(\chi(\sigma))d(X^i(\sigma)) \right)
\]

Again we neglected the overall Jacobian factor for the change of variables from \( A^i_a \) to \( (\alpha^i, \beta^i) \) since it would drop out of the final result anyway.
4.5 Quantum contributions

4.5.1 The original model

The original model is obtained by integrating over the gauge fields and the Lagrange multipliers. We choose the gauge \( \alpha^i = 0 \). As in (2.19) we encounter a contribution \( e^{-W} \) with

\[
W = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \sum_i \left[ \frac{1}{12} R - \frac{1}{4} \Box^{-1} R + \frac{1}{2} R \ln \lambda_i + 2c_1 \Lambda_i^2 \right],
\]

with \( \Lambda_i = \partial_a \lambda_i \).

4.5.2 The dual model

The same calculation as the one leading to (2.23) gives the dual contribution \( e^{-\tilde{W}} \) with

\[
\tilde{W} = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \sum_i \left[ \frac{1}{12} R - \frac{1}{4} \Box^{-1} R + \frac{1}{2} R \ln \lambda_i + 2c_1 \Lambda_i^2 \right].
\]

4.5.3 Dilaton shift

Combining contributions from the original and dual side we arrive at the final result

\[
\int [dX^i] \prod_i [dY^i] \exp(-S[G_{\mu\nu}, B_{\mu\nu}, \Phi] - S_{PV}(Y^i_1)) \]

\[
= \int [dX^i] \prod_i [dY^i_0] \exp(-\tilde{S}[\tilde{G}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{\Phi}] - S_{PV}(Y^i_0)),
\]

where the dual dilaton is given by

\[
\tilde{\Phi} = \Phi - \frac{1}{2} \ln \det(\mathcal{G} + \mathcal{D})
\]

\[
= \Phi - \frac{1}{4} \ln \det f,
\]

where \( f, \mathcal{G} \) and \( \mathcal{D} \) are defined in (4.24).

This agrees with the dilaton transformation usually adopted in the literature [8], [1]. The case of commuting isometries can easily be obtained by putting \( \mathcal{D} = 0 \). One then recovers the abelian results obtained in section 3.

5 Conclusions and outlook

In this paper, we presented an unambiguous derivation of the dilaton transformation under T-duality with respect to both abelian and non-abelian groups of isometries. Although finding the dual metric and torsion is easy enough, obtaining the dual dilaton is subtle since care is needed in defining a correct regularization and measure for the path integral. An important ingredient in our discussion was the use of Pauli-Villars regularization. This method consisting
of adding extra terms to the action is a very convenient way to fix the regulators once and for all and to keep track of them in subsequent calculations.

The main observation is that a careful definition of the path integral measure for the gauge fields leads to a nontrivial (background-dependent) contribution even when going to the original model. This implies that the dilaton transformation comes from a ratio of functional determinants, one from the original and one from the dual side. From this we also deduce that the dilaton transformation itself is a regularization independent effect, since different regulators would give extra contributions on both sides that cancel in the aforementioned ratio of determinants.

In section two we presented a gauge fixing that leads to the target-space covariant form of the Buscher rules for abelian duality. This made it possible, for the first time, to derive the dilaton shift using a manifestly covariant calculation. The key ingredient of our construction was a suitable gauge choice which has no straightforward generalisation to the non-abelian case. Though in this case specific examples have been treated in the literature (see e.g. [8], [9], [17], [18]), an explicit general gauge-fixing procedure is not known, making a general discussion of the dual geometry difficult. It might prove useful to study this problem in a restricted class of backgrounds such as the WZW models or homogeneous spaces[19].

Finally we showed in our discussion of non-abelian duality that gauging is somewhat less restricted than is generally assumed, (see (1.20)) in the literature. At this point the geometrical meaning of this condition is not obvious, nor is it clear whether it permits the gauging of interesting models which have not been considered previously. However it might very well lead to further insights in the gauge fixing procedure for the non-abelian case.

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A Calculation of regularized traces
In this appendix we calculate the regularized traces appearing the derivation of the dilaton transformation. Consider a second order differential operator of the form$^{[7]}$:

$$O(v_i) = \begin{pmatrix} \delta^{ab}(-\partial_a \partial_b + v_{1a} \partial_b + v_{2a} \partial_b) & \varepsilon^{ab} v_{3a} \partial_b \\ -\varepsilon^{ab} v_{3a} \partial_b & \delta^{ab}(-\partial_a \partial_b + v_{4a} \partial_b + v_{5a} v_{5b}) \end{pmatrix}$$

(A.1)

where the $v_i, i = 1 \ldots 5$ are arbitrary world-sheet vectors. We are interested in a quantity $W(\rho, v_i)$ with the property that

$$\frac{\delta W(\rho, v_i)}{\delta \rho} = -\sum_j \frac{c_j}{2} \mathrm{tr} \int_0^\infty d\lambda \rho^{-\lambda} \sum_n \psi_n^*(\sigma) e^{-\frac{M_j^2}{\rho^2} O} \psi_n(\sigma)$$

(A.2)

Epsilon tensors are defined by $\varepsilon^{ab} e^c = -g^{ac}, e^{ab} e^c = -\delta^{ac}$
where tr stands for a trace of 2 by 2 matrices, and the \( \psi_n \) constitute an orthonormal basis for the curved metric: \( \int d^2 \sigma p \psi_n^* \psi_m = \delta_{nm} \). The kernel
\[
\sum_n \psi_n^*(\sigma)e^{-\frac{1}{M_j}O} \psi_n(\sigma) = K\left( \frac{\lambda}{M_j^2}, \sigma, \rho^{-1}O \right) \tag{A.3}
\]
can, following Gilkey \[12\], be expanded for large \( M_j \) as follows:
\[
K\left( \frac{\lambda}{M_j^2}, \sigma, \rho^{-1}O \right) = E_0 \frac{M_j^2}{\lambda} + E_2 + \text{order} \left( M_j^{-2} \right). \tag{A.4}
\]
In casu,
\[
trE_0 = \frac{1}{2\pi}
\]
\[
trE_2 = \frac{1}{4\pi} \left( -v_2^2 - v_3^2 - \frac{1}{4}(v_1^2 + v_4^2 - 2v_3^2) + \frac{1}{2}g^{ab}\partial_a(v_{1b} + v_{4b}) - \frac{R^{(2)}}{3} \right)
\]
so that, using the relations \[2,8\] we obtain \[\text{1}\]
\[
\frac{\delta W(\rho, v_i)}{\delta \rho} = \frac{1}{4\pi} \left( \frac{R^{(2)}}{6} + \frac{1}{2}(v_2^2 + v_3^2) + \frac{1}{8}(v_1^2 + v_4^2 - 2v_3^2) - \frac{1}{4}g^{ab}\partial_a(v_{1b} + v_{4b}) \right). \tag{A.5}
\]

Using the conformal gauge expression for the world-sheet curvature, \( R = -\rho^{-1}\delta^{ab}\partial_a\partial_b \ln \rho \), this can be integrated to give
\[
W(\rho, v_i) = \frac{1}{4\pi} \int d^2 \sigma \sqrt{g} \left( \frac{1}{12}R\Box^{-1}R + \frac{1}{4}\Box^{-1}R(\sigma^2 + v_3^2) \right.
\]
\[
+ \frac{1}{8}(v_1^2 + v_4^2 - 2v_3^2) - \frac{1}{4}g^{ab}\partial_a(v_{1b} + v_{4b}) \bigg) + \text{Weyl inv. terms} \tag{A.6}
\]
We can now apply this general formula to the following special cases:

- In going to the original model, one encounters the operator \( O \) corresponding to \( v_1 = -\Lambda \); \( v_4 = -\Lambda \); \( v_2 = v_3 = v_5 = 0 \). This gives a contribution
\[
W = \frac{1}{4\pi} \int d^2 \sigma \sqrt{g} \left( \frac{1}{12}R\Box^{-1}R + \frac{1}{4}\Box^{-1}R\Lambda^2 + \frac{1}{2}R\ln G_{00} \right) + \text{Weyl inv. terms} \tag{A.7}
\]

- On the dual side, the operator \( \bar{O} \) corresponds to \( v_1 = -\Lambda \); \( v_4 = \Lambda \); \( v_2 = v_3 = v_5 = 0 \), leading to
\[
\bar{W} = \frac{1}{4\pi} \int d^2 \sigma \sqrt{g} \left( \frac{1}{12}R\Box^{-1}R + \frac{1}{4}\Box^{-1}R\Lambda^2 \right) + \text{Weyl inv. terms}. \tag{A.8}
\]

- The calculation indicated by Buscher in \[3\] makes use of a regulator with \( v_1 = v_4 = v_3 = -\Lambda \); \( v_2 = v_5 = \frac{4}{3} \). The full contribution (after partial integration) reads:
\[
W(\rho, v_i) = \frac{1}{4\pi} \int d^2 \sigma \sqrt{g} \left( \frac{1}{12}R\Box^{-1}R + \frac{1}{4}\Box^{-1}R\Lambda^2 - \frac{1}{2}R\ln G_{00} \right) + \text{Weyl inv. terms}. \tag{A.9}
\]

The term proportional to \( \Lambda^2 \) is missing in \[3\].

\[8\] One may wonder where the usual counterterms, to be divergent when \( M_j^2 \to \infty \), have gone. Remarkably, some of these have been swept under the carpet by interchanging the trace and the \( \lambda \) integral. The proper computation (without this unwarranted interchange) yields an extra divergent term \( \frac{1}{2} \sum_c c_i M_j^2 \log M_i^2 \). These terms eventually cancel in the final result and, staying with tradition, we will ignore them from now on.
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