ILL-POSEDNESS FOR THE HAMILTON-JACOBI EQUATION IN BESOV SPACES $B^0_{\infty,q}$

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Abstract. In this paper, we study the Cauchy problem for the following Hamilton-Jacobi equation

$$\begin{cases}
\partial_t u - \Delta u = |\nabla u|^2, & t > 0, \ x \in \mathbb{R}^d, \\
u(0, x) = u_0, & x \in \mathbb{R}^d.
\end{cases}$$

We show that the solution map in Besov spaces $B^0_{\infty,q}(\mathbb{R}^d)$, $1 \leq q \leq \infty$, is discontinuous at origin. That is, we can construct a sequence initial data \{u^N_0\} satisfying $\|u^N_0\|_{B^0_{\infty,q}(\mathbb{R}^d)} \to 0$, $N \to \infty$ such that the corresponding solution \{u^N\} with $u^N(0) = u^N_0$ satisfies

$$\|u^N\|_{L^\infty_T(B^0_{\infty,q}(\mathbb{R}^d))} \geq c_0, \quad \forall \ T > 0, \ N \gg 1,$n with a constant $c_0 > 0$ independent of $N$.

1. Introduction and Main Result

In our paper, we study the viscous Hamilton-Jacobi equation,

$$\begin{cases}
\partial_t u - \Delta u = |\nabla u|^p, & t > 0, \ x \in \mathbb{R}^d, \\
u(0, x) = u_0, & x \in \mathbb{R}^d.
\end{cases}$$

where $d \geq 1$, $\nabla = (\partial_{x_1}, \cdots, \partial_{x_d})$.

Note that the problem (1.1) is the special form of the following viscous Hamilton-Jacobi equation,

$$\begin{cases}
\partial_t u - \Delta u = |\nabla u|, & t > 0, \ x \in \mathbb{R}^d, \\
u(0, x) = u_0, & x \in \mathbb{R}^d.
\end{cases}$$

where $d \geq 1$, $p \geq 1$.

The viscous Hamilton-Jacobi equation owns both mathematical and physical interest. Indeed, from mathematical point of view, it is the simplest example of a parabolic PDE with a nonlinearity depending only on the first order spatial derivatives of $u$, and it describes a model for growing random interfaces, which is known as the Kardar-Parisi-Zhang equation (see [14, 15]).

An equation of the type (1.2) has attracted much attention [1, 2, 3, 7, 8, 9] in recent years. In the earliest of these papers [3], under the assumptions that $p = 1$ and $u_0 \in C^2_0(\mathbb{R}^d)$, results on the existence and uniqueness of a classical solution was obtained. Subsequently, under the hypothesis that $u_0 \in C^2_0(\mathbb{R}^d)$ and $p \geq 1$, the existence of a unique local (and actually global) classical solution of problem (1.2) was established in [2]. This result was recently extended to $u_0 \in C_0(\mathbb{R}^d)$ and $p > 0$ in [9]. Under the much weaker assumptions on $u_0$, the existence of a suitably-defined weak solution when $1 \leq p < 2$ and $u_0$ is a Radon measure was investigated in [1]. In [7], Benachour and Laurençot investigated the existence and uniqueness of nonnegative weak solutions to problem (1.2) with initial data $u_0$ in the space of bounded and non-negative measures when $1 < p < (d + 2)/(d + 1)$ and in $L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ when $(d + 2)/(d + 1) \leq p < 2$

2010 Mathematics Subject Classification. 35F21.

Key words and phrases. Hamilton-Jacobi equation, Besov spaces, Ill-posedness.
provided that $q$ is large enough. The result was generalized by Ben-Artzi, Souplet and Weissler in [8]. For more results of Hamilton-Jacobi equation, we refer the readers to see [6, 10, 12, 17, 20, 21].

To solve the original equations, we may consider the following integral equations:

$$ u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}(|\nabla u|^2)d\tau, $$

where

$$ e^{t\Delta}f = F^{-1}[e^{-|\xi|^2t}\hat{f}(\xi)]. $$

Then we can define the maps $A_n$ for $n = 1, 2, \cdots$ by the recursive formulae

$$ A_1(f) := e^{t\Delta}f, $$

$$ A_n(f) := \sum_{n_1, n_2 \geq 1, n_1 + n_2 = n} \sum_{i=1}^d \int_0^t e^{(t-\tau)\Delta}[\partial_\tau A_{n_1}(f)][\partial_\tau A_{n_2}(f)]d\tau \quad \text{for} \quad n \geq 2. $$

In this paper, motivated by [5, 11, 22], we can construct a sequence $f^N$ such that $||f^N||_{B^0_{\infty,q}(\mathbb{R}^d)} \to 0$ when $N$ tends to infinity. Here, it is easy to check that

$$ ||A_1(f^N)||_{B^0_{\infty,q}(\mathbb{R}^d)} \to 0, \quad N \to \infty. $$

The main key point, we can show that

$$ ||A_2(f^N)||_{B^0_{\infty,q}(\mathbb{R}^d)} \geq c\delta^2 \quad \text{(or} \quad c\delta^3), \quad N \gg 1, \quad \delta \ll 1, $$

for a small constant $c$ independent of $N$ and $q$. Furthermore, we also deduce that

$$ ||u[f^N] - A_1(f^N) - A_2(f^N)||_{B^0_{\infty,q}(\mathbb{R}^d)} \leq C\delta^2 \quad \text{(or} \quad C\delta^4), \quad N \gg 1, \quad \delta \ll 1, $$

for a big constant $C$ independent of $N$ and $q$. This show that (1.1) is ill-posedness in Besov spaces $B^0_{\infty,q}(\mathbb{R}^d)$.

Our main ill-posedness result for Hamilton-Jacobi equation in Besov spaces reads as follows:

**Theorem 1.1.** Let $d \geq 1$ and $s > 1 + \frac{d}{2}$. For $1 \leq q \leq \infty$, Eq. (1.1) is ill-posedness in $B^0_{\infty,q}(\mathbb{R}^d)$ in the sense that the solution map is discontinuous at origin. More precisely, there exist a sequence of initial data $\{f^N\}_{N \geq 1} \subset H^s(\mathbb{R}^d)$ and a sequence of time $\{t_N\}_{N \geq 1}$ with

$$ ||f^N||_{B^0_{\infty,q}(\mathbb{R}^d)} \to 0, \quad t_N \to 0, \quad N \to \infty, $$

such that the corresponding sequence solutions $u[f^N] \in C([0,\infty); H^s(\mathbb{R}^d))$ to initial value problem (1.1) with $u[f^N](0) = f^N$ satisfies

$$ ||u[f^N](t_N)||_{B^0_{\infty,q}(\mathbb{R}^d)} \geq c_0, $$

with a positive constant $c_0$ independent of $N$.

Our paper is organized as follows. In Section 2, we give some preliminaries which will be used in the sequel. In Section 3, we will give the proof of Theorem 1.1.

**Notation.** In the following, since all spaces of functions are over $\mathbb{R}^d$, for simplicity, we drop $\mathbb{R}^d$ in our notations of function spaces if there is no ambiguity. Let $C \geq 1$ and $c \leq 1$ denote constants which can be different at different place. We use $A \lesssim B$ to denote $A \leq CB$. 


2. Preliminaries

In this section we collect some preliminary definitions and lemmas. For more details we refer the readers to [4].

Let \( \varphi : \mathbb{R}^d \to [0, 1] \) be a radial and smooth function satisfying \( \text{Supp} \, \varphi \subset C \triangleq \{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \} \), \( \varphi \equiv 1 \) for \( \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \) and
\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j}|\xi|) = 1,
\]
for all \( |\xi| \neq 0 \). We set \( \Psi(\xi) = 1 - \sum_{j \geq 0} \varphi(2^{-j}|\xi|) \). For \( u \in S' \), \( q \in \mathbb{Z} \), we define the Littlewood-Paley operators: \( \Delta_q u = F^{-1}(\varphi(2^{-q} \cdot) F u) \), \( \Delta_q u = \hat{\Delta}_q u \) for \( q \geq 0 \), \( \Delta_q u = 0 \) for \( q \leq -2 \) and \( \Delta_{-1} u = F^{-1}(\Psi(\xi) F u) \), and \( \hat{S}_q u = F^{-1}(\Psi(2^{-q}\xi) F u) \). Here we use \( F(f) \) or \( \hat{f} \) to denote the Fourier transform of \( f \).

Applying the above decomposition, the Besov spaces \( B^s_{p,r}(\mathbb{R}^d) \), the homogeneous Fourier–Besov spaces \( \dot{B}^s_{p,r}(\mathbb{R}^d) \) and the Chemin–Lerner type spaces \( \widetilde{L}^\lambda(0,T;\dot{B}^s_{p,r}(\mathbb{R}^3)) \) can be defined as follows:

**Definition 2.1.** ([4]) Let \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), the Besov space \( B^s_{p,r}(\mathbb{R}^d) \) or \( \dot{B}^s_{p,r}(\mathbb{R}^d) \) is defined by
\[
B^s_{p,r}(\mathbb{R}^d) = \left\{ f \in S'(\mathbb{R}^d) : \|f\|_{B^s_{p,r}} := \left\| \left( 2^{js} \| \Delta_j u \|_{L^r} \right)_{j \in \mathbb{Z}} \right\|_{L^p} < \infty \right\},
\]

or
\[
\dot{B}^s_{p,r}(\mathbb{R}^d) = \left\{ f \in S'_h(\mathbb{R}^d) : \|f\|_{\dot{B}^s_{p,r}} := \left\| \left( 2^{js} \| \Delta_j u \|_{L^r} \right)_{j \in \mathbb{Z}} \right\|_{L^p} < \infty \right\}.
\]

Here, \( u \in S'_h \) denotes \( u \in S' \) and \( \lim_{j \to -\infty} \|\hat{S}_j u\|_{L^\infty} = 0 \).

**Definition 2.2.** ([11, 18]) Let \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), the space \( \dot{B}^s_{p,r}(\mathbb{R}^d) \) is defined to be the set of all tempered distributions \( f \in S'(\mathbb{R}^d) \) such that \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) and the following norm is finite:
\[
\|f\|_{\dot{B}^s_{p,r}} := \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \| \Delta_j f \|_{L^r} \right)^\frac{1}{r} \quad \text{if} \quad 1 \leq r < \infty,
\]
\[
\sup_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_{L^r} \quad \text{if} \quad r = \infty.
\]

**Definition 2.3.** For \( 0 < T \leq \infty \), \( s \in \mathbb{R} \) and \( 1 \leq p, r, \lambda \leq \infty \), we set (with the usual convention if \( r = \infty \)):
\[
\|f\|_{\widetilde{L}^\lambda(\dot{B}^s_{p,r})} := \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \| \Delta_j f \|_{L^\lambda(0,T;L^r)} \right)^\frac{1}{r}.
\]
We then define the space \( \widetilde{L}^\lambda(0,T;\dot{B}^s_{p,r}(\mathbb{R}^d)) \) as the set of temperate distributions \( f \) over \((0,T) \times \mathbb{R}^3 \) such that \( \lim_{j \to -\infty} S_j f = 0 \) in \( S'(0,T) \times \mathbb{R}^d \) and \( \|f\|_{\widetilde{L}^\lambda(\dot{B}^s_{p,r})} < \infty \).

Next we recall Bony’s decomposition from [4].
\[
uv = \hat{T}_u v + \hat{T}_v u + \hat{R}(u,v),
\]
with
\[
\hat{T}_u v \triangleq \sum_j \hat{S}_{j-1} u \hat{\Delta}_j v, \quad \hat{R}(u,v) \triangleq \sum_j \sum_{|k-j| \leq 1} \hat{\Delta}_j u \hat{\Delta}_k v.
\]
This is now a standard tool for nonlinear estimates. Now we use Bony’s decomposition to prove some nonlinear estimates which will be used in the proof of our theorem.
Lemma 2.5. Let $1 \leq r \leq 2$. Then, there hold
\[
||\hat{T}_u v||_{FB_{1,r}^0} \leq C ||u||_{FB_{1,r}^{-1}} ||v||_{FB_{1,r}^1}, \quad ||\hat{R}(u, v)||_{FB_{1,r}^0} \leq C ||u||_{FB_{1,r}^{-1}} ||v||_{FB_{1,r}^1},
\]
and
\[
||\hat{T}_u v||_{L^1_{r}(FB_{1,r}^0)} \leq C ||u||_{L^1_{r}(FB_{1,r}^{-1})} ||v||_{L^1_{r}(FB_{1,r}^1)},
\]
\[
||\hat{R}(u, v)||_{L^1_{r}(FB_{1,r}^0)} \leq C ||u||_{L^1_{r}(FB_{1,r}^{-1})} ||v||_{L^1_{r}(FB_{1,r}^1)}.
\]

Proof. Note that
\[
|| \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v ||_{FB_{1,r}^0} \leq || \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v ||_{FB_{1,1}^0}
\]
\[
\leq C \sum_{j \in \mathbb{Z}} ||\dot{S}_{j-1} u \dot{\Delta}_j v||_{L^1}
\]
\[
\leq \sum_{j \in \mathbb{Z}} (2^{-j} ||\dot{S}_{j-1} u||_{L^1}) \cdot (2^j ||\dot{\Delta}_j v||_{L^1})
\]
\[
\leq C ||u||_{FB_{1,r}^{-1}} ||v||_{FB_{1,r}^1},
\]
and
\[
|| \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}_j v ||_{FB_{1,r}^0} \leq || \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}_j v ||_{FB_{1,1}^0}
\]
\[
\leq || \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}_j v||_{L^1}
\]
\[
\leq \sum_{j \in \mathbb{Z}} \sum_{|i-j| \leq 1} ||\dot{\Delta}_j u||_{L^1} ||\dot{\Delta}_j v||_{L^1}
\]
\[
\leq C \sum_{j \in \mathbb{Z}} 2^{-2j} ||\dot{\Delta}_j u||_{L^1}^2 ||v||_{FB_{1,2}^1}
\]
\[
\leq C ||u||_{FB_{1,r}^{-1}} ||v||_{FB_{1,r}^1}.
\]

This completes the proof. \hfill \square

Lemma 2.4. Let $d \geq 1$ and $s > 1 + \frac{d}{2}$. Assume that $u \in C([0, T]; H^s)$ be the solution of Eq. (1.1). Then, we have for $i = 1, 2, \cdots, d$,
\[
\||\int_0^t e^{(t-r)\Delta} (\partial_i u \partial_j v) d\tau||_{L^2_T(FB_{1,2}^0)} + ||\int_0^t e^{(t-r)\Delta} (\partial_i u \partial_j v) d\tau||_{L^2_T(FB_{1,2}^3)} \leq C (||u||_{L^2_T(FB_{1,2}^0)} ||v||_{L^2_T(FB_{1,2}^3)} + ||v||_{L^2_T(FB_{1,2}^0)} ||u||_{L^2_T(FB_{1,2}^3)}),
\]
and
\[
\||\int_0^t e^{(t-r)\Delta} (\partial_i u \partial_j v) d\tau||_{X_T \cap L^2_T(BMO)} \leq C ||u||_{X_T} ||v||_{X_T},
\]
where
\[
||u||_{X_T} = \sup_{0 < t \leq T} |t|^\frac{2}{3} ||\nabla u(t, \cdot)||_{L^\infty} + \sup_{x \in \mathbb{R}^d, 0 < R \leq T^\frac{1}{4}} |B(x, R)|^{-\frac{1}{2}} ||\nabla u(t, y)||_{L^2_{t,y}([0, R^2] \times B(x, R))}.
\]

Proof. To prove the first inequality, it is enough to prove the following inequality:
\[
\||\int_0^t e^{(t-r)\Delta} f d\tau||_{L^2_T(FB_{1,2}^0)} \leq C ||f||_{L^2_T(FB_{1,2}^0)},
\]
(2.1)
for all $r \geq 1$. Indeed, setting $f = \partial_t u \partial_d v$ and applying Lemma 2.4, we have
\[
\| \int_0^t e^{(t-r)\Delta}(\partial_t u \partial_d v) d\tau \|_{L^p_T(F^s_{2,1})} + \| \int_0^t e^{(t-r)\Delta}(\partial_t u \partial_d v) d\tau \|_{L^p_T(F^s_{2,1})} \\
\leq C\|\partial_t u \partial_d v\|_{L^p_T(F^s_{1,1})} \\
\leq C(\|\partial_t u\|_{L^p_T(F^s_{1,1})}\|\partial_d v\|_{L^p_T(F^s_{1,1})} + \|\partial_d v\|_{L^p_T(F^s_{1,1})}\|\partial_t u\|_{L^p_T(F^s_{1,1})}) \\
\leq C(\|u\|_{L^p_T(F^s_{1,1})}\|v\|_{L^p_T(F^s_{1,1})} + \|v\|_{L^p_T(F^s_{1,1})}\|u\|_{L^p_T(F^s_{1,1})}).
\]

Now we prove (2.1). By Young’s inequality, we obtain
\[
\| \int_0^t e^{(t-r)\Delta}\widetilde{\Delta}_j f d\tau \|_{L^1_T} \\
\leq C\int_0^t e^{2^{2}(t-r)\|\widetilde{\Delta}_j f\|_{L^1_T}} d\tau \\
\leq C2^{-\frac{2}{p_j}}\|\widetilde{\Delta}_j f\|_{L^1_T}.
\]

This implies that inequality (2.1) holds.

The proof of the second inequality is essentially coming from [16, 19, 23] and we omit it here.

\[\square\]

**Lemma 2.6.** For any $p, \rho, \sigma \in [1, \infty]$, $q \in [1, \infty)$ and $s > \frac{d}{q}$, there exists a constant $C$ depending only on $d, p$ and $q$, but not on $\rho, \sigma$ such that for $u \in B^{\frac{d}{q}}_{p,\rho} \cap B^{s}_{q,\sigma}$, we have
\[
\|u\|_{L^\infty} \leq C(1 + \|u\|_{B^{\frac{d}{q}}_{p,\rho}}) (\ln(1 + \|u\|_{B^{s}_{q,\sigma}}))^{1-\frac{1}{2}}.
\]

**Proof.** See Theorem 2.1 in [13]. \[\square\]

**Lemma 2.7.** Assume $u_0 \in H^s, s > 1 + \frac{d}{2}$, then there exists a positive time $T_{u_0} \geq \frac{1}{C_s\|u_0\|_{H^s}}$ such that Eq. (1.1) have a local solution $u \in C([0, T_{u_0}); H^s)$. Moreover, we have the following blow criterion
\[
\int_0^{T_{u_0}} \|\nabla u\|_{L^\infty}^2 dt = +\infty, \quad \text{or} \quad \int_0^{T_{u_0}} \|\nabla u\|_{L^\infty}^2 dt = +\infty.
\]

**Proof.** We begin by formulating a mollified version of the Cauchy problem (1.1) sa follows:
\[
\partial_t J_n u - \Delta J_n u = |\nabla J_n u|^2, \quad J_n u|_{t=0} = J_n u_0.
\]

Therefore (2.2) defines an ODE on $H^s$ and thus has a unique solution $u_n \in C([0, T_n]; H^s)$. We also have $u_n = J_n u_n$. Applying the operator $D^s$ to both sides of , multiplying both sides of the resulting equation by $D^s u_n$ and then integrating over $\mathbb{R}^d$ yields the following $H^s$ energy of $u_n$ identity:
\[
\frac{1}{2} \frac{d}{dt} \|u_n\|_{H^s}^2 + \|\nabla u_n\|_{H^s}^2 \leq \|\nabla u_n\|_{H^s}^2 \|u_n\|_{H^s} \\
\leq C_s \|\nabla u_n\|_{H^s} \|\nabla u_n\|_{L^\infty} \|u_n\|_{H^s} \\
\leq \frac{1}{2} \|\nabla u_n\|_{H^s}^2 + C_s \|u_n\|_{H^s}^4.
\]

Solving differential inequality gives
\[
\|u_n(t)\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{\sqrt{1 - 2C_s\|u_0\|_{H^s}^2}}.
\]
Setting $T := \frac{1}{4C_s|u_0|_{H^s}}$, we see from that the solution $u_n$ exists for $0 \leq t \leq T$ and satisfies a solution size bound
\[ ||u_n(t)||_{H^s} \leq 2||u_0||_{H^s}, \quad 0 \leq t \leq T. \]

By a standard compactness argument, we deduce that (1.1) has a unique solution $u \in \mathcal{C}([0, T]; H^s)$.

Now, we prove the corresponding blow up criterion. For simplicity, we just prove the second one. Let us assume that
\[ \int_0^{T_0} \|\nabla u\|^2_{B^{0}_{\infty,2}} \, dt < +\infty. \]

Let $u_0 \in H^s$ and $u \in \mathcal{C}([0, T_0]; H^s)$ be the corresponding solution of Eq. (1.1). Then, applying Lemma 2.6 and adopting the similar argument as in (2.3), we have for all $t \in [0, T_0)$,
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^s}^2 + \|\nabla u\|_{H^s}^2 \leq \|\nabla u^2\|_{H^s} \|u\|_{H^s} \leq C_s \|\nabla u\|_{H^s} \|\nabla u\|_{L^\infty} \|u\|_{H^s} \leq \frac{1}{2} \|\nabla u\|_{H^s}^2 + C_s \|\nabla u\|_{L^\infty}^2 \|u\|_{H^s}^2 \leq \frac{1}{2} \|\nabla u\|_{H^s}^2 + C_s \|u\|_{H^s}^2 (1 + \|\nabla u\|^2_{B^{0}_{\infty,2}} \ln(e + \|u\|_{H^s})),
\]

which leads to
\[ \|u(t)\|_{H^s} \leq \|u_0\|_{H^s} \exp\{C_s \int_0^t (1 + \|\nabla u\|^2_{B^{0}_{\infty,2}} \ln(e + \|u\|_{H^s})) \, d\tau\}. \]

Using the Gronwall inequality again, we get for all $t \in [0, T_{u_0})$,
\[ \|u(t)\|_{H^s} \leq (e + \|u_0\|_{H^s}) \exp\{C_s \int_0^t (1 + \|\nabla u\|^2_{B^{0}_{\infty,2}}) \, d\tau\}. \]

Therefore, we can extend the solution past time $T_{u_0}$ which contradicts the assumption. This completes the proof of Lemma 2.7.

\[
\square
\]

3. Proof of the main theorem

In this section, we will give the details for the proof of the theorem.

Define a smooth function $\chi$ with values in $[0, 1]$ which satisfies
\[
\chi(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq \frac{1}{4}, \\
0, & \text{if } |\xi| \geq \frac{1}{2}.
\end{cases}
\]

For the cases $2 < q \leq \infty$. Letting $\chi_j^\pm(\xi) = \chi(\xi \mp 2^j \vec{e})$ for $j \in \mathbb{Z}$, where $\vec{e} = (1, 1, \cdots, 1)$, then we construct a sequence $\{f^N\}_{N=1}^\infty$ by its Fourier transform
\[ \hat{f}^N(\xi) = \frac{\delta}{N^\frac{d}{2}} \sum_{j=N}^{2N} [\chi_j^+(\xi) + \chi_j^-(\xi)]. \]

It is easy to see that
\[ f^N = \frac{2\delta}{N^\frac{d}{2}} \sum_{j=N}^{2N} \cos(2^j x \cdot \vec{e}) \hat{\chi}(x). \]
By the definition of Besov spaces, we have

\[ \|f^N\|_{B^0_{\infty,q}} \leq \|f^N\|_{F^0_{B^1_{\infty,q}}} \leq C \frac{\delta}{N^{\frac{\alpha}{2}}} \left( \sum_{j=0}^{2N} \|\chi(x)\|_{L^q_j} \right)^{\frac{\alpha}{q}} \leq C\delta \frac{1}{N^{\frac{\alpha}{2} - \frac{\alpha}{q}}}, \]

which implies \( \|f^N\|_{F^0_{B^1_{\infty,q}}} \leq C\delta \) and

\[ \|f^N\|_{B^0_{\infty,q}} \to 0, \quad N \to 0. \]

According to Lemma 2.5, we have for all \( t \in [0, T_f \wedge) \),

\[ \|u[f^N]\|_{L^\infty_t(F^0_{B^1_{\infty,q}})} + \|u[f^N]\|_{L^1_t(F^0_{B^1_{\infty,q}})} \leq C\|f^N\|_{F^0_{B^1_{\infty,q}}} + C\|u[f^N]\|_{L^\infty_t(F^0_{B^1_{\infty,q}})} \|u[f^N]\|_{L^1_t(F^0_{B^1_{\infty,q}})}, \tag{3.1} \]

Choosing \( \delta > 0 \) small enough, we infer from (3.1) that

\[ \|u[f^N]\|_{L^\infty_t(F^0_{B^1_{\infty,q}})} + \|u[f^N]\|_{L^1_t(F^0_{B^1_{\infty,q}})} \leq 2C\|f^N\|_{F^0_{B^1_{\infty,q}}} \leq 2C\delta. \]

Therefore, by Lemma 2.7, we deduce that \( u[f^N] \in C([0, \infty); H^s) \). Now, we need to estimate the second iteration. Let us first to rewrite second iteration as follows:

\[ A_2(f^N) = \int_0^t e^{(t-\tau)\Delta} \sum_{i=1}^d [\partial_i e^{\tau \Delta} f^N]^2 \, d\tau. \]

By the definition of Besov spaces, we have

\[ \|\tilde{A}_{-4}[A_2(f^N)]\|_{L^\infty} \geq c \left( \int_{\mathbb{R}^d} \varphi(16\xi) \mathcal{F}[A_2(f^N)](\xi) \, d\xi \right)^{\frac{1}{2}} \geq c \left( \int_{\mathbb{R}^d} \int_0^t e^{-((t-\tau)|\xi|^2)} \varphi(16\xi) \sum_{i=1}^d \mathcal{F} \left( \partial_i e^{\tau \Delta} f^N \right)(\xi) \, d\tau \, d\xi \right)^{\frac{1}{2}}. \tag{3.2} \]

Since \( \text{Supp} \ \chi(\cdot - a) \ast \chi(\cdot - b) \subset B(a + b, 1) \), we see that

\[ \text{Supp} \ \chi^\mu_\ell \ast \chi^\mu_m \cap B(0, \frac{1}{2}) = \emptyset, \quad N \leq \ell, m \leq 2N, \ \mu \in \{+, -\}, \]

\[ \text{Supp} \ \chi^\pm_\ell \ast \chi^\pm_m \cap B(0, \frac{1}{2}) = \emptyset, \quad N \leq \ell, m \leq 2N, \ \ell \neq m. \tag{3.3} \]

According to (3.3), we have

\[ J(t, \xi) := \varphi(16\xi) \sum_{i=1}^d \mathcal{F} \left( [\partial_i e^{\tau \Delta} f^N]^2 \right)(\xi) \]

\[ = -\frac{\delta^2}{N} \varphi(16\xi) \sum_{i=1}^d \int_{\mathbb{R}^d} \eta_i (\xi_i - \eta_i) e^{-\tau(\xi - \eta)^2 + |\eta|^2) \times \sum_{j=N}^{2N} [\chi^+_j(\xi - \eta) + \chi^-_j(\xi - \eta)] \sum_{j=N}^{2N} [\chi^+_j(\eta) + \chi^-_j(\eta)] \, d\eta \]

\[ = -\frac{2\delta^2}{N} \varphi(16\xi) \sum_{j=N}^{2N} \sum_{i=1}^d \int_{\mathbb{R}^d} \eta_i (\xi_i - \eta_i) e^{-\tau(\xi - \eta)^2 + |\eta|^2) \chi^+_j(\xi - \eta) \chi^+_j(\eta) \, d\eta. \]
Therefore, using the Fubini theorem, we have
\[
I(t, \xi) := \int_{0}^{t} e^{-(t-\tau)|\xi|^2} J(\xi) d\tau
\]
\[
= -\frac{2\delta^2}{N} \varphi(16\xi) \sum_{j=1}^{2N} \sum_{i=1}^{d} \int_{\mathbb{R}^d} \eta_i (\xi - \eta_i) \chi_j^+(\xi - \eta) \chi_j^-(\eta) \left( \int_{0}^{t} e^{-t|\xi|^2 - \tau(\xi - \eta)^2 + |\eta|^2 - |\xi|^2} d\tau \right) d\eta
\]
\[
= -\frac{2\delta^2}{N} \varphi(16\xi) e^{-t|\xi|^2} \sum_{j=1}^{2N} \sum_{i=1}^{d} \int_{\mathbb{R}^d} \eta_i (\xi - \eta_i) \chi_j^+(\xi - \eta) \chi_j^-(\eta) \frac{1 - e^{-t|\xi|^2 - |\eta|^2} (\xi - \eta)^2}{|\eta|^2 + |\xi - \eta|^2 - |\xi|^2} d\eta
\]
\[
= \frac{\delta^2}{N} \varphi(16\xi) e^{-t|\xi|^2} \sum_{j=1}^{2N} \int_{\mathbb{R}^d} \chi_j^+(\xi - \eta) \chi_j^-(\eta) (1 - e^{2t(\xi - \eta)^2}) d\eta
\]  \hspace{1cm} (3.4)

Making a change of variable, we can infer form (3.4) that
\[
I(t, \xi) = \frac{\delta^2}{N} \varphi(16\xi) e^{-t|\xi|^2} \sum_{j=1}^{2N} \int_{\mathbb{R}^d} \chi(\xi - \eta) \chi(\eta) (1 - e^{2t(\eta - 2i\xi)(\xi - \eta) + 2i\eta^2}) d\eta.
\]

Letting \( t = 2^{-N} \), then we have \( 1 - e^{2t(\eta - 2i\xi)(\xi - \eta) + 2i\eta^2} \geq \frac{1}{2} \) and
\[
|I(t, \xi)| \geq c\delta^2 \varphi(16\xi) \int_{\mathbb{R}^d} \chi(\xi - \eta) \chi(\eta) d\eta \geq c\delta^2 \varphi(16\xi), \quad \text{for all } |\xi| \leq \frac{1}{4}. \hspace{1cm} (3.5)
\]

Combining (3.2) and (3.5), we show that for \( t = 2^{-N} \),
\[
\left| \int_{\mathbb{R}^d} \int_{0}^{t} e^{-(t-\tau)|\xi|^2} \varphi(16\xi) \sum_{i=1}^{d} \mathcal{F} \left( \partial_{\xi} \varphi \right) (\xi) d\tau d\xi \right| = \int_{\mathbb{R}^d} \int_{0}^{t} e^{-t|\xi|^2} \varphi(16\xi) |I(\xi)| d\xi \geq c\delta^2,
\]
which yields to
\[
|||D_{-4}[A_2(f^N)](t)|||_{L^\infty} \geq c\delta^2.
\]

For any \( T \geq 1 \), we define
\[
D = F B_{1,2}^0, \quad S = \hat{L}_T^\infty (F B_{1,2}^0) \cap \hat{L}_T^1 (F B_{1,2}^2).
\]

It is easy to see that
\[
\begin{aligned}
\begin{cases}
\partial_t (u[f^N] - A_1(f^N)) - \Delta (u[f^N] - A_1(f^N)) = |\nabla u[f^N]|^2, \\
u[f^N] - A_1(f^N)|_{t=0} = 0,
\end{cases}
\end{aligned}
\]
which along with Lemma 2.5 yields
\[
||u[f^N] - A_1(f^N)||_S \leq C||u[f^N]||^2_S \leq C||f^N||^2_D \leq C\delta^2. \hspace{1cm} (3.6)
\]

Using the similar argument as in (3.6), it shows that
\[
||u[f^N] - A_1(f^N) - A_2(f^N)||_S \leq C(||u[f^N]||_S + ||A_1(f^N)||_S)||u[f^N] - A_1(f^N)||_S \leq C\delta^3.
\]
Then, according to the definition of the Besov norm, we have
\[ ||u[f^N](t_N)||_{B^q_{\infty,q}} \geq c||\tilde{\Delta}_4 u[f^N](t_N)||_{L^\infty} \]
\[ \geq c||\tilde{\Delta}_4 (A_2(f^N)(t_N))||_{L^\infty} \]
\[ - ||\tilde{\Delta}_4 (u[f^N] - A_1(f^N) - A_2(f^N))||_{L^\infty}. \]

Therefore, choosing sufficient small \( \delta > 0 \) and large enough \( N \), it follows that
\[ ||u[f^N](t_N)||_{B^q_{\infty,q}} \geq ||\tilde{\Delta}_4 [A_2(f^N)](t_N)||_{L^\infty} - ||u[f^N] - A_1(f^N) - A_2(f^N)||_S \]
\[ \geq c\delta^2 - C\delta^3 \geq c\delta^2. \]

This completes the proof of the cases \( 2 < q \leq \infty \).

Now, we will prove the theorem for the cases \( 1 \leq q \leq 2 \). We set \( \chi_j^\pm(\xi) = \chi(\xi \mp 2^j e_1) \) for \( j \in \mathbb{Z} \), where \( e_1 = \frac{17}{24\sqrt{2}} (1, 1, \ldots, 1) \). Letting \( N \in 16\mathbb{N} = \{16, 32, 48, \ldots\} \) and \( \mathbb{N}(N) = \{k \in 8\mathbb{N} : \frac{1}{4}N \leq k \leq \frac{1}{2}N\} \), we can define a sequence \( \{f_N\}_{N=1}^\infty \) by its Fourier transform
\[ \tilde{f}_N(\xi) = \frac{\delta}{N^q} \sum_{\ell \in \mathbb{N}(N)} [\Phi_\ell^++\Phi_\ell^-+\Phi_\ell^+\Phi_\ell^-], \]
where
\[ \Phi_\ell^+ = e^{i(x\ell_1+2^\ell e_1)} \chi_N(\xi-2^\ell e_1), \quad \Phi_\ell^- = e^{i(x\ell_1+2^\ell e_1)} \chi_N(\xi+2^\ell e_1) \]
\[ \Phi_\ell^- = e^{i(x\ell_1+2^\ell e_1)} \chi_N(\xi-2^\ell e_1), \quad \Phi_\ell^+ = e^{i(x\ell_1+2^\ell e_1)} \chi_N(\xi+2^\ell e_1). \]

Here, we have
\[ \Phi_\ell^+ = e^{i(x\ell_1+2^\ell e_1)}(2^\ell e_1+2^\ell e_1) \tilde{\chi}(x+2^\ell e_1), \quad \Phi_\ell^- = e^{i(x\ell_1+2^\ell e_1)}(2^\ell e_1-2^\ell e_1) \tilde{\chi}(x+2^\ell e_1), \]
\[ \Phi_\ell^- = e^{i(x\ell_1+2^\ell e_1)}(-2^\ell e_1+2^\ell e_1) \tilde{\chi}(x+2^\ell e_1), \quad \Phi_\ell^+ = e^{i(x\ell_1+2^\ell e_1)}(-2^\ell e_1-2^\ell e_1) \tilde{\chi}(x+2^\ell e_1). \]

It is easy to see that
\[ f_N = \frac{2\delta}{N^q} \sum_{\ell \in \mathbb{N}(N)} \left[ \cos(x+2^\ell e_1) \cdot (2^\ell e_1+2^\ell e_1) \right. \]
\[ + \cos(x+2^\ell e_1) \cdot (2^\ell e_1-2^\ell e_1) \left. \right] \tilde{\chi}(x+2^\ell e_1). \]

Since \( \tilde{\chi} \) is a Schwartz function, we have
\[ ||\tilde{\chi}(x)|| \leq C(1+|x|)^{-M}, \quad M \gg 1. \]

Then, we have
\[ ||f_N||_{B^q_{\infty,q}} \leq ||f_N||_{B^q_{\infty,1}} \leq C||f_N||_{L^\infty} \leq C \frac{\delta}{N^q} \left| \sum_{\ell \in \mathbb{N}(N)} \tilde{\chi}(x+2^\ell e_1) \right|_{L^\infty} \]
\[ \leq C \frac{\delta}{N^q} \left| \sum_{\ell \in \mathbb{N}(N)} \frac{1}{(1+|x+2^\ell e_1|)^M} \right|_{L^\infty} \leq C \frac{\delta}{N^q} \rightarrow 0, \quad N \rightarrow \infty. \]

According to Lemma 2.5, we have for all \( t \in [0, T_{f_N}] \),
\[ ||u[f_N]|_{X_t} ||_{BMO} + ||u[f_N]|_{X_t} ||_{X_t}. \]

Choosing \( \delta > 0 \) small enough, we infer from (3.7) that for all \( t \in [0, T_{f_N}] \),
\[ ||u[f_N]|_{X_t} \leq 2C||f_N||_{BMO} \leq 2C \frac{\delta}{N^q}. \]

Therefore, by Lemma 2.7, we deduce that \( u[f^N] \in C([0, \infty); H^s) \).
Next, we need to estimate the second iteration in the Besov norm. To complete our goal, we will use the following lemmas to hackle with.

**Lemma 3.1.** Let $1 \leq q \leq 2$. Then, there exists a positive constant $c$ independent of $N$ and $\delta$ such that

$$|| (\partial f_N)^2 ||_{B_{q,\infty}^{s}(\mathbb{R}^N)} \geq c\delta^2 2^{2N}, \quad N \gg 1.$$  

**Proof.** Since $\text{Supp} \chi(\cdot - a) * \chi(\cdot - b) \subset B(a + b, 1)$, we see that

$$\text{Supp} \left( \Phi^+_{\ell} * \Phi^-_{m} \right) \subset B(\nu 2^{N+1} \epsilon^1, 2^{2+\frac{N}{2}}), \quad \lambda, \mu, \nu \in \{+,-\}.$$  

Moreover, noticing that $\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B((2^\ell - 2^m) \epsilon^1, 1)$, we see that $\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B(0, 1)$. It follows that for any $j \in \mathbb{N}(N)$,

$$\Delta_j[ (\partial f_N)^2 ] = \frac{8\delta^2}{N^2} \Delta_j \sum_{\ell, m \in \mathbb{N}(N)} (\partial_i \Phi^+_{\ell} \partial_i \Phi^-_{m} + \partial_i \Phi^+_{\ell} \partial_i \Phi^+_{m} + \partial_i \Phi^-_{\ell} \partial_i \Phi^-_{m} + \partial_i \Phi^-_{\ell} \partial_i \Phi^-_{m})$$  

(3.8)

Note that if $\max\{\ell, m\} > j$, $\ell, m \in \mathbb{N}(N)$, $\ell \neq m$, we have

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B((2^\ell + 2^m) \epsilon^1, 1) \subset 2^j \mathcal{C}(0, 3, 2^N),$$

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B((2^\ell - 2^m) \epsilon^1, 1) \subset 2^j \mathcal{C}(0, 3, 2^N),$$

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B((-2^\ell + 2^m) \epsilon^1, 1) \subset 2^j \mathcal{C}(0, 3, 2^N),$$

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B((-2^\ell - 2^m) \epsilon^1, 1) \subset 2^j \mathcal{C}(0, 3, 2^N).$$

(3.9)

Else if $\max\{\ell, m\} < j$, $\ell, m \in \mathbb{N}(N)$, $\ell \neq m$, we have

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B((2^\ell + 2^m) \epsilon^1, 1) \subset 2^j B(0, 1/2),$$

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B((2^\ell - 2^m) \epsilon^1, 1) \subset 2^j B(0, 1/2),$$

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B((-2^\ell + 2^m) \epsilon^1, 1) \subset 2^j B(0, 1/2),$$

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{m} \subset B((-2^\ell - 2^m) \epsilon^1, 1) \subset 2^j B(0, 1/2).$$

(3.10)

Moreover, using the facts that

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{j} \subset B(2^{j+1} \epsilon^1, 1) \subset 2^j \mathcal{C}(0, \frac{4}{3}, \frac{3}{2}),$$

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{j} \subset B(0, 1),$$

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{j} \subset B(0, 1),$$

$$\text{Supp} \Phi^+_{\ell} * \Phi^-_{j} \subset B(2^{j+1} \epsilon^1, 1) \subset 2^j \mathcal{C}(0, \frac{4}{3}, \frac{3}{2}),$$

we have

$$\Delta_j(\partial_i \Phi^+_{j} \partial_i \Phi^-_{j} + \partial_i \Phi^+_{j} \partial_i \Phi^-_{j} + \partial_i \Phi^+_{j} \partial_i \Phi^-_{j} + \partial_i \Phi^-_{j} \partial_i \Phi^-_{j})$$

$$= \Delta_j(\partial_i \Phi^+_{j} \partial_i \Phi^+_{j} + \partial_i \Phi^-_{j} \partial_i \Phi^-_{j})$$

$$= \partial_i \Phi^+_{j} \partial_i \Phi^-_{j} + \partial_i \Phi^-_{j} \partial_i \Phi^-_{j}.$$  

(3.11)
Therefore, plugging (3.9)-(3.11) into (3.8), we have
\[
\Delta_j[(\partial_i f_N)^2] = \frac{8\delta^2}{N^*} (\partial_i \Phi_j^{++} \partial_i \Phi_j^{-+} + \partial_i \Phi_j^{+-} \partial_i \Phi_j^{--}) \\
+ \frac{8\delta^2}{N^*} \Delta_j \sum_{k \leq j-1, k \in \mathbb{N}(N)} (\partial_i \Phi_k^{++} \partial_i \Phi_k^{-+} + \partial_i \Phi_k^{+-} \partial_i \Phi_k^{--}) \\
+ \frac{8\delta^2}{N^*} \Delta_j \sum_{k \leq j-1, k \in \mathbb{N}(N)} (\partial_i \Phi_k^{+-} \partial_i \Phi_k^{--} + \partial_i \Phi_k^{++} \partial_i \Phi_k^{+-}) \\
+ \frac{8\delta^2}{N^*} \Delta_j \sum_{k \leq j-1, k \in \mathbb{N}(N)} (\partial_i \Phi_k^{--} \partial_i \Phi_k^{+-} + \partial_i \Phi_k^{-+} \partial_i \Phi_k^{--}) \\
= I_1 + I_2 + I_3 + I_4 + I_5.
\]

Note that
\[
\frac{N_1}{8\delta^2} I_1 = \partial_i \Phi_j^{++} \partial_i \Phi_j^{-+} + \partial_i \Phi_j^{+-} \partial_i \Phi_j^{--}
= (2^{2N} - 2^2j)(\Phi_j^{++} + \Phi_j^{-+} \Phi_j^{--}) + 2 \cos((x + 2j+1 \hat{e}_1) \cdot 2j+1 \hat{e}_1) \partial_i \hat{\chi}(x + 2j+1 \hat{e}_1))^2 \\
\leq \frac{289}{288d} (2^{2N} - 2^2j) \cos((x + 2j+1 \hat{e}_1) \cdot 2j+1 \hat{e}_1) \hat{\chi}^2(x + 2j+1 \hat{e}_1) \\
+ 2 \cos((x + 2j+1 \hat{e}_1) \cdot 2j+1 \hat{e}_1) \partial_i \hat{\chi}(x + 2j+1 \hat{e}_1))^2.
\]

By choosing \( x = -2j+1 \hat{e}_1 \) and using \( \mathcal{F}^{-1}(\xi \hat{\chi})(0) = 0 \), we have
\[
\|I_1\|_{L^\infty} \geq \frac{c}{N^*} \delta^2 2^{2N} \hat{\chi}^2(0). \tag{3.12}
\]

Since \( \text{Supp } \Phi_k^{\mu, \nu} \subset 2^N \mathcal{C}, k \in \mathbb{N}(N), \mu, \nu \in \{+,-\}, \) we have
\[
\|I_2\|_{L^\infty} \leq C \frac{\delta^2}{N^*} \sum_{k \leq j-1, k \in \mathbb{N}(N)} (\partial_i \Phi_k^{++} \partial_i \Phi_k^{-+} + \partial_i \Phi_k^{+-} \partial_i \Phi_k^{--}) \|_{L^\infty} \\
\leq C \frac{\delta^2}{N^*} \sum_{k \leq j-1, k \in \mathbb{N}(N)} 2^{2N} \left( 1 + |x + 2^N \hat{e}_1 + 2^j \hat{e}_1|^2 \right)^{-M} \left( 1 + |x - 2^N \hat{e}_1 + 2^k \hat{e}_1|^2 \right)^{-M} \|_{L^\infty} \\
\leq C \frac{\delta^2}{N^*} 2^{2N} j 2^{-MN} \leq C \frac{\delta^2}{N^*} 2^{2N} N^2 2^{-NM}. \tag{3.13}
\]

Applying similar estimate as in (3.13), we have
\[
\|I_i\|_{L^\infty} \leq C \frac{\delta^2}{N^*} 2^{2N} N^2 2^{-NM}, \text{ for } i = 3,4,5. \tag{3.14}
\]

Combining (3.12)-(3.14), we deduce that for \( N \gg 1 \),
\[
\|\Delta_j[(\partial_i f_N)^2]\|_{L^\infty} \geq \frac{c}{N^*} \delta^2 2^{2N}.
\]
Therefore, by the definition of the Besov norm, we have
\[ \|\partial_t f_N\|^2_{B_{2,\infty}(\mathbb{R}^N)} \geq \left( \sum_{j \in \mathbb{N}(N)} \|\Delta_j[(\partial_t f_N)^2] \|_{L^\infty}^q \right)^{1 \over q} \geq c \delta^2 2^N. \]

This completes the proof of this lemma. \( \square \)

Now, we can rewrite
\[ A_2(f_N) = \int_0^t e^{(t-\tau)\Delta} \sum_{i=1}^d [\partial_i e^{\tau\Delta} f_N]^2 d\tau \]
\[ = t \sum_{i=1}^d (\partial_i f_N)^2 + t \sum_{r \geq 2} \frac{1}{r!} (t\Delta)^{r-1} \sum_{i=1}^d (\partial_i f_N)^2 \]
\[ + \int_0^t e^{(t-\tau)\Delta} \sum_{i=1}^d (e^{\tau\Delta} + 1)\partial_i f_N(e^{\tau\Delta} - 1)\partial_i f_N d\tau. \]

**Lemma 3.2.** Let \( 1 \leq q \leq 2 \) and \( t = \delta 2^{-2N} \). Then there exists a constant \( C \) independent of \( N \) and \( \delta \) such that
\[ t \sum_{r \geq 2} \frac{1}{r!} \| (t\Delta)^{r-1} [(\partial_i f_N)^2] \|_{B_{2,\infty}(\mathbb{N}(N))} \leq C \delta^4. \]

**Proof.** Using the Bernstein’s multiplier estimates, we have
\[ \|F^{-1}(\langle |\xi|^2 \rangle^{r-1}\varphi, F f) \|_{L^\infty} \leq (t 2^j)^{r-1} \|F^{-1}(\langle |\xi|^{2(r-1)}\varphi \rangle) \|_{L^1} \|f\|_{L^\infty} \]
\[ \leq (t 2^j)^{r-1} g^r \|f\|_{L^\infty}. \]

Noticing that \( j \in \mathbb{N}(N) \) implies that \( j \leq {N \over 2} \), one has that
\[ t \sum_{r \geq 2} \frac{1}{r!} \| (t\Delta)^{r-1} [(\partial_i f_N)^2] \|_{B_{2,\infty}(\mathbb{N}(N))} \leq t \sum_{r \geq 2} \frac{1}{r!} \left( \sum_{j \in \mathbb{N}(N)} (t 2^j)^{q(r-1)} \|\partial_i f_N\|_{L^\infty}^{2q} \right)^{1 \over q} \]
\[ \leq \delta \delta^q \sum_{r \geq 2} \frac{g^r}{r!} N^{1 \over 4} N^{-1 \over 2} (t 2^j)^{r-1} \]
\[ \leq \delta^4, \]
which completes the proof of this lemma. \( \square \)

**Lemma 3.3.** Let \( 1 \leq q \leq 2 \), \( \delta \ll 1 \), and \( t = \delta 2^{-2N} \). Then there exists a constant \( C \) independent of \( N \) and \( \delta \) such that
\[ \| \int_0^t e^{(t-\tau)\Delta}(e^{\tau\Delta} + 1)\partial_i f_N(e^{\tau\Delta} - 1)\partial_i f_N d\tau \|_{B_{2,\infty}(\mathbb{N}(N))} \leq C \delta^4. \]

**Proof.** Using the fact that \( \|e^{\tau\Delta} u\|_{L^\infty} \leq C \|u\|_{L^\infty} \), we have
\[ \| \int_0^t e^{(t-\tau)\Delta}(e^{\tau\Delta} + 1)\partial_i f_N(e^{\tau\Delta} - 1)\partial_i f_N d\tau \|_{B_{2,\infty}(\mathbb{N}(N))} \]
\[ \leq t 2^j N^{1 \over 2} \|f_N\|_{L^\infty} \max_{\tau \in [0,t]} \|(e^{\tau\Delta} - 1) f_N\|_{L^\infty} \leq \delta^2 N^{1 \over 2} \max_{\tau \in [0,t]} \|(e^{\tau\Delta} - 1) f_N\|_{L^\infty}. \]  \hspace{1cm} (3.15)

Using Taylor’s expansion, one has that
\[ \|(e^{\tau\Delta} - 1) f_N\|_{L^\infty} \leq \sum_{r \geq 1} \frac{\tau^r}{r!} \|F^{-1}(\langle |\xi|^{2r} \rangle) \|_{L^\infty}. \]
Since $\text{Supp } \hat{f}_N \subset 2^N \mathcal{C}$, we see that
\[ ||\mathcal{F}^{-1}(\xi^{2r} \mathcal{F}(f_N))||_{L^\infty} \lesssim ||\mathcal{F}^{-1}(\xi^{2r} \varphi_N)||_{L^1} ||f_N||_{L^\infty} \lesssim \delta^r \tau^d 2^{2Nr} \lesssim \delta (18)^r 2^{2kr}. \] (3.16)

It follows from (3.16) that
\[ \left| \left| (e^{r \Delta} - 1)f_N \right| \right|_{L^\infty} \lesssim \delta N^{-\frac{1}{2}} \left( \sum_{r \geq 1} \frac{(18 \tau 2^{2N})^r}{r!} - 1 \right) \lesssim \delta N^{-\frac{1}{2}} (e^{18 \tau 2^{2N}} - 1) \] (3.17)
\[ \lesssim \delta N^{-\frac{1}{2}} (e^{18 \delta} - 1) \lesssim \delta^2 N^{-\frac{1}{2}}. \]

Plugging (3.17) into (3.15), we have the result. \qedhere

Collecting Lemmas 3.1-3.3, we immediately have

**Lemma 3.4.** Let $1 \leq q \leq 2$ and $t = \delta 2^{-2N}$. Then, if we choose $\delta$ small enough and $N$ large enough, there exists a positive constant $c$ independent of $N$ and $\delta$ such that
\[ ||A_2(f_N)(t)||_{B^q_{\infty,q}(N(N))} \geq c \delta^3. \]

Now, we will accomplish our proof of the theorem. For any $T \geq 1$, we have
\[ \left| \left| u[f_N] - A_1(f_N) \right| \right|_{Y_T} \leq \left| \left| \int_0^t e^{(t - \tau) \Delta} (||\nabla u[f_N]||^3) \right| \right|_{Y_T} \leq C ||u[f_N]||^2_{X_T} \leq C ||f_N||^2_{B^{\infty}_{\infty,q}} \leq C \frac{\delta^2}{N^{\frac{1}{q}}}, \]
where
\[ ||u||_{Y_T} = ||u||_{X_T} + ||u||_{L^\infty_B(BMO)}. \]

On the other hand, we also have
\[ \left| \left| u[f_N] - A_1(f_N) - A_2(f_N) \right| \right|_{Y_T} \leq (||u[f_N]||_{X_T} + ||A_1(f_N)||_{X_T}) ||u[f_N] - A_1(f_N)||_{X_T} \leq ||f_N||_{B^{\infty}_{\infty,q}} ||u[f_N] - A_1(f_N)||_{X_T} \leq C \frac{\delta^3}{N^{\frac{1}{q}}}, \]
which implies
\[ \left| \left| u[f_N] - A_1(f_N) - A_2(f_N) \right| \right|_{B^q_{\infty,q}(N(N))} \leq C ||u[f_N] - A_1(f_N) - A_2(f_N)||_{Y_T} \leq C \frac{\delta^3}{N^{\frac{1}{q}}}. \]

Then, by the definition of Besov norm, we show that
\[ \left| \left| u[f_N] - A_1(f_N) - A_2(f_N) \right| \right|_{B^q_{\infty,q}(N(N))} \leq C \frac{\delta^3}{N^{\frac{1}{q}}}. \]

Choosing small enough $\delta > 0$, we have
\[ \left| \left| u[f_N](t) \right| \right|_{B^q_{\infty,q}} \geq \left| \left| u[f_N] \right| \right|_{B^q_{\infty,q}(N(N))} \geq ||A_1(f_N)(t)||_{B^q_{\infty,q}(N(N))} - ||A_2(f_N)||_{B^q_{\infty,q}(N(N))} - ||u[f_N] - A_1(f_N) - A_2(f_N)||_{B^q_{\infty,q}(N(N))} \geq c \delta^3 - C \frac{\delta^3}{N^{\frac{1}{q}}} \geq c \delta^3, \quad N \to \infty. \]
Acknowledgements. This work was partially supported by NNSFC (No. 11271382), RFDP (No. 20120171110014), MSTDF (No. 098/2013/A3), Guangdong Special Support Program (No. 8-2015) and the key project of NSF of Guangdong Province (No. 1614050000014).

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