Quantum Affine Algebras
and their Representations

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Abstract. We prove a highest weight classification of the finite-dimensional irreducible representations of a quantum affine algebra, in the spirit of Cartan’s classification of the finite-dimensional irreducible representations of complex simple Lie algebras in terms of dominant integral weights. We also survey what is currently known about the structure of these representations.

1. Introduction

Around 1985, V.G. Drinfel’d and M. Jimbo showed, independently, how to associate to any symmetrizable Kac–Moody algebra $\mathfrak{g}$ over $\mathbb{C}$ a family $U_q(\mathfrak{g})$ of Hopf algebras, depending on a parameter $q \in \mathbb{C}^\times$, and reducing (essentially) to the classical universal enveloping algebra $U(\mathfrak{g})$ when $q = 1$. The introduction of quantum groups has opened up a fascinating new chapter in representation theory; in addition, quantum groups have turned out to have surprising connections with several areas of mathematics (algebraic groups in characteristic $p$, knot theory, …) and physics (two–dimensional integrable systems, conformal field theories, …).

Many of the applications of quantum groups (such as those in knot theory, for example) depend on the fact that, if $\mathfrak{g}$ is finite–dimensional and $q$ is not a root of unity, one can associate to any finite–dimensional representation $V$ of $U_q(\mathfrak{g})$ an operator $R \in \text{End}(V \otimes V)$ which satisfies the quantum Yang–Baxter equation (QYBE):

\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (1) \]

(here, $R_{12}$ means $R \otimes \text{id} \in \text{End}(V \otimes V \otimes V)$, etc.). In fact, if $W$ is another finite–dimensional representation of $U_q(\mathfrak{g})$, it turns out that the tensor products $V \otimes W$ and $W \otimes V$ are isomorphic as representations of $U_q(\mathfrak{g})$, and further that there is a

1991 Mathematics Subject Classification. Primary 17B37, 81R50; Secondary 16W30, 82B23.

The first author was partially supported by NSF Grant #9207701

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The canonical choice of isomorphism $I_{V,W} : V \otimes W \to W \otimes V$. If $V = W$ and $\sigma$ is the flip map $V \otimes V \to V \otimes V$, the matrix $R = \sigma I$ satisfies (1).

In some situations, however, it is important to have a solution of the ‘QYBE with spectral parameters’:

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v).$$

Here, $R(u,v)$ is a family of operators in $\text{End}(V \otimes V)$, for some finite-dimensional vector space $V$, depending on a pair of complex parameters $u,v$. In many cases, possibly after making a change of variable $u \mapsto f(u)$, $v \mapsto f(v)$, $R(u,v)$ becomes a function of $u - v$, which we write as $R(u - v)$.

In the theory of two-dimensional lattice models in statistical mechanics, for example, $R(u)$ is a matrix whose entries are the ‘interaction’ energies of the atoms in the lattice, and $u$ is a parameter on which the properties of the model depend, such as the values of external electric or magnetic fields. From $R(u)$ one constructs the ‘transfer matrices’

$$T(u) = R_{01}(u)R_{02}(u) \ldots R_{0N}(u) \in \text{End}(V \otimes V \otimes \ldots \otimes V)$$

(the first copy of $V$ in $V \otimes V \otimes \ldots \otimes V$ is numbered 0, the others 1, $\ldots$, $N$) and from these the partition function

$$Z = \text{trace}_{V \otimes \ldots \otimes V}(\text{trace}_V(T)^N)$$

(we assume that the lattice is $N$ atoms wide in each direction and that periodic boundary conditions are imposed). It is explained in [1], for example, that the physical properties of the model may be deduced from $Z$. If $R(u)$ is invertible and satisfies (2), it is easy to show that $\text{trace}_V(T(u))$ commutes with $\text{trace}_V(T(v))$ for all $u,v$: for this reason, such models are called ‘integrable’.

One can hope to construct solutions of (2) whenever one has a Hopf algebra $A$ equipped with a family of automorphisms $\tau_u$. For, if $V$ is a finite-dimensional (complex) representation of $A$, pulling back $V$ by $\tau_u$ gives a 1-parameter family of representations $V(u)$. Assume that, for all parameters $u, v, w$, and for some representation $V$ of $A$,

(i) $V(u) \otimes V(v)$ is isomorphic to $V(v) \otimes V(u)$,

(ii) $V(u) \otimes V(v) \otimes V(w)$ is irreducible,

and let $I(u,v) : V(u) \otimes V(v) \to V(v) \otimes V(u)$ be an intertwiner (which, by (i), is well-defined up to a scalar multiple). If $R = \sigma I$, equation (2) is the condition for the equality of the two composites of intertwiners

$$V(u) \otimes V(v) \otimes V(w) \to V(v) \otimes V(u) \otimes V(w) \to V(v) \otimes V(u) \otimes V(w) \to V(v) \otimes V(u) \otimes V(w)$$

and

$$V(u) \otimes V(v) \otimes V(w) \to V(u) \otimes V(w) \otimes V(v) \to V(w) \otimes V(u) \otimes V(v) \to V(w) \otimes V(u) \otimes V(v).$$

Thus, condition (ii) guarantees that (2) is satisfied up to a scalar multiple.
Let \( \hat{\mathfrak{g}} \) be the (untwisted) affine Lie algebra associated to a finite-dimensional complex simple Lie algebra \( \mathfrak{g} \). Recall that \( \hat{\mathfrak{g}} \) is a central extension, with 1-dimensional centre, of the Lie algebra \( \mathfrak{g}[t, t^{-1}] \) of Laurent polynomial maps \( \mathbb{C}^\times \to \mathfrak{g} \), under pointwise operations. There is an obvious multiplicative 1-parameter group of automorphisms of \( \hat{\mathfrak{g}} \), given by rescaling \( t \), which fixes each element of the centre. On the other hand, \( \hat{\mathfrak{g}} \) is a symmetrizable Kac–Moody algebra, so one can define the Hopf algebra \( U_q(\hat{\mathfrak{g}}) \). We shall assume from now on that \( q \) is transcendental. It turns out that \( U_q(\hat{\mathfrak{g}}) \) also has a multiplicative 1-parameter group of automorphisms \( \tau_u \), which reduce, in the limit \( q \to 1 \), to the rescaling automorphisms of \( \hat{\mathfrak{g}} \). According to Drinfel’d [11], property (i) holds for generic values of \( u, v \), and there exists a canonical choice of isomorphism \( I(u, v) \) such that \( R(u, v) = \sigma I(u, v) \) satisfies (2). Moreover, the multiplicative property of \( \tau_u \) implies that \( R(u, v) \) depends only on \( u/v \); reparametrizing by \( u \to e^u, v \to e^v \), we get a solution of (2) which depends only on \( u - v \). Thus, it is of considerable interest to describe the finite-dimensional irreducible representations of \( U_q(\hat{\mathfrak{g}}) \).

The main result proved in this paper (Theorem 3.3) gives a parametrization of the finite-dimensional irreducible representations of \( U_q(\hat{\mathfrak{g}}) \) analogous to Cartan’s highest weight classification of the finite-dimensional irreducible representations of \( \mathfrak{g} \). The role of dominant integral weights in the representation theory of \( \mathfrak{g} \) is played for \( \mathfrak{g} \) by the set of \( \text{rank}(\mathfrak{g}) \)-tuples \( \mathbf{P} \) of polynomials in one variable with constant coefficient 1; let \( V(\mathbf{P}) \) be the representation of \( U_q(\hat{\mathfrak{g}}) \) associated to \( \mathbf{P} \).

To construct explicit solutions of (2), one needs to understand the structure of the representations \( V(\mathbf{P}) \). Now, there is a canonical embedding of Hopf algebras \( U_q(\hat{\mathfrak{g}}) \to U_q(\mathfrak{g}) \) which, in the limit \( q \to 1 \), becomes the embedding \( \mathfrak{g} \to \mathfrak{g} \) given by regarding elements of \( \mathfrak{g} \) as constant maps \( \mathbb{C}^\times \to \mathfrak{g} \). Thus, representations of \( U_q(\hat{\mathfrak{g}}) \) can be regarded as representations of \( U_q(\mathfrak{g}) \). Since finite-dimensional representations of \( U_q(\mathfrak{g}) \) are completely reducible, a first step in understanding \( V(\mathbf{P}) \) would be to describe its decomposition under \( U_q(\mathfrak{g}) \). We shall say that two representations of \( U_q(\hat{\mathfrak{g}}) \) are equivalent if they are isomorphic as representations of \( U_q(\mathfrak{g}) \). Unfortunately, the problem of describing the structure of \( V(\mathbf{P}) \) as a representation of \( U_q(\hat{\mathfrak{g}}) \) appears to be intractable for general \( \mathbf{P} \). However, it is still interesting to understand the representations \( V(\mathbf{P}) \) of some special type.

Any \( V(\mathbf{P}) \) has a unique irreducible \( U_q(\mathfrak{g}) \)-subrepresentation of maximal highest weight. Conversely, given a finite-dimensional irreducible representation \( V \) of \( U_q(\mathfrak{g}) \), one can consider the representations \( V(\mathbf{P}) \) of \( U_q(\hat{\mathfrak{g}}) \) which have \( V \) as their top \( U_q(\mathfrak{g}) \)-component – \( V(\mathbf{P}) \) is then called an affinization of \( V \). (Thus, every \( V(\mathbf{P}) \) is an affinization of its top \( U_q(\mathfrak{g}) \)-component.)

Classically, every finite-dimensional representation \( V \) of \( \mathfrak{g} \) has an affinization (in the obvious sense) which is irreducible under \( \mathfrak{g} \). For, there is an algebra homomorphism \( ev_u : \hat{\mathfrak{g}} \to \mathfrak{g} \), for any \( u \in \mathbb{C}^\times \), which annihilates the centre of \( \hat{\mathfrak{g}} \) and evaluates maps \( \mathbb{C}^\times \to \mathfrak{g} \) at \( u \); note that \( ev_u \) is the identity on \( \mathfrak{g} \). Pulling back \( V \) by \( ev_u \) gives a family of representations \( V(u) \) of \( \hat{\mathfrak{g}} \), which are obviously isomorphic to \( V \) as representations of \( \mathfrak{g} \). In the quantum case, however, there are simple examples...
of irreducible representations of $U_q(\mathfrak{g})$ which have no affinization that is irreducible under $U_q(\mathfrak{g})$. Thus, it is natural to look for the ‘smallest’ affinization(s).

In [4], a natural partial ordering was defined on the set of equivalence classes of finite–dimensional representations of $U_q(\mathfrak{g})$. One can show that a given irreducible representation $V$ of $U_q(\mathfrak{g})$ has only finitely many affinizations, up to equivalence, so it makes sense to look for the minimal one(s). In Section 6, we give necessary and sufficient conditions on $\mathbf{P}$ for $\mathbf{V}(\mathbf{P})$ to be a minimal affinization of its top $U_q(\mathfrak{g})$–component, summarizing results in [7], [9], [4], and [10]. We use these results to describe the $U_q(\mathfrak{g})$–structure of the minimal affinizations in some cases.

2. Quantum affine algebras

We begin by recalling the definition of the Hopf algebras $U_q(\mathfrak{g})$. Let $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Cartan matrix $A = (a_{ij})_{i,j \in I}$. Fix coprime positive integers $(d_i)_{i \in I}$ such that $(d_i a_{ij})$ is symmetric. Let $P = \mathbb{Z}^I$ and let $P^+ = \{ \lambda \in P \mid \lambda(i) \geq 0 \text{ for all } i \in I \}$. For $i \in I$, define $\lambda_i \in P^+$ by $\lambda_i(j) = \delta_{ij}$. Let $R$ (resp. $R^+$) be the set of roots (resp. positive roots) of $\mathfrak{g}$. Let $\alpha_i$ ($i \in I$) be the simple roots and let $\theta$ be the highest root. Let $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \subset \mathfrak{h}^*$ be the root lattice, and set $Q^+ = \sum_{i \in I} \mathbb{N} \alpha_i$. Define a partial order $\geq$ on $P$ by $\lambda \geq \mu$ iff $\lambda - \mu \in Q^+$. If $\eta = \sum_{i \in I} m_i \alpha_i \in Q^+$, define $\text{height}(\eta) = \sum_{i \in I} m_i$. Define a non-degenerate symmetric bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = d_i a_{ij}$, and set $d_0 = \frac{1}{2}(\theta, \theta)$.

Let $q \in \mathbb{C}^*$ be transcendental, and, for $r, n \in \mathbb{N}$, $n \geq r$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$
$$\left[ \begin{array}{c} n \\ r \end{array} \right]_q = \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

Set $q_i = q^{d_i}$.

**Proposition 2.1.** There is a Hopf algebra $U_q(\mathfrak{g})$ over $\mathbb{C}$ which is generated as an algebra by elements $x_i^\pm$, $k_i^{\pm 1}$ ($i \in I$), with the following defining relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \ k_i k_j = k_j k_i,$$
$$k_i x_j^\pm k_i^{-1} = q_i^{\pm a_{ij}} x_j^\pm,$$
$$[x_i^+, x_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_{q_i} (x_i^\pm)^r x_j^- (x_i^\pm)^{1-a_{ij}-r} = 0, \quad i \neq j.$$
The comultiplication $\Delta$, counit $\epsilon$, and antipode $S$ of $U_q(\hat{g})$ are given by

- $\Delta(x_i^+) = x_i^+ \otimes k_i + 1 \otimes x_i^+$,
- $\Delta(x_i^-) = x_i^- \otimes 1 + k_i^{-1} \otimes x_i^-,$
- $\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\mp 1},$
- $\epsilon(x_i^+) = 0, \epsilon(k_i^{\pm 1}) = 1,$
- $S(x_i^+) = -x_i^+ k_i^{-1}, S(x_i^-) = -k_i x_i^-,$

for all $i \in I$.

The generators and relations in 2.1 serve, in fact, to define a Hopf algebra $U_q(\hat{g})$ when $g$ is an arbitrary symmetrically Kac–Moody algebra. In particular, if $\hat{g}$ is the (untwisted) affine Lie algebra associated to $g$, one can define the Hopf algebra $U_q(\hat{g})$ as in 2.1, but replacing $I$ by $\hat{I} = I \amalg \{0\}$ and $A$ by the extended Cartan matrix $\hat{A} = (a_{ij})_{i,j \in I}$ of $g$; we let $q_0 = q_1^0$.

Note that there is a canonical homomorphism $U_q(g) \to U_q(\hat{g})$ such that $x_i^+ \mapsto x_i^+$, $k_i^{\pm 1} \mapsto k_i^{\pm 1}$ for all $i \in I$. Thus, any representation of $U_q(\hat{g})$ may be regarded as a representation of $U_q(g)$.

Now $\hat{g}$ is better understood than an arbitrary infinite–dimensional Kac–Moody Lie algebra because it has another realization as (a central extension of) a space of maps $\mathbb{C}^\times \to g$, as we mentioned in Section 1. In [12], Drinfeld stated (in a slightly different form) a realization of $U_q(\hat{g})$ which, although still in terms of generators and relations, more closely resembles the description of $\hat{g}$ as a space of maps. In the following form, the result was proved by Beck [2]:

**Theorem 2.2.** Let $A_q$ be the algebra with generators $x_{i,r}^\pm$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbb{Z}\{0\}$) and $c^{\pm 1/2}$, and the following defining relations:

- $c^{\pm 1/2}$ are central,
- $k_i k_i^{-1} = 1, c^{1/2}c^{-1/2} = c^{-1/2}c^{1/2} = 1,$
- $k_i k_j = k_j k_i, k_i h_{j,r} = h_{j,r} k_i,$
- $k_i x_{j,r} k_i^{-1} = q_i^{\pm a_{ij}} x_{j,r}^{\pm},$
- $[h_{i,r}, x_{j,s}^\pm] = \pm r [r a_{ij}] q_i^{|r|^{1/2}/2} x_{j,r+s}^\pm,$
- $[x_{i,r+1}^+, x_{j,s}^+] - q_i^{\pm a_{ij}} x_{j,s}^+ x_{i,r+1}^{+} = q_i^{\pm a_{ij}} x_{j,s}^{+} x_{i,r+1}^{+} - x_{j,s+1}^{+} x_{i,r}^+,$
- $[x_{i,r}^+, x_{j,s}^-] = \delta_{ij} \frac{c^{(r-s)/2} q_{i,r+s}^- - c^{(r-s)/2} q_{i,r+s}^+}{q_i - q_i^{-1}},$

for all $i \neq j$, for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}, \Sigma_m$ is the symmetric group on $m$ letters, and the $q_{i,r}$ are determined by equating powers of $u$.
in the formal power series
\[
\sum_{r=0}^{\infty} \phi_{\pm r}^+ u^{\pm r} = k_i^{\pm 1} \exp \left( \pm (q_i - q_i^{-1}) \sum_{s=1}^{\infty} \sum_{x_i, \pm s} u^{\pm s} \right).
\]

If \( \theta = \sum_{i \in I} m_i \alpha_i \), set \( k_0 = \prod_{i \in I} k_i^{m_i} \). Suppose that the root vector \( \pi_0^+ \) of \( \mathfrak{g} \) corresponding to \( \theta \) is expressed in terms of the simple root vectors \( \pi_i^+ \) \((i \in I)\) of \( \mathfrak{g} \) as
\[
\pi_0^+ = \lambda [x_{i_1}^+, [x_{i_2}^+, \cdots, [x_{i_k}^+, x_j^-] \cdots]]
\]
for some \( \lambda \in \mathbb{C}^\times \). Define maps \( w_+^i : U_q(\hat{\mathfrak{g}}) \to U_q(\mathfrak{g}) \) by
\[
w_+^i(a) = x_{i,0}^+ a - k_i^{-1} a k_i x_{i,0}^+.
\]
Then, there is an isomorphism \( f : U_q(\hat{\mathfrak{g}}) \to A_q \) defined on generators by
\[
f(k_0) = k_0^{-1}, \quad f(k_i) = k_i, \quad f(x_i^+) = x_{i,0}^+, \quad (i \in I),
\]
\[
f(x_0^+) = \mu w_{i_1}^- \cdots w_{i_k}^- (x_{j,1}^+) k_0^{-1},
\]
\[
f(x_0^-) = \lambda k_0 w_{i_1}^+ \cdots w_{i_k}^+ (x_{j,-1}^-),
\]
where \( \mu \in \mathbb{C}^\times \) is determined by the condition
\[
[x_0^+, x_0^-] = \frac{k_0 - k_0^{-1}}{q_0 - q_0^{-1}}.
\]

Let \( \hat{U}^\pm \) (resp. \( \hat{U}^0 \)) be the subalgebra of \( U_q(\hat{\mathfrak{g}}) \) generated by the \( x_i^\pm \) (resp. by the \( \phi_{i,r}^\pm \)) for all \( i \in I, r \in \mathbb{Z} \). Similarly, let \( U^\pm \) (resp. \( U^0 \)) be the subalgebra of \( U_q(\mathfrak{g}) \) generated by the \( x_i^\pm \) (resp. by the \( k_i^{\pm 1} \)) for all \( i \in I \). We have the following weak version of the Poincaré–Birkhoff–Witt theorem:

**Proposition 2.3.**

(a) \( U_q(\hat{\mathfrak{g}}) = U^-.U^0.U^+ \).
(b) \( U_q(\hat{\mathfrak{g}}) = \hat{U}^-.\hat{U}^0.\hat{U}^+ \).

See [8] or [14] for details.

### 3. Representation theory of \( U_q(\mathfrak{g}) \) and \( U_q(\hat{\mathfrak{g}}) \)

We begin by summarizing the relevant facts about the representation theory of \( U_q(\mathfrak{g}) \) (we continue to assume that \( q \) is transcendental). For further details, see [8] or [14], for example.

Let \( W \) be a representation of \( U_q(\mathfrak{g}) \). One says that \( \lambda \in P \) is a weight of \( W \) if the weight space
\[
W_\lambda = \{ w \in W | k_i w = q^{\lambda(i)} w \}
\]
is non–zero. We say that \( W \) is of type 1 if
\[
W = \bigoplus_{\mu \in P} W_\mu.
\]
A non-zero vector \( w \in W_\lambda \) is called a highest weight vector if \( x_i^+w = 0 \) for all \( i \in I \), and \( W \) is called a highest weight representation with highest weight \( \lambda \) if \( W = U_q(g).w \) for some highest weight vector \( w \in W_\lambda \). Any highest weight representation is of type 1.

For any \( \lambda \in P \), let \( M(\lambda) \) be the quotient of \( U_q(g) \) by the left ideal generated by \( \{ x_i^+, k_i - q^{\lambda(i)} \} \). Then, \( M(\lambda) \) is a highest weight representation of \( U_q(g) \) with highest weight \( \lambda \), and it follows from 2.3(a) that \( M(\lambda)_\lambda \) is one-dimensional. The standard argument implies that \( M(\lambda) \) has a unique irreducible quotient \( V(\lambda) \), and that every irreducible highest weight representation with highest weight \( \lambda \) is isomorphic to \( V(\lambda) \).

For any \( i \in I \), let \( \sigma_i \) be the algebra automorphism of \( U_q(g) \) such that
\[
\sigma_i(x_j^+) = (-1)^{\delta_{ij}}x_j^+, \quad \sigma_i(k_j) = (-1)^{\delta_{ij}}k_j, \quad \sigma_i(x_j^-) = x_j^-
\]
for all \( j \in I \).

**Proposition 3.1.**

(a) Every finite-dimensional representation of \( U_q(g) \) is completely reducible.

(b) Every finite-dimensional irreducible representation of \( U_q(g) \) can be obtained from a type 1 representation by twisting with a product of the automorphisms \( \sigma_i \).

(c) Every finite-dimensional irreducible representation of \( U_q(g) \) of type 1 is highest weight.

(d) The representation \( V(\lambda) \) is finite-dimensional iff \( \lambda \in P^+ \).

(e) If \( \lambda \in P^+ \), \( V(\lambda) \) has the same character as the irreducible representation of \( g \) of the same highest weight.

(f) The multiplicity \( m_\nu(\lambda \otimes \nu) \) of \( V(\nu) \) in the tensor product \( V(\lambda) \otimes V(\mu) \), where \( \lambda, \mu, \nu \in P^+ \), is the same as in the tensor product of the irreducible representations of \( g \) of the same highest weight (this statement makes sense in view of parts (a), (c) and (d)).

We now turn to the representation theory of \( U_q(\hat{g}) \). A representation \( V \) of \( U_q(\hat{g}) \) is of type 1 if \( c^{1/2} \) acts as the identity on \( V \), and if \( V \) is of type 1 as a representation of \( U_q(g) \). A vector \( v \in V \) is a highest weight vector if
\[
x_i^+v = 0, \quad \Phi_i^\pm v = \Phi_i^\pm v, \quad c^{1/2}v = v,
\]
for some complex numbers \( \Phi_i^\pm \). A type 1 representation \( V \) is a highest weight representation if \( V = U_q(\hat{g}).v \), for some highest weight vector \( v \), and the pair of \((I \times \mathbb{Z})\)-tuples \( (\Phi_i^\pm)_{i \in I, r \in \mathbb{Z}} \) is its highest weight. (In [8], highest weight representations of \( U_q(\hat{g}) \) are called ‘pseudo-highest weight’.)

Note that \( \Phi_i^r = 0 \) (resp. \( \Phi_i^r = 0 \)) if \( r < 0 \) (resp. if \( r > 0 \)), and that \( \Phi_i^0 \Phi_i^0 = 1 \). Conversely, if \( \Phi = (\Phi_i^\pm)_{i \in I, r \in \mathbb{Z}} \) is a set of complex numbers satisfying these conditions, let \( M(\Phi) \) be the quotient of \( U_q(\hat{g}) \) by the left ideal generated by \( \{ x_i^+, \phi_i^\pm - \Phi_i^r \} \) and \( \{ c^{1/2} - 1 \} \). Then, \( M(\Phi) \) is a highest weight representation of \( U_q(\hat{g}) \). It follows from 2.3(b) that, regarding \( M(\Phi) \) as a representation of
$U_q(\mathfrak{g})$, we have $\dim(M(\Phi))_\lambda = 1$, and hence that $M(\Phi)$ has a unique irreducible quotient (as a representation of $U_q(\mathfrak{g})$), say $V(\Phi)$. Clearly, every irreducible highest weight representation of $U_q(\mathfrak{g})$ is isomorphic to some $V(\Phi)$.

Let $\sigma_i (i \in I)$ be the algebra automorphisms of $U_q(\mathfrak{g})$ defined by the formulas in (5), but with the indices $i, j \in \hat{I}$. Also, let $\sigma$ be the algebra automorphism of $U_q(\mathfrak{g})$ given, in terms of the presentation 2.2, by

$$\sigma(e^{1/2}) = -e^{1/2}, \quad \sigma(x_{i,r}^\pm) = (-1)^r x_{i,r}^\pm,$$

$$\sigma(k_i) = k_i, \quad \sigma(h_{i,r}) = h_{i,r}.$$

**Proposition 3.2.** Let $V$ be a finite-dimensional irreducible representation of $U_q(\mathfrak{g})$.

(a) $V$ can be obtained from a type 1 representation by twisting with a product of some of the automorphisms $\sigma_i (i \in \hat{I}), \sigma$.

(b) If $V$ is of type 1 (as a representation of $U_q(\mathfrak{g})$), then $V$ is highest weight.

See Section 12.2 of [8] for the proof.

Thus, to classify the finite-dimensional irreducible representations of $U_q(\mathfrak{g})$, we have only to determine for which $\Phi$ the representation $V(\Phi)$ is finite-dimensional. The answer to this question is the main result of this paper. If $\lambda \in P^+$, let $\mathcal{P}^\lambda$ be the set of all $I$-tuples $(P_i)_{i \in I}$ of polynomials $P_i \in \mathbb{C}[u]$, with constant term 1, such that $\deg(P_i) = \lambda(i)$ for all $i \in I$. Set $\mathcal{P} = \cup_{\lambda \in P^+} \mathcal{P}^\lambda$.

**Theorem 3.3.** Let $\Phi = (\Phi_{i,r})_{i \in I, r \in \mathbb{Z}}$ be a pair of $(I \times \mathbb{Z})$-tuples of complex numbers, as above. Then, the irreducible representation $V(\Phi)$ of $U_q(\mathfrak{g})$ is finite-dimensional iff there exists $P = (P_i)_{i \in I} \in \mathcal{P}$ such that

$$\sum_{r = 0}^{\infty} \Phi_{i,r}^+ u^r = q_i^{\deg(P_i)} P_i(u) = \sum_{r = 0}^{\infty} \Phi_{i,-r}^- u^{-r},$$

in the sense that the left- and right-hand terms are the Laurent expansions of the middle term about 0 and $\infty$, respectively.

By abuse of notation, we denote the finite-dimensional irreducible representation of $U_q(\mathfrak{g})$ associated to $P$ by $V(P)$, and say that $P$ is its highest weight.

The ‘only if’ part of 3.3 is proved in [8], and we shall say no more about it in this paper. The ‘if’ part is proved in the next two sections.

To conclude the present section, however, we describe the behaviour of the $I$-tuples $P$ under tensor products. If $P = (P_i)_{i \in I}$, $Q = (Q_i)_{i \in I} \in \mathcal{P}$, let $P \otimes Q \in \mathcal{P}$ be the $I$–tuple $(P_i Q_i)_{i \in I}$.

**Proposition 3.4.** Let $P, Q \in \mathcal{P}$, and let $v_P$ and $v_Q$ be $U_q(\mathfrak{g})$-highest weight vectors in $V(P)$ and $V(Q)$, respectively. Then, in $V(P) \otimes V(Q)$, we have

$$x_{i,r}^+(v_P \otimes v_Q) = 0, \quad \phi_{i,r}^\pm(v_P \otimes v_Q) = \Psi_{i,r}^\pm(v_P \otimes v_Q)$$
for all \( i \in I, \, r \in \mathbb{Z} \), where the complex numbers \( \Psi_{i,r}^{\pm} \) are related to \( P \otimes Q \) as the \( \Phi_{i,r}^{\pm} \) are related to \( P \) in (6).

See [8] for the proof. The following result is an immediate consequence:

**Corollary 3.5.** Let \( P, Q \in \mathcal{P} \). Then, \( V(P \otimes Q) \) is isomorphic, as a representation of \( \hat{U}_q(\mathfrak{g}) \), to a quotient of the subrepresentation of \( V(P) \otimes V(Q) \) generated by the tensor product of the highest weight vectors in \( V(P) \) and \( V(Q) \).

Since every polynomial is a product of linear polynomials, the last result suggests that we define a representation \( V(P) \) of \( U_q(\mathfrak{g}) \) to be fundamental if, for some \( i \in I \), \( P_j = 1 \) if \( j \neq i \) and \( \text{deg}(P_i) = 1 \). Then, iterating 3.5, we obtain

**Corollary 3.6.** For any \( P \in \mathcal{P} \), the representation \( V(P) \) of \( U_q(\mathfrak{g}) \) is isomorphic to a subquotient of a tensor product of fundamental representations.

This suggests a method of proving the ‘if’ part of Theorem 3.3. For, in view of 3.6, it clearly suffices to prove that the fundamental representations of \( U_q(\mathfrak{g}) \) are all finite-dimensional. Since the fundamentals are the 'simplest' representations of \( U_q(\mathfrak{g}) \), it should be possible to describe them 'explicitly', and, in particular, to prove that they are finite-dimensional. We shall use this approach in the \( \mathfrak{sl}_2 \) case in the next section, and, although we have no doubt that it can be carried through in the general case, we shall use a different, more abstract, approach to complete the proof of 3.3 in Section 5.

4. Proof of the main theorem: \( \mathfrak{sl}_2 \) case

It is easy to construct finite–dimensional representations of the classical affine Lie algebra \( \hat{\mathfrak{g}} \) thanks to the existence of the family of homomorphisms \( \text{ev}_a : \hat{\mathfrak{g}} \to \mathfrak{g} \) which annihilate the centre of \( \hat{\mathfrak{g}} \) and evaluate maps \( \mathbb{C}^* \to \mathfrak{g} \) at \( a \in \mathbb{C}^* \). If \( V \) is a representation of \( \hat{\mathfrak{g}} \), the pull–back of \( V \) by \( \text{ev}_a \) is a representation \( V_a \) of \( \hat{\mathfrak{g}} \). Jimbo [13] defined an analogue of \( \text{ev}_a \) for \( U_q(\mathfrak{sl}_2) \):

**Proposition 4.1.** There is a family of algebra homomorphisms \( \text{ev}_a : U_q(\hat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2) \), defined for all \( a \in \mathbb{C}^* \), such that \( \text{ev}_a(e^{1/2}) = 1 \) and

\[
\text{ev}_a(x_{1,r}^+) = q^{-r}a^{-r}k_1^{-r}x_1^+, \quad \text{ev}_a(x_{1,r}^-) = q^{-r}a^{-r}x_1^+ k_1^-,
\]

for all \( r \in \mathbb{Z} \).

See [6], Proposition 4.1, for the proof.

**Remark.** Jimbo defined an analogue of \( \text{ev}_a \) for \( U_q(\hat{\mathfrak{sl}}_n) \), for all \( n \geq 2 \) (strictly speaking, if \( n > 2 \) Jimbo’s homomorphism takes values in an ‘enlargement’ of \( U_q(\mathfrak{sl}_n) \)). If \( \mathfrak{g} \) is not of type \( A \), there is no homomorphism \( U_q(\hat{\mathfrak{g}}) \to U_q(\mathfrak{g}) \) which is the identity on \( U_q(\mathfrak{g}) \subset U_q(\hat{\mathfrak{g}}) \) (see [8]).

If \( V \) is a type 1 representation of \( U_q(\mathfrak{sl}_2) \), its pull–back \( V_a \) by \( \text{ev}_a \) is obviously a type 1 representation of \( U_q(\hat{\mathfrak{sl}}_2) \); we call \( V_a \) an evaluation representation of \( U_q(\hat{\mathfrak{sl}}_2) \).
Since \( ev_a \) is the identity on \( U_q(\mathfrak{sl}_2) \), \( V_a \) is isomorphic to \( V \) as a representation of \( U_q(\mathfrak{sl}_2) \); in particular, \( V_a \) is irreducible if \( V \) is. The finite-dimensional irreducible type 1 representations of \( U_q(\mathfrak{sl}_2) \) are easy to describe. We know that there is exactly one such representation \( V(r) \) of each dimension \( r + 1 \geq 1 \), since the same is true for \( \mathfrak{sl}_2 \). It is easy to check that, if \( \{ v_0, v_1, \ldots, v_r \} \) is a basis of \( V(r) \), the formulas

\[
k_1 v_k = q^{r-2k} v_k, \quad x_1^+ v_k = [r - k + 1] q v_{k-1}, \quad x_1^- v_k = [k + 1] q v_{k+1}
\]
define the required representation (we set \( v_{-1} = v_{r+1} = 0 \)). Using the relations in 2.2, it follows that \( v_0 \) is a \( U_q(\hat{\mathfrak{g}}) \)-highest weight vector of \( V(r)_a \), and that

\[
\phi^+_1, k, v_0 = a^{-k} q^{k(r-1)} (q^r - q^{-r}) v_0 \\
h_{1, k} v_0 = q^{-k} a - k [r] q^k v_0.
\]

Using these formulas, one finds that \( V(r)_a \cong V(P_{r,a}) \), where

\[
P_{r,a}(u) = \prod_{k=1}^r (1 - a^{-1} q^{-2k+1} u).
\]

The set \( \Sigma_{r,a} = \{ aq^{-r+1}, aq^{-r+3}, \ldots, aq^{-1} \} \) of roots of \( P_{r,a} \) is called the \( q \)-segment of length \( r \) and centre \( a \).

At this point, it is easy to complete the proof of 3.3 in the \( \mathfrak{sl}_2 \) case. As we noted at the end of Section 3, it suffices to prove that the fundamental representations are finite-dimensional. But, since \( P_{1,a}(u) = 1 - a^{-1} u \), it follows that the fundamental representations of \( U_q(\hat{\mathfrak{sl}}_2) \) are precisely the \( V(1)_a \), for arbitrary \( a \in \mathbb{C}^\times \). In particular, they all have dimension 2.

Before turning to the general case of 3.3, however, we shall describe the structure of the representations \( V(P) \) of \( U_q(\mathfrak{sl}_2) \) in more detail:

**Proposition 4.2.** Let \( r_1, r_2, \ldots, r_k \in \mathbb{N}, a_1, a_2, \ldots, a_k \in \mathbb{C}^\times, k \in \mathbb{N} \). Then, the tensor product \( V(r_1)_{a_1} \otimes V(r_2)_{a_2} \otimes \cdots \otimes V(r_k)_{a_k} \) is reducible as a representation of \( U_q(\mathfrak{sl}_2) \) iff at least one pair of \( q \)-segments \( \Sigma_{r_i,a_i}, \Sigma_{r_j,a_j}, 1 \leq i, j \leq k \), are in special position, in the sense that their union is a \( q \)-segment which properly contains them both.

This is proved in [6].

It is now easy to describe the representation \( V(P) \), for any polynomial \( P \in \mathbb{C}[u] \) with constant coefficient 1. The roots of \( P \) form a multiset, i.e. a finite set of non-zero complex numbers (the roots of \( P \)), with a positive integer attached to each element of the set (its multiplicity as a root of \( P \)). It is not difficult to show that every multiset can be written uniquely as a union of \( q \)-segments, no two of which are in special position. (The union is in the sense of multisets: the multiplicity of a complex number in a union of multisets is the sum of its multiplicities in each of them.) We can thus write

multiplet of roots of \( P = \Sigma_{r_1,a_1} \cup \Sigma_{r_2,a_2} \cup \cdots \cup \Sigma_{r_k,a_k} \)
for some $r_1, r_2, \ldots, r_k \in \mathbb{N}$, $a_1, a_2, \ldots, a_k \in \mathbb{C}^\times$, $k \in \mathbb{N}$, and where no pair $\Sigma_{r_i, a_i}$, $\Sigma_{r_j, a_j}$ is in special position. By 3.5 and 4.2, there is an isomorphism of representations of $U_q(\hat{\mathfrak{sl}}_2)$

$$V(r_1)_{a_1} \otimes V(r_2)_{a_2} \otimes \cdots \otimes V(r_k)_{a_k} \cong V( P_{r_1, a_1} P_{r_2, a_2} \cdots P_{r_k, a_k} ).$$

But, the polynomial $P_{r_1, a_1} P_{r_2, a_2} \cdots P_{r_k, a_k}$ has the same roots as $P$, with the same multiplicities, and hence is equal to $P$ (both polynomials having constant coefficient 1). Thus,

$$V(P) \cong V(r_1)_{a_1} \otimes V(r_2)_{a_2} \otimes \cdots \otimes V(r_k)_{a_k}.$$  

We have proved

**Theorem 4.3.** Every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{sl}}_2)$ of type 1 is isomorphic to a tensor product of evaluation representations.

There is an amusing interpretation of $q$-segments in terms of ‘$q$-derivatives’, which will allow us to give a kind of Weyl dimension formula for $V(P)$. We recall that, if $P \in \mathbb{C}[u]$, its $q$-derivative is

$$(D_q P)(u) = \frac{P(q^2 u) - P(u)}{q^2 u - u}.$$  

It is obvious that $D_q P$ is a polynomial in $u$ (and $q$), and that

$$\lim_{q \to 1} D_q P = \frac{dP}{du}.$$  

The interpretation we have in mind is based on the following elementary result, whose proof we leave to the reader.

**Proposition 4.4.** Let $P \in \mathbb{C}[u]$ have non-zero constant coefficient, and let $\Sigma_P$ be its multiset of roots. Then, for each integer $k \geq 2$, the number of $q$-segments of length $k$ in $\Sigma_P$ is equal to the number of common roots of the polynomials $P, D_q P, \ldots, D_q^{k-1} P$.

To clarify the meaning of 4.4, suppose that, in the canonical decomposition of $\Sigma_P$ into a union of $q$-segments, no two of which are in special position, there is one segment of length 2 and one of length 3. Then, the number of $q$-segments of length 2 in $\Sigma_P$ is 3:

$$\circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ$$

Of course, there is one $q$-segment of length 3 in $\Sigma_P$, and none of length $> 3$.

In general, suppose that, for each $k \geq 1$, there are $n_k$ $q$-segments of length $k$ in the canonical decomposition of $\Sigma_P$. Then, there are $N_k$ $q$-segments of length $k$
altogether, where
\[ N_1 = n_1 + 2n_2 + 3n_3 + \cdots + rn_r, \]
\[ N_2 = n_2 + 2n_3 + 3n_4 + \cdots + (r-1)n_r, \]
\[ \vdots \]
\[ N_r = n_r, \]
and \( r = \deg(P) \). Hence,
\[ n_k = N_k - 2N_{k+1} + N_{k+2} \]
(we set \( N_k = 0 \) if \( k > r \)). By 4.3, and the discussion preceding it, it is clear that
\[ \dim(V(P)) = \prod_{k=1}^r (k+1)^{n_k}. \]
A little rearrangement now gives

**Proposition 4.5.** For any \( P \in \mathbb{C}[u] \) with constant coefficient 1,\[ \dim(V(P)) = 2^{\deg(P)} \prod_{k=2}^{\deg(P)} \left( \frac{k^2 - 1}{k^2} \right)^{N_k}, \]
where, for each integer \( k \geq 2 \), \( N_k \) is the number of common roots of \( P, D_qP, \ldots, D_q^{k-1}P \).

It would be interesting to find an analogue of this result for the dimensions of the representations \( V(P) \) of \( U_q(\hat{\mathfrak{g}}) \), for arbitrary \( \mathfrak{g} \).

5. Proof of the main theorem: general case

Let \( P = (P_i)_{i \in I} \in \mathcal{P} \) and let \( v_P \) be a \( U_q(\hat{\mathfrak{g}}) \)-highest weight vector in \( V(P) \). Since \( \phi_{i,0}^\pm = k_i^\pm 1 \), it follows from (6) that, if we define \( \lambda \in P^+ \) by \( \lambda(i) = \deg(P_i) \), then \( k_i.v_P = q_i^{\lambda(i)} v_P \), so \( v_P \in V(P)_{\lambda} \). By 2.3(b), \( V(P)_{\lambda} = \mathbb{C}v_P \) and
\[ V(P) = \bigoplus_{\eta \in Q^+} V(P)_{\lambda - \eta}. \]

Thus, to prove the ‘if’ part of 3.3, it is enough to prove the following assertions:

(a) \( V(P)_{\lambda - \eta} = 0 \) for all except finitely many \( \eta \in Q^+ \).

(b) For all \( \eta \in Q^+ \), \( \dim(V(P)_{\lambda - \eta}) < \infty \).

**Proof of (a).** Let \( 0 \neq v \in V(P)_\mu \), where \( \mu = \lambda - \eta, \eta \in Q^+ \). Let \( U_i \) be the subalgebra of \( U_q(\hat{\mathfrak{g}}) \) generated by \( x_i^\pm \) and \( k_i^\pm 1 \) (\( i \in I \)), \( v_i = U_i.v \).

Note that there is an obvious homomorphism of algebras (actually an isomorphism) \( U_q(sl_2) \to U_i \) which takes \( x_1^\pm \to x_i^\pm \), \( k_1^\pm 1 \to k_i^\pm 1 \), so \( V_i \) may be regarded as a representation of \( U_q(sl_2) \). We claim that, to prove (a), it suffices to prove

(c) If \( 0 \neq v \in V(P)_\mu \) and \( V_i = U_i.v \), then \( \dim(V_i) < \infty \).
To see that (c) implies (a), note that, if \( s_i \) is the \( i \)th fundamental reflection in the Weyl group \( W \) of \( \mathfrak{g} \), the finite-dimensionality of \( V_i \) implies that its set of weights is stable under the action of \( s_i \) (this follows from 3.1(e) and the analogous classical statement). Hence, \( V(P)_{\mu} \neq 0 \) implies \( V(P)_{s_i(\mu)} \neq 0 \) for all \( i \in I \). It follows that, if \( w \in W \) is arbitrary, then \( V(P)_{w(\mu)} \neq 0 \). Since one can choose \( w \) so that \( w(\mu) \in P^+ \), it follows that any \( \mu \in P \) such that \( V(P)_{\mu} \neq 0 \) belongs to the finite set
\[
W \{ \nu \in P^+ \mid \nu \leq \lambda \}.
\]

Thus, we are reduced to proving (c).

Now (c) is clearly a consequence of

(d) If \( V(P)_{\mu} \neq 0 \), there exists \( N > 0 \) such that \( V(P)_{\mu - r_{\alpha_i}} = V(P)_{\mu + r_{\alpha_i}} = 0 \) if \( r > N \).

Indeed, assuming (d), it is clear that \( V_i \) is spanned by \( \{ (x_i^\pm)_r \cdot v \mid 0 \leq r \leq N \} \).

To prove (d), note that it is obvious by 2.3(b) that \( V(P)_{\nu + r_{\alpha_i}} = 0 \) for \( r >> 0 \), since \( \mu + r_{\alpha_i} \leq \lambda \) only for finitely many \( r > 0 \). We shall prove, on the other hand, that \( V(P)_{\mu - r_{\alpha_i}} = 0 \) if \( r > 3h + \lambda(i) \), where \( h = \text{height}(\lambda - \mu) \). Indeed, this follows from

(e) For any \( r > 0 \), \( V(P)_{\mu - r_{\alpha_i}} \) is spanned by vectors of the form
\[
X_1^- x_{i,k_1}^- X_2^- x_{i,k_2}^- \cdots X_h^- x_{i,k_h}^- X_{h+1}^- v \mu, \tag{8}
\]
where \( \lambda - \mu = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_h}, k_1, k_2, \ldots, k_h \in \mathbb{Z} \) are arbitrary, and each \( X_p^- \), \( 1 \leq p \leq h + 1 \), is a product of the form
\[
X_p^- = x_{i,\ell_1,p}^- x_{i,\ell_2,p}^- \cdots x_{i,\ell_p,p}^-;
\]
for some \( \ell_1, \ell_2, \ldots, \ell_p \in \mathbb{Z} \) and \( r_1, r_2, \ldots, r_{h+1} \in \mathbb{N} \) such that
\[
r_1 + r_2 + \cdots + r_{h+1} = r
\]
and
\[
r_1, r_2, \ldots, r_h \leq 3. \tag{9}
\]

To see that (e) implies that \( V(P)_{\mu - r_{\alpha_i}} = 0 \) if \( r > 3h + \lambda(i) \), let \( \hat{U}_i \) be the subalgebra of \( U_q(\hat{\mathfrak{g}}) \) generated by \( \{ x_{i,k}^\pm, \phi_{i,k}^\pm \}_{k \in \mathbb{Z}} \), and set \( \hat{V}_i = \hat{U}_i \cdot v \mu \). There is an obvious homomorphism of algebras (actually an isomorphism) \( U_q(\hat{\mathfrak{s}l}_2) \to \hat{U}_i \) which takes \( x_{i,k}^\pm \mapsto x_{i,k}^\pm, \phi_{i,k}^\pm \mapsto \phi_{i,k}^\pm \), so \( \hat{V}_i \) may be regarded as a representation of \( U_q(\hat{\mathfrak{s}l}_2) \). According to Lemma 2.3 in [9], \( \hat{V}_i \cong V(P_i) \) as a representation of \( U_q(\hat{\mathfrak{s}l}_2) \) (in particular, \( \hat{V}_i \) is irreducible). It follows from 4.2 that \( (V_i)_{\lambda - s_{\alpha_i}} = 0 \) if \( s > \lambda(i) \). On the other hand, 2.3 implies that \( V(P)_{\lambda - s_{\alpha_i}} = (\hat{V}_i)_{\lambda - s_{\alpha_i}} \), for all \( s \geq 0 \). Now, for any vector (8) satisfying the conditions in (e), we have \( r_{h+1} \geq r - 3h > \lambda(i) \), so
\[
X_{h+1}^- v \mu \in V(P)_{\lambda - r_{h+1}\alpha_i} = 0
\]

Thus, (e) implies that \( V(P)_{\lambda - r_{\alpha_i}} = 0 \) if \( r > 3h + \lambda(i) \).

To prove (e), note that it is obvious by 2.3(b) that \( V(P)_{\mu - r_{\alpha_i}} \) is spanned by vectors of the form (8) satisfying all the stated conditions except possibly condition (9). Thus, it suffices to show that any vector \( v \) of the form (8) which does not
satisfy (9) can be written as a linear combination of vectors of the same form which do satisfy (9). We prove this by induction on \( h \).

If \( h = 0 \), there is nothing to prove. Assume that \( h \geq 1 \). By repeated use of relation (4) in 2.2, the product \( X_1^- x_{i_1,k_1}^- \) can be expressed as a linear combination of terms of the form \( Y_1^- x_{i_1,k_1}^- \hat{Y}_1^- \), where \( Y_1^- \) and \( \hat{Y}_1^- \) are of the same form as \( X_1^- \), but where \( Y_3^- \) is a product of \( \leq 3 \) generators \( x_{i,-\ell}^- \), and \( \hat{Y}_1^- \) is a product of \( \geq \ell_i + 3 \) such generators. (If \( \mathfrak{g} \) is simply-laced, we can assume that \( Y_1^- \) is a single \( x_{i,-\ell}^- \), and if \( \mathfrak{g} \) is of type B, C or F, that \( Y_1^- \) is a product of \( \leq 2 \) such generators.) So \( v \) can be expressed as a linear combination of vectors

\[
Y_1^- x_{i_1,k_1}^- \hat{Y}_1^- X_2^- x_{i_2,k_2}^- \cdots X_{h+1}^- v_P. 
\]

By the induction hypothesis,

\[
\hat{Y}_1^- X_2^- x_{i_2,k_2}^- \cdots X_{h+1}^- v_P
\]

can be expressed as a linear combination of vectors

\[
Y_2^- x_{i_2,k_2}^- Y_3^- x_{i_3,k_3}^- \cdots Y_{h+1}^- v_P,
\]

where each of \( Y_2^- , Y_3^- , \ldots , Y_{h+1}^- \) is a product of \( \leq 3 \) \( x_{i,-\ell}^- \)'s, and \( Y_{h+1}^- \) is a product of \( \geq r - 3(h - 1) \) \( x_{i,-\ell}^- \)'s. This completes the inductive step and proves (e).

The proof of (a) is now complete.

**Proof of (b).** We proceed by induction on \( h = \text{height}(\eta) \). If \( \eta = 0 \), there is nothing to prove. If \( \eta = \alpha_i \), we have to show that the vectors \( x_{i,k}^- v_P \) \((k \in \mathbb{Z})\) span a finite-dimensional space. But this space is obviously contained in \( \hat{U}_i.v_P \), and we have already seen that \( \hat{U}_i.v_P \) is finite-dimensional.

Assume now that \( h \geq 2 \), and that (b) has been proved for \( \eta \)'s of height \( < h \). The weight space \( V(P)_{\lambda - \eta} \) is spanned, in view of 2.3(b), by vectors of the form

\[
x_{i_1,k_1}^- x_{i_2,k_2}^- \cdots x_{i_h,k_h}^- v_P,
\]

where \( \eta = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_h} \) and \( k_1, k_2, \ldots, k_h \in \mathbb{Z} \). It clearly suffices to prove that the vectors (10) span a finite-dimensional space for each fixed choice of \( i_1, \ldots, i_h \); denote this space by \( V_{i_1, \ldots, i_h} \). By the induction hypothesis, there exists \( M \in \mathbb{N} \) such that, for all \( i \in \{ i_1, i_2, \ldots, i_h \} \), \( V(P)_{\lambda - \eta + \alpha_i} \) is spanned by vectors of the form

\[
x_{j_2,\ell_2}^- x_{j_3,\ell_3}^- \cdots x_{j_h,\ell_h}^- v_P,
\]

where \( \alpha_{j_2} + \alpha_{j_3} + \cdots + \alpha_{j_h} = \eta - \alpha_i \) and \( |\ell_2|, |\ell_3|, \ldots, |\ell_h| \leq M \). It suffices to prove that \( V_{i_1, \ldots, i_h} \) is contained in the space

\[
W = \sum_{k_2=-M}^{M+1} x_{i_2,k_2}^- V(P)_{\lambda - \eta + \alpha_{i_2}} + x_{i_1,0}^- V(P)_{\lambda - \eta + \alpha_{i_1}},
\]

since \( W \) is finite-dimensional by the induction hypothesis.
For this, we prove, by induction on $k_1$, that the vector (10) lies in $W$ for every $k_2, \ldots, k_h$ (we assume that $k_1 \geq 0$, the proof for $k_1 \leq 0$ being essentially the same). The case $k_1 = 0$ is obvious. For the inductive step, note that we can assume that $|k_2|, |k_3|, \ldots, |k_h| \leq M$. Using relation (3) in 2.2, any vector (10) can be written as a linear combination of the vectors

\[(13)\quad x_{i_2,k_2}^- x_{i_1,k_1}^- x_{i_3,k_3}^- \cdots x_{i_h,k_h}^- \cdot v_P,
\]
\[(14)\quad x_{i_2,k_2+1}^- x_{i_1,k_1-1}^- x_{i_3,k_3}^- \cdots x_{i_h,k_h}^- \cdot v_P,
\]
\[(15)\quad x_{i_1,k_1-1}^- x_{i_2,k_2+1}^- x_{i_3,k_3}^- \cdots x_{i_h,k_h}^- \cdot v_P.
\]

But, vectors of types (13) and (14) obviously belong to $W$, and those of type (15) belong to $W$ by the induction hypothesis on $k_1$. This completes the inductive step. (Note that, by the induction hypothesis again, the vector $x_{i_2,k_2}^- x_{i_1,k_1}^- \cdots x_{i_h,k_h}^- \cdot v_P$ can be written as a linear combination of vectors $x_{i_2,k_2'}^- x_{i_3,k_3}^- \cdots x_{i_h,k_h'}^- \cdot v_P$, where $\alpha_{i_2} + \cdots + \alpha_{i_h} = \alpha_{i_2} + \cdots + \alpha_{i_h}$ and $|k_2'|, \ldots, |k_h'| \leq M$.)

This completes the proof of (b), and hence that of Theorem 3.3.

6. Minimal affinizations

We saw at the beginning of Section 5 that, if $P = (P_i)_{i \in I} \in \mathcal{P}$, and $\lambda \in P^+$ is defined by $\lambda(i) = \deg(P_i)$, then

\[(16)\quad V(P) = \bigoplus_{\eta \in Q^+} V(P)_{\lambda-\eta} \quad \text{and} \quad \dim(V(P)_\lambda) = 1.
\]

Since $V(P)$ is finite-dimensional, it is completely reducible as a representations of $U_q(g)$, and in view of (16) we have

\[V(P) \cong V(\lambda) \oplus \bigoplus_{\mu \in P^+} V(\mu)^{\oplus m_\mu}\]

as a representation of $U_q(g)$, where the multiplicities $m_\mu \in \mathbb{N}$ are zero unless $\mu < \lambda$.

Thus, $V(P)$ gives a way of extending the action of $U_q(g)$ on $V(\lambda)$ to an action of $U_q(g)$, at the expense of enlarging $V(\lambda)$ by the addition of representations of $U_q(g)$ of smaller highest weight. For this reason, we call $V(P)$ an affinization of $V(\lambda)$. We say that two affinizations are equivalent if and only if they are isomorphic as representations of $U_q(g)$, and we denote by $[V(P)]$ the equivalence class of $V(P)$.

There is one situation in which affinizations are unique, up to equivalence:
Proposition 6.1. For any \( i \in I \), \( V(\lambda_i) \) has a unique affinization, up to equivalence.

Proof. If \( V(P) \) is an affinization of \( V(\lambda_i) \), then \( P_j = 1 \) if \( j \neq i \) and \( P_i(u) = 1 - a^{-1}u \), for some \( a \in \mathbb{C}^\times \) (i.e. \( V(P) \) is a fundamental representation of \( U_q(\hat{g}) \)). Denoting this \( V(P) \) by \( V(\lambda_i, a) \), we have to prove that the equivalence class \([V(\lambda_i, a)]\) is independent of \( a \).

We make use of the family of (Hopf) algebra automorphisms \( \tau_t (t \in \mathbb{C}^\times) \) of \( U_q(\hat{g}) \) defined by
\[
\tau_t(x_{i,k}) = t^k x_{i,k}^\pm, \quad \tau_t(\phi_{i,k}^\pm) = t^k \phi_{i,k}^\pm, \quad \tau_t(c^{1/2}) = c^{1/2}.
\]

It is easy to see that, for any \( Q = (Q_i)_{i \in I} \in \mathcal{P} \), the pull-back \( \tau_t^*(V(Q)) \) of \( V(Q) \) by \( \tau_t \) is isomorphic as a representation of \( U_q(\hat{g}) \) to \( V(Q') \), where \( Q' = (Q'_i) \) and \( Q'_i(u) = Q_i(tu) \).

In particular, \( \tau_t^*(V(\lambda_i, a)) \cong V(\lambda_i, 1) \). Since \( \tau_a \) is the identity of \( U_q(\mathfrak{g}) \), it follows that \([V(\lambda_i, a)] = [V(\lambda_i, 1)]\).

Corollary 6.2. For any \( \lambda \in P^+ \), \( V(\lambda) \) has, up to equivalence, only finitely many affinizations.

Proof. By 3.6, any affinization \( V(P) \) of \( V(\lambda) \) is isomorphic as a representation of \( U_q(\hat{g}) \) to a subquotient of a tensor product
\[
\bigotimes_{i \in I} \bigotimes_{j=1}^{\lambda(i)} V(\lambda_i, b_{j,i}),
\]
for some \( b_{j,i} \in \mathbb{C}^\times \) (the order of the factors is unimportant). By 6.1, this tensor product is, up to \( U_q(\mathfrak{g}) \)-isomorphism, independent of the \( b_{j,i} \). It therefore has only finitely many subquotients, regarded as a representation of \( U_q(\mathfrak{g}) \).

In general, a representation \( V(\lambda) \) of \( U_q(\mathfrak{g}) \) has many inequivalent affinizations, and it is natural to ask if one can make a canonical choice among them. To this end, the following partial order on the set of affinizations was introduced in [4].

Proposition 6.3. Let \( \lambda \in P^+ \) and let \( V(P) \) and \( V(Q) \) be affinizations of \( V(\lambda) \). Then, we write \([V(P)] \preceq [V(Q)]\) iff, for all \( \mu \in P^+ \), either
\begin{enumerate}
  \item \( m_\mu(V(P)) \leq m_\mu(V(Q)) \), or
  \item there exists \( \nu > \mu \) with \( m_\nu(V(P)) < m_\nu(V(Q)) \).
\end{enumerate}

Then, \( \preceq \) is a partial order on the set of equivalence classes of affinizations of \( V(\lambda) \).

An affinization \( V(P) \) of \( V(\lambda) \) is minimal if, whenever \( V(Q) \) is an affinization of \( V(\lambda) \) and \([V(Q)] \preceq [V(P)]\), we have \([V(P)] = [V(Q)]\). In view of 6.2, minimal affinizations certainly exist.

If \( \mathfrak{g} = sl_2 \), we explained in Section 4 that the homomorphisms \( ev_a : U_q(\hat{sl}_2) \rightarrow U_q(sl_2) \) enable one to extend the action of \( U_q(sl_2) \) on any representation \( V(\lambda) \) to

an action of $U_q(sl_2)$ on the same space. These evaluation representations obviously provide the unique minimal affinization. We mentioned in Section 4 that there are analogues of the $ev_a$ when $g = sl_n$ for any $n \geq 2$, so the minimal affinizations are also unique, and irreducible under $U_q(g)$, in that case.

The following result, proved in [7], gives the defining polynomials of the minimal affinizations in the type A case.

**Theorem 6.4.** Let $g = sl_{n+1}(\mathbb{C})$ and let $\lambda \in P^+$. Number the nodes of the Dynkin diagram of $g$ as in [3]. Then, $V(\lambda)$ has, up to equivalence, a unique minimal affinization. It is represented by $V(P)$, where $P = (P_i)_{i \in I} \in P^\lambda$, iff, for all $i \in I$ such that $\lambda(i) > 0$, the roots of $P_i$ form a $q$-segment with centre $a_i \in \mathbb{C}^\times$ (say) and length $\lambda(i)$, where

(i) either, for all $i < j$ such that $\lambda(i) > 0$ and $\lambda(j) > 0$,

$$\frac{a_i}{a_j} = q^{\lambda(i) + 2(\lambda(i+1) + \cdots + \lambda(j-1)) + \lambda(j) + j - i},$$

(ii) or, for all $i < j$ such that $\lambda(i) > 0$ and $\lambda(j) > 0$,

$$\frac{a_j}{a_i} = q^{\lambda(i) + 2(\lambda(i+1) + \cdots + \lambda(j-1)) + \lambda(j) + j - i}.$$

To state the corresponding results when $g$ is of type B, C or F, number the nodes of the Dynkin diagram as in [3], and define, for any $\lambda \in P^+$, complex numbers $c_i(\lambda)$ as follows:

$$c_i(\lambda) = \begin{cases} q^{d_i(\lambda(i) + \lambda(i+1) + 1)} & \text{if } a_{i+1}, a_{i+1} = 1, \\ q^{d_i(\lambda(i) + d_{i+1} \lambda(i+1) + 2d_{i+1} - 1)} & \text{if } a_{i+1}, a_{i+1} = 1. \end{cases}$$

**Theorem 6.5.** Let $g$ be non-simply-laced, and let $\lambda \in P^+$. Then, $V(P)$ is a minimal affinization of $V(\lambda)$ iff $P \in P^\lambda$ satisfies the following conditions:

(i) For all $i \in I$, either $P_i = 1$ or the roots of $P_i$ form a $q_i$-segment of length $\lambda(i)$ and centre $a_i$ (say).

(ii) Either, for all $i < j$ such that $\lambda(i) > 0$ and $\lambda(j) > 0$, we have

$$\frac{a_i}{a_j} = \prod_{s=i}^{j-1} c_s,$$

or, for all $i < j$ such that $\lambda(i) > 0$ and $\lambda(j) > 0$, we have

$$\frac{a_i}{a_j} = q^{2d_j - 2d_i} \prod_{s=i}^{j-1} c_s^{-1}.$$

The minimal affinization of $V(\lambda)$ is unique, up to equivalence.

See [10] for the proof. Note that, for any $r$, $I \setminus I_r$ defines a type A subdiagram, so 6.4 gives the precise conditions under which $V(P_{I \setminus I_r})$ is a minimal affinization.
Turning finally to the D and E cases, we introduce the following notation. If \( \emptyset \neq J \subseteq I \), and \( \lambda \in P^+ \), let \( \lambda_J \) be the restriction of \( \lambda : I \rightarrow \mathbb{Z} \) to \( J \). Also, if \( P = (P_i)_{i \in I} \in \mathcal{P} \), let \( P_J \) be the \( J \)-tuple \( (P_j)_{j \in J} \).

Let \( i_0 \) be the unique node of the Dynkin diagram of \( g \) which is linked to three other nodes. Then,

\[
I \setminus \{i_0\} = I_1 \amalg I_2 \amalg I_3,
\]

where \( I_1, I_2 \) and \( I_3 \) define type A subdiagrams.

**Theorem 6.6.** Let \( g \) be of type D or E, let \( \lambda \in P^+ \), and assume that \( \lambda(i_0) \neq 0 \).

If \( \lambda_{r} = 0 \) for some \( r \in \{1, 2, 3\} \), then \( V(\lambda) \) has a unique minimal affinization, up to equivalence. It is represented by \( V(P) \) iff \( V(P_{I \setminus \{i_0\}}) \) is a minimal affinization of \( V(\lambda_{I \setminus \{i_0\}}) \).

If \( \lambda_{r} \neq 0 \) for all \( r \in \{1, 2, 3\} \), then \( V(\lambda) \) has exactly three minimal affinizations, up to equivalence. In fact, \( V(P) \) is a minimal affinization of \( V(\lambda) \) iff there exist \( r \neq s \) in \( \{1, 2, 3\} \) such that \( V(P_{I \setminus \{i_0\}}) \) and \( V(P_{I \setminus \{i_0\}}) \) are minimal affinizations of \( V(\lambda_{I \setminus \{i_0\}}) \) and \( V(\lambda_{I \setminus \{i_0\}}) \), respectively.

See [9] for the proof.

**Remark.** The result of this theorem no longer holds if we drop the assumption \( \lambda(i_0) > 0 \). If \( g \) is of type \( D_4 \), for example, and \( \lambda(i_0) = 0 \), the number of minimal affinizations of \( V(\lambda) \) increases with \( \lambda \) (roughly speaking), and is generally greater than three.

To conclude our discussion of minimal affinizations, we consider their structure as representations of \( U_q(g) \). Except when \( g \) is of type A, when the minimal affinizations are irreducible under \( U_q(g) \), this is not well understood. We give two results.

**Theorem 6.7.** Let \( g \) be of type \( B_2 \), let \( \theta \) be the highest root of \( g \), and assume that \( \alpha_2 \) is the short simple root. Let \( \lambda \in P^+ \) and let \( V(P) \) be a minimal affinization of \( V(\lambda) \). Then, as representations of \( U_q(g) \),

\[
V(P) \cong \bigoplus_{r=0}^{[\frac{1}{2}(\lambda(2))]_r} V(\lambda - r\theta).
\]

See [5] for the proof. Our final result gives the \( U_q(g) \)-structure of most of the fundamental representations of \( U_q(\hat{g}) \).

**Theorem 6.8.** Number the nodes of the Dynkin diagram of \( g \) as in [3].

(a) \( V(\lambda_i, 1) \cong V(\lambda_i) \) under any of the following conditions:

(i) \( g \) is of type A or C and \( i \) is arbitrary;
(ii) \( g \) is of type \( B_n \) (\( n \geq 2 \)) and \( i = 1 \) or \( n \);
(iii) \( g \) is of type \( D_n \) (\( n \geq 4 \)) and \( i = 1, n - 1 \) or \( n \).
(b) If \( \mathfrak{g} \) is of type \( B_n \) or \( D_{n+1} \) \((n \geq 3) \) and \( 1 < i < n \),
\[
V(\lambda_i, 1) \cong \bigoplus_{j=0}^{[i/2]} V(\lambda_{i-2j}).
\]

(c) If \( \mathfrak{g} \) is of type \( E_6 \),
\[
V(\lambda_1, 1) \cong V(\lambda_1), \quad V(\lambda_2, 1) \cong V(\lambda_2) \oplus \mathbb{C},
\]
\[
V(\lambda_3, 1) \cong V(\lambda_3) \oplus V(\lambda_6),
\]
\[
V(\lambda_4, 1) \cong V(\lambda_4) \oplus V(\lambda_1 + \lambda_6) \oplus V(\lambda_2) \oplus V(\lambda_2) \oplus \mathbb{C},
\]
\[
V(\lambda_5, 1) \cong V(\lambda_5) \oplus V(\lambda_1), \quad V(\lambda_6, 1) \cong V(\lambda_6).
\]
(Here and below, \( \mathbb{C} \) denotes the 1-dimensional trivial representation.)

(d) If \( \mathfrak{g} \) is of type \( E_7 \),
\[
V(\lambda_1, 1) \cong V(\lambda_1) \oplus \mathbb{C}, \quad V(\lambda_2, 1) \cong V(\lambda_2) \oplus V(\lambda_7),
\]
\[
V(\lambda_3, 1) \cong V(\lambda_3) \oplus V(\lambda_6) \oplus V(\lambda_1) \oplus V(\lambda_1) \oplus \mathbb{C},
\]
\[
V(\lambda_6, 1) \cong V(\lambda_6) \oplus V(\lambda_1) \oplus \mathbb{C}, \quad V(\lambda_7, 1) \cong V(\lambda_7).
\]

(e) If \( \mathfrak{g} \) is of type \( E_8 \),
\[
V(\lambda_1, 1) \cong V(\lambda_1) \oplus V(\lambda_8) \oplus \mathbb{C},
\]
\[
V(\lambda_7, 1) \cong V(\lambda_7) \oplus V(\lambda_1) \oplus V(\lambda_8) \oplus V(\lambda_8) \oplus \mathbb{C},
\]
\[
V(\lambda_8, 1) \cong V(\lambda_8) \oplus \mathbb{C}.
\]

(f) If \( \mathfrak{g} \) is of type \( F_4 \),
\[
V(\lambda_1, 1) \cong V(\lambda_1) \oplus \mathbb{C},
\]
\[
V(\lambda_2, 1) \cong V(\lambda_2) \oplus V(2\lambda_4) \oplus V(\lambda_1) \oplus V(\lambda_1) \oplus \mathbb{C},
\]
\[
V(\lambda_3, 1) \cong V(\lambda_3) \oplus V(\lambda_1), \quad V(\lambda_4, 1) \cong V(\lambda_4).
\]

(g) If \( \mathfrak{g} \) is of type \( G_2 \),
\[
V(\lambda_1, 1) \cong V(\lambda_1) \oplus \mathbb{C}, \quad V(\lambda_2, 1) \cong V(\lambda_2).
\]

This can be proved using the techniques of [5].

References

1. R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, New York, 1982.
2. J. Beck, *Braid group action and quantum affine algebras*, preprint, MIT (1993).
3. N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4, 5 et 6*, Hermann, Paris, 1968.
4. V. Chari, *Minimal affinizations of representations of quantum groups: the rank 2 case*, preprint (1994).
5. , *Minimal affinizations of representations of quantum groups: \( U_q(\mathfrak{g}) \)-structure*, preprint (1994).
6. V. Chari and A. N. Pressley, *Quantum affine algebras*, Commun. Math. Phys. 142 (1991), 261–283.
7. , *Small representations of quantum affine algebras*, Lett. Math. Phys. 30 (1994), 131–145.
8. A Guide to Quantum Groups, Cambridge University Press, Cambridge, 1994.
9. Minimal affinizations of representations of quantum groups: the simply-laced case, preprint (1994).
10. Minimal affinizations of representations of quantum groups: the non-simply-laced case, preprint (1994).
11. V. G. Drinfel’d, Quantum groups, Proceedings of the International Congress of Mathematicians, Berkeley, 1986, American Mathematical Society, 1987, pp. 798–820.
12. A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. 36 (1988), 212–216.
13. M. Jimbo, A $q$-analog of $U(gl(N + 1))$, Hecke algebra and the Yang–Baxter equation, Lett. Math. Phys. 11 (1986), 247–252.
14. G. Lusztig, Introduction to Quantum Groups, Birkhäuser, Boston, 1993.

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