Local $PT$-symmetric evolutions on separable states and violation of no-signaling

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We show that local $PT$-symmetric evolutions can lead to violation of the no-signaling principle for separable and even classically correlated bipartite shared quantum states. For classically correlated states, specially chosen $PT$-symmetric operations from a set of zero volume can also preserve the principle. The violations can be removed by using a $CPT$ inner product instead of the traditional one.

I. INTRODUCTION

It is usually assumed in textbook quantum mechanics that the observables to be measured are hermitian operators. The possible outcomes of the measurement of these hermitian operators are then the eigenvectors - or clusters of them - and for systematic bookkeeping of what transpired in a measurement, we use the corresponding real eigenvalues. The eigenvectors associated with the eigenvalues form a complete orthonormal set. The evolution corresponding to hermitian Hamiltonians are unitary \[^1\]. However, non-hermitian observables have also been found to have useful applications, and in recent years, such Hamiltonians with real eigenvalues have garnered considerable interest. “$PT$-symmetric” Hamiltonians \[^2\] form an important example of such a class of non-hermitian Hamiltonians, which respect parity and time-reversal symmetry. Although $PT$-symmetric Hamiltonians have real eigenvalues, the eigenvectors corresponding to distinct eigenvalues may not be orthogonal, and evolutions governed by $PT$-symmetric Hamiltonians are non-unitary. See \[^3\] [11] and references therein for further aspects.

Although the interpretation of $PT$-symmetric quantum mechanics remain contentious, it has been used in several areas \[^12\] [27]. Experiments related to such evolutions have been reported in \[^28\] [48]. Recently, Lee et al. in Ref. \[^14\] showed that the local operation of the evolution operator governed by a $PT$-symmetric Hamiltonian, on a bipartite entangled shared state \[^10\], violates the no-signaling principle. This result has been experimentally tested by Tang et al. \[^29\]. A post-selection process was used to simulate the $PT$-symmetric evolution, and without the post-selection, the violation of no-signaling does not appear.

The no-signaling principle requires that instantaneous communication of information at a distance is not allowed \[^50\]. Local non-relativistic quantum evolution on shared quantum states do not allow the violation of this principle. The issue of violation of the no-signaling principle via local $PT$-symmetric evolutions has been re-addressed, and in particular, in Refs. \[^15\] [19, 51], it has been shown that utilizing a “$CPT$ inner product”, instead of the standard inner product of separable Hilbert spaces, can restore the no-signaling principle within $PT$-symmetric quantum mechanics. See also \[^4\] [3, 18, 13, 52–55] in this regard.

The rest of the paper is arranged as follows. In Sec. \[^II\] we discuss the most general two-dimensional $PT$-symmetric Hamiltonian. Violation of the no-signaling principle in $PT$-symmetric quantum mechanics for shared Werner states is considered in Sec. \[^III\]. The case of classically correlated states without shared quantum discord is taken up in Sec. \[^IV\]. The $CPT$ inner product consideration is presented in Sec. \[^V\]. We present a conclusion in Sec. \[^VI\].

II. HAMILTONIAN OF A $PT$-SYMMETRIC SPIN-1/2 SYSTEM

Let us begin by considering the most general $PT$-symmetric Hamiltonian of a system described on a two-dimensional complex Hilbert space. Let it be denoted by $JH_{PT}$, where $H_{PT}$ given by \[^6\]

$$H_{PT} = \begin{pmatrix}
  r + t \cos \xi - is \sin \xi & is \cos \xi + t \sin \xi \\
is \cos \xi + t \sin \xi & r - t \cos \xi + is \sin \xi
\end{pmatrix}.
$$

(1)

The parameters $r, s, t, \xi$ containing in the Hamiltonian are real. The constant, $J$, is a real number that is nonzero and that has the unit of “energy”. This implies that the other parameters are dimension-free. The parity opera-
tor, $P$, is given by

$$P = \begin{pmatrix} \cos \tilde{\phi} & \sin \tilde{\phi} \\ \sin \tilde{\phi} & -\cos \tilde{\phi} \end{pmatrix},$$

(2)

where $\tilde{\phi}$ is real. The time-reversal operator, $T$, is complex conjugation in the computational basis, with the latter being the eigenbasis of the Pauli $\sigma_z$ operator. We are interested in those $H_{PT}$ for which the eigenvalues are real, so that we require $s^2 \leq t^2$. Setting $\sin \sigma = s/t$, we find that the eigenvalues of $JH_{PT}$ are $E_{\pm} = J(r \pm t \cos \alpha)$. The respective eigenstates are given by eigenstates being nonorthogonal. Also, the eigenvectors fall on each other at the “branch points”, $\alpha = \pm \pi/2$. We will consider $PT$-symmetric Hamiltonians away from their branch points. See Refs. 55, 58.

III. WERNER STATES AND VIOLATION OF NO-SIGNALING

Lee and co-authors in Ref. 14 have shown that local evolutions governed by $PT$-symmetric Hamiltonians on bipartite entangled states can result in the violation of the no-signaling principle. In this section, we do the same analysis with the local evolution corresponding to the most general $PT$-symmetric Hamiltonian acting on an arbitrary (but fixed) shared Werner state.

Let Alice and Bob be in two space-like separated locations, sharing the Werner state given by

$$\rho_{ABw} = p|\psi\rangle\langle \psi|_{AB} + \frac{1-p}{4} I_2 \otimes I_2,$$

(4)

where $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and $|0\rangle$ and $|1\rangle$ are eigenstates of the Pauli $\sigma_z$ operator. $I_2$ is the identity operator on the qubit Hilbert space. For $\rho_{ABw}$ to be a quantum state, we require $p$ to lie in the range $[-1/3, 1]$. The no-signaling principle demands that the probability distribution of outcomes of an arbitrary measurement at Bob’s end is unaffected by Alice’s choice of operations. In other words, the no-signaling principle is satisfied if the reduced density matrix of Bob before as well as after the local operation of Alice are the same. Before a local operation, the state at Bob’s end is given by

$$\rho_{Bw} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{I_2}{2}.$$

(5)

Now, let us assume that Alice evolves her part of the shared Werner state by using the $PT$-symmetric Hamiltonian given in Eq. 1. If the non-unitary evolution by Alice is $U_A = e^{-iH_{PT} \tau}$, where $\tau = J\tau'/\hbar$, with $\tau'$ playing the role of “time”, then the composite density matrix reduces to

$$\rho_{ABw}^{U_A} = U_A \otimes I_2 \rho_{ABw} U_A^\dagger \otimes I_2.$$

(6)

In order to check the validity of no-signaling principle, we take the partial trace over Alice’s part. Then the reduced state of Bob, after renormalization, is obtained as

$$\rho_{Bw}^{U_A} = \text{Tr}_A[\rho_{ABw}^{U_A}] = \frac{1}{N_2} \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

(7)

where $N_2 = 2 \sec^2 \alpha \sin^2 t_1 + \cos(2t_1)$, and

$$w_{11} = \frac{1}{2} [N_2 - p \tan \alpha \sin(2t_1) \sin \xi],$$

$$w_{22} = \frac{1}{2} [N_2 + p \tan \alpha \sin(2t_1) \sin \xi],$$

$$w_{21} = 2p \tan \alpha \sin t_1 (\cos t_1 \cos \xi + i \sec \sin t_1),$$

$$w_{12} = w_{21}^*.$$

(8)

Here $t_1 = t \tau \cos \alpha$. To satisfy the no-signaling principle, we require $\rho_{Bw}^{U_A} = I_2/2$. We exclude the branch points at $\alpha = \pm \pi/2$. By comparing the elements of $\rho_{Bw}^{U_A}$ with those of $\rho_{Bw}$, we can see that no-signaling principle is satisfied only if $\alpha = 0$ or $t_1 = 0$ or $p = 0$. For $\alpha = 0$, the Hamiltonian $H_{PT}$ becomes hermitian. And $t_1 = 0$ gives $t = 0$, which in turn leads to a Hamiltonian, $H_{PT}$, that is hermitian, as then $s = 0$, or $\tau = 0$, which implies that the evolution did not occur. We therefore find that all $PT$-symmetric non-hermitian Hamiltonians with real eigenvalues lead to signaling of information by using the pre-shared Werner state (unless $p = 0$), just as for the maximally entangled state shown in [14]. A maximally entangled state is obtained from the family of Werner states by setting $p = 1$.

It is known that the family of Werner states contains entangled as well as separable states. Indeed, it is entangled in the range $p \in (1/3, 1]$, while separable for $p \in [-1/3, 1/3]$ [60]. Therefore, the violation of no-signaling can be afforded by shared states without any entanglement, for local $PT$-symmetric evolutions.

IV. CLASSICALLY CORRELATED STATES AND VIOLATION OF NO-SIGNALING

In the preceding section, we have seen that separable states can also offer violation of the no-signaling principle, for local $PT$-symmetric evolutions. We will see in this section that even weaker correlations suffice for the same violation to occur. To deal with task, we will need some concepts in quantum correlations, and we digress in the next paragraph for a discussion about the same.

A bipartite quantum state, $\rho_{AB}$, shared between two observers, $A$ and $B$, is said to be entangled if it cannot be expressed in separable form, viz.

$$\rho_{AB} = \sum_i p_i |\tilde{\psi}_i\rangle \otimes |\tilde{\psi}_i\rangle.$$

(9)
where \( \{ p_i \} \) form a probability distribution, and \( \tilde{\rho}_A (\tilde{\rho}_B) \) are density matrices of the physical system in possession of \( A \) (\( B \)). It has however been realized that quantum correlations in shared systems can also be conceptualized independent of the entanglement-separability paradigm, with one of the popular ones being quantum discord. It is defined as the difference between quantized versions of two classically equivalent versions of mutual information of measured bipartite quantum states. It was found that while quantum discord is identical with entanglement for pure bipartite states, there exists separable mixed bipartite states with a nonzero quantum discord. States with zero quantum discord have been referred to as “classically correlated states”, and forms a (non-convex and zero-volume) set, strictly within the set of separable states.

In particular, all Werner states for \( p \neq 0 \) can be shown to have a nonzero quantum discord. However, we show in this section that bipartite quantum states having a vanishing quantum discord can also lead to violation of no-signaling when acted locally with \( PT \)-symmetric evolutions.

Let us therefore assume again that Alice and Bob are at two space-like separated locations, sharing the classically correlated state given by

\[
\sigma_{AB_s} = p|00\rangle\langle00|_{AB} + (1-p)|11\rangle\langle11|_{AB}. \tag{10}
\]

where we must now require \( 0 \leq p \leq 1 \). For the case of the state given in Eq. (10), the no-signaling principle is satisfied if the reduced density matrix of Bob before as well as after a local \( PT \)-symmetric operation of Alice, are the same. Before the operation, Bob’s state is

\[
\sigma_{B_s} = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}. \tag{11}
\]

If Alice evolves her part of the classically correlated state by using the \( PT \)-symmetric Hamiltonian given in Eq. (1), then the composite system evolves to

\[
\sigma_{AB_s}^{U_A} = pU|00\rangle\langle00|U^\dagger \otimes |0\rangle\langle0| + (1-p)U|11\rangle\langle11|U^\dagger \otimes |1\rangle\langle1|, \tag{12}
\]

Our next task is to trace out Alice’s part from this evolved state, which results in the post-operation reduced density matrix of Bob given by

\[
\sigma_{B_s}^{U_A} = \frac{1}{N_3} \begin{pmatrix} p|0\rangle\langle0|U|0\rangle \alpha & 0 \\ 0 & (1-p)|1\rangle\langle1|U|1\rangle \end{pmatrix}, \tag{13}
\]

where \( N_3 = (1-2p) \tan \alpha \sin \xi \sin (2t_1) + 2 \sec^2 \alpha \sin^2 t_1 + \cos (2t_1) \). The nonzero elements of the density matrix \( \sigma_{B_s}^{U_A} \) require the following expressions:

\[
\langle 0|U^\dagger U|0 \rangle = 2 \sec^2 \alpha \sin^2 t_1 + \cos (2t_1) - \tan \alpha \sin (2t_1) \sin \xi,
\]

\[
\langle 1|U^\dagger U|1 \rangle = 2 \sec^2 \alpha \sin^2 t_1 + \cos (2t_1) + \tan \alpha \sin (2t_1) \sin \xi.
\]

Here, it should be noted that \( U^\dagger U \neq \mathbb{I}_2 \). Using the expressions for \( \langle 0|U^\dagger U|0 \rangle \) and \( \langle 1|U^\dagger U|1 \rangle \), and comparing the pre- and post-operation density matrices at Bob’s end, we find that the no-signaling requirement is satisfied if \( \xi = 0 \). However, outside the surface given by \( \xi = 0 \), no-signaling is satisfied only if \( \alpha = 0 \) or \( t_1 = 0 \) or \( p = 0, 1 \).

As mentioned before, \( \alpha = 0 \) leads to a hermitian \( H_{PT} \), \( t_1 = 0 \) implies \( t = 0 \), which again leads to a hermitian \( H_{PT} \) as then \( s = 0 \), or \( \tau = 0 \), which means that the evolution did not happen.

We therefore find that even classically correlated states can be used to obtain violation of no-signaling for local \( PT \)-symmetric evolutions. There is however a distinct difference with the considerations for the Werner class. For Werner states, all \( PT \)-symmetric evolutions lead to violation of no-signaling, while for the class of states discussed here, there is a (zero-volume) surface in the parameter space of \( PT \)-symmetric evolutions on which the no-signaling principle is violated.

V. \( CP \) INNER PRODUCT AND NO-SIGNALING PRINCIPLE

It has previously been shown in the literature that using a \( CP \) inner product can remove the violation of no-signaling in the case of shared entangled states and local \( PT \)-symmetric evolutions. In this section, we demonstrate that violation of the no-signaling principle can be avoided both for the Werner state and for the classically correlated state using a \( CP \) inner product. It is known that evolution generated by a non-hermitian \( PT \)-symmetric Hamiltonian is not unitary. This is the basic reason for the violation of the no-signaling principle. However, if a \( CP \) inner product is used appropriately, the time evolution corresponding to a \( PT \)-symmetric Hamiltonian becomes unitary. In order to do so, let us introduce a charge conjugation operator, \( C \), defined as

\[
C = |E_+(\alpha)\rangle\langle E_+ (\alpha)| + |E_-(\alpha)\rangle\langle E_- (\alpha)|
\]

\[
= \frac{1}{N_4} \begin{pmatrix} \cos \xi - i \sin \alpha \sin \xi & i \cos \xi \sin \alpha + \sin \xi \\ i \cos \xi \sin \alpha + \sin \xi & i \sin \alpha \sin \xi - \cos \xi \end{pmatrix}, \tag{14}
\]

where \( N_4 = \cos \alpha \). The eigenstates of \( H_{PT} \) are simultaneous eigenstates of the \( C \) operator. Indeed, the matrix \( C \) commutes with \( H_{PT} \) and \( PT \), and \( C^2 = 1 \). Using the matrix representation of the parity operation defined in Eq. (9) and of the charge conjugation operator, \( C \), in Eq. (13), we obtain

\[
CP = \begin{pmatrix} \sec \alpha & -i \tan \alpha \\ i \tan \alpha & \sec \alpha \end{pmatrix}. \tag{15}
\]

We now define the time evolution operator for the \( PT \)-symmetric Hamiltonian, \( H_{PT} \), as

\[
U = e^{-Q/2}e^{-iH_{PT} \tau}e^{Q/2}, \tag{16}
\]
where $e^Q = CP$. Using the representation of $H_{PT}$ in Eq. (11) and that of $CP$ given in Eq. (15), the evolution operator, $\mathcal{U}$, reduces to

$$
\mathcal{U} = e^{-ir\tau} \begin{pmatrix}
\cos t_1 - i \cos \xi \sin t_1 & -i \sin \xi \sin t_1 \\
-i \sin \xi \sin t_1 & \cos t_1 + i \cos \xi \sin t_1
\end{pmatrix}.
$$

(17)

It can easily be checked that $\mathcal{U}$ is an unitary operator. See e.g. Refs. [52, 59] for further details. Now, if Alice uses $\mathcal{U}$ for the evolution of her local state, then for the case of the shared Werner state, we obtain

$$
\rho_{B_w} = \rho_{B_w}^U = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

(18)

and hence, the no-signaling principle is satisfied. Similarly, for the case of the classically correlated state, if Alice evolves her local state using $\mathcal{U}$, then the state of Bob before and after the measurement are equal:

$$
\sigma_{B_s} = \sigma_{B_s}^U = \begin{pmatrix} p & 0 \\ 0 & 1 - p \end{pmatrix},
$$

(19)

and hence again no-signaling is retrieved. Indeed, for any unitary $\hat{U}_A$ on Alice’s Hilbert space and for any shared quantum state $\rho_{AB}$ between Alice and Bob, $\text{tr}_A \left( \hat{U}_A \otimes \mathbb{I} \rho_{AB} \hat{U}_A^\dagger \otimes \mathbb{I}_B \right) = \text{tr}_A \left( \rho_{AB} \right)$, where $\mathbb{I}_B$ is the identity operator at Bob’s end.

VI. CONCLUSION

$PT$-symmetric quantum mechanics remains a contentious theory and exhibits a host of strange behavior. Violation of the no-signaling principle is one of them. In [14], it was shown that a local $PT$-symmetric operation on a maximally entangled state violates the no-signaling principle. We found that the entanglement content of the shared state is independent of the no-signaling violation, and the same can also be obtained for states with zero entanglement. We went on to show that even classically correlated states that does not possess any quantum discord, a quantum correlation concept that is independent of entanglement, can offer violation of the no-signaling principle for local $PT$-symmetric evolutions. We subsequently found that using a $CPT$ inner product can restate the no-signaling principle, just like it was found to do so for entangled states.

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