Pairing of Cooper Pairs in a Fully Frustrated Josephson Junction Chain

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We study a one-dimensional Josephson junction chain embedded in a magnetic field. We show that when the magnetic flux per elementary loop equals half the superconducting flux quantum $\phi_0 = h/2e$, a local $\mathbb{Z}_2$ symmetry arises. This symmetry is responsible for a nematic Luttinger liquid state associated to bound states of Cooper pairs. We analyze the phase diagram and we discuss some experimental possibilities to observe this exotic phase.

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During the last twenty years, Josephson junction arrays have proved to be very good tools to investigate classical and quantum phase transitions \cite{1}. Recently, much attention has been payed to systems which display highly degenerate classical ground states \cite{2} due to the presence of Aharonov-Bohm cages \cite{3}. Interestingly, a glassy vortex phase without disorder has been predicted some experimental possibilities to observe this exotic phase. We consider the chain of loops shown in Fig. 1, where $J_n = \sum_{n} J_n(b_n + c_n + b_{n+1} - e^{-i\gamma}c_{n+1}) + h.c.$, (1)

We first focus on the special value $\gamma = \pi$ (half a flux quantum per loop). As shown in \cite{2}, this Hamiltonian has, in this case, a single particle spectrum composed of three highly degenerate flat bands $\varepsilon_0 = 0$, $\varepsilon_{\pm} = \pm 2J_1$. The corresponding eigenstates can be chosen as strictly localized (cage states) around each fourfold coordinated site (see Fig. 1). This leads naturally to the notion of Aharonov-Bohm cages discussed in \cite{3}.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Fig1.png}
\caption{The chain of loops and the three (nonnormalized) cage eigenstates corresponding to $\varepsilon_0$ (left) and $\varepsilon_{\pm}$ (right). The dashed line symbolizes the hopping term $-t_J e^{-i\gamma}$.}
\end{figure}

Let us introduce the set of boson operators $A_{\alpha,n}^{\dagger} (A_{\alpha,n})$ that create (destroy) one Cooper pair with energy $\varepsilon_\alpha$ ($\alpha = 0, \pm$) in a cage state localized around the $n$th fourfold site. These operators can be simply expressed as a linear combination of the operators $a_{n}^{\dagger}, b_{n}, c_{n}, b_{n-1}^{\dagger}, c_{n-1}^{\dagger}$ only, whose coefficients are given in Fig. 1 so that we get:

\begin{equation}
H_1 = \sum_{n,\alpha} \varepsilon_\alpha A_{\alpha,n}^{\dagger} A_{\alpha,n} .
\end{equation}

In this present form, $H_1$ clearly exhibits a local $U(1)$ symmetry. We shall now study the effect of boson-boson interaction on this symmetry. Therefore, we consider a real valued function $n \mapsto s_n$, and we construct a unitary operator $U_s$ defined by: $U_s A_{\alpha,n} U_s^{-1} = e^{i s_n} A_{\alpha,n}$ which commutes with $H_1$. Using the precise form of the cage states, we easily obtain:

\begin{align}
U_s a_n^{\dagger} a_n U_s^{-1} &= a_n^{\dagger} a_n \\
U_s b_n^{\dagger} b_n U_s^{-1} &= \cos^2(\Delta_n) b_n^{\dagger} b_n + \sin^2(\Delta_n) c_n^{\dagger} c_n + z_n \\
U_s c_n^{\dagger} c_n U_s^{-1} &= \sin^2(\Delta_n) b_n^{\dagger} b_n + \cos^2(\Delta_n) c_n^{\dagger} c_n - z_n ,
\end{align}

where $\Delta_n = \frac{\phi_n - \phi_0}{\phi_0}$ and $s_n = \frac{\phi_n - \phi_0}{\phi_0}$.
where we have set $z_n = i\sin(2\Delta_n) (b_n^\dagger c_n^\dagger - c_n b_n)/2$ and $\Delta_n = (s_{n+1} - s_n)/2$. From these transformation laws, we can readily see that any interaction term involving the bilinear operators $a_m^\dagger a_n, b_m^\dagger b_n + c_n^\dagger c_n$ preserves the local $U(1)$ symmetry. Physically, this symmetry implies that the total number of bosons in each cage is separately conserved and the system remains an insulator. However, this symmetry is fragile since it is easy to find two-body interactions which break it. For instance, a Hubbard-like interaction term $\sum_n (a_m^\dagger a_n)^2 + (b_m^\dagger b_n)^2 + (c_n^\dagger c_n)^2$, has this effect which is manifested by the appearance of delocalized two-particle bound states discussed in $[3]$. An exciting feature of this system is that this type of interaction still preserves a subgroup of the full $U(1)$ corresponding to a local $\mathbb{Z}_2$ symmetry. This subgroup corresponds to $s_n = 0 \mod \pi$ for all $n$. With this restriction, it is easy to check that the operator $b_m^\dagger b_n + c_n^\dagger c_n$ commutes with $U_s$ for all $(m, n)$. This local $\mathbb{Z}_2$ symmetry has an important physical consequence since it means that the parity of the total number of bosons in each cage is separately conserved. Therefore, if two-particle interactions lead to coherent transport through the chain for a many-boson system, quasi off-diagonal long range order may occur only for composite objects built with an even number of original bosons i.e., here, of Cooper pairs. In other words, a superconducting Josephson junction chain with this geometry and half a flux quantum per loop may realize a quasi-Bose condensate (in fact a LL) of charge 4e composite bosons!

To discuss in more details the physics of this system, it is useful to rephrase these symmetry considerations in the language of quantum rotor models. These offer the advantage of an intuitively simple classical limit defined from the phase of superconducting order parameter. Formally, we introduce three phase fields $\theta_n, \varphi_n, \chi_n$ and their canonically conjugate fields $\Pi_{\theta,n}, \Pi_{\varphi,n}, \Pi_{\chi,n}$, which are related to the local Bose operators by: $a_m^\dagger = \Pi_{\theta,n}^{1/2} e^{i\theta_n}$, $b_m^\dagger = \Pi_{\varphi,n}^{1/2} e^{i\varphi_n}$, $c_n^\dagger = \Pi_{\chi,n}^{1/2} e^{i\chi_n}$. Assuming that the local particle number fluctuations are small, we get the quantum phase Hamiltonian

$$H = \frac{E_C}{2} \sum_n \Pi_{\theta,n}^2 + \Pi_{\varphi,n}^2 + \Pi_{\chi,n}^2 - E_J \sum_n \cos(\theta_n - \varphi_n) + \cos(\theta_n - \chi_n) + \cos(\theta_n - \varphi_{n-1}) + \cos(\theta_n - \chi_{n-1} - \gamma), \quad (6)$$

where $E_C$ is the charging energy and $E_J$ the Josephson coupling between islands. Note that the present modelling of capacitive effects is not meant to be very realistic, since for the sake of simplicity, we have not taken into account off-diagonal elements of the capacitance matrix. This choice corresponds to a local Hubbard-like interaction term between Cooper pairs. For convenience, we set $\sqrt{E_CE_J} = 1$.

The classical ground state of $H$ is easily obtained for any $\gamma$. Indeed, if we set $x_n = \theta_{n+1} - \theta_n - \gamma/2$, and eliminate $\varphi_n$ and $\chi_n$, minimizing $H$ is equivalent to minimize $F(x_n) = -|\cos(x_n/2 + \gamma/4)| - |\cos(x_n/2 - \gamma/4)|$ for all $n$. As shown in Fig. 2 $F$ has two local minima in $x_n = 0$ and $x_n = \pi$. For $0 < \gamma < \pi$, one has $F(0) < F(\pi)$ and the classical ground state is unique (up to a global $U(1)$ degeneracy). By contrast, for $\gamma = \pi$, one has $F(0) = F(\pi)$ so that, for a given plaquette, we obtain two degenerate ground states (up to a global translation of the phase variables) which are illustrated in Fig. 3.

**FIG. 2.** Behaviour of $F(x_n)$ for $\gamma = 0$ (solid line), $\gamma = 3\pi/4$ (dashed line) and $\gamma = \pi$ (dotted line).

These states only differ in the sign of the superconducting currents which circulate around the plaquette. For a chain made up of $N$ loops, we thus get $2^N$ degenerate classical ground states up to a global translation of the phase variables.

**FIG. 3.** Two possible classical ground states of $H$ with different chirality.

This huge degeneracy is a direct consequence of a local $\mathbb{Z}_2$ symmetry of $H$. Note that $H$ is not invariant under the full local $U(1)$ group related to the Aharonov-Bohm cages. This occurs since the Josephson term in $H$ may be written as a strongly non linear expression of the basic local Bose operators. For the $\mathbb{Z}_2$ transformations, it is an easy task to translate the $U_s$ operators in the language of phase variables. An interesting local $\mathbb{Z}_2$ transformation is provided by a kink in the $s_n$'s. Let $U_n$ be the transformation defined by $s_m = 0$ for $m \leq n$.
and $s_m = \pi$ for $m > n$. This transformation does not modify the phase variables for $m \leq n$ whereas it shifts them by $\pi$ if $m > n$ and it permutes $\varphi_n$ and $\chi_n$. Thus, its main physical effect is to change the currents flowing around the plaquette located between $n$ and $n + 1$ into their opposite value. From this description, we deduce that starting from a given classical ground state, we may generate any other ground state by applying a finite sequence of such $U_n$ operators. We also see that, in the classical limit considered here, the local $\mathbb{Z}_2$ symmetry is spontaneously broken, yielding ground states with well-defined local circulating supercurrents. Note that $U_n$ also leaves most conjugate variables unchanged except $\Pi_{\varphi,n}$ and $\Pi_{\chi,n}$ which are exchanged. As a result, we may add to $H$ any term involving these conjugate fields without breaking the local $\mathbb{Z}_2$ symmetry, provided the spatial symmetry between the $\varphi_n$ and $\chi_n$ degrees of freedom is respected. Experimentally, this would require a tight control of offset charges since these may seriously alter an otherwise excellent geometrical symmetry of the chain.

For real systems, it may become important to take into account quantum fluctuations of the phase variables, especially when the superconducting islands are so small that their charging energy $E_C$ can no longer be neglected in comparison to the Josephson coupling energy $E_J$. For a single loop and $\gamma = \pi$, these quantum fluctuations have been shown, theoretically \cite{[11]} and experimentally \cite{[12],[13]}, to induce tunneling between the two degenerate classical ground states shown in Fig. [3]. The true quantum mechanical ground state is therefore a macroscopic linear superposition of these two classical states and provides a simple example of a “Schrödinger cat”. For a system with $N$ loops, eigenstates are classified according to the various irreducible representations of the local $\mathbb{Z}_2$ group which mix all the $2^N$ classical ground states. One of our next goals is to describe how quantum fluctuations lift the degeneracy among these representations, which is an artifact of the classical limit.

At small $g = \sqrt{E_C/E_J}$, the properties of the system are actually very similar to those of a quantum XY model. For small $\gamma$, we thus expect the infinite chain to be in a LL phase for $g < g^*(\gamma)$ and in a gapful insulating (I) phase for $g > g^*(\gamma)$. The transition at $g^*(\gamma)$ is of Berezinskii-Kosterlitz-Thouless (BKT) type \cite{[11]}. Simple spin wave calculations using the harmonic approximation of $H$ around its classical ground state predict $g^*(\gamma) = \sqrt{\cos(\gamma/4)} g^*(0)$ with $g^*(0) = \pi \sqrt{5}/2$. The main effect of the magnetic field, in this simple approximation, is thus to replace $g$ by an effective $g_{eff} = g/\sqrt{\cos(\gamma/4)}$ which controls all the correlation function exponents in the LL phase.

To analyze the effect of the quantum fluctuations in the vicinity of $\gamma = \pi$ where the additional local $\mathbb{Z}_2$ symmetry emerges, it is convenient to eliminate the twofold coordinated islands to get a simple description of the low-energy physics of this system. Therefore, instead of $H$, we now consider the following Hamiltonian:

$$H_{XY} = \sum_n \frac{g'}{2} \Pi_{\varphi,n}^2 - \frac{1}{g'} \left\{ \cos[p(\theta_n - \theta_{n+1} - \gamma/2)] + \epsilon \cos(\theta_n - \theta_{n+1} - \gamma/2) \right\},$$  \hspace{1cm} (7)$$

The parameter $g'$ is provided by fitting the exponent of the correlation function $\langle \exp(i\theta_m - \theta_n) \rangle$ in the semi-classical regime with its value obtained with $H$ in the harmonic approximation. This choice leads to $g' = 5^{5/4} 3^{-1/2} g$. The parameter $\epsilon = 4|\gamma - \pi|$ is determined from the energy splitting between the two local minima of the single loop potential energy. Finally, in our case, we have $p = 2$ but we discuss thereafter the properties of $H_{XY}$ for an arbitrary $p$.

The Hamiltonian $H_{XY}$ has a local $\mathbb{Z}_p$ symmetry at $\epsilon = 0$, corresponding to the local transformations $T_a : \theta_j \mapsto \theta_j + 2\pi a_j/p$ where $a_j$ is an integer. The irreducible representations of this group are easily obtained in a basis which diagonalizes simultaneously the $\Pi_{\theta,j}$’s. For a state $|\psi\rangle$ such that $\Pi_{\theta,j} |\psi\rangle = l_j |\psi\rangle$, where $l_j$ is an integer, we have $T_a |\psi\rangle = \exp(i \sum_j \frac{2\pi l_j}{p} a_j) |\psi\rangle$. Writing $l_j = m_j + np_j$ with $m_j$ and $n_p$ integers and $0 \leq m_j \leq p - 1$, we find that the set of $m_j$’s completely specifies the irreducible representation of the local $\mathbb{Z}_p$ group. For each such representation, the action of the corresponding projector on the approximate gaussian ground state of $H_{XY}$ produces a natural trial wavefunction at least when $g' \ll 1$. We have computed the expectation value of $H_{XY}$ on these states. Doing so, we noticed that $2\pi/p$-tunnel processes occuring on different lattice sites are mostly uncorrelated. Neglecting completely these correlations we get:

$$\langle H_{XY} \rangle = \frac{g' L^2}{2N} - \frac{C e^{-f}}{g'} \sum_j \cos \left[ \frac{2\pi}{p} (m_j - L/N) \right], \hspace{1cm} (8)$$

up to a constant energy independent of the representation and to factors of order $e^{-2f}$. In \cite{[13]}, $L = \sum_j l_j$ is the total angular momentum, $C$ is a number close to $2 \pi^2 - 8$ at small $g'$, and $f \approx 4 \pi/p g'$ the ground state is therefore obtained by choosing the identity representation of the local $\mathbb{Z}_p$ group ($m_j = 0$).

Next, we see that the term proportional to $\epsilon$ couples different irreducible representations of the local $\mathbb{Z}_p$ group. When $p = 2$ the action of this perturbation on the $2^N$ low-energy trial states just discussed, is well described by a quantum Ising model in a transverse magnetic field:

$$H_1 = -\frac{1}{g'} \left( C e^{-f} \sum_n \sigma_n^X + D \sum_n \sigma_n^Z \sigma_{n+1}^Z \right), \hspace{1cm} (9)$$

where $D$ is close to 1 for small $g'$. In terms of these Ising variables, the local $\mathbb{Z}_2$ symmetry corresponds to interchanging the $|+\rangle$ and $|-\rangle$ states on any given subset of
the vortex lattice. It implies that the different nature since it describes the possible melting of fully frustrated Josephson junction arrays. Neverthe-

group ) will leave behind a pair of charge $-2e$ so that a charge $4e$ composite object may propagate along the chain. Another possibility is to close the chain into a large ring. In this geometry, we expect quantum oscillations of the global current with respect to the magnetic flux across the ring with an elementary period $\phi_0/2$ as long as $\gamma \simeq \pi$.

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