LIVŠIC THEOREMS FOR BANACH COCYCLES: EXISTENCE AND REGULARITY

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ABSTRACT. We prove a nonuniformly hyperbolic version Livšic theorem, with cocycles taking values in the group of invertible bounded linear operators on a Banach space. The result holds without the ergodicity assumption of the hyperbolic measure. Moreover, we also prove a $\mu$-continuous solution of the cohomological equation is actually Hölder continuous for the uniform hyperbolic system.

1. INTRODUCTION

For a given dynamical system $f : M \to M$ and a map $A : M \to G$, where $G$ is a topological group, it’s important to determine whether $A$ is a coboundary, that is, whether there exists a map $C : M \to G$ such that

$$A = (C \circ f) \cdot C^{-1}.$$ 

Such a equation is usually called the cohomological equation and $C$ is a solution to the equation.

These problems were first studied by Livšic [25, 24]. He proved that if $f$ is a hyperbolic system, $G = \mathbb{R}$ and $A$ is Hölder continuous, then $A$ is coboundary, if and only if

$$\sum_{i=0}^{n-1} A(f^i p) = 0, \quad \forall p = f^n(p), n \geq 1.$$ 

Due to the interest and the importance of this result, many generalizations have been studied in different directions:

(i) More general groups: Does the Livšic theorem hold for more general groups?

(ii) More general dynamics: For other dynamical systems, e.g., nonuniformly hyperbolic systems, partially hyperbolic systems, etc., is there a Livšic-type theorem?

(iii) Regularity of solutions: If the cohomological equation has a measurable solution, does it coincide almost everywhere with a continuous one? Is a continuous solution actually $C^r$?

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Around these questions, the cohomological equations has been extensively studied in recent decades. We introduce some of the highlights from different dynamics:

- **Expanding systems.** Conze, Guivarc’h [11] and Savchenko [35] proved a non-positive Livšic theorem, that is, if $A : M \to \mathbb{R}$ satisfies
  \[ \sum_{i=0}^{n-1} A(f^i p) \leq 0, \forall p = f^m(p), m \geq 1, \]
  then there exists a Hölder continuous function $C : M \to \mathbb{R}$ such that $A \leq C \circ f - C$.

- **Uniformly hyperbolic systems.** For $G = \mathbb{R}$, Bousch [7] and Lopes and Thieullen [26] proved the non-positive Livšic theorem. Livšic [25] also proved the Livšic theorem when the group $G$ admits a complete bi-invariant distance (e.g. Abelian or compact groups). For the group $G$ not admitting bi-invariant distances, one of the main difficulties is to "control distortions". To do so, initially, many authors [24, 31, 12] assumed that the cocycle is sufficiently close to the constant identity cocycle. A first breakthrough progress, without additional hypotheses, was made by Kalinin [17] in the case when $G = GL(d, \mathbb{R})$. And then, Grabarnik and Guysinsky [15] generalized the result to Banach rings. Navas and Ponce [29] considered the group of germs of analytic diffeomorphisms. For groups of diffeomorphisms, Kocsard and Potrie [23] and Avila, Kocsard and Liu [2] proved the corresponding Livšic theorem. On the regularity of solutions of the cohomological equation, for the connected Lie group, Pollicott and Walkden [31] proved under the "partial hyperbolicity" condition that every measurable solution coincides almost everywhere with a Hölder continuous one. Butler [10] considered the case of the group $G = GL(d, \mathbb{R})$.

- **Flows.** For a transitive Anosov flow and $G = \mathbb{R}$, the classical Livšic theorem was established by Livšic [25]. Pollicott and Walkden generalized the result to connected Lie groups in [31]. The non-positive Livšic theorem for $G = \mathbb{R}$ was proved by Pollicott and Sharp [30] and Lopes and Thieullen [27].

- **Partially hyperbolic systems.** In a partially hyperbolic system, since the periodic orbit may not exist, Katok and Kononenko [22] used the "periodic cycle function" to replace the periodic points, and they gave a sufficient and necessary condition of the coboundary of $A : M \to \mathbb{R}$ when the system $f$ is locally accessible. And then Wilkinson [39] generalized the result to $f$ is accessible, she also considered the regularity of solutions of the cohomological equations. The result for Banach cocycles, i.e. cocycles taking values in the group of invertible bounded linear operators on a Banach space, was proved by Kalinin and Sadovskaya [18].

- **Nonuniformly hyperbolic systems.** For $G = \mathbb{R}$, Katok and Hasselbatt established a nonuniform version Livšic theorem in their book [21]. Recently, Zou and Cao [40] generalized their result to $G = GL(d, \mathbb{R})$. Backes and Poletti [4] also proved a similar result for $G = GL(d, \mathbb{R})$ later independently. In fact, authors for the papers [21, 40, 4] only proved a nonuniform version Livšic theorem for ergodic hyperbolic measures, not for general hyperbolic measures.

In this paper, we prove a nonuniform version Livšic theorem and the Hölder regularity of solutions for $G = GL(X)$, where $X$ is a Banach space, and $GL(X)$ is the group of invertible bounded linear operators on $X$.

### 1.1. A nonuniform version Livšic theorem for $G = GL(X)$

The following closing property is used to replace the nonuniform hyperbolicity.

Recall that a map $C : M \to GL(X)$ is called $\mu$-continuous, if there exists a sequence of compact set $K_n \subset M$ such that $\mu(\bigcup_{n \geq 1} K_n) = 1$ and $C|_{K_n}$ is continuous.

...
for every \( n \). An \( f \)-invariant measure \( \mu \) is called a hyperbolic measure if its Lyapunov exponents are different from zero at \( \mu \)-almost every point.

**Theorem 1.1.** Let \( f \) be a \( C^{1+\gamma} \) diffeomorphism of a compact manifold \( M \), preserving an hyperbolic measure \( \mu \), and let \( A : M \to GL(X) \) be an \( \alpha \)-Hölder continuous map satisfying

\[
A(f^{n-1}p) \cdots A(fp)A(p) = Id, \quad \forall p = f^n(p), \forall n \geq 1.
\]

Then there exists a \( \mu \)-continuous map \( C : M \to GL(X) \) such that

\[
A(x) = C(fx)C(x)^{-1}, \quad \text{for } \mu\text{-almost every } x \in M.
\]

We point out that we do not assume the ergodicity of the hyperbolic measure \( \mu \) in Theorem 1.1. In fact, the proof from ergodicity to non-ergodicity is nontrivial, since the measure \( \mu \) may have uncountably many ergodic components. We need to generalize a result of Fisher, Morris and Whyte [13] to overcome this problem.

1.2. Hölder regularity of solutions. Let \( f : M \to M \) be an Anosov diffeomorphism. An \( f \)-invariant measure \( \mu \) is called having local product structure, if it is locally equivalent to the product of the projections of \( \mu \) to the local stable and unstable manifolds.

**Theorem 1.2.** Let \( f \) be an Anosov diffeomorphism of a compact manifold \( M \), \( \mu \) be an ergodic \( f \)-invariant measure on \( M \) with full support and local product structure, and \( A : M \to GL(X) \) be an \( \alpha \)-Hölder continuous map. Suppose that there exists a \( \mu \)-continuous map \( C : M \to GL(X) \) such that

\[
A(x) = C(fx)C(x)^{-1}, \quad \text{for } \mu\text{-a.e. } x \in M.
\]

Then \( C \) coincides \( \mu \)-a.e. with an \( \alpha \)-Hölder continuous map \( \hat{C} \) satisfying the same equation everywhere.

This theorem gives a positive answer to Question (iii) for \( G = GL(X) \). The case \( G = \mathbb{R} \) was first proved by Livšic [24]. The case \( G = GL(d, \mathbb{R}) \) was proved by Sadovskaya [33] under the fiber bunching condition. And then Bulter [10] improved her result by removing this additional condition. If \( X \) is a Banach space, Sadovskaya [34] proved a similar result under the assumption that \( X \) is a separable Banach space and the cocycle is fiber bunched. We release these assumptions. And it is seen from Theorem 1.1 that the \( \mu \)-continuity condition of \( C \) in Theorem 1.2 is natural.

Since the measure of maximal entropy or more generally the equilibrium states corresponding to Hölder continuous potentials for Anosov diffeomorphisms has full support and local product structure [31]. As a corollary of Theorem 1.1 and 1.2 we obtain immediately the uniform version Livšic theorem which is proved by Grabinik and Guysinsky [15].

**Corollary 1.3.** Let \( f : M \to M \) be a transitive \( C^{1+\gamma} \) Anosov diffeomorphism, and let \( A : M \to GL(X) \) be an \( \alpha \)-Hölder continuous map satisfying

\[
A(f^{n-1}p) \cdots A(fp)A(p) = Id, \quad \forall p = f^n(p), \forall n \geq 1.
\]

Then there exists an \( \alpha \)-Hölder continuous map \( C : M \to GL(X) \) such that

\[
A(x) = C(fx)C(x)^{-1}, \quad \forall x \in M.
\]

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2. Preliminaries and Notations

2.1. Cocycles and Exponents.

**Definition 2.1.** Suppose that \( f : M \to M \) is invertible, and \( A : M \to GL(X) \). A map \( A : M \times \mathbb{Z} \to GL(X) \) is called a linear multiplicative cocycle over \( f \) generated by \( A \); if
\[
A^0 := A(x,0) = I,
\]
\[
A^{n+1} := A(x,n+1) = A(x,n)A(f^n(x)) = A^0 \circ f^n.
\]
Clearly, \( A \) satisfies \( A^{n+k} = A^n \circ A^k \).

We introduce a metric \( d \) on \( GL(X) \) such that \( (GL(X), d) \) is a complete metric space:
\[
d(A, B) = \| A - B \| + \| A^{-1} - B^{-1} \|.
\]
A cocycle \( A \) is called \( \alpha \)-Hölder continuous, if its generator \( A = A(\cdot, 1) : M \to GL(X) \) is \( \alpha \)-Hölder continuous.

Recall that a sequence of functions \( a_n : M \to \mathbb{R} \) over a system \((M, f)\) is called a subadditive cocycle if it satisfies
\[
a_{m+n}(x) \leq a_m(x) + a_n(f^m x)
\]
for any \( m, n \in \mathbb{N} \) and \( x \in M \). If \( f \) is an ergodic measure preserving transformation of a probability space \((M, \mu)\) and \( a_1 \in L^1(M) \), then the Subadditive Ergodic Theorem yields that there exists a set \( \mathcal{R} \) of \( \mu \)-full measure such that for any \( x \in \mathcal{R} \),
\[
\lambda := \lim_{n \to +\infty} \frac{1}{n} \int a_n d\mu = \lim_{n \to +\infty} \frac{1}{n} a_n(x).
\]
The limit \( \lambda \) is called the exponent of the cocycle \( a_n \) with respect to \( \mu \). If we consider the continuous cocycle \( A : M \times \mathbb{Z} \to GL(X) \), since \( \log \| A^n \| \) and \( \log \| (A^{-1})^n \| \) are subadditive, for \( \mu \)-a.e. \( x \in M \), the limits
\[
\lambda_+(A, \mu) := \lim_{n \to +\infty} \frac{1}{n} \int \log \| A^n \| d\mu = \lim_{n \to +\infty} \frac{1}{n} \log \| A^n \|
\]
and
\[
\lambda_-(A, \mu) := -\lim_{n \to +\infty} \frac{1}{n} \int \log \| (A^{-1})^n \| d\mu = -\lim_{n \to +\infty} \frac{1}{n} \log \| (A^{-1})^n \|
\]
exist. \( \lambda_+(A, \mu) \) and \( \lambda_-(A, \mu) \) are called the upper Lyapunov exponent and the lower Lyapunov exponent of \( A \) with respect to \( \mu \), respectively.

2.2. Hyperbolic Measure and Closing Lemma. Let \( f \) be a \( C^{1+\gamma} \) diffeomorphism of a compact manifold \( M \). Recall that an \( f \)-invariant measure \( \mu \) is said to be hyperbolic, if the Lyapunov exponents of the derivative cocycle \( Df \) are non-zero for \( \mu \)-a.e. \( x \in M \).

We will apply the following Katok’s closing lemma [21] Theorem S.4.13 (See also Theorem 15.1.2 of [5]).

**Lemma 2.2 (Katok’s Closing Lemma).** Let \( f \in \text{Diff}^{1+\gamma}(M) \), preserving an ergodic hyperbolic measure \( \mu \). Then there exist \( \lambda > 0 \) and a compact set \( \Lambda \) with \( \mu(\Lambda) > 0 \), such that for any \( \delta > 0 \), there exists \( \beta > 0 \), if \( x, f^n(x) \in \Lambda \) satisfying \( d(x, f^n x) < \beta \), then there exists a periodic point \( p \) with \( p = f^n(p) \) such that \( d(f^i p, f^i x) \leq \delta \cdot e^{-\lambda \min\{i, n-i\}} \).
2.3. Anosov diffeomorphisms and fiber bunching. Recall that a diffeomorphism \( f : M \to M \) is called Anosov, if there exists a \( Df \)-invariant splitting \( TM = E^s \oplus E^u \) on \( M \), and \( \tau > 0 \) such that
\[
\|D_x f(v^s)\| < e^{-\tau} < 1 < e^\tau < \|D_x f(v^u)\|
\]
for any \( x \in M \) and unit vectors \( v^s \in E^s(x) \) and \( v^u \in E^u(x) \).

**Definition 2.3.** An \( \alpha \)-Hölder cocycle \( A \) over an Anosov diffeomorphism \( f \) is called fiber bunched, if there exists \( 0 < \theta < 1 \) and \( L > 0 \) such that for any \( x \in M \) and \( n \in \mathbb{N} \),
\[
\|A^n_x\| \cdot \|(A^n_x)^{-1}\| \cdot e^{-\tau \alpha n} \leq L \theta^n, \quad \text{and} \quad \|A^{-n}_x\| \cdot \|(A^{-n}_x)^{-1}\| \cdot e^{-\tau \alpha n} \leq L \theta^n.
\]
The fiber bunching condition gives existence of stable and unstable holonomies of \( A \) on \( M \). It’s a common assumption in many theories, for instance, the continuity of Lyapunov exponents, the existence of extremal norms, and the cohomology of cocycles. In Theorem 1.2 we do not have the fiber bunching hypothesis. Hence we introduce a related concept. Let \( D(N, \theta) \) be the set of points \( x \) satisfying
\[
(2.1) \quad \prod_{j=0}^{k-1} \|A^N_{f^jN_x}\| \cdot \|(A^N_{f^jN_x})^{-1}\| \leq e^{kN\theta}, \quad \forall k \geq 1,
\]
and
\[
(2.2) \quad \prod_{j=0}^{k-1} \|A^{-N}_{f^{-j}N_x}\| \cdot \|(A^{-N}_{f^{-j}N_x})^{-1}\| \leq e^{kN\theta}, \quad \forall k \geq 1.
\]
It’s known that \( A \) is fiber bunched if and only if \( D(N, \theta) = M \) for some \( \theta < \tau \alpha \) and \( N \geq 1 \). The points in \( D(N, \theta) \) for some \( \theta < \tau \alpha \) also give the existence of stable and unstable holonomies.

**Proposition 2.4** ([37, Proposition 2.5]). Given \( N, \theta \) with \( \theta < \tau \alpha \), there exists \( L > 0 \) such that for any \( x \in D(N, \theta) \) and \( y, z \in W^s_{\text{loc}}(x) \), the limit
\[
H^s_{y,z} = \lim_{n \to \infty} (A^n_z)^{-1} A^n_y
\]
exists and satisfies \( \|H^s_{y,z} - I\| \leq L \cdot d(y, z)^\alpha \) and \( H^s_{x,z} = H^s_{y,z} H^s_{x,y} \). Similarly, for any \( x \in D(N, \theta) \) and \( y, z \in W^u_{\text{loc}}(x) \), the limit
\[
H^u_{y,z} = \lim_{n \to \infty} (A^{-n}_x)^{-1} A^{-n}_y
\]
exists and satisfies \( \|H^u_{y,z} - I\| \leq L \cdot d(y, z)^{\alpha} \) and \( H^u_{x,z} = H^u_{y,z} H^u_{x,y} \). Moreover, \( (y, z) \to H^s_{y,z} \) is continuous for \( * \in \{s, u\} \), where \( y, z \in W^*_{\text{loc}}(x) \) and \( x \in D(N, \theta) \).

3. Proof of Theorem [11]

Let \( f \) be a \( C^{1+\gamma} \) diffeomorphism of a compact manifold \( M \), preserving an hyperbolic measure \( \mu \). Let \( A : M \to GL(X) \) be an \( \alpha \)-Hölder continuous map satisfying [11]. We may first assume that \( \mu \) is an ergodic measure. The general case will be considered in the subsection 3.2.
3.1. The case \( \mu \) is ergodic. Since the cocycle \( A \) generated by \( A \) is continuous, the Subadditive Ergodic Theorem yields that there exists a set \( \mathcal{R} \) of \( \mu \)-full measure such that for any \( x \in \mathcal{R} \),

\[
\lambda_+(A, \mu) = \lim_{n \to +\infty} \frac{1}{n} \int \log \| A^n_x \| d\mu = \lim_{n \to +\infty} \frac{1}{n} \log \| A^n_x \|,
\]

and

\[
-\lambda_-(A, \mu) = \lim_{n \to +\infty} \frac{1}{n} \int \log \| (A^n_x)^{-1} \| d\mu = \lim_{n \to +\infty} \frac{1}{n} \int \log \| A^{-n}_x \| d\mu
\]

\[
= \lim_{n \to +\infty} \frac{1}{n} \log \| A^{-n}_x \|.
\]

By [19, Theorem 1.5], the upper Lyapunov exponent \( \lambda_+(A, \mu) \) and lower Lyapunov exponent \( \lambda_-(A, \mu) \) can be approximated in terms of the norms of its periodic date, that is, for any \( \varepsilon > 0 \), there exists a periodic point \( p = f^n(p) \) such that

\[
|\lambda_+(A, \mu) - \frac{1}{n} \log \| A^n_p \|| < \varepsilon, \quad |\lambda_-(A, \mu) - \frac{1}{n} \log \| (A^n_p)^{-1} \|| 1 < \varepsilon.
\]

Thus \([18]\) implies \( \lambda_+(A, \mu) = \lambda_-(A, \mu) = 0 \). Then for a fixed \( \varepsilon > 0 \) and any point \( x \in \mathcal{R} \), we define the Lyapunov norm \( \| \cdot \|_x = \| \cdot \|_{x, \varepsilon} \) in \( X \) as follows:

\[
\| u \|_x := \sum_{n=-\infty}^{+\infty} \| A^n_x(u) \| e^{-\varepsilon |n|}, \quad \forall u \in X.
\]

By [20] Proposition 3.1, the Lyapunov norm satisfies the following properties:

(i) For any \( x \in \mathcal{R} \),

\[
e^{-\varepsilon} \| u \|_x \leq \| A(x) u \|_x \leq e^\varepsilon \| u \|_x, \quad \forall u \in X.
\]

(ii) There exists an \( f \)-invariant subset \( \mathcal{R}_\varepsilon \subset \mathcal{R} \) with \( \mu(\mathcal{R}_\varepsilon) = 1 \) and a measurable function \( K_\varepsilon(x) \) such that for any \( x \in \mathcal{R}_\varepsilon \),

\[
\| u \| \leq \| u \|_x \leq K_\varepsilon(x) \| u \|, \quad \forall u \in X, \quad \text{and}
\]

\[
K_\varepsilon(x) e^{-\varepsilon} \leq K_\varepsilon(fx) \leq K_\varepsilon(x) e^\varepsilon.
\]

For any \( l \geq 1 \), we define

\[
\mathcal{R}_{\varepsilon, l} = \{ x \in \mathcal{R}_\varepsilon : K_\varepsilon(x) \leq l \}.
\]

Then \( \mu(\mathcal{R}_{\varepsilon, l}) \to 1 \) as \( n \to \infty \). Without loss of generality, we may assume \( \mathcal{R}_{\varepsilon, l} \) is a compact set by using the Lusin’s theorem.

We use the Lyapunov norm to estimate the norm of \( A \) along an orbit segment that is close to a regular one. Denote \( x_i = f^i(x), y_i = f^i(y) \). Let \( \varepsilon_0 = \frac{1}{4} \lambda_\alpha \).

**Lemma 3.1.** Let \( f, A, \mu \) be as above. Then for any \( l > 1, \ 0 < \varepsilon < \varepsilon_0 \), there exist \( \delta_1, c_1 > 0 \), such that for any \( x, f^n(x) \in \mathcal{R}_{\varepsilon, l} \), \( y \in M, 0 < \delta < \delta_1 \) satisfying \( d(f^i(x), f^i(y)) \leq \delta e^{-\lambda \min\{i, n-i\}}, i = 0, \ldots, n \), we have:

\[
c_1^{-1} e^{-2\varepsilon i} \leq m(A^i_y) \leq \| A^i_y \| \leq c_1 e^{2\varepsilon i},
\]

\[
c_1^{-1} e^{-2\varepsilon (n-i)} \leq m(A^{n-i}_y) \leq \| A^{n-i}_y \| \leq c_1 e^{2\varepsilon (n-i)},
\]

where \( m(B) := \inf_{\| v \|=1} \| B v \| = \| B^{-1} \|^{-1} \).
Since \( x,f \) where \( \tilde{x} \) note that \( A \) hence for any \( 0 \leq \varepsilon < \varepsilon_0 \), by (3.1) and (3.2),

\[
\|A(y_j)u\|_{x_{j+1}} = \| (A(y_j) - A(x_j) + A(x_j))u\|_{x_{j+1}} \\
\geq \|A(x_j)u\|_{x_{j+1}} - \| (A(y_j) - A(x_j))u\|_{x_{j+1}} \\
\geq e^{-\varepsilon}\|u\|_{x_{j+1}} - K_\varepsilon(x_{j+1})\|A(y_j) - A(x_j)\| \cdot \|u\|_{x_j}.
\]

Since \( x,f^n x \in \mathcal{R}_{\varepsilon,l} \), it follows from (3.3) that

\[ K_\varepsilon(x_{j+1}) \leq le^{\varepsilon \min\{j+1,n-j-1\}} \leq le^{\varepsilon \varepsilon_0 \min\{j,n-j\}}. \]

Note that \( A(x) \) is \( \alpha \)-Hölder continuous, one has

\[ \|A(y_j) - A(x_j)\| \leq c_0 \cdot d(y_j, x_j)^\alpha \leq c_0 \delta^\alpha e^{-\lambda \alpha \min\{j,n-j\}}. \]

Hence for any \( 0 \leq j \leq n-1 \), \( u \in X \),

\[ K_\varepsilon(x_{j+1})\|A(y_j) - A(x_j)\| \cdot \|u\|_{x_j} \leq c_0 le^{\varepsilon \delta^\alpha e^{\varepsilon \lambda \alpha \min\{j,n-j\}} \|u\|_{x_j}}. \]

Therefore, the previous inequality gives

\[ \|A(y_j)u\|_{x_{j+1}} \geq (e^{-\varepsilon} - c_0 le^{\varepsilon \delta^\alpha e^{\varepsilon \lambda \alpha \min\{j,n-j\}}}) \|u\|_{x_j}. \]

Similarly,

\[ \|A(y_j)u\|_{x_{j+1}} \leq (e^{\varepsilon} + c_0 le^{\varepsilon \delta^\alpha e^{\varepsilon \lambda \alpha \min\{j,n-j\}}}) \|u\|_{x_j}. \]

Thus for any \( v \in X \), we conclude

\[
\|A_y^i(v)\| \geq K_\varepsilon(x_i)^{-1}\|A_y^i(v)\|_{x_i} \\
\geq l^{-1}e^{-\varepsilon i} \prod_{i=0}^{n-1} (e^{\varepsilon} - c_0 le^{\varepsilon \delta^\alpha e^{\varepsilon \lambda \alpha \min\{j,n-j\}}}) \|v\|_{x_0} \\
\geq l^{-1}e^{-2\varepsilon i} \prod_{i=0}^{n} (1 - c_0 le^{2\varepsilon \delta^\alpha e^{\varepsilon \lambda \alpha \min\{j,n-j\}}}) \|v\|.
\]

Take \( \delta_1 > 0 \) small enough such that \( 1 - 2c_0 le^{2\varepsilon \delta^\alpha} > 0 \). Then for any \( 0 < \delta < \delta_1 \), since \( \varepsilon - \lambda \alpha < 0 \), we can estimate

\[
\sum_{j=0}^{n-1} \log \left(1 - c_0 le^{2\varepsilon \delta^\alpha e^{\varepsilon \lambda \alpha \min\{j,n-j\}}}ight) \geq - \sum_{j=0}^{n-1} 2c_0 le^{2\varepsilon \delta^\alpha e^{\varepsilon \lambda \alpha \min\{j,n-j\}}} \\
\geq - \tilde{c}_0,
\]

where \( \tilde{c}_0 = \tilde{c}_0(l, \varepsilon) \) is a constant. It follows that

\[
m(A_y^i) = \inf_{\|v\|=1} \|A_y^i(v)\| \geq l^{-1}e^{-2\varepsilon i} \prod_{i=1}^{n-1} (1 - c_0 le^{2\varepsilon \delta^\alpha e^{\varepsilon \lambda \alpha \min\{j,n-j\}}}) \\
\geq l^{-1}e^{-\tilde{c}_0} e^{-2\varepsilon i} \\
=: c_1 e^{-2\varepsilon i}.
\]
Similarly, we can also obtain
\[
\|A_y^{(i)}(v)\| \leq \|A_y^{(i)}(v)\|_{x_i} = \|A(y_{i-1}) \cdots A(y)v\|_{x_i},
\]
\[
\leq \prod_{j=0}^{i-1} \left( e^\varepsilon + c_0 l e^{\varepsilon} \delta^\alpha (e^{-(\lambda \alpha)} \min(j,n-j)) \right) \|v\|_{x_0},
\]
\[
\leq l e^{\varepsilon i} \prod_{j=0}^{i-1} \left( 1 + c_0 l \delta^\alpha (e^{-(\lambda \alpha)} \min(j,n-j)) \right) \|v\|
\]
\[
\leq c_1 e^{\varepsilon i} \|v\|,
\]
which implies \(\|A_y^{(i)}\| \leq c_1 e^{\varepsilon i}.\) This finishes the proof. \(\square\)

For a fixed \(\varepsilon < \varepsilon_0,\) denote \(G_l = \Lambda \cap R_{\varepsilon,l},\) where \(\Lambda\) is given by Lemma 2.2. Then for \(l\) large enough, we have \(\mu(G_l) > 0.\) Let \(G'_l = \text{supp}(\mu_{G_l}),\) where \(\mu_{G_l}\) is defined by: \(\mu_{G_l}(B) := \mu(B \cap G_l)/\mu(G_l).\)

We need the following lemma to find a dense orbit in \(G'_l.\)

**Lemma 3.2.** \(\text{[20] Lemma 4.1.}\) Let \(f\) be a continuous map of a compact metric space \(M,\) preserving an ergodic measure \(\mu.\) Suppose \(D\) is a closed set with \(\mu(D) > 0,\) and \(E = \{x \in \text{supp}(\mu) : \overline{\text{O}(x)} \cap \text{supp}(\mu_D) = \text{supp}(\mu_D)\},\) where \(\mu_D\) is defined by: \(\mu_D(B) := \mu(B \cap D)/\mu(D).\) Then \(\mu(E) = \mu(\text{supp}(\mu_D)) = \mu(D).\)

Since \(G_l\) is closed, we have \(G'_l \subset G_l.\) By Lemma 3.2, one can find a point \(z \in G'_l \subset \Lambda \cap R_{\varepsilon,l},\) such that \(\overline{\text{O}(z)} \cap G'_l = G'_l.\) We define the map \(C : \overline{\text{O}(z)} \cap G'_l \rightarrow GL(X)\) by \(C(f^n z) = A^{n}_{y_l},\) for any \(f^n z \in G'_l.\) We shall prove that \(C\) is uniformly continuous on \(\overline{\text{O}(z)} \cap G'_l,\) so that \(C\) can be extended to \(G'_l.\)

**Lemma 3.3.** Given any \(l > 1,\) \(0 < \varepsilon < \varepsilon_0,\) there exists \(0 < \delta_2 < 1, c_2 > 0,\) such that for any \(0 < \delta < \delta_2,\) there exists \(\beta > 0,\) if \(x, f^n x \in G_l\) satisfying \(d(x, f^n x) \leq \beta,\) then \(d(A^{n}_{x_l}, Id) \leq c_2 \delta^n.\)

**Proof.** Since \(f\) has the closing property on \(G_l \subset \Lambda,\) there exists \(\lambda > 0,\) such that for any \(0 < \delta < 1,\) there exists \(\beta > 0,\) such that if \(x, f^n x \in G_l\) with \(d(x, f^n x) < \beta,\) then one can find a periodic point \(p = f^n p \in M,\) such that
\[
d(f^i p, f^i x) \leq \delta \cdot e^{-\lambda \min\{i,n-i\}}, \quad \forall i = 0, \ldots, n.
\]

We estimate \(\|A^{n}_{x_l} - Id\|\) first.
\[
A^{n}_{x_l} - A^{n}_{y_l} = A^{n-1}_{x_l} \circ (A(x_0) - A(p_0)) + (A^{n-1}_{x_l} - A^{n-1}_{y_l}) \circ A(p_0)
\]
\[
= A^{n-1}_{x_l} \circ (A(x_0) - A(p_0)) + A^{n-2}_{x_2} \circ (A(x_1) - A(p_1)) \circ A(p_0) +
\]
\[
(A^{n-2}_{x_2} - A^{n-2}_{y_2}) \circ A^{n-1}_{y_l} + \cdots + \sum_{i=0}^{n-1} A^{i-1}_{x_{i+1}} \circ (A(x_i) - A(p_i)) \circ A^{i}_{p_i}
\]

Since \(A^{n}_{p_i} = Id,\) by Lemma 3.1, for any \(0 \leq i \leq n,\)
\[
\|A^{i}_{p_i}\| = \|(A^{n-i}_{p_i})^{-1}\| \leq c_1 e^{2e \min\{i,n-i\}}.
\]
Note that \( \|A(x) - A(p_i)\| \leq c_0 \delta^\alpha e^{-\lambda \alpha \min\{i, n-i\}} \). Denote \( m = \left\lfloor \frac{n}{2} \right\rfloor \), then Lemma 3.1 and the fact \( \varepsilon < \varepsilon_0 \) give
\[
\sum_{i=0}^{m} \|A_{x_{i+1}}^{n-i-1}\| \cdot \|A(x_i) - A(p_i)\| \cdot \|A_p\| \\
\leq \sum_{i=0}^{m} \|A_{x_i}^{n-i}\| \cdot \|((A_x^{i+1})^{-1})\| \cdot \|A(x_i) - A(p_i)\| \cdot \|A_p\| \\
\leq \|A_x^n\| \cdot \sum_{i=0}^{m} c_1 e^{2\varepsilon(i+1)} \cdot c_0 \delta^\alpha e^{-\lambda \alpha i} \cdot c_1 e^{2\varepsilon i} \\
\leq \tilde{\epsilon}_1 \delta^\alpha \cdot \|A_x^n\|,
\]
and
\[
\sum_{i=m+1}^{n} \|A_{x_{i+1}}^{n-i-1}\| \cdot \|A(x_i) - A(p_i)\| \cdot \|A_p\| \\
\leq \sum_{i=m+1}^{n} c_1 e^{2\varepsilon(n-i-1)} \cdot c_0 \delta^\alpha e^{-\lambda \alpha (n-i)} \cdot c_1 e^{2\varepsilon (n-i)} \\
\leq \tilde{\epsilon}_1 \delta^\alpha,
\]
where \( \tilde{\epsilon}_1 \) is a constant. Therefore,
\[
\|A_x^n\| - 1 \leq \|A_x^n - A_x^n\| \\
\leq \sum_{i=0}^{n-1} \|A_{x_{i+1}}^{n-i-1}\| \cdot \|A(x_i) - A(p_i)\| \cdot \|A_p\| \\
\leq \tilde{\epsilon}_1 \delta^\alpha \|A_x^n\| + \tilde{\epsilon}_1 \delta^\alpha.
\]
Take \( \delta_2 \) small enough such that \( \tilde{\epsilon}_1 \delta_2^\alpha < \frac{1}{3} \). Then for any \( 0 < \delta < \delta_2 \), we obtain
\[
\|A_x^n\| \leq \frac{1 + \tilde{\epsilon}_1 \delta_2^\alpha}{1 - \tilde{\epsilon}_1 \delta_2^\alpha} \leq \frac{3}{2}.
\]
Thus equation (3.5) gives
\[
\|A_x^n - Id\| = \|A_x^n - A_x^n\| \\
\leq \tilde{\epsilon}_1 \delta^\alpha \|A_x^n\| + \tilde{\epsilon}_1 \delta^\alpha \\
\leq \frac{5}{2} \tilde{\epsilon}_1 \delta^\alpha.
\]
To estimate \( \|(A_x^n)^{-1} - Id\| \). Let \( Y = Id - A_x^n \), then
\[
(A_x^n)^{-1} = (Id - Y)^{-1} = Id + Y + Y^2 + \cdots.
\]
Since \( \|Y\| = \|Id - A_x^n\| \leq \frac{5}{2} \tilde{\epsilon}_1 \delta_2^\alpha \leq 1/2 \), we have
\[
\|(A_x^n)^{-1} - Id\| \leq \sum_{i=1}^{\infty} \|Y^i\| \leq \sum_{i=1}^{\infty} \left(\frac{5}{2} \tilde{\epsilon}_1 \delta^\alpha\right)^i \leq 5 \tilde{\epsilon}_1 \delta^\alpha.
\]
Therefore, \( d(A_x^n, Id) = \|A_x^n - Id\| + \|(A_x^n)^{-1} - Id\| \leq \frac{15}{2} \tilde{\epsilon}_1 \delta^\alpha =: c_2 \delta^\alpha \). This completes the proof.
Therefore, by Lemma 3.3, for any \( f^n x \in \mathcal{O}(z) \cap G'_i \) satisfying \( d(x, f^n x) < \beta \), one has \( d(C(x), C(f^n x)) \leq c_3 \delta^\alpha \). Suppose \( x = f^k(z) \), then
\[
C(f^n x)C(x)^{-1} = A^n_z(A^k_z)^{-1} = A^n_z.
\]

(3.6)

Therefore,
\[
d(C(x), C(f^n x)) = \|C(x) - C(f^n x)\| + \|C(x)^{-1} - C(f^n x)^{-1}\|
\leq \|Id - A^n_z\| \cdot \|C(x)\| + \|C(x)^{-1}\| \cdot \|Id - (A^n_z)^{-1}\|
\leq (\|C(x)\| + \|C(x)^{-1}\|) \cdot c_2 \delta^\alpha.
\]

It’s enough to prove that \( \|C(x)\| \) and \( \|C(x)^{-1}\| \) are uniformly bounded on \( \mathcal{O}(z) \cap G'_i \). By Lemma 3.3, for any \( \delta < \delta_2 \), there exists \( \beta > 0 \) such that for any \( x, f^n x \in \mathcal{O}(z) \cap G'_i \) with \( d(x, f^n x) < \beta \), we obtain \( d(A^n_z, Id) \leq c_2 \delta^\alpha \). Since \( \mathcal{O}(z) \cap G'_i \) is dense in \( G'_i \), we may choose a segment \( \mathcal{O}_L = \{f^k z\}_{k \in [-L, L]} \) such that \( \mathcal{O}_L \cap G'_i \) forms a \( \beta \)-net of \( G'_i \). Then for any \( f^m z \in \mathcal{O}(z) \cap G'_i \), there exists \( f^k z \in \mathcal{O}_L \cap G'_i \) with \( k \in [-L, L] \) such that \( d(f^k z, f^m z) < \beta \). Then we have \( d(A^m_{f^k z}, Id) \leq c_2 \delta^\alpha \), which implies \( \|A^m_{f^k z} - 1\|, \|(A^m_{f^k z})^{-1}\| \leq 1 + c_2 \delta^\alpha \). Let \( \tilde{c}_3 = \max_{k \in [-L, L]} \{\|A^m_{f^k z}\|, \|(A^m_{f^k z})^{-1}\|\} \), then the equality \( A^m_{f^k z} = (A^m_{f^k z})^{-1} \) gives \( \|A^m_{f^k z}\|, \|(A^m_{f^k z})^{-1}\| \leq \tilde{c}_3 (1 + c_2 \delta^\alpha) \), that is, \( \|C(x)\|, \|C(x)^{-1}\| \) are uniformly bounded on \( \mathcal{O}(z) \cap G'_i \). Therefore, \( C \) can be extended continuously to \( G'_i \). Then by (3.6),
\[
(3.7)
A^n_z = C(f^n x)C(x)^{-1}, \quad \forall x, f^n(x) \in G'_i.
\]

Now we extend \( C \) to \( \bigcup_{i=0}^{\infty} f^i(G'_i) \) as follows: If \( y \in \left( \bigcup_{i=0}^{n} f^i(G'_i) \right) \setminus \bigcup_{i=0}^{n-1} f^i(G'_i) \), then define \( C(y) := A^n_{f^{-n}(y)} C(f^{-n}y) \). To show the map \( C \) satisfies (1.2), for any \( y \in \bigcup_{i=0}^{\infty} f^i(G'_i) \), we may assume \( y \in \left( \bigcup_{i=0}^{n} f^i(G'_i) \right) \setminus \bigcup_{i=0}^{n-1} f^i(G'_i) \) for some \( n \geq 0 \), where \( \bigcup_{i=0}^{n-1} f^i(G'_i) \) denotes the empty set. Then we have \( f^{-n}(y) \in G'_i \), and
\[
f(y) \in \left( \bigcup_{i=1}^{n+1} f^i(G'_i) \right) \setminus \bigcup_{i=1}^{n} f^i(G'_i) \subset \left( \bigcup_{i=0}^{n+1} f^i(G'_i) \right) \setminus \bigcup_{i=1}^{n} f^i(G'_i)
\subset \left( \bigcup_{i=0}^{n+1} f^i(G'_i) \right) \setminus \bigcup_{i=0}^{n} f^i(G'_i)
\subset \left( \bigcup_{i=0}^{n+1} f^i(G'_i) \right) \setminus \bigcup_{i=0}^{n} f^i(G'_i).
\]

If \( f(y) \in \left( \bigcup_{i=0}^{n+1} f^i(G'_i) \right) \setminus \bigcup_{i=0}^{n} f^i(G'_i) \), then by the definition,
\[
C(fy) = A^{n+1}_{f^{-n}(y)} C(f^{-n}y) = A^{n+1}_{f^{-n}(y)} (A^n_{f^{-n}(y)})^{-1} C(y) = A(y)C(y).
\]

If \( f(y) \in G'_i \), then by \( f^{-n}(y) \in G'_i \), (3.7) and the definition of \( C(y) \), we obtain
\[
C(fy) = A^{n+1}_{f^{-n}(y)} C(f^{-n}y) = A(y)A^{n+1}_{f^{-n}(y)} C(f^{-n}y) = A(y)C(y).
\]

Since \( \mu \left( \bigcup_{i=0}^{\infty} f^i(G'_i) \right) = 1 \), we conclude that the map \( C \) is defined almost everywhere and satisfying
\[
A(x) = C(fx)C(x)^{-1}, \quad \text{for } \mu\text{-a.e. } x \in M.
\]
At last, it’s left to prove $C$ is $\mu$-continuous. Note that $GL(X)$ is not separable in general. We can not use the Lusin’s Theorem directly. However, we shall show that the image of $C$ on a set of $\mu$-full measure is contained in a separable complete subspace of $GL(X)$. Let
\begin{equation}
K := \bigcup_{n \in \mathbb{Z}} \{ A^n_x : x \in M \} \subset GL(X).
\end{equation}

Since $x \mapsto A^n_x$ is continuous, $\{ A^n_x : x \in M \}$ is compact for any $n \in \mathbb{Z}$. Therefore $\bigcup_{n \in \mathbb{Z}} \{ A^n_x : x \in M \}$ is a $\sigma$-compact subset and which implies $K$ is separable. We claim that the image of $C$ on $\bigcup_{i=0}^{\infty} f^i(G'_1)$ is contained in $K$. Indeed, for any $y \in \bigcup_{i=0}^{\infty} f^i(G'_1)$, since $f^{-n}(y) \in G'_n$, there exists a sequence $\{ n_k \}_{k \geq 1}$ such that $f^{-n_k}(z) \to f^{-n}(y)$ as $k \to \infty$. By the continuity of $C$ on $G'_1$, $C(f^{-n}y) = \lim_{k \to \infty} C(f^{-n_k}z) = \lim_{k \to \infty} A^n_{n_k} = A^n_{n_k+n} \in K$.

Therefore $C : \bigcup_{i=0}^{\infty} f^i(G'_1) \to K$. Since $\bigcup_{i=0}^{\infty} f^i(G'_1)$ is of $\mu$-full measure and $K$ is a separable complete metric space, by Lusin’s Theorem, for any $k \geq 1$, there exists a compact subset $F_k \subset \bigcup_{i=0}^{\infty} f^i(G'_1)$ with $\mu(F_k) > 1 - \frac{1}{k}$ such that $C|_{F_k}$ is continuous. This shows the $\mu$-continuity of $C$.

3.2. The general case. If $\mu$ is not ergodic, since almost every ergodic component of $\mu$ is an ergodic hyperbolic measure, by subsection 3.1, we reduce the proof of Theorem [3.3] to proving the following Theorem [3.4].

**Theorem 3.4.** Let $f : M \to M$ be a homeomorphism, $\mu$ be an $f$-invariant Borel probability measure, and let $A : M \to GL(X)$ be continuous. If for almost every ergodic component $\nu$ of $\mu$, there exists a $\nu$-continuous map $C_\nu : M \to GL(X)$ such that $A(x) = C_\nu(f x)C_\nu(x)^{-1}$ holds for $\nu$-a.e. $x \in M$. Then there exists a $\mu$-continuous map $C : M \to GL(X)$, such that $A(x) = C(f x)C(x)^{-1}$, for $\mu$-a.e. $x \in M$.

This theorem generalizes Theorem 3.4 in [3.3] in the sense that $GL(X)$ is neither separable nor locally compact. In fact, they proved that if $G$ is a second countable and locally compact group, $A : M \to G$ and $C_\nu : M \to G$ are measurable for almost every ergodic component $\nu$ of $\mu$ such that $A(x) = C_\nu(f x)C_\nu(x)^{-1}$ for $\nu$-a.e. $x \in M$. Then there exists a measurable map $C : M \to G$ such that $A(x) = C(f x)C(x)^{-1}$ for $\mu$-a.e. $x \in M$. In our setting, $GL(X)$ is not separable. We shall prove that the image of $A$ and $C_\nu$ are in a separable subset $K$ (but not a subgroup) of $GL(X)$ almost everywhere. And thanks to the metric of $GL(X)$, the local compactness condition is not needed in our setting.

**Proof of Theorem 3.4.** For any given ergodic component $\nu$ of $\mu$, we shall prove first that $C_\nu : M \to K$, where $K$ is given by (3.8).

Since $C_\nu$ is $\nu$-continuous, there exists a sequence of compact subsets $F_1 \subset F_2 \subset \cdots$ of $M$ with $\nu(\bigcup_{n \geq 1} F_n) = 1$ such that $C_\nu|_{F_n}$ is continuous for every $n \geq 1$. Let
\[ D_n = \text{supp}(\nu_{F_n}), \text{ and let } E_n = \{ x \in D_n : \mathcal{O}(x) \cap D_n = D_n \}, \text{ where } \nu_{F_n} \text{ is defined by } \nu_{F_n}(B) := \nu(B \cap F_n)/\nu(F_n). \] Then by Lemma 3.2, \( \nu(E_n) = \nu(F_n) \) for every \( n \geq 1 \). Since the sequence \( \{ F_n \}_{n \geq 1} \) is increasing, we have \( \nu(\cap_{n \geq 1} E_n) = \nu(F_1) > 0 \). Choose \( z \in \cap_{n \geq 1} E_n \). Then
\[ \mathcal{O}(z) \cap D_n = D_n, \quad \forall n \geq 1. \]

Without loss of generality, we may assume \( C_\nu(z) = \text{Id} \). Indeed, if \( C_\nu(z) \neq \text{Id} \), we may replace the map \( C_\nu(z) \) by the map \( \tilde{C}_\nu(z) := C_\nu(x)C_\nu(z)^{-1} \). Then \( \tilde{C}_\nu(z) = \text{Id} \) and \( A(x) = C_\nu(fx)C_\nu(z)^{-1}C_\nu(x)C_\nu(z)^{-1} = C_\nu(fx)C_\nu(x)^{-1} \) for \( \nu \)-a.e. \( x \in M \).

Now for any \( f^k(z) \in D_n \), we have \( C_\nu(f^k z) = A_k^k C_\nu(z) = A_k^k \). Since \( \mathcal{O}(z) \) is dense in \( D_n \), and \( C_\nu \) is continuous on \( D_n \), we have
\[ C_\nu(y) \in K, \quad \forall y \in D_n, n \geq 1. \]

It follows from \( \nu(\cup_{n \geq 1} D_n) = 1 \) that \( C_\nu(y) \in K \) for \( \nu \)-a.e. \( y \in M \).

To continue the proof Theorem 3.4 we shall make some reductions. We begin by introducing an ergodic decomposition theorem, a theorem of Rohlin and some lemmas of measurability.

For a given measurable partition \( \mathcal{P} \) of \( M \), denote \( \Omega = M/\mathcal{P} \). Let \( \pi : M \to \Omega \) be the corresponding projection, and let \( \hat{\mu} = \pi_* \mu \). We state an ergodic decomposition theorem from the point of Rohlin’s view.

**Theorem 3.5** (Theorem 5.1.3 [38]). Let \( f, \mu \) be as above. Then there exists a measurable subset \( M_0 \subset M \) with \( \mu(M_0) = 1 \), a partition \( \mathcal{P} \) of \( M_0 \) into measurable subsets and a family \( \{ \mu_\omega : \omega \in \Omega = M_0/\mathcal{P} \} \) of probability measures on \( M \) such that
\[ \begin{array}{l}
\mu_\omega(\pi^{-1}(\omega)) = 1 \text{ for } \hat{\mu}-\text{a.e. } \omega \in \Omega, \\
\mu_\omega \text{ is } f-\text{invariant and ergodic for } \hat{\mu}-\text{a.e. } \omega \in \Omega, \\
\omega \mapsto \mu_\omega(E) \text{ is measurable, for every measurable subset } E \subset M, \text{ and}
\end{array} \]
\[ \mu(E) = \int \mu_\omega(E)d\hat{\mu}(\omega). \]

**Definition 3.6.** A Borel probability space \( (S, m) \) is called standard, if \( S \) is Borel isomorphic to a complete separated metric space, and \( m \) is a Borel probability measure on \( S \).

We simplify the problem by using the following Rohlin’s theorem [32]. This statement comes from [13] Proposition 2.21.

**Proposition 3.7** (Rohlin [32]). Assume the notation of Theorem 3.4. Then there is a partition of \( \Omega \) into countably many Borel subsets \( \Omega_1, \Omega_2, \ldots \), such that for any \( k \geq 1 \), there exists a standard Borel probability space \( (S_k, m_k) \) and an isomorphism \( \theta_k : (\pi^{-1}(\Omega_k), \mu) \to (\Omega_k \times S_k, \hat{\mu} \times m_k) \).

**Sketch of the proof.** Two standard Borel probability spaces \( (S_1, \mu_1) \) and \( (S_2, \mu_2) \) are called of the same type if \( S_1 \) is isomorphic Mod 0 to \( S_2 \). Since any standard Borel probability space is isomorphic Mod 0 to the space consisting of an interval with ordinary Lebesgue measure and a sequence of points with positive measure [32] No. 4, §2, there are only countably many equivalent classes. Then by [32] Theorem (I) in §4, there is a measurable decomposition \( \Omega = \Omega_1 \cup \Omega_2 \cup \cdots \) such that for any \( \omega_1, \omega_2 \in \Omega_k \), \( (\pi^{-1}(\omega_1), \mu_{\omega_1}) \) and \( (\pi^{-1}(\omega_2), \mu_{\omega_2}) \) are of the same type. We conclude by using [32] Theorem (II) in §4 that \( (\pi^{-1}(\Omega_k), \mu) \) is isomorphic Mod 0 to \( (\Omega_k \times \pi^{-1}(\omega), \hat{\mu} \times \mu_\omega) \) for any \( \omega \in \Omega_k \).
Let \((S, m)\) be a standard Borel probability space and \((Y, d)\) be a separable metric space. Denote by \(F(S, Y)\) the space of measurable maps \(\phi : S \to Y\), where two maps are identified if they are equal almost everywhere. Then \(F(S, Y)\) is a separable metric space endowed with the metric
\[
d_F(\phi_1, \phi_2) := \inf \{\varepsilon > 0 : m(\{s \in S : d(\phi_1(s), \phi_2(s)) > \varepsilon\}) \leq \varepsilon\}.
\]
Moreover, if \(Y\) is complete, then \(F(S, Y)\) is also complete.

The following three lemmas come from [28] and [13], we shall give a proof here for the completeness.

**Lemma 3.8** ([28]). Let \(\Omega, S\) be standard Borel spaces, \(Y\) be a separable metric space, and \(g : \Omega \times S \to Y\) be a Borel map. Then

(i) For any \(\omega \in \Omega\), the map \(g_\omega : S \to Y\) defined by \(g_\omega(s) = g(\omega, s)\) is Borel.

(ii) The induced map \(\hat{g} : \Omega \to F(S, Y)\) defined by \(\hat{g}(\omega) = g_\omega\) is Borel.

**Proof.** (i) For any \(\omega \in \Omega\) and any open subset \(U \subset Y\), since \(g\) is Borel and
\[
(g_\omega)^{-1}(U) = \{s \in S : (\omega, s) \in g^{-1}(U)\}
\]
is the \(\omega\)-section of \(g^{-1}(U)\), it follows that \(g_\omega\) is Borel.

(ii) It's enough to prove for any \(h \in F(S, Y)\) and any \(\delta > 0\), the set \(\{\omega \in \Omega : d_F(g_\omega, h) < \delta\}\) is measurable. Notice that
\[
\{\omega : d_F(g_\omega, h) < \delta\} = \bigcup_{r_n \in \mathbb{Q}, 0 < r_n < \delta} \{\omega : m(\{s \in S : d(g_\omega(s), h(s)) > r_n\}) \leq r_n\}.
\]
Since \(\{s \in S : d(g_\omega(s), h(s)) > r_n\}\) is the \(\omega\)-section of the measurable set \(\{((\omega, s) \in \Omega \times S : d(g(\omega, s), h(s)) > r_n)\}\), by Fubini's theorem, the measure of the \(\omega\)-section of the measurable set \(\{((\omega, s) \in \Omega \times S : d(g(\omega, s), h(s)) > r_n)\}\) is a measurable function of \(\omega\), that is, \(\omega \mapsto m(\{s \in S : d(g_\omega(s), h(s)) > r_n\})\) is measurable, which implies the set
\[
\{\omega : m(\{s \in S : d(g_\omega(s), h(s)) > r_n\}) \leq r_n\}
\]
is measurable. Thus we conclude the set \(\{\omega : d_F(g_\omega, h) < \delta\}\) is measurable.  

The converse of this lemma is also true.

**Lemma 3.9** ([13]). Let \((\Omega, \mu), (S, m)\) be standard Borel probability spaces, \(Y\) be a separable complete metric space, and \(\Phi : \Omega \to F(S, Y)\) be a Borel map. Then there exists a Borel map \(\hat{\Phi} : \Omega \times S \to Y\), such that for \(\mu\)-a.e. \(\omega \in \Omega\),
\[
\hat{\Phi}(\omega, s) = \Phi(\omega)(s), \text{ for } m\text{-a.e. } s \in S.
\]

**Proof.** Let \(\{P_n = \{P_{n,i}\}_{i \geq 1}\}_{n \geq 1}\) be a sequence of increasing partitions of \(F(S, Y)\) with the diameter of \(P_{n,i}\) less that \(2^{-n}\). Fix any \(\phi_n^i \in P_{n,i}\). Define \(\Phi_n : \Omega \times S \to Y\) by
\[
\Phi_n(\omega, s) = \phi_n^i(\omega), \text{ if } \Phi(\omega) \in P_{n,i}.
\]
Then for any \(n, k \in \mathbb{N}, \omega \in \Omega\), one has
\[
m(\{s : d_F(\Phi_n(\omega, s), \Phi_{n+k}(\omega, s)) > 2^{-n}\}) < 2^{-n}.
\]
Thus the Fubini theorem gives \((\mu \times m)(\{((\omega, s) : d_F(\Phi_n(\omega, s), \Phi_{n+k}(\omega, s)) > 2^{-n}\}) \leq 2^{-n}\), which implies \(d_F(\Phi_n, \Phi_{n+k}) \leq 2^{-n}\). Thus \(\{\Phi_n\}\) converges in \(F(\Omega \times S, Y)\). Denote by \(\hat{\Phi}\) the limit of \(\{\Phi_n\}\). Since \(\{\Phi_n\}\) converges to \(\hat{\Phi}\) in measure, there exists a subsequence \(\{\Phi_{n_k}\}\) converges to \(\hat{\Phi}\) almost everywhere. Let \(\Phi_{n_k, \omega} : S \to X\) and
\[ \hat{\Phi}_\omega : S \to Y \] be gotten by Lemma 3.8. Then for \( \hat{\mu} \)-a.e. \( \omega \in \Omega \), \( \Phi_{n_k, \omega} \) converges to \( \hat{\Phi}_\omega \) almost everywhere, which implies \( d_F(\Phi_{n_k, \omega}, \hat{\Phi}_\omega) \to 0 \). Notice that for any \( \omega \),
\[ d_F(\Phi_{n_k, \omega}, \Phi(\omega)) \leq 2^{-n_k} \to 0 \] as \( k \to \infty \). Therefore \( \hat{\Phi}_\omega = \Phi(\omega) \) for \( \hat{\mu} \)-a.e. \( \omega \in \Omega \).
This proves the Borel map \( \hat{\Phi} \in F(\Omega \times S, Y) \) satisfies
\[ \hat{\Phi}(\omega, s) = \hat{\Phi}_\omega(s) = \Phi(\omega)(s), \] for \( \hat{\mu} \)-a.e. \( \omega \in \Omega \) and m-a.e. \( s \in S \).

\[ \square \]

Let \( (S, m) \) be a standard Borel probability space, and denote by \( \text{Aut}_m(S) \) the group of measure preserving automorphisms on \( S \), where two automorphisms are identified if they are equal almost everywhere. Since \( S \) is a standard Borel space, we may assume \( S \) is a separable and complete metric space, and hence \( \text{Aut}_m(S) \) is a closed subset of \( F(S, S) \).

**Lemma 3.10** (13). Let \( (S, m) \) be a standard Borel probability space, \( Y \) be a separable metric space. Then the natural action \( \text{Aut}_m(S) \times F(S, Y) \to F(S, Y) \) defined by \( (g, \phi) \mapsto \phi \circ g \) is continuous.

**Proof.** Given any \( g_n \to g, \phi_n \to \phi \), we’d prove \( d_F(\phi_n \circ g_n, \phi \circ g) \to 0 \).

Note that \( d_F(\phi_n \circ g_n, \phi \circ g) \leq d_F(\phi_n \circ g_n, \phi \circ g_n) + d_F(\phi \circ g_n, \phi \circ g) \), since
\[ m\{s : d(\phi_n \circ g_n(s), \phi \circ g_n(s)) > \varepsilon\} = m(g_n^{-1}\{s : d(\phi_n, \phi) > \varepsilon\}) \]
\[ = m\{s : d(\phi_n, \phi) > \varepsilon\} \to 0, \]
we have \( d_F(\phi_n \circ g_n, \phi \circ g_n) \to 0 \). Then it remains to show \( d_F(\phi \circ g_n, \phi \circ g) \to 0 \).

For any \( \varepsilon > 0 \), by Lusin’s theorem, there exists a compact subset \( K_\varepsilon \subset S \) such that \( \phi \) is uniformly continuous on \( K_\varepsilon \) and \( m(K_\varepsilon) > 1 - \frac{1}{4}\varepsilon \). Then there exists \( \delta > 0 \) such that for any \( x, y \in K_\varepsilon \) with \( d(x, y) \leq \delta \), one has \( d(\phi(x), \phi(y)) \leq \varepsilon \). Let \( A_n = g_n^{-1}(S \setminus K_\varepsilon) \cup g^{-1}(S \setminus K_\varepsilon) \). Since \( m \) is \( g_n \) and \( g \) invariant, one sees \( m(A_n) \leq \frac{\varepsilon}{2} \).
Then equation
\[ \{s : d(\phi \circ g_n(s), \phi \circ g(s)) > \varepsilon\} \subset A_n \cup \{s : d(g_n(s), g(s)) > \delta\} \]
gives
\[ \limsup_{n \to \infty} m\{s : d(\phi \circ g_n(s), \phi \circ g(s)) > \varepsilon\} \leq \frac{\varepsilon}{2} + 0 < \varepsilon. \]
Since \( \varepsilon \) is arbitrary, we conclude that \( d_F(\phi \circ g_n, \phi \circ g) \to 0 \).

**Reductions.** By Theorem 3.5 and Proposition 3.7, \( M \) is isomorphic Mod 0 to the disjoint union of at most countably many spaces \( \Omega_k \times S_k \). Hence we may assume without loss of generality that \( M = \Omega \times S \) and \( \mu = \hat{\mu} \times m \), where \( (\Omega, \hat{\mu}), (S, m) \) are standard Borel probability spaces. By the ergodic decomposition theorem, every \( \{\omega\} \times S \) is \( f \)-invariant. Denote \( f(\omega, s) = (\omega, f_2(\omega, s)) \). Then by Theorem 3.5 and Lemma 3.8 for \( \hat{\mu} \)-a.e. \( \omega \in \Omega \), the map \( f_\omega = f_{2(\omega)} : S \to S \) defined by \( f_\omega(s) = f_2(\omega, s) \) is Borel and preserves the ergodic measure \( m \). Therefore, the condition of Theorem 3.4 becomes: for \( \hat{\mu} \)-a.e. \( \omega \in \Omega \), there exists a measurable map \( C_\omega : S \to K \), such that
\[ A(\omega, s) = C_\omega(f_\omega(s))C_\omega(s)^{-1}, \] for m-a.e. \( s \in S \).

As a corollary of Lemma 3.10 we have the following result.

**Corollary 3.11.** Let \( \Omega, S, f, \) be as above, and \( K \) be as in 3.8. Then the map defined by \( \Omega \times F(S, K) \to F(S, K) \), \( (\omega, \phi) \mapsto \phi(f_\omega) \) is Borel.
Proof. By Lemma [3.8] the map $\omega \mapsto f_\omega$ is Borel. Thus we obtain the map
\[ \Omega \times F(S, K) \to Aut_m(S) \times F(S, K) \]
\[ (\omega, \phi) \mapsto (f_\omega, \phi) \]
is Borel. By Lemma [3.10] the natural action $Aut_m(S) \times F(S, K) \to F(S, K)$,
\[(g, \phi) \mapsto g \circ \phi \text{ is continuous.} \]
We conclude $(\omega, \phi) \mapsto \phi(f_\omega)$ is Borel. \hfill \square

Denote
\[ K' = \bigcup_{m, n \in \mathbb{Z}} \{ A_{x,y}^m A_{x,y}^n : x, y \in M \} \subset GL(X). \]
Then $K'$ is a separable, complete metric space. We have the following lemma.

Lemma 3.12. The map $\tau_1 : F(S, K) \to F(S, K)$ defined by
\[ \tau_1(\phi)(s) = \phi(s)^{-1} \]
and the map $\tau_2 : F(S, K) \times F(S, K) \to F(S, K')$ defined by
\[ \tau_2(\phi, \psi)(s) = \phi(s) \circ \psi(s) \]
are continuous.

Proof. Suppose that $\phi_n \to \phi \in F(S, K)$, that is,
\[ \inf \{ \varepsilon > 0 : m(\{ s \in S : d(\phi_n(s), \phi(s)) > \varepsilon \}) \leq \varepsilon \} \to 0, \text{ as } n \to \infty. \]
Since
\[ d(\phi_n(s), \phi(s)) = \| \phi_n(s) - \phi(s) \| + \| \phi_n(s)^{-1} - \phi(s)^{-1} \| = d(\phi_n(s)^{-1}, \phi(s)^{-1}), \]
we have $\phi_n^{-1} \to \phi^{-1} \in F(S, K)$ as $n \to \infty$. That is, $\tau_1$ is continuous.

Now given any sequences $\phi_n \to \phi \in F(S, K), \psi_n \to \psi \in F(S, K)$, to show $\phi_n \psi_n \to \phi \psi \in F(S, K')$, it suffices to prove for any $\varepsilon > 0$,
\[ \limsup_{n \to \infty} m(\{ s : d(\phi_n(s)\psi_n(s), \phi(s)\psi(s)) > \varepsilon \}) \leq \varepsilon. \]
Indeed, if this equation holds, then there exists $N \geq 1$, such that for any $n \geq N$, one has
\[ m(\{ s : d(\phi_n(s)\psi_n(s), \phi(s)\psi(s)) > 2\varepsilon \}) \leq m(\{ s : d(\phi_n(s)\psi_n(s), \phi(s)\psi(s)) > \varepsilon \}) \leq 2\varepsilon. \]
Since $\varepsilon$ is arbitrary, it implies $\phi_n \psi_n \to \phi \psi$ as $n \to \infty$.

Notice that
\[ \{ s : d(\phi_n(s)\psi_n(s), \phi(s)\psi(s)) > \varepsilon \} \]
\[ \subseteq \{ s : \| \phi_n(s)\psi_n(s) - \phi(s)\psi(s) \| > \varepsilon \} \cup \{ s : \| \phi_n(s)^{-1} - \psi^{-1} \phi(s)^{-1} \| > \varepsilon \}. \]
Since $\| \phi_n(s)\psi_n(s) - \phi(s)\psi(s) \| \leq \| \phi_n(s) - \phi(s) \| \cdot \| \phi_n(s) \| + \| \psi(s) \| + \| \psi_n(s) - \psi(s) \|$, we have
\[ \{ s : \| \phi_n(s)\psi_n(s) - \phi(s)\psi(s) \| > \varepsilon \} \]
\[ \subseteq \{ s : \| \phi_n(s) - \phi(s) \| \cdot \| \psi_n(s) \| > \varepsilon \} \cup \{ s : \| \phi(s) \| \cdot \| \psi_n(s) - \psi(s) \| > \varepsilon \}. \]
We shall prove: \( \limsup_{n \to \infty} m(\{s : \|\phi_n(s) - \phi(s)\| \cdot \|\psi_n(s)\| > \varepsilon/4\}) \leq \varepsilon/2. \) Let \( A_n = \{s : \|\psi_n(s) - \psi(s)\| \leq 1\}. \) Then we claim that \( \lim_{n \to \infty} m(S \setminus A_n) = 0. \) Indeed, since \( \psi_n \to \psi \) in measure, for any \( 0 < \delta < 1, \) we have
\[
\limsup_{n \to \infty} m(S \setminus A_n) \leq \limsup_{n \to \infty} m(\{s : \|\psi_n(s) - \psi(s)\| > \delta\}) \leq \delta.
\]
Since \( \delta \) is arbitrary, we conclude \( \lim_{n \to \infty} m(S \setminus A_n) = 0. \) Denote \( D_n := \{s : \|\psi(s)\| \leq n\}. \) Since \( \lim_{n \to \infty} m(D_n) = \lim_{n \to \infty} m(\cup_{n \geq 1} D_n) = 1, \) we may take \( N \) large enough such that \( m(D_N) > 1 - \varepsilon/4. \) Then we have
\[
\{s : \|\phi_n(s) - \phi(s)\| \cdot \|\psi_n(s)\| > \varepsilon/4\} \subseteq (S \setminus A_n) \cup \left\{s : \|\phi_n(s) - \phi(s)\| > \frac{\varepsilon}{\|\psi(s)\|} + 1\right\}
\]
\[
\subseteq (S \setminus A_n) \cup (S \setminus D_N) \cup \left\{s : \|\phi_n(s) - \phi(s)\| > \frac{\varepsilon}{N + 1}\right\}.
\]
Since \( \phi_n \to \phi, \) we have \( \limsup_{n \to \infty} m(\{s : \|\phi_n(s) - \phi(s)\| \cdot \|\psi_n(s)\| > \frac{\varepsilon}{4}\}) \leq \varepsilon/2. \) Similarly, we can also get \( \limsup_{n \to \infty} m(\{s : \|\phi(s)\| \cdot \|\psi_n(s) - \psi(s)\| > \frac{\varepsilon}{4}\}) \leq \varepsilon/2. \) Therefore,
\[
\limsup_{n \to \infty} m(\{s : \|\phi_n(s)\psi_n(s) - \phi(s)\psi(s)\| > \frac{\varepsilon}{2}\}) \leq \varepsilon/2.
\]
It can be proved analogously that
\[
\limsup_{n \to \infty} m(\{s : \|\psi_n(s)^{-1}\phi_n(s)^{-1} - \psi(s)^{-1}\phi(s)^{-1}\| > \frac{\varepsilon}{2}\}) \leq \varepsilon/2.
\]
Thus \( \limsup_{n \to \infty} m(\{s : d(\phi_n(s)\psi_n(s), \phi(s)\psi(s)) > \varepsilon\}) \leq \varepsilon. \) This proves \( \phi_n\psi_n \to \phi\psi, \) that is, \( \tau_2 \) is continuous.

We also need the following von Neumann’s selection theorem to find the desired \( \mu \)-continuous map \( C : M \to GL(X). \) Recall that a subset \( \mathcal{F} \) of a standard Borel space \( \Sigma \) is called analytic if there is a standard Borel space \( Y \) and a Borel map \( \psi : Y \to \Sigma \) such that \( \mathcal{F} = \psi(Y). \)

**Theorem 3.13** (von Neumann Selection Theorem [1, Theorem 3.4.3]). Let \((\Omega, \hat{\mu})\) be a standard Borel probability space, \( L \) be a standard Borel space, and \( \mathcal{F} \) be an analytic subset of \( \Omega \times L. \) Denote by \( \Omega_\mathcal{F} \) the projection of \( \mathcal{F} \) to \( \Omega. \) Then there exists a Borel subset \( \Omega_0 \subset \Omega_\mathcal{F} \) of \( \hat{\mu} \)-full measure and a Borel function \( \Phi : \Omega_0 \to L, \) such that \( \text{graph}(\Phi) \subset \mathcal{F}. \)

We now continue the proof of Theorem 3.4.

By the Reductions in this subsection, we assume \( M = \Omega \times S \) and \( \mu = \hat{\mu} \times m, \) where \((\Omega, \hat{\mu}), (S, m)\) are standard Borel probability spaces. Denote \( f(\omega, s) = (\omega, f_\omega(s)). \) Then for any \( \omega \in \Omega, f_\omega(s) \in \text{Aut}_m(S). \) By Lemma 3.18 the map \( A : \Omega \times S \to K \) induces the Borel maps \( A_\omega : S \to K \) and \( \omega \mapsto A_\omega. \) Denote
\[
\tilde{K} = \bigcup_{k,m,n \in \mathbb{Z}} \{A^k_\omega A^n_\varphi A_\omega^n : x, y, z \in M\} \subset GL(X).
\]
Then \( \tilde{K} \) is a separable, complete metric space. We define
\[
\sigma : \Omega \times F(S, K) \to F(S, \tilde{K})
\]
by

\[ \sigma(\omega, \phi)(s) = \phi(f_\omega(s))^{-1} A_\omega(s) \phi(s). \]

Then \( \sigma \) is Borel. Indeed, Corollary 3.11 and Lemma 3.12 show that the map \((\omega, \phi) \mapsto (\phi \circ f_\omega)^{-1}\) is Borel. Since the second term \((\omega, \phi) \mapsto A_\omega\) and the third term \((\omega, \phi) \mapsto \phi\) are also Borel, using the second conclusion of Lemma 3.12 we conclude \( \sigma \) is Borel measurable.

Denote \( F = \sigma^{-1}(\{Id\}) \). Then \( F \) is an analytic subset of \( \Omega \times F(S, K) \). Denote by \( \Omega_F \) the projection of \( F \) onto \( \Omega \). Then by (3.9), \( \Omega_F \) is of \( \mu \)-full measure. Then by Von Neumann selection theorem 3.13 there exists a Borel map \( \Phi : \Omega \to F(S, K) \), such that for \( \mu \)-a.e. \( \omega \in \Omega \), one has \((\omega, \Phi(\omega)) \in F \), that is, for \( \mu \)-a.e. \( \omega \in \Omega \),

\[ A(\omega, s) = \Phi(\omega)(f_\omega(s)) \Phi(\omega)(s)^{-1}, \text{ for } m\text{-a.e. } s \in S. \]

Hence by Lemma 3.9 there exists a Borel map \( C : \Omega \times S \to K \), such that for \( \mu \)-a.e. \( \omega \), we have

\[ C(\omega, s) = \Phi(\omega)(s), \text{ for } m\text{-a.e. } s \in S. \]

Therefore, \( A(x) = C(f(x))C(x)^{-1}, \) for \( \mu = (\hat{\mu} \times m)\)-a.e. \( x = (\omega, s) \in M \). Since \( K \) is separable, by Lusin’s theorem, \( C \) is \( \mu \)-continuous. This completes the proof of Theorem 3.4.

\[ \square \]

4. PROOF OF THEOREM 1.2

We begin by reducing the proof to the topologically mixing case. Since \( \mu \) is an ergodic \( f \)-invariant measure on \( M \) with full support, one has \( f \) is transitive. By the spectral decomposition theorem, there is an integer \( k \geq 1 \) such that \( M = \bigsqcup_{i=1}^{k} \Sigma_i \), satisfying \( f(\Sigma_i) = \Sigma_{i+1} \) and \( f^k|_{\Sigma_i} \) is topologically mixing for \( 1 \leq i \leq k \). Then the normalized restriction \( \mu_{\Sigma_i} \) of \( \mu \) to \( \Sigma_i \) is an ergodic \( f^k \)-invariant measure on \( \Sigma_i \) with full support and local product structure, and \( A^k \) is a Hölder continuous cocycle for \( f^k \) satisfying \( A^k = C(f^k x)C(x)^{-1} \) for \( \mu_{\Sigma_i} \)-a.e. \( x \in \Sigma_i \). Assuming Theorem 1.2 holds for topologically mixing systems \( f^k|_{\Sigma_i} \), then \( C \) coincides \( \mu_{\Sigma_i} \)-a.e. with a Hölder continuous map. Since \( M = \bigsqcup_{i=1}^{k} \Sigma_i \) is a disjoint union, we conclude that \( C \) coincides \( \mu \)-a.e. with a Hölder continuous map satisfying the equation (1.2).

Now we begin the proof of Theorem 1.2 in the topologically mixing case.

4.1. Extending \( C \) to \( D(N, \theta) \). Since there exists a \( \mu \)-continuous map \( C : M \to GL(X) \) satisfying (1.2), we can find a compact set \( \hat{K} \) with positive \( \mu \)-measure on which \( C \) is continuous and thus the norms of \( C \) and \( C^{-1} \) are bounded. Then by the Poincaré’s recurrence theorem, for \( \mu \)-a.e. \( x \in \hat{K} \), there exists infinitely many \( n_k \) with \( f^{n_k}(x) \in \hat{K} \). We may take such a point \( x \in \hat{K} \) whose iterations satisfy (1.2), and we may also assume the point \( x \) is regular, that is, \( x \) satisfies

\[ \lambda_+(A, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \| A^n \| = \lim_{k \to \infty} \frac{1}{n_k} \log \| C(f^{n_k} x)C(x)^{-1} \| = 0. \]

Similarly, \( \lambda_-(A, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \| (A^n)^{-1} \| = 0. \) Then by Lemma 2.2, for any \( \theta > 0 \) and \( \mu \)-a.e. \( x \in \hat{M} \), there exists \( N \geq 1 \) such that

\[ \prod_{j=0}^{k-1} \| A_{f^j}^N x \| \leq e^{k N \theta}, \text{ and } \prod_{j=0}^{k-1} \| (A_{f^j}^N x)^{-1} \| \leq e^{k N \theta}, \quad \forall k \geq 1. \]
Since $C$ with $C$.\n
Analogously, we can also get that for $H$ \ref{4.1} there exists $N \geq 1$ such that $x \in D(N, \theta)$. Take $\theta < \tau \alpha$. Then by Proposition \ref{2.4} the stable and unstable holonomies exist almost everywhere.

**Lemma 4.1.** There exists a set $\Omega$ of full $\mu$-measure such that for any $x, y \in \Omega$ with $y \in W^s_{\text{loc}}(x)$, one has

\[ H^s_{x,y} C(x) = C(y), \]

where $* \in \{s, u\}$.

**Proof.** Fix a $\theta < \tau \alpha$. Then for any given $N \geq 1$, and for $\mu$-a.e. $x, y \in D(N, \theta)$ with $y \in W^s_{\text{loc}}(x)$, one has

\[ H^s_{x,y} = \lim_{n \to \infty} (A^N y)^{-1} A^N x = \lim_{n \to \infty} C(y) C(f^n y)^{-1} C(f^n x) C(x)^{-1}. \]

Since $C$ is $\mu$-continuous, we may take a compact subset $K$ with $\mu(K) > \frac{1}{2}$ such that $C$ is continuous on $K$. By Birkhoff Ergodic Theorem, for $\mu$-a.e. $z \in M$, \( \frac{1}{n} \sum_{i=0}^{n-1} \chi_K(f^i z) \to \mu(K) > \frac{1}{2} \) as $n \to \infty$. Thus for $\mu$-a.e. $x, y \in D(N, \theta)$ with $y \in W^s_{\text{loc}}(x)$, there exists a sub-sequence $\{n_i\}_{i \geq 1}$ such that $f^{n_i}(x), f^{n_i}(y) \in K$ for any $i \geq 1$. Then \ref{4.1} gives

\[ H^s_{x,y} C(x) = C(y), \]

where $* \in \{s, u\}$.

The following lemma shows that the local product structure of $\mu$ and the holonomy invariance of $C$ imply the map $C$ can be extended continuously to $\text{supp}(\mu|D(N, \theta))$.

**Lemma 4.2.** For any $\theta < \tau \alpha$ and $N \geq 1$, there exists an $\alpha$-Hölder continuous map $\tilde{C}$ defined on $\text{supp}(\mu|D(N, \theta))$ which coincides $\mu$-a.e. on $\text{supp}(\mu|D(N, \theta))$ with $C$.

**Proof.** Let $\delta > 0$ be small enough such that for any $y, z \in M$ with $d(y, z) < 2\delta$, $W^s_{\text{loc}}(y)$ intersects $W^u_{\text{loc}}(z)$ at exactly one point $[y, z]$. Given any $x \in D(N, \theta)$, denote

\[ N^u_x(\delta) = N^u_x(N, \theta, \delta) := \{ [x, y] : y \in B(x, \delta) \cap D(N, \theta) \}, \]

\[ N^s_x(\delta) = N^s_x(N, \theta, \delta) := \{ [x, y] : y \in B(x, \delta) \cap D(N, \theta) \}. \]

Then $N^s_x(\delta) \subset W^s_{\text{loc}}(x)$ for $* \in \{s, u\}$. Let $N_x(\delta) = [N^u_x(\delta), N^s_x(\delta)]$ be the image of $N^u_x(\delta) \times N^s_x(\delta)$ under the map $(y, z) \mapsto [y, z]$. Since $\mu$ has local product structure, one has

\[ \text{supp}(\mu|N_x(\delta)) = [\text{supp}(\mu^u|N^u_x(\delta)), \text{supp}(\mu^s|N^s_x(\delta))], \]

where $\mu^u|N^u_x(\delta)$ and $\mu^s|N^s_x(\delta)$ are the projections of $\mu|N_x(\delta)$ to $N^u_x(\delta)$ and $N^s_x(\delta)$ respectively. Notice that $N_x(\delta) \supset D(N, \theta) \cap B(x, \delta)$. It suffices to construct an $\alpha$-Hölder continuous map $\tilde{C}$ on $\text{supp}(\mu|N_x(\delta))$ which coincides $\mu$-a.e. on $\text{supp}(\mu|N_x(\delta))$ with $C$.\n
That is, \( \hat{\mu} \) is invariant under stable holonomies and coincides on \( \Omega \cap \hat{\Omega} \) holonomies on \( \text{supp}(\xi,\eta) \cap \Omega \). Then by the construction, 
\[
\mu(\hat{\Omega} \setminus \Omega_{\mathcal{N}}) = 0.
\]
Fix any such \( \xi \). Let \( \Omega_{\mathcal{N}} \) be the set of points in \( \mathcal{N}_x(\xi) \cap \Omega \) that lie on the local unstable leaves of \( [\xi,\mathcal{N}_x(\xi)] \cap \Omega \). Then we have 
\[
\mu(\hat{\mathcal{N}}_x(\xi) \cap \Omega_{\mathcal{N}}) = 0.
\]
Fix \( x_0 \in [\xi,\mathcal{N}_x(\xi)] \cap \Omega \). For any \( z \in \Omega_{\mathcal{N}} \), let \( \eta = [x_0,z] \). Then by the construction of \( \Omega_{\mathcal{N}} \), one has \( \eta \in \Omega_{\mathcal{N}} \). By Lemma 4.3, \( C(z) = H_{y,z}^u H_{x_0,\eta}^s C(x_0) \). Define \( \hat{C} \) on \( \text{supp}(\mu|\hat{\mathcal{N}}_x(\xi)) \) by 
\[
\hat{C}(z) := H_{y,z}^u \hat{C}(x_0) \quad \forall z \in \text{supp}(\mu|\hat{\mathcal{N}}_x(\xi)), y \in [\mathcal{N}_x(\xi),z].
\]
Then by the construction, \( \hat{C} = C \) almost everywhere on \( \text{supp}(\mu|\hat{\mathcal{N}}_x(\xi)) \). Since the stable and unstable holonomies are continuous, we conclude that \( \hat{C} \) is continuous on \( \Omega_{\mathcal{N}} \). Moreover, by construction, 
\[
\hat{C}(z) = H_{y,z}^u \hat{C}(y) \quad \forall z \in \text{supp}(\mu|\hat{\mathcal{N}}_x(\xi)), y \in [\mathcal{N}_x(\xi),z].
\]
That is, \( \hat{C} \) is invariant under unstable holonomies on \( \text{supp}(\mu|\hat{\mathcal{N}}_x(\xi)) \).

By a dual procedure, we can obtain a continuous map \( \hat{C} \) on \( \text{supp}(\mu|\hat{\mathcal{N}}_x(\xi)) \) which is invariant under stable holonomies and coincides \( \mu \)-a.e. with \( C \) on \( \text{supp}(\mu|\hat{\mathcal{N}}_x(\xi)) \).

Then by the continuity, \( \hat{C} = \hat{C} \). Hence \( \hat{C} \) is invariant under both stable and unstable holonomies on \( \text{supp}(\mu|\hat{\mathcal{N}}_x(\xi)) \). By Proposition [22], \( \|H_{y,z}^u - Id\| \leq L \cdot d(y,z)^{\alpha} \) for \( y,z \in \mathcal{N}_x(\xi) \) and \( y \in W^s_{loc}(z) \). It follows that \( \hat{C} \) is \( \alpha \)-Hölder continuous on every stable and unstable leaf, and thus \( \alpha \)-Hölder continuous on \( \text{supp}(\mu|\hat{\mathcal{N}}_x(\xi)) \). \( \square \)

Since \( \text{supp}(\mu|D(N,\theta)) \) may be a proper subset of \( D(N,\theta) \), we use the following lemma to obtain a continuous map \( \hat{C} \) defined on \( D(N,\theta) \). This lemma resembles Lemma 4.5 of [10]. We give a geometric proof here.

**Lemma 4.3.** For any \( \theta < \tau \alpha \) and \( N \geq 1 \), there exist \( \theta < \theta_* < \tau \alpha \) and \( N_* \geq N \), such that 
\[
D(N,\theta) \subset \text{supp}(\mu|D(N_*,\theta_*)).
\]

**Proof.** For any \( \theta < \tau \alpha \), since \( \lambda_+ (\mathcal{A},\mu) = \lambda_- (\mathcal{A},\mu) = 0 \), we may take \( N \geq 1 \) large enough such that 
\[
\frac{1}{N} \int \log(\|A_N^u\| \cdot \|A_N^s\|^{-1}) d\mu < \theta.
\]
Set \( \varphi(x) = \frac{1}{N} \log(\|A_N^u\| \cdot \|A_N^s\|^{-1}) \). Given any \( \gamma < (\tau \alpha - \theta)/3 \), denote 
\[
K_{\gamma} = \{ y \in M : \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f_i y) \leq \int \varphi d\mu + \gamma < \theta, \forall n \geq J \}.
\]
Then \( K_{\gamma} \subset K_{2,\gamma} \subset \cdots \), and the Birkhoff Ergodic Theorem gives \( \mu(\bigcup_{J \geq 1} K_{J,\gamma}) = 1 \). Thus \( \lim_{j \to \infty} \mu(K_{J,\gamma}) = 1 \). Take \( \delta > 0 \) small enough such that for any \( y,z \in M \) with \( d(y,z) < 3\delta \), one has \( |\varphi(y) - \varphi(z)| < \frac{\theta}{3} \). Then \( W^s_{3\delta}(K_{J,\gamma/2}) \subset K_{J,\gamma} \). Consider 
\[
U_x = [W^s_{\delta}(x),W^s_{2\delta}(x)],
\]
that is, the image of \( W^s_{\delta}(x) \times W^s_{2\delta}(x) \) under the map \( (\xi,\eta) \mapsto [\xi,\eta] \). Then we have 
\[
[y,W^s_{\delta}(z)] \subset K_{J,\gamma}, \quad \forall y \in K_{J,\gamma/2} \cap U_z.
\]
Since $\mu$ has full support, we may fix $J \geq 1$ large enough such that for any $z \in M$,
\begin{equation}
(4.3) \quad \mu(K_{J,\gamma/2} \cap U_z) > 0.
\end{equation}

Now for any $x \in D(N, \theta)$, and any $y \in B_m(x, 2\delta) \cap f^{-m}(K_{J,\gamma})$ for some $m \geq 1$, where $B_m(x, 2\delta) = \{ y : d(f^i x, f^i y) < 2\delta, \forall 0 \leq i \leq m - 1 \}$, we estimate $\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i y)$. If $n \leq m$, then
\[
\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i y) = \frac{1}{n} \sum_{i=0}^{m-1} \varphi(f^i y) + \frac{1}{n} \sum_{i=m}^{n-1} \varphi(f^i y) 
\leq \frac{\gamma}{2} + \theta.
\]

If $n > m$, then
\[
\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i y) = \frac{1}{n} \sum_{i=0}^{m-1} \varphi(f^i y) + \frac{1}{n} \sum_{i=m}^{n-1} \varphi(f^i y) 
\leq \frac{m \gamma}{2} + \frac{m \theta}{n} + \frac{1}{n} \sum_{j=0}^{n-m-1} \varphi(f^j(f^m y)).
\]

Since $f^m(y) \in K_{J,\gamma}$, we have
\[
\frac{1}{n} \sum_{j=0}^{n-m-1} \varphi(f^j(f^m y)) \leq \left( \frac{n-m}{n} (\theta + \gamma) \right), \quad \text{if } n-m \geq J,
\]
and
\[
\frac{1}{n} \sum_{j=0}^{n-m-1} \varphi(f^j(f^m y)) \leq \left( \frac{n-m}{n} \right) \varphi(f^m y), \quad \text{if } n-m < J.
\]

Take $R$ large enough such that $\frac{1}{R} \|\varphi\| \leq \gamma$. Then for any $n \geq R$, one has
\[
\frac{1}{n} \sum_{j=0}^{n-m-1} \varphi(f^j(f^m y)) \leq \frac{n-m}{n} (\theta + \gamma) + \gamma \leq \frac{n-m}{n} \theta + 2\gamma.
\]

Thus we conclude that for any $x \in D(N, \theta)$, $y \in B_m(x, 2\delta) \cap f^{-m}(K_{J,\gamma})$, and any $n \geq R$, we have
\[
\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i y) \leq \theta + 3\gamma.
\]

Let $N_1 = NR$, then for any $x \in D(N, \theta)$, $y \in B_m(x, 2\delta) \cap f^{-m}(K_{J,\gamma})$, we have
\[
\prod_{j=0}^{k-1} \|A_{i,jN_1}^N\| \cdot \|(A_{i,jN_1}^N)^{-1}\| \leq e^{N \sum_{j=0}^{k-1} \varphi(f^j y)} \leq e^{kN_1 (\theta + 3\gamma)}, \quad \forall k \geq 1.
\]

We claim that for any $x \in D(N, \theta)$, $B_m(x, 2\delta) \cap f^{-m}(K_{J,\gamma})$ contains a $\mu$-positive subset which is $W^s$-saturated on $U_x$. Indeed, let
\[
K^u := f^{-m}([K_{J,\gamma/2} \cap U_{f^m x}, f^m x]) \subset W_\delta^u(x).
\]

Then $K_{J,\gamma/2} \cap U_{f^m x} \subset [f^m(K^u), W_\delta(f^m x)]$. Since $[f^m(K^u), W_\delta(f^m x)] \subset [K_{J,\gamma/2} \cap U_{f^m x}, W_\delta(f^m x)]$, by (4.2), we obtain
\[
K_{J,\gamma/2} \cap U_{f^m x} \subset [f^m(K^u), W_\delta(f^m x)] \subset K_{J,\gamma}.
\]

By (4.3) and $\mu$ is $f$-invariant,
\[
\mu(f^{-m}([f^m(K^u), W_\delta(f^m x)])) = \mu([f^m(K^u), W_\delta(f^m x)]) \geq \mu(K_{J,\gamma/2} \cap U_{f^m x}) > 0.
\]
Since \( f^{-m}([f^m(K^u), W^s_d(f^m)]) \) is \( W^s \)-saturated on \( f^{-m}(U_{f^m}) \) and \( \mu \) has local product structure, it implies that
\[
\mu^u(K^u) > 0, \quad \text{and} \quad [K^u, W^s_d(x)] \subset f^{-m}(K_{J, \gamma}).
\]

Since \( K^u \subset f^{-m}(W^s_d(f^m)) \subset B_m(x, \delta) \), one has \([K^u, W^s_d(x)] \subset B_m(x, 2\delta)\). Thus we conclude that
\[
[K^u, W^s_d(x)] \subset B_m(x, 2\delta) \cap f^{-m}(K_{J, \gamma}).
\]

Replace \( f, A \) by \( f^{-1}, A^{-1} \) in the above proof, we can obtain a subset \( K_{J', \gamma} \) and \( N_2 \geq N \) such that for any \( x \in D(N, \theta), y \in B_m(x, 2\delta) \cap f^m(K_{J', \gamma}), \) we have
\[
\prod_{j=0}^{k-1} \|A_{J'^{-1}J_2}^N\| \cdot \|\left(A_{J'^{-1}J_2}^{-N_2}\right)^{-1}\| \leq e^{KN_2(\theta+3\gamma)}, \quad \forall k \geq 1,
\]
where \( B_m^-(x, 2\delta) = \{ y : d(f^i x, f^i y) < 2\delta, \forall -m+1 \leq i \leq 0 \} \). We can also obtain a subset \( K^s \subset W^s_d(x) \) such that
\[
\mu^s(K^s) > 0, \quad \text{and} \quad [W^s_d(x), K^s] \subset B_m^-(x, 2\delta) \cap f^m(K_{J', \gamma}).
\]

Let \( N_* = N_1 N_2, \theta_* = \theta + 3\gamma \), and
\[
E_m(x) = B_m(x, 2\delta) \cap f^{-m}(K_{J, \gamma}) \cap B_m^-(x, 2\delta) \cap f^m(K_{J', \gamma}).
\]

Then we conclude that for any \( x \in D(N, \theta), y \in E_m(x) \), one has \( y \in D(N_*, \theta_*) \). Since \( E_m(x) \supset [K^u, K^s], \mu^u(K^u) > 0, \mu^s(K^s) > 0 \) and \( \mu \) has local product structure, it follows that \( \mu(E_m(x)) > 0 \). Furthermore, by \( \text{[10, p.410]} \) (see also \( \text{[9, Lemma 4.2]} \)), there exists \( 0 < \beta < 1 \) such that \( B_m(x, 2\delta) \cap B_m^-(x, 2\delta) \subset B(x, \beta^{-m}) \). Thus it follows from \( E_m(x) \subset D(N_*, \theta_*) \cap B(x, \beta^{-m}) \) and \( \mu(E_m(x)) > 0 \) for every \( m \geq 1 \) that \( x \in \text{supp}(\mu|D(N_*, \theta_*)) \).

Now we can extend \( C \) to \( D(N, \theta) \).

**Proposition 4.4.** For any \( \theta < \tau \alpha \) and \( N \geq 1 \), there exists an \( \alpha \)-H"older continuous map \( \tilde{C} \) defined on \( D(N, \theta) \) such that for any \( n \geq 1 \) and any \( x, f^n(x) \in D(N, \theta) \), we have
\[
A^n_x = \tilde{C}(f^n x) \tilde{C}(x)^{-1}.
\]

**Proof.** For any \( \theta < \tau \alpha \) and \( N \geq 1 \), Lemma \( \text{[3, Lemma 4.3]} \) gives \( N_*, \theta_* \) such that
\[
D(N, \theta) \subset \text{supp}(\mu|D(N_*, \theta_*)).
\]

By Lemma \( \text{[1, 2]} \) there is an \( \alpha \)-H"older continuous map \( \tilde{C} \) defined on \( \text{supp}(\mu|D(N_*, \theta_*)) \) which coincides \( \mu \)-a.e. on \( \text{supp}(\mu|D(N_*, \theta_*)) \) with \( C \).

Fix a \( \varepsilon < \tau \rho - \theta_* \). Now given any \( x, f^n x \in D(N, \theta) \), take \( N' = N'(n) \) large enough such that \( R^{2n} < e^{N' \varepsilon} \), where \( R = \text{max}_{y \in M}\{\|A(y)\|, \|A(y)^{-1}\|\} \). We may also assume that \( N' \) can be divided by \( N_* \) so that
\[
D(N_*, \theta_*) \subset D(N', \theta_*).
\]

For any \( y \in M \), \( \|A^N_{f^n y}\| = \|A^N_{f^n y} A^N_y (A^N_y)^{-1}\| = \|A^N_{f^n y} A^N_y (A^N_y)^{-1}\| \leq R^{2n} \|A^N_y\| \).
Similarly \( \|A^N_{f^n y}\| \leq R^{2n} \|A^N_y\|^{-1} \). It follows that for any \( y \in D(N', \theta_*) \) and
\( k \geq 1, \)
\[
\prod_{i=0}^{k-1} \|A_{f_i^N (f^n y)}^n \| \| (A_{f_i^N (f^n y)}^n)^{-1} \| \leq R^{4n} \| A_{f_i^N (f^n y)}^n \| \| (A_{f_i^N (f^n y)}^n)^{-1} \|
\]
\[
\leq R^{4nk} e^{kN\theta_*} \leq e^{kN(\theta_* + \varepsilon)}.
\]

Replace \( f, A \) by \( f^{-1}, A^{-1} \), the dual inequality can be proved analogously. Thus we conclude that \( f^n(D(N', \theta_*)) \subset D(N', \theta_* + \varepsilon) \). Hence

(4.4) \( f^n(\text{supp}(\mu|D(N', \theta_*))) = \text{supp}(\mu|f^n(D(N', \theta_*))) \subset \text{supp}(\mu|D(N', \theta_* + \varepsilon)) \).

By Lemma 4.2, there exists a continuous map \( C' \) defined on \( \text{supp}(\mu|D(N', \theta_*)) \) which coincides \( \mu \text{-a.e. on } \text{supp}(\mu|D(N', \theta_* + \varepsilon)) \) with \( C \). Since
\[
\text{supp}(\mu|D(N, \theta_*)) \subset \text{supp}(\mu|D(N', \theta_* + \varepsilon)),
\]
and \( \tilde{C}, C' \) coincide \( \mu \text{-a.e. on } \text{supp}(\mu|D(N, \theta_*)) \) with \( C \), one has

(4.5) \( \tilde{C}(z) = C'(z), \quad \forall z \in D(N, \theta) \subset \text{supp}(\mu|D(N, \theta_*)) \).

By (4.2) and (4.4), for \( \mu \text{-a.e. } y \subset \text{supp}(\mu|D(N', \theta_*)) \), we have

(4.6) \( A_y^n = C(f^n y)C(y)^{-1} = C'(f^n y)C'(y)^{-1} \).

Take a sequence \( \{y_k \}_{k \geq 1} \subset \text{supp}(\mu|D(N', \theta_*)) \) such that \( y_k \rightarrow x \) as \( k \rightarrow \infty \). Then
\[
\text{by (4.6) and (4.5), we obtain } \quad A_x^n = C'(f^n x)C'(x)^{-1} = \tilde{C}(f^n x)\tilde{C}(x)^{-1}.
\]

This completes the proof. \( \square \)

4.2. Periodic obstructions of \( A \). The key proposition we will prove in this subsection is that the measurable coboundary implies the periodic obstructions of \( A \).

**Proposition 4.5.** Suppose that \( A \) is a measurable coboundary, that is, \( A \) satisfies the equation (4.2). Then
\[
A_p^n = Id, \quad \forall p = f^n(p), \forall n \geq 1.
\]

Before the proof of Proposition 4.5, we first estimate the norm of \( A \) along an orbit segment close to a periodic one. Let \( p = f^J(p) \) be a periodic point for some \( J \geq 1 \), denote \( p_j = f^J(p) \), and let \( \mu_p = \frac{1}{J} \sum_{j=0}^{J-1} \delta_{f^j(p)} \) be the corresponding periodic measure. Denote by \( \lambda_+(p) := \lambda_+(A, \mu_p) \) and \( \lambda_-(p) := \lambda_-(A, \mu_p) \) the upper and lower Lyapunov exponent of \( A \) with respect to \( \mu_p \) respectively. For any \( \varepsilon > 0 \), define the Lyapunov norm \( \| \cdot \|_{p_j} \) by
\[
\|u\|_{p_j} = \sum_{i=0}^{\infty} \|A^i_{p_j} u\| e^{-(\lambda_+(p)+\varepsilon)i} + \sum_{i=1}^{\infty} \|A^{-i}_{p_j} u\| e^{(\lambda_-(p)-\varepsilon)i}.
\]
Since \( p \) is a periodic point, we have that \( \| \cdot \|_{p_j} \) is uniformly equivalent to \( \| \cdot \| \) for \( p_j \in O(p) \). Then similar to the proof of Lemma 3.1 (or by Lemma 4.1 of [20]), we have

**Lemma 4.6.** Let \( p \) be a periodic point of \( f \). Then for any \( 0 < \varepsilon < \frac{1}{7} \alpha \), there exist \( \delta = \delta(p, \varepsilon) > 0 \) and \( c = c(p, \varepsilon) > 0 \) such that for any \( x \in M, n \geq 1 \) and \( 0 < \delta < \delta \) satisfying \( d(f^j x, f^j p) \leq \delta e^{-r \min(j, n-j)}, j = 0 \cdots, n \), we have
\[
c^{-1} \cdot e^{(\lambda_-(p)-2\varepsilon)} \leq m(A^j_x) \leq c \cdot e^{(\lambda_+(p)+2\varepsilon)},
\]
\[
d(f^{J}_x, f^{J}_p) = \delta e^{-r \min(J, n-J)} \leq \delta e^{-r \min(J, n-J)}.
\]


\[e^{-1} \cdot e^{(n-j)(\lambda-(p)-2\varepsilon)} \leq m(A_{x_j}^{n-j}) \leq \|A_{x_j}^{n-j}\| \leq c \cdot e^{(n-j)(\lambda+(p)+2\varepsilon)},\]

where \(m(B) := \|B^{-1}\|^{-1}\).

Moreover, if \(\lambda_+(p) = \lambda_-(p) = 0\), we can estimate the distortions.

**Lemma 4.7.** Suppose that \(\lambda_+(p) = \lambda_-(p) = 0\). Then for any \(0 < \varepsilon < \frac{1}{4} \tau \alpha\), there exist \(\delta = \delta(p, \varepsilon) > 0\), such that for any \(0 < \delta < \delta\), \(x \in M\) and any \(n \geq 1\) satisfying \(d(f^j x, f^j p) \leq \delta e^{-\tau \min(j,n-j)}\) for \(j = 0, \cdots, n\),

\[
\frac{1}{2} \leq \frac{\|A_p^n\|}{\|A_p^n\|} \leq 2 \quad \text{and} \quad \frac{1}{2} \leq \frac{\|A_p^{n-1}\|}{\|A_p^{n-1}\|} \leq 2.
\]

**Proof.** Denote \(x_j = f^j(x)\) and \(p_j = f^j(p)\). Then

\[
A_x^n - A_p^n = A_{x_1}^{n-1} \circ (A(x_0) - A(p_0)) + (A_{x_1}^{n-1} - A_{p_1}^{n-1}) \circ A(p_0)
\]

\[
= A_{x_1}^{n-1} \circ (A(x_0) - A(p_0)) + A_{x_2}^{n-1} \circ (A(x_1) - A(p_1)) \circ A(p_0) +
\]

\[
(A_{x_2}^{n-2} - A_{p_2}^{n-2}) \circ A_{p}^2
\]

\[
= \cdots = \sum_{j=0}^{n-1} A_{x_{j+1}}^{n-j-1} \circ (A(x_j) - A(p_j)) \circ A_{p}^j.
\]

Since \(\lambda_+(p) = \lambda_-(p) = 0\), by Lemma 4.6, for any \(0 \leq j \leq n\),

\[
\|A_p^n\| \leq c \cdot e^{2\varepsilon j} \quad \text{and} \quad \|A_p^n\| \leq \|(A_{p_1}^{n-1})^{-1}\| \cdot \|A_p^n\| \leq c \cdot e^{2\varepsilon (n-j)} \|A_p^n\|.
\]

Note that \(\|A(x_j) - A(p_j)\| \leq c_0 \delta^n e^{-\tau \alpha \min(j,n-j)}\). Denote \(m = \left\lfloor \frac{n}{2} \right\rfloor\), then Lemma 4.6 and the fact \(\varepsilon < \frac{1}{4} \tau \alpha\) give

\[
\sum_{j=0}^{m} \|A_{x_{j+1}}^{n-j-1}\| \cdot \|A(x_j) - A(p_j)\| \cdot \|A_{p}^j\|
\]

\[
\leq \sum_{j=0}^{m} \|A_{x_2}^n\| \cdot \|(A_{x_1}^{n-1})^{-1}\| \cdot \|A(x_j) - A(p_j)\| \cdot \|A_{p}^j\|
\]

\[
\leq \|A_{x_1}^n\| \cdot \sum_{j=0}^{m} c e^{2\varepsilon (j+1)} \cdot c_0 \delta^n e^{-\tau \alpha j} \cdot c e^{2\varepsilon j}
\]

\[
\leq \tilde{c} \delta^n \cdot \|A_{x_1}^n\|,
\]

and

\[
\sum_{j=0}^{n-1} \|A_{x_{j+1}}^{n-j-1}\| \cdot \|A(x_j) - A(p_j)\| \cdot \|A_{p}^j\|
\]

\[
\leq \sum_{j=0}^{n-1} c e^{2\varepsilon (n-j-1)} \cdot c_0 \delta^n e^{-\lambda \alpha(n-j)} \cdot c e^{2\varepsilon (n-j)} \|A_p^n\|
\]

\[
\leq \tilde{c} \delta^n \|A_p^n\|,
\]

where \(\tilde{c} = c_0 c^2 e^{2\varepsilon} / (1 - e^{4\varepsilon - \tau \alpha})\). Therefore,

\[
\|A_p^n - A_p^n\| \leq \sum_{j=0}^{n-1} \|A_{x_{j+1}}^{n-j-1}\| \cdot \|A(x_j) - A(p_j)\| \cdot \|A_{p}^j\|
\]

\[
\leq \tilde{c} \delta^n (\|A_p^n\| + \|A_p^n\|).
\]
Take $\tilde{\delta}$ small enough such that $\tilde{\delta}^a < \frac{1}{3}$. Then for any $0 < \delta < \tilde{\delta}$, we obtain

$$\frac{1}{2} \leq \frac{\|A^n_i\|}{\|A^n_i\|} \leq 2.$$

Using

$$(A^n_{x_i})^{-1} - (A^n_{p_i})^{-1} = (A(x_0)^{-1} - A(p_0)^{-1}) \circ (A^n_{p_i}^{-1})^{-1} + A(p_0)^{-1} \circ ((A^n_{x_i}^{-1})^{-1} - (A^n_{p_i}^{-1})^{-1})$$

$$= \cdots = \sum_{j=0}^{n-1} (A^n_{x_j}^{-1})^{-1} \circ (A(x_j)^{-1} - A(p_j)^{-1}) \circ (A^n_{x_{j+1}}^{-1})^{-1},$$

the second conclusion can be proved analogously.

Lemma 4.7 holds under the condition $\lambda_+(p) = \lambda_-(p) = 0$. In general, if $\lambda_+(p) \neq 0$, the conclusion of Lemma 4.7 may not hold for all $n \in \mathbb{N}$. However, we shall show that if the distance of $f^i(x)$ and $f^i(p)$ are much closer, then there exist infinitely many $n$ such that the same conclusion holds. We will use the following result by S. Gouëzel and A. Karlsson [14].

**Proposition 4.8** ([14], Theorem 1.1 and Remark 1.2). Let $\alpha_n(x)$ be an integrable subadditive cocycle with exponent $\lambda$ relative to an ergodic system $(M, f, \nu)$. Then for any $\rho > 0$, there exists a sequence $\varepsilon_i \to 0$, a subset $E \subset M$ with $\nu(E) > 1 - \rho$, and a subset $S \subset \mathbb{N}$ with $\text{Dens}(S) > 1 - \rho$ such that for any $x \in E$ and any $n \in S$,

$$\alpha_n(x) - \alpha_{n-i}(f^i x) \geq (\lambda - \varepsilon_i)i, \forall 0 \leq i \leq n,$$

where $\text{Dens}(S) := \limsup_{N \to \infty} |S \cap [0, N-1]|/N$.

Consider the subadditive cocycles $\alpha_n(x) = \log \|A^n_x\|$ and $\tilde{\alpha}_n(x) = \log \|\langle A^n_x \rangle^{-1}\|$. Then the following corollary can be deduced directly from Proposition 4.8.

**Corollary 4.9.** Let $p$ be a periodic point of $f$. Then for any $\rho > 0$, there exists a sequence $\varepsilon_i \to 0$ and a subset $S_p \subset \mathbb{N}$ with $\text{Dens}(S_p) > 1 - \rho$ such that for any $n \in S_p$,

$$\|A^{n-i}_{p_i}\| \leq \|A^n_p\| e^{-(\lambda_+(p) + \varepsilon_i)i}, \forall 0 \leq i \leq n,$$

$$\|\langle A^{n-i}_{p_i} \rangle^{-1}\| \leq \|\langle A^n_p \rangle^{-1}\| e^{(\lambda_-(p) + \varepsilon_i)i}, \forall 0 \leq i \leq n.$$

Now we estimate the distortion along certain orbit segment.

**Lemma 4.10.** Let $p$ be a periodic point of $f$. Then for any $0 < \varepsilon < \frac{1}{6} \tau \alpha, \rho > 0$, there exist $\delta = \delta(p, \varepsilon, \rho) > 0$ and a subset $S_p \subset \mathbb{N}$ with $\text{Dens}(S_p) > 1 - \rho$ such that for any $n \in S_p$, $0 < \delta < \delta$ and $x \in M$ satisfying $d(f^j x, f^j p) \leq \delta e^{-\frac{\tau j}{6}}$ for $j = 0, \cdots, n$, we have

$$\frac{1}{2} \leq \frac{\|A^n_x\|}{\|A^n_x\|} \leq 2 \quad \text{and} \quad \frac{1}{2} \leq \frac{\|\langle A^n_x \rangle^{-1}\|}{\|\langle A^n_x \rangle^{-1}\|} \leq 2.$$

**Proof.** We only prove the first conclusion, the second one can be proved in a similar fashion.

For any $\rho > 0$, let $\varepsilon_i \to 0$ be given by Corollary 4.9. Then given any $0 < \varepsilon < \frac{1}{6} \tau \alpha$, we may choose $L \geq 1$ large enough such that for any $i \geq L$, one has $\varepsilon_i < \varepsilon$. Let
the subset $S_p \subset \mathbb{N}$ be given by Corollary 4.9 Without loss of generality, we may assume $S_p \subset [L, \infty)$. Then for any $n \in S_p$,

$$\|A^n_p \| \leq \|A^n_\alpha\| e^{-(\lambda_\alpha(p) + \varepsilon)} \quad \forall L \leq i \leq n.$$  

By (4.7)

$$A^n_p - A^n_\alpha = \sum_{j=0}^{n-1} A^{n-j}_p \circ (A(p_j) - A(x_j)) \circ A^j_\alpha.$$  

Note that $\|A(x_j) - A(p_j)\| \leq c_0 \delta^\alpha e^{-\frac{3}{2} \tau_\alpha j}$. By Lemma 4.6

$$\sum_{j=0}^{L-1} \|A^{n-j}_p \| \cdot \|A(p_j) - A(x_j)\| \cdot \|A^j_\alpha\|$$

$$\leq \sum_{j=0}^{L-1} \|A^n_p\| \cdot \|A(p_j) - A(x_j)\| \cdot \|A^j_\alpha\|$$

$$\leq \sum_{j=0}^{L-1} \|A^n_p\| \cdot (c \cdot e^{(j+1)(-\lambda_\alpha(p) + 2\varepsilon)}) \cdot \left(c_0 \delta^\alpha e^{-\frac{3}{2} \tau_\alpha j}\right) \cdot \left(c \cdot e^{(j+1)(\lambda_\alpha(p) + 2\varepsilon)}\right)$$

$$\leq \tilde{c}_1 \delta^\alpha \cdot \|A^n_p\|,$$

where $\tilde{c}_1 = c_0^2 L e^{(\lambda_\alpha(p) - \lambda_\alpha(p) + 4\varepsilon - \frac{3}{2} \tau_\alpha)L}$. By (4.8), Lemma 4.6 and the fact $\varepsilon < \frac{3}{2} \tau_\alpha$,

$$\sum_{j=L}^{n-1} \|A^{n-j}_p \| \cdot \|A(p_j) - A(x_j)\| \cdot \|A^j_\alpha\|$$

$$\leq \sum_{j=L}^{n-1} \|A^n_p\| \cdot e^{-(\lambda_\alpha(p) + \varepsilon)(j+1)} \cdot \left(c_0 \delta^\alpha e^{-\frac{3}{2} \tau_\alpha j}\right) \cdot \left(c \cdot e^{(j+1)(\lambda_\alpha(p) + 2\varepsilon)}\right)$$

$$\leq \|A^n_p\| \cdot e^{-(\lambda_\alpha(p) + \varepsilon)} \cdot \sum_{j=L}^{n-1} e^{(3\varepsilon - \frac{3}{2} \tau_\alpha)j}$$

$$\leq \tilde{c}_2 \delta^\alpha \cdot \|A^n_p\|,$$

where $\tilde{c}_2 = \frac{e^{-(\lambda_\alpha(p) + \varepsilon)}}{1 - e^{(3\varepsilon - \frac{3}{2} \tau_\alpha)j}} \cdot c_0$. Therefore,

$$\|A^n_p - A^n_\alpha\| \leq \sum_{j=0}^{n-1} \|A^{n-j}_p \| \cdot \|A(p_j) - A(x_j)\| \cdot \|A^j_\alpha\| \leq (\tilde{c}_1 + \tilde{c}_2) \delta^\alpha \|A^n_p\|.$$  

Take $\tilde{\delta}$ small enough such that $(\tilde{c}_1 + \tilde{c}_2) \delta^\alpha < \frac{1}{2}$. Then for any $0 < \delta < \tilde{\delta}$, we conclude

$$\frac{1}{2} \|A^n_\alpha\| \leq \frac{2}{3} \|A^n_\alpha\| \leq \|A^n_\alpha\| \leq 2 \|A^n_\alpha\|.$$  

Now we prove Proposition 4.5.

\[\Box\]
Proof of Proposition 4.5. We begin by finding a periodic point $p_1 = f^k(p_1)$ such that $A_{p_1}^k = I$. By Kalinin and Sadovskaya’s result [20] Theorem 1.4, for any $0 < \theta < \tau \alpha$, there exists a periodic point $p_1 = f^k(p_1)$ such that

$$\left| \lambda_+(A_{p_1}, \mu) - \frac{1}{k} \log \| A_{p_1}^k \| \right| \leq \theta, \text{ and } \left| \lambda_-(A_{p_1}, \mu) - \frac{1}{k} \log \| (A_{p_1}^k)^{-1} \|^{-1} \right| \leq \theta.$$ 

Since $\lambda_+(A_{p_1}, \mu) = \lambda_-(A_{p_1}, \mu) = 0$, we have

$$\| A_{p_1}^k \| \leq e^{k\theta}, \text{ and } \| (A_{p_1}^k)^{-1} \| \leq e^{k\theta}.$$ 

It follows that $p_1 \in D(k, \theta)$. Then by Proposition 4.4,

$$A_{p_1}^k = \hat{C}(f^k p_1) \hat{C}(p_1)^{-1} = \hat{C}(p_1) \hat{C}(p_1)^{-1} = I.$$ 

Now for any periodic point $p_2 = f^m(p_2)$, in order to prove $A_{p_2}^m = I$, it’s enough to show $\lambda_+(p_2) = \lambda_-(p_2) = 0$. Indeed, if $\lambda_+(p_2) = \lambda_-(p_2) = 0$, then for any $0 < \theta < \tau \alpha$, there exists $n \in \mathbb{N}$ large enough such that $\| A_{p_2}^{nm} \| \leq e^{nm\theta}$ and $\| (A_{p_2}^{nm})^{-1} \| \leq e^{nm\theta}$, which implies $p_2 \in D(nm, \theta)$. Since $p_2 = f^m(p_2)$, by Proposition 4.4

$$A_{p_2}^m = \hat{C}(f^m p_2) \hat{C}(p_2)^{-1} = I.$$ 

Assume $\lambda_+(p_2) > 0$. To get a contradiction, we will use the fact that the topological mixing Anosov diffeomorphism $f$ satisfies the specification property [8, 21]: For any $\delta > 0$, there exists $N = N(\delta) \geq 1$ such that for any points $x_1, x_2, \ldots, x_n$ and any intervals of integers $I_1, I_2, \ldots, I_n \subset [a, b]$ with $d(I_i, I_j) \geq N$ for $i \neq j$, there exists a periodic point $x = f^{b-n+2N}(x)$ such that $d(f^j(x), f^j(x_1)) < \delta$ for $j \in I_i$. Moreover, by the following lemma, the distance of $f^j(x)$ and $f^j(x_1)$ can be exponentially close.

Lemma 4.11 (Proposition 6.4.16 of [21]). There exists $\delta' > 0$ and $c' \geq 1$ such that for any $0 < \delta < \delta'$ and any $x, y \in M$ with $d(f^i x, f^i y) < \delta$ for $i = 0, \ldots, n$, then in fact

$$d(f^i x, f^i y) < c'\delta e^{-\tau \min(i, n-i)}.$$ 

Now given any $0 < \varepsilon < \frac{1}{6} \min\{\tau \alpha, \lambda_+(p_2)\}$, take $0 < \rho < \frac{1}{3m}$, and let $\delta > 0$ be given such that

$$c'\delta < \min\{\delta(p_2, \varepsilon), \delta(p_1, \varepsilon), \delta(p_2, \varepsilon), \delta(p_2, \varepsilon, \rho), \delta'\},$$

where $\delta(p_2, \varepsilon)$, $\delta(p_1, \varepsilon)$, $\delta(p_2, \varepsilon, \rho)$, $\delta'$ are given by Lemma 4.6, Lemma 4.7, Lemma 4.10 and Lemma 4.11 respectively. Let $S_{p_2} \subset \mathbb{N}$ be given by Lemma 4.10. Since $\text{Dens}(S_{p_2}) > 1 - \rho > 1 - \frac{1}{3m}$, there are infinitely many $b \in \mathbb{N}$ such that $3nb \in S_{p_2}$. Let $c_1 = \max\{\| A(x) \|, \| A(x)^{-1} \|\}$, $c_2 = c(p_2, \varepsilon)$ be given by Lemma 4.6. Note that $\lambda_+(p_2) = \lim_{b \to \infty} \frac{1}{b} \log \| A_{p_2}^{bm} \|$. We may choose $b \in \mathbb{N}$ large enough such that $2bm \in S_{p_2}$ and

$$\| A_{p_2}^{2bm} \| \geq e^{2bm(\lambda_+(p_2) - \varepsilon)} > 4c_1 2^N c_2 \cdot e^{bm(\lambda_+(p_2) + 2\varepsilon)},$$

where $N = N(\delta)$ is given by the specification property. Then choose $a \in \mathbb{N}$ large enough such that

$$2c_1 2^N c_2 \cdot e^{3bm(\lambda_+(p_2) + 2\varepsilon)} \leq e^{2\varepsilon(ak + 2N + 3bm)},$$

and

$$2c_1 2^N c_2 \cdot e^{3bm(-\lambda_-(p_2) + 2\varepsilon)} \leq e^{2\varepsilon(ak + 2N + 3bm)}.$$
Then by the specification property, for $p_1$ and $f^{-ak-N}(p_2)$, there exists a periodic point $q = f^{ak+2N+3bm}(q)$ such that
\[ d(f^i(q), f^i(p_1)) < \delta, \forall 0 \leq i \leq ak, \] and
\[ d(f^j(q), f^j(p_2)) < \delta, \forall 0 \leq j \leq 3bm. \]

Denote $x = f^{ak+N}(q)$. Then by Lemma 4.11
\[ d(f^i(q), f^i(p_1)) < c'\delta e^{-\tau \min\{i,ak-i\}} < \delta(p_1, \varepsilon)e^{-\tau \min\{i,ak-i\}}, \forall 0 \leq i \leq ak, \] and
\[ d(f^j(x), f^j(p_2)) < c'\delta e^{-\tau \min\{j,3bm-j\}} < \delta(p_2, \varepsilon)e^{-\tau \min\{j,3bm-j\}}, \forall 0 \leq j \leq 3bm. \]

Since $A^{ak}_{\nu_i} = Id$, by Lemma 4.7
\[ \|A^{ak}_{\nu_i}\| \leq 2, \] and \(\|(A^{ak}_{\nu_i})^{-1}\| \leq 2\).

By Lemma 4.6
\[ \|A^{3bm}_x\| \leq c_2 \cdot e^{3bm(\lambda_+(p_2)+2\varepsilon)}, \] and \(\|(A^{3bm}_x)^{-1}\| \leq c_2 \cdot e^{3bm(-\lambda_-(p_2)+2\varepsilon)}\).

Therefore, using (4.10),
\[
\|A^{ak+2N+3bm}_q\| \leq \|A^{N}_{f^{3bm}x}\| \cdot \|A^{3bm}_x\| \cdot \|A^{N}_{f^{ak}q}\| \cdot \|A^{ak}_q\|
\leq 2c_1 2^N c_2 \cdot e^{3bm(\lambda_+(p_2)+2\varepsilon)}
\leq e^{2\varepsilon(ak+2N+3bm)}.
\]

Similarly, we can also get \(\|(A^{ak+2N+3bm}_q)^{-1}\| \leq e^{2\varepsilon(ak+2N+3bm)}\). It follows that
\[ q \in D(ak + 2N + 3bm, 2\varepsilon). \]

Then by Proposition 4.1, $A^{ak+2N+3bm}_q = \hat{C}(f^{ak+2N+3bm}(q))\hat{C}(q)^{-1} = Id$, which implies
\[ \|A^{3bm}_x\| \leq \|(A^{ak}_q)^{-1}\| \cdot \|(A^{N}_{f^{ak}q})^{-1}\| \cdot \|(A^{N}_{f^{3bm}x})^{-1}\| \leq 2c_1 2^N. \]

Hence by Lemma 4.6
\[ \|A^{2bm}_x\| \leq \|(A^{3bm}_x)^{-1}\| \cdot \|(A^{bm}_{f^{2bm}x})^{-1}\| \leq 2c_1 2^N c_2 \cdot e^{bm(\lambda_+(p_2)+2\varepsilon)}. \]

Now for any $0 \leq j \leq 2bm$, one has $3bm \geq \frac{j}{2}$. Therefore, for any $0 \leq j \leq 2bm$,
\[ d(f^j(x), f^j(p_2)) < c'd\varepsilon e^{-\tau \min\{j,3bm-j\}} \leq c'd\varepsilon e^{-\frac{\tau j}{2}} \leq \delta(p_2, \varepsilon, \rho)e^{-\frac{\tau j}{4}}. \]

By the choice of $b$ and Lemma 4.10 one has
\[ \|A^{2bm}_{p_2}\| \leq 2\|A^{2bm}_x\| \leq 4c_1 2^N c_2 \cdot e^{bm(\lambda_+(p_2)+2\varepsilon)}. \]

This contradicts (4.9). Hence $\lambda_+(p_2) \leq 0$. It can be proved in a similar fashion that $\lambda_-(p_2) \leq 0$. Then we conclude that $\lambda_+(p_2) = \lambda_-(p_2) = 0$. This completes the proof of Proposition 4.5.

Now we finish the proof of Theorem 1.2. By Proposition 4.5 and Theorem 1.4 of [20], for any ergodic $f$-invariant probability measure $\nu$, $\lambda_+(A, \nu) = \lambda_-(A, \nu) = 0$.

Since
\[
\lambda_+(A, \nu) - \lambda_-(A, \nu) = \lim_{n \to \infty} \frac{1}{n} \int \log \left( \|A^\nu_x\| \cdot \|(A^\nu_x)^{-1}\| \right) d\nu
= \lim_{n \to \infty} \frac{1}{n} \int \log \left( \|A^{-n}_x\| \cdot \|(A^{-n}_x)^{-1}\| \right) d\nu,
\]
it follows from [36] that
\[
\lim_{n \to \infty} \frac{1}{n} \max_{x \in M} \log \left( \|A_x^n\| \right) = \sup \{ \lambda_+(A, \nu) - \lambda_-(A, \nu) : v \in \mathcal{E}(f) \} = 0,
\]
\[
\lim_{n \to \infty} \frac{1}{n} \max_{x \in M} \log \left( \|A_x^{-n}\| \right) = \sup \{ \lambda_+(A, \nu) - \lambda_-(A, \nu) : v \in \mathcal{E}(f) \} = 0,
\]
where \(\mathcal{E}(f)\) denotes the space of ergodic \(f\)-invariant Borel probability measures. Then for any \(\varepsilon < \tau\), there exists \(N \geq 1\), such that
\[
\|A_x^N\| \leq e^{\varepsilon N} \quad \text{and} \quad \|A_x^{-N}\| \leq e^{\varepsilon N}, \quad \forall x \in M.
\]
We conclude that \(D(N, \varepsilon) = M\). Hence by Proposition 4.3 we obtain the desired \(\alpha\)-Hölder continuous map \(\tilde{C}\).

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