ATOMIC DECOMPOSITION FOR BERGMAN SPACES WITH EXPONENTIAL TYPE WEIGHTS

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ABSTRACT. We show that any function in a Bergman space with exponential type weights admits a representation in terms of an infinite series of kernel functions.

1. INTRODUCTION

The fact that any function in a weighted Bergman space with standard weights can be decomposed into a series of very nice functions (called atoms) was obtained by Coifman and Rochberg [3]. This atomic decomposition, whose proof can also be found in the monographs [18, 19], has become a powerful tool in the study of the properties of weighted Bergman spaces having found many applications. In this paper we are going to obtain an atomic decomposition for Bergman spaces with exponential type weights, using reproducing kernels as building blocks.

Let \( D \) denote the open unit disk in the complex plane \( \mathbb{C} \), \( dA(z) = \frac{dx dy}{\pi} \) be the normalized area measure on \( D \), and let \( H(D) \) denote the space of all analytic functions on \( D \). A weight is a positive function \( \omega \in L^1(D, dA) \).

For \( 0 < p < \infty \), the weighted Bergman space \( A^p(\omega) \) consists of those functions \( f \in H(D) \) such that
\[
\|f\|_{A^p(\omega)} = \left( \int_D |f(z)|^p \omega(z) \, dA(z) \right)^{\frac{1}{p}} < \infty.
\]

We are going to obtain the representation mentioned before for a certain class \( \mathcal{E} \) of weights that includes the exponential type weights
\[
\omega_\sigma(z) = \exp \left( \frac{-c}{(1-|z|^2)^\sigma} \right), \quad \sigma > 0, \quad c > 0.
\]

For the weights \( \omega \) considered in this paper, for each \( z \in D \) the point evaluations \( L_z \) are bounded linear functionals on \( A^p(\omega^p) \). In particular, the space \( A^2(\omega) \) is a reproducing kernel Hilbert space: for each \( z \in D \), there exist functions \( K_z \in A^2(\omega) \) such that
\[
L_z f = f(z) = \langle f, K_z \rangle_{\omega},
\]
where
\[
\langle f, g \rangle_{\omega} = \int_D f(z) \overline{g(z)} \omega(z) \, dA(z)
\]
is the natural inner product in \( L^2(D, \omega dA) \). The function \( K_z \) has the property that \( K_z(\xi) = \overline{K_z}(z) \), and is called the reproducing kernel of the Bergman space \( A^2(\omega) \). Several basic properties of the Bergman spaces with exponential type weights are not yet well understood and have attracted some attention in recent years [5, 7, 8, 10, 14]. Sometimes, the typical techniques used in the setting of standard Bergman spaces fail to work in this context, and therefore new tools must be developed. Another difficulty when studying these spaces comes from the fact that we didn’t have an explicit

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expression of the reproducing kernels. When studying properties where the reproducing kernels are involved, the most convenient setting are the spaces \( A^p(\omega^{p/2}) \) (or the weighted Lebesgue spaces\( L^p(\omega^{p/2}) := L^p(\mathbb{D}, \omega^{p/2} dA) \)), and is in these spaces where we obtain the corresponding atomic decomposition. We use the notation \( k_{p,z} \) for the normalized reproducing kernels in \( A^p(\omega^{p/2}) \), that is
\[
k_{p,z} = \frac{K_z}{\| K_z \|_{A^p(\omega^{p/2})}}.
\]

**Theorem 1.1.** Let \( \omega \in \mathcal{E} \) and \( 1 < p < \infty \). Assume that \( A^2(\omega) \) is dense in \( A^1(\omega^{1/2}) \). There exists a sequence \( \{z_n\} \subset \mathbb{D} \) such that:

(i) For any \( \lambda = \{\lambda_n\} \in \ell^p \), the function
\[
f(z) = \sum_{n} \lambda_n k_{p,z_n}(z)
\]
is in \( A^p(\omega^{p/2}) \) with \( \| f \|_{A^p(\omega^{p/2})} \leq C \| \lambda \|_{\ell^p} \).

(ii) For every \( f \in A^p(\omega^{p/2}) \) exists \( \lambda = \{\lambda_n\} \in \ell^p \) such that
\[
f(z) = \sum_{n} \lambda_n k_{p,z_n}(z)
\]
and \( \| \lambda \|_{\ell^p} \leq C \| f \|_{A^p(\omega^{p/2})} \).

Once the right machinery is developed, the proof of the case \( p > 1 \) can be obtained by “standard” methods using duality, but the corresponding result for the case \( 0 < p \leq 1 \) is much more involved. For the case \( 0 < p \leq 1 \), we use the reproducing kernels \( K^*_{z_n} \) of a companion weighted Bergman space \( A^2(\omega_s) \) (see Section 2.2 for the definition).

**Theorem 1.2.** Let \( \omega \in \mathcal{E} \) and \( 0 < p \leq 1 \). There exists a sequence \( \{z_n\} \subset \mathbb{D} \) such that:

(i) For any \( \lambda = \{\lambda_n\} \in \ell^p \), the function
\[
f(z) = \sum_{n} \lambda_n \omega(z_n)^{1/2} \tau(z_n)^{\frac{2(1-p)}{p}} K^*_{z_n}(z)
\]
is in \( A^p(\omega^{p/2}) \) with \( \| f \|_{A^p(\omega^{p/2})} \leq C \| \lambda \|_{\ell^p} \).

(ii) For every \( f \in A^p(\omega^{p/2}) \) exists \( \lambda = \{\lambda_n\} \in \ell^p \) such that
\[
f(z) = \sum_{n} \lambda_n \omega(z_n)^{1/2} \tau(z_n)^{\frac{2(1-p)}{p}} K^*_{z_n}(z)
\]
and \( \| \lambda \|_{\ell^p} \leq C \| f \|_{A^p(\omega^{p/2})} \).

Here \( K^*_{z_n} \) denotes the reproducing kernel of \( A^2(\omega_s) \) with \( \omega_s(z) = \omega(z) \tau(z)^{-\frac{2(1-p)}{p}} \).

Throughout this work, the letter \( C \) will denote an absolute constant whose value may change at different occurrences. We also use the notation \( a \lesssim b \) to indicate that there is a constant \( C > 0 \) with \( a \leq Cb \), and the notation \( a \asymp b \) means that \( a \lesssim b \) and \( b \lesssim a \).

2. Basic properties

In this section we provide the basic tools for the proofs of the main results of the paper.

A positive function \( \tau \) on \( \mathbb{D} \) is said to belong to the class \( \mathcal{L} \) if satisfies the following two properties:

(A) There is a constant \( c_1 > 0 \) such that \( \tau(z) \leq c_1 (1 - |z|) \) for all \( z \in \mathbb{D} \);
(B) There is a constant $c_2 > 0$ such that $|\tau(z) - \tau(\zeta)| \leq c_2 |z - \zeta|$ for all $z, \zeta \in \mathbb{D}$.

We also use the notation

$$m_\tau := \frac{\min(1, c_1^{-1}, c_2^{-1})}{4},$$

where $c_1$ and $c_2$ are the constants appearing in the previous definition. For $a \in \mathbb{D}$ and $\delta > 0$, we use $D(\delta \tau(a))$ to denote the euclidian disc centered at $a$ and radius $\delta \tau(a)$. It is easy to see from conditions (A) and (B) (see [13, Lemma 2.1]) that if $\tau \in \mathcal{L}$ and $z \in D(\delta \tau(a))$, then

$$(2.1) \quad \frac{1}{2} \tau(a) \leq \tau(z) \leq 2 \tau(a),$$

for $\delta \in (0, m_\tau)$. This fact will be used several times in this work.

**Definition 2.1.** We say that a weight $\omega$ belongs to the class $\mathcal{L}^*$ if it is of the form $\omega = e^{-\varphi}$, where $\varphi \in C^2(\mathbb{D})$ with $\Delta \varphi > 0$, and $(\Delta \varphi(z))^{-1/2} \leq \tau(z)$, with $\tau(z)$ being a function in the class $\mathcal{L}$. Here $\Delta$ denotes the classical Laplace operator.

The following lemma is from [13, Lemma 2.2] and gives the boundedness of the point evaluation functionals on $A^p(\omega^\beta)$.

**Lemma A.** Let $\omega = e^{-\varphi} \in \mathcal{L}^*$, and let $\rho \in \mathcal{L}$ such that $\rho(z)^2 \Delta \varphi(z) \leq C$ for some constant $C > 0$. Let $0 < p < \infty$, and $z \in \mathbb{D}$. If $\beta \in \mathbb{R}$ there exists $M \geq 1$ such that

$$|f(z)|^p \omega(z)^\beta \leq \frac{M}{\delta^2 \rho(z)^2} \int_{D(\delta \rho(z))} |f(\xi)|^p \omega(\xi)^\beta \, dA(\xi),$$

for all $f \in H(\mathbb{D})$ and all $\delta > 0$ sufficiently small.

It can be seen from the proof given in [13] that one only needs $f$ to be holomorphic in a neighbourhood of $D(\delta \rho(z))$. It is also clear that we can take $\rho(z) = \tau(z)$. Another consequence of the above result is that the Bergman space $A^p(\omega^\beta)$ is a Banach space when $1 \leq p < \infty$ and a complete metric space when $0 < p < 1$. We also need a result similar to Lemma A with the gradient.

**Lemma 2.1.** Let $\omega \in \mathcal{L}^*$ and $0 < p < \infty$. For any $\delta_0 > 0$ sufficiently small there exists a constant $C(\delta_0) > 0$ such that

$$\left| \nabla |f|^{1/2} \omega^{1/2}(z) \right| \leq \frac{C(\delta_0)}{\tau(z)^{1+\frac{1}{p}}} \left( \int_{D(\delta_0 \tau(z)/2)} |f(\xi)|^p \omega(\xi)^{p/2} dA(\xi) \right)^{\frac{1}{p}},$$

for all $f \in H(\mathbb{D})$.

**Proof.** We follow the method used in [12]. Without loss of generality we can assume $z = 0$. Then, applying the Riesz’s decomposition (see for example [15]) of the subharmonic function $\varphi$ in $D(0, \tau(0))$, we obtain

$$(2.2) \quad \varphi(\xi) = u(\xi) + \int_{D(\xi/2)} G(\xi, \eta) \Delta \varphi(\eta) dA(\eta),$$

where $r = \delta_0 \tau(0)$, $u$ is the least harmonic majorant of $\varphi$ in $D(0, \xi/2)$ and $G$ is the Green function defined for every $\xi, \eta \in D(0, r)$, $\xi \neq \eta$ by

$$G(\xi, \eta) := \log \left| \frac{r(\xi - \eta)}{r^2 - \eta \xi} \right|. $$
For $\xi, \eta \in D(0, \frac{r}{2})$ we have $|\frac{\partial G}{\partial \xi}(\xi, \eta)| \leq \frac{4}{|\xi - \eta|}$. Then

$$\tag{2.3} \left| \frac{\partial \varphi(0)}{\partial \xi} - \frac{\partial u(0)}{\partial \xi} \right| \leq \int_{D(\frac{r}{2})} \left| \frac{\partial G}{\partial \xi}(0, \eta) \right| \Delta \varphi(\eta) dA(\eta) \lesssim \frac{1}{\tau(0)^2} \int_{D(\frac{r}{2})} dA(\eta) = \frac{\delta_0}{\tau(0)}. $$

We pick a function $h \in H(\mathbb{D})$ such that $Re(h) = u$. Since $h'(0) = 2\frac{\partial}{\partial \xi}(0)$, we get

$$\left| \nabla(|f|e^{-\varphi})(0) \right| = \frac{1}{2} \left| f'(0) - 2f(0) \frac{\partial \varphi}{\partial \xi}(0) \right| e^{-\varphi(0)} \leq \frac{1}{2} \left| f'(0) - 2f(0) \frac{\partial u}{\partial \xi}(0) \right| e^{-\varphi(0)} + \left| \frac{\partial u}{\partial \xi}(0) - \frac{\partial \varphi}{\partial \xi}(0) \right| |f(0)|e^{-\varphi(0)} \lesssim \left| \frac{\partial (f e^{-h})(0)}{\partial \xi} \right| e^{-\varphi(0)} + \left| \frac{\partial u}{\partial \xi}(0) - \frac{\partial \varphi}{\partial \xi}(0) \right| |f(0)|e^{-\varphi(0)}. $$

By (2.3) we have

$$\left| \frac{\partial u}{\partial \xi}(0) - \frac{\partial \varphi}{\partial \xi}(0) \right| |f(0)|e^{-\varphi(0)} \lesssim \frac{\delta_0}{\tau(0)} |f(0)|e^{-\varphi(0)}. $$

This gives

$$\tag{2.4} \left| \nabla(|f|e^{-\varphi})(0) \right| \lesssim \left| \frac{\partial (f e^{-h})(0)}{\partial \xi} \right| e^{u(0)-\varphi(0)} + \left| \frac{f(0)}{\tau(0)} \right| e^{-\varphi(0)}. $$

By Lemma A we have

$$\tag{2.5} \frac{|f(0)|}{\tau(0)} e^{-\varphi(0)} \lesssim \frac{1}{\tau(0)^{1+\frac{2}{p}}} \left( \int_{D(\delta_0 \tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} dA(z) \right)^{\frac{1}{p}}. $$

To manage the other term appearing in (2.4), notice that if we use the identity (2.2) with the function $\phi(\xi) = |\xi|^2 - (r/2)^2$ (since $\Delta \phi(\xi) = 4$ and its least harmonic majorant is $u_\phi = 0$), we obtain

$$\int_{D(\frac{r}{2})} G(\xi, \eta) dA(\eta) = \frac{1}{4} (|\xi|^2 - (r/2)^2). $$

Therefore, since $\Delta \varphi(\eta) = \frac{1}{\tau(0)^2} \lesssim \frac{1}{\tau(0)^2} = \Delta \varphi(0)$ and the Green’s function $G \leq 0$, we obtain for every $\xi \in D(0, \frac{r}{2})$

$$u(\xi) - \varphi(\xi) = -\int_{D(\frac{r}{2})} G(\xi, \eta) \Delta \varphi(\eta) dA(\eta) \lesssim \frac{\Delta \varphi(0)}{4} \left( (r/2)^2 - |\xi|^2 \right) = \frac{1}{4\tau(0)^2} \left( (r/2)^2 - |\xi|^2 \right). $$

This gives

$$e^{u(0)-\varphi(0)} \lesssim e^{C\delta_0^2}. $$

Therefore

$$\tag{2.6} \left| \frac{\partial (f e^{-h})(0)}{\partial \xi} \right| e^{u(0)-\varphi(0)} \lesssim \left| \frac{\partial (f e^{-h})(0)}{\partial \xi} \right|. $$
On the other hand, using Cauchy’s estimates, the fact that $\varphi - u \leq 0$ and Lemma A, we get

$$\left| \frac{\partial (fe^{-h})}{\partial \xi}(0) \right| \lesssim \int_{|\eta| = \frac{\delta_\alpha \tau(0)}{4}} \frac{f(\eta)e^{-h(\eta)}}{\eta^2} \, d\eta \lesssim \frac{1}{\delta_\alpha \tau(0)^2} \int_{|\eta| = \frac{\delta_\alpha \tau(0)}{4}} \left| f(\eta) \right| e^{-\varphi(\eta)} e^{\varphi(\eta) - u(\eta)} \, |d\eta| \lesssim \frac{1}{\tau(0)^2} \int_{|\eta| = \frac{\delta_\alpha \tau(0)}{4}} \left( \frac{1}{\tau(\eta)^2} \int_{D(\delta_\alpha \tau(\eta)/4)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{\frac{1}{p}} \, |d\eta| \lesssim \frac{1}{\tau(0)^{1 + \frac{1}{p}}} \left( \int_{D(\delta \alpha \tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{\frac{1}{p}}.$$ 

Finally, using $\tau(\eta) \asymp \tau(0)$, we obtain

$$\left| \frac{\partial (fe^{-h})}{\partial \xi}(0) \right| \lesssim \frac{1}{\tau(0)^2} \int_{|\eta| = \frac{\delta_\alpha \tau(0)}{4}} \left( \frac{1}{\tau(\eta)^2} \int_{D(\delta_\alpha \tau(\eta)/2)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{\frac{1}{p}} \, |d\eta| \lesssim \frac{1}{\tau(0)^{1 + \frac{1}{p}}} \left( \int_{D(\delta \alpha \tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{\frac{1}{p}}.$$ 

Bearing in mind (2.6) this gives

$$\left| \frac{\partial (fe^{-h})}{\partial \xi}(0) \right| e^{u(\eta) - \varphi(\eta)} \lesssim \frac{1}{\tau(0)^{1 + \frac{1}{p}}} \left( \int_{D(\delta \alpha \tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} \, dA(z) \right)^{\frac{1}{p}}.$$ 

Putting this and (2.5) into (2.4) we get the result. \qed

The following lemma on coverings is due to Oleinik, see [11].

**Lemma B.** Let $\tau$ be a positive function in $\mathbb{D}$ in the class $\mathcal{L}$, and let $\delta \in (0, m_\tau)$. Then there exists a sequence of points $\{z_j\} \subset \mathbb{D}$, such that the following conditions are satisfied:

(i) $z_j \notin D(\delta \tau(z_k)), \quad j \neq k$.

(ii) $\bigcup_j D(\delta \tau(z_j)) = \mathbb{D}$.

(iii) $\tilde{D}(\delta \tau(z_j)) \subset D(3\delta \tau(z_j))$, where $\tilde{D}(\delta \tau(z_j)) = \bigcup_{x \in D(\delta \tau(z_j))} D(\delta \tau(z)), \quad j = 1, 2, \ldots$.

(iv) $\{D(3\delta \tau(z_j))\}$ is a covering of $\mathbb{D}$ of finite multiplicity $N$.

The multiplicity $N$ in the previous Lemma is independent of $\delta$, and it is easy to see that one can take, for example, $N = 256$. Any sequence satisfying the conditions in Lemma B will be called a $(\delta, \tau)$-lattice. It is also easy to see that, for a given $(\delta, \tau)$-lattice $\{z_j\}$, and $t > 1$, the covering $\{D(t\delta \tau(z_j))\}$ has multiplicity less than $Ct^2$ with $C$ not depending on $\delta$.

Finally we define the class of weights for which we are going to obtain the corresponding atomic decomposition.

**Definition 2.2.** A weight $\omega$ is in the class $\mathcal{E}$ if $\omega \in \mathcal{L}^*$ and its associated function $\tau$ satisfies the condition

(E) For each $m \geq 1$, there are constants $b_m > 0$ and $0 < t_m < 1/m$ such that

$$\tau(z) \leq \tau(\xi) + t_m |z - \xi|, \quad \text{for} \quad |z - \xi| > b_m \tau(\xi).$$
The prototype of a weight in our class $\mathcal{E}$ are the exponential type weights given by (1.1). Also, an example of a non radial weight in the class $\mathcal{E}$ is given by $\omega_{p,f}(z) = |f(z)|^p \omega(z)$, where $p > 0$, $\omega$ is a radial weight in our class $\mathcal{E}$, and $f$ is a non-vanishing analytic function in $A^p(\omega)$.

2.1. Reproducing kernels estimates. Because the norm of the point evaluation functional equals the norm of the reproducing kernel in $A^2(\omega)$, Lemma A also gives an upper bound for the norm $\|K_z\|_{A^2(\omega)}$. The following result [2, 9, 13] says that (at least for a certain class of weights) this bound yields the right growth of the reproducing kernel.

**Lemma C.** Let $\omega \in \mathcal{E}$. Then

$$\|K_z\|^2_{A^2(\omega)} \omega(z) \asymp \frac{1}{\tau(z)^2}, \quad z \in \mathbb{D}.$$

For weights in the class $\mathcal{E}$, and points close to the diagonal, one has the following well-known estimate (see [10] Lemma 3.6) for example

$$|K_z(\zeta)| \asymp \|K_z\|_{A^2(\omega)} \cdot \|K_\zeta\|_{A^2(\omega)}, \quad \zeta \in D(\delta \tau(z))$$

for all $\delta \in (0, m_\tau)$ sufficiently small. The following pointwise estimate for the reproducing kernel was obtained in [11].

**Theorem A.** Let $K_z$ be the reproducing kernel of $A^2(\omega)$ where $\omega$ is a weight in the class $\mathcal{E}$. For each $M \geq 1$, there exists a constant $C > 0$ (depending on $M$) such that for each $z, \xi \in \mathbb{D}$ one has

$$|K_z(\xi)| \leq C \frac{1}{\tau(z)} \frac{1}{\tau(\xi)} \omega(z)^{-1/2} \omega(\xi)^{-1/2} \left(\frac{\min(\tau(z), \tau(\xi))}{|z - \xi|}\right)^M.$$

As a consequence, we obtain the following integral type estimate involving reproducing kernels (the case $p = 1$ and $\alpha = 0$ was obtained in [11]).

**Lemma 2.2.** Let $\omega \in \mathcal{E}$, and $K_z$ be the reproducing kernel for $A^2(\omega)$. For $0 < p < \infty$ and $\alpha \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |K_z(\xi)|^p \omega(\xi)^{p/2} \tau(\xi)^{\alpha} dA(\xi) \leq C \omega(z)^{-p/2} \tau(z)^{\alpha - 2(p - 1)}.$$

**Proof.** For $0 < \delta_0 \leq m_\tau$ fixed, let

$$A(z) := \int_{|z - \xi| \leq \delta_0 \tau(z)} |K_z(\xi)|^p \omega(\xi)^{p/2} \tau(\xi)^{\alpha} dA(\xi)$$

and

$$B(z) := \int_{|z - \xi| > \delta_0 \tau(z)} |K_z(\xi)|^p \omega(\xi)^{p/2} \tau(\xi)^{\alpha} dA(\xi).$$

By Lemma C and (2.1),

$$A(z) \leq \int_{|z - \xi| \leq \delta_0 \tau(z)} \|K_z\|^p \|K_\xi\|^p \omega(\xi)^{p/2} \tau(\xi)^{\alpha} dA(\xi)$$

$$\asymp \tau(z)^{2 + \alpha - p} \|K_z\|^p \asymp \omega(z)^{-p/2} \tau(z)^{\alpha - 2(p - 1)}.$$

On the other hand, by Theorem A with $M$ taken so that $Mp > 2 + |\alpha - p|$, we have

$$B(z) \lesssim \frac{\omega(z)^{p/2}}{\tau(z)^p} \int_{|z - \xi| > \delta_0 \tau(z)} \frac{1}{\tau(\xi)^{p-\alpha}} \left(\frac{\min(\tau(z), \tau(\xi))}{|z - \xi|}\right)^M dA(\xi).$$
If \( p - \alpha \geq 0 \), then
\[
(2.9) \quad B(z) \lesssim \omega(z)^{-p/2} \tau(z)^{Mp-2p+\alpha} \int_{|z-\xi|>\delta_0 \tau(z)} \frac{dA(\xi)}{|z-\xi|^{Mp}}.
\]
To estimate the last integral, let
\[
R_k(z) = \left\{ \xi \in \mathbb{D} : 2^k \delta_0 \tau(z) < |z-\xi| \leq 2^{k+1} \delta_0 \tau(z) \right\}, \quad k = 0, 1, 2, \ldots
\]
We have
\[
\int_{|z-\xi|>\delta_0 \tau(z)} \frac{dA(\xi)}{|z-\xi|^{Mp}} \leq \sum_{k \geq 0} \int_{R_k(z)} \frac{dA(\xi)}{|z-\xi|^{Mp}} \lesssim \tau(z)^{-Mp} \sum_{k \geq 0} 2^{-Mp} \text{Area}(R_k(z)),
\]
\[
\lesssim \tau(z)^{2-Mp} \sum_{k \geq 0} 2^{-k(Mp-2)} \lesssim \tau(z)^{2-Mp}.
\]
Putting this into (2.9) we get
\[
B(z) \lesssim \omega(z)^{-p/2} \tau(z)^{\alpha-2(p-1)},
\]
which together with (2.8) gives the desired result in that case. If \( p - \alpha < 0 \), then we have
\[
B(z) \lesssim \omega(z)^{-p/2} \tau(z)^{Mp-p} \int_{|z-\xi|>\delta_0 \tau(z)} \frac{\tau(\xi)^{\alpha-p} dA(\xi)}{|z-\xi|^{Mp}}.
\]
Using condition (B), it is easy to see that \( \tau(\xi) \lesssim 2^k \tau(z) \) if \( \xi \in R_k(z) \). Thus we can estimate the previous integral as before to get
\[
\int_{|z-\xi|>\delta_0 \tau(z)} \frac{\tau(\xi)^{\alpha-p} dA(\xi)}{|z-\xi|^{Mp}} \lesssim \tau(z)^{\alpha+2p-Mp} \sum_{k \geq 0} 2^{-k(Mp+p-\alpha-2)} \lesssim \tau(z)^{\alpha+2p-Mp},
\]
Again this gives
\[
B(z) \lesssim \omega(z)^{-p/2} \tau(z)^{\alpha-2(p-1)}.
\]
The proof is complete. \( \square \)

A consequence of Lemma 2.2 is that
\[
(2.10) \quad \|K_z\|_{AP(\omega^{p/2})} \asymp \omega(z)^{-1/2} \tau(z)^{-\frac{2(p-1)}{p}}.
\]
One inequality is Lemma 2.2 and the other follows easily from Lemma [C and (2.7)].

2.2. Companion weighted Bergman spaces. For the proof of the atomic decomposition in the case \( 0 < p < 1 \), we need to consider reproducing kernels \( K^*_z \) of the Bergman space \( A^2_\omega \), where the weight \( \omega_\alpha \) is of the form
\[
\omega_\alpha(z) = \omega(z) \tau(z)^\alpha, \quad \alpha \geq 0.
\]

Lemma 2.3. \( \omega \in \mathcal{L}^*, \beta \in \mathbb{R} \) and \( 0 < p < \infty \). Then
\[
|f(z)|^p \omega_\alpha(z)^\beta \leq \frac{C}{\delta^{2p} \tau(z)^{2}} \int_{D(\delta \tau(z))} |f(\zeta)|^p \omega_\alpha(\zeta)^\beta \, dA(\zeta),
\]
for all \( f \in H(\mathbb{D}) \) and all \( \delta > 0 \) sufficiently small.
Proof. This is an immediate consequence of Lemma A and (2.1). Indeed,

\[ |f(z)^p \omega_s(z)^\beta| \leq |f(z)^p \omega(z)^\beta \tau(z)\beta| \leq \tau(z)^{\beta-2} \int_{D(\delta \tau(z))} |f(\zeta)^p \omega(\zeta)^\beta| dA(\zeta) \]

How to proceed.

Since \( \|K^*_s\|_{A^2(\omega_s)} \) coincides with the norm of the point evaluation functional in \( A^2(\omega_s) \) at the point \( z \), as a consequence of the previous lemma, we have the estimate

(2.11) \[ \|K^*_s\|_{A^2(\omega_s)}^2 \omega_s(z) \leq \tau(z)^{-2}, \quad z \in \mathbb{D}. \]

We need the following result \[1\] Proposition 4.5 on estimates of the solutions of the \( \bar{\partial} \)-equation.

**Proposition A.** Let \( \omega \in \mathcal{E} \) and consider the associated weight

\[ \omega_s(z) = \omega(z) \tau(z)^\alpha, \quad \alpha \in \mathbb{R}. \]

There is a solution \( u \) of the equation \( \bar{\partial} u = f \) satisfying

\[ \int_{\mathbb{D}} |u(z)|^2 \omega_s(z) dA(z) \leq C \int_{\mathbb{D}} |f(z)|^2 \omega_s(z) \tau(z)^2 dA(z). \]

We also need the analogue of Theorem A for the reproducing kernels \( K^*_s \). The result can be found also in \[1\].

**Lemma D.** Let \( \omega \in \mathcal{E} \) and \( K^*_s \) be the reproducing kernel of \( A^2(\omega_s) \) where \( \omega_s \) is the associated weight. For each \( M \geq 1 \), there exists a constant \( C > 0 \) (depending on \( M \)) such that for each \( z, \xi \in \mathbb{D} \) one has

\[ |K^*_s(\xi)| \leq C \frac{1}{\tau(z)} \frac{1}{\tau(\xi)} \omega_s(z)^{-1/2} \omega_s(\xi)^{-1/2} \left( \min(\tau(z), \tau(\xi)) \right)^M. \]

As a consequence, we obtain the following integral estimate involving the reproducing kernels \( K^*_s \). The result is deduced from Lemma D in the same way as we proved Lemma 2.2 using Theorem A and therefore the proof is omitted here.

**Corollary 2.4.** Let \( \omega \in \mathcal{E} \), and \( K^*_s \) be the reproducing kernel for \( A^2(\omega_s) \). For \( 0 < p < \infty \) and \( \beta \in \mathbb{R} \), there exists a constant \( C > 0 \) such that

\[ \int_{\mathbb{D}} |K^*_s(\xi)|^p \omega_s(\xi)^{p/2} \tau(\xi)^{3} dA(\xi) \leq C \omega_s(z)^{-p/2} \tau(z)^{3-2(p-1)}. \]

2.3. **Bounded projections and the reproducing formula.** In \[1\] we proved that, for weights \( \omega \in \mathcal{E} \), the Bergman projection \( P_\omega \) is bounded on \( L^p(\omega^{p/2}) \) for \( p \geq 1 \) (see also \[4\]). Here we must show that the associated Bergman projection \( P_{\omega_s,\alpha} \) given by

\[ P_{\omega_s,\alpha} f(z) = \int_{\mathbb{D}} f(\zeta) \overline{K^*_s(\zeta)} \omega_s,\alpha(\zeta) dA(\zeta) \]

is bounded on \( L^p(\omega_s,\gamma) \) for all \( \alpha, \gamma \in \mathbb{R} \). Recall that, given a weight \( v \), the growth space \( L^\infty(v) \) consists of those measurable functions \( f \) on \( \mathbb{D} \) such that

\[ \|f\|_{L^\infty(v)} := \text{ess sup}_{z \in \mathbb{D}} |f(z)| v(z) < \infty, \]

and \( A^\infty(v) \) is the space of all analytic functions in \( L^\infty(v) \).
Proposition 2.5. Let $\omega \in \mathcal{E}$ and $1 \leq p < \infty$. Then $P_{\omega, \alpha}$ is bounded on $L^p(\omega_{s, \gamma}^{1/p})$ for all $\alpha, \gamma \in \mathbb{R}$. Moreover, $P_{\omega, \alpha}$ is also bounded on $L^\infty(\omega_{s, \gamma}^{1/2})$.

Proof. Let $1 < p < \infty$ and $f \in L^p(\omega_{s, \gamma}^{1/p})$. By Hölder’s inequality and Corollary 2.4 with $\beta = 0$, we have

\[ |P_{\omega, \alpha}f(z)|^p \leq \left( \int_{\mathbb{D}} |f(\zeta)|^p |K^s_\alpha(\zeta)|^{\frac{p+1}{2}} \, dA(\zeta) \right) \left( \int_{\mathbb{D}} |K^s_\alpha(\zeta)|^{\frac{p+1}{2}} \, dA(\zeta) \right)^{-1}
\]

This together with Fubini’s theorem gives

\[ \|P_{\omega, \alpha}f\|_{L^p(\omega_{s, \gamma}^{1/p})} \leq \left( \int_{\mathbb{D}} |f(\zeta)|^p |K^s_\alpha(\zeta)|^{\frac{p+1}{2}} \, dA(\zeta) \right) \left( \int_{\mathbb{D}} |K^s_\alpha(\zeta)|^{\frac{p+1}{2}} \, dA(\zeta) \right)^{-1} \cdot \omega_{s, \gamma}(z)^{\frac{(p-1)}{2}}. \]

Proposition 2.5. Let $\omega \in \mathcal{E}$ and let $\omega_s$ be the associated weight given by $\omega_s(z) = \omega(z) \tau(z)^{\alpha}$, $\alpha \in \mathbb{R}$. For each $f \in A^1(\omega_{s, \gamma}^{1/2})$, one has the reproducing formula $f = P_{\omega_s} f$.

Proof. Let $f \in A^1(\omega_{s, \gamma}^{1/2})$. We begin by constructing functions $f_n \in A^2(\omega_s)$ with $\|f_n\|_{A^1(\omega_{s, \gamma}^{1/2})} \lesssim \|f\|_{A^1(\omega_{s, \gamma}^{1/2})}$ such that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$.

Let $\alpha_n := 1 - 1/n$, and consider a sequence of $C^\infty$ functions $\chi_n$ with compact support on $\mathbb{D}$ such that $\chi_n(z) = 1$ for $|z| \leq 1 - 1/n$, and $|\overline{\partial} \chi_n| \lesssim n$. For each $n$, consider the analytic functions

\[ f_n = P_{\omega_s} (f \chi_n), \]

where $\omega_s$ is the associated weight given by

\[ \omega_s(z) = \omega(z) \tau(z)^2. \]

Since $f \chi_n \in L^2(\omega_s)$ and, by Proposition 2.5, the projection $P_{\omega_s}$ is bounded on $L^p(\omega_{s, \gamma}^{1/p})$, $1 \leq p < \infty$, then the functions $f_n$ belong to $A^2(\omega_s)$, and

\[ \|f_n\|_{A^1(\omega_{s, \gamma}^{1/2})} = \|P_{\omega_s} (f \chi_n)\|_{A^1(\omega_{s, \gamma}^{1/2})} \lesssim \|f \chi_n\|_{L^1(\omega_{s, \gamma}^{1/2})} \leq \|f\|_{A^1(\omega_{s, \gamma}^{1/2})}. \]

Thus, it remains to prove that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Since $|f - f_n| \leq |f - f \chi_n| + |f \chi_n - f_n|$, and, obviously, $f \chi_n \to f$ uniformly on compact subsets of $\mathbb{D}$, it suffices to show that $u_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, with $u_n = f \chi_n - P_{\omega_s} (f \chi_n)$. 
Fix $0 < R < 1$ and let $z \in \mathbb{D}$ with $|z| \leq R$. For $n$ big enough, the function $u_n$ is analytic in a neighborhood of the disc $D(\delta_0 \tau(z))$, with $\delta_0 \in (0, m_\tau)$. Hence, by Lemma 2.3

$$
\tau(z)^4 |u_n(z)|^2 \omega_*(z) \lesssim \tau(z)^2 \int_{D(\delta_0 \tau(z))} |u_n(\zeta)|^2 \omega_*(\zeta) \, dA(\zeta)
$$

(2.12)

$$
\lesssim \int_{D(\delta_0 \tau(z))} |u_n(\zeta)|^2 \omega_*(\zeta) \tau(\zeta)^2 \, dA(\zeta)
$$

$$
\lesssim \int_D |u_n(\zeta)|^2 \omega_*(\zeta) \, dA(\zeta)
$$

Since $u_n$ is the solution of the $\overline{\partial}$-equation $\overline{\partial} v = f \overline{\partial}_\zeta$ with minimal $L^2(\omega_*)$ norm, by Proposition A we have

$$
\int_\mathbb{D} |u_n(\zeta)|^2 \omega_*(\zeta) \, dA(\zeta) \leq C \int_\mathbb{D} |f \overline{\partial}_\zeta(\zeta)|^2 \omega_*(\zeta) \tau(\zeta)^2 \, dA(\zeta).
$$

Since $\overline{\partial}_\zeta$ is supported on $r_n < |\zeta| < 1$ with $|\overline{\partial}_\zeta| \lesssim n$, we get

$$
\int_\mathbb{D} |u_n(\zeta)|^2 \omega_*(\zeta) \, dA(\zeta) \leq C n^2 \int_{|\zeta|>r_n} |f(\zeta)|^2 \omega_*(\zeta) \tau(\zeta)^4 \, dA(z).
$$

By Lemma 2.3 we have the pointwise estimate

$$
|f(\zeta)| \omega_*(\zeta)^{1/2} \tau(\zeta)^2 \lesssim \| f \|_{A^1(\omega^{1/2})}.
$$

This together with $\tau(\zeta) \lesssim (1 - |\zeta|) \leq 1/n$ for $|\zeta| > r_n$, yields

$$
\int_\mathbb{D} |u_n(\zeta)|^2 \omega_*(\zeta) \, dA(\zeta) \leq C \int_{|\zeta|>r_n} |f(\zeta)|^2 \omega_*(\zeta) \tau(\zeta)^2 \, dA(z)
$$

$$
\leq C \| f \|_{A^1(\omega^{1/2})} \int_{|\zeta|>r_n} |f(\zeta)| \omega_*(\zeta)^{1/2} \, dA(z),
$$

and this goes to zero as $n \to \infty$ since $f \in A^1(\omega^{1/2})$. Bearing in mind (2.12), this implies that $u_n \to 0$ uniformly on compact subsets of $\mathbb{D}$.

With help of the constructed functions $f_n$ we are ready to prove the reproducing formula. Note that

$$
|f(z) - P_{\omega_*} f(z)| \leq |f(z) - f_n(z)| + |f_n(z) - P_{\omega_*} f(z)|.
$$

Clearly, the first term goes to zero as $n \to \infty$. For the second term, since $f_n \in A^2(\omega_*)$, the reproducing formula $f_n = P_{\omega_*} f_n$ holds, and therefore

$$
|f_n(z) - P_{\omega_*} f(z)| = |P_{\omega_*} (f_n - f)(z)| \leq \int_D |f_n(\xi) - f(\xi)| |K_\tau^*(\xi)| \omega_*(\xi) \, dA(\xi).
$$

Fix $0 < \delta < m_\tau$ and split the previous integral in two parts: one integrating over the disk $D(\delta \tau(z))$, and the other over $\mathbb{D} \setminus D(\delta \tau(z))$. For the first one, using $|K_\tau^*(\xi)| \leq \| K_\tau^* \|_{A^2(\omega_*)} \| K_\tau^* \|_{A^2(\omega_*)}$ and the estimate (2.11), we have

$$
\int_{D(\delta \tau(z))} |f_n(\xi) - f(\xi)| |K_\tau^*(\xi)| \omega_*(\xi) \, dA(\xi)
$$

$$
\lesssim \frac{\| K_\tau^* \|_{A^2(\omega_*)}}{\tau(\zeta)} \int_{D(\delta \tau(z))} |f_n(\xi) - f(\xi)| \omega_*(\xi)^{1/2} \, dA(\xi)
$$
For the case $p > 1$, let

$$M(z) := \sum_{k=0}^{\infty} \lambda_k p K_k(z).$$

We get

$$\|F\|_{H^p(U)} \leq \sum_{k=0}^{\infty} \lambda_k p K_k(z).$$

Proposition 3.1. Let $\omega \in \mathcal{E}_{0}$, $0 < p < \infty$ and for $\delta \in (0, m)$, let $(s, \ell)$ be a $(s, \ell)$-k-altering on $\mathcal{D}$. The function $F$ belongs to $H^p(\mathcal{D})$. For every sequence $\lambda = \{\lambda_k\}$, let $\lambda_k = 1$ on a finite set of $\mathcal{D}$. We define an analytic function on $\mathcal{D}$ for $\lambda_k > 0$. By taking $0 < \delta < 1$ (close to 1), we can make the last expression as small as desired. Once $F$ converges uniformly on compact subsets of $\mathcal{D}$, this shows that $F$ converges uniformly on compact subsets of $\mathcal{D}$. Therefore, we obtain

$$\int_{0}^{\infty} \left| f_n(x) - f(x) \right|^2 dA < \frac{\min\{\|f\|_{H^p(\mathcal{D})}, 1\}}{1 - \delta}.$$

This proves the proposition.

\[ \square \]
By Hölder’s inequality we have
\[ \|F\|_{A^p(\omega^{p/2})}^p \leq \int_{\mathbb{D}} \left( \sum_{k=0}^{\infty} |\lambda_k| \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} |K_{z_k}(z)| \right)^p \omega(z)^{p/2} dA(z) \]
\[ \lesssim \int_{\mathbb{D}} \left( \sum_{k=0}^{\infty} |\lambda_k|^p \omega(z_k)^{1/2} |K_{z_k}(z)| \right) M(z)^{p-1} \omega(z)^{p/2} dA(z). \]

On the other hand, using Lemma A the lattice properties and Lemma 2.2 we have
\[ M(z) \lesssim \sum_{k=0}^{\infty} \int_{D(\delta z_k))} |K_z(\xi)| \omega(\xi)^{1/2} dA(\xi) \]
\[ \lesssim \int_{\mathbb{D}} |K_z(\xi)| \omega(\xi)^{1/2} dA(\xi) \lesssim \omega(z)^{-1/2}. \]

Therefore, applying Lemma 2.2 again we obtain
\[ \|F\|_{A^p(\omega^{p/2})}^p \lesssim \int_{\mathbb{D}} \left( \sum_{k=0}^{\infty} |\lambda_k|^p \omega(z_k)^{1/2} |K_{z_k}(z)| \right) \omega(z)^{1/2} dA(z) \]
\[ \lesssim \sum_{k=0}^{\infty} |\lambda_k|^p \omega(z_k)^{1/2} \int_{\mathbb{D}} |K_{z_k}(z)| \omega(z)^{1/2} dA(z) \]
\[ \lesssim \sum_{k=0}^{\infty} |\lambda_k|^p. \]

\[ \square \]

**Lemma 3.2.** Let \( \omega \in \mathcal{E} \). There is \( \varepsilon_0 > 0 \) such that if \( \{z_n\} \) is an \((\varepsilon, \tau)\)-lattice on \( \mathbb{D} \), with \( 0 < \varepsilon < \varepsilon_0 \), then
\[ \sum_n |f(z_n)|^p \omega(z_n)^{p/2} \tau(z_n)^2 \gtrsim \|f\|_{A^p(\omega^{p/2})}^p, \]
for all \( f \in A^p(\omega^{p/2}) \) and \( p > 0 \).

**Proof.** Let \( \{z_n\} \) be an \((\varepsilon, \tau)\)-lattice on \( \mathbb{D} \) with \( \varepsilon > 0 \) small enough to be specified later. Let \( f \in A^p(\omega^{p/2}) \) and consider
\[ I_f(n) := \sum_n |f(z_n)|^p \omega(z_n)^{p/2} \tau(z_n)^2. \]

We have
\[ \|f\|_{A^p(\omega^{p/2})}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z)^{p/2} dA(z) \]
\[ \leq C \left[ \sum_n \int_{D(\varepsilon z_n))} \left( |f(z)| \omega(z)^{1/2} - |f(z_n)| \omega(z_n)^{1/2} \right)^p dA(z) + C\varepsilon^2 I_f(n) \right]. \]

For \( z \in D(\varepsilon z_n)) \), there exists \( \xi_{n,z} \in [z, z_n] \) such that
\[ \left( |f(z)| \omega(z)^{1/2} - |f(z_n)| \omega(z_n)^{1/2} \right)^p \leq \nabla(|f| \omega^{1/2})(\xi_{n,z})^p |z - z_n|^p \]
\[ \leq \varepsilon^p \tau(z_n)^p |\nabla(|f| \omega^{1/2})(\xi_{n,z})|^p. \]
This together with Lemma 2.1 with \( \delta_0 \in (0, m_\tau) \) fixed, yields
\[
\int_{D(\varepsilon \tau(z_n))} \left( |f(z)| \omega(z)^{1/2} - |f(z_n)| \omega(z_n)^{1/2} \right)^p dA(z)
\leq C \varepsilon^p \tau(z_n)^p \int_{D(\varepsilon \tau(z_n))} \left( \frac{1}{\tau(z_n)^{p+2}} \int_{D(\delta_0 \tau(z_n))} |f(\eta)|^p \omega(\eta)^{p/2} dA(\eta) \right) dA(z).
\]

Using that \( \tau(x_n, z) \approx \tau(z_n) \) and \( D(\delta_0 \tau(x_n, z)) \subset D(3\delta_0 \tau(z_n)) \) for \( z \in D(\varepsilon \tau(z_n)) \), we obtain
\[
\int_{D(\varepsilon \tau(z_n))} \left( |f(z)| \omega(z)^{1/2} - |f(z_n)| \omega(z_n)^{1/2} \right)^p dA(z)
\leq C \varepsilon^{p+2} \left( \int_{D(3\delta_0 \tau(z_n))} |f(\eta)|^p \omega(\eta)^{p/2} dA(\eta) \right).
\]

Therefore,
\[
\|f\|_{AP(\omega^{p/2})}^p \leq C \varepsilon^{p+2} \sum_n \int_{D(3\delta_0 \tau(z_n))} |f(\eta)|^p \omega(\eta)^{p/2} dA(\eta) + C \varepsilon^2 I_f(n).
\]

By the remark after Lemma 3.2 every point \( z \in \mathbb{D} \) belongs to at most \( C \varepsilon^{-2} \) of the sets \( D(3\delta_0 \tau(z_n)) \), and therefore
\[
\|f\|_{AP(\omega^{p/2})}^p \leq C \varepsilon^p \|f\|_{AP(\omega^{p/2})}^p + C \varepsilon^2 I_f(n).
\]

Thus, taking \( \varepsilon > 0 \) so that \( C \varepsilon^p < 1/2 \), we get the desired result. \( \square \)

Just note that, what actually Lemma 3.2 says, is that an \((\varepsilon, \tau)\)-lattice with \( \varepsilon > 0 \) small enough, is a sampling sequence for the Bergman space \( \text{AP}^{p}(\omega^{p/2}) \). Recall that \( \{z_n\} \subset \mathbb{D} \) is a sampling sequence for the Bergman space \( \text{AP}^{p}(\omega^{p/2}) \) if
\[
\|f\|_{\text{AP}^{p}(\omega^{p/2})}^p \leq \sum_n |f(z_n)|^p \omega(z_n)^{p/2} \tau(z_n)^2
\]
for any \( f \in \text{AP}^{p}(\omega^{p/2}) \). Just note that Lemma 3.2 gives one inequality, and the other follows by standard methods using Lemma A and the lattice properties. Sampling sequences on the classical Bergman space were characterized by K. Seip [16] (see also the monographs [6] and [17]). For sampling sequences on large weighted Bergman spaces we refer to [2].

Now we are ready to prove the result on the atomic decomposition of large weighted Bergman spaces in the case that \( p > 1 \), result that is reformulated next. Recall that \( k_{p, z} \) are the normalized reproducing kernels in \( AP^{p}(\omega^{p/2}) \).

**Theorem 3.3.** Let \( \omega \in \mathcal{E} \) and \( 1 < p < \infty \). There is a lattice \( \{z_n\} \subset \mathbb{D} \) such that:

(i) For any \( \lambda = \{\lambda_n\} \in \ell^p \), the function
\[
f(z) = \sum_n \lambda_n k_{p, z_n}(z)
\]
is in \( AP^{p}(\omega^{p/2}) \) with \( \|f\|_{AP^{p}(\omega^{p/2})} \leq C \|\lambda\|_{\ell^p} \).
(ii) For every \( f \in A^p(\omega^{p/2}) \) exists \( \lambda = \{ \lambda_n \} \in \ell^p \) such that
\[
\lambda(z) = \sum_n \lambda_n k_{p,z_n}(z)
\]
and \( \| \lambda \|_{\ell^p} \leq C \| f \|_{A^p(\omega^{p/2})}. \)

**Proof.** Due to (2.10), part (i) is just Proposition 3.1. In order to prove (ii), we define a linear operator \( S : \ell^p \to A^p(\omega^{p/2}) \) given by
\[
S(\{ \lambda_n \}) := \sum_n \lambda_n k_{p,z_n}.
\]
By (i), the operator \( S \) is bounded. Under the integral pairing \( \langle \cdot, \cdot \rangle_\omega \), the dual space of \( A^p(\omega^{p/2}) \) can be identified with \( A^{p'}(\omega^{p'/2}) \), where \( p' \) is the conjugate exponent of \( p \) (see (1)). Thus, the adjoint operator \( S^* : A^{p'}(\omega^{p'/2}) \to \ell^{p'} \) is defined by
\[
\langle Sx, f \rangle_\omega = \langle x, S^* f \rangle_\ell = \sum_n x_n (S^* f)_n,
\]
for every \( x \in \ell^p \) and \( f \in A^{p'}(\omega^{p'/2}) \). To compute \( S^* \), let \( e_n \) denote the vector that equals 1 at the \( n \)-th coordinate and equals 0 at the other coordinates. Then \( S e_n = k_{p,z_n} \), and using the reproducing formula we get
\[
(S^* f)_n = \langle e_n, S^* f \rangle_\ell = \langle S e_n, f \rangle_\omega = \langle k_{p,z_n}, f \rangle_\omega = \frac{\langle f(z_n) \rangle_{\omega}}{\| K_{z_n} \|_{A^p(\omega^{p/2})}}.
\]
Hence, \( S^* : A^{p'}(\omega^{p'/2}) \to \ell^{p'} \) is given by
\[
S^* f = \{(S^* f)_n\} = \left\{ \frac{f(z_n)}{\| K_{z_n} \|_{A^p(\omega^{p/2})}} \right\}_n.
\]
We must prove that \( S \) is surjective in order to finish the proof of this case. By a classical result in functional analysis, it is enough to show that \( S^* \) is bounded below. By Lemma 3.2 and \( \| K_{z_n} \|_{A^{p'}(\omega^{p'/2})} \sim \omega(z_n)^{p'/2} \tau(z_n)^2 \) we obtain
\[
\| S^* f \|_{\ell^{p'}} \sim \sum_n |f(z_n)|^{p'} \omega(z_n)^{p'/2} \tau(z_n)^2 \geq \| f \|_{A^{p'}(\omega^{p'/2})},
\]
which shows that \( S^* \) is bounded below. Finally, once the surjectivity is proved, the estimate \( \| \lambda \|_{\ell^p} \leq C \| f \|_{A^p(\omega^{p/2})} \) is an standard application of the open mapping theorem. The proof is complete. \( \square \)

4. Atomic decomposition for \( 0 < p \leq 1 \)

If \( \omega \) is a weight we use the associated weight \( \omega_\star \) defined as
\[
\omega_\star(z) = \omega(z) \tau(z)^{\frac{4(1-p)}{p}}, \quad z \in \mathbb{D}.
\]

**Lemma 4.1.** Let \( \omega \in \mathcal{L}_\star \), and \( 0 < p \leq 1 \). Then \( A^p(\omega^{p/2}) \subset A^1(\omega_\star^{1/2}) \) with
\[
\| f \|_{A^1(\omega_\star^{1/2})} \lesssim \| f \|_{A^p(\omega^{p/2})}.
\]
\textbf{Proof.} Let \( f \in A^p(\omega^\beta) \). By Lemma 4.2. we have the pointwise estimate
\[
|f(z)| \omega(z)^{1/2} \lesssim \tau(z)^{-2/p} \|f\|_{A^p(\omega^{p/2})}.
\]

Then, we have
\[
\|f\|_{A^1(\omega^{1/2})} = \int_{\mathbb{D}} |f(z)|^p \cdot (|f(z)| \omega(z)^{1/2})^{1-p} \omega(z)^{p/2} \tau(z)^{2(1-p)/p} \, dA(z)
\lesssim \|f\|_{A^p(\omega^{p/2})} \int_{\mathbb{D}} |f(z)|^p \omega(z)^{p/2} \, dA(z) = \|f\|_{A^p(\omega^{p/2})}.
\]

\[\square\]

\textbf{Lemma 4.2.} Let \( \tau \in \mathcal{L} \) and \( \{z_n\} \) be a \((\delta, \tau)\)-lattice on \( \mathbb{D} \) with \( \delta \in (0, m_\tau) \). For each \( n \) there exists a measurable set \( D_n \) satisfying the following conditions:
(i) \( D(\delta(z_n)) \subset D_n \subset D(\delta \tau(z_n)) \) for every \( n \geq 1 \).
(ii) \( D_k \cap D_j = \emptyset \) for \( k \neq j \).
(iii) \( \mathbb{D} = \bigcup_{n \geq 1} D_n \).

\textbf{Proof.} This is proved in the same way as in Lemma 2.28 of [18]. We omit the details here. \[\square\]

We will also need to consider lattices associated with a function \( \rho \) of the form \( \rho(z) = \tau(z)^\beta \) with \( \beta \geq 1 \). It is clear that, if \( \tau \in \mathcal{L} \), then \( \rho \) is also in the class \( \mathcal{L} \).

\textbf{Lemma 4.3.} Let \( 0 < p \leq 1 \) and \( f \in A^p(\omega^{p/2}) \) with \( \omega \in \mathcal{L}^* \). Let \( \beta \geq 1 \) and \( \rho(z) = \tau(z)^\beta \). For \( \delta \in (0, m_\rho) \) let \( \{z_n\} \) be an \((\delta, \tau)\)-lattice on \( \mathbb{D} \). Then the sequence \( \lambda = \{\lambda_n\} \) defined by
\[
\lambda_n = \int_{D(\delta(z_n))} |f(z)| \omega_n(z)^{1/2} \, dA(z)
\]
belongs to \( \ell^p \) with \( \|\lambda\|_{\ell^p} \lesssim \|f\|_{A^p(\omega^{p/2})} \).

\textbf{Proof.} As \( \tau(z) \leq c_1 (1 - |z|) \), it is clear that \( \rho(z)^2 \Delta \varphi(z) \leq C \) for some positive constant \( C \). Thus, by Lemma 4.2. we have
\[
\|\lambda\|_{\ell^p}^p \leq \sum_n \left( \int_{D(\delta(z_n))} |f(z)| \omega_n(z)^{1/2} \, dA(z) \right)^p
\lesssim \sum_n \left( \int_{D(\delta(z_n))} \frac{1}{\rho(z)^2} \left( \int_{D(\delta(z))} |f(\xi)|^p \omega(\xi)^{p/2} \, dA(\xi) \right)^{1/p} \, dA(z) \right)^p.
\]
Using the properties of being a \((\delta, \rho)\)-lattice and (2.1) we get
\[
\|\lambda\|_{\ell^p}^p \lesssim \sum_n \int_{D(3\delta(z_n))} |f(\xi)|^p \omega(\xi)^p \, dA(\xi) \lesssim \|f\|_{A^p(\omega^{p/2})}^p.
\]

\[\square\]

Now we are ready for the proof of the atomic decomposition result for \( 0 < p \leq 1 \) that, for convenience, is reformulated below.

\textbf{Theorem 4.4.} Let \( \omega \in \mathcal{E} \) and \( 0 < p \leq 1 \). There exists a lattice \( \{z_n\} \subset \mathbb{D} \) such that:
(i) For any \( \lambda = \{ \lambda_n \} \in \ell^p \), the function

\[
f(z) = \sum_n \lambda_n \omega(z_n) \tau(z_n)^{2/(1-p)} K^{\ast}_n(z)
\]

is in \( A^p(\omega^{p/2}) \) with \( ||f||_{A^p(\omega^{p/2})} \leq C ||\lambda||_{\ell^p} \).

(ii) For every \( f \in A^p(\omega^{p/2}) \) exists \( \lambda = \{ \lambda_n \} \in \ell^p \) such that

\[
f(z) = \sum_n \lambda_n \omega(z_n) \tau(z_n)^{2/(1-p)} K^{\ast}_n(z)
\]

and \( ||\lambda||_{\ell^p} \leq C ||f||_{A^p(\omega^{p/2})} \).

Here \( K^{\ast}_n \) denotes the reproducing kernel of \( A^2(\omega_s) \) with \( \omega_s(z) = \omega(z) \tau(z)^{4(1-p)/p} \).

**Proof.** It is easy to see that \( f \) defines an analytic function on \( \mathbb{D} \). Also, since \( 0 < p \leq 1 \), applying Corollary 2.4 we have

\[
||f||_{A^p(\omega^{p/2})} \leq \sum_n |\lambda_n|^p \omega(z_n)^{p/2} \tau(z_n)^{2(1-p)} ||K^{\ast}_n||_{A^p(\omega^{p/2})} \leq C ||\lambda||^p_{\ell^p}.
\]

This proves (i). In order to prove (ii), note that for the exponential weights we have \( \tau(z) = (1 - |z|)^{1+\beta} \), and

\[
|\nabla \varphi(z)| \asymp (1 - |z|)^{-1-\sigma} = \tau(z)^{-\beta}
\]

with \( \beta = 2(1+\sigma)/(2+\sigma) \). In particular, the function

\[
\rho(z) = \tau(z)^{\beta}, \quad z \in \mathbb{D}
\]

is also in the class \( \mathcal{L} \). For \( 0 < \delta_0 < \min(m, m_r) \) fixed, let \( \{ a_j \} \) be a \( (\delta_0, \tau) \)-lattice on \( \mathbb{D} \), and for \( \varepsilon > 0 \) sufficiently small to be specified later, consider another sequence \( \{ z_n \} \) being now an \( (\varepsilon \delta_0, \rho) \)-lattice (note that this is a lattice associated to the function \( \rho \)). Let \( \{ F_n \} \) be the measurable sets associated with the \( (\varepsilon \delta_0, \rho) \)-lattice \( \{ z_n \} \) obtained in Lemma 4.2. Consider the linear operator \( T \) given by

\[
Tf(z) = \sum_n b_n \omega(z_n)^{1/2} \tau(z_n)^{2/(1-p)} K^{\ast}_n(z),
\]

where

\[
b_n = \int_{F_n} f(\xi) \omega_s(\xi)^{1/2} dA(\xi).
\]

Since \( F_n \subset D(\varepsilon \delta_0 \rho(z_n)) \), by Lemma 4.3 and part (i), the operator \( T \) is bounded on \( A^p(\omega^{p/2}) \). Since \( A^p(\omega^{p/2}) \subset A^1(\omega_s^{1/2}) \), by the reproducing formula for \( A^1(\omega_s^{1/2}) \), for every \( f \in A^p(\omega^{p/2}) \) we have

\[
f(z) = \int_\mathbb{D} f(\xi) K^{\ast}_n(\xi) \omega_s(\xi) dA(\xi) = \sum_n \int_{F_n} f(\xi) K^{\ast}_n(\xi) \omega_s(\xi) dA(\xi)
\]

\[
= \sum_j \int_{z_n \in D_j} f(\xi) K^{\ast}_n(\xi) \omega_s(\xi) dA(\xi)
\]

\[
= \sum_j \sum_{z_n \in D_j} f(\xi) K^{\ast}_n(\xi) \omega_s(\xi)^{1/2} \omega(\xi)^{1/2} \tau(\xi)^{2/(1-p)} dA(\xi),
\]
where \( \{D_j\} \) are the sets obtained from Lemma 4.2 associated with the \((\delta_0, \tau)\)-lattice \( \{a_j\} \). On the other hand,
\[
T f(z) = \sum_j \sum_{z_n \in D_j} b_n \omega(z_n)^{1/2} \tau(z_n)^{2(1-p)} K_{\ast n}^s(z)
\]
\[
= \sum_j \sum_{z_n \in D_j} \left( \int_{F_n} f(\xi) \omega_s(\xi)^{1/2} dA(\xi) \right) \omega(z_n)^{1/2} \tau(z_n)^{2(1-p)} K_{\ast n}^s(z).
\]
Then
\[
f(z) - T f(z) = \sum_j \sum_{z_n \in D_j} \int_{F_n} f(\xi) \omega_s(\xi)^{1/2} \left( K_{\ast}^s(\xi) \omega_s(\xi)^{1/2} - K_{\ast n}^s(z_n) \omega_s(z_n)^{1/2} \right) dA(\xi).
\]
(4.2)

Note that, with \( \alpha = 2(1-p)/p \),
\[
|K_{\ast}^s(\xi) \omega_s(\xi)^{1/2} - K_{\ast n}^s(z_n) \omega_s(z_n)^{1/2}| \leq |K_{\ast}^s(z_n)| \omega(z_n)^{1/2} \cdot |\tau(\xi)^\alpha - \tau(z_n)^\alpha| 
\]
(4.3)

Apply the inequality
\[
|x^\alpha - y^\alpha| \leq \alpha \max(x^{\alpha-1}, y^{\alpha-1}) |x - y|
\]
together with (2.11) and conditions (A) and (B) to get that
\[
|\tau(\xi)^\alpha - \tau(z_n)^\alpha| \leq C \tau(z_n)^{\alpha-1} |\tau(\xi) - \tau(z_n)| \leq C \tau(z_n)^{\alpha-1} |\xi - z_n| \leq C \epsilon \tau(z_n)^\alpha.
\]
for \( \xi \in F_n \). This and Lemma A gives
\[
|K_{\ast}^s(z_n)| \omega(z_n)^{1/2} |\tau(\xi)^\alpha - \tau(z_n)^\alpha| \leq \frac{C \epsilon}{\tau(z_n)^2} \left( \int_{D(\delta_0 \tau(z_n))} |K_{\ast}^s(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p}.
\]
(4.4)

On the other hand, for \( \xi \in F_n \), there exists \( \gamma_{\xi, n} \in [\xi, z_n] \) such that
\[
|K_{\ast}^s(\xi) \omega(\xi)^{1/2} - K_{\ast n}^s(z_n) \omega(z_n)^{1/2}| \leq \left| \nabla (K_{\ast n}^s \omega^{1/2})(\gamma_{\xi, n}) \right| \cdot |\xi - z_n|.
\]

Note that we can not apply here Lemma 2.1 because we have \( K_{\ast n}^s \) and not \( |K_{\ast}^s| \). This is the main reason as why we are in need to use lattices associated to the function \( \rho \) instead of \( \tau \). We have
\[
\left| \nabla (K_{\ast n}^s \omega^{1/2}) \right| \leq \left| \partial (K_{\ast n}^s \omega^{1/2}) \right| + \left| \overline{\partial} (K_{\ast n}^s \omega^{1/2}) \right|
\]
\[
\leq \left| (K_{\ast n}^s)' - K_{\ast n}^s \partial \varphi | e^{-\varphi} + |K_{\ast n}^s| |\overline{\partial} \varphi| e^{-\varphi}
\]
\[
\leq \left| (K_{\ast n}^s)' - 2 K_{\ast n}^s \partial \varphi | e^{-\varphi} + |K_{\ast n}^s| |\nabla \varphi| e^{-\varphi}.
\]
The first term has already appeared in the proof of Lemma 2.1. Thus, bearing in mind the estimate obtained in that proof, using Lemma A in order to estimate the second term, and taking into account that \( \tau(\gamma_{\xi, n}) \approx \tau(z_n) \), we obtain
\[
\left| \nabla (K_{\ast n}^s \omega^{1/2})(\gamma_{\xi, n}) \right| \leq C \left( \frac{1}{\tau(z_n)} + |\nabla \varphi(\gamma_{\xi, n})| \right) \left( \frac{1}{\tau(z_n)^2} \int_{D(\delta_0 \tau(z_n))} |K_{\ast n}^s(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p}.
\]
As, for \( \xi \in F_n \), we have
\[
|\nabla \varphi(\gamma_{\xi, n})| \cdot |\xi - z_n| \leq \frac{|\xi - z_n|}{\rho(z_n)} \leq \epsilon,
\]
and (because $\rho(z) \lesssim \tau(z)$ due to condition (A) in the definition of the class $\mathcal{L}$),
\[ \frac{|\xi - z_n|}{\tau(z_n)} \lesssim \frac{\varepsilon \rho(z_n)}{\tau(z_n)} \lesssim \varepsilon, \]
we get
\[ |K^*_x(\xi)\omega(\xi)^{1/2} - K^*_x(z_n)\omega(z_n)^{1/2}| \leq \frac{C \varepsilon}{\tau(z_n)^{2/p}} \left( \int_{D(\delta_0\tau(z_n))} |K^*_z(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p}. \]
Therefore, by (2.1),
\[ |\tau(\xi)|^\alpha |K^*_x(\xi)\omega(\xi)^{1/2} - K^*_x(z_n)\omega(z_n)^{1/2}| \]
(4.5)
\[ \leq \frac{C \varepsilon}{\tau(z_n)^2} \left( \int_{D(\delta_0\tau(z_n))} |K^*_z(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p}. \]
Putting (4.4) and (4.5) into (4.3) we see that
\[ |K^*_x(\xi)\omega(\xi)^{1/2} - K^*_x(z_n)\omega(z_n)^{1/2}| \leq \frac{C \varepsilon}{\tau(z_n)^2} \left( \int_{D(\delta_0\tau(z_n))} |K^*_z(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p}. \]
Bearing in mind (4.2), we have shown that
\[ |f(z) - T f(z)| \leq C \varepsilon \sum_j \sum_{z_n \in D_j} \frac{1}{\tau(z_n)^2} \left( \int_{D(\delta_0\tau(z_n))} |K^*_z(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p} \times \int_{F_n} |f(\xi)| \omega_s(\xi)^{1/2} dA(\xi). \]
Since $z_n \in D_j \subset D(\delta_0\tau(a_j))$, using (2.1) is straightforward to see that $D(\delta_0\tau(z_n)) \subset D(3\delta_0\tau(a_j))$. Then we get
\[ |f(z) - T f(z)| \leq C \varepsilon \sum_j \frac{1}{\tau(a_j)^2} \left( \int_{D(3\delta_0\tau(a_j))} |K^*_z(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p} \times \sum_{z_n \in D_j} \int_{F_n} |f(\xi)| \omega_s(\xi)^{1/2} dA(\xi) \]
\[ \leq C \varepsilon \sum_j \frac{1}{\tau(a_j)^2} \left( \int_{D(3\delta_0\tau(a_j))} |K^*_z(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p} \times \int_{\bigcup_{z_n \in D_j} F_n} |f(\xi)| \omega_s(\xi)^{1/2} dA(\xi). \]
Since, for any $0 < \varepsilon < 1/(4c_1^{\beta-1})$, one has $F_n \subset D(\varepsilon \delta_0 \rho(z_n)) \subset D\left(\frac{3}{2}\delta_0\tau(a_j)\right)$, then
\[ \bigcup_{z_n \in D_j} F_n \subset D\left(\frac{3}{2}\delta_0\tau(a_j)\right), \]
and it follows that
\[
|f(z) - T_f(z)| \leq C \varepsilon \sum_j \frac{1}{\tau(a_j)^2} \left( \int_{D(3\delta_0\tau(a_j))} |K^*_s(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p} \\
\times \int_{D(\frac{3}{2}\delta_0\tau(a_j))} |f(\xi)|\omega_\ast(\xi)^{1/2} dA(\xi).
\]

Because \(0 < p \leq 1\) we get
\[
|f(z) - T_f(z)|^p \leq C \varepsilon^p \sum_j \frac{1}{\tau(a_j)^{2p}} \int_{D(3\delta_0\tau(a_j))} |K^*_s(s)|^p \omega(s)^{p/2} dA(s) \\
\times \left( \int_{D(\frac{3}{2}\delta_0\tau(a_j))} |f(\xi)|\omega_\ast(\xi)^{1/2} dA(\xi) \right)^p.
\]

Therefore, denoting
\[
K(j) = \int_{D} \left( \int_{D(3\delta_0\tau(a_j))} |K^*_s(s)|^p \omega(s)^{p/2} dA(s) \right) \omega(z)^{p/2} dA(z),
\]
we have proved that
\[
\int_{D} |f(z) - T_f(z)|^p \omega(z)^{p/2} dA(z)
\]
\[(4.6)\]
\[
\leq C \varepsilon^p \sum_j \frac{1}{\tau(a_j)^{2p}} \left( \int_{D(\frac{3}{2}\delta_0\tau(a_j))} |f(\xi)|\omega_\ast(\xi)^{1/2} dA(\xi) \right)^p K(j).
\]

Now, we are going to estimate \(K(j)\). By Fubini’s theorem, Corollary 2.4 and (2.1), we obtain
\[
K(j) = \int_{D(3\delta_0\tau(a_j))} \left( \int_{D} |K^*_s(z)|^p \omega(z)^{p/2} dA(z) \right) \omega(s)^{p/2} dA(s)
\]
\[
= \int_{D(3\delta_0\tau(a_j))} \left( \int_{D} |K^*_s(z)|^p \omega_\ast(z)^{p/2} \tau(z)^{2(p-1)} dA(z) \right) \omega(s)^{p/2} dA(s)
\]
\[
\leq C \tau(a_j)^{2p}.
\]

Putting this into (4.6),
\[
\|f - T_f\|^p_{A^p(\omega^{p/2})} \leq C \varepsilon^p \sum_j \left( \int_{D(\frac{3}{2}\delta_0\tau(a_j))} |f(\xi)|\omega_\ast(\xi)^{1/2} dA(\xi) \right)^p
\]
\[(4.7)\]
\[
\leq C \varepsilon^p \sum_j \tau(a_j)^{2(1-p)} \left( \int_{D(\frac{3}{2}\delta_0\tau(a_j))} |f(\xi)|\omega(\xi)^{1/2} dA(\xi) \right)^p.
\]
Using Lemma A and (2.1) again we get
\[
\left( \int_{D(\frac{1}{2} \delta_0 \tau(a_j))} |f(\xi)|^p \omega(\xi)^{1/2} dA(\xi) \right)^p \leq C \cdot \left( \int_{D(\frac{1}{2} \delta_0 \tau(a_j))} \left( \frac{1}{\tau(\xi)^{2}} \right) \int_{D(\frac{1}{2} \delta_0 \tau(\xi))} |f(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p} \leq C \tau(\xi)^{-2(1-p)} \int_{D(3\delta_0 \tau(a_j))} |f(s)|^p \omega(s)^{p/2} dA(s).
\]

Bearing in mind (4.7) and the finite multiplicity of the covering, we finally obtain
\[
\|f-Tf\|^p_{A^p(\omega^{p/2})} \leq C \varepsilon^p \sum_j \int_{D(3\delta_0 \tau(a_j))} |f(s)|^p \omega(s)^{p/2} dA(s) \leq C \varepsilon^p \|f\|^p_{A^p(\omega^{p/2})}.
\]

Taking \(0 < \varepsilon < 1/(4c_1^{\beta-1})\) small enough so that \(C \varepsilon^p < 1/2\), then the operator \(I-T\) acting on \(A^p(\omega^{p/2})\) has norm less than 1, where \(I\) is the identity operator. In this case, it follows from standard functional analysis that the operator \(T\) is invertible on \(A^p(\omega^{p/2})\). Therefore, \(f = T(T^{-1}f)\) for every \(f \in A^p(\omega^{p/2})\), that is, it admits a representation
\[
f(z) = \sum_n \lambda_n \omega(z_n)^{1/2} \tau(z_n)^{-\frac{2(1-p)}{p}} K_{z_n}^*(z),
\]
with \(\lambda = \{\lambda_n\}\) given by
\[
\lambda_n = \int_{F_n} g(\xi) \omega(\xi)^{1/2} dA(\xi)
\]
and \(g = T^{-1}f\). By Lemma 4.3, it follows that \(\lambda \in \ell^p\) with
\[
\|\lambda\|_{\ell^p} \leq C\|g\|_{A^p(\omega^{p/2})} \leq C\|f\|_{A^p(\omega^{p/2})}.
\]
This finishes the proof. \(\square\)

4.1. An application. Next we are going to use the atomic decomposition result just proved in order to identify the dual space of \(A^p(\omega^{p/2})\) when \(0 < p < 1\).

**Theorem 4.5.** Let \(\omega \in \mathcal{E}\) and \(0 < p \leq 1\). Under the integral pairing \(\langle f, g \rangle_{\omega}\), the dual space of \(A^p(\omega^{p/2})\) can be identified (with equivalent norms) with \(A^{\infty}(v)\), where \(v(z) = \omega(z)^{1/2} \tau(z)^{-\frac{2(1-p)}{p}}\).

**Proof.** If \(g \in A^{\infty}(v)\), using Lemma 4.1, it is straightforward to see that \(\Lambda g(f) = \langle f, g \rangle_{\omega}\) defines a bounded linear functional on \(A^p(\omega^{p/2})\) with \(\|\Lambda g\| \leq \|g\|_{A^{\infty}(v)}\).

Conversely, let \(\Lambda \in (A^p(\omega^{p/2}))^*\). In particular, \(\Lambda\) is a bounded linear functional in \(A^2(\omega)\), and hence, there exists a unique function \(g \in A^2(\omega)\) with \(\Lambda(f) = \langle f, g \rangle_{\omega}\) whenever \(f\) is in \(A^2(\omega)\). Then \(g(z) = \langle g, K_z \rangle_{\omega} = \overline{\Lambda(K_z)}\), and by Lemma 2.2 we get
\[
|g(z)| = |\Lambda(K_z)| \leq \|\Lambda\| \cdot \|K_z\|_{A^p(\omega^{p/2})} \leq \|\Lambda\| \omega(z)^{-1/2} \tau(z)^{-\frac{2(1-p)}{p}},
\]
which shows that \(g\) actually belongs to \(A^{\infty}(v)\) with \(\|g\|_{A^{\infty}(v)} \leq \|\Lambda\|\). It remains to see that the identity \(\Lambda(f) = \langle f, g \rangle_{\omega}\) extends to all \(f \in A^p(\omega^{p/2})\). By the atomic decomposition result in Theorem 1.2 there exists a sequence \(\{z_n\} \subset \mathbb{D}\) such that any \(f \in A^p(\omega^{p/2})\) has the form
\[
f = \sum_n \lambda_n \omega(z_n)^{1/2} \tau(z_n)^{-\frac{2(1-p)}{p}} K_{z_n}^*.
\]
for some sequence $\lambda = \{\lambda_n\} \in \ell^p$, with convergence of the series in $A^p(\omega^{p/2})$. Here, $K^*_z$ is the reproducing kernel of the associated weighted Bergman space $A^2(\omega_*)$ with $\omega_*(z) = \omega(z) \tau(z)^{4(1-p)/p}$. Since, by Corollary 2.4, the reproducing kernel $K^*_zn$ is in $A^2(\omega)$, we have

$$\Lambda(f) = \sum_n \lambda_n \omega(z_n)^{1/2} \tau(z_n)^{2(1-p)/p} \Lambda(K^*_zn)$$

$$= \sum_n \lambda_n \omega(z_n)^{1/2} \tau(z_n)^{2(1-p)/p} \langle K^*_zn, g \rangle_\omega = \langle f, g \rangle_\omega.$$

This finishes the proof. □

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