ON MEAN DIVERGENCE MEASURES

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Abstract. Arithmetic, geometric and harmonic means are the three classical means famous in the literature. Another mean such as square-root mean is also known. In this paper, we have constructed divergence measures based on nonnegative differences among these means, and established an interesting inequality by use of properties of Csiszár’s f-divergence. An improvement over this inequality is also presented. Comparison of new mean divergence measures with classical divergence measures such as J-divergence \cite{10,11}, Jensen-Shannon difference divergence measure \cite{3,15} and arithmetic-geometric mean divergence measure \cite{17} are also established.

1. Generalized Mean of Order $t$

Let us consider the following well-known mean of order $t$

\begin{equation}
M_t(a, b) = \begin{cases} 
\left(\frac{a^t + b^t}{2}\right)^{1/t}, & t \neq 0, \\
\sqrt[2]{ab}, & t = 0, \\
\max\{a, b\}, & t = \infty, \\
\min\{a, b\}, & t = -\infty,
\end{cases}
\end{equation}

for all $a, b \in \mathbb{R}$.

It is also well known (ref. Beckenbach and Bellman \cite{11}) that the $M_t(a, b)$ is monotonically non-decreasing function in relation to $t$. This allow us to conclude the following inequality

\begin{equation}
M_{-\infty}(a, b) \leq M_{-1}(a, b) \leq M_0(a, b) \leq M_1(a, b) \leq M_2(a, b) \leq M_{\infty}(a, b),
\end{equation}

where

\begin{align*}
M_{-1}(a, b) &= H(a, b) = \frac{2ab}{a + b} & \text{--- Harmonic mean;}
M_0(a, b) &= G(a, b) = \sqrt{ab} & \text{--- Geometric mean;}
M_1(a, b) &= A(a, b) = \frac{a + b}{2} & \text{--- Arithmetic mean;}
M_2(a, b) &= S(a, b) = \sqrt{\frac{a^2 + b^2}{2}} & \text{--- Square root mean.}
\end{align*}

In view of this we have the following inequality.

\begin{equation}
H(a, b) \leq G(a, b) \leq A(a, b) \leq S(a, b).
\end{equation}

Recently, author \cite{23} improved the above inequality (1.3). Also see Sándor \cite{14} for different kinds of inequalities among the means.

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Let us consider now the following non-negative differences arising due to inequality (1.3):

\[
M_{SA}(a, b) = S(a, b) - A(a, b) = \sqrt{\frac{a^2 + b^2}{2}} - \frac{a + b}{2},
\]

\[
M_{SG}(a, b) = S(a, b) - G(a, b) = \sqrt{\frac{a^2 + b^2}{2}} - \sqrt{ab},
\]

\[
M_{SH}(a, b) = S(a, b) - H(a, b) = \sqrt{\frac{a^2 + b^2}{2}} - \frac{2ab}{a + b},
\]

\[
M_{AH}(a, b) = A(a, b) - H(a, b) = \frac{a + b}{2} - \frac{2ab}{a + b},
\]

\[
M_{AG}(a, b) = A(a, b) - G(a, b) = \frac{a + b}{2} - \sqrt{ab}
\]

and

\[
M_{GH}(a, b) = G(a, b) - H(a, b) = \sqrt{ab} - \frac{2ab}{a + b}.
\]

In view of (1.2), we have the following inequalities among the mean difference measures:

(1.4) \[0 \leq M_{SA}(a, b) \leq M_{SG}(a, b) \leq M_{SH}(a, b)\]

and

(1.5) \[0 \leq M_{AG}(a, b) \leq M_{AH}(a, b)\]

2. **Mean Difference Divergence Measures**

Let

\[
\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) \mid p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n \geq 2,
\]

be the set of all complete finite discrete probability distributions.

Let us take \(a = p_i\) and \(b = q_i\) in the differences given above and sum over all \(i = 1, 2, ..., n\), then for all \(P, Q \in \Gamma_n\), we have the following mean divergence measures:

- **Square root - arithmetic mean divergence**

\[
M_{SA}(P||Q) = \sum_{i=1}^{n} \left( \sqrt{\frac{p_i^2 + q_i^2}{2}} - 1 \right).
\]

- **Square root - geometric mean divergence**

\[
M_{SG}(P||Q) = \sum_{i=1}^{n} \left( \sqrt{\frac{p_i^2 + q_i^2}{2}} - \sqrt{p_i q_i} \right).
\]

- **Square root - harmonic mean divergence**

\[
M_{SH}(P||Q) = \sum_{i=1}^{n} \left( \sqrt{\frac{p_i^2 + q_i^2}{2}} - \frac{2p_i q_i}{p_i + q_i} \right).
\]
• Arithmetic – geometric mean divergence

\[ M_{AG}(P||Q) = 1 - \sum_{i=1}^{n} \sqrt{p_i q_i}. \]

• Arithmetic – harmonic mean divergence

\[ M_{AH}(P||Q) = 1 - \sum_{i=1}^{n} \frac{2p_i q_i}{p_i + q_i}. \]

• Geometric – harmonic mean divergence

\[ M_{GH}(P||Q) = \sum_{i=1}^{n} \left( \sqrt{p_i q_i} - \frac{2p_i q_i}{p_i + q_i} \right) = \sum_{i=1}^{n} \frac{\sqrt{p_i q_i} (\sqrt{p_i} - \sqrt{q_i})^2}{p_i + q_i}. \]

After simplification, we can write

\[ M_{AG}(P||Q) = 1 - B(P||Q) = h(P||Q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2. \]

where \( B(P||Q) \) is the Bhattacharyya distance and \( h(P||Q) \) is the well known Hellinger discrimination.

Also we can write

\[ M_{AH}(P||Q) = 1 - W(P||Q) = \frac{1}{2} \Delta(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{2(p_i + q_i)}, \]

where \( W(P||Q) \) is the harmonic mean divergence and \( \Delta(P||Q) \) is the well known triangular discrimination.

Some studies on square root - arithmetic mean divergence can be seen in Österreicher and Vajda and Dragomir et al.

In view of (1.4) and (1.5), we have the following inequalities

(2.1) \[ 0 \leq M_{SA}(P||Q) \leq M_{SG}(P||Q) \leq M_{SH}(P||Q) \]

and

(2.2) \[ 0 \leq h(P||Q) \leq \frac{1}{2} \Delta(P||Q). \]

In this paper our aim is to obtain an inequality relating the mean divergence measures given above. This shall be done by use of Csiszár’s \( f \)-divergence.

3. Csiszár’s \( f \)-Divergence and Mean Divergence Measures

Given a convex function \( f : (0, \infty) \to \mathbb{R} \), the \( f \)-divergence measure introduced by Csiszár is given by

(3.1) \[ C_f(P||Q) = \sum_{i=1}^{n} q_i f \left( \frac{p_i}{q_i} \right), \]

for all \( P,Q \in \Gamma_n \).

The following theorem is well known in the literature:
Property 3.1. Let the function $f : [0, \infty) \to \mathbb{R}$ be differentiable convex and normalized, i.e., $f(1) = 0$, then the Csiszár $f$–divergence, $C_f(P||Q)$ is nonnegative and convex in the pair of probability distribution $(P, Q) \in \Gamma_n \times \Gamma_n$.

The mean divergence measures given in Section 2 can be written as examples of (3.1) and applying property 3.1, we can check the nonnegativity and convexity of these measures. Here below we shall give these as examples.

Example 3.1. Let us consider

$$f_{SA}(x) = \sqrt{\frac{x^2 + 1}{2}} - \frac{x + 1}{2}, \quad x \in (0, \infty),$$

in (3.1), then we have $C_f(P||Q) = M_{SA}(P||Q)$.

Moreover,

$$f'_{SA}(x) = \frac{x}{\sqrt{2\sqrt{x^2 + 1}} - \frac{1}{2}},$$

and

$$f''_{SA}(x) = \frac{1}{\sqrt{2(x^2 + 1)\sqrt{x^2 + 1}}}.$$

Thus we have $f''_{SA}(x) > 0$ for all $x \in (0, \infty)$. Also, we have $f_{SA}(1) = 0$. In view of this we can say that the square root–geometric mean divergence is nonnegative and convex in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$.

Example 3.2. Let us consider

$$f_{SG}(x) = \sqrt{\frac{x^2 + 1}{2}} - \sqrt{x}, \quad x \in (0, \infty),$$

in (3.1), then we have $C_f(P||Q) = M_{SG}(P||Q)$.

Moreover,

$$f'_{SG}(x) = \frac{1}{\sqrt{2}} \left( \frac{x}{\sqrt{x^2 + 1}} - \frac{1}{\sqrt{x}} \right),$$

and

$$f''_{SG}(x) = \frac{1}{\sqrt{2(x^2 + 1)\sqrt{x^2 + 1}}} + \frac{1}{4x\sqrt{x}}.$$

Thus we have $f''_{SG}(x) > 0$ for all $x \in (0, \infty)$. Also, we have $f_{SG}(1) = 0$. In view of this we can say that the square root–geometric mean divergence is nonnegative and convex in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$.

Example 3.3. Let us consider

$$f_{SH}(x) = \sqrt{\frac{x^2 + 1}{2}} - \frac{2x}{x + 1}, \quad x \in (0, \infty),$$

in (3.1), then we have $C_f(P||Q) = M_{SH}(P||Q)$.

Moreover,

$$f'_{SH}(x) = \frac{x}{\sqrt{2\sqrt{x^2 + 1}}} - \frac{2}{(x + 1)^2},$$

and

$$f''_{SH}(x) = \frac{1}{\sqrt{2(x + 1)\sqrt{x^2 + 1}}} + \frac{4}{(x + 1)^3}.$$

Thus we have $f''_{SH}(x) > 0$ for all $x \in (0, \infty)$. Also, we have $f_{SH}(1) = 0$. In view of this we can say that the square root–geometric mean divergence is nonnegative and convex in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$. 
Example 3.4. Let us consider

\[ f_h(x) = \frac{1}{2}(\sqrt{x} - 1)^2, \ x \in (0, \infty), \]

in (3.1), then we have \( C_f(P||Q) = h(P||Q) \).

Moreover,

\[ f'_h(x) = \frac{\sqrt{x} - 1}{2\sqrt{x}}, \]

and

\[ f''_h(x) = \frac{1}{4x\sqrt{x}}. \]

Thus we have \( f''_h(x) > 0 \) for all \( x \in (0, \infty) \). Also, we have \( f_h(1) = 0 \). In view of this we can say that the square root – geometric mean divergence is nonnegative and convex in the pair of probability distributions \((P, Q) \in \Gamma_n \times \Gamma_n\).

Example 3.5. Let us consider

\[ f_\Delta(x) = \frac{(x - 1)^2}{x + 1}, \ x \in (0, \infty), \]

in (2.1), then we have \( C_f(P||Q) = \Delta(P||Q) \).

Moreover,

\[ f'_\Delta(x) = \frac{(x - 1)(x + 3)}{(x + 1)^2}, \]

and

\[ f''_\Delta(x) = \frac{8}{(x + 1)^3}. \]

Thus we have \( f''_\Delta(x) > 0 \) for all \( x \in (0, \infty) \). Also, we have \( f_\Delta(1) = 0 \). In view of this we can say that the square root – geometric mean divergence is nonnegative and convex in the pair of probability distributions \((P, Q) \in \Gamma_n \times \Gamma_n\).

4. Bounds on Mean Divergence Measures

In this section we shall give bounds on the measures given in Section 2. In order to get these bounds we shall make use of the properties of Csiszár’s \( f \)-divergence due to Dragomir [6].

Property 4.1. Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) be differentiable convex and normalized i.e., \( f(1) = 0 \). If \( P, Q \in \Gamma_n \), then we have

\[ 0 \leq C_f(P||Q) \leq E_{C_f}(P||Q), \tag{4.1} \]

where

\[ E_{C_f}(P||Q) = \sum_{i=1}^{n} (p_i - q_i) f'(\frac{p_i}{q_i}). \]
4.1. Square root – arithmetic mean divergence. We have

\[ 0 \leq M_{SA}(P\parallel Q) \leq E_{SA}(P\parallel Q), \]

where

\[
E_{SA}(P\parallel Q) = \sum_{i=1}^{n} (p_i - q_i) \left( \frac{p_i}{\sqrt{2(p_i^2 + q_i^2)}} \right)
\]

\[
= \sum_{i=1}^{n} \frac{p_i^2 - p_i q_i + q_i^2 - q_i^2}{\sqrt{2(p_i^2 + q_i^2)}}
\]

\[
= M_{SA}(P\parallel Q) + 1 - \sum_{i=1}^{n} \frac{q_i(p_i + q_i)}{\sqrt{2(p_i^2 + q_i^2)}}
\]

\[
= M_{SA}(P\parallel Q) + \sum_{i=1}^{n} q_i \left[ \frac{\sqrt{2(p_i^2 + q_i^2)} - (p_i + q_i)}{\sqrt{2(p_i^2 + q_i^2)}} \right]
\]

\[
= M_{SA}(P\parallel Q) + \xi_{SA}(P\parallel Q),
\]

with

\[
\xi_{SA}(P\parallel Q) = \sum_{i=1}^{n} \frac{\sqrt{2q_i}}{\sqrt{p_i^2 + q_i^2}} \left[ \sqrt{\frac{p_i^2 + q_i^2}{2} - \frac{p_i + q_i}{2}} \right].
\]

In view of (4.1), we can say that \( \xi_{SA}(P\parallel Q) \geq 0 \).

4.2. Square root – geometric mean divergence. We have

\[ 0 \leq M_{SG}(P\parallel Q) \leq E_{SG}(P\parallel Q), \]

where

\[
E_{SG}(P\parallel Q) = \sum_{i=1}^{n} (p_i - q_i) \left( \frac{p_i}{\sqrt{2(p_i^2 + q_i^2)}} - \frac{\sqrt{q_i}}{2\sqrt{p_i}} \right)
\]

\[
= \sum_{i=1}^{n} \frac{(p_i^2 - p_i q_i + q_i^2 - q_i^2)}{\sqrt{2(p_i^2 + q_i^2)}} - \frac{\sqrt{q_i}(p_i - q_i)}{2\sqrt{p_i}}
\]

\[
= M_{SG}(P\parallel Q) + \sum_{i=1}^{n} \left( \frac{\sqrt{q_i}(p_i + q_i)}{\sqrt{p_i}} - \frac{q_i(p_i + q_i)}{\sqrt{2(p_i^2 + q_i^2)}} \right)
\]

\[
= M_{SG}(P\parallel Q) + \sum_{i=1}^{n} \frac{\sqrt{q_i}(p_i + q_i)}{\sqrt{2(p_i^2 + q_i^2)}} \left( \frac{\sqrt{p_i^2 + q_i^2} - \sqrt{2p_i q_i}}{2\sqrt{p_i}} \right)
\]

\[
= M_{SG}(P\parallel Q) + \sum_{i=1}^{n} \frac{\sqrt{q_i}(p_i + q_i)}{\sqrt{2p_i\sqrt{p_i^2 + q_i^2}} + 2q_i} \left( \frac{\sqrt{p_i^2 + q_i^2} - \sqrt{2p_i q_i}}{2\sqrt{p_i}} \right)
\]

\[
= M_{SG}(P\parallel Q) + \xi_{SG}(P\parallel Q),
\]

with

\[
\xi_{SG}(P\parallel Q) = \sum_{i=1}^{n} (p_i + q_i) \sqrt{\frac{q_i}{2p_i(p_i^2 + q_i^2)}} \left( \sqrt{\frac{p_i^2 + q_i^2}{2} - p_i q_i} \right).
\]

In view of (4.1), we can say that \( \xi_{SG}(P\parallel Q) \geq 0 \).
4.3. Square root – harmonic mean divergence. We have

\[ 0 \leq M_{SH}(P||Q) \leq E_{SH}(P||Q), \]

where

\[
E_{SH}(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \left( \frac{p_i}{\sqrt{2(p_i^2 + q_i^2)}} - \frac{2q_i^2}{(p_i + q_i)^2} \right)
\]

\[
= \sum_{i=1}^{n} \left( \frac{p_i^2 - p_iq_i + q_i^2 - q_i^2}{\sqrt{2(p_i^2 + q_i^2)}} - \frac{2q_i^2(p_i - q_i)}{(p_i^2 + q_i^2)^2} \right)
\]

\[
= M_{SH}(P||Q) + \sum_{i=1}^{n} \left( \frac{2p_iq_i(p_i + q_i)}{(p_i + q_i)^2} - \frac{q_i(p_i + q_i)}{\sqrt{2(p_i^2 + q_i^2)}} \right)
\]

\[
= M_{SH}(P||Q) + \sum_{i=1}^{n} q_i \left[ \frac{2(p_i^2 + q_i^2)}{(p_i + q_i)^2} - \frac{(p_i + q_i)}{\sqrt{2(p_i^2 + q_i^2)}} \right]
\]

\[
= M_{SH}(P||Q) + \sum_{i=1}^{n} q_i \left[ \frac{\sqrt{2(p_i^2 + q_i^2)}}{(p_i + q_i)^2 \sqrt{2(p_i^2 + q_i^2)}} \right] - (p_i + q_i)^3
\]

\[
= M_{SH}(P||Q) + \xi_{SH}(P||Q),
\]

with

\[
\xi_{SH}(P||Q) = \sum_{i=1}^{n} q_i \left[ \frac{\sqrt{2(p_i^2 + q_i^2)}}{(p_i + q_i)^2 \sqrt{2(p_i^2 + q_i^2)}} \right].
\]

In view of (4.1), we can say that \( \xi_{SH}(P||Q) \geq 0. \)

4.4. Hellinger discrimination. We have

\[ 0 \leq M_{h}(P||Q) \leq E_{h}(P||Q), \]

where

\[
E_{h}(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i) (\sqrt{p_i} - \sqrt{q_i})}{2 \sqrt{p_i}}
\]

\[
= \sum_{i=1}^{n} \frac{(\sqrt{p_i} - \sqrt{q_i})^2 (\sqrt{p_i} + \sqrt{q_i})}{2 \sqrt{p_i}}
\]

\[
= M_{h}(P||Q) + \frac{1}{2} \sum_{i=1}^{n} \frac{q_i}{p_i} (\sqrt{p_i} - \sqrt{q_i})^2
\]

\[
= M_{h}(P||Q) + \xi_{h}(P||Q),
\]

with

\[
\xi_{h}(P||Q) = \frac{1}{2} \sum_{i=1}^{n} \frac{q_i}{p_i} (\sqrt{p_i} - \sqrt{q_i})^2.
\]

Obviously, \( \xi_{h}(P||Q) \geq 0. \)
4.5. Triangular discrimination. We have
\[ 0 \leq M_\Delta(P||Q) \leq E_\Delta(P||Q), \]
where
\[
E_\Delta(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2(p_i + 3q_i)}{(p_i + q_i)^2}
\]
\[ = \Delta(P||Q) + 2 \sum_{i=1}^{n} q_i \left( \frac{p_i - q_i}{p_i + q_i} \right)^2 \]
\[ = M_\Delta(P||Q) + \xi_\Delta(P||Q), \]
with
\[
\xi_\Delta(P||Q) = 2 \sum_{i=1}^{n} q_i \left( \frac{p_i - q_i}{p_i + q_i} \right)^2.
\]
Obviously, \( \xi_\Delta(P||Q) \geq 0. \)

5. Inequalities among Mean Divergence Measures

In this section we shall obtain inequalities among the measures given in Section 2.

Property 5.1. Let \( f_1, f_2 : I \subset \mathbb{R}_+ \to \mathbb{R} \) be two convex mappings that are normalized, i.e., \( f_1(1) = f_2(1) = 0 \) and suppose the assumptions:
(i) \( f_1 \) and \( f_2 \) are twice differentiable on \((a, b)\);
(ii) there exists the real constants \( \alpha, \beta \) such that \( \alpha < \beta \) and
\[
\alpha \leq \frac{f_1''(x)}{f_2''(x)} \leq \beta, \quad f_2''(x) > 0, \quad \forall x \in (a, b).
\]
Then,
\[
\alpha C_{f_2}(P||Q) \leq C_{f_1}(P||Q) \leq \beta C_{f_2}(P||Q),
\]
and
\[
\alpha [E_{f_2}(P||Q) - C_{f_2}(P||Q)] \leq E_{f_1}(P||Q) - C_{f_1}(P||Q) \leq \beta [E_{f_2}(P||Q) - C_{f_2}(P||Q)]
\]
Proof. Let us consider the functions
\[
p_\alpha(x) = f_1(x) - \alpha f_2(x)
\]
and
\[
p_\beta(x) = \beta f_2(x) - f_1(x),
\]
where \( \alpha \) and \( \beta \) are as given by (5.1).
Since \( f_1(x) \) and \( f_2(x) \) are normalized, i.e., \( f_1(1) = f_2(1) = 0 \), then \( p_\alpha(1) = p_\beta(1) = 0 \). Also, the functions \( f_1(x) \) and \( f_2(x) \) are twice differentiable. Then in view of (5.1), we have
\[
p''_\alpha(x) = f_1''(x) - \alpha f_2''(x) = f_2''(x) \left( \frac{f_1''(x)}{f_2''(x)} - \alpha \right) \geq 0
\]
and
\[
p''_\beta(x) = \beta f_2''(x) - f_1''(x) = f_2''(x) \left( \beta - \frac{f_1''(x)}{f_2''(x)} \right) \geq 0,
\]
for all \( x \in (a, b) \).
In view of (5.8) and (5.9), we can say that the functions \( p_\alpha(\cdot) \) and \( p_\beta(\cdot) \) are convex on \((a, b)\).

According to Property 3.1, we have
\[
C_{p_\alpha}(P||Q) = C_{f_1 - \alpha f_2}(P||Q) = C_{f_1}(P||Q) - \alpha C_{f_2}(P||Q) \geq 0,
\]
and
\[
C_{q_\beta}(P||Q) = C_{\beta f_2 - f_1}(P||Q) = \beta C_{f_2}(P||Q) - C_{f_1}(P||Q) \geq 0.
\]

Combining (5.8) and (5.9) we have the proof of (5.2).

Now, we shall prove the inequalities (5.3). We have seen above that the real mappings \( p_\alpha(\cdot) \) and \( p_\beta(\cdot) \) defined over \( \mathbb{R}_+ \) are normalized, twice differentiable and convex on \((a, b)\). Applying the r.h.s. of the inequalities (5.2), we have
\[
C_{p_\alpha}(P||Q) \leq E_{C_{p_\alpha}}(P||Q)
\]
and
\[
C_{q_\beta}(P||Q) \leq E_{C_{p_\beta}}(P||Q).
\]
Moreover,
\[
C_{p_\alpha}(P||Q) = C_{f_1}(P||Q) - \alpha C_{f_2}(P||Q)
\]
and
\[
C_{p_\beta}(P||Q) = \beta C_{f_2}(P||Q) - C_{f_1}(P||Q).
\]

In view of (5.10) and (5.12), we have
\[
C_{f_1}(P||Q) - \alpha C_{f_2}(P||Q) \leq E_{C_{f_1} - \alpha f_2}(P||Q) = E_{f_1}(P||Q) - \alpha E_{f_2}(P||Q).
\]
This gives,
\[
\alpha \left[ E_{C_{f_2}}(P||Q) - C_{f_2}(P||Q) \right] \leq E_{C_{f_1}}(P||Q) - C_{f_1}(P||Q).
\]
Thus, we have the l.h.s. of the inequalities (5.3).
Again in view of (5.11) and (5.13), we have
\[
\beta C_{f_2}(P||Q) - C_{f_1}(P||Q) \leq E_{C_{f_2} - f_1}(P||Q) = \beta E_{C_{f_2}}(P||Q) - E_{C_{f_1}}(P||Q).
\]
After simplifying, we get
\[
E_{f_1}(P||Q) - C_{f_1}(P||Q) \leq \beta [E_{f_2}(P||Q) - C_{f_2}(P||Q)].
\]
Thus we have the r.h.s. of the inequalities (5.3). This completes the proof of the property. \(\square\)

Theorem 5.1. The following inequalities among the mean difference and auxiliary divergences hold:
\[
M_{SA}(P||Q) \leq \frac{1}{3} M_{SH}(P||Q) \leq \frac{1}{4} \Delta(P||Q) \leq \frac{1}{2} M_{SG}(P||Q) \leq h(P||Q),
\]
and
\[
\xi_{SA}(P||Q) \leq \frac{1}{3} \xi_{SH}(P||Q) \leq \frac{1}{4} \xi_{\Delta}(P||Q) \leq \frac{1}{2} \xi_{SG}(P||Q) \leq \xi_h(P||Q).
\]
The proof is based on the following propositions.

**Proposition 5.1.** The following inequalities hold:

\[(5.16) \quad 0 \leq M_{SA}(P||Q) \leq \frac{1}{3} M_{SH}(P||Q), \]

and

\[(5.17) \quad 0 \leq \xi_{SA}(P||Q) \leq \frac{1}{3} \xi_{SH}(P||Q). \]

**Proof.** Let us consider

\[ g_{SA,SH}(x) = \frac{f''_{SA}(x)}{f''_{SH}(x)} = \frac{(x + 1)^3}{(x + 1)^3 + 4\sqrt{2}(x^2 + 1)^{3/2}}, \quad x \in (0, \infty). \]

This gives

\[(5.18) \quad g'_{SA,SH}(x) = -\frac{24(x - 1)(x^2 + 1)(x + 1)^2}{\sqrt{2}(x^2 + 1) [(x + 1)^3 + 4\sqrt{2}(x^2 + 1)^{3/2}]^2} \begin{cases} \geq 0, & x \leq 1 \\ \leq 0, & x \geq 1 \end{cases}. \]

In view of (5.18), we conclude that the function \( g_{SA,SH}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence

\[(5.19) \quad \beta = \sup_{x \in (0, \infty)} g_{SA,SH}(x) = g_{SA,SH}(1) = \frac{1}{3}. \]

Now (5.19) together with (5.2) and (5.3) give respectively (5.16) and (5.17). \( \square \)

**Proposition 5.2.** The following inequalities hold:

\[(5.20) \quad 0 \leq M_{SA}(P||Q) \leq \frac{1}{4} \Delta(P||Q), \]

and

\[(5.21) \quad 0 \leq \xi_{SA}(P||Q) \leq \frac{1}{4} \xi_{\Delta}(P||Q). \]

**Proof.** Let us consider

\[ g_{SA,\Delta}(x) = \frac{f''_{SA}(x)}{f''_{\Delta}(x)} = \frac{(x + 1)^3}{8\sqrt{2}(x^2 + 1)^{5/2}}, \quad x \in (0, \infty), \]

This gives

\[(5.22) \quad g'_{SA,\Delta}(x) = -\frac{3(x - 1)(x + 1)^2}{8\sqrt{2}(x^2 + 1)^{5/2}} \begin{cases} \geq 0, & x \leq 1 \\ \leq 0, & x \geq 1 \end{cases}. \]

In view of (5.22), we conclude that the function \( g_{SA,\Delta}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence

\[(5.23) \quad M = \sup_{x \in (0, \infty)} g_{SA,\Delta}(x) = g_{SA,\Delta}(1) = \frac{1}{4}. \]

Now (5.23) together with (5.2) and (5.3) give respectively (5.20) and (5.21). \( \square \)

**Proposition 5.3.** The following inequalities hold:

\[(5.24) \quad 0 \leq \frac{1}{2} \Delta(P||Q) \leq M_{SG}(P||Q), \]

and

\[(5.25) \quad 0 \leq \frac{1}{2} \xi_{\Delta}(P||Q) \leq \xi_{SG}(P||Q). \]
Proof. Let us consider
\[ g_{SG,\Delta}(x) = \frac{f''_{SG}(x)}{f''_{H}(x)} = \frac{(x + 1)^3 \left[ 4x^{3/2} + \sqrt{2} (x^2 + 1)^{3/2} \right]}{32 \sqrt{2} (x^2 + 1)^{3/2} x^{3/2}}, \quad x \in (0, \infty), \]

This gives
\[ g'_{SG,\Delta}(x) = \frac{3(x + 1)^4 (x - 1) \left[ \sqrt{2} (x^2 + 1)^{5/2} - 8x^{5/2} \right]}{64 \sqrt{2} [x(x + 1)]^{5/2}}. \]

Since \( x^2 + 1 \geq 2x \), then from (5.30), we conclude that
\[ g'_{SG,\Delta}(x) \begin{cases} \geq 0, & x \geq 1 \\ \leq 0, & x \leq 1 \end{cases}. \]

In view of (5.27), we conclude that the function \( g_{SG,\Delta}(x) \) is decreasing in \( x \in (0, 1) \) and increasing in \( x \in (1, \infty) \), and hence
\[ \alpha = \inf_{x \in (0, \infty)} g_{SG,\Delta}(x) = \min_{x \in (0, \infty)} g_{SG,\Delta}(x) = \frac{1}{2}. \]

Now (5.28) together with (5.2) and (5.3) give respectively (5.24) and (5.25). \( \square \)

**Proposition 5.4.** We have the following bounds:
\[ 0 \leq M_{SG}(P||Q) \leq 2 h(P||Q), \]
and
\[ 0 \leq \xi_{SG}(P||Q) \leq 2 \xi_{H}(P||Q). \]

Proof. Let us consider
\[ g_{SG,\Delta}(x) = \frac{f''_{SG}(x)}{f''_{H}(x)} = \frac{4x^{3/2} + \sqrt{2} (x^2 + 1)^{3/2}}{\sqrt{2} (x^2 + 1)^{3/2}}, \quad x \in (0, \infty). \]

This gives
\[ g'_{SG,\Delta}(x) = -\frac{6(x-1)(x+1)\sqrt{x}}{\sqrt{2}(x^2 + 1)^{5/2}} \begin{cases} \geq 0, & x \leq 1 \\ \leq 0, & x \geq 1 \end{cases}. \]

In view of (5.31), we conclude that the function \( g_{SG,\Delta}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence
\[ \beta = \sup_{x \in (0, \infty)} g_{SG,\Delta}(x) = g_{SG,\Delta}(1) = 2. \]

Now (5.32) together with (5.2) and (5.3) give respectively (5.29) and (5.30). \( \square \)

The inequalities (5.16), (5.20), (5.24) and (5.29) together give (5.14) and the inequalities (5.17), (5.21), (5.25) and (5.30) together give (5.15). This completes the proof of the Theorem 5.1.

**Remark 5.1.**
(i) The divergence measure arising due to geometric–harmonic mean is not studied here because it is not convex.

(ii) The auxiliary measures \( \xi_{(\cdot)}(P||Q) \) can be written in terms of Csiszár f-divergence, but they are not necessarily convex.
6. COMPARISON WITH CLASSICAL DIVERGENCE MEASURES

In this section, we shall present some classical divergence measures. The following Jensen-Shannon divergence measure \[3, 15\] is already known in the literature:

\[
I(P||Q) = \sum_{i=1}^{n} [A(p_i \ln p_i, q_i \ln q_i) - A(p_i, q_i) \ln A(p_i, q_i)].
\]

Taneja [17] presented the following arithmetic and geometric divergence measure arising due to arithmetic and geometric means:

\[
T(P||Q) = \sum_{i=1}^{n} A(p_i, q_i) \ln \frac{A(p_i, q_i)}{G(p_i, q_i)}.
\]

Adding (6.1) and (6.2), we get

\[
I(P||Q) + T(P||Q) = 4J(P||Q),
\]

where \(J(P||Q)\) is the well known Jeffrey-Kullback-Leibler \([11, 10]\) J-divergence given by

\[
J(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln \frac{p_i}{q_i}.
\]

For more studies on the measures (6.2)-(6.4) with their generalizations and some statistical applications refer to Taneja [16, 18, 21, 22]. For new symmetric divergence measure refer to Kumar and Chhina [12].

Recently, author [19, 20] proved an inequality among these divergence measures given by

\[
\frac{1}{4} \Delta(P||Q) \leq I(P||Q) \leq h(P||Q) \leq \frac{1}{8} J(P||Q) \leq T(P||Q).
\]

Finally, combining the inequalities (5.14) and (6.5), we have the following interesting inequalities:

\[
M_{SA}(P||Q) \leq \frac{1}{3} M_{SH}(P||Q) \leq \frac{1}{4} \Delta(P||Q) \leq \frac{1}{2} M_{SG}(P||Q) \leq h(P||Q) \leq \frac{1}{8} J(P||Q) \leq T(P||Q),
\]

and

\[
M_{SA}(P||Q) \leq \frac{1}{3} M_{SH}(P||Q) \leq \frac{1}{4} \Delta(P||Q) \leq I(P||Q) \leq h(P||Q) \leq \frac{1}{8} J(P||Q) \leq T(P||Q).
\]

From the inequalities (5.14), (6.6) and (6.7), we observe that we don’t have relation among the measures \(SG–\text{divergence}\) and \(I–\text{divergence}\). Let us check this by applying Property 5.1.

Let us consider

\[
f_I(x) = \frac{x}{2} \ln x + \frac{x + 1}{2} \ln \left(\frac{2}{x + 1}\right), \quad x \in (0, \infty)
\]

in (3.1), the one gets \(C_f(P||Q) = I(P||Q)\).
Moreover,  

\[ f_1'(x) = \frac{1}{2} \ln \left( \frac{2x}{x+1} \right) \]

and  

\[ f_1''(x) = \frac{1}{2x(x+1)}. \]

Again, let us consider  

\[ g_{SG,x}(x) = \frac{f''_{SG}(x)}{f'_1(x)} = \frac{\left[ 4x^{3/2} + \sqrt{2} \left( x^2 + 1 \right)^{3/2} \right] x(x+1)}{2\sqrt{2} \left( x^2 + 1 \right)^{3/2}x^{3/2}}, \quad x \in (0, \infty). \]

The first order derivative of the function \( g_{SG,x}(x) \) is given by  

\[ g'_{SG,x}(x) = \frac{(x-1)\sigma(x)}{4\sqrt{2} \left( x^2 + 1 \right)^{5/2}x^{3/2}}, \]

where  

\[ \sigma(x) = \sqrt{2} \left( x^2 + 1 \right)^{5/2} - 8x^{3/2} \left( x^2 + 3x + 1 \right). \]

In order to apply Property 5.1 we must prove that \( \sigma(x) \) is either negative or positive for \( x \in (0, \infty) \), but \( \sigma(1) = -32.0 \) and \( \sigma(4.25) = 13.87 \). This implies that we are unable to apply the Property 5.1.

Moreover, if we check the generating functions in both the cases, still the result don’t hold. Let us denote, \( a(x) = f_{SA}(x) \), \( b(x) = \frac{1}{3}f_{SH}(x) \), \( c(x) = \frac{1}{4}f_{\Delta}(x) \), \( d(x) = \frac{1}{4}f_{SG}(x) \), \( e(x) = f_I(x) \) and \( f(x) = f_h(x) \) for all \( x \in (0, \infty) \). Then we have the following values of these two functions:

| \( x \) | 0.1    | 10     | 1000   | 3000   | 3800   | 3900   |
|--------|--------|--------|--------|--------|--------|--------|
| \( a(x) \) | 0.1606 | 1.6063 | 206.6071 | 620.8204 | 786.5058 | 807.2165 |
| \( b(x) \) | 0.1762 | 1.7627 | 235.0363 | 706.4403 | 895.0021 | 918.5723 |
| \( c(x) \) | 0.1840 | 1.8409 | 249.2509 | 749.2503 | 949.2502 | 974.2502 |
| \( d(x) \) | 0.1972 | 1.9720 | 337.7421 | 1033.2741 | 1312.6808 | 1347.6332 |
| \( e(x) \) | 0.2136 | 2.1368 | 342.9660 | 1035.5640 | 1312.7047 | 1347.3491 |
| \( f(x) \) | 0.2337 | 2.3377 | 468.8772 | 1445.7277 | 1838.8558 | 1888.0500 |

We observe from the above table that the values of \( d(x) \) and \( e(x) \) changes in the interval \( x \in [3800, 3900] \), before it \( d(x) \) is always smaller than \( e(x) \).

Let check the same thing by considering particular values of the probability distributions. Let us consider \( n = 2 \), \( p_1 = t \), \( q_1 = 1 - t \), \( p_2 = 1 - t \) and \( q_2 = t \). Then we can write

\[ a(t) = M_{SA}(P||Q) = 2\sqrt{\frac{t^2 + (1-t)^2}{2}} - 1, \]

\[ b(t) = \frac{1}{3} M_{SH}(P||Q) = \frac{2}{3} \sqrt{\frac{t^2 + (1-t)^2}{2}} - \frac{4}{3} t(1-t), \]

\[ c(t) = \frac{1}{4} \Delta(P||Q) = \frac{1}{2} (2t - 1)^2, \]

\[ d(t) = \frac{1}{2} M_{SG}(P||Q) = \sqrt{\frac{t^2 + (1-t)^2}{2}} - \sqrt{t(1-t)}, \]

\[ e(t) = I(P||Q) = t \ln(2t) + (1 - t) \log(2 - 2t) \]
and
\[ f(t) = h(P||Q) = \left( \sqrt{t} - \sqrt{1-t} \right)^2, \]
for all \( t \in [0, 1] \) with the convention that \( 0 \log 0 = 0. \)

Let us compare the measures for some particular values of \( t \).

| \( t \) | 0.0001 | 0.001 | 0.01 | 0.1 | 0.2 | 0.4 |
|------|--------|------|-----|----|----|----|
| \( a(t) \) | 0.4140 | 0.4128 | 0.4001 | 0.2806 | 0.1662 | 0.01980 |
| \( b(t) \) | 0.4712 | 0.4696 | 0.4535 | 0.3068 | 0.1754 | 0.01993 |
| \( c(t) \) | 0.4998 | 0.4980 | 0.4802 | 0.3200 | 0.1800 | 0.02000 |
| \( d(t) \) | 0.6970 | 0.6747 | 0.6005 | 0.3403 | 0.1830 | 0.02004 |
| \( e(t) \) | 0.6921 | 0.6852 | 0.6371 | 0.3680 | 0.1927 | 0.02013 |
| \( f(t) \) | 0.9800 | 0.9367 | 0.8010 | 0.4000 | 0.2000 | 0.02020 |

Here we have considered only the values of \( t \in (0, 1/2] \), since for \( t \in [1/2, 1) \) the values are symmetric. Moreover, all values are zero for \( t = \frac{1}{2} \). From the table we observe that for each \( t \) fixed the values of the functions are monotonically increasing, except for \( t = 0.0001 \), \( d(t) \) is bigger than \( e(t) \).

From the example above we conclude that we are unable to establish an inequality having nine measures in a sequence combining (6.6) and (6.7).

7. **Refinement Inequalities**

Now we shall improve the inequality (5.14). In order to do so, we shall again consider the following *non-negative differences*:

\[ D_{f_k}(P||Q) = \sum_{i=1}^{n} q_i f_k \left( \frac{p_i}{q_i} \right), \quad k = 1, 2, ..., 10, \]

where, for all \( x \in (0, \infty) \), we have

\begin{align*}
    f_1(x) &= f_{AG}(x) - \frac{1}{2} f_{SG}(x), \\
    f_2(x) &= f_{AG}(x) - \frac{1}{2} f_{AH}(x), \\
    f_3(x) &= f_{AG}(x) - \frac{1}{3} f_{SH}(x), \\
    f_4(x) &= f_{AG}(x) - f_{SA}(x), \\
    f_5(x) &= \frac{1}{2} f_{SG}(x) - \frac{1}{2} f_{AH}(x), \\
    f_6(x) &= \frac{1}{2} f_{SG}(x) - \frac{1}{3} f_{SH}(x), \\
    f_7(x) &= \frac{1}{2} f_{SG}(x) - f_{SA}(x), \\
    f_8(x) &= \frac{1}{2} f_{AH}(x) - \frac{1}{3} f_{SH}(x), \\
    f_9(x) &= \frac{1}{2} f_{AH}(x) - f_{SA}(x),
\end{align*}
and
\[ f_{10}(x) = \frac{1}{3} f_{SH}(x) - f_{SA}(x). \]

We can easily verify that
\[ f_1(x) = \frac{1}{2} f_4(x) = f_7(x), \]
and
\[ f_8(x) = \frac{1}{3} f_9(x) = \frac{1}{2} f_{10}(x). \]

For all \( x \in (0, \infty) \), we can write
\[
\begin{align*}
    f_1(x) &= \left( \frac{\sqrt{x} - 1}{2} \right)^2 - \frac{\sqrt{2(x^2 + 1)}}{4} - 2\sqrt{x} = A - \left( \frac{G + S}{2} \right), \\
    f_2(x) &= \left( \frac{\sqrt{x} - 1}{2} \right)^2 - \frac{(x - 1)^2}{4(x + 1)} = \left( \frac{A + H}{2} \right) - G, \\
    f_3(x) &= \left( \frac{\sqrt{x} - 1}{2} \right)^2 - \frac{\sqrt{2(x^2 + 1)}}{6} + \frac{2x}{2(x + 1)} = \frac{1}{3} [3A + H - (S + 3G)], \\
    f_5(x) &= \frac{\sqrt{2(x^2 + 1)}}{2} - \sqrt{x} - \frac{(x - 1)^2}{2(x + 1)} = \frac{1}{2} [S + H - (A + G)], \\
    f_6(x) &= \frac{\sqrt{2(x^2 + 1)}}{12} - \frac{\sqrt{x}}{2} + \frac{2x}{3(x + 1)} = \frac{1}{6} [S + 2H - 3G], \\
\end{align*}
\]

and
\[
\begin{align*}
    f_8(x) &= \frac{(x - 1)^2}{4(x + 1)} - \frac{\sqrt{2(x^2 + 1)}}{6} + \frac{2x}{3(x + 1)} = \frac{1}{6} [3A - (2S + H)], \\
\end{align*}
\]

where \( A = \frac{x + 1}{2} \), \( G = \sqrt{x} \), \( H = \frac{2x}{x + 1} \) and \( S = \sqrt{\frac{x^2 + 1}{2}} \) are respectively arithmetic, geometric, harmonic and square-root means between \( x \) and \( 1 \).

**Theorem 7.1.** The following inequality among the new differences hold:
\[(7.3) \quad D_{f_8}(P||Q) \leq \frac{1}{3} D_{f_1}(P||Q) \leq \frac{1}{4} D_{f_3}(P||Q) \leq \frac{1}{3} D_{f_5}(P||Q) \leq D_{f_6}(P||Q). \]

**Proof.** We shall prove each part of the inequality separately. These inequalities can be proved on similar lines of theorem 5.1 but here we shall give a simpler proof.

(i) We can write
\[
\begin{align*}
    f_6(x) - \frac{1}{3} f_2(x) &= \frac{1}{6} [S + 2H - 3G] - \frac{1}{3} \left[ \left( \frac{A + H}{2} \right) - G \right] \\
    &= \frac{1}{6} [S + H - (A + G)] = \frac{1}{3} f_5(x) \geq 0, \ \forall x \in (0, \infty) \\
\end{align*}
\]

This prove that \( \frac{1}{3} f_2(x) \leq f_6(x), \forall x \in (0, \infty) \), and consequently, we get
\[(7.4) \quad \frac{1}{3} D_{f_2}(P||Q) \leq D_{f_6}(P||Q). \]
(ii) We can write

\[
\frac{1}{3} f_2(x) - \frac{1}{4} f_3(x) = \frac{1}{3} \left[ \frac{A + H}{2} - G \right] - \frac{1}{12} \left[ 3A + H - (S + 3G) \right] \\
= \frac{1}{12} \left[ S + H - (A + G) \right] = \frac{1}{6} f_5(x) \geq 0, \quad \forall x \in (0, \infty).
\]

This proves that \( \frac{1}{4} f_3(x) \leq \frac{1}{3} f_2(x) \), \( \forall x \in (0, \infty) \), and consequently, we get

\[(7.5) \quad \frac{1}{4} D_{f_3}(P||Q) \leq \frac{1}{3} D_{f_2}(P||Q).\]

(iii) We can write

\[
\frac{1}{4} f_3(x) - \frac{1}{3} f_1(x) = \frac{1}{12} \left[ 3A + H - (S + 3G) \right] - \frac{1}{3} \left[ A - \left( \frac{G + S}{2} \right) \right] \\
= \frac{1}{12} \left[ S + H - (A + G) \right] = \frac{1}{6} f_5(x) \geq 0, \quad \forall x \in (0, \infty).
\]

This proves that \( \frac{1}{3} f_1(x) \leq \frac{1}{4} f_3(x) \), \( \forall x \in (0, \infty) \), and consequently, we get

\[(7.6) \quad \frac{1}{3} D_{f_1}(P||Q) \leq \frac{1}{4} D_{f_3}(P||Q).\]

(iv) We can write

\[
\frac{1}{3} f_1(x) - f_8(x) = \frac{1}{3} \left[ A - \left( \frac{G + S}{2} \right) \right] - \frac{1}{6} \left[ 3A - (2S + H) \right] \\
= \frac{1}{6} \left[ S + H - (A + G) \right] = \frac{1}{3} f_5(x) \geq 0, \quad \forall x \in (0, \infty).
\]

This proves that \( f_8(x) \leq \frac{1}{3} f_1(x) \), \( \forall x \in (0, \infty) \), and consequently, we get

\[(7.7) \quad D_{f_8}(P||Q) \leq \frac{1}{3} D_{f_1}(P||Q).\]

Combining (7.4)-(7.7) we get the required result. \( \square \)

**Remark 7.1.**

(i) Simplifying the inequalities given (7.3) and using the nonnegativity of the expression \( f_5(x) \), \( \forall x \in (0, \infty) \), we get the following improvement over the inequalities (5.14):

\[(7.8) \quad M_{GH}(P||Q) \leq M_{SA}(P||Q) \leq \frac{1}{3} M_{SH}(P||Q) \leq \frac{1}{4} \Delta(P||Q) \\
\leq \frac{3 \Delta(P||Q) + 2 M_{SG}(P||Q)}{16} \leq \frac{h(P||Q) + 3 M_{SA}(P||Q)}{4} \\
\leq \frac{h(P||Q) + M_{SH}(P||Q)}{4} \leq \frac{6 M_{SG}(P||Q) + \Delta(P||Q)}{4} \\
\leq \frac{1}{2} M_{SG}(P||Q) \leq h(P||Q) \leq \frac{1}{2} \Delta(P||Q).
\]

We observe that the measure \( M_{GH}(P||Q) \) is not convex in the pair of probability distributions, but even then we are able to relate it in the above inequalities.
(ii) Recently, author [20] also gave an improvement over the inequality (6.7):

\[
\begin{align*}
\frac{1}{4} \Delta(P||Q) & \leq I(P||Q) \leq \frac{2}{3} h(P||Q) + \frac{1}{12} \Delta(P||Q) \\
& \leq \frac{1}{16} J(P||Q) + \frac{1}{2} I(P||Q) \leq \frac{1}{3} T(P||Q) + \frac{2}{3} h(P||Q) \\
& \leq \frac{1}{8} J(P||Q) \leq \frac{2}{3} T(P||Q) + \frac{1}{12} \Delta(P||Q) \leq T(P||Q).
\end{align*}
\]

The above inequality also improves the one studied by Dragomir et al. [7].

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