Spectral properties of the singular Friedrichs-Lee Hamiltonian

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Abstract. We show that the one-excitation sector of a two-level atom interacting with a structured boson field can be modelled by a generalisation of the standard Friedrichs-Lee model which includes the possibility of a singular atom-field coupling. We provide a characterisation of its spectrum and resonances and discuss the inverse spectral theory of the model.

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1. Introduction

The Friedrichs-Lee Hamiltonian is a self-adjoint operator in a Hilbert space which describes the behaviour of an eigenvalue coupled to a continuous spectrum and is a rare example of a solvable model with a rich mathematical structure [11]. It was originally introduced by T. D. Lee [13] as a solvable quantum-field theoretical model suitable for the investigation of the renormalisation procedure. Lee’s Hamiltonian has a conserved quantum number labelling reducing subspaces (excitation sectors). Its reduction to the first nontrivial excitation sector, which we will refer to as the Friedrichs-Lee
Hamiltonian, is the quantum-mechanical model used by Friedrichs in his seminal study of the perturbation of continuous spectra [7].

Since its inception, the Friedrichs-Lee Hamiltonian has proven to be a very useful model in many applications, ranging from quantum field theory of unstable particles [2] to non-relativistic quantum electrodynamics [3], from quantum optics [8] to quantum probability [18], to name a few.

In this paper we aim at a complete study of the mathematical properties of the Friedrichs-Lee operator by extending it to a larger class of couplings (thus providing rigorous foundations to many formal computations usually carried out in the physical literature) and providing a characterisation of its spectrum with respect to the spectrum of the uncoupled operator. The paper is organised as follows:

- in Section 2 we derive the expression of the regular Friedrichs-Lee Hamiltonian as the restriction to the one-excitation sector of Lee’s field-theory model;
- in Section 3 we introduce the singular Friedrichs-Lee Hamiltonian, proving that it includes the case of a regular coupling (Theorem 3.1) and showing that a singularly coupled model can always be obtained as the norm resolvent limit of a proper sequence of regular models (Theorem 3.4);
- in Section 4 we characterise its spectrum: in Theorem 4.1 we find its essential and discrete components, and in Theorem 4.3 we find its absolutely continuous, singular continuous and pure point components, the latter being strictly dependent on a Herglotz function known as the self-energy of the model;
- in Section 5 we apply the results of the previous section to some simple Friedrichs-Lee Hamiltonians;
- in Section 6 we discuss the resonances of the model, showing that they can be characterised as complex eigenvalues of a deformation of the Hamiltonian (Theorem 6.11);
- in Section 7 we introduce the inverse spectral problem for the model, i.e. the choice of a form factor yielding the desired dynamics for a model with given field structure; as an example, we discuss the case of exponential decay in an electromagnetic field.

Future developments may include the generalisation of the singular coupling and spectral characterisation to the \( n \)-atom Friedrichs-Lee Hamiltonian or
to higher excitation sectors, as well as applications to physically interesting systems.

2. Physical model

Let \((X, \mu)\) be a \(\sigma\)-finite measure space, \(L^2(X, \mu)\) the space of square-integrable complex-valued functions with respect to \(\mu\), and \(\mathcal{F}\) the Bose-Fock space on \(L^2(X, \mu)\). Let \(H_{\text{field}}\) be an Hamiltonian operator with formal expression

\[
H_{\text{field}} = \int_X \omega(k) a^*(k) a(k) \, d\mu(k),
\]

where \(\omega : X \to \mathbb{R}\) is a continuous function, and \(a(k), a^*(k)\) are the operator-valued distributions associated with a family of annihilation and creation operators, satisfying the formal relations \([a(k), a^*(k')] = \delta(k - k')\). Physically, \(H_{\text{field}}\) is the operator associated with the energy of a bosonic field, \((X, \mu)\) is the momentum space of the bosons, and \(\omega(k)\) is the dispersion relation, that is the energy of a quantum with momentum \(k\). For example the choices \(X = \mathbb{R}^3, \omega(k) = (||k||^2 + m^2)^{1/2}\) and \(\mu = \) the Lebesgue measure on \(\mathbb{R}^3\), represent a relativistic bosonic field associated with a particle of mass \(m\). On the mathematical level, \(H_{\text{field}}\) is the second quantization of the multiplication operator associated with the function \(\omega\), and is a densely defined self-adjoint operator in \(\mathcal{F}\) [16].

Let us consider a nondegenerate two-level atom in \(\mathbb{C}^2\), with ground state (in Dirac’s notation) \(|\downarrow\rangle\) and excited state \(|\uparrow\rangle\). Let

\[
H_{\text{atom}} = \varepsilon_a |\uparrow\rangle \langle \uparrow|,
\]

be its Hamiltonian, where \(\varepsilon_a\) is the energy of the excited state and the ground state energy is set to zero. The operator \(H_{\text{atom}} \otimes I + I \otimes H_{\text{field}}\), defined in a dense subspace of \(\mathbb{C}^2 \otimes \mathcal{F}\), represents the system atom-field in the absence of mutual interaction. A physically meaningful coupling between the atom and the field can be introduced as follows: given \(g \in L^2(X, \mu)\), let

\[
V_g = \int_X \left( \sigma^+ \otimes \tilde{g}(k) a(k) + \sigma^- \otimes g(k) a^*(k) \right) \, d\mu(k),
\]

where \(\sigma^+ = |\uparrow\rangle \langle \downarrow|\) and \(\sigma^- = |\downarrow\rangle \langle \uparrow|\) are the ladder operators, that is, \(\sigma^+\) raises the ground to the excited state and \(\sigma^-\) lowers the excited to the ground state [3]. The total Hamiltonian \(H_{\text{Lee}}\) associated with the atom-field system is formally given by

\[
H_{\text{Lee}} = H_{\text{atom}} \otimes I + I \otimes H_{\text{field}} + V_g,
\]
where we use the same notation for identity operators acting on different Hilbert spaces. This is a generalisation of the standard Lee model [13]. Physically, \( \mu \) controls and weighs the values of momenta available to the bosons and must be chosen according to the physical setting: for instance, for an electromagnetic field in free space, \( \mu \) is the Lebesgue measure on \( X = \mathbb{R}^3 \), while, for a field confined in an optical cavity, at least one component of the momenta will be discrete.

Summing up, the analytic features of our model will depend on three physically important quantities:

- The space \( (X, \mu) \) of all possible momenta of the field quanta;
- The dispersion relation \( \omega(k) \) that gives the energy of a quantum with momentum \( k \);
- The form factor \( g(k) \) that controls the coupling between a field quantum with momentum \( k \) and the atom.

\( H_{\text{Lee}} \) does not conserve the total number of bosons in the theory: the number operator, formally defined as

\[
    N_{\text{field}} = \int_X a^*(k)a(k) \, d\mu(k),
\]

(5)
does not commute with \( H_{\text{Lee}} \) for any nonzero form factor \( g \). However, the operator

\[
    N_{\text{tot}} = |\uparrow\rangle \langle \uparrow| \otimes I + I \otimes N_{\text{field}},
\]

(6)
representing the total number of excitation in the system, commutes with \( H_{\text{Lee}} \) for every choice of \( g \); since the operator \( N_{\text{tot}} \) has spectrum \( \sigma(N_{\text{tot}}) = \mathbb{N} \), one can study the evolution of the system generated by the restriction of \( H_{\text{Lee}} \) to each eigenspace of \( N_{\text{tot}} \), \( n \in \mathbb{N} \).

The simplest nontrivial choice is the one-excitation sector

\[
    \mathcal{H}_{1-\text{exc}} = \mathbb{C} \oplus L^2(X, \mu).
\]

(7)
The generic normalised element \( \Psi \in \mathcal{H}_{1-\text{exc}} \) may be expressed as

\[
    \Psi = \begin{pmatrix} x \\ \xi \end{pmatrix},
\]

(8)
where \( |x|^2 \) is the probability that the atom is in its excited state and \( \xi \) is the wave function of the boson in the field. In particular, the state

\[
    \Psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

(9)
represents the excited atom interacting with the vacuum. The restriction of $H_{\text{Lee}}$ to the one-excitation sector $H_{1,\text{exc}}$ which we will denote as $H_{\text{FL}}$, is the Friedrichs-Lee Hamiltonian [7, 11]. Its domain is

$$D(H_{\text{FL}}) = \mathbb{C} \oplus D(\Omega) = \left\{ \begin{pmatrix} x \\ \xi \end{pmatrix} \mid x \in \mathbb{C}, \xi \in D(\Omega) \right\},$$

and it acts on a generic vector of its domain as

$$H_{\text{FL}} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \varepsilon_a x + \langle g | \xi \rangle \\ xy + \Omega \xi \end{pmatrix},$$

where $\Omega$ is the multiplication operator associated with the dispersion relation $\omega$, that is, $(\Omega \xi)(k) = \omega(k)\xi(k)$. The action of the Hamiltonian $H_{\text{FL}}$ in (11) can be obtained using a formal matrix representation

$$H_{\text{FL}} = \begin{pmatrix} \varepsilon_a & \langle g | \\ g & \Omega \end{pmatrix},$$

where $\langle g |$ is in Dirac notation the linear functional on $L^2(X, \mu)$ associated with $g$.

3. The singular Friedrichs-Lee Hamiltonian

The Hamiltonian (11) with matrix representation (12) cannot include a singular coupling between field and atom, i.e. a form factor $g \notin L^2(X, \mu)$. This obstruction is relevant at a physical level: for instance, a flat coupling between field and atom (i.e. $g(k) = \text{const.}$) cannot be generally included, thus preventing the description of interesting phenomena (e.g. exponential decay of the state $\Psi_0$).

To extend our model to a (possibly) singular coupling between atom and field, the formalism of Hilbert scales will be extensively used, (see e.g. [1]). Recall that, given a Hilbert space $\mathcal{H}$ and a self-adjoint operator $\Omega$ on it, $\mathcal{H}_s$, for any $s \in \mathbb{R}$, is the (completion of the) domain of $(\Omega - i)^{s/2}$ endowed with the norm $\|\phi\|_s := \|(\Omega - i)^{s/2}\phi\|$; in particular,

- $\mathcal{H}_s \subset \mathcal{H}_{s'}$ for any $s > s'$ and the inclusion is dense with respect to $\|\cdot\|_s$;
- For $s > 0$, $\mathcal{H}_{-s}$ is the algebraic dual of $\mathcal{H}_s$: its elements are functionals with domain $\mathcal{H}_s$;
- $\mathcal{H}_0, \mathcal{H}_1$ and $\mathcal{H}_2$ are, respectively, the original Hilbert space, the form domain of $\Omega$, and the domain of $\Omega$:
• $\Omega$ maps $H_s$ into $H_{s-2}$ and $\frac{1}{\Omega}$ maps it into $H_{s+2}$.

In the following we will consider the Hilbert space $H = L^2(X, \mu)$ and $\Omega$ as the multiplication operator by a real-valued function $\omega$ on $L^2(X, \mu)$.

In this case we have, in particular, \begin{equation}
g \in H_s \quad \text{iff} \quad \int_X |\omega(k) - i|^s |g(k)|^2 \, d\mu(k) < \infty; \quad (13)\end{equation} \begin{equation}g \in H_{-s} \quad \text{iff} \quad \int_X |g(k)|^2 |\omega(k) - i|^s \, d\mu(k) < \infty. \quad (14)\end{equation}

**Theorem 3.1.** Let $\varepsilon \in \mathbb{R}$, $\Omega$ be a multiplication operator in $H = L^2(X, \mu)$ with domain $H_2$, and $g \in H_{-2}$. Consider the operator $H_{g,\varepsilon}$ on the Hilbert space $C \oplus H$ with domain \begin{equation}D(H_{g,\varepsilon}) = \left\{ \left( x \Omega, \xi \right) \left| x \in C, \xi \in H_2 \right. \right\} \quad (15)\end{equation}
such that \begin{equation}H_{g,\varepsilon} \left( x \Omega, \xi \right) = \left( \varepsilon x + \langle g | \xi \rangle, \Omega \xi + x \frac{1}{\Omega^2 + 1} g \right). \quad (16)\end{equation}

Then we have:

(i) If $g \in H$ then $H_{g,\varepsilon}$ reduces to the Friedrichs-Lee Hamiltonian $H_{FL}$ in (11) with atom excitation energy \begin{equation}\varepsilon_a = \varepsilon - \left\langle g \left| \frac{\Omega}{\Omega^2 + 1} g \right. \right\rangle. \quad (17)\end{equation}

(ii) $H_{g,\varepsilon}$ is self-adjoint and, for all $z \in \mathbb{C} \setminus \mathbb{R}$, its resolvent operator is \begin{equation}\frac{1}{H_{g,\varepsilon} - z} \left( x \Omega, \xi \right) = \left( \frac{x - \langle g | \frac{1}{\Omega^2 + 1} \xi \rangle}{z - \langle g | \frac{1}{\Omega^2 + 1} \xi \rangle}, \frac{1}{z - \langle g | \frac{1}{\Omega^2 + 1} \xi \rangle} \right), \quad \left( x \Omega, \xi \right) \in C \oplus H, \quad (18)\end{equation}

where \begin{equation}\Sigma_g(z) := \left\langle g \left| \left( \frac{1}{\Omega^2 + 1} - \frac{\Omega}{\Omega^2 + 1} \right) g \right. \right\rangle. \quad (19)\end{equation}

\footnote{There is no loss of generality in this choice since, by the spectral theorem, every self-adjoint operator is equivalent to a multiplication operator in some $L^2$ space.}
(iii) The evolution group generated by $H_{g,\varepsilon}$ is given by

$$U_{H_{g,\varepsilon}}(t) = \frac{1}{2\pi i} \text{PV} \int_{i\delta - \infty}^{i\delta + \infty} e^{-izt} \frac{1}{H_{g,\varepsilon} - z} \, dz,$$

for all $t > 0$, where $\delta > 0$ is an arbitrary constant and the principal-value integral must be understood in the strong sense. Moreover, if

$$\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = U_{H_{g,\varepsilon}}(t)\Psi_0$$

is the evolution at time $t > 0$ of the initial state $\Psi_0$ in (9), we have

$$x(t) = \frac{1}{2\pi i} \text{PV} \int_{i\delta - \infty}^{i\delta + \infty} \frac{e^{-izt}}{\varepsilon_a - z - \Sigma_g(z)} \, dz,$$

$$\xi(k,t) = -\frac{1}{2\pi i} \text{PV} \int_{i\delta - \infty}^{i\delta + \infty} \frac{g(k)}{\varepsilon_a - z - \Sigma_g(z) \omega(k) - z} \, dz.$$

Proof. (i) If $g \in \mathcal{H}$, then $\frac{\Omega}{\Omega + 1} g \in \mathcal{H}_2 = D(\Omega)$ and hence the domains in Eqs. (10) and (15) coincide, since the additional term can be simply absorbed into $\xi$. Applying $H_{FL}$ to any vector of the form (15) yields the same result as in Eq. (16), hence the two operators coincide.

(ii) Since $D(\Omega)$ is dense in $\mathcal{H}$, $D(H_{g,\varepsilon})$ is dense in $\mathbb{C} \oplus \mathcal{H}$; besides, a direct calculation shows that $H_{g,\varepsilon}$ is symmetric and the bounded operator acting on $\mathbb{C} \oplus \mathcal{H}$ as in Eq. (18) is the inverse of $H_{g,\varepsilon} - z$ for any $z \in \mathbb{C} \setminus \mathbb{R}$; this proves that $H_{g,\varepsilon}$ is self-adjoint.

(iii) Eq. (20) follows from Eq. (18) and from the general link between the resolvent and the evolution group associated with any self-adjoint operator; Eq. (22), in particular, follows by substituting $x = 1$ and $\xi = 0$ in Eq. (18) and applying Eq. (20). \qed

Remark 3.2. We can distinguish three separate cases:

(1) $g \in \mathcal{H}$: the domain does not depend on $g$ and both $\varepsilon_a$ and $\varepsilon$ are finite quantities, representing respectively the “bare” and “dressed” (coupling-dependent) excitation energy of the atom. The formal matrix expression (12) holds.

(2) $g \in \mathcal{H}_1 \setminus \mathcal{H}$: the domain depends on $g$, but again both $\varepsilon_a$ and $\varepsilon$ are finite quantities with the same physical meaning as above, since $\langle g | \Omega | \frac{\Omega}{\Omega + 1} g \rangle$ is finite. Again the model can be written as in Eq. (12).
(3) \( g \in \mathcal{H}_2 \setminus \mathcal{H}_1 \): the domain depends on \( g \) and the bare excitation energy \( \varepsilon_a \) is not defined, since \( \langle g| \frac{\Omega}{\Omega^2 + 1} g \rangle \) is not finite; because of that, Eq. (12) is ill-defined.

The latter situation is reminiscent of the (heuristic) renormalisation procedure of quantum field theory, in which the bare (and hence unobservable) value of a parameter, e.g. the electron charge, diverges in such a way to obtain a finite value of the measurable dressed one. Besides, in the first two cases we may equivalently write

\[
\varepsilon - \Sigma_g(z) = \varepsilon_a - \tilde{\Sigma}_g(z) \tag{24}
\]

where \( \tilde{\Sigma}_g(z) = \langle g| \frac{1}{\Omega - z} g \rangle \) is the “bare” self-energy. In this sense, the extension of the model to the case \( g \in \mathcal{H}_1 \) is straightforward up to an algebraic technicality, i.e. the choice of a convenient representation of the domain, while the further extension to the case \( g \in \mathcal{H}_2 \) requires an “infinite” term to be added to both the bare excitation energy and the bare self-energy.

Finally, the three cases reflect the possible situations in which \( \Psi_0 \) has

(1) finite mean value and variance of energy;

(2) finite mean value of energy, but infinite variance;

(3) infinite mean value of energy and variance.

**Remark 3.3.** There is an interesting connection between the Friedrichs-Lee model and rank-one perturbations of self-adjoint operators. Given a self-adjoint operator \( \Omega \) in \( \mathcal{H} \), consider the following object:

\[
\Omega_{g,\alpha} = \Omega + \alpha |g\rangle \langle g|, \quad \alpha \in \mathbb{R}. \tag{25}
\]

If \( g \in \mathcal{H}_2 \setminus \mathcal{H} \), this is only a formal expression, with which one can associate [1, 17] a well-defined self-adjoint operator. In the case \( g \in \mathcal{H}_1 \), the operator is defined on the domain

\[
D(\Omega_{g,\alpha}) = \left\{ \xi - x \frac{\Omega}{\Omega^2 + 1} g \left| \xi \in D(\Omega), \langle g|\xi\rangle = -x \left( \frac{1}{\alpha} + \langle g| \frac{\Omega}{\Omega^2 + 1} g \rangle \right) \right\}, \tag{26}
\]

where it acts as

\[
\Omega_{g,\alpha} \left( \xi - x \frac{\Omega}{\Omega^2 + 1} g \right) = \Omega \xi + x \frac{1}{\Omega^2 + 1} g. \tag{27}
\]
Indeed, if one applies the formal expression (25) on vectors in the above domain, all terms outside $\mathcal{H}$ cancel out. Interestingly, only the domain depends on $\alpha$.

Notice that, by setting $\varepsilon_a = 1/\alpha$ and hence $\varepsilon = 1/\alpha + \langle g | \Omega_2 g \rangle$, the vectors in $D(\Omega_{a,\alpha})$ correspond to the bosonic components of all the states in $D(H_{g,\varepsilon})$ that are mapped into states with zero atomic component.

If, instead, $g \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, there is an issue, since the action of $\langle g |$ is not defined on $\Omega_2 g$. By choosing some $c \in \mathbb{R}$ and defining the extension $\langle g_c |$ in such a way that

$$
\left\langle g_c \left| \frac{\Omega}{\Omega_2 + 1} g \right. \right\rangle = c,
$$

our desired implementation of the formal object (25) has domain

$$
D(\Omega_{g,a}) = \left\{ \xi - x \frac{\Omega}{\Omega_2 + 1} g \left| \xi \in D(\Omega), \langle g | \xi \rangle = -x \left( \frac{1}{\alpha} + c \right) \right\},
$$

where it acts as

$$
\Omega_{g,a} \left( \xi - x \frac{\Omega}{\Omega_2 + 1} g \right) = \Omega \xi + x \frac{1}{\Omega_2 + 1} g.
$$

The action of the Friedrichs-Lee operator is again recovered if one chooses $\varepsilon = 1/\alpha + c$. Notice that the freedom in the choice of $c$ reflects the fact that, in the Friedrichs-Lee model, a bare excitation energy is not defined for $g \notin \mathcal{H}_{-1}$: the operator really depends only on $1/\alpha + c$, in the same way as the Friedrichs-Lee Hamiltonian ultimately depends on the dressed energy $\varepsilon$ alone.

When $g \notin \mathcal{H}$, we will say that the atom-field coupling is singular, as opposed to the regular case $g \in \mathcal{H}$. Interestingly, every singular Friedrichs-Lee model can be approximated by a proper sequence of regular models:

**Theorem 3.4** (Singular coupling limit). (i) Let $(g_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ be such that $g_n \to g$ in the norm of $\mathcal{H}_{-2}$, then $H_{g_n,\varepsilon} \to H_{g,\varepsilon}$ in the norm resolvent sense as $n \to \infty$.

(ii) Conversely, for every singular Friedrichs-Lee Hamiltonian $H_{g,\varepsilon}$ there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ such that $H_{g_n,\varepsilon} \to H_{g,\varepsilon}$ in the norm resolvent sense as $n \to \infty$.

**Proof.** (i) $g_n \to g$ in the norm of $\mathcal{H}_{-2}$ means that $\| \frac{1}{\Omega_1^{n-1}} (g_n - g) \| \to 0$, and equivalently, by the first resolvent formula, $\frac{1}{\Omega_1^{n-1}} g_n \to \frac{1}{\Omega_1^{n-1}} g$ for every
Remark 3.5. If \( g \in \mathcal{H}_{-2} \), the approximating sequence is characterised by a diverging bare excitation energy \( \varepsilon_{a,n} \to \infty \) \textit{and} a diverging bare self-energy \( \Sigma_{g_n}(z) \to \infty \); their difference converges to a finite limit which depends on the value of the dressed excitation energy \( \varepsilon \); this is clearly a renormalisation procedure, as previously discussed.

Also notice that the theorem holds even if the dressed energy \( \varepsilon \) of the approximating sequence, instead of being kept fixed, is replaced with a converging sequence \( \varepsilon_n \to \varepsilon \).

4. Spectral properties

After having introduced the model, let us characterise its spectral properties with respect to two common decompositions of the spectrum of a self-adjoint operator:

- absolutely continuous, singular continuous and pure point spectrum;
- essential and discrete spectrum,

the discrete spectrum being the set of all isolated eigenvalues of finite multiplicity.

In the absence of coupling (i.e. for \( g = 0 \)), the spectrum of the Friedrichs-Lee Hamiltonian is obviously

\[
\sigma(H_{0,\varepsilon}) = \{\varepsilon\} \cup \sigma(\Omega),
\]

with \( \sigma(\Omega) \) being the spectrum of \( \Omega \), i.e. the \( \mu \)-essential range of \( \omega \), which coincides with the support of the induced measure \( \nu \) defined as

\[
\nu(B) = \int_{\omega^{-1}(B)} d\mu(k)
\]

for every Borel set \( B \subset \mathbb{R} \). Besides, the absolutely continuous (ac), singular continuous (sc) and pure point (pp) components of \( \sigma(\Omega) \) coincide the supports of the ac, sc and pp components of \( \nu \), and

\[
\begin{align*}
\sigma_{ac}(H_{0,\varepsilon}) &= \sigma_{ac}(\Omega); \\
\sigma_{sc}(H_{0,\varepsilon}) &= \sigma_{sc}(\Omega); \\
\sigma_{pp}(H_{0,\varepsilon}) &= \{\varepsilon\} \cup \sigma_{pp}(\Omega),
\end{align*}
\]
with the eigenvector associated with $\varepsilon$ being $\Psi_0$. On the physical level, $\text{supp}(\nu) = \sigma(\Omega)$ is the energy space of the boson. As for the distinction between essential and discrete spectrum, in the most general case we have

- $\sigma_{\text{ess}}(H_{0,\varepsilon}) = \sigma_{\text{ess}}(\Omega)$;

- $\sigma_{\text{dis}}(H_{0,\varepsilon}) \setminus \{\varepsilon\} = \sigma_{\text{dis}}(\Omega),$

with $\varepsilon$ belonging to the discrete spectrum $\sigma_{\text{dis}}(H_{0,\varepsilon})$ if and only if $\varepsilon$ is isolated from the spectrum of $\Omega$.

We want to find a complete characterisation of the spectral properties of $H_{g,\varepsilon}$ with respect to the spectrum of $\Omega$ even for nontrivial coupling. First of all, let us examine the behavior of the discrete/essential decomposition.

**Theorem 4.1.** The essential spectrum of $H_{g,\varepsilon}$ coincides with the essential spectrum of $\Omega$, with the possible exception of the accumulation points of the eigenvalues of $\Omega$.

**Proof.** Suppose $g \in \mathcal{H}$, i.e. consider the regular model. Then, using the matrix representation for $H_{g,\varepsilon}$, we can write

$$H_{g,\varepsilon} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \Omega \end{pmatrix} + \begin{pmatrix} \varepsilon_a(g) - \varepsilon & \langle g \rangle \\ g & 0 \end{pmatrix} \equiv H_{0,\varepsilon} + V_{g,\varepsilon},$$

with $V_{g,\varepsilon}$ being finite-rank and hence, a fortiori, relatively compact with respect to $H_{0,\varepsilon}$; since symmetric, relatively compact perturbations leave the essential spectrum unchanged, we obtain $\sigma_{\text{ess}}(H_{g,\varepsilon}) = \sigma_{\text{ess}}(H_{0,\varepsilon})$, which coincides with $\sigma_{\text{ess}}(\Omega)$.

If $g \in \mathcal{H}_{-2} \setminus \mathcal{H}$, theorem 3.4 ensures that $H_{g,\varepsilon}$ will be the norm resolvent limit of a sequence of regular models sharing the same essential spectrum. Under these conditions, the norm resolvent limit preserves the essential spectrum with the possible exception of accumulation points of the eigenvalues of $\Omega$ [14]. This proves the claim. □

**Remark 4.2.** As a consequence of Theorem 4.1, when $\sigma_{\text{ess}}(\Omega)$ is entirely continuous or dense pure point, it will coincide with the essential spectrum of the corresponding Friedrichs-Lee operator. However, there may be conversion of dense pure point spectrum into continuous spectrum or vice versa.

Now let us study the decomposition of the spectrum $\sigma(H_{g,\varepsilon})$ into its absolutely continuous, singular continuous and pure point components. We
introduce the spectral measure associated with $g$ as follows:

$$
\nu_g(B) = \int_{\omega^{-1}(B)} |g(k)|^2 \, d\mu(k) \quad (34)
$$

for every Borel set $B \subset \mathbb{R}$. In this section we will prove the following result:

**Theorem 4.3.** Let $\mathcal{H}_g = L^2(X_g, \mu)$, with

$$
X_g = \omega^{-1}(\text{supp}(\nu_g)),
$$

and let $\mathcal{H}_g^\perp$ be its orthogonal complement. Define the function $G_g : X_g \to \mathbb{R} \cup \{\infty\}$ by

$$
G_g(\lambda) = \int_{\mathbb{R}} \frac{1}{(\lambda - \lambda')^2} \, d\nu_g(\lambda'),
$$

and set

$$
\Sigma_g^+(\lambda) = \lim_{\delta \downarrow 0} \Sigma_g(\lambda + i\delta). \quad (37)
$$

Then

- $\sigma_{ac}(H_{g,\varepsilon}) = \sigma_{ac}(\Omega)$;
- $\sigma_{sc}(H_{g,\varepsilon}) = \sigma_{sc}(\Omega|_{\mathcal{H}_g^\perp}) \cup \{\lambda \in X_g \mid \varepsilon - \lambda = \Sigma_g^+(\lambda), G_g(\lambda) = \infty\}$;
- $\sigma_{pp}(H_{g,\varepsilon}) = \sigma_{pp}(\Omega|_{\mathcal{H}_g^\perp}) \cup \{\lambda \in X_g \mid \varepsilon - \lambda = \Sigma_g^+(\lambda), G_g(\lambda) < \infty\}$.

Moreover, the restrictions of $\Omega$ and $H_{g,\varepsilon}$ to their absolutely continuous spectra are unitarily equivalent.

**Remark 4.4.** This result can be explained as follows. First of all, the space $X$ of field momenta can be split into a subset $X_g$ of momenta which are effectively coupled to the atom (i.e. on which the form factor $g$ is $\mu$-supported) and a complementary subset of uncoupled momenta; this subdivision induces a correspondent subdivision of the energy space (i.e. the support of $\nu$) into coupled and uncoupled energies, i.e. the support of $\nu_g$ and its complement.

As expected, the uncoupled part of the spectrum is independent of $g$ and hence, in particular, is the same as in the case $g = 0$. As for the coupled one, it turns our that the absolutely continuous spectrum is still unchanged, but the singular (i.e. pure point and singular continuous) spectrum will be, up to a topological closure, the set of solutions of the pole equation

$$
\varepsilon - \lambda = \Sigma_g^+(\lambda). \quad (38)
$$
Moreover, the value of $G(\lambda)$ allows to distinguish between pure point and singular continuous spectrum. Finally, notice that the latter equation admits the unique solution $\lambda = \varepsilon$ only when $g = 0$ and, in this case, necessarily $G(\lambda) = 0$, hence our result is in full agreement with the uncoupled case.

**Remark 4.5.** If one substitutes $g$ with $\beta g$ for some $\beta \in \mathbb{R}$, and hence $\Sigma_{\beta g}(z) = \beta^2 \Sigma_g(z)$, the singular spectrum becomes the set of solutions of the equation $\Sigma_g'(\lambda) = \frac{\varepsilon - \lambda}{2\lambda}$, which will be a different set for every value of $\beta$. By the last point of the theorem we can conclude that the discrete spectrum is highly coupling-dependent. Notice that, if the zeroes depend continuously on $\beta$, they will “move” continuously and monotonically in one direction.

The proof of Theorem 4.3 will be given in Sec. 4.3. First we will need some mathematical preliminaries.

4.1. The self-energy as a Borel transform.

**Definition 4.6.** Let $\nu$ a Borel measure satisfying the growth constraint
\[
\int_{\mathbb{R}} \frac{1}{1 + \lambda^2} \, d\nu(\lambda) < \infty. \tag{39}
\]
Its (regularised) Borel transform is defined as the complex function $B_\nu : \mathbb{C} \setminus \text{supp}(\nu) \to \mathbb{C}$ with
\[
B_\nu(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \, d\nu(\lambda). \tag{40}
\]

**Remark 4.7.** In the literature, the Borel transform is usually defined as follows:
\[
\tilde{B}_\nu(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} \, d\nu(\lambda)
\]
for a smaller class of measure, i.e. measures satisfying the growth condition
\[
\int_{\mathbb{R}} \frac{1}{1 + |\lambda|} \, d\nu(\lambda) < \infty. \tag{41}
\]
For such measures, $\tilde{B}_\nu(z)$ and $B_\nu(z)$ only differ by a finite real constant; as we will show later, this difference is in fact immaterial for our purposes since $\nu$ only depends on the imaginary part of the boundary values of $B_\nu(z)$ on the real line, but the choice (40) is more convenient since it is well-defined for a larger class of measures.
Remark 4.8. For the Friedrichs-Lee Hamiltonian $H_{g,\varepsilon}$, the self-energy defined in Eq. (19), can be written as

$$
\Sigma_g(z) = \int_\mathbb{R} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \, d\nu_g(\lambda),
$$

where $\nu_g$ is the spectral measure (34) associated with $g$ and $\Omega$. Therefore, $\Sigma_g(z) = B_{\nu_g}(z)$, i.e. the self-energy is the regularised Borel transform of $\nu_g$.

Besides,

• $g \in \mathcal{H}$ iff $\int_\mathbb{R} d\nu_g(\lambda) < \infty$;

• $g \in \mathcal{H}_{-1}$ iff $\int_\mathbb{R} \frac{1}{\lambda + |\lambda|} \, d\nu_g(\lambda) < \infty$;

• $g \in \mathcal{H}_{-2}$ iff $\int_\mathbb{R} \frac{1}{1 + \lambda^2} \, d\nu_g(\lambda) < \infty$,

and hence, in particular, the case $g \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ corresponds to the case in which $\nu_g$ does not admit a standard Borel transform (i.e. the bare self-energy $\tilde{\Sigma}_g(z)$), but does have a regularised Borel transform.

The previous remark allows us to obtain useful information on the self-energy, making use of the general theory of Borel transforms.

Proposition 4.9. The self-energy $\Sigma_g(z)$ of a Friedrichs-Lee Hamiltonian satisfies the following properties:

(i) $\Sigma_g(z)$ is a Herglotz function and is analytic in $\mathbb{C} \setminus \text{supp}(\nu_g)$;

(ii) the boundary values of $\Sigma_g(z)$ along $\text{supp}(\nu_g)$ are linked as follows to its Lebesgue decomposition:

- $\text{supp}(\nu_g) = \{ \lambda \in \mathbb{R} : \text{Im} \Sigma_g^+(\lambda) > 0 \}$;

- $\text{supp}(\nu_g^{ac}) = \{ \lambda \in \text{supp}(\nu_g) : \text{Im} \Sigma_g^+(\lambda) < \infty \}$, and $d\nu_g^{ac}(\lambda) = \frac{1}{\pi} \text{Im} \Sigma_g^+(\lambda) \, d\lambda$;

- $\text{supp}(\nu_g^{sc}) = \{ \lambda \in \text{supp}(\nu_g) : \text{Im} \Sigma_g^+(\lambda) = \infty, \delta \text{ Im} \Sigma_g(\lambda + i\delta) \to 0, \text{ as } \delta \downarrow 0 \}$;

- $\text{supp}(\nu_g^{pp}) = \{ \lambda \in \text{supp}(\nu_g) : \text{Im} \Sigma_g^+(\lambda) = \infty, \delta \text{ Im} \Sigma_g(\lambda + i\delta) \to c > 0, \text{ as } \delta \downarrow 0 \}$, and $\nu_g(\{\lambda\}) = c$,

with $\Sigma_g^+$ defined in (37).

(iii) Let $G_g(\lambda)$ be given by (36). The following propositions hold:

- $\lim_{\delta \downarrow 0} \frac{1}{\delta} \text{ Im} \Sigma_g(\lambda + i\delta) = G_g(\lambda)$;

- in addition, if $G_g(\lambda) < \infty$, then $\Sigma_g^+(\lambda)$ is finite and real and $\Sigma_g(\lambda + i\delta) = \Sigma_g^+(\lambda) + i\delta G_g(\lambda) + o(\delta)$, as $\delta \downarrow 0$, i.e. $G_g(\lambda)$ is the upper derivative of $\Sigma_g(z)$ in the direction of the imaginary axis.
Proof. $\Sigma_g(z)$ is the regularised Borel transform of $\nu_g$. In [17] such properties are proven for the standard Borel transform of a measure $\nu$ satisfying the growth condition (41), but the same properties are readily extended to the regularised transform with the same procedure, since the two transforms, when both defined, only differ by a real constant.

4.2. Cyclic subspaces and spectral properties. In the previous subsection we have shown the link between the self-energy $\Sigma_g(z)$ and the properties of $\nu_g$; now we will link the latter with the spectral properties of our model. We will start from some basic definitions.

Definition 4.10. Let $\Omega$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and let $g \in H_{-2}$. The cyclic subspace $\mathcal{H}_g$ spanned by $g$ is defined as follows:

$$ \mathcal{H}_g = \text{Span} \left\{ \frac{1}{\Omega - z} g \mid z \in \mathbb{C} \setminus \mathbb{R} \right\}. $$

(43)

In particular, if $\mathcal{H}_g = \mathcal{H}$, $g$ is called a cyclic vector and $\Omega$ is said to have a simple spectrum.

Proposition 4.11 ([14]). The following properties hold:

- $\mathcal{H}_g$ is a reducing subspace for $\Omega$, and hence $\Omega = \Omega|_{\mathcal{H}_g} \oplus \Omega|_{\mathcal{H}_g^\perp}$;
- $\Omega|_{\mathcal{H}_g}$ is unitarily equivalent to the position operator (multiplication by $x$) in $L^2(\mathbb{R}, \nu_g)$, and $\sigma(\Omega|_{\mathcal{H}_g}) = \text{supp}(\nu_g)$. Moreover, $\sigma_j(\Omega|_{\mathcal{H}_g}) = \text{supp}(\nu_{g,j})$ for $j \in \{\text{ac}, \text{sc}, \text{pp}\}$.

Remark 4.12. As a consequence of Propositions 4.11 and 4.9, the spectrum of $\Omega|_{\mathcal{H}_g}$ and its decomposition may be entirely obtained by studying the boundary values of the (regularised) Borel transform (40) of the spectral measure $\nu_g$.

Remark 4.13. In our case, $\mathcal{H} = L^2(X, \mu)$ and, as a consequence of Stone-Weierstrass theorem, $\mathcal{H}_g = L^2(X_g, \mu)$, with $X_g$ given by Eq. (35); this justifies the use of the same symbol in (43) and in Theorem 4.3. In particular, $\Omega|_{\mathcal{H}_g}$ and $\Omega|_{\mathcal{H}_g^\perp}$ are the multiplication operators associated with the restriction of $\omega$ to $X_g$ and $X \setminus X_g$, respectively. Finally, $g$ is cyclic if it is supported in a set of full measure $\mu$.

Proposition 4.14. Let $H_{g,\varepsilon}$ be the Friedrichs-Lee Hamiltonian with form factor $g \neq 0$. Then:
(i) the cyclic subspace of \(\C \oplus \H\) spanned by \(\Psi_0\) is \(\C \oplus \H_g\), with \(\H_g\) being the cyclic subspace of \(\H\) spanned by \(g\). In particular, if \(g\) is cyclic in \(\H\), \(\Psi_0\) is cyclic in \(\C \oplus \H\).

(ii) \(H_{g,\varepsilon}|_{\C \oplus \H_g^\perp} = H_{0,\varepsilon}|_{(\C \oplus \H_g)^\perp}\), and hence the spectrum of \(H_{g,\varepsilon}|_{(\C \oplus \H_g)^\perp}\) is the same as for the uncoupled case \((g = 0)\).

Proof. By Eq. (18) we have, for any \(z \in \C \setminus \R\),

\[
\frac{1}{H_{g,\varepsilon} - z} \Psi_0 = \frac{1}{\varepsilon - z - \Sigma_g(z)} \left( -\frac{1}{\Omega - z} g \right).
\]

A generic state \(\Psi = (x \xi)\) is orthogonal to all vectors of the form (44) iff \(x = (\xi|\frac{1}{\Omega - z} g)\) for every \(z \in \C \setminus \R\), which happens iff \(x = 0\) and \((\xi|\frac{1}{\Omega - z} g) = 0\) for every \(z\), the latter meaning \(\xi \in \H_g^\perp\); this proves (i). Besides, by Eq. (16), for any \(\xi \in \H_g^\perp \cap \H_2\) we have

\[
H_{g,\varepsilon} \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ \Omega \xi \end{pmatrix},
\]

so that \(0 \oplus \H_g^\perp\) is invariant under \(H_{g,\varepsilon}\) (which is true in general for any cyclic subspace) and, in particular, the action of \(H_{g,\varepsilon}\) on it is independent of \(g\); this proves (ii).

\[\square\]

4.3. Proof of Theorem 4.3. We are now ready to prove Theorem 4.3.

Proof. By Proposition 4.14 we already know that \(\sigma(H_{g,\varepsilon}|_{\C \oplus \H_g^\perp}) = \sigma(H_{0,\varepsilon}|_{\C \oplus \H_g^\perp})\), so we must only find the spectrum of the restriction of \(H_{g,\varepsilon}\) to the cyclic subspace \(\C \oplus \H_g\). To simplify the notation, without loss of generality let us suppose \(g\) cyclic. Again Proposition 4.14 implies that \(g\) is a cyclic vector for \(\Omega\) and \(\Psi_0\) is cyclic for \(H_{g,\varepsilon}\); since both operators are self-adjoint in their Hilbert spaces, by Proposition 4.11 the two operators are equivalent to the position operators respectively in \(L^2(\R, \nu_g)\) and \(L^2(\R, \nu_{\Psi_0})\), with \(\nu_{\Psi_0}\) being the spectral measure associated with \(\Psi_0\); finally, by Proposition 4.9, the ac, sc, pp components of the spectrum of both operators can be inferred through the boundary values of the imaginary parts of their Borel transforms.

Now, the Borel transform of \(\nu_g\) is the self-energy \(\Sigma_g(z)\). The Borel transform of \(\nu_{\Psi_0}\) can be defined as follows:

\[
\Pi_g(z) = \left\langle \Psi_0 \left| \frac{1}{H_{g,\varepsilon} - z} \Psi_0 \right. \right\rangle,
\]

(46)
where we dropped the regularising term, which is both not needed (because $\Psi_0 \in \mathbb{C} \oplus \mathcal{H}$ and hence $\nu_{\Psi_0}$ is a finite measure) and immaterial since it is real. A straightforward calculation yields

$$\Pi_g(z) = \frac{1}{\varepsilon - z - \Sigma_g(z)}, \quad (47)$$

and hence, for any $\lambda \in \mathbb{R}$ and $\delta > 0$,

$$\text{Im } \Pi_g(\lambda + i\delta) \sim \frac{\text{Im } \Sigma_g(\lambda + i\delta)}{|\varepsilon - \lambda - \Sigma_g(\lambda + i\delta)|^2}, \quad \text{as } \delta \downarrow 0. \quad (48)$$

If $\lambda \in \text{supp}(\nu_g^{ac})$, then (up to a topological closure), by Proposition 4.9, $\text{Im } \Sigma_g(\lambda)$ is nonzero and finite, hence in particular $\varepsilon - \lambda \neq \Sigma_g(\lambda)$ and so Eq. (48) yields a finite and nonzero quantity, implying $\lambda \in \text{supp}(\nu_g^{ac})$ too; analogously one proves the converse inclusion, which proves the equality of the absolutely continuous spectra of $\Omega$ and $H_{g,\varepsilon}$. Since both operators act as multiplication by $x$, this also implies the unitary equivalence of their restrictions.

Now suppose that $\lambda$ is in the singular spectrum of $H_{g,\varepsilon}$; by Eq. (48) this happens iff $\varepsilon - \lambda = \Sigma_g^+(\lambda)$ (since, if $\text{Im } \Sigma_g(\lambda + i\delta)$ diverges, the denominator diverges faster), also implying that $\Sigma_g^+(\lambda)$ is real. To distinguish between pure point and singular continuous spectra, we must examine the limiting value of $\delta \Pi_g(\lambda + i\delta)$, as $\delta \downarrow 0$. We have

$$\delta \text{ Im } \Pi_g(\lambda + i\delta) \sim \frac{\frac{1}{2} \text{ Im } \Sigma_g(\lambda + i\delta)}{\varepsilon - \lambda - \text{Re } \Sigma_g(\lambda + i\delta)^2 + \frac{1}{\delta^2} \text{ Im } \Sigma_g(\lambda + i\delta)^2}. \quad (49)$$

Now, when $G_g(\lambda) = \infty$, the denominator diverges faster than the numerator and hence $\delta \Pi_g(\lambda + i\delta) \to 0$. Besides, when $G_g(\lambda) < \infty$, the first term in the denominator vanishes since $\text{Re } \Sigma_g(\lambda + i\delta) \sim \varepsilon - \lambda + o(\delta)$ and hence $(\varepsilon - \lambda - \text{Re } \Sigma_g(\lambda + i\delta))^2 \sim o(\delta^2)$; hence we are left with

$$\delta \text{ Im } \Pi_g(\lambda + i\delta) \sim \frac{\delta}{\text{Im } \Sigma_g(\lambda + i\delta)} \to \frac{1}{G_g(\lambda)} > 0, \quad \text{as } \delta \downarrow 0, \quad (50)$$

and this completes the proof. □

**Remark 4.15.** Notice that $\text{Im } \Sigma_g^+(\lambda) = \infty$ iff $\lim_{\delta \downarrow 0} \text{Im } \Pi_g(\lambda + i\delta) = 0$. In the case in which the singular spectra of both $\Omega$ and $H_{g,\varepsilon}$ are purely discrete, this means that inside the support of the spectral measure, the eigenvalues of $\Omega$ and $H_{g,\varepsilon}$ are completely disjoint: physically, no stable state of the field with energy coupled to the atom preserves its stability.
As for the energy $\varepsilon$ of the excited atom, which is an eigenvalue of the uncoupled operator:

- if $\varepsilon \notin \text{supp}(\nu_g)$, it will be obviously an eigenvalue of the coupled operator too;
- if $\varepsilon \in \text{supp}(\nu_g)$, then it is an eigenvalue of the coupled operator if it satisfies the equation
  \[ \Sigma^+_g(\varepsilon) = 0, \]  
  (51)

and, in particular, it must be a zero for the spectral density associated with the absolutely continuous part of $\nu_g$.

5. Some examples

In this section we discuss spectral properties for some simple examples of Friedrichs-Lee models. The section is organised as follows:

- Examples 5.1–5.3 concern the case in which $\sigma(\Omega)$ is purely absolutely continuous;
- Example 5.4 explores a purely discrete $\sigma(\Omega)$;
- Finally, in Example 5.5 we investigate a pure point $\sigma(\Omega)$, with dense eigenvalues in $[0, 1]$, which becomes singular continuous when the coupling is switched on.

Some considerations are now in order. Suppose that $\sigma(\Omega)$ is purely absolutely continuous in some (possibly unbounded) closed interval $J \subset \mathbb{R}$, and hence

- $\sigma_{ac}(H_{0,\varepsilon}) = J$;
- $\sigma_{pp}(H_{0,\varepsilon}) = \{\varepsilon\}$,

where $\varepsilon$ can also be in $J$. Now, switching on a coupling $g$, we will still have $\sigma_{ac}(H_{g,\varepsilon}) = J$. In particular, $\varepsilon$ is again in the absolutely continuous spectrum of $H_{g,\varepsilon}$ if and only if $\varepsilon \in J$, but generally it will not be in $\sigma_{pp}(H_{g,\varepsilon})$: this happens if and only if $\Sigma^+_g(\varepsilon) = 0$, i.e. if $\varepsilon$ is a zero for the coupling density of $\nu_g$ and in addition $\text{Re} \Sigma^+_g(\varepsilon) = 0$. Physically, the eigenvalue becomes unstable whenever it lies in a set of coupled values of energy, except when the coupling density vanishes at that point; depending on the choice of $\nu_g$, the discrete spectrum may contain other elements. In particular, if the coupling density is nonzero in the whole real line, the
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If some \( \lambda \in \mathbb{R} \) is a zero of the coupling density (and hence \( \text{Im} \Sigma_g^+(\lambda) = 0 \)), a necessary and sufficient condition for it to be in the singular spectrum is that the equation \( \varepsilon - \lambda = \text{Re} \Sigma_g^+(\lambda) \) is fulfilled. In particular, if \( \text{Re} \Sigma_g^+(\lambda) < \infty \), there is a unique value of \( \varepsilon \) such that the previous equation is fulfilled and hence \( \lambda \) is an eigenvalue for the coupled Hamiltonian \( H_{g,\varepsilon} \); this is an example of a resonant state. In particular, the excitation energy \( \varepsilon \) is an eigenvalue for the coupled Hamiltonian if it is a zero for the self-energy.

**Example 5.1** (Lebesgue spectral measure on \( \mathbb{R} \)). Suppose that \( J = \mathbb{R} \) and

\[
d\nu_g(\lambda) = \frac{\beta}{2\pi} d\lambda
\]

for \( \beta > 0 \). Then a simple calculation shows that

\[
\Sigma_g(z) = \begin{cases} 
\frac{i\beta}{2}, & \text{Im } z > 0; \\
-\frac{i\beta}{2}, & \text{Im } z < 0.
\end{cases}
\]

This implies that the pole equation (38) does not have any solution and hence \( \sigma(H_{g,\varepsilon}) \) is purely absolutely continuous; the uncoupled eigenvalue “dissolves” in the continuum for any nonzero value of \( \beta \). Physically, the bound state with energy \( \varepsilon \) becomes unstable: indeed, a simple calculation shows that, if we let evolve the state \( \Psi_0 \), i.e.

\[
\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = U_{H_{g,\varepsilon}}(t)\Psi_0
\]

then \( x(t) = e^{-\left(\frac{\beta}{2} + i\varepsilon\right)t} \), hence \( |x(t)|^2 = e^{-\beta t} \): an exponential decay takes place. Notice that the a purely exponential decay law at both short and large times is possible since

- the initial state \( \Psi_0 \) is not in the domain of \( H_{g,\varepsilon} \), notwithstanding it is in the domain of \( H_{0,\varepsilon} \), since, being \( \nu_g \) Lebesgue, necessarily \( g \in \mathcal{H}_{-2} \setminus \mathcal{H} \) [6].
- \( \Omega \) is unbounded both from below and from above, since \( \sigma(\Omega) = \text{supp}(\nu_g) = \mathbb{R} \), and hence Paley-Wiener’s theorem, which prohibits an exponential decay at large times, does not apply [15, 5, 12].
Example 5.2 (Lebesgue spectral measure on $[0, \infty)$). As a second example suppose now $J = [0, \infty)$, hence with
\[ d\nu_g(\lambda) = \beta \chi_{[0,\infty)}(\lambda) d\lambda \tag{55} \]
for some $\beta > 0$, and hence the uncoupled spectrum is composed of the eigenvalue $\{\varepsilon\}$ and an absolutely continuous part in $J$. Again the self-energy can be evaluated exactly:
\[ \Sigma_g(z) = -\beta \log(-z), \tag{56} \]
with log being the principal value of the complex logarithm, i.e. $\log(-z) = \log|z| + i \arg(-z)$, with $\arg(z) \in (-\pi, \pi)$. One can check that this function is indeed analytic in $\mathbb{C} \setminus [0, \infty)$ and has a branch cut along the support of the measure. Let us search for singular spectrum when the coupling is switched on. Solutions of the pole equation (38) must be searched in $(-\infty, 0)$ since the coupling density is nonzero in $[0, \infty)$; we have
\[ \beta \log(-\lambda) = -\varepsilon + \lambda \]
and hence
\[ -\lambda e^{-\lambda/\beta} = \frac{1}{\beta} e^{-\varepsilon/\beta}, \]
which does admit a unique real solution expressed through the principal branch $W_0$ of Lambert’s $W$-function (or product-log)[4]:
\[ E(\varepsilon, \beta) = -\beta W_0 \left( \frac{1}{\beta} e^{-\varepsilon/\beta} \right). \]
Hence we have a unique eigenvalue $E(\varepsilon, \beta)$ for $H_{g,\varepsilon}$. It is interesting to study the asymptotic behavior of $E(\varepsilon, \beta)$ as a function of the excitation energy $\varepsilon$ of the atom. Since the following properties of the Lambert function hold:
\[ W_0(x) \sim \log x, \quad \text{as } x \to \infty; \quad W_0(x) \sim x, \quad \text{as } x \to 0, \tag{57} \]
we have
\[ E(\varepsilon, \beta) \sim \varepsilon, \quad \text{as } \varepsilon \to -\infty; \quad E(\varepsilon, \beta) \sim -e^{-\varepsilon/\beta}, \quad \text{as } \varepsilon \to \infty. \tag{58} \]
This means that, when $\varepsilon$ is far away from the lowest energy level of the inner Hamiltonian $\Omega$, the coupled eigenvalue $E(\varepsilon, \beta)$ is close to $\varepsilon$ itself and hence the spectrum is nearly unchanged. When $\varepsilon$ approaches and eventually reaches $\sigma(\Omega)$, the approximation $E(\varepsilon, \beta) \sim \varepsilon$ is no longer valid and, as $\varepsilon \to \infty$, $E(\varepsilon, \beta)$ approaches the boundary of $\sigma(\Omega)$. 
Example 5.3 (Sinusoidal spectral measure). Now we consider a model in which the coupling density has support in the whole real line but admits some zeros. Let
\[ d\nu_g(\lambda) = \frac{\beta}{2\pi} (1 - \cos(\tau \lambda)) \, d\lambda \quad (59) \]
for some $\beta > 0$ and $\tau \in \mathbb{R}$, i.e. an absolutely continuous measure with sinusoidal coupling. As in the first example, here the uncoupled operator has spectrum composed of the eigenvalue $\{\varepsilon\}$ embedded in an absolutely continuous spectrum covering the whole real line; however, here the coupling density vanishes at $\lambda_j = \frac{2j\pi}{\tau}$, with $j \in \mathbb{Z}$.

The self-energy is
\[ \Sigma_g(z) = \begin{cases} \frac{i\beta}{2} (1 - e^{iz\tau}) , & \text{Im } z > 0; \\ -\frac{i\beta}{2} (1 - e^{-iz\tau}) , & \text{Im } z < 0, \end{cases} \]
which is indeed discontinuous on the whole real line except for the zeros $\lambda_j$, moreover it is a periodic function. The pole equation (38) reads
\[ \begin{cases} \varepsilon - \lambda = \frac{\beta}{2} \sin \tau \lambda; \\ \cos \tau \lambda = 1. \end{cases} \]
The second equation is solved only when, as told in advance, $\lambda$ is one of the zeros of the coupling density; if so, the first equation simply becomes $\varepsilon - \lambda = 0$. The following phenomenon occurs: the singular spectrum of $H_{g,\varepsilon}$ is empty except for some “resonant” values of the parameter $\varepsilon$, namely $\varepsilon = \frac{2\pi j}{\tau}$ for some $j \in \mathbb{Z}$; when this happens, $\sigma_{\text{sing}}(H_{g,\varepsilon}) = \{\varepsilon\}$.

Example 5.4 (Periodic discrete spectral measure). Consider the case in which
\[ \nu_g = \frac{\beta}{2\pi} \sum_{j \in \mathbb{Z}} \delta_{j\tau} \quad (60) \]
for some fixed $\tau, \beta > 0$, where $\delta_k$ is the Dirac measure at $k \in \mathbb{R}$; i.e. $\nu_g$ is supported on $\tau\mathbb{Z}$. A direct calculation shows that
\[ \Sigma_g(z) = \frac{\beta}{2} \cot \left( \frac{\pi z}{\tau} \right), \]
which can be extended to the real line except for the poles at $\tau\mathbb{Z}$. The spectrum of the uncoupled operator will be purely singular and consisting of the solutions of the equation
\[ \cot \left( \frac{\pi \lambda}{\tau} \right) = \frac{2}{\beta} (\lambda - \varepsilon), \]
which admits a countable set of isolated solutions \( \{E_j(\varepsilon, \beta, \tau)\}_{j \in \mathbb{Z}} \) (hence the spectrum of \( H_{g, \varepsilon} \) is again pure point) where \( E_j(\varepsilon, \beta, \tau) \in (j\tau, (j + 1)\tau) \).

In particular, each \( E_j(\varepsilon, \beta) \) varies smoothly with \( \beta \) and

- \( E_j(\varepsilon, \beta, \tau) \to \tau j \), as \( \beta \to 0 \), i.e. in the limit of small coupling we recover the uncoupled spectrum of \( \Omega \);

- \( E_j(\varepsilon, \beta, \tau) \to (\tau + \frac{1}{2}) j \), as \( \beta \to \infty \), i.e. in the limit of large coupling the spectrum is rigidly shifted by \( \frac{\tau}{2} \).

Differently from the previous cases, the singular spectrum (which is again pure point) is nonempty for every value of the parameters, and indeed contains a countable number of points.

**Example 5.5** (Generation of singular continuous spectrum [17]). For any integer \( n \geq 1 \), let us define the normalised Borel measure \( \nu_n \) as

\[
\nu_n = \frac{1}{2^n} \sum_{j=1}^{2^n} \delta_{j/2^n}, \tag{61}
\]

and then, for some choice of positive sequence \((a_n)_{n \in \mathbb{N}}\),

\[
\nu_g = \sum_{n=1}^{\infty} a_n \nu_n. \tag{62}
\]

This is a pure point measure with support on all the dyadic rationals between 0 and 1, i.e. the numbers which can be written in the form \( j/2^n \) for some \( j \) and \( n \), or equivalently all the numbers whose expansion in base 2 is finite; such numbers are thus dense in \([0, 1]\). Besides, it is a finite measure iff \( \sum_n a_n < \infty \), the latter sum being \( \nu_g(\mathbb{R}) = \nu_g([0, 1]) \).

Consider a Friedrichs-Lee model \( H_{g, \varepsilon} \) with spectral measure \( \nu_g \): the spectrum of this model will be entirely pure point, consisting of the dyadic rationals in \([0, 1]\) plus (if not already dyadic) the atom excitation energy \( \varepsilon \). From the general theory we know that, by switching on the atom–field interaction, the new spectrum will be entirely singular (since the absolutely continuous spectrum is preserved) and will be the closure of set of all solutions of the pole equation (38). By Theorem 4.3, the discriminant between singular continuous and pure point spectrum is given by the value of the function \( G_g(\lambda) \) in (36). In our case,

\[
G_g(\lambda) = \int_{\mathbb{R}} \frac{1}{(\lambda - \lambda')^2} d\nu_g(\lambda') = \sum_{n=1}^{\infty} a_n L_n(\lambda), \tag{63}
\]
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with

\[ L_n(\lambda) = \int_{\mathbb{R}} \frac{1}{(\lambda - \lambda')^2} d\nu_n(\lambda') = \frac{1}{2^n} \sum_{j=1}^{2^n} \frac{1}{(\lambda - \frac{j}{2^n})^2}. \]  

(64)

Now, for any \( n \) and any non-dyadic \( \lambda \in [0, 1] \) (if \( \lambda \) is dyadic, obviously \( G_g(\lambda) \) is infinite), in the previous sum the larger term is the one with \( h \) so that \( h/2^n \) is the dyadic of order \( n \) which is closest to \( \lambda \). Then \(|\lambda - h2^{-n}|\) is smaller than the distance between \( h2^{-n} \) and the consecutive dyadic rationals of the same order, which is \( 2^{-n} \). Hence we have

\[ \frac{1}{2^n} \frac{1}{(\lambda - \frac{j}{2^n})^2} \geq \frac{1}{2^n} 2^{2n} = 2^n. \]  

(65)

In other words, for each \( n \) the sum in the definition of \( L_n(\lambda) \) contains one term which is larger than \( 2^n \). This means that, if we suppose

\[ \sum_{n=1}^{\infty} a_n 2^n = \infty, \]  

then necessarily \( G_g(\lambda) = \infty \) for every \( \lambda \in [0, 1] \), meaning that the spectrum of \( H_g,\varepsilon \) will be fully singular continuous. Interestingly, rescaling the spectral measure through some coupling constant \( \beta > 0 \), the same phenomenon happens for every value of \( \beta \), however small; this is an example of instability of the dense pure point spectrum under perturbations.

6. Resonances

We now complete the discussion about the spectral properties of the Friedrichs-Lee operator, by studying the resonances of the model. Let us recall the definition of a resonance for a self-adjoint operator \( \Omega \) on a Hilbert space \( \mathcal{H} \).

**Definition 6.1.** Let \( z_0 \in \mathbb{C} \) with \( \text{Im} \ z_0 < 0 \). Then \( z_0 \) is a resonance for \( \Omega \) if there is some \( \psi \in \mathcal{H} \) such that the function \( R_\psi(z) \), defined as

\[ z \in \mathbb{C}^+ \setminus \sigma(\Omega) \mapsto R_\psi(z) = \left( \psi \left| \frac{1}{\Omega - z} \psi \right. \right) \in \mathbb{C}, \]  

(67)

with \( \mathbb{C}^+ = \{ z \in \mathbb{C}, \text{Im} \ z > 0 \} \), admits an analytic continuation from the upper to the lower half-plane having a pole at \( z_0 \).
Remark 6.2. All resonances of $\Omega$ are obviously poles of an analytic continuation of the resolvent of $\Omega$; in particular, real resonances of $\Omega$ are its eigenvalues.

Remark 6.3. The function defined in (67) is a Herglotz function: in fact, it is the (non-regularised) Borel transform of the finite spectral measure associated with $\psi$, and hence the properties listed in Proposition 4.9 hold true; in particular, every singularity of $R_\psi(z)$ lies on the real line. In particular, if $\sigma(\Omega)$ admits an absolutely continuous component, for instance in some interval $J \subset \mathbb{R}$, then $R_\psi(z)$ has a branch cut along $J$ with finite boundary values, and will thus admit [9] an analytic extension “through the cut” from the upper to the lower plane. Such a $z_0$ is identified as a resonance since its presence yields a contribution proportional to $e^{-iz_0 t}$ to the survival amplitude of $\psi$; if $z_0$ is close to the real line, this contribution may dominate at large times and make $\psi$ a metastable state with energy $\Re z_0$ and decay rate $|\Im z_0|$.

Now, for the uncoupled Friedrichs-Lee operator, one immediately shows that all nonreal resonances of $H_{g,\varepsilon}$ are nonreal resonances of $\Omega$. As for the coupled case, let us consider a Friedrichs-Lee operator whose spectrum is purely absolutely continuous on some interval $J \subset \mathbb{R}$. By Eq. (18) we can conclude that

- solutions of the analytic continuation of the pole equation $\varepsilon - z = \Sigma_g(z)$ are resonances of the model, since the first component of the resolvent will have a pole in it;
- other singularities of the resolvent are to be searched among resonances of $\Omega$.

We would like to characterise resonances at an algebraic level. Aguilar-Balslev-Combes-Simon theory of resonances (see e.g. [10]) allows us to identify resonances of $\Omega$ as the (complex) eigenvalues of a “deformed” Hamiltonian $\Omega(w)$, with $w$ being a complex parameter. Some hypotheses are required for this purpose. First of all, we need the following definition.

Definition 6.4. A spectral deformation family is a family of bounded invertible operators $(U(w))_{w \in W}$ on $\mathcal{H}$, with $W$ being an open subset of the complex plane, whose restriction to $w \in W \cap \mathbb{R}$ is unitary and $U_0 = I$. $(U(w))_{w \in W}$ admits a dense set $\mathcal{A}$ of analytic vectors (i.e. such that $w \in W \mapsto U(w)\psi$ is analytic for any $\psi \in \mathcal{A}$) with $U(w)\mathcal{A}$ being dense in $\mathcal{H}$ as well.
For any \( w \in W \) define the spectral deformation of \( \Omega \) as
\[
\Omega(w) = U(w)\Omega U(w)^{-1}, \quad \text{on } D(\Omega(w)) = U(w)D(\Omega).
\] (68)

As for \( \Omega \), it must be chosen in such a way that

**Hypothesis 6.5.** \( \Omega \) is a self-adjoint operator having an absolutely continuous spectrum in some (not necessarily bounded) interval \( J \subset \mathbb{R} \), plus possibly a discrete spectrum outside \( J \).

Besides, its continuous spectrum must modify “nicely” under spectral deformations, in the sense that it must be always possible, for any nonreal \( w \), to “move continuously” through \( J \) from the upper to the complex half-plane, so that the matrix elements of the resolvent of \( \Omega(w) \) are analytic continuation of those of \( \Omega \) through \( J \). Technically:

**Hypothesis 6.6.** There exists an open, connected subset \( S \supset J \) of the complex plane, with \( \emptyset \neq S^\pm = S \cap \mathbb{C}^\pm \), and a spectral deformation family \( (U(w))_{w \in W} \), with \( S \subset W \), such that
\begin{itemize}
  \item for every \( w \in W \) with \( \text{Im } w > 0 \), \( \sigma_{\text{ess}}(\Omega(w)) \cap S_+ = \emptyset \);
  \item given \( \delta > 0 \), there is some \( w \in W_{\delta} = \{ w \in W : \text{Im } w > \delta \} \) and some connected \( S_\delta \subset \overline{S}, \) with \( \emptyset \neq S^\pm_\delta = S_\delta \cap \mathbb{C}^\pm \), such that \( \sigma_{\text{ess}}(\Omega(w)) \cap S^-_\delta = \emptyset \).
\end{itemize}

Finally, we must require analyticity of the operator-valued function \( w \mapsto \Omega(w) \). Since \( \Omega \) is generally unbounded and its domain may change under \( U(w) \) for nonreal \( w \), it is convenient to require analyticity of the (bounded) resolvent, thus avoiding any domain issue:

**Hypothesis 6.7.** For every \( z \in S^+ \), the operator-valued function \( w \in W \mapsto \frac{1}{\Omega(w)-z} \) is strongly analytic in \( w \).

Notice that we are taking \( z \in S^+ \) to ensure that, by Hypothesis 6.6, the operator \( \frac{1}{\Omega(w)-z} \) is always well-defined and bounded.

**Remark 6.8.** Notice that it suffices to suppose that \( w \in W \mapsto \frac{1}{\Omega(w)-z_0} \) is strongly analytic in \( w \) for some \( z_0 \in S^+ \). This follows by observing that, for every \( z \) in the resolvent set (and hence, in particular, in \( S^+ \)), the resolvent operator in \( z \) can be written as a norm convergent series of powers of \( \frac{1}{\Omega(w)-z_0} \), each therefore being analytic in \( w \).
Figure 1. Graphical representation of hypothesis 6.6. When choosing any \( w \) with \( \text{Im} w > \delta > 0 \), there must always exist, as in the figure, a set \( S_\delta \) which does not intersect the essential spectrum of \( \Omega(w) \). Notice that, in general, the latter will not be a line.

One gets the following result:

**Theorem 6.9** (Aguilar-Balslev-Combes-Simon [10]). Let \( \Omega \) a self-adjoint operator satisfying Hypotheses 6.5–6.7. Then

- for every \( \psi \in \mathcal{A} \), \( z \in \mathbb{C}^+ \mapsto R_\psi(z) \) has a meromorphic extension \( R_\psi(z,w) \) across \( J \) from \( \mathbb{C}^+ \) to the union of all \( S_\delta^- \): in particular, for every \( w \in W \) with \( \text{Im} w > \delta \),

\[
R_\psi(z,w) = \left\langle \psi(w) \left| \frac{1}{\Omega(w) - z} \psi(w) \right. \right\rangle,
\]

(69)

with \( \psi(w) = U(w)\psi \);

- all resonances of \( \Omega \) are (complex) eigenvalues of the deformed Hamiltonian \( \Omega(w) \) for some \( w \in W \).

Notice that resonances are obviously independent of the particular choice of \( U(w) \). For completeness, we will give a sketch of the proof.

**Proof.** Let \( \psi \in \mathcal{A} \). Fixing any \( z \in S^+ \), for every \( w \in W \cap \mathbb{R} \) the equality \( R_\psi(z) = R_\psi(z,w) \) holds; but, by Hypothesis 6.7, the function
$w \in W \mapsto R_\psi(z, w)$ is analytic and hence, repeating the same argument for any $z \in S^+$, the equality $R_\psi(z) = R_\psi(z, w)$ holds for any $w \in W$ and $z \in S^+$.

Now fix, in particular, $w \in W_\delta$; the equality $R_\psi(z, w) = R_\psi(z)$ holds, in particular, for any $z \in S^\delta$. But we know, by Hypothesis 6.6, that the open set $S^\delta \supset S^\delta$ does not intersect $\sigma(\Omega(w))$, hence the matrix element $R_\psi(z, w)$ of the resolvent of $\Omega(w)$ is well-defined and analytic in the whole $S^\delta$; since, in the subset $S^\delta$, $R_\psi(z, w) = R_\psi(z)$ and both functions are analytic in $z$, we finally conclude that, for any $w \in W_\delta$, $z \in S^\delta \mapsto R_\psi(z, w)$ is a meromorphic extension of $R_\psi(z)$ from $S^\delta$ to $S^\delta$. Finally, since $R_\psi(z, w)$ is the expectation value of the resolvent of $\Omega(w)$ in the state $\psi$, every pole of it is necessarily an eigenvalue of $\Omega(w)$. □

**Remark 6.10.** In the case in which $\Omega$ is a multiplication operator on $L^2(X, \mu)$, a spectral deformation family may be constructed by considering an isometric global flow $R_w$ (for real $w$) acting onto $X$, defining the spectral deformation family as follows:

\[
(U(w)\psi)(k) = J_w(k)^{1/2}\psi(R_w(k)) \quad (70)
\]

for $\psi \in \mathcal{H}$, $J_w(k)$ being the Jacobian of the transformation, extending $R_w(k)$ to an open subset $W \subset \mathbb{C}$, and finding a dense set of analytic vectors on which $w \in W \mapsto U(w)\psi \in \mathcal{H}$ is analytic; besides, for $\psi \in U(w)D(\Omega)$,

\[
(\Omega(w)\phi)(k) = \omega(R_w(k))\psi(k), \quad (71)
\]

hence the “deformed” operator will still be a multiplication operator. The flow must be chosen in such a way that the spectrum of $\Omega(w)$ (i.e. its essential range) is deformed in the desired way to unearth the resonances.

Now we claim that, if $\Omega$ has a spectral deformation family, then the Friedrichs-Lee operator with inner Hamiltonian $\Omega$, under some assumptions, has itself a spectral deformation family and hence Aguilar-Balslev-Combes-Simon theory can be applied:
Theorem 6.11. Let $\Omega$ be a self-adjoint operator with a spectral deformation family $(U(w))_{w \in W}$ and a set of analytic vectors $A$ satisfying hypotheses 6.5–6.7; let $H_{g,\varepsilon}$ be the Friedrichs-Lee operator with inner Hamiltonian $\Omega$ and $g \in H_{-2}$; suppose $\frac{1}{\Omega(z) - z} g \in A$ for any $z \notin \sigma(\Omega(w))$. Then $H_{g,\varepsilon}$ has a spectral deformation family $(\tilde{U}(w))_{w \in W}$ and a set of analytic vectors $\tilde{A}$ satisfying hypotheses 6.5–6.7.

Proof. We will show that hypotheses 6.5–6.7 are satisfied for the Friedrichs-Lee Hamiltonian. First of all, by setting $\tilde{U}(w) := I \oplus U(w)$ and $\tilde{A} = C \oplus A$, it is immediate to show that $(\tilde{U}(w))_{w \in W}$ is a spectral deformation family with a set $\tilde{A}$ of analytic vectors. Besides, Theorem 4.3 ensures that $H_{g,\varepsilon}$ has an absolutely continuous spectrum in $J$ plus possibly some singular spectrum outside $J$ which, by Theorem 4.1, will be discrete, hence Hypothesis 6.5 holds.

To prove that Hypotheses 6.6–6.7 hold, let us evaluate the “deformed” resolvent. Define, for any $w$ with $\text{Im} w > 0$, $g(w) = U(w)g$ and

$$
\Sigma_g(z, w) = \left\langle g(w) \left| \left( \frac{1}{\Omega(w) - z} - \frac{\Omega(w)}{\Omega(w)^2 + 1} \right) g(w) \right. \right\rangle,
$$

which is a meromorphic extension of the self-energy function from the upper half-plane to $S_0^+$, as can be proven following the same steps used in the proof of Theorem 6.9 to prove that $R_\psi(z, w)$ is a meromorphic continuation of $R_\psi(z)$. By the definition of $\tilde{U}(w)$ and Eq. (18) we obtain, for any $x \in \mathbb{C}$ and $\xi \in \mathcal{H}$,

$$
\tilde{U}(w) \frac{1}{H_{g,\varepsilon} - z} \tilde{U}(w)^{-1} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} x - \left\langle g(w) \left| \frac{1}{\Omega(w) - z} \xi \right. \right\rangle & \frac{x - \left\langle g(w) \left| \Sigma_g(z, w) \right. \right\rangle}{\Omega(w) - z} \\
\frac{x - \left\langle g(w) \left| \Sigma_g(z, w) \right. \right\rangle}{\Omega(w) - z} & \frac{x - \left\langle g(w) \left| \Sigma_g(z, w) \right. \right\rangle}{\Omega(w) - z} \end{pmatrix}.
$$

Now, the following properties hold:

- since $\Omega$ satisfies hypothesis 6.6, $\sigma_{\text{ess}}(\Omega(w))$ does not intersect $S^+$ for every $w$ with positive imaginary part;

- since $\Sigma_g(z, w) = \Sigma_g(z)$ in $S^+$, the equation $\varepsilon - z = \Sigma_g(z, w)$ cannot be satisfied for any $z \in S^+$.

---

2 Again (see the remark to Hypothesis 6.7) it suffices to require analyticity for some $z \in S^+$. 
and hence the resolvent in Eq. (73) is well-defined in $S$ for every $w$, and coincides with the resolvent of $H_{g,\varepsilon}$. Again, following the same steps of the proof of Theorem 6.9 one proves that, fixing any $\delta > 0$, the deformed resolvent (as well as its matrix elements) can be meromorphically continued from $S^+_{\delta}$ to $S^-_{\delta}$; besides, by Eq. (73), since by hypothesis $S^-_{\delta}$ has zero intersection with $\sigma_{\text{ess}}(\Omega)$, the only singularities of the resolvent and its matrix elements are either solutions of the equation $\varepsilon - z = \Sigma_g(z,w)$, which are necessarily isolated points, or elements of the discrete spectrum of $H_{g,\varepsilon}(w)$; this implies that $S^-_{\delta}$ does not intersect the essential spectrum of $H_{g,\varepsilon}(w)$ and hence Hypothesis 6.6 holds. The analyticity of the resolvent, i.e. Hypothesis 6.7, is an immediate consequence of the analyticity of the resolvent of $\Omega(w)$ and of $g(w)$. \hfill \Box

**Remark 6.12.** The operator (73) is the resolvent of an operator $H_{g,\varepsilon}(w) = U(w)H_{g,\varepsilon}U(w)^{-1}$ defined as follows:

\[
D(H_{g,\varepsilon}(w)) = \left\{ \left( \frac{x}{\Omega(w)} \right) \left( \frac{x}{\Omega(w)^2 + 1} g(w) \right) \mid x \in \mathbb{C}, \xi \in D(\Omega(w)) \right\}
\] (74)

\[
H_{g,\varepsilon}(w) \left( \frac{x}{\Omega(w)} \right) \left( \frac{x}{\Omega(w)^2 + 1} g(w) \right) = \left( \frac{\varepsilon x}{\Omega(w)} \right) \left( \frac{\varepsilon x}{\Omega(w)} + \frac{\langle g(w) | \xi(w) \rangle}{\Omega(w)} \right) + \frac{\langle g(w) | \xi(w) \rangle}{\Omega(w)} g(w) \right).
\] (75)

Notice that, even if we suppose that $\Omega(w)$ itself is strongly analytic (i.e. $D(\Omega(w)) = D(\Omega)$), the “deformed” Friedrichs-Lee operator has generally a different domain from the “original” one. Also notice that $\langle g(w) | \xi(w) \rangle = \langle g | \xi \rangle$ for every $w \in W$, and hence the first component of the vector does not depend on $w$.

**Example 6.13.** As a simple example, consider a Friedrichs-Lee operator in $L^2(\mathbb{R})$ with dispersion relation $\omega(k) = k$ (hence with $\sigma(H) = \mathbb{R}$) and “flat” singular coupling $g(k) = \sqrt{\beta/2\pi}$, $\beta > 0$; this is a realisation of a Friedrichs-Lee model with flat spectral measure studied in example 5.1. In this case the analytic extension of the self-energy from the upper to the lower half-plane is simply equal to $i\beta/2$, and hence the resonance equation simply reads

\[
z = \varepsilon - i\frac{\beta}{2},
\] (76)

i.e. the eigenvalue $\varepsilon$ of the uncoupled model is transformed in a resonance whose decay rate is $\beta$. 

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To implement this resonance at the operator level, we must find a flow that “pushes” the spectrum below. Define, for any $w > 0$,

$$R_w(k) = k - w, \quad (77)$$

whose Jacobian for real $w$ is 1; hence we obtain

$$\begin{align*}
(U(w)\psi)(k) &= \psi(k - w), \\
(\Omega(w)\psi)(k) &= (k - w)\psi(k),
\end{align*} \quad (78)$$

and the extension to complex values of $w$ is immediate: there is indeed a dense set of analytic vectors in $L^2(\mathbb{R})$ with respect to this transformation, which can be simply taken as the set of analytic square-integrable functions. Besides, $\sigma(\Omega(w))$ will be

$$\sigma(\Omega(w)) = \{x - i \text{Im } w : x \in \mathbb{R}\} = \mathbb{R} - i \text{Im } w; \quad (79)$$

for $\text{Im } w > \beta/2$ the pole in $\varepsilon - i\beta/2$ gets unveiled and is therefore an eigenvalue of $\Omega(w)$.

7. Inverse problem. The case of exponential decay

Theorem 4.3 allows us to fully characterise the spectrum, and hence the dynamics, of a Friedrichs-Lee Hamiltonian with some given momentum space $(X, \mu)$, dispersion relation $\omega$ and form factor $g$. On the other hand, we may want to solve the inverse problem, i.e. finding a Friedrichs-Lee model with some given spectrum, and hence some given dynamics.

As an important consequence of Theorem 4.3, the spectrum of the Friedrichs-Lee Hamiltonian with a fixed $\varepsilon$ depends entirely on the spectral measure $\nu_g$, or equivalently on the self-energy $\Sigma_g(z)$ which can be always reconstructed from the spectral measure through the inversion formula

$$\nu_g((\lambda_0, \lambda]) = \frac{1}{\pi} \lim_{\delta \downarrow 0} \int_{\lambda_0}^\lambda \text{Im } \Sigma_g(\lambda + i\delta) \, d\lambda, \quad (80)$$

different choices of the momentum space $(X, \mu)$, of the dispersion relation $\omega$ and of the form factor $g$, but yielding the same $\nu_g$, are fully equivalent at the spectral level. A trivial solution (although not necessarily physically meaningful) always exists: if $\nu$ is the desired spectral measure, just choose $g(k) = 1$, $\omega(k) = k$ and $(X, \mu) = (\mathbb{R}, \nu)$ and obtain

$$\nu_g(B) = \int_{\omega^{-1}(B)} |g(k)|^2 \, d\mu(k) = \int_B d\nu(\lambda) = \nu(B). \quad (81)$$
However, in the applications, the momentum space \((X, \mu)\) and its dispersion relation \(\omega(k)\) are fixed by the structure of the bosonic bath, but the form factor \(g(k)\) may be unknown a priori. Thus we are led to study the following inverse problem: for a given choice of \((X, \mu)\) and \(\omega\), what choices of \(g(k)\) correspond to a Friedrichs-Lee Hamiltonian whose spectral measure \(\nu_g\) yield our desired dynamics?

As an example, let us analyse the following problem: let us construct a Friedrichs-Lee operator such that the square modulus of \(x(t)\), defined as in Eq. (21), decays with a purely exponential law. By Eq. (22), an exponential law is obtained if \(\nu_g\) is, up to a multiplicative constant, the Lebesgue measure on the whole real line. Indeed, if \(d\nu_g(\lambda) = \beta \frac{\pi}{2} d\lambda\), then by Eq. (19) one readily obtains

\[
\Sigma_g(z) = \begin{cases} 
+i\frac{\beta}{2}, & \text{Im } z > 0; \\
-i\frac{\beta}{2}, & \text{Im } z < 0,
\end{cases} \tag{82}
\]

and hence, by Eq. (22), for \(t > 0\)

\[
x(t) = e^{-(\beta/2+i\epsilon)t}, \tag{83}
\]

implying \(|x(t)|^2 = e^{-\beta t}\), for any value of \(\epsilon\). A Friedrichs-Lee model on a measure space \((X, \mu)\), with dispersion \(\omega\) and form factor \(g\), will be characterised by the following equation:

\[
\frac{\beta}{2\pi}(\lambda - \lambda_0) = \int_{\omega^{-1}([\lambda_0, \lambda])} |g(k)|^2 d\mu(k), \quad \lambda, \lambda_0 \in \mathbb{R}. \tag{84}
\]

Let us find some nontrivial Friedrichs-Lee models satisfying Eq. (84) with space \(X = \mathbb{R}^d\) and \(\mu\) as the Lebesgue measure on \(\mathbb{R}^d\).

**Example 7.1.** Consider a dispersion relation \(\omega\) depending only on the projection of the momentum in some direction; without loss of generality, we fix a reference frame such that

\[
\omega(k_1, \ldots, k_d) = w(k_1), \tag{85}
\]

with \(w : \mathbb{R} \to \mathbb{R}\) being differentiable and strictly increasing. The latter hypothesis ensures

\[
\omega^{-1}([\lambda_0, \lambda]) = [w^{-1}(\lambda_0), w^{-1}(\lambda)] \times \mathbb{R}^{d-1}, \tag{86}
\]

and hence Eq. (84) becomes

\[
\frac{\beta}{2\pi}(\lambda - \lambda_0) = \int_{w^{-1}(\lambda_0)}^{w^{-1}(\lambda)} f(k_1) \, dk_1, \tag{87}
\]
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\[ f(k_1) = \int_{\mathbb{R}^{d-1}} |g(k_1, k_2, \ldots, k_d)|^2 \, dk_2 \ldots dk_d. \]  
\[ \text{(88)} \]

Eq. (87) is satisfied iff \( f(k_1) = \beta^{2/\pi} w'(k_1) \), hence

\[ g(k_1, k_2, \ldots, k_d) = e^{i\phi(k_1)} \sqrt{\frac{\beta}{2\pi} w'(k_1) h(k_2, \ldots, k_d)}, \]  
\[ \text{(89)} \]

with \( \phi: \mathbb{R} \to \mathbb{R} \) and \( h \) a function on \( \mathbb{R}^{d-1} \) such that

\[ \int_{\mathbb{R}^{d-1}} |h(k_2, \ldots, k_d)|^2 \, dk_2 \ldots dk_d = 1. \]  
\[ \text{(90)} \]

This result can be readily generalised to the case in which \( w \) is piecewise monotonically increasing or decreasing: in this case the form factor is

\[ g(k_1, k_2, \ldots, k_d) = e^{i\phi(k_1)} \sqrt{\frac{\beta}{2\pi} |w'(k_1)| h(k_2, \ldots, k_d)}. \]  
\[ \text{(91)} \]

**Example 7.2.** As a second example, consider a dispersion relation \( \omega \) depending only on the modulus of the momentum. For any \( k \in \mathbb{R}^d \) we can write \( k = rn \), where \( r = |k| \) and \( n \in S^1 \), with \( S^1 \) being the unit sphere. We assume that

\[ \omega(k) = w(r), \]  
\[ \text{(92)} \]

with \( w: \mathbb{R}^+ \to \mathbb{R} \) again being differentiable and strictly increasing. Again we obtain an equation analogous to Eq. (87):

\[ \frac{\beta}{2\pi} (\lambda - \lambda_0) = \int_{w^{-1}(\lambda_0)}^{w^{-1}(\lambda)} f(r) r^{d-1} \, dr, \]  
\[ \text{(93)} \]

where

\[ f(r) = \int_{S^1} |g(k)|^2 \, dS(n). \]  
\[ \text{(94)} \]

Eq. (93) is satisfied when \( f(r) = \beta^{2/\pi} w'(r)/r^{d-1} \), and hence the form factor is

\[ g(k) = e^{i\phi(r)} \sqrt{\frac{\beta}{2\pi} \frac{w'(r)}{r^{d-1}}} h(n), \]  
\[ \text{(95)} \]

again with \( \phi: \mathbb{R} \to \mathbb{R} \) arbitrary and \( h \) being a function on \( S^1 \) such that

\[ \int_{S^1} |h(n)|^2 \, dS(n) = 1. \]  
\[ \text{(96)} \]

This procedure can be generalised for a function \( w \) piecewise monotonically increasing or decreasing, in this case we obtain the form factor,

\[ g(k) = e^{i\phi(r)} \sqrt{\frac{\beta}{2\pi} \frac{|w'(r)|}{r^{d-1}}} h(n). \]  
\[ \text{(97)} \]
Acknowledgements

This work is partially supported by Istituto Nazionale di Fisica Nucleare (INFN) through the project “QUANTUM”, and by the Italian National Group of Mathematical Physics (GNFM-INdAM).

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