PULLBACK ATTRACTOR FOR A WEAKLY DAMPED WAVE EQUATION WITH SUP-CUBIC NONLINEARITY

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ABSTRACT. In this paper, the non-autonomous dynamical behavior of weakly damped wave equation with a sup-cubic nonlinearity is considered in locally uniform spaces. We first prove the global well-posedness of the Shatah-Struwe solutions, then establish the existence of the \((H^1_0(\mathbb{R}^3) \times \dot{L}^2_0(\mathbb{R}^3), H^1_0(\mathbb{R}^3) \times \dot{L}^2_0(\mathbb{R}^3))\)-pullback attractor for the Shatah-Struwe solutions process of this equation. The results are based on the recent extension of Strichartz estimates for the bounded domains.

1. Introduction. It is well known that wave phenomena frequently occur in many areas of modern mathematical physics, such as electrodynamics, quantum mechanics, nonlinear elasticity etc., see [40, 39, 36] and the references therein. The wave equation in the form of (1.1) arises as an evolutionary mathematical model which has been extensively studied by lots of authors.

We shall study the global solvability and asymptotic behavior of solutions for the following semilinear damped wave equation with a sup-cubic nonlinearity in locally uniform spaces:

\[
\partial^2_t u + \gamma \partial_t u - \Delta u + f(u) = g(t), \quad \text{in } \mathbb{R}^{N=3} \times (\tau, +\infty)
\]

subject to the initial conditions

\[
u(x, \tau) = u_\tau(x), \quad \partial_t u(x, \tau) = v_\tau(x), \quad \text{in } \mathbb{R}^3,
\]

where \(\gamma > 0\). The nonlinearity \(f \in C^1(\mathbb{R})\) satisfies the following assumptions:

\[|f'(s)| \leq C(1 + |s|^{p-1}), \quad p \in [1, p^{**}), \quad \forall s \in \mathbb{R};\]

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\textit{dissipation condition}\\
\exists k > 0, \mu_1 > 0, \forall \mu \in (0, \mu_1], \exists C_\mu \in \mathbb{R},\\
kF(s) + \mu s^2 - C_\mu \leq sf(s), \forall s \in \mathbb{R}, \quad (1.4)\\
sf(s) \geq -C, \quad \forall s \in \mathbb{R}, \quad (1.5)\\

where \(C\) is a positive constant, \(p^{**} := \frac{N+2}{N-2}\) (\(p^{**} = 5\) for our case \(N = 3\)) and \(F(s) = \int_0^s f(\tau)d\tau\). Using (1.3), we have

\[|f(s)| \leq C(1 + |s|^p), \quad p \in [1, p^{**}).\]  

Suppose that the external force \(g(s) = g(\cdot, s) \in L^1_{loc}(\mathbb{R}; L^2_{lu}(\mathbb{R}^3))\) is translation bounded, i.e.,

\[\|g\|_{L^1_{loc}(\mathbb{R}; L^2_{lu}(\mathbb{R}^3))} := \sup_{y \in \mathbb{R}^3} \sup_{t \in \mathbb{R}} \|g\|_{L^1(t,t+1; L^2(B^1_y))} < +\infty,\]  

where \(B^1_y := \{x \in \mathbb{R}^N \mid |x - y| < 1\}\). Moreover, we can deduce from (1.7) that

\[
\sup_{y \in \mathbb{R}^3} \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\sigma(t-s)} \|g(s)\|_{L^2(B^1_y)} ds \leq \frac{1 - e^{-\sigma}}{1 - e^{-\sigma}} \|g\|_{L^1_{loc}(\mathbb{R}; L^2_{lu}(\mathbb{R}^3))} < +\infty
\]  

for any \(\sigma > 0\).

To some extent, as compared with the case of bounded domains, the case of unbounded domains becomes much more complicated. In this case, some of the methods for bounded domains are no longer valid and the problems that finding the appropriate phase spaces becomes nontrivial.

In order to include the special solutions (e.g., equilibria and relaxation waves) in the attractor, some authors use locally uniform spaces, which enjoy suitable nesting properties. In [30, 20, 31, 18, 44, 2, 15, 16, 17, 13], the authors studied different nonlinear equations and their nonlinear dynamics in locally uniform spaces, e.g., Arrieta et al in [2, 3] studied locally uniform spaces systematically and presented a rather complete linear theory of parabolic equations with initial data in locally uniform spaces, simultaneously, they considered the existence of an attractor of the Cauchy problem for a semilinear second order parabolic equation in the locally uniform space \(W^{s,p}_{loc}(\mathbb{R}^N)\). More recently, Cholewa and Rodríguez-Bernal in [13] have obtained that the linear 2m-th order uniformly elliptic operators which can generate semigroups of bounded linear operators with suitable smoothing properties in scales of locally uniform Bessel’s and Lebesgue’s spaces, which makes the theory of locally uniform spaces become more complete.

The well-posedness and long-time dynamics of semilinear wave equations in bounded domains have been investigated extensively by many authors (see [1, 4, 14] and the references therein), which depends strongly on the growth rate of the nonlinearity. The exponent \(p\) as in (1.6) is called the growth exponent of \(f(u)\), the number \(p^* \equiv N^\frac{N}{N-2}(N \geq 3)\) is called critical exponent relative to the natural energy space \(X = H^1_0(\Omega) \times L^2(\Omega)\), since the uniqueness of the usual energy solutions may be lost when \(p > p^*\), e.g., see Ball [5]. For the problem (1.1)-(1.2), Cholewa and Dlotko [16] proved the existence of \((H^1_{lu}(\mathbb{R}^N) \times \tilde{L}^2_{lu}(\mathbb{R}^N), H^1_p(\mathbb{R}^N) \times L^2_p(\mathbb{R}^N))\)-global attractor for the subcritical case, i.e., \(p < p^*\). In [44], for both autonomous case and non-autonomous case of (1.1)-(1.2), Zelik established the existence of the \((H^1_{lu}(\mathbb{R}^N) \times \tilde{L}^2_{lu}(\mathbb{R}^N), H^1_{loc}(\mathbb{R}^N) \times \tilde{L}^2_{loc}(\mathbb{R}^N))\)-attractor when \(p < p^*\). Recently, Michálek, Pražák and Slavík [29] have studied the well-posedness of a semilinear
wave equation with a nonlinear damping in locally uniform spaces for the critical case, and established existence of a locally compact attractor just only for the autonomous dynamical system by applying the method of $\ell$-trajectories that introduced by Pražák et al.

In the case $3 < p \leq 5 (N = 3)$, only the energy control is not sufficient to treat the nonlinearity properly, but it remains subordinated to the linear part if more delicate space-time integrability properties for the solutions of the linear equation are used. In order to get uniqueness, one needs the so-called Strichartz estimates for the solutions (see [37, 36, 8, 7] and references therein) like

$$u \in L^4_{loc}(\mathbb{R}; L^{12}(\Omega)).$$  \hspace{1cm} (1.9)

Note that it is still an open problem whether or not the extra regularity (1.9) holds for any energy solutions, so we will refer to the energy solutions satisfying (1.9) as Shatah-Struwe solutions. In [23], for (1.1) defined on a compact manifold without boundary, Kapitanski established the existence of global attractor for $f(u)$ satisfies (1.6) with $1 \leq p < p^{**}$ by using the Strichartz type estimates; In [19], Feireisl constructed the global attractor for (1.1) in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ for the sub-quintic case. Lately, under the assumption $p < p^{**}$, the existence of global attractor for autonomous system (1.1) in locally uniform spaces has been considered in [28]. The quintic case $p = 5$ is much more difficult since it is not clear how to “lift” the extra space-time integrability from linear equation to the nonlinear one (at least in a straightforward way). Recently, by applying the Strichartz estimates about the linear wave equation in bounded domain $\Omega \subseteq \mathbb{R}^3$ developed by [8, 7, 9], for (1.1) defined on a bounded smooth domain of $\mathbb{R}^3$ with quintic nonlinearity, the existence and uniqueness of Shatah-Struwe solutions together with the corresponding global attractor have been obtained in [32, 22]. But, in locally uniform spaces, how to obtain the Strichartz estimates for the quintic case is also still an open problem. Since the Strichartz norm may a priori grow uncontrollably as $t \to \infty$ and which prevents the applications to the attractor theory.

For a non-autonomous dynamical system, the solution map does not define a semigroup and, instead, it defines a two-parameter process. In some nonautonomous system the trajectories can be unbounded when time increases to infinity, the classical theory of uniform attractors [12] was not applicable in such systems. In order to handle such problems, Kloeden [25] introduced the theory of pullback attractors which is a natural generalization of the theory of global attractors developed to study autonomous dynamical systems, and it is an appropriate geometric object for describing asymptotic dynamics for the processes of such systems. Whereafter, Carvalho et al. [11] developed the existence theory for pullback attractors in an abstract setting and applied these theory to a range of equations. Caraballo et al. [10] advanced the concept of the pullback $\mathcal{D}$-attractor without uniformly dissipative properties, and the existence of the pullback $\mathcal{D}$-attractor was proved under the assumptions of asymptotic compactness and existence of a family of absorbing sets. This dynamical framework allows us to handle more general non-autonomous time dependence (e.g., the external force needs to be neither almost periodic nor translation compact in time); Such as, Bates and Lu [6] have established the existence of a pullback attractor for a stochastic reaction-diffusion equation on all n-dimensional space, Wang [41] gave a sufficient and necessary condition for existence of pullback attractors for the non-autonomous non-compact dynamical systems. Recently, Wang [42] has studied the multivalued non-autonomous random dynamical system
generated by the non-autonomous stochastic wave equations with a cubic nonlinearity on $\mathbb{R}^3$, and established the existence and measurability of random attractors of this non-autonomous random dynamical system. Whereafter, Wang et al. [43] have proved the existence of $\mathcal{D}$-pullback random attractor for the continuous cocycle of the stochastic reaction-diffusion equation driven by a white noise on $\mathbb{R}^n$.

Hence, a natural and interesting question is how to study the asymptotic dynamics of (1.1)-(1.2) via pullback attractors of the corresponding process in locally uniform spaces, especially when the nonlinearity $f$ has a sup-cubic exponent and when the external force $g$ satisfies (1.7).

Note that all the mentioned results (e.g., [19, 23, 32, 22]) about sup-cubic case were depended crucially on the so-called Strichartz type estimates. However, the locally uniform space is very different from the usual Sobolev space defined on $\mathbb{R}^N$ (e.g., the space considered in [19]) and is also certainly different from the Sobolev spaces defined on bounded domain (e.g., the space considered in [32, 22]), as well as the a priori estimates in such space, but seems contain its mixed difficulties. In addition, the external forces need not be bounded in $L^2(\mathbb{R}; L^p_{lu}(\mathbb{R}^3))$, this will lead to some new difficulties for the establishment of the Strichartz type estimates in locally uniform spaces. Thus how to adjust the Strichartz type estimates to the locally uniform space when the external force $g$ satisfies (1.7) is our first important task. In this paper, by utilizing carefully the recent progress in Strichartz estimates for the case of bounded domains deduced in [8, 7, 9] and the nice properties between the uniform spaces and the locally uniform spaces introduced in [2, 3], we establish the Strichartz type estimates in locally uniform spaces and then obtain the existence and uniqueness of Shatah-Struwe solutions of (1.1) with sup-cubic nonlinearity in locally uniform space. On the other hand, as mentioned above, certain pullback asymptotically compactness of the solutions process is crucial for the attractor theory. Due to the higher growth of the nonlinear term, it is hard to apply the methods used in [44, 16, 29] directly, which forms the second main task of this paper. For this, we utilize the so-called contractive function method introduced in [14, 24, 38] and give some new time-space a priori estimates when the external force satisfies (1.7), and finally deduce the necessary pullback asymptotical compactness.

The framework of this paper is as follows. In Section 2, we first give preliminaries and abstract results on the $(X, X_\rho)$-pullback attractors and the key technical (i.e., Strichartz estimates for the linear wave equation). In Section 3, the Shatah-Struwe solutions are introduced, then the global well-posedness and dissipation of Shatah-Struwe solutions as $1 \leq p < 5$ are obtained, see Theorem 3.3. A pullback absorbing set of the corresponding process is obtained in Section 4. Then, together with some new time-space a priori estimates, the $(H^1_{lu}(\mathbb{R}^3) \times L^2_{lu}(\mathbb{R}^3), H^1_{\rho}(\mathbb{R}^3) \times L^2_{\rho}(\mathbb{R}^3))$-pullback asymptotical compactness is proved by applying the contractive function method. Finally, we obtain the existence of the $(H^1_{lu}(\mathbb{R}^3) \times L^2_{lu}(\mathbb{R}^3), H^1_{\rho}(\mathbb{R}^3) \times L^2_{\rho}(\mathbb{R}^3))$-pullback attractor, see Theorem 5.2.

2. Preliminaries. In this section, we recall some concepts and mathematical preliminaries that will be used later.

2.1. Functional spaces. In this paper, we consider the following weighted space and locally uniform space (see [44, 2, 15, 16, 17] for more details)

$$L^p_{\rho}(\mathbb{R}^N) = \{ \varphi \in L^p_{loc}(\mathbb{R}^N); ||\varphi||_{L^p_{\rho}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \rho(x)|\varphi(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \};$$
\[ L^p_{\text{loc}}(\mathbb{R}^N) = \{ \varphi \in L^p_{\text{loc}}(\mathbb{R}^N) ; \| \varphi \|_{L^p_{\text{loc}}(\mathbb{R}^N)} = \sup_{y \in \mathbb{R}^N} \| \varphi \|_{L^p_{\text{loc}}(\mathbb{R}^N)} < \infty \}, \]

where \( \rho : \mathbb{R}^N \to (0, \infty) \) is a strictly positive integrable weighted function and \( \rho_y(x) := \rho(x-y), 1 \leq p < \infty \). Analogously, the locally uniform spaces \( W^{m,p}_{\text{loc}}(\mathbb{R}^N) \) can be defined, respectively, by \( L^p_{\text{loc}}(\mathbb{R}^N) \) in a way similar to the standard \( W^{m,p}(\mathbb{R}^N) \).

And we consider the following integrable weighted functions \( \rho \in C^2(\mathbb{R}^N), \)

\[
\rho(x) = (1 + \varepsilon |x|^2)^{-l}, \quad \varepsilon > 0, \quad l > \frac{N}{2}, \tag{2.1}
\]

then, obviously, one can obtain the estimates that \( |\nabla \rho| \leq C \sqrt{\varepsilon} \rho \) and \( |\Delta \rho| \leq C \varepsilon \rho \). As shown in [2, 15, 16, 17], the integrable strictly positive functions \( \rho \) in (2.1) will lead to the same locally uniform spaces \( W^{m,p}_{\text{loc}}(\mathbb{R}^N), m = 0, 1, 2, p \geq 1 \), up to equivalent norms.

Now, we recall the uniform spaces \( W^{s,p}_U(\mathbb{R}^N), s \in \mathbb{R}^+ \cup \{0\} \) [30, 20, 18, 2], the Banach spaces consisting of all \( \varphi \in W^{s,p}_U(\mathbb{R}^N) \) such that

\[
\| \varphi \|_{W^{s,p}_U(\mathbb{R}^N)} = \sup_{y \in \mathbb{R}^N} \| \varphi \|_{W^{s,p}(B^1_y)} < \infty,
\]

where \( B^1_y := \{ x \in \mathbb{R}^N \mid |x-y| < 1 \} \).

Next, we recall some Sobolev type embeddings of these spaces.

**Lemma 2.1** ([2]). (i) If \( s_1 \geq s_2 \geq 0, 1 < p_1 \leq p_2 < \infty \) and \( s_1 - \frac{N}{p_1} \geq s_2 - \frac{N}{p_2} \), then

\[
W^{s_1,p_1}_U(\mathbb{R}^N) \hookrightarrow W^{s_2,p_2}_U(\mathbb{R}^N)
\]

is continuous. Moreover,

\[
\| \varphi \|_{W^{s,p}_U} \leq \| \varphi \|_{W^{s_1,p_1}_U} \| \varphi \|_{W^{s_2,p_2}_U}^{1-\theta},
\]

where \( \theta \in [0,1], \frac{1}{p} \leq \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, p, p_1, p_2 \in (1, \infty) \) and

\[
s - \frac{N}{p} \leq \theta \left( s_1 - \frac{N}{p_1} \right) + (1-\theta) \left( s_2 - \frac{N}{p_2} \right).
\]

(ii) If \( \rho \) is given in (2.1), then the embedding \( W^{s_1,p_1}_U(\mathbb{R}^N) \hookrightarrow W^{s_2,p_2}_U(\mathbb{R}^N) \) is compact provided that \( s_2 \in \mathbb{N}, s_1 > s_2, 1 < p_1 \leq p_2 < \infty \) and \( s_1 - \frac{N}{p_1} > s_2 - \frac{N}{p_2} \).

From [2], we know that, for \( k \in \mathbb{N}^+ \cup \{0\} \), the spaces \( W^{k,p}_U(\mathbb{R}^N) \) and \( W^{k,p}_{\text{loc}}(\mathbb{R}^N) \) coincide algebraically and topologically when \( \rho \) as in (2.1). Furthermore, by intermediate spaces we know that the same holds for \( W^{k,p}_U(\mathbb{R}^N) \) and \( W^{k,p}_{\text{loc}}(\mathbb{R}^N) \) with \( s \in \mathbb{R}^+ \cup \{0\} \), and we will use this equivalence frequently in the present paper.

**Notation.** For \( \rho \) in (2.1) and \( N = 3 \), we denote \( H^\rho_U := W^{1,2}_U \) and

\[
\mathcal{E} := H^\rho_U \times L^2_U = H^1_{\text{loc}} \times L^2_{\text{loc}}
\]

with the norm

\[
\| (u,v) \|_\mathcal{E}^2 = \| u \|_{H^\rho_U}^2 + \| v \|_{L^2_U}^2.
\]

And for any \( t \geq \tau \), \((u(t), \partial_t u(t)) \in \mathcal{E} \), we denote \( \xi_u(t) := (u(t), \partial_t u(t)) \).
2.2. Nonautonomous dynamical systems. In the following, we briefly recall some basic concepts related with non-autonomous dynamical systems and pullback attractors in the bi-space setting (see \[4, 12\] and references therein).

First, we define a dynamical process \{U(t, \tau) : t \geq \tau, \tau \in \mathbb{R}\} on a metric space \(X\), i.e., a family of operators \(U(t, \tau) : X \to X, \tau \in \mathbb{R}, t \geq \tau\), satisfying the properties:

\[
U(\tau, \tau) = Id; \quad U(t, \tau) = U(t, s) \circ U(s, \tau), \quad t \geq s \geq \tau, \quad \tau \in \mathbb{R}.
\]

Now, we consider a Cauchy problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u = A(u) + g(t), \\
u|_{t=\tau} = u_\tau,
\end{array} \right.
\end{aligned}
\tag{2.2}
\]

where \(A(u)\) is a non-linear (unbounded) operator and the non-autonomous external force \(g \in L^2_{loc}(\mathbb{R}; L^2)\). Assume that the problem \(2.2\) is globally well-posed in \(X\), i.e., for every \(u_\tau \in \mathcal{X}\) there is a unique solution \(u(t) \in \mathcal{X}\). Then we can obtain a dynamical process through the solution operators by the formula

\[
\begin{aligned}
U(t, \tau) : X \to X, \quad t \geq \tau, \tau \in \mathbb{R}, \\
U(t, \tau)u_\tau := u(t), \quad t \geq \tau, \tau \in \mathbb{R}.
\end{aligned}
\]

As usual, we denote by \(dist(D_1; D_2)\) the Hausdorff semi-distance between \(D_1\) and \(D_2\) in \(X\), defined as

\[
dist(D_1; D_2) := \sup_{u \in D_1} \inf_{v \in D_2} \| u - v \|_X, \quad \text{for } D_1, D_2 \subset X.
\]

**Definition 2.1.** Let \(\{U(t, \tau) : t \geq \tau\}\) be a non-autonomous dynamical process on a Banach space \(X \subseteq X_{\rho}\). The process \(\{U(t, \tau) : t \geq \tau\}\) is said to be \((X, X_{\rho})\)-pullback asymptotically compact if for any \(t \in \mathbb{R}\), any sequences \(\tau_n \to -\infty\) and each bounded sequence \(\{x_n\} \in \mathcal{X}\), the sequence \(U(t, \tau_n)x_n\), \(\tau_n \leq t\) is precompact in \(X_{\rho}\).

**Definition 2.2.** A set \(B \subset X\) is said to be pullback absorbing if for any \(t \in \mathbb{R}\) and each bounded subset \(D\) of \(X\), there exists \(\tau_0 = \tau_0(D) \leq t\) such that

\[
U(t, \tau)D \subset B, \quad \forall \tau \leq \tau_0. \tag{2.3}
\]

**Remark 2.1.** If \(B \subset X\) is pullback absorbing set for a dynamical process \(\{U(t, \tau) : t \geq \tau\}\) and \(B\) is compact in \(X_{\rho}\), then the process \(\{U(t, \tau) : t \geq \tau\}\) is pullback asymptotically compact in \(X_{\rho}\).

**Definition 2.3.** A family \(\{\mathcal{A}(t)\}_{t \in \mathbb{R}}\) is said to be a \((X, X_{\rho})\)-pullback attractor for the process \(\{U(t, \tau) : t \geq \tau\}\) in \(X\) if it satisfies

(i) \(\mathcal{A}(t)\) is compact in \(X_{\rho}\) for all \(t \in \mathbb{R}\);

(ii) \(\mathcal{A}(\cdot)\) is pullback attracts every bounded subset of \(X\) w.r.t. the \(X_{\rho}\)-norm, i.e.,

\[
\lim_{\tau \to +\infty} dist_{X_{\rho}}(U(t, \tau)D, \mathcal{A}(\tau)) = 0,
\]

for all bounded \(D \subset X\) and all \(t \in \mathbb{R}\);

(iii) \(\mathcal{A}(\cdot)\) is invariant, i.e., \(U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)\), for \(-\infty < \tau \leq t < +\infty\).

**Remark 2.2.** Observe that Definition 2.3 does not guarantee the uniqueness of pullback attractor for a non-autonomous dynamical process.

Now we present the following definition and theorem for verifying the \((X, X_{\rho})\)-pullback asymptotic compactness for non-autonomous dynamical systems, which are similar to those in \([10, 14, 24, 38]\).
Definition 2.4. Let \((X, \| \cdot \|_X)\) be a Banach space, \(B\) be a bounded subset of \(X\), we call a function \(\phi(\cdot, \cdot)\) which defined on \(X \times X\) is a contractive function on \(B \times B\) with respect to \(X_\rho\) topology if for any sequence \(\{x_n\}_n \subset B\), there is a subsequence \(\{x_{n_k}\}_k \subset \{x_n\}_n\) such that
\[
\lim_{k \to \infty} \lim_{l \to \infty} \phi(x_{n_k}, x_{n_l}) = 0,
\]
where the double limit for contractive function \(\phi(\cdot, \cdot)\) is taken in the topology of \(X_\rho\). Denoted all such contractive functions on \(B \times B\) with respect to \(X_\rho\) topology by \(\mathcal{C}(B \times B, X_\rho)\).

Theorem 2.2. Let \(\{U(t, \tau) : t \geq \tau\}\) be a process on Banach space \((X, \| \cdot \|)\) that possesses a pullback absorbing set \(B\). Moreover, assume that for any \(\epsilon > 0\) there exist \(T = T(t, B, \epsilon) = t - \tau\) and \(\phi_{t,T}(\cdot, \cdot) \in \mathcal{C}(B \times B, X_\rho)\) such that
\[
\|U(t, t-T)x - U(t, t-T)y\|_{X_\rho} \leq \epsilon + \phi_{t,T}(x, y) \quad \text{for all } x, y \in B,
\]
where \(\phi_{t,T}(\cdot, \cdot)\) depends on \(t\) and \(T\). Then \(\{U(t, \tau)\}_{t \geq \tau}\) is \((X, X_\rho)\)-pullback asymptotically compact.

The existence of a \((X, X_\rho)\)-pullback attractor is usually verified using the following standard result.

Theorem 2.1 ([12, 27]). Let \(\{U(t, \tau) : t \geq \tau\}\) be a process on Banach space \(X\). Assume the bounded set \(B \subset X\) is pullback absorbing for \(\{U(t, \tau) : t \geq \tau\}\) in \(X\) and \(\{U(t, \tau) : t \geq \tau\}\) is \((X, X_\rho)\)-pullback asymptotically compact. Then the process \(\{U(t, \tau) : t \geq \tau\}\) possesses a \((X, X_\rho)\)-pullback attractor \(\mathcal{A}(t)\) for \(t \in \mathbb{R}\), and
\[
\mathcal{A}(t) = \omega(B, t) := \bigcap_{s \leq t} \bigcup_{\tau \leq s} U(t, \tau)B_{X_\rho}.
\]

2.3. Some abstract results. In this subsection, we recall the Strichartz estimates for the following damped linear wave equation (see [8, 7, 9] for more details):
\[
\begin{cases}
\partial_t^2 v + \gamma \partial_t v - \Delta v = F(t), & \text{in } \Omega \times [\tau, +\infty), \\
v|_{\partial \Omega} = 0, & t \in [\tau, +\infty), \\
\xi_v |_{t=\tau} = \xi_\tau, & x \in \Omega,
\end{cases}
(2.4)
\]
where \(\Omega\) is a bounded domain of \(\mathbb{R}^3\) and the initial data \(\xi_v(\tau) = \xi_\tau := (v_\tau, v'_\tau)\) is taken from the standard energy space \(\mathcal{E}_0:\)
\[
\mathcal{E}_0 := H^1_0(\Omega) \times L^2(\Omega), \quad \|\xi_v\|_{\mathcal{E}_0}^2 := \|\nabla v\|_{L^2}^2 + \|\partial_t v\|_{L^2}^2.
\]

Lemma 2.2. Let \(\xi_\tau \in \mathcal{E}_0\), \(F \in L^1([\tau, \tau+T; L^2(\Omega))\) and let \(v(t)\) be a solution of equation (2.4) such that \(\xi_v \in C([\tau, \tau+T; \mathcal{E}_0])\). Then the following estimate holds:
\[
\|\xi_v(t)\|_{\mathcal{E}_0} \leq C \left( \|\xi_\tau\|_{\mathcal{E}_0} e^{-\beta(t-\tau)} + \int_{\tau}^{t} e^{-\beta(t-s)} \|F(s)\|_{L^2} ds \right),
(2.5)
\]
where the positive constants \(\beta, C\) are dependent of \(\gamma > 0\).

Lemma 2.3. Let the assumptions of Lemma 2.2 hold. Then, \(v \in L^4([\tau, \tau+T; L^{12}(\Omega))\) and the following estimate holds:
\[
\|v\|_{L^4([\tau, \tau+T; L^{12}(\Omega))} \leq C(\|\xi_\tau\|_{\mathcal{E}_0} + \|F(s)\|_{L^1([\tau, \tau+T; L^2(\Omega))})),
(2.6)
\]
where \(C\) may depend on \(T\) and \(|\Omega|\), but is independent of \(\tau, \xi_\tau\) and \(F\).
Remark 2.3. Note that, for any $\theta \in [0, 1]$, we have the interpolation inequality
\[ \|v\|_{L^2(\tau, \tau+T; L^\frac{2n}{n+2}(\Omega))} \leq C\|v\|_{L^1(\tau, \tau+T; L^2(\Omega))}^{\theta}\|v\|_{L^\infty(\tau, \tau+T; H^1(\Omega))}^{1-\theta}. \] (2.7)
Taking $\theta = \frac{2}{5}$ in the above interpolation inequality, we can get that
\[ \|v\|_{L^2(\tau, \tau+T; L^{10}(\Omega))} \leq C\|v\|_{L^1(\tau, \tau+T; L^2(\Omega))}^{\frac{2}{5}}\|v\|_{L^\infty(\tau, \tau+T; H^1(\Omega))}^{\frac{3}{5}}. \]

Further, according to (2.5), we can obtain the control of the $L^5(\tau, \tau+T; L^{10}(\Omega))$-norm of the solution $v$.

On the other hand, combining with Lemma 2.2 and Lemma 2.3, we get that
\[ \|\xi(t)\|_C + \|v\|_{L^2(\tau, \tau+T; L^2(\Omega))} \leq C'(\|\xi_0\|_C e^{-\beta(t-\tau)} + \int_\tau^t e^{-\beta(t-s)}\|F(s)\|_{L^2}ds), \] (2.8)
where $\tau(t) = \max\{\tau, t-1\}$ and the constants $C'$, $\beta > 0$ are independent of $\tau$, $t$, $\xi_0$, and $F$, but $C'$ depends on $[\Omega]$.

Lemma 2.4 ([33]). Suppose that a function $h(\cdot) \in C([a, b])$ satisfies $h(a) = 0$, $h(s) \geq 0$, $s \in [a, b]$ and
\[ h(s) \leq C_0 h(s)^\sigma + \varepsilon \] (2.9)
for some $\sigma > 1$, $0 < C_0 < \infty$ and $0 < \varepsilon < \frac{1}{2} \left( \frac{1}{2C_0} \right)^{\frac{1}{\sigma-1}}$. Then, we have
\[ h(s) \leq 2\varepsilon \] for all $s \in [a, b]$.

Lemma 2.5 (Aubin-Dubinskii-Lions Lemma, [26]). Assume that $X$, $Y$, $Z$ is a triple of Banach spaces such that $Y \subset Z$ and $X \rightarrow Y$ with compact embedding.

(i) Let $B$ be a bounded set in $L^p(a, b; X)$, $1 \leq p < \infty$ such that the set
\[ \partial_t B := \{\partial_t u \mid u \in B\} \]
is bounded in $L^q(a, b; Z)$ with $q \geq 1$, where $\partial_t u$ is the derivative in the distributional sense. Then $B$ is relatively compact in $L^p(a, b; Y)$. Moreover, if $q > 1$, then $B$ is relatively compact in $C(a, b; Z)$.

(ii) If $B$ is a bounded set in $L^\infty(a, b; X)$ and $\partial_t B$ is bounded in $L^r(a, b; Z)$ with $r > 1$, then $B$ is relatively compact in $C(a, b; Y)$.

3. Well-posedness. The global well-posedness and dissipation of Shatah-Struwe solutions of the problem (1.1)-(1.2) will be discussed in this section.

3.1. Weak solutions. Let us first introduce several classes of weak solutions for the problem (1.1)-(1.2).

Definition 3.1. A function $u(t)$ is said to be a weak solution of the problem (1.1)-(1.2) if $\xi_u := (u, \partial_t u) \in L^\infty(\tau, \tau+T; E)$ and equation (1.1) is satisfied in the sense of distribution, i.e.
\[ -\int_\tau^{\tau+T} (\partial_t u, \partial_t \phi)dt - \gamma \int_\tau^{\tau+T} (u, \partial_t \phi)dt + \int_\tau^{\tau+T} (\nabla u \cdot \nabla \phi, 1)dt \]
\[ + \int_\tau^{\tau+T} (f(u), \phi)dt = (\gamma u_\tau + v_\tau, \phi(\tau)) + \int_\tau^{\tau+T} (g(t), \phi)dt \] (3.1)
for any $T > 0$ and any function $\phi$ from the class
\[ W_T = \{\phi \mid \phi(x, t) = \varphi(x)\psi(t), \varphi \in \bigcup_{m=1}^\infty C_0^\infty(\Omega_m), \psi \in C^\infty([\tau, \tau+T]), \psi(\tau+T) = 0\}. \]
Here and below, \( \Omega_m = \{ x \in \mathbb{R}^3 \mid |x| < 2m \} \) and \((\cdot, \cdot)\) stands for the usual inner product in \( L^2(\mathbb{R}^3) \).

**Definition 3.2.** A weak solution \( u(t), \ t \in [\tau, \tau + T] \) is a *Shatah-Struwe solution* of the problem (1.1)-(1.2) if the following additional regularity holds:

\[
\| u \|_{L^4(\tau, \tau + T; L^4(\mathbb{R}^3))} := \sup_{y \in \mathbb{R}^3} \| u \|_{L^4(\tau, \tau + T; L^4(B^y_R))}.
\]  

(3.2)

As usual, we denote \( \mathcal{E}_m := H^1_0(\Omega_m) \times L^2(\Omega_m) \). Let \( \eta \in C^\infty \) be a cutoff function, \( \eta(s) : [0, +\infty) \to [0, 1] \),

\[
\eta(s) = 1 \text{ for } 0 \leq s \leq 1, \quad \eta(s) = 0 \text{ for } s \geq 4
\]

and \( |\eta'(s)| + |\eta''(s)| \leq C \) for all \( s \in [0, +\infty) \).

Now we introduce the approximate solutions to (1.1)-(1.2) with the initial data \( \xi_\tau = (u_\tau, v_\tau) \in \mathcal{E} \) and external force \( g \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{R}^3)) \) as follows:

\[
\begin{cases}
\partial_t^2 u^m + \gamma \partial_t u^m - \nabla u^m + f(u^m) = g^m(t), & \text{in } \Omega_m \times (\tau, +\infty), \\
u^m|_{\partial \Omega_m} = 0, \\
(u^m(x, \tau), \partial_t u^m(x, \tau)) = \eta^m(x)(u_\tau(x), v_\tau(x)),
\end{cases}
\]

(3.3)

where \( \eta^m(x) = \eta(\frac{|x|^2}{4m}) \) and \( g^m(t) = \eta^m(x)g(t) \). And, obviously, the initial data \( \xi_{u^m}(\tau) = (u^m(x, \tau), \partial_t u^m(x, \tau)) \in \mathcal{E}_m \) and external force \( g^m \in L^1_{loc}(\mathbb{R}; L^2(\Omega_m)) \).

In the following, we will define the weak solutions of (3.3) in \( \Omega_m \).

**Definition 3.3.** A function \( u^m(t) \) is a *weak solution* of (3.3) if \( \xi_{u^m} := (u^m, \partial_t u^m) \in L^\infty(\tau, \tau + T; \mathcal{E}_m) \) and equation (3.3) is satisfied in the sense of distribution, i.e.

\[
-\int_\tau^{\tau + T} (\partial_t u^m, \partial_t \phi) dt - \gamma \int_\tau^{\tau + T} (u^m, \partial_t \phi) dt + \int_\tau^{\tau + T} (\nabla u^m \cdot \nabla \phi, 1) dt \\
+ \int_\tau^{\tau + T} (f(u^m), \phi) dt = (\gamma u^m(\tau) + \partial_t u^m(\tau), \phi(\tau)) + \int_\tau^{\tau + T} (g^m(t), \phi) dt
\]

(3.4)

for any \( T > 0 \) and any function \( \phi \) from the class \( W_{T, m} = \{ \phi \mid \phi(x, t) = \varphi(x)\psi(t), \ \varphi \in C^\infty(\Omega_m), \ \psi \in C^\infty([\tau, \tau + T]), \ \psi(\tau + T) = 0 \} \). Here \((\cdot, \cdot)\) stands for the usual inner product in \( L^2(\Omega_m) \).

**Definition 3.4.** A weak solution \( u^m(t), \ t \in [\tau, \tau + T] \) is a *Shatah-Struwe solution* of (3.3) if the following additional regularity holds:

\[
u^m \in L^4(\tau, \tau + T; L^4(\Omega_m))
\]

for any \( T > 0 \).

Finally, for any fixed \( m \in \mathbb{Z}^+ \), let \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \) be the eigenvalues of the operator \(-\Delta\) with homogeneous Dirichlet boundary conditions on \( \Omega_m \) and \( e_1, e_2, e_3, \cdots \) be the corresponding eigenfunctions. The eigenfunctions form an orthonormal base in \( L^2(\Omega_m) \) and they are also smooth: \( \{ e_i \}_{i=1}^\infty \subset C^\infty(\Omega_m) \), since \( \Omega_m \) is smooth. Here, for briefness, we omit the index \( m \) of the eigenvalues and eigenfunctions. Then, the Galerkin approximations to the problem (3.3) are defined as follows:

\[
\begin{cases}
\partial_t^2 u^m_N + \gamma \partial_t u^m_N - \nabla u^m_N + P_N f(u^m_N) = P_N g^m(t), \ u^m_N \in P_N L^2(\Omega_m), \\
\xi_{u^m_N}(\tau) = \xi_{u^m}(\tau) := P_N \xi_{u^m}(\tau) \in [P_N L^2(\Omega_m)]^2,
\end{cases}
\]

(3.5)
where \( P_N : L^2(\Omega_m) \rightarrow L^2(\Omega_m) \) is the orthogonal projector from \( L^2(\Omega_m) \) to the linear subspace spanned by the first \( N \) eigenfunctions \( \{e_1, \ldots, e_N\} \).

3.2. Well-posedness of Shatah-Struwe solutions.

**Theorem 3.1.** Assume that \( g(\cdot) \) satisfies the condition (1.7), the nonlinearity \( f \) satisfies the assumptions (1.3)-(1.5). Then, for every \( \xi = (u_\tau, v_\tau) \in \mathcal{E} \), there exists \( T = T(g, \|\xi_T\|_E) > 0 \) such that the problem (1.1)-(1.2) possesses a Shatah-Struwe solution \( u(t) \) on the interval \([\tau, \tau + T]\).

We first give some properties of Shatah-Struwe solutions of (3.3).

**Proposition 3.1.** Let \( g^m \in L^1_{\text{loc}}(\mathbb{R}; L^2(\Omega_m)) \) satisfies (1.7), \( f \) satisfies the condition (1.3). Then, for any \( \xi = (u_\tau, v_\tau) \in \mathcal{E}_m \), we can find \( T = T(g, \|\xi_u(T)\|_{E_m}) > 0 \) such that there is a Shatah-Struwe solution \( u^m(t) \) of (3.3) defined on \([\tau, \tau + T]\). Moreover, there holds

\[
\|u^m\|_{L^4(\tau, \tau + T; L^2(\Omega_m))} \leq Q_1(\|\xi_T\|_{E_m} + T^{\frac{2}{5}} \|g\|_{L^1(\mathbb{R}; L^2(\Omega_m))}),
\]

where \( Q_1(\cdot) \) is an increasing function which depends on \( T \) and \( |\Omega_m| \), but \( Q_1(\cdot) \) does not depend on \( u^m \) and \( \xi_u(T) \).

**Proof.** We will construct a Shatah-Struwe solution \( u^m \) by using the standard Faedo-Galerkin method. If there is no ambiguity, we will omit the superscripts of \( u^m \). Considering the approximating problem (3.5) with the fixed initial data \( \xi_{u^m}(\tau) = P_N \xi_u(T) = P_N \eta^m(x)(u_\tau, v_\tau) \), which is actually an ordinal differential system. Since \( f \in C^1(\mathbb{R}) \), this system has a local classical solution \( u_N(t) \). In order to take the limit on \( N \), we need some uniform estimates for \( u_N(t) \). Then, we decompose the solution \( u_N(t) \) into the sum

\[
u_N(t) = v_N(t) + w_N(t),\]

where \( v_N(t) \) is the solution to the linear problem

\[
\partial_t^2 v_N + \gamma \partial_t v_N - \Delta v_N = P_N g^m(t), \quad \xi_{v_N}(\tau) = \xi_{u_N}(\tau)
\]

and \( w_N(t) \) is a remainder which satisfies

\[
\partial_t^2 w_N + \gamma \partial_t w_N - \Delta w_N = -P_N f(v_N + w_N), \quad \xi_{w_N}(\tau) = 0.
\]

According to (1.3), Hölder inequality and Young inequality, we derive that

\[
\|P_N f(v_N + w_N)\|_{L^1(\tau, \tau + T; L^2(\Omega_m))} \leq C(t + \|v_N\|_{L^p(\tau, \tau + T; L^2(\Omega_m))}^p + \|w_N\|_{L^p(\tau, \tau + T; L^2(\Omega_m))}^p)
\]

\[
\leq C_p,\Omega_m \left( t + \|v_N\|_{L^p(\tau, \tau + T; L^2(\Omega_m))}^p + \|w_N\|_{L^p(\tau, \tau + T; L^2(\Omega_m))}^p \right)
\]

\[
\leq C_p,\Omega_m \left( t + \|v_N\|_{L^p(\tau, \tau + T; L^2(\Omega_m))}^p \right) + \frac{t^{\frac{2p-2}{p}}}{\frac{p-2}{p}} \|w_N\|_{L^2(\tau, \tau + T; L^2(\Omega_m))}
\]

\[
\leq C_p,\Omega_m \left( t + \|v_N\|_{L^p(\tau, \tau + T; L^2(\Omega_m))}^p \right) + \frac{t^{\frac{2p-2}{p}}}{\frac{p-2}{p}} \|w_N\|_{L^2(\tau, \tau + T; L^2(\Omega_m))}.
\]

Here the constant \( C_p,\Omega_m \) depends only on \( p \) and \( |\Omega_m| \). And we suppose \( t \leq 1 \) in the last inequality.

Then, applying the Strichartz estimate (2.8) to (3.6), we have

\[
\|v_N\|_{L^4(\tau, \tau + T; L^2(\Omega_m))} \leq C_{\Omega_m} \left( e^{-\beta_0 t} \|\xi_{v_N}(\tau)\|_{E_m} + \int\limits_{\tau}^{\tau + T} e^{-\beta_0 (T + t - s)} \|g(s)\|_{L^2} ds \right).
\]
Synchronously, by applying the Strichartz estimate \(2.8\) to \(3.7\) and exploiting the interpolation inequality \(2.7\), we deduce that
\[
Y_N(\tau + t) + \|w_N\|_{L^4(\tau, \tau+t; L^{12}(\Omega_m))} \\
\leq C_0 (t + t^{\frac{4}{5-n}}) v_N(\tau, \tau+t; L^{20}(\Omega_m)) + C_0 Y_N^5(\tau + t) \tag{3.9}
\]
for any \(t \in (0, 1]\), where \(Y_N(\tau + t) := \|\xi_w N(\tau + t)\|_{E_m} + \|w_N\|_{L^5(\tau, \tau+t; L^{10}(\Omega_m))}\) and the constant \(C_0\) depends only on \(p, |\Omega_m|\).

Further, combining with \(3.8\) and \(1.8\), we know that for every \(\varepsilon > 0\), there is a time \(T = T(\varepsilon, g, \|\xi_u\|_{E_m})\) such that
\[
C_0 (t + t^{\frac{4}{5-n}}) v_N(\tau, \tau+t; L^{20}(\Omega_m)) \leq \varepsilon, \quad t \leq T.
\]
Thus, fixing \(\varepsilon\) and \(T \in (0, 1]\) being small enough to apply Lemma 2.4 to \(3.9\), we get the uniform estimate
\[
\|\xi_{w_N}\|_{L^\infty(\tau, \tau+t; E_m)} + \|w_N\|_{L^5(\tau, \tau+t; L^{10}(\Omega_m))} \leq C_1,
\]
where the constant \(C_1\) depends only on \(p\) and \(|\Omega_m|\).

Now, together with \(3.9\) and \(3.8\), we can obtain that
\[
\|\xi_{u_N}\|_{L^\infty(\tau, \tau+t; E_m)} + \|u_N\|_{L^5(\tau, \tau+t; L^{12}(\Omega_m))} \\
\leq Q_1(\|\xi_r\|_{E_m} + T^{\frac{4}{5-n}} \|g\|_{L^1_\tau L^4_\Omega}) \leq C_2
\]
for certain increasing function \(Q_1(\cdot)\) and certain positive constant \(C_2\) which depend only on \(p, T\) and \(|\Omega_m|\).

Therefore, we can take the weak limit \(N \to \infty\) in a standard way and get a Shatah-Struwe solution \(u^m(t)\) of \(3.3\) on \([\tau, \tau + T]\). The proof of Proposition 3.1 is completed. 

**Proposition 3.2.** Suppose that \(g^m\) satisfies the condition \(1.7\) and \(f\) satisfies \(1.3\)-(1.5). Then, for every \(\xi_u\) in \(E_m\), there exists a global Shatah-Struwe solution \(u^m(t)\) of the problem \(3.3\).

**Proof.** We will complete the proof by two steps.

**Step 1.** **Dissipative estimate.** Multiplying \(3.3\) by \(\partial_t u + \beta u\) (for brevity, we omit the sup-index \(m\) of \(u^m\) here) and integrating over \(\Omega_m\), we have
\[
(\partial_t^2 u + \gamma \partial_t u - \Delta u + f(u), \partial_t u + \beta u) = (g^m(t), \partial_t u + \beta u).
\]
Approximating the function \(u\) by Faedo-Galerkin method in a standard way, we obtain after some standard formal calculations that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \|\partial_t u(t) + \beta u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \right) - (\gamma - \beta) \beta (\partial_t u(t) + \beta u(t), u(t)) \\
+ (\gamma - \beta) \|\partial_t u(t) + \beta u(t)\|_{L^2}^2 + \beta \|\nabla u(t)\|_{L^2}^2 + (f(u(t)), \partial_t u(t) + \beta u(t)) \\
= (g^m(t), \partial_t u(t) + \beta u(t)). \tag{3.10}
\end{align*}
\]
By virtue of (1.4), for an arbitrary \( \beta > 0 \), we have
\[
(f(u(t)), \partial_t u(t) + \beta u(t)) = \frac{d}{dt} \int_{\Omega_m} F(u(x,t)) dx + \beta(f(u(t)), u(t))
\]
\[
= \frac{d}{dt} \int_{\Omega_m} F(u) dx + k\beta \int_{\Omega_m} F(u) dx
\]
\[
+ \beta \int_{\Omega_m} (f(u)u - kF(u)) dx
\]
\[
\geq \frac{d}{dt} \int_{\Omega_m} F(u(t)) dx + k\beta \int_{\Omega_m} F(u) dx - \beta C_\mu |\Omega_m|.
\]

At the same time, we deduce from (1.5) that
\[
F(s) \geq -C - |s| \geq -C_\eta - \eta |s|^2, \ \forall s \in \mathbb{R},
\]
(3.11)
\[
(\gamma - \beta)\beta(\theta, u) \leq \frac{\gamma - \beta}{4} \|\theta\|_{L^2}^2 + (\gamma - \beta)\beta^2 \|u\|_{L^2}^2
\]
and
\[
(\gamma - \beta)\beta(\theta, u) \leq \|\theta\|_{L^2} \|g^m(t)\|_{L^2},
\]
where \( \theta(t) = \partial_t u(t) + \beta u(t) \).

Substituting the above estimates into (3.10), we can obtain that
\[
\frac{d}{dt} (\|\theta(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + 2 \int_{\Omega_m} F(u(t)) dx) + (\gamma - \beta)\beta(\theta(t), u(t)) + 2\beta \|\nabla u(t)\|_{L^2}^2
\]
\[
+ 2k\beta \int_{\Omega_m} F(u(t)) dx - (\gamma - \beta)\beta^2 \|u\|_{L^2}^2 \leq \|g^m(t)\|_{L^2} \|\theta\|_{L^2} + 2\beta C_\mu |\Omega_m|.
\]
Simultaneously, choosing \( \beta > 0 \) small enough satisfies \( (\gamma - \beta)\beta < \lambda_1 \) (here \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) on \( H^1_0(\Omega_m) \)) and combining with (3.11), we deduce that
\[
\frac{d}{dt} H(t) + k_0 \beta H(t) \leq \|g^m(t)\|_{L^2} \|\theta\|_{L^2} + 2\beta C_\mu |\Omega_m|
\]
\[
\leq \|g^m(t)\|_{L^2} (H(t) + \delta)^{\frac{1}{2}} + 2\beta C_\mu |\Omega_m|.
\]
(3.12)
for a certain \( \delta > 0 \), where \( k_0 = \min\{k, \frac{1}{2}\} \), \( k \) as in (1.4), \( \beta \) depends on \( |\Omega_m| \) and
\[
H(t) = \|\theta(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + 2 \int_{\Omega_m} F(u(t)) dx.
\]
Applying the Gronwall lemma to (3.12), we get that
\[
(H(t) + \delta)^{\frac{1}{2}} \leq e^{-k_0 \beta t} (H(\tau) + \delta)^{\frac{1}{2}} + C_{\beta,\gamma,\delta,\mu}(\|g\|_{L^1(\mathbb{R};L^2)} + 2|\Omega_m|),
\]
(3.13)
where \( \beta \) depends on \( |\Omega_m| \) and the positive constant \( C_{\beta,\gamma,\delta,\mu} \) depends only on \( \beta, \gamma, \delta \) and \( \mu \).

Hence, combining with (1.3), (3.11), (3.13) and the initial conditions, we obtain that
\[
\|\xi_{u_m}(\tau + t)\|_{C_m} \leq Q_2(\|\xi_{u_m}(\tau)\|_{C_m})e^{-k_0 \beta t} + C_{\beta,\gamma,\delta,\mu,m}(\|g\|_{L^1(\mathbb{R};L^2)} + 1)
\]
(3.14)
for the positive constant \( k_0 = \min\{k, \frac{1}{2}\} \), where \( \beta \) depends on \( |\Omega_m| \), the positive constant \( C_{\beta,\gamma,\delta,\mu,m} \) depends only on \( \beta, \gamma, \delta, \mu, m \), and the monotone function \( Q_2(\cdot) \) is independent of \( t \), but depends on \( |\Omega_m| \).

**Step 2. Global existence.** To prove the global existence of the Shatah-Struwe solution, we need to show its \( C_m \)-norm and \( L^1(\tau, \tau + t; L^2(\Omega_m)) \)-norm cannot blow up in any finite time.
In fact, according to the dissipative estimate obtained in Step 1, the energy norm \( \|u^m(t+\tau)\|_{\mathcal{E}_m} \) does not blow up in any finite time. Therefore, the local weak solution \( u^m(\tau+t) \) can be extended globally in time by exploiting Proposition 3.1. Then, together with the dissipative estimate (3.14), an argument similar to the proof of Proposition 3.1 (see also Lemma 3.2 or Theorem 3.3 for details), we get that

\[
\|u^m\|_{L^4(\max\{\tau,\tau+t-1\},\tau+t;L^{12}(\Omega_m))} \leq Q_3(\|\xi_{u^m}(\tau)\|_{\mathcal{E}_m}) e^{-k_0\beta_t} + Q_3(\|g\|_{L^1(R;L^{12}_m)}),
\]

where the positive constant \( k_0 = \min\{k, \frac{1}{2}\} \) and the increasing function \( Q_3(\cdot) \) is independent of \( t \), but depends on \( |\Omega_m| \).

In conclusion, we know that \( \|\xi_{u^m}(\tau+t)\|_{\mathcal{E}_m} \) and \( \|u^m\|_{L^4(\tau,\tau+t;L^{12}(\Omega_m))} \) can not blow up in any finite time, which implies the global existence of the Shatah-Struwe solution. Hence the statements in Proposition 3.2 are proved. \( \Box \)

**Remark 3.1.** As the proof of the above propositions, the condition (1.7) is crucial for establishing the control of the Strichartz-norm of the Shatah-Struwe solutions for the non-autonomous dynamical system (1.1)-(1.2) in locally uniform spaces. And we also find that the dissipative control of the Strichartz-norm which is crucial for the uniqueness and attractors.

Indeed, when the case \( 3 < p < 5 \) is considered, as analyzed in (3.8), (3.25) and (3.26) (or see (3.27)-(3.31)), the proposal of condition (1.7) is natural. And this condition guarantee that there exists a certain control of Strichartz-norm \( (L^4(\tau, \tau+t; L^{12}_m)) \) of the Shatah-Struwe solutions which in terms of the initial data and external force. But, we do not know whether this condition is the optimal condition for the non-autonomous dynamical system as \( 1 \leq p < 5 \). It is not difficult to see that the scope of this condition contains all translation bounded functions which belong to \( L^2(R; L^{12}_m(R^3)) \). However, the goal mentioned above makes the condition (1.7) has some rationality to some extent.

**Remark 3.2.** According to the recent extension of Strichartz estimates to the case of bounded domains, for any fixed \( m \in \mathbb{Z}^+ \), we know that the positive constant \( C_{\Omega_m} \) of Proposition 3.2 depends on \( |\Omega_m| \) in the Strichartz type estimates for (3.3).

Fortunately, we observed that the change in radius of unit ball to any other positive number does not change the space and gives an equivalent norm in \( L^0_t(L^q(\mathbb{R}^N)) \) with the equivalence constants depend only on \( N \) and \( q \).

Based on the above observation, we can establish some uniform Strichartz type estimates (w.r.t. \( m \)) for the Shatah-Struwe solutions \( u^m(t) \) in locally uniform spaces by applying (2.8) to \( u^m(t) \) in a small ball which has a fixed radius.

We now turn to establish some uniform estimates with respect to \( m \) for the Shatah-Struwe solutions \( u^m(t) \) of the problem (3.3).

**Lemma 3.1.** Let the hypotheses of Theorem 3.1 be in force. Then, for the Shatah-Struwe solutions \( u^m(t) \) of (3.3), the following uniform estimate holds:

\[
\|\xi_{u^m}(t)\|_{\mathcal{E}} \leq Q_4(\|\xi_{\tau}\|_{\mathcal{E}}) e^{-\alpha(t-\tau)} + C_{\alpha,\gamma,\mu}(\|g\|_{L^1(R;L^{12}_m)} + 1), \quad \forall t > \tau, \quad (3.15)
\]

where \( \alpha > 0 \), the positive constant \( C_{\alpha,\gamma,\mu} \) depends only on \( \alpha, \gamma, \mu \) and the increasing function \( Q_4(\cdot) \) are independent of \( t \) and \( \Omega_m \).
Proof. For simplicity, we may take $u \equiv u^m$ here and later on. Multiplying $(3.3)$ by $\rho_x \partial_t u$ and integrating over $\mathbb{R}^3$, we have
\begin{equation}
\frac{d}{dt} \left( |\partial_t u|^2 + |\nabla u|^2 + 2F(u), \rho_x \right) + 2\gamma (|\partial_t u|^2, \rho_x) + 2(\nabla u \cdot \nabla \rho_x, \partial_t u) = 2(g^m(t), \rho_x \partial_t u).
\end{equation}

In order to get the dissipative estimate of the Shatah-Struwe solutions in $\mathcal{E}$, we multiply $(3.3)$ by $\alpha \rho_x u$ where $\alpha > 0$ is a small parameter which will be fixed below and integrate over $x \in \mathbb{R}^3$. Then, we obtain that
\begin{equation}
\frac{d}{dt}(2\alpha u \partial_t u + \alpha |u|^2, \rho_x) + 2\beta F(u(t)) + 2\alpha u(t) \partial_t u(t) + \alpha |u(t)|^2, \rho_x).
\end{equation}

Denote $H(t) \equiv (\partial_t u(t)) + |\nabla u(t)|^2 + 2F(u(t)) + 2\alpha u(t) \partial_t u(t) + \alpha |u(t)|^2, \rho_x)$. Summing $(3.16)$ with $(3.17)$, we deduce that
\begin{equation}
\frac{d}{dt} H(t) + \beta H(t) = K(t),
\end{equation}
where
\begin{equation}
K(t) := - (2\gamma - 2\alpha - \beta)(|\partial_t u(t)|^2, \rho_x) - \alpha (|\nabla u(t)|^2, \rho_x) + 2\alpha \beta (u(t) \partial_t u(t), \rho_x)
+ \alpha (\nabla u \cdot \nabla \rho_x, \partial_t u(t)) - 2(\nabla u \cdot \nabla \rho_x, \partial_t u(t))
+ 2(\beta F(u(t)) - \alpha f(u(t)) u(t), \rho_x) + 2(g^m(t)(u(t) + \alpha \partial_t u), \rho_x).
\end{equation}

Here $\beta = k\alpha$ will be fixed below and $k$ is given in $(1.4)$. According to $(1.4)$, $(2.1)$, and then by using H\ölder inequality and Young inequality, we can obtain that
\begin{align*}
&| - 2\alpha (\nabla u \cdot \nabla \rho_x, u)| \leq 2\alpha \varepsilon (|\nabla u||u|, \rho_x) \leq \frac{\alpha}{4} (|\nabla u|^2, \rho_x) + 4\alpha \varepsilon (|u|^2, \rho_x), \\
&| - 2(\nabla u \cdot \nabla \rho_x, \partial_t u)| \leq 2\sqrt{\varepsilon} (|\nabla u||\partial_t u|, \rho_x) \leq \sqrt{\varepsilon} (|\nabla u|^2, \rho_x) + \sqrt{\varepsilon} (|u|^2, \rho_x), \\
&|2\alpha \beta (u \partial_t, \rho_x)| \leq 2\alpha \beta (|u||\partial_t|, \rho_x) \leq \alpha (|\partial_t|^2, \rho_x) + k^2 \alpha^2 (|u|^2, \rho_x), \\
&2(\beta F(u) - \alpha f(u) u) \leq 2\alpha (k F(u) - f(u) u, \rho_x) \leq C_{k, \alpha, \mu} - 2\alpha (|u|^2, \rho_x), \\
&|2(g^m(t) (u + \alpha \partial_t u), \rho_x)| \leq 2\|g^m(t)||L^2_{\rho_x} \||u + \alpha \partial_t u||L^2_{\rho_x}|
\end{align*}

where $\mu$ is given in $(1.4)$, he constant $C_{k, \alpha, \mu}$ depends only on $k, \alpha, \mu, \|\rho\|_{L^1(\mathbb{R}^3)}$.

Substituting the above estimates into $(3.18)$, we can get that
\begin{align*}
\frac{d}{dt} H(t) + k\alpha H(t) &\leq 2C_{k, \alpha, \mu} - (2\gamma - (k + 3)\alpha - \sqrt{\varepsilon})(|\partial_t u(t)|^2, \rho_x) \\
&+ \alpha (4\varepsilon + k\alpha \gamma + k^2 \alpha^2 - 2\mu)(|u(t)|^2, \rho_x) \\
&- (\alpha - \frac{\alpha}{4} + \sqrt{\varepsilon})(|\nabla u(t)|^2, \rho_x) \\
&+ 2\|g^m\|_{L^2_{\rho_x}} \||u + \alpha \partial_t u||L^2_{\rho_x}|
\end{align*}

Synchronously, choosing $\alpha > 0$ and $\varepsilon > 0$ small enough such that
\begin{align*}
\sqrt{\varepsilon} &\leq 2\gamma - (k + 3)\alpha, \quad \frac{\alpha}{4} + \sqrt{\varepsilon} \leq \alpha, \\
\alpha (4\varepsilon + k\alpha \gamma + k^2 \alpha^2 - 2\mu) &\leq 0,
\end{align*}
then we deduce that
\[ K(t) \leq 2C_{k,\alpha,\mu} + 2\|g^m(t)\|_{L_{m}^{2}B_{x_0}^{2}} \|u + \alpha \partial_t u\|_{L_{m}^{2}B_{x_0}^{2}}. \]

Now, let us take
\[ E(t) = (|\partial_{t}u(t)|^{2} + l_{0}|u(t)|^{2} + |\nabla u(t)|^{2}, \rho_{x_{0}}), \]
then, by (3.11), we can find a constant \( l_{0} = l_{0}(\alpha,\gamma) > 0 \) satisfies the following estimate
\[ C_{1}^{-1} E(t) - C_{1} \leq H(t) \]
with the proper constant \( C_{1} \) which depends on \( l_{0}, \eta \) and \( \|\rho\|_{L^{1}(\mathbb{R}^{3})} \). Here we have used the embedding \( H^{1}_{l_{0}}(\mathbb{R}^{3}) \hookrightarrow L^{2}_{m}(\mathbb{R}^{3}) \).

Consequently, by utilizing the Gronwall lemma, we derive from (3.18) that
\[ (H(t) + \delta)^{\frac{1}{2}} \leq (H(t) + \delta)^{\frac{1}{2}} e^{-\kappa_{0}(t-\tau)} + C_{k,\alpha,\delta,\mu}(1 + \|g\|_{L^{1}_{l_{0}}(\mathbb{R};L^{2}_{m})}), \forall t \geq \tau \] for certain positive constant \( \delta > 0 \). Further, combining with (1.3), (3.19) and \( H^{1}_{l_{0}}(\mathbb{R}^{3}) \hookrightarrow L^{p+1}_{m}(\mathbb{R}^{3}), \; p \in [1,5] \), we get that
\[ \begin{align*}
E^{\frac{2}{5}}(t) & \leq (\tilde{Q}_{4}(E(\tau)) + \tilde{Q}_{4}(\|u(\tau)\|_{H^{1}_{l_{0}}})) e^{-\kappa_{0}(t-\tau)} + C_{l_{0}}(\|g\|_{L^{1}_{m}(\mathbb{R};L_{t_{0}}^{2})} + 1) \\
& \leq (\tilde{Q}_{4}(E(\tau)) + \tilde{Q}_{4}(\|u(\tau)\|_{H^{1}_{l_{0}}})) e^{-\kappa_{0}(t-\tau)} + C_{l_{0}}(\|g\|_{L^{1}_{m}(\mathbb{R};L_{t_{0}}^{2})} + 1) \\
& \leq \tilde{Q}_{4}(|\xi_{r}||\xi|) e^{-\kappa_{0}(t-\tau)} + C_{l_{0}}(\|g\|_{L^{1}_{m}(\mathbb{R};L_{t_{0}}^{2})} + 1), \forall t \geq \tau \end{align*} \] for certain monotone increasing function \( \tilde{Q}_{4}(\cdot) \sim \|\cdot\|_{\mathcal{E}} \cdot \|\cdot\|_{\mathcal{E}}^{3} \) which is independent of \( t, m \) and \( \Omega_{m} \).

Finally, taking the supremum with respect to \( x_{0} \in \mathbb{R}^{3} \) in (3.20) and applying the definition of locally uniform spaces, we have
\[ \|\xi_{m}(t)\|_{\mathcal{E}} \leq \tilde{Q}_{4}(\|\xi_{r}\|_{\mathcal{E}}) e^{-\kappa_{0}(t-\tau)} + C_{l_{0}}(\|g\|_{L^{1}_{m}(\mathbb{R};L_{t_{0}}^{2})} + 1), \]
where \( k \) as indicated in (1.4) and the positive constants \( C_{l_{0}}, \alpha \) are independent of \( t, m \) and \( \Omega_{m} \). Here \( l_{0} = l_{0}(\alpha,\gamma) \) is an absolute constant and \( \tilde{Q}_{4}(\cdot) = \frac{2}{\min(1,l_{0})} \tilde{Q}_{4}(\cdot) \) is a monotone increasing function which does not depend on \( t, m \) and \( \Omega_{m} \). \( \square \)

**Lemma 3.2.** Under the conditions of Theorem 3.1, there exists \( T(\|\xi_{r}\|_{\mathcal{E}},\|g\|_{L^{1}_{l_{0}}(\mathbb{R};L_{t_{0}}^{2})}) < 1 \) such that the Shatah-Struwe solutions \( u^{m}(t) \) of (3.3) satisfies the following Strichartz norm estimate:
\[ \|u^{m}\|_{L^{4}(\tau,\tau+T;L^{2}_{l_{0}}(\Omega_{m}))} \leq Q_{5}(\|\xi_{r}\|_{\mathcal{E}}) + Q_{5}(\|g\|_{L^{1}_{l_{0}}(\mathbb{R};L_{t_{0}}^{2})}), \] (3.21)
where the monotone increasing function \( Q_{5}(\cdot) \) is independent of \( m, \Omega_{m} \).

**Proof.** Our goal is to derive the uniform estimate of the Shatah-Struwe solutions \( u^{m}(t) \) to (3.3) in \( L^{4}(\tau,\tau+T;L^{2}_{l_{0}}(\Omega_{m})) \). For convenience, we set \( \tilde{\eta}(x) = \eta(|x-x_{0}|^{2}) \) and \( \overline{\Omega} = \Omega_{m} \cap B_{x_{0}}^{2} \).

Now, let us truncate the Shatah-Struwe solution \( u^{m}(t) \) of (3.3) by the following form
\[ \tilde{u}(t) = \tilde{\eta}u^{m}(t), \]
then we can see that \( \tilde{u}(t) \) solves the linear wave equation
\[ \begin{align*}
\partial_{t}^{2}\tilde{u} + \gamma \partial_{t}\tilde{u} - \Delta\tilde{u} &= g^{m}(t)\tilde{\eta} + G, \quad \text{in } \overline{\Omega} \times [\tau, +\infty), \\
\tilde{u}(\tau) &= \tilde{\eta}u^{m}(\tau), \quad \partial_{t}\tilde{u}(\tau) = \tilde{\eta}\partial_{t}u^{m}(\tau), \end{align*} \] (322)
where \( G(x,t) = -f(u^{m})\tilde{\eta} - u^{m}\Delta\tilde{\eta} - 2\nabla u^{m} \cdot \nabla\tilde{\eta} \).
In the following, we decompose the solution \( \tilde{u}(t) \) of the problem (3.22) into the sum
\[
\tilde{u}(t) = v(t) + w(t),
\]
where \( v(t) \) solves the linear equation
\[
\partial_t^2 v + \gamma \partial_t v - \Delta v = g^m(t)\bar{\eta}, \quad \xi_v(\tau) = \xi_{\bar{\eta}}(\tau)
\] (3.23)
and the remainder \( w(t) \) satisfies
\[
\partial_t^2 w + \gamma \partial_t w - \Delta w = \tilde{G}, \quad \xi_w(\tau) = 0.
\] (3.24)

Applying Strichartz estimate (2.8) to (3.23), we obtain that
\[
\|v\|_{L^4(\tau, \tau + T; L^{12}(\overline{\Omega}))} \leq C_1 (\|\xi_v(\tau)\| \|\tilde{e}\| \|\tilde{e}\|^{-\alpha} + \int_{\tau}^{\tau + T} e^{-\alpha(\tau + t - s)} \|g^m(s)\|_{L^2(\overline{\Omega})} ds)
\leq C_1 (\|\xi_v(\tau)\| \|\tilde{e}\| \|\tilde{e}\|^{-\alpha} + \|g\|_{L^5(\Omega; \mathbb{R}; L^2_{\alpha})}), \quad 0 \leq t \leq 1,
\] (3.25)
where the constant \( C_1 \) depends on \( |\overline{\Omega}| \).

Similarly, applying (2.8) to (3.24), combining with (1.3), (3.15), we can get that
\[
\|w\|_{L^4(\tau, \tau + T; L^{12}(\overline{\Omega}))} \leq C_2 \|G\|_{L^4(\tau, \tau + T; L^2(\overline{\Omega}))}
\leq C_2 \|\tilde{u}\|_{L^4(\tau, \tau + T; H^1(\overline{\Omega}))} + C_2 \|f(\tilde{u})\|_{L^4(\tau, \tau + T; L^2(\overline{\Omega}))}
\leq C'_1 (T + \|\tilde{u}\|_{L^4(\tau, \tau + T; H^1(\overline{\Omega}))} + \|\tilde{u}\|_{L^4(\tau, \tau + T; L^2(\overline{\Omega}))})
\leq C_2 (T + TQ_0(\xi(g), \tilde{e}) + \|\tilde{u}\|_{L^4(\tau, \tau + T; L^2(\overline{\Omega}))}),
\] (3.26)
where \( Q_0(\xi, g) = 1 + 4 \int (\|\xi\| \|\tilde{e}\|) + \int (\|g\|_{L^5(\Omega)}), \quad Q_0(\cdot) \) is defined inLemma 3.1 and \( C_2 \) depends only on \( |\overline{\Omega}| \).

Next, we will establish a \( L^4(\tau, \tau + T; L^{12}(\overline{\Omega})) \) estimate for \( \tilde{u}(t) \) as follows.
If \( p \in [1, 3] \), then \( H^1(\overline{\Omega}) \hookrightarrow L^{2p}(\overline{\Omega}) \), combining with (3.15) and Hölder inequality, we can obtain that
\[
\|u^m\|_{L^1(\tau, \tau + T; L^2(\overline{\Omega}))} \leq C_3 \int_{\tau}^{\tau + T} (\|\xi_u^m(s)\| \|\tilde{e}\| + 1)^p ds \leq C_4 TQ_0^p(\xi(g), \tilde{e}),
\] (3.27)
where the constants \( C_3, C_4 \) depend only on \( |\overline{\Omega}| \).

Therefore, together with (3.26), we can deduce that
\[
\|w\|_{L^4(\tau, \tau + T; L^{12}(\overline{\Omega}))} \leq C_4 T[Q_0(\xi, g) + Q_0^p(\xi(g), \tilde{e})].
\] (3.28)

If \( p \in (3, 5) \), then \( L^{10}(\overline{\Omega}) \hookrightarrow L^{2p}(\overline{\Omega}) \), combining with (2.3), (3.15) and Hölder inequality, we can obtain that
\[
\|u^m\|_{L^1(\tau, \tau + T; L^2(\overline{\Omega}))} \leq C_5 T^{\frac{5-p}{2}} \|u^m\|_{L^5(\tau, \tau + T; L^{10}(\overline{\Omega}))}
\leq C_6 T^{\frac{5-p}{2}} Q_7(\xi(g), \tilde{e}) \|\tilde{u}\|_{L^5(\tau, \tau + T; L^2_{\alpha}(\Omega_m))},
\] (3.29)
where \( Q_7(\xi(g), \tilde{e}) = Q_0^p(\|\xi(g)\|_{L^5(\Omega; \mathbb{R}; L^2_{\alpha})}) \) and the constants \( C_5, C_6 \) depend only on \( |\overline{\Omega}| \). Here we used the definition of locally uniform spaces and \( u^m = \tilde{u} \) in \( \Omega_m \cap B_{\alpha} \).

Further, plugging (3.29) into (3.26), we deduce that
\[
\|u\|_{L^4(\tau, \tau + T; L^{12}(\overline{\Omega}))} \leq Q_8(\xi, g) T^{\frac{4-p}{2}} + Q_8(\xi, g) \|w\|_{L^4(\tau, \tau + T; L^2_{\alpha}(\Omega_m))}
\] (3.30)
where $Q_\varepsilon(\xi, g) = C_2 Q_0(\xi, g) + C_3 Q_0^p(\xi, g) + C_4^p C_2 \varepsilon^{\frac{p}{2}} Q_7(\xi, g)$. Here we suppose $T \leq 1$ in the last inequality. Hence, we have
\[
\|w\|_{L^4(\tau, \tau+T; L^4_{12}((\Omega_m)))} \leq Q_\varepsilon(\xi, g) T^{\frac{5-p}{2}} + Q_\varepsilon(\xi, g) \|w\|_{L^4(\tau, \tau+T; L^4_{12}((\Omega_m)))}^{\frac{p}{2}} \tag{3.31}
\]
with $T \in (0, 1]$ small enough.

Then we can apply the technical Lemma 2.4 to the above inequality which gives
\[
\|w\|_{L^4(\tau, \tau+T; L^4_{12}((\Omega_m)))} \leq 2T^{\frac{5-p}{2}} Q_\varepsilon(\xi, g). \tag{3.32}
\]
According to $u^m(t) = \bar{v}(t) + v(t)$ in $\Omega_m \cap \mathcal{B}^1_{\rho_0}$, we have
\[
\|u^m\|_{L^4(\tau, \tau+T; L^4_{12}((\Omega_m)))} \leq \|v\|_{L^4(\tau, \tau+T; L^4_{12}((\Omega_m)))} + \|w\|_{L^4(\tau, \tau+T; L^4_{12}((\Omega_m)))}
\]
\[
\leq C_\gamma (T^{\frac{5-p}{2}} + 1) Q_\varepsilon(\xi, g), \tag{3.33}
\]
where $C_\gamma$ depends only on $[\Omega]$ and the function $Q_\varepsilon(\cdot)$ does not depend on $m$, $\Omega_m$. Thus, the statements of Lemma 3.2 follows from the uniform estimate (3.33). \qed

In the following, we begin to deduce the well-posedness of the Shatah-Struwe solution to the problem (1.1)-(1.2). Based on the above preliminary propositions and lemmas, we first prove the local existence Theorem 3.1:

**Proof of Theorem 3.1.** For $\xi = (u_\tau, v_\tau)$ and $g$ given in Theorem 3.1, we have
\[
(u^m(\tau), \partial_t u^m(\tau)) \rightarrow (u_\tau, v_\tau) \text{ strongly in } H^1_\rho \times L^2_\rho.
\]
Simultaneously, from (3.15) and (3.21), without loss of generality, we can assume that
\[
u^m \rightarrow u \text{ weakly star in } L^\infty(\tau, \tau+T; H^1_\rho),
\]
\[
\partial_t u^m \rightarrow \partial_t u \text{ weakly star in } L^\infty(\tau, \tau+T; L^2_\rho),
\]
\[
u^m \rightarrow u \text{ weakly in } L^2(\tau, \tau+T; H^1_\rho),
\]
\[
u^m \rightarrow u \text{ weakly in } L^4(\tau, \tau+T; L^2_\rho),
\]
\[
\partial_t u^m \rightarrow \partial_t u \text{ weakly in } L^2(\tau, \tau+T; L^2_\rho),
\]
then, combining with Lemma 2.5 and the compact embedding $H^1_{1u} \hookrightarrow L^2_{1u}$, we get that
\[
u^m \rightarrow u \text{ strongly in } L^2(\tau, \tau+T; L^2_\rho) \tag{3.35}
\]
and $u \in C([\tau, \tau+T]; L^2_\rho)$.

Therefore, together with (1.3), (3.34), (3.35) and Hölder inequality, we can deduce that
\[
\int_\tau^{\tau+T} (f(u^m), \rho \phi) dt \rightarrow \int_\tau^{\tau+T} (f(u), \rho \phi) dt, \quad \forall \phi \in W_T.
\]
All these convergences stated above allow us to take the limit $m \rightarrow \infty$ in (3.4) and show that $u(t)$ is a weak solution of the problem (1.1)-(1.2) which satisfies (3.1). In particular, $(u, \partial_t u) \in L^\infty(\tau, \tau+T; \mathcal{E})$.

Furthermore, by (3.15), (3.34), we derive that $u \in L^4(\tau, \tau+T; L^2_{1u})$. Hence $u(t)$ is a Shatah-Struwe solution of the problem (1.1)-(1.2) on $t \in [\tau, \tau+T]$. \qed

Next, we will prove the uniqueness of the Shatah-Struwe solution of (1.1)-(1.2).
Theorem 3.2. Under the assumptions of Theorem 3.1, the Shatah-Struwe solutions of (1.1) is unique. Moreover, for two Shatah-Struwe solutions $u^i(t)$ $(i = 1, 2)$ with the initial datums $(u^i_0, v^i_0) \in \mathcal{E}$ respectively, the following Lipschitz continuity holds:

$$\|w(t)\|_{H^1} + \|\partial_tw(t)\|_{L^2} \leq C e^{CT} (\|w(\tau)\|_{H^1} + \|\partial_tw(\tau)\|_{L^2}), \quad t \in [\tau, \tau + T],$$  

where $w(t) = u^1(t) - u^2(t)$ and the constant $C$ depends only on the Strichartz norm and $\mathcal{E}$-norm of the two solutions.

Proof. For convenience, without loss of generality, we always assume $\gamma = 1$ here. We see that the function $w(t) = u^1(t) - u^2(t)$ solves the following equation

$$\begin{cases}
\partial_t^2 w + \partial_t w - \Delta w + f(u^1) - f(u^2) = 0, \\
w(\tau) = u^1_0 - u^2_0, \quad \partial_tw(\tau) = v^1_0 - v^2_0.
\end{cases}$$  

(3.37)

Then we can suppose that there exists a smooth domain $\Omega_{x_0} \subset \mathbb{R}^3$ such that $B^1_{x_0} \subset \Omega_{x_0} \subset B^2_{x_0}$ for every point $x_0 \in \mathbb{R}^3$. And we can define a family of cut-off functions $\eta_{x_0} \in C_0^\infty(\mathbb{R}^3)$ in such a way that $\eta_{x_0} = 1$ for $x \in B^1_{x_0}$ and $\eta_{x_0} = 0$ for $x \in \mathbb{R}^3 \setminus \Omega_{x_0}$. Moreover, we may assume that

$$\|\eta_{x_0}\|_{C^2(\mathbb{R}^3)} \leq C_0$$

uniformly with respect to $x_0$. Then, we can make sense of the formal proof of Theorem 3.2 by the following way which similar to the proof of Lemma 3.2.

Note that the function $w_{x_0}(t) = w(t)\eta_{x_0}$ solves the wave equation

$$\begin{cases}
\partial_t^2 w_{x_0} + \partial_t w_{x_0} - \Delta w_{x_0} + w_{x_0} + l(t, x)w_{x_0} = h_{x_0}(t), \\
w_{x_0}|_{\partial\Omega_{x_0}} = 0,
\end{cases}$$  

(3.38)

where $l(t, x) = \int_0^1 f'(su^1(t) + (1-s)u^2(t))ds - 1$ and $h_{x_0}(t) = -w\Delta\eta_{x_0} - 2\nabla w \cdot \nabla \eta_{x_0}$. Multiplying the equation (3.38) by $\partial_tw_{x_0}$ and integrating over $\mathbb{R}^3$, we obtain that

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \left(\|\partial_tw_{x_0}(t)\|_{L^2(\Omega_{x_0})}^2 + \|\nabla w_{x_0}(t)\|_{L^2(\Omega_{x_0})}^2 + \|w_{x_0}(t)\|_{L^2(\Omega_{x_0})}^2\right) \\
+ \|\partial_tw_{x_0}(t)\|_{L^2(\Omega_{x_0})}^2 = -(l(t, x)w_{x_0}, \partial_tw_{x_0}) + (h_{x_0}(t), \partial_tw_{x_0}).
\end{align*}$$  

(3.39)

Then, by virtue of the growth condition (1.3), we obtain that

$$|-(l(t, x)w_{x_0}, \partial_tw_{x_0}(t))| \leq C \int_{\mathbb{R}^3} (1 + |u^1|^{p-1} + |u^2|^{p-1})|w_{x_0}|\|\partial_tw_{x_0}\|dx$$

$$\leq C_1 (1 + \|u^1\|_{L^{12}}^4 + \|u^2\|_{L^{12}}^4)\|w_{x_0}\|_{L^6}\|\partial_tw_{x_0}\|_{L^2}$$

$$\leq C_2 (1 + \|u^1\|_{L^{12}}^4 + \|u^2\|_{L^{12}}^4)\|w_{x_0}\|_{H^1}^2 + \|\partial_tw_{x_0}\|_{L^2}^2,$$

where we used Hölder inequality, Young inequality and Lemma 3.1. Here the constants $C_1, C_2$ depend only on $p$. Simultaneously, we derive that

$$\|w_{x_0}(t), \partial_tw_{x_0}\| \leq C(\|w\|_{L^2(\Omega_{x_0})}^2 + \|\nabla w\|_{L^2(\Omega_{x_0})}^2 + \|\partial_tw\|_{L^2(\Omega_{x_0})}^2).$$

Substituting the above estimates into (3.39) and choosing $\varepsilon > 0$ small enough, we can deduce that

$$\frac{d}{dt}E_{x_0}(t) \leq C_3 (1 + \|u^1(t)\|_{L^{12}(\Omega_{x_0})}^4 + \|u^2(t)\|_{L^{12}(\Omega_{x_0})}^4)E_{x_0}(t)$$

$$+ C_3(\|w(t)\|_{L^2(\Omega_{x_0})}^2 + \|\nabla w(t)\|_{L^2(\Omega_{x_0})}^2),$$

where $E_{x_0}(t)$ is the energy.
where $E_{x_0}(t) = \|\partial_t w_{x_0}(t)\|_{L^2(\Omega_{x_0})}^2 + \|\nabla w_{x_0}(t)\|_{L^2(\Omega_{x_0})}^2 + \|w_{x_0}(t)\|_{L^2(\Omega_{x_0})}^2$ and $C_3$ depends only on $p$.

Then, applying the Gronwall lemma, we get that
$$\sup_{t \in [\tau, \tau + T]} E_{x_0}(t) \leq e^{a(T)} [E_{x_0}(\tau) + \int_{[\tau, \tau + T]} \|w(t)\|_{H^1(\Omega_{x_0})}^2 \, dt] \leq C_4 e^{C_4T} [E_{x_0}(\tau) + \int_{[\tau, \tau + T]} \|w(t)\|_{H^1_{L^2}}^2 \, dt] \leq C_4 e^{C_4T} [E_{x_0}(\tau) + \int_{[\tau, \tau + T]} (\|w(t)\|_{H^1_{L^2}}^2 + \|\partial_t w(t)\|_{L^2_{L^2}}^2) \, dt],$$

here the function $a(T) = C_3(T + \int_{\tau}^{\tau + T} \|u_1(s)\|_{L^2(\Omega_{x_0})}^2 + \|u_2(s)\|_{L^2(\Omega_{x_0})}^2) \, ds$ and the constant $C_4 = e^{C_4(T + \int_{\tau}^{\tau + T} \|u_1(s)\|_{L^2(\Omega_{x_0})}^2 + \|u_2(s)\|_{L^2(\Omega_{x_0})}^2) \, ds}$.

According to Remark 3.1 and Definition 3.2, we can know that $C_4 < \infty$. Now, we applying the definition of locally uniform spaces and taking $T \ll 1$ small enough, and then we have that
$$\|w(t)\|_{H^1_{L^2}}^2 + \|\partial_t w(t)\|_{L^2_{L^2}}^2 \leq C' e^{C'T} (\|w(\tau)\|_{H^1_{L^2}}^2 + \|\partial_t w(\tau)\|_{L^2_{L^2}}^2), \quad t \in [\tau, \tau + T],$$

where the positive constant $C' < \infty$ depends only on $\|u_1\|_{L^4(\tau, \tau + T; L^6_{L^2}(\Omega_{x_0}))}$ and $T$.

Now we are ready to state and prove the main result in this section.

**Theorem 3.3.** Let the hypotheses of Theorem 3.1 be in force. Then, for every initial data $\xi_\tau = (u_\tau, v_\tau) \in \mathcal{E}$ of (1.1), there exists a unique global Shatah-Struwe solution $u(t)$ which satisfies the following estimate:
$$\|\xi_u(t)\|_{\mathcal{E}} + \|u\|_{L^4(\max(\tau, t-1), t; L^6_{L^2})} \leq Q_3(\|\xi_\tau\|_{\mathcal{E}} e^{-\alpha(t-\tau)} + Q_0(\|g\|_{L^4_{L^2}})), \quad (3.40)$$

where the monotone increasing function $Q_3(\cdot)$ is independent of $t$ and $m$.

**Proof.** The local existence and uniqueness of Shatah-Struwe solution to (1.1) follows from Theorem 3.1 and Theorem 3.2 directly. In the following, we only need to show the global existence of the Shatah-Struwe solution.

For every $\xi_\tau = (u_\tau, v_\tau) \in \mathcal{E}$ and every $g$ satisfies (1.7), we want to get that the Strichartz norm of the Shatah-Struwe solution cannot blow up in any finite time. To this end, proceeding as in the proof of Lemma 3.2, we will establish a $L^4(\tau, \tau + T; L^{12}_{L^2}(\Omega))$ estimate for $\bar{u}(t)$ as follows.

Analogously, if $p \in [1, 3]$, we can get that
$$\|\bar{u}\|_{L^4(\tau, \tau + T; L^{12}_{L^2}(\Omega))} \leq \|v\|_{L^4(\tau, \tau + T; L^{12}_{L^2}(\Omega))} + \|\bar{u}\|_{L^4(\tau, \tau + T; L^{12}_{L^2}(\Omega))} \leq C_4(T + 1)[Q_0(\xi_\tau, g) + Q_0(\xi_\tau, g)], \quad \forall T > 0, \quad (3.41)$$

where the monotone increasing function $Q_6(\cdot)$ is independent of $t$ and $m$.

For $p \in (3, 5)$, we can also have that
$$\|w\|_{L^4(\tau, \tau + T; L^{12}_{L^2}(\Omega_m))} \leq T^{\frac{3-p}{p}} (Q_{10}(\|\xi_\tau\|_{\mathcal{E}}) + Q_{10}(\|g\|_{L^4_{L^2}(\Omega_m)})) + Q_8(\xi_\tau, g) \|w\|_{L^4(\tau, \tau + T; L^{12}_{L^2}(\Omega_m))} \quad (3.42)$$

with $T \in (0, 1]$, where $Q_{10}(\cdot) = CQ_4(\cdot) + CQ_4^\xi(\cdot) + CQ_4^\xi(\cdot) + C \cdot |\cdot|_{\mathcal{E}}^p$ and the function $Q_8(\cdot)$ is given in Lemma 3.2.
For any fixed $t > \tau$, $T \in (0,1]$, according to (1.7), we can derive the above estimate on shifted intervals of size $t - t_0 \leq T$ ($t_0 = \max\{\tau, t - T\}$)

\[
\|w\|_{L^4(t_0,t;L^\infty_w(\Omega_m))} \leq |t - t_0|^{\frac{5}{4} - \frac{1}{p}} \left( Q_{10}\left(\|\xi_w(t_0)\|\varepsilon\right) + Q_{10}(\|g\|_{L^1(L^\infty_w)}) \right) \\
+ Q_{8}(\xi_w(t_0),g)\|w\|_{L^4(t_0,t;L^\infty_w(\Omega_m))} \\
\leq |t - t_0|^{\frac{5}{4} - \frac{1}{p}} \left( e^{-\alpha(t-\tau)} e^{\alpha} Q_{10}(\|\xi_\tau\|\varepsilon) + e^{\alpha} Q_{10}(\|g\|_{L^1(L^\infty_w)}) \right) \\
+ Q_{8}(\xi_\tau,g)\|w\|_{L^4(t_0,t;L^\infty_w(\Omega_m))},
\]  

(3.43)

where we used (3.15).

On the other hand, choosing

\[
\sigma_0 = \varepsilon \left( e^{-\alpha(t-\tau)} e^{\alpha} Q_{10}(\|\xi_\tau\|\varepsilon) + e^{\alpha} Q_{10}(\|g\|_{L^1(L^\infty_w)}) \right)^{\frac{1}{5-p}} < 1
\]

with $\varepsilon > 0$ small enough, we can apply Lemma 2.4 to (3.43) which gives

\[
\|w\|_{L^4(\max\{\tau, t - \sigma_0\}, t;L^\infty_w(\Omega_m))} \leq 2e^{2\alpha} \varepsilon^{\frac{5}{5-p}} \left( e^{-\alpha(t-\tau)} Q_{10}(\|\xi_\tau\|\varepsilon) + Q_{10}(\|g\|_{L^1(L^\infty_w)}) \right)
\]

for any $t > \tau$.

Denote that $\tau(t) = \max\{\tau, t - 1\}$, $N = \left\lfloor \frac{1}{\sigma_0} \right\rfloor$. Then, we have the $L^4(L^\infty_w)$-norm estimate for $w(t)$ on the interval $[\tau(t), t]$ as follows:

\[
\|w\|_{L^4(\tau(t), t;L^\infty_w(\Omega_m))} \leq \sum_{i=0}^{N-1} \|w\|_{L^4(\tau(t+i\sigma_0), \tau(t+(i+1)\sigma_0);L^\infty_w)} + \|w\|_{L^4(\tau(t+N\sigma_0), t;L^\infty_w)} \\
\leq 2e^{2\alpha} \varepsilon^{\frac{5}{5-p}} \sum_{i=0}^{N} \left( e^{-\alpha(t-\tau)} Q_{10}(\|\xi_\tau\|\varepsilon) + Q_{10}(\|g\|_{L^1(L^\infty_w)}) \right) \\
= 2e^{2\alpha} \varepsilon^{\frac{5}{5-p}} \left( 1 + \frac{1}{\sigma_0} \right) \\
\leq 4\varepsilon^{\frac{5}{5-p}} \frac{1}{\sigma_0} \\
= 4\varepsilon^{\frac{5}{5-p}} \left( e^{-\alpha(t-\tau)} e^{\alpha} Q_{10}(\|\xi_\tau\|\varepsilon) + e^{\alpha} Q_{10}(\|g\|_{L^1(L^\infty_w)}) \right)^{\frac{1}{5-p}} \\
\leq e^{-\alpha(t-\tau)} Q_{11}(\|\xi_\tau\|\varepsilon) + Q_{11}(\|g\|_{L^1(L^\infty_w)}).
\]

Further, together with (3.25), we have

\[
\|\bar{u}\|_{L^4(\tau(t), t;L^\infty_w(\Omega_m))} \leq e^{-\alpha(t-\tau)} Q_{12}(\|\xi_\tau\|\varepsilon) + Q_{12}(\|g\|_{L^1(L^\infty_w)})
\]

(3.44)

for certain monotone increasing function $Q_{12}(\cdot)$ which is independent of $t$ and $m$.

Hence, exploiting the estimates (3.41) and (3.44), we can complete the proof of Theorem 3.3. \hfill \Box

**Remark 3.3.** Up to the moment, the Strichartz-norm estimate (3.40) is problematic in the quintic case for the autonomous (as mentioned in Remark 5.2 in [28]) and non-autonomous dynamical system (1.1)-(1.2) in locally uniform spaces. To the best of our knowledge, it is proved in $H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$ for the periodic boundary conditions only and its validity for other boundary conditions (e.g., Neumann boundary condition) is an open problem, see [34].
4. Pullback absorbing sets. In what follows, we begin to define the solution process \( U(t, \tau) : \mathcal{E} \to \mathcal{E} \) associated with the problem (1.1).

For every \( \xi_\tau = (u_\tau, v_\tau) \in \mathcal{E} \), we now define the process \( \{ U(t, \tau) : t \geq \tau \} \) by the following formula:

\[
U(t, \tau)\xi_\tau := \xi_u(t), \quad \forall t \geq \tau,
\]

where \( \xi_u(t) = (u(t), \partial_t u(t)) \) and \( u(t) \) is a unique Shatah-Struwe solution of (1.1).

According to Theorem 3.2 and Theorem 3.3, this process is well defined and even locally Lipschitz continuous in \( \mathcal{E} \).

Furthermore, this process satisfies the following dissipative estimate:

\[
\|U(t, \tau)\xi_\tau\|_\mathcal{E} \leq Q_4(\|\xi_\tau\|_\mathcal{E})e^{-\alpha(t-\tau)} + C(\|g\|_{L^4_b(L^2_u)} + 1), \quad \forall t \geq \tau,
\]

where the constant \( C \) depends only on \( \alpha, \gamma, \mu \) and the increasing function \( Q_4(\cdot) \) are independent of \( t \) and \( \Omega_m \).

Utilizing (4.2), we set

\[
R = 2C(\|g\|_{L^4_b(L^2_u)} + 1),
\]

and consider the set \( B \) of closed ball in \( \mathcal{E} \) defined by

\[
B = \{ \xi_u(t) \in \mathcal{E} : \|\xi_u(t)\|_\mathcal{E} \leq R \}.
\]

Now, we give the following conclusion for the Shatah-Struwe solution process.

**Theorem 4.1.** Under the assumptions of Theorem 3.1, the set \( B \subset X \) is pullback absorbing for the Shatah-Struwe solution process \( \{ U(t, \tau) : t \geq \tau \} \). Here \( B \) as defined in (4.4).

**Remark 4.1.** Note that the dissipative control of the Strichartz-norm is necessary for the uniqueness and attractor here, and we established the control of Strichartz-norm via required certain consistent controllability of the external force. In contrast to this, very few is known about the solutions of (1.1)-(1.2) in the critical and super-critical case (i.e., \( p \geq p^\ast \)) especially for the non-autonomous case. Nevertheless, if ignore the possible non-uniqueness, then we can consider the dynamical system related to multivalued semiflows or/and the so-called trajectory dynamical systems and trajectory attractors.

**Remark 4.2.** As mentioned in the previous section, we need a ‘strong’ condition on the external force (i.e., \( g \in L^1_b(\mathbb{R}; L^2_u) \)) to obtain the well-posedness of solutions, which leads us to obtain a ‘better’ absorbing set \( B \) as defined in (4.4). However, may be one can relax the restriction condition (1.7) and still obtain the well-posedness, then the absorbing sets may become more complex (for example, its radius may depends on time explicitly) for this case. And then, one can consider the pullback \( \mathcal{D} \)-attractor of the related dynamical system. Moreover, the conceptions and methods which used in next section about attractor are still valid for such case.

5. Pullback attractor. In this section, we will give some a priori estimates about the energy inequalities first, and then use Theorem 2.1 to establish the \( (H^1_{lu}(\mathbb{R}^3) \times L^2_u(\mathbb{R}^3), H^1_b(\mathbb{R}^3) \times L^2_u(\mathbb{R}^3)) \)-pullback asymptotical compactness of the Shatah-Struwe solution process \( \{ U(t, \tau) : t \geq \tau \} \). Further, we will prove the existence of the \( (H^1_{lu}(\mathbb{R}^3) \times L^2_u(\mathbb{R}^3), H^1_b(\mathbb{R}^3) \times L^2_u(\mathbb{R}^3)) \)-pullback attractor.

From now on, we use the following notations:

\[
E_u(t) = \frac{1}{2} \int_{\mathbb{R}^3} \rho(x)(|\partial_t u(t)|^2 + |\nabla u(t)|^2)dx
\]

and \( T := t - \tau, \quad t \geq \tau \).
5.1. A priori estimates. The main aim of this subsection is to establish the energy inequalities (5.18) and (5.19), which will be used to obtain the necessary pullback asymptotic compactness.

Let \((u^i(t), \partial_t u^i(t))\) \((i = 1, 2)\) be the corresponding Shatah-Struwe solution of the problem (1.1) with the initial data \((u^1_0, v^1_0)\) \(\in B\) and take \(w(t) = u^1(t) - u^2(t)\). Then \(w\) satisfies

\[
\partial_t^2 w + \gamma \partial_t w - \Delta w + f(u^1) - f(u^2) = 0 \quad \text{in } \mathbb{R}^3 \times [\tau, +\infty)
\]

with the initial condition \((w(\tau), \partial_t w(\tau)) = (u^1_0, v^1_0) - (u^2_0, v^2_0)\).

Multiplying (5.1) by \(e^{\sigma t} \rho \partial_t w\), we have

\[
\frac{d}{dt}(e^{\sigma t} E_w(t)) + \gamma e^{\sigma t} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 + e^{\sigma t} \int_{\mathbb{R}^3} \partial_t w \nabla w \cdot \nabla \rho = \sigma e^{\sigma t} E_w(t) + e^{\sigma t} \int_{\mathbb{R}^3} \rho \partial_t w (f(u^2) - f(u^1)),
\]

integrating the above equality on \([s, t]\), we get that

\[
e^{\sigma t} E_w(t) - e^{\sigma s} E_w(s) + \gamma \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 d\mu = \sigma \int_s^t e^{\sigma \mu} E_w(\mu) d\mu + \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_t w (f(u^2) - f(u^1)) d\mu
\]

\[
- \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \partial_t w \nabla w \cdot \nabla \rho d\mu,
\]

then integrating (5.3) from \(\tau\) to \(t\) with respect to \(s\), we obtain that

\[
Te^{\sigma t} E_w(t) - \int_{\tau}^t e^{\sigma s} E_w(s) ds + \gamma \int_{\tau}^t \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 d\mu ds = \sigma \int_{\tau}^t \int_s^t e^{\sigma \mu} E_w(\mu) d\mu ds + \int_{\tau}^t \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \partial_t w \nabla w \cdot \nabla \rho d\mu ds
\]

\[
+ \int_{\tau}^t \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_t w (f(u^2) - f(u^1)) d\mu ds.
\]

Similarly, multiplying (5.1) by \(e^{\sigma t} \rho w\), we deduce that

\[
\frac{d}{dt} \int_{\mathbb{R}^3} e^{\sigma t} \rho w \partial_t w - e^{\sigma t} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 + (\gamma - \sigma) e^{\sigma t} \int_{\mathbb{R}^3} \rho \partial_t w + e^{\sigma t} \int_{\mathbb{R}^3} \rho |\nabla w|^2
\]

\[
+ e^{\sigma t} \int_{\mathbb{R}^3} w \nabla w \cdot \nabla \rho + e^{\sigma t} \int_{\mathbb{R}^3} \rho w (f(u^1) - f(u^2)) = 0.
\]

Firstly, integrating (5.5) over \([s, t]\), we have

\[
e^{\sigma t} \int_{\mathbb{R}^3} \rho w(t) \partial_t w(t) dx = e^{\sigma s} \int_{\mathbb{R}^3} \rho w(s) \partial_t w(s) dx + \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \rho |\nabla w|^2 d\mu
\]

\[
= \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 d\mu - (\gamma - \sigma) \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \rho w \partial_t w dx d\mu
\]

\[
+ \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} \rho w (f(u^2) - f(u^1)) d\mu
\]

\[
- \int_s^t e^{\sigma \mu} \int_{\mathbb{R}^3} w \nabla w \cdot \nabla \rho d\mu,
\]

(5.6)
then integrating (5.6) over $[\tau, t]$ with respect to $s$, we get that

\[
\int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho |\nabla w|^2 dx \, d\mu \, ds
= \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho |\partial_{\tau} w|^2 dx \, d\mu \, ds
- (\gamma - \sigma) \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_{\tau} w dx \, d\mu \, ds
+ \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho w \partial_{\tau} w dx \, d\mu \, ds
- e^{\sigma \tau} \int_{\mathbb{R}^3} \rho \partial_{\tau} w(w(\tau)) dx
- e^{\sigma t} \int_{\mathbb{R}^3} \rho \partial_{\tau} w(t) dx.
\]  

(5.7)

Substituting (5.7) into (5.4), using Hölder inequality, $|\nabla \rho| \leq C \sqrt{\varepsilon} \rho$ and taking $C \sqrt{\varepsilon} \leq \min\{\frac{1}{4}, \frac{\gamma}{4}\}$, we have

\[
Te^{\sigma t} w(t) - \int_{\tau}^{t} e^{\sigma \mu} w(s) ds \leq \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_{\tau} w(f(u^2) - f(u^1)) dx \, d\mu \, ds
- \frac{\sigma + C \sqrt{\varepsilon}}{2} \int_{\tau}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_{\tau} w dx \, d\mu \, ds
+ \frac{\sigma + C \sqrt{\varepsilon}}{2} \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_{\tau} w dx \, d\mu \, ds
- (\gamma - \sigma) \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_{\tau} w dx \, d\mu \, ds
- \frac{\sigma + C \sqrt{\varepsilon}}{2} \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_{\tau} w dx \, d\mu \, ds.
\]  

(5.8)

Secondly, integrating (5.5) over $[\tau, t]$, we obtain that

\[
\int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho |\nabla w|^2 dx \, ds
= e^{\sigma \tau} \int_{\mathbb{R}^3} \rho w(\tau) \partial_{\tau} w(\tau) dx - e^{\sigma t} \int_{\mathbb{R}^3} \rho w(t) \partial_{\tau} w(t) dx
- (\gamma - \sigma) \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_{\tau} w dx \, ds
+ \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho |\partial_{\tau} w|^2 dx \, ds
+ \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho w(f(u^2) - f(u^1)) dx \, ds.
\]  

(5.9)
Inserting (5.9) to (5.8), we get that

\[
T e^{\sigma t} E_w(t) + \int_{\tau}^{t} e^{\sigma s} E_w(s) ds \leq e^{\sigma \tau} \int_{\mathbb{R}^3} \rho w(\tau) \partial_t w(\tau) dx + 2 \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 dx ds \\
- \left( \frac{\sigma + C \sqrt{\varepsilon}}{2} T + 1 \right) e^{\sigma t} \int_{\mathbb{R}^3} \rho w(t) \partial_t w(t) dx - \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} w \nabla w \cdot \nabla \rho dx ds \\
+ \frac{\sigma + C \sqrt{\varepsilon}}{2} \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho w(f(u^2) - f(u^1)) dx d\mu ds \\
- (\gamma - \sigma) \frac{\sigma + C \sqrt{\varepsilon}}{2} \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_t w dx d\mu ds \\
+ \left[ \frac{\sigma + C \sqrt{\varepsilon}}{2} - (\gamma - \sigma) \right] \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t w dx ds \\
- \frac{\sigma + C \sqrt{\varepsilon}}{2} \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} w \nabla w \cdot \nabla \rho dx d\mu ds \\
+ \int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_t w |f(u^2) - f(u^1)| dx d\mu ds \\
+ \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho w(f(u^2) - f(u^1)) dx ds.
\]

On the other hand, integrating (5.2) from \(\tau\) to \(t\), we obtain that

\[
e^{\sigma t} E_w(t) - e^{\sigma \tau} E_w(\tau) + \gamma \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 dx ds \\
= \sigma \int_{\tau}^{t} e^{\sigma s} E_w(s) ds - \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \partial_t w \nabla w \cdot \nabla \rho dx ds \\
+ \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t w (f(u^2) - f(u^1)) dx ds,
\]

then, combining with \(|\nabla \rho| \leq C \sqrt{\varepsilon} \rho\) and Young inequality, we can deduce that

\[
2 \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 dx ds \\
\leq \frac{2}{\gamma(1-a)} e^{\sigma \tau} E_w(\tau) - \frac{2}{\gamma(1-a)} e^{\sigma t} E_w(t) \\
+ \frac{2}{\gamma(1-a)} \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t w (f(u^2) - f(u^1)) dx ds, \\
+ \frac{2a}{1-a} \int_{\tau}^{t} e^{\sigma s} \int_{\mathbb{R}^3} \rho |\nabla w|^2 dx ds \\
+ \frac{2\sigma}{\gamma(1-a)} \int_{\tau}^{t} e^{\sigma s} E_w(s) ds,
\]

where \(a := \frac{C \sqrt{\varepsilon}}{2\gamma}, C \sqrt{\varepsilon} \leq \frac{\gamma}{4}, \sigma \leq \frac{\gamma}{4}\) and \(\frac{1}{\gamma(1-a)}(2a + \frac{\sigma}{\gamma}) \leq 1\).
Substituting (5.12) into (5.10), we deduce that

\[ T e^{\sigma t} E_w(t) \leq \frac{2e^{\sigma \tau} E_w(\tau)}{\gamma(1 - a)} + e^{\sigma \tau} \int_{\mathbb{R}^3} \rho w(\tau) \partial_t w(\tau) \, dx \]

\[ + \frac{2}{\gamma(1 - a)} \int_{\tau}^{\tau + t} \rho \partial_t w(f(u^2) - f(u^1)) \, dx \, ds \]

\[ + \int_{\tau}^{\tau + t} \int_s^{\tau + t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_t w(f(u^2) - f(u^1)) \, dx \, ds \]

\[ + b \int_{\tau}^{\tau + t} \int_s^{\tau + t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho w(f(u^2) - f(u^1)) \, dx \, ds \]

\[ - b(\gamma - \sigma) \int_{\tau}^{\tau + t} \int_s^{\tau + t} \rho w \partial_t w \, dx \, ds \]

\[ + bC \sqrt{\varepsilon} \int_{\tau}^{\tau + t} \int_s^{\tau + t} \rho \theta w |\nabla w| \, dx \, ds \]

\[ + (b + \sigma - \gamma) \int_{\tau}^{\tau + t} \int_{\mathbb{R}^3} \rho w \partial_t w \, dx \]

\[ + \int_{\tau}^{\tau + t} \int_{\mathbb{R}^3} \rho w(f(u^2) - f(u^1)) \, dx \]

\[ - (bT + 1)e^{\sigma t} \int_{\mathbb{R}^3} \rho w(t) \partial_t w(t) \, dx \]

\[ + C \sqrt{\varepsilon} \int_{\tau}^{\tau + t} \int_{\mathbb{R}^3} \rho w |\nabla w| \, dx \, ds, \]

(5.13)

where the constants \(a := \frac{C\sqrt{\varepsilon}}{2\gamma}, b := \frac{\sigma + C\sqrt{\varepsilon}}{2} \).

By the Hölder inequality, the condition (1.7) and Theorem 3.3, we obtain

\[ \int_{\tau}^{\tau + t} \int_{\mathbb{R}^3} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho w \partial_t w \, dx \, ds \]

\[ \leq T e^{\sigma t} \left( \int_{\tau}^{\tau + t} \int_{\mathbb{R}^3} \rho w |\partial_t w|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_{\tau}^{\tau + t} \int_{\mathbb{R}^3} \rho |w|^2 \, dx \, ds \right)^{\frac{1}{2}} \]

\[ \leq T e^{\sigma t} \|w\|_{L^2(\tau, t; L^2_{\rho w})} \left( \int_{\tau}^{\tau + t} \int_{\mathbb{R}^3} \rho w |\partial_t w|^2 \, dx \, ds \right)^{\frac{1}{2}} \]

\[ \leq T e^{\sigma t} \|w\|_{L^2(\tau, t; \mathbb{E})} \left( \int_{\tau}^{\tau + t} \int_{\mathbb{R}^3} \rho w |\partial_t w|^2 \, dx \, ds \right)^{\frac{1}{2}} \]

\[ \leq T e^{\sigma t} H(R, Q_4(\|\xi_\tau\|_{\mathbb{E}}), T) \left( \int_{\tau}^{\tau + t} \int_{\mathbb{R}^3} \rho w |\partial_t w|^2 \, dx \, ds \right)^{\frac{1}{2}}, \]

(5.14)

where the function \(H\) has the form \(H \simeq T(Q_4(\|\xi_\tau\|_{\mathbb{E}}) + R + 1)\). Here we used the condition (1.7) and the estimate (3.15). And we can deal with the term \(\int_{\tau}^{\tau + t} e^{\sigma s} \int_{\mathbb{R}^3} \rho w \partial_t w \, dx \, ds\) in the similar way as above.
Analogously, we can get that

\[
\int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho |w| |\nabla w| dx ds \\
\leq T e^{\sigma t} \left( \int_{\tau}^{t} \int_{\mathbb{R}^3} \rho |\nabla w| dx ds \right)^{\frac{2}{3}} \left( \int_{\tau}^{t} \int_{\mathbb{R}^3} \rho |w|^2 dx ds \right)^{\frac{1}{3}} \\
\leq T e^{\sigma t} \|\nabla w\|_{L^2(\tau,t;L^2_{v})} \left( \int_{\tau}^{t} \int_{\mathbb{R}^3} \rho |w|^2 dx ds \right)^{\frac{1}{2}} \\
\leq T e^{\sigma t} \|\xi_w\|_{L^2(\tau,t;\xi_e)} \left( \int_{\tau}^{t} \int_{\mathbb{R}^3} \rho |w|^2 dx ds \right)^{\frac{1}{2}} \\
\leq T e^{\sigma t} H(R, Q_4(\|\xi_e\|), T) \left( \int_{\tau}^{t} \int_{\mathbb{R}^3} \rho |w|^2 dx ds \right)^{\frac{1}{2}}, \tag{5.15}
\]

where the function $H$ has the same form as mentioned above. At the same time, we can deal with the term $\int_{\tau}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho |w| |\nabla w| dx ds$ in the similar way as above. Utilizing the Hölder inequality and combining with (3.40), we can arrive at

\[
\int_{\tau}^{t} \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \omega(f(u^2) - f(u^1)) dx ds \\
\leq T e^{\sigma t} \int_{\tau}^{t} \left( \int_{\mathbb{R}^3} \rho |f(u^2) - f(u^1)|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \rho |w|^2 dx \right)^{\frac{1}{2}} ds \\
\leq T e^{\sigma t} \int_{\tau}^{t} \left( \int_{\mathbb{R}^3} \rho (1 + |u^1|^{p-1} + |u^2|^{p-1}) |w|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \rho |w|^2 dx \right)^{\frac{1}{2}} ds \\
\leq C T e^{\sigma t} \int_{\tau}^{t} \left( \int_{\mathbb{R}^3} \rho (1 + |u^1|^{2p} + |u^2|^{2p}) dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \rho |w|^2 dx \right)^{\frac{1}{2}} ds \\
\leq C T e^{\sigma t} \left( T^{\frac{5}{2}} + \|u^1\|_{L^p(\tau,t;L^2_{v})}^p + \|u^2\|_{L^p(\tau,t;L^2_{v})}^p \right) \left( \int_{\tau}^{t} \int_{\mathbb{R}^3} \rho |w|^2 dx ds \right)^{\frac{5}{2p}} \\
\leq C T e^{\sigma t} \left( T^{\frac{5}{2}} + \|u^1\|_{L^p(\tau,t;L^6_{v})}^p + \|u^2\|_{L^p(\tau,t;L^6_{v})}^p \right) \left( \int_{\tau}^{t} \int_{\mathbb{R}^3} \rho |w|^2 dx ds \right)^{\frac{5}{2p}} \\
\leq C T e^{\sigma t} \left( T^{\frac{5}{2}} + \sum_{i=1}^{2} \|u^i\|_{L^5(\tau,t;H^1_{v})}^5 \|\xi_e\|_{L^2(\tau,t;L^2_{v})}^2 \right) \left( \int_{\tau}^{t} \int_{\mathbb{R}^3} \rho |w|^2 dx ds \right)^{\frac{5}{2p}} \\
\leq C T e^{\sigma t} \left[ T^{\frac{5}{2}} + K^p(\|g\|_{L^1(\tau,t;L^2)}, \|\xi_e\|_{\xi_e}) \right] \|w\|_{L^2(\tau,t;L^2_{v})}^2, \tag{5.16}
\]

where the function $K \simeq Q_5(\|\xi_e\|) + Q_0(\|g\|_{L^1(\tau,t;L^2)})$ and the constant $C$ depends only on $p, \|\rho\|_{L^1(\mathbb{R}^3)}$. At the same time, we can deal with the term $\int_{\tau}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \omega(f(u^2) - f(u^1)) dx ds$ in the similar way.
Therefore, together with the above estimates, we can derive from (5.13) that
\[
E_w(t) \leq \frac{16e^{-\sigma T}R^2}{7\gamma T} + \frac{e^{-\sigma T}}{T} \int_{\mathbb{R}^3} \rho w(\tau) \partial_t w(\tau) d\tau - (b + \frac{1}{T}) \int_{\mathbb{R}^3} \rho w(t) \partial_t w(t) d\tau \\
+ C(\frac{\gamma}{4} + \frac{1}{T})(T^2 + K^p) \left( \int_{\mathbb{R}^3} \rho|w|^2 dx \right)^{\frac{\gamma}{4}} \\
+ \frac{2}{(1-a)T} e^{-\sigma t} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t w(f(u^2) - f(u^1)) dx ds \\
+ e^{-\sigma t} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t w(f(u^2) - f(u^1)) dx ds \\
+ (\frac{5\gamma^2}{16} + \frac{5\gamma}{4T})H \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho|w|^2 dx ds \right)^{\frac{1}{2}},
\]
(5.17)
where \( T := t - \tau, b := \frac{\sigma + C \gamma}{2} \), the constant \( C \) depends only on \( p, \|\rho\|_{L^1(\mathbb{R}^3)}, R \) as defined in (4.3) and the functions \( H(\cdot), K(\cdot) \) are given in (5.14)-(5.16).

If we set
\[
\phi_{t,T}((u^1, v^1), (u^2, v^2)) = C(\frac{\gamma}{4} + \frac{1}{T})(T^2 + K^p) \left( \int_{\mathbb{R}^3} \rho|w|^2 dx \right)^{\frac{\gamma}{4}} \\
+ \frac{2}{(1-a)T} e^{-\sigma t} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t w(f(u^2) - f(u^1)) dx ds \\
+ e^{-\sigma t} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t w(f(u^2) - f(u^1)) dx ds,
\]
then
\[
E_w(t) \leq \left( \frac{16}{7\gamma} + 1 \right) \frac{e^{-\sigma T}R^2}{T} + \phi_{t,T}((u^1, v^1), (u^2, v^2)),
\]
(5.19)
where \( T := t - \tau \).

Since \( \lim_{T \to \infty} e^{-\sigma T}R^2 = 0 \), for any \( \epsilon > 0 \), we can find a number \( T = T(\epsilon, t, B) \) such that
\[
E_w(t) \leq \epsilon + \phi_{t,T}((u^1, v^1), (u^2, v^2)) \quad \text{for all} \quad (u^1, v^1) \in B \times B.
\]
(5.20)

5.2. Pullback asymptotically compact.

**Theorem 5.1.** Assume that \( f \) satisfies (1.3)-(1.5) and \( q \) satisfies (1.7). Then the Shatah-Struwe solution process \( \{ U(t, \tau) \}_{t \geq \tau} \) is \( (H^1_{lu}(\mathbb{R}^3) \times L^2_{lu}(\mathbb{R}^3), H^1_{\rho}(\mathbb{R}^3) \times L^2_{\rho}(\mathbb{R}^3)) \)-pullback asymptotically compact.

**Proof.** Due to Theorem 2.1 and Theorem 3.3, it is sufficiently to prove that the function \( \phi_{t,T}((u^1, v^1), (u^2, v^2)) \) defined by (5.18) is a contractive function on \( B \times B \) with respect to \( H^1_{\rho}(\mathbb{R}^3) \times L^2_{\rho}(\mathbb{R}^3) \) topology.

Let \( (u^n, \partial_t u^n) \) be the corresponding Shatah-Struwe solutions of Eq.(1.1) corresponding to initial data \( (u^n, v^n) \in B \times B, n = 1, 2, \cdots \). Since \( B \) is a bounded subset in \( H^1_{lu}(\mathbb{R}^3) \times L^2_{lu}(\mathbb{R}^3) \), by the estimate (3.15), we know that
\[
\|u^n(s)\|_{E} \leq C_{t,\tau} < +\infty \quad \text{for all} \quad s \in [\tau, t] \quad \text{and} \quad n \in \mathbb{N},
\]
(5.21)
where \( C_{t, \tau} \) depends on \( t, \tau, \alpha, \gamma \). Then, without loss of generality, we assume that
\[
\begin{align*}
  u^n &\to u \text{ weakly star in } L^\infty(\tau, t; H^1_0(\mathbb{R}^3)), \\
  u^n &\to u \text{ weakly star in } L^\infty(\tau, t; L^6_0(\mathbb{R}^3)), \\
  \partial_t u^n &\to \partial_t u \text{ weakly star in } L^\infty(\tau, t; L^2(\mathbb{R}^3)), \\
  u^n &\to u \text{ strongly in } L^2(\tau, t; L^2(\mathbb{R}^3)) \\
  u^n &\to u \text{ strongly in } L^{\frac{2m}{m-1}}(\tau, t; L^2_0(\mathbb{R}^3))
\end{align*}
\] (5.22) (5.23) (5.24) (5.25) (5.26)

and
\[
  u^n(\tau) \to u(\tau) \text{ and } u^n(t) \to u(t) \text{ in } L^{p+1}_p(\mathbb{R}^3),
\] (5.27)

here we used the continuous embedding \( H^1_0 \hookrightarrow L^6 \), the compact embedding \( H^1_0 \hookrightarrow L^{p+1} \) and Lemma 2.5.

Now, we are ready to deal with the terms in (5.18) one by one as follows.

At first, combining with (5.25)-(5.27), we can get easily that
\[
\begin{align*}
  &\lim_{n \to \infty} \lim_{m \to \infty} \int_\tau^t \int_{\mathbb{R}^3} \rho|u^n(s) - u^m(s)|^2 dx ds = 0, \\
  &\lim_{n \to \infty} \lim_{m \to \infty} \int_\tau^t \left| \int_{\mathbb{R}^3} \rho|u^n(s) - u^m(s)|^2 dx \right|^{\frac{n}{m-1}} ds = 0, \\
  &\lim_{n \to \infty} \lim_{m \to \infty} \int_{\mathbb{R}^3} \rho (\partial_t u^n(t) - \partial_t u^m(t)) u^n(t) - u^m(t) dx = 0.
\end{align*}
\] (5.28) (5.29) (5.30)

Second, note that
\[
\begin{align*}
  &\int_\tau^t e^{\sigma s} \int_{\mathbb{R}^3} \rho (\partial_t u^n(s) - \partial_t u^m(s)) (f(u^n(s)) - f(u^m(s))) dx ds \\
  &= \int_\tau^t e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t u^n f(u^n) + \int_\tau^t e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t u^m f(u^m) \\
  &\quad - \int_\tau^t e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t u^n f(u^m) - \int_\tau^t e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t u^m f(u^n) \\
  &= e^{\sigma t} \int_{\mathbb{R}^3} \rho F(u^n(t)) - e^{\sigma \tau} \int_{\mathbb{R}^3} \rho F(u^n(\tau)) + e^{\sigma t} \int_{\mathbb{R}^3} \rho F(u^m(t)) - e^{\sigma \tau} \int_{\mathbb{R}^3} \rho F(u^m(\tau)) \\
  &\quad - \int_\tau^t e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t u^n f(u^m(s)) - \int_\tau^t e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t u^m f(u^n(s)) \\
  &\quad - \sigma \int_\tau^t e^{\sigma s} \int_{\mathbb{R}^3} \rho F(u^n(s)) - \sigma \int_\tau^t e^{\sigma s} \int_{\mathbb{R}^3} \rho F(u^m(s)).
\end{align*}
\]

By utilizing (1.3), we have
\[
|F(s)| \leq C(1 + |s|^{p+1}), \quad p \in [1, p^{**}), \quad s \in \mathbb{R},
\]
\[
|F(s_1) - F(s_2)| \leq C(1 + |s_1|^p + |s_2|^p)|s_1 - s_2|, \quad p \in [1, p^{**}), \quad s_1, s_2 \in \mathbb{R},
\] (5.31)
then, together with (3.15), (3.40), (5.26) and (5.31), similar to the treatment of (5.16), we can deduce that

\[
\lim_{n \to \infty} \left| \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho(F(u^n(s)) - F(u(s)))dxds \right|
\leq C \lim_{n \to \infty} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho(1 + |u^n(s)|^p + |u(s)|^p)|u^n(s) - u(s)|dxds
\leq C'_t, \tau \lim_{n \to \infty} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho(u^n(s) - u(s))^2 dx \left| \frac{5}{\pi^{(p-1)/p}} ds \right| \right)^{5-p}
\leq 0,
\]

where the constant \( C'_t, \tau \) depends only on \( t, \tau \) and the function \( K(\cdot) \) which is given in (5.16).

In the following, by (5.23)-(5.24), (5.26)-(5.27), (5.32) and (1.3), taking first \( m \to \infty \), then \( n \to \infty \), we can obtain that

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho(\partial_t u^n(s) - \partial_t u^m(s))(f(u^n(s)) - f(u^m(s)))dxds
= e^{\sigma t} \int_{\mathbb{R}^3} \rho F(u(t)) - e^{\sigma \tau} \int_{\mathbb{R}^3} \rho F(u(\tau)) + e^{\sigma t} \int_{\mathbb{R}^3} \rho F(u(t))
- e^{\sigma \tau} \int_{\mathbb{R}^3} \rho F(u(\tau)) - \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t u f(u(s)) - \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho \partial_t f u(s)
- \sigma \lim_{n \to \infty} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho F(u^n(s)) - \sigma \lim_{m \to \infty} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho F(u^m(s))
= \sigma \lim_{n \to \infty} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho (F(u(s)) - F(u^n(s)))
+ \sigma \lim_{m \to \infty} \int_{\mathbb{R}^3} e^{\sigma s} \int_{\mathbb{R}^3} \rho (F(u(s)) - F(u^m(s)))
= 0.
\]

Similarly, we can get that

\[
\int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho(\partial_t u^n(\mu) - \partial_t u^m(\mu))(f(u^n(\mu)) - f(u^m(\mu)))d\mu
= e^{\sigma t} \int_{\mathbb{R}^3} \rho F(u^n(t)) - e^{\sigma s} \int_{\mathbb{R}^3} \rho F(u^n(s)) + e^{\sigma t} \int_{\mathbb{R}^3} \rho F(u(m(t))) - e^{\sigma s} \int_{\mathbb{R}^3} \rho F(u^m(s))
- \sigma \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho F(u^n(\mu)) - \sigma \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho F(u^m(\mu))
- \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_t u^n f(u^n(\mu)) - \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho \partial_t u^m f(u^m(\mu)),
\]

at the same time, \( \left| \int_{s}^{t} e^{\sigma \mu} \int_{\mathbb{R}^3} \rho(\partial_t u^n(\mu) - \partial_t u^m(\mu))(f(u^n(\mu)) - f(u^m(\mu)))d\mu \right| \) is bounded for each \( s \in [\tau, t] \), then according to Lebesgue dominated convergence
In the autonomous case, when the nonlinearity satisfies the quintic growth, the global well-posedness of Shatah-Struwe solutions for non-autonomous dynamical system (1.1) with quintic nonlinearity in locally uniform spaces, is still an open question. Even in autonomous case, when the nonlinearity $f$ satisfies the quintic growth, the global well-posedness of equation (1.1) in locally uniform spaces remains unsolved. By this reason, we only consider the attractor theory of equation (1.1) for the sub-quintic here.

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