A Hierarchical Dynamic Programming Algorithm for Optimal Coalition Structure Generation

Merixtell Vinyals, Thomas Voice, Sarvapali Ramchurn, Nicholas R. Jennings
School of Electronics and Computer Science
University of Southampton, UK

Abstract
We present a new Dynamic Programming (DP) formulation of the Coalition Structure Generation (CSG) problem based on imposing a hierarchical organizational structure over the agents. We show the efficiency of this formulation by deriving DyPE, a new optimal DP algorithm which significantly outperforms current DP approaches in speed and memory usage. In the classic case, in which all coalitions are feasible, DyPE has halved the memory requirements of other DP approaches. On graph-restricted CSG, in which feasibility is restricted by a (synergy) graph, DyPE has either the same or lower computational complexity depending on the underlying graph structure of the problem. Our empirical evaluation shows that DyPE outperforms the state-of-the-art DP approaches by several orders of magnitude in a large range of graph structures (e.g. for certain scale-free graphs DyPE reduces the memory requirements by 10^6 and solves problems that previously needed hours in minutes).

Introduction
A key part of any coalition formation process involves partitioning the set of agents into the most effective coalitions, (i.e. the optimal coalition structure). However, this Coalition Structure Generation problem (CSG) is akin to the set partitioning problem and hence NP-Hard (Sandholm et al. 1999). Over the last few years, several optimal CSG algorithms have been designed to combat this complexity (Service and Adams 2011; Rahwan, Michalak, and Jennings 2012). In most cases, these algorithms were formulated for the classic CSG model in which all coalitions are feasible. In contrast, in this paper, we tackle the problem in which coalition membership is restricted by some kind of (synergy) graph. Such restrictions have been widely studied in the context of cooperative game theory (Greco et al. 2011; Demange 2004) since they naturally reflect many real-life settings, such as communication networks (Myerson 1977) and logistic networks (Johnson and Gilles 2000).

In these restricted settings, Dynamic Programming (DP) approaches are attractive since they can solve the CSG problem by simply assigning an infinite negative value to non-feasible coalitions. To date, all DP algorithms build on the same DP formulation of the CSG problem, due to (Rothkopf, Pekec, and Harstad 1995). As noted in (Rahwan and Jennings 2008), this DP formulation leads to a redundant search of the CSG space although some of this unnecessary calculations are avoided by IDP, the fastest DP algorithm for classic CSG. For (sparse) graph-restricted CSG, the fastest algorithm is DyCE (Voice, Ramchurn, and Jennings 2012), that outperforms IDP by several orders of magnitude in sparse graphs by restricting the DP formulation to feasible coalitions in the graph. Despite these advances, to date, the CSG problem can only be solved optimally for up to 32 agents, even when considering graph restrictions (Voice, Ramchurn, and Jennings 2012).

Against this background, this paper presents DyPE, a new optimal DP algorithm which significantly outperforms current DP approaches and scales to larger problems. DyPE operates on a novel formulation of the CSG problem that imposes a hierarchical structure over the set of agents. In more detail, this paper makes the following contributions:

• We introduce a new DP formulation for the CSG problem that builds its search on a pseudotree hierarchy of the agents’ synergy graph. We further show how this formulation enables a new search of the CSG space in which the coalitions an agent can join are conditioned on the coalitions formed by agents in earlier positions;

• We propose and prove the correctness of our new algorithm, DyPE (Dynamic programming Pseudotree-based optimal coalition structure E valuation) which is an efficient implementation of the hierarchical DP formulation;

• We analyse the complexity of DyPE showing that it has either the same or lower computational complexity (depending on the structure of the synergy graph) than the current state-of-the-art DP algorithms;

• We empirically show that DyPE solves the CSG problem faster than both IDP and DyCE in a range of graph structures, including the classic case (e.g. with a tree-restricted problem with 40 agents it is 10^4 times faster and reduces by 10^2 times the memory requirements). Moreover, for particular graph classes, DyPE solves the CSG problem for hundreds of agents in minutes.

This paper is organised as follows. We proceed with a background section, followed by formulation, algorithm, com-
Enumeration problem returns minus infinity for any non-feasible coalition and one for each partition of $C$ into two sets $C_L, C_R$:

$$P[C] = \max \left( v(C), \max_{(C_L, C_R) \in \Pi^C_k} P[C_L] + P[C_R] \right) \tag{1}$$

Figure 1a shows the trace of this DP formulation specifying the set of evaluated subproblems ($P[C]$), the subspaces evaluated for each subproblem (S2) and the number of subspaces (#) over the classic CSG among three agents.

IDP algorithm by (Rahwan and Jennings 2008) is an improved implementation of Equation 1 that prunes the evaluations of some subspaces that are proven to be redundant during the search. DyCE algorithm by (Voice, Ramchurn, and Jennings 2012), specifically devised for graph-restricted problems, implements a variant of this DP formulation that restricts coalitions in Equation 1 to be feasible in the graph ($C_L, C_R \in F(G)$). Thus, in Figure 1a if the CSG problem is restricted to the L3 graph, DyCE omits the evaluation of subproblem $P[1, 3]$, as well as of subspace s3, since both involve $\{1, 3\} \notin F(G)$ whereas IDP goes through all subspaces independently of the graph.

The operation of these algorithms is typically visualized on the coalition structure (CS) graph. In this graph, nodes stand for coalition structures and, following Equation 1, an edge connects two coalition structures iff one of the coalition structures can be obtained from the other by splitting one of its agents.

Figure 1a depicts the CS graph among three agents with edges numbered with the number of the corresponding subspace that generated it.

A Hierarchical Formulation for CSG

This section presents a novel hierarchical DP formulation of the CSG problem based on a pseudotree of the agents’ synergy graph. This pseudotree structure allows us to define a more efficient search of the CSG space in which the coalitions an agent can join are explored conditioned on the coalitions formed by agents in earlier positions in the hierarchy. We further show how this search can be visualised in a particular graph of coalition structures that we refer to as a hierarchical coalition structure (HCS) graph.

Synergy Graph Pseudotree

A pseudotree (PT) is a directed tree structure commonly used in search and inference procedures (7). A pseudotree PT of synergy graph $G$ is a rooted tree with agents $A$ as nodes and the property that any two agents that share an edge in $G$ are on the same branch in PT. Here we restrict our attention to edge-traversal pseudotrees, namely those whose edges correspond to edges in $G$. An edge-traversal pseudotree of a graph $G$ can be computed by running a depth-first traversal search (DFS) algorithm (7). Specifically, Figure 2
The order of CSG illustrating its operation with two simple examples. We next present the hierarchical DP formulation for classic CSG, restricted CSG with $O_T$ in the graph. We will denote $P_T$ the agents in the path between any agent should be placed before any of its ancestors in the graph. We define $O_T$ as one ordering for pseudotree $PT$. Notice that for the $PT$ in Figure 2a, $O_{PT} = \{2, 1, 3\}$ is the unique ordering that satisfies the $PT$. In contrast, for the $PT$ in Figure 2b, $O_{PT} = \{2, 1, 3\}$ and $O_{PT} = \{2, 3, 1\}$ are both valid orderings. Yet, $O_{PT}$ not only defines an ordering among agents but also on the set of feasible coalitions. Let $i$ be the agent with lowest order included in a coalition $C$. Then, we define the order of $C$, $O(C)$, as the position of $i$ in $O$. Formally, $O(C) = \min\{i \mid O(i) \in C\}$. Thus, given $O_{PT} = \{2, 1, 3\}$ the order of $\{1, 2, 3\}$ is 1 whereas of $\{1, 3\}$ is 2.

Hierarchical DP Formulation

We next present the hierarchical DP formulation for classic CSG, illustrating its operation with two simple examples. First, consider the classic CSG problem among three agents $A = \{1, 2, 3\}$. Solving this problem involves comparing the value of five $CS$ (depicted as nodes in the $CS$ graph in Figure 1b). Now, suppose that agents are organised as in the $PT$ of Figure 2a, with an ordering $O_{PT} = \{2, 1, 3\}$. Given the lowest agent in the ordering, agent 2, the CSG space can be divided into four subspaces, one for each feasible coalition that contains this agent; namely: $s_1$ containing $\{\{1, 2, 3\}\}; s_2$ containing $CS$ that include $\{2\}$ and any $CS$ among $\{1, 3\}; s_3$ containing $CS$ that include $\{1, 2\}$ and any $CS$ among $\{3\};$ and $s_4$ containing $CS$ that include $\{2, 3\}$ and any $CS$ among $\{1\}$. Notice that finding the best $CS$ in each subspace involves solving a CSG subproblem and that the agents in this subproblem depend on the particular coalition that agent 2 formed in this subspace. Thus, for example, the solution of $s_2$ involves computing the best CS among agents not present in coalition $\{2\}$, i.e. computing subproblem $P[1, 3]$. Again, we can solve this problem by taking the agent with lowest order (agent 1), and repeating the above process. Figure 3a shows a complete trace of this example. It is noteworthy that we reduced the number of operations with respect to the current DP operation: we solved 4 subproblems by evaluating 8 subspaces whereas the current DP operation solves 7 problems by evaluating 13 subspaces (compare Fig. 3a with Fig. 1a).

This hierarchical DP search can be similarly applied to any graph-restricted problem. For example, we can follow the same approach in the CSG problem when restricted by the $l_3$ graph (a complete trace is given in Figure 4a). In this case when computing subspace $s_2$, $P[1, 3]$ can be decomposed into two independent subproblems, namely $P[1]$ and $P[3]$, since agents 1, 3 do not interact (are disconnected) given agent 2 formed a coalition without including them. Again, note that this search is more efficient than the current DP operation: it solves 3 subproblems by evaluating 6 subspaces where the graph-restricted current DP operation solves 6 problems by evaluating 10 subspaces (compare Fig. 4a with Fig. 1a).

In general, given an ordering $O$ among agents, the solution of a $G$-restricted CSG problem can be computed as comparing (maximising) over $n$ subspaces where the value of each subspace can be solved by evaluating a feasible coalition and a set of subproblems corresponding to the connected components of the rest of agents not present in this coalition. Formally, we can define our hierarchical DP formulation for the graph-restricted CSG problem as:

$$P[C] = \max_{C_k \subseteq C: O(C_k) = O(C), \quad C_k \in P(G)} \left( v(C_k) + \sum_{C_i \in P(G) \setminus C_k} P[C_i] \right)$$ (2)

Hierarchical Coalition Structure Graph

To discuss how to construct a HCS graph we need first to define the notion of frontier coalitions. Let us define the frontier coalitions of a coalition structure $CS$ as the set of coalitions that are not connected to any other coalitions of

\footnotetext[3]{The connected components of a graph $G$ are the set of the largest subgraphs of $G$ that are connected.}
higher order in CS. That is, the frontier coalitions are the $C \in CS$ such that if $O(C') > O(C)$ for $C' \in CS$ it implies $C \cup C'$ is disconnected. In the HCS graph, nodes stand for feasible coalition structures and, following Equation 2, an edge connects two feasible coalition structures if and only if one of the coalition structures can be obtained from the other by the evaluation of some subspace of one of its frontier coalitions. That is $CS$ is linked to $CS'$ if and only if there exists $C \in CS, C' \in CS'$ with $C' \subseteq C, CS' = (CS \\{C\}) \cup \{C', \Phi(C \\{C\}')\}$ and $C'$ is a frontier coalition of $CS$.

Figure 3b depicts the HCS graph of the 3-agent classic CSG with $O = \{2, 1, 3\}$ and Figure 3b the graph when the problem is restricted by the $1.3$ graph. Frontier coalitions are underlined in the graph. Notice that frontier coalitions correspond to subproblems that would be evaluated during the operation of the hierarchical DP formulation.

**DyPE**

We can now describe the operation of DyPE. First, we lay its formal foundations, namely how its search is efficiently derived from the hierarchical DP formulation, then we move to its algorithmic details and, finally, prove its correctness.

**Formal Foundations**

DyPE implements the hierarchical recursive formulation introduced in the former section. Accordingly, DyPE: (i) enumerates all subproblems in a bottom-up order (from subproblems that appear last in the recursion to the grand coalition); and (ii) computes the value of each enumerated subproblem.

In the hierarchical recursion formulation a subproblem is computed using the results of a set of subproblems of higher order (a subspace of a subproblem $C$ contains a coalition of the same order as $C$ and a set of subproblems of higher order). Thus, to guarantee a valid exploration order (so no problem is evaluated before one of its subproblems) DyPE evaluates subproblems corresponding to feasible coalitions by its order, from highest to lowest. However, not all feasible coalitions are required during the recursion. For example, the DP execution in Figure 5 requires enumerating subproblems $3$, $\{1\}, \{1, 3\}$, and $\{1, 2, 3\}$ but not $\{2, 3\}$, although it corresponds to a feasible coalition.

Detecting which of the subproblems will actually be needed is crucial for the performance of the DP implementation. DyPE exploits the fact that the ordering is based on a synergy pseudotree to detect a necessary condition that any feasible coalition needs to satisfy in order to be evaluated as a subproblem during the recursion. In particular, let $C$ be a feasible coalition. If $C$ contains the root of $PT$, DyPE will only evaluate $C$ if it is the grand coalition ($C = A$). Otherwise, if $C$ does not include the root, DyPE will only evaluate $C$ if the set of remaining agents, $A \setminus C$, is connected. The correctness of these claims is formally proved in next sections, by proving the correctness of DyPE.

---

**Algorithm 1 DyPE**

```plaintext
1: $C \leftarrow \emptyset; S \leftarrow \emptyset; /*Current subproblem, current subspace*/
2: $i \leftarrow |A|; /*Start exploring the last agent in the ordering */
3: while $(C, i) \rightarrow nextSubproblem(C, i)$ do
   4: $P[C] \leftarrow -\infty$;
   5: while $C' \rightarrow nextConnectedSet(C', i, C)$ do
      6: $V \leftarrow v(C') + \sum_{C'' \in \Phi(C \setminus C') \cap P[C]} P[C'']$;
      7: if $P[C] < V$ /*Compare the value of subspaces*/ then
         8: $P[C] \leftarrow V; /*Update subproblem value*/
         9: $B[C] \leftarrow C'; /*Update the best subspace*/
      end if
   end while
   10: end if
   11: end while
12: end while
13: return bestCS($A$);
```

**Algorithm 2 nextSubproblem**

```plaintext
1: if $i = 1$ /*For the root agent*/ then
2: if $C = \emptyset$ then
3: return $\emptyset; /*Only the grand coalition*/
4: end if
5: return $\emptyset; /*All agents explored, return empty set*/
6: else
7: while $C \rightarrow nextConnectedSet(C, i, \{i, \ldots, |A|\})$ do
     8: if $A \setminus C$ is connected then
        9: return $(C, i)$;
     end if
end while
12: return nextSubproblem($\emptyset, i - 1$); /*Recursively call with the previous agent in the ordering*/
13: end if
```

**The Algorithm**

For notational convenience, we use $nextConnectedSet(\cdot, \cdot, \cdot)$ as an iterator function of a connected subgraph enumeration (CSE) algorithm [Voice, Ramchurn, and Jennings 2012]. That is, for any feasible coalitions $C' \subseteq C$ with $i \in C'$, $nextConnectedSet(C', \{i\}, C)$, returns the subset of $C$ that would follow $C'$ during the process of the chosen CSE algorithm as it iterates through all feasible subcoalitions of $C$ that contain $i$. If $C'$ is the last subset to be enumerated by the CSE, the function returns the empty set.

The pseudocode of DyPE is provided in Algorithm 1. As can be seen, DyPE takes as an input a $G$-restricted CSG problem and an ordering $O$ that satisfies a synergy pseudotree of $G$. Let us assume, without loss of generality, that agents are numbered according to an ordering that satisfies $PT$, so the root is $1$. After initialisation, DyPE proceeds to enumerate subproblems, using the iterator function $nextSubproblem(\cdot, \cdot, \cdot)$. For each agent $i$ in the ordering $O$, DyPE goes through all subproblems that need to be evaluated where $i$ is the agent with lowest order.

For each subproblem $C$, Algorithm 1 computes the value of the CSG problem over agents in $C$ (lines 4-11). To do so, it goes over all subspaces of $C$ that need to be evaluated $(S_C)$ by iteratively calling function $nextConnectedSet(\cdot, \cdot, \cdot)$ (line 5). In this way, DyPE evaluates one subspace for each feasible subcoalition $C'$ of $C$ that contains agent $i$. The value of the subspace is computed as the value of coalition $C'$, $v(C')$, plus the value of each subproblem corresponding to each connected component in $C' \setminus C$, $\Phi(C \setminus C')$ (line 6).
value of \( P[C] \) is computed in Algorithm 1 as the maximum between the values of the evaluated subspaces (lines 7-10).

At the end of this process, the solution of the subproblem corresponding to the grand coalition (\( P[A] \)), contains the value of the best CS explored during the execution of the algorithm. To recover the best CS, Algorithm 1 also stores the subspace that maximizes each subproblem \( C \) in \( B[C] \) (line 12) and at the end of its execution calls a recursive procedure \( \text{best}\_CS(C) \) over the grand coalition, where \( \text{best}\_CS(C) \) returns \( C \) if \( C \setminus B[C] = \emptyset \); \( \text{best}\_CS(B[C]) \cup \bigcup_{C' \in \Phi(B[C])} \text{best}\_CS(C') \) otherwise.

The definition of \( \text{nextSubproblem}(\cdot,\cdot) \) is given in Algorithm 2. For agents \( i = [A] \ldots 2 \) (i.e. excluding the root) DyPE, uses nextConnectedSet(\( \cdot,\cdot \)) to enumerate as subproblems every feasible coalition \( C \) consisting of agent \( i \) and any subset of agents placed after \( i \) in the ordering, \( \{i,\ldots,|A|\} \) (line 7). DyPE evaluates subproblem \( C \) only if the rest of the graph, \( A \setminus C \), remains feasible (lines 8-10). Lastly, for the root agent (\( i = 1 \)), DyPE evaluates a single subproblem corresponding to the grand coalition (lines 1-5).

**Correctness of DyPE**

The next theorem proves the correctness of DyPE.

**Theorem 1 (Correctness)** For any given graph-restricted CSG, DyPE computes an optimal coalition structure.

**Proof.** Since the value of the best coalition structure \( CS^* \) returned by DyPE is equal to \( P[A] \), it is sufficient to show that, for every feasible coalition structure \( CS \), on completion of the algorithm, \( P[A] \geq v(CS) \).

Given such \( CS \), let \( L \) be the order of the coalition with highest order in \( CS \). Let \( C_{\leq l} \) be the set of coalitions in \( CS \) with order equal or lower than \( l \). \( C_{\leq l} = \{C \in CS|O(C) \leq l\} \). We prove the result by showing that for all \( l = 1 \ldots L \), \( P[A] \geq V_l \) where \( V_l \) contains the accumulated value (until step \( l \)) of an exploration “path”,

\[
V_l = \sum_{C \in C_{\leq l}} v(C) + \sum_{C' \in \Phi(A|C_{\leq l})} P[C'],
\]

and so \( V_L = v(CS) \). We prove this by induction on \( l \).

In the base case, \( l = 1 \) and \( C_{\leq l} \) is composed of a single coalition, \( C_1 \), the coalition that contains the root agent in \( CS \). Notice that all the subspaces of the grand coalition \( A \) corresponding to feasible coalitions that contain the root are evaluated. Thus, the subspace \( C_1 \) of \( A \) \( v(C_1) + \sum_{C \in \Phi(A\setminus C_1)} P[C] \) is evaluated and so \( P[A] \geq V_1 \).

In the inductive step, consider the coalition in \( CS \) whose level is \( l + 1 \), \( C_{l+1} \), if there is a coalition with order \( l + 1 \), this coalition is unique since it is the one that contains the agent with order \( l + 1 \) and that the induction hypothesis holds for all coalitions in \( CS \) whose level is less or equal than \( l \). Thus, \( C_{\leq l} = \{C \in \Phi(A\setminus C_{\leq l})\} \) is connected through a path to the grand coalition \( A \). Then, there must be one \( CC \in \Phi(A\setminus C_{\leq l}) \) such that \( C_{l+1} \subseteq CC \). Since the ordering follows a pseudotree, the union of coalitions in \( C_{\leq l} \) forms a connected subgraph, \( A \setminus CC \) must be feasible and thus, \( CC \) is evaluated as a subproblem. As all agents in \( CC \) have higher order than \( l \), \( O(C_{l+1}) = O(CC) \) and \( v(C_{l+1}) + \sum_{C' \in \Phi(CC\setminus C_{l+1})} P[C'] \) must be evaluated in the computation of \( P[CC] \). Thus, \( V_{l+1} \leq V_l \), and so, by the inductive hypothesis, \( V_{l+1} \leq P[A] \).

**Complexity Analysis**

Next, we determine the complexity of DyPE and compare it to those of DyCE and IDP. Notice that each of these algorithms, for each evaluated subproblem: stores its value (and possibly its best space) and evaluates a number of subspaces (with a linear number of operations per subspace). Accordingly, we assess their complexity based on the number of subproblems (memory requirements) and the number of subspaces evaluated (computational requirements).

**Memory.** DyPE evaluates, in addition to the grand coalition, one subproblem for \( C \in F(G) \) such that \( A \setminus C \) is feasible and \( C \) does not contain the root. This number is equal to the number of feasible coalition structures composed of one (the grand coalition) or two coalitions (\( |\Pi_2^A| \)). Thus, the memory requirements of DyPE are within \( O(|\Pi_2^A|) \). In classic CSG, \( |\Pi_2^A| = 2^{|A|} - 1 \). In tree-restricted CSG, \( |\Pi_2^A| \) is equal to the number of edges in the tree (removing exactly one edge is the only way to disconnect the tree into two connected subsets), so DyPE has memory requirements within \( O(|A|) \).

Table I shows how the memory requirements of DyPE compares to those of IDP and DyCE on graph-restricted CSG. Table I also highlights the particular cases of classic and tree-restricted CSG. Observe that independently from the graph the complexity of IDP is exponential in the number of agents, whereas of DyCE is linear in the number of feasible coalitions. Since the number of feasible coalitions will always be greater than the number of coalition structures composed of two coalitions, the memory requirements of DyPE are bounded above by those of DyCE. In classic CSG, although of the same order, the memory requirements of DyPE are one half those of IDP or DyCE as among subproblems that contain the root DyPE only stores the grand coalition.

**Computation.** DyPE evaluate subspaces which are subsets of subproblems. Thus, the computational complexity is bounded by a constant times the number of pairs of subsets \( C', C \) \( C' \subseteq C \), which is \( O(3^{|A|}) \). This is the same order of complexity as IDP (and of DyCE in the classic CSG). In classic CSG, DyPE omits the evaluation of all subsets of subproblems that include the root node with exception of those of the grand coalition. Thus, we cannot hope to do better than \( O(3^{|A|}) \). In tree-restricted CSG, for each agent \( i \), DyPE evaluates exactly one subproblem with \( i \) as its lowest order agent. Thus, each feasible coalition can only generate a subspace in one subproblem, namely the one that has the same order. Conversely, the subproblem with \( i \) as its lowest order agent contains agent \( i \) and all agents reachable from \( i \) with higher level, and so all feasible coalitions generate exactly one subspace. Thus, the computational complexity of DyPE in this case is within \( O(|F(G)|) \), and thus expected to be much lower than those of DyCE since the latter evaluates a potentially large set of subspaces for each feasible coalition. Indeed, in the next section we show empirically that this holds true for a wide range of graph structures.
Experimental Evaluation
We evaluate DyPE and compare its performance against IDP and DyCE on a variety of different synergy graph topologies. We then go on to examine the issue of scalability.

Benchmarking DyPE
In our comparison, we take a similar approach to (Voice, Ramchurn, and Jennings 2012), and investigate performance over the following graph classes: random trees (RT), scalefree graphs (SF) (using the standard Barabasi-Albert preferential attachment generation model, with parameters $k = 1, 2, 3$) and complete graphs (CG). Due to long runtimes we extrapolated the results as follows: from 23 agents onwards for IDP, from 27 onwards for DyCE on RT and SF $k = 1$, and on 24 onwards for DyCE on $k = 2$. For each configuration, we run 50 instances recording the number of evaluated subproblems and the running time of each algorithm. We now present the results of this comparison.

Figures 5 (a)-(b) show the results of our performance evaluation over random trees. The memory requirements for DyPE are up to 7 orders of magnitude lower than for DyCE and up to 11 orders of magnitude lower than for IDP (for 40 agents). This is because as the number of agents increases, memory requirements grow exponentially for IDP and DyCE, and only linearly for DyPE. In terms of runtime, DyPE can solve problems of 40 agents in about 20 minutes compared against 20 days for DyCE, and years for IDP. These results are in line with the intuition given by our complexity analysis section.

The results of our performance evaluation over scalefree graphs are depicted in Figures 5 (c)-(d). For $k = 1$ the memory requirements of DyPE for 30 agents are up to 6 orders of magnitude lower than for DyCE and up to 9 orders lower than for IDP. For $k = 2$, these savings are reduced, but still significant; 2 orders of magnitude better with respect to DyCE and 3 orders of magnitude better with respect to IDP in graphs with 30 agents. These results follow the intuition given in our complexity analysis that the computational savings provided by DyPE are more significant on sparse graphs. Turning to execution time, DyPE can solve problems with 30 agents in minutes (or hours when $k = 2$) instead of hours (or days for $k = 2$) for DyCE.

Finally, our results over complete graphs are in line with the complexity analysis, which predicted a similar performance for all algorithms, excepting that: (i) DyCE takes more time than IDP and DyPE due to its less effective pruning; and (ii) DyPE uses half of the memory of the other approaches.

Scalability of DyPE
We have seen that DyPE performs well on sparse graphs (e.g., trees or scale free with $k = 1$). However, as argued in (Voice, Ramchurn, and Jennings 2012), if the degree of agents is not bounded, even trees can lead to an exponential number of coalitions (e.g., a star has $2^{|A|} - 1$). Based on this, we evaluated DyPE on random trees with bounded degree. In particular, Figure 6 shows the execution time where the degree of agents is bounded by $d$, for $d = 2, 3, 4$. Observe that for $d = 3$ and $d = 4$, DyPE is able to run to completion for problems with 50 agents within 15 minutes and 3 hours respectively. For the particular case of $d = 2$, DyPE solves problems with 1000 agents within minutes.

Conclusions
We presented DyPE, a DP algorithm that implements a novel hierarchical DP formulation for CSG using a hierarchy based on pseudotrees. We proved that DyPE is optimal and that it improves upon current DP approaches with savings that go from linear to exponential, depending on the structure of the underlying synergy graph. Our empirical results showed, that DyPE greatly improves on the state-of-the-art, in some cases by several orders of magnitude. Concretely, for random trees with bounded degree, DyPE managed to quickly find the optimal coalition structure for 1000 agents, when even 50 would be intractable for other DP approaches.

As future work, following current trends in the field (Rahwan, Michalak, and Jennings 2012), we are particularly interested in enhancing the presented hierarchical DP approach with anytime properties.
Figure 5: Results for random tree (RT) graphs (a) (b) and scale free (SF) graphs (c)(d).

References

[Demange 2004] Demange, G. 2004. On Group Stability in Hierarchies and Networks. Journal of Political Economy 112(4):754–778.

[Greco et al. 2011] Greco, G.; Malizia, E.; Palopoli, L.; and Scarcello, F. 2011. On the complexity of compact coalitional games. In IJCAI, 147–152.

[Johnson and Gilles 2000] Johnson, C., and Gilles, R. 2000. Spatial social networks. Review of Economic Design 5(3):273300.

[Myerson 1977] Myerson, R. B. 1977. Graphs and cooperation in games. Mathematics of Operations Research 2(3):225–229.

[Rahwan and Jennings 2008] Rahwan, T., and Jennings, N. R. 2008. An improved dynamic programming algorithm for coalition structure generation. In AAMAS, 1417–1420.

[Rahwan, Michalak, and Jennings 2012] Rahwan, T.; Michalak, T. P.; and Jennings, N. R. 2012. A hybrid algorithm for coalition structure generation. In AAAI.

[Rothkopf, Pekec, and Harstad 1995] Rothkopf, M. H.; Pekec, A.; and Harstad, R. M. 1995. Computationally manageable combinatorial auctions. Management Science 44(8):1131–1147.

[Sandholm et al. 1999] Sandholm, T.; Larson, K.; Anderson, M.; Shehory, O.; and Tohmé, F. 1999. Coalition structure generation with worst case guarantees. Artif. Intell. 111(1-2):209–238.

[Service and Adams 2011] Service, T. C., and Adams, J. A. 2011. Constant factor approximation algorithms for coalition structure generation. Autonomous Agents and Multi-Agent Systems 23(1):1–17.

[Voice, Ramchurn, and Jennings 2012] Voice, T.; Ramchurn, S. D.; and Jennings, N. R. 2012. On coalition formation with sparse synergies. In AAMAS, 223–230.
This figure "partialResults_ScaleFree_median.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1310.6704v1
This figure "partialResults_Trees_median.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1310.6704v1
This figure "solvingTimes_ScaleFree_largescalemedian.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1310.6704v1
This figure "solvingTimes_ScaleFree_median.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1310.6704v1
This figure "solvingTimes_Trees_median.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1310.6704v1