On generalized nonholonomic Chaplygin sphere problem

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Abstract

We discuss linear in momenta Poisson structure for the generalized nonholonomic Chaplygin sphere problem and prove that it is non-trivial deformation of the canonical Poisson structure on $e^*(3)$.

1 Introduction

Let us consider a rolling of dynamically asymmetric and balanced spherical rigid body, the so-called Chaplygin ball, over an absolutely rough fixed sphere with radius $a$ [1]. At $a \rightarrow \infty$ one gets a Chaplygin problem on a non-homogeneous sphere rolling over a horizontal plane without slipping [4].

Since slipping at the contact points is absent, its velocity vanishes and we have the following nonholonomic constraint
\[ v + \omega \times r = 0. \tag{1} \]

Here $\omega$ and $v$ are the angular velocity and velocity of the center of mass of the ball, $r$ is the vector joining the center of mass with the contact point and $\times$ means the vector product in $\mathbb{R}^3$. Mass, inertia tensor and radius of the rolling ball will be denoted by $m$, $I = \text{diag}(I_1, I_2, I_3)$ and $b$, respectively.

According to [1], the angular momentum $M$ of the ball with respect to the contact point with the sphere is equal to
\[ M = (I + dE)\omega - d(\gamma, \omega)\gamma, \quad d = mb^2. \tag{2} \]

Here $\gamma$ is the unit normal vector to the fixed sphere at the contact point, $E$ is the unit matrix and $(\cdot, \cdot)$ means the standard scalar product in $\mathbb{R}^3$. All the vectors are expressed in the so-called body frame, which is firmly attached to the ball, its origin is located at the center of mass of the body, and its axes coincide with the principal inertia axes of the body.

After elimination of the Lagrangian multiplier according to [1], one gets the following reduced equations of motion
\[ \dot{M} = M \times \omega, \quad \dot{\gamma} = \kappa \gamma \times \omega, \quad \text{where} \quad \kappa = \frac{a}{a + b}. \tag{3} \]

These equations possess three integrals of motion
\[ H_1 = (M, \omega), \quad H_2 = (M, M), \quad C_1 = (\gamma, \gamma), \tag{4} \]
and invariant measure
\[ \mu = g^{-1}(\gamma)\, d\gamma\, dM, \quad g(\gamma) = \sqrt{1 - d(\gamma, A\gamma)}, \tag{5} \]
where
\[
A = \begin{pmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{pmatrix} = (I + dE)^{-1} = \begin{pmatrix}
\frac{1}{I_1 + d} & 0 & 0 \\
0 & \frac{1}{I_2 + d} & 0 \\
0 & 0 & \frac{1}{I_3 + d}
\end{pmatrix},
\]

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At $\kappa = \pm 1$ one more integral of motion exists

$$C_2 = (\gamma, BM), \quad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} = \text{tr} A^{-1} + (\kappa - 1)A^{-1}.$$  \hfill (6)

It is easy to see, that in the Chaplygin case $a \to \infty$ we have $\kappa = 1$.

At $\kappa = -1$ we have generalised Chaplygin sphere problem or so-called Borisov-Mamaev-Fedorov system, see [1] and [3].

2 The Poisson brackets

At $\kappa = \pm 1$ six equations of motion (3) possess four integrals of motion and an invariant measure. Thus, by the Euler-Jacobi theorem, they are integrable in quadratures. It allows us to suppose that common level surfaces of integrals form a direct sum of symplectic and lagrangian foliations of dual dynamical system which is hamiltonian with respect to the Poisson bivector $P$, so that

$$[P, P] = 0, \quad PdC_{1,2} = 0, \quad (PdH_1, dH_2) \equiv \{H_1, H_2\} = 0.$$  \hfill (1)

Here $[., .]$ is the Schouten bracket. In fact, we suppose that the Euler-Jacobi integrability of non-Hamiltonian system is equivalent to the Liouville integrability of the dual Hamiltonian dynamical system with the same integrals of motion, see [5].

The first equation in (1) guaranties that $P$ is a Poisson bivector. In the second equation we define two Casimir elements $C_{1,2}$ of $P$ and assume that rank$P = 4$. It is a necessary condition because by fixing its values one gets four dimensional symplectic phase space of the desired Hamiltonian system. The third equation provides that integrals $H_{1,2}$ are in involution with respect to the Poisson bracket associated with $P$ and, therefore, that they form a lagrangian foliation.

In order to compare Poisson structures at $\kappa = \pm 1$ we briefly remind some known facts about linear in momenta $M$ solutions $P$ of (1) associated with the Chaplygin sphere problem at $\kappa = 1$ following to [2, 3, 6, 7].

2.1 Chaplygin sphere, $\kappa = 1$.

According to [2], integrals of motion (4-6) are in involution with respect to the following Poisson brackets

$$\{M_i, M_j\}_g = \varepsilon_{ijk} \left( gM_k - \frac{d(M_i, A\gamma)}{g} \gamma_k \right), \quad \{M_i, \gamma_j\}_g = \varepsilon_{ijk} g \gamma_k, \quad \{\gamma_i, \gamma_j\}_g = 0,$$  \hfill (2)

where $\varepsilon_{ijk}$ is a totally skew-symmetric tensor. These brackets have the necessary Casimir functions $C_{1,2}$.

In variables $x = (\gamma_1, \gamma_2, \gamma_3, M_1, M_2, M_3)$ initial equations of motion (3) have the form

$$\frac{dx_k}{dt} = X_k = g^{-1} \{H, x_k\}_g, \quad \text{where} \quad H = \frac{H_1}{2}.$$  \hfill (3)

After a change of time

$$dt \to gdt$$  \hfill (4)

these equations becomes Hamiltonian equations with respect to the Poisson brackets [2]. It means that initial non-Hamiltonian vector field $X$ is the conformally Hamiltonian vector field

$$X = g^{-1}(x) \tilde{X}, \quad \text{where} \quad \tilde{X} = P_g \, dH.$$ 

The Poisson brackets [2] can be easily obtained via trivial deformations of the canonical Poisson brackets and the standard momentum map theory. Namely, let $Q$ be a $n$-dimensional
smooth manifold. Its cotangent bundle $T^*Q$ is naturally endowed with the Liouville 1-form $\theta$ and symplectic 2-form $\Omega = d\theta$, whose associated Poisson bivector will be denoted with $P$. In local symplectic coordinates on $T^*Q$ $z = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ they have the following local expressions

$$\theta = p_1 dq_1 + \ldots p_n dq_n, \quad \Omega = d\theta = p_1 \wedge q_1 + \ldots p_n \wedge q_n.$$ 

Let us substitute the scaling momenta $p_k \rightarrow g(q) p_k \quad k = 1, \ldots, n,$ (5) into the Liouville and symplectic forms

$$\theta_g = g(q) \left(p_1 dq_1 + \ldots p_n dq_n\right), \quad \Omega \rightarrow \Omega_g = d\theta_g.$$ 

The corresponding Poisson bracket

$$\{q_i, q_j\}_g = 0, \quad \{q_i, p_j\}_g = g\delta_{ij}, \quad \{p_i, p_j\}_g = \partial_j g p_i - \partial_i g p_j, \quad (6)$$

is a trivial deformation of canonical Poisson bracket

$$\{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0$$

in the Poisson-Lichnerowicz cohomology [8]. Here $\partial_k = \partial/\partial q_k$.

Now let us identify $Q$ with a two dimensional sphere $S^2$ embedded into $\mathbb{R}^3$, so that $q_i = \gamma_i, \quad i = 1, 2, 3$. The standard momentum map

$$\phi: (p, \gamma) \in T^*S^2 \rightarrow (M, \gamma) \in e^*(3) = \text{so}(3) \ltimes \mathbb{R}^3,$$

defined by the vector product

$$M = \gamma \times p, \quad (7)$$

maps our trivial deformation (6) into the following Poisson brackets on the Lie algebra $e^*(3)$

$$\{M_i, M_j\}_g = \varepsilon_{ijk} \left(g(\gamma) M_k + \gamma_k \sum_{m=1}^3 M_m \partial_m g(\gamma)\right), \quad (8)$$

$$\{M_i, \gamma_j\}_g = \varepsilon_{ijk} g(\gamma) \gamma_k, \quad \{\gamma_i, \gamma_j\}_g = 0.$$ 

As above, here we use an abbreviation $\partial_m = \partial/\partial \gamma_m$.

**Proposition 1** If we identify $g(\gamma)$ with $g(\gamma)$ [6], then the Poisson brackets [8] coincide with the Poisson brackets [6].

So, the Poisson brackets [2] are trivial deformations of canonical ones. Consequently, according to [7], change of variables

$$L_1 = g^{-1} \left(M_1 - \frac{b\gamma_1}{(\gamma, \gamma)} \left(1 + \frac{\gamma_2^2}{\nu}\right)\right) + \frac{c\gamma_1}{\gamma_1^2 + \gamma_2^2},$$

$$L_2 = g^{-1} \left(M_2 - \frac{b\gamma_2}{(\gamma, \gamma)} \left(1 + \frac{\gamma_2^2}{\nu}\right)\right) + \frac{c\gamma_2}{\gamma_1^2 + \gamma_2^2},$$

$$L_3 = g^{-1} \left(M_3 - \frac{b\gamma_3}{(\gamma, \gamma)} \left(1 - \frac{\gamma_1^2 + \gamma_2^2}{\nu}\right)\right), \quad (9)$$
where
\[ b = (\gamma, M), \quad c = (\gamma, L) \quad \text{and} \quad \nu = \gamma_1^2 + \gamma_2^2 - d(\gamma, \gamma)(a_1\gamma_1^2 + a_2\gamma_2^2), \]
allows us to reduce the deformed Poisson brackets \( \{ \cdot, \cdot \} \) to the canonical Lie-Poisson brackets on the Lie algebra \( e^*(3) \)
\[ \{ L_i, L_j \} = \varepsilon_{ijk} L_k, \quad \{ L_i, \gamma_j \} = \varepsilon_{ijk} \gamma_k, \quad \{ \gamma_i, \gamma_j \} = 0. \quad (10) \]
So, we can prove that the original nonholonomic Chaplygin system is trajectory equivalent to the dual integrable dynamical system on two-dimensional sphere \( S^2 \), which is a Hamiltonian system with respect to the canonical Lie-Poisson brackets \( \{ \cdot, \cdot \} \), see details in [7].

2.2 Generalized Chaplygin sphere, \( \kappa = -1 \).

Now let us compare known Poisson structure at \( \kappa = 1 \) with the new Poisson structure obtained for the case \( \kappa = -1 \). It is easy to see that at \( \kappa = 1 \) bivector \( P_g \) associated with the Poisson brackets \( \{ \cdot, \cdot \} \) has the form
\[ P_g = g \begin{pmatrix} 0 & \Gamma \\ -\Gamma^T & M \end{pmatrix} - dg^{-1}(M, A\gamma) \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (11) \]
where
\[ \Gamma = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix}. \]
Of course, canonical Poisson bivector on \( e^*(3) \)
\[ P = \begin{pmatrix} 0 & \Gamma \\ -\Gamma^T & M \end{pmatrix}, \quad (12) \]
is compatible with its trivial deformation \( P_g \) so that
\[ [P, P_g] = 0. \]
The Poisson bivector \( P_b \) for the generalized Chaplygin ball rolling over the sphere, albeit on the similar form, has completely another properties.

**Proposition 2** At \( \kappa = -1 \) integrals of motion \( \{ \cdot, \cdot \} \) are in involution with respect to the Poisson brackets defined by the following Poisson bivector
\[ P_b = g \begin{pmatrix} 0 & \hat{\Gamma} \\ -\hat{\Gamma}^T & \hat{M} \end{pmatrix} + g^{-1}(2d(\gamma, \gamma) - \text{tr} B) \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Gamma} \end{pmatrix}, \quad (13) \]
which is just one linear in \( M \) solution of the equations \( (7) \). Matrix \( \hat{\Gamma} \) depends only on \( \gamma \)
\[ \hat{\Gamma} = (\gamma, \gamma) E - C - \frac{B}{2d} \Gamma_b, \]
where \( E \) is a unit matrix,
\[ C = \begin{pmatrix} \gamma_1^2 & \gamma_1\gamma_2 & \gamma_1\gamma_3 \\ \gamma_2\gamma_1 & \gamma_2^2 & \gamma_2\gamma_3 \\ \gamma_3\gamma_1 & \gamma_3\gamma_2 & \gamma_3^2 \end{pmatrix} \quad \text{and} \quad \Gamma_b = \begin{pmatrix} 0 & b_3\gamma_3 & -b_2\gamma_2 \\ -b_3\gamma_3 & 0 & b_1\gamma_1 \\ b_2\gamma_2 & -b_1\gamma_1 & 0 \end{pmatrix}. \]
Entries of the other two matrices are equal to

\[
\mathbf{M}_{ij} = -\varepsilon_{ijk} \left( \alpha_k \gamma_k - (\gamma, \gamma) b_k M_k + \frac{b_k^2 M_k}{2d} \right),
\]

\[
\bar{\Gamma}_{ij} = -\frac{\varepsilon_{ijk} b_k}{(b_1 + b_2)(b_2 + b_3)(b_1 + b_3)} \left( (b_i + b_j) \alpha_k + (b_k - b_i)(b_k - b_j) M_k \gamma_k \right),
\]

where

\[
\alpha_k = \left( C_2 + b_k(\gamma, M) \right), \quad C_2 = (\gamma, B M).
\]

The corresponding Poisson brackets look like

\[
\{ \gamma_1, \gamma_2 \} = 0,
\]

\[
\{ M_1, \gamma_1 \} = g(b_2 - b_3) \gamma_1 \gamma_2 \gamma_3,
\]

\[
\{ M_1, \gamma_2 \} = g \gamma_3 \left( b_2 (\gamma_1 + \gamma_2) + b_2 \gamma_2 - \frac{b_2 b_3}{2d} \right),
\]

\[
\{ M_1, \gamma_3 \} = -g \gamma_3 \left( b_1 (\gamma_1 + \gamma_2) + b_3 \gamma_3 - \frac{b_2 b_3}{2d} \right),
\]

\[
\{ M_2, \gamma_1 \} = -g \gamma_3 \left( b_1 \gamma_2^2 + b_3 (\gamma_2 + \gamma_3) - \frac{b_1 b_2}{2d} \right),
\]

\[
\{ M_2, \gamma_2 \} = g \gamma_1 \left( b_1 (\gamma_1 + \gamma_2) + b_3 \gamma_3 - \frac{b_1 b_2}{2d} \right),
\]

\[
\{ M_2, \gamma_3 \} = g \gamma_1 \left( b_1 \gamma_1^2 + b_2 \gamma_2 + b_3 \gamma_3 - \frac{b_1 b_2}{2d} \right),
\]

\[
\{ M_3, \gamma_1 \} = g \gamma_2 \left( b_1 \gamma_2^2 + b_3 (\gamma_2 + \gamma_3) - \frac{b_1 b_2}{2d} \right),
\]

\[
\{ M_3, \gamma_2 \} = -g \gamma_1 \left( b_1 \gamma_1^2 + b_3 \gamma_3 + b_2 \gamma_2 - \frac{b_1 b_2}{2d} \right),
\]

and

\[
\{ M_1, M_2 \} = g \left( -\alpha_3 \gamma_3 + (\gamma, \gamma) b_3 M_3 - \frac{b_3^2 M_3}{2d} \right) - g^{-1} \gamma_3 \left( 2d(\gamma, \gamma) - \text{tr} B \right) \times
\]

\[
\times \left( \frac{\alpha_1 b_1}{(b_1 + b_2)(b_3 + b_1)} + \frac{(b_3 - b_1)(b_3 - b_2)b_1 b_3 M_3}{(b_3 + b_2)(b_1 + b_2)(b_3 + b_1)} \right).
\]

Brackets \( \{ M_1, M_3 \}_b \) and \( \{ M_2, M_3 \}_b \) have the same form as \( \{ M_1, M_2 \}_b \) and, therefore, we omit their explicit expressions.

If \( C_2 = 0 \) there are many other linear in momenta \( M \) solutions of the equations \( 1 \) associated with known variables of separation, see details in \( 6 \).

At \( C_2 \neq 0 \) using linear in momenta Poisson brackets \( \{ ..., \}_b \) we can rewrite equations of motion \( 3 \) in the following form

\[
\frac{dx_k}{dt} = X_k = g_1^{-1} \{ H_1, x_k \}_b + g_2^{-1} \{ H_2, x_k \}_b, \quad (14)
\]

where

\[
g_1(\gamma) = \frac{g(\gamma) s(\gamma)}{2d(\gamma, \gamma) - \text{tr} B d}, \quad g_2(\gamma) = \frac{g(\gamma) s(\gamma)}{2d},
\]

and

\[
s(\gamma) = 4d^2(\gamma, \gamma)(\gamma, B \gamma) - 2d((E \text{tr} B - B\gamma, B \gamma) + \det B. \quad (15)
\]

It is easy to see that at \( \kappa = -1 \) equations of motion have a more complicated form in comparison with the original Chaplygin problem \( 2 \) at \( \kappa = 1 \).
Proposition 3 At $\kappa = -1$ the initial non-Hamiltonian vector field $X$ is a sum of two conformally Hamiltonian vector fields

\[ X = g_1^{-1}\dot{X}_1 + g_2^{-1}\dot{X}_2, \]

where $\dot{X}_{1,2}$ are hamiltonian vector fields associated with two commuting integrals of motion

\[ \dot{X}_1 = P_0 \, dH_1, \quad \dot{X}_2 = P_0 \, dH_2, \quad \{H_1, H_2\}_b = 0. \]

On the other hand, this non-Hamiltonian vector field is conformally Hamiltonian vector field

\[ X = g_3^{-1}\dot{X}_3, \quad \text{where} \quad \dot{X}_3 = P_0 \, dH_3 \quad \text{and} \quad g_3(\gamma) = \frac{g(\gamma)s(\gamma)}{d}, \quad (16) \]

but with respect to another Hamiltonian

\[ H_3 = (2d(\gamma, \gamma) - \text{tr}B)H_1 + 2H_2, \quad (17) \]

which is an integral of motion of (3) without any distinguished physical meaning.

At $\kappa = 1$ the similar linear combination $\dot{H}_3 = H_2 - dH_1$ coincides with the Hamiltonian of the Veselova system, which is equivalent to the original Chaplygin ball at the special choice of parameters $\alpha_i$ [7].

Now let us discuss the main difference between bivectors $P_g$ and $P_b$.

Proposition 4 Bivector $P_b$ is nontrivial deformation of the canonical Poisson bivector $P$ on the Lie algebra $e^*(3)$. In contrast with bivector $P_g$ bivector $P_b$ is incompatible with the canonical Poisson bivector $P$ on $e^*(3)$ because

\[ [P, P_g] = 0, \quad \text{whereas} \quad [P, P_b] \neq 0. \]

As sequence bivector $P_b$ can not be trivial deformation of $P$.

Nevertheless, bivector $P_b$ is nontrivial deformation of canonical bivector $P$, because there is change of variables

\[ L_1 = \frac{1}{g(\gamma)s(\gamma)b_1b_2(\gamma_1^2 + \gamma_2^2)} \left( \alpha_1(b_1\gamma_2M_1 - b_2\gamma_1M_2) + \beta_1M_3 + \beta_2M_3 + \frac{b\gamma_1\gamma_2h(\gamma)}{\gamma_1^2 + \gamma_2^2} + \frac{c\gamma_1}{\gamma_1^2 + \gamma_2^2} \right), \]

\[ L_2 = \frac{1}{g(\gamma)s(\gamma)b_1b_2(\gamma_1^2 + \gamma_2^2)} \left( \alpha_2(b_1\gamma_2M_1 - b_2\gamma_1M_2) + \beta_2M_3 + \beta_3M_3 + \frac{b\gamma_1\gamma_3h(\gamma)}{\gamma_1^2 + \gamma_2^2} + \frac{c\gamma_2}{\gamma_1^2 + \gamma_2^2} \right), \]

\[ L_3 = \frac{1}{g(\gamma)s(\gamma)b_1b_2(\gamma_1^2 + \gamma_2^2)} \left( \alpha_3(b_1\gamma_2M_1 - b_2\gamma_1M_2) + \beta_3M_3 + bh(\gamma) \right), \]

\[ b = (B\gamma, M), \quad c = (\gamma, L), \]

which allows us to reduces this bivector to canonical one. Here $s(\gamma)$ is given by (15),

\[ \alpha_1 = 2d\gamma_2(2db_1\gamma_1^2 + 2db_2(\gamma_2^2 + \gamma_3^2) - b_1b_2), \quad \alpha_3 = 4d^2\gamma_1\gamma_2\gamma_3(b_1 - b_2), \]

\[ \alpha_2 = -2d\gamma_1(2db_1(\gamma_1^2 + \gamma_3^2) + 2db_2\gamma_2^2 - b_1b_2), \]

and

\[ \beta_1 = -2d\gamma_1\gamma_3(2d(\gamma_2^2(b_1b_2 - b_1b_3 + b_2b_3) + b_2(\gamma_1^2 + \gamma_3^2)) - b_1b_2b_3), \]

\[ \beta_2 = -2d\gamma_2\gamma_3(2d(\gamma_1^2(b_1b_3 + b_1b_2 - b_2b_3) + b_1(\gamma_2^2 + \gamma_3^2)) - b_1b_2b_3), \]

\[ \beta_3 = 4d^2b_3\gamma_3^2(b_2\gamma_1^2 + b_1\gamma_2^2) + 2db_1b_2(\gamma_1^2 + \gamma_2^2)(2d(\gamma_1^2 + \gamma_2^2) - b_3). \]
For the brevity we omit the explicit expressions for the function $h(\gamma)$, which is a solution of the differential equations $\{L_i, L_j\}^k = \varepsilon_{ijk} L_k$.

Applying this transformation at $c = (\gamma, L) = 0$ to the Hamilton function (17) one gets integrable dynamical system on the two-dimensional sphere, which is a standard Hamiltonian system with respect to canonical Poisson brackets on $T^*S^2$. The generalised nonholonomic Chaplygin sphere is trajectory equivalent to this Hamiltonian system up to change of time defined by (16).

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