SYSTOLES OF HYPERBOLIC 4-MANIFOLDS

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Abstract. We prove that for any $\epsilon > 0$, there exists a closed hyperbolic 4-manifold with a closed geodesic of length $< \epsilon$.

1. INTRODUCTION

It has been known for a long time that closed hyperbolic surfaces and 3-manifolds may have arbitrarily short geodesics. This follows in 2-dimensions by an explicit construction, and holds in 3 dimensions by Thurston’s hyperbolic Dehn surgery theorem [6, Thm. 5.8.2]. In this note, we prove the existence of closed hyperbolic 4-manifolds which have arbitrarily short geodesics. It is conjectured that there is a uniform lower bound on the length of a systole in arithmetic hyperbolic manifolds. This would follow from Lehmer’s conjecture [3, §10]. Examples of non-arithmetic hyperbolic manifolds in higher dimensions come from the method of “inter-breeding” introduced by Gromov and Piatetski-Shapiro [4], by taking hyperbolic manifolds with geodesic boundary produced using arithmetic methods, and gluing them to obtain a non-arithmetic manifold. Our result makes use of a variation on their method, which might best be described as “inbreeding”. The method would extend to all dimensions if a conjecture about the fundamental groups of arithmetic hyperbolic manifolds were known. We speculate on this conjecture in the last section.

2. SUBGROUP SEPARABILITY

Let $G$ be an infinite group, and $A < G$ a subgroup. We say that $A$ is separable in $G$ if $A$ is the intersection of all finite index subgroups of $G$ which contain $A$, that is

\begin{equation}
A = \bigcap_{A \leq B \leq G, [G:B] < \infty} B.
\end{equation}

We say that a discrete group $\Gamma < \text{Isom}(\mathbb{H}^n)$ is GFERF (short for Geometrically Finite Extended Residually Finite) if every geometrically finite subgroup $A < \Gamma$ is separable in $\Gamma$. More generally, a group $G$ is LERF if every finitely generated subgroup $A < G$ is separable in $G$. If $\Gamma$ is LERF, then since geometrically finite subgroups are finitely generated, this would imply that $\Gamma$ is GFERF. Unfortunately, the converse is not necessarily true, since there may be finitely generated subgroups of $\Gamma$ which are not geometrically finite. Scott showed that $A < \Gamma$ is separable if and only if for any compact subset $C \subset \mathbb{H}^4/A$, there exists $\Gamma_1 < \Gamma$, $[\Gamma : \Gamma_1] < \infty$, and $C \hookrightarrow \mathbb{H}^4/\Gamma_1$ embeds under the covering map. We will be using a fact due to Scott [5] that the group generated by reflections in the right-angled 120-cell in $\mathbb{H}^4$ is GFERF; for a proof see [1].
3. SYSTOLES

**Theorem 3.1.** There exist closed hyperbolic 4-manifolds with arbitrarily short geodesics.

*Proof.* The examples will come from cutting and pasting certain covers of a 4-dimensional Coxeter orbifold. Let $\Gamma_D$ be the group generated by reflections in the faces of $D$. Let $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(\sqrt{5})$. Then $\Gamma_D$ is commensurable with $\text{PO}(f; \mathcal{O})$, where $f$ is the 5-dimensional quadratic form $\langle 1, 1, 1, 1, -\phi \rangle$, where $\phi = \frac{1 + \sqrt{5}}{2}$ [1] Lemma 3.3. Let $P \subset \mathbb{H}^4$ be a 3-dimensional geodesic subspace, such that $H = \text{Isom}(P) \cap \Gamma_D$ is a cocompact subgroup of $\text{Isom}(\mathbb{H}^4)$ (where we embed $\text{Isom}(P) < \text{Isom}(\mathbb{H}^4)$ in the natural fashion). If we identify $\mathbb{H}^4$ with a component of the hyperboloid $f(x) = -1$, then we find such a $P$ by letting $P = \mathbb{H}^4 \cap v^\perp$ (with respect to the inner product defined by $f$) where $v \in \mathbb{Q}(\sqrt{5})^5$, $f(v) > 0$. Let $\Gamma$ be a finite index torsion-free subgroup of $\Gamma_D$, which exists by Selberg’s lemma. Now, $\text{Comm}(\Gamma) > \text{PO}(f, \mathbb{Q}(\sqrt{5}))$, so $\text{Comm}(\Gamma)$ is dense in $\text{PO}(f, \mathbb{R}) = \text{Isom}(\mathbb{H}^4)$. Thus, for any $\epsilon > 0$, we may find $\gamma \in \text{Comm}(\Gamma)$ such that $\gamma(P) \cap P = \emptyset$, and $d(P, \gamma(P)) < \frac{\epsilon}{2}$. The plane $\gamma(P)$ is stabilized by $(\gamma H \gamma^{-1}) \cap \Gamma$, which is cocompact in $\text{Isom}(\gamma(P))$, since $[\Gamma : (\gamma \Gamma \gamma^{-1}) \cap \Gamma] < \infty$. Thus, $H_\gamma = \text{Isom}(\gamma(P)) \cap \Gamma$ is cocompact in $\text{Isom}(\gamma(P))$, since $(\gamma H \gamma^{-1}) \cap \Gamma < H_\gamma$. Let $g \subset \mathbb{H}^4$ be a geodesic segment perpendicular to $P$ and $\gamma(P)$, and with endpoints $p_1 = g \cap P$, and $p_2 = g \cap \gamma(P)$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be the function $\rho(x) = \text{artanh}(x)$. Using residual finiteness, choose $H_1 < H$ such that $d(p_1, h(p_1)) > 2\rho(l(g)/2)$, for all $h \in H_1 - \{1\}$. Similarly, choose $H_2 < H_\gamma$, such that $d(p_2, h(p_2)) > 2\rho(l(g)/2)$, for all $h \in H_2 - \{1\}$. Let $\Sigma_1 = P/H_1$, and $\Sigma_2 = \gamma(P)/H_2$.

Claim: $G = \langle H_1, H_2 \rangle \cong H_1 \ast H_2$. Moreover, $\mathbb{H}^4/G$ is geometrically finite.

To prove the claim, let $E_i \subset \mathbb{H}^4$ be a Dirichlet domain about $p_i$ with respect to the group $H_i$. Let $L$ be the 3-plane which is the perpendicular bisector of $g$. Let $pr_1 : \mathbb{H}^4 \to P$, and $pr_2 : \mathbb{H}^4 \to \gamma(P)$. Then $pr_i(L)$ is a disk about $p_i$ of radius $\rho(l(g)/2)$ in $P$ or $\gamma(P)$ (see Figure 1).

Since $E_i$ is a Dirichlet domain, it must contain $pr_i(L)$, and therefore $E_i$ contains $L$. Thus, $\partial E_1 \cap \partial E_2 = \emptyset$ since they are separated by the hyperplane $L$. Thus, $E_1 \cap E_2$ will be a finite-sided fundamental domain for $G$, and thus $G$ is geometrically finite (see [2] for various equivalent notions of geometric finiteness). Topologically, $\mathbb{H}^4/G = (\mathbb{H}^4/H_1)\#_L(\mathbb{H}^4/H_2)$, so $G \cong H_1 \ast H_2$.

Let $U = \Sigma_1 \cup_{p_1} g \cup_{p_2} \Sigma_2$. Then $U$ is an embedded compact spine of $\mathbb{H}^4/G$. Now, we use the fact from theorem 3.1 [1], that $G$ is a separable subgroup of $\Gamma$. By Scott’s separability criterion, we see that we may embed $U$ in $\mathbb{H}^4/\Gamma_1$, for some finite index subgroup $\Gamma_1 < \Gamma$. Thus, we have $\Sigma_1 \cup \Sigma_2 \subset \mathbb{H}^4/\Gamma_1$. Let $N = (\mathbb{H}^4/\Gamma_1)\backslash(\Sigma_1 \cup \Sigma_2)$, and let $M = D_N$, the double of $N$ along its boundary. Since $g \subset N$ is a geodesic arc orthogonal to $\partial N$, we have the double of $g$ $D(g) \subset M$ is a closed geodesic in $M$ of length $< \epsilon$. $\square$

4. Conclusion

**Conjecture 4.1.** There exists closed hyperbolic $n$-manifolds with arbitrarily short geodesics.

Hyperbolic lattices that are subgroup separable on geometrically finite subgroups are called $\text{GFERF}$, short for $\text{Geometrically Finite Extended Residual Finite}$. This conjecture would follow from the following conjecture, by the same proof as the main theorem.
**Conjecture 4.2.** There exist compact arithmetic hyperbolic $n$-manifolds which are defined by a quadratic form, and which are GFERF.

Unfortunately, there does not exist a compact right-angled polyhedron in $\mathbb{H}^n$, $n \geq 5$, so the strategy of proof in [1, 5] will not work in general. By the remark after [1, Lemma 3.4], we know that $O(8,1;\mathbb{Z})$ is GFERF. The above conjecture would hold if we knew $O(n,1;\mathbb{Z})$ were GFERF for all $n$, since one may embed (up to finite index) any cocompact arithmetic lattice defined by a quadratic form into $O(n,1;\mathbb{Z})$ for some $n$ by the restriction of scalars and stabilization. The following theorem is proven the same as Theorem 3.1.

**Theorem 4.3.** There exist finite volume hyperbolic $n$-manifolds with arbitrarily short geodesics for $n \leq 8$.

At most finitely many of the manifolds produced using Theorem 3.1 will be arithmetic. This follows because the groups will lie in $O(f,\mathbb{Q}(\sqrt{5}))$. The integral real eigenvalues of the matrices in $O(f,\mathbb{Q}(\sqrt{5}))$ will be bounded away from 1, since they have an integral minimal polynomial of degree at most 10. Thus, the length of a geodesic of an arithmetic subgroup of $O(f,\mathbb{Q}(\sqrt{5}))$ will be bounded away from 0, which implies that at most finitely many examples from Theorem 3.1 may be arithmetic. This method for proving the existence of non-arithmetic uniform lattices is slightly different than the method of [4], since instead of breeding subgroups of incommensurable arithmetic lattices, it breeds a subgroup of an arithmetic lattice with itself. It’s possible that this “inbreeding” method could produce non-arithmetic lattices in any dimension.

**References**

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