ALGEBRAS OF MULTIPLACE FUNCTIONS FOR SIGNATURES CONTAINING ANTIDOMAIN

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Abstract. We define antidomain operations for algebras of multiplace partial functions. For all signatures containing composition, the antidomain operations and any subset of intersection, preferential union and fixset, we give finite equational or quasiequational axiomatisations for the representation class. We do the same for the question of representability by injective multiplace partial functions. For all our representation theorems, it is an immediate corollary of our proof that the finite representation property holds for the representation class. We show that for a large set of signatures, the representation classes have equational theories that are coNP-complete.

1. Introduction

The scheme for investigating the abstract algebraic properties of functions takes the following form. First choose some sort of functions of interest, for example partial functions or injective functions. Second, specify some set-theoretically-defined operations possible on such functions, for example function composition or set intersection. Finally, study the isomorphism class of algebras that consist of some such functions together with the specified set-theoretic operations.

The study of algebras of so-called multiplace functions started with Menger [7]. Here the objects in the concrete algebras are (usually partial) functions from $X^n$ to $X$ for some fixed $X$ and $n$. Since then, representation theorems—axiomatisations of isomorphism classes via explicit representations—have been given for various cases [1, 10, 9, 2].

For unary functions, the antidomain operation yields the identity function restricted to the complement of a function’s domain. This operation seems first to have been described in [5], where it is referred to as domain complement. Some recent work has been directed towards providing representation theorems in the case of unary functions for signatures including antidomain [6, 4].

In this paper we define, for $n$-ary multiplace functions, $n$ indexed antidomain operations by simultaneous analogy with the indexed domain operations studied on multiplace functions and the antidomain operation studied on unary functions. This definition together with other fundamental definitions we need comprise Section 2.

The majority of this paper, Sections 3–8, consists of representation theorems for multiplace functions for signatures containing composition and the antidomain operations. Much of this is a straightforward translation of [6], where the same is done for unary functions.

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In Sections 3 and 4 we work over the signature containing composition and the antidomain operations. We show that for multiplace partial functions the representation class cannot form a variety and we state and prove the correctness of a finite quasiequational axiomatisation of the class. It follows, as it does for our later representation theorems, that the representation class has the finite representation property.

In Section 5 we use a single quasiequation to extend the axiomatisation of Section 3 to a finite quasiequational axiomatisation for the case of injective multiplace partial functions.

In Section 6 we add intersection to our signature and for both partial multiplace functions and injective partial multiplace functions are able to give finite equational axiomatisations of the representation class.

In Sections 7 and 8 we consider all our previous representation questions with the preferential union and fixset operations, respectively, added to the signature. In all cases we give either finite equational or finite quasiequational axiomatisations of the representation class.

In Section 9 we switch our focus to equational theories. We prove that for any signature containing operations that we mention, the equational theory of the representation class of multiplace partial functions lies in \( \text{coNP} \). If the signature contains the antidomain operations and either composition or intersection then the equational theory is \( \text{coNP} \)-complete.

2. Algebras of Multiplace Functions

In this section we give the fundamental definitions of algebras of multiplace functions and of the various operations that may be included.

Given an algebra \( \mathfrak{A} \), when we write \( a \in \mathfrak{A} \) or say that \( a \) is an element of \( \mathfrak{A} \), we mean that \( a \) is an element of the domain of \( \mathfrak{A} \). We follow the convention that algebras are always nonempty. We use \( n \) to denote an arbitrary nonzero natural number. A bold symbol, \( \mathbf{a} \) say, is either simply shorthand for \( \langle a_1, \ldots , a_n \rangle \) in a term of the form \( \langle a_1, \ldots , a_n \rangle : b \) or denotes an actual \( n \)-tuple \( \langle a_1, \ldots , a_n \rangle \). We may abuse notation, when convenient, by writing \( (x, y) \) for the \((n+1)\)-tuple \( \langle x_1, \ldots , x_n, y \rangle \).

If \( A_1, \ldots , A_n \) are unary operation symbols, the notation \( \langle A^n_1 a \rangle \) is shorthand for \( \langle A_1 a, \ldots , A_n a \rangle \). When a function \( f \) acts on an \( n \)-tuple \( \langle a_1, \ldots , a_n \rangle \) we omit the angle brackets and write \( f(a_1, \ldots , a_n) \). If \( i \) is an index, then ‘for all \( i \)’ or ‘for every \( i \)’ means for all \( i \in \{1, \ldots , n\} \).

First we make clear what we mean by a multiplace function.

**Definition 2.1.** An \( n \)-ary relation is a subset of a set of the form \( X_1 \times \ldots \times X_n \). Without loss of generality we may assume all the \( X_i \)’s are equal. In the context of a given value of \( n \), a **multiplace partial function** is an \((n+1)\)-ary relation \( f \) validating

\[
(x_1, \ldots , x_n, y) \in f \land (x_1, \ldots , x_n, z) \in f \rightarrow y = z.
\]

We may also use the terminology \( n \)-ary partial function for the same concept. We import all the usual terminology for partial functions, for instance if \( (x, y) \in f \) then we may write \( f(x) = y \), say ‘\( f(x) \) is defined’, and so on.

Henceforth, we will use the epithet ‘\( n \)-ary’ in favour of ‘multiplace’ in order to make the arity of the functions in question explicit.
Definition 2.2. Let $\sigma$ be an algebraic signature whose symbols are a subset of \( \{ (\ ), ;, 0, \pi_i, D_i, A_i, F_i, \bowtie, \sqcap, \sqcup \} \), where we write, for example, $A_i$ to indicate that $A_1, \ldots, A_n \in \sigma$ for some fixed $n$. An algebra of $n$-ary partial functions of the signature $\sigma$ is an algebra, $\mathfrak{A}$, of the signature $\sigma$ whose elements are $n$-ary partial functions and that has the following properties.

(i) There is a set $X$, the base, and an equivalence relation $E$ on $X$ with the following property. For all $f \in \mathfrak{A}$ and all $(x_1, \ldots, x_{n+1}) \in f$, we have that $x_i E x_{i+1}$ for every $i$. That is, every partial function in the algebra contains only $(n+1)$-tuples of $E$-equivalent members of $X$.

(ii) The operations are given by the set-theoretic operations on partial functions following property. For all $f \in \mathfrak{A}$ and all $(x_1, \ldots, x_{n+1}) \in f$, we have that $x_i E x_{i+1}$ for every $i$. That is, every partial function in the algebra contains only $(n+1)$-tuples of $E$-equivalent members of $X$.

In an algebra of $n$-ary partial functions

- the $(n+1)$-ary operation $\langle \, \rangle$ is composition, given by:
  \[
  f : g = \{(x, z) \in X^{n+1} \mid \exists y \in X^n : (x, y_i) \in f_i \text{ for each } i \text{ and } (y, z) \in g\},
  \]
- the binary operation $\cdot$ is intersection:
  \[
  f \cdot g = \{(x, y) \in X^{n+1} \mid (x, y) \in f \text{ and } (x, y) \in g\},
  \]
- the constant 0 is the nowhere-defined function:
  \[
  0 = \emptyset = \{(x, y) \in X^{n+1} \mid \bot\},
  \]
- for each $i$ the constant $\pi_i$ is the $i$th projection on the set of all $n$-tuples of $E$-equivalent points:
  \[
  \pi_i = \{(x, x_i) \in X^{n+1} \mid x_1, \ldots, x_n \text{ all } E\text{-equivalent}\},
  \]
- for each $i$ the unary operation $D_i$ is the operation of taking the $i$th projection restricted to the domain of a function:
  \[
  D_i(f) = \{(x, x_i) \in X^{n+1} \mid \exists y \in X : (x, y) \in f\},
  \]
- for each $i$, the unary operation $A_i$ is the operation of taking the $i$th projection restricted to the antidomain of a function—those $n$-tuples of $E$-equivalent points where the function is not defined:
  \[
  A_i(f) = \{(x, x_i) \in X^{n+1} \mid x_1, \ldots, x_n \text{ all } E\text{-equivalent and } \not\exists y \in X : (x, y) \in f\};
  \]
- for each $i$, the unary operation $F_i$, the $i$th fixset operation, is the $i$th projection function intersected with the function itself:
  \[
  F_i(f) = \{(x, x_i) \in X^{n+1} \mid (x, x_i) \in f\},
  \]
- for each $i$, the binary operation $\bowtie_i$, the $i$th tie operation, is the $i$th projection function restricted to those $E$-equivalent $n$-tuples where the two arguments do not disagree, that is, either neither is defined or they are both defined and are equal:
  \[
  f \bowtie_i g = \{(x, x_i) \in X^{n+1} \mid (x, x_i) \in A_i f \cap A_i g \text{ or } \exists y \in X : (x, y) \in f \cap g\},
  \]
- the binary operation $\sqcup$ is preferential union:
  \[
  (f \sqcup g)(x) = \begin{cases} 
  f(x) & \text{if } f(x) \text{ defined} \\
  g(x) & \text{if } f(x) \text{ undefined, but } g(x) \text{ defined} \\
  \text{undefined} & \text{otherwise}
  \end{cases}
  \]
If the equivalence relation $E$ is the universal relation, $X \times X$, then we say that the algebra is square.

**Definition 2.3.** Let $\mathfrak{A}$ be an algebra of one of the signatures specified by Definition 2.2. A **representation of $\mathfrak{A}$ by $n$-ary partial functions** is an isomorphism from $\mathfrak{A}$ to an algebra of $n$-ary partial functions of the same signature. If $\mathfrak{A}$ has a representation then we say it is **representable**.

As we have signified, in this paper the focus is on isomorphs of algebras of $n$-ary partial functions in general, rather than the square ones in particular. However, now is an opportune moment for a brief discussion of the merits of each of these concepts and the relationship between them.

The square algebras of $n$-ary functions have the advantage of being the simpler and more natural concept. However for certain signatures they are not as algebraically well behaved, failing to be closed under direct products. Indeed there are simple examples of pairs of algebras that are each representable as square algebras of functions but whose product is not. The presence of the antidomain operations in the signature will always cause this problem, as the example we now give demonstrates.

**Example 2.4.** Assume $n \geq 2$ and work over any one of the signatures specified by Definition 2.2 containing the $n$ indexed antidomain operations $A_1, \ldots, A_n$. Consider the two-element algebra $\mathfrak{A}$ consisting of both of the $n$-ary partial functions on some base of size one. As $\mathfrak{A}$ is a square algebra of partial functions it is trivially representable as a square algebra of partial functions. We argue that $\mathfrak{A} \times \mathfrak{A}$ is not representable as a square algebra of functions.

Suppose, for contradiction, that $\theta$ is a square representation of $\mathfrak{A} \times \mathfrak{A}$ with base $X$. Since $|\mathfrak{A} \times \mathfrak{A}| = 4$, we know $X$ must contain at least two distinct points, in order that $\theta$ distinguishes all the elements of $\mathfrak{A} \times \mathfrak{A}$. Let $\mathbf{x}$ be any $n$-tuple from $X^n$ not lying on the diagonal. Denote the two elements of $\mathfrak{A}$ by $a$ and $b$ and $A_i a = b$ and $A_i b = a$ for every $i$. So $A_1(a, a) = (b, b)$, and hence the domains of the partial functions $\theta(a, a)$ and $\theta(b, b)$ must partition $X^n$. Without loss of generality we may assume $\mathbf{x}$ is not in the domain of $\theta(a, a)$. But then $\theta(A_1(a, a))(\mathbf{x}) = x_i$ for every $i$. As every $\theta(A_1(a, a))$ is the same function, namely $\theta(b, b)$, all components of $\mathbf{x}$ are equal, contradicting the assumption that $\mathbf{x}$ is not on the diagonal. We conclude that $\mathfrak{A} \times \mathfrak{A}$ cannot be represented as a square algebra of partial functions.

An immediate consequence of not being closed under direct products is that the class of algebras having a square representation cannot be a quasivariety. We note however that these classes always possess universal axiomatisations in first-order logic, for any of the signatures covered by Definition 2.2. This can be seen by appealing to Schein’s fundamental theorem of relation algebra [8]. There are two conditions of Schein’s theorem that need to be checked. The first is that $n$-ary partial functions can be defined as those $(n+1)$-ary relations satisfying a recursive set of sentences in the first-order language with equality and a countable supply of $(n+1)$-ary relation symbols, which is precisely what we did in Definition 2.1 by using (1). The second is that, using the same first-order language, the operations we are considering can each be defined using a formula with $n+1$ free variables. Definitions of the operations for square algebras can be formed from the more general definitions we gave in Definition 2.2 by removing any stipulations of $E$-equivalence. The resulting definitions are of the required form.
Definition 2.5. Let \((A_i)_{i \in I}\) be a family of algebras all of the same signature and \((\theta_i : A_i \rightarrow \mathcal{F}_i)_{i \in I}\) be a corresponding family of homomorphisms to algebras of \(n\)-ary partial functions, with \(\mathcal{F}_i\) having base \(X_i\) and equivalence relation \(E_i\) on \(X_i\).

A disjoint union of \((\theta_i)_{i \in I}\) is any homomorphism \(\theta\) out of \(\prod_{i \in I} A_i\) formed by the following process. First rename the elements of the \(X_i\)'s in such a way that the \(X_i\)'s are pairwise disjoint. Then the codomain of \(\theta\) will be an algebra \(\mathcal{F}\) consisting of all \(n\)-ary partial functions of the form \(\bigcup_{i \in I} \theta_i(a_i)\) for some element \((a_i)_{i \in I}\). The base of \(\mathcal{F}\) will be \(X := \bigcup_{i \in I} X_i\) and the equivalence relation on \(X\) will be \(E := \bigcup_{i \in I} E_i\). The operations on \(\mathcal{F}\) will be given by the concrete operations described in Definition 2.2. Define \(\theta((a_i)_{i \in I}) = \bigcup_{i \in I} \theta_i(a_i)\) for each element \((a_i)_{i \in I}\) of \(\prod_{i \in I} A_i\). The map \(\theta\) is straightforwardly a homomorphism.

A disjoint union of injective homomorphisms will be injective and that is why we remarked that a product of representable algebras can be represented by a disjoint union of representations would be a representation, since the disjoint union of two universal equivalence relations is not universal.

Our final remark about square algebras of partial functions is that it is easily seen that every algebra representable by \(n\)-ary partial functions is a subalgebra of a product of algebras each having a square representation. Hence the general representation class is contained in the quasivariety generated by the square representation class.

For algebras of \(n\)-ary functions, the first representation theorem was provided by Dicker in [1], showing that the equation that has come to be known as the superassociativity law axiomatises the representation class (for total functions, although the equation is valid for partial functions) in the signature consisting only of composition. Trokhimenko gave equational axiomatisations for the signatures of composition and intersection, in [10], and composition and domain, in [9]. In [2], Dudek and Trokhimenko gave a finite equational axiomatisation for the signature of composition, intersection and domain.

The subject of this paper is signatures containing composition and antiderm. Note that \(0, \pi_i\) and \(D_i\) are all definable using composition and antiderm, using \(0 := \langle A_i^n a \rangle : a\), for any \(a\), and then \(\pi_i := A_i 0\) and using \(D_i := A_i^2\) (that is, a double application of \(A_i\)). Further, in the presence of composition and antiderm, the tie operations and intersection are interdefinable. The tie operations are definable as \(a \bowtie b := D_i(a \cdot b) +_i \langle A_i^n a \rangle : A_i b\), where \(\alpha +_i \beta := A_i(\langle A_i^n \alpha \rangle : A_i \beta)\). Intersection is definable as \(a : b := (a \bowtie b) : a\). This leaves \(\cdot\), \(F_i\) and \(\sqcap\) as the only interesting additional operations among those we have mentioned. When intersection is present, the fixset operations are definable as \(F_i f := \pi_i \cdot f\).

We include here, for ease of reference, a summary of the results about representation classes contained in this paper. All classes have finite axiomatisations of the relevant form.
Table 1. Summary of representation classes for $n$-ary functions

| Signature | Partial functions | Injective partial functions |
|-----------|-------------------|-----------------------------|
| $\langle \rangle; A_i$ | proper quasivariety | quasivariety |
| $\langle \rangle; A_i; \cdot$ | variety | variety |
| $\langle \rangle; A_i; \top$ | variety | quasivariety |
| $\langle \rangle; A_i; \bot$ | variety | variety |
| $\langle \rangle; A_i; F_i$ | quasivariety | quasivariety |
| $\langle \rangle; A_i; F_i; \sqcup$ | quasivariety | quasivariety |

Note that whenever a representation class has a finite quasiequational axiomatisation the decision problem of representability of finite algebras is solvable in polynomial time, and if we know such an axiomatisation then we know such an algorithm. We observed earlier that, for each signature, the representation class is contained in the quasivariety generated by the algebras having square representations. Hence another point to note is that our results identify the representation classes as equal to these generated quasivarieties.

Beyond representability, we may also be interested in representability on a finite base. Our final fundamental definition can be invoked in any circumstance where there is a notion of representability.

**Definition 2.6.** The **finite representation property** holds if any finite representable algebra is representable on a finite base.

### 3. Composition and Antidomain

First we examine the signature $\langle \rangle; A_1, \ldots, A_n$ consisting of composition and the antidomain operations. After presenting some equations and one quasiequation that are valid for algebras of $n$-ary partial functions, we deduce some consequences of these (quasi)equations that we use in Section 4 to prove that our (quasi)equations axiomatise the representation class.

In [6], Jackson and Stokes give a finite quasiequational axiomatisation of the representation class of unary partial functions for the signature of composition, antidomain.\(^1\) They call algebras satisfying these laws modal restriction semigroups.

**Definition 3.1.** A **modal restriction semigroup** [6] is an algebra of the signature $\langle , A \rangle$ satisfying the equations

\[
(a ; b ; c = a ; (b ; c) \\
1' ; a = a \\
A(a) ; a = 0 \\
0 ; a = 0 \\
a ; 0 = 0 \\
a ; A(b) = A(a ; b) ; a
\]

(the **twisted law for antidomain**)\(^1\)

\(^1\)Actually, their signature also contains the constants 0 and 1', but these are definable from composition and antidomain.
and the quasiequation
\[ D(a) : b = D(a) : c \land A(a) : b = A(a) : c \rightarrow b = c \]
where \( 0 := A(b) : b \) for any \( b \) (and the third equation says this is a well-defined constant), \( 1' := A(0) \) and \( D := A^2 \).

Note that the definition of modal restriction semigroups given by Jackson and Stokes states they should be monoids, so \( 1' \) should also be a right identity. But this is a consequence of the equations we gave in Definition 3.1, for
\[ a ; 1' = a ; A(0) = A(a ; 0) ; a = A(0) ; a = 1' ; a = a \]
using the twisted law for the second equality.

For \( n \)-ary functions, working over the signature \( (\langle \rangle ; a_1, \ldots, a_n) \), we can try to write down valid \( n \)-ary versions of the (quasi)equations appearing in Definition 3.1. This is easy in every case except that of the twisted law for antidomain, which needs more care.

This is a good point at which to note that we do not need to bracket expressions like \( a ; b ; c \) since this can only mean \( a ; (b ; c) \). When we do write the brackets, we do so only for emphasis.

**Proposition 3.2.** The following equations and quasiequations are valid for the class of \((\langle \rangle ; a_1, \ldots, a_n)\)-algebras representable by \( n \)-ary partial functions.

(2) \[ \langle a ; b_1, \ldots, a ; b_n \rangle ; c = a ; (b ; c) \] \textit{(superassociativity)}

(3) \[ \pi ; a = a \]

(4) \[ \langle a_i^n \rangle ; a = 0 \]

(5) \[ \langle a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n \rangle ; b = 0 \] \textit{for every} \( i \)

(6) \[ a ; 0 = 0 \]

(7) \[ a ; A_i b = \langle A_i^n \rangle (a ; b) ; \langle D_i^n a_1 \rangle ; \ldots ; \langle D_i^n a_n \rangle ; a_i \] \textit{for every} \( i \)

\textit{(the twisted laws for antidomain)}

(8) \[ \langle D_i^n a \rangle ; b = \langle D_i^n a \rangle ; c \land \langle A_i^n a \rangle ; b = \langle A_i^n a \rangle ; c \rightarrow b = c \]

where \( 0 := \langle A_i^n b \rangle ; b \) for any \( b \) (and (4) says this is a well-defined constant), \( \pi_i := A_i 0 \) and \( D_i := A_i^2 \) \textit{(a double application of} \( A_i \)).

**Proof.** We noted in the previous section that every algebra representable by \( n \)-ary partial functions is isomorphic to a subalgebra of a product of algebras having a square representation. As the validity of quasiequations is preserved by taking products and subalgebras, it suffices to prove validity only for algebras having square representations. Further, since representations are themselves isomorphisms, it is sufficient to prove validity for an arbitrary square algebra of \( n \)-ary partial functions. So suppose we have such an algebra, with base \( X \).

The validity of the superassociative law has been recognised since Menger noted it in [7]. We turn next to (4). Given an \( n \)-ary partial function \( a \), if \( \langle A_i^n a \rangle ; a \) is to be defined at an \( n \)-tuple \( x \) then there should be a \( y \) with \( A_i(a)(x) = y_i \) for each \( i \) and with \( a \) defined at \( y \). Since each \( A_i a \) is a restriction of the \( i \)th projection, \( y \) can only be \( x \). But if \( A_i a \) is defined at \( x \) then \( a \) cannot be. Hence \( \langle A_i^n a \rangle ; a \) is the nowhere-defined function. So 0 is well defined, that is, the value of \( \langle A_i^n a \rangle ; a \)
does not depend on the choice of $a$, and so (4) is valid. The validity of (5) and the validity of (6) are now both clear.

Now $\pi_i := A_i0$ is the $i$th projection restricted to those $n$-tuples in $X^n$ where 0 is not defined. So $\pi_i$ is, as the notation indicates, the $i$th projection on the set of all $n$-tuples in $X^n$. The validity of (3) is now clear.

For the twisted laws for antidomain, first suppose that $a : A_i b$ is defined at $x$. Then we know that $a_1, \ldots, a_n$ are all defined at $x$ and that $b$ is not defined at $\langle a_1(x), \ldots, a_n(x) \rangle$. Hence $D_j a_k$ is defined at $x$ for every $j, k$ and $a : b$ is not defined at $x$. It follows that $A_j(a : b)$ is defined at $x$ for every $j$. It is now apparent that $\langle A_i^n(a : b) ; D_i^n a_1 ; \ldots ; D_i^n a_n ; a_i \rangle$ is defined at $x$ with value $a_i(x)$—the same value as $a : A_i b$.

If $a : A_i b$ is not defined at an $n$-tuple $x$, then this is either because $a_j$ is undefined at $x$ for some $j$ or all $a_j$ are defined at $x$, but $A_i b$ is not defined at $\langle a_1(x), \ldots, a_n(x) \rangle$. If $a_j$ is undefined at $x$ then it is clear that $\langle A_i^n(a : b) ; D_i^n a_1 ; \ldots ; D_i^n a_n ; a_i \rangle$ cannot be defined at $x$. In the second case, $b$ must be defined at $\langle a_1(x), \ldots, a_n(x) \rangle$ and so $a : b$ is defined at $x$. Again it is clear that $\langle A_i^n(a : b) ; D_i^n a_1 ; \ldots ; D_i^n a_n ; a_i \rangle$ cannot be defined at $x$.

For (8), suppose the antecedent of the implication is true. Let $x$ be an $n$-tuple in $X^n$. If $a$ is defined on $x$ then $D_i a$ is defined at $x$ for each $i$ and accordingly $\langle D_i^n a ; b = D_i^n a ; c \rangle$ says that either $b(x) = c(x)$ or both $b$ and $c$ are undefined at $x$. If $a$ is undefined at $x$ then $A_i a$ is defined at $x$ for each $i$ and this time

$\langle A_i^n(a ; b) ; D_i^n a_1 ; \ldots ; D_i^n a_n ; a_i \rangle$ says that either $b(x) = c(x)$ or both $b$ and $c$ are undefined at $x$. $\square$

Note that the naive $n$-ary versions of the twisted law for antidomain, namely $\langle a : A_i b \rangle = (A_i^n(a ; b)) ; a_i$, for every $i$, are not valid (except in the unary case). Indeed if at an $n$-tuple, $a_i$ is defined, but $a_j$ is undefined for some $j$ different to $i$, then $a : A_i b$ is undefined, but $\langle A_i^n(a ; b) ; a_i \rangle$ will be defined.

To compensate for the complication with the twisted laws, we introduce as an axiom the equation

$$\langle D_i^n a ; a = a \rangle$$

whose validity is clear and has been noted before; for example it appears as Equation (10) in [2].

In addition we will need one extra indexed set of equations (trivial in the unary case) namely

$$A_i A_j a = A_i A_k a$$

whose validity we now prove.

**Proposition 3.3.** The indexed equations of (10) are valid for the class of $\langle ; A_1, \ldots, A_n \rangle$-algebras representable by $n$-ary partial functions.

**Proof.** As before it is sufficient to prove validity for an arbitrary square algebra of $n$-ary partial functions. So suppose we have such an algebra, with base $X$.

Suppose that $A_i A_j a$ is defined on an $n$-tuple $x$, necessarily with value $x_i$. Then $A_j a$ is not defined on $x$. Hence $a$ is defined on $x$. It follows that $A_k a$ is not defined on $x$ and from there we deduce that $A_i A_k a$ is defined on $x$, necessarily with value $x_i$. Hence the function $A_i A_j a$ is a restriction of $A_i A_k a$. By symmetry the reverse is true and the two functions are equal. $\square$
We are going to prove that (2)–(10) axiomatise the class of \( \langle \rangle; A_1, \ldots, A_n \)-algebras that are representable by \( n \)-ary partial functions and hence the representation class is a quasivariety. But before we do that, we show that the representation class is not a variety.

**Proposition 3.4.** The class of \( \langle \rangle; A_1, \ldots, A_n \)-algebras that are representable by \( n \)-ary partial functions is not closed under quotients and hence is not a variety.

**Proof.** We adapt an example given in [6] to describe an algebra of \( n \)-ary partial functions having a quotient that does not validate (8) and so is not representable by partial functions.

We describe an algebra \( \mathfrak{F} \) of \( n \)-ary partial functions, with base \( \{1, 2, 3\} \). The equivalence relation to which the antidomain operations are relativised partitions the base into \( \{1\} \) and \( \{2, 3\} \). The elements of \( \mathfrak{F} \) are the following 2\((n + 3)\) elements.

- the empty function
- the \( i \)th projection on \( \{2, 3\}^n \), for each \( i \)
- the function with domain \( \{2, 3\}^n \) that is constantly 2
- the function with domain \( \{2, 3\}^n \) that is constantly 3
- each of the aforementioned \( n + 3 \) functions with the pair \( \langle 1, 1 \rangle \) adjoined

It is clear that \( \mathfrak{F} \) is closed under the \( n \) antidomain operations. Checking that \( \mathfrak{F} \) is closed under composition is also straightforward.

It is easy to check, directly, that identifying all the elements with domain \( \{2, 3\}^n \) produces a quotient of \( \mathfrak{F} \). Let \( a \) be any element with domain \( \{2, 3\}^n \), let \( b \) be the element sending \( 1 \) to 1 and constantly 2 elsewhere and let \( c \) be the element sending \( 1 \) to 1 and constantly 3 elsewhere. Then in the quotient

\[
\langle D^n_i[a] \rangle \div [b] = \langle D^n_i[a] \rangle \div [c]
\]

and

\[
\langle A^n_i[a] \rangle \div [b] = \langle A^n_i[a] \rangle \div [c].
\]

but \([b]\) and \([c]\) are not equal. Hence (8) is refuted in the quotient. \( \square \)

Next comes the work of deducing the various consequences of (2)–(10) that are needed to prove their sufficiency for representability.

We noted earlier that the equation \( a \cdot A_i b = \langle A^n_i(a ; b) \rangle ; a \) is not valid, but we can obtain a version in the special case that \( a \) is of the form \( \langle A^n_i a' \rangle \) for some \( a' \).

**Lemma 3.5.** The indexed equations

\[
(11) \quad \langle A^n_i a \rangle ; A_i b = \langle A^n_i (\langle A^n_i a \rangle ; b) \rangle ; A_i a \quad \text{for every } i
\]

are consequences of axioms (2)–(10).

**Proof.** We have

\[
\langle A^n_i a \rangle ; A_i b = \langle A^n_i (\langle A^n_i a \rangle ; b) \rangle ; D^n_i A_1 a ; \ldots ; D^n_i A_n a ; A_i a
\]

\[
= \langle A^n_i (\langle A^n_i a \rangle ; b) \rangle ; D^n_i A_i a ; \ldots ; D^n_i A_i a ; A_i a
\]

\[
= \langle A^n_i (\langle A^n_i a \rangle ; b) \rangle ; A_i a
\]

by first applying the \( i \)th twisted law for antidomain, then applying (10) and then repeatedly applying (9). \( \square \)
We will give (11) the full title: the restricted twisted laws for antidomain, but since these are the twisted laws we apply most frequently, when we refer simply to ‘the ith twisted law’ we will mean the i-indexed version of (11).

In the following lemma and in later proofs an ‘s’ above an equality sign indicates an appeal to superassociativity, a ‘t’ an appeal to the twisted laws and any number an appeal to the corresponding equation.

**Lemma 3.6.** The following equations are consequences of axioms (2)–(10).

\[(12) \quad \langle A^n_1 a \rangle : A_i a = A_i a \quad \text{for every } i\]
\[(13) \quad \langle A^n_i a \rangle : A_i b = \langle A^n_1 b \rangle : A_i a \quad \text{for every } i\]
\[(14) \quad \langle A^n_i a \rangle : \langle A^n_i b \rangle : c = \langle A^n_1 b \rangle : \langle A^n_i a \rangle : c\]
\[(15) \quad D_{ij}(\langle A^n_i a \rangle : A_i b) = \langle A^n_i a \rangle : A_j b \quad \text{for every } i, j\]

**Proof.** We have

\[
\langle A^n_i a \rangle : A_i a = \langle A^n_1 (\langle A^n_1 a \rangle : a) \rangle : A_i a
\]
\[
= \langle A^n_1 0 \rangle : A_i a
\]
\[
= \pi : A_i a
\]
\[
= A_i a
\]

proving (12).

Before proceeding with (13)–(15), we note the following useful consequences of (2)–(10). By (10) then (4) we see that

\[(16) \quad \langle D^n_1 a \rangle : A_i a = \langle A^n_1 A_i a \rangle : A_i a = 0\]

and by first applying superassociativity and then (16) to \(\langle D^n_1 a \rangle : \langle A^n_i a \rangle : b\) we obtain

\[(17) \quad \langle D^n_2 a \rangle : (\langle A^n_1 a \rangle : A_i b) = 0.\]

We will use (8) to prove (13). Firstly

\[
\langle A^n_i a \rangle : (\langle A^n_i a \rangle : A_i b) \overset{s}{=} \langle \langle A^n_1 a \rangle : A_1 a, \ldots, \langle A^n_i a \rangle : A_n a \rangle : A_i b
\]
\[
\overset{12}{=} \langle A^n_1 a \rangle : A_i b
\]

and

\[
\langle A^n_i a \rangle : (\langle A^n_i b \rangle : A_i a) \overset{s}{=} \langle \langle A^n_1 a \rangle : A_1 b, \ldots, \langle A^n_i a \rangle : A_n b \rangle : A_i a
\]
\[
\overset{t}{=} \langle \langle A^n_1 (\langle A^n_1 a \rangle : b) \rangle : A_1 a, \ldots, \langle A^n_1 (\langle A^n_1 a \rangle : b) \rangle : A_n a \rangle : A_i a
\]
\[
\overset{s}{=} \langle A^n_1 (\langle A^n_1 a \rangle : b) \rangle : \langle A^n_1 a \rangle : A_i a
\]
\[
\overset{12}{=} \langle A^n_1 (\langle A^n_1 a \rangle : b) \rangle : A_i a
\]
\[
\overset{t}{=} \langle A^n_1 a \rangle : A_i b
\]

so we see that \(\langle A^n_1 a \rangle : (\langle A^n_1 a \rangle : A_i b)\) and \(\langle A^n_1 a \rangle : (\langle A^n_1 b \rangle : A_i a)\) coincide. We also have

\[
\langle D^n_1 a \rangle : (\langle A^n_1 a \rangle : A_i b) \overset{17}{=} 0
\]
and

\[ \langle D^n_1 a \rangle : (\langle A^n_i b \rangle ; A_i a) = \langle D^n_1 a \rangle : (\langle a \rangle ; D_1 a) \]

\[ = \langle \langle D^n_1 a \rangle b \rangle : D_1 a \]

\[ = (\langle D^n_1 a \rangle b) : a \]

\[ = (D^n_1 a) ; D_1 a \]

\[ = \pi ; D_1 a \]

\[ = D_1 a \]

and so \( \langle D^n_1 a \rangle : (\langle A^n_i a \rangle ; A_i b) \) and \( \langle D^n_1 a \rangle : (\langle A^n_i a \rangle ; A_i a) \) coincide, completing the proof of (13).

Equation (14) is a simple, but useful, consequence of (13). We have

\[ \langle A^n_i a \rangle ; (A^n_1 b) ; c = (\langle A^n_i a \rangle ; A_1 b) \]

\[ = (\langle A^n_1 b \rangle ; A_1 a) \]

\[ = \langle A^n_1 b \rangle ; (A^n_1 a) ; c \]

as required.

To prove (15) we prove that

\[ A_j (\langle A^n_i a \rangle ; A_i b) = A_j (\langle A^n_i a \rangle ; A_j b) \]

for every \( i, j \)

and that

\[ D_j (\langle A^n_i a \rangle ; A_j b) = (\langle A^n_i a \rangle ; A_j b) \]

for every \( j \)

are consequences of (2)–(10).

For (18) we have

\[ \langle A^n_i a \rangle ; A_j (\langle A^n_i a \rangle ; A_i b) = (\langle A^n_i a \rangle ; A_i b) ; A_j a \]

\[ = \langle A^n_i a \rangle ; A_j A_i b \]

\[ = \langle A^n_i a \rangle ; A_j A_j b \]

and in the same way

\[ \langle A^n_i a \rangle ; A_j (\langle A^n_i a \rangle ; A_j b) = \]

\[ = \langle A^n_i a \rangle ; A_j A_j b \]

and we have

\[ \langle D^n_1 a \rangle ; A_j (\langle A^n_1 a \rangle ; A_i b) = (\langle D^n_1 a \rangle ; A_i b) ; D_1 a \]

\[ = (\langle D^n_1 a \rangle 0) ; D_1 a \]

\[ = \pi ; D_1 a \]

\[ = D_1 a \]

and similarly

\[ \langle D^n_1 a \rangle ; A_j (\langle A^n_1 a \rangle ; A_j b) = D_1 a \]

and so from an application of (8) we deduce the required equation.
Equation (19) can be deduced with two applications of (8), composing on the left with \( \langle A^n_i a \rangle \) and \( \langle D^n_i a \rangle \) and with \( \langle A^n_i b \rangle \) and \( \langle D^n_i b \rangle \). One can show that any of the compositions with \( \langle D^n_i a \rangle \) or \( \langle D^n_i b \rangle \) evaluate to 0, for example

\[
\langle D^n_i a \rangle ; \langle A^n_i b \rangle ; D_j (\langle A^n_i a \rangle ; A_j b)
\]

\[
= \langle A^n_i b \rangle ; \langle D^n_i a \rangle ; D_j (\langle A^n_i a \rangle ; A_j b)
\]

by (14)

\[
= \langle A^n_i b \rangle ; \langle D^n_i a \rangle ; A_j ; \langle (A^n_i a) ; A_j b \rangle
\]

by the definition of \( D_j \)

\[
= \langle A^n_i b \rangle ; \langle A^n_i (D^n_i a) ; A_j (\langle A^n_i a \rangle ; A_j b) \rangle ; D_j a
\]

by the \( j \)th twisted law

\[
= \langle A^n_i b \rangle ; \langle A^n_i (\langle (D^n_i a) ; \langle A^n_i a \rangle ; A_j b \rangle) ; D_j a \rangle ; D_j a
\]

by the \( j \)th twisted law

\[
= \langle A^n_i b \rangle ; \langle A^n_i ((A^n_i 0) ; D_j a) \rangle ; D_j a
\]

by (17)

\[
= \langle A^n_i b \rangle ; \langle A^n_i (\pi ; D_j a) \rangle ; D_j a
\]

by the definition of \( \pi \)

\[
= \langle A^n_i b \rangle ; \langle A^n_i D_j a \rangle ; D_j a
\]

by (3)

\[
= \langle A^n_i b \rangle ; 0
\]

by (4)

\[
= 0
\]

by (6)

and the others are similar. The compositions with \( \langle A^n_i a \rangle \) and \( \langle A^n_i b \rangle \) both equal \( \langle A^n_i a \rangle ; A_j ; b \). Observe

\[
\langle A^n_i a \rangle ; \langle A^n_i b \rangle ; D_j (\langle A^n_i a \rangle ; A_j b)
\]

\[
= \langle A^n_i a \rangle ; \langle D^n_i (\langle A^n_i a \rangle ; A_j b) \rangle ; A_j b
\]

by (13)

\[
= \langle D^n_i (\langle A^n_i a \rangle ; A_j b) \rangle ; \langle A^n_i a \rangle ; A_j b
\]

by (14)

\[
= \langle A^n_i a \rangle ; A_j b
\]

by (9)

and

\[
\langle A^n_i a \rangle ; \langle A^n_i b \rangle ; (\langle A^n_i a \rangle ; A_j b)
\]

\[
= \langle A^n_i a \rangle ; \langle A^n_i a \rangle ; \langle A^n_i b \rangle ; A_j b
\]

by (14)

\[
= \langle A^n_i a \rangle ; \langle A^n_i a \rangle ; A_j b
\]

by (12)

\[
= \langle A^n_i a \rangle ; \langle A^n_i b \rangle ; A_j a
\]

by (13)

\[
= \langle A^n_i b \rangle ; \langle A^n_i a \rangle ; A_j a
\]

by (14)

\[
= \langle A^n_i b \rangle ; A_j a
\]

by (12)

\[
= \langle A^n_i a \rangle ; A_j b
\]

by (13)

as claimed.

The equations of (15) now follow easily, for

\[
D_j (\langle A^n_i a \rangle ; A_j b)
\]

\[
= A_j A_j (\langle A^n_i a \rangle ; A_j b)
\]

by definition

\[
= A_j A_j (\langle A^n_i a \rangle ; A_j b)
\]

by definition

\[
= A_j A_j (\langle A^n_i a \rangle ; A_j b)
\]

by (18)

\[
= D_j (\langle A^n_i a \rangle ; A_j b)
\]

by definition

\[
= \langle A^n_i a \rangle ; A_j b
\]

by (19)

as required.

\[\square\]

We will refer to elements of the form \( A_i a \), for any \( a \), as \( A_i \)-elements. For each \( i \) define a product on \( A_i \)-elements by \( A_i a \bullet A_i b := \langle A^n_i a \rangle ; A_j b \). We will omit the subscript and write \( \bullet \) where possible. To prove these are well defined we need to show

\[
A_i a = A_i b \rightarrow A_j a = A_j b
\]

for every \( i, j \)
all hold. But by (15), with $a = 0$, we know that $D_j A_i c = A_j c$ is a consequence of our axioms for all $i$ and $j$. Then assuming $A_i a = A_i b$, we have $A_i a = D_j A_i a = D_j A_i b = A_j b$. Note also that, by (15), every product of $A_i$-elements is an $A_i$-element.

**Lemma 3.7.** It follows from (2)–(10) that the $A_i$-elements with the operation $\bullet$ form a semilattice.

**Proof.** Equations (12) and (13) state that $\bullet$ is idempotent and commutative respectively.

For associativity we have

\[
A_i a \bullet (A_i b \bullet A_i c) = \langle A_i^0 a \rangle : \langle (A_i^0 b) : A_i c \rangle \quad \text{by definition of } \bullet
\]

\[
= \langle (A_i^0 a) : A_i b, \ldots, (A_i^0 a) : A_i b \rangle : A_i c \quad \text{superassociativity}
\]

\[
= D_1(\langle A_i^0 a \rangle : A_i b), \ldots, D_n(\langle A_i^0 a \rangle : A_i b) \rangle : A_i c \quad \text{by (15)}
\]

\[
= \langle A_i^n A_i(\langle A_i^0 a \rangle : A_i b) \rangle : A_i c \quad \text{by (10)}
\]

\[
= A_i A_i(\langle A_i^0 a \rangle : A_i b) \bullet A_i c \quad \text{by definition of } \bullet
\]

\[
= D_i(\langle A_i^0 a \rangle : A_i b) \bullet A_i c \quad \text{definition of } D_i
\]

\[
= (\langle A_i^0 a \rangle : A_i b) \bullet A_i c \quad \text{by (15)}
\]

\[
= (A_i a \bullet A_i b) \bullet A_i c \quad \text{by definition of } \bullet
\]

as required.

**Lemma 3.8.** It follows from (2)–(10) that for every $i$, the $A_i$-elements, with product $\bullet$ and complement given by $A_i$, form a Boolean algebra with top element $\pi_i$ and bottom element $0$.

**Proof.** We already know, by Lemma 3.7, that the $A_i$-elements form a semilattice. Equation (3) says that $\pi_i$ is the top element of the semilattice. We want to show that $0$ is an $A_i$-element, then both (5) and (6) independently say that $0$ is the bottom element of the semilattice. This is easy: $A_i \pi_i = A_i \pi_i \bullet \pi_i = \langle A_i^0 \pi_i \rangle ; \pi_i = 0$.

To complete the proof that we have a Boolean algebra we use the dual of the axiomatisation of Boolean algebras given, for example, in [3, Definition 2.3]. Let $\alpha + \beta$ abbreviate $A_i(\langle A_i \alpha \bullet A_i \beta \rangle)$. We need complement axioms:

\[
A_i A_i \alpha = \alpha
\]

\[
A_i \alpha \bullet \alpha = 0
\]

\[
A_i 0 = \pi_i
\]

and distributivity:

\[
\alpha + \beta \bullet \gamma = (\alpha + \beta) \bullet (\alpha + \gamma)
\]

where Greek letters denote arbitrary $A_i$-elements.

The first complement axiom follows from (15), the second is (4) and the third is true by definition. The distributivity axiom expands to

\[
A_i(\langle A_i \alpha \bullet A_i (\beta \bullet \gamma) \rangle) = A_i(\langle A_i \alpha \bullet A_i \beta \rangle) \bullet A_i(\langle A_i \alpha \bullet A_i \gamma \rangle)
\]
and we prove this using (8). We have by applying the \( i \)th twisted law

\[
\langle D^n_i \alpha \rangle \cdot A_i(A_i \alpha \bullet A_i(\beta \bullet \gamma)) = \langle A^n_i(D_i \alpha \bullet A_i \alpha \bullet A_i(\beta \bullet \gamma)) \rangle \cdot D_i \alpha
\]

\[
= \langle A^n_i \alpha \rangle \cdot D_i \alpha
\]

\[
= D_i \alpha
\]

\[
= \alpha
\]

and again using the \( i \)th twisted law

\[
\langle D^n_i \alpha \rangle \cdot (A_i(A_i \alpha \bullet A_i \beta) \bullet A_i(A_i \alpha \bullet A_i \gamma)) = D_i \alpha \bullet A_i(A_i \alpha \bullet A_i \beta) \bullet A_i(A_i \alpha \bullet A_i \gamma)
\]

\[
= A_i 0 \bullet D_i \alpha \bullet A_i(A_i \alpha \bullet A_i \gamma)
\]

\[
= A_i 0 \bullet A_i 0 \bullet D_i \alpha
\]

\[
= \alpha
\]

and we have

\[
\langle A^n_i \alpha \rangle \cdot A_i(A_i \alpha \bullet A_i(\beta \bullet \gamma)) = A_i(A_i \alpha \bullet A_i(\beta \bullet \gamma)) \cdot A_i \alpha
\]

\[
= A_i \alpha \bullet A_i \alpha \bullet A_i(\beta \bullet \gamma)
\]

\[
= A_i \alpha \bullet D_i(\beta \bullet \gamma)
\]

\[
= A_i \alpha \bullet \beta \bullet \gamma
\]

and

\[
\langle A^n_i \alpha \rangle \cdot (A_i(A_i \alpha \bullet A_i \beta) \bullet A_i(A_i \alpha \bullet A_i \gamma)) = A_i(A_i \alpha \bullet A_i \beta) \bullet A_i(A_i \alpha \bullet A_i \gamma)
\]

\[
= A_i(A_i \alpha \bullet A_i \gamma) \bullet A_i(A_i \alpha \bullet A_i \beta)
\]

\[
= A_i \alpha \bullet A_i \alpha \bullet A_i \gamma \bullet A_i \alpha \gamma
\]

\[
= A_i \alpha \bullet D_i \gamma \bullet D_i \beta
\]

\[
= A_i \alpha \bullet \beta \bullet \gamma
\]

giving the result. \( \square \)

We know that the map \( \theta_{ji} : A_i a \mapsto A_j a \) is well defined for every \( i \) and \( j \). Hence it is a bijection from the \( A_i \)-elements to the \( A_j \)-elements. Then

\[
\theta_{ji}(A_i A_i a) = A_j A_i a
\]

by definition of \( \theta_{ji} \)

\[
= A_j A_j a
\]

by (10)

\[
= A_j \theta_{ji}(A_i a)
\]

by definition of \( \theta_{ji} \)
and
\[
\theta_{ji}(A_i a \cdot_i A_j b) = \theta_{ji}(D_i(A_i a \cdot_i A_j b)) \quad \text{by (15)}
\]
\[
= \theta_{ji}(A_i A_i(A_i a \cdot_i A_j b)) \quad \text{by definition of } D_i
\]
\[
= A_j(A_i(A_i a \cdot_i A_j b)) \quad \text{by definition of } \theta_{ji}
\]
\[
= D_j(A_i a \cdot_i A_j b) \quad \text{by (10)}
\]
\[
= A_j a \cdot_j A_j b \quad \text{by (15)}
\]
\[
= \theta_{ji}(A_i a) \cdot_j \theta_{ji}(A_j b) \quad \text{by definition of } \theta_{ji}
\]
and so \(\theta_{ji}\) is an isomorphism of the Boolean algebras.

Notice that the collection \((\theta_{ji})\) of Boolean algebra isomorphisms commute, that is, each \(\theta_i\) is the identity and \(\theta_{kj} \circ \theta_{ji} = \theta_{ki}\) for all \(i, j\) and \(k\). Hence we may fix a representative of the isomorphism class of these Boolean algebras and fix isomorphisms to the Boolean algebras that commute with the isomorphisms \(\theta_{ji}\). For definiteness we will use the \(A_1\)-elements as the representative Boolean algebra. Then for each \(i\) the isomorphism to the \(A_i\)-elements will be \(\theta_{i1}\).

We will refer to elements of the representative Boolean algebra as \(A\)-elements and use Greek letters to denote arbitrary \(A\)-elements. If \(\alpha\) is an \(A\)-element then \(A\alpha\) is the complement of \(\alpha\) within the Boolean algebra of \(A\)-elements, \(\alpha\) is shorthand for \(\langle \alpha_1, \ldots, \alpha_n \rangle\), consisting of the images of \(\alpha\) in the algebras of \(A_i\)-elements and \(\overline{\alpha}\) is shorthand for \(\langle A_1\alpha_1, \ldots, A_n\alpha_n \rangle\), consisting of the images of \(A\alpha\).

**Lemma 3.9.** The following quasiequations are consequences of axioms (2)–(10).

\[
\begin{align*}
(20) \quad & \langle D_i^n(a; b) : D_i(a; b) = D_i(a; b) \rangle ; a_i \quad \text{for every } i \\
(21) \quad & \langle a; D_i b = \langle D_i^n(a; b) \rangle ; a_i \quad \text{for every } i \\
\end{align*}
\]

(The twisted laws for domain)

\[
(22) \quad a; A_i b = 0 \rightarrow a; A_j b = 0 \quad \text{for every } i, j
\]
\[
(23) \quad \alpha; a = \alpha; b \land \beta; a = \beta; b \rightarrow (\alpha + \beta); a = (\alpha + \beta); b
\]
where \(+\) is the Boolean sum and we have extended notation componentwise to sequences.

**Proof.** Equation (20) is the statement that \(D(a; b) \leq Da_j\) within the Boolean algebra of \(A\)-elements. This is equivalent to \(A(a; b) \geq Aa_j\), that is \(\langle A^n a_j \rangle ; A_1(a; b) = A_1 a_j\). This is true, for

\[
\langle A_i^n a_j \rangle ; A_1(a; b)
\]
\[
\cong \langle A_i^n (\langle A^n a_j \rangle ; a_i ; b) ; A_1 a_j
\]
\[
\equiv \langle A_i^n (\langle A_i^n a_j \rangle ; a_1, \ldots, (A_i^n a_j) ; a_n) ; b) ; A_1 a_j
\]
\[
\equiv \langle A_i^n (\langle A_i^n a_j \rangle ; a_1, \ldots, (A_i^n a_j) ; a_j-1, 0, (A_i^n a_j) ; a_{j+1}, \ldots, (A_i^n a_j) ; a_n) ; b) ; A_1 a_j
\]
\[
\equiv \langle A_i^n 0 ; A_1 a_j
\]
\[
= \pi ; A_1 a_j
\]
\[
\equiv A_1 a_j
\]
and so (20) is valid.

In order to prove the twisted laws for domain we first prove
\[ (D_1^n) c ; d = \langle A_1^n ; (\langle A_1^n c ; d \rangle) ; d \rangle \]
and we do this by an application of (8). We have
\[ \langle D_1^n c ; (\langle D_1^n c ; d \rangle) ; d \rangle = \langle D_1^n c ; d \rangle \]
and
\[ \langle D_1^n c ; (\langle \langle A_1^n c ; d \rangle \rangle ; d) ; d \rangle \]
\[ \quad = \langle \langle A_1^n (\langle D_1^n c ; \langle A_1^n c ; d \rangle) ; d \rangle ; D_1 c, . . . \rangle ; d \rangle \]
\[ \quad = \langle \langle A_1^n ; D_1 c, . . . , A_1^n 0 ; D_n c \rangle ; d \rangle \]
\[ \quad = \langle \pi ; D_1 c, . . . , \pi ; D_n c \rangle ; d \]
\[ \quad = \langle D_1^n c ; d \rangle \]
and we also have
\[ \langle A_1^n c ; (\langle D_1^n c ; d \rangle) ; d \rangle = 0 \]
and
\[ \langle A_1^n c ; (\langle A_1^n c ; d \rangle) ; d \rangle \]
\[ \quad = \langle A_1^n (\langle A_1^n c ; d \rangle) ; d \rangle \]
\[ \quad = \langle A_1^n 0 ; D_1 c, . . . , A_1^n 0 ; D_n c \rangle \]
\[ \quad = \langle \pi ; D_1 c, . . . , \pi ; D_n c \rangle \]
\[ \quad = 0 \]
giving us what we require to deduce (24).

Now to deduce the \( i \)th twisted law for domain, firstly \( a : D_i b = a : A_i A_i b \) by the definition of \( D_i \). Applying the \( i \)th twisted law for antidomain to the right-hand side we get
\[ \langle A_1^n (a ; A_i) ; \langle D_1^n a_1 \rangle ; \ldots ; \langle D_1^n a_n \rangle \rangle ; a_i \]
then by applying the \( i \)th twisted law for antidomain again this equals
\[ \langle A_1^n (\langle \langle A_1^n (a ; A_i) ; \langle D_1^n a_1 \rangle ; \ldots ; \langle D_1^n a_n \rangle \rangle ; a_i \rangle) ; \langle D_1^n a_1 \rangle ; \ldots ; \langle D_1^n a_n \rangle \rangle ; a_i \]
and by setting \( c = a ; b \) and \( d = \langle D_1^n a_1 \rangle ; \ldots ; \langle D_1^n a_n \rangle ; a_i \) in (24), this is equal to
\[ \langle D_1^n (a ; b) ; \langle D_1^n a_1 \rangle ; \ldots ; \langle D_1^n a_n \rangle ; a_i \rangle \]
and this equals \( D_1^n (a ; b) ; a_i \) by repeated application of superassociativity and (20).

For (22), suppose \( a : A_i b = 0 \). Then
\[ \langle D_1^n (a ; A_j b) \rangle ; a_i = a ; D_i A_j b \]
by the \( i \)th twisted law for domain
\[ = a ; A_i A_i A_j b \]
by the definition of \( D_i \)
\[ = a ; A_i A_i A_j b \]
by (10)
\[ = a ; A_i b \]
as \( A_i \) is complement on the \( A_i \)-elements
\[ = 0 \]
by assumption
and so
\[ a : A_j b \overset{9}{=} \langle D_1^n (a : A_j b) \rangle ; a : A_j b \]
\[ \overset{8}{=} \langle \langle D_1^n (a : A_j b) \rangle ; a_1, \ldots, \langle D_1^n (a : A_j b) \rangle ; a_n \rangle ; A_j b \]
\[ = (\ldots, \langle D_1^n (a : A_j b) \rangle ; a_{i-1}, 0, \langle D_1^n (a : A_j b) \rangle ; a_{i+1}, \ldots) ; A_j b \]
\[ \overset{5}{=} 0 \]
hence (22) holds.

For (23), suppose \( \alpha ; a = \alpha ; b \) and \( \beta ; a = \beta ; b \). Then by Boolean reasoning and the assumptions
\[ \alpha ; (\alpha + \beta) ; a = \alpha ; a \]
\[ = \alpha ; b \]
\[ = \alpha ; (\alpha + \beta) ; b \]
and
\[ \overline{\alpha} ; (\alpha + \beta) ; a = \overline{\alpha} ; \beta ; a \]
\[ = \overline{\alpha} ; b \]
\[ = \overline{\alpha} ; (\alpha + \beta) ; b \]
so (23) follows, by (8).

Write \( a \leq b \) to mean \( \langle D_1^n (a) \rangle ; b = a \).

Lemma 3.10. It follows from (2)–(10) that the relation \( \leq \) is a partial order and with respect to this order \( \langle \rangle \); is order preserving in each of its arguments.

Proof. Reflexivity is just (9). For antisymmetry, suppose that \( \langle D_1^n a \rangle ; b = a \) and \( \langle D_1^n b \rangle ; a = b \). Then
\[ \langle D_1^n a \rangle ; a = a \]
by (9)
and
\[ \langle D_1^n a \rangle ; b = a \]
by assumption
and also
\[ \langle A_1^n a \rangle ; a = 0 \]
by (4)
and
\[ \langle A_1^n a \rangle ; b = \langle A_1^n a \rangle ; \langle D_1^n b \rangle ; a \]
by assumption
\[ = \langle D_1^n b \rangle ; \langle A_1^n a \rangle ; a \]
by (14)
\[ = \langle D_1^n b \rangle ; 0 \]
by (4)
\[ = 0 \]
by (6)
and so \( a = b \), by an application of (8).

To prove transitivity, suppose \( \langle D_1^n a \rangle ; b = a \) and \( \langle D_1^n b \rangle ; c = b \). We first claim that
\[ \langle A_1^n b \rangle ; D_i a = 0 \]
for every \( i \)
follows from these assumptions. It suffices to show $D_i a = \langle D^n_1 a \rangle ; D_i b$ for every $i$. Observe that
\[
\langle D^n_i a \rangle ; b = a \Rightarrow D_j (\langle D^n_i a \rangle ; b) = D_j a
\]
for every $j$.

but $\langle D^n_1 (\langle D^n_i a \rangle ; b) \rangle ; D_i a = \langle D^n_i a \rangle ; D_i b$ by the $i$th twisted law for domain, establishing that $D_i a = \langle D^n_1 a \rangle ; D_i b$.

To prove transitivity we now use (8) again. We have
\[
\langle D^n_1 b \rangle ; (\langle D^n_1 a \rangle ; c) = \langle D^n_1 a \rangle ; \langle D^n_1 b \rangle ; c
\]
by (14)
\[
= \langle D^n_1 a \rangle ; b
\]
by assumption
\[
= a
\]
by assumption

and
\[
\langle D^n_1 b \rangle ; a = \langle D^n_1 b \rangle ; \langle D^n_1 a \rangle ; b
\]
by assumption
\[
= \langle D^n_1 a \rangle ; \langle D^n_1 b \rangle ; b
\]
by (14)
\[
= \langle D^n_1 a \rangle ; b
\]
by (9)
\[
= a
\]
by assumption

and we have
\[
\langle A^n_1 b \rangle ; (\langle A^n_1 a \rangle ; c) = \langle\langle A^n_1 b ; D_1 a, \ldots, A^n_1 b \rangle ; D_n a \rangle ; c
\]
by superassociativity
\[
= 0 ; c
\]
by (25)
\[
= 0
\]
by (5)

and
\[
\langle A^n_1 b \rangle ; a = \langle A^n_1 b \rangle ; \langle D^n_1 a \rangle ; b
\]
by assumption
\[
= \langle D^n_1 a \rangle ; \langle A^n_1 b \rangle ; b
\]
by (14)
\[
= \langle D^n_1 a \rangle ; 0
\]
by (4)
\[
= 0
\]
by (6)

from which we may conclude $\langle D^n_1 a \rangle ; c = a$.

To see that $\langle \rangle ;$ is order preserving in its final argument, suppose that $c \leq d$, that is, $\langle D^n_1 c \rangle ; d = c$. Then for an arbitrary $a$ we have
\[
\langle D^n_1 (a ; c) \rangle ; (a ; d) \overset{S}{=} \langle\langle D^n_1 (a ; c) \rangle ; a_1, \ldots, D^n_1 (a ; c) ; a_n \rangle ; d
\]
\[
\overset{21}{=} \langle a ; D_1 c, \ldots, a ; D_n c \rangle ; d
\]
\[
\overset{S}{=} a ; \langle D^n_1 c \rangle ; d
\]
\[
= a ; c
\]
where the last equality holds by the assumption.
To see that \(\langle\;\rangle\) is order preserving in each of its first \(n\) arguments, suppose that \(a_i \leq b_i\) for every \(i\). That is, \((D^i_1 a_i) ; b_i = a_i\) for every \(i\). Then

\[
\langle D^i_1(a ; c) ; (b ; c) \rangle \overset{S}{=} \langle D^i_1(a ; c) ; b_1, \ldots , D^i_1(a ; c) ; b_n \rangle ; c \\
= \langle D^i_1(a ; c) ; D^i_1(a_1) ; b_1, \ldots , D^i_1(a ; c) ; D^i_1(a_n) ; b_n \rangle ; c \\
= \langle D^i_1(a ; c) ; a_1, \ldots , D^i_1(a ; c) ; a_n \rangle ; c \\
\overset{S}{=} \langle D^i_1(a ; c) ; a ; c \rangle \\
\overset{9}{=} a ; c
\]

utilising (20) for the second equality and the assumptions for the third. \(\Box\)

An easy application of laws we have so far shows that the partial order on the entire algebra agrees with the partial orders on each of the embedded Boolean algebras.

Note that

\[\langle A^n_i a \rangle ; b = 0 \rightarrow A_i a \leq A_i b \] for every \(i\)

all hold, for assuming \(\langle A^n_i a \rangle ; b = 0\) gives

\[
\langle A^n_i a \rangle ; A_i b = \langle A^n_i(\langle A^n_i a \rangle ; b) \rangle ; A_i a \\
= \langle A^n_i 0 \rangle ; A_i a \\
= \pi ; A_i a \\
= A_i a
\]

by the ith twisted law

by the assumption

by the definition of \(\pi\)

which says that \(A_i a \cdot A_i b = A_i a\).

4. THE REPRESENTATION

We are now finally ready to start describing our representation. In this section we prove the correctness of our representation for the signature \(\langle\;\rangle ; A_1, \ldots , A_n\), but the representation is the same one we will use for all the expanded signatures that follow.

**Definition 4.1.** Let \(\mathfrak{A}\) be an algebra of a signature containing composition. A right congruence is an equivalence relation \(\sim\) on \(\mathfrak{A}\) such that if \(a_i \sim b_i\) for every \(i\) then \(a ; c \sim b ; c\) for any \(c \in \mathfrak{A}\).

For the remainder of this section, let \(\mathfrak{A}\) be an algebra of the signature \(\langle\;\rangle ; A_1, \ldots , A_n\) validating (2)–(10). Hence all the consequences deduced in Section 3 are true of \(\mathfrak{A}\).

For a filter \(F\) of \(A\)-elements of \(\mathfrak{A}\), define the binary relation \(\sim_F\) on \(\mathfrak{A}\) by \(a \sim_F b\) if and only if there exists \(\alpha \in F\) such that \(\alpha ; a = \alpha ; b\).

**Lemma 4.2.** For any filter \(F\) of \(A\)-elements of \(\mathfrak{A}\), the binary relation \(\sim_F\) is a right congruence.

**Proof.** It is clear that \(\sim_F\) is reflexive and symmetric. To see that \(\sim_F\) is transitive, first note that for any \(A\)-elements \(\alpha\) and \(\beta\) and any \(c\) we have

\[\langle \alpha \bullet \beta \rangle ; c = \langle \alpha_1 ; \beta_1, \ldots , \alpha_n \bullet \beta_n \rangle ; c = \langle \alpha ; \beta_1, \ldots , \alpha ; \beta_n \rangle ; c = \alpha ; (\beta ; c)\]

Now suppose that \(a \sim_F b\) and \(b \sim_F c\) and let \(\alpha \in F\) be such that \(\alpha ; a = \alpha ; b\) and \(\beta \in F\) be such that \(\beta ; b = \beta ; c\). Then \(\alpha \bullet \beta \in F\) since \(F\) is a filter and (27)
and commutativity of the Boolean product operations is precisely what is needed to give \((\alpha \bullet \beta) : a = (\alpha \bullet \beta) : c\). So \(\sim_F\) is transitive.

Suppose now that \(a_i \sim_F b_i\) for every \(i\) and let \(c\) be an arbitrary element of \(\mathfrak{A}\). By hypothesis, for each \(i\) we can find \(\alpha^i \in F\) such that \(\alpha^i : a_i = \alpha^i : b_i\). Then \(\prod_i \alpha_i \in F\) and \((\prod_i \alpha_i) : (a : c) = (\prod_i \alpha_i) : a_1, \ldots, (\prod_i \alpha_i) : a_n) : c = (\prod_i \alpha_i) : (b : c)\). So \(a : c \sim_F b : c\). 

\[\Box\]

The next lemma describes a family of homomorphisms from which we will build a faithful representation.

**Lemma 4.3.** Let \(U\) be an ultrafilter of elements of \(\mathfrak{A}\). Write \([a]\) for the \(\sim_U\)-equivalence class of an element \(a \in \mathfrak{A}\). Let \(X := \{[a] \mid a \in \mathfrak{A}\} \setminus \{[0]\}\) and for each \(b \in \mathfrak{A}\) let \(\theta_U(b)\) be the partial function from \(X^n\) to \(X\) given by

\[
\theta_U(b) : ([a_1], \ldots, [a_n]) \mapsto \begin{cases} 
([a_1], \ldots, [a_n]) : b & \text{if this is not equal to [0]} \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

Then the set \(\{\theta_U(b) \mid b \in \mathfrak{A}\}\) forms a square algebra of \(n\)-ary partial functions, which we will call \(\mathfrak{F}\) and \(\theta_U : \mathfrak{A} \to \mathfrak{F}\) is a (surjective) homomorphism of \((\{\cdot\}, A_1, \ldots, A_n)\)-algebras. Further, if \(a\) is inequivalent to both 0 and \(b\) then \(\theta_U\) separates \(a\) from \(b\).

**Proof.** That \(\sim_U\) is a right congruence says that \(\theta_U(b)\) is well defined for every \(b \in \mathfrak{A}\). If we show that \(\theta_U\) satisfies the conditions for being a homomorphism, then it automatically follows that the domain of \(\mathfrak{F}\) is closed under the operations and so really is an algebra of \(n\)-ary partial functions.

We write \([a]\) for \(([a_1], \ldots, [a_n])\). To see that composition is represented correctly we first argue that \(\theta_U(b ; c)\) is defined if and only if \(\theta_U(b_1), \ldots, \theta_U(b_n)\); \(\theta_U(c)\) is defined. If \(\theta_U(b_1), \ldots, \theta_U(b_n); \theta_U(c)\) is defined at \([a]\) then in particular \([a ; b_1, \ldots, a ; b_n] = [c]\) must be inequivalent to 0. By superassociativity, this equals \([a ; (b ; c)]\) and hence \(\theta_U(b ; c)\) is defined at \([a]\).

If \(\theta_U(b_1), \ldots, \theta_U(b_n); \theta_U(c)\) is undefined at \([a]\) then this is either because \(a ; (b ; c)\) is undefined at \([a]\), or because there is an \(\alpha \in U\) such that \(\alpha ; (a ; b_i) = 0\) for some \(i\). In the second case

\[
\alpha : (a ; (b ; c)) \equiv (\alpha ; a ; b_1, \ldots, a ; b_n) ; c \\
\equiv (\alpha ; a ; b_1, \ldots, \alpha ; a ; b_n) ; c \\
= (\alpha ; a ; b_1, \ldots, \alpha ; a ; b_1, \ldots, 0, \alpha ; a ; b_1, \ldots, \alpha ; a ; b_n) ; c \\
\equiv 0
\]

and so \(\theta_U(b ; c)\) is again undefined at \([a]\).

If \(\theta_U(b ; c)\) and \(\theta_U(b_1), \ldots, \theta_U(b_n); \theta_U(c)\) are both defined at \([a]\) then they both equal \([a ; b ; c]\). We conclude that composition is represented correctly by \(\theta_U\).

We now show that each \(A_i\) is represented correctly by \(\theta_U\). It is helpful to first note that \(\theta_U\) represents 0 correctly, as \(a : 0 = 0\) for any \(a\) and so \(\theta_U(0)\) is undefined everywhere.

Next we will show that \(\theta_U(A_i b)\) is a restriction of the \(i\)th projection, for any \(b \in \mathfrak{A}\) and for any \(i\). Suppose that \(\theta_U(A_i b)\) is defined on \([a]\), so that \(a_1, \ldots, a_n\) and \(a : A_i b\) are all inequivalent to 0. We wish to show that \([a ; A_i b] = [a]\). As \(\langle A_i^n(a ; A_i b)\rangle ; (a ; A_i b) = 0 = \langle A_i^n(a ; A_i b)\rangle ; 0\), we know that \(A_i(a ; A_i b) \notin U\) and
so \( D(a ; A_i b) \in U \) since \( U \) is an ultrafilter. Then
\[
\langle D_i^n(a ; A_i b) \rangle ; (a ; A_i b) = \langle D_i^n(a ; A_i b) \rangle ; (a ; D_i(A_i b)) \\
= \langle D_i^n(a ; A_i b) \rangle ; ((D_i^n(a ; A_i b)) ; a_i) \\
= \langle D_i^n(a ; A_i b) \rangle ; a_i
\]
where the second equality follows by the \( i \)th twisted law for domain. We conclude that \([ a ; A_i b ] = [ a_i ]\), as desired.

Next we will show that where \( \theta_U(b) \) is not defined, \( \theta_U(A_i b) \) is defined. Suppose that \( a_1, \ldots, a_n \) are all inequivalent to 0, so \( D(a_1, \ldots, D(a_n) \in U \), but that \( \theta_U(b) \) is undefined at \([ a ]\), meaning \([ a ; b ] = [0] \). So there is an \( \alpha \in U \) with \( \alpha(a ; b) = \alpha ; 0 = 0 \). Then (26) tells us that \( \alpha \leq A(a ; b) \) and so \( U \), being an ultrafilter, contains \( A(a ; b) \).

Then
\[
\langle A_i^n(a ; b) \rangle ; \langle D_i^n a_1 \rangle ; \ldots ; \langle D_i^n a_n \rangle ; (a ; A_i b) \\
= \langle \langle A_i^n(a ; b) \rangle ; \langle D_i^n a_1 \rangle ; \ldots ; \langle D_i^n a_n \rangle ; a_1, \ldots, \langle A_i^n(a ; b) \rangle ; \langle D_i^n a_1 \rangle ; \ldots ; \langle D_i^n a_n \rangle ; a_n \rangle \\
\langle A_i b \rangle \text{ by superassociativity} \\
= \langle a ; A_1 b, \ldots, a ; A_n b \rangle ; A_i b \text{ by the twisted laws for antidomain} \\
= a ; \langle A_i^n b \rangle ; A_i b \text{ by superassociativity} \\
= a ; A_i b \text{ by (12)} \\
= \langle A_i^n(a ; b) \rangle ; \langle D_i^n a_1 \rangle ; \ldots ; \langle D_i^n a_n \rangle ; a_i \text{ by the } i \text{th twisted law for antidomain}
\]
and hence \([ a ; A_i b ] = [ a_i ] \neq [0] \) and so \( \theta_U(A_i b) \) is defined at \([ a ]\).

It remains to show that \( \theta_U(A_i b) \) cannot be defined when \( \theta_U(b) \) is defined. Suppose for a contradiction that both \( \theta_U(b) \) and \( \theta_U(A_i b) \) are defined on an \( n \)-tuple \([ a ]\). Now (22) tells us that \( a ; A_1 b, \ldots, a ; A_n b \) must be simultaneously equivalent or inequivalent to 0, for if there is an \( \alpha \in U \) with \( \alpha(a ; A_i b) = 0 \) then by superassociativity \( \langle \alpha ; a_1, \ldots, \alpha ; a_n \rangle ; A_i b = 0 \) and so \( \langle \alpha ; A_k b \rangle = \langle \alpha ; a_1, \ldots, \alpha ; a_i \rangle ; A_i b = 0 \). Hence \( \theta_U(A_k b) \), \( \ldots \), \( \theta_U(A_i b) \) are all defined on \([ a ]\), with each \( \theta_U(A_i b) \), being a restriction of the \( i \)th projection, having value \([ a_i ]\). But then
\[
\theta_U(b)([a]) = \theta_U(b)(\theta_U(A_i b)([a]), \ldots, \theta_U(A_n b)([a])) \\
= (\theta_U(A_1 b), \ldots, \theta_U(A_n b)) ; \theta_U(b)([a]) \text{ by the definition of } ( \cdot ); \\
= \theta_U(A_i b, \ldots, A_n b ; b)([a]) \text{ as } ( \cdot ); \text{ represented correctly} \\
= \theta_U(0)([a]) \text{ by (4)}
\]
contradicting our observation that 0 is represented by \( \theta_U \) as the empty function. This completes the proof that the antidomain operations are represented correctly by \( \theta_U \).

For the last part, if \( a \) is inequivalent to both 0 and \( b \), then we know that \( \pi_i \) is inequivalent to 0, for each \( i \), otherwise \( a = \pi ; a \sim_U \pi_1, \ldots, \pi_{i-1}, 0, \pi_{i+1}, \ldots, \pi_n ; a = 0 \). So \( \theta_U(a)([\pi]) = [\pi ; a] = [a] \) and if \( \theta_U(b)([\pi]) \) is defined then it equals \([b]\), which is distinct from \([a]\). \( \square \)

The next lemma shows that there are enough ultrafilters to form a faithful representation.

**Lemma 4.4.** Let \( a, b \in A \) and suppose that \( a \not\sim b \). Then there is an ultrafilter \( U \) of \( A \)-elements for which \( a \sim_U 0 \) and \( a \sim_U b \).
functions within algebras of

The class of \( \equiv \) is a proper quasivariety, finitely axiomatised by (quasi)equations (2)–(10). For each \( \alpha, b \)

Theorem 4.5. The class of \( \langle \rangle; A_1,\ldots, A_n \)-algebras that are representable by \( n \)-ary partial functions is a proper quasivariety, finitely axiomatised by (quasi)equations (2)–(10).

Proof. We continue to let \( \mathfrak{A} \) be an arbitrary \( \langle \rangle; A_1,\ldots, A_n \)-algebra validating (2)–(10). For each \( a, b \in \mathfrak{A} \) with \( a \not\equiv b \), let \( U_{ab} \) be a choice of an ultrafilter of \( n \)-ary partial functions for which \( a \equiv_U 0 \) and \( a \equiv_U b \). Let \( \theta_{ab} \) be the corresponding homomorphism as described in Lemma 4.3, which is guaranteed to separate \( a \) from \( b \). Take a disjoint union, in the sense of Definition 2.5, of the family \( \{\theta_{ab}\}_{a,b \in \mathfrak{A}} \) of homomorphisms and call this \( \varphi \). So \( \varphi \) is a homomorphism from some power \( \mathfrak{A}^S \) of \( \mathfrak{A} \) to an algebra of \( n \)-ary partial functions. Let \( \Delta \) be the diagonal embedding of \( \mathfrak{A} \) into \( \mathfrak{A}^S \). Then the map \( \theta : \mathfrak{A} \rightarrow \text{Im}(\varphi \circ \Delta) \) defined by \( \theta(a) = (\varphi \circ \Delta)(a) \) is a surjective homomorphism from \( \mathfrak{A} \) to an algebra of \( n \)-ary partial functions.

For distinct \( a, b \in \mathfrak{A} \), either \( a \not\equiv b \) or \( b \not\equiv a \) and so \( \theta_{ab} \), and therefore \( \theta \), separates \( a \) and \( b \). Hence \( \theta \) is an isomorphism, so a representation of \( \mathfrak{A} \) by \( n \)-ary partial functions.

Note that whilst Lemma 4.3 only uses square algebras of functions, in Theorem 4.5, by taking a disjoint union of homomorphisms, we require non-square algebras of functions for our representation.\(^2\)

It is clear that if \( \mathfrak{A} \) is finite then the representation described in Theorem 4.5 has a finite base. More specifically the size of the base is no greater than the cube of the size of the algebra.

Corollary 4.6. The finite representation property holds for the signature \( \langle \rangle; A_1,\ldots, A_n \) for representation by \( n \)-ary partial functions.

5. Injective Partial Functions

In this section we present an algebraic characterisation of the injective partial functions within algebras of \( n \)-ary partial functions. This allows us to extend the

\(^2\)It is linguistically convenient to treat the \( \theta \) of Theorem 4.5 as uniquely specified and then refer to ‘our representation’ or ‘the representation’ in defiance of the fact that there is some nonconstructive choice involved in selecting which ultrafilters to use.
axiomatisation of Section 3 to an axiomatisation of the class of \(\langle\cdot; A_1, \ldots, A_n\rangle\)-
algebras representable as injective \(n\)-ary partial functions.

The following definition applies to any algebra with composition in the sign ature
and with the domain operations either in the signature or definable via antidomain
operations.

**Definition 5.1.** We will call an element \(a\) **injective** if it satisfies the indexed
quasiequations

\[
(28) \quad b : a = c : a \rightarrow b : D_i a = c : D_i a \quad \text{for every } i
\]

**Proposition 5.2.** The representation described in Theorem 4.5 represents as
injective functions precisely the injective elements of the algebra.

**Proof.** We first argue that in algebras of \(n\)-ary partial functions injective functions
are injective elements; then if an element of a representable algebra is represented
as an injective function it must be an injective element. To this end, suppose \(a\)
is an injective \(n\)-ary partial function and that \(b : a = c : a\). Suppose further that\((x, z) \in b : D_i a\). Then \(b_1, \ldots, b_n\) are all defined on \(x\), the function \(a\) is defined on\(\langle b_1(x), \ldots, b_n(x) \rangle\) and \(z = b_i(x)\). The first two of these facts tell us that \(b : a\) is
defined on \(x\), with value \(w\) say. Then by assumption, \(c : a\) is defined on \(x\), also
with value \(w\). So \(c_1, \ldots, c_n\) are all defined on \(x\) and \(a(b_1(x), \ldots, b_n(x)) = w = a(c_1(x), \ldots, c_n(x))\). By injectivity of \(a\), we get \(b_j(x) = c_j(x)\), for every \(j\). As
\(c : a\) is defined on \(x\), so is \(c : D_i a\) and it takes value \(c_i(x) = b_i(x) = z\). That is,\((x, z) \in c : D_i a\). We conclude that \(b : D_i a \subseteq c : D_i a\). By symmetry, the reverse
inclusion also holds. Hence \(a\) satisfies (28).

We now prove the converse: that every injective element is represented by our
representation as an injective function. We will argue that, for any ultrafilter \(U\)
of \(A\)-elements, the map \(\theta_U\) described in Lemma 4.3 maps injective elements to
injective functions. Since a disjoint union of injective functions is injective, the
result follows.

Suppose that \(a\) is an injective element and that \(\theta_U(a)(\langle b \rangle) = \theta_U(a)(\langle c \rangle)\). That
is, there is an \(\alpha \in U\) such that \(\alpha : (b : a) = \alpha : (c : a)\) (and neither \(b : a\) nor \(c : a\) is
equivalent to 0). Then
\[
\langle \alpha ; b_1, \ldots, \alpha ; b_n \rangle : a = \langle \alpha ; c_1, \ldots, \alpha ; c_n \rangle : a \quad \text{by superassociativity}
\]
so
\[
\langle \alpha ; b_1, \ldots, \alpha ; b_n \rangle : D_i a = \langle \alpha ; c_1, \ldots, \alpha ; c_n \rangle : D_i a \quad \text{for every } i, \text{ by (28)}
\]
so
\[
\alpha : b : D_i a = \alpha : c : D_i a \quad \text{by superassociativity}
\]
so
\[
\alpha : \langle D_i^n (b : a) \rangle : b_i = \alpha : \langle D_i^n (c : a) \rangle : c_i \quad \text{by twisted laws for domain}
\]
from which we can derive
\[
\alpha : \langle D_i^n (b : a) \rangle : \langle D_i^n (c : a) \rangle : b_i = \alpha : \langle D_i^n (b : a) \rangle : \langle D_i^n (c : a) \rangle : c_i
\]
using superassociativity and the commutativity and idempotency of the \(\bullet_i\) oper-
tions.
Corollary 5.4. As a corollary that the finite representation property holds.

The proof of Proposition 5.2 showed that if an element is represented as an injective function by any representation (not just the one described in Theorem 4.5), then the element is an injective element. Hence the indexed quasiequations of (28) are valid for algebras of injective \(n\)-ary partial functions. So Proposition 5.2 yields the following corollary.

Corollary 5.3. Adding (28) to (2)–(10) gives a finite quasiequational axiomatisation of the class of \(\langle \cdot \rangle; A_1, \ldots, A_n\)-algebras that are representable by injective \(n\)-ary partial functions.

Since Corollary 5.3 uses the same representation as Theorem 4.5, it again follows as a corollary that the finite representation property holds.

Corollary 5.4. The finite representation property holds for the signature \(\langle \cdot \rangle; A_1, \ldots, A_n\) for representation by injective \(n\)-ary partial functions.

6. INTERSECTION

In this section we consider the signature \(\langle \cdot \rangle; A_1, \ldots, A_n, \cdot\). We could search for extensions to the quasiequational axiomatisations of the previous sections. However the presence of intersection in the signature allows us to give equational axiomatisations, deducing the quasiequations that we need.

We first present some valid equations involving intersection.

Proposition 6.1. The following equations are valid for the class of \(\langle \cdot \rangle; A_1, \ldots, A_n, \cdot\)-algebras representable by \(n\)-ary partial functions.

\[
\begin{align*}
(29) & \quad a \cdot a = a \\
(30) & \quad a \cdot b = b \cdot a \\
(31) & \quad a \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c) \\
(32) & \quad \langle D_i^n(a \cdot b) \rangle ; a = a \cdot b
\end{align*}
\]

Proof. Equations (29) and (30) are both well-known properties of intersection. The validity of (31) and the validity of (32) are both easy to see and are noted in [2], where they appear as Equation (29) and Equation (28) respectively. \(\square\)

We will include all the equational axioms of Section 3 in our axiomatisation, that is (2)–(7), (9) and (10), as well as including (29)–(32). All the consequences of Section 3 will follow from our axiomatisation if only we can deduce (8). Next we give three more valid equations whose inclusion enables us to do just that. Notice that (12) was deduced without (8), so is available to us.

We make use of the tie operations. Define \(a \bowtie_i b := D_i(a \cdot b) +_i (A_i^n a); A_i b\), where \(\alpha +_i \beta := A_i((A_i^n \alpha); A_i \beta)\).

Proposition 6.2. The following equations are valid for the class of \(\langle \cdot \rangle; A_1, \ldots, A_n, \cdot\)-algebras representable by \(n\)-ary partial functions.

\[
\begin{align*}
(33) & \quad \langle a \bowtie_i^n b \rangle ; a = \langle a \bowtie_i^n b \rangle ; b \\
(34) & \quad a \cdot D_i(\alpha ; (a \cdot b)) +_i (\alpha ; (A_i^n (\alpha ; a))) ; A_i (\alpha ; b) = \alpha ; (a \bowtie_i b) \quad \text{for every } i \\
(35) & \quad \langle D_i^n a \rangle ; (b \bowtie_i c) +_i (A_i^n a) ; (b \bowtie_i c) = b \bowtie_i c \quad \text{for every } i
\end{align*}
\]
Proof. We first need to convince ourselves that in an algebra of \(n\)-ary functions, \(\bowtie\), as we have defined it, really does give the \(i\)th tie operation on its two arguments. And before we do that we need to see that if \(\alpha\) and \(\beta\) are restrictions of the \(i\)th projection, then \(\alpha +_i \beta\) is the \(i\)th projection on the union of the domains of \(\alpha\) and \(\beta\). It suffices to prove these for the square algebras of \(n\)-ary functions.

In a square algebra of \(n\)-ary partial functions, with base \(X\), the function \(\alpha +_i \beta\) is by definition the \(i\)th projection restricted to where \(\langle A^n_1 \alpha \rangle ; A_i \beta\) is not defined. Now \(\langle A^n_1 \alpha \rangle ; A_i \beta\) is defined precisely where \(A_1 \alpha\) (or indeed any \(A_j \alpha\)) and \(A_i \beta\) are both defined, which is those \(n\)-tuples in the domains of neither \(\alpha\) nor \(\beta\). By De Morgan, \(\alpha +_i \beta\) is as claimed.

Examining the definition of \(a \bowtie b\), we note that \(D_i(a \cdot b)\) is the \(i\)th projection restricted to where \(a\) and \(b\) are both defined and are equal, and \(\langle A^n_1 \alpha \rangle ; A_i b\) is the \(i\)th projection on those \(n\)-tuples where neither \(a\) nor \(b\) are defined. Hence \(a \bowtie b\), being defined as the result of applying the \(+_i\) operation to these projections, is exactly the \(i\)th tie of \(a\) and \(b\).

Now for (33). Suppose that \(\langle a \bowtie_1 b \rangle ; a\) is defined at an \(n\)-tuple \(x\), with value \(z\). This means \(a \bowtie_1 b, \ldots , a \bowtie_n b\) are all defined at \(x\) and \(a\) is defined at \((a \bowtie_1 b)(x), \ldots , (a \bowtie_n b)(x)\) = \(x\), with value \(z\). Then as \(a\) and \(a \bowtie_1 b\) are both defined at \(x\), it must be that \(b\) is also defined at \(x\) with the same value as \(a\), namely \(z\). It follows that \(\langle a \bowtie_1 b \rangle ; b\) is defined at \(x\), with value \(z\). We conclude that \(\langle a \bowtie_1 b \rangle ; a \subseteq \langle a \bowtie_1 b \rangle ; b\). Similarly (utilising the symmetry of the tie operations on \(n\)-ary partial functions) \(\langle a \bowtie_1 b \rangle ; a \supseteq \langle a \bowtie_1 b \rangle ; b\) and so (33) is valid.

We know that in an algebra of \(n\)-ary partial functions the \(A_i\)-elements, with \(-_i\) as in Section 3 as product and \(A_i\) as complement, form a Boolean algebra. We also saw, in the proof of Proposition 6.1, that \(+_i\) acts as the Boolean sum on the \(A_i\)-elements. Then (34) is the statement that

\[
(36) \quad \alpha_i \bullet_i D_i(\alpha ; (a \cdot b)) +_i \alpha_i \bullet_i A_i(\alpha ; a) \bullet_i A_i(\alpha ; b) = \alpha_i \bullet_i (D_i(a \cdot b) +_i A_i a) \bullet_i A_i b
\]

holds for every \(i\). Now \(D_i(\alpha ; (a \cdot b))\) is easily seen to be equal to \(\alpha ; D_i(a \cdot b)\), which is the definition of \(\alpha_i \bullet_i D_i(a \cdot b)\). It is similarly easy to see that \(A_i(\alpha ; a) = A_i \alpha +_i A_i a\) and \(A_i(\alpha ; b) = A_i \alpha +_i A_i b\). After making these substitutions, (36) follows by Boolean reasoning.

Equation (35) is the statement that

\[
A_i A_i a \bullet_i (b \bowtie_i c) +_i A_i a \bullet_i (b \bowtie_i c) = b \bowtie_i c
\]

holds for every \(i\). This follows directly by Boolean reasoning. \(\square\)

The equations (2)–(7), (9), (10) and (29)–(35) will form our axiomatisation. Equation (33) says that the \(A\)-element \(a \bowtie b\) is an ‘equaliser’ of \(a\) and \(b\). In order to deduce (8), we start by showing that \(a \bowtie b\) is the greatest such equaliser.\(^3\)

Lemma 6.3. The following indexed quasiequations are consequences of (2)–(7), (9), (10) and (29)–(35).

\[
(37) \quad \alpha ; a = \alpha ; b \rightarrow \alpha ; (a \bowtie_i b) = \alpha_i \quad \text{for every } i
\]

\(^3\)Note though that we have not yet deduced that the sets of \(A_i\)-elements, for each \(i\), form isomorphic Boolean algebras nor even that they are partially ordered by the \(\bullet_i\) operations of Section 3.
Lemma 6.4. Equation (8) is a consequence of (2)–(7), (9), (10) and (29)–(35).

Proof. Suppose that \( \langle D_1^n a \rangle : b = \langle D_1^n a \rangle : c \) and \( \langle A_1^n a \rangle : b = \langle A_1^n a \rangle : c \). Then we have

\[
\alpha : (a \triangleright_i b)
\]
\[
= \alpha : D_i (\alpha : (a \cdot b)) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : b)
\]
\[
= \alpha : D_i (\alpha : (a \cdot b)) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : a)
\]
\[
= \alpha : D_i ((\alpha : a) \cdot (\alpha : b)) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : a)
\]
\[
= \alpha : D_i (\alpha : a) \cdot (\alpha : a) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : a)
\]
\[
= \alpha : D_i (\alpha : (a \cdot a)) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : a)
\]
\[
= \alpha : (a \triangleright_i a)
\]
\[
= \alpha : (D_i (a \cdot a) +_i \langle A_1^n a \rangle : A_i a)
\]
\[
= \alpha : (D_i (a \cdot a) +_i A_i a)
\]
\[
= \alpha : A_i ((A_1^n D_i a) : A_i a)
\]
\[
= \alpha : A_i ((A_1^n a) : D_i a)
\]
\[
= \alpha : A_i (a)
\]
\[
= \langle A_1^n (\alpha : 0) \rangle : \alpha_i
\]
\[
= \langle A_1^n (0) \rangle : \alpha_i
\]
\[
= \pi ; \alpha_i
\]
\[
= \alpha_i
\]

which is the required conclusion. \(\square\)

Now it is straightforward to deduce (8).

Proof. Assume \( \alpha : a = \alpha : b \). Then we have

\[
\alpha : (a \triangleright_i b)
\]
\[
= \alpha : D_i (\alpha : (a \cdot b)) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : b)
\]
\[
= \alpha : D_i (\alpha : (a \cdot b)) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : a)
\]
\[
= \alpha : D_i ((\alpha : a) \cdot (\alpha : b)) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : a)
\]
\[
= \alpha : D_i ((\alpha : a) \cdot (\alpha : a)) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : a)
\]
\[
= \alpha : D_i (\alpha : (a \cdot a)) +_i \alpha : \langle A_1^n (\alpha : a) \rangle : A_i (\alpha : a)
\]
\[
= \alpha : (a \triangleright_i a)
\]
\[
= \alpha : (D_i (a \cdot a) +_i A_i a)
\]
\[
= \alpha : A_i ((A_1^n D_i a) : A_i a)
\]
\[
= \alpha : A_i ((A_1^n a) : D_i a)
\]
\[
= \alpha : A_i (a)
\]
\[
= \langle A_1^n (\alpha : 0) \rangle : \alpha_i
\]
\[
= \langle A_1^n (0) \rangle : \alpha_i
\]
\[
= \pi ; \alpha_i
\]
\[
= \alpha_i
\]

because

\[
b \triangleright_i c = \pi_i
\]

for every \( i \)

because

\[
b \triangleright_i c = \langle D_1^n a \rangle : (b \triangleright_i c) +_i \langle A_1^n a \rangle : (b \triangleright_i c)
\]
\[
= \langle D_1^n a \rangle : (b \triangleright_i c) +_i A_i a
\]
\[
= D_i a +_i A_i a
\]
\[
= A_i ((A_1^n D_i a) : A_i a)
\]
\[
= A_i ((A_1^n a) : D_i a)
\]
\[
= A_i (a)
\]
\[
= \pi_i
\]

and so

\[
b = \pi ; b
\]
\[
= \langle b \triangleright^n c \rangle : b
\]
\[
= \langle b \triangleright^n c \rangle : c
\]
\[
= \pi ; c
\]
\[
= c
\]
and hence (8) holds.

We are now in a position to state and prove our representation theorem.

**Theorem 6.5.** The class of \( (\simeq); A_1, \ldots, A_n, \cdot \)-algebras that are representable by \( n \)-ary partial functions is a variety, finitely axiomatised by equations (2)–(7), (9) and (10) together with (29)–(35).

**Proof.** Let \( \mathcal{A} \) be an algebra of the signature \( (\simeq); A_1, \ldots, A_n, \cdot \) validating the specified equations. We will show that, for any ultrafilter \( U \) of \( \mathcal{A} \)-elements, the map \( \theta_U \) described in Lemma 4.3 represents intersection correctly. The result follows.

We first show that \( \theta_U(a) \cap \theta_U(b) \subseteq \theta_U(a \cdot b) \), for all \( a, b \in \mathcal{A} \). Suppose that \((c, d) \in \theta_U(a) \cap \theta_U(b)\). Then there is an \( \alpha \in U \) with \( \alpha \cdot (c; a) = \alpha \cdot d \) and a \( \beta \in U \) with \( \beta \cdot (c; b) = \beta \cdot d \). As \( U \) is an ultrafilter we may assume \( \alpha = \beta \). Then

\[
\alpha \cdot (c; (a \cdot b)) = \alpha \cdot ((c \cdot a) \cdot (c \cdot b)) \quad \text{by distributivity of } \cdot; \quad \text{over } \cdot
\]

\[
= (\alpha \cdot (c \cdot a)) \cdot (\alpha \cdot (c \cdot b)) \quad \text{by distributivity of } \cdot; \quad \text{over } \cdot
\]

\[
= (\alpha \cdot d) \cdot (\alpha \cdot d) \quad \text{by equality of the factors}
\]

\[
= (\alpha \cdot d) \cdot (\alpha \cdot d) \quad \text{by idempotency of } \cdot
\]

and hence \([c; (a \cdot b)] = [d]\). This says that \((c, d) \in \theta_U(a \cdot b)\), since we know that \([d] \neq [0]\). We conclude that \( \theta_U(a) \cap \theta_U(b) \subseteq \theta_U(a \cdot b) \).

We now show that the reverse inclusion, \( \theta_U(a \cdot b) \subseteq \theta_U(a) \cap \theta_U(b) \), holds. Suppose that \((c, d) \in \theta_U(a \cdot b)\). This means that \([c; (a \cdot b)] \neq [0]\), equivalently \( \mathcal{D}(c; (a \cdot b)) \in U \), and that \([d] = [c; (a \cdot b)]\). Then

\[
\langle \mathcal{D}_0^\iota(c; (a \cdot b)) \rangle \cdot (c; a) = \langle \mathcal{D}_0^\iota((c \cdot a) \cdot (c \cdot b)) \rangle \cdot (c; a) \quad \text{by (31)}
\]

\[
= (c \cdot a) \cdot (c \cdot b) \quad \text{by (32)}
\]

\[
= c \cdot (a \cdot b) \quad \text{by (31)}
\]

\[
= \langle \mathcal{D}_0^\iota(c; (a \cdot b)) \rangle \cdot (c; (a \cdot b)) \quad \text{by (9)}
\]

and so \([c; a] = [c; (a \cdot b)] = [d] \neq [0]\), which tells us \((c, d) \in \theta_U(a)\). Similarly and using commutativity of \cdot we get \((c, d) \in \theta_U(b)\) and so \((c, d) \in \theta_U(a) \cap \theta_U(b)\). We conclude that \( \theta_U(a \cdot b) \subseteq \theta_U(a) \cap \theta_U(b) \), completing the proof. \(\square\)

With the aid of intersection, we can also replace the indexed quasiequations of (28) to give an equational axiomatisation for the case of injective \( n \)-ary partial functions.

**Proposition 6.6.** The representation used in the proof of Theorem 6.5 represents an element \( a \) as an injective function if and only if it satisfies the following indexed equations.

\[
(39) \quad \langle \mathcal{D}_0^\iota((b; a) \cdot (c \cdot a)) \rangle \cdot A_i(b_i \bowtie_i c_i) = 0 \quad \text{for all } i
\]

**Proof.** We first argue that any injective function \( a \) satisfies (39). Then if an element \( a \) is represented as an injective function it must satisfy (39). To this end, suppose \( a \) is an injective \( n \)-ary partial function and that \( \langle \mathcal{D}_0^\iota((b; a) \cdot (c \cdot a)) \rangle \cdot A_i(b_i \bowtie_i c_i) \) is defined on the \( n \)-tuple \( x \). Then both \( b; a \) and \( c; a \) should be defined on \( x \) and take the same value. This means that \( b_i(x), \ldots, b_n(x) \) and \( c_1(x), \ldots, c_n(x) \) are both defined and \( a(b_i(x), \ldots, b_n(x)) = a(c_1(x), \ldots, c_n(x)) \). By injectivity of \( a \), we get \( b_j(x) = c_j(x) \) for every \( j \). In particular \( b_i(x) = c_i(x) \) and so \( b_i \bowtie_i c_i \) is defined on \( x \).
Hence $A_i(b_i \circlearrowright c_i)$ is not defined on $x$. This contradicts $(D^n_a((b:a) \cdot (c:a))):A_i(b_i \circlearrowright c_i)$ being defined on $x$ and so $(D^n_a((b:a) \cdot (c:a))):A_i(b_i \circlearrowright c_i)$ must be the empty function.

We now prove the converse: that every $a$ satisfying (39) is represented by our representation as an injective function. We will argue that, for any ultrafilter $U$ of $A$-elements, the map $\theta_U$ described in Lemma 4.3 maps elements satisfying (39) to injective functions. Since a disjoint union of injective functions is injective, the result follows.

Suppose $a$ satisfies (39) and suppose for a contradiction that $\theta_U(a)([b]) = \theta_U(a)([c])$ (with both sides defined) and that $[b] \neq [c]$. The second of these statements means that $U$ contains some equaliser of $b:a$ and $c:a$, so $(b:a) \bowtie (c:a) \in U$, as this is the greatest such equaliser. Since both $b:a$ and $c:a$ are inequivalent to $0$ we know that $D(b:a) \in U$ and $D(c:a) \in U$. Since $[b] \neq [c]$, we have $[b_i] \neq [c_i]$ for some $i$. Then $b_i \bowtie c_i \notin U$, so that $A(b_i \bowtie c_i) \in U$. Marshalling all our elements of $U$ we have

$$(b:a) \bowtie (c:a) D(b:a) D(c:a) A(b_i \bowtie c_i) = D((b:a) \cdot (c:a)) A(b_i \bowtie c_i) \in U$$

where we now use juxtaposition for the Boolean meet. We are told by (39) that this element of the ultrafilter $U$ is $0$—a contradiction. We conclude that $\theta_U(a)$ is injective.

**Corollary 6.7.** The class of $((\cdot); A_1, \ldots, A_n, \cdot)$-algebras that are representable by injective $n$-ary partial functions is a variety, finitely axiomatised by the equations specified in Theorem 6.5 together with (39).

**Corollary 6.8.** The finite representation property holds for the signature $((\cdot); A_1, \ldots, A_n, \cdot)$ for representation by $n$-ary partial functions and for representation by injective $n$-ary partial functions.

### 7. Preferential Union

For signatures including composition and the antidomain operations, there is a simple equational characterisation of preferential union in terms of composition and the antidomain operations.

**Proposition 7.1.** In an algebra of $n$-ary partial functions, for signatures containing composition and the antidomain operations, $h$ is the preferential union of $f$ and $g$ if and only if $(D^n_f):h = f$ and $(A^n_f):h = (A^n_f):g$.

**Proof.** First suppose that $h = f \cup g$. If $(D^n_f):h$ is defined on an $n$-tuple $x$ then $f$ is defined on $x$ and so $h$ is defined on $x$ with the same value as $f$. Hence $((D^n_f):h)(x) = f(x)$. Conversely, if $f$ is defined on $x$ then $h$ is too, with the same value. Then $(D^n_f):h$ is defined on $x$ and $((D^n_f):h)(x) = f(x)$. This completes the argument that $(D^n_f):h = f$.

Continuing to suppose that $h = f \cup g$, if $(A^n_f):h$ is defined on an $n$-tuple $x$ then $f$ is not defined on $x$ and $h$ is defined on $x$. As $h$ is the preferential join of $f$ and $g$, this implies that $g$ is defined on $x$ with the same value as $h$. So $(A^n_f):h$ agrees with $(A^n_f):g$ on $x$. Conversely, if $(A^n_f):g$ is defined on $x$ then $f$ is not defined on $x$ and $g$ is. This implies that $h$ is defined on $x$ with the same value as $g$. So again $(A^n_f):h$ agrees with $(A^n_f):g$ on $x$. This completes the argument that $(A^n_f):h = (A^n_f):g$.
We now show that for any \( f, g \) and \( h \) satisfying the two equations, \( h \) is the preferential join of \( f \) and \( g \). Given such an \( f, g \) and \( h \), first suppose that \( h \) is defined on the \( n \)-tuple \( x \). If \( f \) is also defined on \( x \) then \( \langle D^n_1 f \rangle ; h \) is defined on \( x \) with the same value as \( h \). In this case we are told by the equation \( \langle D^n_1 f \rangle ; h = f \) that \( h(x) = \langle D^n_1 f \rangle ; h(x) = f(x) = (f \sqcup g)(x) \). If \( f \) is undefined at \( x \) then \( \langle A^n_1 f \rangle ; h \) is defined on \( x \) with the same value as \( h \). Then the equation \( \langle A^n_1 f \rangle ; h = (\langle A^n_1 f \rangle ; g) \) tells us that \( h(x) = (\langle A^n_1 f \rangle ; h)(x) = (\langle A^n_1 f \rangle ; g)(x) \). So \( g \) must be defined at \( x \) with the same value as \( h \). But \( g(x) = (f \sqcup g)(x) \), as \( f \) is undefined here. Again we have found \( h(x) = (f \sqcup g)(x) \). We conclude that \( h \subseteq f \sqcup g \).

Conversely, suppose that \( f \sqcup g \) is defined on \( x \). If \( f \) is defined on \( x \) then \( (f \sqcup g)(x) = f(x) = \langle D^n_1 f \rangle ; h(x) = h(x) \), utilising the equation \( \langle D^n_1 f \rangle ; h = f \). If \( f \) is not defined on \( x \) then \( g \) must be, since \( f \sqcup g \) is defined, and for the same reason \( A_1 f, \ldots, A_n f \) must be defined on \( x \). Then \( (f \sqcup g)(x) = g(x) = (\langle A^n_1 f \rangle ; g)(x) \) tells us that \( h(x) = (\langle A^n_1 f \rangle ; h)(x) = (\langle A^n_1 f \rangle ; g)(x) \). We conclude that \( h \supseteq f \sqcup g \), completing the proof that \( h = f \sqcup g \). \( \square \)

The content of Proposition 7.1 means we only need add the following two equations in order to extend the axiomatisations of the previous sections so as to include \( \sqcup \) in the signature.

\[
\begin{align*}
(40) & \quad \langle D^n_1 a \rangle ; (a \sqcup b) = a \\
(41) & \quad \langle A^n_1 a \rangle ; (a \sqcup b) = \langle A^n_1 a \rangle ; b
\end{align*}
\]

For the signature \( (\langle \rangle; A_1, \ldots, A_n, \sqcup) \) this gives us quasiequational axiomatisations. However it is possible to replace the quasiequation (8) with a valid equation that trivially implies it.

**Proposition 7.2.** For any signature containing composition, the antidomain operations and preferential union, the following equation is valid for the class of algebras representable by \( n \)-ary partial functions.

\[
(42) \quad (\langle D^n_1 a \rangle ; b) \sqcup (\langle A^n_1 a \rangle ; b) = b
\]

**Proof.** As usual, we prove validity for an arbitrary square algebra of \( n \)-ary partial functions. So let \( a \) and \( b \) be elements of such an algebra, with base \( X \), and let \( x \) be an \( n \)-tuple in \( X^n \).

If \( a \) is defined on \( x \) then \( D_1 a, \ldots, D_n a \) are too. Then \( (\langle D^n_1 a \rangle ; b) \sqcup (\langle A^n_1 a \rangle ; b) \) and \( b \) agree on \( x \), since if \( b \) is defined on \( x \) then \( \langle D^n_1 a \rangle ; b \) is and so \( ((\langle D^n_1 a \rangle ; b) \sqcup (\langle A^n_1 a \rangle ; b))(x) = (\langle D^n_1 a \rangle ; b)(x) = b(x) \) and if \( b \) is not defined on \( x \) then neither \( \langle D^n_1 a \rangle ; b \) nor \( \langle A^n_1 a \rangle ; b \) are and so \( (\langle D^n_1 a \rangle ; b) \sqcup (\langle A^n_1 a \rangle ; b) \) is also not defined on \( x \).

The other case needing consideration is when \( a \) is not defined on \( x \). Then \( \langle D^n_1 a \rangle ; b \) is not defined on \( x \) and \( A_1 a, \ldots, A_n a \) are all defined on \( x \). Again \( (\langle D^n_1 a \rangle ; b) \sqcup (\langle A^n_1 a \rangle ; b) \) and \( b \) agree on \( x \), since if \( b \) is defined then \( ((\langle D^n_1 a \rangle ; b) \sqcup (\langle A^n_1 a \rangle ; b))(x) = (\langle A^n_1 a \rangle ; b)(x) = b(x) \) and if \( b \) is not defined on \( x \) then neither \( \langle D^n_1 a \rangle ; b \) nor \( \langle A^n_1 a \rangle ; b \) are and so \( (\langle D^n_1 a \rangle ; b) \sqcup (\langle A^n_1 a \rangle ; b) \) also is not.

We obtain the following results.

**Theorem 7.3.** The class of \( (\langle \rangle; A_1, \ldots, A_n, \sqcup) \)-algebras that are representable by \( n \)-ary partial functions is a variety, finitely axiomatised by equations (2)–(7), (9) and (10) together with (40), (41) and (42).
Theorem 7.4. The class of \((\langle \_ \rangle; A_1, \ldots, A_n, \sqcup)\)-algebras that are representable by injective \(n\)-ary partial functions is a quasivariety, finitely axiomatised by (2)-(10) together with (28), (40) and (41).

Corollary 7.5. The finite representation property holds for the signature \((\langle \_ \rangle; A_1, \ldots, A_n, \sqcup)\) for representation by \(n\)-ary partial functions and for representation by injective \(n\)-ary partial functions.

For the signature \((\langle \_ \rangle; A_1, \ldots, A_n, \cdot, \sqcup)\) we can simply extend the equational axiomatisations of Section 6.

Theorem 7.6. The class of \((\langle \_ \rangle; A_1, \ldots, A_n, \cdot, \sqcup)\)-algebras that are representable by \(n\)-ary partial functions is a variety, finitely axiomatised by the equations specified in Theorem 6.5 together with (40) and (41).

Corollary 7.7. The class of \((\langle \_ \rangle; A_1, \ldots, A_n, \cdot, \sqcup)\)-algebras that are representable by injective \(n\)-ary partial functions is a variety, finitely axiomatised by the equations specified in Theorem 6.5 together with (28), (40) and (41).

Corollary 7.8. The finite representation property holds for the signature \((\langle \_ \rangle; A_1, \ldots, A_n, \cdot, \sqcup)\) for representation by \(n\)-ary partial functions and for representation by injective \(n\)-ary partial functions.

8. Fixset

As we noted previously, the fixset operations can be expressed using intersection and the antidomain operations as \(F_i f := \pi_i \cdot f\). So, having already given axiomatisations for signatures containing intersection, only the signatures without intersection are interesting to us, namely \((\langle \_ \rangle; A_1, F_i)\) and \((\langle \_ \rangle; A_1, F_i, \sqcup)\).

There is a simple equational axiomatisation of restrictions of the \(i\)th fixset in terms of composition and the domain operations, getting us halfway to axiomatising fixset.

Proposition 8.1. In an algebra of \(n\)-ary partial functions, for signatures containing composition and the antidomain operations, \(g\) is a restriction of \(F_i f\) if and only if \(D_i g = g\) and \((D^n_i g) : f = g\).

Proof. By definition, \(F_i f = \pi_i \cap f\) and so \(g\) is a restriction of \(F_i f\) if and only if \(g\) is both a restriction of \(\pi_i\) and a restriction of \(f\). Being a restriction of the \(i\)th projection is equivalent to satisfying \(D_i g = g\) and being a restriction of \(f\) is equivalent to satisfying \((D^n_i g) : f = g\). \(\square\)

The upshot of Proposition 8.1 is that the following equations are valid and ensure that any representation of a \((\langle \_ \rangle; A_1, \ldots, A_n)\)-reduct represents each \(F_i a\) both as a restriction of the \(i\)th projection and as a restriction of the representation of \(a\).

\[
(43) \quad D_i (F_i a) = F_i a \quad \text{for every } i \\
(44) \quad (D^n_i (F_i a)) : a = F_i a \quad \text{for every } i
\]

Hence adding (43) and (44) as axioms is sufficient to give \(\theta(F_i (a)) \subseteq F_i (\theta(a))\) in Theorem 4.5, for every \(i\). The next proposition presents valid quasiequations that are sufficient for the reverse inclusions to hold.
Proposition 8.2. The following indexed quasi-equations are valid for algebras representable by \( n \)-ary partial functions for any signature containing composition and the fixset operations.

\[
(45) \quad b : a = b_i \rightarrow b : F_i a = b_i \quad \text{for every } i
\]

Further, let \( \mathfrak{A} \) be an algebra of a signature containing composition and the antidual and fixset operations and suppose the \( (\langle \rangle; A_1, \ldots, A_n) \)-reduct of \( \mathfrak{A} \) is representable by \( n \)-ary partial functions. Let \( \theta \) be the representation of the reduct described in Theorem 4.5. If \( \mathfrak{A} \) satisfies the \( i \)-indexed version of (45) then \( \theta(F_i(a)) \supseteq F_i(\theta(a)) \).

Proof. For the first part it is sufficient to prove validity for an arbitrary square algebra of \( n \)-ary partial functions. So let \( a \) and \( b_1, \ldots, b_n \) be elements of such an algebra, with base \( X \), and suppose \( b : a = b_i \). If \( b : F_i a \) is defined on \( x \), with value \( z \), then \( b_1, \ldots, b_n \) are all defined on \( x \) and \( b : F_i a \) is defined on \( \langle b_1(x), \ldots, b_n(x) \rangle \), so \( a \) is too, with value \( b_i(x) = z \). Hence \( b : F_i a \subseteq b_i \).

Conversely, if \( b_i \) is defined on \( x \) then, by the assumption, \( b : a \) is defined on \( x \), with value \( b_i(x) \). Then \( b_1, \ldots, b_n \) are all defined on \( x \) and \( a \) is defined on \( \langle b_1(x), \ldots, b_n(x) \rangle \), also with value \( b_i(x) \). This tells us that \( F_i a \) is defined on \( \langle b_1(x), \ldots, b_n(x) \rangle \) and so \( b : a \) is defined on \( x \), necessarily with the same value as \( b_i \). Hence \( b : F_i a \supseteq b_i \) and we conclude that \( b : F_i a \) and \( b_i \) are equal, so (45) is valid.

For the second part it is sufficient to prove that, for any ultrafilter \( U \) of \( A \)-elements, the homomorphism \( \theta_U \), as defined in Lemma 4.3, satisfies \( \theta_U(F_i(a)) \supseteq F_i(\theta_U(a)) \). So suppose that \( ([b], [c]) \in F_i(\theta_U(a)) \). Then \( [c] = [b_i] \neq [0] \) and \( ([b], [b_i]) \in \theta_U(a) \), that is, there is some \( \alpha \in U \) such that \( \alpha : (b : a) = \alpha : b_i \).

Then by superassociativity

\[
\langle \alpha ; b_1, \ldots, \alpha ; b_n \rangle ; a = \alpha ; b_i
\]

so by (45)

\[
\langle \alpha ; b_1, \ldots, \alpha ; b_n \rangle ; F_i a = \alpha ; b_i
\]

and then by superassociativity

\[
\alpha ; (b : F_i a) = \alpha ; b_i
\]

and so \( b : F_i a = [b_i] \). Hence \( ([b], [c]) = ([b], [b_i]) \in \theta_U(F_i(a)) \) and we are done. \( \square \)

Combining Propositions 8.1 and 8.2, we obtain quasiequational axiomatisations for signatures containing the fixset operations.

Theorem 8.3. The class of \( (\langle \rangle; A_1, \ldots, A_n, F_1, \ldots, F_n) \)-algebras that are representable by \( n \)-ary partial functions is a quasivariety, finitely axiomatised by the (quasi)equations specified in Theorem 4.5 together with (43)–(45).

Corollary 8.4. The class of \( (\langle \rangle; A_1, \ldots, A_n, F_1, \ldots, F_n) \)-algebras that are representable by injective \( n \)-ary partial functions is a quasivariety, finitely axiomatised by the (quasi)equations specified in Theorem 4.5 together with (28) and (43)–(45).

Corollary 8.5. The finite representation property holds for the signature \( (\langle \rangle; A_1, \ldots, A_n, F_1, \ldots, F_n) \) for representation by \( n \)-ary partial functions and for representation by injective \( n \)-ary partial functions.
Theorem 8.6. The class of \((\langle \rangle; A_1, \ldots, A_n, F_1, \ldots, F_n, \sqcup)\)-algebras that are representable by \(n\)-ary partial functions is a quasivariety, finitely axiomatised by the (quasi)equations specified in Theorem 4.5 together with (40), (41) and (43)–(45).

Corollary 8.7. The class of \((\langle \rangle; A_1, \ldots, A_n, F_1, \ldots, F_n, \sqcup)\)-algebras that are representable by injective \(n\)-ary partial functions is a quasivariety, finitely axiomatised by the (quasi)equations specified in Theorem 4.5 together with (28), (40), (41) and (43)–(45).

Corollary 8.8. The finite representation property holds for the signature \((\langle \rangle; A_1, \ldots, A_n, F_1, \ldots, F_n, \sqcup)\) for representation by \(n\)-ary partial functions and for representation by injective \(n\)-ary partial functions.

9. Equational Theories

We conclude with an examination of the computational complexity of equational theories. The following theorem and proof are straightforward adaptations to the \(n\)-ary case of unary versions that appear in [4].

Theorem 9.1. Let \(\sigma\) be any signature whose symbols are a subset of \(\{\langle \rangle; \cdot, 0, \pi, D, A, F, \sqcup\}\). Then the class of \(\sigma\)-algebras that are representable by \(n\)-ary partial functions has equational theory in \(\text{coNP}\). If the signature contains \(A_i\) and either \(\langle \rangle; \cdot\) or \(\cdot\) then the equational theory is \(\text{coNP}\)-complete.

Proof. For the first part we will show that if an equation \(s = t\) is not valid then it can be refuted on an algebra of \(n\)-ary partial functions with a base of size linear in the length of the equation. Then a nondeterministic Turing machine can easily identify invalid equations in polynomial time by nondeterministically choosing an assignment of the variables to \(n\)-ary partial functions and then calculating the interpretations of the two terms.

Suppose \(s = t\) is not valid. Then there is some algebra \(\mathfrak{F}\) of \(n\)-ary partial functions, some assignment \(f\) of elements of \(\mathfrak{F}\) to the variables in \(s = t\) and some \(n\)-tuple \(x\) in the base of \(\mathfrak{F}\) such that \(s[f](x) \neq t[f](x)\), meaning that either both sides are defined and they have different values, or one side is defined and the other not. We will select a subset \(Y\) of the base of \(\mathfrak{F}\), of size linear in the length of the equation, such that in any algebra of \(n\)-ary functions with base \(Y\) and containing the restrictions \(f|_Y\) of \(f\) to \(Y \times Y\), we have \(s[f](x) = s[f|_Y](x)\) and \(t[f](x) = t[f|_Y](x)\) (or both sides are undefined). Then the equation is refuted in any such algebra, for example the algebra generated by the \(f|_Y\).

Define \(Y(r, x)\) by structural induction on the term \(r\) as follows.

- For any variable \(a\),
  \[Y(a, x) := \begin{cases} \{x_1, \ldots, x_n\} \cup \{a[f](x)\} & \text{if } a[f](x) \text{ exists} \\ \{x_1, \ldots, x_n\} & \text{otherwise} \end{cases} \]
- \(Y(u; v, x) := \begin{cases} Y(u_1, x) \cup \ldots \cup Y(u_n, x) \cup Y(v, (u_1[f](x), \ldots, u_n[f](x))) & \text{if } u_1[f](x), \ldots, u_n[f](x) \text{ exist} \\ \{x_1, \ldots, x_n\} & \text{otherwise} \end{cases} \)
- \(Y(0, x) := Y(\pi_i, x) := \{x_1, \ldots, x_n\}\)
- \(Y(D, u, x) := Y(A_i u, x) := Y(F_i u, x) := Y(u, x)\)
- \(Y(u \cdot v, x) := Y(u \circ_i v, x) := Y(u \sqcup v, x) := Y(u, x) \cup Y(v, x)\)
Then it follows by structural induction on terms that for any subset $Y$ of the base of $\mathcal{F}$ that contains $Y(r, x)$, we have $r[f](x) = r[f|Y](x)$. Hence we may take $Y := Y(s, x) \cup Y(t, x)$, which is clearly of size linear in the length of $s = t$.

For the second part, we describe a polynomial time reduction from the coNP-complete problem of deciding the tautologies of propositional logic, to the problem of deciding equational validity in the representation class. To do this, we may assume the propositional formulae are formed using only the connectives $\neg$ and $\land$. Then replace every propositional letter, $p$ say, in a given propositional formula, $\varphi$, with $D_ip$ (for some fixed choice of $i$), every $\neg$ with $A_i$ and every $\land$ with either the product $\cdot$ of Lemma 3.8 or with the operation $\cdot$ of the algebra, depending on availability in the signature. Denoting the resulting term $\varphi^*$, output the equation $\varphi^* = \pi_l$. This reduction is correct since the $A_i$-elements form a Boolean algebra and there are assignments where $D_ip$ is the bottom element and where it is the top.

Note that if we are interested in injective $n$-ary partial functions then the argument in the proof of Theorem 9.1 can be used to give the analogous result for this case so long as preferential union is not in the signature. Since the preferential union of two injective functions is not necessarily injective, restricted functions do not necessarily generate an algebra of injective functions when preferential union is present in the signature, invalidating the argument.

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