Mixing-like properties for some generic and robust dynamics

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Abstract
We show that the whole manifold is a homoclinic class for an open and dense subset of the set of robustly transitive diffeomorphisms far away from homoclinic tangencies. In particular, using the results from Abdenur and Crovisier, we obtain that every diffeomorphism in this subset is robustly topologically mixing. We also show that the set of Bernoulli measures of an isolated topologically mixing homoclinic class of a generic diffeomorphism is a dense subset of the set of invariant measures supported on the class.

Keywords: homoclinic classes, robustly transitive diffeomorphisms, partially hyperbolic diffeomorphisms, homoclinic tangency, Bernoulli measures

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1. Introduction

The study of homoclinic classes gained interest with the advent of Smale’s Spectral Decomposition Theorem. Indeed, this theorem says that for Axiom A (hyperbolic) dynamics the non-wandering set splits into finitely many homoclinic classes. Moreover, each of these classes is isolated: it is the maximal invariant set of a neighbourhood of itself. We recall that a homoclinic class of a periodic point p is the closure of the transversal intersections of the invariant manifolds of the orbit of p. It is well known that such classes are transitive, i.e. they contain a point whose orbit is dense in the class. For transitive Anosov systems the whole manifold is a homoclinic class, and since such systems are stable they are robustly transitive.

The study of such classes in non-hyperbolic situations has attracted the attention of many mathematicians, see [BDV] for a survey on the subject. We can ask for finer properties of their
dynamics, both in the topological sense and in the measure theoretical sense. Furthermore, we can also investigate the robustness of such properties. The purpose of this article is to contribute to two types of such problems. The first one is about the existence of robustness of stronger types of transitivity of such classes. The second one is about the structure of the set of invariant measures supported on the class.

We will postpone the precise definitions for the following sections. The dynamical systems we will consider here are diffeomorphisms in a compact and connected Riemannian manifold and the topology used in the space of diffeomorphisms will be the $C^1$-topology.

A starter point could be to investigate when a manifold is robustly a homoclinic class, as in the transitive Anosov case. In fact, there is a question in [BDV] (problem 7.25, page 144), if this is true for robustly transitive diffeomorphisms (see also [BDU]). Our first result deals with this question far from homoclinic tangencies. Recall that a homoclinic tangency is a non-transversal intersection between the invariant manifolds of a hyperbolic periodic point.

**Theorem A.** There exists an open and dense subset among robustly transitive diffeomorphisms far from homoclinic tangencies formed by diffeomorphisms such that the whole manifold is a homoclinic class.

We can try to explore further this phenomenon. Recall that a system is topologically mixing if given two open sets $U$ and $V$ then the $n$-th iterate of $U$ meets $V$ for every $n$ large enough.

Now, some technical results in section 2 of [AC] can be used to conclude directly the following result.

**Theorem 1.1.** Let $f$ be a generic diffeomorphism. If an isolated homoclinic class of $f$ is topologically mixing then it is robustly topologically mixing.

Hence, this result together with theorem A gives us the following interesting result:

**Theorem B.** There is an open and dense subset among robustly transitive diffeomorphisms far from homoclinic tangencies formed by robustly topologically mixing diffeomorphisms.

This is in fact a partial answer to the following question posed by Abdenur and Crovisier in [AC]: *Is every robustly transitive diffeomorphism topologically mixing?*

Since the mixing property has a higher degree of complexity than the transitivity property it is fair enough to search for finer properties in the measure theoretical sense.

Our next result deals with the denseness of Bernoulli measures in the set of invariant measures of isolated topologically mixing homoclinic classes of generic diffeomorphisms. This is inspired in Sigmund’s [S1] and Abdenur et al [ABC] works. Indeed, Sigmund [S1] shows that the ergodic measures are dense in the set of invariant measures of homoclinic classes of Axiom A diffeomorphisms. This was extended, by Abdenur et al [ABC] for non-hyperbolic isolated homoclinic classes of generic diffeomorphisms. Additionally, Sigmund in [S2] improves the results of [S1], proving that the set of Bernoulli measures is also dense. In the same spirit of [ABC], we show an analogous result of [S2] in the non-hyperbolic setting. We recall that a system is Bernoulli if it is measure theoretically isomorphic to a Bernoulli shift.

**Theorem 1.2.** For any generic diffeomorphism $f$, if the dynamics restricted to an isolated homoclinic class is topologically mixing then the Bernoulli measures are dense in the space of invariant measures supported on the class. In particular, the set of weakly mixing measures contains a residual subset.

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3 We would like to thank Prof Sylvain Crovisier for pointing out this result to us.
This paper is organised as follows. In section 2 we give the precise definitions of the main objects we will be dealing with. In section 3 we state the known results that will be our main tools. In section 4 we prove theorem 1.2, and finally, in section 5 we prove theorem A.

2. Precise definitions

In this section, we give the precise definitions of the objects used in the statements of the results. In this paper $M$ will be a closed and connected Riemannian manifold of dimension $d$. Also, $\overline{\cdot}$ will denote the closure operator.

2.1. Topological dynamics

We recall the notions of transitivity and mixing. We say that $f$ is transitive if there exists a point in $M$ whose forward orbit is dense. This is equivalent to the existence of a dense backward orbit and is also equivalent to the following condition: for every pair $U, V$ of open sets, there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$.

More specifically, we say that $f$ is topologically mixing if for every pair $U, V$ of open sets there exists $N_0 > 0$ such that $n \geq N_0$ implies $f^n(U) \cap V \neq \emptyset$.

2.2. Hyperbolic periodic points

The periodic point is hyperbolic if the eigenvalues of $Df^{\tau(p)}(p)$ do not belong to $S^1$. Where $\tau(p)$ denotes the period of $p$. As usual, $E^s(p)$ (resp. $E^u(p)$) denotes the eigenspace of the eigenvalues with a norm smaller (resp. bigger) than one. This gives a $Df^{\tau(p)}$ invariant splitting of the tangent bundle over the orbit $O(p)$ of $p$. The index of a hyperbolic periodic point $p$ is the dimension of the stable bundle, denoted by $I(p)$.

If $p$ is a hyperbolic periodic point for $f$ then every diffeomorphism $g$, $C^1$–close to $f$ also has a hyperbolic periodic point close to $p$ with the same period and index, which is called the continuation of $p$ for $g$, and is denoted by $p(g)$.

The local stable and unstable manifolds of a hyperbolic periodic point $p$ are defined as follows: given $\varepsilon > 0$ small enough, we set

$$W^s_{\text{loc}}(p) = \{ x \in M; \quad d(f^n(x), f^n(p)) \leq \varepsilon, \quad \text{for every } n \geq 0 \}$$

and

$$W^u_{\text{loc}}(p) = \{ x \in M; \quad (f^{-n}(x), f^{-n}(p)) \leq \varepsilon, \quad \text{for every } n \geq 0 \}.$$ 

They are differentiable manifolds tangent at $p$ to $E^s(p)$ and $E^u(p)$. The stable and unstable manifolds are given by the saturations of the local manifolds. Indeed,

$$W^s(p) = \bigcup_{n \geq 0} f^{-n\tau(p)}(W^s_{\text{loc}}(p))$$

and

$$W^u(p) = \bigcup_{n \geq 0} f^{n\tau(p)}(W^u_{\text{loc}}(p)).$$

The stable and unstable set of a hyperbolic periodic orbit, $O(p)$, are given by:

$$W^s(O(p)) = \bigcup_{j=0}^{\tau(p)-1} W^s(f^j(p))$$

and

$$W^u(O(p)) = \bigcup_{j=0}^{\tau(p)-1} W^u(f^j(p)).$$
2.3. Homoclinic intersections

If \( p \) is a hyperbolic periodic point of \( f \), then its homoclinic class \( H(p) \) is the closure of the transversal intersections of the stable manifold and unstable manifold of the orbit of \( p \):

\[
H(p) = \text{cl}(W^s(O(p)) \cap W^u(O(p))).
\]

A homoclinic class \( H(p) \) is non-trivial if contains elements which are not iterates of \( p \). We say that a hyperbolic periodic point \( q \) is homoclinically related to \( p \) if \( W^s(O(p)) \cap W^u(O(q)) \neq \emptyset \) and \( W^u(O(p)) \cap W^s(O(q)) \neq \emptyset \). It is well known that a homoclinic class coincides with the closure of the hyperbolic periodic points homoclinically related to \( p \). Moreover, it is a transitive invariant set. We say that a homoclinic class \( H(p) \) has a robust property if \( H(p; g) \) also has this property for any diffeomorphism \( g \) sufficiently close to \( f \).

We define the period of a homoclinic class \( H(p) \) as the greatest common divisor of the periods of the hyperbolic periodic points homoclinically related to \( p \), and we denote by \( l(O(p)) \).

We say that the homoclinic class \( H(p) \) is isolated if there exists a neighbourhood \( U \) of \( H(p) \) such that \( Hp \cap f^n(U) = \emptyset \) for all \( n \in \mathbb{Z} \).

On the other hand, we say that a non-transversal intersection between \( W^s(O(p)) \) and \( W^u(O(p)) \) is a homoclinic tangency. We denote by \( HT(M) \) the set of diffeomorphisms exhibiting a homoclinic tangency. We will say that a diffeomorphism \( f \) is far from homoclinic tangencies if \( f \notin HT(M) \).

Given \( p \) and \( q \) hyperbolic periodic points with \( l(p) < l(q) \) we say that they form a heterodimensional cycle if there exists \( x \in W^s(O(p)) \cap W^u(O(q)) \), with \( \dim(T_xW^s(O(p)) \cap T_xW^u(O(q))) = 0 \) and \( W^u(O(p)) \cap W^s(O(q)) = \emptyset \).

2.4. Invariant measures

We will denote by \( \mathcal{M}(f) \) the space of \( f \)-invariant probability measures on \( M \), and by \( \mathcal{M}_e(f) \) the ergodic elements of \( \mathcal{M}(f) \).

For a periodic point \( p \) of \( f \) with period \( \tau(p) \), we let \( \mu_p \) denote the periodic measure associated with \( p \), given by

\[
\mu_p = \frac{1}{\tau(p)} \sum_{x \in O(p)} \delta_x
\]

where \( \delta_x \) is the Dirac measure at \( x \).

Now, let us define the notion of a Bernoulli measure. We first recall the so-called Bernoulli shift. It is the homeomorphism \( \sigma : \{1, ..., n\}^\mathbb{Z} \to \{1, ..., n\}^\mathbb{Z} \) defined by \( \sigma((x_n)) = (x_{n+1}) \). In \( \{1, ..., n\}^\mathbb{Z} \) consider \( m_\mu \) the product measure with respect to the uniform probability in \( \{1, ..., n\} \).

It is easy to see that \( m_\mu \) is invariant under \( \sigma \).

We say that \( \mu \in \mathcal{M}(f) \) is a Bernoulli measure if \( (f, \mu) \) is a measure theoretically isomorphic to \( (\sigma, m_\mu) \).

2.5. Partial hyperbolicity

Let \( \Lambda \subset M \) be invariant under a diffeomorphism \( f \). Let \( E, F \) be subbundles of \( T_xM \) of the tangent bundle over \( \Lambda \) with a trivial intersection at every \( x \in \Lambda \). We say that \( E \) dominates \( F \) if there exists \( N \in \mathbb{N} \) such that

\[
\|Df^N(x)\|_E \|Df^{-N}(f^N(x))\|_F \leq \frac{1}{2},
\]

where \( \| \cdot \|_E \) and \( \| \cdot \|_F \) denote the norms in \( E \) and \( F \).
for every $x \in \Lambda$. We say that $\Lambda$ admits a *dominated splitting* if there exists a decomposition of the tangent bundle $T_x M = \bigoplus_{i=1}^s E_i$ such that $E_i$ dominates $E_{i+1}$.

We say that a $f$-invariant subset $\Lambda$ is *partially hyperbolic* if it admits a dominated splitting $T_x M = E^s \oplus E^c \oplus \cdots \oplus E^c \oplus E^u$, with at least one of the extremal bundles being non-trivial, such that the extremal bundles $E^s$ and $E^u$ have a uniform contraction and expansion, respectively, and the other bundles, which are called centre bundles, do not have a uniform behaviour, i.e. there is neither uniform contraction nor uniform expansion.

If all centre bundles are trivial, then $\Lambda$ is called a *hyperbolic set*. Furthermore, if $\Lambda$ is compact and an isolated set, then it is called a *basic hyperbolic set*. Now, we say $\Lambda$ is *strongly partially hyperbolic* if both extremal bundles and the centre bundle are non-trivial, and moreover such that all of its centre bundles are one-dimensional. In particular, a strongly partially hyperbolic set is not hyperbolic.

We say that a diffeomorphism $f : M \to M$ is *partially hyperbolic* (resp. *strongly partially hyperbolic*) if $M$ is a partially hyperbolic (resp. strongly partially hyperbolic) set of $f$. When $M$ is a hyperbolic set we say that $f$ is *Anosov*.

We remark now that strongly partially hyperbolic diffeomorphisms are by definition far from homoclinic tangencies, since all central sub bundles have one dimension.

**Remark 2.1.** If $f$ is partially hyperbolic, by theorem 6.1 of [HPS] there exist strong stable and strong unstable foliations that integrate $E^s$ and $E^u$. More, precisely, for any point $x \in M$ there is a unique invariant local strong stable manifold $W^s_{\text{loc}}(x)$ which is a smooth graph of a function $\phi^s : E^s \to E^s \oplus E^u$ (in local coordinates), and varies continuously with $x$. In particular, $W^s_{\text{loc}}(x)$ has a uniform size for every $x \in M$. The same holds for $W^u_{\text{loc}}(x)$, integrating $E^u$.

Saturating these local manifolds, we obtain two foliations, which we denote by $\mathcal{F}^s$ and $\mathcal{F}^u$ respectively. Indeed, $\mathcal{F}^s(x) = \bigcup_{n \geq 0} f^{-n} (W^s_{\text{loc}}(f^n(x))$. An analogous definition holds for $\mathcal{F}^u$.

### 2.6. Robustness and genericity

As mentioned before, we are dealing with the space $\text{Diff}^1(M)$ of $C^1$ diffeomorphisms over $M$ endowed with the $C^1$-topology. This is a Baire space. Thus, any residual subset, i.e. a countable intersection of open and dense sets, is dense. When a property $P$ holds for any diffeomorphism in a fixed residual subset, we will say that $P$ holds generically. Or even, that a generic diffeomorphism exhibits the property $P$.

On the other hand, we say that a property holds robustly for a diffeomorphism $f$ if there exists a neighbourhood $\mathcal{U}$ of $f$ such that the property holds for any diffeomorphism in $\mathcal{U}$.

In this way, we say that a diffeomorphism $f \in \text{Diff}^1(M)$ is *robustly transitive* if it admits a neighbourhood entirely formed by transitive diffeomorphisms.

In this paper we let $\mathcal{I}(M)$ denote the open set of $\text{Diff}^1(M)$ formed by robustly transitive diffeomorphisms which are far from tangencies. Note that being far from tangencies is, by definition, an open condition. Also, we define by $T_{\text{SH}}(M)$ as the interior of robustly transitive strongly partially hyperbolic diffeomorphisms, which is a subset of $\mathcal{I}(M)$.

When dealing with properties which involve objects defined by the diffeomorphism itself we need to deal with the continuations of these objects.

For instance, when we say that a homoclinic class of $f$ is robustly topologically mixing, we are fixing a hyperbolic periodic point $p$ of $f$ and a neighbourhood $\mathcal{U}$ of $f$ such that for any $g \in \mathcal{U}$ the continuation $g^n(p)$ of $p$ is defined and the homoclinic class $H(p^n(g), g)$ is topologically mixing, i.e. for any $U$ and $V$ open sets of $H(p^n(g), g)$ there exists $N > 0$ such that for any $n \geq N$ we have $g^n(U) \cap V = \emptyset$. 


3. Some tools

In this section, we collect some results that will be used in the proofs of the main results.

We start with one of the main generic results used in this paper, which is a result of Abdenur and Crovisier, theorem 3 in [AC]. They prove the existence of a decomposition of any generic isolated chain-transitive set. Since we are interested here solely in the study of isolated homoclinic classes, we quote their result only for homoclinic classes.

**Theorem 3.1 (theorem 3 in [AC]).** There exists a residual subset \( \mathcal{R} \subset \text{Diff}^1(M) \) such that for every \( f \in \mathcal{R} \), any isolated homoclinic class \( \mathcal{H}(p, f) \) of a hyperbolic periodic point \( p \) of \( f \), decomposes uniquely as the finite union \( \Lambda_1 \cup \cdots \cup \Lambda_l \) of disjoint compact sets on each of which \( f^l \) is topologically mixing. Moreover, \( l \) is the smallest positive integer such that \( W^s(p) \) has a non-empty transversal intersection with \( W^u(f^l(p)) \).

As an application, they obtain that generically any transitive diffeomorphism is topologically mixing, and in particular there is a non-empty transversal intersection between \( W^u(f(p)) \) and \( W^s(p) \).

The result below, of Bonatti and Crovisier [BC], proves that a large class of transitive diffeomorphisms has the property that the whole manifold coincides with a homoclinic class.

**Theorem 3.2 (corollary 1.3 in [BC]).** There exists a residual subset \( \mathcal{R} \) of \( \text{Diff}^1(M) \) such that for every transitive diffeomorphism \( f \in \mathcal{R} \) if \( p \) is a hyperbolic periodic point of \( f \) then \( M = \mathcal{H}(p, f) \).

Another generic result is the following

**Theorem 3.3 (theorem A, item (1), [CMP]).** There exists a residual subset \( \mathcal{R} \) of \( \text{Diff}^1(M) \) such that for every \( f \in \mathcal{R} \) if two homoclinic classes \( \mathcal{H}(p_1, f) \) and \( \mathcal{H}(p_2, f) \) are either equal or disjoint.

The next result, from [ABCDW], says that generically, homoclinic classes are index complete.

**Theorem 3.4 (theorem 1 in [ABCDW]).** There is a residual subset \( \mathcal{R} \subset \text{Diff}^1(M) \) of diffeomorphisms \( f \) such that, every \( f \in \mathcal{R} \) and any homoclinic class containing hyperbolic periodic points of indices \( i \) and \( j \), also contains hyperbolic periodic points of index \( k \) for every \( i < k < j \).

The next tool we will use is due to Abdenur et al in [ABC], which extends Sigmund’s result [S2] to the non-hyperbolic setting.

**Theorem 3.5 (theorem 3.5, item (a), in [ABC]).** Let \( \Lambda \) be an isolated non-hyperbolic transitive set of a \( C^1 \)–generic diffeomorphism \( f \), then the set of periodic measures supported in \( \Lambda \) is a dense subset of the set \( \mathcal{M}(\Lambda) \) of invariant measures supported in \( \Lambda \).

Crovisier et al in [CSY] showed that for any diffeomorphism \( f \) in an open and dense subset far from homoclinic tangencies, every homoclinic class of \( f \) is a partially hyperbolic set. More precisely, they proved the following:

**Theorem 3.6 (theorem 1.1(2) in [CSY]).** There is an open and dense subset \( \mathcal{A} \subset \text{Diff}^1(M) \) such that for every \( f \in \mathcal{A} \), any homoclinic class \( \mathcal{H}(p) \) is a partially hyperbolic set of \( f \)

\[
T_{\mathcal{H}(p)}M = E^\nu \oplus E^0_1 \oplus \cdots \oplus E^0_k \oplus E^s,
\]
with \( \dim E^s_i = 1 \), \( i = 1, \ldots, k \), and moreover the minimal stable dimension of the periodic points of \( H(p) \) is \( \dim(E^s) \) or \( \dim(E^s) + 1 \). Similarly, the maximal stable dimension of the periodic orbits of \( H(p) \) is \( \dim(E^s) + k \) or \( \dim(E^s) + k - 1 \). For every \( i, 1 \leq i \leq k \) there exists periodic points in \( H(p) \) whose Lyapunov exponent along \( E^c_i \), is arbitrarily close to 0.

It is important to point out that the definition of a partially hyperbolic set in [CSY] is different from the one in this paper. In their definition they also consider a set having a dominated splitting only by one-dimensional sub-bundles as a partially hyperbolic set. This means that the dominated splitting could have no sub-bundles with uniform contraction or expansion. However, as we can see in the next remark, the definitions will coincide for isolated homoclinic classes.

**Remark 3.7.** We would like to remark that generically an isolated partially hyperbolic homoclinic class of a diffeomorphism far from homoclinic tangencies should have hyperbolic periodic points \( p_1 \) and \( p_2 \) such that \( \text{ind } p_1 = E^s \) and \( \text{ind } p_2 = E^u \). In particular, if the homoclinic class is non-trivial then both extremal bundles should be non-trivial. This fact is obtained quite easily from the previous theorem, and generic arguments, since we can change the index of hyperbolic periodic points having Lyapunov exponents close enough to zero by a perturbation.

### 4. Topologically mixing homoclinic classes

#### 4.1. Denseness of Bernoulli measures: proof of theorem 1.2

We recall the following result of Bowen.

**Theorem 4.1** ([Bow2], theorem 34). Let \( \Lambda \) be a topologically mixing isolated hyperbolic set. Then, there exists a Bernoulli measure supported in \( \Lambda \).

**Remark 4.2.** Actually, Bowen constructs a measure such that \( (f|_{\Lambda}, \mu) \) is a \( K \)-automorphism. But, in this case, \( (f|_{\Lambda}, \mu) \) is measure theoretically isomorphic to a mixing Markov chain and according to [FO] it is isomorphic to a Bernoulli shift.

Now, we give the proof of theorem 1.2.

**Proof of theorem 1.2.** Let \( H(p) \) be an isolated topologically mixing homoclinic class of a \( C^1 \) generic diffeomorphism \( f \). Let \( \mu \) be an invariant measure supported in \( H(p) \) and let \( \varepsilon > 0 \) be arbitrarily chosen. By theorem 3.5 there exists a measure \( \mu_{\varepsilon} \), supported on a hyperbolic periodic orbit \( \tilde{O}(\tilde{p}) \), with \( \varepsilon_{\tilde{H}(\tilde{p})} \) close to \( \mu \).

Since \( f \) is \( C^1 \) generic, theorem 3.3 implies that \( H(\tilde{p}) = H(p) \). In particular, we have that \( H(\tilde{p}) \) is topologically mixing, and thus from theorem 3.1 we know that there exists a point \( q \in W^s(\tilde{p}) \cap W^u(f(\tilde{p})) \). Since \( q \) is a transversal homoclinic point it is known that for every small neighbourhood \( U \) of \( O(\tilde{p}) \cup O(q) \) the maximal invariant set is a basic hyperbolic set conjugated to a sub-shift of finite type. Moreover, in this particular situation the next lemma shows that this hyperbolic set is in fact topologically mixing.

We point out that this lemma is probably a folklore result; however, since we have not found any place containing it we will give a proof at the end of this section.

**Lemma 4.3.** Let \( f \) be a diffeomorphism with a hyperbolic periodic point \( p \) such that there exists a point of transverse intersection \( q \in W^s(p) \cap W^u(f(p)) \). Then, for any small enough neighbourhood \( U \) of \( O(p) \cup O(q) \), the restriction of \( f \) to the maximal invariant set \( \Lambda_U = \cap_{n \in \mathbb{Z}} f^n(U) \) is a topologically mixing basic hyperbolic set. \( \square \)
Hence, lemma 4.3 tells us that the maximal invariant set \( \Lambda_U = \bigcap_{n \in \mathbb{Z}} f^n(U) \) is a topologically mixing basic hyperbolic set. Moreover, since \( q \) is a homoclinic point of \( \hat{p} \), by choosing \( U \) sufficiently small we have that the points in \( \Lambda_U \) spent portions of their orbit as large as we please shadowing the orbit of \( \hat{p} \).

Now, take \( \nu \) the Bernoulli measure supported in \( \Lambda_U \), which is given by theorem 4.1. Since a typical point in the support of \( \nu \) spent large portions of its orbit shadowing the orbit of \( \hat{p} \), we can choose \( U \) such that \( \nu \) is \( \varepsilon/2 \) close to \( \mu_p \).

Thus, \( \nu \) is \( \varepsilon \) close to \( \mu \) and we are done. \( \Box \)

**Remark 4.4.** It is worth pointing out that the techniques employed above can also be used to give a new proof of Sigmund’s result on the denseness of Bernoulli measures for hyperbolic topologically mixing basic sets \([S1]\).

To finish this section we prove lemma 4.3.

**Proof of lemma 4.3.** For this proof, we denote \( \tau := \tau(p) \) the period of \( p \).

It is a well known result (see for instance, theorem 4.5 (item b), p 288 in \([Rob]\)) that for any small enough neighbourhood \( U \) of \( O_p \cup O_q \) the maximal invariant set \( \Lambda_U = \bigcap_{n \in \mathbb{Z}} f^n(U) \) is a basic hyperbolic set; indeed, this maximal invariant set is topologically conjugated to a sub-shift of finite type. In particular, this set has a shadowing property.

The proof is finished if we prove the following claim:

**Claim.** Given \( 0 < \varepsilon \) arbitrary small, there exist \( N_0 > 0 \) such that for any \( n > N_0 \) there exists a fixed point \( p \) of \( f^n \) \( 3\varepsilon \) -dense in \( \Lambda_U \).

In fact, for any open sets \( V \) and \( W \) in \( \Lambda_U \), by choosing \( 0 < \varepsilon \) such that \( V \) and \( W \) contain a ball of radius \( \varepsilon \), it is not difficult to see that \( f^n(V) \cap W = \emptyset \) for any \( n \geq N_0 \).

Hence, let us prove the claim.

First, let \( \delta > 0 \) be given by the shadowing lemma for \( \varepsilon \). That is, any \( \delta \) -pseudo orbit should be \( \varepsilon \) -shadowed by real orbits. We will prove that there exists a number \( N_0 \) such that for every \( n \geq N_0 \) it is possible to construct a periodic \( \delta \) -pseudo orbit inside \( U \), with a period exactly equal to \( n \), and whose Hausdorff distance to \( O_p \cup O_q \) is smaller than \( \varepsilon \).

Once we have settled this, the shadowing lemma will produce periodic orbits, which are fixed points for \( f^n \), and whose Hausdorff distance to \( O(p) \cup O(q) \) is \( 2\varepsilon \). In particular, these orbits must be \( 3\varepsilon \) dense in \( \Lambda_U \), with respect to the Hausdorff distance. Then the proof of the claim is finished.

With this goal in mind, we take a large iterate \( x = f^{N_0}(q) \) such that

\[
f^{-r}(x) \in B(p, \delta/2),
\]

for every \( r = 0, ..., \tau - 1 \). Observe that \( f^{-1}(x) \in W^u(p) \), since \( q \in W^u(f(p)) \). This implies that there exists a smallest positive integer \( l \in \mathbb{N} \) such that

\[
f^{-\tau-1}(x) \in B(p, \delta/2).
\]

Now, we can give the number \( N_0 \). For each \( r = 1, ..., \tau - 1 \), let \( k_r = rl \) and take \( L = \prod_{r=1}^{\tau-1} k_r \). We define \( N_0 := L\tau \). Observe that if \( n \geq N_0 \) we can write

\[
n = (a + L)\tau + r = (a + L - k_r)\tau + k_r\tau + r,
\]

for some \( r \in \{1, ..., \tau - 1\} \) and \( a \in \mathbb{N} \).
To complete the proof, we will give the pseudo orbit. It will be given by several strings of orbit, with jumps at specific points. For this reason, and for the sake of clarity, we divide the construction in several steps between each jump.

- The first string: Define $x_0 = f^{-n} \tau(x_0)$, $x_j = f^j(x_0)$, for every $j = 1, \ldots, t_r$.
- The second string: Note that $f(x_{t_r}) = f^{-\tau}(x) \in B(p, \delta/2)$. Put $x_{t_r+1} = f^{-n} \tau(x_{t_r+1})$ for every $j = 1, \ldots, t_r$.
- The procedure continues inductively: Note again that $f^{-n} \tau(x) \in B(p, \delta/2)$, and put $x_{t_r+2} = f^{-n} \tau(x_{t_r+2})$, and proceed with the construction in an analogous way, defining $x_{t_r+j} := f^{-n} \tau(x_{t_r+j})$ and the next $t_r$ terms of the sequence as simply the iterates of this point, all of which belong to $B(p, \delta/2)$.
- The last string: Observe that $f^{-n} \tau(x) \in B(p, \delta/2)$. Hence, we can choose $x_{t_r+2} = x$ and the next $t_r$ terms of the sequence as simply the iterates of this point, all of which belong to $B(p, \delta/2)$.
- The last jump: Finally, we close the pseudo orbit by putting $x_{t_r+2} = x_0$.

This gives a periodic $\delta$-pseudo orbit with period $n$, as required.

5. Robustly large homoclinic class

In this section we will prove theorem A as a consequence of the following result:

**Theorem 5.1.** Let $f \in \text{Diff}^1(M)$ be a robustly transitive strong partially hyperbolic diffeomorphism, with $TM = E^s \oplus E^t \oplus \ldots \oplus E^u$, having hyperbolic periodic points $p_s$ and $p_u$ with index $s$ and $d - u$, respectively, where $s = \dim E^s$ and $u = \dim E^u$. Then, there exists an open subset $V_j$, whose closure contains $f$ such that $M = H(p_j(g)) = H(p_j(g))$ for every $g \in V_j$.

Before we prove theorem 5.1, let us see how it implies theorem A.

**Proof of theorem A.** First we observe that it suffices to deal with the interior of non-hyperbolic robustly transitive diffeomorphisms, since in the transitive Anosov case the whole manifold is robustly a homoclinic class, which is a consequence of the shadowing lemma.

Recall that $T_{\text{SH}}(M) \subset T(M)$ denotes the interior of non-hyperbolic robustly transitive diffeomorphisms far from homoclinic tangencies. Hence, according to theorems 3.2, 3.6 and remark 3.7 there exists a residual subset $R$ in $T_{\text{SH}}(M)$ such that if $f \in R$ then:

(a) $M$ coincides with a homoclinic class;
(b) $f$ is strong partially hyperbolic, with partially hyperbolic decomposition $TM = E^s \oplus E^t \oplus \ldots \oplus E^u$;
(c) There exist hyperbolic periodic points with indices $s = \dim E^s$ and $u = \dim E^u$.

Thus, we can find a dense subset $R_t$ inside $T_{\text{SH}}(M)$ formed by robustly transitive strong partially hyperbolic diffeomorphisms $f$ satisfying the hypothesis of theorem 5.1. Then, considering $V_j$ given by theorem 5.1 for every $f \in R_t$ we have that

$$A = \bigcup_{f \in R_t} V_j,$$
is an open and dense subset of $\mathcal{T}_{NH}(M) \subset \mathcal{T}(M)$. According to theorem 5.1, for every diffeomorphism in $\mathcal{A}$ the whole manifold $M$ coincides with a homoclinic class. This ends the proof.

We now establish some technical results which are key steps in the proof of theorem 5.1.

The following result allows us to find open sets of diffeomorphisms for which the topological dimension of the stable (and unstable) manifold of hyperbolic periodic points is larger than the differentiable dimension.

**Lemma 5.2.** Let $f \in \text{Diff}^1(M)$ be a robustly transitive strong partially hyperbolic diffeomorphism. Suppose there are hyperbolic periodic points $p_j$, $j = i, i + 1, \ldots, k$, with indices $l(p_j) = j$ for $f$. Hence, given any small enough neighbourhood $\mathcal{U}$ of $f$, where the continuation of the hyperbolic periodic points $p_j$ defined, there exists an open set $\mathcal{V} \subset \mathcal{U}$ such that for every $g \in \mathcal{V}$:

$$W^s(p_i(g)) \subset \text{cl}(W^s(p_{k-1}(g))) \subset \ldots \subset \text{cl}(W^s(p_{i+1}(g))) \subset \text{cl}(W^s(p_i(g))).$$

and

$$W^u(p_i(g)) \subset \text{cl}(W^u(p_{i+1}(g))) \subset \ldots \subset \text{cl}(W^u(p_{k-1}(g))) \subset \text{cl}(W^u(p_i(g))).$$

To prove the above lemma we will use the following result, which is a consequence of proposition 6.14 and lemma 6.12 in [BDV], which are the results of Diaz and Rocha [DR]. It is worth pointing out that this result is a consequence of the well-known blender technique, which appears by means of unfolding a heterodimensional co-dimensional one cycle far from homoclinic tangencies.

**Proposition 5.3.** Let $f$ be a $C^1$ diffeomorphism with a heterodimensional cycle associated with saddles $p$ and $\hat{p}$ with indices $i$ and $i + 1$, respectively. Suppose that the cycle is $C^1$ far from homoclinic tangencies. Then there exists an open set $\mathcal{V} \subset \text{Diff}^1(M)$ whose closure contains $f$ such that for every $g \in \mathcal{V}$

$$W^s(\hat{p}(g)) \subset \text{cl}(W^s(p(g))) \text{ and } W^u(p(g)) \subset \text{cl}(W^u(\hat{p}(g))).$$

**Proof of Lemma 5.2.** Since $f$ is a robustly transitive strong partially hyperbolic diffeomorphism, we can assume that every diffeomorphism $g \in \mathcal{U}$ is transitive and is strong partially hyperbolic, reducing $\mathcal{U}$ if necessary. In particular, $\mathcal{U}$ is far from homoclinic tangencies, $\mathcal{U} \subset (\text{cl}(\mathcal{H}(\mathcal{T}(M))))^\circ$. Now, using the transitivity of $f$, there are points $x_n$ converging to the stable manifold of $p_{i+1}$, whose a sequence of iterates $f^m(x_n)$ is converging on the unstable manifold of $p_i$. Hence, we can use Hayashi’s connecting lemma [H] to perturb the diffeomorphism $f$ to $\tilde{f}$ such that $W^s(p_i(\tilde{f}))$ intersects $W^u(p_{i+1}(\tilde{f}))$, which we could assume be transversal after a perturbation, if necessary, since $\dim W^s(p_i(\tilde{f})) + \dim W^u(p_{i+1}(\tilde{f})) > d$. Hence, we can use once more the connecting lemma to find $f_1 \in \mathcal{U}$ close to $\tilde{f}$ exhibiting a heterodimensional cycle between $p_i(f_1)$ and $p_{i+1}(f_1)$, since $\tilde{f}$ is also transitive. Moreover, and in fact this is needed to apply proposition 5.3, after a perturbation we can assume that intersection between $W^s(p_i(f_1))$ and $W^u(p_{i+1}(f_1))$ is quasi-transversal in the sense that $T_0W^s(p_i(f_1)) \cap T_0W^u(p_{i+1}(f_1)) = \{0\}$.

Thus, since $f_1$ is far from homoclinic tangencies, we can use proposition 5.3 to find an open set $\mathcal{V}_1 \subset \mathcal{U}$ such that
for every $g \in V_i$.

Now, since $f_1$ is also robustly transitive we can repeat the above argument to find $f_2 \in V_i$ exhibiting a heterodimensional cycle between $p_{i+1}$ and $p_{i+2}$. Thus, by proposition 5.3 there exists an open set $V_2 \subset V_i$ such that

$$W^s(p_{i+2}(g)) \subset cl(W^s(p_{i+1}(g)))$$

for every $g \in V_2$.

Repeating this argument finitely many times we will find open sets $V_k \subset \cdots \subset V_1$ such that

$$W^s(p_{i+j}(g)) \subset cl(W^s(p_{i+j-1}(g)))$$

for every $g \in V_i$ and $j = 1, \ldots, k - i$.

Taking $V = V_{k-1}$, the result follows. □

Finally, we give a proof of theorem 5.1.

**Proof of theorem 5.1.** Since $p_s$ and $p_u$ are hyperbolic periodic points, we take $U$ small enough such that every diffeomorphism $g \in U$ has defined the continuations $p_s(g)$ and $p_u(g)$. Reducing $U$ if necessary, we could also assume that every $g \in U$ is a strong partially hyperbolic diffeomorphism with the same extremal bundles dimension as in the partially hyperbolic decomposition of $TM$ for $f$, which follows by the continuity of the partially hyperbolicity and the existence of $p_s$ and $p_u$ robustly.

Now, using theorem 3.2 together with theorem 3.4 we can find a residual subset $\mathcal{R}$ in $U$ such that $M$ coincides with a homoclinic class for every $g \in \mathcal{R}$, and moreover $g$ has hyperbolic periodic points of any index in $[s, d - u] \cap \mathbb{N}$.

We fix $g \in \mathcal{R}$, and let $p_j = p_j(g), p_{j+1}, \ldots, p_{k-u} = p_u(g)$ be hyperbolic periodic points of $g$ with indices $s, s+1, \ldots, d-u$, respectively. Also, for all $n \in \mathbb{N}$, let $V_n \subset U$ small neighborhoods of $g$, such that if $g_n \in V_n$, then $g_n$ converges to $g$ in the $C^1$-topology, when $n$ goes to infinity.

Now, since $g$ is still a robustly transitive strong partially hyperbolic diffeomorphism having hyperbolic periodic points of all possible indices, we denote by $\tilde{V}_n \subset V_n$ the open sets given for $g$ and $V_n$ by lemma 5.2. Hence, using the invariance of the stable manifold of hyperbolic periodic points, by lemma 5.2 we have the following:

$$cl(W^s(O(p_j(r)))) \subset cl(W^s(O(p_{j-u}(r)))) \subset \cdots \subset cl(W^s(O(p_s(r))))$$

for every $r \in \tilde{V}_n$.

**Claim.** $W^s(O(p_j(r)))$ and $W^u(O(p_{k-u}(r)))$ are dense in $M$, for every $r \in \tilde{V}_n$.

Since $r$ is transitive, there exist $x \in M$ such that the forward orbit of $x$ is dense in $M$. Now, since $r$ is partially hyperbolic, for remark 2.1 there exists the strong stable foliation that integrates the direction $E^s$. Moreover, these leaves have a local uniform length. Hence, we can take $r^t(x)$ close enough to $p_j(r)$ such that $W^u(x)$, the strong stable leaf containing $x$, intersects the local unstable manifold of $p_j(r), W^u_{loc}(p_j(r))$. Therefore, since points in the same strong stable leaf have the same omega limit set, we have that $W^s(O(p_j(r)))$ is dense in the whole manifold $M$. We can repeat this argument also using the existence of a point $y$ having a dense backward
orbit, and the existence of the strong unstable foliation to conclude that $W^s(O(p_{t-u}(r)))$ is also dense in $M$. Thus, note that by equation (1), the Claim also implies that $W^s(O(p_t(r)))$ and $W^u(O(p_{t-u}(r)))$ is dense in $M$. Now, since $r$ is strong partially hyperbolic, it is easy to conclude from the density of $W^s(O(p_t(r)))$ and $W^u(O(p_t(r)))$, for $i = s$ and $d - u$, that

$$M = H(p_t(r)) = H(p_{t-u}(r)),$$

provided that $W^s(O(p_t))$ and $W^u(O(p_t))$ are invariant manifolds of $f$, and moreover $\text{ind } p_s = s$ and $\text{ind } p_u = d - u$.

Hence, the proof is finished defining $\mathring{V}_g = \bigcup_{g \in \mathcal{R}} \mathring{V}_g$, and

$$V_f = \bigcup_{g \in \mathcal{R}} \mathring{V}_g,$$

which is an open and dense subset of $U$, and hence contains $f$ in its closure. □

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