ULRICH BUNDLES ON BLOWUPS

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Abstract. We construct an Ulrich bundle on the blowup at a point when the original variety is embedded by a sufficiently positive linear system and carries an Ulrich bundle. In particular, we describe the relation between special Ulrich bundles on the blown-up surfaces and the original surfaces.

Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension $n$, embedded by a complete linear system $|O_X(H)|$ for some very ample divisor $H$. An Ulrich bundle on $X$ \cite{Kim08} is a vector bundle $\mathcal{F}$ on $X$ whose twists satisfy a set of vanishing conditions on cohomology

$$H^i(X, \mathcal{F}(-jH)) = 0 \text{ for all } i \text{ and } 1 \leq j \leq n.$$  

Ulrich bundles appeared in commutative algebra in relation with maximally generated maximal Cohen-Macaulay modules \cite{Ulrich}. In algebraic geometry, the notion of Ulrich bundles surprisingly appeared thanks to recent works by Beauville and Eisenbud-Schreyer. The importance is motivated by the relations between the Cayley-Chow forms \cite{Cayley, Chow} and with the cohomology tables \cite{Cohomology}.

Eisenbud and Schreyer made a conjecture that every projective variety admits an Ulrich bundle \cite{Kim08}, which is wildly open even for smooth surfaces. The answer is known for a few cases including: curves \cite{Kim08}, complete intersections \cite{Complete}, Grassmannians \cite{Grassmannian}, del Pezzo surfaces \cite{Kim08} and more rational surfaces with an anticanonical pencil \cite{Anticanonical}, general K3 surfaces \cite{Kim08}, abelian surfaces \cite{Abelian}, Fano polarized Enriques surfaces \cite{Fano}, and surfaces with $q = p_g = 0$ embedded by a sufficiently large linear system \cite{Kollar}.  

In classical algebraic geometry, there are 2 fundamental operations, namely, the hyperplane cut and the linear projection. It is well known that the restriction of an Ulrich bundle to a general hyperplane section is also an Ulrich bundle (cf. \cite{Kim06}). It is also straightforward that the vanishing conditions do not affect on taking a linear projection from a point $P \in \mathbb{P}^N \setminus X$. Hence, the only interesting case occurs from the “projection” from a point inside of $X$ which can be realized as the blowup at a point.

We briefly review the relation between inner projections and blowups. Let $P \in X$ be a point. The linear projection from $P$ gives a rational map $\pi_P : X \dasharrow \mathbb{P}^{N-1}$ defined on $X \setminus \{P\}$. We can eliminate the point of indeterminacy by taking the blow-up $\sigma : \tilde{X} \to X$ at $P$. The complete linear system $|\sigma^*O_X(H) \otimes O_{\tilde{X}}(-E)|$ induces a morphism from $\tilde{X}$ to $\mathbb{P}^{N-1}$ whose image is the closure of $\pi_P(X \setminus \{P\})$, where $E = \sigma^{-1}(P)$ is the exceptional divisor.

In this short note, we construct an Ulrich bundle on the blowup at a point from an Ulrich bundle on the original variety.

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Theorem 1. Assume furthermore that the divisor $\hat{H} := \sigma^*H - E$ is very ample. Suppose we have an Ulrich bundle $\mathcal{F}$ on $X$ with respect to the polarization $\mathcal{O}_X(H)$. Then the vector bundle

$$\tilde{\mathcal{F}} := \sigma^*\mathcal{F} \otimes \mathcal{O}_X(-E)$$

is an Ulrich vector bundle on $\tilde{X}$ with respect to $\mathcal{O}_X(\hat{H})$.

Proof. We have to show that $\tilde{\mathcal{F}}(-j\hat{H}) = \sigma^*(\mathcal{F}(-jH)) \otimes \mathcal{O}_X((j-1)E)$ has no cohomology for every $1 \leq j \leq n$. Note that the push-forward $\sigma_*\mathcal{O}_X(jE) = \mathcal{O}_X$ for every $j \geq 0$. We first claim that the higher direct image $R^i\sigma_*\mathcal{O}_X((j-1)E) = 0$ for every $i > 0$ and $1 \leq j \leq n$. It is enough to show that the stalk vanishes at every point $Q \in X$, which can be computed from the inverse limit

$$(R^i\sigma_*\mathcal{O}_X((j-1)E))^\wedge_Q = \begin{cases} 0 & Q \neq P, \\ \lim H^i(mE, \mathcal{O}_{mE}((j-1)E)) & Q = P. \end{cases}$$

By the short exact sequence

$$0 \to \mathcal{O}_E((-m-1)E) \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(m-1) \to \mathcal{O}_{mE} \to \mathcal{O}_{(m-1)E} \to 0,$$

we have

$$H^i(mE, \mathcal{O}_{mE}((j-1)E)) \simeq H^i((m-1)E, \mathcal{O}_{(m-1)E}((j-1)E))$$

for any $i > 0$, $m \geq 1$ and $1 \leq j \leq n$. Applying the projection formula, we have

$$R^i\sigma_*\tilde{\mathcal{F}}(-j\hat{H})) = \mathcal{F}(-jH) \otimes R^i\sigma_*\mathcal{O}_X((j-1)E) = 0$$

for every $i > 0$ and $1 \leq j \leq n$. Hence, Leray spectral sequence implies that the cohomology group

$$H^i(\tilde{X}, \tilde{\mathcal{F}}(-j\hat{H})) \simeq H^i(X, \sigma_*(\tilde{\mathcal{F}}(-j\hat{H}))) \simeq H^i(X, \mathcal{F}(-jH) \otimes \sigma_*\mathcal{O}_X((j-1)E)) = H^i(X, \mathcal{F}(-jH)) = 0$$

vanishes for every $i$ and $1 \leq j \leq n$, since $\mathcal{F}$ is Ulrich on $(X, H)$. Therefore, we conclude that $\tilde{\mathcal{F}}$ is an Ulrich vector bundle on $(\tilde{X}, \hat{H})$. \hfill $\square$

Particularly interesting case happens when $X$ is a smooth surface. The above construction provides an Ulrich bundle on blown-up surfaces at a few points, by taking consecutive inner projections. Moreover, the procedure also provides a direct application on “special Ulrich bundles”. Eisenbud and Schreyer introduced the notion of special Ulrich bundles on a surface $X$ \cite{EisenbudSchreyer2005}, which are Ulrich bundles $\mathcal{F}$ of rank 2 such that $\det \mathcal{F} = \mathcal{O}_X(K_X + 3H)$, where $K_X$ denotes the canonical divisor of $X$. The existence of special Ulrich bundles yields a very nice presentation of the Cayley-Chow form of $X$, indeed, $X$ admits a Pfaffian Bézout form in Plücker coordinates \cite{EisenbudSchreyer2005}. As an immediate consequence, the procedure with a special Ulrich bundle gives rise to a special Ulrich bundle on the upstairs:
Corollary 2. Let $(X, H)$ be a smooth polarized surface satisfies the assumptions in Theorem 1. If $F$ is a special Ulrich bundle on $X$, then $\tilde{F}$ is also a special Ulrich bundle on $\tilde{X}$.

Proof. It comes from a direct computation
\[ \det \tilde{F} = \sigma^*(\mathcal{O}_X(K_X + 3H)) \otimes \mathcal{O}_{\tilde{X}}(-2E) = \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + 3\tilde{H}). \]

It is also possible to construct a special Ulrich bundle in the converse direction, which reveals the connection between special Ulrich bundles on the upstairs and downstairs:

Theorem 3. Let $(X, H)$ be a smooth polarized surface satisfies the assumptions in Theorem 1 as above. Let $\tilde{F}$ be a special Ulrich bundle on $(\tilde{X}, \tilde{H})$. Then $\sigma_* (\tilde{F}(E))$ is a special Ulrich bundle on $(X, H)$.

Proof. We first claim that $\sigma_*(\tilde{F}(E))$ is a vector bundle on $X$. Since $c_1(\tilde{F}(E)) = K_{\tilde{X}} + 3\tilde{H} + 2E = \sigma^*(K_X + 3H)$, we have $\deg \tilde{F}(E) = 0$. By Grothendieck’s theorem, we have $\tilde{F}(E)|_E \simeq \mathcal{O}_E(a) \oplus \mathcal{O}_E(-a)$ for some $a \geq 0$. Note that $\tilde{F}$ is globally generated since it is 0-regular with respect to $\tilde{H}$. Hence the restriction $\tilde{F}|_E \simeq \mathcal{O}_E(a + 1) \oplus \mathcal{O}_E(-a + 1)$ is also globally generated, so either $a = 0$ or $a = 1$ holds. For the both cases, we obtain $h^0(E, \tilde{F}(E)|_E) = 2$. Therefore $\sigma^{-1}(Q, \tilde{F}(E)|_{\sigma^{-1}(Q)}) = 2$ holds for every $Q \in X$, which implies that $\sigma_*(\tilde{F}(E))$ is locally free of rank 2 by Grauert’s theorem. Also note that a similar computation induces that $R^1\sigma_*(\tilde{F}(E)) = 0$ since $\tilde{F}(E)|_E$ has vanishing $H^1$.

To prove $\sigma_*(\tilde{F}(E))$ is a special Ulrich bundle, it is enough to show that $\det \sigma_*(\tilde{F}(E)) \simeq \mathcal{O}_X(K_X + 3H)$ and partial vanishing conditions $H^*(X, \sigma_*(\tilde{F}(E)) \otimes \mathcal{O}_X(−H)) = 0$.

Indeed, for those vector bundles, we have
\[
H^i(X, \sigma_*(\tilde{F}(E)) \otimes \mathcal{O}_X(−2H)) = H^{2−i}(X, \sigma_*(\tilde{F}(E))^* \otimes \mathcal{O}_X(K_X + 2H))^* = H^{2−i}(X, \sigma_*(\tilde{F}(E)) \otimes \mathcal{O}_X(−H))^* = 0
\]

by Serre duality.

Note that the determinant of a coherent sheaf $G$ is the alternating product $\bigotimes \det(\mathcal{E}_i)^{−1}$, where $\mathcal{E}_i$ define
\[
0 \to \mathcal{E}_r \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to G \to 0
\]
a finite locally free resolution of $G$. It is well-known that such a locally free resolution always exists on a smooth variety, and the determinant is equal to the structure sheaf when $G$ is supported on a subset of codimension at least 2 (cf. [10]). Let $V \subset H^0(\tilde{X}, \tilde{F})$ be a general subspace of dimension 3. Since $\tilde{F}$ is globally generated, the evaluation map $ev : V \otimes \mathcal{O}_{\tilde{X}} \to \tilde{F}$ is surjective possibly except for finitely many points. Hence we have an exact sequence
\[
0 \to \mathcal{O}_{\tilde{X}}(−K_{\tilde{X}}−3\tilde{H}) \simeq \sigma^* \mathcal{O}_X(−K_X−3H) \otimes \mathcal{O}_{\tilde{X}}(2E) \to V \otimes \mathcal{O}_{\tilde{X}} \xrightarrow{ev} \tilde{F} \to R_Z \to 0.
\]
Here, $R_Z = (\ker ev)$ is supported on a finite set of points $Z \subset \tilde{X}$. Since $R_Z$ and $R^1\sigma_*$ terms have supports of codimension at least 2, they don’t affect on
the determinant computation. Twisting by $\mathcal{O}_X(E)$ and taking push-forward, we conclude that

$$\det \sigma_* (\tilde{F}(E)) = \det(V \otimes \sigma_*(\mathcal{O}_X(E))) \otimes (\det \sigma_*(\sigma^*\mathcal{O}_X(-K_X - 3H) \otimes \mathcal{O}_X(3E)))^*$$

$$= (\wedge^3 V \otimes \mathcal{O}_X) \otimes \mathcal{O}_X(-K_X - 3H)^*$$

$$= \mathcal{O}_X(K_X + 3H)$$

as desired.

Apply the projection formula and Leray spectral sequence, we have

$$H^i(X, \sigma_*(\tilde{F}(E)) \otimes \mathcal{O}_X(-H)) \simeq H^i(\tilde{X}, \tilde{F}(E) \otimes \sigma^*\mathcal{O}_X(-H))$$

$$= H^i(\tilde{X}, \tilde{F}(-H))$$

$$= 0$$

which completes the proof. \qed

**Remark 4.** When $X$ is a smooth regular surface, Noma found an equivalence condition for the very ampleness of the analogous line bundle in terms of the position of points for the blowup center [13].

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