THE WIDOM-ROWLINSON MODEL, THE HARD-CORE MODEL
AND THE EXTREMALITY OF THE COMPLETE GRAPH

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Abstract. Let $H_{WR}$ be the path on 3 vertices with a loop at each vertex. D. Galvin [4, 5] conjectured, and E. Cohen, W. Perkins and P. Tetali [2] proved that for any $d$-regular simple graph $G$ on $n$ vertices we have
\[
\text{hom}(G, H_{WR}) \leq \text{hom}(K_{d+1}, H_{WR})^{n/(d+1)}.
\]
In this paper we give a short proof of this theorem together with the proof of a conjecture of Cohen, Perkins and Tetali [2]. Our main tool is a simple bijection between the Widom-Rowlinson model and the hard-core model on another graph. We also give a large class of graphs $H$ for which we have
\[
\text{hom}(G, H) \leq \text{hom}(K_{d+1}, H)^{n/(d+1)}.
\]
In particular, we show that the above inequality holds if $H$ is a path or a cycle of even length at least 6 with loops at every vertex.

1. Introduction

For graphs $G$ and $H$, with vertex and edge sets $V_G, E_G, V_H, E_H$ respectively, a map $\varphi : V_G \to V_H$ is a homomorphism if $(\varphi(u), \varphi(v)) \in E_H$ whenever $(u, v) \in E_G$. The number of homomorphisms from $G$ to $H$ is denoted by $\text{hom}(G, H)$. When $H = H_{\text{ind}}$, an edge with a loop at one end, homomorphisms from $G$ to $H_{\text{ind}}$ correspond to independent sets in the graph $G$, and so $\text{hom}(G, H_{\text{ind}})$ counts the number of independent sets in $G$.

For a given $H$, the set of homomorphisms from $G$ to $H$ correspond to valid configurations in a corresponding statistical physics model with hard constraints (forbidden local configurations). The independent sets of $G$ are the valid configurations of the hard-core model on $G$, a model of a random independent set from a graph. Another notable case is when $H = H_{WR}$, a path on 3 vertices with a loop at each vertex. In this case, we can imagine a homomorphism from $G$ to $H_{WR}$ as a 3-coloring of the vertex set of $G$ subject to the requirement that a blue and a red vertex cannot be adjacent (with white vertices considered unoccupied); such a coloring is called a Widom-Rowlinson configuration of $G$, from the Widom-Rowlinson model of two particle types which repulse each other [12, 1]. See Figure 1.

For a fixed graph $H$, it is natural to study the normalized graph parameter
\[
\rho_H(G) := \frac{\text{hom}(G, H)^{1/|V_G|}},
\]
where $V_G$ denotes the number of vertices of the graph $G$.

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For $H = H_{\text{ind}}$, J. Kahn [7] proved that for any $d$-regular bipartite graph $G$, 
\[ p_{H_{\text{ind}}}(G) \leq p_{H_{\text{ind}}}(K_{d,d}), \]
where $K_{d,d}$ is the complete bipartite graph with classes of size $d$. Y. Zhao [10] showed that one could drop the condition of bipartiteness in Kahn’s theorem. That is, he showed that 
\[ p_{H_{\text{ind}}}(G) \leq p_{H_{\text{ind}}}(K_{d,d}), \]
for any $d$-regular graph $G$. Y. Zhao proved his result by reducing the general case to the bipartite case with a clever trick. He proved that 
\[ p_{H_{\text{ind}}}(G) \leq p_{H_{\text{ind}}}(G \times K_2), \]
where $G \times K_2$ is the bipartite graph obtained by replacing every vertex $u$ of $V_G$ by a pair of vertices $(u,0)$ and $(u,1)$ and replacing every edge $(u,v) \in E_G$ by the pair of edges $((u,0),(v,1))$ and $((u,1),(v,0))$. This is clearly a bipartite graph, and if $G$ is $d$-regular then $G \times K_2$ is still $d$-regular.

D. Galvin [4, 5] conjectured a different behavior for $H = H_{\text{WR}}$: that instead of $K_{d,d}$, the complete graph $K_{d+1}$ maximizes $p_{H_{\text{ind}}}(G)$ among $d$-regular graphs $G$. E. Cohen, W. Perkins and P. Tetali [2] proved that this was indeed the case:

**Theorem 1.1.** [2] For any $d$-regular simple graph $G$ on $n$ vertices we have 
\[ p_{H_{\text{WR}}}(G) \leq p_{H_{\text{WR}}}(K_{d+1}); \]
in other words, 
\[ \text{hom}(G,H_{\text{WR}}) \leq \text{hom}(K_{d+1},H_{\text{WR}})^{n/(d+1)}. \]

One of the goals of this paper is to give a very simple proof of this fact\(^1\), along with a slight generalization. We use a trick similar to that used by Y. Zhao [10, 11]. We will need the following definition:

**Definition 1.2.** The extended line graph $\tilde{H}$ of a (bipartite) graph $H$ has $V_{\tilde{H}} = E_H$; two edges $e$ and $f$ of $H$ are adjacent in $\tilde{H}$ if
(a) $e = f$,
(b) $e$ and $f$ share a common vertex, or
(c) $e$ and $f$ are opposite edges of a 4-cycle in $G$.

Throughout, $V_H$ and $E_H$ refer to the vertex-set and edge-set, respectively, of the graph $H$. If $H$ is bipartite, we use $A_H$ and $B_H$ to refer to the parts of a fixed bipartition. Now we can give a generalization of Theorem 1.1:

**Theorem 1.3.** If $\tilde{H}$ is the extended line graph of a bipartite graph $H$, then for any $d$-regular simple graph $G$ on $n$ vertices we have 
\[ p_{\tilde{H}}(G) \leq p_{\tilde{H}}(K_{d+1}), \]

\(^1\)In fact, Theorem 1.1 follows from a stronger result in [2] that the Widom-Rowlinson occupancy fraction is maximized by $K_{d+1}$. We note that this stronger result also follows from the transformation below and Theorem 1 of [3].
or in other words,
\[ \text{hom}(G, \tilde{H}) \leq \text{hom}(K_{d+1}, \tilde{H})^{n/(d+1)}. \]

To see that Theorem 1.3 is a generalization of Theorem 1.1 it suffices to check that \( H_{WR} \) is precisely the extended line graph of the path on 4 vertices. In Section 3 we will prove a slight generalization of Theorem 1.3 which allows for weights on the vertices of \( H \).

2. Short proof of Theorem 1.1

We are not the first to notice the following connection between the Widom-Rowlinson model and the hardcore model (see, e.g., Section 5 of [1]): Given a graph \( G \), let \( G' \) be the bipartite graph with vertex set \( V_{G'} = V_G \times \{0, 1\} \), where \((u, 0)\) and \((v, 1)\) are adjacent in \( G' \) whenever either \((u, v)\) \( \in E_G \) or \( u = v \). That is, \( G' \) is \( G \times K_2 \) with the extra edges \(((u, 0), (u, 1))\) for all \( u \in V_G \). We will show that
\[ \text{hom}(G, H_{WR}) = \text{hom}(G', H_{ind}). \]

Indeed, consider an independent set \( I \) in \( G' \). Color \( u \in V_G \) blue if \((u, 1)\) \( \in I \), red if \((u, 0)\) \( \in I \), and white if it is neither red or blue. Note that since \( I \) was an independent set and \(((u, 0), (u, 1))\) \( \in E_{G'} \), the color of vertex \( u \) is well-defined and this coloring is in fact a Widom-Rowlinson coloring of \( G \). This same construction also works in the other direction, so
\[ \text{hom}(G, H_{WR}) = \text{hom}(G', H_{ind}). \]

If \( G \) is \( d \)-regular then \( G' \) is \((d + 1)\)-regular, and \( K_{d+1}' = K_{d+1, d+1} \). Applying J. Kahn’s result [7] for \((d + 1)\)-regular bipartite graphs, we see that if \( G \) has \( n \) vertices then
\[ \text{hom}(G, H_{WR}) = \text{hom}(G', H_{ind}) \leq \text{hom}(K_{d+1, d+1}, H_{ind})^{2n/(2(d+1))} = \text{hom}(K_{d+1}, H_{WR})^{n/(d+1)}. \]

We remark that the transformation \( G \to G' \) is also mentioned in [8].

3. Extension

In this section we would like to point out that for every graph \( H \) there is an \( \tilde{H} \) such that
\[ \text{hom}(G, \tilde{H}) = \text{hom}(G', H), \]
where \( G' \) is the bipartite graph defined in the previous section. Exactly the same argument we used for \( H_{WR} \) will work for any graph \( \tilde{H} \) constructed in this manner. Actually, the situation is even better. To give the most general version we need a definition.

Definition 3.1. Let \( G \) be a bipartite graph. Let \( H \) be another bipartite graph equipped with a weight function \( \nu : V_H \to \mathbb{R}_+ \). Let \( \mathbb{I}_{E_H} : A_H \times B_H \to \{0, 1\} \) denote the characteristic function of \( E_H \). Define
\[ Z_0(G, H) = \sum_{\varphi : V_G \to V_H} \prod_{(a, b) \in E_G} \mathbb{I}_{E_H}(\varphi(a), \varphi(b)) \prod_{w \in V_G} \nu(\varphi(w)), \]
(The subscript $b$ stands for bipartite.) If $G$ and $H$ are not necessarily bipartite graphs, but $H$ is a weighted graph we can still define

$$Z(G, H) = \prod_{\varphi: V_G \rightarrow V_H} \prod_{(u,v) \in E_G} \mathbb{I}_{E_H}(\varphi(u), \varphi(v)) \prod_{w \in V_G} \nu(\varphi(w)).$$

In the language of statistical physics, $Z_b(G, H)$ and $Z(G, H)$ are partition functions.

Somewhat surprisingly, J. Kahn’s result holds even in this general case, as shown by D. Galvin and P. Tetali [6].

**Theorem 3.2.** [6] For any bipartite graph $H$ equipped with the weight function $\nu: V_H \rightarrow \mathbb{R}_+$ and $\mathbb{I}_{E_H}: A_H \times B_H \rightarrow \{0,1\}$, and for any $d$-regular simple graph $G$ on $n$ vertices,

$$Z_b(G, H) \leq Z_b(K_{d,d}, H)^{n/(2d)}.$$  

The key observation is that for a bipartite graph $H$ equipped with the weight function $\nu: V_H \rightarrow \mathbb{R}_+$ and characteristic function $\mathbb{I}_{E_H}: A_H \times B_H \rightarrow \{0,1\}$, we can define a weighted graph $\widetilde{H}$ with weight function $\widetilde{\nu}$ and characteristic function $\mathbb{I}_{E_{\widetilde{H}}}$ such that

$$(3.1) \quad Z(G, \widetilde{H}) = Z_b(G', H),$$

for any graph $G$ (where $G'$ is the modification of $G$ defined in the previous section).

Indeed, construct $\widetilde{H}$ with vertex set $A_H \times B_H$, edges

$$\mathbb{I}_{E_{\widetilde{H}}}((a_1, b_1), (a_2, b_2)) = \mathbb{I}_{E_H}(a_1, b_2) \mathbb{I}_{E_H}(a_2, b_1),$$

and weight function

$$\widetilde{\nu}(a, b) = \nu(a)\nu(b)\mathbb{I}_{E_H}(a, b).$$

In effect, the vertex set of $\widetilde{H}$ is only the edges of $H$ (since non-edge pairs are given weight 0). Now, for a map $\varphi: G' \rightarrow H$, we can consider the map $\widetilde{\varphi}: G \rightarrow \widetilde{H}$ given by

$$\widetilde{\varphi}(u) = (\varphi((u, 0)), \varphi((u, 1))).$$

By the construction of the graphs $G'$ and $\widetilde{H}$, the contribution of $\varphi$ to $Z_b(G, H)$ is the same as the contribution of $\widetilde{\varphi}$ to $Z(G, \widetilde{H})$, and the result (3.1) follows.

Finally, applying Theorem 3.2 to the $(d + 1)$-regular graph $G'$ yields

$$Z(G, \widetilde{H}) = Z_b(G', H) \leq Z_b(K_{d,d}, H)^{2n/(2d(d+1))} = Z(K_{d+1, \widetilde{H}})^{n/(d+1)}.$$  

Hence we have proved the following theorem.

**Theorem 3.3.** For a bipartite graph $H = (A, B, E)$ with vertex weight function $\nu: V_H \rightarrow \mathbb{R}_+$ let $\widetilde{H}$ be the following weighted graph: its vertex set is $E(H)$, its edge set is defined by $((a_1, b_1), (a_2, b_2)) \in E(\widetilde{H})$ if and only if $(a_1, b_2) \in E(H)$ and $(a_2, b_1) \in E(H)$, and the weight function on the vertex set is $\widetilde{\nu}(a, b) = \nu(a)\nu(b)$ for $(a, b) \in E(H)$. Then for any $d$-regular simple graph $G$ on $n$ vertices we have

$$Z(G, \widetilde{H}) \leq Z(K_{d+1, \widetilde{H}})^{n/(d+1)}.$$  

We can obtain Conjecture 3 of [2] as a corollary by applying this theorem in the case where $H$ is the path on 4 vertices, $a_1a_2a_2a_2$, with appropriate vertex weights. Indeed, if $\nu(a_1) = 1$, $\nu(b_1) = \lambda_b$, $\nu(a_2) = \frac{\lambda}{\lambda_b}$, $\nu(b_2) = \frac{\lambda_b}{\lambda}$, then $\widetilde{H}$ is precisely the
Widom-Rowlinson graph with vertex weights $\lambda_b, \lambda_r, \lambda_w$. This proves that even for the vertex-weighted Widom-Rowlinson graph we have

$$Z(G, H_{WR}) \leq Z(K_{d+1}, H_{WR})^{n/((d+1)).}$$

Hence we have proved the following theorem.

**Theorem 3.4.** Let $H_{WR}$ be the Widom-Rowlinson graph with vertex weights $\lambda_b, \lambda_w, \lambda_r$. Then for any $d$–regular simple graph $G$ on $n$ vertices we have

$$Z(G, H_{WR}) \leq Z(K_{d+1}, H_{WR})^{n/((d+1)).}$$

Now let us consider the special case when $H$ is unweighted ($\nu \equiv 1$). In this case $\tilde{\nu}$ is just $\mathbb{I}_{E_H}$, so we can think of $\tilde{H}$ as an unweighted graph with vertex set $V_{\tilde{H}} = E_H$. There is an edge in $\tilde{H}$ between edges $e = (a_1, b_1)$ and $f = (a_2, b_2)$ of $H$ whenever $(a_1, b_2)$ and $(a_2, b_1)$ are both also edges of $H$. This is always the case when either $a_1 = a_2$ or $b_1 = b_2$, so in particular every edge $e \in E_{\tilde{H}} = V_{\tilde{H}}$ has a self-loop in $\tilde{H}$, and every pair of incident edges in $H$ are adjacent in $\tilde{H}$. We also get an edge $(e, f) \in E_{\tilde{H}}$ if four vertices $a_1 b_1 a_2 b_2$ are all distinct and form a 4-cycle with $e$ and $f$ as opposite edges. In other words, $\tilde{H}$ is precisely the extended line graph of $H$. Hence as a corollary of Theorem 3.3 we have proved Theorem 1.3.

If $H$ does not contain any 4-cycle, then $\tilde{H}$ is simply the line graph of $H$ with loops at every vertex. In particular, if $H$ is a path (or even cycle of length at least 6) then $\tilde{H}$ is again a path (or even cycle of length at least 6), but now with a loop at every vertex. Letting $H^\circ$ denote the graph obtained by adding a loop at every vertex of the graph $H$, we can write the corollary

**Corollary 3.5.** If $H = C_k^\circ$ (for $k \geq 6$ even) or if $H = P_k^\circ$ (for any $k$), then for any $d$-regular graph $G$

$$p_H(G) \leq p_H(K_{d+1}).$$

It is a good question how to characterize all of the graphs $\tilde{H}$ which can be obtained this way. Note that since $\tilde{H}$ is always fully-looped, this class has no intersection with the class of graphs found by Galvin [4]: the set of graphs $H_q^\ell$ obtained from a complete looped graph on $q$ vertices with $\ell \geq 1$ loops deleted.

**Remark 3.6.** Let $S_k$ be the star on $k$ vertices. One can show (for details see [4]) that, for large enough $d$,

$$p_{S_k}(K_{d+1}) < p_{S_k}(K_{d,d})$$

for $k \geq 6$. From this example we can see that in order to have $p_H(G) \leq p_H(K_{d+1})$ it is not sufficient merely for $H$ to have a loop at every vertex.

L. Sernau [9] introduced many ideas for extending certain inequalities to a larger class of graphs. For instance, recall that the $H_1 \times H_2$ has $V_{H_1 \times H_2} = V_{H_1} \times V_{H_2}$ and $\{(a_1, b_1), (a_2, b_2)\} \in E_{H_1 \times H_2}$ if and only if $(a_1, a_2) \in E_{H_1}$ and $(b_1, b_2) \in E_{H_2}$. Sernau noted that if $H_1$ and $H_2$ are graphs such that

$$p_{H_i}(G) \leq p_{H_i}(K_{d+1}),$$

for $i = 1, 2$, then it is also true that

$$p_{H_1 \times H_2}(G) \leq p_{H_1 \times H_2}(K_{d+1}).$$

This inequality simply follows from the identity

$$\text{hom}(G, H_1 \times H_2) = \text{hom}(G, H_1) \text{hom}(G, H_2),$$

where $\text{hom}$ denotes the homomorphism relation.
which is explained in [9]. Surprisingly, this observation does not allow us to extend our result to any new graphs, because the product of two extended line graphs is again an extended line graph:

\[ \tilde{H}_1 \times \tilde{H}_2 = \tilde{H}_{12}, \]

where \( H_{12} = (A_{H_1} \times A_{H_2}, B_{H_1} \times B_{H_2}, E_{H_1} \times E_{H_2}) \).

4. On a theorem of L. Sernau

Theorem 3 of [9] also provides a class of graphs for which \( K_{d+1} \) is the maximizing graph. Below we explain the relationships between our results and his theorem.

**Definition 4.1.** Let \( H \) and \( A \) be graphs. Then the graph \( H^A \) is defined as follows: its vertices are the maps \( f : V(A) \to V(H) \) and the \( (f_1, f_2) \in E(H^A) \) if \( (f_1(u), f_2(v)) \in E(H) \) whenever \( (u, v) \in E(A) \).

Then Sernau proved the following theorem.

**Theorem 4.2.** [9] Let \( G \) be a \( d \)-regular graph, and let \( F = l(H^B) \), where \( H \) is an arbitrary graph, \( B \) is a bipartite graph, and \( l(H^B) \) is the graph induced by the vertices of \( H^B \) which have a loop. Then

\[ p_F(G) \leq p_F(K_{d+1}). \]

When \( H = H_{\text{ind}}, B = K_2 \) then \( l(H^B) = H_{\text{WR}} \) so this also proves the conjecture of D. Galvin. Note that when \( B = K_2 \) then \( l(H^B) \) is the extended line graph of \( H \times K_2 \). It is not a great surprise that these results are similar, even the proofs behind these results are strongly related to each other.

5. Conjectures

Let \( H \) be a simple graph, i.e., with no multiple edges or loops. Let \( H^o \) denote the graph obtained by adding a loop at each vertex of \( H \) (so for instance \( C_n^o \) denotes the \( n \)-cycle with a loop at each vertex).

**Conjecture 5.1.** Let \( G \) be a \( d \)-regular simple graph. Then for any \( n \geq 4 \)

\[ p_{C_n^o}(G) \leq p_{C_n^o}(K_{d+1}). \]

**Conjecture 5.2.** Let \( G \) be a \( d \)-regular simple graph. Then for any \( d \geq 4 \)

\[ p_{S_k^d}(G) \leq p_{S_k^d}(K_{d+1}). \]

Furthermore, for \( k \geq 6 \)

\[ p_{S_k^d}(G) \leq p_{S_k^d}(K_{d,d}). \]

Finally, for an arbitrary graph \( H \) it is not clear how to characterize the maximizers over all \( d \)-regular graphs \( G \) of \( p_H(G) \). If we restrict to bipartite \( G \), however, D. Galvin and P. Tetali proved that \( p_H(G) \leq p_H(K_{d,d}) \) [6]. We conjecture that this can be extended to the class of triangle-free graphs.

**Conjecture 5.3.** Let \( G \) be a \( d \)-regular triangle-free graph. Then for any graph \( H \) we have

\[ p_H(G) \leq p_H(K_{d,d}). \]

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