Generalized Compactification and Assisted Dynamics of Multi–Scalar Field Cosmologies

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Cosmological models arising from a generalized compactification of Einstein gravity are derived. It is shown that a redefinition of the moduli fields reduces the system to a set of massless fields and a single field with a single exponential potential, independent of the background spacetime. This solution is the unique late–time attractor for an arbitrary spacetime dimensionality. We find that if the number of dimensions is greater than or equal to seven, the scalar fields dominate a relativistic fluid and therefore constitute a potential ‘moduli’ problem.

In standard Kaluza–Klein compactifications it is assumed that the higher–dimensional fields are independent of the compactified coordinates [1]. However, when the action exhibits a global symmetry, consistent lower–dimensional theories may be derived from the more general ansatz of Scherk and Schwarz [2]. The simplest example is the compactification on a circle of a theory containing a massless axion field. Since the field arises in the action only through its derivative, the global symmetry corresponds to a linear shift in its value. A consistent compactification is then possible if the axion has a non–interacting scalar fields, \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m) \), each with an exponential potential, \( V_i(\varphi_i) \propto \exp(-\varphi_i/r_i) \), where \( r_i \) is a constant [3].

The late–time attractor for the spatially flat, Friedmann–Robertson–Walker (FRW) cosmology is then of the form \( a \propto t^\beta \), where \( \beta = \sum_{i=1}^m r_i \) [4]. Thus, inflation is possible, \( \beta > 1 \), even if the potentials are individually too steep to drive inflation (\( r_i < 1 \)).

Cosmologies arising from a generalized compactification of vacuum Einstein gravity minimally coupled to a massless scalar field, \( \Phi \):

\[ S = \int d^{D+1}x \sqrt{-\tilde{g}_{D+1}} \left[ \tilde{R} - \frac{1}{2} \left( \tilde{\nabla} \Phi \right)^2 \right], \]

where \( \tilde{R} \) is the Ricci scalar of the spacetime with metric \( \tilde{g}_{MN} \) and \( \tilde{g} \equiv \det \tilde{g}_{MN} \). Compactification onto a circle may be parametrised in terms of the metric [5]

\[ ds^2_{D+1} = e^{2\alpha z} ds^2_D + e^{-2(D-2)\alpha z} (dz + A_\mu dx^\mu)^2, \]

where \( z \) represents the coordinate of the compactified dimension, \( A_\mu \) is the gauge potential and the numerical constant [6].

\[ \alpha \equiv -\frac{1}{\sqrt{2(D-1)(D-2)}}. \]

\*In this paper, units are chosen such that \( 16\pi G = \hbar = c = 1 \).

\[ ^1 \text{We assume implicitly that the universe does not recollapse before the attractor solution becomes relevant.} \]
value of $\alpha$ is chosen to ensure that the scalar ‘dilaton’ field, $\varphi$, is minimally coupled to the Einstein–frame metric after compactification. For the ansatz

$$\Phi(x^\mu, z) = \Phi(x^\mu) + mz,$$

the reduced action is given by

$$S = \int d^Dx\sqrt{-g_D} \left[ R - \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{4} \varepsilon^{a\varphi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (D\Phi)^2 - \frac{1}{2} m^2 e^{-a\varphi} \right],$$

where $D_\mu \Phi = \partial_\mu \Phi - m A_\mu$, $F_{\mu\nu} = 2\partial_\mu A_\nu$ is the field strength of $A_\mu$ and $a \equiv \sqrt{2(D-1)/(D-2)}$. Action (3) is invariant under the massive gauge transformation $\delta \Phi = m \chi$ and $\delta A_\mu = \partial_\mu \chi$ and this allows the vector field to gain a mass by absorbing the axion $\Phi$. The dilaton field has an exponential self–interaction potential due to the non–trivial slope parameter, $m$, of the higher–dimensional axion.

The dimensional reduction summarised in Eqs. (2)–(3) may be applied to $D$–dimensional Einstein gravity minimally coupled to $m = D - 4$ massless axion fields, $\Phi$. These fields may be interpreted as moduli fields arising from the compactification of $(D+m)$–dimensional, vacuum Einstein gravity on a rectilinear torus. We may then consider a generalised dimensional reduction on a $(D - 4)$–dimensional torus, $T^m = S^1 \times S^1 \times \ldots \times S^1$, in terms of a series of compactifications on successive circles, $S^1$. For each $S^1$, we assume that one of the massless scalar fields has a linear dependence on the coordinate parametrising the circle and that the remaining fields are independent of this coordinate. The massive vector field that arises in the dimensional reduction may then be consistently zero and the process repeated.

The result is that after compactification to four dimensions, the truncated action contains a set of $(D - 4)$ minimally coupled, dilatonic scalar fields, $\varphi_i$, that parametrise the radii of the compactified coordinates. These fields couple exponentially to each other through $(D - 4)$ potentials that originated from the higher–dimensional axions. The action can therefore be expressed in the form

$$S = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \varphi_i)^2 - \frac{1}{2} \sum_{i=1}^n m_i^2 e^{-c_i \varphi_i} \varphi_i \right],$$

where the constant vectors, $c_i$, parametrise the couplings between the fields and $m_i$ are arbitrary constants. The couplings are determined by induction from Eqs. (2)–(3) and are given by

$$c_i = \left(0, 0, \ldots, 0, (D - 1 - i)s_i, s_{i+1}, s_{i+2}, \ldots, s_{D-4}\right),$$

where

$$s_i \equiv \left[\frac{2}{(D - 1 - i)(D - 2 - i)}\right]^{1/2}.$$

By performing linear translations on the values of the scalar fields, the constants, $m_i$, can be rescaled without loss of generality such that $m_i^2 \equiv M_i^2/n$ for all $i$, where $M$ is a constant. The scalar field equations derived from action (6) are then given by

$$\nabla^2 \varphi_i + \frac{M_i^2}{2n} \sum_{i=1}^n c_i e^{-c_i \varphi_i} = 0.$$

The assisted dynamics between the scalar fields arising in action (1) becomes apparent on performing an appropriate field redefinition; following the analysis of Liu and Pope [1] we rotate the fields, $\varphi$, with respect to a unit vector, $\vec{n}$:

$$\varphi = \varphi \vec{n} + \varphi_\perp,$$

for some constant, $c$. Taking the dot product of Eq. (9) with respect to $\vec{n}$ then implies that

$$\nabla^2 \varphi + \frac{c M_i^2}{2n} e^{-c \varphi} \sum_{i=1}^n e^{-c_i \varphi_\perp} = 0,$$

and substituting Eq. (12) into Eq. (4) implies that

$$\nabla^2 \varphi_\perp + \frac{M_i^2}{2n} e^{-c \varphi} \left[ \sum_{i=1}^n c_i e^{-c_i \varphi_\perp} - cn \sum_{i=1}^n e^{-c_i \varphi_\perp} \right] = 0.$$

It follows that if the unit vector satisfies the constraint

$$\vec{n} = \frac{1}{c} \sum_{i=1}^n c_i,$$

a consistent solution to Eqs. (12) and (13) is given by

$$c_i \varphi_\perp = 0 \quad \forall \quad i,$$

so that

$$\nabla^2 \varphi = \frac{1}{2} c M_i^2 e^{-c \varphi} = 0,$$

$$\nabla^2 \varphi_\perp = 0.$$
\[ S = \int d^4 x \sqrt{-g} \left[ R - \frac{1}{2} \left( \nabla \varphi_{\perp} \right)^2 \right. \\
- \frac{1}{2} \left( \nabla \varphi \right)^2 - \frac{1}{2} M^2 e^{-c \varphi} \] \tag{18}

with the fields perpendicular to \( \vec{n} \) behaving as massless scalar fields.

The numerical value of the coupling, \( c \), is evaluated by taking the dot product of Eq. \( (14) \) with respect to the vector \( \vec{c}_j \). We find that

\[ c^2 = \frac{1}{n} \sum_{i=1}^{n} M_{ij}, \tag{19} \]

where the elements of the symmetric \( n \times n \) matrix, \( M_{ij} \), are determined by the dot products of the couplings between the fields:

\[ M_{ij} = \vec{c}_i \cdot \vec{c}_j. \tag{20} \]

Since Eq. \( (19) \) is independent of \( j \), the couplings must be related in a certain way. For the dimensionally reduced model we have derived, Eqs. \( (7) \) and \( (8) \) imply that

\[ \vec{c}_i \cdot \vec{c}_j = \left\{ \begin{array}{ll} \frac{2}{D-s-1} \sum_{k=1}^{D-s-2} \frac{1}{2} \left( \delta_{D-k}(D-1) \right) \left( \delta_{D-k}(D-1) \right), \\
\frac{2}{D-s-2} \sum_{k=s+1}^{D-s-2} \frac{1}{2} \left( \delta_{D-k}(D-1) \right) \left( \delta_{D-k}(D-1) \right), \\
\end{array} \right. \tag{21} \]

where \( s = \max\{i, j\} \) and the top (bottom) line corresponds to \( i = j \) \((i \neq j)\). Employing the relationship

\[ \sum_{k=1}^{n} \frac{1}{kq + (r - q)[kq + r]} = \frac{n}{r(r + nq)}, \tag{22} \]

and substituting Eq. \( (21) \) into Eq. \( (20) \) then yields the remarkably simple result:

\[ M_{ij} = 2 \delta_{ij} + 1. \tag{23} \]

Hence, Eq. \( (19) \) is satisfied and implies that \( c^2 = (D - 2)/(D - 4) \).

Thus far, we have established that Eq. \( (13) \) is a consistent truncation of action \( (1) \), subject to the constraints \( (1) \) and \( (4) \). The question that now arises is whether it is also the late–time attractor for the general system. This can be determined by taking the dot product of Eq. \( (13) \) with \( \vec{c}_i \) and deriving an effective equation of motion for the fields \( \vec{c}_i \cdot \vec{c}_{\perp} \):

\[ \nabla^2 \left( \vec{c}_j \cdot \vec{c}_{\perp} \right) - \frac{M^2}{2n} e^{-c \varphi} \left[ \left( c^2 - 2 \vec{c}_j \cdot \vec{c}_j \right) e^{-\vec{c}_j \cdot \vec{c}_{\perp}} \right. \\
- \sum_{i \neq j} \left( c^2 - 2 \vec{c}_i \cdot \vec{c}_j \right) e^{-\vec{c}_i \cdot \vec{c}_{\perp}} \right] = 0. \tag{24} \]

We now define the scalar field \( y_{jk} = \vec{c}_j \cdot \vec{c}_{\perp} - \vec{c}_k \cdot \vec{c}_{\perp} \). Subtracting the field equation for \( \vec{c}_k \cdot \vec{c}_{\perp} \) from that for \( \vec{c}_j \cdot \vec{c}_{\perp} \) then yields the equation of motion for \( y_{jk} \):

\[ \nabla^2 y_{jk} - \frac{M^2}{n} e^{-c \varphi} \left( \vec{c}_j \cdot \vec{c}_j + \vec{c}_k \cdot \vec{c}_k \right) e^{-(\vec{c}_j \cdot \vec{c}_{\perp} + \vec{c}_k \cdot \vec{c}_{\perp})/2} \times \left[ \sinh(y_{jk}/2) \right] = 0. \tag{25} \]

The second term in Eq. \( (23) \) may be interpreted as the derivative of an effective potential for \( y_{jk} \). The only critical point in this potential is the global minimum that exists at \( y_{jk} = 0 \). This implies that \( y_{jk} \to 0 \) at late times and, consequently, \( \vec{c}_j \cdot \vec{c}_{\perp} \to \vec{c}_k \cdot \vec{c}_{\perp} \) for all \( j \) and \( k \). However, taking the dot product of Eq. \( (13) \) with \( \vec{c}_{\perp} \) implies that \( \sum_{i=1}^{n} \vec{c}_i \cdot \vec{c}_{\perp} = 0 \) and consequently the \( \vec{c}_j \cdot \vec{c}_{\perp} \) can only be equal if they simultaneously vanish: \( \vec{c}_j \cdot \vec{c}_{\perp} = 0 \). This condition is identical (c.f. Eq. \( (15) \)) to that placed on the solution found above. We may conclude, therefore, that the solution given by Eqs. \( (4) \)–\( (7) \) is the unique late–time attractor for the cosmologies derived from Eq. \( (1) \).

The analytical form of the late–time attractor for the spatially flat, FRW cosmology is known when \( c^2 \leq 3 \) and is given by the power law \( a \propto t^{1/c^2} \). The self–interacting scalar field, \( \varphi \), dominates the massless fields, \( \varphi_{\perp} \), since the latter behave collectively as a stiff perfect fluid. Eq. \( (19) \) implies that \( c^2 \leq 3 \) for an arbitrary dimensionality and the unique late–time attractor for the dimensionally reduced, spatially flat FRW cosmology is therefore given by \( a \propto t^3 \), where

\[ r = 1 - \frac{2}{D - 2}. \tag{26} \]

In the five–dimensional model, where only one dimension is compactified, the solution corresponds to that of a massless scalar field, \( r = 1/3 \). As more dimensions are compactified, the assisted dynamics between the scalar fields becomes apparent and the power of the cosmological expansion increases monotonically with the dimensionality of spacetime. We can conclude from Eq. \( (26) \), however, that assisted inflation does not arise in this compactification scheme regardless of the number of compactified dimensions, since \( D \to \infty \), \( r \to 1 \). It is interesting that this upper limit corresponds precisely to the coasting solution, where the scale factor is neither accelerating nor decelerating. In summary, each compactification of an axion field effectively introduces a new mass parameter, but the corresponding increase in the relative expansion rate of the universe is counter–balanced by the new interactions that also arise between the scalar dilaton fields \( (14) \)–\( (17) \).
fields mimics the behaviour of a relativistic fluid \((r = 1/2, \gamma = 4/3)\). For \(D \geq 7\), however, \(c^2 < 2\) and the scalar fields dominate the radiation component. This includes the compactified model derived from the vacuum limit of M-theory. In particular, when \(D = 8\) the expansion rate is equivalent to that of a universe dominated by pressureless dust. In this region of parameter space, therefore, primordial nucleosynthesis may be significantly affected or may not proceed at all unless the scalar field domination is reversed at a sufficiently early epoch. Consequently, this leads to limits on the duration of scalar field domination in such models.

We may compare our results to those of Copeland, Mazumdar and Nunes [16], who studied the action [6] when restricted to the spatially flat FRW cosmology. These authors found a power law solution, \(a \propto t^r\) \((r > 1/3)\), where the exponent is given by

\[
r = \sum_{i,j=1}^{n} (M^{-1})_{ij},
\]

Since \(M_{ij}\) is non-singular, Eq. (19) implies that \(\sum_{i=1}^{n} (M^{-1})_{ij} = 1/(c^2 n)\). Summing over \(j\) then implies that Eqs. (19) and (27) are equivalent.

The above analysis may be readily extended to include the more general class of models with \(n\) exponential potentials satisfying the simultaneous constraints \(\bar{c}_i\bar{c}_i = \alpha + \beta\) and \(\bar{c}_i\bar{c}_j = \beta (i \neq j)\) for arbitrary constants \(\alpha\) and \(\beta\). Eq. (29) then implies that \(M_{ij} = \alpha \delta_{ij} + \beta\) and from Eq. (13), this model admits a power law solution, where the exponent is given by \(r = n/(\alpha + \beta n)\). The condition for inflation may therefore be deduced directly from the form of the coupling parameters.

In conclusion, we have found a wide class of cosmological solutions arising from a generalized Scherk–Schwarz compactification of vacuum, Einstein gravity. If only one potential was present, each would be sufficiently steep for the moduli fields to track a relativistic fluid. However, the assisted dynamics between the fields implies that they dominate the radiation. Moreover, since the assisted dynamics is insufficient to result in an inflationary expansion, regardless of the dimensionality of the higher-dimensional spacetime, this leads to a potential moduli problem for the early universe.

The power law solution of Ref. [16] arises in the special case of the spatially flat FRW cosmology. We have shown that this solution is the unique late–time attractor. Since our analysis is independent of the spacetime geometry, it can be employed to study multi–field cosmology in the more general setting of spatially anisotropic and inhomogeneous models. For example, particular solutions for a single scalar field in a class of inhomogeneous Einstein–Rosen cosmologies have been found previously [20]. It would be interesting to make a detailed study of the nature of the late–time attractors in these models.

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