CHAPTER 1

NONEXTENSIVE ENTROPY APPROACH TO SPACE PLASMA FLUCTUATIONS AND TURBULENCE

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Spatial intermittency in fully developed turbulence is an established feature of astrophysical plasma fluctuations and in particular apparent in the interplanetary medium by in situ observations. In this situation the classical Boltzmann-Gibbs extensive thermo-statistics, applicable when microscopic interactions and memory are short ranged and the environment is a continuous and differentiable manifold, fails. Upon generalization of the entropy function to nonextensivity, accounting for long-range interactions and thus for correlations in the system, it is demonstrated that the corresponding probability distributions (PDFs) are members of a family of specific power-law distributions. In particular, the resulting theoretical bi-kappa functional reproduces accurately the observed global leptokurtic, non-Gaussian shape of the increment PDFs of characteristic solar wind variables on all scales, where nonlocality in turbulence is controlled via a multi-scale coupling parameter. Gradual decoupling is obtained by enhancing the spatial separation scale corresponding to increasing kappa-values in case of slow solar wind conditions where a Gaussian is approached in the limit of large scales. Contrary, the scaling properties in the high speed solar wind are predominantly governed by the mean energy or variance of the distribution, appearing as second parameter in the theory. The PDFs of solar wind scalar field differences are computed from WIND and ACE data for different time-lags and bulk speeds and analyzed within the nonextensive theory, where also a particular nonlinear dependence of the coupling parameter and variance with scale arises for best fitting theoretical PDFs. Consequently, nonlocality in fluctuations, related to both, turbulence and its large scale driving, should be related to long-range interactions in the context of nonextensive entropy generalization, providing fundamentally the physical background of the observed scale dependence of fluctuations in intermittent space plasmas.
1. Introduction

Leptokurtic, long-tailed probability distribution functions (PDF’s) subject to a non-Gaussian core and pronounced halo are a persistent feature in a variety of different astrophysical environments. Those include the thermo-statistical properties of the interplanetary medium where the electron, proton and even heavy ion velocity space distributions show ubiquitously suprathermal halo patterns (see Mendis 2,3 for a general review or Leubner 4,5,6 and references therein), well described by the empirical family of kappa-distributions, a power law in particle speed, recognized first by Vasyliunas 7.

In continuation, significant progress was provided by Treumann 8,9,10 who developed a kinetic theory, demonstrating that power-law velocity space distributions are a particular thermodynamic equilibrium state. Similarly, scale invariant power-law distributions are manifest in any systems relying on self organized criticality (SOC) (Bak 11, Bak et al. 12, Watkins et al. 13, Chapman etal. 14, Chapman and Watkins 15) or gravitationally bound astrophysical stellar systems (Nakamichi et al. 16, Chavanis and Bouchet 17). Moreover, recently Leubner 18,19 developed a theory representing accurately the hot plasma and dark matter density profiles in galaxies and clusters in the context of scale invariant power-law distributions. In all cases the standard Boltzmann-Gibbs-Shannon (BGS) statistics does not apply.

Remarkably, we have to add to this diversity also the PDF’s of the turbulent fluctuations of the magnetic field strength, density and velocity fields in space and astrophysical plasmas (Leubner and Vörös 20,21). In particular, the analysis of the PDFs of the solar wind plasma is of considerable interest to study intermittency and multi-scale statistical properties in fully developed turbulence, since high resolution in situ observations are available. The characteristics of the spectral properties of fluctuations in the incompressible interplanetary medium were provided in classical statistical theory via the phase space distribution obtained from ideal MHD invariants by Matthaeus and Goldstein 22 and followed by a confirmation of the existence of solar wind multifractal structures (Burlaga 23,24). Furthermore, solar wind observations were also able to study the differences between fluid and MHD turbulence (Carbone 25).

The non-Gaussianity of the PDFs of the magnetic field and plasma fluctuations was analyzed and linked to intermittency and the fractal scaling of the solar wind MHD fluctuations (Marsch and Tu 26,27), followed by detailed investigations of the non-Gaussian fractal and/or multifractal characteristics of solar wind and related magnetospheric parameters (Hnat 28).
Their multiscale coupling properties and significance in view of magnetospheric response was analyzed (Vörös et al. [24], Vörös and Jankovičová [25]), including studies of multi-scale intermittency and anisotropy effects in the near-Earth magnetotail dynamics (Vörös et al. [26, 27]).

WIND, ACE and Voyager observations of solar wind multi-scale statistical properties verify that the leptokurtic, long-tailed shapes of the PDFs at small scales represent the characteristics of intermittent turbulence and approach a Gaussian, reflecting a decoupled state, on large scales (Sorriso-Valvo et al. [28], Burlaga et al. [29], Burlaga et al. [30]). In other words, the probability of rare events is raised on small scales, where the spatial separation scale is characterized commonly by the differences $\delta X(t) = X(t+\tau) - X(t)$, $X(t)$ denoting any characteristic solar wind variable at time $t$ and $\tau$ is the time lag. Recently, intermittency was considered to appear as result of an interplay between stochastic Alfvénic fluctuations and coherent 2-D structures (Bruno et al. [31]).

The empirical Castaing model (Castaing et al. [32], Castaing and Dubrulle [33], Castaing [34]) introduces intermittency through fluctuations of log-normal distributed variances based on the idea that for constant energy transfer between spatial scales all variables obey a Gaussian distribution of fluctuations $\delta X$ and hence assuming that fluctuations on different scales are independent. This convolution of Gaussians of different variances was introduced to model the non-Gaussian energy cascade character of intermittency in turbulent flows (Consolini and Michelis [35], Guowei et al. [36], Sorriso-Valvo [28], Schmitt and Marsan [37]) where the log-normal distribution of variances through the inertial scales was found to provide excellent fits to the observed leptokurtic PDFs in solar wind flows. Due to the fitting accuracy the Castaing model achieved high popularity, but appears to be subject to two significant shortcomings on physical grounds: the model provides (1) no link to non-locality and long-range interactions present in turbulence and (2) no justification for direct energy coupling between separated scales. Other known shortcomings of Castaing or log-normal models are related to the predicted features of high-order moments, which violate the fundamental assumptions of turbulence theory (Frisch [71], Arratia et al. [72]).

The global leptokurtic non-Gaussian shape of the increment PDFs requires theoretically a corresponding unique global distribution function. This condition can be formulated on a general level by considering the basic feature of turbulent flows, i.e., multi-scale coupling or nonlocality in physical or in Fourier space, where nonlocality appears due to the presence of...
long-range forces implying direct nonlocal interactions between large scales and small scales. Due to long-range interactions small and large scales are strongly coupled indicating that small-scale fluctuations in each time/space point depend on the large scale motions in the whole time/space domain and vice versa (Tsinober). Accounting for long-range interactions is a particular feature of nonextensive systems and available from pseudo-additive entropy generalization.

The classical Boltzmann-Gibbs extensive thermo-statistics constitutes a powerful tool when microscopic interactions and memory are short ranged and the environment is an Euclidean space-time, a continuous and differentiable manifold. However, in the present situation we are dealing with astrophysical systems, generally subject to spatial or temporal long-range interactions evolving in a non-Euclidean, for instance multi-fractal space-time that makes their behavior nonextensive. A suitable generalization of the Boltzmann-Gibbs-Shannon entropy for statistical equilibrium was first proposed by Renyi, and subsequently by Tsallis, preserving the usual properties of positivity, equiprobability and irreversibility, but suitably extending the standard extensivity or additivity to nonextensivity. The main theorems of the classical Maxwell-Boltzmann statistics admit profound generalizations within nonextensive statistics (sometimes referred to as q-statistics where q characterizes the degree of nonextensivity of the system), wherefore a variety of subsequent analyses were devoted to clarify the mathematical and physical consequences of pseudo-additivity, for an early review see e.g. Tsallis. Those include a reformulation of the classical N-body problem within the extended statistical mechanics (Plastino et al. and the development of nonextensive distributions (Silva et al., Almeida, where a deterministic connection between the generalized entropy and the resulting power-law functionals (Andrade et al.), as well as the duality of nonextensive statistics were recognized (Karlin et al.).

Relating the parameters q and κ by the transformation κ = 1/(1 - q) (Leubner) provided the missing link between nonextensive distributions and κ-functions favored in space plasma physics, leading to the required theoretical justification for the use of κ-distributions from fundamental physics. Since the parameter κ, a measure of the degree of nonextensivity of the system, is not restricted to positive values in the nonextensive context, the commonly observed core-halo twin character of the interplanetary electron and ion velocity space distributions was verified theoretically upon generalization to a bi-kappa distribution, subject to a less pronounced core along with extended tails, as compared to a Maxwellian (Leubner).
Recently, the PDF of the Tsallis ensemble was linked to the analysis of fully developed turbulence providing a relation between the nonextensive parameter $q$ and the intermittency exponent $m$ that is, a manifestation of multifractality of the distribution of eddies (Arimitsu and Arimitsu, 50, 51) as well as of scaling of the velocity structure functions (Arimitsu and Arimitsu, 52). Moreover, the context of generalized thermo-statistics provides analytical formulas for PDFs of distance dependent velocity differences, linking the entropic index to the cascade like structure of the turbulent dynamics (Beck, 53). We relate in the following nonlocality in turbulent flows to the presence of long-range forces in nonextensive systems and demonstrate in the context of entropy generalization the consistency of the theoretically derived bi-kappa distribution (Leubner, 48, Leubner and Vörös, 15) with observed, scale dependent PDFs of characteristic variables in the intermittent, turbulent solar wind, where both, slow and high speed conditions are analyzed separately.

2. Theory

The standard BGS statistics is based on the extensive entropy measure

$$S_B = -k_B \sum p_i \ln p_i$$ (1)

where $p_i$ is the probability of the $i^{th}$ microstate, $k_B$ is Boltzmann’s constant and $S_B$ is extremized for equiprobability. As physical background one assumes that particles move independently from each other, i.e. there are no correlations present in the system considered. This implies isotropy of the velocity directions and thus the entropy appears as additive quantity yielding the standard Maxwellian distribution function. In other words, microscopic interactions are short ranged and we are dealing with an Euclidean space time. The assumptions behind standard BGS statistics are not applicable if one needs to account for nonlocality and long-range interactions in a fractal/multifractal physical environment. It is required to introduce correlation within the system, which is done conveniently in the context of nonextensive entropy generalization leading to scale-free power-law PDFs.

Considering two sub-systems $A$ and $B$ one can illuminate nonextensivity by the property of pseudo-additivity of the entropy such that

$$S_\kappa(A + B) = S_\kappa(A) + S_\kappa(B) + \frac{1}{\kappa}S_\kappa(A)S_\kappa(B)$$ (2)
where the entropic index \( \kappa \) is a parameter quantifying the degree of nonextensivity in the system. For \( \kappa = \infty \) the last term on the right hand side cancels leaving the additive entropy of standard BGS statistics. Hence, nonlocality or long-range interactions are introduced by the multiplicative term accounting for correlations between the subsystems. As a measure for entropy mixing the entropic index \( \kappa \) quantifies the degree of nonextensivity in the system and thus accounts for nonlocality and long-range interactions or coupling and correlations, respectively. In general, the pseudo-additive, \( \kappa \)-weighted term may assume positive or negative definite values indicating a nonextensive entropy bifurcation. Obviously, nonextensive systems are subject to a dual nature since positive \( \kappa \)-values imply the tendency to less organized states where the entropy increases whereas negative \( \kappa \)-values provide a higher organized state of decreased entropy, see Leubner.\(^{14}\)

The general nonextensive entropy consistent with Eq. (2), replacing the classical BGS-statistics for systems subject to long-range interactions, takes the form (Tsallis \(^{40}\), Leubner \(^{48}\))

\[
S_\kappa = \kappa k_B \left( \sum p_i^{1-1/\kappa} - 1 \right) \quad (3)
\]

In order to link the \( \kappa \)-notation defined within \( -\infty \leq \kappa \leq +\infty \), commonly applied in space plasma modeling in terms of the family of \( \kappa \)-distributions, to the Tsallis q-statistics one may perform the transformation \( 1/(1-q) = \kappa \) to Eq. (3) (Leubner \(^{47}\)). \( \kappa = \infty \) corresponds to \( q = 1 \) and represents the extensive limit of statistical independency. Consequently, the interaction term in Eq. (2) cancels recovering with respect to Eq. (3) the classical Boltzmann-Gibbs-Shannon entropy Eq. (1). Eq. (3) applies to systems subject to long-range interactions or memory and systems evolving in a non-Euclidean and multifractal space-time. A further generalization of Eq. (2) for complex systems, composed of an arbitrary number of mutually correlated systems, is provided by Milovanov and Zelenyi \(^{54}\) where appropriate higher order terms in the entropy appear. Once the entropy is known the corresponding probability distributions are available.

In Maxwells derivation the velocity components of the distribution \( f(v) \) are uncorrelated where \( \ln f \) can be expressed as a sum of the logarithms of the one dimensional distribution functions. In nonextensive systems one needs to introduce correlations between the components accounting for the long-range interactions, which is conveniently done by extremizing the entropy under conservation of mass and energy yielding the corresponding
one-dimensional power-law distributions as

\[ f^\pm = A^\pm \left[ 1 + \frac{1}{\kappa} \frac{v^2}{v_t^2} \right]^{-\kappa} \] (4)

where \( v_t \) corresponds to the mean energy or thermal speed of the distribution. Hence, the exponential probability function of the Maxwellian gas of an uncorrelated ensemble of particles is replaced by the characteristics of a power law where the sign of \( \kappa \), indicated by superscripts, governs the corresponding entropy bifurcation. Note that the distribution \( f^\pm \) can be derived entirely using general methods of statistical mechanics without introducing a specific form for long-range interactions. Incorporating the sign of \( \kappa \) into Eq. (4) generates a dual solution with regard to positive and negative \( \kappa \)-values, respectively, resulting also in two different normalizations \( A^\pm \).

The entropy bifurcation appears to be manifested also in higher order moments yielding for the second moments \( \kappa \)-dependent generalized temperatures, see Leubner. Furthermore, the positive solution is restricted to \( \kappa > 3/2 \) whereas the negative solutions are subject to a cutoff in the distribution at \( v_{\text{max}} = v_t \sqrt{\kappa} \). Both functions, \( f^+ \) and \( f^- \) in Eqs. (4) approach the same Maxwellian as \( \kappa \rightarrow \infty \). Fig. 1, left panel, demonstrates schematically the non-thermal behavior of both, the suprathermal halo component and the reduced core distribution, subject to finite support in velocity space, where the case \( \kappa = \infty \) recovers the Maxwellian equilibrium distribution.

Any unique and physically relevant nonextensive PDF must obey the following three conditions: (a) the distribution approaches one and the same Maxwellian as \( \kappa \rightarrow \infty \), (b) a unique, global distribution must be definable by one single density and a unique temperature and (c) upon variation of the coupling parameter \( \kappa \) particle conservation and adiabatic evolution are required, such that a redistribution in a box (a source free environment) can be performed. Subject to these constraints the appropriate mathematical functional, representing observed core-halo structures in nonextensive astrophysical environments, is available from the elementary combination \( f_{ch} = B_{ch}(f_h + f_c) \), \( B_{ch} \) being a proper normalization constant. In this context the full velocity space bi-kappa distribution, compatible with nonextensive entropy generalization and obeying the above constraints, reads

\[ F_{ch}(v; \kappa) = \frac{N}{\pi^{1/2} v_t} G(\kappa) \left\{ \left[ 1 + \frac{1}{\kappa} \frac{v^2}{v_t^2} \right]^{-\kappa} + \left[ 1 - \frac{1}{\kappa} \frac{v^2}{v_t^2} \right]^{-\kappa} \right\} \] (5)
Fig. 1. Left panel: A schematic plot of the characteristics of the nonextensive bi-kappa distribution family: with $\kappa = 3$ the outermost and innermost curve correspond to the halo $f_h$ and core $f_c$ distribution fraction. For increasing $\kappa$-values both sets of curves merge at the same Maxwellian limit, indicated as bold line, $f_h$ from outside and $f_c$ from inside. Right panel: A nonextensive bi-kappa fit (solid line) with $\kappa = 1.8$ of an observed PDF (dashed line) obtained from ACE magnetic field amplitude data. A Gaussian (dashed-dotted line) and a conventional $\kappa$-distribution are provided for comparison.

The last term on the right-hand side denotes an expression subject to a thermal cutoff at the maximum allowed velocity $v_{\text{max}} = \sqrt{\kappa v_t}$, which limits also required integrations. Fig. 1 (right panel) indicates also that the core of the distribution, corresponding to the last term on the right-hand side of Eq. (5), contributes only for small $\delta B$. For details regarding the corresponding second moments or generalized temperatures see Leubner\textsuperscript{48} or Leubner\textsuperscript{49}. The function $G(\kappa)$ is defined by

$$G(\kappa) = \left[ \frac{\kappa^{1/2}\Gamma(\kappa - 1/2)}{\Gamma(\kappa)} + \frac{\kappa^{1/2}\Gamma(\kappa + 1)}{\Gamma(\kappa + 3/2)} \right]^{-1} \quad (6)$$

from the normalization and is subject to a particular weak $\kappa$-dependence where $G(\kappa) \sim 1/2$, see Leubner\textsuperscript{48} for a graphical illustration and discussion. Hence, the normalization is independent of the parameter $\kappa$ and the factor 1/2 reflects consistently the superposition of the two counter-organizing contributions in Eq. (5). For $\kappa = \infty$, $G(\kappa) = 1/2$ and the power laws in the brackets of the right hand side of Eq. (5) turn each into the same Maxwellian exponential.

With regard to this specific mathematical feature the energy levels $E_i$
of the turbulent spectrum can be related to the corresponding kinetic energy of velocity differences $\delta v(t) = v(t + \tau) - v(t)$ between two points of separation $\tau$ allowing to transform the 1D Maxwellian particle distribution of mean energy $v_t$ into the mathematical form of a Gaussian of variance $\sigma$. Upon normalizing the one-dimensional bi-kappa particle distribution \[ P_{ch}(\delta v; \kappa, \sigma) = \frac{1}{2\sqrt{\pi\sigma}} \left\{ \left[ 1 + \frac{\delta v^2}{\kappa^2 \sigma^2} \right]^{-\kappa} + \left[ 1 - \frac{\delta v^2}{\kappa^2 \sigma^2} \right]^{-\kappa} \right\} \] (7)

This two-parameter PDF is applicable to the differences of the fluctuations $\delta X(t) = X(t + \tau) - X(t)$ of any physical variable $X$ in the astrophysical system considered. Here $\kappa$ assumes a physical interpretation defining the degree of nonextensivity or nonlocality in the system, thus being a measure of the degree of organization or intermittency, respectively (Leubner and Vörös\cite{15}) and $\sigma$ denotes the distribution variance. Again, depending on the particular choice of the parameters, the last term on the right-hand side is subject to a cutoff, treated by limiting the integration to the proper interval. Fig. 1, left panel, illuminates also that large values of $\delta X$, corresponding to the tails of the distribution, are represented by the first term on the right-hand side of Equation (7). As $\kappa \to \infty$ the bi-kappa distribution $P_{ch}(\delta X; \kappa, \sigma)$ approaches a single Gaussian.

Note, that Equation 5 represents the full velocity space bi-kappa distribution, while Equation 7 is related to the time shifted differences of any physical variable $X$. Physically, Equation 5 describes particles which motion is controlled by long-range forces and interactions. The basic assumption for deriving the velocity space bi-kappa distribution was the pseudo-additivity of the entropy of particle sub-systems expressed through Equations 2 and 3. It is important to recognize that, the same type of expression for bi-kappa distribution is obtained, if instead of interacting particles we assume interacting coherent structures with the same pseudo-additivity property of the entropy as in Equation 2. In the context of MHD, nonpropagating multi-scale coherent structures or flux tubes can interact, deform and produce new sites of nonpropagating fluctuations. Coherent structures of the same polarity merge into a structure with lower local energetic state, while structures of opposite polarities may repel each other (Chang\cite{64}, Bruno and Carbon\cite{69}). These coherent structures can be considered as discrete
interacting ”particles” in MHD flows, responsible for the particular entropy within the system and validating the analogy to the kinetic level of PDFs (Leubner and Vörös16). A usual way of statistical analysis of intermittency due to the occurrence of coherent structures in turbulence is through two-point differences of fluctuations. Therefore, in the expression of bi-kappa distribution for turbulent interactions (Equation 7) differences of physical variables can be used instead of the velocity notation. Moreover, passive scalars as the magnetic field follow the dynamics of $\delta V$ (see Vörös et al.70).

One of the basic future of turbulent flows is multi-scale redistribution of energy during which interacting coherent structures appear, reducing the entropy of the system and leading also to negative kappa values. At the same time, turbulence enhances dissipation and mixing of the plasma, which increases entropy and can be described in terms of positive kappa values. The presence of processes which increase entropy and those which decrease entropy in turbulence advocates therefore the introduction of bi-kappa distributions.

We proof in the following the relevance of the nonextensive, global bi-kappa PDF (7) on the observed scale dependence of the PDFs of the differences of magnetic field, velocity and density variables in the intermittent, turbulent interplanetary medium.

3. Application: Scale dependent interplanetary PDFs

Based on the nonextensive two parameter bi-kappa distribution $\theta$ we compare the PDFs obtained from slow and fast solar wind data with particular attention to the scale dependent changes of the two physically interpretable parameters ($\kappa$, $\sigma$) involved. We do not consider any occurrence of discontinuities or shocks in the system. The problem of interaction of turbulence with large scale structures as shocks is investigated elsewhere (Vörös et al.70).

For each data set the magnetic field and plasma parameter increments were calculated at a given time lag $\tau$ by $\delta X(t) = X(t + \tau) - X(t)$, where $X$ represents any physical variable considered. For each realization the empirical probability distribution function (histogram) was then computed. $\delta X(t)$ is binned into $n$ equal spaced boxes and the number of elements in each box was computed where the robustness of the histograms against $n$ is tested. $\delta X(t)$ represents characteristic fluctuations at the time scale $\tau$ or, equivalently, across eddies of size $l \sim v \tau$. Hence, by changing $\tau$ it is
possible to analyze the statistical features of fluctuations in different time scales, which roughly correspond to those statistical characteristics across turbulent eddies of size $l \sim v\tau$.

In the following we demonstrate by means of Eq. (7) from first principle statistics that the strong non-Gaussianity of the PDF of small scale fluctuations should be associated physically with long-range interactions provided in nonextensive systems by pseudo-additive entropy generalization. The scale dependence of the PDF in the solar wind can be represented accurately via the tuning parameters $\kappa$ and $\sigma$ of the bi-kappa functional.

In Fig. 1, right panel, we focus on the bi-kappa model demonstrating that the nonextensive context, generates a precise representation for the observed solar wind PDF characterizing the intermittency of the small scale fluctuations. For comparison also a Gaussian and the conventional $\kappa$–function (Leubner [17]), subject to the same $\kappa$–value but not able to reproduce the structure of the PDF of small scale fluctuations, are provided. It is also possible to assume that the smallest fluctuations are random and uncorrelated, and this might be the reason why the central part of the distribution near the maximum is well-fitted by a Gaussian, as in the right panel in Fig. 1. Note, however, that the characteristic scale of the differences of fluctuations is introduced through the time lag $\tau$. Therefore, $\delta B \to 0$ near the maximum of the distribution simply means that, at the scale $\tau$ the differences between the corresponding values of $B(t + \tau)$ and $B(t)$ are very small. Combined with the finite precision of PDFs estimation, small two-point differences can really produce uncorrelated fluctuations near PDFs.
maxima. This does not mean, however, that the Gaussian distribution is the correct answer for the proper description of the central part of the distribution at the small scales $\tau$. If it was true, we should suppose that, at smaller and smaller scales $\tau$, the fluctuations become more and more random and we are getting closer and closer to a Gaussian distribution. Actually the opposite is true, the peakedness of the distribution increases as the scale $\tau$ decreases (see later).

3.1. **Slow speed solar wind**

As an example, in Fig. 2 undisturbed solar wind ACE magnetic field amplitude data of 16s time resolution are analyzed where the dimensionless $\tau$ is multiplied by the resolution to generate an effective time-lag. In particular, the scale dependent PDF evolution of magnetic field fluctuations is subject to a two point separation scale of $\tau = 100, 2000$ and $10000$. The corresponding best fits of the bi-kappa distribution are obtained for $\kappa = 1.8, 3$ and $\infty$, measuring the degree of nonextensivity, or coupling, respectively, through long-range interactions and the dotted lines refer to the standard deviation. The accuracy of the bi-kappa distribution fit demonstrates that non-locality in turbulence, when introduced theoretically by long-range interactions through the nonextensive context, generates a precise representation for the observed PDFs characterizing the intermittency of the fluctuations at all scales.

Based on WIND velocity field magnitude data Fig. 3 presents an analysis of the scale dependence of interplanetary PDFs of the velocity field
Fig. 4. Left panel: The PDF of the increments of observed WIND density fluctuations for $\tau = 10$ and a resolution of 92 sec. as compared to the bi-kappa function with $\kappa = 2$. Based on the same data the central panel provides the characteristics for increase $\tau = 70$ where $\kappa$ assumes a value of 3.5 for the best representation. The PDF of large scale density fluctuations with $\tau = 900$ are well modeled by a Gaussian with $\kappa = \infty$, right panel.

magnitude showing in three plots from left to right the decreasing kurtosis with increasing time-lags, where the observational uncertainty is again indicated by the standard deviation (dotted lines). Constraint by a time resolution of 92 sec the two point time separation $\tau$ assumes the values 10, 70 and 900 from left to right. The solid lines represent best fits to the observed PDFs with nonextensive distributions, where the corresponding $\kappa$ is determined as $\kappa = 2, 3.5, \infty$. The strong non-Gaussian character of the leptokurtic PDFs (left panel), exhibiting pronounced tails associated with solar wind turbulence and intermittency in small-scale fluctuations, finds again an accurate analytical fit and hence a physical background in the nonextensive representation. The non-Gaussian structure is somewhat softened for enhanced $\tau = 70$ (central panel) but again precisely modeled within the pseudo-additive entropy context, turning into the Gaussian shape of large scale fluctuations, which is independent of the increment field.

Fig. 4 provides the corresponding nonextensive analysis of the scale dependence of the density fluctuations obtained from WIND data (92 sec resolution). Evidently, the scale dependent characteristics of the observed PDFs of the increment fields $\delta X(\tau) = X(t+\tau) - X(t)$ for all solar wind variables evolve simultaneously on small scales approaching independency of the increment field in the large scale Gaussian. With separation scales of $\tau = 10, 70, 900$ the corresponding evolution of the PDFs of observed density fluctuations are best represented by the same values of $\kappa = 2, 3.5, \infty$, as for the velocity field magnitude, accounting for nonlocal interactions in the nonextensive theoretical approach.
3.2. High speed solar wind

The four panels in Fig.5 show PDFs of high speed associated magnetic field magnitude fluctuations. The two-point statistics is demonstrated in the subplots from top-left to bottom-right for the scales $\tau = 10, 40, 400, 10000$. The effective time-lag is obtained after multiplying $\tau$ with time resolution of 16 sec. The corresponding best fits reveal differing statistical features of high speed associated magnetic fluctuations. In comparison with low speed data...
Fig. 6. The PDF of the increments of observed ACE high-speed associated density fluctuations (64 s time resolution). Top-left: fluctuations at the scale $\tau = 10$ as compared to the bi-kappa function with $\kappa = 1.5$ and $\sigma = 0.3$; Top-right: $\tau = 40$, $\kappa = 1.5$ and $\sigma = 0.5$; Left-bottom: $\tau = 400$, $\kappa = 1.05$ and $\sigma = 0.9$; Left-right: $\tau = 3000$, $\kappa = \infty$ and $\sigma = 25$ (Gaussian fit).

The degree of nonextensivity does not change during high speed intervals, $\kappa = 1.4, 1.4, 1.2$, over the range of scales $\tau = 10, 40, 400$, and only for $\tau = 10000$, $\kappa$ reaches $\infty$. In contrast to the slow wind data, however, good quality high speed fits can be achieved only when the standard deviation $\sigma$ is changed.

It indicates that the abundance of large scale energy content of high speed flows may facilitate to maintain the degree of nonextensivity and self-organisation unchanged over the considered scales. On the other hand,
we have no clear explanation yet for the observed changes of the standard deviation. We can speculate that changes in sigma appear because of the ample changes in the amplitudes of two-point fluctuations in the solar wind, having solar origin or being generated by local processes absent in the slow wind. Obviously, further comparative case studies on multi-scale fluctuations are needed to answer the question of the relative contribution of local processes versus processes originating in the solar corona to the observed behaviour of statistical moments in the fast and slow solar wind.

The four panels in Figs. 6 and 7 provide the same qualitative behavior for high speed associated ACE density (64 sec resolution) and WIND magnetic field (3 sec resolution).

4. Discussion and conclusions

The Wind and ACE solar wind data analysis unambiguously manifests that the PDFs of large scale density, velocity and magnetic field fluctuations are well represented by a Gaussian, turning into leptokurtic peaked distributions of strong non-Gaussianity in the center along with a pronounced tail structure at smaller scales. In particular, the PDFs of large-scale magnetic field fluctuations, not related to the increment field are known to be subject to relatively small deviations from the Gaussian statistics and are well fitted by the Castaing distribution, a convolution of Gaussians with variances distributed according to a log-normal distribution (Castaing, Padhye et al.). Assuming a constant energy transfer rate between spatial scales all quantities exhibit a Gaussian distribution of fluctuations in this context. Independent of the physical situation considered, the Castaing distribution provides a multi-parameter description of observed PDFs, plausible in this case, since the large-scale fluctuations of the interplanetary magnetic field are generated by a variety of discrete coronal sources. If individual coronal sources evoke Gaussian distributed magnetic fields, the net magnetic fluctuations can be modeled by their superposition with a spread of the corresponding variances.

Contrary, small-scale fluctuations are associated with local intermittent flows where fluctuations are concentrated in limited space volumes. Consequently, the PDFs are scale dependent and intermittency generates long-tailed distributions. It is customary to use $n-th$ order absolute powers of the plasma variables and magnetic field increments ($n-th$ order structure functions (Marsch and T., Pagel and Balogh) allowing to investigate the multi-scale scaling features of fluctuations. Direct studies of observed
Fig. 7. The PDF of the increments of observed ACE high-speed associated magnetic field magnitude fluctuations (3 s time resolution). Top-left: fluctuations at the scale $\tau = 10$ as compared to the bi-kappa function with $\kappa = 1.4$ and $\sigma = 0.003$; Top-right: $\tau = 40$, $\kappa = 1.4$ and $\sigma = 0.006$; Left-bottom: $\tau = 400$, $\kappa = 1.8$ and $\sigma = 0.06$; Left-right: $\tau = 4000$, $\kappa = \infty$ and $\sigma = 1$ (Gaussian fit).

PDFs of the increment fields $\delta X(t) = X(t + \tau) - X(t)$ for any characteristic solar wind variable at time $t$ and time lag $\tau$ revealed departures from a Gaussian distribution over multiple scales (Sorriso-Valvo et al.\cite{28}) and an increase of kurtosis (intermittency) towards small scales (Marsch and Tu\cite{21}). The PDFs are also found to be leptokurtic, which indicates the turbulent character of the underlying fluctuations. Sorriso-Valvo et al.\cite{28} have shown that the non-Gaussian behavior of small-scale velocity and magnetic field fluctuations in the solar wind can also be described well by a Castaing dis-
tribution where the individual sources of Gaussian fluctuations appear at small-scales in turbulent cascades.

From the corresponding nonextensive WIND data analysis of the density and magnetic field fluctuations (Leubner and Vörös [15]) it is evident that the scale dependent characteristics of the observed PDFs of the increment fields \( \delta X(t) = X(t + \tau) - X(t) \) for solar wind variables evolve simultaneously on small scales, approaching independency of the increment field in the large-scale Gaussian. Highly accurately, the overall scale dependence appears as a general characteristic of quiet astrophysical plasma environments indicating a universal scaling dependence between density, velocity and magnetic field intermittency within the experimental uncertainties. This strong correlation implies that the scale dependencies of all physical variables are coupled, where the solar wind Alfvénic fluctuations provide a physical basis of the velocity and magnetic field correlations. On the other hand, according to recent analyses, the magnetic field intensity exhibits a higher degree of intermittency than the solar wind bulk velocity, both in fast and slow winds (Sorriso-Valvo et al. [57]). However, Veltri and Mangeney [58] found that the most intermittent structures in the slow wind are shock waves, displaying similar intermittency in the magnetic field intensity and bulk velocity. Furthermore, the proportionality between density fluctuations and the magnetic field and velocity fluctuations is already maintained in the solar wind by the presence of weak spatial gradients (Spangler [59]).

Fig. 8 (left panel) provides an estimation of the functional dependence between the time lag \( \tau \) and the nonextensive parameter \( \kappa \) for best fitting
bi-kappa functions to the observed PDFs for slow speed solar wind conditions \((v \leq 400 \text{km/s})\). As significant global behavior the scale dependence of the PDFs for quiet conditions appears to be independent of the variance or mean energy of the distribution. The best fitting bi-kappa functions are found for a constant corresponding parameter \(\sigma\), indicating that the parameter \(\kappa\), measuring the degree of coupling within the system, governs primarily the scale dependence of the PDF in the slow solar wind. As \(\tau\) increases from small scales to the intermediate regime a pronounced plateau formation is established, i.e. the relative increase in \(\kappa\)-values with enhanced scales appears reduced. In other words, the PDF shape appears at intermediate scales to be independent of the spatial separation scale. Such a behavior may indicate the presence of a transitional dynamical element characterizing a balance between long and short-range interactions. Contrary, in high speed streams \((v \geq 400 \text{km/s})\) best fits of bi-kappa functions to the observed scale dependent PDFs are found when keeping \(\kappa\) constant and varying only the parameter \(\sigma\), left panel in Fig. 8. Hence the scaling features in the fast wind appear independent of the degree of coupling controlled by \(\kappa\), but rely predominantly on changes of the variance or mean energy. In summary, the theoretical nonextensive context indicates a significant and physically contrary scaling behavior in slow and fast wind. For slow solar wind conditions the correlations/intermittency governed by \(\kappa\) decrease with increasing scale, whereas the characteristic energy governed by \(\sigma\) remains constant. Contrary the scaling properties in high speed streams are characterized by constant correlations \((\kappa)\) but enhanced variance with increasing scale.

\(-\)distributions reproduce the Maxwell-Boltzmann distribution for \(\kappa \to \infty\), a situation identifying \(\kappa\) as an ordering parameter that accounts for correlations within the system. Highly correlated turbulent conditions characterized by kappa distributions represent stationary states far from equilibrium where a generalization of the Boltzmann-Shannon entropy, as measure of the level of organization or intermittency, applies (Goldstein and Lebowitz, Treumann). Physically this can be understood considering a system at a certain nonlinear stage where turbulence may reach a state of high energy level that is balanced by turbulent dissipation. In this environment equilibrium statistics can be extended to dissipative systems, approaching a stationary state beyond thermal equilibrium (Gotoh and Kraichnan). Since turbulence is driven in the solar wind by velocity shears we have chosen for the data analysis intervals of low speed solar wind with limits in velocity space, where the driving and dissipation
A multi-scale cascade mechanism is not the only way for a realization of long range interactions. Let us provide a physical situation where large and small scales are directly coupled and the nonlocal energy transfer is not induced by cascading processes, following e.g. a log-normal model (Frisch63). An example is found in the context of MHD where Chang64 and Consolini and Chang65 proposed an intermittent turbulence model for the solar wind and for the Earth's magnetotail, which comprises neither cascades nor requires local interactions in Fourier space. In this scenario non-propagating or convected fluctuations generate multiscale coherent structures (e.g. flux tubes), which can interact, deform and produce new sites of non-propagating fluctuations. Coherent structures of the same polarity merge into a structure with lower local energetic state, while structures of opposite polarities may repel each other. This coherent structures can be considered as discrete interacting 'particles' in MHD flows, responsible for the particular entropy within the system and validating the analogy to the kinetic level of PDFs. Chang et al.66 have computed PDFs of the intermittent fluctuations from direct numerical simulations of interacting coherent structures. The resulting PDFs have typical leptokurtic shapes, which can be well fitted again by a variety of models, including those with predominantly local interactions in Fourier space. Since all models provide similar fitting accuracy it is required to focus on the underlying physical situation in turbulent flows. Hence, with regard to nonlocal interactions not based on cascade processes the nonextensive entropy approach provides physically a justification for nonlocal interactions and should therefore be favored over cascade models in such processes.

The entropy quantifies the degree of structuring in intermittent turbulence expressed through singular multifractal measures, where also the parameter $\kappa$ (or $q$) is related to the extremes of multifractal distributions (Lyra and Tsallis67, Arimitsu and Arimitsu68, Arimitsu and Arimitsu69, Beck70). Since the nonextensive entropy approach is independent of the mechanism leading to the structures - in both situations, cascading processes and multiscale interacting coherent structures - or even in coexisting situations, the entropy concept can be applied for the analysis and quantification of resulting characteristics in turbulence. In summary, the majority of hitherto existing models of intermittency in the solar wind essentially correspond to the cascade picture of turbulence. Small-scale intermittency,
however, can be associated also by emerging topological complexity of co-
herent structures in turbulence, which might be understood better through entropy concepts, disregarding the goodness criteria of different fits.

We provided specific examples based on multiscale interacting coherent structures where the traditional cascade - and thus e.g. the log-normal ap-
proach beside others - should not be applied for physical reasons (not in terms of PDF fitting accuracy). Therefore the proposed context based on entropy generalization has a potential to describe the underlying physics suitably and thus justifies the nonextensive approach. Certainly this must be viewed as an incomplete concept in view of the complexity of intermit-
tence in turbulence (Cohen 68), in particular regarding the interactions with large scale structures.

Summarizing, a bi-kappa distribution family turns out theoretically as consequence of the entropy generalization in nonextensive thermo-statistics. The two-parameter global bi-kappa function provides theoretically access to the scale dependence of the PDFs observed in astrophysical plasma tur-
bulence. The redistribution of a Gaussian on large scales into highly non-
Gaussian leptokurtic and long-tailed structures, manifest on small scales, is theoretically well described by the family of nonextensive distributions. Pseudo-additive entropy generalization provides the required physical inter-
pretation of the parameter $\kappa$ in terms of the degree of nonextensivity of the system as a measure of nonlocality or coupling due to long-range inter-
actions whereas the variance $\sigma$ measures the mean energy in the system. The scale dependence in the slow speed solar wind is sensitive to variations of $\kappa$ and in high speed streams to variations of $\sigma$. We argue that multi-
scale coupling and intermittency of the turbulent solar wind fluctuations must be related to the nonextensive character of the interplanetary medium accounting for long-range interaction via the entropy generalization.

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