Private life of the Liouville field that causes new anomalies in the Nambu-Goto string

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Abstract

I consider higher-order terms of the Seeley expansion of the heat kernel, which for smooth metrics are suppressed as inverse powers of the UV cutoff $\Lambda$, and demonstrate how they result in an anomalous contribution to the string effective action after doing uncertainties $\Lambda^{-2} \times \Lambda^2$. For the Polyakov string these anomalies precisely reproduce at one loop the result of KPZ-DDK obtained for the Liouville theory by the conformal field theory technique. For the Nambu-Goto string I find a deviation from this result which shows that the two string formulations may differ.

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I. INTRODUCTION

A challenging problem inherited from 1980’s is that of existence of a quantum string in four space-time dimensions. Studies of both the lattice discretization and the Polyakov formulation of continuum bosonic string show that the theory is ill-defined unless the dimension is lower than two. Such a theory is associated with vast models of statistical mechanics whose continuum limits have beautiful description in terms of two-dimensional gravity plus conformal matter. Recently, a progress has been achieved in understanding of the no-go theorem for the lattice strings (see Ref. [1] for a review). It has been shown [2] that the continuum limit of a regularized string should be taken in a very special way to guarantee stringy behavior. In this Paper I shall make an attempt to understand if the results of applying the methods of conformal field theory to the string could be modified.

To be precise I mean the celebrated calculation of the string susceptibility index $\gamma_{str}$ (also known as the gravity anomalous dimension) by Knizhnik-Polyakov-Zamolodchikov [3] and David [4], Distler-Kawai [5] often abbreviated as KPZ-DDK:

$$
\gamma_{str} = (1 - g) \left[ \frac{d - 25 - \sqrt{(25 - d)(1 - d)}}{12} \right] + 2
$$

for a surface of genus $g$ embedded in $d$ Euclidean dimensions. It is seen from Eq. (1) that $\gamma_{str}$ is not the real number for $1 < d < 25$ as it should. In a professional slang this was refer to as the $d = 1$ barrier for the string existence.

The calculation of (1) for the Polyakov string was based on the standard procedure of fixing the conformal gauge for independent metric tensor $\rho_{ab} = e^{2\phi}\delta_{ab}$, integrating over the embedded-space string coordinates $X^\mu$ and considering the resulting Liouville action for the remaining variable $\varphi$ (for a good description of these steps see [6]). DDK assumed that the effective action, describing macroscopic distances, is again of the Liouville type and applied the technique of conformal field theory to obtain (1).

My original motivation for this Paper was to reconsider the calculation for the Nambu-Goto string whose action is just the area of the string world-sheet. Introducing an (imaginary) Lagrange multiplier $\lambda^{ab}$, the Nambu-Goto action can be written as

$$
S_{NG} = K_0 \int \sqrt{\det \partial_a X \cdot \partial_b X} = K_0 \int \sqrt{\det \rho} + \frac{K_0}{2} \int \lambda^{ab} (\partial_a X \cdot \partial_b X - \rho_{ab}) ,
$$

where $K_0$ stands for the bare string tension. Path integrating over $X^\mu$ and $\lambda^{ab}$, we arrive at the
emergent action for $\varphi$

$$S = \frac{1}{16\pi b_0^2} \int \left\{ \partial_a \varphi \partial_a \varphi + \varepsilon e^{-\varphi} \left[ (\partial^2 \varphi)^2 + G \partial_a \varphi \partial_a \varphi \partial^2 \varphi \right] + \mu^2 e^\varphi \right\} + O(\varepsilon^2),$$  \hspace{1cm} (3)

where $\varepsilon$ is a UV cutoff at the world-sheet. The constant $G$ turns out to be different for the Polyakov and Nambu-Goto strings ($G = 0$ for the former and $G \neq 0$, say $G = 1$, for the latter). For smooth $\varphi$ we can drop the term with $\varepsilon$ as $\varepsilon \to 0$, so the difference between the two does not show up and we are left with the Liouville action. However, the additional terms produce interactions for which $\varepsilon$ plays the role of a coupling constant. Accounting them perturbatively results in divergences like powers of $\varepsilon^{-1}$, so uncertainties of the type $\varepsilon \times \varepsilon^{-1}$ appear and have to be done. This looks pretty much similar to the situation in certain non-renormalizable theories, e.g. the sigma model in three dimensions which becomes renormalizable if the coupling is $\sim \Lambda^{-1}$.

In order to study these uncertainties, I shall perform in this Paper an explicit computation of the corresponding effective action which would be again of the type of (3) but with some finite renormalization of the parameters. The computation is performed to the first order of the expansion in $b_0^2$ and the resulting renormalization will be some nonvanishing universal numbers which seemingly do not depend on the form of the higher-order terms in (3) denoted as $O(\varepsilon^2)$. They come from small distances $\sim \sqrt{\varepsilon}$ and look like anomalies in quantum field theory. This is why I say they are due to the private life of the Liouville field.

Our next goal in this paper would be to compare the beautiful intelligent results of KPZ-DDK, obtained for the Polyakov string by using conformal field theory technique, with the performed straightforward brute-force calculation at one loop. Remarkably, I observe a precise agreement for $G = 0$, i.e. for the Polyakov string, and a discrepancy for $G \neq 0$ supposedly associated with the Nambu-Goto string.

This Paper is organized as follows. In Sect. II I present the results obtained for the renormalization of the parameters in (3) and compare them with KPZ-DDK. In Sect. III I describe the setup for the calculation of the effective action and perform it in Sect. IV for the Polyakov string at one loop. The universality of the results, supporting the expectation we are dealing with string anomalies of a new kind, is demonstrated in Sects. V and VII devoted to the renormalization of the metric tensor. Additional arguments in favor of the universality are presented in Sect. VI where the Jacobian associated with the transformation to free fields is calculated. In Sect. VIII I introduce a model which is a simplification to the Nambu-Goto string and yields the action (3) with $G = 1$. The computations are then repeated for the $G \neq 0$ case in Sect. IX. In Sect. X I discuss the results.
II. THE RESULTS AND COMPARISON WITH KPZ-DDK

Among several ways to derive KPZ-DDK I choose the original one \([5]\) based on the background (in)dependence which is beautifully described in \([7]\). If a Weyl factor \(e^\varphi\) is separated in the metric
\[ g_{ab} = e^\varphi \hat{g}_{ab}, \]  
then the curvature changes as
\[ R = e^{-\varphi} \left( \hat{R} - \hat{\Delta} \varphi \right). \]

This produces the linear in \(\varphi\) term in the effective action
\[ S = \frac{1}{8\pi b^2} \int \sqrt{\det \hat{g}} \left[ \frac{1}{2} \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + q \hat{R} \varphi + \mu^2 e^{\alpha \varphi} \right], \]
where the differences of \(b^2\) from \(b_0^2\) and \(q\) from 1 are attributed to the Jacobian for transition to the new field which has the usual Lebesgue measure in the path integral over \(\varphi\). I denote it also as \(\varphi\) because the difference between the original variable with the nonlinear norm
\[ ||\delta \varphi||^2 = \int e^{\varphi} (\delta \varphi)^2 \]
and the new one with the usual norm does not show up in the one-loop calculation.

For \(\hat{g}_{ab} = e^{\hat{\varphi}} \delta_{ab}\) the linear term in \((6)\) can be compensated by the shift
\[ \varphi \to \varphi - q\hat{\varphi}. \]

The requirement for the effective action to be independent on \(\hat{\varphi}\) after this shift then results in two equations
\[ - \frac{6}{b_0^2} + 1 + \frac{6q^2}{b^2} = 0, \quad b_0^2 = \frac{6}{26 - d} \]  
\[ \alpha q - \alpha^2 b^2 = 1. \]  

Equation \((9a)\) implies the vanishing of the total central charge, while Eq. \((9b)\) means that \(e^{\alpha \varphi}\) is a primary field of conformal dimension 1. Both \(\alpha\) and \(b\) changes when \(\varphi\) is multiplied by a constant but the product \(\alpha b\) does not change. There is no need to introduce both \(\alpha\) and \(b\) for the Liouville action \((6)\) but we shall need them both for the action \((3)\) which leads to interactions.

The solution to Eqs. \((9)\) is
\[ \frac{1}{\alpha^2 b^2} = \frac{1}{2} \left[ \frac{1}{b_0^2} - \frac{13}{6} + \sqrt{\left( \frac{1}{b_0^2} - \frac{25}{6} \right) \left( \frac{1}{b_0^2} - \frac{1}{6} \right)} \right] \to \frac{1}{b_0^2} - \frac{13}{6} + \mathcal{O}(b_0^2). \]
This determines the string susceptibility index to be

\[ \gamma_{\text{str}} = (1 - g) \frac{q}{\alpha b^2} + 2, \quad \frac{q}{\alpha b^2} = \frac{1}{b_0^2} - \frac{7}{6} + \mathcal{O}(b_0^2), \]  

(11)
yielding (1). This formula simply follows from a uniform dilatation of space, which means adding a constant value to \( \alpha \varphi \). Then the second term in the brackets in (6) becomes the topological Gauss-Bonett term, explaining why the Euler characteristic \( 2 - 2g \) has appeared in (11).

One of the results of this Paper is an explicit computation of \( b^2 \) and \( \alpha \) to the leading order of the expansion in \( b_0^2 \) (i.e. at one loop). These are given for the Polyakov string by Eqs. (33) and (62) below:

\[ \frac{1}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} + 2 + \mathcal{O}(b_0^2) \]  

(12)

and

\[ \alpha = 1 + 2b_0^2 + \mathcal{O}(b_0^4). \]  

(13)

Combining (12) and (13) we remarkably obtain

\[ \frac{1}{\alpha^2 b^2} = \frac{1}{b_0^2} - \frac{13}{6} + \mathcal{O}(b_0^2), \]  

(14)

exactly reproducing Eq. (10) of DDK to the given order.

Alternatively, for the action (3) with \( G \neq 0 \) we find that \( \alpha \) does not change while

\[ \frac{1}{\alpha^2 b^2} = \frac{1}{b_0^2} - \frac{13}{6} + 2G + \mathcal{O}(b_0^2). \]  

(15)

This gives a clear discrepancy from KPZ-DDK for the simplified model introduced in Sect. VIII which is associated with the Nambu-Goto string and where \( G = 1. \)

### III. SEELEY ET AL. EXPANSION

Let us begin by recalling the structure of the Seeley expansion of the heat kernel

\[ \langle \omega \mid e^{a^2 \Lambda} \omega \rangle = \frac{1}{4\pi a^2} + \frac{1}{24\pi} R(\omega) + \frac{a^2}{120\pi} \left( \Delta R(\omega) + \frac{1}{2} R^2(\omega) \right) + \ldots, \quad a^2 = \frac{1}{4\pi \Lambda^2} \]  

(16)

which is customly used to integrate over \( d \) target-space coordinates \( X^\mu \) in the Polyakov string formulation. Here \( \Delta \) denotes the two-dimensional Laplacian and \( a \) is related to the UV cutoff \( \Lambda \) as shown in (16). Equation (16) was originally derived\(^1\) in \([8, 9]\) for closed curved spaces and generalized to the spaces with boundaries in \([10, 11]\).

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\(^1\) We use the definition of the curvature \( R = -4 e^{-\varphi} \partial \bar{\partial} \varphi \) in the conformal gauge with an opposite sign to that in \([8, 9]\).
Dropping higher terms in $a^2$, which are denoted in the expansion \((16)\) by ..., results in the conformal gauge $\rho_{ab} = \bar{\rho} e^\varphi \delta_{ab}$ in the contribution to the effective action of $\varphi$

$$S_X = \int \left[-\frac{d}{24\pi} \partial \varphi \bar{\partial} \varphi + \frac{d a^2}{30 \pi \bar{\rho}} e^{-\varphi} (\partial \bar{\partial} \varphi)^2\right],$$

(17)

where $\bar{\rho} = 1$ for the classical string ground state, but $\bar{\rho}$ has a nontrivial value for the mean-field ground state \([2]\) which turns out to be stable for $2 < d < 26$. Accounting for ghosts, we get finally

$$S = \frac{1}{4\pi b_0^2} \int \left[\partial \varphi \bar{\partial} \varphi + 4 \varepsilon e^{-\varphi} (\partial \bar{\partial} \varphi)^2\right], \quad b_0^2 = \frac{6}{26 - d},$$

(18)

where $\varepsilon \propto a^2/\bar{\rho}$ and depends in general on the regularization applied. We have dropped in \((18)\) the exponential term because its coefficient (denoted in Eq. \((3)\) by $\mu^2$) vanishes \([2]\) for the stable ground state, minimizing the action in $2 < d < 26$.

The second term in Eq. \((18)\) is usually omitted for smooth metrics when $R \ll \Lambda^2$. However, this term not only changes the propagator but also produces self-interaction of $\varphi$ which results in diagrams with quadratic divergences like powers of $\Lambda^2$. If we treat $\varepsilon$ in \((18)\) as a coupling constant, then terms like $\varepsilon \times \Lambda^2$ appear which are $\sim 1$ for $\varepsilon \sim 1/\Lambda^2$. We shall do below this uncertainty at the one-loop order of the “semiclassical” expansion in $b_0^2$ about the ground state.

Each of the two terms in \((18)\) is invariant under the usual conformal transformation

$$\delta \varphi(z, \bar{z}) = \xi'(z) + \xi(z) \partial \varphi(z, \bar{z}) + \text{h.c.},$$

(19)

For this reason we expect that the action \((18)\) will have conformal structures like KPZ-DDK. We shall explicitly show this below at the one-loop order of the expansion in $b_0^2$.

In the computations we shall use the Pauli-Villars regularization, adding to \((18)\) the action of the regulators

$$S_{\text{reg.}} = \frac{1}{4\pi b_0^2} \int \left\{ \sum_{i=1}^2 \left[ \partial Y_i \bar{\partial} Y_i + \frac{1}{4} M^2 Y_i^2 + 4 \varepsilon e^{-\varphi} (\partial \bar{\partial} Y_i)^2\right] + \left[ \partial Z \bar{\partial} Z + \frac{1}{2} M^2 Z^2 + 4 \varepsilon e^{-\varphi} (\partial \bar{\partial} Z)^2\right] \right\},$$

(20)

where the regulator fields $Y_i$ ($i = 1, 2$) and $Z$ have, respectively, ghost and usual statistics and masses squared $M^2$ and $2M^2$ as is outlined in \([12]\). The total action we shall use in calculations is thus

$$S_{\text{tot.}} = S + S_{\text{reg.}}$$

(21)

It will regularize all divergences that appear.
IV. EFFECTIVE ACTION AT ONE LOOP

A. Divergent part

Let us consider renormalization of the action (18), distinguishing fast (quantum) and slow (classical) fluctuations of $\varphi$ and computing an effective action for the latter. The quadratic divergence of the effective action comes to order $\varphi$ from the tadpole diagrams in Fig. 1. To compare with the previously known results [13–16] for the rigid string, let us start from the diagram in Fig. 1b, where the solid line represents the regulators $Y_i^\mu$ and $Z^\mu (\mu = 1, \ldots, d)$. We have $d = 1$ for our problem, but let us consider an arbitrary $d$. Analogously, let us associate the factor $d$ with the diagram in Fig. 1a like if we have $d$ fields $\varphi$. This is the same as if we have the contribution to the tadpole in Fig. 1b from the field $X^\mu$ with the action like that for $Z^\mu$ in Eq. (20) but zero mass. The right result for the sum of the diagrams in Fig. 1 would be when $d = 1$ in the formulas below, but for the sake of the comparison let us temporary keep $d$ arbitrary.

We then find for the contribution of the diagrams in Fig. 1 to the effective action

$$\text{Fig. 1} = -\frac{d}{2} \int \frac{d^2p}{(2\pi)^2} \varphi(p) \int \frac{d^2k}{(2\pi)^2} \left[ \frac{\varepsilon k^4}{(k^2 + \varepsilon k^4)} - 2 \frac{\varepsilon k^4 - M^2}{(k^2 + M^2 + \varepsilon k^4)} + \frac{\varepsilon k^4 - 2M^2}{(k^2 + 2M^2 + \varepsilon k^4)} \right]$$

$$= -\frac{d}{2} \int \frac{d^2p}{(2\pi)^2} \varphi(p) \Lambda^2$$

with

$$\Lambda^2 = \frac{1}{8\pi\varepsilon} \left[ 4\sqrt{4M^2\varepsilon - 1} \arctan \left( \sqrt{4M^2\varepsilon - 1} \right) - 2\sqrt{8M^2\varepsilon - 1} \arctan \left( \sqrt{8M^2\varepsilon - 1} \right) - \log \frac{M^2\varepsilon}{2} \right].$$

If the quartic in the derivatives term in the action vanishes which means $\varepsilon \rightarrow 0$, we have

$$\Lambda^2 \xrightarrow{\varepsilon \rightarrow 0} \frac{M^2}{2\pi} \log 2,$$
FIG. 2: One-loop diagrams contributing to the effective action to order $\phi^2$. The wavy lines represent $\phi$, while the solid lines represent the regulator fields.

reproducing the correct result for the Polyakov string. Alternatively, for finite $\varepsilon$ and $M \to \infty$ we have

$$\Lambda^2 \to \infty \quad \frac{1}{4} \left[ (2 - \sqrt{2}) \frac{M}{\sqrt{\varepsilon}} - \frac{1}{2\pi} \log(M^2\varepsilon) \right]$$

with the log familiar from the rigid string \[13, 14\].

Finite terms in the effective action come from the diagrams in Fig. 2, which we compute again for an arbitrary $d$. Let us begin with the divergent parts of the diagrams in Fig. 2. We have

$$\left. \text{Fig. 2a} \right|_{\text{div}} + \left. \text{Fig. 2c} \right|_{\text{div}}$$

$$= -\frac{1}{2} \times \frac{d}{2} \int \varphi^2 \int \frac{d^2k}{(2\pi)^2} \left[ \frac{(\varepsilon k^4)^2}{(k^2 + \varepsilon k^4)^2} - 2 \frac{(\varepsilon k^4 - M^2)^2}{(k^2 + M^2 + \varepsilon k^4)^2} + \frac{(\varepsilon k^4 - 2M^2)^2}{(k^2 + 2M^2 + \varepsilon k^4)^2} \right]$$

and

$$\left. \text{Fig. 2b} \right|_{\text{div}} + \left. \text{Fig. 2d} \right|_{\text{div}}$$

$$= \frac{1}{2} \times \frac{d}{2} \int \varphi^2 \int \frac{d^2k}{(2\pi)^2} \left[ \frac{\varepsilon k^4}{(k^2 + \varepsilon k^4)} - 2 \frac{\varepsilon k^4 + M^2}{(k^2 + M^2 + \varepsilon k^4)} + \frac{\varepsilon k^4 + 2M^2}{(k^2 + 2M^2 + \varepsilon k^4)} \right].$$

The results for both (26) and (27) look ugly but their sum is rather simple

$$\left. \text{Fig. 2a} \right|_{\text{div}} + \left. \text{Fig. 2b} \right|_{\text{div}} + \left. \text{Fig. 2c} \right|_{\text{div}} + \left. \text{Fig. 2d} \right|_{\text{div}} = -\frac{d}{4} \Lambda^2 \int \varphi^2$$

with $\Lambda^2$ given by Eq. (23), reproducing together with (22) the expansion of $e^{\varphi^2}$.

**B. Finite part**

To compute the finite part, let us start from the diagram Fig. 2b adding again a part of the diagram in Fig. 2d, whose contribution is the same as if we have the contribution to the diagram Fig. 2d from $d$ massless fields $X^\mu$. It is given by the first term in the curly brackets. This remarkably
reproduces the conformal anomaly

\[ \text{Fig. 2} = \left. -\frac{1}{2} \times \frac{d}{2} \int \frac{d^2 p}{(2\pi)^2} \varphi(-p) \varphi(p) \right|_{\text{div}} \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{[\varepsilon k^2(k-p)^2]^2}{(k^2 + \varepsilon k^4)((k-p)^2 + \varepsilon(k-p)^4)} - \frac{2[k^2 + M^2 + \varepsilon k^4][(k-p)^2 + M^2 + \varepsilon(k-p)^4]}{[\varepsilon k^2(k-p)^2 - 2M^2]^2} \right\} \]

\[ = \text{Fig. 2} = -\frac{d}{96\pi} \int \frac{d^2 p}{(2\pi)^2} \varphi(-p) \varphi(p) + O(M^{-2}). \] (29)

It is not yet the whole story because the diagrams in Figs. 2a, b may also have finite parts. A part of the diagram in Fig. 2a is already taking into account by the above formulas with \( d = 1 \). The additional parts are due to the non-quadratic dependence of the action on \( \varphi \) and do not involve the regulators. Additionally we have

\[ \left. \text{Fig. 2a} \right|_{\text{add}} = -\frac{1}{2} \int (\partial_\alpha \varphi)^2 \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon k^2 \varepsilon k^4}{(k^2 + \varepsilon k^4)^2} \] (30)

and

\[ \left. \text{Fig. 2b} \right|_{\text{add}} = \frac{1}{2} \int (\partial_\alpha \varphi)^2 \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon k^2}{(k^2 + \varepsilon k^4)}. \] (31)

Both (30) and (31) logarithmically diverge but their sum is finite and equals

\[ \left. \text{Fig. 2a} \right|_{\text{add}} + \left. \text{Fig. 2b} \right|_{\text{add}} = \frac{1}{8\pi} \int (\partial_\alpha \varphi)^2. \] (32)

It has the same structure as the conformal anomaly (29) but an opposite sign and also contributes to the effective action. One again, it has appeared as a result of doing uncertainty \( \varepsilon \times \varepsilon^{-1} \) with \( \varepsilon \sim \Lambda^{-2} \rightarrow 0 \). The cancellation of the logs which would otherwise spoil conformal invariance at one loop is a manifestation of the theorem formulated in [17] that is based on the quadratic form of the effective action governing smooth fluctuations essential in the infrared.

Summing (29) with \( d = 1 \) and (32), we find

\[ \frac{1}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} + 2 + O(b_0^2). \] (33)

This formula shows that the “bare” constant \( b_0^2 \) undergoes a finite renormalization because of the interaction implied by the action (18).
V. THE UNIVERSALITY

Let us discuss the universality of the obtained results in the sense of their independence of the exact form of the action. We substitute (18) by a more general action, adding the higher terms,

$$S = -\frac{1}{16\pi b_0^2} \int \sqrt{g} \varphi \Delta F(-\varepsilon \Delta) \varphi$$

and verify whether the results will not depend on the choice of the function

$$F(\varepsilon x) = 1 + \sum_{n \geq 1} f_n \varepsilon^n x^n, \quad f_1 = 1, \quad F(\infty) = \infty.$$  \hspace{1cm} (35)

The next complicated case is \( f_2 \neq 0 \) and \( f_n = 0 \) for \( n \geq 3 \). The action (34) to the quartic order in \( \varphi \) then reads

$$S = \frac{1}{16\pi b_0^2} \int \left[ \varphi \left( -\partial^2 + \varepsilon \partial^2 e^{-\varphi} \partial^2 - f_2 \varepsilon^2 \partial^2 e^{-\varphi} \partial^2 \right) \varphi \right]$$

$$= \frac{1}{16\pi b_0^2} \int \left[ \varphi \left( -\partial^2 + \varepsilon \partial^4 - f_2 \varepsilon^2 \partial^6 \right) \varphi - \varepsilon(\varphi - \frac{1}{2} \varphi^2)(\partial^2 \varphi)^2 + 2f_2 \varepsilon^2(\varphi - \frac{1}{2} \varphi^2)\partial^2 \varphi \partial^4 \varphi$$

$$- f_2 \varepsilon^2 \partial^2 \varphi \partial^4 \varphi + f_2 \varepsilon^2 \partial_0 \varphi \partial_0 \varphi(\partial^2 \varphi)^2 \right] + O(\varphi^5)$$

which generates three- and four-point vertices. For the regulators we have analogously

$$S_{\text{Reg}} = \frac{1}{16\pi b_0^2} \int \left[ \mathcal{Y} \left( -\partial^2 + M^2 e^{-\varphi} + \varepsilon \partial^2 e^{-\varphi} \partial^2 - f_2 \varepsilon^2 \partial^2 e^{-\varphi} \partial^2 \right) \mathcal{Y} \right]$$

$$= \frac{1}{16\pi b_0^2} \int \left[ \mathcal{Y} \left( -\partial^2 + M^2 + \varepsilon \partial^4 - f_2 \varepsilon^2 \partial^6 \right) \mathcal{Y} + M^2(\varphi - \frac{1}{2} \varphi^2)\mathcal{Y}^2 - \varepsilon(\varphi - \frac{1}{2} \varphi^2)(\partial^2 \mathcal{Y})^2$$

$$+ 2f_2 \varepsilon^2(\varphi - \frac{1}{2} \varphi^2)\partial^2 \mathcal{Y} \partial^4 \mathcal{Y} - f_2 \varepsilon^2 \varphi^2 \partial^2 \mathcal{Y} \partial^4 \mathcal{Y} + f_2 \varepsilon^2 \partial_0 \varphi \partial_0 \varphi(\partial^2 \mathcal{Y})^2 \right] + O(\varphi^3).$$

Higher orders in \( \varphi \) in these actions are again not essential at one loop.

The parts of the diagrams in Fig. 2a (with the inclusion a part of the diagram in Fig. 2h as before) and Fig. 2h which reproduce the conformal anomaly are

$$\left. \text{Fig. 2a} \right|_{\text{fin}} = -\frac{1}{2} \times d \times \frac{d^2 p}{(2\pi)^2} \varphi(-p) \varphi(p)$$

$$\times \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{[\varepsilon k^2(k-p)^2 + f_2 \varepsilon^2(k^2(k-p)^4 + k^4(k-p)^2)]^2}{(k^2 + \varepsilon k^4 + f_2 \varepsilon^2 k^6)((k-p)^2 + \varepsilon(k-p)^4 + f_2 \varepsilon^2(k-p)^6)} \right. - \frac{[\varepsilon^2(k^2(k-p)^2 - M^2 + f_2 \varepsilon^2(k^2(k-p)^4 + k^4(k-p)^2)]^2}{(k^2 + M^2 + \varepsilon k^4 + f_2 \varepsilon^2 k^6)((k-p)^2 + M^2 + \varepsilon(k-p)^4 + f_2 \varepsilon^2(k-p)^6)} \right\}$$

and

$$\left. \text{Fig. 2h} \right|_{\text{fin}} = \frac{d}{2} \int \frac{d^2 p}{(2\pi)^2} \varphi(-p) \varphi(p) \int \frac{d^2 k}{(2\pi)^2} \left[ \frac{f_2 \varepsilon^2 k^4}{(k^2 + \varepsilon k^4 + f_2 \varepsilon^2 k^6)} - \frac{f_2 \varepsilon^2 k^4}{(k^2 + M^2 + \varepsilon k^4 + f_2 \varepsilon^2 k^6)} \right].$$

(38)
We have explicitly written here only one regulator with mass \( M \) because the anomaly does not depend on the mass. The two additional terms in Eq. (20) were needed only to regularize the divergent part. The finite part of the sum of (38) and (39) gives precisely the conformal anomaly

\[
\text{Fig. 2c} |^\text{fin} + \text{Fig. 2d} |^\text{fin} = -\frac{d}{96\pi} \int \frac{d^2 p}{(2\pi)^2} p^2 \varphi(-p)\varphi(p) + O(M^{-2})
\]  

(40)

for any \( \varepsilon \) and \( f_2 \).

There is still a bunch of additional diagrams where \( \varphi \) pairs with \( \varphi \) standing in place of \( Y \). We have

\[
\text{Fig. 2h} |^\text{add} = -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} p^2 \varphi(-p)\varphi(p) \int \frac{d^2 k}{(2\pi)^2} \frac{(\varepsilon^2 k^6 + 3f_2\varepsilon^3 k^8 + 2f_2^2\varepsilon^4 k^{10})}{(k^2 + \varepsilon k^4 + f_2^2\varepsilon^2 k^6)^2}
\]

(41)

and

\[
\text{Fig. 2b} |^\text{add} = \frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} p^2 \varphi(-p)\varphi(p) \int \frac{d^2 k}{(2\pi)^2} \frac{(\varepsilon k^2 + 2f_2\varepsilon^2 k^4)}{(k^2 + \varepsilon k^4 + f_2^2\varepsilon^2 k^6)^2}
\]

(42)

The sum of (41) and (42) does not depend on \( \varepsilon \) and \( f_2 \) and is equal to (32). This is therefore a strong argument that we are dealing indeed with a new kind of anomalies. Hopefully, the derivation can be extended to an arbitrary function \( F \) in Eq. (34). An illustration of how it may work is Eq. (59) below.

VI. FIELD REDEFINITION AND JACOBIANS

Let us present yet another derivation of the above result by computing the determinants associated with field redefinition. The action (18) can be deduced from the free action given by the first term in (18) by virtue of the field redefinition

\[
\varphi \to \varphi' = \sqrt{1 - \varepsilon\Delta} \varphi.
\]

(43)

However, a determinant of a nontrivial operator is produced by the measure for \( \varphi \) while changing (43), so we have to investigate how this procedure may reproduce the above results.

To regularize we make the change of the regulators analogously to (43)

\[
Y_i \to Y'_i = \sqrt{1 + \varepsilon M^2 - \varepsilon\Delta} Y_i, \quad Z \to Z' = \sqrt{1 + 2\varepsilon M^2 - \varepsilon\Delta} Z.
\]

(44)

This produces the determinants

\[
\frac{D Y'_i}{D Y_i} = \det^{1/2} (1 + \varepsilon M^2 - \varepsilon\Delta), \quad \frac{D Z}{D Z'} = \det^{-1/2} (1 + 2\varepsilon M^2 - \varepsilon\Delta).
\]

(45)
The positive power of the determinant for \( Y_i \) is because they are Grassmann variables.

It is more complicated with the Jacobian associated with the change (43) because it depends on \( \varphi \) non-linearly and we need the inverse function \( \varphi(\varphi') \). We can simply write

\[
\varphi = \frac{1}{\sqrt{1 - \varepsilon \Delta}} \varphi'
\]  

which gives the correct term of the order \( \varepsilon \), but may differ from exact \( \varphi(\varphi') \) by higher orders in \( \varepsilon \).

But if we believe in the universality, we may think that the trial function (46) will give the right result at least at one loop. To the leading order in \( \varepsilon \) we then write

\[
\frac{\mathcal{D} \varphi}{\mathcal{D} \varphi'} = \frac{\det \sqrt{1 - \varepsilon \Delta}}{\det (1 - \varepsilon \Delta) + \varepsilon R/2}.
\]  

For the total Jacobian we thus write

\[
\frac{\mathcal{D} \varphi}{\mathcal{D} \varphi'} \prod_{i=1}^2 \frac{\mathcal{D} Y_i}{\mathcal{D} Y_i'} \frac{\mathcal{D} Z}{\mathcal{D} Z'} = R_1^{-1} R_2^{-1},
\]  

where

\[
R_1 = \frac{\det (1 - \varepsilon \Delta - \varepsilon R/2)}{\det (1 - \varepsilon \Delta)} = \frac{\det \left( e^{\varphi'} - \varepsilon \partial^2 + \varepsilon \partial^2 \varphi'/2 \right)}{\det \left( e^{\varphi'} - \varepsilon \partial^2 \right)}
\]  

and

\[
R_2 = \frac{\det \sqrt{1 - \varepsilon \Delta} \sqrt{1 + 2 \varepsilon M^2 - \varepsilon \Delta}}{\det (1 + \varepsilon M^2 - \varepsilon \Delta)} = \frac{\det^{1/2} \left( e^{\varphi'} - \varepsilon \partial^2 \right) \det^{1/2} \left( e^{\varphi'} + 2 \varepsilon M^2 - \varepsilon \partial^2 \right)}{\det \left( e^{\varphi'} + \varepsilon M^2 - \varepsilon \partial^2 \right)}. \]

As far as the ratio \( R_2 \) is concerned, it has only (regularized) divergent part and no finite part because the conformal anomalies mutually cancel in the ratio. On the contrary, the divergent parts of the determinants cancel in the ratio \( R_1 \) as well as the usual conformal anomaly does. Naively we expect that \( R_1 \) tends to 1 when \( \varepsilon \to 0 \), but to order \( \varphi^2 \) the result is given by the diagram in Fig. 2c and reads

\[
\log R_1 = \frac{1}{2} \int \partial_a \varphi' \partial_a \varphi' \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon k^2}{(1 + \varepsilon k^2)^2} = \frac{1}{8\pi} \int \partial_a \varphi' \partial_a \varphi'
\]  

which coincides with (32).

The computation of \( R_1 \) can be easily extended to all orders in \( \varphi' \), keeping in mind that we need only the first order of the expansion in \( \varepsilon \partial^2 \varphi \). Higher orders in \( \varepsilon \partial^2 \varphi \) vanish as \( \varepsilon \to 0 \).

The corresponding diagrams are shown in Fig. 3. All of them are of the same order in \( b_0^2 \). The combinatorics is as follows:

\[
\text{Fig. 3} = -\frac{1}{2} \sum_{n=1}^{\infty} \int \left( 1 - e^{\varphi'} \right)^n \partial^2 \varphi' \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon}{(1 + \varepsilon k^2)^{n+1}}
\]  

\[
= -\frac{1}{8\pi} \sum_{n=1}^{\infty} \int \left( 1 - e^{\varphi'} \right)^n \partial^2 \varphi' = \frac{1}{8\pi} \int (\partial_a \varphi')^2,
\]  

\[
12
\]
FIG. 3: Graphical representation of the ratio of the determinants in Eq. (49) to first order in $\varepsilon \partial^2 \phi'$ denoted by a little cross.

FIG. 4: One-loop renormalization of $e^\phi$ whose position is denoted by the dot. The wavy lines represent $\phi$.

reproducing (51). This cancellation of the higher order in $\phi'$ is of cause a consequence of diffeomorphism invariance.

What was actually calculated was the partition function for the following simple modification of the Gaussian model

$$Z = \int \mathcal{D}\phi' e^{-\frac{1}{16\varepsilon^2} \int \partial_\alpha \phi' \partial_\alpha \phi'} R_1^{-1}$$

with the ratio of the determinants $R_1$ given by Eq. (49). Because of the universality argument we expect that it is equivalent to the partition function with the action (18).

VII. RENORMALIZATION OF THE EXPONENTIAL

While Eq. (9a) comes from the requirement for the background $\hat{\phi}$ to disappear in the kinetic part of the action after the shift (8), Eq. (9b) follows from the independence of $\int e^{\alpha \phi}$ on $\hat{\phi}$. The term $-\alpha^2 b^2$ on the heft-hand side of Eq. (9b) results from the renormalization of $e^{\alpha \phi}$ driven at one loop by the diagram in Fig. 4a which gives the logarithmic divergence

$$\text{Fig. 4a} = \frac{e^{\alpha \phi}}{2} \times 8\pi \alpha^2 b^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \varepsilon k^4 + \ldots} = e^{\alpha \phi} \left( \alpha^2 b^2 \log \frac{1}{\varepsilon} + \text{IR divergent} \right).$$

Recalling that the world-sheet cutoff

$$\varepsilon = \frac{a^2}{\bar{\rho}} e^{-\alpha \phi},$$

(55)
where $a^2$ is an invariant cutoff, we find

\[
\text{Fig. 4a} = e^{\alpha \varphi} a^3 b^2 \varphi. \tag{56}
\]

Exponentiating the one-loop contribution, we reproduce the second term on the left-hand side of Eq. (9b).

Now we have at one loop additionally the diagram in Fig. 4b which for the action (18) contributes the same value as (56) expanded in $b^2_0$

\[
\text{Fig. 4b} = e^{\alpha \varphi} a^3 b^2 \varphi. \tag{57}
\]

At this point one may wonder about the term $\partial^2 a \varphi$ which can also appear from the diagram in Fig. 4b. But it comes multiplied by the integral

\[
\partial^2 a \varphi \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon k^4}{(k^2 + \varepsilon k^4)^2} \sim \varepsilon \partial^2 a \varphi \tag{58}
\]

which is negligible as $\varepsilon \to 0$.

It is remarkably simple to show that both (56) and (57) are indeed universal. For (56) it is obvious with logarithmic accuracy. Given the general action (34), Eq. (57) is modified as

\[
\text{Fig. 4b} = e^{\alpha \varphi} a^3 b^2 \varphi \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon k^4}{(k^2 + \varepsilon k^4)^2} = e^{\alpha \varphi} b^2_0 \varphi. \tag{59}
\]

A natural question arises as to the diagram with two additional lines depicted in Fig. 5a whose contribution does not vanish and equals

\[
\text{Fig. 5a} = -\frac{e^{\alpha \varphi}}{2} \times 8\pi b^2_0 \varphi^2 \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon k^4 F(\varepsilon k^2)}{(k^2 + \varepsilon k^4)^2} = -\frac{1}{2} e^{\alpha \varphi} b^2_0 \varphi^2. \tag{60}
\]

It mutually cancels with the diagram in Fig. 5b which has two vertices but is of the same order in $b^2_0$:

\[
\text{Fig. 5b} = \frac{e^{\alpha \varphi}}{2} \times 8\pi b^2_0 \varphi^2 \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon^2 k^8}{(k^2 + \varepsilon k^4)^3} = \frac{1}{2} e^{\alpha \varphi} b^2_0 \varphi^2. \tag{61}
\]
This cancellation is again a consequence of diffeomorphism invariance and holds for all additional powers of $\varphi$.

The sum of Eqs. (56) and (57) implies

$$\alpha = 1 + 2b_0^2 + O(b_0^4)$$

which have the sense of finite renormalization of (the exponent in) the metric tensor. Together with Eq. (33) it remarkably gives Eq. (14), exactly reproducing Eq. (10) of DDK to the given order.

VIII. THE NAMBU-GOTO STRING AND SIMPLIFIED MODEL

To manage the Nambu-Goto action we proceed in the standard way, introducing the (imaginary) Lagrange multiplier $\lambda^{ab}$ and an independent metric tensor $\rho^{ab}$. Path integrating over $X^\mu$, their regulators and ghosts associated with fixing the conformal gauge $\rho^{ab} = \hat{\rho}^{ab}$ with $\hat{\rho}^{ab}$ being a fiducial world-sheet metric, we obtain for the divergent part of the effective action [2]

$$S_{\text{div}} = \int \left[ \frac{K_0}{2} \lambda^{ab} \partial_a X_{cl} \cdot \partial_b X_{cl} + K_0 \rho \left( \sqrt{\det \hat{g}} - \frac{1}{2} \lambda^{ab} \hat{g}_{ab} \right) + \sqrt{\det \hat{g}} \left( - \frac{d \Lambda^2 \rho}{2 \sqrt{\det \lambda}} + \Lambda^2 \rho \right) \right].$$

Here $\Lambda$ is a UV cutoff, $K_0 \sim \Lambda^2$ is the bare string tension and $X_{cl}^\mu$ accounts for the boundary conditions imposed on the world-sheet, e.g. a long cylinder or torus..

Given $\hat{g}_{ab}$, let us split the Lagrange multiplier $\lambda^{ab}$ into the parts parallel and orthogonal to $\hat{g}_{ab}$

$$\lambda^{ab} = \lambda \sqrt{\det \hat{g}} \hat{g}^{ab} + \lambda^{ab}_{\perp}, \quad \lambda^{ab}_{\perp} \hat{g}_{ab} = 0,$$

where $\lambda$ is a scalar and $\lambda^{ab}_{\perp}$ is the orthogonal part. We shall expand near the minimum, substituting

$$\lambda^{ab} = \tilde{\lambda} \sqrt{\det \hat{g}} \hat{g}^{ab} + \delta \lambda^{ab}, \quad \rho = \tilde{\rho} + \delta \rho,$$

so the terms linear in the fluctuations $\delta \lambda^{ab}$ and $\delta \rho$ will vanish in the effective action.

It is convenient to work with complex coordinates $z = \omega^1 + i \omega^2$ and $\bar{z} = \omega^1 - i \omega^2$ when

$$\lambda^{zz} = \lambda^{11} + \lambda^{22}, \quad \lambda^{z\bar{z}} = \lambda^{11} - \lambda^{22} + 2i \lambda^{12}.$$  

Expanding in fluctuations about the minimum, we have for $\hat{g}_{ab} = \delta_{ab}$ the quadratic in $\delta \lambda^{ab}$ part of the action

$$S^{(2)}_{\text{div}} = \int \left[ -\left( K_0 - \frac{d \Lambda^2}{2 \lambda^2} \right) \delta \lambda^{zz} \delta \rho - \frac{d \Lambda^2 (\tilde{\rho} + \delta \rho)}{8 \lambda^3} \left( (\delta \lambda^{zz})^2 + \frac{1}{2} \delta \lambda^{z\bar{z}} \delta \lambda^{\bar{z}z} \right) \right].$$

It is seen from the action (66) that $\delta \lambda^{ab}$ does not propagate to the distances much larger than $1/\Lambda \sqrt{\tilde{\rho}}$. Only $\delta \rho$ propagates to macroscopic distances, so that the variables $\lambda^{zz}$, $\lambda^{z\bar{z}}$ and $\lambda^{\bar{z}z}$...
become localized. This is why the Nambu-Goto string is expected \[6\] to be equivalent to the Polyakov string. We would like to reexamine this issue, having in mind that the anomalies of the type $\Lambda^{-2} \times \Lambda^2$ discussed above might give again a contribution.

Let us first consider the case when $\lambda_{ab}^{\perp} = 0$ in Eq. (64), i.e. $\lambda^{zz} = \lambda^{\bar{z}\bar{z}} = 0$ and $\lambda^{zz} = 2\lambda$. Accounting for the finite part, the action for $\delta\lambda = \lambda - \bar{\lambda}$ and $\delta\rho = \bar{\rho}(e^\varphi - 1)$ reads

$$S^{(2)} = \int \left[ \frac{(26 - d)}{96\pi} \partial_a \varphi \partial_a \varphi - \frac{d}{24\pi\lambda} \partial_a \varphi \partial_a \delta\lambda - K_R \bar{\rho} e^\varphi \delta\lambda - \frac{d\Lambda^2 \bar{\rho} e^\varphi}{2\lambda^3} \delta\lambda^2 \right].$$  \hspace{1cm} (67)

We have dropped here a finite term for $\delta\lambda^2$ because of the presence of $\Lambda^2$, but left it for the mixed term $\delta\rho\delta\lambda$ because $K_R = K_0 - \frac{d\Lambda^2}{2\lambda^2}$ is finite in the scaling regime \[2\].

We can path integrate over $\delta\lambda$. The resulting action for $\varphi$ is given by the substitution of $\delta\lambda$ in (67) by

$$\delta\lambda = \frac{\bar{\lambda}^3}{d\Lambda^2} \left( \frac{d}{24\pi\lambda} \Delta\varphi - K_R \right).$$  \hspace{1cm} (69)

The result would be of the type of the action \[18\], so nothing new appears in comparison with the Polyakov string.

A simplest model where we may expect a deviation from the Polyakov string is the Nambu-Goto string with frozen variable $\lambda^{zz}$ which can be ignored, but still remaining $\lambda^{zz}$ and $\lambda^{\bar{z}\bar{z}}$. We thus concentrate on the simplified model generated by the quadratic action

$$S^{(2)} = \int \left[ \frac{1}{4\pi b_0^2} \partial^a \varphi \partial_a \varphi + d\nu \left( \lambda^{zz} \partial^2 \varphi + \lambda^{\bar{z}\bar{z}} \partial^2 \varphi \right) - d\Lambda^2 \bar{\rho} \lambda^{zz} \lambda^{\bar{z}\bar{z}} \right],$$  \hspace{1cm} (70)

where $\lambda^{ab}$ is imaginary and $\nu$ is a constant. Covariantizing it gives

$$S = \int \left[ \frac{1}{16\pi b_0^2} (\det \hat{\gamma})^{1/2} \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + \frac{1}{8\pi b_0^2} (\det \hat{\gamma})^{1/2} \hat{R} \hat{\varphi} + d\nu \left( \lambda^{zz} \nabla^a \partial_b \varphi - \frac{1}{2} \lambda^{ab} \hat{g}_{cd} \nabla_c \partial_d \varphi \right) \right.$$  

$$-2d\Lambda^2 \bar{\rho} e^\varphi (\det \hat{\gamma})^{-1/2} \left( \hat{g}_{ac} \hat{g}_{bd} - \frac{1}{2} \hat{g}_{ab} \hat{g}_{cd} \right) \lambda^{ab} \lambda^{cd} \right].$$  \hspace{1cm} (71)

For $\hat{g}_{ab} = \delta_{ab}$ Eq. (71) yields the action

$$S = \int \left[ \frac{1}{4\pi b_0^2} \partial^a \varphi \partial_a \varphi + d\nu \left( \lambda^{zz} \nabla^2 \varphi + \lambda^{\bar{z}\bar{z}} \nabla^2 \varphi \right) - d\Lambda^2 \bar{\rho} e^\varphi \lambda^{zz} \lambda^{\bar{z}\bar{z}} \right]$$

$$= \int \left[ \frac{1}{4\pi b_0^2} \partial^a \varphi \partial_a \varphi + d\nu \left[ \lambda^{zz} \left( \partial^2 \varphi - (\partial \varphi)^2 \right) + \lambda^{\bar{z}\bar{z}} \left( \bar{\partial}^2 \varphi - (\bar{\partial} \varphi)^2 \right) \right] - d\Lambda^2 \bar{\rho} e^\varphi \lambda^{zz} \lambda^{\bar{z}\bar{z}} \right].$$  \hspace{1cm} (72)

At the one-loop order we can expand in $\varphi$ to get only cubic and quartic interactions.
An interesting question is what modification of the usual conformal transformation

\[ \delta \varphi = \xi'(z) + \xi(z) \partial \varphi, \] (73a)
\[ \delta \lambda^{ab} = \xi(z) \partial \lambda^{ab} \] (73b)

would be the symmetry of (72)? The first and the last terms on the right-hand side of Eq. (72) are invariant under (73), while the other terms transform as

\[ \delta (\lambda^{zz} \nabla \partial \varphi) = \lambda^{zz} (\xi'' - \xi'' \partial \varphi + \xi' \nabla \partial \varphi), \] (74)
\[ \delta (\lambda^{\bar{z} \bar{z}} \bar{\nabla} \bar{\partial} \varphi) = -\lambda^{\bar{z} \bar{z}} \xi' \bar{\nabla} \bar{\partial} \varphi. \] (75)

The action (72) then remains invariant if

\[ \delta \varphi = \xi' + \xi \partial \varphi, \] (76a)
\[ \delta \lambda^{zz} = \xi \partial \lambda^{zz} - \frac{e^{-\varphi}}{\nu \Lambda^2 \rho} \xi' \bar{\nabla} \bar{\partial} \varphi, \] (76b)
\[ \delta \lambda^{\bar{z} \bar{z}} = \xi \partial \lambda^{\bar{z} \bar{z}} + \frac{e^{-\varphi}}{\nu \Lambda^2 \rho} (\xi'' - \xi'' \partial \varphi + \xi' \nabla \partial \varphi) \] (76c)

and we disregard in the action the terms of order \( \Lambda^{-2} \) as is justified for smooth fields. There exists an analog of this transformation for the Nambu-Goto action as well.

For our simplified model \( \lambda^{ab} \) enters the action (72) quadratically, so we can eliminate it using the equation of motion

\[ \lambda^{zz} = \frac{\nu}{\Lambda^2 \rho} e^{-\varphi} \nabla \bar{\partial} \varphi, \quad \lambda^{\bar{z} \bar{z}} = \frac{\nu}{\Lambda^2 \rho} e^{-\varphi} \nabla \partial \varphi. \] (77)

Now the conformal transformation (76a) of \( \varphi \) generates the transformations (76b) and (76c) of \( \lambda^{zz} \) and \( \lambda^{\bar{z} \bar{z}} \). Using (77) we write for the action (72)

\[ S = \int \left\{ \frac{1}{4\pi b_0} \partial \varphi \bar{\partial} \varphi + \frac{d \nu}{2 \Lambda^2 \rho} e^{-\varphi} (\nabla \partial \varphi) (\bar{\nabla} \bar{\partial} \varphi) \right\}, \]
\[ = \int \left[ \frac{1}{4\pi b_0} \partial \varphi \bar{\partial} \varphi + \frac{d \nu}{2 \Lambda^2 \rho} e^{-\varphi} (\partial^2 \varphi - (\partial \varphi)^2) \left( \bar{\partial}^2 \varphi - (\bar{\partial} \varphi)^2 \right) \right]. \] (78)

The second term in the action (78) modifies the Liouville action. It is negligible for smooth fields \( \varphi \) but, as we have already seen, may contribute in our case where typical virtual \( \varphi \) is not smooth.

Notice the difference between the actions (18) and (78). To show it explicitly we can rewrite (78), integrating by parts, in an equivalent form

\[ S = \frac{1}{4\pi b_0^2} \int \left\{ \partial \varphi \bar{\partial} \varphi + 4 \varepsilon e^{-\varphi} \left[ (\bar{\partial} \partial \varphi)^2 + \partial \varphi \bar{\partial} \varphi \partial \bar{\partial} \varphi \right] \right\}, \quad \varepsilon = \frac{\pi d \nu^2 b_0^2}{\Lambda^2 \rho} \] (79)
by using the identity
\[
e^{-\varphi} \left( \partial^2 \varphi - (\partial \varphi)^2 \right) \left( \partial^2 \varphi - (\partial \varphi)^2 \right) = \left[ (\partial \partial \varphi)^2 + \partial \varphi \partial \varphi \partial \partial \varphi \right] + \partial \left[ e^{-\varphi} \partial \varphi \left( \partial \partial \varphi - (\partial \varphi)^2 \right) \right] - \partial (e^{-\varphi} \partial \varphi \partial \partial \varphi).
\] (80)

The additional terms in the action (79) are transformed under the infinitesimal conformal transformation (73a) as
\[
\delta \int e^{-\varphi} (\partial \partial \varphi)^2 = 0, \quad (81a)
\]
\[
\delta \int e^{-\varphi} \partial \varphi \partial \partial \varphi \partial \partial \varphi = \int e^{-\varphi} \xi'' \partial \varphi \partial \partial \varphi + O(\varepsilon^2). \quad (81b)
\]

Treating \(\varepsilon\) as a coupling constant, the change (81b) can be compensated to order \(\varepsilon\) by the following modification of (73a)
\[
\delta \xi \varphi = \xi' + \xi \partial \varphi + \varepsilon \xi'' e^{-\varphi} \partial \varphi + O(\varepsilon^2). \quad (82)
\]

The action (79) then remains invariant under (82) to order \(\varepsilon\).

Under the transformation (82) we obtain
\[
\delta \xi e^\varphi = \partial (\xi e^\varphi) + \varepsilon \xi'' e^{-\varphi} \partial \varphi + O(\varepsilon^2). \quad (83)
\]

The additional term is the derivative with respect to \(\bar{z}\), thus preserving the invariance of the volume
\[
\delta \xi \int e^\varphi = 0. \quad (84)
\]

A very nice property of the modified conformal transformation (82) is that it preserves to order \(\varepsilon\) the commutation relation
\[
\delta \xi \delta \eta e^\varphi - \delta \eta \delta \xi e^\varphi = \delta \zeta e^\varphi, \quad \zeta = \xi \eta' - \xi' \eta. \quad (85)
\]

We thus expect the Virasoro algebra at the classical level.

We believe that the action (79) of the simplified model captures certain characteristic features of the Nambu-Goto action. For the latter we have additionally a path integral over \(\lambda z \bar{z}\) which was frozen in the simplified model. But this kind of path integration, which we discussed above, can only modify the coefficients of the two terms with quartic derivatives in Eq. (79) (as well as of the higher terms dropped there). We thus may expect that path integrating first over \(X^\mu\) and then over \(\lambda^{ab}\) we obtain for the Nambu-Goto string to the first order in \(\varepsilon\) the action
\[
S = \frac{1}{4\pi b_0^2} \int \left\{ \partial \varphi \partial \varphi + 4\varepsilon e^{-\varphi} \left[ \left( \partial \partial \varphi \right)^2 + G \partial \varphi \partial \varphi \partial \partial \varphi \right] \right\} \quad (86)
\]
with a certain constant $G$. For the simplified model (72), we have $G = 1$. The difference from the action (18), resulting from the Polyakov string, is by the presence of the term with $G$. Higher-order in $\varepsilon$ terms are also possible but will not change the result of the next section if universality holds a la Sect. V.

IX. NEW ANOMALY FOR THE NAMBU-GOTO STRING

It is possible to compute the contribution of the term with $G$ in (86) to the renormalization of the one-loop effective action. There is no contribution like $G^2$ because of the structure of the derivatives, so we have only a mixed contribution

$$\text{Fig. 2a} = \frac{1}{2} G \int (\partial_a \varphi)^2 \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon k^2 \varepsilon k^4}{(k^2 + \varepsilon k^4)^2}$$

and the tadpole

$$\text{Fig. 2b} = \frac{1}{2} G \int (\partial_a \varphi)^2 \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon k^2}{(k^2 + \varepsilon k^4)}$$

which is like (30) and (31). Each of the two is logarithmically divergent, but the sum of (87) and (88) is finite and equals

$$\text{Fig. 2a} + \text{Fig. 2b} = \frac{G}{8\pi} \int (\partial_a \varphi)^2.$$  (89)

Summing with the previous result (33) for the action (18), we obtain

$$\frac{1}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} + 2 + 2G + O(b_0^2).$$  (90)

As far as the renormalization of the metric tensor is concerned, there is obviously no contribution from the term with $G$ to it because of the structure of the derivatives. It is suppressed by $\varepsilon$ similarly to Eq. (58). Therefore $\alpha$ is not affected by $G$ and is still given by Eq. (62). We have thus observed a discrepancy with DDK for $G \neq 0$.

Following the consideration in Sect. VI, we can discuss how the change of the action (86) can be induced by a field redefinition. Let us perform the change

$$\varphi \rightarrow \varphi' = \varphi - \frac{\varepsilon}{2} e^{-\varphi} (\partial_a^2 \varphi + G \partial_a \varphi \partial_a \varphi) + O(\varepsilon^2)$$  (91)

for which

$$\int \partial_a \varphi \partial_a \varphi' \rightarrow \int \partial_a \varphi' \partial_a \varphi' = \int \left\{ \partial_a \varphi \partial_a \varphi + \varepsilon e^{-\varphi} \left[ (\partial_a^2 \varphi)^2 + G \partial_a \varphi \partial_a \varphi \partial_b^2 \varphi \right] \right\} + O(\varepsilon^2).$$  (92)
The change \ref{91} produces the determinant of a nontrivial operator, so we substitute $R_1$ given by Eq. \ref{49} by

$$
R_1 = \frac{\det \left( e^{\varphi'} - \varepsilon \partial_a^2 \varphi' + G \varepsilon \partial_a \varphi' \partial_a \varphi' \right)}{\det \left( e^{\varphi'} - \varepsilon \partial_a^2 \right)}.
$$

As before, we would naively expect that \ref{93} tends to 1 when $\varepsilon \to 0$, but it produces in fact the same anomaly as the sum of \ref{32} and \ref{89}.

X. CONCLUSION

The outcome of the calculation of this paper is twofold. First at all it was shown that straightforward computation for the Liouville theory with the action \ref{18} modified by adding the higher-derivative term emerged for the Polyakov string, which is classically negligible for smooth fields as $\varepsilon \to 0$ but quantumly produces uncertainties like powers of $\varepsilon \times \varepsilon^{-1}$ owing to the attendant interaction, agrees at one loop with KPZ-DDK. The result of doing these uncertainties turned out to be universal and not dependent on the form of possible higher terms like it occurs for quantum anomalies. To my knowledge it is for the first time when KPZ-DDK is reproduced by direct computations without any assumptions.

Secondly, the situation with the Nambu-Goto string is different. Now the second term in the square brackets in Eq. \ref{86} emerges additionally to the first one which was the only one for the Polyakov string. It is also important and changes the results. This seemingly implies that the Nambu-Goto and Polyakov strings may be not equivalent at one loop. For a more definite conclusion we need more calculations, in particular that of $q$. If correct this might explain the deviation from the Alvarez-Arvis string spectrum discovered in \ref{18} \ref{20} for an open outstretched Nambu-Goto string, which is hard to understand for the Polyakov string. It was one of my original motivations for this work. The Alvarez-Arvis formula is associated in our approach with the ground state, while the observed deviation corresponds to one loop in our approach.

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[1] J. Ambjorn, B. Durhuus and T. Jonsson, Quantum geometry. A statistical field theory approach, Cambridge (UK) Univ. Press (1997).
[2] J. Ambjorn and Y. Makeenko, *String theory as a Lilliputian world*, Phys. Lett. B 756, 142 (2016) [arXiv:1601.00540]; *Scaling behavior of regularized bosonic strings*, Phys. Rev. D 93, 066007 (2016) [arXiv:1510.03390]; *Stability of the nonperturbative bosonic string vacuum*, Phys. Lett. B 770, 352 (2017) [arXiv:1703.05382 [hep-th]].

[3] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, *Fractal structure of 2D quantum gravity*, Mod. Phys. Lett. A 3, 819 (1988).

[4] F. David, *Conformal field theories coupled to 2D Gravity in the conformal gauge*, Mod. Phys. Lett. A 3, 1651 (1988).

[5] J. Distler and H. Kawai, *Conformal field theory and 2D quantum gravity*, Nucl. Phys. B 321, 509 (1989).

[6] A.M. Polyakov, *Gauge fields and strings*, Harwood Acad. Pub. (1987), pp. 173, 174.

[7] A. Zamolodchikov and A. Zamolodchikov, *Lectures on Liouville theory and matrix models*, 156pp., [http://qft.itp.ac.ru/ZZ.pdf](http://qft.itp.ac.ru/ZZ.pdf).

[8] B. DeWitt, *Dynamical theory of groups and fields*, in Les Houches 1963, eq. (17.95).

[9] P. B. Gilkey, *The spectral geometry of a Riemannian manifold*, J. Diff. Geom. 10, 601 (1975).

[10] B. Durhuus, P. Olesen and J. L. Petersen, *Polyakov’s quantized string with boundary terms*, Nucl. Phys. B 198, 157 (1982).

[11] O. Alvarez, *Theory of strings with boundaries: fluctuations, topology and quantum geometry*, Nucl. Phys. B 216, 125 (1983).

[12] J. Ambjorn and Y. Makeenko, *The use of Pauli-Villars’ regularization in string theory*, Int. J. Mod. Phys. A 32, 1750187 (2017) [arXiv:1709.00995 [hep-th]].

[13] A. M. Polyakov, *Fine structure of strings*, Nucl. Phys. B 268, 406 (1986).

[14] H. Kleinert, *The membrane properties of condensing strings*, Phys. Lett. B 174, 335 (1986).

[15] P. Olesen and S.-K. Yang, *Static potential in a string model with extrinsic curvatures*, Nucl. Phys. B 283, 73 (1987).

[16] E. Braaten, R. D. Pisarski and S.-M. Tse, *The static potential for smooth strings*, Phys. Rev. Lett. 58, 93 (1987).

[17] Y. Makeenko *Mean field quantization of effective string*, JHEP 1807 (2018) 104 [arXiv:1802.07541 [hep-th]].

[18] S. Dubovsky, R. Flauger and V. Gorbenko, *Effective string theory revisited*, JHEP 1209, 044 (2012) [arXiv:1203.1054 [hep-th]].

[19] O. Aharony and Z. Komargodski, *The effective theory of long strings*, JHEP 1305, 118 (2013) [arXiv:1302.6257 [hep-th]].

[20] S. Hellerman, S. Maeda, J. Maltz and I. Swanson, *Effective string theory simplified*, JHEP 1409, 183 (2014) [arXiv:1405.6197 [hep-th]].