A critical dissipative Schrödinger cat qubit

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(Dated: August 10, 2022)

Encoding quantum information onto bosonic systems is a promising route to quantum error correction. The bosonic Schrödinger’s cat quantum code in particular relies on the stabilization, via two-photon drive and dissipation, of a two-dimensional subspace of the state space of the quantum harmonic oscillator to encode a logical qubit. This encoding ensures autonomous correction of dephasing errors already at moderate displacements of the boson field. Here, we study the cat code in the regime of strong third order Kerr nonlinearity. We show that the nonlinearity gives rise to a first-order dissipative phase transition, with spontaneous breaking of the $Z_2$ strong Liouvillian symmetry, at a finite value of the frequency detuning between the driving field and the oscillator. As a consequence, the encoding is efficient over a wide range of values of the detuning defining the critical region of the phase transition. The performance of the code is greatly enhanced compared to the conventional regime of vanishing Kerr nonlinearity, where the autonomous correction only holds for close-to-zero detuning. We present a thorough analysis of the code, both by studying the spectral properties of the Liouvillian, and by quantifying the enhanced performance for parameters within reach of current experimental setups. We argue that, by efficiently operating over a wide range of detuning values, the critical dissipative cat code should considerably simplify the design of coupled and concatenated qubit operation, thanks to its resilience to frequency shifts originating from dispersively coupled elements.

I. INTRODUCTION

The development of quantum computers capable of efficiently executing useful computational tasks relies on the possibility of taming errors, i.e., the irreversible processes stemming from the interaction of the system with its surrounding environment [1–5]. Quantum error correction has therefore developed as a major research field [6–8].

Quantum error correction schemes redundantly encode quantum information onto multi-level quantum systems, in a way that enables to detect and correct specific types of errors without affecting the stored quantum information. The mainstream quantum error correction paradigm consists in encoding the $|0\rangle$ and $|1\rangle$ states of a logical qubit onto a two-dimensional subspace of the Hilbert space characterizing the states of several physical qubits [1, 6]. Despite its inherent scalability, this type of encoding suffers from its large hardware footprint, and the high connectivity between physical qubits required to execute the actual quantum error correction code.

An alternative paradigm to detect and correct quantum errors consists in encoding the logical states of a qubit onto an appropriately selected subspace of the Hilbert space of a harmonic oscillator [9–14]. These bosonic quantum codes are characterized by a reduced hardware footprint, and essentially eliminate the daunting challenge of simultaneously controlling the multiple degrees of freedom of several physical qubits [15]. In the last two decades several bosonic codes have been proposed [12]. Among the most promising bosonic codes, the Gottesman-Kitaev-Preskill (GKP) [9, 16–20] and Schrödinger cat codes [3, 21–25] stand out for their feasibility on various experimental platforms.

A paramount example of bosonic quantum code is the Schrödinger cat code (hereunder, cat code), whereby the $|0\rangle$ and $|1\rangle$ states of a logical qubit are encoded onto the even and odd quantum superpositions of coherent states of opposite phases [24, 26–34]. Its prototypical feature is to suppress phase-flip errors in the logical qubit originating from pure dephasing of the bosonic field. The phase-flip error rate is suppressed exponentially in the displacement field $\alpha$ – and therefore in the average number of photons – of the coherent state defining the code. The cat code is, however, vulnerable to bit-flip errors originating from photon loss processes, whose error rate increases linearly with the average number of photons. As such, cat qubits are the quintessential example of biased-noise qubits [12, 15, 21–23, 35–40]. The cat code has been generalized in several ways, leading to enhanced error correction capability but more challenging experimental feasibility. Among the most promising extensions are the four-component cat code, capable of protecting from dephasing and photon loss [15, 38, 41, 42], and the squeezed cat code [43, 44], which combines partial protection against photon loss and an improved protection against dephasing. Schrödinger cat qubits have been realized, in particular, on superconducting circuit platforms [45], through an engineered dissipation scheme combining two-photon drive and two-photon dissipation [22, 23, 26, 28, 40], which autonomously stabilizes the code manifold [15, 38, 46].
A different approach for the generation and stabilization of the cat manifold, that partially sidelines the necessity for a quadratic dissipation channel, is the Kerr-cat qubit [36, 47]. Kerr-cat qubits are generated in a purely Hamiltonian, quadratically driven Kerr-nonlinear oscillator. The system can be described by a double-well potential (i.e., two, symmetrically displaced, harmonic oscillators), whose distance in phase space increases with the driving field’s amplitude. The tunneling rate between the two harmonic potentials is exponentially suppressed in the drive intensity, effectively reducing the low-lying eigenspectrum of the system to superpositions of pairs of degenerate states [35]. The cat states of opposite parity, in particular, are the system’s degenerate ground states [35]. The cat states of opposite parity, in the drive intensity, effectively reducing the low-lying energies gap proportional to the amplitude of the drive and to the strength of the Kerr nonlinearity [35]. This separation of energies is pivotal to the realization of fast gates and quantum non-demolition readouts [21]. Leakage to higher excited states outside of the cat manifold induced, e.g., by the strong driving of the resonator, thermal excitations, or photon dephasing, inevitably degrades quantum information and limits the performance of such a purely Hamiltonian encoding [21]. Generalizing the idea of a Kerr stabilization, hybrid Hamiltonian- and dissipative-stabilized cat qubits, combining both confinement mechanisms, have also recently been proposed in Ref. [48], as a possible solution to the leakage problem. There, the Kerr nonlinearity is replaced by a two-photon exchange Hamiltonian, resulting in an energy-level spacing between consecutive eigenstates uniform in the cavity’s occupation.

The autonomous error correction capability of the cat code has recently been studied by Lieu et al. [49] in connection with the notion of dissipative phase transition (DPT). DPTs are critical phenomena, where the system’s steady state manifold discontinuously changes as a function of one or more parameters [50–75]. It was shown that the code space, namely the state manifold where the qubit is encoded, is stabilized thanks to the occurrence of a second-order DPT with spontaneous breaking of the strong $Z_2$ symmetry that characterizes the Liouvillian of the system. DPTs have been the focus of intense theoretical and experimental research in recent years [50–75]. Beyond their fundamental interest, DPTs have also been proposed as a resource for quantum metrology [76–79], quantum information, and quantum computing purposes [49].

While the study in Ref. [49] assumes zero Kerr nonlinearity, we propose to enhance the performance of dissipatively-stabilized cat qubits by introducing a sizeable Kerr nonlinearity on the boson field and operating at nonzero detuning. We dub this bosonic code a critical dissipative cat (CDC) code, while referring to the case at $U = 0$ and $\Delta = 0$ as standard cat. The advantage of this approach, as we will discuss in detail, is rooted in the emergence of a first-order DPT associated with spontaneous symmetry breaking. We show how operating the cat code in the vicinity of the first-order DPT entails an increased robustness of the steady-state manifold of the system. This stems from the possibility of driving the oscillator at finite detuning while maintaining an efficient cat qubit encoding, in contrast to the conventional regime [21–23] in which zero detuning is a strict requirement to avoid phase-flip errors in the logical qubit. This feature makes the qubit resilient to spurious frequency shifts originating from dispersively coupled elements. It also makes it possible to design protocols for two-qubit gates, where the two qubits operate at different frequencies, thereby further suppressing spurious resonance coupling between the two qubits during the gate operation. The enhanced performance comes together with other advantages. The regime of parameters that we propose also allows operating at higher two-photon drive and dissipation rates, thus enabling faster quantum gate operations. The larger average photon number also grants an enhanced protection against dephasing errors.

The article is structured as follows. In Section II, the main results of the paper are summarized. Section III sets the theoretical framework by deriving the effective model of the cat code from the microscopic physics of typical experimental realizations. Section IV reviews the link between the symmetry properties of the Liouvillian, the nature of dissipative phase transitions, and the features of the steady-state manifold where quantum information is encoded. Section V is a study of the Liouvillian spectral properties. In Sec. VI we demonstrate the efficiency and robustness of the encoding for a set of experimentally-viable configurations. We derive our conclusions and provide our future perspectives in Sec. VII.

**II. SUMMARY OF THE MAIN RESULTS**

In this work we focus on dissipatively stabilized cat qubits, whose physics is described by the Liouvillian in Eq. (8). Seminal works have experimentally realized this system, usually by operating a cavity at resonance – namely with a detuning $\Delta = 0$ between its frequency and that of the two-photon drive – and in a regime where the Kerr nonlinearity $U$ is negligible compared to the stabilizing two-photon loss rate $\eta (U \ll \eta)$ [22, 23]. It is important to notice that, for vanishing $U$, a finite detuning is always detrimental, as it induces an effective phase-flip error rate on the logical qubit. For such configurations, the optimal working point is thus always one with $\Delta = 0$.

We propose here a different parameter regime, where $U \approx \eta$ and a non-vanishing detuning $\Delta \neq 0$ can lead to a cat code with enhanced performance. The main conclusion of our work is that, while only at $\Delta = 0$ a qubit’s code space is exactly stabilized, the presence of a sizeable Kerr nonlinearity allows stabilizing the qubit’s manifold over a broad range of positive detunings where only a negligibly small logical phase flip rate is introduced. This feature arises because, in the limit of large $U$, the dissi-
patively stabilized cat qubit displays a first-order DPT, where the spontaneous breaking of the strong $Z_2$ symmetry characterizing the Liouvillian ultimately governs the physics of the cat manifold. The link between spontaneous symmetry breaking in DPTs, and the encoding of quantum information, has been discussed by Lieu et al. in Ref. [49], for the case of the second-order phase transition occurring at $U = 0$ and $\Delta \neq 0$. However, this second-order criticality does not enhance the cat’s performance, as the optimal working point remains $\Delta = 0$. Here, we demonstrate that, for large values of $U$, the first-order DPT is accompanied by a quasi-closure of the Liouvillian spectral gap extending over a wide range of values of $\Delta$, which ultimately enables an efficient and resistant encoding of the qubit.

This regime of operation offers multiple advantages. We show that choosing non-zero interaction leads to larger two-photon dissipation rates, and thus to faster logical operations. The logical gates for the CDC qubit require the same set of physical operations on the resonator as in the conventional regime of zero detunings and interaction. The most appealing advantage of the proposed regime however, resides in the considerable simplification of two-qubit operations. The possibility of operating efficiently over a wide range of detunings makes the qubit resilient against uncontrolled changes in this parameter. Physically, such changes can arise in the form of spurious frequency shifts when in the presence of a second nonlinear element, either a second cat qubit or a dispersively coupled Josephson junction. These configurations are typically required to realize two-qubit entangling gates [38] or concatenated-qubit error-correction protocols [80, 81]. As we show, these frequency shifts can represent a major obstacle to the design of controlled two-qubit operations in the conventional regime, but not for the CDC. Our proposal, therefore, opens venues for the realization of reliable protocols for multiple-qubit operations in view of a scalable bosonic qubit architecture.

In what follows, we thoroughly investigate the properties and performance of a bosonic code operating in this nonlinear regime. In Sec. V, we apply the spectral theory of the Liouvillian superoperator. We prove that, at sizeable values of the Kerr nonlinearity $U \sim \eta$, despite the mixed nature of the encoding states, quantum information can be safely stored and manipulated. The bosonic code admits a noiseless subsystem structure [49, 82] whose form is provided in Eqs. (11) and (16), and relies on the presence of a strong Liouvillian symmetry (see Sec. IV A). We show how this feature arises from the existence of a critical region in the vicinity of a first-order DPT characterized by the spontaneous breaking of the strong $Z_2$ symmetry [49]. For the case of a finite Kerr nonlinearity studied here, the critical properties of the system are detailed in the results displayed in Fig. 2. We demonstrate that, within the broken-symmetry region, the cat qubit can be operated at large photon numbers — a feature which is required to realize the protection against phase flip errors characteristic of the cat code.

In Figs. 3, 4 and 5, we demonstrate that, similarly to the standard cat, the CDC is also characterized by a logical phase-flip error rate which is exponentially suppressed in the photon number. For large values of $U$, the suppression coefficient is enhanced with respect to the standard cat code, making the CDC a better choice for architectures with higher dephasing rates.

In Sec. VI, we demonstrate that the critical dissipative cat code can be realized with the physical parameters of current superconducting hardware. Accounting for the parameter constraints reviewed in Sec. III, we show that, at large $U$ and finite $\Delta$, Eq. (9) can be fulfilled in a much larger region of the parameter space. The advantage of the CDC is discussed in detail in Sec. VI, where in Tab. I we propose several sets of parameters, including those of the standard cat configuration in Ref. [23], used for comparison. In Fig. 6 we numerically study the fidelity of the qubit when subject to random frequency shifts, showing the net superiority of the CDC. In Fig. 7, we further show, for these parameter configurations, the enhanced suppression of phase-flip errors in the CDC. Then, we briefly discuss (and numerically show in Appendix B) protocols for one- and two-qubit quantum operations, again arguing for the increased efficiency of the CDC.

The whole analysis in the main text is carried out in the ideal case where single photon loss events induced by the environment are absent, for the sake of simplicity. The conclusions are unchanged when the analysis is repeated assuming realistic values of the photon-loss rate.

## III. GENERATION AND STABILIZATION OF SCHRODINGER CATS

Schrödinger cat states are the even and odd superpositions of two coherent states of opposite phases

$$|C_{\alpha}^{\pm} \rangle = \frac{|\alpha\rangle \pm |\alpha\rangle}{\sqrt{1 + e^{-|\alpha|^2}}}, \tag{1}$$

where the coherent state $|\alpha\rangle$ is the eigenstate of the annihilation operator $\hat{a}$, i.e. $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ [2, 83]. The average number of photons is expressed as $\langle \hat{n} \rangle = \text{Tr}(\hat{\rho} \hat{a}^\dagger \hat{a})$. In what follows, we will refer to this feature as the cat size. The states $|C_{\alpha}^{\pm}\rangle$, when expressed in the number (Fock) basis, contain only components with, respectively, even and odd number of photons. As such, they are eigenstates of the parity operator $\hat{I} = \exp(i \pi \hat{a}^\dagger \hat{a})$ with eigenvalues $\pm 1$. The logical states of a qubit are encoded on the even and odd cat states as $|C_{\alpha}^{+}\rangle \equiv |0_L\rangle$ and $|C_{\alpha}^{-}\rangle \equiv |1_L\rangle$ [c.f. Fig. 1 (a)]. The cat code is generated and stabilized via reservoir engineering by processes involving exclusively the pairwise exchange of photons between the system and the environment [26, 28, 38]. These processes are modeled by the Lindblad master equation.
\[ \partial_t \hat{\rho}(t) = \mathcal{L} \hat{\rho}(t) = -i \left[ \hat{H}, \hat{\rho}(t) \right] + \eta \mathcal{D} \left[ \hat{a}^2 \right] \hat{\rho}(t), \]

\[ \hat{H} = \frac{G}{2} \left( \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger \right). \]  

Here, \( \hat{\rho}(t) \) (hereafter denoted as \( \hat{\rho} \) for brevity) is the system’s density matrix at time \( t \), \( \mathcal{L} \) is the Liouvillian superoperator, and \( \hat{H} \) is the Hamiltonian. \( G \) is the two-photon driving field’s amplitude, and \( \eta \) the rate at which two photons are incoherently emitted. The dissipator \( \mathcal{D}[\hat{L}] \), with jump operator \( \hat{L} \), acts as

\[ \mathcal{D}[\hat{L}] \hat{\rho} = \hat{L} \hat{\rho} \hat{L}^\dagger - \frac{\hat{L}^\dagger \hat{L} \hat{\rho} + \hat{\rho} \hat{L}^\dagger \hat{L}}{2}. \]  

The stabilization of the manifold spanned by \( |C^+_\alpha \rangle \) is to be understood as follows. The density operators \( |C^+_\alpha \rangle \langle C^+_\alpha | \) and \( |C^-_\alpha \rangle \langle C^-_\alpha | \) are stationary under the action of \( \mathcal{L} \) if \( \alpha = \sqrt{-G/\eta} \). As a consequence of Eq. (2), any initial state \( \hat{\rho}(0) \) exponentially converges toward

\[ \hat{\rho}(t \to \infty) = c_{++} |C^+_\alpha \rangle \langle C^+_\alpha | + c_{--} |C^-_\alpha \rangle \langle C^-_\alpha | + c_{+-} |C^+_\alpha \rangle \langle C^-_\alpha | + c_{-+} |C^-_\alpha \rangle \langle C^+_\alpha |, \]

where the coefficients \( c_{ij} \) are uniquely determined by \( \hat{\rho}(0) \) when the initial state has a well defined parity, namely \( \langle \hat{P} \hat{\rho}(0) \rangle = \pm 1 \), the density matrix converges towards the even (odd) cat \( |C^+_\alpha \rangle \langle C^+_\alpha | \) (\( |C^-_\alpha \rangle \langle C^-_\alpha | \)).

### A. Experimental realization

The setup leading to Eq. (2), sketched in Fig. 1(b), has been proposed in [37, 38], and experimentally realized in [22, 23]. Two superconducting cavities (storage and readout) are coupled through a Josephson junction. Pump and drive microwave tones, with intensities \( \epsilon_p \gg \epsilon_d \) respectively, are applied to the readout. The Liouvillian describing the system’s dynamics reads

\[ \mathcal{L} \hat{\rho} = -i \left[ \hat{H}, \hat{\rho} \right] + \eta \mathcal{D}[\hat{a}\hat{a}^\dagger] \hat{\rho} + \sum_{m=r,s} \kappa_m \mathcal{D}[\hat{a}_m^\dagger \hat{a}_m] \hat{\rho}, \]

\[ H = \sum_{m=r,s} \omega_m \hat{a}_m^\dagger \hat{a}_m - E_J \cos(\varphi) \hat{\rho} + \frac{1}{2} 2 \text{Re} \left[ \epsilon_p e^{-i\omega_p t} + \epsilon_d e^{-i\omega_d t} \right] (\hat{a}_r + \hat{a}_r^\dagger), \]

\[ \varphi = \sum_{m=r,s} \varphi_m (\hat{a}_m + \hat{a}_m^\dagger). \]

Here \( E_J \) denotes the Josephson energy, and \( \omega_{p,r,s,d} \) are the pump, storage, readout, and drive frequencies, respectively. The phase \( \varphi \) across the junction is the sum of the phases \( \varphi_m (\hat{a}_m^\dagger + \hat{a}_m) \) across each mode \( (m = r, s) \), with \( \varphi_m \) representing their zero point fluctuation. \( \kappa_m \) is the single photon loss rate of the modes with annihilation operator \( \hat{a}_m \), while \( \kappa_\phi \) describes the total dephasing rate of the storage mode, including the effects of coupled nonlinear modes such as the reservoir [12].

By appropriately tuning the drive frequencies, specific resonant terms within the expansion of the Josephson non-linearity can be singled out [22, 87]. Adiabatically eliminating the fast readout dynamics, and imposing the frequency matching condition \( \omega_p \approx 2\omega_s - \omega_r \), one finds (see Appendix A for details):

\[ \mathcal{L}_s \hat{\rho} = -i \left[ \hat{H}_s, \hat{\rho} \right] + \eta \mathcal{D}[\hat{a}_s^\dagger \hat{a}_s] \hat{\rho} \]

\[ \hat{H}_s = \Delta_s \hat{a}_s^\dagger \hat{a}_s + \frac{G^*}{2} \hat{a}_s^\dagger \hat{a}_s^\dagger + \frac{G}{2} \hat{a}_s^2 - \frac{U_s}{2} (\hat{a}_s^\dagger)^2 \hat{a}_s^2, \]

where \( \Delta_{rp} = \omega_r - \omega_p, \Delta_s \) is the effective detuning between driving and storage mode [c.f. Eq. (A29)], \( U_m \) the self-Kerr interaction intensity of each mode, \( \chi_{rs} \) the cross-Kerr coupling strength between the two modes, and \( \xi_p \) is the displacement field characterizing the steady state of the reservoir (c.f Appendix A). In what follows, we will mainly refer to the effective model in Eq. (8), and therefore omit, in the absence of ambiguity, the subscript \( s \) from our notation.

### B. Choice of parameters

Comparing Eqs. (8) and (2), the following additional terms are found: the effective pump-to-cavity detuning \( \Delta \), the self-Kerr interaction \( U \), the single-photon loss rate \( \kappa \), and the dephasing rate \( \kappa_\phi \).

It is key to identify a regime of operation, and therefore a set of optimal values of the system’s parameters, that maximize the performance of the qubit. A possible optimization strategy is to make Eq. (8) as close as possible to Eq. (2). This is what characterizes, with few variations, many seminal works on cat qubits [22, 23]. While \( \Delta = 0 \) can be obtained by tuning the drive frequencies, the self-Kerr interaction \( U \) can never be completely eliminated without also switching-off the two-photon drive \( G \) and the dissipation \( \eta \) [c.f. Eq. (9)]. Introducing \( U \) alone, preserves the stationary manifold in Eq. (4). However, the displacement \( \alpha \) in the stationary state is generally smaller than in the weakly nonlinear case, and is given by [51, 71]

\[ \alpha(\Delta = 0) = \sqrt{-\frac{G}{U - i\eta}}. \]
This speed must exceed the single-photon dissipation and tum logical operations can be performed on the qubit. Resulting in spurious cross-talk terms, not accounted for by Eq. (8), and error correction codes based on concatenated architecture coupled qubits, in order to realize two-qubit gates. The first is the possibility to control the parameters. The second criterion concerns the speed at which quantum operations can be reliably performed on a cat qubit with respect to the time scale of bit-flip errors, $\gamma/\eta \ll 1$. (ii) The size $|\alpha|^2$ and structure of the cat states, exponentially suppressing the logical phase-flip error rate $\gamma_0/\eta$. (iii) The possibility to execute controlled two-qubit gates while minimizing random frequency shifts induced by cross-talks effects.

The difficulty in simultaneously meeting these points motivates our search for a different parameter regime, where these criteria can be entirely met. As the main result of this work, hereunder, we present and thoroughly analyze an alternative operating regime, whose key features are a sizeable Kerr nonlinearity $U$, and finite values of the detuning $\Delta$. We will in particular demonstrate that, by operating in the vicinity of the DPT characterizing Eq. (8), it is possible to take advantage of the critical behaviour of the system in a way that transcends the limitations of the conventional scheme and results in considerably better performance of the cat qubit.

### IV. Symmetries and Encoding of Quantum Information

In order to discuss the main results, it is necessary to first review the link between the symmetry properties of the Liouvillian, the nature of dissipative phase transitions, and the features of the steady-state manifold where quantum information is encoded.

Liouvillian symmetries often result in a degenerate steady state spanning a subspace of the Hilbert space, which is immune to the effect of the environment [1]. These subspaces go under the name of decoherence-free subspaces (DFS) or – in a more general case – noiseless
The general density matrix structure in the steady state subspace encapsulating the above properties is

\[ \hat{\rho} = \hat{Q} \otimes \hat{M} = \begin{pmatrix} |a|^2 & a^*b \\ b^*a & |b|^2 \end{pmatrix} \otimes \hat{M}, \]

with $|a|^2 + |b|^2 = 1$. The matrix $\hat{Q}$ encoding the quantum information, is stationary under the action of the bare Liouvillian, while $\hat{M}$ in general evolves in time. Whenever $\hat{M}$ is two- or higher-dimensional, the above is an example of a NS. If $\hat{M}$ is instead a number, then Eq. (11) describes a DFS [82]. In what follows, we discuss how symmetries and spontaneous symmetry breaking result in an emergent NS.

### A. Symmetries

Liouvillian symmetries are distinguished into two classes: weak and strong symmetries [49, 51, 96–103]. A weak symmetry occurs when an operator $\hat{O}$ obeys $L(\hat{O}\hat{\rho}\hat{O}^\dagger) = \hat{O}(L\hat{\rho})\hat{O}^\dagger$. A strong symmetry on the other hand, requires $\hat{O}$ to commute with both the Hamiltonian and all jump operators. A main difference between the two classes, resides in the dimension of the Liouvillian’s steady state manifold. While a weak symmetry only guarantees the existence of a single steady state, the steady state manifold of a strongly symmetric Liouvillian is guaranteed to be at least $n$-dimensional, with $n$ the number of non-equivalent irreducible representations of the symmetry group [103]. A strong symmetry results in the existence of $n$ conserved quantities [46].

The system described in Eq. (2), in particular, is characterized by a strong $\mathbb{Z}_2$ symmetry, as both the Hamiltonian and the jump operators are invariant under the transformation $\hat{a} \to -\hat{a}$, and thus commute with the parity operator $\hat{\Pi}$ [46, 101]. It follows that the system’s Hilbert space can be completely decomposed as $\mathcal{H} = \bigoplus_{\mu \in \{\pm\}} \mathcal{H}_\mu$, with $\mathcal{H}_\mu$ the mutually orthogonal eigenspaces of $\hat{\Pi}$ corresponding to the eigenvalues $\pm 1$. Consequently, operators on $\mathcal{H}$ can also be expanded onto four operator subspaces $\mathcal{B}_{\mu\nu} = \mathcal{H}_\mu \otimes \mathcal{H}_\nu$, each one invariant under the action of the Liouvillian $L\mathcal{B}_{\mu\nu} \subseteq \mathcal{B}_{\mu\nu}$. The same decomposition holds for the Liouvillian, which is thus expressed in the block-diagonal form $L = L_{++} \oplus L_{+-} \oplus L_{-+} \oplus L_{--}$ sketched in Fig. 1(c). The dynamics within each Liouvillian block is therefore independent from that of all others, and is conveniently described in terms of its right eigenoperators

\[ L_{\mu\nu} \hat{\rho}_{\mu\nu} = \lambda_{\mu\nu} \hat{\rho}_{\mu\nu} \quad \text{with} \quad -\text{Re} \{\lambda_{\mu\nu}\} < -\text{Re} \{\lambda_{\mu\nu+1}\}. \]

Due to the strong symmetry, $\hat{\rho}_{\mu\nu}$ are also eigenoperators of the parity operator according to the relation

\[ \hat{\Pi} \hat{\rho}_{\mu\nu} = \mu \hat{\rho}_{\mu\nu}, \quad \text{and} \quad \hat{\rho}_{\mu\nu} \hat{\Pi}^\dagger = \nu \hat{\rho}_{\mu\nu}. \]

From here on out we shall therefore refer to $\hat{\rho}_{0\mu}^\nu$ as steady states, and to $\hat{\rho}_{0\mu}^\nu$ as coherences [39, 46].

The possibility to encode quantum information in a non-decaying manifold is directly linked to the existence of a zero Liouvillian eigenvalue within each of its symmetry sectors, $L\hat{\rho}_{0\mu}^\nu = 0 \forall \mu, \nu \in \{\pm\}$. This condition implies infinitely long lived steady states and coherences, and therefore the preservation of the quantum information stored in the qubit. As demonstrated in Refs. [27, 38], for any finite-size system, the only choice of parameters which exactly realizes this construction is one in which $\Delta = 0$. In this case, for the ideal Liouvillian described in Eq. (2), the steady states and coherences are given by $\hat{\rho}_{0\mu}^\nu = |\mathcal{C}_\mu^\nu\rangle\langle\mathcal{C}_\mu^\nu|$, and the states $|\mathcal{C}_+^\nu\rangle, |\mathcal{C}_-^\nu\rangle$ encode the logical codewords. Within this stable manifold, the physics is equivalent to that of a qubit, and the matrix $M$ in Eq. (11) plays no role. If $\Delta \not= 0$, on the other hand, $L\hat{\rho}_{0\mu}^\nu \neq 0$ and coherences decay throughout the time evolution. Moreover, the exact encoding at $\Delta = 0$ is only valid in the absence of dephasing, since $D[a\dagger a]\hat{\rho}_{0\mu}^{\pm\mp} \neq 0$. In the presence of either detuning or dephasing, the Liouvillian’s symmetric structure is retained, but only two zero-eigenvalues exist and are associated with $L_{++}$ and $L_{--}$. This allows for the asymptotic encoding of a classical bit of information in the two corresponding subspaces, but not of quantum information.

Within this description, the detrimental effect of single photon loss is also of easy interpretation. It results in the loss of the four-block diagonal structure of the Liouvillian as shown in Fig. 1(c), while only the coarser two-block structure $L = L_+ \oplus L_-$, typical of a weak $\mathbb{Z}_2$ symmetry, is preserved [49]. In this case, quantum information can only be reliably encoded and manipulated on timescales much shorter than the decay rate ($t \ll 1/\kappa$). For longer times, both classical and quantum information will decay.

Let us now introduce the left eigenoperators $\hat{J}_{n\mu}^{\nu}$ of the Liouvillian as

\[ J_{n\mu}^{\nu} L_{\mu\nu} = L_{\mu\nu} J_{n\mu}^{\nu} = (\lambda_{n\mu}^{\nu})^* \hat{J}_{n\mu}^{\nu}. \]

The left and right eigenoperators of the Liouvillian are related by the bi-orthogonality relation

\[ \text{Tr}\left\{ (J_{n\mu}^{\nu})^\dagger \hat{\rho}_{m\nu}' \right\} = \delta_{\mu,\nu} \delta_{\nu,\nu}' \delta_{m,n}. \]

The left eigenoperators form a basis to express the time-evolution of observables in the Heisenberg picture. The null-eigenoperators $\{J_{0\mu}^{\nu}\}_{\mu,\nu}$ of $L$ in particular, are conserved quantities, defining the set of all observables which remain constant throughout the evolution [46]. As such, the coefficients of the matrix $\hat{Q}$ for a given initial state $\hat{\rho}(0)$ are defined as

\[ Q_{\mu\nu} = \text{Tr}\left\{ (J_{0\mu}^{\nu})^\dagger \hat{\rho}(0) \right\} = \text{Tr}\left\{ (J_{0\mu}^{\nu})^\dagger \hat{\rho}(t) \right\}. \]

### B. Criticality

The presence of degenerate zeros in the Liouvillian spectrum is not only the hallmark of quantum informa-
ton encoding, but also one of the main signatures of a dissipative phase transition – a non-analytical change in the properties of an open quantum system as a function of one of its parameters \([50, 51, 53, 60, 62, 66, 72, 104, 105]\). As detailed in Ref. [50], two types of critical phenomena can occur in an open quantum system: first- and second-order phase transitions. The former (latter) corresponds to a discontinuous (continuous, but not differentiable) change in the system’s properties. Despite their innate differences, both first- and second-order transitions can present with \([50]\) or without \([66]\) spontaneous symmetry breaking (SSB).

The normal phase of a *weakly symmetric* system is generally characterized by a unique steady state. For a weak \(Z_n\) symmetry, SSB causes the emergence of \(n - 1\) additional zero eigenvalues of the Liouvillian, bringing the dimension of the system’s steady state manifold to \(n\), thus realizing a so called \(1 \rightarrow n\) transition \([49]\). However, no quantum information can be stored into this manifold, as the coherences rapidly decay. As discussed in Ref. [49], quantum information can instead be stored within the *broken symmetry phase* of a system characterized by a *strong symmetry*, which supports classical information encoding in its normal phase. In this case, for a \(Z_n\) symmetry, there always exist \(n\) steady states of the system, which become \(n^2\) upon SSB, thereby realizing a so called \(n \rightarrow n^2\) transition. In the present case of the \(Z_2\) symmetric Kerr resonator described by Eq. (8), the \(n \rightarrow n^2\) transition gives rise to the NS qubit structure in Eq. (11).

Even though criticality is only defined in the thermodynamic limit \([53]\), precursors of DPTs arise in *finite-size* systems. In the case of Eq. (8), these precursors emerge for \(G \gg U, \eta\) \([50, 53]\).

V. THE CRITICAL DISSIPATIVE CAT QUBIT

In this Section we apply the considerations on symmetry and criticality to the dissipative cat qubit.

In the case \(U = 0\) investigated in Ref. [49], criticality emerges from a competition between the opposing Liouvillian terms \(G\) and \(\Delta\). Within this picture, it can be shown that the sign of \(\Delta\) is irrelevant, and the efficiency of the quantum information’s encoding depends on \(|\Delta|\) alone. As such, the system displays *only* second-order phase transitions between vacuum- and cat-like states in the limit of \(G/\eta \gg 1\), \([53, 104, 106, 107]\). The two critical points (one for positive and one for negative detuning) divide the phase space into a normal region, hosting at most a classical qubit structure, and a \(Z_2\)-broken region, where quantum information is protected from dephasing errors. For finite cat size, the efficiency of this encoding increases with increasing values of \(G/\eta\). In actual experimental realizations, however, \(G\) is often a fixed variable, whereas \(\Delta\) can be dynamically modified on demand. Furthermore, random fluctuations in the population of ancillary modes result in uncontrollable frequency shifts of the storage mode.

The yellow curves in Fig. 2 summarize the results for the standard cat, obtained by varying \(\Delta\) at \(U = 0\). We numerically diagonalize the Liouvillian

\[
\mathcal{L}_0 = -i \left[ \hat{H}, \hat{\rho} \right] + \eta \mathcal{D}[\hat{a}^2],
\]

of the system described by the effective model in Eq. (8) with \(\kappa = \kappa_\phi = 0\). In Fig. 2(a) we show that the photon number is maximal at \(\Delta = 0\). The spectral gap of the diagonal symmetry sectors of the Liouvillian, defined as

\[
\Lambda_{\mu\nu} \equiv -\text{Re}\{\lambda^\mu_\nu\},
\]

describe the slowest relaxation rate within each sector. As the off-diagonal sectors are not in general guaranteed to have an eigenoperator with zero eigenvalue, we define the off-diagonal Liouvillian gap as

\[
\Lambda_{\mu\nu} = \Lambda_{\nu\mu} \equiv -\text{Re}\{\lambda^0_{\mu\nu}\} \text{ with } \mu \neq \nu.
\]

This quantity defines the effective phase flip rate \(\gamma_\phi\) of the logical qubit. Quantum information is efficiently encoded if \(\gamma_\phi\) is small compared to \(\eta\). We show these gaps in Fig. 2(b) and (c). \(\Lambda_{\mu\nu}\) shows no signs of criticality. For this choice of parameters, the \(Z_2\)-broken region, signaled by \(\Lambda_{\mu\nu} \ll 1\) is extremely narrow as its boundaries are well approximated by the condition \(|\Delta|/G < 1\).

We conclude that, in this case, increasing \(G\) is the only way to increase the size of the cat, and to extend the reach of the broken symmetry phase. This possibility is fundamentally limited by the fact that a large driving field would unavoidably activate higher order non-linear spurious processes in Eq. (5), thereby degrading the cat code structure.

A. Liouvillian analysis of the critical dissipative cat

This picture drastically changes when \(U \neq 0\). In Fig. 2, several curves are plotted for varying \(U\). In order to provide a thorough analysis of the effect of the nonlinearity, we momentarily forgo the constraints of Eq. (9), and set the values of \(\eta\) and \(U\), for fixed \(G\), as

\[
\eta = W \cos(\theta), \quad U = W \sin(\theta), \quad \eta^2 + U^2 = W^2.
\]

Here \(W\) represents an effective nonlinearity at \(\Delta = 0\). Setting \(W\) constant, results in all curves in Fig. 2(a) having the same photon number at \(\Delta = 0\) \(\text{c.f. Eq. (10)}\). The relative angle \(\theta\) is then used to continuously transition between weak- and strongly-nonlinear configurations. A detailed analysis of the system’s properties under the constraints of Eq. (9) will be provided in the next section.

In Fig. 2(a), the average photon number is plotted as a function of \(\Delta\). At \(U \neq 0\), the critical properties of the system depend on the sign of the ratio \(\Delta/U\). For finite \(U\), the behaviour of the system significantly changes in the regime of positive values of \(\Delta\), as a *first-order* transition...
FIG. 2. Liouvillian spectral analysis. (a) Average occupation of the cavity as a function of the detuning $\Delta$ for different values of the Kerr interaction $U$ and the two-photon dissipation $\eta$ satisfying Eq. (20). As $G/W = 5$, and $|\alpha(\Delta = 0)|^2 = G/W$, all curves intersect at $\Delta = 0$ [c.f. Eq. (10)]. The jump in $\langle a^\dagger a \rangle$ registered for $\Delta > 0$ becomes sharper for larger values of $\theta$, heralding the presence of a first order phase transition. (b) Liouvillian gap of the ++ symmetry sector. Its closure, for increasing values of $U$, confirms the emergence of a first order phase transition. (c) Liouvillian gap of the +- symmetry sector. The width of the "broken symmetry" region progressively increases with $U$. Optimal configurations with non-vanishing detuning appear as local minima of the curves and allow taking advantage of the strong nonlinearity (see text). (d) Trace norm $D(z^+, z^-)$, defined in the main text, as a function of $\Delta$ and for $2\theta/\pi = 0.01, 0.96$. The insets show the structure of the full density matrix for the points indicated in the main figure. (i) exemplifies a DFS, (ii) a NS, and (iii) a configuration where quantum information cannot be encoded.

emerges at $\Delta/G \simeq \sqrt{1 + (U/\eta)^2}$, replacing the second-order phase transition existing at $\Delta = G$. The other phase boundary at negative values of $\Delta$ instead is not significantly modified, and the corresponding phase transition is still of second order. The system still presents the finite-size effect of a nonzero $Z_2$-broken order parameter in the whole region between the two phase boundaries. The extent of this region, however, increases with $U$ far beyond the typical values obtained for $U \simeq 0$.

The spectral gap $\Lambda_{++}$ of the diagonal symmetry sector is plotted in Fig. 2(b). A first-order DPT is heralded by the closure of the diagonal gap. The finite value of the gap for $U \simeq 0$ confirms the absence of first-order DPT in the weakly nonlinear case. When increasing $U$ on the other hand, a clear signature of a closing gap appears at positive values of $\Delta$.

Fig. 2(c) displays the off-diagonal spectral gap $\Lambda_{+-}$. This quantity is exactly zero at $\Delta = 0$ for all values of $U$. 
For increasing $U$, however, the off-diagonal gap becomes progressively smaller over a large region of positive values of $\Delta$ as well. This feature shows that the $Z_2$-broken symmetry region becomes progressively larger for an increasing nonlinearity. This region extends mostly in the range of positive values of $\Delta$. In the vicinity of the second-order DPT \cite{49} at negative values of $\Delta$, in fact, only a very narrow range of detuning values are characterized by a small gap. Furthermore, in this region the photon number is relatively small, reducing the resilience of the code to dephasing (see below). On the other hand, a sizeable increase of the steady state photon number occurs in the symmetry-broken phase, over a wide range of positive values of the detuning close to the boundary of the first-order DPT.

Lastly, in Figure 2(d) we provide numerical evidence of the existence of a NS tensor structure in the strong-broken symmetry phase [c.f. Eq. (11)]. We consider the cases $2\theta/\pi = 0.96$ and $2\theta/\pi = 0.01$ carrying respectively large and vanishing nonlinearities. Following the procedure detailed in Ref. \cite{49}, we diagonalize the four Liouvillian blocks $\mathcal{L}_{\mu\nu}$. The resulting four eigenoperators $\hat{\rho}^{\mu\nu}_0$ present the following block structure

$$
\hat{\rho}^{++}_0 = \left( \begin{array}{cc} z_{++} & 0 \\ 0 & 0 \end{array} \right), \quad \hat{\rho}^{-+}_0 = \left( \begin{array}{cc} 0 & 0 \\ 0 & z_{--} \end{array} \right),
$$

$$
\hat{\rho}^{+-}_0 = \left( \begin{array}{cc} 0 & z_{+-} \\ 0 & 0 \end{array} \right), \quad \hat{\rho}^{--}_0 = \left( \begin{array}{cc} 0 & 0 \\ z_{-+} & 0 \end{array} \right),
$$

(21)

We further assume that the diagonal blocks have been diagonalized by unitary transformations $z_{\mu\nu} \leftarrow U_{\mu}^\dagger z_{\mu\nu} U_\mu$, which implies the transformation $z_{\mu\nu} \leftarrow U_{\nu}^\dagger z_{\mu\nu} U_\nu$ for the off-diagonal blocks. The occurrence of a NS is signalled by the fact that all four blocks are diagonal and identical. In a finite-size system this condition is only approximately met. We quantify the discrepancy from the ideal limiting case using the trace distance $D(A, B) = \sqrt{\text{Tr}[(A - B)^2]}$ between the two diagonal blocks $z_{++,}$ and $z_{--}$. The quantity $D(z_{++,}, z_{--})$ is plotted in Fig. 2(d). It displays small values over the whole symmetry-broken region in the case of large nonlinearity, while for $U \simeq 0$ it only vanishes close to $\Delta = 0$. Similar results were found for $D(z_{++,}, z_{--})$ (not shown). The sketches below Fig. 2(d) are color plots of the absolute value of the matrix elements, for the three cases highlighted in the plot. They provide further evidence of the NS structure of the qubit density matrix.

This analysis shows that the advantages of the CDC, anticipated above, are connected to a first-order DPT.

B. Exponential suppression of phase-flip errors

We now study how this picture changes when in the presence of a dephasing rate $\kappa_\phi$ [c.f. Eq. (5)]. Dephasing is a process that conserves parity, and thus preserves the strong symmetry of the Liouvillian. It results in two steady states $\hat{\rho}^{0\mu}_0$, but not in a stabilized qubit manifold, as the coherences $\hat{\rho}^{\mu\nu}_0 (\mu \neq \nu)$ decay over time even for $\Delta = 0$. Consequently, the Liouvillian admits only two conserved quantities $\hat{J}_0^{\mu\nu}$, while $\hat{J}_0^{\mu\nu} (\nu \neq \mu)$ decay. Within the NS picture, this translates into the qubit’s populations not decaying, and its coherences decaying at a rate $\gamma_\phi$ (phase-flip error rate), given by the combined effect of dephasing and of the non-zero Liouvillian gap due to the finite cat size.

A key feature of cat states at $\Delta = 0$ is the exponential suppression of $\gamma_\phi$ in the size of the cat. For $U = 0$, and for small values of $\kappa_\phi$, the phase-flip rate of the qubit can be found via first-order perturbation theory to be \cite{38}

$$
\gamma_\phi = \kappa_\phi \text{Tr} \left\{ (\hat{J}^{+-})^\dagger \mathcal{D} \left[ a^\dagger a \right] \hat{J}^{-+} \right\} \simeq \kappa_\phi \frac{|\alpha|^2}{\sinh \left( 2|\alpha|^2 \right)}.
$$

(22)

Increasing the two-photon drive $G$ results in larger cat size $|\alpha|^2$ [c.f. Eq. (10)], which in turn exponentially suppresses the phase-flip rate. This feature is displayed in Fig. 3, where $\gamma_\phi$ is plotted as a function of $G$ for $\Delta = U = 0$ and several values of $\kappa_\phi$.

![FIG. 3. Phase-flip error rate $\gamma_\phi$ as a function of the two-photon drive amplitude $G$ for varying dephasing rate $\kappa_\phi$, detuning $\Delta = 0$, and $2\theta/\pi = 0.01$ in Eq. (20). The dashed lines are the first-order approximations resulting from Eq. (22).](image)

EQ. (22) is approximate, and only holds for $\Delta = U = 0$. For simulating the full system, we note that rate $\gamma_\phi$ is the decay rate of the coherences of the density matrix $\hat{\rho}$ in Eq. (11). The relation $\gamma_\phi = \Lambda'_{+-}$ then holds, where $\Lambda'_{+-}$ is the off-diagonal gap of the perturbed Liouvillian $\mathcal{L}' = \mathcal{L}_0 + \kappa_\phi \mathcal{D} [a^\dagger a]$.

We show that in the critical dissipative regime ($\Delta \neq 0$, $U \neq 0$) the exponential suppression of $\gamma_\phi$ still holds, with enhanced performance when compared to the conventional cat code. In Fig. 4(a), we plot $\gamma_\phi$ as a function of $G$, for $2\theta/\pi = 0.88$. For each value of $G$, we select the optimal detuning $\Delta_{\text{min}}$ for which the Liouvillian gap is minimal. We find that $\Delta_{\text{min}}$ approximately obeys

$$
\Delta_{\text{min}}(G, \kappa_\phi, \theta) \simeq \beta(\kappa_\phi, \theta) \times G \cdot \Theta (G - G_c(\kappa_\phi, \theta)),
$$

(23)
where $\beta(\kappa_\phi, \theta)$ does not depend on $G$, $\Theta$ is the Heaviside step function, and $G_c(\kappa_\phi, \theta)$ is the critical value of $G$. These results provide clear evidence of the exponential suppression of $\gamma_\phi$ as a function of the cat size in the critical dissipative regime. Note that in both interacting and non-interacting configurations, the exponential scaling of $\gamma_\phi$ depends strongly on the choice of $\Delta$. For $U \approx 0$ the optimal working point is always $\Delta = 0$. This is not the case for sizable values of $U$. As indicated in Eq. (23), while for $G < G_c(\kappa_\phi, \theta)$ the minimum of $\gamma_\phi(\Delta)$ is in $\Delta_{\text{min}} = 0$, for $G > G_c(\kappa_\phi, \theta)$, $\Delta_{\text{min}} > 0$. This feature is highlighted in Fig. 4(b), where the ratio $\gamma_\phi(\Delta)/\gamma_\phi(0)$ is plotted as a function of $\Delta/G$ for different values of $G$ and $\kappa_\phi/W = 10^{-3}$. The optimal value of $\Delta$ jumps from $\Delta_{\text{min}} = 0$ to $\Delta_{\text{min}} \approx G/2$ upon crossing $G_c = 1.96 W$. The behaviour of $G_c(\kappa_\phi, \theta)$ in $\kappa_\phi$ is shown in the inset of Fig. 4(c) for $2\theta/\pi = 0.88$. Its exponential decay in $\kappa_\phi$ signals that, for increasing dephasing rate, a finite value of the detuning is optimal over an increasingly broader range of values of $G$. These results can be summarized in an empirical expression for the exponential dependence of the qubit's phase-flip error rate on $G$:

$$\gamma_\phi(\Delta_{\text{min}}, G) = \gamma_0 \exp\{-\zeta(\kappa_\phi, \theta)G\}. \quad (24)$$

The quantity $\zeta(\kappa_\phi, \theta)$ can be extracted from the numerical results. In Fig. 4(c), it is plotted as a function of $\tan(\theta) = U/\eta$ for $\kappa_\phi = 10^{-3} W$. The dependence changes abruptly near $\tan(\theta) = 1$, and $\zeta(\kappa_\phi, \theta)$ increases favorably in the limit $\theta \to \pi/2$, corresponding to large Kerr nonlinearity.

In the range of exponential decay, the curves in Fig. 4(a) nearly overlap, showing that $\gamma_\phi$ is almost unaffected by $\kappa_\phi$ in the range of values considered. This is not the case for $\Delta = U = 0$, where $\gamma_\phi$ increases linearly with $\kappa_\phi$, as inferred both from Eq. (22) and from the data in Figure 3. This feature, combined with the increased coefficient $\zeta$, demonstrates the enhanced performance of the code in the critical regime.

The exponential suppression of $\gamma_\phi$ in Eq. (24) still holds, though with a reduced exponential rate $\zeta$, if the detuning is not set to its optimal value. Figure 5 shows the phase-flip rate $\gamma_\phi$ and the average photon occupation, as computed for $\Delta/W = 4$. The sudden increase of the photon occupation in Figure 5(b) reveals the critical point $G_c$ where a first order DPT takes place. The rate $\gamma_\phi$ in Figure 5(a) peaks at $G \approx G_c$. For $G \geq G_c$, Eq. (23) predicts an optimal detuning value $\Delta_{\text{min}}/W \approx 4$, resulting in a steep exponential decay of $\gamma_\phi$ with $G$. The decay is slower for still larger values of $G$, as in this regime the optimal detuning value predicted by Eq. (23) is significantly larger than $\Delta/W = 4$.

VI. IMPLEMENTATION OF THE CRITICAL DISSIPATIVE CAT

We now turn to a realistic implementation of the critical dissipative cat code, by enforcing the conditions defined in Eq. (9). In Tab. I, we propose a choice of six sets of values of the physical parameters fulfilling these conditions.

The first set of values coincides with the experimental regime of Ref. [23], and is characterized by a vanishing
nonlinearity. The remaining five rows correspond to values of the dimensionless nonlinearity $U/\eta$ several orders of magnitude larger than those found in, e.g., Ref. [23].

We notice that the values in Tab. I(b) and (c) also lead to larger values of $\eta$ than in the conventional case in Tab. I(a). This entails a faster performance of the logical operations on the CDC qubit, as argued in Appendix B, where we show that a universal set of gates for the CDC is identical to that of the conventional cat [38].

A. Recovery from frequency shifts

Here we study the robustness of the critical dissipative cat code to spurious frequency shifts. Common sources of shifts in the cavity’s effective eigenfrequency are cross-Kerr interactions originating either from spurious cou-
plings in dissipatively coupled qubits, from stochastic jumps in the reservoir’s population induced by thermal excitations, or from other dispersively coupled elements, such as Josephson junctions. In most cases, these effects are modeled by an interaction term of the form $\chi_\alpha \hat{a} \hat{b} \dot{\hat{b}}$, with $\dot{\hat{b}}$ being any ancillary mode coupled to the storage.

To characterize the robustness of the encoding with respect to these types of events, we adopt the following protocol. The system is initialized in the state $\rho(0) = |+L\rangle\langle+L| + |\pm_L\rangle\langle\pm_L|)/\sqrt{2}$. The logical encoding is taken with respect to the manifold stabilized by the bare Liouvillian $\mathcal{L}_0$ [c.f. Eq. (17)], and the corresponding $\hat{Q}(0)$ matrix is computed using Eq. (11).

The system is then quenched with an additional effective detuning $\Delta_{\text{err}} > 0$ over a time $t_0$. Correspondingly, the system evolves under the effective Liouvillian $\mathcal{L}' = \mathcal{L}_0 - i\Delta_{\text{err}}[\hat{a} \dot{\hat{a}}, \hat{b}]$. After a time $t_0$, $\Delta_{\text{err}}$ is switched off and the system evolves to equilibrium under $\mathcal{L}_0$ for a time $t_1 = 5\eta$. From the final state $\rho(t)$ we extract $\hat{Q}(t)$, with $t = t_0 + t_1$, and compare it to the initial state $\hat{Q}(0)$.

Figure 6 shows the infidelity $1 - F$ computed as a function of both the time $t_0$ and $\Delta_{\text{err}}$, where

$$F(\hat{Q}(0), \hat{Q}(t)) = \text{Tr}[\hat{Q}(0)\hat{Q}(t)] + 2\sqrt{\det[\hat{Q}(0)]\det[\hat{Q}(t)]},$$

is the fidelity between the initial and the final qubit states. Figure 6(a) shows the result for the weakly nonlinear regime in row (a) of Tab. I. Here, moderate values of $\Delta_{\text{err}}$ induce a rapid loss of fidelity, constraining the performance of the code to a narrow region of values around $\Delta = 0$, hence confirming the analysis of the previous Section. Figs. 6(b) and 6(c), display the results of the same analysis, respectively for the parameters in rows (b) and (c) of Tab. I. Here, the system is clearly more resilient to frequency shifts. In particular, a sizeable loss of fidelity only occurs at significantly higher values of $\Delta_{\text{err}}$, and at a much lower rate in terms of $t_0$. These conclusions also hold at very short times, as shown in the inset of Fig. 6(b), where infidelity isolines are plotted for the choices of parameters.

B. Effect of dephasing

In a typical experimental setup, dephasing of the storage cavity is mostly due to random fluctuations associated to the thermal occupation of other dispersively coupled circuit elements, such as Josephson junctions or the readout cavity [12, 23, 108]. Calling $n_{\text{th}} \ll 1$ the average thermal occupation of coupled elements, the induced dephasing rate in the Markov approximation is then given by [109]

$$\kappa_\phi \approx n_{\text{th}} \frac{\kappa_r \chi^{2}_{rs}}{\kappa_r^2 + \chi^2_{rs}},$$

(26)

where $\kappa_r$ and $\chi_{rs}$ respectively denote the loss rate and the dispersive coupling strength of the coupled elements.

For this analysis, we assume the same thermal occupation $n_{\text{th}} = 0.04$ for all parameter settings in Tab. I. In this case, Eq. (26) predicts comparable values of $\kappa_\phi$ for all configurations. In Fig. 7, the qubit phase-flip rate $\gamma_\phi$ is displayed for dephasing rates $\kappa_\phi/n_{\text{th}} \in [0.5, 10]$ MHz, and for the parameter sets in Tab. I(a-d). The finite value of $\kappa_\phi$ results in $\gamma_\phi \neq 0$ for all choices of $U$ and $\Delta$. By comparing the results in Fig. 7(b-d) to Fig. 7(a), a broad range of values of the detuning clearly emerges, where the strongly interacting cases largely outperform the non-interacting one for all values of $\kappa_\phi$. The analysis presented in this Section shows that, under the constraints (9) imposed by the experimental setup for dissipative stabilization of the cat code, operating the qubit in the regime of large nonlinearity entails a better protection of quantum information against dephasing.

VII. CONCLUSIONS

We have investigated the properties of the Schrödinger’s cat code in the regime of large Kerr nonlinearity and finite detuning between the two-photon driving field and the cavity frequency. We have demonstrated that, for large nonlinearity compared to the two-photon loss rate, the system undergoes a first-order dissipative phase transition at a finite value of the detuning, with spontaneous breaking of the strong $Z_2$ symmetry of the Liouvillian. In a broad range of detuning values near the phase boundary, this critical dissipative cat qubit is efficiently stabilized and its density operator displays a vanishing phase flip rate. Through an in-depth analysis of the spectral properties of the Liouvillian, we have shown that in this regime the autonomous correction of the phase-flip error is considerably enhanced compared to the weakly nonlinear case. We have considered the typical experimental setup, where the two-photon drive is engineered by dispersively coupling the qubit cavity to a lossier readout cavity through a Josephson junction. By simulating the system for several sets of values of the physical parameters, we have shown that the advantageous regime of critical dissipative cat qubit lies within reach of state-of-the-art superconducting circuit platforms. Finally, we have studied protocols for one- and two-qubit gates, and shown that the critical dissipative regime achieves better accuracy over a universal set of gates. A key additional advantage of this regime resides its resilience to random frequency shifts that commonly arise from thermal fluctuations of the readout cavity and possibly other dispersively coupled elements.

In most experimental realizations of the cat qubit [22, 23], the Kerr nonlinearity is typically vanishing, ensuring a large average photon occupation of the cat states, which underlies the protection against dephasing errors. In this weakly nonlinear regime, efficient encoding is limited to vanishing detuning, thus severely restricting the viable range of physical parameters. The
The present result highlights a much broader region of the physical parameter space, in which an enhanced encoding performance is achieved. Our analysis suggests that the largely unexplored parameter space of bosonic codes may still offer regions where specific properties of the code are enhanced. A more systematic investigation of the regimes of operation of all bosonic codes – possibly guided by automated optimization procedures [110] – may therefore lead to a significant improvement of such encoding schemes, and to a competitively efficient design of a bosonic quantum code architecture.

ACKNOWLEDGMENTS

We acknowledge useful discussions with Victor V. Albert, David Schlegel, Alexander Grimm, and Carlos Sánchez-Muñoz. This work was supported by the Swiss National Science Foundation through Project No. 200020_185015, and was conducted with the financial support of the EPFL Science Seed Fund 2021.

Appendix A: Full derivation of the Hamiltonian

In this section we present the derivation behind Eq. (8) of the main text. Although we closely follow the arguments presented in the supplementary material of Ref. [22], we shall use this space to expand upon several of its aspects. We start by writing the Hamiltonian of the readout and storage modes coupled through a Josephson junction.

$$\mathcal{L} \dot{\hat{\rho}} = -i \left[ \hat{H}, \hat{\rho} \right] + \kappa_r D[\hat{a}_r] \hat{\rho} + \kappa_s D[\hat{a}_s] \hat{\rho},$$

(A1)

$$\hat{H} = \sum_{m=r,s} \omega_m \hat{a}_m^\dagger \hat{a}_m - E_f \cos(\hat{\varphi}) + 2 \text{Re} \left( \epsilon_p e^{-i\omega_p t} + \epsilon_d e^{-i\omega_d t} \right) \left( \hat{a}_r + \hat{a}_r^\dagger \right),$$

(A2)

$$\dot{\hat{\varphi}} = \sum_{m=r,s} \varphi_m \left( \hat{a}_m + \hat{a}_m^\dagger \right).$$

(A3)

The physical meaning of all terms above is discussed in the main text. The system is illuminated by two microwave tones, a drive and a pump, with complex amplitudes $\epsilon_d$, $\epsilon_p$ and frequencies $\omega_d$, $\omega_p$, respectively. We consider the regime in which the pump is strong and far off-resonant, while the drive is weak and close to resonance with respect to the readout mode. In particular, we have

$$\omega_d \simeq \omega_r \quad \text{and} \quad \epsilon_p \sim (\omega_p - \omega_r) \gg \frac{E_f}{24} \| \hat{\varphi} \|^4, \epsilon_d.$$
We move to a convenient frame via the unitary rotation
$$\hat{U} = \hat{U}_1 \hat{U}_2,$$
with
$$\hat{U}_1 = \exp \left\{ i t \left( \omega_d \hat{a}_s \hat{a}_s + \frac{\omega_d + \omega_r}{2} \hat{a}_s \hat{a}_s \right) \right\} \quad (A5)$$
$$\hat{U}_2 = \exp \left\{ i \xi_p (t) \hat{a}_r - \xi_p (t) \hat{a}_r \right\}.$$  
\(\hat{U}_2\) removes the underlying phononic mean-field population rotating with frequency \(\omega_p\) and supported by the strong pump \(\epsilon_p\). In fact, it can be decomposed as
$$\hat{U}_2 = \left( \hat{U}_2^{(0)} \right)^\dagger \hat{U}_2^{(1)} \hat{U}_2^{(0)}$$
$$\hat{U}_2^{(0)} = \exp \{ i \omega_p \hat{a}_r \hat{a}_r \} \quad (A6)$$
$$\hat{U}_2^{(1)} = \exp \{ i \xi_p \hat{a}_r - \xi_p \hat{a}_r \},$$
i.e., a typical time-independent displacement transformation implemented in a frame rotating at \(\omega_p\). Here, \(\xi_p (t) = \exp \{ i \omega_p t \} \xi_p\), and
$$\xi_p = \frac{-i \epsilon_p}{\omega_r - \omega_p} \quad (A7)$$
is the amplitude of the coherent steady state of the reservoir’s isolated dynamics. \(\hat{U}_1\), on the other hand, is applied in hindsight of the phase matching condition with \(\omega\) \(\omega_r = \omega_p + \epsilon_p = \omega_d + \omega_r\) which, if met, establishes an approximate four-wave mixing channel within the Josephson Junction. This, as we shall see, allows for the parametric process converting two storage photons into one readout and one pump photon. Note that this series of transformations does not remove the time dependence from the Hamiltonian completely, but rather confines it within its nonlinear part. Indeed, in the rotated frame, the Hamiltonian reads
$$\hat{H} = \left( \omega_r - \omega_d \right) \hat{a}_d \hat{a}_d + \left( \omega_r \omega_d \right) \hat{a}_s \hat{a}_s$$
$$+ \left( \epsilon_d \hat{a}_r + \epsilon_d \hat{a}_r \right) - E_J \cos \hat{\varphi},$$
with
$$\hat{\varphi} = \sum_{m=r,s} \varphi_m \left( \hat{a}_m \hat{a}_m \right) + \left( \hat{\xi}_p + \hat{\xi}_p \right) \varphi_r$$
$$\hat{\varphi}_r = \exp \left\{ -i \omega_d t \right\} \hat{\varphi},$$
$$\hat{\varphi}_s = \exp \left\{ -i \left( \omega_d + \omega_r \right) t \right\} \hat{\varphi}.$$  
\(\hat{\varphi}\) is the total phase of the system, \(\hat{\varphi}_r\) and \(\hat{\varphi}_s\) are the phases of the readout and storage fields, respectively.

Finally, the cosine term is expanded up to fourth order as
$$\cos (\hat{\varphi}) = 1 - \hat{\varphi}^2 / 2 - \hat{\varphi}^4 / 24 + O (\hat{\varphi}^6),$$
and the resonant terms alone are retained.

### Rotating wave approximation

Below we develop a general scheme to easily identify the resonant elements within \(\hat{Q}^n\). Let
$$\hat{A} = \sum_{i=1}^3 \hat{A}_i = \varphi_s \hat{a}_s + \varphi_r \hat{a}_r + \varphi_r \hat{\xi}_p,$$  
(A10)
with
$$\hat{A}_1 = \varphi_s \hat{a}_s, \quad \hat{A}_2 = \varphi_r \hat{a}_r, \quad \hat{A}_3 = \varphi_r \hat{\xi}_p,$$  
(A11)
and \(\hat{Q} = \hat{A} \hat{A} + \hat{Q} = \hat{\varphi}^2\). Although here we restrict to the case \(i = 1, \ldots, 3\), the generalization to a higher number of modes \(i = 1, \ldots, m \geq 3\) is immediate. Given that \(C = [\hat{A}, \hat{A}^\dagger] = \varphi_s^2 + \varphi_r^2 \in \mathbb{R}\), the non-commutative binomial theorem [111] can be used to write
$$\hat{Q}^n = \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)! k!} \left( \frac{C}{2} \right)^k : \hat{Q}^{n-2k} :,$$  
(A12)
with
$$: \hat{Q}^n : = \sum_{k=0}^{n} \left( \frac{n}{k} \right)^k \hat{Q}^{n-k} \quad (A13)$$
representing the normal ordered expression of \(\hat{Q}\) [112, 113]. Moreover, the multinomial theorem provides us with an explicit expansion for the terms in the sum:
$$\hat{A}^n = \sum_{\Sigma_\alpha = n} \left( \frac{n}{\alpha} \right) x_\alpha \quad \text{with} \quad \left\{ \begin{array}{l} \alpha = (\alpha_1, \ldots, \alpha_3) \\ \sum_\alpha = n \\ x_\alpha = \prod_{i=1}^3 \hat{A}_i^{\alpha_i} \end{array} \right.,$$  
(A14)
The general term within the expansion of \(\hat{Q}^n\), therefore, will be of the form
$$(\hat{A}^\dagger)^k \hat{A}^{n-k} = \sum_{\Sigma_\alpha = k, \Sigma_\beta = n-k} \left( \frac{k}{\alpha} \right) \left( \frac{n-k}{\beta} \right) \prod_{i=1}^3 \hat{A}_i^{\alpha_i} \prod_{j=1}^3 \hat{A}_j^{\beta_j} \quad (A15)$$
The specifics of the problem under investigation enter in the rotation frequencies of the different \(\hat{A}_i\):
$$\hat{A}_1 \sim \exp \left\{ -i \left( \omega_p + \omega_d t \right) \right\}$$
$$\hat{A}_2 \sim \exp \{ -i \omega_d t \}$$
$$\hat{A}_3 \sim \exp \{ i \omega_p t \}.$$  
(A16)
The resonance condition, in this case, explicated in a set of two coupled equations for the coefficients in Eq. (A15)
$$\begin{cases} \frac{\alpha_1}{2} + \alpha_2 - \beta_1 - \beta_2 = 0 \\ \frac{\alpha_1}{2} - \alpha_3 - \beta_1 + \beta_3 = 0 \end{cases}$$  
(A17)
uniquely determining the possible values of \(D_i = \alpha_i - \beta_i\). Equation (A17) provides therefore an algorithmic way of singling out the resonant terms within a symbolic evaluation of Eq. (A13). We immediately recognize that no resonant terms can exist within \(\hat{Q}^n\): if \(n \% 2 = 1\). Hereunder, we shall thus restrict our considerations to the case \(n = 2m\) with \(m \in \mathbb{N}_+\) and note that Eq. (A13) can be rewritten as

\(\text{Eq. (A13)}\)
\[ :Q^n : \sim \left( \frac{n}{m} \right) (\hat{A}^\dagger)^m \hat{A}^m + \sum_{k=m+1}^{n} \left( \frac{n}{k} \right) (\hat{A}^\dagger)^k \hat{A}^{n-k} + h.c. \]  

(A18)

where the symbol \( \sim \) is used to identify equality up to off-resonant contributions. Note that at this stage all terms of the expansion have been retained except for \( k = 0 \) and \( k = n \) which are evidently off-resonant. Enforcing the conditions in Eq. (A17), the elements of the above sum simplify to

\[
\begin{align*}
(\hat{A}^\dagger)^m \hat{A}^m &\sim \sum_{\alpha_1=0}^{m} \sum_{\alpha_3=0}^{m} C_{m,k}(\alpha_1,\alpha_3) \\
(\hat{A}^\dagger)^{\alpha_1} \hat{A}^{\alpha_1}(\hat{A}_3^{\alpha_3} \hat{A}_3^{\alpha_3}) \approx & \sum_{\alpha_1=2\zeta}^{m} \sum_{\alpha_3=\zeta}^{m} C_{m,k}(\alpha_1,\alpha_3) \\
(\hat{A}_2^{\alpha_1} \hat{A}^{\alpha_1}(\hat{A}_2^{\alpha_3} \hat{A}_2^{\alpha_3}) &\approx (\hat{A}_1^{\alpha_1-2\zeta} \hat{A}_3^{\alpha_3-\zeta} \hat{A}_2^{\alpha_3-\zeta} + h.c.),
\end{align*}
\]

and

\[
\begin{align*}
(\hat{A}^\dagger)^k \hat{A}^{n-k} &\sim \sum_{k=0}^{n-k} \sum_{\alpha_3=0}^{n-k} C_{n,k}(\alpha_1,\alpha_3) \\
(\hat{A}^\dagger)^{\alpha_1} \hat{A}^{\alpha_1}(\hat{A}_3^{\alpha_3} \hat{A}_3^{\alpha_3}) \approx & \sum_{k=0}^{n-k} \sum_{\alpha_3=0}^{n-k} C_{n,k}(\alpha_1,\alpha_3) \\
(\hat{A}_1^{\alpha_1-2\zeta} \hat{A}_2^{\alpha_2} \hat{A}_3^{\alpha_3-\zeta} &\approx (\hat{A}_1^{\alpha_1-2\zeta} \hat{A}_3^{\alpha_3-\zeta} \hat{A}_2^{\alpha_3-\zeta} + h.c.),
\end{align*}
\]

where \( \zeta = k-m, S_{13} = \alpha_1 + \alpha_3, \) and

\[
C_{n,k}(\alpha_1,\alpha_3) = \binom{k}{\alpha_1} \binom{n-k}{\alpha_3} (\alpha_1 - 2\zeta) (\alpha_3 - \zeta). 
\]

(A21)

The terms of interest for the present discussion, \( \hat{Q}^2 \) and \( \hat{Q}^4 \), are straightforwardly derived since the sums in Eq. (A18) contain at most one term:

\[
\begin{align*}
: \hat{Q}^2 : &\sim 2 \sum_i \hat{A}_i^\dagger \hat{A}_i, \\
: \hat{Q}^4 : &\sim \left[ 12(\hat{A}_1^\dagger)^2 \hat{A}_1^\dagger \hat{A}_2 + h.c. \right] + 6 \sum_i (\hat{A}_i^\dagger)^2 \hat{A}_i^2 \\
&+ 24 \sum_{i,j \neq i} \hat{A}_i^\dagger \hat{A}_i^\dagger \hat{A}_j,
\end{align*}
\]

(A22)

(A23)

Substitution into Eq. (A12), consequently yields

\[
- \hat{E}_J \cos \varphi \approx \frac{E_J}{2} \hat{\varphi}^2 - \frac{E_J}{24} \hat{\varphi}^4
\]

\[
\sim \sum_{m=r,s} \omega_m (\hat{a}_m^\dagger \hat{a}_m) \hat{a}_m - \sum_{m=r,s} \frac{U_m}{2} (\hat{a}_m^\dagger)^2 \hat{a}_m - \chi_{rs} \hat{a}_s^\dagger \hat{a}_s^\dagger \hat{a}_r,
\]

(A24)

where

\[
U_m = \frac{E_J \varphi_m^4}{2}, \quad \chi_{rs} = E_J \varphi_s \varphi_r^2, \quad g_2 = \frac{\chi_{rs} \varphi_s^2}{2},
\]

(A25)

with \( m = s, r, \) and

\[
\omega_m^\text{eff} = E_J \varphi_m^2 \left( 1 - \frac{C}{2} - \varphi_r^2 |\varphi|_p^2 \right).
\]

(A26)

All in all, the Liouvillian under examination reads

\[
\hat{L} = -i \left[ \hat{\mathcal{H}}, \hat{\rho} \right] + \kappa_r D[\hat{a}_r] \hat{\rho} + \kappa_s D[\hat{a}_s] \hat{\rho}
\]

\[
\hat{\mathcal{H}} \approx \hat{H}_0 + \hat{\mathcal{H}}_{\text{Kerr}} + \hat{\mathcal{H}}_{\text{param}}
\]

(A27)

with

\[
\begin{align*}
\hat{\mathcal{H}} \approx & \hat{H}_0 + \hat{\mathcal{H}}_{\text{Kerr}} + \hat{\mathcal{H}}_{\text{param}} \\
\hat{\mathcal{H}}_0 &\approx \sum_{m=r,s} \Delta_m \hat{a}_m^\dagger \hat{a}_m \\
\hat{\mathcal{H}}_{\text{Kerr}} &\approx - \sum_{m=r,s} \frac{U_m}{2} (\hat{a}_m^\dagger)^2 \hat{a}_m^2 - \chi_{rs} \hat{a}_s^\dagger \hat{a}_s^\dagger \hat{a}_r, \\
\hat{\mathcal{H}}_{\text{param}} &\approx - \left[ \left( g_2 (\hat{a}_r^\dagger)^2 + \epsilon_d \right) \hat{a}_r + h.c. \right],
\end{align*}
\]

and

\[
\Delta_s = \omega_s + \omega_m^\text{eff} - \frac{\omega_m + \omega_d}{2} \Delta_r = \omega_r + \omega_m^\text{eff} - \omega_d.
\]

(A29)

Here, \( \hat{H}_0 \) incorporates all frequency shifts induced by the normal ordering of the nonlinear terms and the driving tones alike. \( \hat{\mathcal{H}}_{\text{Kerr}} \) all self-Kerr and cross-Kerr coupling terms, and \( \hat{\mathcal{H}}_{\text{param}} \) the seed for the degenerate parametric driving of the storage mode.

Through Eqs. (A19) and (A20), one can also easily evaluate the leading correction to such approximation. The only missing quantity for its evaluation is in fact \( : \hat{Q}^6 : \), which reads

\[
: \hat{Q}^6 : \sim \left[ 180 (\hat{A}_1^\dagger)^2 \hat{A}_3^\dagger \hat{A}_3^\dagger \hat{A}_2 \hat{A}_2 + 180 (\hat{A}_1^\dagger)^2 (\hat{A}_2^\dagger)(\hat{A}_2 \hat{A}_3 \hat{A}_3 \hat{A}_2) \right] + 120 (\hat{A}_1^\dagger)^3 \hat{A}_3^\dagger \hat{A}_1 \hat{A}_2 + h.c. + 720 \prod_i \hat{A}_i^\dagger \hat{A}_i \\
+ 20 \sum_i (\hat{A}_i)^3 \hat{A}_i^\dagger + 180 \sum_{i,j \neq i} (\hat{A}_i)^2 \hat{A}_j^\dagger \hat{A}_j 
\]

(A30)

### Adiabatic elimination of the reservoir mode

Assuming the reservoir mode to be strongly damped, we can approximate the joint density matrix of the two modes as

\[
\rho_{sr} = \rho_{00} |0\rangle \langle 0| + \delta (\rho_{01} |0\rangle \langle 1| + \rho_{10} |1\rangle \langle 0|) + \delta^2 (\rho_{11} |1\rangle \langle 1| + \rho_{20} |2\rangle \langle 0| + \rho_{02} |0\rangle \langle 2|),
\]

(A31)

with \( \delta \ll 1 \). Retaining only up to second order terms in \( \delta \) within the equations of motion stemming from such an ansatz, we can adiabatically eliminate the reservoir mode
in favour of a master equation describing the dynamics of the reduced density matrix $\hat{\rho}_s \approx \text{Tr}_r \{ \hat{\rho}_{sr} \} = \hat{\rho}_{00} + \delta^2 \hat{\rho}_{11}$ of the storage mode alone. In particular, we consider $g_2$, $\epsilon_d$, $U_\alpha$, $\chi_{rs}$, $\Delta_r$, $\kappa_s \sim \delta \kappa_r$. Using that $\partial_t \hat{\rho}_{ij} = (i \mathcal{L} \hat{\rho}_{rs}[j])$, we have

$$\frac{\partial \hat{\rho}_{00}}{\kappa_r \partial t} = -\frac{i}{\kappa_r} \left\{ \left[ \Delta_s \hat{a}_s^\dagger \hat{a}_s - U_\alpha (\hat{a}_s^\dagger)^2 \hat{a}_s^\dagger, \hat{\rho}_{00} \right] + \frac{\kappa_s}{\kappa_r} \mathcal{D}[\hat{a}_s] \hat{\rho}_{00} \right\} + \delta^2 \left[ \hat{\rho}_{11} - (\hat{A}^\dagger \hat{\rho}_{10} - \hat{\rho}_{01} \hat{A}) \right] + \mathcal{O}(\delta^3)$$

\hspace{1cm} (A32)

$$\frac{\partial \hat{\rho}_{10}}{\kappa_r \partial t} = -\left( \frac{1}{2} + i \frac{\Delta_r}{\kappa_r} \right) \hat{\rho}_{10} - i \hat{A} \hat{\rho}_{00} + \mathcal{O}(\delta)$$

\hspace{1cm} (A33)

$$\frac{\partial \hat{\rho}_{11}}{\kappa_r \partial t} = -i \left( \hat{A} \hat{\rho}_{01} - \hat{\rho}_{10} \hat{A}^\dagger \right) - \hat{\rho}_{11} + \mathcal{O}(\delta),$$

\hspace{1cm} (A34)

where $\hat{A} = \frac{1}{\kappa_r} (g_2 \hat{a}_s^2 + \epsilon_d) \sim \mathcal{O}(1)$. At this point the adiabatic approximation consists in setting $\partial_t \hat{\rho}_{01} = \partial_t \hat{\rho}_{11} = 0$ and using the expression for $\hat{\rho}_{01}$ and $\hat{\rho}_{11}$ resulting from the homogeneous solutions to Eqs. (A33) and (A34), as well as Eq. (A32), to obtain

$$\frac{\partial \hat{\rho}_s}{\partial t} = -i \left[ \Delta_s \hat{a}_s^\dagger \hat{a}_s - U_\alpha (\hat{a}_s^\dagger)^2 \hat{a}_s^\dagger, \hat{\rho}_s \right] + \frac{\kappa_s}{\kappa_r} \mathcal{D}[\hat{a}_s] \hat{\rho}_s$$

\hspace{2cm} - $\delta^2 \kappa_r \left\{ \Gamma \left( \hat{A}^\dagger \hat{\rho}_s - \hat{\rho}_s \hat{A} \right) + \text{h.c.} \right\}$,

\hspace{1cm} (A35)

where $\Gamma^{-1} = 1/2 - i \Delta_r / \kappa_r$. Through some rearranging, the above can be cast into the common form $\hat{\rho}_s = \mathcal{L}_s \hat{\rho}_s$ with

$$\mathcal{L}_s \hat{\rho}_s = -i \left[ \hat{H}_s, \hat{\rho}_s \right] + \left\{ \eta \mathcal{D}[\hat{a}^2_s] + \frac{\kappa_s}{\kappa_r} \mathcal{D}[\hat{a}_s], \hat{\rho}_s \right\} + \mathcal{O}(\delta^3)$$

$$\hat{H}_s = \Delta_s \hat{a}_s^\dagger \hat{a}_s + \frac{G^2}{2} \hat{a}_s^\dagger \hat{a}_s^\dagger + \frac{G}{2} (\hat{a}_s^\dagger)^2 - \frac{U_\alpha}{2} (\hat{a}_s^\dagger)^2 \hat{a}_s^\dagger,$$

\hspace{1cm} (A36)

The definition of the different coefficients has been given in Eq. (9) for the case $\Delta_r = 0$. In the general case of $\Delta_r \neq 0$, the only quantities that are changed are $G$ and $\eta$, which read

$$G = -\frac{2i \Gamma \epsilon_d g_2}{\kappa_r}, \quad \eta = \frac{2 \text{Re}[\Gamma]}{\kappa_r} |g_2|^2.$$

\hspace{1cm} (A37)

### Appendix B: Universal gates

#### 1. Single qubit gates

We first focus on performing Rabi oscillations of an arbitrary angle $\theta$ around the $z$-axis of the Bloch sphere, as illustrated in Fig. (8). We study the three parameter settings in rows (a-c) of Tab. (1). For each setting, the gate operation is performed onto the manifold stabilized by the respective parameters. Hence, the logical code words $|0_L\rangle$ and $|1_L\rangle$ denote different states for each of the three settings. Borrowing from [38], the simplest Hamiltonian ensuring this gate operation is that of a linear resonant drive of the storage mode: $\hat{H}_X = F (\hat{a} + \hat{a}^\dagger)$. Indeed, the density operators $|0_L\rangle\langle 0_L|$ and $|1_L\rangle\langle 1_L|$ belong respectively to the $++$ and $--$ diagonal parity manifolds, which can be connected by single-photon exchange. As we will show, the accuracy with which this target state can be generated scales with $(F/\eta)^{-1}$. The driving field amplitude $F$ should then be chosen to be much smaller than $\eta$, in order to minimize the diffusion out of the steady state manifold, induced by higher order terms in the gate operation [38]. The effective frequency (period) $\Omega_X = 2F \text{Re}\{\alpha\} \left( T_X = \pi / \Omega_X \right)$ of the Rabi oscillation is determined by setting $\Delta = 0$ and applying the Hamiltonian $\hat{H}_X$ to the pure cat states $|\psi^+_F\rangle$. We numerically verify that the same value of $\Omega_X$ approximately holds for all configurations explored in Table I(b-f).

In Fig. 8(a), the evolution of the density matrix elements $Q_{\mu\nu}$ ($\mu, \nu \in \{++,--\}$) under the action of Hamiltonian $\hat{H}_X$ are plotted over half a period of the Rabi oscillation. For these data, we chose the parameter settings of Tab. I(f), and set $\Delta = G$ and $F/\eta = 1/40$. Figure 8(b) displays the evolution of the density matrix elements $Q_{\mu\nu}$ ($\mu, \nu \in \{++,--\}$) under the action of Hamiltonian $\hat{H}_X$ are plotted over half a period of the Rabi oscillation. For these data, we chose the parameter settings of Tab. I(f), and set $\Delta = G$ and $F/\eta = 1/40$. Figure 8(b) displays the evolution of the expected state $|+,L\rangle$ and the numerical simulation. All quantities were computed for the settings in row (f) of Tab. I and $\Delta = G$. 

![Single qubit X-gate on a critical dissipative system.](image-url)
We consider the parameter configurations of Table I(a-c) and operate them respectively at $\frac{\Delta}{G} = 0, 0.40, 0.68$. We set $\kappa = \kappa_\phi = 1 \text{kHz}$.

$L'\hat{\rho} = L_0\hat{\rho} - i[\hat{H}_X, \hat{\rho}]$. The two plots clearly indicate that the gate operation can be executed on the critical dissipative cat code. In particular, the accuracy of the operation increases exponentially with $(F/\eta)^{-1}$.

In the presence of single-photon loss with rate $\kappa$ and dephasing with rate $\kappa_\phi$, an upper bound to the ratio $(F/\eta)^{-1}$ arises, as increasing this quantity slows down the gate operation making it more vulnerable to photon-loss and dephasing errors. To gauge the efficiency at which $X$ operations can be performed in the presence of photon loss and dissipation, we set $\kappa = \kappa_\phi = 1 \text{kHz}$, and consider the parameter settings in Tab. I(a-c). We initialize the system in the state $|0_L\rangle$, perform a full Rabi oscillation, and evaluate the overlap between the final and initial states. The results are shown in Fig. 9 as a function of $F/\eta$. For both interacting configurations [Tab. I(b,c)] we observe a significant improvement in the performance of the $X$ gate with respect to the $U = 0$ case [Tab. I(a)].

To complete a universal set of quantum gates, one of the other single-qubit gates, either $Y$ or $Z$, needs to be implemented. As pointed out in Ref. [38], both gates require the exchange of $|+L\rangle$ and $|-L\rangle$. The parity operator $\Pi = \exp\{i\pi a^\dagger a\}$ realizes the $Z$-gate, as the steady states of the $++$ and $--$ symmetry sectors are $\pm 1$ eigenstates of $\Pi$. However, hardware implementation of $\Pi$, require considering that $|\pm L\rangle$ are highly stable under the effect of two photon drive and dissipation. Thus, it is necessary to temporarily switch-off the stabilization process for $Y$ and $Z$ gates to perform properly.

2. Two qubit gate

In the same way in which the Hilbert space of two physical qubit acquires a tensor product structure, so do the logical qubits encoded in the NS. In fact, one can define the two qubit matrix $Q^{(2)}$ using the conserved quantities of the single qubit as

$$Q^{(2)}_{\mu\nu',\nu''} = \text{Tr}\left\{\hat{\rho} \left(\hat{J}_0^{\mu\nu} \hat{J}_0^{\nu''}\right)^\dagger\right\}. \quad (B1)$$
Within this logical Hilbert space, we can describe the effects of entangling gates. For instance, as a witness of entanglement generation upon the action of $\hat{\sigma}_1^x \otimes \hat{\sigma}_2^x$, we consider

$$\left\langle \hat{\Pi}_{B_L^+} \right\rangle = \text{Tr}\left\{\hat{\Pi}_{B_L^+} Q^{(2)}\right\},$$  \hspace{1cm} (B2)

where $\hat{\Pi}_{B_L^+}$ is the projector on the logical Bell states defined as

$$|B_L^+\rangle = \frac{|0_L\rangle \otimes |0_L\rangle \pm |1_L\rangle \otimes |1_L\rangle}{\sqrt{2}}.$$  \hspace{1cm} (B3)

Similarly to the conventional cat qubit, an entangling Hamiltonian of this form between two interacting qubits is realized by the photon hopping Hamiltonian

$$\hat{H}_{XX} = J \left( \hat{a}_1 \hat{a}_1^\dagger + \hat{a}_2 \hat{a}_2^\dagger \right),$$  \hspace{1cm} (B4)

where $a_1$ and $a_2$ are the annihilation operators of the storage modes of the two cat qubits.

Within the effective qubit manifold, it can be shown that this Hamiltonian is approximately recast as \[38\]

$$\hat{H}_{XX} \simeq \Omega_{XX} \hat{\sigma}_1^x \otimes \hat{\sigma}_2^x,$$  \hspace{1cm} (B5)

where $\Omega_{XX} = 2J|\alpha_1 \alpha_2|$. Similarly to the one qubit gate, this approximation is exact in the limit $J/\eta \to 0$. We provide here evidence of the effectiveness of this protocol, when applied to the parameter settings in row (c) of Tab. I. We initialize the system to the logical state $|0_L\rangle \otimes |0_L\rangle$, and let it evolve under the action of the Hamiltonian in Eq. (B4). Figure 10 displays the evolution of the projection $\left\langle \hat{\Pi}_{B_L^+} \right\rangle (t)$ under the Liouvillian \[L^\prime \hat{\rho} = L_0 \hat{\rho} - i[\hat{H}_{XX}, \hat{\rho}]\]. Over time, the system oscillates between the Bell states in Eq. (B3). In the ideal case, in the absence of single-photon loss and dephasing processes, the gate accuracy can be increased at will by decreasing the ratio $J/\eta$. Again, dephasing induced by the finite off-diagonal gap $\Lambda^\prime$ and single photon loss will set a lower bound on $J$.
ran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletic and M. D. Lukin, Probing Many-Body Dynamics on a 51-Atom Quantum Simulator, Nature 551, 579 (2017).

[55] A. Biella, F. Storme, J. Lebreuilly, D. Rossini, R. Fazio, I. Carusotto and C. Ciuti, Phase Diagram of Incoherently Driven Strongly Correlated Photonic Lattices, Phys. Rev. A 96, 033839 (2017).

[56] H. J. Carmichael, Breakdown of Photon Blockade: A Critical Dynamical

[57] W. Casteels, R. Fazio and C. Ciuti, Critical Dynamical Properties of a First-Order Dissipative Phase Transition, Phys. Rev. A 95, 012128 (2017).

[58] W. Casteels and C. Ciuti, Quantum Entanglement in the Spatial-Symmetry-Breaking Phase Transition of a Driven-Dissipative Bose-Hubbard Dimer, Phys. Rev. A 95, 013812 (2017).

[59] T. Fink, A. Schade, S. Höfling, C. Schneider and A. Imamoglu, Signatures of a Dissipative Phase Transition in Photon Correlation Measurements, Nature Phys 14, 365 (2018).

[60] M. Fitzpatrick, N. M. Sundaresan, A. C. Y. Li, J. Koch and A. A. Houck, Observation of a Dissipative Phase Transition in a One-Dimensional Circuit QED Lattice, Phys. Rev. X 7, 011016 (2017).

[61] D. Honda, H. Nakazato and M. Yoshida, Spectral Resolution of the Liouvillian of the Lindblad Master Equation for a Harmonic Oscillator, J. Math. Phys. 51, 072107 (2010).

[62] A. A. Houck, H. E. Türeci and J. Koch, On-Chip Quantum Simulation with Superconducting Circuits, Nature Phys 8, 292 (2012).

[63] J. Kasprzak, M. Richard, S. Kundermann, A. Baas, P. Jeambrun, J. M. J. Keeling, F. M. Marchetti, M. H. Szymańska, R. André, J. L. Staehli, V. Savona, P. B. Littlewood, B. Deveaud and L. S. Dang, Bose-Einstein Condensation of Exciton Polaritons, Nature 443, 409 (2006).

[64] E. M. Kessler, G. Giedke, A. Imamoglu, S. F. Yelin, M. D. Lukin and J. I. Cirac, Dissipative Phase Transition in a Central Spin System, Phys. Rev. A 86, 012116 (2012).

[65] F. Minganti, I. I. Arkhipov, A. Miranowicz and F. Nori, Liouvillian Spectral Collapse in the Scully-Lamb Laser Model, Phys. Rev. E 3, 043197 (2021).

[66] M. Müller, S. Diehl, G. Pupillo and P. Zoller, in Advances In Atomic, Molecular, and Optical Physics, Advances in Atomic, Molecular, and Optical Physics, Vol. 61, edited by P. Berman, E. Arimondo and C. Lin (Academic Press, 2012) pp. 1–80.

[67] C. Sánchez Muñoz, A. Lara, J. Puebla and F. Nori, Hybrid Systems for the Generation of Nonclassical Mechanical States via Quadratic Interactions, Phys. Rev. Lett. 121, 123604 (2018).

[68] V. R. Overbeck, M. F. Maghrebi, A. V. Gorshkov and H. Weimer, Multicritical Behavior in Dissipative Ising Models, Phys. Rev. A 95, 042133 (2017).

[69] R. Rota, F. Storme, N. Bartolo, R. Fazio and C. Ciuti, Critical Behavior of Dissipative Two-Dimensional Spin Lattices, Phys. Rev. B 95, 134431 (2017).

[70] R. Rota, F. Minganti, C. Ciuti and V. Savona, Quantum Critical Regime in a Quadratically Driven Nonlinear Photonic Lattice, Phys. Rev. Lett. 122, 110405 (2019).

[71] A. Rubio-García, R. Molina and J. Dukelsky, From Integrability to Chaos in Quantum Liouvilleans, SciPost Physics Core 5, 026 (2022).

[72] V. Savona, Spontaneous Symmetry Breaking in a Quadratically Driven Nonlinear Photonic Lattice, Phys. Rev. A 96, 033826 (2017).

[73] L. M. Sieberer, S. D. Huber, E. Altman and S. Diehl, Dynamical Critical Phenomena in Driven-Dissipative Systems, Phys. Rev. Lett. 110, 195301 (2013).

[74] H. Weimer, Variational Principle for Steady States of Completely Positive Dynamical Semigroups of N-level Systems, J. Math. Phys. 17, 149 (1976).

[75] G. Lindblad, On the Generators of Quantum Dynamical Semigroups, Commun.Math. Phys. 48, 119 (1976).

[76] F.-X. Sun, Q. He, Q. Gong, R. Y. Teh, M. D. Reid and P. D. Drummond, Schrödinger States in Subharmonic Generation with Kerr Nonlinearities, Phys. Rev. A 100, 033827 (2019).

[77] J. Guillaud and M. Mirrahimi, Error Rates and Resource Overheads of Repetition Cat Qubits, Phys. Rev. A 103, 042413 (2021).

[78] L. Hu, Y. Ma, W. Cai, X. Mu, Y. Xu, W. Wang, Y. Wu, H. Wang, Y. P. Song, C.-L. Zou, S. M. Girvin, L.-M. Duan and L. Sun, Quantum Error Correction and Uni-
versal Gate Set Operation on a Binomial Bosonic Logical Qubit, Nat. Phys. 15, 503 (2019).

[90] V. V. Albert, B. Bradlyn, M. Fras and L. Jiang, Geometry and Response of Lindbladians, Phys. Rev. X 6, 041031 (2016).

[91] B. Buča, J. Tindall and D. Jaksch, Non-Stationary Coherent Quantum Many-Body Dynamics through Dissipation, Nat Commun 10, 1730 (2019).

[92] J. Kempe, D. Bacon, D. A. Lidar and K. B. Whaley, Theory of Decoherence-Free Fault-Tolerant Universal Quantum Computation, Phys. Rev. A 63, 042307 (2001).

[93] E. Knill, R. Laflamme and L. Viola, Theory of Quantum Error Correction for General Noise, Phys. Rev. Lett. 84, 2525 (2000).

[94] D. A. Lidar, I. L. Chuang and K. B. Whaley, Decoherence-Free Subspaces for Quantum Computation, Phys. Rev. Lett. 81, 2594 (1998).

[95] E. M. Fortunato, L. Viola, M. A. Pravia, E. Knill, R. Laflamme, T. F. Havel and D. G. Cory, Exploring Noiseless Subsystems via Nuclear Magnetic Resonance, Phys. Rev. A 67, 062303 (2003).

[96] B. Buča and T. Prosen, A Note on Symmetry Reductions of the Lindblad Equation: Transport in Constrained Open Spin Chains, New J. Phys. 14, 073007 (2012).

[97] M. Cattaneo, G. L. Giorgi, S. Maniscalco and R. Zambrini, Symmetry and Block Structure of the Liouvillian Superoperator in Partial Secular Approximation, Phys. Rev. A 101, 042108 (2020).

[98] T. Prosen, PT-Symmetric Quantum Liouvillian Dynamics, Phys. Rev. Lett. 109, 090404 (2012).

[99] C. Sánchez Muñoz, B. Buča, J. Tindall, A. González-Tudela, D. Jaksch and D. Porras, Symmetries and Conservation Laws in Quantum Trajectories: Dissipative Freezing, Phys. Rev. A 100, 042113 (2019).

[100] J. Thingna and D. Manzano, Degenerated Liouvillians and Steady-State Reduced Density Matrices, Chaos 31, 073114 (2021).

[101] M. van Caspel and V. Gritsev, Symmetry-Protected Coherent Relaxation of Open Quantum Systems, Phys. Rev. A 97, 052106 (2018).

[102] Z. Zhang, J. Tindall, J. Mur-Petit, D. Jaksch and B. Buča, Stationary State Degeneracy of Open Quantum Systems with Non-Abelian Symmetries, J. Phys. A: Math. Theor. 53, 215304 (2020).

[103] V. Y. Mylnikov, S. O. Potashin, G. S. Sokolovskii and N. S. Averkiev, Dissipative Phase Transition in Systems with Two-Photon Drive and Nonlinear Dissipation near the Critical Point, Nanomaterials 12, 2543 (2022).

[104] X. H. H. Zhang and H. U. Baranger, Driven-Dissipative Phase Transition in a Kerr Oscillator: From Semiclassical $\mathcal{PT}$ Symmetry to Quantum Fluctuations, Phys. Rev. A 103, 033711 (2021).

[105] F. Minganti, N. Bartolo, J. Lolli, W. Casteels and C. Ciuti, Exact Results for Schrödinger Cats in Driven-Dissipative Systems and Their Feedback Control, Sci Rep 6, 26987 (2016).

[106] D. Roberts and A. A. Clerk, Driven-Dissipative Quantum Kerr Resonators: New Exact Solutions, Photon Blockade and Quantum Bistability, Phys. Rev. X 10, 021022 (2020).

[107] B. Buča, J. Tindall, A. González-Tudela, D. Jaksch and D. Porras, Symmetry-Protected Coherent Relaxation of Open Quantum Systems, Phys. Rev. A 97, 052106 (2018).

[108] Z. Zhang, J. Tindall, J. Mur-Petit, D. Jaksch and B. Buča, Stationary State Degeneracy of Open Quantum Systems with Non-Abelian Symmetries, J. Phys. A: Math. Theor. 53, 215304 (2020).

[109] V. Y. Mylnikov, S. O. Potashin, G. S. Sokolovskii and N. S. Averkiev, Dissipative Phase Transition in Systems with Two-Photon Drive and Nonlinear Dissipation near the Critical Point, Nanomaterials 12, 2543 (2022).

[110] X. H. H. Zhang and H. U. Baranger, Driven-Dissipative Phase Transition in a Kerr Oscillator: From Semiclassical $\mathcal{PT}$ Symmetry to Quantum Fluctuations, Phys. Rev. A 103, 033711 (2021).

[111] F. Minganti, N. Bartolo, J. Lolli, W. Casteels and C. Ciuti, Exact Results for Schrödinger Cats in Driven-Dissipative Systems and Their Feedback Control, Sci Rep 6, 26987 (2016).

[112] D. Roberts and A. A. Clerk, Driven-Dissipative Quantum Kerr Resonators: New Exact Solutions, Photon Blockade and Quantum Bistability, Phys. Rev. X 10, 021022 (2020).

[113] B. Buča, J. Tindall, A. González-Tudela, D. Jaksch and D. Porras, Symmetry-Protected Coherent Relaxation of Open Quantum Systems, Phys. Rev. A 97, 052106 (2018).