The variant of ADHMN construction associated with $q$-analysis

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Abstract

A $q$-deformation of the ADHMN caloron construction is considered, under which the anti-selfdual (ASD) conditions of the gauge fields are preserved. It is shown that the $q$-dependent Nahm data with certain constraints are crucial to determine the ASD gauge fields, as in the case of ordinary caloron construction. As an application of the $q$-deformed ADHMN construction, we give a $q$-deformed caloron of Harrington-Shepard type. Some limits of the parameters are also considered.

Key words: Calorons, ADHM, $q$-deformation

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1 Introduction

The ADHM construction [1] is a vital tool to find out the exact solutions to the (anti-)selfdual (ASD) Yang-Mills equations in $\mathbb{R}^4$, explicitly. The construction is also effective for the instanton calculus of supersymmetric Yang-Mills theories, e.g.[2].

On the compactified flat space $\mathbb{R}^3 \times S^1$, there exist the solutions to ASD equations with a periodicity in the $S^1$ direction, i.e., the calorons, which have been discussed firstly in finite temperature field theories [3]. Nahm [4] has applied ADHM's approach to the construction of calorons successfully by introducing infinite dimensional functional ($\mathcal{L}^2$) space as a dual space, the Nahm transformation. In the limit that the size of constituent instanton is sufficiently large compared with the circumference of the $S^1$, one can reproduce the monopole solutions to the Bogomolnyi equations [5], the ASD equations in $\mathbb{R}^3$. On the

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other hand, the large circumference limit gives ordinary instanton solutions in $\mathbb{R}^4$. This aspect is strongly supported by the moduli space analysis of calorons [6]. We thus have an interpretation that calorons give the interpolation between instantons and monopoles [7]. There is another perspective of calorons that they may be interpreted as monopoles with a loop group as their gauge group [8,9].

Some years ago, there have been found very interesting types of the calorons through the ADHM/Nahm (ADHMN) constructions. One class of those is the calorons with non-trivial holonomy around $S^1$ at the spatial infinity [10,11,12,13], which brings gauge symmetry breaking through the Wilson loop mechanism. The other class is the symmetric calorons [7], i.e., the multi-caloron solutions with certain spatial symmetries. We expect that there is room for advanced applications of the ADHMN construction to discover new types of ASD solutions.

In a series of papers [14], Kamata and the present author have considered a $q$-analog of the BPS monopole construction, which gives an exact solutions to the ASD equation. They have used a variant formulation of the Nahm construction by introducing an $\ell^2$ functional space as a dual space instead of Nahm’s $L^2$ space, and obtained the solution with a parameter $q$ interpolating the BPS monopole ($q \to 1$) and the pure gauge configuration ($q \to 0$), a $q$-deformed BPS monopole. They also found that the $q$-deformed BPS monopole could be interpreted as a special case of the instantons with axial symmetry [15].

In this paper, we consider the generalization of the previous work by formulating the $q$-deformation of the ADHMN caloron construction, which contains the matrix Nahm data depending on $q$. As a concrete example, we will fix explicitly the vector $\vec{V}_q$ critical to give the $q$-deformed caloron of Harrington-Shepard type [3], which is an exact solution to the ASD equation on $\mathbb{R}^3 \times S^1$.

This paper is organized as follows. In section 2, we give the $q$-deformed ADHMN formalism which yields ASD gauge fields, following a brief introduction of the ADHMN caloron construction. In section 3, we apply the formalism to yield a $q$-deformation of the Harrington-Shepard caloron. Some limits of the parameters for the $q$-deformed caloron are also considered. In the final section, we give concluding remarks and discussion.

2 The ADHMN construction on an $\ell^2$ functional space

We consider the ADHMN construction in $Sp(1)$ formulation, i.e., the gauge group is restricted to $SU(2)$. In Nahm’s caloron construction, the $N \times N$
 Nahm data are crucial to determine the ASD configuration, which data are constrained by the following Nahm equations ($i = 1, 2, 3$) with respect to the continuous variable $z$,

\[
\frac{dT_i}{dz} - \frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} [T_j, T_k] - [T_i, T_0] = \frac{1}{2} \text{tr}_2 \left( \sigma_i \Lambda^\dagger \Lambda \right) \delta(z - z_0),
\]

in addition to the anti-hermite conditions $T_i^\dagger = -T_i$. In the right hand side, the trace is over quaternion and $\Lambda$ is an $N$ component quaternion valued row vector. The distinction from the monopole Nahm data is that they are periodic in $z$ with period $2z_0 := 2\pi/\beta$, where $\beta$ is a circumference of $S^1$. The defining relations (1) are derived from the ASD conditions of the 1-dimensional “Dirac operator”,

\[
\Delta = \begin{bmatrix}
\Lambda \delta(z - z_0) \\
i\partial_z + x + i \sum_{\mu=0}^{3} T_\mu(z) \tau_\mu
\end{bmatrix},
\]

i.e., $\Delta^\dagger \Delta$ is commutative with arbitrary quaternion and has inverse. Here $x_\mu = (x_0, x_1, x_2, x_3)$ is a coordinate of $\mathbb{R}^3 \times S^1$, $x_0$ being that of $S^1$, $\tau_\mu = (1, i\sigma_1, i\sigma_2, i\sigma_3)$ is a quaternion element and $x := \sum_{\mu=0}^{3} x_\mu \tau_\mu$. If the Nahm data are given, we determine a vector $\vec{V}$ of the form

\[
\vec{V} = \begin{bmatrix}
V(x_\mu) \\
v(z, x_\mu)
\end{bmatrix},
\]

where $V(x_\mu)$ is a quaternion valued function, and $v(z, x_\mu)$ is an $N$ component quaternion valued column vector periodic in $z$ with period $2z_0$, that is, $v(z, x_\mu)$ being an element of $L^2[I] \otimes V_N$, where $I := [-z_0, z_0]$. This vector is assumed to solve the differential equation taking into account the discontinuity at the boundary,

\[
\Delta^\dagger \vec{V} = 0,
\]

or writing down (4) definitely,

\[
\Lambda^\dagger \delta(z - z_0)V + (i\partial_z + x^\dagger + i \sum_{\mu=0}^{3} T_\mu \tau_\mu^\dagger)v(z) = 0.
\]

The procedure to find out the solution to (5) is as follows: we solve the differential equation at $z \neq z_0$ to fix $v(z)$ firstly, then determine the top component $V$ by performing a short range integration at the boundary. In addition to (4), we impose a normalization condition $\langle \vec{V}, \vec{V} \rangle = 1$, where the inner product of the vector space with another vector $\vec{U} = \langle U, u(z) \rangle$ is defined as,

\[
\langle \vec{U}, \vec{V} \rangle = U^\dagger V + \int_{-z_0}^{z_0} u^\dagger(z)v(z)dz.
\]
The gauge connection obeying ASD conditions can be determined by this normalized vector up to gauge transformation as,

\[ A = \langle \vec{V}, d\vec{V} \rangle. \]  

(7)

We now consider a one parameter, \( q \), deformation of the ADHMN caloron construction which preserves the ASD condition of the gauge field. In [14], we have made such a deformation of the BPS monopole by introducing an \( \ell^2 \) functional space in place of \( L^2 \) functional space. We apply the same approach to the caloron construction described above. The differences to the previous work are the introduction of the \( N \times N \) periodic Nahm data and the periodicity on \( \vec{V} \).

We introduce an \( \ell^2 \) integrable periodic function \( v(z; q) \) depending on a parameter \( q \in (0, 1) \) in place of the \( L^2 \) integrable function considered in Nahm’s caloron construction. The period is set to be 2\( z_0 \), which is identical to the non-deformed case. Here we define the \( \ell^2 \) integrability of \( v(z; q) \) by the square integrability with the inner product defined on the infinite number of point set \( I_q := \{ z_n, -z_n \mid z_n = q^n z_0, n = 0, 1, 2, \ldots \} \). In the following, we will use the notation \( I_q^{(±)} := \{ ±z_n \} \), namely, \( I_q = I_q^{(+)} \oplus I_q^{(-)} \). We define the \( \ell^2 \) inner product of \( u(z; q) \) and \( v(z; q) \) as,

\[ (u, v)_q = z_0(1 - q) \sum_{n=0}^{\infty} (u^\dagger(z_{n+1}; 1/q)v(z_n; q) + u^\dagger(-z_{n+1}; 1/q)v(-z_n; q)) q^n, \]

(8)

or by using the symbolic notation of the \( q \)-integration and the conjugate vector by \( * \), the description can be simplified into

\[ (u, v)_q = \int_{-z_0}^{z_0} u^*(qz) v(z) d_q z. \]

(9)

Notice that the shift of argument in the conjugate, or left, vector. We, therefore, find that an \( \ell^2[I_q] \) function \( v(z; q) \) can be viewed as an infinite dimensional quaternion valued vector \( v(z; q) = v_+(z; q) \oplus v_-(z; q) \) with components

\[
\begin{align*}
v_+(z; q) &= \langle v(z_0; q), v(z_1; q), v(z_2; q), \cdots \rangle, \\
v_-(z; q) &= \langle v(-z_0; q), v(-z_1; q), v(-z_2; q), \cdots \rangle
\end{align*}
\]

(10)

and the * conjugation of this quaternion valued vector \( v^*(z; q) = v_+^*(z; q) \oplus v_-^*(z; q) \) being defined by the hermite conjugation in addition to the “twist” of \( q \),

\[
\begin{align*}
v_+^*(z; q) &= \langle v^\dagger(z_0; 1/q), v^\dagger(z_1; 1/q), v^\dagger(z_2; 1/q), \cdots \rangle, \\
v_-^*(z; q) &= \langle v^\dagger(-z_0; 1/q), v^\dagger(-z_1; 1/q), v^\dagger(-z_2; 1/q), \cdots \rangle.
\end{align*}
\]

(11)
Under this definition of the $\ell^2$ inner product, we can confirm the “hermiticity” of $q$-difference operator $iD_z$ [14], i.e., $(u, iD_z v)_q = (iD_z u, v)_q$, where $D_z \phi(z) := \phi(z) - \phi(qz)/z - qz$ for a function $\phi(z)$.

By using the $\ell^2[I_q]$ vector space introduced above, we now make a reformulation of the ADHMN construction of calorons. We define the “Dirac operator” by applying the $q$-difference operator, 

$$
\Delta = \begin{bmatrix}
\Lambda \delta(z_0, z) \\
iD_z + x + i \sum_{\mu=0}^{3} T_\mu(z; q) \tau_\mu
\end{bmatrix},
$$

where $\delta(z, z_0)$ is given by a Kronecker delta rather than a delta function,

$$
\delta(z, z_0) := \frac{2}{(1 - q)z} \delta_{z, z_0}.
$$

In analogy with the vector (3), the deformed vector is set to be of the form

$$
\vec{V}_q = \begin{bmatrix}
V(x_\mu; q) \\
v(z, x_\mu; q)
\end{bmatrix},
$$

where $V(x_\mu; q)$ is a quaternion and $v(z, x_\mu; q)$ is a quaternion valued vector of $\ell^2[I_q] \otimes V_N$ rather than $L^2[I] \otimes V_N$, and its conjugation is

$$
\vec{V}_q^* = \begin{bmatrix}
V^\dagger(x_\mu; 1/q), v^\dagger(z, x_\mu; q)
\end{bmatrix},
$$

the second component being defined by (11). We define the inner product of those vectors by using (9) as,

$$
\langle \vec{U}, \vec{V} \rangle_q := U^\dagger(x_\mu; 1/q)V(x_\mu; q) + (u, v)_q.
$$

In accordance with the ADHMN construction, the ASD gauge fields are given by the condition that $\Delta^* \Delta$ is invertible and commutative with quaternion. The explicit form of $\Delta^* \Delta$ is given in the appendix. Here, we show the necessary conditions, the commutativity with quaternion, on the Nahm data

$$
T_i(z_n) = T_i(z_{n+1}), \quad T_i(-z_n) = T_i(-z_{n+1}),
$$

$$
T_i^*(z_n) = -T_i(z_n), \quad T_i^*(-z_n) = -T_i(-z_n),
$$

$$
D_z T_i - \frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} [T_j, T_k] - [T_i, T_0] = \frac{1}{2} \text{tr}_2 (\sigma_i \Lambda^\dagger \Lambda) \delta_q(z, z_0),
$$

where $i = 1, 2, 3$. By the conditions (17), we find the Nahm data are “pseudo-constant” in each interval $I_q^{(\pm)}$, i.e., $T_i(z_n) = T_i(z_0)$, $T_i(-z_n) = T_i(-z_0)$ ($n = 5$).
1, 2, \cdots) so we have omitted the argument in (19). In spite of this, we should reserve the $q$-difference term in (19) to pick up the contribution of the boundary discontinuity, which can exist since $0 \notin I_q$. Obviously, the pseudo-constant conditions do not imply that the $T_i$’s are strictly constant matrices in the continuous interval $I$. The conditions (18) together with (17) are “twisted” anti-hermite conditions,

$$T_i^t(z_0; q) = -T_i(z_0; q^{-1}), \quad T_i^t(-z_0; q) = -T_i(-z_0; q^{-1}). \quad (20)$$

We find that (19) becomes the equations similar to the ADHM equation if $z \neq z_0$,

$$[T_1, T_2] + [T_3, T_0] = 0$$
$$[T_2, T_3] + [T_1, T_0] = 0$$
$$[T_3, T_1] + [T_2, T_0] = 0, \quad (21)$$

where $T_i = T_i(\pm z_0)$ are pseudo-constant matrices on each $I_q^{(\pm)}$, respectively, whereas $T_0 = T_0(z)$ is not constrained. At present, we have no proof that $\Delta^* \Delta$ is invertible for the general Nahm data subject to the constraints (17), (18) and (19), because of the curious $\ell^2$ inner product. The presence of the inverse $f := (\Delta^* \Delta)^{-1}$ must be confirmed case by case after the Nahm data were fixed by the other constraints. For the (non-deformed) Nahm construction of $SU(2)$ monopoles, and also calorons [7], the Nahm data must enjoy the residue conditions which guarantee $\text{dim}_{\mathbb{R}} \ker(\Delta^t) = 1$ [17]. In our deformed construction, the corresponding constraints $\text{dim}_{\mathbb{R}} \ker(\Delta^*) = 1$ have to be confirmed as well.

As in the ordinary caloron construction, if the Nahm data are given, the vector $\vec{V}_q$ is determined by the $q$-difference equation and the normalization condition,

$$\Delta^* \vec{V}_q = \Lambda^* \delta_q(z_0, z) V + (iD_z + x^\dagger + iT_\mu \tau_\mu^\dagger) v(z) = 0 \quad (22)$$

$$\langle \vec{V}_q, \vec{V}_q \rangle_q = V^t(q^{-1}) V(q) + (v, v)_q = 1. \quad (23)$$

The connection one-form is thus given by

$$A = \langle \vec{V}_q, d\vec{V}_q \rangle_q. \quad (24)$$

By construction, this reduces to (7) in the limit $q \to 1$. Finally, since we are considering $\Delta$ in the standard form, $\Delta = \Delta_0 + \lambda[0, x]$ for $\Delta_0$ independent of $x$, the curvature two-form $F = dA + A \wedge A$ is given by the canonical way, with the inverse $f$ and the ASD t’Hooft tensor $\bar{\eta}_{\mu \nu}$ such that,

$$F = \int_{-z_0}^{z_0} dz \int_{-z_0}^{z_0} dw \ v^*(qz) \bar{\eta}_{\mu \nu} f(z, qw) v(w) \ dx^\mu \wedge dx^\nu. \quad (25)$$
We can regard (25) that the ordinary matrix product in the original ADHM construction is replaced by the $q$-integral.

3 Example: a $q$-deformed caloron

In this section, we explicitly apply the $q$-deformed ADHMN caloron construction on $\ell^2$ vector space formulated in the previous section. The Nahm data to be considered here are the simplest case, $N = 1$, in which case the data can be chosen,

$$T_\mu(z; q) = 0, \quad \Lambda = \lambda \cdot 1_2$$

where $\lambda \in \mathbb{R}$ and $1_2$ is a real quaternion element. The first ansatz shows no boundary discontinuity in the Nahm data, which leads to the second one. In this case $\dim \ker(\Delta^*) = 1$ and there exists $f = (\Delta^* \Delta)^{-1}$, whose exact form is the same as the $q$-deformed monopole case [14]. The ASD conditions for the gauge field constructed by the Nahm data (26) are, therefore, fulfilled. We will find that this construction gives the $q$-deformed version of Harrington-Shepard caloron.

Having obtained the Nahm data, we next solve the difference equation (22) to find the unnormalized vector $\vec{V}_q^0 = i(V^0, v^0(z; q))$,

$$\lambda \delta_q(z_0, z)V^0 + (iD_z + x^\dagger)v^0(z; q) = 0.$$ (27)

If $z \neq z_0$, (27) is reduced to a linear homogeneous $q$-difference equation $(iD_z + x^\dagger)v(z) = 0$ so that $v$ is easily solved by,

$$v^0(z, x; q) = e_q(i(1 - q)zx^\dagger)$$

where a $q$-exponential function $e_q(w)$ convergent at $|w| < 1$ is

$$e_q(w) := \sum_{n=0}^\infty \frac{w^n}{(q; q)_n} = \frac{1}{(w; q)_\infty},$$

and the $q$-shifted factorial $(a; q)_n = \prod_{k=1}^n (1 - aq^{k-1})$. It can be shown [16] that the second equality in (29) follows from the $q$-binomial theorem, and that $\lim_{q \to 1^-} e_q((1 - q)w) = e^w$. To find the first component, $V^0$, we implement the short $q$-integral ($\epsilon \to 0$) around the boundary $z_0$ taking into account the periodicity of $v(z; q)$,

$$0 = \int_{z_0 - \epsilon}^{z_0 + \epsilon} \left( \lambda \delta_q(z_0, z)V^0 + (iD_z + x^\dagger)v^0(z; q) \right) dq z = \lambda V^0 + i \int_{z_0 - \epsilon}^{z_0 + \epsilon} D_z v^0(z; q) dq z.$$ (30)

The second term of the right hand side can be evaluated by using the “fundamental theorem of $q$-calculus”, thus,
\[
\int_{z_0-\epsilon}^{z_0+\epsilon} D_z v^0 \, dz = \int_{z_0-\epsilon}^{z_0} D_z v^0 \, dz + \int_{-z_0}^{-z_0+\epsilon} D_z v^0 \, dz
\]
\[
= (v^0(z_0) - v^0(z_0 - \epsilon)) + (v^0(-z_0 + \epsilon) - v^0(-z_0))
\]
\[
= v^0(-z_0 + \epsilon) - v^0(z_0 - \epsilon),
\]
where we have used the periodicity \(v(z + 2z_0) = v(z)\). The boundary value of (28) together with the \(q\)-binomial theorem [16] gives, by using the spacetime variable \(\rho_\pm := x_0 \pm ir\) for \(r^2 = \sum_{i=1}^3 x_i^2\) and a quaternion \(\hat{x} = \sum_{i=1}^3 x_i \sigma_i / r\), that
\[
V^0 = -\frac{i}{\lambda} (R_+(\rho_+; q)(1 - \hat{x}) + R_-(\rho_--; q)(1 + \hat{x})) ,
\]
where
\[
R_\pm = \frac{1}{2} \left( \frac{1}{i\rho_\pm (1-q) z_0; q)_\infty} - \frac{1}{i\rho_\pm (1-q) z_0; q)_\infty} \right),
\]
and whose conjugation is
\[
R^*_\pm = \frac{1}{2} ((i\rho_\pm (1-q) z_0; q)_\infty - (-i\rho_\pm (1-q) z_0; q)_\infty).
\]

So we find that the normalized vector \(\hat{V}_q\) should be of the form
\[
\hat{V}_q = \hat{V}_q^0 \phi^{-1/2} = \left[ -\frac{i}{\lambda} \left( R_+(\rho_+; q)(1 - \hat{x}) + R_-(\rho_--; q)(1 + \hat{x}) \right) / q_1 i x^1 (1 - q) z \right] \phi^{-1/2},
\]
where \(\phi^{-1/2}\) is a quaternion valued normalization function which will be fixed by (23), i.e., \((\phi^{-1/2})^*(\hat{V}_q^0, \hat{V}_q^0) q \phi^{-1/2} = 1\). The procedure to fix \(\phi^{-1/2}\) is in the similar manner to the \(q\)-deformation of BPS monopole case [14]. We finally find,
\[
\phi^{-1/2} = \frac{1}{2} \left\{ (K_+ - K_- + \Lambda_+ - \Lambda_-)^{-1/2} + (K_+ + K_- + \Lambda_+ + \Lambda_-)^{-1/2} \right\} 1
\]
\[
- \frac{1}{2} \left\{ (K_+ - K_- + \Lambda_+ - \Lambda_-)^{-1/2} - (K_+ + K_- + \Lambda_+ + \Lambda_-)^{-1/2} \right\} \hat{x},
\]
where the functions \(\Lambda_\pm\) and \(K_\pm\) are
\[
\Lambda_\pm = \frac{1}{i \rho_+} \sum_{n=0}^{\infty} \left( \frac{q^{\rho_+}; q_{2n}}{(q; q)_{2n+1}} \right)^2 \left( i (1-q) \rho_+ z_0 \right)^{2n+1} \pm (\rho_+ \leftrightarrow \rho_-) \tag{37}
\]
\[
K_\pm = \frac{2}{\lambda^2} (R^*_\pm R_\mp + R^*_\mp R_\pm). \tag{38}
\]
In this way, we have completed fixing the vector \(\hat{V}_q\) by the ADHMN construction on \(\ell^2\) vector space, which yields the \(q\)-deformation of Harrington-Shepard
caloron as will be shown later. The anti-selfduality of the gauge field is obvious from (25) as in the $q$-deformed monopole case [14]. The entire information on the gauge connection and the curvature is included in the vector $\vec{V}_q$, so that they can be obtained by the canonical way but are very complex form. Hereafter we consider $\vec{V}_q$ at some limits of the parameters.

Firstly, it can be shown that the $q \to 1$ limit gives the ordinary Harrington-Shepard caloron. In this limit, the $q$-integral turns into the ordinary integral so that the $L^2$ function getting close to an $L^2$ function. Actually, since

$$\lim_{q \to 1} K_+ = \frac{2}{\lambda^2} (\cosh 2z_0 r - \cos 2z_0 x_0), \quad \lim_{q \to 1} K_- = 0,$$  

and

$$\lim_{q \to 1} \Lambda_+ = \frac{\sinh 2z_0 r}{r}, \quad \lim_{q \to 1} \Lambda_- = 0,$$  

we find the normalization function tends to

$$\lim_{q \to 1} \phi^{-1/2} = \phi^{-1/2}(HS) = \left( \frac{2}{\lambda^2} (\cosh 2z_0 r - \cos 2z_0 x_0) + \frac{\sinh 2z_0 r}{r} \right)^{-1/2}.$$

The vector $\vec{V}_q$, therefore, turns into

$$\lim_{q \to 1} \vec{V}_q = \left[ \frac{2}{\lambda} \left( \sin x_0 z_0 \cosh r z_0 - i \cos x_0 z_0 \sinh r z_0 \hat{x} \right) \right] \phi^{-1/2}(HS),$$

which gives the Harrington-Shepard caloron exactly through the $L^2$ inner product (6). As pointed out in [7], this caloron yields the interpolation between the 1-instanton in $\mathbb{R}^4$ (as $z_0 \to 0$) and the BPS 1-monopole (as $\lambda \to \infty$). On the other hand, the $q$-deformed caloron at general $q$ does not have the instanton limit ($z_0 \to 0$), since the $q$-interval does not include zero, $0 \notin I_q$.

Next, we consider the limit $\lambda \to \infty$, corresponding to the infinite instanton size, which is easily obtained from (35) and (36),

$$\lim_{\lambda \to \infty} \vec{V}_q = \left[ e_q(i \hat{x}^\dagger (1 - q) z) \right] \phi^{-1/2}(q\text{-BPS}),$$

where

$$\phi^{-1/2}(q\text{-BPS}) = \lim_{\lambda \to \infty} \phi^{-1/2} = \frac{1}{2} \left\{ (\Lambda_+ - \Lambda_-)^{-1/2} + (\Lambda_+ + \Lambda_-)^{-1/2} \right\} 1$$

$$- \frac{1}{2} \left\{ (\Lambda_+ - \Lambda_-)^{-1/2} - (\Lambda_+ + \Lambda_-)^{-1/2} \right\} \hat{x},$$
This is exactly the $q$-deformed BPS monopole [14], which becomes the BPS 1-monopole when $q \to 1$, and the pure gauge ($F = 0$) when $q \to 0$.

Finally, we show the $q \to 0$ limit tends to the pure gauge, analogous to the “$q$-BPS” case. In our $q$-deformed caloron, the field strength two form is given by the double $q$-integral (25). In the $q \to 0$ limit, this double integral is reduced to the summation only at the boundary ($z = \pm z_0$),

$$
F \rightarrow z_0^2 \left( v^\dagger(z; 1/q) \eta_{\mu\nu} f(z_0, z_1) v(z_0) + v^\dagger(-z; 1/q) \eta_{\mu\nu} f(-z_0, -z_1) v(-z_0) \right. \\
+ v^\dagger(z; 1/q) \eta_{\mu\nu} f(z_0, -z_1) v(-z_0) \\
\left. + v^\dagger(-z; 1/q) \eta_{\mu\nu} f(-z_0, -z_1) v(-z_0) \right) dx^\mu \wedge dx^\nu,
$$

however, from the boundary condition of $f(z, w)$, this is automatically zero.

In summary, we have had the following diagram.

4 Concluding remarks

In this paper, we have constructed the ADHMN caloron construction on $\ell^2$ vector space, which is a generalization of [14]. We have found the ASD necessary conditions on the matrix Nahm data, which are (17), (18) and (19). As a concrete example, we have made the $q$-deformed Harrington-Shepard caloron, which preserves the ASD conditions for general value of $q \in (0, 1)$.

Further application of the $q$-deformed caloron construction will be possible. It is intriguing to consider the $q$-deformed Nahm data for general matrix dimensions. In contrast to the $N = 1$ case considered in this paper, the $q$-deformed Nahm data of $N > 1$ have no counter part in the ordinary caloron construction at $q \to 1$, due to the pseudo constant condition (17). Such $q$-deformed Nahm data will be corresponding to a new class of gauge fields. As a naive example of $N = 2$ case, we can take an ansatz such as $T_0 = T_1 = T_2 = 0$, and $T_3 = iC(q)\sigma_3$, where $C(q)$ being a pseudo-constant with twist invariance, $C(q^{-1}) = C(q)$, and $\Lambda$ to be a real quaternion, which Nahm data gives the
solution to (17), (18) and (19). The inverse of $\Delta^*\Delta$ can be constructed as in the $N = 1$ case, so that the gauge field is ASD. However, the Nahm data do not have $q \to 1$ counter part obviously. In fact, this leads to $\dim \ker(\Delta^*) = 2$, not to give an $Sp(1) \simeq SU(2)$ gauge field. From this consideration, the generalization to higher rank gauge group must be considered.

Finally, it will be straightforward to construct a $q$-deformed calorons of $N = 1$ with nontrivial holonomy at the spatial infinity by introducing the extra discontinuous points of the Nahm data in the $q$-interval $I_q$, similarly to [11]. This $q$-deformation will bring us the perspective of the $q$-caloron from the constituent $q$-monopoles, which is also quite interesting subject to study.

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A The ASD conditions

In this appendix, we give the explicit form of $\Delta^*\Delta$, which is necessary to be commutative with arbitrary quaternion. By using the formula of $q$-difference,

$$D_z \cdot \phi(z) = D_z \phi(z) + \phi(qz) \cdot D_z,$$  \hspace{1cm} (A.1)

we carry out the calculation of $\Delta^*\Delta$,
\[ \Lambda^* \Lambda \delta_q(z_0, z) + (iD_z \otimes 1_N + 1 \otimes 1_N x^\dagger - 1 \otimes i \sum_{\mu=1}^4 T^*_\mu(z) \tau^\dagger_\mu) \]
\[ \times (iD_z \otimes 1_N + 1 \otimes 1_N x^\dagger + 1 \otimes i \sum_{\mu=1}^4 T_\mu(z) \tau_\mu) \]
\[ = \Lambda^* \Lambda \delta_q(z_0, z) - D^2_z \otimes 1_N + iD_z \otimes 1_N x_0 + 1 \otimes 1_N |x|^2 + 1 \otimes T^*_0(z)T_0(z) \]
\[ - 1 \otimes D_z T_0(z) - D_z \otimes (T_0(qz) - T^*_0(z)) \]
\[ - 1 \otimes \sum_{k=1}^3 D_z T_k(z) \tau_k + 1 \otimes \sum_{j,k=1}^3 T^*_j(z)T_k(z) \tau^\dagger_j \tau_k \]
\[ - D_z \otimes \sum_{k=1}^3 \left( T_k(qz) \tau_k + T^*_k(z) \tau^\dagger_k \right) \]
\[ + 1 \otimes i \sum_{k=1}^3 \left( T_k(z) x^\dagger \tau_k - T^*_k(z) \tau^\dagger_k x \right) \]
\[ + 1 \otimes i \left( T_0(z) x^\dagger - T^*_0(z) x \right) \]
\[ + 1 \otimes \sum_{k=1}^3 \left( T^*_0(z)T_k(z) \tau_k + T^*_k(z)T_0(z) \tau^\dagger_k \right). \]

For the final formula to be commutative with quaternion, we need all of the terms including \( \tau_k \), \( x \) or \( x^\dagger \) vanish. Therefore, we find that it is necessary to hold the pseudo constant condition (17), the twisted anti-hermite condition (18) and the \( q \)-analog of the caloron Nahm equations,

\[ D_z T_i(z) - \sum_{j,k=1}^3 \epsilon_{ijk} T_j(qz) T_k(z) - T_i(qz) T_0(z) + T_0(z) T_i(z) \]
\[ = \Lambda^* \Lambda \delta_q(z_0, z), \quad (A.2) \]

which, together with (17), yields (19). Finally we obtain the operator on \( \ell^2[I_q] \otimes V_N \) without the boundary term,

\[ \Delta^* \Delta = -D^2_z \otimes 1_N + 2ix_0 iD_z \otimes 1_N + 1 \otimes 1_N |x|^2 \]
\[ - 1 \otimes \sum_{\mu=1}^4 (T^*_\mu(z) - 2ix_0) T_\mu(z) - 1 \otimes D_z T_0(z) - D_z \otimes (T_0(qz) + T^*_0(z)) \]
\[ (A.3) \]

in which each entry of the \( N \times N \) matrices is a real quaternion.
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