Cumulant approach to weakly doped antiferromagnets

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We present a new approach to static and dynamical properties of holes and spins in weakly doped antiferromagnets in two dimensions. The calculations are based on a recently introduced cumulant approach to ground–state properties of correlated electronic systems. The present method allows to evaluate hole and spin–wave dispersion relations by considering hole or spin excitations of the ground state. Usually, these dispersions are found from time–dependent correlation functions. To demonstrate the ability of the approach we first derive the dispersion relation for the lowest single hole excitation at half-filling. However, the main purpose of this paper is to focus on the mutual influence of mobile holes and spin waves in the weakly doped system. It is shown that low-energy spin excitations strongly admix to the ground–state. The coupling of spin waves and holes leads to a strong suppression of the staggered magnetization which can not be explained by a simple rigid–band picture for the hole quasiparticles. Also the experimentally observed doping dependence of the spin–wave excitation energies can be understood within our formalism.

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I. INTRODUCTION

The physical properties of high–temperature superconductors are strongly influenced by electronic correlations. The investigation of strongly correlated electronic systems has therefore become a major topic for the understanding of the high–Tc materials. The undoped compounds are antiferromagnetic Mott–Hubbard insulators. Neutron scattering experiments show the existence of spin–wave excitations which can be described by conventional spin–wave theory for the isotropic spin \( S = \frac{1}{2} \) Heisenberg model on a square lattice.\(^{1,2}\) The doped materials show a strong dependence of the magnetic properties on the hole concentration \( \delta \) in the CuO\(_2\) planes. With increasing hole concentration both the Néel temperature and the staggered magnetization decrease and vanish at a critical hole concentration \( \delta_c \) of a few percent before the system becomes paramagnetic and metallic (or superconducting at sufficiently low temperatures). Upon doping also the spin–wave velocity decreases and vanishes at approximately the same hole concentration \( \delta_c \). At the same time the long–wavelength spin–wave modes become overdamped.

The essential aspects of the low–energetic electronic degrees of freedom of the CuO\(_2\) planes are by now believed to be well described by the two–dimensional \( t–J \) model\(^{3–6}\):

\[
H = -t \sum_{\langle ij \rangle} \left( \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \hat{c}_{j\sigma}^\dagger \hat{c}_{i\sigma} \right) + J \sum_{\langle ij \rangle} \left( \mathbf{S}_i \cdot \mathbf{S}_j - \frac{n_i n_j}{4} \right). \tag{1}
\]

Here, \( \mathbf{S}_i \) is the electronic spin operator and \( n_i \) the electron number operator at site \( i \). The symbol \( \langle ij \rangle \) refers to a summation over pairs of nearest neighbors. In the following we denote the two antiferromagnetic sublattices by \( \uparrow \) and \( \downarrow \). In all sums we shall use \( i \in \uparrow \) and \( j \in \downarrow \). Note that the Hamiltonian \( \tag{1} \) is defined in the subspace of the unitary space without double occupations of sites. The electronic creation operators \( \hat{c}_{i\sigma}^\dagger \) are not usual fermion operators but rather exclude double occupancies:

\[
\hat{c}_{i\sigma}^\dagger = c_{i\sigma}^\dagger (1 - n_{i, -\sigma}). \tag{2}
\]

At half–filling the \( t–J \) Hamiltonian reduces to the antiferromagnetic Heisenberg model.

Experiments show that already a few percent of additional holes (away from half–filling) destroy the long–range antiferromagnetic order in the CuO\(_2\) planes. This demonstrates the importance of the interplay between antiferromagnetism and the motion of holes in the high–Tc superconductors. To investigate the mutual influence one may start from the \( t–J \) Hamiltonian \( \tag{1} \). However, due to strong correlations usual diagrammatic techniques based on Wick’s theorem can not be easily applied to \( \tag{1} \). Neither the first nor the second part of \( \tag{1} \) is bilinear in fermion operators, and the creation and annihilation operators \( c_{i\sigma}^\dagger \) and \( c_{i\sigma} \) do not obey simple anticommutation relations. For this reason, non–standard analytical methods like variational wavefunctions, coupled–cluster methods, or slave–boson and slave–fermion techniques\(^{7–10}\) have been employed.
In the following, we present a static approach to evaluate static and dynamical properties in weakly doped antiferromagnets. The calculations are based on a cumulant method for computing the ground–state energy of correlated electronic systems. This is in contrast to the usual approach to dynamical properties which is based on dynamical quantities like time– or frequency–dependent correlation functions. The paper is organized as follows: In Sec. II we shall describe the cumulant method which was recently proposed in refs. 11–14. In Sec. III this formalism is applied to the motion of a single hole generated in the ground state of a quantum antiferromagnet at half–filling. This problem was already investigated by a number of authors9,15,16. Our first aim is to show that our approach is able to reproduce results known from literature, see for instance review articles17,18. Processes leading to the hole motion are described within the concept of path operators which leads to the well–known spin–bag picture. For the case of one hole we obtain a quasiparticle dispersion which has minima at $(\pm\pi/2,\pm\pi/2)$ and a bandwidth of $1.4J\ldots1.5J$ in agreement with several analytical and numerical calculations. Our main aim is to investigate the coupling of spins and holes in the weakly doped regime. This is the subject of Sections IV and V. In Sec. IV we describe the ground state for the case of small doping. Then we evaluate the staggered magnetization from the ground–state energy by introducing an external field coupling to the staggered magnetization. In Sec. V we derive the dispersion relation for spin waves in the doped system. We find that the spin wave energies are strongly renormalized due to the presence of holes. The spin–wave velocity vanishes at a critical hole concentration of a few percent. The strong coupling between spin waves and holes also explains the experimentally observed fast decrease of the staggered magnetization with increasing hole doping.

II. CUMULANT APPROACH

Conventional treatments of systems with electronic correlations start from the uncorrelated limit and include many–body effects by perturbation theory. This includes summation over classes of Feynman diagrams. For a treatment of strongly correlated electrons one would like to proceed the other way round, i.e., by starting from a local picture including the electronic correlations and by expanding with respect to the hybridization interactions. In this case the unperturbed Hamiltonian $H_0$ contains two–particle operators. This leads to the difficulty that Wick’s theorem is no longer applicable. Therefore, usual diagrammatic techniques cannot be used for strongly correlated electronic systems. For this reason other methods for treating correlated electrons have been developed, e.g., slave–boson and slave–fermion techniques19,20.

An alternative approach for calculating expectation values and dynamical correlation functions21,22 is based on the introduction of cumulants. Provided that the Hamiltonian of the system can be split into $H = H_0 + H_1$ with eigenstates and eigenvalues of $H_0$ known, this method uses the decomposition

$$e^{-\lambda H} = e^{-\lambda(H_1 + L_0)} e^{-\lambda H_0}$$

which can be proven by comparing the equations of motion of either side with respect to $\lambda$. The Liouville operator $L_0$ is a superoperator defined by $L_0 A = [H_0, A]$. Let us denote the ground state of the unperturbed Hamiltonian $H_0$ by $|\phi_0\rangle$ and its energy by $\epsilon_0$

$$H_0|\phi_0\rangle = \epsilon_0|\phi_0\rangle.$$  \hspace{1cm} (4)

For the following it is useful to introduce a cumulant bilinear form defined by

$$(A|B) = (\phi_0|A^\dagger B|\phi_0) := (\phi_0|A^\dagger B|\phi_0)^c,$$  \hspace{1cm} (5)

where $\langle...\rangle^c$ denotes a cumulant expectation value. For a detailed discussion of cumulants see e.g. Kubo23.

Our aim is to calculate the ground state energy $E_0$ of $H$:

$$H|\psi_0\rangle = E_0|\psi_0\rangle.$$  \hspace{1cm} (6)

Using (3) one can show24 that $E_0$ is given by

$$E_0 = \epsilon_0 + (H_1|\Omega) = (H|\Omega),$$  \hspace{1cm} (7)

$$\Omega = 1 + \lim_{x \to 0} \frac{1}{x - (L_0 + H_1)} H_1.$$  

The operator $\Omega$ has similarity to the so–called wave operator (or Moeller operator known from scattering theory). It transforms the ground state $|\phi_0\rangle$ of the unperturbed system into the exact ground state $|\psi_0\rangle$ of $H$. 

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The relation (7) can be applied to either weakly or strongly correlated systems because its use is independent of the operator statistics, i.e., it is valid for fermions, bosons or spins. Based on the same approach also dynamical correlation functions can be calculated. For a derivation and detailed discussion of relation (7) see ref. 14.

Treating cumulant expectation values one must distinguish between prime and composite operators. A prime operator is a single entity in the cumulant evaluation procedure. Expanding Ω given in (7) the resulting products of function $f$ have to be chosen in such a way that $\exp(\sum_\nu \lambda_\nu S_\nu)$ (with appropriate parameters $\lambda_\nu$) represents a good approximation of the exact ground state. Following ref. 13 we obtain a set of coupled equations for calculating the ground–state energy $E_0$:

$$\begin{align*}
E_0 &= \langle He^S \rangle,
0 &= \langle S_\mu | He^S \rangle, \quad \mu = 1, \ldots
S &= \sum_\nu \lambda_\nu S_\nu.
\end{align*}$$

(9)

The expansion coefficients $\lambda_\nu$ can be determined from the equation (8). Note that the relation $0 = (A | H \Omega)$ holds for all operators $A$ (see ref. 13).

In several cases the set of non-linear equations (8) can be transformed into an eigenvalue problem. First, the cumulant expectation values with the exponential can be evaluated according to appendix A. This leads to

$$\begin{align*}
E_0 &= \langle He^S \rangle,
\langle S_\mu^+ He^S \rangle &= E_0 \langle S_\mu^+ e^S \rangle, \quad \mu = 1, \ldots
\end{align*}$$

(10)

with $S$ defined in (8). With the formal definition $S_0 = 1$ and the assumption $\langle S_\nu \rangle = 0$ for all operators $S_\nu$ ($\nu = 1, \ldots$) in $S$ we can formally include (10):1

$$\begin{align*}
\langle S_\mu^+ He^S \rangle &= E_0 \langle S_\mu^+ e^S \rangle, \quad \mu = 0, 1, \ldots
\end{align*}$$

(11)

If now consider the case that any products of operators $S_\nu$ can be expressed by linear combinations of other operators from the set $\{S_\nu\}$ we can decompose the exponential into

$$\exp(\sum_\nu \lambda_\nu S_\nu) = \sum_\nu \beta_\nu S_\nu.$$  

(12)

where we have introduced a new set of parameters $\{\beta_\nu\}$. Inserting (12) into (11) leads to a generalized eigenvalue problem:

$$\begin{align*}
\sum_\nu \beta_\nu \langle S_\mu^+ HS_\nu \rangle &= E_0 \sum_\nu \beta_\nu \langle S_\mu^+ S_\nu \rangle, \quad \mu = 0, 1, \ldots
\end{align*}$$

(13)

Note that (11) is a priori size–consistent even if the sum in $S$ is restricted to a finite set of operators $S_\nu$ due to the exponential which contains perturbations of the ground state up to infinite order. Note that, however, (13) with a finite set $\{S_\nu\}$ is no longer size–consistent since the state $\sum_\nu \beta_\nu S_\nu | \phi_0 \rangle$ contains only a finite number of excitations. This is the same case as in a configuration–interaction calculation.

### III. Hole Motion (Revisited)

In this section we use the present formalism to describe one hole moving in an undoped antiferromagnet (for hole concentration $\delta \to 0$). This problem has already been studied in a number of papers. The main contribution to the hole motion is the following: The motion of the hole locally destroys the antiferromagnetic spin order leading to a string of spin defects. However, quantum spin fluctuations can repair pairs of frustrated spins. This process leads to a coherent motion of the hole for each of the two sublattices. Some of the above calculations are based on the motion in an ideal antiferromagnetic background, e.g. ref. 16. However, spin fluctuations in the antiferromagnetic
ground state allow for additional hole motion processes and thus influence the properties of the hole quasiparticles. Several authors have also studied the hole motion using numerical methods like exact diagonalization and Quantum Monte Carlo methods.

In the following we discuss the hole dispersion relation by calculating the ground–state energy of a system at half–filling which contains one additional hole with fixed momentum \( \mathbf{k} \). Contrary to usual methods this approach is based on a static view of the hole motion problem. For a proper description of the one–hole states we use the concept of filling which contains one additional hole with fixed momentum \( \mathbf{k} \). The operators \( A_\uparrow \) operating on the Néel state with one hole, \( \hat{c}_\uparrow \hat{\phi}_{\text{Néel}} \), moves the hole \( n \) steps away and creates a path or string of \( n \) spin defects attached to the transferred hole. Explicitly, the operators \( A_\uparrow \) are defined by

\[
A_{1\uparrow} = -\frac{1}{\sqrt{z_0}} \sum_{ij} \hat{c}_{ji} \hat{c}_{ij} \hat{R}_{ji}, \\
A_{2\uparrow} = \frac{1}{\sqrt{z_0 (z_0 - 1)}} \sum_{i j l} \hat{c}_{i l} S_l^+ \hat{c}_{i l}^\dagger R_{ij}^{(i)} \hat{R}_{ji}, \\
A_{3\uparrow} = \frac{1}{\sqrt{z_0 (z_0 - 1)^2}} \sum_{i j l m} \hat{c}_{m l} S_l^+ S_l^+ \hat{c}_{i l}^\dagger R_{ij}^{(i)} R_{il}^{(j)} \hat{R}_{ji}, \\
\ldots
\]

(\( i \in \uparrow, j \in \downarrow, l \in \uparrow, m \in \downarrow \))

The operators \( A_{n\downarrow} \) for the ‘down’ sublattice are defined analogously with all spins reversed. \( z_0 = 4 \) denotes the number of nearest neighbor sites in the lattice. The matrices \( \hat{R}_{ji} \) and \( R_{ij}^{(i)} \) allow the hole to jump to its four nearest neighbors in the first step and to only three new nearest neighbors by hopping forward in each further step:

\[
\hat{R}_{ji} = \begin{cases} 
1 & i,j \text{ nearest neighbors} \\
0 & \text{otherwise}
\end{cases},
\]

\[
R_{ij}^{(i)} = \begin{cases} 
1 & j,l \text{ nearest neighbors and } l \neq i \\
0 & \text{otherwise}
\end{cases}.
\]

Let us split the Hamiltonian (1) into an unperturbed part \( H_0 \) and into a perturbation \( H_1 \) according to

\[
H_0 = H_{\text{Ising}} = J \sum_{<ij>} (S_i^z S_j^z - \frac{n_i n_j}{4}) + J (N - 2), \\
H_1 = H_{\perp} + H_\perp
\]

\[
= -t \sum_{<ij>,\sigma} (\hat{c}_i^\dagger \hat{c}_j + \hat{c}_j^\dagger \hat{c}_i) + \frac{J}{2} \sum_{<ij>} (S_i^+ S_j^- + S_i^- S_j^+).
\]

The unperturbed Hamiltonian \( H_0 \) is essentially the Ising part of the Heisenberg exchange in (1) whereas \( H_1 \) contains the transverse part as well as the conditional hopping contribution. The ground state \( |\phi_0\rangle \) of \( H_0 \) with one hole with momentum \( \mathbf{k} \) is given by

\[
|\phi_0\rangle = \frac{1}{\sqrt{N/2}} \sum_{i \in \uparrow} e^{i \mathbf{k} \mathbf{R}_i} c_{i \uparrow} |\phi_{\text{Néel}}\rangle.
\]

For the wave operator \( \Omega \) we use the exponential ansatz of Sec. II. In order to include the path operators \( A_n \) described above and also to take into account ground–state spin fluctuations generated by \( H_\perp \) we choose the following form

\[
\Omega = \exp \left( \sum_{n=1}^{\infty} \lambda_n A_n \right) \exp (\mu A_F)
\]

Here, \( A_F \) is a spin–flip operator defined by

\[
A_F = \sum_{<ij>} S_i^- S_j^+ P_{ij} \quad (i \in \uparrow, j \in \downarrow),
\]

\[
P_{ij} = \prod_{l(ij)} (n_{l \uparrow} + n_{l \downarrow}).
\]
It creates pairs of spin flips on nearest neighbor sites and corresponds to the well–known Bartkowski wavefunction which describes spin fluctuations in the ground state of the Heisenberg antiferromagnet alone. The projector $P_{ij}$ prevents spin fluctuations next to the hole ( $l(ij)$ denotes all sites next to the pair of nearest neighbor spins $ij$ ). It is introduced to avoid ambiguities since such an excitation is equivalent to a path of length 2 and is described by the path operator $A_2$. Note that $A_F$ does not commute with the path operators $A_n$ because of the presence of the projector $P_{ij}$. Note also that we have introduced in $\Omega$ a product of two exponentials rather than one. This was done for simplification of further calculations and can be considered as an extension of $\langle 3 \rangle$. For the same reasons we shall also take into account spin fluctuations only up to the first order in $A_F$, i.e.,

$$\exp(\mu A_F) \approx 1 + \mu A_F. \quad (20)$$

Using (20) we shall find $-\mu = \frac{1}{\eta} \ll 1$ which means that higher order terms in $A_F$ are indeed negligible.

Following the method described in Sec. II we obtain the following set of non–linear equations to determine the ground–state energy $E_0$ and the coefficients $\lambda_n$.

$$E_0 = \langle H | \exp(\sum_{n=1}^{\infty} \lambda_n A_n) (1 + \mu A_F) \rangle,$$

$$0 = \langle (1 + \mu A_F) A_i | H \exp(\sum_{n=1}^{\infty} \lambda_n A_n) (1 + \mu A_F) \rangle, \quad i = 1, 2, \ldots, \quad (21)$$

$$0 = \langle A_F | H \exp(\sum_{n=1}^{\infty} \lambda_n A_n) (1 + \mu A_F) \rangle.$$ 

Note that for algebraic reasons we have used composite operators $A$ in the second of these equations (compare (1)). In (21) all cumulant expectation values have to be taken with respect to the unperturbed ground state (17). As described at the end of Sec. II we use the relations given in appendix A to transform the set of coupled non–linear equations (21) into a generalized eigenvalue problem.

After evaluating the cumulants and expanding the exponential one obtains terms with products of path operators. For such a product of two path operators either the two operators couple to different holes creating two different paths, or the second path operator couples to the first one concatenating both paths. In the latter case both paths couple to the same hole. Therefore, their effect can be written as one path operator with both lengths added. Thus, for such a product of two path operators either the two operators couple to different holes creating two different paths, or the second path operator couples to the first one concatenating both paths. In the latter case both paths couple to the same hole. Therefore, their effect can be written as one path operator with both lengths added. Thus, after having evaluated the cumulants we introduce new path coefficients $\beta_n$ instead of the $\lambda_n$. Application of $\Omega$ on the unperturbed ground state leads to

$$\exp(\sum_{n=1}^{\infty} \lambda_n A_n) \exp(\mu A_F) | \phi_0 \rangle = (1 + \sum_{n=1}^{\infty} \beta_n A_n)(1 + \mu A_F) | \phi_0 \rangle \quad (22)$$

There is a non–linear one–to–one correspondence between the $\beta$ and $\lambda$. In the following we only calculate the $\beta_n$ since the knowledge of the original $\lambda_n$ is not needed.

From (21) we find for the energy

$$E_0 = \langle H | \sum_{n=1}^{\infty} \beta_n A_n \rangle + \mu \langle H | A_F \rangle. \quad (23)$$

Note that no mixed contributions in $\beta_n$ and $\mu$ are obtained because $H$ can only repair two spin defects created by $\Omega$. The first term in (23) depends on the hole momentum $K$ and therefore describes the hole dynamics. The second term represents the ground state energy of a Heisenberg antiferromagnet (with one hole). The coefficients $\mu$ and $\beta_i$ have to be determined from (23). By evaluating the cumulants according to Appendix A, expanding the exponentials as described above and reintroducing new cumulants we find

$$\left( (1 + \mu A_F) A_i | H (\sum_{n=1}^{\infty} \beta_n A_n) (1 + \mu A_F) \right)$$

$$= \langle H | \sum_{n=1}^{\infty} \beta_n A_n \rangle \times \left( (1 + \mu A_F) A_i | (\sum_{n=1}^{\infty} \beta_n A_n) (1 + \mu A_F) \right)$$

$$+ \mu (A_i | \sum_{n=1}^{\infty} \beta_n A_n \rangle \times \langle A_F | H (\sum_{n=1}^{\infty} \beta_n A_n) (1 + \mu A_F) \rangle \quad (24)$$

$$+ \mu (A_i | \sum_{n=1}^{\infty} \beta_n A_n \rangle \times \langle A_F | H (\sum_{n=1}^{\infty} \beta_n A_n) (1 + \mu A_F) \rangle \quad (24)$$
and

\[ 0 = (A_F|H(\sum_{n=1}^{\infty} \beta_n A_n)|1 + \mu A_F) + (A_F|H) + \mu (A_F|HA_F). \]  

The set of equations (24) can be considered as a generalized eigenvalue problem for the \( \beta_n \) and has to be solved numerically (note \( A_n|A_m = \delta_{nm} \)). The expectation values \( |A_n| \) were calculated analogously to a recent paper. All important contributions to the hole motion including spiral paths and processes due to ground state spin fluctuations are taken into account. From (22) we obtain \( \mu = -\frac{1}{4} \). This value can also be derived directly from the expression for the wave operator \( \Omega \) given in (7) with \( H_1 = H_\perp \), see ref.\( ^{21} \). After having calculated the values for the coefficients \( \beta_i \) and \( \mu \) we are able to evaluate the ground–state energy \( E_0 \) of the system given by (24). The energy dispersion for different values of \( t/J \) is shown in Fig. 1. This dispersion relation agrees very well with other theoretical results and also with data found from exact diagonalization of small clusters. The energy minima are located at \( (\pm\pi/2, \pm\pi/2) \), the total bandwidth is \( 1.4J \ldots 1.5J \) (see Fig. 2).

The path coefficients strongly decrease in space within a few lattice constants, see e.g.\( ^{22,23} \). The cloud of spin defects surrounding the hole (spin bag) is small with an average radius of approximately 2 lattice constants for \( t/J = 5 \). Therefore it is a reasonable approximation to include only the first path coefficients to describe the quasiparticle.

The present approach can also be extended to finite hole concentrations. If we neglect hole–hole interaction processes we arrive at a rigid–band approximation. The Fermi surface shows the well-known hole pockets (see Fig. 3).

Note that within the method presented here all quantities are calculated consistently. Especially, our results are not based on the calculations of the path coefficients for the localized hole case\( ^{21,22} \). This is an improvement compared to former calculations\( ^{26} \). There the hole dispersion relation problem was tackled in first order perturbation with respect to spin fluctuations with a zero order wave function, i.e., with the path coefficients of the localized hole.

### IV. Ground State and Staggered Magnetization of the Doped System

Now we turn to the description of the spin dynamics in the doped system. We want to focus on the coupling between spin waves and hole motion. In the following \( \delta \) denotes the hole concentration away from half–filling. Our system with \( N \) lattice sites possesses \( M = \delta N \) dopant holes.

The change of the magnetic properties due to the presence of mobile holes was already investigated within the \( t – J \) model by several authors\( ^{26,27} \). In some of the calculations the so-called magnetic polaron model\( ^{28} \) was used which can be derived from the \( t-J \) Hamiltonian\( ^{27} \) by use of slave–fermion methods. In ref.\( ^{27} \) dynamical spin susceptibilities were calculated using projection techniques\( ^{28} \) instead. There a ground state according to a rigid–band approximation for the hole quasiparticles was used to evaluate expectation values. However, such a ground state is not able to explain the observed strong experimental decrease\( ^{29} \) of the sublattice magnetization with increasing hole doping.

In this section we want to determine the ground–state energy and the staggered (sublattice) magnetization of the doped system. In the next section we shall show how spin–wave energies can be obtained from a ground–state calculation. The results demonstrate that this new static approach leads to similar results as obtained from the calculation of dynamical quantities (e.g., correlation functions).

In the Hamiltonian we introduce an additional field \( B_A \) parallel to the \( z \)-axis which couples to the staggered magnetization. We use the following decomposition:

\[
H = H_0 + H_1, \quad H_0 = H_{\text{Ising}} + H_{\text{Zeeman}} = J \sum_{<ij>} (S_i^z S_j^z - \frac{N_i N_j}{4}) + J (N - 2M) + g_J \mu_B B_A (-\sum_i S_i^z + \sum_j S_j^z), \\
H_1 = H_\parallel + H_\perp = -t \sum_{<ij>,\sigma} (\hat{c}_{i\sigma}^+ \hat{c}_{j\sigma} + \hat{c}_{j\sigma}^+ \hat{c}_{i\sigma}) + \frac{J}{2} \sum_{<ij>} (S_i^- S_j^+ + S_i^+ S_j^-).
\]

The ground state \( |\phi_0\rangle \) of the unperturbed Hamiltonian \( H_0 \) is an antiferromagnetically ordered Néel state with \( M \) holes. The holes have fixed momenta \( k_m \) and are located on the sublattice \( \sigma_m \) (\( \sigma_m = \uparrow, \downarrow \)).
\[ |\phi_0\rangle = \hat{c}_{k_1\uparrow} \ldots \hat{c}_{k_M\sigma_M} |\psi_{N\text{\acute{e}el}}\rangle \]
\[ = \frac{1}{(N/2)^{M/2}} \prod_{m=1}^M \left( \sum_{i_m\in\sigma_m} e^{i\mathbf{k}_m \cdot \mathbf{R}_{i_m}} \hat{c}_{i_m\sigma_m} \right) |\psi_{N\text{\acute{e}el}}\rangle. \]  

(27)

The staggered magnetization at zero temperature can be obtained from the ground state energy \( E_0 \) by

\[ M_{\text{eff}} = \langle \psi_0 | g J_M B (\sum_{i\in\uparrow} S_i^z - \sum_{j\in\downarrow} S_j^z) | \psi_0 \rangle = - \frac{\partial E_0}{\partial B_A}. \]  

(28)

where \( |\psi_0\rangle \) denotes the exact ground state.

In the wave operator \( \Omega \) we have to include both parts of \( H_1 \), the spin–flip term \( H_{\perp} \) and the hopping term \( H_t \). Spin fluctuations generated by \( H_{\perp} \) can be described by pairs of spin waves \((S^{-}_{q\uparrow}, S^{+}_{q\downarrow})\) with

\[ S^{-}_{q\uparrow} = \frac{1}{\sqrt{N/2}} \sum_{i\in\uparrow} e^{i\mathbf{q} \cdot \mathbf{R}_i} S_i^-, \]
\[ S^{+}_{q\downarrow} = \frac{1}{\sqrt{N/2}} \sum_{j\in\downarrow} e^{i\mathbf{q} \cdot \mathbf{R}_j} S_j^+. \]  

(29)

defining creation operators of magnons on both sublattices. The momenta \( \mathbf{q} \) have to be taken from the magnetic Brillouin zone. Recently, we have shown\(^{33} \) that the dynamics of the undoped system can well be described by a wave operator containing the magnon creation operators \( \{29\} \). This represents an extension of the Bartkowski wavefunction\(^3 \) used in the previous section. We include spin–flip pairs not only on nearest neighbor sites but on sites which are arbitrarily far away from each other. To treat hole motion processes induced by \( H_t \) we use the path operators \( A_n \) from Sec. III without the projectors \( P_{ij} \). Both types of excitations are included in the wave operator \( \Omega \).

\[ \Omega = \exp \left( \sum_{\mathbf{q}} \nu_{\mathbf{q}} (S^{-}_{\mathbf{q}\uparrow} S^{+}_{\mathbf{q}\downarrow}) + \sum_n \lambda_n A_n \right) \]  

(30)

Note that the path operators \( A_n \) commute with the spin–wave creation operators \( S_{\mathbf{q}} \),\(^{29} \) because they all contain only spin–flip operators destroying the Néel order. The dot \( \cdot \) in the first term of \( \{30\} \) indicates that the quantity inside (...) has to be treated as a single entity in the cumulant formation.

The set of equations for the ground–state energy and the coefficients \( \nu_{\mathbf{q}} \) and \( \lambda_n \) (analogous to \( \{3\} \)) reads

\[ E_0 = \langle H' | \Omega \rangle - \frac{1}{2} Ng J_M B_A (1 - \delta), \]
\[ 0 = \langle (S^{-}_{\mathbf{q}\uparrow} S^{+}_{\mathbf{q}\downarrow}) | H' | \Omega \rangle, \]
\[ 0 = \langle A_n | H' | \Omega \rangle \]  

(31)

with \( \Omega \) given by \( \{27\} \). The cumulant expectation values have to be taken with respect to the unperturbed ground state \( \{27\} \). We have shifted the energy zero level by setting \( H' = H + g J_M B_A (1 - \delta) N/2 \). This leads to \( \langle \phi_0 | H' | \phi_0 \rangle = 0 \) for \( M \) holes in the system which are not on nearest neighbor sites. After evaluating the cumulants and expanding the exponential we again obtain terms containing products of path operators which can again be replaced by new path operators. As discussed above we take care of path concatenations by replacing the set of coefficients \( \{\lambda_n\} \) by new coefficients \( \{\beta_1\} \) which have to be calculated.

To handle \( \{31\} \) we neglect all terms which depend more strongly than linear on the hole concentration \( \delta \). In particular, we do not consider hole–hole interactions which are of order \( \delta^2 \). This is certainly a good approximation for weak doping. For the energy we find

\[ \langle H' | \Omega \rangle = \langle H_{\perp} \sum_{\mathbf{q}} \nu_{\mathbf{q}} S^{-}_{\mathbf{q}\uparrow} S^{+}_{\mathbf{q}\downarrow} \rangle + \beta_1 \langle H_t A_1 \rangle + \beta_2 \langle H_{\perp} A_2 \rangle \]
\[ = \langle \phi_{\text{N\acute{e}el}} | H_{\perp} \sum_{\mathbf{q}} \nu_{\mathbf{q}} S^{-}_{\mathbf{q}\uparrow} S^{+}_{\mathbf{q}\downarrow} | \phi_{\text{N\acute{e}el}} \rangle (1 - \delta)^2 \]
\[ + \sum_{m=1}^{M} \left( \beta_1 \langle m | H_t A_1 | m \rangle (1 - \delta) + \beta_2 \langle m | H_{\perp} A_2 | m \rangle (1 - \delta)^2 \right) \]  

(32)
where we have introduced the notation $|m\rangle$ as abbreviation for an one–hole state:

$$|m\rangle = c_{m,\sigma_m}|\phi_{N\text{\acute{e}el}}\rangle. \quad (33)$$

The brackets $\langle...\rangle$ in the first line of $32$ denote expectation values with the ground state $|\phi_0\rangle$ of $H_0$. In the second equation of $32$ we have assumed that the holes move independently. When calculating expectation values like $\langle|\phi_0\rangle H_L A_2 |\phi_0\rangle$ holes which do not couple to $H_L$ and $A_2$ give rise to a prefactor $(1-\delta)$ for each site where the spin operators from both $H_L$ and $A_2$ act. $(1-\delta)$ describes the probability for finding a spin at such a site. For consistency we explicitly write down all factors $(1-\delta)$ although the actual calculation is only valid up to first order in $\delta$. The energy $32$ consists of a spin–wave part and a hole part. The latter is proportional to the hole concentration $\delta$ (besides the factors $(1-\delta)$ mentioned above).

For the coefficients $\nu_\mathbf{q}$ we obtain from $32$ a set of integral equations for $\nu_\mathbf{q}$ when $\beta_n$ is fixed. Eq. $33$ is a set of coupled non–linear equations for $\beta_n$ with fixed $\nu_\mathbf{q}$. Analogous to the preceding section it can be transformed into a generalized eigenvalue problem. To be short, here we only state the integral equation for the $\nu_\mathbf{q}$:

$$0 = ((S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+) | H \Omega \rangle$$
$$= (\phi_{N\acute{e}el}|S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+\rangle + H_L |\phi_{N\acute{e}el}\rangle (1-\delta)^2$$
$$+ (\phi_{N\acute{e}el}|S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+\rangle (H_{L\text{sing}}(1-\delta) + H_{Zeeman}) \nu_\mathbf{q} (S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+) |\phi_{N\acute{e}el}\rangle (1-\delta)^2$$
$$+ (\phi_{N\acute{e}el}|S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+\rangle + H_\perp \frac{1}{2}\sum_{\mathbf{q}_1} \nu_\mathbf{q}_1 (S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+) |\phi_{N\acute{e}el}\rangle (1-\delta)^4$$
$$+ \sum_{m=1}^{M} (m|(S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+)^+ (H_i \beta_1 A_1 + (H_{L\text{sing}}(1-\delta) + H_{Zeeman}) \beta_2 A_2) |m\rangle (1-\delta)^2$$
$$+ \sum_{m=1}^{M} (m|(S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+)^+ (H_i \beta_1 A_1 + H_\perp \beta_2 A_2 (1-\delta)) \sum_{\mathbf{q}_1} \nu_\mathbf{q}_1 (S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+) |m\rangle (1-\delta)^3$$
$$+ \sum_{m=1}^{M} (m|(S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+)^+ (H_{L\text{sing}}(1-\delta) + H_{Zeeman}) \frac{1}{2!} (\beta_1 A_1)^2 |m\rangle (1-\delta)^2$$
$$+ \sum_{m=1}^{M} (m|(S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+)^+ (H_i \beta_1 A_1 \beta_2 A_2 + H_\perp \frac{1}{2!} (\beta_2 A_2)^2 (1-\delta)) |m\rangle (1-\delta)^3$$
$$+ \sum_{m=1}^{M} (m|(S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+)^+ H_\perp \frac{1}{2!} (\beta_1 A_1)^2 \sum_{\mathbf{q}_1} \nu_\mathbf{q}_1 (S_{\mathbf{q}_1}^- S_{\mathbf{q}_1}^+) |m\rangle (1-\delta)^4. \quad (34)$$

The brackets $\langle\phi_{N\acute{e}el}|...|\phi_{N\acute{e}el}\rangle$ and $\langle m |...| m \rangle$ denote cumulant expectation values with $|\phi_{N\acute{e}el}\rangle$ and $|m\rangle$, respectively. To simplify the evaluation of $34$ we use linear spin–wave approximation and assume independent hole motion. Furthermore we cut the eigenvalue problem for the coefficients $\beta_n$ after the third variable, i.e., we include only paths of lengths 0, 1, and 2. Note that the essential process for the hole motion is taken into account by this approximation, i.e., a hole can hop twice via $H_\epsilon$, then the spin defects are removed via $H_\perp$. The mechanism of hole–spin coupling is also included: A hole hops via $H_\epsilon$, the arising spin defect can be considered as starting point of a spin wave propagating via $H_\perp$. With linear spin–wave approximation $34$ reduces to a quadratic equation for each $\nu_\mathbf{q}$.

To find a solution of $33$ (with fixed hole momenta) we have to proceed iteratively: starting from some fixed values for $\beta_1$ and $\beta_2$ we first calculate the $\nu_\mathbf{q}$ for each $\mathbf{q}$. The obtained values for $\nu_\mathbf{q}$ are then inserted into the eigenvalue problem for the coefficients $\beta_n$. The solution leads to new values of $\beta_n$ for the next step. With good initial values for $\beta_1$ and $\beta_2$ the iteration converges after a few steps.

Taking the derivative of $E_0$ with respect to the external field $B_A$ according to $28$ we can evaluate the doping dependent staggered (sublattice) magnetization. Results for small hole concentrations and different values of $t/J$ are shown in Fig. 4. The magnetization for the undoped case becomes $M_{eff}/M_{eff,N\acute{e}el} \approx 0.606$, i.e., it is the same as in linear spin–wave theory. This was expected since our approximations are equivalent to those of linear spin–wave theory. The magnetization decreases with increasing hole concentration $\delta$. At a critical doping level $\delta_c$, it vanishes indicating a magnetic phase transition to a paramagnetic state. For instance, for $t/J = 5$ the critical concentration $\delta_c$ is approximately 3.1% which is in good agreement with experimental data.

With the coefficients $\nu_\mathbf{q}$ and $\beta_n$ we are able to compute static and dynamical properties of the system. In the next section we want to show how to determine spin–wave energies within our method.
V. SPIN-WAVE ENERGIES

The ground state calculation in the preceding section allows for the derivation of spin–wave energies. The basic idea is to calculate the energy of a state which is the ground state of the system with certain fixed boundary conditions. Such boundary parameters are conserved quantities like the electron number, the total magnetization or the total momentum. The ground state determined above has \((N-M)\) electrons, zero total momentum and zero total magnetization (if we have the same number of holes on each sublattice).

To find an expression for the spin–wave energy we now start from an unperturbed state \(|\phi_0\rangle\) which contains one additional spin excitation

\[
|\phi_0\rangle = S_{\mathbf{K} \uparrow} \hat{c}_{\mathbf{k}, \sigma_1} \cdots \hat{c}_{\mathbf{k}_M \sigma_M} |\phi_{\text{Ncel}}\rangle.
\]

Since the total magnetization \(M_{\text{tot}}\) is a conserved quantity,

\[
[\hat{M}_{\text{tot}}, H]_{-} = 0, \quad [\hat{M}_{\text{tot}}, \Omega]_{-} = 0,
\]

\[
\hat{M}_{\text{tot}} = g J \mu B \left( \sum_i S_i^+ + \sum_j S_j^z \right),
\]

(with \(\Omega\) given by the exponential form \((30)\) ) the perturbed state has a net magnetization of \(g J \mu B\) in \(\mathbf{z}\)-direction and is therefore orthogonal to the ground state calculated in the preceding section. A straightforward calculation based on the wave operator \((30)\) leads to the same equations for the coefficients \(\nu_{\mathbf{q}}\) and \(\beta_{\mathbf{q}}\). This is also due to our use of the spin–wave–like approximations where the spin–wave modes are treated as independent excitations. For the energy of the system we find an expression which differs from the ground state energy \(E_0\) of eq. \((32)\) by \((\hbar=1)\):

\[
\Delta E = \omega_{\mathbf{K}} = 2J (1-\delta) + \frac{J(|\mathbf{K}|)}{2} \nu_{\mathbf{K}} (1-\delta)^2
\]

\[
+ \beta_1 \sum_{m=1}^{M_\uparrow} (m|S_{\mathbf{K} \uparrow} H \sigma_1 A_1 S_{\mathbf{K} \uparrow}^{|m} (1-\delta)
\]

\[
+ \beta_2 \sum_{m=1}^{M} (m|S_{\mathbf{K} \uparrow}^+ H \mathbf{z} \sigma_1 A_2 S_{\mathbf{K} \uparrow}^{|m} (1-\delta)^2.
\]

\(M_\uparrow = M/2\) is the number of holes on the \(\uparrow\) sublattice. The first \(m\)–sum runs over all holes with \(\sigma_m = \uparrow\). Note that a spin excitation \(S_{\mathbf{K} \uparrow}^+\) in \(|\phi_0\rangle\) \((35)\) would lead to the same result because the hole momentum distribution in the ground state should be symmetric with respect to the sublattices. The quantities \(\nu_{\mathbf{K}}\) and \(\beta_{\mathbf{q}}\) depending on the hole concentration \(\delta\) can be taken from the calculation in the last section. \(J(\mathbf{q})\) denotes the Fourier–transformed exchange coupling defined by

\[
J(\mathbf{q}) = J z_0 \gamma(\mathbf{q}) = J \sum_{\Delta} e^{i \mathbf{q} \Delta} = 2J (\cos q_x + \cos q_y).
\]

For the undoped case \((\delta = 0)\) the expression \((38)\) exactly reproduces the result of linear spin-wave theory:

\[
\omega_{\mathbf{K}} = 2J \sqrt{1-\gamma(|\mathbf{K}|)^2}.
\]

The spin–wave energies for different hole concentrations calculated from \((38)\) are shown in Figs. 5 and 6.

For an analytic discussion of the \(\mathbf{K}\)-dependence of the spin–wave energy we calculate \(\nu_{\mathbf{K}}\) for fixed path coefficients \(\beta_1\) and \(\beta_2\) using approximations which are again equivalent to those of linear spin–wave theory. (Note that the results must be inserted iteratively into the eigenvalue problem for the \(\beta_{\mathbf{q}}\) to get a self–consistent solution as discussed in the last section). Eq. \((34)\) is a quadratic equation for \(\nu_{\mathbf{q}}\). Its solution can be inserted into \((38)\). With

\[
t A(\mathbf{q}) := \sum_{m=1}^{M} (m| (S_{\mathbf{q} \uparrow}^+ S_{\mathbf{q} \downarrow}^- ) H \sigma_1 A_1 (S_{\mathbf{q} \downarrow}^- S_{\mathbf{q} \uparrow}^+ ) |m)
\]

we find
\[ \omega_K = 2J(1-\delta) \sqrt{1 - \gamma(K)^2 - \frac{\beta_1 A(K)}{2} \frac{t/J}{J}} + \frac{(\beta_1 A(K) t/J)^2}{16}. \] (42)

The main contribution to the renormalization of the spin–wave energy is determined by \( \beta_1 \) and is therefore due to the coupling between a path of length 1 and a spin wave. Note that terms with \( \beta_2 \) cancel each other. The next process renormalizing \( \omega_K \) is expected to arise from \( \beta_3 \) which we neglect here. However, \( \beta_1 \) is small compared to \( \beta_3 \), see Sec. III. From this fact we expect that the error introduced by including only paths of length 0,1, and 2 in the calculation of the spin–hole coupling is rather small.

To perform the expectation value in \( A(K) \) we need to know the momentum distribution of the hole quasiparticles. It is well established that the minimum energy of the quasiparticle dispersion is located at \( (\pm \pi/2, \pm \pi/2) \). As a simple approximation we neglect the effect of the Fermi surface and assume that the momenta of all holes are \( (\pm \pi/2, \pm \pi/2) \), i.e., all holes are in the centre of the hole pockets. In the following we are only interested in small momenta \( K \). The first non–trivial order of \( K \) leads to

\[ \gamma(K) = \frac{J(K)}{J_{z0}} \approx 1 - \frac{K^2}{4}, \quad A(K) \approx 4 \delta K^2. \] (43)

Inserting this into (42) we end up with the following approximation for the spin–wave energy at small momenta \( K \):

\[ \omega_K = \sqrt{2} J |K| (1-\delta) \sqrt{1 - 4 \delta_1 t/J}. \] (44)

The \( K \)-dependence remains linear even upon doping. For \( K \to 0 \) the spin–wave energy \( \omega_K \) goes to zero in accordance with Goldstone’s theorem. The spin–wave velocity \( v = \omega_K/|K| \) \((|K| \to 0)\) decreases with doping. At a critical hole density \( \delta_{c2} \) the spin–wave velocity vanishes. If we assume \( \beta_1 \) to be independent of the hole concentration and approximate \( 1-\delta \approx 1 \) we obtain

\[ v = v_0 \sqrt{1 - \delta/\delta_{c2}}, \quad \delta_{c2} = \frac{J}{4 t \beta_1}. \] (45)

\( v_0 \) denotes the spin–wave velocity in the undoped antiferromagnet. Using realistic values \( t/J = 5, \beta_1 = 1.25 \) we find \( \delta_{c2} = 4.0\% \). The doping dependence of the spin–wave velocity is in good agreement with experiments \[\text{and as well as with other theoretical results.}^{[4,6]}\] Fig. 7 shows the calculated doping dependence of \( v \) for \( t/J = 5 \). Another important result to be seen in Fig. 5 is that the \( K \)-range, where we find a linear \( K \)-dependence of \( \omega_K \), becomes smaller with increasing hole concentration. This is also in agreement with experimental results, see e.g. Rossat-Mignod \[\text{et al.}^{[4]}\]. Note that the critical concentration \( \delta_{c2} \) where the magnetization vanishes is somewhat smaller than \( \delta_{c2} \) evaluated here. As already mentioned for \( t/J = 5 \) we have \( \delta_{c1} \approx 3.1\% \) whereas \( \delta_{c2} \approx 4.0\% \). This is understandable since the softening of spin–wave excitations should first lead to the vanishing of the staggered magnetization.

The cumulant formalism provides an alternative way to derive the expression (43) for the spin–wave energy. Introducing a transverse wave vector dependent field in the Hamiltonian one obtains the transverse static susceptibility by taking the second derivative of the ground state energy \( E_0 \) with respect to the field. The results for the static susceptibility allow for the calculation of spin–wave energies. The necessary link is given by the dynamical spin susceptibility. The calculation leads exactly to the same expression (43) for the spin-wave energy (!). The detailed derivation has been published in a recent paper \[\text{[28]}\]. The static susceptibility for a staggered field shows a \( K^{-2} \)–divergency for all hole concentrations. This means that the calculation presented in Sec. IV always describes a state with antiferromagnetic long–range order.

VI. CONCLUSION

The aim of this work was to study the hole and spin dynamics of weakly doped antiferromagnets described by the \( t-J \) model. Instead of using dynamical correlation functions our approach is based on the calculation of the ground–state energy and contains a static view of the system.

In the first part we have redervived the quasiparticle dispersion for one hole generated in the ground state at half–filling (Fig. 1,2). The energy minima are found at \( (\pm \pi/2, \pm \pi/2) \) in agreement with other analytical and numerical results. By neglecting hole–hole interaction the calculation can be extended to finite doping by use of a rigid–band approximation. For the Fermi surface (Fig. 3) we obtain the expected picture of hole–pockets around \( (\pm \pi/2, \pm \pi/2) \).

The weakly doped system has been subject of the second part of this paper. We have presented results for the staggered magnetization and for spin–wave energies up to first order in \( \delta \). The staggered magnetization (Fig. 4)
decreases with doping due to spin–hole interactions. The magnetization becomes zero at a hole concentration $\delta_{c_1}$ which indicates the disappearance of antiferromagnetic long–range order. The spin–wave energy is found to be also strongly renormalized with hole doping (Fig. 5 and 6). Both effects are coupled: Due to decreasing spin–wave energies more long–wavelength spin fluctuations are mixed into the ground state which causes the loss of magnetization. The spin–wave velocity has a square–root concentration dependence (Fig. 7) and vanishes for a critical hole concentration $\delta_{c_2}$ given by (\ref{45}). Note that we find $\delta_{c_1} < \delta_{c_2}$ (for instance, $\delta_{c_1} = 3.1\%,$ $\delta_{c_2} = 4.0\%$ for $t/J = 5$), i.e., the magnetization vanishes before the spin-wave velocity becomes zero. This feature was observed in experiments.

The strong renormalization of the spin–wave energies due the presence of holes arises from the interaction of spin waves with spin fluctuations created by hole hopping. This can be interpreted as the decay of spin waves into particle–hole pairs. To first order in $\delta$ only the first path coefficient $\beta_1$ contributes to the renormalized spin–wave energy (\ref{42}). The destruction of the antiferromagnetic state therefore results from the creation and annihilation of additional spin hole pairs. To first order in $\delta$ only the first path coefficient $\beta_1$ contributes to the renormalized spin–wave energy (\ref{42}).

\begin{equation}
\langle A e^S \rangle^c = \langle A e^S \rangle, \\
\langle A B e^S \rangle^c = \langle A B e^S \rangle - \langle A e^S \rangle \langle B e^S \rangle, \\
\langle A e^S B \rangle^c = \langle A e^S B \rangle - \langle A e^S \rangle \langle e^S B \rangle, \\
\langle A B C e^S \rangle^c = \langle A B C e^S \rangle - \langle A B e^S \rangle \langle C e^S \rangle - \langle A e^S \rangle \langle B e^S \rangle - \langle B C e^S \rangle \langle A e^S \rangle + 2 \langle A e^S \rangle \langle B e^S \rangle \langle C e^S \rangle, \\
\ldots
\end{equation}

These relations hold for operators with $\langle S^k \rangle = 0 \forall k > 0.$ $\langle A \rangle = 0, \langle B \rangle = 0$ etc. They can be proven straightforwardly by expanding the exponentials, explicitly evaluating the cumulants and then recollecting all terms. To show this for the second identity we start from the definition of cumulant expectation values for a product of arbitrary operators $A_i$

\begin{equation}
\langle \phi \prod_i A_i^{n_i} | \phi \rangle^c = \left( \prod_i \left( \frac{\partial}{\partial \lambda_i} \right)^{n_i} \right) \ln \langle \phi | \prod_i e^{\lambda_i A_i} | \phi \rangle |_{\lambda_i = 0 \forall i} .
\end{equation}

We find

\begin{equation}
\langle A B S^k \rangle^c = \frac{\partial}{\partial \lambda} \left( \frac{\partial}{\partial \eta} \right)^k \ln \langle e^{\lambda A} e^{\eta B} e^{\eta S} \rangle |_{\lambda = \eta = 0}.
\end{equation}
\[
\begin{align*}
\left( \frac{\partial}{\partial \eta} \right)^k \frac{\langle AB e^{\nu S} \rangle \langle e^{\nu S} \rangle - \langle Ae^{\nu S} \rangle \langle Be^{\nu S} \rangle}{\langle e^{\nu S} \rangle^2} \bigg|_{\nu=0} \\
= \langle ABS^k \rangle - \sum_{i=0}^{k} \binom{k}{i} \langle AS^i \rangle \langle BS^{k-i} \rangle.
\end{align*}
\]

Here we have used \( \langle S^k \rangle = 0 \). Inserting this transformation in the original expression leads to

\[
\langle AB e^{S} \rangle^c = \langle AB \rangle^c + \sum_{k=1}^{\infty} \frac{1}{k!} \langle ABS^k \rangle^c
\]

\[
= \langle AB e^S \rangle - \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \langle AS^i \rangle \langle BS^{k-i} \rangle
\]

\[
= \langle AB e^S \rangle - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i! j!} \langle AS^i \rangle \langle BS^j \rangle
\]

(A3)

In the third equation we have replaced the sum over \( k \) by a sum over \( j \) with \( j = k - i \). Eq. (A4) is the desired result.
FIG. 1. Energy dispersion of the lowest hole excitation for different values of $t/J$. The zero energy was set to the center of mass of the band.

FIG. 2. Total bandwidth of the hole dispersion shown in Fig.1 as function of $t/J$
FIG. 3. Hole Fermi surface obtained from the dispersion relation (Fig. 1) in a rigid band approximation for hole concentrations $\delta=5\%$ and 15\%, $t/J=5$. 
FIG. 4. Staggered magnetization as function of hole concentration $\delta$ for different values of $t/J$.

FIG. 5. Spin–wave energies calculated from (42) as function of momenta $(q,q)$, for $t/J=5$ and different doping concentrations.
FIG. 6. Same as Fig. 5, but for smaller momenta \((q, q)\).

FIG. 7. Spin-wave velocity \(v\) vs. hole concentration \(\delta\) for \(t/J = 5\).