THE BRAUER-MANIN OBSTRUCTION FOR STACKY CURVES

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Abstract. We show that the Brauer-Manin obstruction is the only obstruction to strong approximation for all stacky curves over global fields with finite abelian fundamental groups. This includes all stacky curves of genus $g = \frac{1}{2}$, thus explaining a recent counterexample to the Hasse principle of Bhargava-Poonen. We will furthermore show that the elementary obstruction is the only obstruction to the integral Hasse principle for smooth proper integral models of stacky curves of genus $g < 1$. We then compute the Brauer-Manin obstruction for smooth proper integral models of stacky curves of genus $\frac{1}{2}$.

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1. Introduction

Let $k$ be a global field (a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(T)$). If $X$ is a variety over $k$ then a necessary condition for $X(k)$ to be non-empty is that $X(k_v) \neq \emptyset$ for all places $v$ of $k$. If this is also sufficient we say that $X$ satisfies the Hasse principle.

Recently there has been interest in considering classical arithmetic questions such as the one above when $X$ is an algebraic stack instead of a variety. In particular, Bhargava and Poonen [BP20] have shown that the Hasse principle for a stacky curve holds if the genus is $g < \frac{1}{2}$. And they also gave an explicit example of a stacky curve of genus $g = \frac{1}{2}$ which fails the Hasse principle.

If $X$ has a dense open subscheme, for example if $X$ is a stacky curve, then $X(k)$ is almost equal to the set of rational points $X_{\text{coarse}}(k)$ on its coarse moduli space $X_{\text{coarse}}$, which is a scheme. We will instead look at integral points.

Let $U$ be a Dedekind scheme with fraction field $k$, this is either a smooth curve over a finite field or the spectrum of the ring $\mathcal{O}_{k,S}$ of $S$-integers for a finite set of places $S$ of a number field $k$. An integral model $\mathcal{X}$ of $X$ is a separated finite type algebraic stack $\mathcal{X}$ over $U$ equipped with an isomorphism between the generic fiber $\mathcal{X}_k$ and $X$. The set of integral points on $\mathcal{X}$ is $\mathcal{X}(U)$. We remark that if $\mathcal{X}$ is proper

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over $U$ then we do not necessarily have $\mathcal{X}(U) = X(k)$, unlike the case when $\mathcal{X}$ is a scheme.

A necessary condition for the non-emptiness of $\mathcal{X}(U)$ is that $\mathcal{X}(k_v) \neq \emptyset$ for all places of $k$ which are defined by points of $U$ and $X(k_v) \neq \emptyset$ for all other places. If this condition is also sufficient then we say that $\mathcal{X}$ satisfies the integral Hasse principle.

Although the study of the arithmetic of stacky curves is novel the study of their integral points is not. Indeed, solutions of the generalized Fermat equation $Ax^p + BY^q = CZ^r$ with $X, Y, Z$ coprime can be interpreted as points on a certain stacky curve. This viewpoint is exploited by Poonen, Schaeffer and Stoll in [PSS07] to study the solutions of $X^2 + Y^3 = Z^7$. Furthermore, Darmon and Granville’s work [DG95, Dar97] can be interpreted as the statement that Faltings’s theorem holds for stacky curves, i.e. stacky curves of genus $g > 1$ have only a finite number of points. We also mention work of Beukers [Ben98] which can be done with less explicit computations using stacky methods, the parametrisations in his work correspond to torsors under the geometric fundamental group. The author will expand on these connections in future work.

Recently Mitankin, Nakahara and Streeter [MNS22] studied the Brauer-Manin obstruction for Darmon points on certain quadric orbifolds. The Darmon points on a variety are by definition almost the same as the integral points on a certain root stack, and their orbifold Brauer group is the Brauer group of the corresponding root stack [MNS22, Rem. 3.12]. The Brauer-Manin obstruction on such a quadric orbifold is behaves rather different than the Brauer-obstruction on stacky curves, the Brauer group modulo constants is finite for quadric orbifolds but it is often infinite for stacky curves.

We also mention that there recently has been significant work in proving analogues of the Manin conjecture for modular curves, some of which are stacky curves, see [Phi22] and the references therein. The leading constant, known as Peyre’s constant, in the Manin conjecture for varieties has the order of the Brauer group modulo constants as a factor. It would be interesting to understand what role the Brauer group plays in the analogue of Peyre’s constant for stacky curves of genus $g < 1$. In all the cases for modular curves where the leading constant is known the Brauer group only contains constant elements. In [NX21] the Manin conjecture is studied for a stacky curve $X$ over $\mathbb{Q}$ with signature $(0; 2, 2, 2)$ which has non-constant elements in its Brauer group, but they are unable to obtain a leading constant. Note that the Brauer group modulo constants of this Brauer group is infinite so the role of the Brauer group in Peyre’s constant cannot be the obvious generalization.

A single stack can have many different integral models. But we can describe the subset $\mathcal{X}(U) \subset X(k)$ (it is a subset because $\mathcal{X}$ is separated) using purely local conditions, it is the pre-image of the integral adelic points $\mathcal{X}(\mathbb{A}_U)$ under the map $X(k) \to X(\mathbb{A}_k)$ [BP20, Lemma 13]. To study the Hasse principle on all integral models it thus suffices to study the image of $X(k)$ in $X(\mathbb{A}_k)$.

Two important explanations why a variety fails the Hasse principle are the Brauer-Manin obstruction and the finite descent obstruction [Sto07]. The goal of this paper is to investigate the Brauer-Manin obstruction in the case of stacky curves. One of the main results is.

**Theorem 1.1.** Let $X$ be a tame stacky curve of genus $g < 1$.

1. The finite descent obstruction is the only obstruction to strong approximation. More precisely there exists a torsor $\pi : Y \to X$ under a finite étale
group scheme such that descent along $\pi$ is the only obstruction to strong approximation, i.e. the image of $X(k)$ is dense in $X(\mathbb{A}_k)^n \subset X(\mathbb{A}_k)$.

(2) If $X$ has a finite abelian geometric fundamental group then the only obstruction to strong approximation is the Brauer-Manin obstruction, i.e. the image of $X(k)$ is dense in the Brauer-Manin set $X(\mathbb{A}_k)^{Br} \subset X(\mathbb{A}_k)$.

We will recall the definition of $X(\mathbb{A}_k)^n$ and $X(\mathbb{A}_k)^{Br}$ in §4 and the topology we consider on $X(\mathbb{A}_k)$ is the one constructed by Christensen [Chr20].

If $X$ is a simply connected tame stacky curve then $\text{Br} X = \text{Br} k$ so there is no Brauer-Manin obstruction.

**Theorem 1.2.** Simply connected tame stacky curves $X$ over $k$ satisfy strong approximation, i.e. the image of $X(k)$ in $X(\mathbb{A}_k)$ is dense.

A weaker form of this result is due to Christensen [Chr20, Thm 1.0.1]. He only considers stacky curves of genus $g < \frac{1}{2}$ and shows the weaker statement that strong approximation holds after removing a place in $\mathbb{A}_k$.

All tame stacky curves with genus $g < \frac{1}{2}$ are simply connected due to the Riemann-Hurwitz formula. The above theorem thus give a geometric explanation why the minimal genus of a stacky curve for which the Hasse principle fails is $\frac{1}{2}$.

If $X$ is an integral model of $X$ then $X(\mathbb{A}_U)$ is an open subset of $X(\mathbb{A}_k)$ so we deduce.

**Corollary 1.3.** Let $X$ be an integral model of a stacky curve $X$ with finite abelian geometric fundamental group. The Brauer-Manin obstruction is the only obstruction to the integral Hasse principle for $X$.

The Brauer group of a stacky curve is in general infinite so computing the set $X^{Br}(\mathbb{A}_U)$ is rather difficult in practice. But if $X \to U$ is smooth and proper then it turns out that there is a finite subquotient $B(X)$ of $\text{Br} X$ which is still able to explain all failures of the integral Hasse principle and for which the corresponding Brauer-Manin set $X^{Br}(\mathbb{A}_U)^B(X)$ is easier to compute. The group $B(X)$ of locally constant algebras is defined in §4.

The obstruction coming from $B(X)$ over a number field is known [CTS87, Prop. 3.3.2, 3.6.4] to be related to the vanishing of the elementary obstruction [CTS87, Def. 2.2.1] for varieties. We will extend the definition of the elementary obstruction to integral points using a construction due to Harari and Skorogobatov [HST13, §1] and prove that this relation extends to the integral case and for stacks, this allows us to show our second main result.

**Theorem 1.4.** Let $X \to U$ be a proper smooth map of Deligne-Mumford stacks such that every fiber is a stacky curve. Let $X$ be the generic fiber and assume it has genus $g < 1$.

(1) The elementary obstruction is the only obstruction to the existence of an integral point for $X$.

(2) If there is at least one non-archimedean place of $k$ which is not induced by a point of $U$ then $X(\mathbb{A}_U)^B(X) \neq \emptyset$ implies $X(U) \neq \emptyset$.

(3) In particular, if $X$ has a signature different from $(0)$ and $(0; n, n)$ for all $n > 1$ then $X(U) \neq \emptyset$.

The signature of a stacky curve is defined in §2.

This theorem allows us to give a simple criterion for when a smooth proper integral model of a stacky curve of genus $\frac{1}{2}$ has an integral point in Theorem 7.3.
Structure. We start in §2 with some preliminaries on stacky curves, the analytic topology on stacks and on twisting. We then prove the first simple cases of Theorem 1.4.

We prove Theorem 1.2 in §3 and then apply it and a classical result of Colliot-Thélène and Sansuc [CTS87] which relates the Brauer-Manin and descent obstructions to prove Theorem 1.1 in §4.

We will then compute the Brauer group of a stacky curve in §5 and show that it is generated by certain cup products.

In §6 we introduce the elementary obstruction to integral points and use it to prove the first and last part of Theorem 1.4. We then show that this implies the second part of Theorem 1.4 analogously to [HS13, Cor. 3.6].

In §7 we will use the obstruction induced by \( B(X) \) to give a simple necessary and sufficient criterion for a genus \( \frac{1}{2} \) curve over \( \mathbb{Q} \) to fail the integral Hasse principle. We will finish by using this criterion to reinterpret the example of Bhargava-Poonen.

Conventions. For any stacks \( X, S \) we will use \( X(S) \) to denote the set of equivalence classes of morphisms \( S \to X \) and for a ring \( R \) we write \( X(R) := X(\text{Spec } R) \).

Fix for every field \( k \) a separable closure \( k_s \) and an algebraic closure \( \bar{k} \).

If \( A \) is an abelian group then we write \( A^\vee := \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \) for its dual. We let \( A[n] \subset A \) be the subgroup of \( n \)-torsion elements.

All cohomology/homotopy groups are étale cohomology/homotopy groups. If \( G \) is a smooth group scheme over \( X \) then \( H^1(X, G) \) classifies \( G \)-torsors over \( X \). We will freely use this identification.

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2. Preliminaries

2.1. Stacky curves. Let \( k \) be a field of characteristic \( p \) (possibly 0). We define a stacky curve over \( k \) to be a proper smooth geometrically connected Deligne-Mumford stack of dimension 1 over \( k \) which contains a dense open subscheme. We call a stacky curve tame if \( p \) does not divide the order of any stabilizer. We refer to [VZB22, §5] for basic facts about stacky curves.

For a stacky curve \( X \) we let \( X_{\text{coarse}} \) be its coarse moduli space, which is always a smooth proper curve. Let \( g_{\text{coarse}} \) be the genus of \( X_{\text{coarse}} \). Let \( \mathcal{P} \subset X_{\text{coarse}} \) be the closed subscheme on which the canonical map \( X \to X_{\text{coarse}} \) is not an isomorphism, we will call this the stacky locus. Let \( 1 < e_1, \ldots, e_r \) be the orders of the stabilizers of the finite set of points lying above the points of \( \mathcal{P}(\bar{k}) = \mathcal{P}(k_s) \), which we call the stacky points (the equality follows from [BP20, Lemma 7]). The signature of a stacky curve is the tuple \( (g_{\text{coarse}}; e_1, \ldots, e_r) \), it is well-defined up to a permutation of the \( e_i \). We will also write \( (g; 1, \ldots, 1, e_1, \ldots, e_r) := (g; e_1, \ldots, e_r) \). The genus of a tame stacky curve \( X \) is

\[
g := g_{\text{coarse}} + \frac{1}{2} \left( \frac{e_1 - 1}{e_1} + \cdots + \frac{e_r - 1}{e_r} \right).
\]

The Riemann-Hurwitz formula holds for stacky curves with this choice of genus. This follows from [VZB22 Lemma 5.5.3, Prop. 5.5.6]. If \( g < 1 \), i.e. if \( X \) is Fano, then \( g_{\text{coarse}} = 0 \) and \( r \leq 3 \).
A tame stacky curve is completely determined by its coarse moduli space, its stacky points and the numbers $e_i$ by [VZB22, Lemma 5.3.10]. In particular, every stacky curve is a root stack [Cad07, Def. 2.2.1] over its coarse moduli space $X_{\text{coarse}}$. This implies that dominant maps between stacky curves are the same as dominant maps between their coarse moduli spaces satisfying certain ramification conditions.

2.2. **Fundamental groups.** The fundamental groups of stacky curves over $\mathbb{C}$ have been computed in terms of their signature [BN06, Prop. 5.6]. For a general separably closed field $k$ we can replace the Van Kampen theorem by its analogue [GR03, Thm. IX.5.1] for the étale topology. The analogous argument shows that the fundamental group is a quotient of the profinite completion of the group given in [BN06, Prop. 5.6] by a pro-$p$ group. This group is finite and coprime to $p$ if $X$ is a tame stacky curve of genus $g < 1$, so the description of the fundamental groups of stacky curves in [BN06, Prop 5.5] holds for tame stacky curves of genus $< 1$ over any separably closed field. We will freely use this in what follows.

2.3. **Relative stacky curves.** A relative (tame) stacky curve over a Noetherian scheme $S$ is a smooth proper morphism $\mathcal{X} \to S$ of Deligne-Mumford stacks whose fibers are (tame) stacky curves. It is claimed in [VZB22, Ex. 11.2.1] that relative stacky curves can have certain pathologies, but this is false. In their first example the structure morphism is not well-defined and it is not smooth in their second example. We will prove that all relative tame stacky curves over a regular base are twisted, see [VZB22, Def. 11.2.1].

Let $X$ be a Deligne-Mumford stack, $D \subset X$ a Cartier divisor and $n \in \mathbb{N}$. We will use the notation $X[\sqrt[n]{D}]$ for the root stack which is rooted at $D$, i.e. what is denoted by $X_{D, n}$ in [Cad07, Def. 2.2.4].

**Lemma 2.1.** Let $\mathcal{X} \to S$ be a tame stacky curve. The map $\mathcal{X}_{\text{coarse}} \to S$ is smooth.

If $S$ is regular then $\mathcal{X}$ is isomorphic to a root stack over $\mathcal{X}_{\text{coarse}}$ rooted at irreducible divisors which are pairwise disjoint and étale over $S$.

**Proof.** The map $\mathcal{X}_{\text{coarse}} \to S$ is flat since $\mathcal{X} \to S$ is flat by [AOV08, Cor. 3.3]. By loc. cit. we also have for any point $s \in \mathcal{X} \to S$ an isomorphism $(\mathcal{X}_{\text{coarse}})_s \cong (\mathcal{X})_{\text{coarse}}$. The latter object is the coarse moduli space of a smooth stacky curve and thus smooth.

This implies that $\mathcal{X}_{\text{coarse}} \to S$ is smooth.

Let $D_1, \ldots, D_k \subset \mathcal{X}_{\text{coarse}}$ be the irreducible components of the ramification divisor $D$ of $\mathcal{X} \to \mathcal{X}_{\text{coarse}}$ and let $n_1, \ldots, n_k$ be the corresponding ramification degrees. We get a map $\mathcal{X} \to \mathcal{X}_{\text{coarse}}[\sqrt[n_1]{D_1}] \cdots [\sqrt[n_k]{D_k}]$ to the iterated root stack by the universal property of root stacks.

We now apply [Ols07, Prop. 2.2]. The map $\mathcal{X} \to \mathcal{X}_{\text{coarse}}$ is thus étale locally on $\mathcal{X}_{\text{coarse}}$ given by the projection map of the quotient stack

$$[\text{Spec } \mathcal{O}_{\mathcal{X}_{\text{coarse}}} [z]/[z^n - \pi]/\mu_n] \to \text{Spec } \mathcal{X}_{\text{coarse}}.$$ (2)

Here $\zeta \in \mu_n$ acts via $z \to \zeta z$ and $\pi \in \mathcal{O}_{\mathcal{X}_{\text{coarse}}}$ is such that the induced map $\mathcal{X}_{\text{coarse}} \to \mathbb{A}^1_k$ is étale. The ramification divisor of this map is $\{\pi = 0\}$ which is étale over $S$, so $D$ is étale over $S$. This implies that its irreducible components $D_1, \ldots, D_k$ are disjoint and étale over $S$. The map (2) is the structure map of the root stack $\mathcal{O}_{\mathcal{X}_{\text{coarse}}}[\sqrt[n_i]{\pi = 0}]$ by [Cad07, Ex. 2.4.1]. The map $\mathcal{X} \to \mathcal{X}_{\text{coarse}}[\sqrt[n_1]{D_1}] \cdots [\sqrt[n_k]{D_k}]$ is thus étale locally an isomorphism, hence an isomorphism.

It follows that stacky points on relative stacky curves extend to integral points. To show this we use the relative normalization of schemes [Sta18, Tag 035H].
Theorem 2.2. Let $S$ be a Dedekind scheme with fraction field $K$ and $X \to S$ a relative stacky curve. Let $X$ be the generic fiber. Let $Q \in \mathcal{P} \subset X_{coarse}$ be a point in the stacky locus. Let $T$ be the normalization of $S$ in $Q$. Then $T$ is finite étale over $S$ and $Q$ extends to a $T$-point $Q \in X(T)$.

In particular, if $\mathcal{P}(k) \neq \emptyset$ then $X(S) \neq \emptyset$.

Proof. The map $\mathcal{X}_{coarse} \to S$ is proper so we can extend $Q$ to a $T$-point $Q \to \mathcal{X}_{coarse}$ by the valuative criterion for properness. Let $\mathcal{I} \subset \mathcal{X}_{coarse}$ be its image.

Lemma [2.3] implies that there exist disjoint divisors $D_1, \ldots, D_n \subset \mathcal{X}_{coarse}$ which are finite étale over $S$ and such that $\mathcal{I} \cong \mathcal{X}_{coarse}[\sqrt{D_1}] \cdots [\sqrt{D_n}]$ for $n_i \in \mathbb{N}$. The image $\mathcal{I}$ is irreducible and lies in the closure $\overline{\mathcal{P}} \subset \mathcal{X}_{coarse}$ so has to be equal to one such divisor, assume $\mathcal{I} = D_1$. So $\mathcal{I}$ is finite étale over $S$ and the universal property of normalization implies that $T \cong \mathcal{I}$. The fact that the divisors $D_i$ are disjoint implies by the definition of a root stack [Cad07] Def. 2.2.1 that $Q$ lifts to a point of $\mathcal{I}$.

\[ \square \]

2.4. The analytic topology for points on stacks. Let $R$ be a topological ring which is essentially analytic [Chr20] Def. 5.0.8. Examples of essentially analytic $R$ are local fields $k$ and their rings of integers $\mathcal{O}_k$.

Christensen [Chr20] defines a topology on $X(R)$ for any finitely presented algebraic stack $X$ over $R$ which extends the usual analytic topology for $R$-points on schemes and has all the expected properties.

Let $U$ be a Dedekind scheme whose field of fractions $k$ is a global field and $S$ the set of places of $k$ which do not come from points of $U$. Let $\mathcal{H}_k$ be the ring of adeles and $\mathcal{H}_U$ the subring of integral adeles. Let $\mathcal{X}$ be a separated finite type stack over $U$ with generic fiber $X$. We give $\mathcal{X}(\mathcal{H}_U) = \prod_{v \in S} X(k_v) \times \prod_{v \not\in S} \mathcal{X}(\mathcal{O}_v)$ the product topology.

Christensen also defines a topology on $X(\mathcal{H}_k)$ such that we have a homeomorphism of topological spaces [Chr20] Prop. 13.0.2.

\[ \mathcal{X}(\mathcal{H}_k) \cong \prod_{v \in S} X(k_v) \times \prod_{v \not\in S} ^\prime (X(k_v), \mathcal{X}(\mathcal{O}_v)). \]

The right-hand side is a restricted product. In particular, he shows that the right-hand side is independent of the integral model $\mathcal{X}$.

Remark 2.3. The map $X(k) \to X(\mathcal{H}_k)$ is not necessarily injective when $X$ is a Deligne-Mumford stack, a subtlety which does not happen if $X$ is a scheme. For example, if $X := B\mu_8$ is the classifying stack of $\mu_8$ then the above map is $k^\times /k^{\times 8} \to \mathcal{H}_k^\times /\mathcal{H}_k^{\times 8}$, which is not injective if $k = \mathbb{Q}(\sqrt{7})$, see [NSW08] Thm. 9.1.11]. In practice this seems unimportant.

2.5. Twists of torsors. The following is explained in more detail in [Sko01] §2.2. Let $S$ be a scheme and $G$ a smooth group scheme over $S$. A right $G$-torsor $\mathfrak{a} : T \to S$ is also a left $G\mathfrak{a}$-torsor [Sko01] p. 13, where $G\mathfrak{a}$ is the result of twisting $G$ by the cocycle $\mathfrak{a}$ via the conjugation action $G \to \text{Aut}(G) : g \mapsto (h \mapsto ghg^{-1})$. If $G$ is abelian then $G\mathfrak{a} = G$. We also have the inverse torsor $\mathfrak{a}^{-1} : T \to S$ which has the same underlying scheme as $T$ and which is a right $G\mathfrak{a}$-torsor via the action $t \cdot \mathfrak{a}^{-1} g = g^{-1} \cdot t$. It is analogously a left $G$-torsor.

Let $X$ be an algebraic stack over $S$ and $\pi : Y \to X$ a right $G$-torsor. The twisted torsor [Sko01] p.20, Ex. 2] $\pi_\mathfrak{a} : Y_\mathfrak{a} \to X$ is the right $G\mathfrak{a}$ torsor given by the quotient stack $[T \times_{\mathfrak{a}^{-1}} Y / G]$. The action of $G$ is the diagonal one

\[ G \times_S Y \times_S T \to Y \times_S T : (g, y, t) \mapsto (y \cdot g, g \cdot t^{-1}) = (y \cdot g, t^{-1} \cdot g^{-1}). \]
If $G$ is abelian then the cohomology class of $\pi_a : Y_a \to X$ is equal to $[Y] - a \in H^1(X, G)$.

We can use torsors to study points on $X$ using the following lemma.

**Lemma 2.4.** The images $\pi_a(Y_a(S))$ of the twists $\pi_a : Y_a \to X$ are pairwise disjoint and

$$X(S) = \coprod_{a \in H^1(S, G)} \pi_a(Y_a(S)).$$

**Proof.** If $X$ is a variety then this is [PSS07, Thm 8.4.1]. The proof for stacks is completely analogous. \hfill \Box

3. SIMPLY CONNECTED STACKY CURVES

We recall the definition of the weighted projective line.

**Definition 3.1.** Let $S$ be a stack, and $n, m \in \mathbb{N}^2$. The **weighted projective line** $\mathbb{P}_S(n, m)$ is the quotient stack $[\mathbb{A}_S^2 \setminus \{0\}/G_m]$ where $G_m$ acts on $\mathbb{A}_S^2$ as

$$G_m \times \mathbb{A}_S^2 \to \mathbb{A}_S^2 : (t, x, y) \mapsto (t^n x, t^m y).$$

We will use the traditional notation $\mathbb{P}_R(n, m) := \mathbb{P}_{\text{Spec } R}(n, m)$.

This construction is compatible with base change. The stack $\mathbb{P}_S(n, m)$ has a dense open subscheme if and only if $S$ has a dense open subscheme and $n$ and $m$ are coprime. If $S = \text{Spec } k$ then the resulting stacky curve has signature $(0; n, m)$.

**Lemma 3.2.** The only simply connected tame stacky curves over a field $k$ of characteristic $p$ are the genus 0 curves and the weighted projective spaces $\mathbb{P}_k(n, m)$ for coprime $n, m$ with $p \nmid n, m$.

**Proof.** This is true over a separably closed field by [BN06, Thm. 1.1]. We may thus assume that $X$ has signature $(0; n, m)$ with $n, m$ as in the statement. If $n = m = 1$ then $X$ has genus 0. Otherwise the action of $\text{Gal}(k_s/k)$ on the stacky $k_s$-points has to be trivial since it preserves stabilizers, so the stacky points are $k$-points. It follows that $X \cong \mathbb{P}_k(n, m)$ by [VZB22, Lemma 5.3.10]. \hfill \Box

We now prove Theorem 1.2. Note that the map $\mathbb{A}_k^2 \setminus \{0\} \to \mathbb{P}(n, m)$ is a $G_m$-torsor so Hilbert’s Theorem 90 and Lemma 2.4 imply that $K^2 \setminus \{0\} = \mathbb{A}_k^2 \setminus \{0\}(K) \to \mathbb{P}(n, m)(K)$ is surjective for any field $K$.

**Theorem 3.3.** Simply connected tame stacky curves $X$ over a global field $k$ satisfy strong approximation, i.e. the image of $X(k)$ in $X(\mathbb{A}_k)$ is dense.

**Proof.** For genus 0 curves this is known so we may assume that $X = \mathbb{P}_k(n, m)$ with $n, m$ coprime. We will use the integral model $\mathbb{P}_{O_k}(n, m)$ to describe $\mathbb{P}_k(n, m)(\mathbb{A}_k)$ as a restricted product.

Let $S$ be a finite set of places of $k$ containing all the archimedean ones. For every $v \in S$ let $U_v \subseteq \mathbb{P}_k(n, m)(k_v)$ be a non-empty open subset. Our goal is to find a point $x \in \mathbb{P}_k(n, m)(k)$ such that $x \in U_v$ for all $v \in S$ and $x \in \mathbb{P}_{O_k}(n, m)(O_v)$ for all $v \notin S$.

Let $\pi : \mathbb{A}_k^2 \setminus \{0\} \to \mathbb{P}_k(n, m)$ be the $G_m$-torsor. We will find $(y_1, y_2) \in \mathbb{A}_k^2 \setminus \{0\}(k)$ such that $x := \pi(y_1, y_2)$ works. The map $\pi : k_v^2 \setminus \{0\} = \mathbb{A}_k^2 \setminus \{0\}(k_v) \to \mathbb{P}_k(n, m)(k_v)$ is surjective so we can choose non-empty opens $V_{1,v}, V_{2,v} \subseteq k_v^x$ such that $V_{1,v} \times V_{2,v} \subset \pi^{-1}(U_v)$ for all $v \in S$. Now choose non-empty opens $V'_{1,v}, V'_{2,v} \subset k_v^x$ and $W_{1,v}, W_{2,v} \subset k_v^x$ open neighbourhoods of 1 such that $(W_{i,v})^m \cdot V'_{i,v} \subset V_{i,v}$ for $i = 1, 2$. These exists since multiplication is continuous.
Fix a place $w_0 \notin S$. Apply strong approximation for $A^1$ twice to choose $y'_1, y'_2 \in k^\times$ such that:

- $y'_i \in V'_{i,v}$ for $v \in S$ and $i = 1, 2$
- $y'_1, y'_2 \in \mathcal{O}_v$ and such that at least one of $y'_1, y'_2$ lies in $\mathcal{O}_v^\times$ for $v \notin S \cup \{w_0\}$.

Let $T$ be the finite set of places $v \notin S \cup \{w_0\}$ such that $y'_1y'_2 \notin \mathcal{O}_v^\times$. Fix another place $w_1 \notin S \cup T \cup \{w_0\}$. Choose a $z_1 \in k^\times$ with strong approximation such that:

- $z_1 \in (W_{1,v})^m$ for $v \in S$
- $z_1 \in \mathcal{O}_v^\times$ for $v \in T$
- $n \mid w_0(z_1y_1)$ and $mw_0(z_1y'_1) < nw_0(y'_2)$
- $z_1 \in \mathcal{O}_v$ for $v \notin S \cup T \cup \{w_0, w_1\}$.

Let $T'$ be the finite set of places $v \notin S \cup \{w_0, w_1\}$ such that $y_1y_2 \notin \mathcal{O}_v^\times$. Apply weak approximation to find $z_2 \in k^\times$ such that:

- $z_2 \in W_{2,v}$ for $v \in S$
- $z_2 \in \mathcal{O}_v^\times$ for $v \in T' \cup \{w_0\}$
- $w_1(z_2) > nw_1(z_2)$.

We now claim that $y_1 := z_1y'_2, y_2 := z_2^m y'_2$ works:

- For $v \in S$ we have $y_i \notin (W_{i,v})^m \cdot V'_{i,v} \subset V'_{i,v}$ by construction, so $\pi(y_1, y_2) \in U_v$.
- We have $(y_1, y_2) \in (A^2_{\mathbb{O}_k} \setminus \{0\})(\mathcal{O}_v)$ for $v \in T'$.
- If $\pi$ is a uniformizer in $\mathcal{O}_{w_0}$ then $(\pi^{-w_0(z_1y'_1)}y_1, \pi^{-mw_0(z_1y'_1)/n}y_2) \in \mathcal{O}_v^\times \times \mathcal{O}_v$. Hence $(y_1, y_2) \in \mathbb{G}_m(k_{w_0}) \cdot (A^2_{\mathbb{O}_k} \setminus \{0\})(\mathcal{O}_{w_0})$ which maps into $\mathbb{P}_{\mathbb{O}_k}(\mathcal{O}_{w_0})$.
- Analogously we have $(z_2^{-m}y_1, z_2^{-m}y_2) \in \mathcal{O}_v^\times \times \mathcal{O}_v^\times$ if $v(z_2) < 0$ for $v \notin S \cup T' \cup \{w_0\}$. Otherwise we have $(y_1, y_2) \in \mathcal{O}_v^\times \times \mathcal{O}_v^\times$.

\[
\square
\]

**Remark 3.4.** The projection map $\pi : A^2_k \setminus \{0\} \to \mathbb{P}(k(n, m))$ played an important role in the proof. This is a special case of a more general construction, it is a universal torsor\footnote{CTSS} (2.0.4) of $\mathbb{P}(k(n, m))$. The fact that $\text{Pic}(A^2_k \setminus \{0\}) = 0$ \footnote{CTSS Prop. 2.1.1} imply that it is a universal torsor.

All tame stacky curves of genus $g < \frac{1}{2}$ are simply connected so this is an improvement over \footnote{CTSS Thm. 1.0.1} in two ways, it holds for more stacky curves and it gives strong approximation for all places and not only away from a single place.

4. COHOMOLOGICAL OBSTRUCTIONS FOR STACKS

**Definition 4.1.** Let $X$ be a Deligne-Mumford stack. Its Brauer group is defined as

$$\text{Br } X := H^2(X, \mathbb{G}_m).$$

For the rest of this section let $k$ be a global field unless otherwise mentioned. Let $\Omega_k$ be its set of places. We recall from local class field theory that for each place $v$ of $k$ there exists an injection $\text{inv}_v : \text{Br } k_v \to \mathbb{Q}/\mathbb{Z}$ which is surjective if $v$ is finite.

**Definition 4.2.** Let $X$ be a separated finite type Deligne-Mumford stack over $k$. The Brauer-Manin pairing is defined as the map

$$X(A_k) \times \text{Br } X \to \mathbb{Q}/\mathbb{Z} : (A, (x_v)_{v \in \Omega_k}) \mapsto \sum_{v \in \Omega_k} \text{inv}_v(A(x_v)).$$

The Brauer-Manin set $X(A_k)^{\text{Br}}$ is the left kernel of this paring.
The image $\operatorname{Br}_0(X) := \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X)$ lies in the right kernel of this pairing due to [CTS21, Thm. 12.1.8] so we get an induced pairing

$$X(\mathbb{A}_k) \times \operatorname{Br} X / \operatorname{Br}_0 X \to \mathbb{Q}/\mathbb{Z} : (A, (x_v)_{v \in \Omega_k}) \to \sum_{v \in \Omega_k} \operatorname{inv}_v(A(x_v)).$$

which we also call the Brauer-Manin pairing.

If $B$ is a subgroup of $\operatorname{Br} X$ or $\operatorname{Br} X / \operatorname{Br}_0 X$ then we let $X(\mathbb{A}_k)^B \subset X(\mathbb{A}_k)$ be the set of adelic points which are orthogonal to $B$ for the Brauer-Manin pairing. Important subquotients are the algebraic Brauer group $\operatorname{Br}_1 X := \ker(\operatorname{Br} X \to \operatorname{Br} X_{k_0})$ and its quotient $\operatorname{Br}_a X := \operatorname{Br}_1 X / \operatorname{Br}_0 X$.

If $\pi : X \to Y$ is an integral model over a Dedekind scheme $Y$ then we define the integral Brauer-Manin set of $X$ as $\mathcal{X}(\mathbb{A}_k, Y) := \mathcal{X}(\mathbb{A}_k, Y) \cap X(\mathbb{A}_k)^{\operatorname{Br}}$. If $\mathcal{X}$ is regular then $\operatorname{Br} \mathcal{X}$ is a subgroup of $\operatorname{Br} X$ by [HKT16, Prop. 2]. We then similarly define, $\operatorname{Br}_0 \mathcal{X} := \operatorname{im}(\operatorname{Br} U \to \operatorname{Br} \mathcal{X})$, $\operatorname{Br}_1 \mathcal{X} := \ker(\operatorname{Br} \mathcal{X} \to H^0(U, R^2\pi_*\mathbb{G}_m))$ and $\operatorname{Br}_a \mathcal{X} = \operatorname{Br}_1 \mathcal{X} / \operatorname{Br}_0 \mathcal{X}$.

If $S$ is the set of places of $k$ which do not come from a point in $U$, $S$ must in particular contain all archimedean places, then we define the subgroup of locally constant algebras to be $\mathbb{B}(\mathcal{X}) := \ker(\operatorname{Br}_a \mathcal{X} \to \prod_{v \in S} \operatorname{Br}_a X_{k_v})$.

Some relevant basic properties.

**Proposition 4.3.**

1. The Brauer-Manin pairing is well-defined.
2. The image of $X(k)$ lies in $X(\mathbb{A}_k)^{\operatorname{Br}}$.
3. For all $A \in \operatorname{Br} X$ and every place $v$ the evaluation map $X(k_v) \to \mathbb{Q}/\mathbb{Z} : x_v \to \operatorname{inv}(A(x_v))$ is locally constant.
4. The Brauer-Manin set is closed.

**Proof.** Everything but the third statement follows by changing the word scheme by Deligne-Mumford stack in the proofs of [CTS21, Prop. 12.3.1, Thm. 12.3.2].

For the third statement, let $x_v \in X(k_v)$. There exists a smooth cover $U \to X$ such that $x_v$ lies in the image of $U(k_v)$ by [Chr20, Thm. 7.0.7]. The statement then follows from the analogous statement for schemes, functoriality of the evaluation map and the fact that the morphism of topological spaces $U(k_v) \to X(k_v)$ is open.

Recall that the twists of a $G$-torsor $\pi : Y \to X$ allow us to define the descent obstruction [Sko01, §5.3], another obstruction to strong approximation. We define the obstruction set coming from $\pi$ as

$$X(\mathbb{A}_k)^\pi := \bigcup_{a \in H^1(k, G)} \pi_a(Y_0(\mathbb{A}_k)).$$

It is a classical result of Colliot-Thélène and Sansuc [CTS87] that the Brauer-Manin obstruction is related to the descent obstruction for groups of multiplicative type. We will need a form of this relation which holds for stacks and in positive characteristic. The following proof is analogous to the proof of [HV13, Thm 4.2].

**Proposition 4.4.** Let $X$ be a separated finite type Deligne-Mumford stack over $k$ and $G$ a smooth group scheme of multiplicative type over $k$. Let $\pi : Y \to X$ be a $G$-torsor. Then we have an inclusion

$$X(\mathbb{A}_k)^{\operatorname{Br}_1 X} \subseteq X(\mathbb{A}_k)^\pi = \bigcup_{a \in H^1(k, G)} \pi_a(Y_0(\mathbb{A}_k)).$$
Proof. Let $G' = \text{Hom}(G, \mathbb{G}_m)$ be the group of characters of $G$. It is a $\text{Gal}(k_s/k)$-module which is finitely generated as an abelian group. Let $[Y] \in H^1(X, G)$ be the class corresponding to the torsor $Y \to X$. The cup product coming from the pairing $G \times G' \to \mathbb{G}_m$ defines a morphism

$$H^1(k, G') \to H^2(X, \mathbb{G}_m) = \text{Br } X : \varphi \to \varphi \cup [Y]$$

whose image lands in $\text{Br}_1 X$ because $\varphi = 0$ in $H^1(k_s, G') = 0$.

Let $(x_v)_v \in X(\mathbb{A}_k)^{\text{Br}_1 X}$. Since $X \cong [Y/G]$ as a stack there exists for each place $v$ a class $a_v \in H^1(k_v, G)$ and a $y_v \in Y_{a_v}(k_v)$ such that $x_v = f_{a_v}(y_v)$.

Recall that $[Y_{a_v}] = [Y] - a_v$ so we compute that for an arbitrary $\varphi \in H^1(k, G')$

$$(\varphi \cup [Y])(x_v) = \pi_{a_v}^*(\varphi \cup [Y])(y_v) = \pi_{a_v}^*(\varphi \cup ([Y_{a_v}] + a_v))(y_v) = (\varphi \cup a_v) \in \text{Br } k_v.$$

The group $G'$ has no $p$-torsion because $G$ is smooth. The following sequence is thus exact by Poitou-Tate duality [Mil06, Thm. I.4.20].

$$H^1(k, G) \to \prod_v H^1(k_v, G) \to \pi^{-1}(\varphi \cup [Y]) \to H^1(k, G')^\vee.$$

The assumption $(x_v)_v \in X(\mathbb{A}_k)^{\text{Br}_1 X}$ and $(3)$ imply for all $\varphi \in H^1(k, G')$ that

$$\sum_v \text{inv}_v(\varphi \cup a_v) = \sum_v \text{inv}_v((\varphi \cup [Y])(x_v)) = 0.$$

There thus exists an $a \in H^1(k, G)$ such that the image of $a$ in $H^1(k_v, G)$ is $a_v$ for all $v$. Then $(x_v)_v = \pi_a((y_v)_v)$ as desired.

The following theorem is a generalization of [HST13, Thm. 4.1] to stacks and positive characteristic.

**Theorem 4.5.** Let $X$ be a separated finite type Deligne-Mumford stack over a global field $k$ and let $G$ be a smooth group scheme. Let $Y$ be a $G$-torsor over $X$. If the twist $Y_a$ satisfies strong approximation for all $a \in H^1(k, G)$ then descent along $\pi : Y \to X$ is the only obstruction to strong approximation for $X$.

If $G$ is of multiplicative type then the algebraic Brauer-Manin obstruction is the only obstruction to strong approximation for $X$.

*Proof. We have by definition that*

$$X(\mathbb{A}_k)^\pi = \bigcup_{a \in H^1(k, G)} \pi_a(Y_a(\mathbb{A}_k)).$$

*But we know by Lemma 2.4 that*

$$X(k) = \prod_{a \in H^1(k, G)} \pi_a(Y_a(k)).$$

*Since the $\pi_a$ are continuous in the adelic topology and the image $Y_a(k)$ is dense in $Y_a(\mathbb{A}_k)$ we conclude that the image of $X(k)$ is dense in $X(\mathbb{A}_k)^\pi$.*

*If $G$ has multiplicative type then $X(\mathbb{A}_k)^{\text{Br}_1 X} \subseteq X(\mathbb{A}_k)^\pi$ by Proposition 2.4. The image of $X(k)$ is dense in $X(\mathbb{A}_k)^\pi$ so it is also dense in the subset $X(\mathbb{A}_k)^{\text{Br}_1 X}$. □*

We will identify étale group schemes over $k$ with groups with a $\text{Gal}(k_s/k)$-action. We also recall that for a geometrically connected algebraic stack the exact sequence

$$1 \to \pi_1(X_{k_s}) \to \pi_1(X) \to \text{Gal}(k_s/k) \to 1 \to 1$$

induces a $\text{Gal}(k_s/k)$-action on $\pi_1(X_{k_s})$. 

(4)
Lemma 4.6. Let $X$ be a geometrically connected finite type algebraic stack over a field $k$ such that $X(k) \neq \emptyset$. Let $G$ be an étale group scheme and $f : \pi_1(X_{k_s}) \to G_{k_s}$ a $\text{Gal}(k_s/k)$-equivariant morphism. There then exists a $G$-torsor $\pi : T \to X$ such that $\pi_{k_s} : T_{k_s} \to X_{k_s}$ corresponds to $f \in H^1(\pi_1(X_{k_s}), G_{k_s})$.

Proof. Torsors are classified by cohomology classes so it suffices to find a cocycle $\pi_1(X) \to G_{k_s}$ whose restriction to $\pi_1(X_{k_s})$ is equal to $f$.

Choose a point $P \in X(k)$, this induces a section $P_* : \text{Gal}(k_s/k) \to \pi_1(X)$ of the projection $p : \pi_1(X) \to \text{Gal}(k_s/k)$. We then consider the cochain

$$\varphi : \pi_1(X) \to G_{k_s} : \sigma \to f(\sigma P_*(p(\sigma))^{-1}).$$

Its restriction to $\pi_1(X_{k_s})$ is $f$. It is a cocycle because for $\sigma, \tau \in \pi_1(X)$ we have

$$\varphi(\sigma \tau) = f(\sigma P_*(p(\tau))^{-1} P_*(p(\sigma))^{-1}) = \varphi(\sigma)f(P_*(p(\sigma))\tau P_*(p(\tau))^{-1} P_*(p(\sigma))^{-1}).$$

But the conjugation $P_*(p(\sigma))\tau P_*(p(\tau))^{-1} P_*(p(\sigma))^{-1}$ is by definition the result of $\sigma$ acting on $\tau P_*(p(\tau))^{-1} \in \pi_1(X_{k_s})$. And $f$ is $\text{Gal}(k_s/k)$-equivariant so the right-hand side is equal to $\varphi(\sigma)\sigma f(\tau)$.

Remark 4.7. If $G$ is abelian then this can also be proven using the 5-term exact sequence coming from the Hochschild-Serre spectral sequence.

The $G$-torsor $\pi$ is not unique, but it is unique up to twists.

If $\pi_1(X_{k_s})$ is finite and $f = \text{id} : \pi_1(X_{k_s}) \to \pi_1(X_{k_s})$ is the identity then we will call the morphism $\pi : T \to X$ a universal cover.

Theorem 4.8. If $X$ is a tame stacky curve of genus $g < 1$ over $k$ and $X(\mathbb{A}_k) \neq \emptyset$ then there exists an universal cover $T \to X$, which is a torsor under the finite group scheme $\pi_1(X_{k_s})$. Descent along $T$ is the only obstruction to strong approximation.

If the group $\pi_1(X_{k_s})$ is abelian then the algebraic Brauer-Manin obstruction is the only obstruction to strong approximation.

Proof. If $X(\mathbb{A}_k) \neq \emptyset$ then $X_{\text{coarse}}(\mathbb{A}_k) \neq \emptyset$. The coarse moduli space $X_{\text{coarse}}$ has genus 0 so $X_{\text{coarse}} \cong \mathbb{P}^1$ and thus has a rational point lying on the dense open $U$ on which $X \to X_{\text{coarse}}$ is an isomorphism. So $X(k) \neq \emptyset$ and the group $\pi_1(X_{k_s})$ is finite and has order coprime to $p$ because of [BN06, Prop. 5.5]. We deduce from Lemma 4.6 that $X$ has a universal cover $T \to X$.

The universal cover $T$ and all of its twists are simply connected stacky curves so combining Theorem 3.3 and Theorem 4.5 gives the desired statement.

If $\pi_1(X_{k_s})$ is abelian then $T \to X$ is a torsor under a group of multiplicative type so we can apply the second part of Theorem 4.5.

We have thus proven Theorem 4.1. An interesting corollary is.

Corollary 4.9. If $X$ is stacky curve of genus $g < \frac{5}{6}$ then the only obstruction to strong approximation is the Brauer-Manin obstruction.

Proof. The geometric fundamental group of all tame stacky curves of genus $g < \frac{5}{6}$ is abelian by [BN06, Prop. 5.5].

Remark 4.10. It will follow from the next section that if $X$ is a stacky curve of signature $(0; 2, 3, 5)$ then $Br_1 X = Br_k k$ and that if $k$ is a number field then $Br X = Br_1 X$. There is thus no Brauer-Manin obstruction to strong approximation. But $\pi_1(X_{k_s})$ is isomorphic to the icosahedral group $A_5$ by [BN06, Prop. 5.5]. The stack $X$ thus has a non-trivial étale cover which implies that it does not satisfy strong approximation by [Poo17, Thm. 8.4.10]. (This theorem is only proven for varieties, but the argument also applies to stacks).
5. Brauer groups of stacky curves

Proposition 5.1. Let $X$ be a stacky curve over an algebraically closed field $k$. Then $\text{Br} X = 0$.

Proof. The Brauer group injects into $\text{Br} k(X)$ by [HKT16 Prop. 2], which is trivial by Tsen’s theorem [GS17 Thm. 6.2.8]. □

This implies that $\text{Br}_1 X = \text{Br} X$ over a perfect field $k$. The Hochschild-Serre spectral sequence implies that $\text{Br}_1 X/\text{Br}_0 X$ injects into $H^1(k, \text{Pic} X_{k_s})$ [CTS21 Prop. 4.3.2]. In this section we will compute $H^1(k, \text{Pic} X_{k_s})$ for a tame stacky curve with coarse moduli space $\mathbb{P}^1_k$.

The Picard group $\text{Pic} X$ of a stacky curve $X$ over a field $k$ can be described as a class group of divisors [VZB22 Lemma 5.4.5]. Let $\mathcal{P} \subset X_{\text{coarse}}$ be the stacky locus. For any point $P \in \mathcal{P}$ let $e_P$ be the order of the stabilizer of the point in $X$ lying above it and denote by $[P] \in \text{Pic} X$ the class of the unique integral substack in $X$ lying above it.

There exists a degree map $\text{Pic} X \xrightarrow{\text{deg}} \mathbb{Q}$, which can in general take non-integral values unlike the case of non-stacky curves. We have $\text{deg} [P] = \frac{|k(P) : k|}{e_P}$. Let $d_P$ be the denominator of this fraction in lowest terms, i.e. $d_P := e_P/\gcd(|k(P) : k|, e_P)$. Let $d_X := \text{lcm}_{P \in \mathcal{P}}(d_P)$.

We define $\text{Pic}^0 X$ as the kernel of the degree map $\text{Pic} X \xrightarrow{\text{deg}} \mathbb{Q}$. We then have

Lemma 5.2. Let $X$ be a tame stacky curve with $X_{\text{coarse}} \cong \mathbb{P}^1_k$. The image of deg is $\frac{1}{d_X} \mathbb{Z}$ and we have an exact sequence

$$0 \to \text{Pic}^0 X \to \bigoplus_{P \in \mathcal{P}} \mathbb{Z}/e_P \mathbb{Z}[P] \xrightarrow{\text{deg}[P]} \left( \frac{1}{d_X} \mathbb{Z} \right)/\mathbb{Z} \to 0$$

Proof. We use the description of $\text{Pic} X$ as a divisor class group [VZB22 Lemma 5.4.5].

A stacky curve contains a dense open subscheme so $k(X) = k(X_{\text{coarse}}) = k(\mathbb{A}_k^1)$, where $x, y$ are coordinates on $X_{\text{coarse}}$ such that $\{x = 0\}$ lies outside the stacky locus. For any integral $Z \subset X$ its image $Z_{\text{coarse}} \subset X_{\text{coarse}}$ is defined by an equation $f(x, y)$ of degree $[k(Z_{\text{coarse}}) : k]$. Let $G_Z$ be the stabilizer group of $Z$. The stacky curve $X$ is a root stack over $X_{\text{coarse}}$ so we can see from the local description of root stacks [Cad07 Ex. 2.4.1] that the divisor $\text{div}(f)$ on $X$ corresponding to $f$ is equal to $\#G_Z[Z]$. The fact that $f(x, y)/x^{[k(Z):k]} \in k(X)$ implies that $\#G_Z[Z] = [k(Z) : k] \text{div}(x)$.

These relations generate all relations coming from $k(X)$ because every element is a product of such $f(x, y)$. It follows that $\text{Pic} X$ is equal to the group generated by the divisors $\text{div}(x)$ and $[P]$ for all $P \in \mathcal{P}$ with respect to the relations $e_P [P] = [k(P) : k] \text{div}(x)$. The lemma follows. □

Remark 5.3. If $k$ is separably closed then $d_P = e_P$ for all $P \in \mathcal{P}$. So for a stacky curve $X$ with signature $(0; e_1, \cdots, e_r)$ we have $\text{Pic}^0 X = 0$ if and only if all the $e_i$ are pairwise coprime. One can check that if $g < 1$ the only such curves are the simply connected ones and the stacky curve of signature $(0; 2, 3, 5)$.

Lemma 5.4. Let $X$ be a tame stacky curve with $X_{\text{coarse}} \cong \mathbb{P}^1_k$. Let $U$ be a normal scheme with fraction field $k$ such that the action of $\text{Gal}(k_s/k)$ on $\text{Pic} X_{k_s}$ factors through $\pi_1(U)$, e.g. $U = \text{Spec} k$. We can then identify $\text{Pic} X_{k_s}$ and $\text{Pic}^0 X_{k_s}$ with the...
corresponding locally constant étale sheaves on $U$. We then have an exact sequence

$$0 \to \mathbb{Z}/d_X \mathbb{Z} \to H^1(U, \text{Pic}^0 X_{k_s}) \to H^1(U, \text{Pic} X_{k_s}) \to 0.$$ 

Where the first map sends the generator to the cocycle $\varphi : \pi_1(U) \to \text{Pic}^0 X_{k_s} : \sigma \to \sigma D - D$ for any divisor $D$ on $X_{k_s}$ with degree $1/d_X$.

**Proof.** Note that $H^1(U, \mathbb{Z}) = 0$ so the long exact sequence in cohomology coming from the degree exact sequence $0 \to \text{Pic}^0 X_{k_s} \to \text{Pic} X_{k_s} \to \mathbb{Z}/d_X \mathbb{Z} \to 0$ is

$$(\text{Pic} X_{k_s})^{\pi_1(U)} \xrightarrow{\deg} \mathbb{Z} \to H^1(U, \text{Pic}^0 X_{k_s}) \to H^1(U, \text{Pic} X_{k_s}) \to 0.$$ 

The map $\mathbb{Z}/d_X \mathbb{Z} \to H^1(U, \text{Pic}^0 X_{k_s})$ sends the generator to $\varphi$ by the explicit description of this boundary map. The assumption $X_{\text{coarse}} \cong \mathbb{P}^1_k$ implies that $X(k) \neq \emptyset$ and thus that $(\text{Pic} X_{k_s})^{\pi_1(U)} = \text{Pic} X$, so the image of the degree map is $\mathbb{Z}/d_X \mathbb{Z}$. □

We can now construct an inverse of the map $\text{Br}_1 X \to H^1(k, \text{Pic} X_{k_s})$ using this description of the Picard group and work of Skorogabotov. We will utilise the notion of the type [Sk01] Def. 2.3.1] of a torsor.

Let $T \to X$ be a torsor with type $\text{Pic}^0 X_{k_s} \to \text{Pic} X_{k_s}$. It is a $\text{Hom}(\text{Pic}^0 X_{k_s}, \mathbb{G}_m)$-torsor. We have $X_{\text{coarse}} \cong \mathbb{P}^1_k$ and thus $X(k) \neq \emptyset$. Such a point defines a splitting of the exact sequence [Sk01] Cor. 2.3.9] so such a torsor exists.

**Proposition 5.5.** Let $X$ be a tame stacky curve with $X_{\text{coarse}} \cong \mathbb{P}^1_k$. The group $\text{Br}_1 X$ is generated by $\text{Br} k$ and the Brauer classes obtained by cupping elements of $H^1(k, \text{Pic}^0 X_{k_s})$ with $[T]$.

**Proof.** Follows from [Sk01] Thm. 4.1.1], Lemma 5.4 and the definition of $T$. □

**Example 5.6.** Let $n, m \neq 1$ and $X$ a tame stacky curve over $k$ with signature $(0; n, m)$. Assume that $X_{\text{coarse}} \cong \mathbb{P}^1_k$. Let $P, Q \in X_{\text{coarse}}(k_s)$ be the stacky points which have stabilizers of order $n, m$ respectively and define $r := \gcd(n, m)$. We see from Lemma 5.2 that $\text{Pic}^0 X_{k_s} \cong \mathbb{Z}/r \mathbb{Z}$ as an abstract group. The generator is given by $[\pi_1 P] - [\pi_1 Q]$. Here $P', Q' \in X$ are the unique integral closed substacks lying over $P, Q$ respectively.

If the action of $\text{Gal}(k_s/k)$ on the stacky points is trivial then we may assume that $P = [1 : 0], Q := [0 : 1]$. In this case $\text{Gal}(k_s/k)$ acts trivially on $\text{Pic}^0 X_{k_s} \cong \mathbb{Z}/r \mathbb{Z}$ so $T \to X$ is a $\mu_r$-torsor. One can check via an explicit computation that the $\mu_r$-torsor $T := \mathbb{P}^1_{s,t} \to X$ which on coarse moduli spaces is given by $[s : t] \to [s^r : t^r]$ has type $\text{Pic}^0 X_{k_s} \to \text{Pic} X_{k_s}$.

If the action on the stacky points is non-trivial then $n = m$ and the action becomes trivial after a quadratic extension $K/k$. One can then recover a choice of $[T]$ by descending the above construction. To make this explicit, let $\sigma \in \text{Gal}(K/k)$ be the generator and $f(X, Y) = (X - \alpha Y)(X - \sigma(\alpha) Y) \in k[X, Y]$ a homogenous polynomial of degree 2 defining the stacky locus. The relevant torsor over $K$ is given by the map $\mathbb{P}^1_{K, s,t} \to X_K$ which on the coarse moduli spaces is given by $[s : t] \to [\alpha s^r - \sigma(\alpha) t^r : s^t - t^r]$. The action on $\mathbb{P}^1_K$ which we will use to descend is the map $[s : t] \to [\sigma(t) : \sigma(t)]$. The result of this descent is the curve $T := \{N_{K/k}(U - \alpha V) = W^2\} \subset \mathbb{P}^2$. The map $T \to X$ is given by $[u : v : w] \to [\alpha(u - \sigma(\alpha)v)^r - \sigma(\alpha)w^r : (u - \sigma(\alpha)v)^r - w^r]$. 


Let us assume that \( n = m = r = 2 \). We can use \( w^2 = (u - \alpha v)(u - \sigma(\alpha)v) \) to simplify the image of \([u : v : w]\) to

\[
[(u-\sigma(\alpha)v)(\alpha u-\alpha \sigma(\alpha)v-\sigma(\alpha)u+\sigma(\alpha)v): (u-\sigma(\alpha)v)(u-\sigma(\alpha)v-u+\alpha v)] = [u : v].
\]

This is the same torsor which shows up in the example of Bhargava-Poonen \([BP20]\).

One can do similar constructions for other stacky curves \( X \) with \( X_{\text{coarse}} \cong \mathbb{P}^1 \).

6. The elementary obstruction

Given a Deligne-Mumford stack \( X \) we let \( \mathcal{D}(X) \) be the derived category of sheaves on the (small) étale site of \( X \). Any morphism \( f : Y \to X \) of Deligne-Mumford stacks defines a right derived functor \( Rf_* : \mathcal{D}(X) \to \mathcal{D}(Y) \) and a pullback \( f^* : \mathcal{D}(Y) \to \mathcal{D}(X) \). The global sections functor \( \Gamma \) has a right derived functor \( R\Gamma(X, \cdot) : \mathcal{D}(X) \to \mathcal{D}(\text{Ab}) \) where \( \mathcal{D}(\text{Ab}) \) is the derived category of abelian groups. We denote by \( H^p(X, \cdot) := H^p(R\Gamma(X, \cdot)) \) the hypercohomology. We can extend the constructions of \([HS13, \S1]\) to stacks.

**Construction 6.1.** Let \( \pi : X \to S \) be a morphism of Deligne-Mumford stacks. We denote \( KD(X) := (\tau_{\leq 1} R\pi_* \mathbb{G}_m)[1] \) the shift of the truncation of \( R\pi_* \mathbb{G}_m \). It is concentrated in degrees \(-1 \) and \( H^{-1}(KD(X)) = \pi_* \mathbb{G}_m, H^0(KD(X)) = R^1 \pi_* \mathbb{G}_m \).

The map \( \mathbb{G}_m \to R\pi_* \mathbb{G}_m \) defines a morphism \( \mathbb{G}_m[1] \to KD(X) \). Its cokernel will be denoted \( KD'(X) \). We thus have a distinguished triangle in \( \mathcal{D}(S) \)

\[
\mathbb{G}_m[1] \to KD(X) \to KD'(X) \xrightarrow{w_X} \mathbb{G}_m[2]. \tag{5}
\]

If \( \pi \) has a section \( s : S \to X \) then the map \( R\pi_* \mathbb{G}_m \to R\pi_* R\pi_* \mathbb{G}_m \cong \mathbb{G}_m \) induces a section \( KD(X) \to \mathbb{G}_m[1] \) of \( \mathbb{G}_m[1] \to KD(X) \). This implies that the triangle \([n]\) splits, equivalently that \( w_X = 0 \). We call \( w_X \in \text{Hom}_S(KD'(X), \mathbb{G}_m[2]) = \text{Ext}^2_S(KD'(X), \mathbb{G}_m) \) the elementary obstruction. If \( S \) is the spectrum of a field and \( \pi_* \mathbb{G}_m = \mathbb{G}_m \) then \( w_X \) is the elementary obstruction occurring in \([CTS87, \text{Prop 2.2.4}]\).

A more explicit construction of the complex \( KD(X) \) when \( X \) is a variety smooth over \( S \) is given in \([HV13, \text{App. A}]\).

**Lemma 6.2.**

1. If \( w_X = 0 \) then \( w_{X_{S'}} = 0 \) for all étale morphisms \( f : S' \to S \).
2. If \( f : X \to Y \) is a morphism of \( S \)-Deligne-Mumford stacks then \( w_X = 0 \) implies that \( w_Y = 0 \).

**Proof.** All pushforwards are taken in the étale topology so \( KD(X_{S'}) = f^* KD(X) \), \( KD'(X_{S'}) = f^* KD'(X) \) and \( w_{X_{S'}} = f^* w_X = 0 \).

For the second statement consider the map \( R\pi_* \mathbb{G}_m \to R(\pi \circ f)_* \mathbb{G}_m \), it induces a morphism \( KD(X) \to KD(Y) \) which induces a map of triangles

\[
\begin{array}{ccc}
\mathbb{G}_m[1] & \longrightarrow & KD(Y) \\
\parallel & & \downarrow \\
\mathbb{G}_m[1] & \longrightarrow & KD(X)
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & & \parallel \\
KD(Y) & \longrightarrow & KD'(Y)
\end{array}
\begin{array}{ccc}
& & \longrightarrow \\
\parallel & & \parallel \\
& & \mathbb{G}_m[2]
\end{array}
\begin{array}{ccc}
& & \longrightarrow \\
& & \parallel \\
& & \mathbb{G}_m[2]
\end{array}
\]

So if \( w_X = 0 \) then \( w_Y = 0 \).

The complexes \( KD(X) \) and \( KD'(X) \) are related to the Brauer group as follows

**Lemma 6.3.**

1. \( H^1(S, KD(X)) \cong \text{Br}_1 X := \ker(\text{Br} X \to H^0(S, R^2 \pi_* \mathbb{G}_m)) \).
(2) \( \ker(H^1(S, KD'(X)) \xrightarrow{w_X} H^3(S, \mathbb{G}_m)) \cong Br_\delta X := Br_1 X/\text{im}(Br S \to Br X) \).

Proof. This follows from a standard computation with distinguished triangles, see e.g. [HS08 Lemma 2.1] and [HS13 p. 7]. \( \square \)

If \( T \) is a group of multiplicative type over \( S \) with sheaf of characters \( \hat{T} \) then we have the following exact sequence as in [HS13 Prop. 1.1].

\[
H^1(S, T) \to H^1(X, T) \xrightarrow{\partial} \text{Hom}_S(\hat{T}, KD'(X)) \xrightarrow{\partial} H^2(S, T) \to H^2(X, T). \tag{6}
\]

We can compute the complex \( KD'(X) \) for stacky curves.

**Lemma 6.4.** Let \( S \) be a regular integral scheme. Let \( X \to S \) be a relative stacky curve whose generic fiber \( X \) has a coarse moduli space of genus 0. The complex \( KD'(X) \) is concentrated in degree 0 and is étale locally a constant finitely generated group, i.e. \( \pi_* \mathbb{G}_m = \mathbb{G}_m \) and \( R^1 \pi_* \mathbb{G}_m \) is étale locally a constant finitely generated group. Moreover, the order of the torsion part of \( R^1 \pi_* \mathbb{G}_m \) is invertible on \( S \).

Proof. We have \( \pi_* \mathbb{G}_m = \mathbb{G}_m \) because the map is proper and all fibers are stacky curves and thus geometrically connected.

To show that \( R^1 \pi_* \mathbb{G}_m \) is locally constant we apply [Sta18 Tag 0GKC]. It remains to show that if \( S \) is the spectrum of a regular strictly henselian ring \( R \) with fraction field \( K \) and residue field \( \kappa \) then the specialization map \( \text{Pic} \mathcal{X} = (R^1 \pi_* \mathbb{G}_m)(S) \to (R^1 \pi_* \mathbb{G}_m)(K) = \text{Pic} X_K \) is an isomorphism.

We note that by Lemma 5.2 \( \text{Pic} X_{K_s} \) is finitely generated and the order of its torsion subgroup divides the product of the degrees of the divisors of \( X \) as a root stack. Lemma 2.1 implies that if \( \mathcal{X} \) is tame then all of these degrees are invertible on \( U \) and thus also on \( \mathcal{X} \).

We apply [AV22 Prop. 2.3.2] to see that \( \text{Pic} \mathcal{X} = \text{Pic} X \) since \( \mathcal{X} \) is regular and \( \text{Pic} S = 0 \). It remains to show that \( \text{Pic} X = \text{Pic} X_{K_s} \), which by Lemma 5.2 is equivalent to \( \mathcal{P}(K) = \mathcal{P}(K_s) \) for the stacky locus \( \mathcal{P} \subset X_{\text{coarse}, K} \).

The closure in \( X_{\text{coarse}} \) of \( \mathcal{P} \) is finite étale over \( S = \text{Spec} \, R \) by Lemma 2.1. The fact that \( R \) is strictly henselian implies that the closure is a disjoint union of copies of \( S \) and thus that \( \mathcal{P}(K) = \mathcal{P}(K_s) \). \( \square \)

### 6.1. The elementary obstruction for stacky curves

We will now analyze the elementary obstruction for stacky curves. For the rest of this subsection we will look at the following situation.

Let \( U \) be a Dedekind scheme whose field of fractions \( k \) is a global field. Let \( S \) be the set of places of \( k \) which do not define points of \( U \), it must contain all the archimedean places. Let \( \mathcal{X} \to U \) be a relative tame stacky curve whose generic fiber \( X := \mathcal{X}_k \) has genus \( g < 1 \). Let \( \mathcal{P} \subset X_{\text{coarse}, K} \) be the stacky locus. In this subsection we will prove the first and third part of Theorem 1.4.

By Lemma 1.4 \( KD'(\mathcal{X}) = R^1 \pi_* \mathbb{G}_m \) is locally constant. The stalk at the generic point is \( (R^1 \pi_* \mathbb{G}_m)_k = \text{Pic} X_k \), so \( R^1 \pi_* \mathbb{G}_m \) can be identified with the \( \pi_1(U) \)-module \( \text{Pic} X_k \). Let \( \hat{T}^0 \subset R^1 \pi_* \mathbb{G}_m \) be the subsheaf corresponding to \( \text{Pic} X_k^0 \) and \( T^0 := \text{Hom}(\hat{T}^0, \mathbb{G}_m) \), \( T := \text{Hom}(R^1 \pi_* \mathbb{G}_m, \mathbb{G}_m) \) the dual tori. Let \( T_{\text{deg}} := \ker(T \to T^0) \), it has group of characters \( \text{Pic} X_k^0 / \text{Pic}^0 X_k \cong \mathbb{Z} \) so \( T_{\text{deg}} \cong \mathbb{G}_m \).

The most difficult case is handled in the following lemma.

**Lemma 6.5.** If \( X \) has signature \( (0; 2, 2, 2) \) and \( \text{Gal}(k_s/k) \) acts transitively on \( \mathcal{P}(k_s) \) then \( \mathcal{X}(U) \neq \emptyset \).
Proof. Note that $T^0$ is 2-torsion by Lemma \[5.2\].

Let $\mathcal{P} \subset \mathcal{X}_{\text{coarse}}$ be the stacky locus. The assumption that $\text{Gal}(k_s/k)$ acts transitively on $\mathcal{P}(k_s)$ implies that $\mathcal{P} = \text{Spec } K$ where $K/k$ is a field of degree of

3. Let $L/k$ be the Galois closure of $K/k$.

Let $V, W$ respectively be the normalizations of $K, L$ over $S$. We deduce from Theorem \[2.2\] that $V$ and thus its Galois closure $W$ is finite étale over $U$ and that there exists a point $Q \in \mathcal{X}_V(V) \neq \emptyset$. This point induces a splitting of the exact sequence \[3\] for the torus $T^0$. There thus exists a $T^0$-torsor $\mathcal{Y} \rightarrow \mathcal{X}_V$ with class $[\mathcal{Y}] \in H^1(\mathcal{X}_V, T^0)$ such that $\chi([\mathcal{Y}])$ is the inclusion $T^0 \subset R^1\pi_*G_m$ and $Q^*\mathcal{Y}$ is the trivial torsor over $V$. We thus have $\mathcal{Y}(V) \neq \emptyset$.

There exists a generalization of the corestriction in group cohomology to étale cohomology. It is often called the trace map [NSW08, Tag 03SH]. We will call it corestriction. For any finite étale map of Deligne-Mumford stacks $B \rightarrow A$ we will use the notations $\text{res}_B^A : H^*(A, \cdot) \rightarrow H^*(B, \cdot)$ and $\text{cores}_B^A : H^*(B, \cdot) \rightarrow H^*(A, \cdot)$ for, respectively, the restriction and corestriction map.

Consider the corestriction map $\text{cores}_U^\mathcal{Y} : H^1(\mathcal{X}_V, T^0) \rightarrow H^1(\mathcal{X}, T^0)$ and the torsor $\mathcal{Z} \rightarrow \mathcal{X}$ with class $[\mathcal{Z}] = \text{cores}_U^\mathcal{Y}([\mathcal{Y}])$. The exact sequence \[4\] is functorial and thus commutes with restrictions and corestrictions. We compute by applying [NSW08, Cor. 1.5.7] and using that $\hat{T}^0$ is 2-torsion that

$$\chi([\mathcal{Z}]) = \text{cores}_U^\mathcal{Y}(\hat{T}^0 \subset R^1\pi_*G_m) = \text{cores}_U^\mathcal{Y}(\text{res}_U^\mathcal{Y}(\hat{T}^0 \subset R^1\pi_*G_m))$$

$$= 3(\hat{T}^0 \subset R^1\pi_*G_m) = (\hat{T}^0 \subset R^1\pi_*G_m) \in \text{Hom}_U(\hat{T}^0, R^1\pi_*G_m).$$

Let $Z$ be the generic fiber of $\mathcal{Z} \rightarrow U$. The $T^0$-torsor $Z \rightarrow X$ has injective type so $Z$ is geometrically connected by [Sta18, Tag 0E0N]. We see from Lemma \[5.2\] that $\#\text{Pic}^0(X_{k_s}) = 4$ so the Riemann-Hurwitz formula [VZB22, Prop. 5.5.6] shows that $Z$ has genus $0$. We claim that $Z(K) \neq \emptyset$. The lemma follows from the claim. Indeed, $Z$ is a genus $0$ curve so $[K : k] = 3$ and $Z(K) \neq \emptyset$ imply that $Z(k) \neq \emptyset$. It follows from Lemma \[2.1\] that $Z$ is a scheme and thus by the valuative criterion for properness that $Z(U) = Z(k) \neq \emptyset$. This implies that $\mathcal{X}(U) \neq \emptyset$.

Let $Y$ be the generic fiber of $\mathcal{Y} \rightarrow V$. We have $\mathcal{Y}(V) \neq \emptyset$ so $Y(K) \neq \emptyset$ and $Y \cong \mathbb{P}^1_K$. Let $M/K$ be a field extension. It follows from the exact sequence \[8\] that if $A \rightarrow \mathcal{X}_M$ is any $T^0$-torsor such that $\chi([A]) = \hat{T}^0 \subset R^1\pi_*G_m$ that there exists a unique $\mathcal{C}_A \in H^1(M, T^0)$ such that $[A] = \mathcal{Y}_M + \mathcal{C}_A$. The underlying variety of $A$ is a twist of $\mathcal{Y}_M$ and thus a Severi-Brauer curve. The corresponding class in $H^1(M, \text{PGL}_2) \cong \text{Br } M[2]$ is given by the image of $\mathcal{C}_A$ under the map

$$\alpha : H^1(M, T^0) \rightarrow H^1(M, \text{Aut}_M(Y)) = H^1(M, \text{PGL}_2) \cong \text{Br } M[2].$$

If $A(M) \neq \emptyset$ then $A \cong \mathbb{P}^1_M$ and thus $\alpha(\mathcal{C}_A) = 0$.

Note that for all $\sigma \in \text{Gal}(L/k)$ we have $\sigma(\mathcal{Q}_L) \in \mathcal{Y}(Y_L)$ so $\alpha(\mathcal{C}_{\text{Gal}(L/k)}) = 0$. We will prove that $\alpha(\mathcal{C}_{\text{Gal}(K/k)}) = 0$, from which the claim $Z(K) \neq \emptyset$ follows.

There are two cases to consider, the first case is $L = K$. Then $\text{Gal}(K/k) = \mathbb{Z}/3\mathbb{Z}$. We deduce from [NSW08, Cor. 1.5.7] that

$$[Z_K] = \text{res}_K^\mathcal{X}(\text{cores}_K^\mathcal{X}([\mathcal{Y}])) = \sum_{\sigma \in \text{Gal}(K/k)} \sigma(\mathcal{Y}) + \sum_{\sigma \in \text{Gal}(K/k)} \mathcal{C}_{\sigma(Y)}. $$

We have $3[\mathcal{Y}] = [\mathcal{Y}]$ because $T^0$ is 2-torsion so $\alpha(\mathcal{C}_{Z_K}) = \sum_{\sigma \in \text{Gal}(K/k)} \alpha(\mathcal{C}_{\sigma(Y)}) = 0$. 
The second case is \( L \neq K \). Then \( \text{Gal}(L/k) = S_3 \) and if \( \tau \in \text{Gal}(L/k) \) is any element not in the subgroup \( \text{Gal}(L/K) \) then \[ \text{NSW08} \] Prop. 1.5.6, Cor. 1.5.7 shows that
\[
[Z_K] = \text{res}_{X_K}^X(\text{cores}_{X_K}(Y)) = [Y] + \text{cores}_{X_K}^L(\tau[Y_L])
\]
\[
=[Y] + \text{cores}_{X_K}^L([Y_L]) + \text{cores}_{X_K}^L(\tau[Y_L]) = 3[Y] + \text{cores}_{X_K}^L(\tau[Y_L]) \in H^1(X_K, T^0).
\]
We have \( 3[Y] = [Y] \) because \( T^0 \) is 2-torsion. The map \( \alpha \) commutes with corestriction so \( \alpha(\text{cores}_{X_K}^L(\tau[Y_L])) = \text{cores}_{X_K}^L(\alpha(\tau[Y_L])) = 0 \). \( \square \)

We can now prove Theorem 1.4.(3).

**Theorem 6.6.** Let \( \mathcal{X} \to U \) be a relative stacky curve whose generic fiber \( X \) has genus \( g < 1 \) and which does not have signature \( (0) \) or \( (0; n, n) \) for any \( n > 1 \). Then \( \mathcal{X}(U) \neq \emptyset \).

**Proof.** If the stacky locus \( \mathcal{P} \subset X_{\text{coarse}} \) contains a \( k \)-point then \( \mathcal{X}(U) \neq \emptyset \) by Theorem 2.2. But the action of \( \text{Gal}(k_s/k) \) on \( \mathcal{P}(k_s) \) preserves stabilizers so if \( X \) has a unique point in \( \mathcal{P}(k_s) \) with a certain stabilizer then \( \mathcal{X}(U) \neq \emptyset \). It follows from the list \[ \text{BN06} \] Prop. 5.5] that the only remaining case is when \( X \) has signature \( (0; 2, 2) \) and the action of \( \text{Gal}(k_s/k) \) on \( \mathcal{P}(k_s) \) is transitive. This case is Lemma 6.5. \( \square \)

This allows us to prove 1.4.(1).

**Theorem 6.7.** Let \( \mathcal{X} \to U \) be a relative stacky curve whose generic fiber \( X \) has genus \( g < 1 \). The elementary obstruction is then the only obstruction to the existence of an integral point on \( \mathcal{X} \), i.e. if \( w_X = 0 \) then \( \mathcal{X}(U) \neq \emptyset \).

**Proof.** Due to Theorem 6.4 we only have to deal with the case when \( X \) has signature \( (0) \) or \( (0; n, n) \) for some \( n > 1 \).

If \( X \) has signature \( (0) \) then it is a curve of genus 0. Lemma 6.2 implies that \( w_X = 0 \) and it follows from \[ \text{CTS87} \] Ex. 2.2.11] that \( X \cong \mathbb{P}^1 \). Lemma 2.4 shows that \( \mathcal{X} \) is a scheme so we deduce from the valuative criterion for properness that \( \mathcal{X}(U) = X(k) \neq \emptyset \).

Assume now that \( X \) has signature \( (0; n, n) \). The triangle 3 splits since \( w_X = 0 \). The long exact sequence coming from this triangle splits so there exist a \( T \)-torsor \( \mathcal{V} \to \mathcal{X} \) such that \( \chi([\mathcal{V}]) \) is the identity morphism \( R^1 \pi_* \mathbb{G}_m \to R^1 \pi_* \mathbb{G}_m \). Let \( \mathcal{Y} = [\mathcal{V}/T_{\text{deg}}] \), it is a \( T^0 \)-torsor over \( \mathcal{X} \). We claim that \( \mathcal{Y} \) is a relative stacky curve, the only non-trivial part is to show that the fibers of \( \mathcal{Y} \to U \) are geometrically connected. The generic fiber \( Y := \mathcal{Y}_k \to X := \mathcal{X}_k \) of this torsor is the \( T^0 \)-torsor with injective type \( \text{Pic} X_k^{\text{\circ}} \to \text{Pic} X_k \) and is thus geometrically connected by \[ \text{Sko01} \] (2.6), Lemma 2.3.1.(iii)]. The claim follows from \[ \text{Sta18} \] Tag 0E0N.

The genus of \( Y \) is 0 by the Riemann-Hurwitz formula \[ \text{VZB22} \] Prop. 5.5.6] and the fact that \( \# \text{Pic}^0 X_k = n \) by Lemma 5.2. It remains to show that \( Y \cong \mathbb{P}^1_k \).

The generic fiber \( V \to Y \) of \( \mathcal{V} \to \mathcal{Y} \) is the \( \mathbb{G}_m \)-torsor corresponding to the image of the map \( \text{Pic} X_k \to \text{Pic} Y_k \). The map \( Y \to X \) has degree \( \# \text{Pic}^0 X_k = n \). The following diagram thus commutes
\[
\begin{align*}
\text{Pic} X_k & \longrightarrow \text{Pic} Y_k \\
\downarrow_{\text{deg}} & \cong \\
\mathbb{Z} & \longrightarrow \mathbb{Z} \\
\frac{1}{n} & \mathbb{Z} \longrightarrow \mathbb{Z}
\end{align*}
\]
The degree map \( \text{Pic} X_k \to \frac{1}{n} \mathbb{Z} \) is surjective so \( \text{Pic} X_k \to \text{Pic} Y_k \) is surjective. This implies that the \( \mathbb{G}_m \)-torsor \( V_k \to Y_k \) corresponds to a generator of \( \text{Pic} Y_k \). But the
torsor $V \to Y$ also defines an element of $\mathrm{Pic} Y$ so $\mathrm{Pic} Y = \mathrm{Pic} Y_k$. This implies that $Y \cong \mathbb{P}^1_k$ since $Y$ is a genus 0 curve.

6.2. The elementary obstruction and $\mathfrak{B}(\mathcal{X})$. The goal of this subsection is to extend [HS08 Prop. 3.3], which relates the Brauer-Manin obstruction and the elementary obstruction, to integral models of Deligne-Mumford stacks. This will allow us to prove Theorem 1.4.(2). We use the same notation as the previous subsection except that we let $X$ be any regular Deligne-Mumford stack and $\pi : \mathcal{X} \to U$ any morphism (not necessarily smooth or proper).

We recall the definition of the group $\mathfrak{B}(\mathcal{X}) := \ker(\mathrm{Br}_a \mathcal{X} \to \prod_{v \in S} \mathrm{Br}_a X_{k_v})$ of locally constant algebras.

The evaluation map at a place $v$

$$\text{ev}_v(\alpha; \cdot) : X(k_v)$$

is constant for $\alpha \in \mathfrak{B}(\mathcal{X})$. If $v \in S$ this is because the image of $\alpha$ in $\mathrm{Br}_0 X_{k_v}$ lies in $\mathrm{Br}_0 X_{k_v}$. If $v \notin S$ then its image is an element of $\mathrm{Br}_0 \mathcal{X}_{O_v}$ so $\alpha(x_v) \in \mathrm{Br}_0 \mathcal{O}_v = 0$.

It follows that if $\mathcal{X}(k_U) \neq \emptyset$ then for any choice $(x_v)_{v \in S} \in \prod_{v \in S} \mathcal{X}(k_v)$ the Brauer-Manin pairing $\mathcal{X}(k_U) \times \mathfrak{B}(\mathcal{X}) \to \mathbb{Q}/\mathbb{Z}$ factors through the map

$$\delta : \mathfrak{B}(\mathcal{X}) \to \mathbb{Q}/\mathbb{Z} : \alpha \to \sum_{v \in S} \text{inv}_v(\alpha(x_v)). \quad (7)$$

We can also describe $\delta$ as follows, analogous to [HS08 Lemma 3.1].

**Lemma 6.8.** If $\mathcal{X}(k_v) \neq \emptyset$ for all $v \in S$ then the following commutative diagram has exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathrm{Br} U & \longrightarrow & \mathrm{Br}_1 \mathcal{X} & \longrightarrow & \mathrm{Br}_a \mathcal{X} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{v \in S} \mathrm{Br}_k v & \longrightarrow & \bigoplus_{v \in S} \mathrm{Br}_1 X_{k_v} & \longrightarrow & \bigoplus_{v \in S} \mathrm{Br}_a X_{k_v} & \longrightarrow & 0
\end{array}
$$

The map $\delta$ is equal to the boundary map $\mathfrak{B}(\mathcal{X}) \to \text{coker}(\mathrm{Br} U \to \bigoplus_{v \in S} \mathrm{Br}_k v)$ coming from the snake lemma composed with the map $\bigoplus_{v \in S} \mathrm{Br}_k v \to \mathbb{Q}/\mathbb{Z} : (\alpha_v)_v \to \sum_{v \in S} \text{inv}_v(\alpha_v)$.

**Proof.** Right exactness of the rows is true by the definition of $\mathrm{Br}_a \mathcal{X}$. For $v \in S$ we can take $x_v \in \mathcal{X}(k_v)$ which defines a section $\mathrm{Br}_1 \mathcal{X}_{k_v} \to \mathrm{Br}_k v$ by functoriality of the Brauer group. The fact that $\mathrm{Br} U$ injects into $\bigoplus_{v \in S} \mathrm{Br}_k v$ implies that the top row is also left exact.

Using the construction of the snake lemma we see that the image of $\alpha \in \mathfrak{B}(\mathcal{X})$ under the boundary map is $(\alpha(x_v))_{v \in S} \in \text{coker}(\mathrm{Br} U \to \bigoplus_{v \in S} \mathrm{Br}_k v)$ for any choice of $(x_v)_{v \in S} \in \prod_{v \in S} \mathcal{X}(k_v)$. The lemma follows immediately.

We require the notion of arithmetic étale cohomology with compact support $H^p_c(U, \cdot)$. If $k$ is a function field over a finite field then this is the usual cohomology with compact support. If $k$ is a number field then one has to make certain modifications to deal with the archimedean places. We refer to [Mil06 p. 165] for the details. We denote by $R\Gamma_c(U, \cdot) \in D(\text{Ab})$ the complex of abelian groups computing $H^p_c(U, \cdot)$, i.e., what is referred to as $H_c(U, \cdot)$ in [Mil06 p. 166].

The defining property [Mil06 Prop. II.2.3] of this cohomology with compact support is the existence of the following distinguished triangle in $D(\text{Ab})$

$$R\Gamma_c(U, \mathcal{F}) \to R\Gamma(U, \mathcal{F}) \to \bigoplus_{v \in S} R\Gamma(k_v, j_v^* \mathcal{F}) \to R\Gamma_c(U, \mathcal{F})[1]$$
for any complex $\mathcal{F} \in \mathcal{D}(U)$. Where $j_v : \text{Spec} \ k_v \to U$ is map coming from the inclusion $k \subset k_v$ and $\Gamma(k_v, -)$ is the complex computing Galois or Tate cohomology if $v$ is non-archimedean or archimedean, respectively.

The long exact sequence corresponding to this triangle is

$$\cdots \to H^p_c(U, \mathcal{F}) \to H^p(U, \mathcal{F}) \to \bigoplus_{v \in S} H^p(k_v, j_v^* \mathcal{F}) \to H^{p+1}(U, \mathcal{F}) \to \cdots \tag{9}$$

where $H^p(k_v, -)$ is the Galois or Tate hypercohomology if $v$ is non-archimedean or archimedean, respectively.

If $S$ contains at least one non-archimedean place then $H^3(U, \mathbb{G}_m) = 0$ by [Mil06, Rem. II.2.2]. So Lemma 6.3 and the above long exact sequence imply that

$$\mathcal{B}(\mathcal{X}) \cong \text{im}(H^3_c(U, KD'(\mathcal{X})) \to H^1(U, KD'(\mathcal{X})). \tag{10}$$

**Proposition 6.9.** If $S$ contains at least one non-archimedean place then the following diagram commutes.

$$\begin{array}{ccc}
H^1_c(U, KD'(\mathcal{X})) & \xrightarrow{w_{X*}} & H^1_c(U, \mathbb{G}_m[2]) \\
\downarrow & & \downarrow \cong \\
\mathcal{B}(\mathcal{X}) & \xrightarrow{6} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

Here $w_{X*}$ is induced by $w_X$ in the triangle [5] and $H^2_c(U, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$ is the isomorphism of [Mil06, Prop II.2.6].

**Proof.** This is basically contained in the proof of [HS08, Prop. 3.3], we will expand the argument.

The distinguished triangle [5] induces the following distinguished triangle

$$\Gamma_c(U, \mathbb{G}_m)[1] \to \Gamma_c(U, KD(\mathcal{X})) \to \Gamma_c(U, KD'(\mathcal{X})) \xrightarrow{w_{X*}} \Gamma_c(U, \mathbb{G}_m)[2]. \tag{11}$$

Consider the following sequence of complexes, where the first row has cohomological degree 1

$$\begin{array}{ccc}
H^2(U, \mathbb{G}_m) & \to & H^1(U, KD(\mathcal{X})) \\
\downarrow & & \downarrow \\
\bigoplus_{v \in S} H^2(k_v, \mathbb{G}_m) & \to & \bigoplus_{v \in S} H^1(k_v, j_v^* KD(\mathcal{X})) \\
\downarrow & & \downarrow \\
\bigoplus_{v \in S} H^1(k_v, j_v^* KD(\mathcal{X})) & \to & \bigoplus_{v \in S} H^1(k_v, j_v^* KD'(\mathcal{X}))
\end{array}$$

Using Lemma 6.3 we can identify this with the short exact sequence of complexes [8]. It is in particular a short exact sequence of complexes and thus induces a distinguished triangle $T$ in $\mathcal{D}(\text{Ab})$.

The complex $\Gamma_c(U, -)$ is by definition a shift of the mapping cone of $\Gamma(U, -) \to \bigoplus_{v \in S} \Gamma(k_v, j_v^* -)$. By truncating on both sides we thus get a zigzag of maps from the triangle (11) to the triangle $T$ in $\mathcal{D}(\text{Ab})$, inducing a zigzag of maps between the corresponding long exact sequences.

We now note that the map $H^1_c(U, KD'(\mathcal{X})) \to \mathcal{B}(\mathcal{X})$ is the one induced by this zigzag in cohomology. Similarly, the isomorphism $H^2_c(U, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$ is by definition [Mil06, Prop. II.2.6] given by composing the isomorphism $H^2_c(U, \mathbb{G}_m) \cong \text{coker}(\text{Br} U \to \bigoplus_v \text{Br} k_v)$ coming from the zigzag with the isomorphism $\text{coker}(\text{Br} U \to \bigoplus_v \text{Br} k_v) \cong \mathbb{Q}/\mathbb{Z} : (\alpha_v)_v \to \sum_v \text{inv}_v(\alpha_v)$. 
The boundary map in the long exact sequence coming from the triangle (11), respectively the triangle $T$, is $w_{X, \ast}$, respectively the one coming from the snake lemma. The proposition then follows from Lemma 6.8.

We can now prove Theorem 6.10 [2].

**Theorem 6.10.** Let $U$ be Dedekind scheme whose fraction field $k$ is a global field and such that at least one non-archimedean place of $k$ does not define a point of $U$.

Let $\mathcal{X} \to U$ be a relative stacky curve whose generic fiber $X$ has genus $g < 1$. If $\mathcal{X}(\mathbb{A}_U)^{\text{c}(\mathcal{X})} \neq \emptyset$ then $\mathcal{X}(U) \neq \emptyset$.

**Proof.** The complex $KD'(\mathcal{X})$ is a locally constant finitely generated sheaf due to Lemma 6.4 so the following pairing is perfect by [Mil06, Thm. II.3.1](10)

$$H^1_c(U, KD'(\mathcal{X})) \times \operatorname{Ext}^2(KD'(\mathcal{X}), \mathbb{G}_m) \to H^3(U, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z} : (\alpha, w) \to w_\ast(\alpha).$$

(12)

The assumption $\mathcal{X}(\mathbb{A}_U)^{\text{c}(\mathcal{X})} \neq \emptyset$ implies that $\mathfrak{a} = 0$ and thus by Lemma 6.9 that the map $w_{X, c} : H^2_c(U, KD'(\mathcal{X})) \to H^2(U, \mathbb{G}_m)$ is zero. The fact that the pairing (12) is perfect implies that $w_{X} = 0$. The theorem now follows from Theorem 6.7.

**Definition 7.1.** Genus $\frac{1}{2}$ curves failing the Hasse principle

Let $X$ be a genus $\frac{1}{2}$ tame stacky curve over a global field $k$. It follows from the formula (11) that it has signature $(0; 2, 2)$. Assume that $X(\mathbb{A}_k) \neq \emptyset$, then $X_{\text{coarse}}(\mathbb{A}_k) \neq \emptyset$ and thus $X_{\text{coarse}} \cong \mathbb{P}^1_k$. The stacky locus $\mathcal{P} \subset \mathbb{P}^1$ is defined by a binary quadratic form $f \in k[X, Y]$. It follows from [VZB22, Lemma 5.3.10] that $X \cong \mathbb{P}^1_k[\sqrt{f}]$, where $\mathbb{P}^1_k[\sqrt{f}]$ is the stack of square-roots of $f$ [Cad07, Def. 2.2.1]. We may thus assume that $X = \mathbb{P}^1_k[\sqrt{f}]$.

This stack has a universal cover given by $Y := \{Z^2 = f(X, Y)\}$ with the morphism $Y \to X$ being given by $[z : y : x] \to [x : y]$ on coarse moduli spaces, see Example 5.6 for a computation of this universal cover. This is a $\mu_2$-torsor, denote its class by $[Y/X] \in H^1(X, \mu_2)$. The Brauer group $\operatorname{Br} X$ is generated by $\operatorname{Br} k$ and the cup products $d \cup [Y/X] \in \operatorname{Br} X$ for $d \in k^\times /k^\times 2 \cong H^1(k, \mathbb{Z}/2\mathbb{Z})$ by Proposition 5.5.

Assume that $f = ax^2 + bxy + cy^2 \in \mathcal{O}_k[X, Y]$, this can always be achieved by a projective linear transformation, and let $q := b^2 - 4ac$ be the discriminant of $f$. We will now consider the integral model of $X$ given by the root stack $\mathcal{X} := \mathbb{P}^1_{\mathcal{O}_k[\frac{1}{2q}]}[\sqrt{f}]$.

This is a smooth over $\mathcal{O}_k[\frac{1}{2q}]$ because $\{f = 0\} \subset \mathbb{P}^1_{\mathcal{O}_k[\frac{1}{2q}]}$ is smooth over $\mathcal{O}_k[\frac{1}{2q}]$. Let $U := \{f \neq 0\} \subset X$ be the maximal open subscheme. It follows from the definition of a root stack that $\mathcal{X}(\mathcal{O}_k) \cap U(k)$ is given by those $[x : y] \in U(k)$ such that $v(f(x, y))$ is even for all finite places $v \nmid 2q$ and that we have an analogous description for $\mathcal{O}_v$-points. We have $\mathcal{X}(\mathcal{O}_v) \neq \emptyset$ for all finite places $v \nmid 2q$ because $f$ is smooth over $\mathcal{O}_v$ so we can choose $x, y \in \mathcal{O}_v$ such that $v(f(x, y)) = 0$. For the other places $v$ we have $X(k_v) \supset U(k_v) \neq \emptyset$.

We now specialize to the case $k = \mathbb{Q}$ and will compute when there is an elementary obstruction.

**Lemma 7.1.** We have

$$\mathcal{B}(\mathcal{X}) \cong \left\{ d \in \mathbb{Z}[\frac{1}{2q}]^\times /\{1, q\} \cdot \mathbb{Z}[\frac{1}{2q}]^\times 2 : d \in \{1, q\} \mathbb{Q}^\times 2 \text{ for } p \nmid 2q \text{ and } d > 0 \text{ if } q > 0 \right\}.$$
Proof. The Dedekind scheme $U := \text{Spec } \mathbb{Z}[\frac{1}{2q}]$ does not contain the non-archimedean prime 2 so $H^3(U, \mathbb{G}_m) = 0$ by [Mil06, Rem. II.2.2] so by Lemma 6.3

$$ \mathcal{B}(\mathcal{X}) \cong \ker \left( H^1(U, \text{Pic } \mathcal{X}^U) \to \bigoplus_{v|2q} H^1(\mathbb{Q}_v, \text{Pic } \mathcal{X}^U) \right). \quad (13) $$

It follows from Lemma 5.2 that $\text{Pic}^0 \mathcal{X}^U \cong \mathbb{Z}/2\mathbb{Z}$ with the trivial $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. We deduce via Lemma 5.3 that

$$ H^1(U, \text{Pic}_{k_v}) \cong H^1(U, \mathbb{Z}/2\mathbb{Z})/\varphi \cong \left( \mathbb{Z}[\frac{1}{2q}]^\times / \mathbb{Z}[\frac{1}{2q}]^{\times 2} \right)/q. $$

Where $\varphi : \pi_1(U) \to \mathbb{Z}/2\mathbb{Z}$ is the cocycle which sends $\sigma$ to 0 if $\sigma$ fixes the roots of $f(x, y)$ and to 1 otherwise. Its image under the isomorphism $H^1(U, \mathbb{Z}/2\mathbb{Z}) \cong \times/\mathbb{Z}[\frac{1}{2q}]^{\times 2}$ is $q$.

Completely analogously we get $H^1(\mathbb{Q}_v, \text{Pic}_{k_v}) \cong \mathbb{Q}_v^\times / \{1, q\} \cdot \mathbb{Q}_v^{\times 2}$ for $v \mid 2q\infty$. The lemma then follows by using these descriptions in [13].

An algebra corresponding to $d \in \mathcal{B}(\mathcal{X})$ is by Proposition 5.3 given by the cup product $d \cup [Y/X]$.

Recall that to compute the Brauer obstruction of $\mathcal{B}(\mathcal{X})$ we have to compute the map $6$ defined in (7). Recall [Ser73, §5.2.1] that to each binary quadratic form $f$ and place $v$ we can assign its $\epsilon$-invariant $\epsilon_v := \epsilon_v(f) \in \mu_2$. We will also let the evaluation map $ev_v(d; \cdot)$ take value in $\mu_2$ instead of the isomorphic group $\mathbb{Z}/2\mathbb{Z}$.

**Lemma 7.2.** Let $d \in \mathcal{B}(\mathcal{X})$. Let $v \mid 2q\infty$ be a place.

1. We have $ev_v(d; P_v) = 1$ for all $P_v \in X(\mathbb{Q}_v)$ if $d \in \mathbb{Q}_v^{\times 2}$.
2. If $d \in q\mathbb{Q}_v^{\times 2}$ then we have $ev_v(d; P_v) = \epsilon_v$.

This implies that $6(d) = \sum_{v|2q\infty, d \in q\mathbb{Q}_v^{\times 2}} \epsilon_v$.

**Proof.** The evaluation map is locally constant so we can assume that $P_v \in U(k_v)$. We can thus use that $\alpha|_U = (d, f/X^2) \in \text{Br } U$ is a quaternion algebra. For $P_v = [x : y] \in U(\mathbb{Q}_v)$ we have $ev_v(d; P_v) = (d, f(x, y)/x^2)_v$. If $d \not\in \mathbb{Q}_v^{\times 2}$ then $(d, f(x, y)/x^2)_v = 1$ so assume that $d \in q\mathbb{Q}_v^{\times 2}$. In that case [Ser73, §5.2.2: Cor. of Thm. 6] implies that $(d, f(x, y)/x^2)_v = (q, f(x, y)/x^2)_v = \epsilon_v$ (note that the convention for the discriminant differs by a sign).

The last statement follows from the definition of 6. \qed

The considerations above and Theorem 1.4 immediately lead to

**Theorem 7.3.** Let $f \in \mathbb{Z}[X, Y]$ be a binary quadratic form with discriminant $q$ and $\epsilon$-invariants $\epsilon_v$. The root stack $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}[\frac{1}{2q}]}/\sqrt{f}$ always has local integral points everywhere and has a $\mathbb{Z}[\frac{1}{2q}]$-point if and only if there exists no $d \mid 2q$ such that:

1. For all places $v \mid 2q\infty$ one has $d \in \mathbb{Q}_v^{\times 2}$ or $d \in q\mathbb{Q}_v^{\times 2}$.
2. We have $\prod_{v|2q\infty, d \not\in q\mathbb{Q}_v^{\times 2}} \epsilon_v = \prod_{v|2q\infty, d \in q\mathbb{Q}_v^{\times 2}} \epsilon_v = -1$.

**Proof.** The equality $\prod_{v|2q\infty, d \not\in q\mathbb{Q}_v^{\times 2}} \epsilon_v = \prod_{v|2q\infty, d \in q\mathbb{Q}_v^{\times 2}} \epsilon_v$ follows from the product formula $\prod_v \epsilon_v = 1$ and the fact that $\epsilon_v = 1$ for $v \nmid 2q\infty$ since $f$ has good reduction there.

Everything else follows from Lemma 7.2, Theorem 6.10 and the preceding discussion. \qed

**Example 7.4.** A simple example is $f = 3x^2 + 2xy + 5y^2$ for which $q = -56 = -7 \cdot 8$. Completing the square we find $\epsilon_v = (3, 42)_v$, so $\epsilon_\infty = 1, \epsilon_2 = \epsilon_7 = -1$. We can
thus apply Theorem 7.3 with $d := -7$. Indeed, $q < 0$, $d \in \mathbb{Z}_2^\times$ and $d \in -56\mathbb{Z}_7^\times$ by Hensel’s lemma. We conclude that $\mathcal{X}(\mathbb{Z}(-1)) = \emptyset$.

We can also reinterpret the example of Bhargava-Poonen [BP20, Thm. 1]. Note that unlike them we do not assume that $p$ is a square mod $q$ and that $a$ is a square modulo $p$ and a non-square modulo $q$.

**Example 7.5.** Let $p, q, r$ be primes congruent to 7 (mod 8) such that $p$ and $q$ are squares modulo $r$. Let $f = ax^2 + bxy + cy^2$ be a positive definite binary quadratic form of discriminant $-pqr$ such that $a$ is a non-square modulo $r$. An example is $f = 3x^2 + xy - 850y^2$ with $p = 7, q = 47, r = 31$.

By completing the square one sees that $\epsilon_v = (a, apqr)_v = (a, -pqr)_v$. Since $-pqr \equiv 1$ (mod 8). And due to quadratic reciprocity we get that $\epsilon_\infty = \epsilon_2 = 1, \epsilon_r = -1$. We can then apply Theorem 7.3 to $d = -r$. We have $-r \in \mathbb{Z}_2^\times, \mathbb{Z}_p^\times, \mathbb{Z}_q^\times$ and $-r \in -pqr\mathbb{Z}_r^\times, -pqr\mathbb{Z}_r^\times$. We conclude that $\mathcal{X}(\mathbb{Z}(\mathbb{Z}_{pqr})) = \emptyset$.

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