A generalized blow-up formula for Seiberg–Witten invariants

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Abstract

We prove a gluing formula for Seiberg–Witten invariants which describes in particular the behaviour of the invariant under blow-up and rational blow-down.

1 Introduction

In this paper we use the results of [5, 6] to establish a gluing formula for Seiberg–Witten invariants of certain 4–manifolds containing a negative definite piece. The formula describes in particular the behaviour of the Seiberg–Witten invariant under blow-up and under the rational blow-down procedure introduced by Fintushel–Stern [3]. While the formula will be known in principle to experts, to our knowledge no complete proof has previously been published. As for the classical blow-up formula, this was proved by Bauer [1, Corollary 4.2]. (Earlier, a proof had been announced by Salamon [9], and there is a sketch of a proof in Nicolaescu [8].) In the case of rational blow-down the formula was stated by Fintushel–Stern [3, Theorem 8.5] with a brief outline of a proof. Apart from providing a proof in the general case, the main motivation for writing this paper was to show how the parametrized version of the gluing theorem in [6] can be used to handle at least the simplest cases of obstructed gluing, thereby providing a unified approach to a wide range of gluing problems.

Before stating our results we explain how the Seiberg–Witten invariant, usually defined for closed 4–manifolds, can easily be generalized to compact spin$^c$ 4–manifolds $Z$ whose boundary $Y'' = \partial Z$ satisfies $b_1(Y'') = 0$ and admits a metric $g$ of positive scalar curvature. (By a spin$^c$–manifold we mean as in [5] an oriented smooth manifold with a spin$^c$–structure.) As usual we also assume that $b_2^+(Z) > 1$. Let $\{Y_j\}$ be the components of $Y''$,
which are rational homology spheres. Let $\tilde{Z}$ be the manifold with tubular ends obtained from $Z$ by adding a half-infinite tube $\mathbb{R}_+ \times Y'$. Choose a Riemannian metric on $\tilde{Z}$ which agrees with $1 \times g$ on the ends. We consider the monopole equations on $\tilde{Z}$ perturbed solely by means of a smooth 2–form $\mu$ on $\tilde{Z}$ supported in $Z$ as in [5, Equation 13]. Let $M = M(\tilde{Z})$ denote the moduli space of monopoles over $\tilde{Z}$ that are asymptotic over $\mathbb{R}_+ \times Y'_j$ to the unique (reducible) monopole over $Y'_j$. For generic $\mu$ the moduli space $M$ will be free of reducibles and a smooth compact manifold of dimension

$$\dim M = 2h(Y') + \frac{1}{4}(c_1(L_Z)^2 - \sigma(Z)) - b_0(Z) + b_1(Z) - b_2^+(Z),$$

see [5, Section 9]. Choose a base-point $x \in \tilde{Z}$ and let $M_x$ be the framed moduli space defined just as $M$ except that we now only divide out by those gauge transformations $u$ for which $u(x) = 1$. Let $L \to M$ be the complex line bundle whose sections are given by maps $s : M_x \to \mathbb{C}$ satisfying

$$s(u(\omega)) = u(x) \cdot s(\omega)$$

for all $\omega \in M_x$ and gauge transformations $u$. A choice of homology orientation of $Z$ determines an orientation of $M$, and we can then define the Seiberg–Witten invariant of $Z$ just as for closed 4–manifolds:

$$\text{SW}(Z) = \begin{cases} 
\langle c_1(L^k), [M] \rangle & \text{if } \dim M = 2k \geq 0, \\
0 & \text{if } \dim M \text{ is negative or odd}. 
\end{cases}$$

The use of $L$ rather than $L^{-1}$ prevents a sign in Theorem 1 below. (Another justification is that, although $M_x \to M$ is a principal bundle with respect to the canonical $U(1)$–action, it seems more natural to regard that action as a left action.) This invariant $\text{SW}(Z)$ depends only on the homology oriented spin$^c$–manifold $Z$, not on the choice of positive scalar curvature metric $g$ on $Y'$; the proof of this is a special case of the proof of the generalized blow-up formula, which we are now ready to state.

**Theorem 1** Let $Z$ be a connected, compact, homology oriented spin$^c$ 4–manifold whose boundary $Y' = \partial Z$ satisfies $b_1(Y') = 0$ and admits a metric of positive scalar curvature, and such that $b_2^+(Z) > 1$. Suppose $Z$ is separated by an embedded rational homology sphere $Y$ admitting a metric of positive scalar curvature,

$$Z = Z_0 \cup_Y Z_1,$$

where $b_1(Z_0) = b_2^+(Z_0) = 0$. Let $Z_1$ have the orientation, homology orientation, and spin$^c$ structure inherited from $Z$. Then

$$\text{SW}(Z) = \text{SW}(Z_1) \quad \text{if} \quad \dim M(\tilde{Z}) \geq 0.$$
We will show in Section 2 that $\dim M(\tilde{Z}_0) \leq -1$. (A particular case of this was proved by different methods in [3, Lemma 8.3].) The addition formula for the index then yields

$$\dim M(\tilde{Z}) = \dim M(\tilde{Z}_0) + 1 + \dim M(\tilde{Z}_1) \leq \dim M(\tilde{Z}_1).$$

The following corollary describes the effect on the Seiberg–Witten invariant of both ordinary blow-up and rational blow-down:

**Corollary 1** Let $Z_0, Z'_0, Z_1$ be compact, homology oriented spin$^c$ 4–manifolds with $-\partial Z_1 = \partial Z_0 = \partial Z'_0 = Y$ as spin$^c$ manifolds, where $Y$ is a spin$^c$ rational homology sphere admitting a metric of positive scalar curvature. Suppose $b_2^+(Z_1) > 1$, $b_1(Z_0) = b_1(Z'_0) = 0$, and $b_2(Z_0) = b_2^+(Z'_0) = 0$. Let

$$Z = Z_0 \cup_Y Z_1, \quad Z' = Z'_0 \cup_Y Z_1$$

have the orientation, homology orientation and spin$^c$ structure induced from $Z_0, Z'_0, Z_1$. Then

$$SW(Z) = SW(Z') \quad \text{if} \quad \dim M(Z') \geq 0.$$  

**Proof.** Set $n_\pm = \dim M(\pm \tilde{Z}_0)$ and $W = Z_0 \cup_Y (-Z_0)$. Then

$$-1 = \dim M(W) = n_+ + 1 + n_-,$$

hence $n_\pm = -1$. Thus

$$\dim M(Z) = \dim M(\tilde{Z}_1) \geq \dim M(Z') \geq 0.$$  

The theorem now yields

$$SW(Z) = SW(Z_1) = SW(Z').$$

2 Preliminaries on negative definite 4–manifolds

Let $X$ be a connected spin$^c$ Riemannian 4–manifold with tubular ends $\mathbb{R}_+ \times Y_j, j = 1, \ldots, r$, as in [5, Subsection 1.3]. Suppose each $Y_j$ is a rational homology sphere and $b_1(X) = 0 = b_2^+(X)$. We consider the monopole equations on $X$ perturbed only by means of a 2–form $\mu$ as in [5, Equation 13], where now $\mu$ is supported in a given non-empty, compact, codimension 0 submanifold $K \subset X$. Let $\alpha_j \in \mathcal{R}_{Y_j}$ be the reducible monopole over $Y_j$ and $M_\mu = M(X; \tilde{\alpha}; \mu; 0)$ the moduli space of monopoles over $X$ with asymptotic
limits $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$. This moduli space contains a unique reducible point $\omega(\mu) = [A(\mu), 0]$. Let $\Omega_{X,K}^+$ denote the space of (smooth) self-dual 2–forms on $X$ supported in $K$, with the $C^\infty$ topology. Let $p$ and $w$ be the exponent and weight function used in the definition of the moduli space $M_\mu$, as in [5, Subsection 3.4].

**Lemma 1** Let $R$ be the set of all $\mu \in \Omega_{X,K}^+$ such that the operator

$$D_{A(\mu)} : L_1^{p,w}(S_X^+) \to L_1^{p,w}(S_X^-)$$

(2)

is either injective or surjective. Then $R$ is open and dense in $\Omega_{X,K}^+$.

Of course, whether the operator is injective or surjective for a given $\mu \in R$ is determined by its index, which is independent of $\mu$.

**Proof.** By [5, Proposition 2.2 (ii)] and the proof of [5, Proposition 5.2], the operator

$$d^+ : \ker(d^*) \cap L_1^{p,w} \to L_1^{p,w}$$

is an isomorphism. Therefore, if $A_o$ is a reference connection over $X$ with limits $\alpha_j$ as in [5, Subsection 3.4] then there is a unique (smooth) $a = a(\mu) \in L_1^{p,w}$ with

$$d^+ a = 0, \quad d^+ a = -\hat{F}^+(A_o) - i\mu.$$

Hence we can take $A(\mu) = A_o + a(\mu)$. Since the operator (2) has closed image, it follows by continuity of the map $\mu \mapsto A(\mu)$ that $R$ is open in $\Omega_{X,K}^+$.

To see that $R$ is dense, fix $\mu \in \Omega_{X,K}^+$ and write $A = A(\mu)$. Let $W$ be a Banach space of smooth 1–forms on $X$ supported in $K$ as provided by [5, Lemma 8.2]. Using the unique continuation property of the Dirac operator it is easy to see that 0 is a regular value of the smooth map

$$h : W \times (L_1^{p,w}(S_X^+) \setminus \{0\}) \to L_1^{p,w}(S_X^-),$$

$$(\eta, \Phi) \mapsto D_{A+i\eta}\Phi.$$ 

In general, if $f_1 : E \to F_1$ and $f_2 : E \to F_2$ are surjective homomorphisms between vector spaces then $f_1|_{\ker f_2}$ and $f_2|_{\ker f_1}$ have identical kernels and isomorphic cokernels. In particular, the projection $\pi : h^{-1}(0) \to W$ is a Fredholm map whose index at every point agrees with the index $m$ of $D_A$.

By the Sard-Smale theorem the regular values of $\pi$ form a residual (hence dense) subset of $W$. If $\eta \in W$ is a regular value then we see that $D_{A+i\eta}$ is injective when $m \leq 0$ and surjective when $m > 0$. Since the topology on $W$ is stronger than the $C^\infty$ topology it follows that $R$ contains points of the form $\mu + d^+ \eta$ arbitrarily close to $\mu$. $\quad \square$
Lemma 2 Suppose the metric on each $Y_j$ has positive scalar curvature. Let $R'$ be the set of all $\mu \in \Omega^+_{X,K}$ such that the irreducible part $M^*_\mu$ is empty and the operator $D_{A(\mu)}$ in (2) is injective. Then $R'$ is open and dense in $\Omega^+_{X,K}$.

Proof. Recall that $M^*_\mu$ has expected dimension $2m - 1$, where $m = \text{ind}_C D_{A(\mu)}$.

Suppose $m > 0$. We will show that this leads to a contradiction. Let $R''$ be the set of all $\mu \in \Omega^+_{X,K}$ for which $M^*_\mu$ is regular. (Note that the reducible point is regular precisely when $D_{A(\mu)}$ is surjective.) From Lemma[1] and [5, Proposition 8.2] one finds that $R''$ is dense in $\Omega^+_{X,K}$. (Starting with a given $\mu$, first perturb it a little to make the reducible point regular, then a little more to make also the irreducible part regular.) But for any $\mu \in R''$ the moduli space $M^*_\mu$ would be compact with one reducible point, which yields a contradiction as in [4]. Therefore, $m \leq 0$.

We now see, exactly as for $R''$, that $R'$ is dense in $\Omega^+_{X,K}$. To prove that $R'$ is open we use a compactness argument together with the following fact: For any given $\mu_0 \in R'$ there is a neighbourhood $U$ of $\omega(\mu_0)$ in $B(X; \bar{\alpha})$ such that $M^*_\mu \cap U = \emptyset$ for any $\mu \in \Omega^+_{X,K}$ with $||\mu - \mu_0||_p$ sufficiently small. To prove this we work in a slice at $(A(\mu_0), 0)$, i.e., we represent $\omega(\mu)$ (uniquely) by $(A, 0)$ where $d^*(A - A(\mu_0)) = 0$, and we consider a point in $M^*_\mu$ represented by $(A + a, \phi)$ where $d^*a = 0$. Note that since $b_1(X) = 0$, the latter representative is unique up to multiplication of $\phi$ by unimodular constants.

Observe that there is a constant $C_1 < \infty$ such that if $||\mu - \mu_0||_p$ is sufficiently small then

$$||\psi||_{L^1_{p,w}} \leq C_1 ||D_A \psi||_{L^p,w}$$

for all $\psi \in L^1_{p,w}$. Hence if $L = (d^* + d^+, D_A)$ then for such $\mu$ one has

$$||s||_{L^1_{p,w}} \leq C_2 ||Ls||_{L^p,w}$$

for all $s \in L^1_{p,w}$. Denoting by $SW_\mu$ the Seiberg–Witten map over $X$ for the perturbation form $\mu$ we have

$$0 = SW_\mu(A + a, \phi) - SW_\mu(A, 0) = (d^+a - Q(\phi), D_A \phi + a\phi),$$

where $Q$ is as in [5]. Taking $s = (a, \phi)$ we obtain

$$||s||_{L^1_{p,w}} \leq C_2 ||Ls||_{L^p,w} \leq C_3 ||s||^2_{L^2_{p,w}} \leq C_4 ||s||^2_{L^1_{p,w}}.$$
Since \( s \neq 0 \) we conclude that
\[
\|s\|_{L^1_{p,w}} \geq C_4^{-1}.
\]

Choose \( \delta \in (0, C_4^{-1}) \) and define
\[
U = \{ [A(\mu_0) + b, \psi] : \| (b, \psi) \|_{L^1_{p,w}} < \delta, \ d^* b = 0 \}.
\]

If \( \|\mu - \mu_0\|_p \) is so small that \( \|A - A(\mu_0)\|_{L^1_{p,w}} \leq C_4^{-1} - \delta \) then
\[
\|(A + a - A(\mu_0), \phi)\|_{L^1_{p,w}} \geq \|s\|_{L^1_{p,w}} - \|A - A(\mu_0)\|_{L^1_{p,w}} \geq \delta,
\]
hence \([A + a, \phi] \notin U\). \( \square \)

3 The extended monopole equations

We now return to the situation in Theorem 1. Set \( X_j = \tilde{Z}_j \) for \( j = 0, 1 \).

Choose metrics of positive scalar curvature on \( Y \) and \( Y' \) and a metric on the disjoint union \( X = X_0 \cup X_1 \) which agrees with the corresponding product metrics on the ends. Let \( Y \) be oriented as the boundary of \( Z_0 \), so that \( X_0 \) has an end \( \mathbb{R}_+ \times Y \) and \( X_1 \) an end \( \mathbb{R}_+ \times (-Y) \). Gluing these two ends of \( X \) we obtain as in [5, Subsection 1.4] a manifold \( X(T) \) for each \( T > 0 \).

Choose smooth monopoles \( \alpha \) over \( Y \) and \( \alpha'_j \) over \( Y'_j \) (these are reducible, and unique up to gauge equivalence). Let \( \tilde{S}_o = (\tilde{A}_o, \tilde{\Phi}_o) \) be a reference configuration over \( X \) with these limits over the ends, and \( S'_o \) the associated reference configuration over \( X(T) \). Adopting the notation introduced in the beginning of [6, Subsection 2.2], let \( C \) be the corresponding \( L^1_{p,w} \) configuration space over \( X \) and \( C' \) the corresponding \( L^1_{p,w} \) configuration space over \( X(T) \). For any finite subset \( b \subset X_0 = Z_0 \cup Z_1 \) let \( G_b, G'_b \) be the corresponding groups of gauge transformations that restrict to 1 on \( b \).

As in Section 2 we first consider the monopole equations over \( X \) and \( X(T) \) perturbed only by means of a self-dual 2–form \( \mu = \mu_0 + \mu_1 \), where \( \mu_j \) is supported in \( Z_j \). The corresponding moduli spaces will be denoted \( M(X) \) and \( M(T) = M(X(T)) \). Of course, \( M(X) \) is a product of moduli spaces over \( X_0 \) and \( X_1 \):
\[
M(X) = M(X_0) \times M(X_1).
\]

By Lemma 2 we can choose \( \mu_0 \) such that \( M(X_0) \) consists only of the reducible point (which we denote by \( \omega_{\text{red}} = [A_{\text{red}}, 0] \)), and such that the operator
\[
D_{A_{\text{red}}} : L^1_{p,w}(S^+_{X_0}) \to L^1_{p,w}(S^-_{X_0})
\] (3)
is injective. By [5, Proposition 8.2] and unique continuation for self-dual closed 2–forms we can then choose \( \mu_1 \) such that
• $M(X_1)$ is regular and contains no reducibles, and
• the irreducible part of $M(T)$ is regular for all natural numbers $T$.

Set

$$k = -\text{ind}_C(D_{A_{\text{red}}}) \geq 0.$$  

If $k > 0$ then $\omega_{\text{red}}$ is not a regular point of $M(X_0)$ and we cannot appeal to the gluing theorem [6, Theorem 2.1] for describing $M(T)$ when $T$ is large. We will therefore introduce an extra parameter $z \in \mathbb{C}^k$ into the Dirac equation on $Z_0$, to obtain what we will call the “extended monopole equations”, such that $\omega_{\text{red}}$ becomes a regular point of the resulting parametrized moduli space over $X_0$. This will allow us to apply the gluing theorem for parametrized moduli spaces, [6, Theorem 5.1].

We are going to add to the Dirac equation an extra term $\beta(A, \Phi, z)$ which will be a product of three factors:

(i) a holonomy term $h_A$ (to achieve gauge equivariance)

(ii) a cut-off function $g(A, \Phi)$ (to retain an apriori pointwise bound on $\Phi$)

(iii) a linear combination $\sum z_j \psi_j$ of certain negative spinors (to make $\omega_{\text{red}}$ regular).

We will now describe these terms more precisely.

(i) Choose an embedding $f : \mathbb{R}^4 \to \text{int}(Z_0)$, and set $x_0 = f(0)$ and $U_0 = f(\mathbb{R}^4)$. For each $x \in U_0$ let $\gamma_x : [0,1] \to U_0$ be the path from $x_0$ to $x$ given by

$$\gamma_x(t) = f(tf^{-1}(x)).$$

For any spin$^c$ connection $A$ over $U_0$ define the function $h_A : U_0 \to U(1)$ by

$$h_A(x) = \exp \left( - \int_{[0,1]} \gamma_x^*(A - A_{\text{red}}) \right),$$

cf. [6] Equation 1. Note that $h_A$ depends on the choice of $A_{\text{red}}$, which is only determined up to modification by elements of $\mathcal{G}$.

(ii) Set $K_0 = f(D^4)$, where $D^4 \subset \mathbb{R}^4$ is the closed unit disk. Choose a smooth function $g : \mathcal{B}^*(K_0) \to [0,1]$ such that $g(A, \Phi) = 0$ when $\|\Phi\|_{L^\infty(K_0)} \geq 2$ and $g(A, \Phi) = 1$ when $\|\Phi\|_{L^\infty(K_0)} \leq 1$. Extend $g$ to $\mathcal{B}(K_0)$ by setting $g(A,0) = 1$ for all $A$.

(iii) By unique continuation for the formal adjoint $D_{A_{\text{red}}}^*$ there are smooth sections $\psi_1, \ldots, \psi_k$ of $S_{X_0}^-$ supported in $K_0$ and spanning a linear complement of the image of the operator $D_{A_{\text{red}}}$ in [6].
For any configuration \((A, \Phi)\) over \(X\) and \(z = (z_1, \ldots, z_k) \in \mathbb{C}^k\) define
\[
\beta(A, \Phi, z) = g(A, \Phi) h_A \sum_{j=1}^{k} z_j \psi_j.
\]

Note that for gauge transformations \(u\) over \(X\) one has
\[
u(x_0) h_u(A) = uh_A.
\]
Since \(g\) is gauge invariant, this yields
\[
\beta(u(A), u(\Phi), u(x_0) z) = u \cdot \beta(A, \Phi, z).
\]

The following lemma is useful for estimating the holonomy term \(h_A\):

**Lemma 3** Let \(a = \sum a_j dx_j\) be a 1-form on the closed unit disk \(D^n\) in \(\mathbb{R}^n\), \(n > 1\). For each \(x \in D^n\) let \(J(x)\) denote the integral of \(a\) along the line segment from \(0\) to \(x\), i.e
\[
J(x) = \sum_{j=1}^{n} x_j \int_{0}^{1} a_j(tx) \, dt.
\]
Then for any \(q \geq 1\) and \(r > qn\) and non-negative integer \(k\) there is a constant \(C < \infty\) independent of \(a\) such that
\[
\|J\|_{L^q(D^n)} \leq C \|a\|_{L^r(D^n)}.
\]

**Proof.** If \(b\) is a function on \(D^n\) and \(\chi\) the characteristic function of the interval \([0,1]\) then
\[
\int_{D^n} \int_{0}^{1} b(tx) \, dt \, dx = \int_{D^n} b(x) \int_{0}^{1} t^{-n} \chi(t^{-1}|x|) \, dt \, dx
\]
\[
= \frac{1}{n-1} \int_{D^n} (|x|^{1-n} - 1) b(x) \, dx.
\]
From this basic calculation the lemma is easily deduced. □

It follows from the lemma that \(a \mapsto h_{A,\omega(d+a)}\) defines a smooth map \(L^p_1(K_0; i\Lambda^1) \to L^q_1(K_0)\) provided \(p > 4q > 16\). Hence, if \(p > 16\) (which we henceforth assume) then
\[
\mathcal{C}(K_0) \times \mathbb{C}^k \to L^p(K_0; \mathbb{S}^-), \quad ((A, \Phi), z) \mapsto \beta(A, \Phi, z)
\]
is a smooth map whose derivative at every point is a compact operator. Here $\mathcal{C}(K_0)$ is the $L^p_1$ configuration space over $K_0$.

The extended monopole equations for $((A, \Phi), z) \in \mathcal{C} \times \mathbb{C}^k$ are

\begin{align*}
\hat{F}_A^+ + i\mu - Q(\Phi) &= 0, \\
D_A\Phi + \beta(A, \Phi, z) &= 0.
\end{align*}

(Cf. the holonomy perturbations of the instanton equations constructed in \cite{2} 2 (b).) We define actions of $G$ and $G'$ on $\mathcal{C} \times \mathbb{C}^k$ and $\mathcal{C}' \times \mathbb{C}^k$ respectively by

$$u(S, z) = (u(S), u(x_0)z).$$

Then the left hand side of (4) describes a $G$–equivariant smooth map $\mathcal{C} \times \mathbb{C}^k \to L^{p,w}$. For $\epsilon > 0$ let $B^{2k}_\epsilon \subset \mathbb{C}^k$ denote the open ball of radius $\epsilon$ about the origin, and $D^{2k}_\epsilon$ the corresponding closed ball. For $0 < \epsilon \leq 1$ set

$$\epsilon M_b(X) = \{\text{solutions } ((A, \Phi), z) \in \mathcal{C} \times B^{2k}_\epsilon \text{ to (4)}\} / G_b,$$

This moduli space is clearly a product of moduli spaces over $X_0$ and $X_1$:

$$\epsilon M_b(X) = \epsilon M_{b_0}(X_0) \times M_{b_1}(X_1),$$

where $b_j = b \cap X_j$.

Noting that the equations (4) also make sense over $X^{(T)}$ we define

$$\epsilon M^{(T)}_b = \{\text{solutions } ((A, \Phi), z) \in \mathcal{C}' \times B^{2k}_\epsilon \text{ to (4)}\} / G'_b.$$

We define $\epsilon M_b(X)$ and $\epsilon M^{(T)}_b$ in a similar way as $\epsilon M_b(X)$ and $\epsilon M^{(T)}_b$, but with $D^{2k}_\epsilon$ in place of $B^{2k}_\epsilon$.

Choose a base-point $x_1 \in Z_1$. We will only consider the cases when $b$ is a subset of $\{x_0, x_1\}$, and we indicate $b$ by listing its elements (writing $\epsilon M_{x_0,x_1}$ and $\epsilon M$ etc).

**Lemma 4** Any element of $^{1}M(X_0)$ or $^{1}M^{(T)}$ has a smooth representative.

**Proof.** Given Lemma 3 this is proved in the usual way. \[\square\]

**Lemma 5** There is a $C < \infty$ independent of $T$ such that $\|\Phi\|_\infty < C$ for all elements $[A, \Phi, z]$ of $^{1}M(X)$ or $^{1}M^{(T)}$. 

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Proof. Suppose $|\Phi|$ achieves a local maximum $\geq 2$ at some point $x$. If $x \not\in K_0$ then one obtains a bound on $|\Phi(x)|$ using the maximum principle as in [7 Lemma 2]. If $x \in K_0$ then the same works because then $g(A, \Phi) = 0$. \hfill \Box

**Lemma 6** $1M(X)$ and $1M(T)$ are compact for all $T > 0$.

Proof. Given Lemmas 3 and 5 the second approach to compactness in [5] carries over.

We identify $M_b(X)$ with the set of elements of $1M_b(X)$ with $z = 0$, and similarly for moduli spaces over $X, X^{(T)}$. It is clear from the definition of $\beta(A, \Phi, z)$ that $\omega_{\text{red}}$ is a regular point of $1M(X)$. Since $1M_{x_0}(X_0)$ has expected dimension 0, it follows that $\omega_{\text{red}}$ is an isolated point of $1M_{x_0}(X_0)$. Because $1M_{x_0}(X_0)$ is compact, there is an $\epsilon$ such that $\epsilon M_{x_0}(X_0)$ consists only of the point $\omega_{\text{red}}$. Fix such an $\epsilon$ for the remainder of the paper.

**Lemma 7** If $\omega_n \in \epsilon M(T_n)$ with $T_n \to \infty$ then a subsequence of $\{\omega_n\}$ chain-converges to $(\omega_{\text{red}}, \omega)$ for some $\omega \in M(X_1)$.

Proof. Again, this is proved as in [5] using the second approach to compactness. \hfill \Box

**Corollary 2** If $T \gg 0$ then $\epsilon M(T)$ contains no element which is reducible over $Z_1$. \hfill \Box

4 Applying the gluing theorem

Let $\text{Hol} = \text{Hol}_1$ be defined as in [6 Equation 1] in terms of a path in $X^{(T)}$ from $x_0$ to $x_1$ running once through the neck.

By [6] Proposition 2.3, if $K_1 = (X_1)_t$ with $t \gg 0$ then there is a $U(1)$–invariant open subset $V_1 \subset B_{x_1}^* (K_1) = B_{x_1} (K_1)$ containing $R_{K_1} (M_{x_1} (X_1))$, and a $U(1)$–equivariant smooth map

$$q_1 : V_1 \to M_{x_1} (X_1)$$

such that $q_1(\omega|_{K_1}) = \omega$ for all $\omega \in M_{x_1} (X_1)$. Here $R_{K_1}$ denotes restriction to $K_1$. It follows from Lemma 7 that if $T$ is sufficiently large then $\omega|_{K_1} \in V_1$ for all $\omega \in \epsilon M_{x_1}^{(T)}$. 

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Proposition 1. For all sufficiently large $T$ the moduli space $\mathcal{M}_{x_1}^{(T)}$ is regular and the map

$$\mathcal{M}_{x_1}^{(T)} \to M_{x_1}(X_1), \quad \omega \mapsto q_1(\omega|_{K_1}) \quad (5)$$

is an orientation preserving $U(1)$–equivariant diffeomorphism.

Proof. We will apply the version of [6, Theorem 5.1] with (in the notation of [6]) $T$ acting non-trivially on $W$. Set

$$G = \epsilon M_{x_0,x_1}(X) = \{\omega_{\text{red}}\} \times M_{x_1}(X_1),$$

$$K = K_0 \cup K_1,$$

$$V = \mathcal{B}_{x_0}(K_0) \times V_1 \times B^2_{\epsilon}.$$ 

Note that $G$ is compact and $\tilde{\mathcal{G}}_b(K) = \mathcal{G}_b(K)$. Define

$$q : V \to G, \quad (\omega_0, \omega_1, z) \mapsto (\omega_{\text{red}}, q_1(\omega_1)).$$

In general, an element $(u_0, u_1) \in U(1)^2$ acts on appropriate configuration and moduli spaces like any gauge transformation $u$ with $u(x_j) = u_j, \ j = 0, 1$, and it acts on $B^2_{\epsilon}$ by multiplication with $u_0$. Then clearly, $q$ is $U(1)^2$–equivariant, so by the gluing theorem there is a compact, codimension 0 submanifold $K' \subset X$ containing $K$ and a $U(1)^2$–equivariant open subset $V' \subset \mathcal{B}_{x_0,x_1}(K') \times B^2_{\epsilon}$ containing $R_{K'}(G)$ and satisfying $R_{K'}(V') \subset V$ and such that for all sufficiently large $T$ the space

$$G^{(T)} = \{(\omega, z) \in \epsilon M_{x_0,x_1}^{(T)} : (\omega|_{K'}, z) \in V'\}$$

consists only of regular points, and the map

$$G^{(T)} \to U(1) \times \epsilon M_{x_0,x_1}(X), \quad (\omega, z) \mapsto (\text{Hol}(\omega), (\omega_{\text{red}}, q_1(\omega|_{K_1}))) \quad (6)$$

is a $U(1)^2$–equivariant diffeomorphism. But it follows from Lemma [7] that $G^{(T)} = \epsilon M_{x_0,x_1}^{(T)}$ for $T \gg 0$, and dividing out by the action of $U(1) \times \{1\}$ in [6] we see that $\mathcal{M}^{(T)}$ is a $U(1)$–equivariant diffeomorphism.

We now discuss orientations. Given $\delta = \pm 1$ we will say a map is $\delta$–preserving if it changes orientations by the factor $\delta$. Set

$$c := b_1(X) + b_2^+(X).$$

By [6] Proposition 4.3 (ii) and Theorem 5.1 the map [6] is $(-1)^{c+1}$–preserving. Using [6] Proposition 4.4] it is a simple exercise to show that $\epsilon M_{x_0,x_1}(X) \to M_{x_1}(X_1)$ is $(-1)^c$–preserving. On the other hand, $(u_0, 1) \in U(1) \times \{1\}$ acts
on $U(1)$ in (6) by multiplication with $u_0^{-1}$. Thus we have got three signs, which cancel each other since $(c + 1) + c + 1$ is even. Therefore, the map (5) does preserve orientations.

Proof of Theorem 1 For large $T$ let $L \to \epsilon M(T)$ be the complex line bundle associated to the base-point $x_1$ as in Section 1. For $j = 1, \ldots, k$ the map

$$s_j : \epsilon M^{(T)}_{x_1} \to \mathbb{C}, \quad [A, \Phi, z] \mapsto \text{Hol}(A) \cdot z_j$$

is $U(1)$–equivariant in the sense of (1) and therefore defines a smooth section of $L$. The sections $s_j$ together form a section $s$ of the bundle $E = \oplus^k L$ whose zero set is the unparametrized moduli space $M^{(T)}$. It is easy to see that $s$ is a regular section precisely when $M^{(T)}$ is a regular moduli space, which by Corollary 2 and the choice of $\mu_1$ holds at least when $T$ is a sufficiently large natural number. In that case $s^{-1}(0) = M^{(T)}$ as oriented manifolds. Set

$$\ell = \frac{1}{2} \dim M^{(T)} \geq 0,$$

so that $\dim M(X_1) = 2(k + \ell)$. If $\ell$ is not integral then $\text{SW}(Z_1) = 0 = \text{SW}(Z)$ and we are done. Now suppose $\ell$ is integral and let $T$ be a large natural number. Choose a smooth section $s'$ of $E' = \oplus^\ell L$ such that $\sigma = s'|_{M^{(T)}}$ is a regular section of $E'|_{M^{(T)}}$, or equivalently, such that $s \oplus s'$ is a regular section of $E \oplus E' = \oplus^{k+\ell} L$. Then

$$\text{SW}(Z_1) = #(s \oplus s')^{-1}(0) = #\sigma^{-1}(0) = \text{SW}(Z),$$

where the first equality follows from Proposition 11 and # as usual means a signed count.

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