TWISTED EQUIVARIANT ELLIPTIC COHOMOLOGY FROM GAUGED PERTURBATIVE SIGMA MODELS I: FINITE GAUGE GROUPS

DANIEL BERWICK-EVANS

Abstract. We use the geometry of gauged 2|1-dimensional sigma models to construct cocycles for the twisted equivariant elliptic cohomology with complex coefficients of a smooth manifold with an action by a finite group. Our twists come from classical Chern-Simons theory of the group, and we construct induction functors using Freed-Quinn quantization of finite gauge theories.

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1. Introduction

In this paper we use the geometry of gauged 2|1-dimensional sigma models to construct twisted equivariant elliptic cohomology with complex coefficients for smooth manifolds equipped with an action by a finite group. This provides a differential geometric counterpart to the homotopical construction of J. Devoto [Dev96], the algebraic geometric construction of I. Grojnowski [Gro07], and the recent derived algebraic geometric construction of J. Lurie [Lur09]. We focus on twists coming from classes in $H^4(BG; \mathbb{Z})$, and explain their relationship to Chern-Simons theory for a finite group. One upshot is a construction of induction functors (following N. Ganter [Gan09]) using the quantization maps constructed by D. Freed and F. Quinn [FQ93] for finite gauge theories.

We will also present a parallel construction of twisted equivariant K-theory with complex coefficients using the geometry of gauged 1|1-dimensional sigma models. The induction functors defined using Freed-Quinn style quantization recover the standard Frobenius character formula for an induced (projective) representation. This lower-dimensional case is a bit simpler and helps clarify the more complicated geometry of the 2-dimensional construction.

The connection between gauged 2-dimensional field theories and equivariant elliptic cohomology goes back to G. Segal’s early investigations, e.g., see the discussion preceding Theorem 5.3 of [Seg88]. In view of the higher character theory of M. Hopkins, N. Kuhn and D. Ravenel [HKR00], the description of $G$-bundles on the torus as pairs of commuting elements in $G$ is a striking indication of a relationship between 2-dimensional field theory and elliptic cohomology. This paper demonstrates an explicit connection in a simple case.

1.1. Statement of results. In our prior construction of elliptic cohomology with complex coefficients ([BE13], Section 3) the main object of study was the stack of classical vacua, denoted $\Phi^2_0(X)$ consisting of pairs $(\Lambda, \phi)$ for $\Lambda \subset \mathbb{R}^2 \subset \mathbb{R}^{2|1}$ a lattice defining a super torus.
for some \( i \) stack is precisely the moduli stack of \( \text{bundle} \) over the moduli of \( G \).

Remark 1.1. We showed that these maps have classical action zero, and hence are an important object in the study of perturbative quantization (see also [Cos10, Cos11]).

There are three distinct ways to generalize \( \Phi_0^{21}(\lambda') \) for orbifold targets \( \lambda' \) depending on the factorization of the map \( \phi \):

\[
\begin{align*}
\Phi_0^{21}(\lambda') & \cong \mathbb{R}^{21}/\mathbb{Z}^2 \to \mathbb{R}^{01} \to \lambda', \\
\Phi_0^{21}(\lambda') & \cong \mathbb{R}^{21}/\mathbb{Z}^2 \to \mathbb{R}^{01}/\mathbb{Z} \to \lambda', \\
\Phi_0^{21}(\lambda') & \cong \mathbb{R}^{21}/\mathbb{Z}^2 \to \mathbb{R}^{01}/\mathbb{Z}^2 \to \lambda',
\end{align*}
\]

where we have identified \( \Lambda \cong \mathbb{Z}^2 \) via a choice of generators. When \( \lambda' = X/\!/G \), these three conditions are related to Borel equivariant elliptic cohomology over \( \mathbb{C} \), an elliptic version of delocalized equivariant K-theory, and equivariant elliptic cohomology over \( \mathbb{C} \) in the sense of Devoto [Dev96], respectively. We focus on the third choice, as it strictly contains the equivariant information of the first two, and denote by \( \Phi_0^{21}(\lambda') \) the stack consisting of pairs \( (\Lambda, \phi) \) where \( \phi \) is a map that factors according to the third option in (1).

Remark 1.2. Physically, this third choice is also the correct one. This leads to some desirable constructions such as induction functors in equivariant elliptic cohomology coming from quantization of gauge theories (see subsection 3.6). This construction is physically unmotivated and mathematically less interesting for the other choices.

Most of our computations will concern quotient orbifolds, \( \lambda' \simeq X/\!/G \). The stack \( \Phi_0^{21}(\star/\!/G) \) is essentially the moduli of \( G \)-bundles over spin elliptic curves. As such, it supports a few geometrically interesting line bundles. The first, denoted \( \omega^{1/2} \), pulls back along the functor \( \Phi_0^{21}(\star/\!/G) \to \Phi_0^{21}(\star) \) induced by the trivial homomorphism \( G \to \{e\} \), i.e., the functor that forgets the \( G \)-bundle. Over \( \Phi_0^{21}(\star), \omega^{1/2} \) is the line bundle whose fiber at a super torus \( T^{21}_\Lambda \) is the space of holomorphic sections of the square root of the canonical line of the reduced torus, \( |T^{21}_\Lambda| \), where the spin structure determines the square root.

The second class of line bundles over \( \Phi_0^{21}(\star/\!/G) \) comes from a construction of D. Freed and F. Quinn in their study of Chern-Simons theory of a finite group, whereby a correspondence defined by the projection and evaluation maps

\[
\begin{align*}
T^{21}_\Lambda \times \text{Map}(T^{21}_\Lambda, \star/\!/G) & \xrightarrow{p} \star/\!/G \\
\text{Map}(T^{21}_\Lambda, \star/\!/G) & \xrightarrow{\text{ev}} \star/\!/G
\end{align*}
\]

is used to transgress a normalized\(^2\) cocycle \( \alpha \) representing \( [\alpha] \in H^3(BG;U(1)) \) to a line bundle over the moduli of \( G \)-bundles on tori. By varying the lattice \( \Lambda \) above, the mapping stack is precisely the moduli stack of \( G \)-bundles on tori, and pulling back a 3-cocycle on \( \star/\!/G \) along the evaluation map, \( \text{ev} \), then pushing forward along the projection, \( p \), gives a 1-cocycle on this moduli stack, \( p_\star \circ \text{ev}^* \alpha \). This defines the line bundle we denote by \( \mathcal{CS}(\alpha) \), which is often called the Chern-Simons prequantum line bundle, whence our notation. For any \( G \)-

manifold \( X \), the canonical map \( X/\!/G \to \star/\!/G \) induces a functor \( \Phi_0^{21}(X/\!/G) \to \Phi_0^{21}(\star/\!/G) \) along which we may pullback the line bundle \( \mathcal{CS}(\alpha) \).

Our cocycles for twisted equivariant elliptic cohomology come from supersymmetric sections of \( \omega^{k/2} \otimes \mathcal{CS}(\alpha) \), denoted \( \Gamma_{\text{sup}}(\Phi_0^{21}(X/\!/G); \omega^{k/2} \otimes \mathcal{CS}(\alpha)) \). A section is supersymmetric when it extends to a section over a larger stack \( \Phi_0^{21}(X/\!/G) \supset \Phi_0^{21}(X) \) consisting of pairs \( (\Lambda, \phi) \) for \( \Lambda \subset \mathbb{R}^{21} \) an arbitrary (super) lattice and \( \phi: T^{21}_\Lambda \to X \) a map factoring through \( \mathbb{R}^{01}/\mathbb{Z}^2 \); see Definition 3.11.

\(^2\)A degree \( k \) cocycle \( \alpha: G \times \cdots \times G \to \mathbb{C}^\times \) is normalized if \( \alpha(g_1, \ldots, g_k) = 1 \) on all \( k \)-tuples for which \( g_i = e \) for some \( i \). Any cocycle is cohomologous to a normalized one.
Theorem 1.3. Let $G$ be a finite group acting on $X$ and $\alpha$ a normalized cocycle representative of $[\alpha] \in H^3(BG; U(1))$. There are natural isomorphisms of graded abelian groups

$$Ell^{\pm \alpha}_G(X) \cong \Gamma_{\text{usy}}(\Phi_{G/\Sigma}^{2\alpha}(X); \omega^{\pm} \otimes CS(\alpha))/\text{concordance}.$$ 

Tensor product of sections is compatible with the product structure on twisted equivariant elliptic cohomology, $Ell^{k+\alpha}_G(X) \times Ell^{l+\beta}_G(X) \to Ell^{k+l+\alpha+\beta}_G(X)$. 

In the above, $Ell^{\pm \alpha}_G(X)$ is an equivariant refinement of elliptic cohomology with complex coefficients that reduces to the theory defined by J. Devoto [Dev90] when $G$ has odd order and the twist is trivial; $Ell^{\pm \alpha}_G(\cdot)$ is the twisted coefficient ring over $\mathbb{C}$ defined by N. Ganter [Gan09] for all finite $G$ and normalized 3-cocycles $\alpha$. We give a self-contained definition of $Ell^{\pm \alpha}_G(X)$ in Section 3.1 which is rather straightforward given our desire to work with complex coefficients.

Remark 1.4. Our construction of twisted equivariant elliptic cohomology with complex coefficients depends on the orbifold underlying $X//G$ and the 2-gerbe representing a torsion class $[\alpha] \in H^3(X; U(1))$. Thus, one can also view Theorem 1.3 as justification for a candidate definition of $\alpha$-twisted orbifold elliptic cohomology, or—in fancier language—the elliptic cohomology of certain smooth 2-stacks.

1.2. An equivariant refinement of the Stolz-Teichner conjecture. J. Lurie has highlighted [Lur09] a remarkable feature the universal elliptic cohomology theory of topological modular forms (TMF) is expected to possess that he calls $\mathbb{E}$-equivariance. Very roughly, one expectation is that for nice enough 2-stacks one can define an equivariant version of TMF, much in the same way that for nice enough stacks one can define an equivariant version of K-theory. Lurie goes on to explain how this $\mathbb{E}$-equivariance is at the heart of many salient features of TMF, such as the string orientation and chromatic height 2 phenomena. Any reasonable geometric model for TMF should have similar equivariant features. In this subsection we describe one possible such geometric model gotten from a modest generalization of the main conjecture of Stolz and Teichner (compare their discussion of equivariant field theories in [ST11], Section 1.7) following the initial work of Segal and Witten [Wit87, Seg88].

The (non-equivariant) Stolz-Teichner conjecture concerns a category of degree $k$, 2|1-dimensional Euclidean field theories over a fixed smooth manifold $X$. The ultimate definition of this category remains under investigation, but the three main ingredients are: (1) a source 3-category $2|1$-$\text{EBord}(X)$ with $m$-morphisms $m$|1-dimensional Euclidean manifolds with a map to $X$ for $m = 0, 1, 2$ and 3-morphisms isometries of 2|1-dimensional Euclidean manifolds; (2) a target 3-category $\text{Alg}$ that deloops the bicategory of topological algebras, bimodules and bintertwiners; and (3) twist functors $T^{\otimes k}: 2|1$-$\text{EBord}(X) \to \text{Alg}$ generated by tensor powers of a fixed symmetric monoidal functor $T$ constructed out of the free fermion field theory. These ingredients have been carefully constructed as bicategories [ST11], and it remains to pin down an appropriate 3-categorical refinement. For example, Douglas and Henriques are currently pursuing an option for $\text{Alg}$ in the language of conformal nets [DH11].

With these caveats in place, a degree $k$, 2|1-dimensional Euclidean field theory is a symmetric monoidal natural transformation from $T^{\otimes k}$ to the constant functor $1: 2|1$-$\text{EBord}(X) \to \text{Alg}$. Letting $2|1$-$\text{EFT}^k(X)$ denote the category of such natural transformations, the conjecture is

$$\text{TMF}^k(X) \cong 2|1$-$\text{EFT}^k(X)/\text{concordance}$$

(conjectural).

All of these ingredients make sense when one replaces $X$ by the quotient groupoid $X//G^T$ for $G$ a compact Lie group acting on $X$, where maps $S \to X//G^T$ are principal $G$-bundles with connection over $S$ whose total space is equipped with a $G$-equivariant map to $X$. Roughly, this bordism category encodes (a higher-categorical version of) the fields of the gauged 2|1-dimensional sigma model with target $X$ with its $G$-action. When $X = *$ this is a pure gauge theory, and we claim there is a new supply of twist functors associated to any choice of level $[\alpha] \in H^3(BG; \mathbb{Z})$. These are the composition of the forgetful functor from super Euclidean bordisms to (spin) topological bordisms, $2|1$-$\text{EBord}(*//G^T) \to$
2-Bord(*//G\text{\textsuperscript{\text{\textdegree}}}), with the functor $\mathcal{CS}(\alpha): 2\text{-Bord}(*//G\text{\textsuperscript{\text{\textdegree}}}) \to \text{Alg}$ gotten from restriction of classical Chern-Simons theory for the group $G$ and level $\alpha$ to manifolds of dimension $\leq 2$. Precomposition with the functor $2|1\text{-EBord}(X//G\text{\textsuperscript{\text{\textdegree}}}) \to 2|1\text{-EBord}(*//G\text{\textsuperscript{\text{\textdegree}}})$ induced by the canonical map $X \to *$ promotes these to twist functors also denoted $\mathcal{CS}(\alpha)$ with source $2|1\text{-EFT}(X//G\text{\textsuperscript{\text{\textdegree}}})$ for any $X$ with $G$-action. Define the category of $\alpha$-twisted field theories of degree $k$ as the category of natural transformations $F$,

\[
\begin{array}{c}
2|1\text{-EBord}(X//G\text{\textsuperscript{\text{\textdegree}}}) \downarrow F \\
\text{Alg}
\end{array}
\]

\[\Downarrow T^k \otimes \mathcal{CS}(\alpha)\]

Remark 1.5. The above definition can be extended to field theories over a proper stack $X$ with a choice of cocycle representing a class in $H^3(X; U(1))$: Hepworth has shown that locally $X$ is the quotient of a manifold by a compact Lie group [Hep09], and a degree 3 cocycle defines a 3-dimensional topological field theory (related to classical Chern-Simons theory) whose restriction to bordisms of dimension $\leq 2$ defines a twist.

This leads us to hope for an isomorphism

$\text{TMF}_G^{k+\alpha}(X) \cong 2|1\text{-EFT}^{k+\alpha}(X//G\text{\textsuperscript{\text{\textdegree}}})/\text{concordance}$ (conjectural)

for a suitable equivariant refinement of TMF. Assuming the Stolz-Teichner conjecture (or some variant thereof), one might even be tempted to define equivariant TMF as above. To give credence to this suggestion, in this paper we essentially compute the restriction of the right hand side to closed 2|1-dimensional bordisms in the case that $G$ is finite. More precisely, there is a restriction map

$2|1\text{-EFT}^{*,\alpha}(X//G) \to \Gamma_{\text{susy}}(\Phi_0^{2|1}(X//G); \omega^{*2} \otimes \text{CS}(\alpha)),$

that evaluates a field theory on the super tori over $X//G$—viewed as bordisms from the empty set to itself—that comprise $\Phi_0^{2|1}(X//G)$; see [BE13], Section 1.4 for a detailed discussion of the nonequivariant case, which immediately generalizes. In this way Theorem 1.3 establishes a link between 2|1-EFTs over quotient stacks and twisted equivariant elliptic cohomology.

There is also an expected type of Mackey structure on $\alpha$-twisted field theories: naturality produces restriction functors, and we expect quantization of gauge theories to produce induction functors. In the case that $G$ is finite, aspects of this structure were observed by N. Ganter [Gan13] using the Freed-Quinn quantization maps for finite gauge theory. We explain this type of induction in the context of Theorem 1.3 in Section 3.6.

Remark 1.6. A 2-group is a symmetric monoidal groupoid whose monoidal structure has (weak) inverses; these are also called categorical groups. Sinha [Sin75] showed that any essentially finite 2-group is determined up to isomorphism by a finite group $G$ and a normalized cocycle representing a class in $H^3(BG; U(1))$. The cocycle condition guarantees that the pentagon axiom is satisfied, and the cocycle being normalized implies the triangle axioms. This gives a way of interpreting our construction of twisted equivariant elliptic cohomology in terms of equivariance with respect to a 2-group.

1.3. Notation and conventions. A supermanifold will be a manifold with a sheaf of $\mathbb{Z}/2$-graded $\mathcal{C}$-algebras locally isomorphic to functions on $\mathbb{R}^n$ tensored with an exterior algebra. Global sections of these sheaves are denoted $C^\infty(M)$ for a supermanifold $M$. Morphisms $M \to N$ between supermanifolds are $\mathbb{Z}/2$-graded algebra homomorphisms $C^\infty(N) \to C^\infty(M)$. This comprises the category of cs-manifolds in the survey [DM99]; apart from this altered terminology, our notation and conventions agree with theirs. A generalized (super) manifold is a presheaf of sets on the category of (super) manifolds and smooth maps.
A vector bundle over a supermanifold $M$ is a finitely generated projective module over the structure sheaf $C^\infty(M)$. There is an annoying subtlety illustrated by the following example: sections of the trivial line bundle over a supermanifold $S$ differ from sections of the projection $S \times \mathbb{C} \to S$. However, there is a map between these two reasonable notions of section: given a morphism $S \to \mathbb{C}$, we can pull back the function $z \in C^\infty(\mathbb{C})$ to one on $S$, and identify this function with an element of the trivial rank one module over $C^\infty(S)$, i.e., a section of the trivial line on $S$. Similarly, isomorphisms of the trivial line bundle over $S$ are elements of $C^\infty(S)^\times$, and these differ from maps $S \to \mathbb{C}^\times$. Again there is a map gotten by pulling back $z \in C^\infty(\mathbb{C}^\times)$, but it does not induce a bijection. In fact, the assignment $S \mapsto C^\infty(S)^\times$ is not even a representable supermanifold! When defining line bundles over supermanifolds, we will freely use the above maps without comment.

Many of our constructions take place in the category of (smooth) stacks, denoted $\text{SmSt}$. A stack will mean a presheaf of groupoids on the site of super manifolds satisfying descent. Throughout, $S$ will denote a test supermanifold. The $S$-points of a stack therefore form a groupoid natural and local in $S$. Any (generalized) super Lie groupoid determines a smooth stack, and we will freely make this identification without comment; see [Bor08] for an account of this perspective. An orbifold will be a stack for which there exists a representing Lie groupoid that is proper and étale. For stacks $\mathcal{Y}$ and $Z$, $\text{SmSt}(\mathcal{Y}, Z)$ is the mapping stack that assigns to $S$ the groupoid $\text{SmSt}(S \times \mathcal{Y}, Z)$.

All the relevant geometry on supermanifolds will be defined in terms of rigid geometries in the spirit of F. Klein. This requires that we specify a model super space $\mathbb{M}$ and a super group of isometries, $\text{Iso}(\mathbb{M})$. From this data one can define $S$-families of super manifolds with the given model geometry and isometries of a family over $S$, i.e., a stack of manifolds with model geometry $\mathbb{M}$. See [HST10] Section 6.3 relevant details. Presently we shall define the two model geometries relevant to this paper.

Let $\mathbb{R}^{1|1}$ denote the Lie super group with multiplication

$$(t, \theta) \cdot (t', \theta') = (t + t' + i\theta \theta', \theta + \theta'), \quad (t, \theta), (t', \theta') \in \mathbb{R}^{1|1}(S).$$

Define an action of $\mathbb{Z}/2 = \{ \pm 1 \}$ on $\mathbb{R}^{1|1}$ by the reflection

$$(t, \theta) \mapsto (t, \pm \theta), \quad (t, \theta) \in \mathbb{R}^{1|1}(S).$$

This defines the group $\mathbb{R}^{1|1} \times \mathbb{Z}/2$. The left action of $\text{Euc}(\mathbb{M}^{1|1}) := \mathbb{R}^{1|1} \times \mathbb{Z}/2$ on $\mathbb{M}^{1|1} := \mathbb{R}^{1|1}$ defines a 1|1-dimensional Euclidean model space, where the isometries of $\mathbb{R}^{1|1}$ are the supergroup $\mathbb{R}^{1|1} \times \mathbb{Z}/2$. Next consider $\mathbb{R}^{2|1}$ with multiplication

$$(z, \bar{z}, \theta) \cdot (z', \bar{z}', \theta') = (z + z', \bar{z} + \bar{z}' + i\theta \theta', \theta + \theta'), \quad (z, \bar{z}, \theta), (z', \bar{z}', \theta') \in \mathbb{R}^{2|1}(S)$$

where we have identified $\mathbb{R}^2 \cong \mathbb{C}$, and a map $S \to \mathbb{R}^2 \cong \mathbb{C}$ allows us to pull back the functions $z, \bar{z} \in C^\infty(\mathbb{C})$ to $S$. The map $S \to \mathbb{R}^2$ is in fact determined by the pullback of these functions, which in an abuse of notation we again denote by $z$ and $\bar{z}$. We caution the reader that $z$ and $\bar{z}$ are not related by complex conjugation in $C^\infty(S)$, since in general such a star structure only exists on the reduced manifold of $S$. There is an action of $\mathbb{C}^\times \cong \text{Spin}(2) \times \mathbb{R}^\times$ on $\mathbb{R}^{2|1}$ by

$$(q, \bar{q}) \cdot (z, \bar{z}, \theta) = (q^2 z, q^2 \bar{z}, q \theta), \quad (q, \bar{q}) \in \mathbb{C}^\times(S), \quad (z, \bar{z}, \theta) \in \mathbb{R}^{2|1}(S),$$

where, again, $(q, \bar{q}) \in \mathbb{C}^\times(S) \subset C^\infty(S)$ are the pullbacks of coordinates on $\mathbb{C}^\times$ under a map $S \to \mathbb{C}^\times$. From this we form the supergroup $\text{CEuc}(\mathbb{M}^{2|1}) := \mathbb{R}^{2|1} \times \mathbb{C}^\times$ which acts on $\mathbb{M}^{2|1} := \mathbb{R}^{2|1}$ and defines a 2|1-dimensional rigid conformal model space.

We follow the usual convention in geometry that symmetry groups act on the left. This has an important consequence in the case of a mapping space $\text{Map}(\Sigma, X)$: a diffeomorphism $f: \Sigma \to \Sigma$ acts on $\phi \in \text{Map}(\Sigma, X)$ by $\phi \mapsto \phi \circ f^{-1}$.

1.4. Acknowledgements. It is a pleasure to thank Ralph Cohen, Kevin Costello, Chris Douglas, Owen Gwilliam, André Henriques, Dmitri Pavlov, Nat Stapleton, Stephan Stolz and Peter Teichner for helpful conversations during the development of this work.
2. Warm-up: twisted equivariant K-theory over $\mathbb{C}$

In this section we prove an analogous result to Theorem 1.3 for twisted equivariant K-theory. Starting with the 1|1-dimensional gauged sigma model with target an orbifold $\mathcal{X}$, we define a stack of classical vacua, denoted $\Phi^{|1|}_{0}(\mathcal{X})$, that consists of super Euclidean circles equipped with a map to $\mathcal{X}$ that factors through $\mathbb{R}^{0|1}//\mathbb{Z}$. We construct a sequence of line bundles $\kappa^\beta$ over this stack, and in the case that $\mathcal{X} = X//G$, we consider lines $T(\beta)$ for each normalized 2-cocycle $\beta: G \times G \to U(1)$. We compute the space of supersymmetric sections (see Definition 2.14) denoted $\Gamma_{\text{susy}}(\Phi^{|1|}_{0}(X//G); \kappa^\beta \otimes T(\beta))$.

**Theorem 2.1.** There is a natural isomorphism of graded abelian groups

$$K^*_{\beta}(X) \otimes \mathbb{C} \cong \Gamma_{\text{susy}}(\Phi^{|1|}_{0}(X//G); \kappa^* \otimes T(\beta))/\text{concordance}$$

where the left hand side denotes the $\beta$-twisted $G$-equivariant K-theory of $X$ tensored with $\mathbb{C}$. Furthermore, the tensor product of line bundles is compatible with the product structure on twisted equivariant K-theory.

### 2.1. The delocalized Chern character in twisted equivariant K-theory

For $G$ a finite group acting on a compact manifold $X$ and $\beta: G \times G \to U(1)$ a normalized 2-cocycle, let $K^\beta_{G}(X)$ denote the $\beta$-twisted $G$-equivariant K-theory of $X$. The **delocalized Chern character** is the natural map

$$K^\beta_{G}(X) \to K^\beta_{G}(X) \otimes \mathbb{C}.$$ 

An explicit description of the codomain of the delocalized Chern character in the case that $G$ is a finite group was given by Adem and Ruan (see also the result of Freed, Hopkins and Teleman [[FHT08], Proposition 3.11]) generalizing the untwisted calculation by Atiyah and Segal ([AS89], Theorem 2).

**Theorem 2.2** ([AR03] Theorem 7.4). Let $\beta: G \times G \to U(1)$ be a normalized 2-cocycle. For a manifold $X$ with $G$-action, the $\beta$-twisted $G$-equivariant K-theory of $X$ with complex coefficients can be computed as

$$K^\beta_{G}(X) \otimes \mathbb{C} \cong \bigoplus_{[g]} \left( \text{H}^{\text{ev/odd}}_{dR}(X^g) \otimes \chi^\beta_{G}(g) \right)$$

where the sum ranges over conjugacy classes of $g \in G$, $C_{G}(g)$ denotes the centralizer of $g$, and $\chi^\beta_{G}$ denotes the 1-dimensional representation of $C_{G}(g)$ given by $h \mapsto \beta(h, g)\beta(g, h)^{-1}$.

By the Atiyah-Segal completion theorem [AS69], the Chern character with codomain Borel equivariant de Rham cohomology first completes $K^\beta_{G}(X)$ at the augmentation ideal and then tensors with $\mathbb{C}$. The map from the delocalized Chern character to the Borel equivariant Chern character projects to the summand indexed by $[e]$ above. In particular, the Borel equivariant target is oblivious to twistings.

### 2.2. The stacks $\Phi^{|1|}_{0}(X//G)$ and $\Phi^{|1|}_{e}(X//G)$

The projection $\mathbb{R} \to *$ gives a morphism of Lie groupoids, and hence stacks:

$$\mathbb{R}//(R \cdot \mathbb{Z}) \xrightarrow{\mathbb{R}} \mathbb{R}//(R \cdot \mathbb{Z}) \to *//(R \cdot \mathbb{Z}) \cong *//\mathbb{Z},$$

where $R \in \mathbb{R}_{>0} \subset \mathbb{R}^{|1|}_{0}$. This induces a morphism of mapping stacks

$$\text{SmSt}(*/\mathbb{Z}, \mathcal{X}) \to \text{SmSt}(\mathbb{R}//(R \cdot \mathbb{Z}), \mathcal{X}),$$

In the case that $\mathcal{X} = X$ is an ordinary manifold, this is precisely the inclusion of the constant loops into the free loop space of of $X$.

**Definition 2.3.** The stack $\text{SmSt}(*/\mathbb{Z}, \mathcal{X})$ is the **the inertia stack** of $\mathcal{X}$, denoted $\mathcal{I}\mathcal{X}$. We observe that $\mathcal{I}\mathcal{X}$ is again an orbifold.
Consider the composition of morphisms of stacks

\[(S \times \mathbb{R}^{1|1})/(R \cdot \mathbb{Z}) \overset{\cong}{\to} (S \times \mathbb{R}^{1|1})/(R \cdot \mathbb{Z}) \to (S \times \mathbb{R}^{0|1})/(R \cdot \mathbb{Z}) \cong (S \times \mathbb{R}^{0|1})/\mathbb{Z}\]

where the second arrow is induced by the projection \(\mathbb{R}^{1|1} \to \mathbb{R}^{0|1}\). We denote this composition by \(\text{proj}: S \times_\mathbb{R} \mathbb{R}^{1|1} \to (S \times \mathbb{R}^{0|1})/\mathbb{Z}\).

**Definition 2.4.** For \(\mathcal{X}\) an orbifold, the stack of classical vacua over \(\mathcal{X}\), denoted \(\Phi_0^{1|1}(\mathcal{X})\) is the stack associated to the prestack whose objects over \(S\) consist of pairs \((R, \phi)\) where \(R \in \mathbb{R}_{>0}\) determines a family of super circles \((S \times \mathbb{R}^{1|1})/(R \cdot \mathbb{Z})\) and \(\phi\) is an object in the groupoid of maps \(S \times_\mathbb{R} \mathbb{R}^{1|1} \to \mathcal{X}\) with a chosen factorization \(\phi = \phi_0 \circ \text{proj}\). Morphisms over \(S\) consist of 2-commuting triangles

\[
\begin{array}{ccc}
S \times_\mathbb{R} \mathbb{R}^{1|1} & \overset{\cong}{\sim} & S \times_\mathbb{R} \mathbb{R}^{1|1} \\
\phi & \Downarrow & \phi' \\
\mathcal{X} & \end{array}
\]

where the horizontal arrow is an isomorphism of \(S\)-families of super Euclidean 1|1-manifolds.

**Remark 2.5.** We emphasize that the 2-commutative property in the above is extra data: maps into \(\mathcal{X}\) form a groupoid, and we require the maps comprising the triangle to be isomorphic with a specified isomorphism. For example, when the horizontal arrow is the identity super Euclidean isometry, the 2-commuting data is an isomorphism of the maps \(\phi\) and \(\phi'\) in the groupoid of maps \(S \times_\mathbb{R} \mathbb{R}^{1|1} \to \mathcal{X}\).

In order to define supersymmetric sections over \(\Phi_0^{1|1}(\mathcal{X})\) as in the statement of Proposition 2.1, we require a stack \(\Phi_e^{1|1}(\mathcal{X})\) consisting of arbitrary supercircles with maps to \(\mathcal{X}\) that factor through \(\mathbb{R}^{0|1}/\mathbb{Z}\). To this end, we define the map

\[\text{proj}_R: \mathbb{R}^{1|1} \times S \to \mathbb{R}^{0|1} \times S, \quad \text{proj}_R(t, \theta, s) = (\theta - \frac{t}{\rho R}, s).\]

Direct computation shows that the map \(\text{proj}_R\) is invariant under the action of \(R \cdot \mathbb{Z}\), so defines a morphism of stacks \((S \times \mathbb{R}^{1|1})/(R \cdot \mathbb{Z}) \to (S \times \mathbb{R}^{0|1})/\mathbb{Z}\), which we also call \(\text{proj}_R\).

**Definition 2.6.** Let \(\Phi_e^{1|1}(\mathcal{X})\) denote the stack associated to the prestack whose objects over \(S\) consist of pairs \((R, \phi)\) where \(R \in \mathbb{R}_{>0}(S)\) and \(\phi\) is an object in the groupoid of maps \(S \times_\mathbb{R} \mathbb{R}^{1|1} \to \mathcal{X}\) with a chosen factorization \(\phi = \phi_0 \circ \text{proj}_R\). The morphisms in this groupoid are 2-commuting triangles

\[
\begin{array}{ccc}
S \times_\mathbb{R} \mathbb{R}^{1|1} & \overset{\cong}{\sim} & S \times_\mathbb{R} \mathbb{R}^{1|1} \\
\phi & \Downarrow & \phi' \\
\mathcal{X} & \end{array}
\]

where the horizontal arrow is an isomorphism of \(S\)-families of super Euclidean 1|1-manifolds.

### 2.3. Groupoid presentations.

**Definition 2.7.** A surjective map of stacks is a morphism of stacks that on \(S\)-points induces an essentially surjective, full morphism of groupoids.

**Proposition 2.8.** Choose a groupoid presentation of \(\mathcal{I}\mathcal{X}\) with objects \(\text{Ob}(\mathcal{I}\mathcal{X})\) and morphisms \(\text{Mor}(\mathcal{I}\mathcal{X})\). There is a surjective morphism of stacks

\[
\left( \begin{array}{c}
\mathbb{R}_{>0}^{1|1} \times \text{Euc}(\mathbb{R}^{1|1}) \times \pi T(\text{Mor}(\mathcal{I}\mathcal{X})) \\
\mathbb{R}_{>0}^{1|1} \times \pi T(\text{Ob}(\mathcal{I}\mathcal{X}))
\end{array} \right) \to \Phi_e^{1|1}(\mathcal{X}).
\]
Proof. Factoring through $\text{proj}_R$ means the stack $\Phi^1_{\epsilon}(\mathcal{X})$ is equivalent to one whose objects over $S$ are pairs $(R, \phi_0)$ for $R \in \mathbb{R}^{11}_{>0}$ and $\phi_0$ an object in the groupoid of maps $S \times \mathbb{R}^{01}///\mathbb{Z} \to \mathcal{X}$, where the $\mathbb{Z}$-action on $S \times \mathbb{R}^{01}$ is trivial. The equivalence $(S \times \mathbb{R}^{01})///\mathbb{Z} \simeq (S \times \mathbb{R}^{01}) \times (*///\mathbb{Z})$ along with the hom-tensor adjunction gives an equivalence,

$$\text{SmSt}(S \times \mathbb{R}^{01}//\mathbb{Z}, \mathcal{X}) \simeq \text{SmSt}(S \times \mathbb{R}^{01}, \text{SmSt}(*///\mathbb{Z}, \mathcal{X})) \simeq \text{SmSt}(S, \pi T(\mathcal{X})).$$

An object of $\Phi^1_{\epsilon}(\mathcal{X})$ over $S$ can therefore be identified with the pair $(R, \phi_0)$ for $R \in \mathbb{R}^{11}_{>0}(S)$ and $\phi_0 \in \pi T(\text{Ob}(\mathcal{X}))(S)$. This gives the supermanifold of objects claimed in the proposition.

The morphisms in $\Phi^1_{\epsilon}(\mathcal{X})$ consist of pairs $(F, h)$ where $F \colon S \times_R \mathbb{R}^{11} \rightarrow S \times_R' \mathbb{R}^{11}$ is a family of super Euclidean isometries and $h \colon \phi \sim \phi' \circ F$ is an isomorphism in the groupoid of maps $S \times_R \mathbb{R}^{11} \rightarrow \mathcal{X}$. Choosing a lift of $F$ to the universal cover gives a super Euclidean isometry $\tilde{F} \colon S \times \mathbb{R}^{11} \rightarrow S \times \mathbb{R}^{11}$, so $\tilde{F} \in \text{Euc}(\mathbb{R}^{11})(S) \cong (\mathbb{R}^{11} \times \mathbb{Z}/2)(S)$. Conversely, any super Euclidean isometry $\tilde{F} = (u, \nu, \pm 1) \in (\mathbb{R}^{11} \times \mathbb{Z}/2)(S)$ descends to an isometry $F \colon S \times_R \mathbb{R}^{11} \rightarrow S \times_R' \mathbb{R}^{11}$, since the left action by isometries commutes with the right $R \cdot \mathbb{Z}$-action. However, this isometry changes the family of basepoints so conjugates $R$; a simple computation finds that $R' = (r \pm 2i\nu \rho, \pm \rho)$ for $R = (r, \rho)$. Hence there is a surjection on hom sets over $S$ sending the pair $(\tilde{F}, h)$ to $(F, h)$.

To connect this observation to the claimed groupoid presentation, we identify the pair $(\tilde{F}, h)$ with a pair $(F_0, h_0)$ for $F_0 \colon S \times \mathbb{R}^{01} \rightarrow S \times \mathbb{R}^{01}$ and $h_0 \colon \phi_0 \sim \phi'_0 \circ F_0$ where $\phi_0, \phi'_0 \colon S \times \mathbb{R}^{01} \rightarrow \text{Ob}(\mathcal{X}(S))$. By the discussion in the first paragraph of the proof, $h$ is determined by a morphism in the groupoid of maps $S \times \mathbb{R}^{01} \rightarrow \mathcal{X}$, and since $S \times \mathbb{R}^{01}$ is a discrete groupoid this is locally determined by a map $h_0 \colon S \times \mathbb{R}^{01} \rightarrow \text{Mor}(\mathcal{X})$, i.e., an $S$-point of $\pi T(\text{Mor}(\mathcal{X}))$, recovering a piece of the morphism super manifold claimed in the proposition. Abstractly, this proves we obtain a surjection as in the statement of the proposition, but it will be useful to characterize the morphisms between objects coming from super Euclidean isometries.

We shall define $F_0$ as the dotted arrow

$$\begin{array}{ccc}
\mathbb{R}^{11} \times S & \xrightarrow{id \times \text{proj}_R} & \mathbb{R}^{01} \times S \\
\tilde{F} & \downarrow & \tilde{F}_0 \\
\mathbb{R}^{11} \times S & \xrightarrow{\text{proj}_{R'}} & \mathbb{R}^{01} \times S \\
\end{array}$$

that makes the square commute. For $\tilde{F} = (u, \nu, \pm 1)$ and $\theta$ a coordinate on $\mathbb{R}^{01}$, direct computation shows

$$\theta \mapsto \left( \pm 1 \left( \theta + \nu - \frac{u + i\theta \nu}{\rho} \right), s \right), \quad \tilde{F} = (u, \nu, \pm 1) \in \text{Euc}(\mathbb{R}^{11})(S)$$

is the unique such map. This provides the desired translation between super Euclidean isometries and morphisms $(R, \phi_0) \rightarrow (R', \phi'_0)$, competing the proof. 

Upon restriction to $\Phi^1_{\epsilon}(\mathcal{X})$, we obtain a similar result.

**Proposition 2.9.** There is a surjection of stacks

$$\begin{array}{c}
\left( \begin{array}{c}
\mathbb{R}_{>0} \times \mathbb{R}^{11} \times \mathbb{Z}/2 \times \pi T(\text{Mor}(\mathcal{X})) \\
\mathbb{R}_{>0} \times \pi T(\text{Ob}(\mathcal{X}))
\end{array} \right) \\
\downarrow \\
\mathbb{R}_{>0} \times \mathbb{R}^{01} \times \pi T(\text{Ob}(\mathcal{X}))
\end{array} \rightarrow \Phi^1_{\epsilon}(\mathcal{X}).$$

**Remark 2.10.** The failure of the maps in the above pair of propositions to induce equivalences arises from the possible kernel of the map from $\text{Euc}(\mathbb{R}^{11})(S)$ to the super Euclidean isometries of a given family of super circles. However, this kernel is always discrete and we will only be concerned with objects invariant under the $\mathbb{R}^{11}$-action, so working with the groupoid in Proposition 2.8 will do the job. We say a bit more about this discrete stabilizer in Remark 2.11 below.
In the case that $\mathcal{X} = X//G$, the inertia stack has groupoid presentations,

\begin{equation}
\mathcal{I}(X//G) := \text{SmSt}(\ast//\mathbb{Z}, X//G) \cong \left( \prod_{g \in G} X^g \right)//G \cong \prod_{[g]} X^g//C_G(g)
\end{equation}

where $G$ acts by conjugation on the fixed point sets in the first presentation, and the coproduct in the second presentation is indexed by conjugacy classes. Plugging these presentations into the previous propositions will be our main tool in computations.

Remark 2.11. One way to specify the data of an $S$-family of loops in $X//G$ is a pair $\gamma : S \times \mathbb{R} \to X$ and $g \in G$ such that $g \cdot \gamma(t) = \gamma(t + 1)$. For a fixed $g$, there is an action on the set of $S$-families of paths $(\gamma, g)$ by the pushout in groups

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{g} & C_G(g) \\
\downarrow & & \downarrow \\
\mathbb{R} & \to & \text{Aut}(g\text{-paths})
\end{array}
\]

where the map $g$ sends $1 \in \mathbb{Z}$ to $g \in C_G(g)$. When $G$ is the trivial group this pushout is $\mathbb{R}/\mathbb{Z} \cong S^1$, which acts by rotation of loops. A similar discussion applies to the case of super loops in $X//G$. Below, categorical consideration will cause us to consider objects that are strictly invariant under the action of (super) loop rotation, so this subtlety doesn’t feature. If one wishes to consider objects that are only invariant up to isomorphism, a more careful analysis along the lines of Nat Stapleton’s work on transchromatic character maps \cite{Sta13} is required.

The diagram \[\begin{array}{c}
\end{array}\] and the displayed equation immediately following it define an action of the super Euclidean group, $\text{Euc}(\mathbb{R}^{1|1}) \cong \mathbb{R}^{1|1} \times \mathbb{Z}/2$, on $\mathbb{R}^{1|1}_2 \times \pi TX$ that sends $R \mapsto R'$ and acts on $\pi TX \cong \text{SMid}(\mathbb{R}^{0|1}, X^g)$ by precomposing with the action on $\mathbb{R}^{0|1}_2$ defined by $F_0$. It will be helpful below to have an explicit formula for this action on $C^\infty(\mathbb{R}^{1|1}_2 \times \pi TX^g)$ for each $g \in G$. The $\mathbb{Z}/2$-action is $f(r, \rho) \otimes \alpha \mapsto f(r, -\rho) \otimes (-1)^{\text{deg}(\alpha)} \alpha$ for $(r, \rho)$ coordinates on $\mathbb{R}^{1|1}_2$ and $\alpha \in \Omega^\bullet(X^g) \cong C^\infty(\pi TX^g)$. Since the Lie algebra of $\mathbb{R}^{1|1}$ is free on a single odd generator, the action of $(u, \nu) \in \mathbb{R}^{1|1}(S) < \text{Euc}_{S}(\mathbb{R}^{1|1})$ is given by $\exp(\nu \partial + Q)$; a tedious but straightforward computation (see \cite{BE13}, Section 2.5) gives the following characterization.

**Lemma 2.12.** The action of an $S$-family of super Euclidean isometries of $\mathbb{R}^{1|1}$ on $C^\infty(\mathbb{R}^{1|1}_2 \times \pi TX^g) \cong (C^\infty(\mathbb{R}^{1|1}_2) \rho) \otimes \Omega^\bullet(X^g)$ is determined by the infinitesimal generator

\[
Q := 2\rho \frac{d}{dr} \otimes \text{id} - \text{id} \otimes d + i\rho \frac{d}{r} \otimes \text{deg}
\]

where $d$ is the de Rham operator and $\text{deg}$ is the degree endomorphism on differential forms.

2.4. Line bundles over $\Phi_{e}^{1|1}(\mathcal{X})$ and $\Phi_{0}^{1|1}(\mathcal{X})$. We consider two sorts of line bundles over $\Phi_{e}^{1|1}(\mathcal{X})$: the first come from tensor powers of a line bundle over the moduli of super Euclidean circles, and the second come from the transgression of gerbes over $\ast//G$ to line bundles over $\mathcal{I}(\ast//G)$.

We start by defining the line bundles coming from the moduli of circles. We observe that $\Phi_{e}^{1|1}$ and $\Phi_{0}^{1|1}$ can be promoted to functors from orbifolds to stacks: all definitions are natural with respect to orbifold morphisms $\mathcal{X} \to \mathcal{Y}$. This naturality will allow us to pull back line bundles along the arrows of the commutative diagram

\[
\begin{array}{ccc}
\Phi_{0}^{1|1}(\mathcal{X}) & \to & \Phi_{0}^{1|1}(\ast) \\
\downarrow & & \downarrow \\
\Phi_{e}^{1|1}(\mathcal{X}) & \to & \Phi_{e}^{1|1}(\ast)
\end{array}
\]

As a diagram in the bicategory of stacks this is 2-commutative; however, if we can choose Lie groupoid representatives for all orbifolds, our Lie super groupoid presentations of $\Phi_{e}^{1|1}(\mathcal{X})$ and $\Phi_{0}^{1|1}(\mathcal{X})$ allow one to understand this as a strictly commuting diagram where the arrows are Lie super groupoid homomorphisms.
where the downward arrows are inclusions of stacks and the horizontal arrows are induced by the canonical map \( X \to \ast \). We define line bundles \( \kappa_\rho \) over \( \mathbb{R}^{1\!\!1}_{>0} / (\mathbb{R}^{1\!\!1} \times \mathbb{Z}/2) \) by the homomorphism

\[
\rho: \mathbb{R}^{1\!\!1} \times \mathbb{Z}/2 \to \mathbb{Z}/2 \subset \mathbb{C}^\infty \cong \text{Aut}(\mathbb{C}^{0\!\!1}),
\]

where \( p \) is the projection. In the above, \( \mathbb{C}^{0\!\!1} \) is regarded as a super vector space (or perhaps better, as a finitely generated projective module over \( C^\infty(\ast) \cong \mathbb{C} \)). Since the super group of translations is in the kernel of \( \rho \), the above defines a line bundle \( \kappa \) over \( \Phi^\dagger_\epsilon(\ast) \) that pulls back to \( \kappa_\rho \) along the map in Proposition 2.8. Furthermore, we obtain an isomorphism on spaces of sections

\[
\Gamma(\mathbb{R}^{1\!\!1}_{>0} / (\mathbb{R}^{1\!\!1} \times \mathbb{Z}/2); \kappa_\rho^\ast) \cong \Gamma(\Phi^\dagger_\epsilon(\ast), \kappa^\ast).
\]

Pulling back \( \kappa \) along the inclusion \( i: \Phi^\dagger_0(\ast) \hookrightarrow \Phi^\dagger_\epsilon(\ast) \) defines a line bundle over \( \Phi^\dagger_0(\ast) \) that we also denote by \( \kappa_\rho \).

Next we define line bundles coming from gerbes over \( \ast//G \). Let \( \beta \) be a normalized 2-cocycle representing a class \( [\beta] \in H^2(BG;U(1)) \). The transgression of \( \beta \) to the inertia stack of \( \ast//G \) is a flat line bundle we denote by \( T(\beta) \). Explicitly, the line on \( T(\ast//G) \simeq G//G \) is determined by the cocycle \( \tau^\beta: G \times G \to U(1) \), 

\[
\tau^\beta(h,g) = \frac{\beta(hgh^{-1},h)}{\beta(h,g)} \in U(1) \subset \text{Aut}(\mathbb{C}).
\]

To clarify the notation with respect to objects and morphisms in the groupoid \( G//G \), in the above we view \( g \) as the holonomy of \( G \)-bundle on the circle with a trivialization over a basepoint (i.e., an object), and \( h \) as change of basepoint (i.e., a morphism \( g \to hgh^{-1} \)). Concretely, the cocycle \( \tau^\beta \) defines the trivial line bundle on \( G \) with fiber at \( g \in G \) denoted \( T^\beta_g \), that is equivariant for the action of \( h \in G \) on \( g \in G \) via \( \tau^\beta(g,h): T^\beta_g \to T^\beta_{hgh^{-1}} \). Associativity of this action follows from the cocycle condition on \( \beta \). We may pullback \( T(\beta) \) along the projection

\[
(\mathbb{R}^{1\!\!1}_{>0} \times G) / (\mathbb{R}^{1\!\!1} \times \mathbb{Z}/2 \times G) \to G//G.
\]

By construction, the action of \( \mathbb{R}^{1\!\!1} \) is trivial on the resulting line bundle over the source, and so can be identified with the pullback of a line over \( \Phi^\dagger_\epsilon(\ast//G) \). We shall also denote this line by \( T(\beta) \). When \( \beta = 1 \)—the constant function on \( G \times G \)—the line bundle \( T(\beta) \) is trivial. Cohomologous cocycles, \( [\beta] = [\beta'] \), give rise to isomorphic line bundles, \( T(\beta) \cong T(\beta') \), and such an isomorphism is specified by an \( \alpha \) with \( \beta/\beta' = d\alpha \).

Now, consider the 2-commutative diagram

\[
\begin{array}{ccc}
\Phi^\dagger_0(X//G) & \to & \Phi^\dagger_0(\ast//G) \\
\downarrow & & \downarrow \\
\Phi^\dagger_\epsilon(X//G) & \to & \Phi^\dagger_\epsilon(\ast//G)
\end{array}
\]

where the vertical arrows are inclusions of stacks and the horizontal arrows are induced by the canonical map \( X//G \to \ast//G \). The line \( T(\beta) \) over \( \Phi^\dagger_\epsilon(\ast//G) \) can be pulled back to one over \( \Phi^\dagger_\epsilon(X//G) \) that we also denote by \( T(\beta) \).

Pulling back along the maps described above, we obtain line bundles \( \kappa_\epsilon \otimes T(\beta) \) over \( \Phi^\dagger_\epsilon(X//G) \) and \( \Phi^\dagger_0(X//G) \) that are natural with respect to maps between \( G \)-manifolds.

**Remark 2.13.** A \( \beta \)-projective representation is a map \( \rho: G \to \text{End}(V) \) with the property

\[
\rho(gh) = \beta(g,h)\rho(g)\rho(h), \quad \rho(\epsilon) = \text{id}.
\]

For such representations, one can compute explicitly that

\[
\text{Tr}_\rho(hgh^{-1}) = \frac{\beta(hgh^{-1},h)}{\beta(h,g)} \text{Tr}_\rho(g) = \tau^\beta(g,h)\text{Tr}_\rho(g)
\]

so that sections of \( T(\beta) \) are in bijection (as functions on \( G \) with properties) with the vector space generated by characters of \( \beta \)-projective representations of \( G \). We can also view these
concretely, the following inclusion induces an equivalence of groupoids and hence stacks, $T\to \mathcal{E}$ for the line bundle $\alpha$ such that $\beta^\ast \alpha = \alpha$ implies the closure of $\alpha$. Since the $r$-dependence is completely determined by the degree of the form, we may identify supersymmetric sections with a closed form of even or odd degree.

2.5. The proof of Theorem 2.14. First we single out preferred sections of the lines $\kappa^l \otimes \mathcal{T}(\beta)$ that we will compute below.

Definition 2.14. For a line bundle $\mathcal{L}$ over $\Phi_0^{1|1}(X)$, a supersymmetric section of $\mathcal{L}$ is a section in the image of the restriction map $i^* : \Gamma(\Phi_0^{1|1}(X); \mathcal{L}) \to \Gamma(\Phi_0^{1|1}(\mathcal{X}); \mathcal{L})$. We denote the vector space of supersymmetric sections by $\Gamma_{\text{susy}}(\Phi_0^{1|1}(X); \mathcal{L})$.

Proof of Theorem 2.14. Cocycles in twisted equivariant K-theory with complex coefficients can be described as closed differential forms on $\bigcup_g X^g$ with properties, as in Theorem 2.12. We will prove that supersymmetric sections of $\kappa^l \otimes \mathcal{T}(\beta)$ are in bijection with such forms.

We compute supersymmetric sections of $\kappa^l \otimes \mathcal{T}(\beta)$ as functions on $\mathbb{R}_{>0} \times \prod \pi TX^g$ that are invariant under the action of $\mathbb{R}^{1|1}$, equivariant with respect to the action of $\mathbb{Z}/2 \times G$, and extend to equivariant sections over $\prod_{g=0}^{l} \pi TX^g$ (i.e., are sup er symmetric). We write such a function as $\sum_i f_i(r) \otimes \alpha_i \in C^\infty(\mathbb{R}_{>0} \times \prod \pi TX^g)$ for $\alpha_i \in C^\infty(\pi TX^g)$ for some $g \in G$ and $f_i(r) \in C^\infty(\mathbb{R}_{>0})$.

The action of $\mathbb{R}^{1|1}$ factors through the $\mathbb{R}^{0|1}$-action on the components $\pi TX^g \cong \text{SMfld}(\mathbb{R}^{0|1}, X^g)$, which is generated by the de Rham operator on forms. Hence, $\mathbb{R}^{1|1}$-invariance implies that the $\alpha_i$ are closed. The action of $\mathbb{Z}/2$ is trivial on $\mathbb{R}_{>0}$ and is the degree endomorphism on forms, so being equivariant with respect to this action implies the $\alpha_i$ are of even degree (respectively, of odd degree) if $l$ is even (respectively, $l$ is odd).

A section being supersymmetric means that it is closed under the operator $Q$ of Lemma 2.12 when we view it as an element of $C^\infty(\mathbb{R}_{>0} \times \prod \pi TX^g) \supset C^\infty(\mathbb{R}_{>0} \times \prod \pi TX^g)$. This requires

$$\sum_i f_i(r) \otimes \alpha_i = \sum_i e^{\deg(\alpha_i)/2} \otimes \alpha_i.$$  

Since the $r$-dependence is completely determined by the degree of the form, we may identify supersymmetric sections with a closed form of even or odd degree.

Under the equivalence of stacks $G//G \cong \bigcup_{[g]} //C_G(g)$ where $[g]$ runs over conjugacy classes, the line bundle $\mathcal{T}(\beta)$ pulls back to a line bundle over the coproduct whose cocycle is determined by the character $\chi_{\beta}^{-1}$ in the formula of Theorem 2.12. This is the universal case for the line bundle $\mathcal{T}(\beta)$ over $\Phi_0^{1|1}(X//G)$, from which the Theorem follows. More concretely, the following inclusion induces an equivalence of groupoids and hence stacks,

$$\prod_{[g]} (\mathbb{R}_{>0} \times \pi TX^g) // (C_G(g) \times \mathbb{Z}/2) \hookrightarrow \prod_{g} (\mathbb{R}_{>0} \times \pi TX^g) // (G \times \mathbb{Z}/2)$$

which allows us to compute sections of $\kappa^l \otimes \mathcal{T}(\beta)$ as closed forms of degree $l$ mod 2 for which the $G$-action on the component of the form supported on $\pi TX^g$ is through the character $\mathcal{T}(\beta)(g, h) = \beta(h, g)\beta(h, h)^{-1} = (\chi_{\beta}^{-1})^{-1}$, for $h \in C_G(g)$. This concludes the proof.

Remark 2.15. Supersymmetric sections of the line $\kappa^l$ over the stack consisting of pairs $(R, \phi)$ for $R \in \mathbb{R}_{>0}(S)$ and $\phi : S \times R \mathbb{R}^{1|1} \to X//G$ a map factoring through $S \times \mathbb{R}^{0|1}$ can be identified with $G$-invariant differential forms on $X$, which we identify with cocycles for Borel equivariant de Rham cohomology.
2.6. Restriction, induction, and 2-dimensional Yang-Mills theory. For a homomorphism \( \lambda: H \to G \) that is compatible with given \( H \)- and \( G \)-actions on \( X \), we obtain a morphism of groupoids \( X//H \to X//G \) which induces a morphism of stacks \( \Phi_{\text{0}}^{11}(\lambda): \Phi_{\text{0}}^{11}(X//H) \to \Phi_{\text{0}}^{11}(X//G) \). For \( \beta \) a 2-cocycle on \( G \), naturality gives a restriction map
\[
\Phi_{\text{0}}^{11}(\lambda)^*: \Gamma_{\text{susy}}(\Phi_{\text{0}}^{11}(X//G); \kappa^f \otimes \mathcal{T}(\beta)) \to \Gamma_{\text{susy}}(\Phi_{\text{0}}^{11}(X//H); \kappa^f \otimes \mathcal{T}(\lambda^* \beta)).
\]
The focus of this subsection is the construction of pushforwards along \( \Phi_{\text{0}}^{11}(\lambda) \) using methodology from Freed-Quinn quantization of finite gauge theories \[\text{[FQ93]}\]. However, two entirely different types of gauge theories are present in our set up: 2-dimensional Yang-Mills theories define twists (and in turn, line bundles over the moduli of super circles) and twisted 1-dimensional gauge theories determine sections of these line bundles. Notions of quantization native to these respective field theories lead to two different flavors of pushforward. In terms of representation theory, the pushforward coming from the 1-dimensional theory leads to the Frobenius character formula for an induced representation: the pushforward coming from the 2-dimensional theory simply views a projective representation of a group as a module over a twisted group algebra. We describe this second version primarily to clarify a generalization in the next section involving Chern-Simons theory.

We start with the case related to quantization of 1-dimensional gauge theories. Fix a homomorphism \( \lambda: H \to G \) that induces a morphism of Lie groupoids also denoted \( \lambda: X//H \to X//G \), and 2-cocycle \( \beta: G \times G \to U(1) \). The homotopy fiber of the induced functor \( \Phi_{\text{0}}^{11}(X//H) \to \Phi_{\text{0}}^{11}(X//G) \) at the \( S \)-point \((R, \phi)\) of \( \Phi_{\text{0}}^{11}(X//G) \) has as objects
\[
\text{Ob}(\text{hofib}(\lambda)(R, \phi)) \cong \{(R, \tilde{\phi}) \in \Phi_{\text{0}}^{11}(X//H), \ t \mid \lambda \circ \tilde{\phi} \cong \phi\}
\]
and morphisms are isomorphisms \( \tilde{\phi} \cong \tilde{\phi}' \) in the groupoid of maps \( S \times_{R} \mathbb{R}^{11} \to X//H \).

Since \( H \) and \( G \) are finite the homotopy fiber has finitely many isomorphism classes, and a candidate measure comes from a choice of a finite groupoid presentation and an assignment of a weight to each object. To define this weight, identify \((R, \tilde{\phi})\) with a principal \( H \)-bundle \( P \) over \( S \times_{R} \mathbb{R}^{11} \) equipped with an \( H \)-equivariant map to \( X \). Following Freed-Quinn \[\text{[FQ93]}\], Section 2.1, let the weight \( d\mu \) of \((R, \tilde{\phi})\) be \( 1/|\text{Aut}(R, \tilde{\phi})| \), where \( \text{Aut}(R, \tilde{\phi}) \) denotes the automorphisms of \( P \) in the given groupoid presentation. They showed that this measure is well-behaved under equivalence of finite groupoids (see also \[\text{[DD01]}\]), which allows for the following definition.

**Definition 2.16.** Define \( \lambda_1: \Gamma_{\text{susy}}(\Phi_{\text{0}}^{11}(X//H); \kappa^\bullet \otimes \mathcal{T}(\lambda^* \beta)) \to \Gamma_{\text{susy}}(\Phi_{\text{0}}^{11}(X//G); \kappa^\bullet \otimes \mathcal{T}(\beta)) \) by
\[
\lambda_1(s)(R, \phi) := \sum_{\text{Ob}(\text{hofib}(\lambda)(R, \phi))} t^* s(R, \tilde{\phi}) \cdot d\mu, \quad s \in \Gamma_{\text{susy}}(\Phi_{\text{0}}^{11}(X//H); \kappa^\bullet \otimes \mathcal{T}(\lambda^* \beta)),
\]
where \( \lambda_1(s)(R, \phi) \) and \( s(R, \tilde{\phi}) \) denote the respective values of the sections \( \lambda_1(s) \) and \( s \) at the \( S \)-point determined by \((R, \phi)\) and \((R, \tilde{\phi})\), respectively.

To give an explicit formula for \( \lambda_1 \), we identify a section \( s \in \Gamma(\Phi_{0}(X//H); \mathcal{T}(\beta) \otimes \kappa^\bullet) \) with a section over the groupoid
\[
\left( \coprod_h \mathbb{R}_{>0} \times \pi TX^h \right)//(H \times \mathbb{R}^{11} \times \mathbb{Z}/2),
\]
and we will write \( \lambda_1(s) \) as a section over the groupoid
\[
\left( \coprod_g \mathbb{R}_{>0} \times \pi TX^g \right)//(G \times \mathbb{R}^{11} \times \mathbb{Z}/2).
\]
Identifying sections over the groupoid as functions on objects with transformation properties, let \( s(h) \) denote the function on \( \mathbb{R}_{>0} \times \pi TX^h \) gotten by restriction and similarly let \( \lambda_1(s)(g) \) denote the function on \( \mathbb{R}_{>0} \times \pi TX^g \) gotten by restriction. Since the homomorphism \( \lambda \) induces a morphism of Lie groupoids, we have induced diffeomorphisms on fixed
point sets, \(X^h \to X^\lambda(h)\). We also have diffeomorphisms \(X^g \cong X^{g_0g_0^{-1}}\) from the left action of \(g_0\). Hence, we may identify the function \(s(h)\) with a function on \(\mathbb{R}_{>0} \times \pi TX^g\) when \(\lambda(h)\) is conjugate to \(g \in G\); this is the function \(\iota^*s(h)\). We compute

\[
\lambda_1(s)(g) = \frac{1}{|H|} \sum_{h \in H, g_0 \in G, g_0\lambda(h)g_0^{-1} = g} \iota^*s(h)
\]

where the sum is over \(h \in H\) whose image \(\lambda(h) \in G\) is conjugate to \(g\), i.e., principal \(H\)-bundles \(P \to S \times_R \mathbb{R}^{1|1}\) with holonomy \(h\) for which the associated bundle \(P \times_H G\) is isomorphic to a \(G\)-bundle with holonomy \(g\). Since the 2-cocycle \(\lambda^*\beta\) on \(H\) is the pullback of a cocycle on \(G\), the sum indeed has the appropriate equivariance property for the action of \(G\), so gives the claimed map on supersymmetric sections. In the case that \(X = *\) and \(H \to G\) is an inclusion, we observe that the above is the Frobenius character formula of an induced representation. In the case that \(X = *\) and \(H \to \{e\}\), we obtain a sum over all \(H\)-bundles, which is a quantization-type formula for 1|1-dimensional gauge theory.

Now we turn our attention to pushforwards

\[
\lambda_1^{YM}: \Gamma_{\text{susy}}(\Phi_0^{1|1}(X//H); \kappa^* \otimes T(\beta)) \to \Gamma_{\text{susy}}(\Phi_0^{1|1}(X//G); \kappa^* \otimes \lambda_* T(\beta))
\]

for a 2-cocycle \(\beta: H \times H \to U(1)\). The target is sections of a vector bundle over \(\Phi_0^{1|1}(X)\) — typically with fiber dimension greater than 1—gotten from taking sections of \(T(\beta)\) along the homotopy fibers of the map \(\Phi_0^{1|1}(\lambda)\); explicitly, this homotopy fiber is a groupoid (as defined above) and global sections along the fibers are the invariant sections over this groupoid of the pullback of \(T(\beta)\) to the fiber. Given this definition, the value of \(\lambda_1^{YM}\) on a given section is tautological: we simply view \(s\) restricted to a homotopy fiber as defining a vector in the space of sections along the fiber.

**Example 2.17.** In the case \(X = *\) and \(\lambda: G \to \{e\}\) we have as source

\[
\Gamma_{\text{susy}}(\Phi_0^{1|1}(*)//G; T(\beta)) \cong \Gamma(G//G; T(\beta)) \cong \mathbb{C}^{|G|^G}
\]

where the final vector space is the center of the twisted group algebra of \(G\) for the cocycle \(\beta\). The map \(\lambda_1^{YM}\) takes global sections,

\[
\lambda_1^{YM}: \Gamma_{\text{susy}}(\Phi_0^{1|1}(*)//G; \lambda_* T(\beta)) \cong \Gamma_{\text{susy}}(\Phi_0^{1|1}(*)//G; \sum^\beta |G|^G) \cong \mathbb{C}^{|G|^G}
\]

where \(\sum^\beta |G|^G\) denotes the trivial vector bundle with fiber \(\mathbb{C}^{|G|^G}\). Since \(\lambda_1^{YM}\) is an isomorphism of vector spaces, the value of \(\lambda_1^{YM}\) on sections is completely determined: one simply views a section of \(T(\beta)\) as an element of \(\mathbb{C}^{|G|^G}\).

**Remark 2.18.** The vector space \(\mathbb{C}^{|G|^G}\) is the vector space of states of 2-dimensional (quantum) Yang-Mills theory for the group \(G\) and cocycle \(\beta\); in terms of a functor from the 2-dimensional bordism category to the bicategory of algebras, the vector space \(\mathbb{C}^{|G|^G}\) is value on the circle of the 2-dimensional TFT associated to the twisted group algebra \(\sum^\beta |G|^G\). Viewing this vector space as the global sections of \(T(\beta)\) over \(G//G\) is part of the quantization of Yang-Mills theory described by Quinn and Freed: the space of classical solutions is \(L(*//G) \cong G//G\), and the quantum Hilbert space is the global sections of the prequantum line bundle \(T(\beta)\) over this space.

### 3. Twisted Equivariant Elliptic Cohomology over \(\mathbb{C}\)

For \(G\) a finite group of odd order, Devoto \[Dev96\] used equivariant Thom spectra to define an equivariant refinement of the elliptic cohomology of Landweber, Ravenel and Stong \[LRS95\]. Using a prior construction of Freed and Quinn \[FQ93\], Ganter \[Gan09\] showed that an element \(H^4(BG; \mathbb{Z})\) can be used to twist a generalization of Devoto’s coefficient ring for arbitrary finite groups. She explained how this twisted coefficient ring can be interpreted as sections of the Freed-Quinn line bundle over the moduli stack of \(G\)-bundles on elliptic curves. Below we apply methods analogous to the previous section to obtain cocycles for twisted equivariant elliptic cohomology over manifolds with \(G\)-action that reduces
in the case $X = *$ to Devoto and Ganter’s coefficient ring. We begin the section by giving a quick-and-easy definition of twisted equivariant elliptic cohomology in the footsteps of these previous authors. Many subtleties are avoided by our working over $\mathbb{C}$.

### 3.1. Twisted equivariant elliptic cohomology with complex coefficients.

Let $L$ denote the smooth manifold of based, oriented lattices $\Lambda$, meaning pairs of points $\ell_1, \ell_2 \in \mathbb{C}$ that are linearly independent over $\mathbb{R}$ and whose ratio is in the upper half plane, $\ell_1/\ell_2 \in \mathfrak{h}$. Let $\mathcal{O}^j(L)$ denote holomorphic functions $f$ on $L$ such that $f(q^j \Lambda) = q^{-j}f(\Lambda)$ for $\Lambda \in L$. Then weak modular forms of weight $j/2$ are the elements of $\mathcal{O}^j(L)$ that are invariant under the $\text{SL}_2(\mathbb{Z})$-action on $L$. A convenient description of elliptic cohomology of a manifold $X$ with complex coefficients is

\[
\text{Ell}^k(X) \cong \bigoplus_{i+j=k} (\mathcal{H}^i_{\text{dR}}(X) \otimes \mathcal{O}^j(L))^{\text{SL}_2(\mathbb{Z})},
\]

where the $\text{SL}_2(\mathbb{Z})$-action is trivial on the de Rham cohomology groups, and acts on $\mathcal{O}^j(L)$ through its action on $L$.

For $G$ a finite group, define $\mathcal{C}(G)$ to be the set of pairs of commuting elements of $G$. There is an action of $G$ on $\mathcal{C}(G)$ by conjugation, $(g_1, g_2) \mapsto (h g_1 h^{-1}, h g_2 h^{-1})$. Let $\mathcal{C}(g_1, g_2)$ denote the stabilizer of $(g_1, g_2)$ under this action, and $\mathcal{C}[G] := \mathcal{C}(G)/G$ denote the quotient, writing $[g_1, g_2] \in \mathcal{C}[G]$ for the point in the image of $(g_1, g_2) \in \mathcal{C}(G)$. The sets $\mathcal{C}(G)$ and $\mathcal{C}[G]$ carry a right action of $\text{SL}_2(\mathbb{Z})$ determined by

\[
(g_1, g_2) \mapsto (g_1^d \ell_2^{-b} g_2^{-c} \ell_1^e), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}).
\]

**Definition 3.1.** Let $\alpha : G \times G \times G \to U(1)$ be a normalized 3-cocycle. Define the abelian group $\text{Ell}^k_G(\mathcal{C}[G])$ as the set of holomorphic functions on $L \times \mathcal{C}[G]$ that are invariant under the diagonal $\text{SL}_2(\mathbb{Z})$-action and equivariant for the $\mathcal{C}(g_1, g_2) \times C^\times$-actions in the following sense:

\[
f(q^j \Lambda, [g_1, g_2]) = q^{-k} \frac{\alpha(h, g_1, g_2)\alpha(g_2, h, g_1)\alpha(g_1, g_2, h)}{\alpha(g_1, h, g_2)\alpha(h, g_2, g_1)\alpha(g_2, g_1, h)} f(\Lambda, [g_1, g_2]), \quad (h, q) \in \mathcal{C}(g_1, g_2) \times C^\times.
\]

For $\alpha, \beta$ a pair of normalized cocycles, multiplication of functions on $L \times \mathcal{C}[G]$ gives the graded multiplication

\[
\text{Ell}^{k+\alpha}_G(\mathcal{C}[G]) \otimes \text{Ell}^{l+\beta}_G(\mathcal{C}[G]) \to \text{Ell}^{k+l+\alpha \boxplus \beta}_G(\mathcal{C}[G]).
\]

This is a repackaged form of the ring defined by N. Ganter in [Gan09]. In the untwisted case ($\alpha \equiv 1$) and when the cardinality of $G$ is odd, the following definition reduces to Devoto’s equivariant elliptic cohomology, taken with complex coefficients; compare Part 3 of Corollary 2.7 in [Dev98] and Theorem 5.3 of [Dev96].

**Definition 3.2.** For $X$ a manifold with the action of a finite group $G$, let

\[
\text{Ell}_{G}^{k+\alpha}(X) := \bigoplus_{i+j=k} \left( \bigoplus_{[g_1, g_2] \in \mathcal{C}[G]} (\mathcal{H}^i_{\text{dR}}(X^{[g_1, g_2]}) \otimes \mathcal{O}^j(L) \otimes \chi_{[g_1, g_2]}^{\alpha})^{\text{SL}_2(\mathbb{Z})} \right),
\]

where $\chi_{[g_1, g_2]}^{\alpha}$ is the character of $C(g_1, g_2) \subset G$ defined by $h \mapsto \frac{\alpha(h, g_1, g_2)\alpha(g_2, h, g_1)\alpha(g_1, g_2, h)}{\alpha(g_1, h, g_2)\alpha(h, g_2, g_1)\alpha(g_2, g_1, h)}$.

### 3.2. The stacks $\Phi_{\ell, \sigma}^{211}(X)$ and $\Phi_{\ell', \sigma'}^{211}(X)$.

An $S$-family of (oriented, based) super lattices, denoted $\Lambda$, is given by a pair $\ell = (\ell, \ell, \sigma)$, $\ell = (\ell, \ell', \sigma') \in \mathbb{R}^{211}(S)$ satisfying

1. $\ell \cdot \ell = \ell' \cdot \ell'$ where $\cdot$ denotes multiplication in $\mathbb{R}^{211}(S)$ (see subsection 1.3);
2. For any map $f : * \to S$, the complex numbers $f^*\ell$ and $f^*\ell'$ are linearly independent over $\mathbb{R}$ and $f^*\ell'/f^*\ell \in \mathfrak{h}$ where $\mathfrak{h} \subset \mathbb{C}$ is the upper half plane.

We denote the generalized supermanifold with these $S$-points by $sL$, the generalized supermanifold of based lattices. We observe that the first condition is equivalent to $\sigma\sigma' = 0$, from which one can deduce that $sL$ is not representable. However, $sL$ is affine in the sense...
that maps $S \to sL$ can be identified with algebra maps $C^\infty(L)[\sigma, \sigma']/(\sigma\sigma') \to C^\infty(S)$. Let $S \times_\Lambda \mathbb{R}^{2|1}$ denote the quotient of $S \times \mathbb{R}^{2|1}$ by the action of $\Lambda \in sL(S)$.

For $\Lambda \in L(S) \subset sL(S)$ an $S$-family of based oriented lattices, consider the composition of morphisms of stacks

$$S \times_\Lambda \mathbb{R}^{2|1} \xrightarrow{\sim} (S \times \mathbb{R}^{2|1})/\Lambda \to (S \times \mathbb{R}^{0|1})/\Lambda \cong (S \times \mathbb{R}^{0|1})/\mathbb{Z}^2$$

where the second arrow is induced by the projection $\mathbb{R}^{2|1} \to \mathbb{R}^{0|1}$ and the third by the isomorphism $\Lambda \cong \mathbb{Z}^2$ (by definition $\Lambda$ comes with an ordered choice of generators). We denote the above composition by $\text{proj}: S \times_\Lambda \mathbb{R}^{2|1} \to (S \times \mathbb{R}^{0|1})/\mathbb{Z}^2$.

**Definition 3.3.** For $\mathcal{X}$ an orbifold, the *stack of classical vacua over $\mathcal{X}$*, denoted $\Phi_0^{2|1}(\mathcal{X})$, is the stack associated to the prestack whose objects over $S$ consist of pairs $(\Lambda, \phi)$ where $\Lambda \in L(S) \subset sL(S)$ defines a family of super tori $S \times_\Lambda \mathbb{R}^{2|1}$ and $\phi$ is an object in the groupoid of maps $S \times_\Lambda \mathbb{R}^{2|1} \to \mathcal{X}$ with a chosen factorization $\phi = \phi_0 \circ \text{proj}$. Morphisms over $S$ consist of 2-commuting triangles,

$$
\begin{array}{ccc}
S \times_\Lambda \mathbb{R}^{2|1} & \xrightarrow{\sim} & S \times_\Lambda' \mathbb{R}^{2|1} \\
\phi & \xleftarrow{\sim} & \phi' \\
\mathcal{X} & \to & \mathcal{X}
\end{array}
$$

where the horizontal arrow is an isometry of S-families of rigid conformal 2|1-manifolds.

For a fixed $\Lambda \in sL(S)$ an $S$-family of super lattices, we define a $\Lambda$-invariant morphism $\text{proj}_\Lambda: S \times \mathbb{R}^{2|1} \to S \times \mathbb{R}^{0|1}$ by

$$\text{proj}_\Lambda: \mathbb{R}^{2|1} \times S \to \mathbb{R}^{0|1}, \quad \text{proj}_\Lambda(z, \bar{z}, \theta, s) := \theta - \frac{\bar{z}\ell' - z\ell}{\ell' - \ell} - \frac{\sigma'\bar{z}\ell - \bar{z}\ell}{\ell' - \ell}.$$

This map is invariant under translations in the lattice,

$$\text{proj}_\Lambda(z + \ell, \bar{z} + \bar{\ell} + \theta\sigma, \theta + \sigma, s) = \theta + \sigma - \frac{(\bar{z} + \bar{\ell})\ell' - (z + \ell)\bar{\ell}}{\ell' - \ell} - \frac{\sigma'(\bar{z} + \bar{\ell})\ell - (z + \ell)\bar{\ell}}{\ell' - \ell} = \text{proj}_\Lambda(z, \bar{z}, \theta, s) + \sigma - \sigma,$$

$$\text{proj}_\Lambda(z + \ell', \bar{z} + \bar{\ell}' + \theta\sigma', \theta + \sigma', s) = \theta + \sigma' - \frac{(\bar{z} + \bar{\ell}')\ell' - (z + \ell')\bar{\ell}}{\ell' - \ell} - \frac{\sigma'((z + \ell')\bar{\ell} - (z + \ell)\bar{\ell})}{\ell' - \ell} = \text{proj}_\Lambda(z, \bar{z}, \theta, s) + \sigma' - \sigma',$$

where we use that $\sigma^2 = \sigma'^2 = \sigma\sigma' = 0$. Hence, $\text{proj}_\Lambda$ descends to a map we also denote by $\text{proj}_\Lambda$,

$$\text{proj}_\Lambda: S \times_\Lambda \mathbb{R}^{2|1} \to S \times \mathbb{R}^{0|1}/\mathbb{Z}^2.$$

**Definition 3.4.** Define the stack $\Phi_0^{2|1}(\mathcal{X})$ as the stack associated to the prestack whose objects over $S$ consist of pairs $(\Lambda, \phi)$ where $\Lambda \in sL(S)$ defines a family of super tori $S \times_\Lambda \mathbb{R}^{2|1}$ and $\phi$ is an object in the groupoid of maps $S \times_\Lambda \mathbb{R}^{2|1} \to \mathcal{X}$ with a chosen factorization $\phi = \phi_0 \circ \text{proj}_\Lambda$. Morphisms over $S$ consist of 2-commuting triangles,

$$
\begin{array}{ccc}
S \times_\Lambda \mathbb{R}^{2|1} & \xrightarrow{\sim} & S \times_\Lambda' \mathbb{R}^{2|1} \\
\phi & \xleftarrow{\sim} & \phi' \\
\mathcal{X} & \to & \mathcal{X}
\end{array}
$$

where the horizontal arrow is an isomorphism of S-families of rigid conformal 2|1-manifolds.
3.3. Groupoid presentations. For an orbifold $\mathcal{X}$ define the double inertia stack as the internal hom in stacks, $\mathcal{I}^2\mathcal{X} := \mathbf{SmSt}(*/\mathbb{Z}^2, \mathcal{X})$. This stack inherits a left action of $\text{SL}_2(\mathbb{Z})$ from the standard left action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$, following our discussion in subsection 1.3 regarding actions on mapping spaces.

Remark 3.5. As a word of warning, other authors have taken the double inertia stack to mean the fibered product, $\mathcal{I}\mathcal{X} \times \mathcal{X} \mathcal{I}\mathcal{X}$. Geometrically, this fibered product corresponds to pairs of ghost loops in $\mathcal{X}$ meeting at a point rather than ghost tori in $\mathcal{X}$.

Proposition 3.6. Choose a groupoid presentation of $\mathcal{I}^2\mathcal{X}$ with objects $\text{Ob}(\mathcal{I}^2\mathcal{X})$ and morphisms $\text{Mor}(\mathcal{I}^2\mathcal{X})$. There is a surjective morphism of stacks

$$
\left( \begin{array}{c}
\text{SL}_2(\mathbb{Z}) \times \mathbf{EuC}(\mathbb{R}^{2|1}) \times sL \times \pi T(\text{Mor}(\mathcal{I}^2\mathcal{X})) \\
sL \times \pi T(\text{Ob}(\mathcal{I}^2\mathcal{X}))
\end{array} \right) \to \Phi^{2|1}_s(\mathcal{X}).
$$

Proof. Factoring through $\text{proj}_\Lambda$ means the stack $\Phi^{2|1}_s(\mathcal{X})$ is equivalent to one whose objects over $S$ are pairs $(\Lambda, \phi_0)$ for $\Lambda \in sL(S)$ and $\phi_0 : S \times \mathbb{R}^{0|1}//\mathbb{Z}^2 \to \mathcal{X}$, where the $\mathbb{Z}^2$-action on $S \times \mathbb{R}^{0|1}$ is trivial. The equivalence $(S \times \mathbb{R}^{0|1})//\mathbb{Z}^2 \simeq (S \times \mathbb{R}^{0|1}) \times (*/\mathbb{Z}^2)$ along with the hom-tensor adjunction gives an equivalence of groupoids

$$
\mathbf{SmSt}(S \times \mathbb{R}^{0|1}//\mathbb{Z}^2, \mathcal{X}) \simeq \mathbf{SmSt}(S \times \mathbb{R}^{0|1}, \mathbf{SmSt}(*//\mathbb{Z}^2, \mathcal{X})).
$$

An object of $\Phi^{2|1}_s(\mathcal{X})$ over $S$ can therefore be identified with the pair $(\Lambda, \phi_0)$ for $\Lambda \in sL(S)$ and $\phi_0 \in \pi T(\text{Ob}(\mathcal{I}^2\mathcal{X}))(S)$. This gives the supermanifold of objects claimed in the proposition.

The morphisms in $\Phi^{2|1}_s(\mathcal{X})$ consist of pairs $(F, h)$ where $F : S \times \Lambda \mathbb{R}^{2|1} \to S \times \Lambda \mathbb{R}^{2|1}$ and $h : \phi \mapsto \phi' \circ F$ is an isomorphism in the groupoid of maps $S \times \Lambda \mathbb{R}^{2|1} \to \Lambda \mathcal{X}$. Choosing a lift of $F$ to the universal cover gives a super Euclidean isometry $\tilde{F} : S \times \mathbb{R}^{2|1} \to S \times \mathbb{R}^{2|1}$ and an isomorphism of lattices determined by $A \in \text{SL}_2(\mathbb{Z})$. Hence, $F$ is determined (possibly non-uniquely) by $\tilde{F} \in \mathbf{CEuc}(\mathbb{R}^{2|1})(S) \cong (\mathbb{R}^{2|1} \times \mathbb{C}^\times)(S)$ and $A \in \text{SL}_2(\mathbb{Z})$. Conversely, a simple computation shows that any super Euclidean isometry $\tilde{F} = (u, \ddot{u}, \nu, q, \ddot{q}) \in \mathbb{R}^{2|1} \times \mathbb{C}^\times$ and $A \in \text{SL}_2(\mathbb{Z})$ uniquely determines an isometry $F : S \times \Lambda \mathbb{R}^{2|1} \to S \times \Lambda \mathbb{R}^{2|1}$ where the target for the isometry induced by $\tilde{F}$ is determined by the lattice

$$
\Lambda' = (q^2, \ell, q^2(\ell + 2i\nu), \ddot{q}, \ell, q^2(\ell + 2i\nu), \ddot{q}^2, \sigma, \ell', \ddot{\ell}, \sigma'),
$$

and the target for the isometry induced by $A$ is determined by the left action of $\text{SL}_2(\mathbb{Z})$ on $\Lambda$.

To connect with our description of objects, we will identify the triple $(\tilde{F}, A, h)$ with a morphism $(\Lambda, \phi_0) \to (\Lambda', \phi'_0)$ between objects in our desired description. For a fixed triple, the discussion of the previous paragraph determines one datum of the target, namely $\Lambda'$ in the above notation, so we need to compute the remaining datum, namely $\phi'_0$. By the discussion in the first paragraph of the proof, we may identify $h$ with a morphism in the groupoid of maps $S \times \mathbb{R}^{0|1} \to \mathcal{I}^2\mathcal{X}$, and since $S \times \mathbb{R}^{0|1}$ is discrete this is locally determined by a map $h_0 : S \times \mathbb{R}^{0|1} \to \text{Mor}(\mathcal{I}^2\mathcal{X})$, i.e., $h_0 \in \pi T\text{Mor}(\mathcal{I}^2\mathcal{X})(S)$, recovering a component of the morphism supermanifold claimed in the proposition. Abstractly, this proves we obtain a surjection as in the statement of the proposition, but it will be useful to characterize the morphisms between objects coming from rigid conformal isometries.

We shall define $F_0$ as the arrow that makes the diagram commute

$$
\begin{array}{c}
S \times \mathbb{R}^{2|1} \\
\downarrow F_0 \\
(S \times \mathbb{R}^{2|1})
\end{array}
\xymatrix{ S \times \mathbb{R}^{2|1} \ar[r]^-{\text{id} \times \text{proj}_\Lambda} & S \times \mathbb{R}^{0|1} \ar[d]^-{F} \\
S \times \mathbb{R}^{0|1} \ar[r]_-{\text{proj}_\Lambda'} & S \times \mathbb{R}^{0|1}.}
$$

where the vertical left arrow is an isometry associated to an $S$-point $\tilde{F}$ of $\mathbf{CEuc}(\mathbb{R}^{2|1})(S)$, and $\text{proj}_\Lambda'$ is the projection associated to the lattice $\Lambda'$ that is the image of $\Lambda$ under the
change of lattice induced by $\bar{F}$. Direct (if somewhat tedious) computation shows that the map

$$
\theta \mapsto \bar{q} \left( \theta + \nu - \frac{(\sigma \ell' - \alpha \ell')(i\theta + \nu)}{\ell' - \ell} \right)
$$

is the unique one making (7) commute for $(u, \bar{u}, \nu, q) \in (\mathbb{R}^{2|1} \times \mathbb{C}^\times)(\mathbb{S}) \cong CEuc(\mathbb{R}^{2|1})(\mathbb{S}).$

This uniquely determines $\phi_0 = \phi_0(\bar{F}_0)^{-1}$ in case, and we obtain a unique morphism $(\Lambda, \theta) \rightarrow (\Lambda', \phi_0')$ from the original isometry $\bar{F}$.

Turning attention to an isomorphism induced by $A \in SL_2(\mathbb{Z})$, a simple computation shows that $\text{proj}_{\Lambda'} = \text{proj}_{\Lambda}$, where $\Lambda' = A \cdot \Lambda$. Hence, the effect of $A$ on $\phi_0: \mathbb{S} \times \mathbb{R}_{0|1} // \mathbb{Z}^{2} \rightarrow \mathcal{X}$ is purely through the action of $A$ on $\mathbb{Z}^{2}$. Therefore $A$ defines a unique morphism $(\Lambda, \phi_0) \rightarrow (\Lambda', \phi_0')$ determined by the standard action on $\Lambda$ and $\theta$ (and consequently on $\pi T \text{Ob}(\mathcal{I}^2(\mathcal{X}))$). This completes the proof. □

By restriction to $\Lambda \in \mathcal{L}(\mathcal{S}) \subset sL(\mathcal{S})$, we obtain the following.

**Proposition 3.7.** There is a surjective map of stacks

$$
\left(\begin{array}{cc}
SL_2(\mathbb{Z}) \times \mathbb{R}^{2|1} \times \mathbb{C}^\times \times L \times \pi T(\text{Mor}(\mathcal{I}^2(\mathcal{X})))
\end{array}\right)
\downarrow
\begin{array}{c}
\text{proj}_{\Lambda} = \text{proj}_{\Lambda'}
\end{array}
\downarrow
\begin{array}{c}
L \times \pi T(\text{Ob}(\mathcal{I}^2(\mathcal{X})))
\end{array}
\rightarrow
\Phi_{01}^{2|1}(\mathcal{X}).
$$

**Remark 3.8.** The failure of the map in the previous pair of propositions to induce an equivalence comes from the possible kernel of the map from $CEuc(\mathbb{R}^{2|1})(\mathbb{S})$ to the isometries of an $S$-family of super tori. This kernel is always discrete, and since we will only be concerned with quantities invariant under the action of these isometries, it suffices to compute using the groupoids in the previous propositions. To understand the precise structure of $\Phi_{01}^{2|1}(\mathcal{X}/\mathcal{G})$, ideas similar to those in Remark 2.11 apply.

In the case that $\mathcal{X} = \mathcal{X}/\mathcal{G}$, the double inertia stack has a groupoid presentation,

$$
\mathcal{I}^2(\mathcal{X}/\mathcal{G}) := \text{SmSt}(//\mathbb{Z}^{2}, \mathcal{X}/\mathcal{G}) \simeq \left( \bigotimes_{g_1, g_2 \in \mathcal{G}(\mathcal{S})} X^{(g_1, g_2)} \right)/\mathcal{G}.
$$

Plugging this presentation into the previous propositions will be our main tool in computations.

The diagram (7) defines an action of the rigid conformal group, $CEuc(\mathbb{R}^{2|1}) \cong \mathbb{R}^{2|1} \times \mathbb{C}^\times$, on $sL \times \pi T \mathcal{X}$ that sends $\Lambda \mapsto \Lambda'$ and acts on $\pi T \mathcal{X}^{(g_1, g_2)} \cong \text{SmSt}(//\mathcal{Z}^{2}, X^{(g_1, g_2)})$ by precomposing with the action on $\mathcal{R}_{0|1}$ determined by $\bar{F}_0$. We will require explicit formulas for this action on $C^\infty(sL \times \pi T \mathcal{X})$. The $C^\infty$-action is

$$
f(\ell, \bar{\ell}, \sigma, \bar{\ell}', \sigma') \mapsto f(q^2 \bar{\ell}, q^2 \bar{\ell}', q \sigma, q^2 \bar{\ell}', q \bar{\sigma}) \otimes \frac{\alpha}{\bar{q}^{\text{deg}(\alpha)}},
$$

for $(\ell, \bar{\ell}, \sigma, \bar{\ell}', \sigma')$ coordinates on $sL$ and $\alpha \in \Omega^*(\mathcal{S}) \cong \mathcal{C}^{\infty}(\pi T \mathcal{X})$. The Lie algebra of $\mathbb{R}^{2|1}$ is generated by one odd and one even element, $\partial_\ell$ and $\partial_0 - i\partial_\ell$, that commute with each other. Hence, to specify an $\mathbb{R}^{2|1}$-action we need to specify the infinitesimal generators associated to these generators of the Lie algebra. A tedious but straightforward computation (see [BLR], Section 3.5) yields the following characterization.

**Lemma 3.9.** The $\mathbb{R}^{2|1}$-action on $f \in C^{\infty}(sL) \otimes \Omega^*(\mathcal{X})$ can be expressed as

$$
f \mapsto \exp(uR + i\bar{u}Q^2 + \nu Q)f,
$$

for infinitesimal generators

$$
R := \sigma \bar{\ell}' - \alpha \bar{\ell} \otimes d, \quad Q = 2i\sigma \partial_\ell \otimes id + 2i\alpha' \partial_\ell' \otimes id - id \otimes d + \frac{\sigma \bar{\ell}' - \alpha \bar{\ell}}{\ell' - \ell} \otimes \text{deg},
$$

where $d$ is the de Rham $d$ and deg is the degree endomorphism on differential forms.
3.4. Line bundles over $\Phi_2^{21}(\mathcal{X})$. The line bundles over $\Phi_2^{21}(\mathcal{X})$ we wish to consider come from two sources: line bundles over the moduli stack of super tori, and transgression of 2-gerbes over $\star//G$ to line bundles over $\mathcal{I}^2(\star//G)$. Tensor powers of the former give the grading on equivariant elliptic cohomology, whereas the latter give twists of the theory.

First we describe line bundles that come from lines over the moduli stack of super tori. We have the commuting square

$$
\begin{array}{ccc}
\Phi_0^{21}(\mathcal{X}) & \to & \Phi_0^{21}(\star) \\
\downarrow & & \downarrow \\
\Phi_2^{21}(\mathcal{X}) & \to & \Phi_2^{21}(\star)
\end{array}
$$

(9)

where the vertical arrows are inclusions of stacks, and the horizontal arrows are induced by the canonical map $\mathcal{X} \to \star$. This square allows us to promote any line over $\Phi_2^{21}(\star)$ to a naturally defined line over $\Phi_2^{21}(\mathcal{X})$. Let $\omega_{\rho}^{1/2}$ be the line bundle over $sL//\left( SL_2(\mathbb{Z}) \times (\mathbb{R}^{21} \rtimes \mathbb{C}^\times) \right)$ defined from the homomorphism

$$
\rho: SL_2(\mathbb{Z}) \times (\mathbb{R}^{21} \rtimes \mathbb{C}^\times) \to \mathbb{C}^\times \cong \text{Aut}(\mathbb{C}^{01})
$$

given by the projection composed with the map $z \mapsto z^{-1}$, $z \in \mathbb{C}^\times$. Since the super group of translations is in the kernel of $\rho$, this defines a line bundle $\omega_{\rho}^{1/2}$ over $\Phi_2^{21}(\star)$ that pulls back to $\omega_{\rho}^{1/2}$ along the surjective map of Proposition 3.6. Furthermore, we obtain an isomorphism on spaces of sections,

$$
\Gamma \left( sL//\left( SL_2(\mathbb{Z}) \times (\mathbb{R}^{21} \rtimes \mathbb{C}^\times) \right); \omega_{\rho}^{k/2} \right) \cong \Gamma \left( \Phi_2^{21}(\star); \omega_{\rho}^{k/2} \right).
$$

From the commuting square (9) this gives a functor from orbifolds to graded abelian groups, $\mathcal{X} \mapsto \Gamma_{\text{susy}}(\Phi_0^{21}(\mathcal{X}); \omega^{\star//2})$.

Next we define line bundles coming from 2-gerbes over $\star//G$. Let $\alpha$ be a normalized 3-cocycle. Double transgression of $\alpha$ gives rise to an $SL_2(\mathbb{Z})$-equivariant flat line bundle $\mathcal{C}S(\alpha)$ on $\mathcal{I}^2(\star//G)$, which is $\mathcal{C}S(\alpha)$-equivariant. Explicitly $\alpha$ determines $\mathcal{C}S(\alpha)$ via the cocycle $\rho_{g_1,g_2}^\alpha: G \to U(1)$ on $\mathcal{C}(G)//G$ (see Willerton [Wil08], Section 3.4)

$$
\rho_{g_1,g_2}^\alpha(h) = \frac{\alpha(h,g_2,g_1)\alpha(hg_1h^{-1},h,g_2)\alpha(hg_2h^{-1},hg_1h^{-1},h)}{\alpha(h,g_1,g_2)\alpha(hg_2h^{-1},h,g_1)\alpha(hg_1h^{-1},hg_2h^{-1},h)}
$$

where $(g_1,g_2) \in \mathcal{C}(G)$ is an object and $h \in G$ determined a morphism. A computation of Freed-Quinn shows that this line bundle is $SL_2(\mathbb{Z})$-invariant, by which we mean the line descends to $\mathcal{C}(G)//(G \times SL_2(\mathbb{Z}))$ with a cocycle determined by the same formula as above (apply the orbit-stabilizer theorem to the equation just before 5.10 in Proposition 5.8 of [FQ93]). We pull this line bundle back to $\Phi_2^{21}(\mathcal{X})$ along the map

$$
sL \times \mathcal{C}(G)//(G \times SL_2(\mathbb{Z}) \times (\mathbb{R}^{21} \rtimes \mathbb{C}^\times)) \to \mathcal{C}(G)//(G \times SL_2(\mathbb{Z})),
$$

induced from the map $sL \to \star$ and homomorphism $G \times SL_2(\mathbb{Z}) \times (\mathbb{R}^{21} \rtimes \mathbb{C}^\times) \to G \times SL_2(\mathbb{Z})$. Since by construction the cocycle is $\mathbb{R}^{21}$-invariant, this constructs a line bundle $\mathcal{C}S(\alpha)$ over $\Phi_2^{21}(\star//G)$.

Consider the commutative square

$$
\begin{array}{ccc}
\Phi_0(X//G) & \to & \Phi_0(\star//G) \\
\downarrow & & \downarrow \\
\Phi_2^{21}(X//G) & \to & \Phi_2^{21}(\star//G)
\end{array}
$$

(10)

where the vertical arrows are inclusions of stacks and the horizontal arrows are induced by the canonical map $X//G \to \star//G$. The line bundle $\mathcal{C}S(\alpha)$ over $\Phi_2^{21}(\star//G)$ can be pulled back to one over $\Phi_2(X//G)$ that we also denote by $\mathcal{C}S(\alpha)$. As constructed, this line is natural for maps of $G$-manifolds.

**Remark** 3.10. Willerton [Wil08] has observed that the line $\mathcal{C}S(\alpha)$ over $\mathcal{C}(G)//G$ is precisely the line bundle in which characters of $\alpha$-projective representations of the loop groupoid...
\[ L(\ast//G) \cong G//G \] 

takes values. Hence, the above line is a (super) conformal enlargement of such a loop groupoid character theory.

3.5. The proof of Theorem 1.3

**Definition 3.11.** For a line bundle \( \mathcal{L} \) over \( \Phi_2^{11}(X) \), a supersymmetric section of \( \mathcal{L} \) over \( \Phi_0^{21}(X) \) is a section in the image of the restriction map \( i^* : \Gamma(\Phi_2^{11}(X); \mathcal{L}) \to \Gamma(\Phi_0^{21}(X); \mathcal{L}) \). We denote the vector space of supersymmetric sections by \( \Gamma_{\text{susy}}(\Phi_0^{21}(X); \mathcal{L}) \).

**Proof of Theorem 1.3.** Cocycles in twisted equivariant elliptic cohomology over \( \mathbb{C} \) can be described as compatible closed differential forms on the coproduct of \( X^{(g_1,g_2)} \) indexed by \( \mathbb{C}[G] \) with values in functions on lattices satisfying certain properties. We will prove that supersymmetric sections of \( \omega^{j/2} \otimes \mathcal{CS}(\alpha) \) are in bijection with such forms.

Explicitly, we compute supersymmetric sections as functions on \( L \times \prod \pi TX^{(g_1,g_2)} \) that are invariant under the action of \( \mathbb{R}^{21} \times \text{SL}_2(\mathbb{Z}) \), equivariant with respect to the action of \( \mathbb{C}^\times \times G \), and extend to equivariant sections over \( sL \times \prod \pi TX^{(g_1,g_2)} \). We write such a function as \( \sum_i f_i(\ell, \bar{\ell}, \ell', \bar{\ell}') \otimes \alpha_i \in \mathcal{C}^\infty(L \times \prod \pi TX^{(g_1,g_2)}) \) for \( f_i(\ell, \bar{\ell}, \ell', \bar{\ell}') \in \mathcal{C}^\infty(L) \) and \( \alpha_i \in \mathcal{C}^\infty(\pi TX^{(g_1,g_2)}) \).

The action of \( \mathbb{R}^{21} \) factors through the action of \( \mathbb{R}^{01} \) on \( \prod \pi TX^{(g_1,g_2)} \), where it is generated by the action of the de Rham d on differential forms. Hence the \( \alpha_i \) must be closed. A section being supersymmetric means that it is closed under the operators \( Q \) and \( R \) of Lemma 3.9 when we view it as an element of \( \mathcal{C}^\infty(sL \times \prod \pi TX^{(g_1,g_2)}) \supset \mathcal{C}^\infty(L \times \prod \pi TX^{(g_1,g_2)}) \). Invariance under \( R \) follows from the \( \alpha_i \) being closed; for \( \alpha_i \) of homogenous degree, invariance under \( Q \) requires

\[
\sum_i f_i(\ell, \bar{\ell}, \ell', \bar{\ell}') \otimes \alpha_i = \sum_i (\bar{\ell}' - \bar{\ell})^{\text{deg}(\alpha_i)/2} F(\ell, \ell') \otimes \alpha_i,
\]

where \( F(\ell, \ell') \) is a holomorphic function on \( L \), i.e., a function independent of \( \bar{\ell} \) and \( \bar{\ell}' \). Since the dependence of \( \bar{\ell} \) and \( \bar{\ell}' \) is completely determined by the degree of the form, we may identify supersymmetric sections with a closed form with values in holomorphic functions on \( L \).

Equivariance under the \( G \)-action implies that a section is determined on components indexed by \( \mathbb{C}[G] \), for any choice of representatives. This immediately yields

\[
\Gamma_{\text{susy}}(\Phi_2^{11}(X); \omega^{j/2} \otimes \mathcal{CS}(\alpha)) = \bigoplus_{i+j=l} \left( \bigoplus_{[i,j]} \left( \left( \Omega^i_{\text{cl}}(X^{(g_1,g_2)}) \otimes \mathcal{O}^j(L) \otimes (\rho_{[g_1,g_2]}^\alpha)^{-1} \right)^{\mathcal{C}(g_1,g_2)} \right)^{\text{SL}_2(\mathbb{Z})} \right)
\]

where in the above \( \rho_{[g_1,g_2]}^\alpha \) denotes the restriction of \( \rho_{[g_1,g_2]}^\alpha \) to \( C(g_1,g_2) \). But this restriction is precisely the character \( (\chi_{\alpha_{g_1,g_2}})^{-1} \), which concludes the proof of the theorem.

3.6. Restriction, induction, and Chern-Simons theory. For groupoids \( X//H \) and \( X//G \) and a homomorphism \( \lambda : H \to G \) inducing a morphism of Lie groupoids \( X//G \to X//H \), naturality yields restriction functors

\[
\Phi_0^{21}(\lambda)^* : \Gamma_{\text{susy}}(\Phi_0^{21}(X//G); \omega^{j/2} \otimes \mathcal{CS}(\alpha)) \to \Gamma_{\text{susy}}(\Phi_0^{21}(X//H); \omega^{j/2} \otimes \mathcal{CS}(\lambda^* \alpha)).
\]

The focus of this subsection is the construction of pushforwards along \( \Phi_0^{21}(\lambda) \). As in the 11-dimensional case, there are two options related to Freed-Quinn quantization of 21-dimensional gauge theories (which runs in parallel to the construction of N. Ganter [Gan09], Section 5.2) and the quantization of Chern-Simons theory as a 3-dimensional gauge theory. The former recovers the Hopkins-Kuhn-Ravenel character formula when \( \lambda \) is an inclusion of groups.

We start with the case related to 21-dimensional field theories. Fix \( \alpha : G \times G \times G \to U(1) \) and let \( \lambda^* \alpha \) denote the pullback 3-cocycle on \( H \). The homotopy fiber of the induced functor
\[ \Phi_{0}^{21}(X//H) \to \Phi_{0}^{21}(X//G) \] at the S-point \((\Lambda, \phi)\) of \(\Phi_{0}^{21}(X//G)\) has as objects
\[ \text{Ob}(\text{hofib}(\lambda(\Lambda, \phi))) \cong \{ (\Lambda, \tilde{\phi}) \in \Phi_{0}^{21}(X//H), \quad \iota \circ \lambda \circ \tilde{\phi} \cong \phi \} \]
and morphisms are isomorphisms \(\tilde{\phi} \cong \tilde{\phi}'\) in the groupoid of maps \(S \times \Lambda \mathbb{R}^{21} \to X//H\). Since \(H\) and \(G\) are finite, the homotopy fibers have finitely many isomorphism classes. To define a measure on the homotopy fibers it suffices to choose a finite groupoid presentation and assign a weight to each object. Following Freed-Quinn [FQ93] Section 2.1, define the measure \(d\mu\) of \(\tilde{\phi}\) to be \(1/|\text{Aut}(\Lambda, \tilde{\phi})|\), where \(\text{Aut}(\Lambda, \tilde{\phi})\) denotes the automorphisms in the given groupoid presentation. They showed that this measure is well-behaved under equivalence of finite groupoids, which allows for the following definition.

**Definition 3.12.** Define \(\lambda : \Gamma_{\text{susy}}(\Phi_{0}^{21}(X//H); \omega^{*}/\otimes \mathcal{C}(\alpha)) \to \Gamma_{\text{susy}}(\Phi_{0}^{21}(X//G); \omega^{*} \otimes \mathcal{C}(\lambda^* \alpha))\) by
\[ \lambda(s)(\Lambda, \phi) := \sum_{\text{Ob}(\text{hofib}(\lambda(\Lambda, \phi)))} \iota^* s(\Lambda, \tilde{\phi}) \cdot d\mu, \quad s \in \Gamma_{\text{susy}}(\Phi_{0}^{21}(X//H); \omega^{*}/\otimes \mathcal{C}(\lambda^* \alpha)), \]
where \(\lambda(s)(\Lambda, \phi)\) and \(s(\Lambda, \tilde{\phi})\) denote the respective values of the sections \(\lambda(s)\) and \(s\) at the S-point determined by \((\Lambda, \phi)\) and \((\Lambda, \tilde{\phi})\), respectively.

To give an explicit formula for \(\lambda\), we identify a section \(s \in \Gamma(\Phi_{0}(X//H); \mathcal{C}(\alpha) \otimes \omega^{*}/\otimes)\) with a section over the groupoid
\[ \left( \prod_{(h_{1}, h_{2}) \in \mathcal{C}(H)} L \times \pi TX^{(h_{1}, h_{2})} \right) \mathcal{C}(H) \rightarrow (\text{SL}_{2}(\mathbb{Z}) \times H \times \mathbb{R}^{21} \times \mathbb{C}^\times), \]
and we will write \(\lambda(s)(\Lambda, h_{1}, h_{2})\) as a section over the groupoid
\[ \left( \prod_{(g_{1}, g_{2}) \in \mathcal{C}(G)} L \times \pi TX^{(g_{1}, g_{2})} \right) \mathcal{C}(G) \rightarrow (\text{SL}_{2}(\mathbb{Z}) \times G \times \mathbb{R}^{21} \times \mathbb{C}^\times). \]
Identifying sections over the groupoid as functions on objects with transformation properties, let \(s(h_{1}, h_{2})\) denote the function on \(L \times \pi TX^{(h_{1}, h_{2})}\) gotten from restriction and similarly denote \(\lambda(s)(g_{1}, g_{2})\) as the restriction to \(L \times \pi TX^{(g_{1}, g_{2})}\). Since \(\lambda\) induces a homomorphism of Lie groupoids, we have induced diffeomorphisms \(X^{(h_{1}, h_{2})} \to X^{(\lambda(h_{1}), \lambda(h_{2}))}\). We also have diffeomorphisms \(X^{(g_{1}, g_{2})} \cong X^{(g_{1}g_{2}, g_{1}g_{2})}\) from the left action of \(g\) on \(X\). Hence, the function \(s(h_{1}, h_{2})\) uniquely determines a function on \(\mathbb{R}^{21} \times \pi TX^{(g_{1}, g_{2})}\) when \((\lambda(h_{1}), \lambda(h_{2}))\) is conjugate to \((g_{1}, g_{2})\) via \(g\); this is precisely \(\iota^* s(h_{1}, h_{2})\). We compute
\[ \lambda(s)(g_{1}, g_{2}) = \frac{1}{|H|} \sum_{(h_{1}, h_{2}) \in \mathcal{C}(H), g \in \mathbb{R}^{21}, \lambda(h_{1})g^{-1}=g, \lambda(h_{2})g^{-1}=g_{2}} \iota^* s(h_{1}, h_{2}) \]
where the sum is over \((h_{1}, h_{2}) \in \mathcal{C}(H)\) whose image in \(\mathcal{C}(G)\) is conjugate to \((g_{1}, g_{2})\), i.e., principal \(H\)-bundles \(P \to S \times \mathbb{R}^{21}\) with holonomy \((h_{1}, h_{2})\) for which the associated bundle \(P \times \Lambda G\) is isomorphic to a \(G\)-bundle with holonomy \((g_{1}, g_{2})\). Since the 3-cocycle \(\lambda^* \alpha\) on \(H\) is the pullback of a cocycle on \(G\), the sum indeed has the appropriate equivariance property for the action of \(G\), so gives the claimed map on supersymmetric sections. In the case that \(X = *\) and \(H \to G\) is an inclusion, we observe the the above agrees with the Hopkins-Kuhn-Ravenel higher character formula [HKR00], Theorem D. When \(X = *\) and \(H \to \{e\}\), we obtain a sum over all \(H\)-bundles on super tori, which is a version of quantization for a 2-dimensional gauge theory.

**Remark 3.13.** The pullback of a 3-cocycle \(\alpha : G \times G \times G \to U(1)\) along an inclusion \(i : H \to G\) allows one to form a 2-group from \(i^* \alpha\) and \(H\) with a faithful functor to the 2-group defined by \(\alpha\) and \(G\). From this perspective, the above constructs induction functors for faithful morphisms between essentially finite 2-groups. However, induction for arbitrary morphisms of 2-groups would require different techniques. For example, the above does not construct
a pushforward along the terminal map from a general 2-group determined by $H$ and $\alpha$ to the trivial 2-group: the only 3-cocycle on the trivial group is the trivial cocycle, which need not pull back to $\alpha$. It remains unclear if this is a bug or a feature of our construction.

Now we turn our attention to pushforwards
\[
\lambda^\text{CS}_H: \Gamma_{\text{susy}}(\Phi^0_0(2)(X//H); \omega^{*2} \otimes CS(\alpha)) \to \Gamma_{\text{susy}}(\Phi^0_0(2)(X//G); \omega^{*2} \otimes \lambda_\alpha CS(\alpha))
\]
for 3-cocycles $\alpha: H \times H \times H \to U(1)$. The target is sections of a vector bundle over $\Phi^0_0(2)(X)$—typically with fiber dimension greater than 1—gotten from taking sections of $CS(\alpha)$ along the homotopy fibers of the map $\Phi^0_0(2)(\lambda)$. Explicitly, this homotopy fiber is a groupoid and global sections along the fibers are the invariant sections over this groupoid of the pullback of $CS(\alpha)$ to the fiber. Given this definition, the value of $\lambda^\text{CS}_H$ on a given section $s$ is tautological: we simply view $s$ restricted to a homotopy fiber as defining a vector in the sections along the fiber.

**Example 3.14.** In the case $X = \ast$, and $\lambda: G \to \ast$ we have
\[
\Gamma_{\text{susy}}(\Phi^0_0(2)(\ast//G); CS(\alpha)) \cong \Gamma(\mathcal{C}(G)//G; CS(\alpha))
\]
In this case, the map $\lambda^\text{CS}_H$ takes global sections which we identify as the space of conformal blocks over the moduli of tori for Chern-Simons theory of the group $G$ and 3-cocycle $\alpha$.

**Remark 3.15.** It is unclear if this second sort of pushforward is useful in applications; instead we view it as clarifying one aspect of the role of 2-group representation theory in equivariant elliptic cohomology that we explain presently. By analogy to the 1|1-dimensional case, 2|1-dimensional field theories twisted by Chern-Simons theory of $G$ are the elliptic analog of modules over the twisted group algebra of $G$. These twisted field theories strictly generalize previous authors’ definitions of 2-group representations, e.g., Bartlett [Bar11], Frenkel and Zhu [FZ11], Ganter and Usher [GU13], Ostrrik [Ost03], and Willerton [Wil08]. This suggests a possible avenue that connects equivariant elliptic cohomology with a version of 2-group representation theory for which higher characters are elements of $\text{Ell}^\alpha_G(\ast)$.

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