A MAUREY TYPE RESULT FOR OPERATOR SPACES

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Abstract. The little Grothendieck theorem for Banach spaces says that every bounded linear operator between $C(K)$ and $\ell_2$ is 2-summing. However, it is shown in [7] that the operator space analogue fails. Not every cb-map $v : K \to OH$ is completely 2-summing. In this paper, we show an operator space analogue of Maurey’s theorem: Every cb-map $v : K \to OH$ is $(q, cb)$-summing for any $q > 2$ and hence admits a factorization $\|v(x)\| \leq c(q)\|v\|_cb\|axb\|_q$ with $a, b$ in the unit ball of the Schatten class $S_{2q}$.

1. Introduction

The theory of operator spaces investigates subspace of $C^*$-algebras with their inherited matricial structure. Many concepts from Banach space theory can be formulated in the setting of so-called “quantized Banach spaces”. In particular Grothendieck’s fundamental work on tensor norms leads to many interesting new problems in the context of operator algebras and operator spaces. Let us mention in particular Shlyakhtenko and Pisier’s version of Grothendieck’s theorem for operator spaces, Haagerup and Musat’s the very recent completion of Grothendieck’s theorem for $C^*$-algebras and the results in [7, 19, 20]. A fundamental object in the theory of operator spaces is Pisier’s operator space $OH$, the only operator space completely isometric to its anti-dual. Using their version of Grothendieck’s theorem for operator spaces, Pisier-Shlyakhtenko obtained a characterization of completely bounded maps $u : A \to OH$ for every $C^*$-algebra $A$: Indeed $u$ is completely bounded if and only if there exists a state $\phi$ and a constant $C > 0$ such that

$$\|u(x)\| \leq C[\phi(x^*x)\phi(xx^*)]^{\frac{1}{4}}.$$  

This characterization should be considered as analogue of the little Grothendieck’s theorem in the theory of Banach spaces. It is shown in [7] that a straightforward translation

$$\pi_2^u(u) \leq C\|u\|_{cb}$$  

does not hold in general, and not even uniformly for finite dimensional $C^*$-algebra’s $A$. We will define the completely 2-summing norm $\pi_2^u$ below.

In this paper we will approach from a different angle. Let us first recall the classical Banach space theory. Let $X$ be a Banach space with cotype $q$, i.e.

$$\left(\sum_k \|Tx_k\|_X^q\right)^{\frac{1}{q}} \leq c_q(X)E\sum_k \varepsilon_k x_k \|x\|_X$$

holds for all finite families $x_1, ..., x_n \in X$, where $(\varepsilon_k)_{k\geq 1}$ is the classical Rademacher sequence and $E$ is the corresponding expectation. Maurey ([14]) showed that for a Banach space with cotype $q$ we have

$$L(C(K), X) = \Pi_p(C(K), X)$$  

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holds for every \( p > q \) and every space of continuous functions \( C(K) \). Even for \( X = L_q([0,1]) \) and \( q > 2 \) the result is not true for \( p = q \). Let us recall that a map \( T : X \to Y \) is \( p \)-summing if

\[
\left( \sum_{k=1}^n \|Tv(e_k)\|_Y^p \right)^{\frac{1}{p}} \leq C\|v : \ell_p^p \to X\|.
\]

Then \( \pi_p(T) = \inf C \), where the infimum is taken over all constants satisfying the inequality above for arbitrary \( u \). In the setting of operator spaces we can easily adapt this notation and say that \( T : E \to F \) is \((p,cb)\)-summing if

\[
\left( \sum_{k=1}^n \|Tv(e_k)\|_F^p \right)^{\frac{1}{p}} \leq C\|v : \ell_p^p \to E\|_{cb}.
\]

holds for some constant \( C \). As above we define \( \pi_{p,cb}(T) = \inf C \) and \( \Pi_{p,cb}(E,F) \) as the space of \((p,cb)\)-summing maps. Very little is known about the right concept of cotype \( q \), although some attempts have been made in the literature (see [15, 9] and [11, 12]). Clearly, we should expect that \( OH \) has cotype 2. In this sense our main result is an operator space version of Maurey’s theorem:

**Theorem 1.1.** Let \( 2 < q < \infty \). Then

\[
CB(B(H), OH) \subseteq \Pi_{q,cb}(B(H), OH).
\]

The factorization theory for \((q, cb)\)-maps is very satisfactory (see [5, 17]). In the finite dimensional setting, the result reads as follows: Let

\[
u : M_m \to OH
\]

be a completely bounded map. Then there are positive elements \( a, b \) with

\[
\|a\|_{S_{2q}}, \|b\|_{S_{2q}} \leq 1
\]

such that

\[
\|u(x)\| \leq c(q)\|u\|_{cb}\|axb\|_q.
\]

Note that the statement fails for \( q = 2 \) and indeed we have \( c(q) \leq c_0(\frac{q^2}{q-2})^\frac{1}{2} \) for some constant \( c_0 \).

The definition of \((p, cb)\)-summing maps lies in between Banach space and operator space theory. In operator space theory a map is called completely \( p \)-summing if

\[
\|Tv(e_{ij})\|_{S_p^p(F)} \leq C\|v : S_p^p \to E\|_{cb}.
\]

Then \( \pi_p^c(T) = \inf C \), and \( \Pi_p^c(E,F) \) is the space of completely \( p \) summing maps between operator spaces \( E \) and \( F \). Let us recall that for a matrix \( x = [x_{ij}] \) with values in \( E \) the norm in \( S_p^p(E) \) is defined as

\[
\|x\|_{S_p^p(E)} = \inf_{x_{ij} = \sum_{a_i, b_i, y_{ik}, b_{kj}} |a|\|a\|_2\|y_{ij}\|_{M_{n}(E)}\|b\|_{2p},
\]

where \( a = [a_{ij}] \) and \( b = [b_{ij}] \). Note that every operator space carries a natural family of matrix norms \( M_n(E) \). We refer to [17] for more details and properties of the vector-valued noncommutative \( L_p \) spaces. It is well-known that completely \( p \)-summing maps are completely bounded. Therefore it is tempting to formulate the following strengthening of our result.

**Problem:** Let \( 2 < q < \infty \). It is true that

\[
CB(B(H), OH) = \Pi_q^c(B(H), OH)? \tag{1.3}
\]

Our approach to Theorem 1.1 uses duality. We first show that the conclusion is equivalent to

\[
\ell_p(\ell_2) = \Pi_q^c(OH, \ell_p). \tag{1.4}
\]
Let us note that also (1.3) is equivalent to
\[ S_p(OH) = \Pi_p^o(OH, S_p). \]
Here \( \frac{1}{p} + \frac{1}{q} = 1 \) is the conjugate index. Following the general theory of completely 1-summing maps we can realize the space \( \Pi_p^o(OH, \ell_p) \) as a subspace of a noncommutative \( L_1 \) space. Here we invoke the results and methods from the recent paper [10] which shows that \( \ell_p \) is completely isomorphic to subspace of a noncommutative \( L_1 \) space with respect to a von Neumann algebra with QWEP. Recall that a \( C^* \)-algebra \( A \) has WEP (Lance’s weak expectation property) if the inclusion map \( i_A : A \hookrightarrow A^{**} \) factors completely positively and completely contractively through \( B(H) \) for some Hilbert space \( H \). A \( C^* \)-algebra \( B \) has QWEP if there is a WEP \( C^* \)-algebra \( A \) and two sided ideal \( I \subseteq A \) such that \( B \cong A/I \).

Based on recent results of Xu on embedding results using tools from real interpolation theory (see [27]) and Pisier’s concrete embedding of \( OH \) using generalized free gaussian variables (see [20] and [19]), we can identify a rather concrete em- interpolation theory (see [27]) and Pisier’s concrete embedding of \( OH \) which shows that \( \ell_p \) is completely isomorphic to subspace of a noncommutative \( L_1 \) space. Here we invoke the results and methods from the recent paper [10] which shows that \( \ell_p \) is completely isomorphic to subspace of a noncommutative \( L_1 \) space with respect to a von Neumann algebra with QWEP. Recall that a \( C^* \)-algebra \( A \) has WEP (Lance’s weak expectation property) if the inclusion map \( i_A : A \hookrightarrow A^{**} \) factors completely positively and completely contractively through \( B(H) \) for some Hilbert space \( H \). A \( C^* \)-algebra \( B \) has QWEP if there is a WEP \( C^* \)-algebra \( A \) and two sided ideal \( I \subseteq A \) such that \( B \cong A/I \).

Based on recent results of Xu on embedding results using tools from real interpolation theory (see [27]) and Pisier’s concrete embedding of \( OH \) using generalized free gaussian variables (see [20] and [19]), we can identify a rather concrete embedding of \( \Pi_p^o(S_p, OH) \). We are then able to show that at least for the identity \( id : \ell_p^n \rightarrow \ell_2^n = OH_n \) we have
\[ \pi_1^n(id : \ell_p^n \rightarrow \ell_p^n) \sim \left( \frac{q}{q^2 - 2} \right)^{\frac{1}{2}} n^{\frac{1}{p'}}. \] (1.5)

Unfortunately, calculating this norms turns out to be rather delicate and requires a detailed case by case analysis in a 8-term quotient space. Using further properties of our concrete realization, we can then find an intermediate vector-valued Orlicz norm estimating the completely 1-summing from above and the norm in \( \ell_p(\ell_2) \) from below. Testing the Orlicz norm on the sum of the unit vectors we obtain the full result from [13]. Using the theory of tensor norms in operator space, we can formulate the following application.

**Corollary 1.2.** Let \( 1 < p < 2 \). Then
\[ \Pi_p^o(OH, \ell_p) = \Pi_p^o(OH, \ell_p) \]
with equivalent norms.

The paper is organized as follows. We collect some preliminaries in section 2. In section 3 we present the dual formulation (1.4). This requires us several embedding results into a noncommutative \( L_1 \) space, which will be given in the following section. In section 4 we combine the ideas of Junge, Xu, Pisier and Junge & Parcet of embedding \( OH \) and \( S_p \) \((1 < p < 2)\) into noncommutative \( L_1 \) spaces. In section 5 we use the information from the previous section to find a concrete embedding of \( \Pi_p^o(S_p, OH) \). In section 6 we do the calculation for the identity, which is crucial to our conclusion. In the last section we apply the “Orlicz space argument” by Junge and Xu to explain that the result for the identity is enough to show our main result.

2. Preliminaries and Notations

We assume that the reader is familiar with standard concepts in operator algebra ([23, 24]) and operator space theory ([3, 18]).

For two operator spaces \( E_0 \) and \( E_1 \) we denote their \( \ell_p \)-direct sum by \( E_0 \oplus_p E_1 \) for \( 1 \leq p \leq \infty \) ([18]). If \((E_0, E_1)\) is a pair of operator spaces which is a compatible pair in the Banach space sense, then \( E_0 \oplus_p E_1 \) refers to the quotient operator space of \( E_1 \oplus_p E_2 \) by the subspace \( \{(x_0, x_1) : x_0 + x_1 = 0\} \). Similarly \( E_0 \cap_p E_1 \) refers to the diagonal subspace of \( E_0 \oplus_p E_1 \). Note that \( E_0 \oplus_p E_1 \)'s are all completely isomorphic for \( 1 \leq p \leq \infty \) with a universal constant and so are \( E_0 \oplus_p E_1 \)'s and \( E_0 \cap_p E_1 \)'s. When \( p = 1 \) we simply write \( E_0 +_p E_1 \) as \( E_0 + E_1 \). We will prefer \( E_0 +_2 E_1 \) and
For all \( E_0 \cap E_1 \) in section 4 to be more precise in constant, while we prefer \( E_0 + E_1 \) in the following sections since we have

\[
(E_0 + E_1) \otimes (F_0 + F_1) \cong (E_0 \otimes F_0) + (E_1 \otimes F_1)
\]

completely isometrically, where \( \otimes \) is the projective tensor product of operator spaces.

For a Hilbert space \( H \) we denote the column, the row and the operator Hilbert space on \( H \) by \( H^c \), \( H^r \) and \( H^{ab} \), respectively. For \( 1 \leq p \leq \infty \) and \( n \in \mathbb{N} \) we denote \( R_p^n = [R_n, C_n]_{\frac{1}{p}} \), where \( [\cdot, \cdot]_{\frac{1}{p}} \) implies complex interpolation in the operator space sense ([16]).

We will frequently use noncommutative \( L_1 \) spaces in this paper. For a \( \sigma \)-finite von Neumann algebra \( A \) with a distinguished normal faithful state \( \phi \) with density \( D \) the noncommutative \( L_1 \)-space in the sense of Haagerup is denoted by \( L_1(A) \) (= \( L_1(A, \phi) \)). There is a natural operator space structure on \( L_1(A) \) as the predual of \( A \).

Vector valued \( L_1 \)-spaces can be defined for \( R_1^n, C_1^n \) and \( OH_n \) as follows.

\[
L_1(A; R_1^n) := \left\{ \sum_{i=1}^n x_i \otimes e_{i1} \right\} \subseteq L_1(\mathcal{A} \otimes M_n),
\]

\[
L_1(A; C_1^n) := \left\{ \sum_{i=1}^n x_i \otimes e_{i1} \right\} \subseteq L_1(\mathcal{A} \otimes M_n)
\]

and

\[
L_1(A; OH_n) := \left[ L_1(A; R_1^n), L_1(A; C_1^n) \right]_{\frac{1}{p}}.
\]

Let \( A \) be a sub-von Neumann algebra of \( \mathcal{A} \) and \( E : \mathcal{A} \to A \) a normal faithful conditional expectation satisfying

\[
\phi = \phi|_A \circ E.
\]

Then, the space \( L_1^c(A, E) \) and \( L_1^c(A, E) \) ([2]) are defined by the completions of \( DA \) and \( AD \) under the norms

\[
\|Dx\|_{L_1^c(A, E)} = \|DE(xx^*)D\|_{L_1(A)}^{\frac{1}{2}} \quad \text{and} \quad \|xD\|_{L_1^c(A, E)} = \|DE(x^*x)D\|_{L_1(A)}^{\frac{1}{2}},
\]

respectively.

Since \( L_2(A) \) is a Hilbert space we can consider \( L_2^c(A) \) and \( L_2^c(A) \) endowed with operator space structures in the sense of \( R_1 = C \) and \( C_1 = R \), then their operator space structure can be described as follows. Let \( \text{tr}_A \) the unique tracial functional on \( L_1(A) \) satisfying

\[
\phi(a) = \text{tr}_A(aD)
\]

for all \( a \in A \). Then we have

\[
\left\| (I_{S_1^n} \otimes D^\frac{1}{2})a \right\|_{S_1^n(L_1^c(A))} = \left\| (I_{S_1^n} \otimes \text{tr}_A)((I_{S_1^n} \otimes D^\frac{1}{2})aa^*(D^\frac{1}{2} \otimes I_{S_1^n})) \right\|_{S_1^n}
\]

\[
= \left\| (I_{M_m} \otimes \phi)(aa^*) \right\|_{S_1^n}
\]

and

\[
\left\| b(D^\frac{1}{2} \otimes I_{S_1^n}) \right\|_{S_1^n(L_1^c(A))} = \left\| (I_{S_1^n} \otimes \text{tr}_A)((I_{S_1^n} \otimes D^\frac{1}{2})b^*b(I_{S_1^n} \otimes D^\frac{1}{2})) \right\|_{S_1^n}
\]

\[
= \left\| (I_{M_m} \otimes \phi)(b^*b) \right\|_{S_1^n}
\]

for \( a, b \in S_1^n \otimes A \) and \( m \in \mathbb{N} \).

We use the symbol \( a \lesssim b \) if there is a \( C > 0 \) such that \( a \leq Cb \) and \( a \sim b \) if \( a \lesssim b \) and \( b \lesssim a \).
3. The dual problem

We present a dual formulation of the original problem, which enables us to do
concrete calculations. For a linear map \( v : E \to F \) between operator spaces we
consider \( \Gamma_{\infty} \)-norm and \( \gamma_{\infty} \)-norm of \( v \) defined by
\[
\Gamma_{\infty}(v) = \inf \|\alpha\|_{cb} \|\beta\|_{cb},
\]
where the infimum is taken over all Hilbert space \( H \) and the factorization
\[
i_{F}v : E \overset{\alpha_{0}}{\to} B(H) \overset{\beta_{0}}{\to} F^{**}, \text{ where } i_{F} \text{ is the inclusion } F \hookrightarrow F^{**}
\]
and
\[
\gamma_{\infty}(v) = \inf \|\alpha\|_{cb} \|\beta\|_{cb},
\]
where the infimum is taken over all \( m \in \mathbb{N} \) and the factorization
\[
v : E \overset{\alpha}{\to} M_{m} \overset{\beta}{\to} F.
\]
See section 4 of [7] or [2] for the details.

**Theorem 3.1.** Let \( 1 < p < 2 \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then, the following conditions are
equivalent.

1. For any Hilbert space \( H \) we have
\[
CB(B(H), OH) \subseteq \Pi_{p', cb}(B(H), OH).
\]

2. There is a constant \( C > 0 \) such that
\[
\pi_{1}^{p}(T_{x} : OH \to \ell_{p}) \leq C \|x\|_{\ell_{p}(OH)}
\]
for all \( x \in \ell_{p}(OH) \) and \( T_{x} : OH \to \ell_{p} \), the linear map naturally associated
to \( x \).

3. \( \Pi_{p}^{\circ}(OH, \ell_{p}) \subseteq \Pi_{1}^{p}(OH, \ell_{p}). \)

**Proof.** \( (1) \Rightarrow (2) \)
By a standard density argument it is enough to consider \( n \)-dimensional case,
\( n \in \mathbb{N} \), \( \ell_{p}^{n}(OH_{n}) \) instead of \( \ell_{p}(OH) \). Then, since \( \Gamma_{\infty} = \gamma_{\infty} \) for linear maps between
finite dimensional spaces (see [2]) and \( \gamma_{\infty} \) is the trace dual of \( \pi_{1}^{p} \), (2) is equivalent to
\[
\|y\|_{\ell_{p}^{n}(OH_{n})} \leq C \cdot \Gamma_{\infty}(T^{y} : \ell_{p}^{n} \to OH_{n}) \tag{3.1}
\]
for all \( y \in \ell_{p}^{n}(OH_{n}) \) and \( T^{y} : \ell_{p}^{n} \to OH_{n} \), the linear map naturally associated to \( y \).

Now for any \( \epsilon > 0 \) we have a factorization \( T^{y} : \ell_{p}^{n} \overset{\alpha}{\to} B(H) \overset{\beta}{\to} OH_{n} \) with
\[
\|\alpha\|_{cb} \|\beta\|_{cb} \leq (1 + \epsilon) \Gamma_{\infty}(T^{y}).
\]
Then, for \( y = \sum_{i=1}^{n} e_{i} \otimes y_{i} \in \ell_{p}^{n}(OH_{n}) \) we have
\[
\|y\|_{\ell_{p}^{n}(OH_{n})} = \left( \sum_{i=1}^{n} \|y_{i}\|_{OH_{n}}^{p'} \right)^{\frac{1}{p'}} = \left( \sum_{i=1}^{n} \|T^{y}e_{i}\|_{OH_{n}}^{p'} \right)^{\frac{1}{p'}}
\leq \pi_{p', cb}(T^{y}) \left\| \sum_{i=1}^{n} e_{i} \otimes e_{i} \right\|_{\ell_{p'}^{n} \otimes \ell_{p}^{n}}
= \pi_{p', cb}(T^{y}) \left\| \ell_{p}^{n} \rightarrow \ell_{p'}^{n}, \ e_{i} \mapsto e_{i} \right\|_{cb}
= \pi_{p', cb}(T^{y}) \leq \pi_{p', cb}(\beta) \|\alpha\|_{cb}
\leq C \|\beta\|_{cb} \|\alpha\|_{cb} \leq C(1 + \epsilon) \Gamma_{\infty}(T^{y})
\]
for some constant \( C > 0 \) coming from the inclusion (1).

\( (2) \Rightarrow (1) \)
With the same reason as above it is enough to consider $OH_n$ instead of $OH$. Let $u : B(H) \to OH_n$. Then for any $(x_i)_{i=1}^n \subseteq B(H)$ and $v : \ell_p^n \to B(H)$, $e_i \mapsto x_i$ we have by (3.1)
\[
\left( \sum_{i=1}^n \|ux_i\|_{OH_n}^p \right)^{\frac{1}{p}} = \left( \sum_{i=1}^n \|uve_i\|_{OH_n}^p \right)^{\frac{1}{p}} \leq C \cdot \Gamma_\infty(uv)
\]
\[
\leq C \|u\|_{cb} \|v\|_{cb} = C \|u\|_{cb} \left\| \sum_{i=1}^m e_i \otimes x_i \right\|_{\ell_p^m \otimes_{\min} B(H)} ,
\]
which implies $\pi_{p',cb}(u) \leq C \|u\|_{cb}$.

(2) $\iff$ (3)
Again, we are enough to consider finite dimensional cases. Note that there is a completely isomorphic embedding $OH_n \hookrightarrow L_p(M)$ for a von Neumann algebra with QWEP. This noncommutative $L_p$ space is understood in the sense of Haagerup. Then, by Corollary 10 of [28] we have
\[
\pi_p^n(T_x : OH_n \to \ell_p^n) \sim \pi_p^n(T_x \circ i^* : i(OH_n)^* \to \ell_p^n)
\]
\[
= \left\| f_p \otimes i(x) \right\|_{\ell_p^n(L_p(M))} \sim \|x\|_{\ell_p^n(OH_n)}
\]
for any $x \in \ell_p^n(OH_n)$.

\[\square\]

**Remark 3.2.** By a similar argument as the above theorem we can show that for $1 < p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$ the followings are equivalent.

1. For any Hilbert space $H$ we have
\[
CB(B(H), OH) \subseteq \Pi_p^n(B(H), OH).
\]
2. There is a constant $C > 0$ such that
\[
\pi_p^n(T_x : OH \to S_p) \leq C \|x\|_{S_p(OH)}
\]
for all $x \in S_p(OH)$ and $T_x : OH \to S_p$, the linear map naturally associated to $x$.
3. $\Pi_p^n(OH, S_p) \subseteq \Pi_1^n(OH, S_p)$.

At the time of this writing we could not answer this question.

If we look at the condition (3'), then (3) of Theorem 3.1 is a particular case the above question, which we are dealing with diagonals. Thus, it is natural to consider columns and rows as the next candidate of particular cases. That is to say we are interested in the following question.

3. $\Pi_p^n(OH, C_p) \subseteq \Pi_1^n(OH, C_p)$. (resp. $\Pi_p^n(OH, R_p) \subseteq \Pi_1^n(OH, R_p)$)

which is true and can be explained in a similar way yet the calculation is much simpler.

Now we focus on the $n$-dimensional ($n \in \mathbb{N}$) case of (2) of Theorem 3.1. The right-hand side term $\|x\|_{\ell_p^n(OH_n)}$ is easy to describe, so the point is to describe the left-hand side term $\pi_p^n(T_x : OH_n \to \ell_p^n)$ in a concrete way.

Suppose there are embeddings
\[
OH \hookrightarrow E \subseteq L_1(M) \text{ and } \ell_p \hookrightarrow F \subseteq L_1(N)
\]
for some von Neumann algebras $M$ and $N$ with QWEP and cb-projections
\[
P : L_1(M) \to E \text{ and } Q : L_1(N) \to F
\]
with \( P|_E = I_E \) and \( Q|_F = I_F \), then by Lemma 4.4 and 4.5 in [7] we have
\[
\pi_1^0(T_x : OH_n \to \ell_p^n) \sim \pi_0^0(j \circ T_x \circ \iota^* : i(OH_n)^* \to j(\ell_p^n)) \\
= \|i \otimes j(x)\|_{L_1(M) \otimes L_1(N)}
\]
for all \( x \in OH_n \otimes \ell_p^n \). Thus, it would be the first task to find such embeddings with \( E \) and \( F \) are concrete spaces, which will be considered in the following section.

4. Embeddings of various spaces into the noncommutative \( L_1 \) space with respect to a von Neumann algebra with QWEP

4.1. Some aspects of real interpolation approach. When we want to embed \( OH \) into the predual of a von Neumann algebra it is very important to observe that it is completely isomorphic to a subspace of quotient of \( R \oplus C \) ([7] [19] [27]). Similarly, the embedding of \( S_p \) (\( 1 < p < 2 \)) ([19]) starts with the observation that \( C_p \) and \( R_p \) are completely isomorphic to a subspace of quotient of \( R \oplus OH \) and \( C \oplus OH \), respectively. In this section we review the real interpolation approach by Xu ([26] [27]) to the above observations.

Let \( 1 < p < \infty \), \( \theta = \frac{1}{p} \) and \( \alpha \in \mathbb{R} \). For a Banach space \( X \) we denote \( X \)-valued \( L_2(\mathbb{R}^+, t^{2\alpha \theta \beta}) \) space by \( L_2(t^{2\alpha \theta \beta}; X) \). Now we let
\[
K_0 = L_2^h(t^{-\theta}; \ell_2) + L_2^n(t^{1-\theta}; \ell_2) \quad \text{and} \quad J_0 = L_2^h(t^{-\theta}; \ell_2) \cap L_2^n(t^{1-\theta}; \ell_2).
\]

Let \( C_{\theta;K} \) be the subspace of \( K_\theta \) consisting of constant functions and \( C_{\theta;J} \) be the quotient space of \( J_\theta \) by the subspace of mean zero functions. If we look at the Banach space level then \( C_{\theta;K} \) and \( C_{\theta;J} \) are nothing but the interpolation of \( \ell_2 \) with itself, so that we clearly recover \( \ell_2 \) regardless of \( \theta \). However by posing column and row Hilbert space structure in the above way we get a completely isomorphic copy of \( C_p \), which now depends on \( \theta = \frac{1}{p} \). Note that \((C_{\theta;J})^* = C_{1-\theta;K} \) completely isometrically.

**Proposition 4.1.** Let \( 1 < p < \infty \) and \( \theta = \frac{1}{p} \). Then, \( C_p \) and \( C_{\theta;K} \) are completely isomorphic allowing constant depending only on \( \theta \). More precisely, we have
\[
\left\| \sum_{i,j=1}^n x_{ij} \otimes 1 \otimes e_{ij} \right\|_{M_m(C_{\theta;K})} \sim \theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}} \left\| \sum_{i,j=1}^n x_{ij} \otimes e_{ij} \right\|_{M_m(C_p)},
\]
where \( 1 \) implies the constant scalar function with value 1.

**Proof.** See Theorem 3.3 of [27]. Note that the factor of \( \theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}} \) was ignored in the proof, which should have appeared when we were dealing with the interpolation of two \( L_p \) spaces with different measures (see [17]).

For \( 1 < p < 2 \) we can consider two variations of the above interpolation. Now we pose row and operator (resp. column and operator) Hilbert space structure as follows, so that we get \( C_p \) (resp. \( R_p \)).

For \( 0 < \theta < 1 \) we let
\[
K_{c,\theta} = L_2^h(t^{-\theta}; \ell_2) + L_2^n(t^{1-\theta}; \ell_2), \quad K_{r,\theta} = L_2^h(t^{1-\theta}; \ell_2) + L_2^n(t^{-\theta}; \ell_2), \\
J_{c,\theta} = L_2^h(t^{-\theta}; \ell_2) \cap L_2^n(t^{1-\theta}; \ell_2) \quad \text{and} \quad J_{r,\theta} = L_2^h(t^{1-\theta}; \ell_2) \cap L_2^n(t^{-\theta}; \ell_2).
\]

Let \( C_{\theta;K} \) (resp. \( R_{\theta;K} \)) be the subspace of \( K_{\theta} \) (resp. \( K_{\theta} \)) consisting of constant functions and \( R_{\theta;J} \) (resp. \( C_{\theta;J} \)) be the quotient space of \( J_{\theta} \) (resp. \( J_{\theta} \)) by the subspace of mean zero functions. Note that
\[
(R_{\theta;J})^* = C_{\theta;K} \quad \text{(resp.} \quad (C_{\theta;J})^* = R_{\theta;K} \text{)}
\]
completely isometrically.

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Proposition 4.2. Let $1 < p < 2$, $\frac{1}{q} + \frac{1}{p} = 1$ and $\theta = \frac{2}{p}$. Then, $C_p$ and $C_{c,\theta,K}$ (resp. $R_p$ and $R_{r,\theta,K}$) are completely isomorphic allowing constant depending only on $\theta$. More precisely, we have

$$\left\| \sum_{i,j=1}^{n} x_{ij} \otimes 1 \otimes e_{ij} \right\|_{M_m(C_{c,\theta,K})} \sim \theta^{-\frac{1}{p}}(1-\theta)^{-\frac{1}{q}} \left\| \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \right\|_{M_m(C_p)}.$$  

The situation for $R_{r,\theta,K}$ is similar.

Proof. The following proof is similar to that of Theorem 3.3 of [27]. Recall that (Theorem 8.4 of [16]) for $x = (x_k) \in M_m(C_p)$ we have

$$\|x\|_{M_m(C_p)} = \sup \left\{ \left( \sum_{k \geq 1} \|ax_k b\|_2^{2} \right)^{\frac{1}{2}} : \|a\|_{S_p^m}, \|b\|_{S_p^{m'}} \leq 1, \ a, b > 0 \right\}.$$  

For fixed $a$ and $b$ with $\|a\|_{S_p^m}, \|b\|_{S_p^{m'}} \leq 1$ and $a, b > 0$ we consider

$$A_0 = L_{ap^\theta} \text{ and } A_1 = L_{a^{\frac{p}{p'}}} R_{b^{\frac{p'}{p}}},$$  

where $L_{\alpha}$ and $R_{\beta}$ implies left and right multiplications by $\alpha$ and $\beta$, respectively, on $H = \ell_2(S_p^m)$. Then $A_0$ and $A_1$ are commuting invertible positive bounded operators on $H$, and $A_i$ induces an equivalent norm $\|\cdot\|_i$ on $H$ as follows:

$$\|x\|_i := \|A_i x\|, \ i = 0, 1.$$  

Let $H_i$ be $H$ equipped with $\|\cdot\|_i$. Then $(H_0, H_1)$ becomes a compatible couple of Hilbert spaces, which can be identified as a couple of weighted $L_2$ spaces. Then by real interpolation of $L_2$ spaces with different weight (see [1]) we have

$$\left( \sum_{k \geq 1} \|ax_k b\|_2^{2} \right)^{\frac{1}{2}} = \|A_0^{1-\theta} A_1^\theta x\| \sim c_\theta^{-1} \|x\|_{(H_0, H_1)_{2,\theta,K}}$$  

for some $c_\theta \sim \theta^{-\frac{1}{p}}(1-\theta)^{-\frac{1}{q}}$.

Now we suppose $\|x\|_{M_m(C_{c,\theta,K})} < 1$. Then, there are $f \in M_m(L_2(t^{-\theta};\ell_2))$ and $g \in M_m(L_2^{h}(t^{1-\theta};\ell_2))$ such that $x = f(t) + g(t)$ for almost all $t \in (0, \infty)$,

$$\left\| \int_0^\infty \sum_{k \geq 1} f_k(t)f_k^*(t)t^{-2\theta} \frac{dt}{t} \right\|_{M_m} < 1$$  

and

$$\left\| \int_0^\infty \sum_{k \geq 1} g_k(t) \otimes g_k(t)t^{2(1-\theta)} \frac{dt}{t} \right\|_{M_m \otimes_{\min} M_m} < 1.$$  

Moreover, we have

$$\|f\|_{L_2(t^{-\theta};H_0)}^2 = \int_0^\infty \|f(t)\|_{H_0}^2 t^{-2\theta} \frac{dt}{t} = \int_0^\infty \sum_{k \geq 1} \text{tr}_m (a^p f_k(t)f_k^*(t)a^p) t^{-2\theta} \frac{dt}{t} = \text{tr}_m \left( a^{2p} \int_0^\infty \sum_{k \geq 1} f_k(t)f_k^*(t)t^{-2\theta} \frac{dt}{t} \right) \leq \|a^{2p}\|_1 \left\| \int_0^\infty \sum_{k \geq 1} f_k(t)f_k^*(t)t^{-2\theta} \frac{dt}{t} \right\|_{M_m} < 1.$$  

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and by (7.3) of [18]
\[
\|g\|_{L^2(t^{-\theta}, H_1)}^2 = \int_0^\infty \|g(t)\|_{H_1}^2 t^{2(1-\theta)} \frac{dt}{t} \\
= \int_0^\infty \sum_{k \geq 1} \text{tr}_m(a_k^* g_k(t)b_k g_k^*(t)a_k^*) t^{2(1-\theta)} \frac{dt}{t} \\
= \text{tr}_m \left( \int_0^\infty \sum_{k \geq 1} a^p g_k(t)b_k g_k^*(t) t^{2(1-\theta)} \frac{dt}{t} \right) \\
\leq \|a^p\|_2 \|b^p\|_2 \left\| \int_0^\infty \sum_{k \geq 1} g_k(t) \otimes g_k^*(t) t^{2(1-\theta)} \frac{dt}{t} \right\|_{M_m \otimes M_m} < 1
\]

Thus, we have \( \|x\|_{(H_0, H_1)_{\theta; K}} < \sqrt{\theta} \), and consequently
\[
C_{c,\theta; K} \subseteq C_p \quad \text{with cb-norm} \quad \leq c_\theta^{-1} \sqrt{\theta}.
\]

Using J-method we can similarly show that
\[
R_{c,\theta; I} \subseteq R_p \quad \text{with cb-norm} \quad \leq c_\theta \sqrt{\theta}.
\]

In this case we need to take \( A_0 = R_{\delta^n} \) and \( A_1 = L_n \otimes R_{\delta^n} \). Then, by duality we get the desired cb-isomorphism.

The proof for \( R_p \) and \( R_{c,\theta; K} \) is similar. \hfill \( \square \)

4.2. The case of \( OH \). In this section we consider the case of \( OH \), which was first done by Junge ([7]) and explained in different forms by Pisier ([19]) and Xu ([27]).

We will continue to employ the real interpolation approach as in the previous section. Now we set \( \theta = \frac{1}{2} \) and consider a discretization \( K_{\frac{1}{2}, \delta} \) (\( 1 < \delta \leq 2 \)) of \( K_{\frac{1}{2}} \) defined by
\[
K_{\frac{1}{2}, \delta} := \ell_2^\delta(\delta^{-\frac{1}{2}}; \ell_2) + \frac{1}{2} \ell_2^\delta(\delta^{\frac{1}{2}}; \ell_2),
\]
where \( \ell_2(\delta^{k\alpha}; \ell_2) \) denotes the weighted \( \ell_2(\mathbb{N}) \)-valued \( \ell_2 \) space on \( \mathbb{Z} \) with respect to the weight \( (\delta^{2k\alpha})_{k \in \mathbb{Z}} \). Then, \( K_{\frac{1}{2}, \delta} \) is \( \delta \)-completely isomorphic to \( K_{\frac{1}{2}} \). In order to show that \( K_{\frac{1}{2}, \delta} \) can be embedded into the predual of a von Neumann algebra we need some tools from free probability.

Let \( \mathcal{H} \) be a Hilbert space with Hilbert space basis \( (e_{\pm n})_{n \geq 1} \). Then we consider the full Fock space \( \mathcal{F}(\mathcal{H}) = \mathbb{C} \Omega \oplus_{n \geq 1} \mathcal{H}^\otimes n \), the left creation operator \( \ell(e) \) and the left annihilation operator \( \ell^*(e) \) on \( \mathcal{F}(\mathcal{H}) \) associated to \( e \in \mathcal{H} \). Let
\[
g_n = \lambda_n^{\frac{1}{2}} \ell(e_n) + \lambda_n^{\frac{1}{2}} \ell^*(e_{-n})
\]
for some sequence \((\lambda_n)_{n \geq 1}\) of strictly positive real numbers. These \( g_n \)‘s are called “generalized circular elements” by Shlyakhtenko ([22] [20]), and it is well known that the von Neumann algebra \( \mathcal{M} \) generated by \( \{g_n : n \geq 1\} \) has QWEP. Moreover, if we let \( D_\Phi \) be the density of the vector state \( \Phi \) on \( \mathcal{M} \) determined by the vacuum vector \( \Omega \), then \( D_\Phi^\frac{1}{2} g_n D_\Phi^\frac{1}{2} \in L_1(\mathcal{M}) \) and the operator space
\[
G_* = \overline{\text{span}}\{D_\Phi^\frac{1}{2} g_n D_\Phi^\frac{1}{2} : n \geq 1\} \subseteq L_1(\mathcal{M})
\]
is 2-completely isomorphic to \( \ell_2^\delta(\mathbb{N}, \lambda_n^{\frac{1}{2}}) + \ell_2^\delta(\mathbb{N}, \lambda_n^{\frac{1}{2}}) \) and is 2-completely complemented in \( L_1(\mathcal{M}) \). Note that we have ([22]
\[
D_\Phi^\frac{1}{2} g_n D_\Phi^\frac{1}{2} = \lambda_n g_n D_\Phi = \lambda_n^{-1} D_\Phi g_n.
\]

Now we go back to our original concern \( K_{\frac{1}{2}, \delta} \). If we set
\[
\mathcal{H} = \ell_2(\mathbb{Z}; \ell_2) \oplus_2 \ell_2(\mathbb{Z}; \ell_2)
\]
with basis \( \{e_k \otimes e_j : k \in \mathbb{Z}, j \in \mathbb{N}\} \cup \{f_k \otimes e_j : k \in \mathbb{Z}, j \in \mathbb{N}\} \) and 
\[ \lambda_{k,j} = \delta^k \text{ for } k \in \mathbb{Z} \text{ and } j \in \mathbb{N}, \]
then the corresponding \( \mathcal{M}_\mathbb{N} (= \mathcal{M}_\mathbb{N}^0) = \{g_{k,j} : k \in \mathbb{Z}, j \in \mathbb{N}\}' \), where 
g_{k,j} (= g_{k,j}^\delta) = \delta^k \ell(e_k \otimes e_j) + \delta^j \ell^*(f_k \otimes f_j),
and 
\[ G_n^\delta (= G_n^\delta(\delta)) = \varprojlim \{D_{k,j}^\delta \begin{pmatrix} \rho_{k,j}g_{k,j}D_{k,j}^\delta \end{pmatrix} : k \in \mathbb{Z}, j \in \mathbb{N}\} \subseteq L_1(\mathcal{M}_\mathbb{N}) \]
is our desired embedding.

More precisely if we set \( M(j) = \{g_{k,j} : k \in \mathbb{Z}\}' \), then \( M_j \)'s are all isomorphic and free each other. Let \( \phi_j \) be the restriction of \( \Phi \) on \( M(j) \), and we set 
\[ (\mathcal{M}_n, \Phi) = *_{j=1}^n (M(j), \phi_j). \]
Note that \( \mathcal{M}_\infty = \mathcal{M}_\mathbb{N} \). Now we denote 
\[ M(1), \phi_1 \text{ and } (g_{k,1})_{k \in \mathbb{Z}} \text{ by simply } M, \phi \text{ and } (g_k)_{k \in \mathbb{Z}}, \]
respectively, and let \( \rho_j : M \hookrightarrow \mathcal{M}_n = *_{j=1}^n M_j \) be the natural embedding into the \( j \)-th component. Then since 
\[ \rho_j(g_k) = g_{k,j} \text{ and } \rho_j(D_{\phi}^k x D_{\phi}^j) = D_{\phi}^k \rho_j(x) D_{\phi}^j, \]
for the density \( D_{\phi} \) of \( \phi \) and \( x \in M \) we have the following with the help of Proposition 4.1. This observation is a combination of the ideas in [27] and [19].

**Proposition 4.3.** Let \( n \in \mathbb{N} \cup \{\infty\} \) and \( 1 < \delta \leq 2 \). Then \( OH_n \) is cb-embedded in a completely complemented subspace 
\[ G_n^\delta = \varprojlim \{D_{k,j}^\delta g_{k,j}D_{k,j}^\delta : k \in \mathbb{Z}, 1 \leq j \leq n\} \subseteq L_1(\mathcal{M}_n) \]
with the constants independent of \( \delta \) and \( n \) by the following embedding.
\[ v_n^\delta : OH_n \rightarrow G_n^\delta \subseteq L_1(*_{j=1}^n M_j), \quad e_j \mapsto \rho_j \left( \sum_{k \in \mathbb{Z}} D_{\phi}^k g_{k,j} D_{\phi}^j \right) = \sum_{k \in \mathbb{Z}} D_{\phi}^k g_{k,j} D_{\phi}^j. \]

4.3. **The case of \( S_p \) (\( 1 < p < 2 \)).** In this section we consider the case of \( S_p \) (\( 1 < p < 2 \)) following the very recent work of Junge and Parcet ([10]). The starting point of this embedding is the factorization 
\[ S_p = C_p \otimes h R_p \]
and cb-embeddings
\[ C_p \hookrightarrow (R \oplus_2 OH)/(R \cap_2 \ell_2^{oh}(\lambda))' \text{ and } R_p \hookrightarrow (C \oplus_2 OH)/(C \cap_2 \ell_2^{oh}(\lambda))' \]
obtained by a generalized version of “Pisier’s exercise”, where \( \ell_2^{oh}(\lambda) \) means the operator Hilbert space on the weighted \( \ell_2 \) space with respect to the weight \( \lambda^2 \) for a sequence of strictly positive real numbers \( \lambda = (\lambda_k)_{k \geq 1} \).

The next step is to consider a diagonal operator \( d_{\lambda} = \sum_k \lambda_k \delta_{kk}, \) which can be regarded as the density \( D_{\psi} \) associated to a normal strictly semifinite faithful (n.s.s.f. in short) weight \( \psi \) on \( B(\ell_2) \). Let \( q_n \) be the projection \( \sum_{k \leq n} \delta_{kk} \) and \( \psi_n \) be the restriction of \( \psi \) to the subalgebra \( M_n = q_n B(\ell_2) q_n \). Now we set 
\[ k_n = \psi_n(q_n) = \sum_{k=1}^n \lambda_k^2, \]
and let \( \varphi_n \) and \( \varphi_n \) be states on \( M_n \) and \( M_n \oplus M_n \), respectively, defined by 
\[ \varphi_n = \psi_n/k_n \text{ and } \varphi_n(x,y) = \frac{1}{2}(\varphi_n(x) + \varphi_n(y)) \]
for \( x, y \in M_n \).

If \( k_n \) is an integer, then we have a nice embedding of
\[ K_{1,2}(\psi_n) = [(R_n \oplus_2 OH_n)/(R_n \cap_2 \ell_2^{oh}(\lambda))'] \otimes h [(C_n \oplus_2 OH_n)/(C_n \cap_2 \ell_2^{oh}(\lambda))'] \]
as follows.

**Proposition 4.4.** Assume that $k_n = \sum_{k=1}^n \lambda_k^4$ is an integer and define

$$A_n = \bigoplus_{j=1}^{k_n} (M_n \oplus M_n, \varphi_n).$$

If $\pi_j : M_n \oplus M_n \to A_n$ is the natural embedding into the $j$-th component of $A_n$, then the mapping

$$w_n : K_{1,2}(\psi_n) \to L_1(A_n; OH_{k_n}), \ x \mapsto \frac{1}{k_n} \sum_{j=1}^{k_n} \pi_j(x, -x) \otimes e_j$$

is a cb-embedding with constants independent of $n$.

**Proof.** See Lemma 2.11 of [10].

Combining with Proposition 4.3 we get an embedding $K_{1,2}(\psi_n) \hookrightarrow L_1(A_n \overline{\otimes} \mathcal{M}_{k_n})$ by $(I_{L_1(A_n)} \otimes \psi) \circ w_n$. Now we consider the embedding for

$$K_{1,2}(\psi) = [(R \oplus_2 OH) / (R \cap_2 \ell_2(\lambda)^{oh})] \otimes_h [(C \oplus_2 OH) / (C \cap_2 \ell_2(\lambda)^{oh})].$$

Note that we may assume that $k_n = \sum_{k=1}^n \lambda_k^4$ are non-decreasing positive integers since we may approximate each $k_n$ by its closest integer. This allows us to recover $K_{1,2}(\psi)$ by a completely isometric embedding

$$K_{1,2}(\psi) = \bigcup_{n \geq 1} K_{1,2}(\psi_n) \hookrightarrow \prod_{n,d} K_{1,2}(\psi_n).$$

Thus, according to [21] we get a cb-embedding

$$K_{1,2}(\psi) \hookrightarrow L_1(B)$$

with $B = \left( \prod_{n,d} (A_n \overline{\otimes} \mathcal{M}_{k_n})^* \right)^*,$

and by the stability of QWEP with respect to free product, tensor product and ultraproduct ([6, 7]) $B$ also satisfies QWEP.

However the embedding above is not appropriate for our purpose, since we do not know whether $K_{1,2}(\psi)$ itself is cb-complemented in $L_1(B)$ or not, so that we need to find another embedding of $K_{1,2}(\psi)$ which is cb-complemented in the noncommutative $L_1$ space with respect to a von Neumann algebra with QWEP. We will use the following noncommutative version of Rosenthal’s inequality for identically distributed random variables in $L_1$ from [8] and [10].

Let $\mathcal{N}$ and $\mathcal{A}$ be $\sigma$-finite von Neumann algebras with a normal faithful conditional expectation $E_{\mathcal{N}} : \mathcal{A} \to \mathcal{N}$. We recall that a family of von Neumann algebras $(\mathcal{A}_k)_{k \geq 1}$ satisfying $\mathcal{N} \subseteq \mathcal{A}_k \subseteq \mathcal{A}$ is a system of symmetrically independent copies over $\mathcal{N}$ (s.t.c. in short) when

1. If $a \in (A_1, \cdots, A_{k-1}, A_{k+1}, \cdots)$ and $b \in A_k$, then we have

   $$E_{\mathcal{N}}(ab) = E_{\mathcal{N}}(a)E_{\mathcal{N}}(b).$$

2. There is a von Neumann algebra $\mathcal{A}$ containing $\mathcal{N}$, a normal faithful conditional expectation $E_0 : \mathcal{A} \to \mathcal{N}$ and isomorphisms $\pi_k : \mathcal{A} \to \mathcal{A}_k$ such that

   $$E_{\mathcal{N}} \circ \pi_k = E_0$$

   and the following holds for every permutation $\alpha$ of the integers

   $$E_{\mathcal{N}}(\pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m)) = E_{\mathcal{N}}(\pi_{\alpha(j_1)}(a_1) \cdots \pi_{\alpha(j_m)}(a_m)).$$

3. There is a normal faithful conditional expectation $E_k : \mathcal{A} \to \mathcal{A}_k$ such that

   $$E_{\mathcal{N}} = E_0 \pi_k^{-1} E_k.$$
Proposition 4.5. Let $\mathcal{N}$, $\mathcal{A}$ and $(A_k)_{k \geq 1}$ are as before and $(A_k)_{k \geq 1}$ is a system of s.i.c. over $\mathcal{N}$. Then for $x \in L_1(A)$ with $E_0(x) = 0$ we have

$$\left\| \sum_{k=1}^{n} \pi_k(x) \right\|_{L_1(A)} \sim \inf_{x = z_1 + z_2 + z_3} n \| x_1 \|_{L_1(A)} + n^{\frac{1}{2}} \| x_2 \|_{L_1(A,E_0)} + n^{\frac{1}{2}} \| x_3 \|_{L_1(A,E_0)}.$$ 

Proof. See Theorem 6.11 of [8] and Lemma 4.9 of [10].

Now we turn our attention back to $\mathcal{K}_{1,2}(\psi_n)$ and assume that $k_n = \sum_{k=1}^{n} \lambda_k^n$ is an integer as before. Then it is clear that

$$(\pi_j(M_n \oplus M_n) \otimes \rho_j(M))_{j=1}^{k_n}$$

is s.i.c. over $\mathcal{C}$ with

$$\mathcal{A} = A_n \otimes M_{k_n}, \quad \mathcal{A} = (M_n \oplus M_n) \otimes M,$$

$$E_{\mathcal{C}} = \sum_{j=1}^{k_n} \varphi_n \otimes \varphi_n \otimes \varphi_n \otimes \varphi_n$$

and $\varphi_n = \varphi_n \otimes \varphi_n \otimes \varphi_n \otimes \varphi_n$, so that we can calculate the norm of the image of $(I_{L_1(A_n)} \otimes \varphi_n) \circ \varphi_n$ as follows.

Proposition 4.6. Assume that we are in the same situation as in Proposition 4.5 and let $\gamma_1 = \sum_{k \in \mathbb{Z}} D_{\varphi}^2 g_k D_{\varphi}^2 \in L_1(M)$, where $M$, $\varphi$ and $(g_k)_{k \in \mathbb{Z}}$ are from [4.3]. Then for $x \in L_1(M_n)$ we have

$$\left\| \sum_{k=1}^{n} \pi_j(x - x) \otimes \rho_j(\gamma_1) \right\|_{L_1(A')}
\sim \inf_{x \otimes \gamma_1 = x_1 + x_2 + x_3} k_n \| x_1 \|_{L_1(A')} + \frac{1}{2} k_n \| x_2 \|_{L_1'(A';E_1)} + \frac{1}{2} k_n \| x_3 \|_{L_1'(A';E_1)},$$

where $A' = M_n \otimes M$ and $E_1 = \varphi_n \otimes \varphi_n \otimes \varphi_n \otimes \varphi_n$.

Proof. This is a direct application of Proposition 4.5 taking the completely contractive map $L_1(A) \to L_1(A')$, $(x,y) \mapsto \frac{1}{2}(x-y)$ into account.

Now we consider a cb-embedding of $\mathcal{K}_{1,2}(\psi_n)$ into

$$\mathcal{K}_{1,2}^{1}(\psi_n \otimes \varphi) = k_n L_1(A') + k_n^{\frac{1}{2}} L_1^2(A') + k_n^{\frac{1}{2}} L_1^2(A').$$

More precisely, we have

$$\|x\|_{S_n^{m}(\mathcal{K}_{1,2}^{1}(\psi_n \otimes \varphi))} = \inf \left\{ k_n \| x_1 \|_{S_n^{m}(L_1(A'))} + k_n^{\frac{1}{2}} \| x_2 \|_{S_n^{m}(L_1^2(A'))} + k_n^{\frac{1}{2}} \| x_3 \|_{S_n^{m}(L_1^2(A'))} \right\},$$

where the infimum runs over all possible decompositions

$$x = x_1 + (IS_n \otimes D_{\varphi_n \otimes \varphi}) x_2 + x_3 (IS_n \otimes D_{\varphi_n \otimes \varphi}^2),$$

where $D_{\varphi_n \otimes \varphi}$ is the density of $\varphi_n \otimes \varphi$.

Theorem 4.7. Assume that we are in the same situation as in the Proposition 4.6, then the mapping

$$u_n : \mathcal{K}_{1,2}(\psi_n) \to \mathcal{K}_{1,2}^{1}(\psi_n \otimes \varphi), \quad x \mapsto \frac{1}{k_n} x \otimes \gamma_1$$

is a cb-embedding with constants independent of $n$. Furthermore, $\mathcal{K}_{1,2}(\psi_n \otimes \varphi)$ is completely complemented in $L_1(*_{j=1}^{k_n} (A' \oplus A'))$ with constants independent of $n$. 

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Proof. We consider \( M_n(A), M_n(A) \) and \( L_{M_n} \otimes \mathbb{E}_0 \) instead of \( A, A \) and \( \mathbb{E}_0 \), respectively, and apply Proposition 4.4 taking the contractive map
\[
S_I^n(L_1(A)) \to S_I^n(L_1(A')), \quad (x, y) \mapsto \frac{1}{2}(x - y)
\]
into account. Note that we have
\[
\left\| (I_{S_I^n} \otimes D_{\varphi_n \otimes \phi})a \right\|_{S_I^n(L_1(A'))} = m \left\| (I_{S_I^n} \otimes D_{\varphi_n \otimes \phi})a \right\|_{L_1^n(M(A'), L_{M_n} \otimes \mathbb{E}_1)}
\]
and
\[
\left\| b(D_{\varphi_n \otimes \phi} \otimes I_{S_I^n}) \right\|_{S_I^n(L_1(A'))} = m \left\| b(D_{\varphi_n \otimes \phi} \otimes I_{S_I^n}) \right\|_{L_1^n(M(A'), L_{M_n} \otimes \mathbb{E}_1)}.
\]
The second statement is from Corollary 7.10 of [7]. \( \square \)

Remark 4.8. The above approach is the same as that of [10], which was used in constructing the embedding of \( S_p \) into the predual of a hyperfinite von Neumann algebra. However, we are using \( A_n \), the free product of \( M_n \oplus M_n \) to be consistent with Proposition 4.4 instead of the tensor product of \( M_n \oplus M_n \).

We can describe the operator space structure of \( u_n(K_{1,2}(\psi_n)) \) more precisely. Let
\[
K_n^\delta = R_n \otimes C_n \otimes (\ell_2^\varphi(\delta^2) \otimes \ell_2(\delta^{-2})) + R_n \otimes \ell_2(\lambda^{-2}) \otimes \ell_2(\delta^2) \otimes C_n \otimes \varphi \ell_2(\delta^{-2})
\]
where \( \lambda^{-2} \) means the sequence \( (\lambda_n^{-2})_{n \geq 1} \).

Proposition 4.9. Assume that we are in the same situation as in the Proposition 4.4. Let \( 1 < \delta \leq 2 \) and \( P = P_2 : L_1(M) \to G_1^2 \) be the canonical projection onto \( G_1^2 \). Then
\[
(I_{S_I^n} \otimes P)K_{RC_1}(\psi_n \otimes \phi) \to K_n^\delta, \quad \frac{1}{k_n} x \otimes D_{\varphi_n \otimes \phi}^\delta y_k D_{\varphi_n \otimes \phi}^\delta \mapsto x \otimes e_k
\]
is a complete isomorphism with constants independent of \( \delta \) and \( n \).

Proof. Let \( A' = M_n \otimes M \). Then, for \( x \in S^n_1(K_{RC_1}(\psi_n \otimes \phi)) \) we have
\[
\frac{1}{k_n} \left\| x \right\|_{S^n_1(K_{RC_1}(\psi_n \otimes \phi))} = \inf \left\{ \left\| x_1 \right\|_{S^n_1(L_1(A'))} + k_n^{1/2} \left\| x_2 \right\|_{S^n_1(L_2^2(A'))} + k_n^{1/2} \left\| x_3 \right\|_{S^n_1(L_1^2(A'))} : x = x_1 + (I_{S_I^n} \otimes D_{\varphi_n \otimes \phi}^\delta)x_2 + x_3(I_{S_I^n} \otimes D_{\varphi_n \otimes \phi}^\delta) \right\}
\]
\[
= \inf \left\{ \left\| y_1 \right\|_{S^n_1(L_1(A'))} + k_n^{1/2} \left\| (I_{S_I^n} \otimes D_{\varphi_n \otimes \phi}^\delta)y_2 \right\|_{S^n_1(L_2^2(A'))} + k_n^{1/2} \left\| y_3(I_{S_I^n} \otimes D_{\varphi_n \otimes \phi}^\delta) \right\|_{S^n_1(L_1^2(A'))} : x = y_1 + y_2 + y_3 \right\}
\]
\[
= \inf \left\{ \left\| y_1 \right\|_{S^n_1(L_1(A'))} + \left\| (I_{S_I^n} \otimes D_{\varphi_n \otimes \phi}^\delta)y_2 \right\|_{S^n_1(L_2^2(A'))} + \left\| y_3(I_{S_I^n} \otimes D_{\varphi_n \otimes \phi}^\delta) \right\|_{S^n_1(L_1^2(A'))} : x = y_1 + y_2 + y_3 \right\},
\]
where \( D_{\varphi_n \otimes \phi}^\delta \) is the density of \( \psi_n \otimes \phi \).
Let $y_i = \sum_k y_{i,k} \otimes D_{\phi,k}^\frac{1}{2} g_k D_{\phi,k}^\frac{1}{2}$ for $i = 1, 2, 3$. For the first term we have

$$\left\| \sum_k y_{1,k} \otimes D_{\phi,k}^\frac{1}{2} g_k D_{\phi,k}^\frac{1}{2} \right\|_{S_m^\infty(L_1(\mathcal{A}'))} = \left\| \sum_k y_{1,k} \otimes D_{\phi,k}^\frac{1}{2} g_k D_{\phi,k}^\frac{1}{2} \right\|_{S_m^\infty(S_1^m(G_1^1))} \sim \left\| \sum_k y_{1,k} \otimes e_k \right\|_{S_m^\infty \otimes \mathcal{K}_n(L_1)}.$$ 

For the second term we recall that $g_k D_{\phi} = \delta^{-2k} D_{\phi} g_k$ by [1.2], then we have

$$\left\| (I_{S_1^m} \otimes D_{\phi,n} D_{\phi}) \sum_k y_{2,k} \otimes D_{\phi,k}^\frac{1}{2} g_k D_{\phi,k}^\frac{1}{2} \right\|_{S_m^\infty(L_1^2(\mathcal{A}'))} = \left\| (I_{S_1^m} \otimes d_{\lambda-2}) \sum_k y_{2,k} \otimes D_{\phi,k}^\frac{1}{2} g_k D_{\phi,k}^\frac{1}{2} \right\|_{S_m^\infty(S_1^m(L_1^2(\mathcal{A}')))} = \left\| (I_{S_1^m} \otimes \text{tr}_{k'}) \left( \sum_{k,l} (I_{S_1^m} \otimes d_{\lambda-2}) y_{2,k} y_{2,l} (I_{S_1^m} \otimes d_{\lambda-2}) \otimes g_k D_{\phi} g_l^* \right)^\frac{1}{2} \right\|_{S_m^\infty} = \left\| (I_{S_1^m} \otimes \text{tr}_{k'}) \left( \sum_{k,l} (I_{S_1^m} \otimes d_{\lambda-2}) y_{2,k} y_{2,l} (I_{S_1^m} \otimes d_{\lambda-2}) \otimes \delta^{-2k} D_{\phi} g_k g_l^* \right)^\frac{1}{2} \right\|_{S_m^\infty} = \left\| \sum_{k,l} (I_{S_1^m} \otimes d_{\lambda-2}) y_{2,k} y_{2,l} (I_{S_1^m} \otimes d_{\lambda-2}) \otimes \delta^{-2k} D_{\phi} g_k g_l^* \right\|_{S_m^\infty} = \left\| \sum_{k,l} (I_{S_1^m} \otimes d_{\lambda-2}) y_{2,k} y_{2,l} (I_{S_1^m} \otimes d_{\lambda-2}) \otimes \delta^{-k} e_{kk} \right\|_{S_m^\infty} = \left\| \sum_{k} y_{2,k} \otimes e_k \right\|_{S_m^\infty \otimes \mathcal{K}_n(e)},$$

where $d_{\lambda-2}$ is the diagonal operator $\sum_k \lambda^{-2k} e_{kk}$.

Similarly, we have

$$\left\| \sum_k y_{3,k} \otimes D_{\phi,k}^\frac{1}{2} g_k D_{\phi,k}^\frac{1}{2} (I_{S_1^m} \otimes D_{\phi,n} D_{\phi}) \right\|_{S_m^\infty(L_1^2(\mathcal{A}'))} = \left\| \sum_k y_{3,k} \otimes e_k \right\|_{S_m^\infty \otimes \mathcal{K}_n(r)},$$

where $\mathcal{K}_{1,2}^{\infty}(\psi)$ is the completely isometric embedding $\mathcal{K}_{1,2}^{\infty}(\psi) = \bigcup_{n \geq 1} \mathcal{K}_{1,2}^{\infty}(\psi_n) \to \prod_{n \in \mathbb{N}} \mathcal{K}_{1,2}^{\infty}(\psi_n)$.

Thus, according to [21] we get a cb-embedding

$$\mathcal{K}_{1,2}^{\infty}(\psi) \to \prod_{n \in \mathbb{N}} (I_{S_1} \otimes P) \mathcal{K}_{RC_1}^{\infty}(\psi_n \otimes \phi) \subseteq L_1(\mathcal{B})$$

with $\mathcal{B} = \prod_{n \in \mathbb{N}} (I_{S_1} \otimes P) \mathcal{K}_{RC_1}^{\infty}(\psi_n \otimes \phi)$, where $\mathcal{A}' = M_n \mathcal{O} M$, and by the stability of QWEP with respect to free product, tensor product and ultraproduct $\mathcal{B}$ also satisfies QWEP. Moreover, since each $(I_{S_1} \otimes P) \mathcal{K}_{RC_1}^{\infty}(\psi_n \otimes \phi)$ is cb-complemented in $L_1(\mathcal{B})$, and $(I_{S_1} \otimes P) \mathcal{K}_{RC_1}^{\infty}(\psi_n \otimes \phi)$ is also cb-complemented in $L_1(\mathcal{B})$. 

[1.2]
Furthermore, by Proposition 4.9 we have the cb-isomorphism
\[ K_{1,2}(\psi) \cong R \otimes C \otimes (\ell'_2(\delta_{\frac{1}{2}}) + \ell'_2(\delta_{\frac{1}{2}})) \]
\[ + R \otimes \ell'_2(\lambda^{-2}) \otimes \ell'_2(\delta_{\frac{1}{2}}) + \ell'_2(\lambda^{-2}) \otimes C \otimes \ell'_2(\delta_{\frac{1}{2}}). \]

5. THE CHANGE OF DENSITY

In this section we present a concrete embedding of \( \Pi^h_1(OH, S_p) \) using the materials in the previous section. As was pointed out in Section 3 we need to consider embeddings of \( OH \) and \( S_p \). In the case of \( OH \) we have by Proposition 4.9

\[ \psi^h : OH \to \mathbb{C}^N, \quad e_j \mapsto \sum_{k \in \mathbb{Z}} D^2_{k,j} \delta^2_{k} \]

for a fixed \( \delta = 2 \). Moreover, \( \psi^h \) is 2-completely complemented in \( L_1(M_\mathbb{N}) \) and cb-isomorphic to

\[ L_2^2(t^{-\frac{1}{2}}; \ell_2) + L_2^2(t^{\frac{1}{2}}; \ell_2). \]

Now we consider the case of \( S_p \). Then we start with the observation

\[ S_p = C_p \otimes_h R_p \to K_{r,\theta} \otimes_h K_{r,\theta} \]
\[ = \left( L_2^h(t^{-\theta}; \ell_2) + L_2^h(t^{1-\theta}; \ell_2) \right) \otimes_h \left( L_2^h(s^{-\theta}; \ell_2) + L_2^h(s^{1-\theta}; \ell_2) \right). \]

Thus, we need to consider the situation \( (R + \ell^h(\lambda)) \otimes_h (C + \ell^h(\lambda)) \) by a suitable identification. However, we have

\[ \|x\|_{M_\mathbb{N}(\ell^h(\lambda) + \ell^h(\lambda))} \sim \inf_{x=x_1+x_2} \|x_1\|_{M_\mathbb{N}(\ell^h(\lambda))} + \|x_2\|_{M_\mathbb{N}(\ell^h(\lambda))} \]
\[ = \inf_{x=x_1+x_2} \|x_1\|_{M_\mathbb{N}(\ell^h(\lambda))} + \|x_2\|_{M_\mathbb{N}(\ell^h(\lambda))} \]
\[ \sim \|x\|_{M_\mathbb{N}(\ell^h(\lambda)+\ell^h(\lambda))} \]

and similarly \( \|x\|_{M_\mathbb{N}(C+\ell^h(\lambda))} \sim \|x\|_{M_\mathbb{N}(C+\ell^h(\lambda))} \)

for any \( m \in \mathbb{N} \). Thus, we have a complete isomorphism

\[ (R + \ell^h(\lambda)) \otimes_h (C + \ell^h(\lambda)) \cong K_{1,2}(\psi^h), \]

where \( \psi^h \) is the weight associated to \( \sum_k \lambda_k \frac{1}{2} \delta_{k} \). By combining 4.9, 5.1 and 5.2 we can guess that \( S_p \) can be embedded in the space \( K_{S_p} \) defined by

\[ K_{S_p} = L_2^h(s^{-\theta}; \ell_2) \otimes L_2^h(t^{\frac{1}{2}}; \ell_2) \otimes L_2^h(u^{\frac{1}{2}}; \ell_2) + L_2^h(s^{\theta}; \ell_2) \otimes L_2^h(t^{-\frac{1}{2}}; \ell_2) \otimes L_2^h(u^{-\frac{1}{2}}; \ell_2) \]
\[ + L_2^h(s^{-\theta}; \ell_2) \otimes L_2^h(t^{-\theta}; \ell_2) \otimes L_2^h(u^{\frac{1}{2}}; \ell_2) + L_2^h(s^{\theta}; \ell_2) \otimes L_2^h(t^{\frac{1}{2}}; \ell_2) \otimes L_2^h(u^{-\frac{1}{2}}; \ell_2), \]

which is a 4-term sum of vector valued function space with 3 variables \( (s, t, u) \in \mathbb{R}^3_+ \). It is worth of mention that we can observe a nontrivial change of density between 5.1 and 5.3.

**Theorem 5.1.** Let \( 1 < p < 2, \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \theta = \frac{\lambda}{p} \). Then we have the following cb-embedding

\[ C_p \otimes_h R_p \to K_{S_p}, \quad e_{ij} \otimes e_{ij} \mapsto (1 \otimes e_i) \otimes (1 \otimes e_j) \otimes 1. \]

More precisely, for any \( m \in \mathbb{N} \) we have

\[ \left\| \sum_{i,j=1}^n x_{ij} \otimes e_{ij} \right\|_{M_\mathbb{N}(C_p \otimes_h R_p)} \]
\[ \sim \theta(1-\theta) \left\| \sum_{i,j=1}^n x_{ij} \otimes (1 \otimes e_i) \otimes (1 \otimes e_j) \otimes 1 \right\|_{M_\mathbb{N}(K_{S_p})}. \]
Moreover, \( K_{S_p} \) is completely complemented in the noncommutative \( L_1 \) space with respect to a von Neumann algebra with QWEP.

**Proof.** For \( 1 < \delta \leq 2 \) and \( \alpha \in \mathbb{R} \) we consider the following maps

\[
\Phi_{\delta,\alpha} : \ell_2(\delta^k) \to L_2(t^\alpha), \quad (x_k)_{k \in \mathbb{Z}} \mapsto (\log \delta)^{-1} \sum_{k \in \mathbb{Z}} x_k 1_{[\delta^k, \delta^{k+1}]}(t)
\]

and

\[
\Psi_{\delta,\alpha} : L_2(t^\alpha) \to \ell_2(\delta^k), \quad f \mapsto (\log \delta)^{-1} \int_{\delta^k}^{\delta^{k+1}} f(t) \frac{dt}{t} \quad \text{for } k \in \mathbb{Z}.
\]

Then we have \( \Psi_{\delta,\alpha} \circ \Phi_{\delta,\alpha} = I_{\ell_2(\delta^k)} \) and

\[
\|\Phi_{\delta,\alpha}\| \leq \max(1, \delta^\alpha) \quad \text{and} \quad \|\Psi_{\delta,\alpha}\| \leq \max(1, \delta^{-\alpha}).
\]

Note that \( \Phi_{\delta,\alpha} \) (resp. \( \Psi_{\delta,\alpha} \)) is uniformly bounded for \(-1 < \alpha < 2\) (in particular, for \( \alpha \in \{0, (1-\theta), (2-\theta)\} \)), and it is actually the same map regardless of \( \alpha \), so that we just denote by \( \Phi_\delta \) and \( \Psi_\delta \).

Now we fix \( m \in \mathbb{N} \) and \( x \in M_m(C_p \otimes_h R_p) \). Since \( \cup_{1 < \delta \leq 2} \{ \text{ran} \Phi_{\delta,\alpha} \} \) is dense in \( L_2(t^\alpha) \) we can choose \( 1 < \delta \leq 2 \) with \( \delta - 1 \) small enough so that there is

\[
y = I_{M_m} \otimes \left[ (\Phi_\delta \otimes I_{\ell_2}) \otimes (\Phi_\delta \otimes I_{\ell_2}) \right] (z) \in M_m(K_{c,\theta} \otimes_h K_{r,\theta})
\]

with very small

\[
\left\| \sum_{i,j=1}^n x_{ij} \otimes (1 \otimes e_i) \otimes (1 \otimes e_j) - y \right\|_{M_m(K_{c,\theta} \otimes_h K_{r,\theta})}
\]

and

\[
\left\| \sum_{i,j=1}^n x_{ij} \otimes (1 \otimes e_i) \otimes (1 \otimes e_j) \otimes 1 - y \otimes 1 \right\|_{M_m(K_{S_p})},
\]

where \( z \in M_m(B_\delta) \) and

\[
B_\delta = \ell_2^c(\delta^{-\theta k}; \ell_2) + \ell_2^h(\delta^{(1-\theta)k}; \ell_2) \otimes_h \left( \ell_2^c(\delta^{-\theta k}; \ell_2) + \ell_2^h(\delta^{(1-\theta)k}; \ell_2) \right).
\]

By applying \( 4.4 \) (in this case \( \delta^{2k} \) is the weight) and \( 5.2 \) to \( B_\delta \) we get the following cb-embedding with constant independent of \( \delta \).

\[
B_\delta \hookrightarrow C_\delta = \ell_2^c(\delta^{-\theta k}; \ell_2) \otimes \ell_2^c(\delta^{-\theta k}; \ell_2) \otimes \left( \ell_2^c(\delta^{-\theta k}; \ell_2) + \ell_2^c(\delta^{-\theta k}; \ell_2) \right)
\]

\[
+ \ell_2^c(\delta^{-\theta k}; \ell_2) \otimes \ell_2^c(\delta^{(2-\theta)k}; \ell_2) \otimes \ell_2^c(\delta^{-\theta k}; \ell_2)
\]

\[
+ \ell_2^c(\delta^{(2-\theta)k}; \ell_2) \otimes \ell_2^c(\delta^{-\theta k}; \ell_2) \otimes \ell_2^c(\delta^{-\theta k}; \ell_2),
\]

\( w \mapsto w \otimes \sum_{k \in \mathbb{Z}} e_k \).

Note that \( 1 = \Phi_\delta(\sum_{k \in \mathbb{Z}} e_k) \) and

\[
(\Phi_\delta \otimes I_{\ell_2}) \otimes (\Phi_\delta \otimes I_{\ell_2}) \otimes \Phi_\delta : C_\delta \to K_{S_p}
\]

and

\[
(\Psi_\delta \otimes I_{\ell_2}) \otimes (\Psi_\delta \otimes I_{\ell_2}) : K_{c,\theta} \otimes_h K_{r,\theta} \to B_\delta
\]
are cb-maps with uniformly bounded cb-norms, so by Proposition 4.2 we have

$$
\theta^{-1}(1-\theta)^{-\frac{1}{2}} \left\| \sum_{i,j=1}^{n} x_{ij} \otimes e_{i} \otimes e_{j} \right\|_{M_{n}(C_{p} \otimes h, R_{p})}
\sim \left\| \sum_{i,j=1}^{n} x_{ij} \otimes (1 \otimes e_{i}) \otimes (1 \otimes e_{j}) \right\|_{M_{n}(K_{c,\theta} \otimes h, K_{c,\theta})}
\sim \| y \|_{M_{n}(K_{c,\theta} \otimes h, K_{c,\theta})} \sim \| z \|_{M_{n}(B_{1})} \sim \left\| z \otimes \sum_{k \in \mathcal{K}} e_{k} \right\|_{M_{n}(C_{1})}
\sim \| y \otimes 1 \|_{M_{n}(K_{c,\theta})} \sim \left\| \sum_{i,j=1}^{n} x_{ij} \otimes (1 \otimes e_{i}) \otimes (1 \otimes e_{j}) \otimes 1 \right\|_{M_{n}(K_{c,\theta})}.
$$

Note that all equivalences above are independent of the choice of $\delta$.

Moreover, for any $1 < \delta \leq 2$

$$
E_{\delta} = \left[ (\Phi_{\delta} \otimes I_{\ell_{2}}) \otimes (\Phi_{\delta} \otimes I_{\ell_{2}}) \otimes \Phi_{\delta} \right] (C_{\delta}) \cong C_{\delta}
$$
completely isometrically and by Proposition 4.3 and the following argument we have a cb-embedding

$$
C_{\delta} \hookrightarrow D_{\delta} \subseteq L_{1}(N_{\delta}),
$$
where $N_{\delta}$ satisfies QWEP and $D_{\delta}$ is completely complemented in $L_{1}(N_{\delta})$ with constants independent of $\delta$.

Let $\mathcal{U}'$ be a free ultrafilter on the collection of subsets of $(1, 2]$ containing all $(1, \delta]$ for $1 < \delta \leq 2$. Then we have

$$
\mathcal{K}_{S_{\mathcal{P}}} = \bigcup_{1 < \delta \leq 2} \mathcal{K}_{S_{\mathcal{P}}} = \prod_{\delta \in \mathcal{U}'} D_{\delta} \subseteq L_{1}(C), \text{ with } C = \left( \prod_{\delta \in \mathcal{U}'} L_{1}(N_{\delta}) \right)^{\ast}.
$$

By the stability of QWEP with respect to free product, tensor product and ultraproduct $C$ also satisfies QWEP. Moreover, since each $D_{\delta}$ is cb-complemented in $L_{1}(N_{\delta})$ with uniformly bounded cb-norms $\prod_{\delta \in \mathcal{U}'} D_{\delta}$ is also cb-complemented in $L_{1}(C)$.

By combining the above two embeddings for $OH$ and $S_{\mathcal{P}}$ we get an embedding of $\Pi_{1}^{\text{q}}(OH_{n}, S_{p}^{n})$ to the following space $\mathcal{K}_{\Pi_{1}^{\text{q}}(OH_{n}, S_{p}^{n})}$, which is a 8-term sum of vector valued function space with 4 variables $(s, t, u, v) \in \mathbb{R}_{+}^{4}$! Let $\mathcal{K}_{S_{\mathcal{P}}}$ be the space $\mathcal{K}_{S_{\mathcal{P}}}$ using $\ell_{2}^{2}$ instead of $\ell_{2}$. Then we define

$$
\mathcal{K}_{\Pi_{1}^{\text{q}}(OH_{n}, S_{p}^{n})} = \mathcal{K}_{S_{p}^{n}} \otimes (L_{2}^{s}(v^{-\frac{1}{2}}; \ell_{2}^{2}) + 2\, L_{2}^{s}(v^{\frac{1}{2}}; \ell_{2}^{2}))
\quad = L_{2}^{s}(s^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(t^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(u^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(v^{2}; \ell_{2}^{2})
\quad + L_{2}^{s}(s^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(t^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(u^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(v^{2}; \ell_{2}^{2})
\quad + L_{2}^{s}(s^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(t^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(u^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(v^{2}; \ell_{2}^{2})
\quad + L_{2}^{s}(s^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(t^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(u^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(v^{2}; \ell_{2}^{2})
\quad + L_{2}^{s}(s^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(t^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(u^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(v^{2}; \ell_{2}^{2})
\quad + L_{2}^{s}(s^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(t^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(u^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(v^{2}; \ell_{2}^{2})
\quad + L_{2}^{s}(s^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(t^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(u^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(v^{2}; \ell_{2}^{2})
\quad + L_{2}^{s}(s^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(t^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(u^{2}; \ell_{2}^{2}) \otimes L_{2}^{s}(v^{2}; \ell_{2}^{2}).
$$

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Corollary 5.2. Let $1 < p < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\theta = \frac{2}{p}$. Then we have the following cb-embedding with constants independent of $n$.

\[ \Pi^1_p(OH_n, S^n_p) \rightarrow K_{\Pi^1_p(OH_n, S^n_p)}, \quad T_{e_k \otimes e_{ij}} \mapsto (1 \otimes e_i) \otimes (1 \otimes e_j) \otimes (1 \otimes e_k) \otimes 1. \]

Moreover, for $a = \sum_{i,j,k=1}^{n} a_{i,j,k} e_k \otimes e_{ij} \in OH_n \otimes S^n_p$ we have

\[ \pi^n_1(T_a) \sim \theta(1 - \theta) \|1 \otimes a\|_{K_{\Pi^1_p(OH_n, S^n_p)}}. \]

6. A RESULT FOR THE IDENTITY

In this section we calculate the $K_{\Pi^1_p(OH_n, S^n_p)}$-norm of $1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i$ which corresponds to the formal identity map $I_n : OH_n \rightarrow \ell^n_p$. First, we rearrange $K_{\Pi^1_p(OH_n, S^n_p)}$ as follows.

\[ K_{\Pi^1_p(OH_n, S^n_p)} = L^r_2(s^{2-\theta} t^{-\theta} u^{-\frac{\theta}{2}} v^{-\frac{\theta}{2}}; \ell^n_2 \otimes 2 \ell^n_2) + L^r_2(s^{2-\theta} t^{-\theta} u^{-\frac{\theta}{2}} v^{-\frac{\theta}{2}}; \ell^n_2 \otimes \ell^n_2) \]

\[ + L^r_2(s^{2-\theta} t^{-\theta} u^{-\frac{\theta}{2}} v^{-\frac{\theta}{2}}; \ell^n_2 \otimes 2 \ell^n_2) + L^r_2(s^{2-\theta} t^{-\theta} u^{-\frac{\theta}{2}} v^{-\frac{\theta}{2}}; \ell^n_2 \otimes \ell^n_2) \]

\[ + L^r_2(s^{2-\theta} t^{-\theta} u^{-\frac{\theta}{2}} v^{-\frac{\theta}{2}}; \ell^n_2 \otimes 2 \ell^n_2) \]

\[ + L^r_2(s^{2-\theta} t^{-\theta} u^{-\frac{\theta}{2}} v^{-\frac{\theta}{2}}; \ell^n_2 \otimes \ell^n_2) \]

\[ = F_1 + F_2 + \cdots + F_8. \]

Let $\mu_1, \mu_2$ be the measures

\[ d\mu_1(s, t, u, v) = s^{4-2\theta} t^{-2\theta} u^{-1} v^{-1} dsdtdu \] and \[ d\mu_2(s, t, u, v) = s^{-2\theta} t^{4-2\theta} uv dsdtdu \]

corresponding to $F_1$ and $F_2$. We also let $\mu_{3, 1}$ and $\mu_{3, 2}$ be the measures

\[ d\mu_{3, 1}(s, t, u) = s^{4-2\theta} t^{-2\theta} u^{-1} dsdtdu \] and \[ d\mu_{3, 2}(v) = v dv \]

corresponding to $F_3$, and we define $\mu_{k, l}$ for $4 \leq k \leq 8$ and $l = 1, 2$ similarly.

If we look at the Banach space level of $F_l$ it is easier to understand. For example, we have

\[ F_1 \cong L_2(\mu_1; \ell^n_2 \otimes 2 \ell^n_2) \]

and

\[ F_3 \cong L_2(\mu_{3, 1}; \ell^n_2 \otimes 2 \ell^n_2) \otimes L_2(\mu_{3, 2}; \ell^n_2) \]

isometrically, where $\otimes_\pi$ implies the projective tensor product in the Banach space category.

In the case of identity we can make the calculation depend only on the decomposition of constant 1 function by scalar-valued functions. This will be proved in the following section.

Lemma 1.

\[ \left\| \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i \right\|_{K_{\Pi^1_p(OH_n, S^n_p)}} \]

\[ \sim \inf_{1 \leq f_1 + \cdots + f_8} n^{\frac{1}{2}} \|f_1\|_{L_2(\mu_1)} + n^{\frac{1}{2}} \|f_2\|_{L_2(\mu_2)} + n \|f_3\|_{L_2(\mu_{3, 1}) \otimes L_2(\mu_{3, 2})} + \cdots + n \|f_8\|_{L_2(\mu_8, 1) \otimes L_2(\mu_8, 2)}. \]

Note that the above infimum is the norm of $1$ in the following function space.

\[ L_2(n\mu_1) + L_2(n\mu_2) + L_2(n\mu_{3, 1}) \otimes L_2(n\mu_{3, 2}) + \cdots + L_2(n\mu_8, 1) \otimes L_2(n\mu_8, 2). \]

Now we do the calculation for the identity.
Theorem 6.1. Let $1 < p < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\theta = \frac{2}{p'}$. Then

$$
\left\| 1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i \right\|_{K^{\infty}(\xi H_n, \mathbb{S}^p)} \sim \theta^{-1}(1 - \theta)^{-\frac{3}{4}p}.
$$

Proof. First we consider the lower bound. Recall that the formal identity

$$
L_2(\nu) \otimes \pi X \rightarrow L_2(\nu; X)
$$

is a contraction for any measure $\nu$ and Banach spaces $X$ and

$$
L_2(f(t)dt) + L_2(g(t)dt) \equiv L_2(\min\{f(t), g(t)\}dt)
$$

isomorphically. Then by Lemma [1] we have

$$
\left\| 1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i \right\|_{K^{\infty}(\xi H_n, \mathbb{S}^p)}^2 \sim \left\| 1 \right\|_{L_2(n\mu_1 + L_2(n\mu_2) + L_2(n\mu_3, 1) \otimes \pi L_2(n\mu_3, 2) + \cdots + L_2(n\mu_8, 1) \otimes \pi L_2(n\mu_8, 2)}
$$

$$
\geq \left\| 1 \right\|_{L_2(n\mu_1 + L_2(n\mu_2) + L_2(n\mu_3, 1) \otimes \pi L_2(n\mu_3, 2) + \cdots + L_2(n\mu_8, 1) \otimes \pi L_2(n\mu_8, 2)}
$$

$$
\sim \int_{\mathbb{R}^4_+} \min(ns^{4-2\theta}t^{-2\theta}u^{-1}v^{-1}, ns^{4-2\theta}u^{-1}v^{-1}, n^2s^{4-2\theta}t^{-2\theta}u^{-1}v^{-1}, n^2s^{4-2\theta}u^{-1}v^{-1}, n^2s^{4-2\theta}u^{-1}v^{-1}, n^2s^{4-2\theta}u^{-1}v^{-1}) \frac{dtsdudsdu}{stuuv}
$$

$$
= \int_{\mathbb{R}^4_+} n^2s^{4-2\theta}t^{-1-2\theta} \min(n^{-1}s^{4}u^{-2}v^{-2}, n^{-1}t^{4}u^{4}v^{-2}, n^{-1}u^{-2}v^{-2}, n^{-1}t^{4}v^{-2}, u^{-2}v^{-2}, u^{-2}, v^{-2})
$$

$$
\frac{dtsdudsdu}{stuuv}
$$

Now we divide $\mathbb{R}^4_+$ into the regions according to the values of the minimum used in the integral above. First we consider 8 regions $A_1, \cdots, A_8 \subseteq \mathbb{R}^4_+$ according to the values of $\min(u^{-2}v^{-2}, u^{-2}, 1, v^{-2})$, and we further divide $A_i$’s ($1 \leq i \leq 8$) into 3 sub-regions $A_{i,j}$ ($1 \leq j \leq 3$) according to the behavior of $s$ and $t$. See TABLE 1 in the next page for the details. Note that if we take the transform $(s, t, u, v) \mapsto (t, s, u^{-1}, v^{-1})$ then the regions $A_5, \cdots, A_8$ and the associated integrand correspond to those of $A_1, \cdots, A_4$, respectively, so that we are only to consider the cases $A_1, \cdots, A_4$. The integrals over each regions are calculated in TABLE 2 in page 21. Note that the integrals over $A_{2,1}$ and $A_{1,1}$ are dominant with values $n^{1-\frac{2}{p}}(1 - \theta)^{-\frac{3}{4}p}$ when $\theta$ goes to 0, and the integrals over $A_{2,3}$, $A_{4,2}$ and $A_{4,3}$ are dominant with values $n^{1-\frac{2}{p}}(1 - \theta)^{-\frac{3}{4}p}$ when $\theta$ goes to 1. Thus, by combining all these calculations and $1 - \frac{\theta}{2} = \frac{1}{p}$ we get the desired lower estimate $n^{1-\frac{2}{p}}(1 - \theta)^{-\frac{3}{4}p}.$
Table 1. Regions

| $A_1$  | $0 < u < 1$, $0 < v < n^{-1/2}$ | $A_{1,1}$ | $s \geq u^{1/2}$, $t \geq n^{1/2}$ |
|--------|----------------------------------|-----------|-------------------------------|
|        |                                  | $A_{1,2}$ | $s < u^{1/2}$, $t \geq n^{1/2}u^{-1/2}s$ |
|        |                                  | $A_{1,3}$ | $s \geq n^{-1/2}u^{1/2}t$, $t < n^{1/2}$ |
| $A_2$  | $0 < u < 1$, $n^{-1/2} \leq v < 1$ | $A_{2,1}$ | $s \geq n^{1/2}u^{1/2}v$, $t \geq n^{1/2}$ |
|        |                                  | $A_{2,2}$ | $s < n^{1/2}u^{1/2}v$, $t \geq u^{-1/2}u^{-1/2}s$ |
|        |                                  | $A_{2,3}$ | $s \geq u^{1/2}u^{1/2}t$, $t < n^{1/2}$ |
| $A_3$  | $0 < u < 1$, $n^{1/2} \leq v$    | $A_{3,1}$ | $s \geq n^{1/2}u^{1/2}$, $t \geq 1$ |
|        |                                  | $A_{3,2}$ | $s < n^{1/2}u^{1/2}$, $t \geq n^{-1/2}u^{-1/2}s$ |
|        |                                  | $A_{3,3}$ | $s \geq n^{1/2}u^{1/2}t$, $t < 1$ |
| $A_4$  | $0 < u < 1$, $v < n^{1/2}$       | $A_{4,1}$ | $s \geq n^{1/2}u^{1/2}v$, $t \geq n^{1/2}v^{-1/2}$ |
|        |                                  | $A_{4,2}$ | $s < n^{1/2}u^{1/2}v$, $t \geq u^{-1/2}v^{-1/2}s$ |
|        |                                  | $A_{4,3}$ | $s \geq u^{1/2}v^{1/2}t$, $t < n^{1/2}v^{-1/2}$ |

$A_{5, a}$: $1 \leq u$, $n^{1/2} \leq v$

| $A_{6}$ | $1 \leq u$, $1 \leq v < n^{1/2}$ |
| $A_{7}$ | $1 \leq u$, $0 < v < n^{-1/2}$    |
| $A_{8}$ | $1 \leq u$, $n^{-1/2} \leq v < 1$ |

$A_{a, 1}, \ldots, A_{a, 3}$ are similarly determined but omitted.

Now we consider the upper estimate. We use the same regions and fortunately that is enough. Indeed, we have

$$
\|1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i\|_{K^{p}(OH_{n}, s^p)}
= \left\| (1_{A_{1,1}} + \cdots + 1_{A_{4,3}}) + 1_{A_{5, a}} + \cdots + 1_{A_{a, 3}} \right\|_{K^{p}(OH_{n}, s^p)}
\leq \sum_{A \in R_1} 1_{A} \|_{L_2(n_{L_2})} + \sum_{A \in R_2} 1_{A} \|_{L_2(n_{L_2})} + \cdots
+ \sum_{A \in R_3} 1_{A} \|_{L_2(n_{L_2})} + \cdots + \sum_{A \in R_8} 1_{A} \|_{L_2(n_{L_2})},
$$

where

$R_l := \{A_{i,j}: A_{i,j} \text{ corresponds to } L_2(n_{L_2}) \text{ in } (6.1)\}$

for $l = 1, 2$ and

$R_l := \{A_{i,j}: A_{i,j} \text{ corresponds to } L_2(n_{L_2}) \otimes_{\sigma} L_2(n_{L_2}) \text{ in } (6.1)\}$

for $3 \leq l \leq 8$. Thus, we get the upper bound of $\|1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i\|_{K^{p}(OH_{n}, s^p)}$, namely the sum of norms of $1_{A_{1,1}}$’s calculated in the corresponding function spaces in (6.1). However, this is the same as the lower bound which is nothing but the sum of norms of $1_{A_{1,1}}$’s calculated in the corresponding function spaces in (6.2).
Indeed, the terms corresponding to $L_2(\mu_1)$ or $L_2(\mu_2)$ are no problem since we calculate the norm in the same space. For the remaining problematic terms we observe the following. For example, if we consider the region

$$A_{1,2} = \{0 < u < 1, s < u^{\frac{3}{2}}, t \geq n^{\frac{1}{4}} u^{-\frac{1}{2}} s\} \times \{0 < v < n^{-\frac{1}{2}}\},$$

then we need to compare two norms calculated in

$$L_2(\mu_{3,1}) \otimes \pi L_2(\mu_{3,2}) = L_2(s^{4-2\theta} t^{-2\theta} u^{-1} \frac{dsdtdu}{stu}) \otimes \pi L_2(v \frac{dv}{v}),$$

and

$$L_2(\mu_{3,1} \times \mu_{3,2}) = L_2(s^{4-2\theta} t^{-2\theta} u^{-1} \frac{dsdtdu}{stu}) \otimes_2 L_2(v \frac{dv}{v}),$$

which are the same since we have the separation of variables $(s, t, u)$ and $v$ and then the norms are just the product of two $L_2$-norms.

Let’s check another one. If we consider the region

$$A_{4,1} = \{0 < u < 1, s \geq n^{\frac{3}{4}} u^{\frac{1}{2}} \} \times \{1 \leq v < n^{\frac{1}{2}}, t \geq n^{\frac{1}{4}} v^{-\frac{1}{2}}\},$$

then we need to compare two norms calculated in

$$L_2(\mu_{8,1}) \otimes \pi L_2(\mu_{8,2}) = L_2(s^{-2\theta} u \frac{dsdu}{stu}) \otimes \pi L_2(t^{-2\theta} v^{-1} \frac{dtdv}{tv})$$

and

$$L_2(\mu_{8,1} \times \mu_{8,2}) = L_2(s^{-2\theta} u \frac{dsdu}{stu}) \otimes_2 L_2(t^{-2\theta} v^{-1} \frac{dtdv}{tv}),$$

Table 2. Integrals over the regions

| Region $A_{i,j}$ | $\left( \int_{A_{i,j}} G \, dsdtduv \right)^{\frac{1}{2}}$ | Corresponding Function Space in (6.2) | Corresponding Function Space in (6.1) |
|------------------|-------------------------------------------------|--------------------------------------|--------------------------------------|
| $A_{1,1}$        | $n^{\frac{3}{2}-\theta} \theta^{-1} (1 - \theta)^{-\frac{1}{2}}$ | $L_2(n^{\mu_7,1} \times \mu_7,2)$ | $L_2(n^{\mu_7,1} \otimes \pi L_2(n^{\mu_7,2})$ |
| $A_{1,2}$        | $n^{\frac{3}{2}-\theta} \theta^{-\frac{1}{2}} (1 - \theta)^{-1}$ | $L_2(n^{\mu_3,1} \times \mu_3,2)$ | $L_2(n^{\mu_3,1} \otimes \pi L_2(n^{\mu_3,2})$ |
| $A_{1,3}$        | $n^{\frac{3}{2}-\theta} \theta^{-\frac{1}{2}} (1 - \theta)^{-1}$ | $L_2(n^{\mu_3,1} \times \mu_3,2)$ | $L_2(n^{\mu_3,1} \otimes \pi L_2(n^{\mu_3,2})$ |
| $A_{2,1}$        | $n^{\frac{3}{2}-\theta} \theta^{-1} (1 - \theta)^{-1}$ | $L_2(n^{\mu_7,1} \times \mu_7,2)$ | $L_2(n^{\mu_7,1} \otimes \pi L_2(n^{\mu_7,2})$ |
| $A_{2,2}$        | $n^{\frac{3}{2}-\theta} \theta^{-\frac{1}{2}} (1 - \theta)^{-1}$ | $L_2(n^{\mu_3,1} \times \mu_3,2)$ | $L_2(n^{\mu_3,1} \otimes \pi L_2(n^{\mu_3,2})$ |
| $A_{2,3}$        | $n^{\frac{3}{2}-\theta} \theta^{-\frac{1}{2}} (1 - \theta)^{-1}$ | $L_2(n^{\mu_3,1} \times \mu_3,2)$ | $L_2(n^{\mu_3,1} \otimes \pi L_2(n^{\mu_3,2})$ |
| $A_{3,1}$        | $n^{\frac{3}{2}-\theta} \theta^{-1} (1 - \theta)^{-\frac{1}{2}}$ | $L_2(n^{\mu_8,1} \times \mu_8,2)$ | $L_2(n^{\mu_8,1} \otimes \pi L_2(n^{\mu_8,2})$ |
| $A_{3,2}$        | $n^{\frac{3}{2}-\theta} \theta^{-\frac{1}{2}} (1 - \theta)^{-1}$ | $L_2(n^{\mu_3,1} \times \mu_3,2)$ | $L_2(n^{\mu_3,1} \otimes \pi L_2(n^{\mu_3,2})$ |
| $A_{3,3}$        | $n^{\frac{3}{2}-\theta} \theta^{-\frac{1}{2}} (1 - \theta)^{-1}$ | $L_2(n^{\mu_3,1} \times \mu_3,2)$ | $L_2(n^{\mu_3,1} \otimes \pi L_2(n^{\mu_3,2})$ |
| $A_{4,1}$        | $n^{\frac{3}{2}-\theta} \theta^{-1} (1 - \theta)^{-1}$ | $L_2(n^{\mu_8,1} \times \mu_8,2)$ | $L_2(n^{\mu_8,1} \otimes \pi L_2(n^{\mu_8,2})$ |
| $A_{4,2}$        | $n^{\frac{3}{2}-\theta} \theta^{-\frac{1}{2}} (1 - \theta)^{-\frac{1}{2}}$ | $L_2(n^{\mu_3,1} \times \mu_3,2)$ | $L_2(n^{\mu_3,1} \otimes \pi L_2(n^{\mu_3,2})$ |
| $A_{4,3}$        | $n^{\frac{3}{2}-\theta} \theta^{-\frac{1}{2}} (1 - \theta)^{-1}$ | $L_2(n^{\mu_3,1} \times \mu_3,2)$ | $L_2(n^{\mu_3,1} \otimes \pi L_2(n^{\mu_3,2})$ |
which are the same since we have the separation of variables \((s, u)\) and \((t, v)\) as we wanted.

Similarly we can easily check that this separation of variables happens in every problematic terms, which leads us to the desired upper bound.

\[ \square \]

**Remark 6.2.** When \(\theta = 1\) we recover the well known \(\sqrt{1 + \log n}\) factor (Proposition 4.9 of [7]) in the integral over every subregion of \(A_2, A_4, A_6\) and \(A_8\).

7. An application of Orlicz spaces

In this section we will show that the result for the identity in the previous section is enough to conclude our final goal. First we will look at the diagonal part to see that it is equivalent to an Orlicz sequence space, and for the whole matrix we will consider its vector valued case. This “Orlicz space argument” goes back to an unpublished result of Junge and Xu and is also used by K. L. Yew in [28].

We consider the function \(\Psi\) defined on \([0, \infty)\) by

\[
\Psi(x) = \inf_{f_1 + \cdots + f_n} x^2 \left\| f_1 \right\|_{L_2(\mu_1)}^2 + x^2 \left\| f_2 \right\|_{L_2(\mu_2)}^2 + x \left\| f_3 \right\|_{L_2(\mu_3, 1) \otimes \pi L_2(\mu_3, 2)} + \cdots \\
+ x \left\| f_8 \right\|_{L_2(\mu_8, 1) \otimes \pi L_2(\mu_8, 2)}.
\]

**Lemma 2.** \(\Psi\) is equivalent to a Orlicz function \(\tilde{\Psi}\).

**Proof.** Clearly we have \(\Psi(0) = 0\) and \(\lim_{x \to \infty} \Psi(x) = \infty\). Since we have

\[
\begin{align*}
\frac{\Psi(x)}{x} &= \inf_{f_1 + \cdots + f_n} x \left\| f_1 \right\|_{L_2(\mu_1)}^2 + x \left\| f_2 \right\|_{L_2(\mu_2)}^2 + x \left\| f_3 \right\|_{L_2(\mu_3, 1) \otimes \pi L_2(\mu_3, 2)} + \cdots \\
&\quad + x \left\| f_8 \right\|_{L_2(\mu_8, 1) \otimes \pi L_2(\mu_8, 2)},
\end{align*}
\]

it is also clear that \(\frac{\Psi(x)}{x}\) is an increasing function.

Now we consider the convex function \(\Psi(x) = \inf\{f(x) : f \in F_x\}\), where \(F_x\) is the set of all linear functions intersecting at least two distinct points with the graph of \(\Psi\). Then by Lemma 1.e.7 of [13] we have

\[
\frac{\Psi(x)}{4} \leq \frac{x}{2} \leq \Psi(x) \leq \Psi(x).
\]

Due to the previous lemma we can consider the Orlicz sequence space \(\ell_{\tilde{\Psi}}\) defined by

\[
\ell_{\tilde{\Psi}} = \{(a_n) : \sum_{n \geq 1} \tilde{\Psi}\left(\frac{|a_n|}{\rho}\right) < \infty \text{ for some } \rho > 0\}
\]

and

\[
\|(a_n)\|_{\tilde{\Psi}} = \inf_{\rho > 0} \sum_{n \geq 1} \tilde{\Psi}\left(\frac{|a_n|}{\rho}\right) \leq 1.
\]

We recover a similar form of our function space by a standard argument.

**Lemma 3.**

\[
\|(a_n)\|_{\tilde{\Psi}} \sim \inf\{\left\| g_1 \right\|_{L_2(\mu_1, \pi_2)} + \left\| g_2 \right\|_{L_2(\mu_2, \pi_2)} + \left\| g_3 \right\|_{L_2(\mu_3, 1) \otimes \pi L_2(\mu_3, 2) \otimes \pi \ell_1} + \cdots + \left\| g_8 \right\|_{L_2(\mu_8, 1) \otimes \pi L_2(\mu_8, 2) \otimes \pi \ell_1} \},
\]

where the infimum runs over all possible \(g_1 = (g_1^n)_n, \ldots, g_8 = (g_8^n)_n\) with

\[
1 \otimes a_n = g_1^n + \cdots + g_8^n.
\]
Proof. Let $R((a_n))$ be the right side. Suppose we have $\|(a_n)\|_{\tilde{g}} < 1$, then, by Lemma 2 we can choose
\[
1 = f_1^n + \cdots + f_8^n
\]
for each $n$ satisfying
\[
\sum_n \left[ |a_n|^2 \| f_1^n \|_{L_2(\mu_1)}^2 + |a_n|^2 \| f_2^n \|_{L_2(\mu_2)}^2 + |a_n| \| f_3^n \|_{L_2(\mu_3,1) \otimes_\pi L_2(\mu_3,2)}^2 \right] < 4.
\]
Then, we have
\[
\sum_n |a_n|^2 \| f_1^n \|_{L_2(\mu_1)}^2, \sum_n |a_n|^2 \| f_2^n \|_{L_2(\mu_2)}^2, \sum_n |a_n| \| f_3^n \|_{L_2(\mu_3,1) \otimes_\pi L_2(\mu_3,2)}^2,
\]
\[
\cdots, \sum_n |a_n| \| f_8^n \|_{L_2(\mu_8,1) \otimes_\pi L_2(\mu_8,2)}^2 < 4
\]
which implies
\[
R((a_n)) \leq \left( \sum_n |a_n|^2 \| f_1^n \|_{L_2(\mu_1)}^2 \right)^{\frac{1}{2}} + \left( \sum_n |a_n|^2 \| f_2^n \|_{L_2(\mu_2)}^2 \right)^{\frac{1}{2}}
\]
\[
+ \sum_n |a_n| \| f_3^n \|_{L_2(\mu_3,1) \otimes_\pi L_2(\mu_3,2)}^{\frac{1}{2}} + \cdots + \sum_n |a_n| \| f_8^n \|_{L_2(\mu_8,1) \otimes_\pi L_2(\mu_8,2)} < 32
\]
by setting $g_l^n = a_n \otimes f_l^n$ for $1 \leq l \leq 8$ and $n \geq 1$. Thus, we get
\[
R((a_n)) \leq 32 \|(a_n)\|_{\tilde{g}}.
\]
For the converse we assume that $R((a_n)) < 1$. Then we can choose
\[
1 \otimes a_n = g_1^n + \cdots + g_8^n
\]
such that
\[
\|g_1\|_{L_2(\mu_2;\ell_2)} + \|g_2\|_{L_2(\mu_2;\ell_2)} + \|g_3\|_{L_2(\mu_3,1) \otimes_\pi L_2(\mu_3,2) \otimes_\pi \ell_1} + \cdots + \|g_8\|_{L_2(\mu_8,1) \otimes_\pi L_2(\mu_8,2) \otimes_\pi \ell_1} < 1,
\]
which means
\[
\sum_n \|g_1^n\|_{L_2(\mu_1)}^2, \sum_n \|g_2^n\|_{L_2(\mu_2)}^2, \sum_n \|g_3^n\|_{L_2(\mu_3,1) \otimes_\pi L_2(\mu_3,2)}^2,
\]
\[
\cdots, \sum_n \|g_8^n\|_{L_2(\mu_8,1) \otimes_\pi L_2(\mu_8,2)}^2 < \frac{1}{8}.
\]
Thus, by observing $1 = a_n^{-1} g_1^n + \cdots + a_n^{-1} g_8^n$ for non-zero $a_n$, we have
\[
\sum_{n \geq 1} \Psi\left( \frac{|a_n|}{8} \right) \leq \sum_{n \geq 1} \Psi\left( \frac{|a_n|}{8} \right)
\]
\[
\leq \sum_{n \geq 1} \left( \frac{\|g_1^n\|_{L_2(\mu_1)}^2}{8^2} + \frac{\|g_2^n\|_{L_2(\mu_2)}^2}{8^2} + \frac{\|g_3^n\|_{L_2(\mu_3,1) \otimes_\pi L_2(\mu_3,2)}^2}{8} \right) + \cdots + \frac{\|g_8^n\|_{L_2(\mu_8,1) \otimes_\pi L_2(\mu_8,2)}^2}{8} < 1,
\]
which means
\[
\|(a_n)\|_{\ell_{\tilde{g}}} < 8.
\]
In the case of identity we can further simplify the calculation by the averaging trick.
Lemma 4.
\[ \left\| \sum_{i=1}^{n} e_i \right\|_{\hat{\Psi}} \sim \inf_{1 = f_1, \ldots, f_8} n^{\frac{1}{\mu}} \left\| f_1 \right\|_{L_2(\mu_1)} + n^{\frac{1}{\mu}} \left\| f_2 \right\|_{L_2(\mu_2)} + \sum_{\ell = 3}^{8} \left\| f_\ell \right\|_{L_2(\mu_1) \otimes \cdots \otimes L_2(\mu_8)} \cdot \]

Proof. Let
\[ A := \|g_1\|_{L_2(\mu_1; \ell_2)} + \|g_2\|_{L_2(\mu_2; \ell_2)} + \|g_3\|_{L_2(\mu_3; \ell_2)} + \cdots + \|g_8\|_{L_2(\mu_8; \ell_2)} \]
for fixed \( g_1 = (g_1^n)_{n=1}^8, \ldots, g_8 = (g_8^n)_{n=1}^8 \) with \( 1 = g_1^n + \cdots + g_8^n \). Now we set
\[ f_l = \frac{1}{|S_n|} \sum_{\sigma \in S_n} g_l^{\sigma(i)}, \quad 1 \leq l \leq 8, \]
where \( S_n \) is the permutation group of \( \{1, \ldots, n\} \). Then for \( l = 1, 2 \) we have
\[ n^{\frac{1}{\mu}} \left\| f_l \right\|_{L_2(\mu_\ell)} = \left\| \sum_{i=1}^{n} f_i \otimes e_i \right\|_{L_2(\mu_\ell)} \leq \frac{1}{|S_n|} \sum_{\sigma \in S_n} \left\| \sum_{i=1}^{n} g_l^{\sigma(i)} \otimes e_i \right\|_{L_2(\mu_\ell)} \leq \|g_l\|_{L_2(\mu_\ell)}, \]
Similarly, we have
\[ n \left\| f_l \right\|_{L_2(\mu_1) \otimes \cdots \otimes L_2(\mu_2)} \leq \|g_l\|_{L_2(\mu_1) \otimes \cdots \otimes L_2(\mu_2) \otimes \ell_1} \]
for \( 3 \leq l \leq 8 \).
Consequently, we have
\[ A \geq n^{\frac{1}{\mu}} \left\| f_1 \right\|_{L_2(\mu_1)} + n^{\frac{1}{\mu}} \left\| f_2 \right\|_{L_2(\mu_2)} + n \left\| f_3 \right\|_{L_2(\mu_3) \otimes L_2(\mu_2)} + \cdots + n \left\| f_8 \right\|_{L_2(\mu_8) \otimes L_2(\mu_8)} ; \]
which leads us to the desired conclusion by Lemma \( 8 \). \( \square \)

Now we prove Lemma \( 11 \)
(proof of Lemma \( 11 \))
\[ \left\| 1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i \right\|_{K_{N_0}(O_{N_0}, S_0^p)} = \inf \left\{ \sum_{l=1}^{8} \left\| h_l \right\|_{F_1} : \right\}, \]
where the infimum runs over all possible decomposition
\[ 1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i = h_1 + \cdots + h_8. \]

For a given \( \epsilon > 0 \) we consider a decomposition \( (h_l)_{l=1}^{8} \) with
\[ \sum_{l=1}^{8} \left\| h_l \right\|_{F_1} \leq (1 + \epsilon) \left\| 1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i \right\|_{K_{N_0}(O_{N_0}, S_0^p)}, \]
and let
\[ h_l = \sum_{i,j,k=1}^{n} h_l^{(i,j,k)} \otimes e_i \otimes e_j \otimes e_k \]
with scalar-valued \( h_l^{(i,j,k)} \) for \( 1 \leq l \leq 8 \).
If we consider the diagonal projection
\[ P : L_2^0 \otimes L_2^0 \otimes L_2^0 \to L_2^0 \otimes L_2^0 \otimes L_2^0, \quad e_i \otimes e_j \otimes e_k \mapsto \delta_{i,j,k} e_i \otimes e_i \otimes e_k, \]
which leads us to the desired conclusion by Lemma \( 8 \). \( \square \)
then we have
\[ 1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i = (I \otimes P) \left( 1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i \right) \]
\[ = (I \otimes P) \sum_{l=1}^{8} h_l = \sum_{l=1}^{8} \sum_{i=1}^{n} h_{l(i,i,i)} \otimes e_i \otimes e_i \otimes \delta_i \]
and
\[ \sum_{l=1}^{8} \left\Vert (I \otimes P) h_l \right\Vert_{F_l} \leq \sum_{l=1}^{8} \left\Vert h_l \right\Vert_{F_l} \leq (1 + \epsilon) \left\Vert 1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i \right\Vert_{K_{\mathcal{H}^2}}. \]
Indeed, we are only to check that \( P \) is completely contractive as mappings on \( C_n \otimes C_n \otimes C_n, R_n \otimes R_n \otimes R_n, C_n \otimes C_n \otimes R_n \) and \( R_n \otimes R_n \otimes C_n \). The first two cases are clear since column and row Hilbert spaces are homogeneous, i.e., every bounded maps are completely bounded with the same cb-norm.
For \( P : C_n \otimes C_n \otimes R_n \to C_n \otimes C_n \otimes R_n \), we consider the factorization
\[ P : C_n \otimes C_n \otimes R_n \xrightarrow{Q \otimes I_{R_n}} C_n \otimes C_n \otimes R_n \xrightarrow{I_{C_n} \otimes Q} C_n \otimes C_n \otimes R_n, \]
where
\[ Q : \ell_2^n \otimes \ell_2^n \to \ell_2^n \otimes \ell_2^n, \quad e_i \otimes e_j \mapsto \delta_{i,j} e_i \otimes e_i. \]
Since \( Q \) is completely contractive as mappings on \( C_n \otimes R_n \) and \( C_n \otimes R_n \) we get the desired conclusion. The last case is obtained similarly.
By looking at the coefficient of \( e_i \otimes e_i \otimes \delta_i \) we observe that
\[ \sum_{l=1}^{8} h_{l(i)} = 1 \]
for all \( 1 \leq i \leq n \), where \( h_{l(i)} = h_{l(i,i,i)} \). If we set
\[ \rho = \left\Vert 1 \otimes \sum_{i=1}^{n} e_i \otimes e_i \otimes \delta_i \right\Vert_{K_{\mathcal{H}^2}}, \]
then we have
\[ \sum_{i=1}^{n} \Psi \left( \frac{1}{8 \rho} \right) \leq \sum_{i=1}^{n} \Psi \left( \frac{1}{8 \rho} \right) \]
\[ \leq \sum_{i=1}^{n} \left( \frac{1}{64 \rho^2} \left\Vert h_{1(i)} \right\Vert_{L_2(\mu_1)}^2 + \frac{1}{64 \rho^2} \left\Vert h_{2(i)} \right\Vert_{L_2(\mu_2)}^2 \right. \]
\[ + \frac{1}{8 \rho} \left\Vert h_{3(i)} \right\Vert_{L_2(\mu_3,1) \otimes L_2(\mu_3,2)} + \cdots + \frac{1}{8 \rho} \left\Vert h_{8(i)} \right\Vert_{L_2(\mu_{8,1}) \otimes \cdots \otimes L_2(\mu_{8,2})} \right) \]
\[ \leq 1 + \epsilon, \]
since we have
\[ \left\Vert (1 \otimes P) h_l \right\Vert_{F_l} = \sum_{i=1}^{n} \left\Vert h_{l(i)} \right\Vert_{L_2(\mu_1)}^2 \]
for \( l = 1, 2 \) and
\[ \left\Vert (1 \otimes P) h_l \right\Vert_{F_l} = \sum_{i=1}^{n} \left\Vert h_{l(i)} \right\Vert_{L_2(\mu_{1,1}) \otimes \cdots \otimes L_2(\mu_{1,2})} \]
for \( 3 \leq l \leq 8. \)
Thus, by Lemma 4 we have
\[
\inf_{\mathbf{f}_1, \ldots, \mathbf{f}_n} n^\frac{1}{2} \|f_1\|_{L_2(\mu_1)} + n^\frac{1}{p} \|f_2\|_{L_2(\mu_2)} + n \|f_3\|_{L_2(\mu_3, \xi_3)} + \cdots + n \|f_s\|_{L_2(\mu_s, \xi_s)} \\
= \| \sum_{i=1}^n c_i \|_{\mathcal{K}_{U}^{p}(\mathcal{H}, S_p)} \leq 8 \|1 \otimes \sum_{i=1}^n e_i \otimes e_i \otimes \delta_i\|_{\mathcal{K}_{U}^{p}(\mathcal{H}, S_p)}.
\]

The converse inequality is clear.

**Proposition 7.1.** Let \(1 < p < 2, \frac{1}{p} + \frac{1}{p'} = 1\) and \(\theta = \frac{2}{p'}\). Then we have the inclusion \(\ell_p \subseteq \ell_{\Phi}\) with norm \(\lesssim (1 - \theta)^{-\frac{3}{2}}\).

**Proof.** Note that \(\ell_p\) and \(\ell_{\Phi}\) are both Orlicz sequence spaces. Thus, by Proposition 4.a.5. in [13] it is enough to check that if there is a constant \(C > 0\) such that
\[
\left\| \sum_{i=1}^n c_i \right\|_{\Phi} \leq C \left\| \sum_{i=1}^n c_i \right\|_{\ell_p} = C(1 - \theta)^{-\frac{3}{2}} n^\frac{1}{2}
\]
for any \(n \in \mathbb{N}\), which is assured by Theorem 6.1. \(\square\)

Finally we prove our main result.

**Theorem 7.2.** Let \(1 < p < 2\) and \(\frac{1}{p} + \frac{1}{p'} = 1\). Then, for any Hilbert space \(H\) we have
\[
CB(B(H), \mathcal{O}H) \subseteq \Pi_{p', cb}(B(H), \mathcal{O}H)
\]
with the norm \(\lesssim \left(\frac{p'}{p' - 2}\right)^{\frac{1}{2}}\). Equivalently, we have
\[
\pi^n(T_x : \mathcal{O}H \to \ell_p) \lesssim \left(\frac{p'}{p' - 2}\right)^{\frac{1}{2}} \|x\|_{\ell_p(\mathcal{O}H)}
\]
for all \(x \in \ell_p(\mathcal{O}H)\) and \(T_x : \mathcal{O}H \to \ell_p\), the linear map naturally associated to \(x\).

**Proof.** We focus on the \(n\)-dimensional case as before. Let
\[
a = \sum_{i,j=1}^n a_{ij} e_j \otimes e_i \in \mathcal{O}H_n \otimes S_p^n.
\]
Suppose
\[
\|(a_{ij})\|_{\ell_p' (\ell_p^2)} = \left\| \left( \sum_{i=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}} \right\|_{\ell_p'} < 1.
\]
Then there are \(g_i = (g^i_1)_{i=1}^n, \ldots, g_8 = (g^i_8)_{i=1}^n\) with
\[
1 = g^i_1 + \cdots + g^i_8
\]
such that
\[
4 > \sum_{i=1}^n \left[ \sum_{j=1}^n |a_{ij}|^2 \right]^{\frac{1}{2}} \|g^i_1\|_{L_2(\mu_1)} + \sum_{j=1}^n |a_{ij}|^2 \left\| g^i_2 \right\|_{L_2(\mu_2)} + \cdots + \left( \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}} \left\| g^i_3 \right\|_{L_2(\mu_3, \xi_3)}
\]

\[
+ \cdots + \left( \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}} \left\| g^i_8 \right\|_{L_2(\mu_8, \xi_8)}.
\]

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If we set $f_{ij}^l = g_i^l \otimes a_{ij}$ for $1 \leq l \leq 8$, then we have

$$4 > \sum_{i=1}^{n} \left[ \left\| (f_{ij}^1)^n_{j=1} \right\|^2_{L_2(\mu_1; \ell_2^n)} + \left\| (f_{ij}^2)^n_{j=1} \right\|^2_{L_2(\mu_2; \ell_2^n)} ight. \\
+ \left( f_{ij}^3 \right)^n_{i,j=1} \left\| L_{2(\mu_3,1) \otimes L_{2(\mu_3,2) \otimes \ell_2^n}} + \cdots + \left( f_{ij}^8 \right)^n_{i,j=1} \right\|_{L_2(\mu_8,1) \otimes L_{2(\mu_8,2) \otimes \ell_2^n}} \\
\left. = \left( f_{ij}^1 \right)^n_{i,j=1} \left\| L_{2(\mu_1,1) \otimes L_{2(\mu_2,2) \otimes \ell_2^n}} \right\|_{L_2(\mu_1,1) \otimes L_{2(\mu_2,2) \otimes \ell_2^n}} + \left( f_{ij}^2 \right)^n_{i,j=1} \right\|_{L_2(\mu_2,1) \otimes L_{2(\mu_2,2) \otimes \ell_2^n}} \\
+ \cdots + \left( f_{ij}^8 \right)^n_{i,j=1} \right\|_{L_2(\mu_8,1) \otimes L_{2(\mu_8,2) \otimes \ell_2^n}}. \\
$$

Now we have by Corollary 5.2 that

$$\pi^n(T_a) \sim (1 - \theta) \| 1 \otimes a \|_{K_{P(T(a), \ell_2^n)}} \leq (1 - \theta) \inf \sum_{l=1}^{8} \| f_l \|_{F_l},$$

where $\theta = \frac{2}{p}$ and the infimum above runs over all possible

$$1 \otimes a = f_1 + \cdots + f_8.$$

Note that the formal identities

$$L_2(\mu) \otimes \pi X \rightarrow L_2(\mu; X)$$

and $\ell^n(\ell_2^n) = \ell^n \otimes \pi \ell_2^n \rightarrow \ell^n \otimes \pi \ell_2^n$ are contractions for any Banach space $X$. Then, we have

$$\sum_{l=1}^{8} \| f_l \|_{F_l} \leq \| f_1 \|_{L_2(\mu_1,1) \otimes L_{2(\mu_2,2) \otimes \ell_2^n}} + \| f_2 \|_{L_2(\mu_2,1) \otimes L_{2(\mu_2,2) \otimes \ell_2^n}} \\
+ \| f_3 \|_{L_2(\mu_3,1) \otimes L_{2(\mu_3,2) \otimes \ell_2^n}} + \| f_4 \|_{L_2(\mu_4,1) \otimes L_{2(\mu_4,2) \otimes \ell_2^n}} + \| f_5 \|_{L_2(\mu_5,1) \otimes L_{2(\mu_5,2) \otimes \ell_2^n}} + \cdots + \| f_8 \|_{L_2(\mu_8,1) \otimes L_{2(\mu_8,2) \otimes \ell_2^n}}$$

are all less than $28$.

If we set $f_l = (f_{ij}^l)_{i,j=1}^n$, then we have

$$\sum_{l=1}^{8} \| f_l \|_{F_l} < 28.$$

Thus, we have

$$\pi^n(T_a) \leq (1 - \theta) \left\| (a_{ij})_{i,j=1}^n \right\|_{\ell_\theta(\ell_2^n)}.$$

Finally, by Proposition 7.1 we have

$$\pi^n(T_a) \leq (1 - \theta)^{-\frac{1}{p}} \left\| \sum_{i,j=1}^{n} a_{ij} e_i \otimes e_j \right\|_{\ell_p(\ell_2^n)}.$$

\[\square\]

**Remark 7.3.** A similar argument as above can be used to prove (3’’) of Remark 3.2. Let’s describe it briefly. Let $1 < p < 2$ and $\theta = \frac{1}{p}$. First, we consider the embedding of

$$C_p \hookrightarrow L_2^p(t^{-\theta}; \ell_2) + L_2^p(t^{1-\theta}; \ell_2), \quad e_i \mapsto 1 \otimes e_i,$$

By a similar argument as in section 1.2 it is well known that

$$L_2^p(t^{-\theta}; \ell_2) + L_2^p(t^{1-\theta}; \ell_2)$$
is completely complemented in the predual of a von Neumann algebra with QWEP. For \( OH \) we use the same embedding as before. Then, we have

\[
\pi_1^0(T_x : OH \to C_p) \sim \| 1 \otimes x \|_{\mathcal{K}_n^0(OH,C_p)},
\]

where

\[
\mathcal{K}_n^0(OH,C_p) = (L_2^p(t^{-\frac{1}{2}}; \ell_2) + L_2^p(t^{\frac{1}{2}}s^{-\theta}; \ell_2) \otimes (L_2^p(s^{-\theta}; \ell_2) + L_2^p(s^{1-\theta}; \ell_2))
\]

\[
= L_2^p(t^{-\frac{1}{2}}; \ell_2) \otimes L_2^p(s^{-\theta}; \ell_2) + L_2^p(t^{\frac{1}{2}}; \ell_2) \otimes L_2^p(s^{1-\theta}; \ell_2)
\]

\[
+ L_2^p(t^{-\frac{1}{2}}; \ell_2) \otimes L_2^p(s^{-\theta}; \ell_2) + L_2^p(t^{\frac{1}{2}}; \ell_2) \otimes L_2^p(s^{0}; \ell_2).
\]

When \( x = \sum_{i=1}^n e_i \otimes e_i \), we can calculate

\[
\| 1 \otimes x \|_{\mathcal{K}_n^0(OH,C_p)} \sim (1 - \theta)^{-\frac{1}{2}}(2\theta - 1)^{-\frac{1}{2}} n^{\frac{4p}{p+2}}
\]

as before. (We divide \( \mathbb{R}_+^2 \) into four regions according to the minimum, then we get the lower bound and the upper bound is the same since we have separation of variables for all problematic terms.)

Since we have \( S_p(OH) = C_p \otimes_h OH \otimes_h R_p \) under the mapping

\[
e_{ij} \otimes e_k \mapsto e_{i1} \otimes e_{k} \otimes e_{1j}
\]

we are only to compare \( \| 1 \otimes x \|_{\mathcal{K}_n^0(OH,C_p)} \) and \( \| x \|_{C_p \otimes_h OH} \). Note that for any unitaries \( U \) and \( V \) we have

\[
\| 1 \otimes UXV \|_{\mathcal{K}_n^0(OH,C_p)} = \| 1 \otimes x \|_{\mathcal{K}_n^0(OH,C_p)}
\]

and

\[
\| UXV \|_{C_p \otimes_h OH} = \| x \|_{C_p \otimes_h OH},
\]

since

\[
C_p \otimes_h OH = [C,R]^{1/2}_h \otimes_h OH = [C \otimes_h OH, R \otimes_h OH]^{1/2}_h
\]

\[
= \left[ [C \otimes_h C, C \otimes_h R]^{1/4}_h, [R \otimes_h C, R \otimes_h R]^{1/4}_h \right]^{1/2}_h \cong S_r
\]

isometrically for \( r = \frac{4p}{p+2} \).

Thus it is enough to consider the case when \( x \) is a diagonal matrix. Since the closed linear span of \( 1 \otimes x \) and \( x \) for diagonal \( x \) in \( \mathcal{K}_n^0(OH,C_p) \) and \( C_p \otimes_h OH \), respectively, are equivalent to Orlicz sequence spaces we are only to compare norms \( \| 1 \otimes x \|_{\mathcal{K}_n^0(OH,C_p)} \) and \( \| x \|_{C_p \otimes_h OH} \) for \( x = \sum_{i=1}^n e_i \otimes e_i \), which is already done above.

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