New results for the $\varepsilon$-expansion of certain one-, two- and three-loop Feynman diagrams

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Abstract

For certain dimensionally-regulated one-, two- and three-loop diagrams, problems of constructing the $\varepsilon$-expansion and the analytic continuation of the results are studied. In some examples, an arbitrary term of the $\varepsilon$-expansion can be calculated. For more complicated cases, only a few higher terms in $\varepsilon$ are obtained. Apart from the one-loop two- and three-point diagrams, the examples include two-loop (mainly on-shell) propagator-type diagrams and three-loop vacuum diagrams. As a by-product, some new relations involving Clausen function, generalized log-sine integrals and certain Euler–Zagier sums are established, and some useful results for the hypergeometric functions of argument $\frac{1}{4}$ are presented.

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1 Introduction

One of the most powerful tools used in loop calculations is dimensional regularization [1]. In some cases, one can derive results valid for an arbitrary space-time dimension $n = 4 - 2\varepsilon$, usually in terms of various hypergeometric functions. For practical purposes the coefficients of the expansion in $\varepsilon$ are important. In particular, in multiloop calculations higher terms of the $\varepsilon$-expansion of one- and two-loop functions are needed, since one can get contributions where these functions are multiplied by poles in $\varepsilon$. Such poles may appear not only due to factorizable loops, but also as a result of application of the integration by parts [2] or other techniques [3].

One of the important examples are massless propagator-type diagrams occurring in the renormalization group calculations. By now, the structure of the terms of the $\varepsilon$-expansion of such diagrams is well understood. As a rule, the occurring transcendental numbers can be expressed in terms of multiple Euler–Zagier sums [4],

$\zeta(s_1, \ldots, s_k; \sigma_1, \ldots, \sigma_k) = \sum_{n_1 > n_2 > \ldots > n_k > 0} \prod_{j=1}^{k} \frac{(\sigma_j)^{n_j}}{n_j^{s_j}}$, \hspace{1cm} (1.1)

where $\sigma_j = \pm 1$ and $s_j > 0$. For lower cases, these sums correspond to ordinary $\zeta$-functions. For studying the higher-order sums analytical and numerical methods have been recently developed [5]. It was demonstrated [6] that there is a “link” between the quantum field theory and the knot theory: some Feynman diagrams can be connected with knots, so that the values (Euler–Zagier sums) of Feynman diagrams are also associated with knots.

In multiloop massive calculations in QED and QCD [7] within the on-shell scheme, new constants appear, which are related to polylogarithms. A detailed description of the basis of this type is presented in Ref. [8]. Two-loop vacuum diagrams with equal masses [9, 10] yield the transcendental number $\text{Cl}_2 \left( \frac{\pi}{3} \right)$, where $\text{Cl}_j (\theta)$ is the Clausen function (A.4). Some useful properties of this function are collected in Appendix A (see also in [11]). The same constant $\text{Cl}_2 \left( \frac{\pi}{3} \right)$ appears in the one-loop off-shell three-point diagram with massless internal lines, in the symmetric case when all external momenta squared are equal, see in [12].

To classify new constants appearing in single-scale massive diagrams, Broadhurst has introduced in Ref. [13] the “sixth root of unity” basis connected with

$\zeta \left( \begin{array}{cccc} s_1 & \ldots & s_k \\ \lambda^{p_1} & \ldots & \lambda^{p_k} \end{array} \right) = \sum_{n_1 > n_2 > \ldots > n_k > 0} \prod_{j=1}^{k} \frac{\lambda^{p_j n_j}}{n_j^{s_j}}$, \hspace{1cm} (1.2)

where $\lambda = \exp \left( i \frac{\pi}{3} \right)$ and $p_j \in \{0, 1, 2, 3, 4, 5\}$. For particular cases $p_j \in \{0, 3\}$ it coincides with the Euler–Zagier sums (1.1). For both definitions (1.1) and (1.2), the weight can be defined as $\sum_{i=1}^{k} s_i$, whereas the value of $k$ can be associated with the depth (see in [8, 13]).

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1 An on-line calculator for the Euler sums, with an accuracy of 100 decimals, is available at [http://www.cecm.sfu.ca/projects/EZFace/](http://www.cecm.sfu.ca/projects/EZFace/).

2 A collection of results related to this connection can be found on Dirk Kreimer’s home page, [http://dipmza.physik.uni-mainz.de/~kreimer/ps/knft.html](http://dipmza.physik.uni-mainz.de/~kreimer/ps/knft.html) or on David Broadhurst’s home page [http://yan.open.ac.uk/~dbroadhu/knft.html](http://yan.open.ac.uk/~dbroadhu/knft.html).
One of the remarkable results of Ref. [13] is that all finite parts of three-loop vacuum integrals without subdivergences, with an arbitrary distribution of massive and massless lines, can be expressed in terms of four weight-4 constants: $\zeta_4$, $\left[\text{Cl}_2\left(\frac{2}{3}\right)\right]^2$, $U_{3,1} = -2\text{Li}_4\left(\frac{1}{2}\right) + \frac{1}{2} \zeta_4 - \frac{1}{12} \ln^4 2 + \frac{1}{2} \zeta_2 \ln^2 2$ (1.3) (see Eq. (55) of [13]) and, finally,

$$V_{3,1} = \sum_{p>k>0} \frac{(-1)^p}{p^2 k} \cos\left(\frac{2}{3} \pi k\right),$$

(1.4)

which is an essentially new constant. The same constants appear in the $(2-2\varepsilon)$-dimensional Feynman integrals [14], as one should expect due to algebraic relations between diagrams with shifted dimension [3]. We also note that $\left[\text{Cl}_2\left(\frac{2}{3}\right)\right]^2$ appears in the two-loop non-planar three-point diagram [15], when internal lines are massless, whereas all external momenta squared are off shell and equal. Unfortunately, a large number of elements (more than 4000) makes it difficult to define the complete basis of “sixth root of unity” at the weight 4. So far, only the cases with the depth $k \leq 2$ have been examined [13].

In Refs. [16, 17], the central binomial sums were considered,

$$S(a) \equiv \sum_{n=1}^{\infty} \frac{(n!)^{2} 1}{(2n)! n^a}.$$  

(1.5)

It was shown that these sums are connected with the multiple Clausen values, which are defined as the real or imaginary part of the multi-dimensional polylogarithm [18],

$$\text{Li}_{a_1,\ldots,a_k}(z) = \sum_{n_1>n_2>\ldots>n_k>0} \prod_{j=1}^{k} \frac{z^{n_j}}{n_1^{a_1} \ldots n_k^{a_k}},$$

(1.6)

taken at $z = \exp\left(i\frac{\pi}{3}\right)$ (i.e., the “sixth root of unity”)$^3$. In particular (see Theorem 1 in [17]),

$$S(8) = -4\pi \text{Im} \left[\text{Li}_{6,1}\left(e^{i\pi/3}\right)\right] + \frac{3462601}{234230} \zeta_8 - \frac{14}{15} \zeta_5 \zeta_3 - \frac{38}{3} \zeta_5 \zeta_3 + \frac{2}{3} \zeta_2 \zeta_3^2,$$

(1.7)

where we use the following short-hand notation (see Eq. (1.1))

$$\zeta_{s_1,\ldots,s_k} = \zeta(s_1, \ldots, s_k; 1, \ldots, 1).$$

(1.8)

The constant $\zeta_{5,3}$ has occurred in the 6-loop calculation of the $\beta$-function in $\phi^4$-theory [19] (this was recently confirmed in [20]). At the 7-loop order, a new transcendental number arises, $\zeta_{3,5,3}$ [19]. The $\zeta_{5,3}$ and another constant, $\zeta_{7,3}$, appear in the calculation of anomalous dimensions at $\mathcal{O}(1/N^3)$ in the large-$N$ limit [21]. A remarkable property of these constants is

$^3$An on-line calculator for the multiple Clausen values, with an accuracy of 100 decimals, can be found at http://www.cecm.sfu.ca/projects/ezface+. 
their connection with knots\(^4\) [6, 19, 21]: namely, the torus knots \(8_{19}\) and \(10_{124}\) are associated with \(\{29\zeta_5 - 12\zeta_{5,3}\}\) and \(\{94\zeta_{10} - 793\zeta_{7,3}\}\), respectively, whereas \(\zeta_{3,5,3}\) is associated with a certain hyperbolic knot [19, 21].

To predict types of functions (and the values of their arguments) which may appear in higher orders of the \(\varepsilon\)-expansion, a geometrical approach [22] happens to be very useful. Using this approach, the results for all terms of the \(\varepsilon\)-expansion have been obtained for the one-loop two-point function with arbitrary masses [23, 24]. Moreover, all terms have been also obtained for the \(\varepsilon\)-expansion of one-loop three-point integrals with massless internal lines and arbitrary (off-shell) external momenta and two-loop vacuum diagrams with arbitrary masses [24, 25], which are related to each other, due to the magic connection [26]. All these results have been represented in terms of the log-sine integrals (see in [11] and Appendix A.1 of this paper), whose angular arguments have a rather transparent geometrical interpretation (angles of certain triangles). In more complicated cases, like, e.g., the three-point function with general values of the momenta and masses, an arbitrary term of the \(\varepsilon\)-expansion can be represented in terms of one-fold angular integrals whose parameters can be related to the angles associated with a four-dimensional simplex.

In Ref. [27], the on-shell values of two-loop massive propagator-type integrals have been studied, and it was observed that the finite (as \(\varepsilon \to 0\)) parts of all such integrals without subdivergences can be expressed in terms of three weight-3 constants, two for the real part, \(\zeta_3\) and \(\pi \text{Cl}_2\left(\frac{\pi}{3}\right)\), and one for the imaginary part, \(\pi \zeta_2\).

Furthermore, in Ref. [28] an ansatz was elaborated for constructing the “irrationalities” occurring in the \(\varepsilon\)-expansion of single-scale diagrams involving cut(s) with two massive particles. This construction is closely related to the geometrically-inspired all-order \(\varepsilon\)-expansion of the one-loop propagator-type diagrams [23, 24], which was also useful when fixing the normalization factor \(\frac{1}{\gamma_j}\). The procedure of constructing the ansatz is as follows: for each given weight \(j\) the set \(\{b_j\}\) of the basic transcendental numbers contains (i) all products of the lower-weight elements \(\{b_j-kb_k\}, k = 1, 2, \ldots, j - 1\) and (ii) a set of new (non-factorizable) elements \(\{\tilde{b}_j\}\), which are associated with the quantities arising in the real and imaginary parts of the polylogarithms \(\text{Li}_j(e^{i\theta})\) and \(\text{Li}_j(1 - e^{i\theta})\), with \(\theta = \frac{\pi}{3}\) or \(\theta = \frac{2\pi}{3}\).

The real and imaginary parts of such polylogarithms can be expressed in terms of the Clausen function \(\text{Cl}_j(\theta)\) (A.4), log-sine integrals \(\text{Ls}_j(\theta)\) (A.5) and generalized log-sine integrals \(\text{Ls}_j^{(k)}(\theta)\) (A.6) (see also in [11]). Note that \(\text{Ls}_2(\theta) = \text{Cl}_2(\theta)\). The relevant relations are collected in Appendix A.1, Eqs. (A.2) and (A.7). Therefore, in our case the non-factorizable part of the basis can be expressed in terms of the functions (A.4), (A.5) and (A.6) of two possible angles, \(\theta = \frac{\pi}{3}\) and \(\theta = \frac{2\pi}{3}\). It should be noted that this basis is not uniquely defined, since there are several relations between polylogarithmic functions \(\text{Cl}_j(\theta)\), \(\text{Ls}_j(\theta)\) and \(\text{Ls}_j^{(k)}(\theta)\) of these arguments (see, e.g., Eqs. (A.9)–(A.11) and (A.14) of this paper). After excluding all linearly-dependent terms, the basis contains the following non-factorizable constants: \(\text{Ls}_j\left(\frac{2\pi}{3}\right)\), where \(j = 3, 4, 5\); \(\text{Ls}_j\left(\frac{\pi}{3}\right)\) for \(j = 2, 4, 5\); \(\text{Ls}_j^{(1)}\left(\frac{2\pi}{3}\right)\) for \(j = 4, 5\); and \(\text{Ls}_5^{(2)}\left(\frac{2\pi}{3}\right)\). This set of elements will be called the odd basis [28]. The numerical values of

\(^4\)Below we shall discuss connection of these constants with the odd basis.
these constants are given in Appendix A of [28].

One can use the PSLQ algorithm [29] to search for a linear relation between the given term of the ε-expansion of the diagram of interest and the set of numbers \{b_j\}. Using this procedure, it was possible to find results for several two- and three-loop single-scale diagrams [28], some of them have been also calculated analytically in [24, 30]. The constructed odd basis has an interesting property: the number \(N_j\) of the basic irrational constants of a weight \(j\) satisfies a simple “empirical” relation \(N_j = 2^j\), which has been checked up to weight 4. We note that the constant \(V_{3,1}\) (given in Eq. (1.4)) can be expressed in terms of the weight-4 elements of the odd basis,

\[
V_{3,1} = \frac{1}{3} \left[ \text{Cl}_2 \left( \frac{\pi}{3} \right) \right]^2 - \frac{1}{4} \pi \text{Ls}_3 \left( \frac{2\pi}{3} \right) + \frac{13}{24} \zeta_3 \ln 3 - \frac{259}{108} \zeta_4 + \frac{3}{8} \text{Ls}_4^{(1)} \left( \frac{2\pi}{3} \right),
\]
(see Eq. (12) in Ref. [28]). For the weight 5, 32 linearly-independent elements were elaborated in Ref. [28]. It should be noted, however, that some of these irrational numbers, like \(\text{Ls}_5^{(1)} \left( \frac{2\pi}{3} \right)\), so far have not appeared in the results for Feynman integrals\(^5\). So, the question about the completeness of the set of weight-5 basis elements is still open. We are going to re-analyze this basis below.

By analogy with the odd basis introduced in [28], it is possible to consider the even basis connected with the angles \(\frac{\pi}{2}\) and \(\pi\). Apart from the well-known elements \(\pi, \ln 2, \zeta_j\) and the Catalan’s constant\(^6\) \(G\), this basis also contains (up to the weight 5) \(\text{Li}_j \left( \frac{1}{2} \right)\) \((j = 4, 5, \text{see also in [8]})\), \(\text{Ls}_j \left( \frac{\pi}{2} \right)\) \((j = 3, 4, 5)\), \(\text{Cl}_4 \left( \frac{\pi}{2} \right)\) and \(\text{Ls}_5^{(2)} \left( \frac{\pi}{2} \right)\). Instead of \(\text{Li}_4 \left( \frac{1}{2} \right)\) and \(\text{Ls}_5 \left( \frac{1}{2} \right)\), one could take, e.g., \(\text{Ls}_4^{(1)} \left( \frac{\pi}{2} \right)\) and \(\text{Ls}_5^{(1)} \left( \frac{\pi}{2} \right)\) (or \(\text{Ls}_4^{(1)} (\pi)\) and \(\text{Ls}_5^{(1)} (\pi)\)), using the relations between these elements presented in Appendix A, Eq. (A.14). For the even basis, the same “empirical” relation \(N_j = 2^j\) is valid up to weight 4. However, at the weight 5 only 30 independent elements have been found so far. This issue will be also discussed below.

Later, in Ref. [33], an interesting connection between the function \(S(a; z)\) associated with the central binomial sums (1.5) and the generalized log-sine integrals was established,

\[
S(a; z) \equiv \sum_{n=1}^{\infty} \left( \frac{n!}{(2n)!} \right)^2 \frac{z^n}{n^a} = -\sum_{j=0}^{a-2} \frac{(-2)^j}{(a - 2 - j)! j!} (\ln z)^{a-2-j} \text{Ls}_{j+2}^{(1)} \left( 2 \arcsin \sqrt{\frac{z}{2}} \right), \quad (1.9)
\]

where \(a \geq 2\). In particular, the sums (1.5) can be represented as

\[
S(a) \equiv S(a; 1) = \sum_{n=1}^{\infty} \left( \frac{n!}{(2n)!} \right)^2 \frac{1}{n^a} = -\frac{(-2)^{a-2}}{(a - 2)!} \text{Ls}_a^{(1)} \left( \frac{\pi}{3} \right), \quad (1.10)
\]

whereas \(S(a; 3)\) is related to \(\text{Ls}_3^{(1)} \left( \frac{2\pi}{3} \right)\) (plus a combination of lower terms with \(\ln 3\)). Therefore, \(S(a; 1)\) and \(S(a; 3)\) are connected with the terms of the odd basis. Furthermore, considering \(S(a; 2)\) and \(S(a; 4)\) we obtain \(\text{Ls}_a^{(1)} \left( \frac{\pi}{2} \right)\) and \(\text{Ls}_a^{(1)} (\pi)\) (plus a combination of lower terms with \(\ln 2\)), respectively, which are related to the even basis.

\(^5\)The only known integral containing \(\text{Ls}_4^{(1)} \left( \frac{2\pi}{3} \right)\) is \(\text{D}_5(1, 1, 1, 1, 1, 0)\) [28]. The integral \(\text{D}_5(1, 1, 1, 1, 1, 1)\) is reduced to it by using recurrence relations [31]. It is natural to expect that its \(\varepsilon\)-term may contain \(\text{Ls}_5^{(1)} \left( \frac{2\pi}{3} \right)\).

\(^6\)Note that \(G = \text{Cl}_2 \left( \frac{\pi}{2} \right) = \text{Ls}_2 \left( \frac{\pi}{2} \right)\), see in [11].

\(^7\)An example of a physical calculation where the constant \(\text{Ls}_3 \left( \frac{\pi}{2} \right)\) arises is given in [32].
Moreover, in Ref. [33] a more general case of multiple binomial sums has been examined, 
\[ \sum_{i_1, \ldots, i_p; j_1, \ldots, j_q} \equiv \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{k^n}{n} [S_a(n-1)]^{i_1} \ldots [S_{a_p}(n-1)]^{i_p} [S_b(2n-1)]^{j_1} \ldots [S_{b_q}(2n-1)]^{j_q}, \]
where \( S_a(n) = \sum_{j=1}^{n} j^{-a} \) is the harmonic sum. In [33], mainly the case \( k = 1 \) (related to the odd basis) has been treated. A PSLQ-based analysis has shown that not all of sums (1.11) are separately expressible in terms of the basis elements.

This paper is organized as follows. In Section 2 we examine in detail the \( \varepsilon \)-expansion of the two-point function with arbitrary masses, including the problem of analytic continuation of the corresponding functions. In Section 3 we consider some examples of the \( \varepsilon \)-expansion of one-loop three-point functions, and study which functions and transcendental constants may occur in the cases considered. In Section 4 we consider some physically relevant two- and three-loop diagrams. In Section 5 we discuss the results obtained in this paper. There are also two appendices containing some further technical details and useful formulæ. In Appendix A we collect relevant results for the polylogarithms and associated functions. In particular, we discuss the relation of the Euler–Zagier sums and the generalized log-sine functions. In Appendix B we discuss the expansion of the occurring hypergeometric functions with respect to their parameters.

## 2 One-loop two-point function

### 2.1 General case

Consider the one-loop two-point function with the external momentum \( k \) and masses \( m_1 \) and \( m_2 \),
\[ J^{(2)}(n; \nu_1, \nu_2) \equiv \int \frac{d^n q}{[q^2 - m_1^{2\nu_1}][[(k - q)^2 - m_2^{2\nu_2}]. \]

Using, for instance, a geometrical approach [22, 23], one can obtain the following result (for unit powers of the propagators, \( \nu_1 = \nu_2 = 1 \)):
\[ J^{(2)}(4-2\varepsilon; 1, 1) = i \pi^{\varepsilon-1} \Gamma(\varepsilon) \frac{1}{2k^2} \left\{(k^2 + m_1^2 - m_2^2)m_1^{-2\varepsilon} \left. F_1 \left( \frac{1}{3}, \frac{\varepsilon}{2} \mid \cos^2 \tau'_{01} \right) \right\} \right. \]
\[ \left. +(k^2 - m_1^2 + m_2^2)m_2^{-2\varepsilon} \left. F_1 \left( \frac{1}{3}, \frac{\varepsilon}{2} \mid \cos^2 \tau'_{02} \right) \right\} \right. \]

where \( F_1 \) is the Gauss hypergeometric function. The angles \( \tau'_{0i} \) are defined (see in [24]) via
\[ \cos \tau'_{01} = \frac{k^2 + m_1^2 - m_2^2}{2m_1 \sqrt{k^2}}, \quad \cos \tau'_{02} = \frac{k^2 - m_1^2 + m_2^2}{2m_2 \sqrt{k^2}}. \]

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8 Another notation for \( S_a(n) \), used by mathematicians, is \( H_a^{(a)} \). In particular, for \( a = 1 \), \( S_1(n) = H_n \). When there are no sums of the type \( S_a(n-1) \) or \( S_a(2n-1) \) on the r.h.s. of Eq. (1.11), we shall put a “−” sign instead of the indices \((a,i)\) or \((b,j)\) of \( \Sigma \), respectively. If the argument \((k)\) is omitted in (1.11), this means that the case \( k = 1 \) is understood.

9 They are related to the angles \( \tau_{0i} \) used in Ref. [22] as \( \tau'_{0i} = \frac{\pi}{2} - \tau_{0i} \).
In particular, for angle $\tau_{12}$ (defined so that $\tau_{12} + \tau'_{01} + \tau'_{02} = \pi$) we have
\[
\cos \tau_{12} = \frac{m_1^2 + m_2^2 - k^2}{2m_1 m_2}, \quad \sin \tau_{12} = \frac{\sqrt{\Delta(m_1^2, m_2^2, k^2)}}{2m_1 m_2}.
\] (2.4)

Here the “triangle” function $\Delta$ is defined as
\[
\Delta(x, y, z) = 2xy + 2yz + 2zx - x^2 - y^2 - z^2 = -\lambda(x, y, z),
\] (2.5)

where $\lambda(x, y, z)$ is the well-known Källen function. The result (2.2) can be related to those presented in Ref. [22] by a simple transformation of the occurring $2F_1$ functions.

Using Kummer relations for the contiguous $2F_1$ functions, we get
\[
(1 - 2\varepsilon)\ _2F_1\left(\begin{array}{c}1, \varepsilon \\ \frac{3}{2} \end{array} \bigg| z\right) = 1 - 2\varepsilon(1 - z)\ _2F_1\left(\begin{array}{c}1, 1 + \varepsilon \\ \frac{3}{2} \end{array} \bigg| z\right).
\] (2.6)

The resulting $2F_1$ function can be represented as
\[
_2F_1\left(\begin{array}{c}1, 1 + \varepsilon \\ \frac{3}{2} \end{array} \bigg| \sin^2 \theta\right) = \frac{1}{\sin \theta (\cos \theta)^{1+2\varepsilon}} f_\varepsilon(\theta),
\] (2.7)

with (see in [11, 24])
\[
f_\varepsilon(\theta) \equiv \int_0^\theta d\phi (\cos \phi)^{2\varepsilon} = 2^{-1-2\varepsilon} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} [L_{s,j+1}(\pi - 2\theta) - L_{s,j+1}(\pi)] .
\] (2.8)

As a result, we reproduce the $\varepsilon$-expansion of the two-point integral obtained in Ref. [24],
\[
J^{(2)}(4-2\varepsilon; 1, 1) = i\pi^{2-\varepsilon} \frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)} \left\{ \frac{m_1^{-2\varepsilon} + m_2^{-2\varepsilon}}{\varepsilon} + \frac{m_2^2 - m_1^2}{\varepsilon k^2} \left( \frac{m_1^{-2\varepsilon} - m_2^{-2\varepsilon}}{2\varepsilon} \right) \right. \\
+ \frac{\Delta(m_1^2, m_2^2, k^2)^{1/2-\varepsilon}}{(k^2)^{1-\varepsilon}} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} \sum_{i=1}^{2} [L_{s,j+1}(\pi) - L_{s,j+1}(2\tau'_{0i})] \left. \right\}.
\] (2.9)

The expansion (2.9) is directly applicable in the region where $\Delta(m_1^2, m_2^2, k^2) \geq 0$, i.e. when $(m_1 - m_2)^2 \leq k^2 \leq (m_1 + m_2)^2$. To obtain results valid in the region $\Delta(m_1^2, m_2^2, k^2) \leq 0$ (i.e., $\lambda(m_1^2, m_2^2, k^2) \geq 0$) the proper analytic continuation of the occurring $L_{s,j}(\theta)$ should be constructed.

An important special case of Eqs. (2.7)-(2.8) is $\theta = \frac{\pi}{6}$ ($z = \frac{1}{4}$). It corresponds to the on-shell value $k^2 = m^2$ of the integral (2.9) with $m_1 = m_2 \equiv m$. In this case, we obtain $L_{s,j+1}(\frac{2\pi}{3})$ which correspond to the odd basis discussed in the introduction. Expansion of more general $2F_1$ functions of argument $z = \frac{1}{4}$ is discussed in Appendix B.2.

We shall also need another representation for the two-point function (see, e.g., in [34]),
\[
J^{(2)}(4-2\varepsilon; 1, 1) = i\pi^{2-\varepsilon} \Gamma(\varepsilon) \frac{1}{k^2} \left\{ \frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \lambda^{1/2-\varepsilon}(k^2)^{\varepsilon} e^{i\pi \varepsilon} \\
\right. \\
\left. \right\}.
\]
where
\[ z_1 = \left[ \frac{\sqrt{\lambda(m_1^2, m_2^2, k^2) + m_1^2 - m_2^2 - k^2}}{4m_2^2k^2} \right]^2, \quad z_2 = \left[ \frac{\sqrt{\lambda(m_1^2, m_2^2, k^2) - m_1^2 + m_2^2 - k^2}}{4m_2^2k^2} \right]^2, \]
(2.11)
with \( \lambda(m_1^2, m_2^2, k^2) \) defined in Eq. (2.5).

Employing Kummer’s relations for contiguous functions, one can transform the \( \text{$_2F_1$} \) function from Eq. (2.10) into
\[ \text{$_2F_1$} \left( \begin{array}{c} 1, \varepsilon \\ 2 - \varepsilon \end{array} \bigg| z \right) = \frac{1 - \varepsilon}{2(1-2\varepsilon)z} \left\{ 1 + z - (1-z)^2 \text{$_2F_1$} \left( \begin{array}{c} 1, 1+\varepsilon \\ 1 - \varepsilon \end{array} \bigg| z \right) \right\} . \] (2.12)
The resulting \( \text{$_2F_1$} \) function can be expressed in terms of a simple one-fold parametric integral,
\[ \text{$_2F_1$} \left( \begin{array}{c} 1, 1+\varepsilon \\ 1 - \varepsilon \end{array} \bigg| z \right) = (1-z)^{1-2\varepsilon} \left\{ 1 - \varepsilon \int_0^1 \frac{dt}{t} t^{-\varepsilon} \left[ (1-zt)^{2\varepsilon} - 1 \right] \right\} . \] (2.13)

Expanding the integrand in \( \varepsilon \), we get
\[ \text{$_2F_1$} \left( \begin{array}{c} 1, \varepsilon \\ 2 - \varepsilon \end{array} \bigg| z \right) = \frac{1 - \varepsilon}{2(1-2\varepsilon)z} \left\{ 1 + z - (1-z)^{1-2\varepsilon} - 2(1-z)^{1-2\varepsilon} \sum_{j=1}^{\infty} \varepsilon^j \sum_{k=1}^{\infty} (-2)^{j-k} S_{k,j-k+1}(z) \right\} , \]
(2.14)
where \( S_{a,b}(z) \) is the Nielsen polylogarithm (see, e.g., in Ref. [35]), whose definition (A.17) and some properties are collected in Appendix A.1.

### 2.2 Analytic continuation

To relate results in different regions, it is convenient to introduce the variables
\[ z_j \equiv e^{i\sigma \theta_j}, \quad \ln(-z - i\sigma 0) = \ln(z) - i\sigma \pi, \] (2.15)
where we put \( \theta_j \equiv 2\pi \delta_j \), and the choice of the sign \( \sigma = \pm 1 \) is related to the causal “+i0” prescription for the propagators. Whenever possible, we shall keep \( \sigma \) undetermined, since one may need different signs in different situations.

Let us note that, transforming from variable \( z \) to \( 1/z \), we get the same \( \text{$_2F_1$} \) function,
\[ \text{$_2F_1$} \left( \begin{array}{c} 1, \varepsilon \\ 2 - \varepsilon \end{array} \bigg| \frac{1}{z} \right) = \frac{1}{z} \text{$_2F_1$} \left( \begin{array}{c} 1, \varepsilon \\ 2 - \varepsilon \end{array} \bigg| \frac{1}{z} \right) + \frac{(1-\varepsilon) \Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)} (z)^{-\varepsilon} \left( 1 - \frac{1}{z} \right)^{1-2\varepsilon} . \] (2.16)
In particular, we can replace each \( \genfrac{2}{1}{.}{.}{2}{1} \) function in Eq. (2.10) by a linear combination of l.h.s. and r.h.s. of Eq. (2.16), with the sum of the corresponding coefficients equal to one (say, \( \rho_{1,2} \) and \( (1 - \rho_{1,2}) \)),

\[
J^{(2)}(4 - 2\varepsilon; 1, 1) = i\pi^{2 - \varepsilon} \Gamma(1 + \varepsilon) \frac{m_1^{-2\varepsilon} + m_2^{-2\varepsilon}}{2(1 - 2\varepsilon)} + \frac{m_1^2 - m_2^2}{\varepsilon k^2} \left( m_1^{-2\varepsilon} - m_2^{-2\varepsilon} \right)
\]

\[
+i\sigma \left[ \frac{\Delta(m_1^2, m_2^2, k^2)^{1/2 - \varepsilon}}{(k^2)^{1 - \varepsilon}} \frac{2\Gamma^2(1 - \varepsilon)}{\varepsilon(1 - 2\varepsilon)} (1 - \rho_1 - \rho_2) + \sum_{i=1}^{2} \varepsilon_i \left[ \rho_i(-z_i)^{-\varepsilon} - (1 - \rho_i)(-z_i)^{\varepsilon} \right] \right]
\]

\[
+ 2 \sum_{i=1}^{2} \varepsilon_i \sum_{j=1}^{\infty} \varepsilon_j \sum_{k=1}^{j} (-2)^{j-k} S_{k,j-k+1}(z_i)
\]

\[
- (1 - \rho_1)(-z_i)^{\varepsilon} \sum_{j=1}^{\infty} \varepsilon_j \sum_{k=1}^{j} (-2)^{j-k} S_{k,j-k+1}(1/z_i) \right] \right). \tag{2.17}
\]

Comparing (2.9) and (2.17) for \( \rho_1 = \rho_2 = \frac{1}{2} \), we arrive at the following analytic continuation of the functions involved in the \( \varepsilon \)-expansion:

\[
i\sigma [Ls_{j}(\pi) - Ls_{j}(\theta)] = \frac{1}{2j} \ln'(-z) \left[ 1 - (-1)^{j} \right]
\]

\[
+ (-1)^{j}(j - 1) \sum_{p=0}^{j-2} \frac{\ln^p(-z)}{2^p p!} \sum_{k=1}^{p} (-2)^{-k} [S_{k,j-k-p}(z) - (-1)^p S_{k,j-k-p}(1/z)] . \tag{2.18}
\]

where \( z_i \) and \( \sigma \) are defined in (2.15). In particular,

\[
i\sigma \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} \left[ Ls_{j+1}(\pi) - Ls_{j+1}(\theta) \right] = \frac{1}{2\varepsilon} \left[ (-z)^{\varepsilon} - (-z)^{-\varepsilon} \right]
\]

\[
- (-z)^{-\varepsilon} \sum_{j=1}^{\infty} \varepsilon_j \sum_{k=1}^{j} (-2)^{j-k} S_{k,j-k+1}(z) + (-z)^{\varepsilon} \sum_{j=1}^{\infty} \varepsilon_j \sum_{k=1}^{j} (-2)^{j-k} S_{k,j-k+1}(1/z) . \tag{2.19}
\]

Since \( Ls_{1}(\theta) = -\theta, Ls_{2}(\pi) = 0 \) we get

\[
i\sigma [Ls_{1}(\pi) - Ls_{1}(\theta)] = \ln(-z), \quad i\sigma [Ls_{2}(\pi) - Ls_{2}(\theta)] = -\frac{1}{2} [Li_{2}(z) - Li_{2}(1/z)] . \tag{2.20}
\]

We note that for higher values of \( j \) the number of generalized polylogarithms involved in Eq. (2.18) can be reduced. For even \( j = 2l \ (l \geq 2) \), we have

\[
i\sigma [Ls_{2l}(\pi) - Ls_{2l}(\theta)] =
\]

\[
- \frac{(2l - 1)!}{2} \sum_{p=0}^{l-2} \frac{\ln^p(-z)}{p!} \sum_{k=0}^{l-p-2} (-1)^k a_{l,p+k} [S_{k+1,2l-k-p-1}(z) - (-1)^p S_{k+1,2l-k-p-1}(1/z)]
\]

\[
+ (-1)^l \frac{(2l - 1)!}{2^{2l-1}} \sum_{k=0}^{2l-2} \frac{(-1)^k}{k!} \ln^k(-z) \left[ Li_{2l-k}(z) - (-1)^k Li_{2l-k}(1/z) \right] , \tag{2.21}
\]
whereas for odd $j = 2l + 1$ we have
\[
\frac{i}{l+1} (2l)! \sum_{l=0}^{p-1} \frac{\ln^p(-z)}{p!} l^{p-1} \sum_{k=0}^{l-p-1} (-1)^k a_{l+1,p+k} \left[ S_{l+k+1,2l-k-p}(z) - (-1)^p S_{l+k+1,2l-k-p}(1/z) \right] \\
- \frac{1}{4^d l!} \sum_{k=0}^{j} \frac{(-2)^k (2l - k)!}{k!(l - k)!} \ln^k(-z) \left[ \text{Li}_{l-k+1}(z) - (-1)^k \text{Li}_{l-k+1}(1/z) \right], \tag{2.22}
\]
where
\[
a_{l,p} = \frac{p}{4^p} \sum_{q=0}^{l-p-2} \frac{(p + 2q - 1)!}{4^q q!(p + q)!}, \quad a_{l,0} = 1. \tag{2.23}
\]

### 2.3 Massless limit

As a simple example we can consider the limit when one of the masses vanishes, $m_1 = 0$ ($m_2 \equiv m$). Using hypergeometric representation (see, e.g., Eq. (10) of [44]),
\[
J^{(2)}(4-2\varepsilon; 1, 1)|_{m_1=0, m_2=m} = i\pi^{2-\varepsilon} m^{-2\varepsilon} \frac{\Gamma(1 + \varepsilon)}{\varepsilon(1 - \varepsilon)} \left( \begin{array}{c} 1, \varepsilon \\ 2 - \varepsilon \end{array} \right) \frac{k^2}{m^2} \tag{2.24}
\]
and Eq. (2.14), the result for an arbitrary term of the $\varepsilon$-expansion can be obtained [25]
\[
J^{(2)}(4-2\varepsilon; 1, 1)|_{m_1=0, m_2=m} = i\pi^{2-\varepsilon} m^{-2\varepsilon} \frac{\Gamma(1 + \varepsilon)}{(1 - 2\varepsilon)} \left\{ \frac{1}{\varepsilon} - \frac{1 - u}{2u\varepsilon} \left[ (1 - u)^{-2\varepsilon} - 1 \right] - \frac{(1-u)^{1-2\varepsilon}}{u} \sum_{j=1}^{\infty} \varepsilon^j \sum_{k=1}^{j} (-2)^{j-k} S_{k,j-k+1}(u) \right\}, \tag{2.25}
\]
with $u = k^2/m^2$.

Note that for this limit the terms up to order $\varepsilon^3$ can be extracted from Eq. (A.3) of Ref. [36]. Our expressions are in agreement with their results. We would like to mention, that each term of the expansion (2.25) can be obtained from general results, (2.9) or (2.17). Expanding them with respect to $m_1^2$ and $\ln(m_1^2)$ (see details in [25]), we reveal, that there is a limit $m_1^2 = 0$. This procedure allows to check the unambiguity of $\sigma$: the limit $m_1^2 = 0$ must exist in each order of $\varepsilon$. So, the finite part gives relation between (2.15) and $\ln(-m_1^2/k^2)$, whereas linear one gives rise to correlation between sign of the analytical continuation of function $\lambda(m_1^2, m_2^2, k^2)$ and $\ln(-m_1^2/k^2)$. Since, for Feynman propagator, $\ln(-m_1^2/k^2) = \ln(m_1^2/k^2) + i\pi$, we have $\sigma = -1$. 

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3 Three-point examples

3.1 General remarks

In this section we shall consider $\varepsilon$-expansion of one-loop three-point integrals, shown in Fig. 1,

$$J^{(3)}(n; \nu_1, \nu_2, \nu_3) \equiv \int \frac{d^n q}{\left[(p_2 - q)^2 - m_1^2\right]^{\nu_1}\left[(p_1 + q)^2 - m_2^2\right]^{\nu_2}\left[q^2 - m_3^2\right]^{\nu_3}}.$$  (3.1)

![Figure 1: One-loop three-point function with masses $m_i$ and momenta $p_i$ ($p_1 + p_2 + p_3 = 0$)](image)

We shall mainly be interested in the case of positive integer powers of the propagators $\nu_i$. Using the integration-by-parts approach [2] (for details, see in [37]), all such integrals can be algebraically reduced to $J^{(3)}(n; 1, 1, 1)$ and two-point integrals. Therefore, we shall concentrate on the case $\nu_1 = \nu_2 = \nu_3 = 1$.

To construct terms of the $\varepsilon$-expansion of $J^{(3)}(n; 1, 1, 1)$ with general masses and external momenta, the geometrical description seems to be rather instructive. The geometrical approach to the three-point function is discussed in section V of [22] (see also in [23]). This function can be represented as an integral over a spherical (or hyperbolic) triangle, as shown in Fig. 6 of [22], with a weight factor $1/\cos^{1-2\varepsilon}\theta$ (see eqs. (3.38)–(3.39) of [22]). This triangle 123 is split into three triangles 012, 023 and 031. Then, each of them is split into two rectangular triangles, according to Fig. 9 of [22]. We consider the contribution of one of the six resulting triangles, namely the left rectangular triangle in Fig. 9. Its angle at the vertex 0 is denoted as $\frac{1}{2}\varphi_{12}^+$, whereas the height dropped from the vertex 0 is denoted $\eta_{12}$.
The remaining angular integration is (see eq. (5.16) of [22])

\[
\frac{1}{2^\varepsilon} \int_0^{\varphi_0^{1/2}} \ln^{j+1} \left( 1 + \frac{\tan^2 \eta_{12}}{\cos^2 \varphi} \right) \times \sum_{j=0}^{\infty} \frac{(-\varepsilon)^j}{(j+1)!} \int_0^{\varphi_0^{1/2}} \ln^{j+1} \left( 1 + \frac{\tan^2 \eta_{12}}{\cos^2 \varphi} \right)
\]

First of all, we note that the l.h.s. of Eq. (3.2) yields a representation valid for an arbitrary \(\varepsilon\) (i.e., in any dimension). To get the result for the general three-point function, we need to consider a sum of six such integrals. The resulting representation is closely related to the representation in terms of hypergeometric functions of two arguments [38] (see also in [39] for some special cases).

In the limit \(\varepsilon \to 0\) we get a combination of Cl functions\(^{10}\), eq. (5.17) of [22]. Collecting the results for all six triangles, we get the result for the three-point function with arbitrary masses and external momenta, corresponding (at \(\varepsilon = 0\)) to the analytic continuation of the well-known formula presented in [41]. The higher terms of the \(\varepsilon\)-expansion correspond to the angular integrals on the r.h.s. of Eq. (3.2). We note that the \(\varepsilon\)-term of the three-point function with general masses has been calculated in [42] in terms of \(\text{Li}_3\).

An important special case is when all internal masses are equal to zero, whereas the external momenta are off shell. This case was considered in detail in Refs. [24, 25]. The following result was obtained:

\[
J^{(3)}(4 - 2\varepsilon; 1, 1, 1)_{m_1=m_2=m_3=0} = 2\pi^{2-\varepsilon} i^{1+2\varepsilon} \Gamma(1+\varepsilon)\Gamma(1-\varepsilon) \frac{[\Delta(p_1^2, p_2^2, p_3^2)]^{-1/2+\varepsilon}}{(p_1^2 p_2^2 p_3^2)^\varepsilon} \times \sum_{j=0}^{\infty} \frac{(-\varepsilon)^j}{(j+1)!} \left[ L_{s_{j+2}}(\pi) - \sum_{i=1}^{3} L_{s_{j+2}}(\pi) - L_{s_{j+2}}(2\phi_i) \right],
\]

(3.3)

where the angles \(\phi_i\) \((i = 1, 2, 3)\) are defined via

\[
\cos \phi_1 = \frac{p_2^2 + p_3^2 - p_1^2}{2\sqrt{p_2^2 p_3^2}}, \quad \cos \phi_2 = \frac{p_3^2 + p_1^2 - p_2^2}{2\sqrt{p_1^2 p_3^2}}, \quad \cos \phi_3 = \frac{p_1^2 + p_2^2 - p_3^2}{2\sqrt{p_1^2 p_2^2}},
\]

(3.4)

so that \(\phi_1 + \phi_2 + \phi_3 = \pi\). Therefore, the angles \(\phi_i\) can be understood as the angles of a triangle whose sides are \(\sqrt{p_1^2}, \sqrt{p_2^2}\) and \(\sqrt{p_3^2}\), whereas its area is \(\frac{1}{2} \sqrt{\Delta(p_1^2, p_2^2, p_3^2)}\). Introducing \(\theta_i = 2\phi_i\), we can use the same procedure of analytic continuation as discussed in Section 2.2 (with \(\sigma = 1\)), in terms of the Nielsen polylogarithms \(S_{a,b}(z)\).

Note that the integral (3.3) is closely related to the two-loop vacuum integral with different masses, due to the magic connection [26]. The result for this integral, which is called \(I(n; \nu_1, \nu_2, \nu_3)\), is also presented in Refs. [24, 25], in an arbitrary order in \(\varepsilon\). For the analytic continuation, we can also apply the procedure of Section 2.2 (with \(\sigma = -1\)). Such integrals will also occur in Section 4.

\(^{10}\)For \(\varepsilon = \frac{1}{2}\) \((n = 3)\) we reproduce the well-known result of [40] in terms of elementary functions (for further details, see Section VA of [22]).
3.2 On-shell triangle with two different masses

Consider a three-point integral (3.1) with one massless propagator \( m_3 = 0 \) and two adjacent legs on shell \( (p_3^2 = m_1^2, p_1^2 = m_2^2) \). Such integrals are important, e.g., for studying corrections to the muon decay in the Fermi model. Using Feynman parametric representation, one can show that in the \( \nu_1 = \nu_2 = \nu_3 = 1 \) case this integral can be reduced to a two-point integral with the masses of internal particles \( m_1 \) and \( m_2 \) and the external momentum \( k \equiv p_3 \).

\[
J^{(3)}(4 - 2\varepsilon; 1, 1, 1) \bigg|_{m_3=0, p_3^2=m_1^2, p_1^2=m_2^2} = \frac{\pi}{2\varepsilon^2} J^{(2)}(2 - 2\varepsilon; 1, 1) .
\]

Note that all powers of propagators are equal to one, whereas the space-time dimension is \( (2 - 2\varepsilon) \) in the two-point integral.

Using Eq. (6) of [43], we can represent it in terms of the \( (4 - 2\varepsilon) \)-dimensional integrals,

\[
J^{(2)}(2 - 2\varepsilon; 1, 1) = -\frac{1}{\pi} \left[ J^{(2)}(4 - 2\varepsilon; 2, 1) + J^{(2)}(4 - 2\varepsilon; 1, 2) \right] .
\]

Then, using the integration-by-parts technique [2] (see also Eq. (A.17) of [34]), we can express the integrals with second power of one of the propagators in terms of \( J^{(2)}(4 - 2\varepsilon; 1, 1) \) and tadpole integrals. As a result, we get

\[
J^{(2)}(2 - 2\varepsilon; 1, 1) = -\frac{1}{\pi\Delta(m_1^2, m_2^2, k^2)} \left\{ 2(1 - 2\varepsilon)k^2 J^{(2)}(4 - 2\varepsilon; 1, 1) - i\pi^{2-\varepsilon} \Gamma(\varepsilon) \left[ \left(k^2 + m_1^2 - m_2^2\right)m_1^{-2\varepsilon} + (k^2 - m_1^2 + m_2^2)m_2^{-2\varepsilon} \right] \right\},
\]

with \( \Delta(m_1^2, m_2^2, k^2) \) defined in Eq. (2.5).

In particular, the \( \varepsilon \)-expansion of this integral is

\[
J^{(2)}(2 - 2\varepsilon; 1, 1) = -i\pi^{1-\varepsilon} \frac{\Gamma(1 + \varepsilon)(k^2)^\varepsilon}{\Delta(m_1^2, m_2^2, k^2)^{1/2+\varepsilon}} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} \sum_{i=1}^{2} \left[ \text{L}_{s_{j+1}}(\pi) - \text{L}_{s_{j+1}}(2\tau_{0i}) \right] .
\]

This gives us the \( \varepsilon \)-expansion of the on-shell triangle diagram considered. Extracting the infrared (on-shell) singularity, it can be presented as

\[
J^{(3)}(4 - 2\varepsilon; 1, 1, 1) \bigg|_{m_3=0, p_3^2=m_1^2, p_1^2=m_2^2} = i\pi^{2-\varepsilon} \frac{(k^2)^\varepsilon}{\Delta(m_1^2, m_2^2, k^2)^{1/2+\varepsilon}} \times \left\{ \frac{\tau_{12}}{\varepsilon} - \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{(j+1)!} \sum_{i=1}^{2} \left[ \text{L}_{s_{j+2}}(\pi) - \text{L}_{s_{j+2}}(2\tau_{0i}) \right] \right\},
\]

where \( \tau_{12} = \pi - \tau_{01}' - \tau_{02}' \).

Since the functions are of the same type as in the two-point example, we can use Eq. (2.18) for the analytic continuation.
3.3 A more complicated example

Let us consider triangle integral (3.1) with \( m_1 = m_2 = m_3 = m \), \( p_1^2 = p_2^2 = 0 \), with an arbitrary (off-shell) value of \( p_3^2 \). Such diagrams occur, for example, in Higgs decay into two photons or two gluons via a massive quark loop. Following the notation of Ref. [44], we shall denote this integral (with unit powers of propagators) as \( J_3(1,1,1; m) \). According to Eq. (40) of Ref. [44], the result in an arbitrary space-time dimension \( n = 4 - 2\varepsilon \) is

\[
J_3(1,1,1; m) \bigg|_{p_1^2=p_2^2=0} = -\frac{1}{2}i\pi^{2-\varepsilon}(m^2)^{-1-\varepsilon}\Gamma(1+\varepsilon)\ _3F_2\left(1,1,1+\varepsilon; \frac{3}{2}, 2; \frac{p_3^2}{4m^2}\right). \tag{3.10}
\]

For the occurring \(_3F_2\) function, various one-fold integral representations can be constructed (see, e.g., in [45]). Below we list some of them:

\[
\ _3F_2\left(1,1,1+\varepsilon; \frac{3}{2}, 2; z\right) = -\frac{\Gamma(1-\varepsilon)}{2\varepsilon\Gamma(1+\varepsilon)}\frac{1}{\Gamma(1-2\varepsilon)}\int_0^1 \frac{dt}{t}\sqrt{1-t}\ln(1-tz) \left[ t - \frac{t}{4(1-t)} \right]^{\varepsilon}, \tag{3.11}
\]

\[
= \frac{1}{z\varepsilon} \int_0^\infty d\tau \left[ \left(1 - \frac{z}{\cosh^2\tau}\right)^{-\varepsilon} - 1 \right], \tag{3.12}
\]

\[
= \frac{1}{2z\varepsilon} \int_0^1 \frac{dx}{x} \left\{ [1 - 4zx(1-x)]^{-\varepsilon} - 1 \right\}. \tag{3.13}
\]

In fact, to get representations of the terms of the \( \varepsilon \)-expansion in terms of Clausen and log-sine functions, the following two-fold angular integral representation appears to be rather convenient:

\[
\sin^2 \theta \ _3F_2\left(1,1,1+\varepsilon; \frac{3}{2}, 2; \sin^2 \theta\right) = 2\int_0^\theta d\phi (\cos \phi)^{-2\varepsilon} \int_0^\phi d\tilde{\phi} (\cos \tilde{\phi})^{2\varepsilon} \equiv \sum_{j=0}^\infty \varepsilon^j h_j(\theta) \tag{3.14}
\]

It is easy to see that

\[
h_0(\theta) = \theta^2, \tag{3.15}
\]

\[
h_1(\theta) = 2\text{Cl}_3 (\pi - 2\theta) - 2\text{Cl}_3 (\pi) - 2\theta\text{Cl}_2 (\pi - 2\theta). \tag{3.16}
\]

Moreover, in terms of the function \( f_\varepsilon \) whose definition and the \( \varepsilon \)-expansion are given in Eq. (2.8), we can represent the integral in Eq. (3.14) as

\[
2\int_0^\theta d\phi (\cos \phi)^{-2\varepsilon} \int_0^\phi d\tilde{\phi} (\cos \tilde{\phi})^{2\varepsilon} = 2\int_0^\theta f_\varepsilon (\phi) \ d f_{-\varepsilon}(\phi) = 2 f_\varepsilon (\theta) f_{-\varepsilon}(\theta) - 2\int_0^\theta f_{-\varepsilon}(\phi) \ d f_\varepsilon (\phi) = f_\varepsilon (\theta) f_{-\varepsilon}(\theta) + \int_0^\theta \left[ f_\varepsilon (\phi) \ d f_{-\varepsilon}(\phi) - f_{-\varepsilon}(\phi) \ d f_\varepsilon (\phi) \right]. \tag{3.17}
\]

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The last representation is nothing but splitting the function (3.14) into the even and odd parts. The first term, \( f_\varepsilon(\theta)f_{-\varepsilon}(\theta) \), is symmetric with respect to \( \varepsilon \leftrightarrow -\varepsilon \) and therefore contains only even powers of \( \varepsilon \). The second (integral) term is antisymmetric and contains only odd powers.

Thus we have obtained results for all \( h_j(\theta) \) with even values of \( j \) \((j = 2l)\). One needs just to pick up the \( \varepsilon^{2l} \) term of the expansion of \( f_\varepsilon(\theta)f_{-\varepsilon}(\theta) \), with \( f_{\pm \varepsilon} \) given explicitly in Eq. (2.8). For example,

\[
\begin{align*}
h_2(\theta) & = -[Ls_2(\pi - 2\theta)]^2 + 2\theta [Ls_3(\pi - 2\theta) - Ls_3(\pi)] , \\
h_4(\theta) & = [Ls_3(\pi - 2\theta) - Ls_3(\pi)]^2 - \frac{4}{3}Ls_2(\pi - 2\theta) [Ls_4(\pi - 2\theta) - Ls_4(\pi)] \\
& + \frac{4}{5}\theta [Ls_5(\pi - 2\theta) - Ls_5(\pi)] .
\end{align*}
\]

(3.18)
(3.19)

However, the calculation of the odd terms of the expansion \((j = 2l + 1)\) is less trivial, starting from \( j = 3 \) (the result for \( h_1(\theta) \) is given in Eq. (3.15)). It appears that \( h_3(\theta) \) (and the higher odd functions) cannot be expressed in terms of the \( Ls_j \) functions and their simple generalizations.

Of course, for a given value of \( \theta \) there is no problem to calculate \( h_j(\theta) \) with very high precision, using integral representation (3.13). As an illustration, let us consider a special value of \( p_3^2, p_3^2 = m^2 \) \((z = \frac{1}{4}, \theta = \frac{\pi}{6})\). Then we have

\[
h_j\left(\frac{\pi}{6}\right) = \frac{(-1)^{j+1}}{2(j + 1)!} \int_0^1 \frac{dx}{x} \ln^{j+1}(1 - x + x^2) .
\]

(3.20)

In particular, for \( h_3\left(\frac{\pi}{6}\right) \) we obtain a number which, with the help of the PSLQ program [29] can be identified as

\[
h_3\left(\frac{\pi}{6}\right) = \frac{1}{12}\chi_5 + \frac{71}{58!}\zeta_2\zeta_3 + \frac{401}{72!}\zeta_5 - \frac{23}{24!}\pi Ls_4\left(\frac{\pi}{6}\right) + \frac{1}{8!}\pi\zeta_2 Ls_2\left(\frac{\pi}{6}\right) ,
\]

(3.21)

where

\[
\chi_5 \equiv \Sigma_{i_1;\ldots;i_2}^{\zeta_1;\ldots;\zeta_2}(1) = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n^2} [S_1(n-1)]^3 = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n^2} [\psi(n) + \gamma]^3 \\
\sim 0.0678269619272092908692628002256569482253389 \ldots
\]

(3.22)

is a special case of binomial sum [17, 33]. Therefore, the obtained \( \varepsilon \)-expansion of the \( 3F_2 \) function at \( z = \frac{1}{4} \) is

\[
3F_2\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{4} + \varepsilon \right) = \frac{2}{3}\zeta_2 + \varepsilon\left\{ \frac{401}{72!}\zeta_2\zeta_3 - \frac{401}{24!}\pi Ls_2\left(\frac{\pi}{6}\right) \right\} + \varepsilon^2\left\{ 10\zeta_4 + \frac{4}{3}\pi Ls_3\left(\frac{2\pi}{3}\right) - \frac{10}{9} Ls_2\left(\frac{\pi}{3}\right)^2 \right\} \\
+ \varepsilon^3\left\{ \frac{5}{3}\chi_5 + \frac{71}{58!}\zeta_2\zeta_3 + \frac{401}{72!}\zeta_5 - \frac{92}{24!}\pi Ls_4\left(\frac{2\pi}{3}\right) + \frac{1}{8!}\pi\zeta_2 Ls_2\left(\frac{\pi}{6}\right) \right\} \\
+ \varepsilon^4\left\{ 4 \left[ Ls_3\left(\frac{2\pi}{3}\right) \right]^2 - \frac{32}{9}\pi Ls_2\left(\frac{\pi}{3}\right) Ls_4\left(\frac{2\pi}{3}\right) + \frac{4}{9}\pi Ls_5\left(\frac{2\pi}{3}\right) \\
+ 4\pi\zeta_2 Ls_3\left(\frac{2\pi}{3}\right) + \frac{16}{3}\pi\zeta_3 Ls_2\left(\frac{\pi}{6}\right) + \frac{110}{21}\zeta_6 \right\} + \mathcal{O}(\varepsilon^5).
\]

(3.23)
In Appendix B we present several terms of the \( \varepsilon \)-expansion of the \( 3F_2 \) function with more general parameters. For another special case, \( p_3^2 = 3m^2 \) \( (z = \frac{3}{4}, \theta = \frac{\pi}{3}) \), we find

\[
h_3 \left( \frac{\pi}{3} \right) = -\frac{2}{3} \pi \zeta_2 L_4 \left( \frac{\pi}{3} \right) - \frac{4}{9} \pi L_4 \left( \frac{\pi}{3} \right) + \frac{2}{3} \zeta_2 \zeta_3 + \frac{2}{9} \zeta_5 .
\] (3.24)

Note that here we do not need that extra basis element \( \chi_5 \).

Let us discuss the appearance of the constant \( \chi_5 \) (3.22) and its relation to transcendental numbers examined in [28]. A detailed analysis of hypergeometric functions of argument \( z = \frac{1}{4} \) (see Appendix B) and Feynman diagrams (see Eqs. (3.10), (4.5) and (4.10)) allows us to state that at the level \( 5 \) there is a new (with respect to the constants defined in [28]) independent irrationality related to the binomial sum (3.22). The set of sums or their linear combinations involving this new irrationality (for a more detailed discussion, see Appendix B.1) consists of six elements: \( \Sigma_{1;1;2}^{21} \), \( \Sigma_{1;1;3}^{11} \), \( \Sigma_{1;1;3}^{21} \) and three linear combinations given in Appendix B, Eq. (B.7). For example,

\[
\chi_5 + 2\Sigma_{1;1;3}^{21}(1) = \frac{4}{9} \pi L_4 \left( \frac{\pi}{3} \right) - \frac{13}{9} \zeta_2 \zeta_3 - \frac{47}{9} \zeta_5 .
\]

Of course, instead of (3.22) one could choose any other element of this set. The linear independence of \( \chi_5 \) (3.22) and other weight-5 elements of the odd basis was established by PSLQ-analysis of all these constants calculated with 800-decimal accuracy\(^{11}\).

Introduction of a new constant (3.22) means that the ansatz for the odd basis of weight 5 elaborated in Ref. [28] (see Table 1 of [28]) should be re-analyzed. We note that one of the elements, \( \frac{1}{\sqrt{3}} \text{Cl}_5 \left( \frac{\pi}{3} \right) \), is proportional to \( \frac{1}{\sqrt{3}} \zeta_5 \). Although this element is linearly independent, its appearance contradicts one of the mnemonic principles of constructing the ansatz: any non-factorizable element may appear only once, with or without the normalization factor \( \frac{1}{\sqrt{3}} \). Since we already have \( \zeta_5 \) (without \( \frac{1}{\sqrt{3}} \)), this element can be omitted. Including, instead of it, the new element \( \chi_5 \), we are able to save the rule \( N_j = 2^j \), whereas the simplicity of the ansatz gets spoiled, since, as far as we know, the new element is not expressible in terms of the Clausen or log-sine functions of \( \frac{\pi}{3} \) or \( \frac{2\pi}{3} \).

Let us also consider cases when \( h_j(\theta) \) can be expressed in terms of the even basis. The threshold case, \( p_3^2 = 4m^2 \) \( (z = 1, \theta = \frac{\pi}{2}) \), is trivial:

\[
3F_2 \left( \frac{1,1,1+\varepsilon}{\frac{3}{2},2} \right) \left( 1 \right) = -\frac{1}{\varepsilon} \int_0^1 d\lambda \frac{1-\lambda^{-2\varepsilon}}{1-\lambda^2} = \frac{1}{2\varepsilon} \left[ \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1}{2} - \varepsilon \right) \right] = \frac{1}{2} \sum_{j=0}^{\infty} \varepsilon^j (2^{j+2} - 1) \zeta_{j+2} .
\] (3.25)

Therefore, we get

\[
h_j \left( \frac{\pi}{2} \right) = \frac{1}{2} (2^{j+2} - 1) \zeta_{j+2} .
\] (3.26)

In particular, \( h_3 \left( \frac{\pi}{3} \right) = \frac{31}{2} \zeta_5 \).

\(^{11}\)The algorithm for high precision calculation of \( L_{s,j}(\theta) \) and \( L_{s,j}^{(1)}(\theta) \) functions is discussed in Appendix A of Ref. [33]. The corresponding programs can be found on Oleg Veretin’s home page at http://www.ifh.de/veretin/MPFUN/mpfun.html.
Another case of interest is $\theta = \frac{\pi}{4}$. Here, the situation is rather similar to the case $\theta = \frac{\pi}{6}$. Again, the calculation of $h_3 \left( \frac{\pi}{4} \right)$ happens to be non-trivial. With the help of the PSLQ program, we obtain

$$h_3 \left( \frac{\pi}{4} \right) = \frac{1}{12} \tilde{\chi}_5 - \frac{3}{4} \pi C_l_4 \left( \frac{\pi}{2} \right) + \frac{1}{16} \pi \zeta_2 C_l_2 \left( \frac{\pi}{2} \right) + \frac{1333}{812} \zeta_5 - \frac{91}{256} \zeta_3 \zeta_3,$$

(3.27)

where

$$\tilde{\chi}_5 \equiv \Sigma_{1; -2}(2) = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)! n^2} [S_1(n-1)]^3 = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)! n^2} [\psi(n) + \gamma]^3 \approx 0.4951660770553611208168992757694958708378 \ldots$$

(3.28)

One can see that this multiple binomial sum is nothing but the $k = 2$ version of the odd element $\chi_5$ given in Eq. (3.22). Therefore, this element $\tilde{\chi}_5$ should be added to the set of weight-5 elements of the even basis. A further discussion of the even-basis elements is given in Appendix B.1.

4 Two- and three-loop examples

4.1 General remarks

Below we present the $\varepsilon$-expansion of several master integrals arising in FORM [46] packages for calculation of two-loop on-shell self-energy diagrams [47] and three-loop vacuum integrals [31, 48][12]. Our method of calculation is a combination of the Mellin–Barnes technique [44] and the PSLQ analysis [29]: we present results for the master integrals in terms of hypergeometric functions, whose expansions, given in Appendix B, are obtained by means of PSLQ. In some cases we give the result for arbitrary mass and/or momentum[13]. Sometimes, the integrals with higher powers of some propagators happen to be simpler from the $\varepsilon$-expansion point of view (for instance, the integrals having an ultraviolet-divergent subgraph, like $V_{1001}$ or $J_{011}$ in Fig. 2). In such cases we derive an expansion of these “simple” integrals and then reduce them to the master integrals by means of packages [47] or [31].

The integrals under consideration and notations (indices and mass distributions) are shown in Fig. 2. We are working in Minkowski space-time with dimension $n = 4 - 2\varepsilon$. Moreover, each loop is multiplied by the normalization factor $\left[ i \pi^{n/2} \Gamma(1 + \varepsilon) \right]^{-1}$. For example, the two-loop vacuum integrals with different masses, which were denoted in Refs. [10, 26, 24] as $I(n; \nu_1, \nu_2, \nu_3; m_1, m_2, m_3)$, are in the case of equal masses normalized as

$$I(n; \nu_1, \nu_2, \nu_3; m, m, m)|_{m=1} = -\pi^n \Gamma^2 (1 + \varepsilon) \textbf{VL111}(\nu_1, \nu_2, \nu_3)$$

(4.1)

where we adopt the notation $\textbf{VL111}$ from Ref. [28][14] (and also put $m = 1$).
We can mention an example of physical calculations [53], where the finite parts of $E_3$, $D_3$ (which was found in [28, 30]) and $D_4$ (found in Ref. [13]) have been used.

4.2 F10101

This integral is a good illustration of the application of general expressions given in Appendix B. The off-shell result for this integral in arbitrary dimension was presented in [54] (where it was called $I_3$, see Eq. (22) of [54]). For unit powers of propagators, the result reads

$$m^{2+4\varepsilon}(1-2\varepsilon)F_{10101}(p^2, m) = \frac{1}{(1-\varepsilon^2)(1+2\varepsilon)}4F_3\left(\frac{1}{3}, \frac{1}{2} + \varepsilon, 2 + \varepsilon, 2 - \varepsilon \left| \frac{p^2}{4m^2} \right. \right)$$

$$-\frac{1}{2\varepsilon(1+\varepsilon)}3F_2\left(\frac{1}{3}, \frac{1}{2} + \varepsilon \left| \frac{p^2}{4m^2} \right. \right)$$

$$+\frac{1}{2\varepsilon \Gamma(1-2\varepsilon)}\left(\frac{m^2}{p^2}\right)\varepsilon 3F_2\left(1, 1, 1 + \varepsilon \left| \frac{p^2}{4m^2} \right. \right). \quad (4.2)$$

The finite part of $F_{10101}$ is given in [55, 56], whereas the term linear in $\varepsilon$ has been considered in [33]. An algorithm for the small momentum expansion of such diagrams (involving a “zero-threshold”) has been constructed in Ref. [57] where also explicit results for a few terms have been given, for general values of the masses of massive particles, in the limit $\varepsilon \to 0$. All these calculations agree with the result (4.2).

For $p^2 > 0$ this diagram has imaginary part, due to the factor $(-m^2/p^2)^\varepsilon$, which is related to the massless two-particle cut. If $p^2 < 4m^2$, this is the only contribution to the imaginary
part, whereas for $p^2 > 4m^2$ we get additional contributions from the cuts involving two massive particles. Therefore, for $0 < p^2 < 4m^2$ we get

$$m^2 + 4\varepsilon (1 - 2\varepsilon) \text{Im } F_{10101}(p^2, m) = \frac{\pi}{2} \frac{\Gamma(1 - \varepsilon)}{\Gamma(1 + \varepsilon)\Gamma(1 - 2\varepsilon)} \left( \frac{m^2}{p^2} \right)^\varepsilon \left( \begin{array}{c} 1, 1, 1 + \varepsilon \\ \frac{3}{2}, 2 \end{array} \right) \frac{p^2}{4m^2} ,$$

(4.3)

with the same $3F_2$ function as in the three-point example in section 3.3. This is expected, since, cutting the considered two-loop diagram across the two massless lines, we get nothing but that triangle diagram. Therefore, all results for the $\varepsilon$-expansion ($h_j(\theta)$ functions, etc.) presented in section 3.4 are directly applicable to the imaginary part of $F_{10101}$.

In particular, in the on-shell limit $p^2 = m^2$ (we also put $m = 1$) we follow the notation of Ref. [27] and denote

$$F_{10101} \equiv F_{10101}(p^2, m) \bigg|_{p^2 = m^2; \ m = 1} ,$$

(4.4)

where it is understood that all powers of all propagators are equal to one. This integral is used as one of the master integrals in the ONShEL2 package [47].

Using expressions given in Appendix B, we can now present the $\varepsilon$-expansion of the real part up to the $\varepsilon^2$ term,

$$(1 - 2\varepsilon) \text{Re } F_{10101} = \left[ -4\zeta_3 + 2\pi Ls_2\left( \frac{\pi}{3} \right) \right] + \varepsilon \left\{ \frac{16}{9} \left[ Ls_2\left( \frac{\pi}{3} \right) \right]^2 - 7\pi Ls_3\left( \frac{2\pi}{3} \right) - \frac{488}{9} \zeta_4 \right\}$$

$$+ \varepsilon^2 \left\{ 6\pi Ls_4\left( \frac{2\pi}{3} \right) - \frac{5}{2} \chi_5 - \frac{2111}{54} \zeta_2 \zeta_3 - \frac{501}{18} \zeta_5 + \frac{56}{27} \pi^2 Ls_2\left( \frac{\pi}{3} \right) - \frac{50}{81} \pi Ls_4\left( \frac{\pi}{3} \right) \right\} + O(\varepsilon^3) .$$

(4.5)

The finite and linear in $\varepsilon$ parts coincide with those given in [27] and [33], respectively. According to Eq. (4.3), the imaginary part can be extracted from Eq. (3.23),

$$(1 - 2\varepsilon) \text{Im } F_{10101} = \frac{\pi}{2} \frac{\Gamma(1 - \varepsilon)}{\Gamma(1 + \varepsilon)\Gamma(1 - 2\varepsilon)} \text{r.h.s. of Eq. (3.23)} ,$$

(4.6)

which yields its $\varepsilon$-expansion up to $\varepsilon^4$.

4.3 $D_4$

The integral $D_4(1, 1, 1, 1, 1)$ is another master integral used in [31, 48]. The finite part of this integral was given in [13]. Here we are going to obtain result for the $\varepsilon$-term. It is easy to see that the integral $D_4(1, 1, 1, 1, \nu)$ can be obtained by integrating the off-shell two-loop diagram $F_{10101}$ (see [54] and Eq.(4.2)) with a massive propagator raised to a power $\nu$. Using

\[\text{We take into account the Erratum to [27] where Eq. (1) and Table 1 of [27] were corrected. In fact, there is another misprint in Table I of Ref. [27]: for the master integral $F_{11111}$ the signs of the coefficients $a_1$ and $a_2$ should be changed: $a_1 = -1$, $a_2 = \frac{9}{2}$. The corrected result is presented in p. 541 of [47] (see also in [58]).}\]
the following property (in our case $\mu^2 = 4m^2$)
\[
\frac{1}{\pi^{n/2}} \int \frac{d^n p}{(p^2 - M^2)^\nu} \quad pF_Q \left( \begin{array}{c} a_1, \ldots, a_P \\ b_1, \ldots, b_Q \end{array} \right| \frac{p^2}{\mu^2} \right)^{\nu} 
\]
\[
\frac{\Gamma\left(\nu - \frac{n}{2}\right)}{\Gamma\left(\nu\right)} \quad p_{+1}F_{+1} \left( \begin{array}{c} a_1, \ldots, a_P, \frac{n}{2} \\ b_1, \ldots, b_Q, \frac{n}{2} - \nu + 1 \end{array} \right| \frac{M^2}{\mu^2} \right) 
\]
\[
+ (\mu^2)^{n/2-\nu} \quad \prod_{k=1}^{Q} \prod_{i=1}^{P} \frac{\Gamma(b_k) \Gamma\left(\nu + a_i - \frac{n}{2}\right)}{\Gamma\left(\nu + b_k - \frac{n}{2}\right)} 
\]
\[
\times p_{+1}F_{+1} \left( \begin{array}{c} \nu + a_1 - \frac{n}{2}, \ldots, \nu + a_P - \frac{n}{2}; \nu \\ \nu - \frac{n}{2} + b_1, \ldots, \nu - \frac{n}{2} + b_Q, \nu - \frac{n}{2} + 1 \end{array} \right| \frac{M^2}{\mu^2} \right) ,
\]
where $\{a_i, b_j\} \neq 0, -1, -2, \ldots$, we arrive at a result which contains (for an arbitrary $\nu$) three $4F_3$ functions and three $3F_2$ functions. For $\nu = 1$, two of the three $4F_3$ functions reduce to the $3F_2$ functions of the type considered in Appendix B. For some $3F_2$ functions we use the relation
\[
pF_Q \left( \begin{array}{c} a_1, \ldots, a_P \\ b_1, \ldots, b_Q \end{array} \right| z \right) = 1 + z \frac{a_1 \ldots a_P}{b_1 \ldots b_Q} p_{+1}F_{+1} \left( \begin{array}{c} 1, 1 + a_1, \ldots, 1 + a_P \\ 2, 1 + b_1, \ldots, 1 + b_Q \end{array} \right| z \right). \quad (4.7)
\]
Then the master integral becomes
\[
(1 - \varepsilon)(1 - 2\varepsilon)D_4(1, 1, 1, 1, 1, 1) =
\]
\[
\frac{1}{\varepsilon(1 - \varepsilon)(1 + \varepsilon)(1 + 2\varepsilon)} \quad 4F_3 \left( \begin{array}{c} 1, 1 + \varepsilon, 1 + \varepsilon, 1 + 2\varepsilon \\ \frac{3}{2} + \varepsilon, 2 + \varepsilon, 2 - \varepsilon \end{array} \right| \frac{1}{4} \right)
\]
\[
+ \quad \frac{\Gamma(1 - \varepsilon)\Gamma^2(1 + 2\varepsilon)\Gamma(1 + 3\varepsilon)}{\varepsilon(1 + 2\varepsilon)(1 + 4\varepsilon)\Gamma^2(1 + \varepsilon)\Gamma(1 + 4\varepsilon)} \quad 4F_3 \left( \begin{array}{c} 1, 1 + 2\varepsilon, 1 + 2\varepsilon, 1 + 3\varepsilon \\ \frac{3}{2} + 2\varepsilon, 2 + 2\varepsilon, 2 \end{array} \right| \frac{1}{4} \right)
\]
\[
+ \quad \frac{1}{2\varepsilon^2(1 + \varepsilon)} \quad 3F_2 \left( \begin{array}{c} 1, 1 + \varepsilon, 1 + \varepsilon \\ \frac{3}{2}, 2 + \varepsilon \end{array} \right| \frac{1}{4} \right) - \quad \frac{\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)}{4\varepsilon^2\Gamma(1 + \varepsilon)} \quad 3F_2 \left( \begin{array}{c} 1, 1, 1 + \varepsilon \\ \frac{3}{2}, 2 \end{array} \right| \frac{1}{4} \right)
\]
\[
- \quad \frac{1}{2\varepsilon^2(1 + 2\varepsilon)^2} \quad 3F_2 \left( \begin{array}{c} 1, 1 + 2\varepsilon, 1 + 2\varepsilon \\ \frac{3}{2} + \varepsilon, 2 + 2\varepsilon \end{array} \right| \frac{1}{4} \right) + \quad \frac{\Gamma(1 - \varepsilon)\Gamma^2(1 + 2\varepsilon)\Gamma(1 + 3\varepsilon)}{4\varepsilon^4\Gamma(1 + \varepsilon)\Gamma(1 + 4\varepsilon)} \quad 3F_2 \left( \begin{array}{c} 1, 1 + 2\varepsilon, 1 + 3\varepsilon \\ \frac{3}{2} + 2\varepsilon, 2 + 2\varepsilon \end{array} \right| \frac{1}{4} \right) - \quad \frac{1}{4\varepsilon^4} . \quad (4.9)
\]
Now all $pF_Q$ functions belong to the types considered in Appendix B. Using those results, we obtain the following result for the lowest terms of the $\varepsilon$-expansion:
\[
(1 - \varepsilon)(1 - 2\varepsilon)D_4(1, 1, 1, 1, 1, 1) =
\]
\[
\frac{2\zeta_3}{\varepsilon} - \left\{ \frac{77}{12}\zeta_4 + 6 \left[ \text{Li}_2\left(\frac{5}{3}\right)\right]^2 \right\}
\]
\[
+ \varepsilon \left\{ \frac{21}{8}\chi_5 - \frac{2615}{2} \zeta_2 \zeta_3 + \frac{2047}{16} \zeta_5 + \frac{387}{512} \pi^2 \text{Li}_2\left(\frac{5}{3}\right) - \frac{161}{44} \pi \text{Li}_4\left(\frac{2\pi}{3}\right) + 7\pi \text{Li}_4\left(\frac{2\pi}{3}\right) \right\} + \mathcal{O}(\varepsilon^2). \quad (4.10)
\]
The finite term is in full agreement with the result of [13], whereas the result for the $\varepsilon$-term is new. As we see, the new constant $\chi_5$, Eq. (3.22), also appears in the $\varepsilon$-part of this integral.

20
4.4 V1001

The off-shell integral $V_{1001}$ with two different masses and unit powers of propagators was considered in [59]. The result, Eq. (46) of [59] is presented as a sum of $F_2$ and $F_4$ hypergeometric functions of two variables \(^{16}\).

Again, we define

$$V_{1001}(\alpha, \sigma_1, \sigma_2, \beta) \equiv V_{1001}(\alpha, \sigma_1, \sigma_2, \beta; p^2, m, m) \bigg|_{p^2=m^2, \ m=1} . \quad (4.11)$$

The finite part of on-shell master integral has been calculated in [61], while the linear in $\varepsilon$ term is presented in [27]. The on-shell diagram $V_{1001}(1, 1, 1)$ belongs to the set of master integrals used in the \textsc{Onshell2} package [47]. Employing the Mellin–Barnes technique [44] one can find the following result for an arbitrary set of the indices (remember that we put $m = 1$):

$$V_{1001}(\alpha, \sigma_1, \sigma_2, \beta) = \frac{\Gamma \left( \frac{n}{2} - \sigma_1 \right) \Gamma \left( \frac{n}{2} - \sigma_2 \right) \Gamma \left( \sigma_1 + \sigma_2 - \frac{n}{2} \right)}{\Gamma(\sigma_1)\Gamma(\sigma_2)\Gamma(n - \sigma_1 - \sigma_2)\Gamma^2 \left( 3 - \frac{n}{2} \right)}$$

$$\times \left\{ \frac{\Gamma(\alpha + \beta + \sigma_1 + \sigma_2 - n)\Gamma(2n - 2\beta - \alpha - 2\sigma_1 - 2\sigma_2)}{\Gamma(2\sigma_1 - \alpha - \beta - \sigma_1 - \sigma_2)} \right\}$$

$$\times 3F_2 \left( \left. \begin{array}{l} \beta, \alpha + \beta + \sigma_1 + \sigma_2 - n, 1 + \alpha + \beta + \sigma_1 + \sigma_2 - \frac{3}{2}n, 1 + \frac{\alpha}{2} + \beta + \sigma_1 + \sigma_2 - n, \frac{1}{2}(\alpha + 1) + \beta + \sigma_1 + \sigma_2 - n \end{array} \right| 1 \right)$$

$$\times \frac{\Gamma \left( \alpha + 1 \right) \Gamma \left( \frac{\alpha - 1}{2} + \beta + \sigma_1 + \sigma_2 - n \right) \Gamma(n - \sigma_1 - \sigma_2 - \frac{\alpha - 1}{2})}{2\Gamma(\beta)\Gamma \left( \frac{n - \alpha - 1}{2} \right)}$$

$$\times 3F_2 \left( \left. \begin{array}{l} \frac{1}{2}(\alpha + 1), \frac{1}{2}(3 + \alpha - n), n - \sigma_1 - \sigma_2 + \frac{1}{2}(1 - \alpha) \end{array} \right| 1 \right) \right\}. \quad (4.12)$$

Let us consider the case $\sigma_1 = \sigma_2 = \beta = 1$. For $\alpha = 1$ (the master integral), and also for $\alpha = 2$, the result (4.12) can be expressed in terms of the $\_2F_1$ functions. For $\alpha = 1$, they can be transformed using the relation (2.6) and another combination of Kummer relations for contiguous functions,

$$\_2F_1 \left( \left. \begin{array}{l} 1, -1 + 3\varepsilon \end{array} \right| \frac{1}{2} + 2\varepsilon \right) = 1 - \frac{2(1 - 3\varepsilon)z}{(1 - 2\varepsilon)} + \frac{12\varepsilon(1 - 3\varepsilon)z(1 - z)}{(1 - 2\varepsilon)(1 + 4\varepsilon)} \_2F_1 \left( \left. \begin{array}{l} 1, 1 + 3\varepsilon \end{array} \right| \frac{3}{2} + 2\varepsilon \right) \frac{1}{z} . \quad (4.13)$$

\(^{16}\)We note that the occurring $F_4$ function can be reduced to $F_1$ function of the type (2.7), whereas the $F_2$ function can be transformed into $F_1$ function of two variables (see Eq. (A.64) in [60]).
The integral with \( \alpha = 2 \) can be reduced by the **ONShEll2** package [47] to this master integral, plus some combination of \( \Gamma \)-functions:

\[
V_{1001}(2, 1, 1, 1) = -\frac{1}{3}(1 - 2\varepsilon)V_{1001}(1, 1, 1, 1) + \frac{\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)}{3\varepsilon^2(1 - 2\varepsilon)\Gamma(1 + \varepsilon)} \\
- \frac{(1 - 4\varepsilon)\Gamma^3(1 - \varepsilon)\Gamma(1 - 4\varepsilon)\Gamma(1 + 2\varepsilon)}{6\varepsilon^2(1 - 3\varepsilon)(1 - 2\varepsilon)\Gamma(1 - 2\varepsilon)\Gamma(1 - 3\varepsilon)\Gamma(1 + \varepsilon)}. \tag{4.14}
\]

This connection provides a non-trivial check on the results for the \( \varepsilon \)-expansion of the occurring integrals. The result for the master integral can be presented as

\[
(1 - 2\varepsilon)^2V_{1001}(1, 1, 1, 1) = -\frac{3\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)}{8\varepsilon\Gamma(1 + \varepsilon)} {}_2F_1\left(\frac{1}{2}, 1 + \varepsilon; \frac{1}{4}; \frac{1}{4}\right) \\
+ \frac{9\Gamma^2(1 - \varepsilon)\Gamma(1 - 4\varepsilon)\Gamma(1 + 2\varepsilon)}{8\varepsilon(1 + 4\varepsilon)\Gamma(1 - 2\varepsilon)\Gamma(1 - 3\varepsilon)\Gamma(1 + \varepsilon)} {}_2F_1\left(\frac{3}{2}, 1 + 3\varepsilon; \frac{1}{4}; \frac{1}{4}\right) \\
+ \frac{(1 - \varepsilon)^2\Gamma^2(1 - \varepsilon)\Gamma(1 - 4\varepsilon)\Gamma(1 + 2\varepsilon)}{4\varepsilon^2(1 - 3\varepsilon)\Gamma(1 - 2\varepsilon)\Gamma(1 - 3\varepsilon)\Gamma(1 + \varepsilon)} + \frac{\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)}{4\varepsilon^2\Gamma(1 + \varepsilon)} \\
+ \frac{\pi 3^{3/4}\Gamma^3(1 - \varepsilon)\Gamma(1 - 4\varepsilon)\Gamma(1 + 4\varepsilon)}{2\varepsilon\Gamma^3(1 - 2\varepsilon)\Gamma(1 + 2\varepsilon)\Gamma(1 + \varepsilon)}. \tag{4.15}
\]

Here we have two \( {}_2F_1 \) hypergeometric functions of the argument \( \frac{1}{4} \). The \( \varepsilon \)-expansion of the first function can be extracted from [24] (see also Eqs. (2.7)–(2.8) of this paper),

\[
{}_2F_1\left(\frac{1}{2}, 1 + \varepsilon; \frac{1}{4}; \frac{1}{4}\right) = \frac{2}{3^{3/4+\varepsilon}} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} \left[ L_{s_{j+1}}\left(\frac{2\pi}{3}\right) - L_{s_{j+1}}(\pi) \right], \tag{4.16}
\]

whereas the second function can be represented as

\[
{}_2F_1\left(\frac{3}{2}, 1 + 3\varepsilon; \frac{1}{4}; \frac{1}{4}\right) = \frac{2^{2+6\varepsilon}(1 + 4\varepsilon)}{3^{3/4+\varepsilon}} \int_0^{\pi/6} d\phi (\sin \phi)^{4\varepsilon} (\cos \phi)^{2\varepsilon}. \tag{4.17}
\]

When we expand in \( \varepsilon \) we get integrals of the products of various powers of \( \ln(\sin \phi) \) and \( \ln(\cos \phi) \). An integral representation of this function for an arbitrary argument is given in Eq. (B.13). Its \( \varepsilon \)-expansion can be written in terms of \( \text{Lsc} \) function (for details, see Appendix A.2). The expansion up to \( \varepsilon^4 \) looks like

\[
\frac{3^{3/4+\varepsilon}}{4(1 + 4\varepsilon)} {}_2F_1\left(\frac{1}{2}, 1 + 3\varepsilon; \frac{1}{4}; \frac{1}{4}\right) = \frac{1}{6} \pi - \frac{(2\varepsilon)^2}{3} L_{s_{2}}\left(\frac{\pi}{3}\right) + (2\varepsilon)^2 \left[ \frac{5}{72} \pi \zeta_2 - \frac{1}{2} L_{s_{3}}\left(\frac{2\pi}{3}\right) \right] \\
+ (2\varepsilon)^3 \left[ \frac{10\pi}{108} \zeta_3 - \frac{1}{2} L_{s_{4}}\left(\frac{\pi}{3}\right) - \frac{1}{6} L_{s_{4}}\left(\frac{2\pi}{3}\right) \right] \\
+ (2\varepsilon)^4 \left[ -\frac{1}{1152} \pi \zeta_4 + \frac{4}{3} \zeta_2 L_{s_{3}}\left(\frac{2\pi}{3}\right) - \frac{22}{72} L_{s_{5}}\left(\frac{\pi}{3}\right) - \frac{1}{24} L_{s_{5}}\left(\frac{2\pi}{3}\right) - \frac{7}{8} \pi \zeta_4 L_{s_{4}}\left(\frac{2\pi}{3}\right) + \frac{15}{32} L_{s_{5}}^{(1)}\left(\frac{2\pi}{3}\right) \right] + \mathcal{O}(\varepsilon^5). \tag{4.18}
\]
These expressions allow us to fix an error in the $\varepsilon$ part, presented in Eq. (2) of [27]. The correct expression is:

$$V_{1001}(1,1,1,1) = \frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{5}{2} - \frac{\pi}{\sqrt{3}}\right) + \frac{19}{2} + \frac{3\zeta_2}{2} - 4\frac{\pi}{\sqrt{3}} - 7\frac{Ls_2(\frac{\pi}{3})}{\sqrt{3}} + \frac{\pi}{\sqrt{3}} \ln 3
+ \varepsilon\left\{\frac{65}{2} + 8\zeta_2 + \frac{3}{2}\zeta_3 - 12\frac{\pi}{\sqrt{3}} - 28\frac{Ls_2(\frac{\pi}{3})}{\sqrt{3}} + 7\frac{Ls_2(\frac{\pi}{3})}{\sqrt{3}} \ln 3
+ 4\frac{\pi}{\sqrt{3}} \ln 3 - \frac{1}{2} \frac{\pi}{\sqrt{3}} \ln^2 3 - \frac{21}{2} \frac{\pi}{\sqrt{3}} \zeta(2) - \frac{21}{2} Ls_3(\frac{2\varepsilon}{3})\right\} + O(\varepsilon^2). \tag{4.19}\) 

Our expressions (4.15)–(4.18) provide two new terms, $\varepsilon^2$ and $\varepsilon^3$, of the $\varepsilon$-expansion of $V_{1001}(1,1,1,1)$.

4.5 $E_3$

The three-loop vacuum integral $E_3$ with all indices equal to one is one of the master integrals used in Avdeev’s package [31] and in MATAD [48]. The result for general values of the indices is

$$E_3(\sigma_1, \sigma_2, \rho_1, \rho_2, \nu) = \frac{\Gamma\left(\frac{n}{2} - \sigma_1\right)\Gamma\left(\frac{n}{2} - \sigma_2\right)\Gamma(\sigma_1 + \sigma_2 - \frac{n}{2})}{\Gamma(\sigma_1)\Gamma(\sigma_2)\Gamma(\frac{n}{2})\Gamma^3\left(3 - \frac{n}{2}\right)} \times \left\{ \frac{\Gamma(\rho_1 + \rho_2 - \frac{n}{2})\Gamma(\nu + \sigma_1 + \sigma_2 - n)}{\Gamma(\nu)\Gamma(\rho_1 + \rho_2)} 4f_{\rho_3}\left(\frac{\rho_1, \rho_2, \rho_1 + \rho_2 - \frac{n}{2}, n - \sigma_1 - \sigma_2}{\frac{1}{2}(\rho_1 + \rho_2)\frac{1}{2}(\rho_1 + \rho_2 + 1), n - \nu - \sigma_1 - \sigma_2 + 1}\right) \right\} + \frac{\Gamma(\nu + \rho_1 + \sigma_1 + \sigma_2 - n)\Gamma(\nu + \rho_2 + \sigma_1 + \sigma_2 - n)}{\Gamma(\nu)\Gamma(\rho_1 + \rho_2)} 4f_{\rho_3}\left(\frac{\nu + \rho_1 + \sigma_1 + \sigma_2 - n, \nu + \rho_2 + \sigma_1 + \sigma_2 - n}{\frac{1}{2}(\rho_1 + \rho_2) - n, \nu + \sigma_1 + \sigma_2 + \frac{1}{2}(\rho_1 + \rho_2 + 1) - n, \nu + \sigma_1 + \sigma_2 - n + 1}\right). \tag{4.20}\) 

For the purpose of expanding in $\varepsilon$, the integral $E_3(1,1,1,2,1)$ is the simplest one,

$$E_3(1,1,1,2,1) = -\frac{1}{4\varepsilon^2(1 - \varepsilon)(1 - 2\varepsilon)} \frac{\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)}{\Gamma(1 + \varepsilon)} 2f_1\left(\frac{1}{2} + \varepsilon, \frac{1}{4}\right) - \frac{2\Gamma(1 + 2\varepsilon)\Gamma(1 + 3\varepsilon)}{3\varepsilon(1 + \varepsilon)\Gamma(1 + 4\varepsilon)} 2f_1\left(\frac{1}{2} + 2\varepsilon, \frac{1}{4}\right) \right\}. \tag{4.21}\) 

\footnote{In Eq. (2) and Table 2 of [27], results for two integrals, $V_{1111}(1,1,1,1)$ (with all massive lines) and $V_{1001}(1,1,1,1)$, are combined. The result for $V_{1111}(1,1,1,1)$ is correct. In order to correct the result for the $\varepsilon$-term of $V_{1001}(1,1,1,1)$, the following changes should be done: (i) the term $+9s_3\frac{\pi}{\sqrt{3}}$ (with $s_3 = \frac{4}{9\sqrt{3}}Ls_2(\frac{\pi}{3})$) should read $+\frac{b_3}{b_1}Ls_2(\frac{\pi}{3})$, where $b_1 = 0$ for $V_{1001}(1,1,1,1)$, according to Table 2; (ii) the coefficient $b_3$ of the $\zeta_3$ term (given in Table 1) should be $-\frac{3}{2}$, rather than $-\frac{13}{6}$. The result for $V_{1001}(1,1,1,1)$ on p. 546 of [47] should also be corrected, according to Eq. (4.19)}
The \( \varepsilon \)-expansion of the first \( 2F_1 \) function is given in Eq. (4.16), whereas the second one can be reduced to (4.18) via Kummer relation

\[
2F_1 \left( \frac{1}{3} + \frac{2}{\varepsilon}, \frac{1}{2} + 2\varepsilon \right) = 1 + \frac{6\varepsilon z}{1 + 3\varepsilon} \left( 2F_1 \left( \frac{1}{3} + \frac{3\varepsilon}{2} + 2\varepsilon \right), \right),
\]

which is a particular case of (4.8).

Then, using recurrence relations [31], \( \mathbf{E}_3(1,1,1,2,1) \) can be related to to the master integral \( \mathbf{E}_3(1,1,1,1,1) \) as

\[
(1 - 2\varepsilon) \mathbf{E}_3(1,1,1,1,1) = -3 \mathbf{E}_3(1,1,1,2,1) - \frac{\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)}{2\varepsilon^3(1 - \varepsilon)(1 - 2\varepsilon)\Gamma(1 + \varepsilon)}
\]

\[
+ \frac{(1 - 4\varepsilon)\Gamma(1 - \varepsilon)\Gamma^2(1 + 2\varepsilon)\Gamma(1 + 3\varepsilon)}{3\varepsilon^3(1 - \varepsilon)(1 - 2\varepsilon)(1 - 3\varepsilon)\Gamma(1 + 4\varepsilon)\Gamma^2(1 + \varepsilon)}. \tag{4.23}
\]

In this way we get

\[
(1 - 2\varepsilon)^2 \mathbf{E}_3(1,1,1,1,1) = \frac{3\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)}{4\varepsilon^2(1 - \varepsilon)\Gamma(1 + \varepsilon)}
\]

\[
2F_1 \left( \frac{1}{3} + \frac{3\varepsilon}{2} + 2\varepsilon \right) \frac{1}{4}
\]

\[
- \frac{3\Gamma(1 - \varepsilon)\Gamma^2(1 + 2\varepsilon)\Gamma(1 + 3\varepsilon)}{4\varepsilon^2(1 - \varepsilon)(1 + 4\varepsilon)\Gamma(1 + 4\varepsilon)\Gamma^2(1 + \varepsilon)}
\]

\[
2F_1 \left( \frac{1}{3} + \frac{3\varepsilon}{2} + 2\varepsilon \right) \frac{1}{4}
\]

\[
\frac{\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)}{2\varepsilon^3(1 - \varepsilon)\Gamma(1 + \varepsilon)} - \frac{\Gamma(1 - \varepsilon)\Gamma^2(1 + 2\varepsilon)\Gamma(1 + 3\varepsilon)}{6\varepsilon^3(1 - 3\varepsilon)\Gamma(1 + 4\varepsilon)\Gamma^2(1 + \varepsilon)}. \tag{4.24}
\]

with the same \( 2F_1 \) functions as in the case of \( \mathbf{V}_{1001} \). The \( \varepsilon \)-expansion of these \( 2F_1 \) functions is given in (4.16) and (4.18). Thus, for this integral we have also produced two new (\( \varepsilon^2 \) and \( \varepsilon^3 \)) terms of the \( \varepsilon \)-expansion, reaching the level of 6-loop calculations (see in [53]). We also find an error in the \( \varepsilon \)-part of the result given in Eq. (10) of [28], which should read (remember that we put \( m = 1 \) and \( L_{S_2} \left( \frac{2\pi}{3} \right) = \frac{9\sqrt{3}}{4} S_2 \))

\[
\mathbf{E}_3(1,1,1,1,1) = -\frac{2}{3\varepsilon^3} - \frac{11}{3\varepsilon^2} + \frac{1}{\varepsilon} \left\{ -14 + 6 \frac{L_{S_2} \left( \frac{2\pi}{3} \right)}{\sqrt{3}} - \zeta_2 \right\}
\]

\[
+ \left\{ -\frac{139}{3} + 30 \frac{L_{S_2} \left( \frac{2\pi}{3} \right)}{\sqrt{3}} - 6 \frac{L_{S_2} \left( \frac{2\pi}{3} \right)}{3\sqrt{3}} \ln 3 - 5\zeta_2 - \frac{1}{3}\zeta_3 + \frac{2}{3} \frac{\pi}{\sqrt{3}} \zeta_2 + 9 \frac{L_{S_3} \left( \frac{2\pi}{3} \right)}{\sqrt{3}} \right\}
\]

\[
+ \varepsilon \left\{ -\frac{430}{3} + 102 \frac{L_{S_2} \left( \frac{2\pi}{3} \right)}{\sqrt{3}} - 30 \frac{L_{S_2} \left( \frac{2\pi}{3} \right)}{3\sqrt{3}} \ln 3 + 3 \frac{L_{S_2} \left( \frac{2\pi}{3} \right)}{\sqrt{3}} \ln^2 3
\]

\[
+ 45 \frac{L_{S_4} \left( \frac{2\pi}{3} \right)}{\sqrt{3}} - 9 \frac{L_{S_3} \left( \frac{2\pi}{3} \right)}{\sqrt{3}} \ln 3 + \frac{80}{3} \frac{L_{S_4} \left( \frac{2\pi}{3} \right)}{\sqrt{3}} + 6 \frac{L_{S_4} \left( \frac{2\pi}{3} \right)}{3\sqrt{3}} - 17\zeta_2
\]

\[
+ \frac{25}{3} \frac{\pi}{\sqrt{3}} \zeta_2 - \frac{5}{3} \frac{\pi}{\sqrt{3}} \zeta_2 \ln 3 - \frac{13}{3} \zeta_3 - \frac{94}{9} \frac{\pi}{\sqrt{3}} \zeta_3 + \frac{2}{3} \zeta_4 + 4 \frac{L_{S_2} \left( \frac{2\pi}{3} \right)}{\sqrt{3}} \right\} + O(\varepsilon^2). \tag{4.25}
\]
4.6 $D_3$ and $J011$

The off-shell result for the sunset-type integral $J_{011}$ with arbitrary powers of propagators has been obtained in [62, 54], by using the Mellin–Barnes technique [44]:

$$J_{011}(\sigma, \nu_1, \nu_2; p^2, m) = (m^2)^{n-\sigma-\nu_1-\nu_2-\nu} \frac{\Gamma(\nu_1+\nu_2+\sigma-n)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu_1+\sigma-\frac{n}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma\left(\frac{n}{2}\right)\Gamma(\nu_1+\nu_2+2\sigma-n)\Gamma^2\left(\frac{3}{2}\right)} \times _4F_3\left(\begin{array}{c}
\frac{n}{2}, \sigma, \nu_1+\nu_2+\sigma-n, \nu_2+\sigma-\frac{n}{2}, \nu_1+\sigma-\frac{n}{2} \\
\frac{1}{2}, \sigma+\frac{1}{2}(\nu_1+\nu_2-n), \sigma+\frac{1}{2}(\nu_1+\nu_2+1-n) \\
\end{array}
\right| \frac{p^2}{4m^2}\right).$$

The three-loop vacuum integral $D_3$ can be obtained by integrating (4.26) over $p$,

$$D_3(0, \sigma, 0, \nu_1, \nu_2, \nu_3) = \left. \frac{1}{i\pi^{n/2} \Gamma\left(\frac{3}{2}\right)} \int \frac{dp}{(p^2-m^2)^{\nu_3}} J_{011}(\sigma, \nu_1, \nu_2; p^2, m) \right|_{m=1}.$$  

In such a way we find

$$D_3(0, \sigma, 0, \nu_1, \nu_2, \nu_3) = \frac{\Gamma\left(\frac{n}{2} - \sigma\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\nu_1+\sigma-\frac{n}{2}\right)\Gamma\left(\nu_2+\sigma-\frac{n}{2}\right)\Gamma(\nu_1+\nu_2+\sigma-n)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}-\nu_3\right)\Gamma(\nu_1+\nu_2+2\nu_3+2\sigma-2n)}$$

$$\times \left(\begin{array}{c}
\Gamma\left(\nu_1+\sigma-\frac{n}{2}\right)\Gamma\left(\nu_2+\sigma-\frac{n}{2}\right)\Gamma(\nu_1+\nu_2+\sigma-n)\Gamma\left(\frac{3}{2}\right) \\
\Gamma\left(\frac{n}{2}-\nu_3\right)\Gamma(\nu_1+\nu_2+2\nu_3+2\sigma-2n) \\
\end{array}
\right) \times _4F_3\left(\begin{array}{c}
\nu_1+\sigma-\frac{n}{2}, \nu_1+\nu_3+\sigma-n, \nu_2+\nu_3+\sigma-n, \nu_1+\nu_2+\nu_3+\sigma-\frac{3n}{2} \\
\nu_3-\frac{n}{2}+1, \nu_3+\frac{1}{2}(\nu_1+\nu_2)+\sigma-n, \nu_3+\frac{1}{2}(\nu_1+\nu_2+1)+\sigma-n \\
\end{array}
\right| \frac{1}{4}\right).$$

The integral $D_3(0, 1, 0, 1, 1, 1)$ is also one of the master integrals used in the packages [31, 48]. The result including the $\varepsilon$-term is available in [28] and [30]. Here we are going to provide some further terms of the $\varepsilon$-expansion.

Again, when using the general result (4.28) it is simpler to consider an integral with shifted indices, namely $D_3(0, 1, 0, 2, 2, 2)$, which is related to the master one through a simple relation [31]

$$9D_3(0, 1, 0, 2, 2, 2) = -\frac{4-15\varepsilon}{\varepsilon^3(1-\varepsilon)} + 2(1-3\varepsilon)(1-2\varepsilon)(2-3\varepsilon)D_3(0, 1, 0, 1, 1, 1).$$

Considering Eq. (4.28) at $\sigma = 1$, $\nu_1 = \nu_2 = \nu_3 = 2$, we get

$$(1-\varepsilon)D_3(0, 1, 0, 2, 2, 2) = \frac{1}{\varepsilon(1+2\varepsilon)} 3F_2\left(\begin{array}{c}
1, 1+\varepsilon, 1+2\varepsilon \\
\frac{1}{2}+\varepsilon, 1-\varepsilon \\
\end{array}
\right| \frac{1}{4}\right)$$

$$- \frac{1}{\varepsilon(1+4\varepsilon)} \frac{\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1+4\varepsilon)} \times _2F_1\left(\begin{array}{c}
1+2\varepsilon, 1+3\varepsilon \\
\frac{3}{2}+2\varepsilon \\
\end{array}
\right| \frac{1}{4}\right).$$
Moreover, the occurring \( _2F_1 \) function can be reduced to a product of \( \Gamma \)-functions:

\[
_2F_1 \left( \frac{1 + 2\varepsilon, 1 + 3\varepsilon}{\frac{3}{2} + 2\varepsilon} \left| \frac{1}{4} \right. \right) = \frac{2\pi}{3\varepsilon^{3 + 3\varepsilon}} \frac{\Gamma(2 + 4\varepsilon)}{\Gamma(1 + 2\varepsilon) \Gamma^2(1 + \varepsilon)}, \tag{4.31}
\]

see (B.14). The \( _3F_2 \) function in Eq. (4.30) belongs to one of the types considered in Appendix B. Its \( \varepsilon \)-expansion is given by

\[
_3F_2 \left( \frac{1, 1 + \varepsilon, 1 + 2\varepsilon}{\frac{3}{2} + \varepsilon, 1 - \varepsilon} \left| \frac{1}{4} \right. \right) = \frac{1 + 2\varepsilon}{3\varepsilon^{3 + 3\varepsilon}} \left\{ \frac{2}{3}\pi + 4\varepsilon Ls_2 \left( \frac{\pi}{3} \right) \right. \\
+ \varepsilon^2 \left[ \frac{28}{3}\pi\zeta_2 + 18\text{Ls}_3 \left( \frac{2\pi}{3} \right) \right] - \varepsilon^3 \left[ \frac{112}{3}\pi\zeta_3 + \frac{32}{3}\text{Ls}_4 \left( \frac{\pi}{3} \right) - 36\text{Ls}_4 \left( \frac{2\pi}{3} \right) \right] \\
+ \varepsilon^4 \left[ 216\zeta_2\text{Ls}_3 \left( \frac{2\pi}{3} \right) + \frac{679}{6}\pi\zeta_4 - \frac{196}{3}\text{Ls}_5 \left( \frac{\pi}{3} \right) + 54\text{Ls}_5 \left( \frac{2\pi}{3} \right) - 108\pi\text{Ls}_4 \left( \frac{2\pi}{3} \right) + 81\text{Ls}_5 \left( \frac{2\pi}{3} \right) \right] \\
+ \mathcal{O}(\varepsilon^5) \right\}. \tag{4.32}
\]

Finally, we obtain for the master integral:

\[
2\varepsilon(1 - \varepsilon)(1 - 2\varepsilon)(1 - 3\varepsilon)(2 - 3\varepsilon)D_3(0, 1, 0, 1, 1, 1) = \frac{9}{(1 + 2\varepsilon)^3} \left[ \frac{1}{4} \right] \\
+ \frac{(4 - 15\varepsilon)}{\varepsilon^2} - \frac{6\pi}{3\varepsilon^{3 + 3\varepsilon}} \frac{\Gamma(1 + 2\varepsilon) \Gamma(1 - \varepsilon) \Gamma(1 + 3\varepsilon)}{\Gamma(1 + \varepsilon)^4}. \tag{4.33}
\]

Considering result (4.28) with a different set of indices, \( \sigma = \nu_1 = 1 \) and \( \nu_2 = \nu_3 = 2 \), we get

\[
D_3(0, 1, 0, 1, 2, 2) = \frac{1}{(1 - \varepsilon)^2(1 + 2\varepsilon)\varepsilon} \left[ \frac{1}{4} \right] \\
+ \frac{1}{3(1 - \varepsilon)^2} \frac{\Gamma^2(1 + 2\varepsilon) \Gamma(1 - \varepsilon) \Gamma(1 + 3\varepsilon)}{\Gamma(1 + 4\varepsilon) \Gamma^2(1 + \varepsilon)^2} \left[ \frac{1}{4} \right] - 1 \tag{4.34}\number{4.34}
\]

On the other hand, using the recurrence package [31] we obtain a much simpler result for the same integral,

\[
D_3(0, 1, 0, 1, 2, 2) = \frac{1}{3\varepsilon^3(1 - \varepsilon)}. \tag{4.35}
\]

Therefore, we arrive at the following non-trivial relation between hypergeometric functions:

\[
_3F_2 \left( \frac{1 + \varepsilon, 1 + 2\varepsilon}{\frac{3}{2} + \varepsilon, 2 - \varepsilon} \left| \frac{1}{4} \right. \right) = \frac{(1 - \varepsilon)(1 + 2\varepsilon)}{3\varepsilon^2} \left[ \frac{\Gamma^2(1 + 2\varepsilon) \Gamma(1 - \varepsilon) \Gamma(1 + 3\varepsilon)}{\Gamma^2(1 + \varepsilon) \Gamma(1 + 4\varepsilon)} \right] \left[ \frac{1}{4} \right]. \tag{4.36}\number{4.36}
\]

The same relation can be found by considering the integral (A.23) from Ref. [63], which they denote as \( G_0(1; 1, 1; 1, 2) \). This integral is nothing but \( E_4(1, 1, 1, 2, 1) \) in notations of

\footnote{Eq. (4.31) can be obtained via a simple transformation of Eq. (31) on p. 495 of [45]. A standard way to obtain formulae of such type is to use Bayley’s cubic transformations of hypergeometric functions.}
The result for \( \text{VL111}(1, 1, 1) \) in terms of hypergeometric function is given by Eq. (33) from Ref. [54],

\[
\text{VL111}(1, 1, 1) = -\frac{1}{(1 - \varepsilon)(1 + 2\varepsilon)\varepsilon^2} \left[ \frac{\pi \varepsilon}{2} 3^{\frac{1}{2} - \varepsilon} \frac{\Gamma(1 + 2\varepsilon)}{\Gamma^2(1 + \varepsilon)} + \frac{3}{2}(1 - 2\varepsilon)_{2} F_{1}\left( \frac{1, \varepsilon}{2} \left| \frac{1}{4} \right. \right) \right].
\]

(4.38)

Considering the difference of the two results, one given in [63] and another presented in Eqs. (4.37)–(4.38), we reproduce the relation (4.36).

It is interesting that the same function (4.36) can be obtained from Eq. (4.26), at special values of the indices \( \sigma = 1, \nu_1 = \nu_2 = 2 \):

\[
\textbf{J}^{11}(1, 2, 2) = \frac{1}{(1 - \varepsilon)(1 + 2\varepsilon)} 3_{2} F_{2}\left( \frac{1, 1 + \varepsilon, 1 + 2\varepsilon}{\frac{3}{2} + \varepsilon, 2 - \varepsilon} \left| \frac{1}{4} \right. \right)
\]

\[
= \frac{1}{3\varepsilon^2} \left[ \frac{\Gamma^2(1 + 2\varepsilon)\Gamma(1 - \varepsilon)}{\Gamma^2(1 + \varepsilon)\Gamma(1 + 3\varepsilon)}_{2} F_{1}\left( \frac{2\varepsilon, 3\varepsilon}{\frac{3}{2} + 2\varepsilon} \left| \frac{1}{4} \right. \right) - 1 \right],
\]

(4.39)

where

\[
\textbf{J}^{11}(\sigma, \nu_1, \nu_2) \equiv J_{111}(\sigma, \nu_1, \nu_2; p^2, m)\bigg|_{p^2 = m^2, m = 1},
\]

and we have used relation (4.36). For the same integral \( \textbf{J}^{11}(1, 2, 2) \), applying Padé approximants calculated from the small momentum expansion [64] with the PSLQ-based analysis (for details, see [27, 28]), we restore several terms of the \( \varepsilon \)-expansion:

\[
\textbf{J}^{11}(1, 2, 2) = \frac{2}{3} \zeta_2 - \frac{\varepsilon}{3} \zeta_3 + \varepsilon^2 3\zeta_4 - \varepsilon^3 \left\{ 2\zeta_5 + \frac{4}{3}\zeta_2 \zeta_3 \right\} + \varepsilon^4 \left\{ \frac{61}{6}\zeta_6 + \frac{2}{3}\zeta_3^2 \right\}
\]

\[
- \varepsilon^5 \left\{ 6\zeta_7 + 4\zeta_2 \zeta_5 + 6\zeta_3 \zeta_4 \right\} + \varepsilon^6 \left\{ \frac{1201}{36}\zeta_8 + 4\zeta_5 \zeta_3 + \frac{4}{3}\zeta_2 \zeta_3^2 \right\}
\]

\[
- \varepsilon^7 \left\{ \frac{170}{9}\zeta_9 + 12\zeta_2 \zeta_7 + \frac{61}{3}\zeta_3 \zeta_6 + 18\zeta_4 \zeta_5 + \frac{4}{9}\zeta_3^3 \right\}
\]

\[
+ \varepsilon^8 \left\{ \frac{4077}{40}\zeta_{10} + 12\zeta_3 \zeta_7 + 6\zeta_5^2 + 6\zeta_5 \zeta_4 + 8\zeta_2 \zeta_3 \zeta_5 \right\} + \mathcal{O}(\varepsilon^9).
\]

(4.41)

Using (4.41), the result for the \( _2F_1 \) function occurring in (4.39) can be deduced as:

\[
_{2} F_{1}\left( \frac{2\varepsilon, 3\varepsilon}{\frac{3}{2} + 2\varepsilon} \left| \frac{1}{4} \right. \right) = \frac{\Gamma(1 + \varepsilon)\Gamma(1 + 4\varepsilon)}{\Gamma(1 + 2\varepsilon)\Gamma(1 + 3\varepsilon)}.
\]

(4.42)

Therefore, we get

\[
\frac{1}{(1 - \varepsilon)(1 + 2\varepsilon)^3} 3_{2} F_{2}\left( \frac{1, 1 + \varepsilon, 1 + 2\varepsilon}{\frac{3}{2} + \varepsilon, 2 - \varepsilon} \left| \frac{1}{4} \right. \right) = \frac{1}{3\varepsilon^2} \left[ \frac{\Gamma(1 + 2\varepsilon)\Gamma(1 - \varepsilon)}{\Gamma(1 + \varepsilon)} - 1 \right],
\]

(4.43)
\[ J_{011}(1, 2, 2) = \frac{1}{3 \varepsilon^2} \left[ \frac{\Gamma(1 + 2\varepsilon)\Gamma(1 - \varepsilon)}{\Gamma(1 + \varepsilon)} - 1 \right]. \quad (4.44) \]

We note that the results similar to Eq. (4.42) are usually obtained using Bailey’s cubic transformations for hypergeometric functions.

Let us remind that for the on-shell integral \( J_{011} \) there are two master integrals [66] of this type, \( J_{011}(1, 1, 1) \) and \( J_{011}(1, 1, 2) \). Using the fact that the integral (4.44) is a linear combination of these integrals, we obtain

\[ J_{011}(1, 1, 2) = -\frac{\Gamma(1 + 2\varepsilon)\Gamma(1 - \varepsilon)}{6\varepsilon^2(1 - 2\varepsilon)\Gamma(1 + \varepsilon)} - \frac{2 - 3\varepsilon}{3} J_{011}(1, 1, 1). \quad (4.45) \]

Alternatively, one can consider other independent combinations of these master integrals: for example, \( J_{011}(1, 2, 2) \) and \([J_{011}(1, 2, 2) + 2J_{011}(2, 1, 2)]\) (see in [65]). While \( J_{011}(1, 2, 2) \) is given in Eq. (4.44), the second combination is less trivial and cannot be represented in terms of the \( \Gamma \)-function. Using relation (see, e.g., in [45])

\[ _3F_2 \left( \begin{array}{c} 2, 1 + \varepsilon, 1 + 2\varepsilon \\ \frac{3}{2} + \varepsilon, 2 - \varepsilon \end{array} \right) - \varepsilon _3F_2 \left( \begin{array}{c} 1, 1 + \varepsilon, 1 + 2\varepsilon \\ \frac{3}{2} + \varepsilon, 2 - \varepsilon \end{array} \right) = (1 - \varepsilon) \quad _3F_2 \left( \begin{array}{c} 1, 1 + \varepsilon, 1 + 2\varepsilon \\ \frac{3}{2} + \varepsilon, 1 - \varepsilon \end{array} \frac{1}{z} \frac{1}{4} \right) \quad (4.46) \]

we obtain

\[ J_{011}(1, 2, 2) + 2J_{011}(2, 1, 2) = -\frac{1}{\varepsilon(1 + 2\varepsilon)} \quad _3F_2 \left( \begin{array}{c} 1, 1 + \varepsilon, 1 + 2\varepsilon \\ \frac{3}{2} + \varepsilon, 1 - \varepsilon \frac{1}{4} \right). \quad (4.47) \]

This is the same \( _3F_2 \) function as in the case of \( D_3 \), whose \( \varepsilon \)-expansion is given in (4.32). The lowest terms of the \( \varepsilon \)-expansion of Eq. (4.47) coincide with Eq. (9) of Ref. [28], which has been obtained via PSLQ analysis, based on the small momentum expansion.

### 5 Conclusions

In this paper we have studied some important issues related to the \( \varepsilon \)-expansion of Feynman diagrams in the framework of dimensional regularization [1].

For the one-loop two-point function with different masses \( m_1 \) and \( m_2 \), the analytic continuation (2.17) of known results [23, 24] to other regions of interest has been constructed. In particular, explicit formulae, (2.18) and (2.21)–(2.23), relating the log-sine integrals and the generalized (Nielsen) polylogarithms have been obtained to an arbitrary order.

Then, we have examined some physically-important examples of the \( \varepsilon \)-expansion of the one-loop three-point function. For the cases of the off-shell massless triangles, as well as for a specific on-shell triangle diagram with two different masses, the \( \varepsilon \)-expansion and the procedure of analytic continuation is, basically, similar to those for the two-point function.

However, the situation for another interesting example, a massive triangle loop with \( p_1^2 = p_2^2 = 0 \), appears to be much more complicated. We have shown that all even terms of the expansion, i.e. the coefficients of \( \varepsilon^{2l} \), can be presented in terms of the log-sine integrals. In
the meantime, for the odd terms (starting from $\varepsilon^3$) the result does not seem to be expressible in terms of known functions (related to the polylogarithms), although its one-fold integral representation looks reasonably simple. Nevertheless, for some special values of $p_2^3$ we have identified the corresponding $\varepsilon^3$ contributions in terms of known transcendental numbers, expanding the hypergeometric functions and using the PSLQ procedure [29].

In particular, we have found that one extra term should be added to the odd weight-5 basis considered in [28]. This new constant (3.22) can be related to a special case of the multiple binomial sums (1.11). We also constructed the even basis up to weight 5 (see Introduction and Appendix B.1).

Then, we have obtained a number of new results for the higher terms of the $\varepsilon$-expansion for certain two-loop (mainly, on-shell) two-point functions (see Eqs. (4.5), (4.15)) and three-loop vacuum diagrams, Eqs. (4.10), (4.24), (4.33). The considered examples serve as the master integrals in the analytic computer packages [31, 47, 48]. In particular, the obtained $\varepsilon$-terms of three-loop vacuum diagrams are important in the four-loop-order calculations.

Within these calculations, a new relation (4.36) between $3F_2$ and $2F_1$ hypergeometric functions of argument $\frac{1}{4}$ was found. Moreover, the analytical solution Eq. (4.44) makes it possible to express one of the on-shell integrals, $J_{011}(1, 2, 2)$, in terms of $\Gamma$-functions. In this way, we have reduced the number of non-trivial master integrals (4.45) used in the package [47].

One of the interesting issues related to the odd and even basis construction is the connection between the generalized log-sine integrals and the multiple $\zeta$ values which have appeared earlier in Refs. [19, 21], in the study of the connection between knot and quantum field theory. Namely, our Eq. (A.10) shows that $\zeta_{5,3}$, $\zeta_{7,3}$ and $\zeta_{3,7,3}$ are connected with the combinations $\left[L_{3j}^{(1)}\left(\frac{\pi}{3}\right) - \frac{\pi}{3}L_{3j-1}\left(\frac{\pi}{3}\right)\right]$ (plus ordinary $\zeta$-functions), where $j$ is the weight of the corresponding multiple $\zeta$ value. Besides this, we have found that the constant $U_{5,1}$ (see in [8]) can be expressed in terms of the even basis (A.16).

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Note added (May 9, 2017)
Over the years, a few typos were found in the published version of this paper (Nucl. Phys. B 605 (2001) 266). For completeness, we list them below.

- Eq. (2.7) and Eq. (B.11):
The power of $\cos \theta$ should be $1 + 2\varepsilon$ rather than $1 - 2\varepsilon$.
The correct expression has been given in Eq. (2.10) of A. I. Davydychev and M. Yu. Kalmykov, Nucl. Phys. B 699 (2004) 3 [hep-th/0303162].

- Eq. (4.10):
The coefficient at $\pi \text{L}_4 \left(\frac{\pi}{3}\right)$ should be $\frac{161}{54}$ instead of $\frac{161}{154}$.
This typo was mentioned in footnote 1 of M. Yu. Kalmykov, Nucl. Phys. B 718 (2005) 276 [hep-ph/0503070].

- Eq. (4.25):
The finite part of master-integral should include $\frac{5}{3} \frac{\pi}{\sqrt{3}} \zeta_2$ rather than $\frac{5}{3} \frac{\pi}{\sqrt{3}}$.
The correct expression has been given in Eq. (10) of J. Fleischer and M. Yu. Kalmykov, Phys. Lett. B 470 (1999) 168 [hep-ph/9910223].

We are indebted to Andrey Grozin for discussions and independent checks.
A Polylogarithms and related functions

A.1 Polylogarithms and log-sine integrals

The polylogarithm \( \text{Li}_j(z) \) is defined as

\[
\text{Li}_j(z) = \frac{(-1)^j}{(j-1)!} \int_0^1 \frac{1}{z} \frac{\ln^{j-1} \xi}{\xi - z} \, d\xi = \sum_{k=1}^\infty \frac{z^k}{k^j}.
\]  

(A.1)

When the argument \( z \) belongs to the unit circle in the complex \( z \)-plane, \( z = e^{i\theta} \), we get (see, e.g., in [11])

\[
\text{Li}_{2l}(e^{i\theta}) = \text{Gl}_{2l}(\theta) + i\text{Cl}_{2l}(\theta), \quad \text{Li}_{2l+1}(e^{i\theta}) = \text{Cl}_{2l+1}(\theta) + i\text{Gl}_{2l+1}(\theta),
\]  

(A.2)

where \( \text{Gl}_j(\theta) \) is proportional to Bernoulli polynomial of the order \( j \), \( B_j(\theta/(2\pi)) \) (see Eq. (22) in p. 300 of [11]), whereas \( \text{Cl}_j(\theta) \) is the Clausen function,

\[
\text{Cl}_{2l}(\theta) = -\frac{\sin \theta}{(2l-1)!} \int_0^1 \frac{1}{1 - 2\xi \cos \theta + \xi^2} \ln^{2l-1} \xi \, d\xi = \sum_{k=1}^\infty \frac{\sin(k\theta)}{k^{2l}},
\]  

(A.3)

\[
\text{Cl}_{2l+1}(\theta) = -\frac{1}{(2l)!} \int_0^1 \frac{\xi - \cos \theta}{1 - 2\xi \cos \theta + \xi^2} \ln^{2l} \xi \, d\xi = \sum_{k=1}^\infty \frac{\cos(k\theta)}{k^{2l+1}}.
\]  

(A.4)

Below we list some useful properties of the Clausen function (for details see in [11]):

\[
\text{Cl}_{2l+1} \left( \frac{\pi}{3} \right) = \frac{1}{2} \left( 1 - 2^{-2l} \right) \left( 1 - 3^{-2l} \right) \zeta_{2l+1}, \quad \text{Cl}_{2l+1} \left( \frac{2\pi}{3} \right) = -\frac{1}{2} \left( 1 - 3^{-2l} \right) \zeta_{2l+1},
\]

and

\[
\text{Cl}_{2l} \left( \frac{\pi}{3} \right) = \left( 1 + 2^{1-2l} \right) \text{Cl}_{2l} \left( \frac{2\pi}{3} \right).
\]

When we consider the imaginary and real parts of \( \text{Li}_j(1-e^{i\theta}) \), also the log-sine function,

\[
\text{Ls}_j(\theta) = -\int_0^\theta d\phi \ln^{j-1} \left| 2 \sin \frac{\phi}{2} \right|, \quad \text{Ls}_j^{(k)}(\theta) = -\int_0^\theta d\phi \phi^k \ln^{j-k-1} \left| 2 \sin \frac{\phi}{2} \right|
\]  

(A.5)

(A.6)

get involved,

\[
\text{Re} \text{Li}_3(1-e^{i\theta}) = \frac{1}{2} [\zeta_3 - \text{Cl}_3(\theta)] + \frac{1}{4} \theta^2 l_\theta,
\]

31
Im $\text{Li}_3(1-e^{i\theta}) = \frac{1}{24} \theta^3 - \frac{1}{2} \theta t_\theta^2 - \text{Cl}_2(\theta) t_\theta + \frac{1}{2} \text{Li}_3(\theta),$

Re $\text{Li}_4(1-e^{i\theta}) = \frac{1}{4} \text{Li}_4^{(1)}(\theta) - \frac{1}{4} \theta \text{Li}_3(\theta) + \frac{1}{8} \theta^2 t_\theta^2 + \frac{1}{2} [\zeta_3 - \text{Cl}_3(\theta)] l_\theta - \frac{1}{192} \theta^4,$

Im $\text{Li}_4(1-e^{i\theta}) = -\frac{1}{6} \text{Li}_4(\theta) + \frac{1}{2} \text{Li}_3(\theta) l_\theta - \frac{1}{2} \theta^3 t_\theta - \frac{1}{4} \text{Cl}_2(\theta) l_\theta^2 + \frac{1}{24} \theta^3 l_\theta - \frac{1}{4} \text{Cl}_4(\theta) + \frac{1}{4} \theta \zeta_3,$

Re $\text{Li}_5(1-e^{i\theta}) = -\frac{1}{12} \text{Li}_5^{(1)}(\theta) + \frac{1}{8} [\zeta_5 - \text{Cl}_5(\theta)] + \frac{1}{4} \text{Li}_4^{(1)}(\theta) l_\theta + \frac{1}{4} [\zeta_3 - \text{Cl}_3(\theta)] l_\theta^2$

\[\text{Re} \text{Li}_5(1-e^{i\theta}) = \frac{1}{24} \theta^3 + \frac{1}{16} \theta^2 \zeta_3 + \frac{1}{24} \theta^2 l_\theta^2 - \frac{192}{192} \theta^4,\]

Im $\text{Li}_5(1-e^{i\theta}) = \frac{1}{24} \text{Li}_5(\theta) - \frac{1}{16} \text{Li}_5^{(1)}(\theta) - \frac{1}{24} \theta^3 l_\theta - \frac{1}{4} \text{Li}_4(\theta) l_\theta + \frac{1}{2} \text{Li}_3(\theta) l_\theta^2 - \frac{1}{6} \text{Cl}_2(\theta) l_\theta^3$

\[\text{Im} \text{Li}_6(1-e^{i\theta}) = \frac{1}{48} \text{Li}_6^{(1)}(\theta) - \frac{1}{96} \text{Li}_6^{(3)}(\theta) + \frac{1}{12} [\zeta_3 - \text{Cl}_3(\theta)] l_\theta^2 + \frac{1}{8} \theta^3 l_\theta^3 + \frac{1}{16} \text{Li}_4(\theta) l_\theta^2 - \frac{1}{16} \text{Li}_5^{(2)}(\theta) l_\theta^2$

\[\text{Re} \text{Li}_6(1-e^{i\theta}) = \frac{1}{48} \text{Li}_6^{(2)}(\theta) - \frac{1}{120} \text{Li}_6(\theta) - \frac{1}{16} \text{Cl}_6(\theta) + \frac{1}{24} \text{Li}_5(\theta) l_\theta - \frac{1}{16} \text{Li}_5^{(2)}(\theta) l_\theta$

\[\text{Im} \text{Li}_6(1-e^{i\theta}) = \frac{1}{48} \text{Li}_6^{(2)}(\theta) - \frac{1}{120} \text{Li}_6(\theta) - \frac{1}{16} \text{Cl}_6(\theta) + \frac{1}{24} \text{Li}_5(\theta) l_\theta - \frac{1}{16} \text{Li}_5^{(2)}(\theta) l_\theta$

where $l_\theta \equiv \ln \left| 2 \sin \frac{\theta}{2} \right|$, $-\pi \leq \theta \leq \pi$. Results for $\text{Li}_j(1-e^{i\theta})$ with $j \leq 4$, as well as an outline how to get results for higher $j$'s, can be found in [11].

According to the definition (A.5), the following integral and differential relations hold:

$\text{Cl}_2(\theta) = \frac{1}{2} \theta \zeta_2 - \frac{1}{2} \theta \zeta_3 + \frac{1}{2} \theta l_\theta,$

$\text{Cl}_2(\theta) = \frac{1}{2} \theta \zeta_2 + \frac{1}{2} \theta \zeta_3 - \frac{1}{2} \theta l_\theta,$

$\text{Cl}_2(\theta) = \frac{1}{2} \theta \zeta_2 - \frac{1}{2} \theta \zeta_3 + \frac{1}{2} \theta l_\theta,$

$\text{Cl}_2(\theta) = \frac{1}{2} \theta \zeta_2 + \frac{1}{2} \theta \zeta_3 - \frac{1}{2} \theta l_\theta.$

(A.7)

In addition, using the fact that $\text{Li}_2(\theta) = \text{Cl}_2(\theta)$ and taking into account the integration rules

$\text{Cl}_2(\theta) = \int_0^\theta \text{Cl}_2(\phi) \, d\phi,$

one can obtain Eqs. (7.52) and (7.53) of [11]. In particular, the function $\text{Li}_3^{(k)}(\theta)$ is always expressible in terms of Clausen functions,

$\text{Li}_3^{(1)}(\theta) = \theta \text{Cl}_2(\theta) + \text{Cl}_3(\theta) - \zeta_3,$

$\text{Li}_3^{(2)}(\theta) = \theta^2 \text{Cl}_2(\theta) + 2 \theta \text{Cl}_3(\theta) - 2 \text{Cl}_4(\theta),$

$\text{Li}_3^{(3)}(\theta) = \theta^3 \text{Cl}_2(\theta) + 3 \theta^2 \text{Cl}_3(\theta) - 6 \theta \text{Cl}_4(\theta) - 6 \text{Cl}_5(\theta) + 6 \zeta_5,$

$\text{Li}_3^{(4)}(\theta) = \theta^4 \text{Cl}_2(\theta) + 4 \theta^3 \text{Cl}_3(\theta) + 6 \theta^2 \text{Cl}_4(\theta) + 4 \theta \text{Cl}_5(\theta) + 6 \text{Cl}_6(\theta),$

\[\text{Li}_3^{(5)}(\theta) = \theta^5 \text{Cl}_2(\theta) + 5 \theta^4 \text{Cl}_3(\theta) + 10 \theta^3 \text{Cl}_4(\theta) + 10 \theta^2 \text{Cl}_5(\theta) + 10 \theta \text{Cl}_6(\theta) + 6 \zeta_5,

\[\text{Li}_3^{(6)}(\theta) = \theta^6 \text{Cl}_2(\theta) + 6 \theta^5 \text{Cl}_3(\theta) + 15 \theta^4 \text{Cl}_4(\theta) + 20 \theta^3 \text{Cl}_5(\theta) + 15 \theta^2 \text{Cl}_6(\theta) + 6 \text{Cl}_7(\theta) + 6 \zeta_5,

\[\text{Li}_3^{(7)}(\theta) = \theta^7 \text{Cl}_2(\theta) + 7 \theta^6 \text{Cl}_3(\theta) + 21 \theta^5 \text{Cl}_4(\theta) + 35 \theta^4 \text{Cl}_5(\theta) + 35 \theta^3 \text{Cl}_6(\theta) + 21 \theta^2 \text{Cl}_7(\theta) + 6 \text{Cl}_8(\theta) + 6 \zeta_5.

(A.8)

\[\text{Li}_3^{(k)}(\theta) = \theta^k \text{Cl}_2(\theta) + \text{Cl}_3(\theta) - \zeta_3,$

$\text{Li}_3^{(2)}(\theta) = \theta^2 \text{Cl}_2(\theta) + 2 \theta \text{Cl}_3(\theta) - 2 \text{Cl}_4(\theta),$

$\text{Li}_3^{(3)}(\theta) = \theta^3 \text{Cl}_2(\theta) + 3 \theta^2 \text{Cl}_3(\theta) - 6 \theta \text{Cl}_4(\theta) - 6 \text{Cl}_5(\theta) + 6 \zeta_5,$

In eq. (7.67) of [11], as well as in eq. (36) on p. 301, the coefficient of $\log^2 (2 \sin \frac{\theta}{2}) \text{Cl}_2(\theta)$ should be $-\frac{1}{2}$ (rather than $+\frac{3}{2}$).
etc.

It is also known that the values of $L_{s_j}(\pi)$ can be expressed in terms of $\zeta$-function, for any \( j \) (see in [11]):

\[
L_{s_2}(\pi) = 0, \quad L_{s_3}(\pi) = -\frac{1}{2} \pi \zeta_2, \quad L_{s_4}(\pi) = \frac{3}{2} \pi \zeta_3, \quad L_{s_5}(\pi) = -\frac{27}{8} \pi \zeta_4, \quad L_{s_6}(\pi) = \frac{45}{2} \pi \zeta_5 + \frac{15}{2} \pi \zeta_2 \zeta_3,
\]

etc. Moreover,

\[
L_{s_3}(\frac{\pi}{3}) = -\frac{7}{18} \pi \zeta_2, \quad L_{s_4}^{(1)}(\frac{\pi}{3}) = -\frac{17}{12} \zeta_4.
\]

For \( \theta = \frac{\pi}{3} \) there are some relations among the Clausen function and log-sine integrals,

\[
\begin{align*}
\text{Cl}_4(\frac{\pi}{3}) &= \frac{2}{3} L_{s_4}(\frac{\pi}{3}) - \frac{1}{9} \pi \zeta_3, \\
\text{Cl}_6(\frac{\pi}{3}) &= \frac{2}{15} L_{s_6}(\frac{\pi}{3}) - \frac{1}{5} \pi \zeta_5 - \frac{17}{81} \pi \zeta_2 \zeta_3.
\end{align*}
\]

The result for \( \text{Cl}_4(\frac{\pi}{3}) \) was given in [28]. Moreover, the quantities \( L_{s_j}^{(1)}(\frac{\pi}{3}) \) can be expressed in terms of $\pi L_{s_{j-1}}(\frac{\pi}{3})$ and Euler–Zagier sums,

\[
\begin{align*}
L_{s_5}^{(1)}(\frac{\pi}{3}) &= \frac{1}{3} \pi L_{s_4}(\frac{\pi}{3}) - \frac{19}{4} \zeta_5 - \frac{1}{2} \zeta_2 \zeta_3, \\
L_{s_6}^{(1)}(\frac{\pi}{3}) &= \frac{1}{3} \pi L_{s_5}(\frac{\pi}{3}) + \frac{2029}{96} \zeta_6 + 2 \zeta_3^2, \\
L_{s_7}^{(1)}(\frac{\pi}{3}) &= \frac{1}{3} \pi L_{s_6}(\frac{\pi}{3}) - \frac{2465}{32} \zeta_7 - \frac{15}{2} \zeta_5 \zeta_2 - \frac{205}{8} \zeta_4 \zeta_3 \\
L_{s_8}^{(1)}(\frac{\pi}{3}) &= \frac{1}{3} \pi L_{s_7}(\frac{\pi}{3}) + \frac{148789}{256} \zeta_8 + \frac{285}{2} \zeta_5 \zeta_3 + \frac{15}{2} \zeta_2^2 \zeta_2 + \frac{21}{2} \zeta_5 \zeta_3, \\
L_{s_9}^{(1)}(\frac{\pi}{3}) &= \frac{1}{3} \pi L_{s_8}(\frac{\pi}{3}) - \frac{487235}{192} \zeta_9 - \frac{945}{4} \zeta_7 \zeta_2 - \frac{71015}{16} \zeta_6 \zeta_3 - \frac{12915}{16} \zeta_5 \zeta_4 - 35 \zeta_3^3, \\
L_{s_{10}}^{(1)}(\frac{\pi}{3}) &= \frac{1}{3} \pi L_{s_9}(\frac{\pi}{3}) + \frac{1405963}{512} \zeta_{10} + \frac{51765}{8} \zeta_7 \zeta_3 + \frac{57105}{16} \zeta_5^2 + \frac{4305}{4} \zeta_3^2 \zeta_3 + 630 \zeta_2 \zeta_3 \zeta_5 + \frac{3075}{16} \zeta_7 \zeta_3, \\
L_{s_{11}}^{(1)}(\frac{\pi}{3}) &= \frac{1}{3} \pi L_{s_{10}}(\frac{\pi}{3}) - \frac{76115403}{512} \zeta_{11} - \frac{2677}{2} \zeta_9 \zeta_2 - \frac{24875377}{128} \zeta_8 \zeta_3 - \frac{348705}{8} \zeta_7 \zeta_4 - \frac{1917405}{32} \zeta_6 \zeta_5 \\
&\quad - \frac{17955}{2} \zeta_5 \zeta_3^2 - \frac{315}{2} \zeta_5 \zeta_2 - \frac{1323}{2} \zeta_5 \zeta_3 - \frac{1323}{2} \zeta_5 \zeta_3, \quad (A.10)
\end{align*}
\]

where $\zeta_{a,b}$ and $\zeta_{a,b,c}$ are defined in Eq. (1.8). The result for \( L_{s_5}^{(1)}(\frac{\pi}{3}) \) was given in [28]. We also list some other results relevant for the weights 5 and 6:

\[
\begin{align*}
L_{s_5}^{(2)}(\frac{\pi}{3}) &= \frac{2}{3} \pi L_{s_5}(\frac{\pi}{3}) + \frac{253}{144} \pi \zeta_4, \\
L_{s_6}^{(2)}(\frac{\pi}{3}) &= -\frac{1}{10} \pi L_{s_6}(\frac{\pi}{3}) + \frac{2}{5} \zeta_2 L_{s_4}(\frac{\pi}{3}) + \frac{5}{6} \pi \zeta_5 + \frac{11}{3} \pi \zeta_2 \zeta_3, \\
L_{s_6}^{(3)}(\frac{\pi}{3}) &= \frac{2}{3} \pi L_{s_5}(\frac{\pi}{3}) + \frac{18887}{432} \zeta_6 + 4 \zeta_3^2. \quad (A.11)
\end{align*}
\]

The results for \( L_{s_5}^{(1)}(\frac{\pi}{3}) \) and \( L_{s_6}^{(2)}(\frac{\pi}{3}) \) were given in [28]. All these results (A.9)–(A.11) have been obtained by the \texttt{PSLQ} procedure.

Using one-fold series representation (1.10) for \( L_{s_j}^{(1)}(\frac{\pi}{3}) \) and the corresponding expression for \( L_{s_j}(\frac{\pi}{3}) \) (see Appendix A in [33]20),

\[
L_{s_j}(\frac{\pi}{3}) = (-1)^j (j - 1)! \sum_{k=0}^{\infty} \frac{2k!}{(k!)(2k+1)!} \left( \frac{1}{16} \right)^k, \quad (A.12)
\]

\footnote{20A general factor, $(-1)^n$, in definition of $L_n(\theta)$ is missing in Appendix A in [33].}
we obtain rapidly-convergent series, which can be used for the high-precision calculations of $\zeta_{5,3}$, $\zeta_{7,3}$ and $\zeta_{3,5,3}$. For example,

$$\zeta_{5,3} = -\frac{2^5 \cdot 3 \cdot 5}{7} \left[ \frac{\pi}{2^n} \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} \frac{1}{k^8} - \frac{\pi}{3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^7} \frac{(2k)!}{(k!)^2} \left( \frac{1}{16} \right)^k \right] - \frac{17 \cdot 8287 \cdot \pi^8}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7^2} - \frac{5 \cdot 19}{7} \zeta_5 \zeta_3 = \frac{5 \cdot \pi^2}{2 \cdot 3 \cdot 7 \zeta_3}. \quad (A.13)$$

For the calculation of $\zeta$-function with an arbitrary accuracy, an algorithm elaborated in Ref. [67] can be used.

Considering $L_{s_j}^{(l)} \left( \frac{\pi}{2} \right)$ and $L_{s_j}^{(l)}(\pi)$ ($j = 4, 5; l = 1, 2$) we find that most of them (except for $L_{5s}^{(2)} \left( \frac{\pi}{2} \right)$) which is one of the even-basis elements) are connected with $\text{Li}_j \left( \frac{1}{2} \right)$,

$$L_{s_4}^{(1)} \left( \frac{\pi}{2} \right) = -\frac{5}{96} \ln^4 2 + \frac{5}{16} \zeta_2 \ln^2 2 - \frac{35}{32} \zeta_3 \ln 2 + \frac{125}{32} \zeta_4 + \frac{1}{2} \pi \text{Li}_3 \left( \frac{\pi}{2} \right) - \frac{5}{2} \text{Li}_4 \left( \frac{\pi}{2} \right),$$

$$L_{s_5}^{(1)} \left( \frac{\pi}{2} \right) = -\frac{1}{16} \ln^5 2 + \frac{5}{16} \zeta_2 \ln^3 2 - \frac{105}{128} \zeta_3 \ln^2 2 - \frac{15}{8} \text{Li}_4 \left( \frac{1}{2} \right) \ln 2 - \frac{9}{8} \zeta_2 \zeta_3 + \frac{5}{4} \pi \text{Li}_3 \left( \frac{\pi}{2} \right) - \frac{1209}{256} \zeta_5 - \frac{15}{8} \text{Li}_5 \left( \frac{1}{2} \right),$$

$$L_{s_4}^{(1)}(\pi) = -\frac{1}{6} \ln^4 2 + \zeta_2 \ln^2 2 - \frac{7}{2} \zeta_3 \ln 2 + \frac{19}{8} \zeta_4 - 4 \text{Li}_4 \left( \frac{1}{2} \right),$$

$$L_{s_5}^{(1)}(\pi) = -\frac{2}{5} \ln^5 2 + 2 \zeta_2 \ln^3 2 - \frac{21}{4} \zeta_3 \ln^2 2 - 12 \text{Li}_4 \left( \frac{1}{2} \right) \ln 2 + \frac{9}{2} \zeta_2 \zeta_3 + \frac{93}{32} \zeta_5 - 12 \text{Li}_5 \left( \frac{1}{2} \right),$$

$$L_{s_5}^{(2)}(\pi) = -\frac{1}{3} \pi \ln^4 2 + 2 \pi \zeta_2 \ln^2 2 - 7 \pi \zeta_3 \ln 2 + \frac{27}{4} \pi \zeta_4 - 8 \pi \text{Li}_4 \left( \frac{1}{2} \right). \quad (A.14)$$

All relations (A.9)–(A.14) have been obtained using the PSLQ procedure [29]21. We give some relevant numerical values,

$$L_{s_3} \left( \frac{\pi}{2} \right) \approx -2.03357650607205460091206896970 \ldots,$$

$$L_{s_4} \left( \frac{\pi}{2} \right) \approx 6.003109556529006567309305614033 \ldots,$$

$$C_{14} \left( \frac{\pi}{2} \right) \approx 0.9889445517411053361084226332284 \ldots,$$

$$L_{s_5} \left( \frac{\pi}{2} \right) \approx -24.0143377201598359235946799181446 \ldots,$$

$$L_{s_5}^{(2)} \left( \frac{\pi}{2} \right) \approx -0.126813242835588697100232996611 \ldots.$$ 

As an example of application of the even basis, we consider

$$U_{a,b} \equiv \zeta(a, b; -1, -1). \quad (A.15)$$

The lowest basis element of alternating Euler sums [8] which cannot be expressed in terms of $\zeta_j$, $\ln 2$ and $\text{Li}_j \left( \frac{1}{2} \right)$ or their products22 is $U_{5,1}$. This constant appears in the $\varepsilon$-expansion of

21The result for $L_{s_4}^{(1)}(\pi)$ coincides with Eq. (7.71) of [11], whereas in Eq. (7.145) for $L_{s_5}^{(2)}(\pi)$ the term $+2 \pi \zeta_2 \ln^2 2$ is missing, and the term $+37 \pi^5/360$ should read $3 \pi^5/40$. There is also an error in Eq. (7.144) of [11]: $L_{s_5}^{(2)}(2\pi)$ should be equal to $-13 \pi^5/45$, rather than to $7 \pi^5/30$.

22Two other constants are $\zeta(5,1,-1)$ and $\zeta(3,3,-1)$, see Eq. (17) in [8].
the large-$N$ (Nielsen) polylogarithms which are defined as (see, e.g., in Ref. [35])

accuracy of 300 decimals, yields the following relation:

For the procedure of the analytic continuation (in Section 2.2) we need the generalized

The following integration formula is useful:

In particular,

The following integration formula is useful:

Starting from Eq. (2.18) and applying relations (A.8) and (A.19), we arrive at

Substituting $\theta \to \pi - i \sigma \ln(-z)$ (see Eq. (2.15)) and using Eq. (2.18) we can rewrite this result in a different form,

$$\text{Ls}_{j+1}^{(1)}(\theta) - \text{Ls}_{j+1}^{(1)}(\pi) = \pi \left[ \text{Ls}_j(\theta) - \text{Ls}_j(\pi) \right] + \frac{1}{2^j(j+1)} \ln^{j+1}(-z) \left[ 1 - (-1)^j \right]$$

$$-(-1)^j(j-1)! \sum_{p=0}^{j-2} \frac{\ln^p(-z)}{2^p p!} \sum_{k=1}^{j-1-p} \frac{(-1)^k}{2^k} k \left[ S_{k+1,j-k-p}(z) + (-1)^p S_{k+1,j-k-p}(1/z) \right]$$

$$+2(-1)^j(j-1)! \sum_{k=1}^{j-1} \frac{(-1)^k}{2^k} k S_{k+1,j-k}(1)$$

Substituting $\theta \to \pi - i \sigma \ln(-z)$ (see Eq. (2.15)) and using Eq. (2.18) we can rewrite this result in a different form,

$$\text{Ls}_{j+1}^{(1)}(\theta) - \text{Ls}_{j+1}^{(1)}(\pi) = \pi \left[ \text{Ls}_j(\theta) - \text{Ls}_j(\pi) \right] + \frac{1}{2^j(j+1)} \ln^{j+1}(-z) \left[ 1 - (-1)^j \right]$$

$$+2(-1)^j(j-1)! \sum_{k=1}^{j-1} \frac{(-1)^k}{2^k} k S_{k+1,j-k}(1)$$

$$+(-1)^j(j-1)! \sum_{r=1}^{j-2} \frac{\ln^r(-z)}{2^r r!} \sum_{k=1}^{j-1-r} \frac{(-1)^k}{2^k} (p-k) \left[ S_{k+1,j-k-p}(z) + (-1)^p S_{k+1,j-k-p}(1/z) \right]$$

$$+(-1)^j(j-1)! \sum_{k=1}^{j-1} \frac{\ln^k(-z)}{2^k k!} \left[ S_{1,j-r}(z) + (-1)^r S_{1,j-r}(1/z) \right].$$

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A.2 Auxiliary function

It is useful to define another generalization of the log-sine integrals as

\[ \text{Lsc}_{i,j}(\theta) = -\int_0^\theta d\phi \ln^{i-1} \left| 2 \sin \frac{\phi}{2} \ln^{j-1} \left| 2 \cos \frac{\phi}{2} \right| \right. \]  \hspace{1cm} (A.22) \]

When \( i = 1 \) or \( j = 1 \), it reduces to ordinary Ls functions,

\[ \text{Lsc}_{i,1}(\theta) = \text{Ls}_i(\theta) \quad \text{and} \quad \text{Lsc}_{1,j}(\theta) = -\text{Ls}_j(\pi - \theta) + \text{Ls}_j(\pi) . \]  \hspace{1cm} (A.23) \]

There exists an obvious symmetry property,

\[ \text{Lsc}_{i,j}(\theta) = -\text{Lsc}_{j,i}(\pi - \theta) + \text{Lsc}_{j,i}(\pi) , \]  \hspace{1cm} (A.24) \]

which implies that the functions Lsc\(_{i,j}(\theta)\) with \( i > j \) are not independent. The values of Lsc\(_{i,j}(\pi)\) can be extracted from Lewin’s book [11], Eqs. (7.114)–(7.118). We present here some of them

\[ \begin{align*}
\text{Lsc}_{2,2}(\pi) &= \frac{1}{4}\pi \zeta_2, \\
\text{Lsc}_{2,3}(\pi) &= -\frac{1}{2}\pi \zeta_3, \\
\text{Lsc}_{3,3}(\pi) &= -\frac{9}{16}\pi \zeta_4, \\
\text{Lsc}_{2,4}(\pi) &= \frac{21}{16}\pi \zeta_4 .
\end{align*} \]  \hspace{1cm} (A.25) \]

Using \( \sin \phi = 2 \sin \frac{\phi}{2} \cos \frac{\phi}{2} \), the following duplication formula can be derived:

\[ \text{Ls}_k(2\theta) = 2(k - 1)! \sum_{i=0}^{k-1} \frac{1}{i!(k - 1 - i)!} \text{Lsc}_{i+1,k-i}(\theta) . \]  \hspace{1cm} (A.26) \]

For special cases \( k = 2, 3, 4, 5 \) it gives, respectively,

\[ \begin{align*}
\text{Ls}_2(\theta) - \text{Ls}_2(\pi - \theta) &= \frac{1}{2}\text{Ls}_2(2\theta) , \\
2\text{Lsc}_{2,2}(\theta) &= \frac{1}{2}\text{Ls}_3(2\theta) - \text{Ls}_3(\theta) + \text{Ls}_3(\pi - \theta) - \text{Ls}_3(\pi) , \\
3 \left[ \text{Lsc}_{2,3}(\theta) - \text{Lsc}_{2,3}(\pi - \theta) \right] &= \frac{1}{2}\text{Ls}_4(2\theta) - \text{Ls}_4(\theta) + \text{Ls}_4(\pi - \theta) - \frac{1}{2}\text{Ls}_4(\pi) , \\
6\text{Lsc}_{3,3}(\theta) + 4 \left[ \text{Lsc}_{2,4}(\theta) - \text{Lsc}_{2,4}(\pi - \theta) \right] &= \frac{1}{2}\text{Ls}_5(2\theta) - \text{Ls}_5(\theta) + \text{Ls}_5(\pi - \theta) + \frac{15}{8}\pi \zeta_4 .
\end{align*} \]  \hspace{1cm} (A.27) \]

We can see that for odd values of \( k \) one would always obtain representations of the functions Lsc\(_{(k+1)/2,(k+1)/2}(\theta)\) in terms of the Lsc functions with \( i \neq j \). This means that the functions Lsc\(_{i,j}(\theta)\) with \( i = j \) are not independent. Therefore, it is enough to consider only the functions with \( i < j \). In particular, up to the level \( k = 5 \) only two new functions are needed, in addition to ordinary Ls\(_j(\theta)\): Lsc\(_{2,3}(\theta)\) and Lsc\(_{2,4}(\theta)\).

For a particular point \( \theta = \frac{\pi}{2} \), Eqs. (A.27) yield

\[ \begin{align*}
\text{Lsc}_{2,2}\left(\frac{\pi}{2}\right) &= -\frac{1}{4}\text{Ls}_3(\pi) = \frac{1}{8}\pi \zeta_2 , \\
\text{Lsc}_{2,3}(\pi) &= -\frac{1}{2}\text{Ls}_4(\pi) = -\frac{1}{4}\pi \zeta_3 , \\
\text{Lsc}_{3,3}\left(\frac{\pi}{2}\right) &= -\frac{2}{3}\text{Lsc}_{2,4}(\pi) - \frac{1}{12}\text{Ls}_5(\pi) = -\frac{9}{32}\pi \zeta_4 ,
\end{align*} \]  \hspace{1cm} (A.28) \]
whereas for $\theta = \frac{\pi}{3}$ we get

$$
\text{Lsc}_{2,2}\left(\frac{\pi}{3}\right) = \frac{3}{4}\text{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{1}{2}\text{Ls}_3\left(\frac{\pi}{3}\right) + \frac{1}{4}\pi\zeta_2 ,
$$

$$
3 \left[\text{Lsc}_{2,3}\left(\frac{\pi}{3}\right) - \text{Lsc}_{2,3}\left(\frac{2\pi}{3}\right)\right] = \frac{3}{2}\text{Ls}_4\left(\frac{2\pi}{3}\right) - \text{Ls}_4\left(\frac{\pi}{3}\right) - \frac{3}{4}\pi\zeta_3 ,
$$

$$
6\text{Lsc}_{3,3}\left(\frac{\pi}{3}\right) + 4 \left[\text{Lsc}_{2,4}\left(\frac{\pi}{3}\right) - \text{Lsc}_{2,4}\left(\frac{2\pi}{3}\right)\right] = \frac{3}{2}\text{Ls}_5\left(\frac{2\pi}{3}\right) - \text{Ls}_5\left(\frac{\pi}{3}\right) + \frac{15}{8}\pi\zeta_4 . \quad (A.29)
$$

For these particular values of $\theta$, the PSLQ procedure yields

$$
\text{Lsc}_{2,3}\left(\frac{\pi}{3}\right) = -\frac{59}{108}\pi\zeta_3 - \frac{2}{27}\text{Ls}_4\left(\frac{\pi}{3}\right) + \frac{1}{2}\text{Ls}_4\left(\frac{2\pi}{3}\right) ,
$$

$$
\text{Lsc}_{2,3}\left(\frac{2\pi}{3}\right) = -\frac{8}{27}\pi\zeta_3 + \frac{7}{27}\text{Ls}_4\left(\frac{\pi}{3}\right) ,
$$

$$
\text{Lsc}_{2,4}\left(\frac{\pi}{3}\right) = \frac{595}{192}\pi\zeta_4 + \frac{3}{4}\zeta_2\text{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{3}{8}\pi\text{Ls}_4^{(1)}\left(\frac{2\pi}{3}\right) - \frac{1}{24}\text{Ls}_5\left(\frac{\pi}{3}\right) + \frac{3}{8}\text{Ls}_5\left(\frac{2\pi}{3}\right) + \frac{9}{32}\text{Ls}_5^{(2)}\left(\frac{2\pi}{3}\right) ,
$$

$$
\text{Lsc}_{2,4}\left(\frac{2\pi}{3}\right) = \frac{431}{432}\pi\zeta_4 + \frac{27}{8}\zeta_2\text{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{7}{8}\pi\text{Ls}_4^{(1)}\left(\frac{2\pi}{3}\right) - \frac{14}{48}\text{Ls}_5\left(\frac{\pi}{3}\right) + \frac{81}{64}\text{Ls}_5^{(2)}\left(\frac{2\pi}{3}\right) ,
$$

$$
\text{Lsc}_{3,3}\left(\frac{\pi}{3}\right) = -\frac{599}{676}\pi\zeta_4 + \frac{7}{4}\zeta_2\text{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{7}{8}\pi\text{Ls}_4^{(1)}\left(\frac{2\pi}{3}\right) - \frac{23}{72}\text{Ls}_5\left(\frac{\pi}{3}\right) + \frac{21}{32}\text{Ls}_5^{(2)}\left(\frac{2\pi}{3}\right) ,
$$

$$
\text{Lsc}_{2,3}\left(\frac{\pi}{2}\right) = \frac{1}{8}\pi\zeta_3 - \frac{1}{3}\text{Ls}_4\left(\frac{\pi}{2}\right) + 2\text{Cl}_4\left(\frac{\pi}{2}\right) . \quad (A.30)
$$

These results obey the conditions (A.28)–(A.29). However, no relation has been found for

$$
\text{Lsc}_{2,4}\left(\frac{\pi}{2}\right) \simeq 0.30945326106363459854315773429895417071 \ldots , \quad (A.31)
$$

which represents a new transcendental constant appearing at the weight-5 level of the even basis.

If we introduce a variable $z = e^{i\phi}$, we see that

$$
\ln \left(2 \sin \frac{\phi}{2}\right) \leftrightarrow \ln(1 - z) - \frac{1}{2} \ln(-z), \quad \ln \left(2 \cos \frac{\phi}{2}\right) \leftrightarrow \ln(1 + z) - \frac{1}{2} \ln z ,
$$

Therefore, the analytic continuation of $\text{Lsc}_{i,j}(\theta)$ is related to the integrals of the type

$$
\int \frac{dz}{z} \ln^\alpha(1 + z) \ln^\beta(1 - z) \ln^\gamma z .
$$

This is nothing but a particular case of the harmonic polylogarithms [72]. It is known that some of such functions of weight 4 cannot be expressed in terms of polylogarithms (or Nielsen polylogarithms) of specific types of arguments, like $\pm z$ and $\frac{1}{2}(1 \pm z)$ (see in Ref. [72]). For example, in our case the analytic continuation of $\text{Lsc}_{2,3}(\theta)$ involves an integral

$$
\int \frac{dz}{z} \ln^2(1 + z) \ln(1 - z) .
$$

**B Hypergeometric functions**

**B.1 Procedure of the $\varepsilon$-expansion**

Let us briefly describe some technical details of obtaining terms of the $\varepsilon$-expansion of hypergeometric functions $\text{2F}_1$, $\text{3F}_2$ and $\text{4F}_3$ given below. All of them belong to the type

$$
\text{pF}_Q \left( \begin{array}{c} A_1 + a_1\varepsilon, \ldots, A_P + a_P\varepsilon \\ B + \frac{1}{2} + b\varepsilon, C_1 + c_1\varepsilon, \ldots, C_{Q-1} + c_{Q-1}\varepsilon \end{array} \right) \left( \begin{array}{c} 1 \\ 4 \end{array} \right) , \quad (B.1)
$$
where \( A_j = B = 1 \), whereas \( C_j \) take the values 1 or 2. More generally, we shall assume that \( A_j, B \) and \( C_j \) are positive and integer. This function (B.1) can be presented as

\[
\sum_{j=0}^{\infty} \frac{1}{j!} \left( A_1 + a_1 \varepsilon \right) \cdots \left( A_p + a_p \varepsilon \right) \left( C_1 + c_1 \varepsilon \right) \cdots \left( C_{Q-1} + c_{Q-1} \varepsilon \right) \left( B + 1 + b \varepsilon \right)^{2j},
\]

where \((\alpha)_j \equiv \Gamma(\alpha + j)/\Gamma(\alpha)\) is the Pochhammer symbol, and we have used the duplication formula \((2\beta)_j = 4^j (\beta)(\beta + \frac{1}{2})_j\).

To perform the \( \varepsilon \)-expansion we use the well-known representation

\[
(1 + a \varepsilon)_j = j! \exp \left[ - \sum_{k=1}^{\infty} \frac{(-a \varepsilon)^k}{k} S_k(j) \right],
\]

which, for \( A > 1 \), yields

\[
(A + a \varepsilon)_j = (A)_j \exp \left\{ - \sum_{k=1}^{\infty} \frac{(-a \varepsilon)^k}{k} \left[ S_k(A + j - 1) - S_k(A - 1) \right] \right\},
\]

where \( S_k(j) = \sum_{i=1}^{j} i^{-k} \) is the harmonic sum satisfying the relation \( S_k(j) = S_k(j - 1) + j^{-k} \).

In Ref. [33] it was demonstrated that the multiple binomial sums (1.11), or their certain linear combinations, can be expressed in terms of transcendental constants elaborated in [28]. The only values of \( A, B \) and \( C \) for which each term of the \( \varepsilon \)-expansion of (B.2) can be expressed in terms of the multiple binomial sums (1.11) are \( B = 1, A = 1 \) or 2, \( C = 1 \) or 2. For other values of the indices binomial sums of a different type appear, which are connected with series like

\[
\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)! (2n + p)^q} \prod_{a,b,i,j} \left[ S_a(n - 1) \right]^i \left[ S_b(2n - 1) \right]^j,
\]

where \( p \) and \( q \) are positive integers. In such series, new transcendental constants appear. For example, for \( p = 1, q = 2, i = j = 0 \) we get (see in [70])

\[
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)! (2n + 1)^2} = \frac{8}{3} G - \frac{1}{3} \pi \ln(2 + \sqrt{3}),
\]

where \( G \) is the Catalan’s constant.

The values of multiple binomial sums (B.2) or their combinations up to weight 4 are presented in Appendix B of Ref. [33]. Most of the sums can be expressed in terms of the basis elements. There are, however, a few weight-3 and weight-4 sums which are not separately expressible, but only certain linear combinations of them, namely:

\[
\Sigma_{-1:1}^{-1}, \Sigma_{-1:2}^{-1}, \Sigma_{-1:2}^{-1} + \Sigma_{-2:2}^{-1}, \Sigma_{1:1}^{1} + \Sigma_{1:2}^{1}, \Sigma_{-1:1}^{-3} + 3 \Sigma_{-1:1,1,1}^{-1,1} + 2 \Sigma_{-1:3,1}^{-1,1}.
\]

The situation at the weight-5 level is similar. Again, most of the sums can separately be expressed in terms of the basis elements, except for a few ones, which are expressible only in certain linear combinations. There are 4 linear combinations independent of \( \chi_5 \) (3.22),

\[
\Sigma_{1:1}^{2}, \Sigma_{1:2}^{2}, \Sigma_{2:1}^{1}, \Sigma_{2:2}^{1}, \Sigma_{1:1}^{1}, \Sigma_{1:2}^{1}, \Sigma_{1:1}^{1}, \Sigma_{1:2}^{1}, 3 \Sigma_{1:1,1}^{1,1} + 2 \Sigma_{1:3,1}^{1,1},
\]

\[
\Sigma_{1:1}^{1} + 3 \Sigma_{-1:2}^{-1} + 6 \Sigma_{-1:4}^{-1} + 8 \Sigma_{-1:1,3,1}^{-1,1} + 6 \Sigma_{-1:1,2,1}^{-1,1}.
\]
and 3 combinations which involve $\chi_5$,

$$
\Sigma_{-1;3}^2 + \Sigma_{-1;2;3}^1, \quad \Sigma_{-1;2}^3 + 2\Sigma_{-1;3}^1 + 3\Sigma_{-1;1;2;2}^1, \quad \Sigma_{1;1;2}^1 + \Sigma_{1;2}^1.
$$

Using relation (1.9) and the results of Ref. [33], it is natural to expect that the sums (1.11), not only with $k = 1$ but also with $k = 3$, are connected with the odd basis, whereas the sums (1.11) with $k = 2$ and $k = 4$ should be associated with the even basis.

To check this conjecture, we have performed PSLQ-analysis of the sums (1.11) with $k = 2$ up to weight $5^{23}$. At the weight-$5$ level, one needs to add two new elements to the even basis of [28]. One of them can be associated with $L_{s,4}^2 \left( \frac{\pi}{2} \right)$ (see Eq. (A.31) in Appendix A.2), whereas the second one, $\chi_5$, is given in Eq. (3.28). In this way we restore, accidentally, the “empirical” relation $N_j = 2^j$ for the $j = 5$ level of the even basis. Then, we find that the expressions for the sums (1.11) with $k = 2$, contain all elements of the even basis$^{24}$. However, the question about possibility to express all elements of the even basis in terms of only binomial sums (1.11) is still open. As in the $k = 1$ case, most of the $k = 2$ sums are separately expressible in terms of the even basis elements, except for the same set of sums, which can be expressed only in certain linear combinations$^{25}$: namely, for the weights $3$ and $4$ the combinations given in Eq. (B.5) and at the weight $5$ the familiar combinations (B.6) and (B.7), everything with $k = 2$. Worth noting is that the combinations (B.6) do not contain $\chi_5$, whereas the sums (B.7) do not involve $L_{s,4}^2 \left( \frac{\pi}{2} \right)$. In total, we have found 8 elements which involve $\chi_5$: besides the combinations given in (B.7) (with $k = 2$) and the sums $\Sigma_{1;1;2}^2(2)$, $\Sigma_{1;2}^3(2)$, $\Sigma_{1;1;3}^2(2)$ (which have already occurred in the $k = 1$ case), there are two other sums, $\Sigma_{1;1;4}^2(2)$ and $\Sigma_{1;1;4}^2(2)$ (for $k = 1$ these two sums are expressible in terms of $\zeta_5$, $\zeta_2 \zeta_3$ and $\pi L_s^4 \left( \frac{\pi}{2} \right)$).

We would like to mention some issues related to the high-precision calculation$^{26}$ of the elements of our basis, needed for the PSLQ-analysis. Most of the constants of the basis up to weight $5$ can be produced, with desirable accuracy of 1000 or more digits, using the algorithm described in Appendix A of Ref. [33]. The $\chi_5$ and $\bar{\chi}_5$ constants are originally defined in terms of rapidly-convergent series, see Eqs. (3.22) and (3.28). Therefore, only three constants, $L_s^{(2)} \left( \frac{2\pi}{3} \right)$, $L_{s,4}^2 \left( \frac{\pi}{2} \right)$ and $L_s^{(2)} \left( \frac{\pi}{2} \right)$, may require additional consideration. In principle, the corresponding binomial sums may be used for precise numerical calculation. However, specifically for $L_s^{(2)} \left( \frac{2\pi}{3} \right)$, the following representation in terms of inverse tangent

$^{23}$The values of some sums (1.11) with $k = 4$ up to weight $3$ can be extracted from the results of [71].

$^{24}$This is connected with the fact that both $L_{s,j}^{(1)} \left( \frac{\pi}{2} \right)$ and $L_{s,j}^{(1)}(\pi)$, for $j = 4, 5$, are expressible in terms of the same constants (see Appendix A.1).

$^{25}$Such interrelation is not trivial. For example, for $k = 3$ the number of such combinations of sums is reduced, so that the first irreducible combination arises only at the weight $4$. Besides this, the element $\chi_5$ is not reproduced by series like (3.22) with $k = 3$.

$^{26}$All such computations have been performed with the help of David Bailey’s MPFUN routines. The latest version of these codes, including the documentation and original papers, is available from http://www.nersc.gov/~dhb/mpdist/mpdist.html.
integral (see in [11]) happens to be very useful:

\[
\operatorname{Ti}_5\left(\frac{1}{\sqrt{3}}\right) \\
\equiv \frac{1}{\sqrt{3}} \sum_{j=0}^{\infty} \frac{(-1)^j}{3^j (2j+1)^3} = \frac{1}{2304} \ln^4 3 + \frac{1}{18} \ln^2 3 + \frac{59}{144} \ln 3 + \frac{1273}{2304} \ln 4
\]

Expanding and using (arcsin \(y\) \(\equiv\) 1) in (A.6) happens to be very useful:

\[
\sin \phi \\
\equiv \sqrt{1 - 3} \sum_{k=0}^{\infty} \frac{\pi^j L_2 \left(\frac{\pi}{3}\right)}{5^{2k} L_4 \left(\frac{\pi}{3}\right)} \ln 3 - \frac{5}{48} \ln 4 \left(\frac{\pi}{3}\right) \ln 3
\]

\[
\equiv \frac{7}{16} \pi \ln 2 \left(\frac{\pi}{3}\right) + \frac{11}{288} \ln 5 \left(\frac{\pi}{3}\right) + \frac{5}{96} \ln 5 \left(\frac{\pi}{3}\right) + \frac{7}{32} \pi \ln 4 \left(\frac{\pi}{3}\right) - \frac{21}{128} \ln 5 \left(\frac{\pi}{3}\right) (\text{B.8})
\]

For numerical calculation of \(L_{j}^{(k)}(\theta)\) the following algorithm is useful. Substituting \(y = \sin \frac{\theta}{2}\), we get from the original integral representation (A.6)

\[
L_{j}^{(k)}(\theta) = -2^{k+1} \int_0^{\sin(\theta/2)} \frac{dy}{\sqrt{1-y^2}} (\arcsin y)^k \ln^{j-k-1}(2y).
\]

Expanding

\[
\frac{1}{\sqrt{1-y^2}} = \sum_{r=0}^{\infty} \frac{(2r)! y^{2r}}{(r!)^2 4r}
\]

and \((\arcsin y)^k\) (see below) in \(y\) and using

\[
\int x^a \ln^b x \, dx = (-1)^b b! x^{a+1} \sum_{p=0}^{b} \frac{(-\ln x)^{b-p}}{(b-p)!(a+1)^{p+1}},
\]

we obtain the multiple series representation suitable for numerical evaluation.

The expansion of \((\arcsin y)^k\) is discussed in Ref. [70]. The generating expression is \(\exp(a \arcsin y) = \sum_{p=0}^{\infty} (b_p y^p/p!)\), where \(b_0 = 1, b_1 = a\) and

\[
b_{2k+1} = a(a^2+1)(a^2+3^2) \cdots (a^2+(2k-1)^2), \quad k \geq 1;
\]

\[
b_{2k} = a^2(a^2+2^2)(a^2+4^2) \cdots (a^2+(2k-2)^2), \quad k \geq 1.
\]

Expanding \(\exp(a \arcsin y)\) in a power series in \(a\) and equating coefficients of \(a^k\) on both sides, the Taylor series for \((\arcsin y)^k\) can be deduced. For example,

\[
\frac{1}{2!} (\arcsin y)^2 = \sum_{r=0}^{\infty} \frac{4^r (r!)^2 y^{2r+2}}{(2r+2)!},
\]

\[
\frac{1}{3!} (\arcsin y)^3 = \sum_{r=1}^{\infty} \left(1 + \frac{1}{3^2} + \cdots + \frac{1}{(2r-1)^2}\right) \frac{(2r)!}{(r!)^2} \frac{y^{2r+1}}{4^r (2r+1)!},
\]

\[
\frac{1}{4!} (\arcsin y)^4 = \sum_{r=1}^{\infty} \left(1 + \frac{1}{4^2} + \cdots + \frac{1}{(2r)^2}\right) \frac{4^r (r!)^2 y^{2r+2}}{(2r+2)!}.
\]
B.2 The $2F_1$ function

Here we present some relevant results for the hypergeometric function of the type

$$2F_1 \left( \frac{1}{2} + a_1 \varepsilon, 1 + a_2 \varepsilon \mid \frac{z}{2} + b \varepsilon \right). \quad (B.9)$$

Especially we are interested in its expansion in $\varepsilon$, which basically corresponds to the calculation of the derivatives of $2F_1$ function with respect to the parameters.

There are some special cases when the result is known exactly. For instance, when $a_1 = -a_2 \equiv a$ and $b = 0$ we have (see Eq. (94) in p. 469 of [45]),

$$2F_1 \left( \frac{1}{2} + a \varepsilon, 1 - a \varepsilon \mid \sin^2 \theta \right) = \frac{\sin (2a \varepsilon \theta)}{a \varepsilon \sin (2\theta)}. \quad (B.10)$$

When $a_1 = b = 0$ ($a_2 \equiv a$) we get (see Eqs. (2.7)–(2.8))

$$2F_1 \left( \frac{1}{2} + \frac{a \varepsilon}{2}, \frac{1}{2} + \frac{a \varepsilon}{2} \mid \sin^2 \theta \right) = \frac{1 + a \varepsilon}{2 \sin (2\theta)} \sum_{j=0}^{\infty} \frac{(a \varepsilon)^j}{j!} [L_{s_j+1}(\pi - 2\theta) - L_{s_j+1}(\pi)] , \quad (B.11)$$

i.e., we know all orders of the $\varepsilon$-expansion, in terms of the log-sine integrals (A.5). For $z = \frac{1}{4}$ ($\theta = \frac{\pi}{6}$) we get Eq. (4.16). Another interesting case is $a_1 = 0$, $a_2 = 2b \equiv a$, for which we find

$$2F_1 \left( \frac{1 + a \varepsilon}{2}, 1 + a \varepsilon \mid \sin^2 \theta \right) = -\frac{1 + a \varepsilon}{2 \sin (2\theta)} \sum_{j=0}^{\infty} \frac{(a \varepsilon)^j}{j!} L_{s_j+1}(4\theta). \quad (B.12)$$

Moreover, in a more general case, when $a_1 = 0$ while $a_2 \equiv a$ and $b$ are arbitrary, the $2F_1$ function (B.9) can be reduced to an incomplete Beta-function (see in [45]). As a result, we get the integral representation

$$2F_1 \left( \frac{1 + a \varepsilon}{2}, 1 + a \varepsilon, 1 + a \varepsilon \mid \frac{z}{2} + b \varepsilon \right) = \frac{1 + 2b \varepsilon}{(\sin \theta)^{1+2b \varepsilon} (\cos \theta)^{1+2a \varepsilon - 2b \varepsilon}} \int_0^\theta d\phi (\sin \phi)^{2b \varepsilon} (\cos \phi)^{2a \varepsilon - 2b \varepsilon} . \quad (B.13)$$

The $\varepsilon$-expansion can be written in terms of Lsc function (see Appendix A.2).

There are some further results available for the case when $z = \frac{1}{4}$ ($\theta = \frac{\pi}{6}$). When $a_1 = b \equiv a$ and $a_2 = \frac{3}{2} a$ we get

$$2F_1 \left( \frac{1}{2} + \frac{a \varepsilon}{2}, 1 + \frac{3}{2} a \varepsilon \mid \frac{1}{4} \right) = \frac{2\pi}{3^2 (1 + a \varepsilon) \Gamma(1 + a \varepsilon) \Gamma(1 + \frac{3}{2} a \varepsilon)} \frac{\Gamma(2 + 2a \varepsilon)}{\Gamma(1 + a \varepsilon) \Gamma^2(1 + \frac{1}{2} a \varepsilon)} . \quad (B.14)$$

Using the procedure described in Appendix B.1, we obtain a few terms of the $\varepsilon$-expansion for general values of $a_1$, $a_2$ and $b$:

$$2F_1 \left( 1 + a_1 \varepsilon, 1 + a_2 \varepsilon \mid \frac{1}{4} \right) = \frac{2(1 + 2b \varepsilon)}{3^2 + A_1 \varepsilon - b \varepsilon}$$
× \left\{ \frac{1}{3} \pi + \varepsilon (2A_1 - 5b) \sum_{j=0}^{\infty} \frac{(-2\varepsilon)^j}{(j+1)!} (b - A_1)^j \left[ \text{Ls}_{j+2}(\frac{2\varepsilon}{3}) - \text{Ls}_{j+2}(\pi) \right] \\
+ \varepsilon^2 \left[ \frac{19}{18} b^2 - \frac{1}{34} (A_1^2 - A_2) - \frac{\pi}{18} b A_1 \right] \pi \zeta_2 + \varepsilon^3 \left[ \frac{1}{9} (A_1^2 - A_2) (2A_1 - 5b) \zeta_2 \text{Ls}_2 \left( \frac{\pi}{3} \right) \\
+ \left( \frac{103}{162} A_1 A_1^2 - \frac{17}{27} b^3 - \frac{541}{162} b A_1^2 + \frac{409}{162} b A_2 + \frac{13}{9} b^2 A_1 \right) \pi \zeta_3 \\
+ \left( -\frac{32}{81} (2A_1 - 5b) (A_1^2 - A_2) + \frac{4}{9} b (6A_2 + 9b A_1 - 42b^2) \right) \text{Ls}_4 \left( \frac{\pi}{3} \right) \right] \\
+ \varepsilon^4 \left[ \frac{11}{27} (A_1^2 - A_2) (2A_1 - 5b) (11A_1 - 35b) \zeta_3 \text{Ls}_2 \left( \frac{\pi}{3} \right) - \frac{1}{9} (A_1^2 - A_2) (2A_1 - 5b)^2 \varepsilon \left[ \text{Ls}_2 \left( \frac{\pi}{3} \right) \right]^2 \\
+ \frac{1}{9} (b - A_1) \left( -\frac{17}{27} (2A_1 - 5b) (A_1^2 - A_2) + 6b A_2 + 9b^2 A_1 - 42b^3 \right) \zeta_2 \text{Ls}_3 \left( \frac{2\pi}{3} \right) \\
- \frac{1}{6} (b - A_1) \left( -8 (2A_1 - 5b) (A_1^2 - A_2) + 6b A_2 + 9b^2 A_1 - 42b^3 \right) \left[ \pi \text{Ls}_4 \left( \frac{2\pi}{3} \right) - \frac{3}{4} \text{Ls}_5 \left( \frac{2\pi}{3} \right) \right] \\
+ \left( \frac{209}{36} b^4 - \frac{1753}{144} b^3 A_1 + \frac{10399}{648} b^2 A_2 - \frac{11069}{1296} b^2 A_1^2 + \frac{10031}{648} b A_1^3 - \frac{10799}{648} b A_1 A_2 \\
+ \frac{305}{72} A_1^2 A_2 - \frac{5491}{1296} A_1^4 + \frac{1}{1296} A_2^2 \right) \pi \zeta_4 + \left( \frac{11}{9} b^4 - \frac{53}{18} b^3 A_1 + \frac{293}{84} b^2 A_2 - \frac{433}{162} b^2 A_1^2 \\
+ \frac{379}{81} b A_1^3 - \frac{361}{81} b A_1 A_2 - \frac{104}{81} (A_1^2 - A_2) A_1^2 \right) \text{Ls}_5 \left( \frac{\pi}{3} \right) \right] + \mathcal{O} (\varepsilon^5) \right\},
(B.15)

where $A_j = \sum_{i=1}^{2} a_i^j$, and we take into account that there are some relations among $A_j$,

$$2A_3 + A_1^3 - 3A_2 A_1 = 0, \quad 2A_4 + A_1^4 - 2A_1^2 A_2 - A_2^2 = 0.$$  

At the weight-5 level of the odd basis, we have observed that the constants $\pi \text{Ls}_4 \left( \frac{2\pi}{3} \right)$ and $\text{Ls}_5 \left( \frac{2\pi}{3} \right)$ appear only in the combination $\left[ \pi \text{Ls}_4 \left( \frac{2\pi}{3} \right) - \frac{3}{4} \text{Ls}_5 \left( \frac{2\pi}{3} \right) \right]$. Moreover, this combination arises only in the sums like $\sum_{a_i b_1}$. It is natural to expect that similar “junctions” also happen for higher generalized log-sine integrals.

We also mention some useful results for the contiguous $2F_1$ functions. One of them, Eq. (4.42), is rather essential for the results of this paper. Another interesting case,

$$2F_1 \left( \begin{array}{c} \varepsilon, 1 - 3\varepsilon \\ \frac{3}{2} - 2\varepsilon \end{array} \bigg| \frac{1}{4} \right) = \frac{2^{1-4\varepsilon} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{3}{2} - 2\varepsilon \right)}{3^{1-3\varepsilon} \Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{4}{3} - 2\varepsilon \right)},
(B.16)$$

can be obtained by a simple transformation of Eq. (114) on p. 461 of [45].

### B.3 The $3F_2$ function

Here we present some results for the hypergeometric function of the type

$$3F_2 \left( \begin{array}{c} 1 + a_1 \varepsilon, 1 + a_2 \varepsilon, 1 + a_3 \varepsilon \\ \frac{3}{2} + b \varepsilon, C + \varepsilon \end{array} \bigg| z \right),
(B.17)$$

with $C = 1$ or $C = 2$.  

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There are a few special cases, when the considered function (B.17) can be expressed in terms of \( _2F_1 \) functions. First of all, when \( C = 1 \) and \( c \) is equal to one of the \( a_i \) (for example \( c = a_3 \)), we obtain the function (B.9). Another reduction case is \( C = 2 \) and \( c = a_3 = 0 \) (see Eq. (7) in p. 497 of [45] and Eq. (4.8) of this paper).

There is also a less trivial relation, valid for the case when \( C = 2 \) and \( c = 2b = 2a_3 = a_1 + a_2 \) (see, e.g., Eq. (22) on p. 498 of [45]),

\[
3F_2 \left( \frac{1}{3} + a_1 \varepsilon, 1 + a_2 \varepsilon, 1 + \frac{1}{2}(a_1 + a_2)\varepsilon \bigg| z \right) = (1-z) \left[ 2F_1 \left( \frac{1}{3} + \frac{1}{2}a_1 \varepsilon, 1 + \frac{1}{2}a_2 \varepsilon \bigg| z \right) \right]^2,
\]

which reduces the \( 3F_2 \) function to a square of the \( 2F_1 \) function of the type (B.9).

Another interesting relation (which follows from Eq. (20) on p. 498 of [45]) reads

\[
3F_2 \left( \frac{1}{3} + \frac{1}{2}(a_1 + a_2)\varepsilon, 1 + (a_1 + a_2)\varepsilon \bigg| z \right) = 2F_1 \left( \frac{1}{3} + \frac{1}{2}a_1 \varepsilon, 1 + \frac{1}{2}a_2 \varepsilon \bigg| z \right)
\times \left\{ 1 + \frac{a_1 a_2 \varepsilon^2}{2(1+(a_1 + a_2)\varepsilon)} \right\} 3F_2 \left( \frac{1}{3} + \frac{1}{2}a_1 \varepsilon, 1 + \frac{1}{2}a_2 \varepsilon, 1 \bigg| z \right).
\]

This relation gives a possibility to cross-check results for the \( \varepsilon \)-expansion of both \( 3F_2 \) functions presented below, together with the result for the \( 2F_1 \) function.

We note that the results for some other values of the parameters can be obtained using relations between contiguous \( 3F_2 \) functions, like (see Eq. (26) on p. 440 of [45])

\[
(1 + a_3 \varepsilon) 3F_2 \left( \frac{1}{3} + a_1 \varepsilon, 1 + a_2 \varepsilon, 2 + a_3 \varepsilon \bigg| z \right) = (1 + c \varepsilon) 3F_2 \left( \frac{1}{3} + a_1 \varepsilon, 1 + a_2 \varepsilon, 1 + a_3 \varepsilon \bigg| z \right) + (a_3 - c) \varepsilon 3F_2 \left( \frac{1}{3} + a_1 \varepsilon, 1 + a_2 \varepsilon, 1 + a_3 \varepsilon \bigg| z \right).
\]

This connection makes it possible to use, as a check, another special case (see case (iii) in Table I of [73]),

\[
3F_2 \left( \frac{1}{3} + \frac{1}{2}(a_1 + a_2)\varepsilon, 2 + \frac{1}{2}(a_1 - a_2)\varepsilon \bigg| \frac{1}{4} \right) = \frac{2^{2+\frac{2}{3}a_1 \varepsilon} \Gamma \left( \frac{1}{2} + \frac{1}{2}(a_1 + a_2)\varepsilon \right) \Gamma \left( 2 + \frac{1}{2}(a_1 - a_2)\varepsilon \right)}{\Gamma (1 + a_1 \varepsilon) \Gamma \left( 1 + \frac{1}{6}(a_1 - 3a_2)\varepsilon \right) \Gamma \left( \frac{1}{6} + \frac{1}{6}(a_1 + 3a_2)\varepsilon \right)} \times \left\{ \Gamma \left( \frac{1}{2} + \frac{1}{2}(a_1 - a_2)\varepsilon \right) \Gamma \left( \frac{1}{6} + \frac{1}{6}(a_1 + 3a_2)\varepsilon \right) \right\}.
\]

Again, using the procedure described in Appendix A.1, we obtain a few terms of the \( \varepsilon \)-expansions for general values of \( a_i \), \( b \) and \( c \). For the case \( C = 2 \) we find

\[
3F_2 \left( \frac{1}{3} + b \varepsilon, 2 + c \varepsilon \bigg| \frac{1}{4} \right) = 2(1 + 2b \varepsilon)(1 + c \varepsilon)
\]
\[
\frac{1}{72} \zeta_2 + \varepsilon \left[ \frac{1}{15} \zeta_3 \left( 11 A_1 - 35 b + c \right) - \frac{2}{9} \pi L s_2 \left( \frac{2 \pi}{4} \right) \left( 2 A_1 - 5 b + c \right) \right] \\
+ \varepsilon^2 \left[ \frac{1}{3} \pi L s_3 \left( \frac{2 \pi}{3} \right) \left( 2 A_1 - 5 b + c \right) (A_1 - b - c) - \frac{2}{9} \left[ L s_2 \left( \frac{2 \pi}{3} \right) \right]^2 \left( 2 A_1 - 5 b + c \right) \right] \\
+ \zeta_4 \left( -\frac{139}{9} b A_1 - \frac{109}{54} c A_1 + \frac{377}{108} b c + \frac{160}{9} b^2 - \frac{1108}{216} c^2 + \frac{1085}{216} A_1^2 - \frac{55}{216} A_2 \right) \\
+ \varepsilon^3 \left( \frac{1}{15} \chi_5 \left( 2 A_1 - 5 b + c \right) (2 A_1 - 5 b - 2 c) (A_1 - b - c) - \frac{1}{3} \pi L s_4 \left( \frac{2 \pi}{3} \right) b \left( 2 A_1 - 5 b + c \right) (A_1 - b - c) \right. \\
- \frac{1}{102} \pi \zeta_4 L s_4 \left( \frac{2 \pi}{3} \right) (2 A_1 - 5 b + c) \left( 109 b^2 - 23 b A_1 + 23 b c + 2 c^2 - 2 A_2 \right) \\
+ \zeta_4 \left( \frac{433}{102} b A_1^2 - \frac{2095}{648} b c A_1 - \frac{69}{8} b^2 A_1 + \frac{13}{36} c A_1^2 + \frac{23}{162} A_1 A_2 - \frac{167}{162} b A_2 + \frac{31}{162} b c A_2 \right. \\
+ \frac{8}{27} A_3 - \frac{13}{81} c^2 A_1 - \frac{773}{648} b c^2 + \frac{833}{108} b^2 c + \frac{3361}{648} b^3 + \frac{32}{32} c^3) \\
+ \zeta_5 \left( \frac{1093}{72} b^2 A_1 - \frac{5606}{648} b c A_1 - \frac{529}{102} b A_1^2 + \frac{47}{36} c A_1^2 + \frac{17}{81} c A_2 + \frac{53}{162} A_1 A_2 + \frac{58}{27} A_3 \right. \\
+ \frac{164}{81} c^2 A_1 - \frac{1271}{102} b b A_2 - \frac{4217}{648} b c^2 + \frac{1445}{108} b^2 c - \frac{22907}{648} b^3 + \frac{103}{324} c^3) \\
+ \pi L s_4 \left( \frac{2 \pi}{3} \right) \left( \frac{259}{486} b c A_1 - \frac{41}{18} b^2 A_1 + \frac{116}{243} b A_1^2 - \frac{20}{243} c^2 A_1 + \frac{136}{243} b A_2 - \frac{8}{243} c A_2 \right. \\
- \frac{4}{27} A_3 - \frac{10}{243} A_1 A_2 - \frac{1}{9} c A_1^2 - \frac{53}{486} b c^2 - \frac{29}{81} b^2 c + \frac{911}{486} b^3 - \frac{7}{243} c^3) \right) \\
+ O(\varepsilon^4) \right\}.
\] (B.22)

where \( A_j \equiv \sum_{i=1}^{3} a^i_j \), while \( \chi_5 \) is defined in Eq. (3.22). In particular, we have checked that this result obeys the condition (B.18), where the \( \varepsilon \)-expansion of the resulting \( 2F_1 \) function is given in Eq. (B.15)).

For another case of interest, \( C = 1 \), we find

\[
3 \text{F}_2 \left( 1 + a_1 \varepsilon, 1 + a_2 \varepsilon, 1 + a_3 \varepsilon \left| \frac{1}{2} + b \varepsilon, 1 + c \varepsilon \right. \right) = \frac{2(1 + 2b \varepsilon)}{3^2 + A_1 \varepsilon - b \varepsilon - c \varepsilon} \\
\times \left\{ \frac{1}{72} \pi + \varepsilon (2 A_1 - 5 b - 2 c) \sum_{j=0}^{\infty} \frac{(2 \varepsilon)^j}{(j + 1)!} (A_1 - b - c)^j \left[ L s_{j+2} \left( \frac{2 \pi}{3} \right) - L s_{j+2} (\pi) \right] \\
+ \varepsilon^2 \pi \zeta_2 \left[ A_1^2 - (A_1 + c - A_1)(15b + 2c) + 57b^2 \right] \\
+ \varepsilon^3 \left[ \pi \zeta_3 \left( \frac{541}{81} A_1 b c + \frac{13}{9} A_1 b^2 + \frac{242}{81} A_1 c^2 - \frac{49}{162} A_1 A_2 - \frac{541}{162} A_1^2 b - \frac{445}{324} A_1 c + \frac{1445}{162} A_1^3 \right) \\
- \frac{475}{81} b c^2 + \frac{409}{162} b A_2 - \frac{13}{9} b^2 c - \frac{17}{27} b^3 + \frac{49}{486} c A_2 - \frac{49}{81} c^3 - \frac{16}{27} A_3 \right) \right. \\
+ \zeta_4 L s_2 \left( \frac{2 \pi}{3} \right) \left( \frac{10}{9} A_1 b c + \frac{2}{9} A_1 c^2 - \frac{1}{9} A_1 A_2 - \frac{2}{9} A_1^2 b - \frac{3}{9} A_1^2 c + \frac{1}{9} A_1^3 - \frac{10}{9} b c^2 + \frac{2}{9} b A_2 - \frac{1}{9} c A_2 + \frac{2}{9} c^3 - \frac{2}{9} A_3 \right) \\
+ \zeta_4 L s_4 \left( \frac{2 \pi}{3} \right) \left( \frac{4}{9} A_1 b^2 - \frac{320}{81} A_1 b c - \frac{148}{81} A_1 c^2 + \frac{10}{81} A_1 A_2 + \frac{160}{81} A_1^2 b + \frac{46}{27} A_1^2 c - \frac{46}{81} A_1^3 \right. \\
+ \frac{296}{81} b c^2 - \frac{136}{81} b A_2 - \frac{1}{9} b^2 c - \frac{56}{27} b^3 - \frac{10}{81} c A_2 + \frac{20}{81} c^3 + \frac{4}{9} A_3 \right) \right. \\
+ \zeta_4 L s_3 \left( \frac{2 \pi}{3} \right) \left( A_1 - b - c \right) \left( 4 A_1^3 - \frac{85}{6} A_1 b - 12 A_1^2 c + \frac{85}{3} A_1 b c - 3 A_1 b^2 \\
- \frac{2}{3} A_1 A_2 + \frac{38}{3} A_1 c^2 - \frac{79}{3} b c^2 + 3 b^2 c + 14 b^3 + \frac{2}{3} c A_2 + \frac{73}{6} b A_2 - \frac{10}{3} A_3 - \frac{4}{3} c^3 \right) \right. \\
\]
Finally, we present some results for the hypergeometric function $4\!F_3$ in p. 497 of [45]),

$$
\varepsilon(b-a) \ 4\!F_3 \left( \begin{array}{c} 1+a_1 \varepsilon, 1+a_2 \varepsilon, 1+a_3 \varepsilon, 1+a_4 \varepsilon \\ \frac{3}{2}+b \varepsilon, 2+a_1 \varepsilon, 2+a_2 \varepsilon \\
\end{array} \right) z
$$

(B.24)

There is an interesting special case, $c_1 = a_1$ and $c_2 = a_2$, when this function (B.24) reduces to a combination of two $3\!F_2$ functions of the type (B.17) with $C = 2$ (see, e.g., Eq. (4) in p. 497 of [45]),

$$
\varepsilon(a_2-a_1) \ 3\!F_2 \left( \begin{array}{c} 1+a_1 \varepsilon, 1+a_2 \varepsilon, 1+a_3 \varepsilon, 1+a_4 \varepsilon \\ \frac{3}{2}+b \varepsilon, 2+a_1 \varepsilon, 2+a_2 \varepsilon \\
\end{array} \right) z
$$

(B.25)
where $A_j \equiv \sum_{i=1}^{4} a^i_j$, $C_j \equiv \sum_{i=1}^{2} c^i_j$, while $\chi_5$ is defined in Eq. (3.22).

References

[1] G. 't Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189;
C.G. Bollini and J.J. Giambiagi, Nuovo Cimento 12B (1972) 20;
J.F. Ashmore, Lett. Nuovo Cim. 4 (1972) 289;
G.M. Cicuta and E. Montaldi, Lett. Nuovo Cim. 4 (1972) 329.

[2] F.V. Tkachov, Phys. Lett. B100 (1981) 65;
K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159.

[3] O.V. Tarasov, Phys. Rev. D54 (1996) 6479.

[4] L. Euler, Novi Comm. Acad. Sci. Petropol. 20 (1775) 140;
D. Zagier, Values of zeta functions and their generalizations, in: A. Joseph et al., ed.: Proceedings of the First European Congress of Mathematics, Paris, vol. II, Birkhäuser, 1994 (Progress in Mathematics, vol. 120), p. 497–512.

[5] D.H. Bailey, J.M. Borwein and R. Girgensohn, Experimental Math. 3 (1994) 17;
D. Borwein, J.M. Borwein and R. Girgensohn, Proc. Edinburgh Math. Soc. 38 (1995) 277;
R.E. Crandall and J. Buhler, Experimental Math. 4 (1994) 275;
J.M. Borwein and R. Girgensohn, Electronic J. Combinatorics 3 (1996) R23;
M.E. Hoffman and C. Moen, J. Number Theory 60 (1996) 329;
J.M. Borwein, D.M. Bradley and D.J. Broadhurst, Electronic J. Combinatorics 4 (1997) R5 (hep-th/9611004);
P. Flajolet and B. Salvy, Experimental Math. 7 (1998) 15;
O.M. Ogreid and P. Osland, J. Comput. Appl. Math. 98 (1998) 245.

[6] D. Kreimer, Phys. Lett. B354 (1995) 117; J. Knot Th. Ram. 6 (1997) 479.
[7] N. Gray, D.J. Broadhurst, W. Grafe and K. Schilcher, Z. Phys. \textbf{C48} (1990) 673;
S. Laporta and E. Remiddi, Phys. Lett. \textbf{B379} (1996) 283;
K. Melnikov and T. van Ritbergen, Phys. Lett. \textbf{B482} (2000) 99; Nucl. Phys. \textbf{B591} (2000) 515.

[8] D.J. Broadhurst, Open University preprint OUT-4102-62 (hep-th/9604128).

[9] J. van der Bij and M. Veltman, Nucl. Phys. \textbf{B231} (1984) 205;
C. Ford, I. Jack and D.R.T. Jones, Nucl. Phys. \textbf{B387} (1992) 373; B \textbf{504} (1997) 551(E).

[10] A.I. Davydychev and J.B. Tausk, Nucl. Phys. \textbf{B397} (1993) 123.

[11] L. Lewin, \textit{Polylogarithms and associated functions} (North-Holland, Amsterdam, 1981).

[12] W. Celmaster and R.J. Gonsalves, Phys. Rev. \textbf{D20} (1979) 1420.

[13] D.J. Broadhurst, Eur. Phys. J. \textbf{C8} (1999) 311.

[14] S. Groote, J. G. Körner and A. A. Pivovarov, Phys. Rev. \textbf{D60} (1999) 061701.

[15] N.I. Ussyukina and A.I. Davydychev, Phys. Lett. \textbf{B332} (1994) 159.

[16] D.H. Bailey and D.J. Broadhurst, math.NA/9905048.

[17] J.M. Borwein, D.J. Broadhurst and J. Kamnitzer, Experimental Math. \textbf{10} (2001) 25
(hep-th/0004153).

[18] J.M. Borwein, D.M. Bradley, D.J. Broadhurst and P. Lisonek, Trans. Amer. Math. Soc. \textbf{353} (2001) 907
(math.CA/9910045).

[19] D.J. Broadhurst and D. Kreimer, Int. J. Mod. Phys. \textbf{C6} (1995) 519; Phys. Lett. \textbf{B393}
(1997) 403.

[20] O. Schnetz, hep-th/9912149.

[21] D.J. Broadhurst, J.A. Gracey and D. Kreimer, Z. Phys. \textbf{C75} (1997) 559.

[22] A.I. Davydychev and R. Delbourgo, J. Math. Phys. \textbf{39} (1998) 4299.

[23] A.I. Davydychev, Proc. Workshop “AIHENP-99”, Heraklion, Greece, April 1999
(Parisianou S.A., Athens, 2000), p. 219 (hep-th/9908032).

[24] A.I. Davydychev, Phys. Rev. \textbf{D61} (2000) 087701.

[25] A.I. Davydychev and M.Yu. Kalmykov, Nucl. Phys. B (Proc. Suppl.) \textbf{89} (2000) 283
(hep-th/0005287).

[26] A.I. Davydychev and J.B. Tausk, Phys. Rev. \textbf{D53} (1996) 7381.
[27] J. Fleischer, M.Yu. Kalmykov and A.V. Kotikov, Phys. Lett. B462 (1999) 169; B467 (1999) 310(E).

[28] J. Fleischer and M. Yu. Kalmykov, Phys. Lett. B470 (1999) 168.

[29] H.R.P. Ferguson and D.H. Bailey, RNR Technical Report, RNR-91-032 (1991); H.R.P. Ferguson, D.H. Bailey, and S. Arno, Math. Comput. 68 (1999) 351; D.H. Bailey, Computing in Science & Engineering 2, No. 1 (2000) 24.

[30] K.G. Chetyrkin and M. Steinhauser, Nucl. Phys. B573 (2000) 617.

[31] L.V. Avdeev, Comput. Phys. Commun. 98 (1996) 15.

[32] A.V. Kotikov and L.N. Lipatov, Nucl. Phys. B582 (2000) 19.

[33] M.Yu. Kalmykov and O. Veretin, Phys. Lett. B483 (2000) 315.

[34] F.A. Berends, A.I. Davydychev and V.A. Smirnov, Nucl. Phys. B478 (1996) 59.

[35] K.S. Köhlbig, J.A. Mignaco and E. Remiddi, B.I.T. 10 (1970) 38; R. Barbieri, J.A. Mignaco and E. Remiddi, Nuovo Cim. A11 (1972) 824; A. Devoto and D.W. Duke, Riv. Nuovo Cim. 7, No.6 (1984) 1; K.S. Köhlbig, SIAM J. Math. Anal. 17 (1986) 1232.

[36] J. Fleischer, F. Jegerlehner, O.V. Tarasov and O.L. Veretin, Nucl. Phys. B539 (1999) 671; B571 (2000) 511(E).

[37] A.I. Davydychev, J. Phys. A25 (1992) 5587.

[38] O.V. Tarasov, Nucl. Phys. B (Proc. Suppl.) 89 (2000) 237 (hep-ph/0102271)

[39] L.G. Cabral-Rosetti and M.A. Sanchis-Lozano, J. Comput. Appl. Math. 115 (2000) 93 (hep-ph/9809213).

[40] B.G. Nickel, J. Math. Phys. 19 (1978) 542.

[41] G. ’tHooft and M. Veltman, Nucl. Phys. B153 (1979) 365.

[42] U. Nierste, D. Müller and M. Böhm, Z. Phys. C57 (1993) 605.

[43] A.I. Davydychev, Phys. Lett. B263 (1991) 107.

[44] E.E. Boos and A.I. Davydychev, Teor. Mat. Fiz. 89 (1991) 56 [Theor. Math. Phys. 89 (1991) 1052].

[45] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, Integrals and Series, v.3: More Special Functions, Gordon and Breach, New York, 1990.

[46] J.A.M. Vermaseren, Symbolic manipulation with FORM, Amsterdam, Computer Algebra Nederland, 1991.
[47] J. Fleischer and M.Yu. Kalmykov, Comput. Phys. Commun. 128 (2000) 531.

[48] M. Steinhauser, Comput. Phys. Commun. 134 (2001) 335.

[49] R. Harlander and M. Steinhauser, Prog. Part. Nucl. Phys. 43 (1999) 167.

[50] P.A. Baikov, Phys. Lett. B385 (1996) 404; A.G. Grozin, J. High Energy Phys. 03 (2000) 013.

[51] J.M. Chung and B.K. Chung, Phys.Rev. D59 (1999) 105014; A.V. Kotikov, J. High Energy Phys. 09 (1998) 001.

[52] L.V. Avdeev, J. Fleischer, M.Yu. Kalmykov and M.N. Tentyukov, Comput. Phys. Commun. 107 (1997) 155; L.V. Avdeev and M.Yu. Kalmykov, Nucl. Phys. B502 (1997) 419.

[53] K.G. Chetyrkin and M. Steinhauser, Phys. Rev. Lett. 83 (1999) 4001.

[54] D.J. Broadhurst, J. Fleischer and O.V. Tarasov, Z. Phys. C60 (1993) 287.

[55] D.J. Broadhurst, Z. Phys. C47 (1990) 115.

[56] J. Fleischer, A.V. Kotikov and O.L. Veretin, Nucl. Phys. B547 (1999) 343.

[57] F.A. Berends, A.I. Davydychev, V.A. Smirnov and J.B. Tausk, Nucl. Phys. B439 (1995) 536.

[58] V. Borodulin and G. Jikia, Phys. Lett. B391 (1997) 434.

[59] S. Bauberger, F.A. Berends, M. Böhm and M. Buza, Nucl. Phys. B434 (1995) 383.

[60] C. Anastasiou, E.W.N. Glover and C. Oleari, Nucl. Phys. B572 (2000) 307.

[61] P.N. Maher, L. Durand and K. Riesselmann, Phys. Rev. D48 (1993) 1061; D52 (1995) 553(E).

[62] A.I. Davydychev, “Loop calculations in QCD with massive quarks”, talk at Int. Conf. “Relativistic Nuclear Dynamics” (Vladivostok, Russia, September 1991), http://wwwthep.physik.uni-mainz.de/~davyd/preprints/vladiv.ps.gz

[63] A. Pelissetto and E. Vicari, Nucl. Phys. B575 (2000) 579.

[64] J. Fleischer and O. V. Tarasov, Z. Phys. C64 (1994) 413.

[65] A.I. Davydychev and V.A. Smirnov, Nucl. Phys. B554 (1999) 391.

[66] O.V. Tarasov, Nucl. Phys. B502 (1997) 455.

[67] D.J. Broadhurst, math.CA/9803067.
[68] D.J. Broadhurst, Z. Phys. C54 (1992) 599.

[69] D.J. Broadhurst and A.V. Kotikov, Phys. Lett. B441 (1998) 345.

[70] B.C. Berndt, Ramanujan’s Notebooks: Part I, Springer-Verlag, 1985; D.M. Bradley, The Ramanujan J. 3 (1999) 159.

[71] J. Fleischer, A.V. Kotikov and O.L. Veretin, Phys. Lett. B417 (1998) 163.

[72] E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. A15 (2000) 725; S. Moch and J.A.M. Vermaseren, Nucl. Phys. B573 (2000) 853.

[73] P.W. Karlsson, J. Math. Anal. Appl. 196 (1995) 172.