SOME REMARKS ON THE CLASSICAL PRIME SPECTRUM OF MODULES

Alireza Abbasi and Mohammad Hasan Naderi

Faculty of Science, Department of Mathematics, University of Qom, Qom, Iran, P.O. Box 37161-46611

Abstract. Let $R$ be a commutative ring with identity and let $M$ be an $R$-module. A proper submodule $P$ of $M$ is called a classical prime submodule if $abm \in P$, for $a, b \in R$, and $m \in M$, implies that $am \in P$ or $bm \in P$. The classical prime spectrum of $M$, $\text{ClSpec}(M)$, is defined to be the set of all classical prime submodules of $M$. We say $M$ is classical primeful if $M = 0$, or the map $\psi$ from $\text{ClSpec}(M)$ to $\text{Spec}(R/\text{Ann}(M))$, defined by $\psi(P) = (P : M)/\text{Ann}(M)$ for all $P \in \text{ClSpec}(M)$, is surjective. In this paper, we study classical primeful modules as a generalization of primeful modules. Also, we investigate some properties of a topology that is defined on $\text{ClSpec}(M)$, named the Zariski topology.

Keywords: Classical prime, Classical primeful, Classical top module

1. Introduction

Throughout the paper all rings are commutative with identity and all modules are unital. Let $M$ be an $R$-module. If $N$ is a submodule of $M$, then we write $N \leq M$. For any two submodules $N$ and $K$ of an $R$-module $M$, the residual of $N$ by $K$ is denoted by $(N : K) = \{ r \in R : rK \subseteq N \}$. A proper submodule $P$ of $M$ is called a prime submodule if $am \in P$, for $a \in R$ and $m \in M$, implies that $m \in P$ or $a \in (P : M)$. Also, a proper submodule $P$ of $M$ is called a classical prime submodule if $abm \in P$, for $a, b \in R$ and $m \in M$, implies that $am \in P$ or $bm \in P$ (see for example [5]). The set of prime (resp. classical prime) submodules of $M$ is denoted by $\text{Spec}(M)$ (resp. $\text{ClSpec}(M)$). The class of prime submodules of modules was introduced and studied in 1992 as a generalization of...
the class of prime ideals of rings. Then, many generalizations of prime submodules were studied such as primary, classical prime, classical primary and classical quasi primary submodules, see [1, 8, 16, 4] and [7].

For a proper submodule \( N \) of an \( R \)-module \( M \), the prime radical of \( N \) is 
\[
\sqrt{N} = \bigcap \{ P \mid P \in \mathcal{V}(N) \},
\]
where \( \mathcal{V}(N) = \{ P \in \text{Spec}(M) \mid N \subseteq P \} \). Also the classical prime radical of \( N \) is 
\[
\sqrt{N} = \bigcap \{ P \mid P \in \mathcal{V}(N) \}, \quad \mathcal{V}(N) = \{ P \in \text{Cl.Spec}(M) \mid N \subseteq P \}. \]
If there are no such prime (resp. classical prime) submodules, \( \sqrt{N} \) (resp. \( \sqrt{N} \)) is \( M \). We say \( N \) is a radical (resp. classical radical) submodule, if \( \sqrt{N} = \emptyset \)(resp. \( \sqrt{N} = N \)).

The set of all maximal submodules of \( M \) is denoted by \( \text{Max}(M) \). A Noetherian module \( M \) is called a semi-local (resp. a local) module if \( \text{Max}(M) \) is a non-empty finite (resp. a singleton) set. A non-Noetherian commutative ring \( R \) is called a quasisemilocal (resp. a quasilocal) ring if \( R \) has only a finite number (resp. a singleton) of maximal ideals. An \( R \)-module \( M \) is called a multiplication (resp. weak multiplication) module if for every submodule (resp. prime submodule) of \( M \), there exists an ideal \( I \) of \( R \) such that \( N = IM \)(see [14] and [2]). If \( N \) is a prime submodule of a multiplication \( R \)-module \( M \), then \( N_1 \cap N_2 \subseteq N \), where \( N_1, N_2 \subseteq M \), implies that \( N_1 \subseteq N \) or \( N_2 \subseteq N \) (see for more detail [11] and [19]). An \( R \)-module \( M \) is called compatible if its classical prime submodules and its prime submodules coincide. All commutative rings and multiplicative modules are examples of compatible modules, (see for more detail [8]). A submodule \( N \) of \( M \) is said to be strongly irreducible if for submodules \( N_1 \) and \( N_2 \) of \( M \), the inclusion \( N_1 \cap N_2 \subseteq N \) implies that either \( N_1 \subseteq N \) or \( N_2 \subseteq N \). Strongly irreducible submodules have been characterized in [13].

Let \( M \) be an \( R \)-module. For any subset \( E \) of \( M \), we consider classical varieties denoted by \( \mathcal{V}(E) \). We define \( \mathcal{V}(E) = \{ P \in \text{Cl.Spec}(M) \mid E \subseteq P \} \). Then

(a) If \( N \) is a submodule generated by \( E \), then \( \mathcal{V}(E) = \mathcal{V}(N) \).
(b) \( \mathcal{V}(0_M) = \text{Cl.Spec}(M) \) and \( \mathcal{V}(M) = \emptyset \).
(c) \( \bigcap_{i \in J} \mathcal{V}(N_i) = \mathcal{V}(\sum_{i \in J} N_i) \), where \( N_i \leq M \)
(d) \( \mathcal{V}(N) \cup \mathcal{V}(L) \subseteq \mathcal{V}(N \cap L) \), where \( N, L \leq M \).

Now, we assume that \( \mathcal{C}(M) \) denotes the collection of all subsets \( \mathcal{V}(N) \) of \( \text{Cl.Spec}(M) \). Then, \( \mathcal{C}(M) \) contains the empty set and \( \text{Cl.Spec}(M) \), and also \( \mathcal{C}(M) \) are closed under arbitrary intersections. However, in general, \( \mathcal{C}(M) \) is not closed under finite union. An \( R \)-module \( M \) is called a classical top module if \( \mathcal{C}(M) \) is closed under finite unions, i.e., for every submodules \( N \) and \( L \) of \( M \), there exists a submodule \( K \) of \( M \) such that \( \mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(K) \), for in this case, \( \mathcal{C}(M) \) satisfies the axioms for the closed subsets of a topological space, then in this case, \( \mathcal{C}(M) \) induce a topology on \( \text{Cl.Spec}(M) \). We call the induced topology the classical quasi-Zariski topology (see [9]).

In this paper, we introduce the notion of classical primeful modules and also we investigate some properties of classical quasi-Zariski topology of \( \text{Cl.Spec}(M) \). In Section 2, we introduce the notion of classical primeful modules as a generalization of primeful modules. In particular, in Proposition 2.3, it is proved that if \( M \) is
a classical primeful $R$-module, then $\text{Supp}(M) = \text{V}(\text{Ann}(M))$. Then we get some properties of classical top modules. In Section 3, we get some properties of classical quasi-Zariski topology of $\text{Cl.Spec}(M)$ and also we get some properties of classical top modules.

2. Classical primeful module

The notion of primeful modules was introduced by Chin P. Lu in [18] as follows:

**Definition 2.1.** An $R$-module $M$ is primeful if either $M = (0)$, or $M \neq (0)$ and the map $\phi : \text{Spec}(M) \rightarrow \text{Spec}(R/\text{Ann}(M))$, defined by $\phi(P) = (P : M)/\text{Ann}(M)$ for all $P \in \text{Spec}(M)$, is surjective.

Now, we extend the notion of primeful modules to classical primeful modules.

**Definition 2.2.** Suppose $\text{Cl.Spec}(M) \neq \emptyset$, then the map $\psi$ from $\text{Cl.Spec}(M)$ to $\text{Spec}(R/\text{Ann}(M))$ defined by $\psi(P) = (P : M)/\text{Ann}(M)$ for all $P \in \text{Cl.Spec}(M)$, will be called the natural map of $\text{Cl.Spec}(M)$.

An $R$-module $M$ is classical primeful if either

(i) $M = (0)$, or

(ii) $M \neq (0)$ and the map $\psi : \text{Cl.Spec}(M) \rightarrow \text{Spec}(R/\text{Ann}(M))$ from above is surjective.

**Lemma 2.1.** Let $M$ be a classical top $R$-module. Then the natural map $\psi : \text{Cl.Spec}(M) \rightarrow \text{Spec}(R/\text{Ann}(M))$ is injective.

**Proof.** Let $P, Q \in \text{Cl.Spec}(M)$. If $\psi(P) = \psi(Q)$, then 

$$(P : M)/\text{Ann}(M) = (Q : M)/\text{Ann}(M).$$

So $(P : M) = (Q : M)$ and then $P = Q$. □

**Theorem 2.1.** Let $M$ be a classical top $R$-module. Then, If $R$ satisfies ACC on prime ideals, then $M$ satisfies ACC on classical prime submodules.

**Proof.** Let $N_1 \subseteq N_2 \subseteq \ldots$ be an ascending chain of classical prime submodules of $M$. This induces the following chain of prime ideals, $\psi(N_1) \subseteq \psi(N_2) \subseteq \ldots$, where $\psi$ is the natural map

$$\psi : \text{Cl.Spec}(M) \rightarrow \text{Spec}(R/\text{Ann}(M)).$$

Since $R$ satisfies ACC on prime ideals, there exists a positive integer $k$ such that for each $i \in \mathbb{N}$, $\psi(N_k) = \psi(N_{k+i})$. Now by Lemma 2.1, we have $N_k = N_{k+i}$ as required. □
Remark 2.1. ([8, Proposition 5.3]) Let $S$ be a multiplicatively closed subset of $R$, $p$ a prime ideal of $R$ such that $p \cap S = \emptyset$ and let $M$ be an $R$-module. If $P$ is a classical $p$-prime submodule of $M$ with $P \neq M$, then $P_s$ is also a classical $p_s$-prime submodule of $M_s$. Moreover if $Q$ is a prime $R_s$-submodule of $M_s$, then $Q^c = \{ m \in M : f(m) \in Q \}$ is a classical prime submodule of $M$.

Let $p$ be a prime ideal of a ring $R$, $M$ an $R$-module and $N \subseteq M$. By the saturation of $N$ with respect to $p$, we mean the contraction of $N_p$ in $M$ and designate it by $S_p(N)$. It is also known that

$$S_p(N) = \{ e \in M | es \in N \text{ for some } s \in R \setminus p \}.$$ 

Saturations of submodules were investigated in detail in [17].

Proposition 2.1. For any nonzero $R$-module $M$, the following are equivalent:

1. The natural map $\psi : ClSpec(M) \rightarrow Spec(R/Ann(M))$ is surjective;
2. For every $p \in V(Ann(M))$, there exists $P \in ClSpec(M)$ such that $(P : M) = p$;
3. $pM_p \neq M_p$, for every $p \in V(Ann(M))$;
4. $S_p(pM)$, the contraction of $pM_p$ in $M$, is a classical $p$-prime submodule of $M$ for every $p \in V(Ann(M))$;
5. $ClSpec_p(M) \neq \emptyset$; for every $p \in V(Ann(M))$.

Proof. (1)$\iff$(2): It is clear by Definition 2.2.

(2)$\implies$(3): Let $p \in V(Ann(M))$ and let $N$ be a classical $p$-prime submodule of $M$. Then $N_p$ is a classical $pR_p$-prime submodule of $M_p$ by Remark 2.1. Now, since $pM_p \subseteq N_p \subseteq M_p$, we conclude that $pM_p \neq M_p$.

(3)$\implies$(4): Since $pR_p$ is the maximal ideal of $R_p$ and $pM_p \neq M_p$, $pM_p = (pR_p)M_p$ is a $pR_p$-prime, and therefore classical $pR_p$-prime, submodule of $M_p$. Then $S_p(pM) = (pM_p)^c$, the contraction of $pM_p$ in $M$, is a classical $p$-prime submodule of $M$ by Remark 2.1.

(4)$\implies$(5) and (5)$\implies$(2) are easy. □

Proposition 2.2. Every finitely generated $R$-module $M$ is classical primeful.

Proof. If $M = 0$, evidently the results is true. Now, let $M$ be a nonzero finitely generated $R$-module. Then $\text{Supp}(M) = V(Ann(M))$, so for every $p \in V(Ann(M))$, $M_p$ is a nonzero finitely generated module over the local ring $R_p$. Then by virtue
of Nakayama’s Lemma, \( pM_p \neq M_p \), for every \( p \in V(\text{Ann}(M)) \). Therefore by Proposition 2.1, \( M \) is classical primeful. \( \Box \)

For every finitely generated module \( M \), \( \text{Supp}(M) = V(\text{Ann}(M)) \). The next proposition proves that the equality holds even if \( M \) is only a classical primeful module.

**Proposition 2.3.** (see [18, Proposition 3.4]) If \( M \) is a classical primeful \( R \)-module, then \( \text{Supp}(M) = V(\text{Ann}(M)) \).

**Proof.** If \( M = (0) \), then \( \text{Supp}(M) = V(\text{Ann}(M)) = \emptyset \). Now let \( M \) be a nonzero classical primeful \( R \)-module, so \( V(\text{Ann}(M)) \neq \emptyset \). By Proposition 2.1, if \( p \in V(\text{Ann}(M)) \), then \( S_p(pM) \) is a classical \( p \)-prime submodule of \( M \), so \( S_p(pM) \neq M \). Since \( S_p(0) \subseteq S_p(pM) \), then \( M \neq S_p(0) \), from which we can see that \( M_p \neq (0) \). Thus \( V(\text{Ann}(M)) \subseteq \text{Supp}(M) \). The other inclusion is always true.

For every prime, ideal \( p \) of \( R \), \( R_p \) is always a quasilocal ring. However, for an arbitrary \( R \)-module \( M \), \( M_p \) is not necessarily a local \( R_p \)-module. But by the next proposition, if \( M \) is a nonzero classical top classical primeful \( R \)-module, then \( R/\text{Ann}(M) \) is a quasilocal ring.

**Proposition 2.4.** Let \( M \) be a nonzero classical top classical primeful \( R \)-module. If \( M \) is a semi-local (resp. local) module, then \( R/\text{Ann}(M) \) is a quasisemilocal (resp. a quasilocal) ring.

**Proof.** Let \( M \) be a local module with unique maximal submodule \( P \). Then \( p := (P : M) \in \text{Max}(R) \). Now let \( \text{Ann}(M) \subseteq q \in \text{Max}(R) \). It is enough to prove \( q = p \). To prove this, we note that \( S_q(qM) \) is a classical \( q \)-prime submodule of \( M \) by Proposition 2.1. Now we show that \( S_q(qM) \in \text{Max}(M) \). Let \( S_q(qM) \subseteq K \) for some submodule \( K \) of \( M \). Then we have \( q = (S_q(qM) : M) = (K : M) \). Hence \( S_q(qM) = K \) by Lemma 2.1. This implies that \( S_q(qM) = P \) and therefore \( q = p \). For the semi-local case we argue similarly. \( \Box \)

In the rest of this section, we get some properties of classical top modules. First note that every classical top module is a top module ([9, Proposition 2.4]). In the next theorem, we introduce some modules that they are classical top modules.

**Theorem 2.2.** Let \( M \) be an \( R \)-module. Then \( M \) is a classical top module in each of the following cases:

1. \( M \) is a multiplication \( R \)-module.

2. \( M \) be a module that every classical prime submodule of \( M \) is strongly irreducible.
(3) \( M \) is an \( R \)-module with the property that for any two submodules \( N \) and \( L \) of \( M \), \((N : M)\) and \((L : M)\) are comaximal.

**Proof.** (1). Let \( P \in \mathcal{V}(N_1 \cap N_2) \) and so \( N_1 \cap N_2 \subseteq P \). Since \( M \) is compatible, then \((N_1 \cap N_2 : M) \subseteq (P : M)\), so \( N_1 \subseteq P \) or \( N_2 \subseteq P \). Therefore \( P \in \mathcal{V}(N_1) \) or \( P \in \mathcal{V}(N_2) \). This implies that \( M \) is a classical top module.

(2). Let \( P \in \mathcal{V}(N \cap L) \). Since \( \mathcal{V}(N) \cup \mathcal{V}(L) \subseteq \mathcal{V}(N \cap L) \), for each submodules \( N \) and \( L \) of \( M \), then \( N \cap L \subseteq P \). Now, since \( P \) is strongly irreducible, then \( N \subseteq P \) or \( L \subseteq P \). Therefore \( P \in \mathcal{V}(N) \cup \mathcal{V}(L) \). Thus \( \mathcal{C}(M) \) is closed under finite unions. Hence \( M \) is a classical top module.

(3). Let \( P \) be a classical prime submodule of \( M \) with \( N \cap L \subseteq P \). Then \((N : M) \cap (L : M) \subseteq (P : M) \in Spec(R) \). We may assume that \((N : M) \subseteq (P : M)\). Then clearly \((L : M) \nsubseteq (P : M)\) by assumption. Hence \( N \subseteq P \). Therefore \( P \) is strongly irreducible. This implies that \( M \) is a classical top module by (2). \( \square \)

If \( Y \) is a nonempty subset of \( \text{Cl.Spec}(M) \), then the intersection of the members of \( Y \) is denoted by \( \mathfrak{S}(Y) \). Thus, if \( Y_1 \) and \( Y_2 \) are subsets of \( \text{Cl.Spec}(M) \), then \( \mathfrak{S}(Y_1 \cup Y_2) = \mathfrak{S}(Y_1) \cap \mathfrak{S}(Y_2) \). An \( R \)-module \( M \) is said to be distributive if \((A+B) \cap C = (A \cap C) + (B \cap C)\), for all submodules \( A \), \( B \) and \( C \) of \( M \)(see for example [12]).

**Theorem 2.3.** Let \( M \) is a classical top module and \( \sqrt[\mathfrak{S}]{E} = E \) for each submodule \( E \) of \( M \). Then \( M \) is a distributive module.

**Proof.** Let \( A, B \) and \( C \) be any submodules of \( M \). Then,
\[
(A+B) \cap C = \sqrt[\mathfrak{S}]{(A+B) \cap C} \\
= \cap \{P \in \text{Cl.Spec}(M)|(A+B) \cap C \subseteq P\} \\
= \cap \{P|P \in \mathcal{V}((A+B) \cap C)\} \\
= \mathfrak{S}(\mathcal{V}((A+B) \cap C)) \\
= \mathfrak{S}(\mathcal{V}(A+B) \cup \mathcal{V}(C)) \\
= \mathfrak{S}((\mathcal{V}(A) \cap \mathcal{V}(B)) \cup \mathcal{V}(C)) \\
= \mathfrak{S}((\mathcal{V}(A) \cup \mathcal{V}(B)) \cap \mathcal{V}(C)) \\
= \mathfrak{S}((\mathcal{V}(A \cap C)) \cap (\mathcal{V}(B \cap C))) \\
= \sqrt[\mathfrak{S}]{(A \cap C) \cap (B \cap C)} \\
= (A \cap C) + (B \cap C)
\]

Hence \( M \) is a distributive module. \( \square \)

**Proposition 2.5.** Let \( M \) be a classical top module. Then for every two submodules \( A \) and \( B \) of \( M \) the equality \( \sqrt[\mathfrak{S}]{A \cap B} = \sqrt[\mathfrak{S}]{A} \cap \sqrt[\mathfrak{S}]{B} \) holds.

**Proof.** By definition, \( \sqrt[\mathfrak{S}]{A \cap B} = \mathfrak{S}(\mathcal{V}(A \cap B)) = \mathfrak{S}(\mathcal{V}(A) \cup \mathcal{V}(B)) \\
= \mathfrak{S}(\mathcal{V}(A)) \cap \mathfrak{S}(\mathcal{V}(B)) = \sqrt[\mathfrak{S}]{A} \cap \sqrt[\mathfrak{S}]{B} \). \( \square \)
3. Some properties of topological space \( \text{ClSpec}(M) \)

In this section, we study some properties of topological space \( \text{ClSpec}(M) \). The closure of \( Y \) in \( \text{ClSpec}(M) \) with respect to the classical quasi-Zariski topology denoted by \( \overline{Y} \).

**Lemma 3.1.** Let \( M \) be a classical top module and let \( Y \) be a nonempty subset of \( \text{ClSpec}(M) \). Then \( \overline{Y} = \mathcal{V}(\overline{Y}) \). Hence, for every \( N \leq M \), \( \mathcal{V}(\mathcal{V}(N)) = \mathcal{V}(N) \).

**Proof.** Suppose \( \mathcal{V}(E) \) is a closed set of \( \text{ClSpec}(M) \) containing \( Y \). Then for every classical prime submodule \( P \) in \( Y \), \( E \subseteq P \). Therefore \( E \subseteq \overline{Y} \) and so \( \mathcal{V}(\overline{Y}) \subseteq \mathcal{V}(E) \). Since \( Y \subseteq \mathcal{V}(\overline{Y}) \), then \( \mathcal{V}(\overline{Y}) \) is the smallest closed subset of \( \text{ClSpec}(M) \) containing \( Y \). Thus \( \overline{Y} = \mathcal{V}(\overline{Y}) \).

Finally, since \( \mathcal{V}(\overline{\mathcal{V}(N)}) = \overline{\mathcal{V}(N)} \), and since \( \mathcal{V}(N) \) is a closed subset of \( \text{ClSpec}(M) \), then \( \overline{\mathcal{V}(N)} = \mathcal{V}(N) \). Consequently \( \mathcal{V}(\overline{\mathcal{V}(N)}) = \mathcal{V}(N) \). \( \square \)

Let \( X \) be a topological space and let \( x \) and \( y \) be two points of \( X \). We say that \( x \) and \( y \) can be separated if each lies in an open set which does not contain the other point. \( X \) is a \( T_1 \)-space if any two distinct points in \( X \) can be separated. A topological space \( X \) is a \( T_1 \)-space if and only if the singleton set \( \{x\} \) is a closed set, for any \( x \) in \( X \).

**Theorem 3.1.** Let \( M \) be an \( R \)-module. Then \( \text{ClSpec}(M) \) is \( T_1 \)-space if and only if each classical prime submodule is maximal in the family of all classical prime submodules of \( M \). i.e, \( \text{Max}(M) = \text{ClSpec}(M) \).

**Proof.** Let \( P \) be maximal in \( \text{ClSpec}(M) \) with respect inclusion. Then \( \{P\} = \mathcal{V}(\{P\}) = \mathcal{V}(P) \), but \( P \) is maximal in \( \text{ClSpec}(M) \), so \( \{P\} = \{P\} \). Then \( \{P\} \) is a closed set in \( \text{ClSpec}(M) \). Thus \( \text{ClSpec}(M) \) is a \( T_1 \)-space, and vice versa. \( \square \)

**Definition 3.1.** Let \( X \) be a topological space and \( Y \subseteq X \). Then:

1. \( X \) is irreducible if \( X \neq \emptyset \) and for every decomposition \( X = A_1 \cup A_2 \) with closed subsets \( A_i \subseteq X, i = 1, 2 \), we have \( A_1 = X \) or \( A_2 = X \).

2. \( Y \) is irreducible if \( Y \) is irreducible as a space with the relative topology. For this to be so, it is necessary and sufficient that, for every pair of sets \( F \), \( G \) which are closed in \( X \) and satisfy \( Y \subseteq F \cup G \), then \( Y \subseteq F \) or \( Y \subseteq G \)[10, Ch. II, p. 119].

**Lemma 3.2.** Let \( M \) be an \( R \)-module. Then for every \( P \in \text{ClSpec}(M) \), \( \mathcal{V}(P) \) is irreducible.

**Proof.** Let \( \mathcal{V}(P) \subseteq Y_1 \cup Y_2 \), for some closed sets \( Y_1 \) and \( Y_2 \). Since \( P \in \mathcal{V}(P) \), either \( P \in Y_1 \) or \( P \in Y_2 \). Suppose that \( P \in Y_1 \). Then \( Y_1 = \bigcap_{i \in I} \bigcup_{j=1}^{n_i} \mathcal{V}(N_{ij}) \), for some \( I \), \( n_i (i \in I) \) and \( N_{ij} \leq M \). Then for all \( i \in I \), \( P \in \bigcup_{j=1}^{n_i} \mathcal{V}(N_{ij}) \). Thus for all \( i \in I \), \( \mathcal{V}(P) \subseteq \bigcup_{j=1}^{n_i} \mathcal{V}(N_{ij}) \), so \( \mathcal{V}(P) \subseteq Y_1 \). Thus \( \mathcal{V}(P) \) is irreducible. \( \square \)
M. Behboodi and M. R. Haddadi show that if \( Y \subseteq \text{Spec}(M) \) and \( \mathfrak{T}(Y) \) is a prime submodule of \( M \) and \( \mathfrak{T}(Y) \in \mathcal{Y} \), then \( Y \) is irreducible[6, Theorem 3.4]. In the next proposition, we extend this fact to classical prime submodules.

**Proposition 3.1.** Let \( M \) be a classical top module and \( Y \subseteq \text{ClSpec}(M) \). Then \( \mathfrak{T}(Y) \) is a classical prime submodule of \( M \) if and only if \( Y \) is an irreducible space.

**Proof.** Let \( P = \mathfrak{T}(Y) \) be a classical prime submodule of \( M \) and \( P \in Y \), so \( \mathcal{Y} \subseteq \mathcal{Y}(P) \). If \( Y \subseteq Y_1 \cup Y_2 \), for closed sets \( Y_1 \) and \( Y_2 \), then \( \mathcal{Y} \subseteq Y_1 \cup Y_2 \). Since \( \mathcal{Y}(P) \subseteq Y_1 \cup Y_2 \) and by Lemma 3.2, \( \mathcal{Y}(P) \) is irreducible, then \( \mathcal{Y}(P) \subseteq Y_1 \) or \( \mathcal{Y}(P) \subseteq Y_2 \). Now, since \( Y \subseteq \mathcal{Y}(P) \), then either \( Y \subseteq Y_1 \) or \( Y \subseteq Y_2 \). Thus \( Y \) is irreducible. For the converse, we can apply [6, Theorem 3.4]. \( \square \)

**Corollary 3.1.** Let \( M \) be a classical top module. Then for every classical prime submodule \( P \), \( \mathcal{Y}(P) \) is an irreducible subspace of \( C\text{lSpec}(M) \). Consequently, \( \mathcal{Y}(N) \) is irreducible if and only if \( \sqrt[\mathfrak{T}]{N} \) is a classical prime submodule.

**Proof.** First note that \( \mathfrak{T}(\mathcal{Y}(P)) = \bigcap\{P|P \in \mathcal{Y}(P)\} = \sqrt[\mathfrak{T}]{P} = P \). Then \( \mathcal{Y}(P) \) is an irreducible subspace of \( \text{ClSpec}(M) \), by Proposition 3.1. Finally, it is enough to note that \( \sqrt[\mathfrak{T}]{\mathcal{Y}(N)} = \mathfrak{T}(\mathcal{Y}(N)) \). \( \square \)

**Proposition 3.2.** Let \( M \) be a classical top \( R \)-module, \( \mathcal{R} = R/\text{Ann}(M) \) and let \( \psi: \text{ClSpec}(M) \rightarrow \text{Spec}(\mathcal{R} \text{Ann}(M)) \) be the natural map of \( \text{ClSpec}(M) \). Then \( \psi \) is continuous in the classical quasi-Zariski topology.

**Proof.** It suffices to prove that \( \psi^{-1}(\mathcal{V}(\mathcal{T})) = \mathcal{V}(IM) \), for every \( I \in \mathcal{V}(\text{Ann}(M)) \). Let \( P \in \text{ClSpec}(M) \), then \( P \in \psi^{-1}(\mathcal{V}((\mathcal{T})) \), so \( \psi(P) \in \mathcal{V}(\mathcal{T}) \), therefore \( (P : M) \in \mathcal{V}(\mathcal{T}) \). Then \( (P : M) \in \text{Spec}(\mathcal{R}) \) and \( I \subseteq (P : M) \), so \( (P : M) \in \text{Spec}(\mathcal{R}) \) and \( I/\text{Ann}(M) \subseteq (P : M)/\text{Ann}(M) \). Hence \( (P : M) \in \text{Spec}(\mathcal{R}) \) and \( \text{Ann}(M) \subseteq I \subseteq (P : M) \). Now, since \( IM \subseteq (P : M)M \subseteq P \), then \( P \in \mathcal{V}(IM) \), which it shows that \( \psi^{-1}(\mathcal{V}(\mathcal{T})) \subseteq \mathcal{V}(IM) \). In similar way, we can show \( \mathcal{V}(IM) \subseteq \psi^{-1}(\mathcal{V}(\mathcal{T})) \) and hence \( \psi^{-1}(\mathcal{V}(\mathcal{T})) = \mathcal{V}(IM) \). \( \square \)

**Lemma 3.3.** Let \( M \) be a classical top \( R \)-module, \( \mathcal{R} = R/\text{Ann}(M) \) and let \( \psi \) be the natural map of \( \text{ClSpec}(M) \). If \( M \) is classical primeful, then \( \psi \) is both closed and open; more precisely, for every submodule \( N \) of \( M \), \( \psi(\mathcal{V}(N)) = \mathcal{V}((N : M)) \) and \( \psi(\text{ClSpec}(M) \setminus \mathcal{V}(N)) = \text{ClSpec}(\mathcal{R}/\text{Ann}(M)) \setminus (\mathcal{V}(N : M)) \).

**Proof.** First we show that \( \psi(\mathcal{V}(N)) = \mathcal{V}((N : M)) \), for every \( N \subseteq M \), whenever \( M \) is classical primeful. Since \( \psi \) is continuous, as we have seen in Proposition 3.2, \( \psi^{-1}(\mathcal{V}(N : M)) = \mathcal{V}((N : M)M) = \mathcal{V}(N) \). Hence, \( \psi(\mathcal{V}(N)) = \psi \circ \psi^{-1}(\mathcal{V}(N : M)) = \mathcal{V}(N : M) \), since \( \psi \) is surjective and \( M \) is classical primeful. Consequently: \( \psi(\text{ClSpec}(M) \setminus \mathcal{V}(N)) = \text{Spec}(\mathcal{R}/\text{Ann}(M)) \setminus (\mathcal{V}(N : M)) \). \( \square \)
Corollary 3.2. Let $M$ be a classical top $R$-module, $\overline{R} = R/\text{Ann}(M)$ and let $\psi$ be the natural map of $\text{ClSpec}(M)$. Then $\psi$ is bijective if and only if it is a homeomorphism.

Proof. This follows from Proposition 3.2 and Lemma 3.3. □

Proposition 3.3. Let $M$ be a classical top $R$-module and let $Y$ be a subset of $\text{ClSpec}(M)$. If $Y$ is irreducible, then $T = \{(P : M) | P \in Y\}$ is an irreducible subset of $\text{Spec}(R)$, with respect to Zariski topology.

Proof. Let $\overline{R} = R/\text{Ann}(M)$, $\psi$ the natural map of $\text{ClSpec}(M)$ and let $Y$ be a subset of $\text{ClSpec}(M)$. Since $\psi$ is continuous by proposition 3.2, then $\psi(Y) = Y$ is an irreducible subset of $\text{Spec}(R/\text{Ann}(M))$. Therefore

$$
\exists(Y) = (\exists(Y) : M)\text{Ann}(M) \in \text{Spec}(R/\text{Ann}(M)).
$$

Therefore $\exists(T) = (\exists(Y) : M)$ is a prime ideal of $R$, then by Proposition 3.1, $T$ is an irreducible subset of $\text{Spec}(R)$. □

Clearly the next lemma is true (see for example [8], page 10).

Lemma 3.4. If $\{P_i\}_{i \in I}$ is a chain of classical prime submodules of an $R$-module $M$, then $\bigcap_{i \in I} P_i$ is a classical prime submodule of $M$.

Let $Y$ be a closed subset of a topological space. An element $y \in Y$ is called a generic point of $Y$ if $Y = \text{Cl}(|\{y\}|)$, where $\text{Cl}(|\{y\}|)$ is the closure of $\{y\}$ in $Y$. Note that a generic point of a closed subset $Y$ of a topological space is unique if the topological space is a $T_0$-space.

Theorem 3.2. Let $M$ be a classical primeful $R$-module. If $M$ is a classical top module, then a subset $Y$ of $\text{ClSpec}(M)$ is an irreducible closed subset if and only if $Y = \mathcal{V}(P)$, for some $P \in \text{ClSpec}(M)$. Thus every irreducible closed subset of $\text{ClSpec}(M)$ has a generic point.

Proof. By Corollary 3.1, for every $P \in \text{ClSpec}(M)$, $Y = \mathcal{V}(P)$ is an irreducible closed subset of $\text{ClSpec}(M)$. Conversely, if $Y$ is an irreducible closed subset of $\text{ClSpec}(M)$, then $Y = \mathcal{V}(N)$, for some $N \leq M$. Now, since $Y = \mathcal{V}(N) = \mathcal{V}(\sqrt{N})$, then $\exists(Y) = \exists(\mathcal{V}(N)) = \sqrt{\exists(N)}$ is a classical prime submodule of $M$ by Lemma 3.4. Then $\mathcal{V}(\exists(Y)) = \mathcal{V}(\exists(\mathcal{V}(N))) = \mathcal{V}(\sqrt{\exists(N)})$, so by Theorem 3.1, $Y = \mathcal{V}(N) = \mathcal{V}(\sqrt{\exists(N)})$, with $\sqrt{\exists(N)} \in \text{ClSpec}(M)$. □

A maximal irreducible subset $Y$ of $X$ is called an irreducible component of $X$ and it is always closed. In the next theorem, we show that there exists a bijection map from the set of irreducible components of $\text{ClSpec}(M)$ to the set of minimal classical prime submodules of $M$. 

Theorem 3.3. Let $M$ be a classical top $R$-module. Then the map $\mathcal{V}(P) \mapsto P$ is a bijection from the set of irreducible components of $\text{ClSpec}(M)$ to the set of minimal classical prime submodules of $M$.

Proof. Let $Y$ be an irreducible component of $\text{ClSpec}(M)$. By Theorem 3.2, each irreducible component of $\text{ClSpec}(M)$ is a maximal element of the set $\{\mathcal{V}(Q) | Q \in \text{ClSpec}(M)\}$, so for some $P \in \text{ClSpec}(M)$, $Y = \mathcal{V}(P)$. Obviously, $P$ is a minimal classical prime submodule of $M$. Suppose $T$ is a classical prime submodule of $M$ that $T \subseteq P$, then $\mathcal{V}(P) \subseteq \mathcal{V}(T)$, so $P = T$. Now, let $P$ be a minimal classical prime submodule of $M$, so for every $Q \in \text{ClSpec}(M)$, $P \subseteq Q$. Then for all $Q \in \text{ClSpec}(M)$, $\mathcal{V}(Q) \subseteq \mathcal{V}(P)$. Thus $\mathcal{V}(P)$ is a maximal irreducible subset of $\text{ClSpec}(M)$. $\square$

Theorem 3.4. Consider the following statements for a nonzero classical top primeful $R$-module $M$:

1. $\text{ClSpec}(M)$ is an irreducible space.
2. $\text{Supp}(M)$ is an irreducible space.
3. $\sqrt{\text{Ann}(M)}$ is a prime ideal of $R$.
4. $\text{ClSpec}(M) = \mathcal{V}(pM)$, for some $p \in \text{Supp}(M)$.

Then $(1) \implies (2) \implies (3) \implies (4)$. In addition, if $M$ is a multiplication module, then all of the four statements are equivalent.

Proof. $(1) \implies (2)$: By Proposition 3.2, the natural map $\psi$ is continuous and by assumption $\psi$ is surjective. Therefore $\text{Im}(\psi) = \text{Spec}(R/\text{Ann}(M))$ is also irreducible. Now by Proposition 2.3, $\text{Supp}(M) = \mathcal{V}(\text{Ann}(M))$ is homeomorphic to $\text{Spec}(R/\text{Ann}(M))$. Therefore $\text{Supp}(M)$ is an irreducible space.

$(2) \implies (3)$: By Proposition 3.1, $\mathfrak{S}(\text{Supp}(M))$ is a prime ideal of $R$. Then $\mathfrak{S}(\text{Supp}(M)) = \mathfrak{S}(\mathcal{V}(\text{Ann}(M))) = \sqrt{\text{Ann}(M)}$ is a prime ideal of $R$.

$(3) \implies (4)$: Let $a \in \sqrt{\text{Ann}(M)}$. So for some integer $n \in N$, $a^nM = 0$. Therefore for every classical prime submodule $P$ of $M$, $a \in (P : M)$. Then for each $P \in \text{ClSpec}(M)$, $\text{Ann}(M) \subseteq \sqrt{\text{Ann}(M)} \subseteq (P : M)$. Since $M$ is classical primeful, there exists a classical prime submodule $Q$ of $M$ such that $(Q : M) = \sqrt{\text{Ann}(M)}$. Then, 

\[ \text{ClSpec}(M) = \{ P \in \text{ClSpec}(M) | (Q : M) \subseteq (P : M) \} = \mathcal{V}((Q : M)M) = \mathcal{V}(\sqrt{\text{Ann}(M)}M). \]

It is clear that $p := \sqrt{\text{Ann}(M)} \in \text{Supp}(M)$. Therefore $\text{ClSpec}(M) = \mathcal{V}(pM)$.

Now, let $M$ be a multiplication module and let $\text{ClSpec}(M) = \mathcal{V}(pM)$, for some $p \in \text{Supp}(M)$. Since $M$ is classical primeful, there exists $P \in \text{ClSpec}(M)$, such that $(P : M) = p$. Since $M$ is multiplication, we have $\text{ClSpec}(M) = \mathcal{V}(pM) = \mathcal{V}((P : M)M) = \mathcal{V}(P)$. This implies that $\text{ClSpec}(M)$ is an irreducible space by Corollary 3.1. $\square$
Let \( M \) be an \( R \)-module. For each subset \( N \) of \( M \), we denote \( \text{ClSpec}(M) - V(N) \) by \( \mathcal{U}(N) \). Further for each element \( m \in M \), \( \mathcal{U}(\{m\}) \) is denoted by \( \mathcal{U}(m) \). Hence

\[
\mathcal{U}(m) = \text{ClSpec}(M) - V(\{m\}).
\]

Moreover, for any family \( \{N_i\}_{i \in I} \) of submodules of \( M \), we have

\[
\mathcal{U}(\sum_{i \in I} N_i) = \mathcal{U}(\bigcup_{i \in I} N_i).
\]

**Theorem 3.5.** Let \( M \) be a classical top module. Then for every \( m \in M \), the sets \( \mathcal{U}(m) \) form a base for \( \text{ClSpec}(M) \).

**Proof.** Let \( \mathcal{U}(N) \) be an open set in \( \text{ClSpec}(M) \), where \( N \) is a submodule of \( M \). Then:

\[
\mathcal{U}(N) = \mathcal{U}(\bigcup_{n \in N} \{n\}) = \text{ClSpec}(M) - V\left(\bigcup_{n \in N} \{n\}\right)
\]

\[
= \text{ClSpec}(M) - \bigcap_{n \in N} V(\{n\})
\]

\[
= \bigcup_{n \in N} \left(\text{ClSpec}(M) - V(\{n\})\right)
\]

\[
= \bigcup_{n \in N} \mathcal{U}(n).
\]

Then for every \( m \in M \), the sets \( \mathcal{U}(m) \) form a base of \( \text{ClSpec}(M) \). \( \square \)

For a submodule \( N \) of an \( R \)-module \( M \), we put:

\[
\mathcal{F}(N) := \{L|L \subseteq N \text{ and } L \text{ is finitely generated}\}
\]

**Lemma 3.5.** Let \( M \) be an \( R \)-module and \( N \) be a submodule of \( M \). Then \( V(N) = \bigcap_{L \in \mathcal{F}(N)} V(L) \) and \( \mathcal{U}(N) = \bigcup_{L \in \mathcal{F}(N)} \mathcal{U}(L) \).

**Proof.** Suppose that \( P \in V(N) \). If \( L \in \mathcal{F}(N) \), then \( L \subseteq N \subseteq P \). Then \( P \in V(L) \), and \( V(N) \subseteq \bigcap_{L \in \mathcal{F}(N)} V(L) \). Now, let for every \( L \in \mathcal{F}(N) \), \( P \in V(L) \) and \( P \not\in V(N) \). Since \( N \not\subseteq P \), there exists \( x \in N \setminus P \). Then \( Rx \subseteq N \) and \( Rx \) is finitely generated, hence \( Rx \in \mathcal{F}(N) \). Therefore \( x \in Rx \subseteq P \), a contradiction. Thus \( \bigcap_{L \in \mathcal{F}(N)} V(L) \subseteq V(N) \). \( \square \)

**Theorem 3.6.** Let \( M \) be a classical top \( R \)-module. Then every quasi-compact open subset of \( \text{ClSpec}(M) \) is of the form \( \mathcal{U}(N) \), for some finitely generated submodule \( N \) of \( M \).
Proof. Suppose $U(B) = \text{Cl}(\text{Spec}(M)) \setminus \text{V}(B)$ is a quasi-compact open subset of $\text{Cl}(\text{Spec}(M))$. Then by Lemma 3.5, $U(B) = \bigcup_{L \in \mathcal{P}(B)} U(L)$. Now, since $U(B)$ is quasi-compact, then every open covering of $U(B)$ has a finite subcovering, therefore $U(B) = U(L_1) \cup \ldots \cup U(L_n) = U(\bigcap_{i=1}^{n} L_i)$. □

Proposition 3.4. Let $M$ be a classical top $R$-module. If $\text{Spec}(R)$ is a $T_1$-space, then $\text{Cl}(\text{Spec}(M))$ is also a $T_1$-space.

Proof. Suppose $Q$ is a classical prime submodule of $M$. Then $\text{Cl}(\{Q\}) = \text{V}(Q)$. If $P \in \text{V}(Q)$, then by Theorem 3.1, every prime ideal of $R$ is a maximal ideal, so $(Q : M) = (P : M)$, then by Lemma 2.1, $Q = P$. Therefore $\text{Cl}(\{Q\}) = \{Q\}$ and this implies that $\text{Cl}(\text{Spec}(M))$ is a $T_1$-space. □

Definition 3.2. A topological space $X$ is Noetherian provided that the open (respectively, closed) subsets of $X$ satisfy the ascending (respectively, descending) chain condition (see for example [3], page 79 or [10], §4.2).

Theorem 3.7. An $R$-module $M$ has Noetherian classical spectrum if and only if the ACC for classical radical submodules of $M$ holds.

Proof. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots$ be an ascending chain of classical radical submodules of $M$. Since for all $i \in \mathbb{N}$, $\sqrt{N_i} = N_i$, then equivalently

$$\sqrt{N_1} \subseteq \sqrt{N_2} \subseteq \sqrt{N_3} \subseteq \ldots$$

is an ascending chain of classical radical submodules of $M$. Then equivalently

$$\mathcal{V}(\mathcal{V}(N_1)) \subseteq \mathcal{V}(\mathcal{V}(N_2)) \subseteq \mathcal{V}(\mathcal{V}(N_3)) \subseteq \ldots$$

is an ascending chain of classical radical submodules of $M$. Therefore

$$\mathcal{V}(N_1) \supseteq \mathcal{V}(N_2) \supseteq \mathcal{V}(N_3) \supseteq \ldots$$

is a descending chain of closed sets $\mathcal{V}(N_i)$ of $\text{Cl}(\text{Spec}(M))$. Now, $R$-module $M$ has Noetherian spectrum if and only if $\text{Cl}(\text{Spec}(M))$ is a Noetherian topological space if and only if there exists a positive integer $k$ such that $\mathcal{V}(N_k) = \mathcal{V}(N_{k+n})$ for each $n = 1, 2, \ldots$ if and only if $\sqrt{N_k} = \sqrt{N_{k+n}}$ if and only if $N_k = N_{k+n}$ if and only if the ACC for classical radical submodules of $M$ holds. □

Theorem 3.8. Let $M$ be a classical top $R$-module such that $\text{Cl}(\text{Spec}(M))$ is a Noetherian space. Then the following statements are true.

1. Every ascending chain of classical prime submodules of $M$ is stationary.

2. The set of minimal classical prime submodules of $M$ is finite. In particular, $\text{Cl}(\text{Spec}(M)) = \bigcup_{i=1}^{n} \mathcal{V}(P_i)$, where $P_i$ are all minimal classical prime submodules of $M$. 
Proof. (1). Suppose $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots$ is an ascending chain of classical prime submodules of $M$. Therefore $V(N_1) \supseteq V(N_2) \supseteq \ldots$ is a descending chain of closed subsets of Cl.Spec$(M)$, which is stationary by assumption. There exists an integer $k \in \mathbb{N}$ such that $V(N_k) = V(N_{k+i})$, for each $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, $N_k = N_{k+i}$.

(2). This follows from Theorem 3.3 and the fact that if $X$ is a Noetherian space, then the set of irreducible components of $X$ is finite (see for example [10, Proposition 10]). □

Recall that if $M$ is a Noetherian module, then each open subset of Spec$(M)$ is quasi-compact (see for example [15, Theorem 3.3]). The next theorem shows that the same result is true for Cl.Spec$(M)$ in Noetherian classical top modules.

Theorem 3.9. Let $M$ be a Noetherian classical top module. Then each open subset of Cl.Spec$(M)$ is quasi-compact.

Proof. Let for every submodule $N$ of $M$, $U(N)$ be an open subset of Cl.Spec$(M)$. Also, let $\{U(n_i)\}_{n_i \in N}$ be a basic open cover for $U(N)$. We show that there exist a finite subfamily of $\{U(n_i)\}_{n_i \in N}$ which covers Cl.Spec$(M)$. Since $U(N) \subseteq \bigcup_{n_i \in N} U(n_i) = U(\bigcup_{n_i \in N} n_i)$, then for every submodule $K$ of $M$ that is generated by the set $A = \{n_i\}_{i \in I}$, $U(N) \subseteq U(K)$. Since $M$ is a Noetherian module, then $K = \langle k_1, k_2, \ldots, k_n \rangle$, for some $k_i \in K$, therefore $b_i = \sum_{j=1}^{n} r_{ij} n_{ij}$, where $i = 1, \ldots, n$ and $n_{ij} \in A$. Thus there exists the subset $\{n_1, \ldots, n_m\} \subseteq A$ such that $K = \langle n_1, \ldots, n_m \rangle$. So $U(N) \subseteq U(K) = U(\langle n_1, \ldots, n_m \rangle)$. Then

$$U(N) \subseteq U(\bigcup_{i=1}^{n} n_i) = \bigcup_{i=1}^{n} U(n_i).$$

consequently, $U(N)$ is quasi-compact. □

Recall that a function $\Phi$ between two topological spaces $X$ and $Y$ is called an open map if, for any open set $U$ in $X$, the image $\Phi(U)$ is open in $Y$. Also, $\Phi$ is called a homeomorphism if it has the following properties

(i) $\Phi$ is a bijection;
(ii) $\Phi$ is continuous;
(iii) $\Phi$ is an open map

A spectral space is a topological space homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. By Hochster’s characterization [15], a topology $\tau$ on a set $X$ is spectral if and only if the following axioms hold:

(i) $X$ is a $T_0$-space.
(ii) $X$ is quasi-compact and has a basis of quasi-compact open subsets.
(iii) The family of quasi-compact open subsets of $X$ is closed under finite intersections.

(iv) $X$ is a sober space; i.e., every irreducible closed subset of $X$ has a generic point.

For any ring $R$, it is is well-known that $\text{Spec}(R)$ satisfies these conditions (cf. [10], Chap. II, 4.1 - 4.3). We show that $\text{ClSpec}(M)$ is necessarily a spectral space in the classical quasi-Zariski topology for every module $M$.

We remark that any closed subset of a spectral space is spectral for the induced topology.

**Theorem 3.10.** Let $M$ be a classical top primful $R$-module, $\overline{R} = R/\text{Ann}(M)$ and let $\psi$ be the natural map of $\text{ClSpec}(M)$. Then $\psi$ is a homeomorphism.

**Proof.** It is clear by Lemma 2.1, Proposition 3.2, Lemma 3.3 and Corollary 3.2. □

**Corollary 3.3.** Let $M$ be a classical top primful $R$-module. Then $\text{ClSpec}(M)$ with classical quasi-Zariski topology is a spectral space.

**Lemma 3.6.** Let $M$ be a classical top $R$-module. Then the following statements are equivalent:

(a) the natural map $\psi : \text{ClSpec}(M) \rightarrow \text{Spec}(R/\text{Ann}(M))$ is injective.

(b) $\text{ClSpec}(M)$ is a $T_0$-space.

**Proof.** We recall that a topological space is $T_0$ if and only if the closures of distinct points are distinct. Now, the result follows from

$$ P = Q \iff V(P) = V(Q), \quad \forall P, Q \in \text{ClSpec}(M). $$

**Corollary 3.4.** Let $M$ be a Noetherian classical primeful top module. Then the following statements are holed:

(i) $\text{ClSpec}(M)$ is a $T_0$-space.

(ii) $\text{ClSpec}(M)$ is quasi-compact and has a basis of quasi-compact open subsets.

(iii) The family of quasi-compact open subsets of $\text{ClSpec}(M)$ is closed under finite intersections.

(iv) $\text{ClSpec}(M)$ is a sober space; i.e., every irreducible closed subset of $\text{ClSpec}(M)$ has a generic point.

**Proof.** It is clear by Lemma 3.6, Theorem 3.5, Theorem 3.9, Theorem 3.2. □

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