NONCLASSICAL DIFFUSION WITH MEMORY
LACKING INSTANTANEOUS DAMPING

Dedicated to our colleague and friend Professor Tomás Caraballo
on the occasion of his sixtieth birthday

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Abstract. We consider the nonclassical diffusion equation with hereditary memory

\[ u_t - \Delta u_t - \int_0^\infty \kappa(s) \Delta u(t-s) \, ds + f(u) = g \]

on a bounded three-dimensional domain. The main feature of the model is that the equation does not contain a term of the form \(-\Delta u\), contributing as an instantaneous damping. Setting the problem in the past history framework, we prove that the related solution semigroup possesses a global attractor of optimal regularity.

1. Introduction. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary \( \partial \Omega \). We consider the nonclassical diffusion equation with memory

\[ u_t - \Delta u_t - \int_0^\infty \kappa(s) \Delta u(t-s) \, ds + f(u) = g, \tag{1.1} \]

in the unknown variable \( u = u(x,t) : \Omega \times \mathbb{R} \to \mathbb{R} \), subject to Dirichlet boundary conditions

\[ u|_{\partial \Omega} = 0. \tag{1.2} \]

The problem is supplemented with the initial conditions

\[ u(0) = u_0 \quad \text{and} \quad u(-s)|_{s>0} = \varphi_0(s), \tag{1.3} \]

where \( u_0 : \Omega \to \mathbb{R} \) and \( \varphi_0 : \Omega \times \mathbb{R}^+ \to \mathbb{R} \) are prescribed data.

We make the following assumptions on the structural quantities of the problem:

- \( g \in L^2(\Omega) \) is a time-independent external force.
- The nonlinearity \( f \), with \( f(0) = 0 \), fulfills the growth restriction

\[ |f(u) - f(v)| \leq k_f |u-v|(1 + |u|^4 + |v|^4), \tag{1.4} \]

for some \( k_f > 0 \), along with the dissipation conditions

\[ F(u) \geq -c_f, \tag{1.5} \]

\[ f(u)u \geq \nu F(u) - c_f, \tag{1.6} \]

for some \( c_f \geq 0 \) and \( \nu > 0 \), having set \( F(u) = \int_0^u f(y) \, dy \).

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• The memory kernel $\kappa$ is a nonnegative summable function (we take it of unitary mass) of the form

$$\kappa(s) = \int_{s}^{\infty} \mu(\sigma) \, d\sigma,$$

where $\mu \in W^{1,1}(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is a nonnegative function such that

$$\mu'(s) + \delta \mu(s) \leq 0,$$  \hspace{1cm} (1.7)

for some $\delta > 0$ and almost every $s > 0$. In particular, these assumptions imply that

$$\kappa(0) = \int_{0}^{\infty} \mu(\sigma) \, d\sigma < \infty,$$

and

$$\mu(0) < \infty,$$

namely, $\mu$ can be continuously extended to the origin.

**Remark 1.1.** We mention here (but see the final Section 6), that $\mu$ can be taken to be unbounded in the origin; also, it is allowed to exhibit downward jumps, even countably many.

Equation (1.1) is an integro-differential relaxation of the nonclassical diffusive equation

$$u_t - \Delta u_t - \Delta u + f(u) = g,$$  \hspace{1cm} (1.8)

often referred to in the literature as pseudo-parabolic equation (in spite of its full hyperbolic character). Equation (1.8) appears in connection with the description of diffusion in solids, and is recovered from (1.1) in the limit case when $\kappa$ is equal to the Dirac mass at $0^+$. For instance, in the paper [1], equation (1.8) serves as a model of the diffusion of a linearly viscous fluid into a solid matrix, $u$ representing the density of the fluid; the same equation is derived in [2] to describe the flow of liquids in fissured rocks; in [4], it appears within the theory of heat conduction with two temperatures. Nonetheless, (1.1) provides a more accurate description of the diffusive process in certain materials, such as high-viscosity liquids at low temperatures and polymers (see e.g. [21]). Indeed, the convolution term allows to take into account the influence of the past history of $u$ on its future evolution, a fact which is experimentally observed. The model with memory corresponding to the nonclassical diffusion equation (1.8) with the additional instantaneous dissipation term $-\Delta u$, that is,

$$u_t - \Delta u_t - \Delta u - \int_{0}^{\infty} \kappa(s) \Delta u(t - s) \, ds + f(u) = g,$$  \hspace{1cm} (1.9)

has been studied in [10, 12, 26, 27]. In particular, a complete analysis of the asymptotic behavior of solutions has been given in [12], in terms of global attractors of optimal regularity for the associated dynamical system, under very mild assumptions on the memory kernel. Indeed, in [12] the only requirement on $\mu$ is

$$\kappa(s) \leq \Theta \mu(s),$$  \hspace{1cm} (1.10)

for some $\Theta > 0$ and every $s > 0$, which can be equivalently expressed in the form

$$\mu(\sigma + s) \leq M e^{-\delta \sigma} \mu(s),$$

for some $M \geq 1$, $\delta > 0$, every $\sigma \geq 0$ and almost every $s > 0$. Assumption (1.10) is much weaker than (1.7), and it is actually the most general condition that one might ask within the class of decreasing (hence physical) memory kernels. In particular, its
failure prevents the uniform decay of solutions for systems with memory, no matter the types of equations involved, see [6]. For equation (1.9), the analysis under the sole assumption (1.10) is made possible by the presence of the instantaneous diffusion $-\Delta u$. On the contrary, here we focus on the more delicate case where such a term is not present, and the whole dissipation is contributed by the convolution term only. Then, the weaker condition (1.10) no longer suffices to ensure the decay of the energy, as the following example shows.

**Example 1.2.** For $\ell > 0$, we denote by $\chi_{[0,\ell)}$ the characteristic function of the interval $[0,\ell)$, and we consider the kernel

$$\kappa(s) = \int_{s}^{\infty} \chi_{[0,\ell)}(\sigma) \, d\sigma.$$ 

Accordingly,

$$\mu(s) = \chi_{[0,\ell)}(s),$$

which clearly complies with (1.10) (cf. Remark 1.1), but not with (1.7). For $\omega > 0$, set

$$u(t) = v \sin(\omega t),$$

where $v$ is the eigenvector (normalized in $L^2(\Omega)$) of the Laplace-Dirichlet operator corresponding to the first eigenvalue $\lambda_1 > 0$, so that

$$-\Delta v = \lambda_1 v \quad \text{and} \quad v|_{\partial \Omega} = 0.$$ 

We claim that it is possible to choose $\ell$ and $\omega$ in such a way that $u$ is a solution to the linear homogeneous version of (1.1), i.e. with $f(u) = 0$ and $g = 0$, with the memory kernel above. It is then clear that corresponding energy, one of whose (nonnegative) addends is the oscillating term

$$\|\nabla u(t)\|_{L^2(\Omega)}^2 = \lambda_1 \sin^2(\omega t),$$

cannot enjoy any decay. To see that, defining

$$I = -\int_{0}^{\infty} \kappa(s) \Delta u(t-s) \, ds,$$

we observe that, for $\kappa$ and $u$ as above,

$$I = \int_{0}^{\ell} (\ell - s) \lambda_1 v \sin(\omega(t-s)) \, ds,$$

and an integration by parts yields

$$I = -\frac{\ell \lambda_1}{\omega} v \cos(\omega t) - \frac{\lambda_1}{\omega^2} v[\sin(\omega t - \omega \ell) - \sin(\omega t)].$$

Choosing $\omega = 2\pi/\ell$, the second term disappears, and we get

$$I = -\frac{\ell^2 \lambda_1}{2\pi} v \cos\left(\frac{2\pi t}{\ell}\right).$$

Therefore, the linear homogeneous version of (1.1) becomes

$$\left[\frac{2\pi}{\ell} + \frac{2\pi \lambda_1}{\ell} - \frac{\ell^2 \lambda_1}{2\pi}\right] v \cos\left(\frac{2\pi t}{\ell}\right) = 0,$$

which is satisfied by taking

$$\ell = \sqrt{4\pi^2 \left(1 + \frac{1}{\lambda_1}\right)}.$$
The discussion above motivates us to assume that $\mu$ satisfies the inequality (1.7), firstly introduced by Dafermos [15], and successfully employed in most of the literature thereafter. In this note, after rephrasing the equation in the so-called past history framework (Section 2), we show that the associated solution semigroup $S(t)$, acting on a suitable phase space accounting for the past values of $u$, is indeed dissipative when (1.7) holds true (Section 3). In particular, in the linear homogeneous case corresponding to $f(u) = 0$ and $g = 0$, the energy is driven to zero exponentially fast. This is a crucial point in the mathematical analysis, since the lack of instantaneous damping introduces new difficulties with respect to the previous literature. Accordingly the employment of suitable additional multipliers is needed.

We complete the picture by proving the existence of the global attractor of optimal regularity. This is done in two steps (Sections 4 and 5). First, the existence is obtained in a quite direct way, by exploiting an appropriate decomposition of the semigroup, along with time-dependent higher-order energy estimates. At this point, we take advantage of the abstract scheme devised in [14], which provides a tool apt to establish the regularity of the attractor without uniform-in-time estimates. We conclude the paper with some remarks on possible advances in the study of the model (Section 6), discussing in particular the possibility of weakening the assumptions on the memory kernel.

**Functional setting.** To avoid the presence of unnecessary constants, from now on we assume

$$\kappa(0) = \int_0^\infty \mu(\sigma) \, d\sigma = 1,$$

which can be always obtained by rescaling the memory kernel. In order to incorporate the boundary condition (1.2) into the functional framework, we introduce the strictly positive selfadjoint Laplace-Dirichlet operator

$$A = -\Delta \quad \text{with domain} \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega).$$

Then, for $r \in \mathbb{R}$, we consider the hierarchy of compactly nested Hilbert spaces ($r$ will be always omitted whenever zero)

$$H^r = \mathcal{D}(A^{r/2}), \quad \langle u, v \rangle_r = \langle A^{r/2}u, A^{r/2}v \rangle_{L^2(\Omega)}, \quad \|u\|_r = \|A^{r/2}u\|_{L^2(\Omega)}.$$

As customary, the symbol $\langle \cdot, \cdot \rangle$ will also stand for the duality product between $H^r$ and its dual space $H^{-r}$. In particular,

$$H = L^2(\Omega), \quad H^1 = H_0^1(\Omega), \quad H^2 = H^2(\Omega) \cap H^1_0(\Omega).$$

We recall the Poincaré inequalities

$$\lambda_1 \|u\|_{r+1}^2 \leq \|u\|_r^2, \quad \forall u \in H^{r+1},$$

where $\lambda_1 > 0$ is the first eigenvalue of $A$. Next, we define the (Hilbert) history spaces

$$\mathcal{M}^r = L^1_\mu(\mathbb{R}^+; H^{r+1}), \quad \langle \eta, \xi \rangle_{\mathcal{M}^r} = \int_0^\infty \mu(s) \langle \eta(s), \xi(s) \rangle_{r+1} \, ds,$$

and the extended memory spaces

$$\mathcal{H}^r = H^{r+1} \times \mathcal{M}^r,$$

endowed with the standard (Euclidean) product norm (again, $r$ is omitted if zero).
General agreements. Given a Banach space $X$, we denote by $B_X(R)$ the closed ball in $X$ of radius $R \geq 0$ centered at the origin, i.e.

$$B_X(R) = \{ x \in X : \| x \|_X \leq R \}.$$ 

Throughout the paper, $C \geq 0$ and $Q : [0, \infty) \to [0, \infty)$ will stand for a generic constant and a generic increasing function, respectively, depending only on the structural parameters of the problem. Further dependencies, if any, will be specified from case to case. We will also use, often without explicit mention, the Sobolev embeddings, as well as the Young, Hölder and Poincaré inequalities.

2. The solution semigroup. In the same spirit of [15], we introduce the auxiliary variable

$$\eta^t(s) = \int_0^s u(t-y)dy,$$

containing all the information on the (integrated) past history of $u$, and satisfying the “boundary condition” $\eta^t(0) = 0$. Exploiting the initial condition (1.3), the variable $\eta^t$ takes the form

$$\eta^t(s) = \begin{cases} 
\int_0^s u(t-y)dy, & 0 < s \leq t, \\
\eta_0(s-t) + \int_0^t u(t-y)dy, & s > t,
\end{cases} \tag{2.1}$$

where

$$\eta_0(s) = \int_0^s \varphi_0(y)dy.$$ 

Then, integrating by parts (or exchanging the order of integration), we rewrite (1.1)-(1.2) as

$$u_t + Au_t + \int_0^\infty \mu(s)A\eta(s)ds + f(u) = g. \tag{2.2}$$

Indeed, equations (2.1)-(2.2) above indicate the correct strategy to translate the original boundary value problem (1.1)-(1.2) into an evolution problem in the extended memory space $\mathcal{H}$.

**Theorem 2.1.** For every initial datum $z = (u_0, \eta_0) \in \mathcal{H}$, there is a unique

$$(u(t), \eta^t) \in C_b([0, \infty), \mathcal{H})$$

satisfying (2.1)-(2.2) with $u(0) = u_0$, and $\eta^0 = \eta_0$ by construction. Besides,

$$u_t \in L^\infty(\mathbb{R}^+; H^1).$$

Accordingly, writing

$$S(t)z = (u(t), \eta^t),$$

the one-parameter family $S(t) : \mathcal{H} \to \mathcal{H}$ turns out to be a strongly continuous semigroup.

In the statement above, $C_b([0, \infty), \mathcal{H})$ is the space of bounded continuous functions from $[0, \infty)$ to $\mathcal{H}$ with the supremum norm. The proof of the result follows by an application of a Galerkin scheme, based on the energy estimates of the forthcoming Theorem 3.1.
It is worth noting that Theorem 2.1 can be equivalently formulated by saying that there is a unique weak solution to the problem in $H$

$$
\begin{cases}
  u_t + Au_t + \int_0^\infty \mu(s)A\eta(s)\,ds + f(u) = g, \\
  \eta_t = T\eta + u,
\end{cases}
$$

(2.3)

where $T$ is the infinitesimal generator of the right-translation semigroup on $M$, namely,

$$
T\eta = -\eta' \quad \text{with domain} \quad D(T) = \{\eta \in M : \eta' \in M, \eta(0) = 0\},
$$

the prime standing for weak derivative of $\eta(s)$ with respect to $s$. The following equality from [6, 18] will be needed in the sequel:

$$
\langle T\eta, \eta \rangle_M = \frac{1}{2} \int_0^\infty \mu'(s)\|\eta(s)\|_1^2\,ds \leq 0, \quad \forall \eta \in D(T). \quad (2.4)
$$

**Remark 2.2.** The notion of solution implicitly contained in Theorem 2.1, and first devised in [7], is completely equivalent to the one of problem (2.3). Nonetheless, this different point of view, besides being more closely related to the transport nature of the memory component, has the main advantage to render much easier the computation in the Galerkin approximation scheme, especially when passing to the limit.

3. **Dissipativity.** In this section we show that semigroup $S(t)$ is dissipative on $H$. Namely, it admits a bounded absorbing set $B_0 \subset H$ such that for any $R > 0$, there is an entering time $t_R \geq 0$ for which

$$
S(t)B_H(R) \subset B_0, \quad \forall t \geq t_R.
$$

The argument is based on uniform-in-time estimates for the natural energy $E(t)$ of the solution $(u, \eta)$ defined by

$$
E(t) = \frac{1}{2} \left[ \|u(t)\|_1^2 + \|u\|_1^2 + \|\eta\|_M^2 \right].
$$

More precisely, the following result holds.

**Theorem 3.1.** There exist structural constants $C_0 \geq 0$ and $\nu_0 > 0$ such that

$$
E(t) \leq Q(E(0))e^{-\nu_0 t} + C_0,
$$

for all initial data in $H$. Furthermore, $C_0 = 0$ whenever $g \equiv 0$ and $f$ satisfies (1.5)-(1.6) with $c_f = 0$. In that case, the energy $E$ decays exponentially to zero.

**Remark 3.2.** It is clear from Theorem 3.1 that for any $R_0 > 0$ sufficiently large the ball $B_0 = B_H(R_0)$ is an absorbing set for $S(t)$. Besides, Theorem 3.1 proves the exponential decay of the energy whenever no external force $g$ is acting on the system for a wide class of nonlinearities satisfying the sign condition $f(u)u \geq \nu F(u) \geq 0$, and, in particular, for the linear model corresponding to $f \equiv 0$.

The proof of Theorem 3.1 exploits in a crucial way the following technical lemma.

**Lemma 3.3.** Assume that $(u, \eta)$ is a sufficiently regular solution to (2.3), satisfying in particular $\eta \in D(T)$. Then, the functional

$$
N_0(t) = -\langle u(t), \eta' \rangle_M
$$
fulfills the differential inequality
\[
\frac{d}{dt} N_0(t) + \frac{1}{2} \| u(t) \|^2_1 \leq \frac{1}{4} \| u_t(t) \|^2_1 + \| \eta' \|^2_M + \frac{\mu(0)}{2} \int_0^\infty -\mu'(s) \| \eta'(s) \|^2 ds.
\]
Besides, we have the control
\[
|N_0(t)| \leq \frac{1}{2} \| (u(t), \eta') \|^2_M \leq \mathcal{E}(t).
\] (3.1)

**Proof.** Since \( \mu \) has unitary total mass, (3.1) is an immediate consequence of
\[|N_0| \leq \| u \|_1 \| \eta \|_M.\]
The time-derivative on \( N_0 \) reads
\[
\frac{d}{dt} N_0 = -\langle u, \eta \rangle_M - \langle u_t, \eta \rangle_M.
\]
The second term in the right-hand side can be estimated as
\[
-\langle u_t, \eta \rangle_M \leq \| u_t \|_1 \left( \int_0^\infty \mu(s) \| \eta(s) \|^2 ds \right)^{1/2} \leq \frac{1}{4} \| u_t \|^2_1 + \| \eta \|^2_M.
\]
Concerning the first term, in light of the second equation in (2.3) we get the identity
\[
-\langle u, \eta \rangle_M = -\langle u, T \eta \rangle_M - \| u \|^2_1.
\]
Integrating by parts with respect to \( s \) (the boundary terms disappear due to the decay of \( \mu \) and the equality \( \eta(0) = 0 \)), we obtain
\[
-\langle u, T \eta \rangle_M = -\int_0^\infty \mu'(s) \langle u, \eta(s) \rangle_1 ds
\]
\[
\leq \sqrt{\mu(0)} \| u \|_1 \left( \int_0^\infty -\mu'(s) \| \eta(s) \|^2 ds \right)^{1/2}
\]
\[
\leq \frac{1}{2} \| u \|^2_1 + \frac{\mu(0)}{2} \int_0^\infty -\mu'(s) \| \eta(s) \|^2 ds.
\]
The desired differential inequality follows by collecting the estimates above. \(\square\)

**Proof of Theorem 3.1.** Multiplying the first equation of (2.3) by \( u \) in \( \mathcal{H} \) and the second one by \( \eta \) in \( \mathcal{M} \), and exploiting (2.4), we obtain the equality
\[
\frac{d}{dt} \mathcal{E} + \frac{1}{2} \int_0^\infty -\mu'(s) \| \eta(s) \|^2_1 ds + \langle f, u \rangle = \langle g, u \rangle.
\]
Invoking (1.7), we get
\[
\frac{d}{dt} \mathcal{E} + \frac{1}{4} \int_0^\infty -\mu'(s) \| \eta(s) \|^2_1 ds + \frac{\delta}{4} \| \eta \|^2_M + \langle f, u \rangle \leq \langle g, u \rangle. \tag{3.2}
\]
We now multiply the first equation of (2.3) by \( u_t \) in \( \mathcal{H} \), yielding
\[
\| u_t \|^2_1 + \| u_t \|^2_1 + \frac{d}{dt} \langle F(u), 1 \rangle = \langle g, u_t \rangle - \int_0^\infty \mu(s) \langle \eta(s), u_t \rangle_1 ds.
\]
Note that
\[
- \int_0^\infty \mu(s) \langle \eta(s), u_t \rangle_1 ds \leq \frac{1}{2} \| u_t \|^2_1 + \frac{1}{2} \| \eta \|^2_M,
\]
and
\[
\langle g, u_t \rangle \leq \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| g \|^2.
\]
Therefore,
\[
\frac{d}{dt} \langle F(u), 1 \rangle + \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| u \|^2 \leq \frac{1}{2} \| \eta \|^2_{\mathcal{M}} + \frac{1}{2} \| g \|^2. \tag{3.3}
\]
At this point, for \( \varepsilon \in (0, 1) \) to be fixed later, we define the functional
\[
\Phi_\varepsilon(t) = \mathcal{E}(t) + \varepsilon \langle F(u(t)), 1 \rangle + \varepsilon N_0(t).
\]
In light of (3.2), (3.3) and Lemma 3.3, we see that
\[
\frac{d}{dt} \Phi_\varepsilon(t) \leq \frac{d}{dt} \Phi_\varepsilon(t) + \langle f(u), u \rangle + \frac{\varepsilon}{4} \| u_t \|^2 + \frac{\varepsilon}{2} \| u \|^2 + \frac{\varepsilon}{2} \| u \|^2
\]
Using the control
\[
\langle g, u \rangle \leq \frac{\varepsilon}{4} \| u \|^2 + \frac{1}{\varepsilon \lambda_1} \| g \|^2,
\]
and choosing \( \varepsilon > 0 \) small enough, we obtain
\[
\frac{d}{dt} \Phi_\varepsilon + 2\varepsilon^2 \mathcal{E} + \langle f(u), u \rangle \leq \left( \frac{\varepsilon}{2} + \frac{1}{\varepsilon \lambda_1} \right) \| g \|^2 \leq \frac{2}{\varepsilon \lambda_1} \| g \|^2. \tag{3.4}
\]
Then, in light of (1.6) and (3.1), for a possibly smaller \( \varepsilon > 0 \) we end up with
\[
\frac{d}{dt} \Phi_\varepsilon + \varepsilon^2 \Phi_\varepsilon \leq \frac{k}{\varepsilon}, \tag{3.5}
\]
where
\[
k = k(f, g) = \frac{2}{\lambda_1} \| g \|^2 + |\Omega| c_f,
\]
being \( |\Omega| \) the (finite) measure of \( \Omega \). A direct application of the Gronwall lemma gives
\[
\Phi_\varepsilon(t) \leq \Phi_\varepsilon(0) e^{-\varepsilon^2 t} + \frac{k}{\varepsilon^3}.
\]
In light of (1.4), (1.5) and (3.1), up to fixing \( \varepsilon \) sufficiently small, we also have
\[
\frac{1}{2} \mathcal{E}(t) - |\Omega| c_f \leq \Phi_\varepsilon(t) \leq C \mathcal{E}^3(t) + C, \tag{3.6}
\]
for some structural constant \( C > 0 \). The conclusion follows.

4. **The global attractor.** A relevant object in the description of the asymptotic dynamics of \( S(t) \) is the **global attractor**, namely, the unique compact set \( \mathfrak{A} \subset \mathcal{H} \) which is at the same time:
- **fully invariant**, i.e \( S(t) \mathfrak{A} = \mathfrak{A} \) for every \( t \geq 0 \); and
- **attracting**, i.e. for every \( R \geq 0 \),
\[
\lim_{t \to \infty} \text{dist}_{\mathcal{H}}(S(t)B_{\mathcal{H}}(R), \mathfrak{A}) = 0,
\]
where \( \text{dist}_{\mathcal{H}} \) denotes the standard Hausdorff semidistance in \( \mathcal{H} \).

Recall that the Hausdorff semidistance in \( \mathcal{H} \) between two (nonempty) sets \( B_1, B_2 \subset \mathcal{H} \) is given by
\[
\text{dist}_{\mathcal{H}}(B_1, B_2) = \sup_{z_1 \in B_1} \inf_{z_2 \in B_2} \| z_1 - z_2 \|_{\mathcal{H}}.
\]
The aim of this section is to prove the following theorem.

**Theorem 4.1.** The semigroup \( S(t) \) possesses the global attractor \( \mathfrak{A} \).
The strategy consists in showing that the semigroup is asymptotically compact. It amounts to saying that for any $t \geq 0$ there exists a compact set $\mathcal{K}(t) \subset \mathcal{H}$ such that

$$\lim_{t \to \infty} \text{dist}_\mathcal{H}(S(t)\mathbb{B}_0, \mathcal{K}(t)) = 0,$$

(4.1)

where $\mathbb{B}_0$ is a bounded absorbing set according to Theorem 3.1. It is well known that such a condition is sufficient for the existence of the global attractor, see e.g. [19, 20, 25]. In what follows, the generic constant $C \geq 0$ will depend on the absorbing set $\mathbb{B}_0$, besides on the structural quantities of the problem.

We start by showing that for any $t \geq 0$ there is a closed bounded set $\mathcal{B}(t) \subset \mathcal{H}^{1/3}$ satisfying

$$\text{dist}_\mathcal{H}(S(t)\mathbb{B}_0, \mathcal{B}(t)) \leq Ce^{-\omega t},$$

(4.2)

for some $\omega > 0$. To this end, we decompose the nonlinearity $f$ into the sum

$$f(u) = f_0(u) + f_1(u),$$

for some continuous $f_0, f_1$ with $f_0(0) = 0$ such that

$$f_0(u)u \geq 0, \quad F_0(u) = \int_0^u f_0(y)dy \geq 0,$$  

(4.3)

$$|f_0(u) - f_0(v)| \leq c|u - v|(|u|^4 + |v|^4),$$

(4.4)

$$|f_1(u)| \leq c,$$  

(4.5)

for some $c > 0$ depending only on $f$. Indeed, having in mind (1.4)-(1.6), we can take

$$f_0(u) = \sigma(u)\left[f(u) + \frac{(\nu + 1)c_1}{u}\right] \quad \text{and} \quad f_1(u) = f(u) - f_0(u),$$

where $\sigma : \mathbb{R} \to [0, 1]$ is a Lipschitz function with $\sigma(u) = 0$ if $|u| \leq 1$ and $\sigma(u) = 1$ if $|u| > 2$. Then, for $z \in \mathbb{B}_0$, we split the solution $S(t)z = (u(t), \eta^t)$ into the sum

$$S(t)z = (\hat{v}(t), \hat{\xi}^t) + (\hat{w}(t), \hat{\zeta}^t),$$

where $(\hat{v}(t), \hat{\xi}^t)$ and $(\hat{w}(t), \hat{\zeta}^t)$ are the solutions to

$$\begin{cases}
\hat{v}_t + A\hat{v}_t + \int_0^\infty \mu(s)A\hat{\xi}(s)\,ds + f_0(\hat{v}) = 0,
\hat{\xi}_t = T\hat{\xi} + \hat{v},
(\hat{v}(0), \hat{\xi}^0) = z,
\end{cases}$$

(4.6)

and

$$\begin{cases}
\hat{w}_t + A\hat{w}_t + \int_0^\infty \mu(s)A\hat{\zeta}(s)\,ds = \hat{\gamma},
\hat{\zeta}_t = T\hat{\zeta} + \hat{w},
(\hat{w}(0), \hat{\zeta}^0) = 0,
\end{cases}$$

(4.7)

respectively, where

$$\hat{\gamma} = -f_0(u) + f_0(\hat{v}) - f_1(u) + g.$$  

We now claim that

$$\|((\hat{v}(t), \hat{\xi}^t))\|_{\mathcal{H}} \leq C\|z\|_{\mathcal{H}}e^{-\omega t},$$

(4.8)

for some $\omega > 0$, and

$$\|((\hat{w}(t), \hat{\zeta}^t))\|_{\mathcal{H}^{1/3}} \leq R(t) \quad \text{where} \quad R(t) = Ce^{Ct}. $$

(4.9)
Once the latter two inequalities are established, the conclusion follows by noting that the closed ball 
\[ B(t) = B_{\mathcal{H}^{1/3}}(R(t)) \]
fulfills (4.2).

We now proceed to verify (4.8) and (4.9). The exponential decay (4.8) is obtained arguing exactly as in the proof of Theorem 3.1, replacing \( f \) with \( f_0 \) and taking \( g \equiv 0 \). The only difference is in the passage from (3.4) to (3.5), since this time \( f_0 \) does not satisfy (1.6). Nonetheless, thanks to (4.3), the differential inequality (3.2) yields 
\[ \frac{d}{dt} \mathcal{E} \leq 0, \]
where now 
\[ \mathcal{E}(t) = \frac{1}{2} \left[ \| \hat{\gamma}(t) \|_{r}^2 + \| \hat{\nu}(t) \|_{r}^2 + \| \hat{\xi} \|^2_{\mathcal{M}} \right], \]
implying \( \mathcal{E}(t) \leq C \). This, together with (4.3) and (4.4) tell that, instead of (3.6), we have the controls 
\[ \frac{1}{2} \mathcal{E}(t) \leq \Phi_\epsilon(t) \leq C \mathcal{E}(t). \]
Therefore, (3.5) holds true with \( k = 0 \) and the desired conclusion follows from the Gronwall lemma. In order to show (4.9), we multiply the first equation of (4.7) by \( \hat{w} \) in \( \mathcal{H}^r \) and the second one by \( \hat{\zeta} \) in \( \mathcal{M}^r \) with \( r = 1/3 \). On account of (2.4), we obtain the estimate 
\[ \frac{d}{dt} \hat{\mathcal{E}}_r = \langle \hat{\gamma}, A^r \hat{w} \rangle + \langle TA^{r/2} \hat{\zeta}, A^{r/2} \hat{\zeta} \rangle_{\mathcal{M}} \leq \langle \hat{\gamma}, A^r \hat{w} \rangle, \]
having set 
\[ \hat{\mathcal{E}}_r(t) = \frac{1}{2} \left[ \| \hat{\nu}(t) \|_{r+1}^2 + \| \hat{\nu}(t) \|_{r}^2 + \| \hat{\xi} \|^2_{\mathcal{M}} \right]. \]
In light of (4.4) and (4.5),
\[ |\hat{\gamma}| \leq c |\hat{w}| (|\nu|^4 + |\hat{\nu}|^4) + |g| + c. \]
The Sobolev embeddings
\[ \mathcal{H}^{2/3} \subset L^{18/5}(\Omega) \quad \text{and} \quad \mathcal{H}^{4/3} \subset L^{18}(\Omega), \]
jointly with Theorem 3.1 and (4.8), entail
\[ \langle \hat{\gamma}, A^r \hat{w} \rangle \leq c (\| u \|^4_{L^6} + \| \hat{\nu} \|^2_{L^6}) \| \hat{w} \|_{L^{18}} \| A^r \hat{w} \|_{L^{18/5}} + (|\Omega|^{1/2} c + \| g \|) \| \hat{w} \|^2_{2/3} + C. \quad (4.10) \]
Thus, we arrive at the differential inequality 
\[ \frac{d}{dt} \hat{\mathcal{E}}_r \leq C \hat{\mathcal{E}}_r + C. \]
Since \( \hat{\mathcal{E}}_r(0) = 0 \), the Gronwall lemma yields
\[ \left( \| \hat{\nu}(t) \|_{\mathcal{H}^{1/3}} \right)^2 \leq 2 \hat{\mathcal{E}}_r(t) \leq Ce^{Ct}, \]
which is nothing but (4.9).

At this point, we observe that the set \( B(t) \) appearing in (4.2), although bounded in \( \mathcal{H}^{1/3} \), is not compact in \( \mathcal{H} \), due to the lack of compactness of the embedding \( \mathcal{H}^{1/3} \subset \mathcal{H} \) (see e.g. [24]). This obstacle is overcome by considering the decomposition above, and defining
\[ \Xi_t = \bigcup_{z \in \mathcal{B}_0} \hat{\xi}^t. \]
As a consequence, the set
\[ K(t) = \mathcal{B}_{H^{1/3}}(R(t)) \times \Xi_t \subset \mathcal{B}(t), \]
the bar standing for closure in \( \mathcal{M} \), is compact in \( \mathcal{H} \). Due to (4.8)-(4.9), the attracting property (4.1) holds.

5. Regularity of the global attractor. At this stage, no regularity of the attractor \( \mathfrak{A} \) has been obtained, since the size of the compact set in (4.1) depends on time. To deepen our analysis, we shall exploit the abstract result [14, Theorem 3.1], written here in a form specifically tailored for our scopes.

Lemma 5.1. Let \( \mathcal{B}_0 \subset \mathcal{H} \) be an absorbing set for \( S(t) \), and let \( r > 0 \). For every \( z \in \mathcal{B}_0 \), assume there exist two operators \( V_z(t) \) and \( U_z(t) \) acting on \( \mathcal{H} \) and \( \mathcal{H}^r \), respectively, such that

(i) given any \( y \in \mathcal{B}_0 \) and any \( x \in \mathcal{H}^r \) satisfying the relation \( y + x = z \),
\[ S(t)z = V_z(t)y + U_z(t)x; \]

(ii) there exists a nonnegative decreasing function \( d_1 \) with \( d_1(\infty) < 1 \) such that, for any \( y \in \mathcal{B}_0 \),
\[ \sup_{z \in \mathcal{B}_0} \| V_z(t)y \|_{\mathcal{H}} \leq d_1(t)\| y \|_{\mathcal{H}}; \]

(iii) there exist a nonnegative decreasing function \( d_2 \) with \( d_2(\infty) < 1 \) and a positive increasing function \( J \) such that, for any \( x \in \mathcal{H}^r \),
\[ \sup_{z \in \mathcal{B}_0} \| U_z(t)x \|_{\mathcal{H}^r} \leq d_2(t)\| x \|_{\mathcal{H}^r} + J(t). \]

Then, \( \mathcal{B}_0 \) is exponentially attracted by a closed ball of \( \mathcal{H}^r \) centered at zero; namely, there exist positive constants \( R, K, \kappa \) such that
\[ \text{dist}_{\mathcal{H}}(S(t)\mathcal{B}_0, \mathcal{B}_{\mathcal{H}^r}(R)) \leq Ke^{-\kappa t}. \]

Relying on Lemma 5.1, we can prove the optimal regularity of the attractor.

Theorem 5.2. The global attractor \( \mathfrak{A} \) is bounded in \( \mathcal{H}^1 \).

Proof. We divide the proof into two steps. Again, the generic constant \( C \geq 0 \) will depend on the absorbing set \( \mathcal{B}_0 \).

Step 1. We first show that the attractor \( \mathfrak{A} \) is bounded in \( \mathcal{H}^{1/3} \). The strategy consists in proving that
\[ \text{dist}_{\mathcal{H}}(S(t)\mathcal{B}_0, \mathcal{B}_{\mathcal{H}^{1/3}}(R_*)) \leq c_*e^{-\omega_*t}, \quad (5.1) \]
for some positive \( R_* \), \( c_* \) and \( \omega_* \) (depending on \( \mathcal{B}_0 \)). Since \( \mathfrak{A} \) is fully invariant, it is contained in every closed attracting set, hence
\[ \mathfrak{A} \subset \mathcal{B}_{\mathcal{H}^{1/3}}(R_*). \]

To prove (5.1), we apply Lemma 5.1 with \( r = 1/3 \). In order to verify the assumptions therein, let \( z \in \mathcal{B}_0 \) be fixed and let \( S(t)z = (u(t), \eta^t) \) be the solution of (2.3) departing from \( z \). For any \( y \in \mathcal{B}_0 \) and \( x \in \mathcal{H}^{1/3} \) such that \( y + x = z \), we define
\[ V_z(t)y = (v(t), \xi^t) \quad \text{and} \quad U_z(t)x = (w(t), \zeta^t), \]
where \((v(t), \xi^t)\) and \((w(t), \zeta^t)\) solve the problems (4.6)-(4.7) without the hats, i.e.

\[
\begin{aligned}
v_t + Av_t + \int_0^\infty \mu(s)A\xi(s) \, ds + f_0(v) &= 0, \\
\xi_t &= T\xi + v, \\
(v(0), \xi^0) &= y,
\end{aligned}
\]

and

\[
\begin{aligned}
w_t + Aw_t + \int_0^\infty \mu(s)A\zeta(s) \, ds &= \gamma, \\
\zeta_t &= T\zeta + w, \\
(w(0), \zeta^0) &= x,
\end{aligned}
\]

respectively, with \(\gamma\) given by

\[
\gamma = -f_0(u) + f_0(v) - f_1(u) + g.
\]

In this way condition (i) of Lemma 5.1 is satisfied. Note also that condition (ii) is a consequence of the exponential decay (4.8), that now reads

\[
\|\!(v(t), \xi^t)\!\|_H \leq C\|w\|_H e^{-\omega t}.
\]

We are left to verify condition (iii). We define (recall that \(\varepsilon, r\)

\[
E_r(t) = \frac{1}{2} \left[ \|w(t)\|^2_{r+1} + \|w(t)\|^2_r + \|\zeta^t\|^2_{\mathcal{M}^r} \right].
\]

Multiplying the first equation of (5.3) by \(w + \varepsilon w_t\) in \(H^r\) (with \(\varepsilon \in (0, 1)\)) and the second one by \(\zeta\) in \(\mathcal{M}^r\), on account of (2.4) we obtain

\[
\frac{d}{dt} E_r - \frac{1}{2} \int_0^\infty \mu'(s)\|\zeta(s)\|^2_{r+1} \, ds + \varepsilon \|w_t\|^2_r + \varepsilon \|w_t\|^2_{r+1} = \langle \gamma, A^r w \rangle + \varepsilon \langle \gamma, A^r w_t \rangle - \varepsilon \int_0^\infty \mu(s)\langle \zeta(s), w_t \rangle_{r+1} \, ds \\
\leq \langle \gamma, A^r w \rangle + \varepsilon \langle \gamma, A^r w_t \rangle + \frac{\varepsilon}{4} \|w_t\|^2_{r+1} + \varepsilon \|\zeta\|^2_{\mathcal{M}^r}.
\]

We now introduce the functional

\[
N_r(t) = -\langle \xi^t, w(t) \rangle_{\mathcal{M}^r},
\]

satisfying (cf. the proof of Lemma 3.3)

\[
\frac{d}{dt} N_r + \frac{1}{2} \|w\|^2_{r+1} + \frac{1}{4} \|w_t\|^2_{r+1} + \|\zeta\|^2_{\mathcal{M}^r} + \frac{\mu(0)}{2} \int_0^\infty -\mu'(s)\|\zeta(s)\|^2_{r+1} \, ds.
\]

Finally, we define the energy-like functional

\[
\Psi_{\varepsilon,r}(t) = E_r(t) + \varepsilon N_r(t),
\]

which is easily seen to be equivalent to \(E_r\) for all \(\varepsilon > 0\) small enough, that is,

\[
\frac{1}{2} E_r(t) \leq \Psi_{\varepsilon,r}(t) \leq 2E_r(t).
\]

Then, \(\Psi_{\varepsilon,r}\) fulfills the differential inequality

\[
\frac{d}{dt} \Psi_{\varepsilon,r} + \frac{\varepsilon}{2} \|w\|^2_{r+1} + \frac{1}{2} (1 - \varepsilon \mu(0)) \int_0^\infty -\mu'(s)\|\zeta(s)\|^2_{r+1} \, ds \\
- 2\varepsilon \|\zeta\|^2_{\mathcal{M}^r} + \varepsilon \|w_t\|^2_r + \frac{\varepsilon}{2} \|w_t\|^2_{r+1} \\
\leq \langle \gamma, A^r w \rangle + \varepsilon \langle \gamma, A^r w_t \rangle.
\]
Exploiting (1.7), provided that $\varepsilon > 0$ is small enough, we deduce
\[
\frac{1}{2}(1 - \varepsilon\mu(0))\int_0^\infty -\mu'(s)||\zeta||_{r+1}^2 ds - 2\varepsilon\|\zeta\|_{\Lambda^r}^2 \geq \frac{\delta}{4}\|\zeta\|_{\Lambda^r}^2.
\]
Hence, for a possibly smaller $\varepsilon$,
\[
\frac{d}{dt}\Psi_{\varepsilon,r} + 2\varepsilon^2 E_r + \varepsilon^2 \|w\|_{r+1}^2 + \frac{\varepsilon}{2} \|w_t\|_{r+1}^2 \leq \langle \gamma, A^r w \rangle + \varepsilon\langle \gamma, A^r w_t \rangle. \tag{5.6}
\]
We are left to estimate the terms involving the nonlinearity $\gamma$. Since $u = \hat{v} + \hat{w} = v + w$, we can write
\[
|\gamma| \leq c|w|(\|w\|^4 + |v|^4) + |g| + c
\]
\[
\leq 8c|w|(\|w\|^4 + |\hat{v}|^4) + 8c(|u| + |v|)|\hat{w}|^4 + |g| + c.
\]
With analogous computations as those in (4.10), we obtain
\[
\langle \gamma, A^r w \rangle \leq C(||v||_1 + ||\hat{v}||_1)^4||w||_{4/3}^2 + C(||u||_1 + ||v||_1)||\hat{w}||_{4/3}^2||w||_{4/3} + C||w||_{4/3}.
\]
Similarly,
\[
\langle \gamma, A^r w_t \rangle \leq C(||v||_1 + ||\hat{v}||_1)^4||w||_{4/3}||w_t||_{4/3} + C(||u||_1 + ||v||_1)||\hat{w}||_{4/3}||w_t||_{4/3} + C
\]
\[
\leq \frac{1}{2}||w_t||_{4/3}^2 + C(||v||_1 + ||\hat{v}||_1)^8||w||_{4/3}^2 + C(||u||_1 + ||v||_1)^2||\hat{w}||_{4/3}^2 + C.
\]
Exploiting now the decay estimates (4.8) and (5.4) together with (4.9), we conclude that
\[
\langle \gamma(t), A^r w(t) \rangle + \varepsilon\langle \gamma(t), A^r w_t(t) \rangle \leq \frac{\varepsilon}{2}||w_t(t)||_{4/3}^2 + \left(\frac{\varepsilon}{4} + Ce^{-4\omega t}\right)||w(t)||_{4/3}^2 + \frac{1}{\varepsilon}q(t),
\]
for some positive increasing function $q$ depending on $\mathbb{B}_0$. Accordingly, (5.6) becomes
\[
\frac{d}{dt}\Psi_{\varepsilon,r}(t) + 2\varepsilon^2 E_r(t) \leq h(t)E_r(t) + \frac{1}{\varepsilon}q(t),
\]
with $h(t) = Ce^{-4\omega t}$. In light of (5.5), we arrive at
\[
\frac{d}{dt}\Psi_{\varepsilon,r}(t) \leq (2h(t) - \varepsilon^2)\Psi_{\varepsilon,r}(t) + \frac{1}{\varepsilon}q(t).
\]
At this point, a direct application of the Gronwall lemma, together with (5.5), gives
\[
\frac{1}{2}E_r(t) \leq I(t)e^{-\varepsilon^2 t}\Psi_{\varepsilon,r}(0) + \frac{1}{\varepsilon}I(t)q(t)e^{-\varepsilon^2 t} \int_0^t e^{\varepsilon^2 \tau} d\tau
\]
\[
\leq 2I(t)e^{-\varepsilon^2 t}E_r(0) + \frac{1}{\varepsilon^3}I(t)q(t),
\]
with
\[
I(t) = \exp \left[ \int_0^t 2h(\tau) d\tau \right].
\]
Noting that $I(t)$ is bounded, the conclusion is
\[
\|U_z(t)x\|_{\mathbb{H}^{1/3}} \leq C\|x\|_{\mathbb{H}^{1/3}}e^{-\varepsilon^2 t/2} + J(t),
\]
for some positive increasing function $J$. This establishes condition (iii) of Lemma 5.1 and finishes the proof of the first step.

**Step 2.** We split the solution $S(t)z = (u(t), \eta')$ with $z \in \mathcal{A}$ into the sum
\[
S(t)z = L(t)z + K(t)z,
\]
where \( L(t)z = (v(t), \xi^t) \) and \( K(t)z = (w(t), \zeta^t) \) solve the problems

\[
\begin{cases}
v_t + Av_t + \int_0^\infty \mu(s) A\xi(s) \, ds = 0, \\
\xi_t = T\xi + v, \\
(v(0), \xi^0) = z,
\end{cases}
\]

and

\[
\begin{cases}
w_t + Aw_t + \int_0^\infty \mu(s) A\zeta(s) \, ds + f(u) = g, \\
\zeta_t = T\zeta + w, \\
(w(0), \zeta^0) = 0,
\end{cases}
\]

respectively. From Theorem 3.1, the linear semigroup \( L(t) \) decays exponentially to 0 in \( \mathcal{H} \). In addition, being \( \mathfrak{A} \) fully invariant and bounded in \( \mathcal{H}^{1/3} \), it is possible to show that

\[\|K(t)z\|_{\mathcal{H}^1} \leq \rho,\]

for some \( \rho = \rho(\mathfrak{A}) > 0 \). The latter control can be obtained by recasting the argument in the first step with \( r = 1 \), leading to higher order estimates similar to those in [12]. The details are left to the reader. Owing once more to the fact that \( \mathfrak{A} \) is fully invariant, we arrive at

\[\text{dist}_{\mathcal{H}}(\mathfrak{A}, B_{\mathcal{H}^1}(\rho)) = \text{dist}_{\mathcal{H}}(S(t)\mathfrak{A}, B_{\mathcal{H}^1}(\rho)) \rightarrow 0\]

as \( t \rightarrow \infty \). Hence the inclusion \( \mathfrak{A} \subset B_{\mathcal{H}^1}(\rho) \) holds, providing the desired boundedness of the global attractor \( \mathfrak{A} \) in \( \mathcal{H}^1 \).

6. **Final remarks.** We conclude the paper by mentioning possible generalizations of the hypotheses and further developments.

**I.** Assumption (1.6) on the nonlinearity can be weakened by simply asking that \( f(u)u \geq -c_f \).

This introduces an additional difficulty in the proof of Theorem 3.1, as one can no longer reconstruct the energy functional \( \Phi_\varepsilon \) in the differential inequality (3.4) to get (3.5). Nonetheless, the problem can be circumvented by exploiting a nonlinear generalization of the Gronwall lemma from [3], although this strategy requires an extra technical effort.

**II.** In our assumptions on the memory kernel \( \mu \) we ask, among others, that \( \mu(0) < \infty \). This enters in a structural way in Lemma 3.3, through the functional \( N_\varepsilon \). However, it is actually possible to include in the analysis kernels \( \mu \) which are weakly singular (i.e. integrable) about zero. To this end, one has to suitably modify the functional \( N_\varepsilon \), via the introduction of a proper cut-off on the kernel. We address the reader to [8, 23], where this situation is taken into account.

**III.** In our hypotheses on \( \mu \), the (absolute) continuity is required. In fact, as already mentioned in Remark 1.1, it is enough to take \( \mu \) piecewise absolutely continuous, hence possessing (downward) jumps. The jump points can be either finite, or even infinitely many, provided that the set of their limit points is discrete. This generalization does not affect significantly the proofs [5, 6].

**IV.** As we saw in the Introduction, memory kernels \( \mu \) complying with the weaker assumption (1.10), in place of (1.7), do not provide the equation with enough dissipation, a fact highlighted in Example 1.2. Still, assumption (1.10) becomes sufficient
in order to drive the system uniformly to equilibrium if, roughly speaking, the kernel $\mu$ is not too flat. The precise condition, devised in [23], reads as follows: the flatness rate of $\mu$ must be less than $1/2$, that is,
\[
F_\mu = \frac{1}{\kappa(0)} \int_{\mathcal{F}_\mu} \mu(s) \, ds < \frac{1}{2},
\]
where the flatness set $\mathcal{F}_\mu$ of $\mu$ is defined as
\[
\mathcal{F}_\mu = \{ s \in \mathbb{R}^+ : \mu(s) > 0 \text{ and } \mu'(s) = 0 \}.
\]
Note that in Example 1.2 we have $F_\mu = 1$. Anyway, it is worth pointing out that handling kernels of this type is quite delicate, and the level of complication in the estimates increases exponentially.

V. Following the lines of [10], based on an abstract result from [16] specifically tailored for equations with memory, it is possible to prove that the semigroup $S(t)$ possesses exponential attractors of finite fractal dimension (see [22] for more details on the theory of exponential attractors). As a consequence the global attractor $\mathcal{A}$, being contained in any exponential attractor, has finite fractal dimension as well.

VI. Lastly, we mention that the analysis of this paper can also be carried out within a different theoretical scheme, the so-called minimal state framework, in place of the past history one. This alternative approach, introduced in [17], is based on the notion of minimal state, namely, an additional variable accounting for the past history which contains the necessary and sufficient information determining the future dynamics. In contrast, the past history is overdetermined: two different initial past histories may lead to the same evolution. Accordingly, in principle, the past history cannot be recovered by the evolution itself. This scheme has been successfully employed in several works (e.g. [9, 11, 13, 16]). In particular, in [16] it is shown that the results obtained working with the (integrated) past history variable can be somehow automatically transferred in the minimal state framework.

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