Topological Extensions of Rough Approximation Spaces

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Abstract: In this work, we extend Pawlak approximation spaces by topological spaces. Also, Rough Membership, equality and inclusion relations are extended using topological near open sets. In addition, new extended measures of accuracy and quality of approximations are defined and studied. An application example of data reduction of information systems is introduced.

Keywords: Topological Spaces; Rough Sets; Rough Approximations; Order Relations; Data Reduction.

MSC: 54A05; 54B05; 54D35; 03B70.

1. Introduction

The problem addressed in this paper is how to construct knowledge from datasets by a topological generalization of rough set theory. In the classical approach of Pawlak knowledge theory, there is an extended and wealthy history. For understanding, representing, and manipulating knowledge, there are a variety of opinions and approaches in this area. The process of extracting knowledge based on the ability to classify objects. Knowledge in this approach is necessarily connected with the selection of classification patterns, related to specific parts of the real or abstract world. This view of knowledge and information is more near to the theory of abstract topological spaces, which is based on a topological structure, consists of a class of subsets of the universe classified according to its satisfaction to some axioms. Consequently, the abstract topological space is not a metric model for most types of information representations.

There are many methods for studying data in information systems; a recent one which related to topology is rough set. This theory depends on a special class of topological spaces known by quasi-discrete topological space in which every open set is closed. The basic assumption of this theory concerns the fundamental problem of knowledge representation: in this theory, one assumes that objects under study are described in terms of functions which map these objects into the corresponding value sets.

The starting point of this paper is any general binary relation. We used that relation to generate two classes to use them for generating two spaces of two topologies. Then we used these topologies to generalize Pawlak approximation space to topological approximation space. Also, we defined and studied membership, equality and inclusion relations in this biological space. Some measures of accuracy also defined and studied. Finally, we determined the reject of the multivalued information system as an application. This paper is organized as follows:

Section 2 introduced the previous work of this research. In Section 3, we discussed the needed fundamentals of classical rough set theory and topological spaces. In Section 4, we introduced and investigated the concept of extended approximation space. Section 5 is devoted to introduce the basic concepts of topological approximations. An application example of biological information systems is introduced in Section 6. The paper's conclusion is given in Section 7.
2. Previous Works

Rough set theory introduced by Pawlak in [1] is a mathematical tool for imperfect knowledge, decision analysis, and knowledge discovery from databases. This theory is paying attention, of a lot of researchers and practitioners all over the world, who contributed fundamentally to its improvement and applications. Usually rough sets used together with other methods such as fuzzy sets [2, 3, 4, 5, 6, 7], covering and fuzzy covering [8, 9, 10], statistical methods (probabilistic space) [11], tolerance and similarity relations [12, 13, 14, 15], topological generalizations [16, 17, 18, 19, 20, 21, 22] to locate accurate approaches used in applications. Recently, many generalizations of this theory have developed to help in applications such as in information retrieval [23], reduction of finite information systems [24, 25], establishment rules of interval-valued fuzzy information systems [26, 27], and foundation attributes missing values in incomplete information systems [28].

The original rough set theory depended totally on equivalence relations to approximate concepts. But these types of relations are still restrictive for many applications. Many researchers studied this issue and several interesting and meaningful generalizations to equivalence relation have been proposed. For instance, binary, tolerance, similarity, reflexive, and transitive relations are some solutions on this issue [29, 30, 31, 32, 33]. Some researchers extended rough sets by combining fuzzy sets with rough sets [34, 35, 36, 37]. Another group has characterized a measure of uncertainty by the concept of fuzzy relations [38]. Wiweger in [20] is the first researcher that defined a topological rough set that classified to be the very important topological generalizations of rough sets. Different approaches extended approximation space, extended rough set models and using relational interpretations for approximate operators using rough sets [39, 40, 41, 42]. In [43] the necessary and sufficient conditions for the lower and upper approximations are considered to formulate rough sets to certain families of exact sets.

3. Basic Concepts of Rough Sets and Topological Spaces

The major profit of rough set theory in data reduction is that it does not need any opening or additional information about data.

The basic tool of Pawlak theory is the approximation space [30]. An approximation space is a pair \( A = (U, R) \), where \( U \) is a set called the universe and \( R \) is an equivalence relation. The blocks \([x]_R \subseteq U\) made by the equivalence relation \( R \) are the elementary sets of this theory that are used in approximations. Any subset of the universe can approximated using these elementary sets from lower and from upper. Classical rough lower approximation of \( X \subseteq U \) is defined by Pawlak by \( \overline{R}(X) = \bigcup\{[x]_R : [x]_R \subseteq X\} \) and the classical upper is defined by. The lower and upper approximations are the keys to define other regions using the subset. The optimistic, unenthusiastic and border regions of the subset \( X \subseteq U \) are defined as follows:

1) The positive region of \( X \subseteq U \) is defined by \( POS_R(X) = \overline{R}(X) \).
2) The negative region of \( X \subseteq U \) is defined by \( NEG_R(X) = U - \overline{R}(X) \).
3) The boundary region of \( X \subseteq U \) is defined by \( BN_R(X) = \overline{R}(X) - R(X) \)

Now the subset \( X \) is called exact set with respect to \( R \) if the boundary region is
empty namely \( BN_R(X) = \varnothing \). It called rough set if the boundary region is non-empty namely \( BN_R(X) \neq \varnothing \).

The degree of accuracy of the subset \( X \subseteq U \) is defined by \( \alpha_R(X) = \frac{|R(X)|}{|\overline{R}(X)|} \) where \(|X|\) denotes the cardinality of \( X \). Obviously, \( 0 \leq \alpha_R(X) \leq 1 \). If \( \alpha_R(X) = 1 \) then \( X \) is exact, otherwise if \( \alpha_R(X) < 1 \) then \( X \) is rough.

The pair \((U, \tau)\) of a non empty set \( U \) and a family \( \tau \) of subsets of \( U \) bing a topological space when \( \varnothing, U \in \tau \) and the family \( \tau \) is closed under arbitrary union and finite intersection [44]. The equation \( cl_\tau(X) = \bigcap\{F \subseteq U : X \subseteq F, F \in \tau^c\} \) is for the closure of \( X \subseteq U \). But this \( \text{int}_\tau(X) = \bigcup\{G \subseteq U : G \subseteq X, G \in \tau\} \) is in the interior.

Suppose that the class \( S_R = \{[x]_R : x \in U\} \) is a base of a topology \( \tau_R \) on \( U \). If we used equivalence relations, then the classical rough approximations are identical with the interior and the closure operations in the equivalent topology.

In this paper, we define two classes by a general binary relation (not equivalence) \( R \) on \( U \). The class \( S_R = \{R_x : x \in U\} \), such that \( R_x = \{y \in U : xRy\} \) generates the topology \( \tau_1 \) and the class \( S_{R,\cup} = \{R_{x,\cup} : x \in U\} \), such that \( R_{x,\cup} = \{y \in U : yRx\} \) generates the topology \( \tau_2 \).

Rough approximations of \( X \subseteq U \) using \( S_R \) defined as follows:
\[
S_R(X) = \bigcup\{G \in S_R : G \subseteq X\} \quad \text{and} \quad \overline{S_R}(X) = \bigcap\{G \in S_R, G \cap X \neq \varnothing\}.
\]
Also, if \( \tau = \tau_1 \cap \tau_2 \) we can define two other lower and upper approximations as follows:
\[
R_\tau(X) = \bigcup\{G \in \tau : G \subseteq X\} \quad \text{and} \quad \overline{R}_\tau(X) = \bigcap\{F \in P(U) - \tau : F \cap X \neq \varnothing\}.
\]

4. Extended Rough Approximations by Topological Spaces
In this section, we introduce and investigate the concept of extended approximation space \( BIOApp = (U, R, \tau_1, \tau_2) \). Also, we introduce the concepts of extended lower and extended upper approximations and study their properties.

Consider the topological space \((U, \tau)\) and since \( X \subseteq U \) we define the set:
\[
\varphi(X) = \{Y \subseteq U : Y \subseteq cl(\text{int}(cl(X)))\}.
\]

**Definition 4.1** In the extended approximation space \( BIOApp = (U, R, \tau_1, \tau_2) \) we define for any subset \( X \subseteq U \):
\[
\varphi_{12}(X) = \{Y \subseteq U : Y \subseteq cl_{\tau_1}(\text{int}_{\tau_2}(cl_{\tau_1}(X)))\},
\]
\[
\varphi_{21}(X) = \{Y \subseteq U : Y \subseteq cl_{\tau_2}(\text{int}_{\tau_1}(cl_{\tau_2}(X)))\}.
\]

We can define the \( \tau_1 \tau_2 \) - extended lower and \( \tau_1 \tau_2 \) - extended upper approximations as follows:

**Definition 4.2** In the extended approximation space \( BIOApp = (U, R, \tau_1, \tau_2) \) and for any \( X \subseteq U \) we define:
\[
R_{12}(X) = \bigcup\{G : \varphi_{12}(G) \subseteq X\},
\]

\[
\overline{R}_{12}(X) = \bigcap\{F : \varphi_{12}(F) \cap X \neq \varnothing\}.
\]
\[ \overline{R}_{12}(X) = \bigcap \{ G : \varphi_{12}(G) \cap X \neq \varphi \} , \]

By the same manner we can define the \( \tau_2 \tau_1 \)- extended lower and \( \tau_2 \tau_1 \)- extended upper approximations as follows:

\[ \overline{R}_{21}(X) = \bigcup \{ G : \varphi_{21}(G) \subseteq X \} \]

\[ \overline{R}_{21}(X) = \bigcap \{ G : \varphi_{21}(G) \cap X \neq \varphi \} \]

**Definition 4.3** Let \( BIOApp = (U, R, \tau_1, \tau_2) \) be a extended approximation space, the mixed extended lower and mixed extended upper approximations for any subset \( X \subseteq U \) defined as follows:

\[ \overline{R}_m(X) = \overline{R}_{12}(X) \cup \overline{R}_{21}(X), \]

\[ \overline{R}_m(X) = \overline{R}_{12}(X) \cap \overline{R}_{21}(X). \]

**Definition 4.4** Let \( BIOApp = (U, R, \tau_1, \tau_2) \) be an extended approximation space. Then we define the following degrees of completeness for a subset \( X \neq \varphi \) as follows:

1- Pawlak accuracy measure \( \alpha_R(X) = \frac{|R(X)|}{|R(X)|} \),

2- Topological accuracy measure \( \alpha_t(X) = \frac{|R_t(X)|}{|R_t(X)|} \), \( \tau = \tau_1 \) or \( \tau = \tau_2 \).

3- Extended accuracy measure \( \alpha_s(X) = \frac{|R_s(X)|}{|R_s(X)|} \), \( s = 12 \) or \( 21 \).

4- Mixed extended accuracy measure \( \alpha_m(X) = \frac{|R_m(X)|}{|R_m(X)|} \).

**Example 4.1** Given a universe of four classes \( U = \{1, 2, 3, 4\} \) that have some features that we used in defining the binary relation \( R = \{(1,1), (1,3), (1,4), (2,2), (2,4), (3,1), (3,2), (3,4), (4,1)\} \). Then by Definition 3.1 we can generate the following:

1\( R = \{1,3,4\} \), 2\( R = \{2,4\} \), 3\( R = \{1,2,4\} \) and 4\( R = \{1\} \). Also \( R1 = \{1,3,4\} \), \( R2 = \{2,3\} \), \( R3 = \{1\} \) and \( R4 = \{1,2,3\} \). Then \( S_R = \{1,3,4\}, \{2,4\}, \{1,2,4\}, \{1\} \} \) is the base of the topology \( \tau_1 = \{U, \varphi, \{1\}, \{4\}, \{1,4\}, \{2,4\}, \{1,2,4\}, \{1,3,4\}\} \) and \( S_{R_{U}} = \{1,3,4\}, \{2,3\}, \{1\}, \{1,2,3\}\} \) is the base of the topology \( \tau_2 = \{U, \varphi, \{1\}, \{3\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \{1,3,4\}\} \). Then applying Definitions 3.3 & 3.4 & 3.5 of the chosen subsets of Table 1 then we have a comparison among the degree of accuracy measure, \( \alpha_t(X) \), \( \alpha_s(X) \) and \( \alpha_m(X) \) as given in Table 1.
Using the mixed extended accuracy measure the results of Table 1 have been improved. Consequently the mixed extended accuracy measure is more accurate than other measures.

5. Properties of the extended approximations

In this section, we study the basic concepts of extended lower and the mixed extended upper approximations. We introduce six membership relations using these approximations and study their properties. Also, we introduce the equality and inclusion relations with studying some basic properties.

**Definition 5.1** In the extended approximation space \( BIOApp = (U, R, \tau_1, \tau_2) \) we define for every \( X \subseteq U \), we have the following:

1) \( x \in X \) if and only if \( x \in R(X) \),
2) \( x \notin X \) if and only if \( x \in R(X) \),
3) \( x \in \tau_1 X \) if and only if \( x \in R_\tau(X) \),
4) \( x \in \tau_2 X \) if and only if \( x \in R_\tau(X) \),
5) \( x \in \phi m X \) if and only if \( x \in R_m(X) \),
6) \( x \in \phi m X \) if and only if \( x \in R_m(X) \).

**Remark 5.1** According to Definition 4.1, mixed extended lower and mixed extended upper approximations of a set \( X \subseteq U \) can be rewritten as: \( R_m(X) = \{x \in X : x \in \phi m X\} \), \( R_m(X) = \{x \in X : x \in \phi m X\} \).

**Remark 5.2** Let \( BIOApp = (U, R, \tau_1, \tau_2) \) be a extended approximation space. For any subset \( X \subseteq U \), \( x \in X \land x \notin \tau_1 X \Rightarrow \neg x \in \phi m X \) and \( x \in X \land x \notin \tau_2 X \Rightarrow x \in \phi m X \).

**Example 5.1** In Example 4.1, if \( X = \{a, b, c\} \), then \( R(X) = \{a\}, R_\tau(X) = \{a, c\} \) and \( R_m(X) = \{a, b, c\} \), hence \( b \in \phi m X \), \( b \notin \tau_1 X \) and \( b \notin \phi m X \). Also \( c \notin \tau_1 X \), but \( c \notin \phi m X \).

**Example 5.2** In Example 4.1, if \( X = \{4\} \), then \( R(X) = \{2, 3, 4\}, R_\tau(X) = \{2, 4\} \) and \( R_m(X) = \{4\} \). So \( 2 \in X \), \( 2 \in \tau_1 X \), \( 2 \notin \phi m X \) and \( 3 \in X \), \( 3 \notin \tau_1 X \) and \( 3 \notin \phi m X \).
We investigate mixed extended rough equality and mixed extended rough inclusion based on rough equality and inclusion.

**Definition 5.2** In the extended approximation space \( BIOApp = (U, R, \tau_1, \tau_2) \) and since \( A, B \subseteq U \) we define:

(i) Roughly bottom equal by extending mixed \( (A \sqsubseteq_m B) \) if and only if \( \overline{R}_m(A) = \overline{R}_m(B) \).

(ii) Roughly top equal by extending mixed \( (A \sqsupseteq_m B) \) if and only if \( \underline{R}_m(A) = \underline{R}_m(B) \).

(iii) Roughly equal by extending mixed \( (A \approx_m B) \) if and only if \( \overline{R}_m(A) \sqsubseteq \overline{R}_m(B) \) and \( \underline{R}_m(A) \sqsupseteq \underline{R}_m(B) \).

**Example 5.3** According to Example 4.1 the set \( \{2\} \) and \( \emptyset \) are roughly bottom equal by extending mixed. But the set \( \{1, 3, 4\} \) and \( U \) are roughly top equal by extending mixed.

**Definition 5.3** In the extended approximation space \( BIOApp = (U, R, \tau_1, \tau_2) \) and since \( A, B \subseteq U \) we define:

(i) \( A \) is roughly bottom included by extended mixed in \( B \) (\( A \sqsubseteq_m B \)) if \( \overline{R}_m(A) \subseteq \overline{R}_m(B) \).

(ii) \( A \) is roughly top included by extended mixed in \( B \) (\( A \sqsupseteq_m B \)) if \( \underline{R}_m(A) \subseteq \underline{R}_m(B) \).

(iii) \( A \) is roughly included by extended mixed in \( B \) (\( A \sqsubseteq \overline{R}_m B \)) if \( A \subseteq B \) and \( \overline{R}_m(A) \subseteq \overline{R}_m(B) \).

**Example 5.4** In Example 4.1, we have \( X_1 = \{2\}, X_2 = \{3\}, Y_1 = \{1, 2, 4\} \) and \( Y_2 = \{1, 3, 4\} \), then \( X_1 \) is mixed extended roughly bottom included in \( X_2 \) and \( Y_1 \) is mixed extended roughly top included in \( Y_2 \).

**Definition 5.4** Let \( BIOApp = (U, R, \tau_1, \tau_2) \) be an extended approximation space. The subset \( A \subseteq U \) is called:

(i) Extended mixed definable, if \( \overline{R}_m(A) = \overline{R}_m(A) \).

(ii) Extended mixed rough, if \( \overline{R}_m(A) \neq \overline{R}_m(A) \).

(iii) Extended roughly mixed definable, if \( \overline{R}_m(A) \neq \emptyset \) and \( \overline{R}_m(A) \neq U \).

(iv) Extended internally mixed undefinable, if \( \overline{R}_m(A) = \emptyset \) and \( \overline{R}_m(A) \neq U \).

(v) Extended externally mixed undefinable, if \( \overline{R}_m(A) \neq \emptyset \) and \( \overline{R}_m(A) = U \).

(vi) Extended totally mixed undefinable, if \( \overline{R}_m(A) = \emptyset \) and \( \overline{R}_m(A) = U \).

**Proposition 5.1** Let \( BIOApp = (U, R, \tau_1, \tau_2) \) be an extended approximation space. Then we have:

(i) Every exact set in \( U \) is mixed extended exact.

(ii) Every \( \tau \) – exact set in \( U \) is mixed extended exact.

(iii) Every mixed extended rough set in \( U \) is rough.

(iv) Every mixed extended rough set in \( U \) is \( \tau \) – rough.

**Proof.** Obvious.

The converse of Proposition 5.1, may not be true in general and we declare this by the following example.
Example 5.5 In Example 4.1, the sets \( \{ 3 \} , \{ 1, 2 \} , \{ 2, 3 \} , \{ 1, 2, 3 \} \) and \( \{ 1, 2, 4 \} \) are mixed extended exact, but neither \( \tau - \text{exact} \) nor exact.

Remark 5.2 Let \( BIOApp = (U, R, \tau_1, \tau_2) \) be an extended approximation space. Then we have:
(i) The intersection of two mixed extended exact sets need not be mixed extended exact set.
(ii) The union of two mixed extended exact sets need not be mixed extended exact set.

The following example illustrates the above remark.

Example 5.6 According to Example 4.1, suppose \( X_1 = \{ 1 \} \), \( X_2 = \{ 3, 4 \} \), \( Y_1 = \{ 2, 3 \} \) and \( Y_2 = \{ 2, 4 \} \) are mixed extended exact. Then \( Y_1 \cap Y_2 \) and \( X_1 \cup X_2 \) are not mixed extended exact.

6. An Application Example

The structure \( (NP, At, \{ V_a : a \in At \}, f_a, \{ R_a : B \subseteq At \}) \) is a biological attribute system. The set \( NP \) is the system universe that we selected it from the field identification of the 5 most common plant families in temperate regions. The set \( At \) is the attributes of this plant family. The set \( V_a \) is values of each attribute \( a \in At \). Finally, \( f_a : NP \rightarrow V_a \) is the information function such that \( f_a(x) \in V_a \).

For any subset \( B \in At \), we define the relation:
\[
R_a = \{ (x, y) : \{ V_a, f_a(x) : a \in B \} \subseteq \{ f_a(y) : a \in B \}, x, y \in NP \}, \quad \text{for } a \in At, \text{we define the class } A_{R_a}
\]

as follows: \( A_{R_a} = \{ R_a(x) : x \in NP \} \), where \( R_a(x) = \{ y : xR_a y \} \).

The structure \( (NP, \{ At, D \}, \{ V_a : a \in At \}, f_a, \{ R_a : B \subseteq At \}) \) is a decision table, where \( D \) is the set of decisions.

We define the relation of the decision attribute \( D \) by:
\[
R_D = \{ (x, y) : \{ f_a(x) : a \in D \} = \{ f_a(y) : a \in D \}, x, y \in NP \}.
\]

The class of this relation is \( R_D(x) = \{ y : xR_D y \} \). The set of all classes is \( A_{R_D} = \{ R_D(x) : x \in U \} \).

Table 2 contains the following shortcuts:
A= Often with bulb,
B = Narrow leaves in a basal rosette,
C= Inflorescence a terminal umbel,
D= Ovary superior
E = Flowers small, Unisexual, sitting behind a bract
F = Ovary superior, often inside a bottle-shaped structure.

In Table 2, the value 1 means that the plant has the attribute property and 0 means that this plant has not this attribute property. Decision have three values, Yes to means that the plant name surely belongs to the corresponding family. The value No means that it surely not belong to this family. Finally, the value maybe for not sure belong case of the family.
| Plant Family | Plant Name | Attributes | Decision |
|-------------|------------|------------|----------|
| Onion       | Garlic     | A: 1, B: 0, C: 0, D: 1, E: 0, F: 0 | No |
|             | Onion      | A: 1, B: 1, C: 1, D: 1, E: 0, F: 0 | Yes |
|             | Leek       | A: 0, B: 1, C: 1, D: 1, E: 0, F: 0 | Yes |
|             | Chives     | A: 1, B: 0, C: 1, D: 0, E: 0, F: 0 | Maybe |
| Amaranth    | Beet       | A: 1, B: 0, C: 1, D: 0, E: 0, F: 0 | No |
|             | Amaranth   | A: 1, B: 1, C: 1, D: 1, E: 0, F: 0 | Yes |
|             | Quinoa     | A: 0, B: 0, C: 1, D: 0, E: 0, F: 0 | No |
| Aloe        | Aloe       | A: 1, B: 1, C: 1, D: 1, E: 0, F: 0 | Yes |
|             | Haworthia  | A: 0, B: 1, C: 1, D: 1, E: 0, F: 0 | Yes |
|             | Asphodel   | A: 0, B: 1, C: 0, D: 0, E: 0, F: 0 | Maybe |
|             | Poker      | A: 1, B: 0, C: 0, D: 1, E: 0, F: 1,1 | Maybe |
| Barrage     | Myosotis   | A: 0, B: 0, C: 1, D: 0, E: 0, F: 0 | No |
|             | Symphytum  | A: 1, B: 0, C: 0, D: 0, E: 1, F: 0 | Yes |
|             | Pulmonaria | A: 0, B: 1, C: 0, D: 0, E: 0, F: 0 | No |
|             | Echium     | A: 1, B: 0, C: 0, D: 0, E: 0, F: 0 | Maybe |
| Sedge       | Carex      | A: 1, B: 1, C: 0, D: 0, E: 1, F: 1 | Yes |
|             | Cyprus     | A: 1, B: 1, C: 0, D: 0, E: 1, F: 1 | Yes |

**Table 2: Decision Information System**

Table 3 is the multivalued form obtained by transformation of Table 2 by collecting the values 1 in Positive attribute and the values 0 in Negative attribute.
Table 3: Multivalued Biological Information System

|          | P11=poker | P12=Myosotis | P13=Symphytum | P14=Pulmonaria | P15=Echium | P16=Carex | P17=Cyperus |
|----------|-----------|--------------|---------------|---------------|------------|-----------|------------|
| Borage   |           | {A,C,F}      | {B,D,E}       | Maybe         |           | {C}       | Yes        |
| P12=Myosotis |          | {A,B,C,E}    | {D,F}         | No            |           |           |            |
| P13=Symphytum |        | {A,B,D,E}    | {C,F}         | Yes           |           |           |            |
| P14=Pulmonaria |        | {B,E}        | {A,C,D,F}     | No            |           |           |            |
| P15=Echium |           | {A,C,F}      | {B,D,E}       |               |            |           | Maybe      |
| Sedge    |           | {A,B,D,E,F}  | {C}           | Yes           |            |           |            |
| P16=Carex |           |              |               |               |            |           |            |
| P17=Cyperus |          | {A,B,C,E,F}  | {D}           | Yes           |            |           |            |

To obtain the spaces of the two topologies we define the following relations: $R_{Pos} = \{(x,y): f(x) \subseteq f(y), \forall x,y \in PN\}$ and $R_{Neg} = \{(x,y): f(x) \nsubseteq f(y), \forall x,y \in PN\}$

We define the following two bases using the above relations:

$\beta_{Pos} = \{R_{Pos}(x) : x \in PN\}$, where $R_{Pos}(x) = \{y : xRy\}$,

$\beta_{Neg} = \{R_{Neg}(x) : x \in PN\}$, where $R_{Neg}(x) = \{y : yRx\}$.

The categories of the positive relation form a base for the positive topology and their blocks are given as the following:

$\beta_{Pos} = \{R_{Pos}(x) : x \in NP\}$

$R_{Pos}(P1) = \{P1, P2, P4, P6, P8, P13, P16\}$

$R_{Pos}(P2) = \{P2, P6, P8\}$

$R_{Pos}(P3) = \{P2, P3, P6, P8, P9\}$

$R_{Pos}(P4) = \{P2, P4, P6, P8\}$

$R_{Pos}(P5) = \{P2, P5, P6, P7, P8, P13, P16, P17\}$

$R_{Pos}(P6) = \{P2, P6, P8\}$

$R_{Pos}(P7) = \{P7, P16, P17\}$

$R_{Pos}(P8) = \{P8\}$

$R_{Pos}(P9) = \{P8, P9\}$

$R_{Pos}(P10) = \{P8, P9, P10, P17\}$

$R_{Pos}(P11) = \{P11, P15, P17\}$

$R_{Pos}(P12) = \{P12, P16\}$

$R_{Pos}(P13) = \{P8, P13, P16\}$

$R_{Pos}(P14) = \{P7, P8, P9, P10, P13, P14, P16, P17\}$

$R_{Pos}(P15) = \{P11, P15, P17\}$

$R_{Pos}(P16) = \{P16\}$

$R_{Pos}(P17) = \{P17\}$

The categories of the negative relation form a base for the negative topology and their blocks are given as the following:

$\beta_{Neg} = \{R_{Neg}(x) : x \in NP\}$
With respect to the decision attribute we define the relation:

\[ R_{Decision} = \{(x, y) : f(x) = f(y), \forall x, y \in PN\} \]

Then we have three decision categories are the base of the decision topology as the following:

- **Yes**

  \[ R_{Decision}(Yes) = \{P2, P3, P6, P8, P9, P13, P16, P17\} = D_1 \]

- **No**

  \[ R_{Decision}(No) = \{P1, P5, P7, P12, P14\} = D_2 \]

- **Maybe**

  \[ R_{Decision}(Maybe) = \{P4, P10, P11, P15\} = D_3 \]

Now we reduct Table 3 using the Positive (\( \tau_{Pos} \)) and the Negative (\( \tau_{Neg} \)) topologies. First, we calculate the approximations for the three categories of the decision attributes () with respect to positive and negative topologies:

\[ \phi(D_1) = \{G \in P(U) : G \subseteq cl_{\tau_{Pos}}(int_{\tau_{Neg}}(cl_{\tau_{Pos}}(D_1)))\}, \]

\[ \phi(D_2) = \{G \in P(U) : G \subseteq cl_{\tau_{Pos}}(int_{\tau_{Neg}}(cl_{\tau_{Pos}}(D_2)))\}, \]

\[ \phi(D_3) = \{G \in P(U) : G \subseteq cl_{\tau_{Pos}}(int_{\tau_{Neg}}(cl_{\tau_{Pos}}(D_3)))\}, \]
Then after many topological calculations, we have the results in Table 4:

| The decision category (D) | \(\phi(D)\) | \(R_{Pos-Neg}(D)\) | \(\overline{R}_{Pos-Neg}(D)\) | \(\alpha_{Pos-Neg}(D)\) |
|---------------------------|-------------|-------------------|-----------------|-------------------|
| \(R_{Decision}(Yes)\)    | \{P2,P3,P6, P8,P9,P12, P13,P16,P17\} | \{P2,P3,P6,P8, P9,P12,P13,P16\} | \{P2,P3,P6,P8, P9,P12,P1\} | 88.9% |
| \(R_{Decision}(No)\)     | \{P1,P5,P7, P11,P12,P14,P16\} | \{P1,P5,P7,P11, P12,P16\} | \{P1,P5,P7,P8,P9,P11,P12,P14,P16,P17\} | 60% |
| \(R_{Decision}(Maybe)\)  | \{P4,P6,P10, P11,P15\} | \{P4,P10,P11, P15\} | \{P4,P6,P8,P10,P11,P15\} | 66.7% |

Table 4: Results due topological approximations

From Table 4, we concluded that the decision taken such that the plants have decision “Yes” are belong to the selected family by accuracy 88.9%. The plants that have decision value “No” have a 40% error of chosen in their family. Finally, the plants that maybe in some different families have accuracy 66.7%.

7. Conclusion

The objectives of this work are to generalize rough set theory using new alternative methods via topological generalizations. These generalizations achieved using two topological spaces. The advantages of these generalizations are using general binary relations to define two spaces that generate two topologies.

Our proposed different methods to achieve the main aim of generalizations of rough set and using them to reduce the boundary region and to increase the accuracy of approximations. We applied the current approach to reduce biological data by approximate the boundary region and measure the accuracy of it.

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