A Moment-Matching Approach to Testable Learning and a New Characterization of Rademacher Complexity

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ABSTRACT

A remarkable recent paper by Rubinfeld and Vasilyan (2022) initiated the study of testable learning, where the goal is to replace hard-to-verify distributional assumptions (such as Gaussianity) with efficiently testable ones and to require that the learner succeed whenever the unknown distribution passes the corresponding test. In this model, they gave an efficient algorithm for learning halfspaces under testable assumptions that are provably satisfied by Gaussians.

In this paper we give a powerful new approach for developing algorithms for testable learning using tools from moment matching and metric distances in probability. We obtain efficient testable learners for any concept class that admits low-degree sandwiching polynomials, capturing most important examples for which we have ordinary agnostic learners. We recover the results of Rubinfeld and Vasilyan as a corollary of our techniques while achieving improved, near-optimal sample complexity bounds for a broad range of concept classes and distributions.

Surprisingly, we show that the information-theoretic sample complexity of testable learning is tightly characterized by the Rademacher complexity of the concept class, one of the most well-studied measures in statistical learning theory. In particular, uniform convergence is necessary and sufficient for testable learning. This leads to a fundamental separation from (ordinary) distribution-specific agnostic learning, where uniform convergence is sufficient but not necessary.

CCS CONCEPTS

• Theory of computation → Boolean function learning. Sample complexity and generalization bounds. Models of learning.

KEYWORDS

PAC learning, moment matching, sandwiching polynomials, generalization, Rademacher complexity

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1 INTRODUCTION

In the fundamental model of agnostic learning [38, 65], a learner tries to output the best-fitting function from a concept class C with respect to an unknown labeled distribution D in the following sense: given sufficiently many labeled examples, with high probability it must produce a hypothesis with error at most opt(C, D) + ε over D, where opt(C, D) denotes the optimal error achievable over D by any concept in C. No assumptions are made on the labels.

Agnostic learning is known to be computationally intractable for even the simplest function classes without making assumptions on the marginal [19, 20, 26, 29, 37, 38, 42]. There is now a substantial literature of efficient agnostic learning algorithms under various distributional assumptions, the most common being that the marginal is Gaussian or Unif{±1}d (see e.g. [18, 33, 35, 41, 49]). The problem of directly verifying this distributional assumption from samples, however, is often computationally infeasible (such as for Unif{±1}d or fundamentally ill-posed (as for N(0, Ld)).

Since in the agnostic model we make no assumptions on the labels, we have no a priori estimate of opt(C, D), the error of the best-fitting classifier. Thus, a major (and often overlooked) issue with the agnostic learning model is that it is unclear how to verify that the agnostic learner has actually succeeded. Note that while we can estimate the true error of the output hypothesis on a hold-out set (a.k.a. validation), we do not know its relationship to opt(C, D).

With this motivation in mind, very recent work of Rubinfeld and Vasilyan [55] introduced the elegant model of testable agnostic learning, or just testable learning for short. In this model, no assumptions are made on D, but there is a tester responsible for verifying whether the unknown marginal is suitably well-behaved. Whenever the tester accepts, the learner must succeed at producing a hypothesis with error at most opt(C, D) + ε (with high probability). And to ensure nontriviality, whenever the unknown marginal is indeed a certain well-behaved target marginal D∗, the tester must accept (with high probability). We say the class C is testably learnable with respect to a target marginal D∗ if there is a tester-learner pair meeting these conditions (see Definition 2.1).

In this model, [55] showed that halfspaces can be testably learned with respect to Gaussians in time and sample complexity 2Ω(1/ε4). Their proof involves checking that the low-degree moments of the unknown marginal are close to those of a Gaussian. They show that this implies concentration and anticoncentration properties of the unknown marginal and further prove that any distribution (including the empirical distribution on samples) that satisfies such...
properties admits low-degree polynomial approximators for halfspaces. Their analysis, however, is catered specifically to the case of halfspaces and Gaussian marginals.

We note that in independent and concurrent work, Rubinfeld and Vasičkan [56] have found a testable learning algorithm for halfspaces over the uniform distribution on the hypercube with the same \(d^{O(1/\epsilon^4)}\) sample complexity as in the Gaussian case. As we’ll discuss, their techniques are quite different from ours.

### 1.1 Our Results

Our main algorithmic contribution is a general framework that yields efficient testable learning algorithms for broad classes of functions and distributions (both continuous and discrete). As we discuss in more detail below, our framework departs from the focus on constructing low-degree polynomial approximations with respect to absolute loss as in [55], which appears hard to extend to classes beyond a single halfspace. Instead, we rely on a new connection to a stronger type of approximator — sandwiching polynomials — that arises naturally in constructing pseudorandom generators for classes of Boolean functions.

As it turns out, many interesting and well-studied concept classes admit both sandwiching approximators and ordinary low-degree polynomial approximators of essentially the same degree, even though sandwiching is a formally stronger notion. As a result, we derive testable learning algorithms for halfspaces and more generally arbitrary functions of a bounded number of halfspaces with respect to any fixed strongly logconcave distribution. For the uniform distribution on the hypercube, we obtain algorithms for halfspaces, degree-2 PTFs, and constant-depth circuits. For each of these applications, our running times and sample complexity guarantees match the best known results for ordinary agnostic learning, thus showing that testable learning can often be achieved at no additional cost (see Theorem 4.2 and Theorem 5.2 for precise statements).

In particular, for the special case of testably learning a single halfspace with respect to the Gaussian, our results improve the \(d^{O(1/\epsilon^4)}\) running time and sample complexity guarantee shown in [55] to \(d^{O(1/\epsilon^4)}\), matching the best known (and conditionally optimal) results for ordinary agnostic learning. Moreover, our analysis extends to a broad family of distributions including strongly logconcave distributions.

We now describe our results and discuss our techniques in more detail.

**Sandwiching Polynomial Approximation.** Our starting point is a relationship between testable learning and a certain stronger notion of polynomial approximation that arises naturally in building pseudorandom generators for Boolean function classes. Specifically, a concept class \(C\) admits *sandwiching* approximations of degree \(d\) and error \(\epsilon\) on \(D_X\) if for every function \(f \in C\), there exist two degree-\(d\) polynomials \(p_1, p_2\) such that for every \(x\), \(p_1(x) \leq f(x) \leq p_2(x)\), and moreover \(\mathbb{E}_{D_X}[|p_1 - f|], \mathbb{E}_{D_X}[|p_2 - f|] \leq \epsilon\). Observe that this is a stronger requirement than the existence of approximating polynomials for \(C\) with respect to absolute loss, which only requires that for every \(f\), there be a degree \(d\) polynomial \(p\) such that \(\mathbb{E}_{D_X}[|p - f|] \leq \epsilon\).

The main theorem underlying our framework shows that unlike the existence of polynomial approximators with respect to absolute loss, the existence of sandwiching approximations universally translates into testable learning algorithms:

**Theorem 1.1** (Testable learning using approximate moment matching; see Theorem 4.6). Let \(D_X\) be a distribution on \(X\), and let \(C\) be a concept class mapping \(X\) to \(\{\pm 1\}\). Let \(k \in \mathbb{N}, \delta \in \mathbb{R}_+\) be degree and slack parameters, and let \(\epsilon > 0\) be the error parameter. Suppose that for every \(f \in C\) admits degree-\(k\) \(\epsilon\)-sandwiching polynomials \(p_1 \leq f \leq p_2\) w.r.t. \(D_X\) such that \(\|p_1\|, \|p_2\|_1 \leq \epsilon / \delta\), where \(\|p_2\|\) (resp. \(\|p_1\|\)) refers to the \(\ell_1\) norm of the coefficients of \(p_2\) (resp. \(p_1\)). Suppose also that with high probability over a sample of size \(d^{O(k)}\), the empirical moments of degree at most \(k\) of \(D_X\) are within \(\delta\) of their true moments. Then \(C\) can be testably learned w.r.t. \(D_X\) up to excess error \(O(\epsilon)\) in sample and time complexity \(d^{O(k)} / \text{poly}(\epsilon)\).

Theorem 1.1 relies on a simple tester: verify that the empirical moments of degree at most \(k\) are close enough to the those of \(D_X\). The correctness of the tester relies on the claim that sandwiching polynomials for \(C\) under \(D_X\) are also sandwiching polynomials for \(C\) under the uniform distribution \(\hat{D}\) on a large enough sample from \(D_X\), with an additional error that scales proportional to the \(\ell_1\) norm of the coefficients of the polynomial approximators. Thus, whenever we have sandwiching approximators with appropriate bounds on the \(\ell_1\) norm of the coefficient vectors, our testable learner can simply use the now-standard degree-\(k\) (absolute-loss) polynomial regression algorithm of [33].

The work of [33] showed that the existence of low-degree (not necessarily sandwiching) polynomial approximators with respect to absolute loss suffices for *ordinary* agnostic learning. Our theorem on sandwiching polynomial approximators can be thought of as the natural counterpart to their condition but for testable agnostic learning.

We stress that merely having ordinary polynomial approximators for \(C\) with respect to absolute loss under \(D_X\) — as opposed to sandwiching polynomials — would not have sufficed for Theorem 1.1. This is because such approximators do not readily translate to approximators for \(C\) under a different distribution \(D'\) that only approximately matches low-degree moments with \(D_X\) (i.e., passes our test for \(D_X\)). The crucial role played by sandwiching approximation is that it allows such a transfer principle: sandwiching approximators for \(C\) under \(D_X\) (with sufficiently small coefficients) are also sandwiching approximators for \(C\) under such a \(D'\) (see Corollary 3.3).

Our main task now reduces to constructing sandwiching polynomials with sufficiently small coefficients. Since proofs of existence of sandwiching polynomials are sometimes nonconstructive (e.g., for constant-depth circuits over the hypercube, where the existence of such polynomials follows from LP duality [12]), we require new techniques to prove bounds on the \(\ell_1\) norm of the coefficients. We make progress by crucially exploiting a form of *approximate* duality between sandwiching polynomials and moment matching.

**Moment Matching and Sandwiching Polynomials.** The duality between fooling using moment matching and the existence of sandwiching polynomials is well-known in the setting of the Boolean hypercube [12], where moment matching \(\text{Unif}\{\pm 1\}^d\) up to degree
$k$ is equivalent to $k$-wise independence. We need an approximate version of this duality in order to derive a bound on the coefficients of the approximating polynomials. Moreover, we need duality to hold over continuous domains for our applications to non-discrete settings (such as Gaussian and strongly logconcave distributions). A duality relating exact moment matching and sandwiching approximations over general domains was proved in [40].

We derive the following general duality result, which tells us that approximate moment matching fools a class $C$ over $D_X$ if and only if every concept in $C$ admits a pair of sandwiching polynomials with sufficiently small coefficients. Our proof relies on establishing a strong duality result for a certain semi-infinite linear program using tools from general conic duality [60].

**Theorem 1.2** (Fooling using approximate moment matching $\iff$ sandwiching approximation; see Theorem 3.2). Let $D_X$ be a distribution on $X$, and let $C$ be a concept class mapping $X$ to $\{\pm 1\}$. Let $k \in \mathbb{N}$, $\delta \in \mathbb{R}_+$ be degree and slack parameters, and let $\epsilon > 0$ be the error parameter. The following are equivalent:

- (Approximate moment matching fools $C$.) For all $f \in C$ and for all distributions $D'$ whose moments of degree at most $k$ are within $\delta$ of those of $D_X$, we have $\|E_{D'}[f] - E_{D_X}[f]\|_1 \leq \epsilon$.

- (Existence of sandwiching polynomials with bounded coefficients for $C$.) For all $f \in C$, there exist degree-$k$ polynomials $p_1, p_u$ such that $p_1 \leq f \leq p_u$ (pointwise over $\mathbb{R}^d$), and $E_{D_X}[p_u - f] + \delta \|p_u\|_1 \leq \epsilon$, $E_{D_X}[f - p_1] + \delta \|p_1\|_1 \leq \epsilon$.

where $\|p_u\|_1$ (resp. $\|p_1\|_1$) refers to the $\ell_1$ norm of the coefficients of $p_u$ (resp. $p_1$).

**Applications:** Testably Learning Functions of Halfspaces and More. Combining Theorems 1.1 and 1.2, we obtain a clean framework for testable learning that reduces the task to establishing that approximate low-degree moment matching fools the target concept class over the target marginal. As our main application, we show that any function of a constant number of halfspaces over $\mathbb{R}^d$ can be testably learned up to excess error $\epsilon$ in sample and time complexity $d^{O(1/\epsilon^2)}$ with respect to any distribution whose directional projections are sufficiently anticoncentrated and have strictly sub-exponentially decaying tails.

**Definition 1.3.** We say a distribution $D_X$ on $\mathbb{R}^d$ is anticoncentrated and has $\alpha$-strictly subexponential tails if the following hold:

(a) $\alpha$-strictly subexponential tails: For all $\|u\| = 1$, $P \{ |(x,u)| > t \} \leq \exp(-\alpha t^{1+\alpha})$ for some constant $\alpha$.

(b) Anticoncentration: For all $\|u\| = 1$ and continuous intervals $T \subset \mathbb{R}$, we have $\mathbb{P}(x,u) \in T \leq C'TT$ for some constant $C'$.

**Theorem 1.4** (Testably learning functions of halfspaces; see Theorem 5.2). Let $C$ be the class of functions of a constant number of halfspaces over $\mathbb{R}^d$. Let $D_X$ be a distribution that is anticoncentrated and has $\alpha$-strictly subexponential tails (Definition 1.3). Then $C$ can be testably learned w.r.t. $D_X$ up to excess error $\epsilon$ using sample and time complexity $d^{O(1/(\epsilon^{(1+\alpha)})^{1/2})}$.

We note that even for ordinary agnostic learning, the above result is an exponential improvement in the dependence on $\epsilon$ in the degree of the sandwiching polynomial compared to prior constructions of [40].

On the flip side, note that even though our framework handles $D_X$ that come from a fairly broad family, our tester must know the low-degree moments of the particular $D_X$ with respect to which it is expected to succeed. This is true for the approach of [55] as well, and it is an interesting open question whether this can be relaxed.

The class of distributions that are anticoncentrated and have strictly subexponential tails is fairly general and includes Gaussians, the uniform distribution on the unit sphere, and more generally, any strongly logconcave distribution [57] (and in fact all of these examples have $\alpha = 1$). This latter class includes the uniform distribution on any convex body with smooth boundary [2] and in particular, additive Gaussian smoothing of any convex body. Theorem 1.4 already matches the upper bound of [33] as well as known statistical-query (SQ) lower bounds [23, 25, 28] for ordinary agnostic learning of a single halfspace with respect to the Gaussian distribution. It also generalizes and improves the main algorithmic result of [55], who showed such a result for a single halfspace with time and sample complexity $d^{O(1/(\epsilon^2))}$.

The key technical result underlying Theorem 1.4 is a proof that any distribution that approximately matches degree-$O(\epsilon^{-1+\alpha}/\alpha)$ moments with a distribution $D_X$ which is anticoncentrated and has $\alpha$-strictly subexponential tails fools functions of halfspaces with respect to $D_X$ (see Theorem 5.6). Similar to the approach of [40], we rely on powerful methods arising from the classical theory of moments [39] and metric distances in probability [51, 68] to show that whenever the moments of $D_X$ are strictly sub-exponential, moment closeness implies distribution closeness in the so-called $\lambda$-metric (see Section 5.1).

We also apply our framework to immediately obtain testable learning results with respect to the Unif($\pm 1$) distribution in time $d^{O(k)}$ for classes $C$ that are fooled by $k$-wise independence, including halfspaces [22], degree-2 PTFs [24], and constant-depth circuits [17], with running time and sample complexity that matches their ordinary agnostic counterparts; see Theorem 4.2. Over the hypercube, the fact that approximate moment matching — i.e. almost $k$-wise independence — suffices to fool such classes is immediate by a result due to [1].

**Moments vs Anticoncentration.** Theorem 1.4 immediately implies that one can test anticoncentration properties of all directional marginals of a broad family of distributions by checking only the low-degree moments.

**Corollary 1.5** (Anticoncentration from approximate moment matching; see Corollary 5.7). Fix $\epsilon > 0$ and a distribution $D_X$ that is anticoncentrated and has $\alpha$-strictly subexponential tails. Let $D'$ be any distribution whose moments of degree at most $k = O(\epsilon^{-1+\alpha}/\alpha)$ match those of $D_X$ up to an additive slack of $d^{O(k)}$. Then for any $\|u\| = 1$ and any continuous interval $T \subset \mathbb{R}$, $P_{x \sim D'} \{ (x,u) \in T \} \leq P_{x \sim D_X} \{ (x,u) \in T \} + \epsilon$.

This statement relates anticoncentration phenomena to structure in low-degree moments. In particular, any distribution that matches the first degree-$O(1/\epsilon^2)$ moments of a strongly logconcave distribution must have all its directional marginals anticoncentrated up to an additive error of $\epsilon$. In addition to being a basic result in probability, such a connection relates to verifying anticoncentration of all directional marginals from a small sample. Finding verification
subroutines that extend beyond Gaussian (and the uniform distribution on the sphere) have a host of applications in algorithmic robust statistics and immediately yield efficient robust algorithms for list-decodable linear regression \cite{RubinfeldV20,IVP21} and covariance estimation \cite{BNS19,IVP21}, and robust clustering \cite{BNS19,IVP19} of mixtures for broader families of distributions than currently known.

For the specific case of Gaussian distributions (and the uniform distribution on the unit $d$-dimensional sphere), such a property for the case when $T$ is an origin centered interval was first proved in a sequence of works that introduced certifiable anticoncentration in the context of algorithmic robust statistics \cite{RubinfeldV20,IVP21}. Their proofs use a polynomial approximator for the “box function” (see e.g. \cite[Appendix A]{IBS21}) and show that degree-$\tilde{O}(1/e^2)$ moments are enough to ensure $\epsilon$-approximate anticoncentration for origin centered intervals $T$. A similar argument based on approximations for the box function was used by \cite{RS21} to show that matching degree-$\tilde{O}(1/e^4)$ moments of Gaussian implies $\epsilon$-approximate anticoncentration for all intervals $T$ as above. This quartic dependence in the order of moments required appears necessary in a proof that constructs polynomial approximations for the box function. Our argument above circumvents this bottleneck in these previous techniques and recovers the $\epsilon$-additive error anticoncentration from matching just the degree-$\tilde{O}(1/e^2)$ moments.

Comparison to the Algorithmic Technique of \cite{RS21}. As their main algorithmic result, Rubinfeld and Vasilyan \cite{RS21} gave a testable learning algorithm for halfspaces that uses $d^{O(1/\epsilon^4)}$ time and samples. At a high level, the chief difference between our approach and theirs is that theirs relies on explicitly constructing polynomial approximators for any distribution $D$ that matches low-degree moments with the target $D_X$, whereas ours uses a general transfer principle showing that sandwiching approximators under $D_X$ yield sandwiching approximators under $D$.

In more detail, their algorithm uses the fact that halfspaces admit a low-degree polynomial approximator with respect to a distribution $D$ whenever $D$ is anticoncentrated and has subgaussian low-degree moments. In order to verify that the empirical distribution on a large enough Gaussian sample possesses these two properties, they relate anticoncentration to low-degree moments via polynomial approximators for the box function as described above.

In contrast, an appeal to sandwiching approximation allows us to extend our testable learning results to non-anticoncentrated discrete distributions such as the uniform distribution on the hypercube, more expressive concept classes such as constant depth circuits on the hypercube and functions of halfspaces on continuous distributions, and to a broad family of distributions including all strongly logconcave distributions.

Sample Complexity of Testable Learning and Rademacher Complexity. One of our main contributions is a complete characterization of the sample complexity of testable learning. Similar to how VC-dimension corresponds to the sample complexity of distribution-free agnostic learning, we show that Rademacher complexity is the key quantity that controls the sample complexity of testable learning. Recall that the Rademacher complexity of a class $C$ w.r.t. $D_X$ at sample size $m$ is given by

$$R_m(C, D_X) = \mathbb{E}_{\{x_i\}_{i\in[m]} \sim D_X^m} \mathbb{E}_{\sigma \sim \pm 1^m} \sup_{f \in C} \left| \frac{1}{m} \sum_{i \in [m]} \sigma_i f(x_i) \right|.$$ 

This measure plays an important role in statistical learning theory since it controls the uniform convergence of empirical losses to true losses over all $f \in C$ (see Theorem 2.3). To our knowledge, our result is the first natural model-based characterization of Rademacher complexity. We obtain precise upper and lower bounds on the sample complexity of testable learning within excess error $\epsilon$ purely in terms of Rademacher complexity:

**Theorem 1.6** (Rademacher complexity characterizes testable learning, see Theorems 6.1 and 6.2). Let $D_X$ be a distribution on $X$, let $C$ be a concept class mapping $X$ to $\{\pm 1\}$, and let $\epsilon > 0$ be the error parameter.

- **(Upper bound.)** Let $m$ be such that $R_m(C, D_X) \leq \epsilon/5$. Then $C$ can be testably learned w.r.t. $D_X$ up to excess error $\epsilon$ using sample complexity $m + O(1/\epsilon^2)$.

- **(Lower bound.)** Let $M$ be such that $R_M(C, D_X) \geq 5\epsilon$, and assume $M \geq \Theta(1/\epsilon^2)$. Then the sample complexity required to testably learn $C$ w.r.t. $D_X$ up to excess error $\epsilon$ is at least $\Omega(\sqrt{M})$.

This characterization yields an interesting separation between ordinary distribution-specific agnostic learning and testable learning. For the former, while uniform convergence is always a sufficient condition, it is not necessary, as witnessed by examples such as convex sets in Gaussian space \cite{RST19} and monotone Boolean functions \cite{ADW18} (see Section 6.2.1). Indeed, the sample complexity of distribution-specific agnostic learning is known to be characterized by the $L^1(D_X)$ metric entropy rather than the Rademacher complexity \cite{BNS19}. In contrast, it is the Rademacher complexity that provides the right characterization for testable learning. Thus, testable learning is a natural supervised learning model for which bounded Rademacher complexity, and hence uniform convergence, provides a necessary and sufficient condition for learning. For further discussion, see Section 7.

1.2 Concurrent Work

In independent and concurrent work, Rubinfeld and Vasilyan \cite{RS22} have extended their algorithm for halfspaces with respect to Gaussian target marginals to the uniform distribution over the hypercube. They do so by reusing their approximator for the box function and showing that it yields a polynomial approximator for regular halfspaces with respect to almost $k$-wise independent distributions. They then utilize the "critical index" framework of \cite{BR17} to reduce the case of general halfspaces to the regular case. Their tester and its analysis rely on $\ell_1$-approximating polynomials for (regular) halfspaces (instead of sandwiching approximations as in our work) and incurs a suboptimal $d^{O(1/\epsilon^4)}$ time and sample complexity as opposed to the (conjecturally) optimal $d^{O(1/\epsilon^2)}$ bound obtained by our approach.
1.3 Related Work

The duality between fooling using bounded independence and sandwiching approximation is a fundamental tool in the pseudo-randomness literature for showing that \( k \)-wise independence fools various classes [12, 17, 22]. Its more general statement in terms of moment matching was observed by [40] (see also [34]), who used it to obtain low-degree sandwiching polynomials for functions of halfspaces w.r.t. logconcave distributions (their constructions do not give any insight on the \( \ell_1 \) norm of the coefficients). We build on their approach for our main application, namely testably learning functions of halfspaces with respect to Gaussians, and obtain exponentially improved degree bounds in terms of \( \epsilon \) along with effective bounds on the size of the coefficients.

In statistical learning theory and nonparametric regression, one of the basic objectives is to place tight bounds on the error risk \( L(f) - \inf_{f \in C} L(f) \) and on the generalization gap \( \ell_{\text{inst}}(f) - \hat{L}_m(f) \) of an estimator \( \hat{f} \) in various settings (see e.g. [15, 62, 64]). In particular, there is a long line of work studying data-dependent bounds on these quantities in terms of measures such as the Rademacher complexity and various refinements and variants thereof [7, 8, 10, 43–45]. Our sample complexity upper bound applies a simple such data-dependent bound to the testable learning setting. In terms of lower bounds, [47] have studied bounds on the minimal error of any ERM estimator, in the additive Gaussian noise setting, in terms of the Gaussian complexity. None of these works, however, consider a model similar to testable learning.

Statistical characterizations of PAC learning have also been well-studied. In the distribution-free setting, it is very well-known that the sample complexity is characterized fully by the VC-dimension, and equivalent to uniform convergence [65]. For distribution-specific agnostic setting, [14] obtained a characterization in terms of the \( L^1(D_X) \) metric entropy, i.e. the log covering number w.r.t. the metric \( \rho(f, g) = \mathbb{E}_{D_X}[|f - g|] \). Work by [46] (see also [63]) proposed a characterization of efficient agnostic learning using the so-called refutation complexity, and interpreted it as a computational analog of Rademacher complexity. Results of [59] showed that in Vapnik’s General Setting of Learning, the sample complexity is in general characterized by notions of algorithmic stability rather than uniform convergence. In modern deep learning theory, the failures of the uniform convergence paradigm in the overparameterized regime have been much studied (see e.g. [50, 67]); we refer the reader to [11, 13] for surveys.

2 PRELIMINARIES

2.1 Notation and Conventions

We denote the domain by \( X \), which for us is always either \( \mathbb{R}^d \) or \( \{|\pm 1\}|^d \), and labels always lie in \( \{\pm 1\} \). We use \( C \) to denote a concept class mapping \( X \) to \( \{\pm 1\} \). We use \( D_X \) to denote a well-behaved distribution on \( X \) (i.e. the target marginal, such as \( \mathcal{N}(0, I_d) \) or \( \text{Unif}(\{\pm 1\})^d \)), and we use the calligraphic \( D \) to denote labeled distributions on \( X \times \{\pm 1\} \). We denote a size-\( m \) (labeled) sample drawn from \( D \) by \( S \sim D^\text{opt}_m \). If \( S = \{(x_i, y_i)\}_{i=1}^m \), then we use \( S_X = \{x_i\}_{i=1}^m \) to denote its “marginal”, i.e. the unlabeled sample.

Our loss function throughout will be the 0/1 loss function, \( \ell(y, y') = \mathbbm{1}[y \neq y'] \). Given a labeled distribution \( D \), we denote the population loss by \( L(f, D) = \mathbb{P}(x, y \sim D)[f(x) \neq y] \) (or just \( L(f) \) when \( D \) is implicit), and the empirical loss on a size-\( m \) sample \( S \sim D^\text{opt}_m \) by \( \hat{L}_m(f, D) = \mathbb{P}(x, y \sim \text{opt}_m)[f(x) \neq y] \) (or just \( L_m(f) \) when \( S \) is implicit). We follow the convention of denoting empirical quantities using a hat and a subscript to denote the sample size (as in \( \hat{L}_m \)). We use \( \text{opt}(C, D) \) to denote \( \inf_{f \in C} L(f, D) \).

We follow the following conventions when working with monomials over \( X \). For any multi-index \( I \in \mathbb{N}^d \), let \( |I| = \sum_{j} I_j \) denote its degree (or sometimes order), and let \( x_I \) denote the monomial \( \prod_{j=1}^d x_j^{I_j} \). We use \( I(k, d) = \{I \in \mathbb{N}^d \mid |I| \leq k \} \) to denote the set of multi-indices of degree at most \( k \). For a vector \( A \in \mathbb{R}_{+}^{I(k,d)} \) and a degree-k polynomial \( p : \mathbb{R}^d \to \mathbb{R} \) given by \( p(x) = \sum_{I \in I(k,d)} A_I x_I \), we use \( \langle A, p \rangle \) to denote \( \sum_{I \in I(k,d)} A_I \Delta_I \). This may be thought of as the \( \Delta \)-weighted \( \ell_1 \) norm of the coefficients of \( p \).

We use \( a \leq b \) and \( a \leq b \) to denote equalities and inequalities up to constants. It will be convenient to state Stirling’s approximation in the following form [54]: for all \( n \geq 1 \), \( n! \leq e^{-\frac{1}{2}} \sqrt{2\pi n} n^{n+\frac{1}{2}} \). We will also make use of the double factorial \( n!! = (n-2) \cdots 2 \cdot 1 \), satisfying \( n!! = n!(n-1)!! \) for even \( n \).

Throughout this paper, we use the term “with high probability” to mean “with probability at least 0.99” (or any other sufficiently large constant) for simplicity. In all cases, confidence may be amplified using standard repetition arguments.

2.2 Testable Learning

Definition 2.1 (Testable agnostic learning, [55]). We say a tester-learner pair \( (T, A) \) testably learns \( C \) w.r.t. \( D_X \) up to excess error \( \epsilon \) if for any distribution \( D \) on \( X \times \{\pm 1\} \), the following conditions are met:

- (Soundness/composability.) If \( D \) is such that the tester \( T \) accepts with high probability over a sample drawn from \( D \), then the learner \( A \) succeeds in agnostically learning \( C \) w.r.t. \( D \), i.e. with high probability it produces a hypothesis \( h \) such that \( L(h) \leq \text{opt}(C, D) + \epsilon \).
- (Completeness.) Whenever \( D \) truly has marginal \( D_X \) on \( X \), the tester \( T \) accepts with high probability.

Again, here “with high probability” may be taken to be “with probability at least 0.99” for simplicity, and the confidence in each step may be amplified using standard repetition arguments.

2.3 Rademacher Complexity

The Rademacher complexity is one of the most well-studied measures of the complexity of a function class in statistical learning theory, and may be intuitively thought of as measuring the ability of a function class to fit a randomly-labeled sample. The following definitions and theorems are now standard in the literature (see e.g. [6, 10, 11, 16] and references therein).

Definition 2.2 (Rademacher complexity). Consider a sample of \( m \) points \( S_X = \{x_{i}\}_{i=1}^{m} \sim D_X^\text{opt}_m \). The empirical Rademacher complexity of the class \( C \) w.r.t. \( D_X \) at sample size \( m \) is defined to be

\[
\hat{\Delta}_m(C, S_X) = \mathbb{E}_{\sigma \sim \{-1,1\}^m} \sup_{f \in C} \frac{1}{m} \sum_{i=1}^{m} \sigma_i f(x_i).
\] (2.1)

Note that this is a random variable depending on \( S_X \). The (expected) Rademacher complexity of \( C \) w.r.t. \( D_X \) at sample size \( m \) is defined
to be
\[
\mathcal{R}_m(C, D_X) = \mathbb{E}_{S_X \sim D_X^m} \mathcal{R}_m(C, S_X).
\] (2.2)

Sometimes we simply say \( \mathcal{R}_m(C) \) (resp. \( \mathcal{R}_m(C) \)) when \( D_X \) (resp. \( S_X \)) is clear from context.

The next theorem states that the Rademacher complexity of a class tightly controls uniform convergence, i.e. bounds on the quantity \( \sup_{f \in C} |L(f) - \tilde{L}_m(f)| \), where \( L \) and \( \tilde{L}_m \) are the population and empirical loss functionals. The upper bound here follows from a so-called symmetrization argument, while the lower bound follows from a desymmetrization argument. In our statement, we specialize to the case of the 0-1 loss.

**Theorem 2.3 (see e.g. [6]).** Let \( C \) be a class of functions mapping \( X \) to \( \{\pm 1\} \). Let \( D \) be a distribution on \( X \times \{\pm 1\} \) with marginal \( D_X \) on \( X \), and let \( S \sim D^\otimes m \) be a random sample of size \( m \) drawn from \( D \). For any \( f \in C \), let \( L(f) = \mathbb{P}_{(x,y) \sim D} [f(x) \neq y] \), and let \( \tilde{L}_m(f) = \mathbb{P}_{(x_1, \ldots, x_m) \sim D} [f(x_i) \neq y_i] \). Then with probability \( 1 - \delta \) over the draw of \( S \), we have
\[
\frac{1}{4} \mathcal{R}_m(C) - \Theta \left( \sqrt{\frac{\log(1/\delta)}{m}} \right) \leq \sup_{f \in C} \left| L(f) - \tilde{L}_m(f) \right| \leq \mathcal{R}_m(C) + \Theta \left( \sqrt{\frac{\log(1/\delta)}{m}} \right).
\]

3 DUALITY

In this section we state the duality between fooling using approximate moment matching and sandwiching polynomials. This is a generalization of duality over the hypercube [12] to continuous domains and more general distributions. Note that a version of duality over \( \mathbb{R}^d \), albeit only for exact moment matching, was stated in [40, Lemma 3.3].

**Definition 3.1 (Approximate moment matching).** Let \( k \in \mathbb{N} \) be a degree parameter, and let \( \Delta \in \mathbb{R}^{I(k,d)} \) be a slack parameter, satisfying \( \Delta_0 \coloneqq \Delta_{\{0,\ldots,0\}} = 0 \) and \( \Delta_I > 0 \) for all other \( I \in I(k,d) \). We say that two distributions \( D, D' \) on \( X \) match moments of degree (or order) at most \( k \) up to slack \( \Delta \) if \( \| E[D_x] - E[D'_x] \| \leq \Delta_I \) for all \( I \in I(k,d) \).

The reason for allowing the slack \( \Delta_I \) to depend on \( I \) is that in general we expect the scale of the moments to vary widely with \( I \) (as with the Gaussian, for example). The empty index \( I_0 = 0 = (0, \ldots, 0) \) plays a special role, since \( x_k = 1 \) and \( E[D]\{1\} = 1 \) for any valid distribution, meaning we may assume \( \Delta_0 = 0 \) without loss of generality.

We can now state the main theorem. We prove this theorem using conic LP duality [60], taking care to establish strong duality, but the essential argument is similar to [12, Thm A.1].

**Theorem 3.2.** Let \( k \in \mathbb{N}, \Delta \in \mathbb{R}^{I(k,d)} \) be the degree and slack parameters, as in Definition 3.1. Let \( f : X \to \mathbb{R} \) be a function, and let \( D \) be a distribution on \( X \). The following are equivalent:

(a) (Approximate moment matching fools \( f \) w.r.t. \( D \)) For any distribution \( D' \) whose moments up to order \( k \) match those of \( D \) up to \( \Delta \), we have \( \| E[D_f] - E[D'_f] \| \leq \epsilon \).

(b) (Existence of sandwiching polynomials with bounded coefficients for \( f \) w.r.t. \( D' \)) There exist degree-\( k \) polynomials \( p_1, p_u \) such that \( p_1 \leq f \leq p_u \) (pointwise over \( \mathbb{R}^d \)), and
\[
\mathbb{E}_D[p_u - f] + \langle \Delta, |p_u| \rangle \leq \epsilon, \quad \mathbb{E}_D[f - p_1] + \langle \Delta, |p_1| \rangle \leq \epsilon.
\]

(Recall that for a degree-\( k \) polynomial \( p(x) = \sum_l p_l x^l \), we use \( \langle \Delta, |p| \rangle \) to denote \( \sum_{l \in I(k,d)} |p_l| \Delta_l \).)

**Proof.** Let \( \sigma_l = \mathbb{E}_D[x_l] \). Let \( P_D \) be the set of all Borel probability measures on \( \mathbb{R}^d \). Consider the following semi-infinite linear program, which seeks to maximize \( \mathbb{E}_D[f] \) over all probability distributions \( D' \) on \( \mathbb{R}^d \) that approximately match moments with \( D' \):
\[
\sup_{D' \in P_D} \mathbb{E}_D[f]
\]
subject to
\[
\sigma_I - \Delta_I \leq \mathbb{E}_{D'}[x_I] \leq \sigma_I + \Delta_I \quad \forall I \in I(k,d)
\]

The case of \( I = (0, \ldots, 0) \) is special: here \( \sigma_0 = \mathbb{E}_D[1] = 1 \) and \( \Delta_0 = 0 \), so the corresponding constraint becomes simply \( \mathbb{E}_{D'}[1] = 1 \), which is equivalent to requiring that \( D' \) be a valid probability measure.

The dual LP turns out to be equivalent to the following, with variable \( \beta \in \mathbb{R}^{I(k,d)} \):
\[
\inf_{\beta \in \mathbb{R}^{I(k,d)}} \sum_{I \in I(k,d)} \beta_I \sigma_I + \sum_{I \in I(k,d)} |\beta_I| \Delta_I
\]
subject to
\[
\sum_{I \in I(k,d)} \beta_I x_I \geq f(x) \quad \forall x \in \mathbb{R}^d
\]

Notice that the primal LP (Eq. (3.1)) is feasible (indeed, by \( D' = D \)). Moreover, it can be shown using results from general conic LP duality [60] that strong duality holds. Denote the common optimum of Eqs. (3.1) and (3.3) by \( \gamma \). The claim that approximate moment matching fools \( f \) (in a one-sided fashion) w.r.t. \( D \) is the same as asserting \( \gamma \leq \mathbb{E}_D[f] + \epsilon \). Take \( \beta \) to be an optimal solution to the dual, and let \( p_u(x) = \sum_{I \in I(k,d)} \beta_I x_I \). (In fact, this correspondence between degree-\( k \) polynomials and their coefficient vectors allows us to equivalently view the dual as optimizing over such polynomials instead of their coefficients.) The dual then tells us that \( p_u \geq f \) pointwise, and
\[
\gamma = \sum_{I \in I(k,d)} \beta_I \sigma_I + \sum_{I \in I(k,d)} |\beta_I| \Delta_I = \mathbb{E}_D[p_u] + \langle \Delta, |p_u| \rangle \leq \mathbb{E}_D[f] + \epsilon,
\]
establishing the existence of the upper sandwiching polynomial. To obtain the lower sandwiching polynomial, we replace the objective of the primal with \( -\mathbb{E}_D[f] \) and repeat the same argument, this time using the fact that the common optimum \( \gamma' \) satisfies \( \gamma' \leq -\mathbb{E}_D[f] + \epsilon \) (i.e., effectively replacing \( f \) with \( -f \) throughout). This establishes the desired equivalence.

□

For the purposes of testing using moment matching, one can only ever hope to check that the unknown marginal (\( D' \), say) approximately matches moments with the target marginal. Approximate duality — and specifically the appearance of the quantities \( \langle \Delta, |p_u| \rangle \) and \( \langle \Delta, |p_1| \rangle \) — turns out to be precisely what we need to guarantee sandwiching polynomials even w.r.t. such a \( D' \).

**Corollary 3.3.** Let \( f, D, k, \Delta, \epsilon \) satisfy the conditions of Theorem 3.2, and let \( p_1 \leq f \leq p_u \) be the resulting sandwiching polynomials for \( f \) w.r.t. \( D \). Consider any particular \( D' \) whose moments up to order \( k \)
match those of $D$ up to $\Delta$. Then $p_1, p_2$ are sandwiching polynomials for $f$ w.r.t. $D$ as well, satisfying

$$\mathbb{E}_D[|p_2 - f|] \leq 2\epsilon, \quad \mathbb{E}_D[|f - p_1|] \leq 2\epsilon.$$  

**Proof.** By the first part of Theorem 3.2, we know $|\mathbb{E}_D[f] - \mathbb{E}_D[p_1]| \leq \epsilon$. Thus

$$\left|\mathbb{E}_D[p_2 - f] - \mathbb{E}_D[p_1 - f]\right| \leq \left|\mathbb{E}_D[p_2 - p_1] - \mathbb{E}_D[f] - \mathbb{E}_D[f]\right|$$

$$\leq \langle \Delta, |p_1| \rangle + \epsilon. \quad (3.5)$$

Applying the second part of Theorem 3.2, this means

$$\mathbb{E}_D[p_2 - f] \leq \mathbb{E}_D[p_1 - f] + \langle \Delta, |p_1| \rangle + \epsilon \leq 2\epsilon. \quad (3.6)$$

The argument for $p_1$ is exactly the same. $\square$

## 4 TESTABLE LEARNING VIA MOMENT MATCHING

### 4.1 Warm-up: Testable Learning over the Hypercube via $k$-wise Independence

The main ideas of our approach are already illustrated in the setting of the Boolean hypercube. A key technical ingredient for us will be the fact that an almost $k$-wise independent distribution is statistically close to being truly $k$-wise independent [1].

**Definition 4.1.** We say a distribution $D$ on $\{\pm 1\}^d$ is $(\delta, k)$-independent if for all $|I| \leq k$, $|\mathbb{E}_D[x_I]| \leq \delta$. When $\delta = 0$, we simply call $D$ a $k$-wise independent distribution.

We say that a concept class $C$ is $\epsilon$-fooled by $(\delta, k)$-independence (resp. $k$-wise independence) if for every $f \in C$ and any $(\delta, k)$-independent (resp. $k$-wise independent) $D$, $|\mathbb{E}_D[f] - \mathbb{E}_U[f]| \leq \epsilon$.

**Theorem 4.2.** Let $C$ be any concept class that is $\xi$-fooled by $k$-wise independence. Then $C$ can be tested by learning w.r.t. Unif$[\{\pm 1\}]^d$ up to excess error $\epsilon$ with time and sample complexity $d^{O(k)}/\epsilon^2$.

**Proof.** Let the unknown labeled distribution be $D$. Let $S \sim D^m$ be the labeled sample given to $(T, A)$ (where the sample size $m$ will be picked later), and let $S_X$ be its (unlabeled) marginal. Let $\hat{D}_m$ be the induced empirical distribution, i.e. the uniform distribution over $S_X$.

The tester $T$ and algorithm $A$ are simple: the tester checks that the empirical moments (or biases) up to degree $k$ are all no larger than $\delta = ed^{-k}/4$ in magnitude (i.e. that the empirical distribution is $(\delta, k)$-independent), and the algorithm runs degree-$k$ polynomial regression [33] over the sample.

It is clear that when $D$ indeed has marginal exactly Unif$[\{\pm 1\}]^d$ (or indeed any $(\delta/2, k)$-independent distribution), then by taking $m = d^k/\delta^2 = d^{O(k)}/\epsilon^2$ sufficiently large, we can ensure with high probability all the empirical moments of order at most $k$ concentrate about their true moments up to $\delta$ (by a standard Hoeffding plus union bound). That is, with high probability $\hat{D}_m$ will indeed be $(\delta, k)$-independent, and the tester will accept. This verifies completeness.

To verify soundness, suppose that $\hat{D}_m$ is indeed $(\delta, k)$-independent. By [1, Thm 2.1], this means that $\hat{D}_m$ has TV distance at most $\delta^k = \epsilon/4$ from a truly $k$-wise independent distribution. This in turn means that $\hat{D}_m$ (and indeed any $(\delta, k)$-independent distribution) $\xi$-fools $C$. We now appeal to duality, stated here in some generality as Theorem 3.2, although in the setting of the hypercube this theorem reduces exactly to the form in [12]. Formally, observe that for every $f \in C$, condition (a) of Theorem 3.2 is satisfied (with $D = \text{Unif}[\{\pm 1\}]^d$, and where the slack parameter $\Delta$ is now simply $\delta$ in every coordinate). This allows us to apply Corollary 3.3 to conclude that there exist $\epsilon$-sandwiching polynomials for $C$ w.r.t. $\hat{D}_m$. By [33], this ensures the learner succeeds at learning $C$ up to error $\text{opt}(C, D) + \epsilon$ with high probability. This proves the theorem. $\square$

We may apply this theorem to obtain testable learning w.r.t. Unif$[\{\pm 1\}]^d$ for halfspaces, degree-2 PTFs, and constant-depth circuits.

**Corollary 4.3.** Let $C$ be the class of halfspaces over $\{\pm 1\}^d$. Let $\epsilon > 0$, and let $k = O(1/\epsilon^2)$. Then $C$ is $\epsilon$-fooled by $k$-wise independence [22], and hence it can be testably learned w.r.t. Unif$[\{\pm 1\}]^d$ up to excess error $\epsilon$ with time and sample complexity $d^{O(k)}$.

**Corollary 4.4.** Let $C$ be the class of degree-$2$ polynomial threshold functions over $\{\pm 1\}^d$. Let $\epsilon > 0$, and let $k = \Theta(1/\epsilon^2)$. Then $C$ is $\epsilon$-fooled by $k$-wise independence [24], and hence it can be testably learned w.r.t. Unif$[\{\pm 1\}]^d$ up to excess error $\epsilon$ with time and sample complexity $d^{O(k)}$.

**Corollary 4.5.** Let $C$ be the class of depth-$4$ AC$^0$ circuits of size $s$ over $\{\pm 1\}^d$. Let $\epsilon > 0$, and let $k = (\log s)^{O(1)} \log(1/\epsilon)$. Then $C$ is $\epsilon$-fooled by $k$-wise independence [17, 30, 61], and hence it can be testably learned w.r.t. Unif$[\{\pm 1\}]^d$ up to excess error $\epsilon$ with time and sample complexity $d^{O(k)}$.

### 4.2 A General Algorithm using Moment Matching

We now give a more general algorithm for testable learning that does not need the target distribution to be $k$-wise independent. In this case, our tester will check that the low-degree moments of the empirical distribution are close to those of the target distribution. The correctness of our tester is a consequence of duality (Theorem 3.2).

**Theorem 4.6.** Let $D_X$ be a distribution on $X$, and let $C$ be a concept class mapping $X$ to $\{\pm 1\}$. Let $k \in \mathbb{N}$, $\Delta \in \mathbb{R}_+[f(k,d)]$ be the degree and slack parameters, as in Definition 3.1, and let $\epsilon > 0$ be the error parameter. Suppose the following conditions hold:

(a) (Empirical moments concentrate around true moments.) There exists $m$ large enough that with high probability over a sample $S_X \sim D_X^{m}$, the corresponding empirical distribution $\hat{D}_m$ matches moments of degree at most $k$ with $D_X$ up to slack $\Delta$.

(b) (Existence of sandwiching polynomials with bounded coefficients for $C$, or equivalently approximate moment matching fools $C$.) For every $f \in C$, there exist degree-$k$ sandwiching polynomials $p_l \leq f \leq p_u$ such that

$$\mathbb{E}_{D_X}[|p_u - f| + \langle \Delta, |p_u| \rangle] \leq \frac{\epsilon}{2}, \quad \mathbb{E}_{D_X}[|f - p_l| + \langle \Delta, |p_l| \rangle] \leq \frac{\epsilon}{2}.$$
(Recall that for a degree-k polynomial \( p(x) = \sum_i p_i x_i \), we use \( \langle \Delta, |p| \rangle \) to denote \( \sum_{i \in [k,d]} |p_i| |\Delta_i| \). Equivalently, for every \( f \in \mathcal{C} \) and for any distribution \( D' \) whose moments up to order \( k \) match those of \( D \) up to \( \Delta \), we have \( |E_D[f] - E_{D'}[f]| \leq \frac{\Delta}{k} \).

Then \( C \) can be testably learned w.r.t. \( D \) up to excess error \( \epsilon \) using time and sample complexity \( m + O(k^2) \). Moreover, the tester \( T \) and learner \( A \) are simple: \( T \) tests whether the empirical moments up to order \( k \) match those of \( D \) up to \( \Delta \), and \( A \) performs degree-\( k \) polynomial regression over the sample [33].

PROOF. Let the unknown labeled distribution be \( \mathcal{D} \), and let \( S \sim \mathcal{D} \circ m \) be the sample given to \( (T,A) \). First we verify completeness. By assumption, when \( \mathcal{D} \) indeed has marginal \( D \), then \( m \) is large enough that with high probability over \( S \), the empirical moments concentrate about the true moments up to order \( \Delta \), and hence \( T \) accepts.

As for soundness, suppose that \( T \) accepts, i.e., that the empirical distribution \( \hat{D} \) indeed matches order-\( k \) moments with \( \Delta \). We observe that our condition (b) is the same as condition (b) of Theorem 3.2 is satisfied. Thus we may apply Corollary 3.3 (with \( D = \hat{D} \) and \( D' = \mathcal{D} \)) to conclude that there exist degree-\( k \) \( e \)-sandwiching polynomials for \( C \) w.r.t. \( \hat{D} \). By [33], we have that degree-\( k \) polynomial regression achieves error \( \text{opt}(C, \mathcal{D}) + \epsilon \) with high probability. (This implicitly assumes that the degree-\( k \) polynomial fitting \( S \) will generalize to \( \mathcal{D} \), which will be true by classic VC theory whenever \( m \geq O(k)/\epsilon^2 \) since the VC dimension of degree-\( k \) polynomials (with bounded coefficients, as here) is at most \( O(k) \). If this is not the case, we may replace \( m \) with \( m + O(k)/\epsilon^2 \).) \( \square \)

To see how Theorem 1.1 may be recovered from this, simply set \( \Delta \) to be \( \delta \) in every coordinate, and also rescale \( \epsilon \) appropriately.

5 TESTABLY LEARNING FUNCTIONS OF HALFSPACES OVER STRICTLY SUBEXPONENTIAL DISTRIBUTIONS

In this section we apply Theorem 4.6 to prove that we can testably learn functions of halfspaces w.r.t. a target marginal \( D_X \) on \( X = \mathbb{R}^d \) that is anticoncentrated and has strictly subexponential tails in the sense of Definition 1.3 from the introduction.

Definition 5.1 (Restatement of Definition 1.3). We say a distribution \( D_X \) on \( \mathbb{R}^d \) is anticoncentrated and has \( \alpha \)-strictly subexponential tails if the following hold:

(a) For all \( \|u\|_2 = 1 \), \( \mathbb{P}[|(x,u)| > t] \leq \exp(-C_1 t^{1+\alpha}) \) for some constant \( C_1 \).

(b) For all \( \|u\|_2 = 1 \) and \( k \in \mathbb{N} \), \( \mathbb{E}[\|(x,u)\|^k] \leq C_2 k^{1/(1+\alpha)} \) for some constant \( C_2 \).

(c) For all \( \|u\|_2 = 1 \) and continuous intervals \( T \subset \mathbb{R} \), we have \( \mathbb{P}[\{x \in T \} \leq C_3 |T| \) for some constant \( C_3 \).

The first two conditions are a strengthening of the usual definition of subexponential distributions (see e.g. [66]), and standard arguments show that the two are actually equivalent. The third asks directions marginals of \( D_X \) to be anticoncentrated. Examples include all strongly logconcave distributions, which satisfy this definition with \( \alpha = 1 \) (see e.g. [57, \$5.1] or [48, Thm 2.15]). This class includes the standard Gaussian distribution, the uniform distribution on the unit \( d \)-dimensional sphere and more generally, uniform distribution on any convex body with smooth boundaries (e.g., Gaussian smoothing of arbitrary convex bodies).

Throughout this section, let \( \mathcal{C} \) be the class of functions of \( p \) halfspaces over \( \mathbb{R}^d \), i.e., functions \( f : \mathbb{R}^d \to \mathbb{R} \) of the form

\[
f(x) = g(\text{sign}(w^\top x + \theta_1), \ldots, \text{sign}(w^p x + \theta_p))
\]

for some \( w^1, \ldots, w^p \in \mathbb{R}^d \) (where we use superscripts to avoid confusion with coordinate notation), \( \theta_1, \ldots, \theta_p \in \mathbb{R} \), and \( g : \{|\pm 1\}^p \to \{|\pm 1\} \). We focus on the setting where \( p \) is a constant. Also let \( D_X \) be some fixed distribution that is anticoncentrated and \( \alpha \)-strictly subexponential. We will prove the following theorem, stated earlier as Theorem 1.4.

Theorem 5.2. Let \( C \) be the class of functions of \( p \) halfspaces over \( \mathbb{R}^d \), as above. Assume that \( p = O(1) \). Let \( D_X \) be a distribution that is anticoncentrated and \( \alpha \)-strictly subexponential. Then \( C \) can be testably learned w.r.t. \( D_X \) up to excess error \( \epsilon \) using \( O((e^{-1+\alpha}/\epsilon^2)) \) sample and time complexity.

In particular whenever \( \alpha = 1 \), as for strongly logconcave distributions including \( N(0, I_d) \), we obtain an \( O((e^{-1}/\epsilon^2)) \)-time algorithm.

We now describe our proof plan. To use Theorem 4.6, we must show that approximately matching the low-degree moments of \( D_X \) fools functions of halfspaces. Work due to [40] introduced an argument for this problem based on general techniques from the classical theory of moments and the method of metric distances in probability [39, 51]. Their broad proof approach was to use [39, Thm 2] to show that closeness in moments of two distributions implies closeness in the \( \lambda \)-distance (Definition 5.3), and then relate this to the CDF distance, which directly relates to fooling halfspaces. For our purposes, a direct application of [39, Thm 2] does not suffice. Instead, we directly analyze the \( \lambda \)-distance under the assumption that the moments of \( D_X \) grow in a strictly subexponential fashion. We begin with the technical lemmas we need, and then prove Theorem 5.2 in the final subsection.

5.1 Moment Closeness Implies Distribution Closeness

Definition 5.3 (\( \lambda \)-distance, see e.g. [68], [51, Chap 10]). For a distribution \( P \) on \( \mathbb{R}^d \), let \( \varphi_P : \mathbb{R}^d \to \mathbb{C} \) given by \( \varphi_P(t) := \mathbb{E}_{x \sim P} e^{i t^\top x} \) be its characteristic function. For two distributions \( P, P' \) on \( \mathbb{R}^d \), define the \( \lambda \)-distance between them as follows:

\[
d_\lambda(P, P') = \min_{T > 0} \left\{ \max_{\|t\| \leq T} \left| \varphi_P(t) - \varphi_{P'}(t) \right|, \frac{1}{T} \right\}.
\]

We prove an approximate version of [39, Thm 1] (see also [51, Thm 10.3.4]), bounding the \( \lambda \)-distance between two distributions whose moments approximately match and grow with the degree \( k \) in a strictly subexponential fashion, i.e. as \( k^{1/(1+\alpha)} \).

Lemma 5.4. Let \( k \in \mathbb{N} \) be even. Let \( P \) be a distribution on \( \mathbb{R}^d \) such that for all \( \|u\| \leq 1 \),

\[
\mathbb{E}_{z \sim P} \left| \langle u, z \rangle^k \right| \leq M_k := k^{k/2} C_2^k k^{1/(1+\alpha)}
\]

for some constant \( C_2 \). Let \( P' \) be a distribution that approximately matches moments up to order \( k \) with \( P \) in the following strong sense:
for all \( j \leq k \) and \( \|u\| \leq 1 \),
\[
\left| \mathbb{E}_{z \sim P} (u, z)^j \right| - \mathbb{E}_{\tau \sim P^*} (u, z')^j \right| \leq \eta_j := \frac{j!}{2^k} \left( \frac{6M_k}{k!} \right)^{j+1/(k+1)} \]
for some constant \( C_4 \) depending only on \( C_2 \). Then
\[
d_3(P, P^*) \leq \sqrt{p \cdot k^{-a/(1+\alpha)}}.
\]

**Proof.** To control \( d_3(P, P^*) \), we need to control
\[
\max_{\|u\| \leq 1} \left| \mathbb{E}_{\tau \sim P} (\varphi_{P^*}(\tau) - \varphi_P(\tau)) \right|
\]
as a function of \( T \). To this end, fix any direction \( u \in \mathbb{R}^P \) with \( \|u\| = 1 \), and let \( t = \tau u \) for \( \tau \in [0, T] \) be a vector in that direction satisfying \( \|t\| \leq T \). Let \( \varphi_1(\tau) = \varphi_P(\tau u) \) and \( \varphi_2(\tau) = \varphi_{P^*}(\tau u) \) be the characteristic functions of \( P \) and \( P^* \) along \( u \). We may Taylor expand \( \varphi_1 - \varphi_2 \) up to degree \( k \) as follows:
\[
\varphi_1(\tau) - \varphi_2(\tau) = \sum_{0 \leq j \leq k} \frac{\varphi_1^{(j)}(0)}{j!} t^j - \frac{\varphi_2^{(j)}(0)}{j!} t^j \leq \frac{\eta_j}{k!} \]
for some \( \tau' \in [0, \tau] \).

The crucial fact we use now is that the derivatives of the characteristic function encode its moments. Indeed, for any \( \tau \),
\[
\varphi_1(\tau) = \mathbb{E}_{z \sim P} e^{i\tau^T(z, u)} \implies \varphi_1^{(j)}(\tau) = \mathbb{E}_{z \sim P} [i^j(z, u)^j e^{i\tau^T(z, u)}],
\]
so that in particular \( \varphi_1^{(j)}(0) = \mathbb{E}_{\tau \sim P} (\tau, u)^j \) for all \( j \) (and similarly for \( \varphi_2 \)). This means \( \varphi_1^{(j)}(0) - \varphi_2^{(j)}(0) = 0 \), and for each \( 1 \leq j < k \), by our assumption that \( P^* \) approximately moment matches \( P \), we have
\[
\varphi_1^{(j)}(0) - \varphi_2^{(j)}(0) \leq \eta_j.
\]
At degree \( k \), we have
\[
\varphi_1^{(k)}(\tau') = \mathbb{E}_{z \sim P} [i^k(z, u)^k e^{i\tau^T(z, u)}] \leq \mathbb{E}_{z \sim P} [(z, u)^k] \leq M_k.
\]
And since \( \mathbb{E}_{\tau \sim P^*} [(\tau, u)^k] \leq \mathbb{E}_{z \sim P^*} [(z, u)^k] + \eta_k \), we similarly have
\[
\varphi_2^{(k)}(\tau') \leq M_k + \eta_k \ll 2M_k.
\]
Substituting Eqs. (5.3) into Eq. (5.2), we obtain
\[
\varphi_1(\tau) - \varphi_2(\tau) \leq \sum_{1 \leq j \leq k} \frac{\eta_j}{j!} + \frac{3M_k}{k!} \frac{k^j}{j!} =: F(\tau),
\]
where we have denoted the expression on the RHS by \( F(\tau) \) for convenience. Since \( F(\tau) \) is clearly increasing in \( \tau \) and independent of \( u \), we have \( \max_{\|\tau\| \leq T} \left| \varphi_P(\tau) - \varphi_P(t) \right| < F(T) \). This means that
\[
d_3(P, P^*) \leq \max_{\|\tau\| \leq T} \left| \varphi_P(\tau) - \varphi_P(t) \right| \leq \max \left\{ F(T), \frac{1}{T} \right\},
\]
and our job now is to pick \( T > 0 \) that minimizes the RHS.

This is equivalent to picking the largest \( T \) such that \( F(T) \leq \frac{1}{T} \), i.e.
\[
TF(T) = \sum_{1 \leq j \leq k} \frac{\eta_j}{j!} T^{j+1} + \frac{3M_k}{k!} T^{k+1} \leq 1.
\]
Let us divide this further into two sufficient conditions:
\[
\sum_{1 \leq j \leq k} \frac{\eta_j}{j!} T^{j+1} \leq \frac{1}{2} \quad \text{and} \quad \frac{3M_k}{k!} T^{k+1} \leq \frac{1}{2}.
\]
The second condition is equivalent to
\[
T = \left( \frac{k!}{6M_k} \right)^{1/(k+1)} = \left( \frac{6M_k}{k!} \right)^{1/(k+1)} \leq \frac{\kappa^a}{(1+\alpha)} \sqrt{p},
\]
by Stirling’s approximation. As for the first, we have picked \( \eta_j \) exactly such that when we plug in this value of \( T \), for each \( 1 \leq j < k \) we have
\[
\eta_j = \frac{j!}{2^k} T^{-j+1} \leq \frac{j!}{2^k} \left( \frac{6M_k}{k!} \right)^{j+1/(k+1)} \implies \frac{\eta_j}{j!} T^{j+1} = \frac{1}{2k}.
\]
Summing over \( 1 \leq j < k \) verifies the first condition. Thus for this \( T \), we have
\[
d_3(P, P^*) \leq \max \{ F(T), \frac{1}{T} \} \leq \frac{1}{T} \leq \sqrt{p \cdot k^{-a/(1+\alpha)}},
\]
proving the lemma.

The following lemma is a convenient distillation of the rest of the argument from [40], where the \( \lambda \)-distance is related to the Levy distance (using (27)), which in turn is related to the CDF distance (using anticoncentration), and which leads finally to the desired conclusion.

**Lemma 5.5 (Implicit in [40, §3.3]).** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a function of \( p \) halfspaces as above, and also let distributions \( D, D' \) on \( \mathbb{R}^d \) and \( P, P' \) on \( \mathbb{R}^d \) be as above. Assume that for any continuous interval \( T \subset \mathbb{R} \), each coordinate \( z_j \) of \( z \) satisfies \( \mathbb{P}[z_j \in T] \leq \Theta(|T|) \). Suppose that \( d_3(P, P^*) \leq \delta \). Let \( N(\delta) \) be such that \( \mathbb{E}_{z \sim P} \|z\|_\infty > N(\delta) \leq \delta \) and \( \mathbb{P}[z \sim P] \|z\|_\infty > N(\delta) \leq \delta \). Then
\[
\left| \mathbb{E}_D[f] - \mathbb{E}_{D'}[f] \right| \leq O(2^p \delta \log N(\delta) + 2 \log(1/\delta))^{1/2}.
\]

**5.2 Approximate Low-Degree Moment Matching Fools Functions of Halfspaces**

We now prove our main structural result, which is that any distribution that approximately matches the low-degree moments of \( D \) fools functions of halfspaces.

**Theorem 5.6.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be of the form in Eq. (5.1). Let \( D_X \) be a distribution that is anticoncentrated and \( \alpha \)-strictly subexponential. For any \( k \in \mathbb{N} \), let \( \Delta_k = \mathbb{E}_{x \sim D^k} \left[ \frac{1}{C_k k^{a/(1+\alpha)}} \right]^{k+1} \)
for some constant \( C_4 \). Then for any distribution \( D' \) whose moments up to order \( k \) match those of \( D_X \) up to \( k \), we have
\[
\left| \mathbb{E}_{D}[f] - \mathbb{E}_{D'}[f] \right| \leq k^{-a/(1+\alpha)} \sqrt{p} \left( C \log \left( \frac{\sqrt{p} \cdot k^{-a/(1+\alpha)}}{1} \right) \right)^{2p}.
\]

**Proof.** Let \( D = D_X \). Assume without loss of generality that \( w^1, \ldots, w^p \) are unit vectors, and let \( W \in \mathbb{R}^{P \times d} \) be the matrix with \( w^i \) as its rows. Let \( P \) be the distribution (on \( \mathbb{R}^d \)) of \( Wx \) for \( x \sim D \), and define \( P' \) similarly. We would like to apply Lemma 5.4 to \( P \) and \( P' \). To do so, we must first verify moment closeness. Let \( u \in \mathbb{R}^P \) be
a unit vector, and let \( u = W^T u \in \mathbb{R}^d \). For any multi-index \( I \in \mathbb{N}^d \), let \( v_I \) denote \( \prod_{j \in [d]} |v_j|^{l_j} \). Then for any \( j \),

\[
\mathbb{E}_{z \sim p'}[(z, u^j)^j] = \mathbb{E}_{x \sim D}[(x, v)^j]
\]

so that in particular for each \( I \) with \( |I| = j \), \( v_I = \prod_{j \in [d]} |v_j|^{l_j} \) \( \leq \|v\|_\infty \leq p^{j/2} \). Thus

\[
\begin{aligned}
\left| \mathbb{E}_{z \sim p'}[(z, u^j)^j] - \mathbb{E}_{x \sim D}[(x, v^j)^j] \right| &= \left| \mathbb{E}_{x \sim D}[(x, v^j)^j] - \mathbb{E}_{x' \sim D}[(x', v^j)^j] \right| \\
&\leq \mathbb{E}_{x \sim D} \left[ \mathbb{E}_{x' \sim D}[|x - x'|^2] \right]^{1/2} \sup_{|I| = j} \Delta_I \\
&\leq d! p^{j/2} \sup_{|I| = j} \Delta_I \\
&\leq \eta_j,
\end{aligned}
\]

where \( \eta_j \) is as defined in Lemma 5.4, and the final inequality follows since we have picked \( \Delta \) in the theorem statement precisely such that \( \sup_{|I| = j} \Delta_I = d^{-1} p^{-j/2} \eta_j \). Also observe that

\[
\mathbb{E}_{x \sim D}[u(x^j)^k] = \mathbb{E}_{x \sim D}[(x, v^j)^k] = \|v\|_\infty^2 \mathbb{E}_D[(x, v^j)^k] \leq p^{k/2} \mathbb{E}_D[(x, v^j)^k].
\]

Now we apply Lemma 5.4 to conclude that \( d_j(P, P') \leq \sqrt{p^{-\alpha/(1+\alpha)}} \).

To finish the proof, we appeal to Lemma 5.5. For this we must first verify anticoncentration of \( P \) and estimate \( N(\delta) \) as defined in that lemma. Observe first that for any \( i \in [d] \), the \( i \)th coordinate of \( x \sim P \) (resp. \( x' \sim P' \)) is precisely \( (w^i, x) \) for \( x \sim D \) (resp. \( (w^i, x') \) for \( x' \sim D' \)). Anticoncentration of each coordinate of \( z \) follows immediately from Definition 5.1(c). To estimate \( N(\delta) \), we will use a simple Chebyshev-style bound. For any coordinate \( i \in [d] \) and any even degree \( j \leq k \), we have

\[
\mathbb{P}_{D'}[|z_i| > t] \leq \frac{\mathbb{E}_D[(w_i^i, x_i)^j]}{t^j} \leq \frac{c_j^j 2^{j/2}}{t^j}.
\]

And since \( D' \) approximately matches moments with \( D \), by a similar calculation as earlier (now with \( \|w^i\|_\infty \leq 1 \) in place of \( \|w\|_\infty \leq \sqrt{p} \)),

\[
\mathbb{E}_D[(w_i^i, x_i)^j] \leq \mathbb{E}_D[(w_i^i, x_i)^j] + \sum_{|l| = j} |w_I| \Delta_I \\
\leq \mathbb{E}_D[(w_i^i, x_i)^j] + \eta_j/p^{j/2} \\
\leq 2 \mathbb{E}_D[(w_i^i, x_i)^j],
\]

and so

\[
\begin{aligned}
\mathbb{P}_{D'}[|z_i| > t] &\leq \frac{\mathbb{E}_D[(w_i^i, x_i)^j]}{t^j} \leq \frac{c_j^j 2^{j/2}}{t^j} = \left( \frac{C_2 j^{j/(1+\alpha)}}{t} \right)^j. \\

\end{aligned}
\]

We need \( t \) such that the RHS is at most \( \delta/p \). For this it suffices to set \( j = 2 \log(p/\delta) \) and \( t = C_2 j^{j/(1+\alpha)} \) for this. By a union bound over the \( p \) coordinates of \( z \sim P \) (similarly \( z' \sim P' \)), we see that we may take \( N(\delta) = t = O(\log(p/\delta)^{1/(1+\alpha)}) \).

We are now ready to apply Lemma 5.5 with this \( N(\delta) \) and \( \delta = \sqrt{p^{-\alpha/(1+\alpha)}} \). Substituting these expressions in, we get that

\[
\begin{aligned}
\mathbb{E}_D[f] - \mathbb{E}_D'[f] &\leq O(\delta \log(N(\delta)) + 2 \log(1/\delta))^p) \\
&\leq p^\delta \left( C \log(p/\delta)^2 \right)^p \leq p^\delta \left( C \log(p/\delta)^2 \right)^p \\
&\leq p^\delta \left( C \log(p/\delta)^2 \right)^p \leq p^\delta \left( C \log(p/\delta)^2 \right)^p.
\end{aligned}
\]

as claimed.

\[
\Box
\]

We pause to note an interesting corollary of this theorem, stated informally earlier as Corollary 1.5, which states that any \( D' \) that approximately matches low-degree moments with \( D_X \) must be anticoncentrated.

**Corollary 5.7.** Let \( \epsilon > 0 \), \( k = O(\epsilon^{-1/(1+\alpha)}) \), and \( \Delta \) be as in Theorem 5.6, with \( p = 2 \). Let \( D' \) be any distribution whose moments up to order \( k \) match those of \( D_X \) up to \( \Delta \). Then for any \( \|u\|_1 = 1 \) and any continuous interval \( T \subset \mathbb{R} \), \( \mathbb{P}_{x \sim D'}[(x, u) \in T] \leq \mathbb{P}_{x \sim D_X}[(x, u) \in T] + \epsilon \).

**Proof.** Write \( T = [\theta, \theta'] \) for some \( \theta < \theta' \in \mathbb{R} \), and consider the function \( f(x) = \text{sign}(x, u - \theta) \wedge \text{sign}(\theta' - x, u) \) (where \( b_1 \wedge b_2 = 1 \) if \( b_1 = b_2 = 1 \)). Clearly \( (x, u) \in T \) iff \( f(x) = 1 \). But \( f \) is an intersection of two halfspaces, and we know by Theorem 5.6 that \( |\mathbb{E}[f] - \mathbb{E}_{D_X}[f]| \leq \epsilon \). Since \( \mathbb{E}_{D_X}[f] \leq C_3 |T| \) by Definition 5.1(c), the statement follows.

\[
\Box
\]

In fact, the same reasoning tells us that for any collection of \( p = O(1) \) intervals \( T \) and any \( \|u\|_1 = 1 \), \( \mathbb{P}_{D'}[(x, u) \in T] = \mathbb{P}_{D_X}[(x, u) \in T] \pm \epsilon \).

### 5.3 Proof of Theorem 5.2

The final ingredient for the proof of Theorem 5.2 is the following lemma, which gives a bound on the sample complexity required for the empirical moments of \( D_X \) to concentrate about their true moments.

**Lemma 5.8.** Let the degree parameter be \( k \), and the slack parameter \( \Delta \in \mathbb{R}^{[f, (k,d)]} \) be as in Theorem 5.6. Assume \( p = O(1) \). Then drawing a sample of size \( m = q^O(k) \) from \( D_X \) is sufficient to ensure that with high probability, the empirical moments of order at most \( k \) match those of \( D_X \) up to slack \( \Delta \).
Proof. Let \(D = D_X\), and let \(\hat{D}_m\) denote the empirical distribution on a sample \(S\) of size \(m\) drawn from \(D\). We would like to ensure that for every \(I \in \mathcal{I}(k, \Delta)\), \(|E_{\hat{D}_m} [x_I] - E_D [x_I]| \leq \Delta I\). It suffices to consider the case when \(x_I\) has the weakest concentration, and this is clearly when \(|I| = k\) and in fact when \(I = (k, 0, \ldots, 0)\) (without loss of generality), so that \(x_I = x_I^k\).

Let \(Z\) denote the random variable \(E_{\hat{D}_m}[x_I^k] - E_D[x_I^k]\). For our purposes it is sufficient to use a crude Chebyshev bound, although higher moment analogs will give a slightly better bound. We have \(\text{Var}[Z] = \frac{1}{m} \text{Var}[x_I^k] \leq \frac{1}{m} 2^k 2^{k(1+\alpha)}\).

\[
\mathbb{P}[Z > \Delta I] \leq \frac{\text{Var}[Z]}{\Delta_I^2} \leq \frac{1}{m} \left( \frac{2^k k^{2k/(1+\alpha)}}{\Delta_I^2} \right) \leq \frac{1}{m} \left( \frac{2^k k^{2k/(1+\alpha)}}{\Delta_I^2} \right),
\]

after plugging in the value of \(\Delta_I\) when \(|I| = k\) from Eq. (3.7) and some manipulation. This is at most \(\delta\) if \(m \geq (d\Delta I/(1+\alpha))\).

For moment closeness to hold simultaneously for all \(I \in \mathcal{I}(k, \Delta)\) with high probability, we set \(\delta = \Theta(1/|\mathcal{I}(k, \Delta)|) = d^{-\Theta(k)}\) and apply a union bound. For this \(\delta, m\) may be simplified to \(d^{O(k)}\), as desired.

We are now ready to prove Theorem 5.2 using our general algorithm, Theorem 4.6.

Proof of Theorem 5.2. We need to pick \(m, k, \Delta\) suitably as functions of \(\epsilon, d\), and verify that the conditions in Theorem 4.6 hold. Let \(\Delta\) be as defined in Theorem 5.6. By Lemma 5.8, it suffices to take \(m = d^{O(k)}\) to ensure that with high probability, the empirical distribution \(\hat{D}_m\) matches moments of order at most \(k\) with \(N(0, L_k)\) up to \(\Delta\). This verifies condition (a) of Theorem 4.6. Now for any \(f \in C\), we can combine Theorem 5.6 with Theorem 3.2 to obtain degree-\(k\) sandwiching polynomials satisfying condition (b) of Theorem 4.6, with \(\epsilon = O(k^{-\alpha/(1+\alpha)})\), or equivalently \(k = O(\epsilon^{-1/(1+\alpha)})\). Applying Theorem 4.6 completes the proof.

6 SAMPLE COMPLEXITY OF TESTABLE LEARNING

In this section we show that the sample complexity of testably learning a class is characterized by its Rademacher complexity. Throughout this section, let \(D_X\) be the target marginal, and let \(C\) be the concept class (mapping \(A\) to \(\{\pm 1\}\)) that we wish to testably learn w.r.t. \(D_X\). We remind the reader that we use the term “with high probability” to mean “with probability at least 0.99” for simplicity.

6.1 Upper Bound

We begin with the upper bound, which is essentially just the observation that the empirical Rademacher complexity provides a generalization guarantee that can be estimated to high accuracy from the sample itself. Note that this is an information-theoretic upper bound.

**Theorem 6.1.** Let \(\epsilon > 0\), and let \(m'\) be such that \(\mathcal{R}_m(C, D_X) \leq \frac{\epsilon}{2}\). Then \(C\) can be testably learned w.r.t. \(D_X\) up to excess error \(\epsilon\) with sample complexity \(m' + O(1/\epsilon^2)\).

Proof. Let \(m = m' + O(1/\epsilon^2)\). Let \(S = \{(x_i, y_i)\}_{i \in [m]} \sim D^\otimes m\) be a sample of \(m\) labeled points drawn from \(D\), and let \(S_X\) denote \(\{x_i\}_{i \in [m]}\). Our tester \(T\) accepts iff \(\tilde{R}_m(C, S_X) \leq \frac{\epsilon}{2}\). Whenever the tester accepts, the learner \(A\) simply performs ERM over \(S\) w.r.t. \(C\).

To see why completeness is satisfied, suppose that the true marginal is in fact \(D_X\). Since \(\mathcal{R}_m(C, D_X) \leq \mathcal{R}_m(C, D_X) \leq \frac{\epsilon}{2}\), we can ensure that with high probability over \(S_X, \tilde{R}_m(C, S_X) \leq \frac{\epsilon}{2}\), and so \(T\) will accept.

Soundness holds by a standard argument showing generalization using uniform convergence. Formally, suppose that the tester accepts \(S_X\), i.e., \(\tilde{R}_m(C, S_X) \leq \frac{\epsilon}{2}\). Consider an ERM hypothesis

\[
\hat{f}_m = \arg\min_{f \in C} \frac{1}{m} \sum_{i \in [m]} \ell(f(x_i), y_i)
\]

as well as an optimal hypothesis

\[
f^* = \arg\min_{f \in C} \mathbb{E}_{(x,y) \sim D} \ell(f(x), y).
\]

Then with high probability over \(S\) we have

\[
L(\hat{f}_m) \leq \tilde{L}_m(\hat{f}_m) + \tilde{R}_m(C) + O\left(\sqrt{\frac{1}{m}}\right)
\]

(6.1)\]

\[
\leq \tilde{L}_m(f^*) + \tilde{R}_m(C) + O\left(\sqrt{\frac{1}{m}}\right)
\]

(6.2)\]

\[
\leq L(f^*) + 2\tilde{R}_m(C) + O\left(\sqrt{\frac{1}{m}}\right)
\]

(6.3)\]

\[
\leq L(f^*) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = L(f^*) + \epsilon,
\]

(6.4)\]

by our choice of \(m\). This proves the theorem.

\[
6.2 \text{ Lower Bound}\n\]

Now we state the lower bound, which matches the upper bound up to a quadratic factor. Our lower bound can be viewed as a generalization of the argument of [55], who proved lower bounds for testable learning for the specific cases of convex sets and monotone functions. We obtain the full range of lower-bounds for any value of \(\epsilon\) purely in terms of Rademacher complexity. The idea here is that no tester with bounded sample complexity \(m\) can distinguish between a distribution \(D_X\) and the uniform distribution on a sufficiently large sample of size \(M \gg m\) drawn from \(D_X\). If the Rademacher complexity w.r.t. \(D_X\) at sample size \(M\) is somewhat large, then the large sample can be labeled randomly and still admit nontrivial optimal error, but of course the learner cannot do well on unseen data.

**Theorem 6.2.** Let \(\epsilon > 0\), and let \(M\) be such that \(\mathcal{R}_m(C, D_X) \geq 5\epsilon\). Assume that \(M \geq \Theta(1/\epsilon^2)\) is sufficiently large. Then testably learning \(C\) up to excess error \(\epsilon\) requires sample complexity at least \(\Omega(\sqrt{M})\).
Proof. Suppose we had a tester-learner \((T, A)\) requiring sample complexity only \(m\) where \(m \leq \frac{\sqrt{M}}{100}\). We will show how to "foil" \((T, A)\) into failing its guarantee by constructing a (random) labeled distribution \(\mathcal{D}\) such that (with high probability over the randomness of our construction),

(a) \(\text{opt}(\mathcal{D}, C) \leq \frac{1}{2} - 2\epsilon\);
(b) with high probability, \(T\) will accept a sample of size \(m\) drawn from \(\mathcal{D}\); and yet
(c) with high probability, \(A\)'s output will have error greater than \(\frac{1}{2} - \epsilon\) on \(\mathcal{D}\).

For such a \(\mathcal{D}\), it is clear that the tester-learner pair \((T, A)\) fails its guarantee in that with high probability, despite \(T\) accepting, \(A\) cannot produce a hypothesis with error at most \(\text{opt}(\mathcal{D}, C) + \epsilon\).

We construct \(\mathcal{D}\) as follows. Draw a sample of \(M\) randomly labeled points \(S = \{(x_i, y_i)\}_{i \in [M]} \sim (D_X \times \text{Unif}(\pm 1))^\otimes M\), and let \(S_X\) denote \(\{x_i\}_{i \in [M]}\). Define \(\mathcal{D}\) to be the uniform distribution over \(S\). We now show that with high probability over the draw of \(S\) (including its random labeling), the distribution \(\mathcal{D}\) satisfies the required properties.

Denote the size-\(m\) sample given to \((T, A)\) by \(S' \sim \mathcal{D}^\otimes m\), and let \(S'_X\) denote its marginal. That is, \(S'\) is an iid subsample of \(S\). Notice that unless we sample the same element of \(S\) twice, \(S'_X\) is distributed exactly as a sample of \(m\) points drawn directly from \(D_X\). In other words, the statistical distance between \(S'_X\) formed in this way vs by drawing \(m\) points directly from \(D_X\) is at most the collision probability, which is \(m^2/M \leq 10^{-4}\). Thus any tester satisfying completeness must accept \(S'\) with high probability. This gives us property (b).

Let us see why property (a) holds with high probability over \(S\). The idea is that because \(R_M(C) \geq \Omega(\epsilon)\), we expect that there exists a classifier in \(C\) that achieves error at most \(\frac{1}{2} - \Omega(\epsilon)\) on the randomly labeled sample \(S\). Indeed, observe that the random labels are exactly equivalent to Rademacher random variables. Assuming that \(M\) is sufficiently large, one can show that with high probability over the sample \(S\) (together with the realization of the random labels),

\[
\left| R_M(C) - \sup_{f \in C} \frac{1}{M} \sum_{i=1}^{M} y_i f(x_i) \right| \leq \epsilon.
\]

In particular, since \(R_M(C) \geq 5\epsilon\), there exists \(f^* \in C\) such that

\[
\frac{1}{M} \sum_{i=1}^{M} y_i f^*(x_i) \geq 4\epsilon,
\]

or equivalently

\[
\frac{1}{M} \sum_{i=1}^{M} \mathbb{1}[f^*(x_i) \neq y_i] = 1 - \frac{1}{2M} \sum_{i=1}^{M} y_i f^*(x_i) \leq \frac{1}{2} - 2\epsilon.
\]

In other words, \(\text{opt}(\mathcal{D}, C) \leq \frac{1}{2} - 2\epsilon\).

For property (c), the idea is that the learner, having only seen a minuscule fraction of the randomly labeled \(D\), cannot possibly output a hypothesis with error substantially better than \(\frac{1}{2}\) on all of \(D\). Formally, observe that any classifier \(h\) that \(A\) outputs is stochastically independent of \(S', S'\). This means that in expectation over \(S\) and the randomness of \(A\),

\[
\mathbb{P}_{(x,y) \sim \mathcal{D}}[h(x) \neq y] \geq 0.5 - \frac{m}{M} + \frac{1}{2} - \frac{M-m}{M} = \frac{1}{2} - \frac{m}{2M} \geq \frac{1}{2} - \epsilon/2,
\]

since the fact that \(M \geq \Theta(1/\epsilon^2)\) and \(m \leq \sqrt{M}/100\) mean that \(\frac{m}{M} \leq \frac{1}{100\sqrt{M}} < \epsilon\). It is clear that for sufficiently large \(M \geq \Theta(1/\epsilon^2)\), we will actually have \(\mathbb{P}_{(x,y) \sim \mathcal{D}}[h(x) \neq y] > \frac{1}{2} - \epsilon\) with high probability over \(S\) and \(A\). (Such an argument is also formalized as [55, Lemma 25].)

6.2.1 Applications. The lower bounds of [55] may be viewed as applications of Theorem 6.2. The first application is the class of convex sets w.r.t. \(N(0, I_2)\), and the second is the class of monotone Boolean functions w.r.t. Unif\((\pm 1)^d\). Recall that \(C\) is said to shatter an unlabeled set \(S_X\) if every possible labeling of \(S_X\) can be achieved by some \(f \in C\), or equivalently \(R_M(C, S_X) = 1\).

Theorem 6.3 (Implicit in [55, Theorem 22]). Let \(X = \mathbb{R}^d, D_X = N(0, I_d)\), and \(C\) be the class of \((\pm 1)\)-valued indicator functions of convex sets in \(\mathbb{R}^d\). Let \(M = 2^{Cd}\) for some small constant \(C > 0\). Then with probability \(1 - \exp(-\Omega(d))\) over the draw of a size-\(M\) sample \(S \sim D_X^\otimes M\), \(C\) can shatter \(S\).

Theorem 6.4 (Implicit in [55, Theorem 23]). Let \(X = \{\pm 1\}^d, D_X = \text{Unif}\((\pm 1)^d\), and \(C\) be the class of monotone Boolean functions. Let \(M = 2^{Cd}\) for some small constant \(C > 0\). Then with probability \(1 - \exp(-\Omega(d))\) over the draw of a size-\(M\) sample \(S \sim D_X^\otimes M\), \(C\) can shatter \(S\).

Interestingly, these examples add to what has been called "the emerging analogy between symmetric convex sets in Gaussian space and monotone Boolean functions"; see [21] and references therein.

7 DISCUSSION

As observed in [55], an interesting consequence of the lower bounds in Section 6.2.1 is that they demonstrate a strict separation between distribution-specific agnostic learning and testable learning. In the case of both convex sets over \(N(0, I_d)\) and monotone functions over \text{Unif}(\{\pm 1\}^d), Fourier-theoretic arguments are known to require agnostic learners requiring sample complexity only \(2^{O(\sqrt{d}/\text{poly}(\epsilon))}\) to learn up to excess error \(\epsilon\) [18, 41]. In particular, they require only sample complexity \(2^{O(\sqrt{d})}\) to learn up to excess error \(\epsilon = 0.1\) (say), which is much smaller than the lower bounds of \(2^{\Omega(d)}\) for testably learning these classes up to \(\epsilon = 0.1\).

But we have just characterized testable learning in terms of Rademacher complexity, which we know in turn tightly characterizes uniform convergence (Theorem 2.3). We draw the following implications from this:

- Uniform convergence is always sufficient for distribution-specific agnostic learning but it is not necessary, as witnessed by the examples of convex sets and monotone functions.
- Uniform convergence is both necessary and sufficient for testable learning.

That is, not only is there a strict separation between distribution-specific agnostic learning and testable learning, it is the latter that is in fact characterized by uniform convergence.

Uniform Convergence in Distribution-Free vs Distribution-Specific Learning. In the distribution-free setting, uniform convergence is well-known to be necessary and sufficient for agnostic (as well as realizable) learning, by classic VC theory (see e.g. [58, Chapter 6]). Let us clarify that in the distribution-free setting the term "uniform convergence" now means a uniform bound on the generalization
functions over the Boolean hypercube. These classes $C$ are also capable of interpolating a random sample of size $m = 2^\Theta(d\varepsilon)$, and yet there exist estimators $\hat{f}$ that achieve error opt$(C, D) + \varepsilon$ using sample complexity only $2^\Theta(\sqrt{d\varepsilon}/\poly(e))$ [18, 41]. These estimators are not based on ERM and do not lie strictly in $C$; instead, they are low-degree (specifically, degree-$O(\sqrt{d\varepsilon}/\poly(e))$) polynomial approximators of functions in $C$ (as in [33]). Such polynomial approximators essentially constitute a small (improper) cover of the class $C$ (w.r.t. the $L_1(D\varepsilon)$ metric $\rho(f, p) = \mathbb{E}_{x \sim D}(|f(x) - p(x)|)$). The improved sample complexity we obtain by such methods may be explained by the fact that to obtain generalization, it suffices to have uniform convergence over this cover as opposed to all of $C$ (as in the metric entropy characterization of [14]).

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