Scattering of a Klein-Gordon particle by a smooth barrier

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Abstract
We present the study of the one-dimensional Klein-Gordon equation by a smooth barrier. The scattering solutions are given in terms of the Whittaker $M_{\kappa,\mu}(x)$ function. The reflection and transmission coefficients are calculated in terms of the energy, the height and the smoothness of the potential barrier. For any value of the smoothness parameter we observed transmission resonances.

Keywords: Hypergeometric functions, Klein-Gordon equation, Scattering theory.

1 Introduction

In this article we computed the scattering solutions of the one-dimensional Klein-Gordon equation in presence of a smooth barrier. This is a mathematical interesting problem because the solutions of the Klein-Gordon equation
are given in terms of the Whittaker $M_{\kappa,\mu}(x)$ function, whose asymptotic behavior is well-known.

This smooth barrier is a short-range potential which presents scattering states. This potential is interesting because varying the smoothness of the curve can be represented from the potential barrier to the cusp potential barrier, in all cases we observed transmission resonances. These potential barriers have applications in several topics of the solid state physics.

The Klein-Gordon equation is used to describe spin-0 particles. This equation have been widely study in the literature for different physical systems both time-independent [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and time-dependent [14, 15, 16, 17] Klein-Gordon equation. The analytical solution of the time-independent Klein-Gordon equation for different potentials has been caused of a lot of interest in recent years, for both bound states [2, 3, 6, 9, 12] and scattering solutions [1, 4, 5, 7, 8, 9, 10, 11, 13]. It has allowed the understanding of several physical phenomena of Relativistic Quantum Mechanics such as the antiparticle bound state [18, 19], transmission resonances [1, 4, 5], and superradiance [20, 7, 21, 22].

This article is organized as follow. In section 2, we present the one-dimensional Klein-Gordon equation. In section 3, we present the smooth barrier. In section 4 we study the solutions for scattering states, and calculate the transmission and reflection coefficients. The discussion of our results are given in section 5. Finally, in section 6 we give the concluding remarks.

## 2 The Klein-Gordon equation

The Klein-Gordon equation for free particles, in natural units $\hbar = c = m = 1$, is given by [23],

$$\hat{p}^\mu \hat{p}_\mu \varphi = \varphi,$$  \hspace{1cm} (1)

being $\hat{p}^\mu = i \left( \frac{\partial}{\partial t}, -\vec{\nabla} \right)$, Eq. (1) becomes:

$$(\Box + 1) \varphi = 0.$$  \hspace{1cm} (2)

We need to solve the Klein-Gordon equation interacting with a spatially one-dimensional potential, then we start finding the form of the Klein-Gordon equation with the interaction of an electromagnetic field.
The electromagnetic field is described by the four-vector [23]:

\[ A^\mu = (A_0, \vec{A}), \quad (3) \]
\[ A_\mu = \eta_{\mu\nu} A^\nu = (A_0, -\vec{A}), \quad (4) \]

where \( \eta_{\mu\nu} = \text{diag}(1, -1) \).

The minimal coupling of the electromagnetic field is expressed in the form,

\[ \hat{A}^\mu \to \hat{p}^\mu - eA^\mu, \quad (5) \]
\[ \hat{A}_\mu \to \hat{p}_\mu - eA_\mu. \quad (6) \]

The one-dimensional Klein-Gordon equation minimally coupled to a vector potential \( A^\mu \) can be written as [23]:

\[ (\hat{p}^\mu - eA^\mu)(\hat{p}_\mu - eA_\mu)\varphi = \varphi. \quad (7) \]

Consider a spatially one-dimensional potential \( eA_0 = V(x) \), \( \vec{A} = 0 \), and a stationary solution of the Klein-Gordon equation \( \varphi(x, t) = \phi(x)e^{-iEt} \), Eq. (7) can be written as:

\[ \frac{d^2\phi(x)}{dx^2} + \left\{ [E - V(x)]^2 - 1 \right\} \phi(x) = 0, \quad (8) \]

where \( E \) is the energy of the particle.

3 The smooth barrier

The smooth barrier is given by:

\[ V(x) = \begin{cases} 
V_0e^{(x-x_0)/a}, & \text{for } x < x_0, \\
V_0, & \text{for } x_0 \leq x \leq 0, \\
V_0e^{-x/a}, & \text{for } x > 0, 
\end{cases} \quad (9) \]

where \( V_0 \) represents the height of the barrier and \( a \) gives the smoothness of the curve. The form of the potential (9) is showed in the Fig. (1). From
Fig. 1 we can note that the smooth barrier reduces to the square potential barrier for $a \to 0$. Also this potential reduces to the cusp potential barrier for $a \gg 0$ and $x_0 = 0$.

Figure 1: Smooth barrier for $V_0 = 2$ with $a = 0.5$ (dotted line) and square potential barrier for $a = 0.001$ (solid line).

4 Scattering States

4.1 Scattering solutions for $x < x_0$

The scattering solutions for $x < x_0$ are obtained by solving the differential equation

$$\frac{d^2\phi_I(x)}{dx^2} + \left\{ \left[ E - V_0e^{(x-x_0)/a} \right]^2 - 1 \right\} \phi_I(x) = 0. \quad (10)$$

On making the change of variable $y = 2iaV_0e^{(x-x_0)/a}$, Eq. (10) becomes

$$y \frac{d}{dy} \left( y \frac{\phi_I}{dy} \right) - \left[ (iaE - y/2)^2 + a^2 \right] \phi_I = 0. \quad (11)$$

Putting $\phi_I = y^{-1/2}f(y)$ we obtain the Whittaker differential equation
\[ f(y)'' + \left[ -\frac{1}{4} + \frac{\text{i}aE}{y} + \frac{1/4 - a^2(1 - E^2)}{y^2} \right] f(y) = 0, \quad (12) \]

which general solution is given by

\[ f(y) = c_1 M_{\kappa,\mu}(y) + d_1 W_{\kappa,\mu}(y), \quad (13) \]

where \( M_{\kappa,\mu}(y) \), \( W_{\kappa,\mu}(y) \) are the Whittaker functions, \( \kappa = \text{i}aE \) and \( \mu = \sqrt{E^2 - 1}a \). Then the solution to the Eq. (11) is given by,

\[ \phi_I(y) = c_1 y^{-1/2} M_{\kappa,\mu}(y) + d_1 y^{-1/2} W_{\kappa,\mu}(y). \quad (14) \]

In terms of the variable \( x \), Eq. (14) becomes

\[ \phi_I(x) = c_1 (2\text{i}aV_0)^{(-1/2)} e^{-(x-x_0)/2a} M_{\kappa,\mu} \left[ 2\text{i}aV_0 e^{(x-x_0)/a} \right] + d_1 (2\text{i}aV_0)^{(-1/2)} e^{-(x-x_0)/2a} W_{\kappa,\mu} \left[ 2\text{i}aV_0 e^{(x-x_0)/a} \right]. \quad (15) \]

Because the asymptotic behavior of the Whittaker functions we only keep the solutions with the \( M_{\kappa,\mu}(x) \) function. Then, from Eq. (15), the incident and reflected waves are,

\[ \phi_{\text{inc}}(x) = c_1 (2\text{i}aV_0)^{(-1/2)} e^{-(x-x_0)/2a} M_{\kappa,\mu} \left[ 2\text{i}aV_0 e^{(x-x_0)/a} \right], \]
\[ \phi_{\text{ref}}(x) = b_1 (2\text{i}aV_0)^{(-1/2)} e^{-(x-x_0)/2a} M_{\kappa,-\mu} \left[ 2\text{i}aV_0 e^{(x-x_0)/a} \right]. \quad (16) \]

which are solutions of the differential equation (10).

### 4.2 Scattering solutions for \( x_0 \leq x \leq 0 \)

The scattering solutions for \( x_0 < x < 0 \) are obtained by solving the differential equation

\[ \frac{d^2\phi_{\text{II}}(x)}{dx^2} + \left[ (E - V_0)^2 - 1 \right] \phi_{\text{II}}(x) = 0, \quad (17) \]

Eq. (17) has the general solution

\[ \phi_{\text{II}}(x) = b_2 e^{-iqx} + c_2 e^{iqx}, \quad (18) \]

where \( q = \sqrt{(E - V_0)^2 - 1}. \)
4.3 Scattering solutions for $x > 0$

The scattering solutions for $x > 0$ are obtained by solving the differential equation

$$\frac{d^2 \phi_{\text{III}}(x)}{dx^2} + \left[ (E - V_0 e^{-x/a})^2 - 1 \right] \phi_{\text{III}}(x) = 0. \tag{19}$$

On making the change of variable $z = 2iaV_0 e^{-x/a}$, Eq. (19) becomes

$$z \frac{d}{dz} \left( z \frac{\phi_{\text{III}}}{dz} \right) - \left[ (iaE - z/2)^2 + a^2 \right] \phi_{\text{III}} = 0. \tag{20}$$

Putting $\phi_{\text{III}} = z^{-1/2} g(z)$ we obtain the Whittaker differential equation

$$g(z)^{\prime\prime} + \left[ -\frac{1}{4} + \frac{iaE}{z} + \frac{1/4 - a^2(1 - E^2)}{z^2} \right] g(z) = 0, \tag{21}$$

which solution is given by

$$g(z) = c_3 M_{\kappa,-\mu}(z) + d_3 W_{\kappa,-\mu}(z), \tag{22}$$

Finally, the solution of Eq. (20) becomes

$$\phi_{\text{III}}(z) = c_3 z^{-1/2} M_{\kappa,-\mu}(z) + d_3 z^{-1/2} W_{\kappa,-\mu}(z), \tag{23}$$

In terms of the variable $x$, (23) becomes

$$\phi_{\text{III}}(x) = b_3 (2iaV_0)^{(-1/2)} e^{x/2a} M_{\kappa,-\mu} \left( 2iaV_0 e^{-x/a} \right) + d_3 (2iaV_0)^{(-1/2)} e^{x/2a} W_{\kappa,-\mu} \left( 2iaV_0 e^{-x/a} \right). \tag{24}$$

From Eq. (25) the transmitted wave is:

$$\phi_{\text{trans}}(x) = b_3 (2iaV_0)^{(-1/2)} e^{x/2a} M_{\kappa,-\mu} \left( 2iaV_0 e^{-x/a} \right).$$
4.4 Transmission and reflection coefficients

For compute the transmission and reflection coefficient we need to use the definition of the electrical current. The electrical current density for the one-dimensional Klein-Gordon equation (8) is defined as

\[ j^\mu = \frac{i}{2} (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \]  
(25)

The current as \( x \to -\infty \) can be decomposed as \( j_L = j_{\text{in}} - j_{\text{refl}} \) where \( j_{\text{in}} \) is the incident current and \( j_{\text{refl}} \) is the reflected one. Analogously we have that, on the right side, as \( x \to \infty \) the current is \( j_R = j_{\text{trans}} \), where \( j_{\text{trans}} \) is the transmitted current.

Using the reflection coefficient \( R \), and the transmission coefficient \( T \), are calculated by

\[ R = \frac{j_{\text{refl}}}{j_{\text{inc}}} \]  
(26)

\[ T = \frac{j_{\text{trans}}}{j_{\text{inc}}} \]  
(27)

for which we need to identify the incident, reflected and transmitted wave. The quantities \( R \) and \( T \) are not independent, they are related via the unitary condition \( R + T = 1 \).

Using the asymptotic behavior of the Whittaker function \( M_{\kappa,\mu} \to e^{-y/2} y^{1/2+\mu} \), as \( y \to 0 \) \[24\], we can write the incoming solution, the reflected solution and the transmitted solutions like a plane wave.

From Eq. (15), as \( x \to -\infty \), the incident and reflected waves are given by

\[ \phi_{\text{inc}}(x) = c_1 (2iaV_0)^\mu e^{i\sqrt{E^2-1}x} \]  
(28)

\[ \phi_{\text{refl}}(x) = b_1 (2iaV_0)^{-\mu} e^{-i\sqrt{E^2-1}x} \]  
(29)

and from Eq. (25) as \( x \to \infty \) the transmitted wave has the form,

\[ \phi_{\text{trans}}(x) \to b_3 (2iaV_0)^{-\mu} e^{i\sqrt{E^2-1}x} \]  
(30)

Then

\[ R = \left| \frac{(2iaV_0)^{-\mu}}{(2iaV_0)^{\mu}} \right|^2 \left| \frac{b_1}{c_1} \right|^2 \]  
(31)
\[ T = \left| \frac{(2iaV_0)^{-\mu}}{(2iaV_0)^{\mu}} \right| \left| \frac{b_3}{c_1} \right|^2. \]  

(32)

In order to find \( R \) and \( T \), the wave functions and their first derivatives must be matched at \( x = x_0 \) and \( x = 0 \). The coefficients \( b_1 \) and \( b_3 \) are calculated numerically in terms of \( c_1 \) from the following system of equations,

\[
\begin{align*}
 b_1 M_{\kappa,\mu}(2iaV_0) + c_1 M_{\kappa,-\mu}(2iaV_0) &= b_2 e^{-iqx_0} + c_2 e^{iqx_0}, \\
 b_1 \left[ \left( -\frac{1}{2a} + iV_0 + \frac{\kappa}{a} \right) M_{\kappa,\mu}(2iaV_0) - \frac{1}{a} \left( \frac{1}{2} + \mu + \kappa \right) M_{\kappa+1,-\mu}(2iaV_0) \right] + \\
 c_1 \left[ \left( -\frac{1}{2a} + iV_0 - \frac{\kappa}{a} \right) M_{\kappa,-\mu}(2iaV_0) + \frac{1}{a} \left( \frac{1}{2} - \mu + \kappa \right) M_{\kappa+1,-\mu}(2iaV_0) \right] &= -b_2 q e^{-iqx_0} + c_2 q e^{iqx_0}, \\
 b_2 + c_2 &= b_3 M_{\kappa,-\mu}(2iaV_0), \\
 ib_2 q + ic_2 q &= b_3 \left[ \left( \frac{1}{2a} - iV_0 + \frac{\kappa}{a} \right) M_{\kappa,-\mu}(2iaV_0) - \frac{1}{a} \left( \frac{1}{2} - \mu + \kappa \right) M_{\kappa+1,-\mu}(2iaV_0) \right].
\end{align*}
\]

(33) 

(34) 

(35) 

(36)

5 Results and discussion

Fig. (2) shows the transmission and reflection coefficients for \( x_0 = -1 \), \( a = 0.5 \) and \( E = 2 \) respect to the height of the barrier \( V_0 \). Fig. (3) shows the transmission and reflection coefficients for \( x_0 = -2 \), \( a = 0.5 \) and \( E = 2 \) respect to the height of the barrier \( V_0 \). When the value of \( x_0 \) increases, more peaks appear in the same range of \( V_0 \). Using the values \( a = 0.5 \) with \( x_0 = 0 \) in Fig. (4) and \( a = 0.001 \) in Fig. (5) we recover the results obtained by the cusp potential barrier and the square potential barrier, respectively. In all cases we observed transmission resonances. The relation \( T + R = 1 \) also is satisfied.

In the non-relativistic limit the Klein-Gordon equation reduces to the Schrödinger equation \([23]\). We wish to compare the scattering solutions of the Schrodinger equation with those of the Klein Gordon equation for the square potential barrier.
For the square potential barrier showed in Fig. 1 (solid line), the reflection $R$ and transmission $T$ coefficients are obtained by:
Figure 4: Transmission and reflection coefficients in function of the height of the barrier $V_0$ for $x_0 = 0$, $a = 0.5$ and $E = 2$. The dashed line represents the transmission coefficient, and the solid line represents the reflection coefficient.

Figure 5: Transmission and reflection coefficients in function of the height of the barrier $V_0$ for $x_0 = -2$, $a = 0.001$ and $E = 2$. The dashed line represents the transmission coefficient, and the solid line represents the reflection coefficient.

$$R_{\text{barrier}} = \frac{(k_1^2 - k_2^2)^2 \sin^2 (k_2 x_0)}{4k_1 k_2 + (k_1^2 - k_2^2)^2 \sin^2 (k_2 x_0)}.$$  \hspace{1cm} (37)
\[ T_{\text{barrier}} = \frac{4k_1^2k_2^2}{4k_1^2k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(k_2x_0)}, \]  

where \( k_1 = \sqrt{2E} \), \( k_2 = \sqrt{2(E - V_0)} \) for the Schrödinger equation and \( k_1 = \sqrt{E^2 - 1}, \) \( k_2 = \sqrt{(E - V_0)^2 - 1} \) for the Klein-Gordon equation.

In Figs. 6 and 7 we illustrated the behaviour of the reflection and transmission coefficients in both cases. We can observed that for the Klein-Gordon equation more peaks appears in the transmission coefficient that for the Schrödinger equation. In both cases the relation \( T + R = 1 \) is accomplished.

![Figure 6: The reflection R and transmission T coefficients varying energy V0 of the Schrödinger equation with the square potential barrier for E = 3 and x0 = -3. The dashed line represents the transmission coefficient, and the solid line represents the reflection coefficient.](image)

6 Conclusions

In this paper we have studied the scattering solutions of the Klein-Gordon equation by a smooth barrier. We have calculated the transmission \( T \) and reflection \( R \) coefficients in function of the height of the potential \( V_0 \) for three different widths of the barrier. For certain values of the smoothness and weight of the potential barrier we recover the results for the cusp potential...
Figure 7: The reflection $R$ and transmission $T$ coefficients varying energy $V_0$ of the Klein-Gordon equation with the square potential barrier for $E = 3$ and $x_0 = -3$. The dashed line represents the transmission coefficient, and the solid line represents the reflection coefficient.

4 and for the square potential barrier 23, 25. In all cases we observe transmission resonances and the relationship $T + R = 1$ is accomplished. For future research we are going to consider the bound-states solutions of this smooth potential barrier.

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