SPANIER-WHITEHEAD $K$-DUALITY FOR $C^*$-ALGEBRAS

JEROME KAMINKER AND CLAUDE L. SCHOCHET

ABSTRACT. Classical Spanier-Whitehead duality was introduced for the stable homotopy category of finite CW complexes. Here we provide a comprehensive treatment of a noncommutative version, termed Spanier-Whitehead $K$-duality, which is defined on the category of $C^*$-algebras whose $K$-theory is finitely generated and that satisfy the UCT, with morphisms the $KK$-groups. We explore what happens when these assumptions are relaxed in various ways. In particular, we consider the relationship between Paschke duality and Spanier-Whitehead $K$-duality.

1. Introduction

Classical Spanier-Whitehead duality is a generalization of Alexander duality, which relates the homology of a space to the cohomology of its complement in a sphere. Ed Spanier and J.H.C. Whitehead [42], [43], noting that the dimension of the sphere did not play an essential role, adapted it to the context of stable homotopy theory. Its history and its relation to other classical duality ideas are described in depth by Becker and Gottlieb [4]. To be more precise, given a finite complex $X$ there is another finite complex, the Spanier-Whitehead dual of $X$, denoted $DX$. 

2010 Mathematics Subject Classification. 46L80, 46L87, 46M20, 55P25.

Key words and phrases. Spanier-Whitehead duality, K-theory, Operator algebras.
and a duality map $\mu : X \wedge DX \to S^n$ such that slant product with the pull-back of the generator, $\mu^*([S^n]) \in H^*(X \wedge DX)$ induces isomorphisms

$$\mu^*([S^n]) : H_*(X) \to H_*(DX).$$

Moreover, $DX$ is stably homotopy equivalent to $X$. Note that there is no need for any sort of orientability requirement, in contrast to Poincaré duality. Spanier-Whitehead duality turns out to be an interesting and fairly universal notion which generalizes to many contexts.

Spanier-Whitehead duality extends in a natural way to generalized cohomology theories such as K-theory. For a finite complex $X$, a dual finite complex $DX$ turns out to be a K-theoretic dual as well [20]. Since K-theory and ordinary cohomology detect torsion differently this result requires proof. The essential fact is that $X \to K^*(DX)$ defines a homology theory naturally equivalent on finite complexes to $K_*(X)$. We shall refer to such a dual as a Spanier-Whitehead K-dual.

The bivariant version of K-theory introduced by Kasparov [23] is closely related to duality. One has, for $X$ and $Y$ finite complexes,

$$KK^*(C(X), C(Y)) \cong KK^*(C, C(DX \wedge Y)) \cong K_*(C(DX \wedge Y)) \cong K^*(DX \wedge Y)$$

and, in fact, this can be taken as a definition of KK-theory for finite complexes, [35].

Turning to duality for $C^*$-algebras, the subject of this paper, we see that there are several points which must be considered:

1. The $C^*$-algebras which arise naturally in applications to topology, dynamics, and index theory are not simply $C(X)$ for $X$ a finite complex. They are generally noncommutative, and the topological spaces commonly associated with them may be completely uninteresting or intractable.

2. The cohomology theories that have been used successfully on $C^*$-algebras are K-theory and its various relatives. These do generalize topological K-theory but have less structure. There is no natural product structure when the algebras are noncommutative, and the Adams operations do not extend to the noncommutative case.

3. For a separable, nuclear $C^*$-algebra $A$ represented on a Hilbert space, the commutant of its projection into the Calkin algebra has some of the properties of a Spanier-Whitehead K-dual. This is the Paschke dual of $A$, which we denote $P(A)$. It satisfies

$$K_*(P(A)) \cong K^*(A).$$

However, in general $P(A)$ is not separable or nuclear, the Kasparov product is not defined, there is no analogous description for $K_*(A)$ and one cannot simply take the Paschke dual of the Paschke dual.

Keeping this in mind, we shall see what can be done. There are several different arenas to investigate:

1. If we stay within the bootstrap category [36] and restrict to $C^*$-algebras whose K-theory groups are finitely generated, then there is a very satisfactory duality situation. Spanier-Whitehead K-duals exist, they are suitably unique, and “everything” works out as one would expect from considering the category of finite cell complexes.
(2) If we stay within the bootstrap category and allow $K_*(A)$ to be countable but not necessarily finitely generated, then there exist $C^*$-algebras which cannot have Spanier-Whitehead $K$-duals for algebraic reasons.

(3) If we keep the finite generation hypothesis but no longer require the bootstrap hypothesis, then various things can happen, most of which are bad.

(4) We will find separable and nuclear substitutes for the Paschke dual which may be useful in various analytic contexts. cf. \[19\]

We now give a more formal summary of our results.

**Section 2** provides the basic definitions and basic properties of Spanier-Whitehead $K$-duality. Our purpose here is to clarify the various and sometimes contradictory definitions that appear in the literature. Our definitions require separability because we want the full power of the Kasparov pairing.

In **Section 3** we explain the relationship between classical Spanier-Whitehead duality and Spanier-Whitehead $K$-duality. In a word, the first implies the second for finite complexes, but this is not automatic from the axioms; it requires a spectral sequence comparison theorem that we established many years ago.

Spanier-Whitehead $K$-duality arises in several different areas of mathematics. **Section 4** discusses how this type of duality arises naturally even when the algebras are simple, hence are very far from commutative ones. We discuss examples drawn from hyperbolic dynamics, the Baum-Connes conjecture, and others.

In **Section 5** we start a discussion of the relationship of Poincaré duality as used in noncommutative geometry and the traditional notion from topology.

**Section 6** is devoted to establishing a very important and basic result. Every separable nuclear $C^*$-algebra in the bootstrap category with finitely generated $K$-theory groups has a Spanier-Whitehead $K$-dual that is suitably unique. We show further that this is the largest category of $C^*$-algebras with this property.

Then comes the bad news. In **Section 7** we give a concrete example of a $C^*$-algebra that is separable, nuclear, bootstrap and yet has no Spanier-Whitehead $K$-dual. The example we provide is pretty basic: it is an AF-algebra with $K_0 = \mathbb{Q}$.

**Section 8** provides an interesting application of the theory to mod-p $K$-theory. Indeed, these issues led the second author to initiate the current study.

**Section 9** is devoted to Paschke duality. We show how this differs in basic ways from Spanier-Whitehead $K$-duality but resembles it in other ways. The main problem is that the Paschke dual is typically not separable or nuclear. **Sections 10 and 11** develop some tools to help us replace non-separable, non-nuclear $C^*$-algebras with smaller versions of themselves. We are motivated morally (though not at all in a technical sense) by the fact that any topological space is weakly equivalent to a CW-complex.

In a future paper we will see what can be done when separability and nuclearity are not assumed. We have in mind the possibility of replacing $KK$ by the Brown-Douglas-Fillmore $Ext$ groups, which agree with the $KK$ groups when $A$ is separable nuclear. A generalization of Spanier-Whitehead $K$-duality would be very useful here and could yield insight on the following conjecture.

Suppose that $A$ is a separable $C^*$-algebra in the bootstrap category with $K_*(A)$ finitely generated. Then it has a Spanier-Whitehead $K$-dual $DA$ and it also has a Paschke dual $P(A)$. The UCT implies that there is an element $u \in KK_0(DA, P(A))$ inducing an isomorphism

$$u_* : K_*(DA) \xrightarrow{\cong} K_*(P(A)).$$
We may regard \( u \in \text{Ext}(DA, S\mathbb{F}(A)) \) since \( DA \) is separable nuclear.

**Conjecture 1.1.** There exists an element \( v \in \text{Ext}(S\mathbb{F}(A), DA) \) and enough of the \( KK \)-pairing transfers over to \( \text{Ext} \) so that one can say that \( DA \) and \( S\mathbb{F}(A) \) are “\( \text{Ext} \) equivalent” via the duality classes \( u \) and \( v \), in a suitable categorical setting.

**Some technical notes:**

1. Signs: In the classical Spanier-Whitehead duality pairing \( X \wedge DX \to S^n \), the number \( n \) is determined by the dimension of the sphere in which \( X \) is initially embedded. It is thus not intrinsic to the problem. It does, however, control the shift in dimension that occurs when passing from the homology of \( X \) to the cohomology of \( DX \) and hence \( DX \) is frequently denoted \( D_n X \) or \( D_{n-1} X \). Working in periodic \( K \)-theory the number is even less important, since all that matters is its parity. The result is that either the duality classes \( \mu \) and \( \nu \) both appear in \( KK_1 \) or both appear in \( KK_0 \). In the case of \( KK_0 \) no attention to signs is required. In the case of \( KK_1 \) (and this is the case in the paper of Putnam-Kaminker-Whittaker \[22\], for example) there are various changes in sign forced by the parity requirement. We will stay away from this case for simplicity, confident that the reader can see the necessary changes needed from the Putnam-Kaminker-Whittaker paper.

2. When we say that “\( A \) satisfies the UCT” we mean that for all \( C^* \)-algebras \( B \) with countable approximate unit, the Kasparov groups \( KK_*(A, B) \) satisfy the Universal Coefficient Theorem \[34\]. We conjectured at the Kingston conference (1980) that every separable nuclear \( C^* \)-algebra was equivalent to a \( C^* \)-algebra in the bootstrap category \[36\] and hence satisfied the UCT; this conjecture is still open and more plausible than ever.

3. The analogy between the stable homotopy category and the category of \( C^* \)-algebras with \( KK \)-theory as morphisms has been developed by several people, e.g. \[27, 28\]. In that context Spanier-Whitehead \( K \)-duality and classical Spanier-Whitehead duality arise in similar ways \[27, 4\]. The fact that there are geometric and dynamical instances in the noncommutative setting perhaps enhances their interest. Nevertheless, we will not develop this aspect in the present paper.

It is a pleasure to acknowledge assistance from Heath Emerson, Peter Landweber, Lenny Makar-Limonov, Orr Shalit, and Baruch Solel in the creation of this article. Special thanks go to Ilan Hirshberg and Jonathan Rosenberg for their substantial contributions to Section 10. A special thanks goes to the referee for a meticulous and very helpful report. Claude Schochet is also very conscious of a bridge game that he played as a graduate student in 1968 with Ed Spanier, G.W. Whitehead, and N.E. Steenrod, sitting in for his advisor Peter May and thinking that his whole mathematical career was on the line.

2. **Spanier-Whitehead \( K \)-Duality**

The existing literature is somewhat confused regarding the proper definition of Spanier-Whitehead \( K \)-duality. The basic idea is natural and seems to have
appeared first in [23]. Connes considered a noncommutative version of Poincaré duality which refers to algebras dual to their opposite algebras, but some of his examples have the important additional structure of a fundamental class, which make them especially interesting. They were precursors to his notion of spectral triple as a noncommutative manifold. Basically, he proved that the existence of the fundamental class yielded what we are calling Spanier-Whitehead K-duality classes. In [21] Kaminker and Putnam referred to it as Spanier-Whitehead duality explicitly. The definitions used are essentially the same, but there are some technical points which we will clarify in this section.

**Definition 2.1.** Suppose given separable C*-algebras $A$ and $DA$ together with $KK$-classes

$$
\mu \in KK_*(\mathbb{C}, A \otimes DA).
$$

$$
\nu \in KK_*(DA, A \otimes DA, \mathbb{C})
$$

with the property that

$$
\mu \otimes_A \nu = \pm 1_{DA} \in KK_0(DA, DA)
$$

and

$$
\mu \otimes_{DA} \nu = \pm 1_{A} \in KK_0(A, A).
$$

Then $A$ and $DA$ are said to be Spanier-Whitehead K-dual with duality classes $\mu$ and $\nu$.

The separability condition is to ensure that the $KK$-products are defined. (We discuss weakening this condition later in the paper.) Note that this definition is symmetric. If both classes have even parity then the sign is +1 in both cases; in the odd case one introduces signs as in [22, 17].

**Theorem 2.2.** (1) Suppose given Spanier-Whitehead K-dual C*-algebras $A$ and $DA$. Then each of the associated slant product maps

- $(-) \otimes_A \nu : K_*(A) \rightarrow K^*(DA)$
- $(-) \otimes_{DA} \nu : K_*(DA) \rightarrow K^*(A)$
- $\mu \otimes_{DA} (-) : K^*(DA) \rightarrow K_*(A)$
- $\mu \otimes_A (-) : K^*(A) \rightarrow K_*(DA)$

is an isomorphism, and the compositions

$$
K_*(A) \rightarrow K^*(DA) \rightarrow K_*(A)
$$

and

$$
K_*(DA) \rightarrow K^*(A) \rightarrow K_*(DA)
$$

are each $\pm 1$.

(2) Conversely, given separable C*-algebras $A$ and $DA$ together with classes $\mu$ and $\nu$, if the indicated compositions are $\pm 1$ then $A$ and $DA$ are Spanier-Whitehead K-dual.
Proof. (Although versions of this appear in the literature, we include a proof for completeness.)

We are given duality classes
\[ \mu \in KK_0(\mathbb{C}, A \otimes DA) \quad \nu \in KK_0(A \otimes DA, \mathbb{C}) \]
so that we are considering the case where no signs appear. Instead of assuming that \( x \in KK_0(DA, \mathbb{C}) \), which would suffice for the first part of the proof, we assume that we are given auxiliary separable \( C^* \)-algebras \( F \) and \( G \) and that
\[ x \in KK_0(DA \otimes F, \mathbb{C} \otimes G). \]

We shall prove that the composite map
\[
\begin{align*}
KK_0(DA \otimes F, \mathbb{C} \otimes G) & \xrightarrow{\mu \otimes DA (-)} KK_0(C \otimes F, A \otimes G) \xrightarrow{(-) \otimes A \nu} KK_0(DA \otimes F, C \otimes G) \\
\end{align*}
\]
is the identity map. By symmetry, it follows that the dual composite map
\[
KK_0(C \otimes F, A \otimes G) \rightarrow KK_0(C \otimes F, A \otimes G)
\]
is an isomorphism and this proves the proposition.

Let \( 1_A \in KK_0(A, A) \) denote the class of the identity map, and then let
\[ 1_A \otimes w = 1_A \otimes_{\mathbb{C}} w \in KK_0(A \otimes Y, A \otimes Z) \]
denote the external product of \( 1_A \) with some class \( w \in KK_0(Y, Z) \). Then:
\[
\begin{align*}
(\mu \otimes DA x) \otimes_A \nu &= \\
= (\mu \otimes A \otimes DA (1_A \otimes x)) \otimes A \nu \\
= [\mu \otimes A \otimes DA (1_A \otimes x)] \otimes A \otimes DA \nu \\
= \mu \otimes A \otimes DA [1_A \otimes x \otimes 1_{DA}] \otimes A \otimes DA \nu \\
= \mu \otimes A \otimes DA (x \otimes_{\mathbb{C}} \nu) \quad \text{(because} \otimes_{A \otimes DA} \text{is associative)} \\
= \mu \otimes A \otimes DA (\nu \otimes_{\mathbb{C}} x) \quad \text{(because} \otimes_{\mathbb{C}} \text{is commutative)} \\
= (\mu \otimes A \nu) \otimes DA x = 1_{DA} \otimes DA x = x.
\end{align*}
\]

Conversely, suppose that the composite \( KK_s(DA \otimes F, \mathbb{C} \otimes G) \rightarrow KK_s(DA \otimes F, \mathbb{C} \otimes G) \) is the identity. This translates into the formula
\[
(\mu \otimes DA x) \otimes_A \nu = x
\]
for all \( x \in KK_0(DA \otimes F, \mathbb{C} \otimes G) \). Set \( F = \mathbb{C}, G = DA \) and \( x = 1_{DA} \). Then we have
\[ 1_{DA} = (\mu \otimes DA 1_{DA}) \otimes_A \nu = \mu \otimes DA \nu \]
as desired. By symmetry,
\[ 1_A = \nu \otimes_A \mu \]
and the proof is complete.
\[ \square \]
Corollary 2.3. Suppose given two pairs of Spanier-Whitehead dual algebras \( A \) and \( DA \) and also \( B \) and \( DB \) with associated duality classes \( \mu_A, \nu_A, \mu_B, \nu_B \). Then these classes determine canonical isomorphisms
\[
KK^\ast(A, B) \cong KK^\ast(DB, DA).
\]

Proof. The natural map
\[
KK^\ast(A, B) \xrightarrow{(-) \otimes B \nu_B} KK^\ast(A \otimes DB, C) \xrightarrow{\mu_A \otimes A(-)} KK^\ast(DB, DA)
\]
is obtained by taking special cases of equation (2) and its dual.

Corollary 2.4. Suppose with the notation above that we are given \( \mu \otimes_A \nu = u \in KK^0(DA, DA) \) and \( \mu \otimes DA \nu = v \in KK^0(A, A) \) where \( u \) and \( v \) are KK-invertible elements, not necessarily \( \pm 1 \). Then the four slant products listed in Theorem 2.2 will be isomorphisms, and the composites
\[
K_s(A) \to K_s(A)
\]
and
\[
K_s(DA) \to K_s(DA)
\]
will be the isomorphisms \( v_* \) and \( u_* \) respectively.

Conversely, if \( A \) and \( DA \) satisfy the UCT and the composites \( u_* \) and \( v_* \) are isomorphisms then \( u \) and \( v \) are KK-invertible.

Proof. This is mostly immediate from the Theorem. The missing link is provided by the following proposition, which is of independent interest.

Proposition 2.5. Suppose that \( A \) is a \( C^\ast \)-algebra satisfying the UCT and there is an element \( u \in KK^0(A, A) \) such that
\[
u_* : K_s(A) \to K_s(A)
\]
is an isomorphism. Then \( u \) is KK-invertible. If \( u_* = \pm 1 \) then \( u = \pm 1 + k \) for some \( k \in Ker(\gamma_\infty) \), where
\[
\gamma_\infty : KK_s(A, A) \to Hom(K_s(A), K_s(A))
\]
is the index map in the UCT.

Proof. (Thanks to L. Makar-Limanov for help with this proof.) The UCT sequence
\[
0 \to Ker(\gamma_\infty) \to KK^\ast(A, A) \xrightarrow{\sim} End(K_s(A)) \to 0
\]
splits as rings, and \( Ker(\gamma_\infty)^2 = 0, [34] \). Write
\[
u = w + k
\]
for some \( k \in Ker(\gamma_\infty) \) and \( w \in KK^\ast(A, A) \) an invertible element coming from an invertible in \( End(K_s(A)) \) via the splitting. (If \( u_* = \pm 1 \) then \( w = \pm 1 \).) Write \( x = w^{-1} \). Then
\[
u = w + k = (wx)(w + k) = w(1 + xk)
\]
and hence
\[
[(1 - xk)x]u = (1 - xk)xw(1 + xk) = 1 - (xk)^2 = 1
\]
since \((x^k)^2 = 0\). Thus \(u\) is \(KK\)-invertible.

Here are some basic properties of Spanier-Whitehead \(K\)-duality, cf. [21].

**Theorem 2.6.** Suppose that \(A\) and \(DA\) are Spanier-Whitehead \(K\)-dual and both satisfy the UCT. Then:

1. \(D(DA)\) is \(KK\)-equivalent to \(A\).
2. \(K_\ast(A)\) is finitely generated.
3. If \(Q\) and \(R\) satisfy the UCT then slant pairing with the Spanier-Whitehead \(K\)-duality classes yield natural inverse isomorphisms

\[
\nu : KK_\ast(Q \otimes DA, R) \rightarrow KK_\ast(Q, R \otimes A)
\]

4. If \(A\) is \(KK\)-equivalent to \(B\), then \(B\) has a Spanier-Whitehead \(K\)-dual \(DB\) and \(DA\) is \(KK\)-equivalent to \(DB\).

**Theorem 2.7.** Suppose that \(A\), \(B\), and \(A \otimes B\) each have Spanier-Whitehead \(K\)-duals. Then there is a natural \(KK\)-equivalence

\[
D(A \otimes B) \simeq DA \otimes DB
\]

**Proof.** Under natural duality class maps (which are isomorphisms by (2))

\[
KK_0(D(A \otimes B) \otimes A \otimes B, \mathbb{C}) \cong KK_0(D(A \otimes B) \otimes A, DB) \cong KK_0(D(A \otimes B), DA \otimes DB)
\]

the class \(\nu_{A \otimes B} \in KK_0(D(A \otimes B) \otimes A \otimes B, \mathbb{C})\) is sent to a class which we designate

\[
\Psi \in KK_0(D(A \otimes B), DA \otimes DB)).
\]

We can similarly produce a class \(\Phi \in KK_0(DA \otimes DB, D(A \otimes B))\) simply by using [2] and its dual a few times. Then a proof similar to the proof of Theorem 2.2 shows that \(\Phi = \Psi^{-1}\).

3. Fitting Classical Spanier-Whitehead duality into the Spanier-Whitehead \(K\)-duality framework

Classical Spanier-Whitehead duality actually lives in the world of stable homotopy theory. Thus its beautiful properties need some modification before the relationship with Spanier-Whitehead \(K\)-duality emerges.

We borrow the following exposition from Becker-Gottlieb [4], §4. Given a polyhedron \(X\) in \(S^{n+1}\), Spanier-Whitehead define an \(n\)-dual, \(D_nX\), to be a polyhedron contained in \(S^{n+1} - X\) which has the property that some suspension of \(D_nX\) is a deformation retract of the corresponding suspension of \(S^{n+1} - X\).

Now suppose that \(X^* \subset (S^{n+1} - X)\) is a polyhedron which is actually a deformation retract, hence an \(n\)-dual. Following Spanier, remove a point of \(S^{n+1}\) that is neither in \(X\) nor in \(X^*\). Then one can regard both spaces as embedded in \(\mathbb{R}^{n+1}\). Define

\[
\mu^X : X \times X^* \rightarrow S^n
\]
by
\[ \mu_X(x, x^*) = \frac{(x - x^*)}{|x - x^*|}. \]
The restriction of \( \mu^X \) to \( X \vee X^* \) is null-homotopic and so one obtains a map
\[ \mu^X : X \wedge X^* \to S^n. \]
Slant product with this class induces an isomorphism
\[ \mu^X / (-)^* : H_q(X) \cong H^{n-q}(X^*). \]
Spanier, following work of Wall, Freyd, and Husemoller (see [4] for details and references) shows that the whole duality theory can be expressed in terms of the duality map \( \mu \). The space \( X^* \) depends upon the choice of \( n \), the choice of embedding, and the choice of the deformation retraction. It turns out, though, that for \( n \) large the stable homotopy type of \( X^* \) is independent, up to suspension, of the choices of the embedding and of the deformation retraction. The resulting (stable) space \( DX \), defined for any finite complex \( X \), is called the Spanier-Whitehead dual of \( X \), in honor of the people who discovered it and determined its primary properties [42], [43]. Taking \( n \) large enough to be in the stable range we have a duality pairing as
\[ \mu_{C(X)} : C(S^{2n}) \to C(X) \otimes C(DX). \]
We use even-dimensional spheres to control the parity of the degree of the duality class. The associated candidate for a duality class
\[ \nu_{C(X)} : C(X) \otimes C(DX) \to C(S^{2k}) \]
may be obtained by taking the stable dual \( \nu^X \) of the map \( \mu^X : X \times DX \to S^{2k} \).

**Theorem 3.1.** Suppose that \( X \) is a finite CW complex and that \( DX \) is a Spanier-Whitehead dual for \( X \). Then \( C(X) \) and \( C(DX) \) are Spanier-Whitehead \( K \)-dual. Indeed,
\[ D(C(X)) \cong C(DX). \]

**Proof.** This result is non-trivial, since an algebraic isomorphism in homology does not imply an isomorphism in \( K \)-theory. However, this result was established previously with D.S. Kahn in [20]. It was shown there that it follows from the identification of \( K_* (X) \) with \( K^*(DX) \) as discussed in the introduction.

\[ \square \]

**Example 3.2.** When defining duality, one might be tempted to always require that the classes \( \mu \) and \( \nu \) actually be \( KK \)-inverses of one another. Here is an example to show that this is a bad idea.

Suppose that \( X \) is a mod \( p \) Moore space. That is, its reduced homology is zero except in degree one, and \( H_1(X; \mathbb{Z}) \cong \mathbb{Z}/p \). This space is self-dual in the classical Spanier-Whitehead sense. In fact, the reduced cohomology of \( X \) is zero except in dimension two, and \( H_1(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \).

There are stable duality maps \( S^r \to X \wedge X \to S^t \) such that slant product with these maps yields isomorphisms in reduced homology and cohomology
\[ H_*(X) \cong H^*(X) \quad H^*(X) \cong H_*(X) \]
with degree shifts. However, the composite
\[ H_*(X \wedge X) \to H_*(S^r) \to H_*(X \wedge X) \]
cannot possibly be the identity map, since $H_*(X \wedge X)$ has torsion and $H_*(S^*)$ is torsionfree.

Write $A = C_0(X - pt)$ so that

$$K_0(A) = \mathbb{Z}/p \quad K_1(A) = 0$$

Then the Künneth Theorem \[36\] implies that there are isomorphisms

$$K_0(A) \otimes K_0(A) \xrightarrow{\cong} K_0(A \otimes A)$$

and

$$K_1(A \otimes A) \xrightarrow{\cong} Tor(K_0(A), K_0(A))$$

so that

$$K_0(A \otimes A) = \mathbb{Z}/p \quad K_1(A \otimes A) = \mathbb{Z}/p$$

and the UCT \[34\] implies that

$$KK_0(A \otimes A, C) = \mathbb{Z}/p \quad KK_0(C, A \otimes A) = \mathbb{Z}/p$$

The resulting pairing

$$KK_0(C, A \otimes A) \times KK_0(A \otimes A, C) \xrightarrow{\otimes_{A \otimes A}} KK_0(C, C)$$

is evidently trivial since $KK_0(A \otimes A, C)$ and $KK_0(C, A \otimes A)$ are both torsion groups, while $KK_0(C, C) \cong \mathbb{Z}$. Thus the classical Spanier-Whitehead duality classes \[42, 43\] give us $K$-duality classes but do NOT give us invertible $KK$-classes.

4. Examples of noncommutative duality

The results of the previous section seem to suggest, at least when K-theory is finitely generated, that Spanier-Whitehead K-duality is a commutative phenomenon. However, many of the algebras providing natural examples of duality owe this property to underlying geometry and dynamics and are very far from being commutative. Indeed, many are simple algebras. We will survey some of these in this section.

The importance of finite complexes in algebraic topology is the fact that they are constructed systematically out of basic building blocks which are determined by their homology, e.g. spheres. This information can be assembled to compute homology and cohomology for general finite complexes.

In the noncommutative case one is often confronted with simple algebras, i.e. ones with no nontrivial ideals. It is natural to look for building blocks in $KK_F$ which are of this type and, because of the results above, one may choose to consider simple $C^*$-algebras which have Spanier-Whitehead K-duals. We will discuss two examples of this phenomenon—the first coming from the study of hyperbolic dynamics and the second from the study of hyperbolic groups. We will then briefly consider additional instances of noncommutative duality.

4.1. Hyperbolic dynamics. We refer to \[22\] for precise statements and details. A Smale space is a compact metric space, $X$, along with an expansive homeomorphism, $\phi$, which has similar properties to that of an Anosov diffeomorphism of a torus. By this we mean that there are two equivalence relations defined on $X$ called stable and unstable equivalence. Each defines a locally compact groupoid with Haar system and hence one may associate $C^*$-algebras to them. Let us denote them by $\mathcal{S}$ and $\mathcal{U}$. Both can be represented on $L^2(X)$ and the groupoids can be viewed as “transverse” because each stable equivalence class meets an unstable
class in a countable set. This implies that the product of an element of \( S \) and an element of \( U \) is a compact operator.

Using the automorphisms induced by \( \phi \) on \( S \) and \( U \) one constructs the crossed product algebras, \( R^u = U \rtimes_\phi \mathbb{Z} \) and \( R^s = S \rtimes_\phi \mathbb{Z} \), called Ruelle algebras. They can be shown to be Spanier-Whitehead K-dual, \([22]\). It is interesting to consider the construction of the duality classes. One first obtains a projection in \( S \otimes U \) and from that a unitary in \( R^u \otimes R^s \) which yields a class \( \delta \in KK^1(\mathbb{C}, R^u \otimes R^s) \). Then, strongly using the hyperbolic properties of the dynamics, one constructs an extension which yields an element \( \Delta \in KK^1(R^u \otimes R^s, \mathbb{C}) \). These classes are the required duality classes.

An example of a Smale space is a subshift of finite type associated to a matrix \( A \). Associated to this data are the Cuntz-Krieger algebras \( O_A \) and \( O_{AT} \). It turns out that the Ruelle algebras \( R^u \) and \( R^s \) are isomorphic to \( O_A \otimes K \) and \( O_{AT} \otimes K \), and so the Cuntz-Krieger algebras \( O_A \) and \( O_{AT} \) are (stably) Spanier-Whitehead K-dual.

### 4.2. Baum-Connes conjecture

Let \( \Gamma \) be a torsion free and non-elementary Gromov hyperbolic group. It has been shown by de la Harpe \([19]\) that \( C^*_r(\Gamma) \) is a simple \( C^* \)-algebra. We will assume that there is a model for the classifying space \( B\Gamma \) which is a closed smooth manifold. The Baum-Connes conjecture, which is known to hold in this case \([29]\), asserts that there is an isomorphism,

\[
\mu : KK(C(B\Gamma), \mathbb{C}) \rightarrow KK(\mathbb{C}, C^*_r(\Gamma)).
\]

In the present setting the map \( \mu \) can be obtained via Kasparov product with the class in \( \Psi_\Gamma \in KK(\mathbb{C}, C^*_r(\Gamma) \otimes C(B\Gamma)) \) determined by the Mishchenko line bundle,

\[
C^*_r(\Gamma) \rightarrow E\Gamma \times_\Gamma C^*_r(\Gamma) \rightarrow B\Gamma.
\]

This is the first duality class \( \mu \). As in the dynamical situation above, very little special structure is needed to define it. However, as above, the other duality class \( \nu \) makes use of the hyperbolic structure of the group. That class is the dual-Dirac class

\[
\kappa_\Gamma \in KK(C^*_r(\Gamma) \otimes C(B\Gamma), \mathbb{C})
\]

introduced by Kasparov. Thus, in this context, the Baum-Connes conjecture is the same as \( C(\partial \Gamma) \) being Spanier-Whitehead K-dual to \( C^*_r(\Gamma) \).

There is a possible connection between these examples. The hyperbolic group \( \Gamma \) acts amenably on its Gromov boundary, \( \partial \Gamma \). If we choose a quasi-invariant measure on \( \Gamma \) then, by a result of Connes, Feldman, Weiss \([11]\) that action is orbit equivalent to a \( \mathbb{Z} \) action. Although this result is in a measure theoretic setting, in certain cases, such as a Fuchsian group of the first kind acting on \( S^1 \), the transformation generating the \( \mathbb{Z} \) action can be taken to be a piecewise homeomorphism which can be studied using hyperbolic dynamics. Indeed, both of the \( C^* \)-algebras associated to this hyperbolic dynamical system in the first example are isomorphic to the crossed product, \( C(\partial \Gamma) \rtimes \Gamma \). This has been generalized to \( SL(2, \mathbb{Z}) \) acting on \( S^1 \) \([24]\) but in this case the isomorphism between the dynamical algebras and the crossed products is obtained by computing K-theory and applying the classification result of Kirchberg and Phillips. Duality in general for hyperbolic groups acting on their boundary has been studied in detail by Emerson \([17]\). This suggests the question of whether the proof of the Connes-Feldman-Weiss theorem, in the case of a hyperbolic group acting on its boundary, can be refined so that one
obtains a hyperbolic dynamical system for which the associated Ruelle algebras are isomorphic to the crossed product.

A general theory of duality on the level of groupoids with hyperbolic structure has been developed by Nekrashevych [32]. There is a setting in which analogs of the stable and unstable groupoids can be defined, but as of yet there is no general K-theory result involving the associated $C^*$-algebras. It would be interesting to show that they are Spanier-Whitehead K-dual.

4.3. Mukai transform. The actual Mukai transform is studied in the context of algebraic geometry and relates the derived category of coherent sheaves on an abelian variety to that of its dual variety [31]. However, the formula for the transform can be identified with the map in the Baum-Connes example above, and hence can be viewed as an instance of Spanier-Whitehead K-duality. The fact that it yields an isomorphism was first proved by Lusztig [26], which was one of the origins of the K-theoretic approach to such problems. We mention this here because it indicates the sense that this type of duality is like a “transform”.

Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice and $T^n = \mathbb{R}^n/\Lambda$ the associated torus. Let $\hat{T}^n = \hat{\mathbb{R}}^n/\hat{\Lambda}$ be the dual torus, where $\hat{\Lambda} = \{ \alpha \in \hat{\mathbb{R}}^n | \alpha(x) \in \mathbb{Z}, \text{ for } x \in \Lambda \}$. The Poincaré line bundle, $\mathcal{P}^\Lambda$, over $T^n \times \hat{T}^n$ is determined by the property that $\mathcal{P}^\Lambda|_{T^n \times \alpha} = L_\alpha$, where $L_\alpha = L_{\alpha_1} \otimes \ldots \otimes L_{\alpha_n}$. The Mukai transform is obtained as

$$K^*(T^n) \xrightarrow{\mathcal{P}^\Lambda} K^*(T^n \times \hat{T}^n) \xrightarrow{\mathcal{P}^\Lambda \otimes} K^*(T^n \times \hat{T}^n) \xrightarrow{(p_\hat{T}^n)_!} K^*(\hat{T}^n)$$

We also have the Mishchenko line bundle, $C^*(\Lambda) \to \Psi^\Lambda \to T^n$. There is a map induced by the Gelfand transform

$$1 \otimes G : KK(\mathbb{C}, C(T^n) \otimes C^*(\Lambda)) \to KK(\mathbb{C}, C(T^n) \otimes C(\hat{T}^n)),$$

with the property that $1 \otimes G([\Psi^\Lambda]) = [\mathcal{P}^\Lambda]$. The diagram below expresses the relation between the Baum-Connes map and the Mukai transform in this setting. We assume $n$ is even to simplify the diagram.

5. Poincaré duality

We will assume in this section that our algebras are unital and are in $KK_F$. We also avoid formulating statements for odd Poincaré duality.
In [9] Connes (see also [12, 23]) discussed a notion of Poincaré duality for a C\(^*\)-algebra. It states that an algebra \(A\) satisfies Poincaré duality if it is Spanier-Whitehead K-dual to its opposite algebra, \(A^{op}\). This yields a class \(\vartheta \in KK(A, \mathbb{C})\) by setting \(\vartheta = 1 \otimes \mu\), where \(1 \in KK(\mathbb{C}, A^{op})\) and \(\mu \in KK(A^{op} \otimes A, \mathbb{C})\) is the duality class. In the commutative case \(\vartheta\) would correspond to a K-theory fundamental class and taking cap product with it would yield an isomorphism

\[\cap \vartheta : KK(\mathbb{C}, A) \rightarrow KK(A, \mathbb{C}).\]

Since such an algebra \(A\) is Morita equivalent to its opposite, we may just as well formulate Poincaré duality in terms of \(A\) alone.

If \(A\) is not commutative there is, in general, no cap product in K-theory. We will present here a slightly weaker condition which will allow a version of a cap product to be defined so that one could obtain a Poincaré duality isomorphism of the usual form. Note that we are using the convention that \(1 \in KK(\mathbb{C}, A)\) is the class of the identity element in \(A\), while \(1_A \in KK(A, \mathbb{C})\) is the class of the identity homomorphism.

Let

\[\tau^A : KK(B, D) \rightarrow KK(B \otimes A, D \otimes A)\]

and

\[\tau_A : KK(B, D) \rightarrow KK(A \otimes B, A \otimes D)\]

denote the standard homomorphisms.

**Definition 5.1.** A C\(^*\)-algebra \(A\) is *K-commutative* if there is a class \(m \in KK(A \otimes A, A)\) with the property that one has

\[(9)\quad \tau_A \otimes A \otimes A m = 1_A, \quad \tau^A \otimes A \otimes A m = 1_A.\]

Recall that, when \(A\) is commutative, \(m\) plays the role of the class determined by the diagonal map and it also agrees with the class determined by the multiplication in \(A\). We will call \(m\) a *K-commutative product*. If such a class exists one defines the usual cup and cap products via the following diagrams.

**Cup product:**

\[\begin{array}{ccc}
KK(\mathbb{C}, A) \times KK(\mathbb{C}, A) & \xrightarrow{\cup} & KK(\mathbb{C}, A) \\
\downarrow \otimes & & \downarrow (-) \otimes A \otimes A m \\
KK(\mathbb{C}, A \otimes A) & &
\end{array}\]

**Cap product:**

\[\begin{array}{ccc}
KK(\mathbb{C}, A) \times KK(A, \mathbb{C}) & \xrightarrow{\cap} & KK(A, \mathbb{C}) \\
(id, m \otimes A)(-)) & & \otimes A \\
KK(\mathbb{C}, A) \times KK(A \otimes A, \mathbb{C}) & &
\end{array}\]

**Definition 5.2.** Let \(A\) be an algebra with a K-commutative product. A *fundamental class* is an element \(\vartheta \in KK(A, \mathbb{C})\) such that

\[\cap \vartheta : KK(\mathbb{C}, A) \rightarrow KK(A, \mathbb{C})\]

is an isomorphism.
Proposition 5.3. Let $A$ be a $K$-commutative algebra satisfying Poincaré duality with duality classes $\nu$ and $\mu$. Then for any $u \in KK(C, A)$ which is invertible with respect to cup product, the class $u \otimes_A \nu$ is a fundamental class.

Proof. We must show that if $x \in KK(C, A)$ has an inverse with respect to cup product then the map $x \mapsto x \cap (u \otimes_A \nu)$ is an isomorphism. Unraveling the definitions and using properties of the Kasparov product as in Theorem 2.2, one obtains the formula

$$x \cap (u \otimes_A \nu) = \tau^A(x) \otimes_{A \otimes A} (m \otimes_A (\tau^A(u) \otimes_{A \otimes A} \nu))$$

$$= (\tau^A(x) \otimes_{A \otimes A} (m \otimes_A (\tau^A(u))) \otimes_{A \otimes A} \nu)$$

$$= (x \cup u) \otimes_A \nu.$$ Since $x \mapsto x \cup u$ and $x \mapsto x \otimes_A \nu$ are isomorphisms the result follows. \hfill \Box

Additional aspects of this topic, such as the study of noncommutative algebras which are $K$-commutative, will be developed in further work.

6. Existence of Spanier-Whitehead $K$-Duals

In this section we show that if $A$ is a separable $C^*$-algebra satisfying the UCT and if $K_*(A)$ is finitely generated then $A$ has a Spanier-Whitehead $K$-dual. This result is analogous to the classical theorem that any space of the homotopy type of a finite CW-complex has a classical Spanier-Whitehead dual.

Proposition 6.1. Suppose given a countable \(\mathbb{Z}/2\)-graded abelian group $G_*$. Then there exists a sequence

$$A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n \rightarrow \cdots$$

of $C^*$-algebras and $C^*$-maps such that

1. The unitalization $A_n^+$ satisfies that $A_n^+ \cong C(X_n)$ for some finite CW complex $X_n$.
2. Each map $K_*(A_n) \rightarrow K_*(A_{n+1})$
3. There is an isomorphism $\lim_{\rightarrow} K_*(A_n) \cong G_*$. 
4. Let $A = \lim_{\rightarrow} A_n$. Then $A \cong C_0(X)$ is a separable commutative $C^*$-algebra in the bootstrap category, and

$$K_*(A) \cong G_*.$$ 

Proof. Write $G_*$ as the union of an increasing sequence of finitely generated $\mathbb{Z}/2$-graded abelian groups $G_n^*$. Then apply [39] Theorem 5.1. \hfill \Box

Theorem 6.2. Suppose that $A$ is a separable $C^*$-algebra that satisfies the UCT and $K_*(A)$ is finitely generated. Then there exists a finite CW-complex (or finite minus a point) $X$ such that $A$ is Spanier-Whitehead $K$-dual to $C(X)$ (or $C_0(X \setminus \text{pt})$).
Proof. Let $Y$ be a finite complex (or finite minus a point) such that $K^*(Y) \cong K^*_s(A)$. The space $Y$ has a classical Spanier-Whitehead dual; pick one that has a duality map $X \times Y \to S^{2n}$. Theorem 3.1 implies that $C(X)$ and $C(Y)$ are Spanier-Whitehead $K$-dual. Now $A$ and $C(Y)$ are $KK$-equivalent, by the UCT [34], and so Proposition 2.6 implies that $A$ and $C(X)$ are Spanier-Whitehead $K$-dual. □

Remark 6.3. If $A$ is separable, satisfies the UCT, but $K^*_s(A)$ is not finitely generated then separability implies that $K^*_s(A)$ is countable, and we may apply the previous result to obtain a locally compact space $X$ such that $K^*_s(A) \cong \pi_0(C_0(Y))$. Then $A$ and $C_0(Y)$ are $KK$-equivalent by the UCT. The problem now is topological: how do you take the Spanier-Whitehead dual of a compact space that is not of the homotopy type of a finite CW-complex? (The situation is analogous to Paschke duality, which we discuss in Section 7). It turns out that if $X$ is finite-dimensional then one may use functional Spanier-Whitehead duals as in [20]. However, the resulting Spanier-Whitehead dual must be treated as a spectrum rather than a space. In principle one could move to a larger category at this point, but we refrain.

Remark 6.4. It is often useful to view the category $\mathcal{KK}$, with objects separable $C^*$-algebras and with morphisms $KK(A,B)$, as analogous to the stable homotopy category of countable CW-complexes and stable homotopy classes of maps, $\mathcal{SH}$, cf. [27, 28]. In the stable homotopy setting there is a result of Boardman [3] which implies that the largest full subcategory of $\mathcal{SH}$ closed under Spanier-Whitehead duality is that determined by stable homotopy types of finite CW-complexes. It is interesting that the results of Section 6 lead to a noncommutative version of Boardman’s theorem.

Let $\mathcal{KK}^*$ be the full subcategory of $\mathcal{KK}$ with objects nuclear $C^*$-algebras in the bootstrap category. The algebras in $\mathcal{KK}^*$ will satisfy the UCT [34] and are all $KK$-equivalent to $C(X)$ or $C_0(X \setminus pt)$, for $X$ a compact Hausdorff space.

Let $\mathcal{KK}_F$ be the full subcategory of $\mathcal{KK}^*$ with objects that have finitely generated $K$-theory.

Proposition 6.5. The category $\mathcal{KK}_F$ is the largest subcategory of $\mathcal{KK}^*$ closed under Spanier-Whitehead $K$-duality.

Proof. First we note that Theorem 6.2 shows that any object, $A$, in $\mathcal{KK}_F$ is $KK$-equivalent to $C(X)$, for $X$ a finite complex. Thus, $A$ has a dual which is $KK$-equivalent to $C(Y)$ with $Y$ a finite complex. Hence, $\mathcal{KK}_F$ is closed under taking Spanier-Whitehead $K$-duals.

To complete the proof we must show that any object in $\mathcal{KK}^*$ which has a Spanier-Whitehead $K$-dual in $\mathcal{KK}^*$ will have finitely generated $K$-theory, hence will be in $\mathcal{KK}_F$. This is proved in [22], Section 4.4(d). The hypothesis there is that there is an odd Spanier-Whitehead $K$-duality, but the proof works in the even case as well. □

7. Non-existence of Spanier-Whitehead $K$-Duals

Not every nice $C^*$-algebra in the bootstrap category has a separable bootstrap $KK$-dual. Here is an example. The following proposition is actually an instant consequence of Theorem 2.6 but we give a direct proof to illustrate what goes wrong.
Proposition 7.1. Suppose that $A$ is separable, satisfies the UCT, $K_0(A) \cong \mathbb{Q}$ and $K_1(A) = 0$. Then $A$ cannot have a separable Spanier-Whitehead $K$-dual that satisfies the UCT.

Note that $A$ may be taken to be an AF-algebra, the direct limit of finite dimensional matrix rings, and (by the UCT) is unique up to $KK$-equivalence. One may use this $C^*$-algebra to localize $K$-theory, so it should not be thought of as bizarre.

Proof. Suppose that $A$ has a $K$-dual $DA$ that is separable and satisfies the UCT, so that $K^0(DA) = \mathbb{Q}$ and $K^1(DA) = 0$. We apply the UCT to $K_1(DA)$,

$$0 \to Ext(K^0(DA), \mathbb{Z}) \to K_1(DA) \to Hom(K^1(DA), \mathbb{Z}) \to 0.$$  (12)

But, $Hom(K^1(DA), \mathbb{Z}) = 0$ and one has
\[ K_1(DA) \cong Ext(K^0(DA), \mathbb{Z}) = Ext(\mathbb{Q}, \mathbb{Z}). \]

This leads to a contradiction since it is known that $Ext(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{R}$, [47, 49], but since $DA$ is separable $K_1(DA)$ is a countable group. \qed

8. Mod-$p$ $K$-theory

There are two standard constructions of topological mod-$p$ $K$-theory $K_*(A; \mathbb{Z}/p)$.

The first construction, which appears in Schochet [38], is to select a $C^*$ algebra $N$ in the bootstrap category with $K_0(N) = \mathbb{Z}/p$ and $K_1(N) = 0$, and then for any $C^*$-algebra $A$ define
\[ K_j(A; \mathbb{Z}/n) = K_j(A \otimes N). \]

In [38] we initially built $N$ from a Moore space (a space whose reduced homology is zero except in one degree, where it is $\mathbb{Z}/p$) and then subsequently showed that any bootstrap choice for $N$ gave an isomorphic theory.

The second construction, the kernel of which appears in Dadarlat-Loring [13], is to select a $C^*$ algebra $N$ in the bootstrap category with $K_0(N) = \mathbb{Z}/p$ and $K_1(N) = 0$, and then for any $C^*$-algebra $A$ define
\[ K_j(A; \mathbb{Z}/n) = KK_{j-1}(N, A). \]

Dadarlat-Loring used a dimension-drop algebra with suitable $K$-theory, but it is clear that any bootstrap choice will work equally well. Note that the dimension-shift comes from the UCT isomorphism
\[ \mathbb{Z}/p \cong Ext(K_0(N), K_0(\mathbb{C})) \xrightarrow{\cong} KK_1(N, \mathbb{C}). \]

We were asked by Jeff Boersema whether these two constructions are equivalent. The second construction is defined on a somewhat smaller category than the first, but with that caveat we shall demonstrate that the two constructions are equivalent.

Let us fix $N$ as above. Since it is in the bootstrap category we know that $DN$ exists, and using the UCT we obtain
\[ K_0(DN) = 0 \quad K_1(DN) = \mathbb{Z}/p. \]

Since $DN$ is also in the bootstrap category, we conclude at once that $SDN$ is $KK$-equivalent to $N$. Assume that $A$ is separable so that the $KK$-pairing is available. Then we have the result:
\[ K_j(A \otimes N) \cong KK_j(\mathbb{C}, A \otimes N) \cong KK_j(DN, A) \cong KK_{j-1}(SDN, A) \cong KK_{j-1}(N, A) \]
and we have proved the following theorem:
Theorem 8.1. Suppose that $A$ is separable and that $N$ is chosen in the bootstrap category with $K_0(N) = \mathbb{Z}/p$ and $K_1(N) = 0$. Then the two different constructions of mod-$p$ $K$-theory

$$K_j(A; \mathbb{Z}/n) = K_j(A \otimes N) \quad \text{and} \quad K_j(A; \mathbb{Z}/n) = KK_{j-1}(N, A)$$

are naturally equivalent. \hfill \square

Remark 8.2. The same argument shows that $K_*(A; G)$ is uniquely defined for any finite abelian group. However if one were dealing with a group such as $\mathbb{Q}/\mathbb{Z}$, for instance, then much more care is required. Torsion will be governed by the behavior of the functor $Ext(-, \mathbb{Z})$ and the torsionfree part of this group will bring us to the same difficulty illustrated by the case where $K_0(N) = \mathbb{Q}$.

Remark 8.3. In the proof of our result we show that $SDN$ is $KK$-equivalent to $N$. This is actually stronger than Poincaré duality, as it corresponds to the statement that the Moore space is actually stably homotopy equivalent to its dual. We may obtain the requisite duality maps in $KK(N \otimes N, C)$ by first creating the maps at the level of Moore spaces, moving them to $KK$, and then using the $KK$-equivalences.

9. Paschke Duality

We have seen that not every separable $C^*$-algebra has a Spanier-Whitehead $K$-dual, even if we make bootstrap hypotheses. In [33], Paschke developed a different sort of duality that is

1. better, because it is defined for every separable $C^*$-algebra;
2. worse, because the resulting dual is in general non-separable, we cannot form the double dual, and only one of the two duality maps is present.

After describing the Paschke dual, $P(A)$, we discuss the possibility of substituting more tractable $C^*$-algebras in place of $P(A)$.

These results are due to Paschke [33] as refined by Higson and Roe [19].

Let $H$ be a separable Hilbert space. Let $\mathcal{L}(H)$ denote the $C^*$-algebra of bounded operators on $H$ and $K = K(H)$ denote the compact operators. Let $\pi : \mathcal{L}(H) \to \mathcal{L}(H)/K \cong Q = Q(K)$ be the projection of the bounded operators to the Calkin algebra. Suppose that $A$ is a separable, unital $C^*$-algebra with an ample representation $\rho : A \to \mathcal{L}(H)$. Define

$$D_{\rho}(A) = \{ T \in \mathcal{L}(H) : \pi(T \rho(a) - \rho(a)T) = 0 \quad \forall a \in A \}. $$

The projection of this algebra in the Calkin algebra is $P(A) = \pi(D_{\rho}(A))$, the \textit{Paschke dual} of $A$. Since $P(A)$ is independent of the choice of ample representation by Voiculescu’s Theorem [46], we shall drop $\rho$ from the notation. In general $P(A)$ is unital, but it is typically neither separable nor nuclear. Paschke’s theorem is the following ([33], Theorem 2).

Theorem 9.1. Let $A$ be a separable, unital $C^*$-algebra with an ample representation $\rho : A \to \mathcal{L}(H)$. Then one has that

$$K_0(P(A)) \cong Ext^1(A)$$

\footnote{A representation $\rho : A \to \mathcal{L}(H)$ is ample if it is non-degenerate and if $\rho(A) \cap K = 0$.}
and hence, if $A$ is nuclear, that

$$K_0(\mathbb{P}(A)) \cong K^1(A).$$

and similarly for $K_1$.

We note that there is a canonical $^*$-homomorphism

$$\Psi : A \otimes \mathbb{P}(A) \to \mathbb{Q}$$

given by

$$\Psi(x \otimes y) = \pi(\rho(x))y$$

which is well-defined because $\pi\rho(A)$ commutes with each element of $\mathbb{P}(A)$. The Kasparov groups $KK_\ast(A \otimes \mathbb{P}(A), \mathbb{Q})$ are defined and so we have

$$\nu = [\Psi] \in KK_0(A \otimes \mathbb{P}(A), \mathbb{Q}).$$

Although the full Kasparov product is not available (since $A \otimes \mathbb{P}(A)$ is not separable), the slant product with the map $\Psi$ still makes sense and gives us a well-defined map

$$K_0(\mathbb{P}(A)) \xrightarrow{(-) \otimes \mathbb{P}^A} KK_0(A, \mathbb{Q}) \xrightarrow{\delta} KK^1(A, \mathcal{K}) \cong K^1(A)$$

which Paschke shows is an isomorphism. Thus Paschke’s duality result is a one-sided duality.

The simplest case is actually of interest. Take $A = \mathbb{C}$. Then $\mathbb{P}(A) = \mathbb{Q}$, $\Psi = 1_{\mathbb{Q}}$,

$$\nu = [1_{\mathbb{Q}}] \in KK_0(\mathbb{Q}, \mathbb{Q}).$$

$$\delta : KK_0(\mathbb{Q}, \mathbb{Q}) \to KK_1(\mathbb{Q}, \mathcal{K})$$

and the UCT index map

$$\gamma_\infty : KK_1(\mathbb{Q}, \mathcal{K}) \xrightarrow{\cong} Hom(K_1(\mathbb{Q}), K_0(\mathcal{K})) \cong \mathbb{Z}$$

gives the Paschke isomorphism

$$K_1(\mathbb{Q}) \xrightarrow{\cong} K_0(\mathbb{C}) \cong \mathbb{Z}.$$

If we regard $K^0(\mathbb{C}) \cong K^1(S\mathbb{C}) = K^1(C_0(\mathbb{R}))$ then we have a way to realize a map in the other direction. Let

$$\tau : C(S^1) \to \mathbb{Q}$$

be the map that takes $z$ to the image of the adjoint of the unilateral shift $U^*$. This map classifies the extension\footnote{This is the storied extension that started the BDF work on the classification of essentially normal operators.}
Restrict \( \tau \) to \( C_0(\mathbb{R}) \). We then have the pullback diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\mathcal{K} & \longrightarrow & \mathcal{K} \\
\downarrow & & \downarrow \\
\mathcal{E} & \longrightarrow & \mathcal{L}(\mathcal{H}) \\
\downarrow & & \downarrow \\
C_0(\mathbb{R}) & \rightarrow & Q \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

The right column generates a (very!) canonical extension \( \Upsilon \in Ext(Q, K) \) and

\[
[\tau] = \tau^*(\Upsilon) \in Ext(C_0(\mathbb{R}), K) \cong KK^1(C_0(\mathbb{R}), K).
\]

Further,

\[
\gamma_\infty([\tau]) : K_1(C_0(\mathbb{R})) \xrightarrow{\cong} K_0(K)
\]

and this map is in a sense the inverse to the Paschke isomorphism. This example is the basis for our hope for the Conjecture at the end of Section 1.

The Paschke dual is not a Spanier-Whitehead \( K \)-dual, in general, for several related reasons. It is usually (perhaps always) non-separable, its \( K \)-theory is not necessarily finitely generated and may well be uncountable even for \( A \) an AF-algebra, and there does not seem to be a duality class \( C \rightarrow A \otimes P(A) \). We discuss what can be done in future sections.

10. \( C^* \)-substitutes I: \( K_*(A) \) countable

In this section we show that if \( A \) is a (nuclear) \( C^* \)-algebra with \( K_*(A) \) countable then there exists a separable (nuclear) sub-\( C^* \)-algebra \( \theta A \subseteq A \) which is weak \( K \)-equivalent to \( A \).

**Definition 10.1.** A \( * \)-homomorphism \( f : A \rightarrow B \) is a weak \( K \)-equivalence if the induced map \( f_* : K_*(A) \rightarrow K_*(B) \) is an isomorphism, [34].

Note that if \( A \) satisfies the UCT then a weak \( K \)-equivalence \( f : A \rightarrow B \) lifts to a \( KK \)-class \( \mu \in KK_0(A, B) \). If \( B \) is also in the UCT class then this class may be chosen to be \( KK \)-invertible, so that \( A \) is \( KK \)-equivalent to \( B \).

In the other direction, if \( \mu \in KK_0(A, B) \) is an invertible class then it induces an isomorphism \( \mu/ : K_*(A) \xrightarrow{\cong} K_*(B) \) but it does not necessarily arise from a map \( A \rightarrow B \). Here are two examples:

1. \( M_3(\mathbb{C}) \) and \( M_2(\mathbb{C}) \) are \( KK \)-equivalent but there is no map \( M_3(\mathbb{C}) \rightarrow M_2(\mathbb{C}) \) inducing this equivalence.

2. \( C(\mathbb{C}P^2) \) and \( C(S^2 \vee S^4) \) are \( KK \)-equivalent, but there is no map of spaces that can induce this equivalence, since \( K^*(\mathbb{C}P^2) \) and \( K^*(S^2 \vee S^4) \) are not isomorphic as rings. The authors learned the cohomology version of this
example as students from an unpublished paper of Steenrod, since published as [45].

**Proposition 10.2.** Let $A$ be a $C^*$-algebra and suppose that $K_*(A)$ is countable. Then there exists a separable subalgebra $F$ of $A$ such that the inclusion map $\iota : F \to A$ induces a surjection $\iota_* : K_*(F) \to K_*(A)$.

**Proof.** Since $K_*(A)$ is countable we may list a countable family of projections and unitaries that generate $K_0$ and $K_1$ respectively. Each of these lies in some $A \otimes M_n$. Take the (countable) collection of elements of $A$ that are the matrix entries of this family and let $F$ be the subalgebra of $A$ that they generate. Then it is clear that the map $\iota_* : K_*(F) \to K_*(A)$ is surjective. \qed

The map $\iota_* : K_*(F) \to K_*(A)$ probably is not injective in general. To remedy this problem we use the following construction, due to Ilan Hirshberg.

**Lemma 10.3.** Suppose given a $C^*$-algebra $A$ and a $C^*$-subalgebra $\iota : B \to A$. Suppose $x \in K_0(B)$ and $\iota_*(x) = 0$. Then there are elements $\{a_1, \ldots, a_n\}$ of $A$ with the property that if $B'$ is the $C^*$-subalgebra generated by $B \cup \{a_1, \ldots, a_n\}$ with inclusion map $\iota' : B \to B'$, then $\iota'_*(x) = 0 \in K_0(B').$

**Proof.** Represent $x = [p] - [q]$ where $p$ and $q$ are projections in matrix rings over $B$. The fact that $\iota_*(x) = 0$ means that we have

$$[p] - [q] = [t] - [t]$$

for some trivial projection $t$. Unraveling this leads us to the equation

$$upu^* \oplus h = w(vqv^* \oplus h)w^*$$

for some unitaries $u, v, w$ and some projection $h$, where $u, v, w,$ and $h$ lie in matrix rings over $B$. Take the set $\{a_1, \ldots, a_n\}$ to be the (finite!) collection of matrix coefficients in the matrices $u, v, w, h$. Then it is obvious that the same calculations that took place in $A$ can take place in $B'$, and so $\iota'_*(x) = 0$ as desired. \qed

**Lemma 10.4.** Suppose given a $C^*$-algebra $A$ and a $C^*$-subalgebra $\iota : B \to A$. Suppose $x \in K_1(B)$ and $\iota_*(x) = 0$. Then there are elements $\{a_1, \ldots, a_n\}$ of $A$ with the property that if $B'$ is the $C^*$-subalgebra generated by $B \cup \{a_1, \ldots, a_n\}$ with inclusion map $\iota' : B \to B'$, then $\iota'_*(x) = 0 \in K_1(B').$

**Proof.** Represent $x$ by $u \in U_n(B)$. The fact that $\iota_*(x) = 0$ translates into the existence of a continuous path of unitaries $u_t \in U_{n+k}(A)$ for some $k$ such that $u_0 = u \oplus I$ and $u_1 = I$. Pick a finite sequence of elements $a_j$ on this path with $a_0 = u_0, a_n = I,$ and with the property that $|a_j^{-1}a_{j+1}| < 1$. Then we may construct a path in $U_{n+k}(B')$ connecting these same elements, and hence $u \oplus I$ is in the path component of the identity of $U_{n+k}(B')$, showing that $\iota'_*(x) = 0$. \qed

**Lemma 10.5.** Suppose given a $C^*$-algebra $A$ and a $C^*$-subalgebra $\iota : B \to A$ with associated map

$$\iota_* : K_0(B) \to K_0(A).$$

Suppose that $\text{Ker}(\iota_*)$ is countable. Then there exists a countable number of elements $\{a_j\}$ of $A$ such that if we let $B'$ denote the $C^*$-algebra generated by $B$ and by the $\{a_j\}$ and let $\iota' : B \to B'$ denote the inclusion, then

$$\text{Ker}(\iota_*) = \text{Ker}(\iota'_*).$$
If $\text{Ker}(\iota_\ast)$ is finitely generated then only a finite number of additional elements are needed.

Proof. This follows immediately from the previous two lemmas- we simply choose generators for $\text{Ker}(\iota_\ast)$ and kill them off by adding all of the needed additional elements at once. □

**Theorem 10.6.** (I. Hirshberg) Suppose that $A$ is a $C^\ast$-algebra with $K_\ast(A)$ countable. Then there exists an ascending sequence of separable sub-$C^\ast$-algebras of $A$

$$F_1 \subset F_2 \subset F_3 \subset \ldots$$

with coherent inclusion maps $\iota_n : F_n \to A$ such that each map $\iota_{n_\ast} : K_\ast(F_n) \to K_\ast(A)$ is surjective. Let $\theta A = \lim F_j$. Then $\theta A$ is separable and the induced inclusion map $\iota : \theta A \to A$ yields an isomorphism

$$\iota_\ast : K_\ast(A) \xrightarrow{\cong} K_\ast(A).$$

Proof. We use Lemma [10.2] to construct $F_1$ together with the map

$$\iota_1 : K_\ast(F_1) \to K_\ast(A)$$

which induces a surjection in $K$-theory. Then repeatedly use Lemma [10.6] to construct the higher $F_n$. This gives us an ascending sequence of sub-$C^\ast$-algebras

$$F_1 \subset F_2 \subset F_3 \subset \ldots$$

with coherent inclusion maps $\iota_n : F_n \to A$ and

$$\text{Ker}(\iota_{n_\ast}) \subseteq \text{Ker}[K_\ast(F_n) \to K_\ast(F_{n+1})].$$

Since the map

$$\iota_1 : K_\ast(F_1) \to K_\ast(A)$$

is surjective the induced map

$$\iota_\ast : K_\ast(\theta A) \to K_\ast(A)$$

is surjective. Finally, we claim that $\iota_\ast$ is injective, and hence an isomorphism. Suppose that $\iota_\ast(y) = 0$. Then the class $x$ must arise in some $K_\ast(F_n)$ with $\iota_{n_\ast}(x) = 0$. But then $x \in \text{Ker}(\iota_{n_\ast})$ and so $x = 0 \in K_\ast(F_{n+1})$. Thus $x = 0 \in K_\ast(A)$ and the proof is complete. □

**Corollary 10.7.** In Theorem [10.6] if $A$ is nuclear then $\theta A$ may be constructed to be separable and nuclear.

Proof. We construct inductively an increasing sequence of separable subalgebras $F_n$ of $A$, as follows. $F_1$ will be the one described as in the proof of Theorem [10.6].

Choose a countable dense subset of the unit ball of $F_1$, call it $S_1$. Regard $S_1$ as a sequence. Since $A$ is nuclear, we can find completely positive contractions $\psi : A \to M_k$, $\omega : M_k \to A$ for some $k$ such that

$$||\omega(\psi(a)) - a|| < 1,$$

where $a$ is the first element in $S_1$.

Now, let $F_2$ be the subalgebra generated by $F_1$, all the elements which are added according to the proof above, and the image of the map $\psi$ (which is finite dimensional, so it is still separable). Now choose a dense subset $S_2$ of the unit ball of $F_2$, again ordered as a sequence.
Suppose we constructed
\[ F_1 \subset F_2 \cdots \subset F_n, \]
along with dense sequences \( S_1, S_2, \ldots, S_n \) of the respective unit balls. Pick the first \( n \) elements of each of the sets \( S_1, \ldots, S_n \), and call this set \( S \) (it has at most \( n^2 \) elements). Pick completely positive contractions \( \psi : A \to M_j \), \( \omega : M_j \to A \) for some \( j \) such that
\[ \|\omega(\psi(a)) - a\| < 1/n \]
for all \( a \in S \). Now, modify the definition of \( F_{n+1} \) to be generated by the elements as in the proof of the Theorem along with \( \omega(M_j) \).

The closure of the union, \( \theta A \), is now nuclear. To see this, one needs to verify that \( \theta A \) has the Completely Positive Approximation Property, CPAP ([25], p. 170). One may start with a finite subset \( X \) of the unit ball and an \( \epsilon > 0 \). It can be assumed that \( X \) is in the union of the \( S_n \)'s, since they are dense. If one goes far enough out in the sequence of inclusions (e.g. find an \( N \) so that \( 1/N < \epsilon \) and \( X \) is contained in the union of the first \( N \) elements of each of \( S_1, \ldots, S_N \)), then the maps \( \psi \) (restricted to \( F \)) and \( \omega \) (whose image is in \( \theta A \)) which were used to define \( F_{n+1} \) now witness the CPAP for the finite set \( X \) to within tolerance \( \epsilon \). (None of the \( F_n \)'s need be nuclear themselves, but the union is.)

\[ \square \]

Remark 10.8. Our construction of the subalgebra \( \theta A \) in\[ \text{Theorem 10.6} \]
involves many choices and hence there is no reason to think that \( \theta A \) is uniquely defined. At best one might hope that any two choices would be \( KK \)-equivalent. This would follow at once if \( \theta A \) satisfied the UCT.

11. \( C^* \)-substitutes II: bootstrap entries

We would like to know that every \( C^* \)-algebra \( A \) has a commutative (or at least a bootstrap) \( C^* \)-algebra that is weakly \( K \)-equivalent to it. In the previous section we showed that if \( K_*(A) \) is countable then up to weak \( K \)-equivalence we can replace \( A \) by a separable subalgebra. If \( K_*(A) \) is uncountable then obviously any substitute will be non-separable, but still we could hope for commutativity. In this section we demonstrate that it is almost possible to have a commutative substitute.

If \( A \) satisfies the UCT then \( A \) is \( KK \)-equivalent to a commutative \( C^* \)-algebra \( C \), but the invertible \( KK \)-elements that link them are not necessarily implemented by maps \( C \to A \) or vice versa. In this section we prove that if \( A \) satisfies the UCT then there exists a 2-step solvable (hence bootstrap) \( C^* \)-algebra \( \beta A \) and an auxiliary \( C^* \)-algebra \( T \) together with maps \( \beta A \to T \leftarrow S^3 A \) that are weak \( K \)-equivalences.

The following lemmas and the theorem are variants of the original argument of the second author, ([36], Lemma 3.1) used in the proof of the Künneth formula and also the revised argument due to Blackadar ([5], Theorem 23.51).

Lemma 11.1. Suppose that \( K_1(A) \cong \mathbb{Z}^s \) with \( s \) finite, countably infinite, or uncountable. Then there exists a map
\[ f : \oplus_s C_0(\mathbb{R}) \to A \otimes K \]
such that the induced map
\[ f_* : K_1(\oplus_s C_0(\mathbb{R})) \to K_1(A \otimes K) \]
is an isomorphism (and the induced map on \( K_0 \) is trivial).
Proof. Choose unitaries \( \{u_1, u_2, \ldots \} \subset (A \otimes K)^+ \) which represent a minimal set of generators of \( K_1(A) \). Without loss of generality we may take these generators to be mutually orthogonal. They induce the obvious map

\[ \oplus_s C(S^1) \longrightarrow (A \otimes K)^+ \]

which is an isomorphism on \( K_1 \). Define \( f \) to be the restriction of this map to \( \oplus_s C_0(\mathbb{R}) \); it factors through \( A \otimes K \) and the result follows. \( \square \)

Lemma 11.2. Suppose that \( K_0(A) \cong \mathbb{Z}^r \) with \( r \) finite, countably infinite, or uncountable. Then there exists a map

\[ f : \oplus_s C_0(\mathbb{R}) \longrightarrow SA \otimes K \]

such that the induced map

\[ f_* : K_1(\oplus_s C_0(\mathbb{R})) \longrightarrow K_1(SA \otimes K) \]

is an isomorphism. Suspending, we obtain a map \( g \),

\[ g : \oplus_s C_0(\mathbb{R}^2) \cong S(\oplus_s C_0(\mathbb{R})) \longrightarrow S^2 A \otimes K \]

such that the induced map

\[ g_* : K_0(\oplus_s C_0(\mathbb{R}^2)) \longrightarrow K_0(S^2 A \otimes K) \cong K_0(A) \]

is an isomorphism, and the induced map on \( K_1 \) is trivial.

Combining these two lemmas gives us the desired result.

Theorem 11.3. Suppose that \( A \) is a \( C^* \)-algebra with \( K_*(A) \) free abelian. Then

1. There is a commutative \( C^* \)-algebra \( C \) which is a direct sum of copies of \( C_0(\mathbb{R}^2) \) and \( C_0(\mathbb{R}^1) \) and a map

\[ h : C \longrightarrow SA \otimes K \]

such that the induced map

\[ h_* : K_*(C) \longrightarrow K_*(SA \otimes K) \cong K_{*-1}(A) \]

is an isomorphism.

2. Suspending, there is a a commutative \( C^* \)-algebra \( SC \) which is a direct sum of copies of \( C_0(\mathbb{R}^3) \) and \( C_0(\mathbb{R}^2) \) and a map

\[ h : SC \longrightarrow S^2 A \otimes K \]

such that the induced map

\[ h_* : K_*(SC) \longrightarrow K_*(S^2 A \otimes K) \cong K_*(A) \]

is an isomorphism.

Proof. For the first statement, take

\[ Sf \oplus g : ( \oplus_s C_0(\mathbb{R}^2)) \oplus ( \oplus_s C_0(\mathbb{R}^1)) \longrightarrow SA \otimes K. \]

For the second part, simply suspend.

\[ h = S^2 f \oplus Sg : ( \oplus_s C_0(\mathbb{R}^3)) \oplus ( \oplus_s C_0(\mathbb{R}^2)) \longrightarrow S^2 A \otimes K. \]

\( \square \)

Here is a restatement of the previous results couched in terms of \( \beta A \).
Theorem 11.4. Suppose given a $C^*$-algebra $A$ with $K_*(A)$ free abelian. Then there exists a $C^*$-algebra $\beta A$ with the following properties:

1. There is a map $h: \beta A \to S^2 A \otimes K$ which induces an isomorphism
   $$ h_*: K_*(\beta A) \cong K_*(A) $$
   so that $\beta A$ is weakly $K$-equivalent to $S^2 A \otimes K$.
2. If $A$ is separable (or, more generally, if $K_*(A)$ is countable) then $\beta A$ is separable.
3. $\beta A$ is commutative and is the direct sum of copies of $C_0(\mathbb{R}^3)$ and $C_0(\mathbb{R}^2)$.
4. If $K_*(A)$ is countable then $\beta A$ is in the bootstrap category.

Proof. Take $\beta A = SC$ as above. □

If $K_*(A)$ is not free abelian then our results are unfortunately not so neat. Here is what happens:

Theorem 11.5. Let $A$ be a $C^*$-algebra. Then there exists a $C^*$-algebra $\beta A$ with the following properties:

1. $K_*(\beta A) \cong K_*(A)$.
2. If $A$ is separable then $\beta A$ is separable.
3. If $K_*(A)$ is countable then $\beta A$ is in the bootstrap category.
4. $\beta A$ fits into a short exact sequence of the form
   $$ 0 \to C_0(X_1) \otimes K \to \beta A \to C_0(X_2) \otimes K \to 0 $$
   where $X_j$ consist of disjoint unions of lines, planes, and their suspensions. Thus $\beta A$ is a solvable $C^*$-algebra. If $K_*(A)$ is countable (resp. finitely generated) then the $X_j$ are disjoint unions of countable (resp. finite) number of components.
5. There exists an auxiliary $C^*$-algebra $T$ and maps
   $$ \beta A \xrightarrow{h} T \xleftarrow{j} S^3 A $$
   with the following properties:
   (a) The map $h$ is a weak $K$-equivalence.
   (b) The map $j$ is the inclusion of an ideal, and $T/S^3 A$ is a contractible $C^*$-algebra. In particular, $j$ is also a weak $K$-equivalence.

Remark 11.6. It is interesting to compare the properties of $\beta A$ with the properties of $\theta A$ in Theorem 10.6 under the assumption that $K_*(A)$ is countable. On the one hand, $\beta A$ is a better behaved approximation for $A$ than $\theta A$ because it is solvable and satisfies the UCT. On the other hand, the inclusion $\theta A \to A$ is a weak $K$-equivalence, whereas for $\beta A$ the best we can do is a sequence of $K$-equivalences
   $$ \beta A \xrightarrow{h} T \xleftarrow{j} S^3 A, $$
   one of which points in the wrong direction!

Proof. We may assume without loss of generality that $A$ is stable, i.e. $A \cong A \otimes K$. The case where $K_*(A)$ is free abelian is covered by the previous proposition. Consider the general case. There is a stably commutative $C^*$-algebra $N$ with $K_*(N)$ free abelian, and a map
   $$ f: N \to SA \otimes K $$
inducing a surjection
\[ K_*(N) \overset{f_\ast}{\to} K_*(SA) \to 0. \]

Form the mapping cone sequence
\[ 0 \to S^2A \to Cf \overset{\pi}{\to} N \to 0. \]

We may assume that \( Cf \) is stable. The associated \( K \)-theory sequence corresponds via the suspension isomorphism to the sequence
\[ 0 \to K_*(Cf) \to K_*(N) \overset{f_\ast}{\to} K_*(SA) \to 0. \]

Thus \( K_*(Cf) \) is free abelian, and the sequence above is a free resolution of \( K_*(SA) \).

Proposition 11.5 tells us that there is a stably commutative \( C^* \)-algebra \( M \) and a weak \( K \)-equivalence \( g : M \to SCf \) with associated mapping cone sequence
\[ 0 \to S^2Cf \to Cg \to M \to 0. \]

Note that \( K_*(Cg) = 0 \) since \( g \) is a weak \( K \)-equivalence, and hence there is a natural diagram
\[ \begin{array}{cccccc}
0 & \to & K_* (SCf) & \overset{(S\pi)_\ast \cdot g_\ast}{\to} & K_* (SN) & \overset{f_\ast}{\to} & K_* (S^2A) & \to & 0 \\
& & \uparrow \cong & & \uparrow 1 & & \uparrow 1 & & \\
0 & \to & K_* (M) & \overset{(S\pi)_\ast \cdot g_\ast}{\to} & K_* (SN) & \overset{f_\ast}{\to} & K_* (S^2A) & \to & 0
\end{array} \]

Now consider the composition
\[ M \xrightarrow{g} SCf \xrightarrow{S\pi} SN \]

and define the mapping cone of the composition by
\[ \beta A = C((S\pi)g). \]

The mapping cone sequence takes the form
\[ 0 \to S^2N \to \beta A \to M \to 0 \]

and fits into a natural diagram
\[ \begin{array}{cccccc}
0 & \to & S^2N & \to & \beta A & \to & M & \to & 0 \\
& & \downarrow 1 & & \downarrow h & & \downarrow g & & \\
0 & \to & S^2N & \to & C(S\pi) & \to & SCf & \to & 0
\end{array} \]

Applying \( K \)-theory to this diagram yields the following diagram, with exact rows:
\[ \begin{array}{cccccc}
\to K_* (\beta A) & \to & K_* (M) & \overset{(S\pi)_\ast \cdot g_\ast}{\to} & K_{*-1}(S^2\bar{N}) & \to & K_{*-1}(\beta A) & \to \\
\downarrow g_\ast & & \downarrow \cong & & \downarrow h_\ast & & \\
\to K_* (C(S\pi)) & \to & K_* (SCf) & \overset{(S\pi)_\ast}{\to} & K_{*-1}(S^2\bar{N}) & \to & K_{*-1}(C(S\pi)) & \to
\end{array} \]
The map $g_*$ is an isomorphism and the map $(S\pi)_*$ is mono, and so the diagram simplifies to the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & K_*(M) & \stackrel{(S\pi)_*g_*}{\longrightarrow} & K_{*-1}(S^2N) & \longrightarrow & K_{*-1}(\beta A) & \longrightarrow & 0 \\
\downarrow g_* & & \downarrow \cong & & \downarrow h_* & \\
0 & \longrightarrow & K_*(SCf) & \stackrel{(S\pi)_*}{\longrightarrow} & K_{*-1}(S^2N) & \longrightarrow & K_{*-1}(C(S\pi)) & \longrightarrow & 0
\end{array}
$$

The Five Lemma implies that the map

$$h_* : K_*(\beta A) \to K_*(C(S\pi))$$

is an isomorphism.

Recall that the map $\pi : Cf \to N$ fits into the sequence

$$0 \to S^2A \to Cf \stackrel{\pi}{\longrightarrow} N \to 0.$$ 

Suspending yields the exact sequence

$$0 \to S^3A \to SCf \stackrel{S\pi}{\longrightarrow} SN \to 0.$$ 

Since $S\pi$ is surjective, its cone sequence fits into the following diagram, by [37] Proposition 2.3,

$$
\begin{array}{cccccc}
0 & \longrightarrow & S^3A & \longrightarrow & SN & \longrightarrow & C(S\pi) & \longrightarrow & SCf & \longrightarrow & 0 \\
\downarrow & & \downarrow j & & \downarrow & & \downarrow & & \downarrow & & \downarrow 0 \\
0 & & CN & & 0
\end{array}
$$

where $CN$ denotes the cone on $N$, which is contractible. In particular, the natural map $j : S^3A \to C(S\pi)$ is a weak $K$-equivalence. Let $T = C(S\pi)$ for brevity.

To summarize, we have constructed $C^*$-maps

$$S^3A \stackrel{j}{\longrightarrow} T \stackrel{h}{\longrightarrow} \beta A$$

which are both weak $K$-equivalences. The $C^*$-algebra $\beta A$ fits in a sequence of the form

$$0 \to C_0(X_1) \otimes K \to \beta A \to C_0(X_2) \otimes K \to 0$$

and is hence two-step solvable. This completes the proof. □

**Remark 11.7.** If $K_*(A)$ is countable then $\beta A$ may be chosen to be in the bootstrap category, and then the UCT implies that any two choices will be $KK$-equivalent. If $K_*(A)$ is free abelian then its maximal ideal space is uniquely determined up to homeomorphism, simply by counting components.
References

[1] J. F. Adams, Stable homotopy and generalized homology. Reprint of the 1974 original. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995. x+373 pp.
[2] M. F. Atiyah, Thom complexes, Proc. London Math. Soc. (3) 11 (1961), 291-310.
[3] J. M. Boardman, Stable homotopy theory is not self-dual, Proc. Amer. Math. Soc. 26 1970 369-370
[4] J.C. Becker and D.H. Gottlieb, A History of duality in algebraic topology, History of topology, 725-745, North-Holland, Amsterdam, 1999.
[5] B. Blackadar, K-Theory for Operator Algebras, 2nd edition, MSRI Publ. 5. Cambridge University Press, Cambridge, 1998. xx+300 pp.
[6] L. G. Brown, R.G. Douglas, and P.A. Fillmore, Extensions of C*-algebras and K-homology, Ann. of Math. (2) 105:2 (1977), 265-324.
[7] R. Bowen and C. Series, Markov maps associated with Fuchsian groups, Inst. Hautes Etudes Sci. Publ. Math. No. 50 (1979), 153-170.
[8] R. Busby, Double centralizers and extensions of C*-algebras, Trans. Amer. Math. Soc. 132 (1968), 79-99.
[9] A. Connes, Noncommutative Geometry, Academic Press, Inc., San Diego, CA, 1994. xiv+661 pp.
[10] A. Connes, Gravity coupled with matter and the foundation of non-commutative geometry, Comm. Math. Phys. 182 (1996), no. 1, 155-176.
[11] A. Connes, J. Feldman, and B. Weiss, An amenable equivalence relation is generated by a single transformation, Ergodic Theory Dynamical Systems 1 (1981), no. 4, 431-450 (1982).
[12] A. Connes and G. Skandalis, The longitudinal index theorem for foliations, Publ. Res. Inst. Math. Sci. 20:6 (1984), 1139-1183.
[13] M. Dadarlat and T. Loring, A universal multicoefficient theorem for the Kasparov groups, Duke Math. J. 84 (1996), 355-377.
[14] C. Debord and J.-M. Lescure, K-duality for pseudomansfolds with isolated singularities, Journal of Functional Analysis 219 (2005), 109-133.
[15] C. Debord and J.-M. Lescure, Index theory and groupoids, Geometric and topological methods for quantum field theory, 86-158, Cambridge Univ. Press, Cambridge, 2010.
[16] P. de la Harpe, Groupes hyperboliques, algèbres d’opérateurs et un théorème de Jolissaint, C.R. Acad. Sc. Paris, Série I 307 (1988), 771-774.
[17] H. Emerson, Noncommutative Poincaré duality for boundary actions of hyperbolic groups, J. Reine Angew. Math. 564 (2003), 1-33.
[18] H. Emerson, Lefschetz numbers for C*-algebras, Canad. Math. Bull 54 (2011), 82-99.
[19] N. Higson and J. Roe, Analytic K-homology, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. xviii+405 pp.
[20] D.S. Kahn, J. Kaminker, and C. Schochet, Generalized homology on compact metric spaces, Michigan Math J. 24 (1977), 203-224.
[21] J. Kaminker and I. Putnam, K-theoretic Duality for Shifts of Finite Type, Comm. Math. Phys. 187 (1997), 509-522.
[22] J. Kaminker, I. Putnam, and M.F. Whittaker, K-Theoretic Duality for Hyperbolic Dynamical Systems, J. Reine Angew. Math. , DOI 10.1515/crelle-2014-0126.
[23] G. Kasparov, Equivariant KK-theory and the Novikov conjecture, Invent. Math. 91 (1988), 147-201.
[24] M. Laca and J. Spielberg, Purely infinite C*-algebras from boundary actions of discrete groups, J. Reine Angew. Math. 480 (1996), 125-139. 46L55 (22D25 46L05 46L80).
[25] E. C. Lance, On Nuclear C*-algebras, J. Functional Analysis 12 (1973), 157-176.
[26] G. Lusztig, Novikov’s higher signature and families of elliptic operators, J. Diff. Geom. 7 (1972), 229-256
[27] R. Meyer, Categorical aspects of bivariant K-theory, K-theory and noncommutative geometry, 139, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008.
[28] R. Meyer and R. Nest, Homological algebra in bivariant K-theory and other triangulated categories, I. (English summary) Triangulated categories, 236-289, London Math. Soc. Lecture Note Ser., 375, Cambridge Univ. Press, Cambridge, 2010.
[29] I. Mineyev and G. Yu, The Baum-Connes conjecture for hyperbolic groups, (English summary) Invent. Math. 149 (2002), no. 1, 97-122.
[30] C.C. Moore and C. Schochet, *Global Analysis on Foliated Spaces*, Math. Sci. Res. Inst. Publ. 9, Second Edition, Cambridge Univ. Press, 2005, 293 pages.

[31] S. Mukai, *Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves*, Nagoya Math. J. 81 (1981), 153-175.

[32] V. V. Nekrashevych, *Hyperbolic groupoids and duality*, (English summary) Mem. Amer. Math. Soc. 237 (2015), no. 1122, v+105 pp.

[33] W. L. Paschke, *K-theory for commutants in the Calkin algebra*, Pacific J. Math 95 (1981), 427-434.

[34] J. Rosenberg and C. Schochet, *The Künneth Theorem and the Universal coefficient Theorem for Kasparov’s Generalized $K$-functor*, Duke Math J., 55 (1987), 431-474.

[35] C. Schochet, *The Kasparov groups for commutative $C^*$-algebras and Spanier-Whitehead duality*, Operator theory: operator algebras and applications, Part 2 (Durham, NH, 1988), 307321, Proc. Sympos. Pure Math., 51, Part 2, Amer. Math. Soc., Providence, RI, (1990).

[36] C. Schochet, *Topological methods for $C^*$-algebras II: geometric resolutions and the Künneth formula*, Pacific J. Math. 98 (1982), 443-458.

[37] C. Schochet, *Topological methods for $C^*$-algebras III: axiomatic homology*, Pacific J. Math. 99 (1983), 459-483.

[38] C. Schochet, *Topological methods for $C^*$-algebras IV: mod $p$ homology*, Pacific J. Math. 114 (1984), 447-468.

[39] C. Schochet, *The UCT, the Milnor Sequence, and a canonical decomposition of the Kasparov Groups*, K-theory 10 (1996), 49 -72.

[40] C. Schochet, *A Peix primer: pure extensions and lim$^1$ for infinite abelian groups*, (English summary) NYJM Monographs, 1. State University of New York, University at Albany, Albany, NY, 2003. ii+67 pp. The book is available electronically at http://nyjm.albany.edu:8000/m/indexr.htm.

[41] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[42] E.H. Spanier and J.H.C. Whitehead, *Duality in homotopy theory*, Mathematika 2 1955, 56-80.

[43] E.H. Spanier and J.H.C. Whitehead, *Duality in relative homotopy theory*, Ann. of Math. (2) 67 1958, 203-238

[44] J. Spielberg, *Cuntz-Krieger algebras associated with Fuchsian groups*, (English summary) Ergodic Theory Dynam. Systems 13 (1993), no. 3, 581-595.

[45] N. Steenrod, *Cohomology operations, and obstructions to extending continuous functions*, Advances in Math. 8, 371416. (1972).

[46] D. V. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Rev. Roumaine Math. Pures Appl. 21 (1976), no. 1, 97-113.

[47] J. Wiegold, *Ext$(\mathbb{Q},\mathbb{Z})$ is the additive group of real numbers*, Bull. Austral. Math. Soc. 1 (1969), 341-343.

[48] G. W. Whitehead, *Generalized homotopy theories*, Trans. Amer. Math. Soc. 102 1962, 227-283.

[49] G. W. Whitehead, *Elements of Homotopy Theory*, Springer Verlag, New York, 1978.

Department of Mathematics, University of California, Davis, Davis, CA 95616

E-mail address: kaminker@math.ucdavis.edu

Department of Mathematics, Technion, Haifa 32000, Israel

E-mail address: clsmath@gmail.com