KOROVKIN-TYPE APPROXIMATION OF SET-VALUED AND VECTOR-VALUED FUNCTIONS

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Abstract. We establish some general Korovkin-type results in cones of set-valued functions and in spaces of vector-valued functions. These results constitute a meaningful extension of the preceding ones.

1. Introduction and notation. After the pioneering papers by Keimel and Roth [11, 12], many Korovkin-type results have been obtained both in spaces of vector-valued and set-valued functions.

In [4, 5], the notion of convexity monotone operator has been used in order to extend different characterizations of Korovkin subcones and subspaces. A more recent and detailed analysis has been performed in [9] even in the case where the limit operator is not assigned.

In this paper we continue the study of the approximation of set-valued operators and establish some more general results which improve those obtained in [9].

We begin by introducing some notation.

We shall denote by $E$ a Fréchet space and by $B$ a base of open convex neighborhoods of 0 in $E$. Moreover, we shall denote by $\mathcal{K}(E)$ the cone of all non empty compact convex subsets of $E$ endowed with the natural addition and multiplication by positive scalars.

If $X$ is a Hausdorff topological space, we shall deal with the cone $C(X, \mathcal{K}(E))$ of all continuous set-valued functions; namely, $f \in C(X, \mathcal{K}(E))$ if and only if for every $x_0 \in X$ and $V \in B$, there exists a neighborhood $U$ of $x_0$ such that $f(x) \subset f(x_0) + V$ and $f(x_0) \subset f(x) + V$ whenever $x \in U$.

The space $C(X, \mathcal{K}(E))$ is naturally ordered by inclusion, that is

$$f \leq g \iff \forall x \in X : f(x) \subset g(x),$$

whenever $f, g \in C(X, \mathcal{K}(E))$.

An operator $L : C(X, \mathcal{K}(E)) \to C(X, \mathcal{K}(E))$ is called linear if preserves addition and multiplication by positive scalars and is called monotone if it preserves inclusions.

If $\varphi \in C(X, E)$, it will be useful to denote by $\{\varphi\}$ the set-valued function (in $C(X, \mathcal{K}(E))$) defined by setting $\{\varphi\}(x) = \{\varphi(x)\}$ for every $x \in X$.

2020 Mathematics Subject Classification. Primary: 41A65, 41A36; Secondary: 41A25, 41A63.
Key words and phrases. Korovkin approximation, approximation of vector-valued functions, approximation of set-valued functions, convexity-monotone operators, linear positive continuous operators.

Work performed under the auspices of G.N.A.M.P.A. (I.N.d.A.M.) and the UMI Group TAA “Approximation Theory and Applications”.

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Moreover if $\varphi_1, \ldots, \varphi_n \in C(X, E)$ are set-valued functions on $X$, we can consider the set-valued function $\text{co}(\varphi_1, \ldots, \varphi_n)$ defined by

$$\text{co}(\varphi_1, \ldots, \varphi_n)(x) = \text{co}(\varphi_1(x), \ldots, \varphi_n(x)), \quad x \in X,$$

where $\text{co}(\varphi_1(x), \ldots, \varphi_n(x))$ denotes the convex hull of $\varphi_1(x), \ldots, \varphi_n(x)$.

Moreover, if $f \in C(X, \mathcal{K}(E))$, we shall denote by $\text{Sel}(f)$ the convex subset of $C(X, \mathcal{K}(E))$ consisting of all continuous selections of $f$.

It is well-known that (see e.g. [11, Lemma 4.1. and Theorem 3.2] or [3, Proposition 1.1])

$$f = \bigcup_{\varphi \in \text{Sel}(f)} \{\varphi\}$$

for every $f \in C(X, \mathcal{K}(E))$, that is $f(x) = \bigcup_{\varphi \in \text{Sel}(f)} \{\varphi(x)\}$ for every $x \in X$.

We recall that a subset $M$ of $C(X, E)$ is called a Korovkin system in $C(X, E)$ if for every equicontinuous net $(T_i)_{i \in I}$ of positive linear operators from $C(X, E)$ into itself satisfying $\lim_{i \in I}^\leq T_i(\varphi) = \varphi$ for every $\varphi \in M$, we also have $\lim_{i \in I}^\leq T_i(\psi) = \psi$ for every $\psi \in C(X, E)$.

If the limit operator is not the identity operator but a linear positive operator $T : C(X, E) \to C(X, E)$, we shall say that $M$ is called a $T$-Korovkin system (or a Korovkin system for $T$) in $C(X, E)$.

In the space $C(X, \mathcal{K}(E))$ we have a similar definition (see [11, 3]). Namely, let $\mathcal{C}$ be a subcone of $C(X, \mathcal{K}(E))$; a subset $H$ of $\mathcal{C}$ is called a Korovkin system in $\mathcal{C}$ if for every equicontinuous net $(L_i)_{i \in I}^{\leq}$ of linear monotone operators from $\mathcal{C}$ into itself satifying $\lim_{i \in I}^\leq L_i(h) = h$ for every $h \in H$, we also have $\lim_{i \in I}^\leq L_i(f) = f$ for every $f \in H$.

Also in this case we shall call $H$ a $L$-Korovkin system (or a Korovkin system for $L$) in $\mathcal{C}$ in the case where the limit operator is not the identity operator but a continuous linear monotone operator $L : \mathcal{C} \to \mathcal{C}$.

\section{Korovkin approximation of set-valued functions.} While the existence of Korovkin systems in $C(X, \mathbb{R}^d)$ (and in particular in $C(X, \mathbb{R})$) has been widely studied (a complete treatment can be found in [1, 2]), we do not have a complete analysis of Korovkin systems in $C(X, \mathcal{K}(E))$ and in $C(X, E)$.

In this section we give some contribution to this topic.

First, we have to point out an extension of Korovkin systems to the case of set-valued functions obtained by Keimel and Roth in [11]; their result shows in particular that the functions

$$x \mapsto B, \quad x \mapsto x \cdot B, \quad x \mapsto x^2 \cdot B$$

constitute a Korovkin system in $C(X, \mathcal{K}(\mathbb{R}^d))$, where $B$ denotes the closed unit ball of center 0 in $\mathbb{R}^d$.

In [3] it was obtained a refinement of the result by Keimel and Roth in terms of upper and lower envelopes.

In [4] it has been established a more meaningful extension of the preceding result in a more general setting and even in the case where the limit operator is not necessarily the identity operator.

The following extension is the starting point of our analysis and therefore it may be useful to recall it explicitly [4, Theorem 2.4 and Corollary 2.5].
**Theorem 2.1.** ([4, Theorem 2.4]) Let $X$ be a compact Hausdorff topological space, $C$ a subcone of $C(X, K(\mathbb{R}^d))$ containing the single-valued functions and $L : C \to C$ a continuous monotone linear operator satisfying the following conditions

a) For every $\varphi \in C(X, E)$, $L(\{\varphi\})$ is single-valued;  
b) For every $f \in C$ and $x \in X$:

$$L(f)(x) = \bigcup_{\varphi \in \text{Sel}(f)} L(\{\varphi\})(x).$$

Let $H$ be a subset of $C$ such that, for every $f \in C$, $x_0 \in X$ and $V \in B$, there exists $h \in H$ satisfying the following conditions

$$f \leq h, \quad L(h)(x_0) \subset L(f)(x_0) + V.$$

Then $H$ is a Korovkin system for $L$ in $C$.

As observed in [4, Remark 2.6], if $H$ contains the constant set-valued functions, then condition (1) can be weakened with the following condition

$$f \leq h + V, \quad L(h)(x_0) \subset L(f)(x_0) + V.$$  

If $X$ is a compact metric space, we can obtain the following result, which has been established in [9, Theorem 2.4] only in the particular case of the identity operator.

Here, we obtain a more general result with respect to a continuous monotone linear operator $L$ satisfying the conditions a) and b) in Theorem 2.1 and suitable further conditions.

We need to consider the setting of finite dimensional spaces.

**Theorem 2.2.** Let $(X, \sigma)$ be a compact metric space, $C$ a subcone of $C(X, K(\mathbb{R}^d))$ containing the single-valued functions and $L : C \to C$ a continuous monotone linear operator satisfying the following conditions

a) For every $\varphi \in C(X, \mathbb{R}^d)$, $L(\{\varphi\})$ is single-valued;  
b) For every $f \in C$ and $x \in X$:

$$L(f)(x) = \bigcup_{\varphi \in \text{Sel}(f)} L(\{\varphi\})(x).$$

c) $L$ preserves the constant set-valued functions, i.e. for every $K \in K(\mathbb{R}^d)$, if we denote by $K$ the constant set-valued function of constant value $K$, then $K \in C$ and $L(K) = K$.

d) If $x_0 \in X$ then $x \mapsto \sigma(x, x_0) \cdot B \in C$ and

$$L(\sigma(x, x_0) \cdot B)(x_0) = \{0\}.$$  

Let $H$ be a subset of $C$ which contains the constant set-valued functions and the set-valued functions

$$x \mapsto \sigma(x, x_0) \cdot B$$

for every $x_0 \in X$.

Then $H$ is a Korovkin system for $L$ in $C$.

**Proof.** Let $f \in C$ and $x_0 \in X$. Since $f$ is Hausdorff continuous and $X$ is compact, we can find $M > 0$ such that, for every $x \in X$, $f(x) \subset f(x_0) + M \cdot B$.

Now, let $\varepsilon > 0$; from the continuity of $f$, we can find $\delta > 0$ such that, for every $x \in X$ satisfying $\sigma(x, x_0) \leq \delta$, we have $f(x) \subset f(x_0) + \frac{\varepsilon}{2} \cdot B$. Now, consider the function $h : X \to K(\mathbb{R}^d)$ defined by setting, for every $x \in X$,

$$h(x) := f(x_0) + \frac{\varepsilon}{2} \cdot B + M \frac{\sigma(x, x_0)}{\delta} \cdot B,$$
The function $h$ is obviously continuous and our assumptions ensure that $h \in H \subset C$.

Now, we show that $f \leq h$. Indeed, if $\sigma(x, x_0) \leq \delta$ we have

$$f(x) \subset f(x_0) + \frac{\varepsilon}{2} \cdot B \subset h(x)$$

and similarly, if $\sigma(x, x_0) > \delta$, we have

$$f(x) \subset f(x_0) + M \cdot B \subset f(x_0) + \varepsilon \cdot B + M \cdot \frac{\sigma(x, x_0)}{\delta} \cdot B = h(x).$$

Finally our assumptions on $H$ and conditions c) and d) ensure that

$$L(h)(x_0) := L(f(x_0))(x_0) + \frac{\varepsilon}{2} \cdot L(B)(x_0) + L\left(\frac{\sigma(x, x_0)}{\delta} \cdot B\right)(x_0)$$

$$= f(x_0) + \frac{\varepsilon}{2} \cdot B$$

and hence the subset $H$ satisfies the assumptions in Theorem 2.1 and therefore is a Korovkin system for $L$ in $C$. \[\Box\]

Obviously, the identity operator $I : C \to C$ satisfies conditions a)-d) of Theorem 2.1.

The proof of the following result is similar by considering the function $k : X \to K(\mathbb{R}^d)$ defined by setting, for every $x \in X$,

$$k(x) := f(x_0) + \frac{\varepsilon}{2} \cdot B + \frac{\sigma(x, x_0)^2}{\delta^2} \cdot B$$

in place of $h$. We omit the details for the sake of brevity.

**Theorem 2.3.** Let $(X, \sigma)$ be a compact metric space, $C$ a subcone of $C(X, K(\mathbb{R}^d))$ containing the single-valued functions and $L : C \to C$ a continuous monotone linear operator satisfying the following conditions

a) For every $\varphi \in C(X, \mathbb{R}^d)$, $L(\{\varphi\})$ is single-valued;

b) For every $f \in C$ and $x \in X$:

$$L(f)(x) = \bigcup_{\varphi \in \text{Sel}(f)} L(\{\varphi\})(x).$$

c) $L$ preserves the constant set-valued functions.

d) If $x_0 \in X$ then $x \mapsto \sigma(x, x_0)^2 \cdot B \in C$ and

$$L(\sigma(x, x_0)^2 \cdot B)(x_0) = \{0\}.$$

Let $H$ be a subset of $C$ which contains the constant set-valued functions and the set-valued functions

$$x \mapsto \sigma(x, x_0)^2 \cdot B$$

for every $x_0 \in X$.

Then $H$ is a Korovkin system for $L$ in $C$.

We can state the following consequences of the preceding Theorems 2.2 and 2.3 in the case where $X$ is a compact subset of $\mathbb{R}$.

**Corollary 1.** Let $X$ be a compact subset of $\mathbb{R}$, $C$ a subcone of $C(X, K(\mathbb{R}^d))$ containing the single-valued functions and $L : C \to C$ a continuous monotone linear operator satisfying the following conditions

a) For every $\varphi \in C(X, \mathbb{R}^d)$, $L(\{\varphi\})$ is single-valued;
b) For every \( f \in \mathcal{C} \) and \( x \in X \):
\[
L(f)(x) = \bigcup_{\varphi \in \text{Sel}(f)} L(\{\varphi\})(x).
\]

c) \( L \) preserves the constant set-valued functions.

d) If \( x_0 \in X \) then \( x \mapsto |x - x_0| \cdot B \in \mathcal{C} \) and
\[
L(|x - x_0| \cdot B)(x_0) = \{0\}.
\]
(or alternatively, \( x \mapsto (x - x_0)^2 \cdot B \in \mathcal{C} \) and
\[
L((x - x_0)^2 \cdot B)(x_0) = \{0\}.
\]

Let \( H \) be a subset of \( \mathcal{C} \) which contains the constant set-valued functions and the set-valued functions \( x \mapsto |x - x_0| \cdot B \) (or alternatively, \( x \mapsto (x - x_0)^2 \cdot B \)) for every \( x_0 \in X \).

Then \( H \) is a Korovkin system for \( L \) in \( \mathcal{C} \).

In particular, if \( X = [0, 1] \) and \( d = 1 \) and if \( L : C([0, 1], \mathcal{K}(\mathbb{R})) \rightarrow C([0, 1], \mathcal{K}(\mathbb{R})) \) is a continuous monotone linear operator satisfying conditions a)-d) of Corollary 1, then the subcone \( H \) of \( C([0, 1], \mathcal{K}(\mathbb{R})) \) containing the constant set-valued functions and the set-valued functions
\[
x \mapsto |x - x_0| \cdot [-1, 1],
\]
for every \( x_0 \in [0, 1] \), is a Korovkin system for \( L \) in \( C([0, 1], \mathcal{K}(\mathbb{R})) \).

In the alternative formulation, we have to require that the functions
\[
x \mapsto (x - x_0)^2 \cdot [-1, 1],
\]
belong to \( H \). Hence we find the classical Korovkin system in \( C([0, 1], \mathcal{K}(\mathbb{R})) \) consisting of the functions
\[
x \mapsto B, \quad x \mapsto x \cdot B, \quad x \mapsto x^2 \cdot B.
\]

The following result has been established in [10, Theorem 2.6] only in the setting of finite dimensional spaces (in the case of the identity operator see also [7, Theorem 1.3]). Here we consider the general setting of Fréchet spaces.

**Theorem 2.4.** Let \( X \) be a compact Hausdorff topological space, \( \mathcal{C} \) a subcone of \( C(X, \mathcal{K}(\mathbb{E})) \) containing the set
\[
C(X, \mathbb{R}) \otimes \mathcal{K}(\mathbb{E}) := \{\varphi \cdot A \mid \varphi \in C(X, \mathbb{R}), \ A \in \mathcal{K}(\mathbb{E})\}
\]
and \( L : \mathcal{C} \rightarrow \mathcal{C} \) a continuous monotone linear operator such that
a) For every \( \varphi \in C(X, \mathbb{E}) \), \( L(\{\varphi\}) \) is single-valued;

b) For every \( f \in \mathcal{C} \) and \( x \in X \):
\[
L(f)(x) = \bigcup_{\varphi \in \text{Sel}(f)} L(\{\varphi\})(x).
\]

If \( H \) is a subset of \( \mathcal{C} \) such that, for every \( f \in \mathcal{C} \), \( x_0 \in X \) and \( V \in \mathcal{B} \), there exist \( h_1, \ldots, h_m \in H \) such that
\[
f \leq h_j, \ j = 1, \ldots, m, \quad \bigcap_{j=1}^m L(h_j)(x_0) \subseteq L(f)(x_0) + V,
\]
then \( H \) is a Korovkin set for \( L \) in \( \mathcal{C} \).
Proof. Let \((L_i)_{i \in I}\) be an equicontinuous net of monotone linear operators such that \(\lim_{i \in I} L_i(h) = L(h)\) for every \(h \in H\). Let \(f \in C\).

**First step.** Assume that \(f = \varphi\) with \(\varphi \in C(X,E)\). Let \(V \in \mathcal{B}\) and \(x_0 \in X\). The assumptions on \(H\) ensure the existence of \(h_1, \ldots, h_m \in H\) such that

\[
f \leq h_j, \ j = 1, \ldots, m, \quad \bigcap_{j=1}^m L(h_j)(x_0) \subset L(f)(x_0) + \frac{1}{4} V.
\]

Since \(L(f)\) and each \(L(h_j), j = 1, \ldots, m\) are continuous at \(x_0\), there exists a neighborhood \(U\) of \(x_0\) such that, for every \(x \in U\),

\[
L(f)(x_0) \subset L(f)(x) + \frac{1}{4} V, \quad L(h_j)(x) \subset L(h_j)(x_0) + \frac{1}{4} V, \quad j = 1, \ldots, m.
\]

Hence, for every \(x \in U\), we have

\[
\bigcap_{j=1}^m L(h_j)(x) \subset \bigcap_{j=1}^m L(h_j)(x_0) + \frac{1}{4} V \subset L(f)(x_0) + \frac{1}{2} V \subset L(f)(x) + \frac{3}{4} V.
\]

Since \(\lim_{i \in I} L_i(h_j) = L(h_j)\) for every \(j = 1, \ldots, m\), there exists \(\alpha \in I\) such that, for every \(i \in I, i \geq \alpha, j = 1, \ldots, m\) and \(x \in X\),

\[
L_i(h_j)(x) \subset L(h_j)(x) + \frac{1}{4} V, \quad L(h_j)(x) \subset L_i(h_j)(x) + \frac{1}{4} V.
\]

It follows, for every \(i \in I, i \geq \alpha, j = 1, \ldots, m\) and \(x \in U\),

\[
L_i(f)(x) \subset L_i(h_j)(x) \subset L(h_j)(x) + \frac{1}{4} V,
\]

and hence

\[
L_i(f)(x) \subset \bigcap_{j=1}^m L(h_j)(x) + \frac{1}{4} V \subset L(f)(x) + V.
\]

Since \(X\) is compact, we can deduce the existence of \(\beta \in I\) such that \(L_\beta(f)(x) \subset L(f)(x) + V\) for every \(i \in I, i \geq \beta\). Since \(f\) is single-valued we have also \(L(f)(x) \subset L_i(f)(x) + V\) for every \(i \in I, i \geq \beta\) and the proof is complete in this case.

**Second step.** Assume that \(f = \varphi \cdot A\) with \(\varphi \in C(X,\mathbb{R})\) and \(A \in \mathcal{K}(E)\). Let \(V \in \mathcal{B}\) and \(x_0 \in X\). Since \(A\) is compact, there exist \(y_1, \ldots, y_p \in A\) such that

\[
f(x_0) = \varphi(x_0) \cdot A \subset \bigcup_{s=1}^p \varphi(x_0) \cdot \left(\{y_s\} + \frac{1}{4} V\right)
\]

for every \(s = 1, \ldots, p\), we consider the set-valued function \(g_s = \{\varphi \cdot y_s\}\) which satisfies \(g_s \leq f\).

From the assumptions on \(H\), there exist \(h_1, \ldots, h_m \in H\) such that

\[
f \leq h_j, \ j = 1, \ldots, m, \quad \bigcap_{j=1}^m L(h_j)(x_0) \subset L(f)(x_0) + \frac{1}{4} V.
\]

Since \(L(f)\) and each \(L(g_s), s = 1, \ldots, p\) and \(L(h_j), j = 1, \ldots, m\) are continuous at \(x_0\), we can apply the same argument of the first step and obtain a neighborhood \(U\) of \(x_0\) such that, for every \(x \in U\),

\[
L(f)(x) \subset \bigcup_{s=1}^p L(g_s)(x) + \frac{3}{4} V, \quad \bigcap_{j=1}^m L(h_j)(x) \subset L(f)(x) + \frac{3}{4} V.
\]
For every \( s = 1, \ldots, p \), the function \( g_s \) is single-valued and therefore the net \((L_i(g_s))_{i \in I}\) converges to \( L(g_s) \). Moreover \( \lim_{i \in I} L_i(h_j) = L(h_j) \) for every \( j = 1, \ldots, m \) and therefore we can find \( \alpha \in I \) such that, for every \( i \in I, i \geq \alpha, s = 1, \ldots, p, j = 1, \ldots, m \) and \( x \in X \),
\[
L_i(h_j)(x) \leq L(h_j)(x) + \frac{1}{4} V, \quad L(g_s)(x) \leq L_i(g_s)(x) + \frac{1}{4} V \subset L_i(f)(x) + \frac{1}{4} V.
\]
It follows, for every \( i \in I, i \geq \alpha, j = 1, \ldots, m \) and \( x \in U \),
\[
L(f)(x) \subset \bigcup_{s=1}^{p} L(g_s)(x) + \frac{3}{4} V \subset L_i(f)(x) + V.
\]
Similarly, since \( L_i(f)(x) \subset L_i(h_j)(x) \subset L(h_j) + \frac{3}{4} V \), we have
\[
L_i(f)(x) \subset \bigcap_{j=1}^{m} L(h_j)(x) + \frac{1}{4} V \subset L(f)(x) + V.
\]
Since \( X \) is compact, we can conclude the proof as in the first step.

**Third step.** Let \( f \in C \). From [7, Lemma 1.2], for every \( \varepsilon > 0 \) we can find \( \varphi_1, \ldots, \varphi_p \in C(X, \mathbb{R}) \) and \( A_1, \ldots, A_p \in \mathcal{K}(\mathbb{R}^d) \) such that
\[
f \leq \sum_{s=1}^{p} \varphi_s \cdot A_s + V, \quad \sum_{s=1}^{p} \varphi_s \cdot A_s + V \leq f,
\]
and from the second step we easily obtain the convergence of \((L_i(f))_{i \in I}\) to \( L(f) \) also in this case. \( \square \)

### 3. Korovkin approximation of vector-valued functions

The results in the preceding Section 2 have some interesting applications concerning the Korovkin approximation of single vector-valued functions.

In this section we consider a suitable concept of monotonicity and establish different Korovkin-type results for these operators.

Let \( X \) be a compact Hausdorff topological space. Let \( T : C(X, E) \to C(X, E) \) and \( L : C(X, \mathcal{K}(E)) \to C(X, \mathcal{K}(E)) \) be continuous linear operators.

It is easy to show that we can associate a continuous linear operator \( L_T : C(X, \mathcal{K}(E)) \to C(X, \mathcal{K}(E)) \) to \( T \) and a continuous linear operator \( T_L : C(X, E) \to C(X, E) \) to \( L \). This result has been established in [9, Lemma 2.3] only in the case \( E = [0, 1] \) and \( E = \mathbb{R}^d \).

More precisely, if \( L : C(X, \mathcal{K}(E)) \to C(X, \mathcal{K}(E)) \) is a monotone continuous linear operator which satisfies conditions a) and b) of Theorem 2.1, we can consider the operator \( T_L : C(X, E) \to C(X, E) \) defined by setting
\[
T_L(\varphi) := L(\varphi), \quad \varphi \in C(X, E).
\]

The operator \( T_L \) is well-defined since \( L \) maps single-valued functions into single-valued functions. Moreover, \( T_L \) is clearly linear and continuous and since \( L \) is also monotone, the operator \( T_L \) satisfies the following condition: for every \( f, g \in C(X, \mathcal{K}(E)) \)
\[
f \leq g \implies \forall x \in X : T_L(f)(x) \subset T_L(g)(x).
\]

We shall refer to condition 4 as the monotonicity property of the operator \( T_L \).
Conversely, let \( T : C(X, E) \to C(X, E) \) be a continuous linear operator satisfying (4) and define the operator \( L_T : C(X, \mathcal{K}(E)) \to C(X, \mathcal{K}(E)) \) by setting, for every \( f \in C(X, \mathcal{K}(E)) \) and \( x \in X \),

\[
L_T(f)(x) = \bigcup_{\varphi \in \text{Sel}(f)} \{ L(\varphi)(x) \} .
\] (5)

Then \( L_T \) is a continuous monotone linear operator which satisfies conditions a) and b) of Theorem 2.1.

At this point, the results obtained in the preceding Section 2 can be used to study the Korovkin approximation of vector-valued continuous functions.

**Theorem 3.1.** Let \( (X, \sigma) \) be a compact metric space and \( T : C(X, \mathbb{R}^d) \to C(X, \mathbb{R}^d) \) a continuous monotone linear operator satisfying the following conditions

i) \( T \) preserves the constant vector-valued functions, i.e. for every \( y \in \mathbb{R}^d \), if we denote by \( y \) the constant vector-valued function of constant value \( y \), then

\( L(y) = y \).

ii) If \( x_0 \in X \) and \( y \in \mathbb{R}^d \), then

\[ T(\sigma(x, x_0) \cdot y)(x_0) = 0. \]

Let \( M \) be a subset of \( C(X, \mathbb{R}^d) \) which contains the constant vector-valued functions and the functions

\[ x \mapsto \sigma(x, x_0) \cdot y \]

for every \( x_0 \in X \) and \( y \in \mathbb{R}^d \).

Then \( M \) is a Korovkin system for \( L \) in \( C(X, \mathbb{R}^d) \).

**Proof.** Consider the operator \( L_T : C(X, \mathcal{K}(\mathbb{R}^d)) \to C(X, \mathcal{K}(\mathbb{R}^d)) \) defined by (5). We have already observed that \( L_T \) is a monotone continuous linear operator satisfying conditions a) and b) of Theorem 2.2. Moreover assumption i) ensures that \( L_T \) maps single-valued functions into single-valued functions and therefore also condition c) of Theorem 2.2 is satisfied.

Finally, let \( x_0 \in X \) and consider the set-valued function \( f : C(X, \mathcal{K}(\mathbb{R}^d)) \to C(X, \mathcal{K}(\mathbb{R}^d)) \) defined by setting, for every \( x \in X \), \( f(x) := \sigma(x, x_0) \cdot \mathbf{B} \). If \( \varphi \in \text{Sel}(f) \) then the monotonicity of \( L_T \) ensures that

\[
L_T(\{ \varphi \}) \leq L \left( \bigcup_{y \in \mathbf{B}} \{ \sigma(x, x_0) \cdot y \} \right)
\]

and therefore \( L_T(\{ \varphi \})(x_0) = \{ 0 \} \). Hence from (5) we have also \( L(\sigma(x, x_0) \cdot \mathbf{B})(x_0) = \{ 0 \} \) and condition d) in Theorem 2.2 is also satisfied.

It follows that the subset \( H \) of \( C(X, \mathcal{K}(\mathbb{R}^d)) \) consisting of the constant set-valued functions and the set-valued functions \( x \mapsto \sigma(x, x_0) \cdot \mathbf{B} \), \( x_0 \in X \), is a Korovkin system for \( L_T \) in \( C(X, \mathcal{K}(\mathbb{R}^d)) \). Since \( M \) consists of all single-valued functions in \( H \), we conclude that \( M \) is a Korovkin system for \( T \) in \( C(X, \mathbb{R}^d) \).

Also in this case condition ii) in the preceding Theorem 3.1 can be formulated as follows

ii)' If \( x_0 \in X \) and \( y \in \mathbb{R}^d \), then

\[ T(\sigma(x, x_0)^2 \cdot y)(x_0) = 0. \]

We have also the following consequences in the case where \( X \) is a compact subset of \( \mathbb{R} \).
Corollary 2. Let $X$ be a compact subset of $\mathbb{R}$ and $T : C(X, \mathbb{R}^d) \to C(X, \mathbb{R}^d)$ a continuous monotone linear operator satisfying the following conditions

i) $T$ preserves the constant vector-valued functions, i.e. for every $y \in \mathbb{R}^d$, if we denote by $y$ the constant vector-valued function of constant value $y$, then $L(y) = y$.

ii) If $x_0 \in X$ and $y \in \mathbb{R}^d$, then

$$T((|x - x_0| \cdot y)(x_0) = 0 \quad \text{or alternatively, } T((x - x_0)^2 \cdot y)(x_0) = 0).$$

Let $M$ be a subset of $C(X, \mathbb{R}^d)$ which contains the constant vector-valued functions and the functions

$$x \mapsto |x - x_0| \cdot y \quad \text{or alternatively, } x \mapsto (x - x_0)^2 \cdot y.$$ 

for every $x_0 \in X$ and $y \in \mathbb{R}^d$.

Then $M$ is a Korovkin system for $L$ in $C(X, \mathbb{R}^d)$.

Finally, we state the following result, which is the analogous of Theorem 2.4 and can be proved using the same arguments in the proof of Theorem 3.1.

Theorem 3.2. Let $X$ be a compact Hausdorff topological space and $T : C(X, E) \to C(X, E)$ a continuous monotone linear operator.

If $M$ is a subset of $C(X, E)$ such that, for every $x \in C(X, E)$, $x_0 \in X$ and $\varepsilon > 0$, there exist $\eta_1, \ldots, \eta_m \in M$ such that, for every $x \in X$,

$$\varphi(x) \in \text{co}(\eta_1(x), \ldots, \eta_m(x)), \quad T(\eta_j)(x_0) \in T(\varphi)(x_0) + \varepsilon \cdot B; \quad j = 1, \ldots, m.$$ 

then $M$ is a Korovkin set for $T$ in $C(X, E)$.

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Received August 2021; revised October 2021; early access November 2021.

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