ON THE \( p \)-ADIC PERIODS OF THE MODULAR CURVE
\( X(\Gamma_0(p) \cap \Gamma(2)) \).

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ABSTRACT. We prove a variant of Oesterlé’s conjecture describing \( p \)-adic periods of the modular curve \( X_0(p) \), with an additional \( \Gamma(2) \)-structure (and also \( \Gamma(3) \cap \Gamma_0(p) \) if \( p \equiv 1 \pmod{3} \)). We use de Shalit’s techniques and \( p \)-adic uniformization of curves with semi-stable reduction.

1. Introduction

Let \( p \geq 5 \) be a prime. Let \( X_0(p) \) be a regular, proper and flat model over \( \mathbb{Z}_p \) of the modular curve of level \( \Gamma_0(p) \), obtained by several blowing-up of the coarse moduli space of the algebraic stack \( \mathcal{M}_{\Gamma_0(p)} \) parametrizing generalised elliptic curves with \( \Gamma_0(p) \)-level structure [1]. Deligne and Rapoport proved that the special fiber of \( X_0(p) \) is semi-stable. Let \( K \) be the quadratic unramified extension of \( \mathbb{Q}_p \), \( \mathcal{O}_K \) be the ring of integers of \( K \), \( k \) be the residue field of \( \mathcal{O}_K \) and \( X_0^{rig}(p)_K \) be the rigid variety associated to the proper flat curve \( X_0(p)_K \). Since the supersingular points of the special fiber of \( X_0(p)_{\mathbb{F}_p} \) are \( k \)-rational, as well as its irreducible components, Mumford’s theorem [2] implies that \( X_0^{rig}(p)_K \) is the quotient of a \( p \)-adic half plane \( \mathcal{H}_\Gamma = \mathbb{P}_K^1 - \mathcal{L} \) by a Schottky group \( \Gamma \), where \( \mathcal{L} \) is the set of the limits points of \( \Gamma \). Manin and Drinfeld constructed a pairing \( \Phi : \Gamma^{ab} \times \Gamma^{ab} \to K^\times \) in [4] and explained how this pairing gives a \( p \)-adic uniformization of the Jacobian \( J_0(p)_K \) of \( X_0(p)_K \).

Let \( \Delta \) be the dual graph of the special fiber of \( X_0(\mathcal{O}_K) \). Mumford’s construction shows that \( \Gamma \) is isomorphic to the fundamental group \( \pi_1(\Delta) \). The abelianization of \( \Gamma \) is isomorphic to the augmentation subgroup of the free \( \mathbb{Z} \)-module with basis the isomorphism classes of supersingular elliptic curves over \( \overline{\mathbb{F}}_p \). Oesterlé conjectured that the pairing \( \Phi \) can be expressed, modulo the principal units, in terms of the modular invariant \( j \). E. de Shalit proved this conjecture in [8] (up to a sign if \( p \equiv 3 \pmod{4} \)). The aim of this paper is to prove a variant of Oesterlé conjecture for modular curves with a \( \Gamma_0(p) \cap \Gamma(2) \)-level structure by replacing \( j \) by the modular invariant \( \lambda \).
Notation

(i) For any algebraic stack $X$ over $\text{Spec } A$ and for any morphism $A \to B$, we denote by $X_B$ the fiber product $X \times_{\text{Spec } A} \text{Spec } B$.

(ii) For any algebraic extension $k$ of the field $\mathbb{Z}/p\mathbb{Z}$, we denote by $W(k)$ the ring of Witt vectors associated to $k$ and by $\bar{k}$ the separable closure of $k$.

(iii) For any congruence subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$, we denote by $M_{\Gamma}$ the stack over $\mathbb{Z}$ whose $S$-points classify generalized elliptic curves over $S$ with a $\Gamma$-level structure.

(iv) For any congruence subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ and any $c \in \mathbb{P}^1(\mathbb{Q})$, we denote by $[c]_\Gamma$ the cusp of $\Gamma \setminus (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}))$ corresponding to the class of $c$, where $\mathcal{H}$ is the (complex) upper-half plane.

(v) For two congruence subgroups $\Gamma$ and $L$, we denote by $\mathfrak{m}_{\Gamma \cap L}$ for the fiber product of algebraic stacks $\mathcal{M}_\Gamma \times_{\mathfrak{m}_L} \mathcal{M}_L$, where $\mathcal{M}$ is the stack over $\mathbb{Z}$ whose $S$-points classify generalized elliptic curves over the scheme $S$.

(vi) For any algebraic stack $\mathcal{M}_\Gamma$ over $\mathbb{Z}$, we denote by $M_{\Gamma}$ the coarse moduli space attached to $\mathcal{M}_\Gamma$ ($\mathcal{M}_\Gamma$ is an algebraic space).

(vii) For any proper and flat scheme $X$ over $\mathcal{O}_K$, we denote by $X_{\text{an}}$ the rigid analytic space given by the generic fiber of the completion of $X$ along its special fiber (we have a Galois-equivariant isomorphism $X_{K}(\bar{K}) \simeq X_{\text{an}}(\bar{K})$).

We refer the reader to [1, IV] for more details about the stacks defined above.

Let $\mathfrak{X}$ be the proper flat normal coarse moduli space over $\text{Spec } \mathcal{O}_K$ associated to the algebraic stack $\mathcal{M}_{\Gamma_{(p)\cap \Gamma(2)}}$. Deligne-Rapoport proved in [1, VI.6.9] that the singularities of $\mathfrak{X}$ are ordinary, and that the special fiber of $\mathfrak{X}$ is an union of two copies of $M_{\Gamma(2)\otimes k}$ meeting transversally at the supersingular points, and such that a supersingular point $x$ of the first copy is identified with the point $x^p = \text{Frob}_p(x)$ of the second copy (the supersingular points of the special fiber of $\mathfrak{X}$ are $k$-rational).

Theorem 1.1.

(i) The scheme $\mathfrak{X}$ is regular and the irreducible components of its special fiber are isomorphic to $\mathbb{P}^1_k$.

(ii) Let $S := \{e_i\}$ ($i \in \{0, \ldots, g\}$ where $g = \frac{p-3}{2}$ is the genus of $\mathfrak{X}$) be the set of supersingular points of $\mathfrak{X}_k$ and let $\Gamma \subset \text{PGL}_2(K)$ be a Schottky group such that there exists an isomorphism of rigid spaces $\mathfrak{N}_\Gamma/\Gamma \simeq \mathfrak{X}^\text{an}_K$, where $\mathfrak{N}_\Gamma = \mathbb{P}^1_K - \mathcal{L}$ and $\mathcal{L}$ are the limits points of $\Gamma$. Then the Drinfeld pairing $\Phi : \Gamma^{ab} \times \Gamma^{ab} \to K^\times$ takes values in $\mathbb{Q}^\times_p$. 
(iii) Let $\Phi$ be the residual pairing modulo the principal units $U_1(\mathbb{Q}_p^\times) \otimes \mathbb{Q}_p^\times$, then, after the identification $\Gamma^{ab} \simeq H_1(\Delta, \mathbb{Z}) \simeq \mathbb{Z}[S]^0$ (the augmentation subgroup of $\mathbb{Z}[S]$), $\Phi$ extends to a pairing (still denoted $\Phi$) $\mathbb{Z}[S] \otimes \mathbb{Z}[S] \to K^\times / U_1(K)$ such that:

$$\Phi(e_i, e_j) = \begin{cases} 
(\lambda(e_i) - \lambda(e_j))^{p+1} & \text{if } i \neq j; \\
p \cdot \prod_{k \neq i} (\lambda(e_i) - \lambda(e_k))^{-(p+1)} & \text{if } i = j.
\end{cases}$$

Remark 1. One can also prove an analogue of the theorem above when $N = 3$ and $p \equiv 1 \pmod{3}$, for a suitable model $X$ over $\mathbb{Z}_p$ of the modular curve of level $\Gamma_0(p)$ (c.f. 7.2 in appendix).

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2. Coarse moduli spaces of moduli stacks of generalized elliptic curves with level structure

Let $p \geq 5$ be a prime number and $N \geq 2$ be a squarefree integer prime to $p$. Let $\mathfrak{M}_{\Gamma_0(p)\cap\Gamma(N)}$ be the stack over $\mathbb{Z}[1/N]$ whose $S$-points are the isomorphism classes of generalized elliptic curves $E/S$, endowed with a locally free subgroup $A$ of rank $p$ such that $A + E[N]$ meets each irreducible component of any geometric fiber of $E$ ($E[N]$ is the subgroup of $N$-torsion points of $E$) and a basis of the $N$-torsion (i.e. an isomorphism $\alpha_N : E[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$). Deligne and Rapoport proved in [1] that $\mathfrak{M}_{\Gamma_0(p)\cap\Gamma(N)}$ is a regular algebraic stack, proper, of pure dimension 2 and flat.

Let $M'_{\Gamma_0(p)\cap\Gamma(N)}$ be the coarse space of the algebraic stack $\mathfrak{M}_{\Gamma_0(p)\cap\Gamma(N)}$ over $\mathbb{Z}[1/N]$. Since $M'_{\Gamma_0(p)\cap\Gamma(N)}$ is smooth and proper over $\mathbb{Z}[1/Np]$, $M'_{\Gamma_0(p)\cap\Gamma(N)}[1/Np]$ is a scheme of relative dimension one over $\mathbb{Z}[1/Np]$ ($M'_{\Gamma_0(p)\cap\Gamma(N)}[1/Np]$ is a quasi-projective scheme). Let $M_{\Gamma_0(p)\cap\Gamma(N)}$ be the model over $\mathbb{Z}[1/N]$ of $M'_{\Gamma_0(p)\cap\Gamma(N)}[1/Np]$ given by the normalization along the modular invariant $j : M'_{\Gamma_0(p)\cap\Gamma(N)}[1/Np] \to \mathbb{P}^1_{\mathbb{Z}[1/N]}$ (see [1 IV.3.3]). By using the same arguments as in Theorems [1 V.1.6], [1 IV.3.4], and Proposition [1
IV.3.10], we deduce that \( M_{\Gamma_0(p) \cap \Gamma(N)} \) is the coarse moduli space of the algebraic stack \( \mathfrak{M}_{\Gamma_0(p) \cap \Gamma(N)} \) (c.f. variante [1, V.1.14]).

The results of [1] show that the scheme \( M_{\Gamma_0(p) \cap \Gamma(N)} \) is smooth over \( \text{Spec } \mathbb{Z}[1/N] \) outside the points associated to supersingular elliptic curves in characteristic \( p \).

The cusps of \( M_{\Gamma_0(p) \cap \Gamma(N)} \) correspond to \( N \)-gons or \( Np \)-gons and are given by sections \( \text{Spec } \mathbb{Z}[1/N, \zeta_N] \rightarrow \mathfrak{M}_{\Gamma_0(p) \cap \Gamma(N)} \) composed with the coarse moduli map \( \mathfrak{M}_{\Gamma_0(p) \cap \Gamma(N)} \rightarrow M_{\Gamma_0(p) \cap \Gamma(N)} \).

Meanwhile, the work [6] is a reference about moduli of elliptic curves, and much of the material of [1] is best dealt in [6].

**Remark 2.** The genus of the complex modular curve \( \Gamma(N) \setminus \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) \) is 0 if and only if \( N \in \{1, 2, 3, 4, 5\} \).

In order to apply Mumford’s uniformization theorem, we need the following result.

**Proposition 2.1.**

(i) The special fiber of \( M_{\Gamma(2)} \otimes \mathcal{O}_K \) is isomorphic to \( \mathbb{P}^1_k \).

(ii) The scheme \( \mathfrak{X} \) is regular.

(iii) The lambda modular invariant induces an isomorphism

\[
\lambda : M_{\Gamma(2)} \otimes \mathbb{Q} \simeq \mathbb{P}^1_{\mathbb{Q}}.
\]

**Proof.**

i) The scheme \( M_{\Gamma(2)} \) is smooth, proper and of relative dimension one over \( \text{Spec } \mathbb{Z}[1/2] \), so it is flat and the genus of the fibers is constant (see [2, 7.9]). Corollaries [1, IV.5.5] and [1, IV.5.6] show that the geometric fibers of \( M_{\Gamma(2)} \) over \( \text{Spec } \mathbb{Z}[1/2] \) are connected and smooth, hence irreducible. Thus, the fact that the coarse moduli space formation commutes with flat base change implies that

\[
M_{\Gamma(2)} \otimes \mathbb{C} \simeq \Gamma(2) \setminus (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})).
\]

Hence the genus of the special fiber of \( M_{\Gamma(2)} \) at \( p \) is 0. On the other hand, the cusps of \( M_{\Gamma(2)} \) correspond to 2-polygons or \( 2p \)-polygons and are given by sections \( \text{Spec } \mathbb{Z}[1/2] \rightarrow \mathfrak{M}_{\Gamma(2)} \), then \( M_{\Gamma(2)} \otimes k \) has a \( k \)-rational point. Thus, the curve \( M_{\Gamma(2)} \otimes k \) is isomorphic to the projective line \( \mathbb{P}^1_k \).

ii) Note that a local noetherian ring is regular if and only if its strict henselianization is regular. Let \( x : \text{Spec}(k(x)) \rightarrow \mathfrak{X} \) be a singular geometric point (i.e. it corresponds to a supersingular elliptic curve). Thanks to Theorem [1, VI.6.9], the completion of the strict
henselianization of the local ring of $\mathcal{X}$ at $x$ is isomorphic to $W(\overline{k})[[X,Y]]/(XY - p^n)$, where $n$ is the cardinality of the automorphism group mod $\{-1, 1\}$ of the pair $(E, \alpha_2)$, $E$ and $\alpha_2$ are respectively a supersingular elliptic curve and a basis of the 2-torsion of $E$ associated to the geometric points $x$. But any automorphism of $E$ different from $\{-1, 1\}$ acts non trivially on $\alpha_2$, so $n = 1$.

iii) Let $E_\lambda$ be the elliptic curve over $\mathbb{A}^1_{\mathbb{Q}}$ given by the equation

$$Y^2 = X(X - 1)(X - \lambda)$$

where $\lambda \in \mathbb{Q}[T]$. It is clear that $E_\lambda$ induces a natural morphism $\mathbb{A}^1_{\mathbb{Q}} \to M_{\Gamma(2)} \otimes \mathbb{Q}$ and after composing with the coarse moduli map, we get a morphism $g : \mathbb{A}^1_{\mathbb{Q}} \to M_{\Gamma(2)} \otimes \mathbb{Q}$ which extends to a morphism $\tilde{g} : \mathbb{P}^1_{\mathbb{Q}} \to M_{\Gamma(2)} \otimes \mathbb{Q}$.

Moreover, the universal properties of the coarse moduli space of $M_{\Gamma(2)}$ imply that $\mathfrak{M}_{\Gamma(2)}(K) \simeq M_{\Gamma(2)}(K)$ for any field $K$. Hence, $\tilde{g}$ is an isomorphism and $\lambda$ is the inverse of $\tilde{g}$. □

**Remark 3.** When $N = 3$, the algebraic stack $\mathfrak{M}_{\Gamma_0(p) \cap \Gamma(N)}$ is rigid. Hence it will be represented by a scheme of relative dimension 1 over $\mathbb{Z}[1/N]$, which is smooth over $\mathbb{Z}[1/Np]$.

The geometric fibers of the morphism $M_{\Gamma(3)}[1/3] \to \text{Spec} \mathbb{Z}[1/3, \zeta_3]$ have genus 0 and are smooth and irreducible. However, they are not geometrically irreducible over $\text{Spec} \mathbb{Z}[1/3]$ (see [1]). On the other hand, Deligne and Rapoport proved in [1, VI.6.8] that for any subgroup $K \subset \text{SL}_2(\mathbb{Z}/3\mathbb{Z})$, the geometric fibers of the morphism $M_K[1/3] \to \text{Spec} \mathbb{Z}[1/3, \zeta_3]^K$ are of genus 0, irreducible and smooth. Thus, if we choose $K$ such that $\mathbb{Z}[1/3, \zeta_3]^K = \mathbb{Z}[1/3]$ and $M_{K[1/3]}(\mathbb{C}) = \Gamma(3) \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}))$, then Theorem [1, VI.6.9] and the same arguments as in the above propositions show that $M_{\Gamma \cap \Gamma_0(p)} \otimes \mathcal{O}_K$ is regular, semi-stable and the irreducible components of its special fiber are $k$-rational.

### 3. $p$-adic Uniformization of $\mathcal{X}^{an}_K$ and the Reduction Map

Mumford’s Theorem [7] shows the existence of a free discrete subgroup $\Gamma \subset \text{PGL}_2(K)$ (i.e. a Schottky group) and of a $\text{Gal}(\overline{K}/K)$-equivariant morphism of rigid spaces:

$$\tau : \mathfrak{H}_\Gamma \to \mathcal{X}^{an}_K$$

inducing an isomorphism $\mathcal{X}^{an}_K \simeq \mathfrak{H}_\Gamma / \Gamma$, where $\mathfrak{H}_\Gamma = \mathbb{P}^1_K - \mathcal{L}$ and $\mathcal{L}$ is the set of limit points of $\Gamma$. Note that $\mathfrak{H}_\Gamma$ is an admissible open of the rigid projective line $\mathbb{P}^1_K$. 
Let $T_{\Gamma}$ be the subtree of the Bruhat–Tits tree for $\text{PGL}_2(K)$ generated by the axes whose ends correspond to the limit points of $\Gamma$. Mumford constructed in [7] a continuous map $\rho: \mathcal{S}_{\Gamma} \to T_{\Gamma}$ called the reduction map, satisfying the following properties:

**Proposition 3.1.**

(i) The inverse image of any vertex $v$ of $T_{\Gamma}$ is isomorphic to the closed unit disk minus the union of some open maximal rational sub-disks.

(ii) The inverse image of any edge is an open annulus of width $r \in |K^\times|_p$, i.e. $D(0, r[-D(0, 1)]$.

(iii) If we endow the graph $T_{\Gamma}$ with the natural topology in which we identify an edge to the interval $[0, 1]$, then $\rho$ is continuous.

(iv) If $\Delta$ is the dual graph of the special fiber of $X$, then $\Delta \simeq T_{\Gamma}/\Gamma$ and $H_1(\Delta, \mathbb{Z}) \simeq \Gamma_{ab} \simeq \mathbb{Z}[S]_0$, where $\mathbb{Z}[S]_0$ is the augmentation subgroup of $\mathbb{Z}[S]$.

The following proposition follows immediately from the proof of the main theorem of Mumford in [7].

**Proposition 3.2.** Let $v$ be a vertex of $T_{\Gamma}$ having $k + 1$ neighbours denoted by $v_i$ for $1 \leq i \leq k + 1$. Then $\rho^{-1}(v)$ is isomorphic to the closed unit disk $D(0, 1]$ minus $k$ maximal open disks $D(a_i, 1]$. Each disk $D(a_i, 1] \subset D(a_i, r_i) \subset D(a_i, 1]$ for a unique neighbour $v_i$ of $v$. There also exists a unique neighbour $v_{i_0}$ of $v$ such that $r_i \subset D(0, r_{i_0}]$ where $r_{i_0} < 1$, and the annuli $D(a_i, 1] - D(a_i, r_i]$ for $i \neq i_0$ or $D(0, 1] - D(0, r_{i_0}]$ reduce respectively to the edges connecting $v$ to $v_i$ and $v$ to $v_{i_0}$.

The special fiber of $X$ has two components, and each component has 3 cusps. One of these components, which we call the étale component, classifies elliptic curves or $2p$-sided Néron polygons over $\bar{k}$ with an étale subgroup of order $p$ and a basis of the 2-torsion. The other component, which we call the multiplicative component, classifies elliptic curves or 2-sided Néron polygons over $\bar{k}$ with a multiplicative subgroup of order $p$ and a basis of the 2-torsion.

The involution $w_p$ sends a $2p$-gon to a 2-gon, since the quotient of a $2p$-gon by its unique cyclic étale subgroup of order $p$ (i.e. $\mathbb{Z}/p\mathbb{Z}$) in its smooth locus gives 2-sided polygone ($\mathbb{Z}/p\mathbb{Z}$ acts by rotations).

Let $c$ and $c' = w_p(c)$ be two cusps of $M_{\Gamma_0(p)\cap\Gamma(2)}(C)$ defined in Appendix, section 7.1.1. They are $\mathbb{Q}$-rational and $M_{\Gamma_0(p)\cap\Gamma(2)}$ is proper over $\mathbb{Z}[1/2]$. The valuative criterion shows
that there exists two cusps \( \xi_c : \text{Spec} \mathbb{Z}_p \to \mathcal{X} \) (resp. \( \xi_{c'} : \text{Spec} \mathbb{Z}_p \to \mathcal{X} \)) corresponding to the cusps \( c \) and \( c' \) of \( M_{\Gamma_0(p) \cap \Gamma(2)}(\mathbb{C}) \) after taking the generic fiber. The cusp \( \xi_c \) corresponds to a 2\( p \)-gon and \( \xi_{c'} = w_p(\xi_c) \) corresponds to a 2-gon.

The dual graph \( \Delta \) of the special fiber of \( \mathcal{X} \) has two vertices \( v_{c'} \) and \( v_c \) indexed respectively by the cusps \( \xi_{c'} \) and \( \xi_c \). There are \( g + 1 \) edges \( e_i \) (\( i \in \{0, \ldots, g\} \)) corresponding to supersingular elliptic curves with a \( \Gamma(2) \)-structure. We orient these edges so that they point out of \( v_{c'} \).

The Atkin-Lehner involution \( w_p \) exchanges the two vertices \( v_{c'} \) and \( v_c \) and also acts on edges (reversing the orientation). More precisely, if \( E_i \) is a supersingular elliptic curve corresponding to \( e_i \), then \( w_p(e_i) = e_j \) where \( e_j \) is the elliptic curve associated to \( E_i^{(p)} = w_p(E_i) \) (here \( w_p \) is the Frobenius). Thanks to Lemma 3.3 below, one can identify the generators \( \{ \gamma_i \}_{1 \leq i \leq g+1} \) of \( \Gamma \) with \( (e_i - e_0)_{1 \leq i \leq g+1} \).

Let \( \tilde{v}_c \) and \( \tilde{v}_{c'} \) be neighbour vertices of \( \mathcal{T}_\Gamma \) reducing to \( v_{c'} \) and \( v_c \) respectively, such that the edge linking \( \tilde{v}_c \) to \( \tilde{v}_{c'} \) reduces to \( e_0 \) modulo \( \Gamma \). For \( 0 \leq i \leq g \), let \( \tilde{e}_i' \) be an edge pointing out of \( \tilde{v}_{c'} \) and reducing to \( e_i \) modulo \( \Gamma \). Let \( \tilde{e}_i \) be oriented edges of \( \mathcal{T}_\Gamma \) lifting \( e_i \) and pointing to \( \tilde{v}_c \). Note that \( \tilde{e}_0 = \tilde{e}_0' \).

Let \( A = \rho^{-1}(\tilde{v}_c) \) and \( A' = \rho^{-1}(\tilde{v}_{c'}) \). Proposition 3.2 implies that \( A \) (resp. \( A' \)) is the complement of \( g + 1 \) open disks in \( \mathbb{P}^1_k \), hence \( \mathbb{P}^1_k - A = \bigsqcup_{0 \leq i \leq g} B_i \) and \( \mathbb{P}^1_k - A' = \bigsqcup_{0 \leq i \leq g} C_i' \) where \( 0 \leq i \leq g \). We index \( B_i \) and \( C_i' \) such that \( A \subset C_i'' \), \( A' \subset B_0 \), \( B_i \) and \( C_i'' \) are associated to \( \tilde{e}_i' \) and the inverse of \( \tilde{e}_i \) respectively.

For all \( 0 \leq i \leq g \), \( \rho^{-1}(\tilde{e}_i') = c_i \) is an annulus of \( C_i' \) and \( C_i = C_i' - \rho^{-1}(\tilde{e}_i') \) is a closed disk; we also have \( \mathbb{P}^1_k - C_0 = B_0 \). We have

\[
\mathbb{P}^1_k - \rho^{-1}(\bigsqcup_{0 \leq i \leq g} \tilde{e}_i' \cup \tilde{v}_{c'}) = \bigsqcup_{0 \leq i \leq g} C_i.
\]

Note that \( \tilde{v}_c \cup \tilde{v}_{c'} \cup \{ \tilde{e}_i' \} \) is a fundamental domain of \( \mathcal{T}_\Gamma \), so

\[
D = \mathbb{P}^1_k - \bigsqcup_{1 \leq i \leq g} B_i \cup \bigsqcup_{1 \leq i \leq g} C_i
\]

is a fundamental domain of \( \mathfrak{B}_\Gamma \).

Using the same techniques as in [8], §1, we get the following lemma:

**Lemma 3.3 (Normalization of \( \Gamma \)).** We can choose \( \Gamma \) such that there is a Schottky basis \( \alpha_1, \ldots, \alpha_g \) of \( \Gamma \), and a fundamental domain \( D \) satisfying:

(i) \( B_i \) is the open residue disk in the closed unit disk of \( \mathbb{P}^1_K \) which reduces to \( \lambda(e_i)^p \), \( \forall 0 \leq i \leq g \).
(ii) For $1 \leq i \leq g$, $\alpha_i$ corresponds, under the identification $\Gamma_{ab} = \mathbb{Z}[S]^0$, to $e_i - e_0$.

(iii) $\alpha_i$ sends bijectively $P^1_k - B_i$ to $C_i$ and $\alpha_i^{-1}$ sends bijectively $P^1_k - C_i$ to $B_i$.

(iv) The annulus $c_i$ is isomorphic, as a rigid analytic space, to $\{z, |p| < |z| < 1\}$.

Proof.

(i) The standard reduction sends $P^1_k$ to $P^1_k$ (see [3, II.2.4.2]). If we restrict the standard reduction to $A$, we get back $\rho$, so $\rho$ sends $A$ to $G_m$ minus the $\rho(B_i \cap \mathcal{H}_\Gamma) = \lambda(e_i)$'s.

(ii) $\mathcal{H}_\Gamma$ is the universal covering of $\Delta$, so $\pi_1(\Delta) \cong \Gamma$. Let $\tilde{v}_i$ be a neighbour of $\tilde{v}_{c'}$ whose edge to $\tilde{v}_{c'}$ lifts $e_i$. By monodromy, we can choose $\alpha_i$ which lifts $e_i - e_0$, such that $\alpha_i(\tilde{v}_i) = \tilde{v}_c$, and since the action of $\Gamma$ on the graph is continuous, we see that $\alpha_i(P^1_k - B_i) = C_i$ (i.e. $\alpha_i^{-1}$ sends neighbours of $\tilde{v}_i$ to neighbours of $\tilde{v}_c$). This also shows (iii).

4. Extension of $\Phi$ to $\mathbb{Z}[S] \times \mathbb{Z}[S]$}

For $a, b \in \mathcal{H}_\Gamma$, define the meromorphic function $\theta(a,b;z) = \theta((a) - (b); z)$ ($z \in \mathcal{H}_\Gamma$) by the convergent product

$$\theta(a,b;z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma a}{z - \gamma b}.$$

See [4] for the basic properties of these theta functions.

For all $a, b \in \mathcal{H}_\Gamma$, the theta series $\theta(a,b,.)$ converges and defines a rigid meromorphic function on $\mathcal{H}_\Gamma$ (which is modified by a constant if we conjugate $\Gamma$). We extend $\theta$ to degree zero divisors $D$ of $\mathcal{H}_\Gamma$. The series $\theta(D,.)$ is entire if and only if $\tau_*(D) = 0$, where we recall that $\tau : \mathcal{H}_\Gamma \rightarrow X^{an}_K$ is the uniformization.

The proposition below follows from [4] (see also [8]).

**Proposition 4.1.** [4]

(i) $\theta(a,b,z) = c(a,b,\alpha)\theta(a,\alpha a,z)$, where $\alpha \in \Gamma$ and $c(a,b,\alpha \beta) = c(a,b,\alpha)c(a,b,\beta)$.

(ii) The function $u_\alpha(z) = \theta(a,\alpha a,z)$ does not depend on $a$, and $u_{\alpha\beta} = u_\alpha u_\beta$.

(iii) $c(a,b,\alpha) = u_\alpha(a)/u_\alpha(b)$.

(iv) $\theta(a,b,z)/\theta(a,b,z') = \theta(z,z',a)/\theta(z,z',b)$.

We recall that $\Phi : \mathbb{Z}[S]^0 \times \mathbb{Z}[S]^0 \rightarrow K^*$ is defined by:

$$\Phi(\alpha,\beta) = \theta(a,\alpha a,z)/\theta(a,\alpha a,\beta z) = u_\alpha(z)/u_\alpha(\beta z).$$
We can identify $\Gamma^{ab}$ with $\mathbb{Z}[S]^0$ since $\mathcal{T}_\Gamma$ is the universal covering of the graph $\Delta$.

Manin and Drinfeld proved that $v_K \circ \Phi$ is positive definite ($v_K$ is the $p$-adic valuation of $K$).

**Lemma 4.2.** The pairing $\Phi$ takes values in $\mathbb{Q}_p^\times$.

**Proof.** Since $p \equiv 1 \pmod{2}$, the Atkin–Lehner automorphism $w_p$ is an involution which is the Frobenius on the supersingular points. The proof is now the same as in Lemma [80 0.4].\[\square\]

Note that our pairing $\Phi : \mathbb{Z}[S]^0 \times \mathbb{Z}[S]^0 \to \mathbb{Q}_p^\times$ does not depend on the choice of $\lambda : X(2)_{\mathbb{Q}} = M_{\Gamma(2)} \otimes \mathbb{Q} \sim \mathbb{P}^1_{\mathbb{Q}}$. We are going to define an extension $\Phi : \mathbb{Z}[S] \times \mathbb{Z}[S] \to K^\times$ (which depends on the choice of $\lambda$).

### 4.1. Extension of $\Phi$ to $\mathbb{Z}[S]^0 \times \mathbb{Z}[S]$.

For all $0 \leq i \leq g$, we choose $\xi_{c}^{(i)}$ (resp. $\xi_{c'}^{(i)}$) in $\mathcal{H}_\Gamma$ which reduces modulo $\Gamma$ to the cusp $\xi_{c} \otimes \mathbb{Q}_p$ (resp. $\xi_{c'} \otimes \mathbb{Q}_p$), and such that $\xi_{c}^{(i)}$ and $\xi_{c'}^{(i)}$ are separated by an annulus reducing to $e_i$.

Let $\tilde{v}_{c}^{(i)}$ and $\tilde{v}_{c'}^{(i)}$ be two neighbour vertices of $\mathcal{T}_\Gamma$ above $v_c$ and $v_{c'}$ respectively, separated by an edge reducing to $e_i$. We fix $\tilde{v}_{c}^{(0)} = \tilde{v}_{c}$ and $\tilde{v}_{c'}^{(0)} = \tilde{v}_{c'}$. We then choose $\xi_{c}^{(i)}$ (resp. $\xi_{c'}^{(i)}$) in $\rho^{-1}(\tilde{v}_{c}^{(i)})$ (resp. $\rho^{-1}(\tilde{v}_{c'}^{(i)})$). If we choose, for all $0 \leq i \leq g$, $\xi_{c}^{(i)} = z_0 \in A$, then the $\xi_{c'}^{(i)}$ are uniquely determined, and satisfy

$$
\xi_{c'}^{(i)} = \alpha_i^{-1}(\xi_{c}^{(0)}) \in B_i .
$$

Indeed, we have $\xi_{c'}^{(0)} \in \rho^{-1}(\tilde{v}_{c'}) = A'$ and $\alpha_i^{-1}(A') \subset \alpha_i^{-1}(P^1 - C'_i) \subset B_i$.

Note that for all $i$, the pair $(\xi_{c}^{(i)}, \xi_{c'}^{(i)})$ is uniquely determined modulo $\Gamma$. We can assume that $z_0 \neq \infty$.

Let $a \in \mathcal{H}_\Gamma$. We then define, for all $\alpha \in \Gamma$,

$$
\Phi(\alpha, e_i) = \frac{\theta(a, \alpha(a), \xi_{c}^{(i)})}{\theta(a, \alpha(a), \xi_{c'}^{(i)})} = \frac{u_{\alpha}(\xi_{c}^{(i)})}{u_{\alpha}(\xi_{c'}^{(i)})} = \frac{u_{\alpha}(\xi_{c}^{(i)})}{u_{\alpha}(z_0)} .
$$

This definition does not depend on the choice of $a$ and $(\xi_{c}^{(i)}, \xi_{c'}^{(i)})$, by Proposition [4.1]. Since $K$ is complete and $\xi_{c}^{(i)}$ and $\xi_{c'}^{(i)}$ are defined over $K$, $\Phi$ takes values in $K^\times$.

**Lemma 4.3.** The pairing $\Phi$ defined above extends the previous pairing $\Phi$ on $\mathbb{Z}[S]^0 \times \mathbb{Z}[S]^0$. 
Proof. Using Proposition 4.1 we have:

\[ \frac{\Phi(\alpha, e_i)}{\Phi(\alpha, e_0)} = \frac{u_\alpha(\xi^{(i)}_c)}{u_\alpha(\xi^{(0)}_c)} = \frac{u_\alpha(\xi^{(i)}_e)}{u_\alpha(\alpha_i(\xi^{(i)}_c'))} = \Phi(\alpha, \alpha_i) = \Phi(\alpha, e_i - e_0). \]

\[ \square \]

5. Extension of \( \Phi \) to \( \mathbb{Z}[S] \times \mathbb{Z}[S] \)

Let \( \lambda \) be the Hauptmodul for \( M_{\Gamma(2)} \); recall that \( \lambda \) induces an isomorphism

\[ M_{\Gamma(2)} \otimes \mathbb{Q} \simeq \mathbb{P}^1_\mathbb{Q}. \]

We define \( \lambda' : X^w_\mathbb{K} \to \mathbb{P}^1_\mathbb{K} \) by \( \lambda' = \lambda \circ w_p \)

5.1. Atkin–Lehner involution on \( \mathfrak{H}_\mathbb{K} \). The Atkin–Lehner involution acts on \( \Gamma \backslash \mathcal{T}_\mathbb{K} \) and lifts to an orientation reversing involution \( w_p \) of \( \mathcal{T}_\mathbb{K} \) (by the universal covering property). By [5] ch. VII Sect. 1, there is a unique class in \( N(\Gamma)/\Gamma \) (where \( N(\Gamma) \) is the normalizer of \( \Gamma \) in \( \text{PGL}_2(K) \)) inducing \( w_p \) on \( \mathcal{T}_\mathbb{K} \). We denote by \( w_p \) the induced map of \( \mathfrak{H}_\mathbb{K} \) (it is only unique modulo \( \Gamma \)).

5.2. Definition of \( \Phi \). Fix \( 0 \leq i, j \leq g \).

Let \( z \in \mathfrak{H}_\mathbb{K} \) near \( \xi^{(i)}_e \) and \( z' \) near \( \xi^{(i)}_c' \) such that \( \tau(z) = w_p(\tau(z')) \). Recall that by hypothesis, \( \xi^{(i)}_c = z_0 \) is independent of \( i \).

For \( 0 \leq i, j \leq g \), we let:

\[ (2) \quad \Phi(e_i, e_j) = \lim \lambda'(\tau(z))^2 \cdot \frac{\theta(z', z, \xi^{(j)}_c)}{\theta(z', z, \xi^{(j)}_c')} \]

where \( z \) and \( z' \) approach \( \xi^{(i)}_e \) et \( \xi^{(i)}_c' \) respectively. Since at \( z = z_0 \), \( \lambda' \circ \tau \) has a simple pole and the numerator and denominator have a simple zero and simple pole respectively, \( \Phi(e_i, e_j) \) is finite, and is in \( K^\times \) since \( K \) is complete (we choose \( z \) and \( z' \) in \( K \) to compute the limit). We bilinearly extend \( \Phi \) to \( \mathbb{Z}[S] \times \mathbb{Z}[S] \).

Lemma 5.1. The pairing \( \Phi \) defined above extends the previous pairing on \( \mathbb{Z}[S] \times \mathbb{Z}[S]^0 \).

Proof. Since \( \xi^{(i)}_c = \xi^{(0)}_c = z_0 \), we have:

\[ \frac{\Phi(e_i, e_j)}{\Phi(e_0, e_j)} = \lim \frac{\theta(z', z, \xi^{(j)}_c)}{\theta(z', z, \xi^{(j)}_c')} \cdot \frac{\theta(z', z, \xi^{(j)}_c)}{\theta(z', z, \xi^{(j)}_c')} \]
where $\zeta' = \alpha(z')$ approaches $\xi^{(0)}$. Since
\[
\frac{\theta(a, b, z)}{\theta(\gamma(a), b, z)} = \theta(a, \gamma(a), z)
\]
(which is obvious by the infinite product definition of $\theta$), we get:
\[
\frac{\Phi(e_i, e_j)}{\Phi(e_0, e_j)} = \lim \frac{\theta(z', \alpha_i(z'), \xi^{(j)}_\gamma)}{\theta(z', \alpha_i(z'), \xi^{(j)}_{\xi'})} = \Phi(\alpha_i, e_j).
\]

\[\square\]

6. PROOF OF THE MAIN THEOREM

Let $\hat{e} = \sum_{e_i \in S} e_i \in \mathbb{Z}[S]$ be the Eisenstein element. Let $\Phi$ be the reduction of $\Phi$ modulo principal units. Let $d = \gcd(p - 1, 12)$.

**Theorem 6.1.** If $i \neq j$, we have:
\[
\Phi(e_i, e_j) \equiv \left(\lambda(e_i) - \lambda(e_j)\right)^{(p+1)} \Phi(e_0, e_j) 
\]

Else, we have:
\[
\Phi(e_i, \hat{e}) \equiv p \Phi(e_0, \hat{e}) \mod p
\]

Theorem 1.1 follows from Theorem 6.1 since $\mathbb{Q}_p^\times / U_1(\mathbb{Q}_p)$ has no $\frac{12}{d}$-torsion. In the rest of this article, we prove Theorem 6.1.

6.1. Case $i \neq j$. Let’s first show (3), which is easier. Fix $0 \leq i \leq g$. For simplicity, but without loss of generality, we assume that $j = 0$.

Proposition 4.1 (iv) shows that
\[
\Phi(e_i, e_0) = \lim \frac{\lambda'(\tau(z))}{\lambda'(\tau(z))} \cdot \frac{\theta(\xi^{(0)}_\gamma, \xi^{(0)}_{\xi'\gamma}, z')}{\theta(\xi^{(0)}_\gamma, \xi^{(0)}_{\xi'}, z)}
\]

Recall that, by definition, $\xi^{(0)}_{\xi'} = z_0 \in A$. For each $\gamma \in \Gamma$, the corresponding term in the infinite product defining the above expression is:
\[
(z' - \gamma(z_0)) \cdot (z - \gamma(z_0))
\]

**Lemma 6.2.** The map $\lambda' \circ \tau : \mathcal{F}_\Gamma \to \mathbf{P}^1_K$ stabilizes $A$ and $A \cap U$ where $U$ is the closed unit disk. Furthermore, the restriction of $\lambda' \circ \tau$ to $A \cap U$ is the identity modulo $p$. 
Proof. We have a map $A \to \mathbf{P}^1_k - S$ given by composing the uniformization $\tau$ with the reduction on the special fiber $X_k$. This map coincides with the naive reduction of $A$ to $\mathbf{P}^1_k$ induced by the standard reduction $\mathbf{P}^1_k \to \mathbf{P}^1_k$ by the lemma of normalization of $\Gamma$, $\lambda' \circ \tau$ stabilizes $A$ and is the identity modulo $p$. □

Since $\lambda(\xi_c) = \infty$, $\xi_c^{(0)} = z_0 \in A$ reduces to $\infty$ modulo $p$, i.e. is not $p$-integral as an element of $K$. Since $z$ is near $z_0$, $z$ reduces to $\infty$ modulo $p$.

6.1.1. Case $\gamma = 1$. If $\gamma = 1$,
\[
\frac{(z' - \gamma(\xi_c^{(0)})) \cdot (z - \gamma(z_0))}{(z' - \gamma(z_0)) \cdot (z - \gamma(\xi_c^{(0)}))}
\]
is equivalent, modulo principal units, to
\[
\frac{(\lambda(e_i)p - \lambda(e_0)p) \cdot (z - z_0)}{-z_0^2} = \frac{(\lambda(e_i) - \lambda(e_0))p \cdot (z - z_0)}{-z_0^2}.
\]
Note that this equality mixes terms in $k^\times$ and $K^\times$. In fact, this equality (and all the similar equalities below) is viewed in $K^\times/U_1(K)$.

6.1.2. Case $\gamma \neq 1$. Let $\gamma \neq 1 \in \Gamma$, written in reduced form as $\alpha_{i_1}^1 \alpha_{i_2}^2 \cdots$ with $i_k \in \{1, \ldots, g\}$ and $\epsilon_{i_k} \in \mathbb{Z}\{0\}$. Then for all $u$ in the fundamental domain
\[
D = \mathbf{P}^1_K - \bigcup_{1 \leq i \leq g} B_i \cup \bigcup_{1 \leq i \leq g} C_i,
\]
we have $\gamma(u) \in B_i$ if $\epsilon_i < 0$ and $\gamma(u) \in C_i$ if $\epsilon_i > 0$. Note that $\gamma(z_0)$ is not in $A$, so it is not in the residue disk containing $\infty$ (i.e. the residue disk containing $z_0$). Hence $\gamma(z_0)$ does not reduce to $\infty$ modulo $p$.

In particular, $\gamma(D) \subset B_i$ or $C_i$. Thus
\[
\frac{z - \gamma(z_0)}{z - \gamma(\xi_c^{(0)})}
\]
is a principal unit.

Assume first that $\alpha_{i_j}^k$ with $i_j \neq j$ or $\alpha_{i_i}^n$ with $n > 0$ occurs as the first term in the reduced form of $\gamma$. In that case, $\gamma(\xi_c^{(0)})$ and $\gamma(z_0)$ are both in $B_j$, $C_j$ or $C_i$ and $z' \in B_i$, so
\[
\frac{z' - \gamma(\xi_c^{(0)})}{z' - \gamma(z_0)}
\]
is a principal unit. So in this case, the factor associated to $\gamma$ in the definition of $\phi(e_i, e_0)$ is a principal unit.
Thus, in what follows, assume that \( \gamma = \alpha_i^{-1} \cdot \gamma' \), where \( \gamma' \) is such that its reduced expression in terms of the \( \alpha_i \)'s does not begin with \( \alpha_i \). By invariance of the cross-ratio by the action \( \text{PGL}_2(K) \) (here by the action of \( \alpha_i \)), we have:

\[
\frac{(z' - \gamma(\xi_{\alpha_i}^{(0)})) \cdot (z - \gamma(z_0))}{(z' - \gamma(z_0)) \cdot (z - \gamma(\xi_{\alpha_i}^{(0)}))} = \frac{(\alpha_i(z') - \gamma'(\xi_{\alpha_i}^{(0)})) \cdot (\alpha_i(z) - \gamma'(z_0))}{(\alpha_i(z') - \gamma'(z_0)) \cdot (\alpha_i(z) - \gamma'(\xi_{\alpha_i}^{(0)}))}.
\]

**First subcase.** We first assume that the decomposition of \( \gamma' \) does not begin by \( \alpha_i^{-1} \). We distinguish the cases \( \lambda(e_0) \in \mathbb{F}_p \) and \( \lambda(e_0) \in k \setminus \mathbb{F}_p \).

**Case \( \lambda(e_0) \in \mathbb{F}_p \).** Assume first that \( \lambda(e_0) \in \mathbb{F}_p \). We choose an element of \( N(\Gamma) \) (normalizer of \( \Gamma \) in \( \text{PGL}_2(K) \)) inducing \( w_p \) on \( X_K^\text{an} \) and such that the induced automorphism on \( T_\Gamma \) (still denoted by \( w_p \)) fixes (by reversing the vertices) the edge \( \tilde{c}_0 \) between \( \tilde{c} \) and \( \tilde{\nu}_\gamma \). Then, by definition, \( w_p(A) = A' \) and \( w_p(C_i') = B_{i'} \) where \( i' \in \{0, \ldots, g\} \) is such that \( \lambda(e_{i'}) = \lambda(e_i)p \). Since \( w_p \) fixes the edge \( \tilde{c}_0 \), \( w_p^2 = 1 \) (the stabilizer of \( \tilde{c}_0 \) is compact and \( \Gamma \) is discrete, so \( w_p^2 \) is torsion but \( \Gamma \) is torsion-free). We have \( w_p(\tilde{c}_\gamma) = \tilde{\nu}_\gamma \), hence \( w_p(\tilde{\nu}_\gamma) = \tilde{c}_\gamma \).

Therefore \( w_p(z_0) = \xi_{\gamma}^{(0)} \) and \( w_p(\xi_{\gamma}^{(0)}) = z_0 \). We also have \( w_p(\alpha_i(z')) = z \) (both terms are near \( z_0 \) and \( \lambda'(\tau(w_p(\alpha_i(z')))) = \lambda'(\tau(z')) = \lambda'(\tau(z)) \)). By invariance of the cross-ratio by the action of \( \text{PGL}_2(K) \) (hence by \( w_p \)) and if \( \gamma' = 1 \), we have:

\[
\frac{(\alpha_i(z') - \xi_{\gamma}^{(0)}) \cdot (\alpha_i(z) - z_0)}{(\alpha_i(z') - z_0) \cdot (\alpha_i(z) - \xi_{\gamma}^{(0)})} = \frac{(z - z_0) \cdot (w_p(\alpha_i(z)) - \xi_{\gamma}^{(0)})}{(z - \xi_{\gamma}^{(0)}) \cdot (w_p(\alpha_i(z)) - z_0)}.
\]

This last term equals, modulo principal units, to:

\[
\frac{(z - z_0)}{-z_0^2} \cdot (\lambda(e_i) - \lambda(e_0)).
\]

Otherwise, \( \gamma' \neq 1 \) and we get similarly:

\[
\frac{(\alpha_i(z') - \gamma'(\xi_{\gamma}^{(0)})) \cdot (\alpha_i(z) - \gamma'(z_0))}{(\alpha_i(z') - \gamma'(z_0)) \cdot (\alpha_i(z) - \gamma'(\xi_{\gamma}^{(0)}))} = \frac{(z - w_p(\gamma'(\xi_{\gamma}^{(0)}))) \cdot (w_p(\alpha_i(z)) - w_p(\gamma'(z_0)))}{(z - w_p(\gamma'(z_0))) \cdot (w_p(\alpha_i(z)) - w_p(\gamma'(\xi_{\gamma}^{(0)})))}.
\]

If the reduced form of \( \gamma' \) is \( \alpha_j^k \alpha_i z_2 \cdots \) with \( i \neq j \) and \( k \geq 1 \), then the right-hand side of the equality is seen to be a principal unit, otherwise the reduced form of \( \gamma' \) is \( \alpha_j^{-k} \alpha_i z_2 \cdots \) with \( k \geq 1 \), so the left-hand side is seen to be a principal unit.

To summarize, in the case \( \lambda(e_0) \in \mathbb{F}_p \), if \( \gamma' = 1 \), the factor associated to \( \gamma \) is

\[
\frac{(z - z_0)}{-z_0^2} \cdot (\lambda(e_i) - \lambda(e_0)).
\]

Else, if \( \gamma' \neq 1 \), the factor associated to \( \gamma \) is a principal unit.
Case $\lambda(e_0) \in k \setminus F_p$. Assume now that $\lambda(e_0) \in k \setminus F_p$, and without loss of generality that $\lambda(e_0)^p = \lambda(e_g)$. In that case, we choose $w_p$ such that $w_p(e_0) = e_g$. Since $w_p^2 \in \Gamma$, we have $w_p^2 = \alpha_g$ (this follows from $w_p^2(\tilde{v}_c) = \tilde{v}_g = \alpha_g(\tilde{v}_c)$ and the fact that the stabilizer of a vertex in $T_\Gamma$ is trivial). With the same notation as in the previous case, we have $w_p(B_i) = C_i'$, $w_p(z_0) = \xi_i^{(0)}$, $w_p^{-1}(z_0) \in B_g$ et $w_p^{-1}(\alpha_i(z')) = z$. We again use the invariance of the cross-ratio by $w_p^{-1}$ when $\gamma' = 1$:

$$\frac{(\alpha_i(z') - \xi_{i'}^{(0)}) \cdot (\alpha_i(z) - z_0)}{(\alpha_i(z') - \gamma'(z_0)) \cdot (\alpha_i(z) - \gamma'(z_0))} = \frac{(z - z_0) \cdot (w_p^{-1}(\alpha_i(z)) - \alpha_g^{-1}(\xi_{i'}^{(0)}))}{(z - \alpha_g^{-1}(\xi_{i'}^{(0)})) \cdot (w_p^{-1}(\alpha_i(z)) - z_0)}.$$

This last term equals, modulo principal units, to:

$$\frac{(z - z_0)}{-z_0^2} \cdot (\lambda(e_i) - \lambda(e_0)).$$

Otherwise, $\gamma' \neq 1$ and by using the invariance of the cross-ratio by the action of $\text{PGL}_2(K)$ (hence by $w_p^{-1}$), we get:

$$\frac{(\alpha_i(z') - \gamma'(\xi_{i'}^{(0)})) \cdot (\alpha_i(z) - \gamma'(z_0))}{(\alpha_i(z') - \gamma'(z_0)) \cdot (\alpha_i(z) - \gamma'(z_0))} = \frac{(z - w_p^{-1}(\gamma'(\xi_{i'}^{(0)}))) \cdot (w_p^{-1}(\alpha_i(z)) - w_p^{-1}(\gamma'(z_0)))}{(z - w_p^{-1}(\gamma'(z_0))) \cdot (w_p^{-1}(\alpha_i(z)) - w_p^{-1}(\gamma'(\xi_{i'}^{(0)})))}.$$ 

Note that $w_p^{-1}\alpha_i z \in B_{\gamma}$. Thus, if the reduced form of $\gamma'$ is $\alpha_{j'}^k \alpha_{i_1'}^{r_1} \cdots$ with $i \neq j$ and $k \geq 1$, then the right-hand side of the equality is seen to be a principal unit, otherwise the reduced form of $\gamma'$ is $\alpha_{j'}^{-k} \alpha_{i_1'}^{r_1} \cdots$ with $k \leq -1$, so the left-hand side is seen to be a principal unit.

To summarize, in the case $\lambda(e_0) \in k \setminus F_p$, if $\gamma' = 1$, the factor associated to $\gamma$ is

$$\frac{(z - z_0)}{-z_0^2} \cdot (\lambda(e_i) - \lambda(e_0)).$$

Else, if $\gamma' \neq 1$, the factor associated to $\gamma$ is a principal unit.

Second subcase. If the reduced form of $\gamma$ is $\alpha_i^{-n} \gamma'$ with $n \geq 2$, then by using again the invariance of the cross-ratio by the action of $\text{PGL}_2(K)$ (hence by $\alpha_i^n$), we get:

$$\frac{(z' - \gamma'(\xi_{i'}^{(0)})) \cdot (z - \gamma'(z_0))}{(z' - \gamma'(z_0)) \cdot (z - \gamma'(\xi_{i'}^{(0)}))} = \frac{(\alpha_i^n(z') - \gamma'(\xi_{i'}^{(0)})) \cdot (\alpha_i^n(z) - \gamma'(z_0))}{(\alpha_i^n(z') - \gamma'(z_0)) \cdot (\alpha_i^n(z) - \gamma'(\xi_{i'}^{(0)}))}.$$ 

In the case where the first term of $\gamma'$ is $\alpha_j^k$ with $k \leq -1$, a direct computation shows that the right-hand side is a principal unit (since $\alpha_i^n z' \in B_0$). Otherwise, $k \geq 1$ or $\gamma' = 1$ and by applying $w_p$ (or $w_p^{-1}$) is the case where $\lambda(e_0) \in k \setminus F_p$ to the left-hand side above we find a principal unit (in the case where $\lambda(e_0) \in k \setminus F_p$, $w_p^{-1}(C_g) \subset B_0$).

Thus in this second subcase, the factor associated to $\gamma$ is a principal unit.
To summarize the various cases, there are exactly two elements $\gamma \in \Gamma$ for which the factor associated to $\gamma$ is not a principal unit, namely $\gamma = 1$ and $\gamma = \alpha_i^{-1}$. In the first case, the factor is equivalent modulo principal units to

$$\frac{(z - z_0)}{-z_0^2} \cdot (\lambda(e_i) - \lambda(e_0))^p$$

and in the second case to

$$\frac{(z - z_0)}{-z_0^2} \cdot (\lambda(e_i) - \lambda(e_0)) .$$

Thus, in order to conclude the proof of (3), it suffices to show:

**Proposition 6.3.** We have

$$\lim \frac{(\lambda' \circ \tau(z)) \cdot (z_0 - z)}{z_0^2} = 1 .$$

**Proof.** This follows from Lemma 6.2 and [8] Lemma 2.1. $\square$

6.2. Case $i = j$. To conclude this article, we prove the formula (1) of Theorem 6.1. Let $d = \gcd(p - 1, 12)$. Let $X = M_{\Gamma_0(p) \cap \Gamma(2)}$ over $\mathbb{Z}^{[\frac{1}{2}]}$.

Let $\mu$ in the function field of $X_{\mathbb{Q}} = M_{\Gamma_0(p) \cap \Gamma(2)} \otimes \mathbb{Q}$ as in Appendix section 7.1.2. We see $\mu$ as a meromorphic function on $X_{\mathbb{Q}}$ (or on $H_{\mathbb{Q}}$ via the uniformization $\tau : H_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$, as the context will make clear). It is shown in the appendix that the divisor of $\mu$ is $6 \cdot \frac{p-1}{d} \cdot ((\xi_c') - (\xi_c))$.

For $z \in H_{\mathbb{Q}}$, let

$$\nu(z) = \left( \prod_{0 \leq \ell \leq g} \theta(\xi_{c\ell}^{(t)}, z_0, z)^{\frac{i \mu}{d}} \right) .$$

**Proposition 6.4.** The two functions $\nu$ et $\mu$ are proportionals.

Let us show how we conclude the proof of Theorem 6.1 before we prove the above proposition. Let $z$ and $z'$ be as in the definition $\Phi$. By definition, we have:

$$\Phi(e_i, \hat{e})^{\frac{i \mu}{d}} = \lim \ (\lambda' \circ \tau(z))^{\frac{i \mu}{d} \cdot \frac{2 \cdot p - 1}{d}} \cdot \frac{\nu(z')}{\nu(z)} .$$

Proposition 6.4 shows that:

$$\Phi(e_i, \hat{e})^{\frac{i \mu}{d}} = \lim \ (\lambda' \circ \tau(z))^{\frac{i \mu}{d} \cdot \frac{2 \cdot p - 1}{d}} \cdot \frac{\mu(z')}{\mu(z)} .$$

The function $\mu$ is algebraic in $\tau(z)$, and since $\tau(z') = w_p(\tau(z))$, $\frac{\mu(z')}{\mu(z)}$ is algebraic in $\tau(z)$. This is also the case for $\lambda'$. Therefore, in (5), it suffices to do a Fourier expansion
computation at the cusp $\xi_c$ of $X$ (recall that $\lambda'(\xi_c) = \lambda(\xi_c) = \infty$, by section 7.1.1 of Appendix)

We know that the Fourier expansion of $\lambda'$ begins with $\frac{\sqrt{16}}{16q_c}$, where $q_c$ is a local parameter at $\xi_c$ (c.f. Appendix, section 7.1.1). Since $\mu(z') = p^{-12/d} \cdot \mu(z)^{-1}$ by Lemma 7.2 and the Fourier expansion of $\mu$ at $\xi_c$ begins with $p^{-12/d} \cdot q_c^{-\frac{(p-1)}{2}}$ by Lemma 7.3, the limit we seek is $\left(\frac{-1}{16}p^{-1} \cdot p\right)^{\frac{12}{d}} = p^{\frac{12}{d}}$ modulo principal units. This concludes the proof of Theorem 6.1.

Let us now prove Proposition 6.4.

We have seen that both functions have divisor $\frac{12}{d} \cdot (g + 1) \cdot (\Gamma \cdot \xi_c - \Gamma' \cdot \xi_c) = \frac{12}{d} \cdot \frac{p - 1}{2} \cdot (\Gamma \cdot \xi_c - \Gamma' \cdot \xi_c)$.

Hence $\frac{\mu}{\nu}$ is entire with no zeroes in $\mathcal{H}_\Gamma$. To conclude, we are going to use the criterion of Corollary 2.4 of [8]. If $\epsilon$ is in an oriented annulus of $\mathcal{H}_\Gamma$, and $f$ is an analytic function on $\mathcal{H}_\Gamma$, the degree $\deg_\epsilon(f)$ of the restriction of $f$ to $\epsilon$ was defined by de Shalit in section 2.2 of [8]. It suffices to show that for any edge $e$ of $T_\Gamma$ (with the orientation of $T_\Gamma$ we have fixed before), $\mu$ and $\nu$ have the same degree on the annulus $\rho^{-1}(e)$. Let $\epsilon$ be such an annulus.

It is clear that $\deg_\epsilon(\nu) = \frac{12}{d}$. Let $F = \tau(A)$ and $F' = \tau(A')$ be affinoids of $X$. We first show that $|\mu(z)| = p^{-\frac{12}{d}}$ for all $z \in F$ and that $|\mu(z')| = 1$ for all $z' \in F'$. Recall that $\mu \circ w_p = p^{-12/d} \cdot \mu^{-1}$ by Lemma 7.2. Therefore it suffices to prove the first norm equality. The affinoid $F$ parametrizes the triples $(E, C, \alpha)$ where $E$ is an elliptic curve with ordinary good reduction at $p$, and $C$ extends to an étale group-scheme over some ring of integer. Likewise $F'$ parametrizes triples where $C$ is of multiplicative type.

If $z \in F$ corresponds to $(E, C, \alpha = (P_1, P_2))$, and $\omega$ is differential on $E$, we let $\omega'$ be the unique differential on $E/C$ whose pullback to $E$ is $\omega$. We then have

$$\frac{\Delta(pz)}{\Delta(z)} = \frac{\Delta(E/C, p^{-1} \cdot \omega')}{\Delta(E, \omega)}.$$ 

Note that the right hand side is independent of the choice of $\omega$, since if we change $\omega$ by $c \cdot \omega$ then both the numerator and the denominator are multiplied by $c^{12}$.

We now choose $\omega$ to be a Néron differential on $E$. Then $\Delta(E, \omega)$ is a unit. Since $C$ extends to an étale group scheme over some integer ring, the isogeny $E \to E/C$ is étale and $\omega'$ is also a Néron differential on $E/C$. Therefore both $\Delta(E, \omega)$ and $\Delta(E/C, \omega')$ are units, and $|u(z)| = p^{-\frac{12}{d}}$ where $u$ is defined in Appendix 7.1.2. The complex correspondance $z \mapsto \frac{z + 1}{2}$ has a moduli interpretation (see the proof of Lemma 7.2) which
shows that the numerator of \( \mu \) has norm \( |p|^{-\frac{24}{d}} \) on \( F \) and its denominator has norm \( |p|^{-\frac{12}{d}} \). Thus \( |u(z)| = |p|^{-\frac{24}{d}} \), as wanted.

Let us finish the proof of Proposition 6.4. Let \( f \) be a rigid analytic function on the annulus \( \epsilon = \{z, |p| < z < 1\} \). Then we see, by definition of the degree, that:
\[
\deg_\epsilon(f) = \lim_{|z| \to |p|} \operatorname{ord}(f(z)) - \lim_{|z| \to |p|} \operatorname{ord}(f(z))
\]
(where \( \operatorname{ord} \) is the \( p \)-adic valuation normalized by \( \operatorname{ord}(p) = 1 \)). Therefore \( \deg_\epsilon(\mu) = 0 - (-\frac{24}{d}) = \frac{12}{d} \), which concludes the proof of Proposition 6.4.

7. Appendix: Some basic facts about \( M_{\Gamma(N)\cap \Gamma_0(p)}(\mathbb{C}) \)

7.1. Case \( N = 2 \).

7.1.1. The cusps. Let \( X = M_{\Gamma_0(p)\cap \Gamma(2)} \) over \( \mathbb{Z}_{[\frac{1}{2}]} \). There is an isomorphism
\[
\Gamma_0(p) \cap \Gamma(2) \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})) \simeq X(\mathbb{C})
\]
induced by the map sending \( z \in \mathcal{H} \) to the triple \((\mathbb{C}/(\mathbb{Z} + z \cdot \mathbb{Z}), <\frac{1}{p}>, (\frac{1}{2}, \frac{1}{2})\)). Let \( \epsilon \) be the involution of \( X \) defined by
\[
\epsilon(E, C, (P, Q)) = (E, C, (-Q, P)).
\]
This induces, with the above identification, an automorphism of \( \Gamma_0(p) \cap \Gamma(2) \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})) \).

The Atkin–Lehner involution on \( X(\mathbb{C}) \) is induced by the automorphism \( z \mapsto \epsilon(\frac{1}{p\cdot z}) \) of \( \Gamma_0(p) \cap \Gamma(2) \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})) \). Indeed, \( \frac{1}{p\cdot z} \) corresponds to the triple
\[
(\mathbb{C}/(\mathbb{Z} + \frac{-1}{p\cdot z} \cdot \mathbb{Z}), <\frac{1}{p}>, (\frac{-1}{2p\cdot z}, \frac{1}{2})) = (\mathbb{C}/(\mathbb{Z} + \frac{1}{p} \cdot \mathbb{Z}), <\frac{z}{p}>, (\frac{-1}{2p}, \frac{z}{2}))
\]
\[
= \epsilon(w_p(\mathbb{C}/(\mathbb{Z} + z \cdot \mathbb{Z}), <\frac{1}{p}>, (\frac{z}{2}, \frac{1}{2}))).
\]

The curve \( X(\mathbb{C}) \) has \( 6 = 2 \cdot \text{Card}(\mathbb{P}^1(\mathbb{F}_2)) \) cusps, namely \([1]_{\Gamma_0(p)\cap \Gamma(2)}, [\frac{1}{2}]_{\Gamma_0(p)\cap \Gamma(2)}, [0]_{\Gamma_0(p)\cap \Gamma(2)}, [\frac{1}{2}]_{\Gamma_0(p)\cap \Gamma(2)}, [\frac{2}{p}]_{\Gamma_0(p)\cap \Gamma(2)} \) and \([\infty]_{\Gamma_0(p)\cap \Gamma(2)} \) (the first three are above the cusp 0 of \( M_{\Gamma_0(p)}(\mathbb{C}) \), and the last three are above \( \infty \); \( w_p \) exchanges the cusps of the first groups and the cusps of the second group). The genus of \( X \) is \( g = \frac{p-3}{2} \), so \( X \) has \( g + 1 = \frac{p-1}{2} \) supersingular points in its special fiber at \( p \). The Hauptmodul \( \lambda : X(2) \to \mathbb{P}^1 \) (defined over \( \mathbb{Z}_{[\frac{1}{2}]} \)) induced over \( \mathbb{C} \) the classical meromorphic modular function for \( \Gamma(2) \):
\[
\lambda(z) = 16q \prod_{n \geq 1} \left( \frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8
\]
where \( q = e^{i \pi z} \). The curve \( X(2) \) has three cusps, namely \([0]_{\Gamma(2)}, [1]_{\Gamma(2)} \) and \([\infty]_{\Gamma(2)} \). We have \( \lambda(0) = 1 \), \( \lambda(i \infty) = 0 \) and \( \lambda(1) = \infty \) (c.f. [10] p. 115).

Let \( c' \) be the (unique) cusp of \( X(\mathbb{C}) \) above \([1]_{\Gamma(2)} \) on \( X(2)(\mathbb{C}) \) and above \([\infty]_{\Gamma_0(p)} \) on \( X_0(p)(\mathbb{C}) \) (via the forgetful maps). Let \( c = w_p(c') \).

The function \( \lambda \) satisfies, for all \( z \in \mathcal{N} \),

\[
\lambda(z + 1) = \frac{\lambda\left(\frac{1}{z}\right) - 1}{\lambda\left(\frac{1}{z}\right)}
\]

(c.f. [10] p.115). In particular, its Fourier expansion at the cusp \([1]_{\Gamma(2)} \) of \( X(2)(\mathbb{C}) \) begins with \( \frac{1}{16q} \), where \( q_1 = e^{\frac{i \pi}{12}} \) is a local parameter at \([1]_{\Gamma(2)} \).

From now on, if \( x \in \mathbb{P}^1(\mathbb{Q}) \), \( q_x \) will denote a local parameter at \([x]_{\Gamma_0(p) \cap \Gamma(2)} \) on \( X(\mathbb{C}) \).

**Proposition 7.1.** Let \( q_0' = e^{-\frac{2i \pi}{3p}} \) and \( q_{\infty}' = e^{2i \pi z} \) be local parameters at the cusps \([0]_{\Gamma_0(p)} \) and \([\infty]_{\Gamma_0(p)} \) respectively on \( X_0(p)(\mathbb{C}) \). Let \( \pi, \pi' : X \to X_0(p) \) given on \( \mathbb{C} \) by \( z \mapsto z \) and \( z \mapsto \frac{z+1}{2} \) respectively. Then we have the following relations:

- \( q_0' \circ \pi = q_0^2 \) et \( q_0' \circ \pi' = q_1^4 \) at \( 1 \),
- \( q_{\infty}' \circ \pi = q_1^4 \) et \( q_{\infty}' \circ \pi' = q_0^1 \) at \( \frac{1}{p} \),
- \( q_0' \circ \pi = q_2^p \) et \( q_0' \circ \pi' = q_2^p \) at \( \frac{1}{2} \),
- \( q_0' \circ \pi = q_0^2 \) et \( q_0' \circ \pi' = q_0^2 \) at \( \frac{1}{2} \),
- \( q_0' \circ \pi = q_0^1 \) et \( q_0' \circ \pi' = q_0^1 \) at \( 0 \),
- \( q_{\infty}' \circ \pi = q_{\infty}^2 \) et \( q_{\infty}' \circ \pi' = q_{\infty} \) at \( \infty \).

Let \( \psi \) be the correspondence \( X \to X_0(p) \) given on complex divisors by \( (z) \mapsto 2(\frac{z+1}{2}) \) \(- (z) \). Then

\[
\psi^{-1}([\infty]_{\Gamma_0(p)}) - ([0]_{\Gamma_0(p)}) = 6([\Gamma_0(p) \cap \Gamma(2)] - ([1]_{\Gamma_0(p) \cap \Gamma(2)}) .
\]

**Proof.** The relations between local parameters are direct (but tedious) computations. The reader can find an example in the proof of Proposition 1 of [9]. \( \square \)

7.1.2. The modular unit \( \mu \). Let \( \eta(z) = q^{\frac{1}{12}} \cdot \prod_{i \geq 1}(1 - q^i) \) be the Dedekind eta function, where \( q = e^{2i \pi z} \), and \( \Delta(z) = \eta(z)^{24} \). Let \( u(z) = \left( \frac{\Delta(pz)}{\Delta(z)} \right)^{\frac{1}{3}} \); this is a rational function of \( X_0(p) = M_{\Gamma_0(p)} \otimes \mathbb{Q} \) (c.f. [10] p. 99 for the fact that we can extract the \( d \)-th root). Let \( \mu \) in the function field of \( X_\mathbb{Q} = M_{\Gamma_0(p) \cap \Gamma(2)} \otimes \mathbb{Q} \) be such that over \( \mathbb{C} \) we have:

\[
\mu(z) = \frac{u(z)^{\frac{z+1}{2}}}{u(z)} .
\]
This is a modular unit: \( \mu \) is meromorphic on \( M_{\Gamma_0(p) \cap \Gamma(2)}(\mathbb{C}) = X(\mathbb{C}) \) with zeros and poles at the cusps only. Since the divisor of \( u \) on \( X_0(p) \) is \( \frac{\ell - 1}{d} \cdot (\infty_{\Gamma_0(p)}) - (0_{\Gamma_0(p)}) \), Proposition 7.1 shows that \( \mu \)'s divisor is \( 6 \cdot \frac{\ell - 1}{d} \cdot ((c') - (c)) \).

**Lemma 7.2.** We have

\[
\mu \circ w_p = p^{-\frac{12}{d}} \cdot \mu^{-1}.
\]

**Proof.** The map \( X \to X_0(p) \) induced on complex points by \( z \mapsto z + \frac{1}{2} \) commutes with \( w_p \). Indeed, since \( p > 2 \), it is given by \( (E,C,(P,Q)) \to (E/ < P + Q>,(C+ < P + Q>/ < P + Q)> \) in moduli terms (here \( < P + Q> \) is the subgroup of \( E \) generated by \( P + Q \)). We then readily see that it commutes with \( w_p \). The lemma follows from the fact that:

\[
u \circ w_p = p^{-\frac{12}{d}} \cdot u^{-1}.
\]

This is true since on complex points we have:

\[
u\left(\frac{1}{pz}\right) = \left(\frac{\Delta\left(\frac{1}{pz}\right)}{\Delta\left(\frac{1}{p^2z}\right)}\right)^\frac{1}{d} = \frac{z^{\frac{12}{d}}}{(pz)^{\frac{12}{d}}} \cdot u^{-1} = p^{-\frac{12}{d}} \cdot u^{-1}(z).
\]

**Lemma 7.3.** The first term of the Fourier expansion of \( \mu \) at the cusp \( c \) of \( X \) is \( p^{-\frac{12}{d}} \cdot q_1 \frac{\sigma_{10}}{\sigma_d} \).

**Proof.** Note that \( c = [1]_{\Gamma_0(p) \cap \Gamma(2)} \), since \([1]_{\Gamma_0(p) \cap \Gamma(2)} \) is a pole of \( \mu \) (see for example the computation below). We have, for \( z = 1 + h \in \mathcal{H} \):

\[
\mu(z) = \frac{u\left(1 + \frac{h}{2}\right)^2}{u(1 + h)} = \frac{u\left(\frac{h}{2}\right)^2}{u(h)}
\]

\[
= p^{-\frac{12}{d}} \cdot \frac{(u \circ w_p)(h)}{(u \circ w_p)(\frac{h}{2})^2}.
\]

The assertion now follows from the fact that the expansion of \( u(z) \) at the cusp \( [\infty]_{\Gamma_0(p)} \) begins with \( (e^{2\pi i z})^{\frac{n-1}{d}} \) (a quick computation shows that \( q_1(z) = e^{\frac{2\pi i z}{p-1}} \)).

**Lemma 7.4.** The function \( \lambda \circ w_p \) of \( X \) has a simple pole at \( c \).

**Proof.** We saw that \( \lambda \) has a simple pole at the cusp \( [1]_{\Gamma(2)} \). A computation shows that \( X \) is unramified at \( c' \) over \( X(2) \) (relatively to the standard projection \( z \mapsto z \) of upper-half planes). Since \( c' \) maps to \( [1]_{\Gamma(2)} \), \( \lambda \) (viewed as a function on \( X \)) also has a simple pole at \( c' \).
7.2. Case $N = 3$.

7.2.1. The cusps. We assume $p \equiv 1 \pmod{3}$. Let $X = M_{\Gamma_0(p) \cap K}$ over $\mathbb{Z}[\frac{1}{3}]$, where $K$ is congruence subgroup of $\text{SL}_2(\mathbb{Z})$ corresponding to the diagonal matrices modulo 3. By Remark 3, $X$ is geometrically irreducible and we have a natural isomorphism

$$X(\mathbb{C}) \simeq (\Gamma_0(p) \cap \Gamma(3)) \setminus (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})).$$

The curve $X(\mathbb{C})$ has $8 = 2 \cdot \text{Card}(\mathbb{P}^1(\mathbb{F}_3))$ cusps, namely $[0]_{\Gamma_0(p) \cap \Gamma(3)}$, $[1]_{\Gamma_0(p) \cap \Gamma(3)}$, $[\frac{1}{3}]_{\Gamma_0(p) \cap \Gamma(3)}$, $[\frac{2}{3}]_{\Gamma_0(p) \cap \Gamma(3)}$, $[\infty]_{\Gamma_0(p) \cap \Gamma(3)}$, $[\frac{2}{p}]_{\Gamma_0(p) \cap \Gamma(3)}$, $[\frac{1}{p}]_{\Gamma_0(p) \cap \Gamma(3)}$, $[\frac{2}{p}]_{\Gamma_0(p) \cap \Gamma(3)}$ (the first three are above the cusp $[0]_{\Gamma_0(p)}$ of $M_{\Gamma_0(p)}(\mathbb{C})$, and the last three are above $[\infty]_{\Gamma_0(p)}$; $w_p$ exchanges the first group of three cusps with the second group). The genus of $X$ is $g = p - 2$, so $X$ has $g + 1 = p - 1$ supersingular points in its special fiber $p$. We choose the Hauptmodul $H : M_K \xrightarrow{\sim} \mathbb{P}^1$ (defined over $\mathbb{Z}[\frac{1}{3}]$) such that over $\mathbb{C}$ we get the classical meromorphic function for $\Gamma(3)$:

$$H(z) = \left( \frac{\eta(\frac{z}{3(1 - z)})}{\eta(\frac{3z}{1 - z})} \right)^3$$

where $\eta$ is the Dedekind eta function.

Let $c'$ be the (unique) cusp of $X(\mathbb{C})$ above $[1]_{\Gamma(3)}$ on $X(3)(\mathbb{C})$ and above $[\infty]_{\Gamma_0(p)}$ on $X_0(p)(\mathbb{C})$ (via the forgetful maps). Let $c = w_p(c')$.

The function $H$ has a pole of order 1 at the cusp $[1]_{\Gamma(3)}$ of $M_H$ (because a local parameter at $[1]_{\Gamma(3)}$ is $e^{\frac{2\pi iz}{3(g - 1)}}$). For $0 \leq i \leq 2$, let $\pi_i(z) = \frac{z + i}{3}$, $\pi_{-1}(z) = z$ and $\pi_2(z) = 3z$. Then the $\pi_i$’s induce maps $X \to X_0(p)$. We computed in Table 1 the ramification index of these maps at the cusps of $X$ above $[0]_{\Gamma_0(p)}$ (the result being the same for the cusps above $[\infty]_{\Gamma_0(p)}$, thanks to $w_p$).

7.2.2. The modular unit $\mu$. We define

$$\mu(z) = \frac{u(\pi_2(z))^3}{u(\pi_{-1}(z))} = \frac{u(\frac{z + 2}{3})^3}{u(z)}$$

where, as in the case $N = 2$, $u(z) = \left( \frac{\Delta(pz)}{\Delta(z)} \right)^{\frac{1}{2}}$ (with $d = \gcd(p - 1, 12)$). The divisor of $\mu$ is $24 \cdot \frac{p - 1}{d} \cdot ((c') - (c))$. Its degree on the various annuli is $\frac{24}{d}$. We have

$$\mu \circ w_p = p^{-\frac{12}{d}} \cdot \mu^{-1}.$$ 

Finally, $H$ (viewed as a function on $X$) has a double pole at $c'$. 


ON THE p-ADIC PERIODS OF THE MODULAR CURVE $X(\Gamma_0(p) \cap \Gamma(2))$.

Table 1. Ramification index at the cusps

| Cusp of X | Local parameter | $\pi_{-1}$ | $\pi_{0}$ | $\pi_{2}$ | $\pi_{3}$ |
|-----------|-----------------|------------|------------|------------|------------|
| 0         | $e^{\frac{-3p\pi}{4}}$ | 3          | 1          | 1          | 9          |
| $\frac{1}{3}$ | $e^{\frac{-2\pi}{9p(3z-1)}}$ | 3          | 1          | 1          | 9          |
| 1         | $e^{\frac{2\pi}{9p(2z-1)}}$ | 3          | 1          | 1          | 9          |
| $\frac{1}{2}$ | $e^{\frac{-3\pi}{9p(2z-1)}}$ | 3          | 1          | 9          | 1          |

We define:

$$\nu(z) = \left( \prod_{0 \leq \ell \leq g} \theta(\xi^{(l)}_{\ell}, z_0, z) \right)^{\frac{24}{d}}.$$  

The same proof as in Proposition 6.4 shows that $\mu$ and $\nu$ are proportionals. A Fourier expansion computation shows the analogue of Theorem 1.1 in the case $N = 3$, with $\lambda$ replaced by $H$.

References

[1] P. Deligne and M. Rapoport, *Les Schémas de Modules de Courbes Elliptiques*, Modular Functions of One Variable II Volume 349 of the series Lecture Notes in Mathematics pp 143-316
[2] A. Grothendieck and J. Dieudonné, *étude cohomologique des faisceaux cohérents*, J. Publ. Math. IHES, 4,8,11,17,20,24,28,32
[3] W. Lütkebohmert, *Rigid Geometry of Curves and Their Jacobians*, A Series of Modern Surveys in Mathematics Volume 61 (2016).
[4] Y. Manin and V. Drinfeld, *Periods of p-adic Schottky groups*, Journal Crelle, 0262-0263 (1973):239-247.
[5] L. Gerritzen and M. Van der Put, *Schottky Groups and Mumford Curves*. Notes in Mathematics-Volume 817-1980.
[6] N. Katz and B. Mazur, *Arithmetic moduli of elliptic curves*. Annals of Math. Studies 108, Princeton Univ. Press, Princeton. 1985.
[7] D. Mumford, *An analytic construction of degenerating curves over complete local rings*. Compositio Mathematica 24.2 (1972):129-174.
[8] E. de Shalit, *On the p-adic periods of $X_0(p)$*. Mathematische Annalen (1995). Volume: 303, Issue: 3, page 457-472
[9] Merel, *L’accouplement de Weil entre le sous-groupe de Shimura et le sous-groupe cuspidal de $J_0(p)$*. J. Reine Angew. Math (1996). Volume: 477, pages 71 – 115.
[10] J. Borwein and P. Borwein, *Pi and the AGM*. Canadian Mathematical Society Series of Monographs and Advanced Texts, 4 (1998), pages xvi+414.
[11] Mazur, B., *Modular curves and the Eisenstein ideal*. Inst. Hautes Études Sci. Publ. Math. (1977). Volume: 47, pages 33–186.
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