On the Relativity in Configuration Space: A Renewed Physics In Sight

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Abstract

The idea that possible configurations of a physical system can be represented as points in a multidimensional configuration space \( C \) is further explored. Matter configurations are not considered as living in a 4-dimensional spacetime. The notion of spacetime itself, without \( C \), does not exist in this theory. Spacetime is associated with the degrees of freedom of a chosen single particle within a considered configuration, and is thus a subspace of \( C \). Finite dimensional configuration spaces of point particles, and infinite dimensional configuration spaces of strings and branes are considered. Quenched description of extended objects in terms of a finite number of degrees of freedom is also considered. It is pointed out that the multidimensionality of a configuration space has for a consequence the existence of not only the 4-dimensional gravity, but also of other interactions. All those interactions are incorporated in the metric, connection and the curvature of \( C \).

1 Introduction

Occasionally a fresh look at a well established theory may bring surprises. In this paper I will discuss and further develop an approach to description of many particle and extended systems which was considered in ref. [1, 2]. Related work has been done in refs. [3]–[7]. Paraphrasing Feynman\(^2\) a full understanding of one and the same physics requires at least six or seven different representation. According to that famous remark it should be thus desirable to discover some alternative ways of describing a system of point particles or a system of strings and branes. The latter objects are amongst the hottest topics of current research in fundamental theoretical physics.

Usually, all those objects are considered to live in a background spacetime. In spacetime we thus have “matter” consisting of all sorts of physical objects, such as branes of various dimensionalities, including point particles and strings. But we can look at the situation from another angle. We can consider spacetime as a space of all possible positions of a chosen single particle (a “test particle”) while keeping fixed positions of all other particles. In other words, spacetime can be considered as the configuration space of a single point particle relative to the (assumed) fixed position of the remaining particles within the ‘full’ configuration.

The configuration space of a single point particle is just a start in a construction of physical theories. We can include other particles and extended objects into the description as well, and consider a multidimensional configuration space of a system of particles or extended objects. Usually, an action for a system of (free) point particles or, in general, branes, is written as the sum of one particle (brane) action. But there is a fascinating possibility to go beyond the existing physics which takes place in spacetime. We can formulate physics in configuration space and take the latter space as the arena for physics. Similar approaches were previously proposed within the context of an infinite dimensional space of branes, called \( M \)-space [1], and within the context of 16-dimensional space of points, areas and volumes, called Clifford space, or shortly, \( C \)-space [9]–[12],[1]. Both spaces are particular cases of configuration space. The former one is an infinite dimensional space of all possible brane (or many brane) configurations, while the latter one is a space of all possible polyvectors (superpositions of \( r \)-vectors) associated with extended

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\(^2\)...every theoretical physicist who is any good knows six or seven different theoretical representations of exactly the same physics [8].\(^3\).
objects. Clifford space $C$ is a manifold whose tangent space at any point is a Clifford algebra. $C$ can be flat, but in general it can be curved. Flat Clifford space ($C$-space) provides a possible generalization of special relativity, whilst curved $C$-space provides a generalization of general relativity. Metric of curved 16-dimensional $C$-space can describe, à la Kaluza-Klein, the ordinary 4-dimensional gravitational field and gauge fields due to other interactions [12].

2 System of point particles

A system of free relativistic point particles in $N$-dimensional spacetime can be described by the action

$$I^{(i)}[X^\mu] = \sum_{i=1}^{n} m_i \int d\tau_i (\dot{X}_i^\mu \dot{X}_i^\mu)^{1/2}, \quad i = 1, 2, ..., n; \quad \mu = 0, 1, 2, N-1$$

which is the sum of single particle actions. Here $n$ denotes the number of particles and $m_i$ the $i$-th particle mass. Equations of motion derived from (1) are

$$m_i \frac{d}{d\tau_i} \left( \frac{\dot{X}_i^\mu}{\sqrt{\dot{X}_i^2}} \right) = 0,$$

where $\dot{X}_i^2 \equiv \dot{X}_i^\mu \dot{X}_i^\nu = \dot{X}_i^2(\tau_i)$. Each particle within the system moves as free particle, its mass being $m_i$.

Let us now consider the following action:

$$I^{(i)}[X^\mu] = M \int d\tau (\dot{X}_1^\mu \dot{X}_1^\nu \eta_{\mu\nu} + \dot{X}_2^\mu \dot{X}_2^\nu \eta_{\mu\nu} + \dot{X}_3^\mu \dot{X}_3^\nu \eta_{\mu\nu} + ... + \dot{X}_n^\mu \dot{X}_n^\nu \eta_{\mu\nu})^{1/2},$$

where $M$ is a constant and $\tau$ and arbitrary monotonically increasing parameter.

Writing $\dot{X}_i^\mu \equiv \dot{X}^{\mu(i)} \equiv \dot{X}^M$, $M \equiv (\mu i)$, $\mu = 0, 1, 2, 3$; $i = 1, 2, ..., n$, then the action (3) becomes

$$I[X^M] = M \int d\tau (\dot{X}^M \dot{X}^N \eta_{MN})^{1/2},$$

where

$$\eta_{MN} \equiv \eta_{(\mu i)(\nu j)} = \eta_{\mu\nu} \delta_{ij}.$$  

Eq. (4) is the minimal length action in flat configuration space $C$ spanned by a system of free point particles, $\eta_{MN}$ being the diagonal metric of $C$. The corresponding equations of motion are

$$M \frac{d}{d\tau} \left( \frac{\dot{X}^M}{(\dot{X}^M \dot{X}^M)^{1/2}} \right) = 0.$$  

Action (4) is invariant with respect to reparametrizations of $\tau$. Taking a gauge in which $\dot{X}^N \dot{X}_N \equiv \dot{X}^{\mu(i)} \dot{X}^{\nu(j)} \eta_{(\mu i)(\nu j)}$ is constant, we have $\dot{X}^M \equiv \dot{X}^{\mu(i)} = 0$, which implies that $\dot{X}^{\mu(i)}$ is constant. As a consequence, also the quadratic form $\dot{X}^{\mu(i)} \dot{X}^{\nu(j)} \eta_{(\mu i)(\nu j)}$ for a single particle, labeled by $i$, is constant, which is just a gauge fixing condition for the ordinary equation of motion (2). Such choice of gauge in eq. (2) also gives $\dot{X}^{\mu(i)} = 0$.

We thus see that the second action (4) gives the same equations of motion for the $i$-th particle as the usual action (1). For free particles we may use either the usual action which is the sum of point particle actions, or we may use the action (4) which is proportional to the length in configuration space $C$. The difference occurs when we consider interactions. This will be explored in next sections. The form of the action (4) suggests that we have now the theory of relativity in configuration space, quite analogous to the theory of relativity in spacetime.
In the case of flat configuration space \( \mathcal{C} \) the law of motion is given by eq. (3) which says that a configuration, represented by a point in \( \mathcal{C} \), traces a flat worldline in \( \mathcal{C} \). This means that in spacetime, every particle traces a flat worldline.

Just as the ordinary Lorentz transformations preserve the quadratic form

\[
(x^\mu - x_0^\mu)(x^\nu - x_0^\nu)\eta_{\mu\nu} = (x^\mu - x_0^\mu)(x^\nu - x_0^\nu)\eta_{\mu\nu}
\]

so in the configuration space we have analogous transformations which preserve

\[
(x^M - x_0^M)(x^N - x_0^N)\eta_{MN} = (x^M - x_0^M)(x^N - x_0^N)G_{MN}.
\]

Thus \((x^M - x_0^M) = L^M_J(x^J - x_0^J)\), where the transformation matrix has to satisfy \(L^M_J L^N_K \eta_{MN} = \eta_{JK}\). Here \(x^M - x_0^M \equiv x^\mu - x_0^\mu\) is the difference of coordinates of two configurations. The group of Lorentz transformations in multidimensional space \( \mathcal{C} \) contains a subgroup of the ordinary Lorentz transformations that preserve the 4-dimensional quadratic form \( \eta \). According to this picture Lorentz transformations in spacetime are just particular transformations, whereas in general we have Lorentz transformations in \( \mathcal{C} \). A consequence is that the spacetime Lorentz symmetry is violated, because of the presence of the extra degrees of freedom which are also involved in the transformation \( \eta \).

### 3 Point particle in curved configuration space

We will assume that, in general, a space \( \mathcal{C} \) need not be flat, but may have non vanishing curvature. Instead of the flat space action (1) we have now the action in the presence of a background metric field \( G_{MN}(X) \) which depends on points \( x^M \) of \( \mathcal{C} \):

\[
I[X^M] = M \int d\tau \left( \dot{X}^M(\tau) \dot{X}^N(\tau) G_{MN} \right)^{1/2}.
\]

From the point of view of the underlying 4-dimensional spacetime \( M_4 \) (which is a subspace of \( \mathcal{C} \)) we have a system of worldlines, described by functions \( X^M(\tau) \equiv X^{\mu}(\tau) \). If there are no interactions between the particles, then the worldlines are straight lines in \( M_4 \); a point in \( \mathcal{C} \) traces a straight line.

Configuration space \( \mathcal{C} \) is then flat and its metric \( G_{MN} \) is that of a flat space, considered in previous section. One can choose a coordinate system in which \( G_{MN} \equiv G_{(i\mu)(j\nu)} \) is diagonal metric at all points of \( \mathcal{C} \):

\[
G_{MN} = \eta_{MN}.
\]

In general this need not be the case: The metric \( G_{MN} \) of the configuration space can have non vanishing off diagonal terms that cannot be transformed away by a choice of coordinates.

The off diagonal terms \( G_{(i\mu)(j\nu)} \), \( i \neq j \) are responsible for the interactions between the particles which according to this novel theory exist besides the ordinary gravitational interaction incorporated in the metric \( G_{(i\mu)(j\nu)} \equiv g_{\mu\nu} \). In ref. [2] we provided arguments that such approach might explain on the cosmological and astrophysical scales the puzzles of “dark matter” or “missing mass”, and on the microscopic scale the existence of electroweak and color interactions.

From the action (9) we obtain the equation of geodesic in the presence of a metric \( G_{MN} \):

\[
\sqrt{X^2} \frac{d}{d\tau} \left( \frac{\dot{X}^M}{\sqrt{X^2}} \right) + \Gamma^M_{JK} \dot{X}^J \dot{X}^K = 0.
\]

The configuration space metric \( G_{MN} \) causes that a worldline \( X^M(\tau) \) in general is not a straight line in \( \mathcal{C} \) and thus also the worldlines of particles are not straight lines in \( M_4 \). The term with

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\[3\] In analogous way a 2-dimensional rotation symmetry is violated if we allow for the transformations in 3-dimensions: a distance in 2-dimensions is then just a projection of the distance in 3-dimensions, and is not preserved under a general rotation in 3-dimensions.
connection $\Gamma$ occurring in the geodesic equation (11) has a role of a “force” in $\mathcal{C}$, and manifests itself in $M_4$ as the interactions between the particles.

In this approach a configuration of a system of particles is considered as a whole, namely as a point in configuration space, which is the space of all possible configurations. We postulate that the latter space is endowed with metric, connection, and curvature. Metric $G_{MN}$ should be considered as a dynamical quantity, its kinetic term being given by the Einstein-Hilbert action in $\mathcal{C}$ whose dimension is $D = N \times n$:

$$I[G_{MN}] = \frac{1}{16\pi} \int d^Dx \sqrt{|G|} R,$$

where $R$ is the Ricci scalar in $\mathcal{C}$, and $G \equiv \det G_{MN}$. The total action is the sum of $I[X^M]$ and $I[G_{MN}]$. It is invariant under general coordinate transformations in $\mathcal{C}$.

We will leave aside a detailed study of solutions to such a system. For the purpose of the present paper it suffices if we make a plausible assumption that within a set of solutions there exist solutions with isometries. Let us therefore suppose that as a solution to our dynamical system there can exist a space $\mathcal{C}$ which admits $K$ Killing vector fields $k^\alpha_M$, $\alpha = 1, 2, ..., K$, satisfying $D_M k^\alpha_N + D_N k^\alpha_M = 0$, where the covariant derivative $D_M$ is defined with respect to the metric $G_{MN}$ of configuration space $\mathcal{C}$.

Let us split the indices according to $M = (\mu, 1, \bar{M})$, where $\mu = 0, 1, 2, 3$ are indices of coordinates of a chosen single particle, say a particle No. 1 (i.e., with $i = 1$), whilst $\bar{M}$ are indices of coordinates of all the remaining particles within the system. Then the metric can be written as

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} - \phi^M k_\alpha M k^\beta \bar{N} A^\alpha_\mu A^\beta_\nu, & k_\alpha \bar{M} A^\alpha_\mu \\ k_\alpha \bar{N} A^\alpha_\mu, & \phi^M \bar{N} \end{pmatrix},$$

where $\phi^M \bar{N}$ is the inverse of $\phi^M N$ in the “internal space”, and where a coordinate system in which $k_\alpha^\mu = 0$ and $k_\alpha M \neq 0$ has been used.

Using metric (13), a quadratic form in $\mathcal{C}$ can be split into a 4-dimensional part plus the part due to the remaining dimensions of $\mathcal{C}$. We will now apply this to the action (9). Since it is a reparametrization invariant action, there exists a constraint

$$P^M P^N G_{MN} - M^2 = 0,$$

where

$$P^M = \frac{M X^M}{(X^N X_N)^{1/2}}$$

are contravariant components of the momentum conjugate to coordinates $X^M$. Inserting eq. (13) into eq. (14) we have

$$M^2 = g_{\mu\nu} p^\mu p^\nu + \phi^M \bar{N} p^M p_N,$$

where $g_{\mu\nu} = G_{\mu\nu} - \phi^M k_\alpha M k^\beta \bar{N} A^\alpha_\mu A^\beta_\nu$. From eq. (16) we find that the 4-dimensional mass is

$$m \equiv \sqrt{g_{\mu\nu} p^\mu p^\nu} = \sqrt{M^2 - \phi^M N p^M p_N}.$$

According to the latter relation, a mass $m$ of a single particle, defined by means of the 4-dimensional momentum quadratic form, depends on the momenta $p_M \equiv p_{\mu i}$, $i \neq 1$, of all the remaining particles within the considered system, which could be the entire universe. This is reminiscent of Mach’s principle.

Now let us investigate whether the 4-dimensional mass $m$ can be a constant of motion. Obviously, the configuration space mass $M$ is constant, whatever the metric $G_{MN}$. In a trivial case, if $G_{MN} = \eta_{MN}$ at all points of $\mathcal{C}$, then $m$ is a constant of motion. We will show that $m$ can be a constant of motion in the case of a more general metric as well, if the space $\mathcal{C}$ admits suitable isometries.
The metric $\phi^{MN}$ of the internal space can be rewritten in terms of a metric $\varphi^{\alpha\beta}$ in the space of isometries:

$$
\phi^{MN} = \varphi^{\alpha\beta} k^{N}_{\alpha} k^{M}_{\beta} + \phi_{\text{extra}}^{MN}.
$$

Here $\phi_{\text{extra}}^{MN}$ are additional terms due to the directions that are orthogonal to isometries. For particular internal spaces $\tilde{C}$, those additional terms may vanish. Let us assume that this is the case.

Inserting eq. (18) into eq. (17) we have

$$
m = \left( M^2 - \varphi^{\alpha\beta} p_{\alpha} p_{\beta} \right)^{1/2},
$$

where $p_{\alpha} \equiv k^{M}_{\alpha} p_{M}$ is a constant of motion due to the $\alpha$-th isometry. So also 4-dimensional mass $m$ is a constant of motion in this particular case of appropriate isometries.

A consequence of the latter property is that a particle accelerated by means of “forces” due to the metric (13) of a configuration space $C$ which admits the above isometries cannot exceed the speed of light in 4-dimensional spacetime.

This can be shown by considering the momentum (15) and the relation

$$
\dot{X}^{M} \dot{X}^{N} G_{MN} = \dot{X}^{\mu} \dot{X}^{\nu} g_{\mu\nu} + \dot{X}_{M} \dot{X}_{N} \phi^{MN}
$$

and eq. (16). Here

$$
g_{\mu\nu} = G_{\mu\nu} - \phi^{MN} k^{M}_{\alpha} k^{N}_{\beta} A^{\alpha}_{\mu} A^{\beta}_{\nu}.
$$

is 4-dimensional metric. Using eq. (20), the momentum (15) can be rewritten as

$$
p^{M} = \frac{m \dot{X}^{M}}{(\dot{X}^{\rho} \dot{X}^{\sigma} g_{\rho\sigma})^{1/2}},
$$

For the components $M = \mu 1 \equiv \mu = 0, 1, 2, 3$ of a single particle within our multiparticle system we have thus

$$
p^{\mu} = \frac{m \dot{X}^{\mu}}{(\dot{X}^{\rho} \dot{X}^{\sigma} g_{\rho\sigma})^{1/2}}
$$

If $m$ is a constant, which is indeed the case in the presence of the considered isometries, then, according to eq. (24), the condition for $p^{\mu}$ to remain real is

$$
\dot{X}^{\rho} \dot{X}^{\sigma} g_{\rho\sigma} \geq 0.
$$

In other words, in spite of the fact that the particle is being accelerated (i.e., moving along a geodesic of the configuration space $C$), its limiting speed in the subspace $M_4$ is the speed of light. This is not the case in a more general configuration space, which does not admit Killing vector fields. Then the general expression for momentum eq. (15) cannot be reduced to the form (23) with $m$ being a constant of motion. A prediction of this theory is thus that the speed of light in $M_4$ is the limiting speed for a particle which is accelerated by gauge fields $A^{\alpha}_{\mu}$ (including the

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4One possibility is to choose isometries $k^{M}_{\alpha}$ in the full configuration space $C$. Then the projections $p_{\mu}$ of momenta $P_{M}$ onto the Killing vectors $k^{M}_{\alpha}$ are constants of motion. Another possibility is to consider the isometries of the ‘internal’ subspace of $C$. Then $p_{\alpha} = k^{M}_{\alpha} p_{M}$ are not constants in general, whereas the quadratic form $\varphi^{\alpha\beta} p_{\alpha} p_{\beta}$ can be a constant.
electromagnetic field $A_\mu$) that arise in the presence of isometries, but is not a limiting speed in a more general case when isometries are absent.

The above refers to the limiting speed in 4-dimensional spacetime, which is a subspace of the configuration space $C$. In the latter larger space, because of the relation \[ \dot{X}^M \dot{X}^N G_{MN} > 0 \] regardless of whether there are isometries or not. That is, in $C$ there is a limiting speed, determined by the condition \[ \dot{X}^2 > 0. \] The latter limiting speed involves not only four spacetime components due to a single particle, but also components $\dot{X}^M$ due to the presence of other particles.

Let us now explicitly show that in the presence of isometries, which imply that $\varphi_{\alpha\beta} p_\alpha p_\beta$ is constant, conditions \[ \varphi_{\alpha\beta} p_\alpha p_\beta = \eta^{\alpha\beta} p_M p_N = \frac{M^2 \varphi_{\alpha\beta} X_{\alpha} X_{\beta}}{X^\mu X^\nu g_{\mu\nu} + \varphi_{\mu\nu} X_M X_N} \] (27) in which we denote $\dot{X}^\mu X^\nu g_{\mu\nu} \equiv X$, $\varphi_{\alpha\beta} X_{\alpha} X_{\beta} \equiv Y$ and $\varphi_{\alpha\beta} p_\alpha p_\beta / M^2 \equiv C$ we have that for a fixed chosen constant $C$ there is a proportionality between $X$ and $Y$:

\[ Y = \frac{C}{1 - C} X. \] (28)

Therefore $X$ and $Y$ cannot change independently; if $X$ approaches zero, also $Y$ approaches zero, and so does the sum $X + Y \equiv G_{MN} X^M X^N$. This proves consistency of the conditions \[ \varphi_{\alpha\beta} p_\alpha p_\beta = \eta^{\alpha\beta} p_M p_N \] and \[ \varphi_{\alpha\beta} X_{\alpha} X_{\beta} \equiv Y \] does not hold. However, condition \[ \varphi_{\alpha\beta} p_\alpha p_\beta / M^2 \equiv C \] which imposes a restriction on velocities in configuration space remains valid.

The signature of the configuration space $C$ is in general $(p, q)$, and thus in $C$ there is no separation between different regions, that could be identified with past, present and future. And yet, if $C$ admits isometries, as described above, then the concept of light cone in a subspace $M_4$, with distinction between past, present and future, makes sense. This is so, because a particle’s 4-dimensional mass $m$ is then a constant of motion, and no particle can pass the light barrier during its motion in $M_4$.

The ordinary relativity in 4-dimensional spacetime is thus embedded in the more general relativity that holds in a multidimensional space $C$.

4 Configuration space for strings and branes

String and brane theories are very elegant and promising in explaining the origin and inter-relationship of the fundamental interactions, including gravity \[ 15, 14]. But such theories are still far from being finished. One of the unsettled problems is a question of the geometric principle behind the string and brane theories \[ 15]. For a recent serious criticism see \[ 16]. In the following we will consider the possibility that string/brane theories should take into account the concept of configuration space.

A brane configuration can be described by the set of functions $X^\mu(\xi^a)$, where $\xi^a$, $a = 1, 2, ..., n$, is a set of parameters on the brane. We will consider a brane configuration as a point in an infinite dimensional configuration space, called brane space $\mathcal{M}$. Following refs. \[ 1, 2], we will therefore use a condensed notation

\[ X^\mu(\xi^a) \equiv X^\mu(\xi) \equiv X^M. \] (29)

We assume that the branes within classes of tangentially deformed branes are in principle physically distinct objects. All such objects are represented by different points of $\mathcal{M}$-space.

Instead of one brane we can take a 1-parameter family of branes $X^\mu(\tau, \xi^a) \equiv X^\mu(\xi)(\tau) \equiv X^M(\tau)$, i.e., a curve (trajectory) in $\mathcal{M}$. In principle every trajectory is kinematically possible. A particular dynamical theory then selects which amongst those kinematically possible branes
and trajectories are dynamically possible. We assume that dynamically possible trajectories are geodesics in $\mathcal{M}$ determined by the minimal length action [1]:

$$I[X^M] = \int d\tau \, (\rho_{MN} \dot{X}^M \dot{X}^N)^{(1/2)}. \quad (30)$$

Here $\rho_{MN}$ is the metric of $\mathcal{M}$.

In particular, if metric is

$$\rho_{MN} \equiv \rho_{\xi(\xi')\nu(\xi'')} = \kappa \frac{\sqrt{f(\xi')}}{\sqrt{X^2(\xi')}} \delta(\xi' - \xi'') \eta_{\mu\nu}, \quad (31)$$

where $f_{ab} \equiv \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$ is the induced metric on the brane, $f \equiv \det f_{ab}$, $X^2 \equiv X^\mu \dot{X}^\nu g_{\mu\nu}$, $(\eta_{\mu\nu}$ being the Minkowski metric of the embedding spacetime), then the equations of motion derived from (30) are precisely those of a Dirac-Nambu-Goto brane [1]. Although we started from a brane configuration space in which tangentially deformed branes are considered as distinct objects, the dynamical theory, based on the action (30) and the particular choice of metric (31), has for solutions the branes which satisfy such constraints which imply that only the transversal excitations are physical, whereas the tangential excitations are nothing but reparametrizations of $\xi^a$ and $\tau$. For more details see ref. [1].

In this theory we assume that metric (31) is just one particular choice amongst many other possible metrics of $\mathcal{M}$. But dynamically possible metrics are not arbitrary. We assume that they must be solutions of the Einstein equations in $\mathcal{M}$ [1].

We take the brane space $\mathcal{M}$ as an arena for physics. The arena itself is a part of the dynamical system, it is not prescribed in advance. The theory is thus background independent. It is based on the geometric principle which has its roots in the brane space $\mathcal{M}$.

To sum up, the infinite dimensional brane space $\mathcal{M}$ has in principle any metric that is a solution to the Einstein’s equations in $\mathcal{M}$. For the particular diagonal metric (31) we obtain the ordinary branes, including strings. But it remains to be checked whether such particular metric is a solution of this generalized dynamical system at all. If not, then this would mean that the ordinary string and brane theory is not exactly embedded into the theory based on dynamical $\mathcal{M}$-space. The proposed theory goes beyond that of the usual strings and branes. It resolves the problem of background independence and the geometric principle behind the string theory. Geometric principle behind the string theory is based on the concept of brane space $\mathcal{M}$, i.e., the configuration space for branes. Occurrence of gauge and gravitational fields in string theories is also elucidated. Such fields are due to string configurations. They occur in the expansion of a string state functional in terms of the Fock space basis. A novel insight is that they occur even within the classical string theory based on the action (30) with $\mathcal{M}$-space metric $\rho_{MN}$, which is dynamical and satisfies the Einstein equations in $\mathcal{M}$. Multidimensionality of $\rho_{MN}$ allows for extra gauge interactions, besides gravity. In the following we will discuss how in the infinite dimensional space $\mathcal{M}$ one can factor out a finite dimensional subspace.

5 Finite dimensional description of extended objects

When considering the motion of macroscopic extended objects such as planets, we usually take into account a finite set of degrees of freedom only, e.g., the coordinates of the center of mass, and neglect all the remaining many degrees of freedom. Similarly, when considering, e.g., a closed string, we can describe it in the first approximation by four coordinates $X^\mu$ of the center of mass. In the next approximation we can describe it in terms of the coordinates $X^{\mu_1\mu_2}$ of the oriented area enclosed by the string. If the string has finite thickness and thus it actually is not a string but a 2-brane, then we can also consider the corresponding volume degrees of freedom $X^{\mu_1\mu_2\mu_3}$.

In general, an extended object in 4-dimensional spacetime can be described by 16 coordinates

$$X^M \equiv X^{\mu_1 \cdots \mu_r}, \quad r = 0, 1, 2, 3, 4. \quad (32)$$
They are the projections of \(r\)-dimensional volumes (areas) onto the coordinate planes, and they denote a point in a 16-dimensional space \(C\), which is a subspace of the full infinite dimensional space \(\mathcal{M}\), the configuration space of the considered extended object.

Oriented \(r\)-volumes can be elegantly described by Clifford algebra [17]. Let us illustrate this on the example of a 2-surface \(\Sigma\) bounded by a loop \(B\). An infinitesimal surface element is given by the wedge product of two infinitesimal vectors \(d\xi_1\) and \(d\xi_2\), expanded in terms of the basis tangent vectors \(e_a\), \(a = 1, 2\):

\[
d\Sigma = d\xi_1 \wedge d\xi_2 = d\xi_1^a d\xi_2^b e_a \wedge e_b = \frac{1}{2} d\xi^{ab} e_a \wedge e_b, \tag{33}
\]

where \(d\xi^{ab} = d\xi_1^a d\xi_2^b - d\xi_2^a d\xi_1^b\). Inserting the relation \(e_a = \partial_a X^\mu \gamma_\mu\) between the basis vector \(e_a\) tangent to the surface \(\Sigma\) and the basis vectors \(\gamma_\mu\) of the embedding space (time), and integrating over \(d\Sigma\), we have

\[
\int_{\Sigma_B} d\Sigma = \frac{1}{2} X^{\mu \nu} \gamma_\mu \wedge \gamma_\nu, \tag{34}
\]

where

\[
X^{\mu \nu} = \frac{1}{2} \int_{\Sigma_B} d\xi^{ab} (\partial_a X^\mu \partial_b X^\nu - \partial_a X^\nu \partial_b X^\mu), \tag{35}
\]

which, by the Stokes theorem, gives

\[
X^{\mu \nu} = \frac{1}{2} \oint_{\Sigma_B} ds \left( X^\mu \frac{\partial X^\nu}{\partial s} - X^\nu \frac{\partial X^\mu}{\partial s} \right). \tag{36}
\]

Here \(X^\mu(s)\) are embedding functions of the boundary loop \(B\), \(s\) being a parameter along the loop. Eq. (36) tells us that there is a mapping

\[
X^\mu(s) \rightarrow X^{\mu \nu} \tag{37}
\]

from infinite dimensional objects \(X^\mu(s)\), describing loops, into the finite dimensional objects \(X^{\mu \nu}\).

The above arrangement can describe two physically distinct situations:

(i) A loop \(B\) can be a closed string. Then \(X^{\mu \nu}\) are bivector coordinates associated with the closed string.

(ii) A surface \(\Sigma\) can correspond to an open 2-brane whose boundary is \(B\). Then \(X^{\mu \nu}\) are bivector coordinates associated with the open 2-brane.

Analogous setup holds for objects of arbitrary dimensions:

\[
X^{\mu_1 \mu_2 ... \mu_3} = \frac{1}{2} \int_{B_r} d\xi^{\alpha_1 ... \alpha_r} \partial_{[\alpha_1} X^{\mu_1} ... \partial_{\alpha_r]} X^{\mu_3}. \tag{38}
\]

Instead of the usual relativity, formulated in spacetime in which the interval is

\[
ds^2 = \eta_{\mu \nu} dx^\mu dx^\nu, \tag{39}
\]

one can consider the theory in which the interval is extended to the space of \(r\)-volumes, called pandimensional continuum [18] or Clifford space [5, 11]:

\[
ds^2 = G_{MN} dx^M dx^N. \tag{40}
\]

Coordinates of Clifford space can be used to model extended objects [5, 11, 6, 7]. They are a generalization of the concept of center of mass. Instead of describing an extended object in “full detail”, we can describe it in terms of the center of mass, area and volume coordinates. In particular, the extended object can be a fundamental string/brane.
Dynamics. Taking also a time like parameter $\tau$, our object can be described by 16 functions $X^M(\tau)$. Let the action for an extended object described in terms of the coordinates of Clifford space be

$$I = \int d\tau \left( G_{MN} \dot{X}^M \dot{X}^N \right)^{1/2}. \quad (41)$$

If $G_{MN} = \eta_{MN}$ is Minkowski metric, then the equations of motion are

$$\ddot{X}^M \equiv \frac{d^2X^M}{d\tau^2} = 0. \quad (42)$$

They hold for tensionless branes. For the branes with tension one has to replace $\eta_{MN}$ with a generic metric $G_{MN}$ with non vanishing curvature. Eq. (42) then generalizes to the corresponding geodesic equation

$$\frac{1}{\sqrt{X^2}} \left( \frac{\dot{X}^M}{\sqrt{X^2}} \right) + \Gamma^{M}_{JK} \frac{\dot{X}^J \dot{X}^K}{X^2} = 0. \quad (43)$$

As an example let us consider the Dirac membrane, described by four embedding functions of three parameters $\xi^a = (\tau, \vartheta, \phi)$:

$$X^\mu(\xi^a) = (X^0, r \sin \vartheta \cos \phi, r \sin \vartheta \sin \phi, r \cos \vartheta). \quad (44)$$

The induced metric on the worldsheet swept by the membrane is

$$f_{ab} = \begin{pmatrix} \dot{X}^2_0 - \dot{r}^2 & 0 & 0 \\ 0 & -r^2 & 0 \\ 0 & 0 & -r^2 \sin^2 \vartheta \end{pmatrix}. \quad (45)$$

The action is

$$I = \int d\tau d\vartheta d\phi \sqrt{|f|} = \int d\tau 4\pi r^2 \sqrt{\dot{X}^2_0 - \dot{r}^2}, \quad (46)$$

where $\sqrt{|f|} \equiv \sqrt{|\det f|} = \sqrt{\dot{X}^2_0 - \dot{r}^2} r^2 \sin \vartheta$. Variation of the above action with respect to $r$ and $X^0 = X_0$ gives the following equations of motion:

$$\frac{d}{d\tau} \left( \frac{\dot{r}}{\sqrt{\dot{X}^2_0 - \dot{r}^2}} \right) + \frac{2 \dot{X}^2_0}{r \sqrt{\dot{X}^2_0 - \dot{r}^2}} = 0 \quad (47)$$

$$\frac{d}{d\tau} \left( \frac{r^2 \dot{X}_0}{\sqrt{\dot{X}^2_0 - \dot{r}^2}} \right) = 0. \quad (48)$$

If we now introduce the new variable according to eq. (38)

$$X^{123} = \frac{1}{4\pi r^2} \int dr d\vartheta d\phi \partial_a X^1 \partial_b X^2 \partial_c X^3 = \frac{4\pi r^3}{3}$$

$$\dot{X}^{123} = 4\pi r^2 \dot{r}$$

$$\frac{dX^{123}}{dS} = \frac{\dot{X}^{123}}{4\pi r^2 \sqrt{\dot{X}^2_0 - \dot{r}^2}} = \frac{\dot{r}}{\sqrt{\dot{X}^2_0 - \dot{r}^2}},$$

where $dS = d\tau 4\pi r^2 \sqrt{\dot{X}^2_0 - \dot{r}^2}$, then the equation of motion (47) becomes

$$\frac{d^2X^{123}}{dS^2} + \frac{2}{3X^{123}} \left( 1 + \left( \frac{dX^{123}}{dS} \right)^2 \right) = 0, \quad (51)$$
whereas eq. (48), due to the reparametrization invariance of our action (46), is redundant.

The equation of motion (51) can be considered as the geodesic equation (43) derived from the $C$-space action (41) for the case of a subspace described by two coordinates $X^M = (X^0, X^{123})$ with the metric

$$G_{MN} = \begin{pmatrix} C \hat{X}^{4/3} & 0 \\ 0 & -1 \end{pmatrix},$$

where $C$ is a constant, and where we have denoted $\hat{X} \equiv X^{123}$. Namely, if we insert the particular metric (52) into the equation of geodesic (43), then we obtain eq. (51).

We can show that the above $C$-space description is equivalent to the Dirac membrane by directly comparing the actions. In the 2-dimensional subspace with coordinates $X^M = (X^0, X^{123})$, $X^{123} \equiv \hat{X}$, and the metric (52) we have the following line element

$$dS^2 = G_{00}(dX^0)^2 + G_{\hat{X}\hat{X}}d\hat{X}^2 = C\hat{X}^{4/3}(dX^0)^2 - d\hat{X}^2$$

Using

$$\hat{X} = \frac{4\pi r^3}{3}, \quad d\hat{X} = 4\pi r^3 dr$$

$$\hat{X}^{4/3} = \left(\frac{4\pi}{3}\right)^{4/3} r^4$$

$$C \left(\frac{4\pi}{3}\right)^{4/3} = (4\pi)^2,$$

we have

$$dS^2 = (4\pi r^2)^2 (d(X^0)^2 - dr^2).$$

Inserting the latter line element into the action

$$I[X^M] = \int dS = \int d\tau (G_{MN}\dot{X}^M \dot{X}^N)^{1/2},$$

we obtain

$$I = \int d\tau (4\pi r^2)^2 \sqrt{(\dot{X}^0)^2 - \dot{r}^2}.$$ 

which is the action for the Dirac membrane.

The above example explicitly shows why description of branes with non-vanishing tension requires non-trivial metric of the brane configuration space $C$. The $C$-space metric (52) that gives the usual membrane action (57) is just one particular case. Other more general $C$-space metrics are possible in this theory. Such higher dimensional configuration space, associated with branes, enables unification of fundamental interactions à la Kaluza-Klein [12]. For alternative, although related approaches see [19].

### 6 Conclusion

We have considered a theory in which spacetime is replaced by a larger space, namely the configuration space associated with a system under consideration. For strings/branes the configuration space is infinite dimensional, but it can be reduced to a corresponding finite dimensional space, the so called Clifford space $C$. Since configuration space has extra dimensions, its metric provides a description of additional interactions, besides the 4-dimensional gravity, just as in Kaluza-Klein theories. In this theory there is no need for extra dimensions of spacetime. The latter space is a subspace of the configuration space $C$, and all dimensions of $C$ are physical. Therefore, there is no need for a compactification of the extra dimensions of $C$.

The notion of configuration space has as well been considered by Barbour [3]. In those works matter configurations have been described in the intrinsic terms, without recourse to an embedding space or spacetime, therefore usage of coordinates has been avoided. Also in our approach every possible matter configuration is represented as a point in a configuration space. But, following general relativity, we adopt the view that to points of $C$ we can assign arbitrary coordinates. And, like “house numbers”, the set of coordinates assigned to a point in $C$ can be changed into another set of coordinates. So our approach retains the crucial feature of general relativity, such as diffeomorphism invariance and background independence.
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