Communication complexity and the reality of the wave-function

Alberto Montina

Facoltà di Informatica, Università della Svizzera Italiana, Via G. Buffi 13, 6900 Lugano, Switzerland

In this review, we discuss a relation between quantum communication complexity and a long-standing debate in quantum foundation concerning the interpretation of the quantum state. Is the quantum state a physical element of reality as originally interpreted by Schrödinger? Or is it an abstract mathematical object containing statistical information about the outcome of measurements as interpreted by Born? Although these questions sound philosophical and pointless, they can be made precise in the framework of what we call classical theories of quantum processes, which are a record of quantum phenomena in the language of classical probability theory. In 2012, Pusey, Barrett and Rudolph (PBR) proved, under an assumption of preparation independence, a theorem supporting the original interpretation of Schrödinger in the classical framework. The PBR theorem has attracted considerable interest revitalizing the debate and motivating other proofs with alternative hypotheses. Recently, we showed that these questions are related to a practical problem in quantum communication complexity, namely, quantifying the minimal amount of classical communication required in the classical simulation of a two-party quantum communication process. In particular, we argued that the statement of the PBR theorem can be proved if the classical communication cost of simulating the communication of $n$ qubits grows more than exponentially in $n$. Our argument is based on an assumption that we call probability equipartition property. This property is somehow weaker than the preparation independence property used in the PBR theorem, as the former can be justified by the latter and the asymptotic equipartition property of independent stochastic sources. The probability equipartition property is a general and natural hypothesis that can be assumed even if the preparation independence hypothesis is dropped. In this review, we further develop our argument into the form of a theorem.

I. INTRODUCTION

One of the main objectives of quantum information theory is to understand when quantum devices outperform their classical counterpart in terms of computational resources and amount of communication. The goal can be achieved by finding the most efficient model that classically simulates the quantum device. Besides the practical interest, the study of optimal classical simulators can also have important implications in the context of a long-standing quantum foundational debate concerning the interpretation of the quantum state. Is the quantum state an element of reality, as initially interpreted by Schrödinger, or is it a mere abstract mathematical object of the theory? As we will discuss in this review, some results in quantum communication complexity support a realistic interpretation of the quantum state in the framework of what we call classical theories of quantum processes. Let us first discuss the foundational and practical motivations that are at the basis of this debate.

Quantum theory provides a consistent framework for computing the probabilities of the outcomes of measurements given some previous knowledge, which is mathematically represented by the quantum state. Quantum theory has successfully been applied in fields like atom physics, particle physics, condensed matter and cosmology. In spite of its success, quantum theory still suffers from an interpretational issue known as the measurement problem. Whenever a particle like an electron is spatially delocalized, the formalism does not provide any description of its actual position. Although, this feature could be fine in the microscopic world, it becomes problematic when it is extrapolated to the macroscopic domain of the every-day experience, as illustrated by Schrödinger cat’s paradox. In the standard interpretation, the problem is generally fixed by marking a boundary between the fuzzy microscopic quantum world and the macroscopic well-defined world made of objective observations. Besides other interpretations such as the consistent-histories and many-worlds interpretations, a possible alternative solution of the issue is to fit quantum theory into the framework of classical probability theory, that is, to remove the boundary by phagocytizing the quantum domain into the classical one. In this framework, the state of a system would be described by a set of classical variables evolving according to some deterministic or stochastic law. These variables should account for the definiteness of the macroscopic reality by containing the actual value of what we can observe non-invasively, such as the position of a pen on a desk.

As quantum systems can be simulated through classical resources this reduction to a classical framework is in principle possible. The simplest way to realize it is to identify the classical variables with the quantum state, now regarded as an element of reality, as initially interpreted by Schrödinger. The wave-function would be as real as the waves on the ocean, a particle being a spatially localized wave-packet. However, as the wave-function can spread out, this interpretation needs some active mechanism that spontaneously localizes the wave-function. This is the route to realism taken in a collapse theory a la Ghirardi, Rimini and Weber. An alternative approach that does not need a collapse mechanism is taken in pilot-wave theories, where the quantum state is
supplemented by additional auxiliary variables, such as the actual position of the particles. Both collapse theories and pilot-wave theories have the common feature of promoting the wave-function to the rank of an ontologically objective field. For this reason, they are often called ψ-ontic in the quantum foundation community. Curiously, ψ-ontic theories are the only currently available classical reformulation of quantum theory. Unfortunately, this classical reword of quantum theory does not provide any practical advantage in terms of computation of quantum processes. As in the standard formulation, the computation of a process in the classical model passes through the solution of the Schrödinger equation. Thus, unless ψ-ontic theories are not exactly equivalent to quantum theory and they can predict observations detectably different from the standard formulation, their content remains merely philosophical.

More interestingly, we could wonder if these theories are the only available option. As an evidently necessary condition, the classical variables should contain at least the values of what can be observed non-invasively. As the quantum state of a single system cannot be directly measured, there is no evident reason to take it as part of the classical description. After all, the quantum state can only be recovered from the statistical distribution of the outcomes of measurements performed on many identically prepared systems. Thus, the quantum state looks more similar to a probability distribution, representing our knowledge of what is the actual state of affairs of a system. Bearing this in mind, in a more general classical formulation of quantum theory, quantum states could be mapped to overlapping probability distributions over the classical space, so that the actual values of the classical variables in a single realization would not contain the full information about the quantum state. In other words, a single statistical realization of the classical variables would be compatible with many different quantum states. These hypothetical theories are called ψ-epistemic, since the quantum state is not part of the ontological description, but it merely represents our statistical knowledge about the classical variables. The question whether this statistical representation is actually possible has attracted growing interest in the recent years. [4–17]

One possible advantage of ψ-epistemic theories is the fact that the statistical role of the quantum state makes them potentially less exposed to the principle of Occam’s razor than ψ-ontic theories. For example, the information required to describe the classical state of a single system can turn out to be finite on average, whereas the classical information required to define exactly a quantum state is infinite. Thus, ψ-epistemic theories could be supported by the law of parsimony, as suggested in Ref. [3] and, more recently, in Refs. [10, 12]. The relevant point is to understand if ψ-epistemic theories exist and if they provide some descriptive advantage over their ψ-ontic counterpart. We will see that these questions could have a negative answer and ψ-epistemic theories could collapse to ψ-ontic theories in the asymptotic limit of infinite qubits.

In 2012, Pusey, Barrett and Rudolph (PBR) provided the first proof, under a hypothesis of preparation independence, that ψ-epistemic theories are incompatible with the predictions of quantum theory. These findings fed considerable interest and motivated other proofs using alternative hypotheses, like in Refs. [11, 13]. Subsequently, Lewis, Jennings, Barrett and Rudolph reported a counterexample showing that the PBR theorem can be evaded once the preparation independence hypothesis is dropped. [14] However, their findings do not solve definitely our questions. Indeed, although the reported model is formally ψ-epistemic, it has still some unwanted properties that make it not completely ψ-epistemic, according to the definition given in Ref. [18] and recalled later in this review. For example, it occurs that some statistical realizations of the classical state can still contain the full information about the quantum state. This implies that the quantum state can be inferred from the classical state with a finite probability of success. Furthermore, the model collapses to a ψ-ontic model in the limit of infinite qubits.

In Ref. [18], we showed that the question about the existence of completely ψ-epistemic theories is equivalent to the quantum communication complexity problem of quantifying the minimal amount of classical communication required to simulate a two-party quantum communication process. More precisely, we showed that a completely ψ-epistemic theory exists if and only if the communication of qubits can be simulated by a classical protocol employing a finite amount of classical communication (hereafter, more concisely, finite communication protocol or FC protocol). As the communication of a single qubit can be classically simulated by a FC protocol, [19, 20] completely ψ-epistemic models for single qubits exist. At the present, both FC protocols and completely ψ-epistemic models are known only for single qubits, as the two problems of extending the communication protocols and the ψ-epistemic models to more general cases are equivalent.

Provided that this extension is actually possible, we can still wonder how much the ψ-epistemic theory differs from a ψ-ontic theory. Indeed, it could occur that the classical variables turn out to contain the information about the quantum state up to an error that goes to zero as the number of involved qubits is increased. In this case, the ψ-epistemic theory would collapse to a ψ-ontic theory in the asymptotic limit of infinite qubits (as it occurs with the model in Ref. [14]). In Refs. [21, 22], we argued that this is the case if the minimal communication cost of a FC protocol grows more than exponentially with the number of qubits. This statement can be proved under an assumption that we call probability equipartition property. As we will discuss later, this property is somehow weaker than the preparation independence hypothesis used in the PBR theorem. Using two mathematical conjectures, we also proved that the communication cost grows at least as n2^n [22]. An exact proof of this lower
bound without conjectures would provide a proof of the PBR theorem by replacing the preparation independence hypothesis with the aforementioned equipartition property. We will also see that this lower bound implies that a \( \psi \)-epistemic theory does not provide a descriptive advantage over \( \psi \)-ontic theories, even if the equipartition property is dropped.

This review is organized as follows. In section \( \text{II} \) we introduce the framework of a classical theory of quantum processes and provide a mathematical definition of \( \psi \)-ontic, \( \psi \)-epistemic and completely \( \psi \)-epistemic theories. Section \( \text{III} \) is devoted to classical protocols simulating a two-party communication process and to the definition of communication cost of a simulation. In section \( \text{IV} \) we establish a relationship between completely \( \psi \)-epistemic theories and protocols with a finite communication cost. In section \( \text{V} \) we show that a \( \psi \)-epistemic theory collapses to a \( \psi \)-ontic theory in the limit of infinite qubits, under the assumption that the probability equipartition property holds and the minimal communication cost of a FC protocol grows more than exponentially in the number of qubits. Finally, the conclusions are drawn. This short review is mainly focused on some recent results of the Author. An extensive review of other results in the field can be found in Ref. 23.

II. CLASSICAL REFORMULATION OF QUANTUM THEORY

Let us introduce the general framework of a classical theory of quantum systems. By classical theory, we just mean a classical probability theory of quantum processes. The theory does not necessarily have a structure resembling classical mechanics. Determinism is neither required. For our purposes, it is sufficient to consider the simple scenario of state preparation and subsequent measurement.

In the classical theory, a system is described by a set of variables, which we denote by \( x \). As these variables are meant to be ontologically objective, let us call their actual value the ontic state of the system. When the system is prepared in some quantum state \( |\psi\rangle \), the preparation procedure modifies the variable \( x \) through some process that sets its value according to a probability distribution \( \rho(x|\psi) \) depending on the procedure, that is, on \( |\psi\rangle \). To simplify the notation, hereafter the ket \( |\psi\rangle \) is concisely denoted by \( \psi \). The bra-ket notation will be used only for scalar products. More generally, the probability distribution could depend on additional parameters specifying the preparation context, but this is irrelevant for our discussion. Thus, there is a mapping

\[
\psi \rightarrow \rho(x|\psi) \tag{1}
\]

that associates each quantum state with a probability distribution on the classical space. The mapping must be injective. Note that this mapping could be achieved with a classical space much smaller than the Hilbert space. Indeed, the minimal number of classical states required to have an injective mapping is finite and equal to the double of the Hilbert space dimension, whereas the number of quantum states is infinite. However this minimal requirement is not sufficient to provide an effective classical simulation of the overall process of state preparation and measurement.

In quantum theory, a general measurement is described by a positive-operator valued measure (POVM), which is defined by a set of positive semidefinite operators, \( \{\hat{E}_1, \hat{E}_2, \ldots\} \equiv \mathcal{M} \). Each operator \( \hat{E}_i \) labels an event of the measurement \( \mathcal{M} \). In the framework of the classical theory, the probability of \( \hat{E}_i \) is conditioned by the ontic state \( x \). Thus, each measurement \( \mathcal{M} \) is associated with a probability distribution \( P(\hat{E}_i|x, \mathcal{M}) \),

\[
\mathcal{M} \rightarrow P(\hat{E}_i|x, \mathcal{M}) \tag{2}
\]

Also in this case, the conditional probability can depend on additional parameters specifying the measurement context, but we can safely ignore them. Finally, the classical theory is equivalent to quantum theory if the probability of having \( \hat{E}_i \) given the preparation \( \psi \) is equal to the quantum probability, that is,

\[
\int dx P(\hat{E}_i|x, \mathcal{M})\rho(x|\psi) = \langle \psi|\hat{E}_i|\psi\rangle \tag{3}
\]

It is worth to underline that the integral symbol stands for integral over some manifold. Here and hereafter, we could indifferently replace the manifold with a more abstract measurable space.

A. \( \psi \)-ontic and \( \psi \)-epistemic theories

As said in the introduction, the simplest (and trivial) way to fit quantum theory into the framework of classical probability theory is to identify the classical state with the quantum state. In this case, the classical state \( x \) is a vector in the Hilbert space. Thus, the mapping \( \text{II} \) is

\[
\psi \rightarrow \rho(x|\psi) = \delta(x - \psi) \tag{4}
\]

The conditional probability of getting \( \hat{E}_i \) given \( x \) and the measurement \( \mathcal{M} \) is trivially

\[
P(\hat{E}_i|x, \mathcal{M}) = \langle x|\hat{E}_i|x\rangle \tag{5}
\]

Like in a pilot-wave theory, the model can be made deterministic by adding some auxiliary classical variables. In this kind of models, the quantum state takes part in the classical description and it is regarded as ontologically objective. For this reason, such theories are called \( \psi \)-ontic. 23 In a \( \psi \)-ontic theory, the ontic state \( x \) always contains the full information about the quantum state.

**Definition 1** (strong definition) A theory is \( \psi \)-ontic in strong sense if the quantum state can be inferred from the classical state, that is, if \( \rho(x|\psi) \) is a delta distribution in \( \psi \) for every \( x \).
There is another definition that is somehow weaker and is widely employed, such as in the PBR paper. 

**Definition 2 (weak definition)** A theory is ψ-ontic in weak sense if the distributions ρ(x|ψ) and ρ(x|ψ2) are not overlapping for every ψ̸= ψ2.

The difference between these two definitions could look marginal, but it is not. According to the second definition, a theory is ψ-ontic if it is possible to infer one of two given quantum states, once the classical state is known. However, this does not imply that a quantum state can be inferred from the classical state if we have no a priori information about the quantum state. Indeed, the model for single qubits in Ref. [10] is ψ-ontic only according to the second definition. Under the preparation independence hypothesis, the PBR theorem proves that a classical theory is ψ-ontic in weak sense, but not in strong sense. As the weak definition is the most popular one, we also employ it in this review.

Theories that are not ψ-ontic are called ψ-epistemic. These theories are less trivial than their counterpart and are object of this review. In a ψ-epistemic theory, the information about the quantum state is encoded only in the statistical behaviour of x, that is, in the distribution ρ(x|ψ). It is only required that the mapping (1) is injective. This feature takes to the possibility of a reduction of information required to specify the ontic state. For example, whereas the amount of information required to specify the classical state of a ψ-ontic theory is obviously infinite, this amount could be finite on average in a ψ-epistemic theory.

1. Completely ψ-epistemic theories

Let us introduce a subclass of ψ-epistemic theories that satisfy a very reasonable condition. These theories are particularly relevant for the present discussion and for their role in quantum communication complexity. The condition that we are going to introduce is a weaker consequence of the following two natural conditions. First, we assume that the probability density ρ(x|ψ) is bounded by some constant for every x and ψ. For example, this property is satisfied if the distribution is some smooth function. We also assume that the supports of ρ(x|ψ) and ρ(x) = ∫ dψ ρ(x|ψ)ρ(ψ) have a finite measure for every distribution ρ(ψ). In particular, this is true if the space x is compact. Under these reasonable conditions, the entropy H(x|ψ) of x given ψ is finite, as well as the entropy H(x) [Note the abuse of notation. H(x|ψ) is not a function of x and ψ]. The entropy of x (which can be a set of continuous variables) is not well-defined, as it depends on the measure taken on the classical space. A measure-independent quantity is

\[ I(x; ψ) = H(x|ψ) - H(x), \]  

which is also finite for every ψ, that is, the quantity

\[ C(ψ → x) = \max_{ρ(ψ)} I(x; ψ) \]

is finite. In information theory, I(x; ψ) is known as the mutual information between x and ψ. It quantifies the degree of dependence between two stochastic variables. The quantity C(ψ → x) is the capacity of the communication channel ψ → x associated with the conditional probability ρ(x|ψ), \[ ψ \text{ and } x \text{ being the input and outcome of the channel, respectively.}\] Let us recall that a channel y → x is a stochastic process from an input variable y to an output variable x described by a conditional probability ρ(x|y). In information theory, a channel represents a physical device, such as a wire, carrying information from a sender to a receiver. The information-theoretic interpretation of the channel capacity is provided by the noisy-channel coding theorem. Roughly speaking, the capacity of a channel is the rate of information that can be transmitted through the channel. Now we define a completely ψ-epistemic model by keeping only property (7).

**Definition 3** The classical model defined by the maps (12) is completely ψ-epistemic if the capacity of the channel ψ → x is finite.

This definition has been justified by assuming that ρ(x|ψ) is bounded and the space of x is compact, but the defined class is actually broader, as it includes some models such that ρ(x|ψ) is not bounded and the supports of ρ(x|ψ) and ρ(x) do not have finite measure. It is not hard to show that a completely ψ-epistemic theory is ψ-epistemic. Furthermore, it will become clear in section [V] that the two classes are equivalent if the assumption of probability equipartition holds.

2. Example: Kochen-Specker model

The Kochen-Specker model [24] is an example of completely ψ-epistemic model working for single qubits. The ontic state is given by a unit three-dimensional vector, \( \vec{x} \). Let us represent a pure quantum state through the unit Bloch vector, \( \vec{v} \). Given the quantum state \( \vec{v} \), the probability distribution of \( \vec{x} \) is

\[ ρ(\vec{x} | \vec{v}) = π^{-1} \vec{v} \cdot \vec{x} \theta(\vec{v} \cdot \vec{x}), \]  

where \( \theta \) is the Heaviside step function. As shown in Ref. [18], the capacity of the channel \( \vec{v} → \vec{x} \) is \( C(\vec{v} → \vec{x}) = 2 - (2 \log_2 2)^{-1} \approx 1.28 \) bits. At the present, no other completely ψ-epistemic model is known for higher dimensional quantum systems.

### III. COMMUNICATION COMPLEXITY OF A TWO-PARTY COMMUNICATION PROCESS

In the previous section, we have introduced the general structure of a classical model that simulates the quantum process of state preparation and subsequent measurement. This process can be regarded as the following
A sender, say Alice, chooses a quantum state $\psi$ and sends it to another party, say Bob, who performs a measurement chosen by him. A problem in quantum communication complexity is to quantify the minimal amount of classical communication required to simulate the two-party quantum process through a classical protocol. The classical protocol has the same structure of the classical models introduced in the previous section, once the ontic variable $x$ is identified with the communicated variable and some possible additional stochastic variables shared between the sender and the receiver. A classical protocol is as follows. Alice chooses a state $\psi$ and generates a variable $k$ with a probability $\rho(k|\chi,\psi)$ depending on $\psi$ and a possible random variable, $\chi$, shared with Bob. The variable $\chi$ is generated according to the probability distribution $\rho_{\chi}(\chi)$. Note the $\chi$ is independent of $\psi$. It can be regarded as a distributed key that is generated before Alice and Bob choose the state $\psi$ and the measurement, respectively, and the protocol is initiated. Then, Alice communicates the value of $k$ to Bob. Finally, Bob chooses a measurement $\mathcal{M}$ and generates an outcome $\hat{E}_i$ with a probability $P(\hat{E}_i|k,\chi,\mathcal{M})$. The protocol simulates exactly the quantum channel if the probability of $\hat{E}_i$ given $\psi$ is equal to the quantum probability, that is, if

$$\sum_k \int d\chi P(\hat{E}_i|k,\chi,\mathcal{M})\rho(k|\chi,\psi)\rho_{\chi}(\chi) = \langle \psi|\hat{E}_i|\psi \rangle. \quad (9)$$

As said, the protocol is equivalent to the model introduced in the previous section, the variable $x$ corresponding to the pair $(k,\chi)$. The distribution $\rho(x|\psi)$ in the mapping (11) corresponds to the distribution $\rho(k|\chi,\psi) \equiv \rho(k|\chi,\psi)\rho_{\chi}(\chi)$.

There are different definitions of communication cost of a classical simulation. Without loss of generality, we can assume that $k$ is deterministically generated from $\psi$ and $\chi$. If this is not the case, we can make the protocol deterministic by adding auxiliary stochastic variables that we include in $\chi$. The variable $k$ can be regarded as a sequence of bits whose number depends on $\chi$ and $\psi$. Let $C(\psi,\chi)$ be the number of bits sent by Alice when the state $\psi$ is chosen with the shared noise $\chi$. The worst-case cost is the maximum of $C(\psi,\chi)$ over every possible value taken by $\chi$ and $\psi$. As an alternative, denoting by $C(\psi)$ the average of $C(\psi,\chi)$ over $\chi$, we can define the cost as the maximum of $C(\psi)$ over $\psi$. Denoting by $\bar{C}$ this quantity, we have

$$\bar{C} \equiv \max_{\psi} \int d\chi \rho_{\chi}(\chi)C(\psi,\chi). \quad (10)$$

There is also an entropic definition. For our purposes, the average and entropic cost can be indifferently used. Here, we will refer to the average cost $\bar{C}$.

**Definition 4** We define the communication complexity $C_{\text{min}}$ of a quantum communication process as the minimal amount of classical communication required by an exact classical simulation of the process.

### A. Parallel simulations

If a parallel simulation of $N$ quantum processes are performed, it is possible to envisage a larger set of communication protocols, where the probability of generating $k$ can depend on the full set of quantum states, say $\psi_{i=1,2,...,N}$, prepared in each single process. In other words, the distribution $\rho(k|\chi,\psi)$ becomes $\rho(k|\chi,\psi_1,\psi_2,\ldots,\psi_N)$. The asymptotic communication cost, $C_{\text{asym}}$, is the cost of the parallelized simulation divided by $N$ in the limit of large $N$.

**Definition 5** We define the asymptotic communication complexity $C_{\text{asym}}$ of a quantum process as the minimum of $C_{\text{asym}}$ over the class of parallel protocols that simulate the process.

Since the set of protocols working for parallel simulations is larger than the set of single-shot protocols, it is clear that

$$C_{\text{asym}} \leq C_{\text{min}}. \quad (11)$$

However, the difference between $C_{\text{asym}}$ and $C_{\text{min}}$ is tiny and not bigger than the logarithm of $C_{\text{asym}}$. [20]

### IV. $\psi$-Epistemic Theories and Communication Complexity

A finite communication protocol (FC protocol) of a quantum process is a protocol that simulates the process with a finite amount of classical communication. Using the data processing inequality and the chain rule for the mutual information, it is possible to show that a FC protocol corresponds to a completely $\psi$-epistemic classical model. Let us denote by $I(x;y|z)$ the conditional mutual information between $x$ and $y$ given $z$, which is the mutual information between $x$ and $y$ given $z$ and averaged on $z$. From the chain rule it is

$$I(k,\chi;\psi) = I(\chi;\psi) + I(k;\psi|\chi) \quad (12)$$

and the fact that $\psi$ and $\chi$ are uncorrelated, we have that

$$I(k,\chi;\psi) = I(k;\psi|\chi). \quad (13)$$

From the data-processing inequality, we have that $I(k;\psi|\chi)$ is smaller than or equal to the communication cost $\bar{C}$ for any $\rho(\psi)$, that is,

$$\bar{C} \geq C \{ \psi \rightarrow (k,\chi) \}, \quad (14)$$

where $C \{ \psi \rightarrow (k,\chi) \}$ is the capacity of the channel $\psi \rightarrow (k,\chi)$. Thus, if the communication cost is finite, the protocol corresponds to a completely $\psi$-epistemic model with $(k,\chi)$ as ontic variables. Thus, we have the following.

**Lemma 1** A finite-communication protocol is a completely $\psi$-epistemic classical model.

In Ref. [18], we showed that also the opposite is true in some sense. Namely, we showed that there is procedure turning a completely $\psi$-epistemic model into a finite-communication protocol.
A. FC protocols from completely ψ-epistemic models

We now describe the procedure introduced in Ref. [18] for generating a FC protocol from a completely ψ-epistemic classical model. This procedure is a consequence of the reverse Shannon theorem [27] and its one-shot version [28]. Given M identical a noisy channels \( x \to y \), defined by the conditional probability \( \rho(y|x) \) and with capacity \( C_{ch} \), the reverse Shannon theorem states that the channels can be simulated through a noiseless channel with a communication cost equal to \( MC_{ch} + o(M) \), provided that the sender and receiver share some random variable. In other words the asymptotic communication cost of a parallel simulation of many copies of a channel \( x \to y \) is equal to \( C_{ch} \). A one-shot version of this theorem was recently reported in Ref. [28]. The communication cost \( \bar{C} \) of simulating a single channel \( x \to y \) satisfies the bounds

\[
C_{ch} \leq \bar{C} \leq C_{ch} + 2 \log_2(C_{ch} + 1) + 2 \log_2 e. \tag{15}
\]

Thus, the communication cost is \( C_{ch} \) plus a possible small additional cost that does not grow more than the logarithm of \( C_{ch} + 1 \).

These results have an immediate application to the problem of deriving FC protocols from completely ψ-epistemic model. Let \( x \) be the classical variable in the completely ψ-epistemic model (see section [11]). In general, the direct communication of this variable can require infinite bits. A strategy for making the communication finite and as small as possible is as follows. Instead of communicating directly the variable \( x \) Alice can communicate an amount of information that allows Bob to generate \( x \) according to the probability distribution \( \rho(x|\psi) \). By Eq. (15), the minimal amount of required communication is essentially equal to the capacity \( C_{ch} \) of the channel \( \psi \to x \). Since \( C_{ch} \) is finite in a completely ψ-epistemic model, the communication cost of the simulation protocol is finite. If many simulations are performed in parallel, the reverse Shannon theorem implies that there is a classical simulation such that the asymptotic communication cost is strictly equal to \( C_{ch} \).

Theorem 1 There is a procedure that turns a completely ψ-epistemic model into a FC protocol whose communication cost \( \bar{C} \) is bounded by the Ineqs. (15), where \( C_{ch} \) is the capacity of the channel \( \psi \to x \). In the case of a parallel simulation of many instances, there is a protocol whose asymptotic communication cost is strictly equal to \( C_{ch} \).

1. Communication cost of simulating the communication of a single qubit

Theorem 1 can be immediately applied to the Kochen-Specker model, introduced in section [11A2]. We have seen that the capacity of the channel \( \psi \to x \) in this model is equal to about 1.28 bits. Thus, theorem 1 implies that there is a parallel simulation of many instances of a single qubit communication process such that the asymptotic communication cost is equal to 1.28 bits [18]. This value is lower than the upper bound 1.85 bits proved by Toner and Bacon [20] in the case of parallel simulations.

V. ψ-EPISTEMIC THEORIES IN THE LIMIT OF INFINITE QUBITS

In the previous section we have seen that completely ψ-epistemic theories and finite-communication protocols are two sides of the same coin. To find a completely ψ-epistemic theory means to find a FC protocol and viceversa. Indeed, note that both completely ψ-epistemic theories and FC protocols are not yet known, apart from the case of single qubits. Suppose that a completely ψ-epistemic theory actually exists for any finite number of qubits, it could turn out that the difference from a ψ-ontic theory is actually small and the overlap between \( \rho(x|\psi) \) and \( \rho(x|\psi') \) could go to zero in the limit of infinite qubits for every pair \( (\psi, \psi') \). In this case, the ψ-epistemic theory would collapse to a ψ-ontic theory in this limit. In this section, we will prove that this is the case if the communication complexity of a quantum communication process grows more than exponentially in the number of communicated qubits, provided that a suitable reasonable equipartition property is satisfied. This property will be discussed later in the section. In the proof, we employ the weak definition [2] of a ψ-ontic theory, which is also used in the PBR paper [13].

Let \( C(n) \) be the asymptotic communication complexity of a quantum communication process where the sender Alice can prepare any quantum state of \( n \) qubits and the receiver Bob can perform any measurement. We could also consider the one-shot communication complexity, as it differs by a small amount that is irrelevant for the following discussion. As defined in section [11] let \( \rho(x|\psi) \) be the conditional probability of a generic classical model of \( n \) qubits. Theorem 1 implies the inequality

\[
C(n) \leq C(\psi \to x), \tag{16}
\]

\( C(\psi \to x) \) being the capacity of the process \( \psi \to x \). By definition of channel capacity, we have that

\[
C(\psi \to x) = \max_{\rho(\psi)} \int dx \int d\psi \rho(x, \psi) \log_2 \frac{\rho(x|\psi)}{\rho(x)}. \tag{17}
\]

In a general ψ-epistemic theory, the conditional probability \( \rho(x|\psi) \) can be a mixture of a broad smooth function and very narrow functions. If the position of these narrow peaks does not depend on the quantum state, we can remove them with a change of the measure on the classical space manifold. In the antipodal case that their position is a bijective function of the quantum state, it turns out that the quantum state can be inferred with a very small error and a finite probability of success. In this case, the theory is somehow partially ψ-ontic. For example, the model in Ref. [14], which is formally ψ-epistemic
(but not completely ψ-epistemic), displays this feature. Although this kind of theory is, technically speaking, ψ-epistemic, it looks very artificial and unpalatable, especially if we are not interested to find a mere classical simulation of quantum processes with possible practical interest in quantum information theory, but we pretend to find a classical theory picturing what actually occurs in the backstage of quantum phenomena. A classical theory of a complex system made of a high number of qubits should satisfy a natural requirement that we call probability equipartition property (or, more exotically, ontological equipartition property). As we will see, this property is related to the asymptotic equipartition property known in information theory, and it is stated as follows.

**Definition 6** A classical theory satisfies the probability (or ontological) equipartition property if, given ψ, there is a typical set of classical states with probability close to one such that the probability distribution ρ(x|ψ) is approximately a constant independent of ψ, in the limit of a high number of qubits.

For our purposes, the probability equipartition property can be satisfied very roughly and some deviation from the uniformity can be acceptable. It is sufficient that ρ(x|ψ) has the same order of magnitude on the typical set of classical states. This property is introduced to discard theories displaying huge narrow fluctuations in the probability distribution. Clearly, the model in Ref. [14] does not satisfy the equipartition property, as the corresponding distribution is the mixture of a broad function and a delta distribution. Furthermore, we assume that the marginal distribution ρ(x) = ∫ dψ ρ(x|ψ)ρ(ψ) satisfies the uniformity property of ρ(x|ψ) for a uniform distribution ρ(ψ) of quantum states. This assumption is very reasonable, as ρ(x) is the probability distribution of the classical state provided that nothing is known about the quantum state. Again, the uniformity can be satisfied roughly.

The ontological equipartition property is somehow weaker than the preparation independence property used in the PBR theorem. Indeed, the latter justifies the former. A procedure for preparing a general quantum state of n qubits is as follows. First, we prepare each qubit in the same quantum state, then we let them evolve according to some suitable unitary evolution. The unitary evolution can be implemented through some quantum circuit. Let us consider the first stage of this procedure. Under the preparation independence hypothesis of the PBR theorem, each qubit is associated with a classical variable xi and the collection of these variables is the overall classical state x. Furthermore the preparation independence property claims that the variables xi are independent stochastic variables, provided that the qubits are prepared in a factorized quantum state. Thus, we can conclude that the probability distribution ρ0(x) after the first stage of the quantum state preparation satisfies the uniformity property of definition. Indeed, this is a consequence of the asymptotic equipartition property of independent stochastic processes. Let us consider the second stage of the quantum state preparation. As a unitary evolution is a reversible conservative process, we can argue that also the associated classical process describing the evolution of x is somehow conservative, in the sense that the volume of sets in the classical space is somehow conserved during the evolution. More generally, we can argue that the process conserves the uniformity property of the initial distribution ρ0(x), implying that the distribution ρ(x|ψ) satisfies the uniformity property for every ψ. That is, ρ(x|ψ) is approximately equal to a constant independent of ψ in the typical set. The uniformity property is a general feature of complex systems with a high number of variables and can be reasonably assumed even if the preparation independence property is dropped.

The probability equipartition property and the uniformity of ρ(x) for a uniform ρ(ψ) imply that the maximum in Eq. (17) is achieved for a uniform distribution ρ(ψ). This can be verified by using the Karush-Kuhn-Tucker conditions for optimality. Let us choose a measure in the space of ψ such that the uniform distribution is ρ(ψ) = 1. At this point, by taking a suitable measure on the manifold of x so that the function ρ(x|ψ) is equal to one on its support, it is easy to prove that

$$\int dψ' ω(ψ, ψ') ≃ 2^{C(ψ→x)}$$

where

$$\omega(ψ, ψ') = \int dψ x ρ(x|ψ')$$

is the overlap between ρ(x|ψ) and ρ(x|ψ'). The overlap definition can be recast in the measure-independent form

$$\omega(ψ, ψ') = \int dψ x \min[ρ(x|ψ), ρ(x|ψ')]$$

Note that Eq. (19) implies that a ψ-epistemic theory is equivalent to a completely ψ-epistemic theory if the probability equipartition property holds. Using Ineq. (16), we have that

$$\int dψ' ω(ψ, ψ') ≃ 2^{C(n)}$$

Thus, the overlap ω(ψ, ψ'), averaged over ψ', goes to zero for n → ∞, regardless of how fast the communication complexity C(n) grows by increasing the number of qubits. In other words, most of the pairs of probability distributions are almost non-overlapping in high dimension. This feature is still compatible with ψ-epistemic theories. Indeed, two orthogonal quantum states can be always distinguished by a measurement, implying that their associated probability distributions cannot overlap.
This can be easily inferred from the general structure of a classical theory introduced in section 11. Now, by the principle of the concentration of measure, most of the quantum states \( \psi' \) are almost orthogonal to \( \psi \) in high dimension, implying that the overlap \( \omega(\psi, \psi') \) is almost zero for most of the pairs.

To prove that a \( \psi \)-epistemic theory collapses to a \( \psi \)-ontic theory, we have to show that the overlap \( \omega(\psi, \psi') \) goes to zero for \( n \to \infty \) regardless how close the states \( \psi \) and \( \psi' \) are. To do this, we single out pairs of quantum states whose distance is bounded above by any given constant, that is, whose scalar product is not smaller than any given constant. Let \( S(\theta) \) be a set of quantum states \( \psi' \) satisfying the constraint \( |\langle \psi | \psi' \rangle|^2 \geq \cos^2 \theta \) for some given vector \( \psi \) and angle \( \theta \). This set is a kind of cap whose angular aperture is \( 2\theta \). The volume, say \( \Omega_S \), of the set \( S \) is

\[
\Omega_S = 2^{(2^n-1) \log_2 \sin \theta} \quad (23)
\]

(Note that the volume of the whole quantum state manifold is equal to 1).

Now, we perform the integral in Ineq. (22) only on the set \( S(\theta) \). Obviously, the inequality is still satisfied, that is,

\[
\int_{S(\theta)} d\psi' \rho(\psi') \omega(\psi, \psi') \lesssim 2^{-C(n)} \quad (24)
\]

for every \( \theta \in [0, \pi/2] \). Dividing both sides by the integral of \( \rho(\psi') \) over the set \( S \) and bearing in mind that \( \rho(\psi) = 1 \) [see Eq. (13)], we have from Eq. (23) and Ineq. (24) that

\[
\bar{\omega}(\theta) \lesssim 2^{-C(n)+(1-2^n) \log_2 \sin \theta}, \quad (25)
\]

where \( \bar{\omega}(\theta) \) is the average value of \( \omega(\psi, \psi') \) over the set \( S(\theta) \) of vectors \( \psi' \).

This inequality implies that the \( \psi \)-epistemic theory collapses to a \( \psi \)-ontic theory if the communication complexity grows faster than \( 2^n \) (according to the weak definition 2 of \( \psi \)-ontic theory). Indeed, in this case, the right-hand side of the inequality goes to zero for every \( \theta \), that is,

\[
\lim_{n \to \infty} \bar{\omega}(\theta) = 0, \quad \forall \theta \in [0, \pi/2]. \quad (26)
\]

In other words, the overlap \( \omega(\psi, \psi') \) converges to zero over \( S(\theta) \) for every \( \theta \in [0, \pi/2] \) in the limit \( n \to \infty \) (mathematically speaking, the convergence is uniform). Regardless of how close two quantum states \( \psi \) and \( \psi' \) are, their overlap goes to zero in the limit of infinite qubits. Summarizing, we have the following.

**Theorem 2** If the probability equipartition property is satisfied, then a \( \psi \)-epistemic theory collapses to a \( \psi \)-ontic theory (in the weak sense of definition 2), provided that

\[
\lim_{n \to \infty} 2^{-n} C(n) = \infty.
\]

What is known about \( C(n) \)? Brassard, Cleve and Tapp proved the lower bound \( 0.01 \times 2^n \) for the communication complexity. 29 Subsequently, the bound was increased to \( 0.293 \times 2^n \) and, with a mathematical conjecture, to \( 2^n \) 30. Unfortunately, these bounds are at the border of the condition stated in theorem 2. Very recently, we showed that \( C(n) \) scales at least as \( n2^n \) if two suitable mathematical conjectures hold. 31 The proof and the conjectures are quite technical, thus we will not discuss them in this review. The details can be found in the cited paper. As implied by theorem 2 an exact proof of this lower bound would provide a proof of the PBR theorem by replacing the independence hypothesis with the probability equipartition hypothesis. It is worth to underline that the lower bound \( n2^n \) has another relevant consequence. It is known that an approximate classical description of the quantum state requires an amount of information that grows as \( n2^n \) up to some factor that grows as the logarithm of the inverse of the error. Thus, if the lower bound \( n2^n \) holds, then a \( \psi \)-epistemic theory would not provided a significant descriptional advantage over an error-bounded \( \psi \)-ontic theory, regardless of the probability equipartition hypothesis.

Our work has a relation with a recent result by Leifer. 32 If we assume that \( C(n) \) grows as \( 2^n \), then Ineq. (22) is similar to an inequality derived by Leifer. As \( \lim_{n \to \infty} 2^{-n} C(n) \) is finite, the result of Leifer is not sufficient to prove that the overlap \( \omega(\psi, \psi') \) goes to zero for every pair of quantum states. It is interesting to observe that Leifer’s result comes from the Frankl-Rödl theorem 33, which is also used to prove the bound \( C(n) > 0.01 \times 2^n \). 29

VI. CONCLUSION

In principle, quantum theory can be reformulated in the framework of classical probability theory. The simplest way to reward quantum phenomena in a classical language is to employ the quantum state as part of the classical description, possibly supplemented by auxiliary variables. This is done in the so-called \( \psi \)-ontic theories. However, these reformulations do not provide any substantial new content or improvement, unless they can predict observations detectably different from the standard formulation.

A classical theory of quantum processes becomes interesting if it provides substantial descriptional advantages. In this review, we have discussed about a possible alternative class of theories called \( \psi \)-epistemic. In spite of their exotic name, these theories are related to certain classical protocols studied in quantum communication complexity. This relation was first noted in Ref. 18. As the quantum state is not part of the classical description, \( \psi \)-epistemic theories can potentially introduce some simplification in the description of quantum systems. However, under the assumption of preparation independence, the PRB theorem implies that such reformulations are incompatible with quantum theory. Here, we have shown that it is possible to reach the same conclusion by re-
placing the preparation independence property with the somehow weaker equipartition property. As necessary requirement of the proof, the minimal amount of classical communication $C(n)$ required to replace the communication of $n$ qubits should increase faster than $2^n$. Interestingly, some recent results suggest that $C(n)$ increases as $n2^n$. An exact proof of this partial result would provide a strong suggestion that $\psi$-epistemic theories are actually incompatible with quantum theory.

We conclude by noting that the model in Ref. [14] collapses to a $\psi$-ontic theory in the limit of infinite qubits, even if the model does not satisfy the probability equipartition property. This leads us to wonder if the collapse to a $\psi$-ontic theory can be proved without any assumption.

Acknowledgments

The Author acknowledge useful discussions with Matthew Leifer, Jonathan Barrett and Stefan Wolf. This work is supported by the COST action on Fundamental Problems in Quantum Physics.

[1] R. Omnès, *Understanding Quantum Mechanics* (Princeton University Press, 1999).
[2] H. Everett, *Rev. Mod. Phys.* 29, 454 (1957).
[3] G. C. Ghirardi, A. Rimini, T. Weber, *Phys. Rev. D* 34, 470 (1986).
[4] L. Hardy, *Stud. Hist. Phil. Sci.* B 35, 267 (2004).
[5] A. Montina, *Phys. Rev. Lett.* 97, 180401 (2006).
[6] R. W. Spekkens, *Phys. Rev. A* 75, 032110 (2007).
[7] A. Montina, *Phys. Rev. A* 77, 022104 (2008).
[8] N. Harrigan, R. W. Spekkens, *Found. Phys.* 40, 125 (2010).
[9] A. Montina, *Phys. Rev. A* 83, 032107 (2011).
[10] A. Montina, *Phys. Lett. A* 375, 1385 (2011).
[11] S. D. Bartlett, T. Rudolph, R. W. Spekkens, *Phys. Rev. A* 86, 012103 (2012).
[12] A. Montina, *Phys. Rev. Lett.* 108, 160501 (2012).
[13] M. F. Pusey, J. Barrett, T. Rudolph, *Nature Physics* 8, 476 (2012).
[14] P. G. Lewis, D. Jennings, J. Barrett, T. Rudolph, *Phys. Rev. Lett.* 109, 150404 (2012).
[15] R. Colbeck, R. Renner, *Phys. Rev. Lett.* 108, 150402 (2012).
[16] M. Schlosshauer, A. Fine, *Phys. Rev. Lett.* 108, 260404 (2012).
[17] L. Hardy, arXiv:1205.1439 (2012).
[18] A. Montina, *Phys. Rev. Lett.* 109, 110501 (2012).
[19] N. J. Cerf, N. Gisin, and S. Massar, *Phys. Rev. Lett.* 84, 2521 (2000).
[20] B. F. Toner, D. Bacon, *Phys. Rev. Lett.* 91, 187904 (2003).
[21] A. Montina, S. Wolf, *Int. Symp. on Information Theory (ISIT)*, 1484, (IEEE, 2014).
[22] A. Montina, S. Wolf, *Phys. Rev. A* 90, 012309 (2014).
[23] M. S. Leifer, *Quanta* 3, 67 (2014).
[24] S. Kochen, E. Specker, *J. Math. Mech.* 17, 59 (1967).
[25] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley, New York, 1991).
[26] A. Montina, M. Pfaffhauser, S. Wolf, *Phys. Rev. Lett.* 111, 160502 (2013).
[27] C. H. Bennett, P. Shor, J. Smolin, and A. V. Thapliyal, IEEE Trans. Inf. Theory 48, 2637 (2002).
[28] P. Harsha, R. Jain, D. McAllester, J. Radhakrishnan, IEEE Trans. Inf. Theory 56, 498 (2010).
[29] G. Brassard, R. Cleve, A. Tapp, Phys. Rev. Lett. 83, 1874 (1999).
[30] A. Montina, *Phys. Rev. A* 84, 060303(R) (2011).
[31] A. Montina, *Phys. Rev. A* 90, 012309 (2014).
[32] M. S. Leifer, *Phys. Rev. Lett.* 112, 160404 (2014).
[33] P. Frankl and V. Rödl, *Trans. Amer. Math. Soc.* 300, 259 (1987).