Phase-space analysis of Hořava-Lifshitz cosmology

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We perform a detailed phase-space analysis of Hořava-Lifshitz cosmology, with and without the detailed-balance condition. Under detailed-balance we find that the universe can reach a bouncing-oscillatory state at late times, in which dark-energy, behaving as a simple cosmological constant, is dominant. In the case where the detailed-balance condition is relaxed, we find that the universe reaches an eternally expanding, dark-energy-dominated solution, with the oscillatory state preserving also a small probability. Although this analysis indicates that Hořava-Lifshitz cosmology can be compatible with observations, it does not enlighten the discussion about its possible conceptual and theoretical problems.

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I. INTRODUCTION

Recently, a power-counting renormalizable, ultraviolet (UV) complete theory of gravity was proposed by Hořava in \cite{1, 2, 3, 4}. Although presenting an infrared (IR) fixed point, namely General Relativity, in the UV the theory possesses a fixed point with an anisotropic, Lifshitz scaling between time and space of the form $x \rightarrow \ell x$, $t \rightarrow \ell^z t$, where $\ell$, $z$, $x$ and $t$ are the scaling factor, dynamical critical exponent, spatial coordinates and temporal coordinate, respectively.

Due to these novel features, there has been a large amount of effort in examining and extending the properties of the theory itself \cite{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21}. Additionally, application of Hořava-Lifshitz gravity as a cosmological framework gives rise to Hořava-Lifshitz cosmology, which proves to lead to interesting behavior \cite{22, 23}. In particular, one can examine specific solution subclasses \cite{24, 25, 26, 27, 28, 29}, the perturbation spectrum \cite{30, 31, 32, 33, 34, 35, 36}, the gravitational wave production \cite{37, 38, 39}, the matter bounce \cite{40, 41, 42}, the black hole properties \cite{43, 44, 45, 46, 47, 48, 49, 50, 51, 52}, the dark energy phenomenology \cite{53, 54, 55, 56, 57}, the astrophysical phenomenology \cite{58, 59} etc. However, despite this extended research, there are still many ambiguities if Hořava-Lifshitz gravity is reliable and capable of a successful description of the gravitational background of our world, as well as of the cosmological behavior of the universe \cite{11, 12, 13}.

Although the discussion about the foundations and the possible conceptual and phenomenological problems of Hořava-Lifshitz gravity and cosmology is still open in the literature, it is worth investigating in a systematic way the possible cosmological behavior of a universe governed by Hořava gravity. Thus, in the present work we perform a phase-space and stability analysis of Hořava-Lifshitz cosmology, with or without the detailed-balance condition, and we are interesting in investigating the possible late-time solutions. In these solutions we calculate various observable quantities, such as the dark-energy density and equation-of-state parameters. As we see, indeed Hořava-Lifshitz cosmology can be consistent with observations and in addition it can give rise to a bouncing universe. Furthermore, the results seem to be independent of the specific form of the dark matter content of the universe. This analysis however does not enlighten the discussion about possible conceptual problems and instabilities of Hořava-Lifshitz gravity, which is the subject of interest of other studies.

The paper is organized as follows: In section \textbf{II} we present the basic ingredients of Hořava-Lifshitz cosmology, extracting the Friedmann equations, and describing the dark matter and dark energy dynamics. In section \textbf{III} we perform a systematic phase-space and stability analysis for various cases under the detailed-balance condition, including a flat or non-flat geometry in the presence or not of a cosmological constant. In section \textbf{IV} we extend the phase-space analysis in the case where the detailed-balance condition is relaxed. In section \textbf{V} we discuss the corresponding cosmological implications and the effects on observable quantities. Finally, our results are summarized in section \textbf{VI}.

II. HOŘAVA-LIFSHITZ COSMOLOGY

We begin with a brief review of Hořava-Lifshitz gravity. The dynamical variables are the lapse and shift functions, $N$ and $N_i$ respectively, and the spatial metric $g_{ij}$ (roman letters indicate spatial indices). In terms of these fields the full metric is

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

where indices are raised and lowered using $g_{ij}$. The scaling transformation of the coordinates reads ($z=3$):

$$t \rightarrow \ell^3 t \quad \text{and} \quad x^i \rightarrow \ell x^i.$$

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Decomposing the gravitational action into a kinetic and a potential part as $S_g = \int dt d^3x \sqrt{g} N (L_K + L_V)$, and under the assumption of detailed balance [3] (the extension beyond detail balance will be performed later on), which apart form reducing the possible terms in the Lagrangian it allows for a quantum inheritance principle [1] (the $D + 1$ dimensional theory acquires the renormalization properties of the $D$-dimensional one), the full action of Hořava-Lifshitz gravity is given by

$$S_g = \int dt d^3x \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \Lambda K^2) - \frac{\kappa^2}{2a^2} C_{ij} C^{ij} + \frac{\kappa^2 \mu}{2a^2} \frac{\epsilon^{ijk}}{\sqrt{g}} R_i \nabla_j R_k - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left[ 1 - 4\lambda \frac{R^2}{\kappa^2} + \Lambda R - 3\Lambda^2 \right] \right\} , \quad (3)$$

where $K_{ij} = \frac{1}{2N} (g_{ij} - \nabla_i N_j - \nabla_j N_i)$, (4)

is the extrinsic curvature and

$$C^{ij} = \frac{\epsilon^{ijk}}{\sqrt{g}} \nabla_k (R^i_j - \frac{1}{4} R^i) \quad (5)$$

the Cotton tensor, and the covariant derivatives are defined with respect to the spatial metric $g_{ij}$. $\epsilon^{ijk}$ is the totally antisymmetric unit tensor, $\lambda$ is a dimensionless constant and $\Lambda$ is a negative constant which is related to the cosmological constant in the IR limit. Finally, the variables $\kappa$, $w$ and $\mu$ are constants with mass dimensions $-1$, 0 and 1, respectively.

In order to add the dark-matter content in a universe governed by Hořava gravity, a scalar field is introduced \cite{22, 23} with action:

$$S_M \equiv S_\phi = \int dt d^3x \sqrt{g} N \left\{ \frac{3\lambda - 1}{4} \dot{\phi}^2 + m_1 m_2 \phi \nabla^2 \phi - \frac{1}{2} m_2^2 \phi \nabla^4 \phi + \frac{1}{2} m_3^2 \phi \nabla^6 \phi - V(\phi) \right\} , \quad (6)$$

where $V(\phi)$ acts as a potential term and $m_i$ are constants. Although one could just follow a hydrodynamical approximation and introduce straightaway the density and pressure of a matter fluid \cite{12}, the field approach is more robust, especially if one desires to perform a phase-space analysis.

Now, in order to focus on cosmological frameworks, we impose the so called projectability condition \cite{11} and use an FRW metric,

$$N = 1 \, , \quad g_{ij} = a^2(t) \gamma_{ij} \, , \quad N^i = 0 \, , \quad (7)$$

with

$$\gamma_{ij} dx^i dx^j = \frac{dt^2}{1 - kr^2} + r^2 d\Omega_2^2 \, , \quad (8)$$

where $k = -1, 0, 1$ correspond to open, flat, and closed universe respectively. In addition, we assume that the scalar field is homogenous, i.e $\phi \equiv \phi(t)$. By varying $N$ and $g_{ij}$, we obtain the equations of motion:

$$H^2 = \frac{\kappa^2}{6(3\lambda - 1)} \left[ \frac{3\lambda - 1}{4} \dot{\phi}^2 + V(\phi) \right] + \frac{\kappa^2}{6(3\lambda - 1)} \left[ -3\kappa^2 \mu^2 k^2 \frac{3\kappa^2 \mu^2 \Lambda}{8(3\lambda - 1)} \right] + \frac{\kappa^4 \mu^2 \Lambda}{8(3\lambda - 1)^2 a^2} , \quad (9)$$

$$\dot{H} + \frac{3}{2} H^2 = -\frac{\kappa^2}{4(3\lambda - 1)} \left[ \frac{3\lambda - 1}{4} \dot{\phi}^2 - V(\phi) \right] - \frac{\kappa^2}{4(3\lambda - 1)} \left[ -\kappa^2 \mu^2 k^2 + \frac{3\kappa^2 \mu^2 \Lambda}{8(3\lambda - 1)} \right] + \frac{\kappa^4 \mu^2 \Lambda}{16(3\lambda - 1)^2 a^2} , \quad (10)$$

where we have defined the Hubble parameter as $H \equiv \frac{\dot{a}}{a}$. Finally, the equation of motion for the scalar field reads:

$$\ddot{\phi} + 3H \dot{\phi} + \frac{2}{3\lambda - 1} \frac{dV(\phi)}{d\phi} = 0. \quad (11)$$

At this stage we can define the energy density and pressure for the scalar field responsible for the matter content of the Hořava-Lifshitz universe:

$$\rho_M \equiv \rho_\phi = \frac{3\lambda - 1}{4} \dot{\phi}^2 + V(\phi) \quad (12)$$

$$p_M \equiv p_\phi = \frac{3\lambda - 1}{4} \dot{\phi}^2 - V(\phi). \quad (13)$$

Concerning the dark-energy sector we can define

$$\rho_{DE} \equiv -\frac{3\kappa^2 \mu^2 k^2}{8(3\lambda - 1)a^2} - \frac{3\kappa^2 \mu^2 \Lambda}{8(3\lambda - 1)} \quad (14)$$

$$p_{DE} \equiv -\frac{\kappa^2 \mu^2 k^2}{8(3\lambda - 1)a^4} + \frac{3\kappa^2 \mu^2 \Lambda}{8(3\lambda - 1)} . \quad (15)$$

The term proportional to $a^{-4}$ is the usual “dark radiation term”, present in Hořava-Lifshitz cosmology \cite{22, 23}. Finally, the constant term is just the explicit (negative) cosmological constant. Therefore, in expressions \cite{14, 15} we have defined the energy density and pressure for the effective dark energy, which incorporates the aforementioned contributions.

Using the above definitions, we can re-write the Friedmann equations \cite{10, 14} in the standard form:

$$H^2 = \frac{\kappa^2}{6(3\lambda - 1)} [\rho_M + \rho_{DE}] + \frac{\beta k}{a^2} \quad (16)$$

$$\dot{H} + \frac{3}{2} H^2 = -\frac{\kappa^2}{4(3\lambda - 1)} [p_M + p_{DE}] + \frac{\beta k}{2a^2} . \quad (17)$$
In these relations we have defined $\beta \equiv \frac{\kappa^2 \rho M}{(3\lambda - 1)H^3}$, which is the coefficient of the curvature term. Additionally, we could also define an effective Newton’s constant and an effective light speed \cite{22, 23}, but we prefer to keep $\kappa^2 (\partial 3\lambda - 1)$ in the expressions, just to make clear the origin of these terms in Hořava-Lifshitz cosmology. Finally, note that using \cite{11} it is straightforward to see that the aforementioned dark matter and dark energy quantities verify the standard evolution equations:

$$\dot{\rho}_M + 3H(\rho_M + p_M) = 0 \quad (18)$$

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0. \quad (19)$$

The aforementioned formulation of Hořava-Lifshitz cosmology has been performed under the imposition of the detailed-balance condition. However, in the literature there is a discussion whether this condition leads to reliable results or if it is able to reveal the full information of Hořava-Lifshitz gravity \cite{22, 23}. Thus, for completeness, we add here the Friedmann equation in the case where detailed balance is relaxed. In such a case one can in general write \cite{11, 12, 13}:

$$H^2 = \frac{2\sigma_0}{(3\lambda - 1)} \left[ \frac{3\lambda - 1}{4} \dot{\phi}^2 + V(\phi) \right] +$$

$$+ \frac{2}{(3\lambda - 1)} \left[ \frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{6a^4} + \frac{\sigma_4 k}{a^6} \right] +$$

$$+ \frac{\sigma_2}{3(3\lambda - 1) a^2} \frac{k}{a^2} \quad (20)$$

$$\dot{H} + \frac{3}{2} H^2 = - \frac{3\sigma_0}{(3\lambda - 1)} \left[ \frac{3\lambda - 1}{4} \dot{\phi}^2 - V(\phi) \right] -$$

$$- \frac{3}{(3\lambda - 1)} \left[ - \frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{18a^4} + \frac{\sigma_4 k}{6a^6} \right]$$

$$+ \frac{\sigma_2}{6(3\lambda - 1) a^2} \frac{k}{a^2}, \quad (21)$$

where $\sigma_0 \equiv \kappa^2/12$, and the constants $\sigma_i$ are arbitrary (although one can set $\sigma_2$ to be positive too). Thus, the effect of the detailed-balance relaxation is the decoupling of the coefficients, together with the appearance of a term proportional to $a^{-6}$. This term has a negligible impact at large scale factors, however it could play a significant role at small ones. Finally, in the non-detailed-balanced case, the energy density and pressure for matter coincide with those of detailed-balance scenario (expressions \cite{12, 13}), since the detailed-balance condition affects only the gravitational sector of the theory and has nothing to do with the matter content of the universe. However, the corresponding quantities for dark energy are generalized to:

$$\rho_{DE|_{\text{non-db}}} \equiv \frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{6a^4} + \frac{\sigma_4 k}{a^6} \quad (22)$$

$$p_{DE|_{\text{non-db}}} \equiv -\frac{\sigma_1}{6} + \frac{\sigma_3 k^2}{18a^4} + \frac{\sigma_4 k}{6a^6}. \quad (23)$$

Having presented the cosmological equations of a universe governed by Hořava-Lifshitz gravity, under detailed-balance condition or not, we can investigate the possible cosmological behaviors and discuss the corresponding physical implications by performing a phase-space analysis. This is done in the following two sections, for the detailed and non-detailed balance cases separately.

**III. DETAILED BALANCE: PHASE-SPACE ANALYSIS**

In order to perform the phase-space and stability analysis of the Hořava-Lifshitz universe, we have to transform the cosmological equations into an autonomous dynamical system \cite{59, 60, 61, 62}. This will be achieved by introducing the auxiliary variables:

$$x = \frac{\kappa \dot{\phi}}{2\sqrt{6H}}, \quad (24)$$

$$y = \frac{\kappa \sqrt{V(\phi)}}{6\sqrt{3\lambda - 1}}, \quad (25)$$

$$z = \frac{\kappa^2 \mu}{4(3\lambda - 1)a^2H}, \quad (26)$$

$$u = \frac{\kappa^2 \Lambda \mu}{4(3\lambda - 1)H}. \quad (27)$$

together with $M = \ln a$. Thus, it is easy to see that for every quantity $F$ we acquire $\dot{F} = H \frac{dF}{da}$. Using these variables we can straightforwardly obtain the density parameters of dark matter and dark energy (through expressions \cite{12, 13}) as:

$$\Omega_M \equiv \frac{\kappa^2}{6(3\lambda - 1)H^2} \rho_M = x^2 + y^2, \quad (28)$$

$$\Omega_{DE} \equiv \frac{\kappa^2}{6(3\lambda - 1)H^2} \rho_{DE} = -k^2 z^2 - u^2, \quad (29)$$

and in addition we can calculate the corresponding equation-of-state parameters:

$$w_M \equiv \frac{p_M}{\rho_M} = \frac{x^2 - y^2}{x^2 + y^2}, \quad (30)$$

$$w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} = \frac{k^2 z^2 - 3u^2}{3k^2 z^2 + 3u^2}. \quad (31)$$

We mention that these relations are always valid, that is independent of the specific state of the system (they are valid in the whole phase-space and not only at the critical points). Finally, for completeness, and observing \cite{10}, we can define the curvature density parameter as:

$$\Omega_k \equiv \frac{\beta k}{H^2 a^2} = 2kuz. \quad (32)$$

Using the auxiliary variables \cite{24, 25, 26, 27} the cosmological equations of motion \cite{12, 17, 18} and \cite{19}, can be transformed into an autonomous form $X' = \text{...}$. 


\( f(\mathbf{X}) \), where \( \mathbf{X} \) is the column vector constituted by the auxiliary variables, \( f(\mathbf{X}) \) the corresponding column vector of the autonomous equations, and prime denotes derivative with respect to \( M = \ln a \). The critical points \( \mathbf{X}_c \) are extracted satisfying \( \mathbf{X}' = 0 \), and in order to determine the stability properties of these critical points we expand around \( \mathbf{X}_c \), setting \( \mathbf{X} = \mathbf{X}_c + \mathbf{U} \) with \( \mathbf{U} \) the perturbations of the variables considered as a column vector. Thus, up to the first order we acquire \( \mathbf{U}' = \mathbf{Q} \cdot \mathbf{U} \), where the matrix \( \mathbf{Q} \) contains the coefficients of the perturbation equations. Thus, for each critical point, the eigenvalues of \( \mathbf{Q} \) determine its type and stability.

In the following we perform a phase-space analysis of the cosmological system at hand. As we can see from the Friedmann equations (9), (10) one can have a zero or non-zero cosmological constant, in a flat or non-flat universe. Thus, we observe, this phase plane is not compact since \( z \) is in general unbounded. However, the system is integrable and the orbit in the plane \( \Psi \), passing initially through \((x_0, z_0)\), can be obtained explicitly and it is given by the graph

\[
\begin{align*}
\dot{x} &= 3x - \sqrt{6q} \\
\dot{z} &= (3x^2 - 2)z.
\end{align*}
\]

We mention that for simplicity we have set \( q = \frac{1}{\kappa V(\phi) \left( \frac{dV(\phi)}{d\phi} \right)} \) and we have assumed it to be a constant, that is we are investigating the usual exponential potentials. However, as we will see this is not necessary, since the most important results of the present work are independent of the matter sector.

The autonomous system (35) is defined in the phase space

\[
\Psi = \{ (x, z) : -1 \leq x \leq 1, z \in \mathbb{R} \}.
\]

As we observe, this phase plane is not compact since \( z \) is in general unbounded. However, the system is integrable and the orbit in the plane \( \Psi \), passing initially through \((x_0, z_0)\), can be obtained explicitly and it is given by the parameter graph

\[
z(x) = z_0 \left( \frac{3x - \sqrt{6q}}{3x_0 - \sqrt{6q}} \right)^{\frac{1}{3}} \left( \frac{x^2 - 1}{x_0^2 - 1} \right)^{\frac{1}{3}} \cdot \exp \left\{ \frac{\sqrt{6q} \left( \tanh^{-1}(x) - \tanh^{-1}(x_0) \right)}{6q^2 - 9} \right\}.
\]

The critical points \((x_c, z_c)\) of the autonomous system (35) are obtained by setting the left hand sides of the equations to zero. They are displayed in table I where we also present the necessary conditions for their existence. In addition, for each critical point we calculate the values of \( w_M \) (given by relation (30)), of \( \Omega_{DE} \) (given by (29)), and of \( w_{DE} \) (given by (31)). Note that in this case, \( w_{DE} \) remains unspecified and the results hold independently of its value. Finally, The 2 \( \times \) 2 matrix \( \mathbf{Q} \) of the linearized perturbation equations writes:

\[
\mathbf{Q} = \begin{bmatrix}
9x^2 - 2\sqrt{6q}x - 3 & 0 \\
6xz & 3x^2 - 2
\end{bmatrix},
\]

and in table II we display its eigensystems (eigenvalues and associated eigenvectors) evaluated at each critical point, as well as their type and stability, acquired by examining the sign of the real part of the eigenvalues.

For hyperbolic critical points (all the eigenvalues have non-zero real parts different from zero) one can easily extract their type (source (unstable) for positive real parts, saddle for real parts of different sign and sink (stable) for negative real parts). However, if at least one eigenvalue has a zero real part (non-hyperbolic critical point) one is not able to obtain conclusive information about the stability from linearization and needs to resort to other tools like Normal Forms calculations [63, 64], or numerical experimentation. Thus, in the case at hand, \( P_1 \) and \( P_2 \) are
TABLE I: The critical points of a flat universe with $\Lambda = 0$ (case 1) and their behavior.

| Cr. P $\pm 1$ 0 Existence | Eigensystem | Stable for $w_M$ | $\Omega_{DE}$ | $w_{DE}$ |
|-----------------------------|-------------|-----------------|-------------|----------|
| $P_{1,2}$                  | All $q$     | $\mp 2\sqrt{3}q + 6$ | $\{0, 1\}$ | unstable | 1        | 0        | arbitrary |
| $P_3$ $\sqrt{\frac{3}{2}} q$ 0 $-\sqrt{\frac{3}{2}} < q < \sqrt{\frac{3}{2}}$ | $-3 + 2q^2$ | $\{0, 1\}$ | $1$ | $-1 < q < 1$ | $\frac{3}{4}q^2 - 1$ | 0 | arbitrary |
| $P_4$ $\sqrt{\frac{3}{2}} z_c$ | $q = \pm 1$ | $\frac{-1}{\mp 2\sqrt{3}z_c, 1}$ | $\{0, 1\}$ | nonhyperbolic | $1/3$ | 0 | arbitrary |

nonhyperbolic for $q = \sqrt{3/2}$, while $P_3$ is nonhyperbolic for $q^2 \in \{3/2, 1\}$. Finally, note that in the special case where $q = \pm 1$, the system admits an extra curve of critical points $P_4$. Each point in $P_4$ is non-hyperbolic, with center manifold tangent to the $z$-axis, but the curve $P_4$ is actually “normally hyperbolic” [63]. This means that we can indeed analyze the stability by analyzing the sign of the real parts of the non-null eigenvalues. Therefore, since the non zero eigenvalue is negative, $P_4$ is a local attractor.

In order to present the aforementioned behavior more transparently, we evolve the autonomous system numerically for the choice $q = 0.6$, and the results are shown in figure [11] As we can see, in this case the critical point $P_3$ is the global attractor of the system.

B. Case 2: non-flat universe with $\Lambda = 0$

Under this scenario, and using the auxiliary variables [24, 25, 26, 27], the Friedmann equations [9, 10] become:

$$1 = x^2 + y^2 - z^2$$

$$\frac{H'}{H} = -3x^2 + 2z^2$$

while the autonomous system writes:

$$x' = x \left(3x^2 - 2z^2 - 3\right) + \sqrt{6q} \left(-x^2 + z^2 + 1\right)$$

$$z' = z \left[3x^2 - 2 \left(z^2 + 1\right)\right]$$

It is defined in the phase space $\Psi = \{(x, z) : x^2 - z^2 \leq 1, z \in \mathbb{R}\}$ and as before the phase space is not compact. Finally, the matrix $Q$ of the linearized perturbation equations is:

$$Q = \begin{bmatrix}
9x^2 - 2\sqrt{6q} - 2z^2 - 3 & 2 \left(\sqrt{6q} - 2x\right)z \\
6xz & 3x^2 - 6z^2 - 2
\end{bmatrix}.$$

The critical points, the eigensystems, the conditions for their existence and stability, and the physical quantities are presented in table [11]. Thus, $P_{1,2,3}$ are exactly the same as in case 1, while $P_{4,5,6}$ are saddle points except if $q^2 \to 1$, where they are nonhyperbolic. It is interesting to notice that this scenario admits two more unstable critical points, namely $P_{7,8}$, in which $z_c^2 = -1$. These points are of great physical importance, as we are going to see in the next section.

In order to present the results more transparently, in fig. [2] we present the numerical evolution of the system for the choice $q = \sqrt{3}$. In this specific realization of the scenario the critical points $P_3$ and $P_{5,6,7,8}$ do not exist. We find only the source $P_1$ and the saddle $P_2$, and we indeed observe that there is one orbit approaching $P_2$ (the solution with $z \equiv 0$). Finally, note that the divergence of the orbits towards the future is typical and suggests that the future attractor of the system can be located at infinite regions. In fig. [3] we depict the phase-space graph for the choice $q = 0.6$. In this case the critical points $P_{1,2}$ are unstable (sources), while $P_3$ is a local attractor. The points $P_{5,6}$ are saddle ones, and thus we observe that some orbits coming from infinity spend a large amount of time near them before diverge again in a finite time.
with $\Lambda = 0$

**Phase plane for a non-flat universe**

**FIG. 3:** (Color Online)

**TABLE II:** The critical points of a non-flat universe with $\Lambda = 0$ (case 2) and their behavior. We use the notations $\mu_0 = \sqrt{-15 + \frac{16}{q^2}}$, $\mu_1 = \frac{-9q^2 - \sqrt{6q^2 - 15q^4 + 8}}{4\sqrt{q^2 - 6q}}$ and $\mu_2 = \frac{-9q^2 + \sqrt{6q^2 - 15q^4 + 8}}{4\sqrt{q^2 - 6q}}$.

In this specific scenario the critical point $P_3$ is a local attractor, while $P_{1,2}$ and $P_{5,6}$ are saddle ones.

**C. Case 3: flat universe with $\Lambda \neq 0$**

In this case the Friedmann equations (9), (10) write as

$$1 = x^2 + u^2 - a^2$$

$$\frac{H'}{H} = -3x^2,$$ (40)

and the autonomous system becomes:

$$x' = \sqrt{6q} (u^2 - x^2 + 1) + 3x (x^2 - 1),$$

$$u' = 3a x^2,$$ (42)

defined in the phase space $\Psi = \{(x, u) : x^2 - u^2 \leq 1, u \in \mathbb{R}\}$. As before the phase space is not compact. The matrix $Q$ of the linearized perturbation equations is:

$$Q = \begin{bmatrix} 9x^2 - 2\sqrt{6qx} - 3 & 2\sqrt{6qu} \\ 6ux & 3x^2 \end{bmatrix}.$$ (43)

The critical points, the eigensystems, the conditions for their existence and stability, and the physical quantities are presented in table III. Note that the critical point $P_{13}$ is nonhyperbolic if $q^2 \in \{0, 3/2\}$, while it is a saddle otherwise, with stable (unstable) manifold tangent to the $x$- ($u$-) axis. Finally, the system admits two more nonhyperbolic critical points, namely $P_{12,13}$, in which $u_c^2 = -1$.

**FIG. 4:** (Color Online) Phase plane for a flat universe with $\Lambda \neq 0$ (case 3), for the choice $q = \sqrt{3}$. In this specific scenario the critical point $P_{13}$ does not exists. $P_{10}$ is unstable (source), while $P_9$ is a saddle one.

In fig. 4 we present the phase-space graph of the system for the choice $q = \sqrt{3}$. In this case the critical point...
Finally, in fig. 5 we display the phase-space graph for the attractor of the system will be located at infinite regions. wards the future is typical and suggests that the future orbits to-
are presented in table IV.

The critical points, and their corresponding information, are presented in table IV.

\[
Q = \begin{bmatrix}
9x^2 - 2\sqrt{6}qz + 2(u - z)z - 3 & -2\sqrt{6}kqu + 2xu + 2\sqrt{6}qz - 4xz & 2 [xz + \sqrt{6}q(u - k)] \\
6xz & 3x^2 - 6z^2 + 4uz - 2 & 2z^2 \\
6ux & 2u(u - 2z) & 3x^2 - 2z^2 + 4uz
\end{bmatrix}.
\]

The critical point \( P_{14} \) is nonhyperbolic if \( q = \sqrt{3/2} \), it is a source if \( q < \sqrt{3/2} \) or a saddle otherwise, while \( P_{15} \)

TABLE III: The critical points of a flat universe with \( \Lambda \neq 0 \) (case 3) and their behavior. NH stands for nonhyperbolic.

| Cr. P | \( x_c \) | \( u_c \) | Existence | Eigensystem | Stable for | \( w_M \) | \( \Omega_{DE} \) | \( w_{DE} \) |
|-------|--------|--------|-----------|-------------|------------|---------|------------|---------|
| \( P_{9,10} \) | \( \pm 1 \) | 0 | All \( q \) | \( \pm 2\sqrt{6}q + 6 \) | unstable | 1 | 0 | arbitrary |
| \( P_{11} \) | \( \sqrt{\frac{2}{3}}q \) | 0 | \(-\sqrt{\frac{2}{3}} < q < \sqrt{\frac{2}{3}} \) | \(-3 + 2q^2 \) | unstable | \( \frac{3}{2}q^2 - 1 \) | 0 | arbitrary |
| \( P_{12,13} \) | 0 | \( \pm i \) | always | \(-3 \) | unstable | \( 2i\sqrt{\frac{2}{3}}q, 1 \) | NH | arbitrary | 1 | -1 |

\( P_{11} \) does not exists, while \( P_9 \) and \( P_{10} \) are unstable (source and saddle respectively). The divergence of the orbits towards the future is typical and suggests that the future attractor of the system will be located at infinite regions. Finally, in fig. 5 we display the phase-space graph for the choice \( q = 0.6 \). In this case the critical point \( P_{11} \) is a saddle one (with stable manifold tangent to the x-axis), while \( P_9 \) and \( P_{10} \) are unstable (sources). There are two orbits, one joining \( P_{10} \) with \( P_9 \) and one joining \( P_9 \) with \( P_{11} \), both of them overlapping the x-axis. Note that some orbits remain close to \( P_{11} \) before finally diverge towards the future, and this suggests that the future attractor of the system is located at infinite regions.

D. Case 4: \( k \neq 0, \Lambda \neq 0 \)

Under this scenario, and using the auxiliary variables \( \varphi_1, \varphi_2, \varphi_3, \varphi_4 \), the Friedmann equations \( (9), (10) \) become:

\[
1 = x^2 + y^2 - (u - k)z^2 \quad (43)
\]

\[
\frac{H'}{H} = -3x^2 + 2z(-u + z). \quad (44)
\]

while the autonomous system writes:

\[
x' = \sqrt{6}q \left[ -x^2 + (u - z)^2 + 1 \right] + x \left[ 3x^2 + 2(u - z)z - 3 \right],
\]

\[
z' = z \left[ 3x^2 + 2(u - z)z - 2 \right],
\]

\[
u' = u \left[ 3x^2 + 2(u - z)z \right],
\]

defined in the phase space \( \Psi = \{ (x, z, u) : x^2 - (u - k)z^2 \leq 1, u, z \in \mathbb{R} \} \), which is not compact. The linearized perturbation matrix \( Q \) reads:

\[
Q = \begin{bmatrix}
9x^2 - 2\sqrt{6}qz + 2(u - z)z - 3 & -2\sqrt{6}kqu + 2xu + 2\sqrt{6}qz - 4xz & 2 [xz + \sqrt{6}q(u - k)] \\
6xz & 3x^2 - 6z^2 + 4uz - 2 & 2z^2 \\
6ux & 2u(u - 2z) & 3x^2 - 2z^2 + 4uz
\end{bmatrix}.
\]
is nonhyperbolic \( q = -\sqrt{3/2} \), it is a source if \( q > -\sqrt{3/2} \) or a saddle otherwise. \( P_{16} \) is nonhyperbolic if \( q^2 \in \{0, 1, 3/2\} \) and saddle otherwise, while \( P_{17,18} \) are nonhyperbolic if \( q^2 \rightarrow 1 \), and saddle otherwise. The points \( P_{19,20} \) have the eigenvalues \( \{-3, -2, 0\} \) with associated eigenvectors \( \{1, 0, 0\}, \{0, 1, 1\}, \{\pm 2i \sqrt{q}, 0, 1\} \). Hence, they are nonhyperbolic possessing a 2-dimensional stable manifold. Finally, \( P_{21,22} \) are unstable because they give rise to the eigenvalues \( \{4, 2, -1\} \), with associated eigenvectors \( \{\pm \frac{3}{4} i \sqrt{6q}, 1, 0\}, \{0, 1, 1\}, \{1, 0, 0\} \).

### IV. BEYOND DETAILED BALANCE: PHASE SPACE ANALYSIS

In this section we extend the phase-space analysis to a universe governed by Hořava gravity in which the detailed balance condition has been relaxed. In order to transform the corresponding cosmological equations into an autonomous dynamical system, we use the auxiliary variables \( x \) and \( y \) defined in (49), (50), and furthermore we define the following four new ones:

\[
\begin{align*}
x_1 &= \frac{\sigma_1}{3(3\lambda - 1)H^2}, \\
x_2 &= \frac{k\sigma_2}{3(3\lambda - 1)a^2H^2}, \\
x_3 &= \frac{\sigma_3}{3(3\lambda - 1)a^4H^2}, \\
x_4 &= \frac{2k\sigma_4}{(3\lambda - 1)a^6H^2}. \\
\end{align*}
\]

Thus, using these variables and the definitions (22) and (41), we can express the dark energy density and equation-of-state parameters respectively as:

\[
\begin{align*}
\Omega_{DE}\big|_{\text{non-db}} &= \frac{2}{(3\lambda - 1)H^2} \left( \frac{\sigma_1 + \sigma_3 k^2}{6a^4} + \frac{\sigma_4 k}{a^6} \right) = x_1 + x_3 + x_4, \\
w_{DE}\big|_{\text{non-db}} &= \frac{\sigma_1}{6a^4} + \frac{\sigma_3 k^2}{6a^4} + \frac{\sigma_4 k}{a^6} = \frac{6x_1 - 2x_3 - x_4}{6(x_1 - x_3 + x_4)}. \\
\end{align*}
\]

Note that the corresponding quantities for dark matter coincide with those of the detailed balance case (expressions (28) and (30)).

Using the aforementioned auxiliary variables, the Friedmann equations (20), (21) become:

\[
\begin{align*}
1 &= x_1 + x_2 + x_3 + x_4 + x^2 + y^2, \\
H' &= -3x_2 - 2x_3 - 3x_4. \\
\end{align*}
\]

Thus, after using the first of these relations in order to eliminate one variable, the corresponding autonomous system writes:

\[
\begin{align*}
x_2' &= 2x_2 \left( 3x_2^2 + x_2 + 2x_4 + 3x_4 - 1 \right), \\
x_3' &= 2x_3 \left( 3x_2^2 + x_2 + x_3 + 3x_4 - 2 \right), \\
x_4' &= 2x_4 \left( 3x_2^2 + x_2 + 2x_3 + 3x_4 - 3 \right), \\
x' &= 3x^3 + (x_2 + 2x_4 + 3x_4 - 3)x + \sqrt{6q}y, \\
y' &= \left( 3x^2 - \sqrt{6q}x + x_2 + 2x_3 + 3x_4 \right)y. \\
\end{align*}
\]

defining a dynamical system in \( \mathbb{R}^5 \). Its critical points and their properties are displayed in table V and in table VI we present the corresponding observable cosmological quantities.

The curve of nonhyperbolic critical points denoted by \( P_{23} \) is “normally hyperbolic” [65]. Thus, examining the sign of the real parts of the non-null eigenvalues, we find that they are always local sources provided \( q_{xc} > \sqrt{6}/2 \). \( P_{27,28} \) are saddle points and their stable manifold can be 4-dimensional provided \( -\sqrt{\frac{3}{2}} < q < \sqrt{\frac{3}{2}} \). \( P_{29} \) has a 2-dimensional unstable manifold tangent to the \( x_2-y \) plane and its stable manifold is always 3-dimensional. \( P_{30,31} \) have a 4-dimensional stable manifold provided \( q^2 > \frac{3}{2} \) or \( -\sqrt{\frac{3}{2}} \leq q \leq -\sqrt{\frac{3}{2}} \), or \( \sqrt{\frac{3}{2}} < q \leq \sqrt{\frac{3}{2}} \). Finally, \( P_{33,34} \) has a 3-dimensional stable manifold if \( q^2 > \frac{16}{15} \) or \( -\frac{1}{\sqrt{15}} \leq q \leq -1 \) or \( 1 < q \leq \sqrt{\frac{16}{15}} \).

Amongst all these critical points \( P_{26} \) although nonhyperbolic, proves to be a stable one. To see that we use techniques such as the Normal Forms theorem [63, 64], which allows to obtain a simplified system by successive nonlinear transformations. In particular, we

| Cr. P | \( x_c \) | \( z_c \) | \( u_c \) | Existence | Stable for | \( w_M \) | \( \Omega_{DE} \) | \( w_{DE} \) |
|-------|---------|---------|---------|-----------|-----------|-------|-----------|-------|
| \( P_{14,15} \) | \( \pm 1 \) | 0 | 0 | All \( q \) | unstable | 1 | 0 | arbitrary |
| \( P_{16} \) | \( \sqrt{\frac{3}{2}}q \) | 0 | 0 | \( -\sqrt{\frac{3}{2}} < q < \sqrt{\frac{3}{2}} \) | unstable | \( \frac{2}{3}q^2 - 1 \) | 0 | arbitrary |
| \( P_{17,18} \) | \( \sqrt{\frac{3}{2}}q \) | \( \pm \sqrt{1 + \frac{1}{q^3}} \) | 0 | \( -1 < q \leq 1, q \neq 0 \) | unstable | 2 | 1 - \( \frac{1}{q^2} \) | 1/3 |
| \( P_{19,20} \) | 0 | 0 | \( \pm i \) | 0 | always | nonhyperbolic | arbitrary | 1 | -1 |
| \( P_{21,22} \) | 0 | \( \pm i \) | 0 | always | unstable | arbitrary | 1 | 1/3 |

**TABLE IV**: The critical points of a non-flat universe with \( \Lambda \neq 0 \) (case 4) and their behavior.
TABLE V: The critical points of a universe governed by Hořava gravity beyond detailed balance (system (51)) and their behavior. NH stands for nonhyperbolic.

| Cr. P | $x_2c$ | $x_3c$ | $x_{4c}$ | $x_c$ | $y_c$ | Existence | Eigenvalues | Stable for |
|-------|--------|--------|---------|------|------|-----------|------------|----------|
| $P_{23}$ | 0 | 0 | $1 - x_c^2$ | $x_c$ | 0 | All $q$ | $6 \ 0 \ 2 \ 4 \ 3 - \sqrt{6}q x_c$ | NH |
| $P_{24,25}$ | 0 | 0 | 0 | $\pm 1$ | 0 | All $q$ | $6 \ 4 \ 2 \ 0 \ 3 + \sqrt{6}q$ | NH |
| $P_{26}$ | 0 | 0 | 0 | 0 | 0 | All $q$ | $-6 \ -4 \ -3 \ -2 \ 0$ | stable |
| $P_{27,28}$ | 0 | 0 | 0 | $\sqrt{\frac{3}{2}}q \pm \sqrt{1 - \frac{2q^2}{3}}$ | $q^2 \leq \frac{3}{2}$ | $4q^2 - 3 + 2q^2 \ 2(-3 + 2q^2) \ 4(-1 + q^2) \ 2(-1 + 2q^2)$ | unstable |
| $P_{29}$ | 1 | 0 | 0 | 0 | 0 | All $q$ | $-4 \ -2 \ -2 \ 2 \ 1$ | unstable |
| $P_{30,31}$ | $1 - \frac{\sqrt{3}q}{2q^2}$ | 0 | 0 | $\sqrt{\frac{3}{2}q} \pm \sqrt{\frac{3}{4}q}$ | $q \neq 0$ | $-4 \ -2 \ 2 \ -1 - \sqrt{-3 + \frac{6}{q^2}} \ -1 + \sqrt{-3 + \frac{6}{q^2}}$ | unstable |
| $P_{32}$ | 0 | 1 | 0 | 0 | 0 | All $q$ | $-4 \ -2 \ 2 \ 2 \ -1$ | unstable |
| $P_{33,34}$ | $1 - \frac{\sqrt{3}}{3}q^2$ | 0 | 0 | $\frac{\sqrt{3}q}{q} \pm \sqrt{\frac{3}{4}q}$ | $q \neq 0$ | $4 \ -2 \ 2 \ -\frac{1}{2} \left(1 - \sqrt{-15 + \frac{16}{q^2}}\right) \ -\frac{1}{2} \left(1 + \sqrt{-15 + \frac{16}{q^2}}\right)$ | unstable |
| $P_{35}$ | 0 | 0 | 1 | $1 - \frac{3}{2q^2}$ | $\frac{\sqrt{3}q}{q}$ | 0 | All $q$ | $6 \ 0 \ 0 \ 2 \ 4$ | NH |

TABLE VI: Observable cosmological quantities of a universe governed by Hořava gravity beyond detailed balance.

| Cr. P | $w_M$ | $\Omega_M$ | $\Omega_{DE}$ | $w_{DE}$ |
|-------|-------|----------|-------------|--------|
| $P_{23}$ | 1 | $x_c^2$ | $1 - x_c^2$ | $1/6$ |
| $P_{24,25}$ | 1 | 1 | 0 | arbitrary |
| $P_{26}$ | arbitrary | 0 | 1 | -1 |
| $P_{27,28}$ | $\frac{2q^2 - 1}{3}$ | 1 | $\frac{1 - 2q^2 + \sqrt{9 - 6q^2} - 3}{-2q^2 + \sqrt{9 - 6q^2} + 3}$ | -1 |
| $P_{29}$ | 1 | 1 | 0 | arbitrary |
| $P_{30,31}$ | $-\frac{1}{3}$ | $\frac{2q^2}{3q^2 + \sqrt{3q} + 1}$ | -1 |
| $P_{32}$ | arbitrary | 0 | 1 | $1/3$ |
| $P_{33,34}$ | $\frac{1}{3}$ | $\frac{1}{q}$ | $1 + \frac{3}{\sqrt{3 - 3q} + 1} \frac{2 - q(4 + q)}{\sqrt{3 - 3q} + 2}$ | |
| $P_{35}$ | 1 | $\frac{3}{2q^2}$ | $1 - \frac{3}{2q^2}$ | $1/6$ |

As a result, we get the simplified system in the new variables:

$x'_2 = -6x_2 + O(4),$
$x'_3 = -4x_3 + O(4),$
$x'_4 = 4x_4 (-3 + 4q^2y^2) + O(4),$

First make the linear transformation $(x_2, x_3, x_4, x, y) \rightarrow (x, x_3, x_2, x_4, y)$ in order to transform the matrix of linear perturbations $Q$ evaluated in $(0,0,0,0,0)$ to its Jordan real form: $\text{diag} (-6, -4, -3, -2, 0).$ The next step is to perform a quadratic coordinate transformation given by
\[ x' = -2x + O(4), \]
\[ y' = -2q^2 y^3 + O(4), \]  
\[ y < \varepsilon \]}
\[ \varepsilon > 0 \]

where \( O(4) \) denotes terms of fourth order with respect to the vector norm. The center manifold, \( W_{\text{loc}}^c \), is tangent to the y-axis at the origin, and it can be represented locally, up to an error \( O(4) \), as the graph

\[ W_{\text{loc}}^c = \left\{ (x_2, x_3, x_4, x, y) \in \mathbb{R}^5 : x_2 = x_{20} e^{-\frac{1}{2y^2\varepsilon^2}}, \right. \]
\[ x_3 = x_{30} e^{-\frac{1}{2y^2\varepsilon^2}}, \]
\[ x_4 = x_{40} y^2 e^{-\frac{1}{2y^2\varepsilon^2}}, \]
\[ x = x_{10} e^{-\frac{1}{2y^2\varepsilon^2}}, \]
\[ y < \varepsilon \]}
\[ \varepsilon > 0 \]

where \( \varepsilon \) is a positive, sufficiently small constant. From \[ \varepsilon > 0 \] we deduce that \( y \), under the initial condition \( y(0) = y_0 \), evolves as \( y(t) = y_0 (1 + 4q^2 y_0^2 M)^{-1/2} \). Hence, as time passes the origin is approached and thus the critical point \( P_{26} \) is definitely an attractor. Note also that since it has a 1-dimensional center manifold tangent to the y-axis, then the stable manifold of \( P_{26} \) is 4-dimensional, which also proves that \( P_{26} \) is a late-time attractor.

V. COSMOLOGICAL IMPLICATIONS

Since we have performed a phase-space analysis of a universe governed by Hořava-Lifshitz gravity, with or without the detailed-balance condition, we can now discuss the corresponding cosmological behavior.

A. Detailed balance

1. Case 1: flat universe with \( \Lambda = 0 \)

In this scenario the critical points \( P_{1,2} \) are not relevant from a cosmological point of view, since apart from being unstable they correspond to complete dark matter domination, with the matter equation-of-state parameter being unphysically stiff. However, point \( P_3 \) is more interesting since it is stable for \(-1 < q < 1\) and thus it can be the late-time state of the universe. If additionally we desire to keep the dark-matter equation-of-state parameter being unphysically stiff, however, point \( P_3 \) is more interesting since it is stable for \(-1 < q < 1\) and thus it can be the late-time state of the universe. If additionally we desire to keep the dark-matter equation-of-state parameter being unphysically stiff, however, point \( P_3 \) is more interesting since it is stable for \(-1 < q < 1\) and thus it can be the late-time state of the universe.

2. Case 2: non-flat universe with \( \Lambda = 0 \)

In this scenario, the first three critical points are identical with those of case 1, and thus the physical implications are the same. The critical points \( P_{5,6} \) are unstable, corresponding to a dark-matter dominated universe. This was expected since in the absence of the cosmological constant \( \Lambda \), the curvature role is downgrading as the scale factor increases and thus in the end this case tends to the case 1 above. Note however that at early times, where the scale factor is small, the behavior of the system will be significantly different than case 1, with the dark energy playing an important role. This different behavior is observed in the corresponding phase-space figures \[ [2][3] \] comparing with figure \[ [1] \].

The case at hand admits another solution sub-class, namely points \( P_{7,8} \). In these points \( s_2^2 = -1 \), and thus using \[ (26) \] we straightforwardly find the late-time solution \( a(t) = e^{\gamma t} \), with \( \gamma = |\kappa^2\mu/[4(3\lambda - 1)]| \). This solution corresponds to an oscillatory universe \[ [48][49] \] and in the context of Hořava-Lifshitz cosmology it has already been studied in the literature \[ [40][11][42] \]. However, as we see, these critical points are unstable and thus this solution subclass cannot be a late-time attractor in the case of a non-flat universe with zero cosmological constant. This situation will change in the case where the cosmological constant is switched on.

3. Case 3: flat universe with \( \Lambda \neq 0 \)

Under this scenario, the Hořava-Lifshitz universe admits two unstable critical points \( P_{9,10} \), completely dominated by stiff dark matter. Point \( P_{11} \) exhibits a more physical dark matter equation-of-state parameter, but still with negligible dark energy at late times. The case at hand admits the two nonhyperbolic points \( P_{12,13} \) possessing \( w^2_M = -1 \), and thus (as can be seen by \[ (27) \]) they correspond to the oscillatory solution \( a(t) = e^{\delta t} \), with \( \delta = |\kappa^2\mu/[4(3\lambda - 1)]| \). We mention that these points are nonhyperbolic, with a negative eigenvalue, and thus they have a large probability to be a late-time solution of Hořava-Lifshitz universe. Additionally, they correspond to dark-energy domination, with dark-energy equation-of-state parameter \(-1\) and an arbitrary \( w_M \). These features make them good candidates to be a realistic description of the universe. We mention that this result is independent from the parameter \( q \) which comes from the dark matter sector. Thus, we conclude that it is valid independently of the matter-content of the universe. Indeed, this behavior is novel, and arises purely by the extra terms that are present in Hořava gravity.

4. Case 4: non-flat universe with \( \Lambda \neq 0 \)

This case admits the unstable critical points \( P_{14,15,16} \) which correspond to a dark-matter dominated universe,
and the unstable points $P_{17,18}$ which are unphysical since they possess $w_M = 2$. As expected, the system admits also the unstable points $P_{21,22}$ which correspond to oscillatory universes with $a(t) = e^{x_0 t}$ ($\gamma = \left| \kappa^2 \mu / 4(3\lambda - 1) \right|$). However, we find two more oscillatory critical points, namely $P_{19,20}$, which correspond to $a(t) = e^{i\omega t}$, with $\delta = \left| \kappa^2 \mu / 4(3\lambda - 1) \right|$. These points are nonhyperbolic, with a negative eigenvalue, and thus they have a large probability to be the late-time state of the universe, and additionally this result is independent of the specific form of the dark-matter content. Furthermore, they correspond to a dark-energy dominated universe, with $w_{DE} = -1$ and arbitrary $w_M$. Thus, they are good candidates for a realistic description of the universe.

**B. Beyond detailed balance**

Let us now discuss about the cosmological behavior of a Hořava-Lifshitz universe, in the case where the detailed balance condition is abandoned. In this case the system admits the unstable critical points $P_{27,28,29}$ which correspond to dark matter domination, the unstable point $P_{32}$ corresponding to an unphysical dark-energy dominated universe, and the unstable $P_{30,31,33,34}$ which have physical $w_M, w_{DE}$ but dependent on the specific dark-matter form. The system admits also the critical points $P_{23}, P_{35}$ which are nonhyperbolic with positive non-null eigenvalues, thus unstable, with furthermore unphysical cosmological quantities. Additionally, points $P_{24,25}$ are also dark-matter dominated, unstable nonhyperbolic ones.

It is interesting to notice that since $\sigma_3$ has an arbitrary sign, $P_{33,34}$ could also correspond to an oscillatory universe, for a wide region of the parameters $\sigma_3$ and $q$. However, this oscillatory behavior has a small probability to be the late-time state of the universe because it is not stable (with at least two positive eigenvalues). Additionally, the fact that it depends on $q$ means that this solution depends on the matter form of the universe.

The scenario at hand admits a final critical point, namely $P_{26}$. As we showed in detail in section IV using Normal Forms techniques, it is indeed stable and thus it can be a late-time attractor of Hořava-Lifshitz universe beyond detailed balance. Using the definition of the auxiliary variables, we can straightforwardly show that it corresponds to an eternally expanding solution. Additionally, it is characterized by complete dark energy domination, with dark-energy equation-of-state parameter $-1$ and arbitrary $w_M$. Note also that this result is independent of the specific form of the dark-matter content. These feature make it a very good candidate for the description of our universe. We mention that according to the initial conditions, this universe on its way towards this late-time attractor can be just an expanding universe with a non-negligible dark matter content, which is in agreement with observations, and this can be verified also by numerical investigation. This fact makes the aforementioned result more concrete.

**VI. CONCLUSIONS**

In this work we performed a detailed phase-space analysis of Hořava-Lifshitz cosmology, with and without the detailed-balance condition. In particular, we examined if a universe governed by Hořava gravity can have late-time solutions compatible with observations.

In the case where the detailed-balance condition is imposed, we find that the universe can reach a bouncing-oscillatory state at late times, in which dark-energy, behaving as a simple cosmological constant, will be dominant. Such solutions were already investigated in the context of Hořava-Lifshitz cosmology [40,41,42] as possible ones, but now we see that they can indeed be the late-time attractor for the universe. They arise purely from the novel terms of Hořava-Lifshitz cosmology, and in particular the dark-radiation term proportional to $a^{-4}$ is responsible for the bounce, while the cosmological constant term is responsible for the turnaround.

In the case where the detailed-balance condition is abandoned, we find that the universe reaches an eternally expanding solution at late times, in which dark-energy, behaving like a cosmological constant, dominates completely. Note that according to the initial conditions, the universe on its way to this late-time attractor can be an expanding one with non-negligible matter content. We mention that this behavior is independent of the specific form of the dark-matter content. Thus, the aforementioned features make this scenario a good candidate for the description of our universe, in consistency with observations. Finally, in this case the universe has also a probability to reach an oscillatory solution at late times, if the initial conditions lie in its basin of attraction (in this case the eternally expanding solution will not be reached).

Although this analysis indicates that Hořava-Lifshitz cosmology can be compatible with observations, it does not enlighten the discussion about possible conceptual and phenomenological problems and instabilities of Hořava-Lifshitz gravity, nor it can interfere with the questions concerning the validity of its theoretical background, which is the subject of interest of other studies. It just faces the problem from the cosmological point of view, and thus its results can been taken into account only if Hořava gravity passes successfully the aforementioned theoretical tests.

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**Note added**

While this work was being typed, we became aware of [68], which presents also an analysis of the phase-space of Hořava-Lifshitz cosmology, though in a different framework. We agree with [68] on the regions of overlap.
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