LOCAL LIMIT THEOREM IN NEGATIVE CURVATURE

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Abstract. Consider the heat kernel \( \varphi(t, x, y) \) on the universal cover \( \tilde{M} \) of a closed Riemannian manifold of negative curvature. We show the local limit theorem for \( \varphi \):

\[
\lim_{t \to \infty} t^{3/2} e^{-\lambda_0 t} \varphi(t, x, y) = C(x, y),
\]

where \( \lambda_0 \) is the bottom of the spectrum of the geometric Laplacian and \( C(x, y) \) is a positive \((-\lambda_0)\)-eigenfunction which depends on \( x, y \in \tilde{M} \).

We show that the \( \lambda_0 \)-Martin boundary of \( \tilde{M} \) is equal to its topological boundary. The Martin decomposition of \( C(x, y) \) gives a family of measures \( \{\mu_{\lambda_0}^x\} \) on \( \partial \tilde{M} \). We show that \( \{\mu_{\lambda_0}^x\} \) is the unique family minimizing the energy or the Rayleigh quotient of Mohsen.

We use the uniform Harnack inequality on the boundary \( \partial \tilde{M} \) and the uniform three-mixing of the geodesic flow on the unit tangent bundle \( SM \) for suitable Gibbs-Margulis measures.

1. Introduction

Let \((M, g)\) be an \(m\)-dimensional closed connected Riemannian manifold of negative curvature, and \((\tilde{M}, \tilde{g})\) its universal cover endowed with the lifted Riemannian metric. Let us denote by \(d\) the distance on \(M\), \(\tilde{M}\), as well as on their unit tangent bundles \(\pi : SM \to M\) and \(\tilde{\pi} : S\tilde{M} \to \tilde{M}\) (see [PPS] for various distances on \(M\) and on \(SM\) and the equivalences between them). We denote the natural projection \((\tilde{M}, \tilde{g}) \to (M, g)\) and its derivative by \(p\). The fundamental group \(\Gamma = \pi_1(M)\) acts on \(\tilde{M}\) as isometries such that \(M = \tilde{M}/\Gamma\). Let \(M_0\) be a bounded fundamental domain for this action.

We consider the Laplace-Beltrami operator \(\Delta := \text{Div}\nabla\) for smooth functions on \(\tilde{M}\) and the corresponding heat kernel function \(\varphi(t, x, y), t \in \mathbb{R}_+, x, y \in \tilde{M}\), which is the probability density defined as the fundamental solution of the heat equation, i.e. the function which satisfies \(\frac{\partial \varphi}{\partial t} = \Delta \varphi\) and \(\lim_{t \to 0} \varphi(t, x, y) = \delta_x(y)\). The function \(\varphi\) is clearly \(\Gamma\)-invariant and symmetric.

Denote by \(\lambda_0\) the bottom of the spectrum of the geometric Laplacian \(-\Delta\) on \(L^2(\tilde{M}, \text{Vol})\), where \(d\text{Vol}(z)\) is the Riemannian volume form on \(\tilde{M}\). Since \(\Gamma\) is not amenable, \(\lambda_0\) is
positive \([B]\). For all \(x, y \in \widetilde{M}\), we have
\[
\lambda_0 = \lim_{t \to \infty} -\frac{1}{t} \ln \varphi(t, x, y)
\]
by the spectral theorem.

Our main result is a local limit theorem which refines \([EL]\).

**Theorem 1.1** (Local Limit Theorem). There exists a positive function \(C\) on \(\widetilde{M} \times \widetilde{M}\) such that for all \(x, y \in M\),
\[
\lim_{t \to \infty} t^{3/2} e^{\lambda_0 t} \varphi(t, x, y) = C(x, y).
\]

When \(\widetilde{M}\) is the hyperbolic space \(\mathbb{H}^3\), there is an explicit expression for \(\varphi(t, x, y)\) (\([\text{DGM}]\)) and Theorem 1.1 is clear, with
\[
C(x, y) = (4\pi)^{-3/2} \frac{d(x, y)}{\sinh d(x, y)}.
\]

More generally, if \(\widetilde{M}\) is a symmetric space of non-compact type, then Theorem 1.1 with \(t^{k/2}\) instead of \(t^{3/2}\) was proven by Bougerol (\([B]\)) using representation theory of the group \(G\) of orientation-preserving isometries of \(M\). The integer \(k\) is given by the rank plus twice the number of positive indivisible roots, which depends only on the Lie algebra of the semi-simple group \(G\). In particular, \(k = 3\) for all rank one symmetric spaces and this explains why one might expect \(t^{3/2}\) for negatively curved manifolds. Bougerol proved the theorem for all random walks on semi-simple groups with a distribution that is left and right \(K\)-invariant, which implies the same result for Brownian motions on \(M = G/K\).

The limit function \(C(x, y)\) is symmetric by Theorem 1.1 and it is a positive \((-\lambda_0)\)-eigenfunction in \(y\): we further give a formula in Theorem 1.7 below. However, it was already known that if the limit
\[
\lim_{t \to \infty} \frac{\varphi(t, x, y)}{\varphi(t, x, x)} = \frac{C(x, y)}{C(x, x)}
\]
exists on a Riemannian manifold, then \(C(x, y)\) is a \((-\lambda_0)\)-eigenfunction in \(y\) \([\text{ABJ}]\) (Theorem 1.2). It is indeed a conjecture by Davies (\([\text{Da}]\)) that the limit \([1.3]\) always exists (see \([\text{Ko}]\) for a recent counter-example for the analogous question on graphs). Our result can be stated as:

**Corollary 1.2.** The universal cover of a compact Riemannian manifold with negative sectional curvature satisfies Davies conjecture.

See \([\text{ABJ}]\) for further discussion and applications of Davies conjecture.

A similar local limit theorem was first observed by Gerl \([\text{Ge}]\) and Woess \([\text{GW}]\) for random walks on a free group which are supported by the generators of the group. It was then proven by Lalley for random walks with finite support on a finitely generated free group \([\text{La}]\). This was extended by Gouëzel and Lalley to symmetric random walks with finite support on cocompact Fuchsian groups \([\text{GL}]\) and finally by Gouëzel to symmetric random walks with finite support on hyperbolic groups \([\text{G1}]\). Our proof follows the
strategy and ideas of GL and G1. By G2, this general strategy works for measures of infinite support and with superexponential moments.

Two main new ingredients of the proof of Theorem 1.1 are the uniform rapid-mixing of the geodesic flow generalizing Dolgopyat theorem and the generalised Patterson-Sullivan conformal family whose Radon-Nikodym derivative is the Martin kernel \( k_{\lambda_0}^2(x, y, \xi) \), which is defined in Theorem 1.4 below.

As in G1, we obtain several subsequent results which have their own interest. Let us introduce more notation to describe these results. For any real \( \lambda < \lambda_0 \), we define the \( \lambda \)-Green function \( G_\lambda \): for all \( x \neq y \in \tilde{M} \),

\[
G_\lambda(x, y) := \int_0^\infty e^{\lambda t} \varphi(t, x, y) dt.
\]

The integral converges at \( \infty \) thanks to the spectral theorem (1.1). The integral converges at 0 since as \( t \to 0 \), \( \varphi(t, x, y) \sim C t^{-d(x,y)/4} e^{\frac{d(x,y)^2}{4t}} \). The function \( G_\lambda(x, \cdot) \) is positive \( (\Delta + \lambda) \)-harmonic outside \( \{x\} \).

Guivarc’h observed that the \( \lambda_0 \)-Green function is finite as well: for all \( x \neq y \in \tilde{M} \),

\[
G_{\lambda_0}(x, y) := \int_0^\infty e^{\lambda_0 t} \varphi(t, x, y) dt < \infty.
\]

Indeed, if \( G_{\lambda_0}(x, y) = \infty \), then the Doob \( \varphi \)-process is recurrent for a positive \( (\Delta + \lambda_0) \)-harmonic function \( \varphi \) [S], which implies that \( \Gamma \) is virtually \( \mathbb{Z}, \mathbb{Z}^2 \) or trivial [V].

In Section 3 we show (see Proposition 3.11, where we relate \( \tau \) with other dynamical properties)

**Theorem 1.3.** There are positive constants \( \tau \) and \( C \) such that, for \( x, y \in \tilde{M} \) with \( d(x, y) \geq 1 \),

\[
G_{\lambda_0}(x, y) \leq C e^{-\tau d(x,y)}.
\]

Two geodesic rays in \( \tilde{M} \) are said to be **equivalent** if they remain a bounded distance apart. The geometric boundary \( \partial M \) is defined as the space of equivalence classes of unit speed geodesic rays. A sequence \( \{y_n\}_{n \in \mathbb{N}} \) in \( \tilde{M} \) converges to a point in \( \partial \tilde{M} \) if, and only if, for some (hence, for all) \( x \in \tilde{M} \),

\[
d(x, y_n) + d(x, y_m) - d(y_n, y_m) \to \infty \quad \text{as} \quad n, m \to \infty.
\]

We now describe the Martin boundary of the operator \( \Delta + \lambda_0 \).

**Theorem 1.4** (\( \lambda_0 \)-Martin boundary). Fix \( x \in \tilde{M} \) and assume that the sequence \( \{y_n\}_{n \in \mathbb{N}} \) converges to a point \( \xi \in \partial \tilde{M} \). Then, there exist a positive \( (-\lambda_0) \)-eigenfunction \( k_{\lambda_0}(x, y, \xi) \) of the Laplacian, which we call Martin kernel, such that

\[
\lim_{n \to \infty} \frac{G_{\lambda_0}(y_n, y)}{G_{\lambda_0}(x, y)} = k_{\lambda_0}(x, y, \xi).
\]
Moreover, the Martin boundary of $\Delta + \lambda_0$ coincides with the geometric boundary. In particular, for any positive harmonic function $F$ of $\Delta + \lambda_0$, there is a finite measure $\nu_F$ on $\partial \tilde{M}$, depending on $x$, such that

$$F(y) = \int_{\partial \tilde{M}} k_{\lambda_0}(x, y, \xi) d\nu_F(\xi).$$

See Section 4 for the proof and more properties of the Martin kernel $k_{\lambda_0}(x, y, \xi)$. Martin kernel plays the role of a conformal density for a family of measures on the boundary $\partial \tilde{M}$.

**Theorem 1.5.** For all $y \in \tilde{M}$, there is a finite measure $\mu^\lambda_y$ on $\partial \tilde{M}$ such that

1) the family $y \mapsto \mu^\lambda_y$ is $\Gamma$-equivariant: $\mu^\lambda_{\gamma y} = \gamma^* (\mu^\lambda_y)$ for $\gamma \in \Gamma$ and
2) for $\mu^\lambda_y$-a.e. $\xi \in \partial \tilde{M}$, all $y \in \tilde{M}$,

$$\frac{d\mu^\lambda_y}{d\mu^\lambda_x}(\xi) = k^2_{\lambda_0}(x, y, \xi).$$

The family is unique if we normalize by $\int_M \mu^\lambda_x(\partial \tilde{M}) d\text{Vol}(x) = 1$.

For any $\Gamma$-equivariant families $\nu = \{\nu_x\}_{x \in \tilde{M}}$ of measures on $\partial \tilde{M}$ with cocycle $k(x, y, \xi) := \frac{d\nu_x}{d\nu_y}(\xi)$ and normalized by $\int_{M_0} \nu^\lambda_x(\partial \tilde{M}) d\text{Vol}(x) = 1$, we define the energy of $\nu$ as follows:

$$E(\nu) := \int_{M_0} \left( \int_{\partial \tilde{M}} \| \nabla_y = x \ln k(x, y, \xi) \|^2 d\nu_x(\xi) \right) d\text{Vol}(x),$$

It is equal to 4 times the Rayleigh quotient

$$R(\nu) := \int_{M_0} \left( \int_{\partial \tilde{M}} \| \nabla_x \sqrt{k(x_0, x, \xi)} \|^2 d\nu^\lambda_{x_0}(\xi) \right) d\text{Vol}(x)$$

defined by O. Mohsen in [Mo]. Mohsen showed that $\lambda_0 = \inf \nu R(\nu)$ and asked whether the minimum is achieved. Using Rayleigh quotient, it is easy to see that $\mu^\lambda_x$ has energy $E = 4\lambda_0$.

**Corollary 1.6.** The family $\mu^\lambda_x$ is the unique conformal family of measures which achieves the minimum energy.

Mohsen proved the uniqueness for the manifolds with constant negative curvature.

The family $\mu^\lambda_x$ is a fourth natural $\Gamma$-equivariant family $\nu = \nu_x$ of measures on $\partial \tilde{M}$ with regular cocycles, alongside with the Lebesgue visual measures, the Margulis-Patterson-Sullivan measures and the harmonic measures. For rank one symmetric spaces, the Lebesgue measure on $\partial \tilde{M}$ achieves the minimum. It is easy to see that the family $\mu^\lambda_x$ is proportional to the harmonic measures or to the Margulis-Patterson-Sullivan measures only when the manifold $\tilde{M}$ is a symmetric space [L3].

\[1\text{Observe that the energy of Margulis-Patterson-Sullivan measure is the volume entropy squared, and the energy of harmonic measure is Kaimanovich entropy [H2], [K1], [L3].}\]
The last result we would like to emphasize is a formula of the function $C(x, y)$ in Theorem 1.1. In case of symmetric spaces, the function $C(x, y)$ is proportional to the positive $(-\lambda_0)$-eigenfunction invariant under the stabilizer $K_x$ of the point $x$, a.k.a. the Harish-Chandra function, or the ground state, centered at $x$.

**Theorem 1.7.** Fix $x \in \bar{M}$. There is a constant $\Upsilon = \Upsilon_{\lambda_0}$, given by (2.17), such that the positive $(-\lambda_0)$-eigenfunction $C(x, y)$ satisfies

$$C(x, y) = \frac{\sqrt{T}}{2\sqrt{\pi}} \int_{\partial M} k_{\lambda_0}(x, y, \xi)d\mu^{\lambda_0}_x(\xi) = \frac{\sqrt{T}}{2\sqrt{\pi}} \int_{\partial M} \sqrt{d\mu^{\lambda_0}_x(\xi)}\sqrt{d\mu^{\lambda_0}_y(\xi)}.$$ 

Here, $\int_{\partial M} \sqrt{d\mu^{\lambda_0}_x(\xi)}\sqrt{d\mu^{\lambda_0}_y(\xi)}$ as used in unitary representation of $\Gamma$ associated to its action on $(\partial \bar{M}, \mu^{\lambda_0})$.

The article is organized along the path of the proof of Theorem 1.1.

In Section 2, we recall the consequences of Ancona’s boundary Harnack inequality for $\lambda < \lambda_0$ ([An6]), in conjunction with the thermodynamic formalism for the geodesic flow (following [K1], [H3] and [L2]). Using mixing properties of the geodesic flow on the unit tangent bundle $SM$ for suitable $\Gamma$-invariant Gibbs measures, we show that there is a function $P(\lambda)$ of $\lambda$ and a positive function $D(x, \lambda)$ such that, for $\lambda < \lambda_0$, as $R \to \infty$

$$e^{-P(\lambda)R} \int_{S(x, R)} G^2_\lambda(x, z)dz \to D(x, \lambda),$$

where $S(x, R)$ is the sphere of radius $R$ centered at $x$. (see Proposition 2.4).

We also recall from [H3] Corollary 5.5.1 that $\int_{S(x, R)} G^2_{\lambda_0}(x, z)dz$ is bounded independently from $R$ (Proposition 2.12).

In Section 3, we use this bound to establish the uniform Harnack inequality at the boundary, i.e. the Ancona-Gouëzel inequality (Theorem 3.2). Theorem 1.4 follows and the other applications of thermodynamic formalism hold equally at $\lambda = \lambda_0$.

In Section 4, we discuss limits of measures on large spheres using uniform mixing of the geodesic flow. For $\lambda \in [0, \lambda_0], x \in M$ and $R > 2$, we define the measure $m_{x, \lambda, R}$ on $SM$ by first lifting the measure $e^{-P(\lambda)R}G^2_\lambda(x, z)dz$ on $S(x, R)$ to the set of unit vectors pointing towards $x$, and then project to $SM$ by $p$. One consequence of our result is that the measures $m_{x, \lambda, R}$ converge towards some measure $m_x$ on $SM$ as $R \to \infty$ and $\lambda \to \lambda_0$ (see Corollary 4.7). Another consequence is that the measures $\mu_{x, \lambda, R}$ on the spheres $S(x, R)$ with density $e^{-RP(\lambda_0)}G^2_{\lambda_0}(x, y)$ converge to some measure $\mu^{\lambda_0}_x$ as $R \to \infty$ (Corollary 1.8). The measures $\mu^{\lambda_0}_x$ turns out to be a $\Gamma$-equivariant family with regular cocycle $e^{-P(\lambda_0)\beta(x, y, \xi)}k^2_{\lambda_0}(x, y, \xi)$, where $\beta(x, y, \xi)$ is the Busemann function (see the equation (2.6)).

Once we prove that $P(\lambda_0) = 0$ in Section 5 the family of measures $\mu^{\lambda_0}_y$ satisfies the statements of Theorem 1.3 and Corollary 1.6. We also obtain that for $x, y \in \bar{M}$, $\lim_{\lambda \to \lambda_0} P(\lambda) \frac{\partial}{\partial \lambda} G_\lambda(x, y)$ is proportional to $C(x, y)$.
By a precise study of the second derivative $\frac{\partial^2}{\partial \lambda^2} G_\lambda(x, y)$ in Section 6.1, we obtain that both
$$\frac{P(\lambda)}{\sqrt{\lambda_0 - \lambda}}$$
and
$$\sqrt{\lambda_0 - \lambda} \frac{\partial}{\partial \lambda} G_\lambda(x, y)$$
converge towards positive numbers as $\lambda \to \lambda_0$. In Section 6.2, we conclude the proof of Theorem 1.1 from Theorem 6.1 and a Tauberian Theorem as in [GL]. Theorem 1.7 follows as well.

In Section 7, we prove a uniform version of Dolgopyat’s rapid-mixing for hyperbolic flows which is of independent interest.

**Remark 1.8.** In this text, $C$ stands for a number depending only on the geometry of $M$ and $\Gamma$. However, its actual value may change from line to line. For the sake of clarity, we specify $C_0, \ldots, C_9, C_\varepsilon, C(R)$ when the same number is used in another computation. Note that $C_1, C_4, C_7$ in Section 7 have the same role as in [Me]. Likewise, we consider spaces of $\alpha$-Hölder continuous functions for some $\alpha$ of which the actual value may vary.

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### 2. Potential theory and thermodynamic formalism

We recall in this section the results obtained by applying classical potential theory to the Laplacian on $\widetilde{M}$ and thermodynamic formalism to the geodesic flow.

**Proposition 2.1.** We have, for $\lambda \in [0, \lambda_0)$, for any two points $x \neq y \in \widetilde{M}$:

$$\frac{\partial^k}{\partial \lambda^k} G_\lambda(x, y) = k! \int_{\widetilde{M}^k} G_\lambda(x, x_1) G_\lambda(x_1, x_2) \cdots G_\lambda(x_k, y) d\text{Vol}(x, \cdots, x_k).$$

**Proof.** It follows directly from computation (See [La] for discrete case), e.g. for $k = 1$,

$$\int_{\widetilde{M}} G_\lambda(x, z) G_\lambda(z, y) d\text{Vol}z = \int_0^\infty \int_0^\infty \int_{\widetilde{M}} e^{\lambda(t+u)} \varphi(t, x, z) \varphi(u, z, y) d\text{Vol}z dt du$$

$$= \int_0^\infty \int_0^\infty e^{\lambda(t+u)} \varphi(t + u, x, y) dt du$$

$$= \int_0^\infty \int_0^s e^{\lambda s} \varphi(s, x, y) ds = \int_0^\infty e^{\lambda s} \varphi(s, x, y) ds = \frac{\partial}{\partial \lambda} G_\lambda(x, y).$$

Since the Green function is positive, by (2.1) for $k = 1$ and 2, the map $\lambda \mapsto G_\lambda(x, y)$ is a convex increasing function. Since $G_\lambda(x, y)$ is analytic outside the spectrum as a
resolvent, its derivative is finite as well, i.e.

\[(2.2) \quad \text{for all } \lambda < \lambda_0, \text{ all } x \neq y \in \tilde{M}, \int_{\tilde{M}} G_\lambda(x, z)G_\lambda(z, y) d\text{Vol}(z) < +\infty.\]

For each \( x \in \tilde{M} \) and \( \xi \) in the geometric boundary \( \partial \tilde{M} \), there is a unique unit speed geodesic \( \gamma_{x,\xi} \) starting from \( x \) in the equivalent class \([\xi]\) of \( \xi \). The mapping \( \xi \mapsto \dot{\gamma}_{x,\xi}(0) \) is a homeomorphism, which we denote by \( \pi_{x,\xi}^{-1} \), from \( \partial \tilde{M} \) to the unit sphere \( S_{\tilde{M}} \) in the tangent space at \( x \) to \( \tilde{M} \). Thus we will identify the unit tangent bundle \( S_{\tilde{M}} \) with \( \tilde{M} \times \partial \tilde{M} \).

For each \( x \in \tilde{M} \), \( \partial \tilde{M} \) is endowed with the Gromov metric

\[ d_x(\xi, \eta) = e^{-a(\xi|\eta)_x}, \]

where \( 0 < a \leq 1 \), is such that the sectional curvature \( \kappa \) satisfies \( \kappa \leq -a^2 \) on \( \tilde{M} \) and \( (\xi|\eta)_x \) is the Gromov product

\[(2.3) \quad (\xi|\eta)_x = \lim_{y \to \xi, z \to \eta} \frac{1}{2} (d(x, y) + d(x, z) - d(y, z)). \]

The following properties follow from pinched negative curvature:

**Proposition 2.2** (\[Anc\]). For all \( \lambda \in [0, \lambda_0) \), every \( \xi \in \partial \tilde{M} \) there exist a positive \((-\lambda)\)-eigenfunction \( k_\lambda(x, y, \xi) \) of the Laplacian such that for each \( x, y \in \tilde{M} \),

\[(2.4) \quad \lim_{z \to \xi} G_\lambda(y, z)G_\lambda(z, x) = k_\lambda(x, y, \xi). \]

For any positive \((-\lambda)\)-eigenfunction \( F \), there is a measure \( \nu_F \) on \( \partial \tilde{M} \) such that

\[ F(y) = \int_{\partial \tilde{M}} k_\lambda(x, y, \xi) d\nu_F(\xi). \]

**Proposition 2.3** (\[H1\]). Moreover, for all \( \lambda \in [0, \lambda_0) \), there are constants \( \alpha(\lambda) > 0, C(\lambda) > 0 \) such that

\[ \frac{\|\nabla_y \ln k_\lambda(x, y, \xi) - \nabla_y \ln k_\lambda(x, y, \eta)\|}{(d_x(\xi, \eta))^{\alpha}} \leq C. \]

**Proposition 2.4** (\[K1\]). For three distinct points \( x, y, z \in \tilde{M} \cup \partial \tilde{M} \), consider the function

\[ \theta^\lambda_{x}(y, z) := \frac{G_\lambda(y, z)}{G_\lambda(y, x)G_\lambda(x, z)}. \]

There is a \( \tau > 0 \), and a \( \tau \)-Hölder continuous positive function \( \theta^\lambda_{x}(\xi, \eta) \) on \( \partial \tilde{M} \times \partial \tilde{M} \setminus \Delta \) such that

\[ \theta^\lambda_{x}(\xi, \eta) = \lim_{y \to \xi, z \to \eta} \theta^\lambda_{x}(y, z). \]
The function $\vartheta^\lambda_\bar{\gamma}(\xi,\eta)$, when it is finite as it is here, is called the Naïm kernel in potential theory \cite{N}. Compare with the definition of the Gromov product \cite{GM}.

Fix $x \in \tilde{M}$. For $\xi \in \partial \tilde{M}, y \in \tilde{M}$, the Busemann function $\beta(x, y, \xi)$ is defined by

$$\beta(x, y, \xi) = \lim_{y_n \to \xi} d(y, y_n) - d(x, y_n). \tag{2.6}$$

As $\tilde{M}$ is the universal cover of a closed manifold of negative curvature, we also use the thermodynamic formalism of the geodesic flow as in \cite{KL, H1, L2}.

The geodesic flow $g = \{g_t\}_{t \in \mathbb{R}}$ is defined on the unit tangent bundles $SM$ and $\tilde{SM}$. On $SM$, the geodesic flow is an Anosov flow. For a $g$-invariant probability measure $m$ on $SM$, denote by $h_m(g)$ the measure-theoretic entropy of time-1 map $g_t$ with respect to $m$ (see e.g. \cite{W}) . For any continuous function $\varphi$, define the topological pressure $P(\varphi)$ of $\varphi$ by

$$P(\varphi) := \sup_m \left( h_m(g) + \int_{SM} \varphi dm \right),$$

where the supremum is taken over all $g$-invariant probability measures on $SM$.

Consider $v \in SM$. For a lift $\tilde{v}$ in $\tilde{SM}$, consider the geodesic $\gamma_{\tilde{v}}(t)$ with initial tangent vector $\gamma_{\tilde{v}}(0) = \tilde{v}$. We will denote $\tilde{v}^- = \gamma_{\tilde{v}}(-\infty)$ and $\tilde{v}^+ = \gamma_{\tilde{v}}(+\infty)$. For all $\lambda \in [0, \lambda_0)$, the potential function associated to $\lambda$ is the function on $SM$ defined as

$$\varphi_\lambda(v) := -2 \frac{d}{dt} \ln k_{\lambda}(\gamma_{\tilde{v}}(0), \gamma_{\tilde{v}}(t), \tilde{v}) \bigg|_{t=0}.$$ \hspace{1cm} \text{(2.7)}

We set $P(\lambda) := P(\varphi_\lambda)$ for $0 \leq \lambda \leq \lambda_0$. Let $m_\lambda$ be the unique equilibrium probability measure of $\varphi_\lambda$, which attains the supremum in (2.7). It is mixing for the geodesic flow $g$ of $M$. The measure $m_\lambda$ can be constructed from the generalized family of Patterson-Sullivan measures associated to the potential function $\varphi_\lambda$, characterized by the following proposition.

**Proposition 2.5 (\cite{L2}).** Fix $x \in \tilde{M}$. For all $\lambda \in [0, \lambda_0)$, all $y \in \tilde{M}$, there is a finite measure $\mu_\lambda^y$ on $\partial M$ such that

1) the family $y \mapsto \mu_\lambda^y$ is $g$-equivariant: $\mu_\lambda^y g_* = g_*(\mu_\lambda^y)$ for $g \in \Gamma$ and

2) for all $\lambda \in [0, \lambda_0)$, $\mu_\lambda^y$-a.e. $\xi \in \partial M$,

$$\frac{d\mu_\lambda^y}{d\mu_\lambda^x}(\xi) = k_\lambda^y(x, y, \xi) e^{-P(\lambda)\beta(x, y, \xi)}.$$

The family is unique if we normalize by $\int_{\tilde{M}_0} \mu_\lambda^y(\partial \tilde{M}) d\text{Vol}(y) = 1$.

Using Harnack inequality, the measures $\mu_\lambda^y$ defined above have finite total mass:

**Proposition 2.6 (Harnack inequality \cite{M}).** There is a constant $C = C(R)$ such that for any $0 \leq \lambda \leq \lambda_0$, any positive function $f$ $(\Delta + \lambda)$-harmonic on $B(y, 2R)$, and any
y, y′ ∈ ˜M such that d(y, y′) < R, we have
\[ f(y) = C. \]

**Corollary 2.7.** There exists a constant C > 0, such that for all \( \lambda \in [0, \lambda_0) \), all \( x \in M_0 \),
\[ C^{-1} \leq \mu_\lambda^x (\partial \tilde{M}) \leq C. \]

**Proof.** By Proposition 2.6 applied to \( k_\lambda(x, y, \xi) \), for \( x, y \in M_0 \), \( |\ln k_\lambda^x(x, y, \xi)| \) is bounded. The pressure \( P(\lambda) \) is bounded because \( \varphi_\lambda \) is bounded [CY]. Thus, the Radon-Nikodym derivatives \( \frac{d\mu_\lambda}{d\mu} \) is bounded for \( x, y \in M_0 \) uniformly in \( \lambda \). Since the total measure is 1, the corollary follows.

Fix \( x_0 \in ˜M \). By Hopf parametrization, i.e. by associating \((v^-, v^+, \beta(x_0, \gamma_0(0), v^+))\) to \( v \), we identify \( SM \) with \((\partial \tilde{M} \times \partial \tilde{M} \setminus \Delta(\partial \tilde{M})) \times \mathbb{R} \), where \( \Delta(\partial \tilde{M}) \) is the diagonal embedding. Since \( \theta_x^\lambda(\xi, \eta)e^{2P(\lambda)(\xi, \eta)} \mu_x(\xi)d\mu_x(\eta) \) is independent of \( x \), we define a \( G_\text{-invariant}, g_\text{-invariant measure} \( \tilde{m}_\lambda \) by
\[ \tilde{m}_\lambda(\xi, \eta, t) = \Omega_\lambda(\xi, \eta)e^{2P(\lambda)(\xi, \eta)} \mu_x(\xi) \times \mu_x(\eta) \times dt \]
on \( \tilde{M} \), which does not depend on \( x \) (it does not depend on \( x_0 \) either). Here, \( \Omega_\lambda \) is the normalizing constant chosen so that the measure \( \tilde{m}_\lambda \) is equal to the \( G \)-invariant lift of the probability measure \( m_\lambda \) to \( SM \).

**Remark 2.8.** Note that we have a symmetric measure thanks to the fact that \( \varphi_\lambda \) is cohomologous to \( \varphi_\lambda \circ \iota \) where \( \iota \) is the flip map \( v \mapsto -v \) (compare with asymmetric measure in [PPS] Section 3.7).

We can also identify the orthogonal two frame bundle \( S^2 \tilde{M} \) with the triples of distinct points in \( \partial \tilde{M} \times \partial \tilde{M} \) by associating \((v, w \in v^+)\) to \((v^+, v^-, w^+)\). The measure
\[ d\rho_\lambda^x(a, b, c) := \varpi_\lambda(\xi) \theta_x^\lambda(a, b)\theta_x^\lambda(b, c)\theta_x^\lambda(c, a) \mu_x(\xi) \times \mu_x(\eta) \times dt \]
does not depend on \( x \) and \( G \)-invariant. Here \( \varpi_\lambda \) is a normalizing constant so that the total measure satisfies
\[ \rho_\lambda^x(S^2 \tilde{M}) = 1. \]

Let us recall dynamical foliations of \( T^1 \tilde{M} \) in order to define measures associated to \( \mu_\lambda^x \).

For every \( v \in T^1 \tilde{M} \), define the strong stable manifold, strong unstable manifold, weak (or central) stable manifold and weak (or central) unstable manifold of \( v \) as follows:
\[ W^{ss}(v) = \{ w \in T^1 \tilde{M} : \lim_{t \to +\infty} d(g_t v, g_t w) = 0 \}, \]
\[ W^{uu}(v) = \{ w \in T^1 \tilde{M} : \lim_{t \to -\infty} d(g_t v, g_t w) = 0 \}, \]
\[ W^{cs}(v) = \{ w \in T^1 \tilde{M} : \exists s, \lim_{t \to +\infty} d(g_{t+s} v, g_{t+w}) = 0 \}, \]
\[ W^{cu}(v) = \{ w \in T^1 \tilde{M} : \exists s, \lim_{t \to -\infty} d(g_{t+s} v, g_{t+w}) = 0 \}. \]
The generalized Patterson-Sullivan measures $\mu^x_\lambda$ obtained in Proposition 2.5 induce measures on strong unstable manifolds and more generally any transversal to $W^{cs}$ via the homeomorphism $(\pi^{-1})_*\mu^x_\lambda$: first, on $W^{uu}(v)$, we define
\[
d\mu^{uu}_\lambda(w) = \frac{k^2(\gamma\bar{w}(0), \gamma\bar{w}(t), w^+)}{k^2(x_0, \gamma\bar{w}(t), w^+)} e^{P(\lambda)\beta(\pi(x), \gamma\bar{w}(t), w^+)} d\mu^x_\lambda(w),
\]
which is independent of $x_0$ and is $\Gamma$-equivariant (see [PPS] Section 3.9 for example).

By extending the family $\mu^{uu}_\lambda$ on $W^{uu}(v)$, we define a family of measures $\mu^{uu}_\lambda$ on all transversals to the central stable manifolds on $\tilde{S}\tilde{M} \simeq \tilde{M} \times \partial\tilde{M}$ in such a way that for two transversals through $\pi^{-1}_x(\xi)$ and $\pi^{-1}_y(\xi)$, respectively, the Radon-Nikodym derivative $\rho_\lambda(\pi^{-1}_x(\xi), \pi^{-1}_y(\xi))$ of the holonomy from $\pi^{-1}_x(\xi)$ to $\pi^{-1}_y(\xi)$ along the leaf $\tilde{M} \times \{\xi\}$ is given by
\[
\rho_\lambda(\pi^{-1}_x(\xi), \pi^{-1}_y(\xi)) = k^2_\lambda(x, y, \xi)e^{-P(\lambda)\beta(x, y, \xi)}.
\]
Observe that $\mu^{uu}_\lambda$ on $S_x\tilde{M}$ is $(\pi^{-1})_*\mu^x_\lambda$: note that
\[
\frac{d\mu^{uu}_\lambda}{dg\mu^{uu}_\lambda}(v) = e^{-P(\lambda)}k^2_\lambda(\gamma\bar{v}(t), \gamma\bar{v}(0), \gamma\bar{v}(\infty)),
\]
and
\[
\begin{align*}
\int_{S_p\tilde{M}} f(v) d\mu^{uu}_\lambda(v) &= \int_{\partial\tilde{M}} f(p \circ \pi^{-1}_x) d\mu^{\lambda_0}_\lambda(\xi), \\
\int_{S_p\tilde{M}} h(-u) d\mu^{uu}_\lambda(u) &= \int_{\partial\tilde{M}} h(p \circ \pi^{-1}_y) d\mu^{\lambda_0}_\lambda(\xi).
\end{align*}
\]

By a direct generalization of Margulis argument [ML] to Gibbs measures, one obtains the following proposition (see Section 3 for details and [L1] for the case $\lambda = 0$).

**Proposition 2.9.** There exists a positive continuous function $D : (\tilde{M} \times [0, \lambda_0]) \to \mathbb{R}_+$ such that
\[
\lim_{R \to \infty} e^{-P(\lambda)} \int_{S(x, R)} G^2_\lambda(x, z) dz = D(x, \lambda).
\]

Clearly, $x \mapsto D(x, \lambda)$ is $\Gamma$-invariant and depends only on $p(x) \in M$. The function $D(x, \lambda)$ will be described in Corollary 4.9.

**Corollary 2.10.** For all $\lambda \in [0, \lambda_0)$, we have $P(\lambda) < 0$.

**Proof.** Indeed, otherwise, we have by Proposition 2.8 and Proposition 2.9
\[
\int_{\tilde{M}} G_\lambda(x, z)G_\lambda(z, y)d\text{Vol}(z) \geq \frac{1}{C} \int_{1+d(x, y)}^{+\infty} \left( \int_{S(x, R)} G^2_\lambda(x, z) dz \right) \, dR \geq \frac{D(x, \lambda)}{C} \int_{1+d(x, y)}^{+\infty} dR.
\]
The integral diverges, which is a contradiction with (2.13). \[\square\]

**Corollary 2.11.** For all $\lambda \in [0, \lambda_0)$,
\[
\lim_{y \to z} \frac{\partial}{\partial \lambda} G_\lambda(x, y) = \int_{\tilde{M}} G^2_\lambda(x, z)d\text{Vol}(z)
\]
and
\[
\lim_{y \to x} \frac{\partial}{\partial \lambda} \left( \frac{\partial}{\partial \lambda} G_\lambda(x, y) \right) = 2 \int_{\tilde{M}^2} G_\lambda(x, z) G_\lambda(z, w) G_\lambda(w, x) d\text{Vol}^2(z, w).
\]

By abusing notation, we write \( \frac{\partial}{\partial \lambda} G_\lambda(x, x) := \int_M G_\lambda^2(x, z) d\text{Vol}(z) \) and \( \frac{\partial}{\partial \lambda} \left( \frac{\partial}{\partial \lambda} G_\lambda(x, x) \right) = 2 \int_{\tilde{M}^2} G_\lambda(x, z) G_\lambda(z, w) G_\lambda(w, x) d\text{Vol}^2(z, w) \).

Proof. For \( d(x, z) \) small, approximating by Euclidean case, we have \( G_\lambda(x, z) \leq Cd(x, z)^{-m-1} \). Therefore, given \( \varepsilon > 0 \), there is \( \delta > 0 \) such that for \( d(x, y) < \delta/2 \),
\[
\int_{B(x, \delta)} G_\lambda(x, z) G_\lambda(y, z) d\text{Vol}(z) < \varepsilon/3.
\]
For \( d(x, z) > \delta \) and \( d(x, y) < \delta/2 \), by Harnack inequality,
\[ G_\lambda(y, z) G_\lambda(x, z) < C G_\lambda(x, z)^2. \]
By Proposition 2.9, \( G_\lambda(x, z)^2 \) is integrable on \( \tilde{M} \setminus B(x, \delta) \). Since \( G_\lambda(y, z) \to G_\lambda(x, z) \) pointwise as \( y \to x \), it follows that as \( y \to x \),
\[
\int_{\tilde{M} \setminus B(x, \delta)} G_\lambda(y, z) G_\lambda(x, z) d\text{Vol}(z) \to \int_{\tilde{M} \setminus B(x, \delta)} G_\lambda^2(x, z) d\text{Vol}(z).
\]
The second equation follows from the first equation. \( \square \)

The rest of the section is devoted to the proof of the following proposition due to Hamenstädter.

Proposition 2.12 (H3, Corollary 5.5.1). There is a constant \( C > 0 \) such that for \( R \geq 1 \),
\[
\int_{S(x, R)} G_{\lambda_0}^2(x, z) dz \leq C.
\]
Proof. (cf. G1, Lemma 2.5. We repeat, with our notations, the proof in H3. One reason is the comfort of the reader.) Harnack inequality (Proposition 2.6) says that there exists \( C_0 \) such that for \( 0 \leq \lambda \leq \lambda_0 \) and \( x, y, z \) such that \( d(x, z) \geq 1, d(x, y) \geq 1 \), we have \( G_\lambda(x, z) G_\lambda(x, y) \leq C_0 G_\lambda(z, y) \). Thus for any \( \lambda \in [0, \lambda_0] \), any \( x, z \in \tilde{M} \) and any \( y \in M \) with \( d(x, z) \geq 1, d(x, y) \geq 1 \),
\[
G_\lambda(x, z) = \frac{G_\lambda(x, z) G_\lambda(x, y)}{G_\lambda(x, y)} \leq C_0 \frac{G_\lambda(z, y)}{G_\lambda(x, y)}.
\]
Letting \( y \) go to \( \xi \in \partial \tilde{M} \), we obtain
\[
G_{\lambda}^2(x, z) e^{-P(\lambda)d(x, z)} \leq C_0^2 k_0^2(x, z, \xi) e^{-P(\lambda)d(x, z)}.
\]
A transversal \( T \) to the central stable manifold \( W^{cs} \) is said to be \( \varepsilon \)-transversal if the maximum angle between \( T \) and the strong unstable manifold \( W^{uu} \) is smaller than \( \varepsilon \). Observe that \( g_R(S_x M) \) is \( \varepsilon \)-transversal to \( W^{cs} \) for \( R \geq R_0 \) for some \( R_0 > 0 \).

Lemma 2.13. Let \( T \) be \( \varepsilon \)-transversal. There is \( C_\varepsilon \) such that for all \( w \in T \), \( \lambda < \lambda_0 \),
\[
\mu_\lambda^{uu}(B(w, 1) \cap T) \geq C_\varepsilon^{-1}.
\]
Proof. Let \( B = B(w, 1) \cap T \). By the minimality of \( W^{cs} \) on \( SM \), there is \( r \) large enough that \( \cup_{b \in R} B^{cs}(b, r) \) covers at least one fundamental domain of \( SM \), where \( B^{cs}(b, r) \) is the ball of center \( b \) and radius \( r \) in \( W^{cs} \). By Harnack inequality, \( \rho_\lambda(b, z) \) is bounded from above on \( B^{cs}(b, r) \) uniformly in \( \lambda \) and \( b \). Therefore there is \( C_\varepsilon \) such that for all \( \lambda \)

\[
\int_B \int_{B^{cs}(b, r)} \rho_\lambda(b, z) d\text{Vol}(z) d\mu_\lambda^u(b) \leq C_\varepsilon \mu_\lambda^u(B).
\]

On the other hand, by our choice of normalization in Proposition 2.5, where \( C_\lambda \) above on \( B \) and radius \( 1 \), we obtain a constant \( C_\lambda \) uniformly in \( \lambda \) and \( b \). Therefore there is a constant \( C_\varepsilon \) such that for all \( \lambda \)

\[
\int_B \int_{B^{cs}(b, r)} \rho_\lambda(b, z) d\text{Vol}(z) d\mu_\lambda^u(b) \geq \int_{M_0} \int_{\partial M} \frac{d\mu_x}{d\mu_{x_0}}(\xi) d\mu_{x_0}(\xi) dm(x) = 1.
\]

\( \square \)

Using all the above estimates, for any \( R \geq R_0 \), \( \lambda < \lambda_0 \) and a ball \( B(w, 1) \) of radius 1 in \( S(x, R) \),

\[
\int_{B(w, 1)} G^2_\lambda(x, z) e^{-P(\lambda) R} dz \leq C_2 G^2_\lambda(x, w) e^{-P(\lambda) R} \leq C_2 C_0 \frac{dg R \mu_\lambda^u}{d\mu_\lambda^u}(w)
\]

\[
\leq \frac{C_2 C_0^2 C_3}{\mu_\lambda^u(B(w, 1))} \int_{B(w, 1)} \frac{dg R \mu_\lambda^u}{d\mu_\lambda^u}(z) d\mu_\lambda^u(z)
\]

\[
\leq C_2 C_0^2 C_3 \frac{g R \mu_\lambda^u(B(w, 1))}{\mu_\lambda^u B(w, 1)}
\]

\[
\leq C_2 C_3 C_0^3 \mu_\lambda^u g - R(B(w, 1)),
\]

where \( C_2 = C(1), C_3 \) are constants in Harnack inequality (Proposition 2.6).

By covering \( S(x, R) \) by balls of radius 1, the intersection of which is of bounded multiplicity, we obtain a constant \( C \) such that for all \( \lambda < \lambda_0 \) and for \( R > R_0 \),

\[
\int_{S(x, R)} G^2_\lambda(x, z) e^{-P(\lambda) R} dz \leq C.
\]

(2.13)

Since \( P(\lambda) < 0 \) for all \( \lambda < \lambda_0 \), there is a constant \( C > 0 \) such that for all \( \lambda \in [0, \lambda_0) \), all \( x \in \tilde{M}, \) all \( R \geq 1 \),

\[
\int_{S(x, R)} G^2_\lambda(x, z) dz \leq C.
\]

Proposition 2.12 follows by letting \( \lambda \) go to \( \lambda_0 \).

\( \square \)

**Corollary 2.14.** For \( T > 0 \), \( P_T(\lambda) \) be the Pressure of the function \( \frac{T}{2} \phi_\lambda \). Then there exists a constant \( C(T) \) and \( R_0 \) such that for all \( \lambda \in [0, \lambda_0), R \geq R_0, x \in \tilde{M}, \)

\[
e^{-R P_T(\lambda)} \int_{S(x, R)} G^T_\lambda(x, z) dz \leq C(T).
\]

**Proof.** We have as above

\[
G^T_\lambda(x, z) e^{-P_T(\lambda)d(x, z)} \leq C_0^T k^T_\lambda(x, z, \xi) e^{-P_T(\lambda)d(x, z)}.
\]


We can also apply Proposition 2.5 to the Hölder continuous function $\frac{T}{2} \varphi_\lambda$ instead of $\varphi_\lambda$. We obtain a family of measures $\mu^{\lambda,T}_x$ on $\partial \tilde{M}$ such that for all $\lambda \in [0, \lambda_0)$, $\mu^{\lambda,T}_x$-a.e. $\xi \in \partial \tilde{M}$,

$$\frac{d\mu^{\lambda,T}_x}{d\mu^{\lambda,T}_y}(\xi) = k^{\lambda,T}_x(x, y, \xi)e^{-P_T(\lambda)\beta(x, y, \xi)}$$

and $\int_{M_0} \mu^{\lambda,T}_y(\partial \tilde{M})d\text{Vol}(y) = 1$. We can therefore associate measures $\mu^{\lambda,T}_{\mu,\nu}$ on transversals to the central stable manifolds such that the holonomy from $\pi_x^{-1}(\xi)$ to $\pi_y^{-1}(\xi)$ along the leaf $\tilde{M} \times \{\xi\}$ is given by

$$\rho_\lambda(\pi_x^{-1}(\xi), \pi_y^{-1}(\xi)) = k^{\lambda,T}_x(x, y, \xi)e^{-P_T(\lambda)\beta(x, y, \xi)}.$$

The same computation yields the analog of (2.13).

\[\square\]

3. Ancona-Gouëzel inequality

Definition 3.1. Let $v \in S\tilde{M}$. The cone $\mathcal{C}(v)$ based on $v$ is defined by:

$$\mathcal{C}(v) := \{y; y \in \tilde{M}, \angle_x(v, y) \leq \pi/2\},$$

where $\angle_x(v, y)$ denotes the angle between $v \in T_x\tilde{M}$ and the geodesic going from $x$ to $y$.

We denote $\partial \mathcal{C}(v) := \{y; y \in \tilde{M}, \angle_x(v, y) = \pi/2\}$, Observe that $\tilde{M} = \mathcal{C}(v) \cup \mathcal{C}(-v)$ and $\partial \mathcal{C}(v) = \mathcal{C}(v) \cap \mathcal{C}(-v)$.

The relative Green function $G_\lambda(x, y : A)$ is the Green function of the process killed when entering $A^c$. It can be defined as the Green function of the operator $\Delta + \lambda$ on $A$ with Dirichlet conditions at the boundary. Clearly, for all set $A, B, A \subset B$ all $\lambda \leq \lambda_0$, and all $x, y \notin A, G_\lambda(x, y : A) \leq G_\lambda(x, y : B)$.

3.1. Ancona-Gouëzel inequality. The key property of the $\lambda$-Green functions for $0 \leq \lambda \leq \lambda_0$ is the following uniform Ancona inequality, which we call Ancona-Gouëzel inequality.

Theorem 3.2. With the above notations, there are constants $C_4, R_0$ such that for all $\lambda \in [0, \lambda_0]$, all points $(x, y, z)$ such that $y$ is on the geodesic segment $[xz]$ from $x$ to $z$ and $d(x, y) \geq R_0, d(y, z) \geq R_0$,

$$C_4^{-1}G_\lambda(x, y : A)G_\lambda(y, z : A) \leq G_\lambda(x, z : A) \leq C_4G_\lambda(x, y : A)G_\lambda(y, z : A)$$

for all set $A$ containing $\mathcal{C}(v)$, where $v \in S_x\tilde{M}$ is the initial vector of the geodesic $[xz]$.

Theorem 3.2 was proven by A. Ancona for $\lambda < \lambda_0$ ([Anc]). The first inequality in (3.1) follows from Harnack inequality uniformly in $0 \leq \lambda \leq \lambda_0$. The new fact here is that the second inequality (3.1) holds when $\lambda = \lambda_0$ as well, with the same constant $C_4$, so that the consequences of Theorem 3.2 are now uniform in $\lambda \in [0, \lambda_0]$ as in [G1]. The Ancona inequality follows from the pre-Ancona inequality in the following Proposition.
Proposition 3.3. Let $x, y, z$ be points on a geodesic $\gamma$ in this order, $v$ the tangent vector to $\gamma$ at $x$. Then, there exists $\varepsilon > 0, R_2$ such that if $r \geq R_2$ and $d(x, y) \geq r, d(y, z) \geq r$, 

$$G_{\lambda_0}(x, z : B(y, r)^c \cap C(v)) \leq 2^{-e^{\varepsilon r}}.$$ 

Proof. As in [G1], we will construct $N = e^{\varepsilon r}$ barriers, for a positive constant $\varepsilon$ which we will specify as follows.

Figure 1. Ancona-Gouëzel inequality

Let $X_i = ((2(N + i) - 1)\pi/4N, 2(N + i)\pi/4N) \subset [\pi/2, \pi]$. Choose $\theta_i$ from $X_i$, for $i = 1, \ldots, N$. The circumference of a great circle of a ball $B(y, r)$ is at least $e^{a_0}r$, where $a_0$ depends on the least upper bound of the curvature, which is negative. Thus the intersections $\{A_i\}$’s of $B(y, r)^c$ and the cones $\{w : \angle_y(x, w) = \theta_i\}$ of angle $\theta_i$ at $y$, are of distance between them bounded below by $e^{(a_0 - \varepsilon)r}$ for all $r$ large enough. Moreover, the sets $A_i$ do not intersect $\partial C(v)$ (see Figure 1).

Now for fixed $\theta_i$’s, we can write

$$G_{\lambda_0}(x, z : B(y, r)^c \cup C(v)) = \int_{A_i} \cdots \int_{A_N} G_{\lambda_0}(x, u_1)G_{\lambda_0}(u_1, u_2) \cdots G_{\lambda_0}(u_N, z) du_N \cdots du_1$$

$$= \langle \delta_x, L_1 \cdots L_N \delta_z \rangle \leq ||L_1 \cdots L_N|| \leq \prod ||L_i||,$$

where $L_i : L^2(A_{i+1}) \rightarrow L^2(A_i)$ is defined by $L_i f(u_i) = \int G_{\lambda_0}(u_i, u_{i+1}) f(u_{i+1}) du_{i+1}.$

Thus it suffices to show that there exist $\theta_1, \ldots, \theta_N$ for which $f_i(\theta_1, \ldots, \theta_N) = ||L_i||^2 \leq \int \int G_{\lambda_0}^2(u_i, u_{i+1}) du_{i+1} du_i < 1/4$ for all $i$.

Now choose $\theta_i$ uniformly from $X_i$. We claim that the expectation of $f_i$ with respect to normalized measures $\frac{1}{\pi} N d\theta_i$ satisfies

$$\mathbb{E}(f_i) = \frac{e^{-\varepsilon r}}{\frac{y}{5}}.$$
if \( \varepsilon \) is small enough. It would imply that \( \mathbb{E}(\sum f_i) \leq Ne^{-\varepsilon r} < 1/4 \), which would in turn imply that \( \sum f_i(\theta_1, \cdots, \theta_N) < 1/4 \) for some \( \{\theta_1, \cdots, \theta_N\} \), thus \( f_i(\theta_1, \cdots, \theta_N) < 1/4 \) for all \( i \).

Now it remains to prove the claim. For \( a \in A_i, b \in A_{i+1} \), let \( \Phi(a, b, \theta_i, \theta_{i+1}) = \gamma_0^{-1}\gamma_1 \in \Gamma \) where \( \gamma_0 \) and \( \gamma_1 \) are the first elements of \( \Gamma \) in the lexicographical order such that

\[
d(\gamma_0 y, a) < \text{diam } M \text{ and } d(\gamma_1 y, b) < \text{diam } M.
\]

Denoting by \( d\mu(a, b, \theta_i, \theta_{i+1}) \) the product of the Lebesgue measures on \( A_i, A_{i+1} \) and of \( \frac{\mu}{\pi} N^2 d\theta_i d\theta_{i+1} \), we have

\[
\mathbb{E}(f_i) = \int G_{\lambda_0}^2(a, b)d\mu(a, b, \theta_i, \theta_{i+1})
\leq C \sum_{\gamma \in \Gamma} G_{\lambda_0}^2(y, \gamma y)\mu(\{(a, b, \theta_i, \theta_{i+1}) : \Phi(a, b, \theta_i, \theta_{i+1}) = \gamma\})
\leq C \int_M G_{\lambda_0}^2(y, w)A(w)dVol(w),
\]

where \( A(z) = \mu(\{(a, b, \theta_i, \theta_{i+1}) : z \in \Phi(a, b, \theta_i, \theta_{i+1})M_0\})/\text{vol}(M) \) and \( C = C(\text{diam } M) \) in Proposition 2.12 (Here \( M_0 \) is a fundamental domain containing \( y \).) The first inequality holds since

\[
G_{\lambda_0}(a, b) = G_{\lambda_0}(\gamma_0^{-1}a, \gamma_0^{-1}b) \leq CG_{\lambda_0}(y, \gamma_0^{-1}\gamma_1 y).
\]

Observe that the angles \( \angle_y(\gamma_0 y, a), \angle_y(\gamma_1 y, b) \) are at most \( e^{-a_0 r} \). If \( \varepsilon \) is chosen small enough, this implies that \( \angle_y(\gamma_0 y, \gamma_1 y) \geq 1/2e^{-\varepsilon r} \), so that \( \gamma_0 \) and \( \gamma_1 \) have at most \( Ce^{a_0 e^{-\varepsilon r}} \) letters in common, for some constant \( a_0 \) depending on the inner diameter of the fundamental domain \( M_0 \). Therefore, for a fixed \( \gamma \in \Gamma \), there are at most \( |\gamma|Ce^{a_0 e^{-\varepsilon r}} \) possible values of \( \gamma_0 \). For fixed \( \gamma, \gamma_0 \),

\[
\mu(\{(a, b, \theta_i, \theta_{i+1}) : \Phi(a, b, \theta_i, \theta_{i+1}) = \gamma, d(a, \gamma_0 y) \leq \text{diam } M, d(b, \gamma_0 y) \leq \text{diam } M\}) \leq CN^2 e^{-a_0(d(y, a)+d(y, b))} \leq CN^2 e^{-a_0|\gamma|}.
\]

Since word metric on \( \Gamma \) and the hyperbolic metric are quasi-isometric, \( |\gamma| \geq a_1 d(y, w) - 2 \). Thus

\[
A(w) \leq d(y, w)Ce^{a_1 e^{-\varepsilon r}}e^{2\varepsilon r}e^{-a_0 a_1 d(y, w)}.
\]

It follows that, if \( \varepsilon \) is chosen small enough and \( r \geq R_2 \),

\[
\mathbb{E}(f_i) \leq Ce^{(2+a_1)e^{-r/5}} \int_{\mathbb{R}} \int_{S(y, r)} G_{\lambda_0}^2(y, z)dR \leq Ce^{(2+a_1)e^{-r/5}} \int_{\mathbb{R}} \int_{S(y, r)} Ce^{((2+a_1)\varepsilon - a_0 a_1)r} < e^{-\varepsilon r/5},
\]

where we used Proposition 2.12 for the second inequality.

Theorem 3.2 follows from Proposition 3.3 by an inductive nesting argument (see the proof of Theorems 4.1–4.3 in [GL], where Lemma 4.4 plays the role of our Proposition 3.3).
Corollary 3.4. There are constants $C_5, R_1$ such that, for all $\lambda \in [0, \lambda_0]$, all $v \in \tilde{M}$, all $y, y' \not\in \mathcal{C}(g_{-2R}, v)$ and all $z \in \mathcal{C}(g_{2R}, v)$,

$$
G_\lambda(y, z) \sim C_5 \frac{G_\lambda(y, \gamma_v(-R_1))}{G_\lambda(y', \gamma_v(-R_1))}.
$$

Proof. If $R$ is large enough, it follows from Theorem 3.2 that

$$
G_\lambda(y, z) = \int_{\partial C(v)} G_\lambda(y, x) G_\lambda(x, z : C(v)) dx \sim C^2 R^4
$$

$$
G_\lambda(y, \gamma_v(-R)) G_\lambda(\gamma_v(R), z : \mathcal{C}(v)) \int_{\partial C(v)} G_\lambda(\gamma_v(-R), x) G_\lambda(x, \gamma_v(R) : \mathcal{C}(v)) dx,
$$

where we used that if $R$ is large enough, the geodesic from any $x \in \partial C(v)$ to $z$ passes at distance less than 1 from $\gamma_v(R)$, and the geodesic from $y$ to $x$ passes at distance less than 1 from and $\gamma_v(-R)$. Therefore, for $y, y' \not\in \mathcal{C}(g_{-2R}v)$, the ratio $\frac{G_\lambda(y, z)}{G_\lambda(y', z)}$ is, up to a fixed multiplicative factor, independent of $z \in \mathcal{C}(g_{2R}v)$.

3.2. Martin boundary. We now follow Section 6 of [AnS] simultaneously for all $\lambda \in [0, \lambda_0]$ to obtain Propositions 2.2, 2.3, 3.7 uniformly in $\lambda \leq \lambda_0$. For $x, y, z \in \tilde{M}, \lambda \in [0, \lambda_0]$, set

$$
k_\lambda(x, y, z) := \frac{G_\lambda(y, z)}{G_\lambda(x, z)}.
$$

The function $k_\lambda(x, y, z)$ is clearly $(\Delta + \lambda)$-harmonic in $y$ on $\tilde{M} \setminus \{z\}$.

Lemma 3.5. There are constants $C > 1, K < 1$ such that for all geodesic $\gamma$ and all $x, y \not\in \mathcal{C}(\gamma(-2R_1 - T)), z, w \in \mathcal{C}(\gamma(2R_1)), \lambda \in [0, \lambda_0], T > 0$, 

$$
\left| \ln \frac{k_\lambda(x, y, z)}{k_\lambda(x, y, w)} \right| \leq CK^T.
$$

Proof. It suffices to prove the case $T = 2nR_1$ for $n \in \mathbb{N}$. Fix a geodesic $\gamma$ and points $z, w \in \mathcal{C}(\gamma(2R_1))$. Denote

$$
k_\lambda(x, y, z; n) = \frac{G_\lambda(y, z : \mathcal{C}(\gamma(-2nR_1)))}{G_\lambda(x, z : \mathcal{C}(\gamma(-2nR_1)))}.
$$

The following numbers $\bar{\theta}(n), \bar{\vartheta}(n)$ are well defined for $n \in \mathbb{N}$ and a priori smaller than $C_5^2$ (and larger than $(C_5^2)^{-1}$) independently of $\lambda, \gamma, z, w$ by (3.2):

$$
\bar{\vartheta}(n) := \sup_{x, y \in \partial \mathcal{C}(\gamma(-2nR_1))} \frac{k_\lambda(x, y, z; n)}{k_\lambda(x, y, w; n)}, \quad \bar{\theta}(n) := \inf_{x, y \in \partial \mathcal{C}(\gamma(-2nR_1))} \frac{k_\lambda(x, y, z; n)}{k_\lambda(x, y, w; n)}.
$$
Denoting $G(a, b) := \mathcal{G}(\gamma(-2n + 1) R_1))$ and $H(a, b) := \mathcal{G}(\gamma(-2n R_1))$, we have
\[
\frac{k_{\lambda}(x, y; z; 2n + 1)}{k_{\lambda}(x, y, w; n + 1)} - \bar{\theta}(n) = \frac{G(y, z)G(x, w) - \bar{\theta}(n)G(x, z)G(y, w)}{G(x, z)G(y, w)}
\]
\[
= \int_{a, b \in \partial \mathcal{E}(\gamma(-2n R_1))} G(y, a)G(x, b) [H(a, z)H(b, w) - \bar{\theta}(n)H(b, z)H(a, w)] 
\]
dadb, where $x_1 := \gamma(-2n + 1) R_1$. The last line above follows from the fact that the function
\[
H(a, z)H(b, w) - \bar{\theta}(n)H(b, z)H(a, w)
\]
is nonnegative and that the geodesics from $y$ to $a$ and from $x$ to $b$ pass through a point of distance at most 1 from $x_1$. Since the last line above doesn’t depend on $x$ and $y$, we have,
\[
\bar{\theta}(n + 1) - \bar{\theta}(n) = \sup\left\{ \frac{k_{\lambda}(x, y; z; 2n + 1)}{k_{\lambda}(x, y, w; 2n + 1)} - \bar{\theta}(n) \right\}
\leq C_5^2 \inf\left\{ \frac{k_{\lambda}(x, y, z; 2n + 1)}{k_{\lambda}(x, y, w; 2n + 1)} - \bar{\theta}(n) \right\}
\leq C_5^2 (\bar{\theta}(n + 1) - \bar{\theta}(n)).
\]

Applying an analogous argument to the function $\bar{\theta}(n) - \frac{k_{\lambda}(x, y; z; 2n + 1)}{k_{\lambda}(x, y, w; 2n + 1)}$, we get
\[
\bar{\theta}(n) - \bar{\theta}(n + 1) \leq C_5^2 (\bar{\theta}(n) - \bar{\theta}(n + 1)).
\]

Therefore, by adding the two inequalities and multiplying the results,
\[
\bar{\theta}(n) - \bar{\theta}(n + 1) \leq \left( \frac{C_5^4 - 1}{C_5^4 + 1} \right)^n (\bar{\theta}(0) - \bar{\theta}(0)) \leq C_5^2 \left( \frac{C_5^4 - 1}{C_5^4 + 1} \right)^n.
\]

Since both $k(x, y, z)$ and $k(x, y, w)$ are 1 for $x = y$, $\bar{\theta} \leq 1 \leq \bar{\theta}$. Since the difference $\bar{\theta}(n) - \bar{\theta}(n)$ is small, they are both close to 1 and the ratio is between $\ln \bar{\theta}$ and $\ln \bar{\theta}$, which are of the same order as $\max\{\bar{\theta} - 1, 1 - \bar{\theta}\} \leq \bar{\theta} - \bar{\theta}$. Finally, we obtain constants $C$ and $K < 1$ such that, for all geodesic $\gamma$, all $\lambda \in [0, \lambda_0]$, all $x, y \in \partial \mathcal{E}(\gamma(-2n R_1))$ and $z, w \in \mathcal{E}(\gamma(-2n R_1))$
\[
\ln \left| \frac{k_{\lambda}(x, y; z; 2n)}{k_{\lambda}(x, y, w; 2n + 1)} \right| \leq CK^n.
\]

We extend \[\text{LM}^{\star}\] to all $x, y \in \mathcal{E}(-\gamma(-2n R_1))$ by writing as above
\[
\frac{k_{\lambda}(x, y; z)}{k_{\lambda}(x, y, w)} = \int_{a, b \in \partial \mathcal{E}(\gamma(-2n R_1))} G(y, a)G(x, b)H(a, w)H(b, z) \frac{k_{\lambda}(a, b; z; 2n R_1)}{k_{\lambda}(a, b, w; 2n + 1)} 
\]
dadb.\]

$\square$
Proposition 3.6. (1) Let $\xi \in \partial \widetilde{M}$ and $\lambda \leq \lambda_0$. The following limit exists and defines a positive $(\Delta + \lambda)$-harmonic function

$$k_\lambda(x, y, \xi) = \lim_{z \to \xi} k_\lambda(x, y, z),$$

which we call the Martin kernel.

(2) Fix $x, y \in \widetilde{M}$. There exist $\alpha$ and $C = C(d(x, y)) > 0$ such that for any $\lambda \in [0, \lambda_0]$,

$$|\ln \frac{k_\lambda(x, y, \xi)}{k_\lambda(x, y, \eta)}| \leq C(d(x, \xi, \eta))^\alpha,$$

where $d_x$ is the Gromov metric on $\partial \widetilde{M}$. Moreover, for $\beta < \alpha$, the function $\lambda \mapsto k_\lambda(x, y, \xi)$ is continuous from $[0, \lambda_0]$ into the space of $\beta$-Hölder continuous functions on $\partial \widetilde{M}$.

Proof. (1) It suffices to show it for a fixed $x = x_0$ and a sequence $z_n \to \xi$. Let $\gamma$ be the geodesic going from $x_0$ to $\xi$. There is $T$ such that $x_0, y \notin \mathcal{C}(\gamma(T - 2R_1))$. As $n \to \infty$, $z_n \to \gamma(T_n + 2R_1))$, with $T_n \to \infty$. By Lemma 3.5, the sequence $k_\lambda(x_0, y, z_n)$ converges.

(2) Let $\gamma$ be the geodesic such that $\gamma(0) = x, \gamma(\infty) = \xi$. There is $\delta_0$ depending only on the curvature bound such that if the Gromov distance $d_x(\xi, \eta)$ is smaller than $\delta_0$, then there is $T = -C \ln d_x(\xi, \eta)$ such that $\xi, \eta$ lie in the closure of $\mathcal{C}(\gamma(T))$. We choose $\delta_0$ small enough so that $T > d(x, y) + 4R_1$. Then, Lemma 3.5 applies to the limits $k_\lambda(x, y, \xi)$ and $k_\lambda(x, y, \eta)$ so that

$$|\ln \frac{k_\lambda(x, y, \xi)}{k_\lambda(x, y, \eta)}| \leq C(d(x, y))K(-C \ln d_x(\xi, \eta)) = C(d_x(\xi, \eta))^\alpha,$$

where $\alpha = -C \ln K > 0$.

As $\lambda$ varies, by Lemma 3.5, the functions $k_\lambda(x, y, z)$ are uniformly $\alpha$-Hölder continuous on a neighborhood of $\xi$ in $M \cup \partial \widetilde{M}$. The continuity in $\lambda$ follows for any $\beta < \alpha$. \qed

For $x, y, z \in \widetilde{M}$, denote

$$\theta_\lambda^x(y, z) := \frac{G_\lambda(y, z)}{G_\lambda(y, x)G_\lambda(x, z)}.$$

Proposition 3.7. Fix $x \in \widetilde{M}, \xi \neq \eta \in \partial \widetilde{M}$. As $y \to \xi, z \to \eta$, the following limit exists and defines the Naim kernel $\theta^x_\lambda(\xi, \eta):$

$$\theta^x_\lambda(\xi, \eta) := \lim_{y \to \xi, z \to \eta} \theta^x_\lambda(y, z) = \lim_{y \to \xi, z \to \eta} \frac{G_\lambda(y, z)}{G_\lambda(y, x)G_\lambda(x, z)}.$$

The limit is uniform in $\lambda$ on the set of triples $(x, \xi, \eta)$ with $d_x(\xi, \eta)$ bounded away from 0. For $v \in SM$, set $\theta_\lambda(v) := \theta^x_\lambda(0)(\tilde{v}^-, \tilde{v}^+)$. Then there is $\beta$ such that the mapping $\lambda \mapsto \theta_\lambda$ is continuous from $[0, \lambda_0]$ to the space of $\beta$-Hölder continuous functions on $SM$.

Proof. Observe first that, by (3.22), for $d_x(y, z) = d(x, z) + d(x, y) - d(x, z)$ bounded away from 0, the functions $\theta^x_\lambda(y, z)$ are uniformly bounded. As before, by (3.5), the functions $y, z \mapsto \theta^x_\lambda(y, z)$ are uniformly $\alpha$-Hölder continuous in $y$ and in $z$ as long as $d_x(y, z)$ remains bounded away from 0. The convergence and the continuity follow. Observe also
that the function $\theta^\lambda_{\tilde{\gamma}(0)}(\tilde{v}^-, \tilde{v}^+)$ is $\Gamma$-invariant and so $\theta^\lambda_\gamma$ is indeed a function on $SM$.

Since $d_{\tilde{\gamma}(0)}(\tilde{v}^-, \tilde{v}^+)=1$, the mapping $\lambda \mapsto \theta^\lambda_{\tilde{\gamma}(0)}(\tilde{v}^-, \tilde{v}^+)$ is continuous from $[0, \lambda_0]$ to the space of $\beta'$-Hölder continuous functions on $\tilde{SM}$ endowed with the metric coming from the identification of $\partial M \times \partial M \times \mathbb{R}$ for some $\beta' < \alpha$. This identification being itself Hölder continuous ([AnS] Proposition 2.1), the last statement of Proposition 3.7 follows.

\[ \square \]

For $v \in SM$, $x \in \tilde{M}$, $\xi, \eta \in \partial \tilde{M}$, we set

\[ (3.4) \quad \theta(v) := \theta^v_0(v), \quad \theta_x(\xi, \eta) := \theta^\lambda_x(\xi, \eta). \]

Fix $x, z \in \tilde{M}, d(x, z) \geq 1$. The functions $x \mapsto k_\lambda(x, y, \xi)$ and $y \mapsto k_\lambda(x, y, z), \xi \in \partial \tilde{M}$ are $\Delta + \lambda$ harmonic in $y$ in a neighborhood of $x$. Let $v \in S\tilde{M}$. The directional derivative $\partial_v k_\lambda(x, ., z)$ exists. Since $k(x, y, z)$ is a harmonic function of $y$ away from $z$, by [CY] (see also [AS]), $|\partial_v \ln k_\lambda(x, y, z)|_{y=x} = |\partial_v k_\lambda(x, y, z)|_{y=x} \leq C_8$ where the constant $C_8$ does not depend on $\lambda \in [0, \lambda_0]$. Following [H1] Lemma 3.2, we have:

**Proposition 3.8.** For fixed $x_0 \in \tilde{M}$ and $\tilde{v} \in S\tilde{M}$, the mapping $\xi \mapsto \partial_\xi k_\lambda(x_0, y, \xi)$ is $\alpha$-Hölder continuous, uniformly in $\lambda \in [0, \lambda_0], \tilde{v} \in S\tilde{M}$. In particular, there is $\beta > 0$ such that the function $\lambda \mapsto \Phi_\lambda$ is continuous from $[0, \lambda_0]$ to the space of $\beta$-Hölder continuous functions on $SM$, where, for $v \in SM$, $\Phi_\lambda(v)$ is defined by

\[ \Phi_\lambda(v) := \partial_{\tilde{v}} \ln k_\lambda(\tilde{v}_\lambda(0), ., \tilde{v}_\lambda(+\infty)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \ln k_\lambda(\tilde{v}_\lambda(0), \tilde{v}_\lambda(\varepsilon), \tilde{v}_\lambda(+\infty)), \]

where $\tilde{v}$ is a lift of $v$.

**Proof.** Let $x_\varepsilon := \exp_\lambda(\varepsilon v)$ and take $z, d(x, z) \geq 1$. We have, for $v \in S\tilde{M}$:

\[ \partial_v k_\lambda(x, ., z) + C_8 = \lim_{\varepsilon \to 0} \frac{\varepsilon^{-1} (G_\lambda(x, z) - G_\lambda(x, z)) + C_8G_\lambda(x, z)}{G_\lambda(x, z)}. \]

Let $\gamma$ be a geodesic with $\gamma(0) = x$. For $T > 1$ such that $z, w \in C(\gamma(T))$, and for $\varepsilon < 1$, we can write

\[ \frac{G_\lambda(x, z) - G_\lambda(x, z)}{\varepsilon} + C_8G_\lambda(x, z) \]

\[ = \int_{\partial C(\gamma(1))} \left[ \frac{G_\lambda(x, a) - G_\lambda(x, a)}{\varepsilon} + C_8G_\lambda(x, a) \right] G_\lambda(a, z : C(\gamma(1))) da \]

and

\[ \frac{G_\lambda(x, w) - G_\lambda(x, w)}{\varepsilon} + C_8G_\lambda(x, w) \]

\[ \int_{\partial C(\gamma(1))} \left[ \frac{G_\lambda(x, a) - G_\lambda(x, a)}{\varepsilon} + C_8G_\lambda(x, a) \right] G_\lambda(a, w : C(\gamma(1))) da. \]

As $\varepsilon \to 0$ the expression between brackets is positive, converge towards $\partial_v G_\lambda(x, a) + C_8G_\lambda(x, a)$ and is dominated by $2C_8G_\lambda(x, a)$, which is integrable with respect to $G_\lambda(a, z : C(\gamma(T)),$
invariant probability measures. Then, Proposition 3.11.

Proposition 3.6 and Proposition 3.8, Corollary follows. The argument is the same for $\Upsilon$.

Indeed, the measures $\mu_\lambda$ satisfies the expression (2.8). Since all the functions involved are continuous by Proposition 3.8, the mapping $\lambda \mapsto \varphi_\lambda$ is continuous at $\lambda_0$.

Corollary 3.9. Set $P(\lambda_0)$ for the pressure of the function $\varphi_{\lambda_0} := -2\Phi_{\lambda_0}$. Then, $P(\lambda_0) \leq 0$.

Indeed we know by Corollary 2.10 that the pressure of the function $\varphi_\lambda$ is negative, and by Proposition 3.8, the mapping $\lambda \mapsto \varphi_\lambda$ is continuous at $\lambda_0$.

Corollary 3.10. The measures $\mu_\lambda$ and the normalising constants $\Omega_\lambda, \Upsilon_\lambda$ are continuous functions of $\lambda$ as $\lambda \to \lambda_0$ in $[0, \lambda_0]$.

Proof. Indeed, the measures $\mu_{\lambda_0}$ satisfies the conditions in Proposition 2.5 and $\Omega_{\lambda_0}$ satisfies the expression (2.8). Since all the functions involved are continuous by Proposition 3.8, Corollary follows. The argument is the same for $\Upsilon_\lambda$.

We can now prove Theorem 1.3 giving the exponential decay of $G_{\lambda_0}(x, y)$ with the distance. More precisely, we have:

Proposition 3.11. Let $\tau_0 := \inf\{\int \Phi_{\lambda_0} \, dm\}$, where the infimum is taken over all $g$-invariant probability measures. Then, $\tau_0 > 0$ and

$$\lim_{R \to \infty} -\frac{1}{R} \ln \max\{G_{\lambda_0}(x, y) : d(x, y) = R\} = \tau_0.$$ 

Proof. First we prove that $\tau_0 > 0$. Suppose $m_1$ attains $\sup \int \varphi_\lambda dm$. Suppose $\int \varphi_\lambda dm_1 \geq 0$. Then $h_{m_1} + \int \varphi_\lambda dm_1 \geq 0$. However, we know that $P(\varphi_{\lambda_0}) \leq 0$, thus $h_{m_1} = 0$ and $\int \varphi_\lambda dm_1 = 0$, thus $m_1$ is the equilibrium state. Thus is a contradiction since $h_{m_1} > 0$. 

$$\int C_{\varphi_\lambda} \, dm$$
if \( m_1 \) is the equilibrium state of the Hölder continuous function \( \varphi_{\lambda_0} \), thus \( \tau_0 \) is positive. Let \( 0 < \tau' < \tau_0 \). We have
\[
\lim_{t \to -\infty} \frac{1}{t} P(-t\Phi_{\lambda_0}) = -\tau_0,
\]
so we can find \( T \) large enough that \(-P(-T\Phi_{\lambda_0}) > T\tau'\). By Corollary 2.14, we get that there exists a constant \( C(T) \) such that for all \( R \geq 1, x \in \bar{M} \),
\[
e^{-RP(-T\Phi_{\lambda_0})} \int_{S(x,R)} G^T_{\lambda_0}(x,z)dz \leq C(T).
\]
Set \( C \) there exists a constant \( \lambda \) and therefore
\[
\liminf_{R \to \infty} \frac{1}{R} \ln G_{\lambda_0}(x,z) : d(x,z) = R \geq \tau'.
\]
By compactness, there exist \( x, y \) with \( d(x,y) = R \) and \( G_{\lambda_0}(x,y) = e^{-R\tau(R)} \). We have, for \( z \in S(x,R) \),
\[
G_{\lambda_0}(x,z) \geq C^{-1}e^{-R\tau(R)} \quad \text{and thus} \quad G^T_{\lambda_0}(x,z) \geq C^{-T}e^{-TR\tau(R)}.
\]
Therefore, we have for all \( R \geq 1 \),
\[
C(T) \geq e^{RT\tau'} \int_{S(x,R) \cap B(y,1)} G^T_{\lambda_0}(x,z)dz \geq C^{-T}e^{RT(\tau'-\tau)}.
\]
This is possible only if \( \liminf_{R \to \infty} \frac{1}{R} \ln \max\{G_{\lambda_0}(x,y) : d(x,y) = R\} \geq \tau_0 \).

Conversely, for all \( \varepsilon > 0 \), there exists a closed geodesic of length \( \ell \), say, such that for \( \tilde{v} \) tangent to that geodesic,
\[
\int_0^\ell \Phi_{\lambda_0}(\tilde{g}_s,\tilde{v}) ds \leq (\tau_0 + \varepsilon)\ell.
\]
Let \( \tilde{v} \) be a lift of \( v \). The geodesic \( \gamma_{\tilde{v}} \) is a periodic axis and for all \( j \in \mathbb{N} \),
\[
k_{\lambda_0}(\gamma_{\tilde{v}}(j\ell), \gamma_{\tilde{v}}((j+1)\ell) , \gamma_{\tilde{v}}(+\infty)) \leq e^{(\tau_0+\varepsilon)\ell}.
\]
By Lemma 3.5, we have:
\[
\frac{G_{\lambda_0}(\gamma_{\tilde{v}}(j\ell), \gamma_{\tilde{v}}(N\ell))}{G_{\lambda_0}(\gamma_{\tilde{v}}((j+1)\ell), \gamma_{\tilde{v}}(N\ell))} \geq \left( 1 + CK^{(N-j)\ell} \right)^{(-1)} e^{-(\tau_0+\varepsilon)\ell}
\]
and therefore
\[
\frac{G_{\lambda_0}(\gamma_{\tilde{v}}(0) , \gamma_{\tilde{v}}(N\ell))}{G_{\lambda_0}(\gamma_{\tilde{v}}((0)\ell), \gamma_{\tilde{v}}(N\ell))} = \prod_{j=0}^{N-2} \frac{G_{\lambda_0}(\gamma_{\tilde{v}}(j\ell), \gamma_{\tilde{v}}(N\ell))}{G_{\lambda_0}(\gamma_{\tilde{v}}((j+1)\ell), \gamma_{\tilde{v}}(N\ell))} \geq Ce^{-(\tau_0+\varepsilon)\ell}.
\]
This shows that, for all \( \varepsilon > 0 \),
\[
\limsup_{R \to \infty} \frac{1}{R} \ln \max\{G_{\lambda_0}(x,y) : d(x,y) = R\} \leq \tau_0 + \varepsilon.
\]
\(\square\)

Now let us prove Theorem 1.4. Recall that Martin compactification of the operator \( \Delta + \lambda_0 \) is given by all possible limits of \( k_{\lambda_0}(x,y,z) \) as \( z \to \infty \). Proposition 3.6 and its proof show that there is a continuous mapping from the geometric compactification of \( \bar{M} \) onto the Martin compactification. So it suffices to show that this mapping is one-to-one.
The proposition above says that $G_{\lambda_0}(x_0,\cdot)$ goes to 0 at infinity uniformly, it follows from Theorem 3.2 that $k_{\lambda_0}(x,y,\xi)$ goes to $+\infty$ as $y \to \xi$. Let $\eta \in \partial M$, $\eta \neq \xi$, and $y_n \to \eta$. We show that $k_{\lambda_0}(x,y_n,\xi) \to 0$ as $y_n \to \eta$, so that $k_{\lambda_0}(x,.,\xi)$ will not coincide with $k_{\lambda_0}(x,.,\eta)$: choose $x_0$ on the geodesic between $\xi$ and $\eta$. As $y_n \to \eta, x_m \to \xi$, there is some $d_0$ such that $x_0$ is at a distance less than $d_0$ of the geodesic segment $[y_n, x_m]$ for $n, m$ large enough. It follows from Harnack inequality Proposition 2.6 and Theorem 3.2 that $k_{\lambda_0}(x_0, y_n, x_m)$ is smaller than $C G_{\lambda_0}(x_0, y_n)$, which goes to 0 as $y_n \to \eta$.

The decomposition of positive $(\Delta + \lambda_0)$-harmonic functions follows then by general Martin theory.

4. Renewal theory

In this section, we use uniform mixing of the geodesic flow $g_t$ we obtained from Section 7 to control the convergence in Proposition 2.2 as $\lambda$ goes to $\lambda_0$. We use the following notation throughout this section: $\sim^a$ means that the ratios between the two sides are bounded by $a$.

Throughout the section, let us denote $\chi(t) := 1$ for $|t| \leq 1/2$ and 0 otherwise. Let $\chi_{\partial}(t) = \chi(t/\partial')$. Let $\psi(t) := \max\{1 - |t|, 0\}$.

Thanks to Proposition 3.8, for $\lambda$ close to $\lambda_0$, the functions $\varphi_{\lambda}$ are close to $\varphi_{\lambda_0}$ in the space $\mathcal{K}_\alpha$ of $\alpha$-Hölder continuous functions, for some $\alpha > 0$ (see Section 7.1 for definition of $\mathcal{K}_\alpha$). We can therefore apply Proposition 7.3 and Corollary 7.4 to the equilibrium measure $\mu_{\lambda}$ associated to $\varphi_{\lambda}$ to obtain the following propositions.

Proposition 4.1. There exist $\alpha > 0$ and $\delta_0 > 0$ with the following property. For every $\varepsilon > 0, f, h \in \mathcal{K}_\alpha$ positive $\alpha$-Hölder continuous functions, there exists $t_0 = t_0(f, h, \varepsilon)$, such that for $t \geq t_0$, for any $\lambda \in [\lambda_0 - \delta_0, \lambda_0]$, $\int h \circ g_t \, dm_{\lambda} \sim^{1+\varepsilon} \int f \, dm_{\lambda} \int h \, dm_{\lambda}$.

Indeed, $t_0$ depends only on $(\varepsilon, |f|_{\alpha}, |h|_{\alpha}, \inf \lambda \int f \, dm_{\lambda}, \inf \lambda \int h \, dm_{\lambda})$, in particular is independent of $\lambda \in [\lambda_0 - \delta_0, \lambda_0]$.

Proposition 4.2. There exist $\alpha' > 0$ and $\delta'_0 > 0$ with the following property. For every $\varepsilon > 0, f, u, h \in \mathcal{K}_{\alpha'}$ positive $\alpha'$-Hölder continuous functions, there exists $t'_0 = t'_0(f, u, h, \varepsilon)$, such that for $t \geq t'_0$, for any $\lambda \in [\lambda_0 - \delta_1, \lambda_0]$, $\frac{1}{t} \int_0^t \left[ \int f \circ (u \circ g_s) \cdot (h \circ g_t) \, ds \right] ds \sim^{1+\varepsilon} \int f \, dm_{\lambda} \int u \, dm_{\lambda} \int h \, dm_{\lambda}$.

Indeed, $t'_0$ depends only on $(\varepsilon, |f|_{\alpha'}, |u|_{\alpha'}, |h|_{\alpha'}, \inf \lambda \int f \, dm_{\lambda}, \inf \lambda \int u \, dm_{\lambda}, \inf \lambda \int h \, dm_{\lambda})$, in particular is independent of $\lambda \in [\lambda_0 - \delta'_0, \lambda_0]$.

4.1. Integral on large spheres with respect to Green functions. Let us introduce some more notations: for $x \neq z \in \tilde{M}$, denote by $v_x^z$ the unit vector in $S_z \tilde{M}$ pointing towards $x$. The mapping $z \mapsto v_x^z$ identifies $\tilde{M} \setminus \{x\}$ with a subset of $SM$. 

Theorem 4.3. Given $\varepsilon' > 0$ and positive Hölder continuous functions $f, h$ on $SM$, there exist $R(f, h, \varepsilon')$ and $\delta(f, h, \varepsilon')$ such that if $R > R(f, h, \varepsilon')$ and $\lambda \in [\lambda_0 - \delta(f, h, \varepsilon'), \lambda_0]$, for all $x \in \tilde{M}$,

\begin{equation}
(4.1) \quad e^{-RP(\lambda)} \int_{S(x, R)} f(p v^y)h(p v^y)G^2_\lambda(x, y)dy \sim (1+\varepsilon')^3
\end{equation}

\begin{equation}
\Omega_{\lambda} \int_{\partial M} f(p \circ \pi^{-1}_x(\xi))d\mu^\lambda(y) \left( \int_{\partial M} h(p \circ \pi^{-1}_x(\xi))d\mu^\lambda(y) \right) dVol(y).
\end{equation}

The rest of Section 4.1 is devoted to the proof of Theorem 4.3.

Fix $f, h$ positive and Hölder continuous. We choose $\delta'_0 > 0$ such that, if $R > 1$ and $|R - R'| < \delta'_0$, then, for all $x \in \tilde{M}$ and $\lambda \in [\lambda_0 - \delta(f, h, \varepsilon'), \lambda_0]$,

\begin{equation}
(4.2) \quad e^{-RP(\lambda)} \int_{S(x, R)} f(p v^y)h(p v^y)G^2_\lambda(x, y)dy \sim (1+\varepsilon) e^{-RP(\lambda)} \int_{S(x, R')} f(p v^y)h(p v^y)G^2_\lambda(x, y)dy.
\end{equation}

Then

\begin{equation}
\delta' \sim (1+\varepsilon)^2 \int_R \chi \delta(x - R)e^{-sP(\lambda)} \left( \int_{S(x, s)} f(p v^y)h(p v^y)G^2_\lambda(x, y)dy \right) ds.
\end{equation}

\begin{align*}
\lambda' \sim & \int_{\tilde{M}} \chi \delta(x - R)e^{-d(x, y)P(\lambda)} f(p v^y)h(p v^y)G^2_\lambda(x, y)dVol(y) \\
= & \int_{\tilde{M}} \left( \sum_{\{(v, t) : v \in S_{px} \cap \gamma(t) \neq S_{py} \cap \gamma(t)\}} \chi \delta(t - R)f(v)h(-g_tv)e^{-tP(\lambda)}G^2_\lambda(\gamma(t), \gamma(t)) \right) dVol(y).
\end{align*}

Recall from Section 2 that

\begin{equation}
\frac{d\mu^{uu}_{\lambda}}{dg_{-t}(\mu^{uu}_{\lambda})}(v) = e^{-tP(\lambda)}k_{\lambda}^2(\gamma(t), \gamma(0), \gamma(\infty)).
\end{equation}

By Proposition 5.7, for given $\varepsilon$, if $R$ is large enough (depending on $\varepsilon$) and $|t - R| \leq \delta' \leq 1$,

\begin{equation}
\frac{d\mu^{uu}_{\lambda}}{dg_{-t}(\mu^{uu}_{\lambda})}(v) = e^{-tP(\lambda)} \lim_{z \to v^+} G_{\lambda}^2(\gamma(0), z)G_{\lambda}^2(\gamma(t), \gamma(t))G_{\lambda}^2(\gamma(t), \gamma(t)) \sim (1+\varepsilon) e^{-tP(\lambda)}G_{\lambda}^2(g_tv, \gamma(t), \gamma(t)),
\end{equation}

where the approximation is uniform in $g_tv$ and $\lambda$. Therefore, when replacing

\begin{equation}
e^{-tP(\lambda)}G_{\lambda}^2(\gamma(t), \gamma(t)) \quad \text{by} \quad \frac{1}{\theta_{\lambda}(g_tv)} \frac{d\mu^{uu}_{\lambda}}{dg_{-t}(\mu^{uu}_{\lambda})}(v),
\end{equation}

for $R$ large enough and $|t - R| \leq \delta'$, uniformly in $x, y$ and $\lambda$, we lose a factor $(1+\varepsilon')^{1+1}$.

Remark that $\theta_{\lambda}(v) = \theta_{\lambda}(-v)$.

We are reduced to show:
Proposition 4.4. Given $\varepsilon' > 0$ and positive Hölder continuous functions $f, h$ on $SM$, there exist $R_0 = R_0(f, h, \varepsilon')$, $\delta = \delta(f, h, \varepsilon') > 0$ and $\delta', 0 < \delta' < \delta'$, such that for $R \geq R_0$, all $x, y \in \tilde{M}$ and all $\lambda \in [\lambda_0 - \delta, \lambda_0]$,

$$
\sum_{\{(v, t): v \in S_{px} \cap g^{-1}S_{py}M\}} \chi_{d'v}((R - t)f(v)(\theta^{-2}h)(-gtv))\frac{d\mu^u_{\lambda}}{dg_{\theta_{-t}}\mu^u_{\lambda}}(v) \\
\sim 1 + \varepsilon'
$$

The last line in Proposition 4.4 is the same as

$$
\delta' \Omega \int_{\partial M} f(p \circ \pi^{-1}_x \xi) d\mu_\lambda^\alpha(\xi) \int_{\partial M} h(p \circ \pi^{-1}_y \xi) d\mu_\lambda^\alpha(\xi)
$$

by (2.11) and (2.12).

Theorem 4.3 follows from Proposition 4.4 and the previous discussion by integrating the approximation in $y$ over a fundamental domain $M_0$.

Proof. We combine ideas of [11] and Section III in [11]. Choose $\varepsilon$ such that $(1 + \varepsilon)^{22} \leq 1 + \varepsilon'$. Proposition follows from Proposition 4.1 applied to the non-negative Hölder continuous functions $F^\pm_{\theta}, H^+_{\theta}$ with the property that there exist constants $C, \alpha, \beta_0, \beta'_0, \gamma'$ such that for all $x, y \in \tilde{M}$ and all $\lambda \in [0, \lambda_0]$, the following (1)-(5) holds.

1. $\|F^\pm_{\theta}\|_\alpha < C, \|H^+_{\theta}\|_\alpha < C$,
2. $\int S_{py}M F^\pm_{\theta} dm_{\lambda} > C^{-1}, \int H^+_{\theta} dm_{\lambda} > C^{-1}$.
3. $\Omega \chi_{d'v}(1 + \varepsilon)^{-14} \int_{S_{px}M} f(v) d\mu^u_{\lambda}(v) \leq \int F^-_{\lambda} dm_{\lambda}$
   \leq \int F^+_{\lambda} dm_{\lambda} \leq \Omega \chi_{d'v}(1 + \varepsilon)^{14} \int_{S_{px}M} f(v) d\mu^u_{\lambda}(v).
4. $\Omega \gamma \beta'_0(1 + \varepsilon)^{-14} \int_{S_{py}M} h(-u) d\mu^s_{\lambda}(u) \leq \int H^-_{\lambda} dm_{\lambda}$
   \leq \int H^+_{\lambda} dm_{\lambda} \leq \Omega \gamma \beta'_0(1 + \varepsilon)^{14} \int_{S_{py}M} h(-u) d\mu^s_{\lambda}(u).
5. There is $R(\varepsilon)$ such that for $R \geq R(\varepsilon)$,
   $$(1 + \varepsilon)^{-32} \int F^-_{\lambda} H^-_{\lambda} \circ g_R dm_{\lambda}$$
   \leq \Omega \chi_{d'v}(1 + \varepsilon)^{32} \sum_{\{(v, t): v \in S_{px} \cap g^{-1}S_{py}M\}} \chi_{d'v}((R - t)f(v)(\theta^{-2}h)(-gtv))\frac{d\mu^u_{\lambda}}{dg_{\theta_{-t}}\mu^u_{\lambda}}(v)$
   \leq (1 + \varepsilon)^3 \int F^+_{\lambda} H^+_{\lambda} \circ g_R dm_{\lambda}.$$
We choose $\delta' < \delta'_0$ and such that, for all $\xi \in \partial \tilde{M}$, all $\lambda \in [0, \lambda_0]$, all
\[
\frac{\theta^2(v')}{\theta^2(v)}, k_\lambda(x, x', \xi), e^{-P(x, x', \xi)} \sim 1 + \epsilon \quad \text{for} \quad d(v, v') < 2\delta', d(x, x') < 2\delta',
\]
where $P := \inf_{\lambda \in [0, \lambda_0]} P(\lambda) < 0$.

The functions $F^\pm_\lambda(v), H^\pm_\lambda(u)$ will approximate $\theta^{-2}_\lambda f(v), \theta^{-2}_\lambda h(-u)$ respectively, on the $\delta$-neighborhoods $N_\delta(S_{px}M), N_\delta(S_{py}M)$ of $S_{px}M, S_{py}M$, respectively.

For $w \in N_\delta(S_{px}M)$, there exist a unique $v \in S_{px}M$, and $v' \in W^s_{loc}(v), t$ such that $v' = \gamma_t w$. Similarly, if $w \in N_\gamma(S_{py}M)$, then there exists a unique triple $(u, u', s), u \in S_{py}M, u' \in W^u_{loc}(u)$ such that $u' = g_s(w)$.

By the Hölder regularity of the strong stable and the strong unstable foliations, the systems of coordinates $(v, v', t)$ (respectively $(u, u', t)$) are Hölder continuous, uniformly in $x$ and $y$.

**Step 1.** There exist $\beta_0, \beta'_0 > 0$ and non-negative Hölder continuous functions $a_\pm, b_\pm$ supported on $N_\delta S_{px}M, N_\delta S_{py}M$, respectively, such that for all $v \in S_{px}M$ and $u \in S_{py}M$,
\[
(4.3) \quad \int_{W^s_{loc}(v)} a_\pm(w) \, d\mu^s_\lambda(w) = \beta_0(1 + \epsilon)^{\pm 1}, \quad \int_{W^u_{loc}(u)} b_\pm(w) \, d\mu^u_\lambda(w) = \beta'_0(1 + \epsilon)^{\pm 1}.
\]

Moreover, the Hölder exponent and the Hölder coefficient of $a_\pm, b_\pm$ are bounded uniformly in $x, y, \lambda$.

**Lemma 4.5.** Let
\[
h_{r,v,\lambda} = \int_{W^s_{loc}(v)} \psi \left( \frac{d_{ss}(v, v')}{r} \right) \mu^s_\lambda(v').
\]

The map $(r, v, \lambda) \mapsto h_{r,v,\lambda}$ is continuous in $r, v$ and $\lambda$. For a fixed $r$, the function $v \mapsto h_{r,v,\lambda}$ is Hölder continuous, uniformly in $\lambda \in [0, \lambda_0]$. As $r$ varies from 0 to $\delta'$, the function $r \mapsto h_{r,v,\lambda}$ is increasing and admits right and left derivatives that are are bounded below by a positive constant uniformly in $v, \lambda$ and $r$ away from zero.

**Proof.** The continuity is as in Margulis’s Lemma 7.1 in [M2] (p. 51). The proof also yields Hölder continuity in $r$. Indeed, $W^s_{loc}(v)$ depends on $v$ in a Hölder continuous way and if $v_1, v_2$ are close, the holonomy $H^s_1$ from $W^s_{loc}(v_1)$ to $W^s_{loc}(v_2)$ is Hölder continuous, and satisfies for $v'_1, v''_1 \in W^s_{loc}(v_1),$
\[
d(v_2, H^s_1 v_1) \leq C(d(v_1, v_2))^{\alpha}, \quad \text{and} \quad |d(H^s_1 v'_1, H^s_1 v''_1) - d(v'_1, v''_1)| \leq C(d(v'_1, v''_1))^{\alpha}.
\]

Moreover the logarithm of the Radon Nikodym derivatives of the measure $(H^s_1)^* \mu^s_\lambda(v'_2)$ with respect to $\mu^s_\lambda(v'_2)$ is also, uniformly in $\lambda$, proportional to $(d(v_2, H^s_1 v_1))^{\alpha}$ (Proposition 4.3). Since $d(v_2, H^s_1 v_1) \leq C(d(v_1, v_2))^{\alpha}$, we can report in the definition of $h_{r,v,\lambda}$ and see that, for $v_1, v_2$ close,
\[
|h_{r,v_1,\lambda} - h_{r,v_2,\lambda}| \leq C(r)(d(v_1, v_2))^{\alpha},
\]
where the constant $C(r)$ is uniform in $\lambda \in [0, \lambda_0]$ and goes to infinity as $r \to 0$. 

[M2] Margulis, G. A. (1985). Ergodic Theory and Ergodic Problems of Geometric Group Theory. Springer-Verlag.
Direct computation shows that, as \( r \) varies from 0 to \( \delta' \), the function \( r \mapsto h_{r,v,\lambda} \) is increasing and admits left and right derivatives given by
\[
\frac{\partial}{\partial r} h_{r,v,\lambda}|_{r-} = \lim_{r' \to r, r' > r} \int_{W_{loc}^{ss}(v)} \frac{1}{r'} d_{ss}(v,v') \chi_{d(v, \cdot) \leq r'}(v') \, d\mu_{\lambda}(v')
\]
and
\[
\frac{\partial}{\partial r} h_{r,v,\lambda}|_{r+} = \int_{W_{loc}^{ss}(v)} \frac{1}{r} d_{ss}(v,v') \chi_{d(v, \cdot) \leq r}(v') \, d\mu_{\lambda}(v').
\]
In particular, the derivatives are bounded from below by a positive constant uniformly in \( v, \lambda \) and \( r \) away from 0. \( \square \)

For given \( \beta \), choose \( r_{\lambda}^{\pm}(v,\beta) \) such that \( h_{r,\lambda}^{\pm}(v,\beta;v,\lambda) = \beta(1 + \varepsilon)^{\pm 1} \). Now choose \( \beta_0 \) so that \( r_{\lambda}^{\pm}(v,\beta_0) < \varepsilon \delta'/2 \) for all \( v \) and \( \lambda \). Set \( r_{\lambda}^{\pm}(v) := r_{\lambda}^{\pm}(v,\beta_0) \). By the Implicit function theorem with Hölder coefficients, the functions \( r_{\lambda}^{\pm}(v) \) are Hölder continuous uniformly in \( \lambda \) for \( \lambda \in [\lambda_0 - \delta(\varepsilon), \lambda_0] \) and \( v \).

Now for \( w = (v,v',t) \in N_{\delta}(S_{py}M), \lambda \in [\lambda_0 - \delta(\varepsilon), \lambda_0] \), define
\[
a_{\lambda}^{\pm}(w) = \psi \left( \frac{d_{ss}(v,v')}{r_{\lambda}^{\pm}(v)} \right).
\]

Properties similar to Lemma 4.3 hold for the function
\[
(r,u,\lambda) \mapsto h_{r,u,\lambda} = \int_{W_{loc}^{uu}(u)} \psi \left( \frac{d_{uu}(u,u')}{r} \right) d\mu_{\lambda}^{uu}(u'),
\]
thus we can define \( r_{\lambda}^{\pm}(u) \) analogously: \( \beta_0' \) is chosen so that \( r_{\lambda}^{\pm}(u,\beta_0') < \varepsilon \delta'/2 \) and \( r_{\lambda}^{\pm}(u) \) is such that \( h_{r,\lambda}^{\pm}(u,\beta_0;u,\lambda) = \beta_0'(1 + \varepsilon)^{\pm 1} \). For \( w = (u,u',s) \in N_{s}(S_{py}M) \), define
\[
b_{\lambda}^{\pm}(w) = \psi \left( \frac{d_{uu}(u,u')}{r_{\lambda}^{\pm}(u)} \right).
\]
The functions \( a^{\pm}, b^{\pm} \) satisfy the properties of Step 1. \( \square \)

Step 2. Definition of \( F_{\lambda}^{\pm}, H_{\lambda}^{\pm} \) and Property (1)

Consider Lipschitz continuous \( \chi_{\pm}(t) \) on \( \mathbb{R} \) such that, for all \( t \in \mathbb{R} \),
\[
\chi_{(1+\varepsilon)^{-3}}(t) \leq \chi_{-}(t) \leq \chi_{(1+\varepsilon)^{-1}}(t) \leq \chi_{+(t)} \leq \chi_{(1+\varepsilon)^{2}}(t).
\]
Now for \( w = (v,v',t) \), define
\[
F_{\lambda}^{\pm}(w) = \chi_{\pm}(t/\delta') a_{\pm}(v')(\theta_{-2} f)(v)
\]
and for \( w = (u,u',s) \),
\[
H_{\lambda}^{\pm}(w) = \chi_{\pm}(s/\gamma) b_{\pm}(u')(\theta_{-2} h)(-u),
\]
for some \( \gamma < \delta'/2 \).

\(^2\)We have \( h_{r(v),v} = \beta = h_{r(v'),v'} \) so that \( |h_{r(v),v} - h_{r(v'),v'}| = |h_{r(v'),v} - h_{r(v'),v'}| \leq C(d(v,v'))^\alpha \), with uniform \( C, \alpha \). But \( |h_{r(v),v} - h_{r(v'),v'}| \) is greater than \( |r(v) - r(v')| \) times the derivative at \( r \) of \( r \mapsto h_{r,v} \) and the derivative is bounded from below.
Recall that the systems of coordinates \((v, v', t)\) and \((u, u', s)\) are Hölder continuous uniformly in \(x\) and \(y\). The functions \(F^\pm_\lambda, H^\pm_\lambda\) in those coordinates are compositions of Hölder continuous functions \((\psi, f, h)\) and of the functions \(r^\pm_\lambda\) that depend on \(v\) in a Hölder continuous way, uniformly in \(\lambda \in [0, \lambda_0]\) by Step 1, which proves Property (1).

**Step 3. Properties (2), (3) and (4)**

Recall that under Hopf parametrisation introduced in Section 2 if we let \(x_0 = x\), the lift \(\tilde{m}_\lambda\) of \(m_\lambda\) to \(SM\) is given by

\[
d\tilde{m}_\lambda(x, \eta, t) = \Omega_\lambda \theta^2_\lambda(x, \eta) e^{2P(\lambda)(x, \eta)} d\mu^\lambda_x((x) \times d\mu^\lambda_\eta(\eta) \times dt).
\]

Consider \(\tilde{w} = \tilde{w}(x, \eta, t)\) close to \(S_x \tilde{M}\) and write the coordinates \((v, v', t)\) of \(w = p \tilde{w}\) as:

\[
v = p(\pi^{-1}_x(\eta)), \quad v' = p(W^{ss}(\pi^{-1}_x(\eta)) \cap \gamma(\eta)), \quad t = t.
\]

In particular, \(w\) is close to \(v\) and

\[
\theta^\lambda_x(x, \eta) = \theta_\lambda(v) k_\lambda(x, p(w), \xi_\lambda(x, p(w), \eta) \sim (1 + \varepsilon)^2 \theta_\lambda(v) \sim (1 + \varepsilon)^3 \theta_\lambda(v),
\]

and

\[
e^{-P(\lambda)(x, \eta)} \sim (1 + \varepsilon)^2 1.
\]

We see that the measure \(\tilde{m}_\lambda\) has a density \(~(1 + \varepsilon)^8 \Omega_\lambda \theta^{-2}_\lambda(v)\) with respect to the product measure \(d\mu^\lambda_x((x) \times d\mu^\lambda_\eta(\eta) \times dt\). The mapping \(\xi \mapsto v'\) sends the measure \(d\mu^\lambda_x\) to a measure with density \(~(1 + \varepsilon)^4 1\) with respect to the measure \(d\mu^\lambda_x(v')\). This implies that we can write, in the neighborhood of \(S_{px} M\), the measure \(m_\lambda\) in the coordinates \((v, v', t)\) as

\[
dm_\lambda(v, v', t) \sim (1 + \varepsilon)^{12} \Omega_\lambda \theta^2_\lambda(v) [d\mu^\lambda(v) \times d\mu^\lambda(v') \times dt].
\]

Since \(\delta'(1 + \varepsilon)^{-1} \leq \int \chi_- (\frac{x}{\delta'}) dt \leq \int \chi_+ (\frac{x}{\delta'}) dt \leq \delta'(1 + \varepsilon)\), it follows that

\[
\int F^+_\lambda(w, \lambda) dm_\lambda \leq (1 + \varepsilon)^{12} \Omega_\lambda \int \chi_\pm (\frac{x}{\delta'}) dt \int_{S_{px} M} \left(\int_{W^{ss}_loc(v)} a^+_\lambda(v') d\mu^\lambda(v')\right) f(v) d\mu^\lambda(v)
\]

\[
\leq (1 + \varepsilon)^{14} \Omega_\lambda \delta'_0 \int_{S_{px} M} f(v) d\mu^\lambda(v),
\]

and

\[
\int F^-_\lambda(w, \lambda) dm_\lambda \geq (1 + \varepsilon)^{-14} \Omega_\lambda \delta'_0 \int_{S_{px} M} f(v) d\mu^\lambda(v).
\]

Similarly, in the \(\delta'\)-neighborhood of any lift of \(S_{py} M\), we have, in the \((u, u', s)\) coordinates, \(u \in S_y \tilde{M}, u' \in W^{uu}_{loc}(u), |s| \leq 2\delta',\)

\[
dm_\lambda(u, u', t) \sim (1 + \varepsilon)^{12} \Omega_\lambda \theta^2_\lambda(u) [d\mu^\lambda(u) \times d\mu^\lambda(u') \times ds].
\]
The analog computation yields that
\[
(1 + \varepsilon) - 14 \Omega \gamma \beta_0 \int_{S_{py} M} h(-u) d\mu^s_{\lambda}(u) \leq \int \tilde{H}_0 dm_{\lambda}
\]
\[
\leq \int \tilde{H}_0 dm_{\lambda} \leq (1 + \varepsilon) - 14 \Omega \gamma \beta_0 \int_{S_{py} M} h(-u) d\mu^s_{\lambda}(u).
\]
This shows Properties (3) and (4). Property (2) follows as \( \int f d\mu^u_{\lambda} \) and \( \int h d\mu^s_{\lambda} \) are bounded away from 0, uniformly in \( x, y \) and \( \lambda \in [0, \lambda_0] \) by Corollary 2.7.

**Step 4. Property (5)**

Suppose \( F_{\lambda}(w)H_{\lambda}(g Rw) \neq 0 \). Set \( w = (v, v', t) \) and \( g Rw = (u, u', s) \), where \( v \in \text{supp } f, u \in \text{supp } h \), \( v' \) is in the \( r^+_{\lambda}(v) \)-neighborhood of \( v \) in \( W^s(v) \) and \( u' \) is in the \( r^+_{\lambda}(u) \)-neighborhood of \( u' \) in \( W^u(u) \) with \( |t| < \delta'/2 \) and \( |s| < \gamma < \delta'/2 \).

Since \( w \) is close to \( x \) and \( g Rw \) is close to \( y \), the geodesic from \( x \) to \( y \) remains close to the geodesic segment \( \{g_t w : 0 \leq t \leq R \} \). In other words, there is a unique \( v_0 \in S_{px} M \) such that \( \{g_t v_0 : 0 \leq t \leq r \} \) remains close to \( \{g_t w : 0 \leq t \leq R \} \).

![Figure 2. M near x and y](image)

Thus
\[
\int F_{\lambda}^\pm(w) H_{\lambda}^\pm(g Rw) dm_{\lambda} = \sum_{(v_0, r)} \int_{\{g_t w : 0 \leq t \leq r \}} \theta_\lambda^{-2} f(v) \theta_\lambda^{-2} h(-u) \chi_{\pm}(t/\delta') \chi_{\pm}(s/\gamma) a_{\pm}(v') b_{\pm}(u') dm_{\lambda}
\]

By triangular inequalities on points in Figure 2
\[
|R - r| < |s - t| + \delta' \varepsilon < (1 + 3 \varepsilon/2) \delta'.
\]
Now fix such \((v_0, r)\) and calculate the contribution of \( \int F_{\lambda}^\pm(w) H_{\lambda}^\pm(g Rw) \) for \( w \) close to \( v_0 \).

Consider the closed path in \( SM \) near \( y \) in Figure 3 consisting of vectors
\[
g_{Rv_0}, g_{Rv}, g_{Rv'}, g_{Rw}, g_{Rw'}, u, u, g_{Rv_0}.
\]

Then
\[
d(g_{Rv_0}, g_{Rw}) \leq d(g_{Rv}, g_{Rv'}) + |t| + |s| + d(u', u) + d(u, g_{Rv_0}) + d(g_{Rv_0}, g_{Rw})
\]
\[
\leq e^{-R\alpha} r^+_{\lambda}(v) + |t| + |s| + r^+_{\lambda}(u) + \text{diam}(S_{py} M) + |R - r|
\]
\[
\leq e^{-R\alpha} \varepsilon \delta'/2 + \delta' + \gamma + \varepsilon \delta'/2 + C + \delta'(1 + \varepsilon) \leq C
\]
It follows that the distance between $v_0$ and $v$ along the unstable direction is bounded above by $e^{-a_0 R} C$. Since $S_{px} M$ is $\varepsilon_1$-transversal to $W^{ss}(v)$ (for some $\varepsilon_1$ independent of $v$),

$$d(v_0, v) < e^{-a_0 R} C'.$$

Let us choose $R_0$ large enough so that if $R \geq R_0$, then $d(v_0, v) \leq \delta'$. Similarly, considering a closed path in Figure 4, $d(g_{-R} v_0, g_{-R} u) < C$, thus

$$d(g_{-R} v_0, u) < e^{-R_0} C' < \delta'.$$

Choose another system of coordinates $W^{uu} \times W^{ss} \times \mathbb{R}$ for $w$ near $v_0$ as follows:

$$w = (u'', v'', t),$$

where $u''$ and $v''$ are the projections of $v'$ on $W^{uu}(v_0)$ and $W^{ss}(v_0)$, respectively, and $t$ is defined as above (see Figure 4). Since $v, v'$ are $\delta'$-close to $u'', v''$, respectively, the measure $m_\lambda$ satisfies

$$(4.4) \quad dm_\lambda(u'', v'', t) \sim (1+\varepsilon)^{16} \Omega_\lambda^2(\phi)[d\mu_\lambda^{uu}(u'') \times d\mu_\lambda^{ss}(v'') \times dt].$$
Note that, for fixed $v, v', u, u'$, the distances $d(\pi v, \pi u) = r$ and $d(\pi v', \pi u') = s + R - t$ are fixed. We are reduced to compute

$$\Omega_{\lambda} \int \int \left[ \int \chi_\pm(t/\delta') \chi_\pm(s/\gamma) dt \right] a_\pm(v') b_\pm(u') f(v) \theta^{-2} h(u) d\mu^{ss}(v') d\mu^{uu}(u').$$

Let us first estimate the integral with respect to $t$. From $F^{-}_\lambda(w) H^+_\lambda(g_R w) \neq 0$, by triangular inequality,

$$d(v', g-R u') < d(v', w) + d(w, g-R u') \leq \delta'/2(1 + \varepsilon) + \delta'\varepsilon/2(1 + \varepsilon) = \delta'/2.$$ 

Clearly $\int \chi_-(t/\delta') \chi_-(s/\gamma) dt \leq \int \chi_+(s/\gamma) ds = (1 + \varepsilon)^2 \gamma$.

On the other hand, $d(v', g-R u') < \delta'/2$ implies that

$$\int \chi_+(t/\delta') \chi_+(s/\gamma) dt \geq \int \chi_-(s/\gamma) ds = (1 + \varepsilon)^{-2} \gamma.$$

Since $v''$ is $\delta'$-close to $v'$, we have

$$\int a_\pm(v') d\mu^{ss}(v') \sim (1 + \varepsilon)^4 \int a_\pm(v') d\mu^{ss}(v').$$

Indeed, the change of variables from $v''$ to $v'$ is done by sliding along the central unstable manifolds between two close pieces of strong stable manifolds and the Radon-Nikodym derivative is

$$k^2_\lambda(v'', v', \cdot) e^{-P(\lambda)h(v'', v', \cdot)} \sim (1 + \varepsilon)^4 1.$$ 

We obtain, independently of $u'$ and $t$ that

$$\int a_-(v') d\mu^{ss}(v') \leq (1 + \varepsilon)^5 \beta_0, \quad (1 + \varepsilon)^{-5} \beta_0 \leq \int a_+(v') d\mu^{ss}(v').$$

In the same way we have, independently of $u''$, $t$:

$$\int_{W^{uu}_{loc}(v_0)} b_\pm(u') d\mu^{uu}_{\lambda}(u'') = \int_{W^{uu}_{loc}(v_0)} b_\pm(u') d\mu^{uu}_{\lambda}(u') \frac{d\mu^{uu}_{\lambda}(u'')} {dg_{-R} \mu^{uu}_{\lambda}(v_0)} \int_{W^{uu}_{loc}(v_0)} b_\pm(u') d\mu^{uu}_{\lambda}(u'),$$

since $u'', u'$ are $\delta'$-close to $v_0$ and $g_{-R} v_0$, respectively. Altogether, we get for each $(v, t)$ with $v \in S_{px} M \cap g_{-t} S_{py} M$ for some $t, |t - R| \leq \delta'/2$ a contribution $C_{\pm}(v, t)$ to the integral of $F^{-}_\lambda H^- \circ g_R, F^+_\lambda H^+ \circ g_R$ satisfying:

$$(4.5) \quad (1 + \varepsilon)^{-32} C_-(v, t) \leq \Omega_{\lambda} \beta_0 f(v)(\theta^{-2} h)(-g_R v) \frac{d\mu^{uu}_{\lambda} (v)} {dg_{-t} \mu^{uu}_{\lambda}(v)} \leq (1 + \varepsilon)^{32} C_+(v, t).$$

Summing all these contributions for all relevant pairs $(v, t)$, we obtain Property (5).

*Step 5. End of the proof of Proposition 4.4*
By Properties (1), (2) we can apply Proposition 4.4 and find $R_0, \delta_0$ independent of $\lambda, x, y$ such that for $R > R_0, \lambda \in [\lambda_0 - \delta_0, \lambda_0]$,

$$\int F^\lambda_\gamma H_\gamma \circ g_R \, dm_\lambda \sim (1+\varepsilon) \int F^\lambda_\gamma \, dm_\lambda \int H_\gamma \, dm_\lambda$$
$$\int F_+ H_+ \circ g_R \, dm_\lambda \sim (1+\varepsilon) \int F_+ \, dm_\lambda \int H_+ \, dm_\lambda.$$

We get

$$\Omega_\lambda \gamma_{\beta_0}^\prime \sum_{\{(v, t) : v \in S_{\beta_0} \cap S_{\beta_0 - 2\lambda} \}} \chi_{\beta_0}(R - t) f(v)(\theta^2 h(-g_\theta v)) \frac{d\mu^u_\lambda}{d\mu^u_\lambda}(v)$$

$$\sim (1+\varepsilon)^{61} \Omega_\lambda \gamma_{\beta_0}^\prime \int_{S_{\beta_0} \cap S_{\beta_0 - 2\lambda}} f(v) d\mu^u_\lambda(v) \int_{S_{\beta_0} \cap S_{\beta_0 - 2\lambda}} h(-v) d\mu^u_\lambda(v),$$

which the statement of Proposition 4.4 after dividing both members by $\Omega_\lambda \gamma_{\beta_0}^\prime$. □

4.2. Convergence of measures. We state in this subsection several consequences and variants of Theorem 4.3 which will be used in the next Sections. Set $\Omega := \Omega_{\lambda_0}$ and recall that $\Upsilon := \Upsilon_{\lambda_0}$.

First, observe that the expression (4.1) is continuous in $\lambda$ as $\lambda \rightarrow \lambda_0$ by Corollary 5.10. By choosing $\delta_1 = \delta_1(f, h, \varepsilon)$ such that for $\lambda \in [\lambda_0 - \delta_1, \lambda_0]$

$$\Omega \int_{\partial M} f(p \circ 0^{1-x} \xi) d\mu_\lambda(x)(\xi) \int_{M_0} \left(\int_{\partial M} h(p \circ 0^{1-x} \xi) d\mu_\lambda(y)(\xi)\right)$$

$$\sim (1+\varepsilon) \Omega_\lambda \int_{\partial M} f(p \circ 0^{1-x} \xi) d\mu_\lambda(x)(\xi) \int_{M_0} \left(\int_{\partial M} h(p \circ 0^{1-x} \xi) d\mu_\lambda(y)(\xi)\right),$$

we obtain, with the notations of Theorem 4.3.

**Corollary 4.6.** Given $\varepsilon' > 0$ and positive Hölder continuous functions $f, h$ on $S M$, there is $R(f, h, \varepsilon')$ and $\delta(f, h, \varepsilon')$ such that if $R > R(f, h, \varepsilon')$ and $\lambda_0 - \lambda < \delta(f, h, \varepsilon')$, for all $x \in M$,

$$e^{-R P(\lambda)} \int_{S(x, R)} f(p v^y_p)(h(p v^x_p)) G^2_\lambda(x, y) \, dy \sim (1+\varepsilon')^4$$

$$\Omega \int_{\partial M} f(Dp \circ 0^{1-x} \xi) d\mu_\lambda(x)(\xi) \int_{M_0} \left(\int_{\partial M} h(p \circ 0^{1-x} \xi) d\mu_\lambda(y)(\xi)\right) \, d\text{Vol}(y).$$

We have to take $\delta(f, h, \varepsilon') < \delta_1(f, h, \varepsilon')$ and apply Theorem 4.3.

Letting $f = 1$ in Corollary 4.6 we obtain the convergence announced in the introduction.

**Corollary 4.7.** Fix $x \in \tilde{M}$. As $R \rightarrow \infty$ and $\lambda \rightarrow \lambda_0$, the measures $m_{x, \lambda, R}$ of the introduction converge to the measure $m_x$ on $S M$ given by, for any continuous function $h$ on $C(S M)$,

$$\int_{S M} h \, dm_x = \Omega \mu_{\lambda_0}(\partial \tilde{M}) \int_{M_0} \left(\int_{\partial M} h(p \circ 0^{1-x} \xi) d\mu_{\lambda_0}(\xi)\right) \, d\text{Vol}(y).$$
Letting $h = 1$ in Corollary 4.6 we get:

**Corollary 4.8.** Fix $x \in \tilde{M}$. Given $\varepsilon > 0$ and a positive Hölder continuous functions $f$ on $SM$, there is $R(f, \varepsilon')$ and $\delta(f, \varepsilon')$ such that if $R > R(f, \varepsilon')$ and $\lambda_0 - \lambda < \delta(f, \varepsilon')$,

$$e^{-RP(\lambda)} \int_{S(x, R)} f(p v_x^y G_\lambda^2(x, y) dy \sim (1 + \varepsilon')^4 \int_{\partial M} f(p \circ \pi_x^{-1} \xi) d\mu^\lambda(x, \xi).$$

In particular, for $\lambda = \lambda_0$, 

$$\lim_{R \to \infty} e^{-RP(\lambda_0)} \int_{S(x, R)} f(p v_x^y G_{\lambda_0}^2(x, y) dy = \Omega \int_{\partial M} f(p \circ \pi_x^{-1} \xi) d\mu^\lambda(x, \xi).$$

Observe also that in the proof of Theorem 4.3, the choice of $\delta(f, h, \varepsilon')$ is only made in Step 5, when we want to use the uniform mixing of Proposition 4.1. For a fixed $\lambda$, we can use instead the regular mixing of $m_\lambda$ for Hölder continuous functions and obtain a proof of Proposition 2.9. We can write, taking $f = h = 1$:

**Corollary 4.9.** In Proposition 2.9, the limit $D(x, \lambda)$ is given by

$$D(x, \lambda) = \Omega \mu^\lambda(\tilde{\partial M}) \int_{M_0} \int_{\partial M} d\mu^\lambda(y) d\text{Vol}(y) = \Omega \mu^\lambda(\tilde{\partial M}).$$

As a Corollary of the proof of Theorem 4.3 and Corollary 4.6 we state a generalization which will be needed in Section 6.1.

**Proposition 4.10.** Given $\varepsilon > 0$ and positive Hölder continuous functions $f, u$ on $SM$, there is $R(f, u, \varepsilon)$ and $\delta(f, u, \varepsilon)$ such that if $R > R(f, u, \varepsilon)$ and $\lambda_0 - \lambda < \delta(f, u, \varepsilon)$,

$$e^{-RP(\lambda)} \int_{S(x, R)} f(p v_x^y) \left( \frac{1}{R} \int_0^R u(g_s p v_x^y) ds \right) G_\lambda^2(x, y) dy \sim 1 + \varepsilon \int_{\partial M} f(p \pi_x^{-1} \xi) d\mu^\lambda(\xi) \int_{SM} u \ d\mu_\lambda.$$ 

By the same argument as in the proof of Theorem 4.3 and Corollary 4.6, it is enough to show:

**Proposition 4.11.** Given $\varepsilon > 0$ and positive Hölder continuous functions $f, u$, there exist $R_1 = R_1(f, u, \varepsilon)$, $\delta_1(f, u, \varepsilon) > 0$ and $\delta', 0 < \delta' < \delta_0$, such that for $R \geq R_1$, all $x, y \in M$ and all $\lambda \in [\lambda_0 - \delta_1, \lambda_0]$,

$$\sum_{\{(v, t) : v \in S_{\mu} M \cap \gamma^{-1} S_{\mu} M\}} \lambda'' \gamma(R - t) f(v) \left( \frac{1}{R} \int_0^R u(g_s v) ds \right) (\theta^{-2} - g_v) \frac{d\mu^s_{\lambda}}{d\gamma_{\mu}^s}(g_R v) \sim 1 + \varepsilon' \lambda'' \Omega \Gamma \left( \int_{\partial M} f(p \pi_x^{-1} \xi) d\mu^\lambda(\xi) \right) \left( \int_{SM} u \ d\mu_\lambda \right).$$

**Proof.** We choose the same $\varepsilon$ such that $(1 + \varepsilon)^62 \leq 1 + \varepsilon'$. We choose $\delta'_1 < \delta'$ small enough that, for all $t > 0$, if $v, w \in SM$ are such that $d(v, w) < \delta'_1$ and $d(g_v v, g_w w) < \delta'_1$, then

$$\int_0^t u(g_s v) ds \sim 1 + \varepsilon \int_0^t u(g_s v) ds.$$
This is possible because $u$ is Hölder continuous and the two geodesics $g_xv, g_xw$ satisfy

$$d_{SM}(g_xv, g_xw) \leq C \delta'_1 \max\{e^{-a_0s}, e^{a_0(s-t)}\},$$

where $C, a_0$ are positive geometric constants. We then construct $F^\pm_\lambda, H^\pm_\lambda$ in the same way, with this new $\delta'_1$ (and accordingly possibly new $\beta_0, \beta_0', \gamma$). Properties (1) to (4) still hold. In the relation (4.5), we multiply the middle term by $\frac{1}{R} \int_0^R u(g_xv)ds$. We then loose one more factor $1 + \varepsilon$ in relation (4.5) if we replace $C_\pm(v,t)$ by the corresponding contributions to the integrals

$$\int F^\pm_\lambda(w) \left(\frac{1}{R} \int_0^R u(g_xw)ds\right) H^\pm_\lambda(w) dm_\lambda(w).$$

For $R$ large enough, the new Property (5) reads as:

$$\frac{(1 + \varepsilon)^{-33}}{\Omega_\lambda \gamma \beta_0 \beta'_0} \int F^-(w) \left(\frac{1}{R} \int_0^R u(g_xw)ds\right) H_\lambda^{-}(g_Rw) dm_\lambda(w) \leq \sum_{\{(v,t) \in S_{pg} M \cap g_{-t} S_{py} M\}} \chi_{\phi}(R - t)f(v) \left(\frac{1}{R} \int_0^R u(g_xw)ds\right) (\theta^{-2})(-g_tv) \frac{d\mu^{\alpha \Omega_\lambda}}{d\mu^{\alpha \lambda}}(v) \leq \frac{(1 + \varepsilon)^{33}}{\Omega_\lambda \gamma \beta_0 \beta'_0} \int F^+_\lambda(w) \left(\frac{1}{R} \int_0^R u(g_xw)ds\right) H^+_\lambda(g_Rw) dm_\lambda(w).$$

We conclude as above, using Proposition 4.2 instead of Proposition 4.1. \hfill \Box

5. Topological pressure at $\lambda_0$

In this section, we show that $P(\lambda_0) = 0$ and show direct consequences. We already know that $P(\lambda_0) \leq 0$ by Corollary 3.9. As in Section 7 in [GL], we show below in Proposition 5.1 that if $P(\lambda_0) < 0$, then $\int_{S(x,R)} G_\lambda^2(x,y) dy$ decays exponentially with $R$ and conclude that $G_{\lambda_0 + \varepsilon}(x,y)$ is finite, contradicting the definition of $\lambda_0$. Remark that instead of constructing a branching random walk as in [GL], we have a more direct argument.

**Proposition 5.1.** $P(\lambda_0) = 0$.

**Proof.** Assume that $P(\lambda_0) < 0$. We claim that for all $x \neq y$, there exists $\varepsilon > 0$ such that the function $\lambda \mapsto G_\lambda(x,y)$ admits a real analytic extension on an $\varepsilon$-neighborhood of $\lambda_0$. In particular, for $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, the extension $G_\lambda(x,y)$ satisfies $G_\lambda(x,y) = \int_0^\infty e^{\lambda t} \phi(t, x, y) dt$, a contradiction with the definition of $\lambda_0$.

Let us now prove our claim. Fix $x \neq y \in \tilde{M}$. By Proposition 2.1

$$\frac{\partial^k}{\partial \lambda^k} G_\lambda(x,y) = k! \int_{\tilde{M}^k} G_\lambda(x_1)G_\lambda(x_1, x_2) \cdots G_\lambda(x_k, y) d\text{Vol}^k(x_1, x_2, \ldots, x_k).$$
The claim follows with $\varepsilon = 1/\rho$, if we show that there are positive numbers $\delta, C$ and $\rho$ such that:

\[(5.1)\quad F_k := \int_{M^k} G_{\lambda_0}^k(x_1, x_2) \cdots G_{\lambda_0}(x_k, y) e^{\delta d(x, x_k)} \, d\text{Vol}^k(x_1, x_2, \ldots, x_k) \leq C \rho^k.
\]

Since $P(\lambda_0) < 0$, by Theorem 4.3 there is $C, \delta > 0$ such that, for all $x \in \tilde{M}$, all $R > 1$,

\[\int_{S(x, R)} G_{\lambda_0}^2(x, z) \, dz \leq C e^{-\delta R} \quad \text{and thus} \quad \int_{\{y \in \tilde{M} : d(x, y) \geq 2\}} G_{\lambda_0}^2(x, y) \, d\text{Vol}(y) < +\infty.
\]

By possibly choosing a smaller $\delta > 0$, we prove in the same way the following relations:

\[(5.2)\quad \int_{\tilde{M}} G_{\lambda_0}(x, z) G_{\lambda_0}(z, y) e^{\delta d(x, z)} \, d\text{Vol}(z) \leq B(x, y)
\]

\[(5.3)\quad \int_{\{y \in \tilde{M} : d(x, y) \geq 2\}} G_{\lambda_0}^2(x, y) e^{\delta d(x, y)} \, d\text{Vol}(y) \leq B(x),
\]

for some function $B(x, y)$ and a bounded function $B(x)$. For this choice of $\delta$, we prove (5.1) by induction on $k$. Relation (5.2) is $F_1 < +\infty$. We are going to show that $F_{k+1}/F_k$ is bounded independently of $k$ (compare [GL], Proposition 4.7). We write:

\[F_{k+1} = \int_{\tilde{M}} \int_{\tilde{M}^k} G_{\lambda_0}(x, x_1) G_{\lambda_0}(x_1, x_2) \cdots G_{\lambda_0}(x_k, z) G_{\lambda_0}(z, y) e^{\delta d(x, x_k)} \, d\text{Vol}^k(x_1, \ldots, x_k) \, d\text{Vol}(z).
\]

Relation (5.1) follows from Lemma 5.2. \hfill \Box

**Lemma 5.2.** There is $\rho > 0$ such that, for all $x_k \in \tilde{M}$,

\[\int_{\tilde{M}} G_{\lambda_0}(x_k, z) G_{\lambda_0}(z, y) e^{\delta d(x, z)} \, d\text{Vol}(z) \leq \rho \ G_{\lambda_0}(x_k, y) e^{\delta d(x, x_k)}.
\]

**Proof.** Let $L$ be the geodesic segment going from $y$ to $x_k$ (the RHS is infinite if $x_k = y$). Let $pr(z)$ be the point of $L$ realizing $d(z, pr(z)) = d(z, L)$.

If $d(z, pr(z)) \leq 2$, we have $G_{\lambda_0}(x_k, z) G_{\lambda_0}(z, y) \leq CG_{\lambda_0}(x_k, y)$ by Harnack inequality and the easy side of the Ancona inequality. On the other hand, it is clear that

\[\int_{\{z, d(pr(z), L) \leq 2\}} e^{\delta d(x, z)} \, d\text{Vol}(z) \leq C' e^{\delta d(y, x_k)} \leq C' e^{\delta d(x, y)} e^{\delta d(x, x_k)}.
\]

For $d(z, pr(z)) \geq 2$, consider the thin geodesic triangles $(y, z, pr(z))$ and $(x_k, z, pr(z))$. The distance from $pr(z)$ to both geodesics $[z, y]$ and $[z, x_k]$ is bounded by the hyperbolicity constant. Using Harnack inequality and the hard side of the Ancona-Gouëzel inequality, we get

\[G_{\lambda_0}(x_k, z) \leq CG_{\lambda_0}(x_k, pr(z)) G_{\lambda_0}(pr(z), z)
\]

\[G_{\lambda_0}(z, y) \leq CG_{\lambda_0}(z, pr(z)) G_{\lambda_0}(pr(z), y).
\]
Therefore, we have
\[
\int_{\{z \in M : d(pr(z), L) \geq 2\}} G_{\lambda_0}(x, z)G_{\lambda_0}(z, y)e^{\delta d(x, z)} \, d\text{Vol}(z)
\]
\[
\leq C \int_{\{z \in M : d(pr(z), L) \geq 2\}} G_{\lambda_0}(x, pr(z))G_{\lambda_0}(pr(z), y)e^{\delta d(x, pr(z))}G_{\lambda_0}^2(z, pr(z))e^{\delta d(z, pr(z))} \, d\text{Vol}(z)
\]
\[
\leq CG_{\lambda_0}(x, y) \int_{\{z \in M : d(pr(z), L) \geq 2\}} e^{\delta d(x, pr(z))}G_{\lambda_0}^2(z, pr(z))e^{\delta d(z, pr(z))} \, d\text{Vol}(z).
\]

We use Fubini Theorem and the function \(\psi : \mathbb{R} \to \mathbb{R}, \psi(t) = \max(1 - |t|, 0)\). We obtain
\[
\int_{\{z \in M : d(pr(z), L) \geq 2\}} e^{\delta d(x, pr(z))}G_{\lambda_0}^2(z, pr(z))e^{\delta d(z, pr(z))} \, d\text{Vol}(z)
\]
\[
= \int_{\{z \in M : d(pr(z), L) \geq 2\}} \int_{-1}^{d(y, x_k)+1} \psi(t - d(y, pr(z)))e^{\delta d(x, pr(z))}G_{\lambda_0}^2(z, pr(z))e^{\delta d(z, pr(z))} \, dt \, d\text{Vol}(z)
\]
\[
= \int_{-1}^{d(y, x_k)+1} \int_{\{z \in M : d(pr(z), L) \geq 2\}} \psi(t - d(y, pr(z)))e^{\delta d(x, pr(z))}G_{\lambda_0}^2(z, pr(z))e^{\delta d(z, pr(z))} \, d\text{Vol}(z) \, dt
\]
\[
\leq C \int_{-1}^{d(y, x_k)+1} e^{\delta t}B(w(t)) \, dt \leq Ce^{\delta d(y, x_k)} \leq Ce^{\delta d(x, x_k)},
\]
where the first inequality comes from the fact that \(d(x, pr(z)) \leq d(x, y) + d(y, pr(z)) = C + t\).

\[\square\]

**Proposition 5.3.** For \(x, y \in \tilde{M}\),
\[
\lim_{\lambda \to \lambda_0} -P(\lambda) \frac{\partial}{\partial \lambda} G_{\lambda}(x, y) = \Omega c(x, y),
\]
where \(c(x, y)\) is given by
\[
c(x, y) = \int k_{\lambda_0}(x, y, \xi)d\mu_{\lambda_0}^x(\xi).
\]

**Proof.** We have:
\[
-P(\lambda) \frac{\partial}{\partial \lambda} G_{\lambda}(x, y) = -P(\lambda) \int_{\tilde{M}} G_{\lambda}(x, z)G_{\lambda}(y, z) \, d\text{Vol}(z)
\]
\[
= -P(\lambda) \int_0^\infty e^{P(\lambda)R} \left( \int_{S(x, R)} e^{-P(\lambda)R}k_{\lambda}(x, y, z)G_{\lambda}^2(x, z) \, dz \right) \, dR.
\]

On the other hand, as \(R \to \infty\), the function \(k_{\lambda}(x, y, z)\) is close to \(k_{\lambda}(x, y, (v_z^*)^+)\) uniformly in \(\lambda\) (Theorem 1.4). By Proposition 1.3 and Corollary 1.8 for \(R\) large enough and \(\lambda\) close enough to \(\lambda_0\),
\[
\int_{S(x, R)} e^{-P(\lambda)R}k_{\lambda}(x, y, z)G_{\lambda}^2(x, z) \, dz \sim^{1 + \epsilon} \Omega \int k_{\lambda}(x, y, \xi)d\mu_{\lambda_0}^x(\xi) = \Omega c(x, y).
\]
As \( \lambda \to \lambda_0 \), \( P(\lambda) \to P(\lambda_0) = 0 \), it follows that

\[
\lim_{\lambda \to \lambda_0} - P(\lambda) \frac{\partial}{\partial \lambda} G_\lambda(x, y) = \lim_{\lambda \to \lambda_0, R \to \infty} \int_{S(x, R)} e^{-P(\lambda)R} k_\lambda(x, y, z) G_\lambda^2(x, z) dz = \Omega c(x, y).
\]

Recall that \( \frac{\partial}{\partial \lambda} G_\lambda(x, x) \) was defined in Corollary 2.11. \( \square \)

In particular, since \( \Omega \) and \( c(x, y) \) are positive numbers, \( \frac{\partial}{\partial \lambda} G_\lambda(x, y) \) goes to infinity as \( \lambda \to \lambda_0 \).

Theorem 1.5 follows from Proposition 2.5, proof of Corollary 3.10 and Theorem 5.1 by letting \( \lambda \to \lambda_0 \).

Theorem 1.6 follows by the following lemma.

Lemma 5.4. The energy functional \( \mathcal{E} \) is strictly convex.

Proof. Let \( \mu \) be a conformal family of measures with Radon-Nikodym derivative \( \frac{d\mu}{d\mu_x}(\xi) = k(x, y, \xi) \). Let \( \mu^t = t f_1 \mu + (1 - t) f_2 \mu \) for non-negative function \( f_i \) and \( 0 \leq t \leq 1 \). Then

\[ k_t(x, y, \xi) := \frac{d\mu^t}{d\mu_x}(\xi) = \frac{tf_1(y, \xi) + (1 - t) f_2(y, \xi)}{tf_1(x, \xi) + (1 - t) f_2(x, \xi)} k(x, y, \xi). \]

\[ \nabla_y \ln k_t|_{y=x} = \ln \frac{t \nabla f_1(y, \xi)|_{y=x} + (1 - t) \nabla f_2(y, \xi)|_{y=x}}{tf_1(x, \xi) + (1 - t) f_2(x, \xi)} + \nabla_y \ln k(x, y, \xi)|_{y=x}. \]

Let us omit \( y = x \) notation. We have

\[
\int_{\partial M} ||\nabla \ln k_t(x, y, \xi)||^2 d\mu^t_x(\xi) = \int_{\partial M} ||\nabla \ln k(x, y, \xi)||^2 d\mu^t_x(\xi) + \frac{2}{t} \int (\nabla f_1, \nabla \ln k) + (1 - t) (\nabla f_2, \nabla \ln k) d\mu(\xi)
\]

\[
+ \int \frac{||t \nabla f_1(x, \xi) + (1 - t) \nabla f_2(x, \xi)||^2}{tf_1(x, \xi) + (1 - t) f_2(x, \xi)} d\mu(\xi)
\]

The first two terms are linear and the third term is strictly convex. \( \square \)

Finally with Corollary 4.8, we obtain

**Proposition 5.5.** For \( x, y \in \tilde{M} \), as \( R \to \infty \), we have, with the above notations

\[
\int_{S(x, R)} G_\lambda^2(x, z) dz \to \Omega \mu_x^{\lambda_0}(\partial \tilde{M}), \quad \int_{S(x, R)} G_\lambda(x, z) G_{\lambda_0}(y, z) dz \to \Omega c(x, y),
\]

and

\[
\int_{S(x, R)} f(v_x^2) G_\lambda^2(x, z) dz \to \Omega \int_{\partial \tilde{M}} f(p \pi_x^{-1}(\xi)) \mu_x^{\lambda_0}(\xi).
\]

Observe that the last limit can serve as another definition of the \( \mu_x^{\lambda_0} \).
6. Proof of Theorem 1.1

6.1. Derivative of the Green function. In this subsection, we establish Theorem 6.1. Fix $x \in \Tilde{M}$. There is a constant $\Upsilon = \Upsilon_{\lambda_0}$, given by (2.11), such that, as $\lambda \to \lambda_0$,

$$\frac{\partial}{\partial \lambda} G_{\lambda}(x, y) \sim \frac{\sqrt{\Upsilon}}{2\sqrt{\lambda_0 - \lambda}} c(x, y).$$

where $c(x, y)$ is given by (5.4).

Theorem 6.1 follows from the following Proposition.

Proposition 6.2. For $x, y \in \Tilde{M}$, as $\lambda \to \lambda_0$,

$$-\frac{1}{2} P^3(\lambda) \frac{\partial^2}{\partial \lambda^2} G_{\lambda}(x, y) \text{ converges to } 2\Omega^3 / \Upsilon c(x, y).$$

Proof. (Compare with [G1] Section 3.6.) Recall that we defined $\frac{\partial}{\partial \lambda} \left( \frac{\partial}{\partial \lambda} G_{\lambda}(x, x) \right)$ in Corollary 2.11. By Proposition 2.1, $-\frac{1}{2} P^3(\lambda) \frac{\partial^2}{\partial \lambda^2} G_{\lambda}(x, y)$ is equal to

$$-P^3(\lambda) \int_{\Tilde{M}} \int_{\Tilde{M}} G_{\lambda}(x, z) G_{\lambda}(w, y) d\text{Vol}(w) d\text{Vol}(z)$$

$$= -P^3(\lambda) \int_{\Tilde{M}} G_{\lambda}^2(x, z) G_{\lambda}(w, y) \left( \int_{\Tilde{M}} G_{\lambda}(w, z) G_{\lambda}(y, z) d\text{Vol}(w) \right) d\text{Vol}(z)$$

$$= P^2(\lambda) \int_0^\infty \text{Re} P^3 R \left( \int_{S(x, R)} e^{-P^3 R} G_{\lambda}^2(x, z) k_{\lambda}(x, y, z) \Psi_{\lambda}(x, y, z) dz \right) dR,$$

where

$$\Psi_{\lambda}(x, y, z) = \frac{1}{d(x, z)} \left( -P(\lambda) \int_{\Tilde{M}} G_{\lambda}(w, z) G_{\lambda}(y, z) d\text{Vol}(w) \right).$$

As before, as $\lambda \to \lambda_0$, $P(\lambda) \to 0$ and the above integral converges towards

$$\lim_{R \to \infty, \lambda \to \lambda_0} \int_{S(x, R)} e^{-P^3 R} G_{\lambda}^2(x, z) k_{\lambda}(x, y, z) \Psi_{\lambda}(x, y, z) dz$$

if the limit exists uniformly. We study this limit. First we study $\Psi_{\lambda}(x, y, z)$.

Lemma 6.3. There is a Hölder continuous positive functions $u$ on $SM$ such that for large $d(x, z)$ and $\lambda$ close to $\lambda_0$,

$$\Psi_{\lambda}(x, y, z) \sim \frac{1}{1 + \varepsilon} \Omega \frac{1}{d(x, z)} \int_0^{d(x, z)} u(g_s v^2_s) ds.$$

Proof. For $w \in \Tilde{M}$, write $pr(w)$ for the projection of $w$ on the geodesic segment from $x$ to $z$ and $s(w)$ for the distance $d(pr(w), x)$. Set $\psi(v, w) := \psi(d(pr(w), \pi(v))) = \max\{|1 -$
\[ d(pr(w), \pi(v)), 0 \}. \] We can write
\[
-P(\lambda) \int_{pr(w) \in N_1(z)} \frac{G_\lambda(w, z)G_\lambda(w, y)}{G_\lambda(z, y)} d\text{Vol}(w)
= -P(\lambda) \left( \int_{pr(w) \in N_1(z)} + \int_{pr(w) \notin N_1(z)} \right) \frac{G_\lambda(w, z)G_\lambda(w, y)}{G_\lambda(z, y)} d\text{Vol}(w),
\]
where \( K = K(x, y, z) \) is such that the angle between the vectors \( u_{pr(w)}^x \) and \( v_{pr(w)}^y \) is small enough if \( pr(w) \notin N_K(x) \). The first integral is bounded by Ancona inequality: for example, for \( w \) such that \( pr(w) \in N_1(z) \) of \( z \), we have
\[
-P(\lambda) \int_{pr(w) \in N_1(z)} \frac{G_\lambda(w, z)G_\lambda(w, y)}{G_\lambda(z, y)} dw
= -P(\lambda) \int_{pr(w) \in N_1(z)} G_\lambda^2(w, z) \frac{G_\lambda(w, y)}{G_\lambda(z, y)} G_\lambda(z, w) dw
\leq -P(\lambda) C \int_{pr(w) \in N_1(z)} G_\lambda^2(w, z) dw,
\]
which is bounded by Proposition 5.3. The argument is similar for \( pr(w) \in N_K(x) \).

The second part is
\[
-P(\lambda) \int_{-1/2}^{d(z, x)+1/2} \left( \int_{M} \psi(s - s(w)) \frac{G_\lambda(w, z)G_\lambda(w, y)}{G_\lambda(z, y)} d\text{Vol}(w) \right) ds
= \int_{1/2}^{d(z, x)-1/2} \bar{\pi}_\lambda(g_s v^z_x) ds
\]
where \( \bar{\pi}_\lambda \) is the \( \Gamma \)-invariant function on \( S\bar{M} \) given by
\[
\bar{\pi}_\lambda(v) = -P(\lambda) \int_M \psi(v, w) \frac{G_\lambda(w, z)G_\lambda(w, y)}{G_\lambda(z, y)} d\text{Vol}(w)
= -P(\lambda) \int_0^\infty e^{-P(\lambda) R} \left( e^{-P(\lambda) R} \int_{S(\pi(v), R)} G_\lambda^2(\pi(v), w) \psi(v, w) \frac{G_\lambda(w, z)G_\lambda(w, y)}{G_\lambda(z, y)} G_\lambda(z, w) d\text{Vol}(w) \right) dR.
\]
We are reduced to show that \( \bar{\pi}_\lambda(v) \to \Omega u(v) \) as \( \lambda \to \lambda_0 \).

By Corollary 3.73, given \( \varepsilon > 0 \), our choice of \( K > 0 \) is such that if \( pr(w) \notin N_K(x) \cup N_1(z) \) and \( pr(w) \) is far from \( w \), then
\[
\frac{G_\lambda(w, z)G_\lambda(w, y)}{G_\lambda(z, y)G_\lambda^2(\pi(v), w)} \sim \frac{\theta_{\pi(v)}^\lambda(w, z)\theta_{\pi(v)}^\lambda(w, y)}{\theta_{\pi(v)}^\lambda(y, z)} \sim 1 + \varepsilon \frac{\theta_{\pi(v)}^\lambda(\zeta, v^+)\theta_{\pi(v)}^\lambda(\zeta, v^-)}{\theta_{\pi(v)}^\lambda(v^-, v^+)} ,
\]
where \( \zeta = (v^w_{\pi(v)})^+ \) (See Figure 5).

Extend the projection \( pr \) to the boundary \( \partial M \). Then for \( w \) such that \( pr(w) \notin N_K(x) \cup N_1(z) \), if \( R \) is large enough, \( \psi(v, w) \sim 1 + \varepsilon \psi(v, \zeta) \). Also, the functions \( d_{\pi(v)}(\zeta, v^+) \) are bounded away from 0 and the function \( \theta_{\pi(v)}^\lambda(\zeta, v^+)\theta_{\pi(v)}^\lambda(\zeta, v^-) \) is uniformly Hölder and
bounded away from 0. The denominator $\theta_{\lambda_0}(v)$ is also Hölder. By Corollary 4.8 centered at $\pi(v)$, there is $R(\varepsilon)$ and $\delta(\varepsilon)$ such that for $R \geq R(\varepsilon)$, $\lambda \in [\lambda_0 - \delta(\varepsilon), \lambda_0]$, setting

$$u_{\lambda,R}(v) := e^{-P(\lambda)R} \int_{S(\pi(v), R)} G_\lambda^2(\pi(v), w) \psi(v, w) \frac{G_\lambda(w, z)G_\lambda(z, y)}{G_\lambda(\pi(v), w)} dw,$$

we have $u_{\lambda,R}(v) \sim^{1+\varepsilon} \Omega u(v)$, where

$$u(v) = \int_{\partial M} \psi(v, \zeta) \frac{\theta_{\pi(v)}(\zeta, v^+)}{\theta(v)} \frac{\theta_{\pi(v)}(\zeta, v^-)}{\theta(v)} d\mu_{\pi(v)}(\zeta).$$

The Lemma follows, since $P(\lambda) \to 0$ and $\pi_{\lambda}(v) = -P(\lambda) \int_0^\infty e^{P(\lambda)R} u_{\lambda,R}(v) dR$. □

Note that

$$\int_{S_{M_0}} u(v) dm_{\lambda}(v) = \int \int_{\partial M} \psi(v, \zeta) \frac{\theta_{\pi(v)}(\zeta, v^+)}{\theta(v)} \frac{\theta_{\pi(v)}(\zeta, v^-)}{\theta(v)} d\mu_{\pi(v)}(\zeta) dm_{\lambda}(v)$$

$$= \int \int_{\partial M} \psi(v, \zeta) \frac{\theta_{\pi(v)}(\zeta, v^+)}{\theta(v)} \frac{\theta_{\pi(v)}(\zeta, v^-)}{\theta(v)} d\mu_{\pi(v)}(\zeta) \Omega \theta^2(v) d\mu_{\pi(v)}(v^-) d\mu_{\pi(v)}(v^+) dt$$

$$= \Omega/\Upsilon_{\lambda} \int_{\partial M} \int_{(v^-, v^+, t) \in S_{M_0}} \psi(v, \zeta) dt d\rho_{\pi(v)}(v^+, v^-, \zeta)$$

$$= \Omega/\Upsilon_{\lambda} \rho^2_{\pi}(S^2 M) = \Omega/\Upsilon_{\lambda}.$$

We are reduced to understanding the limit

$$\lim_{R \to \infty, \lambda \to \lambda_0} \int_{S(x, R)} e^{-P(\lambda)R} G_\lambda^2(x, z) k_{\lambda}(x, y, z) \left( \frac{1}{R} \int_0^R u(g_s v_z^z) ds \right) dz.$$

As before, we can replace $k_{\lambda}(x, y, z)$ by $k_{\lambda_0}(x, y, \pi_x(v_z^z))$ for $R$ sufficiently large and $\lambda$ close to $\lambda_0$. By Proposition 4.10 for $R$ large and $\lambda_0 - \lambda$ small,

$$\int_{S(x, R)} e^{-P(\lambda)R} G_\lambda^2(x, z) k_{\lambda_0}(x, y, \pi_x(v_z^z)) \left( \frac{1}{R} \int_0^R u(g_s v_z^z) ds \right) dz \sim \Omega^2 c(x, y) \int_{S_{M_0}} u dm_{\lambda_0}$$

Figure 5. Approximating by Naim kernels.
by \([6.4]\). This achieves the proof of Proposition \([6.2]\). □

Now to finish the proof of Theorem \([6.1]\) for \(x \neq y\), set \(F(\lambda) = \frac{\partial}{\partial \lambda} G_\lambda(x, y)\). By Proposition \([5.3]\) and Proposition \([6.2]\),

\[
\lim_{\lambda \to \lambda_0} - P(\lambda) F(\lambda) = \Omega \ c(x, y) \quad \lim_{\lambda \to \lambda_0} - P^2(\lambda) F'(\lambda) = 2 \frac{\Omega^3}{\lambda} c(x, y).
\]

It follows that \(\frac{2F'(\lambda)}{F(\lambda)^2}\) converge towards \(\frac{4}{\lambda} (c(x, y))^{-2}\). Since \(F(\lambda)\) goes to \(\infty\) as \(\lambda \to \lambda_0\), we conclude that \(F(\lambda) \sim \frac{\sqrt{T}}{\sqrt{\lambda_0 - \lambda}}\). This is Theorem \([6.1]\).

For \(x = y\), we define \(F(\lambda) = \frac{\partial}{\partial \lambda} G_\lambda(x, x) := \lim_{y \to x} \frac{\partial}{\partial \lambda} (x, y)\), then the proof of Proposition \([6.2]\) gives \(\lim_{\lambda \to \lambda_0} - P^2(\lambda) F'(\lambda) = (2\Omega^3/\lambda) c(x, x)\). Note also that \(\lim_{\lambda \to \lambda_0} - P(\lambda) F(\lambda) = \Omega c(x, x)\). Theorem \([6.1]\) follows.

6.2. Proof of Theorem \([1.1]\) and Theorem \([1.7]\) Let \(F\) be a nonnegative continuous function on \(\bar{M}\), with compact support. Set \(\mu_F\) for the spectral measure of \(F\), i.e. the Borel finite measure on the spectrum \([0, +\infty)\) of \(-\Delta - \lambda_0\) such that, for all \(m \geq 0\),

\[
\int_{\bar{M}} F(x) \Delta^m F(x) \, d\Vol(x) = \int_0^{+\infty} (\omega - \lambda_0)^m \, d\mu_F(\omega).
\]

The function\( c_F(t) := \int_{\bar{M} \times \bar{M}} e^{\lambda_0 t} \varphi(t, x, y) F(x) F(y) \, d\Vol(x) \, d\Vol(y) = \int_0^{+\infty} e^{-\omega t} \, d\mu_F(\omega)\) is nonincreasing in \(t\). By Theorem \([6.1]\) we have, as \(s \to 0\),

\[
\int_0^{+\infty} e^{-st} t c_F(t) \, dt = \int_{\bar{M} \times \bar{M}} \int_0^{+\infty} t e^{(\lambda_0 - s)t} \varphi(t, x, y) \, dF(x) F(y) \, d\Vol(x) \, d\Vol(y)
\]

\[
= \int_{\bar{M} \times \bar{M}} \frac{\partial G}{\partial \lambda} \big|_{\lambda = \lambda_0 - s} (x, y) F(x) F(y) \, d\Vol(x) \, d\Vol(y)
\]

\[
\sim \frac{\sqrt{T}}{2\sqrt{\lambda_0 - \lambda}} \int_{\bar{M} \times \bar{M}} c(x, y) F(x) F(y) \, d\Vol(x) \, d\Vol(y).
\]

By Hardy-Littlewood Tauberian Theorem (\([F]\) p. 445), we have

\[
\int_0^T t c_F(t) \sim \frac{\sqrt{T}}{2\Gamma(3/2)} \sqrt{T} \int_{\bar{M} \times \bar{M}} c(x, y) F(x) F(y) \, d\Vol(x) \, d\Vol(y).
\]

Since the function \(t \mapsto c_F(t)\) is nonincreasing, it follows that (\([GL]\) Lemma 9.5)

\[
c_F(t) \sim \frac{\sqrt{T}}{4\Gamma(3/2)T^{3/2}} \int_{\bar{M} \times \bar{M}} c(x, y) F(x) F(y) \, d\Vol(x) \, d\Vol(y).
\]

Since \(c(x, y) = \int k_{\lambda_0}(x, y) \, d\mu_x\), and \(k_{\lambda_0}(x, y)\) is smooth as a \((-\lambda_0)\)-eigenfunction of the Laplacian, the function \(c(x, y)\) is smooth (see the paragraph after Proposition \([3.7]\)). We claim that \(\varphi(t, x, y)\) has bounded gradient. Indeed, if \(u(x, t)\) is a solution of the heat
equation on $\tilde{M}$ with Ricci curvature bounded below, then by a sharp gradient estimate by Souplet and Zhang [SZ], on $\{(x,t) : x \in B(x_0, R/2), s \in [t_0 - T/2, t_0]\}$,
\[
\frac{|\nabla u(x,t)|}{u(x,t)} \leq C \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \right) \left( 1 + \ln \frac{\max u(x,t)}{\min u(x,t)} \right),
\]
where the maximum and minimum are taken on the set $\{(x,t) : x \in B(x_0, R), t \in [t_0 - T, t_0]\}$. Thus it remains to show that for some $T_0 > 0$, $
abla u(x_0)$ is bounded above and below for $x \in B(x_0, 1)$ and for all $t > T_0$. We use parabolic Harnack inequality:

Note that
\[
\rho(\phi(t,x,y)) \geq \int_{B(x_0,1)} \phi(s - 1, x, z) \phi(1, z, y) dz \geq C \int_{B(x_0,1)} \phi(s - 1, x, z) dz,
\]
where $C$ is the minimum of $\phi(1, z, y)$.

Since $\phi(t, x, y)$ has bounded gradients, we can take functions $F$ with compact support converging to $\{x_0\}$ that approximate the Dirac measure $\delta_{x_0}$ to get
\[
\lim_{t \to \infty} t^{3/2} e^{\lambda_0 t} \phi(t, x_0, x_0) = \frac{\sqrt{T}}{2\sqrt{\pi}} c(x_0, x_0).
\]
We get the general case of $x_0 \neq x_1$ (and thus Theorem 1.1) in the same way by considering functions that approximate $\delta_{x_0} + \delta_{x_1}$.

Theorem 1.7 follows from the above equation and definition of $c(x, y)$ (5.4). Finally, by Proposition 5.3 and Theorem 6.1, we obtain

**Corollary 6.4.** As $\lambda \to \lambda_0$,
\[
\frac{P(\lambda)}{\sqrt{\lambda_0 - \lambda}} \to \frac{2\Omega}{\sqrt{T}}.
\]

7. **Uniform mixing**

In this section, we establish a uniform power mixing of the geodesic flow for Gibbs measures, when the potential varies in a neighbourhood of the space $K_0$ of functions which will be defined shortly. The proof combines the ideas from [P1 and P2], with a slightly different framework. For the comfort of the reader, we recall the different steps in our notations.

7.1. **Uniform mixing and three-mixing.** Let $\mathbb{X} := (X, \mathcal{A}, m; \varphi_t, t \in \mathbb{R})$ be a system with one parameter group $\{\varphi_t, t \in \mathbb{R}\}$ of measurable transformations of the space $(X, \mathcal{A})$ preserving a probability measure $m$. For bounded measurable functions $f, h, u$ we define the correlations functions for $s, t \geq 0$:

\[
\rho_{f,h,m}(t) = \int f(x)h(\varphi_x x)dm(x) - \int f dm \int h dm
\]
\[
\rho_{f,u,h,m}(s,t) = \int f(x)u(\varphi_s x)h(\varphi_{s+t} x)dm(x) - \int f dm \int u dm \int h dm
\]
\[
\overline{\rho}_{f,u,h,m}(t) = \frac{1}{t} \int_0^t \left[ \int f(x)u(\varphi_s x)h(\varphi_t x)dm(x) \right] ds - \int f dm \int u dm \int h dm
\]
The system \( X \) is called *mixing* if \( \lim_{t \to \infty} \rho_{f,h,m}(t) = 0 \) for all bounded functions \( f, h \), 3-mixing if \( \lim_{s,t \to \infty} \rho_{f,u,h,m}(s,t) = 0 \) for all bounded functions \( f, u, h \) and *average 3-mixing* if \( \lim_{t \to \infty} \rho_{f,u,h,m}(t) = 0 \) for all bounded functions \( f, u, h \). It is a well-known open problem whether mixing implies 3-mixing. It is easy to see that mixing implies average 3-mixing.

Let us consider the rate of mixing. A system \( X \) is called power mixing for a class \( \mathcal{K} \) of functions if for \( f, h, k \in \mathcal{K}, \rho_{f,h,m}(t) \) decays polynomially (see Theorem 7.2 for a precise statement). Below, we will show a uniform version of power mixing of the geodesic flow for the class \( \mathcal{K} = \mathcal{K}_\alpha \) which we define now.

Let \( \alpha > 0 \). We denote \( \mathcal{K}_\alpha \) the space of functions \( f \) on \( X \) such that \( \|f\|_\alpha < \infty \), where

\[
\|f\|_\alpha := \sup_x |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{(d(x,y))^{\alpha}}.
\]

Let \( \varphi \in \mathcal{K}_\alpha \). There is a unique invariant probability measure \( m_\varphi \) such that:

\[
h_{m_\varphi}(g_1) + \int \varphi \, dm_\varphi = P(\varphi) := \sup_{m_\varphi = \mu} \left\{ h_m(g_1) + \int \varphi \, dm \right\},
\]

where \( h_m(g_1) \) denotes the measure theoretic entropy of the \( g \)-invariant probability measure \( m \) (see e.g. [PP]). The mapping \( \varphi \mapsto m_\varphi \) is continuous from \( \mathcal{K}_\alpha \) to the space of measures on \( X \) endowed with the weak* topology.

The following property is important in Dolgopyat’s approach to the speed of mixing.

**Definition 7.1.** A system \( \mathcal{K} \) is topologically power mixing if there exists \( t_0, \delta > 0 \) such that for any \( r \) and \( t > \max\{\frac{1}{\delta}, t_0\} \),

\[
g_t(B(x,r)) \cap B(y,r) \neq \emptyset.
\]

We now establish a local uniform power mixing for topological power mixing Anosov flows, for Gibbs measures associated to potentials \( \varphi \), and for functions in \( \mathcal{K}_\alpha \). The mixing rate is uniform as we vary the potential \( \varphi \) in a small neighbourhood in \( \mathcal{K}_\alpha \), for \( \alpha \) sufficiently small.

**Theorem 7.2.** Let \( \mathcal{K} \) to be a topological power mixing Anosov flow. Let \( \varphi_0 \in \mathcal{K}_\alpha \) be a potential. There exist \( \varepsilon > 0 \) and \( C'_0, c'_0 > 0 \) such that for all \( \varphi \) in the \( \varepsilon \)-neighborhood of \( \varphi_0 \) and all \( f, u, h \in \mathcal{K}_\alpha \), we have, for all positive \( s, t \):

\[
|\rho_{f,u,h,m_\varphi}(s,t)| \leq C'_0\|f\|_\alpha\|u\|_\alpha\|h\|_\alpha[(1 + s)^{-c'_0} + (1 + t)^{-c'_0}].
\]

**Proposition 7.3.** Let \( \mathcal{K} \) to be a topological power mixing Anosov flow. Let \( \varphi_0 \in \mathcal{K}_\alpha \) be a potential. There exist \( \varepsilon > 0 \) and \( C_0, c_0 > 0 \) such that for all \( \varphi \) in the \( \varepsilon \)-neighborhood of \( \varphi_0 \) and all \( f, h \in \mathcal{K}_\alpha \), we have, for all positive \( t \):

\[
|\rho_{f,h,m_\varphi}(t)| \leq C_0\|f\|_\alpha\|h\|_\alpha(1 + t)^{-c_0}.
\]

Although it appears as if Proposition 7.3 follows from Theorem 7.2, we will prove both results simultaneously.
Corollary 7.4. Let $\mathcal{X}$ be a topological power mixing Anosov flow and $\varphi_0 \in K_\alpha$ be a potential. There exist $\varepsilon > 0$ and $C'_0, c'_0 > 0$ such that for all $\varphi$ in the $\varepsilon$-neighborhood of $\varphi_0$ and all $f, u, h \in K_\alpha$, we have, for all positive $s, t$:

$$\left| \rho_{f, u, h, m_\varphi}(t) \right| \leq C'_0 \|f\|_\alpha \|u\|_\alpha \|h\|_\alpha (1 + t)^{-c'_0}. \tag{7.3}$$

We assume now that the system $\mathcal{X}$ is the geodesic flow $g_t, t \in \mathbb{R}$ on the unit tangent bundle $\mathcal{X} = SM$, where $M$ is a closed negatively curved manifold.

Liverani proved exponential mixing for contact Anosov flows for the Liouville measure, which implies exponential mixing for the geodesic flow on manifolds of negative curvature for the Liouville measure $\mathcal{L}_\text{Li}$. It implies that the geodesic flow is topologically power mixing. Thus we can apply the above theorems to the geodesic flow and the Gibbs measure associated to $\varphi_{\lambda_0}$ to obtain Propositions 4.1 and 4.2.

7.2. Proof of Theorem 7.2 and Proposition 7.3. First, following Bowen and Ruelle $[BR]$, $[PP]$ Chapter 9 and Appendix III, in such a way that H"older continuous functions on $SM$ correspond to H"older continuous functions on the symbolic system.

We may thus assume that there is a subshift of finite type $(\Sigma, \sigma)$ and a positive $\alpha$-H"older continuous function $\tau$ on $\Sigma$ such that the system $\mathcal{X}$ is the suspension flow $\sigma_t(x, r) = (x, r + t)$ on the set $\Sigma^\tau := \{(x, r) : x \in \Sigma, 0 \leq r < \tau(x)\} \cup \{(x, r) : (x, \tau(x)) \sim (sx, 0)\}$. Let us denote by $[a_0, \ldots, a_k]$ the cylinder set $\{x : x_i = a_i, i = 0, \ldots, k\}$. Let us also define $d_\alpha$ on the space $\Sigma_+$ of one-sided sequences with the left-shift by $d_\alpha(x, y) = \alpha^k$, where $k$ is the first index for which $x_k, y_k$ are not equal. Let us denote by $K_\alpha(\Sigma_+)$ the space of $d_\alpha$-Lipschitz functions on the space $\Sigma_+$ of one-sided sequences. Let $\varphi \in K_\alpha(\Sigma^\tau)$ be a potential function on $\Sigma^\tau$.

We may assume that the function $\tau$ is a function on $\Sigma_+$ in the sense that $\tau(x) = \tau(y)$ if the points $x$ and $y$ in $\Sigma$ have the same nonnegative coordinates. Moreover, by possibly modifying $\alpha$, we may assume that the function $\tau$ is a $d_\alpha$-Lipschitz function on $\Sigma_+$. We may also assume that the functions $\phi(x) := \int_0^{\tau(x)} \varphi(x, r) dr$ are $d_\alpha$-Lipschitz functions on $\Sigma_+$ as well and that $\phi$ are normalized so that $L_\phi 1 = 1$, where

$$L_\phi F(x) := \sum_{y : \sigma y = x} e^{\phi(y)} F(y) \tag{7.4}$$

is the transfer operator associated to $\phi$.

It follows from these assumptions that the map $\mathcal{J} : \varphi \mapsto \phi$ is continuous from $K_\alpha(\Sigma^\tau)$ into $K_\alpha(\Sigma_+)$. The equilibrium measure $m_\varphi$ for the function $\varphi$ is of the form

$$m_\varphi = \frac{1}{\int \tau d\nu_\phi (\mathcal{F}_\phi \otimes dr)|_{\Sigma^\tau},}$$
where \( \nu_\phi \) is the unique \( \sigma \)-invariant probability measure on \( \Sigma \) such that its projection \( \nu_\phi \) to \( \Sigma_+ \) satisfies, for all functions \( F \in C(\Sigma_+) \),

\[
\int \mathcal{L}_\phi F \, d\nu_\phi = \int F \, d\nu_\phi.
\]

Let us denote \( \phi_k(x) = \phi(x) + \phi(\sigma(x)) + \cdots + \phi(\sigma^k(x)) \). We choose an \( \varepsilon_1 \)-neighborhood of \( \phi_0 \) so that there exists a constant \( C_1 \geq 1 \) with

\[
\frac{e^{\phi_k(x)} - 1}{e^{\phi_k(y)} - 1} \leq C_1 \alpha^{-k} d_\alpha(x, y) \quad \text{and} \quad C_1^{-1} \leq \frac{\nu_\phi[a_0, \ldots, a_k]}{e^{\phi_k(x)}} \leq C_1
\]

for all \( \phi \) in \( \varepsilon_1 \)-neighborhood of \( \phi_0 \) and all \( x, y \in [a_0, \cdots, a_k-1] \). With these choices, for all \( \phi \), 1 is an isolated eigenvalue of \( \mathcal{L}_\phi \) with eigenfunction the constant 1 (see [PP] for this and all the above standard reductions).

**Remark 7.5.** The constants \( C_1, C_6, C_7, \gamma_3 \) that appear in this section coincide with the constants \( C_1, C_6, C_7, \gamma_3 \) in [Me].

### 7.2.1. Properties of the complex transfer operator

In this subsection, we will denote the space of complex \( d_\alpha \)-Lipschitz continuous functions on \( \Sigma_+ \) by \( \mathcal{K}_\alpha(\Sigma_+) \) again. We define the complex transfer operator \( \mathcal{L}_{\phi+s\tau}, s \in \mathbb{C} \) on \( \mathcal{K}_\alpha(\Sigma_+) \) by

\[
\mathcal{L}_{\phi+s\tau} F(x) := \sum_{y, \sigma y = x} e^{\phi(y) + s\tau(y)} F(y).
\]

Following [Me], set \( s = a + ib \).

**Proposition 7.6.** (Lemma 3.5 of [Me]) Let \( \mathcal{X} \) be a topological power mixing Anosov flow. Let \( \varphi_0 \) be a \( \alpha \)-Hölder continuous function. There are \( \varepsilon > 0, \theta > 0, C > 0 \) such that for all \( s = a + ib \) with \( |b| > 1 \) and \( |a| < C^{-1}|b|^{-\theta} \),

\[
||| (1 - \mathcal{L}_{\phi+s\tau})^{-1} |||_\alpha \leq C|b|^\theta,
\]

for all \( \phi = \mathcal{T}(\varphi) \) with \( ||| \varphi - \varphi_0 |||_\alpha < \varepsilon \).

**Proof.** As in [Me], we carry the calculations for \( 0 \leq a \leq 1 \) and \( b > 1 \). They are analogous for \( b < -1 \) and \( -1 \leq a \leq 0 \). We find a neighborhood \( \mathcal{U} \) of \( \phi_0 \) where (7.7) holds and choose \( \varepsilon \) such that \( ||| \varphi - \varphi_0 |||_\alpha < \varepsilon \) implies that \( \phi = \mathcal{T}(\varphi) \in \mathcal{U} \). We first have the preliminary estimate of \( \mathcal{K}_\alpha \) in a uniform way.

**Lemma 7.7.** (Lemma 3.7 of [Me]) There exist \( C_6, C_7, \gamma_3, \varepsilon_2 > 0 \) such that for all \( \phi \) in the \( \varepsilon_2 \)-neighborhood of \( \phi_0 \),

\[
(1) \ |\mathcal{L}_{\phi+ib\tau}\|_\infty \leq 1,
(2) \ ||| \mathcal{L}_{\phi+ib\tau} F |||_\alpha \leq C_6 \{ |F|_\infty + \alpha^n |F|||_\alpha \} \quad \text{for all } n \leq 1 \text{ and } F \in \mathcal{K}_\alpha(\Sigma_+),
(3) \ ||| \mathcal{L}_{\phi} F - \int_{\Sigma_+} F d\nu_\phi |||_\alpha \leq C_7 \gamma_3 |\alpha| |F|||_\alpha \quad \text{for all } n \leq 1 \text{ and } F \in \mathcal{K}_\alpha(\Sigma_+).
\]

**Proof.** Part (2) comes from the fundamental inequality, thus \( C_6 \) is uniform in \( \phi \). Part (3) comes from the spectral gap of \( \mathcal{L}_\phi \) thus \( C_7 \) and \( \gamma_3 \) can be chosen uniformly in a neighborhood of \( \phi_0 \) (see e.g. Kato [Ka] Theorem IV.3.1). \( \square \)
As in [Me], define
\[ \|f\|_b := \max \left\{ \|f\|_\infty, \frac{1}{2C_6^b} \sup_{x \neq y} \frac{|f(x) - f(y)|}{(d(x,y))^\alpha} \right\}. \]

Since one may assume that \(2C_6^b > 1\), we have
\[ \|F\|_b \leq \|F\|_\alpha \leq (2C_6^b + 1)\|F\|_b, \]
which implies that \(\|L\|_\alpha/\|L\|_b\) lies between \(2C_6^b + 1\) and \((2C_6^b + 1)^{-1}\).

Let \(M_b F = e^{-ib\tau} F \circ \sigma\).

**Definition 7.8.** The operator \(M_b\) has no approximate eigenfunction if there exists \(N \in \mathbb{N}\) such that for every triple \((\theta \geq N, \beta > 0, C \geq 1)\), there exists \(k = k(\theta, \beta, C)\) such that for all \((b, \rho, F)\) with \(\|F\| = 1\) and \(|b| > k, \)
\[ |M_{\beta}^{\alpha,\ln |b|} F(y) - e^{i\rho} F(y)| \geq C|\ln |b||^{-\theta}, \]
for some \(y\).

**Lemma 7.9.** [uniform version of Section 3.2 of [Me]] Consider the following conditions.

1. \(M_b\) has no approximate eigenfunction.
2. There exists \(\theta > 0, \varepsilon > 0\) such that \(\|(I - L_{\phi+ib\tau})\|_b = O(|b|^\theta)\) as \(|b| \to \infty\), for all \(\phi \in B_\varepsilon(\phi_0)\).
3. There exist \(\theta, \varepsilon > 0, \varepsilon > 0\) and \(C \geq 1\) such that \(\|(I - L_{\phi+ib\tau})^{-1}\|_b \leq C|\ln |b||^{-\theta}\) for all \(s = a + ib\) with \(|b| > 1, |a| < \varepsilon|b|^{-\theta}\) and all \(\phi \in B_\varepsilon(\phi_0)\).

With the above notations, (1) implies (2) and (2) implies (3).

**Proof.** See Section 3.2 of [Me]. Let \(\varepsilon_1\) be a constant such that \(C_1\) in equation (7.6) and \(\alpha_1, \alpha_2\) in [Me] are uniform in \(\phi\) in \(\varepsilon_1\)-neighborhood of \(\phi_0\). Now let \(\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}\), where \(\varepsilon_2\) is chosen as in Lemma 7.7. □

Topological power mixing of \(X\) implies that \(M_b\) has no approximate eigenfunction by Sections 3 and 5 of [D2], thus Proposition 7.6 follows. □

We also recall that, by mixing of the geodesic flow, \(\|L_{\phi+ib\tau}\|_\alpha < 1\) for \(b \neq 0\) (see [PP] Proposition 6.2). It follows that:

**Proposition 7.10.** There is \(\delta = \delta_{\phi_0} > 0, \varepsilon > 0\) such that, for all \(\phi\) in a \(\varepsilon\)-neighborhood of \(\phi_0\), the mapping \(s \mapsto \sum_n L^n_{\phi+ib\tau}\) is meromorphic on \(V\)
\[ V := \{s = a + ib : |b| < 2, |a| < \delta\} \]
with a simple pole at \(s = 0\). Moreover, for a function \(K \in \mathcal{K}_\alpha(\Sigma_+),\) the residue at \(s = 0\) of the meromorphic function \(s \mapsto \sum_n L^n_{\phi+ib\tau}K\) (with values in \(\mathcal{K}_\alpha\)) is a constant function with value \(\nu_\phi(K)\).
Proof. For a fixed \( \phi \), this follows from [PP], Proposition 6.2 and Theorem 10.2, with a fixed \( \delta = \delta_\phi \). By [Ka] Theorem IV.3.1 and compactness of the closure \( \overline{V} \), there is a neighborhood \( U_0 \) of \( \phi_0 \) such that for \( \phi \in U_0 \), the rest of the spectrum of \( \mathcal{L}_{\phi,t+\tau} \), \( s \in \overline{V} \), is separated from 1 with \( \delta = \delta_{\phi_0} \).

7.2.2. One-sided smooth functions. We start by proving Theorem [D2] for a particular space of functions. For \( \alpha > 0 \) and \( M \in \mathbb{N} \), let \( \mathcal{K}^+_{\alpha,M} \) be the set of functions \( f \) on \( \Sigma^+ \) with the following properties:

- for all \( x \in \Sigma \), \( f(x, r) = 0 \) for \( r \) outside the interval \([\inf \tau, \frac{2\inf \tau}{3}]\),
- for all \( x \in \Sigma \), \( r \to f(x, r) \) is of class \( C^M \),
- for all \( r \in [\frac{\inf \tau}{3}, \frac{2\inf \tau}{3}] \), \( x \to f(x, r) \) depends only on the nonnegative coordinates of \( x \) and
- the functions \( \frac{\partial^k}{\partial r^k}(x, r) \), for \( 0 \leq k \leq M \) are \( \alpha \)-Hölder continuous in \( x \in \Sigma \) and continuous in \( r \).

For \( f \in \mathcal{K}^+_{\alpha,M} \), we denote \( ||f||_{\alpha,M} := \sup_{r,k \leq M} ||\frac{\partial^k}{\partial r^k}(., r)||_\alpha \). The heart of the proof uses the arguments of [D2] to establish:

**Proposition 7.11.** Let \( \varphi_0 \in \mathcal{K}_\alpha(\Sigma_+) \) as above. There exist \( \varepsilon, C, c > 0 \) and \( M \) such that for all \( \varphi, ||\varphi - \varphi_0|| < \varepsilon \), all \( f, u, h \in \mathcal{K}^+_{\alpha,M} \), we have, for all positive \( t_1, t_2 \):

\[
|\rho_{f,u,h,\alpha}(t_1,t_2)| \leq C||f||_{\alpha,M}||u||_{\alpha,M}||h||_{\alpha,M}[(1 + t_1)^{-\varepsilon} + (1 + t_2)^{-\varepsilon}].
\]

**Proof.** Choose \( \varepsilon \) so that Proposition [7.6] and Proposition [7.10] holds for all \( \varphi \) with \( ||\varphi - \varphi_0|| < \varepsilon \). Fix \( f, u, h, \varphi \) and write \( \rho_{f,u,h,\alpha}(t_1,t_2) \) for \( \rho_{f,h,g,\alpha}(t_1,t_2) \). Assume first that \( \int f \, dm_\lambda = \int g \, dm_\lambda = 0 \). We consider the Laplace transform

\[
\tilde{\rho}(s_1, s_2) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \rho(t_1,t_2) e^{-s_1 t_1} e^{-s_2 t_2} \, dt_1 \, dt_2
\]

which makes sense a priori for \( a_j > 0 \), where \( s_j = a_j + ib_j, j = 1, 2 \). The following computation is valid for \( a_j > 0 \) and will allow us to extend analytically \( \tilde{\rho}(s_1, s_2) \) to a larger domain and deduce the decay of \( \rho(t_1,t_2) \) as \( t_1, t_2 \) go to infinity.

**Lemma 7.12.** Consider the Laplace transforms \( F, U \) and \( H \) of the functions \( f, u \) and \( h \) given by:

\[
F(x, s) = \int_\mathbb{R} e^{-sr} f(x, r) dr, \quad U(x, s) = \int_\mathbb{R} e^{-sr} u(x, r) dr, \quad G(x, s) = \int_\mathbb{R} e^{-sr} g(x, r) dv.
\]

Then, we have, for \( a_1, a_2 > 0 \):

\[
\tilde{\rho}(s_1, s_2) = \sum_{n,m} \int_\Sigma H(x, s_2) \mathcal{L}_{\phi,s_1}^n [U(., s_1 - s_2) \mathcal{L}_{\phi,s_1}^m F(., -s_1)(.)] (x) \, d\nu_\phi(x).
\]
Proof. We develop:

\[ \bar{\rho}(s_1, s_2) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \int \Sigma \Gamma f(x, r) u(\sigma_{t_1}(x, r)) h(\sigma_{t_1 + t_2}(x, r)) e^{-s_1 t_1} e^{-s_2 t_2} \, dm_\varphi(x, r) \, dt_1 \, dt_2 \]

\[ = \sum_{n, m} \int_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+} \int \Sigma \Gamma f(x, r) u(\sigma^n x, r + t_1 - \tau^n(x)) h(\sigma^{n+m} x, r + t_2 + t_1 - \tau^{n+m}(x)) e^{-s_1 t_1} e^{-s_2 t_2} \, dr \, d\nu_\varphi(x) \, dt_1 \, dt_2, \]

where \( \tau^n(x) := \sum_{k=0}^{n-1} \tau(\sigma^k(x)) \). Observe that for all fixed positive \( n, m \) the integral in \( t_1, t_2, r \) is also an integral over \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \). Then using the variables \( w = r + t_1 - \tau^n(x) \) and \( z = w + t_2 - \tau^m(\sigma^n x) \), the integral (*) can be written as

\[ (*) = \int_{\Sigma} H(\sigma^{n+m} x, -s_2) e^{s_2 \tau^m(x)} U(\sigma^n x, s_2 - s_1) e^{s_1 \tau^n(x)} F(x, s_1) \, d\nu_\varphi(x). \]

Using now the invariance of \( \nu_\varphi \) under \( L_\phi \), and the fact that \( L^n(HK \circ \sigma^n)(x) = K(x)L(H)(x) \), we obtain:

\[ (*) = \int_{\Sigma} H(\sigma^n x, -s_2) e^{s_2 \tau^m(x)} U(x, s_2 - s_1) L^n_{\phi+s_1} F(\cdot, s_1)(x) \, d\nu_\varphi(x) \]

\[ = \int_{\Sigma} H(x, -s_2) L^n_{\phi+s_1} [U(\cdot, s_2 - s_1) L^n_{\phi+s_1} F(\cdot, s_1)(\cdot)](x) \, d\nu_\varphi(x). \]

The Lemma follows for \( a_j = \Re s_j > 0 \).

By Proposition 7.6 and our choice of \( \varepsilon \), we conclude that there exist constants \( \delta, \beta \) such that, for all \( \varphi, \| \varphi - \varphi_0 \|_\alpha < \varepsilon \), the series of operators \( \sum_n L^n_{\phi+s_1} \) converges and is analytic on the region \( U \)

\[ U := \{ s, s = a + ib; |b| > 1, |a| < \frac{2\delta}{|b|^{\beta/2}} \} \]

and that there is a constant \( D_0 \) such that, for \( s \in U \),

\[ \| \sum_n L^n_{\phi+s_1} \|_\alpha \leq D_0 |b|^{D_0}. \]

Moreover, by Proposition 7.10, there is \( \delta > 0 \) such that the series of operators \( \sum_n L^n_{\phi+s_1} \) converges and is meromorphic on the region \( V \); has a simple pole at 0 and has residue at 0 the projection on the constant function \( \nu_\varphi(\cdot) \).

On the other hand, since \( f, u \) and \( h \) belong to \( \mathcal{X}^+_{\alpha, M} \), the functions \( s \mapsto F(\cdot, s), s \mapsto U(\cdot, s) \) and \( s \mapsto H(\cdot, s) \) are analytic into \( \mathcal{X}_\alpha(\Sigma_+) \). Moreover, the functions \( \| F(\cdot, s) \|_\alpha, \| U(\cdot, s) \|_\alpha \) and \( \| H(\cdot, s) \|_\alpha \) decay at infinity like \( |b|^{-M} \) and

\[ \nu_\varphi(F(\cdot, 0)) = \int_{\Sigma^+} \left( \int_{\mathbb{R}} f(x, r) \, dr \right) \, d\nu_\varphi(x) = \int_{\Sigma^+} f \, dm_\varphi = 0. \]

It follows that the function

\[ J(x, s) := \sum_n L^n_{\phi+s_1} F(\cdot, s)(x) \]
is analytic from \( U \cup V \) into \( \mathcal{K}_\alpha \) and that its \( \mathcal{K}_\alpha \)-norm is bounded by \( C\|f\|_{\alpha,M}(1 + |b|)^{D_0-M} \) as \( |b| \to \infty \). Summarizing, the function \( \tilde{\rho}(s_1, s_2) \) admits an analytic extension to \( \{(s_1, s_2); s_1 \in U \cup V, a_2 > 0\} \) and this extension satisfies:

\[
\tilde{\rho}(s_1, s_2) = \sum_m \int_{\Sigma^+} H(x, -s_2) \mathcal{L}^m_{\rho+s_2^*}[U(\cdot, s_2 - s_1)J(\cdot, s_1)](x) \, d\nu_\phi(x).
\]

As before, for each fixed \( s_1 \in U \cup V \), the mapping \( s_2 \mapsto \sum_m \mathcal{L}^m_{\rho+s_2^*}[U(\cdot, s_2 - s_1)J(\cdot, s_1)](x) \) is meromorphic from \( U \cup V \) with a unique simple pole at \( s_2 = 0 \) and a residue a constant function on \( \Sigma^+ \) with value \( C_0(s_1) \). Therefore, for all \( s_1 \in U \cup V \), \( s_2 \mapsto \tilde{\rho}(s_1, s_2) \) admits a meromorphic extension to \( U \cup V \) of the form

\[
\tilde{\rho}(s_1, s_2) = \frac{C_0(s_1) \int_{\Sigma^+} H(x, 0) \, d\nu_\phi(x)}{2\pi is_2} + \tilde{\rho}(s_1, s_2),
\]

where \( \tilde{\rho}(\xi, \eta) \) is an analytic function on \( (U \cup V) \times (U \cup V) \) such that

\[
|\tilde{\rho}(s_1, s_2)| \leq C\|h\|_{\alpha,M} \|u\|_{\alpha,M} \|f\|_{\alpha,M}(1 + |b|)^{D_0-M}(1 + |b_1 - b_2|)^{-M}(1 + |b|)^{D_0-M}.
\]

We again have \( \int_{\Sigma^+} H(x, 0) \, d\nu_\phi(x) = 0 \) by our condition that \( \int h \, d\mu_\phi = 0 \) and finally, the function \( \tilde{\rho}(s_1, s_2) \) admits an analytic extension to \( (U \cup V) \times (U \cup V) \) and satisfies:

\[
|\tilde{\rho}(s_1, s_2)| \leq C\|h\|_{\alpha,M} \|u\|_{\alpha,M} \|f\|_{\alpha,M}(1 + |b_2|)^{D_0-M}(1 + |b_1 - b_2|)^{-M}(1 + |b|)^{D_0-M}.
\]

We now compute \( \rho(t_1, t_2) \) as the Laplace inverse of \( \tilde{\rho}(s_1, s_2) \) by integrating on the imaginary axis in \( s_2 \) and in \( s_1 \). For a fixed \( s_1 \in U \cup V \), we can move the curve of integration in \( s_2 \) to the curve

\[
\Gamma := \{-\delta \max\{1, \frac{1}{|b|}\} + ib; b \in \mathbb{R}\}.
\]

We obtain that the function \( \tilde{\rho}(s_1, t_2) \)

\[
\tilde{\rho}(s_1, t_2) := \frac{-1}{4\pi^2} \int_{\mathbb{R}} \tilde{\rho}(s_1, b_2) e^{ib_2t_2} \, db_2
\]

\[
= \frac{-1}{4\pi^2} \left( \int_{-1}^{+1} \tilde{\rho}(s_1, -\delta + ib_2) e^{ib_2t_2} e^{-\delta t_2} \, db_2 + \int_{\mathbb{R}\setminus[-1,1]} \tilde{\rho}(s_1, -\delta \frac{1}{|b_2|^2}) e^{ib_2t_2} e^{-\delta t_2/|b_2|^2} \, db_2 \right)
\]

is, as a function of \( s_1 \), an analytic function on \( U \times V \) and satisfies

\[
|\tilde{\rho}(s_1, t_2)| \leq C\|h\|_{\alpha,M} \|u\|_{\alpha,M} \|f\|_{\alpha,M} \left( \frac{2e^{-\delta t} + \int_{\mathbb{R}\setminus[-1,1]} \frac{e^{-\delta t/|b|^2}}{(1 + |b|)^{D_0-M}} \, db}{(1 + |b_1|)^{M-D_0}} \right),
\]

as soon as \( M > D_0 + 2 \). We are interested in \( \rho(t_1, t_2) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \tilde{\rho}(s_1, t_2) e^{-ib_1t_1} \, db_1 \). In the same way, by moving the curve of integration in \( s_1 \) to \( \Gamma \), we obtain (recall that we have assumed that \( \int f \, dm_\lambda = \int g \, dm_\lambda = 0 \)):

\[
\rho(t_1, t_2) \leq C\|h\|_{\alpha,M} \|u\|_{\alpha,M} \|f\|_{\alpha,M} \left( (1 + t_1)^\beta + (1 + t_2)^\beta \right).
\]
Proposition 7.13. Let \( \varphi_0 \in \mathcal{K}_\alpha(\Sigma^+) \) as above. For \( \varepsilon, C, \theta > 0 \) and \( M \) as above, for all \( \varphi, \|\varphi - \varphi_0\| < \varepsilon \), all \( f, h \in \mathcal{K}_{\alpha,M}^+ \), we have, for all positive \( t \):

\[
\|\rho_{f,h,m}(t)\| \leq C\|f\|_\alpha,M\|h\|_\alpha,M[(1 + t)^{-c'}].
\]

Indeed, if we assume \( \int f \, dm_\varphi = 0 \), this is exactly the same computation, with only one variable \( s \). But (7.10) holds for \( f \) as soon as it holds for \( f - \int f \, dm_\varphi \). By the same token, using Proposition 7.13, we can replace in (7.8) \( f \) and \( h \) by \( f - \int f \, dm_\varphi \) and \( h - \int h \, dm_\varphi \). This achieves the proof of Proposition 7.11. □

7.2.3. From one-sided to two-sided smooth functions. This part goes back to Ruelle (\([R]\)), we present it here for completeness. We consider a new space of functions: for \( \alpha > 0 \) and \( M \in \mathbb{N} \), let \( \mathcal{K}_{\alpha,M}' \) be the set of functions \( f \) on \( \Sigma^+ \) with the following properties:

- for all \( x \in \Sigma \), \( f(x,r) = 0 \) for \( r \) outside the interval \([\inf r, 2\inf r]\),
- for all \( x \in \Sigma \), \( r \mapsto f(x,r) \) is of class \( C^M \) and
- the functions \( \frac{\partial^k f}{\partial r^k}(x,r) \), for \( 0 \leq k \leq M \) are \( \alpha \)-Hölder continuous on \( \Sigma \) and continuous in \( s \).

For \( f \in \mathcal{K}_{\alpha,M}' \), we still denote \( \|f\|_{\alpha,M} := \sup_{r,k \leq M} \|\frac{\partial^k f}{\partial r^k}(.,r)\|_\alpha \). We show in this subsection

Proposition 7.14. There exist \( \varepsilon', C', c' > 0 \) and \( M \) such that for all \( \varphi \) such that \( \|\varphi - \varphi_0\|_\alpha < \varepsilon' \), all \( f, u, h \in \mathcal{K}_{\alpha,M}' \), we have, for all positive \( t_1, t_2 \):

\[
\|\rho_{f,u,h,m}(t_1,t_2)\| \leq C\|f\|_{\alpha,M}\|u\|_{\alpha,M}\|h\|_{\alpha,M}[(1 + t_1)^{-c'} + (1 + t_2)^{-c'}].
\]

Proof. Assume first that \( \int f \, dm_\lambda = \int u \, dm_\lambda = \int h \, dm_\lambda = 0 \).

The following construction reduces the proof of Proposition 7.14 to a direct extension of the proof of Proposition 7.11. Let \( A(x) \) be a function in \( \mathcal{K}_{\alpha}(\Sigma) \); then (see e.g. \([P1]\)), there exists a decomposition \( A = \sum_{j=0}^\infty A_j \), where

\begin{enumerate}
  \item \( x \mapsto A_j(x) \) depends only on the coordinates \( (x_{-j}, x_{-j+1}, \cdots) \) of \( x \),
  \item \( \sup_x \|A_j(x)\| \leq \alpha^j\|A\|_\alpha \) and
  \item \( \|A_j\|_\alpha \leq \|A\|_\alpha \).
\end{enumerate}

We also associate to \( A \) and \( s \in \mathbb{C} \) the functions \( \tilde{A}_j^s(x) := e^{-sr^j}(A_j(\sigma^j x)) \). Then, by \([R]\) (see also \([D1]\) and \([P1]\)), there is \( \alpha', 0 < \alpha' < \alpha \), and \( \theta, 0 < \theta < 1 \), such that, with \( s = a + ib \),

\begin{enumerate}
  \item \( x \mapsto \tilde{A}_j^s(x) \) depends only on the coordinates \( (x_0, x_1, \cdots) \) of \( x \),
  \item \( \sup_x \|\tilde{A}_j^s(x)\| \leq Ce^{\theta^j}\|A\|_\alpha \) and
  \item \( \|\tilde{A}_j^s\|_\alpha \leq Ce^{\theta^j}\|b\|\theta^j\|A\|_\alpha \).
\end{enumerate}

Finally, we set \( \tilde{A}^s(x) := \sum_j \tilde{A}_j^s(x) \); we have, if \( |a| \) is small enough,
(1) $x \mapsto \tilde{A}^x(x)$ depends only on the coordinates $(x_0, x_1, \cdots)$ of $x$,
(2) $\sup_{x} |\tilde{A}^x(x)| \leq C |A|_\alpha$,
(3) $\|A^x\|_\alpha \leq C |b| |A|_\alpha$ and
(4) $\int \tilde{A}^0(x) \, d\nu(x) = \int A(x) \, d\nu(x)$ for any shift invariant measure $\nu$ on $\Sigma$.

Property (4) is clear since $\tilde{A}^0(x) = \sum_j \tilde{A}^0_j(x) = \sum_j A_j(x)$, whereas $A(x) = \sum_j A_j(x)$ and both series of functions converge uniformly.

Choose $\varepsilon'$ so that for all $\varphi$, $\|\varphi - \varphi_0\|_\alpha < \varepsilon'$. Proposition 7.6 and Proposition 7.10 apply on $\mathcal{K}_{\alpha'}$. Fix $f, u, h, \varphi \in \mathcal{K}_{\alpha,M}$ and write $\rho(t_1, t_2)$ for $\rho_{f, h, g, m, \varphi}(t_1, t_2)$ We now write as before the Laplace transform $\tilde{\rho}(s_1, s_2)$ of $\rho(t_1, t_2)$ as:

$$\tilde{\rho}(s_1, s_2) = \sum_{n,m} \int_{\Sigma} H(\sigma^{n+m}x, -s_2) e^{s_2 \tau^{n+m}(x)} U(\sigma^n x, s_2 - s_1) e^{(s_1 - s_2) \tau^n(x)} F(x, s_1) \, d\nu(x).$$

We consider it as a series in the sense of tempered distributions: for any $B(s, t)$ in the Schwartz space of $\mathbb{R}^2$, $\int B(ib_1, ib_2) \tilde{\rho}(ib_1, ib_2) \, db_1 db_2$ makes sense and is equal to $-4\pi^2 \int B(t_1, t_2) \rho(t_1, t_2) \, dt_1 dt_2$. The series of integrals $\int B(t_1, t_2) \rho_{n,m}(t_1, t_2) \, dt_1 dt_2$ converges absolutely. It still does if one consider the sum over $n, m$ in $\mathbb{Z}$ instead of $\mathbb{Z}_+$. For each $(n, m) \in \mathbb{Z} \times \mathbb{Z}$, we write, using the decompositions $H(x, s) = \sum_j H_j(x, s)$, $U(x, s) = \sum_k U_k(x, s)$, $F(x, s) = \sum_\ell F_\ell(x, s)$ and the above $\tilde{A}^x_j$ notation:

$$\tilde{\rho}_{n,m}(s_1, s_2) = \int_{\Sigma} H(\sigma^{n+m}x, -s_2) e^{s_2 \tau^{n+m}(x)} U(\sigma^n x, s_2 - s_1) e^{(s_1 - s_2) \tau^n(x)} F(x, s_1) \, d\nu(x)$$

$$= \sum_{j,k,\ell} \int_{\Sigma} H_j(\sigma^{n+m}x, -s_2) e^{s_2 \tau^{n+m}(x)} U_k(\sigma^n x, s_2 - s_1) e^{(s_1 - s_2) \tau^n(x)} F_\ell(x, s_1) \, d\nu(x)$$

$$= \sum_{j,k,\ell} \int_{\Sigma} H_j(\sigma^{n+m}x, -s_2) e^{s_2 \tau^{n+m}(x)} \widetilde{U}_k(\sigma^n x, s_2 - s_1) e^{(s_1 - s_2) \tau^n(x)}$$

$$\tilde{F}_\ell(\sigma^{-\ell} x, s_1) e^{s_1 \tau(\sigma^{-\ell} x)} \, d\nu(x),$$

where we used the cocycle relation $\tau^{n+m}(x) = \tau^n(x) + \tau^m(\sigma^n x)$ valid for all $n, m \in \mathbb{Z}$ and wrote $\tilde{F}(x, s)$ for $\tilde{F}^s(x, s)$, $\tilde{G}_j(x, s)$ for $\tilde{G}_j^s(x, s)$ and $\tilde{U}_k(x, s)$ for $\tilde{U}_k^s(x, s)$.

We now replace the summation in $(n, m)$ by a summation in $(p, q)$, where $p := n - k + \ell, q := m + k - j$. Assume for example $p \geq 0, q \geq 0$ (and then $p + q = n + m - j + \ell \geq 0$). We write, using the invariance of $\nu$, the integral

$$\int_{\Sigma} H_j(\sigma^{n+m-j}x, -s_2) e^{s_2 \tau^{n+m-j}(\sigma^n x)} \widetilde{U}_k(\sigma^{-k} x, s_2 - s_1) e^{s_1 \tau^{-k}(\sigma^{-k} x)} \tilde{F}_\ell(\sigma^{-\ell} x, s_1) \, d\nu(x),$$

and
as:
\[
\int_{\mathbb{R}^n} \frac{1}{(s + m - j + \ell x, -s_2 - s_1) e^{x_2 \tau^{m+k-j}(s_1-k+\ell x)} U_k(s_2-s_1 e^{x_2 \tau^{n-k+\ell x}}) F_\ell(x, s_1) \, d\nu_\phi(x)},
\]
where we replaced \( \mathcal{P}_0 \) by \( \nu_\phi \) since the integrand now depends only on the non-negative coordinates of \( x \). As before, we can write these integrals using the transfer operators as
\[
\int_{\mathbb{R}^n} \frac{1}{(s + m - j + \ell x, -s_2) e^{x_2 \tau^{m+k-j}(s_1-k+\ell x)} U_k(s_2-s_1) L^m_{\phi+s\tau}(F_\ell(s_1))(x) \, d\nu_\phi(x) = \int_{\mathbb{R}^n} \frac{1}{(s + m - j + \ell x, -s_2 - s_1) e^{x_2 \tau^{m+k-j}(s_1-k+\ell x)} U_k(s_2-s_1) L^m_{\phi+s\tau}(F_\ell(s_1))(x) \, d\nu_\phi(x).}
\]
If \(|a_1|, |a_2|, |a_1 - a_2| \) are small enough, one can sum in \( j, k, \ell \) for the same value of \((p, q) \); we obtain, when \( p, q \geq 0 \),
\[
\int_{\mathbb{R}^n} \frac{1}{(s + m - j + \ell x, -s_2 - s_1) e^{x_2 \tau^{m+k-j}(s_1-k+\ell x)} U_k(s_2-s_1) L^m_{\phi+s\tau}(F_\ell(s_1))(x) \, d\nu_\phi(x).
\]
The other possible signs of \( p, q \) and \( p + q \) are treated in the same way.

By applying Proposition 7.6 to \( \mathcal{K}_\alpha' \), we conclude that there are positive numbers \( \delta', \beta' \) such that, for all \( \delta, |\| \varphi - \varphi_0\|_{\alpha'} < \delta' \) the series of operators \( \sum_n L^n_{\phi+s\tau} \) converges and is analytic on the region \( U' \)
\[
U' := \{ s = a + ib; |b| > 1, |a| < \frac{2\delta'}{|b|/\beta'^{1/2}} \}
\]
and that there is a constant \( D_0' \) such that, for \( s \in U' \),
\[
(7.12) \quad \| \sum_n L^n_{\phi+s\tau} \| \leq D_0'|b|D_0'.
\]
Moreover, there is \( \delta' > 0 \) such that on the series of operators \( \sum_n L^n_{\phi+s\tau} \) converges and is meromorphic on the region \( V' \),
\[
V' := \{ s = a + ib; |b| < 2, |a| < 2\delta' \}
\]
with a simple pole at 0 and residue the projection of the constant function \( \nu_\phi(\cdot) \). We conclude as above (but with a different argument for each one of the six sums over \((p, q), (-p, p + q), (-p, -p + q), (-p, -q, q), (p, p - q), (-p, -q)\) in \((\mathbb{Z} \times \mathbb{Z}_+)\)) that \( \tilde{\rho}(s_1, s_2) \) is given by an analytic function defined on the region where \( s_1, s_2 \) and \( s_1 - s_2 \) all belong to \( U' \cup V' \) (and have a real part smaller than \( \delta_0 \)) and satisfying
\[
|\tilde{\rho}(s_1, s_2)| \leq C\|h\|_{\alpha, M}\|u\|_{\alpha, M}\|f\|_{\alpha, M}(1 + |b_1|)D_0'^{-\alpha-M}(1 + |b_1 - b_2|)D_0'^{-\alpha-M}(1 + |b_2|)D_0'^{-\alpha-M} - M,
\]
where \( D_0' = D_0' + 1 \).

If \( M \) has been chosen greater than \( D_0' + 2 \), we obtain Proposition 7.14 (for functions with integral 0) by the same argument as before, provided one chooses in each of the six cases contours \( \Gamma \) of integration with the right sign.

The extension of Proposition 7.13 to functions \( f, h \in \mathcal{K}_\alpha' \) with \( \int f \, dm_\phi = 0, \int h \, dm_\phi = 0 \) goes again by the same computation, without the function \( u \). Again, (7.10) holds for \( f \) as soon as it holds for \( f - \int f \, dm_\phi \). This justifies the reduction to functions with integral 0 in the proof of proposition 7.14.
7.2.4. Hölder continuous functions. We conclude the proof of Theorem 7.2 and of Proposition 7.3 by approximating any Hölder continuous function by regular functions. We have proven (7.1) for functions in \( \mathcal{K}_{\alpha,M} \) with some constants \( C', c' \); (7.1) holds also if \( f, u, h \) are such that \( f \circ \sigma_1, u \circ \sigma_2, h \circ \sigma_3 \in \mathcal{K}_{\alpha,M} \) for bounded \( t_i, i = 1, 2, 3 \). There is 

\[ C_0 = 10 + 6 \sup_{\tau(x)} \frac{\tau(x)}{\inf_{\tau(x)}} \]

such that any function which is of class \( C_M \) along the trajectories of the special flow \((\Sigma^\tau, \sigma_t, t \in \mathbb{R})\) and such that the first \( M \) derivatives along the flow are \( \alpha \)-Hölder continuous functions can be written as a sum of less than \( C_0 \) functions in \( \mathcal{K}_{\alpha,M} \). Using the projection from the manifold to \( \Sigma^\tau \), we conclude that there exist \( \varepsilon, C', c' > 0, \alpha, M \) such that for all for all \( \varphi, \| \varphi - \varphi_0 \|_\alpha < \varepsilon \), all \( f, u, h \) that are of class \( C_M \) along the trajectories of the flow and such that all the derivatives along the flow up to order \( M \) belongs to \( \mathcal{K}_\alpha(SM) \), we have, for all \( t_1, t_2 \geq 0 \):

\[
|\rho_{f,u,h,\varphi}(t_1, t_2)| \leq C'|f\|_{\alpha,M} |u\|_{\alpha,M} |h\|_{\alpha,M} [(1 + t_1)^{-c'} + (1 + t_2)^{-c'}],
\]

where \( \|f\|_{\alpha,M} \) is the maximum of the \( \|f\|_\alpha \) norms of the first \( M \) derivatives along the flow.

We conclude by smoothing all functions in \( \mathcal{K}_\alpha \). Let \( \overline{\psi} \) be a \( C^0 \) nonnegative function on \( \mathbb{R} \), with support in \([-1, +1]\) and integral 1. For \( \varepsilon > 0 \) and a function \( f \in \mathcal{K}_\alpha \), set

\[
\overline{\psi}_\varepsilon(t) := \frac{1}{\varepsilon} \overline{\psi}\left(\frac{t}{\varepsilon}\right) \text{ and } f_\varepsilon(x) := \int_\mathbb{R} \overline{\psi}_\varepsilon(t) f(x, t) \, dt.
\]

We have \( \sup_x |f(x) - f_\varepsilon(x)| \leq \varepsilon^\alpha \|f\|_\alpha \) and \( \|f_\varepsilon\|_{\alpha,M} \leq \varepsilon^{-M-1} \|f\|_\alpha \).

Fix \( t_1, t_2 > 0 \), choose \( \varepsilon = [1/3(1 + t_1)^{-c'} + 1/3(1 + t_2)^{-c'}]^{-1} \) and replace \( f, u, h \) by \( f_\varepsilon, u_\varepsilon, h_\varepsilon \). One obtains (7.1) for \( f, u, h \) with some constant \( C'_0 \) and \( c'_0 = \frac{c'_\alpha}{\alpha + 3M + 3} \).

References

[ABJ] J.-P. Anker, P. Bougerol and T. Jeulin, The infinite Brownian loop on a symmetric space, Rev. Mat. Iberoam., 18 (2002), 41–97.

[Anc] A. Ancona, Negatively curved manifolds, elliptic operators and the Martin boundary, Ann. Math. (2) 125 (1987) 495–536.

[Ano] D. Anosov, Theory of dynamical systems. Part I: Ergodic theory, Lectures held in Warsaw, 1966, Lecture Notes Series, No. 23 Matematisk Institut, Aarhus Universitet, Aarhus 1970.

[AS] D. Anosov, J. Sinai, Certain smooth ergodic systems. (Russian), Uspehi Mat. Nauk 22 (1967) no. 5 (137), 107–172.

[AnS] M.T. Anderson and R. Schoen, Positive harmonic functions on complete manifolds of negative curvature, Annals Math. (2) 121 (1985), 429–461.

[B] P. Bougerol, Théorème central limite local sur certains groupes de Lie, Ann. Sci. École Norm. Sup. (4) 14 (1981), 403–432.

[Br] R. Brooks, The fundamental group and the spectrum of the Laplacian, Comment. Math. Helv. 56 (1981), 581–598.

[BR] R. Bowen and D. Ruelle, The ergodic theory of Axiom A flows, Invent. Math., 29 (1975), 181–202.

[CY] S. Y. Cheng and S. T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Commun. Pure Appl. Math. 28 (1975) 333–354.

[Da] E. B. Davies, Non-Gaussian aspects of heat kernel behavior, J. London Mat. Soc. (2) 55 (1997), 105–125.

[D1] D. Dolgopyat, On decay of correlations in Anosov flows, Ann. Math., 147 (1998), 357–390.
D. Dolgopyat, Prevalence of rapid mixing in hyperbolic flows, Ergod. Th. & Dynam. Sys. 18 (1998), 1097–1114.

A. Debiard, B. Gaveau and E. Mazet, Théorèmes de comparaison en géométrie riemannienne, Publ. Kyoto Univ. 12 (1976), 391–425.

W. Feller, An introduction to probability theory and its applications. Vol. II. John Wiley & Sons.

P. Gerl, Ein Gleichverteilungssatz auf $F_2$, in Probability Measures on Groups, Springer Lecture Notes in Math. 706 Springer-Verlag, Berlin-Heidelberg-New York, (1979), 126–130.

S. Gouëzel, Local limit theorem for symmetric random walks in Gromov-hyperbolic groups, J. Amer. Math. Soc., 27 (2014), 893–928.

S. Gouëzel, Martin boundary of random walks with unbounded jumps in hyperbolic groups, to appear (Ann. Probab.).

S. Gouëzel and S. Lalley, Random walks on co-compact Fuchsian groups, Ann. Sci. École Norm. Sup. (4) 46 (2013).

P. Gerl and W. Woess, Local limits and Harmonic Functions for Nonisotropic Random Walks on Free Groups, Proba. Th. Rel. Fields 71 (1986), 341–355.

U. Hamenstädt, An explicite description of harmonic measure, Math. Z. 205 (1990), 287–299.

U. Hamenstädt, Harmonic measures for compact negatively curved manifolds, Acta Math. 178 (1997), 39–107.

U. Hamenstädt, Harmonic measures, Hausdorff measures and positive eigenfunctions, J. Diff. Geom. 44 (1996), 1–31.

T. Kato, Perturbation theory of linear operators, Grund. math. Wissen. 132 Springer Verlag, Berlin-Heidelberg-New York (1980).

V. A. Kaimanovich, Invariant measures of the geodesic flow and measures at infinity on negatively curved manifolds, Ann. Inst. H. Poincaré, A, Phys. Théor., 53 (1990), 361–393.

G. Kozma, A graph counter-example to Davies’s conjecture, Rev. Mat. Iberoam. 30 (2014), 1–12.

S. Lalley, Finite range random walks on free groups and homogeneous trees, Ann. Prob. 21 (1993), 2087–2130.

F. Ledrappier, A renewal theorem for the distance in negative curvature, Stochastic analysis (Ithaca, New York, 1993) Proc. Symp. Pure Math. 57 (1995), 351–360.

F. Ledrappier, Structure au bord des variétés à courbure négative, Séminaire Théorie Spec. Géom. Grenoble, 13 (1995), 97-122.

F. Ledrappier, Applications of dynamics to compact manifolds of negative curvature, Proceedings of the International Congress of Mathematicians, Vol 1,2 (Zürich, 1994) Birkhäuser, Basel (1995), 1195–1202.

C. Liverani, On Contact Anosov flows, Annals of Math., 159 (2004), 1275–1312.

J. Moser, On Harnack theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961), 577-591.

G. Margulis, Applications of ergodic theory to the investigation of manifolds of negative curvature, Functional Anal. Appl. 3 (1969), 335–336.

G. Margulis, On Some Aspects of the Theory of Anosov Systems, Springer Monographs in Math. Springer-Verlag, Berlin Heidelberg (2004).

I. Melbourne, Rapid decay of correlations for nonuniformly hyperbolic flows, Trans. AMS, 359 (2007), 2421–2441.

O. Mohsen, Le bas du spectre d’une variété hyperbolique est un point selle, Ann. Sci. École Norm. Sup. (4), 40 (2007), 191–207.

L. Naim, Sur le rôle de la frontière de R.S. Martin dans la théorie du potentiel, Ann. Inst. Fourier, Grenoble 7 (1957), 183–281.

M. Pollicott, Multiple mixing for hyperbolic flows, preprint

M. Pollicott, Uniform Dolgopyat super-polynomial mixing, in preparation; personal communication.

W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics. Astérisque 187–188 (1990).

F. Paulin, M. Pollicott, B. Schapira, Equilibrium states in negative curvature, to appear (Astérisque).
[R] D. Ruelle, *Resonances for Axiom A flows*, J. Diff. Geometry 25 (1987), 99–116.

[SZ] P. Souplet, Q. Zhang, *Sharp gradient estimates and Yau Liouville theorem for the heat equation on non-compact manifolds*, Bull. London Math. Soc. 38 (2006) 1045–1053.

[S] D. Sullivan, *Related aspects of positivity in Riemannian geometry*, J. Diff. Geom. 25 (1987), 327–351.

[V] N. Varopoulos, *Théorie du potentiel sur des groupes et des variétés*, C.R. Acad. Sci. Paris Sér. I Math. 302 (1986), 203–205.

[W] P. Walters, *An introduction to ergodic theory*, Graduate texts in Mathematics 79, Springer-Verlag, New York-Berlin, (1982).

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