Exact Finite-Size-Scaling Corrections to the Critical Two-Dimensional Ising Model on a Torus.

II. Triangular and hexagonal lattices

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Abstract

We compute the finite-size corrections to the free energy, internal energy and specific heat of the critical two-dimensional spin-1/2 Ising model on a triangular and hexagonal lattices wrapped on a torus. We find the general form of the finite-size corrections to these quantities, as well as explicit formulas for the first coefficients of each expansion. We analyze the implications of these findings on the renormalization-group description of the model.

Key Words: Ising model; finite-size scaling; corrections to scaling; renormalization group; irrelevant operators; scaling functions.

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1 Introduction

It is well-known that phase transitions in statistical-mechanical systems can occur only in the infinite-volume limit. In any finite system, all thermodynamic quantities (such as the magnetic susceptibility and the specific heat) are analytic functions of all parameters (such as the temperature and the magnetic field); but near a critical point they display peaks whose height increases and whose width decreases as the volume $V = L^d$ grows, yielding the critical singularities in the limit $L \to \infty$. For bulk experimental systems (containing $V \sim 10^{23}$ particles) the finite-size rounding of the phase transition is usually beyond the experimental resolution; but in Monte Carlo simulations ($V \lesssim 10^6$–$10^7$) it is visible and is often the dominant effect.

Finite-size scaling theory [1, 2, 3, 4] provides a systematic framework for understanding finite-size effects near a critical point. The idea is simple: the only two relevant length scales are the system linear size $L$ and the correlation length $\xi_\infty$ of the bulk system at the same parameters, so everything is controlled by the single ratio $\xi_\infty/L$. If $L \gg \xi_\infty$, then finite-size effects are negligible; for $L \sim \xi_\infty$, thermodynamic singularities are rounded and obey a scaling Ansatz $O \sim L^{p_O} F_O(\xi_\infty/L)$ where $p_O$ is a critical exponent and $F_O$ is a scaling function. Finite-size scaling is the basis of the powerful phenomenological renormalization group method (see ref. [3] for a review); and it is an efficient tool for extrapolating finite-size data coming from Monte Carlo simulations so as to obtain accurate results on critical exponents, universal amplitude ratios and subleading exponents [1, 3, 4, 5, 6] and references therein]. In particular, in systems with multiplicative and/or additive logarithmic corrections (as the two-dimensional 4-state Potts model [4]), a good understanding of finite-size effects is crucial for obtaining reliable estimates of the physically interesting quantities.

In finite-size-scaling theory for systems with periodic boundary conditions, three simplifying assumptions have frequently been made:

(a) The regular part of the free energy, $f_{\text{reg}}$, is independent of the lattice size $L$ [4] (except possibly for terms that are exponentially small in $L$).

(b) The scaling fields associated to the temperature $T$ and magnetic field $h$ (i.e., $\mu_t$ and $\mu_h$, respectively) are independent of $L$ [11].

(c) The scaling field $\mu_L$ associated to the lattice size equals $L^{-1}$ exactly, with no corrections $L^{-2}, L^{-3}, \ldots$ [4].

Moreover, in the nearest-neighbor spin-1/2 two-dimensional Ising model, it was further assumed for many years that there are no irrelevant operators [12, 13]; indeed this assumption was confirmed numerically through order $(T - T_\text{c})^3$ at least as regards the bulk behavior of the susceptibility in the isotropic square-lattice Ising model [13]. However, several authors have recently found overwhelming evidence that there

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1 This is true only for systems below the upper critical dimension $d_c$. For Ising models with short-range interaction, $d_c = 4$.

2 Finite-size scaling has also been successfully applied to data coming from transfer-matrix computations [3].
are indeed irrelevant operators playing a role in the two-dimensional Ising model \[14, 15, 16, 17, 18, 19, 20\]. In particular, for the square-lattice Ising model they have found by studying the bulk magnetic susceptibility that there is one irrelevant operator contributing to order \((T - T_c)^4\) and there is (at least) one irrelevant operator contributing to order \((T - T_c)^6\).

An interesting way to test assumptions (a)–(c) and see the effect of the irrelevant operators is to compute the asymptotic expansion (in powers of \(L^{-1}\)) of the free energy and its derivatives with respect to the temperature at the critical point. The square-lattice Ising model is the best understood case.

In a classic paper, Ferdinand and Fisher \[21\] considered the energy and the specific heat of the square-lattice Ising model on a torus of width \(L\) and aspect ratio \(\rho\), and obtained the first two (resp. three) terms of the large-

\[L\] asymptotic expansion of the energy (resp. specific heat) at fixed \(x \equiv L(T - T_c)\) [this is the finite-size-scaling regime] and fixed \(\rho\). In particular, at criticality \((T = T_c)\) they computed the finite-size corrections to both quantities to order \(L^{-1}\). Their results have been improved at the critical point by several authors \[22, 23, 24, 25\]. Their results can be summarized as follows:

\[
\begin{align*}
  f_{\text{sq}}(L, \rho) &= f_{\text{sq}}^\text{bulk} + \sum_{m=1}^{\infty} \frac{f_{\text{sq}}^{2m}(\rho)}{L^{2m}} \\
  E_{\text{sq}}(L, \rho) &= E_0 + \sum_{m=0}^{\infty} \frac{E_{\text{sq}}^{2m+1}(\rho)}{L^{2m+1}} \\
  C_{H,c}^{\text{sq}}(L, \rho) &= C_{H,c}^{\text{sq}} + C_0^{\text{sq}}(\rho) + \sum_{m=1}^{\infty} \frac{C_m^{\text{sq}}(\rho)}{L^m}
\end{align*}
\]

where \(f_c, E_c\) and \(C_{H,c}\) are respectively the critical free energy, internal energy and specific heat.

The first important observation is that there are no logarithmic corrections except for the specific-heat leading term \(C_{00}^{\text{sq}} \log L\). Secondly, the finite-size corrections are integer powers of \(L^{-1}\), which is consistent with irrelevant operators taking integer exponents. Furthermore, not all the powers of \(L^{-1}\) occur: in the large-

\[L\] expansion of the free energy (resp. internal energy) only even (resp. odd) powers of \(L^{-1}\) can occur. In the specific-heat expansion all powers of \(L^{-1}\) can appear. In addition, the

\[3\]Janke and Kenna \[26\] have studied similar expansions for the square-lattice Ising model with Branscamp-Kunz boundary conditions. The analytic structure is similar to (1.1) but additional terms arise due to the boundary conditions. For instance, there is a term \(\sim \log L/L\) in the specific heat. On the other hand, Lu and Wu \[27\] studied the critical free energy for the square-lattice Ising model on non-orientable surfaces (namely, the Möbius strip and the Klein bottle). They found the first terms of the large-

\[L\] expansion of \(f_c(L, \rho)\); although they did not give details about the analytic structure of such expansion. In particular, there is an additional term \(\sim L^{-1}\) in the expansion for the Möbius strip (due to “surface” effects) which is absent in the Klein bottle. They also explicitly showed that the coefficient \(f_2^{\text{sq}}(\rho)\) depends on the boundary conditions [even if the expansion (1.14) holds true].

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coefficients \( C_{m}^{sq} \) and \( E_{m}^{sq} \) satisfy the relation

\[
\frac{E_{m}^{sq}(\rho)}{C_{m}^{sq}(\rho)} = \begin{cases} 
-1/\sqrt{2} & \text{for } m \text{ odd} \\
0 & \text{for } m \text{ even}
\end{cases}
\tag{1.2}
\]

The authors of ref. [20] classified (using conformal field theory) all possible irrelevant operators that may occur in the two-dimensional Ising model and found that all their results (in the thermodynamic limit and at criticality on a finite torus) can be explained in terms of the following conjecture

**Conjecture 1.1** [20, Conjecture (d2)] *The only irrelevant operators which appear in the two-dimensional nearest-neighbor Ising model are those due to the lattice breaking of the rotational symmetry*

In particular, for the square-lattice Ising model the first operator that breaks rotational invariance is the spin-four operator \( T^2 + \bar{T}^2 \) (where here \( T \) is the energy-momentum operator) whose renormalization-group exponent is \( y = -2 \). In ref. [20] they showed that this operator can give rise to all the observed corrections in (1.1).

In this paper we extend the above results to the triangular and hexagonal lattices. We will obtain the large-\( L \) asymptotic expansions for the critical free energy, internal energy and specific heat for such lattices wrapped on a torus of width \( L \) and fixed aspect ratio \( \rho \). The interest of this computation is three-fold. First, we can make a new test of Conjecture 1.1. In the triangular lattice, the first irrelevant operator (belonging to the identity family) that breaks rotational invariance is \( T^3 + \bar{T}^3 \) with \( y = -6 \) [20]. If Conjecture 1.1 is true, then several coefficients in the finite-size-scaling expansions (1.1) should vanish. Second, we can directly check whether the ratio (1.2) is universal or not, that is, if (1.2) depends or not on the microscopic details of the lattice. Finally, the asymptotic expansions could be useful to check Monte Carlo simulations.

A first study of the triangular-lattice Ising model partition function on a finite torus was done by Nash and O’Connor [32]. They obtained (among other interesting results) the exact expression of such partition function with anisotropic nearest-neighbor couplings and extracted its scaling limit. They computed the bulk contribution to the free energy \( f_{\text{bulk}} \) and the first finite-size correction \( f_{2}(\rho) \). Here we will extend their results at the critical point.

The main results of this paper can be summarized as follows:

\[
f_{c}(L, \rho) = f_{\text{bulk}} + \sum_{m=1}^{\infty} \frac{f_{2m}(\rho)}{L^{2m}} \tag{1.3a}
\]

\[
E_{c}(L, \rho) = E_{0} + \sum_{m=0}^{\infty} \frac{E_{2m+1}(\rho)}{L^{2m+1}} \tag{1.3b}
\]

\[\text{4 A similar finite-size scaling analysis was carried out for the one-dimensional Ising quantum chain which belongs to the same universality class of the two-dimensional Ising model [28, 29, 30, 31].}\]
\[ C_{\text{H,c}}(L, \rho) = C_{00} \log L + C_0(\rho) + \sum_{m=1}^{\infty} \frac{C_m(\rho)}{L^m} \]  

(1.3c)

\[ f^{(3)}_c(L, \rho) = A_1(\rho)L + A_{00} \log L + A_0(\rho) + \sum_{m=1}^{\infty} \frac{A_m(\rho)}{L^m} \]  

(1.3d)

\[ f^{(4)}_c(L, \rho) = B_{00} \log L \]  

(1.3e)

where \( f^{(3)}_c \) (resp. \( f^{(4)}_c \)) is the third derivative (resp. the logarithmic contribution to the fourth derivative) of the free energy with respect to the inverse temperature \( \beta \) evaluated at the critical point. We have also found explicit formulas for the coefficients \( f_2(\rho), f_6(\rho), E_1(\rho), E_5(\rho), C_{00}, C_0(\rho), C_1(\rho), C_4(\rho), C_5(\rho), A_1(\rho), A_{00}, A_0(\rho), A_1(\rho), \) and \( B_{00} \) (Indeed, \( f_4 = f_8 = E_3 = E_7 = C_2 = C_3 = A_2 = 0 \)).

Our results on the general analytic structure of the finite-size corrections to these models are:

- The analytic structure of the finite-size-scaling corrections of the quantities considered here is exactly the same for the triangular and the hexagonal lattices.

- The finite-size corrections to the free energy, internal energy and specific heat are always integer powers of \( L^{-1} \), unmodified by logarithms (except of course for the leading \( \log L \) term in the specific heat).

- In the finite-size expansion of the free energy, only even integer powers of \( L^{-1} \) occur. The only exceptions are the powers \( L^{-4} \) and \( L^{-8} \) whose coefficients vanish.

- In the finite-size expansion of the energy, we only find odd integer powers of \( L^{-1} \). In this case, the coefficients associated to the powers \( L^{-3} \) and \( L^{-7} \) vanish.

- In the finite-size expansion of the specific heat, any integer powers of \( L^{-1} \) can occur, except the terms \( L^{-2} \) and \( L^{-3} \). In addition, the non-zero coefficients of the odd powers of \( L^{-1} \) in this expansion are proportional to the corresponding coefficients in the internal energy expansion as in the square lattice.

- In the finite-size expansion of \( f^{(3)}_c \) we find that the expected leading term \( L \log L \) is missing, and the actual leading term is simply \( L \). We find that all powers of \( L^{-1} \) appear in such expansion, except \( L^{-2} \).

- In the finite-size expansion of the fourth derivative of the free energy \( f^{(4)}_c \) we find that there is only a logarithmic term \( \sim \log L \), even though we expect two additional logarithmic contributions of order \( L \log L \) and \( L^2 \log L \) respectively.

The above results are very useful to gain new insights on the renormalization-group description of the two-dimensional Ising model. Our conclusions on this topic are

- Some irrelevant operators should vanish at criticality. This happens, in particular to the less irrelevant one \( TT \) with renormalization-group exponent \( y = -2 \).
In order to give account of all the finite-size corrections, we should include at least two irrelevant operators, in agreement with the results of [17, 18].

The scaling function $\widehat{W}(x)$ (which is the responsible for the logarithmic corrections to the derivatives of the free energy) vanishes at criticality $x = 0$. Its first derivatives at criticality satisfy

$$\left. \frac{\partial^n W(x)}{\partial x^n} \right|_{x=0} = \begin{cases} 0 & \text{for } n = 1, 3, 4 \\ 1/(\lambda \pi \sqrt{3}) & \text{for } n = 2 \end{cases} \quad (1.4)$$

where $\lambda = 1$ (resp. 2) for the triangular (resp. hexagonal) lattice. These equations motivate the conjecture that $\widehat{W}(x) = x^2/(2\lambda \pi \sqrt{3})$.

The non-linear scaling field associated to the temperature can be computed for both lattices and it is given by

$$\mu_t(\tau) = \tau - \frac{1}{24} \tau^3 + O(\tau^5) \quad (1.5)$$

This result provides a cross-check of the analysis of infinite-volume quantities [20].

The plan of this paper is as follows: In Section 2 we present our definitions and notation. In Sections 3, 4, 5 and 6 we present the computation of the asymptotic expansions for the free energy, internal energy, specific heat, and higher derivatives of the free energy, respectively. In Section 7 we discuss the consequences of our results on the renormalization-group description of the models. In particular, we will focus on the irrelevant operators of the model and on the finite-size-scaling functions. Finally, in Section 8 we present our conclusions and discuss the results. We have summarized the technical details in the appendixes: in Appendix A we recall the Euler-MacLaurin formula, and in Appendix B (resp. Appendix C) we collect the definitions and properties of the $\theta$-functions (resp. Kronecker’s double series).

## 2 Basic definitions

Let us first consider an Ising model on a triangular lattice wrapped on a torus of size $N \times M$ at zero magnetic field. The Hamiltonian is given by

$$\mathcal{H} = -\beta \sum_{<i,j>} \sigma_i \sigma_j \quad (2.1)$$

The partition function is given by

$$Z_{NM}(\beta) = \sum_{\{\sigma = \pm 1\}} e^{-\mathcal{H}} \quad (2.2)$$

The dual of such triangular lattice is an hexagonal lattice wrapped on a torus of size $N \times M$ and containing $2NM$ sites (i.e., the hexagonal lattice can be viewed
as a triangular lattice with a two-point basis). The Hamiltonian and the partition function of the Ising model on this lattice are also given by (2.1) / (2.2).

If one brushes aside some subtleties about boundary conditions, one can relate the partition function (2.2) of a triangular-lattice Ising model at coupling $\beta$ to the partition function of the Ising model on the dual (i.e., hexagonal) lattice at a “dual” coupling $\beta^*$ \[33, 34\]:

$$Z_{\text{tri}}^{NM}(\beta) = Z_{\text{hc}}^{2NM}(\beta^*) 2^{1-2NM} (2 \sinh 2\beta)^{3NM/2}$$

(2.3)

where $\beta^*$ is defined by

$$\tanh \beta^* = e^{-2\beta}$$

(2.4)

Using eq. (2.3) and the star-triangle equation \[35\] we can obtain the critical values of the couplings for both models

$$\beta_c = \begin{cases} \frac{1}{4} \log 3 & \text{triangular} \\ \frac{1}{2} \log(2 + \sqrt{3}) & \text{hexagonal} \end{cases}$$

(2.5)

However, this argument is strictly valid only in the infinite-volume limit; it gives the correct relation

$$f_{\text{tri}}(\beta) = 2f_{\text{hc}}(\beta^*) - 2 \log 2 + \frac{3}{2} \log(2 \sinh 2\beta)$$

(2.6)

between infinite-volume free energies and the correct critical points (2.5), but the identity (2.3) for finite-lattice partition functions does not in general hold. This is because a periodic lattice is non-planar, so that the correct duality formula also involves a pair of “homological” modes arising from the two directions of winding around the torus \[33\]. Or put it another way: high-temperature graphs that wind around the lattice do not necessarily correspond to low-temperature graphs on the dual lattice. Therefore, on a finite lattice — which is the subject of this paper — we need to be more careful.

We begin by computing the exact partition function of both models on a torus of size $N \times M$. One way to do this is by relating the Ising model to a dimer model \[37\]. The same computation leading to the square-lattice partition function can be used to obtain the hexagonal-lattice partition function \[38\] by changing the weights of the different dimer configurations. Though the triangular-lattice Ising partition function cannot be derived from the hexagonal-lattice partition function using duality (2.3), for the reasons given above, we can instead use the star-triangle transformation \[35\]. Then, the triangular-lattice partition function $Z_{\text{tri}}^{MN}$ is related to the hexagonal-lattice partition function $Z_{\text{hc}}^{MN}$ (containing twice as much sites) by the formula

$$Z_{\text{tri}}^{MN}(\beta) = R(\beta)^{-MN} Z_{\text{hc}}^{2MN}(\tilde{\beta})$$

(2.7)

\[5\] We thank Alan Sokal for useful clarifications about this point.
where the $\tilde{\beta}$ and $R(\beta)$ are given by

\[
\sinh 2\tilde{\beta} = \frac{1}{\kappa(\beta)} \frac{1}{\sin 2\beta} \quad (2.8a)
\]

\[
R(\beta)^2 = \frac{2}{\kappa(\beta)^2 \sinh^3 2\beta} \quad (2.8b)
\]

and $\kappa$ (which depends on $\beta$ through the parameter $v = \tanh \beta$) is equal to

\[
\kappa(\beta) = \frac{(1 - v^2)^3}{4\sqrt{(1 + v^3)v^3(1 + v)^3}} \quad (2.9)
\]

After straightforward (but lengthy) algebra we find that the partition function for both lattices can be written in a very similar way in the ferromagnetic regime:

\[
Z_V(\beta) = \frac{1}{2} (2 \sin 2\beta)^{V/2} \sum_{\alpha,\beta = 0,1/2} Z_{\alpha,\beta}(\mu) \quad (2.10)
\]

where $V$ is the number of spins in the lattice (e.g., $V = NM$ in the triangular lattice and $V = 2NM$ in the hexagonal lattice). The functions $Z_{\alpha,\beta}(\mu)$ are given by

\[
Z_{\alpha,\beta}(\mu)^2 = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left\{ \sin^2 \left( \frac{\pi(n + \alpha)}{N} \right) + \sin^2 \left( \frac{\pi(m + \beta)}{M} \right) \right. \\
+ \left. \sin^2 \left( \frac{\pi(m + \beta)}{M} - \frac{\pi(n + \alpha)}{N} \right) + 2 \sinh^2 \mu \right\} \quad (2.11)
\]

where the “mass” term $\mu$ is given by

\[
e^{2\mu} = \begin{cases} 
\frac{1}{2} (e^{4\beta} - 1) & \text{triangular} \\
2 \sinh^2 \beta & \text{hexagonal}
\end{cases} \quad (2.12)
\]

The critical point corresponds to the vanishing of the mass, thus giving (2.5).

**Remark.** The fact that the partition function of both lattices depends on the same functions $Z_{\alpha,\beta}(\mu)$ can be explained by noting that the translational symmetry of both lattices is the same (i.e., they have the same underlying Bravais lattice). This issue explains why the finite-size expansions are so similar in both lattices.

The functions $Z_{\alpha,\beta}(\mu)$ can be expanded in powers of $\mu$. In particular, when $(\alpha, \beta) \neq (0,0)$ the functions are even in $\mu$, while $Z_{0,0}(\mu)$ is an odd function of $\mu$:

\[
Z_{\alpha,\beta}(\mu) = Z_{\alpha,\beta}(0) + \frac{1}{2!} Z''_{\alpha,\beta}(0) \mu^2 + \cdots \quad (\alpha, \beta) \neq (0,0) \quad (2.13)
\]

\[
Z_{0,0}(\mu) = \mu Z'_{\alpha,\beta}(0) + \frac{1}{3!} Z'''_{\alpha,\beta}(0) \mu^3 + \cdots \quad (2.14)
\]

This is similar to what happens in the square-lattice Ising model [25].
We are interested in computing the asymptotic expansions for large \( N \) and \( M \) with fixed aspect ratio (e.g. length to width ratio):

\[
\rho = \frac{M}{N}
\]

of the free energy \( f(\beta; N, \rho) \), internal energy \( E(\beta; N, \rho) \) and specific heat \( C_H(\beta; N, \rho) \) at the critical point \( \beta = \beta_c \). These quantities are defined as follows

\[
f(\beta; N, \rho) = \frac{1}{V} \log Z_V(\beta)
\]

\[
E(\beta; N, \rho) = -\frac{\partial}{\partial \beta} f(\beta; N, \rho)
\]

\[
C_H(\beta; N, \rho) = \frac{\partial^2}{\partial \beta^2} f(\beta; N, \rho)
\]

In Section 6 we will also consider higher derivatives of the free energy at criticality

\[
f_c^{(k)}(N, \rho) = \frac{\partial^k}{\partial \beta^k} f(\beta; N, \rho) \bigg|_{\beta = \beta_c}
\]

with \( k = 3, 4 \).

**Remark.** The definition of the specific heat (2.16c) is somewhat non-standard as it does not contain the factor \( \beta^2 \).

### 3 Finite-size-scaling corrections to the free energy

Let us start with the the basic quantity \( Z_{\alpha, \beta} \) (2.11) and write it in the form

\[
Z_{\alpha, \beta}(\mu) = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left\{ \cosh 2\mu + \sin^2 \left( \frac{\pi(n+\alpha)}{N} \right) \right. \\
- \cos \left( \frac{\pi(n+\alpha)}{N} \right) \cos \left( \frac{2\pi(m+\beta)}{M} - \frac{\pi(n+\alpha)}{N} \right) \left\} \right. 
\]

(3.1)

The product over \( m \) in (3.1) can be exactly performed with the help of the following identity [32]:

\[
\prod_{m=0}^{M-1} \left[ \zeta - \lambda \cos \left( \frac{2\pi(m+\beta)}{M} \right) \right] = \left( \frac{\lambda z_+}{2} \right)^M |1 - z_- e^{-2\pi i \beta}|^2
\]

(3.2)

where \( \zeta \) and \( \lambda \) are any two real numbers such that \( |\zeta/\lambda| \geq 1 \) and the quantities \( z_\pm \) are given by

\[
z_\pm = \frac{\zeta}{\lambda} \pm \sqrt{\left( \frac{\zeta}{\lambda} \right)^2 - 1}
\]

(3.3a)

\[
z_+ z_- = 1
\]

(3.3b)
We can finally write $Z_{\alpha,\beta}(\mu)$ as

$$Z_{\alpha,\beta}(\mu) = 2^{NM/2} \prod_{n=0}^{N-1} \left( \cosh 2\mu + \sin^2 \phi_{n+\alpha} \right. \right.$$  

$$+ \sqrt{\left[ \cosh 2\mu + \sin^2 \phi_{n+\alpha} \right]^2 - \cos^2 \phi_{n+\alpha}} \right)^{M/2} \left. \times \prod_{n=0}^{N-1} |1 - z-(n + \alpha, N, \mu)M e^{-2\pi i\beta + M\phi_{n+\alpha}}| \right) (3.4)$$

where we have used the shorthand notations

$$z_{\pm}(k, N, \mu) = \cosh 2\mu + \sin^2 \phi_{k} \pm \sqrt{\left[ \cosh 2\mu + \sin^2 \phi_{k} \right]^2 - \cos^2 \phi_{k}}$$  

$$\phi_{k} = \frac{\pi k}{N} \quad (3.5a)$$  

Let us now evaluate the functions $Z_{\alpha,\beta}(0)$ for $(\alpha, \beta) \neq (0, 0)$. We follow here the procedure used in ref. [25], which proved to be very efficient for extracting the large-$N$ asymptotic expansions of the quantities of interest. We first compute the sum

$$f_1 = \frac{M}{2} \sum_{n=0}^{N-1} N - 1 \sum_{n=0}^{N-1} \log \left[ 1 + \sin^2 \phi_{n+\alpha} + \sin \phi_{n+\alpha} \sqrt{3 + \sin^2 \phi_{n+\alpha}} \right] = \frac{M}{2} \sum_{n=0}^{N-1} \omega_1(\phi_{n+\alpha}) \quad (3.6)$$

where

$$\omega_1(k) = \log \left[ 1 + \sin^2 k + \sin k \sqrt{3 + \sin^2 k} \right] = \lambda k + \sum_{k=2}^{\infty} \frac{k^p}{p!} \lambda_p \quad (3.7)$$

The function $\omega_1$ and all its derivatives are integrable over $[0, \pi]$, and in addition,

$$\omega_1^{(k)}(\pi) - \omega_1^{(0)}(0) = \begin{cases} -2\omega_1^{(k)}(0) & k = 2, 6, 10, 12, 14, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (3.8)$$

We can now use the Euler-MacLaurin summation formula (A.6) to obtain

$$\frac{1}{N} \sum_{n=0}^{N-1} \omega_1(\phi_{n+\alpha}) = \frac{1}{\pi} \int_0^{\pi} \omega_1(x) dx - \frac{\lambda}{\pi N^2} B_2(\alpha)$$

$$- \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{B_{2m+2}(\alpha)}{(2m+2)!} \lambda_{2m+1} \quad (3.9)$$

The first coefficients $\lambda_k$ are

$$\lambda = \sqrt{3}; \quad \lambda_3 = \lambda_7 = 0; \quad \lambda_5 = \frac{16}{\sqrt{3}}; \quad \lambda_9 = 1792\sqrt{3}; \quad \lambda_{11} = -\frac{51200}{\sqrt{3}} \quad (3.10)$$
The final result for $f_1$ is

$$f_1 = \frac{NM}{2\pi} \int_0^\pi \omega_1(x)dx - \frac{\pi \lambda \rho}{2} B_2(\alpha) - \pi \rho \sum_{m=1}^\infty \left( \frac{\pi}{N} \right)^{2m} \frac{B_{2m+2}(\alpha)}{(2m+2)!} \lambda_{2m+1}$$ (3.11)

Let us now consider the quantity $f_2$

$$f_2 = \sum_{n=0}^{N-1} \log |1 - z_-(n + \alpha, N, 0)|^M e^{-2\pi i \beta + M i \phi_{n+\alpha}}$$ (3.12)

We first note that when $n + \alpha = N/2$, the factor $z_-(n + \alpha, N, 0) = 0$, so this term does not contribute to the sum (3.12). In the other cases $z_-(n + \alpha, N, 0)$ does not vanish and we can use (3.3b) to write (3.12) as

$$f_2 = \sum_{n=0}^{N-1} \log \left| 1 - e^{-M \log z_+(n + \alpha, N, 0)} - 2\pi i \beta + M i \phi_{n+\alpha} \right|$$ (3.13)

where $\sum'$ means that we have taken out the term with $n + \alpha = N/2$ (if such term exists).

We now proceed as in ref. [25]: we first write $\log |1 - e^{-A}| = \text{Re} \log(1 - e^{-A})$ and then we expand $\log(1 - e^{-A})$ as a power series in $e^{-A}$:

$$f_2 = -\text{Re} \sum_{p=1}^{\infty} \sum_{n=0}^{\lfloor N/2 \rfloor - 1} \frac{1}{p} e^{-2p \left[ \frac{M \log z_+(n + \alpha, N, 0) - i \phi_{n+\alpha}}{2 + \pi i \beta} \right]}$$ (3.14)

It is convenient to write the function $\log z_+(k, N, 0)$ as

$$\log z_+(k, N, 0) \equiv \omega_2(\phi_k) = \omega_1(\phi_k) - \log \cos \phi_k$$ (3.15)

where $\omega_1(k)$ is the function (3.7). We then split the sum over $n$ into two parts: $n \in [0, \lfloor N/2 \rfloor - 1]$, and $n \in [\lceil N/2 \rceil, N - 1]$. By making the substitution $n \rightarrow N - 1 - n$ in the second sum, we finally obtain

$$f_2 = -\text{Re} \sum_{p=1}^{\infty} \sum_{n=0}^{\lfloor N/2 \rfloor - 1} \frac{1}{p} e^{-2p \left[ M \omega_2(\phi_{n+\alpha}) - i \phi_{n+\alpha} \right] + i \pi \beta}$$

$$-\text{Re} \sum_{p=1}^{\infty} \sum_{n=0}^{N - \lfloor N/2 \rfloor - 1} \frac{1}{p} e^{-2p \left[ M \omega_2(\phi_{n+1-\alpha}) - i \phi_{n+\alpha} \right] - i \pi \beta}$$ (3.16)

We now expand the function $\omega_2(k)$ as a power series in $k$

$$\omega_2(k) = \lambda k + \sum_{m=1}^\infty \frac{\lambda_{2m+1}}{(2m+1)!} k^{2m+1}$$ (3.17)
where the $\lambda_k$ are exactly those of the function $\omega_1$ (3.10). We obtain an expression of the form

$$f_2 = - \text{Re} \sum_{p=0}^{\infty} \frac{1}{p} \sum'_{n=0} e^{-2p[\pi \tau_0 (n+\alpha) + i\beta]}$$

$$\times \exp \left\{ -\pi p\rho \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{\lambda_{2m+1}}{(2m+1)!} (n+\alpha)^{2m+1} \right\}$$

$$- \text{Re} \sum_{p=0}^{\infty} \frac{1}{p} \sum'_{n=0} e^{-2p[\pi \tau_0 (n+\alpha-\alpha) - i\beta]}$$

$$\times \exp \left\{ -\pi p\rho \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{\lambda_{2m+1}}{(2m+1)!} (n+1-\alpha)^{2m+1} \right\}$$

(3.18)

where $\tau_0$ is a complex number equal to

$$\tau_0 = \frac{\lambda - i}{2} = \frac{\sqrt{3} - i}{2} = e^{-i\pi/6}$$

(3.19)

The next step consists in expanding the exponentials in powers of $N^{-k}$. By following the procedure introduced in [25, Appendix B] we obtain

$$f_2 = - \text{Re} \sum_{p=0}^{\infty} \frac{1}{p} \sum'_{n=0} \left\{ 1 - p\pi\rho \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{\Lambda_{2m+1}}{(2m+1)!} (n+\alpha)^{2m+1} \right\}$$

$$\times \exp \left\{ -2p[\pi \tau_0 (n+\alpha) + i\beta] \right\}$$

$$- \text{Re} \sum_{p=0}^{\infty} \frac{1}{p} \sum'_{n=0} \left\{ 1 - p\pi\rho \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{\Lambda_{2m+1}}{(2m+1)!} (n+1-\alpha)^{2m+1} \right\}$$

$$\times \exp \left\{ -2p[\pi \tau_0 (n+1+\alpha) - i\beta] \right\}$$

(3.20)

where the $\Lambda_k$ are certain differential operators. The first ones are

$$\Lambda_3 = \Lambda_7 = 0$$

(3.21a)

$$\Lambda_5 = \lambda_5$$

(3.21b)

$$\Lambda_9 = \lambda_9 + \frac{63}{5} \lambda_5^2 \frac{\partial}{\partial \lambda}$$

(3.21c)

$$\Lambda_{11} = \lambda_{11}$$

(3.21d)

We can now extend the sum over $n$ to $n = \infty$ as the error is exponentially small. On the other hand, the contribution of the term with $n+\alpha = N/2$ is also exponentially small, so we can take out this constraint. Then, after rearranging the sums, we obtain

$$f_2 = \sum_{n=0}^{\infty} \log \left| 1 - e^{-2\pi[\rho \tau_0 (n+\alpha) + i\beta]} \right| + \sum_{n=0}^{\infty} \log \left| 1 - e^{-2\pi[\rho \tau_0 (n+1-\alpha) - i\beta]} \right|$$

$$+ \pi \rho \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{\Lambda_{2m+1}}{(2m+1)!} \text{Re} \sum_{p=1}^{\infty} \sum_{n=0}^{\infty} \left\{ (n+\alpha)^{2m+1} e^{-2p\pi[\rho \tau_0 (n+\alpha) + i\beta]} \right\}$$

$$+ (n+1-\alpha)^{2m+1} e^{-2p\pi[\rho \tau_0 (n+1-\alpha) + i\beta]}$$

(3.22)
The desired result can be obtained by plugging in (B.13)/(C.2):

\[
f_2 = \log \left| \frac{\theta_{\alpha,\beta}(i\tau_0\rho)}{\eta(i\tau_0\rho)} \right| + \frac{\pi \lambda \rho}{2} B_2(\alpha) + \pi \rho \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{\Lambda_{2m+1}}{(2m+2)!} \left[ B_{2m+2}(\alpha) - \Re K_{2m+2}^{\alpha,\beta}(i\tau_0\rho) \right] (3.23)
\]

where the elliptic \( \theta \)-function \( \theta_{\alpha,\beta} \) and the Dedekind’s \( \eta \)-function are defined in Appendix B, the objects \( B_p(\alpha) \) are Bernoulli polynomials defined in Appendix A, and \( K_{2m+2}^{\alpha,\beta} \) are Kronecker’s double series defined in Appendix C. Then, the value of \( Z_{\alpha,\beta}(0) \) is given by

\[
\log Z_{\alpha,\beta}(0) = \frac{NM}{2} \log 2 + \frac{NM}{2\pi} \int_0^\pi \omega_1(t) \, dt + \log \left| \frac{\theta_{\alpha,\beta}(i\tau_0\rho)}{\eta(i\tau_0\rho)} \right| - \pi \rho \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{\Lambda_{2m+1}}{(2m+2)!} \Re K_{2m+2}^{\alpha,\beta}(i\tau_0\rho) (3.24)
\]

The free energy at the critical point can be computed directly from (2.10):

\[
f_c(N, M) = -\frac{1}{V} \log 2 + \frac{1}{2} \log(2 \sinh 2\beta_c) + \frac{1}{V} \log \sum_{\alpha,\beta} Z_{\alpha,\beta}(0) (3.25)
\]

The result (3.24) means that the free energy for both lattices can be written as

\[
f_c(N, \rho) = f_{\text{bulk}} + \sum_{m=1}^{\infty} \frac{f_{2m}(\rho)}{N^{2m}} (3.26)
\]

Thus, only even powers of \( N^{-1} \) can occur, and in contrast to what happens in the square-lattice, we find some even powers whose coefficient vanishes (e.g., \( f_4 = f_8 = 0 \)). The above result agrees with the formula found by Izmailian and Hu \[41\] for an Ising model on a \( N \times \infty \) hexagonal (or triangular) lattice with periodic boundary conditions.

The first coefficients for the triangular lattice are given by

\[
\begin{align*}
f_{\text{tri}} = & \frac{1}{2} \log \frac{4}{\sqrt{3}} + \frac{1}{2\pi} \int_0^\pi \omega_1(t) \, dt \approx 0.8795853861 \ldots (3.27a) \\
f_2^{\text{tri}}(\rho) = & \frac{1}{\rho} \log \frac{|\theta_2| + |\theta_3| + |\theta_4|}{2|\eta|} (3.27b) \\
f_4^{\text{tri}}(\rho) = & f_8^{\text{tri}}(\rho) = 0 (3.27c) \\
f_6^{\text{tri}}(\rho) = & -\frac{\pi^5}{45\sqrt{3}} \Re \frac{|\theta_4| K_6^{\alpha,\beta} + |\theta_2| K_6^{\alpha,\beta} + |\theta_3| K_6^{\alpha,\beta}}{|\theta_2| + |\theta_3| + |\theta_4|} (3.27d)
\end{align*}
\]

where the \( \theta_i \) are the standard \( \theta \)-functions defined in (B.10) and the functions \( K_6^{\alpha,\beta} \) are given in terms of \( \theta \)-functions in (C.4). As explained in the Appendix B all the
functions $\theta_i$, $\eta$ and $K^{\alpha,\beta}_p$ are evaluated at $z = 0$ and $\tau = i\tau_0\rho$ (B.11). The numerical values of these coefficients for several values of $\rho$ can be found in Table 1.

The coefficients of the hexagonal-lattice expansion are found to be

$$f^\text{hc}_{\text{bulk}} = \frac{1}{2} \log 2\sqrt{6} + \frac{1}{4\pi} \int_0^\pi \omega_1(t) \, dt \approx 1.0250590964 \ldots$$  \hspace{1cm} (3.28a)

$$f^\text{hc}_{2m}(\rho) = \frac{1}{2} f^\text{tri}_{2m}(\rho)$$  \hspace{1cm} (3.28b)

The numerical values of the coefficients $f^\text{hc}_2$ and $f^\text{hc}_6$ can be obtained from Table 1 with the help of (3.28b).

Remarks. 1. The values of the bulk critical free energy (3.27a)/(3.28a) indeed coincide with the values obtained from the well-known results in the thermodynamic limit \[39, 40, 38\] when $\beta = \beta_c$:

$$f^\text{tri}_{\text{bulk}}(\beta) = \frac{1}{2} \int_0^\pi \int_0^\pi dx dy \log \left[ \cosh^3 2\beta + \sinh^3 2\beta - \omega(x, y) \sinh 2\beta \right]$$

$$+ \log 2$$  \hspace{1cm} (3.29a)

$$f^\text{hc}_{\text{bulk}}(\beta) = \frac{1}{4} \int_0^\pi \int_0^\pi dx dy \log \left[ 1 + \cosh^3 2\beta - \omega(x, y) \sinh^2 2\beta \right]$$

$$+ \frac{3}{4} \log 2$$  \hspace{1cm} (3.29b)

where

$$\omega(x, y) = \cos x + \cos y + \cos(x - y)$$  \hspace{1cm} (3.30)

2. The limiting values of the coefficients $f_2$ and $f_6$ as $\rho \to \infty$ are easily found to be [c.f., (B.12)]

$$\lim_{\rho \to \infty} f^\text{tri}_2(\rho) = \frac{\sqrt{3\pi}}{24}$$  \hspace{1cm} (3.31a)

$$\lim_{\rho \to \infty} f^\text{tri}_6(\rho) = \frac{31\pi^5}{60480\sqrt{3}}$$  \hspace{1cm} (3.31b)

The corresponding limiting values for the hexagonal lattice are one half of the above values [c.f., (3.28b)].

3. Using the properties of the $\theta$-functions (B.20)/(B.21) and of the functions $K^{\alpha,\beta}_6$ (C.5)/(C.6) we can easily check that the terms (3.27b)/(3.27d) have the correct behavior under the transformation $N \leftrightarrow M (\rho \to 1/\rho)$. In particular,

$$f_2(\rho) = \frac{f_2(1/\rho)}{\rho^2}$$  \hspace{1cm} (3.32)

$$f_6(\rho) = \frac{f_6(1/\rho)}{\rho^6}$$  \hspace{1cm} (3.33)

4. From (3.24)/(3.21) we see that there is in general a non-zero contribution to $\log Z_{\alpha,\beta}(0)$ at any order $N^{-2m}$ with $m \geq 4$. However, we cannot rule out cancellations leading to the vanishing of any of the coefficients $f_{2m}(\rho)$ with $m \geq 5$ in (3.26). Similar arguments apply to the other large-$N$ expansions in the next sections.
4 Finite-size-scaling corrections to the internal energy

Now we will deal with the internal energy (2.16b). Using (2.10)/(2.13) we can write the critical internal energy as follows:

\[ -E_c(N, \rho) = \coth 2\beta_c + \frac{1}{V} \left. \frac{d\mu}{d\beta} \right|_{\beta=\beta_c} \sum_{\alpha,\beta} Z_{\alpha,\beta}(0) \]  

(4.1)

The derivative \( d\mu/d\beta \) can be easily computed from eq. (2.12). Thus, the only unknown object is \( Z'_{0,0}(0) \), which can be written as

\[ Z'_{0,0}(0) = 2M 2^{NM/2} \prod_{n=0}^{N-1} \left( 1 + \sin^2 \phi_n + \sin \phi_n \sqrt{3 + \sin^2 \phi_n} \right)^{M/2} \]

\[ \times \prod_{n=1}^{N-1} \left| 1 - z_-(n, N, 0)^M e^{Mi\phi_n} \right| \]  

(4.2)

Table 1: Values of the coefficients \( f_{2}^{\text{tri}}(\rho) \) and \( f_{6}^{\text{tri}}(\rho) \) for several values of the torus aspect ratio \( \rho \).
By noting that the first product is nothing more than $f_1(3.10)$ with $\alpha = 0$, we can write (4.2) as

$$\log Z'_{0,0}(0) = \frac{NM}{2} \log 2 + \log M + \frac{NM}{2\pi} \int_0^\pi \omega_1(t) dt$$

$$-\frac{\rho \lambda}{2} B_2(0) - \pi \rho \sum_{m=1}^\infty \left( \frac{\pi}{N} \right)^{2m} \frac{B_{2m+2}(0)}{(2m+2)!} \lambda_{2m+1}$$

$$+ \sum_{n=1}^{N-1} \log \left| 1 - z_-(n, N, 0) e^{i\phi_n} \right|$$

(4.3)

The last sum in (4.3) is equal to the definition of $f_2(3.12)$ with $\alpha = 0$, except for the fact that the sum in (4.3) starts at $n = 1$ rather than at $n = 0$. We can follow step by step the same procedure leading to (3.22): the result coincides with (3.22) when $\alpha = 0$ except that the first sum in (3.22) now starts at $n = 1$. Using (B.16)/(C.2) we obtain the final result

$$\log Z'_{0,0}(0) = \frac{NM}{2} \log 2 + \log M + \frac{NM}{2\pi} \int_0^\pi \omega_1(t) dt + 2 \log |\eta(i\tau_0 \rho)|$$

$$-\pi \rho \sum_{m=1}^\infty \left( \frac{\pi}{N} \right)^{2m} \frac{A_{2m+1}}{(2m+2)!} \operatorname{Re} K_{2m+2}^0(i\tau_0 \rho)$$

(4.4)

This equation implies that the critical internal energy can be written as a power series in $N^{-1}$:

$$-E_c(N, \rho) = E_0 + \sum_{m=0}^\infty \frac{E_{2m+1}(\rho)}{N^{2m+1}}$$

(4.5)

For the triangular lattice we find that

$$E_{0}^{\text{tri}} = 2$$

$$E_{1}^{\text{tri}}(\rho) = \frac{3|\theta_2 \theta_3 \theta_4|}{|\theta_2| + |\theta_3| + |\theta_4|}$$

$$E_{3}^{\text{tri}}(\rho) = E_{7}^{\text{tri}}(\rho) = 0$$

$$E_{5}^{\text{tri}}(\rho) = -\frac{\pi^5 \rho}{15\sqrt{3}} \frac{|\theta_2 \theta_3 \theta_4|}{(|\theta_2| + |\theta_3| + |\theta_4|)^2} \operatorname{Re} \left\{ \left( |\theta_2| + |\theta_3| + |\theta_4| \right) K_{6}^{0,0} \right.$$

$$- |\theta_4| K_{6}^{1,0} - |\theta_2| K_{6}^{0,\frac{1}{2}} - |\theta_3| K_{6}^{1,\frac{1}{2}} \left. \right\}$$

(4.6d)

where we have used (3.13)/(C.4). The numerical values of these coefficients can be found in Table 2. In the hexagonal-lattice case we obtain

$$E_{0}^{\text{hc}} = \frac{2}{\sqrt{3}}$$

$$E_{2m+1}^{\text{hc}}(\rho) = \frac{E_{2m+1}^{\text{tri}}(\rho)}{2\sqrt{3}}$$

(4.7b)
The numerical values of the coefficients $E_{1}^{hc}$ and $E_{5}^{hc}$ can be obtained from Table 2 by using (4.7b).

**Remarks.** 1. The limiting values of the coefficients $E_{1}$ and $E_{5}$ as $\rho \to \infty$ are easily found to be [c.f., (B.12)]

$$
\lim_{\rho \to \infty} E_{1}(\rho) = \lim_{\rho \to \infty} E_{5}(\rho) = 0 \quad (4.8)
$$

This formula is valid for the triangular and hexagonal lattices. In particular, we expect that all the coefficients $E_{2m+1}(\rho)$ will vanish in the limit $\rho \to \infty$ due to the existence of the factor $|\theta_{2}\theta_{3}\theta_{4}|$ which vanishes exponentially fast. Thus, on an infinitely long torus, the internal energy for any finite width $N$ is equal to the bulk value $E_{0}$ with no finite-size corrections.

2. Using the properties of the $\theta$-functions (B.20)/(B.21) and of the functions $K_{6}^{\alpha,\beta} (C.5)/(C.6)$ we can easily check that the coefficients $E_{1}$ and $E_{5}$ (4.6b)/(4.6d)/(4.7b) have the correct behavior under the transformation $\rho \to 1/\rho$. In particular,

$$
E_{1}(\rho) = \frac{E_{1}(1/\rho)}{\rho} \quad (4.9)
$$

$$
E_{5}(\rho) = \frac{E_{5}(1/\rho)}{\rho^{5}} \quad (4.10)
$$

## 5 Finite-size-scaling corrections to the specific heat

The specific heat at criticality is given by the following formula

$$
C_{H,c} = -\frac{2}{\sinh^{2}2\beta_{c}} + \frac{1}{V} \left. \frac{d^{2}\mu}{d\beta^{2}} \right|_{\beta=\beta_{c}} \sum_{\alpha,\beta} \frac{Z'_{0,\beta}(0)}{Z_{\alpha,\beta}(0)}
$$

$$
+ \frac{1}{V} \frac{d\mu}{d\beta} \left|_{\beta=\beta_{c}} \left[ \sum_{\alpha,\beta} \frac{Z''_{\alpha,\beta}(0)}{Z_{\alpha,\beta}(0)} - \left( \frac{Z'_{0,\beta}(0)}{Z_{\alpha,\beta}(0)} \right)^{2} \right] \right. \quad (5.1)
$$

The main goal of this section is to compute the ratio

$$
\sum_{\alpha,\beta} \frac{Z''_{\alpha,\beta}(0)}{Z_{\alpha,\beta}(0)} \quad (5.2)
$$

where the sums go over $(\alpha, \beta) \neq (0, 0)$. After some algebra, we can write the derivative $Z''_{\alpha,\beta}(0)$ as follows

$$
Z''_{\alpha,\beta}(0) = \frac{4MN}{\pi \sqrt{3}} Z_{\alpha,\beta}(0) \left[ S_{\alpha}^{(1)} + 2S_{\alpha,\beta}^{(2)} + \frac{\pi \sqrt{3}}{4} \rho \delta_{\alpha,0} \right] \quad (5.3)
$$
Table 2: Values of the coefficients $E_{1}^{\text{tri}}(\rho)$ and $E_{5}^{\text{tri}}(\rho)$ for several values of the torus aspect ratio $\rho$.

| $\rho$ | $E_{1}^{\text{tri}}(\rho)$ | $E_{5}^{\text{tri}}(\rho)$ |
|-------|-----------------------------|-----------------------------|
| 1     | 1.017408797595956           | -0.359705063388737          |
| 2     | 0.612513647162813           | -0.178088378079924          |
| 3     | 0.345040108164264           | -0.168599461543254          |
| 4     | 0.185288835745847           | -0.127979167922216          |
| 5     | 0.096804501605795           | -0.086206117890971          |
| 6     | 0.049827662298672           | -0.054108487929080          |
| 7     | 0.025447703091251           | -0.032506071497113          |
| 8     | 0.012944169002509           | -0.018975959727317          |
| 9     | 0.006570580061525           | -0.010859541565321          |
| 10    | 0.003331786807789           | -0.006125084912961          |
| 11    | 0.001688570266906           | -0.003416526641140          |
| 12    | 0.000855546533105           | -0.00188941526185           |
| 13    | 0.000433419665204           | -0.001036828005370          |
| 14    | 0.000219555049642           | -0.00056566233279           |
| 15    | 0.000112214898315           | -0.000307012198010          |
| 16    | 0.000056334542069           | -0.00016588347105           |
| 17    | 0.000028535313425           | -0.00008278100310           |
| 18    | 0.000014454016292           | -0.00004782460122           |
| 19    | 0.000007321388062           | -0.000025601385603          |
| 20    | 0.000003708495908           | -0.000013650380771          |

where the sums $S^{(j)}$ are given by

$$S_{\alpha}^{(1)} = \frac{\pi \sqrt{3}}{2N} \sum_{n=\delta_{\alpha,0}}^{N-1} \frac{1}{\sin \phi_{n+\alpha} \sqrt{3 + \sin^2 \phi_{n+\alpha}}}$$

$$S_{\alpha,\beta}^{(2)} = \Re \sum_{n=\delta_{\alpha,0}}^{N-1} \frac{1}{\sin \phi_{n+\alpha} \sqrt{3 + \sin^2 \phi_{n+\alpha}}} \frac{z_{M+\frac{M}{2}} e^{-2\pi i \beta + M i \phi_{n+\alpha}}}{1 - z_{M+\frac{M}{2}} e^{-2\pi i \beta + M i \phi_{n+\alpha}}}$$

The variables $\phi_{n+\alpha}$ and $z_{-} = z_{-}(n+\alpha, N, 0)$ are given by (3.5) and $\delta_{\alpha,0}$ is the usual Kronecker’s delta.

The first step is to compute the sum $S_{\alpha}^{(1)}$ (5.4a). We will follow a procedure similar to the one used in ref. [24] for the square lattice. Let us define the function

$$\omega_3(k) = \frac{\sqrt{3}}{\sin k \sqrt{3 + \sin^2 k}} - \frac{1}{k} + \frac{1}{k - \pi}$$

This function and all its derivatives are integrable over the interval $[0, \pi]$ so we can
apply the Euler-MacLaurin formula (A.6). The final result is

$$S_\alpha^{(1)}(N) = \sum_{n=\delta_{\alpha,0}}^{N-1} \frac{1}{n+\alpha} + \frac{1}{2N} \delta_{\alpha,0} + \frac{1}{2} \int_0^\pi \omega_3(t) \, dt - \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{B_{2m}(\alpha)}{(2m)!} \tilde{\gamma}_{2m-1}$$

(5.6)

where the coefficients $\tilde{\gamma}_{2m-1}$ come from the expansion of $\omega_3(k)$ in powers of $k$:

$$\omega_3(k) = \sum_{m=0}^{\infty} \tilde{\gamma}_m \frac{k^m}{m!}$$

(5.7a)

$$= -\sum_{m=0}^{\infty} \frac{k^m}{\pi^{m+1}} + \sum_{m=1}^{\infty} \frac{\gamma_{2m+1}}{(2m+1)!} \frac{2m+1}{k^{2m+1}}$$

(5.7b)

In general, the coefficient $\tilde{\gamma}_m$ contains two contributions: one comes from the term $1/(k-\pi)$ which gives the (trivial) coefficient $-\pi^{-(m+1)}m!$, and the other contribution comes from the first two terms in the l.h.s. of (5.5). We will denote by $\gamma_m$ this latter (non-trivial) contribution. In particular, only the coefficients $\gamma_{2m+1}$ with $m = 1, 2, 3, \ldots$ are non-zero. The first non-vanishing coefficients $\gamma_m$ are

$$\gamma_3 = \frac{8}{15}; \quad \gamma_5 = -\frac{80}{21}; \quad \gamma_7 = \frac{448}{5}$$

(5.8)

On the other hand, the value of the integral in (5.6) is

$$\frac{1}{2} \int_0^\pi \omega_3(t) \, dt = \log \frac{\sqrt{3}}{\pi}$$

(5.9)

In computing the sums $\sum_{n=\delta_{\alpha,0}}^{N-1} (n+\alpha)^{-1}$ we will use the result (See e.g., [12])

$$\sum_{n=1}^{N} \frac{1}{N} = \log N + \gamma_E + \frac{1}{2N} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{N^{2k}}$$

(5.10)

(where $\gamma_E \approx 0.5772156649$ is the Euler constant) and take into account that $\alpha = 0, 1/2$. In the simplest case $\alpha = 0$ we have

$$S_0^{(1)}(N) = \log N + \gamma_E + \log \frac{\sqrt{3}}{\pi} - \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{B_{2m}}{(2m)!} \tilde{\gamma}_{2m-1} + \frac{1}{2N}$$

$$+ \frac{1}{2(N-1)} + \log \left( 1 - \frac{1}{N} \right) - \sum_{m=1}^{\infty} \frac{B_{2m}}{2m} \frac{1}{(N-1)^{2m}}$$

(5.11)

This expression can be simplified by expanding it in powers of $N^{-1}$, and then using formulas (A.3)/(A.10). A further simplification can be made if we take into account (5.7b). The final result for $\alpha = 0$ is

$$S_0^{(1)}(N) = \log N + \gamma_E + \log \frac{\sqrt{3}}{\pi} - \sum_{m=2}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{B_{2m}}{(2m)!} \gamma_{2m-1}$$

(5.12)
The value for $\alpha = 1/2$ can be obtained using similar arguments in addition to (A.3). The final result for $S_{\alpha}^{(1)}$ is

$$S_{\alpha}^{(1)}(N) = \log N + \gamma_E + \log \frac{4\sqrt{3}}{\pi} - \log 4 \delta_{\alpha,0} - \sum_{m=2}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{B_{2m}^{(1)}(\alpha)}{(2m)!} \gamma_{2m-1}$$

(5.13)

In the above result only enter the non-trivial Taylor coefficients of the function $\omega_3$.

The second step is to compute the sums $S_{\alpha,\beta}^{(2)}$ (5.4b). The procedure is similar to the ones already done in Sections 3 and 4. We first write

$$S_{\alpha,\beta}^{(2)} = \frac{\pi \sqrt{3}}{2N} \text{Re} \left[ \sum_{n=\delta_{\alpha,0}}^{[N/2]-1} \frac{1}{\sin \phi_{n+\alpha} \sqrt{3 + \sin^2 \phi_{n+\alpha}}} e^{-2[M(\omega_2(\phi_{n+\alpha}) - i\phi_{n+\alpha})/2/\pi i \beta] \right] - e^{-2[M(\omega_2(\phi_{n+\alpha}) - i\phi_{n+\alpha})/2/\pi i \beta]}

+ \sum_{n=0}^{N-[N/2]-1} \left( \alpha \to 1 - \alpha \beta \to -\beta \right)$$

(5.15)

where the second term is the same as the first one with $(\alpha, \beta)$ replaced by $(1 - \alpha, -\beta)$. Now we perform several Taylor expansions: first, we expand the denominator $1 - e^{-2A}$ in powers of $e^{-2A}$:

$$S_{\alpha,\beta}^{(2)} = \frac{\pi \sqrt{3}}{2N} \text{Re} \left[ \sum_{n=\delta_{\alpha,0}}^{[N/2]-1} \sum_{p=1}^{\infty} \frac{e^{-2p[M(\omega_2(\phi_{n+\alpha}) - i\phi_{n+\alpha})/2/\pi i \beta]}}{\sin \phi_{n+\alpha} \sqrt{3 + \sin^2 \phi_{n+\alpha}}} \right] - \sum_{n=0}^{N-[N/2]-1} \sum_{p=1}^{\infty} \frac{e^{-2p[M(\omega_2(\phi_{n+1-\alpha}) - i\phi_{n+1-\alpha})/2/\pi i \beta]}}{\sin \phi_{n+1-\alpha} \sqrt{3 + \sin^2 \phi_{n+1-\alpha}}}$$

(5.16)

Secondly, we expand $e^{-2p(M\omega_2/2)}$ as we did in (3.18) and finally, we expand the function

$$\frac{\sqrt{3}}{\sin k \sqrt{3 + \sin^2 k}} = \omega_3(k) + \frac{1}{k} - \frac{1}{k - \pi} = \frac{1}{k} + \sum_{m=1}^{\infty} \frac{\gamma_{2m+1}}{(2m+1)!} k^{2m+1}$$

(5.17)

in powers of $k$. After rearranging the series, extending the sums over $n$ to $\infty$ (as the error is exponentially small) and using (3.14)/(C.2) we obtain

$$S_{\alpha,\beta}^{(2)} = - \text{Re} \log \theta_{\alpha,\beta} + \left[ \log 2 - \frac{\pi \rho \sqrt{3}}{8} \right] \delta_{\alpha,0}

+ \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{\pi}{N} \right)^{2k+2} \frac{\gamma_{2k+1}}{(2k+2)!} \left[ B_{2k+2}(\alpha) - \text{Re} K_{2k+2}(\tau_0 \rho) \right]$$
\[-\frac{\pi \rho}{2} \sum_{k,m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m+2k+2} \frac{\Lambda_{2m+1}}{(2m+1)!} \frac{\gamma_{2k+1}}{(2k+2)!} W^{\alpha,\beta}_{2m+2k+2}(i\tau_0 \rho)\]

\[-\frac{\pi \rho}{2} \sum_{m=1}^{\infty} \left( \frac{\pi}{N} \right)^{2m} \frac{\Lambda_{2m+1}}{(2m+1)!} W^{\alpha,\beta}_{2m}(i\tau_0 \rho)\]  

where the function \( W^{\alpha,\beta}_m(\tau) \) is defined as follows:

\[
W^{\alpha,\beta}_m(\tau) = \text{Re} \sum_{n=0}^{\infty} \left[ (n+\alpha)^m \frac{e^{2\pi i (\tau n+\alpha) - \beta}}{1 - e^{2\pi i (\tau n+\alpha) - \beta}} \right] + (n+1-\alpha)^m \frac{e^{2\pi i (\tau (n+1-\alpha) + \beta)}}{1 - e^{2\pi i (\tau (n+1-\alpha) + \beta)}}\]

The ratio \( Z''_{\alpha,\beta}(0)/Z_{\alpha,\beta}(0) \) can be written as a power series in \( N^{-1} \):

\[
\frac{1}{MN} \frac{Z''_{\alpha,\beta}(0)}{Z_{\alpha,\beta}(0)} = \frac{4}{\pi \sqrt{3}} \left[ \log N + \gamma_E + \log \frac{4\sqrt{3}}{2} - 2 \text{Re} \log \theta_{\alpha,\beta} \right] + \sum_{m=2}^{\infty} \frac{d_{2m}(\rho)}{N^{2m}}
\]

This series contains only even powers of \( N^{-1} \) and it starts at \( N^{-4} \) (i.e., \( d_{2m}(\rho) = 0 \)). The first non-vanishing coefficient \( \tilde{d}_{2m}(\rho) \) is

\[
\tilde{d}_{2m}(\rho) = -\frac{4\pi^4}{45} \text{Re} \mathcal{K}_{\alpha,\beta}^{4}(i\tau_0 \rho) - \frac{2\pi^5 \rho}{15\sqrt{3}} W^{\alpha,\beta}_{4}(i\tau_0 \rho)
\]

It is worth noticing that the terms with \( \delta_{\alpha,0} \) in \((5.3)/(5.13)/(5.18)\) cancel out exactly.

The computation of the ratio \((5.2)\) is straightforward from \((3.24)/(5.20)\). The leading term grows like \( \log N \) and the rest can be expressed as a power series in \( N^{-1} \) where only even powers of \( N^{-1} \) enter:

\[
\frac{1}{MN} \sum_{\alpha,\beta} \frac{Z''_{\alpha,\beta}(0)}{Z_{\alpha,\beta}(0)} = \frac{4}{\pi \sqrt{3}} \log N + d_0(\rho) + \sum_{m=2}^{\infty} \frac{d_{2m}(\rho)}{N^{2m}}
\]

The coefficient associated to \( N^{-2} \) vanishes, so the first two non-zero coefficients \( d_m(\rho) \) are

\[
d_0(\rho) = \frac{4}{\pi \sqrt{3}} \left[ \gamma_E + \log \frac{4\sqrt{3}}{\pi} - \sum |\theta_i| \text{Re} \log \theta_i \right]
\]

\[
d_4(\rho) = -\frac{4\pi^3}{45} \left\{ \frac{2\pi^2 \rho}{3} \left( \sum |\theta_i| \text{Re} \log \theta_i \right) \left( \sum |\theta_{\alpha,\beta}| \text{Re} \mathcal{K}_{\alpha,\beta}^{6} \right) \right. \\
- \frac{\sum |\theta_{\alpha,\beta}| \text{Re} \mathcal{K}_{\alpha,\beta}^{6} \text{Re} \log \theta_{\alpha,\beta}} {\sum |\theta_i|} \left. \right\} \\
+ \frac{1}{\sqrt{3}} \sum |\theta_{\alpha,\beta}| \text{Re} \mathcal{K}_{\alpha,\beta}^{4} \left( \sum |\theta_i| \right) + 2\pi \rho \sum |\theta_{\alpha,\beta}| W^{\alpha,\beta}_{4}(\rho)
\]
where we have denoted by $\theta_i$ the $\theta$-functions in the standard notation \[(B.10)\]. The numerical values of these coefficients can be found in Table $3$.

**Remarks.**

1. The limiting values of the coefficients $d_0$ and $d_4$ as $\rho \to \infty$ are easily found to be [c.f., (B.12)]

\[
\begin{align*}
\lim_{\rho \to \infty} d_0(\rho) &= \frac{4}{\pi \sqrt{3}} \left[ \gamma_E + \log \frac{4\sqrt{3}}{\pi} \right] \\
\lim_{\rho \to \infty} d_4(\rho) &= -\frac{7\pi^3}{2700\sqrt{3}}
\end{align*}
\]

2. Using the properties of the $\theta$-functions \[(B.20)/(B.21)\] we can easily verify that $d_0(\rho)$ has the right behavior under the transformation $N \leftrightarrow M$ ($\rho \to 1/\rho$):

\[
\frac{4}{\pi \sqrt{3}} \log \rho + d_0(1/\rho) = d_0(\rho)
\]

The behavior of $d_4(\rho)$ under this transformation can be checked numerically to be the right one:

\[
d_4(\rho) = \frac{d_4(1/\rho)}{\rho^4}
\]

The specific heat for the triangular and hexagonal lattices can be obtained from \[(5.1)\] and using the results of this section and of Section $4$. In particular, we can write for both lattices

\[
C_{H,c}(N, \rho) = C_{00} \log N + C_0(\rho) + \sum_{m=1}^{\infty} C_m(\rho) \frac{N^m}{\rho^m}
\]

For the triangular lattice the first coefficients are given by

\[
\begin{align*}
C_{00}^{\text{tri}} &= \frac{12\sqrt{3}}{\pi} \\
C_0^{\text{tri}}(\rho) &= 9d_0(\rho) - 6 - \rho E_1^{\text{tri}}(\rho)^2 \\
C_1^{\text{tri}}(\rho) &= -2E_1^{\text{tri}}(\rho) \\
C_2^{\text{tri}}(\rho) &= C_3^{\text{tri}}(\rho) = 0 \\
C_4^{\text{tri}}(\rho) &= 9d_4(\rho) - 2\rho E_1^{\text{tri}}(\rho) E_5^{\text{tri}}(\rho) \\
C_5^{\text{tri}}(\rho) &= -2E_5^{\text{tri}}(\rho)
\end{align*}
\]

and for the hexagonal lattice the corresponding coefficients are

\[
\begin{align*}
C_{00}^{\text{hc}} &= \frac{2\sqrt{3}}{\pi} \\
C_0^{\text{hc}}(\rho) &= \frac{3}{2}d_0(\rho) - \frac{2}{3} - 2\rho E_1^{\text{hc}}(\rho)^2
\end{align*}
\]
| $\rho$   | $d_0(\rho)$ | $d_4(\rho)$ |
|---------|-------------|-------------|
| 1       | 0.993000152525293 | -0.034652876469773 |
| 2       | 1.205930021583709  | -0.084727027938228  |
| 3       | 1.233520243783654  | -0.1462954292750869 |
| 4       | 1.189798214112785  | -0.167434330275211  |
| 5       | 1.134144577781982  | -0.157595538508037  |
| 6       | 1.08841663135744    | -0.13456242800881   |
| 7       | 1.056420958518946   | -0.110373125092552  |
| 8       | 1.035808103247928   | -0.090138741047513  |
| 9       | 1.023167108495542   | -0.075083168632513  |
| 10      | 1.015661512376353   | -0.064637113563536  |
| 11      | 1.011305166086030   | -0.057721974432342  |
| 12      | 1.008818898345350   | -0.053296998167206  |
| 13      | 1.007418256678703   | -0.050537602364217  |
| 14      | 1.006637342854852   | -0.048851575843533  |
| 15      | 1.006205629259680   | -0.04788335036578   |
| 16      | 1.005968648007326   | -0.0472774722882    |
| 17      | 1.005839340365928   | -0.046885868437925  |
| 18      | 1.005769147640350   | -0.046681800486309  |
| 19      | 1.005731215221337   | -0.04656452961825   |
| 20      | 1.005710797002295   | -0.046497492807615  |
| $\infty$ | 1.005687333437919   | -0.046411250116879  |

Table 3: Values of the coefficients $d_0(\rho)$ and $d_4(\rho)$ for several values of the torus aspect ratio $\rho$.

\[ C_1^{hc}(\rho) = -\frac{2}{\sqrt{3}} E_1^{hc}(\rho) \]  
\[ C_2^{hc}(\rho) = C_3^{hc}(\rho) = 0 \]  
\[ C_4^{hc}(\rho) = \frac{3}{2} d_4(\rho) - 4 \rho E_1^{hc}(\rho) E_5^{hc}(\rho) = \frac{1}{6} C_4^{tri}(\rho) \]  
\[ C_5^{hc}(\rho) = -\frac{2}{\sqrt{3}} E_5^{hc}(\rho) \]

The numerical values of the coefficients $C_0^{tri}$, $C_4^{tri}$ and $C_0^{hc}$ can be found in Table 4. The values of the coefficients $C_1$ and $C_5$ can be obtained from Table 2, and the value of $C_4^{hc}$ can be read from Table 4 with the help of (5.29e).

Remarks. 1. The fact that the coefficients $C_1$ and $C_3$ are proportional respectively to $E_1$ and $E_5$ for the triangular (5.28) and hexagonal (5.29) lattices is not accidental. In fact, from (5.1) / (7.22) we conclude that all the odd coefficients in the specific-heat expansion are proportional to the corresponding coefficients of the internal-energy.
\[
E_{2m+1} = \frac{d\mu}{d\beta} \left. \left( \frac{d^2 \mu}{d\beta^2} \right) \right|_{\beta = \beta_c}^{\beta = \beta_c}^{-1}
\]

(5.30)

Indeed, for \( m = 1, 3 \) this ratio is indeterminate as both coefficients vanish.

2. The limiting values of the coefficients \( C_m(\rho) \) as \( \rho \to \infty \) are easily found to be [c.f., (B.12)]

\[
\lim_{\rho \to \infty} C_0^{\text{tri}}(\rho) = \frac{12\sqrt{3}}{\pi} \left[ \gamma_E + \log \frac{4\sqrt{3}}{\pi} - \frac{\pi}{2\sqrt{3}} \right]
\]

(5.31a)

\[
\lim_{\rho \to \infty} C_1^{\text{tri}}(\rho) = \lim_{\rho \to \infty} C_5^{\text{tri}}(\rho) = 0
\]

(5.31b)

\[
\lim_{\rho \to \infty} C_4^{\text{tri}}(\rho) = -\frac{7\pi^3}{300\sqrt{3}}
\]

(5.31c)

Table 4: Values of the coefficients \( C_0^{\text{tri}}(\rho) \), \( C_4^{\text{tri}}(\rho) \) and \( C_0^{\text{hc}}(\rho) \) for several values of the torus aspect ratio \( \rho \).
\[
\lim_{\rho \to \infty} C_{0}^{hc}(\rho) = \frac{2\sqrt{3}}{\pi} \left[ \gamma_E + \log \frac{4\sqrt{3}}{\pi} - \frac{\pi}{3\sqrt{3}} \right] \\
\lim_{\rho \to \infty} C_{1}^{hc}(\rho) = \lim_{\rho \to \infty} C_{5}^{hc}(\rho) = 0 \\
\lim_{\rho \to \infty} C_{4}^{hc}(\rho) = -\frac{7\pi^3}{1800\sqrt{3}} 
\]

3. The behavior of the coefficients \( C_m(\rho) \) under the transformation \( \rho \to 1/\rho \) is the expected one
\[
C_{0}(\rho) = C_{00} \log \rho + C_{0}(1/\rho) \\
C_{m}(\rho) = \frac{C_{m}(1/\rho)}{\rho^m} \quad \text{for } m \geq 1
\]

4. From Table 4 it is clear that \( C_{4}^{\text{tri}} \) should vanish at a value between 1 and 2. Actually, due to (5.29e), \( C_{4}^{hc} \) should also vanish at the same value of \( \rho \). We have found numerically that \( C_{4} \) vanishes at
\[
\rho_{\min} \approx 1.4688897779
\]

Indeed, due to the transformation properties of \( C_{4}(\rho) \) under the transformation \( \rho \to 1/\rho \), \( C_{4} \) also vanishes at \( \rho_{\min}^{-1} \approx 0.6807862748 \). This is similar to what happens in the square lattice [24].

6 Higher derivatives of the free energy

6.1 Finite-size-scaling corrections to \( f_{c}^{(3)} \)

In this section we will consider the third derivative of the free energy (2.17) at criticality. Even though this observable is not relevant in practice, its computation is interesting as it provides new insights into the finite-size-scaling function \( \hat{W} \) defined in Section 4. The observable \( f_{c}^{(3)} \) (2.17) can be written as follows:
\[
f_{c}^{(3)} = \frac{8 \cosh 2\beta_c}{\sinh^3 \beta_c} + \frac{1}{V} \frac{d^3 \mu}{d\beta^3} \bigg|_{\beta=\beta_c} \sum \frac{Z_{00}(0)}{Z_{\alpha,\beta}(0)} + \frac{1}{V} \left( \frac{d\mu}{d\beta} \right)^3_{\beta=\beta_c} \\
\times \left[ \frac{Z_{00}(0)}{Z_{\alpha,\beta}(0)} - 3 \sum \frac{Z_{\alpha,\beta}(0)}{Z_{\alpha,\beta}(0)} \sum Z_{\alpha,\beta}(0) + 2 \left( \sum \frac{Z_{00}(0)}{Z_{\alpha,\beta}(0)} \right)^3 \right] \\
+ \frac{3}{V} \frac{d^2 \mu}{d\beta^2} \bigg|_{\beta=\beta_c} \frac{d\mu}{d\beta} \bigg|_{\beta=\beta_c} \left[ \sum \frac{Z_{\alpha,\beta}(0)}{Z_{\alpha,\beta}(0)} - \left( \sum \frac{Z_{00}(0)}{Z_{\alpha,\beta}(0)} \right)^2 \right]^{(6.1)}
\]

The only unknown object is the derivative \( Z_{00}''(0) \), which can be written in the following way
\[
\frac{Z_{00}''(0)}{Z_{00}(0)} = M^2 + \frac{12MN}{\pi \sqrt{3}} \left[ S_0^{(1)} + 2S_0^{(2)} \right] 
\]
where the sums $S^{(j)}$ were defined in (5.4). By following step by step the procedure developed in Section 5 and leading to (5.20), we can compute the finite-size expansion of the ratio (6.2)

$$
\frac{1}{MN} \frac{Z_{0,0}''(0)}{Z_{0,0}'(0)} = \frac{12}{\pi \sqrt{3}} \log N + \tilde{A}(\rho) + \sum_{m=2}^{\infty} \frac{A_{2m}}{N^{2m}}
$$

By plugging in (6.1) the above result (6.3) and the results already obtained in Sections 4 and 5, we obtain

$$
f_c^{(3)}(N, \rho) = A_1(\rho) N + A_0 \log N + A_0(\rho) + \sum_{m=1}^{\infty} \frac{A_m(\rho)}{N^m}
$$

In this expansion, the coefficient $A_2$ is identically zero.

| $\rho$ | $A_1^{\text{tri}}(\rho)$ | $A_0^{\text{tri}}(\rho)$ |
|-------|-----------------|-----------------|
| 1     | -16.556352382598901 | 0.588715732188061 |
| 2     | -16.439945008735128 | -12.618145549991986 |
| 3     | -6.161165303236093  | -16.467144991961212 |
| 4     | 2.808024778142930   | -15.425136698441144 |
| 5     | 6.829759193109778   | -12.962673854292640 |
| 6     | 7.259238447062151   | -10.685119155844771 |
| 7     | 6.078697042860241   | -9.019531651330200 |
| 8     | 4.522637873985072   | -7.925595102852174 |
| 9     | 3.134987142065343   | -7.248692542526644 |
| 10    | 2.072903487004694   | -6.845055620123084 |
| 11    | 1.324966640033683   | -6.610290784855592 |
| 12    | 0.825436595920901   | -6.476167809538218 |
| 13    | 0.503936252639239   | -6.400571208146704 |
| 14    | 0.302641624812243   | -6.358412464990734 |
| 15    | 0.179285740783589   | -6.335102868634869 |
| 16    | 0.104987265970099   | -6.322306687731851 |
| 17    | 0.06087084500045    | -6.31532496705190 |
| 18    | 0.034988709413814   | -6.311533950015708 |
| 19    | 0.019959520236906   | -6.309486515841503 |
| 20    | 0.011309756764775   | -6.308383036473573 |
| $\infty$ | 0 | -6.307116005647652 |

Table 5: Values of the coefficients $A_1^{\text{tri}}(\rho)$ and $A_0^{\text{tri}}(\rho)$ for several values of the torus aspect ratio $\rho$.

The most important result contained in (6.4) is that the coefficient associated to the expected leading term $\sim L \log L$ vanishes. This is a highly non-trivial fact and...
we will discuss its physical meaning in Section 7. We have obtained the first four non-vanishing coefficients for the triangular lattice

\[ A^{\text{tri}}_1(\rho) = 2\rho E^{\text{tri}}_1(\rho) \left( \rho E^{\text{tri}}_1(\rho)^2 + \frac{36\sqrt{3}}{\pi} \left( \frac{\sum |\theta_j| \text{Re} \log \theta_j}{\sum |\theta_j|} - \log 2|\eta| \right) \right) \] (6.5a)

\[ A^{\text{tri}}_0 = -\frac{216}{\pi \sqrt{3}} \] (6.5b)

\[ A^{\text{tri}}_0(\rho) = 48 - 54d_0(\rho) + 6\rho E^{\text{tri}}_1(\rho)^2 \] (6.5c)

\[ A^{\text{tri}}_1(\rho) = 16E^{\text{tri}}_1(\rho) \] (6.5d)

where the function \( d_0(\rho) \) is defined in (5.23a). The numerical values of \( A^{\text{tri}}_1 \) and \( A^{\text{tri}}_0 \) can be found in Table 5, while the numerical values of \( A^{\text{tri}}_1 \) can be computed from Table 2. The coefficients for the hexagonal lattice are

\[ A^{\text{hc}}_1(\rho) = 2\rho E^{\text{hc}}_1(\rho) \left( 4\rho E^{\text{hc}}_1(\rho)^2 + \frac{36}{\pi \sqrt{3}} \left( \frac{\sum |\theta_j| \text{Re} \log \theta_j}{\sum |\theta_j|} - \log 2|\eta| \right) \right) \] (6.6a)

\[ A^{\text{hc}}_0 = -\frac{12}{\pi} \] (6.6b)
\[ A_0^{hc}(\rho) = \frac{16}{3\sqrt{3}} - 3\sqrt{3}d_0(\rho) + 4\sqrt{3}\rho E_1^{hc}(\rho)^2 \]  
(6.6c)
\[ A_1^{hc}(\rho) = 4E_1^{hc}(\rho) \]  
(6.6d)

and their numerical values can be found in Table 4.

Remarks 1. The coefficients (6.3)/(6.4) have the right behavior under the transformation \( \rho \to 1/\rho \). In particular, they satisfy
\[ A_i(\rho) = \rho A_i(1/\rho) \]  
(6.7a)
\[ A_0(\rho) = A_00 \log \rho + A_0(1/\rho) \]  
(6.7b)
\[ A_1(\rho) = \frac{A_1(1/\rho)}{\rho} \]  
(6.7c)

2.- In the limit \( \rho \to \infty \), both \( A_1(\rho) \) and \( A_1(\rho) \) go to zero as in this limit \( E_1(\rho) \to 0 \) exponentially fast. The limit of the coefficients \( A_0(\rho) \) can be computed from (6.7c)/(6.6c) and (5.24a)
\[ \lim_{\rho \to \infty} A_0^{tri}(\rho) = 48 - \frac{72\sqrt{3}}{\pi} \left( \gamma_E + \log \frac{4\sqrt{3}}{\pi} \right) \]  
(6.8a)
\[ \lim_{\rho \to \infty} A_0^{hc}(\rho) = \frac{16}{3\sqrt{3}} - \frac{12}{\pi} \left( \gamma_E + \log \frac{4\sqrt{3}}{\pi} \right) \]  
(6.8b)

3.- From Table 3 we see that \( A_1^{tri}(\rho) \) vanishes at at value \( \rho_{min} \) between 3 and 4. Actually, \( A_1^{tri} \) is zero at
\[ \rho_{tri,min,1}^{tri} \approx 3.6249264261 \]  
(6.9)
We also find that \( A_0^{tri}(\rho) \) vanishes at
\[ \rho_{tri,min,2}^{tri} \approx 1.0300773853 \]  
(6.10)

In the hexagonal lattice, \( A_1^{hc}(\rho) \) only vanishes in the limit \( \rho \to \infty \), while \( A_0^{hc} \) is zero at
\[ \rho_{hc,min,2}^{hc} \approx 2.6367691963 \]  
(6.11)

6.2 Logarithmic finite-size corrections to \( f_c^{(4)} \)

The full finite-size-scaling corrections to the fourth derivative of the free energy at criticality \( f_c^{(4)} \) are rather cumbersome to compute. However, we can extract with much less effort the terms including logarithms. This is what we really need in the renormalization-group analysis of Section 7.

The first step is the computation of the full expression for \( f_c^{(4)} \) in terms of the derivatives of the basic objects \( Z_{\alpha,\beta} \). We should keep only those terms in which
where the derivatives of $\mu$ with respect to $\beta$ have been represented for short by $\mu', \mu''$, etc. The first contribution (6.12a) is clearly non-zero and of order $\log N$. The second contribution (6.12b) does not actually contain any logarithm, as the logarithmic contributions of $Z''_{00}(0)/Z'_{00}(0)$ and $-3 \sum Z_{\alpha,\beta}(0)' \sum Z_{\alpha,\beta}(0)$ cancel out exactly [See e.g., (5.22)/(6.3)]. The same argument applies to the second line of (6.12c).

In order to compute the contribution of the first two terms in (6.12c), we have to consider the fourth derivative of $Z_{\alpha,\beta}(\mu)$ at $\mu = 0$ when $(\alpha, \beta) \neq (0, 0)$. After some algebra, we find that the logarithmic contributions to that derivative are

$$Z_{\alpha,\beta, \log}(0) = \frac{12 M N}{\sqrt{3}} Z_{\alpha,\beta}(0) \left( S^{(1)}_\alpha + 2 S^{(2)}_{\alpha,\beta} \right) + M^2 Z''_{\alpha,\beta}(0) \delta_{\alpha,0}$$

$$+ \frac{8 M^3 N}{\sqrt{3}} Z_{\alpha,\beta}(0) \log N \delta_{\alpha,0} \quad (6.13)$$

where the sums $S^{(j)}$ are defined in (5.4). After some more algebra we find that the contribution of (6.12c) does not contain any logarithms.

In conclusion, we find that the finite-size-scaling expansion for the observable $f^{(4)}_c$ contains a single logarithmic term

$$f^{(4)}_{c, \log} = B_{00} \log N \quad (6.14)$$

where $B_{00}$ can be read from (6.12a). Its numerical value is

$$B_{00} = \begin{cases} 2736/(\pi \sqrt{3}) & \text{triangular} \\ 120/(\pi \sqrt{3}) & \text{hexagonal} \end{cases} \quad (6.15)$$

The leading term in the large-$N$ expansion of $f^{(4)}_c$ is expected to be $\sim N^2$, and we also expect a term of order $\sim N$. 

29
7 Irrelevant operators in the two-dimensional Ising model

Let us first collect the main results of the previous sections

\[ f_c(N, \rho) = f_{\text{bulk}} + \sum_{m=1}^{\infty} \frac{f_{2m}(\rho)}{N^{2m}} \]  
\[ E_c(N, \rho) = E_0 + \sum_{m=0}^{\infty} \frac{E_{2m+1}(\rho)}{N^{2m+1}} \]  
\[ C_{H,c}(N, \rho) = C_{00} \log N + C_0(\rho) + \sum_{m=1}^{\infty} \frac{C_m(\rho)}{N^m} \]  
\[ f_c^{(3)}(N, \rho) = A_1(\rho)N + A_{00} \log N + A_0(\rho) + \sum_{m=1}^{\infty} \frac{A_m(\rho)}{N^m} \]  
\[ f_{c,\log}(N, \rho) = B_{00} \log N \]

It is also important to mention that the coefficients \( f_4, f_8, E_3, E_7, C_2, C_3, \) and \( A_2 \) vanish. In this Section we will use these results to study the irrelevant operators in the two-dimensional Ising model and the finite-size-scaling function \( \tilde{W} \) defined below. The results will be applicable to both the triangular and hexagonal lattices as the analytic structure of the corresponding asymptotic expansions is the same. To our knowledge, there are no predictions based on conformal field theory for the hexagonal-lattice Ising model. In this section we will follow basically the notation of ref. [20].

Let us start with the basic finite-size-scaling Ansatz for a system defined on a torus of linear size \( L \) (the aspect ratio is also fixed and plays no role in this discussion), zero magnetic field and reduced temperature \( \tau \) \([20, 44]\)

\[ f(\tau; L) = f_b(\tau) + \frac{1}{L^2} W(\{\mu_j(\tau)L^{y_j}\}) + \frac{\log L}{L^2} \tilde{W}(\{\mu_j(\tau)L^{y_j}\}) \]  

where \( f_b(\tau) \) is a regular function of \( \tau \) and the scaling functions \( W \) and \( \tilde{W} \) depend on the non-linear scaling fields \( \mu_j(\tau) \) belonging to the identity and energy conformal families. Among them the only relevant field is the one associated to the temperature \( \mu_t(\tau) \) (See Table [7]). In this Ansatz we have explicitly used the assumptions (a)-(c) introduced in Section [4].

The reduced temperature \( \tau \) measures the distance to the critical point\(^6\) and it is defined such that \( \tau = 0 \) at \( \beta = \beta_c \) and \( \tau > 0 \) (resp. \( \tau < 0 \)) for \( \beta < \beta_c \) (resp. \( \beta > \beta_c \)). In the Ising model on the triangular and hexagonal lattices this parameter takes the

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\(^6\) This parameter should not be confused with the torus modular parameter. In this section \( \tau \) will mean the reduced temperature, while in the rest of the paper it will denote the usual modular parameter.
form
\[ \tau = \begin{cases} \frac{1 + v^2 - 4v}{(1 - v)\sqrt{2v}} & \text{triangular} \\ \frac{1 - 3v^2}{v\sqrt{2(1 - v^2)}} & \text{hexagonal} \end{cases} \] (7.3)

where as usual \( v = \tanh \beta \). Under the transformations that map the high-temperature phase onto the low-temperature phase and viceversa
\[ v \rightarrow v' = \begin{cases} \left( \sqrt{1 - v} + \frac{v}{\sqrt{2}} - \sqrt{v} \right)^2 & \text{triangular} \\ \sqrt{\frac{1 - v^2}{1 + 3v^2}} & \text{hexagonal} \end{cases} \] (7.4)

the reduced temperature simply maps as \( \tau \rightarrow -\tau \). Equations (7.3)/(7.4) in the triangular-lattice case were introduced in [20].

The non-linear scaling fields \( \mu_j(\tau) \) can be written as a power series in \( \tau \)
\[ \mu_j(\tau) = \mu_j(0) + \tau \mu_{1,j} + \frac{1}{2} \tau^2 \mu_{2,j} + \cdots \] (7.5)

and we usually take the normalization \( \mu_j(0) = 1 \) for the identity-family fields, and \( \mu_{1,j} = 1 \) for the energy-family fields (These latter scaling fields are odd under the transformation \( \tau \rightarrow -\tau \), thus they satisfy \( \mu_j(0) = 0 \)).

As explained in ref. [20], both scaling functions satisfy
\[ W(\{ \mu_j(-\tau)(-L)^{y_j} \}) = W(\{ \mu_j(\tau)L^{y_j} \}) \] (7.6)

(and analogously for \( \tilde{W} \)). Thus, even (resp. odd) derivatives of \( W \) and \( \tilde{W} \) with respect to \( \tau \) will contain only even (resp. odd) powers of \( L \). This fact explains the structure found for the internal-energy and specific-heat expansions:

\[ -E_c(L) = \left. \frac{\partial}{\partial \beta} \right|_{\beta = \beta_c} \left. \frac{\partial f}{\partial \tau} \right|_{\tau = 0} \] (7.7a)
\[ C_{H,c}(L) = \left. \frac{\partial^2 \tau}{\partial \beta^2} \right|_{\beta = \beta_c} - \left. \frac{\partial f}{\partial \tau} \right|_{\tau = 0} + \left. \frac{\partial \tau}{\partial \beta} \right|_{\beta = \beta_c} \left. \frac{\partial f^2}{\partial \tau^2} \right|_{\tau = 0} \] (7.7b)

In particular, (7.7) shows why the odd powers of the specific-heat expansion are related to those of the internal energy. We will also make the following assumption, which is motivated by the absence of terms \( L^{-m} \log L \) for any \( m > 0 \) in the expansions (7.1)

(d) The scaling function \( \tilde{W} \) only depends on the scaling field associated to the temperature
\[ \tilde{W}(\{ \mu_j(\tau)L^{y_j} \}) = \tilde{W}(\mu_t(\tau)L) \] (7.8)

As we know that there are no logarithmic contributions to the free and internal energies (7.1a)/(7.1b), the scaling function \( \tilde{W} \) should satisfy
\[ \tilde{W}(0) = \left. \frac{\partial \tilde{W}(x)}{\partial x} \right|_{x=0} = 0 \] (7.9)
Conformal field theory [20] provides a list of irrelevant operators that may appear in the two-dimensional Ising model (See Table 7). By comparing the finite-size-scaling Ansätze for the free energy, internal energy and specific heat obtained from (7.2) to the corresponding exact results (7.1) we may get new insights about the operator content of the model.

| Family | $j$ | $\mu_j$ | $s$ | $y$ |
|--------|-----|---------|-----|-----|
| $[I]$  | 0   | $I$     | 0   | 2   |
|        | 1   | $T\bar{T}$ | 0   | -2  |
|        | 2   | $T^3 + T^3$ | 6   | -4  |
|        | 3   | $Q_4^I\bar{Q}_4^I$ | 0   | -6  |
|        | 4   | $Q_6^I Q_6^I + Q_8^I \bar{Q}_8^I$ | 6   | -8  |
|        | 5   | $Q_{12}^I + \bar{Q}_{12}^I$ | 12  | -10 |
| $[\epsilon]$ | $\epsilon$ | $\epsilon$ | 0   | 1   |
|        | 6   | $Q_6^\epsilon + \bar{Q}_6^\epsilon$ | 6   | -5  |
|        | 7   | $Q_6^\epsilon \bar{Q}_4^\epsilon$ | 0   | -7  |

Table 7: Operators in the two-dimensional Ising model according to ref. [20]. For each conformal family, we have listed the primary and quasiprimary fields belonging to it. For each scaling field $\mu_j$, we show the notation used in ref. [20], its spin $s$ and its renormalization-group exponent $y$. We have included the most relevant fields (i.e., $y \geq -10$) with spin $s = 6$. We have omitted the conformal family $[\sigma]$ as it is irrelevant in this discussion. Only the primary fields $I$ and $\epsilon$ are relevant.

Let us start with the free energy. At the critical point $\tau = 0$ this can be written as

$$f_c(L) = f_b(0) + \frac{1}{L^2} W([x_j])$$

(7.10)

where $W$ depends only on the identity-family fields through the variables $x_j = \mu_j(0)L^{y_j}$, as the energy-family scaling fields vanish at criticality. This expression can be Taylor expanded for large $L$, so we obtain a power series in $L^{-1}$. The exact result (7.1a) tells us that only corrections of order $L^{2m}$ can occur, except for the terms of order $L^{-4}$ and $L^{-8}$. From Table 7, we see that the scaling fields $T\bar{T}$ and $\mu_3$ precisely give corrections of order $L^{-4}$ and $L^{-8}$ to the expansion of (7.10). Hence, we need to impose the conditions

$$\mu_{TT}(0) = \mu_3(0) = 0$$

(7.11)

The derivative of the free energy with respect to $\tau$ can be written as [20]

$$\left. \frac{\partial f}{\partial \tau} \right|_{\tau=0} = \left. \frac{\partial f_b}{\partial \tau} \right|_{\tau=0} + \frac{1}{L^2} \sum_{j \in [\epsilon]} L^{y_j} \mu_{1,j} W_j([x_k])$$

(7.12a)

$$= \left. \frac{\partial f_b}{\partial \tau} \right|_{\tau=0} + \frac{1}{L} \mu_{1,1} W_1([x_k]) + \frac{1}{L^2} \mu_{1,6} W_6([x_k])$$

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\[ + \frac{1}{L \mu} \mu_{1,7} W_7(\{x_k\}) + \cdots \quad (7.12b) \]

Each function \( W_j(\{x_k\}) \) can be expanded as we did for the free energy, giving a power series in \( L^{-2m} \) with no contribution to orders \( L^{-4} \) and \( L^{-8} \). From the exact solution (7.1b), we see that only corrections of the type \( L^{-2m-1} \) can appear except for the powers \( L^{-3} \) and \( L^{-7} \). This implies that the scaling field \( \mu_0 \) cannot play any role, thus

\[ \mu_{1,6} = 0 \quad (7.13) \]

The second derivative of the free energy at criticality is given by [20]:

\[ \frac{\partial^2 f}{\partial \tau^2} \bigg|_{\tau=0} = \frac{\partial^2 f_b}{\partial \tau^2} \bigg|_{\tau=0} + \frac{1}{L^2} \sum_{i,j \in [\epsilon]} L^{y_i+y_j} W_{ij}(\{x_k\}) + \frac{1}{L^2} \sum_{j \in [I]} \mu_{2,j} L^y W_j(\{x_k\}) \]

\[ + 2 \log L \frac{\partial^2 \hat{W}(x)}{\partial x^2} \bigg|_{x=0} \quad (7.14) \]

where we have used the standard normalization. The second term in the r.h.s. of (7.14) can be written as

\[ \frac{1}{L^2} \sum_{i,j \in [\epsilon]} L^{y_i+y_j} W_{ij}(\{x_k\}) = W_6(\{x_k\}) + \frac{1}{L^6} W_7(\{x_k\}) + \cdots \quad (7.15) \]

These two terms alone give all even powers of \( L^{-1} \) except \( L^{-2} \) in agreement with the exact expansion (7.1c). The third term in the r.h.s. of (7.14) is equal to

\[ \frac{1}{L^2} \sum_{j \in [I]} \mu_{2,j} L^y W_j(\{x_k\}) = \frac{1}{L^4} \mu_{2,1} W_1(\{x_k\}) + \frac{1}{L^6} \mu_{2,2} W_2(\{x_k\}) + \cdots \quad (7.16) \]

Again, this a power series containing all even powers of \( L^{-1} \) except \( L^{-2} \) in agreement with (7.1c).

The coefficient of the leading term should be equal to \( C_{00} \)

\[ C_{00} = \left( \frac{d \tau}{d \beta} \bigg|_{\beta=\beta_c} \right)^2 \hat{W}''(0) \quad (7.17) \]

Hence we can determine the numerical value of \( \hat{W}''(0) \) by using (5.28a)/(5.29a) and the definition of \( \tau \) (7.3). The result is

\[ \hat{W}''(0) = \begin{cases} \frac{1}{(\pi \sqrt{3})} & \text{triangular} \\ \frac{1}{(2\pi \sqrt{3})} & \text{hexagonal} \end{cases} \quad (7.18) \]

where we have considered the standard normalization for \( \mu_1(\tau) \). The value (7.18) for the triangular lattice agrees with the result obtained in [20, eq. (3.34)].
Remarks. 1. The irrelevant scaling fields belonging to the identity family that may play a role have non-zero spin (namely, \( s = 6, 12 \)). The spin-zero fields belonging to this family should vanish at criticality (e.g., \( \mu_j(0) = 0 \)). This result agrees with Conjecture 1.1.

2. The vanishing of the field \( \mu_1 = T\bar{T} \) at criticality supports Conjecture (d0) of [20]. However, our results do not imply their stronger Conjecture (d1): the scaling field \( T\bar{T} \) decouples (i.e., \( \mu_{T\bar{T}}(\tau) = 0 \) for all \( \tau \)).

3. In the internal-energy analysis, we concluded that the irrelevant field \( \mu_6 \) should vanish at criticality (7.13). This operator has spin six, therefore this result is not implied by Conjecture 1.1. In other words, there are cancellations also in the non-scalar sector. On the other hand, we find no constraint on the spin-zero irrelevant field \( \mu_7 \). However, if Conjecture 1.1 is true, then we should have \( \mu_1 = \mu_7 = 0 \).

4. In order to obtain the exact solutions (7.1) we need to include at least two irrelevant operators. This result agrees with the findings of [17, 18] for the square-lattice model. It is worth noticing that we can formally obtain the exact solutions (7.1) by including the spin-6 irrelevant scaling field \( \mu_2 = T^3 + \bar{T}^3 \) with \( y = -4 \) and the spin-12 field \( \mu_5 \) with \( y = -10 \).

\[
f_c(L) = f_b(0) + \frac{1}{L^2} W(\mu_2(0)L^{-4}, \mu_5(0)L^{-10}) \quad (7.19a)
\]
\[
\frac{\partial f}{\partial \tau} \bigg|_{\tau=0} = \frac{\partial f_b}{\partial \tau} \bigg|_{\tau=0} + \frac{1}{L} W_t(\mu_2(0)L^{-4}, \mu_5(0)L^{-10}) \quad (7.19b)
\]
\[
\frac{\partial^2 f}{\partial \tau^2} \bigg|_{\tau=0} = 2\hat{W}''(0) \log L + \frac{\partial^2 f_b}{\partial \tau^2} \bigg|_{\tau=0} + W_{t\tau}(\mu_2(0)L^{-4}, \mu_5(0)L^{-10}) + \frac{1}{L^6} \mu_{2,2} W_2(\mu_2(0)L^{-4}, \mu_5(0)L^{-10}) + \frac{1}{L^{12}} \mu_{2,5} W_5(\mu_2(0)L^{-4}, \mu_5(0)L^{-10}) \quad (7.19c)
\]

Let us now consider the observable \( f_c^{(3)} \) (2.17). We are interested here only in the terms containing logarithms, which are directly related to derivatives of the scaling function \( \hat{W}(x) \). The contribution of this scaling function to this observable can be written as

\[
f_{c,\text{log}}^{(3)} = L \log L \hat{W}'''(0) \left( \frac{\partial \tau}{\partial \beta} \right)^3 \bigg|_{\beta=\beta_c} + 3 \log L \hat{W}'''(0) \frac{\partial \tau}{\partial \beta} \bigg|_{\beta=\beta_c} \frac{\partial^2 \tau}{\partial \beta^2} \bigg|_{\beta=\beta_c} \quad (7.20)
\]

The exact result (5.4) shows that

\[
\frac{\partial^3 \hat{W}(x)}{\partial x^3} \bigg|_{x=0} = 0 \quad (7.21)
\]

\footnote{It is worth mentioning that the authors of [20] showed by considering the large-distance behavior of the triangular-lattice Ising model two-point function that \( \mu_{T\bar{T}}(\tau) = o(\tau^4) \). This result strongly supports their Conjecture (d1).}
This result is consistent with the conjecture put forth by the authors of ref. [20] who claimed that the scaling function \( \hat{W} \) is quadratic in its argument (i.e., \( \hat{W}(x) = Ax^2 \)).

On the other hand, the coefficient of the logarithmic term in \( f_c^{(3)} \) (i.e., \( A_{00} \)) is proportional to \( \hat{W}''(0) \). This observation provides another way to compute the quantity \( \hat{W}''(0) \) and a direct mean to test the predictions (7.18). By using the exact results (6.5b)/(6.6b)/(7.3), we arrive at the same values as in (7.18) supporting the correctness of our results.

Finally, we will discuss the observable \( f_c^{(4)} \). The contribution of the scaling function \( \hat{W} \) to this observable is given by

\[
\begin{align*}
\tilde{f}_{c,\log}(4) &= L^2 \log L \hat{W}(4)(0) \left( \frac{\partial \tau}{\partial \beta} \right) \\
&+ 3 \log L \hat{W}''(0) \left[ 4 \frac{\partial \tau}{\partial \beta} \frac{\partial^3 \tau}{\partial \beta^3} + 3 \left( \frac{\partial^2 \tau}{\partial \beta^2} \right)^2 + 4 \left( \frac{\partial^2 \tau}{\partial \beta^2} \right)^4 \mu_{3,t} \right] 
\end{align*}
\]

(7.22)

where all the derivatives of \( \tau \) with respect to \( \beta \) should be evaluated at \( \beta = \beta_c \). By comparing the above formula to (7.14)/(6.15), we conclude that

\[
\left. \frac{\partial^4 \hat{W}(x)}{\partial x^4} \right|_{x=0} = 0
\]

(7.23)

This result is compatible with \( \hat{W}(x) \) being a quadratic function of \( x \). On the other hand, as we know the numerical values of the derivatives of \( \tau \) w.r.t. \( \beta \) for the triangular and hexagonal lattices, we can use equations (6.15)/(7.22) to deduce the value of \( \mu_{3,t} \). The result is the same for both lattices \( \mu_{3,t} = -1/4 \), so the non-linear scaling field \( \mu_t \) depends on \( \tau \) in the following way

\[
\mu_t(\tau) = \tau - \frac{1}{24} \tau^3 + \mathcal{O}(\tau^5)
\]

(7.24)

This relation coincides with the function \( a(\tau) \) obtained in ref. [20] for the triangular lattice:

\[
a(\tau) = \tau - \frac{1}{24} \tau^3 + \frac{47}{10368} \tau^5 - \frac{161}{248832} \tau^7 + \mathcal{O}(\tau^9)
\]

(7.25)

The equality between \( a(\tau) \) and \( \mu_t(\tau) \) is important because it provides support to Conjecture [1.1] if this conjecture is correct, then both function should coincide [20].

We can summarize the results obtained on the scaling function \( \hat{W} \) in the following conjecture (which is a natural extension of the conjecture \( \hat{W}(x) = x^2/(2\pi) \) for the square-lattice model [21]):

\[\text{It is not hard to realize that the function } a(\tau) \text{ is the same for the hexagonal lattice.} \]

The key observation is that the function \( a(\tau) \) (7.23) is the same for the hexagonal lattice. The key observation is that the free energy for this lattice in the thermodynamic limit (3.29b) can be written as

\[
f^{\text{bc}}_{\text{bulk}} = \frac{1}{4} \int_0^\pi dx dy \int_0^\pi 4\pi^2 \log \left[ 3 + \tau^2 - \omega(x, y) \right] + \text{cnt.}
\]

where \( \tau \) is given by (7.3). This equation is equivalent to the definition used in [20] to define \( a(\tau) \) for the triangular lattice.
Conjecture 7.1 In the Ising model on the triangular and hexagonal lattices with toroidal boundary conditions, the scaling function $\hat{W}$ is a function solely of the argument $x = \mu(\tau)L$ and this function is equal to

$$
\hat{W}(x) = \begin{cases} 
  x^2/(2\pi\sqrt{3}) & \text{triangular} \\
  x^2/(4\pi\sqrt{3}) & \text{hexagonal}
\end{cases}
$$

The coefficient of $\hat{W}$ should coincide with the constant $A$ obtained in the infinite-volume limit analysis of the triangular-lattice model [20, Eq. (2.34)]. The agreement between those coefficients adds support to this conjecture.

8 Further remarks and conclusions

We have obtained the asymptotic expansions for the free energy, internal energy, specific heat and $f^{(3)}$ of a critical Ising model on the triangular and hexagonal lattices wrapped on a torus of width $N$ and aspect ratio $\rho$. These expansions are given in (7.1). In particular, we have found the exact coefficients $f_{\text{bulk}}, f_2, f_4 = f_8 = 0, f_6, E_0, E_1, E_3 = E_7 = 0, E_5, C_{00}, C_0, C_1, C_2 = C_3 = 0, C_4, C_5, A_1, A_{00}, A_0, A_1, A_2 = 0,$ and $B_{00}$ for both lattices.

The first important observation is that the analytic structure of the finite-size corrections of the observables considered in this paper is the same for the triangular- and the hexagonal-lattice models. The reason of this coincidence is that both lattices have the same underlying Bravais lattice. This agrees with the physical content of Conjecture 1.1: as they have the same rotational symmetry group, they should have the same irrelevant operators, hence leading to the same finite-size corrections.

As it can be seen in (7.1), all the corrections are integer powers of $N^{-1}$. The only exceptions are the logarithmic terms in the specific heat (7.1c), $f^{(3)}$, and (7.1d), and $f^{(4)}$ (7.1e). In the first case this term is the leading one, while in the other ones it is subleading. In the free-energy expansion (7.1a) only even powers of $N^{-1}$ can occur, while in the internal-energy expansion (7.1b) only odd powers of $N^{-1}$ appear. In the specific-heat expansion even and odd powers of $N^{-1}$ occur. Furthermore, the odd coefficients in this latter expansion are proportional to the corresponding odd coefficients in the internal-energy expansion. The constant depends on how the mass $\mu$ (2.12) depends on the temperature (5.30). In the expansion of the observable $f^{(3)}$, we find corrections with all powers of $N^{-1}$ except for the term $N^{-2}$. Indeed, the coefficients $f_m, E_m, C_m$ and $A_m$ do depend on the lattice structure of the model, hence they are not universal.

The fact that $E_{2m+1}/C_{2m+1}$ is a $\rho$-independent number for the square lattice ($= -1/\sqrt{2}$) [24, 23], suggested the idea that this ratio might be universal (e.g., it does not depend on the lattice structure)\[\text{[9]}\]. However, our results show that this is not the case.

\[\text{[9]}\] Izmailian and Hu [41] (see also [43]) computed the finite-size expansion of the free energy $f(N) = f_{\text{bulk}} + \sum_{k=1}^{\infty} f_k/N^{2k}$ and the inverse correlation length $\xi^{-1}(N) = \sum_{k=1}^{\infty} b_k/N^{2k-1}$ for a critical Ising model on several $N \times \infty$ lattices (i.e., square, hexagonal and triangular) with periodic boundary conditions.
Thus, the proportional constant does depend on the lattice structure, hence it is not universal. We can write (8.1) and the corresponding square-lattice relation (1.2) in an unified way by realizing that the constant is just $-1/E_0$.

It is important to note that one key ingredient in this discussion is the fact that there is an exact transformation (7.4) mapping the high-temperature phase onto the low-temperature phase, so we can define a parameter $\tau$ (7.3) transforming as $\tau \rightarrow -\tau$. It is not clear whether this transformation exists or not for an Ising model defined on a general lattice. However, if that transformation does exist, then we can define $\tau$ so eq. (7.3) holds, leading to (7.7)/(8.1). We can summarize all these observations in the following conjecture:

**Conjecture 8.1** Let us consider a critical Ising system on a regular two-dimensional lattice with toroidal boundary conditions. Let us further assume that there is an exact mapping $v \rightarrow v'$ from the high-temperature phase onto the low-temperature phase such that $\tau \rightarrow -\tau$. Then, the internal energy and specific heat can be expanded in power series of $N^{-1}$ as in (7.1b)/(7.1c) and the coefficients $E_m(\rho)$ and $C_m(\rho)$ satisfy

$$
\frac{E_m(\rho)}{C_m(\rho)} = \begin{cases} 
-1/2 & \text{triangular} \\
-\sqrt{3}/2 & \text{hexagonal}
\end{cases}
\quad (8.2)
$$

where $E_0$ is the bulk internal energy [See (7.1b)]. Indeed, we understand that this ratio is not defined whenever $E_m = 0$.

If this conjecture is true, then we could define the expansions

$$
E_c(N, \rho) = E_0 \left[ 1 + \sum_{m=0}^{\infty} \frac{\tilde{E}_{2m+1}(\rho)}{N^{2m+1}} \right] \quad (8.3a)
$$

$$
C_{H,c}(N, \rho) = E_0^2 \left[ \tilde{C}_{00} \log N + \tilde{C}_0(\rho) + \sum_{m=1}^{\infty} \frac{\tilde{C}_m(\rho)}{N^m} \right] \quad (8.3b)
$$

and then the new ratios would be universal

$$
\frac{\tilde{E}_m(\rho)}{\tilde{C}_m(\rho)} = \begin{cases} 
-1 & \text{for } m \text{ odd} \\
0 & \text{for } m \text{ even}
\end{cases}
\quad (8.4)
$$

In Section 4 we mentioned that the results contained in this paper could serve also to test Monte Carlo simulations. Indeed, the expressions (8.4)/(1.2)/(5.3) provide a way to compute the exact values of the internal energy and specific heat for any finite boundary conditions. They found lattice-dependent coefficients $f_k$ and $b_k$, but universal ratios $b_k/f_k = (2^{2k} - 1)/(2^{2k-1} - 1)$.

10We thank Andrea Pelissetto for useful comments on this matter.
torus of size $N \times M$. For very large lattices one could also use the (easier to evaluate) asymptotic expansions (7.1b)/(7.1c).

On the other hand, by taking the exact values of any observable for fixed aspect ratio $\rho$ and several values of the torus width $N$, we can check whether the asymptotic expansions (7.1) are correct or not. In particular, by fitting the exact values to the corresponding Ansatz, we can verify whether the numerical coefficients coincide with the estimates coming from the fits. We have performed such an analysis and we have confirmed that the numerical values of the coefficients $f_m, E_m, C_m$ and $A_m$ for several values of $\rho$ coincide with the estimates coming from the fits. In addition, this procedure allows us to obtain crude estimates of the next coefficients in each expansion. For instance, we obtain for $\rho = 1$ (which is the case most frequently considered in the literature) the following values

$$f_{10}^{\text{tri}}(1) \approx 1.932, \quad f_{10}^{\text{hc}}(1) \approx 0.966 \quad (8.5a)$$
$$E_{9}^{\text{tri}}(1) \approx -7.821, \quad E_{9}^{\text{hc}}(1) \approx -2.258 \quad (8.5b)$$
$$C_{6}^{\text{tri}}(1) \approx -0.722, \quad C_{6}^{\text{hc}}(1) \approx -0.120 \quad (8.5c)$$
$$A_{3}^{\text{tri}}(1) \approx 9.124, \quad A_{3}^{\text{hc}}(1) \approx 0.878 \quad (8.5d)$$

A The Euler-MacLaurin formula

The Euler-MacLaurin formula is one important tool we need to compute asymptotic series. Here we will use the version of ref. [45, formula 23.1.32]. Let $F(x)$ be a function whose first $2n$ derivatives are continuous in the interval $(a, b)$. If we divide the interval into $m$ equal parts (so that $h = (b - a)/m$), then we have

$$\sum_{k=0}^{m-1} F(a + kh + \alpha h) = \frac{1}{h} \int_a^b F(t) dt$$
$$+ \sum_{k=1}^{p} \frac{h^{k-1}}{k!} B_k(\alpha)[F^{(k-1)}(b) - F^{(k-1)}(a)]$$
$$- \frac{h^p}{p!} \int_0^1 \hat{B}_p(\alpha - t) \left\{ \sum_{k=0}^{m-1} F^{(p)}(a + kh + th) \right\} dt \quad (A.1)$$

where $p \leq 2n$, $0 \leq \alpha \leq 1$, $\hat{B}_n(x) = B_n(x - \lfloor x \rfloor)$ and $B_n(x)$ are the Bernoulli polynomials defined in terms of the Bernoulli numbers $B_k$ by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (A.2)$$

Indeed, $B_n(0) = B_n$. The Bernoulli polynomials satisfy the identity [45, eq. 23.1.21]:

$$B_n(1/2) = \left( \frac{1}{2^{n-1}} - 1 \right) B_n \quad (A.3)$$
We are mainly interested in sums of the form

\[ \frac{1}{L} \sum_{n=0}^{\gamma L - 1} F \left( \frac{2\pi}{L} (n + \alpha) \right) \]

(A.4)

The asymptotic expansion of the sum (A.4) in the limit \( L \to \infty \) with \( \gamma \) fixed can be obtained from (A.1). If we assume that all the derivatives of \( F(t) \) are integrable over the interval \([0, 2\pi\gamma]\) we can formally extend the sum in (A.1) to \( k = \infty \) and drop the remainder term [namely, the integral in (A.1)]. In this case we can write the Euler-MacLaurin formula as follows

\[ \frac{1}{L} \sum_{n=0}^{\gamma L - 1} F \left( \frac{2\pi}{L} (n + \alpha) \right) = \frac{1}{2\pi} \int_0^{2\pi\gamma} F(t) \, dt \]

\[ + \frac{1}{2\pi} \sum_{k=1}^{\infty} \left( \frac{2\pi}{L} \right)^k \frac{B_k(\alpha)}{k!} \left[ F^{(k-1)}(2\pi\gamma) - F^{(k-1)}(0) \right] \]

(A.5)

In this paper we need the above formula in the particular case \( L = 2N \) and \( \gamma = 1/2 \). Then (A.5) reads

\[ \frac{1}{N} \sum_{n=0}^{N-1} F \left( \frac{\pi}{N} (n + \alpha) \right) = \frac{1}{\pi} \int_0^{\pi} F(t) \, dt \]

\[ + \frac{1}{\pi} \sum_{k=1}^{\infty} \left( \frac{\pi}{N} \right)^k \frac{B_k(\alpha)}{k!} \left[ F^{(k-1)}(\pi) - F^{(k-1)}(0) \right] \]

(A.6)

In the computation of the specific heat we also need formula (A.1) in the particular case \( \alpha = 0 \) and \( h = 1 \). In this case we can formally write (A.1) for a function \( F \) whose derivatives are all integrable over \([a, b] \) in the following form [42]

\[ \sum_{k=a}^{b-1} F(k) = \int_a^b F(t) \, dt - \frac{1}{2} [F(b) - F(a)] \]

\[ + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} [F^{(2k-1)}(b) - F^{(2k-1)}(a)] \]

(A.7)

where we have used the fact that [42]

\[ B_{2k+1} = \begin{cases} -\frac{1}{2} & k = 0 \\ 0 & k > 0 \end{cases} \]

(A.8)

As we did in [24], we can apply (A.7) to the function \( F(x) = x^{2m} \) with \( a = 0 \) and \( b = 1 \). We then obtain the identity

\[ \sum_{k=1}^{m} \frac{B_{2k}}{2k} \binom{2m}{2k - 1} = \frac{1}{2} - \frac{1}{2m + 1} \]

(A.9)

If we apply (A.7) to the case \( F(x) = x^{2m-1} \) with the same endpoints as before, we obtain

\[ \sum_{k=1}^{m-1} \frac{B_{2k}}{2k} \binom{2m - 1}{2k - 1} = \frac{1}{2} \left( 1 - \frac{1}{m} \right) \]

(A.10)
B Theta functions

In this appendix we gather all the definitions and properties of the Jacobi’s $\theta$-functions needed in this paper. We first introduce the object $\theta_{\alpha,\beta}(z, \tau)$ \((\alpha, \beta = 0, 1/2)\):\(^{\text{11}}\)

$$\theta_{\alpha,\beta}(z, \tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2-\alpha)^2} \exp \left\{ 2\pi i \left( n + \frac{1}{2} - \alpha \right) \left( z + \beta - \frac{1}{2} \right) \right\}$$  \quad (B.1)

where the nome $q$ is defined in terms of the modular parameter $\tau$ as follows

$$q = e^{\pi i \tau}$$  \quad (B.2)

Using the identity (proved in [16])

$$\prod_{n=0}^{\infty} \left[ 1 + q^{2n-1} \right] \left[ 1 + q^{2n-1} t^{-1} \right] \left[ 1 - q^{2n} \right] = \sum_{n \in \mathbb{Z}} q^n t^n$$  \quad (B.3)

we can write (B.1) as

$$\theta_{\alpha,\beta}(z, \tau) = \eta(\tau) q^{B_2(\alpha)} e^{2\pi i (1/2-\alpha)(z+\beta-1/2)} \times \prod_{n=0}^{\infty} \left[ 1 - q^{2(n+1-\alpha)} e^{2\pi i (z+\beta)} \right] \left[ 1 - q^{2(n+\alpha)} e^{-2\pi i (z+\beta)} \right]$$  \quad (B.4)

where $\eta(\tau)$ is Dedekind $\eta$-function

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} \left[ 1 - e^{2\pi i n\tau} \right]$$  \quad (B.5)

and $B_2(\alpha)$ is the Bernoulli polynomial [c.f.,(A.2)]

$$B_2(\alpha) = \alpha^2 - \alpha + \frac{1}{6}$$  \quad (B.6)

The relation of the functions $\theta_{\alpha,\beta}$ with the usual $\theta$-functions $\theta_i(z, \tau)$ \(i = 1, \ldots, 4\) is the following

$$\theta_{0,0}(z, \tau) = \theta(1, \tau) = -i \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi iz(n+1/2)+\pi i(n+1/2)^2}$$  \quad (B.7)

$$\theta_{0,\frac{1}{2}}(z, \tau) = \theta_{\frac{1}{2}}(z, \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi iz(n+1/2)+\pi i(n+1/2)^2}$$  \quad (B.8)

$$\theta_{\frac{1}{2},0}(z, \tau) = \theta_{\frac{1}{2}}(z, \tau) = \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi izn+\pi i n^2}$$  \quad (B.9)

$$\theta_{\frac{1}{2},\frac{1}{2}}(z, \tau) = \theta_{\frac{3}{2}}(z, \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi izn+\pi i n^2}$$  \quad (B.10)

\(^{11}\) This object is almost identical to the one introduced in ref. [25]. However, this latter one gives the wrong sign to $\theta_{1}(z, \tau)$ [c.f. (B.10)], although this is not important as we are only interested in the case $z = 0$ where $\theta_{1}(0, \tau) = 0$.\[\]
In this paper we will only need these functions evaluated at \( z = 0 \) and \( \tau = i\tau_0\rho \) where \( \tau_0 \) is given by (3.19). To simplify the notation we will use the shorthands
\[
\theta_{\alpha,\beta} = \theta_{\alpha,\beta}(i\tau_0\rho) = \theta_{\alpha,\beta}(z = 0, \tau = i\tau_0\rho) \quad (B.11a)
\]
\[
\theta_i = \theta_i(i\tau_0\rho) = \theta_i(z = 0, \tau = i\tau_0\rho) \quad (B.11b)
\]
\[
\eta = \eta(i\tau_0\rho) = \eta(\tau = i\tau_0\rho) \quad (B.11c)
\]
We also need the limits of the \( \theta \)-functions in the limit \( \rho \to \infty \). These limits are given by
\[
\lim_{\rho \to \infty} \theta_3(i\tau_0\rho) = \lim_{\rho \to \infty} \theta_4(i\tau_0\rho) = 1 \quad (B.12a)
\]
\[
\lim_{\rho \to \infty} \theta_2(i\tau_0\rho) = \lim_{\rho \to \infty} 2e^{-\pi\tau_0\rho/4} = 0 \quad (B.12b)
\]
From eq. (B.4) we arrive at the following identity valid when \((\alpha, \beta) \neq (0, 0)\):
\[
\log \left| \frac{\theta_{\alpha,\beta}(i\tau_0\rho)}{\eta(i\tau_0\rho)} \right| + \pi \rho \Re(\tau_0)B_2(\alpha) = \sum_{n=0}^{\infty} \left\{ \log \left| 1 - e^{-2\pi\tau_0\rho(n+1-\alpha)-i\beta} \right| \right. \\
+ \left. \log \left| 1 - e^{-2\pi\tau_0\rho(n+\alpha)+i\beta} \right| \right\} \quad (B.13)
\]
Another useful relation involving \( \log \theta_{\alpha,\beta}(0, \tau) \) is the following
\[
\sum_{n=\delta_{\alpha,0}}^{\infty} \sum_{p=1}^{\infty} e^{2\pi pi(\tau(n+\alpha)-\beta)} n/2 \alpha = - \left[ \log \theta_{\alpha,\beta}(\tau) - \left( \frac{i\pi\tau}{4} + \log 2 \right) \delta_{\alpha,0} \right] \quad (B.14)
\]
We have proved this identity by considering each case \( \alpha, \beta = 0, 1/2 \) [with \((\alpha, \beta) \neq (0,0)\)] separately and by a careful rearrangement of the corresponding series.

Dedekind’s \( \eta \)-function satisfies the following identity
\[
\eta(\tau)^3 = \frac{1}{2} \theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \quad (B.15)
\]
The analogue of (B.13) when \((\alpha, \beta) = (0,0)\) is given in the particular case \( \tau = i\tau_0\rho \) by
\[
\sum_{n=1}^{\infty} \log \left| 1 - e^{-2\pi\tau_0\rho n} \right| = \log |\eta| + \frac{\pi \rho}{12} \Re(\tau_0) \quad (B.16)
\]
We also need the behavior of the \( \theta \) functions under the Jacobi transformation
\[
\tau \to \tau' = -1/\tau \quad (B.17)
\]
The result when \( z = 0 \) is given in ref. [10]
\[
\theta_3(0, \tau') = (-i\tau)^{1/2} \theta_3(0, \tau) \quad (B.18a)
\]
\[
\theta_{2,4}(0, \tau') = (-i\tau)^{1/2} \theta_{4,2}(0, \tau) \quad (B.18b)
\]
In particular, if \( \tau = i\tau_0^*\rho \) where
\[
\tau_0^* = \frac{\sqrt{3} + i}{2} = \frac{1}{\tau_0}
\]  
(B.19)
is the complex conjugate of \( \tau_0 \) (3.19), the \( \theta \)-functions transforms under (B.17) as follows
\[
\theta_3(0, i\tau_0^*/\rho) = (\tau_0\rho)^{1/2} \theta_3(0, i\tau_0) \quad \text{(B.20a)}
\]
\[
\theta_2, 4(0, i\tau_0^*/\rho) = (\tau_0\rho)^{1/2} \theta_2, 4(0, i\tau_0) \quad \text{(B.20b)}
\]
Finally, we should mention that the absolute value of the above \( \theta \)-functions does not depend on the sign of \( \text{Im} \tau_0 \). Thus,
\[
|\theta_i(0, i\tau_0/\rho)| = |\theta_i(0, i\tau_0^*/\rho)| \quad \text{(B.21)}
\]

C \quad Kronecker’s double series

In this appendix we collect a few properties of the Kronecker’s double series [48]. These series are defined as
\[
K_p^{\alpha, \beta}(\tau) = \frac{p!}{(-2\pi i)^p} \sum_{m,n\in\mathbb{Z}}\frac{e^{-2\pi i(n\alpha + m\beta)}}{(n + \tau m)^p} \quad \text{(C.1)}
\]
The basic property we need is the following
\[
B_{2p}(\alpha) - \text{Re} K_{2p}^{\alpha, \beta}(\tau) = 2p \text{Re} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left( (n + \alpha)^{2p-1} e^{2\pi i m[\tau(n+\alpha)-\beta]} + (n + 1 - \alpha)^{2p-1} e^{2\pi i m[\tau(n+1-\alpha)-\beta]} \right) \quad \text{(C.2)}
\]
in the particular case \( \tau = i\tau_0\rho \) with \( \tau_0 \in \mathbb{C} \) [c.f., (B.19)] and \( \rho \in \mathbb{R} \). Eq. (C.2) can be easily proved using the same arguments as in ref. [25, Appendix D] where they consider the particular case \( \tau = i\rho, \rho \in \mathbb{R} \).

In this paper we also need certain values of the \( K_p^{\alpha, \beta} \) obtained in [25, Appendix E]
\[
K_4^{0, 1/2}(\tau) = \frac{1}{30} \left( 7/8 \theta_2^4 - \theta_2^4 \theta_4^4 \right) \quad \text{(C.3a)}
\]
\[
K_4^{1, 0}(\tau) = \frac{1}{30} \left( 7/8 \theta_2^4 - \theta_2^4 \theta_3^4 \right) \quad \text{(C.3b)}
\]
\[
K_4^{1/2, 1/2}(\tau) = \frac{1}{30} \left( 7/8 \theta_3^4 + \theta_2^4 \theta_4^4 \right) \quad \text{(C.3c)}
\]
\[ K_{6}^{0,0}(\tau) = \frac{1}{84}(\theta_{2}^{4} + \theta_{3}^{4})(\theta_{4}^{4} - \theta_{2}^{4})(\theta_{3}^{4} + \theta_{4}^{4}) \quad (C.4a) \]
\[ K_{6}^{0,\frac{1}{2}}(\tau) = \frac{1}{84}(\theta_{3}^{4} + \theta_{4}^{4}) \left( \frac{31}{16} \theta_{2}^{8} + \theta_{3}^{4} \theta_{4}^{4} \right) \quad (C.4b) \]
\[ K_{6}^{\frac{1}{2},0}(\tau) = -\frac{1}{84}(\theta_{2}^{4} + \theta_{3}^{4}) \left( \frac{31}{16} \theta_{4}^{8} + \theta_{2}^{4} \theta_{3}^{4} \right) \quad (C.4c) \]
\[ K_{6}^{\frac{1}{2},\frac{1}{2}}(\tau) = \frac{1}{84}(\theta_{2}^{4} - \theta_{4}^{4}) \left( \frac{31}{16} \theta_{3}^{8} - \theta_{2}^{4} \theta_{4}^{4} \right) \quad (C.4d) \]

The behavior of the functions \( K_{6}^{\alpha,\beta} \) under the Jacobi transformation \((B.17)\) can be obtained using \((B.20)\) and taking into account that \( \tau_{0}^{6} = -1 \)

\[ K_{6}^{0,\frac{1}{2}}(0, i\tau_{0}^{6}/\rho) = \rho^{6} K_{6}^{\frac{1}{2},0}(0, i\tau_{0}/\rho) \quad (C.5a) \]
\[ K_{6}^{\frac{1}{2},0}(0, i\tau_{0}^{6}/\rho) = \rho^{6} K_{6}^{0,\frac{1}{2}}(0, i\tau_{0}/\rho) \quad (C.5b) \]
\[ K_{6}^{\frac{1}{2},\frac{1}{2}}(0, i\tau_{0}^{6}/\rho) = \rho^{6} K_{6}^{\frac{1}{2},\frac{1}{2}}(0, i\tau_{0}/\rho) \quad (C.5c) \]
\[ K_{6}^{0,0}(0, i\tau_{0}^{6}/\rho) = \rho^{6} K_{6}^{0,0}(0, i\tau_{0}/\rho) \quad (C.5d) \]

Finally we mention that the value of \( \text{Re} K_{6}^{\alpha,\beta} \) does not depend on the sign of \( \text{Im} \tau_{0} \):

\[ \text{Re} K_{6}^{\alpha,\beta}(0, i\tau_{0}^{*}/\rho) = \text{Re} K_{6}^{\alpha,\beta}(0, i\tau_{0}/\rho) \quad (C.6) \]

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