THE FOURTH-ORDER TYPE LINEAR ORDINARY DIFFERENTIAL EQUATIONS

W.N. EVERITT, D. J. SMITH, AND M. VAN HOEIJ

Abstract. This note reports on the recent advancements in the search for explicit representation, in classical special functions, of the solutions of the fourth-order linear ordinary differential equations named Bessel-type, Jacobi-type, Laguerre-type, Legendre-type.

Contents

1. Introduction 1
2. Bessel-type differential equation 2
3. Laguerre-type differential equation 3
4. Legendre-type differential equation 5
5. Jacobi-type differential equation 6
6. Maple files 10
7. Acknowledgements 10
References 10

1. INTRODUCTION

The four fourth-order type linear ordinary differential equations are named after the four classical second-order equations of Bessel, Jacobi, Laguerre, Legendre.

The three fourth-order equations Jacobi-type, Laguerre-type, Legendre-type, and their associated orthogonal polynomials, were first defined by H.L. Krall in 1940, see [10] and [11], and later studied in detail by A.M. Krall in 1981, see [9], and by Koornwinder in 1984, see [12].

The structured definition of the general-even order Bessel-type special functions is dependent upon the Jacobi and Laguerre classical orthogonal polynomials, and the Jacobi-type and Laguerre-type orthogonal polynomials; see [8].

The properties of the fourth-order Bessel-type functions have been studied in the papers [3], [5] and [6].

This short paper reports on the recent progress that has been made in the search for representation of solutions of these four type differential equations in terms of classical special functions.

Date: 21 March 2006.

2000 Mathematics Subject Classification. Primary: 33C15, 33D15, 33F10; Secondary: 33C05, 34B05.

Key words and phrases. Ordinary linear differential equations, special functions, Bessel-type, Jacobi-type, Laguerre-type, Legendre-type equations.
For the Bessel-type equation two independent solutions were obtained, see [8] and [3], and then a complete set of four independent solutions in the papers [14] and [4]; these solutions are dependent upon the classical Bessel functions \( J_r, Y_r, I_r, K_r \) for \( r = 0, 1 \).

Subsequently, using the techniques developed in [13], solutions of the three Jacobi-type, Laguerre-type, Legendre-type equations have been obtained in terms of the solutions of the corresponding classical differential equations.

In all these four cases the verification of these representations as solutions of the type differential equations, is a formal process using the computational methods in the computer program Maple. Copies of the corresponding Maple .mws files, for the Jacobi-type, Laguerre-type, Legendre-type equations, are available; see Section 6 below.

There are four subsequent sections devoted to reporting on the form of these solutions, for each of the type fourth-order differential equations.

The four type differential equations are all written in the Lagrange symmetric (formally self-adjoint) form with spectral parameter \( \Lambda \) or \( \lambda \); if further properties of the solutions of these equations, involving the theory of special functions, are to be studied, then these spectral notations may have to be changed.

2. Bessel-type differential equation

The fourth-order Bessel-type differential equation takes the form

\[
(xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \text{ for all } x \in (0, \infty)
\]

where \( M \in (0, \infty) \) is a positive parameter and \( \Lambda \in \mathbb{C} \), the complex field, is a spectral parameter. The differential equation (2.1) is derived in the paper [8, Section 1, (1.10a)]. The many spectral properties of this equation in the weighted Hilbert function space \( L^2((0, \infty); x) \) are studied in the papers [3], [5] and [6]; for a collected account see the survey paper [4].

Our knowledge of the special function solutions of the Bessel-type differential equation (2.1) is now more complete than at the time the paper [8] was written. However, the results in [8, Section 1, (1.8a)], with \( \alpha = 0 \), show that the function defined by

\[
J_{\lambda}^0, M(x) := [1 + M(\lambda/2)^2]J_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}J_1(\lambda x) \text{ for all } x \in (0, \infty),
\]

is a solution of the differential equation (2.1), for all \( \lambda \in \mathbb{C} \), and hence for all \( \Lambda \in \mathbb{C} \), and all \( M > 0 \). Here:

(i) the parameter \( M > 0 \)
(ii) the parameter \( \lambda \in \mathbb{C} \)
(iii) the spectral parameter \( \Lambda \) and the parameter \( M \), in the equation (2.1), and the parameters \( M \) and \( \lambda \), in the definition (2.2), are connected by the relationship

\[
\Lambda = \Lambda(\lambda, M) = \lambda^2(\lambda^2 + 8M^{-1}) \text{ for all } \lambda \in \mathbb{C} \text{ and all } M > 0
\]

(iv) \( J_0 \) and \( J_1 \) are the classical Bessel functions (of the first kind), see [15, Chapter III].

Similar arguments to the methods given in [8] show that the function defined by

\[
Y_{\lambda}^{0, M}(x) := [1 + M(\lambda/2)^2]Y_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}Y_1(\lambda x) \text{ for all } x \in (0, \infty),
\]
FOURTH-ORDER TYPE EQUATIONS

is also an independent solution of the differential equation (2.1), for all \( \lambda \in \mathbb{C} \), and hence for all \( \Lambda \in \mathbb{C} \) and all \( M > 0 \); here, again, \( Y_0 \) and \( Y_1 \) are classical Bessel functions (of the second kind), see [15, Chapter III].

These earlier studies of the fourth-order differential equation (2.1) failed to find any explicit form of two linearly independent solutions, additional to the solutions \( J_0^0, M \) and \( Y_0^0, M \). However, results of van Hoeij, see [13] and [14], using the computer algebra program Maple have yielded the required two additional solutions, here given the notations of \( I_0^0, M \) and \( K_0^0, M \), with explicit representation in terms of the classical modified Bessel functions \( I_0 \) and \( K_0 \), and \( I_1 \), \( K_1 \).

These two additional solutions are defined as follows, where as far as possible we have followed the notation used for the solutions \( J_0^0, M \) and \( Y_0^0, M \),

\[(i)\] given \( \lambda \in \mathbb{C} \), with \( \text{arg}(\lambda) \in [0, 2\pi) \), \( M \in (0, \infty) \) and using the principal value of \( \sqrt{\cdot} \),

\[
c \equiv c(\lambda, M) := \sqrt{\lambda^2 + 8M^{-1}} \quad \text{and} \quad d \equiv d(\lambda, M) := 1 + M(\lambda/2)^2
\]

\[(ii)\] define the solution, for all \( x \in (0, \infty) \),

\[
I_0^0, M(x) := -dI_0(cx) + \frac{1}{2}cMx^{-1}I_1(cx)
\]

\[(iii)\] define the solution, for all \( x \in (0, \infty) \),

\[
K_0^0, M(x) := dK_0(cx) + \frac{1}{2}cMx^{-1}K_1(cx)
\]

Remark 2.1. We have

(1) The four linearly independent solutions \( J_0^0, M \), \( Y_0^0, M \), \( I_0^0, M \), \( K_0^0, M \) provide a basis for all solutions of the original differential equation (2.1), subject to the \((\Lambda, \lambda)\) connection given in (2.3).

(2) These four solutions are real-valued on their domain \((0, \infty)\) for all \( \lambda \in \mathbb{R} \).

(3) The domain \((0, \infty)\) of the solutions \( J_0^0, M \) and \( I_0^0, M \) can be extended to the closed half-line \([0, \infty)\) with the properties

\[
J_0^0, M(0) = I_0^0, M(0) = 1 \quad \text{for all} \quad \lambda \in \mathbb{R} \quad \text{and} \quad M \in (0, \infty).
\]

3. LAGUERRE-TYPE DIFFERENTIAL EQUATION

The Laguerre-type differential equation was discovered by H.L. Krall, see [10] and [11], and subsequently studied by other authors, see [9], [12] and [7].

The differential equation may be written in two forms; here the parameter \( A \in (0, \infty) \) and the spectral parameter \( \lambda \in \mathbb{C} \):
The special functions involved are the Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ as defined in the compendium [1, Chapter 13, Sections 13.1.31 to 13.1.34]. Define the functions $\Phi$ and $\Psi$, for the range of variables given in [1, Chapter 13]:

\begin{equation}
\Phi(\kappa, \mu; z) := M_{\kappa,\mu}(z)
\end{equation}

\begin{equation}
\Psi(\kappa, \mu; z) := W_{\kappa,\mu}(z).
\end{equation}

Given the parameters $A \in (0, \infty)$ and $\lambda \in \mathbb{C}$ define

\begin{equation}
\Gamma(\lambda, A) := \sqrt{4A^2 + 4A + 1 + 4\lambda}.
\end{equation}

We designate the four linearly independent solutions of (3.1), equivalently (3.2), again the parameters $A \in (0, \infty)$ and $\lambda \in \mathbb{C}$, by

$L_0(\lambda, A; x)$ for all $x \in (0, \infty)$ and $r = 1, 2, 3, 4$.

These solutions are defined as follows, where $d/dx$ denotes differentiation with respect to the independent variable $x$,

\begin{equation}
L_1(\lambda, A; x) := \left( \frac{1}{2} + \frac{1}{2} \Gamma(\lambda, A) \right) x^{-1/2} \exp\left( \frac{1}{2} x \right) \Phi(-A - \frac{1}{2} \Gamma(\lambda, A), 0; x)
- \frac{d}{dx} \left( x^{-1/2} \exp\left( \frac{1}{2} x \right) \Phi(-A - \frac{1}{2} \Gamma(\lambda, A), 0; x) \right).
\end{equation}

\begin{equation}
L_2(\lambda, A; x) := \left( \frac{1}{2} - \frac{1}{2} \Gamma(\lambda, A) \right) x^{-1/2} \exp\left( \frac{1}{2} x \right) \Phi(-A + \frac{1}{2} \Gamma(\lambda, A), 0; x)
- \frac{d}{dx} \left( x^{-1/2} \exp\left( \frac{1}{2} x \right) \Phi(-A + \frac{1}{2} \Gamma(\lambda, A), 0; x) \right).
\end{equation}

\begin{equation}
L_3(\lambda, A; x) := \left( \frac{1}{2} + \frac{1}{2} \Gamma(\lambda, A) \right) x^{-1/2} \exp\left( \frac{1}{2} x \right) \Psi(-A - \frac{1}{2} \Gamma(\lambda, A), 0; x)
- \frac{d}{dx} \left( x^{-1/2} \exp\left( \frac{1}{2} x \right) \Psi(-A - \frac{1}{2} \Gamma(\lambda, A), 0; x) \right).
\end{equation}
FOURTH-ORDER TYPE EQUATIONS

\[ L_4 \]
\[ L_4(\lambda, A; x) := \left( \frac{1}{2} - \frac{1}{2} \Gamma(\lambda, A) \right) x^{-1/2} \exp(\frac{x}{2}) \Psi(-A + \frac{1}{2} \Gamma(\lambda, A), 0; x) \]
\[ - \frac{d}{dx} \left( x^{-1/2} \exp(\frac{x}{2}) \Psi(-A + \frac{1}{2} \Gamma(\lambda, A), 0; x) \right). \]

4. LEGENDRE-TYPE DIFFERENTIAL EQUATION

The Legendre-type differential equation was discovered by H.L. Krall, see [10] and [11], and subsequently studied by other authors, see [9], [12] and [7].

The differential equation may be written in two forms; here the parameter \( A \in (0, \infty) \) and the spectral parameter \( \lambda \in \mathbb{C} \):

1. The Frobenius form:
   \[ (x^2 - 1)^2 y^{(4)}(x) + 8x(x^2 - 1)y^{(3)}(x) + (4A + 12)(x^2 - 1)y''(x) + 8Ax y'(x) = \lambda y(x) \]
   for all \( x \in (-1, +1) \).

2. The Lagrange symmetric form (formally self-adjoint form):
   \[ ((1 - x^2)y''(x))'' - ((8 + 4A(1 - x^2))y'(x))' = \lambda y(x) \]
   for all \( x \in (-1, +1) \).

The spectral properties of the equation (4.2) are considered in the Hilbert function space \( L^2(-1, +1) \).

The van Hoeij method of searching for factors and solutions of this differential equation depends on the use of the computer algebra program Maple; see [13] and [14].

Following the solutions of the fourth-order Bessel-type obtained in [8, Section 2], and then in [4] on using the methods of [14], it has proved possible to express all solutions of the equations (4.1) and (4.2) in terms of the confluent hypergeometric functions.

The special functions involved are the Legendre functions \( P_{\nu}^{\mu}(z) \) and \( Q_{\nu}^{\mu}(z) \) as defined in the compendium [1, Chapter 8, Section 8.1].

Define the functions \( P(\nu, x) \) and \( Q(\nu, x) \), for the range of variables given in [1, Chapter 8], by:

\[ P(\nu, x) := P_{\nu}^{0}(x) \]
\[ Q(\nu, x) := Q_{\nu}^{0}(z). \]

Given the parameters \( A \in (0, \infty) \) and \( \lambda \in \mathbb{C} \) define, noting the use of the ± symbol,

\[ \Gamma_{\pm}(\lambda, A) := \pm \sqrt{4A^2 - 4A + 1 + \lambda} \]
\[ \Omega_{\pm}(\lambda, A) := \sqrt{5 - 8A + 4\Gamma_{\pm}(\lambda, A)} \]

We designate the four linearly independent solutions of (4.1), equivalently (4.2), again the parameters \( A \in (0, \infty) \) and \( \lambda \in \mathbb{C} \), by

\[ L_{e_r}(\lambda, A; x) \] for all \( x \in (-1, +1) \) and \( r = 1, 2, 3, 4 \).
These solutions are defined as follows, where \( d/dx \) denotes differentiation with respect to the independent variable \( x \),

\[
(4.7) \quad Le_1(\lambda, A; x) := -\frac{1}{2}(1 + \Omega^+(\lambda, A))xP\left(\frac{3}{2}\sqrt{9 - 8A + 4\Gamma^+(\lambda, A) + 4\Omega^+(\lambda, A) - \frac{1}{2}}, x\right) + \left[-(\lambda + 3 - 4A + 4A^2) + \Omega^+(\lambda, A) - 3\Gamma^+(\lambda, A) + \Omega^+(\lambda, A)\Gamma^+(\lambda, A) + (\lambda + 4A + 4A^2)x^2\right] \\
\times (\lambda + 4A + 4A^2)^{-1}\frac{d}{dx}P\left(\frac{3}{2}\sqrt{9 - 8A + 4\Gamma^+(\lambda, A) + 4\Omega^+(\lambda, A) - \frac{1}{2}}, x\right).
\]

\[
(4.8) \quad Le_2(\lambda, A; x) := -\frac{1}{2}(1 + \Omega^-(\lambda, A))xP\left(\frac{3}{2}\sqrt{9 - 8A + 4\Gamma^-(\lambda, A) + 4\Omega^-(\lambda, A) - \frac{1}{2}}, x\right) + \left[-(\lambda + 3 - 4A + 4A^2) + \Omega^-(\lambda, A) - 3\Gamma^-(\lambda, A) + \Omega^-(\lambda, A)\Gamma^-(\lambda, A) + (\lambda + 4A + 4A^2)x^2\right] \\
\times (\lambda + 4A + 4A^2)^{-1}\frac{d}{dx}P\left(\frac{3}{2}\sqrt{9 - 8A + 4\Gamma^-(\lambda, A) + 4\Omega^-(\lambda, A) - \frac{1}{2}}, x\right).
\]

\[
(4.9) \quad Le_3(\lambda, A; x) := -\frac{1}{2}(1 + \Omega^+(\lambda, A))xQ\left(\frac{3}{2}\sqrt{9 - 8A + 4\Gamma^+(\lambda, A) + 4\Omega^+(\lambda, A) - \frac{1}{2}}, x\right) + \left[-(\lambda + 3 - 4A + 4A^2) + \Omega^+(\lambda, A) - 3\Gamma^+(\lambda, A) + \Omega^+(\lambda, A)\Gamma^+(\lambda, A) + (\lambda + 4A + 4A^2)x^2\right] \\
\times (\lambda + 4A + 4A^2)^{-1}\frac{d}{dx}Q\left(\frac{3}{2}\sqrt{9 - 8A + 4\Gamma^+(\lambda, A) + 4\Omega^+(\lambda, A) - \frac{1}{2}}, x\right).
\]

\[
(4.10) \quad Le_4(\lambda, A; x) := -\frac{1}{2}(1 + \Omega^-(\lambda, A))xQ\left(\frac{3}{2}\sqrt{9 - 8A + 4\Gamma^-(\lambda, A) + 4\Omega^-(\lambda, A) - \frac{1}{2}}, x\right) + \left[-(\lambda + 3 - 4A + 4A^2) + \Omega^-(\lambda, A) - 3\Gamma^-(\lambda, A) + \Omega^-(\lambda, A)\Gamma^-(\lambda, A) + (\lambda + 4A + 4A^2)x^2\right] \\
\times (\lambda + 4A + 4A^2)^{-1}\frac{d}{dx}Q\left(\frac{3}{2}\sqrt{9 - 8A + 4\Gamma^-(\lambda, A) + 4\Omega^-(\lambda, A) - \frac{1}{2}}, x\right).
\]

(1) It has been verified, using Maple 9, that \( Le_1, Le_2, Le_3, Le_4 \) are solutions of the differential equations (4.1) and (4.2) for the domain \( x \in (-1, +1) \), and the parameter values \( A \in (0, \infty) \) and \( \lambda \in \mathbb{C} \); this verification follows from the original results of van Hoeij.

(2) The factor \( (\lambda + 4A + 4A^2)^{-1} \) which appears in the derivative term of all four of the solutions \( Le_r \), for \( r = 1, 2, 3, 4 \), presents an apparent singularity in these solutions when \( \lambda = -(4A + 4A^2) \); however Maple 9 determines that the other factors

\[
[-(\lambda + 3 - 4A + 4A^2) + \Omega^+(\lambda, A) - 3\Gamma^+(\lambda, A) + \Omega^+(\lambda, A)\Gamma^+(\lambda, A) + (\lambda + 4A + 4A^2)x^2]
\]

both have a zero at this value \( \lambda = -(4A + 4A^2) \).

5. Jacobi-type differential equation

This note concerns information about the van Hoeij solutions to the fourth-order Jacobi-type linear ordinary differential equation.
The Jacobi-type differential equation was discovered by H.L. Krall, see [10] and [11], and subsequently studied by other authors, see [9], [12] and [7].

The differential equation may be written in two forms; here the parameters $A \in (0, \infty)$ and $\alpha \in (-1, \infty)$, and the spectral parameter $\lambda \in \mathbb{C}$:

(1) The Frobenius form:

\begin{equation}
\begin{aligned}
(1 - x^2)^2 y^{(4)}(x) - 2(1 - x^2)((\alpha + 4)x + \alpha)y^{(3)}(x) \\
+ (1 + x)((4A2^2 + \alpha^2 + 9\alpha + 14)x + (-4A2^2 + \alpha^2 - 3\alpha - 10)) y''(x) \\
+ ((4\alpha 2^2 + 8A2^2 + 2\alpha^2 + 6\alpha + 4)x + (4A2^2 + 2\alpha^2 + 6\alpha + 4)y(x) \\
= \lambda y(x) \text{ for all } x \in (-1, +1).
\end{aligned}
\end{equation}

(2) The Lagrange symmetric form (formally self-adjoint form):

\begin{equation}
\begin{aligned}
J[y](x) := ((1 - x)\alpha^2 + 2(1 + x)y''(x))'' \\
- ((1 - x)\alpha + 1) - (4A2^2 + 2\alpha + 2)x + 4A2^2 + 2\alpha + 6) y'(x)' \\
= \lambda(1 - x)^\alpha y(x) \text{ for all } x \in (-1, +1).
\end{aligned}
\end{equation}

The spectral properties of the equation (5.2) are considered in the weighted Hilbert function space $L^2((-1, +1); (1 - x)^\alpha)$.

The van Hoeij method of searching for factors and solutions of this differential equation depends on the use of the computer algebra program Maple; see [13] and [14].

Following the solutions of the fourth-order Bessel-type obtained in [8] Section 2, and then in [14] on using the methods of [14], it has proved possible to express all solutions of the equations (5.1) and (5.2) in terms of hypergeometric functions.

The special functions involved are the JacobiP functions; these functions are defined in terms of the hypergeometric functions $_{p}F_{q}$; for certain values of the parameters involved in this definition, they reduce to the classical Jacobi polynomials $P_{n}^{(\alpha, \beta)}$. The notation JacobiP is not to be found in [11] Chapter 22] but is given in the program Maple9; here we shorten the notation JacobiP and define the function $JP$ as follows, in terms of the classical $\Gamma$ and $2F_1$ functions:

\begin{equation}
\text{JacobiP}(\nu, \alpha, \beta; z) \equiv JP(\nu, \alpha, \beta; z) = \frac{\Gamma(\nu + \alpha + 1)}{\Gamma(\nu + 1)\Gamma(\alpha + 1)} _{2}F_{1}(-\nu, \nu + \alpha + \beta + 1; \alpha + 1; (1 - z)/2).
\end{equation}

Here, the independent variables $\nu, \alpha, \beta, z$ are in the complex field $\mathbb{C}$, but for the results in this note we make the restrictions

$\nu \in \mathbb{C}, \alpha \in (-1, \infty), \beta = 0, z \in \mathbb{R}$.

When $\nu = n \in \mathbb{N}_0$ then we have the connection, see [11] Section 22.2] and [2] Lecture 2, (2.2)],

$P_{n}^{(\alpha, \beta)}(x) = JP(n, \alpha, \beta; x)$ for all $\alpha, \beta \in (-1, \infty)$ and all $x \in (-1, +1)$.

The Jacobi-type orthogonal polynomials were discovered by H.L. Krall, see [10] and [11], and later studied by A.M. Krall, T.H. Koornwinder and L.L. Littlejohn. These polynomials are denoted by $P_{n,A}^{\alpha}(x)$ for $\alpha \in (-1, \infty), A \in (0, \infty), n \in \mathbb{N}_0$ and $x \in (-1, +1); each P_{n,A}^{\alpha}$ is real-valued on $(-1, +1)$ and of degree $n$.

These polynomials have the properties:
(i) For all \( n \in \mathbb{N}_0 \) the polynomial \( P_{\alpha}^{n,A} \) is a solution of the differential equation, see (5.2)

\[
J[P_{\alpha}^{n,A}](x) = \lambda_n(\alpha, A)(1 - x)\alpha P_{\alpha}^{n,A}(x) \quad \text{for all } x \in (-1, 1)
\]

where the eigenvalue \( \lambda_n(\alpha, A) \) is determined by

\[
\lambda_n(\alpha, A) = n(n + \alpha + 1)(n^2 + (\alpha + 1)n + 4A^2\alpha + \alpha) \quad \text{for all } n \in \mathbb{N}_0.
\]

(ii) The collection \( \{P_{\alpha}^{n,A}(\cdot) : n \in \mathbb{N}_0\} \) is a complete, orthogonal set in the Lebesgue-Stieltjes function space \( L^2((-1, 1); \mu_{Jac}) \), where the Borel measure \( \mu_{Jac} \) is determined by the monotonic non-decreasing function \( \hat{\mu}_{Jac} \), for all \( \alpha \in (-1, \infty) \) and all \( A \in (0, \infty) \),

\[
\begin{align*}
\hat{\mu}_{Jac}(x) &:= \begin{cases} 
-1/2 & \text{for all } x \in (-\infty, -1) \\
A/2(\alpha + 1) & \text{for all } x \in [-1, 1] \\
A/2(\alpha + 1) & \text{for all } x \in (+1, \infty).
\end{cases}
\end{align*}
\]

The orthogonality of the set \( \{P_{\alpha}^{n,A} : n \in \mathbb{N}_0\} \) then appears as

\[
\int_{[-1,1]} P_{m,A}^{\alpha}(x)P_{n,A}^{\alpha}(x) \, d\mu_{Jac}(x) = \begin{cases} 
0 & \text{for all } m, n \in \mathbb{N}_0 \text{ with } m \neq n.
\end{cases}
\]

The van Hoeij solutions for the general differential equation (5.2) begin with solutions to the set of special equations (5.4) when the general spectral parameter \( \lambda \) takes the value of one of the eigenvalues \( \{\lambda_n(\alpha, A) : n \in \mathbb{N}_0\} \).

Given \( n \in \mathbb{N}_0 \) define, for all \( x \in (-1, 1) \), with \( JP \) given by (5.3):

\[
S_{1,n}(x) := (n\alpha + 2A^2\alpha + n + n^2)JP(n, \alpha, 0; x) + (1 - x)\frac{d}{dx}JP(n, \alpha, 0; x).
\]

Then the van Hoeij methods determine that \( S_{1,n} \) is a solution of (5.1) for this value of \( n \), i.e. for \( \lambda = \lambda_n(\alpha, A) \).

**Remark 5.1.** From the definition (5.8) it would appear the this solution \( S_{1,n} \) is linearly dependent upon the Jacobi-type polynomial \( P_{\alpha}^{n,A} \).

To proceed further, given \( \lambda \in \mathbb{C} \), let \( \{\rho_r(\lambda) \in \mathbb{C} : r = 1, 2, 3, 4\} \) denote the roots of the quartic polynomial

\[
\rho(\rho + \alpha + 1)(\rho^2 + (\alpha + 1)\rho + 4A^2\alpha + \alpha) - \lambda = 0.
\]
Maple gives explicit representations for these roots as follows:

\[
\begin{align*}
\rho_1(\alpha, A, \lambda) &= \frac{1}{2} \left(-\alpha - 1 + \sqrt{\alpha^2 + 1 - 8A^2\alpha + 2\xi(\alpha, A, \lambda)}\right), \\
\rho_2(\alpha, A, \lambda) &= \frac{1}{2} \left(-\alpha - 1 - \sqrt{\alpha^2 + 1 - 8A^2\alpha + 2\xi(\alpha, A, \lambda)}\right), \\
\rho_3(\alpha, A, \lambda) &= \frac{1}{2} \left(-\alpha - 1 + \sqrt{\alpha^2 + 1 - 8A^2\alpha - 2\xi(\alpha, A, \lambda)}\right), \\
\rho_4(\alpha, A, \lambda) &= \frac{1}{2} \left(-\alpha - 1 - \sqrt{\alpha^2 + 1 - 8A^2\alpha - 2\xi(\alpha, A, \lambda)}\right),
\end{align*}
\]

(5.10)

where

\[
\xi(\alpha, A, \lambda) := \sqrt{\alpha^2 + 8\alpha A^2 + 16A^22^\alpha + 4\lambda}.
\]

The van Hoeij solutions to the Jacobi-type equation, see (5.10),

\[
J[y](x) = \lambda(1 - x)^\alpha y(x) \text{ for all } x \in (-1, +1)
\]

(5.11)

now develop as follows.

With \(\lambda \in \mathbb{C}\), replace \(n\) in the solution \(S_{1,n}\) with the first root \(\rho_1(\lambda)\) taken from (5.10); this yields a solution \(J_1\) in terms of the function \(JP\), for all \(\alpha \in (-1, \infty)\) and all \(A \in (0, \infty)\),

\[
\begin{align*}
J_1(\alpha, A, \lambda; x) := \frac{1}{2} [\alpha - \xi(\alpha, A, \lambda)] JP(\rho_1(\alpha, A, \lambda), \alpha, 0; x) \\
- (1 - x) \frac{d}{dx} JP(\rho_1(\alpha, A, \lambda), \alpha, 0; x)
\end{align*}
\]

(5.12)

for all \(x \in (-1, +1)\).

Similarly, for \(\lambda \in \mathbb{C}\), there is a solution \(J_2\) derived from the third root \(\rho_3(\lambda)\), for all \(\alpha \in (-1, \infty)\) and all \(A \in (0, \infty)\),

\[
\begin{align*}
J_2(\alpha, A, \lambda; x) := \frac{1}{2} [\alpha + \xi(\alpha, A, \lambda)] JP(\rho_3(\alpha, A, \lambda), \alpha, 0; x) \\
- (1 - x) \frac{d}{dx} JP(\rho_3(\alpha, A, \lambda), \alpha, 0; x)
\end{align*}
\]

(5.13)

for all \(x \in (-1, +1)\).

Given \(n \in \mathbb{N}_0\) define, for all \(x \in (-1, +1)\), there is a second solution \(S_{2,n}\), similar to \(S_{1,n}\) of (4.1), defined by

\[
S_{2,n}(x) := \left[(n + 1)\alpha + A2^{\alpha+1} + n + n^2\right](x - 1)^{-\alpha} JP(-1 - n, -\alpha, 0; x) \\
- (x - 1)^{1-\alpha} \frac{d}{dx} JP(-1 - n, -\alpha, 0; x).
\]

(5.14)

Then the van Hoeij methods determine that \(S_{2,n}\) is a solution of (5.1) for this value of \(n\), i.e. for \(\lambda = \lambda_n(\alpha, A)\).

With \(\lambda \in \mathbb{C}\), replace \(n\) in the solution \(S_{2,n}\) with the first root \(\rho_1(\lambda)\) taken from (5.10); this yields a solution \(J_3\) in terms of the function \(JP\), for all \(\alpha \in (-1, \infty)\) and all \(A \in (0, \infty)\),

\[
\begin{align*}
J_3(\alpha, A, \lambda; x) := -\frac{1}{2} [\alpha + \xi(\alpha, A, \lambda)](-1 + x)^{-\alpha} JP(-1 - \rho_1(\alpha, A, \lambda), -\alpha, 0; x) \\
+ (-1 + x)^{1-\alpha} \frac{d}{dx} JP(-1 - \rho_1(\alpha, A, \lambda), -\alpha, 0; x)
\end{align*}
\]

(5.15)

for all \(x \in (-1, +1)\).
Similarly, for \( \lambda \in \mathbb{C} \), there is a solution \( J_4 \) derived from the third root \( \rho_3(\lambda) \), for all \( \alpha \in (-1, \infty) \) and all \( A \in (0, \infty) \),

\[
\begin{align*}
J_4(\alpha, A, \lambda; x) & := -\frac{1}{2}[\alpha - \xi(\alpha, A, \lambda)](-1 + x)^{-\alpha} J P(-1 - \rho_3(\alpha, A, \lambda), -\alpha, 0; x) \\
& \quad + (-1 + x)^{1-\alpha} \frac{d}{dx} J P(-1 - \rho_3(\alpha, A, \lambda), -\alpha, 0; x) \text{ for all } x \in (-1, 1).
\end{align*}
\]

6. Maple files

The formal validity of these solution results were originally obtained by van Hoeij, following the methods given in [13] and [14], using the computer program Maple. These results have been subsequently confirmed by David Smith in the Maple program at the University of Birmingham.

The Maple .mws files for the following three cases are named as follows:

- Jacobi-type-solutions.mws
- Laguerre-type-solutions.mws
- Legendre-type-solutions.mws

Copies of these files may be obtained on application to David Smith at smithd@for.mat.bham.ac.uk

7. Acknowledgements

The authors indicate here that the results reported on in this note follow from the combined efforts of the many collaborators whose names are given in the references below.

The author Norrie Everitt is particularly grateful for the advice and help received from Dr David Smith, School of Mathematics at the University of Birmingham, on the application and use of the computer program Maple.

The author Mark van Hoeij is supported by National Science Foundation grant 0511544.

References

[1] M. Abramowitz and I.A. Stegun. Handbook of mathematical functions. (Dover Publications, New York: 1965.)
[2] R. Askey. Orthogonal Polynomials and Special Functions. (SIAM, Philadelphia, USA: 1975.)
[3] J. Das, D.B. Hinton, L.L. Littlejohn and C. Markett. The fourth-order Bessel-type differential equation. Applicable Analysis. 83 (2004), 325-362.
[4] W.N. Everitt. Fourth-order Bessel-type special functions: a survey. (Submitted for proceedings of the International Conference on Difference Equations, Special Functions and Applications; Technical University Munich, Germany: 25 to 30 July 2005.)
[5] W.N. Everitt, H. Kalf, L.L. Littlejohn and C. Markett. Additional properties of the fourth-order Bessel-type differential equation. Math. Nachr. 278 (2005), 1538-1549.
[6] W.N. Everitt, H. Kalf, L.L. Littlejohn and C. Markett. The fourth-order Bessel equation; eigenpackets and a generalised Hankel transform. (Submitted to Integral Transforms and Special Transforms.)
[7] W.N. Everitt, A.M. Krall, L.L. Littlejohn and V.P. Onyango-Otieno. Differential operators and the Laguerre type polynomials. SIAM J. Math. Anal. 23 (1992), 722–736.
[8] W.N. Everitt and C. Markett. On a generalization of Bessel functions satisfying higher-order differential equations. Jour. Computational Appl. Math. 54 (1994), 325-349.
FOURTH-ORDER TYPE EQUATIONS

[9] A.M. Krall. Orthogonal polynomials satisfying fourth order differential equations. *Proc. Roy. Soc. Edinburgh* (A) 87 (1981), 271-288.

[10] H.L. Krall. Certain differential equations for Tchebycheff polynomials. *Duke Math. J.* 4 (1938) 705-718.

[11] H.L. Krall. On orthogonal polynomials satisfying a certain fourth order differential equation. *The Pennsylvania State College Studies: No. 6.* (The Pennsylvania State College, State College, PA.: 1940.)

[12] T.H. Koornwinder. Orthogonal polynomials with weight function $(1-x)\alpha(1+x)\beta+M\delta(x+1)+N\delta(x-1)$. *Canad. Math. Bull.* 27 (2) (1984), 205-214.

[13] M. van Hoeij. Formal solutions and factorization of differential operators with power series coefficients. *J. Symbolic Comput.* 24 (1997), 1-30.

[14] M. van Hoeij. Personal contribution. (International Conference on Difference Equations, Special Functions and Applications; Technical University Munich, Germany: 25 to 30 July 2005.)

[15] G.N. Watson. *A treatise on the theory of Bessel functions.* (Cambridge University Press; second edition: 1950.)

W.N. EVERITT, SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF BIRMINGHAM, EDBASTON, BIRMINGHAM B15 2TT, ENGLAND, UK

E-mail address: w.n.everitt@bham.ac.uk

D. J. SMITH, SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF BIRMINGHAM, EDBASTON, BIRMINGHAM B15 2TT, ENGLAND, UK

E-mail address: smithd@for.mat.bham.ac.uk

M. VAN HOEIJ, DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, 208 JAMES J. LOVE BUILDING, TALLAHASSEE, FL 32306-4510, USA

E-mail address: hoeij@math.fsu.edu