LOGARITHMIC SUBMAJORISATIONS INEQUALITIES FOR OPERATORS IN A FINITE VON NEUMANN ALGEBRA

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ABSTRACT. The aim of this paper is to study the logarithmic submajorisations inequalities for operators in a finite von Neumann algebra. Firstly, some logarithmic submajorisations inequalities due to Garg and Aulja are extended to the case of operators in a finite von Neumann algebra. As an application, we get some new Fuglede-Kadison determinant inequalities of operators in that circumstance. Secondly, we improve and generalize to the setting of finite von Neumann algebras, a generalized Hölder type generalized singular numbers inequality.

1. Introduction

The views of determinants and majorization inequalities in mathematical analysis are based on the convexity(or concavity) of function and the singular values equations and inequalities. For instance, Weyl’s Theorem in matrix analysis involves that idea in the proof. Using singular values inequalities and log convexity of determinants inequalities, Key Fan obtained two different forms of generalizations of Weyl’s theorem in [6, 7]. The singular values of $A \in \mathbb{M}_n$, the space of $n \times n$ complex matrices, i.e., the eigenvalues of the matrix $|A| := (A^* A)^{1/2}$, enumerated in decreasing order, will be denoted by $s_j(A)$, $j = 1, 2, \ldots, n$.

Rotfel’d [17] proved a determinants inequality:

$$\det(I_n + r|A + B|^p) \leq \det(I_n + r|A|^p)\det(I_n + r|B|^p),$$

(1)

where $A, B \in \mathbb{M}_n$, $r > 0$, $0 < p \leq 1$ and $I_n$ is the identity matrix. Subsequently, Garg-Aujla [11] proposed the following singular value inequality for matrices:

$$\Pi_{j=1}^k s_j(I_n + f(|A + B|)) \leq \Pi_{j=1}^k s_j(I_n + f(|A|))s_j(I_n + f(|B|)),$$

(2)

where $1 \leq k \leq n$ and $A, B \in \mathbb{M}_n$ and $f : [0, +\infty) \to [0, +\infty)$ is an operator concave function with $f(0) = 0$. This is a refinement of the inequality (1). They also proved that for $A, B \in \mathbb{M}_n$ and $1 \leq r \leq 2$,

$$\Pi_{j=1}^k s_j(|A + B|^r) \leq \Pi_{j=1}^k s_j(I_n + |A|^r)s_j(I_n + |B|^r)$$

(3)

holds for $1 \leq k \leq n$. Very recently, a new proof for inequalities (2) and (3) were gave by Zhao in [19]. Meanwhile, they showed that inequality (2) also holds when
\(f\) is a nonnegative concave function. Liu-Poon-Wang [18] proved a generalized Hölder type eigenvalue inequality:

\[
\prod_{j=1}^{k} (1 - s_j(|A_1 \cdots A_m|)^r) \geq \prod_{j=1}^{k} \Pi_{i=1}^{m} (1 - s_j(|A_i|)^{r_p}) \frac{1}{p},
\]

(4)

where \(A_1, \ldots, A_m\) are \(n \times n\) contractive matrices and \(p_1, \ldots, p_m > 0\), with \(\sum_{i=1}^{m} \frac{1}{p_i} = 1\), for each \(k = 1, 2, \ldots, n\) and \(r \geq 1\).

With the help of generalized singular numbers’ method, Fack [8] and Fack-Kosaki [9] gave the inequality (1) and (2) for operators in a semi-finite von Neumann algebra. It is our intention in this paper to indicate that the logarithmic submajorisations inequalities of Garg-Aulja may be extended to the general setting of operators in a finite von Neumann algebra. And we prove a generalized Hölder type generalized singular numbers inequality in the case of operators in a finite von Neumann algebra.

This article is organized as follows. In section 2, we setup the background for our discussion. Along with setting up notation, we present a primer on the theory of von Neumann algebras, two kinds of generalized singular numbers, Fuglede-Kadison determinant of measure operators affiliated with a finite von Neumann algebras. In section 3, we collect some lemmas of logarithmic submajarisations inequalities under the case of the monotone concave function on \([0, +\infty)\) and extend Garg and Aulja’s results to operator case. The crux of the discussion is in the final section where some basic equations and inequalities of two kinds of generalized singular numbers are proved. And we propose a generalized Hölder type generalized singular numbers inequality in this section.

2. Preliminaries

Suppose that \(\mathcal{H}\) is a separable Hilbert space over the field \(\mathbb{C}\) and \(I\) is the identity operator in \(\mathcal{H}\). We will denote by \(\mathcal{B}(\mathcal{H})\) the *-algebra of all linear bounded operators in \(\mathcal{H}\). Let \(\mathcal{M}\) be a *-subalgebra of \(\mathcal{B}(\mathcal{H})\) containing the identity operator \(I\). Then \(\mathcal{M}\) is called a von Neumann algebra if \(\mathcal{M}\) is weak*-operator closed. Let \(\mathcal{M}\) be a finite von Neumann algebra, with a finite normal faithful trace \(\tau\), acting on the separable Hilbert space \(\mathcal{H}\), and \(\mathcal{M}_+\) its positive part. We refer to [9, 16] for noncommutative integration.

Let \(x\) be a closed densely defined operator and \(x = u|x|\) its polar decomposition, where \(|x| = (x^*x)^{\frac{1}{2}}\) and \(u\) is a partial isometry. Then \(x\) is affiliated with \(\mathcal{M}\) iff \(u \in \mathcal{M}\) and \(|x|\) is affiliated with \(\mathcal{M}\). For convenience, we assume \(\tau(I) = 1\) in the following.

For \(x \in \mathcal{M}\), we define the generalized singular numbers by

\[\mu_t(x) = \inf\{\lambda > 0 : \tau(e_{\lambda}(|x|) \leq t)\}, \quad t > 0,\]

where the operators \(e_{\lambda}(|x|)\) are the spectral projection of \(|x|\). We denote simply by \(\mu_t(x)\) the function \(t \to \mu_t(x)\). If we consider the algebra \(\mathcal{M} = L^\infty([0, 1])\) of all Lebesgue measurable essentially bounded functions on \([0, 1]\). For \(f \in L^\infty([0, 1])\), the decreasing rearrangement \(f^*\) of the function \(f\) is given by

\[f^*(t) = \inf\{s \in \mathbb{R} : m(\{h \in [0, 1] : |f(h)| > s\}) \leq t\}, 0 < t < 1.\]

Then \(\mu_t(f) = f^*(t)\).
As a matter of convenience, we state some properties of generalized singular numbers as follows without proof (see [9]). The function $t \to \mu_t(x)$ is a non-increasing right-continuous function on $(0, \infty)$ and

\[ \mu_t(x^*x) = \mu_t(xx^*) \quad \text{and} \quad \mu_t(u xv) \leq \|u\|\mu_t(x)\|v\|, \]  

where $x, u, v \in \mathcal{M}$. If $f$ is a continuous increasing function on $[0, \infty)$ with $f(0) \geq 0$, then

\[ \mu_t(f(x)) = f(\mu_t(x)) \]  

and

\[ \tau(f(x)) = \int_0^{\tau(1)} f(\mu_t(x))dt. \]

See [9] for basic properties and detailed information on generalized singular number of $x$.

For $x \in \mathcal{M}$, we now introduce the non-increasing left-continuous function

\[ \mu^l_t(x) = \inf\{\lambda > 0 : \tau(e^\lambda(|x|) < t)\}, \quad t > 0. \]

Except for continuity, $\mu^l(x)$ and $\mu(x)$ have many similar properties. See [14] and [15] for basic properties and detailed information of this non-increasing left-continuous function.

For $x \in \mathcal{M}$, we define

\[ \Lambda_t(x) = \exp\left(\int_0^t \log \mu_s(x)ds\right), \quad t > 0. \]  

From the definition of $\Lambda_t(x)$ and the properties of $\mu_t(x)$, we obtain

\[ \Lambda_t(x) = \Lambda_t(x^*) = \Lambda_t(|x|), \quad t > 0 \]

and

\[ \Lambda_t(x^\alpha) = \Lambda_t(x)^\alpha, \quad t > 0, \quad \text{if} \quad \alpha > 0 \quad \text{and} \quad x > 0. \]  

Moreover, it follows from [9, Theorem 4.2] that

\[ \Lambda_t(xy) \leq \Lambda_t(x)\Lambda_t(y), \quad t > 0 \]

holds for all $x, y \in \mathcal{M}$.

**Definition 2.1.** Let $x, y \in \mathcal{M}$. We say that $x$ is logarithmically submajorized by $y$ and write $x \preceq_{\text{wlog}} y$ if and only if

\[ \Lambda_t(x) \leq \Lambda_t(y) \quad \text{for all} \quad t \geq 0. \]

**Definition 2.2.** Let $\mathcal{M}$ be a finite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ with a normal faithful finite trace $\tau$. For $x \in \mathcal{M}$, the Fuglede-Kadison determinant of $x$ is defined by

\[ \Delta(x) = \exp \tau(\log |x|) \quad \text{if} \quad |x| \quad \text{is invertible}; \]

and otherwise, the Fuglede-Kadison determinant $\Delta(x) = \inf \Delta(|x| + \varepsilon \mathbb{I})$, the infimum takes over all scalars $\varepsilon > 0$.

Furthermore, let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function with $f(0) = 1$. If $x \in \mathcal{M}_+$, then $t \to \log f(t)$ is a continuous increasing function with $\log f(0) = 0$. From [9, Lemma 2.5] and the definition of determinant, we have

\[ \Delta(f(x)) = \exp \int_0^{\tau(1)} \log(f(\mu_t(x)))dt = \Lambda_{\tau(1)}(f(x)). \]
See [1, 2, 5] for basic properties and detailed information on Fuglede-Kadison determinant and logarithmic submajorisations of \( x \in \mathcal{M} \).

Let \( \mathbb{M}_2(\mathcal{M}) \) denote the linear space of \( 2 \times 2 \) matrices

\[
x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}
\]

with entries \( x_{ij} \in \mathcal{M} \), \( i, j = 1, 2 \). Let \( \mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H} \), then \( \mathbb{M}_2(\mathcal{M}) \) is a von Neumann algebra on the Hilbert space \( \mathcal{H}^2 \). For \( x \in \mathbb{M}_2(\mathcal{M}) \), we define \( \tau_2(x) = \sum_{i=1}^{2} \tau(x_{ii}) \). Then \( \tau_2 \) is a normal faithful finite trace on \( \mathbb{M}_2(\mathcal{M}) \). The direct sum of operators \( x_1, x_2 \in \mathcal{M} \), denoted by \( x_1 \oplus x_2 \), is the block-diagonal operator matrix defined on \( \mathcal{H}^2 \) by

\[
x_1 \oplus x_2 = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}
\]

3. Some logarithmic submajorisations inequalities

To achieve our main results, we state for easy reference the following fact, obtained from [4, 13], which will be applied below.

**Lemma 3.1.** Let \( x, y \in \mathcal{M} \).

1. If \( 0 < x \in \mathcal{M} \), then

\[
\mu_t(\mathbb{1} + x) = 1 + \mu_t(x).
\]

2. Let \( a, b \in \mathcal{M} \) be two positive operators. Then the matrix \( \begin{pmatrix} a & x \\ x^* & b \end{pmatrix} \) is a positive semidefinite operator if and only if \( x = a^{\frac{1}{2}} w b^{\frac{1}{2}} \) for some contraction \( w \).

**Lemma 3.2.** Let \( x, y \in \mathcal{M} \) be two positive operators. Then

\[
\Lambda_t((xy)^p) \leq \Lambda_t(y^px^py^p), \ 0 \leq p \leq 1
\]

and

\[
\Lambda_t((xy)^p) \geq \Lambda_t(y^px^py^p), \ p \geq 1.
\]

**Proof.** It follows from [2, Proposition 1.11] and [12, Lemma 2.5].

In the following theorem, we extend inequality (3) to the case of operators of finite von Neumann algebra.

**Theorem 3.3.** Let \( x, y \in \mathcal{M} \). Then

\[
\Lambda_t(|x + y|^r) \leq \Lambda_t(\mathbb{1} + |x|^r) \Lambda_t(\mathbb{1} + |y|^r)
\]

holds for \( t \geq 0 \) and \( 1 \leq r \leq 2 \). Moreover, we have

\[
\Delta(|x + y|^r) \leq \Delta(\mathbb{1} + |x|^r) \Delta(\mathbb{1} + |y|^r)
\]

holds for \( 1 \leq r \leq 2 \).
Proof. Since
\[
\begin{pmatrix}
1 + xx^* & x + y \\
(x + y)^* & 1 + y^*y
\end{pmatrix}
= \begin{pmatrix}
1 & x \\
y^* & 1
\end{pmatrix}
\begin{pmatrix}
1 & y \\
x^* & 1
\end{pmatrix}
\geq 0,
\]
by Lemma 3.1 (2), there exists a contraction \( w \) with
\[
x + y = (1 + xx^*)^{\frac{1}{2}} w (1 + y^*y)^{\frac{1}{2}} = (1 + |x|^2)^{\frac{1}{2}} w (1 + |y|^2)^{\frac{1}{2}}.
\]
Thus
\[
|x + y|^{2r} = [(1 + |y|^2)^{\frac{1}{2}} w (1 + |x|^2)^{\frac{1}{2}} w (1 + |y|^2)^{\frac{1}{2}}]^{2r},
\]
for \( t \geq 0 \). Since \( w \) is a contraction, then
\[
\mu_t(1 + |x|^2)w \leq \mu_t(1 + |x|^2), \quad t > 0.
\]
Inequalities above and (10) imply
\[
\Lambda_t(|x + y|^{2r}) \leq \Lambda_t((1 + |y|^2)^{\frac{1}{2}} w (1 + |x|^2)^{\frac{1}{2}} w (1 + |y|^2)^{\frac{1}{2}})^{2r}
= \Lambda_t((1 + |y|^2)^{\frac{1}{2}} w (1 + |x|^2)^{\frac{1}{2}} w (1 + |y|^2)^{\frac{1}{2}})^{2r}, \quad t > 0,
\]
the above equality holds due to the unitarily equivalent of \(|x|^2\) and \(|x|^2\). When
\( r = 1 \), by inequality above, we have
\[
\Lambda_t(|x + y|^{2r}) \leq \Lambda_t((1 + |y|^2)^{\frac{1}{2}} w (1 + |x|^2)^{\frac{1}{2}} w (1 + |y|^2)^{\frac{1}{2}})^{2r}, \quad t \geq 0.
\]
On the other hand, \( f(t) = t^2 (1 \leq r < 2) \) is a concave function on \([0, +\infty)\),
then \( f(s) + f(t) \leq f(s + t) \), for \( s, t \in [0, +\infty) \). It follows that
\[
(1 + \mu_t(|x|)^2)^{\frac{1}{2}} \leq 1 + \mu_t(|x|)^r, \quad x \in \mathcal{M} \quad \text{and} \quad t \geq 0.
\]
Combining (6), (7), (8), (11) with (13) and noting that \( \mu_t(1 + |x|^2) = 1 + \mu_t(|x|)^2 \)
\( (t \geq 0) \), for \( x \in \mathcal{M} \), we get
\[
\Lambda_t(|x + y|^{2r}) \leq \Lambda_t((1 + |y|^r)^{\frac{1}{2}} w (1 + |x|^r)^{\frac{1}{2}} w (1 + |y|^r)^{\frac{1}{2}})^{2r}
\]
which means that
\[
\Lambda_t(|x + y|^{2r}) \leq \Lambda_t((1 + |y|^r)^{\frac{1}{2}} w (1 + |x|^r)^{\frac{1}{2}} w (1 + |y|^r)^{\frac{1}{2}})^{2r}, \quad t \geq 0 \quad \text{and} \quad 1 \leq r < 2.
\]
Thus, the result follows from inequalities (12) and (14). This completes the proof. \( \square \)

4. A generalized Hölder type eigenvalue inequality

To prove the finial theorem, we start with the following equations of the two kinds of singular numbers.

**Lemma 4.1.** Suppose \( x \in \mathcal{M} \) is a contractive operator. Then
\[
\mu_s(1 - |x|) = 1 - \mu_{1-s}(|x|), \quad 0 < s < 1
\]
and
\[
\mu_s^t(1 - |x|) = 1 - \mu_{1-s}(|x|), \quad 0 < s < 1.
\]
Proof. Let
\[ K = \left\{ x : x = \sum_{k=1}^{n} c_{k} e_{k}, c_{k} \in \mathbb{C}, \text{ for any positive integer } n \right\}. \]
Since \( K \) is dense in \( M \) with the operator norm, it is sufficient to show the lemma holds for \( x = \sum_{k=1}^{N} c_{k} e_{k} \in K \). It is clear that \(|x| = \sum_{k=1}^{N} |c_{k}| e_{k} \). Without loss of generality, we suppose \( 1 \geq |c_1| > |c_2| > \cdots > |c_N| \). Let \( d_j = \sum_{k=1}^{j} \tau(e_k), 1 \leq j \leq N, d_0 = 0 \) and \( \chi \) denote the indicative function. Then
\[
\mu_s(x) = \mu_s(|x|) = |c_1| \chi_{(d_0, d_1)}(s) + \sum_{j=2}^{N} |c_j| \chi_{(d_{j-1}, d_j)}(s), s \in (0, d_N)
\]
and \( \mu_s(x) = \mu_s(|x|) = 0, s \in [d_N, 1) \). Similarly,
\[
\mu_s^l(x) = \mu_s^l(|x|) = \sum_{j=1}^{N} |c_j| \chi_{(d_{j-1}, d_j)}(s), s \in (0, d_N)
\]
and \( \mu_s^l(x) = \mu_s^l(|x|) = 0, s \in (d_N, 1) \). Let \( f(t) = 1 - \mu_l(x) \). Then
\[
\mu_l(f) = \chi_{(0, 1-d_0)} + \sum_{i=1}^{N} (1 - |c_i|) \chi_{[1-d_i, 1-d_{i-1})} = 1 - \left\{ 0 \chi_{(0, 1-d_0)} + \sum_{i=1}^{N} |c_i| \chi_{[1-d_i, 1-d_{i-1})} \right\} = 1 - \mu_l^l(x),
\]
which implies that the first equation holds. The proof of the other containment is similar. \( \square \)

**Lemma 4.2.** Let \( x \in M \) be a self-adjoint contractive operator, then
\[
\mu_t(\mathbb{I} - |x|) \leq \mu_t(\mathbb{I} - x).
\]

*Proof.* Let \( \xi \in H \), we have
\[
\langle \xi, (\mathbb{I} - x)\xi \rangle = \langle \xi, \xi \rangle - \langle \xi, x\xi \rangle \geq \langle \xi, \xi \rangle - \langle \xi, |x|\xi \rangle = \langle \xi, (\mathbb{I} - |x|)\xi \rangle.
\]
By the monotonicity of \( \mu_t \) proved by [9, Lemma 2.5], we immediately get the conclusion. \( \square \)

**Lemma 4.3.** Let \( x, y \in M \) be two contractive operators and \( r \geq 1 \). Then
\[
\int_{0}^{t} \log(1 - \mu_s(|xy|^r))ds \geq \int_{0}^{t} \log(1 - \mu_s(|x|^r|y|^r))ds.
\]
Moreover,
\[
\int_{1-t}^{1} \log\mu_s^l(\mathbb{I} - |xy|^r)ds \geq \int_{1-t}^{1} \log\mu_s^l(\mathbb{I} - |x|^r|y|^r)ds.
\]

*Proof.* Let \( g(t) = -\log(1 - t) \), then \( g(x) \) is increasing and convex on \((0, 1)\). For \( t > 0 \), set
\[
x_t = \frac{x + t\mathbb{I}}{1 + 2t} \quad \text{and} \quad y_t = \frac{y + t\mathbb{I}}{1 + 2t},
\]
then $x_t, y_t$ are invertible and $\|x_t\| < 1, \|y_t\| < 1$. From the property of polar decomposition of $x_1$ and $x_2$, we obtain $\mu(x_1, x_2) = \mu(\|x_1\| \|x_2\|)$. Using [12, proposition 2.4], we get
\[
\int_0^h \mu_s(g(\|x_t y_t\|)) ds \leq \int_0^h \mu_s(g(\|x_t^r y_t^r\|)) ds.
\]
Since $-\int_0^h \log(1 - \mu_t(\|x_t y_t\|)) ds \leq -\int_0^h \log(1 - \mu_t(\|x_t^r y_t^r\|)) ds$, we have
\[
\int_0^h \log(1 - \mu_t(\|x_t y_t\|)) ds \geq \int_0^h \log(1 - \mu_t(\|x_t^r y_t^r\|)) ds.
\]
Taking limits with $t \to 0^+$, $x_t \to x$, $y_t \to y$, the general case follows that $\mu_s(xy) = \mu_s(x_t y_t)$. Then we get the results from the Lemma 4.1. \qed

**Lemma 4.4.** Let $r \geq 1$ and $f$ be a continuous increasing function on $[0, \infty)$ such that $f(0) = 0$ and $t \to f(e^t)$ is convex. For $0 \leq x, y \in L_0(\mathcal{M})$, we have
\[
\int_0^t \mu_s^i(f(\|xy\|^r)) ds \leq \int_0^t \mu_s^i(f(\|x^r y^r\|)) ds.
\]

**Proof.** Replacing left-continuous with right-continuous, $\mu_s^i(|x|)$ have the same property as $\mu_i(x)$. By slightly modifying the proof of in [12, Proposition 2.4], we can prove the lemma and omit the details. \qed

**Lemma 4.5.** Let $x, y \in \mathcal{M}$ be two contractive operators and $r \geq 1$. Then
\[
\int_0^t \log(1 - \mu_s^i(\|xy\|^r)) ds \geq \int_0^t \log(1 - \mu_s^i(\|x^r\| \|y^r\|)) ds.
\]
Moreover,
\[
\int_{1-t}^1 \log \mu_s(1 - \|xy\|^r) ds \geq \int_{1-t}^1 \log \mu_s(1 - \|x^r\| \|y^r\|) ds.
\]
Thus $\Delta(1 - \|xy\|^r) \leq \Delta(1 - \|x^r\| \|y^r\|)$.

**Proof.** Using Lemma 4.4, the proof can be done similarly to Lemma 4.3. The details are omitted. \qed

A generalized Hölder type eigenvalue inequality (4) be extended to the following theorem.

**Theorem 4.6.** Suppose $x_1, \ldots, x_m \in \mathcal{M}$ are contractive operators, $r \geq 1$ and $p_1, \ldots, p_m > 0$ with $\sum_{i=1}^m (1/p_i) = 1$. Then for all $t > 0$, we have
\[
\int_0^t \log(1 - \mu_s(\|x_1 \cdots x_m\|^r)) ds \geq \sum_{i=1}^m \int_0^t \log(1 - \mu_s(\|x_i\|)^{p_i}) \frac{1}{p_i} ds. \tag{15}
\]
Moreover,
\[
\int_{1-t}^1 \log \mu_s^i(1 - \|x_1 \cdots x_m\|^r) ds \geq \sum_{i=1}^m \int_{1-t}^1 \log \mu_s^i(1 - \|x_i\|)^{p_i} \frac{1}{p_i} ds. \tag{16}
\]
Proof. Fixed $x, y \in \mathcal{M}$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. By [10, Theorem 3.3], we have $\mu(xy^*) \leq \mu(\frac{1}{p}|x|^p + \frac{1}{q}|x|^q)$. Then $\lambda_t(xy^*) \leq \lambda_t(\frac{1}{p}|x|^p + \frac{1}{q}|x|^q)$. It follows from [3, Lemma 3.3] that there is a unitary operator $U \in \mathcal{M}$ such that

$$U|xy^*|U^* \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q + \varepsilon_1 I,$$

for every $\varepsilon_1 > 0$. Let $x, y \in \mathcal{M}$ be contractive operators. Then

$$\mathbb{1} - |x|^p \geq 0, \quad \mathbb{1} - |y|^q \geq 0.$$

Thus there exists a unitary operator $V \in \mathcal{M}$ such that

$$\mathbb{1} - U|xy^*|U^* \geq \frac{1}{p}(\mathbb{1} - |x|^p) + \frac{1}{q}(\mathbb{1} - |y|^q) - \varepsilon_1 \mathbb{1} \geq V[(\mathbb{1} - |x|^p)\frac{1}{p}(I - |y|^q)^\frac{1}{q}]V^* - \varepsilon_2 \mathbb{1} - \varepsilon_1 \mathbb{1},$$

for a fixed $\varepsilon_2 > 0$. By [4, Lemma 2.4] and the property of rearrangements, we get

$$\mu_s(\mathbb{1} - |xy^*|) = \mu_s(\mathbb{1} - U|xy^*|U^*) \geq \mu_s((\mathbb{1} - |x|^p)\frac{1}{p}(\mathbb{1} - |y|^q)^\frac{1}{q}) - (\varepsilon_2 + \varepsilon_1)\mu_s(\mathbb{1}).$$

Letting $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, we deduce

$$\mu_s(\mathbb{1} - |xy^*|) \geq \mu_s((\mathbb{1} - |x|^p)\frac{1}{p}(\mathbb{1} - |y|^q)^\frac{1}{q}).$$

On the other hand, it is clear that

$$\tau(\chi(s, \infty)(\mathbb{1} - x)) = \tau(\chi(s, \infty)(\mathbb{1} - x)) = \int_0^{\infty} \chi(s, \infty)(1 - t) dE_t(x) = \int_0^{1-s} \chi(s, \infty)(t) dE_t(x) + \int_{1+s}^{\infty} \chi(s, \infty)(t) dE_t(x) = \tau(\chi(0, 1-s)(x)) + \tau(\chi(1+s, \infty)(x))$$

and

$$\tau(\chi(s, \infty)(x)) = m\{r > 0 : \mu_r(x) > s\}.$$ 

Since $\tau(\mathbb{1}) = 1$, it from the normality of $\tau$, we have

$$\tau(\chi(s, \infty)(x)) = m\{r \geq 0 : \mu_r(x) \geq s\}.$$ 

Hence, for $s \geq 0$, we have

$$\tau(\chi(s, \infty)(\mathbb{1} - x)) = m\{r \geq 0 : 0 \leq r \leq \tau(\mathbb{1}) \text{ and } \mu_r(x) < 1 - s\} + m\{r \geq 0 : \mu_r(x) > 1 + s\} = m\{r \geq 0 : \mu_r(\mathbb{1}) - \mu_r(x) > s\}.$$ 

Consequently, let $f(t) = \mu_t(\mathbb{1}) - \mu_t(|xy|)$, we get

$$\mu_s(\mathbb{1} - |xy|) = \mu_s(f(t)). \quad (17)$$

Since $\mu_t(\mathbb{1}) - \mu_t(|xy|) = 1 - \mu_t(|xy|)$, $0 < t < 1$, by Lemma 4.1 and Equation (17),

$$\mu_s(\mathbb{1} - |xy|) = 1 - \mu_{1-s}^t(|xy|), \quad 0 < s < 1.$$
and

$$\mu_s^t(\| - |xy|) = 1 - \mu_{1-s}(|xy|), \ 0 < s < 1.$$  

Combining this with Lemma 4.3, we show that

$$\int_0^t \log(1 - \mu_s(|xy|)) ds = \int_0^t \log(\mu_{1-s}^t(\| - |xy|)) ds$$

$$\geq \int_0^t \log(\mu_{1-s}^t((\| - |x|^r)\frac{1}{r} (\| - |y|^r)\frac{1}{r})) ds$$

$$\geq \int_0^t \log((1 - \mu_s(|x|)^{r})^{\frac{1}{r}} (1 - \mu_s(|y|)^{r})^{\frac{1}{r}})) ds$$

Replacing $x$ and $y$ by $|x_1|^r$ and $|x_2|^r$, we have

$$\int_0^t \log(1 - \mu_s(|x_1|^r|x_2|^r)) ds \geq \int_0^t \log((1 - \mu_s(|x_2|^r)^{\frac{1}{r}} (1 - \mu_s(|x_2|^r)^{\frac{1}{r}})) ds.$$  

Using the property of polar decomposition of $x_1$ and $x_2$, we obtain $\mu(x_1,x_2) = \mu(|x_1||x_2|)$. Hence, by Lemma 4.3 and (5) and (6), we deduce

$$\int_0^t \log(1 - \mu_s(|x_1x_2|^r)) ds = \int_0^t \log(1 - \mu_s(|x_1|^r|x_2|^r)) ds$$

$$\geq \int_0^t \log(1 - \mu_s(|x_1|^r|x_2|^r)) ds$$

$$\geq \int_0^t \log((1 - \mu_s(|x_1|^r)^{\frac{1}{r}} (1 - \mu_s(|x_2|^r)^{\frac{1}{r}})) ds$$

$$= \int_0^t \log((1 - \mu_s(|x_1|^r)^{\frac{1}{r}} (1 - \mu_s(|x_2|^r)^{\frac{1}{r}})) ds.$$  

So, the inequality (15) holds in $m = 2$ case.

Suppose the inequality (15) holds for a fixed $m(\geq 2)$ and $r \geq 1$. Let $x_1, \ldots, x_m$, $x_{m+1} \in \mathcal{M}$ are contraction operators, $r \geq 1$ and $p_1, \ldots, p_m$, $p_{m+1} > 0$ with $\sum_{i=1}^{m+1} \frac{1}{p_i} = 1$. Let $p = \sum_{i=1}^{m} \frac{1}{p_i}$, i.e., $\frac{1}{p} + \frac{1}{p_{m+1}} = 1$. By the above discussion, we have

$$\int_0^t \log(1 - \mu_s(|x_1 \cdots x_m x_{m+1}|^r)) ds$$

$$= \int_0^t \log((1 - \mu_s(|x_1 \cdots x_m|^r)^{\frac{1}{r}} (1 - \mu_s(|x_{m+1}|^{r_{m+1}})^{\frac{1}{r_{m+1}}}) ds$$

$$\geq \sum_{i=1}^{m} \int_0^t \log((1 - \mu_s(|x_i|^r)^{\frac{1}{r}}) ds + \int_0^t \log((1 - \mu_s(|x_{m+1}|^{r_{m+1}})^{\frac{1}{r_{m+1}}}) ds$$

$$= \sum_{i=1}^{m+1} \int_0^t \log((1 - \mu_s(|x_{m+1}|^{r_{m+1}})^{\frac{1}{r_{m+1}}}) ds.$$
So, the inequality (15) holds for $m+1$ case. That is, (15) holds for every finite $m$.

Finally, inequality (16) can be obtained by direct calculation and Lemma 4.1. □

**Corollary 4.7.** Suppose $x_1, x_2, \ldots, x_m \in \mathcal{M}$ are contractive operators and $p_1, \ldots, p_m > 0$ satisfies $\sum_{i=1}^{m} \frac{1}{p_i} = 1$. Then for all $t > 0$, we have

$$\int_0^t \log(1 - \mu_s(|x_1 \cdots x_m|)) ds \geq \sum_{i=1}^{m} \int_0^t \log(1 - \mu_s(|x_i|^{p_i})^{\frac{1}{p_i}}) ds.$$  

Moreover, if $x_1 \cdots x_m$ is self-adjoint, we have

$$\int_{1-t}^1 \log \mu_s^t(\mathbb{I} - x_1 \cdots x_m) ds \geq \int_{1-t}^1 \log \mu_s^t(\mathbb{I} - |x_1 \cdots x_m|) ds$$

$$\geq \sum_{i=1}^{m} \int_{1-t}^1 \log \mu_s^t(\mathbb{I} - |x_i|^{p_i})^{\frac{1}{p_i}} ds.$$  

**Proof.** By the proof of theorem 4.6, we have

$$\int_0^t \log(1 - \mu_s(|x_1 \cdots x_m|)) ds = \int_0^t \log \mu_{1-s}^t(\mathbb{I} - |x_1 \cdots x_m|) ds$$

$$= \int_0^t \log(1 - \mu_s(|x_1 \cdots x_m|)) ds$$

$$\geq \sum_{i=1}^{m} \int_0^t \log(1 - \mu_s(|x_i|^{p_i})^{\frac{1}{p_i}}) ds.$$  

By Lemma 4.2, we get

$$\mu_s(\mathbb{I} - x_1 \cdots x_m) \geq \mu_s(\mathbb{I} - |x_1 \cdots x_m|).$$  

Then we obtain the results from the Lemma 4.1. □

**Remark 4.8.** From Theorem 4.6 and 4.7, we have

$$\Delta(\mathbb{I} - |x_1 \cdots x_m|^r) \geq \Pi_{i=1}^{m} \Delta(\mathbb{I} - |x_i|^{r_{p_i}})^{\frac{1}{r_{p_i}}}$$

and

$$\Delta(\mathbb{I} - |x_1 \cdots x_m|) \geq \Pi_{i=1}^{m} \Delta(\mathbb{I} - |x_i|^{p_i})^{\frac{1}{p_i}}.$$  

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