Global Existence and Nonlinear Stability for the Coupled CGL–Burgers Equations for Sequential flames in $\mathbb{R}^N$

Boling Guo  
Institute of Applied Physics and Computational Mathematics  
Beijing, 100088, P. R. China *

Xinglong Wu  
Institute of Applied Physics and Computational Mathematics  
100088, Beijing, China †

Abstract

The present paper is devoted to the study of the global solution and nonlinear stability to the coupled complex Ginzburg–Landau and Burgers (CGL–Burgers) equations for sequential flames which describe the interaction of the excited oscillatory mode and the damped monotonic mode and are derived from the nonlinear evolution of the coupled long-scale oscillatory and monotonic instabilities of a uniformly propagating combustion wave governed by a sequential chemical reaction, having two flame fronts corresponding to two reaction zones with a finite separation distance between them. We first obtain a priori estimation in homogeneous Besov spaces to a heat equation, thanks to the lemma of priori estimation, we show the global solution in a critical Besov space for the Cauchy problem of Eq.(2.8) if the initial data is small. Next, we obtain the nonlinear stability by the linearized method as the coefficient satisfies certain condition. Finally, thanks to two lemmas, we show the nonlinear instability for the Cauchy problem of Eq.(3.4), if we choose certain constants.

Keywords: The coupled CGL–Burgers equations, sequential flames, Global solution, Besov spaces, the Bony decomposition, the nonlinear stability and instability, the plane wave, the linearized method.

*E-mail: gbl@mail.iapcm.ac.cn  
†E-mail: wxl8758669@yahoo.com.cn
1 Introduction

In this paper, we study the Cauchy problem of the coupled CGL–Burgers equations for sequential flames [9]

\[
\begin{align*}
\frac{\partial P}{\partial t} + \nabla Q \cdot \nabla P - (1 + iu)\Delta P &= \xi P - (1 + iv)|P|^2 P - r_1 \Delta Q P, \\
\frac{\partial Q}{\partial t} + \frac{1}{2} |\nabla Q|^2 - m\Delta Q + \kappa|P|^2 &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
P(0, x) &= P_0(x), \quad Q(0, x) = Q_0(x), \quad x \in \mathbb{R}^N, 
\end{align*}
\]

which describe the interaction of the excited oscillatory mode and the damped monotonic mode, where \(P\) denotes the rescaled complex amplitude of the flame oscillations, and \(Q\) is the deformation of the first front, the constants \(u = \frac{\delta_i}{\delta_r}, v = \frac{\lambda_i}{\lambda_r}, \xi \in \mathbb{R}, m = -\frac{\mu}{\chi_r}, \) and \(\kappa = \frac{\eta}{\chi_r}\delta_r\) are real, otherwise \(r_1 = v_1\) is complex. These coefficients as well as the other parameters in Eq.(1.1) were derived from the original model of flames governed by a sequential reaction in [9]. Applying \(\nabla\) to Eq.(1.1)\(_2\), letting \(\Omega = \nabla Q\). Substitute \(\Omega\) into Eq.(1.1)\(_1\), we obtain

\[
\begin{align*}
\frac{\partial P}{\partial t} + \Omega \cdot \nabla P - (1 + iu)\Delta P &= \xi P - (1 + iv)|P|^2 P - r_1 P \text{div} \Omega, \\
\frac{\partial \Omega}{\partial t} + \Omega \cdot \nabla \Omega - m\Delta \Omega + \kappa \nabla(|P|^2) &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
P(0, x) &= P_0(x), \quad \Omega(0, x) = \Omega_0(x), \quad x \in \mathbb{R}^N, 
\end{align*}
\]

where \(\Omega \cdot \nabla \Omega = \sum_{i=1}^{N} \Omega_i \partial_{x_i} \Omega, \text{div} \Omega = \sum_{i=1}^{N} \partial_{x_i} \Omega_i\).

As well known, uniformly propagating planar premixed flame fronts can become unstable leading to a lot of spatial or spatio-temporal flame structures, such as cellular flames, pulsating flames, spinning and spiral flames, flames with traveling or standing waves on the flame fronts and many others can be found in [23, 25] as well as the references cited therein. Premixed gaseous flames generally involve many reactions and reactants. In premixed gaseous flames with a one-stage chemical reaction (the case of a single limiting reactant which reacts to form products), there exist two kinds of diffusional thermal instability [5], which occur depending on the value of Lewis number \(L = \frac{\kappa_T}{D}\), where \(\kappa_T\) denotes thermal diffusivity of the reactive mixture, and \(D\) is the diffusivity of the deficient reacting component. If \(L < L_{c1} < 1\), the flame front is unstable to monotonic long-scale perturbations that develop into cellular structures whose spatio-temporal evolution is governed by the Kuramoto–Sivashinsky (KS) equation [22]. KS equation describes stationary spatially periodic patterns as well as further transitions to increasingly complex spatio-temporal evolution, eventually leading to chaotic or weakly turbulent behavior. If \(1 < L_{c2} < L\), an oscillatory instability occurs with a nonzero wavenumber and frequency at the instability threshold, and its nonlinear evolution is described by the complex Ginzburg-Landau (CGL) equations [17].

However, the chemical kinetics governing the structure of flames can be quite complex and can lead to qualitative flames behavior, which can not
be accounted for by a one-stage kinetic model. The effect of complex chemical kinetics on the dynamics of propagating flame fronts is governed by the two-stage sequential reaction, which leads to the occurrence of two flame fronts with various regimes of propagation depending on the heat releases, the reaction rate constants, and the activation energies of the chemical reactions \[4, 12, 16, 20\]. One of the possible regimes of propagation is that of two planar uniformly propagating fronts with a constant separation distance between them. A linear stability analysis of this regime for the sequential reaction problem proved that, as in the case of a one-stage reaction, the flame fronts can become unstable with respect to either monotonic or oscillatory instabilities \[19\]. The nonlinear interaction between the monotonic and oscillatory modes of instability was investigated in reaction diffusion systems \[14, 21\], in Rayleigh–Benard and Benard–Marangoni convection \[7, 8\]. If the monotonic mode was excited with a nonzero wavenumber, the resulting nonlinear evolution was described by coupled real and CGL equations.

The nonlinear interaction between the monotonic and oscillatory modes of instability of the two uniformly propagating flame fronts can be governed by the following coupled KS-CGL equations \[9\]

\[
\begin{aligned}
\partial_t P + \nabla Q \cdot \nabla P - (1 + iv)\Delta P &= \xi P - (1 + iv)|P|^2 P - (r_1 \Delta Q + r_2 g \Delta^2 Q)P, \\
\partial_t Q + \frac{1}{2} |\nabla Q|^2 + \kappa |P|^2 &= m \Delta Q - g \Delta^2 Q, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
P(0, x) &= P_0(x), \quad Q(0, x) = Q_0(x), \quad x \in \mathbb{R}^N,
\end{aligned}
\]

(1.3)

which describes the interaction of the evolving monotonic and oscillatory modes of instability of the uniform flame fronts. where \( r_2 = \nu_2 \delta_r, \quad g = \frac{\nu_r (l_A - l_0 A)}{l}, \) and other coefficients present as Eq.(1.1). The oscillatory and monotonic modes are excited as \( m < 0 \) and \( \xi = 1 \). In the case \( m < 0 \) and \( \xi = -1 \), the oscillatory mode is damped and the monotonic mode is excited. If \( m > 0 \) and \( \xi = 1 \), the oscillatory mode is excited and the monotonic mode is damped, in this case of letting \( g = 0 \), then Eq.(1.3) becomes the coupled CGL–Burgers equations. In contrast to the coupled KS-CGL equations, the coupled CGL–Burgers equations which describes the interaction between the excited oscillatory mode and the damped monotonic mode is an asymptotically valid description of the system behavior. Eq.(1.1) is CGL equation that describes the weakly nonlinear evolution of a long-scale instability if no coupled monotonic mode \( Q \) \[1, 2, 6\], while Eq.(1.2)\[2\] becomes the Burgers equation if we let the coefficient \( \kappa = 0 \). There are some works concerning the coupled CGL–Burgers equations. In \[9\], for the 1D version of Eq.(1.1), they obtain that the coupling with the Goldstone mode leads to new types of dynamics of a planar pulsating flame resulting from the Hopf bifurcation at zero wavenumber by the method of numerical computation. In \[10\], for 2D of Eq.(1.1), the authors focus on the effect of the coupling between the two modes, \( P \) and \( Q \), on the evolution of traveling waves, they show that the coupled system exhibits new types of instabilities as well as
new dynamical behavior, including bound states of two or four spirals, liquid spiral states, superspiral structures, oscillating cellular structures separated by chaotically merging and splitting domain walls.

The remainder of this paper is organized as follows. In Section 2, we first recall the Littlewood–Paley decomposition, the Bony decomposition, the definition and properties of Besov space. Next, we obtain a priori estimation in homogeneous Besov spaces to Eq.(2.1). Thanks to these lemmas, we establish global solution for the Cauchy problem of Eq.(2.8) in Besov spaces by Theorem 2.1. By virtue of some remarks, we get the global solution to Eq.(1.1). In Section 3, we first recall the definition of spectral stability. Next, we obtain the nonlinear stability by linearization to Eq.(3.1). In Section 4, by virtue of two lemmas, we get the the nonlinear instability to the Cauchy problem of Eq.(3.4) if we choose certain constants.

Notation: Let \( \widetilde{L}^\sigma_T(\dot{B}^\sigma_{p,r}) \) denote the set of functions \( u \) such that

\[
\|u\|_{\widetilde{L}^\sigma_T(\dot{B}^\sigma_{p,r})} := \left( \sum_{k \in \mathbb{Z}} (2^{k\sigma} \| \hat{\Delta}_k u \|_{L^r_p})^r \right)^{1/r},
\]

for \( \sigma \in \mathbb{R}, T > 0, \) and \( (p, r, \rho) \in [1, \infty]^3 \). Then via the Minkowski inequality, we have

\[
\|u\|_{\widetilde{L}^\sigma_T(\dot{B}^\sigma_{p,r})} \leq C \|u\|_{L^\sigma_p(\dot{B}^\sigma_{p,r})} \quad \text{if} \quad 1 \leq \rho \leq r,
\]

otherwise,

\[
\|u\|_{\widetilde{L}^\sigma_T(\dot{B}^\sigma_{p,r})} \leq C \|u\|_{\widetilde{L}^\rho_p(\dot{B}^\sigma_{p,r})} \quad \text{if} \quad 1 \leq r \leq \rho
\]

If for any \( T > 0, u \in \widetilde{L}^\sigma_T(\dot{B}^\sigma_{p,r}) \), then we have \( u \in \widetilde{L}^\rho(\dot{B}^\sigma_{p,r}) \), where

\[
\|u\|_{\widetilde{L}^\rho_p(\dot{B}^\sigma_{p,r})} := \left( \sum_{k \in \mathbb{Z}} (2^{k\sigma} \| \hat{\Delta}_k u \|_{L^\rho_p(\mathbb{R}^+;L^r_p)})^r \right)^{1/r}.
\]

2 Global existence of the solution

In this section, in order to establish the global solution of the Cauchy problem for Eq.(1.1) in Besov spaces. First, for the convenience of the readers, we recall some facts on the Littlewood–Paley decomposition, the Bony decomposition and some useful lemmas.

Proposition 2.1 \[3\] There exists a couple of \( C^\infty \) functions \( (\chi, \varphi) \) valued in \([0, 1]\), such that \( \chi \) is supported in the ball \( \mathcal{B} = \{ \xi \in \mathbb{R}^N; |\xi| \leq \frac{4}{3}\} \), and \( \varphi \) is supported in the annulus \( \mathcal{C} = \{ \xi \in \mathbb{R}^N; \frac{4}{3} \leq |\xi| \leq \frac{8}{3}\} \). Moreover,

\[
\chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^N,
\]
\[ q \geq 1 \implies \text{supp } \chi(\cdot) \cap \text{supp } \varphi(2^{-q}\cdot) = \emptyset, \]
\[ \text{supp } \varphi(2^{-q}\cdot) \cap \text{supp } \varphi(2^{-p}\cdot) = \emptyset, \text{ if } |p-q| \geq 2 \]
and
\[ \frac{1}{3} \leq \chi^2(\xi) + \sum_{q \in \mathbb{N}} \varphi^2(2^{-q}\xi) \leq 1, \forall \xi \in \mathbb{R}^N. \]

Let \( \tilde{h} = \mathcal{F}^{-1}\chi \) and \( h = \mathcal{F}^{-1}\varphi. \) Then the nonhomogeneous dyadic blocks \( \Delta_q \) and the nonhomogeneous low-off operators \( S_q \) can be defined as follows:

\[ \Delta_{-1} u = S_0 u \quad \text{and} \quad \Delta_q u = 0, \quad \text{if } q \leq -2, \]
\[ \Delta_q u = \varphi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} h(2^qy)u(x-y)dy, \quad \text{if } q \geq 0, \]
\[ S_q u = \sum_{p \geq -1} \Delta_p u = \chi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} \tilde{h}(2^qy)u(x-y)dy. \]

Moreover, if \( u, v \in \mathcal{S}'(\mathbb{R}^N), \) then we have
\[ \Delta_p \Delta_q = 0 \quad \text{if } |p-q| \geq 2, \]
\[ \Delta_q(S_{p-1}u \Delta_p v) = 0 \quad \text{if } |p-q| \geq 5. \]
Furthermore, for all \( u \in \mathcal{S}'(\mathbb{R}^N), \) one can easily check that
\[ u = \sum_{q \in \mathbb{Z}} \Delta_q u \quad \text{in } \mathcal{S}'(\mathbb{R}^N). \]

The homogeneous dyadic blocks \( \tilde{\Delta}_q \) and the homogeneous low-off operators \( \tilde{S}_q \) are defined for all \( q \in \mathbb{Z} \) by
\[ \tilde{\Delta}_q u = \varphi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} h(2^qy)u(x-y)dy, \]
\[ \tilde{S}_q u = \sum_{p \geq q-1} \Delta_p u = \chi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} \tilde{h}(2^qy)u(x-y)dy. \]

Then the nonhomogeneous Besov space \( B^s_{p,r}(\mathbb{R}^N) \) and the homogeneous Besov space \( B^s_{p,r}(\mathbb{R}^N) \) is defined as follows:

**Definition 2.1** Let \( s \in \mathbb{R}, \ p, r \in [1, \infty], \) we set
\[ B^s_{p,r} = \left\{ u \in \mathcal{S}'(\mathbb{R}^N); \|u\|_{B^s_{p,r}} = \left( \sum_{k=-1}^{\infty} 2^{ksr}\|\Delta_k u\|_{L^p}^r \right)^{1/r} < \infty \right\}, \]
where \( \Delta_k \) are the nonhomogeneous dyadic blocks. If \( s = \infty, \) \( B^\infty_{p,r} = \cap_{\sigma \in \mathbb{R}} B^\sigma_{p,r}. \)
\[ \tilde{B}^s_{p,r} = \left\{ u \in \mathcal{S}'(\mathbb{R}^N); \|u\|_{\tilde{B}^s_{p,r}} = \left( \|2^k \tilde{\Delta}_k u\|_{L^p}\right)_{k \in \mathbb{Z}} \right\}, \]
where \( \tilde{\Delta}_k \) is the homogeneous dyadic blocks. If \( s = \infty, \) \( \tilde{B}^\infty_{p,r} = \cap_{\sigma \in \mathbb{R}} \tilde{B}^\sigma_{p,r}. \)
For $u, v \in \mathcal{S}'(\mathbb{R}^N)$, we have the Bony decomposition as follows:

**Definition 2.2** Let $u, v \in \mathcal{S}'(\mathbb{R}^N)$. Denote

$$T_u v = \sum_{q \geq 1} \sum_{p \geq 1} \Delta_p u \Delta_q v = \sum_{q \geq 1} S_{q-1} u \Delta_q v$$

and

$$R(u, v) = \sum_{q \geq 1} \Delta_q u \tilde{\Delta}_q v \text{ with } \tilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$ 

Then formally, we have the nonhomogeneous Bony decomposition

$$uv = T_u v + T_u + R(u, v).$$

Similarly, the homogeneous Bony decomposition gives rise by

$$uv = \hat{T}_u v + \hat{T}_u + \hat{R}(u, v),$$

where $\hat{T}_u v = \sum_{q \in \mathbb{Z}} \sum_{p \leq q-2} \Delta_p u \Delta_q v = \sum_{q \in \mathbb{Z}} \hat{S}_{q-1} u \Delta_q v$ and

$$\hat{R}(u, v) = \sum_{q \in \mathbb{Z}} \Delta_q u \tilde{\Delta}_q v \text{ with } \tilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$ 

We now state the result concerning continuity of the inhomogeneous para-product operator $T$ and the remainder operator $R$.

**Lemma 2.1** There exists a constant $C$ such that for any couple of real numbers $(s, t)$ with $t$ negative and any $(p, r, r_1, r_2) \in [1, \infty]^4$, we have, for any $(u, v) \in L^\infty \times B^s_{p,r}$,

$$\|T_u v\|_{B^s_{p,r}} \leq C^{1+s} \|u\|_{L^\infty} \|v\|_{B^s_{p,r}},$$

for any $(u, v) \in B^s_{\infty,r_1} \times B^s_{p,r_2}$ and $\frac{1}{p} := \min\{1, \frac{1}{p_1} + \frac{1}{p_2}\}$

$$\|T_u v\|_{B^s_{p,r}^+} \leq \frac{C^{1+|s+t|}}{-t} \|u\|_{B^s_{\infty,r_1}} \|v\|_{B^s_{p,r_2}}.$$ 

Moreover, assume $(s_1, s_2)$ be real, and $(p_1, p_2, r_1, r_2) \in [1, \infty]^4$ such that

$$\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{r} := \frac{1}{r_1} + \frac{1}{r_2} \leq 1.$$ 

If $s_1 + s_2 > 0$, then we imply, for any $(u, v) \in B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2},$

$$\|R(u, v)\|_{B^{s_1+s_2}_{p_1,r_1}^+} \leq \frac{C^{1+|s_1+s_2|}}{s_1 + s_2} \|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}}.$$ 

If $r = 1$ and $s_1 + s_2 \geq 0$, we have, for any $(u, v) \in B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2},$

$$\|R(u, v)\|_{B^{s_1+s_2}_{p_1,\infty}^+} \leq C^{1+|s_1+s_2|} \|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}}.$$ 

6
Remark 2.1 Similar to the case of inhomogeneous, for the homogeneous paraproduct operator $\hat{T}$ and the remainder operator $\hat{R}$, we can get the same result.

Lemma 2.2 The following properties hold:

i) Density: if $p, r < \infty$, then $S(\mathbb{R}^N)$ is dense in $B^s_{p,r}$. Let the space $S_0(\mathbb{R}^N)$ denotes the function in $\mathcal{S}(\mathbb{R}^N)$ whose Fourier transforms are supported away from 0. Then the space $S_0(\mathbb{R}^N)$ is dense in $B^s_{p,r}$.

ii) Generalized derivatives: let $f \in C^\infty(\mathbb{R}^N)$ be a homogeneous function of degree $m \in \mathbb{R}$ away from a neighborhood of the origin. There exists a constant $C$ depending only on $f$ and such that $\|f(D)u\|_{B^s_{p,r}} \leq C\|u\|_{B^{s+m}_{p,r}}$.

iii) Sobolev embeddings: if $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B^s_{p_1,r_1} \hookrightarrow B^s_{p_2,r_2}$. If $s_1 < s_2, 1 \leq p \leq \infty$ and $1 \leq r_1, r_2 \leq \infty$, then the embedding $B^s_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$ is locally compact.

iv) Algebraic properties: for $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. Moreover ($B^s_{p,r}$ is an algebra) $\Leftrightarrow (B^s_{p,r} \hookrightarrow L^\infty) \Leftrightarrow (s > N/p$ or $(s \geq N/p$ and $r = 1)$).

v) Fatou property: if $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$ which tends to $u$ in $S'$, then $u \in B^s_{p,r}$ and $\|u\|_{B^s_{p,r}} \leq \liminf_{n \to \infty} \|u_n\|_{B^s_{p,r}}$.

vi) Complex interpolation: if $u \in B^s_{p,r} \cap B^{s^\prime}_{p^\prime,r^\prime}$ and $\theta \in [0,1]$, $p, r \in [1, \infty]$, then $u \in B^{s \theta + (1-\theta)s_1}_{p^\theta r^\theta}$ and

$$\|u\|_{B^{s \theta + (1-\theta)s_1}_{p^\theta r^\theta}} \leq \|u\|_{B^s_{p,r}}^{\theta} \|u\|_{B^{s_1}_{p^\prime r^\prime}}^{1-\theta}.$$

vii) Let $m \in \mathbb{R}$ and $f$ be a $S^m$-multiplier, i.e. $f : \mathbb{R}^n \to \mathbb{R}$ is smooth and satisfies that for all multi-index $\alpha$, there exists a constant $C_\alpha$ such that $\forall \xi \in \mathbb{R}^n, |\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}$. Then for all $s \in \mathbb{R}$ and $p, r \in [1, \infty]$, the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.

Remark 2.2 Properties ii), v), vii) hold for the homogeneous spaces $B^s_{p,r}$, and the following properties hold for the homogeneous spaces; Sobolev embeddings: if $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B^s_{p_1,r_1} \hookrightarrow B^s_{p_2,r_2}$. Moreover, for $2 \leq p < \infty$, $B^0_{p,2} \hookrightarrow L^p$. Algebraic properties: for $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. Moreover ($B^s_{p,r}$ is an algebra) $\Leftrightarrow (B^s_{p,r} \hookrightarrow L^\infty) \Leftrightarrow (s = N/p$ and $r = 1)$.

By the Bony decomposition, we can infer the following estimate.

Lemma 2.3 For any positive real number $s$ and any $p, r \in [1, \infty]$, the space $L^\infty \cap B^s_{p,r}$ is an algebra, and there exists a constant $C$ such that

$$\|uv\|_{B^s_{p,r}} \leq \frac{C^{s+1}}{s} (\|u\|_{L^\infty} \|v\|_{B^s_{p,r}} + \|v\|_{L^\infty} \|u\|_{B^s_{p,r}}).$$
Lemma 2.4 Assume $f$ is a smooth function such that $f(0) = 0$, $s > 0$ and $p, r \in [1, \infty]$. If $u$ belongs to the space $L^\infty \cap B^s_{p,r}$, then we have
\[
\|f(u)\|_{B^s_{p,r}} \leq C(s, f', \|u\|_{L^\infty}) \|u\|_{B^s_{p,r}}.
\]
Moreover, if $f$ belongs to $C_b^\infty(\mathbb{R})$ and $u$ belongs to $B^{-1}_{\infty,\infty}$, then we also obtain
\[
\|f(u)\|_{B^s_{p,r}} \leq C(s, f, \|\nabla u\|_{B^{-1}_{\infty,\infty}}) \|u\|_{B^s_{p,r}}.
\]
Finally, taking advantage of
\[
f(u) - f(v) = (u - v) \int_0^1 f'(v + \theta(u - v))d\theta,
\]
we can get the following lemma.

Lemma 2.5 Assume $f$ is a smooth function such that $f(0) = 0$, $s > 0$ and $p, r \in [1, \infty]$. For any couple $(u, v)$ belongs to the space $L^\infty \cap B^s_{p,r}$, we have the function $f(u) - f(v)$ belongs to $L^\infty \cap B^s_{p,r}$ and
\[
\|f(u) - f(v)\|_{B^s_{p,r}} \leq C(s, f'', \|u\|_{L^\infty}, \|v\|_{L^\infty})(\sup_{\theta \in [0,1]} \|u + \theta(v - u)\|_{L^\infty})
\]
\[
\|u - v\|_{B^s_{p,r}} + \|u - v\|_{L^\infty} \sup_{\theta \in [0,1]} \|u + \theta(v - u)\|_{B^s_{p,r}}.
\]

Consider the following $n$-dimension linear equation
\[
\begin{cases}
\partial_t f - \mu(1 + iu)\Delta f = g, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
f(0, x) = f_0(x), \quad x \in \mathbb{R}^N,
\end{cases}
\]
(2.1)
where $f(t, x) = (f_1, \cdots, f_n)$ is the vector field, the external source term $g = g(t, x)$ and the initial $f_0$ are known data. The diffusion constant $\mu$ is positive. Applying the partial Fourier transformation with respect to the space variable, we have
\[
\hat{f}(t, \xi) = e^{-\mu(1+iu)|\xi|^2t}\hat{f}_0(\xi) + \int_0^t e^{-\mu(1+iu)|\xi|^2(t-\tau)}\hat{g}(\tau, \xi)d\tau.
\]
(2.2)
If we introduce the semi-group $\{e^{t\Delta}\}_{t \geq 0}$, then (2.2) is equivalent to
\[
f(t, x) = e^{\mu(1+iu)t\Delta} f_0(x) + \int_0^t e^{\mu(1+iu)(t-\tau)\Delta} g(\tau, x)d\tau.
\]
(2.3)
Analogy to the proof of Lemma 2.4 on page 54 in [3], we can obtain the following result.

Lemma 2.6 Assume $p \in [1, \infty]$, $C$ be an annulus, and supp $\hat{f}$ be the support set of $f$. If supp $\hat{f} \subset \lambda C$, for some $\lambda > 0$. Then there exists constants $c$ and $C$, for any positive number $t, \mu$, we obtain
\[
\|e^{\mu(1+iu)t\Delta} f\|_{L^p} \leq Ce^{-c\mu t^2} \|f\|_{L^p}.
\]
By Lemma 2.6, we can get an important lemma to Eq.(2.1).

**Lemma 2.7** Given \((p, p, r) \in [1, \infty]^3\) and \(\sigma \in \mathbb{R}\), Assume that the initial data \(f_0 \in \hat{B}^p_{p,r}\) and the external source term \(g \in \hat{L}^p_T(\hat{B}^{\sigma+\frac{2}{r}}_{p,r})\), for a fixed \(T > 0\). Then for all \(\rho_1 \in [p, \infty]\), Eq.(2.1) has a unique solution \(f \in \hat{L}^p_T(\hat{B}^{\sigma+\frac{2}{r}}_{p,r}) \cap \hat{L}^\infty_T(\hat{B}^p_{p,r})\) with the initial data \(f_0\). Moreover, there exists a constant \(C\) such that

\[
\mu^{\frac{1}{p}}\|f\|_{\hat{L}^p_T(\hat{B}^{\sigma+\frac{2}{r}}_{p,r})} \leq C\|f_0\|_{\hat{B}^p_{p,r}} + C\mu^{\frac{1}{p}-1}\|g\|_{L^p_T(\hat{B}^{\sigma-\frac{2}{r}}_{p,r})}. \tag{2.4}
\]

In addition, if \(r\) is finite, then the solution \(f \in \mathcal{C}([0, T]; \hat{B}^p_{p,r})\).

**Proof.** Applying the dyadic blocks \(\hat{\Delta}_k\) to Eq.(2.1) to yield

\[
\begin{align*}
\partial_t \hat{\Delta}_k f - \mu (1 + iu) \hat{\Delta}_k f &= \hat{\Delta}_k g, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
\hat{\Delta}_k f(0, x) &= \hat{\Delta}_k f_0(x), \quad x \in \mathbb{R}^N,
\end{align*}
\tag{2.5}
\]

Using formula (2.3) to Eq.(2.5) to obtain

\[
\hat{\Delta}_k f(t, x) = e^{\mu(1+iu)\hat{\Delta}} \hat{\Delta}_k f_0(x) + \int_0^t e^{\mu(1+iu)(t-\tau)\hat{\Delta}} \hat{\Delta}_k g(\tau, x) d\tau. \tag{2.6}
\]

Taking advantage of Lemma 2.6 to (2.6), for some \(c > 0\), we have

\[
\|\hat{\Delta}_k f\|_{L^p_t(L^p)} \leq C\|e^{-c\mu p_22kT}\|\hat{\Delta}_k f_0\|_{L^p} + \|e^{-c\mu p_22kT} * \|\hat{\Delta}_k g(t)\|_{L^p}\|_{L^p_t}
\]

\[
\leq C \left(1 - \frac{e^{-c\mu p_12kT}}{c\mu p_22k}\right)^{\frac{1}{p_1}} \|\hat{\Delta}_k f_0\|_{L^p} + C \left(1 - \frac{e^{-c\mu p_12kT}}{c\mu p_22k}\right)^{\frac{1}{p_2}} \|\hat{\Delta}_k g\|_{L^p_t(L^p)}, \tag{2.7}
\]

where we used Young’s inequality and \(1 + \frac{1}{p_1} = \frac{1}{p} + \frac{1}{p_2}\).

Therefore, we get (2.4) by taking the \(L'((Z))\) norm to (2.7). Moreover, one can easily get the solution \(f \in \mathcal{C}([0, T]; \hat{B}^p_{p,r})\), if \(r\) is finite. \(\square\)

Next, we consider the following modified CGL–Burgers equations

\[
\begin{align*}
(\partial_t - (1 + iu)\Delta)P &= -\Omega \cdot \nabla P - (1 + iu)|P|^2P - r_1P div \Omega + f_1,
(\partial_t - m\Delta)\Omega &= -\Omega \cdot \nabla \Omega - \kappa \nabla(|P|^2) + f_2, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N,
\end{align*}
\tag{2.8}
\]

\[
P(0, x) = P_0(x), \quad \Omega(0, x) = \Omega_0(x), \quad x \in \mathbb{R}^N,
\]

where \(m > 0\) and other coefficients present as Eq.(1.1), \(f_1\) and \(f_2\) are the external source terms.

Now, we present the main result of this paper.
Theorem 2.1 Assume $N \geq 2$, $p \in [1, 2N]$, and $(P_0, \Omega_0) \in (\dot{B}^N_{p,1})^2$. Then there exists a constant $s$ for all external force $f_i \in \dot{L}^1(\dot{B}^N_{p,1})$, $i = 1, 2$ such that
\[
\| (P_0, \Omega_0) \|_{\dot{B}^N_{p,1}} + \sum_{i=1}^2 \| f_i \|_{\dot{L}^1(\dot{B}^N_{p,1})} \leq s,
\]
for all external force
\[
\text{and initial data} \quad (P_0, \Omega_0) \in (\dot{B}^N_{p,1})^2.
\]
Moreover, system (2.8) has a unique global solution $(P, \Omega)$ belongs to $\dot{L}^\infty(\dot{B}^N_{p,1}) \cap \dot{L}^1(\dot{B}^N_{p,1})$ with the initial data $(P_0, \Omega_0)$ which satisfies
\[
\| (P, \Omega) \|_{\dot{L}^\infty(\dot{B}^N_{p,1}) \cap \dot{L}^1(\dot{B}^N_{p,1})} \leq 2s.
\]
Moreover, the solution $(P, \Omega)$ belongs to $\mathcal{C}([0, T^*), \dot{B}^N_{p,1})$.

Remark 2.3 The spaces $\dot{L}^\infty(\dot{B}^N_{p,1}) \cap \dot{L}^1(\dot{B}^N_{p,1})$ are the scaling invariance in the following transformation. If $(P(t, x), \Omega(t, x))$ is the solution to Eq. (2.8) with the initial $(P_0(x), \Omega_0(x))$ and external force term $f_i$, then for any $\lambda > 0$, $P(t, x) = \lambda P(\lambda^2 t, \lambda x)$ and $\Omega(t, x) = \lambda \Omega(\lambda^2 t, \lambda x)$ solve Eq. (2.8) with initial data $(P_0, \Omega_0) = (\lambda P_0(\lambda x), \lambda \Omega_0(\lambda x))$ and $f_i = \lambda^3 f_i(t \lambda^2, \lambda x)$. Moreover, we have
\[
\| (\lambda P, \lambda \Omega) \|_{\dot{L}^\infty(\dot{B}^N_{p,1}) \cap \dot{L}^1(\dot{B}^N_{p,1})} = \| (P, \Omega) \|_{\dot{L}^\infty(\dot{B}^N_{p,1}) \cap \dot{L}^1(\dot{B}^N_{p,1})}
\]
and
\[
\| f_i \lambda \|_{\dot{L}^1(\dot{B}^N_{p,1})} = \| f_i \|_{\dot{L}^1(\dot{B}^N_{p,1})}.
\]

Remark 2.4 If we choose a compact domain $K$ of $\mathbb{R}^N$. Let $\dot{B}^N_{p,r}(K)$ (resp., $B^N_{p,r}(K)$) denote the set of distributions $f$ in $\dot{B}^N_{p,r}(\mathbb{R}^N)$ (resp., $B^N_{p,r}(K)$), the support of which is included in $K$. By Proposition 2.9.3 on page 108 in [3], if $p < N$, then the spaces $\dot{B}^N_{p,1}(K)$ and $B^N_{p,1}(K)$ coincide. Furthermore, we have
\[
\| (P_1 - P_2) \|_{\dot{L}^\infty(\dot{B}^N_{p,1}(K))} \leq \| P_1 - P_2 \|_{\dot{L}^1(\dot{B}^N_{p,1}(K))}.
\]
Therefore, if $m > 0$, we can get the same result of Theorem 2.1 in the spaces
\[
\dot{L}^\infty(\dot{B}^N_{p,1}(K)) \cap \dot{L}^1(\dot{B}^N_{p,1}(K))
\]
to Eq. (1.2).

Remark 2.5 Due to the estimation (2.4) is true for the norm $L^p_T(\dot{B}^N_{p,r})$. Therefore, if we take place $\dot{L}^\infty(\dot{B}^N_{p,1}), \dot{L}^1(\dot{B}^N_{p,1})$ by $L^\infty(\dot{B}^N_{p,1}), L^1(\dot{B}^N_{p,1})$ in Theorem 2.1, respectively, the result is also right. Moreover,
\[
\dot{L}^\infty(\dot{B}^N_{p,1}) \cap \dot{L}^1(\dot{B}^N_{p,1}) \hookrightarrow L^\infty(\dot{B}^N_{p,1}) \cap L^1(\dot{B}^N_{p,1}).
\]
We introduce the solution \((P_F, \Omega_F)\) of the following linear equation
\[
\begin{align*}
\partial_t P_F - (1 + iu) \Delta P_F &= f_1, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
\partial_t \Omega_F - m \Delta \Omega_F &= f_2, \\
(P_F, \Omega_F)|_{t=0} &= (P_0, \Omega_0), \\
\end{align*}
\]
(2.9)

By the virtue of Duhamel’s formula, \((P, \Omega)\) is a solution of Eq. (2.8) if and only if
\[
\left\{ \begin{align*}
\partial_t - (1 + iu) \Delta P &= -\Omega \cdot \nabla P - (1 + iv)|P|^2 P + r_1 P \text{div} \Omega, \\
\partial_t - m \Delta \Omega &= -\Omega \cdot \nabla \Omega - \kappa \nabla(|P|^2), \\
(F(P), F(\Omega))|_{t=0} &= (0, 0), \\
\end{align*} \right. 
\]
(2.10)

The existence of solution to Eq. (2.8) in Theorem 2.1 is a straightforward corollary of the following result.

**Lemma 2.8** Given \(\beta \geq 1, N \geq 2\) and \(p \in [1, 2N]\), define the set \(X_p^\beta(\beta)\) of the functions \(u\) such that for any \(T > 0\),
\[
u \in \tilde{L}^\infty(\tilde{B}^{N/p}_{p,1} \cap \tilde{L}^1(\tilde{B}^{N/p+1}_{p,1}) \cap \tilde{L}^1(\tilde{B}^{N/p+1}_{p,1})) \quad \text{and} \quad \|u\|_{\tilde{L}^\infty(\tilde{B}^{N/p}_{p,1} \cap \tilde{L}^1(\tilde{B}^{N/p+1}_{p,1}))} \leq \beta s
\]
if the initial data \(u_0\) satisfies \(\|u_0\|_{\tilde{B}^{N/p}_{p,1}} \leq s\), for some small enough \(s\).

Then the mapping \(\left( \begin{array}{c} P \\ \Omega \end{array} \right) \mapsto \left( \begin{array}{c} P_F + F(P) \\ \Omega_F + F(\Omega) \end{array} \right)\) maps \(X_p^\beta(\beta)\) into \(X_p^\beta(\beta)\), if the initial data \((P_0, \Omega_0) \in (X_p^\beta(\beta))^2\). More precisely, for any solutions \((P_1, \Omega_1)\) and \((P_2, \Omega_2)\) in \(X_p^\beta(\beta)\), the following result holds
\[
\| (F(P_1) - F(P_2), F(\Omega_1) - F(\Omega_2)) \|_{\tilde{L}^\infty(\tilde{B}^{N/p-1}_{p,1} \cap \tilde{L}^1(\tilde{B}^{N/p+1}_{p,1}))} \leq \frac{1}{2} \| (P_1 - P_2, \Omega_1 - \Omega_2) \|_{\tilde{L}^\infty(\tilde{B}^{N/p-1}_{p,1} \cap \tilde{L}^1(\tilde{B}^{N/p+1}_{p,1}))}.
\]
(2.11)

**Proof.** Since \((P_1, \Omega_1)\) and \((P_2, \Omega_2)\) are two solutions to Eq. (2.8), it is obviously that \((F(P_1) - F(P_2), F(\Omega_1) - F(\Omega_2))\) solves the following equation
\[
\left\{ \begin{align*}
(\partial_t - (1 + iu) \Delta)(F(P_1) - F(P_2)) &= Q(P_1, P_2, \Omega_1, \Omega_2), \\
(\partial_t - m \Delta)(F(\Omega_1) - F(\Omega_2)) &= W(P_1, P_2, \Omega_1, \Omega_2), \\
(F(P_1) - F(P_2), F(\Omega_1) - F(\Omega_2))|_{t=0} &= (0, 0), \\
\end{align*} \right. 
\]
(2.12)

where \(Q(P_1, P_2, \Omega_1, \Omega_2) = -(1 + iv)(|P_1|^2 P_1 - |P_2|^2 P_2) - (\Omega_1 \cdot \nabla P_1 - \Omega_2 \cdot \nabla P_2) + r_1 (\text{div} \Omega_1 P_1 - \text{div} \Omega_2 P_2)\) and \(W(P_1, P_2, \Omega_1, \Omega_2) = -(\Omega_1 \cdot \nabla \Omega_1 - \Omega_2 \cdot \nabla \Omega_2) - \kappa \nabla(|P_1|^2 - |P_2|^2)\).

Taking advantage of Lemma 2.7 to Eq. (2.12) with \(\sigma = \frac{N}{p} - 1\), \(\rho = 1\) and \(r = 1\), we obtain
\[
\| (F(P_1) - F(P_2)) \|_{\tilde{L}^p(\tilde{B}^{N/p-1+2/p_1}_{p,1})} \leq c \| Q \|_{\tilde{L}^{1}(\tilde{B}^{N/p-1}_{p,1})} \lesssim (I + II + III)
\]
(2.13)
with \( I = \| P_1^2 P_1 - |P_2|^2 P_2 \|_{L^1(B_{p,1}^{N/p-1})} \), \( II = \| \Omega_1 \cdot \nabla P_1 - \Omega_2 \cdot \nabla P_2 \|_{L^1(B_{p,1}^{N/p-1})} \)
and \( III = \| \text{div}\Omega_1 P_1 - \text{div}\Omega_2 P_2 \|_{L^1(B_{p,1}^{N/p-1})} \).

We now estimate \( I, II, III \) respectively. Due to
\[
|P_1|^2 P_1 - |P_2|^2 P_2 = |P_1|^2 (P_1 - P_2) + P_1 P_2 (\tilde{P}_1 - \tilde{P}_2) + |P_2|^2 (P_1 - P_2)
\]
and the Bony decomposition
\[
|P_1|^2 (P_1 - P_2) = \tilde{T}_1|P_1|^2 (P_1 - P_2) + \tilde{T}_2 (P_1 - P_2)|P_1|^2 + \tilde{R}(|P_1|^2, (P_1 - P_2)).
\]

Thanks to Lemma 2.1, we have
\[
\| \tilde{T}_1|P_1|^2 (P_1 - P_2) \|_{B_{p,1}^{N/p-1}} \leq \| P_1 \|_{L^\infty}^2 \| P_1 - P_2 \|_{B_{p,1}^{N/p-1}},
\]
\[
\leq \| P_1 \|_{B_{p,1}^{N/p}}^2 \| P_1 - P_2 \|_{B_{p,1}^{N/p-1}},
\]
\[
\| \tilde{T}_2 (P_1 - P_2)|P_1|^2 \|_{B_{p,1}^{N/p-1}} \leq \| P_1 - P_2 \|_{B_{p,1}^{N/p}} \| |P_1|^2 \|_{B_{p,1}^{N/p}}
\]
\[
\leq \| P_1 \|_{B_{p,1}^{N/p}}^2 \| P_1 - P_2 \|_{B_{p,1}^{N/p-1}},
\]
\[
\| \tilde{R}(|P_1|^2, (P_1 - P_2)) \|_{B_{p,1}^{N/p-1}} \leq \| P_1 \|_{B_{p,1}^{N/p}}^2 \| P_1 - P_2 \|_{B_{p,1}^{N/p-1}}, \text{ if } 1 \leq p < N,
\]
\[
\| \tilde{R}(|P_1|^2, (P_1 - P_2)) \|_{B_{p,1}^{2N/p-1}} \leq \| \tilde{R}(|P_1|^2, (P_1 - P_2)) \|_{B_{p,1}^{2N/p-1}}
\]
\[
\leq \| P_1 \|_{B_{p,1}^{2N/p}}^2 \| P_1 - P_2 \|_{B_{p,1}^{N/p-1}}, \text{ if } 2 \leq p < 2N.
\]

Therefore, by virtue of the above inequalities, applying interpolation theorem, it follows that
\[
\| |P_1|^2 (P_1 - P_2) \|_{B_{p,1}^{N/p-1}} \leq \| P_1 \|_{B_{p,1}^{N/p}} \| P_1 \|_{B_{p,1}^{N/p+1}} \| P_1 - P_2 \|_{B_{p,1}^{N/p-1}}. \quad (2.14)
\]

Similarly, along the same lines to estimate above inequality, we immediately imply that
\[
\left\{
\begin{array}{l}
\| P_1 P_2 (\tilde{P}_1 - \tilde{P}_2) \|_{B_{p,1}^{N/p-1}} \leq \| P_1 \|_{B_{p,1}^{N/p}} \| P_2 \|_{B_{p,1}^{N/p}} \| P_1 - P_2 \|_{B_{p,1}^{N/p-1}}, \\
\| P_1^2 (P_1 - P_2) \|_{B_{p,1}^{N/p-1}} \leq \| P_1 \|_{B_{p,1}^{2N/p}} \| P_1 - P_2 \|_{B_{p,1}^{N/p-1}}.
\end{array}
\right. \quad (2.15)
\]

Therefore, combining (2.14) with (2.15), as \( N \geq 2, p \in [1, 2N] \), we deduce
\[
I \leq \left( \sum_{j=1}^{2} \| P_j \|_{L^\infty(B_{p,1}^{N/p-1})} \| P_j \|_{L^1(B_{p,1}^{N/p+1})} \right) \| P_1 - P_2 \|_{L^\infty(B_{p,1}^{N/p-1})}. \quad (2.16)
\]

Note that \( \Omega_1 \cdot \nabla P_1 - \Omega_2 \cdot \nabla P_2 = \Omega_1 \cdot \nabla (P_1 - P_2) + (\Omega_1 - \Omega_2) \cdot \nabla P_2 \) and
\( \text{div}\Omega_1 P_1 - \text{div}\Omega_2 P_2 = P_1 \text{div}(\Omega_1 - \Omega_2) + \text{div}\Omega_2 (P_1 - P_2) \).
Now, argument as we did in proving (2.14), we end up with

\[ II \leq \|\Omega_1 - \Omega_2\|_{L^\infty(B_{p,1}^{N/p-1})} \|P_2\|_{L^1(B_{p,1}^{N/p+1})} + \|\Omega_1\|_{L^\infty(B_{p,1}^{N/p-1})} \|P_1 - P_2\|_{L^1(B_{p,1}^{N/p+1})}, \]

(2.17)

\[ III \leq \|P_1\|_{L^\infty(B_{p,1}^{N/p-1})} \|\Omega_1 - \Omega_2\|_{L^1(B_{p,1}^{N/p+1})} + \|\Omega_2\|_{L^1(B_{p,1}^{N/p+1})} \|P_1 - P_2\|_{L^\infty(B_{p,1}^{N/p-1})}, \]

(2.18)

Combining (2.13), (2.16), (2.17) with (2.18), for \(\rho_1 = 1\) and \(\rho_1 = \infty\), one can easily check that

\[
\|F(P_1) - F(P_2)\|_{X_1 \cap X_2} \leq \|\Omega_1\|_{X_1} \|P_1 - P_2\|_{X_2} + \|\Omega_1 - \Omega_2\|_{X_1} \|P_1 - P_2\|_{X_2} + \left(\sum_{j=1}^{2} \|P_j\|_{X_1} \|P_j\|_{X_2} + \|\Omega_2\|_{X_2}\right) \|P_1 - P_2\|_{X_1},
\]

(2.19)

where \(X_1 = \tilde{L}^\infty(B_{p,1}^{N/p-1})\) and \(X_2 = \tilde{L}^1(B_{p,1}^{N/p+1})\).

Taking advantage of Lemma 2.7 to Eq.(2.12) with \(\sigma = \frac{N}{p} - 1, \rho = 1\) and \(r = 1\), we obtain

\[
\|(F(P_1) - F(P_2))\|_{L^\rho_1(B_{p,1}^{N/p+1})} \leq c\|W\|_{L^\rho_1(B_{p,1}^{N/p+1})} \leq c(IV + V), \quad (2.20)
\]

where \(IV = \|\Omega_1 \nabla \Omega_1 - \Omega_2 \nabla \Omega_2\|_{L^1(B_{p,1}^{N/p+1})}\), and \(V = \|\nabla (|P_1|^2 - |P_2|^2)\|_{L^1(B_{p,1}^{N/p+1})}\).

Along the same lines to estimate (2.14), we immediately imply that

\[
IV \leq \|\Omega_1\|_{X_1} \|\Omega_1 - \Omega_2\|_{X_2} + \|\Omega_1 - \Omega_2\|_{X_1} \|\Omega_2\|_{X_2}. \quad (2.21)
\]

Since \(|P_1|^2 - |P_2|^2 = P_1(\bar{P}_1 - \bar{P}_2) + (P_1 - P_2)\bar{P}_2\), and \(B_{p,1}^{N/p} \hookrightarrow L^\infty\) is an algebra. Applying interpolation theorem and Cauchy–Schwarz inequality, we have

\[
\|\nabla (|P_1|^2 - |P_2|^2)\|_{B_{p,1}^{N/p+1}} \leq \|P_1\|_{B_{p,1}^{N/p}}^2 - \|P_2\|_{B_{p,1}^{N/p}}^2 \leq \left(\sum_{j=1}^{2} \|P_j\|_{B_{p,1}^{N/p+1}}\right) \|P_1 - P_2\|_{B_{p,1}^{N/p+1}} \quad (2.22)
\]

Thus, we can get the estimation of \(V\) as follows

\[
V \leq (\|P_1\|_{X_1} + \|P_2\|_{X_1}) \|P_1 - P_2\|_{X_2} + (\|P_1\|_{X_2} + \|P_2\|_{X_2}) \|P_1 - P_2\|_{X_1}. \quad (2.23)
\]
where $X_1 = \tilde{L}^\infty(\tilde{B}^{N/p}_{p,1})$ and $X_2 = \tilde{L}^1(\tilde{B}^{N/p+1}_{p,1})$.

Plugging (2.21), (2.23) into (2.20), taking $\rho_1 = 1$ and $\rho_1 = \infty$, we imply that

$$
\|F(\Omega_1) - F(\Omega_2)\|_{X_1 \cap X_2} \lesssim \|\Omega_1\|_{X_1} \|\Omega_1 - \Omega_2\|_{X_2} + \left( \sum_{j=1}^2 \|P_j\|_{X_1} \right) \|P_1 - P_2\|_{X_2}
+ \|\Omega_2\|_{X_2} \|\Omega_1 - \Omega_2\|_{X_1} + \left( \sum_{j=1}^2 \|P_j\|_{X_2} \right) \|P_1 - P_2\|_{X_1}.
$$

(2.24)

Adding (2.19) with (2.24) to obtain

$$
\|(F(P_1) - F(P_2), F(\Omega_1) - F(\Omega_2))\|_{X_1 \cap X_2} \leq c \left( \sum_{j=1}^2 (\|P_j\|_{X_1 \cap X_2} + \|\Omega_j\|_{X_1 \cap X_2} + \|P_j\|_{X_1} \|P_j\|_{X_2}) \right) \|P_1 - P_2, \Omega_1 - \Omega_2\|_{X_1 \cap X_2},
$$

(2.25)

where $\|(A, B)\|_{X_1 \cap X_2} = (\|A\|_{X_1} + \|A\|_{X_2}) + (\|B\|_{X_1} + \|B\|_{X_2})$.

By the assumption of Lemma 2.8, one can easily get

$$
c \left( \sum_{j=1}^2 (\|P_j\|_{X_1 \cap X_2} + \|\Omega_j\|_{X_1 \cap X_2} + \|P_j\|_{X_1} \|P_j\|_{X_2}) \right)
\leq 2c\beta s(2 + \beta s) = \alpha < 1,
$$

if we let $s$ small enough. The above inequality implies that

$$
\|(F(P_1) - F(P_2), F(\Omega_1) - F(\Omega_2))\|_{X_1 \cap X_2} \leq \alpha \|(P_1 - P_2, \Omega_1 - \Omega_2)\|_{X_1 \cap X_2}.
$$

Next, we shall prove the existence of solution to Eq.(2.8), i.e. if $(P, \Omega) \in (X_p^s(\beta))^2$, so does $(P_F + F(P), \Omega_F + F(\Omega)) \in (X_p^s(\beta))^2$.

On one hand, by Lemma 2.7 to (2.9)_1 and (2.10)_1, we have

$$
\begin{align*}
\|P_F\|_{X_1 \cap X_2} &\leq c(\|P_0\|_{X_1} + \|f_1\|_{\tilde{L}^1(\tilde{B}^{N/p-1}_{p,1})}); \\
\|F(P)\|_{X_1 \cap X_2} &\leq c(\|\Omega\|_{X_1} \|P\|_{X_2} + \|P\|_{X_1} \|P\|_{X_2}).
\end{align*}
$$

Therefore, without loss generality, we let $f_1 = 0$. Then

$$
\|P_F + F(P)\|_{X_1 \cap X_2} \leq c(\|P_0\|_{X_1} + \|\Omega\|_{X_1} \|P\|_{X_2} + \|P\|_{X_1} \|P\|_{X_2})
\leq c(s + (\beta s)^2 + (\beta s)^3) \quad (2.26)
$$

(2.26)
where we let $s$ small enough, and $\beta$ is a fix real number.

On the other hand, by Lemma 2.7 to (2.9) and (2.10), we imply

\[
\begin{align*}
\| \Omega F \|_{X_1 \cap X_2} &\leq c(\| \Omega \|_{X_1} + \| \Omega \|_{X_2} + \| P \|_{X_1} \| P \|_{X_2}), \\
\| F(\Omega) \|_{X_1 \cap X_2} &\leq c(\| \Omega \|_{X_1} \| \Omega \|_{X_2} + \| P \|_{X_1} \| P \|_{X_2}).
\end{align*}
\]

Therefore, with no loss generality, we let $f_2 = 0$. Then

\[
\begin{align*}
\| \Omega F \|_{X_1 \cap X_2} &\leq c(\| \Omega \|_{X_1} + \| \Omega \|_{X_2} + \| P \|_{X_1} \| P \|_{X_2}) \\
&\leq c(s + 2(\beta s)^2) \\
&\leq \beta s,
\end{align*}
\]

(2.27)

where we let $s$ small enough, and $\beta$ is a fix real number. This completes the proof of Lemma 2.8. □

Proof of Theorem 2.1. Thanks to Lemma 2.8, we can get the existence of solution to Eq.(2.8) by the contraction mapping argument, we will complete the proof of Theorem 2.1, if we prove the uniqueness. Indeed, let $(P_1, \Omega_1)$, $(P_2, \Omega_2)$ be two solutions to Eq.(2.8) with initial $(P_{1,0}, \Omega_{1,0})$, $(P_{2,0}, \Omega_{2,0})$, respectively. Then, similar the process of (2.26) and (2.27), by virtue of (2.11), we obtain

\[
(P_1 - P_2, \Omega_1 - \Omega_2) = (P_1 F - P_2 F, \Omega_1 F - \Omega_2 F) + (F(P_1) - F(P_2), F(\Omega_1) - F(\Omega_2))
\]

satisfies the following inequality

\[
\begin{align*}
\|(P_1 - P_2, \Omega_1 - \Omega_2)\|_{X_1 \cap X_2} &\leq \|(P_1 F - P_2 F, \Omega_1 F - \Omega_2 F)\|_{X_1 \cap X_2} \\
&\quad + \|(F(P_1) - F(P_2), F(\Omega_1) - F(\Omega_2))\|_{X_1 \cap X_2} \\
&\leq C\|(P_{1,0} - P_{2,0}, \Omega_{1,0} - \Omega_{2,0})\|_{X_1 \cap X_2} + \frac{1}{2}\|(P_1 - P_2, \Omega_1 - \Omega_2)\|_{X_1 \cap X_2},
\end{align*}
\]

(2.28)

where $X_1 = \tilde{L}^\infty(\dot{B}_{p,1}^{N/p-1})$ and $X_2 = \tilde{L}^1(\dot{B}_{p,1}^{N/p+1})$. One can easily check the uniqueness from the term (2.28). This completes the proof of Theorem 2.1. □

3 Nonlinear Stability by Linearization

In this section, we will establish the the nonlinear stability of the Cauchy problem for Eq.(1.1) by linearization. First, for the convenience of the readers, we recall an useful definition.

Definition 3.1 The plane wave is said to be spectrally stable if the spectrum is bounded in the complex plane by the curve $\text{Re}\lambda = -C(\text{Im}\lambda)^2$, for some $C > 0$. 

15
Next, letting $P = r(t, x)e^{i\theta(t, x)}$, we can obtain from (1.2) that the following CGL–Burgers equations
\[
\begin{cases}
  r_t + \Omega \cdot \nabla r = \Delta r - u(2\nabla r \cdot \nabla \vartheta + r\Delta \vartheta) + r(1 - r^2 - |\nabla \vartheta|^2 - s_1 \text{div} \Omega), \\
  r \vartheta_t + r \Omega \cdot \nabla \vartheta = u(\Delta r - r|\nabla \vartheta|^2) + 2\nabla r \cdot \nabla \vartheta + r(\Delta \vartheta + vr^2 - s_2 \text{div} \Omega), \\
  \Omega_t = m\Delta \Omega - \Omega \cdot \nabla \Omega - \kappa \nabla (r^2),
\end{cases}
\]
where the constant $m > 0$, and the other coefficients are the function of $r$.

Consider the plane wave solution of the CGL–Burgers equations of one-dimension, which is given by
\[
\begin{cases}
  P = r_0 e^{i\theta_0 x}, \\
  \Omega = \omega_0.
\end{cases}
\]
Substituting it into (3.1), by simply calculation, we imply that
\[
\begin{cases}
  r_0^2 + \vartheta_0^2 = 1, \\
  u\vartheta_0^2 + vr_0^2 = 0.
\end{cases}
\]
If we consider the following perturbation of the wave which is assumed to satisfy
\[
\begin{cases}
  r = r_0 + \rho, \\
  \vartheta = \vartheta_0 x + \phi, \\
  \Omega = \omega_0 + h.
\end{cases}
\]
Then, we have the following result.

**Theorem 3.1** If the domain $\mathbb{T} \in \mathbb{R}$ is bounded. Assume $m > 0$. Then there exists constants $r_0, \vartheta_0$ and functions $u, v$ such that the plane wave is spectrally stable. Moreover, letting $\pi = (\rho, \phi, h)$, if $\|\pi_0\|_{L^1} + \|\pi_0\|_{H^{s+1}}, s > \frac{1}{2}$ is sufficiently small and $u = 0$, then the perturbations satisfy
\[
\|\pi\|_{H^{s+1}} \leq C(1 + t)^{-\frac{1}{2}(s+\frac{3}{2})}\|\pi_0\|_{H^{s+1}}.
\]
We break the argument into several lemmas.

**Lemma 3.1** Suppose $m > 0$. Then there exists constants $r_0, \vartheta_0$ such that the plane wave is spectrally stable.

**Proof.** Plugging (3.2) into (3.1), letting $\kappa = 0$, it follows that
\[
\begin{cases}
  \rho_t = \rho_{xx} - w_0\rho_{xx} - h\rho_x - u[2\rho_x(\vartheta_0 + \phi_x) + (r_0 + \rho)\varphi_{xx}] + (r_0 + \rho)[1 - (r_0 + \rho)^2 - (\vartheta_0 + \varphi_x)^2 - s_1 h_x], \\
  (r_0 + \rho)\varphi_t + (r_0 + \rho)(w_0 + h)(\vartheta_0 + \phi_x) = 2\rho_x(\vartheta_0 + \phi_x) + u[\rho_{xx} - (r_0 + \rho)(\vartheta_0 + \varphi_x)^2] + (r_0 + \rho)[\varphi_{xx} - v(r_0 + \rho)^2 - s_2 h_x], \\
  h_t = mh_{xx} - (w_0 + h)h_x.
\end{cases}
\]

Assume \( u(r) = c_0 + c_1 r \). By the formulation of Taylor expansion

\[
\begin{aligned}
m(r) &= m(r_0) + m'(r_0)r + m''(r_0)r^2/2 + \mathcal{O}(r^3), \\
s_1(r) &= s_1(r_0) + s_1'(r_0)r + s_1''(r_0)r^2/2 + \mathcal{O}(r^3), \\
s_2(r) &= s_2(r_0) + s_2'(r_0)r + s_2''(r_0)r^2/2 + \mathcal{O}(r^3).
\end{aligned}
\]

Therefore, inserting (3.3) into (3.2), by simply computation, it follows that

\[
\begin{aligned}
\rho_t &= \rho_{xx} - (c_0 + c_1 r_0)r_0 \phi_{xx} - [w_0 + 2\vartheta_0(c_0 + c_1 r_0)]\rho_x - 2\vartheta_0 r_0 \phi_x \\
- r_0 s_1(r_0)h_x - 2r_0^2 \rho - 2\vartheta_0 c_1 \rho \rho_x - 2w_0 \rho_x \rho_x - (c_0 + c_1 r_0) \rho \phi_{xx} - h \rho_x \\
+ r_0 (s_1(r_0) - s_1(r)) h_x - \rho^2 - \phi_x^2] - \rho [2r_0 \rho + \rho^2 + 2\vartheta_0 \rho_x + \phi_x^2 + s_1 h_x], \\
\phi_t &= c_1 \rho_{xx} + \phi_{xx} - [2\vartheta_0(c_0 + c_1 r_0) + w_0] \phi_x - [c_1 \vartheta_0^2 + r_0^2 \vartheta'(r_0) + 2r_0 v(r_0)] \rho \\
- s_2(r_0) h_x - \vartheta_0 h - h \rho x - w_0 \vartheta_0 - (c_0 + c_1 r_0)(\vartheta_0^2 + \phi_x^2) \\
+ \frac{\vartheta_0}{r_0 + \rho} \rho_{xx} - c_1 (2\vartheta_0 \rho_x + \phi_x^2) \rho - \frac{2\vartheta_0}{r_0 + \rho} (\vartheta_0 + \phi_x) + [s_2(r_0) - s_2(r)] h_x \\
- \vartheta_0 \rho^2 - r_0^2 [\vartheta(r) - \vartheta'(r_0) \rho] - 2r_0 \rho [\vartheta(r) - \vartheta(r)], \\
h_t &= m h_{xx} - w_0 h_x - h h_x. 
\end{aligned}
\]

Note that (3.4) gives the following linearized equation

\[
\pi_t = A \pi_{xx} + B \pi_x + C \pi,
\]

where the vector \( \pi \), and the 3 \times 3 matrices \( A, B, C \) are

\[
\begin{aligned}
\pi &= \begin{pmatrix} \rho \\ \phi \\ h \end{pmatrix}, A &= \begin{pmatrix} 1 & -r_0(c_0 + c_1 r_0) & 0 \\ c_1 & 1 & 0 \\ 0 & 0 & m \end{pmatrix}, \\
B &= \begin{pmatrix} -w_0 - 2\vartheta_0(c_0 + c_1 r_0) & -2\vartheta_0 r_0 & -s_1(r_0) r_0 \\ 0 & -2\vartheta_0 (c_0 + c_1 r_0) - w_0 & -s_2(r_0) \\ 0 & 0 & -w_0 \end{pmatrix}, \\
C &= \begin{pmatrix} -[c_1 \vartheta_0^2 + r_0^2 \vartheta'(r_0) + 2r_0 v(r_0)] & 0 & 0 \\ 0 & -\vartheta_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\]

Define the operator \( \mathcal{L} \) as follows

\[
\mathcal{L} = A \partial_x^2 + B \partial_x + C.
\]

As is well known [11], the spectrum of \( \mathcal{L} \) in any \( L^p \) space, \( p \in [1, \infty) \), is bounded by the curve

\[
C_\lambda = \{ \lambda; | -k^2 A + i k B + (C - \lambda I_3) | = 0, k \in \mathbb{T} \}.
\]

A lengthy calculation shows that for \( \lambda \in C_\lambda \), such that

\[
\lambda_1(k) = -k^2 m - w_0 ki,
\]

17
\[ \lambda_{2,3}(k) = -[r_0^2 + k^2 + (w_0 + 2\vartheta_0(c_0 + c_1r_0))ki] \pm \sqrt{r_0^4 - c_1r_0(c_0 + c_1r_0)k^4 + r_0[c_0\vartheta_0^2 + r_0^2\vartheta'(r_0) + 2r_0v(r_0)]k + 2r_0\vartheta_0c_1k^3i}, \]

(3.7)

from which one can easily check that

\[ \text{Re}(\lambda_{2,3}(k)) = -(r_0^2 + k^2) \pm \frac{[\sqrt{a^2 + b^2} + a]^{1/2}}{\sqrt{2}}, \]  

(3.8)

where \( a = r_0^4 - c_1r_0(c_0 + c_1r_0)k^4 + r_0[c_0\vartheta_0^2 + r_0^2\vartheta'(r_0) + 2r_0v(r_0)]k, \) and \( b = 2r_0\vartheta_0c_1k^3. \)

If \( k^2m > 0, 2(r_0^2 + k^2)^2 \geq a, \) and \( 4(r_0^2 + k^2)^4 - 4a(r_0^2 + k^2)^2 > b^2, \) then we deduce that

\[ \text{Re}\lambda_1(k) < 0, \quad \text{and} \quad \text{Re}\lambda_{2,3}(k) < 0. \]

Therefore, the plane wave of the CGL-Burgers is spectrally stable by Definition 3.1.  \( \square \)

**Remark 3.1** We give some certain constants satisfies the assumption of Theorem 3.1.

1). If \( m > 0, r_0 = 1, \vartheta_0 = v = 0 \) and \( u = 0 \) or \( u = 1, \) then we have

\[ \lambda_1(k) = -k^2m - w_0ki, \quad \lambda_2(k) = -k^2 - w_0ki, \quad \lambda_3(k) = -2 - k^2 - w_0ki. \]

2). Assume \( m > 0, r_0^2 + \vartheta_0^2 = 1 \) and \( u = v = 0. \) Then we have

\[ \lambda_1(k) = -k^2m - w_0ki, \quad \lambda_2(k) = -k^2 - w_0ki, \quad \lambda_3(k) = -(2r_0 + k^2 + w_0ki). \]

3). If \( m > 0, r_0 = u = 0 \) and \( \vartheta_0 = 1, \) then we have

\[ \lambda_1(k) = -k^2m - w_0ki, \quad \lambda_{2,3}(k) = -k^2 - w_0ki. \]

**Remark 3.2** Assume that \( \kappa = \kappa(r), \) processing as (3.4), from (3.6). Then we imply

\[
\begin{vmatrix}
 k^2 + 2r_0^2 + [\vartheta_0 + \lambda] - k^2r_0(c_0 + c_1r_0) + 2\vartheta_0r_0ki & r_0s_1(r_0)ki \\
 c_1k^2 + \Gamma & k^2 + \vartheta_0 + s_2(r_0)ki \\
 2r_0\vartheta_0(r_0) & 0 & k^2m + w_0ki + \lambda
\end{vmatrix} = 0,
\]

where \( \Gamma = c_1\vartheta_0^2 + r_0^2\vartheta'(r_0) + 2r_0v(r_0), \) and \( [\vartheta] = 2\vartheta_0(c_0 + c_1r_0) + w_0. \) We now consider a simply case as follows: Letting \( r_0 = 1, \vartheta_0 = v = 0, \kappa = m = 1, \) and \( s_1(r_0) = \frac{1}{2}. \) By a long computation, as \( u = c_0 + c_1r = 0, \) it follows that

\[ \lambda_1(k) = -k^2 - w_0ki, \quad \text{and} \quad \text{Re}\lambda_{2,3}(k) = -(k^2 + 1) \pm \frac{\sqrt{2}}{4} \sqrt{4 + \sqrt{16 + k^2}} < 0. \]

18
On the other hand, if $r_0 = m = 1$, $v_0 = v = 0$, $\kappa = 1/2$, and $u = c_0 + c_1 r = 1$, then we have

$$\lambda_1(k) = -\frac{1}{3}(2 + 3k^2 + 3kw_0i) + \frac{2^{1/3}(4 + 3ks_1i)}{3 \left[a + \sqrt{a^2 - 4b^3}\right]^{1/3}}, \quad (3.9)$$

$$\lambda_2(k) = -\frac{1}{3}(2 + 3k^2 + 3kw_0i) - \frac{(1 + \sqrt{3}i)(4 + 3ks_1i)}{3 \times 2^{2/3} \left[a + \sqrt{a^2 - 4b^3}\right]^{1/3}} \left[-\frac{1}{6 \times 2^{1/3}} \left(a + \sqrt{a^2 - 4b^3}\right)^{1/3}, \quad (3.10)\right.$$

$$\lambda_3(k) = -\frac{1}{3}(2 + 3k^2 + 3kw_0i) - \frac{(1 - \sqrt{3}i)(4 + 3ks_1i)}{3 \times 2^{2/3} \left[a + \sqrt{a^2 - 4b^3}\right]^{1/3}} \left[-\frac{1}{6 \times 2^{1/3}} \left(a + \sqrt{a^2 - 4b^3}\right)^{1/3}, \quad (3.11)\right.$$

with $a = -16 + (27k^3s_2 - 18ks_1)i$ and $b = (4 + 3ks_1i)$. Therefore, one can easily check from (3.9), (3.10) and (3.11) that

If \( s_1 = 0, \ s_2 = 0, \) then \( \lambda_1(k) = -2 - k^2 - kw_0i, \quad \lambda_2, 3(k) = -k^2 - kw_0i. \) \( (3.12) \)

If \( s_1 = 1, \ s_2 = 0, \) then \( \lambda_1(k) = -k^2 - kw_0i, \quad \lambda_2(k) = -1 - k^2 - \sqrt{1 + ki} - kw_0i, \quad \lambda_3(k) = -1 - k^2 + \sqrt{1 + ki} - kw_0i, \) \( (3.13) \)

and

If \( s_1 = -1, \ s_2 = 0, \) then \( \lambda_1(k) = -k^2 - kw_0i, \quad \lambda_2(k) = -1 - k^2 - \sqrt{1 - ki} - kw_0i, \quad \lambda_3(k) = -1 - k^2 + \sqrt{1 - ki} - kw_0i, \) \( (3.14) \)

By virtue of (3.13) and (3.14), it follows that

$$\lambda_{2, 3}(k) = -\left(1 + k^2\right) \pm \left(1 + \frac{\sqrt{1 + k^2}}{2}\right) < 0.$$

Next, we give a useful lemma which comes from [13, 24].

**Lemma 3.2** Assume the operator $L$ is simply a lower order perturbation of the Laplacian, $S(t)$ is an analytic semigroup generated by $L$. Then the semigroup $S(t)$ satisfies

$$(i) \quad \|\partial_x^k S(t)u\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}\left(rac{1}{2} + k\right)}\|u\|_{L^1} + Ce^{-\beta t}\|\partial_x^k u\|_{L^2},$$

19
\[(ii) \quad \|\partial_x^k S(t)u\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}(\frac{k}{2}+k)}\|u\|_{L^1} + C\|\partial_x^{k-1} u\|_{L^2},\]

for some $\beta > 0$.

The following lemma will yield the last part of Theorem 3.1.

**Lemma 3.3** Assume the initial data $\pi_0$ satisfy $(\|\pi_0\|_{L^1} + \|\pi_0\|_{H^{s+1}})$ is sufficiently small for $s > \frac{1}{2}$. Given $u = 0$. If the domain $\Omega \in \mathbb{R}$ is bounded, then the solution $\pi$ to Eq. (3.4) satisfies

\[\|\pi\|_{H^{s+1}} \leq C(1 + t)^{-\frac{1}{2}(\frac{k}{2}+s)}(\|\pi_0\|_{L^1} + \|\pi_0\|_{H^{s+1}}).\]

**Proof.** Assume $u = c_0 + c_1 r = 0$, by (3.4), we have

\[\rho_t = L_{11}\rho - (2\partial_0 r_0)\phi_x - r_0 s_1(r_0)h_x + \psi_1(\rho, h, h_x, \phi_x), \quad (3.15)\]

where $L_{11} = \partial_x^2 - w_0 \partial_x - 2\omega^2$, and $\psi_1(\rho, h, h_x, \phi_x) = -h\rho_x - r_0[\rho^2 + \phi_x^2 + (s_1(r) - s_1(r_0))h_x] - \rho[2\rho \phi + \rho^2 + 2\partial_0 \phi_x + \phi_x^2 + s_1(r)h_x].$

Since the operator $L_{11}$ is the laplacian with a lower order perturbation. Thanks to the perturbation theorem in [15] to obtain $L_{11}$ generates an analytic semigroup $S_{11}(t)$, and we imply from (3.15) that

\[\rho = S_{11}(t)\rho_0 + \int_0^t S_{11}(t - \tau) [-2\partial_0 r_0(\phi_x - r_0 s_1(r_0)h_x + \psi_1)] d\tau. \quad (3.16)\]

Recall that $\|\partial_x^k S_{11}(t)u\|_{L^2} \leq e^{-\frac{\rho^2}{2}t}\|\partial_x^k u\|_{L^2}$, one can easily check that

\[
\begin{cases}
\|S_{11}(t)\rho_0\|_{\dot{H}^k} \leq e^{-\frac{\rho^2}{2}t}\|\rho_0\|_{\dot{H}^k}, \\
\int_0^t \|S_{11}(t - \tau)\phi_x\|_{\dot{H}^k} \leq \int_0^t e^{-\frac{\rho^2}{2}(t-\tau)}\|\phi_x\|_{\dot{H}^k}d\tau.
\end{cases} \quad (3.17)
\]

Applying $\dot{H}^k$ norm to (3.16), in conjunction with (3.17), we imply

\[\|\rho\|_{\dot{H}^k} \leq e^{-\frac{\rho^2}{2}t}\|\rho_0\|_{\dot{H}^k} + C\int_0^t e^{-\frac{\rho^2}{2}(t-\tau)}(\|\phi_x\|_{\dot{H}^k} + \|h_x\|_{\dot{H}^k} + \|\psi_1\|_{\dot{H}^k})d\tau. \quad (3.18)\]

Let $L_{22} = \partial_x^2 - w_0 \partial_x$. Then the operator $L_{22}$ generates a semigroup $S_{22}(t)$, using (3.4) to yield

\[\phi = S_{22}(t)\phi_0 + \int_0^t S_{22}(t - \tau)[-(s_2(r_0)v'(r_0) + 2r_0 v(r_0))\rho - (s_2(r_0)\partial_x + \partial_0)h + \psi_2]d\tau, \quad (3.19)\]

where $\psi_2(\rho, h, \rho_x, h_x, \phi_x) = -h\phi_x - w_0 \phi_0 - \frac{2\rho}{r_0} (\phi_0 + \phi_x) - \frac{s_2(r) - s_2(r_0)}{h_x - v(r) + v'(r_0)\rho} - 2r_0 \rho(v(r) - v(r_0)).$

In view of lemma 3.2, we obtain

\[
\begin{align*}
\|S_{22}(t)\phi_0\|_{\dot{H}^k} & \leq C(1 + t)^{-\frac{1}{2}(\frac{k}{2}+k)}\|\phi_0\|_{L^1} + Ce^{-\beta t}\|\phi_0\|_{\dot{H}^k} \\
& \leq C(1 + t)^{-\frac{1}{2}(\frac{k}{2}+k)}(\|\phi_0\|_{L^1} + \|\phi_0\|_{\dot{H}^k}), \quad (3.20)
\end{align*}
\]
as \( t \) is large enough.

Similarly, we can get the following estimation

\[
\begin{aligned}
\{ & \|S_{22}(t - \tau)h_x\|_{\dot{H}^k} \leq C(1 + t - \tau)^{-\frac{k}{2}(\frac{3}{2} + k + 1)} ||h||_{L^1} + e^{-\beta(t-\tau)}||h||_{\dot{H}^{k+1}} \\
& \quad \leq C(1 + t - \tau)^{-\frac{k}{2}(\frac{3}{2} + k + 1)}(||h||_{L^1} + ||h||_{\dot{H}^{k+1}}), \\
& \|S_{22}(t - \tau)\psi_2\|_{\dot{H}^k} \leq C(1 + t - \tau)^{-\frac{k}{2}(\frac{3}{2} + k + 1)}(||\psi_2||_{L^1} + ||\psi_2||_{\dot{H}^k}). \\
\}
\]

(3.21)

Therefore, applying \( \dot{H}^k \) norm to (3.19), combining (3.20) with (3.21), it follows that

\[
\|\phi\|_{\dot{H}^k} \leq (1 + t)^{-\frac{k}{2}(\frac{3}{2} + k)}(||\phi_0||_{L^1} + ||\phi_0||_{\dot{H}^k}) + \int_0^t (1 + t - \tau)^{-\frac{k}{2}(\frac{3}{2} + k + 1)}
\times (||h||_{L^1} + ||h||_{\dot{H}^{k+1}})d\tau + \int_0^t (1 + t - \tau)^{-\frac{k}{2}(\frac{3}{2} + k)}
\times (||\rho||_{L^1} + ||\rho||_{\dot{H}^k} + ||h||_{L^1} + ||h||_{\dot{H}^k} + ||\psi_2||_{L^1} + ||\psi_2||_{\dot{H}^k})d\tau.
\]

(3.22)

Similarly, we can get the estimation to \( h \) as follows

\[
\|h\|_{\dot{H}^k} \leq (1 + t)^{-\frac{k}{2}(\frac{3}{2} + k)}(||h_0||_{L^1} + ||h_0||_{\dot{H}^k}) + \int_0^t (1 + t - \tau)^{-\frac{k}{2}(\frac{3}{2} + k)}
\times (||\psi_3||_{L^1} + ||\psi_3||_{\dot{H}^k})d\tau
\]

(3.23)

with \( \psi_3(\rho, h, \rho_x, h_x) = -hh_x - 2\kappa\rho_x\).

Define \( \pi = (\rho, \phi, h) \), if the domain \( T \) is bounded, by Poincaré inequality, then we have \( ||\pi||_{L^2} \leq C||\pi||_{L^1} \) and \( ||\pi||_{\dot{H}^k} \sim ||\pi||_{H^k} \). Moreover, as \( k > \frac{1}{2} \), the space \( H^k \) is an algebra. A long calculation shows that \( \psi = (\psi_1, \psi_2, \psi_3) \) satisfies

\[
||\psi||_{L^1} \leq C||\psi||_{\dot{H}^k} \leq C||\pi||^2_{H^{k+1}}.
\]

(3.24)

Adding up (3.18), (3.22) with (3.23), by virtue of (3.24), it follows that

\[
\|\pi\|_{\dot{H}^k} \leq (1 + t)^{-\frac{k}{2}(\frac{3}{2} + k)}||\pi_0||_{\dot{H}^k} + \int_0^t (1 + t - \tau)^{-\frac{k}{2}(\frac{3}{2} + k)}||\pi||_{\dot{H}^{k+1}}d\tau
\]

\[
+ \int_0^t (1 + t - \tau)^{-\frac{k}{2}(\frac{3}{2} + k)}(||\pi||_{H^k} + ||\pi||^2_{\dot{H}^{k+1}})d\tau.
\]

(3.25)

Analogy to the process of proof to(3.25), one can easily check that

\[
\|\pi\|_{\dot{H}^{k+1}} \leq (1 + t)^{-\frac{k}{2}(\frac{3}{2} + k)}(||\pi_0||_{L^1} + ||\pi_0||_{\dot{H}^{k+1}}) + \int_0^t (t - \tau)^{-\frac{k}{2}e^{-\beta(t-\tau)}}||\pi||^2_{\dot{H}^{k+1}}d\tau
\]

\[
+ \int_0^t (1 + t - \tau)^{-\frac{k}{2}(\frac{3}{2} + k)}||\pi||^2_{\dot{H}^{k+1}}d\tau.
\]

(3.26)
Define
\[ N(t) = \sup_{0 \leq \tau \leq t} \{(1 + \tau)^{\frac{1}{2}(\frac{3}{2} + k)}\|\pi\|_{H^{k+1}}\}. \]

Thanks to (3.26). Using the definition of \( N(t) \), then we have
\[
\|\pi\|_{H^{k+1}} \leq (1 + t)^{-\frac{1}{2}(\frac{3}{2} + k)}E_0 + N^2(t) \int_0^t (1 + t - \tau)^{-\frac{1}{2}(\frac{3}{2} + k)}(1 + \tau)^{-\frac{3}{2} + k}d\tau
\]
\[ + N^2(t) \int_0^t (t - \tau)^{-\frac{1}{2}}e^{-\beta(t-\tau)}(1 + \tau)^{-\frac{3}{2} + k}d\tau. \]  
(3.27)
where \( E_0 = (\|\pi_0\|_{L^1} + \|\pi_0\|_{H^{k+1}}) \).

Note that
\[
\int_0^t (1 + t - \tau)^{-\frac{1}{2}(\frac{3}{2} + k)}(1 + \tau)^{-\frac{3}{2} + k}d\tau \leq C(1 + t)^{-\frac{1}{2}(\frac{3}{2} + k)},
\]
\[
\int_0^t (t - \tau)^{-\frac{1}{2}}e^{-\beta(t-\tau)}(1 + \tau)^{-\frac{3}{2} + k}d\tau \leq C(1 + t)^{-\frac{1}{2}(\frac{3}{2} + k)}.
\]
Therefore
\[ N(t) \leq CE_0 + C_1N^2(t). \]  
(3.28)
In view of the assumption stated in the lemma, we imply that \( N(t) \leq CE_0 \). This completes the proof the lemma. □

Remark 3.3 Assume that the initial \( \pi_0 = (\rho_0, \phi_0, h_0) \in (H^{s+1})^3, \ s > \frac{1}{2}. \) Then Eq.(3.4) becomes \( \pi_t = L\pi + \psi, \) and the operator \( L \) generates a semigroup \( S(t) \). Due to \( \|\partial_t^s S(t)\pi\|_{L^2} \leq C t^{-\frac{s}{2}}\|\pi\|_{L^2}, \) \( H^s \) is an algebra, and the higher order term \( \psi \) is continuous in \( H^s \), by the semigroup theory [18], there exists a positive \( T > 0, \) such that the solution \( \pi \in C([0, T]; H^{s+1}) \cap C^1([0, T]; H^s). \) This ensures the existence of solution to Eq.(3.4) in Lemma 3.3.

4 Nonlinear Instability

In this section, we present the instability of the equilibrium solution of the CGL–Burgers equations by the principle of linearized instability in [15]. We consider the following equation
\[
\begin{align*}
\begin{cases}
\pi_t = L\pi + \psi, & (t, x) \in \mathbb{R}^+ \times T, \\
\pi|_{t=0} = \pi_0 = (\rho_0, \phi_0, h_0), & x \in T,
\end{cases}
\end{align*}
\]  
(4.1)
where the operator $\mathbf{L} = \mathbf{A}\partial_x^2 + \mathbf{B}\partial_x + \mathbf{C}$, the functions $\pi = (\rho, \phi, h)$ and $\psi = (\psi_1, \psi_2, \psi_3)$,
\[
\begin{align*}
\psi_1(\rho, h, h_x, \phi_x) = -2\partial_0 c_1 \rho \rho_x - 2(c_0 + c_1 r_0) \phi \rho_x - (c_0 + c_1 r_0) \rho \phi_{xx} - h \rho_x \\
-\rho_0 [\rho^2 + \phi_x^2 + (s_1(r) - s_1(r_0)) h_x] - \rho [2\rho_0 \rho + \rho_x^2 + 2\partial_0 \phi_x + \phi_{xx}^2 + s_1(r) h_x],
\end{align*}
\]
\[
\begin{align*}
\psi_2(\rho, h, \rho_x, h_x, \phi_x) = -h \phi_x - w_0 \partial_0 - (c_0 + c_1 r_0) (\phi^2 + \phi_x^2) + \frac{s_0}{r_0 + \rho} \rho_{xx} \\
-c_1 (2\partial_0 \phi_x + \phi_x^2) - \frac{2\phi_x}{r_0 + \rho} (\partial_0 + \phi_x) - v(r) \rho^2 - [s_2(r) - s_2(r_0)] h_x \\
-h_0 [\rho (v(r) - v'_0) \rho] - 2\rho_0 \rho [v(r_0) - v(r)],
\end{align*}
\]
\[
\begin{align*}
\psi_3(\rho, h, \rho_x, h_x) = -h h_x - 2\kappa \rho \rho_x.
\end{align*}
\]

We shall prove an instability result under the assumption
\begin{align}
\begin{cases}
\sigma_+ (\mathbf{L}) = \sigma (\mathbf{L}) \cap \{ \lambda \in \mathbb{C}: \text{Re} \lambda > 0 \} \neq \emptyset, \\
\inf \{ \text{Re} \lambda : \lambda \in \sigma_+(\mathbf{L}) \} = \omega_+ > 0.
\end{cases}
\end{align}

**Theorem 4.1** Under the condition (4.3). There exists a bounded domain $\Omega \in \mathbb{R}$ such that the equilibrium solution of the coupled CGL–Burgers equations is unstable.

By virtue of Theorem 9.1.3 of page 344 in [15], we will break the result into two lemmas.

**Lemma 4.1** The operator $\mathbf{L} = \mathbf{A}\partial_x^2 + \mathbf{B}\partial_x + \mathbf{C}$ is sectorial and the graph norm of $\mathbf{L}$ is equivalent to the norm of $D(\mathbf{L})$. Moreover, there exists domain $\Omega$ and a positive number $\delta_0$ such that
\begin{align}
\begin{cases}
\sigma_+ (\mathbf{L}) = \sigma (\mathbf{L}) \cap \{ \lambda \in \mathbb{C}: \text{Re} \lambda > 0 \} \neq \emptyset, \\
\inf \{ \text{Re} \lambda : \lambda \in \sigma_+(\mathbf{L}) \} = \delta_0 > 0.
\end{cases}
\end{align}

**Proof.** From (3.7) and (3.8), we have $\text{Re} \lambda_2(k) < 0$. Case 1: If $k^2 m > 0$, $2(r_0^2 + k^2) < a$, or $4(r_0^2 + k^2)^2 - 4a(r_0^2 + k^2)^2 < b^2$, then we deduce that
\[
\text{Re} \lambda_1(k) < 0, \quad \text{and} \quad \text{Re} \lambda_3(k) > 0.
\]

Therefore, there exists $\Omega \in \mathbb{R}$ such that
\[
\inf \{ \text{Re} \lambda : \lambda \in \sigma_+(\mathbf{L}) \} > 0.
\]

Case 2: If $m < 0$, then it follows that $\text{Re} \lambda_1(k) > 0$. Therefore, there also exists $\Omega \in \mathbb{R}$ such that
\[
\inf \{ \text{Re} \lambda : \lambda \in \sigma_+(\mathbf{L}) \} > 0.
\]

Since the Laplacian is a sectorial operator, and the operator $\mathbf{L}$ is a lower order perturbation of the Laplacian, by the perturbation theorem, it is easy to show that $\mathbf{L}$ is a sectorial operator and the graph of $\mathbf{L}$ is equivalent to the norm of $D(\mathbf{L})$. 

In order to obtain Theorem 4.1, we only to prove the following lemma by virtue of Theorem 9.1.3 of page 344 in [15].
Lemma 4.2 Suppose $\mathcal{D}$ be a neighborhood of the origin in $D(L)$. Then there exists constants $r_0, \vartheta_0$ and the functions $u, v$ such that the map $\psi: \mathcal{D} \rightarrow L^2$ is a $C^1$ function with locally Lipschitz continuous derivative and satisfies

$$\psi(0) = 0, \quad \psi'(0) = 0,$$

where $\psi'(0)$ is the Fréchet derivative of $\psi(\pi)$ at origin.

Proof. First, by virtue of definition of $\psi(\pi)$, we have $\psi(0) = (0, -w_0 \vartheta_0 - (c_0 + c_1 r_0) \vartheta_0^2, 0)$. Therefore, we deduce $w_0 \vartheta_0 + (c_0 + c_1 r_0) \vartheta_0^2 = 0$. If $D_1, D_2, D_3$ denote the Fréchet derivative of $\rho, \phi, h$, respectively. Then we obtain

$$
\begin{cases}
D_1 \psi_1 = -2 \vartheta_0 c_1 \rho_x - 2 \phi \rho_x - (c_0 + c_1 r_0) \phi_{xx} - r_0[2 \rho + s'_1(r) h_x] \\
\quad - [4 \vartheta_0 \rho + 2 \vartheta_0 \phi_x + \phi_x^2 + s_1(r) h_x + \rho s'_1(r) h_x] - [2 \vartheta_0 c_1 \rho] \\
\quad + 2(c_0 + c_1 r) \phi + h] \vartheta_x \\
= H_{11} + J_{11} \vartheta_x,
\end{cases}
$$

$$
D_2 \psi_1 = -2(c_0 + c_1 r) \rho_x - (2r_0 \phi_x + 2 \vartheta_0 \rho + 2 \rho \phi_x) \vartheta_x - (c_0 + c_1 r_0) \rho \vartheta_x^2 \\
= H_{12} + J_{12} \vartheta_x + K_{12} \vartheta_x^2,
$$

$$
D_3 \psi_1 = -\rho_x + [r_0(s_1(r_0) - s_1(r))] - \rho s_1(r) \vartheta_x \\
= -\rho_x + J_{33} \vartheta_x.
$$

(4.5)

$$
\begin{cases}
D_1 \psi_2 = -\frac{c_0 \vartheta_x}{(r_0 + \rho)^2} - c_1(2 \vartheta_0 \phi_x + \phi_x^2 + 2 \frac{\vartheta_0 \phi_x}{r_0 + \rho}) \vartheta_x - v'(r) \rho^2 \\
\quad - 2 v(r) \rho - r_0^2 [v'(r) - v'(r_0)] - 2 r_0 [v(r_0) - v(r)] + 2 r_0 v'(r) \rho \\
\quad - 2 \frac{\vartheta_0 \rho + \vartheta_0 \phi_x}{r_0 + \rho} \vartheta_x + \frac{\vartheta_0^2}{r_0 + \rho} \\
= H_{21} + J_{21} \vartheta_x + K_{21} \vartheta_x^2,
\end{cases}
$$

$$
D_2 \psi_2 = -\left[ h + 2(c_0 + c_1 r_0) \phi_x + 2 c_1 \rho (\vartheta_0 + \phi_x) + \frac{2 \rho \rho_x}{r_0 + \rho} \right] \vartheta_x \\
= J_{22} \vartheta_x,
$$

$$
D_3 \psi_2 = -\phi_x + [s_2(r_0) - s_2(r)] \vartheta_x \\
= -\phi_x + J_{33} \vartheta_x.
$$

(4.6)

$$
\begin{cases}
D_1 \psi_3 = -2 \kappa \rho_x - 2 \kappa \rho \vartheta_x \\
D_2 \psi_3 = 0 \\
D_3 \psi_3 = -h_x - \rho \vartheta_x.
\end{cases}
$$

(4.7)

Therefore,

$$
\psi'(\pi)|_{\pi=0} = \begin{pmatrix}
D_1 \psi_1 \\
D_2 \psi_1 \\
D_3 \psi_1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

Then, we have $\vartheta_0 = c_0 = 0$ and $v = 0, r_0^2 = 1$.

In view of (4.5), (4.6) and (4.7), we can write $\psi'(\pi)$ as follows

$$
\psi'(\pi) = \begin{pmatrix}
H_{11} & H_{12} & -\rho_x \\
H_{21} & 0 & -\phi_x \\
-2 \kappa \rho_x & 0 & -h_x
\end{pmatrix} + \begin{pmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
-2 \kappa \rho & 0 & -h
\end{pmatrix} \frac{\partial}{\partial x}.
$$

24
\[
\begin{pmatrix}
0 & K_{12} & 0 \\
K_{21} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\partial^2 \frac{\partial}{\partial x^2} = H(\pi) + J(\pi) \frac{\partial}{\partial x} + K(\pi) \frac{\partial^2}{\partial x^2}
\]

Let \( \pi_1 = (\rho_1, \phi_1, h_1) \in \mathbb{D} \), \( \pi_2 = (\rho_2, \phi_2, h_2) \in \mathbb{D} \), for any \( R = (f, g, \eta) \in \mathbb{D} \), we imply that

\[
\begin{align*}
&\|\psi'(\pi_1) - \psi'(\pi_2)\| R \leq \|H(\pi_1) - H(\pi_2)\| R + \\
&\|J(\pi_1) - J(\pi_2)\| R_x + \|K(\pi_1) - K(\pi_2)\| R_{xx}.
\end{align*}
\]

(4.8)

Because of

\[
\begin{align*}
\|H(\pi_1) - H(\pi_2)\| R &= \sum_{i=1}^{3} \|(H_i(\pi_1) - H_i(\pi_2))f\| + \\
\|J(\pi_1) - J(\pi_2)\| R_x &= \sum_{i=1}^{3} \|(J_i(\pi_1) - J_i(\pi_2))f_x\| + \\
\|K(\pi_1) - K(\pi_2)\| R_{xx} &= \sum_{i=1}^{3} \|(K_i(\pi_1) - K_i(\pi_2))f_{xx}\| +
\end{align*}
\]

(4.9) (4.10)

and

\[
\begin{align*}
\|H(\pi_1) - H(\pi_2)\| R &= \sum_{i=1}^{3} \|(H_i(\pi_1) - H_i(\pi_2))f\| + \\
\|J(\pi_1) - J(\pi_2)\| R_x &= \sum_{i=1}^{3} \|(J_i(\pi_1) - J_i(\pi_2))f_x\| + \\
\|K(\pi_1) - K(\pi_2)\| R_{xx} &= \sum_{i=1}^{3} \|(K_i(\pi_1) - K_i(\pi_2))f_{xx}\| +
\end{align*}
\]

(4.11)

where we used the integrating by parts and Sobolev’s embedding theorem.

Combining (4.8), (4.9), (4.10) with (3.11), it follows that

\[
\|\psi'(\pi_1) - \psi'(\pi_2)\| H^{-1} \leq C(\|\pi\| H^{1+*}) \|\pi_1 - \pi_2\| H^1.
\]

This completes the proof Lemma 4.2. \(\square\)

**Acknowledgments**

This work was partially supported by CPSF (Grant No.: 2012M520007). The authors thank the references for their valuable comments and constructive suggestions.
References

[1] I.S. Aranson, L. Kramer, The world of the complex Ginzburg-Landau equation, *Rev. Mod. Phys.* 74 (2002) 99–143.

[2] I. Aranson, L. Kramer and A. Weber, Core instability and spatiotemporal intermittency of spiral waves in oscillatory media, *Phys. Rev. Lett.* 72 (1994) 2316–2319.

[3] H. Bahouri, J.Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations. Springer–Verlag Berlin Heidelberg, 2011.

[4] V.S. Berman and Yu.S. Ryazantsev, Asymptotic analysis of stationary propagation of the front of a two-stage exothermic reaction in a gas, *J. Appl. Math. Mech. (PPM)* 37 (1973) 1049–1058.

[5] J.D. Buckmaster and G.S.S. Ludford, Theory of Laminar Flames, Cambridge University Press, Cambridge, 1982.

[6] H. Chate and P. Manneville, Phase diagram of the two-dimensional complex Ginzburg–Landau equation, *Physica A* 224 (1996) 348–368.

[7] P. Colinet, P. Georis, J.C. Legros and G. Lebon, Spatially quasiperiodic convection and temporal chaos in two-layer thermocapillary instabilities, *Phys. Rev. E* 54 (1996) 514–524.

[8] K. Fujimura and Y.Y. Renardy, The 2/1 steady Hopf mode interaction in the two-layer Benard problem, *Physica D* 85 (1995) 25–65.

[9] A.A. Golovin, B.J. Matkowsky, A. Bayliss and A.A. Nepomnyashchy, Coupled KS-CGL and coupled Burgers-CGL equations for flames governed by a sequential reaction, *Physica D* 129 (1999) 253–298.

[10] A.A. Golovin, A.A. Nepomnyashchy and B.J. Matkowsky, Traveling and spiral waves for sequential flames with translation symmetry: coupled CGL–Burgers equations, *Physica D* 160 (2001) 1–28.

[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics No. 840, Springer-Verlag, Berlin-New York, 1981.

[12] A.K. Kapila and G.S.S. Ludford, Two-step sequential reactions for large activation energies, *Combust. Flame* 29 (1977) 167–176.

[13] S. Kawashima, Large-time behavior of solutions to hyperbolic-parabolic systems of conservation laws and applications, *Proc. Roy. SOC. Edinburgh Sect. A* 106 (1987) 169–194.

[14] H. Kidachi, On mode interactions in reaction diffusion equation with nearly degenerate bifurcations, *Prog. Theor. Phys.* 63 (1980) 1152–1169.

[15] A. Lunardi, Analytic Semigroup and Optimal Regularity in Parabolic Problem, Birkhöuser, 1995.
[16] S.B. Margolis and B.J. Matkowsky, Steady and pulsating modes of sequential flame propagation, *Comb. Sci. Technol.* **27** (1982) 193–213.

[17] D.O. Olagunju and B.J. Matkowsky, Coupled complex Ginzburg-Landau type equations in gaseous combustion, *Stability Appl. Ana. Continu. Media* **2** (1992) 31–58.

[18] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Spring-Verlag, New York, 1983.

[19] J. Peláez, Stability of premixed flames with two thin reaction layers, *SIAM J. Appl. Math.* **47** (1987) 781–799.

[20] J. Peláez and A. Liñán, Structure and stability of flames with two sequential reactions, *SIAM J. Appl. Math.* **45** (1985) 503–522.

[21] J.J. Perraud, A.De Wit, E. Dulos, P.De Kepper, G. Dewel and P. Borckmans, One-dimensional “spirals”: novel asynchronous chemical wave source, *Phys. Rev. Lett.* **71** (1993) 1272–1275.

[22] G.I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames I. Derivation of basic equations, *Acta Astronautica* **4** (1977) 1177–1206.

[23] F.A. Williams, Combustion Theory, Benjamin Cummings, Menlo Park, 1985.

[24] J. X. Xin, Multidimensional stability of travelling waves in a bistable reaction-diffusion system, I, *Comm. Partial Differential Equations* **17** (1992) 1889–1900.

[25] Y.B. Zeldovich, G.I. Barenblatt, V.B. Librovich and G.M. Makhviladze, The Mathematical Theory of Combustion and Explosion, Consultants Bureau, New York, 1985.