Renormalization Group Approach to $c = 1$
Matrix Model on a circle and D-brane Decay

Satabhisa Dasgupta

Department of Physics and Astronomy, Rutgers University,
Piscataway, NJ 08854, U.S.A.

Tathagata Dasgupta

Department of Physics, New York University,
4 Washington Place, New York, NY 10003, U.S.A.

Abstract

Motivated by the renormalization group (RG) approach to $c = 0$ matrix model of Brezin and Zinn-Justin, we develop a RG scheme for $c = 1$ matrix model on a circle and analyze how the two coupling constants in double scaling limit with critical exponent flow with the change in length scale. The RG flow equations produce a non-trivial fixed point with the correct string susceptibility exponent and the expected logarithmic scaling violation of the $c = 1$ theory. The change of world-sheet free energy with length scale indicates a sign change as we increase the temperature, indicating a phase transition due to liberation of the non-singlet states. At low temperature, the RG analysis also lead to T-duality of the singlet sector free energy. The RG flow to the $c = 1$ fixed point can be understood as the decay of unstable $D0$-branes with open string rolling tachyon to the $2D$ closed string theory described by the end point of the flow. The amplitude of the decay is extracted from the change of the world-sheet free energy described by the RG process and is in accordance with the prediction from boundary Liouville theory.
## 1 Introduction

Back in the late 80’s and early 90’s, there had been extensive work on low dimensional string theories, namely the $c \leq 1$ non-critical (or $D \leq 2$ critical) bosonic string theories, both in the discretized large $N$ matrix model approach \[1, 2, 3, 4\] and in continuum Liouville theory.
approach [5, 6, 7] by looking at them as 2D quantum gravity coupled to $c \leq 1$ matter. One of the motivations was to learn something about the non-perturbative effects of 2D quantum gravity and string theory and to apply that knowledge to higher dimensions. Studying 2D gravity with simple matter was thought to be useful to probe issues of topology change in a finite and tame gravitational theory and to construct a string theoretical basis of QCD. A remarkable fact is that all the ideas of developing non-perturbative techniques using branes and dualities, string field theory etc. is now going back to matrix model. According to the recent idea, initiated in [8], the eigenvalues of $c = 1$ matrix model represent unstable D-branes in the dual two-dimensional string theory. Before going into the details of this idea in the context of this paper, we describe briefly some of the difficulties in understanding $c = 1$ matrix model on compact space to sketch the main motivation of this paper.

In the discrete approach (see [9, 10, 12, 13, 14] for review), the Euclidean path-integral for two dimensional gravity can be written as a sum over discretized random surfaces and is evaluated using the large $N$ or the planar limit of the appropriate random matrix models. However the large $N$ limit, the lowest order in the string perturbation theory, is not the only thing that we can extract from the expansion of the world-sheet free energy. In the continuum limit, for any genus, the area of the surface is large compared to the elementary polygons of the discretized surface. Hence for any fixed genus $G$, i.e., at order $N^{-2G}$ in the $1/N$-expansion, we should stay in the vicinity of the singular point $g = g_c$ of the free energy at which the mean area diverges. Thus the limit of interest involves large $N$ and small $(g - g_c)$ with $g_s^2 = \frac{1}{N^2(g - g_c)^{2 - \gamma_{str}}}$ (where $\gamma_{str}$ is the string susceptibility exponent) fixed. This double scaling limit [15, 16, 17] led to the solution of the matrix models to all orders in string perturbation theory. The genus expansion of the world-sheet free energy can be written as

$$F(N, g) = \sum_{G=0}^{\infty} N^{2-2G} (g - g_c)^{(1-G)(2-\gamma_{str})}.$$  \hspace{1cm} (1.1)

Also the genus expansion becomes universal, i.e. almost entirely independent of how the surface is discretized.

The simplest model involving integral over one matrix variable has been solved to describe pure two-dimensional gravity (the $c = 0$ case) [13, 16, 17]. The one-dimensional hermitian matrix chain models have been solved in the double scaling limit and identified with the $c < 1$ minimal models coupled to two-dimensional gravity [13, 16, 20, 21]. For this general case of $(p, q)$ minimal models coupled gravity, Douglas has proposed a solution in terms of the generalized KdV equations [21], where the non-perturbative partition function is given by the square of the $\tau$-function of the KdV hierarchy, satisfying the string equation. The hermitian matrix quantum mechanics, describing $c = 1$ theory, has been solved [22, 23, 24, 25] and is interpreted in terms of two-dimensional string theory [26, 27], where the role of the extra dimension is played by the
conformal factor of the world-sheet quantum gravity.

The solvability of the \( c \leq 1 \) matrix models of \( N^2 \) degrees of freedom is mainly due to the fact that only \( N \) eigenvalues contribute, and the other \( N(N - 1) \) angular degrees of freedom decouple. The \( SU(N) \) singlet spectrum is described by \( N \) free fermions moving in a potential. The non-singlet states, coming from the angular degrees of freedom have energy diverging as logarithm of the cut-off. So they are pushed to infinity in the continuum limit. As a result, the singlet wave functions of the \( c = 1 \) matrix model are sufficient to account for all physical states of the continuum string theory in a non-compact dimension, or in a circle bigger than a critical size, \( R > R_c \) \cite{28}. This decoupling corresponds to the confinement of Berezinskii-Kosterlitz-Thouless (BKT) vortices \cite{29, 30}, which wind around the target space plaquette of lattice size. Although each of the vortices is suppressed by a divergent action in the continuum limit, its entropy factor, given by the number of places it can be found, grows and for sufficiently small circle \( R < R_c \), a Kosterlitz-Thouless phase transition occurs. As a result, although exact results in matrix model have been proved to be very powerful in describing low dimensional string theory in uncompactified dimension, understanding physics from very small compact target space is lacking. Although, for \( c = 1 \) case, \( X \) and the extra hidden dimension or the Liouville field \( \phi \) are \textit{a priori} spatial dimensions, it is more convenient to continue \( X \rightarrow iX \), and denoting \( t = iX \) as the Euclidean time. If we have \( t \) wrapped on a circle, \( t \sim t + 2\pi nR \), we have \( c = 1 \) string at finite temperature.

For \( c > 1 \), all the \( N^2 \) angular degrees of freedom become relevant and the situation becomes difficult to pursue in the matrix model framework, until we understand considerably how to deal with the non-singlet states. At this moment the predictions of continuum theories for the well-understood \( c \leq 1 \) cases turn out to be meaningless for \( c > 1 \), as the KPZ-DDK formula \cite{5, 6, 7} for the string susceptibility exponent and for scaling dimensions of matter operators lead to nonphysical complex values. Having said all these, the difficulties with \( c > 1 \) models are not related to just these technical issues. The continuum approach indicates that these models are tachyonic \cite{31, 32}. Light-cone quantization of certain \( c = 2 \) matrix model gave some partial results \cite{33}, but at the same time faces potential difficulties as the non-singlet states do not confine even if non-compact target space is considered.

The motivation of this paper is to initiate an approach to deal with the less understood non-singlet sector in matrix quantum mechanics on a compact space. We develop a renormalization group (RG) scheme, generalizing the work of Brézin and Zinn-Justin \cite{34} on \( c = 0 \) to the \( c = 1 \) model on a circle. The motivation is to develop a scheme which would reasonably reproduce the known cases that are solvable and to understand, at least qualitatively, the physically interesting situations which can not be simplified because of the lack of the solvable structure. The RG analysis of the \( c = 0 \) model, the simplest of the solvable examples, studied in \cite{34} has been extended to various general cases (see for example \cite{35, 36, 37, 38}). In \cite{38}, reparametrization
invariance and the the loop equations are used to eliminate some of the induced interactions which appear in the RG transformation at higher orders in coupling. However extension of this method to matrix quantum mechanics on circle is not obvious. We instead directly generalize \([34]\) by explicitly evaluating the determinant obtained by integrating out part of the matrices to get a RG flow in the space of actions in general representation. This enables us to study the models which do not allow angular integration, such as the \(c = 1\) model on small circle, or \(c > 1\) matter coupled to gravity.

As an illuminating interpretation and feedback from our RG analysis, we will digress to the recent progress in understanding the dynamics of the boundaries in the 2\(D\) string theory, that realizes the duality between \(c = 1\) model and 2\(D\) quantum gravity coupled to \(c = 1\) matter as an exact open/closed string duality \([8, 39, 40, 41, 42, 43, 44]\). According to this idea, the quantum mechanics of \(SU(N)\) invariant matrix variables in an inverted oscillator potential is visualized as the quantum mechanics of open string tachyons attached to \(N\) unstable \(D0\) branes that decays into (i.e. dual to) Liouville theory coupled to \(c = 1\) matter describing 2\(D\) closed string theory together with its \(D0\) branes. In matrix model, an unstable brane corresponds to a free fermion excited to the top of the potential (in presence of the rest forming the static Fermi sea). It then decays to the closed string vacuum by rolling down to the Fermi level as an unstable eigenvalue trajectory

\[
z(t) = \sqrt{2 \mu \alpha'} \hat{\lambda} e^t.
\]

The amplitude of this decay calculated by continuum methods can be read from the matrix model analysis considering the bosonization of the relativistic fermions in the asymptotic limit \([40]\). Alternatively as proposed in \([41]\), one can formally view the Vandermonde determinant corresponding to the \((N + 1)\)-th unstable eigenvalues in presence of the rest \(N\), to be related to the bosonization field and thus to lead to the decay amplitude.

To understand from our RG analysis, we consider integrating out one of the eigenvalues in presence of the static Fermi sea of the rest of the \(N\) eigenvalues. The decay amplitude is shown to be contained in the change of the world-sheet free energy given by the ratio \(Z_{N+1}/Z_N\) of the partition functions. The nice thing is that the end point of such a decay is explicitly a \(c = 1\) fixed point of the flow. We compute the above-mentioned Vandermonde determinant, the entropy arising out of the loss of information due to integrating out one of the eigenvalues, taking the unstable eigenvalue trajectory to be that corresponding to the ZZ-boundary state \([45]\) tensored with Sen’s boundary state \([46]\). This leads to the explicit relation between the two different views in \([40]\) and \([41]\) to compute decay amplitude and reproduces the expected time of decay

\[
\Delta t \sim \ln \hat{\lambda}.
\]

Understanding the dynamics of the boundaries in the 2\(D\) string theory, i.e. the \(D\)-objects
of the boundary Liouville theory, from the RG analysis of general \((N \times N)\) matrices on a circle is much more interesting. In this case, by tuning the matrix coupling constant and the mass parameter with \(N \to \infty\) in the double scaling limit, one tunes the bulk and boundary cosmological constants respectively to arrive at a relation between the two. This is given by the integration of the flow hitting the \(c = 1\) fixed point. Such a relation presumably represent Dirichlet or Neumann boundaries that are present in the Liouville theory coupled to \(c = 1\) matter. We will show that for large circle we arrive at such trajectories with correct scaling between the two cosmological constants. However a detail analysis of such trajectories for large and small circle will be extremely interesting and will be studied in detail in a future publication.

The organization of the paper is as follows. In section 2, we present the details of the RG calculation of matrix quantum mechanics on a circle with a cubic potential and obtain the set of beta function equations for the couplings. In section 3, we analyze the flow, the fixed points and the critical exponents. As understood previously, the usual KPZ-DDK scaling laws \([8, 9, 10]\) for \(c < 1\) are obeyed with a logarithmic scaling violation for the \(c = 1\) model \([22, 23, 24, 25, 28]\). With the understanding of the RG flows, here we are able to obtain the correct critical exponents of the \(c = 1\) strings, and to reproduce the scaling behavior for the free energy with expected scaling violation for the singlet sector. Also the RG trajectory near the nontrivial fixed point shows the correct scaling behavior between the bulk and boundary cosmological constants. From the running of the prefactor of the partition function written in the renormalized couplings, analogous to the running due to the wave function renormalization, the free energy is observed to change sign near \(R = 1\) for small value of the critical coupling. This is reminiscent of the Kosterlitz-Thouless transition at self-dual radius triggered by the liberation of the world-sheet vortices. In section 4, we discuss that in what sense the RG analysis can capture the T-duality respected by the singlet partition function. In section 5, we discuss how the change of the world-sheet free energy in integrating out one matrix eigenvalue can be seen in the recent context to lead to the amplitude of the closed string emission from the decaying brane. In section 6, we conclude with some open questions.

## 2 The Basic Set Up and the World-sheet RG Calculation

The existence of double scaling limit indicates that a change in length scale induces flow in the coupling constants of the theory in a way that one reaches the continuum limit with desired critical exponents. In this continuum limit, as the matrix coupling constant approaches a critical value \(g_c\), the average number of triangles \(\langle n_G \rangle\) in triangulations at any genus \(G\) diverges with the exponent \(-1\) while the length of the triangles or the regularized spacing of the random lattice \(a\) is scaled to zero by taking \(N \to \infty\) to keep the physical area \(a^2 \langle n_G \rangle\) or equivalently the string...
coupling $g_s$ fixed,

$$g \rightarrow g_c \Rightarrow \langle n_G \rangle \sim (1 - G)(\gamma_0 - 2)(1 - g/g_c)^{-1} \rightarrow \infty,$$

$$N \rightarrow \infty \Rightarrow a \sim N^{-\frac{1}{1-\gamma_0}} \rightarrow 0,$$

with

$$a^2\langle n_G \rangle \sim N^{-\frac{2}{1-\gamma_0}}(1 - g/g_c) = \text{const. or, } g_s^{-2} \equiv N^2(g - g_c)^{2-\gamma_0} = \text{const.} \quad (2.1)$$

Thus one would naturally try to understand how the two parameters of the theory, the size of the matrices $N$ and the cosmological constant (mapped into the matrix coupling $g$), evolve at the constant long distance physics with the rescaling of the regularization length in the triangulation of the world-sheet. The flow equations will automatically give rise to the correct scaling laws and the critical exponents around the nontrivial IR fixed points governing the continuum physics. In the Wilsonian sense this can be achieved by changing $N \rightarrow N + \delta N$ by integrating out some of the matrix elements, which is like integrating over the momentum shell $\Lambda - d\Lambda < |p| < \Lambda$, and compensating it by enlarging the space of the coupling constants $g \rightarrow g + \delta g$. Here the space of coupling constants will contain both the matrix coupling $g$ and the mass parameter $M^2$.

Following the RG scheme of Brézin and Zinn-Justin we construct the flow equations by integrating out a column and a row of an $(N + 1) \times (N + 1)$ matrix, reducing it to an $N \times N$ matrix. One expects the process to lead to the following key relation satisfied by the matrix partition function

$$Z_{N+1}(g, M, R) = [\lambda(g, M, R)]^{N^2} Z_N(g', M', R') \quad (2.2)$$

with

$$g' = g + \frac{1}{N} \beta(g, M, R) + O\left(\frac{1}{N^2}\right),$$

$$M'^2 = M^2 + \frac{1}{N} \beta(g, M, R) + O\left(\frac{1}{N^2}\right), \quad (2.3)$$

and

$$\lambda(g, M, R) = 1 + \frac{1}{N} \gamma(g, M, R) + O\left(\frac{1}{N^2}\right). \quad (2.4)$$

Then the string partition function

$$\mathcal{F}(N, g, M, R) = \frac{1}{N^2} \ln Z_N(g, M, R) \quad (2.5)$$

satisfies the Callan-Symanzik equation.
\[ F(N, g, M, R) = r(g, M, R). \] (2.6)

### 2.1 Integrating out a row and a column

Let us consider the \((N + 1) \times (N + 1)\) matrices \(\phi_{N+1}(t)\) and decompose them into \(N \times N\) matrices \(\phi_N(t)\), \(N\)-vectors \(v_a(t)\) and \(v^*_a(t)\) \((a = 1, \ldots, N)\) and a scalar \(\alpha\). For the time being we can choose \(\alpha = 0\) as they are of relative order \(1/N\) and can be ignored in the double scaling limit.

\[
\phi_{N+1}(t) = \begin{pmatrix} \phi_N(t) & v_a(t) \\ v^*_a(t) & 0 \end{pmatrix}.
\] (2.7)

For simplicity, we consider a cubic potential, which will serve as a wall stabilizing the inverted oscillator potential. In terms of \((N + 1)\) dimensional matrix variable, the action reads

\[
S_{N+1}[\phi_{N+1}(t), g, M, R] = (N + 1) \int_0^{2\pi R} dt \, \text{Tr} \left[ \frac{1}{2} \dot{\phi}_{N+1}^2(t) + \frac{1}{2} M^2 \phi_{N+1}^2(t) - \frac{g}{3} \phi_{N+1}^3(t) \right].
\] (2.8)

The parametrization (2.7) gives the simple relations,

\[
\begin{align*}
\text{Tr} \phi_{N+1}^{2k} &= \text{Tr} \phi_N^k + 2k \, v^* \phi_N^{2k-2} v + O(v^4), \\
\text{Tr} \phi_{N+1}^{2k+1} &= \text{Tr} \phi_N^k + (2k + 1) \, v^* \phi_N^{2k-1} v + O(v^4 \phi_N^2 v).
\end{align*}
\] (2.9)

The higher order terms in \(v^* v\) can be neglected as they are supressed by powers of \(O(1/N)\).

The resulting partition function can be written as

\[
Z_{N+1}[g, M, R] = \int_{\phi_N(2\pi R) = \phi_N(0)} D^{N^2} \phi_N(t) \, e^{-(N+1) \int_0^{2\pi R} dt \, \text{Tr} \left[ \frac{1}{2} \dot{\phi}_N^2(t) + \frac{1}{2} M^2 \phi_N^2(t) - \frac{g}{3} \phi_N^3(t) \right]} \times \int_{v, v^*(2\pi R) = v, v^*(0)} D^N v(t) D^N v^*(t) \, e^{-\int_0^{2\pi R} dt \, \left[ v^*(t)[-\partial^2 + M^2 - g \phi_N(t)] v(t) \right]}. \] (2.10)

The above partition function is identical to the one considered in [17, 18], where the \(c = 1\) matrix model suitable for Veneziano type QCD has been considered to study open strings. Both color \(N\) and fermion (quark) flavor \(N_f\) has been taken to be large. Here the quarks are bosonic (the vectors). It is precisely these fields in the fundamental representation of the global
SU(\(N\)) group, which generate boundary terms in the Feynman diagrams. As usual, we will use adiabatic treatment of first integrating out the quark loops. Integrating over the quarks, we get

\[
\mathcal{Z}_{N+1}[g, M, R] = \left( \frac{\pi}{N+1} \right)^N \int_{\phi_N(2\pi R) = \phi_N(0)}^N D^{N^2} \phi_N(t) \times \exp \left\{ -\int_0^{2\pi R} dt \left[ (N+1)\text{Tr}\left\{ \frac{1}{2} \dot{\phi}_N^2(t) + \frac{1}{2} M^2 \phi_N^2(t) - \frac{g}{3} \phi_N^3(t) \right\} \right] + N \text{Tr} \log \left\{ -\partial_t^2 + M^2 - g \phi_N(t) \right\} \right\}.
\]

(2.11)

Logarithm with minus sign arise if the integration is performed on \(N\) flavors of fermions. Although such model has been considered as matrix model for open strings, where the logarithmic term has the effect of generating boundaries in the world-sheet, there are some differences. Unlike the zero dimensional case [49, 50, 51], this logarithmic term does not arise by integrating out fields (quarks), which are \(1 \times N\) matrices that couple to \(\phi_N\). As a result, in those open string models, the couplings in front of the logarithm and in its argument are introduced by hand, rather than being determined by the original closed string action. One tunes the couplings to some appropriate values and the adiabatic treatment of first integrating out the quark loops rise to tearing phenomena. Interesting critical behavior of getting phases with torn surface in the \(c = 0\) case [49] also occur in the one-dimensional case [47] if the dynamical loops are generated by bosons without kinetic term. Absence of the kinetic term makes fermion loops uncorrelated in time. One can ignore the derivative term inside the logarithm if the mass and the couplings are large enough. But there are interesting critical behavior when the argument of the logarithm without the kinetic term approaches zero [52, 47]. In [48], \(c = 1\) model with explicit expression for the fundamental fields has been considered and is the one which is closest to our model obtained after integrating out one row and one column of flavor degrees of freedom. If one considers infinite line, only ground state is relevant. For some choices of the coupling and the mass of the particle moving around the boundary of the holes, it has been possible to find the ground state [48], but the exact spectrum is not known. We will return to this discussion in section 5, when we will interpret the results from the RG analysis in the recent context of time dependent tachyonic decay of \(D\)-branes in two-dimensional string theory.

Now we return to evaluate the determinant by standard Feynman expansion. In general it is a non-local object. For simplicity one could consider the constant \(\phi_N\) mode, which is equivalent to studying the effective potential to determine the phase structure. But here we will treat more general case of evaluating the determinant with flavors coupling to general \(\phi_N\) by performing the calculation in Fourier transformed variables with discrete momenta arising from the compact target space. The complicated induced interactions arising from the logarithmic
term are ignored in small field approximation in order to get back the beta functions that depend only on the original couplings.

Rescaling the vectors \( v(t) \rightarrow \frac{v(t)}{\sqrt{2\pi R(N+1)}} \), the \( v \) dependent part of the partition function turns out to be

\[
I[g, M, \phi_N, R] = \frac{1}{\sqrt{2\pi R(N+1)}}^{2N} \int_{v,v^*} D^N v(t) D^N v^*(t) \exp \left[ -\int_0^{2\pi R} \frac{dt}{2\pi R} \left\{ \dot{v}^* \dot{v}^* + v^*(M^2 - g\phi_N) v \right\} \right],
\]

(2.12)

Before integrating out the vectors to get the determinant \( \det | -\partial^2_t + M^2 - g\phi_N | \)^{-1} and expanding it in Feynman diagrams, it is convenient to Fourier transform all the fields as

\[
O(t) = \sum_{m=-\infty}^{\infty} O_m e^{\frac{i m}{R} t}, \quad O_m = \int_0^{2\pi R} \frac{dt}{2\pi R} e^{-\frac{i m}{R} t} O(t),
\]

(2.13)

with

\[
\delta_{mn} = \int_0^{2\pi R} \frac{dt}{2\pi R} e^{\frac{i (n-m)}{R} t}, \quad \delta(t-t') = \frac{1}{2\pi R} \sum_{m=-\infty}^{\infty} e^{\frac{i m}{R} (t-t')}.
\]

In terms of the Fourier modes, the \( v \)-integration can be expressed as

\[
I[g, M, \phi_N, R] = \frac{1}{\sqrt{2\pi R(N+1)}}^{2N} \int (\prod_n dv_n dv_n^*) e^{-\sum_m v_m^*(\frac{m^2}{R^2} + M^2)v_m + g\sum_{m,l} v_m \phi_{m-l} v_l},
\]

(2.14)

where we have neglected the \( O(1/N) \) terms.

### 2.2 One loop Feynman Diagrams

In order to carry out the \( v \) integration diagrammatically, let us now define the following operators

\[
\mathcal{O}_{mn} = \left( \frac{m^2}{R^2} + M^2 \right) \delta_{mn}, \quad \mathcal{O}_{m-l}(g, \phi) = g\phi_{m-l},
\]

(2.15)

The inverse of these operators define various propagators and vertices according to figure [1].

Hence the integral becomes

\[
I[g, M, \phi_N, R, N] = \frac{1}{[2\pi R(N+1)]^N} \exp \left[ -\sum_{m_1, l_1} \mathcal{O}_{m_1-l_1}(g, \phi) I_0(R, M, N) \right],
\]

(2.16)
where the gaussian part is as follows

\[
I_0[R, M, N] = \int \left( \prod_j dv_j^* dv_j \right) \exp \left[ - \sum_m v_m^* \mathcal{O}^* v_m \right].
\] (2.17)

In order to perform the gaussian integration, we rescale \( v_n \) as \( v_n \rightarrow v_n / (n^2/R^2 + M^2)^{1/2} \) and use (2.13). Performing the gaussian integration, we get

\[
I_0[R, M, N] = \left( \frac{2\pi^4 R^3}{\sinh^2 \pi RM} \right)^N \prod_{n=1}^{\infty} \left( \frac{R}{n} \right)^{4N},
\] (2.18)

where we have used the standard relation

\[
\prod_{n=1}^{\infty} \left( 1 + \frac{x^2}{n^2} \right)^{-1} = \frac{\pi x}{\sinh \pi x}
\]

Inserting this into (2.16), the \( v \)-integration becomes

\[
I[g, M, \phi_N, R, N] = C(R, M, N) \Sigma[g, M, \phi_N, R, N],
\] (2.19)

where

\[
C(R, M, N) = \left[ \frac{\pi^3 R^2}{(N+1) \sinh^2 \pi RM} \right]^N \prod_{n=1}^{\infty} \left( \frac{R}{n} \right)^{4N}.
\] (2.20)

In (2.19), \( \Sigma[g, M, \phi_N, R, N] \) represents sum of one loop Feynman diagrams as shown in figure 2. The sum \( \Sigma[g, M, \phi_N, R, N] \) can be expressed as...
\[
\Sigma[g, \phi_N, R, N] = 1 + g \left[ \sum_n \frac{1}{\frac{m_n^2}{R^2} + M^2} \phi_0 \right] \\
+ \frac{g^2}{2} \left[ \sum_{m,n} \frac{1}{\left( \frac{m_n^2}{R^2} + M^2 \right) \left( \frac{n_n^2}{R^2} + M^2 \right)} \phi_{m-n} \phi_{n-m} \right] \\
+ \frac{g^3}{3!} \left[ \sum_{m,n,l} \frac{1}{\left( \frac{m_n^2}{R^2} + M^2 \right) \left( \frac{n_n^2}{R^2} + M^2 \right) \left( \frac{l_l^2}{R^2} + M^2 \right)} \phi_{m-l} \phi_{l-n} \phi_{n-m} \right] + O(g^4) 
\] (2.21)

### 2.3 Evaluation of the diagrams

In order to evaluate the one loop correction to the effective action, we inverse transform the Fourier modes according to the rule (2.13) and sum-up the set of infinite series using the formulae

\[
\text{sum}_{m=-\infty}^{\infty} \frac{\exp[i(m/R)t]}{\frac{m^2}{R^2} + M^2} = \frac{\pi R \cosh(\pi MR - Mt)}{M \sinh \pi MR}, \quad 0 \leq t \leq 2\pi R, 
\] (2.22)

and,

\[
\sum_{m=-\infty}^{\infty} \frac{m/R \exp[i(m/R)t]}{\frac{m^2}{R^2} + a^2} = \frac{i\pi R \sinh(\pi aR - at)}{\sinh \pi aR}, \quad 0 < t < 2\pi R, \\
\text{as } t \to 0, \quad t \to 2\pi R, \\
0, \quad t = 0 = 2\pi R. 
\] (2.23)

These summations give the corrections to the coefficients of the various terms in the action, after \(\Sigma[g, M, \phi_N, R, N]\) is exponentiated and log-expanded using small field approximation. Since after performing the inverse Fourier transform, the various terms has nonlocal integrals over several one dimensional dummy time variables, we breakup the variables into center of mass and relative coordinates. Then we expand the functions about the center of mass coordinates,
assuming the relative coordinates to be small enough, and consider integration over the relative coordinates.

The details of the diagram evaluation are given in the appendix taking into account all the above considerations and evaluating the integrations over center of mass and relative time variables. Collecting all the contributions up to $O(\phi\phi\phi)$ from the appendix, the expression for $\Sigma[g, M, \phi_N, R, N]$ becomes

$$
\Sigma[g, M, \phi_N, R, N] \simeq 1 + F_{g_1}(R, M) g \int_0^{2\pi R} dt \text{Tr}\phi_N(t) + F_{g_2}(R, M) g^2 \int_0^{2\pi R} dt \text{Tr}\phi_N^2(t) + \hat{F}_{g_2}(R, M) g^2 \int_0^{2\pi R} dt \frac{1}{2} \text{Tr}\phi_N^3(t) + F_{g_3}(R, M) g^3 \int_0^{2\pi R} dt \frac{1}{3} \text{Tr}\phi_N^3(t).
$$

Note that, in the evaluation of $\Sigma$ we keep the contribution from the nonlocal terms of the action up to the kinetic term and the contribution from the higher order terms in the matrix field up to the cubic term. All other operators are redundant for our purpose and are negligible due to the small field approximation.

In the above expression, $F(R)$s are defined as follows

$$
F_{g_1}(R, M) = \frac{1}{2M} \coth \pi MR,
$$

$$
F_{g_2}(R, M) = \frac{1}{M^3 \sinh^2 \pi MR} \left(\frac{1}{2} \pi MR \cosh 2\pi MR + \frac{1}{8} \sinh 4\pi MR\right),
$$

$$
\hat{F}_{g_2}(R, M) = \frac{1}{M^5 \sinh^2 \pi MR} \left( \frac{\pi MR}{16} \cosh 4\pi MR - \frac{\pi^3 M^3 R^3}{6} \cosh 2\pi MR 
- \frac{1}{64} (1 + 8\pi^2 M^2 R^2) \sinh 4\pi MR \right),
$$

$$
F_{g_3}(R, M) = \frac{\pi MR}{64 M^5 \sinh^3 \pi MR \left( \cosh 2\pi MR + \cosh 4\pi MR \right) \left[ 4\pi MR (3 \cosh \pi MR + 2 \cosh 3\pi MR + 2 \cosh 5\pi MR + \cosh 7\pi MR) 
+ \sinh \pi MR + \sinh 3\pi MR + \sinh 5\pi MR + 2 \sinh 7\pi MR + \sinh 9\pi MR \right]}.
$$

2.4 Elimination of the tadpole term

The term proportional to $\int_0^{2\pi R} dt \phi_N(t)$ is unwanted. As usual, in order to remove this term, we change the background $\phi_N(t) \to \phi_N(t) + f$, and set the net coefficient of the term linear in $\phi$ to zero. This fixes the value of $f$ as
\[ f = \frac{1}{2} \left( g + \frac{g}{N} + \frac{g^3 F_{g^3}}{N} \right)^{-1} \left[ \left( M^2 + \frac{M^2}{N} + \frac{g^2 F_{g^2}}{N} \right) \pm \left( M^2 + \frac{M^2}{N} + \frac{g^2 F_{g^2}}{N} \right)^2 \right. \\
+ \left. \frac{4g F_{g^1}}{N} \left( g + \frac{g}{N} + \frac{g^3 F_{g^3}}{N} \right) \right] ^{1/2} \approx -g M^2 F_{g^1} / N + O \left( \frac{1}{N^2} \right), \quad (2.25) \]

and accordingly modifies the coefficients of all the terms in the action. After accommodating all the changes, the expression for \( Z_{N+1} \) turns out to be

\[
Z_{N+1} = C(R, M, N) \exp[2\pi R N^2 \mathcal{F}(g, M, R, N)] \int D^N \phi_N \exp \left[ -N \text{Tr} \int_0^{2\pi R} dt \left\{ \left( 1 + \frac{1}{N} \right) \phi^2(t) + \frac{g^2 \hat{F}_{g^2}}{N} \phi^2(t) + \left( 1 + \frac{M^2}{N} + \frac{g^2 F_{g^2}}{N} - \frac{g^2 M^2 F_{g^1}}{N} \right) \phi^2(t) \right. \right. \\
- \left. \left. \left( g + \frac{g}{N} + \frac{g^3 F_{g^3}}{N} \right) \phi^3(t) \right\} \right], \quad (2.26)\]

where the expression for \( \mathcal{F}(g, M, R, N) \) is given by

\[
\mathcal{F}(g, R, N) = \frac{g F_{g^1}}{N} f + \left( M^2 + \frac{M^2}{N} + \frac{g^2 F_{g^2}}{N} \right) f^2 + \left( g + \frac{g}{N} + \frac{g^3 F_{g^3}}{N} \right) f^3, \quad (2.27)\]

which is of \( O\left( \frac{1}{N^2} \right) \).

### 2.5 Rescaling of the fields and the variables

In order to restore the original cut-off, we perform the following rescalings so that the effective action is of the same form as the bare one but with the renormalized strength of the couplings.

\[
\phi_N(t) \rightarrow \rho \phi'_N(t'), \quad t' \rightarrow t(1 - h \, dl), \quad R' \rightarrow R(1 - h \, dl), \quad (2.28) \]

where \( dl = 1/N \), and set the overall coefficient of the kinetic term to zero. Here we can assume that

\[
h = \sum_{i,j} c_{ij} g^4 M^4 h_{ij}(R). \quad (2.29)\]

The functional form of \( h \) can be guessed from the contribution of the Feynman diagrams in the behavior of the flow near the fixed points. This sets the value of \( \rho \) to

\[
\rho = 1 + \frac{1}{2} \left( h - 1 + g^2 \hat{F}_{g^2} \right) dl + O(dl^2). \quad (2.30)\]
2.6 The beta function equations

The effective action is of the same form as the bare one, but with a renormalized strength of coupling. The resulting partition function is given by

\[ Z_{N+1} = \lambda'^{N^2} \int D^{N^2}\phi'_{N}(t') \exp \left[ -N \text{Tr} \int_0^{2\pi R'} dt' \left( \frac{\phi'^2_{N}(t')}{2} + M'^2 \phi'^2_{N}(t') - g'\phi'^3_{N}(t') \right) \right], \quad (2.31) \]

where,

\[ \lambda'^{N^2} = C(R, M, N) \exp [-2\pi RN^2 \mathcal{F}(g, M, N, R)] \rho^{N^2}. \quad (2.32) \]

Neglecting the \( O(dl^2) \) terms, the renormalized coupling and mass are expressed in terms of the bare quantities as follows:

\[ g' = g + \left( \frac{5}{2}h - \frac{1}{2} \right) g dl + \left[ F_{g3}(R, M) - \frac{3}{2} \hat{F}_{g2}(R, M) \right] g^3 dl, \]
\[ M'^2 = M^2 + \left[ 2hM^2 + g^2(1 - M^2)F_{g2}(R, M) - g^2M^2\hat{F}_{g2}(R, M) \right] dl, \]
\[ \lambda' = 1 + \ln \left[ \frac{\pi^3 R^2}{\sinh \pi RM} \right] dl + \frac{1}{2} \left[ (h - 1) + g^2\hat{F}_{g2}(R, M) \right] dl. \quad (2.33) \]

Here, in simplifying the part \( C(R, M, N)^{\frac{1}{N}} \) in the expression for \( \lambda' \), we have assumed that for any value of \( R \),

\[ C(R, M, N)^{\frac{1}{N}} = \exp \left[ \frac{1}{N} \ln \left( \frac{\pi^3 R^2}{\sinh \pi RM} \right) \right] \left[ \frac{\pi^3 R^{4n}}{(N + 1)(n!)^4} \right] \]
\[ \simeq 1 + \frac{1}{N} \ln \left( \frac{\pi^3 R^2}{\sinh \pi RM} \right) + O(1/N^2). \quad (2.34) \]

Also, the term \( \exp [-2\pi RN^2 \mathcal{F}(g, M, N, R)] \) contributes only a factor of 1 as \( \mathcal{F}(g, M, N, R) \sim O(dl^2) \). The resulting beta function equations are given by

\[ \beta_g = \frac{dg}{dl} = \left( \frac{5}{2}h - \frac{1}{2} \right) g dl + \left[ F_{g3}(R, M) - \frac{3}{2} \hat{F}_{g2}(R, M) \right] g^3, \]
\[ \beta_{M^2} = \frac{dM^2}{dl} = 2hM^2 + g^2(1 - M^2)F_{g2}(R, M) - g^2M^2\hat{F}_{g2}(R, M), \]
\[ \beta_{\lambda} = \frac{d\lambda}{dl} = \ln \left[ \frac{\pi^3 R^2}{\sinh \pi RM} \right] + \frac{1}{2} \left[ (h - 1) + g^2\hat{F}_{g2}(R, M) \right]. \quad (2.35) \]

The relation \( (2.28) \) indicates in some sense a renormalization of the radius \( R \) as

\[ \beta_R = \frac{dR}{dl} = -hR. \quad (2.36) \]
Now before going to the detail analysis of the fixed points let us try to understand few things about the structure of the beta function equations. The much of the structure depends on understanding the quantity $h$. One can clearly see for $g = 0$ the gaussian model is never expected to flow and hence for such a fixed point $h = 0$ (i.e. corresponding to the trivial rescaling $t' = t$ and $R' = R$). The situation is different for a non-vanishing $h$. Demanding the mass parameter to be always at some fixed value of $M$ (we will use $M^2 = 1$ for simplicity) and $\beta M_2 = 0$, one can easily determine some $h = h(R)$ for nontrivial fixed points $g^* \neq 0$, as there are two independent equations ($\beta_g = 0$, $\beta M_2 = 0$) and two unknowns ($h(R)$ and $g$). This is extremely interesting as it could indicate a phase transition at certain radius due to turning on (i.e. being relevant) the operator coupled to the cosmological constant. Keeping $M^2 = 1$ for simplicity is consistent with the value of the mass parameter ($M^2 = \frac{1}{\alpha'}$) one originally works with in the matrix partition function to visualize the matrix path integral as the generator of the discretized version of the Polyakov path integral of 2D bosonic string. In recent identification of the matrix quantum mechanics with the quantum mechanics of open string tachyon on unstable D0-branes the mass parameter $M^2 = \frac{1}{\alpha'}$ is identified with the open string tachyon mass. However, we will use a framework of the flow of a general $M^2$ (i.e. $h = 0$ )to discuss the presence of the boundaries.

Another point to be mentioned is that in this general analysis though we will see the signature of the singlet sector to be more apparent, the non-singlet sector does render indirect signature on the flow. This is reflected in the $R$ dependence of the change of the world sheet free energy with the scale, i.e. in the $\beta \lambda$. For very small $g^*$ it exhibits a sign change at certain radius which is reminiscent of the tendency of the non-singlet sector to be liberated at certain critical temperature ($R_c = 1$). To capture this effect one needs to turn on the right operator coupled to the fugacity of the vortices. In a separate analysis we will work with a gauged matrix model with a gauge breaking term that couples to the fugacity of vortices and drives the system to the phase where the vortices are liberated, above a critical temperature.

### 3 The $c = 1$ Fixed point

We will now analyze the fixed points of the flow equations given by the simultaneous solutions of $\beta_g = \beta M_2 = 0$. As we discussed before, for the trivial rescaling of the coordinates and the momenta, $t' = t$ and $1/R' = 1/R$, one can assume $h = 0$. In this case, the gaussian fixed point $\Lambda_1^* = (0, 0)$ satisfies both the equations $\beta_g = 0$ and $\beta M_2 = 0$ trivially. The nontrivial fixed point $\Lambda_2^*$ is given by,

$$g^* = \pm \sqrt{\frac{1}{2 F_{g3}(R, M^*) - \tilde{F}_{g2}(R, M^*)}},$$  

(3.1)
where $M^*$ is determined by the equation,

$$g^* \left( (1 - M^{*2})F_{g2}(R, M^*) - M^{*2} \hat{F}_{g2}(R, M^*) \right) = 0.$$ \hspace{1cm} (3.2)

We will see in the next section that this nontrivial value of $g^*$ will always characterize $c = 1$ fixed point as it gives the critical exponent of $c = 1$ for any $R$ for $h = 0$. Using the value of $g^*$ at the nontrivial fixed point (3.1) in the equation (3.2), one has

$$\frac{(1 - M^{*2})F_{g2}(R, M^*) - M^{*2} \hat{F}_{g2}(R, M^*)}{2F_{g3}(R, M^*) - 3\hat{F}_{g2}(R, M^*)} = 0 \hspace{1cm} (3.3)$$

For any value of $R$, this equation trivially has solution for very small value of $M^*$. However, for large $R$ there are also solutions for large $M^*$. In the large $R$ limit the equation (3.3) takes the form

$$M^{*2} \left( \frac{8\pi^2 M^{*2} R^2 - 8M^{*2} - 4\pi M^* R + 9}{24\pi^2 M^{*2} R^2 - 10\pi M^* R + 3} \right) = 0 \hspace{1cm} (3.4)$$

The solution for large $M^*$ corresponds to the situation that the denominator is much larger than the numerator (which solves with a precision of 70%). This gives

$$M^* \gg m_+, \quad \text{or} \quad M^* \ll m_-,$$

where, \(m_\pm = \frac{3\pi R \pm \sqrt{48 + 105\pi^2 R^2}}{8(4\pi^2 R^2 + 1)}\). \hspace{1cm} (3.5)

As $R$ becomes large, $m_\pm$ becomes smaller as $1/R$ and $M^*$ accesses all possible values ranging from small to large. Thus the nontrivial solutions of $\beta_g = 0$ and $\beta_{M^2} = 0$ gives dense lines of fixed points occupying a region. Depending on all these values of $M^*$, $g^* \neq 0$ will have different values from (3.1). Note that all these nontrivial fixed points corresponds to $c = 1$ as they have the string susceptibility exponent of $c = 1$ ($\gamma_0 = 0$) irrespective of the different values of $g^*$ and $M^*$. It can be mentioned here that as $R \to \infty$ the magnitude of $g^*$ becomes smaller and smaller and eventually all the pair of fixed points for \(\pm g^* \neq 0\) coincides with the Gaussian fixed point.

Now let us look at the general shape of the trajectories flowing to these $c = 1$ nontrivial fixed points $g^* \neq 0$. The RG equations $\beta_g = 0$ and $\beta_{M^2} = 0$ can be combined to give

$$\frac{\partial M^2}{\partial g} = \frac{2g(F_{g2} - M^2(F_{g2} + \hat{F}_{g2}))}{-1 + g^2(2F_{g3} - 3\hat{F}_{g2})}, \quad (g \neq 0). \hspace{1cm} (3.6)$$

For large $R$ this equation takes the following form where the variables are easily separable

$$\frac{\partial M}{\partial g} \sim \frac{(F_{g2} - M^2(F_{g2} + \hat{F}_{g2}))}{gM(2F_{g3} - 3\hat{F}_{g2})} \hspace{1cm} (3.7)$$

Using the explicit functional form at large $R$ this becomes

$$\frac{\partial M}{\partial g} \sim \frac{M(8\pi^2 R^2 M^2 - 4\pi RM - 8M^2 + 9)}{g(24\pi^2 R^2 M^2 - 10\pi RM + 3)}. \hspace{1cm} (3.8)$$
The leading behavior (considering $1/R$ as small parameter) indicates,

$$3 \frac{dM}{M} \sim \frac{dg}{g}, \quad \Rightarrow \quad M^2 \sim g^{0.6}. \quad (3.9)$$

In the last section we will come back to this behavior in the context of Neumann and/or Dirichlet boundaries of 2D string theory inserted by the integration of the fundamental fields (quarks) $v^*$ and $v$. For Dirichlet boundaries one expects large value of $M^*$. Then the boundary fields become uncorrelated in time. One can consider the scaling function of $M^2$ as the boundary cosmological constant $\mu_B$. Since $\mu_B$ is expected to scale with the bulk cosmological constant $\mu$ as

$$\mu_B \sim \sqrt{\mu}, \quad (3.10)$$

and since the scaling function of $g$ gives the renormalized bulk cosmological constant, the above scaling behavior in (3.9) is nothing but an indication of the presence of various boundaries of the 2D string theory.

### 3.1 The critical exponents

Let us now go back to the matrix partition function we started with. After completing the RG transformations, it obeys the relation

$$Z_{N+1}[g, M, R] \simeq [\lambda(N, g, M, R)]^{N^2} Z_N[g' = g + \delta g, M' = M + \delta M, R' = R + \delta R]. \quad (3.11)$$

This leads to the Callan-Symanzik equation

$$\left[ N \frac{\partial}{\partial N} - \beta_g \frac{\partial}{\partial g} - \beta_M \frac{\partial}{\partial M} - \beta_R \frac{\partial}{\partial R} + \gamma \right] F[g, M, R] \approx r[g, M, R]. \quad (3.12)$$

for the string partition function (or the world-sheet free energy)

$$F[g, M, R] = \frac{1}{N^2} \ln Z[g, M, R], \quad (3.13)$$

with

$$\gamma = 2. \quad (3.14)$$

The singular part of the world-sheet free energy $F_s$ is given by the solution of the homogeneous Callan-Symanzik equation. The inhomogeneous part defined by the change in the prefactor $\lambda$, contributes to subtleties in the free energy.

Let us now discuss the critical exponents for the scaling variables, the renormalized bulk cosmological constant $\Delta = 1 - g/g^*$, the renormalized mass (or in the context of boundaries, the renormalized boundary cosmological constant) $\mathcal{M} = 1 - M/M^*$. Introducing the matrix
\[ \Omega_{k,l} = \frac{\partial \beta_k (\Lambda^*)}{\partial \Lambda_l} , \] 

the homogeneous part of the Callan-Symanzik equation, satisfied by the singular part of the free energy, can be rewritten as:

\[
\left[ N \frac{\partial}{\partial N} - \Omega_1 \Delta \frac{\partial}{\partial \Delta} - \Omega_2 \mathcal{M} \frac{\partial}{\partial \mathcal{M}} + h R \frac{\partial}{\partial R} + 2 \right] F_s [ \Delta, \mathcal{M}, R ] = 0 ,
\] 

where, \( \Omega_i \)s are eigenvalues of the matrix \( \Omega_{k,l} \). They are nothing but the scaling dimensions of the relevant operators. The general expressions of \( \Omega_{k,l} \)s and the eigenvalues \( \Omega_1, \Omega_2 \), for different fixed points (both Gaussian and the nontrivial one) for a general nonzero \( h \) are evaluated in the Appendix B. For \( h = 0 \), the \( \beta_R \) term drops out from the Callan Symanzik equation. The singular behavior with respect to the renormalized cosmological constant goes as,

\[ F_s \sim \Delta^{2/\Omega_1} f_1 [N^{\Omega_1} \Delta ] f_2 [N^{\Omega_2} \mathcal{M} ] . \] 

Comparing the above expression of \( F_s \) with the matrix model result \( F_s \sim \Delta^{(2-\gamma_0)} f [N^{2/\gamma_3} \Delta] \), or using the standard definition of the susceptibility \( \Gamma \sim \frac{\partial^2 F}{\partial \Delta^2} |_{\mathcal{M} = 0} \sim \Delta^{-\gamma_0} \), the string susceptibility exponent \( \gamma_0 \) is given by

\[ \gamma_0 \sim (2 - 2/\Omega_1) . \]

Note that in our analysis \( 2/\gamma_1 \sim \Omega_1 \), i.e. \( \gamma_1 \sim 2/\Omega_1 \) is consistent with the matrix model relation \( \gamma_0 + \gamma_1 = 2 \). This relation is independent of the explicit values of \( \gamma_0 \) and \( \gamma_1 \) and is easily obtainable from the consideration of the torus. The string susceptibility exponent at genus \( G \) is defined by

\[ \gamma_G = \gamma_0 + G \gamma_1 . \]

Referring to the Appendix B, we observe that, for our nontrivial fixed point, \( \Omega_1 = 1 - 5h = 1 \) \((h = 0)\), and hence \( \gamma_0 = 0 \). This shows that our nontrivial fixed point is a \( c = 1 \) fixed point at any \( R \) for \( h = 0 \). This \( c = 1 \) nontrivial fixed points are repulsive with respect to the flow of the parameter \( g \) while the gaussian fixed point is attractive. Similarly one can analyze the critical exponents for the scaling function \( \mathcal{M} \), the boundary cosmological constant, which will characterize different boundary conditions. Such an analysis will be useful to understand all possible boundaries and to identify the Neumann and Dirichlet case.

Note that in this RG, typically the nontrivial fixed point is always situated close to the Gaussian fixed point. Hence, \( \Omega_{12}, \Omega_{22}, \Omega_{21} \) components of the scaling dimension matrix are small. Thus \( \Omega_{11} \sim \Omega_1 \) and \( \Omega_2 \) is also small.
3.2 The logarithmic scaling violation of the singlet free energy

In the previous section, the behavior $F_s = \Delta^2$ of the singular part of the free energy near the $c = 1$ nontrivial fixed point ($\gamma_0 = 0$) as a function of the renormalized couplings or the scaling variables is consistent with the continuum prediction, the KPZ-DDK scaling law [5, 6, 7]. This power law dependence on $\Delta$ is present in all known $c < 1$ theories. However, for $c = 1$, matrix model predicts a logarithmic deviation to the usual power law scaling [22, 23, 24, 25, 28]. This is known as the logarithmic scaling violation of the $c = 1$ matrix model. In terms of the singlet free energy or the ground state energy $E_0$,

$$-N^2 F_s = E_0 = -N^2 \pi^2 \frac{\Delta^2}{\ln \Delta} + \ldots,$$

or

$$E_0 = -N^2 \frac{\mu^2 \ln \mu}{4\pi} + \ldots,$$

where $\Delta = -\frac{1}{2\pi \mu \ln \mu}$. (3.20)

In the inverse Laplace Transform with respect to the area $A$, the sum over surfaces of fixed area $A$ behaves as

$$\mathcal{F}(A) \sim \frac{1}{A^3 (\ln A)^2}$$

(3.21)

The continuum methods seem to be insensitive to this peculiar dependence.

These scaling violations of the $c = 1$ matrix model can be naturally explained considering the lack of the translational invariance in the Liouville dimension $\phi$ of the 2D string theory, that is manifested by the $\phi$ dependence of the background fields and is needed to maintain the conformal invariance. Here, the Liouville dimension arises indirectly from the semiclassical dynamics of the matrix eigenvalues while the Euclidean time of the matrix quantum mechanics provides the other dimension. The logarithmic factors in the expression of energy actually reflects the logarithmic divergence of the Liouville volume as the critical point is approached. The scaling violation was argued to arise from the unusual dependence of the tachyon potential on the Liouville field, $T(\phi) \sim \phi e^\phi$ [27]. Later it was shown that [23, 54] considering interactions that represent touching random surfaces give rise to new critical behavior. This corresponds to the other branch where tachyon potential has the ordinary Liouville form, $T(\phi) \sim e^\phi$, with simpler scaling

$$F_s(\Delta) \sim \Delta^2 \ln \Delta, \quad F(A) \sim \frac{1}{A^3}.$$  

(3.22)

We would now like to see how one can get this behavior peculiar to $c = 1$ from the Callan-Symanzik equation. The usual scaling violation, $F_s \sim \Delta^2 / \ln \Delta$, directly follows from the solution of Callan-Symanzik equation (3.12) near the $c = 1$ nontrivial fixed point

$$\left[ N \frac{\partial}{\partial N} - \Delta \frac{\partial}{\partial \Delta} \right] N^2 F_s = 0,$$

up to a term that vanishes in the scaling limit.
Also it is observed that the renormalization of $\lambda$ giving rise to the right hand side of the inhomogeneous Callan-Symanzik equation (3.12) can contribute logarithmic terms in the behavior of the singlet free energy [37]. The reason right hand side is capable of contributing to the free energy (universal physics) is that it actually measures the change of the world sheet free energy with the change of scale and can be expressed in the same footing as the terms in the left hand side of the Callan-Symanzik equation

$$\beta_\lambda \frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial \left(\frac{1}{N}\right)}.$$ (3.23)

Thus, using our Callan-Symanzik equation (3.12) and the linearized beta function equations (2.35) around the nontrivial fixed point (3.1) we have,

$$\left[N \frac{\partial}{\partial N} - \Delta \frac{\partial}{\partial \Delta}\right] N^2 F = \beta_\lambda \frac{\partial (N^2 F)}{\partial \lambda} = -\Delta \hat{F}_{g^3}(R) g^* + \Delta^2 \hat{F}_{g^3} \frac{1}{2}$$ (3.24)

This gives

$$2F - \Delta \frac{\partial F}{\partial \Delta} = N^{-2} \left(-\Delta \hat{F}_{g^3}(R) g^* + \Delta^2 \hat{F}_{g^3} \frac{1}{2}\right),$$

$$\Rightarrow \quad \partial (F/\Delta^2) = N^{-2} \left(\frac{\hat{F}_{g^3}(R)}{2\Delta} - \frac{\hat{F}_{g^3}(R) g^*}{\Delta^2}\right) \partial \Delta.$$ (3.25)

Integrating both sides gives rise to the scaling

$$E_0 = -N^2 F \sim -\frac{1}{2} \hat{F}_{g^3}(R) \Delta^2 \ln \Delta.$$ (3.26)

However, this dependence does not give rise to the logarithm in the inverse Laplace transform with respect to the area. This is rather the behavior corresponding to the other branch of the tachyon of the Liouville theory. Perhaps this reflects the fact that even though we are not explicitly considering the branching interactions representing the touching random surfaces generated by the redundant higher order terms, they have indirect effect in the change of the world sheet free energy.

Note that in this analysis we are not looking at the $M$ dependence of the free energy since typically $\Omega_2$ is very small and hence can be ignored in (3.12) or in (3.16). Alternatively, one can also consider a fixed mass parameter $M$ in doing the analysis.

3.3 The Radius Dependence of the free energy

As we have discussed before in context of the analysis of the structure of the beta function equations, one can observe an indirect impact of the world sheet vortices on the free energy in
the sense that it flips its sign at the self-dual radius \( R = 1 \). This radius is known to be the inverse temperature for the Kosterlitz-Thouless phase transition due to the liberation of the world-sheet vortices \cite{29}. To observe this we would demand that the mass parameter \( M \) is fixed to a certain value (for simplicity we choose \( M = 1 \)) while \( \beta_{M^2} \) is set to zero. Then for nontrivial fixed points of \( \beta_g = 0 \), the parameter \( h \) can be determined as a function of \( R \) by solving the two simultaneous equations \( \beta_g = 0 \) and \( \beta_{M^2}|_{M=1} = 0 \) for the two unknowns \( h(R) \) and \( g^* \). This gives,

\[
h(R) = \frac{1}{\left( 4 \frac{F_{g3}(R)}{\dot{F}_{g2}(R)} - 1 \right)}.
\]

(3.27)

In low \( R \) approximation \( h(R) \sim 1/R^2 \) and thus the change of the world-sheet free energy around the nontrivial fixed point becomes (considering \( g^* \) to be very small)

\[
\beta_\lambda \approx \frac{1}{2} (1/R^2 - 1).
\]

(3.28)

This indicates a phase transition at \( R = 1 \).

![Figure 3: Behavior of \( h(R) - 1 \) at relatively large \( R \).](image)

However, a plot of \( \beta_\lambda \approx \frac{1}{2}(h(R) - 1) \) using a simplified \( h(R) \) (\( h(R) \sim \frac{1}{-1+1.27/R} \)) in relatively large \( R \) expansion shows the discontinuity at \( R = 1.2 \) (Figure 3). In this discussion the logarithmic constant in the equation for \( \beta_\lambda \) (2.35) is ignored as it merely indicates some vacuum normalization and can be conveniently absorbed. Another important point to be mentioned is that the high \( R \) approximation for \( h(R) \) is not quite efficient to compute the correct exponent of the \( c = 1 \) fixed point (in the sense that \( h(R) \) does not go to zero for large \( R \) instead approaches \(-1\) in that limit). Since this is purely dependent on the contribution of the single line Feynman diagrams in integrating out the quark loops, we need further improvement in the computation of the diagrams.
4 The T-duality of the singlet partition function

In this section we would analyze that in what sense our RG can capture the T-duality respected by the singlet sector of the MQM. Let us consider the matrix partition function in the eigenvalue representation by diagonalizing the matrices by $SU(N)$ transformation and suitably integrating out the angular degrees of freedom. The diagonalization of $\Phi(t)$ gives,

$$\Phi_{ij}(t) = \sum_{k=1}^{N} \Omega_{ik}^\dagger(t) \lambda_k(t) \Omega_{kj}(t).$$  \hspace{1cm} (4.1)

Hence,

$$\text{Tr} \Phi^2 = \sum_{k=1}^{N} \lambda_k^2 + \sum_{k \neq j} |\lambda_k - \lambda_j|^2 |A_{kj}(t)|^2, \quad A_{kj}(t) = (\Omega_{ij}^\dagger(t) \Omega(t))_{kj}. \hspace{1cm} (4.2)$$

Here the gauge field $A$ acts as a lagrange multiplier that projects onto $SU(N)$ singlet wave functions which depend on the $N$ matrix eigenvalues only. In the recent works on $N \times N$ matrix quantum mechanics as the quantum mechanics of string tachyon field on the $N$ unstable $D0$-branes, this gauge field is identified as the non-dynamical gauge field in the open string spectrum corresponding to the vertex operator $i \cdot t$. Due to the periodic boundary condition on the $\Phi(t)$ and on the $\Omega(t)$-s the gauge field $A_{ij}(t)$-s are constrained:

$$\hat{T} \exp i \int_0^{2\pi R} A(t) \ dt = I. \hspace{1cm} (4.3)$$

Taking into account of the above constraint (which actually projects to the singlet sector) in the measure of the path integral, the (singlet) partition function can be formally written as (see the review [8]):

$$Z_N(g, R) = \sum_{r} d_r \ d_r^{(0)} \int \prod \mathcal{D} \lambda_k(t) e^{- \int_0^{2\pi R} dt [N \sum_i^{N} (\frac{1}{2} \lambda_i^2 + V(\lambda_i)) + \frac{C_r^{(2)}}{2N^2} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j^2}]}, \hspace{1cm} (4.4)$$

where, the $d_R$ and $d_R^{(0)}$ are the dimensions of the $R$-th representation of $U(N)$ and that of the subspace of vectors with the zero weights in $R$ and $C_r^{(2)}$ is the value of the quadratic Casimir operator in the $r$-th representation.

$$C_r^{(2)} \simeq N. \hspace{1cm} (4.5)$$

Note that, in this process, all the Vandermonde determinants are gone but the fermionic statistics is still maintained for the eigenvalues are now periodic:
\[ \lambda_i(0) = \lambda_i(2\pi R), \quad i = 1, \ldots, N. \] (4.6)

In a similar sense to our scheme, let us now begin with the partition function of \((N + 1)\) eigenvalues and observe the effect of integrating out one of the eigenvalues in presence of the rest of the \(N\) eigenvalues. Physically this would mean integrating out one of the fermions in the presence of the interaction due to the rest of the \(N\) fermions. Then the Fermi sea readjusts its height and this renormalizes the parameters of the theory like the cosmological constant. In recent works a similar picture has been discussed in context of closed string radiation due to the rolling open string tachyon on unstable \(D_0\) branes represented by the unstable matrix eigenvalues. We will come to that in the next chapter.

Now the partition function of \((N + 1)\) eigenvalues can be split as,

\[
Z_{N+1}(g,R) = \sum_r d_r \int \prod_{k=1}^N \mathcal{D} \lambda_k(t)e^{-\int_0^{2\pi R} dt \left[ (N+1) \sum_i^N \left( \frac{1}{2} \dot{\lambda}_i^2 + \frac{1}{2} \lambda_i^2 - \frac{g^3}{2(N+1)^2} \sum_{i \neq j} \frac{1}{(\lambda_i - \lambda_j)^2} \right) \right]}
\]

\[
\int \mathcal{D} \lambda_{N+1}(t)e^{-\int_0^{2\pi R} dt \left[ (N+1) \left( \frac{1}{2} \dot{\lambda}_{N+1}^2 + \frac{1}{2} \lambda_{N+1}^2 - \frac{g^3}{2(N+1)^2} \sum_{j=1}^N \frac{1}{(\lambda_{N+1} - \lambda_j)^2} \right) \right]}. \] (4.7)

Let us consider the simplest form of real (Hermitian Matrix model) and periodic eigenvalue,

\[ \lambda_{N+1}(t) = \lambda_{N+1} \cos t/R. \] (4.8)

Then the measure over the \((N + 1)\)-th eigenvalue trajectory (to be integrated out) becomes,

\[ \mathcal{D} \lambda_{N+1}(t) = \prod_{k=1}^N d\lambda_{N+1} \cos(2\pi k/N) = \prod_{k=1}^N d\lambda_{N+1} (1 - O(1/N^2)) \sim \prod_{k=1}^N d\lambda_{N+1}. \] (4.9)

Hence the integration over \(\lambda_{N+1}\) gives,

\[
I_{N+1} = \prod_{k=1}^N \int d\lambda_{N+1} e^{-(N+1) \left[ \frac{1}{2} \dot{\lambda}_{N+1}^2 + \int_0^{2\pi R} dt \left( \frac{1}{R} \sin^2 t/R + \cos^2 t/R \right) - \frac{g^3}{2(N+1)^2} \int_0^{2\pi R} dt \cos^3 t/R \right]}
\]

\[
eq \prod_{k=1}^N \int d\lambda_{N+1} e^{-\frac{1}{2(N+1)^2} \left[ \sum_{j=1}^N \frac{1}{(\lambda_{N+1} - \lambda_j)^2} \int_0^{2\pi R} dt \cos^2 t/R \right]} = \left( \frac{2}{(N+1)(R + 1/R)} \right)^{N/2} \lambda_{N+1}^N. \] (4.10)

As before, considering the rescaling of the fields

\[ \lambda_i(t) \rightarrow \rho \lambda_i(t), \quad i = 1, \ldots, N, \] (4.11)

and setting the coefficient of \(\frac{1}{2} \dot{\lambda}_i^2(t)\) to one (which in this simplest case simultaneously sets the coefficient of \(\frac{1}{2} \lambda_i^2(t)\) to one too, and the mass term does not run with the scale change) we have,
\[ \rho = (1 + \frac{1}{N})^{-\frac{3}{2}} \simeq (1 - \frac{1}{2N} + O(1/N^2)) , \]  

(4.12)

Hence, the partition function can be rewritten as,

\[ Z_{N+1}(g, R) = \left(\frac{2}{(N+1)(R+1/R)}\right)^{\frac{2N}{N+1}} (1 - \frac{1}{2N} + O(1/N^2))^N \]

\[ \sum_r d_r (d_r^{(0)})^N \prod_{k=1}^N D\lambda'_k(t) e^{-\int_0^{2\pi R} dt \left[N \sum_i^N (\frac{1}{4} \lambda_i'^2 + \frac{1}{4} \lambda_i''^2 - \frac{2g}{N} \lambda_i') + \frac{g}{2N} \sum_{i \neq j} (\lambda_i - \lambda_j)^2 \right]} , \]

\[ g' = g - \frac{g}{2N} + O(\frac{1}{N^2}) , \quad C' = 1 - \frac{1}{N} + O(\frac{1}{N^2}). \]

(4.13)

This leads to,

\[ \frac{Z_{N+1}(g, R)}{Z_N(g', R)} = \left(\frac{2}{(N+1)(R+1/R)}\right)^{\frac{2N}{N+1}} (1 - \frac{1}{2N} + O(1/N^2))^N = \left(\frac{1}{N^{2N}} \left(\frac{2}{R+1/R}\right)^{\frac{2N}{N+1}} (1 - \frac{1}{N})^{\frac{1}{N}}\right)^N. \]

(4.14)

Note that, due to the cosine, which is the simplest choice for the eigenvalue \( \lambda_{N+1} \), both the interaction terms corresponding to the cubic self-interaction of \( \lambda_{N+1} \) and the mutual (repulsive coulomb) interaction with the rest of the \( N \) eigenvalues drop out in the integration \( I_{N+1} \). As a result in the equation (4.7), the integration over the \( N + 1 \)-th eigenvalue becomes just a overall prefactor (contributing to the change of the world-sheet free energy associated with the readjustment of the Fermi level in response to loosing one of the fermions) as in (4.13), unlike the more general situations studied in the previous chapters where the integration over a part of the matrix degrees of freedom does produce nontrivial one loop correction terms that adds up to the left over \( N \)-fermion partition \( Z_N(g', R) \) besides producing the overall prefactor. Even in simple eigenvalue representation, a more general choice of the functional form of \( \lambda_{N+1} \) would produce such correction terms. In those cases the beta function for \( g \), \( \beta_g = \frac{\delta g}{\delta(\frac{1}{g})} = -\frac{g}{2} + \ldots \), will have higher order terms in \( g \) (which can give nontrivial fixed points). Note that the leading term in the beta function, \( -\frac{g}{2} \), is the same in any case.

Now the equation (4.14) implies the Callan-Symanzik equation for the world sheet free energy \( F[g, R] = \frac{1}{N^2} \ln Z[g, R] \):

\[ \left[ N \frac{\partial}{\partial N} - \beta_g \frac{\partial}{\partial g} + \gamma \right] F[g, R] \simeq \frac{1}{2} \ln \left(\frac{2}{R+1/R}\right) + O(1/N) , \quad \beta_g \simeq -\frac{g}{2} , \quad \gamma = 2. \]

(4.15)

In terms of the scaling variable \( \Delta = (1 - g/g^*) \), the Callan-Symanzik equation in the Double scaling limit is given by,

\[ \left[ N \frac{\partial}{\partial N} - 2 \Delta \frac{\partial}{\partial \Delta} + \gamma \right] F[\Delta, R] \simeq \frac{1}{2} \ln \left(\frac{2}{R+1/R}\right) . \]

(4.16)
Solution of the homogeneous equation is same as before. Using the same ansatz as before, \( \mathcal{F}[^{\Delta, R}] = N^{-2}f_1(\Delta)f_2(R) \), and solving the inhomogeneous part we observe that the world sheet free energy has a functional form of \( f(R + 1/R) \) with respect to \( R \):

\[
\mathcal{F}[\Delta, R] = -\ln \Delta \ln \left( \frac{2}{R + 1/R} \right). \tag{4.17}
\]

This shows that the large \( N \) RG is capable of capturing the \textit{T-duality} property respected by the singlet partition function. However, two comments are notable here. First of all, in this simplest case, the beta function does not receive higher order corrections in the coupling constant. Hence the fixed point is a trivial Gaussian fixed point with inadequate critical exponent \((\Omega_1 = -\frac{1}{2}, \gamma_0 = 2 - 2/\Omega_1 = 6)\) to describe the \( c = 1 \) fixed point \((\gamma_0 = 0)\). Actually, from the results of the more general set up in the previous chapters, this Gaussian fixed point could be thought of to be overlapping with a pair of double zero of the beta function at infinitesimal distance from it for large \( R \) (where the singlet free fermion picture is meaningful). With a more general choice of the functional form of the eigenvalue, the gaussian fixed point will be resolved into a nontrivial double zero of the beta function with appropriate critical index to describe \( c = 1 \), as obtained in the previous chapters. This \( c = 1 \) fixed point would explicitly show \textit{T-duality} in exactly the same manner from the gaussian part of the integration over the \( N + 1 \)-th eigenvalue in presence of the rest of the \( N \) eigenvalues. Secondly, we note that the T-duality arises in the usual sense from the kinetic and the quadratic term of the action in the wave function of the \( N + 1 \)-th fermion. This wave function can be compared with the free fermion wave functions for the singlet sector of \( c = 1 \) on large circle. A small observation is that in the more general set up (with \( M = 1 \) as in here), the gaussian part of the integration over a part of the matrix degrees of freedom produces the functional form \( f(\sinh \frac{\pi R}{\pi R^2}) \approx f \left( \frac{1}{\pi R} + \frac{\pi R}{3} O(R^2) \right) \) (to compare see equations (2.34)), instead of producing \( f(R + 1/R) \). This shows how the T-duality is broken at this level.

## 5 Comments on D-brane Decay and Rolling Tachyon

The recent observations \([8, 40, 41, 39]\) show that the matrix model can be realized as the effective dynamics of \( D \)-branes in \( c = 1 \) non-critical string theory. We briefly review the basic picture before going into the matrix model RG interpretation of the open string rolling tachyon in string theory.

### 5.1 Review of D-brane Decay in 2D String Theory

According to this conjecture, the matrix \( \Phi_N \) itself can be seen as an open string tachyon field on \( D0 \)-branes and the matrix potential as the tachyon potential. The \( SU(N) \) symmetric matrix
quantum mechanics in an upside down inverted harmonic oscillator potential, i.e. in the double scaling limit, exactly describes open strings on $N$ $D0$-branes and is dual to Liouville theory coupled to $c = 1$ matter, that describes the two dimensional closed string theory together with its $D0$-branes. The $D0$-branes, on which the tachyon field resides, are described by the localized boundary states for the Liouville coordinate $\phi$ introduced by A.B. and Al.B. Zamolodchikov \cite{15} tensored with Sen’s rolling tachyon boundary states with Neumann boundary condition for the free time direction $t$. They are localized in the strong coupling region $\phi \to \infty$ far from the bulk region of large and negative $\phi$ (signalled by the absence of ”bulk poles” in the one-point function of the closed string vertex operator on the disc). These branes are parametrized by the pair of integers $(m, n)$. Among them only the set $(1, n)$ has smooth behavior with no singularity in the classical limit $b \to 0$, where $b$ measures the ”rigidity” of the 2$D$ surface to quantum fluctuations of the metric. The one labelled as $(1, 1)$ contains operator that matches with the massless tachyon in the open string spectrum and is the only one which has been found to be consistent with the standard loop perturbation theory and consequently is identified as the boundary state at one-dimensional infinity or the ”absolute” of Euclidean $AdS_2$ (i.e. classical Lobachevskiy plane) \cite{15}. It is still not clear what the other boundary states with $m \neq 1, n \neq 1$ correspond to.

There is also another class of Liouville boundary states \cite{55}, extended in the bulk weak coupling region of $\phi \to -\infty$ (exhibited by the poles in the bulk one point function). They correspond to $D$-strings when the time direction is taken to be Neumann. The continuous families of such boundary states are parametrized by the uniformization parameter $s$, given by

$$\mu_B = \sqrt{\mu} \cosh \pi s.$$  \hspace{1cm} (5.1)

Here $\mu_B$ is the renormalized boundary cosmological constant and $\mu$ is the renormalized bulk cosmological constant in the Liouville theory:

$$\mu_B \equiv (\pi \mu_{B0} \gamma(b^2)),$$

$$\mu \equiv (\pi \mu_0 \gamma(b^2)),$$

$$\gamma(b^2) = \frac{\Gamma(b^2)}{\Gamma(1-b^2)},$$  \hspace{1cm} (5.2)

in the sense that, in order to get finite amplitude, they are kept finite in the limit $b \to 1$, i.e. $c_L \to 25$. All such $D$-branes in this family have a continuous spectrum of open string states, namely, the massless open string tachyons. For the choice of $s = \frac{i}{2}(2n + 1)$, $m \in \mathbb{Z}$, one can set $\mu_B = 0$. The work of \cite{8} uses the minimal values $s = \pm \frac{i}{2}$. Increasing $m$ gives rise to increasingly tachyonic open string modes.
As the tachyon is rolling down the potential, one can identify the boundary cosmological constant $\mu_B$ of the Liouville theory with the time dependent tachyon potential $\lambda \cosh t$ as follows:

$$\mu_B(t) = \sqrt{\mu} \cosh s(t) = \lambda \cosh t. \quad (5.3)$$

As $t \to -\infty$, $\mu_B \to 0$, which corresponds to unstable $D$-string picture. As $t \to \infty$, $\mu_B$ and $s$ grow to large values. In this process, the unstable $D$-string one starts with, decays into closed string vacuum of Liouville theory with its localized $D0$-branes and all the open string excitations on the $D$-string are pushed to $\phi \to -\infty$. In the matrix model picture, an unstable brane corresponds to a free fermion on the top of the inverted harmonic oscillator potential rolling down as an unstable eigenvalue $z(t) = \lambda e^t$ from the top to the Fermi level $\mu_F$.

The amplitude for such $D$-brane tachyonic decay with momentum $P$ and energy $\omega = |P|$ is given by the expectation value of a normalizable bulk operator $v_{\omega,P} = \exp[(2 + iP)\phi + i|P|t]$ inside the disk. As was shown in [40], this disk one-point function agrees with the outgoing radiation amplitude derived from $c = 1$ matrix model (taking into account the leg-pole factors). The disk one-point function in the two dimensional string theory is given as follows:

$$A(\omega, P) = A_t(\omega)A_L(P) = \langle v_{\omega}|B_s\rangle\langle v_P|B_s\rangle, \quad (5.4)$$

where the two factors denote the time part (obtained in [56]) and the Liouville part (obtained in [53] and [45] for the two kinds of branes) respectively. Although the end result of tachyonic decay for both the branes are the same and the time part of the amplitude is identical, the Liouville part or the bulk one point functions for the two cases differs with the absence of pole for the localized brane.

The time part of the amplitude $A_t(\omega) = \langle v_{\omega}|B_s\rangle$ depends on the choice of contour [56] and is described by the boundary interaction $\hat{\lambda} \cos t/\alpha'$. Note that, in our convention, $t$ denotes the Euclidean time. In the Lorentzian coordinate $X^0 = it$, the boundary interaction is given by $\hat{\lambda} \cosh X^0/\alpha'$. The choice of Hartle-Hawking contour gives rise to the following result [56]:

$$A_t(\omega) = \langle \exp[i\omega t]\rangle = \frac{\pi \exp[-i\sqrt{\alpha'} \omega \log \hat{\lambda}]}{\sinh \pi \sqrt{\alpha'} \omega}, \quad (5.5)$$

having pole at $\omega = 0$, signaling the presence of continuous spectrum in the on-shell open string channel [57, 41].

The Liouville part of the amplitude for $D1$-brane decay (for the choice of $s = \pm i/2$) can be evaluated from the on-shell CFT calculation of the boundary state constructed by Fateev, Zamolodchikov and Zamolodchikov [54] resulting the total amplitude to be

\footnote{Here $\hat{\lambda} = \sin \pi \tilde{\lambda}$ in order to match with the trajectory considered in [10].}
\[ A_{D1}(\omega, P) = 2\pi^2 e^{-i\delta(P)} e^{-i\sqrt{\alpha'}|P|\log \lambda} \frac{\cos \pi \sqrt{\alpha'}sP}{\sinh \pi \sqrt{\alpha'}\omega \sinh \pi \sqrt{\alpha'}P}. \] (5.6)

The corresponding phase factor is given by
\[ e^{-i\delta(P)} = i\mu^{-i\sqrt{\alpha'}P/2} \frac{\Gamma(i\sqrt{\alpha'}P)}{\Gamma(-i\sqrt{\alpha'}P)}. \] (5.7)

The corresponding disk one-point function for the D0-brane has no pole [40]. The Liouville part is described by the Zamolodchikov boundary state [45] and the total amplitude is given by
\[ A_{D0}(\omega, P) = -\text{sgn}(P)\pi e^{-i\delta(P)} e^{-i\sqrt{\alpha'}|P|\log \lambda}. \] (5.8)

Let us now try to understand this from matrix quantum mechanics picture. The physics of the matrix model in eigenvalue representation is that of \( N \) non-interacting fermions, moving in the potential \( V(\lambda) \). In the double scaling limit, the potential is that of an inverted harmonic oscillator. Each fermion occupies a volume of \( 2\pi/\beta = 2\pi g^2/N \) in phase space, where \( 1/\beta \) plays the role of \( \hbar \). All levels with single-particle energy \( \frac{1}{2}v^2 + V < \mu_F \) are filled. As \( g \) increases or \( \beta \) decreases, the phase space volume occupied by the fermionic states grow. At the critical value, \( g \to g_c \), the fermi level reaches the top of the potential, \( \mu_F \to \mu_c \), where the fermions start to spill over the barrier. The free energy is singular at this point. In the double scaling limit, \( N \to \infty \) and \( g \to g_c \), holding the string coupling \( g_s \equiv \bar{\mu}^{-1} = [\beta(\mu_c - \mu_F)]^{-1} \) fixed [10]. In other words, the ratio of the gap between fermionic levels (which is of order \( \hbar \sim 1/\beta \)) to \( \mu = \mu_c - \mu_F \) remains constant in the double scaling limit. In this limit the physics is that of the Fermi sea in inverted harmonic oscillator with the hamiltonian \( H(v, \lambda) = (v^2 - \lambda^2)/2 + \bar{\mu} \), where \( \lambda, v \) are canonical conjugate variables. The collective motions of the fermions are classically described in terms of a time dependent Fermi surface that separates the filled and the empty regions in the phase space. The fermions on the surface moving freely in the inverted harmonic oscillator potential are described by,
\[ D_t v = \lambda, \quad D_t \lambda = v, \] (5.9)
where \( D_t \) denotes the co-moving derivative following a phase point on the surface. Hence the free fermions on the Fermi surface execute simple hyperbolic orbits. However the time dependent profile of the Fermi surface is more complicated. For small perturbations, the profile can be described by the positions \( v_\pm(\lambda, t) \) of the upper and lower surfaces of the Fermi sea at each time, satisfying,
\[ \partial_t v_\pm(\lambda, t) = \lambda - v_\pm(\lambda, t)\partial_\lambda v_\pm(\lambda, t). \] (5.10)

\[ 2 \text{In terms of } \Delta = g - g_c \simeq \mu \ln \mu, \text{ we have kept } g_s = [N \Delta^{(2-\gamma_0)/2}]^{-1} = (N \Delta^{1/\Omega_{11}})^{-1} \text{ fixed in our RG analysis in previous sections.} \]
Therefore, the static Fermi sea satisfying the equation of motion (5.10) is given by,

\[ v_{\pm}(\lambda) = \pm \sqrt{(\lambda^2 - 2\bar{\mu})}, \quad \text{i.e.} \quad \frac{1}{2}(v^2 - \lambda^2) + \bar{\mu} = 0, \tag{5.11} \]

which also given by the ground state of the hamiltonian \( H(v, \lambda) = (v^2 - \lambda^2)/2 + \bar{\mu} \). For simplicity, we assume that, in the ground state, the local maximum of the potential is at zero energy level (\( \mu_c = 0 \)), and the energy at the Fermi level is \( \mu_F = -\mu \), measured from the local maximum. Identifying

\[ v(\lambda) = \frac{\partial \lambda}{\partial \tau} = \sqrt{(\lambda^2/\alpha' - 2\bar{\mu})} \tag{5.12} \]

as the velocity of the classical trajectory of a particle at the Fermi level, we define a new spatial coordinate \( \tau \), the classical time of motion at the Fermi level.

A single D0-brane state of the continuum theory corresponds to a single eigenvalue or a fermion excited from the Fermi surface to higher energy, \textit{e.g.} to the top of the upside down potential. This corresponds to putting the Neumann boundary condition on the world sheet field \( \tilde{X}(\sigma_1, \sigma_2) \). Turning on an exactly marginal boundary operator that interpolates between the Neumann and Dirichlet boundary conditions on \( \tilde{X}(\sigma_1, \sigma_2) \) makes the unstable brane to roll down to D-objects localized at large Liouville direction \( \phi \). This corresponds to considering an unstable eigenvalue as the probe fermion decaying in the presence of the \( N \) free fermions, as it executes classical Euclidean motion in the forbidden region along the trajectory

\[ \lambda(t) = \sqrt{2\alpha'/\bar{\mu}} \hat{\lambda} \cos(t/\alpha'). \tag{5.13} \]

In order to relate to the picture of the rolling tachyon in \([46]\), the rolling boundary state parametrized by \( \hat{\lambda} \), where \( \sin \pi \hat{\lambda} = \hat{\lambda} \), is identified with the decaying eigenvalue starting at

\[ \lambda = -\sqrt{2\alpha'/\bar{\mu}} \hat{\lambda}, \tag{5.14} \]

where the energy of the state is given by

\[ E = \bar{\mu} \cos^2 \pi \hat{\lambda}. \tag{5.15} \]

The time delay in the classical evolution of the trajectory relative to the classical trajectory at the Fermi level is therefore

\[ -\Delta t = -\int^\tau \frac{d\tau'}{\sqrt{\alpha'}} = \int \frac{d\hat{\lambda}}{\sqrt{\hat{\lambda}^2 - 1}} \approx \ln \hat{\lambda}, \tag{5.16} \]

for large \( \hat{\lambda} \). The parameter \( \hat{\lambda} \in [0, 1] \). The case \( \hat{\lambda} = 0 \), \textit{i.e.} \( \hat{\lambda} = 1 \), describes an eigenvalue at the top of the inverted harmonic oscillator potential \((V = 0, E = -\bar{\mu})\) and \( \tilde{\lambda} = 1 \), \textit{i.e.} \( \tilde{\lambda} = \frac{1}{2} \), describes and eigenvalue on the Fermi sea \((V = -\bar{\mu}, E = 0)\).
To relate such physical process in the free fermion theory to that of the $D = 1$ noncritical string theory in the dilaton and tachyon background, one needs to bosonize the non-relativistic free fermions in a suitable way. One way to view this is to express the small fluctuations of the time dependent Fermi surface in terms of the massless scalar $X(\tau, t)$, where $\tau = \ln \hat{\lambda}$ \[14\]. In this respect equation (5.10) can be generalized to the time dependent case as

$$v_\pm(\lambda, t) = \mp \lambda \pm \frac{1}{\lambda} \epsilon_\pm(\tau, t),$$

where

$$\frac{1}{\sqrt{\pi}} \epsilon_\pm(\tau, t) = \pm \partial_t X(\tau, t) - \partial_\tau X(\tau, t).$$

(5.17)

The right and left moving fluctuations $\epsilon_\pm(\tau, t)$ are proportional to the right and left moving currents, $J_R = : \psi_R^\dagger \psi_R :$ and $J_L = : \psi_L^\dagger \psi_L :$, generated by the chiral fermionic variables as given by the bosonization relation

$$
\begin{align*}
\psi_R &= \frac{1}{\sqrt{2\pi}} : \exp \left[ i\sqrt{\pi} \int d\tau' (\partial_t X - \partial_\tau X) \right] : \\
\psi_L &= \frac{1}{\sqrt{2\pi}} : \exp \left[ i\sqrt{\pi} \int d\tau' (\partial_t X + \partial_\tau X) \right] : \\
\end{align*}
$$

(5.19)

(This is because $(J_L + J_R) = \sqrt{\pi} \partial_\tau X$ and $(J_L - J_R) = \sqrt{\pi} \partial_t X$). The second quantized hamiltonian for the free fermions is written in collective fields $\psi(\lambda, t) = \sum_i a_i \psi_i(\lambda) e^{-i\epsilon_i t}$, where $\psi_i$ are the single particle wave function and $a_i$ are the corresponding annihilation operators. One can expand the fermionic field $\psi(\lambda, t)$ in $\psi_R$ and $\psi_L$ as

$$
\psi(\lambda, t) = \frac{e^{i\mu t}}{\sqrt{2v(\lambda)}} \left[ \exp[-i \int d\lambda' v(\lambda') + i\pi/4] \psi_L(\lambda, t) + \exp[i \int d\lambda' v(\lambda') - i\pi/4] \psi_R(\lambda, t) \right].
$$

(5.20)

and rewrite the second quantized hamiltonian

$$
H \sim \int d\tau \left[ i(\psi_R^\dagger \partial_\tau \psi_R - \psi_L^\dagger \partial_\tau \psi_L) + \frac{1}{2v^2} (\partial_\tau \psi_L^\dagger \partial_\tau \psi_L + \partial_\tau \psi_R^\dagger \partial_\tau \psi_R) \\
+ \frac{1}{4} \left( \frac{v''}{v^3} - \frac{5v'^2}{2v^4} \right) (\psi_L^\dagger \psi_L + \psi_R^\dagger \psi_R) \right].
$$

(5.21)

Here $v' \equiv dv/d\tau$. Solving the equation (5.16) for $\hat{\lambda}$, $\hat{\lambda} = \cosh(\tau/\sqrt{\alpha'}) \simeq \exp \tau$, $\tau \to \infty$. Hence $v = \sqrt{2\bar{\mu}} \sinh \tau \simeq \sqrt{2\bar{\mu}} \exp \tau$. This shows that the $O(1/v^2)$ terms are not negligible in the double scaling limit where we keep $\bar{\mu}$ fixed, unless we go to the asymptotic or the $\tau \to \infty$ region. This maps the second quantized hamiltonian of $N$ non-relativistic free fermions to the two dimensional Dirac hamiltonian. This shows that $\tau$ is the natural spatial coordinate in terms of which the fermionic system has a standard Dirac action to leading order in $\beta$. Also in
terms of the bosonization variables the hamiltonian in the $\tau \to \infty$ limit takes the form of the canonically normalized free scalar hamiltonian

$$H \sim \frac{1}{2} \int d\tau \left[ (\partial_t X)^2 + (\partial_\tau X)^2 + e^{-\tau} O(X^3) \right], \quad (5.22)$$

with the equation of motion $(\partial_t^2 - \partial_\tau^2)X = e^{-\tau} O(X^2)$. This also shows that $\tau$ can be identified with the zero mode of the Liouville field [23]. One can restrict $\lambda$ to lie between the two turning points of the classical motion $(\pm \sqrt{2\alpha'} \mu)$, i.e. $\hat{\lambda} \in [-1,1]$, or equivalently restricting $\tau$ to lie between 0 and $T/2$ where $T$ is the period of classical motion ($\oint d\tau = 2\pi T$). One then imposes a standard bag-like boundary conditions on the chiral fields to get a consistent picture. Introducing the cut-off on $\tau$, the energy levels are approximated by

$$\epsilon_n \sim -\mu + n/T + \delta'(n/T, \mu) + ... \quad (5.23)$$

where the last two terms can be dropped in the limit $T \to \infty$.

Thus, if a single fermion starts near the top of the upside down potential it will roll down to infinity. In the region $\tau \gg 1$, $\hat{\lambda} \gg 1$, it becomes approximately relativistic (since $v \gg 1$). The final state could be expressed in terms of the chiral fields bosonized as

$$\psi_{L,R}(\tau,t)|0\rangle = e^{i \frac{2}{\sqrt{2}\pi} X_{L,R}(\tau, t)}, \quad (5.24)$$

(where, $X_L + X_R = X$). Since the bosonization scalars have the mode expansion

$$X_{L,R}(\tau+t) = \int \frac{dp}{2\pi \sqrt{2E}} \left[ a_{L,R,p}^\dagger \exp \left( i|p|t + ip\tau \right) + \text{h.c.} \right], \quad (5.25)$$

a final state of the decay would be given by

$$\int d\tau \Psi(\tau)\psi_L|0\rangle = \int d\tau \Psi(\tau)e^{i \frac{2}{\sqrt{2}\pi} X_L(\tau, t)}|0\rangle = \int d\tau \Psi(\tau)e^{i2\sqrt{\pi} \int \frac{dp}{2\pi \sqrt{2E}} e^{-ip\tau} a_{L,p}^\dagger}|0\rangle. \quad (5.26)$$

Thus keeping in mind the form of the coherent state obtained in the tree level string theory, $|\psi\rangle \sim \exp \left( \int \frac{dp}{\sqrt{2E}} a_{p}^\dagger A \ |0\rangle \right)$, one can extract the decay amplitude from the bosonization scalars as

$$A \sim i2\sqrt{\pi} e^{-iE\tau}, \ |p| = E. \quad (5.27)$$

This is the same as the amplitudes calculated from the continuum method.

### 5.2 The RG Picture of Tachyonic Decay

One can view such a process of decay of an unstable eigenvalue describing the open string tachyon roll-down to 2D closed string vacuum, as a Large N Wilsonian RG in the singlet sector
of the matrix quantum mechanics leading to the \( c = 1 \) fixed point (in a similar fashion to our calculations in the previous sections). The RG equations are nothing but a statement of response of the free fermion system as successively one of the fermion is integrated out in the presence of the rest of the free fermions. Hence, following our calculation, by performing the Large \( N \) world sheet RG for the singlet Matrix quantum mechanics in the eigenvalue representation (with \( \lambda_{N+1}(t) = z(t) = \sqrt{2\alpha'\mu} \lambda \cos t/\sqrt{\alpha'} \)), one can compare the change of the world sheet free energy in the Large \( N \) RG with that due to the closed string emission picture. One can do this comparison both for the singlet MQM on infinite line and that on a large circle. In the later case one would have to consider closed string emission amplitude for compact time on large circle. This comparison, on one hand gives rise to a physical picture of the Large \( N \) RG we considered in this paper, on the other hand it provides a way to test our results. Note that the desired \( c = 1 \) end point, corresponding to the 2D closed string vacuum, always emerges in a very nice and robust way in the RG calculation with all it’s critical properties.

We would now try to understand the emergence of this amplitude from the point of view of the world sheet RG by turning on an unstable eigenvalue

\[
z(t) = \sqrt{2\alpha'\mu} \lambda \cos(t/\alpha'),
\]

and integrating it out in the presence of the rest of the \( N \) free fermions. The Fermi sea, in response to this, would readjust its height infinitesimally which is reflected in the renormalization of the couplings. Later we will focus on the more general context of the Wilsonian world sheet RG of the \((N \times N)\) arbitrary matrices on circle rather than working in the eigenvalue representation only, which is good for the large radius.

The partition functions before and after turning on this unstable probe eigenvalue respectively, are given by,

\[
Z_N[g,t] = \int \prod_{i=1}^{N} d\lambda_i(t) \Delta_N(t) \exp \left[ - \frac{N}{g^2} \int dt' \left( \frac{1}{2} \dot{\lambda}_i(t')^2 + V(\lambda_i(t')) \right) \right],
\]

\[
Z_{N+1}[g,t] = \int \prod_{i=1}^{N} d\lambda_i(t) \Delta_N(t) \exp \left[ - \frac{(N+1)}{g^2} \int dt' \left( \frac{1}{2} \dot{\lambda}_i(t')^2 + V(\lambda_i(t')) \right) \right]
\int dz(t) \exp \left[ - \frac{(N+1)}{g^2} \int dt' \left\{ \frac{1}{2} \dot{z}^2(t') + V(z(t')) \right\} \right] \exp \left[ \frac{1}{\hbar} \hat{\phi}(z(t)) \right],
\]

where the operator

\[
\frac{1}{\hbar} \hat{\phi}(z(t)) = \log \left[ \frac{\Delta_{N+1}(t)}{\Delta_N(t)} \right] = \sum_{i=1}^{N} \log \left[ \lambda_i(t) - z(t) \right], \quad N \sim 1/\hbar.
\]
The Laplace transform of the operator $\hat{\phi}(z(t))$ can be seen as the loop operator $O(l)$ that creates a closed (Dirichlet) boundary of length $l$ on the world sheet. Its expectation value gives the loop amplitude. In the limit of the vanishing boundary length, the operator becomes tachyonic and its excitations correspond to the excitations of the closed string tachyon:

$$\hat{\phi}(\tau, t) = \int_{-\infty}^{\infty} dq \ e^{-iqt} \int_{0}^{\infty} dl \ e^{-l\lambda(\tau)} \ O(l, q),$$

$$O(l, q) \sim \frac{1}{2} \int dt \ e^{iqt} \int d\tau \ e^{-l\lambda(\tau)} \ \partial_{\tau} X.$$ (5.31)

The mode expansion of the massless scalar field $X$ is related to the amplitude $A$ of the closed string decay in the sense of final state of the evolution being somewhat like a coherent state, as shown in (5.26). Thus in the matrix model, one can also compute the decay amplitude by computing $\hat{\phi}(\tau, t)$. (The reference [41] describes a similar thought to relate $\hat{\phi}(\tau, t)$ to the amplitude. However, they chose $\hat{\phi}(\tau, t)$ to be in the same footing as $X$, instead of (5.31)).

Let us now consider the partition function for the $N+1$ eigenvalues (5.29) and rewrite the right hand side in terms of the renormalized coupling $g'$

$$Z_{N+1}[g, t] = \int \prod_{i=1}^{N} d\lambda_i(t) \ \Delta_N(t) \ e^{-\frac{N}{g^2} \int_{t'}^{t} dt' \{\frac{1}{2} \dot{z}^2(t') + V(z(t'))\}} \ e^{\frac{1}{\hbar} \hat{\phi}(z(t))},$$

(5.32)

where $g' = g - \frac{g^2}{2N} + O(1/N^2)$. As before, extracting the partition function of $N$ eigenvalues, we have

$$\frac{Z_{N+1}[g, t]}{Z_N[g', t]} = \int dz(t) \ \exp \left[-\frac{N}{g^2} \int_{t'}^{t} dt' \{\frac{1}{2} \dot{z}^2(t') + V(z(t'))\}\right] \ \exp \left[\frac{1}{\hbar} \hat{\phi}(z(t))\right] \ = \psi_t(N, g') \left\langle \exp \left[\frac{1}{\hbar} \hat{\phi}(z(t))\right]\right\rangle,$$ (5.33)

where $\psi_t(N, g')$ is given by

$$\psi_t(N, g') = \int dz(t) \ \exp \left[-\frac{N}{g^2} \int_{t'}^{t} dt' \{\frac{1}{2} \dot{z}^2(t') + V(z(t'))\}\right].$$ (5.34)

Then from the RG point of view the change of the world-sheet free energy ($\delta F = \log Z_{N+1}[g, t]/N^2 - \log Z_N[g', t]/N^2$) in the process of D-brane decay, given by the ratio

$$\frac{Z_{N+1}[g, t]}{Z_N[g', t]} = \psi_t(N, g') \left\langle \exp \left[\frac{1}{\hbar} \hat{\phi}(\tau, t)\right]\right\rangle,$$ (5.35)

encodes the contribution from the amplitude $A$ through a knowledge of $\hat{\phi}(\tau, t)$. In terms of the discrete Callan-Symanzik equation (2.10), the left hand side indicates a flow from $g \to g'$ as
\((N+1) \rightarrow N\). The right hand side computes the contribution of the amplitude \(A\) to the change of the world sheet free energy as an \(O(1/N)\) effect.

\[
\left[ N \frac{\partial}{\partial N} - \beta(g) \frac{\partial}{\partial g} + \gamma \right] \mathcal{F}[g, t] = \frac{1}{N} \langle \dot{\phi} \rangle + \frac{1}{N^2} \ln \psi_t, \quad \gamma = 2.
\] (5.36)

Here \(\psi_t\) is the single particle wave function corresponding to the probe eigenvalue. Note that for the singlet MQM on a large circle, the \(\psi_t\) thus calculated by the Large \(N\) RG is a function of \((R + 1/R)\) which is a manifestation of the anticipated \(T\)-duality property.

We would now calculate the right hand side of the relation \((5.35)\). Let us assume that the rest of the \(N\) eigenvalues belong to the static Fermi sea forming a closed string background given by

\[
\lambda_i = -\sqrt{2\alpha' \bar{\mu}} \hat{\lambda}, \quad i = 1 \ldots N.
\] (5.37)

Hence from \((5.30)\) and \((5.28)\) and considering \(\hat{\lambda} \sim \exp(\tau/\sqrt{\alpha'})\) for \(\tau \gg 1\) and also \(N \sim 1/\hbar\),

\[
\dot{\phi}(\tau, t) = \ln \left[ \sqrt{2\alpha' \bar{\mu}} \left(1 - \cos \frac{t}{\sqrt{\alpha'}}\right) \right] + \tau/\sqrt{\alpha'}.
\] (5.38)

Thus,

\[
\frac{Z_{N+1}[g, t]}{Z_N[g', t]} = \psi_t(N, g') \left( \sqrt{2\alpha' \bar{\mu}} \left(1 - \cos \frac{t}{\sqrt{\alpha'}}\right) \right)^N \langle \exp \frac{\tau}{\hbar} \sqrt{\alpha'} \rangle.
\] (5.39)

Here the \(t\) dependent part could be absorbed in the overall prefactor of the ratio. As in \((5.35)\) effectively \(\dot{\phi} \sim \tau/\sqrt{\alpha'}\) contains the contribution from the decay amplitude to the change of the world sheet free energy.

We will now go back to the relation of \(\dot{\phi}\) with the bosonization field \(X\) (and hence \(A\)) and verify that the expression of \(\dot{\phi}\) we obtained in \((5.38)\) does lead to the standard answer for \(A\). Using \((5.31)\) and performing the delta function integration over \(t\),

\[
\dot{\phi}(\tau, t) = \pi \int d\tau' \frac{1}{\sqrt{2\alpha' \bar{\mu}} \left(\exp \tau/\sqrt{\alpha'} + \exp \tau'/\sqrt{\alpha'}\right)} \left(\partial_\tau X(\tau, t)\right).
\] (5.40)

This implies,

\[
(\partial_\tau \dot{\phi}) = \frac{\pi (\partial_\tau X)}{\sqrt{2\alpha' \bar{\mu}} \left(\exp \tau/\sqrt{\alpha'} + \exp \tau'/\sqrt{\alpha'}\right)}.
\] (5.41)

Inverting and solving for \(X\),

\[
X = \frac{1}{\pi} \int d\tau' \sqrt{2\alpha' \bar{\mu}} \left(\exp \tau/\sqrt{\alpha'} + \exp \tau'/\sqrt{\alpha'}\right)(\partial_\tau \dot{\phi}).
\] (5.42)

Considering \((\partial_\tau \dot{\phi}) = \frac{1}{\sqrt{\alpha}} \delta(\tau' - \tau)\), we have \(X = \frac{2}{\pi} \sqrt{2\bar{\mu}} e^{\tau'\sqrt{\alpha'}}\). Comparing with the relation of the mode expansion of \(X\) to the amplitude \(A\) we have,

\[
A \sim \frac{2}{\pi} \sqrt{2\bar{\mu}} e^{\tau'\sqrt{\alpha'}} \sim e^{\ln \lambda}.
\] (5.43)
The time delay $(\tau)$ is thus given by $|\Delta t| \sim \ln \lambda$ which is consistent. Thus we conclude that in our RG, integration of one eigenvalue in the presence of the others describes the decay of an unstable $D0$-brane with open string tachyon attached to it to the $2D$ closed string theory (with its $D0$-branes) with an amplitude $A$ given by

$$
\left[ N \frac{\partial}{\partial N} - \beta(g) \frac{\partial}{\partial g} + \gamma \right] F[g, t] = \frac{1}{N} \ln A + \frac{1}{N^2} \ln \psi_t, \quad A \sim e^{\ln \lambda}, \quad \gamma = 2.
$$

(5.44)

In our set up it is clear that there is a flow, $g \to g'$ as $(N + 1) \to N$, corresponding to the decay process. However, as we have discussed in the previous chapter, the eigenvalue representation is too simple to explicitly produce a nontrivial fixed point of the flow $(\beta(g) = -g/2)$ from the beta function equation. The flow in the eigenvalue representation only gives the gaussian fixed point corresponding to the inverted harmonic oscillator. However, still one can say that the flow is hitting the $c = 1$ fixed point situated at the infinitesimal distance from the Gaussian fixed point. Working in a more general set up of $MQM$ of general $N \times N$ matrices, one can explicitly show from the beta function that the endpoint of the flow is indeed a pair of $c = 1$ nontrivial fixed point $(\beta(g) = -g/2 + 3F(g, R) g^3/2)$ dual to the $2D$ closed string theory the $D$-brane decay leads to. Also it is shown that for $R \to \infty$ or even for large finite $R$ (where the singlet free fermion picture is meaningful) the pair of $c = 1$ fixed points are drawn infinitesimally close to each other and to the Gaussian fixed point such that they are overlapping. Thus it is consistent to think that even in the simple eigenvalue representation the flow corresponding to the $D$-brane decay does end at the pair of overlapping $c = 1$ fixed point in the infinitesimally close neighborhood of the gaussian fixed point. This is derived in detail in the previous sections and is the key observation of this paper.

We will now discuss the decay process in the context of RG analysis of the general $N \times N$ matrices $\phi_N(t)$. Consider the usual partition function $Z_{N+1}$ of the matrix field $\phi_{N+1}(t)$ that is decomposed into $\phi_N(t)$ and the $(1 \times N)$ and $(N \times 1)$ row and column vectors $v^*(t)$ and $v(t)$ respectively,

$$
Z_{N+1}[g, M] = \int \mathcal{D}^N \phi_N(t) \mathcal{D}^N v^*(t) \mathcal{D}^N v(t)
\exp \left[ -N \text{Tr} \int dt \left( \frac{1}{2} (D_\phi \phi_N)^2 + \frac{1}{2} M^2 \phi_N^2 - \frac{g}{3} \phi_N^3 + Dv^* Dv + M^2 v^* v - g v^* \phi_N v \right) \right].
$$

(5.45)

As usual, the covariant derivative is defined in terms of the non-dynamical gauge field $A$ in the open string spectrum corresponding to the vertex operator $\hat{t}$,

$$
D \phi_N = \partial_t \phi_N - [A, \phi_N].
$$

(5.46)
The gauge field $A$ projects onto the $SU(N)$ singlet wave functions by acting as a Lagrange multiplier.

Now, one can see $Z_{N+1}$ as the dual large $N$ description of the 2D string theory in the presence of the space-time filling $D1$-branes, the extended Liouville boundary states tensored with Neumann boundary state for the (Euclidean) time. One can similarly understand $Z_{N+1}$ for the (Euclidean) time taken on a (large) circle. Motivated from previous works [47, 48, 58], the authors of [40] pointed out that integration over the vectors $v^*(t), v(t)$ inserts into the random surfaces dynamical boundaries (quark loops) wandering in the time direction. Although their model is essentially the same as (5.45), in the former case the couplings in the flavour part are introduced by hand rather than being determined by the original closed string action due to the parametrization (2.7).

As we have discussed in previous sections, precisely this is happening in our RG. The $(N+1) \times (N+1)$-th matrix element gives the position of the wandering boundaries, which for simplicity, have been set to zero in our analysis. The sum of the ‘single line’ one loop Feynman diagrams, generated by the integration over the vectors $v^*(v)$ (see equations (2.21) and (2.24)), inserts in the path integral different operators coupled to the dynamical boundaries. The relevant operators coupled to these boundaries are of the form $1/N(\text{Tr} \phi^n - g_n v^* \phi^{(n-2)} v)$. The massless open string tachyon would correspond to the operators $\int dt e^{iqt} v^* v(t)$. For a model without the kinetic term for $(v^*, v)$, integration over the vectors generates the macroscopic loops, the boundaries with the Dirichlet boundary condition on time. This happens when both the parameters $M^2$ and $g$ acquire large values due to tuning. As pointed out in [40], tuning the parameters $M^2$ and $g$ together with $N$ one should be able to get a scaling model with two independent parameters $\mu_B$ and $\mu$, the renormalized boundary and bulk cosmological constants, given by the relation (5.1). In our RG analysis we arrive at an analogous relation (3.9) with usual scaling between $\mu_B$ and $\mu$ (with an accuracy of 80 %). This presumably indicate the presence of various boundaries in 2D string theory. However to realize whether these boundaries correspond to Neumann or Dirichlet case needs further investigation. Here, tuning of the matrix coupling constant $g$ gives rise to a renormalized bulk cosmological constant $\mu$. The scaling variable $\Delta = (1 - g/g^*)$ depends nontrivially on $\mu$

$$\Delta \sim \frac{1}{2\pi} \mu \ln \mu, \quad (5.47)$$

as they are mutually coupled through the Fermi-Dirac distribution of the fermions in the grand canonical ensemble [28]. On the other hand one can think of tuning the mass parameter $M^2$ to control the boundary cosmological constant $\mu_B$ as

$$\mu_B \sim 1 - M^2/M^*2. \quad (5.48)$$
Hence solving the equation
\[ \frac{\beta_g}{\beta_M} = \frac{\partial g}{\partial M} = f(g, M, T), \quad (5.49) \]
the RG flows \( g = g(M, T) \) in the Double scaling limit near the nontrivial fixed points is expected to give rise to relations analogous to the trajectory \((5.1)\). In equation \((5.49)\) we have solved this trajectory for large \( R \) (for matrix quantum mechanics on circle) and got a consistent scaling. We expect the fixed points of large \( M^* \) obeying such scaling could be Dirichlet boundaries. However, since in our analysis we are dealing with bosons on a circle and do have kinetic term for the quarks, these could be Neumann boundaries as well. We hope to come back to these interesting questions in close future.

6 Discussion

In our analysis we have seen the following remarkable fact. The general framework of world-sheet RG analysis of the matrix quantum mechanics on a circle is seen to be capable of reproducing the critical exponent and the expected scaling laws of the \( c = 1 \) fixed point and the \( T \)-duality property respected by its singlet sector. Moreover, it indirectly reflects the presence of the non-singlet sector from the tendency of the change of the world-sheet free energy to change sign at a critical radius. It also indicates the presence of the boundaries in the 2D string theory and gives a flavor of capturing their dynamics. All these serve to be an initial understanding and progress towards our main goal, to develop a scheme for dealing with the non-singlet sectors (or the world-sheet vortices) which govern the physics of 2D string theory on small circle. In a separate publication \[63\] we will describe how the behavior of the non-singlet sector can be explicitly captured in our RG scheme with a gauged matrix model, by introducing an appropriate gauge breaking term and tuning the fugacity of the vortices coupled to it (along with the usual tuning of the other parameters in the double scaling limit). In this case, the RG flows will turn out to be capable to explore the interesting fixed points beyond the \( c = 1 \) universality class.

As we have observed, by tuning the parameters \( M^2 \) and \( g \) we arrive at RG trajectories with appropriate scaling of the bulk and the boundary cosmological constants, suggesting the presence of the various boundaries of the underlying 2D theory. To explicitly see these boundaries for each of the Dirichlet and Neumann cases one needs to study the behavior of the operator creating boundaries of finite length. The most relevant operator that controls this length is the boundary cosmological constant. Such finite length boundary operators are computed from the free fermion wave function. Using our RG scheme one can study the behavior of these operators for the \( c = 1 \) fixed point at any radius and try to see the all possible boundaries. The results at large \( R \) can be compared with the free fermion wave functions as a test. The study can be
further extended to include the effects of the non-singlets. We have work in progress in this area.

We would now comment on the subtle issues on dealing the nonlocal terms in the action, in computing the determinant emerging from the adiabatic integration over the vector row and column of the original matrix variable. In our method, we are expanding the determinant in the Fourier space about $\phi(t) = 0$ (small field approximation). We tackle the non-local integrals by introducing 'center of mass' (large) and 'relative' (small) coordinates, expanding all the functions around the latter and then integrating over the 'relative' coordinates. We keep the contribution from the non-localities in the action up to the kinetic term $\Pi^2(t)$, and the contribution from the higher order terms in $\phi(t)$ up to the $\phi^3(t)$ term. The kinetic term is the most important nonlocal term to capture the effect of the world sheet vortices. As by the parametrization (2.7), it also gives rise to the kinetic term of the vector fields, it plays an important role in understanding the nonlocal Neumann boundaries, the $D1$ brane. The higher order terms like $\text{Tr}(\Pi^m(t)\phi^n(t))$ generated by expanding the determinant give rise to powers of traces and are redundant for the RG analysis. The higher order terms are also negligible due to small field approximation. However, it will be interesting to study the scheme with more couplings to see the contribution from the higher order terms and the convergence of the scheme. Also it is an interesting question to understand the relevant operators of the flow from the matrix quantum mechanics point of view.

An interesting solution of the two-dimensional theory, apart from the flat space background with linear dilaton, is the two-dimensional black hole [59, 60, 61]. It is partially believed that the nonperturbative formulation of two-dimensional string theory in terms of an integrable theory of noninteracting nonrelativistic fermions of the matrix quantum mechanics can deal issues like black hole evaporation and gravitational collapse. In any case, the condensation of world-sheet vortices of the two-dimensional noncritical string theory should describe two-dimensional black hole background, which had been a long standing challenge to obtain from matrix model. The recent work by Kazakov, Kostov and Kutasov gives proposals for a matrix model for two-dimensional black hole [62]. In our forthcoming paper [63], we will address this issue in more detail. Also in this context, one can consider integrating out several vector rows and columns simultaneously to see the effect of inserting multiple boundaries. This is analogous to the situation of the decay of several unstable branes and is anticipated by some authors to give black-hole like phase at the end point of the decay [13, 14]. However, the multiple row and column integration presumably would insert redundant operators of the form $(\text{Tr}\phi_N^n)^n$, which are not likely to give rise to black hole fixed point.
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Appendix

A The Feynman Diagrams

Here we evaluate and discuss the terms in different orders of the series \( \sum[g, M, \phi_N, R, N] \) \((2.21)\) using the summation rules discussed in section (3.3) and the relation \((2.13)\) for the inverse Fourier Transform.

A.1 The terms of order \( \mathcal{O}(\phi) \)

\[
g \text{Tr} \left[ \sum_n \frac{1}{\left( \frac{n^2}{R^2} + M^2 \right)} \phi_0 \right] = \frac{g}{2M} \coth(\pi MR) \int_0^{2\pi R} dt \text{ Tr} \phi(t). \tag{A.1}
\]

A.2 The terms of order \( \mathcal{O}(\phi \phi) \)

\[
\frac{g^2}{2} \text{Tr} \left[ \sum_{m,n} \frac{1}{\left( \frac{m^2}{R^2} + M^2 \right) \left( \frac{n^2}{R^2} + M^2 \right)} \phi_{m-n} \phi_{n-m} \right] = \frac{g^2}{2} \int \frac{dt_1 dt_2}{(2\pi R)^2} \text{Tr} \left( \phi(t_1) \phi(t_2) \right) \left[ \sum_{m,n} \frac{\exp \left( i(n-m) t_1/R \right) \exp \left( i(m-n) t_2/R \right)}{\left( \frac{m^2}{R^2} + M^2 \right) \left( \frac{n^2}{R^2} + M^2 \right)} \right]. \tag{A.2}
\]

Now, changing the variables to 'center of mass' and 'relative' coordinates defined respectively by

\[
T = \frac{t_1 + t_2}{2}, \quad \tau = \frac{t_1 - t_2}{2}, \tag{A.3}
\]

we have,

\[
dt_1 \ dt_2 = J \left( \frac{t_1, t_2}{T, \tau} \right) \ dT \ d\tau = 2 \ dT \ d\tau. \tag{A.4}
\]
Hence,

\[
\begin{align*}
\frac{g^2}{2} \text{Tr} \left[ \sum_{m,n} \frac{1}{(m^2 + M^2)(n^2 + M^2)} \phi_{m-n} \phi_{n-m} \right] \\
= \frac{g^2}{4\pi^2 R^2} \int_0^{2\pi R} dT \int_{-\pi R}^{\pi R} d\tau \left( \phi(T + \tau)\phi(T - \tau) \right) \sum_{m,n} \frac{\exp\left(i(n-m) \frac{2\tau}{R}\right)}{\left(\frac{m^2}{R^2} + 1\right)\left(\frac{n^2}{R^2} + 1\right)} \\
\approx \frac{g^2}{4\pi^2 R^2} \int_0^{2\pi R} dT \int_{-\pi R}^{\pi R} d\tau \left( \phi(T)^2 - \tau^2 \phi'(T)^2 \right) \sum_{m,n} \frac{\exp\left(i(n-m) \frac{2\tau}{R}\right)}{\left(\frac{m^2}{R^2} + 1\right)\left(\frac{n^2}{R^2} + 1\right)} \\
\approx g^2 F_{g2}(R, M) \int_0^{2\pi R} dT \left( \phi(T)^2 / 2 \right) + g^2 \hat{F}_{g2}(R, M) \int_0^{2\pi R} dT \left( \phi(T)^2 / 2 \right), \hspace{1cm} (A.5)
\end{align*}
\]

where,

\[
\begin{align*}
F_{g2}(R, M) &= \frac{1}{2\pi^2 R^2} \int_{-\pi R}^{\pi R} d\tau \sum_{m,n} \frac{\exp\left(i(m-n) \frac{2\tau}{R}\right)}{\left(\frac{m^2}{R^2} + M^2\right)\left(\frac{n^2}{R^2} + M^2\right)} \\
&= \frac{1}{2\pi^2 R^2} \int_0^{2\pi R} d\tau' \sum_{m,n} \frac{\exp\left(i(m-n) \frac{\tau'}{R}\right)}{\left(\frac{m^2}{R^2} + M^2\right)\left(\frac{n^2}{R^2} + M^2\right)}, \hspace{1cm} (\tau' = 2\tau) \\
&= \frac{1}{2M^2 \sinh^2 \pi MR} \int_0^{2\pi R} d\tau' \cosh M(\pi R + \tau') \cosh M(\pi R - \tau') \\
&= \frac{1}{M^3 \sinh^2 \pi MR} \left[ \frac{1}{2} \pi MR \cosh 2\pi MR + \frac{1}{8} \sinh 4\pi MR \right], \hspace{1cm} (A.6)
\end{align*}
\]

and,

\[
\begin{align*}
\hat{F}_{g2}(R, M) &= -\frac{1}{2\pi^2 R^2} \int_{-\pi R}^{\pi R} d\tau \sum_{m,n} \frac{\exp\left(i(m-n) \frac{2\tau}{R}\right)}{\left(\frac{m^2}{R^2} + M^2\right)\left(\frac{n^2}{R^2} + M^2\right)} \\
&= -\frac{1}{8\pi^2 R^2} \int_0^{2\pi R} d\tau' \sum_{m,n} \frac{\exp\left(i(m-n) \frac{\tau'}{R}\right)}{\left(\frac{m^2}{R^2} + M^2\right)\left(\frac{n^2}{R^2} + M^2\right)}, \hspace{1cm} (\tau' = 2\tau) \\
&= -\frac{1}{8M^2 \sinh^2 \pi MR} \int_0^{2\pi R} d\tau' \cosh M(\pi R + \tau') \cosh M(\pi R - \tau') \\
&= \frac{1}{M^5 \sinh^2 \pi MR} \left[ -\frac{1}{64} (1 + 8\pi^2 M^2 R^2) \sinh 4\pi MR - \frac{\pi^3 M^3 R^3}{6} \cosh 2\pi MR + \frac{\pi MR}{16} \cosh 4\pi MR \right]. \hspace{1cm} (A.7)
\end{align*}
\]
A.3 The terms of order $O(\phi \phi \phi)$

\[ \frac{g^2}{6} \text{Tr} \left[ \sum_{m,n,k} \frac{1}{(m^2/R^2 + M^2)(n^2/R^2 + M^2)(k^2/R^2 + M^2)} \phi_{m-n} \phi_{n-k} \phi_{k-m} \right] \]

\[ = \frac{g^2}{6} \int \frac{dt_1 dt_2 dt_3}{(2\pi R)^3} \text{Tr} \left( \phi(t_1)\phi(t_2)\phi(t_3) \right) \]

\[ \left[ \sum_{m,n} \exp \left( i(n-m) t_1/R \right) \exp \left( i(k-n) t_2/R \right) \exp \left( i(m-k) t_1/R \right) \right] \]

\[ = \frac{g^2}{48\pi^3 R^3} \int_0^{2\pi R} dT \int_{-\pi R}^{\pi R} d\tau_1 \int_{-\pi R}^{\pi R} d\tau_2 \]

\[ \text{Tr} \left( \phi(T + \frac{\tau_1 + \tau_2}{3}) \phi(T - \frac{2}{3} \tau_1 + \frac{1}{3} \tau_2) \phi(T + \frac{1}{3} \tau_1 - \frac{2}{3} \tau_2) \right) \]

\[ \left[ \sum_{m,n,k} \exp \left( im\tau_1 / R \right) \exp \left( - im\tau_2 / R \right) \exp \left( ik(\tau_2 - \tau_1) / R \right) \right] \]

\[ \approx g^3 F_{g3}(R, M) \int_0^{2\pi R} dT \text{Tr}(\phi(T)^3 / 3) \]  

(A.8)

Using redefinition of the variables into the "center of mass" and the "relative coordinates",

\[ T = \frac{1}{3} (t_1 + t_2 + t_3), \quad \tau_1 = (t_1 - t_2), \quad \tau_2 = (t_1 - t_3), \]

\[ dt_1 dt_2 dt_3 = J \left( \frac{t_1, t_2, t_3}{T, \tau_1, \tau_2} \right) dT d\tau_1 d\tau_2 = dT d\tau_1 d\tau_2. \]

Considering $\tau_1$ and $\tau_2$ to be small and keeping the order $O(\phi^3)$ term, above series could be evaluated as,

\[ \frac{g^3}{6} \text{Tr} \left[ \sum_{m,n,k} \frac{1}{(m^2/R^2 + M^2)(n^2/R^2 + M^2)(k^2/R^2 + M^2)} \phi_{m-n} \phi_{n-k} \phi_{k-m} \right] \]

\[ = \frac{g^3}{48\pi^3 R^3} \int_0^{2\pi R} dT \int_{-\pi R}^{\pi R} d\tau_1 \int_{-\pi R}^{\pi R} d\tau_2 \]

\[ \text{Tr} \left( \phi(T + \frac{\tau_1 + \tau_2}{3}) \phi(T - \frac{2}{3} \tau_1 + \frac{1}{3} \tau_2) \phi(T + \frac{1}{3} \tau_1 - \frac{2}{3} \tau_2) \right) \]

\[ \left[ \sum_{m,n,k} \exp \left( im\tau_1 / R \right) \exp \left( - im\tau_2 / R \right) \exp \left( ik(\tau_2 - \tau_1) / R \right) \right] \]

\[ \approx g^3 F_{g3}(R, M) \int_0^{2\pi R} dT \text{Tr}(\phi(T)^3 / 3) \]  

(A.9)
where,

\[ F_{g^3}(g, R) = \frac{1}{16\pi^3 R^3} \int_{-\pi R}^{\pi R} d\tau_1 \int_{-\pi R}^{\pi R} d\tau_2 \left[ \sum_{m,n,k} \frac{\exp\left(\frac{im\tau_1}{R}\right) \exp\left(-\frac{im\tau_2}{R}\right) \exp\left(\frac{i(k\tau_2 - \tau_1)}{R}\right)}{(\frac{m^2}{R^2} + M^2)(\frac{n^2}{R^2} + M^2)(\frac{k^2}{R^2} + M^2)} \right] \]

\[ = \frac{1}{16M^2 \pi R \sinh^2 \pi MR} \int_{-\pi R}^{\pi R} d\tau_1 \int_{-\pi R}^{\pi R} d\tau_2 \cosh(\pi MR - \tau_1) \cosh(\pi MR + \tau_2) \sum_k \frac{\exp\left(i(k\tau_2 - \tau_1)/R\right)}{\left(\frac{k^2}{R^2} + M^2\right)} \]

\[ = \frac{1}{64M^2 \pi R \sinh^2 \pi MR} \int_{-2\pi R}^{2\pi R} d\hat{\tau} \int_{-2\pi R}^{2\pi R} d\hat{\tau} \left[ \cosh(2\pi MR + \hat{\tau}) + \cosh M\hat{\tau} \right] \sum_k \frac{\exp ik\hat{\tau}/R}{\left(\frac{k^2}{R^2} + M^2\right)} \]

\[ = \frac{\pi R}{16M^3 \sinh^3 \pi MR} \int_0^{2\pi R} d\hat{\tau} \left[ \cosh(2\pi MR + \hat{\tau}) + \cosh M\hat{\tau} \right] \cosh(\pi MR - M\hat{\tau}) \]

\[ + \left( \cosh(2\pi MR - M\hat{\tau}) + \cosh M\hat{\tau} \right) \cosh(\pi MR + M\hat{\tau}) \]

\[ = \frac{\pi MR}{64M^3 \sinh^3 \pi MR \cosh(2\pi MR + \cosh 4\pi MR)} \left[ 4\pi MR(3 \cosh \pi MR + 2 \cosh 3\pi MR + 2 \cosh 5\pi MR + \sinh \pi MR + \sinh 3\pi MR + \sinh 5\pi MR + 2 \sinh 7\pi MR + \sinh 9\pi MR) \right] \]

(A.10)

**B The Scaling Dimensions**

Here we will summarize the general expression of the matrix \( \Omega_{k,l} = \frac{\partial \beta_k(\Lambda^*)}{\partial \Lambda_l} \) and its eigenvalues \( \Omega_1 \) and \( \Omega_2 \), the scaling dimensions of the relevant operators, at different fixed points.

\[ \Omega_{11} = \frac{1}{2}(5h - 1) + 3g^{*2}(F_{g^3}^* - \frac{3}{2}\hat{F}_{g^2}^*) \]

\[ \Omega_{12} = g^{*3} \left( \frac{\partial F_{g^3}^*}{\partial M} - \frac{3}{2} \frac{\partial \hat{F}_{g^2}^*}{\partial M} \right) \]

\[ \Omega_{21} = g^* \left( \frac{1}{M^* - M^*} F_{g^2}^* - M^* \hat{F}_{g^2}^* \right) \]

\[ \Omega_{22} = h + g^{*2} \frac{\partial}{\partial M} \left( \frac{1}{2} M^* - M^* / 2 \right) F_{g^2}^* - M^* / 2 \hat{F}_{g^2}^* \]

(B.11)

For the gaussian fixed point

\[ \Omega_{11} = (5h - 1)/2 = -1/2, \quad \Omega_{22} = h = 0, \quad \Omega_{12} = \Omega_{21} = 0. \]

(B.12)

For the nontrivial fixed point

\[ \Omega_{11} = 1 + 5h/2 = 1, \quad \Omega_{22} \simeq h \simeq 0, \quad \Omega_{12} \simeq \Omega_{21} \simeq 0. \]

(B.13)
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