Exact Solutions of the Equations of Motion of Liquid Helium with a Charged Free Surface

N. M. Zubarev

Institute of Electrophysics, Ural Branch, Russian Academy of Sciences,
106 Amundsen Street, 620016 Ekaterinburg, Russia
tel.: +7 343 2678776; fax: +7 343 2678794.

The dynamics of the development of instability of the free surface of liquid helium, which is charged by electrons localized above it, is studied. It is shown that, if the charge completely screens the electric field above the surface and its magnitude is much larger then the instability threshold, the asymptotic behavior of the system can be described by the well-known 3D Laplacian growth equations. The integrability of these equations in 2D geometry makes it possible to described the evolution of the surface up to the formation of singularities, viz., cuspidal point at which the electric field strength, the velocity of the liquid, and the curvature of its surface assume infinitely large values. The exact solutions obtained for the problem of the electrocapillary wave profile at the boundary of liquid helium indicate the tendency to a charge in the surface topology as a result of formation of charged bubbles.

∗ Electronic address: nick@ami.uran.ru
I. INTRODUCTION

It is well known [1, 2] that the liquid helium surface may be charged to high values of the surface density of a negative electric charge. This is due to the fact that, on the one hand, electrons are attracted to the surface by weak electrostatic image forces and, on the other hand, the liquid helium boundary is a potential barrier for electrons, which prevents their penetration in the bulk. An important feature of liquid helium as a dielectric with a low polarizability is the relative weakness of the image forces, as a result of which the mean distance between localized electrons and the surface is much larger than the atomic spacing. Consequently, the electrons are not bound to individual atoms of the substance and form a two-dimensional conducting system.

The ability of electrons to move freely over the surface of liquid helium ensures the equipotential nature of this surface over characteristic hydrodynamic times and scales. A charged surface of a conducting liquid also possesses this property, the only difference being that the electric field cannot penetrate into a conducting medium, while liquid helium is not subjected to such a limitation. This enabled Gor’kov and Chernikova [3, 4] to extend a number of classical results from the theory of instability of a liquid metal surface in an external electric field [5, 6, 7] to the case of the charged boundary of liquid helium (the geometry of the system is shown schematically in Fig. 1). For example, a natural generalization of the dispersion relation for linear waves on the surface of a conducting liquid is the following dispersion relation for liquid helium:

\[ \omega^2 = gk + \frac{\alpha}{\rho} k^3 - \frac{E'^2 + E^2}{4\pi\rho} k^2, \]  

(1)

where \(\omega\) is the frequency, \(k\) is the wave number, \(g\) is the acceleration due to gravity, \(\alpha\) is the surface tension, \(\rho\) is the density of the medium, and \(E'\) and \(E\) are the electric field strengths above the liquid and in the bulk of it, respectively (\(E = 0\) for a conducting medium). It follows hence that for

\[ E'^2 + E^2 < E_c^2 = 8\pi\sqrt{g\alpha\rho}, \]

the inequality \(\omega^2 > 0\) holds for any \(k\) and, hence, small perturbations of the surface do not build up with time. In the case when the sum of the squares of the fields \(E'^2 + E^2\), which plays the role of an extrinsic controlling parameter, exceeds the critical value \(E_c^2\), a region of wave numbers \(k\) for which \(\omega^2 < 0\) is formed. This corresponds to an aperiodic instability.
FIG. 1: Schematic diagram of the surface of liquid helium, charged by electrons, in a parallel-plate capacitor.

of the liquid boundary.

The buildup of perturbations of the surface inevitably transforms the system to a state in which its evolution is determined by nonlinear processes. The nature of their effect can be estimated most easily in the vicinity of the instability threshold, i.e., for a small supercriticality $\varepsilon = (E^2 + E'^2 - E_c^2)/E_c^2$, when only perturbations with wave numbers close to $k_0 = \sqrt{g\rho/\alpha}$ increase and we can pass to envelopes in the equations of motion. For example, Gor’kov and Chernikova \[8\] proved that, in the case of 2D symmetry of the problem, the complex amplitude $A(x,t)$ of perturbation of the surface obeys the nonlinear Klein-Gordon equation

$$(gk_0)^{-1} A_{tt} = 2\varepsilon A + k_0^{-2} A_{xx} + \left(2S^2 - 5/8\right) A|A|^2,$$  \hspace{1cm} \text{(2)}$$

where $S = (E^2 - E'^2)/E_c^2$ is the dimensionless parameter characterizing the surface charge density. It can be seen from Eq. (2) that, depending on the value of parameter $S$, the nonlinearity either saturates the instability, or, conversely, facilitates a burst of the perturbation amplitude. A similar conclusion can also be drawn in the general (3D) case with a correction taking into account the fact that the nonlinearity in the first nonvanishing order plays a destabilizing role due to the interaction of three waves forming the hexagonal struc-
ture. As in 2D case, cubic nonlinearities produce a stabilizing effect for small values of $S$. Consequently, for a low surface charge density (when the values of $E$ and $E'$ are close), a steady-state relief of the liquid helium boundary may be formed. In this case, the standard perturbation theory in the small parameter, viz., the characteristic slope of the surface, can be used for studying the structures being generated (see [11] and the literature cited therein).

The processes occurring in the supercritical region of electric fields and for relatively large electron surface charge screening the field above the liquid surface to a considerable extent have not been investigated in detail theoretically. This is due to the fact that, in these cases, the development of instability violates the small-angle approximation. For example, the analysis of the behavior of the charged boundary of liquid helium by high-speed microphotography carried out by Vololdin et al. [12] proved that the dimples appearing on the surface are sharpened over a finite time (the bubbles which are subsequently formed at the tips carry the charge from the helium surface to the positive plate of the capacitor). In view of the considerable nonlinearity of such processes, their description requires the construction of solutions to the fundamental equations of the electrohydrodynamics of liquid helium.

In the present work, it will be shown that, when the condition $E \gg E'$, which corresponds to complete screening of the field above the liquid by the surface electron charge, is satisfied along with the condition that the electric field strength considerably exceeds the critical value, $E \gg E_c$, the equations of motion of liquid helium have an infinitely large number of exact analytic solutions. Their analysis has facilitated a considerable advance in the analysis of unsolved problems in the electrohydrodynamics of liquids with a free surface, which are associated with the formation of singularities (cusps) and with considerable changes in the surface geometry (formation of bubbles).

In Section 2, the equations of a vortex-free flow of liquid helium with a free surface charge are considered. In the limit of a strong electric field, when the effect of the force of gravity and capillary forces can be neglected, the approach to an analysis of the liquid helium dynamics proposed in our earlier work [13] is developed. This approach is based on the separation of two branches corresponding to solutions increasing and decreasing with time in the equations of motion. In Section 3, it is shown that the asymptotic behavior of the system is given by the well-known equations describing the Laplacian growth in the 3D
geometry (the motion equipotential boundary with the velocity determined by the normal
derivative of the harmonic potential). Section 4 is devoted to an analysis of the dynamics of
the formation of cuspidal dimples on the helium surface in 2D geometry, when the Laplacian
growth equations have an unlimited number of exact nontrivial solutions. The propagation
of nonlinear surface waves in the short-wave region in which the surface pressure must be
taken into account along with the electrostatic pressure is considered in Section 5. It is
shown that the problem of the profile of a progressive electrocapillary wave at the liquid
helium boundary has exact analytic solutions similar to the Crapper solutions for capillary
waves [14]. These solution are used for obtaining a nonlinear dispersion relation for surface
waves of an arbitrary amplitude, whose analysis led to a number of conclusions concerning
the stability of the charged surface of liquid helium to finite-amplitude perturbations and
the domain of the existence of wave solutions to the electrohydrodynamic equations. In
Section 6, the simplest axisymmetric solutions of the equations of motion, describing the
pulling of the surface into the bulk of the liquid at a constant rate, are analyzed.

II. INITIAL EQUATIONS: THE LIMIT OF A STRONG FIELD

Let us consider the potential motion of an ideal dielectric liquid (liquid helium) with a
free surface charged by electrons in an electric field. We assume that, in the unperturbed
state, the boundary of the liquid is a flat horizontal surface \( z = 0 \) and the field vector is
directed along the \( z \) axis of our system of coordinates (Fig. II). We introduce a function
\( \eta(x, y, t) \) specifying the deviation of the boundary from the plane. Then, the shape of the
perturbed surface of liquid helium is described by the equation \( z = \eta(x, y, t) \). The velocity
potential \( \Phi \) for an incompressible liquid satisfied the Laplace equation

\[
\nabla^2 \Phi = 0,
\]

which must be supplemented with the dynamic boundary condition

\[
\Phi_t + \frac{(\nabla \Phi)^2}{2} = \frac{E^2 - (\nabla \varphi)^2}{8\pi \rho} + \frac{\alpha}{\rho} \nabla_\perp \cdot \frac{\nabla_\perp \eta}{\sqrt{1 + (\nabla_\perp \eta)^2}} - g \eta, \quad z = \eta(x, y, t),
\]

where \( \varphi \) is the electric potential in the liquid (we assume that the charge completely screens
the field above the helium surface). The first term on the right-hand side of the time-
dependent Bernoulli equation (4) is responsible for electrostatic pressure, the second is re-
sponsible for capillary pressure, and the third takes into account the effect of the field of
gravity. We assume that the characteristic spatial of surface perturbations is smaller than the size of the region occupied by the liquid. In this case, we can write

\[ \Phi \to 0, \quad z \to -\infty, \]

i.e., the motion of the liquid attenuates at infinity. The time evolution of the free surface is determined by the kinematic relation (the condition that the liquid does not flow through its boundary):

\[ \eta_t = \Phi_z - \nabla \eta \cdot \nabla \Phi, \quad z = \eta(x, y, t). \]

Finally, the electric potential \( \varphi \) in the absence of space charges satisfies the Laplace equation

\[ \nabla^2 \varphi = 0, \]

which must be solved under the condition that the liquid helium boundary is equipotential and the field is uniform at an infinitely large distance from the surface:

\[ \varphi = 0, \quad z = \eta(x, y, t), \]

\[ \varphi \to -Ez, \quad z \to -\infty. \]

It should be noted that, in zero electric field (\( E = 0 \) and, hence, \( \nabla \varphi = 0 \)) the above equations coincide with the equation of motion for a thick layer of liquid in the field gravity.

Let the electric field strength exceed considerably its critical value (\( E \gg E_c \)), and let the following relation hold for the characteristic wavelength \( \lambda \) of surface waves: \( \alpha E^{-2} \ll \lambda \ll E^2/(g\rho) \). It follows from the dispersion relation (1) that, in an analysis of small-amplitude surface perturbation, we can disregard the effect of both the capillary forces and the force of gravity. In Secton 4, we will prove that this statement also holds for finite-amplitude surface perturbations. This means that we can omit the last two terms on the right-hand side of the boundary condition (4) and take into account the electrostatic pressure alone.

Now we pass to the dimensionless notation, assuming that the unit of length is equal to \( \lambda \), the unit of electric field strength is \( E \), and the unit of time is \( \lambda E^{-1}(4\pi\rho)^{1/2} \). In this case, the equations of motion (3)–(9) assume the form

\[ \nabla^2 \varphi = 0, \quad \nabla^2 \Phi = 0, \]

\[ \nabla^2 \varphi = 0, \quad \nabla^2 \Phi = 0, \]
\[ \Phi_t + (\nabla \Phi)^2/2 + (\nabla \varphi)^2/2 = 1/2, \quad z = \eta(x, y, t), \tag{11} \]
\[ \eta_t = \Phi_z - \nabla_\perp \cdot \nabla_\perp \Phi, \quad z = \eta(x, y, t), \tag{12} \]
\[ \varphi = 0, \quad z = \eta(x, y, t), \tag{13} \]
\[ \Phi \to 0, \quad z \to -\infty, \tag{14} \]
\[ \varphi \to -z, \quad z \to -\infty. \tag{15} \]

Let us write these equations in the form which does not contain function \( \eta \) explicitly and introduce the perturbed harmonic potential \( \tilde{\varphi} = \varphi + z \) attenuating at infinity (\( \tilde{\varphi} \to 0 \) as \( z \to -\infty \)). At the boundary, we have \( \tilde{\varphi}|_{z=\eta} = \eta \). This readily leads to relations

\[ \eta_t = \frac{\tilde{\varphi}_t}{1 - \tilde{\varphi}_z|_{z=\eta}}, \quad \nabla_\perp \eta = \frac{\nabla_\perp \tilde{\varphi}|_{z=\eta}}{1 - \tilde{\varphi}_z|_{z=\eta}}, \]

which allow us to eliminate \( \eta \) from Eq. (12). The kinematic and dynamic boundary conditions (11) and (12) can be transformed to

\[ \tilde{\varphi}_t - \Phi_z = -\nabla \tilde{\varphi} \cdot \nabla \Phi, \quad z = \eta(x, y, t), \]
\[ \Phi_t - \tilde{\varphi}_z = -(\nabla \Phi)^2/2 - (\nabla \tilde{\varphi})^2/2, \quad z = \eta(x, y, t). \]

Adding and subtracting these equations, we obtain

\[ (\tilde{\varphi} + \Phi)_t - (\tilde{\varphi} + \Phi)_z = -(\nabla (\tilde{\varphi} + \Phi))^2/2, \quad z = \eta(x, y, t), \]
\[ (\tilde{\varphi} - \Phi)_t + (\tilde{\varphi} - \Phi)_z = + (\nabla (\tilde{\varphi} - \Phi))^2/2, \quad z = \eta(x, y, t), \]
i.e., the boundary conditions can be specified separately for the sum and the difference of the harmonic potentials \( \tilde{\varphi} \) and \( \Phi \). It is convenient to introduce a pair of auxiliary potentials

\[ \phi^{(\pm)}(x, y, z, t) = (\tilde{\varphi} \pm \Phi)/2. \]

Using these potentials, we can write the equations of motion in the following symmetric form:

\[ \nabla^2 \phi^{(\pm)} = 0, \tag{16} \]
\[ \phi_t^{(\pm)} = \pm \phi_z^{(\pm)} \mp (\nabla \phi^{(\pm)})^2, \quad z = \eta(x, y, t), \quad (17) \]

\[ \phi^{(\pm)} \to 0, \quad z \to -\infty, \quad (18) \]

while the shape of the liquid helium boundary is determined from the relation

\[ \eta = (\phi^{(+)} + \phi^{(-)})\big|_{z=\eta}. \quad (19) \]

Thus, the equations of motion can be split into two systems of equations for potentials \( \phi^{(+)} \) and \( \phi^{(-)} \), the relation between which is given by the implicit equation for the shape of the surface (19). It is important that these equations are compatible with the condition \( \phi^{(-)} = 0 \) or with the condition \( \phi^{(+)} = 0 \). In the next section, we will show that the former condition corresponds to the solutions of the problem whose amplitude increases with time, while the latter (which is of no interest to us), to damped solutions.

The possibility of separating equations into individual branches is due to the symmetries of the electrohydrodynamic equations, which can be easily seen when the Hamilton formalism is used. Indeed, the equations of motion (10)–(15) for a liquid with a free surface possess a Hamilton structure, the function \( \eta(x, y, t) \) and \( \psi(x, y, t) = \Phi \big|_{z=\eta} \) being canonically conjugate quantities [15],

\[ \psi_t = -\frac{\delta H}{\delta \eta}, \quad \eta_t = \frac{\delta H}{\delta \psi}, \]

where the Hamiltonian \( H \) coincides to within constants with the total energy of the system:

\[ H = K + P, \quad K = \int_{z \leq \eta} \frac{(\nabla \Phi)^2}{2} d^3r, \]

\[ P = \int_{z \leq \eta} \frac{1 - (\nabla \varphi)^2}{2} d^3r = -\int_{z \leq \eta} \frac{(\nabla \tilde{\varphi})^2}{2} d^3r. \]

It should be recalled that the harmonic potentials \( \Phi \) and \( \tilde{\varphi} \) attenuate for \( z \to -\infty \) and their values on the surface are defined by the functions \( \psi \) and \( \eta \), respectively. Consequently, if \( \psi = \eta \), then \( \Phi = \tilde{\varphi} \), and the kinetic energy functional \( K \) coincides, expect for the sing, with the potential energy functional \( P \). This allows us to write the Hamilton equations of motion using the functional \( K \) alone:

\[ \psi_t = -\frac{\delta K}{\delta \eta} + \left( \frac{\delta K}{\delta \eta} + \frac{\delta K}{\delta \psi} \right) \bigg|_{\psi=\eta}, \quad \eta_t = \frac{\delta K}{\delta \psi}. \]
It can be seen that, if we set $\psi = \eta$ in these equations, they will coincide. This means that the condition $\psi = \eta$ or (which is the same) the condition $\phi^{-} = 0$ is compatible with the equations of motion for liquid helium. Similarly, we can prove that the Hamilton equations coincide for $\psi = -\eta$, which corresponds to the condition $\phi^{+} = 0$. It should also be noted that the equations describing the evolution of the system on the branches $\phi^{+} = 0$ and $\phi^{-} = 0$ coincide except for the substitution $t \to -t$, which is associated with the time reversibility in the Hamilton equations of motion. In this case, the conditions $\phi^{(\pm)} = 0$ single out the solutions of the problem for which $H$ is equal to zero.

III. INCREASING BRANCH: STABILITY

In the linear approximation whose applicability is limited by the condition of the smallness of the slopes of the surface $|\nabla_{\perp} \eta| \ll 1$, the boundary conditions (17) assume the form

$$\phi^{(\pm)}_t = \pm \phi^{(\pm)}_z, \quad z = 0,$$

and Eqs. (16)–(19) split into two independent systems. The dispersion relations for these systems can be found by substituting potentials in the form $\phi^{(\pm)} \sim e^{kz \mp i kr_{\perp} \pm i \omega t}$. This gives

$$\omega^{(\pm)} = \pm i k$$

(the same result follows directly from the dispersion relation (1) considered in the strong field limit). It can be seen that, for one branch, small periodic perturbations of the surface increase exponentially with the characteristic times $k^{-1}$, while, for the other branch, these perturbations attenuate. In this case, for large periods of time, we can assume that $\phi^{-} = 0$ and consider only equations for potential $\phi^{+}$. Let us prove that this statement is also valid in the general case, when the evolution of the surface is described by nonlinear equations (16)–(19).

We assume that, in the nonlinear equations of motion (16)–(19),

$$\phi^{+} = \varphi + z, \quad \phi^{-} = 0,$$

which, in accordance with the results of linear analysis, isolates the solutions increasing with time. Passing to the moving frame of reference $\{x, y, z'\} = \{x, y, z - t\}$ in which the plane unperturbed surface of the liquid moves downwards (i.e., in the direction opposite to the $z'$
axis) at a constant velocity, after simple transformations, we obtain

$$\nabla^2 \varphi = 0,$$  \hspace{1cm} (20)

$$\eta_t' = \partial_n \varphi \sqrt{1 + (\nabla_\perp \eta')^2}, \quad z' = \eta'(x, y, t).$$  \hspace{1cm} (21)

$$\varphi = 0, \quad z' = \eta'(x, y, t)$$  \hspace{1cm} (22)

$$\varphi \to -z', \quad z' \to -\infty,$$  \hspace{1cm} (23)

where $\eta'(x, y, t) = \eta - t$ and $\partial_n$ denotes the derivative along the normal to the boundary of the liquid. These equations define explicitly the motion of the free charged surface of liquid helium $z' = \eta'(x, y, t)$. They coincide with the equations describing the so-called Laplacian growth, viz., the motion of the phase boundary with a velocity directly proportional to the normal derivative of a certain harmonic scalar field ($\varphi$ in our case). Depending on the chosen frame of reference, this field may have the meaning of temperature (Stefan’s problem in the quasi-stationary limit), electrostatic potential (electrolytic deposition), or pressure (flow through a porous medium).

Let us prove that the solutions of Eqs. (10)–(15) corresponding to system (20)–(23) are stable to small perturbations of potential $\phi^{(-)}$. It should be noted that the motion of the liquid boundary described by Eqs. (20)–(23) is always directed inwardly; this is associated with the principle of the extremum for harmonic functions. Let function $\eta'$ at the initial instant $t = 0$ be a single-valued function of variables $x$ and $y$. In this case, for $t > 0$, the following inequality holds: $\eta'(x, y, t) \leq \eta'(x, y, 0)$. In the original notation, we have

$$\eta(x, y, t) \leq \eta(x, y, 0) + t$$  \hspace{1cm} (24)

for any $x$ and $y$. This inequality remains valid for small perturbations of $\phi^{(-)}$ also, when the effect of potential $\phi^{(-)}$ in relation (19) can be disregarded as compared to the effect of potential $\phi^{(+)}$, and the motion of the boundary is described by the same Eqs. (20)–(23).

As regards the evolution of potential $\phi^{(-)}$, it is described, for small $|\nabla \phi^{(-)}|$, by Eqs. (16)–(18), where it is sufficient to consider the condition (17) at the boundary in the linear approximation:

$$\phi^{(-)}_t = -\phi^{(-)}_z, \quad z = \eta(x, y, t).$$
Let us suppose that, at the initial instant \( t = 0 \), the potential distribution is described by the following expression:

\[
\phi(-)|_{t=0} = \phi_0(x, y, z),
\]

where \( \phi_0 \) is a certain function which is harmonic for \( z \leq \eta(x, y, 0) \) and attenuating for \( z \to -\infty \). In this case, the temporal dynamics of potential \( \phi(-) \) is described by the expression

\[
\phi(-) = \phi_0(x, y, z - t).
\]

It can be seen from this expression that the singularities of the function \( \phi(-) \) are displaced in the direction of the \( z \) axis and can exist only in the region

\[
z > \eta(x, y, 0) + t.
\]  

A comparison of this inequality with (24) shows that the singularities of potential \( \phi(-) \) do not approach the liquid helium boundary \( z = \eta(x, y, t) \) and, hence, the value of the potential at the surface does not increase with time. It should be noted that, otherwise, the solutions obtained by us for \( \phi(-) \) would be inapplicable.

In view of incompressibility of the liquid, the level of its surface (the value of function \( \eta \) averaged over the spatial variables) is not displaced. On the other hand, the boundary of the region defined by inequality (25) and averaged over \( x \) and \( y \), in which singularities of the function \( \phi(-) \) occurs, moves upwards at a constant velocity. This means that the singularity moves away from the surface of liquid helium and the perturbation of \( \phi(-) \) relaxes to zero.

Thus, we have proved that, as \( t \to \infty \), we have

\[
\varphi(x, y, z, t) + z \to \Phi(x, y, z, t),
\]

and Eqs. (20)–(23) describe the asymptotic behavior of liquid helium with a charged surface in a strong electric field.

**IV. SOLUTIONS OF 2D EQUATIONS OF MOTION**

In the previous section, we proved that the analysis of the 3D potential motion of liquid helium in a strong electric field can be reduced to analysis of Eqs. (20)–(23) describing the three-dimensional Laplacian growth. The exact solvability of these equations in the 2D
geometry will allow us to effectively study the dynamics of the development of instability of the charged surface of a liquid, including the formation of singularities in it.

We assume that, in the system of equations (20)–(23), all quantities are independent of variable \( y \) (variable \( y' \)). We introduce the function \( w = v - i\varphi \) of the complex argument \( Z = x + iz' \), which is analytic for \( z' \leq \eta'(x, t) \) (this is the so-called complex potential of the field correct to a constant factor). Here, \( v \) is a function harmonically conjugate to \( \varphi \) and such that the condition \( v = \text{const} \) defines the electric field lines in the medium. Clearly, \( w \to Z \) as \( Z \to x - i\infty \).

It is convenient for the subsequent analysis to pass to a system of coordinates in which the role of the independent variable is played by quantity \( w \) and the role of the unknown function is played by function \( Z \) which is analytic in the lower half-plane of the complex variable \( w \) (i.e., for \( \varphi > 0 \)). It follows from condition (23) that the following condition holds at infinity:

\[
Z \to w, \quad w \to v - i\infty. \tag{26}
\]

We can also obtain the condition for \( Z \) at the boundary \( \varphi = 0 \) of the half-plane. The profile of the liquid helium surface can be specified by the parametric relations

\[
z' = z'(v, t) = \eta'(x(v, t), t), \quad x = x(v, t).
\]

Using these relations, we can easily express the normal velocity of the surface and the electric field strength appearing in formulas (21) in terms of the functions \( z'(v, t) \) and \( x(v, t) \):

\[
\frac{\eta'_t}{\sqrt{1 + \eta'_x^2}} = \frac{z'_v x_v - x_t z'_v}{\sqrt{z'_v^2 + x'_v^2}}, \quad \partial_n \varphi = -\frac{1}{\sqrt{z'_v^2 + x'_v^2}}.
\]

Substituting these relations into the condition (21) at the surface, we obtain

\[
z'_v x_v - x_t z'_v = -1,
\]

or, which is the same,

\[
\text{Im}(Z'^* Z_w) = 1, \quad w = v. \tag{27}
\]

Thus, we arrive at the problem of determining the function \( Z \), which is analytic in the lower half-plane of the complex variable \( w \) and satisfies conditions (26) and (27). The nonlinear condition (27) is the so-called Laplacian growth equation which is widely used for describing the 2D motion of the boundary between two liquids with noticeably different
viscosities \[16, 17\], the evolution of the free surface of a liquid in the field of gravity \[18, 19\], and so on. The Laplacian growth equation is integrable in the sense that it has an infinitely large number of particular solutions of the form \[20\]

\[
Z(w) = w - it - i \sum_{n=1}^{N} a_n \ln (w - w_n(t)) + i \left( \sum_{n=1}^{N} a_n \right) \ln (w - w_0(t)).
\]  

(28)

Here, \(a_n\) are complex constants, and the functions of time \(w_n\) satisfy the condition \(\text{Im}(w_n) > 0\) (singularities of the function \(Z\) can only be in the upper half-plane of the complex variable \(w\)). The last term in expression (28) was supplemented to ensure the fulfillment of condition (26) and, hence, the condition of localization of the perturbation of the surface in a certain region: \(\eta \to 0\) for \(|x| \to \infty\). We can set \(\text{Im}(w_0) \gg \text{Im}(w_n)\); in this case, the effect of this term on the evolution of the surface is negligibly small.

Substituting expression (28) into Eq. (27) and decomposing the obtained expression into simple fractions, we obtain a system of \(N\) ordinary differential equations for \(w_n(t)\):

\[
\dot{w}_n + i + i \sum_{m=1}^{N} a_m^* \frac{\dot{w}_n - \dot{w}_m^*}{w_n - w_m^*} = 0.
\]

Integration with respect to \(t\) leads to the following \(N\) transcendental equations:

\[
w_n + it + i \sum_{m=1}^{N} a_m^* \ln (w_n - w_m^*) = C_n,
\]

where \(C_n\) are arbitrary complex constants.

Let us consider the simplest solutions of this type, which correspond to \(N = 1\), \(\text{Re}(w_1) = 0\) and \(a_1 = \pm 1\):

\[
Z(w) = w - it \mp i \ln(w - iq(t)),
\]  

(29)

\[
q(t) \pm \ln q(t) = 1 + t_c - t,
\]  

(30)

where \(q = \text{Im}(w_1)\) and \(t_c\) is a real constant. The form of a solitary perturbation corresponding to Eqs. (29) and (30) is specified by the parametric expressions

\[
z(v, t) = z'(v, t) + t = \mp \ln \sqrt{v^2 + q^2(t)},
\]

\[
x(v, t) = v \pm \arctan (v/q(t)).
\]
Let us suppose that $a_1 = +1$ and we are dealing with a solitary perturbation of the surface, which is directed "upwards". It can be seen from Eq. (30) that for large values of $t$, the quantity $q \sim e^{-t}$ and, hence, the surface perturbation amplitude increases linearly with time: $z|_{v=0} \to t$ as $t \to \infty$. This is the "one-finger" solution of the Laplacian growth equation (see Fig. 2). It can easily be proved that similar solutions are possible in the 3D case also. It can be seen from Eqs. (20)–(23) describing the three-dimensional Laplacian growth that, if the surface initially contains a region in which the field strength $\partial_n \phi$ is small (e.g., in the vicinity of the apex of a 3D fingerlike perturbation of the surface), its velocity in the coordinates $\{x, y, z'\}$ is also small. In the laboratory reference frame, this corresponds to a jet flowing at a constant velocity in the direction of the $z$ axis.
FIG. 3: The profile of the liquid helium surface at the instant of formation of a singularity (cusp); $a_1 = -1$, $q = 0.8$, and $w_0 = i4$.

Let us now consider a solitary perturbation of the surface, which is directed "downwards" ($a_1 = -1$, $q(t) \geq 1$). This solution exists only during a finite period of time, leading to the formation of a singularity on the liquid surface, viz., cuspidal point of the first kind (Fig. 3), at instant $t = t_c$. Indeed, expanding $z$ and $x$ into power series in $v$ and $\tau$ taking into account the fact that the function $q(t)$ in the vicinity of the $t_c$ satisfies the relation

$$q(t) \approx 1 + \sqrt{2\tau}, \quad \tau = t_c - t,$$

we obtain the following expressions in the main order:

$$z = v^2/2 + \sqrt{2\tau}, \quad x = v^3/3 + v\sqrt{2\tau}. \quad (31)$$

It can be seen that, at instant $\tau = 0$ (i.e., for $t = t_c$), the shape of the surface in the vicinity of a singularity point is defined by the relation $2z = |3x|^{2/3}$, which corresponds to a cusp [31]. It was indicated in [17, 23] that the singularities of $z^3 \sim x^2$ are general-position singularities for processes described by the Laplacian growth equation. Similar solutions of the equations of motion for liquid helium with the charged boundary reflect the experimentally observed tendency [12, 24] to the emergence of dimples on the surface, which become sharpened over a finite time. From the mathematical point of view, the emergence of singularity on the liquid
surface is associated with vanishing of the Jacobian of the transformation \( \{x, z'\} \rightarrow \{v, \phi\} \) for \( \varphi = v = \tau = 0 \). At a cusp, the electric field strength increases indefinitely along with the velocity of the surface over a finite time interval:

\[
|\nabla \varphi| \sim x_v^{-1} \big|_{v=0} \sim \tau^{-1/2}, \quad |\nabla \Phi| \sim z_t \big|_{v=0} \sim \tau^{-1/2}.
\]

It is important to note that the singular solution of the problem described by expressions (31) is also valid in the case when the field above the surface is not screened completely; i.e., the condition \( E' \ll E \) does not hold. As a matter of fact, in the vicinity of a singularity, the condition of the smallness of the field above the surface as compared to the field in the bulk of the liquid naturally holds. In addition, the condition \( \lambda \ll E^2/(g\rho) \) is not necessary. This is due to the fact that the amplitude of surface perturbations remains finite, and the effect of the gravity forces is always negligibly small in the vicinity of the cusp.

Let us now consider the capillary effects. The surface and the electrostatic pressure in the vicinity of a singularity can be estimated easily:

\[
\begin{align*}
P_S & \sim \alpha \eta_{ex} \sim \alpha \rho^{1/2} E^{-1} \tau^{-1}, \\
P_E & \sim (\nabla \phi)^2 \sim \lambda \rho^{1/2} E \tau^{-1}.
\end{align*}
\]

Here, we have returned to the dimensional notation. It can be seen from these expressions that, when the condition \( \lambda \gg \alpha E^{-2} \) is satisfied, the capillary forces are small as compared to electrostatic forces and, hence, can be disregarded at the stage of formation of cusps. This is the only necessary condition of the applicability of the Laplacian growth equation and its solutions (31) in the vicinity of singularities.

**V. ELECTROCAPILLARY WAVES**

Let us consider the case when the characteristic length of the surface waves is comparable with the value of \( \alpha E^{-2} \) and the capillary effects must be taken into consideration. We assume that condition \( E \gg E_c \) is satisfied; in this case, the effect of the force of gravity can be neglected. The dispersion relation (1) for electrocapillary waves at the charged boundary of liquid helium for \( E' = 0 \) in the dimensionless notation introduced in Secton 2 assumes the form

\[
\omega^2(k) = k^3 - k^2,
\]

where the value of \( \lambda = 4\pi \alpha E^{-2} \) is taken for unit length. It can be seen from Eq. (32) that \( \omega^2 < 0 \) for \( k < 1 \) and, hence, aperiodic electrohydrodynamic instability of the liquid
surface develops. If, however, the condition \( k > 1 \) holds, the frequency \( \omega \) is real-valued, which corresponds to the propagation of linear dispersive waves.

The approach to the study of the evolution of a charged liquid surface based on the analysis of relation (32) is obviously applicable only in the case of small-amplitude perturbations of the boundary: \( A \ll k^{-1} \). For finite-amplitude waves, the nonlinear effect may consist in the dependence of the dispersion relation on \( A \) (see, for example, [25]):

\[
\omega = \omega(k, A).
\]

The amplitude dependence of frequency is usually sought in the form of a power series in \( A \) (Stokes expansion), which limits the analysis to the weak-nonlinearity limit. Let us prove that, for electrocapillary waves, an exact solution to the nonlinear dispersion relation can be found.

The equations describing a progressive wave (whose profile does not change in the reference frame attached to the wave) can be obtained from the electrohydrodynamic equations (3)–(9) with the help of the following substitutions:

\[
\varphi = \varphi(x', z), \quad \Phi = \Phi'(x', z) + Cx', \quad \eta = \eta(x'),
\]

where \( x' = x - Ct \) and constant \( C \) has the meaning of the velocity of a wave moving in the direction of the \( x \) axis. This gives

\[
\Phi'_{x'x'} + \Phi'_{zz} = 0,
\]

\[
\varphi_{x'x'} + \varphi_{zz} = 0,
\]

\[
\frac{\Phi'_{x'}^2 + \Phi'_{z}^2 - C^2}{2} + \frac{\varphi_{x'}^2 + \varphi_{z}^2 - 1}{2} = \frac{\eta_{x'x'}}{(1 + \eta_{x'}^2)^{3/2}}, \quad z = \eta(x'),
\]

\[
\Phi'_{z} = \eta_{x'} \Phi'_{x'}, \quad z = \eta(x'),
\]

\[
\varphi = 0, \quad z = \eta(x'),
\]

\[
\Phi' \rightarrow -Cx', \quad z \rightarrow -\infty,
\]
\[ \phi \to -z, \quad z \to -\infty. \]  

(39)

These equations can be simplified by introducing the function of current \( \Psi(x', z) \), which is harmonically conjugate to potential \( \Phi' \):

\[ \Psi_{x'} = -\Phi'_z, \quad \Psi_z = \Phi'_{x'}. \]

This function satisfies the Laplace equation

\[ \Psi_{x'x'} + \Psi_{zz} = 0 \]  

(40)

with the boundary conditions

\[ \Psi = 0, \quad z = \eta(x'), \]  

(41)

\[ \Psi \to -Cz, \quad z \to -\infty, \]  

(42)

which follow from relations (36) and (38). It can easily be seen that Eqs. (40)–(42) coincide with the Eqs. (34), (37) and (39) for the electric potential. Consequently, the following functional relation exists:

\[ \Psi = C\phi. \]

Using this relation, we can considerably simplify the Bernoulli equation (35), which assume the form

\[ \frac{C^2 + 1}{2} \left( \phi_{x'}^2 + \phi_z^2 - 1 \right) = \frac{\eta_{x'x'}}{(1 + \eta_{x'}^2)^{3/2}}, \quad z = \eta(x'). \]  

(43)

In combination with relations (34), (37), and (39), this condition completely defines the shape of a wave propagating in the coordinate system \( \{x', z\} \).

Equations (34), (37), (39) and (43) coincide except for constant factors with the equations describing the shape of a progressive capillary wave [14] and an equilibrium configuration of the charged surface of the liquid metal [20]. These equations have exact periodic solutions for which the boundary of the liquid is defined by the parametric expressions

\[ z = \frac{4k^{-2}}{2(C^2 + 1)^{-1} + A\cos(kp)} + z_0, \]  

(44)

\[ x' = p - \frac{2Ak^{-1}\sin(kp)}{2(C^2 + 1)^{-1} + A\cos(kp)} + x'_0, \]  

(45)
where \( z_0 \) and \( x_0' \) are constants, \( p \) is a parameter (the value of \( p \) changes over a period by \( 2\pi/k \)), and the quantity \( A \) has the meaning of the amplitude of a surface perturbation; i.e., \( A = (z_{\text{max}} - z_{\text{min}})/2 \). The dependence of \( A \) on \( k \) and \( C \) is specified by the relation

\[
A = \left[ \frac{4}{(C^2 + 1)^2} - \frac{4}{k^2} \right]^{1/2}.
\]

(46)

It was mentioned in [14] that solutions (44) and (45) exist only for \( 1 \leq k/(C^2 + 1) \leq \gamma \), where \( \gamma \approx 1.52 \).

Considering that \( C \) is the phase velocity of the wave, we set \( C = \omega/k \) in relation (46). Solving the obtained equation for frequency \( \omega \), we arrive the exact nonlinear dispersion relation

\[
\omega^2(k, A) = \frac{k^3}{\sqrt{1 + A^2 k^2/4}} - k^2,
\]

(47)

and the conditions of its applicability

\[
k^3 \gamma^{-1} \leq \omega^2 - k^2 \leq k^3.
\]

(48)

It can be seen that, in the limit of infinitely small amplitudes \( (A \to 0) \), expression (47) is transformed into the linear dispersion relation (32). Let us consider the consequences of this nonlinearity. It can be seen from relation (47) that, for a fixed wave number \( k \geq 1 \), the maximum value of the surface perturbation amplitude \( A_{\text{max}}(k) \) corresponds to the minimum possible value of \( \omega^2 \). It follows from conditions (48) that, for \( 1 \leq k \leq \gamma \), the value of \( \omega_{\text{min}}^2 = 0 \), which corresponds to a wave with zero velocity. In this case, expressions (44) and (45) define the solution of the problem on the steady-state profile of the charged surface of the liquid helium. For \( k > \gamma k_1 \), the amplitude has the maximum value for electrocapillary waves propagating at the velocity \( C = \sqrt{k \gamma^{-1} - 1} \); in this case, \( \omega_{\text{min}}^2 = k^3 \gamma^{-1} - k^2 \). This gives

\[
A_{\text{max}}(k) = \begin{cases} 
0, & 0 \leq k < 1 \\
2\sqrt{1 - k^{-2}}, & 1 \leq k \leq \gamma \\
2k^{-1}\sqrt{\gamma^2 - 1}, & k > \gamma
\end{cases}
\]

(the curve describing this dependence is presented in Fig. 4). If the amplitude exceeds this value, expressions (44) and (45) describe a self-intersecting surface, which cannot be realized from the physical considerations, or \( \omega^2 < 0 \), which corresponds to incorrectly formulated problem in the context wave propagation. This leads to the assumption that the condition
FIG. 4: The maximum value of amplitude $A_{\text{max}}$ of an electrocapillary wave on the charged surface of liquid helium as a function of the wave number $k$. For $k < \gamma$, the peak corresponds to the value of $\omega = 0$, while, for $k > \gamma$, the frequency differs from zero.

$A(k) > A_{\text{max}}(k)$ is the criterion of hard excitation of electrohydrodynamic instability of a plane charged surface of liquid helium, which generalizes the simplest linear instability criterion $k < 1$ to the case of finite-amplitude perturbation.

It should be noted that peak of the function $A_{\text{max}}(k)$ corresponds to $k = \gamma$. The shape of the liquid surface corresponding to this value of the wave number is depicted in Fig. 5. It can be seen that the liquid acquires cavities. Such solutions reflect the tendency to the formation of charged bubbles (referred to as bubblons in the experimental work [12]) on cuspidal dimples of the liquid helium boundary. The main mechanism of departure of electrons from the surface is associated with the generation of such bubbles.
VI. AXISYMMETRIC SOLUTIONS

Let us consider the evolution of the charged surface of liquid helium in an important case of the axial symmetry of the problem. The equations of motion (20)–(23) corresponding to the increasing branch of the solutions to system (10)–(15) in the cylindrical coordinaties
\{r, z'\} = \{r, z - t\} assume the form

\[ \varphi_{rr} + r^{-1} \varphi_r + \varphi_{z'z'} = 0, \]

\[ \eta'_t = -(\varphi_r^2 + \varphi_z^2)^{\frac{1}{2}}(1 + \eta'_r^2)^{\frac{1}{2}}, \quad z' = \eta'(r, t), \]

\[ \varphi = 0, \quad z' = \eta'(r, t), \]

\[ \varphi \to -z', \quad z' \to -\infty. \]

Here, \( r = \sqrt{x^2 + y^2} \) and we have taken into account the fact that \( \partial_n \varphi = -|\nabla \varphi| \) at the equipotential boundary in condition (21).

At essentially nonlinear stages of the formation of a dimple on the surface of a liquid, we can assume that the electric field in the region of a large curvature of the surface is much stronger than the external field, \( |\nabla \varphi| \gg 1 \). In this case, the dynamics of the boundary \( \eta' = \eta - t \) is completely determined by the intrinsic field rapidly attenuating with increasing distance, which allows us to use the condition \( |\nabla \varphi| \to 0 \) for \( z \to -\infty \) instead of the condition of field uniformity. We will also assume that the velocity of the liquid surface is considerably higher than the velocity of the origin in the laboratory reference frame (i.e., \( |\eta_t| \gg 1 \)). In this case, we can substitute \( \eta \) for \( \eta' \) and \( z \) for \( z' \). This gives

\[ \varphi_{rr} + r^{-1} \varphi_r + \varphi_{zz} = 0, \quad z < \eta(r, t), \] 

(49)

\[ \varphi_t = -\varphi_r^2 - \varphi_z^2, \quad z = \eta(r, t), \] 

(50)

\[ \varphi = 0, \quad z = \eta(r, t), \] 

(51)

\[ \varphi_r^2 + \varphi_z^2 \to 0, \quad r^2 + z^2 \to \infty. \] 

(52)

In relation (50), we have used the following conditions at the boundary of the liquid:

\[ \eta_t = -\varphi_t/\varphi_z, \quad \eta_r = -\varphi_r/\varphi_z. \]

The conditions of the applicability of this approximation will be considered at the end of this section.
A particular solution of Eqs. (49)–(52) can be obtained by using a substitution similar to that used in [27] for constructing the axisymmetric solutions of the Stefan problem:

\[ \varphi(r, z, t) = f(u(r, z, t)), \]

\[ u(r, z, t) = -z - Vt + \sqrt{r^2 + (z + Vt)^2}, \]

where the constant \( V \) has the meaning of the inward-directed velocity of the liquid surface. It can easily be seen that the equipotential surfaces corresponding to Eqs. (53) and (54) form a family of confocal paraboloids of revolution:

\[ r^2 = 2u(z + Vt) + u^2 \]

with the focus at the point \( r = 0 \) and \( z = -Vt \).

Substituting expressions (53) and (54) into Eq. (49), we arrive at the following ordinary differential equation:

\[ uf_{uu} + f_u = 0. \]

It follows from Eqs. (50) and (51) that the boundary conditions to this equation have the form

\[ f_u(u_0) = V/2, \quad f(u_0) = 0. \]

Here, \( u_0 \) is the value of parameter \( u \) at the surface of the liquid. Henceforth, we will use the quantity \( K = 1/u_0 \) which, in accordance with Eq. (55), defines the curvature of the liquid surface at the symmetry axis. Solving Eqs. (56) and (57), we obtain

\[ f(u) = V \ln(Ku)/(2K), \]

which, together with Eqs. (53) and (54), describes the time evolution of the electric potential. It should be noted that condition (52) is naturally satisfied. The shape of the surface for the given exact solution of the equations of motion (49)–(52) is defined by the relation

\[ \eta(r, t) = Kr^2/2 - Vt - (2K)^{-1}, \]

which corresponds to needle-shaped dimple pulled into the bulk of the liquid velocity \( V \). Such a geometry of the surface perturbation can be regarded as the simplest (paraboloidal)
approximation of the shape of the liquid boundary at essentially nonlinear stages of the
development of instability of the charged boundary of the liquid.

It should be recalled that the applicability of approximation (49)–(52) of the initial system
(20)–(23) is limited by the conditions $|\eta_t| \gg 1$ and $|\nabla \varphi| \gg 1$. Since $\eta_t = -V$ in solutions
(59) for any $r$ and $t$, the first condition is reduced to the inequality $V \gg 1$ (in the dimensional
notation, $V \gg E \sqrt{4\pi \rho}$). As regards the second condition, we can find the characteristic size $D$
of the region in which the electric field created by a charged paraboloidal surface exceeds
the external field. It follows from relations (53), (54) and (58) that the field distribution in
the liquid is described by the relation

$$|\nabla \varphi| = \frac{V}{K \sqrt{2Ru}}.$$

Here, $R = \sqrt{r^2 + (z + Vt)^2}$ is the distance to the focus of the paraboloid; i.e., the field
attains it maximum value equal to $V$ at the point $r = 0$ and $z = -Vt - (2K)^{-1}$. Since
the field strength generally decreases in proportion to $R^{-1}$ with increasing distance from
the focus, the scale of $D$ can be estimated as $D \sim V/K$. It should be noted that such a
conclusion makes sense only if the value of $D$ is much larger than the radius of the curvature
$K^{-1}$ of the surface perturbation. This requirement again leads us to the inequality $V \gg 1$.

Thus, we have obtained partial axisymmetric solutions to the equations of motion of
liquid helium with a charged surface, which describe the evolution of a localized perturbation
of the surface with a considerable curvature, and have determined the conditions of their
applicability. However, the obtained solutions should not be regarded as general-position
solutions. In all probability, solutions of the burst type, for which the surface becomes
indefinitely cuspidate over a finite time interval, will dominate as in the 2D case.

VII. CONCLUDING REMARKS

In the absence of a surface charge, the equations of electrohydrodynamics of liquid helium
considered by us are transformed into the well-known equations of a vortex-free flow of
an incompressible liquid with a free boundary. These equations are extremely difficult to
analyze, and the methods for the solution have not been developed at present. In this work,
we succeeded in proving that the inclusion of the electrostatic pressure does not complicate
the analysis of these equations. On the contrary, the emergence of an additional term in
the dynamic boundary condition introduces a certain symmetry into the equations so that they become compatible with the conditions \( \varphi + z = \pm \Phi \). The emerging functional relation between the potentials of velocity and of electric field makes it possible to reduce by half the number of equations required for describing the motion of the surface and, in the long run, to find a wide class of exact solutions of the equations of motion of liquid helium with the boundary charged by electrons. It is important that the solutions obtained by us are not limited by the condition of smallness of surface perturbations; they describe the evolution of the liquid boundary up to the formation of cuspidal points in it.

The dynamics of the formation of singularities in the case when the characteristic scale \( \lambda \) of surface perturbations is comparable with the value of \( \alpha E^{-2} \) and the capillary effects must be taken into consideration has not been considered by us here. In 2D geometry, such an analysis can be carried out using the methods of investigations of 2D potential flows with a free boundary, which was proposed in [28, 29] and is based on conformal mapping of the region occupied by the liquid to a half-plane. In terms of the present work, such a transformation corresponds to the use of the field potential \( \varphi \) and its harmonically conjugate function \( v \) as independent variables. In the case of the axial symmetry of the problem (such a geometry reflects the experimentally observed phenomena [12, 24] most correctly), the formation of singularities can be described by self-similar solutions of the electrohydrodynamic equations, which are analogous to those considered in the recent publication [30] devoted to the formation of conic tips on the surface of a liquid metal in an external electric field. In accordance with the self-similar scenario of the development of instability, conical dimples with an angle of 98.6° appear on the surface over a finite time. A detailed analysis of these processes calls for further investigations.

The author is grateful to V. E. Zakharov and E. A. Kuznetsov for their interest in this research.

---

[1] M. W. Cole, M. H. Cohen, Phys. Rev. Lett. 23, 1238 (1969).

[2] V. B. Shikin, Zh. Eksp. Teor. Fiz., 58, 1748 (1970) [Sov.Phys. JETP 31, 936 (1970)].

[3] L.P. Gor’kov and D.M. Chernikova, Pis’ma Zh. Eksp. Teor. Fiz., 18, 119 (1973) [JETP Lett. 18, 68 (1973)].
[4] D.M. Chernikova, Fiz. Nizk. Temp. 2, 1374 (1976) [Sov. J. Low Temp. Phys. 2, 669 (1976)].
[5] L. Tonks, Phys. Rev. 48, 562 (1935).
[6] Ya.I. Frenkel’, Zh. Eksp. Teor. Fiz., 6, 347 (1936).
[7] L.D. Landau and E.M. Lifshitz, Course of Theoretical Physics, Vol. 8: Electrodynamics of Continuous Media (Nauka, Moscow, 1982; Pergamon, New York, 1984).
[8] L.P. Gor’kov and D.M. Chernikova, Dokl. Akad. Nauk SSSR, 228, 829 (1976) [Sov. Phys. Dokl., 21, 328 (1976)].
[9] H. Ikezi, Phys. Rev. Lett. 42, 1688 (1979).
[10] D.M. Chernikova, Fiz. Nizk. Temp. 6, 1513 (1980) [Sov. J. Low Temp. Phys. 6, 737 (1980)].
[11] V. B. Shikin and Yu. P. Monarkha, Two-Dimensional Charged Systems in Helium, (Nauka, Moskow, 1989).
[12] A. P. Volodin, M. S. Haikin, and V. S. Edel’man, Pis’ma Zh. Eksp. Teor. Fiz. 26, 707 (1977) [JETP Lett. 26, 543 (1977)].
[13] N. M. Zubarev, Pis’ma Zh. Eksp. Teor. Fiz. 71, 534 (2000) [JETP Lett. 71, 367 (2000)].
[14] G. D. Crapper, J. Fluid Mech. 2, 532 (1957).
[15] V.E. Zakharov, Prikl. Mekh. Tekh. Fiz., No. 2, 86 (1968).
[16] P. Ya. Polubarinova-Kochina, Dokl. Akad. Nauk SSSR, 47, 254 (1945).
[17] D. Bensimon, L. P. Kadanoff, Sh. Liang, et al., Rev. Mod. Phys. 58, 977 (1986).
[18] A. I. Dyachenko, V.E. Zakharov and E.A. Kuznetsov, Phys. Plazmy 22, 916 (1996).
[19] V. E. Zakharov and A. I. Dyachenko, Physica D 98, 652 (1996).
[20] M. B. Mineev-Weinstein and S. P. Dawson, Phys. Rev. E 50, R24 (1994).
[21] N. M. Zubarev, Phys. Lett. A 243, 128 (1998).
[22] N. M. Zubarev, Zh. Eksp. Teor. Fiz., 114, 2043 (1998) [Sov. Phys. JETP 87, 1110 (1998)].
[23] S. D. Howison, SIAM J. Appl. Math. 46, 20 (1986).
[24] V. S. Edel’man, Usp. Fiz. Nauk, 130, 675 (1980) [Sov. Phys. Usp., 23, 227 (1980)].
[25] G. B. Whitham, Linear and Nonlinear Waves, (Wiley, New York, 1974; Mir, Moskow, 1977).
[26] N. M. Zubarev, Zh. Eksp. Teor. Fiz., 116, 1990 (1999) [JETP 89, 1078 (1999)].
[27] G. P. Ivantsov, Dokl. Akad. Nauk SSSR 58, 567 (1947).
[28] A. I. Dyachenko, E. A. Kuznetsov, M. D. Spector, and V. E. Zakharov, Phys. Lett. A 221, 73 (1996).
[29] A. I. Dyachenko, Dokl. Akad. Nauk 376, 27 (2001).
[30] N. M. Zubarev, Pis’ma Zh. Eksp. Teor. Fiz., 73, 613 (2001) [JETP Lett. 73, 544 (2001)].

[31] On the charged surface of a conducting liquid, for which $E = 0$ and $E' \neq 0$, in the limit of a strong field, weak root singularities of the type $z \sim |x|^{3/2}$ are formed, for which curvature is equal to infinity, while the surface itself remains smooth [21, 22].