A flat plane that is not the limit of periodic flat planes

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Abstract We construct a compact nonpositively curved squared 2-complex whose universal cover contains a flat plane that is not the limit of periodic flat planes.

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1 Introduction

Gromov raised the question of which “semi-hyperbolic spaces” have the property that their flats can be approximated by periodic flats [4, §6.B3]. In this note we construct an example of a compact nonpositively curved squared 2-complex $Z$ whose universal cover $\tilde{Z}$ contains an isometrically embedded flat plane that is not the limit of a sequence of periodic flat planes.

A flat plane $E \hookrightarrow \tilde{Z}$ is periodic if the map $E \rightarrow Z$ factors as $E \rightarrow T \rightarrow Z$ where $E \rightarrow T$ is a covering map of a torus $T$. Equivalently, $\pi_1 Z$ contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$ which stabilizes $E$ and acts cocompactly on it. A flat plane $f: E \hookrightarrow \tilde{Z}$ is the limit of periodic flat planes if there is a sequence of periodic flat planes $f_i: E \hookrightarrow \tilde{Z}$ which converge pointwise to $f: E \rightarrow \tilde{Z}$. In our setting, $\tilde{Z}$ is a 2-dimensional complex, and so $E \hookrightarrow \tilde{Z}$ is the limit of periodic flat planes if and only if every compact subcomplex of $E$ is contained in a periodic flat plane.

In Section 2 we describe a compact nonpositively curved 2-complex $X$ whose universal cover contains a certain aperiodic plane called an “anti-torus”. In Section 3 we construct $Z$ from $X$ by strategically gluing tori and cylinders to $X$ so that $\tilde{Z}$ contains a flat plane which is a mixture of the anti-torus and periodic planes. This flat plane is not approximable by periodic flats because it contains a square that does not lie in any periodic flat. Our example $Z$ is a $K(\pi, 1)$ for a negatively curved square of groups, and in Section 4 we describe an interesting related triangle of groups.
Figure 1: The figure above indicates the gluing pattern for the six squares of $X$. The three vertical edges colored white, grey, and black are denoted $a$, $b$, and $c$ respectively. The two horizontal edges, single and double arrow, are denoted $x$ and $y$ respectively.

2 The anti-torus in $X$

2.1 The 2-complex $X$

Let $X$ denote the complex consisting of the six squares indicated in Figure 1. The squares are glued together as indicated by the oriented labels on the edges. Note that $X$ has a unique 0-cell, and that the notion of vertical and horizontal are preserved by the edge identifications. Let $H$ denote the subcomplex consisting of the 2 horizontal edges, and let $V$ denote the subcomplex consisting of the 3 vertical edges.

The complex $X$, which was first studied in [8], has a number of interesting properties that we record here: The link of the unique 0-cell in $X$ is a complete bipartite graph. It follows that the universal cover $\tilde{X}$ is the product of two trees $\tilde{H} \times \tilde{V}$ where $\tilde{H}$ and $\tilde{V}$ are the universal covers of $H$ and $V$. In particular, the link contains no cycle of length $< 4$ and so $X$ satisfies the combinatorial nonpositive curvature condition for squared 2-complexes [3, 1] which is a special case of the $C(4)\cdot T(4)$ small-cancellation condition [6].

The 2-complex $X$ was used in [8] to produce the first examples of non-residually finite groups which are fundamental groups of spaces with the above properties. The connection to finite index subgroups arises because while $\tilde{X}$ is isomorphic to the cartesian product of two trees, $X$ does not have a finite cover which is the product of two graphs.

2.2 The anti-torus $\Pi$

The exotic behavior of $X$ can be attributed to the existence of a strangely aperiodic plane $\Pi$ in $\tilde{X}$ that we shall now describe. Let $\tilde{x} \in \tilde{X}^0$ be the basepoint of $\tilde{X}$. Let $c^\infty$ denote the infinite periodic vertical line in $\tilde{X}$ which is the based component of the preimage of the loop labeled by $c$ in $X$. Define $y^\infty$
Figure 2: The Anti-Torus $\Pi$: The plane $\Pi$ above is the convex hull of two periodically labeled lines in $\tilde{X}$. A small region of the northeast quadrant has been tiled by the squares of $X$.

analogously. Let $\Pi$ denote the convex hull in $\tilde{X}$ of the infinite geodesics labeled by $c^\infty$ and $y^\infty$, so $\Pi = y^\infty \times c^\infty$. The plane $\Pi$ is tiled by the six orbits of squares in $\tilde{X}$ as in Figure 2. The reader can extend $c^\infty \cup y^\infty$ to a flat plane by successively adding squares wherever there is a pair of vertical and horizontal edges meeting at a vertex. From a combinatorial point of view, the existence and uniqueness of this extension is guaranteed by the fact that the link of $X$ is a complete bipartite graph.

The “axes” $c^\infty$ and $y^\infty$ of $\Pi$ are obviously periodic, and using that $X$ is compact, it is easy to verify that for any $n \in \mathbb{N}$, the infinite strips $[-n, n] \times \mathbb{R}$ and $\mathbb{R} \times [-n, n]$ are periodic. However, the period of these infinite strips increases exponentially with $n$. Thus, the entire plane $\Pi$ is aperiodic. Note that to say that $[-n, n] \times \mathbb{R}$ is periodic means that the immersion $([-n, n] \times \mathbb{R}) \looparrowright X$ factors as $([-n, n] \times \mathbb{R}) \rightarrow C \looparrowright X$ where $([-n, n] \times \mathbb{R}) \rightarrow C$ is the universal covering map of a cylinder. The map $\Pi \looparrowright X$ is aperiodic in the sense that it does not factor through an immersed torus.

We conclude this section by giving a brief explanation of the aperiodicity of $\Pi$. A complete proof that $\Pi$ is aperiodic is given in [8]. Let $W_n(m)$ denote the word corresponding to the length $n$ horizontal positive path in $\Pi$ beginning at the endpoint of the vertical path $c^m$. Thus, $W_n(m)$ is the label of the side opposite $y^n$ in the rectangle which is the combinatorial convex hull of $y^n$ and $c^m$. Equivalently, $W_n(m)$ occupies the interval $\{m\} \times [0, n]$. For each $n$, the words $\{W_n(m) \mid 0 \leq m \leq 2^n - 1\}$ are all distinct! Consequently every positive length $n$ word in $x$ and $y$ is $W_n(m)$ for some $m$. This implies that the infinite
Figure 3: The complex $Y$ is formed by gluing four cylinders to a square.

strip $[0,n] \times \mathbb{R}$ has period $2^n$, and in particular $\Pi$ cannot be periodic.

We refer to $\Pi$ as an \textit{anti-torus} because the aperiodicity of $\Pi$ implies that $c$ and $y$ do not have nonzero powers which commute. Indeed, if $c^p$ and $y^q$ commuted for $p, q \neq 0$ then the flat torus theorem (see [1]) would imply that $c^\infty$ and $y^\infty$ meet in a periodic flat plane, which would contradict that $\Pi$ is aperiodic.

3 \hspace{1em} \textbf{The 2-complex $Z$ with a nonapproximable flat}

We first construct a new complex $Y$ as follows: Start with a square $s$, and then attach four cylinders each of which is isomorphic to $S^1 \times I$. One such cylinder is attached along each side of $s$. The resulting complex $Y$ containing exactly five squares is illustrated in Figure 3.

Let $T^2$ denote the torus $S^1 \times S^1$ with the usual product cell structure consisting of one 0-cell, two 1-cells, and a single square 2-cell. We let $\tilde{T}^2$ denote the universal cover and we shall identify $\tilde{T}^2$ with $\mathbb{R}^2$.

At each corner of $s \subset Y$, there is a pair of intersecting circles in $Y^1$, which are boundary circles of distinct cylinders. Note that they meet at an angle of $\frac{3\pi}{2}$ in $Y$. At each of three (NW, SW, & SE) corners of $s \subset Y$ we attach a copy of $T^2$ by identifying the pair of circles in the 1-skeleton of $T^2$ with the pair of intersecting circles noted above at the respective corner of $s$. At the fourth (NE) corner of $s$, we attach a copy of the complex $X$. Here we identify the pair of circles meeting at the corner of $s$ with the pair of perpendicular circles $c$ and $y$ of $X$. We denote the resulting complex by $Z$. Thus, $Z = T^2 \cup T^2 \cup T^2 \cup Y \cup X$. See Figure 4 for a depiction of the 8 squares of $Z - X$ and their gluing patterns.

\textbf{Definition 3.1} \hspace{0.5em} \textit{Infinite cross} An infinite cross is a squared 2-complex isomorphic to the subcomplex of $\tilde{T}^2$ consisting of $([0,1] \times \mathbb{R}) \cup (\mathbb{R} \times [0,1])$. The base square of the infinite cross is the square $[0,1] \times [0,1]$.
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Figure 4: $Z - X$ and $Z$: The eight squares of the figure on the left are glued together following the gluing pattern to form $Z - X$. To form $Z$, we add a copy of $X$ at the NE corner, identifying the loops in $X$ labeled by $c$ and $y$, with the black single and double arrows of the diagram. The figure on the right represents an infinite cross whose convex hull in $Z$ is not approximable by any periodic plane. Note that while the NW, SW, and SE quarters of this plane are periodic, the NE quarter is an aperiodic quarter of $\Pi$.

The planes containing $s$: Observe that $Y$ contains various immersions of an infinite cross whose base square maps to $s$. In particular, there are exactly 16 distinct immersed infinite crosses $C \twoheadrightarrow Y$ that pass through $s$ exactly once. Each of these infinite crosses extends uniquely to an immersed flat plane in $Z$. Each such flat plane fails to be periodic because its four quarters map to distinct parts of $Z$. Our main result is that these immersed flat planes are not approximable by periodic flat planes because of the following:

**Theorem 3.2** (No periodic approximation) There is no immersion of a torus $T^2 \rightarrow Z$ which contains $s$. Equivalently, there is no periodic plane in $\tilde{Z}$ containing $\tilde{s}$.

**Proof** We argue by contradiction. Suppose that there is an immersed periodic plane $\Omega$ containing $s$. We shall now produce a rectangle as in Figure 5 that will yield a contradiction. We may assume that a copy of $s$ in $\Omega$ is oriented as in Figure 4. We begin at this copy of $s$ and travel north inside the northern cylinder until we reach another copy $s_n$ of $s$. The existence of $s_n$ is guaranteed by our assumption that $\Omega$ is periodic. Similarly, we travel east from $s$ to reach a square $s_e$. Travelling north from $s_e$ and east from $s_n$, we trace out the boundary of a rectangle whose NE corner is a square $s_{ne}$ (see Figure 5).
This yields a contradiction because the inside of this rectangle is tiled by squares in $X$, yet the boundary of this rectangle is a commutator $[c^{\pm n}, y^{\pm m}]$. As explained in Section 2, such a word cannot be trivial in $\pi_1 X$ because of the anti-torus.

**Remark 3.3** Using an argument similar to the above proof, one can show that these sixteen planes are the only flat planes in $\tilde{Z}$ containing $\tilde{s}$. One considers the pair of “axes” intersecting at $\tilde{s}$ in a plane containing $\tilde{s}$. If this plane is different from each of the 16 mentioned above, then some translate of $\tilde{s}$ must appear along one of these “axes”. The infinite strip in the plane whose corners are these two $s$ squares yields a contradiction similar to the one obtained above.

**Remark 3.4** While $X$ is a rather pathological complex, we note that every flat plane in $\tilde{X}$ is the limit of periodic flat planes. Indeed this holds for any compact 2-complex $X$ whose universal cover is isomorphic to the product of two trees [8].

4 Polygons of groups

4.1 The algebraic angle versus the geometric angle

Since the elements $c$ and $y$ have axes which intersect perpendicularly in a plane in $\tilde{X}$, the natural geometric angle between the subgroups $\langle c \rangle$ and $\langle y \rangle$
of $\pi_1 X$ is $\frac{\pi}{2}$. However, the algebraic Gersten-Stallings angle (see [7]) between these subgroups is $\leq \frac{\pi}{2}$. To see this, we must show that there is no non-trivial relation of the form $c^k y c^m y^n = 1$.

Since $\tilde{X}$ is isomorphic to the cartesian product $\tilde{V} \times \tilde{H}$, of two trees and $c$ and $y$ correspond to distinct factors, it follows that the only relations that must be checked are rectangular (i.e., $|k| = |m|$ and $|l| = |n|$). However, these are easily ruled out by the anti-torus $\Pi$ and the fact that $X$ is nonpositively curved.

### 4.2 Square of groups and triangle of groups

The complex $Z$ can be thought of in a natural way as a $K(\pi, 1)$ for a negatively curved square of groups (see [7, 5, 2]) with cyclic edge groups and trivial face group.

Because the algebraic angle between $\langle c \rangle$ and $\langle y \rangle$ in $\pi_1 X$ is $\leq \frac{\pi}{2}$, it is tempting to form an analogous nonpositively curved triangle of groups $D$. The face group of $D$ is trivial, the edge groups of $D$ are cyclic, the vertex groups of $D$ are isomorphic to $\pi_1 X$, and each edge group of $D$ is embedded on one (clockwise) side as $\langle c \rangle$ and on the other (counter-clockwise) side as $\langle y \rangle$. This can be done so that the resulting triangle of groups $D$ has $\mathbb{Z}_3$ symmetry. The tension between the algebraic and geometric angles should endow $\pi_1 D$ with some interesting properties. For instance, I suspect that $\pi_1 D$ fails to be the fundamental group of a compact nonpositively curved space, but it fails for reasons different from the usual types of problems.

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## References

1. Martin R Bridson, André Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin (1999)
2. Jon Michael Corson, *Complexes of groups*, Proc. London Math. Soc. (3) 65 (1992) 199–224
3. M Gromov, *Hyperbolic groups*, from: “Essays in group theory”, Math. Sci. Res. Inst. Publ. 8, Springer, New York (1987) 75–263
4. M Gromov, *Asymptotic invariants of infinite groups*, from: “Geometric group theory, Vol. 2 (Sussex, 1991)”, Cambridge Univ. Press, Cambridge (1993) 1–295

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[5] André Haefliger, *Complexes of groups and orbihedra*, from: “Group theory from a geometrical viewpoint (Trieste, 1990)”, (É Ghys, A Haefliger, A Verjovsky, editors), World Sci. Publishing, River Edge, NJ (1991) 504–540

[6] Roger C Lyndon, Paul E Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin (1977), Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89

[7] John R Stallings, *Non-positively curved triangles of groups*, from: “Group theory from a geometrical viewpoint (Trieste, 1990)”, (É Ghys, A Haefliger, A Verjovsky, editors), World Sci. Publishing, River Edge, NJ (1991) 491–503

[8] Daniel T Wise, *Non-positively curved squared complexes, aperiodic tilings, and non-residually finite groups*, Ph.D. thesis, Princeton University (1996)

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