Finite groups in which every self-centralizing subgroup is a TI-subgroup or subnormal or has $p'$-order *

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Abstract

We first give complete characterizations of the structure of finite group $G$ in which every subgroup (or non-nilpotent subgroup, or non-abelian subgroup) is a TI-subgroup or subnormal or has $p'$-order for a fixed prime divisor $p$ of $|G|$. Furthermore, we prove that every self-centralizing subgroup (or non-nilpotent subgroup, or non-abelian subgroup) of $G$ is a TI-subgroup or subnormal or has $p'$-order for a fixed prime divisor $p$ of $|G|$ if and only if every subgroup (or non-nilpotent subgroup, or non-abelian subgroup) of $G$ is a TI-subgroup or subnormal or has $p'$-order. Based on these results, we obtain the structure of finite group $G$ in which every self-centralizing subgroup (or non-nilpotent subgroup, or non-abelian subgroup) is a TI-subgroup or subnormal or has $p'$-order for a fixed prime divisor $p$ of $|G|$. 

Keywords: self-centralizing; TI-subgroup; subnormal; $p'$-order; non-nilpotent subgroup; non-abelian subgroup

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1 Introduction

Throughout this paper all groups are assumed to be finite. Suppose that $G$ is a group and $H$ a subgroup of $G$, then $H$ is termed to be a TI-subgroup of $G$ if $H^g \cap H = 1$ or $H$ for each $g \in G$. It is clear that the TI-subgroup and the subnormal subgroup are two relatively independent concepts in group theory. For TI-subgroups, Walls [12] classified groups in which every subgroup is a TI-subgroup. As a generalization, Shi and Zhang [6, Theorem 2] characterized groups of even order in which every subgroup of even order is a TI-subgroup. As a generalization, Shi and Zhang [6, Theorem 2] characterized groups of even order in which every subgroup of even order is a TI-subgroup. For subnormal subgroups, [4, Theorem 5.2.4] indicated that a group $G$ is nilpotent if and only if every subgroup of $G$ is subnormal. In [2] Ebert and Bauman characterized groups in which every subgroup is subnormal or abnormal. Kurdachenko and Smith [3] investigated general groups in which every subgroup is either subnormal or self-normalizing. In [1] Theorem 1] Ballester-Bolinches and Cossey described groups in which every subgroup is supersolvable or subnormal.

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Combined the TI-property and the subnormality of subgroups together Shi and Zhang [5, Theorem 1] gave a complete characterization of groups in which every subgroup is a TI-subgroup or subnormal. Combined the nilpotence, the normality and the order of subgroups together Shi, Li and Shen [9, Theorems 1.3, 1.4 and Theorem 1.7] investigated group $G$ in which every maximal subgroup is nilpotent or normal or has $p'$-order for a fixed prime divisor $p$ of $|G|$. Furthermore, Shi [10, Theorem 1.1] obtained a complete characterization of group $G$ in which every maximal subgroup is nilpotent or a TI-subgroup or has $p'$-order for a fixed prime divisor $p$ of $|G|$.

In this paper, motivated by above researches, combining the TI-property, the subnormality and the order of subgroups together we have the following result whose proof is given in Section 2.

**Theorem 1** Suppose that $G$ is a group and $p$ a fixed prime divisor of $|G|$. Then every subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order if and only if one of the following statements holds:

1. every subgroup of $G$ of order divisible by $p$ is subnormal in $G$;
2. $p = 2$, $G = Z_q \rtimes \langle a \rangle$ is a Frobenius group with kernel $Z_q$ and complement $\langle a \rangle$, where $q$ is an odd prime and $o(a)$ is an even number;
3. $p > 2$, $G = Z_q^r \rtimes (P \times H)$ is a Frobenius group with kernel $Z_q^r$ and complement $P \times H$, where $q \neq p$ and $r \geq 1$, $P \in Syl_p(G)$ and $P$ is cyclic, $H$ is either a cyclic group or a direct product of a quaternion group $Q_8$ and a cyclic group of odd order, and every non-identity subgroup of $P$ acts irreducibly on $Z_q^r$;
4. $p > 2$, $G = Z_q^r \rtimes (Z_p \rtimes H)$ is a Frobenius group with kernel $Z_q^r$ and complement $Z_p \rtimes H$, where $q \neq p$ and $r > 1$, $Z_p \in Syl_p(G)$, $H$ is either a cyclic group or a direct product of a quaternion group $Q_8$ and a cyclic group of odd order such that $[Z_p, H] \neq 1$, and $Z_p$ acts irreducibly on $Z_q^r$.

In [7, Theorem 1 and Corollary 2] Shi characterized groups in which every non-abelian subgroup is a TI-subgroup or subnormal. In this paper, as a further generalization and extension, assume that every non-nilpotent subgroup (or non-abelian subgroup) of a group $G$ is a TI-subgroup or subnormal or has $p'$-order for a fixed prime divisor $p$ of $|G|$, arguing as in the proof of Theorem 1 we can obtain the following Theorem 2 and Theorem 4 here we omit their proofs.

**Theorem 2** Suppose that $G$ is a group and $p$ a fixed prime divisor of $|G|$. Then every non-nilpotent subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order if and only if one of the following statements holds:

1. every non-nilpotent subgroup of $G$ of order divisible by $p$ is subnormal in $G$;
2. $p > 2$, $G = Z_q^r \rtimes (Z_p \rtimes H)$ is a Frobenius group with kernel $Z_q^r$ and complement $Z_p \rtimes H$, where $q \neq p$ and $r > 1$, $Z_p \in Syl_p(G)$, $H$ is either a cyclic group or a direct product of a quaternion group $Q_8$ and a cyclic group of odd order.
product of a quaternion group $Q_8$ and a cyclic group of odd order such that $[Z_p, H] \neq 1$, and $Z_p$ acts irreducibly on $Z_q^r$.

The following corollary is a direct consequence of Theorem 2.

**Corollary 3** Suppose that $G$ is a group and $p$ the smallest prime divisor of $|G|$. Then every non-nilpotent subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order if and only if every non-nilpotent subgroup of $G$ of order divisible by $p$ is subnormal in $G$.

**Theorem 4** Suppose that $G$ is a group and $p$ a fixed prime divisor of $|G|$. Then every non-abelian subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order if and only if one of the following statements holds:

1. every non-abelian subgroup of $G$ of order divisible by $p$ is subnormal in $G$;
2. $p > 2$, $G = Z_q^r \rtimes (P \times H)$ is a Frobenius group with kernel $Z_q^r$ and complement $P \times H$, where $q \neq p$ and $r > 1$, $P \in \text{Syl}_p(G)$ and $P$ is cyclic, $H$ is a direct product of a quaternion group $Q_8$ and a cyclic group of odd order, and every non-identity subgroup of $P$ acts irreducibly on $Z_q^r$;
3. $p > 2$, $G = Z_q^r \rtimes (Z_p \times H)$ is a Frobenius group with kernel $Z_q^r$ and complement $Z_p \rtimes H$, where $q \neq p$ and $r > 1$, $Z_p \in \text{Syl}_p(G)$, $H$ is either a cyclic group or a direct product of a quaternion group $Q_8$ and a cyclic group of odd order such that $[Z_p, H] \neq 1$, and $Z_p$ acts irreducibly on $Z_q^r$.

Let $G$ be a group and $H$ a subgroup of $G$, then $H$ is said to be self-centralizing in $G$ if $C_G(H) \leq H$. As an extension of [7, Theorem 1 and Corollary 2], Sun, Lu and Meng [11, Theorem 1.1] proved that if every self-centralizing non-abelian subgroup of a group $G$ is a TI-subgroup or subnormal then every non-abelian subgroup of $G$ is subnormal. Furthermore, Shi and Li [8, Theorem 1 and Theorem 2] investigated groups in which every self-centralizing non-nilpotent subgroup is a TI-subgroup or subnormal.

According to above results, it is natural and interesting to characterize the structure of group $G$ in which every self-centralizing subgroup (or non-nilpotent subgroup, or non-abelian subgroup) is a TI-subgroup or subnormal or has $p'$-order for a fixed prime divisor $p$ of $|G|$. In this paper, we obtain the following Theorem 5 which can indicate the equivalent relation between group $G$ in which every self-centralizing subgroup is a TI-subgroup or subnormal or has $p'$-order for a fixed prime divisor $p$ of $|G|$ and group $G$ in which every subgroup is a TI-subgroup or subnormal or has $p'$-order for a fixed prime divisor $p$ of $|G|$.

**Theorem 5** Suppose that $G$ is a group and $p$ a fixed prime divisor of $|G|$. Then every self-centralizing subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order if and only if every subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order.

The proof of Theorem 5 is given in Section 3.

Arguing as in proof of Theorem 5, we can also obtain the following two results, here we omit their proofs.
Theorem 6  Suppose that $G$ is a group and $p$ a fixed prime divisor of $|G|$. Then every self-centralizing non-nilpotent subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order if and only if every non-nilpotent subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order.

Theorem 7  Suppose that $G$ is a group and $p$ a fixed prime divisor of $|G|$. Then every self-centralizing non-abelian subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order if and only if every non-abelian subgroup of $G$ is a TI-subgroup or subnormal or has $p'$-order.

Remark 8  Theorems 1, 2, 4, 5, 6, and 7 give the structure of group $G$ in which every self-centralizing subgroup (or non-nilpotent subgroup, or non-abelian subgroup) is a TI-subgroup or subnormal or has $p'$-order for a fixed prime divisor $p$ of $|G|$.

2  Proof of Theorem 1

Proof.  We first prove the necessity part.

Assume that $G$ has at least one subgroup of order divisible by $p$ that is not subnormal in $G$. Let $M$ be the largest subgroup of $G$ of order divisible by $p$ that is not subnormal in $G$, one must have $M = N_G(M)$. Since $M$ is a TI-subgroup of $G$ by the hypothesis, we get that $G$ is a Frobenius group with complement $M$. Let $N$ be the kernel, then $G = N \rtimes M$.

Let $P \in \text{Syl}_p(M)$, obviously $P \in \text{Syl}_p(G)$. Note that the order of the subgroup $N \rtimes P$ is divisible by $p$. By the hypothesis, $N \rtimes P$ is a TI-subgroup of $G$ or subnormal in $G$. For the case when $N \rtimes P$ is a TI-subgroup of $G$, one has $N \rtimes P \leq G$ since $(N \rtimes P)^g \cap (N \rtimes P) = (N^g \rtimes P^g) \cap (N \rtimes P) = (N \rtimes P^g) \cap (N \rtimes P) \geq N \neq 1$ for each $g \in G$. It follows that $P = (N \cap M)P = (N \rtimes P) \cap M \leq M$. For another case when $N \rtimes P$ is subnormal in $G$, one has that $P$ is subnormal in $M$ and then $P \leq M$ since $P \in \text{Syl}_p(M)$. Thus we always have $P \leq M$. By Schur-Zassenhaus theorem (see [4, Theorem 9.1.2]), $M$ has a subgroup $H$ such that $M = PH$ and $P \cap H = 1$, that is $M = P \rtimes H$.

Claim 1: $M$ is maximal in $G$. It is clear that $M \leq N_G(P) < G$. Let $K$ be a maximal subgroup of $G$ such that $M \leq N_G(P) \leq K$. If $M < K$, by the choice of $M$, one has that $K$ is subnormal in $G$. Then $K \leq G$. By Frattini-argument, one has $P \leq G$, a contradiction. Thus $M = N_G(P) = K$ is maximal in $G$.

By Claim 1, $N$ is a minimal normal subgroup of $G$. Since $N$ is nilpotent by [4, Theorem 10.5.6(i)], one has that $N$ is an elementary abelian group. Assume $N = Z_q^r$, where $q \neq p$ and $r \geq 1$.

Claim 2: $H$ is nilpotent. If $H = 1$, $H$ is obvious nilpotent. Next assume $H > 1$. For any maximal subgroup $H_1$ of $H$, $N \rtimes (P \rtimes H_1)$ is a maximal subgroup of $G$ of order divisible by $p$. By the hypothesis, $N \rtimes (P \rtimes H_1)$ is a TI-subgroup of $G$ or subnormal in $G$. Arguing as above, we can get $H_1 \leq H$. By the choice of $H_1$, one has that $H$ is nilpotent.
Claim 3: By conjugation every non-identity subgroup of $P$ acts irreducibly on $N$. Otherwise, assume that $P_i$ is a non-identity subgroup of $P$ and $N_i$ is a non-trivial subgroup of $N$ such that $P_i$ normalizes $N_i$. Then $N_i \rtimes P_i$ is a subgroup of $G$ of order divisible by $p$. By the hypothesis, $N_i \rtimes P_i$ is a TI-subgroup of $G$ or subnormal in $G$. One has that $N_i \rtimes P_i$ is also a TI-subgroup of $N \rtimes P_i$ or subnormal in $N \rtimes P_i$. For the case when $N_i \rtimes P_i$ is a TI-subgroup of $N \rtimes P_i$. Note that $N_i \trianglelefteq N \rtimes P_i$ since $N$ is abelian. Then $(N_i \rtimes P_i)^g \cap (N_i \rtimes P_i) = (N_i^g \rtimes P_i^g) \cap (N_i \rtimes P_i) = (N_i \rtimes P_i^g) \cap (N_i \rtimes P_i) \trianglerighteq N_i \neq 1$ for each $g \in N \rtimes P_i$. It follows that $N_i \rtimes P_i \trianglelefteq P_i \trianglelefteq N \rtimes P_i$, one has either $N_i \rtimes P_i \leq N$ or $N < N_i \rtimes P_i$. It is obvious that both of them are impossible. For another case when $N_i \rtimes P_i$ is subnormal in $N \rtimes P_i$. Assume that $N_i \rtimes P_i = L_0 \triangleleft L_1 \triangleleft \cdots \triangleleft L_s = N \rtimes P_i$ is a subnormal subgroups series, where $s \geq 1$. Observing that $N \ntrianglelefteq L_0$ but $N \trianglelefteq L_s$. Let $m$ be the smallest number from 1 to $s$ such that $N \leq L_m$ but $N \ntrianglelefteq L_{m-1}$. Since $L_0 \leq L_m$ but $L_0 \ntrianglelefteq N$, one has $N \lhd L_m$. Then $L_m = N \rtimes (L_m \cap P_i)$ is also a Frobenius group with kernel $N$ and complement $L_m \cap P_i \trianglerighteq 1$. Since $L_{m-1} \trianglelefteq L_m$, one has either $L_{m-1} \leq N$ or $N < L_{m-1}$. It is also obvious that both of them are impossible. Hence every non-identity subgroup of $P$ acts irreducibly on $N$.

Case I: Assume $p = 2$. Take an element $e$ of order 2 in $P$. Since $C_G(N) = N$ as $G$ being a Frobenius group, $e$ induces a fixed-point-free automorphism of $N$ of order 2. One has $g^e = g^{-1}$ for each non-identity element $g \in N$. Note that every non-identity subgroup of $P$ acts irreducibly on $N$. It follows that $N = Z_q$. By $N/C$-theorem, $M \cong G/N = N_G(N)/C_G(N) \trianglelefteq \text{Aut}(N)$ is a cyclic group. One has $G = Z_q \rtimes \langle a \rangle$, where $o(a)$ is an even number.

Case II: Assume $p > 2$.

Subcase (1): Assume that the order of every maximal subgroup of $M$ is divisible by $p$. Arguing as above, one can get that every maximal subgroup of $M$ is normal in $M$. Then $M$ is nilpotent. One has $M = P \times H$ and then $G = Z_q^x \rtimes (P \times H)$. Since $p > 2$, $P$ is cyclic by [4, Theorem 10.5.6(ii)]. Let $M_0$ be any non-trivial subgroup of $P \times H$ of order divisible by $p$, then $M_0 = (M_0 \cap P) \times (M_0 \cap H)$, where $M_0 \cap P > 1$. By the hypothesis, $M_0$ is a TI-subgroup of $G$ or subnormal in $G$. Arguing as above, $M_0$ cannot be subnormal in $G$. Then $M_0$ is a TI-subgroup of $G$ and so $M_0$ is also a TI-subgroup of $M$. If $M_0$ is not normal in $M$, then $N_M(M_0) < M$. Let $M_1 \trianglelefteq M$ such that $N_M(M_0)$ is maximal in $M_1$, then $N_M(M_0) \trianglelefteq M_1$. Take $x \in M_1 \setminus N_M(M_0)$, one has $M_0 \cap M_0^x = 1$. Note that $M_0^x \trianglelefteq (N_M(M_0))^x = N_M(M_0)$. Thus $M_0 M_0^x = M_0 \times M_0^x$. It follows that $(M_0 \cap P)(M_0 \cap P)^x = (M_0 \cap P) \times (M_0 \cap P)^x$, this contradicts that $P$ is cyclic. Thus every non-trivial subgroup of $M$ of order divisible by $p$ is normal in $M$. It follows that every non-trivial subgroup of $H$ is normal in $H$. Then $H$ is a Dedekind group (see [4, Theorem 5.3.7]). Moreover, since every Sylow $t$-subgroup of $H$ is cyclic if $t > 2$ and cyclic or a generalized quaternion group if $t = 2$ by [4, Theorem 10.5.6(ii)])], it follows that $H$ is cyclic or a direct product of a quaternion group $Q_8$ and a cyclic group of odd order.
Subcase (2): Assume that $M$ has a maximal subgroup $M_2$ of $p'$-order and $M_2 \not\unlhd M$. Then $M = P \rtimes M_2$. Note that $M = P \rtimes H$, we can assume $M_2 = H$. By the maximality of $H$, one has that $P$ is a minimal normal subgroup of $M$. Since $P$ is cyclic by [4, Theorem 10.5.6(ii)]), it follows that $P = Z_p$. Then $M = Z_p \rtimes H$. Arguing as in Subcase (1), one can get that $H$ is cyclic or a direct product of a quaternion group $Q_8$ and a cyclic group of odd order.

In the following we prove the sufficiency part.

(a) Suppose that $G$ is a group belonging to case (1), the proof is trivial.

(b) Suppose that $G$ is a group belonging to case (2). Let $A$ be any subgroup of $G$ of even order. If $Z_q < A$, then $A = A \cap (Z_q \times \langle a \rangle) = Z_q \times (A \cap \langle a \rangle) \subseteq Z_q \times \langle a \rangle = G$. If $Z_q \not\subseteq A$, then $(q, |A|) = 1$. Since $\langle a \rangle$ is a Hall-subgroup of $G$ and $G$ is obvious solvable, we can assume $A \leq \langle a \rangle$ by [4, Theorem 9.1.7]. It is easy to see that $N_G(A) = \langle a \rangle$ since $\langle a \rangle$ is maximal in $G$. Then $A^g \cap A \leq \langle a \rangle^g \cap \langle a \rangle = 1$ for each $g \in G \setminus N_G(A) = G \setminus \langle a \rangle$, one has that $A$ is a TI-subgroup of $G$.

(c) Suppose that $G$ is a group belonging to case (3). Let $A$ be any subgroup of $G$ of order divisible by $p$. If $Z_q^r < A$, then $A = A \cap (Z_q^r \times (P \times H)) = Z_q^r \times (A \cap (P \times H)) \subseteq Z_q^r \times (P \times H) = G$. If $Z_q^r \not\subseteq A$, since $A \cap P \neq 1$ and every non-identity subgroup of $P$ acts irreducibly on $Z_q^r$, one has that $Z_q^r \cap A = 1$ and $P \times H$ is maximal in $G$. Then we can assume $A \leq P \times H$. Note that $P \times H$ is a Dedekind-group and $A \not\unlhd G$. One has $N_G(A) = P \times H$. Therefore, $A^g \cap A \leq (P \times H)^g \cap (P \times H) = 1$ for each $g \in G \setminus N_G(A) = G \setminus (P \times H)$. It implies that $A$ is a TI-subgroup of $G$.

(d) Suppose that $G$ is a group belonging to case (4). Let $A$ be any subgroup of $G$ of order divisible by $p$. If $Z_q^r < A$, then $A = A \cap (Z_q^r \times (Z_p \times H)) = Z_q^r \times (A \cap (Z_p \times H)) = Z_q^r \times (Z_p \times (A \cap H)) \subseteq Z_q^r \times (Z_p \times H)$ since $H$ is a Dedekind-group. If $Z_q^r \not\subseteq A$, arguing as above, one has that $Z_q^r \cap A = 1$ and $Z_p \times H$ is a maximal subgroup of $G$. We can assume $A \leq Z_p \times H$. Then $A = A \cap (Z_p \times H) = Z_p \times (A \cap H) \subseteq Z_p \times H$. It follows that $N_G(A) = Z_p \times H$. One has that $A^g \cap A \leq (Z_p \times H)^g \cap (Z_p \times H) = 1$ for each $g \in G \setminus N_G(A) = G \setminus (Z_p \times H)$. It shows that $A$ is a TI-subgroup of $G$.

\[ \square \]

### 3 Proof of Theorem 5

**Proof.** We only need to prove the necessity part.

Suppose that the theorem is false. Assume that $K$ is the largest subgroup of $G$ of order divisible by $p$ that is neither a TI-subgroup of $G$ nor subnormal in $G$, then for any subgroup $L$ of $G$ satisfying $L > K$ we have that $L$ is a TI-subgroup of $G$ or subnormal in $G$. It follows that $K$ is not self-centralizing in $G$ by the hypothesis and then $K < N_G(K)$.

Considering the following subgroups series: $K < N_G(K) = K_1 \leq N_G(K_1) = K_2 \leq N_G(K_2) = K_3 \leq N_G(K_3) = \cdots = K_{t-1} \leq N_G(K_{t-1}) = K_t \leq N_G(K_t) \leq \cdots$. 

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6
(1) Suppose that there exists a positive integer \( t \geq 1 \) such that \( N_G(K_t) = G \). It follows that \( K \) is subnormal in \( G \), a contradiction.

(2) Suppose that for any positive integer \( t \geq 1 \) we have \( N_G(K_t) < G \). Note that \( G \) is a finite group. It follows that there must exist a positive integer \( t \geq 1 \) such that \( K_t = N_G(K_t) < G \), which implies that \( K_t \) is self-centralizing in \( G \). It is obvious that \( K_t > K \). Arguing as in (1), \( K_t \) cannot be subnormal in \( G \). Then \( K_t \) is a non-normal TI-subgroup of \( G \) by the hypothesis. Moreover, since \( K_t = N_G(K_t) \), one has that \( G \) is a Frobenius group with complement \( K_t \). Let \( N \) be the kernel of \( G \), then \( G = N \rtimes K_t \).

Claim I: \( K_t \) is either nilpotent or non-nilpotent and \( K_t = Z_p \rtimes M \), where \( Z_p \in \text{Syl}_p(K_t) \), \( M \) is nilpotent and \( M \) has \( p' \)-order satisfying \([Z_p, M] \neq 1\).

Suppose that \( K_t \) is non-nilpotent, then \( K_t \) has at least one non-normal maximal subgroup. Let \( M \) be a non-normal maximal subgroup of \( K_t \), then \( N \rtimes M \) is a non-normal maximal subgroup of \( G \). It follows that \( N \rtimes M \) is self-centralizing in \( G \) since \( N \rtimes M = N_G(N \rtimes M) \). Assume \( p \mid |M| \), then \( p \mid |N \rtimes M| \). By the hypothesis, \( N \rtimes M \) is a TI-subgroup of \( G \) or subnormal in \( G \). If \( N \rtimes M \not\trianglelefteq G \), then \( N \rtimes M \) is a non-normal TI-subgroup of \( G \). However, since \((N \rtimes M)^g \cap (N \rtimes M) = (N^g \rtimes M^g) \cap (N \rtimes M) = (N \rtimes M^g) \cap (N \rtimes M) \geq N > 1 \) for each \( g \in G \), it follows that \( N \rtimes M \trianglelefteq G \), a contradiction. Therefore \( N \rtimes M \trianglelefteq G \). One has \( M = (N \cap K_t)M = (N \rtimes M) \cap K_t \trianglelefteq K_t \), a contradiction. Thus \( p \nmid |M| \). Let \( P \in \text{Syl}_p(K_t) \), arguing as above, one has \( P \trianglelefteq K_t \). Then \( K_t = P \rtimes M \). Since \( M \) is maximal in \( K_t \), \( P \) is a minimal normal subgroup of \( K_t \), which implies that \( P \) is an elementary abelian group. Moreover, since \( K_t \) is a complement of Frobenius group \( G \), by [4, Theorem 10.5.6(ii)] one can get that \( P \) must be a cyclic group of prime order. Let \( P = Z_p \), then \( K_t = Z_p \rtimes M \). Moreover, arguing as above, one can obtain that every maximal subgroup of \( M \) is normal in \( M \) and then \( M \) is nilpotent.

Claim II: \( K \trianglelefteq K_t \).

Case (i): Assume that \( K_t \) is nilpotent. If \( K_t \) is cyclic, then \( K \trianglelefteq K_t \). Next assume that \( K_t \) is non-cyclic. Let \( P \in \text{Syl}_p(K_t) \).

If \( p = 2 \), take an element \( x \) of \( P \) of order \( 2 \). Then \( x \) induces a fixed-point-free automorphism of \( N \). It follows that \( N \) is an abelian group of odd order and \( g^2 = g^{-1} \) for each non-identity element \( g \in N \). Let \( g \) be an element of \( N \) of prime order, then \( \langle g \rangle \rtimes \langle x \rangle \) is also a Frobenius group of order divisible by \( 2 \). Since \( C_G(\langle g \rangle \rtimes \langle x \rangle) \leq C_G(\langle g \rangle) \cap C_G(\langle x \rangle) = N \cap C_G(\langle x \rangle) = C_N(\langle x \rangle) = 1 < \langle g \rangle \rtimes \langle x \rangle \), \( \langle g \rangle \rtimes \langle x \rangle \) is self-centralizing in \( G \). By the hypothesis, \( \langle g \rangle \rtimes \langle x \rangle \) is a TI-subgroup of \( G \) or subnormal in \( G \). It follows that \( \langle g \rangle \rtimes \langle x \rangle \) is a TI-subgroup of \( N \rtimes \langle x \rangle \) or subnormal in \( N \rtimes \langle x \rangle \). (a) Assume that \( \langle g \rangle \rtimes \langle x \rangle \) is a TI-subgroup of \( N \rtimes \langle x \rangle \). Since \( (\langle g \rangle \rtimes \langle x \rangle)^y \cap (\langle g \rangle \rtimes \langle x \rangle) = (\langle g^y \rangle \rtimes \langle x^y \rangle) \cap (\langle g \rangle \rtimes \langle x \rangle) = (\langle g \rangle \rtimes \langle x^y \rangle) \cap (\langle g \rangle \rtimes \langle x \rangle) \geq \langle g \rangle \neq 1 \) for each \( y \in N \rtimes \langle x \rangle \), one has \( \langle g \rangle \rtimes \langle x \rangle \trianglelefteq N \rtimes \langle x \rangle \). It follows that \( N < \langle g \rangle \rtimes \langle x \rangle \) as \( \langle g \rangle \rtimes \langle x \rangle \not\trianglelefteq N \). Therefore \( N = N \cap (\langle g \rangle \rtimes \langle x \rangle) = \langle g \rangle (N \cap \langle x \rangle) = \langle g \rangle \). By \( N/C \)-theorem, \( K_t \cong G/N = N_G(N)/C_G(N) \leq \text{Aut}(N) \) is cyclic. It follows that \( K_t \) is cyclic and then \( K \trianglelefteq K_t \). (b) Assume that \( \langle g \rangle \rtimes \langle x \rangle \) is subnormal in \( N \rtimes \langle x \rangle \). We claim \( N \leq \langle g \rangle \rtimes \langle x \rangle \).
Otherwise, assume \( N \not\leq \langle g \rangle \times \langle x \rangle \). Let \( \langle g \rangle \times \langle x \rangle = L_0 < L_1 < \cdots < L_{u-1} < L_u = N \times \langle x \rangle \) be a subnormal subgroups series, where \( u \geq 1 \). Since \( N \not\leq L_0 \) and \( N \leq L_u \), there must exist a positive integer \( v \) where \( 1 \leq v \leq u \) such that \( N \not\leq L_{v-1} \) and \( N \leq L_v \). Note that \( \langle g \rangle \times \langle x \rangle \not\leq N \), one has \( N < L_v \). Then \( L_v = N \times (L_v \cap K_t) \), where \( L_v \cap K_t > 1 \). It follows that \( L_v \) is a Frobenius group with complement \( N \). Since \( L_{v-1} < L_v \) and \( N \not\leq L_{v-1} \), one has \( L_{v-1} < N \), which implies that \( \langle g \rangle \times \langle x \rangle = L_0 < N \), a contradiction. Thus \( N \not\leq \langle g \rangle \times \langle x \rangle \).

Arguing as above, one has \( N = \langle g \rangle \) and then \( K_t \) is cyclic. One also has \( K \leq K_t \).

If \( p > 2 \), let \( P_2 \in \text{Syl}_2(K_t) \). Since \( K_t \) is non-cyclic, \( P_2 \) is a generalized quaternion group of order \( 4n \) by [1] Theorem 10.5.6(ii)], where \( n \geq 2 \). Then \( K_t = R \times P_2 \), where \( R \) is a cyclic group of order divisible by \( p \). Let \( P_2 = \langle a, b \mid a^n = b^2, a^{2n} = 1, b^{-1}ab = a^{-1} \rangle \), one has \( Z(P_2) = a^n = b^2 \). For any non-identity subgroup \( Q \) of \( P_2 \), it is easy to see that \( C_G(R \times Q) = R \times Z(P_2) \leq R \times Q \) and then \( R \times Q \) is self-centralizing in \( G \). By the hypothesis, \( R \times Q \) is a TI-subgroup of \( G \) or subnormal in \( G \). Arguing as above, \( R \times Q \) cannot be subnormal in \( G \). Then \( R \times Q \) is a TI-subgroup of \( G \). It follows that \( R \times Q \) is a TI-subgroup of \( K_t \). If \( R \times Q \not\subseteq K_t \), then there exists \( g \in K_t \) such that \( (R \times Q)^g \cap (R \times Q) = 1 \). However, \( (R \times Q)^g \cap (R \times Q) = (R \times Q) \cap (R \times Q) = (R \times Q^g) \cap (R \times Q) \geq R \not\leq 1 \), a contradiction. Therefore \( R \times Q \leq K_t \). It follows that \( Q \leq P_2 \). By the choice of \( Q \), one has that \( P_2 \) is a Dedekind-group. Then \( K_t = R \times P_2 \) is a Dedekind-group. It follows that \( K \leq K_t \).

Case (ii): Assume \( K_t = Z_p \times M \), where \( M \) is nilpotent, \( p \nmid |M| \) and \( [Z_p, M] \neq 1 \). Since \( p \nmid |K| \), one has \( Z_p \leq K \). Then \( K = K \cap (Z_p \times M) = Z_p \times (K \cap M) \). If \( M \) is cyclic, it is obvious that \( Z_p \times (K \cap M) \leq Z_p \times M \), that is \( K \leq K_t \). If \( M \) is non-cyclic, then \( M = T \times P_2 \), where \( T \) is cyclic and \( P_2 \) is a generalized quaternion group. For any non-identity subgroup \( Q \) of \( P_2 \), \( Z_p \times (T \times Q) \) is a subgroup of \( K_t \) of order divisible by \( p \). Moreover, it is clear that \( C_G(Z_p \times (T \times Q)) \leq Z_p \times (T \times Q) \), that is \( Z_p \times (T \times Q) \) is self-centralizing in \( G \). By the hypothesis, \( Z_p \times (T \times Q) \) is a TI-subgroup of \( G \) or subnormal in \( G \). Arguing as in case (i), one can get that \( P_2 \) must be a quaternion group of order 8 and then \( M = T \times P_2 \) is a Dedekind-group. It follows that \( K = Z_p \times (K \cap M) \leq Z_p \times M = K_t \).

By claim II, one has \( K_t \leq N_G(K) \). It follows that \( N_G(K) = K_t \) as \( N_G(K) = K_t \leq K_t \). Since \( K^g \cap K \leq K^g \cap K_t = 1 \) for each \( g \in G \backslash N_G(K) = G \backslash K_t = G \backslash N_G(K_t) \), one has that \( K \) is a TI-subgroup of \( G \), a contradiction.

It implies that our assumption is not true and so every subgroup of \( G \) is a TI-subgroup or subnormal or has \( p' \)-order.

\[ \square \]

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