REGULARITY OF ELLIPTIC SYSTEMS IN DIVERGENCE FORM WITH DIRECTIONAL HOMOGENIZATION

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ABSTRACT. In this paper, we study regularity of solutions of elliptic systems in divergence form with directional homogenization. Here directional homogenization means that the coefficients of equations are rapidly oscillating only in some directions. We will investigate the different regularity of solutions on directions with homogenization and without homogenization. Actually, we obtain uniform interior $W^{1,p}$ estimates in all directions and uniform interior $C^{1,\gamma}$ estimates in the directions without homogenization.

1. Introduction. In the present paper, we study the following elliptic systems in divergence form

\[ \mathcal{L}_{\varepsilon} u_{\varepsilon} := D_i \left( a_{ij}^{ij}(x', x''_{\varepsilon}) D_j u_{\varepsilon}^j \right) = D_i f_i^\alpha =: \text{div} f \quad \text{in } \Omega. \tag{1.1} \]

Here $\Omega$ is a domain in $\mathbb{R}^n$ with $n \geq 2$, $f = (f_i^\alpha) \ (1 \leq i \leq n \text{ and } 1 \leq \alpha \leq N)$ are suitable functions and $x \in \Omega$ is separated into two parts

\[ x' = (x^1, \ldots, x^q) \text{ and } x'' = (x^{q+1}, \ldots, x^n) \tag{1.2} \]

for some integer $0 < q < n$. Observe that the coefficients $(a_{ij}^{ij})$ of (1.1) are rapidly oscillating only in $x''$ as $\varepsilon \to 0^+$. We call this directional homogenization and call $x''$ the homogenization direction. Our main target is to investigate the different regularity of solutions of (1.1) on $x'$ and on $x''$.

We will assume that the coefficient matrix $a(x', y'') = (a_{ij}^{ij}(x', y'')) \ (1 \leq i, j \leq n, 1 \leq \alpha, \beta \leq N)$ satisfies the following conditions:

$(H_1)$ Directional periodicity:

\[ a(x', y'' + z'') = a(x', y'') \]

for any $(x', y'') \in \mathbb{R}^n$ and any $z'' \in \mathbb{Z}^{n-q}$;

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(H2) Ellipticity:
\[ \lambda |\xi|^2 \leq a_{ij}(x', y')\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \]
for any \((x', y') \in \mathbb{R}^n\), any \(\xi = (\xi^\alpha) \in \mathbb{R}^{n \times N}\) and some \(\lambda \in (0, 1]\):

(H3) Smoothness: \(a = (a_{ij}) \in C^\gamma(\mathbb{R}^n)\) and
\[ |a|_{C^\gamma(\mathbb{R}^n)} \leq M \]
for some \(0 < \gamma < 1\) and \(M > 0\).

Throughout this paper, we will use the summation convention. In (1.1), \(1 \leq i, j \leq n\) and \(1 \leq \alpha, \beta \leq N\), where \(n\) is the dimension of variables and \(N\) is the dimension of unknowns. \(n - q\) (see (1.2)) is defined to be the dimension of homogenization and is fixed throughout this paper.

Such directional homogenization problems arise in linearly elastic laminates, composite materials and wave-guides (cf. [6], [12], [4] and [14]). The differences between the general homogenization and this directional homogenization are that, for this directional homogenization (1.1), it is intuitively plausible that the regularity of solutions in \(x'\) and in \(x''\) are different. In this paper, we will investigate the differences and demonstrate the different regularity in \(x'\) and in \(x''\) quantitatively. Our results are optimal theoretically (cf. Remark 1.3 (ii)).

The following are the main results of the paper, where in the first one, we assume that \(f\) is Hölder continuous only in the non-homogenization directions \(x'\) and in the second one, we assume that \(f\) is Hölder continuous in both the homogenization directions and the non-homogenization directions. We obtain uniform interior \(W^{1,p}\) estimates in all directions and moreover, higher regularity in directions without homogenization. (cf. Notations in the end of this section).

**Theorem 1.1.** Suppose (H1)-(H3) hold and \(\varepsilon > 0\). Let \(f = (f^\alpha) \in C^{\tilde{\gamma}, \tilde{\gamma}}(\Omega)\) with \(0 < \tilde{\gamma} < \gamma\). If \(u_\varepsilon = (u_\varepsilon^\alpha) \in W^{1,2}(\Omega)\) is a weak solution of (1.1), then for any \(\Omega' \subset \Subset \Omega\), we have

(i) \(u_\varepsilon \in W^{1,p}(\Omega')\) for any \(2 \leq p < \infty\) and
\[ \|D u_\varepsilon\|_{L^p(\Omega')} \leq C(\|u_\varepsilon\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}), \tag{1.3} \]
where \(C\) depends only on \(n, N, M, \lambda, \gamma, p, \Omega'\) and \(\Omega\);

(ii) \(D x' u_\varepsilon \in C^{\tilde{\gamma}}(\Omega')\) and
\[ |D x' u_\varepsilon|_{\gamma, \Omega'} \leq C(\|u_\varepsilon\|_{L^2(\Omega)} + |f|_{x', \tilde{\gamma}, \Omega}), \tag{1.4} \]
where \(C\) depends only on \(n, N, M, \lambda, \gamma, \tilde{\gamma}, \Omega'\) and \(\Omega\).

**Theorem 1.2.** Suppose (H1)-(H3) hold and \(\varepsilon > 0\). Let \(f = (f^\alpha) \in C^{\gamma, \delta}(\Omega) \cap C^{\tilde{\gamma}, \tilde{\gamma}}(\Omega)\) with \(0 < \delta < 1\). If \(u_\varepsilon = (u_\varepsilon^\alpha) \in W^{1,2}(\Omega)\) is a weak solution of (1.1), then for any \(\Omega' \subset \Subset \Omega\), we have

(i) \(u_\varepsilon \in W^{1,\infty}(\Omega')\) and
\[ \|D u_\varepsilon\|_{L^\infty(\Omega')} \leq C(\|u_\varepsilon\|_{L^2(\Omega)} + |f|_{\tau, \Omega}), \tag{1.5} \]
where \(\tau = \min\{\gamma, \delta\}\) and \(C\) depends only on \(n, N, M, \lambda, \gamma, \delta, \Omega'\) and \(\Omega\);

(ii) \(D x' u_\varepsilon \in C^{\gamma}(\Omega')\) and
\[ |D x' u_\varepsilon|_{\gamma, \Omega'} \leq C(\|u_\varepsilon\|_{L^2(\Omega)} + |f|_{x', \gamma, \Omega} + |f|_{x''', \delta, \Omega}), \tag{1.6} \]
where \(C\) depends only on \(n, N, M, \lambda, \gamma, \delta, \Omega'\) and \(\Omega\).
**Remark 1.3.** (i) Compared to the general homogenization results (cf. [1] and [2]), Theorem 1.1 (i) and 1.2 (i) give the same regularity in all directions, which is optimal by the arguments in [1] and [11]. Observe \(a, f\) and \(u_\varepsilon\) may not depend on \(x'\) and then our directional homogenization becomes the general homogenization. Theorem 1.1 (ii) and 1.2 (ii) give the higher regularity in the directions without homogenization, which is also optimal since even if \(a, f\) and \(u_\varepsilon\) do not depend on \(x''\), our results can not be improved. Observe if \(a, f\) and \(u_\varepsilon\) do not depend on \(x''\), the results reduce to the classical regularity theory.

(ii) Compared to the so called partial regularity results (cf. [9]), Theorem 1.1 and 1.2 reveal higher regularities. For example, as \(N = 1\), under the same assumptions in \(x'\) and \(L^\infty\) assumptions in \(x''\) for \(a\), by [9], we have (1.4) holds with \(0 < \hat{\gamma} < \min\{\gamma_0, \gamma\}\), where \(\gamma_0\) is the H"older exponent appearing in the De Giorgi-Nash-Moser's estimate, which may be very small. That is, the homogenization results are better than the results that we only assume \(a\) is \(L^\infty\) in \(x''\). Actually, one of the target of studying homogenization problem is to discover this kind of hidden higher regularity.

(iii) If the homogenization direction is one-dimensional, from [7], better results can be obtained. (See also [12].) However, if the homogenization directions are multi-dimensional, the results in [7] are unadapted here.

As far as to the authors’ knowledge scope, there is not too much literatures concerning the directional homogenization problems. In [13], Suslina considered the following elliptic equation in a strip \(\Pi = (0,a) \times \mathbb{R}^2:\)

\[
D_1\left(g_1(x_1, \frac{x_2}{\varepsilon})D_1u_\varepsilon\right) + D_2\left(g_2(x_1, \frac{x_2}{\varepsilon})D_2u_\varepsilon\right) + u_\varepsilon = f. \tag{1.7}
\]

Here the coefficient matrix is a diagonal matrix. If \(g_j\ (j = 1, 2)\) are bounded, periodic in \(x_2\) and Lipschitz in \(x_1\), it was proved that

\[
\|u_\varepsilon - u_0\|_{L^2(\Pi)} \leq C\varepsilon\|f\|_{L^2(\Pi)},
\]

where \(u_0\) solves the homogenized problem

\[
D_1\left(g_0^1(x_1)D_1u_0\right) + D_2\left(g_0^2(x_1)D_2u_0\right) + u_0 = f.
\]

If the lower-order term \(u_\varepsilon\) in (1.7) is replaced by \(b(x_1, \frac{x_2}{\varepsilon})u_\varepsilon\), a similar result was established in [5]. In Section 2, we will give the weak convergence of \(u_\varepsilon\) to \(u_0\) in \(H^1\).

We now briefly describe the ideas of the proofs of our main results. Since the behaviour of variables on the homogenization directions and the non-homogenization directions are essentially different, we deal with them by different methods. To obtain the regularity of \(u_\varepsilon\) on the homogenization directions \(x''\), we use a three-step compactness method, which was demonstrated by Avellaneda and Lin in [1], while for the regularity on the non-homogenization directions \(x'\), the frozen coefficients method is exploited. Conclusion (i) in Theorem 1.1 and 1.2 will be used in the proofs of conclusion (ii) in Theorem 1.1 and 1.2 respectively.

The remaining sections are organized in the following way. In section 2, it is given the homogenized operator \(L_0\) of \(L_\varepsilon\) as \(\varepsilon \to 0^+\), whose properties will be used in the compactness method, and whose coefficients, different from the general homogenization, are not constant. In section 3, we prove Theorem 1.1 (i) and 1.2 (i), that is, the regularity of solutions of (1.1) in all directions. Theorem 1.1 (ii) and 1.2 (ii), that is, the higher regularity in directions without homogenization is derived in section 4.
Throughout the paper, we use standard notations. Before the end of this section, we list them in the following.

$C$: a positive constant independent of $\varepsilon$ and the exact value may change from line to line.

$\Omega$: a domain in $\mathbb{R}^n$.

$[u]_{x',\gamma;\Omega} := \sup_{(x',x'')(y',x'')\in\Omega, x'\neq y'} \frac{|u(x',x'') - u(y',x'')|}{|x' - y'|}$, this is called directional Hölder semi-norm with respect to $x'$.

$[u]_{x'',\gamma;\Omega} := \sup_{(x',x''), (y',x'')\in\Omega, x'' \neq y''} \frac{|u(x',x'') - u(y',x'')|}{|x'' - y''|}$, this is called directional Hölder semi-norm with respect to $x''$.

$|u|_{x',\gamma;\Omega} := [u]_{x',\gamma;\Omega} + ||u||_{\infty;\Omega}$ and $C_{x'}^\gamma(\Omega) := \{u : |u|_{x',\gamma;\Omega} < \infty\}$.

$|u|_{x'',\gamma;\Omega} := [u]_{x'',\gamma;\Omega} + ||u||_{\infty;\Omega}$ and $C_{x''}^\gamma(\Omega) := \{u : |u|_{x'',\gamma;\Omega} < \infty\}$.

$[D_\alpha u]_{\gamma;\Omega} := \sup_{\alpha' \in \mathbb{Z}^n, ||\alpha'||=1} [D^\alpha u]_{\gamma;\Omega}$, where $D^\alpha := D_1^{\alpha_1} \cdots D_q^{\alpha_q}$.

$B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$, $B_r = B_r(0)$.

$\Omega_r(x) := \Omega \cap B_r(x)$.

$\langle u \rangle_{x,r} := \int_{\Omega_r(x)} u(y)dy = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} u(y)dy$.

2. Homogenization and weak convergence. In this section, we give the homogenized operator $L_0$ of $L_\varepsilon$ and a weak convergence result as $\varepsilon \to 0^+$. Different from the general homogenization problems, here we will see the homogenized operator $L_0$ has non-constant coefficients.

We first introduce $\chi = \chi(x',y'') = \chi^i_{\ell}(x',y'')$, the so called the matrix of correctors for $L_\varepsilon$, where $x' \in \mathbb{R}^q$ and $y'' \in \mathbb{R}^{n-q}$. Throughout this section, we confine the indices $1 \leq i,j,\ell \leq n$, $q + 1 \leq s, t \leq n$ and $1 \leq \alpha, \beta, \iota \leq N$. Actually, $\chi(x',y'')$ is obtained by the following way. For any fixed $x'$, we solve $\chi(x',y'')$ as a periodic function of $y''$ by

$$
\begin{cases}
D_{y'} \left( a_{\alpha \beta}^i(x',y'') D_{y''} \chi^i_{\ell}(x',y'') \right) = -D_y a_{\alpha \beta}^i(x',y''), \\
\chi^i_{\ell}(x',y'') \text{ is periodic with respect to } \mathbb{Z}^{n-q}, \\
\int_{[0,1]^{n-q}} \chi^i_{\ell}(x',y'')dy'' = 0.
\end{cases}
$$

(2.1)

Now we set

$$
\hat{a}_{\alpha \beta}^i(x') = \int_{[0,1]^{n-q}} \left( a_{\alpha \beta}^i(x',y'') + a_{\alpha \beta}^i(x',y'')D_{y''} \chi^i_{\ell}(x',y'') \right)dy''
$$

(2.2)

and

$$
L_0 := D_i \left( \hat{a}_{\alpha \beta}^i(x') D_j \right).
$$

(2.3)

We have the following convergent result as $\varepsilon \to 0^+$.

**Theorem 2.1** (Weak convergence). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $u_\varepsilon \in W^{1,2}(\Omega)$ be a weak solution of

$$
\begin{cases}
L_\varepsilon u_\varepsilon = \text{div } f \text{ in }\Omega, \\
u_\varepsilon = 0 \text{ on } \partial \Omega,
\end{cases}
$$

(2.4)
where the coefficients of \( L_\varepsilon \) satisfy \((H_1)-(H_3)\) and \( f \in L^2(\Omega) \). Then
\[
u_\varepsilon \to u_0 \text{ in } W^{1,2}_0(\Omega) \text{ weakly as } \varepsilon \to 0^+,
\]
where \( u_0 \) solves 
\[
\begin{cases}
L_0 u_0 = \text{div}\ f \text{ in } \Omega, \\
u_0 = 0 \text{ on } \partial\Omega
\end{cases}
\]
with \( L_0 \) is given by \((2.3)\).

**Proof.** Suppose \((\tilde{a}^{ij}_{\alpha\beta}(x,y))\) are elliptic and continuous in \( \Omega \times \mathbb{R}^n \), periodic with respect to \( y \). The following elliptic system are the so called non-uniformly oscillating coefficients homogenization.

\[
\begin{cases}
\tilde{L}_\varepsilon u_\varepsilon := D_i \left( \tilde{a}^{ij}_{\alpha\beta}(x,\frac{x}{\varepsilon}) D_j u_\varepsilon^2 \right) = \text{div}\ f \text{ in } \Omega, \\
u_\varepsilon = 0 \text{ on } \partial\Omega,
\end{cases}
\]

where we assume that \( f \in L^2(\Omega) \).

From [3], one has that the solution \( u_\varepsilon \) of \((2.5)\) converges weakly in \( W^{1,2}_0(\Omega) \) to \( u_0 \) satisfying 
\[
\begin{cases}
\tilde{L}_0 u_0 := D_i \left( \hat{a}^{ij}_{\alpha\beta}(x) D_j u_0^2 \right) = \text{div}\ f \text{ in } \Omega, \\
u_0 = 0 \text{ on } \partial\Omega,
\end{cases}
\]

where \( \hat{a}^{ij}_{\alpha\beta}(x) \) are given by
\[
\hat{a}^{ij}_{\alpha\beta}(x) = \int_{[0,1]^n} \left( \tilde{a}^{ij}_{\alpha\beta}(x,y) + \tilde{a}^{ij}_{\alpha\beta}(x,y) D_y \chi^j_{i\beta}(x,y) \right) dy
\]
and \( \chi^j_{i\beta} \) are solutions to the cell problems
\[
\begin{cases}
D_y \left( \tilde{a}^{ij}_{\alpha\beta}(x,y) D_y \chi^j_{i\beta}(x,y) \right) = -D_y \hat{a}^{ij}_{\alpha\beta}(x,y), \\
\chi^j_{i\beta}(x,y) \text{ is periodic with respect to } \mathbb{Z}^n, \\
\int_{[0,1]^n} \chi^j_{i\beta}(x,y) dy = 0.
\end{cases}
\]

Observe that if \((\tilde{a}^{ij}_{\alpha\beta}(x,y))\) do not depend on \( x'' \) and \( y' \), then System \((2.5)\) is deduced to be System \((2.4)\). And it is easy to see that in this case \((2.1)\) and \((2.6)\) are the same. Thus, our conclusion follows immediately.

**Remark 2.2.** \( L_0 \) given by \((2.3)\) is called the homogenized operator of \( L_\varepsilon \) as \( \varepsilon \to 0^+ \).

We claim that although the coefficient matrix of \( L_0 \), \((\hat{a}^{ij}_{\alpha\beta}(x'))\), is not a constant matrix (depending only on \( x' \)), it satisfies the assumption \((H_2)\) and \((H_3)\). Actually, we have the following proofs.

Let \( \tilde{x}' \) be a point other than \( x' \) and \( \chi(\tilde{x}', y'') \) satisfy
\[
\begin{cases}
D_{y_\nu} \left( a^{\nu\lambda}_{\alpha\lambda}(\tilde{x}',y'') D_{y_\nu} \chi^j_{i\beta}(\tilde{x}',y'') \right) = -D_{y_\nu} a^{\nu\lambda}_{\alpha\lambda}(\tilde{x}',y''), \\
\chi^j_{i\beta}(\tilde{x}',y'') \text{ is periodic with respect to } \mathbb{Z}^{n-q}, \\
\int_{[0,1]^{n-q}} \chi^j_{i\beta}(\tilde{x}',y'') dy'' = 0.
\end{cases}
\]
From this and (2.1), we obtain
\[
D_y \left[ a_{\alpha \beta}^\ell (x', y') D_y \left( \chi_{\alpha \beta}^j (x', y') \right) - \chi_{\alpha \beta}^j (x', y') \right] = -D_y \left[ a_{\alpha \beta}^\ell (x', y') - a_{\alpha \beta}^\ell (x', y') \right] \chi_{\alpha \beta}^j (x', y') ,
\]
\[
\chi_{\alpha \beta}^j (x', y') \quad \text{is periodic with respect to } \mathbb{Z}^{n-q}
\]
and
\[
\int_{[0,1]^{n-q}} \left( \chi_{\alpha \beta}^j (x', y') - \chi_{\alpha \beta}^j (x', y') \right) dy'' = 0.
\]

Then, by $W^{1,2}$ estimates, we have
\[
\|D_y \chi(x', y') - D_y \chi(x', y')\|_{L^2([0,1]^{n-q})}
\leq C \left( \|a(x', y') - a(x', y')\|_{L^2([0,1]^{n-q})} + \|a(x', y') - a(x', y')D_y \chi(x', y')\|_{L^2([0,1]^{n-q})} \right)
\leq C \left( 1 + \|D_y \chi(x', y')\|_{L^2([0,1]^{n-q})} \right)
\leq C|x' - \tilde{x}'|^{\gamma},
\]
where $C$ depends only on $n$, $N$, $\lambda$ and $M$. By this and Hölder's inequality, we get
\[
\left| \int_{[0,1]^{n-q}} \left( a(x', y')D_y \chi(x', y') - a(\tilde{x}', y')D_y \chi(\tilde{x}', y') \right) dy'' \right|
= \left| \int_{[0,1]^{n-q}} \left( a(x', y')D_y \chi(x', y') - a(\tilde{x}', y')D_y \chi(x', y') \right) dy'' \right|
+ \left| \int_{[0,1]^{n-q}} \left( a(\tilde{x}', y')D_y \chi(x', y') - a(\tilde{x}', y')D_y \chi(\tilde{x}', y') \right) dy'' \right|
\leq \int_{[0,1]^{n-q}} |a(x', y') - a(\tilde{x}', y')| \|D_y \chi(x', y')\| dy''
+ \int_{[0,1]^{n-q}} |a(\tilde{x}', y')| \|D_y \chi(x', y') - D_y \chi(\tilde{x}', y')\| dy''
\leq C|x' - \tilde{x}'|^{\gamma} \|D_y \chi(x', y')\|_{L^2([0,1]^{n-q})}
+ C \|D_y \chi(x', y') - D_y \chi(\tilde{x}', y')\|_{L^2([0,1]^{n-q})}
\leq C|x' - \tilde{x}'|^{\gamma},
\]
where $C$ depends only on $n$, $N$, $\lambda$ and $M$. This implies that $\left( a_{\alpha \beta}^j (x') \right)$ defined in (2.2) satisfies ($H_3$).

Next, we prove that $\left( a_{\alpha \beta}^j (x') \right)$ satisfies ($H_2$). Let $1 \leq m \leq n$ and $1 \leq \rho \leq N$. Note that (2.1) is equivalent to
\[
\int_{[0,1]^{n-q}} a_{\rho \sigma}^m (x', y') \left( D_{\rho m} P_{\ell \beta}^j + D_{\rho m} \chi_{\beta j} (x', y') \right) D_y \varphi(y'') dy'' = 0 \tag{2.7}
\]
for any $\varphi$ with the properties that $\int_{[0,1]^{n-q}} \varphi(y'') dy'' = 0$ and $\varphi$ is periodic with respect to $\mathbb{Z}^{n-q}$, where $P_{\ell \beta}^j = y_j (0, ..., 1, ..., 0)$ with 1 in the $\beta^{th}$ position and $P_{\ell \beta}^j$ is
the $i^{th}$ component of $P^j_\beta$. (Here we use $D_{y m} P^j_\beta = \delta_{m j} \delta_{\beta \beta}.$) Observe (2.2) can be rewritten as
\[ a^{ij}_{\alpha \beta}(x') = \int_{[0,1]^{n-q}} a^{m}_{\rho u}(x', y') \left( D_{y m} P^j_\beta + D_{y m} \chi_{i \beta}(x', y') \right) d y'. \]
By taking $\varphi = \chi^i_{\rho u}(x', y'')$ in (2.7), we deduce
\[ \tilde{a}^{ij}_{\alpha \beta}(x') = \int_{[0,1]^{n-q}} a^{m}_{\rho u}(x', y') \left( D_{y m} P^j_\beta + D_{y m} \chi_{i \beta}(x', y'') \right) d y'. \]
It follows that for any $\xi = (\xi^\alpha) \in \mathbb{R}^{n \times N},$
\[ \tilde{a}^{ij}_{\alpha \beta}(x') \xi^\alpha \xi^\beta = \int_{[0,1]^{n-q}} a^{m}_{\rho u}(x', y') \left( \xi^\beta P^j_\alpha + \xi^\beta \chi_{i \beta}(x', y'') \right) d y'. \] (2.8)
Since
\[ \int_{[0,1]^{n-q}} D_{y u} \chi(x', y'') d y'' = 0 \]
and $a(x', y'')$ satisfies the condition (H2), we obtain from (2.8) that
\[ \tilde{a}^{ij}_{\alpha \beta}(x') \xi^\alpha \xi^\beta \geq \lambda \int_{[0,1]^{n-q}} D \left( \xi^\beta P^j_\alpha + \xi^\beta \chi_{i \beta}(x', y'') \right) d y'' \]
\[ = \lambda |\xi|^2 + \lambda \int_{[0,1]^{n-q}} D \left( \xi^\beta \chi_{i \beta}(x', y'') \right) d y'' \]
\[ \geq \lambda |\xi|^2. \]
This implies that $(\tilde{a}^{ij}_{\alpha \beta}(x'))$ satisfies (H2).

The observation that $(\tilde{a}^{ij}_{\alpha \beta}(x'))$ satisfies (H2) and (H3) will be used in Section 3 as applying the compactness method.

3. Regularity in all directions. In this section, we give the proofs of uniform interior $W^{1,p}$ estimates of all variables with $2 \leq p \leq \infty$ for solutions of (1.1), thus the result (i) in both Theorem 1.1 and 1.2.

Proof of Theorem 1.1 (i). Let $d = \text{dist}(\Omega', \partial \Omega)$ and fix any $x_0 \in \Omega'$. Choose $0 < R \leq d/2$ and $\eta \in C^\infty_0(B_R(x_0)), which will be determined later. Observe $B_R(x_0) \subset \Omega.$ From (1.1), one computes that $\eta u_\varepsilon$ satisfies
\[ D_i \left( a^{ij}_{\alpha \beta}(x', \frac{x''}{\varepsilon}) D_j (\eta u_\varepsilon^\beta) \right) = D_i F_\alpha^i + G_\alpha, \] (3.1)
where
\[ G_\alpha = a^{ij}_{\alpha \beta}(x', \frac{x''}{\varepsilon}) D_i \eta D_j u_\varepsilon^\beta - f^i_\alpha D_i \eta \]
and
\[ F_\alpha^i = a^{ij}_{\alpha \beta}(x', \frac{x''}{\varepsilon}) u_\varepsilon^\beta D_j \eta + f^i_\alpha \eta. \]
We first assume that $2 \leq p \leq 2^* = 2n/(n - 2)$. Let $v_\varepsilon \in W^{1,2}_0(B_R(x_0))$ solve
\[ \Delta v_\varepsilon^\alpha = G_\alpha. \]
Then we have
\[
\|Dv_\varepsilon\|_{L^p(B_R(x_0))} \leq C\|v_\varepsilon\|_{W^{2,2}(B_R(x_0))} \leq C\|G\|_{L^2(B_R(x_0))} \leq C(\|Du_\varepsilon\|_{L^2(B_R(x_0))} + \|f\|_{L^2(B_R(x_0))}),
\]
(3.2)
where \(C\) depends only on \(n, N, M, p\) and \(|D\eta|\)\(L^\infty\).

From (3.1), it follows that
\[
D_i\left(a_{\alpha\beta}^{ij}(x_0, x', \varepsilon)D_j(\eta u^\beta_\varepsilon)\right) = D_i\left((a_{\alpha\beta}^{ij}(x_0, x', \varepsilon) - a_{\alpha\beta}^{ij}(x', \varepsilon)) D_j(\eta u^\beta_\varepsilon) + F_\alpha^i + D_i u^\beta_\varepsilon\right).
\]
(3.3)
Since \(x_0\) is fixed, (3.3) can be regarded as a general homogenization problem. By Theorem C in [2], we obtain that
\[
\|D(\eta u_\varepsilon)\|_{L^p(B_R(x_0))} \leq C\left\{\|R_i[a, x', \gamma; B_R(x_0)]\|_{L^p(B_R(x_0))} + \|F\|_{L^p(B_R(x_0))} + \|Du_\varepsilon\|_{L^p(B_R(x_0))}\right\},
\]
where \(C\) depends only on \(n, N, M, \lambda, \gamma\) and \(p\). We choose \(R\) to be sufficiently small such that \(C R_i[a, x', \gamma; B_R(x_0)] \leq 1/2\) and then
\[
\|D(\eta u_\varepsilon)\|_{L^p(B_R(x_0))} \leq C\left(\|F\|_{L^p(B_R(x_0))} + \|Du_\varepsilon\|_{L^p(B_R(x_0))}\right),
\]
(3.4)
where \(C\) depends only on \(n, N, M, \lambda, \gamma\) and \(p\).

By Sobolev imbedding inequality,
\[
\|F\|_{L^p(B_R(x_0))} \leq C\|Du_\varepsilon\|_{L^2(B_R(x_0))} + \|f\|_{L^p(B_R(x_0))},
\]
for some constant \(C\) depending only on \(n, N, M, p\) and \(|D\eta|\)\(L^\infty\). Plug this and (3.2) into (3.4). Then we have
\[
\|D(\eta u_\varepsilon)\|_{L^p(B_R(x_0))} \leq C\left(\|Du_\varepsilon\|_{L^2(B_R(x_0))} + \|f\|_{L^p(B_R(x_0))}\right)
\leq C\left(\|u_\varepsilon\|_{L^2(B_R(x_0))} + \|f\|_{L^p(B_R(x_0))}\right),
\]
where Caccioppoli’s inequality is used to derive the last inequality and \(C\) depends only on \(n, N, M, \lambda, \gamma, p, |D\eta|_{L^\infty}\) and \(R\). Now we choose \(\eta \equiv 1\) in \(B_{R/2}(x_0)\) such that \(|D\eta|_{L^\infty} \leq \frac{2}{R}\). Then the above inequality becomes
\[
\|Du_\varepsilon\|_{L^p(B_{R/2}(x_0))} \leq C\left(\|u_\varepsilon\|_{L^2(B_{R/2}(x_0))} + \|f\|_{L^p(B_{R/2}(x_0))}\right),
\]
where \(C\) depends only on \(n, N, M, \lambda, \gamma, p, R\). After a standard covering, the estimate (1.3) is proved for the case \(2 \leq p \leq 2^* = 2n/(n-2)\).

As for \(p > 2^* = 2n/(n-2)\), we use a standard bootstrap argument. Actually, we set \(p_k = p_{k-1}^*\) for \(k = 1, 2, \ldots\) with \(p_0 = 2\). And then there exists a finite \(k\) such that \(p_k < p \leq p_k^*\). Using the proof above, we derive (1.3) holds with \(p\) replaced by \(p_1, p_2, \ldots\). Hence, by \(k\) times, we will obtain (1.3).

To prove Theorem 1.2 (i), we apply the three-step compactness method introduced by Avellaneda and Lin in [1]. The first and second steps are given by the following two lemmas.

**Lemma 3.1.** Suppose \((H_1)\)-(\(H_3\)) hold. For each \(0 < \tau < 1\), there exist \(0 < \theta < 1\) and \(\varepsilon_0 > 0\) depending only on \(n, N, M, \lambda, \gamma\) and \(\tau\) such that if \(u_\varepsilon\) and \(f\) satisfy
\[
\mathcal{L}_\varepsilon u_\varepsilon = \text{div} \ f \text{ in } B_1,
\]
then for any $0 < \varepsilon < \varepsilon_0$, we have
\[
\sup_{|x| < \theta} \left| u_{\varepsilon_k}(x) - u_{\varepsilon_k}(0) - \left( x + \varepsilon \chi(x', \frac{x''}{\varepsilon}) \right)(Du_{\varepsilon_k})_{0, \theta} \right| \leq \theta^{1+\tau/2},
\]
(3.5)
where $\tau = \min\{\gamma, \tau\}$ and $\chi$ is the cell function defined by (2.1).

Proof. We prove (3.5) by contradiction. Suppose, to the contrary, that (3.5) is not true. There exists a sequence $\varepsilon_k \to 0^+$ and sequences $u_{\varepsilon_k}$ and $f_k$ so that
\[
\mathcal{L}_{\varepsilon_k} u_{\varepsilon_k} = \text{div} f_k \text{ in } B_1,
\]
and
\[
[f_k]_{\tau; B_1} \leq 1,
\]
but
\[
\sup_{|x| < \theta} \left| u_{\varepsilon_k}(x) - u_{\varepsilon_k}(0) - \left( x + \varepsilon_k \chi(x', \frac{x''}{\varepsilon_k}) \right)(Du_{\varepsilon_k})_{0, \theta} \right| > \theta^{1+\tau/2},
\]
(3.6)
for any $0 < \theta < 1$.

Note that Theorem 1.1 (i) and the Sobolev imbedding theorem imply $\{||u_{\varepsilon_k}||_{C^1} \}$ is bounded. After extraction of a subsequence (same notation for subsequence), there exist $u_0 \in L^\infty(B_1)$ and $f_0 \in C^1(B_1)$ so that
\[
u_{\varepsilon_k} \to u_0 \text{ in } B_\theta \text{ uniformly},
\]
\[
f_k \to f_0 \text{ in } B_1 \text{ uniformly}
\]
ad
\[
Du_{\varepsilon_k} \to Du_0 \text{ in } L^2(B_\theta) \text{ weakly},
\]
with $u_0$ and $f_0$ satisfying
\[
\mathcal{L}_0 u_0 = \text{div} f_0 \text{ in } B_1,
\]
(3.7)
where $\mathcal{L}_0$ is given by (2.3) and satisfies $(H_2)$ and $(H_3)$ (cf. Remark 2.2).

From (3.7) and the Schauder estimates, we have $u_0 \in C^{1,\tau}_{\text{loc}}(B_1)$ and
\[
\sup_{|x| < \theta} \left| u_0(x) - u_0(0) - x(Du_0)_{0, \theta} \right| \leq C\theta^{1+\tau} \left( ||u_0||_{L^\infty(B_1)} + [f_0]_{\tau; B_1} \right)
\]
for any $0 < \theta \leq \frac{1}{2}$, where $C$ depends only on $n$, $N$, $\lambda$, $\gamma$ and $\tau$. It follows that
\[
\sup_{|x| < \theta} \left| u_{\varepsilon_k}(x) - u_{\varepsilon_k}(0) - x(Du_{\varepsilon_k})_{0, \theta} \right| \leq C\theta^{1+\tau} \left( ||u_0||_{L^\infty(B_1)} + [f_0]_{\tau; B_1} \right)
\]
for any $k$ large enough. Hence we can and we do choose $\theta > 0$ to be small enough such that
\[
\sup_{|x| < \theta} \left| u_{\varepsilon_k}(x) - u_{\varepsilon_k}(0) - x(Du_{\varepsilon_k})_{0, \theta} \right| \leq \frac{1}{4} \theta^{1+\tau/2}.
\]
Fix this $\theta$ and since, by (2.1), $\chi(x', \frac{x''}{\varepsilon_k})$ is bounded, we have
\[
\sup_{|x| < \theta} \left| u_{\varepsilon_k}(x) - u_{\varepsilon_k}(0) - \left( x + \varepsilon_k \chi(x', \frac{x''}{\varepsilon_k}) \right)(Du_{\varepsilon_k})_{0, \theta} \right| \leq \frac{1}{2} \theta^{1+\tau/2}
\]
as $k$ is large enough. This contradicts with (3.6).
Lemma 3.2. Let $\tau$, $\tilde{\tau}$, $\theta$, $\varepsilon_0$ and $\chi$ be as in Lemma 3.1 and set
\[
J = \|u_\varepsilon\|_{L^\infty(B_1)} + |f|_{L^1(B_1)}. \tag{3.8}
\]
Suppose (H1)-(H3) hold, and $u_\varepsilon \in L^\infty(B_1)$ and $f \in C^\tau(B_1)$ satisfying
\[
\mathcal{L}_\varepsilon u_\varepsilon = \text{div} f \text{ in } B_1.
\]
Then, for all $k \in \mathbb{N}$ so that $\varepsilon/\theta^k < \varepsilon_0$, there exist $b^*_k \in \mathbb{R}^N$ and $B^*_k \in \mathbb{R}^{n \times N}$ such that
\[
|b^*_k| + |B^*_k| \leq C, \tag{3.9}
\]
and
\[
\sup_{|x| < \theta^k} \left| u_\varepsilon(x) - u_\varepsilon(0) - \varepsilon b^*_k - \left( x + \varepsilon \chi(x', \frac{x''}{\varepsilon}) \right) B^*_k \right| \leq \theta^{k(1+\tilde{\tau}/2)} J, \tag{3.10}
\]
where $C$ depends only on $n$, $N$, $M$, $\lambda$, $\gamma$ and $\tau$.

Proof. The proof is by induction on $k$. By Lemma 3.1, estimate (3.10) holds for $k = 1$ with $b^*_1 = 0$ and $B^*_1 = (Du_\varepsilon)_{0,\theta}$.

Suppose that (3.10) holds for some $k$ such that $\varepsilon/\theta^k < \varepsilon_0$. Let
\[
w_\varepsilon(x) = \theta^{-k(1+\tilde{\tau}/2)} J^{-1} \left[ u_\varepsilon(\theta^k x) - u_\varepsilon(0) - \varepsilon b^*_k - \left( \theta^k x + \varepsilon \chi(\theta^k x', \frac{\theta^k x''}{\varepsilon}) \right) B^*_k \right].
\]
With the aid of (2.6), we have
\[
\mathcal{L}_\varepsilon/\theta^k w_\varepsilon = D_i \left( a_{ij}^i(\theta^k x', \frac{x''}{\theta^k \varepsilon}) D_j w^2_\varepsilon \right) = \text{div} \hat{f} \text{ in } B_1, \tag{3.11}
\]
where $\hat{f} = J^{-1} \theta^{-k\tilde{\tau}/2} f(\theta^k x)$. Since, by (3.10) and (3.8),
\[
\|w_\varepsilon\|_{L^\infty(B_1)} \text{ and } |\hat{f}|_{L^1(B_1)} \leq 1,
\]
we have, by Lemma 3.1,
\[
\sup_{|x| < \theta} \left| w_\varepsilon(x) - w_\varepsilon(0) - \left( x + \frac{\varepsilon}{\theta^k} \chi(\theta^k x', \frac{\theta^k x''}{\varepsilon}) \right) (Du_\varepsilon)_{0,\theta} \right| \leq \theta^{1+\tilde{\tau}/2}.
\]
Rewriting this inequality in terms of $u_\varepsilon$, we get
\[
\sup_{|x| < \theta} \left| u_\varepsilon(\theta^k x) - u_\varepsilon(0) + \varepsilon \chi(0) B^*_k - \left( \theta^k x + \varepsilon \chi(\theta^k x', \frac{\theta^k x''}{\varepsilon}) \right) B^*_k \right|
\]
\[
- J \theta^{(k+1)(1+\tilde{\tau}/2)} \left( x + \frac{\varepsilon}{\theta^k} \chi(\theta^k x', \frac{\theta^k x''}{\varepsilon}) \right) (Du_\varepsilon)_{0,\theta} \leq J \theta^{(k+1)(1+\tilde{\tau}/2)}
\]
or
\[
\sup_{|x| < \theta^{k+1}} \left| u_\varepsilon(x) - u_\varepsilon(0) + \varepsilon \chi(0) B_k^\varepsilon - \left( x + \varepsilon \chi(x', \frac{x''}{\varepsilon}) \right) B_k^\varepsilon \right|
\]
\[
- J \theta^{k\tilde{\tau}/2} \left( x + \varepsilon \chi(x', \frac{x''}{\varepsilon}) \right) (Du_\varepsilon)_{0,\theta} \leq J \theta^{(k+1)(1+\tilde{\tau}/2)}.
\]
Set
\[
b_k^{\varepsilon} = - \chi(0) B_k^\varepsilon \text{ and } B_k^{\varepsilon} = B_k^\varepsilon + J \theta^{k\tilde{\tau}/2} (Du_\varepsilon)_{0,\theta}.
\]
Observe
\[
|(Du_\varepsilon)_{0,\theta}| = \int_{B_\theta} Dw_\varepsilon = \frac{1}{|B_\theta|} \int_{\partial B_\theta} w_\varepsilon \frac{\partial y}{|y|} ds_y \leq C \|w_\varepsilon\|_{L^\infty(B_1)} \leq C
\]
for some constant $C$ depending only on $n$ and $\theta$. We deduce from iterating the recursive formulae above that

$$|b_{k+1}^\varepsilon| + |B_{k+1}^\varepsilon| \leq CJ,$$

where $C$ depends only on $n$, $N$, $M$, $\lambda$, $\gamma$ and $\tau$. $\square$

**Proof of Theorem 1.2 (i).** We only consider the case $\varepsilon < \varepsilon_0$, since the case $\varepsilon \geq \varepsilon_0$ follows from the classical Schauder estimates. Suppose $B_2 \subset \Omega$ and to prove (1.5), it is sufficient to prove

$$\|Du_\varepsilon\|_{L^\infty(B_{\varepsilon/2\varepsilon_0})} \leq C\left(\|u_\varepsilon\|_{L^2(B_2)} + |f|_{\gamma,B_2}\right),$$

(3.12)

where $\tau = \min\{\gamma, \delta\}$ and $C$ depends only on $n$, $N$, $M$, $\lambda$, $\gamma$ and $\delta$.

Let $\theta$ be given by Lemma 3.2 and $k \in \mathbb{N}$ be such that

$$\theta^{k+1} \leq \frac{\varepsilon}{\varepsilon_0} < \theta^k.$$

The assumed condition $f \in C^\gamma_\varepsilon(\Omega) \cap C^\delta_\varepsilon(\Omega)$ implies $f \in C^\tau(B_2)$ with $\tau = \min\{\gamma, \delta\}$. By Lemma 3.2,

$$\sup_{|x| < 1/\varepsilon_0} \left| \frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(0) - \varepsilon b_k^\varepsilon - (x + \varepsilon \chi(x', \frac{x''}{\varepsilon})) B_k^\varepsilon}{\varepsilon} \right| \leq C(\varepsilon \varepsilon_0^{(1+\tau)/2})J,$$

where $C$ depends only on $\theta$ and $J$ is given by (3.8). Rescaling the above inequality, we obtain by (3.9) that

$$\sup_{|x| < 1/\varepsilon_0} \left| \frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(0)}{\varepsilon} \right| \leq |b_k^\varepsilon - (x + \chi(\varepsilon x', x'')) B_k^\varepsilon| + C\varepsilon_0^{-(1+\tau)/2}J \leq CJ$$

for some constant $C$ depending only on $n$, $N$, $M$, $\lambda$, $\gamma$ and $\delta$.

For $|x| < 1/\varepsilon_0$, define the function

$$v_\varepsilon(x) = \frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(0)}{\varepsilon}$$

and then we have

$$L_1 v_\varepsilon = D_i \left( a_{ij} \left( \varepsilon x', x'' \right) D_j v_\varepsilon^{ij} \right) = \text{div}^\varepsilon f$$

with $\hat{f} = f(\varepsilon x)$. It is clear that

$$\|v_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} \leq C J, \quad \|v_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} \leq C J.$$
4. Higher regularity in directions without homogenization. This section is devoted to Hölder continuity of derivatives of variables without homogenization. Observe here the derivatives are taken in directions without homogenization. However, the Hölder continuity is in all variables. The conclusions (ii) in Theorem 1.1 and 1.2 shall be proved respectively.

First, we recall a result that characterizes Hölder continuity by Campanato spaces (cf. Theorem 1.2 in [10]).

**Lemma 4.1.** Let Ω be a bounded Lipschitz domain in \( \mathbb{R}^n \). For each \( \gamma, 0 < \gamma < 1 \), there exist positive constants \( C_1 \) and \( C_2 \), which depend only on \( \gamma, n \), and the geometry of Ω such that, for all \( u \in C^\gamma(\overline{\Omega}) \), we have

\[
C_1[u]_{\gamma, \overline{\Omega}} \leq \sup_{x \in \Omega} \sup_{0 < r < R} \left[ r^{-2\gamma} \int_{\Omega_r(x)} |u - (u)_{x,r}|^2 \right]^{1/2} \leq C_2[u]_{\gamma, \overline{\Omega}}.
\]

Next, we introduce a technical lemma (cf. Lemma 2.1 in [10]).

**Lemma 4.2.** Let \( \phi(t) \) be a nonnegative and nondecreasing function on \( (0, R_0] \). Suppose that

\[
\phi(r) \leq A \left[ \left( \frac{r}{R} \right)^\alpha + \epsilon \right] \phi(R) + Br^\beta
\]

for all \( 0 < r \leq R \leq R_0 \), with \( A, B, \alpha, \beta \) nonnegative constants, \( \beta < \alpha \). Then there exists a constant \( \epsilon_0 \) depending only on \( A, \alpha \) and \( \beta \) so that if \( \epsilon < \epsilon_0 \), for all \( 0 < r \leq R_0 \), we have

\[
\phi(r) \leq C \left[ \left( \frac{r}{R_0} \right)^\beta \phi(R_0) + Br^\beta \right],
\]

where \( C \) depends only on \( A, \alpha \) and \( \beta \).

Now, we are in a position to prove Theorem 1.1 (ii) and 1.2 (ii).

**Proof of Theorem 1.1 (ii).** Let \( d = \text{dist}(\Omega', \partial\Omega) \) and fix any \( x_0 \in \Omega' \). Set \( 0 < r < R/4 \) and \( 0 < R \leq d/2 \). By (1.1), we have

\[
D_i \left( a_{ij}^{ij}(x_0, \frac{x''}{\varepsilon}) D_j u \varepsilon \right) = D_i \left( (a_{ij}^{ij}(x_0, \frac{x''}{\varepsilon}) - a_{ij}^{ij}(x', \frac{x''}{\varepsilon})) D_j u \varepsilon \right) + D_i f^i \alpha(x) \text{ in } B_R(x_0).
\]

Decompose \( u_\varepsilon \) into \( v_\varepsilon + w_\varepsilon \) with \( v_\varepsilon \) and \( w_\varepsilon \) satisfying

\[
\begin{align*}
D_i \left( a_{ij}^{ij}(x_0, \frac{x''}{\varepsilon}) D_j v_\varepsilon \right) &= D_i f^i \alpha(x_0, x'') \text{ in } B_R(x_0), \\
v_\varepsilon &= u_\varepsilon \text{ on } \partial B_R(x_0),
\end{align*}
\]

and

\[
\begin{align*}
D_i \left( a_{ij}^{ij}(x_0, \frac{x''}{\varepsilon}) D_j w_\varepsilon \right) &= D_i f^i \alpha(x - f^i \alpha(x_0, x'')) \\
+ D_i \left( (a_{ij}^{ij}(x_0, \frac{x''}{\varepsilon}) - a_{ij}^{ij}(x', \frac{x''}{\varepsilon})) D_j u \varepsilon \right) \text{ in } B_R(x_0), \\
w_\varepsilon &= 0 \text{ on } \partial B_R(x_0)
\end{align*}
\]

respectively.

Notice that the \( a_{ij}^{ij}(x_0, \frac{x''}{\varepsilon}) \) and \( f^i \alpha(x_0, x'') \) in (4.1) are independent of \( x' \). We differentiate the first equation in (4.1) with respect to \( x' \) and obtain that

\[
D_i \left( a_{ij}^{ij}(x_0, \frac{x''}{\varepsilon}) D_j \varepsilon \right) = 0 \text{ in } B_{R/2}(x_0),
\]

where
where $\hat{v}_\varepsilon := D_x^\varepsilon v_\varepsilon$. It follows that

$$D_i\left(a_{ij}(x_0, \frac{x''}{\varepsilon})D_j\hat{v}_\varepsilon\right) = 0 \text{ in } B_{R/2}(x_0),$$

where $\tilde{v}_\varepsilon := \hat{v}_\varepsilon - (\hat{v}_\varepsilon)_{x_0,R}$. Take any $\mu \in (\hat{\gamma}, 1)$. By Hölder estimate (cf. Lemma 9 in [1]), we obtain

$$\int_{B_r(x_0)} |D_x^\varepsilon v_\varepsilon - (D_x^\varepsilon v_\varepsilon)_{x_0,R}|^2 = \int_{B_r(x_0)} |\tilde{v}_\varepsilon - (\hat{v}_\varepsilon)_{x_0,R}|^2$$

$$\leq C(\frac{R}{r})^{n+2\mu} \int_{B_{R/2}(x_0)} |\tilde{v}_\varepsilon|^2 = C(\frac{R}{r})^{n+2\mu} \int_{B_{R/2}(x_0)} |\hat{v}_\varepsilon - (\hat{v}_\varepsilon)_{x_0,R}|^2,$$

where $C$ depends only on $n, N, M, \lambda$ and $\hat{\gamma}$.

It follows that

$$\int_{B_r(x_0)} |D_x^\varepsilon u_\varepsilon - (D_x^\varepsilon u_\varepsilon)_{x_0,R}|^2$$

$$= \int_{B_r(x_0)} |D_x^\varepsilon v_\varepsilon - (D_x^\varepsilon v_\varepsilon)_{x_0,R} + D_x^\varepsilon w_\varepsilon - (D_x^\varepsilon w_\varepsilon)_{x_0,R}|^2$$

$$\leq C \int_{B_r(x_0)} |D_x^\varepsilon v_\varepsilon - (D_x^\varepsilon v_\varepsilon)_{x_0,R}|^2 + C \int_{B_r(x_0)} |D_x^\varepsilon w_\varepsilon|^2$$

$$\leq C(\frac{R}{r})^{n+2\mu} \int_{B_{R/2}(x_0)} |D_x^\varepsilon v_\varepsilon - (D_x^\varepsilon v_\varepsilon)_{x_0,R}|^2$$

$$+ C \int_{B_r(x_0)} |Dw_\varepsilon|^2$$

$$\leq C(\frac{R}{r})^{n+2\mu} \int_{B_{R/2}(x_0)} |D_x^\varepsilon v_\varepsilon - (D_x^\varepsilon v_\varepsilon)_{x_0,R}|^2$$

$$+ C \int_{B_r(x_0)} |Dw_\varepsilon|^2,$$

where $C$ depends only on $n, N, M, \lambda$ and $\hat{\gamma}$.

Next, we estimate $\int_{B_R(x_0)} |Dw_\varepsilon|^2$. In fact, from (4.2),

$$\int_{B_R(x_0)} |Dw_\varepsilon|^2$$

$$\leq C \int_{B_R(x_0)} |f(x) - f(x_0', x'') + (a(x_0', \frac{x''}{\varepsilon}) - a(x', \frac{x''}{\varepsilon}))Du_\varepsilon|^2$$

$$\leq C \left(R^{n+2\gamma}[f]_{x', \hat{\gamma}; B_R(x_0)} + R^{2\gamma}[a]_{x', \gamma; B_R(x_0)} \int_{B_R(x_0)} |Du_\varepsilon|^2\right),$$

where $C$ depends only on $\lambda$ and $n$. Apply Theorem 1.1 (i) to $u_\varepsilon$ with $p = n/(\gamma - \hat{\gamma}) > n$ and then

$$\int_{B_R(x_0)} |Du_\varepsilon|^2 \leq CR^{n-2/p} \|Du_\varepsilon\|_{L^p(B_{R/2}(x_0))}^2$$

$$\leq CR^{n-2(\gamma - \hat{\gamma})} \left(\|u_\varepsilon\|_{L^2(B_R(x_0))}^2 + \|f\|_{L^\infty(B_R(x_0))}^2\right),$$

where $C$ depends only on $n, N, M, \lambda, \gamma, \hat{\gamma}$ and $d$. Recall $d = \text{dist}(\Omega', \partial\Omega)$. This and the above inequality imply

$$\int_{B_R(x_0)} |Dw_\varepsilon|^2 \leq CR^{n+2\hat{\gamma}} \left(\|u_\varepsilon\|_{L^2(B_R(x_0))}^2 + \|f\|_{L^2(B_R(x_0))}^2\right),$$

where $C$ depends only on $n, N, M, \lambda, \gamma, \hat{\gamma}$ and $d$. 

In the above, we have used the following fact:

$$\int_{B_R(x_0)} |Dw_\varepsilon|^2 \leq CR^{n+2\hat{\gamma}} \left(\|u_\varepsilon\|_{L^2(B_R(x_0))}^2 + \|f\|_{L^2(B_R(x_0))}^2\right),$$
Inserting (4.4) into (4.3), one has
\[
\int_{B_r(x_0)} |D_x^r u_\varepsilon - (D_x^r u_\varepsilon)_{x_0,r}|^2 \leq C\left(\frac{r}{R}\right)^{n+2\mu} \int_{B_R(x_0)} |D_x^r u_\varepsilon - (D_x^r u_\varepsilon)_{x_0,R}|^2 \leq C R^{n+2\gamma} \left(\|u_\varepsilon\|_{L^2(B_d(x_0))}^2 + |f|^2_{x_0^r,\gamma;B_d(x_0)}\right).
\]

In Lemma 4.2, we set \(\alpha = n + 2\mu, \beta = n + 2\gamma\) and
\[
\phi(r) = \int_{B_r(x_0)} |D_x^r u_\varepsilon - (D_x^r u_\varepsilon)_{x_0,r}|^2.
\]

Here \(\phi(r)\) is nondecreasing. Actually, since for any function \(h, \int_{\Omega} |h - c|^2\) takes minimum as \(c = \int_{\Omega} h\), we have
\[
\phi(r_1) = \int_{B_{r_1}(x_0)} |D_x^r u_\varepsilon - (D_x^r u_\varepsilon)_{x_0,r_1}|^2 \leq \int_{B_{r_2}(x_0)} |D_x^r u_\varepsilon - (D_x^r u_\varepsilon)_{x_0,r_2}|^2 \leq \int_{B_{r_3}(x_0)} |D_x^r u_\varepsilon - (D_x^r u_\varepsilon)_{x_0,r_3}|^2 = \phi(r_2)
\]
for any \(0 < r_1 \leq r_2\). Then it follows from Lemma 4.2, (4.5) and (1.3) that
\[
\int_{B_r(x_0)} |D_x^r u_\varepsilon - (D_x^r u_\varepsilon)_{x_0,r}|^2 \leq C R^{n+2\gamma} \left(\|u_\varepsilon\|_{L^2(B_d(x_0))}^2 + |f|^2_{x_0^r,\gamma;B_d(x_0)}\right),
\]
where \(C\) depends only on \(n, N, M, \lambda, \gamma, \hat{\gamma}, \hat{\delta}\) and \(d\). From Lemma 4.1, we have Theorem 1.1 (ii) holds.

**Proof of Theorem 1.2 (ii).** Let \(d = \text{dist}(\Omega', \partial\Omega)\) and fix any \(x_0 \in \Omega'\). Set \(0 < r < R/4\) and \(0 < R \leq d/2\). The same as the proof of Theorem 1.1 (ii), we also decompose \(u_\varepsilon = v_\varepsilon + w_\varepsilon\), where \(v_\varepsilon\) and \(w_\varepsilon\) satisfy (4.1) and (4.2) respectively. Take any \(\mu \in (\gamma, 1)\). Then we have (4.3) still holds.

Next, we estimate \(w_\varepsilon\) by the following way. From (4.2),
\[
\int_{B_R(x_0)} |Dw_\varepsilon|^2 \leq C \int_{B_R(x_0)} |f(x) - f(x_0, x')| + (a(x_0, \frac{x'}{\varepsilon}) - a(x', \frac{x'}{\varepsilon})) Du_\varepsilon|^2 \leq C \left(R^{n+2\gamma} |f|^2_{x_0^r,\gamma;B_R(x_0)} + R^{2\gamma} |a|^2_{x_0^r,\gamma;B_R(x_0)} \right) \left(\int_{B_R(x_0)} |Du_\varepsilon|^2\right),
\]
where \(C\) depends only on \(\lambda\) and \(n\). In view of Theorem 1.2 (i), we obtain \(u_\varepsilon \in W^{1,\infty}(B_{d/2}(x_0))\) and
\[
\int_{B_R(x_0)} |Du_\varepsilon|^2 \leq CR^n \|Du_\varepsilon\|_{L^2(B_{d/2}(x_0))}^2 \leq CR^n \left(\|u_\varepsilon\|_{L^2(B_d(x_0))}^2 + |f|^2_{x_0^r,\gamma;B_d(x_0)}\right),
\]
where \( \tau = \min\{\gamma, \delta\} \) and \( C \) depends only on \( n, N, M, \lambda, \gamma, \delta \) and \( d \). Then

\[
\int_{B_R(x_0)} |Dw_\varepsilon|^2 \leq CR^{n+2\gamma} \left( \|u_\varepsilon\|^2_{L^2(B_d(x_0))} + \|f\|^2_{L^2;\gamma;B_d(x_0)} \right),
\]

where \( C \) depends only on \( n, N, M, \lambda, \gamma, \delta \) and \( d \).

Putting (4.6) into (4.3), we have

\[
\int_{B_r(x_0)} |Dx^\varepsilon u_\varepsilon - (Dx^\varepsilon u_\varepsilon)_{x_0,r}|^2 \leq C R^{n+2\mu} \int_{B_R(x_0)} |Dx^\varepsilon u_\varepsilon - (Dx^\varepsilon u_\varepsilon)_{x_0,R}|^2 + CR^{n+2\gamma} \left( \|u_\varepsilon\|^2_{L^2(B_d(x_0))} + \|f\|^2_{L^2;\gamma;B_d(x_0)} \right),
\]

By Lemma 4.2, where we set \( \alpha = n + 2\mu, \beta = n + 2\gamma \) and

\[
\phi(r) = \int_{B_r(x_0)} |Dx^\varepsilon u_\varepsilon - (Dx^\varepsilon u_\varepsilon)_{x_0,r}|^2,
\]

we deduce from (4.7) and (1.3) that

\[
\int_{B_r(x_0)} |Dx^\varepsilon u_\varepsilon - (Dx^\varepsilon u_\varepsilon)_{x_0,r}|^2 \leq C R^{n+2\gamma} \left( \|u_\varepsilon\|^2_{L^2(B_d(x_0))} + \|f\|^2_{L^2;\gamma;B_d(x_0)} \right),
\]

where \( C \) depends only on \( n, N, M, \lambda, \gamma, \delta \) and \( d \). By Lemma 4.1, we have the conclusion (ii) in Theorem 1.2.

**Remark 4.3.** In Theorem 1.1, if \( a^{ij}_{\alpha\beta} \) are independent of \( x' \) and \( f \in C_c^\infty(\Omega) \), we can prove that \( Dx^\varepsilon u_\varepsilon \in C^\gamma(\Omega') \) and

\[
[Dx^\varepsilon u_\varepsilon]_{\gamma;\Omega'} \leq C \left( \|u_\varepsilon\|_{L^2(\Omega)} + |f|_{L^2;\gamma;\Omega} \right),
\]

where \( C \) depends only on \( n, N, M, \lambda, \gamma, \Omega' \) and \( \Omega \). The proof is similar to that of Theorem 1.1 (ii). Note that here \( Dx^\varepsilon u_\varepsilon \) is Hölder continuous in all variables. We point out that, in [8], Dong and Kim showed that if \( N = 1 \), the coefficients are independent of \( x' \), \( L^\infty \) in \( x'' \) and the data are Hölder continuous in \( x' \), then for any weak solution \( u \) of divergence form scalar elliptic equation, \( D^\varepsilon u \) is Hölder continuous only in \( x' \). However, to obtain (4.8), the periodicity in \( x'' \) is essential and in [8], no periodicity is assumed.

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