Conformal Deformation to Scalar Flat Metrics with Constant Mean Curvature on the Boundary in Higher Dimensions

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On a closed Riemannian manifold of dimension \( n \geq 3 \), every metric is conformal to a constant scalar curvature metric. This problem, called the Yamabe problem, was proved by Yamabe [20], Trudinger [19], Aubin [1] and Schoen [18].

To extend the conformal deformation problem to manifolds with boundary, Escobar proposed two types of formulations. Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 3 \) with boundary \( \partial M \). We denote by \( R_g \) the scalar curvature of the manifold and by \( \kappa_g \) the mean curvature of the boundary. The first type is to find a metric \( \tilde{g} \) in the conformal class of \( g \) such that \( R_{\tilde{g}} \) is constant and \( \kappa_{\tilde{g}} \) is zero. This was studied by Escobar [12] and recently by Brendle and the author [6].

The second type is to find a metric \( \tilde{g} \) in the conformal class of \( g \) such that \( R_{\tilde{g}} \) is zero and \( \kappa_{\tilde{g}} \) is constant. This problem, as Escobar remarked [11], is a higher dimensional generalization of the Riemann mapping theorem. The problem is studied by Escobar [11], [13] and Marques [16], [17]. (For analysis background for both problems, see [9]).

In this paper, we will study the second formulation; that is the existence of a conformal metric with zero scalar curvature and constant mean curvature on the boundary. The problem turns out to be finding a critical point of the functional

\[
E_g(\phi) = \frac{\int_M \left( \frac{4(n-1)}{n-2} |\nabla_g \phi|^2 + R_g \phi^2 \right) dV_g + \int_{\partial M} 2\kappa_g \phi^2 d\sigma_g}{(\int_{\partial M} \phi^\frac{2(n-1)}{n-2} d\sigma_g)^\frac{n-2}{n-1}},
\]

where \( \phi \) is a positive smooth function on \( M \). The exponent \( \frac{2(n-1)}{n-2} \) is critical for the trace Sobolev embedding \( H^1(M) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\partial M) \). This embedding is not compact and the functional \( E_g \) does not satisfy the Palais-Smale condition. For this reason, standard variational methods cannot be applied.

To study the problem, we consider the Sobolev quotient, introduced in [11],

\[
Q(M, \partial M, g) = \inf_{0 < \phi \in C^\infty} E_g(\phi).
\]

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This is known that $Q(M,\partial M,g)$ is a conformal invariant and $Q(M,\partial M,g) \leq Q(B^n,\partial B^n)$, where $Q(B^n,\partial B^n)$ is the Sobolev quotient of the unit ball $B^n$ in $\mathbb{R}^n$ equipped with the flat metric. It was proved by Escobar that

**Theorem 1.** (Escobar [11]) If $Q(M,\partial M,g) < Q(B^n,\partial B^n)$, then there exists a metric $\tilde{g}$ in the conformal class of $g$ such that $R_{\tilde{g}}$ is zero and $\kappa_{\tilde{g}}$ is constant.

For $n \geq 6$, when $\partial M$ is not umbilic, Escobar showed that $Q(M,\partial M,g) < Q(B^n,\partial B^n)$. He also proved the inequality holds when $n = 3$, and when $n = 4, 5$ and $\partial M$ is umbilic, provided $M$ is not conformally equivalent to the unit ball. When $n = 4, 5$, and $\partial M$ is not umbilic, Marques verified that the inequality holds.

Consequently, it remains to consider the case that $n \geq 6$ and $\partial M$ is umbilic (some special case was considered in [16]). As in [4], [6], we denote by $Z$ the set of points $p \in M$ such that

$$\limsup_{x \to p} d(p, x)^{2-d}|W_g|(x) = 0,$$

where $d = \left[\frac{n-2}{2}\right]$ and $W_g$ is the Weyl tensor of $g$. We note that $p \in Z$ if and only if $\nabla^m W_g(p) = 0$ for $m = 0, \ldots, d-2$. Moreover, the set $Z$ is conformally invariant.

Our main result is

**Theorem 2.** Let $(M,g)$ be a compact Riemannian manifold of dimension $n \geq 6$ with umbilic boundary. Suppose there exists a point $p \in \partial M$ such that $p \notin Z$, then $Q(M,\partial M,g) < Q(B^n,\partial B^n)$. As a result, there exists a metric $\tilde{g}$ in the conformal class of $g$ such that $R_{\tilde{g}}$ is zero and $\kappa_{\tilde{g}}$ is constant.

We now discuss the case that $p \in Z$ for all $p \in \partial M$. In Section 4 we consider a flux integral $\mathcal{I}(p,\delta)$ introduced in [6] in a small neighborhood of $p \in \partial M$. When $p \in Z$, it was shown in [6] that $\lim_{\delta \to 0} \mathcal{I}(p,\delta) \exists$ and is equal to a positive multiple of ADM mass of certain scalar flat asymptotically flat manifold; see Section 4. We reduce the case to positivity of mass.

**Theorem 3.** Let $(M,g)$ be a compact Riemannian manifold of dimension $n \geq 6$ with umbilic boundary. Suppose there exists a point $p \in \partial M$ such that $p \in Z$ and $\lim_{\delta \to 0} \mathcal{I}(p,\delta) > 0$, then $Q(M,\partial M,g) < Q(B^n,\partial B^n)$. As a result, there exists a metric $\tilde{g}$ in the conformal class of $g$ such that $R_{\tilde{g}}$ is zero and $\kappa_{\tilde{g}}$ is constant.

We give the outline of the proof. By Marques [16], we may choose conformal Fermi coordinates around a boundary point $p$. In these coordinates, we define

$$v_\epsilon = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + \sum_{1 \leq a \leq n-1} x_a^2}\right)^{\frac{n-2}{n-1}}.$$

We note that $v_\epsilon$ is the extremal function for the sharp trace Sobolev inequality on the half plane; see [10], [2]. By conformal invariance, it holds

$$Q(B^n,\partial B^n) \left(\int_{\partial R^n_+} v_\epsilon^{\frac{2(n-1)}{n-2}} d\sigma\right)^{\frac{n-2}{n-1}} = \frac{4(n-1)}{n-2} \int_{R^n_+} |\nabla v_\epsilon|^2 dx.$$
It is then understood that \( v \) is the model function on \( \mathbb{R}_+^n \).

We now consider the function \( v + \psi \) defined in a small neighborhood of \( p \), where \( \psi \) satisfies

\[
\Delta \psi = \sum_{i,k=1}^n \left( \frac{n-2}{4(n-1)} v_i \partial_i \partial_k S_{ik} + \partial_k (\partial_i v \partial_k S_{ik}) \right) \quad \text{in } B_\delta \cap \mathbb{R}_+^n, \tag{1}
\]

\[
\partial_n \psi = -\frac{1}{2(n-1)} \partial_n v \partial_{nn} + \frac{n}{n-2} v^{-1} \partial_n v \psi \quad \text{on } B_\delta \cap \partial \mathbb{R}_+^n. \tag{2}
\]

In the above equations, the tensor \( S_{ij} \) comes from applying the conformal killing operator to certain vector field we solve; see Section 2. The equation (1) corresponds to a linear approximation of the scalar curvature equation of \((v + \psi)^{n+2} g\). However, in our construction, the boundary condition (2) is not the linear approximation of the mean curvature equation of \((v + \psi)^{n+2} g\); the "linear mean curvature equation" should be

\[
\partial_n \psi = \frac{n}{n-2} v^{-1} \partial_n v \psi.
\]

We emphasize that the Sobolev quotient \( Q(M, \partial M, g) \) is normalized by the volume of the boundary (not the volume of the manifold). Our deformation of the metric does not fix the volume of the boundary locally. As a consequence, in order to get the energy functional small enough, the term \(-\frac{1}{2(n-1)} \partial_n v \partial_{nn} \) is important because it cancels out to the right order the change of the volume of the boundary. This is the reason that the linear approximation of the mean curvature equation does not work here. This turns out to be the delicate part of the proof. Finally, to define a test function globally, we glue the function \( v + \psi \) with the Green’s function of the conformal Laplacian centered at \( p \).

To show the above test function has the energy functional less than \( Q(B^n, \partial B^n) \), we use the method and techniques developed by Brendle [4] (see also [6]). In [4], these nice techniques were used to prove a convergence theorem for the Yamabe flow. In [6], these techniques were used to study the problem of first type described at the beginning. To be more precise, let \( u_\epsilon = \epsilon^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{-\frac{1}{2}} \). In [4], one considers the function \( u_\epsilon + w \) in normal coordinates, where \( w \) satisfies \( \Delta w + n(n+2) u_\epsilon^{\frac{4}{n+2}} w = \frac{n-2}{4(n-1)} u_\epsilon \partial_i \partial_k S_{ik} + \partial_k (\partial_i u_\epsilon S_{ik}) \). In [6], one considers the function \( u_\epsilon + w \) in Fermi coordinates together with the boundary condition \( \partial_n w = 0 \). We refer the readers to [3], [5], [15], [7], [8] for other related works concerning the Yamabe problem.

We introduce the notation in this paper. We denote by \( dx \) the volume element in \( \mathbb{R}^n \), by \( d\sigma \) the area element of a hypersurface in \( \mathbb{R}^n \) and by \( d\mu \) the area element of an \((n-2)\)-dimensional surface in \( \mathbb{R}^n \). We also denote by \( \mathbb{R}_+^n \) the area plane \( \{ x : x_n \geq 0 \} \). Let \( B_r(x) \) be the ball of radius \( r \) centered at \( x \). When \( x \) is at the origin, we simply denote by \( B_r \).

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1 Background

Let \( v_\epsilon(x) = \epsilon^{n^2/2}((\epsilon + x_n)^2 + \sum_{a=1}^{n-1} x_a^2)^{-n^2/2} = \frac{1}{n-2} v_\epsilon \partial_i v_\epsilon \partial_k v_\epsilon = -\frac{1}{n-2} |dv_\epsilon|^2 \delta_{ik} \) for \( x \in \mathbb{R}_+^n \), \( \Delta v_\epsilon = 0 \) for \( x \in \mathbb{R}_+^n \), \( \partial_n v_\epsilon = -(n-2)v_\epsilon^{n/2} \) for \( x \in \partial \mathbb{R}_+^n \).

By integration, we get
\[
\int_{\mathbb{R}_+^n} |\nabla v_\epsilon|^2 dx = (n-2) \int_{\partial \mathbb{R}_+^n} v_\epsilon^{2(n-1)/n-2} d\sigma.
\]

Moreover, \( v_\epsilon \) satisfies the following inequalities:
\[
\epsilon^{n^2/2}(\epsilon + |x|)^{-n^2+2} \leq v_\epsilon(x) \leq C(n)\epsilon^{n^2/2}(\epsilon + |x|)^{-n^2+2} \quad \text{for} \quad x \in \mathbb{R}_+^n;
\]
\[
|\partial v_\epsilon(x)| \leq C(n)\epsilon^{n^2/2}(\epsilon + |x|)^{-n^2+1} \quad \text{for} \quad x \in \mathbb{R}_+^n;
\]
and
\[
|v_\epsilon - \epsilon^{n^2/2} |x|^{-n^2+2}| \leq C(n)\epsilon^{n^2/2} |x|^{-n^2+1} \quad \text{for} \quad x \in \mathbb{R}_+^n, \text{ and } |x| \geq 2\epsilon,
\]
where \( C(n) \) is a positive constant depending only on \( n \).

Let \( V \) be a smooth vector field and \( H_{ik} \) be a trace-free symmetric two-tensor. We define
\[
S_{ik} = \partial_i V_k + \partial_k V_i - \frac{2}{n} div V \delta_{ik},
\]
\[
T_{ik} = H_{ik} - S_{ik},
\]
\[
P_{ik,l} = v_\epsilon \partial_l T_{ik} - \frac{2}{n-2} \partial_i v_\epsilon T_{kl} - \frac{2}{n-2} \partial_k v_\epsilon T_{il} + \frac{2}{n-2} \sum_{p=1}^{n} \partial_p v_\epsilon T_{ip} \delta_{kl} + \frac{2}{n-2} \sum_{p=1}^{n} \partial_p v_\epsilon T_{kp} \delta_{il},
\]
\[
\psi = \partial_i v_\epsilon V_i + \frac{n-2}{2n} v_\epsilon div V.
\]

In [4], [6], a similar notation was introduced with \( v_\epsilon \) replaced by \( u_\epsilon \).

The following formula is a revision of the formula in [4] Proposition 5, 6. The formula in [4] corresponds to the second variation of the scalar curvature on the sphere. Similarly, the formula here corresponds to the second variation of the scalar curvature on the ball in \( \mathbb{R}^n \).

**Proposition 1.** Let \( H_{ik} \) be a trace-free symmetric two-tensor, and \( V \) be a smooth vector field. Then \( \psi \) satisfies
\[
\Delta \psi = \sum_{i,k=1}^{n} \left( \frac{n-2}{4(n-1)} v_i \partial_i \partial_k S_{ik} + \partial_k (\partial_i v_\epsilon S_{ik}) \right).
\]
Moreover,

\[
\frac{1}{4} |P|^2 - \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n (v_i \partial_k T_{ik} + \frac{2n}{n-2} \partial_k v_i T_{ik})^2
\]

\[
= \sum_{i,k,l=1}^n \left( \frac{1}{4} v_i^2 \partial_l H_{ik} \partial_l H_{ik} - \frac{1}{2} v_i^2 \partial_k H_{ik} \partial_l H_{il} - 2v_i \partial_k v_i H_{ik} \partial_l H_{il} - \frac{2(n-1)}{n-2} \partial_k v_i \partial_l v_i H_{ik} H_{il} \right)
\]

\[
+ \sum_{i,k,l=1}^n (-2v_i \psi \partial_l H_{ik} + \frac{8(n-1)}{n-2} \partial_i v_i \partial_k \psi H_{ik}) - \frac{4(n-1)}{n-2} |d\psi|^2 + \sum_{i=1}^n \partial_i \xi_i,
\]

where

\[
\xi_i = \sum_{k=1}^n (2v_i \psi \partial_k H_{ik} - 2v_i \partial_k \psi H_{ik} - 2\partial_k v_i \psi H_{ik} - v_i \partial_k S_{ik} + \partial_k (v_i \psi) S_{ik})
\]

\[
+ \sum_{k,l=1}^n (2v_i \partial_k v_i S_{kl} H_{ki} - \frac{1}{2} v_i^2 \partial_k S_{kl} H_{ki} + v_i^2 \partial_l S_{kl} H_{ki} + \frac{1}{4} v_i^2 \partial_i S_{kl} S_{ik} - \frac{1}{2} v_i^2 \partial_k S_{ik} S_{kl})
\]

\[
+ \sum_{k,l=1}^n (-v_i \partial_k v_i S_{kl} S_{ld} - \frac{2}{n-2} v_i \partial_k v_i T_{ik} T_{il}) + \frac{4(n-1)}{n-2} (-\sum_{k=1}^n \partial_k v_i \psi S_{ik} + \psi \partial_i \psi).
\]

**Proof.** Since the proof is similar, we only point out the difference. In [4] Proposition 5, it was shown that

\[
v_i \partial_i \partial_k S_{ik} + \frac{4(n-1)}{n-2} \partial_k (\partial_i v_i S_{ik}) = \frac{4(n-1)}{n-2} \Delta (\sum_{l=1}^n \partial_l v_i V_l + \frac{n-2}{2n} v_i \text{div}V)
\]

\[
- \frac{4(n-1)}{n-2} (\sum_{l=1}^n \partial_l \Delta v_i V_l + \frac{n+2}{2n} \Delta v_i \text{div}V)
\]

(with $v_i$ replaced by $u_i$ but the formula holds in general). By [3], then (6) follows.

For the second identity, by [4] Proposition 5, it holds

\[
\frac{1}{4} v^2 \partial T^2 - \frac{1}{2} v^2 |\text{div}T|^2 - \sum_{i,k,l=1}^n (2v_i \partial_k v_i T_{ik} \partial_l T_{il} + \frac{2(n-1)}{n-2} \partial_k v_i \partial_l v_i T_{ik} T_{il})
\]

\[= I_1 - 2I_2 + I_3,
\]

where

\[
I_1 = \sum_{i,k,l=1}^n \left( \frac{1}{4} v^2 \partial_l H_{ik} \partial_l H_{ik} - \frac{1}{2} v^2 \partial_k H_{ik} \partial_l H_{il} - 2v \partial_k v \partial_l H_{ik} H_{il} - \frac{2(n-1)}{n-2} \partial_k v \partial_l H_{ik} H_{il} \right),
\]
\( I_2 = \sum_{i,k=1}^{n} (v_\epsilon \partial_i \partial_k H_{ik} - \frac{4(n-1)}{n-2} \partial_i v_\epsilon \partial_k \psi H_{ik} - \partial_k (v_\epsilon \partial_i \psi H_{ik}) + \partial_k (v_\epsilon \partial_i \psi H_{ik})} \)
\[+ \sum_{i,k=1}^{n} \partial_k (\partial_i v_\epsilon \psi H_{ik}) + \sum_{i,k,l=1}^{n} \left( \frac{1}{4} \partial_i (v_\epsilon^2 \partial_l S_{ik} H_{ik}) - \frac{1}{2} \partial_k (v_\epsilon^2 \partial_l S_{il} H_{ik}) - \partial_k (v_\epsilon \partial_l v_\epsilon S_{il} H_{ik}) \right) \]
\[+ \sum_{i,k,l=1}^{n} (v_\epsilon \partial_k \partial_l v_\epsilon - \frac{n}{n-2} \partial_k v_\epsilon \partial_l v_\epsilon) (\partial_i V_l - \partial_i V_i) H_{ik} \]
\[- \sum_{i,k,l=1}^{n} \partial_l [(v_\epsilon \partial_l \partial_k v_\epsilon - \frac{n}{n-2} \partial_l v_\epsilon \partial_k v_\epsilon) V_i] H_{ik}, \]

and

\( I_3 = \sum_{i,k=1}^{n} (v_\epsilon \partial_i \partial_k S_{ik} - \frac{4(n-1)}{n-2} \partial_i v_\epsilon \partial_k \psi S_{ik} - \partial_k (v_\epsilon \partial_i \psi S_{ik}) + \partial_k (v_\epsilon \partial_i \psi S_{ik})) \)
\[+ \sum_{i,k=1}^{n} \partial_k (\partial_i v_\epsilon \psi S_{ik}) + \sum_{i,k,l=1}^{n} \left( \frac{1}{4} \partial_i (v_\epsilon^2 \partial_l S_{ik} S_{ik}) - \frac{1}{2} \partial_k (v_\epsilon^2 \partial_l S_{il} S_{ik}) - \partial_k (v_\epsilon \partial_l v_\epsilon S_{il} S_{ik}) \right) \]
\[+ \sum_{i,k,l=1}^{n} (v_\epsilon \partial_k \partial_l v_\epsilon - \frac{n}{n-2} \partial_k v_\epsilon \partial_l v_\epsilon) (\partial_i V_l - \partial_i V_i) S_{ik} \]
\[- \sum_{i,k,l=1}^{n} \partial_l [(v_\epsilon \partial_l \partial_k v_\epsilon - \frac{n}{n-2} \partial_l v_\epsilon \partial_k v_\epsilon) V_i] S_{ik}. \]

And in \[4\] Proposition 6, it holds

\[\frac{1}{4} v_\epsilon^2 |\partial T|^2 - \frac{1}{2} v_\epsilon^2 |\text{div} T|^2 - \sum_{i,k,l=1}^{n} (2 v_\epsilon \partial_k v_\epsilon T_{ik} \partial_l T_{il} + \frac{2(n-1)}{n-2} \partial_k v_\epsilon \partial_l v_\epsilon T_{ik} T_{il}) \]
\[= \frac{1}{4} |P|^2 - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} (v_\epsilon \partial_k T_{ik} + \frac{2(n-1)}{n-2} \partial_k v_\epsilon T_{ik})|^2 - \frac{2}{(n-2)^2} |\partial v_\epsilon|^2 |T|^2 \]
\[+ \sum_{i,k,l=1}^{n} (- \frac{2}{n-2} (v_\epsilon \partial_k \partial_l v_\epsilon - \frac{n}{n-2} \partial_k v_\epsilon \partial_l v_\epsilon) T_{ik} T_{il} + \frac{2}{n-2} \partial_l (v_\epsilon \partial_k v_\epsilon T_{ik} T_{il})) \quad (7) \]

(with \( v_\epsilon \) replaced by \( u_\epsilon \) but the formula holds in general). Using (4) in \( I_2 \), (4) and (6) in \( I_3 \) and using (4) in (7) give the identity.

\[\square\]

## 2 Construction

We first state some properties about conformal Fermi coordinates that we will use later. Then we construct the correction term \( \psi \) and compute some formulas on the boundary. Let \( n \geq 6 \). We assume \( \partial M \) is totally geodesic.
In this section, we assume $g$ is the metric in conformal Fermi coordinates. We write $g = \exp h$. By Marques [16], we have $tr h(x) = O(|x|^{2d+2})$ for $x \in \mathbb{R}^n_+$, where $d = \lfloor \frac{n-2}{2} \rfloor$. Moreover, $h_{in}(x) = 0$ for $x \in \partial \mathbb{R}^n_+$ and $i = 1, \ldots, n$. We also have $\partial_n h_{ab}(x) = \sum_{i=1}^n h_{ai}(x) x_i = 0$ for $x \in \partial \mathbb{R}^n_+$ and $a, b = 1, \ldots, n - 1$. In this case, $\det g(x) = 1 + O(|x|^{2d+2})$ for $x \in \mathbb{R}^n_+$.

Let $H_{ij}$ be the Taylor expansion of $h_{ij}$ up to the order $d$

$$H_{ij} = \sum_{2 \leq |\alpha| \leq d} h_{ij,\alpha}(0) x^\alpha,$$

where $\alpha$ is a multi-index. Then $h_{ik} = H_{ik} + O(|x|^{d+1})$. It follows that

$$tr H(x) = H_{in}(x) = 0 \quad \text{for all } x \in \mathbb{R}^n_+ \text{ and } i = 1, \ldots, n,$$

and

$$\partial_n H_{ab}(x) = \sum_{i=1}^n H_{ai}(x) x_i = 0 \quad \text{for all } x \in \partial \mathbb{R}^n_+ \text{ and } a, b = 1, \ldots, n - 1.$$

We define algebraic Schouten tensor and algebraic Weyl tensor of $H_{ij}$ as in [4]:

$$A_{ij} = \partial_i \partial_m H_{mj} + \partial_m \partial_j H_{im} - \Delta H_{ij} - \frac{1}{n-1} \partial_m \partial_p H_{mp} \delta_{ij},$$

$$Z_{ijkl} = \partial_i \partial_k H_{jl} - \partial_i \partial_l H_{jk} - \partial_j \partial_k H_{il} + \partial_j \partial_l H_{ik} + \frac{1}{n-2} (A_{jl} \delta_{ik} - A_{jk} \delta_{il} - A_{il} \delta_{jk} + A_{ik} \delta_{jl}).$$

**Proposition 2.** [9] If $Z_{ijkl} = 0$ for all $x \in \mathbb{R}^n_+$, then $H_{ij} = 0$ for all $x \in \mathbb{R}^n_+$.

**Proposition 3.** [6] The scalar curvature $R_g$ satisfies

$$|R_g - \partial_i \partial_i H_{ik}| \leq C \sum_{i,j} \sum_{2 \leq |\alpha| \leq d} |h_{ij,\alpha}| |x|^{|\alpha|} + C |x|^{d-1},$$

and

$$|R_g - \partial_i \partial_j h_{ik} + \partial_k (H_{ik} \partial_i H_{il}) - \frac{1}{2} \partial_k H_{ik} \partial_l H_{il} + \frac{1}{4} \partial_l H_{ik} \partial_l H_{ik}|$$

$$\leq C \sum_{i,j} \sum_{2 \leq |\alpha| \leq d} |h_{ij,\alpha}| |x|^{2|\alpha|} + C \sum_{i,j} \sum_{2 \leq |\alpha| \leq d} |h_{ij,\alpha}| |x|^{|\alpha|+d-1} + C |x|^{2d}$$

for $|x|$ sufficiently small.

Let $V$ be a smooth vector field. We next define as in Section 1 that

$$S_{ik} = \partial_i V_k + \partial_k V_i - \frac{2}{n} div V \delta_{ik},$$

$$T_{ik} = H_{ik} - S_{ik},$$

$$P_{ik,l} = v_e \partial_l T_{ik} - \frac{2}{n-2} \partial_i v_e T_{kl} - \frac{2}{n-2} \partial_k v_e T_{il} + \frac{2}{n-2} \sum_{p=1}^n \partial_p T_{ip} \delta_{kl} + \frac{2}{n-2} \sum_{p=1}^n \partial_p v_e T_{kp} \delta_{il}.$$
Proposition 4. Let $V$ be a smooth vector field. Then

$$
\sum_{i,j} \sum_{2 \leq |\alpha| \leq d} |h_{ij,\alpha}|^2 \epsilon^{n-2} \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|+2-2n} \, dx \leq C(n) \int_{B_\delta \cap \mathbb{R}^n_+} |P|^2 \, dx
$$

for $\delta \geq 2 \epsilon > 0$.

Proof. In [4] Proposition 9, it was shown that

$$
\sum_{i,j,k,l=1}^n \{ \partial_j (\partial_l T_{ik} - \frac{2}{n-2} v_\epsilon^{-1} \partial_k v_\epsilon T_{il}) + \frac{2}{n-2} v_\epsilon^{-1} \partial_k v_\epsilon (\partial_j T_{il} - \frac{2}{n-2} v_\epsilon^{-1} \partial_l v_\epsilon T_{ji}) \\
+ \frac{2}{n-2} v_\epsilon^{-2} (v_\epsilon \partial_j \partial_k v_\epsilon - \frac{n}{n-2} \partial_j v_\epsilon \partial_k v_\epsilon) T_{il} + \frac{4}{(n-2)^2} v_\epsilon^{-2} \partial_k v_\epsilon (\partial_i v_\epsilon T_{jl} + \partial_j v_\epsilon T_{il}) \} Z_{ijkl}
$$

$$
= \sum_{i,j,k,l=1}^n \partial_j \partial_l H_{ik} Z_{ijkl}
$$

(with $v_\epsilon$ replaced by $u_\epsilon$ but the formula holds in general). Then by [1], we have

$$
\sum_{i,j,k,l=1}^n (\partial_j (v_\epsilon^{-1} P_{ik,l}) Z_{ijkl} + \frac{2}{n-2} v_\epsilon^{-2} \partial_k v_\epsilon P_{il,j} Z_{ijkl}) = \frac{1}{4} |Z|^2.
$$

From this, the assertion follows easily by the proof in [6] Proposition 7 and Corollary 8 using $\epsilon^{n-2} (\epsilon + |x|)^{-n+2} \leq v_\epsilon(x) \leq C(n) \epsilon^{n-2} (\epsilon + |x|)^{-n+2}$ and $|\partial v_\epsilon|(x) \leq C(n) \epsilon^{n-2} (\epsilon + |x|)^{-n+1}$.

We next construct the correction term $\psi$. We fix a positive smooth function $\eta(t)$ such that $\eta(t) = 1$ for $t \leq \frac{4}{3}$ and $\eta(t) = 0$ for $t \geq \frac{5}{3}$. For $\delta > 0$, we define $\eta_\delta(x) = \eta(\frac{|x|}{\delta})$, $x \in \mathbb{R}^n_+$. Notice that $\partial_n \eta_\delta(x) = 0$ for all $x \in \partial \mathbb{R}^n_+$. By Proposition 12 in Appendix, there exists a smooth vector field $V$ which solves

$$
\begin{cases}
\sum_{k=1}^n \partial_k [v_\epsilon^{\frac{2n}{n-2}} (\eta_\delta H_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} \text{div} V \delta_{ik})] = 0 & \text{in } \mathbb{R}^n_+ \\
\partial_a V_a = 0 & \text{on } \partial \mathbb{R}^n_+ \\
V_n = 0 & \text{on } \partial \mathbb{R}^n_+
\end{cases}
$$

for $i = 1, \cdots, n$ and $a = 1, \cdots, n-1$. Moreover, $V$ satisfies

$$
|\partial^\beta V(\epsilon, \delta)(x)| \leq C(n, |\beta|) \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| (\epsilon + |x|)^{|\alpha|+1-|\beta|}.
$$

By the equation,

$$
\sum_{k=1}^n (v_\epsilon \partial_k T_{ik} + \frac{2n}{n-2} \partial_k v_\epsilon T_{ik}) = 0
$$
for \( x \in B_\delta \cap \mathbb{R}^n_+ \) and \( i = 1, \ldots, n \). We next define
\[
\psi = \sum_{i=1}^{n} \partial_i v_i V_i + \frac{n-2}{2n} v_\epsilon \text{div}V.
\]

**Proposition 5.** It holds \( S_{an}(x) = 0 \),
\[
\partial_n S_{nn}(x) = -\frac{2n}{n-2} v_\epsilon(x)^{-1} \partial_n v_\epsilon(x) S_{nn}(x) = 2n v_\epsilon(x)\frac{n-2}{2n} S_{nn}(x),
\]
and
\[
\partial_n S_{ab}(x) = -\frac{2n}{n-1} v_\epsilon(x) \frac{n-2}{2n} S_{nn}(x) \delta_{ab}
\]
for \( x \in \partial \mathbb{R}^n_+ \) and \( a, b = 1, \ldots, n-1 \). As a consequence, for \( x \in \partial \mathbb{R}^n_+ \),
\[
\partial_n \psi(x) = -\frac{1}{2(n-1)} \partial_n v_\epsilon(x) S_{nn}(x) + \frac{n}{n-2} v_\epsilon(x)^{-1} \partial_n v_\epsilon(x) \psi(x).
\]

**Proof.** By assumptions, \( V_n = \partial_n V_a = 0 \) for \( x \in \partial \mathbb{R}^n_+ \) and \( a = 1, \ldots, n-1 \). Thus, \( S_{na} = T_{na} = \partial_n V_a - \partial_a V_n = 0 \) on \( \partial \mathbb{R}^n_+ \) for \( a = 1, \ldots, n-1 \) and
\[
\partial_n \partial_a V_b = 0 \quad \text{for} \quad x \in \partial \mathbb{R}^n_+ \quad \text{and} \quad a, b = 1, \ldots, n-1.
\]

We next consider the equation (8). It gives
\[
\sum_{k=1}^{n} (v_\epsilon \partial_k (\eta_b H_{nk} - S_{nk}) + \frac{2n}{n-2} \partial_k v_\epsilon (\eta_b H_{nk} - S_{nk})) = 0.
\]
Since \( H_{nk}(x) = 0 \) for all \( x \in \mathbb{R}^n_+ \) and \( k = 1, \ldots, n \), we have
\[
\sum_{k=1}^{n} (v_\epsilon \partial_k S_{nk} + \frac{2n}{n-2} \partial_k v_\epsilon S_{nk}) = 0.
\]
for all \( x \in \mathbb{R}^n_+ \). Therefore, using (5)
\[
\partial_n S_{nn} = -\sum_{a=1}^{n-1} \partial_a S_{na} - \frac{2n}{n-2} v_\epsilon^{-1} \sum_{k=1}^{n} \partial_k v_\epsilon S_{nk} = -\frac{2n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon S_{nn} = 2n v_\epsilon \frac{n-2}{2n} S_{nn}.
\]
Moreover, by (11), it follows that
\[
\partial_n S_{ab} = \partial_n \partial_a V_b + \partial_n \partial_b V_a - \frac{2n}{n-2} \partial_n \text{div}V \delta_{ab} = -\frac{2n}{n-2} \partial_n \partial_a V_n \delta_{ab}
\]
\[
= -\frac{1}{n-1}(2\partial_n V_n - \frac{2n}{n-2} \partial_n \text{div}V) \delta_{ab} = -\frac{1}{n-1} \partial_n S_{mn} \delta_{ab} = -\frac{2n}{n-2} v_\epsilon \frac{n-2}{2n} S_{nn} \delta_{ab}.
\]
We now compute \( \partial_n \psi \).
\[
\partial_n \psi = \sum_{i=1}^{n} (\partial_n \partial_i v_i V_i + \partial_i v_i \partial_n V_i) + \frac{n-2}{2n} \partial_n v_\epsilon \text{div}V + \frac{n-2}{2n} v_\epsilon \partial_n \text{div}V
\]
\[
= \sum_{i=1}^{n} (\partial_n \partial_i v_i - \frac{n}{n-2} v_\epsilon^{-1} \partial_i v_\epsilon \partial_n v_i) V_i + \sum_{i=1}^{n} \partial_i v_i \partial_n V_i
\]
\[
+ \frac{n}{n-2} (\sum_{i=1}^{n} \partial_i v_i + \frac{n-2}{2n} v_\epsilon \text{div}V) v_\epsilon^{-1} \partial_n v_\epsilon - \frac{1}{n} \text{div}V \partial_n v_\epsilon + \frac{n-2}{2n} v_\epsilon \partial_n \text{div}V.
\]
By \((4)\) and \(\partial_n \text{div} V = \frac{n}{2(n-1)} \partial_n S_{nn}\), we get

\[
\partial_n \psi = \frac{n}{n-2} \left( \sum_{i=1}^{n} \partial_i v_i V_i + \frac{n-2}{2n} v_i \text{div} V \right) v_i^{-1} \partial_n v_i - \frac{1}{n} \text{div} V \partial_n v_i
\]

\[-\frac{1}{n-2} v_i^2 V_i + \sum_{i=1}^{n} \partial_i v_i \partial_n V_i + \frac{n-2}{4(n-1)} v_i \partial_n S_{nn}.\]

Since \(\partial_n S_{nn} = -\frac{2n}{n-2} v_i^{-1} \partial_n v_i S_{nn}\) and \(\partial_n V_a = V_n = 0\) on \(\partial \mathbb{R}^n_+\) for \(a = 1, \cdots, n-1\), then

\[
\partial_n \psi = \frac{n}{n-2} v_i^{-1} \partial_n v_i - \frac{1}{n} \text{div} V \partial_n v_i + \partial_n v_i \partial_n V_n - \frac{n}{2(n-1)} \partial_n v_i S_{nn}
\]

\[-\frac{n}{n-2} \partial_n v_i v_i^{-1} \psi - \frac{1}{2(n-1)} \partial_n v_i S_{nn}.\]

\[\square\]

**Proposition 6.** Let \(\xi_i\) be defined as in Proposition \([4]\). It follows for \(x \in \partial \mathbb{R}^n_+\),

\[
\xi_i(x) = -\frac{n+2}{2(n-2)} v_i(x) \partial_n v_i(x) S_{nn}(x)^2 + \frac{4n(n-1)}{(n-2)^2} v_i(x)^{-1} \partial_n v_i(x) \psi(x)^2.
\]

**Proof.** Since \(H_{in} = 0\) for \(i = 1, \cdots, n\) and \(x \in \mathbb{R}^n_+\), and \(S_{na} = T_{na} = 0\) for \(a = 1, \cdots, n-1\) and \(x \in \partial \mathbb{R}^n_+\), we have

\[
\xi_n = -\frac{1}{2} v_i^2 \sum_{a,b=1}^{n-1} \partial_n S_{ab} H_{ab} - v_i \psi \partial_n S_{nn} + v_i \partial_n \psi S_{nn} + \partial_n v_i \psi S_{nn}
\]

\[+ \frac{1}{4} v_i^2 \left( \sum_{a,b=1}^{n-1} \partial_n S_{ab} S_{ab} + \partial_n S_{nn} S_{nn} \right) - \frac{1}{2} v_i^2 \partial_n S_{nn} S_{nn} - v_i \partial_n v_i S_{nn} S_{nn}
\]

\[- \frac{2}{n-2} v_i \partial_n v_i S_{nn} S_{nn} + \frac{4(n-1)}{n-2} (-\partial_n v_i \psi S_{nn} + \psi \partial_n \psi).\]

By \(\partial_n S_{nn} = -\frac{2n}{n-2} v_i^{-1} \partial_n v_i S_{nn}\) and \(\partial_n S_{ab} = -\frac{2n}{n-1} v_i^{-2} S_{nn} \delta_{ab}\), we get

\[
\xi_n = \frac{n}{n-1} v_i^2 v_i^{-2} S_{nn} \sum_{a=1}^{n-1} H_{aa} + \frac{2n}{n-2} v_i \psi \partial_n v_i S_{nn} + v_i \partial_n \psi S_{nn} + \partial_n v_i \psi S_{nn}
\]

\[- \frac{1}{2} v_i^2 \left( \frac{n}{n-1} v_i^{-2} S_{nn} \sum_{a=1}^{n-1} S_{aa} + \frac{n}{n-2} v_i^{-1} \partial_n v_i S_{nn}^2 \right) + \frac{n}{n-2} v_i \partial_n v_i S_{nn}^2
\]

\[- v_i \partial_n v_i S_{nn} S_{nn} + \frac{4(n-1)}{n-2} (-\partial_n v_i \psi S_{nn} + \psi \partial_n \psi) - \frac{2}{n-2} v_i \partial_n v_i S_{nn} S_{nn}.\]
Thus,

\[
\xi_n = \frac{n}{n-1} v^2 v^{n-2} S_{nn} \sum_{a=1}^{n-1} H_{aa} - \psi \partial_n v \psi S_{nn} + v \partial_n \psi S_{nn}
\]

\[
= \frac{1}{2} v^2 \left( \frac{n}{n-1} v^{n-2} S_{nn} \sum_{a=1}^{n-1} S_{aa} + \frac{n}{n-2} v^{-1} \partial_n v \psi^2 S_{nn} \right) + \frac{4(n-1)}{n-2} \psi \partial_n \psi.
\]

By \( \sum_{a=1}^{n-1} H_{aa} = \sum_{i=1}^{n} S_{ii} = 0 \) and (5), we get

\[
\xi_n = -\partial_n v \psi S_{nn} + v \partial_n \psi S_{nn} - \frac{1}{2} v \left( \frac{n}{n-1} v \psi S_{nn} + \frac{n}{n-2} v^{-1} \partial_n v \psi^2 S_{nn} \right)
\]

\[
+ \frac{4(n-1)}{n-2} \psi \partial_n \psi
\]

\[
= -\partial_n v \psi S_{nn} + v \partial_n \psi S_{nn} - \frac{n^2}{2(n-1)(n-2)} v \partial_n \psi S_{nn} + \frac{4(n-1)}{n-2} \psi \partial_n \psi.
\]

Finally, by \( \partial_n \psi = -\frac{1}{2(n-1)} \partial_n v \psi S_{nn} + \frac{n}{n-2} v^{-1} \partial_n v \psi \), we arrive at

\[
\xi_n = -\partial_n v \psi S_{nn} + v \left( \frac{1}{2(n-1)} \partial_n v \psi S_{nn} + \frac{n}{n-2} v^{-1} \partial_n v \psi S_{nn} \right)
\]

\[
- \frac{n^2}{2(n-1)(n-2)} v \partial_n v \psi^2 S_{nn} + \frac{4(n-1)}{n-2} \psi \left( \frac{1}{2(n-1)} \partial_n v \psi S_{nn} + \frac{n}{n-2} v^{-1} \partial_n v \psi S_{nn} \right)
\]

\[
= -\frac{n+2}{2(n-2)} v \partial_n v \psi^2 S_{nn} + \frac{4n(n-1)}{(n-2)^2} v^{-1} \partial_n v \psi^2.
\]

\[\square\]

3 Main estimates

In this section, we assume \( g \) is the metric in conformal Fermi coordinates as described in Section 2. Suppose \( V \) is a smooth vector field which satisfies (8) and (9). We adopt the notation in Section 2.
Proposition 7. There exist positive numbers $\theta, C$ and $\delta_0$ such that

$$\int_{B_\delta \cap \mathbb{R}^n_+} \left( \frac{4(n-1)}{n-2} |d(v_\epsilon + \psi)|_g^2 + R_g(v_\epsilon + \psi)^2 \right) dx$$

$$\leq 4(n-1) \int_{B_\delta \cap \partial \mathbb{R}^n_+} v_\epsilon^{2}(v_\epsilon^2 + 2v_\epsilon \psi + \frac{n}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} v_\epsilon^2 |S_n|^2) ds$$

$$+ \int_{\partial B_\delta \cap \mathbb{R}^n_+} \sum_{i=1}^n \left( \frac{4(n-1)}{n-2} v_\epsilon \partial_i v_\epsilon + v_\epsilon^2 \partial_k h_{ik} - \partial_k v_\epsilon^2 h_{ik} \right) \frac{x_i}{|x|} dx$$

$$- \sum_{i,k,l=1}^n |h_{ik,\alpha}|^2 \varepsilon^{n-2} \int_{B_\delta \cap \mathbb{R}^n_+} (\varepsilon^2 + |x|)^2 |v_\epsilon|^{2+n} dx$$

$$+ Ce^{n-2} \sum_{i,k,l=1}^n |h_{ik,\alpha}| \delta^{2-n} + Ce^{n-2} \delta^{2d+4-n}$$

for $0 < 2\varepsilon \leq \delta \leq \delta_0$, where $\theta = \theta(n), C = C(n, g)$ and $\delta_0 = \delta_0(n, g)$.

Proof. We write

$$\frac{4(n-1)}{n-2} |d(v_\epsilon + \psi)|_g^2 + R_g(v_\epsilon + \psi)^2 = \frac{4(n-1)}{n-2} |\partial v_\epsilon|^2 + J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 = \frac{8(n-1)}{n-2} \sum_{i=1}^n \partial_i v_\epsilon \partial_i \psi + \sum_{i,k=1}^n \left( \frac{-4(n-1)}{n-2} \partial_i v_\epsilon \partial_k h_{ik} + v_\epsilon^2 \partial_i \partial_k h_{ik} \right)$$

$$- \sum_{i,k,l=1}^n (v_\epsilon^2 \partial_k (H_{ik \partial_l H_d}) + \partial_k v_\epsilon^2 H_{ik \partial_l H_d}).$$

$$J_2 = \sum_{i,k,l=1}^n \left( -\frac{1}{4} v_\epsilon^2 \partial_l \partial_k H_{ik} \partial_l H_{ik} + \frac{1}{2} v_\epsilon^2 \partial_k H_{ik} \partial_l H_{ik} + \partial_k v_\epsilon^2 H_{ik} \partial_l H_{ik} + \frac{2(n-1)}{n-2} \partial_k v_\epsilon \partial_l v_\epsilon H_{ik} H_{kl} \right)$$

$$+ \sum_{i,k=1}^n \left( 2v_\epsilon \psi \partial_i \partial_k H_{ik} - \frac{8(n-1)}{n-2} \partial_i v_\epsilon \partial_k \psi H_{ik} \right) + \frac{4(n-1)}{n-2} |d\psi|^2,$$

$$J_3 = \frac{4(n-1)}{n-2} \sum_{i,k=1}^n (g^{ik} - \delta_{ik} + h_{ik} - \frac{1}{2} \sum_{l=1}^n H_{il} H_{kl}) \partial_i v_\epsilon \partial_k v_\epsilon$$

$$+(R_g - \sum_{i,k=1}^n \partial_i \partial_k h_{ik} + \sum_{i,k,l=1}^n \partial_k (H_{ik \partial_l H_d}) - \frac{1}{2} (div H)^2 + \frac{1}{4} |\partial H|^2) v_\epsilon^2,$$

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Moreover, by Proposition 4 there exists \( \theta > 0 \) such that

\[
J_1 = \frac{8(n-1)}{n-2} \sum_{i,k=1}^n (g^{ik} - \delta_{ik}) \partial_i \partial_k \psi + 2(R_g - \sum_{i,k=1}^n \partial_i \partial_k H_{ik}) \psi
\]

\[
+ R_g \psi^2 + \frac{4(n-1)}{n-2} \sum_{i,k=1}^n (g^{ik} - \delta_{ik}) \partial_i \psi \partial_k \psi.
\]

We compute

\[
J_1 = \frac{8(n-1)}{n-2} \sum_{i=1}^n \partial_i (\partial_i \psi) - \frac{8(n-1)}{n-2} \Delta \psi + \sum_{i,k=1}^n (\partial_i (v_i^2 \partial_k H_{ik}) - \partial_k (\partial_i v_i^2 h_{ik}))
\]

\[
+ 2 \sum_{i,k=1}^n (v_i \partial_i \partial_k v_i - \frac{n}{n-2} \partial_i v_i \partial_k v_i) h_{ik} - \sum_{i,k,l=1}^n \partial_k (v_i^2 H_{ik} \partial_l H_{il}).
\]

By (3) and (4),

\[
J_1 \leq \frac{8(n-1)}{n-2} \sum_{i=1}^n \partial_i (\partial_i \psi) + \sum_{i,k=1}^n (\partial_i (v_i^2 \partial_k h_{ik}) - \partial_k (\partial_i v_i^2 h_{ik})) - \sum_{i,k,l=1}^n \partial_k (v_i^2 H_{ik} \partial_l H_{il})
\]

\[
+ C \epsilon^{n-2}(\epsilon + |x|)^{2d+4-n}.
\]

Thus, integrating \( J_1 \) over \( B_\delta \cap \mathbb{R}_+^n \) and using (9),

\[
\int_{B_\delta \cap \mathbb{R}_+^n} J_1 \, dx \quad \leq \quad \int_{\partial B_\delta \cap \mathbb{R}_+^n} \sum_{i,k=1}^n (v_i \partial_i \partial_k h_{ik} - \partial_k v_i^2 h_{ik}) \frac{x_i}{|x|} \, d\sigma - \int_{B_\delta \cap \partial \mathbb{R}_+^n} \frac{8(n-1)}{n-2} \partial_n \psi \, d\sigma
\]

\[
+ \quad C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| \delta^{2|\alpha|+2-n} \epsilon^{n-2} + C \delta^{2d+4-n} \epsilon^{n-2}.
\]

For \( J_2 \), we first note that by Proposition 1 and (10), \( J_2 = -\frac{1}{2} |P|^2 + \sum_{i=1}^n \partial_i \xi_i \). And by (9)

\[
\int_{B_\delta \cap \mathbb{R}_+^n} \xi_i \frac{x_i}{|x|} \, d\sigma \leq C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| \delta^{2|\alpha|+2-n} \epsilon^{n-2}.
\]

Moreover, by Proposition 4 there exists \( \theta > 0 \) such that

\[
8\theta \sum_{i,j} \sum_{2 \leq |\alpha| \leq d} |h_{ij,\alpha}|^2 \epsilon^{n-2} \int_{B_\delta \cap \mathbb{R}_+^n} (\epsilon + |x|)^{2|\alpha|+2-2n} \, dx \leq \int_{B_\delta \cap \mathbb{R}_+^n} |P|^2 \, dx.
\]
Hence, using Proposition [6]

\[
\int_{B_\delta \cap \mathbb{R}^n_+} J_3 dx = -\int_{B_\delta \cap \mathbb{R}^n_+} \frac{1}{4} |P|^2 dx + \int_{\partial B_\delta \cap \mathbb{R}^n_+} \frac{\xi_i}{|x|} dx - \int_{B_\delta \cap \partial \mathbb{R}^n_+} \xi_n d\sigma
\]

\[
\leq \int_{B_\delta \cap \partial \mathbb{R}^n_+} \left( \frac{n+2}{2(n-2)} v_\epsilon \partial_n v_\epsilon |S_{nn}|^2 - \frac{4n(n-1)}{(n-2)^2} v_\epsilon^{-1} \partial_n v_\epsilon \psi^2 \right) d\sigma
\]

\[
- 2\theta \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}|^2 \epsilon^{n-2} \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|+2-2n} dx
\]

\[
+ C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}|^2 \epsilon^{2|\alpha|+2-2n} \epsilon^{n-2}.
\]

For \(J_3\) and \(J_4\), by (9), Proposition 3 and Cauchy inequality,

\[
J_3 + J_4 \leq C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} \left( |h_{i,k,\alpha}|^2 (\epsilon + |x|)^{2|\alpha|+4-2n} + |h_{i,k,\alpha}| (\epsilon + |x|)^{|\alpha|+d+3-2n} \right) \epsilon^{n-2}
\]

\[
+ C (\epsilon + |x|)^{2d+4-2n} \epsilon^{n-2}
\]

\[
\leq \theta \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}|^2 \epsilon^{n-2} (\epsilon + |x|)^{2|\alpha|+2-2n} + C (\epsilon + |x|)^{2d+4-2n} \epsilon^{n-2}.
\]

Thus,

\[
\int_{B_\delta \cap \mathbb{R}^n_+} (J_3 + J_4) dx \leq \theta \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}|^2 \epsilon^{n-2} \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|+2-2n} dx + C \epsilon^{2d+4-n} \epsilon^{n-2}.
\]

Finally, by (3) we compute

\[
\int_{B_\delta \cap \mathbb{R}^n_+} \frac{4(n-1)}{n-2} |dv_\epsilon|^2 dx = \frac{4(n-1)}{n-2} \left( \int_{B_\delta \cap \partial \mathbb{R}^n_+} -v_\epsilon \partial_n v_\epsilon d\sigma + \int_{\partial B_\delta \cap \mathbb{R}^n_+} \sum_{i=1}^n v_\epsilon \partial_i v_\epsilon \frac{x_i}{|x|} d\sigma \right).
\]
Combining the above, we obtain
\[
\int_{B_\delta \cap \mathbb{R}^n_+} \left( \frac{4(n-1)}{n-2} |d(v_\epsilon + \psi)|^2 + R_g(v_\epsilon + \psi)^2 \right) dx \\
\leq - \frac{4(n-1)}{n-2} \int_{B_\delta \cap \partial \mathbb{R}^n_+} (v_\epsilon \partial_n v_\epsilon + 2 \partial_n v_\epsilon \psi + \frac{n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon \psi^2) d\sigma \\
+ \frac{n + 2}{2(n-1)} \int_{B_\delta \cap \partial \mathbb{R}^n_+} v_\epsilon \partial_n v_\epsilon |S_{nn}|^2 d\sigma \\
+ \int_{\partial B_\delta \cap \mathbb{R}^n_+} \sum_{i=1}^n (\frac{4(n-1)}{n-2} v_\epsilon \partial_i v_\epsilon + v_\epsilon^2 \partial_k h_{ik} - \partial_k v_\epsilon^2 h_{ik}) x_i d\sigma \\
- \theta \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 \epsilon^{n-2} \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|+2} d\sigma \\
+ C \epsilon^{n-2} \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 n + C \epsilon^{n-2} \delta^{2d+4-n}.
\]

Finally, by (5) and \( v_\epsilon \partial_n v_\epsilon |S_{nn}|^2 \leq 0 \) for \( x \in \partial \mathbb{R}^n_+ \),
\[
- \frac{4(n-1)}{n-2} \int_{B_\delta \cap \partial \mathbb{R}^n_+} (v_\epsilon \partial_n v_\epsilon + 2 \partial_n v_\epsilon \psi + \frac{n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon \psi^2) d\sigma \\
+ \frac{n + 2}{2(n-1)} \int_{B_\delta \cap \partial \mathbb{R}^n_+} v_\epsilon \partial_n v_\epsilon |S_{nn}|^2 d\sigma \\
\leq - \frac{4(n-1)}{n-2} \int_{B_\delta \cap \partial \mathbb{R}^n_+} (v_\epsilon \partial_n v_\epsilon + 2 \partial_n v_\epsilon \psi + \frac{n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon \psi^2) d\sigma \\
+ \frac{1}{2(n-1)} \int_{B_\delta \cap \partial \mathbb{R}^n_+} v_\epsilon \partial_n v_\epsilon |S_{nn}|^2 d\sigma \\
= 4(n-1) \int_{B_\delta \cap \partial \mathbb{R}^n_+} v_\epsilon^2 (v_\epsilon^2 + 2 v_\epsilon \psi + \frac{n}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} v_\epsilon^2 |S_{nn}|^2) d\sigma.
\]

This completes the proof.

\[ \square \]

**Proposition 8.**
\[
4(n-1) \int_{B_\delta \cap \partial \mathbb{R}^n_+} v_\epsilon^{n-2} (v_\epsilon^2 + 2 v_\epsilon \psi + \frac{n}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} v_\epsilon^2 |S_{nn}|^2) d\sigma \\
\leq O(B, \partial B) \left( \int_{B_\delta \cap \partial \mathbb{R}^n_+} (v_\epsilon + \psi)^{2(n-1)} d\sigma \right)^{\frac{n-2}{n-1}} + C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 n + C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 \epsilon^{n-1} \delta^2 \int_{B_\delta \cap \partial \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|-2n+2} d\sigma
\]

for \( 0 < 2 \epsilon \leq \delta \leq \delta_0 \) and \( \delta_0 \) sufficiently small.
Thus, putting above together and using Holder inequality, we get
\[ \frac{4(n-1)}{n-2} \int_{\mathbb{R}_+^n} |\nabla v_\epsilon|^2 \, dx \]
and
\[ \int_{\mathbb{R}_+^n} |\nabla v_\epsilon|^2 \, dx = (n-2) \int_{\partial \mathbb{R}_+^n} v_\epsilon^{\frac{2(n-1)}{n-2}} \, d\sigma. \]

Then it follows that
\[ 4(n-1) \left( \int_{\partial \mathbb{R}_+^n} v_\epsilon^{\frac{2(n-1)}{n-2}} \, d\sigma \right)^{\frac{1}{n-1}} = \mathcal{Q}(B, \partial B). \]

Besides, since \( V_n = 0 \) on \( \partial \mathbb{R}_+^n \), we have
\[
\psi = \frac{n-2}{2(n-1)} \epsilon^{-\frac{n-2}{n-1}} \sum_{a=1}^{n-1} \partial_a (v_\epsilon^{\frac{2(n-1)}{n-2}} V_a) + \frac{n-2}{4(n-1)} v_\epsilon (2 \partial_a V_n - \frac{2}{n} \text{div} V)
\]
\[ = \frac{n-2}{2(n-1)} \epsilon^{-\frac{n-2}{n-1}} \sum_{a=1}^{n-1} \partial_a (v_\epsilon^{\frac{2(n-1)}{n-2}} V_a) + \frac{n-2}{4(n-1)} v_\epsilon S_{nn} \]
for \( x \in \partial \mathbb{R}_+^n \). Moreover, by (9)
\[
\int_{\partial B_\beta \cap \partial \mathbb{R}_+^n} \frac{2(n-1)}{n-2} v_\epsilon \sum_{a=1}^{n-1} V_a \frac{x_a}{|x|} \, d\mu \leq C \sum_{i,k=1}^{n} \sum_{1 \leq |\alpha| \leq d} |h_{ik,\alpha}| |\delta^{\alpha} - \epsilon^{\alpha}| \epsilon^{n-1}.
\]
Thus,
\[
\int_{B_\beta \cap \partial \mathbb{R}_+^n} \frac{2(n-1)}{n-2} v_\epsilon \psi \, d\sigma - \int_{B_\beta \cap \partial \mathbb{R}_+^n} \frac{n-2}{2(n-1)} v_\epsilon^{\frac{2(n-1)}{n-2}} S_{nn} \, d\sigma \leq C \sum_{i,k=1}^{n} \sum_{1 \leq |\alpha| \leq d} |h_{ik,\alpha}| |\delta^{\alpha} - \epsilon^{\alpha}| \epsilon^{n-1}.
\]
Putting above together and using Holder inequality, we get
\[
4(n-1) \int_{B_\beta \cap \partial \mathbb{R}_+^n} v_\epsilon^{\frac{2(n-1)}{n-2}} (v_\epsilon^2 + 2v_\epsilon \psi + \frac{n-2}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} v_\epsilon^2 \omega_{nn}) \, d\sigma
\]
\[
\leq 4(n-1) \int_{B_\beta \cap \partial \mathbb{R}_+^n} v_\epsilon^{\frac{2(n-1)}{n-2}} (v_\epsilon^2 + \frac{n-2}{2(n-1)} v_\epsilon^2 S_{nn} + \frac{n}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} v_\epsilon^2 S_{nn}^2) \, d\sigma
\]
\[ + C \sum_{i,k=1}^{n} \sum_{1 \leq |\alpha| \leq d} |h_{ik,\alpha}| |\delta^{\alpha} - \epsilon^{\alpha}| \epsilon^{n-1}
\]
\[
\leq \mathcal{Q}(B, \partial B) \left( \int_{B_\beta \cap \partial \mathbb{R}_+^n} (v_\epsilon^2 + \frac{n-2}{2(n-1)} v_\epsilon^2 S_{nn} + \frac{n}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} v_\epsilon^2 S_{nn}^2) \, d\sigma \right)^{\frac{n-2}{n-1}}
\]
\[ + C \sum_{i,k=1}^{n} \sum_{1 \leq |\alpha| \leq d} |h_{ik,\alpha}| |\delta^{\alpha} - \epsilon^{\alpha}| \epsilon^{n-1}.
\]
We next notice that by Taylor expansion, there exists a constant $C_0 = C_0(n)$ such that
\[
(1 + \frac{n-2}{2(n-1)} y + \frac{n}{n-2} z^2 - \frac{n-2}{8(n-1)^2} y^2)^{\frac{n-1}{n-2}} - (1 + y)^{\frac{2(n-1)}{n-2}} + \frac{2(n-1)}{n-2} z - \frac{1}{2} y
\leq C_0(|y|^3 + |z|^3)
\]
for $|y|, |z| \leq \frac{1}{2}$. By (3), $|S_{nn}| \leq \frac{1}{2}$ and $|\psi| \leq \frac{1}{2} \epsilon_\epsilon$ for $|x| \leq \delta$. Hence,
\[
(v^2 + \frac{n-2}{2(n-1)} v^2 S_{nn} + \frac{n}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} v^2 S^2_{nn})^{\frac{n-1}{n-2}} - (v + \psi)^{\frac{2(n-1)}{n-2}}
\]

\[
+ \frac{2(n-1)}{n-2} \frac{n}{v^2} \psi - \frac{1}{2} \frac{n}{v^2} S_{nn}
\]
\[
\leq C_0 v^{-n-2} (|v|^3 + |S_{nn}|^3) \leq C v^{-n-2} (|v|^2 + |S_{nn}|^2) \delta^2
\]
\[
\leq C \sum_{i,k=1}^{n} \sum_{2 |\alpha| \leq d} |h_{ik,\alpha}|^2 (\epsilon + |x|)^{2|\alpha|-2n+2} \epsilon^{n-1} \delta^2.
\]

Thus,
\[
\int_{B_{\delta} \cap \partial \mathbb{R}_+^n} (v^2 + \frac{n-2}{2(n-1)} v^2 S_{nn} + \frac{n}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} v^2 S^2_{nn})^{\frac{n-1}{n-2}} d\sigma
\]
\[
\leq \int_{B_{\delta} \cap \partial \mathbb{R}_+^n} (v + \psi)^{\frac{2(n-1)}{n-2}} d\sigma + \int_{B_{\delta} \cap \partial \mathbb{R}_+^n} \frac{2(n-1)}{n-2} v^{-n-2} \psi d\sigma - \int_{B_{\delta} \cap \partial \mathbb{R}_+^n} \frac{1}{2} v^{-n-2} S_{nn} d\sigma
\]

\[
+ C \sum_{i,k=1}^{n} \sum_{2 |\alpha| \leq d} |h_{ik,\alpha}|^2 \epsilon^{n-1} \delta^2 \int_{B_{\delta} \cap \partial \mathbb{R}_+^n} (\epsilon + |x|)^{2|\alpha|-2n+2} d\sigma
\]
\[
\leq \int_{B_{\delta} \cap \partial \mathbb{R}_+^n} (v + \psi)^{\frac{2(n-1)}{n-2}} d\sigma + C \sum_{i,k=1}^{n} \sum_{2 |\alpha| \leq d} |h_{ik,\alpha}|^2 \epsilon^{n-1} \delta^2 \int_{B_{\delta} \cap \partial \mathbb{R}_+^n} (\epsilon + |x|)^{2|\alpha|-2n+2} d\sigma
\]

\[
+ C \sum_{i,k=1}^{n} \sum_{2 |\alpha| \leq d} |h_{ik,\alpha}| \delta^{|\alpha|-n+1} \epsilon^{n-1}.
\]

This completes the proof. \qed

4 Proof of the main theorems

In this section, we construct a test function $\phi_{(\epsilon, \delta)}$ with energy functional less than $Q(B, \partial B)$ and prove Theorem 2 and 3. Since the case that $Q(M, \partial M, g) \leq 0$ is trivial, it suffices to consider $Q(M, \partial M, g) > 0$.

After a conformal change of the metric, we may assume $\partial M$ is totally geodesic. Let $p \in \partial M$ and let $(x_1, \cdots, x_n)$ be the conformal Fermi coordinates around $p$ described in
Section 2. We denote by $G$ the Green’s function of the conformal Laplacian with pole at $p$ which satisfies the Neumann boundary condition. We assume that $G$ is normalized such that $\lim_{|x|\to 0} |x|^{n-2}G(x) = 1$. Then $G$ satisfies \[ |G(x) - |x|^{2-n}| \leq C \sum_{i,k=1}^{n} \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| |x|^{|\alpha|+2-n} + C|x|^{d+3-n}. \tag{12} \]

Moreover, we define as in [6] a flux integral
\[ \mathcal{I}(p, \delta) = \frac{4(n-1)}{n-2} \int_{\partial B_{\delta} \cap \mathbb{R}^{n}_{+}} \sum_{i=1}^{n} (|x|^{2-n} \partial_{i} G - G \partial_{i} |x|^{2-n}) \frac{x_i}{|x|} d\sigma \]
\[ - \int_{\partial B_{\delta} \cap \mathbb{R}^{n}_{+}} \sum_{i,k=1}^{n} |x|^{2-n} (|x|^{2} \partial_{k} h_{ik} - 2n x_{k} h_{ik}) \frac{x_i}{|x|} d\sigma \]

for $\delta > 0$ sufficiently small.
We define
\[ \phi(\epsilon, \delta) = \eta_{\delta}(v_{\epsilon} + \psi) + (1 - \eta_{\delta}) \epsilon^{\frac{n-2}{2}} G, \]
where $\psi$ is the function constructed in Section 2. We recall that
\[ \epsilon^{\frac{n-2}{2}} (\epsilon + |x|)^{-n+2} \leq v_{\epsilon}(x) \leq C(n) \epsilon^{\frac{n-2}{2}} (\epsilon + |x|)^{-n+2} \quad \text{for} \quad x \in \mathbb{R}^{n}_{+}; \]
\[ |\partial v_{\epsilon}|(x) \leq C(n) \epsilon^{\frac{n-2}{2}} (\epsilon + |x|)^{-n+1} \quad \text{for} \quad x \in \mathbb{R}^{n}_{+}; \]
and
\[ |v_{\epsilon} - \epsilon^{\frac{n-2}{2}} |x|^{-n+2}| \leq C(n) \epsilon^{\frac{n}{2}} |x|^{-n+1} \quad \text{for} \quad x \in \mathbb{R}^{n}_{+}, \quad \text{and} \quad |x| \geq 2\epsilon. \tag{13} \]

**Proposition 9.**
\[ \int_{M} \left( \frac{4(n-1)}{n-2} |d\phi(\epsilon, \delta)|^{2} + R_{g} \phi_{(\epsilon, \delta)}^{2} \right) dV_{g} \]
\[ \leq \mathcal{Q}(B, \partial B)(\int_{\partial M} \phi_{(\epsilon, \delta)}^{2} d\sigma_{g})^{\frac{n-2}{n-1}} - \frac{\theta}{2} \sum_{i,k=1}^{n} \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^{2} \epsilon^{n-2} \int_{B_{\delta} \cap \mathbb{R}^{n}_{+}} (\epsilon + |x|)^{2|\alpha|+2-2n} d\sigma \]
\[ - \epsilon^{n-2} \mathcal{I}(p, \delta) + C \epsilon^{n-2} \sum_{i,k=1}^{n} \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^{\delta^{-n+2+|\alpha|}} + C \epsilon^{n-2} \delta^{2d+4-n} + C \delta^{-n+1} \epsilon^{-n+1} \]
for $0 < 2\epsilon \leq \delta \leq \delta_{0}$ and $\delta_{0}$ sufficiently small.

**Proof.** Let $\Omega_{\delta}$ be the coordinates ball of radius $\delta$ in Fermi coordinates. In other words, $(x_{1}, \ldots, x_{n})$ satisfies $x_{1}^{2} + \cdots + x_{n}^{2} < \delta^{2}$ and $x_{n} \geq 0$. By divergence theorem
\[ \int_{M \setminus \Omega_{\delta}} \left( \frac{4(n-1)}{n-2} |\nabla_{g} \phi(\epsilon, \delta)|^{2} + R_{g} \phi_{(\epsilon, \delta)}^{2} \right) dV_{g} \]
\[ = \int_{M \setminus \Omega_{\delta}} \left( \frac{4(n-1)}{n-2} \Delta_{g} \phi(\epsilon, \delta) - R_{g} \phi(\epsilon, \delta) \right) (\phi(\epsilon, \delta) - \epsilon^{\frac{n-2}{2}} G) dV_{g} \]
\[ + \frac{4(n-1)}{n-2} \int_{\partial (M \setminus \Omega_{\delta})} (\nabla_{\nu_{g}} \phi(\epsilon, \delta) \phi(\epsilon, \delta) + \epsilon^{\frac{n-2}{2}} (\phi(\epsilon, \delta) \nabla_{\nu_{g}} G - G \nabla_{\nu_{g}} \phi(\epsilon, \delta))) d\sigma_{g}, \]

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where $\nu_g$ is the unit outer normal on $\partial(M \setminus \Omega_\delta)$ with respect to $g$. Notice that

$$\partial(M \setminus \Omega_\delta) = (\partial M \setminus \Omega_\delta) \cup (\partial \Omega_\delta \setminus \partial M).$$

We will compute the above integral in several steps.

We first notice that for $x \in M \setminus \Omega_\delta$, we have $\phi_{(\epsilon, \delta)} - \epsilon^{\frac{n-2}{2}} G = \eta_\delta(v_\epsilon + \psi - \epsilon^{\frac{n-2}{2}} G)$. In particular, $\phi_{(\epsilon, \delta)} - \epsilon^{\frac{n-2}{2}} G = 0$ in $M \setminus \Omega_{2\delta}$. By (12) and (9),

$$\sup_{M \setminus \Omega_\delta} (|\phi_{(\epsilon, \delta)} - \epsilon^{\frac{n-2}{2}} G| + \epsilon^2 |\frac{4(n-1)}{n-2} \Delta_g \phi_{(\epsilon, \delta)} - R_g \phi_{(\epsilon, \delta)}|)
\leq C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}| \delta^{2|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C \delta^{d+3-n} \epsilon^{\frac{n-2}{2}} + C \delta^{-n+1} \epsilon^{\frac{n}{2}}.
$$

Thus,

$$-\int_{M \setminus \Omega_\delta} (\frac{4(n-1)}{n-2} \Delta_g \phi_{(\epsilon, \delta)} - R_g \phi_{(\epsilon, \delta)})(\phi_{(\epsilon, \delta)} - \epsilon^{\frac{n-2}{2}} G) dV_g
\leq C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}| \delta^{2|\alpha|+2-n} \epsilon^{n-2} + C \delta^{2d+4-n} \epsilon^{n-2} + C \delta^{-n} \epsilon^n.
$$

We now compute the boundary terms on $\partial M \setminus \Omega_\delta$. Since $\nabla_{\nu_g} G = 0$ on $\partial M$, by (5), Proposition 5 and (9),

$$\sup_{\partial M \cap (\Omega_{2\delta} \setminus \Omega_\delta)} |\nabla_{\nu_g} \phi_{(\epsilon, \delta)}| \leq \sup_{\partial M \cap (\Omega_{2\delta} \setminus \Omega_\delta)} |\partial_n v_\epsilon + \partial_n \psi| \leq C \epsilon^{\frac{n}{2}} \delta^{-n} + C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}| \delta^{2|\alpha|-2n-1} \epsilon^{\frac{n}{2}}.
$$

Hence,

$$\int_{\partial M \setminus \Omega_\delta} (\nabla_{\nu_g} \phi_{(\epsilon, \delta)} \phi_{(\epsilon, \delta)} + \epsilon^{\frac{n-2}{2}} (\phi_{(\epsilon, \delta)} \nabla_{\nu_g} G - G \nabla_{\nu_g} \phi_{(\epsilon, \delta)})) d\sigma_g
= \int_{\partial M \setminus \Omega_\delta} \nabla_{\nu_g} \phi_{(\epsilon, \delta)}(\phi_{(\epsilon, \delta)} - \epsilon^{\frac{n-2}{2}} G) d\sigma_g
\leq C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}| \delta^{2|\alpha|+1-n} \epsilon^{n-1} + C \delta^{2d+4-n} \epsilon^{n-2} + C \delta^{-n} \epsilon^n.
$$

We next compute the boundary terms on $\partial \Omega_\delta \setminus \partial M$.

$$\int_{\partial \Omega_\delta \setminus \partial M} \nabla_{\nu_g} \phi_{(\epsilon, \delta)} \phi_{(\epsilon, \delta)} d\sigma_g \leq \int_{\partial B_{\delta} \cap \mathbb{R}^n_+} \sum_{i=1}^n (-\partial_i v_\epsilon v_\epsilon + \sum_{k=1}^n v_\epsilon \partial_k v_\epsilon h_{i,k}) \frac{x_i}{|x|} d\sigma
+ C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}| \delta^{2|\alpha|+2-n} \epsilon^{n-2} + C \delta^{2d+4-n} \epsilon^{n-2}.
$$
Also,

\[
\int_{\partial \Omega \setminus \partial M} (\phi(\epsilon, \delta) \nabla v_\gamma G - GV_\gamma \phi(\epsilon, \delta)) d\sigma_g \leq - \int_{\partial B_\delta \cap \mathbb{R}^n_+} \sum_{i=1}^n (v_\epsilon \partial_i G - G \partial_i v_\epsilon) \frac{x_i}{|x|} d\sigma_g
\]

\[+ C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| \epsilon |\alpha|^{2-n} \epsilon^{n-2} + C \delta^{2d+4-n} \epsilon^{n-2}.
\]

Combining the above, we obtain

\[
\int_{M \setminus \Omega} \left( \frac{4(n-1)}{n-2} |d\phi(\epsilon, \delta)|^2_g + R_g \phi(\epsilon, \delta)^2 \right) dV_g \\
\leq - \frac{4(n-1)}{n-2} \int_{\partial B_\delta \cap \mathbb{R}^n_+} \sum_{i=1}^n (\partial_i v_\epsilon v_\epsilon - \epsilon \partial_i v_\epsilon + \epsilon \frac{n-2}{2} (v_\epsilon \partial_i G - G \partial_i v_\epsilon)) \frac{x_i}{|x|} d\sigma_g
\]

\[+ C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| \epsilon |\alpha|^{2-n} \epsilon^{n-2} + C \delta^{2d+4-n} \epsilon^{n-2}.
\]

On the other hand, by Proposition 7 and 8

\[
\int_{\Omega} \left( \frac{4(n-1)}{n-2} |d\phi(\epsilon, \delta)|^2_g + R_g \phi(\epsilon, \delta)^2 \right) dV_g \\
\leq Q(B, \partial B_\delta) \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| \epsilon |\alpha|^{2-n} \epsilon^{n-2} \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|) |\alpha|^{2|\alpha|+2-2n} d\sigma_g
\]

\[+ \frac{n \theta}{2} \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| \epsilon |\alpha|^{2-n} \epsilon^{n-2} \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|)^2 |\alpha|^{2|\alpha|+2-2n} d\sigma_g
\]

\[+ C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| \epsilon |\alpha|^{2-n} \epsilon^{n-2} + C \delta^{2d+4-n} \epsilon^{n-2}.
\]
Adding the above two inequalities, we get
\[
\int_M \left( \frac{4(n-1)}{n-2} |d\phi_{(\epsilon, \delta)}|^2 + R_g(\phi_{(\epsilon, \delta)})^2 \right) dV_g \\
\leq \mathcal{Q}(B, \partial B) \left( \int_{\partial M} \frac{2(n-1)}{n-2} \phi_{(\epsilon, \delta)} \right) d\sigma_g + \int_{\partial B \cap \mathbb{R}^n_+} \sum_{i=1}^n (v_i^2 \partial_k h_{ik} + \frac{n}{n-2} \partial_k v_i^2 x_i) \frac{x_i}{|x|} d\sigma \\
- \frac{4(n-1)}{n-2} \int_{\partial B \cap \mathbb{R}^n_+} \sum_{i=1}^n \frac{n-2}{n} \left( v_i \partial_i G - G \partial_i v_i \right) x_i \frac{x_i}{|x|} d\sigma \\
- \theta \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 \epsilon^{n-2} \int_{B_\epsilon \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \\
+ C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 \epsilon^{n-2} \int_{B_\epsilon \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|-2n+2} d\sigma \\
+ C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 \epsilon^{-n+2+|\alpha|} \epsilon^{n-2} + C \delta^{2d+4-n} \epsilon^{n-2} + C \delta^{-n} \epsilon^n.
\]

Since
\[
\epsilon^{n-1} \delta^2 \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|-2n+2} d\sigma = \epsilon^{2|\alpha|} \delta^2 \int_0^\delta (1 + t)^{2|\alpha|-2n+2} t^{n-2} dt
\]
and
\[
\epsilon^{n-2} \int_{B_\epsilon \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|+2-2n} dx = \epsilon^{2|\alpha|} \int_0^\delta (1 + t)^{2|\alpha|-2n+2} t^{n-1} dt,
\]
then for \( \delta \) sufficiently small and \( 2\epsilon \leq \delta \), we have
\[
C \epsilon^{n-1} \delta^2 \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|-2n+2} d\sigma < \frac{\theta}{2} \epsilon^{n-2} \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|+2-2n} dx.
\]
Moreover, by (12) and (13)
\[
\int_{\partial B \cap \mathbb{R}^n_+} \left( \sum_{i=1}^n (v_i^2 \partial_k h_{ik} + \frac{n}{n-2} \partial_k v_i^2 x_i) - \frac{4(n-1)}{n-2} \sum_{i=1}^n \frac{n-2}{n} (v_i \partial_i G - G \partial_i v_i) \right) x_i \frac{x_i}{|x|} d\sigma \\
\leq -\epsilon^{n-2} \mathcal{I}(p, \delta) + C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 \epsilon^{-n+2} + C \epsilon^{n-1} \delta^{-n+1}.
\]
From these the assertion follows.

We are ready to prove Theorem 2.
Proof of Theorem 2. Since \( p \notin \mathcal{Z} \), we have \( \sum_{i,k=1}^{n} \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}| > 0 \). Thus, by Proposition 9

\[
\int_M \left( \frac{4(n-1)}{n-2} |d\phi(\epsilon,\delta)|^2_g + R_g \phi^2(\epsilon,\delta) \right) dV_g < Q(B, \partial B)(\int_{\partial M} \phi^2(\epsilon,\delta) d\sigma_g)^{\frac{n-4}{n-2}}
\]

for \( \epsilon > 0 \) sufficiently small. This completes the proof. \( \square \)

Now we consider the case that \( p \in \mathcal{Z} \). We recall a result about \( \mathcal{I}(p, \delta) \).

**Proposition 10.** Let \( p \in \partial M \). Suppose \( p \in \mathcal{Z} \).

(i) The limit \( \lim_{\delta \to 0} \mathcal{I}(p, \delta) \) exists.

(ii) The doubling of \( (M \setminus \{p\}, G^{\frac{4}{n-2}}) \) has a well-defined mass which equals \( \lim_{\delta \to 0} \mathcal{I}(p, \delta) \) up to a positive factor.

Proof of Theorem 2. Since \( p \in \mathcal{Z} \), we have \( \sum_{i,k=1}^{n} \sum_{2 \leq |\alpha| \leq d} |h_{i,k,\alpha}| = 0 \). By Proposition 9

\[
\int_M \left( \frac{4(n-1)}{n-2} |d\phi(\epsilon,\delta)|^2_g + R_g \phi^2(\epsilon,\delta) \right) dV_g < Q(B, \partial B)(\int_{\partial M} \phi^2(\epsilon,\delta) d\sigma_g)^{\frac{n-4}{n-2}}
\]

for \( 0 < 2\epsilon \leq \delta \). By assumption \( \lim_{\delta \to 0} \mathcal{I}(p, \delta) > 0 \), we may choose \( \delta \) sufficiently small such that \( \mathcal{I}(p, \delta) - C\delta^{2d+4-n} \epsilon^{n-2} + C\delta^{-n+1} \epsilon^{n-1} \). We next choose \( 0 < \epsilon < \frac{\delta}{2} \) sufficiently small such that \( \mathcal{I}(p, \delta) - C\delta^{2d+4-n} \epsilon^{n-2} + C\delta^{-n+1} \epsilon^{n-1} < 0 \). Then

\[
\int_M \left( \frac{4(n-1)}{n-2} |d\phi(\epsilon,\delta)|^2_g + R_g \phi^2(\epsilon,\delta) \right) dV_g < Q(B, \partial B)(\int_{\partial M} \phi^2(\epsilon,\delta) d\sigma_g)^{\frac{n-4}{n-2}}.
\]

\( \square \)

**Appendix: An elliptic system in \( \mathbb{R}^n_+ \)**

In the appendix, we solve a boundary value problem for an elliptic system in \( \mathbb{R}^n_+ \).

Let \( \mathcal{B}_2^+ \) be the ball of radius \( \frac{1}{2} \) equipped with the flat metric \( g \). We denote by \( \mathcal{X} \) the space of vector fields \( V \in H^1(\mathcal{B}_2^+) \) such that \( \langle V, \nu \rangle = 0 \) on \( \partial \mathcal{B}_2^+ \), where \( \nu \) is the unit outer normal on \( \partial \mathcal{B}_2^+ \). We also denote by \( \mathcal{Y} \) the space of trace-free symmetric two-tensors on \( \mathcal{B}_2^+ \) of class \( L^2 \). Let \( \mathcal{D} : \mathcal{X} \to \mathcal{Y} \) be the conformal killing operator, which satisfies

\[
(\mathcal{D}V)_{ik} = V_{i,k} + V_{k,i} - \frac{2}{n} \text{div} V g_{ik}.
\]

By stereographic projection, \( \mathcal{B}_2^+ \) is conformal to the hemisphere \( \mathbb{S}^n_+ \) with standard metric \( g_c \). The metric \( g_c \) satisfies \( g_c = u^{-1} g \), where \( u = \left( \frac{2}{4+|x|^2} \right)^{\frac{n-2}{2}} \) for \( |x| \leq \frac{1}{2} \). We may define similarly \( \mathcal{X}^* \) the space of vector fields \( V \in H^1(\mathbb{S}^n_+) \) such that \( \langle V, \nu \rangle = 0 \) on \( \partial \mathbb{S}^n_+ \), where \( \nu \) is the unit outer normal on \( \partial \mathbb{S}^n_+ \), \( \mathcal{Y}^* \) the space of trace-free symmetric two-tensors on \( \mathbb{S}^n_+ \) of class \( L^2 \) and \( \mathcal{D}^* : \mathcal{X}^* \to \mathcal{Y}^* \) the conformal killing operator on the hemisphere. Then it follows that \( V \in H^1(\mathbb{S}^n_+) \) if and only if \( V \in H^1(\mathcal{B}_2^+) \), and \( \mathcal{D}^* V = 0 \) if and only if \( \mathcal{D} V = 0 \).

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Lemma 1. \( \ker \mathcal{D} \) is finite dimensional.

Proof. In [6], it was shown (after Lemma 21) that \( \ker \mathcal{D}^* \) is finite dimensional. Then the assertion follows easily. \( \square \)

We now define \( \mathcal{X}_0 = \{ V \in \mathcal{X} : \langle V, U \rangle_{L^2(B_{1/2}^n)} = 0 \text{ for all } U \in \ker \mathcal{D} \} \).

Lemma 2. For all \( V \in \mathcal{X}_0 \), it holds \( \| V \|^2_{H^1(B_{1/2}^n)} \leq C \| \mathcal{D}V \|^2_{L^2(B_{1/2}^n)} \), where \( C = C(n) \).

Proof. Suppose the inequality does not hold, then there exist a sequence of vector fields \( V^{(j)} \in \mathcal{X}_0 \) such that \( \| V^{(j)} \|_{H^1(B_{1/2}^n)} = 1 \) for all \( j \) and \( \| \mathcal{D}V^{(j)} \|_{L^2(B_{1/2}^n)} \to 0 \) as \( j \to \infty \). By passing to a subsequence, \( V^{(j)} \to V^{(0)} \) weakly in \( H^1(B_{1/2}^n) \) for some \( V^{(0)} \in \mathcal{X}_0 \). It follows that \( \mathcal{D}V^{(0)} = 0 \), and as a result \( V^{(0)} = 0 \). Notice that \( V^{(j)} \to V^{(0)} \) strongly in \( L^2(B_{1/2}^n) \). Thus, \( \| V^{(j)} \|_{L^2(B_{1/2}^n)} \to 0 \). Therefore, \( \| V^{(j)} \|_{L^2(\mathbb{R}_+^n)} \to 0 \). By [6] Lemma 21, \( \| V^{(j)} \|_{H^1(\mathbb{R}_+^n)} \to 0 \) as \( j \to \infty \). Hence, \( \| V^{(j)} \|_{H^1(B_{1/2}^n)} \to 0 \) as \( j \to \infty \). This gives a contradiction. \( \square \)

Proposition 11. Let \( h \) be a two-tensor in \( \mathcal{Y} \). Then there exists a unique vector field \( V \in \mathcal{X}_0 \) such that \( \langle h - \mathcal{D}V, \mathcal{D}U \rangle_{L^2(B_{1/2}^n)} = 0 \) for all \( U \in \mathcal{X} \).

Moreover, \( \| V \|^2_{H^1(B_{1/2}^n)} \leq C \| h \|^2_{L^2(B_{1/2}^n)} \), where \( C = C(n) \).

Proof. It follows by the same argument in [6] Proposition 23, and Lemma 2 above that the minimizer of \( \| h - \mathcal{D}V \|^2_{L^2(B_{1/2}^n)} \) exists in \( \mathcal{X}_0 \), which satisfies the required properties. \( \square \)

We now consider another conformal map. The ball \( B_{1/2}^n \) is conformal to \( \mathbb{R}_+^n \cup \{ \infty \} \).

The metric \( g \) satisfies \( g = v^{n+2} \delta \), where

\[
v = \left( \frac{1}{(1 + x_n)^2 + \sum_{1 \leq a \leq n-1} x_a^2} \right)^{\frac{n-2}{2}}.
\]

Proposition 12. Let \( h \) be a smooth trace-free symmetric two-tensor on \( \mathbb{R}_+^n \) with compact support. Then there exists a smooth vector field \( V \) on \( \mathbb{R}_+^n \) such that

\[
\begin{aligned}
&\sum_{k=1}^{n} \partial_k \left[ v^{\frac{2n}{n-2}} (h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} \text{div} V \delta_{ik}) \right] = 0 \quad \text{in } \mathbb{R}_+^n \\
&\partial_n V_a - h_{an} = 0 \quad \text{on } \partial \mathbb{R}_+^n \\
&V_n = 0 \quad \text{on } \partial \mathbb{R}_+^n
\end{aligned}
\]

for \( i = 1, \cdots, n \) and \( a = 1, \cdots, n-1 \). Moreover,

\[
\int_{\mathbb{R}_+^n} v^{\frac{2(n+2)}{n-2}} |V|^2 \, dx \leq C \int_{\mathbb{R}_+^n} v^{\frac{2n}{n-2}} |h|^2 \, dx,
\]

where \( C = C(n) \).
Proof. By Proposition [11] there exists a smooth vector field $V$ such that

$$\int_{\mathbb{R}_+^n} v^{2(n+1)} (h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} \text{div} V \delta_{ik}) \partial_k U_t dx = 0$$

for all $U \in \mathcal{X}$ and $V_n = 0$ on $\partial \mathbb{R}_+^n$. By elliptic regularity ([14] pp.245-249), $V$ is smooth. Hence, $\sum_{k=1}^n \partial_k [v^{\frac{2n}{n-2}} (h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} \text{div} V \delta_{ik})] = 0$ on $\mathbb{R}_+^n$ and $\partial_n V_a - h_{an} = 0$ on $\partial \mathbb{R}_+^n$. 

Proposition 13. Let $h_{ik} = \eta(\frac{|x|}{\rho}) \sum_{|\alpha|=2}^d h_{ik,\alpha} x^\alpha$ be a trace-free symmetric two-tensor, where $d = \lfloor \frac{n-2}{2} \rfloor$, $\rho \geq 1$ and $\eta(t)$ be a fixed cut-off function which satisfies $\eta(t) = 0$ for $t \geq 2$. Suppose $V$ is the vector field constructed in Proposition 12. Then for $x \in \mathbb{R}_+^n$,

$$|\partial^\beta V|^2(x) \leq C(n, |\beta|) \sum_{i,k} \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 (1 + |x|)^2|\alpha|+2-2|\beta|$$

for every multi-index $\beta$.

Proof. The proof is similar to [4] Proposition 23 and Corollary 24.

Without loss of generality we may assume $h_{ik} = \eta(\frac{|x|}{\rho}) \sum_{|\alpha|=l} h_{ik,\alpha} x^\alpha$, where $2 \leq l \leq d$. We first prove that

$$\sup_{r \geq 1} r^{-2l-n-2} \int_{(B_{2r} \setminus B_r) \cap \mathbb{R}_+^n} |V|^2 dx \leq C \int_{\mathbb{R}_+^n} ((1 + x_n)^2 + \sum_{a=1}^{n-1} x_a^2)^{-n-2} |V|^2 dx$$

$$+ C \sup_{r \geq 1} r^{-2l-n} \int_{(B_{2r} \setminus B_r) \cap \mathbb{R}_+^n} |h|^2 dx. \quad (14)$$

Suppose (14) does not hold, there exist sequences $h^{(s)}_{ik}$ and $V^{(s)}$ such that

$$\sup_{r \geq 1} r^{-2l-n-2} \int_{(B_{2r} \setminus B_r) \cap \mathbb{R}_+^n} |V^{(s)}|^2 dx = 1,$$

$$\lim_{s \to \infty} \int_{\mathbb{R}_+^n} ((1 + x_n)^2 + \sum_{a=1}^{n-1} x_a^2)^{-n+2} |V^{(s)}|^2 dx = 0,$$

and

$$\limsup_{s \to \infty} r^{-2l-n} \int_{(B_{2r} \setminus B_r) \cap \mathbb{R}_+^n} |h^{(s)}|^2 dx = 0.$$

Therefore, there exists a sequence $\rho^{(s)} \to \infty$ such that

$$(\rho^{(s)})^{-2l-n-2} \int_{(B_{2\rho^{(s)}} \setminus B_{\rho^{(s)}}) \cap \mathbb{R}_+^n} |V^{(s)}|^2 dx \geq \frac{1}{2}.$$
Let \( \bar{h}_{ik} = (\bar{\rho}^{(s)})^{-1} [x]^{-4} \left( |x|^2 \delta_{ij} - 2x_i x_j \right) |x|^2 \delta_{il} - 2x_k x_l \right) \bar{h}^{(s)}_{jl} \left( \frac{\bar{\rho}^{(s)}}{|x|^2} \right) \) and \( \bar{V}_j = (\bar{\rho}^{(s)})^{-1} \left( |x|^2 \delta_{ij} - 2x_i x_j \right) V_i^{(s)} \left( \frac{\bar{\rho}^{(s)}}{|x|^2} \right) \). Then they satisfy

\[
\sum_{k=1}^{n} \partial_k \left[ \left( 1 + \frac{x_n}{\rho(x)} \right)^2 + \sum_{a=1}^{n-1} \left( \frac{x_a}{\rho(x)} \right)^2 \right]^{-n} \left( \bar{h}_{ik} - \partial_i \bar{V}_k - \partial_k \bar{V}_i + \frac{2}{n} \text{div} \bar{V} \delta_{ik} \right) = 0
\]

in \( \mathbb{R}^n_+ \) for \( i = 1, \cdots, n \), and \( \bar{V}_n = \partial_n \bar{V}_a - \bar{h}_{an} = 0 \) on \( \partial \mathbb{R}^n_+ \). Thus, by passing to a subsequence, \( \bar{V}_j \) converges weakly to a vector field \( V \in W^{1,2}(\mathbb{R}^n_+ \setminus \{0\}) \). \( V \) satisfies

\[
\sum_{k=1}^{n} \partial_k [-\partial_i \bar{V}_k - \partial_k \bar{V}_i + \frac{2}{n} \text{div} \bar{V} \delta_{ik}] = 0
\]

weakly in \( \mathbb{R}^n_+ \setminus \{0\} \) for \( i = 1, \cdots, n \), and \( V_n = \partial_n V_a = 0 \) on \( \partial \mathbb{R}^n_+ \setminus \{0\} \). By elliptic regularity theory, \( V \) is smooth in \( \mathbb{R}^n_+ \setminus \{0\} \). Thus, \( V \) satisfies \( \Delta V_j + \frac{n-2}{n} \partial_j \text{div} V = 0 \). This implies \( \Delta \text{div} V = 0 \). Moreover, on \( \partial \mathbb{R}^n_+ \setminus \{0\} \) we have \( 0 = \Delta V_n + \frac{n-2}{n} \partial_n \text{div} V = 2 \frac{n-1}{n} \partial_n V_n \). Therefore, \( \partial_n \text{div} V = 0 \) on \( \partial \mathbb{R}^n_+ \setminus \{0\} \). We now define the function \( \text{div} V \) on \( \mathbb{R}^n_+ \setminus \{0\} \) by standard reflection. Then \( \text{div} V \) is a \( C^{2,1} \) harmonic function in \( \mathbb{R}^n_+ \setminus \{0\} \). Since \( \sup_{\mathbb{R}^n_+ \setminus \{0\}} |x|^2 |\text{div} V|^2 \) is bounded, we obtain \( \text{div} V = 0 \) in \( \mathbb{R}^n_+ \setminus \{0\} \). Thus, \( \Delta V_j = 0 \) in \( \mathbb{R}^n_+ \setminus \{0\} \). By the same reflection argument applied to the function \( V_a \), we get \( V_a \) is a \( C^{2,1} \) harmonic function in \( \mathbb{R}^n_+ \setminus \{0\} \). Since \( \sup_{\mathbb{R}^n_+ \setminus \{0\}} |x|^{2l-2} |V|^2 < \infty \), we have \( V_a = 0 \) in \( \mathbb{R}^n_+ \setminus \{0\} \). Finally, since \( \partial_n V_a = \text{div} V = 0 \), using the same reflection argument again we obtain \( V_n = 0 \) in \( \mathbb{R}^n_+ \setminus \{0\} \). This contradicts to \( \int_{(B_1 \setminus B_{\frac{1}{2}}) \cap \mathbb{R}^n_+} |V|^2 \) \( dx > 0 \). Thus, (14) holds.

Now since we have

\[
\int_{\mathbb{R}^n_+} \left( (1 + x_n)^2 + \sum_{a=1}^{n-1} x_a^2 \right) |V|^2 dx \leq C \int_{\mathbb{R}^n_+} \left( (1 + x_n)^2 + \sum_{a=1}^{n-1} x_a^2 \right) |h|^2 dx
\]

\[
\leq C \sum_{i,k=1}^{n} \sum_{|\alpha|=l} |h_{ik,\alpha}|^2,
\]

then by (14) \( \sup_{r \geq 1} r^{-2l-2} \int_{(B_2 \setminus B_r) \cap \mathbb{R}^n_+} |V|^2 dx \leq C \sum_{i,k=1}^{n} \sum_{|\alpha|=l} |h_{ik,\alpha}|^2 \). Finally, by elliptic regularity \( |\partial^\beta V|^2 (x) \leq C(n, |\beta|) \sum_{i,k=1}^{n} \sum_{|\alpha|=l} |h_{ik,\alpha}|^2 (1 + |x|)^{2|\alpha|+2-2|\beta|} \). \( \square \)

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