Variation of rest mass scale in a gravitational field

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I argue that the rest mass scales associated with different locally flat regions of a curved spacetime differ from each other. If, Planck's constant (h) and the velocity of light in vacuum (c), are considered to be given fundamental (constant) scales in local measurements in both of these locally flat regions, then the rest mass scale varies exactly inversely as the time scale varies. As the time scale variation gives rise to gravitational redshift, the rest mass scale variation will give rise to an apparent change in the Compton length scale associated with an elementary particle due to the gravitational potential difference in a suitable coordinate choice. This is verifiable by performing the same Compton scattering experiment in two such different locally flat spacetime regions.

I. INTRODUCTION

Three most important physical entities that one perceives of in the daily life are mass, length and time, which are measured with scales (or units) such as kilogram, metre and second, respectively [1, 2]. In special relativity, Einstein showed that the notion of length scale and time scale are not absolute concepts i.e. the notion of ‘1 metre’ and ‘1 second’ varies in two frames with constant relative velocity [3]. What is fundamental in special relativity, by postulate, is the velocity scale denoted by c, which is the ‘velocity’ of light in vacuum [3]. Another scale that is absolute, but not fundamental, in special relativity is the rest mass scale \( m_0 \) associated with a particle. It is ‘absolute’ in the sense that it is invariant under Lorentz transformations i.e. \( p^I p_I = -m_0^2 c^2 \) where \( p^I \) is the four-momentum [7, 8]. It is not ‘fundamental’ because it is arbitrary. \( m_0 \) appears as the rest mass scale in the four force representation of Newton’s laws of motion (e.g. see [9]). On the other hand \( m_0 \) represents the rest mass scale of an elementary particle, which is rather understood as the Lorentz invariant mass scale associated with a quantum field (e.g. see [25]). While the former theory is devoid of Planck’s constant (h), the later involves and relies on it. Nonetheless, there is no hard and fast rule that dictates the association of h with the rest mass scale. It is only through experiment and experience, one comes to a decision about which theory is suitable to describe some particular rest mass scale.

In ref. [5], just by considering equivalence and mass-energy conservation principles, Einstein concluded that the energy scale differs for two observers sitting in two frames that differ by a gravitational potential. In the sequence of arguments, Einstein used only Newton’s laws of motion and special relativity; h did not play any explicit role\(^1\). However, while writing down the formula for gravitational redshift, Einstein explicitly used, without mentioning, a fundamental angular momentum scale. The experimental verification of gravitational redshift by Pound and Rebka [10, 11] confirmed that this fundamental scale is none other than h. In the language of curved spacetime geometry, Einstein’s result is a manifestation of energy scale variation between two locally flat spacetime regions when the local observations are compared in a suitable coordinate system for the Newtonian potential to represent the curvature [12].

In this article I shall argue that if one considers h and c to be given, rather than determined, fundamental scales, then the rest mass scale associated with an elementary particle varies exactly inversely as the time scale varies in two different locally flat regions of a curved spacetime (or there is a gravitational potential and it follows directly from Einstein’s results obtained from equivalence and mass-energy conservation principles [5]. I describe a thought process to realize this effect, which is just a modification of that of Einstein’s. Then, I provide a concrete result regarding the variation of Compton length scale of an electron from a scattering experiment that can be readily verified.

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\(^1\) The role of h is special relativity is implicit. See [26] for an explanation
II. A REVIEW OF EINSTEIN’S THOUGHT PROCESS

With an aim to shed light on the mass scale variation due to gravity, I briefly mention the relevant results which were reported by Einstein in [5]. Einstein considered the following thought process:

1. Two observers $O_1$ and $O_2$, relatively at rest with respect to each other, are equipped with a set of identical measuring instruments (implies identical scales for measurements).

2. Now, these observers, along with their corresponding set of instruments, sit at different potentials of a homogeneous gravitational field or equivalently have a relative non-zero acceleration (equivalence principle).

3. A certain amount of energy $E_1$, as measured by $O_1$ locally in its rest frame, is emitted, in the form of radiation, towards $O_2$.

4. $O_2$ receives this energy and performs a local measurement in its rest frame to yield the result $E_2$.

Einstein showed that, if energy conservation principle holds, then, approximately up to the first order in $\Phi/c^2$, $E_1$ and $E_2$ are related by the following equation

$$E_2 = E_1 \left(1 + \frac{\Phi}{c^2}\right)$$  \hspace{1cm} (1)

where $\Phi$ is the potential difference between $O_1$ and $O_2$.

A. Redshift

Then, Einstein went on to write down the gravitational redshift formula. To do that, Einstein simply assigned ‘frequency’ corresponding to radiation energy (and not ‘intensity’): “If the radiation emitted ...... had the frequency....” (see the very beginning of section 3 of [5]). He arrived at the formula

$$\nu_2 = \nu_1 \left(1 + \frac{\Phi}{c^2}\right).$$  \hspace{1cm} (2)

Einstein’s argument to interpret the frequency shift was based on the fact that the time scale (unit) for $O_1$ and that for $O_2$ do not remain identical due to the difference in gravitational potential. Eq.(2) is the formula that was experimentally verified by Pound and Rebka [10, 11].

1. In the spacetime geometry language

In the above thought experiment, one can equivalently consider $O_1$ and $O_2$ freely falling along two time-like geodesics at different constant values of the radial coordinate of a Schwarzschild spacetime, both the coordinate values being much larger than unity while measured with the Schwarzschild length scale (see how gravitational redshift is explained in [12]). Therefore, $O_1$ and $O_2$, individually, only experience flat spacetime (i.e. special relativity). Considerations of $h$ and $c$ to be fundamental scales by both the observers and comparison of their measurements, provide them with the realization of the curved spacetime through the gravitational redshift.

B. Fundamental scales in local measurement

1. Einstein’s untold assumption

In the whole sequence of arguments those Einstein provided to reach the formula for the gravitational redshift, the most important step is the assumption, without explanation, of proportionality of radiation energy to its frequency. It is interesting to note that Einstein did not use or mention, explicitly, the involvement of any fundamental angular momentum scale, that remains unaffected by gravity [13]. However, without this assumption it is impossible to pass on from eq.(1) to eq.(2). The experimental verification of Pound and Rebka [10, 11], indeed relied on methods, namely Mossbauer effect [20], that involve a fundamental angular
momentum scale in the form of $h$. Therefore, Einstein’s *untold assumption* was that, besides $c$, $h$ is another fundamental scale that remains identical for $O_1$ and $O_2$ while performing local measurements. In fact such a fundamental angular momentum scale is indispensable to make sense of propagation of energy along a null geodesic [26].

2. *de Broglie hypothesis*

I have remarked earlier that $O_1$ and $O_2$, individually, can be considered as just falling freely along two different timelike geodesics experiencing flat spacetime and the rules of special relativity are valid individually for both. Now, as far as special relativity is concerned, a Lorentz invariant quantity appears in the theory, that is, the rest mass scale associated with a point particle ($m_0$). However, unlike $c$, $m_0$ is not a fundamental constant (or scale). Therefore, the energy scale $E_0 = m_0c^2$ is also not a fundamental one, although it *is* locally Lorentz invariant. Now, energy scale is affected by gravity (or curvature of the spacetime) when two local Lorentz frames at different spacetime regions are compared (i.e. while the measurements made by $O_1$ and $O_2$ are compared), as shown by Einstein in the form of eq. (2).

Since $h$ needs to be considered as a fundamental scale in local measurement, then one is free to associate it with massive particles following de Broglie hypothesis [22]. The situation where this association has any physical relevance is that of elementary particles, which are considered as excitations of quantum fields. Therefore, one can not physically get hold of the rest frame associated with an elementary particle due to Heisenberg uncertainty principle [14]. Nevertheless, the theoretical principle that is valid for any massive particle, irrespective of whether ‘point’ or ‘elementary’, is that, there is a rest frame attached to it and one can, in principle, perform calculations in this rest frame.

III. REALIZING MASS SCALE VARIATION WITH A MODIFIED THOUGHT PROCESS

To explain the effect of gravity on the rest mass scale of an elementary particle, I shall slightly modify and refine Einstein’s thought process. Since $h$ and $c$ are fundamental scales for an elementary particle, following de Broglie, I consider that its rest frame is associated with an energy scale $E_0$ and a frequency scale ($\nu_0$) by the following relation: $E_0 = m_0c^2 = h\nu_0$ [21, 22]. Then, the description goes as follows:

1. An observer $O_1$ considers $h$ and $c$ to be given scales and define all other scales in terms of those two. $O_1$ makes measurements with these scales.

2. $O_1$ studies the decay of a massive elementary particle in its rest frame, e.g. the decay of a neutral pion to two photons: $\pi^0 \rightarrow \gamma + \gamma$, and measures the amount of released energy in the form of photons with the derived scales.

3. Let, $O_1$ measures $\alpha_1$ units of energy with the derived scale $E_1$, i.e. $\alpha_1E_1$ amount of energy is released in the form of radiation as measured by $O_1$. For $O_1$, it implies, $\alpha_1$ amount of mass in the scale $m_1 = E_1/c^2$ is the rest mass of the pion, which follows from four momentum conservation. Another observer $O_2$ notes down the whole procedure while relatively at rest with $O_1$.

4. Then $O_2$ goes to a frame which differs by a gravitational potential from that of $O_1$. $O_1$ repeats the same experiment and sends the resulting photons from the pion decay to $O_2$.

5. $O_2$ receives the photons and measured the energy with the predefined scales and found that he did not receive $\alpha_1E_1$, but an amount $\alpha_2E_1$.

6. $O_2$ finds that the energy measurement yields $\alpha_1$ if the scale is redefined to be $E_2$ which is related to $E_1$ by the eq.(1).

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2 Of course, the characteristic length scale of those regions are much smaller than the length scale of separation of those regions as pointed out by Einstein [5]
7. $O_2$ concludes that the rest mass of the pion is $\alpha_2 m_1$ or $\alpha_1 m_2$ where

$$m_2 = m_1 \left(1 + \frac{\Phi}{c^2}\right).$$ (3)

This implies, if $O_2$ studies the decay of a pion by bringing it to relatively at rest with respect to him/her, then the rest mass will come out to be $\alpha_1$ if measured with the scale $m_2$, but not $m_1$.

In the above thought process, the decay of a massive elementary particle to radiation energy through a physical phenomenon involving Planck constant, is the modification that I made to that of Einstein’s. Although it sufficed for the above thought process to consider only the kinematics (the four momentum conservation), the study of the dynamics of a neutral pion decay suggests that the fine structure constant also needs to be considered as a fundamental constant alongside $\hbar$ and $c$ [25]. I may emphasize that the ratios of rest mass scales associated with different elementary particles, remain the same, individually for $O_1$ and $O_2$.

To mention, exactly like the gravitational redshift has been discussed in terms of geodesics in [12], one can think of a similar process here. For example, imagine two space stations $S_1$ and $S_2$ following two different timelike geodesics. $O_1$ performs the experiment in $S_1$ and transmits the radiation through a window towards $S_2$. $O_2$ receives the radiation through a window at $S_2$. Both $O_1$ and $O_2$ consider $\hbar$ and $c$ to be given. Rest of the steps of the procedure remain alike.

IV. COMPTON SCATTERING AND EFFECT OF GRAVITY

Now, let me discuss a table-top experiment, that can be performed in two frames that differ by a gravitational potential. Consider the Compton effect i.e. scattering of x-ray by an electron [18]. If x-ray, with an associated length scale, $\lambda_0$, is incident on an electron, then it is scattered at an angle $\theta$ with the incident direction. The scattered x-ray is associated with an increased length scale $\lambda_0 (> \lambda_0)$. The increment in the length scale of the x-ray is given by

$$\Delta \lambda := \lambda_0 - \lambda_0 = \lambda_e (1 - \cos \theta).$$ (4)

where $\lambda_e := h/m_e c$ is a length scale associated with the electron, known as Compton wavelength (see [19] for a remark), $m_e$ is the rest mass scale associated with the electron. Now, consider the same Compton scattering experiment is performed by $O_1$ and $O_2$. By ‘same’, I mean, $O_1$ and $O_2$ incident same $\lambda_0$, in their respective frames, on an electron and study the same $\lambda_0$. Now, according to eq.(3), the rest mass of an electron are different for $O_1$ and $O_2$. Therefore, one has the following result:

$$\Delta \lambda = \lambda_1 (1 - \cos \theta_1) = \lambda_2 (1 - \cos \theta_2)$$ (5)

where $\lambda_1 = \frac{h}{m_1 c}$, $\lambda_2 = \frac{h}{m_2 c}$ are the Compton wavelengths of the electron and $m_1, m_2$ are the rest masses of the electron for $O_1$ and $O_2$ respectively. Combining eq.(3) and eq.(5), one has the following result

$$\frac{m_2}{m_1} = \frac{1 - \cos \theta_2}{1 - \cos \theta_1} = \left(1 + \frac{\Phi}{c^2}\right)$$ (6)

$$\Rightarrow \frac{\Phi}{c^2} = \frac{(\cos \theta_1 - \cos \theta_2)}{(1 - \cos \theta_1)}.$$ (7)

A. In a Schwarzschild spacetime

Now, let me bring in the spacetime language and set the observers and radiation propagation along geodesics, which I have already mentioned earlier. Let me consider $O_1$ and $O_2$ freely falling along two timelike geodesics at different constant values of the radial coordinate of a Schwarzschild spacetime [12]; $O_1$ and $O_2$ have radial coordinates $r_1$ and $r_2$ respectively, with $r_2 = r_1 + H$ and $H > 0$. Then, one has the following approximate expression:

$$\frac{\Phi}{c^2} \simeq r_0 \left[\frac{1}{r_1 + H} - \frac{1}{r_1}\right] = \left[\frac{1}{x + \Delta x} - \frac{1}{x}\right] \simeq -\frac{\Delta x}{x^2}.$$ (8)
where $r_0 = \frac{GM}{c^2}$, $M$ is the mass scale associated with the Schwarzschild spacetime \cite{24}, $r_1 = xr_0$, $H = (\Delta x)r_0$, $x \gg 1$, $\Delta x \gg 1$ and the last step of eq. (8) holds for $\Delta x \ll x$. Now, let $\theta_2 = \theta_1 + \delta$, where $\delta$ is a function of $\Delta x$. Then, the right hand side of eq. (7) can be approximated, up to the leading order in $\delta$, to obtain

\[
\frac{(\cos \theta_1 - \cos \theta_2)}{(1 - \cos \theta_1)} \simeq \delta \cot \frac{\theta_1}{2}.
\]

Therefore, using the results of eq. (8) and eq. (9), back in eq. (7), one obtains the result

\[
\delta \simeq -\frac{\Delta x}{x^2} \tan \frac{\theta_1}{2}.
\]

Just as an example, if one considers $O_1$ to be at earth’s radius (6370 km \cite{27}) and $O_2$ on the MICROSCOPE satellite, which is further 700 km radially outward \cite{28}, then $\Delta x/x^2 \simeq 700/(6370)^2 \simeq 1.72 \times 10^{-5}$. Then,

\[
\delta_{\text{MICRO}} \simeq -1.72 \times 10^{-5} \tan \frac{\theta_E}{2}.
\]

Here, $\theta_E$ is the deflection angle of the scattered x-ray photon on earth’s surface and $\delta_{\text{MICRO}}$ is the change in the deflection angle while the experiment is performed on the MICROSCOPE.

\section{V. CONCLUSION}

The variation of rest mass scale associated with an elementary particle, in a gravitational field, follows from the equivalence and mass-energy conservation principles and the Planck’s constant plays a crucial role in its realization as it does for gravitational redshift. It is an introductory lesson that two vectors at different regions in curved spacetime geometry can not be compared with an elementary particle at two different regions of a curved spacetime geometry can not be compared either. Therefore, it does not seem correct to say that “the rest mass of an electron is same everywhere in the universe”, although it is a correct statement to make that “ratio of the rest mass of an electron to that of any other elementary particle is the same in any accessible region of the universe”. While the former is a statement about the comparison between two physical scales at different locally flat regions of a curved spacetime, the later is a statement about the comparison of a pure number.

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\begin{thebibliography}{99}

\bibitem{1} \textit{The International System of Units}, 8th edition (2006)
\bibitem{2} Let me clarify why I associate the word ‘scale’ with all the physical dimensions in the whole discussion. The quantities like mass, length, time, etc. are measured with some predefined scales set by the experimenter. Otherwise, within the theory, one can not assign values to the parameters that appear in the equations. For example, when one says that “the rest mass of a particle $m_0 = 3$ kilograms” it implies that the person has already set a predefined notion of ‘1 kilogram’. Without such a predefined notion, $m_0$ is just a parameter of the theory – an intrinsic ‘scale’ associated with the particle, which is assigned the physical dimension of ‘mass’.
\bibitem{3} A. Einstein, Annalen der Physik, 17 (1905): 891-921
\bibitem{4} A. Einstein, Annalen der Physik, 18 (1905): 639-641
\bibitem{5} A. Einstein, Annalen der Physik 35 (1911)
\bibitem{6} A. Einstein, Königlich Preußische Akademie der Wissenschaften (Berlin). Sitzungsberichte (1914)
\bibitem{7} L. D. Landau, E. M. Lifshitz, \textit{The classical theory of fields}, Volume 2-Butterworth-Heinemann (1994)
\bibitem{8} J. D. Jackson, \textit{Classical electrodynamics}, Wiley (1999)
\bibitem{9} J. B. Hartle, \textit{Gravity: an introduction to Einstein’s general relativity}, Addison-Wesley (2003)
\bibitem{10} Pound, R. V.; Rebka Jr. G. A., \textit{Physical Review Letters}, 3 (9): 43944 (1959).
\bibitem{11} Pound, R. V.; Rebka Jr. G. A., \textit{Physical Review Letters}, 4 (7): 337341 (1960).
\bibitem{12} R. M. Wald, \textit{General relativity}, University of Chicago Press (1984)
\bibitem{13} It is interesting, as well as ironical, because Einstein’s own paper concerning the photoelectric effect \cite{15} and Planck’s paper on black body radiation spectrum \cite{16, 17}, all of which contained the idea of light quanta, leading to the proportionality of energy and frequency, were published before ref.\cite{5} appeared in the literature. The proportionality constant is now known as Planck’s constant.
\end{thebibliography}
[14] W. Heisenberg, *The physical content of quantum kinematics and mechanics*, page-62 in *Quantum Theory and Measurement* Edited by J. A. Wheeler and W. H. Zurek, Princeton Series in Physics, Princeton University Press Princeton, New Jersey (1983)

[15] A. Einstein, *Annalen der Physik*. 17 (6): 132148

[16] M. Planck, *Ann. Physik* 1,99 (1900)

[17] M. Planck, *Ann. Physik* 4, 561 (1901)

[18] Compton, Arthur H, *Physical Review*. 21 (5): 483502 (1923).

[19] There is another length scale associated with a mass scale \( m \) and it is \( Gm/c^2 \) where \( G \) is the gravitational constant. This is not used here. This type of length scale involving \( G \) can not be associated with radiation.

[20] Mössbauer, R. L.,*Zeitschrift fr Physik* A. 151 (2): 124143 (1958).

[21] This equation should not be interpreted as “photon has mass”. It just means that the rest mass scale associated with a particle corresponds to an energy scale, which in turn corresponds to a frequency scale, if \( h \) and \( c \) are considered as two fundamental constants. Also, see eq.(1.1.5) of ref.[22].

[22] L. de Broglie, *Ph.D. thesis* (English translation by A. F. Kracklauer), online link.

[23] K. Schwarzschild, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 7: 189196 (1916).

[24] In the Schwarzschild metric there are two mass scales involved. The one which appears in the metric itself, generally denoted by \( M \), is called the ‘mass’ associated with the spacetime. It is a scale that appears in the solution of the Einstein equation as an integration constant [23]. The other one is the mass scale associated with the test particles, say \( m_{\text{test}} \), which are used as probes to study the time-like geodesics. One must have \( m_{\text{test}} \ll M \) so that the metric remains undistorted.

[25] Michael E. Peskin, Daniel V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley (1995)

[26] A. Majhi, *An angular momentum scale and null geodesic* (in preparation).

[27] Approximate average value taken from E. E. Mamajek et al, *IAU 2015 Resolution B3, passed by the XXIXth IAU General Assembly in Honolulu, 13 August 2015*, arXiv:1510.07674

[28] https://microscope.cnrs.fr/en/MICROSCOPE/GP-mission.htm