Space-times which are asymptotic to certain Friedman-Robertson-Walker space-times at timelike infinity

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We define space-times which are asymptotic to radiation dominant Friedman-Robertson-Walker space-times at timelike infinity and study the asymptotic structure. We discuss the local asymptotic symmetry and give a definition of the total energy from the electric part of the Weyl tensor.

I. INTRODUCTION

According to the present observations we are living in an expanding universe and we know that the the space-time is approximately described by Friedman-Robertson-Walker(FRW) space-times. So far only the structure of asymptotically Minkowskian space-time is studied. The formulation is adequate to understand the structure and evolution of compact objects and the stability of space-time. However, if we wish to study the formation of very early objects such as primordial black holes, we should define asymptotically FRW space-times. Thus, it is important to define and investigate the structure and stability of asymptotically FRW space-times.

There is another reason to investigate asymptotic structure of asymptotically FRW space-times. Standard inflation predicts flat FRW universe. Since observations of baryon suggest the small density parameter $\Omega_0 \sim 0.3$, a positive cosmological constant is needed for the standard inflation scenario. However, the existence of the cosmological constant is not clear in aspects of observations and fundamental particle physics at present. A few years ago the creation of an open FRW universe aided by one bubble inflation has been proposed. Unfortunately, the scenario demands a fine-tuning on the potential of inflaton. Recently, Hawking and Turok proposed a scenario that an open FRW universe can be created from ‘nothing’. Since the Hawking-Turok instanton has timelike singularity, the issue is still under debate. However, their scenario of the quantum creation of an open universe is very attractive because the model needs no strong fine-tuning. Thus, it is important to obtain the basic structure of asymptotically open FRW universe in order to investigate the nature of the instanton.

In this paper, we will define space-times which are asymptotic to radiation dominant FRW space-times at timelike infinity (for brevity, we will call them by AFRWTI space-times hereafter), except for closed FRW universe. This is because closed FRW universes terminate at the big-crunch and the notion of AFRWTI space-times is lost at the singularity. We use the mathematical tool which was developed by Ashtekar and Romano for spatial infinity and extended by Gen and Shiromizu to timelike infinity in the asymptotically flat space-times.
The covariant analysis of asymptotic structure has been studied so far by using the conformal completion of space-times. The conformal completion is useful for the investigation of the global causal structure and can treat simultaneously null and spatial infinities. However, the method leads to a point spatial infinity and this results in complicated differential structure at spatial infinity. To overcome this difficulty, Ashtekar & Romano have proposed a new completion to investigate asymptotic structure of universe in time.

Let us consider FRW space-times in order to obtain the essence of the new completion. The metric of spatially open FRW space-times is given by

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \gamma_{ij}dx^idx^j\right],$$

(2.1)

where $\gamma_{ab}$ is the metric of the three-dimensional unit hyperboloid space. The scale factor $a(\eta)$ can be written as $a(\eta) = \sinh(\eta/\eta_0)$ and $a(\eta) = \cosh(\eta/\eta_0) - 1$ for radiation and matter dominant universe, respectively.

We introduce a coordinate $\Omega = a^{-1}$ to define the asymptotic region at the timelike infinity. In terms of $\Omega$ the metric becomes

$$\hat{g}_{\text{FRW}} = -F_0^{-1}\Omega^{-4}\left[1 - \Omega^2 + O(\Omega^3)\right]d\Omega^2 + \Omega^{-2}\gamma_{ij}dx^idx^j,$$

(2.2)

and

$$\hat{g}_{\text{FRW}} = -F_0^{-1}\Omega^{-4}\left[1 - 2\Omega + O(\Omega^2)\right]d\Omega^2 + \Omega^{-2}\gamma_{ij}dx^idx^j,$$

(2.3)

for radiation and matter dominant open FRW universe, respectively. In the above $F_0^{-1} = \eta_0^2$. In the coordinate system $(\Omega, x^i)$ the metric is singular at the timelike infinity $\Omega \to 0$. However,

$$n^a = \Omega^{-4}\hat{g}^{ab}\nabla_b\Omega = -F(\partial\Omega)^a$$

(2.4)

and

$$q_{ab} = \Omega^2a^2\gamma_{ij}dx^idx^j = \gamma_{ij}dx^idx^j.$$  

(2.5)

are smooth tensors at the infinity and contain all informations of gravitational field. Hence the gravitational field of asymptotic region can be investigated in terms of the pair $(n^a, q_{ab})$.

As we can see, the leading behaviour of the metric does not depend on the equation of state. Since the curvature dominant open universe approaches to Milne universe which is a submanifold of Minkowski space-time covered by the hyperbolic coordinate, we can apply directly our previous study on the timelike infinity of asymptotically flat space-times for spatially open FRW space-times.

Next, we consider the spatially flat FRW space-times whose metric is given by

$$ds^2 = a^2\left[-d\eta^2 + \gamma_{ij}dx^idx^j\right],$$

(2.6)

where $ad\eta = dt$, $a = a_0(t/t_0)^{2/3\gamma}$ and $\gamma_{ab}$ is the metric of the 3-Euclid space. $\gamma = 4/3, 0$ correspond to radiation and matter dominant universe, respectively.

In the same way as the open FRW cases, we introduce a coordinate $\Omega = (a/a_0)^{-1}$ to define the asymptotic region at the timelike infinity. In terms of $\Omega$ the metric becomes

II. FRW SPACE-TIMES

The rest of this paper is organised as follows. In Sec. II, we will treat FRW space-times to look for a better definition of AFRWTI space-times. In Sec. III we define AFRWTI space-times and show that spacelike hypersurfaces of space-times approach conformally to three-dimensional Euclid space or hyperbolic space. In Sec. IV, we consider the asymptotic symmetry. In Sec. V, we define the total energy of AFRWTI space-times from the electric part of the Weyl tensor. Finally we give a summary and discuss about the relation between the total energy and the gravitational instanton in Sec. VI. In the appendix, we will give pedagogical examples for AFRWTI space-times. We basically follow the notation of Wald’s text [3].
\[ \tilde{g}_{\text{frw}} = -F^{-1} \Omega^{-3\omega-5} d\Omega^2 + \Omega^{-2} \gamma_{ij} dx^i dx^j, \]  

where \( F^{-1} = (9/4)\hat{\Omega}^2(1 + \omega)^2 \) and \( \omega = 1/3, \) \( 0 \) correspond to \( \gamma = 4/3, 1. \) In the coordinate system \( (\Omega, x^r) \) the metric is singular at the timelike infinity \( \Omega \to 0. \) However, 

\[ n^a = \Omega^{-3\omega-5} \hat{g}^{ab} \hat{\nabla}_b \Omega = -F(\partial_1)^a \] 

and 

\[ q_{ab} = \Omega^2 a^2 \gamma_{ij} dx^i dx^j = \gamma_{ij} dx^i dx^j. \] 

are smooth tensors at the infinity and contain all informations of gravitational field. From eq. (2.8) the function \( F \) can be expressed by \( F = -\mathcal{E}_n \Omega \) which will be turned out to contain the information of the energy in Sec. V.

We can see easily that the case of spatially open FRW space-times are accidentally combined to eq.(2.7) with \( \omega = -1/3. \)

### III. AFRWTI SPACE-TIMES

From the study of the completion of FRW space-times in the previous section, we may define space-times which are asymptotic to radiation dominant FRW space-times at timelike infinity as follows;

**Definition:** A space-time \( (\hat{M}, \hat{g}_{ab}) \) will be said to be asymptotic to radiation dominant spatially flat (or open) FRW space-times at timelike infinity \( \hat{I}^+ \) (AFRWTI space-times) if there exists a smooth function \( \Omega \) satisfying the following features (i) and (ii) and the energy momentum tensor satisfies the fall off condition (iii);

(i) \( \Omega |_{\mathcal{I}^+} = 0 \) and \( d\Omega |_{\mathcal{I}^+} \neq 0 \)

(ii) The following quantities have smooth limits on \( \mathcal{I}^+ \).

\[ n^a = \Omega^{-3\omega-5} \hat{g}^{ab} \hat{\nabla}_b \Omega \] 

\[ q_{ab} = \Omega^2 (\hat{g}_{ab} + F^{-1} \Omega^{-3\omega-5} \hat{\nabla}_a \hat{\nabla}_b \Omega) = \Omega^2 \hat{g}_{ab}, \]

where \( F = -\mathcal{E}_n \Omega \) and \( \omega = 1/3 \) for flat (or \( \omega = -1/3 \) for open).

(iii) \( \hat{T}^\mu_{\hat{g}} := \hat{T}^\mu_{\hat{g}}(\hat{e}_a)^a(\hat{e}_b)^b = O(\Omega^4) \) near \( \mathcal{I}^+ \), where \( \{(\hat{e}_a)^a\}_{a=0,1,2,3} \) is a tetrad of the metric \( \hat{g}_{ab}. \)

In the above definition, we excluded the matter dominant case \( (\omega = 0) \) because we realise that one cannot obtain the comprehensive asymptotic structure. The detail will be discussed in below and Sec. VI. In the above formulation, the \((3 + 1)-\)decomposition split is implicitly included. Since we are interested in asymptotic structure at timelike infinity, the treatment is plausible.

In terms of \( (n^a, q_{ab}) \) the extrinsic curvature of the spacelike hypersurface \( \Omega = \text{constant} \) is written as 

\[ \hat{K}_{ab} = \frac{1}{2} \mathcal{E}_n \hat{g}_{ab} = F^{1/2} \Omega^{3\omega+1} q_{ab} + \frac{1}{2} F^{-1/2} \Omega^{3\omega+1} \mathcal{E}_n q_{ab} \]  

Since the tensor is singular at \( \Omega = 0, \) we define a smooth tensor \( K_{ab} \) by 

\[ K_{ab} = \Omega^{-3/4(3\omega-1)} \hat{K}_{ab} = F^{1/2} q_{ab} + \frac{1}{2} F^{-1/2} \Omega \mathcal{E}_n q_{ab} \]

The time-space components of the Einstein equation is written as 

\[ \Omega^{-3/4(3\omega+5)} (\hat{e}_a)^a(\hat{e}_b)^b \hat{G}_{ab} = (D_a K^n_b - D_b K)(\hat{e}_i)^i = 0, \]  

where \( \{(\hat{e}_a)^a\}_{a=1,2,3} \) is a tetrad of the metric \( \hat{g}_{ab} \) and \( \{(\hat{e}_i)^a\}_{i=1,2,3} \) is a smooth triad of the metric \( q_{ab}. \) Substituting eq. (5.4) into (5.3) and imposing the fall-off condition (iii), we find \( F \equiv \text{constant}, \) where \( \equiv \) denotes the evaluation on \( \mathcal{I}^+. \) By virtue of the remaining freedom of the conformal rescaling we can always set \( F \equiv 1 \) without loss of generality.

The space-space components of the Einstein equation is written as 

\[ \Omega^{-2} \hat{R}_{ab}(\hat{e}_i)^a(\hat{e}_j)^b = \left[ 3 R_{ab} - F^{1/2} D_a D_b F^{-1/2} - \frac{1}{2} (3\omega - 1) \Omega^{3\omega+1} F^{1/2} K_{ab} + \Omega^{3\omega+1} (-2 K^i_b K_{ac} + K_{ab} K) \right] (\hat{e}_i)^a(\hat{e}_j)^b. \]
Substituting eq. (3.4) and $F = 1$ into eq. (3.3) and imposing the fall-off condition (iii), we find

$$3R_{ab} = 2\kappa_{ab} \iff \otimes_{(0)} q_{ab} = \omega_{ab} \quad (3.7)$$

where $K = 0$ and $K = -1$ for $\omega = 1/3$ and $\omega = -1/3$, respectively. In other words, $q_{ab}$ is locally the metric of 3-dimensional Euclid and unit hyperboloid space at timelike infinity for the asymptotically flat and open FRW space-times, respectively. These features, $F = 1$ and $\otimes_{(0)} q_{ab} = \omega_{ab}$, belong to the 0th order structure in the words of the ref. [7].

In terms of $(\rho^a, q_{ab})$, the electric part of the Weyl tensor is written as

$$\hat{E}_{ab} = F^{1/2}D_aD_bF^{-1/2} + \Omega^{3\omega+1}K_{ac}K_{bd} + \frac{1}{2}(3\omega - 1)F^{1/2}\Omega^{3\omega+1}K_{ab}$$

$$- F^{-1/2}\Omega^{3\omega+2}L_aK_b + \frac{1}{2}(\hat{q}_{ab}\hat{q}_{cd} - \hat{q}_{ab}\hat{r}_{cd})\hat{L}_{cd}, \quad (3.8)$$

where $\hat{L}_{ab} = \hat{R}_{ab} - \frac{1}{n}\hat{q}_{ab}\hat{R}$. By using expansion series $F = 1 + (3)F\Omega + \cdots$ and $\hat{L}_{ab}(\hat{e}_a)\theta(\hat{e}_b)^b = (3)\hat{L}_{\mu\nu}\Omega^3 + (4)\hat{L}_{\mu\nu}\Omega^4 + \cdots$, which follows from the condition (iii), around $\Omega = 0$, we obtain the expression

$$\hat{E}_{ab} = -\frac{1}{2}(\hat{q}_{ab}\hat{q}_{cd} - \hat{q}_{ab}\hat{r}_{cd})\hat{L}_{cd} + O(\Omega^2). \quad (3.9)$$

Although the condition (iii) implies $(3)\hat{L}_{\mu\nu} = 0$ we leave the term $(3)\hat{L}_{\mu\nu}$ in the above equation in order to give a comment on the matter dominant universes below. To discuss the leading behaviour of gravitational field we define the tensor field, $E_{ab} := \hat{E}_{ab}\Omega^{-1}$. This tensor field $E_{ab}$ satisfies the field equation

$$D_{[a}E_{b]c} = 2D_{[a}[\delta_{(1)}(\hat{e}_i)\theta(\hat{e}_j)c((3)\hat{L}_{12} + \delta_{(1)}(3)\hat{L}_{00})] \quad (3.10)$$

From the tracelessness of $E_{ab}$ we find that $(1)F$ obeys the differential equation,

$$(D^2 + 3K)(1)F = 2(3)\hat{L}_{00} + \frac{1}{2}(3)\hat{L}_{11}. \quad (3.11)$$

In AFRWTI space-times, $(3)\hat{L}_{\mu\nu}$ vanishes on $\mathcal{I^+}$, and we obtain

$$E_{ab} = -\frac{1}{2}(D_aD_b + Kq_{ab})(1)F \quad (3.12)$$

$$D_{[a}E_{b]c} = 0 \quad \Rightarrow \quad D_aE^{ab} = 0 \quad (3.13)$$

$$D^2 + 3K(1)F = 0 \quad (3.14)$$

The behaviours of the function $(1)F$ and the tensor field $E_{ab}$ are completely determined regardless of the leading behaviour of the energy-momentum tensor $(4)\hat{L}_{\mu\nu}$. These features belong to the 1st order structure in the words of the ref. [7].

If one wants to consider asymptotically matter dominant FRW space-times $(\omega = 0)$, one should demand $\hat{T}_{\mu\nu} = O(\Omega^3)$ instead of the fall-off condition (iii) because of the behaviour of $\hat{T}_{\mu\nu}$ in the exact matter dominant FRW space-time. In this case we obtain $F = 1$ and eq. (3.7) but not eqs. (3.12)–(3.14). That is, asymptotically matter dominant FRW space-times possess 0th order asymptotic structure but not the 1st order structure. $(1)F$ depends on the behaviour of $(3)\hat{L}_{\mu\nu}$. Among them the case with $(3)\hat{T}_{\mu\nu} = \rho_0\delta_{\mu\rho}\delta_{\nu\rho}$, where $\rho_0 = \text{const.}$. may be worth discussing here. In this case

$$E_{ab} = -\frac{1}{2}(D_aD_b + Kq_{ab})(1)F + \frac{1}{4}\rho_0q_{ab} \quad (3.15)$$

$$D_{[a}E_{b]c} = 0 \quad \Rightarrow \quad D_aE^{ab} = 0 \quad (3.16)$$

$$(D^2 + 3K)(1)F = \rho_0 \quad (3.17)$$

hold instead of eqs. (3.12)–(3.14).
IV. LOCAL ASYMPTOTIC SYMMETRY

The FRW space-times have a timelike conformal Killing vector (CKV) field $\hat{\xi}_{(t)} = \partial_t$ and spacelike Killing vector fields. Here, we seek its asymptotic correspondences in AFRWTI space-times. Since CKV $\hat{\xi}$ is defined by $\mathcal{L}_{\hat{\xi}} \tilde{g}_{ab} = f \tilde{g}_{ab}$, where $f$ is a smooth function, CKV is a Killing vector field of a conformal metric $\tilde{g}_{ab} = \Omega^2 \hat{g}_{ab}$ with $f = -2 \mathcal{L}_{\hat{\xi}} \ln \Omega$, that is, $\mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = 0$.

Simple calculations lead us to

$$(e_a)^a (e_b)^b \mathcal{L}_{\hat{\xi}} (\Omega^2 \hat{g}_{ab}) = 3(1 + \omega) \Omega^{-1} \hat{\xi} \cdot \Omega - F^{-1} \mathcal{L}_{\hat{\xi}} F + 2 F^{-1} \mathcal{L}_{\hat{\xi} \cdot \Omega}$$

$$(e_i)^a (e_b)^b \mathcal{L}_{\hat{\xi}} (\Omega^2 \hat{g}_{ab}) = \left[ F^{-1/2} D_a (\hat{\xi} \cdot \Omega) - \frac{\Omega^2}{3(1 + \omega)} F^{-1/2} [\hat{\xi}, n]_a \right] (e_i)^a$$

$$(e_i)^a (e_j)^b \mathcal{L}_{\hat{\xi}} (\Omega^2 \hat{g}_{ab}) = (e_i)^a (e_j)^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab},$$

where $\hat{\xi}$ is a vector field and $\{ (e_{\mu})^a \}_{\mu = 0, 1, 2, 3}$ is a tetrad of the metric $\hat{g}_{ab}$.

The vector field which induces the timelike translation in the conformally transformed space-times with metric $\tilde{g}_{ab} = \Omega^2 \hat{g}_{ab}$ can be written as $\hat{\xi} = \alpha \Omega^{(1 + \omega)} \partial_t$, where $\alpha$ is a smooth function. Using this expression we see that the above equations become

$$(e_a)^a (e_b)^b \mathcal{L}_{\hat{\xi}} (\Omega^2 \hat{g}_{ab}) = 3(1 + \omega) \Omega^{-1} \hat{\xi} \cdot \Omega - 2 \alpha + \cdots$$

$$(e_i)^a (e_b)^b \mathcal{L}_{\hat{\xi}} (\Omega^2 \hat{g}_{ab}) = \frac{\Omega^2}{3(1 + \omega)} D_i \alpha + \cdots$$

$$(e_i)^a (e_j)^b \mathcal{L}_{\hat{\xi}} (\Omega^2 \hat{g}_{ab}) = \Omega^{(1 + \omega)} D_{ij} + \cdots,$$

where we used the expansion series $\alpha = (0) \alpha + (1) \alpha + \cdots$ and $q_{ab} = \gamma_{ab} + \Omega^{(1)} q_{ab} + \cdots$ around $\Omega = 0$. Hence, $(e_a)^a (e_b)^b \mathcal{L}_{\hat{\xi}} (\Omega^2 \hat{g}_{ab}) = O(\Omega^2 (1 + \omega))$ holds in general. Thus, the vector field defined by $\hat{\xi} = \alpha \Omega^{(1 + \omega)} \partial_t$ deserves to be called asymptotically timelike conformal Killing vector fields.

On the other hand, the vector field which induces the spacelike translation in the physical space-times with the metric $\hat{g}_{ab}$ can be written as $\hat{\xi} = \xi^i (e_i)^a$. In this case the Lie derivatives of the physical metric become

$$\hat{n}^a n^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = - \Omega \hat{\xi} \cdot \hat{F} + \cdots$$

$$(\hat{e}_i)^a n^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = - \Omega^2 [\hat{\xi}_i, n]_a + \cdots$$

$$(\hat{e}_i)^a (\hat{e}_j)^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = (e_i)^a (e_j)^b \mathcal{L}_{\hat{\xi}} (q_{ab}) + \cdots.$$  

The vector field which generates the isometry always exists in the hypersurface with the metric $\gamma_{ab}$. If we adopt $\hat{\xi}$ as such vector fields, $(e_a)^a (e_b)^b \mathcal{L}_{\hat{\xi}} (\hat{g}_{ab}) = O(\Omega)$ holds. Thus, AFRWTI space-times have asymptotic spacelike Killing vector fields.

We can obtain more comprehensive results for cases with $\omega = -1/3$ because such cases is belonging to space-times defined in ref. See the ref. for the detail.

V. ENERGY

In this section we define the total energy of AFRWTI space-times from the electric part of the Weyl tensor. The coefficient of the monopole component of $1^1 F$, $q_{00}$, is expected to express the total conserved ‘energy’ because $E_{ab}$, which plays the tidal force in the equation of geodesic congruence, is written in terms of $1^1 F$ as eq. When one considers the total energy, one need not take account of the magnetic part of the Weyl tensor. The part contains the information of the total angular momentum or/and the local energy of a sort of gravitational wave.

First of all, we consider $\omega = 1/3$ case. In this case, the ‘metric’ of the timelike infinity $\mathcal{I}^+$ becomes the Euclid metric

$$q_{ab} = \hat{d} (dr)^a (dr)^b + r^2 [(d\theta)^a (d\theta)^b + \sin^2 (d\phi)^a (d\phi)^b].$$

From the regularity condition on the spatial infinity ($r \to \infty$), the general solution of eq. with $K = 0$ is given by
Substituting eq. (5.2) into eq. (3.12), we obtain
\[ \int dS_{rr} = \frac{16\pi}{r}a_{00}. \] (5.3)
Since the ‘energy’ is expected to be contained in the monopole component of \( E_{ab} \), we may define the energy, \( \mathcal{E} \), as
\[ \mathcal{E} = a_{00} = \frac{r}{16\pi} \int dS_{rr}. \] (5.4)
Second, let us consider \( \omega = -1/3 \) case. In this case, the ‘metric’ of the timelike infinity \( \mathcal{I}^+ \) becomes
\[ q_{ab} = \hat{\omega}(d\chi)_a(d\chi)_b + \sinh^2 \chi[(d\theta)_a(d\theta)_b + \sin^2 \theta(d\phi)_a(d\phi)_b]. \] (5.5)
From the regularity condition on the spatial infinity (\( \chi \to \infty \)), the general solution of eq. (3.14) with \( K = -1 \) is given by
\[ ^{(1)}F = \sum_{\ell,m} a_{\ell m} \frac{1}{\sinh \chi} P^{\ell+\frac{1}{2}}_{\frac{3}{2}}(\cosh \chi)Y_{\ell m}, \] (5.6)
where \( P^{\ell+\frac{1}{2}}_{\frac{3}{2}}(\cosh \chi) \) denotes the associated Legendre function. In the same way as \( \omega = 1/3 \), the ‘energy’ may be defined as
\[ \mathcal{E} = a_{00} = \frac{\sinh \chi}{8\pi} \int E_{\chi\chi}dS. \] (5.7)
Note that this expression is same as the energy defined in ref. [1]. If \( ^{(3)}T_{\mu\nu} = \rho_0 \delta_{\mu0}\delta_{\nu0} \) with \( \rho_0 = \text{const.} \), using eqs. (3.15)-(3.17), we see \( ^{(1)}F = \sum_{\ell,m} r^{-(\ell+1)} a_{\ell m}Y_{\ell m} + \frac{\rho_0}{8\pi r} \) for \( \omega = 1/3 \) and \( ^{(1)}F = \sum_{\ell,m} a_{\ell m} \frac{1}{\sinh \chi} P^{\ell+\frac{1}{2}}_{\frac{3}{2}}(\cosh \chi)Y_{\ell m} - \frac{1}{4}\rho_0 \) for \( \omega = -1/3 \), which lead us to the result \( \mathcal{E} = a_{00} \).

VI. SUMMARY AND DISCUSSION

In this paper, we defined space-times which are asymptotic to radiation dominant universes and showed that time slices of these space-times approach conformally to Euclid space or hyperboloid space. Furthermore, we found that the asymptotic timelike conformal Killing vector and spacelike Killing vector fields exist locally. Finally, we defined the physical Hamiltonian \( H \) should be finite and given by \( H_P = H - H_0 \), where \( H_0 \) is the Hamiltonian of the background space-time that is stationary or static [13]. From the definition we naively see that the ground state with the minimum energy is the background space-time. Given a solution, only the surface term in \( H_P \) lefts and then \( H_P = -(1/8\pi) \int_{S_\infty} dS(k - k_0) \), where \( k \) and \( k_0 \) are the trace of the extrinsic curvatures of \( S_\infty \) in full and background space-times; \( S_\infty \) is a 2-dimensional surface at infinity. Here we adopted the gauge such that the lapse function \( N = 1 \) and the shift vector \( N^a = 0 \). The value supplies the ‘total energy’. In particular, this is the same as ADM energy for
asymptotically flat and Abbott-Deser energy \[14\] for asymptotically anti-deSitter space-times. The each ground state is Minkowski and anti-deSitter space-times. On the other hand, it is likely that our energy \( E \) defined in the previous section coincides to the case in which the background space-times is FRW space-times. In fact, \( E \) has approximately the expression
\[
E \sim \int \delta \rho \sqrt{q} d^3 x
\]
for the linear perturbation in the flat FRW universe. The plausible construction of the physical Hamiltonian of the dynamical background space-times should be established in order to study deeply non-linear version.

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APPENDIX A:

We give two examples of AFRWTI space-times which belong to Bianchi type I and V. The matter components is perfect fluid having the equation of state, \( P = (1/3) \rho \).

First, we take an exact solution of Bianchi type I found in \[15\] [17]. The metric is given by
\[
ds^2 = -[A(t)]^{2/3}dt^2 + t^{2p_1}[A(t)]^{4/3-2p_1}dx^2 + t^{2p_2}[A(t)]^{4/3-2p_2}dy^2 + t^{2p_3}[A(t)]^{4/3-2p_3}dz^2
\]
(A1)
where \( A(t) = (\alpha + \beta t^{2/3})^{3/2} \) and \( p_1, p_2, p_3 \) are constants satisfying \( p_1 + p_2 + p_3 = 1 \) and \( p_1^2 + p_2^2 + p_3^2 = 1 \). The energy density is
\[
\rho = \frac{4\beta}{3t^{4/3}A^{4/3}}.
\]
(A2)

We define the function \( \Omega \) as follows,
\[
\Omega := t^{-2/3}.
\]
(A3)

In terms of the new coordinate \( \Omega \), the metric is written as
\[
ds^2 = -F^{-1}\Omega^6(d\Omega)^2 + \Omega^{-2}\left[(\beta + \alpha\Omega)^{2-3p_1}dx^2 + (\beta + \alpha\Omega)^{2-3p_2}dy^2 + (\beta + \alpha\Omega)^{2-3p_3}dz^2\right],
\]
(A4)
where
\[
F = \frac{4}{9\beta + \alpha\Omega}.
\]
(A5)
The energy density is written as
\[
\rho = \frac{4\beta}{3(\beta + \alpha\Omega)^2}\Omega^4.
\]
(A6)

It is easy to see that the above solution satisfies the condition (i) ~ (iii) of AFRWTI space-times with \( \omega = 1/3 \).

Next we consider an exact solution of Bianchi type V found in \[16\] [17]. The metric is given by
\[
ds^2 = A(t)B(t)\left[-dt^2 + dx^2 + e^{2x}\left\{\left(\frac{A}{B}\right)^2 dy^2 + \left(\frac{B}{A}\right)^2 dz^2\right\}\right],
\]
(A7)
where \( A = \sinh t \) and \( B = \alpha \cosh t + \beta \sinh t \) and the energy density is
\[
\rho = \frac{3\beta}{(AB)^2}.
\]
(A8)

We define the function \( \Omega \) as follows,
\[
\Omega := e^{-t}.
\]
(A9)
In term of $\Omega$, the metric is written as

$$ds^2 = -F^{-1} \Omega^{-4} (d\Omega)^2 + \Omega^{-2} \left[ \frac{1}{4} (1 - \Omega^2)(\alpha_+ + \alpha_+ \Omega^2) dx^2 + \frac{1}{4} e^{2x} \frac{(1 - \Omega^2)^{\sqrt{3}+1}}{(\alpha_+ + \alpha_+ \Omega^2)^{\sqrt{3}+1}} \left\{ dy^2 + \left( \frac{\alpha_+ + \alpha_+ \Omega^2}{1 - \Omega^2} \right)^3 dz^2 \right\} \right],$$

(A10)

where $\alpha_\pm = \alpha \pm \beta$ and

$$F = \frac{4}{(1 - \Omega^2)(\alpha_+ + \alpha_+ \Omega^2)}.$$

The energy density is

$$\rho = \frac{48 \beta}{(1 - \Omega^2)^2 (\alpha_+ + \alpha_+ \Omega^2)^2} \Omega^4.$$

(A12)

We see again that the solution satisfies the condition $(i) \sim (iii)$ of AFRWTI space-times with $\omega = -1/3$. 

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