Meshfree Finite Volume Element Method for Constrained Optimal Control Problem Governed by Random Convection Diffusion Equations

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Abstract. In this paper, we investigate a stochastic meshfree finite volume element method for an optimal control problem governed by the convection diffusion equations with random coefficients. There are two contributions of this paper. Firstly, we establish a scheme to approximate the optimality system by using the finite volume element method in the physical space and the meshfree method in the probability space, which is competitive for high-dimensional random inputs. Secondly, the a priori error estimates are derived for the state, the co-state and the control variables. Some numerical tests are carried out to confirm the theoretical results and demonstrate the efficiency of the proposed method.

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1 Introduction

Optimal control problems governed by partial differential equations (PDEs) are crucial to many real-life applications, and have become an important research field in applied science. Numerical methods for PDEs have been a major research topic in applied mathematics and control theory. Much research has been carried out for these optimal control problems on theoretical analysis and numerical computation, see such as [12,15–19,27,29,30,33].

In many complex physical and engineering models, uncertainties arise from various sources such as the coefficients, boundary conditions, forcing term or parameter of the partial differential equations. These models are often represented by stochastic partial differential equations (SPDEs). The quantities in these models are usually provided by measurement data, which will lead to errors and impact the computational efficiency of the optimal control problems. So it is important to take into account the uncertainties for optimal control problems governed by SPDEs.

Although theoretical analysis and numerical approximation have been investigated for many years in dealing with the deterministic optimal control problems governed by PDEs, optimal control problems governed by SPDEs with random coefficient have become popular research field only in recent years. There exist many methods for solving these stochastic optimal control problem, such as Monte Carlo method, stochastic Galerkin method, stochastic collocation method and so on. The Monte Carlo (MC) method is one of the most commonly used methods for dealing with simulating elliptic SPDEs. The MC method is very robust and can deal with SPDEs of high dimensions, and has been used to solve optimal control problem governed by SPDEs in [5]. However, the convergence of MC method is slow for the SPDEs as reported and verified in [2]. The stochastic Galerkin method is another method which has been widely used to solve stochastic problems by using the Wiener-Hermite polynomial chaos expansion. In [13,24–26,28,31], the stochastic Galerkin method has been applied to solve a series of optimal control problems governed by SPDEs. However, the stochastic Galerkin method could produce a fully coupled system of linear equations for the SPDEs and this would bring difficulty for solving the problems of high dimensions. The stochastic collocation is another efficient method for SPDEs, which leads to uncoupled linear systems and could be solved parallelly. In [9–11,32], the stochastic collocation method is used to solve the optimal control problems with SPDEs constraints.

Since it is difficult to give the mesh partition in a high-dimensional space, the meshfree method has been developed. There are several meshfree methods
for the deterministic problems such as element-free Galerkin method, moving least squares method, kernel methods, RBF method, etc. The RBF method is a powerful method for scattered data interpolation problems of high-dimensional. In recent years, the RBF method has been used to solve the SPDEs, and we refer to [6–8] and references therein. In this paper, we study numerical methods for the optimal control problem governed by the convection diffusion equations with random field coefficients. A new approximation scheme is developed for the optimality system by using the RBF method for the random space and the finite volume element method for the physical space for the first time. We then derive the a priori error estimates for the state, the co-state and the control variables. Finally, some numerical tests for random dimensions are carried out to verify the theoretical results.

The plan of the paper is as follows: In Section 2, we introduce some function spaces and the stochastic optimal control problem. In Section 3, we discuss the finite dimensional truncations of the stochastic fields. In Section 4, the RBF and finite volume element approximation scheme are given for the optimal control problem. The a priori error estimates are presented for the state, the co-state and the control variables in Section 5. Finally, numerical examples are presented to illustrate our theoretical results in Section 6.

2 Notations and model control problem

2.1 Function spaces and notations

Let \( D \subset \mathbb{R}^d (d \leq 3) \) be a convex bounded polygonal spatial domain in \( \mathbb{R}^d (1 \leq d \leq 3) \) with boundary \( \partial D \) and \( B(D) \) be the Borel \( \sigma \)-algebra generated by the open subset of \( D \). Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space, where \( \Omega \) is a set of outcomes, \( \mathcal{F} \) is a \( \sigma \)-algebra of events and \( P: \mathcal{F} \to [0,1] \) is a probability measure. Throughout this paper, we use standard notations for Sobolev spaces on \( D \) as in [1]. For examples, \( L^2(D) \) and \( H^1(D) \) are Hilbert spaces with norms \( \| \cdot \|_{L^2(D)} \) and \( \| \cdot \|_{H^1(D)} \), respectively; \( H^1_0(D) \) is the subspace of \( H^1(D) \) whose function value is zero on \( \partial D \). With these standard Sobolev spaces, the stochastic Sobolev spaces are defined as follows:

\[
L^2(\Omega; H^m(D)) = \{ v: D \times \Omega \to \mathbb{R} \| v \|_{L^2(\Omega; H^m(D))} < +\infty \},
\]

with

\[
\| v \|_{L^2(\Omega; H^m(D))} = \mathbb{E}[\| v \|_{H^m(D)}^2] = \int_\Omega \| v \|_{H^m(D)}^2 dP,
\]
\[ \mathcal{L}^2(\Omega; L^2(D)) = \{ v : D \times \Omega \to \mathbb{R} \|v\|_{L^2(\Omega; L^2(D))} < +\infty \}, \]

with \[ \|v\|^2_{L^2(\Omega; L^2(D))} = \mathbb{E} \|v\|^2 = \int_{\Omega} \|v\|^2 dP. \]

Note that these stochastic Sobolev spaces are Hilbert spaces.

### 2.2 Stochastic optimal control problem

We will consider the following control problem governed by the random convection diffusion equations with constrained control:

\[ \min_{u \in K} \mathcal{J}(u) = \min_{u \in K} \mathbb{E} \left[ \frac{1}{2} \int_D (y - y_d)^2 dx + \frac{\alpha}{2} \int_D u^2 dx \right] \quad (2.1) \]

subject to

\[
\begin{cases}
- \text{div}(\mu(x,\omega) \nabla y(x,\omega)) + \vec{b}(x,\omega) \cdot \nabla y(x,\omega) \\
+ a(x,\omega) y(x,\omega) = u(x), & (x,\omega) \in D \times \Omega, \\
y(x,\omega) = 0, & (x,\omega) \in \partial D \times \Omega,
\end{cases}
\]

(2.2)

where the operator \( \nabla \) and \( \text{div} \) mean derivatives with respect to the spatial variable \( x \in D \) only, \( \mathcal{J} \) is a cost functional, \( y : D \times \Omega \to \mathbb{R} \) is the state variable, \( \vec{b}(x,\omega) : D \times \Omega \to \mathbb{R} \) is the convection velocity which is a random vector function with continuous and bounded covariance function, \( \mu : D \times \Omega \to \mathbb{R} \) is the diffusion coefficient with continuous and bounded covariance function, \( a : D \times \Omega \to \mathbb{R} \) is a random function with continuous and bounded covariance function, \( u : D \to \mathbb{R} \) is a deterministic control. The convex admissible set \( K \) is given by

\[ K = \{ u \in L^2(D) : u(x) \geq 0, \forall x \in D \}. \]

(2.3)

In the following, we take the state space \( Y = L^2(\Omega; H^1_0(D)) \) and the control space \( U = L^2(D) \). We introduce the following bilinear forms for the weak formulation of our stochastic convection diffusion PDEs

\[ A^s[y,v] = \mathbb{E} \int_D \mu \nabla y \cdot \nabla v dx + \mathbb{E} \int_D \vec{b} \cdot \nabla y \cdot v dx + \mathbb{E} \int_D ay v dx, \quad \forall y, v \in Y, \quad (2.4) \]

\[ [u,v] = \mathbb{E} \int_D uv dx, \quad \forall u \in U, \ v \in Y. \]

(2.5)
In order to guarantee the existence and uniqueness of the solution for (2.2), we assume the diffusion coefficient $\mu$ is bounded and uniformly coercive, i.e. there exist positive constants $\mu_{\text{min}}$ and $\mu_{\text{max}}$ such that

$$
\mu_{\text{min}} \leq \mu(x, \omega) \leq \mu_{\text{max}}, \quad \text{a.e. } \omega \in \Omega. \quad (2.6)
$$

We also assume that there exists a positive constant $a_0 > 0$ such that the following coercivity condition holds:

$$
a(x, \omega) - \frac{1}{2} \text{div} \bar{b}(x, \omega) \geq a_0 > 0, \quad \text{a.e. } \omega \in \Omega. \quad (2.7)
$$

Then, with the assumption (2.6), the existence and uniqueness for the solution $y$ of (2.2) can be proved [23]. Then, we can derive the weak formulation for the state equation (2.2) as follows: find $y \in Y$, such that

$$
A^s[y, v] = [u, v], \quad \forall v \in Y. \quad (2.8)
$$

Therefore, the optimal control problem (2.1)-(2.2) can be restated as:

$$
\min_{u \in K} J(u) = \min_{u \in K} \mathbb{E} \left[ \frac{1}{2} \int_D (y - y_d)^2 dx + \frac{\alpha}{2} \int_D u^2 dx \right] \quad (2.9)
$$

subject to

$$
A^s[y, v] = [u, v], \quad \forall v \in Y. \quad (2.10)
$$

3 Finite dimensional representation of optimal control problem

In this paper, we use the truncated Karhunen-Loève(K-L) expansion to construct the stochastic field $\mu(x, \omega), a(x, \omega), \bar{b}(x, \omega)$ with a finite number of random variables, and convert the stochastic optimal control problem to a finite dimensional deterministic PDEs constrained optimization problem.

3.1 K-L expansions and finite expansion of random field

As supposed in Section 2, the random fields $\mu(x, \omega), a(x, \omega), \bar{b}(x, \omega)$ have the continuous and bounded covariance function. Taking $\mu(x, \omega)$ for example, we can define the covariance as

$$
\text{Cov}[\mu](x_1, x_2) = \mathbb{E}[(\mu(x_1, \cdot) - \mathbb{E}[\mu](x_1))(\mu(x_2, \cdot) - \mathbb{E}[\mu](x_2))], \quad \forall x_1, x_2 \in D.
$$
From the theory of Karhunen-Loeve expansion (KLE) [20, 21], we can give a expansion of the stochastic coefficient \( \mu \) based on random variables as follows.

\[
\mu(x, \omega) = \mathbb{E}[\mu](x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n} \phi_n(x) \xi_n(\omega), \quad \forall x \in D,
\]

where the real random variables \( \{\xi_n\}_{n=1}^{\infty} \) are mutually uncorrelated with zero mean and unit variance, and \( \{\lambda_n, \phi_n(x)\}_{n=1}^{\infty} \) denotes the sequence of eigen pairs corresponding to \( \text{Cov}[\mu](x_1, x_2) \), which implies that

\[
\int_D \text{Cov}[\mu](x_1, x_2) \phi_n(x_1) dx_1 = \lambda_n \phi_n(x_2).
\]

The eigenvalues \( \{\lambda_n\}_{n=1}^{\infty} \) form a sequence of nonnegative real numbers decreasing to zero with \( \sum_{n=1}^{\infty} \lambda_n = \int_D \text{Var}[\mu](x) dx \), and the corresponding eigenfunctions \( \{\phi_n\}_{n=1}^{\infty} \) form a complete orthonormal basis in \( L^2(D) \). Then the truncated K-L expansion of the stochastic process \( \mu(x, \xi) \) is defined by

\[
\mu_N(x, \omega) = \mathbb{E}[\mu](x) + \sum_{n=1}^{N} \sqrt{\lambda_n} \phi_n(x) \xi_n(\omega), \quad \forall x \in D.
\]

Also, we have

\[
\sup_{x \in D} \mathbb{E}[(\mu - \mu_N)^2](x) = \sup_{x \in D} \sum_{n=N+1}^{\infty} \lambda_n \phi_n^2(x) \rightarrow 0, \quad N \rightarrow \infty.
\]

### 3.2 Finite expansion assumption of random field

As shown in Section 3.1, a random field \( \mu(x, \omega), a(x, \omega), \vec{b}(x, \omega) \) can be approximated just by a few of uncorrelated random variables. And it is necessary to reduce the infinite expansion to a finite expansion for the numerical methods. Here we use the finite noise assumption to construct \( \mu(x, \omega), a(x, \omega), \vec{b}(x, \omega) \) as follows:

\[
\begin{align*}
\mu &= \mu(x, \omega) = \mu(x, \xi(\omega)) = \mu(x, \xi_1(\omega), \xi_2(\omega), \ldots, \xi_N(\omega)), \\
a &= a(x, \omega) = s(x, \xi(\omega)) = a(x, \xi_1(\omega), \xi_2(\omega), \ldots, \xi_N(\omega)), \\
\vec{b} &= \vec{b}(x, \omega) = \vec{b}(x, \xi(\omega)) = \vec{b}(x, \xi_1(\omega), \xi_2(\omega), \ldots, \xi_N(\omega)),
\end{align*}
\]

where the real random variables \( \{\xi_n(\omega)\}_{n=1}^{N} \) are mutually independent with zero mean and unit variance. And we assume that the images \( \Gamma_n = \xi_n(\Omega), n=1,2,\ldots,N \)
are bounded intervals in $\mathbb{R}$ and that each $\xi_n$ has a density function $\rho_n: \Gamma \rightarrow \mathbb{R}^+$ for $n = 1, 2, \cdots, N$. Then we use the notation $\rho(\xi) = \prod_{n=1}^{N} \rho_n(\xi_n)$ for the joint probability density function of random vector $\xi$ with the support $\Gamma = \prod_{n=1}^{N} \subset \mathbb{R}^N$.

### 3.3 Finite dimensional representation of control problem

Following from the above assumptions, we can reformulate the stochastic optimal control problem (2.1)-(2.2) as a deterministic PDE-constrained optimization problem as follows:

$$
\min_{u \in K} J(u) = \min_{u \in K} \int_{\Gamma} \left( \frac{1}{2} \int_{D} (y - y_d)^2 \, dx + \frac{\alpha}{2} \int_{D} u^2 \, dx \right) \rho(\xi) \, d\xi
$$

subject to

$$
A^s[y, v]_{\rho} = [u, v]_{\rho}, \quad \forall u \in U, \ v \in Y_{\rho},
$$

We redefine the probabilistic Hilbert space $L^2_{\rho}(\Gamma; H^1_0(D))$ and $L^2_{\rho}(\Gamma; L^2(D))$ in which the random processes based upon the random variables $\xi$ reside. By denoting the deterministic state space $Y_{\rho} = L^2_{\rho}(\Gamma; H^1_0(D))$, and from (2.4)-(2.5), we have

$$
A^s[y, v]_{\rho} = \int_{\Gamma} \int_{D} \mu(x, \xi) \nabla y(x, \xi) \cdot \nabla v(x, \xi) \, dx \rho(\xi) \, d\xi
$$

$$
+ \int_{\Gamma} \int_{D} a(x, \xi) yv \, dx \rho(\xi) \, d\xi, \quad \forall y, v \in Y_{\rho},
$$

and

$$
[u, v]_{\rho} = \int_{\Gamma} \int_{D} uv \, dx \rho(\xi) \, d\xi, \quad \forall u \in U, \ v \in Y_{\rho}.
$$

Then the optimal control problem (3.1)-(3.2) can be reformulated as

$$
\min_{u \in K} J(u) = \min_{u \in K} \int_{\Gamma} \left( \frac{1}{2} \int_{D} (y - y_d)^2 \, dx + \frac{\alpha}{2} \int_{D} u^2 \, dx \right) \rho(\xi) \, d\xi
$$

subject to

$$
A^s[y, v]_{\rho} = [u, v]_{\rho}, \quad \forall v \in Y_{\rho}.
$$
Then it is trivial to prove that the optimal control problem (3.5)-(3.6) has a unique solution \((y, u) \in Y_\rho \times K\). Furthermore, the pair \((y, u)\) is the solution of (3.5)-(3.6) iff there is a co-state variable \(p \in Y_\rho\), such that the triplet \((y, p, u)\) satisfies the following optimality system:

\[
\begin{align*}
A^s[y, v]_\rho &= [u, v]_\rho, \quad \forall v \in Y_\rho, \\
A^a[p, q]_\rho &= [y - y_d, q]_\rho, \quad \forall q \in Y_\rho, \\
[p + \alpha u, w - u]_\rho &\geq 0, \quad \forall w \in K,
\end{align*}
\]

where

\[
A^a[p, q]_\rho = \int_\Gamma \int_D \mu(x, \xi) \nabla p \cdot \nabla q dx \rho(\xi) d\xi - \int_\Gamma \int_D (\tilde{b}(x, \xi) \cdot \nabla p) \cdot q dx \rho(\xi) d\xi + \int_\Gamma \int_D (a(x, \xi) - \text{div} \tilde{b}) pq dx \rho(\xi) d\xi.
\]

4 Stochastic Galerkin using RBF and finite volume element method

In this section, the stochastic Galerkin using the RBF is considered to discretize the optimality system (3.7) in the random space and the finite volume element method is used to discretize the optimality system (3.7) in the physical space. First of all, we consider the finite volume element approximation defined on spatial domain \(D \subset \mathbb{R}^d\). Let \(T_h\) be a family of regular partition of \(D\), such that \(\bar{D} = \bigcup_{\tau \in T_h} \bar{\tau}\). Let \(h = \max_{\tau \in T_h} h_\tau\), where \(h_\tau\) denotes the diameter of the element \(\tau\). For the given partition \(T_h\), we construct the dual mesh \(T^*_h\) whose elements are called control volumes. Each element \(\tau \in T_h\) can be divided into sub-domains by connecting an midpoints point of the element. Around each \(P_i \in D\), we associate a control volume \(\tau^*_i = \tau^*_p\), which consists of the union of subregions with \(P_i\) as a vertex. For the vertex \(P_i \in \partial D\), we can define its control volume in a similar way. Then we define the dual partition \(T^*_h = \{\tau^*_p, P_i \in D\}\) to be the union of all the control volumes. We choose the inner points as the barycenters and assume all the inner angles of each element are not larger than \(\frac{\pi}{2}\).

Associated with \(T_h\) and \(T^*_h\), we define the following finite dimensional space:

\[
V_{h} = \{v \in C(D) : v|_{\tau} \text{ is linear for all } \tau \in T_h \text{ and } v|_{\partial D} = 0\},
\]

it is easy to see that \(V_{h} \subset Y = H^1_0(D)\), and its dual volume element space \(V^*_{h}\)

\[
V^*_{h} = \{v \in L^2(D) : v|_{\tau^*_i} \text{ is constant for all } \tau^*_p \in T^*_h \text{ and } v|_{\partial D} = 0\}.
\]
Let \( W_{h_s} \subset L^2(D) \) be another finite element space defined on \( T_h \),

\[
W_{h_s} = \{ v \in L^2(D) : v|_\tau \text{ is constant for all } \tau \in T_h \}.
\]

Set \( K^h = W_{h_s} \cap K \), and we have \( K^h \subset K \).

In the following, we consider a finite dimensional space defined on \( \Gamma \subset \mathbb{R}^N \). Let \( \Xi = \{ \xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(r)} \} \) be a set of distinct points in \( \Gamma \). We give the approximation of stochastic function on \( \Gamma \) from RBFs as

\[
S_{h_r} = \text{span}\{ \Phi(\cdot - \xi^{(1)}), \Phi(\cdot - \xi^{(2)}), \ldots, \Phi(\cdot - \xi^{(r)}) + P_1^N \},
\]

where \( \Phi : \mathbb{R}^N \rightarrow \mathbb{R} \) is at least a \( C^1 \) function, \( P_1^N \) denotes the space of polynomials of degree less than \( l \). Then the finite dimensional spaces \( Y_h = S_{h_r} \times V_{h_s} \) and \( U_h = W_{h_s} \) are used to approximate spaces \( Y_\rho \) and \( U \), respectively.

Multiplying (3.2) by test function \( z \in Y_h \) and integrating by parts yield

\[
B_1[b;y,z]_\rho + B_2[b;y,z]_\rho + C[\mu;w,z]_\rho + D[a;y,z]_\rho = [u;z]_\rho, \quad \forall z \in V_{h_s}^* \times L^2_\rho(\Gamma),
\]

where

\[
\begin{align*}
B_1[b;y,z]_\rho &= -\int_{\Gamma} \sum_{P_i \in D} z(P_i) \cdot \int_{\tau_i^*} (\text{div } b) \omega dx \rho(\xi) d\xi, \\
B_2[b;y,z]_\rho &= \int_{\Gamma} \sum_{P_i \in D} z(P_i) \cdot \int_{\partial \tau_i^*} (\bar{b} \cdot v) \omega ds \rho(\xi) d\xi, \\
C[\mu;w,z]_\rho &= -\int_{\Gamma} \sum_{P_i \in D} z(P_i) \cdot \int_{\partial \tau_i^*} \mu (\nabla w \cdot v) ds \rho(\xi) d\xi, \\
D[a;y,z]_\rho &= \int_{\Gamma} \sum_{P_i \in D} z(P_i) \int_{\tau_i^*} a \omega dx \rho(\xi) d\xi,
\end{align*}
\]

here \( \nu \) is the unit outward normal vector of \( \partial \tau_i^* \).

Letting \( \phi_i(x) \) and \( \psi_i(x) \) be the nodal basis functions of \( V_{h_s} \) and the characteristic function of \( V_{h_s}^* \) respectively, we define \( I_h \) and \( I_{h_s}^* \) are the interpolation operators from \( C(D) \) to \( V_{h_s} \) and \( V_{h_s}^* \) respectively. Then for any \( v \in C(D) \),

\[
I_h v = \sum_{P_i \in D} v(x_i) \phi_i(x),
\]

and

\[
I_{h_s}^* v = \sum_{P_i \in D} v(x_i) \psi_i(x).
\]
Then the stochastic Galerkin with RBF and finite volume scheme of (3.2) is as follows:

\[ A_h^\rho[y_h, I_h^\rho w_h] = B_1[y_h, I_h^\rho w_h] + B_2[y_h, I_h^\rho w_h] + C[y_h, I_h^\rho w_h] + D[y_h, I_h^\rho w_h] \]

Finally, we use the stochastic Galerkin method with RBFs and finite volume element to approximate the state equation and co-state equation in (3.7), and use the stochastic Galerkin method with RBFs and standard finite element to approximate the inequality in (3.7). The approximation scheme of the optimality conditions (3.7) in the physical space reads: find \( y_h \in Y_h, p_h \in Y_h, u_h \in K^h \) such that

\[
\begin{align*}
A_h^\rho[y_h, I_h^\rho w_h] &= [u_h, I_h^\rho w_h], \quad \forall w_h \in Y_h, \\
A_h^\rho[p_h, I_h^\rho q_h] &= [y_h - y_d, I_h^\rho q_h], \quad \forall q_h \in Y_h, \\
[a u_h + p_h, v_h - u_h] &\geq 0, \quad \forall v_h \in K^h,
\end{align*}
\]

where

\[ A_h^\rho[p_h, I_h^\rho q_h] = B_1[p_h, I_h^\rho q_h] + B_2[-\tilde{b}; p_h, I_h^\rho q_h] + C[p_h, I_h^\rho q_h] + D[a - \text{div} \tilde{b}; p_h, I_h^\rho q_h]. \]

Following from [11], we have that there exists a unique solution for the scheme (4.5).

5 A priori error estimation

In this section, we will derive the error estimates between the optimal control condition (3.7) and the stochastic Galerkin method with the RBF and finite volume element approximation (4.5). From [31], in order to obtain the separate error estimates in \( D \) and \( \Gamma \), we define the following projection operators. Let \( R_{h_s}: H^1_0(D) \rightarrow V_{h_s} \) be the \( H^1_0(D) \)-projection operator as

\[ (R_{h_s} r, r_{h_s})_{H^1_0(D)} = (r, r_{h_s})_{H^1_0(D)}, \quad \forall r_{h_s} \in V_{h_s}, \quad \forall r \in H^1_0(D). \]

And for all \( r \in H^2(D) \cap H^1_0(D) \),

\[ \|r - R_{h_s} r\|_{H^1_0(D)} \leq C_{h_s} \|r\|_{H^2(D)}. \]
Lemma 5.1. Let \(|z|\) for RBFs and using the Fourier transform, we have for each \(z \in H^r(\Gamma)\)

\[
(\pi_{h_r} z, z_{h_r})_{L^2_0(\Gamma)} = (z, z_{h_r})_{L^2_0(\Gamma)}, \quad \forall z_{h_r} \in S_{h_r}, \quad \forall z \in L^2_0(\Gamma).
\]

From [34] for RBFs and using the Fourier transform, we have for each \(z \in H^r(\Gamma)\)

\[
\|z - z_{h_r}\|_{H^i(\Gamma)} \leq C h_r^{j-i} \|z\|_{H^j(\Gamma)},
\]

where \(h_r = \sup_{\xi \in \Gamma} (1 \leq j \leq (r)) \|\xi - \xi^{(i)}\|_2\) is sufficiently small.

**Lemma 5.1.** Let \((y, p, u)\) and \((y_{h_r}, p_{h_r} u_{h_r})\) be the solution of (3.7) and (4.5), respectively. Assume that \(K^h \subset K, u \in H^1(D)\). Then we have

\[
\|u - u_{h_r}\|_{L^2(D)} \leq C (h_s + \|p(u_{h_r}) - p_{h_r}\|_{L^2_0(\Gamma, L^2(D))}),
\]

where \((p(u_{h_r}) \in Y_{p_r}, y(u_{h_r}) \in Y_{p_r})\) is the solution of the following equations:

\[
A^2[y(u_{h_r}), w]_{p_r} = [u_{h_r}, w]_{p_r}, \quad \forall w \in Y_{p_r},
\]

(5.4)

\[
A^2[p(u_{h_r}), q]_{p_r} = [y(u_{h_r}) - y_{d_r}, q]_{p_r}, \quad \forall q \in Y_{p_r}.
\]

(5.5)

**Proof.** Let \(u_1 \in K^h\) be the integral average of \(u\) on each element such that

\[
u_1|_{\tau} = \frac{\int_{\tau} u dx}{\int_{\tau} 1 dx}.
\]

It follows from (3.7) and (4.5) that

\[
\begin{align*}
&c \|u - u_{h_r}\|^2_{L^2(D)} \\
&\leq \int_D \int_D \alpha(u - u_{h_r})^2 d\rho d\xi \\
&= [\alpha(u - u_{h_r}), u - u_{h_r}]_p \\
&\leq [\alpha(u - u_{h_r}), u - u_{h_r}]_p + [y - y(u_{h_r}), y - y(u_{h_r})]_p \\
&= [\alpha(u - u_{h_r}), u - u_{h_r}]_p + A^2[p - p(u_{h_r}), y - y(u_{h_r})]_p \\
&= [\alpha(u - u_{h_r}), u - u_{h_r}]_p + A^2[y - y(u_{h_r}), p - p(u_{h_r})]_p \\
&= [\alpha(u - u_{h_r}), u - u_{h_r}]_p + [u - u_{h_r}, p - p(u_{h_r})]_p.
\end{align*}
\]
where $\delta$ is an arbitrary positive number. Note that $u_l \in U^h$ be the integral average of $u$ on each element, then it is easy to obtain that

$$
\|u-u_l\|_{L^2(D)} \leq Ch_2 \|u\|_{H^1(D)},
$$

(5.7)

Moreover, since $u \in H^1(D)$ and $p \in L^2(\Gamma; H^1(D))$, we have

$$
[p+au,u_l-u]_\rho = \int_\Gamma \sum_{\tau \in T^h} \int_{\tau} (p+au)(u_l-u)d\sigma(\xi)d\xi
$$

$$
= \int_\Gamma \sum_{\tau \in T^h} \int_{\tau} (p+au-(p+au)_1)(u_l-u)d\sigma(\xi)d\xi
$$

$$
\leq C \|p+au-(p+au)_1\|_{L^2(\Gamma; L^2(D))} \|u_l-u\|_{L^2(D)}
$$

$$
\leq Ch_2 \|p+au\|_{L^2(\Gamma; H^1(D))} \|u\|_{H^1(D)}
$$

$$
\leq Ch_2^2.
$$

(5.8)

Then it follows from (5.6)-(5.8) that

$$
\|u-u_h\|_{L^2(D)}^2 \leq Ch_2^2 + C \|p(u_h) - p_h\|_{L^2(\Gamma; L^2(D))}^2
$$

$$
+ C\delta \|p(u_h) - p\|_{L^2(\Gamma; L^2(D))}^2 + C\delta \|u-u_h\|_{L^2(D)}^2,
$$

(5.9)

From (3.7), (5.5) and (2.7), we can obtain that

$$
A^a[p-p(u_h),q]_\rho = [y-y(u_h),q]_\rho, \quad \forall q \in Y_\rho.
$$

Let $q = p-p(u_h)$, we get that

$$
\|p-p(u_h)\|_{L^2(\Gamma; H^1(D))} \leq C \|y-y(u_h)\|_{L^2(\Gamma; L^2(D))},
$$
Similarly, we can prove
\[
\|y - y(u_h)\|_{L^2_h(\Gamma;H^1(D))} \leq C\|u - u_h\|_{L^2(D)}.
\]

Therefore, we have
\[
\|p - p(u_h)\|_{L^2_h(\Gamma;L^2(D))} \leq C\|u - u_h\|_{L^2(D)}.
\]

Thus set \(\delta = \frac{1}{\sqrt{C}}\), from (5.9) and (5.10), we have
\[
\|u - u_h\|_{L^2(D)}^2 \leq Ch^2_s + C\|p(u_h) - p_h\|^2_{L^2_h(\Gamma;L^2(D))}.
\]

This proves (5.3).

Using Lemma 5.1, we obtain the following theorem.

**Theorem 5.1.** Let \((y, p, u)\) be the solution of the optimal control problem (3.7), \((y_h, p_h, u_h)\) be the solution of the discretized problem (4.5). Under the conditions in Lemma 5.1, the following a priori error estimate holds:
\[
\|y - y_h\|_{L^2_h(\Gamma;H^1(D))} + \|p - p_h\|_{L^2_h(\Gamma;H^1(D))} + \|u - u_h\|_{L^2(D)} \leq Ch_s + Ch_r.
\]

**Proof.** From coercivity condition (2.7) and coercion of \(A^a(\cdot, \cdot)\), we have that
\[
C\|p(u_h) - p_h\|_{L^2_h(\Gamma;H^1(D))} \leq A^a[p(u_h) - p_h, p(u_h) - p_h],
\]

\[
= A^a[p(u_h) - p_h, (p(u_h) - p_h) - \pi_h(p(u_h) - p_h)]
\]

\[
+ A^a[p(u_h) - p_h, \pi_h(p(u_h) - p_h) - R_{h\ast}\pi_h(p(u_h) - p_h)]
\]

\[
+ A^a[p(u_h) - p_h, R_{h\ast}\pi_h(p(u_h) - p_h) - I_{h\ast}^r R_{h\ast}\pi_h(p(u_h) - p_h)]
\]

\[
+ A^a[p(u_h) - p_h, I_{h\ast}^r R_{h\ast}\pi_h(p(u_h) - p_h)].
\]

Using the results of [14] and the estimation for \(R_{h\ast}\) and \(\pi_{h\ast}\) in (5.1) and (5.2), we obtain the following error estimate:
\[
\|p(u_h) - p_h\|_{L^2_h(\Gamma;H^1(D))} \leq Ch_s + Ch_r + C\|y(u_h) - y_h\|_{L^2_h(\Gamma;H^1(D))}.
\]

Similarly, for the term \(y(u_h) - y_h\), we have
\[
\|y(u_h) - y_h\|_{L^2_h(\Gamma;H^1(D))} \leq Ch_s + Ch_r.
\]

Using (5.13)-(5.14) and Lemma 5.1, the estimate (5.11) can be obtained.
6 Numerical examples

In this section, a numerical example similar as the one given in [31] will be performed. We consider the optimal control problem of SPDEs (2.1) with physical space $D = [-1, 1]$, random space $\Gamma = \prod_{i=1}^{N} \Gamma_i = [-\sqrt{3}, \sqrt{3}]^N$ and the observation $y_d = \sin(4\pi x)$. Moreover, we assume that each probability density function on $\Gamma_i$ is uniform with $\rho_i(\xi_i) = \frac{1}{2\sqrt{3}}$, and that the joint probability density function $\rho(\xi)$ of $(\xi_1, \xi_2, \cdots, \xi_N)$ is $\frac{1}{(2\sqrt{3})^N}$. As the K-L expansion given in [13] and for simple, we consider the same expansion for the random coefficients as $\mu(x, \xi) = b(x, \xi) = a(x, \xi)$. The covariance function is given as $\text{Cov}(\mu)(x_1, x_2) = e^{-|x_1 - x_2|}$ and the eigenpairs $(\lambda_i, \phi_i)_{1 \leq i \leq N}$ is obtained by solving the corresponding eigenvalue problem:

$$\int_D e^{-|x_1 - x_2|} \phi_i(x_1) dx_1 = \lambda_i \phi_i(x_2).$$

In the following computation, we use the linear finite volume element given in [14] to discretize the physical space and use the RBFs given as (4.1) to discretize the stochastic space, where $\Phi$ is defined by $\Phi(\xi) = \phi(\|\xi - \xi_i\|_2)$, and $\|\cdot\|_2$ is the Euclidean norm and $\phi(\|\cdot\|)$ is the compactly supported functions given in [34].

In this paper, the gradient projection algorithm [31] is used to solve the stochastic optimal control (2.1). The partition for physical space and stochastic space (for two dimensional random space as an example) is shown in Fig. 1. First, we consider the two dimension random space as $N = 2$ and $\Gamma = \prod_{i=1}^{2} \Gamma_i = [-\sqrt{3}, \sqrt{3}]^2$. We choose the regular parameter $\alpha = 10^{-2}$ and Table 1 shows that $\mathbb{E}(\|y_h - y_d\|_{L^2(D)})$ and $J(u_h)$ get smaller as the partition points $L$ for stochastic space become larger and the mesh size $h_r$ for physical space become smaller.

![Figure 1: The partition for physical space (left) and stochastic space (right).](image-url)
We also consider the six dimensions random space as $N=6$ and $\Gamma = \prod_{i=1}^{6} \Gamma_i = [-\sqrt{3}, \sqrt{3}]^2$. We choose the regular parameter $\alpha = 10^{-4}$ and Table 2 shows that $\mathbb{E}(\|y_h - y_d\|_{L^2(D)})^2$ and $\mathcal{J}(u_h)$ get smaller as the partition points $L$ for stochastic space become larger and the mesh size $h_r$ for physical space become smaller.

The results presented in Theorem 5.1 shows us a description for the convergence of the stochastic Galerkin method with finite volume element and RBFs, and this result can be confirmed by the results in Table 1 and Table 2. Although it is difficult to provide the convergence order of the proposed method due to the fact that we do not know the exact solutions of the control problem, from Table 1, we see that the solution $y_h$ get closer to the observation $y_d$ as the mesh size become smaller and partition points become larger.

### 7 Conclusion and future work

In this work, we combine the RBF and finite volume element methods to solve the optimal control problem governed by the stochastic convection diffusion equations. We establish a scheme to approximate the optimality system through the discretization by the finite volume element method for the physical space, and by the RBF method for the probability space. The a priori error estimation is derived for the state, the co-state and the control variables. The numerical tests verify the
convergence of the theoretical results. Compared with the stochastic Galerkin method, the RBF method proposed in this paper could handle moderate high dimension of stochastic space $\Gamma$ and the finite volume element could easily be used for the fluid computation. Thus, it could be a good choice for approximating the control problems of stochastic fluid equations with moderate dimensional random inputs. And in the future work, we could use the RBF and other finite volume element methods to improve the efficiency of the proposed method for these problems.

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