On Convergence of Difference Approximations to Problems of Optimal Control in the Coefficients of Elliptic Equations with Mixed Derivatives and Unbounded Non-Linearity

Aigul Manapova
Bashkir State University, Zaki Validi Street 32, 450076 Ufa, Republic of Bashkortostan, Russia
E-mail: aygulrm@yahoo.com

Abstract. This paper deals with optimization problems for elliptic PDEs with mixed derivatives and nonlinearities of unlimited growth in the coefficients (i.e. the conditions imposed on the equation coefficients are assumed to hold only in a neighborhood of the exact solution to the original problem). The controls are contained in the leading part of the state equation. Convergence and approximation analysis for the optimization problems is investigated. The convergence of the approximations with regard to the state, functional and control is established, the approximations are regularized.

1. Introduction
Optimization processes take place in many fields of science and engineering. The study of such processes leads to problems that are commonly subject to partial differential equations (PDEs) depending on several parameters. While the PDE describes the underlying system, the parameters are used to identify or specify particular configurations such as material properties, the position of sensors and actuators, initial conditions, among others [1].

This paper is concerned with nonlinear optimal control problems for equations of elliptic type with mixed derivatives and unbounded nonlinearity. The coefficients of the state equation multiplying the highest derivatives are control functions. In the case of unbounded nonlinearity the state equation coefficients, depending on the exact solution, are proposed to satisfy the required properties (boundedness of $u$-derivatives, positive definiteness) only in the small neighborhood of the exact solution $u$ (see [2]-[7]).

When solving the above problems on a computer it is necessary to replace them by a finite dimensional problem. Accordingly, the main issues here are algorithms construction, their convergence analysis with regard to the state, functional, and control; and the regularization of discretizations (see, e.g., [8]-[10], [15], [17]).

There exist several papers dealing with control problems with state and control constraints. In contrast to our work, these contributions do not consider convergence issues for control problems for PDEs with controls in the leading part of the state equation along with nonlinearities of unlimited growth in the coefficients. Among recent articles we mention, for instance, the works of [1], [18]-[19] and the references given therein. In [18] the authors deal with an error analysis...
of a variational approximation of the control problem subject to a semilinear elliptic equation of the type $-\Delta y + \theta(y) = f$, $x \in \Omega$, where certain growth conditions are imposed on the nonlinearity $\theta$ of the state equation and the control is in the r.h.s. of the equation. Paper [1] is dedicated to greedy and weak greedy algorithms development in the context of optimal control problems. With the present work we complement our investigations of [19] in that we investigate the similar optimal control problems, but with different sets of admissible controls (the difference in the sets influences on how to prove convergence of approximations) and where only discretization accuracy regarding the state are estimated. The author is not aware of other convergence results for optimal control problems governed by partial differential equations with mixed derivatives and nonlinearities of unlimited growth other than those proved in this paper and those proved in [19] for a different optimization problem.

In the present paper, finite-difference approximations for optimal control problems for elliptic equations with mixed derivatives and unbounded nonlinearity are proposed. At first, the first objective is to prove their well-posedness (see §2) and analyze the approximation accuracy with respect to the state. Our studies concerning the difference schemes for this type of problems lead us to the conclusion that the convergence analysis is a challenging technical problem even for smooth solutions. This is due to the fact that the problem for the error of the grid method with respect to the state is nonlinear. In §3 we thus prove that the solution to the difference analogue of the boundary value problem is within the domain of values or its small neighborhood of values of the exact solution. This requires the convergence rate of the grid method with respect to the state for the optimization problem be analyzed in the norm $C(\omega)$. In §4, we estimate the convergence rate of the approximations with respect to the cost functional, prove a weak convergence with respect to the control, and regularize the approximations by A. N. Tikhonov [8]. It is the second objective of the paper. Note that the regularization allows to build minimizing sequences for the objective functionals of the nonlinear optimal control problems that strongly converge in control spaces to the sets of minima of functionals of the original formulations.

The nonlinear optimization models considered in this paper include, as particular versions of their formulations, a wide range of applied optimization problems of the theory of elasticity, thermal conductivity, convection-diffusion-reaction (with appropriate specification of the equations of state, control actions, constraints on controls and target functionals corresponding to optimization of a final number of a quality criteria). For example, many problems arise in the theory of elasticity such that they can be naturally formulated as problems of optimal control. In this case, the role of controlling actors in these problems can be performed, for example, by the coefficients multiplying the highest derivatives of a differential operator and defining the internal structure, i.e., the distribution of the material elastic characteristics. Differential equations involving mixed derivatives appear, in particular, in problems of heat distribution in anisotropic media, fluid flows of underground waters, in financial mathematics for option pricing in stochastic volatility models, or in numerical mathematics when coordinate transformations are applied, etc.

The results obtained will be used later while constructing numerical methods for solving the approximating optimization problems and conducting computational experiments.

2. Setting of the Optimal Control Problem and its Well-Posedness
Consider in a domain $\Omega = \{ x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_\alpha < l_\alpha, \alpha = 1, 2 \}$ with a boundary $\Gamma = \partial \Omega$ a Dirichlet problem for a second-order nonlinear differential equation:

$$
- \sum_{\alpha,\beta=1}^{2} k_{\alpha,\beta}(x) \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + q(u)u = f(u), \quad x = (x_1, x_2) \in \Omega,
$$

$$
u(x) = 0, \quad x \in \partial \Omega = \Gamma,
$$
Define the set of admissible controls:
\[ g = (k_{11}, k_{22}, k_{12}, k_{21}) \in U = \left\{ k_{\alpha \beta} \equiv g_{\alpha \beta} \in W^1_{\infty}(\Omega) : 0 < \nu \leq g_{\alpha \alpha}(x) \leq \nu, \ \alpha = 1, 2, \ g_{12} = g_{21}, \right\} \]
\[ |g_{12}(x)| \leq \nu_s = \nu - \delta_s, \ 0 < \delta_s < \nu, \ \forall x \in \Omega, \ \left| \frac{\partial g_{\alpha \beta}}{\partial x_1} \right| \leq R_1, \ \left| \frac{\partial g_{\alpha \beta}}{\partial x_2} \right| \leq R_2, \alpha, \beta = 1, 2 \].

Let an objective functional \( J : U \to \mathbb{R}^1 \) be of the form:
\[ g \to J(g) = \int_{\Omega} |u(r; g) - u_0^{(1)}(r)|^2 \, d\Omega. \]

Here, \( k_{12}(x) = k_{21}(x), \ x \in \Omega \), i.e. the symmetry condition is satisfied; \( u_0 \in W^1_2(\Omega) \) and \( q(\eta), f(\eta) \) are given functions of \( \eta \).

It is a priori assumed that the problem (1)-(2) is uniquely solvable in the class \( W^m_{2,0}(\Omega) = W^2_{2,0}(\Omega) \cap W^m_1(\Omega), 3 < m \leq 4 \) (see [7]) and the following estimate
\[ \sup_{g \in U} \|u(g)\|_{W^2_2(\Omega)} \leq M, \quad M = \text{Const} > 0 \]
holds. We denote by \( M_u = \{ u : M_1 \leq u(x) \leq M_2, x \in \Omega \} \) a range of values of the exact solution to the problem (1)-(2) (which is a bounded set by the assumption that the solution of the original problem is smooth). By a neighborhood \( D_u = (\delta - \text{neighborhood}) \) of the exact solution values domain \( M_u \), we mean the set
\[ D_u = \{ \pi : \pi_1 = M_1 - \delta \leq \pi(x) \leq M_2 + \delta = \pi_2, \ x \in K \subset \Omega, \delta > 0 \}, \]
where \( \delta > 0 \) is an arbitrary constant and can be quite small.

It is supposed that
\[ 0 \leq q_0 \leq q(\eta) \leq \eta_0, \quad |f(\eta)| \leq f_0, \ \forall \eta \in D_u; \]
\[ |q(\eta_1) - q(\eta_2)| \leq L_q|\eta_1 - \eta_2|, \quad |f(\eta_1) - f(\eta_2)| \leq L_f|\eta_1 - \eta_2|, \ \forall \eta_1, \eta_2 \in D_u; \]
\[ \begin{align*}
2\nu(\max l_\alpha)^2 & \left( \frac{\nu_0^\frac{1}{2}}{\nu_0^\frac{1}{2} - \nu_0(\max l_\alpha)^2} \right) \left( L_f + \frac{\nu f_0 L_q (\max l_\alpha)^2}{\nu_0^\frac{1}{2} - \nu_0(\max l_\alpha)^2} \right) = q_0^\frac{1}{2}, \\
\frac{\nu_0^\frac{1}{2}}{\nu_0^\frac{1}{2} - \nu_0(\max l_\alpha)^2} > 0, \quad q_1 = \frac{q_0^\frac{1}{2}}{2} < 1.
\end{align*} \]

Here, \( \nu, \nu_0, R_\alpha, \alpha = 1, 2, \ q_0, \eta_0, \ L_q, \ L_f, \ f_0 \) are given constants. Note that the conditions imposed on the equation coefficients are assumed only in a neighborhood of the exact solution values that indicates the presence of unbounded non-linearity in the coefficients.

**Remark 1.** In connection with the a priori assumption that the problem is uniquely solvable in the class \( W^m_{2,0}(\Omega), 3 < m \leq 4 \) and taking into account the embedding \( W^m_2(\Omega) \to C^2(\Omega) \), under the solution of the boundary value problem (1)-(2) we mean a function \( u = u(x), x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \ u \in W^m_{2,0}(\Omega), m > 3 \), satisfying the equation (1) in a classical sense.

Consider the question on the solvability of optimal control problem (1)-(4).

**Theorem 1.** The problem (1)-(4) is well posed in a weak sense, i.e., the cost functional \( J(g) \) is bounded from below, the set of minimizers of the cost functional \( U_* = \{ g_* : \ U(g_*) = J_* \} \) is not empty, it is weakly compact in \( H = (W^1_2(\Omega))^4 \), and any minimizing sequence \( \{ g^{(n)} \}_{n=1}^\infty \subset U \) of \( J(g) \) converges weakly in \( H \) to the set \( U_* \).

**Proof.** One can show that \( U \) is a convex, a closed and bounded subset of \( H \). \( U \) is therefore weakly compact in \( H \). Furthermore, the functional \( J(g) \) is weakly semicontinuous on \( U \). Combining these two facts regarding \( U \) and \( J(g) \) with the results of [8], p. 505, Theorem 4, we derive all assertions of Theorem 1.
3. A Difference Approximation of the Optimization Problem. Its Well-Posedness and the Approximation Accuracy with Respect to the State

We tackle the problem (1)-(4) by applying the finite difference method. In contrast to other methods (for example, finite element and finite volume approaches [11]), the application of the finite difference approximation allows utilizing more ordered data structures and having a smaller RAM footprint. Our approach can also be parallelized to reduce the runtime.

We introduce the following grids on \([0, l_α], \alpha = 1, 2\) and in \(Ω\): \(\bar{ω}_α = \{x_α = x_α^{(l_α)} = i_α h_α \in [0, l_α] : i_α = 0, N_α, N_α h_α = l_α\}\), \(\alpha = 1, 2\); \(ω_α = \bar{ω}_α \cap [0, l_α]\); \(ω_α^+ = \bar{ω}_α \cap [0, l_α]\), \(ω_α^- = \bar{ω}_α \cap [0, l_α]\), \(α = 1, 2\); \(\bar{ω} = \bar{ω}_1 \times \bar{ω}_2\); \(ω = ω_2^+ \times ω_2, ω(±2) = ω_1^+ \times ω_2^+\); \(γ = \bar{ω} \setminus ω\); \(|h|^2 = h_1^2 + h_2^2\).

Let \(V\) be a set of grid functions defined on the grid \(\bar{ω} = \bar{ω}_1 \times \bar{ω}_2\), and let \(\overset{0}{V}\) be its subset consisting of all grid functions vanishing on \(γ = \bar{ω} \setminus ω\). For grid functions from \(\overset{0}{V}\), the inner products, norms, and seminorms are defined as [11]-[14]:

\[
(y, v)_{L_2(ω)} = \sum_ω h_1 h_2 y(x) v(x),
\]

\[
\|y\|_{L_2(ω)}^2 = \|y\|^2_{L_2(ω)} = \sum_ω h_1 h_2 y^2(x),
\]

\[
\|y\|^2_{W^1_2(ω)} = \|y_{x1}\|^2_{L_2(ω)} + \|y_{x2}\|^2_{L_2(ω)} = \|y\|^2_{W^1_2(ω)} = \sum_α=1,2 \sum_ω^+(ω) h_1 h_2 y^2_{xα}, \|∇y\|^2_{L_2(ω)},
\]

\[
\|y\|^2_{W^2_2(ω)} = \|y_{x1, x1}\|^2_{L_2(ω)} + \|y_{x2, x2}\|^2_{L_2(ω)} + 2\sum_α=1,2 \sum_ω^+(ω) h_1 h_2 y^2_{xα, xα} + 2\sum_ω^+(ω) h_1 h_2 y^2_{xα, x2} = |y|^2_{W^2_2(ω)},
\]

\[
\|y\|_{C(ω)} = \|y\|_{L_∞(ω)} = \max_{ω} |y(x)|.
\]

Here, \(|·|_{W^1_2(ω)}\) and \(|·|_{W^2_2(ω)}\) denote the seminorms in \(W^1_2(ω)\) and \(W^2_2(ω)\), respectively.

Propose the following difference approximations: to minimize the grid functional

\[
J_h(Φ_h) = \sum_{x∈ω} |y(x; Φ_h) - u_0h(x)|^2 h_1 h_2 = \|y(Φ_h) - u_0h\|^2_{L_2(ω)},
\]

subject to the difference analogue of problem (1)-(2):

\[
A(y) y = -\sum_{α=1,2} \Phi_{αα}(x) y_{xα, xα} - 2Φ_{12}(x) Q(x) + q(y) y = f(y), \quad x ∈ ω,
\]

\[
y(x) = 0, \quad x ∈ γ,
\]

where

\[
Q(y)(x) = \frac{y_{x1, x2}(x) + y_{x1, x2}(x) + y_{x1, x2}(x) + y_{x1, x2}(x)}{4} = \frac{y(x_1 + h_1, x_2 + h_2) + y(x_1 - h_1, x_2 + h_2) + y(x_1 + h_1, x_2 - h_2) + y(x_1 - h_1, x_2 - h_2)}{4h_1 h_2} = y_{x1, x2}(x_1, x_2),
\]

and the grid controls

\[
Φ_h = (Φ_{11}, Φ_{22}, Φ_{12}, Φ_{21}) ∈ U_h = \{Φ_{αβ}(x) ∈ W^1_∞(ω) : 0 < ν ≤ Φ_{αα}(x) ≤ τ, α, β = 1, 2, \quad Φ_{12}(x) = Φ_{21}(x), \quad |Φ_{12}(x)| ≤ ν_α - ν - δ_α, 0 < δ_α < τ, \quad \forall x ∈ ω, \quad |Φ_{αβ}(x_1)| ≤ R_1, x ∈ [ω_1 \setminus l_1 - h_1] × ω_2, \quad |Φ_{αβ}(x_2)| ≤ R_2, x ∈ ω_1 × [ω_2 \setminus l_2 - h_2], \quad α, β = 1, 2\}.
\]
The problem (6)-(7) is a system of nonlinear equations for the grid function \( y = y(x) \), \( x \in \mathbb{R} \). In this section we show that the numerical solution of such boundary value problems can be effectively carried out using iterative methods. To problem (6)-(7) we associate the following iterative process:

\[
A(y^{(s)})y^{(s+1)} = \sum_{a=1}^{2} \Phi_{\alpha a}(x)y_{x_{\alpha},a}^{(s+1)} - 2\Phi_{12}(x)Q(y^{(s+1)}) + q(y^{(s)})y^{(s+1)} = f(y^{(s)}), x \in \omega, \tag{9}
\]

\[
y^{(s+1)}(x) = 0, \quad x \in \gamma. \tag{10}
\]

Moreover, we get estimates for the grid solution to scheme (6)-(7) in various grid norms:

\[
\delta \quad \text{and we have the estimate of the rate of convergence:}
\]

\[
\| y^{(s)} - y \|_{C(\omega)} \leq \frac{\beta}{1 - q_1^*} (q_1^*)^s.
\]

Moreover, we get estimates for the grid solution to scheme (6)-(7) in various grid norms:

\[
\| y \|_{W^2_2(\omega)} \leq C_5, \quad \| y \|_{C(\omega)} \leq C_2 C_5, \quad \| y \|_{L_2(\omega)} \leq C_3 C_5, \quad \| y \|_{W^2_2(\omega)} \leq C_4 C_5,
\]

where positive constants \( C_k, k = 2, 3, 4, 5 \) are defined as:

\[
C_2 = \frac{(\max l_\alpha)^2}{2(l_1 l_2)^{1/2}}, \quad C_3 = \frac{(\max l_\alpha)^2}{2}, \quad C_4 = \left( \frac{l_1^2 + l_2^2}{32} \right)^{1/2},
\]

\[
C_5 = C_0 f_0 (l_1 l_2)^{1/2}, \quad C_0 = \frac{2\mu}{\nu^2 - \mu q_0 (\max l_\alpha)^2}.
\]

**Theorem 3.** The sequence of grid (difference) optimization problems (5)-(8) is well-posed in a weak sense, i.e.,

\[
J_{h^*} = \inf \{ J_h(\Phi_h) : \Phi_h \in U_h \} > -\infty, \quad U_{h^*} = \{ \Phi_{h^*} \in U_h : J_h(\Phi_{h^*}) = J_{h^*} \} \neq \emptyset.
\]
This theorem is proved similarly to Theorem 1 under the condition that (5)-(8) is a grid analogue of problem (1)-(4) (see also [8], p. 502, Theorem 1 and the comments on pp. 565-566).

In the following theorem we analyze approximation convergence with respect to the state.

**Theorem 4.** Suppose that the solution to problem (1)-(2) belongs to the class $W^m_2(\Omega)$, $3 < m \leq 4$, and let the initial approximation $y^{(0)}$ of the iterative process (9)-(10) be in $S_u^*$.

Then, under the conditions from Section 1, for sufficiently small $h < h_0$, we establish accuracy estimate of the grid method with respect to the state:

$$
\|z\|_{W^2_0(\omega)} = \|y - u\|_{W^2_0(\omega)} \leq C_6|h|^{m-2}\|u\|_{W^m_2(\Omega)} +
$$

$$+ C_7\left\{ \sum_{\alpha=1}^2 \|k_{2\alpha} - \Phi_{2\alpha}\|_{L^\infty(\omega)} + \|k_{12} - \Phi_{12}\|_{L^\infty(\omega)} \right\} \|u\|_{W^2_0(\omega)}, \quad 3 < m \leq 4,
$$

where $C_6$ and $C_7$ are independent of $h$ and $u(x)$.

The proofs of Theorems 2 and 4 are based on applying the iterative process (9)-(10). It is important to mention that proving the convergence of iterative methods in the case of unbounded nonlinearities can be a very difficult task in itself.

4. Convergence Analysis of the Approximating Optimization Problems.

Regularization of Approximations

We devote this section to the discussion underpinning the general method of studying approximating optimal control problems.

**Theorem 5.** Let $g \in U$ and $\Phi_h \in U_h$ be arbitrary controls, and let $u(g)$ and $y(\Phi_h)$ be the corresponding solutions of state problems in (1)-(4) and (5)-(8). Then, the following statements are fulfilled:

1) the family of grid problems (5)-(8) depending on the mesh size $h$ of the grid $\mathcal{W}$ approximates the original optimization problem (1)-(4) with respect to the functional as $|h| \to 0$, and we establish the convergence rate estimate

$$
|J_{h^*} - J_*| \leq M|h|,
$$

where $J_* \text{ and } J_{h^*}$ are the infima of the functionals $J(g) \text{ and } J_h(\Phi_h)$, respectively.

2) the approximations (5)-(8) converge weakly to original problem (1)-(4) with respect to control, namely, if $\{\Phi_{h^*}\} \subset U_h$ is a sequence of grid controls determined by conditions

$$
J_{h^*} \leq J_h(\Phi_{h^*}) \leq J_{h^*} + \epsilon_h, \quad \Phi_{h^*} \in U_h,
$$

where the sequence $\epsilon_h \geq 0$, $\epsilon_h \to 0$ as $|h| \to 0$, and characterizes the accuracy of solving the minimization problem for the grid functional $J_h(\Phi_h)$, then the sequence of controls $F_h\Phi_{h^*}$ is a minimizing sequence for the cost functional $J(g)$, i.e. $\lim J(F_h\Phi_{h^*}) = J_*$ as $|h| \to 0$ and we have the convergence rate estimate

$$
0 \leq J(F_h\Phi_{h^*}) - J_* \leq C|h| + \epsilon_h,
$$

where $\{F_h\Phi_{h^*}\} = \{F_{abh}\Phi_{ah}\}_{a=1}^4$ is a sequence of piecewise constant extensions of grid controls $\Phi_{ah}, \alpha = 1,4$. Moreover, the sequence $\{F_h\Phi_{h^*}\}$ converges weakly in $H$ to the set $U_* = \{g \in U : J(g) = J_*\} \neq \emptyset$.

**Proof.** The statements are proved by using the above results and applying the ideas of [8], [15]. The foremost item of the technique here is to construct mappings $R_h : H \to H_h$ and $N_h : H_h \to H$, and study their properties. In particularly, we define the mappings as $R_h \Phi = \Phi_h,$
where \( g = (k_{11}, k_{22}, k_{12}, k_{21}) \), and \( N_h \Phi_h = g \), where \( g = (F_{1h} \Phi_{11}, F_{2h} \Phi_{22}, F_{3h} \Phi_{12}, F_{4h} \Phi_{21}) \),

where \( R^\alpha_h \), \( \alpha = 1, 4 \), - some discretizations of the functions of continuous argument, and \( F^\alpha_h \), \( \alpha = 1, 4 \), - some piecewise constant extensions of the grid functions.

We further note that optimization problem (1)-(4) is ill-posed minimization problem in Tikhonov’s terminology. Similarly to paper [16], pp.1722–1723, a Tikhonov regularization algorithm is applied to regularize the family of grid problems (5)-(8). It generates a minimizing sequence (based on the difference approximations) that converges strongly to the set of \( \Omega \)-normal optimal controls in problem (1)-(4). The set of \( \Omega \)-normal solutions of (1)-(4) is defined as \( U^{\ast \ast} = \{ g^{\ast \ast} \in U_{\ast} : \Omega(g^{\ast \ast}) = \inf \{ \Omega(g_{\ast}) : g_{\ast} \in U_{\ast} \} = \Omega_{\ast} \} \), where \( \Omega(g) = \| g \|^2_H \), \( g \in U \), is a stabilizing functional.

References

[1] Hernández-Santamaría V, Lazar M and Zuazua E, Greedy optimal control for elliptic problems and its application to turnpike problems 2019 Numerische Mathematik 141 2 455–493

[2] Abrashin V N, Difference schemes for nonlinear hyperbolic equations 1975 Differ. Uravn. 11 2 294–308

[3] Abrashin V N and Asmolik V A, Locally one-dimensional difference schemes for multidimensional quasilinear hyperbolic equations 1982 Differ. Uravn. 18 7 1107–17

[4] Matus P P, Unconditional convergence of some difference schemes for gas dynamics problems 1995 Differ. Uravn. 21 7 122738

[5] Matus P P, Moskalkov M N and Shcheglik V S, Consistent estimates for rate of convergence of the grid method for a second-order nonlinear equation with generalized solutions 1995 Differ. Uravn. 31 7 1198–1206

[6] Shcheglik V S, An analysis of a difference scheme approximating a third boundary-value problem for a second-order nonlinear differential equation 1997 Comput. Math. Math. Phys. 37 8 919–925

[7] Lubyshev F V, Fairuzov M E and Manapova A R, Accuracy of difference schemes for nonlinear elliptic equations with non-restricted nonlinearity 2017 Zhurnal SVMO 19 3 41–52

[8] Vasilev F P 2002 Optimization Methods (Moscow: Faktorial)

[9] Potapov M M 1985 Approximation of Optimization Problems in Mathematical Physics (Hyperbolic Equations) (Moscow: Mosk. Gos. Univ.)

[10] Ishmukhametov A Z 1999 Stability and Approximation of Optimal Control Problems (Moscow: Vychisl. Tsentr Ross. Akad. Nauk)

[11] Samarskii A A 2001 The Theory of Difference Schemes (New York: Marcel Dekker)

[12] Samarskii A A and Andreev V B 1976 Difference Schemes for Elliptic Equations (Moscow: Nauka)

[13] Samarskii A A and Vabishchevich P N 2009 Computational Heat Transfer (New York: Wiley)

[14] Samarskii A A, Lazarov R D and Makarov V L 1987 Difference Schemes for Differential Equations with Weak Solutions (Moscow: Vysshaya Shkola)

[15] Lubyshev F V and Manapova A R On some optimal control problems and their finite difference approximations and regularization for quasilinear elliptic equations with controls in the coefficients 2007 Comput. Math. Math. Phys. 47 3 361–380

[16] Lubyshev F V, Manapova A R and Fairuzov M E, Approximations of optimal control problems for semilinear elliptic equations with discontinuous coefficients and solutions and with control in matching boundary conditions 2014 Comput. Math. Math. Phys. 54 11 1700–24

[17] Lubyshev F V, Manapova A R, An approximation of optimal control on the coefficients of elliptic convection-diffusion equations with an imperfect contact matching condition 2019 Zhurnal SVMO 21 2 187–214

[18] Ali A A and Deckelnick K and Hinze M, Error analysis for global minima of semilinear optimal control problems 2017 arXiv preprint arXiv:1705.01201

[19] Manapova A, Lubyshev F, About convergence of difference approximations for optimization problems described by elliptic equations with mixed derivatives and unbounded nonlinearity 2018 AIP Conf. Proc. 1997 020011-1 –020011-5