ISOMORPHISM CLASSES OF FINITE DIMENSIONAL CONNECTED HOPF ALGEBRAS IN POSITIVE CHARACTERISTIC

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Abstract. We classify all finite-dimensional connected Hopf algebras with large abelian primitive space. We show that they are extensions of restricted enveloping algebras with some restrictions. For a fixed action, we construct a cohomological type group, which classifies all the extensions up to equivalence. Moreover, we present a 1-1 correspondence between the isomorphism classes and a group quotient of the cohomology type group deleting some bad points, where the group action respects the automorphisms of the extension and the bad points respect those restricted Lie algebra extensions.

1. Introduction

1.1. Background. The study of connected Hopf algebras of finite Gelfand-Kirillov dimension (GK-dimension), over an algebraically closed field of characteristic 0, was initiated by Zhuang in [35]. The classification is well known for such Hopf algebras of GK-dimension less than or equal to 4, see [3, 4, 35, 27]. For continued and relevant works, see for example, [34, 26, 5].

In comparison, we like to study connected Hopf algebras of finite dimension, or GK-dimension 0. Since finite-dimensional connected Hopf algebras only appear in positive characteristic, throughout, we work over an algebraically closed field $k$ of characteristic $p > 0$ for classification problem. Henderson [9] classified graded co-commutative connected Hopf algebras of dimension up to $p^3$ by using Singer’s theory [20] of extensions of connected Hopf algebras. Recently, in [30], the author of this paper classified all connected Hopf algebras of dimension $p^2$ by using the theories of restricted Lie algebras and Hochschild cohomology of coalgebras. Furthermore, pointed Hopf algebras of dimension $p^2$ were completely classified in [28]. For connected Hopf algebras of dimension $p^3$, most of them have been classified in [29], except the case when the primitive space of the connected Hopf algebra is an abelian restricted Lie algebra of dimension 2.

1.2. Motivation. The motivation of this paper is to classify the remaining case in [29], i.e., $p^3$-dimensional connected Hopf algebras with abelian primitive space of dimension 2, by studying a larger family of connected Hopf algebras with similar
properties. We first give the description of the family as below. Let $\mathcal{H}$ be the set of all finite-dimensional connected Hopf algebras $H$ satisfying

1. $\dim H = p^d + 1$ for some $d \geq 1$;
2. primitive space $P(H)$ is an abelian restricted Lie algebra of dimension $d$.

In this paper, we classify all such Hopf algebras in $\mathcal{H}$. Note that in the above description if we let $d = 2$, it becomes the classification of the special case in dimension $p^3$.

1.3. Approach. For any Hopf algebra $H$ in $\mathcal{H}$, as shown in the paper, it is an extension of restricted enveloping algebras:

$$1 \longrightarrow u(\mathfrak{h}) \longrightarrow H \longrightarrow u(\mathfrak{g}) \longrightarrow 1,$$

where $\mathfrak{h} = P(H)$ and $\mathfrak{g}$ is a one-dimensional restricted Lie algebra. For simplicity, we write $A = u(\mathfrak{h})$ and $B = u(\mathfrak{g})$.

Firstly, we study all extensions of $A$ by $B$. It is shown that each extension gives rise to a special restricted Lie algebra representation $\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$, which is called algebraic representation. Choose a basis $x_1, x_2, \cdots, x_d$ for $\mathfrak{h}$, and $z$ for $\mathfrak{g}$. Now, by fixing the type $T = (\mathfrak{h}, \mathfrak{g}, \rho)$, we prove that every extension of $u(\mathfrak{h})$ by $u(\mathfrak{g})$, which gives rise to the same representation $\rho$, shares the similar structure: it is generated, as an algebra, by $x_i$’s and $x$, subject to the relations in $A$ and

$$[x, x_i] = \rho_z(x_i), \quad x^p + \lambda x + \Theta = 0,$$

where $\Theta \in A^+$ and the coefficient $\lambda$ is determined by the restricted map on $\mathfrak{g}$, i.e., $z^p + \lambda z = 0$. Moreover, the coalgebra structure is given by

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i, \quad \Delta(x) = x \otimes 1 + 1 \otimes x + \chi,$$

where $\chi$ is a cocycle in the cobar construction $\Omega A$. The collection of all pairs $(\chi, \Theta)$ forms an abelian group via componentwise addition. We denote it by $E^2(B, A)$. Regarding equivalence of extensions, we define an equivalence relation on $E^2(B, A)$, and we set $\mathcal{H}^2(B, A) = E^2(B, A)/ \sim$.

Secondly, note that not all extensions in $E^2(B, A)$ belong to $\mathcal{H}$. The exceptions are extensions that are primitively generated. We denote them by $L^2(B, A)$, which is a subset of $E^2(B, A)$. In particular, the subgroup $\mathcal{H}^2(\mathfrak{g}, \mathfrak{h}) = L^2(B, A)/ \sim$ classifies all restricted Lie algebra extensions of $\mathfrak{h}$ by $\mathfrak{g}$. Thus, we set $\mathcal{H}^2(T) = \mathcal{H}^2(B, A) \setminus \mathcal{H}^2(\mathfrak{g}, \mathfrak{h})$.

Finally, we take the automorphisms of the algebraic representation $\rho$ into account. We define the automorphism group $\text{Aut}(T)$ to be pairs of automorphisms of $\mathfrak{h}$ and $\mathfrak{g}$, which are compatible with $\rho$. It has a natural action on $\mathcal{H}^2(T)$, whose orbits give the isomorphism classes of Hopf algebras of type $T$ in $\mathcal{H}$. 

1.4. Primary results. Our first main result establishes the bijections between the groups we have constructed so far and the extensions we try to classify up to equivalence. In particular, part (iii) implies that there exists a natural bijection between the isomorphism classes of Hopf algebras, in which we are interested, and the group orbits in $H^2(B, A)$ deleting the subgroup $H^2(g, h)$ regarding restricted Lie algebra extensions.

**Theorem A.** Let $\mathfrak{h}$ be a finite-dimensional abelian restricted Lie algebra, and $\mathfrak{g}$ be a one-dimensional restricted Lie algebra. Denote by $A, B$ the restricted enveloping algebra of $\mathfrak{h}, \mathfrak{g}$ respectively. Suppose $\rho$ is an algebraic representation of $\mathfrak{g}$ on $\mathfrak{h}$. Then, we have following 1-1 correspondences.

(i) View $\mathfrak{h}$ as a (left) restricted $\mathfrak{g}$-module via $\rho$. Thus,

$$\{\text{equivalence classes of restricted Lie algebra extensions of } \mathfrak{h} \text{ by } \mathfrak{g}\} \longleftrightarrow \{\text{elements of } H^2(\mathfrak{g}, \mathfrak{h})\}.$$ 

(ii) Fix the type $T = (\mathfrak{h}, \mathfrak{g}, \rho)$. Thus,

$$\{\text{equivalence classes of Hopf algebra extensions of type } T\} \longleftrightarrow \{\text{elements of } H^2(B, A)\}.$$ 

(iii) In $\mathcal{H}$,

$$\{\text{isomorphism classes of Hopf algebras of type } T\} \longleftrightarrow \{\text{Aut}(T)\text{-orbits in } H^2(T)\}.$$ 

Next, we try to find all the isomorphism classes in $\mathcal{H}$. We will let the type $T$ vary through all possible types. The best way to think about the scenario, is to construct a functor $p_{\mathcal{H}}$ from the category of $\mathcal{H}$ to the category $\mathcal{I}$ of all triples $T = (\mathfrak{h}, \mathfrak{g}, \rho)$. For the second category, we do not require dim $\mathfrak{g} = 1$ in an arbitrary object $T$. The morphisms between two objects are pairs of restricted Lie algebra maps, which are compatible with the algebraic representations. Therefore, we can have automorphism groups and isomorphism classes in the category $\mathcal{I}$. In particular, all types are considered to be special objects of $\mathcal{I}$ where dim $\mathfrak{g} = 1$.

**Theorem B.** Isomorphism classes in $\mathcal{H}$ are in 1-1 correspondence with elements of the disjoint union

$$\coprod_T H^2(T)/\text{Aut}(T),$$

where $T$ runs through isomorphism classes of all types in $\mathcal{I}$.

1.5. Secondary results. In order to prove the main results, we use the following techniques. They will give us the Hopf algebra structures of the extensions in precious section. The reader can skip them first and refer to them later in the paper.

In Section 3, we consider a general construction of finite-dimensional connected Hopf algebras from the data

$$\mathcal{D} = (T, z, \chi, \Theta),$$
where we choose a triple $T = (\mathfrak{h}, g, \rho)$ in $\mathcal{T}$, and let $z$ be a nonzero element of $g$. The other ingredients $\chi$ and $\Theta$ are related to the coalgebra and algebra structures in the construction.

**Theorem C.** For the constructed $u(\mathcal{D})$, we have

(i) $u(\mathcal{D})$ is a connected Hopf algebra of dimension $p^d + n$, where $d = \dim \mathfrak{h}$ and $p^n$ is the dimension of the subalgebra generated by $z$ in $u(g)$;

(ii) the primitive space of $u(\mathcal{D})$ is isomorphic to $\mathfrak{h}$ if and only if $\{[D^i_\mathcal{D}(\chi)]|0 \leq i \leq n - 1\}$ are linearly independent in $H^2(\Omega A)$.

The next result builds a bridge between the general construction we have and the extensions we try to classify.

**Theorem D.** Any extension of type $T$ is isomorphic to $u(\mathcal{D})$, for some $\mathcal{D} = (T, z, \chi, \Theta)$.

### 1.6. A better understanding of the group action.

Let $T = (\mathfrak{h}, g, \rho)$ be some type in $\mathcal{T}$. In Section [1], we study explicitly the $\text{Aut}(T)$-action on $H^2(T)$. Firstly, we view $H^2(T)$ via some embedding as a quotient space of the affine space. Secondly, we construct a projection $q$ from $H^2(T)$ to $H^2(\Omega A)$, which is the second cohomology group of the cobar construction on $A$. We call the images of $H^2(T)$ under $q$ admissible $z$-characteristic elements of $H^2(\Omega A)$. Next, we show that the fiber of $q$ over any such point are all isomorphic to $\text{Ker}\rho_z/\text{Im}\Phi_z$, where $\Phi_z$ is the $z$-operator defined on $\Omega A$. Thus, if $\text{Ker}\rho_z = \text{Im}\Phi_z$, we conclude that $q$ induces a bijection between $H^2(T)$ and nonzero admissible $z$-characteristic elements in $H^2(\Omega A)$. Finally, with an explicit formula of $H^2(\Omega A)$, we can compute the group action in a more practical way.

### 1.7. Examples.

At last, three concrete examples are provided to help us better understand the results in the paper. As we deal with the examples, it explains why we formulate the classification of $\mathcal{H}$ in such a way:

(1) for any fixed type $T$, we can directly compute $H^2(T)$ and the $\text{Aut}(T)$-orbits via some embedding;

(2) we can recover the structure of the representative Hopf algebra by generators and relations from each point in the orbit;

(3) we can equip the group action with geometric meanings to involve more interesting results.

In first example, we classify all semisimple Hopf algebras in $\mathcal{H}$. Denote by $\mathbb{F}$ the finite field with exactly $p$ elements. We use $\Lambda(V)$ to denote the exterior algebra over the vector space $V$. And any quadratic curve in $\Lambda(V)$ is considered to be a one-dimensional subspace in $V \oplus (V \wedge V)$.

**Proposition.** The following are in 1-1 correspondence with each other.
The isomorphism classes of semisimple connected Hopf algebras of dimension $p^{d+1}$ with $\dim P(H) = d$.

(ii) The isomorphism classes of quadratic curves in $\mathbb{P}^{d-1}_K$ ($p = 2$) or $\Lambda(V)$ with $\dim V = d$ ($p > 2$) up to the automorphism group $\text{PGL}(d, K)$.

(iii) The isomorphism classes of $p$-groups of order $p^{d+1}$, whose Frattini group is isomorphic to $C_p$.

In second example, we choose a type $T = (\mathfrak{h}, \mathfrak{g}, \rho)$, where we set up $\dim \mathfrak{h} = 2$ with basis $x, y$ and $\dim \mathfrak{g} = 1$ with basis $z$ satisfying $x = 0, y^p = y, z^p = 0$. The algebraic representation $\rho$ is given by $\rho_z(x) = y$ and $\rho_z(y) = 0$. Through this example, we work out the $\text{Aut}(T)$-action on $H^2(T)$ explicitly. As a result, isomorphism classes of Hopf algebras in $\mathcal{H}$ of type $T$ are represented by $H_1$, $H_2$, and $H(\lambda)$ for $\lambda \in k^\times$.

Example. As algebras, they are isomorphic to $k\langle x, y, z \rangle / (x^p, y^p - y, z^p + x^{p-1}y - x, [x, y], [z, x] - y, [z, y])$.

For coalgebra structures, $x$ and $y$ are primitive and the comultiplication of $z$ is given by

$H_1$: $\Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y$.

$H_2$: $\Delta(z) = z \otimes 1 + 1 \otimes z + \sum_{1 \leq i \leq p-1} (p \choose i)_i/p x^i \otimes x^{p-i}$.

$H(\lambda)$: $\Delta(z) = z \otimes 1 + 1 \otimes z + \lambda x \otimes y + \sum_{1 \leq i \leq p-1} (p \choose i)_i/p x^i \otimes x^{p-i}$.

Moreover, we have $H(\lambda) \cong H(\lambda')$ if and only if $\lambda, \lambda'$ are in the same orbit of $k^\times / G$, where $G$ is the multiplicative group of $(p^2 - 1)/2$th roots of unity. They appear to be new examples of $p^3$-dimensional connected Hopf algebras.

In third example, we discuss the concept of admissibility of $z$-characteristic elements defined in Section 6.4. We give a sufficient and necessary condition for a special element in $H^2(\Omega A)$ to be $z$-characteristic and further admissible.

1.8. Questions. D.-G. Wang, J.J. Zhang, and G. Zhuang in [27] studied connected Hopf algebras over an algebraically closed field of characteristic 0 with finite GK-dimension (equal to the dimension of the primitive space plus one), see [27] Theorem 0.5 or Theorem 2.7]. Such an infinite-dimensional connected Hopf algebra can be thought of almost primitively generated (i.e., generated by primitives plus a non-primitive element). Moreover, as an algebra, it is isomorphic to $U(\mathfrak{h})$ for some Lie algebra $\mathfrak{h}$. A similar statement does not hold in positive characteristic. That is, a finite-dimensional almost primitively generated connected Hopf algebra may not be isomorphic to $u(\mathfrak{g})$ for some restricted Lie algebra $\mathfrak{g}$. Since any Hopf algebra in $\mathcal{H}$ satisfies the same description, we ask the following question:

1The Frattini group of a $p$-group $G$ is the smallest normal subgroup $N$, where $G/N$ is an elementary abelian $p$-group.
Question A. When is a finite-dimensional connected Hopf algebra with large abelian primitive space isomorphic, as an algebra, to some restricted enveloping algebra?

Let \( h, g \) be two restricted Lie algebras provided \( h \) is finite abelian and \( \dim g = 1 \). We can use cohomology theory of Hopf algebra extensions to study the extensions of \( A = u(h) \) by \( B = u(g) \). Note that any algebraic representation \( \rho \) makes \( A \) into a \( B \)-module algebra (Proposition 2.5). Let us fix the action \( \rho \). Then generally, we have a cohomology group \( \mathcal{H}^2(B, A) \), which classifies all the extensions up to equivalence; see example [23, 6, 15] for the case when \( A \) is commutative and \( B \) is cocommutative. Moreover, we can compute the cohomology group from the total complex of a double complex, and recover the structure by some bicrossed product.

Question B. Is our construction \( \mathcal{H}^2(B, A) \) isomorphic to \( \mathcal{H}^2(B, A) \) as cohomology groups?

Compared with finite-dimensional Hopf algebras arising from matched pairs of groups [25] and Lie algebras [14] in characteristic zero, we like to study those extensions arising from finite-dimensional restricted Lie algebras. Moreover, we want to generalize our results in the paper.

Question C. Can we generalize our results for connected Hopf algebras \( H \) satisfying that \( h = P(H) \) is abelian and \( H \) is an extension of \( u(h) \) by some \( u(g) \), where we do not require \( \dim g = 1 \)?

2. Preliminary results

Throughout, we work over a base field \( k \) of characteristic \( p > 0 \). We reserve the notation \( \mathbb{K} \) for the finite field of exactly \( p \) elements, which can be thought as a subfield of \( k \). Any vector space, tensor product, homomorphism, etc., are over \( k \), unless it is stated otherwise. Let \((H, m, u, \Delta, \varepsilon, S)\) denote a Hopf algebra \( H \) in the sense of [17]. The primitive space of \( H \) is the set \( P(H) := \{ x \in H | \Delta(x) = x \otimes 1 + 1 \otimes x \} \), which is a restricted Lie algebra and its restricted enveloping algebra is denoted by \( u(P(H)) \) [12, Chapter V, §7]. The coradical \( H_0 \) of \( H \) is the sum of all simple subcoalgebras of \( H \). For each \( n \geq 1 \), inductively define

\[
H_n = \Delta^{-1}(H \otimes H_{n-1} + H_0 \otimes H).
\]

The chain of subcoalgebras \( H_0 \subseteq H_1 \subseteq \ldots \subseteq H_{n-1} \subseteq H_n \subseteq \ldots \) is called the coradical filtration of \( H \). Moreover, \( H \) is said to be connected if \( H_0 \) is one-dimensional. The augmentation ideal of \( H \) is denoted by \( H^+ := \ker \varepsilon \). When \( H \) is finite-dimensional, we denote by \( H^* \) the dual Hopf algebra of \( H \). Sweedler’s notation is used to write \(^2\)In those circumstances, we work over \( \mathbb{K} \).
the comultiplication of $H$, i.e., $\Delta(h) = \sum h_1 \otimes h_2$, for any $h \in H$. For simplicity, we write $H^n$ for the $n$-fold tensor product $H \otimes^n$ and 1 for the identity map on $H$.

2.1. **Module Hopf algebras.** A Hopf algebra $A$ is an $H$-module Hopf algebra if $A$ is a (left) $H$-module satisfying that

(i) $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$,

(ii) $h \cdot 1_A = \epsilon(h)1_A$,

(iii) $\Delta(h \cdot a) = \sum (h_1 \cdot a_1) \otimes (h_2 \cdot a_2)$,

(iv) $\epsilon(h \cdot a) = \epsilon(h)\epsilon(a)$,

(v) $S(h \cdot a) = h \cdot S(a)$,

for all $h \in H$ and $a, b \in A$. Note that any algebra $A$, which is a (left) $H$-module satisfying (i) and (ii) is said to be a $H$-module algebra [17, Definition 4.1.1].

2.2. **Cobar construction.** For any Hopf algebra $A$, the cobar construction on $A$ is the differential graded algebra $\Omega A$ defined as follows:

(i) As a graded algebra, $\Omega A$ is the tensor algebra $T(A^+)$,

(ii) The differentials are given by $d^i = (-1)^i \sum \sigma \otimes 1^{n-i-1}$, where $\sigma(a) = \Delta(a) - a \otimes 1 - 1 \otimes a$ for any $a \in A^+$.

By definition, we have $d^1(a) = -\Delta(a)$ and $d^2(a \otimes b) = -\Delta(a) \otimes b + a \otimes \Delta(b)$ for any $a, b \in A^+$. See [8, §19] for basic properties of bar and cobar constructions.

2.3. **Algebraic representations.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be two restricted Lie algebras. An algebraic representation of $\mathfrak{g}$ on $\mathfrak{h}$ is a linear map $\rho : \mathfrak{g} \to \text{End}_k(\mathfrak{h})$ such that

(i) $\rho_{[x,y]} = \rho_x \rho_y - \rho_y \rho_x$,

(ii) $\rho_{(xp)} = (\rho_x)^p$,

(iii) $\rho_x([a,b]) = [\rho_x(a), b] + [a, \rho_x(b)]$,

(iv) $\rho_x(a^p) = \rho_x(a)(\text{ad } a)^{p-1}$,

for any $x, y \in \mathfrak{g}$ and $a, b \in \mathfrak{h}$. Note that (i) and (ii) makes $\mathfrak{h}$ into a restricted $\mathfrak{g}$-module via $\rho$. The representation is said to be abelian if $\mathfrak{h}$ is an abelian restricted Lie algebra, where we have an extension of restricted Lie algebras [33, 7.4.9]

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{h} \rtimes \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow 0.$$
2.4. The category $\mathcal{T}$. We use $\mathcal{T}$ to denote the collection of all abelian algebraic representations for finite-dimensional restricted Lie algebras. To be more precise, we define the category $\mathcal{T}$, whose objects consist of all triples $(\mathfrak{h}, \mathfrak{g}, \rho)$ containing two finite-dimensional restricted Lie algebras $\mathfrak{h}, \mathfrak{g}$ provided $\mathfrak{h}$ is abelian, and an algebraic representation $\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$. The morphisms between two objects $(\mathfrak{h}, \mathfrak{g}, \rho)$ and $(\mathfrak{h}', \mathfrak{g}', \rho')$ are pairs $(\phi, \psi)$ of restricted Lie algebra maps, where $\phi : \mathfrak{h} \to \mathfrak{h}'$ and $\psi : \mathfrak{g} \to \mathfrak{g}'$ satisfy the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{g} \otimes \mathfrak{h} & \xrightarrow{\rho} & \mathfrak{h} \\
\psi \otimes \phi & & \phi \\
\mathfrak{g}' \otimes \mathfrak{h}' & \xrightarrow{\rho'} & \mathfrak{h}'
\end{array}
\]

The automorphism group $\text{Aut}(T)$, for any object $T = (\mathfrak{h}, \mathfrak{g}, \rho)$ in $\mathcal{T}$, consists of pairs $(\phi, \psi)$ of automorphisms of $\mathfrak{h}$ and $\mathfrak{g}$ respectively, which is compatible with $\rho$ as described by the above commutative diagram. The group multiplication in $\text{Aut}(T)$ is composition of maps. Moreover, we say $T$ is a type in $\mathcal{T}$ if $\dim \mathfrak{g} = 1$.

2.5. Algebraic representations and module Hopf algebras. We will investigate the basic facts regarding the three concepts above. Firstly, let $\mathfrak{h}$ and $\mathfrak{g}$ be two restricted Lie algebras. In characteristic $p > 0$, it is clear that all the derivations on $u(\mathfrak{h})$ form a restricted Lie algebra through the commutator and the $p$-th power map in $\text{End}_k(u(\mathfrak{h}))$. For any $\delta \in \text{Der}(u(\mathfrak{h}))$ and $a, b \in u(\mathfrak{h})$, direct computation shows that

\[
\delta[a, b] = [\delta(a), b] + [a, \delta(b)], \quad \delta(a^p) = \delta(a)(\text{ad } a)^{p-1}.
\]

Now, consider an algebraic representation $\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$. By Definition 2.3 and Relation (2.3), we see $\rho$ induces a restricted Lie algebra map from $\mathfrak{g}$ to $\text{Der}(u(\mathfrak{h}))$. It is straightforward to check that this makes $u(\mathfrak{h})$ into a $u(\mathfrak{g})$-module Hopf algebra according to Definition 2.1. Moreover, we have the following bijection between algebraic representations and module Hopf algebras for their restricted enveloping algebras.

**Proposition.** Algebraic representations of $\mathfrak{g}$ on $\mathfrak{h}$ are in 1-1 correspondence with $u(\mathfrak{g})$-module Hopf algebra structures on $u(\mathfrak{h})$.

**Proof.** Here, we only prove one direction. Suppose $u(\mathfrak{h})$ is a $u(\mathfrak{g})$-module Hopf algebra via the action $\rightarrow$. We define the representation $\rho$ by $\rho_x(a) = x \rightarrow a$ for any $x \in \mathfrak{g}$.
and $a \in h$. By Definition 2.1(iii), we have
\[
\Delta(\rho_x(a)) = \sum (x \rightarrow a_1) \otimes (x \rightarrow a_2)
\]
\[
= (x \rightarrow a) \otimes (1 \rightarrow 1) + (x \rightarrow 1) \otimes (1 \rightarrow a) + (1 \rightarrow a) \otimes (x \rightarrow 1) + (1 \rightarrow 1) \otimes (x \rightarrow a)
\]
\[
= (x \rightarrow a) \otimes 1 + 1 \otimes (x \rightarrow a).
\]
The last equality follows from Definition 2.1(ii). Hence, we have $\rho_x : g \rightarrow \text{End}_k(h)$. Moreover, since $g$ is the primitive space of $u(g)$, we see
\[
\rho_x(ab) = x \rightarrow (ab) = (x \rightarrow a)(1 \rightarrow b) + (1 \rightarrow a)(x \rightarrow b) = \rho_x(a)b + a\rho_x(b),
\]
for any $a, b \in u(h)$. It follows that $\rho : g \rightarrow \text{Der}(u(h))$. Therefore, in Definition 2.3, it is easy to see that (iii) and (iv) come from Relation (2.3). And it is direct to check for (i) and (ii) by the construction of $u(g)$. Then, we show that $\rho$ is an algebraic representation. Finally, the bijection comes from the explicit construction. This completes the proof. ■

2.6. Module Hopf algebras and cohomology ring of cobar constructions.
Secondly, suppose we have an $H$-module Hopf algebra $A$. Note that the cobar construction $\Omega A$ is the tensor algebra $T(A^+)$ and $A^+$ is invariant under the $H$-action by Definition 2.1(iv). Hence, we can consider $\Omega A$ as an $H$-module algebra via the comultiplication of $H$. In details,
\[
h \cdot 1_{\Omega A} = \epsilon(h)1_{\Omega A},
\]
\[
h \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \sum (h_1 \cdot a_1) \otimes (h_2 \cdot a_2) \otimes \cdots \otimes (h_n \cdot a_n),
\]
for any $h \in H$ and $a_i \in A^+$. Moreover, we can pass the $H$-module algebra structure on to the cohomology ring of $\Omega A$.

**Proposition.** The $H$-action commutes with the differentials of $\Omega A$. Thus, the cohomology ring $H^*(\Omega A)$ becomes an $H$-module algebra.

**Proof.** Observe that the cobar construction $\Omega A$ is a differential graded algebra, which is generated in degree one. Thus, it suffices to show that the $H$-action commutes with $d^1$. Suppose $a \in A^+$ and $h \in H$. We have
\[
h[d^1(a)] = h[1 \otimes a - \Delta(a) + a \otimes 1]
\]
\[
= \sum h_1 1 \otimes h_2 a - \sum h_1 a_1 \otimes h_2 a_2 + \sum h_1 a \otimes h_2 1
\]
\[
= 1 \otimes ha - \sum (ha)_1 \otimes (ha)_2 + ha \otimes 1
\]
\[
= d^1(ha).
\]
This completes the proof. ■
2.7. Algebraic representations and cohomology ring of cobar constructions. In a conclusion, we have the following result about algebraic representations of $g$ on $h$ and the corresponding $u(g)$-module algebra structures on $H^*(\Omega u(h))$.

**Proposition.** Let $\rho$ be an algebraic representation of $g$ on $h$. Then $H^*(\Omega u(h))$ becomes a $u(g)$-module algebra via the representation $\rho$.

2.8. Convention. Throughout this paper, let $T = (h, g, \rho)$ be an object in $\mathcal{T}$. We choose a basis for $h$, denoted by $x_1, x_2, \ldots, x_d$, and keep the notations $A$, $B$ for the restricted enveloping algebras of $h$, $g$ respectively. Let $z$ be a nonzero element of $g$, where the minimal relation among $z, z^p, z^{p^2} \cdots$ is denoted by

$$z^{p^n} + \lambda_{n-1}z^{p^{n-1}} + \cdots + \lambda_1 z^p + \lambda_0 z = 0.$$  

Except in the next section, we always assume that $\dim g = 1$. Therefore, the element $z$ is fixed as a basis for $g$. The minimal relation is written as $z^p + \lambda z = 0$ for some $\lambda \in k$. Note that it also gives the restricted map in $g$.

3. A general construction

In this section, our aim is to construct a connected Hopf algebra $u(\mathcal{D})$ from the following data

$$\mathcal{D} = (T, z, \chi, \Theta).$$

We will first explain all the elements in $\mathcal{D}$. Let $T = (h, g, \rho)$ and $z$ be described in Convention 2.8. For the general construction, we do not require that $\dim g = 1$.

3.1. Characteristic operators. In order to explain all the other elements $\chi$ and $\Theta$ in (3.1), we need to introduce some new concepts. We use $\text{Hom}_{\text{grK}}(\Omega A, \Omega A)$ to denote all the $K$-linear graded maps from $\Omega A$ to itself. Note that by Proposition 2.7, $\Omega A$ is a $B$-module algebra via $\rho$, where the action commutes with the differentials. Therefore, we can consider $\rho_z$ as a degree zero cochain map from $\Omega A$ to itself.

**Definition.** We define three degree zero cochain maps in $\text{Hom}_{\text{grK}}(\Omega A, \Omega A)$.

(i) For any element $a = \sum (a) a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in $(A^+)^n$, the $p$-th map $P$ is given by $P(a) = \sum (a) a_1^p \otimes a_2^p \otimes \cdots \otimes a_n^p$. 

(ii) Inductively, we define $D_z^0 = \text{Id}$ and $D_z^m = P \circ D_z^{m-1} + \rho_z^{p^m-p^{m-1}} \circ D_z^{m-1}$ for any $m \geq 1$.

(iii) The $z$-operator on $\Omega A$ is given by $\Phi_z := D_z^n + \lambda_{n-1} D_z^{n-1} + \cdots + \lambda_1 D_z^1 + \lambda_0 D_z^0$. 

The reason why we give the definitions above is from the next lemma, which is important for later use. Let $F$ be the algebra generated by $A$ and an indeterminate $x$, subject to the relations $[x, x_i] = \rho_z(x_i)$ for all $1 \leq i \leq d$.

**Lemma.** Let $f \in A^+ \otimes A^+$. In the tensor algebra $F \otimes F$, we have

$$(x \otimes 1 + 1 \otimes x + f)^p = x^p \otimes 1 + 1 \otimes x^p + D_z^m(f),$$

for all $m \geq 0$.

**Proof.** We will prove the statement by induction on $m \geq 0$. The statement is trivial for $m = 0$. Suppose it is true for $m = n$. Write $X = x \otimes 1 + 1 \otimes x$ and observe that

$$(\text{ad}X)^p = (\text{ad}X)^{p+1} - p^n = (\text{ad}x \otimes 1 + 1 \otimes \text{ad}x)^{p+1} - p^n = \rho_z^{p+1} - p^n.$$

Hence by [30, Lemma A.1] ($A$ is commutative), we have

$$(X + f)^{p+1} = [X^p + D_z^n(f)]^p
\quad = X^{p+1} + (D_z^n(f))^p + (\text{ad}X)^p D_z^n(f)
\quad = X^{p+1} + (\mathcal{P} \circ D_z^m + \rho_z^{p+1} - p^n \circ D_z^n)(f)
\quad = x^{p+1} \otimes 1 + 1 \otimes x^{p+1} + D_z^{n+1}(f).$$

Thus, the statement is true for $m = n + 1$, and the proof of the induction step is complete. \[\square\]

Before we move on to define $\chi$, we list some basic facts regarding those maps we just defined.

**Proposition.** The following are true:

(i) All maps $\mathcal{P}$, $D_z^m$ and $\Phi_z$ commute with the differentials of $\Omega A$.

(ii) $D_z^m = \mathcal{P} \circ D_z^{m-1} + \rho_z^{p_{m-1}}$ for all $m \geq 1$.

(iii) $\rho_z \circ \Phi_z = 0$.

**Proof.** (i) We first show that $\mathcal{P}$ commutes with the differential $d$. Because $A$ is commutative and the comultiplication $\Delta$ in $A$ is an algebra map, it follows that

3Throughout the paper, by abuse of notations, we will also consider $\mathcal{P}, D_z^m, \Phi_z$ as maps from $(A^+)^m$ to itself for any integer $m \geq 0$. 
\[ \mathcal{P} \circ \Delta = \Delta \circ \mathcal{P}. \] By the definition of \( d \) (see Equation (2.1)), we have
\[
\mathcal{P} \circ \left[ \sum_{i=0}^{n-1} (-1)^{i+1} 1^i \otimes \Delta \otimes 1^{n-i-1} \right] = \sum_{i=0}^{n-1} (-1)^{i+1} 1^i \otimes \mathcal{P} \circ \Delta \otimes 1^{n-i-1} \\
= \sum_{i=0}^{n-1} (-1)^{i+1} 1^i \otimes \mathcal{P} \otimes 1^{n-i-1} \\
= \left[ \sum_{i=0}^{n-1} (-1)^{i+1} 1^i \otimes \Delta \otimes 1^{n-i-1} \right] \circ \mathcal{P}.
\]
This shows that \( \mathcal{P} \circ d = d \circ \mathcal{P} \). Next, we prove inductively that \( \mathcal{D}_z^m \circ d = d \circ \mathcal{D}_z^m \) for \( m \geq 0 \). It is true for \( m = 0 \) by definition. Suppose it holds for \( m = n \). By Proposition 2.6, we know \( \rho_z d = d \rho_z \). Then
\[
d \circ \mathcal{D}_z^{n+1} = d \circ (\mathcal{P} \circ \mathcal{D}_z^n + \rho_z^{p^{n+1}-p^n} \circ \mathcal{D}_z^n) \\
= (\mathcal{P} \circ \mathcal{D}_z^n + \rho_z^{p^{n+1}-p^n} \circ \mathcal{D}_z^n) \circ d \\
= \mathcal{D}_z^{n+1} \circ d.
\]
Thus, the induction step is complete. Finally, \( \Phi_z \circ d = d \circ \Phi_z \) since the \( z \)-operator \( \Phi_z \) is a linear combination of \( \mathcal{D}_z^m \)'s.

(ii) We prove the statement by induction on \( m \geq 1 \). When \( m = 1 \), the statement is just the definition. Suppose it is true for \( m = n \). In characteristic \( p \), it is clear that any derivation vanishes on \( A^p \), which implies that \( \rho_z \circ \mathcal{P} = 0 \). Thus
\[
\mathcal{D}_z^{n+1} = \mathcal{P} \circ \mathcal{D}_z^n + \rho_z^{p^{n+1}-p^n} \circ \mathcal{D}_z^n \\
= \mathcal{P} \circ \mathcal{D}_z^n + \rho_z^{p^{n+1}-p^n} \circ (\mathcal{P} \circ \mathcal{D}_z^{n-1} + \rho_z^{p^{n-1}}) \\
= \mathcal{P} \circ \mathcal{D}_z^n + \rho_z^{p^{n+1}-1}.
\]
Then, the statement is true for \( m = n + 1 \), and it completes the induction step.

(iii) By definition and (ii), we have
\[
\rho_z \circ \Phi_z = \rho_z \circ (\mathcal{D}_z^n + \cdots + \lambda_1 \mathcal{D}_z^1 + \lambda_0 \mathcal{D}_z^0) \\
= \rho_z \circ (\mathcal{P} \circ \mathcal{D}_z^{n-1} + \cdots + \lambda_1 \mathcal{P} \circ \mathcal{D}_z^0) + \rho_z (\rho_z^{p^{n-1}} + \cdots + \lambda_1 \rho_z^{p-1} + \lambda_0) \\
= (\rho_z)^p + \lambda_{n-1} (\rho_z)^{p^{n-1}} + \cdots + \lambda_1 (\rho_z)^p + \lambda_0 \rho_z \\
= \rho (z^{p^n} + \lambda_{n-1} z^{p^{n-1}} + \cdots + \lambda_1 z^p + \lambda_0) \\
= 0.
\]
This concludes the proof. □
3.2. \textit{Z-cocycles and characteristic elements.} By Proposition 3.1(i), we can view $\Phi_z$ as a map from $H^\ast(\Omega A)$ to itself preserving the degree. Now, we are able to give the description of $\chi$ in (3.1).

\textbf{Definition.} Let $\chi \in A^+ \otimes A^+$. We say that $\chi$ is a $z$-cocycle if

(i) $\chi \in Z^2(\Omega A)$,

(ii) $\Phi_z(\chi) \in B^2(\Omega A)$.

We say any cohomology class $\xi \in H^2(\Omega A)$ is $z$-characteristic if $\Phi_z(\xi) = 0$. It is clear that $\chi$ is a $z$-cocycle if and only if $[\chi]$ is $z$-characteristic in $H^2(\Omega A)$.

3.3. \textbf{Construction.} In the data $D$ of (3.1), we let $\chi \in (A^+)^2$ be a $z$-cocycle, and $\Theta \in A^+$ satisfying

(3.2) \quad $\Phi_z(\chi) = d^1(\Theta), \quad \rho_z(\Theta) = 0$.

We construct the connected Hopf algebra $u(D)$ explicitly. As an algebra, it is generated by $A$, that is, by $x_i$'s satisfying the relations in the restricted enveloping algebra, and an indeterminate $x$, subject to the relations

(3.3) \quad $[x, x_i] = \rho_z(x_i)$, for all $1 \leq i \leq d$,

\quad $x^p + \lambda_{n-1}x^{p-1} + \cdots + \lambda_1x + \lambda_0x + \Theta = 0$.

The coalgebra structure is given by

(3.4) \quad $\Delta(x) = x \otimes 1 + 1 \otimes x + \chi, \quad \Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$,

for all $1 \leq i \leq d$.

3.4. \textbf{PBW Theorem.} In order to prove Theorem C, we first show a PBW Theorem for $u(D)$. This will also cover the dimensionality in statement (i).

\textbf{Lemma.} The following set is a basis for $u(D)$:

\[ \{ x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_d^{\sigma_d} x^{\sigma_{d+1}} \mid 0 \leq \sigma_1, \cdots, \sigma_d \leq p-1, \ 0 \leq \sigma_{d+1} \leq p^n-1 \} \]

\textbf{Proof.} By the construction above, it is clear that, as an algebra, $u(D)$ is isomorphic to the quotient algebra $k\langle x_1, x_2, \cdots, x_d, x \rangle/I$, where the relation $I$ is generated by

$x_j x_i = x_i x_j$, for $1 \leq i < j \leq d$,

$x_i^p = \sum_{j=1}^d a_{ij} x_j$, for $1 \leq i \leq d$,

$xx_i = x_i x + \rho_z(x_i)$, for $1 \leq i \leq d$,

$x^{pn} = - \sum_{i=0}^{n-1} \lambda_i x^{pi} - \Theta$. 

The coefficients $a_{ij}$ above are determined by the restricted map on $h$. Therefore, we can find all the possible ambiguities as below.

(a) $(x_k x_j) x_i = x_k (x_j x_i)$, for $1 \leq i < j < k \leq d$,
(b) $x_j (x_i^p) = (x_j x_i) x_i^{p-1}$, for $1 \leq i \leq j \leq d$,
(c) $x(x_j x_i) = (x x_j) x_i$, for $1 \leq i < j \leq d$,
(d) $x(x_i^p) = (x x_i) x_i^{p-1}$,
(e) $x(x^p) = (x^2) x^{p-1}$,
(f) $(x^p) x_i = x^{p-1} (x x_i)$.

It is easy to check that ambiguities (a), (b) and (c) are resolvable. For (d), by Definition 2.3 we have

\[
x(x_i^p) = x(\sum_{j=1}^{d} a_{ij} x_j) = (\sum_{j=1}^{d} a_{ij} x_j) x + \rho_z (\sum_{j=1}^{d} a_{ij} x_j) = (\sum_{j=1}^{d} a_{ij} x_j) x + \rho_z (a_i^p) = (\sum_{j=1}^{d} a_{ij} x_j) x.
\]

On the other hand,

\[
(x x_i) x_i^{p-1} = x_i (x x_i^{p-1}) + \rho_z (x_i) x_i^{p-1} = \cdots = (x_i^p) x + \rho_z (x_i) x_i^{p-1} = (\sum_{j=1}^{d} a_{ij} x_j) x.
\]

Hence, it is resolvable. For ambiguity (e), by the condition (3.2), we have

\[
x(x^p) = x\left(-\sum_{i=0}^{n-1} \lambda_i x^{p^i} - \Theta\right) = (-\sum_{i=0}^{n-1} \lambda_i x^{p^i} - \Theta) x - \rho_z (\Theta)
\]

\[
= (-\sum_{i=0}^{n-1} \lambda_i x^{p^i} - \Theta) x.
\]

On the other hand,

\[
(x^2) x^{p-1} = (x^p) x = (-\sum_{i=0}^{n-1} \lambda_i x^{p^i} - \Theta) x.
\]

Thus, it is resolvable. Finally, we turn to the last ambiguity (f). By induction, it is easy to show that $x^{s-1} (x x_i) = \sum_{j=0}^{s} \binom{s}{j} \rho_z^j (x_i) x^{s-j}$, for all $s \geq 1$. Hence,

\[
x^{p-1} (x x_i) = x_i (x^p) + \rho_z^p (x_i) = x_i (-\sum_{i=0}^{n-1} \lambda_i x^{p^i} - \Theta) + \rho_z^p (x_i).
\]
On the other hand,

\[(x^{p^n})x_i = (-\sum_{i=0}^{n-1} \lambda_i x^{p^i} - \Theta)x_i = x_i(-\sum_{i=0}^{n-1} \lambda_i x^{p^i} - \Theta) - \sum_{i=0}^{n-1} \lambda_i \rho_z^{p^i}(x_i)\]

\[= x_i(-\sum_{i=0}^{n-1} \lambda_i x^{p^i} - \Theta) + \rho(-\sum_{i=0}^{n-1} \lambda_i x^{p^i})(x_i) = x_i(-\sum_{i=0}^{n-1} \lambda_i x^{p^i} - \Theta) + \rho_z^{p^n}(x_i).\]

Therefore, it is resolvable, and the result follows from the Diamond Lemma [2]. ■

3.5. **Proof of Theorem C.** (i) It remains to show that \(u(\mathcal{O})\) is a connected Hopf algebra. Firstly, we prove that it is a bialgebra. Denote by \(F\) the algebra generated by \(A\) and \(x\), subject to the relations \([x, x_i] = \rho_z(x_i)\), for all \(1 \leq i \leq d\). Suppose the comultiplication is defined on the generators of \(F\) by Equation (3.4). Since

\[\Delta([x, x_i] - \rho_z(x_i)) = [\Delta(x), \Delta(x_i)] - \Delta(\rho_z(x_i))\]

\[= [x \otimes 1 + 1 \otimes x + \chi, x_i \otimes 1 + 1 \otimes x_i] - \rho_z(x_i) \otimes 1 - 1 \otimes \rho_z(x_i)\]

\[= \{[x, x_i] - \rho_z(x_i)\} \otimes 1 + 1 \otimes \{[x, x_i] - \rho_z(x_i)\},\]

it follows that \(\Delta\) extends to an algebra map from \(F\) to \(F \otimes F\). Regarding the coassociativity, since all \(x_i\)'s are primitive, and

\[(1 \otimes \Delta)\Delta(x) - (\Delta \otimes 1)\Delta(x) = 1 \otimes x - (\chi \otimes 1)(\chi) + (1 \otimes \Delta)(\chi) - \chi \otimes 1 = d^2(\chi) = 0,\]

we know it holds for the generators, hence it is true for all \(F\). Moreover, we define \(\epsilon(x) = \epsilon(x_i) = 0\), for all \(1 \leq i \leq d\). Then, it is direct to check that \(\epsilon\) is the counit, and \((F, m, u, \Delta, \epsilon)\) becomes a bialgebra.

Next, we denote \(X = x^{p^n} + \lambda_{n-1}x^{p^{n-1}} + \cdots + \lambda_0x + \Theta\) as an element in \(F\). Because \(u(\mathcal{O}) \simeq F/(X)\), in order to show that \(u(\mathcal{O})\) is a bialgebra, it suffices to show that \((X)\) is a bi-ideal in \(F\). According to Lemma 3.1, we have

\[\Delta(x^{p^n}) = (x \otimes 1 + 1 \otimes x + \chi)^{p^n} = x^{p^n} \otimes 1 + 1 \otimes x^{p^n} + D^{p^n}_z(\chi),\]

for all \(s \geq 0\). Thus

\[\Delta(X) = X \otimes 1 + 1 \otimes X + [D^{p^n}_z(\chi) + \lambda_{n-1}D^{p^{n-1}}_z(\chi) + \cdots + \lambda_0\chi] + [\Delta(\Theta) - \Theta \otimes 1 - 1 \otimes \Theta]\]

\[= X \otimes 1 + 1 \otimes X + \Phi_z(\chi) - d^2(\Theta)\]

\[= X \otimes 1 + 1 \otimes X.\]

The last equality comes from (3.2). Also, it is easy to see that \(\epsilon(X) = 0\) since \(\Theta \in A^+\), which gives the conclusion.

Secondly, we prove that \(u(\mathcal{O})\), as a coalgebra, is connected, thus the antipode exists automatically [24, Lemma 14]. Choose an integer \(s \geq 1\) such that \(\chi \in \sum_{0 \leq i \leq s} A_i \otimes A_{s-i}\), where \(\{A_i\}\) denotes the coradical filtration of \(A\). By Lemma 3.4
we can define an exhausted filtration of $u(\mathcal{D})$, whose $m$-th term $u(\mathcal{D})_m$ is spanned by
\[ \{ x^{\sigma_1}_1 x^{\sigma_2}_2 \cdots x^{\sigma_d}_d x^{\sigma_{d+1}} \mid \sigma_1 + \cdots + \sigma_d + s\sigma_{d+1} \leq m \}. \]
By the relations \((3.3)\), it is definitely an algebra filtration. Furthermore, by \([17, \text{Proposition 5.5.3}]\), it is clear that $A_m \subseteq u(\mathcal{D})_m$ for all $m \geq 0$. Hence
\[ \Delta(x) \in \sum_{i=0}^{s} u(\mathcal{D})_i \otimes u(\mathcal{D})_{s-i}. \]
Then, direct computation shows that it is also a coalgebra filtration. Note that $u(\mathcal{D})_0 = k$, and the result follows from \([17, \text{Lemma 5.3.4}]\).

(ii) In order to find all the primitives in $u(\mathcal{D})$, we need the following lemma.

\textbf{Lemma.} Let $a \in u(\mathcal{D})$. Thus, $\Delta(a) - a \otimes 1 - 1 \otimes a \in A \otimes A$ if and only if $a \in A + \sum_{i=0}^{n-1} kx^p$.

\textbf{Proof.} By Lemma \(3.4\), we can view $u(\mathcal{D})$ as a free left $A$-module equipped with the basis $\{x^i \mid 0 \leq i \leq p^n - 1\}$. Hence, we can write every element $a \in u(\mathcal{D})$ as
\[ a = \sum_{i=0}^{p^n-1} a_i x^i, \]
for some $a_i \in A$ in a unique way. Moreover, the tensor algebra $u(\mathcal{D}) \otimes u(\mathcal{D})$ becomes a free left $A \otimes A$-module with the basis $\{x^i \otimes x^j \mid 0 \leq i, j \leq p^n - 1\}$. According to Lemma \(3.1\), we know
\[ \Delta(a_i x^p) = \Delta(a_i)(x^p \otimes 1 + 1 \otimes x^p + b_i), \]
for some $b_i \in A \otimes A$ when $0 \leq i \leq n - 1$. Now, one direction of the proof is clear, and we will prove the other direction.

Define the index set $S = \{1, 2, \cdots, p^n - 1\} \setminus \{1, p, \cdots, p^{n-1}\}$. Firstly, we show that $a_i = 0$, whenever $i \in S$, by contradiction. The contradiction is obtained by looking at the coefficient of the possible highest term in $\Delta(a) - a \otimes 1 - 1 \otimes a$. Suppose it is not true. Then, we can find the maximal index $m \in S$ such that $a_m \neq 0$. By definition, we can write $m = pl$, where $l > 1$ and $l \not\equiv 0 \pmod{p}$. Hence, by \([11, \text{Lemma 5.1}]\), $\binom{m}{p^l} \equiv l \pmod{p}$. Note that in $\Delta(a) - a \otimes 1 - 1 \otimes a$, the coefficient for $\omega^{p^l} \otimes \omega^{m-p^l}$ is $l\Delta(a_m) = 0$. Since $A$ is counital, we have $a_m = 0$. This implies a contradiction.

Secondly, since $a_s = 0$ for all $s \in S$, Equation \((3.5)\) can be simplified as
\[ a = a_0 + \sum_{i=0}^{n-1} a_{p^i} x^{p^i}. \]
It remains to show that all \( a_p \in k \). By Lemma 3.1, it is easy to see that the term \( x_p \otimes 1 \) in \( \Delta(a) - a \otimes 1 - 1 \otimes a \) has coefficient \( \Delta(a_p^i) - a_p^i \otimes 1 \). Again, since \( A \) is counital, we have \( a_p^i \in k \). This completes the proof.

We continue our proof for (ii). Let \( a \) be a primitive element in \( u(D) \). By Lemma 3.5, we can write

\[
a = \sum_{(\sigma)} \mu_\sigma x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_d^{\sigma_d} + \sum_{i=0}^{n-1} \mu_i x_p^i,
\]

for some coefficients in \( k \). Note that \( d_1(x_p^i) = x_p^i \otimes 1 + 1 \otimes x_p^i - \Delta(x_p^i) = x_p^i \otimes 1 + 1 \otimes x_p^i - (x \otimes 1 + 1 \otimes x + \chi) x_p^i = -D_z^i(\chi) \) by Lemma 3.1. Since \( d_1(a) = 0 \), we have

\[
d_1(\sum_{(\sigma)} \mu_\sigma x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_d^{\sigma_d}) = -\sum_{i=0}^{n-1} \mu_i d_1(x_p^i) = \sum_{i=0}^{n-1} \mu_i D_z^i(\chi).
\]

Hence, if we pass to the homology \( H^2(\Omega A) \), we have

\[
\sum_{i=0}^{n-1} \mu_i [D_z^i(\chi)] = 0.
\]

For one direction, suppose \( \{ [D_z^i(\chi)] | 0 \leq i \leq n - 1 \} \) are linearly independent in \( H^2(\Omega A) \). By Equation (3.6), we have all \( \mu_i = 0 \). Hence, \( a \in A \), which yields that the primitive space of \( u(D) \) is equal to \( \text{P}(A) \simeq \mathfrak{h} \). On the other hand, suppose they are linearly dependent in \( H^2(\Omega A) \). Thus, there are coefficients \( \mu_i \in k \), not all zero, such that \( \sum_{i=0}^{n-1} \mu_i D_z^i(\chi) = d_1(b) \) for some element \( b \in A^+ \). Direct computation shows that \( \sum_{i=0}^{n-1} \mu_i x_p^i + b \) is primitive, which is certainly not in \( A \). This shows the other direction, and completes the proof.

4. Extensions of connected Hopf algebras

In this section, we classify up to equivalence the following short exact sequence of finite-dimensional connected Hopf algebras

\[
1 \longrightarrow u(\mathfrak{h}) \xrightarrow{i} H \xrightarrow{\pi} u(\mathfrak{g}) \longrightarrow 1,
\]

by an abelian group. We keep the notations in Convention 2.8, where \( \mathfrak{h} \) is finite abelian and \( \dim \mathfrak{g} = 1 \) with basis \( z \) satisfying \( z^p + \lambda z = 0 \). We recall some basic facts regarding extensions.
4.1. **Hopf algebra extensions.** A sequence of Hopf algebras

\[ 1 \longrightarrow A \overset{\iota}{\longrightarrow} C \overset{\pi}{\longrightarrow} B \longrightarrow 1, \]

where 1 denotes the Hopf algebra \( k \), is **exact** if \([1\ Proposition\ 1.2.3]\)

(i) \( \iota \) is injective. Identify then \( A \) with its image,

(ii) \( \pi \) is surjective,

(iii) \( \pi \iota = \epsilon \),

(iv) \( \ker \pi = CA^+ \),

(v) \( A = \{ x \in C : (\pi \otimes \text{Id})\Delta(x) = 1 \otimes x \} \).

We also say that \( C \) is an **extension** of \( A \) by \( B \), and two extensions \( C, C' \) are **equivalent** if we have an isomorphism \( \vartheta : C \rightarrow C' \) such that the following diagram commutes:

\[ \begin{array}{ccc}
1 & \longrightarrow & A \\
\downarrow & & \downarrow \iota \\
A & \overset{\iota'}{\longrightarrow} & C' \\
\downarrow & & \downarrow \pi' \\
1 & \longrightarrow & B \\
\end{array} \]

By \([13\ Lemma-Definition\ 1.1]\), in finite-dimensional case, sequence \((4.1)\) is exact if and only if

(i) \( u(\mathfrak{h}) \) is a normal Hopf subalgebra \( H \) via the injection \( \iota \),

(ii) \( H/u(\mathfrak{h}) + H \simeq u(\mathfrak{g}) \) via the projection \( \pi \).

Moreover, we have \( \dim H = \dim u(\mathfrak{h}) \dim u(\mathfrak{g}) = \rho^{\dim \mathfrak{h} + \dim \mathfrak{g}} \) by \([19\ p.\ 290]\).

4.2. **Restricted Lie algebra extensions.** If \( H \) is primitively generated in sequence \((4.1)\), we have a short exact sequence of restricted Lie algebras.

\[ 1 \longrightarrow \mathfrak{h} \overset{\iota}{\longrightarrow} \text{P}(H) \overset{\pi}{\longrightarrow} \mathfrak{g} \longrightarrow 1. \]

Choose some element \( x \) in \( \text{P}(H) \) satisfying \( \pi(x) = z \). Since \( \dim \mathfrak{g} = 1 \), it suffices to define the representation \( \rho \) of \( \mathfrak{g} \) on \( \mathfrak{h} \) by giving its value on \( z \). Let \( \rho_z(h) = [x, h] \) for any \( h \in \mathfrak{h} \). Since \( \mathfrak{h} \) is abelian, the value \( \rho_z(h) \) is independent of the choice of \( x \). Moreover, it is easy to see that \( \rho \) is indeed an algebraic representation of \( \mathfrak{g} \) on \( \mathfrak{h} \).

**Remark.** In particular, when \( \mathfrak{h} \) is \( p \)-nilpotent, we know \( \mathfrak{h} \) is a restricted \( \mathfrak{g} \)-module via \( \rho \) in the sense of \([33\ Exercise\ 7.6.4]\). Hence, equivalence classes of extensions of \( \mathfrak{h} \) by \( \mathfrak{g} \) are in 1-1 correspondence with elements in the second Hochschild cohomology of \( \mathfrak{g} \) with coefficients in \( \mathfrak{h} \) \([33\ Definition\ 7.2.2]\).
4.3. **Algebraic representation type.** More generally, for any extension $H$ described in sequence (4.1), we can define a type $T = (h, g, \rho) \in \mathcal{T}$, which is invariant under the equivalence of extensions. Identify $A = u(h)$ with its image in $H$ via $\iota$. Let $n \geq 1$ be the minimal integer such that $A_n \subseteq H_n$ (cf. [30, Definition 2.3(2)]). Note that $H$ induces an extension for their associated graded Hopf algebras [30, Definition 2.1]:

$$1 \longrightarrow \text{gr}A \longrightarrow \text{gr}H \longrightarrow \text{gr}B \longrightarrow 1.$$ 

By [30, Theorem 3.1], we know $\text{gr}A$ is isomorphic to $k[x_1, x_2, \ldots, x_d]/\langle x_1^p, x_2^p, \ldots, x_d^p \rangle$, as algebras, and $\text{gr}H \simeq \text{gr}A[x]/(x^p)$, where $x \in H_n \setminus A_n$. Moreover, through the projection $\pi$, we have natural isomorphisms $H_n/A_n \simeq g \simeq P(H/A^+)H$ as one-dimensional vector spaces (cf. [30, Lemma 4.1]). As a consequence, we can choose $x \in H_n \setminus A_n$ satisfying $x \in H^+$ and $\pi(x) = z$. We define the representation $\rho$ associated to the extension $H$ by $\rho_z(h) = [x, h]$ for all $h \in h$. It is clear that the definition is independent of the choice of $x$ since $h$ is abelian.

**Proposition.** The representation $\rho$ is an algebraic representation of $g$ on $h$.

**Proof.** Firstly, we show that $\rho$ is well-defined. We know $H_{n-1} = A_{n-1}$ by the choice of $n$. Then by [17, Lemma 5.3.2], we have

$$\Delta(x) - x \otimes 1 - 1 \otimes x \in H_{n-1} \otimes H_{n-1} = A_{n-1} \otimes A_{n-1} \subseteq A \otimes A.$$ 

So we can write $\Delta(x) = x \otimes 1 + 1 \otimes x + \chi$ for some $\chi \in A^+ \otimes A^+$. For any $h \in h$, direct computation shows that $[x, h]$ is primitive. Thus, by [30, Lemma 4.2], we have $[x, h] \in P(H) \cap A = h$ since $A$ is normal. It proves that $\rho_z \in \text{End}_k(h)$.

Secondly, by Theorem [30, Theorem 4.5], we have relation $x^p + \sigma x + \Theta = 0$ for some $\sigma \in k$ and $\Theta \in A^+$. Thus, $\pi(x^p + \sigma x + \Theta) = z^p + \sigma z = 0$, which yields that $\sigma = \lambda$. In Definition [2.3 (i)] is trivial and (iii), (iv) come from the fact that $\rho_z = \text{ad}x \in \text{Der}(A)$. For (ii), we have

$$(\rho_z)^p(h) = (\text{ad}x)^p(h) = [x^p, h] = [-\lambda x - \Theta, h] = [-\lambda x, h] = \rho_{(-\lambda z)}(x) = \rho_{(z^p)}(h).$$

This proves that $\rho$ is an algebraic representation of $g$ on $h$, which completes the proof. □

4.4. **Invariance.** The fact that $\rho$ is invariant under the equivalence of extensions is implied by the following lemma, which is also used in Section 5.2.
Lemma. In the commutative diagram below

\[
\begin{array}{ccc}
1 & \xrightarrow{1} & u(h) \\
\phi & \downarrow & \phi \\
1 & \xrightarrow{1} & u(h')
\end{array}
\quad
\begin{array}{ccc}
H & \xrightarrow{\pi} & H \\
\psi & \downarrow & \psi \\
H' & \xrightarrow{\pi'} & H'
\end{array}
\quad
\begin{array}{ccc}
1 & \xrightarrow{1} & 1 \\
\phi & \downarrow & \phi \\
1 & \xrightarrow{1} & 1
\end{array}
\]

suppose both rows are exact satisfying \( \dim g = \dim g' = 1 \). Moreover, let \( \rho \) and \( \rho' \) be the corresponding algebraic representations. Thus, \( (\phi, \psi) \), when restricted to their primitive spaces, is a morphism from \( (h, g, \rho) \) to \( (h', g', \rho') \) in \( \mathcal{T} \).

Proof. If \( \psi \) is zero, then there is nothing to prove. In the remaining, suppose \( \psi \neq 0 \). Let \( n \) be the minimal integer such that \( A_n \subset H_n \). Following the definition of the algebraic representation \( \rho \) in the top row, we choose some \( x \in H_n \setminus A_n \) such that \( \pi(x) = z \). Thus, \( \rho_z(h) = [x, h] \) for any \( h \in g \).

Now, we turn to the second row. Denote \( z' = \psi(z) \), which is nonzero since \( \dim g = 1 \) and \( \psi \neq 0 \). Similarly, we define \( A' \) and the integer \( n' \) regarding \( A' \subset H' \). Note that the commutative diagram in the statement yields another commutative diagram with respect to their associated graded Hopf algebras. Hence, we have, by abuse of the notations for \( \pi, \psi \), etc., the following commutative square

\[
\begin{array}{ccc}
H_n/A_n & \xrightarrow{\pi} & g \\
\phi & \downarrow & \psi \\
H'_n/A'_n & \xrightarrow{\pi'} & g'.
\end{array}
\]

If we denote \( x' = \psi(x) \), then we know \( x' \in H'_n \setminus A'_n \setminus A \) and \( \pi'(x') = z' \). Still by definition, we have \( \rho_z(h') = [x', h'] \) for any \( h' \in h \). Since \( \dim g = 1 \), it suffices to check the Diagram (2.2) commutes for \( z \in g \) and any \( h \in h \), which is clear from the above setup. This completes the proof. \( \blacksquare \)

4.5. Classification. Given any short exact sequence in (4.1), then we have a type \( T \) in \( \mathcal{T} \), which is called the type of the extension \( H \). Now, we fix the type \( T = (g, h, \rho) \). We want to classify all extensions of type \( T \). Recall that the \( z \)-operator is given by

\[
\Phi_z = D_z^1 + \lambda D_z^0 = P + \lambda \circ \text{Id} + \rho_z^{p-1}.
\]

Let \( H \) be any extension of type \( T \). Summarize the results we have known so far.

There exists some \( x \in H^+ \setminus A \) satisfying that

(i) \( \Delta(x) = 1 \otimes x + 1 \otimes x + \chi \), for some \( \chi \in A^+ \otimes A^+ \).
(ii) \( \pi(x) = z \) and \( x^p + \lambda x + \Theta = 0 \), for some \( \Theta \in A^+ \).
(iii) \( [x, x_i] = \rho_z(x_i) \), for all \( 1 \leq i \leq d \).

Moreover,
Lemma. We know $\chi$ is a $z$-cocycle satisfying $\Phi_z(\chi) = d^1(\Theta)$ and $\rho_z(\Theta) = 0$.

Proof. We first show that $\chi$ is a cocycle. By the coassociativity of $\Delta$, we have $(1 \otimes \Delta)\Delta(z) = (\Delta \otimes 1)\Delta(z)$. Direct computation yields that $1 \otimes \chi + (1 \otimes \Delta)(\chi) = (\Delta \otimes 1)(\chi) + \chi \otimes 1$. By writing $\chi = \sum (\chi_i) \chi_1 \otimes \chi_2 \in A^+ \otimes A^+$, we have

$$d^2(\chi) = - (\Delta \otimes 1)(\chi) + (1 \otimes \Delta)(\chi) = - (\Delta \otimes 1)(\chi) + 1 \otimes \chi + (1 \otimes \Delta)(\chi) - \chi \otimes 1 = 0.$$  

Hence, $\chi \in Z^2(\Omega A)$. Moreover, by Lemma 3.1 we have $\Delta(x^p + \lambda x + \Theta) = (x \otimes 1 + x \otimes 1 + \chi)^p + \lambda(x \otimes 1 + 1 \otimes x + \chi) + \Delta(\Theta)$

$$= (x^p + \lambda x) \otimes 1 + 1 \otimes (x^p + \lambda x) + [D^1_z(\chi) + \lambda \chi] + \Delta(\Theta)$$

$$= - \Theta \otimes 1 - 1 \Theta + \Phi_z(\chi) + \Delta(\Theta).$$

The last equality comes from the definition $d^1(\Theta) = -\Delta(\Theta)$. So we have $\Phi_z(\chi) = d^1(\Theta)$. Moreover, by Summary (ii) and (iii), we have

$$\rho_z(\Theta) = [x, \Theta] = [x, -x^p - \lambda x] = 0.$$  

This completes the proof.  

4.6. Proof of Theorem D. In the data $\mathcal{D}$, let $\chi$ and $\Theta$ be as in the summary (i) and (ii). Note that the data $\mathcal{D}$ is well-defined by previous lemma. Moreover, in sequence (4.1), we have $\dim H = p^{\dim h + \dim g} = p^{d+1}$. Hence, the isomorphism comes from the construction of $u(\mathcal{D})$ and the degree argument (see Lemma 3.4). This completes the proof.

4.7. The cohomological type group. We will define a cohomological type group $\mathcal{H}^2(B, A)$, which classifies all the extensions of type $T$ up to equivalence.

Definition. In the set $Z^2(\Omega A) \times A^+$, we define

(i) a subset $\mathcal{E}^2(B, A)$, where $(\chi, \Theta)$ belongs to $\mathcal{E}^2(B, A)$ if $\Phi_z(\chi) = d^1(\Theta), \rho_z(\Theta) = 0$;

(ii) an equivalence relation $\sim$, where $(\chi, \Theta) \sim (\chi', \Theta')$ if there exists some $a \in A^+$ such that $d^1(a) = \chi - \chi', \Phi_z(a) = \Theta - \Theta'$. 
Since the differential $d^1$ and the $z$-operator $\Phi_z$ are $\mathbb{K}$-linear, and vanishing on zero, it follows that the equivalence relation is well-defined in the set $Z^2(\Omega A) \times A^+$. Moreover, $Z^2(\Omega A) \times A^+$ becomes an abelian group via $(\chi, \Theta) + (\chi', \Theta') = (\chi + \chi', \Theta + \Theta')$.

**Lemma.** The subset $\mathcal{E}^2(B, A)$ is a subgroup of $Z^2(\Omega A) \times A^+$ and it is invariant under the equivalence relation.

**Proof.** It is easy to see that $\mathcal{E}^2(B, A)$ is a subgroup of $Z^2(\Omega A) \times A^+$. Here, we only check that it is invariant under the equivalence relation. Let $(\chi, \Theta)$ be any element of $\mathcal{E}^2(B, A)$. Suppose it is equivalent to some element $(\chi', \Theta')$ of $Z^2(\Omega A) \times A^+$. We need to show that $(\chi', \Theta')$ is also in $\mathcal{E}^2(B, A)$. By definition, there is some $a \in A^+$ such that

$$\chi' = \chi - d^1(a), \quad \Theta' = \Theta - \Phi_z(a).$$

Thus, $\Phi_z(\chi') = \Phi_z(\chi) - \Phi_z[d^1(a)] = d^1[\Theta - \Phi_z(a)] = d^1(\Theta')$, and, by Proposition 3.1

$$\rho_z(\Theta') = \rho_z[\Theta - \Phi_z(a)] = \rho_z(\Theta) - \rho_z \circ \Phi_z(a) = 0.$$

So, $(\chi', \Theta')$ belongs to $\mathcal{E}^2(B, A)$, which completes the proof. □

Therefore, we can define

$$\mathcal{H}^2(B, A) := \mathcal{E}^2(B, A)/\sim.$$ 

**4.8. The subgroup.** In particular, when the extension $H$ is primitively generated, it includes all the restricted Lie algebra extensions of $\mathfrak{h}$ by $\mathfrak{g}$.

**Definition.** In the set $B^2(\Omega A) \times A^+$, we define a subset $\mathcal{L}^2(\mathfrak{g}, \mathfrak{h})$, where $(\chi, \Theta)$ belongs to $\mathcal{L}^2(\mathfrak{g}, \mathfrak{h})$ if

$$\Phi_z(\chi) = d^1(\Theta), \quad \rho_z(\Theta) = 0.$$

We can view $\mathcal{L}^2(\mathfrak{g}, \mathfrak{h})$ as a subset of $\mathcal{E}^2(B, A)$ by considering $B^2(\Omega A) \times A^+$ as a subset of $Z^2(\Omega A) \times A^+$. Moreover, it is invariant under the equivalence relation $\sim$ in $Z^2(\Omega A) \times A^+$. Thus, we can define

$$\mathcal{H}^2(\mathfrak{g}, \mathfrak{h}) := \mathcal{L}^2(\mathfrak{g}, \mathfrak{h})/\sim.$$ 

By definition, it is a subgroup of $\mathcal{H}^2(B, A)$. 
4.9. Proof of Theorem A(ii). Firstly, we define a map from $\mathcal{E}^2(B, A)$ to all the extensions of $A$ by $B$ of type $T$. For any point $(\chi, \Theta) \in \mathcal{E}^2(B, A)$, take the data $\mathcal{D} = (T, z, \chi, \Theta)$. We show that $u(\mathcal{D})$ is an extension of $A$ by $B$, which is of type $T$.

Let us recall the construction of $u(\mathcal{D})$. As an algebra, it is generated by $x_1, x_2, \cdots, x_d$ and $x$, subject to the relations in $A$ and

$$[x, x_i] = \rho_i(x_i), \text{ for all } 1 \leq i \leq d,$$

$$x^p + \lambda x + \Theta = 0. \quad (4.4)$$

The coalgebra structure is given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \chi, \quad \Delta(x_i) = 1 \otimes x_i + 1 \otimes x_i, \text{ for all } 1 \leq i \leq d. \quad (4.5)$$

It directly follows from the construction (4.3) that $A^+ u(\mathcal{D}) = u(\mathcal{D}) A^+$. Hence $A$ is normal in $u(\mathcal{D})$. Now, consider the following sequence:

$$1 \longrightarrow A \longrightarrow u(\mathcal{D}) \xrightarrow{\pi} B \longrightarrow 1,$$

where $\iota$ is the natural injection, and $\pi(x_i) = 0$ for all $1 \leq i \leq d$ and $\pi(x) = z$. Because of Relation (4.4), we have $\pi(x^p + \lambda x + \Theta) = z^p + \lambda z = 0$. And $(\pi \otimes \pi) \Delta(x) = \Delta(z) = \Delta[\pi(x)]$ by the comultiplications given in (4.3). Hence, $\pi$ is a well-defined Hopf algebra projection. It is clear that $u(\mathcal{D})/u(\mathcal{D})^+ A \simeq B$ via $\pi$. Hence $u(\mathcal{D})$ is an extension of $A$ by $B$. Moreover, the extension is of type $T$ by Relation (4.3).

Therefore, we have a map from $\mathcal{E}^2(B, A)$ to all the extensions of $A$ by $B$ of type $T$.

Secondly, by Theorem D, we know the map above is surjective. It remains to show that the equivalence relation defined in $\mathcal{E}^2(B, A)$ respects the equivalence of extensions. It is implied by the following lemma, which completes the proof.

**Lemma.** Let $(\chi, \Theta)$ and $(\chi', \Theta')$ be two points in $\mathcal{E}^2(B, A)$. Any equivalence $\vartheta : u(\mathcal{D}) \mapsto u(\mathcal{D}')$ can be written as

$$\vartheta(x_i) = x_i, \quad \vartheta(x) = x' - a, \quad (4.6)$$

for all $1 \leq i \leq d$ and some $a \in A^+$. Moreover, $\vartheta$ is an equivalence if and only if

$$d^1(a) = \chi - \chi', \quad \Phi_z(a) = \Theta - \Theta'. \quad (4.7)$$

**Proof.** Firstly, we prove one direction. Let $\vartheta : u(\mathcal{D}) \mapsto u(\mathcal{D}')$ be an equivalence. Hence, it is an isomorphism of Hopf algebras, which yields the following commutative diagram:

$$\begin{array}{ccc} 1 & \longrightarrow & A & \longrightarrow & u(\mathcal{D}) & \longrightarrow & B & \longrightarrow & 1 \\
\vartheta \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & A & \longrightarrow & u(\mathcal{D}') & \longrightarrow & B & \longrightarrow & 1. \end{array} \quad (4.8)$$
Suppose \( u(\mathcal{D}') \) is generated by \( x_1, x_2, \ldots, x_d \) and \( x' \) satisfying \( \pi'(x') = z \) as we did for \( u(\mathcal{D}) \). Note that \( \vartheta = \text{Id} \) on \( A \). Since \( \vartheta \) is a coalgebra map, we have

\[
\Delta[\vartheta(x)] = (\vartheta \otimes \vartheta)\Delta(x).
\]

This implies that \( \Delta[\vartheta(x)] - \vartheta(x) \otimes 1 - 1 \otimes \vartheta(x) \in A \otimes A \). Thus, we can write \( \vartheta(x) = \gamma x' - a \) for some \( \gamma \in \mathbb{k}^\times \) and \( a \in A^+ \) by Lemma 3.5. Because of the commutativity of the Diagram (4.8), \( \pi(x) = \pi'\vartheta(x) = z \), which yields that \( \gamma = 1 \).

Hence, \( \vartheta \) must be in the form of (4.6).

Moreover, direct computation in Equation (4.9) shows that \( d^1(a) = a \otimes 1 + 1 \otimes a - \Delta(a) = \chi - \chi' \). Also since \( \vartheta \) is an algebra map, we have

\[
\vartheta(x^p + \lambda x + \Theta) = x^p + \lambda x' + \Theta - (D_z^1 + \lambda \circ \text{Id})(a) = \Theta - \Theta' - \Phi_z(a) = 0.
\]

Here, we use the fact that \( (x' - a)^p = x^p - a^p - (\text{ad}x')^{p-1}(a) = x^p - a^p - \rho_z^{p-1}(a) = x^p - D_z^1(a) \). Therefore, we have the conditions in (4.7).

Secondly, for the other direction, let \( \vartheta \) be in the from of (4.6) satisfying (4.7). By the same calculation, it is easy to see that \( \vartheta \) is a well-defined bialgebra map from \( u(\mathcal{D}) \) to \( u(\mathcal{D}') \). Then, it is a Hopf algebra map by [7, Proposition 4.2.5]. Finally, according to the form of \( \vartheta \), it is certainly a bijection, which makes the Diagram (4.8) commutative. This completes the proof.

4.10. **Proof of Theorem A(i).** We can view \( h \) as a restricted \( g \)-module via \( \rho \). Then, any restricted Lie algebra extension \( l \) of \( h \) by \( g \) yields a primitively generated extension of \( A \) by \( B \) given by \( u(l) \). It is clear that it is of type \( T \) and vice versa. Hence, equivalence classes of such extensions correspond to a subgroup of \( \mathcal{H}^2(B, A) \), which describes all primitively generated extensions. By Theorem C(ii), the subgroup is represented by points \( (\chi, \Theta) \) in \( \mathcal{E}^2(B, A) \), where \( [\chi] = 0 \) in \( H^2(\Omega A) \) or \( \chi \in B^2(\Omega A) \). Then, the result follows from the definition of \( \mathcal{L}^2(g, h) \) in Section 4.8.

5. **Group quotient of the cohomological type group**

In this section, we classify all finite-dimensional connected Hopf algebras with large abelian primitive space according to their types. We will first describe the type for each object in \( \mathcal{H} \).

5.1. **The types for \( \mathcal{H} \).** We construct a functor from \( \mathcal{H} \) to \( \mathcal{T} \), i.e., \( p: \mathcal{H} \to \mathcal{T} \), and we say that an object \( H \in \mathcal{H} \) has type \( T \) if \( p(H) = T \). As a consequence, isomorphic objects in \( \mathcal{H} \) must have isomorphic images in \( \mathcal{T} \) via the functor \( p \). In other words, objects in \( \mathcal{H} \) are classified, at first step, by their types in \( \mathcal{T} \). We start by showing that \( H \) is always an extension in the sense of (4.1).
Lemma. Let $H$ be a Hopf algebra in $\mathcal{H}$, and $A$ be the Hopf subalgebra of $H$ generated by $P(H)$. Then, $A$ is normal in $H$. Moreover, we have the following short exact sequence:

\[
\begin{array}{cccccc}
1 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & H/A^+H \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & u(\mathfrak{h}) & \longrightarrow & H & \longrightarrow & u(\mathfrak{g}) & \longrightarrow & 1,
\end{array}
\]

where $\mathfrak{h} = P(H)$, and $\mathfrak{g} = P(H/A^+H)$ which is one-dimensional.

Proof. Firstly, we show that $A$ is a normal Hopf subalgebra of $H$. Let $n$ be the minimal integer such that $A_n \subseteq H_n$. Choose any $x \in H_n \setminus A_n$. Following the argument in Proposition 4.3, we know $\Delta(x) - x \otimes 1 - 1 \otimes x \in A \otimes A$. Since $A$ is commutative, it is easy to check that $[x, P(H)] \subseteq P(H) \subseteq A$. Thus, $[x, A] \subseteq A$ for $A$ is generated by $P(H)$. It follows from $\dim H/\dim A = p$ that $\dim (H_n/K_n) = 1$ by [30, Lemma 4.1]. This implies that $H_n$ is spanned by $A_n$ and $x$. Hence, $[H_n, A] \subseteq A$, and $A$ is normal in $H$ by [30, Lemma 4.2].

Secondly, we can make a natural identification $A = u(P(H))$ since $A$ is primitively generated by $P(H)$ according to [30, Proposition 2.2(5)]. Regrading the dimension of the quotient Hopf algebra, we have $\dim (H/A^+H) = p$. Then by [30, Theorem 7.1], we know $H/A^+H$ must be primitively generated by $P(H/A^+H)$, which is one-dimensional. This completes the proof. ■

5.2. The functor $p_\mathcal{H}$. For any object $H \in \mathcal{H}$, its type is defined to be the type of the extension as in Diagram (5.1), i.e.,

\[
p_\mathcal{H}(H) := (P(H), P(H/A^+H), \rho).
\]

It remains to show that $p_\mathcal{H}$ respects the identity map and composition of maps in $\mathcal{H}$. Suppose $\phi : H \to H'$ is a Hopf algebra map between two objects $H, H' \in \mathcal{H}$. We denote restricted Lie algebras $\mathfrak{h}$ (resp. $\mathfrak{h}'$) and $\mathfrak{g}$ (resp. $\mathfrak{g}'$) regarding $H$ (resp. $H'$) as in Lemma 5.1. Since every Hopf algebra map sends primitive elements to primitive elements, it follows that there is a restriction $\phi : u(\mathfrak{h}) \to u(\mathfrak{h}')$. Moreover, by passing to the quotient, the following diagram commutes:

\[
\begin{array}{cccccc}
1 & \longrightarrow & u(\mathfrak{h}) & \longrightarrow & H & \longrightarrow & u(\mathfrak{g}) & \longrightarrow & 1 \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \\
1 & \longrightarrow & u(\mathfrak{h}') & \longrightarrow & H' & \longrightarrow & u(\mathfrak{g}') & \longrightarrow & 1.
\end{array}
\]

Therefore, by Lemma 4.4, we conclude that $p_\mathcal{H}$ is a well-defined functor.
5.3. The subset $\mathcal{H}^2(T)$. Let $T = (h, g, \rho)$ be as in Convention 2.8. We are interested in constructing a set, whose points naturally are in 1-1 correspondence with all the isomorphism classes in $\mathcal{H}$ of type $T$. By using the definitions in Section 4.7 and 4.8, we define

$$\mathcal{H}^2(T) = \mathcal{H}^2(B, A) \setminus \mathcal{H}^2(g, h).$$

5.4. The $G$-action. Next, we define a group action of $\text{Aut}(T)$ on $\mathcal{H}^2(T)$. Recall that any automorphism $\phi \in \text{Aut}(T)$ consists of two automorphisms of $h$ and $g$, which extend to automorphisms of $A$ and $B$ respectively. For simplicity, we will keep the same notation $\phi$ for any of these automorphisms. It is easy to see that there exists a group character $\gamma : \text{Aut}(T) \to k^\times$ such that $\gamma \phi$ is given by

$$\phi(z) = \gamma \phi z,$$

for any $\phi \in \text{Aut}(T)$. We first define the group action on the set $Z_2(\Omega A) \times A^+$ as

$$\phi.(\chi, \Theta) = (\phi.\chi, \phi.\Theta) := (\gamma^{-1}_\phi(\phi \otimes \phi)(\chi), \gamma^{-p}_\phi(\phi(\Theta))),$$

for any $\phi \in \text{Aut}(T)$. We claim that it is well-defined. Choose any point $(\chi, \Theta) \in Z_2(\Omega A) \times A^+$. We need to show that $\phi.\chi \in Z_2(\Omega A)$ and $\phi.\Theta \in A^+$. For the first part, since $\Delta$ is an algebra map, we have $\Delta \phi = (\phi \otimes \phi) \Delta$ on $A^+$. Hence,

$$d^2(\phi.\chi) = (\Delta \otimes 1 + 1 \otimes \Delta)[\gamma^{-1}_\phi(\phi \otimes \phi)(\chi)] = \gamma^{-1}_\phi(\phi \otimes \phi \otimes \phi)[d^2(\chi)] = 0.$$

This shows that $\phi.\chi \in Z_2(\Omega A)$, and certainly $\phi.\Theta \in A^+$. Then, it is direct to check that the above definition gives a group action, which proves the claim. Moreover,

**Lemma.** The subsets $E^2(B, A)$, $L^2(g, h)$ are $\text{Aut}(T)$-invariant. Furthermore, the equivalence relation is preserved by the $\text{Aut}(T)$-action.

**Proof.** Firstly, we show that $E^2(B, A)$ is $\text{Aut}(T)$-invariant. Let $\phi \in \text{Aut}(T)$. Since $z^p + \lambda z = 0$, it follows that $\phi(z^p) = \phi(z)^p = \phi(-\lambda z)$. Thus, $\lambda \gamma^p_\phi = \lambda \gamma_\phi$ by the definition of the group character $\gamma$. By the commutative diagram 2.2, it is easy to
see that \( \rho_{(\phi \otimes \phi)}((\phi \otimes \phi)(\chi)) = (\phi \otimes \phi)[\rho_z(\chi)] \). Then for any \((\chi, \Theta) \in \mathcal{E}^2(B, A)\), we have
\[
\Phi_z(\phi, \chi) = (\phi, \chi)^p + \lambda(\phi, \chi) + \rho_z^{p-1}(\phi, \chi)
\]
\[
= [\gamma^{-1}_\phi(\phi \otimes \phi)(\chi)]^p + \lambda \gamma^{-1}_\phi(\phi \otimes \phi)(\chi) + \rho_z^{p-1}[\gamma^{-1}_\phi(\phi \otimes \phi)(\chi)]
\]
\[
= \gamma^{-p}_\phi(\phi \otimes \phi)(\chi^p) + \gamma^{-p}_\phi(\phi \otimes \phi)(\lambda \chi) + \gamma^{-p}_\phi \rho^{p-1}_{\phi(z)}[(\phi \otimes \phi)(\chi)]
\]
\[
= \gamma^{-p}_\phi(\phi \otimes \phi)[\chi^p + \lambda \chi + \rho_z^{p-1}(\chi)]
\]
\[
= \gamma^{-p}_\phi(\phi \otimes \phi)[\Phi_z(\chi)]
\]
\[
= \gamma^{-p}_\phi(\phi \otimes \phi)[d^1(\Theta)]
\]
\[
= d^1[\gamma^{-p}_\phi \rho(\Theta)]
\]
\[
= d^1(\phi, \Theta).
\]
Moreover, \( \rho_z(\phi, \Theta) = \gamma^{-p-1}_\phi \rho_{(\phi \otimes \phi)}(\phi(\Theta)) = \gamma^{-p-1}_\phi \rho_z(\Theta) = 0 \). Therefore, \((\phi, \Theta, \chi) \in \mathcal{E}^2(B, A)\) by definition, which proves that \( \mathcal{E}^2(B, A) \) is \( \text{Aut}(T) \)-invariant. It is similar to check that \( \mathcal{L}^2(\mathfrak{g}, \mathfrak{h}) \) is \( \text{Aut}(T) \)-invariant. Let \((\chi, \Theta) \in \mathcal{L}^2(\mathfrak{g}, \mathfrak{h})\) such that \( \chi \in \mathbb{B}^2(\Omega A) \). Thus, we can write \( \chi = d^1(a) \) for some \( a \in A^+ \). Since \( \phi \) is an automorphism of \( A \) as Hopf algebras, following the definition of \( d^1 \), we have
\[
d^1(\gamma^{-1}_\phi \phi(a)) = \gamma^{-1}_\phi \phi(a) \otimes 1 + 1 \otimes \gamma^{-1}_\phi \phi(a) - \Delta[\gamma^{-1}_\phi \phi(a)]
\]
\[
= \gamma^{-1}_\phi(\phi \otimes \phi)[a \otimes 1 + 1 \otimes a - \Delta(a)]
\]
\[
= \gamma^{-1}_\phi(\phi \otimes \phi)[d^1(a)]
\]
\[
= \gamma^{-1}_\phi(\phi \otimes \phi)(\chi)
\]
So we have \( \phi, \chi \in \mathbb{B}^2(\Omega A) \), which proves that \( \mathcal{L}^2(\mathfrak{g}, \mathfrak{h}) \) is also \( \text{Aut}(T) \)-invariant.

Secondly, we need to show that the equivalence relation defined on \( Z^2(\Omega A) \times A^+ \) respects the \( \text{Aut}(T) \)-action. Suppose we have \((\chi, \Theta) \sim (\chi', \Theta')\). Thus, there exists some \( a \in A^+ \) such that
\[
d^1(a) = \chi - \chi', \quad \Phi_z(a) = \Theta - \Theta'.
\]
Let \( b = \gamma^{-1}_\phi(\phi(a)) \in A^+ \). Direct computation shows that \( d^1(b) = \phi \cdot \chi - \phi \cdot \chi' \) and
\[
\Phi_z(b) = b^p + \lambda b + \rho_z^{p-1}(b)
\]
\[
= \gamma^{-p}_\phi \phi(a)^p + \lambda \gamma^{-1}_\phi \phi(a) + \gamma^{-p}_\phi \rho^{p-1}_{\phi(z)}[\phi(a)]
\]
\[
= \gamma^{-p}_\phi \phi[a^p + \lambda a + \rho_z^{p-1}(a)]
\]
\[
= \gamma^{-p}_\phi \phi[\Phi_z(a)]
\]
\[
= \gamma^{-p}_\phi \phi(\Theta - \Theta')
\]
\[
= \phi \cdot \Theta - \phi \cdot \Theta'.
\]
Hence $\phi.(\chi, \Theta) \sim \phi.(\chi', \Theta')$, which completes the proof. ■

As a consequence of the previous lemma, we have an induced $\text{Aut}(T)$-action on $\mathcal{H}^2(T) = \mathcal{H}^2(B, A) \setminus \mathcal{H}^2(\mathfrak{g}, \mathfrak{h})$.

5.5. **Proof of Theorem A(iii).** We first show that the Hopf algebras of type $T$ in $\mathcal{H}$ are naturally in 1-1 correspondence with the points in $\mathcal{E}^2(B, A) \setminus \mathcal{L}^2(\mathfrak{g}, \mathfrak{h})$. Let $H$ be any Hopf algebra in $\mathcal{H}$ of type $T$. By Lemma 5.1, we know it is an extension of $A$ by $B$. According to Theorem D, we can assume that $H$ is generated, as an algebra, by all $x_i$’s and $x$, subject to the relations in $A$ and

$$[x, x_i] = \rho_z(x_i), \quad x^p + \lambda x + \Theta = 0.$$  

The coalgebra structure is given by

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i, \quad \Delta(x) = x \otimes 1 + 1 \otimes x + \chi,$$

where $(\chi, \Theta)$ lies in $\mathcal{E}^2(B, A)$. Moreover, by Theorem C(ii), $H \in \mathcal{H}$ if and only if $\chi \not\in B^2(\Omega A)$. This shows the 1-1 correspondence.

Secondly, in order to find isomorphism classes in $\mathcal{H}$ of type $T$, we need to further consider the $\text{Aut}(T)$-action on $\mathcal{H}^2(T)$. Therefore, the result comes from the following lemma.

**Lemma.** Let $(\chi, \theta)$ and $(\chi', \Theta')$ be two points in $\mathcal{E}^2(B, A) \setminus \mathcal{L}^2(\mathfrak{g}, \mathfrak{h})$, and $H$ and $H'$ be the corresponding objects in $\mathcal{H}$. Then, for any map $f : H \to H'$, we have $\phi \in \text{Mor}(T, T)$ with $\phi(z) = \gamma z$ for some $\gamma \in k$, and $f$ can be written as

$$f(x_i) = \phi(x_i), \quad f(x) = \gamma x' - a,$$

for some $a \in A^+$. Moreover, $f$ is an isomorphism if and only if $\phi \in \text{Aut}(T)$ and

$$d^1(\gamma^{-1}a) = \phi.\chi - \chi', \quad \Phi_z(\gamma^{-1}a) = \phi.\Theta - \Theta'.$$

**Proof.** By the functor $p_\mathcal{H} : \mathcal{H} \to \mathcal{T}$, any map $f : H \to H'$ induces a morphism $p_\mathcal{H}(f) = : \phi \in \text{Mor}(T, T)$. Then, we can mimic the proof of Lemma 4.9 with appropriate adjustment in $\phi$, which is not in general the identity map. Since the calculation is straightforward and similar, we omit the details here. ■

5.6. **Remark.** Apparently, the construction of the $\text{Aut}(T)$-orbits in $\mathcal{H}^2(T)$ depends on the choice of the basis of $\mathfrak{g}$. But, if we multiple the basis $z$ by some nonzero scalar $\gamma$, the resulting subset $\mathcal{E}^2(B, A)$ is isomorphic to the previous one via $(\chi, \Theta) \to (\gamma^{-1}\chi, \gamma^{-p}\Theta)$, and so is $\mathcal{L}^2(B, A)$. Moreover, the isomorphism is compatible with the equivalence relation and the $\text{Aut}(T)$-action. Since $\mathfrak{g}$ is one-dimensional, it ensures that our definition is unique up to isomorphisms.
5.7. **Proof of Theorem B.** Because the type of any object in $\mathcal{H}$ is defined by the functor $p_{\mathcal{H}} : \mathcal{H} \to \mathcal{T}$, two Hopf algebras in $\mathcal{H}$ can not be isomorphic if their types are not isomorphic in $\mathcal{T}$. Thus, it suffices to show that for any two isomorphic types $T, T' \in \mathcal{T}$, there is a bijection between their fibers in $\mathcal{H}$ via Hopf algebra isomorphisms, i.e., there is a bijection $\Psi : p_{\mathcal{H}}^{-1}(T) \to p_{\mathcal{H}}^{-1}(T')$ where $\Psi(H) \simeq H$ for all $H \in p_{\mathcal{H}}^{-1}(T)$.

Suppose $T$ is isomorphic to $T' = (h', g', \rho')$ via $\phi$. We let $\phi(z)$ be the basis for $g'$. From Section 5.5, we know that elements of $p_{\mathcal{H}}^{-1}(T)$ are in 1-1 correspondence with elements of $E^2(B, A) \setminus L^2(g, h)$. Also, it is true for $p_{\mathcal{H}}^{-1}(T')$. Thus, we can define the bijection

$$\Psi(\chi, \Theta) := ((\phi \otimes \phi)(\chi), \phi(\Theta)),$$

for any point $(\chi, \Theta) \in E^2(B, A) \setminus L^2(g, h)$. Direct computation shows that $\Psi$ can be extended to isomorphisms of their corresponding Hopf algebras in $\mathcal{T}$ and $\mathcal{T}'$. This completes the proof.

6. **A realization of the group quotient**

We like to realize the isomorphism classes in $\mathcal{H}$ of a given type as group quotient of some geometric space. Throughout this section, let $T = (h, g, \rho)$ be some type in $\mathcal{T}$ as in Convention 2.8. For simplicity, suppose the base field $k$ is perfect.

6.1. **The cohomology ring** $H^*(\Omega A)$. Recall that the cobar construction on $A$ is the following differential graded algebra

$$k \overset{0}{\longrightarrow} A^+ \overset{d^1}{\longrightarrow} (A^+)^2 \overset{d^2}{\longrightarrow} (A^+)^3 \longrightarrow \cdots,$$

where the differentials $d^1$ and $d^2$ are given by

$$d^1(a) = a \otimes 1 + 1 \otimes a - \Delta(a), \quad d^2(a \otimes b) = 1 \otimes a \otimes b - \Delta(a) \otimes b + a \otimes \Delta(b) - a \otimes b \otimes 1.$$

for any $a, b \in A^+$. It is clear that $H^1(\Omega A) = P(A) = \mathfrak{h}$. We define a map $\omega : H^1(\Omega A) \to H^2(\Omega A)$ as

$$\omega(x) = \left[ \sum_{1 \leq i \leq p-1} \binom{p}{i} / p \ x^i \otimes x^{p-i} \right],$$

for any $x \in \mathfrak{h}$. When $A$ is commutative, it is easy to check that $\omega$ is semi-linear with respect to the Frobenius map of $k$ (cf. [30, Lemma 6.3]). Moreover, by [30, Proposition 6.2, we have the vector space isomorphism

$$H^2(\Omega A) = \begin{cases} S^2(\mathfrak{h}) & p = 2, \\ \Lambda^2(\mathfrak{h}) \oplus \omega(\mathfrak{h}) & p > 2, \end{cases}$$

5By abuse of language, we also consider $\omega(x) = \sum_{i=1}^{p-1} \binom{p}{i} / p \ x^i \otimes x^{p-i}$ as an element in $(A^+)^2$. 


where \( S^2(\mathfrak{h}) \) and \( \Lambda^2(\mathfrak{h}) \) are the degree two part of the polynomial and exterior algebra respectively. In general, we claim that, as cohomology ring,

\[
H^\bullet(\Omega A) \simeq \begin{cases} 
S(\mathfrak{h}) & p = 2, \\
\Lambda(\mathfrak{h}) \otimes S(\omega(\mathfrak{h})) & p > 2.
\end{cases}
\]

Let \( C^d_p \) be the elementary abelian \( p \)-group of rank \( d \). Note that \( A^* \) is isomorphic, as an algebra, to the group algebra \( k[C^d_p] \). It is a well-known fact that (e.g., see [21, Proposition 1.4]):

\[
H^\bullet(\Omega A) \simeq HH^\bullet(A^*, k) \simeq H^\bullet(C^d_p, k),
\]

where the right side is the group cohomology of \( C^d_p \) with coefficients in \( k \). Moreover, the isomorphism is on the complex level, and it is compatible with the cup product in the group cohomology and the tensor product in the cobar construction. Finally, by using the well-known group cohomology ring \( H^\bullet(C^d_p, k) \) (e.g., see [18, Section 4]) and Equation (6.1), we get our result.

### 6.2. An embedding

In this section, we assume the characteristic \( p > 2 \). We can do the similar embedding for \( p = 2 \) by changing the formula of \( H^2(\Omega A) \) according to Equation (6.1). Firstly, we embed the affine space \( \mathbb{A}^{d(d+1)/2}_k \times \mathbb{A}^d_k \) into \( Z^2(\Omega A) \times \mathfrak{h} \) by sending any point

\[
P = (a_{ij}, b_k, c_l)_{1 \leq i < j \leq d, 1 \leq k, l \leq d}
\]

to some element \((\chi_P, \Theta_P)\) such that

\[
\chi_P = \sum_{1 \leq i < j \leq d} a_{ij} x_i \otimes x_j + \omega \left( \sum_{1 \leq i \leq d} b_i x_i \right), \quad \Theta_P = \sum_{1 \leq i \leq d} c_i x_i.
\]

Secondly, we know \( A^+ \), which is the restricted enveloping algebra of \( \mathfrak{h} \), has the following basis

\[
\{ x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_d^{\sigma_d} | 0 \leq \sigma_1, \sigma_2, \cdots, \sigma_d \leq p - 1 \}
\]

by the PBW Theorem. We denote by \( A_{\geq 2} \) the subspace of \( A \) spanned by all these bases satisfying \( \sigma_1 + \sigma_2 + \cdots + \sigma_d \geq 2 \). Thus, we have a vector space decomposition \( A^+ = A_{\geq 2} \oplus \mathfrak{h} \).

Thirdly, we define the subset \( S_T \) of \( \mathbb{A}^{d(d+1)/2}_k \times \mathbb{A}^d_k \), where \( P \in S_T \) if

(i) \( \chi_P \notin \mathcal{B}^2(\Omega A) \);
(ii) \( \Phi_2(\chi_P) = d^1(a) \);
(iii) \( \rho_2(a + \Theta_P) = 0 \),

\[\text{[6] This decomposition depends on the choice of the basis of } \mathfrak{h} \text{.} \]
for some $a \in A_{\geq 2}$. Note that the element $a \in A_{\geq 2}$ in the definition above is uniquely determined by $\chi_P$. It is easy to see the uniqueness by taking the difference of two possible solutions. Hence, we will denote $a$ by $\Psi_P$ for any $P \in S_T$.

Next, we identify $A_k^d = h$. Thus, the $z$-operator $\Phi_z$, when restricted to $h$, becomes a regular map from $A_k^d$ to itself. Moreover, we view $h$ as an abelian group via the vector space addition. Let $h$ act on $A_k^d$ by subtraction, i.e., $\Theta x := x - \Phi_z(\Theta)$ for all $x \in A_k^d$ and $\Theta \in h$.

Finally, we denote the quotient space by $A_k^d/h$, and there is a quotient map

$$\pi: A_k^{d(d+1)/2} \times A_k^d \rightarrow A_k^{d(d+1)/2} \times (A_k^d/h).$$

**Proposition.** Every equivalence class in $E^2(B, A) \setminus L^2(g, h)$ can be represented by $(\chi_P, \Psi_P + \Theta_P)$ for some $P \in S_T$. Moreover, elements of $H^2(T)$ are in 1-1 correspondence with points in the image of $S_T$ via the quotient map $\pi$.

**Proof.** Let $(\chi, \Theta)$ be any point in $E^2(B, A) \setminus L^2(g, h)$. By definition, we know $\chi$ is a $z$-cocycle. Thus, we can write $\chi = \chi' + d^1(a)$ for some $a \in A^+$, where

$$\chi' = \sum_{1 \leq i < j \leq d} a_{ij} x_i \otimes x_j + \omega \left( \sum_{1 \leq i \leq d} b_i x_i \right)$$

according to Equation (6.1). We denote $\Theta' = \Theta - \Phi_z(a)$. It is direct to check that $(\chi', \Theta') \sim (\chi, \Theta)$, which is in $E^2(B, A) \setminus L^2(g, h)$.

Now, we will show that $(\chi', \Theta')$ corresponds to some point in $S_T$. By the vector space decomposition $A^+ = A_{\geq 2} \oplus h$, we can write $\Theta' = \Theta_2' + \Theta_1'$, where $\Theta_2' \in A_{\geq 2}$ and $\Theta_1' = \sum_{1 \leq i \leq d} c_i x_i$. Let $P = (a_{ij}, b_i, c_i)$ be the point. By definition, it is clear that

$$(\chi_P, \Theta_P) = (\chi', \Theta_1').$$

Thus, direct computation shows that, in the definition of $S_T$, (ii) and (iii) are satisfied with $a = \Theta_2'$. Since (i) is obvious, it follows that $P \in S_T$. This proves the first half of the statement.

Next, we define a map $f: S_T \rightarrow H^2(T)$ by

$$f(P) = [(\chi_P, \Psi_P + \Theta_P)].$$

That is, it sends any point $P$ of $S_T$ to the equivalence class of $(\chi_P, \Psi_P + \Theta_P)$ in $Z^2(\Omega A) \times A^+$. Moreover, the image of $S_T$ is contained in $E^2(B, A) \setminus L^2(g, h)$ by the definition of $S_T$. By the previous discussion, $f$ is surjective. Hence, it remains to show that $f$ factors through the quotient map $\pi$ and the factorization is injective. Let $P, Q$ be two points in $S_T$. We will prove that $f(P) = f(Q)$ if and only if $\pi(P) = \pi(Q)$, which will complete the proof. By Definition 4.7(ii), $f(P) = f(Q)$ if and only if there exists some $a \in A^+$ such that

$$d^1(a) = \chi_P - \chi_Q, \quad \Phi_z(a) = (\Psi_P + \Theta_P) - (\Psi_Q + \Theta_Q).$$
Note that \( d^1(a) = \chi_P - \chi_Q \) implies that the two cohomology classes represented by them are the same in \( H^2(\Omega A) \). Thus, it follows that \( \chi_P = \chi_Q \) by the explicit expressions of \( \chi_P \) and \( \chi_Q \) according to Equation (6.1). Therefore, we have \( f(P) = f(Q) \) if and only if \( \chi_P = \chi_Q \) and there exists some \( a \in \mathfrak{h} \) \( (d^1(a) = 0) \) such that \( \Phi_z(a) = \Theta_P - \Theta_Q \). In other words, the second condition means that \( \Theta_P \) and \( \Theta_Q \) are in the same orbit of the \( \mathfrak{h} \)-action on \( A^d_{k} \) via \( \Phi_z \). This proves the statement.

### 6.3. Characteristic elements.

We still assume the basis field has characteristic \( p > 2 \). Recall that any \( \xi \in H^2(\Omega A) \) is \( z \)-characteristic if

\[
\Phi_z(\xi) = \xi^p + \lambda \xi + \rho_z^{p-1}(\xi) = 0.
\]

Firstly, by Equation (6.1), we can write every element in \( H^2(\Omega A) \) as

\[
\xi = \Lambda + \omega(x),
\]

where \( \Lambda = \sum_{1 \leq i < j \leq d} \mu_{ij} x_i \wedge x_j \) and \( x = \sum_{1 \leq i \leq d} \mu_i x_i \) for some coefficients in \( k \). Note that the two subspaces \( \Lambda^2(\mathfrak{h}) \) and \( \omega(\mathfrak{h}) \) are all \( \Phi_z \)-invariant. Hence, \( \xi \) is \( z \)-characteristic if and only if

\[
\Phi_z(\Lambda) = 0, \quad \Phi_z[\omega(x)] = 0.
\]

Secondly, we show that the \( \rho \)-action is trivial on \( \omega(\mathfrak{h}) \).

**Lemma.** For any \( x \in \mathfrak{h} \), we have \( \rho_z[\omega(x)] = d^1[-x^{p-1}\rho_z(x)] \). In particular, the \( \rho \)-action is trivial on \( \omega(\mathfrak{h}) \).

**Proof.** Since \( \mathfrak{h} \) is abelian, we have \( \rho_z(x^i) = ix^{i-1}\rho_z(x) \) for any \( x \in \mathfrak{h} \). Thus,

\[
\rho_z[\omega(x)] = \sum_{i=0}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i \rho_z(x) \otimes x^{p-i} + \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i \otimes \rho_z(x^{p-i})
\]

\[
= \sum_{i=1}^{p-1} \frac{(p-1)!}{(i-1)!(p-i)!} x^{i-1} \rho_z(x) \otimes x^{p-i} + \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-1-i)!} x^i \otimes x^{p-1-i} \rho_z(x)
\]

\[
= \sum_{i=0}^{p-2} \binom{p-1}{i} x^i \rho_z(x) \otimes x^{p-1-i} + \sum_{i=1}^{p-1} \binom{p-1}{i} x^i \otimes x^{p-1-i} \rho_z(x)
\]

\[
= (x \otimes 1 + 1 \otimes x)^{p-1} [\rho_z(x) \otimes 1 + 1 \otimes \rho_z(x)] - [x^{p-1} \rho_z(x)] \otimes 1 - 1 \otimes [x^{p-1} \rho_z(x)]
\]

\[
= d^1[-x^{p-1} \rho_z(x)].
\]

It implies that \( \rho_z[\omega(x)] = \epsilon(z) \omega(x) = 0 \) in \( H^2(\Omega A) \). This completes the proof.
Next, applying previous results, we have

\[ \Phi_z(\lambda) = \sum_{1 \leq i < j \leq d} \mu_{ij} x_i^p \wedge x_j^p + \sum_{1 \leq i < j \leq d} \lambda \mu_{ij} x_i \wedge x_j + \sum_{1 \leq i < j \leq d} (p - 1) \left( \frac{p - 1}{k} \right) \mu_{ij} \rho_z^k(x_i) \wedge \rho_z^l(x_j). \]

Moreover, we know \( \omega \) is semi-linear with respect to the Frobenius map of \( k \). Thus, by previous lemma,

\[ \Phi_z(\omega(x)) = \omega(x)^p + \lambda \omega(x) + \rho_z^{p-1}(\omega(x)) = \omega(x^p) + \omega(\lambda^{1/p} x) = \omega(x^p + \lambda^{1/p} x). \]

**Proposition.** By above notations, \( \xi \) is \( z \)-characteristic if and only if \( x^p + \lambda^{1/p} x = 0 \), and the following equality holds in \( \Lambda^2(\mathfrak{h}) \).

\[ \sum_{1 \leq i < j \leq d} \mu_{ij}^p x_i^p \wedge x_j^p + \sum_{1 \leq i < j \leq d} \lambda \mu_{ij} x_i \wedge x_j + \sum_{1 \leq i < j \leq d} (p - 1) \left( \frac{p - 1}{k} \right) \mu_{ij} \rho_z^k(x_i) \wedge \rho_z^l(x_j) = 0. \]

6.4. **Admissible cocycles.** Another way to understand \( H^2(T) \) is to consider the following commutative diagram

\[
\begin{array}{ccc}
H^2(T) & \xrightarrow{q} & A^{d(d+1)/2} \times (A_k^d/\mathfrak{h}) \\
\downarrow q & & \downarrow p_1 \\
H^2(\Omega A) & \xrightarrow{p_1} & A_k^{d(d+1)/2},
\end{array}
\]

where \( p_1 \) is the projection of the first component. Moreover, \( q \) sends every equivalence class \([\chi, \Theta]\) to the cohomology class in \( H^2(\Omega A) \) represented by \( \chi \). It is clear that \( q \) maps every equivalence class into a nonzero \( z \)-characteristic element in \( H^2(\Omega A) \). We are interested in the inverse problem: when does a \( z \)-characteristic element have preimage in \( H^2(T) \). In order to answer this question, we give the following definition.

**Definition.** Let \( \xi \) be \( z \)-characteristic in \( H^2(\Omega A) \), which is represented by some cocycle \( \chi \). We say \( \xi \) is **admissible** if there exists some \( a \in A^+ \) such that \( \Phi_z(\chi) = d^1(a) \) and \( \rho_z(a) = 0 \).

We need to show that the definition does not depend on the choice of the representative cocycle \( \chi \). Suppose \( \chi' \) is another cocycle representing \( \xi \). Then, there is some \( X \in A^+ \) satisfying \( \chi' = \chi + d^1(X) \). If we have found some \( a \in A^+ \) satisfying \( \Phi_z(\chi) = d^1(a) \) and \( \rho_z(a) = 0 \). Let \( b = a + \Phi_z(X) \). It is easy to see that \( \Phi_z(\chi') = d^1(b) \) and \( \rho_z(b) = \rho_z(a) + \rho_z \circ \Phi_z(X) = 0 \) by Proposition 3.1(iii). So, the property of admissibility is well-defined for any \( z \)-characteristic element in \( H^2(\Omega A) \).
Remark. We make some observations concerning the above definition.

(i) A $z$-characteristic element has preimage in $\mathcal{H}^2(T)$ if and only if it is nonzero admissible.

(ii) Let $\chi$ be a $z$-cocycle, and $a \in A_{\geq 2}$ be the unique element determined by $\Phi_a(\chi) = d^1(a)$. Thus, $[\chi]$ is admissible if and only if $\rho_z(a) \in \text{Im} \rho_z$.

(iii) The admissibility is preserved by base field extension.

In [22], a finite-dimensional restricted Lie algebra $L$ is called a torus if it is abelian and every element of $L$ is semisimple in $u(L)$, i.e., generates a semisimple subalgebra.

Proposition. If either $h$ or $g$ is a torus, (either $A$ or $B$ is semisimple), thus, every $z$-characteristic element is admissible.

Proof. Without loss of generality, we can assume the base field is algebraically closed. Here, we only treat the case when $p = 2$. For the other case $p > 2$, the argument is similar. In the following, let $\xi$ be a $z$-characteristic element in $H^2(\Omega A)$. Since the admissibility does not depend on the choice of the representing cocycle, we can write the cocycle as

$$\chi = \sum_{1 \leq i \leq j \leq d} \mu_{ij} x_i \otimes x_j$$

for some coefficients $\mu_{ij} \in k$ according to Equation (6.1).

(i) $h$ is a torus. By [10], we can further assume that $x_i^p = x_i$ for $1 \leq i \leq d$. Hence, by Definition 2.3(iv), we know $\rho_z = 0$. Therefore,

$$\Phi_z(\chi) = \chi^p + \lambda \chi + \rho_z^{p-1}(\chi) = \sum_{1 \leq i \leq j \leq d} (\mu_{ij}^p + \lambda \mu_{ij}) x_i \otimes x_j.$$

So $\chi$ is a $z$-cocycle if and only if all the coefficients $\mu_{ij}^p + \lambda \mu_{ij} = 0$ since $[x_i \otimes x_j]$ where $1 \leq i \leq j \leq d$ is a basis for $H^2(\Omega A)$. Then, we can take $a = 0$ in the above definition, which implies that $\xi$ is admissible.

(ii) $g$ is a torus. Then, in the relation $z^p + \lambda z = 0$, we have $\lambda \neq 0$. By Definition 2.3(iv), we know $\rho_z(h^p) = 0$. Since the base field is algebraically closed, it follows that $h^p$ is a subspace, and hence $\rho_z$ is diagonalizable on $h/h^p$. Without loss of generality, we can assume that in the basis $x_1, x_2, \ldots, x_d$ of $h$, the first $s$ elements form a basis for the subspace $h^p$, and the images of the remaining $d - s$ elements are eigenvectors in the quotient space $h/h^p$. In other words, we can set up $\rho_z(x_i) = 0$ for all $1 \leq i \leq s$, and $\rho_z(x_j) = \sigma_i x_j + y_j$ for some $y_j \in h^p$, when $s + 1 \leq j \leq d$. It is easy to see that, if the eigenvalue $\sigma_j = 0$, we have $y_j = 0$. Therefore, by replacing $x_j$ by $x_j + y_j/\sigma_j$ when $\sigma_j \neq 0$, we can always assume in the above setup $y_j = 0$ for all $s + 1 \leq j \leq d$. Moreover, if we write $\sigma_i = 0$ for all $1 \leq i \leq s$, thus $\rho_z(x_i) = \sigma_i x_i$ for all $1 \leq i \leq d$. 

Now, we consider the Hopf subalgebra of $A$, which is generated by the restricted Lie subalgebra $h^p$. We denote it by $C = u(h^p)$. Thus,

\[(6.2) \quad \Phi_z(\chi) = \chi^p + \sum_{1 \leq i \leq j \leq d} \lambda \mu_{ij} x_i \otimes x_j + \sum_{1 \leq i \leq j \leq d} \left( \binom{p-1}{k} \right) \mu_{ij} \rho_z^k(x_i) \wedge \rho_z^l(x_j)\]

\[= \chi^p + \sum_{1 \leq i \leq j \leq d} \mu_{ij} [\lambda + (\sigma_i + \sigma_j)^{p-1}] x_i \otimes x_j.\]

Since the $p$-th map in $\Omega A$ commutes with the differential by Proposition 3.1(i), we can view $\chi^p$ as a cocycle in the subcomplex $\Omega C$. Therefore, there exists some $X \in C^+$, where we can write

\[\chi^p = d^1(X) + \sum_{1 \leq i \leq j \leq d} \tau_{ij} x_i \otimes x_j.\]

Combine it with the above Equation (6.2), we get

\[\Phi_z(\chi) = d^1(X) + \sum_{1 \leq i \leq j \leq d} \phi_{ij} x_i \otimes x_j,\]

for some new coefficients $\phi_{ij} \in k$. Hence, we have $\chi$ is a $z$-cocycle if and only if all the coefficients $\phi_{ij} = 0$ since $[x_i \otimes x_j]$ where $1 \leq i \leq j \leq d$ is a basis for $H^2(\Omega A)$. Then, we can take $a = X$ in the above definition. It is clear that $\rho_z(X) = 0$ for $X \in A^p$, which completes the proof. 

6.5. Fibers over $q$. Let $\xi$ be a nonzero admissible $z$-cocycle in $H^2(\Omega A)$. We like to know what does its fiber $q^{-1}(\xi)$ look like in $H^2(T)$. As discussed in Section 6.2, the additive group $h$ acts on the affine space $A_k^{d+1} = h$ by $\Theta.x = x - \Phi_z(\Theta)$. Also, we can think $\rho_z$ as a map from $A_k^{d+1}$ to itself, whose kernel is a subspace. We denote it by $\text{Ker} \rho_z$. Because of Proposition 3.1(iii), the $h$-action can be restricted to the subspace $\text{Ker} \rho_z$.

**Lemma.** Points in the fiber $q^{-1}(\xi)$ are in 1-1 correspondence with $h$-orbits in $\text{Ker} \rho_z$. Moreover, the map $q$ is injective if and only if $\text{Ker} \rho_z = \text{Im} \Phi_z$.

**Proof.** Consider the following commutative diagram:

\[\begin{array}{ccc}
S_T & \xrightarrow{\pi} & A_k^{d(d+1)/2} \times (A_k^d/h) \\
\downarrow{f} & & \downarrow{g} \\
\mathcal{H}^2(T) & \xrightarrow{q} & H^2(\Omega A).
\end{array}\]
As a consequence of Proposition 6.2, points in \( q^{-1}(\xi) \) are in 1-1 correspondence with points in \( \pi(fg)^{-1}(\xi) \). Following the notations in Section 6.2, we have an embedding from \( S_T \) to \( Z^2(\Omega A) \times h \), where the image of any point \( P \) is written by \((\chi_P, \Theta_P)\). Recall that \( \Psi_P \) denotes the unique element in \( A_{\geq 2} \) such that

\[
\Phi_z(\chi_P) = d^1(\Psi_P), \quad \rho_z(\Psi_P + \Theta_P) = 0.
\]

Since the nonzero \( z \)-characteristic element \( \xi \) is admissible, by Remark 6.4(i), there exists some point \( P \) in the fiber \( (fg)^{-1}(\xi) \). It is clear, by definition, any other point \( Q \) belongs to \((fq)^{-1}(\chi)\) if and only if \( \chi_Q = \chi_P =: \chi \), \( \rho_z(\Psi_Q + \Theta_Q) = 0 \).

Since \( \Psi_P = \Psi_Q \), it follows that \( \rho_z(\Theta_P - \Theta_Q) = 0 \). Thus,

\[
(fg)^{-1}(\xi) = \{ Q \in S_T | (\chi_Q, \Theta_Q) \in (\chi, Q_P + \text{Ker} \rho_z) \}.
\]

Therefore, by definition, \( \pi(fg)^{-1}(\xi) = \pi(\chi) \times (Q_P + \text{Ker} \rho_z/h) \simeq \pi(\chi) \times (\text{Ker} \rho_z/h) \). This proves the first statement. The second statement is a direct application of the first one, since \( q \) is injective if and only if \( h \)-orbit in \( \text{Ker} \rho_z \) is single if and only if \( \text{Ker} \rho_z = \text{Im} \Phi_z \). ■

We combine the results we have obtained so far in this section and the previous one.

**Proposition.** Let either \( h \) or \( g \) be a torus, and assume that \( \text{Ker} \rho_z = \text{Im} \Phi_z \). Then, elements in \( H^2(T) \) are in 1-1 correspondence with nonzero \( z \)-characteristic elements in \( H^2(\Omega A) \).

**Proof.** Firstly, since \( \text{Ker} \rho_z = \text{Im} \Phi_z \), by previous lemma, we know \( H^2(T) \) can be embedded in \( H^2(\Omega A) \) via \( q \). Secondly, by Remark 6.4(i), \( q(H^2(T)) \) contains exactly those nonzero admissible \( z \)-characteristic elements. Finally, the result follows from Proposition 6.4. ■

6.6. **The \( G \)-action.** Now, we will consider the \( \text{Aut}(T) \)-action on \( H^2(T) \). Firstly, we assume that the base \( p > 2 \). Let \( \phi \in \text{Aut}(T) \). Recall that the group character \( \gamma : \text{Aut}(T) \to k^\times \) is defined by \( \phi(z) = \gamma_\phi z \). According to Equation (6.1), we can write any cohomology class \( \xi \in H^2(\Omega A) \) as

\[
\xi = \sum_{1 \leq i < j \leq d} \mu_{ij} x_i \wedge x_j + \omega \left( \sum_{1 \leq i \leq d} \mu_i x_i \right).
\]

Regarding Equation (5.2), the action of \( \phi \in \text{Aut}(T) \) on \( H^2(\Omega A) \) is given by

\[
\phi(\xi) = \sum_{1 \leq i < j \leq d} \gamma_\phi^{-1} \mu_{ij} \phi(x_i) \wedge \phi(x_j) + \omega \left( \sum_{1 \leq i \leq d} \gamma_\phi^{-1/p} \mu_i \phi(x_i) \right).
\]
The group action can be restricted to those nonzero admissible \( z \)-characteristic elements. And it is compatible with the \( G \)-action on \( H^2(T) \) via \( q \).

Furthermore, if the base field is \( \mathbb{K} \), then we can identify \( H^2(\Omega A) = \Lambda^1(h) \oplus \Lambda^2(h) \), i.e., the degree one and two part of the exterior algebra \( \Lambda(h) \). Thus, we can rewrite

\[
\xi = \sum_{1 \leq i \leq d} \mu_i x_i + \sum_{1 \leq i < j \leq d} \mu_{ij} x_i \wedge x_j.
\]

Then, we have a group homomorphism from \( \text{Aut}(T) \) to the automorphism group of \( \Lambda(h) \), which is given by

\[
\phi(\xi) = \sum_{1 \leq i \leq d} \gamma^{-1}_\phi \mu_i \phi(x_i) + \sum_{1 \leq i < j \leq d} \gamma^{-1}_\phi \mu_{ij} \phi(x_i) \wedge \phi(x_j).
\]

The discussion for \( p > 2 \) is similar, so we omit it here.

7. Examples

7.1. Semisimple Hopf algebras in \( \mathcal{F} \). In this example, we classify semisimple connected Hopf algebras in \( \mathcal{H} \). For simplicity, suppose the base field \( k \) is algebraically closed. We recall the criteria for semisimple connected Hopf algebras.

**Theorem.** [16, 31] Let \( H \) be a finite-dimensional connected Hopf algebra. The following are equivalent:

(i) \( H \) is semisimple.

(ii) \( H \) is commutative and semisimple.

(iii) \( P(H) \) is a torus.

(iv) \( H^* \simeq k[G] \), for some \( p \)-group \( G \).

Let \( H \) be a semisimple connected Hopf algebra in \( \mathcal{H} \). Following Section 4.3, denote \( h = P(H) \), and \( g = P(H/A^+H) \), where \( A = u(h) \). By the previous theorem, we know \( H \) is semisimple if and only if \( h \) is a torus. Note that \( H \) is commutative. Thus, the Hopf algebra \( H \) has type \( T = (h, g, 0) \in \mathcal{F} \). Moreover, since quotient of any semisimple Hopf algebra is semisimple, it follows that \( g \) is a one-dimensional torus. We call type \( T = (h, g, 0) \) such that \( h, g \) are tori a *semisimple type*. By Theorem A (iv), isomorphism classes of semisimple connected Hopf algebras in \( \mathcal{H} \) are in 1-1 correspondence with points in

\[
\prod_T H^2(T)/\text{Aut}(T),
\]

where \( T \) runs through isomorphism classes of all semisimple types in \( \mathcal{F} \).

Firstly, we classify all the semisimple types in \( \mathcal{F} \). Since the base field is algebraically closed, it is clear that isomorphism classes of semisimple types are parametrized by \( \dim h \). For any \( d \geq 1 \), we choose a representative \( T_d = (h, g, 0) \), where we fix a
basis \( x_1, x_2, \ldots, x_d \) for \( \mathfrak{h} \), and a basis \( z \) for \( \mathfrak{g} \) satisfying \( x_i^p = x_i \) for all \( 1 \leq i \leq n \) and \( z^p = z \).

Secondly, we find \( \mathcal{H}^2(T_d) \) for each \( T_d \). It is direct to check that

\[
\Phi_z(x) = x^p - x = \sum_{1 \leq i \leq d} (\mu_i^p - \mu_i)x_i,
\]

for any \( x = \sum_{1 \leq i \leq d} \mu_ix_i \) in \( \mathfrak{h} \). Hence, \( \text{Im}\Phi_z = \mathfrak{h} = \text{Ker}\rho_z \) since \( k = \overline{k} \). Then by Lemma 6.3, we can embed \( \mathcal{H}^2(T_d) \) into \( \mathcal{H}^2(\Omega A) \) via the map \( q \). Moreover by Proposition 6.5, elements in \( \mathcal{H}^2(T_d) \) are in 1-1 correspondence with nonzero \( z \)-characteristic elements in \( \mathcal{H}^2(\Omega A) \). Then, we follow Section 6.3 to find all the \( z \)-characteristic elements in \( \mathcal{H}^2(\Omega A) \). When \( p = 2 \), we can write any cohomology class by

\[
\xi = \sum_{1 \leq i \leq j \leq d} \mu_{ij} x_ix_j.
\]

Thus, direct computation shows that

\[
\Phi_z(\xi) = \xi^p - \xi
= \sum_{1 \leq i \leq j \leq d} \mu_{ij}^p x_i^p x_j^p - \sum_{1 \leq i \leq j \leq d} \mu_{ij} x_i x_j
= \sum_{1 \leq i \leq j \leq d} (\mu_{ij}^p - \mu_{ij}) x_i x_j.
\]

So \( \xi \) is \( z \)-characteristic if and only if all the coefficients are in the finite field \( \mathbb{K} \). When \( p > 2 \), we can write any cohomology class by

\[
\xi = \sum_{1 \leq i < j \leq d} \mu_{ij} x_i x_j + \omega \left( \sum_{1 \leq i \leq d} \mu_i x_i \right).
\]

Similar computation shows the same result. Hence, elements in \( \mathcal{H}^2(T) \) are in 1-1 correspondence with points in \( \mathbb{A}_{\mathbb{K}}^{d(d+1)/2} \setminus \{0\} \).

Thirdly, we find the automorphism group \( \text{Aut}(T_d) \). It is clear that \( \text{Aut}(T_d) = \text{Aut}(\mathfrak{h}) \times \text{Aut}(\mathfrak{g}) \) since \( \rho_z = 0 \). We can consider \( \text{Aut}(\mathfrak{h}) \) as a subgroup of \( \text{GL}(d, k) \). Direct computation shows that \( \text{Aut}(\mathfrak{h}) = \text{GL}(d, \mathbb{K}) \), and the same argument works for \( \text{Aut}(\mathfrak{g}) \) as well. Hence, we have \( \text{Aut}(T_d) = \text{GL}(d, \mathbb{K}) \times \mathbb{K}^\times \).

In a conclusion, for the \( \text{Aut}(T_d) \)-quotient of \( \mathcal{H}^2(T_d) \), it is equivalent to consider the group action of \( \text{GL}(d, \mathbb{K}) \times \mathbb{K}^\times \) on \( \mathbb{A}_{\mathbb{K}}^{d(d+1)/2} \setminus \{0\} \) correspondingly. By Equation (5.2), the subgroup \( \mathbb{K}^\times \) acts on \( \mathbb{A}_{\mathbb{K}}^{d(d+1)/2} \setminus \{0\} \) by inverse multiplication, whose quotient space is the projective space \( \mathbb{P}_{\mathbb{K}}^{d(d-1)/2} \). Furthermore, the group action of \( \text{GL}(d, \mathbb{K}) \) factors through \( \text{PGL}(d, \mathbb{K}) \). Hence in general, \( \mathcal{H}^2(T)/\text{Aut}(T) \) are in 1-1
correspondence with
\[ \mathbb{P}^{d(d-1)/2}_K / \text{PGL}(d, \mathbb{K}). \]

Finally, we will describe the group action and the quotient space in a more explicit way. In the following, we work over \( \mathbb{K} \).

When \( p = 2 \), points of \( \mathbb{P}^{d(d-1)/2}_K \) are in 1-1 correspondence with quadratic curves in \( \mathbb{P}^{d-1}_K \) by
\[ [\mu_{ij}]_{1 \leq i \leq j \leq d} \mapsto \sum_{1 \leq i \leq j \leq d} \mu_{ij} X_i X_j. \]

Note that \( \text{Aut}(\mathbb{P}^{d-1}_K) = \text{PGL}(d, \mathbb{K}) \). Moreover, by Equation (5.2), \( \text{PGL}(d, \mathbb{K}) \) acts on quadratic curves by automorphisms of the projective space \( \mathbb{P}^{d-1}_K \). Hence, \( \text{Aut}(T_d) \)-orbits in \( H^2(T_d) \) are in 1-1 correspondence with isomorphism classes of quadratic curves in \( \mathbb{P}^{d-1}_K \).

When \( p > 2 \), we denote by \( \Lambda(\mathfrak{h}) \) the exterior algebra over \( \mathfrak{h} \), and \( W = \Lambda^1(\mathfrak{h}) \oplus \Lambda^2(\mathfrak{h}) \) denotes its degree one and two part. As in Section 6.6, points of \( \mathbb{P}^{d(d-1)/2}_K \) are in 1-1 correspondence with one-dimensional subspaces of \( W \) by
\[ [\mu_i, \mu_{jk}]_{1 \leq i \leq d, 1 \leq j < k \leq d} \mapsto \sum_{1 \leq i \leq d} \mu_i x_i + \sum_{1 \leq j \leq d} \mu_{jk} x_j \land x_k. \]

Moreover, \( \text{PGL}(d, \mathbb{K}) \) acts on \( \mathbb{P}(W) \) by affine automorphisms of \( \Lambda(\mathfrak{h}) \). If we consider those one-dimensional subspaces of \( W \) as “noncommutative” quadratic curves in the exterior algebra, then \( \text{Aut}(T_d) \)-orbits in \( H^2(T_d) \) are in 1-1 correspondence with isomorphism classes of quadratic curves in \( \Lambda(\mathfrak{h}) \) by affine automorphisms.

Let \( C \) be a quadratic curve in \( \mathbb{P}^{d-1}_K \) (\( p = 2 \)) or \( \Lambda(\mathfrak{h}) \) (\( p > 2 \)) with coefficients \( \mu_{ij} \) (up to a scalar) for \( 1 \leq i \leq j \leq d \). The corresponding semisimple connected Hopf algebra can be realized as
\[ k[x_1, x_2, \cdots, x_d, x]/(x_1^p - x_1, \cdots, x_d^p - x_d, x^p - x). \]

The coalgebra structure is given by
\[ \Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i \]
for all \( 1 \leq i \leq d \) and
\[ \Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{1 \leq i \leq j \leq d} \mu_{ij} x_i \otimes x_j \]
when \( p = 2 \) and
\[ \Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{1 \leq i < j \leq d} \mu_{ij} x_i \otimes x_j + \omega \left( \sum_{1 \leq i \leq d} \mu_{ii} x_i \right) \]
when \( p > 2 \).
Regarding part (iv) of the previous theorem, semisimple connected Hopf algebras are in 1-1 correspondence with $p$-groups. In other words, to classify all semisimple connected Hopf algebras (over an algebraically closed field of characteristic $p > 0$) are equivalent to classify all the $p$-groups. We recall that the Frattini group of a $p$-group $G$ is the smallest normal subgroup $N$, where $G/N$ is an elementary abelian $p$-group. Let $G$ be a $p$-group of order $p^{d+1}$ with Frattini group isomorphic to $C_p$. Then, we have the following $p$-group extension:

$$1 \rightarrow C_p \rightarrow G \rightarrow C_p^d \rightarrow 1.$$ 

By taking their group algebras, then we have a short exact sequence of finite-dimensional local Hopf algebras:

$$1 \rightarrow k[C_p] \rightarrow k[G] \rightarrow k[C_p^d] \rightarrow 1.$$ 

According to [10], we can write the dual Hopf algebra of $k[C_p^d]$ as $u(h)$ for some torus $h$ of dimension $d$. Similarly, we write $(k[C_p])^*$ as $u(g)$ for some one-dimensional torus $g$. Therefore by dualising the above sequence (see [6, Lemma 4.1]), we will end up with a short exact sequence in the sense of (4.1). Moreover, the dual Hopf algebra $(k[G])^*$ corresponds to some extension in (4.1), whose primitive space is isomorphic to $h$, i.e., some semisimple connected Hopf algebra in $\mathcal{H}$. Hence, semisimple connected Hopf algebra in $\mathcal{H}$ are in 1-1 correspondence with $p$-groups whose Frattini group is isomorphic to $C_p$. Combining all the results, we get Proposition 1.7 in the Introduction.

7.2. Some $p^3$-dimensional Hopf algebras in $\mathcal{H}$. In this example, we work over a special type $T = (h, g, \rho) \in \mathcal{T}$, where explicit computation is given for $\mathcal{H}^2(T)$ and the Aut$(T)$-orbits in $\mathcal{H}^2(T)$. Suppose $h$ is two-dimensional with a basis $x, y$, and let $z$ be a basis for $g$. Moreover, suppose we have $x^p = 0, y^p = y$ and $z^p = 0$, and the algebraic representation $\rho$ is given by $\rho_z(x) = y, \rho_z(y) = 0$. In the following, let the base field $k$ be algebraically closed of characteristic $p > 2$.

Firstly, the $z$-operator is given by $\Phi_z = \mathcal{P} + \rho_z^{p-1}$. Hence, for any $t = \alpha x + \beta y \in h$, we have

$$\Phi_z(t) = t^p + \rho_z^{p-1}(t) = \beta^p y.$$ 

Thus, it is clear that $\text{Im} \Phi_z = \text{Ker} \rho_z$ on $h$. By Lemma 3.5, points in $\mathcal{H}^2(T)$ are in 1-1 correspondence with nonzero admissible $z$-characteristic elements in $\mathcal{H}^2(\Omega A)$.

Secondly, every element $\xi$ of $H^2(\Omega A)$ can be represented by some cocycle

$$\chi = ax \otimes y + \omega(bx + cy),$$

for some $a, b, c \in k$. Direct computation shows that

$$\Phi_z(\xi) = \xi^p + \rho_z^{p-1}(\xi) = \omega(c^p y) + \rho_z^{p-1}\omega(bx + cy).$$
By Lemma 6.3, we know $\xi$ is $z$-characteristic if and only if $c = 0$. In the remaining, let $c = 0$. Hence, we have
\[ \rho_z^{p-1}\omega(bx) = \rho_z^{p-2}[d^1(X)] = d^1[\rho_z^{p-2}(X)], \]
where $X = -(bx)^{p-1}\rho_z(bx)$. Thus,
\[ \rho_z^{p-2}(X) = -\rho_z^{p-2}(b^p x^{p-1}y) = -(p - 1)!b^p xy^{p-1} = b^p xy^{p-1}. \]

We denote $\Theta = b^p(xy^{p-1} - x)$. By previous computation, It is clear to see that
\[ \Phi_z(\chi) = d^1(\Theta), \quad \rho_z(\Theta) = b^p \rho_z(xy^{p-1} - x) = b^p(y^p - y) = 0. \]
Then, it follows that every $z$-characteristic element is admissible. Hence, we can identify $\mathbb{A}_k^2 \setminus \{0\}$ with $H^2(T)$ by sending $(a, b)$ to the element
\[ \xi = ax \wedge y + \omega(bx), \]
in $H^2(\Omega A)$ as in Section 6.3.

Thirdly, direct computation shows that
\[ \text{Aut}(T) = \{(\alpha, \beta, \gamma) \in k^\times \times \mathbb{K}^\times \times k^\times | \alpha\gamma = \beta\}. \]
Let $\phi = (\alpha, \beta, \gamma) \in k^\times \times \mathbb{K}^\times \times k^\times$. Thus, the automorphism $\phi$ is given by
\[ \phi(x) = \alpha x, \phi(y) = \beta y, \phi(z) = \gamma z. \]
Therefore, by Equation (5.2), we have
\[ \phi(\xi) = \gamma^{-1}\{a\phi(x) \wedge \phi(y) + \omega[b\phi(x)]\} = (\gamma^{-1}\alpha\beta\alpha)x \wedge y + \omega[(\gamma^{-1/p}\alpha b)x]. \]
Finally, $\text{Aut}(T)$-orbits in $H^2(T)$ are in 1-1 correspondence with $k^\times \times \mathbb{K}^\times \times k^\times$-orbits in $\mathbb{A}_k^2 \setminus \{0\}$, where the group action is given by
\[ \phi(a, b) = (\gamma^{-1}\alpha\beta a, \gamma^{-1/p}\alpha b) = (\alpha^2 a, \gamma^{-1/p}\alpha b). \]
for any $\phi = (\alpha, \beta, \gamma) \in k^\times \times \mathbb{K} \times k^\times$ satisfying $\alpha\gamma = \beta$.

It is clear to see that the group orbits in $\mathbb{A}_k^2 \setminus \{0\}$ are represented by two points $(1, 0)$ and $(0, 1)$ and one line quotient $(\lambda, 1)/G$ for $\lambda \neq 0$, where the finite group $G$ is the multiplicative group of the $(p^2 - 1)/2$th roots of unity. Finally, we have these $p^3$-dimensional connected Hopf algebras listed in Example 1.7 in the Introduction.

7.3. Non-admissible cocycles. In Section 6.4 we prove that every $z$-characteristic element is admissible if either $\mathfrak{h}$ or $\mathfrak{g}$ is a torus. Now, we give an example to show that there exists some $z$-cocycle, whose representing cohomology class is not admissible. We first state a result on derivations of commutative algebras over a base field $k$ of characteristic $p > 0$. Let $A$ be a commutative algebra, and $\delta \in \text{Der}(A)$ satisfying $\delta^p = 0$. Then, $\delta^{p-1}[x^{p-1}\delta(x)] = \delta(x)^p$ for any $x \in A$. It can be proved easily by combinatorics.
Let $T = (\mathfrak{h}, \mathfrak{g}, \rho)$ be a type in $\mathcal{T}$, and let $\mathfrak{g}$ be $p$-nilpotent. Choose a basis $z$ for $\mathfrak{g}$. By assumption, we have $z^p = 0$. For any $x \in \mathfrak{h}$, we will study the cocycle

$$\omega(x) = \sum_{1 \leq i \leq p-1} \binom{p}{i} \rho^{p-i} x^i \otimes x^{p-i}.$$ 

We first find the condition when $\omega(x)$ becomes a $z$-cocycle, and then determine when it represents an admissible cohomology class. Note that in this case, the $z$-operator is given by $\Phi_z(\chi) = \chi^p + \rho_z^{p-1}(\chi)$ for any cocycle $\chi \in (u(\mathfrak{h}))^2$. Thus, by Lemma 6.3, we have

$$\Phi_z[\omega(x)] = \omega(x)^p + \rho_z^{p-1}[\omega(x)]$$

$$= \omega(x^p) + \rho_z^{p-2}\{d^1[-x^{p-1}\rho_z(x)]\}$$

$$= \omega(x^p) + d^1\{\rho_z^{p-2}[-x^{p-1}\rho_z(x)]\}.$$ 

Therefore, we see that $\omega(x)$ is a $z$-cocycle if and only if $x$ is $p$-nilpotent, or $x^p = 0$. In the remaining, suppose $x^p = 0$.

We briefly write $X = \rho_z^{p-2}[x^{p-1}\rho_z(x)]$. Thus, we have $\Phi_z[\omega(x)] = d^1(X)$. By definition 6.4, $[\omega(x)]$ is admissible if there exists some $Y \in u(\mathfrak{h})^+$ such that $\Phi_z(\chi) = d^1(Y)$ and $\rho_z(Y) = 0$. Suppose such $Y$ exists, then it would imply that $d^1(X-Y) = 0$. Thus, we can denote $Y = X + a$ for some $a \in \mathfrak{h}$. Hence, $\rho_z(X) + \rho_z(a) = 0$. Note that $\rho_z$ is a derivation on $u(\mathfrak{h})$. By applying the above result on derivations, we get

$$\rho_z(X) = -\rho_z^{p-1}[x^{p-1}\rho_z(x)] = -[\rho_z(x)]^p = -\rho_z(a).$$ 

So we see that $[\rho_z(x)]^p \in \text{Im}\rho_z$. On the contrary, if $[\rho_z(x)]^p \in \text{Im}\rho_z$. Then, we can write $[\rho_z(x)]^p = \rho_z(a)$ for some $a \in \mathfrak{h}$. By taking $Y = X + a$, it is clear that $\Phi_z[\omega(x)] = d^1(Y)$ and $\rho_z(Y) = 0$. So $[\omega(x)]$ is admissible.

In a conclusion, we have $\omega(x)$ is $z$-characteristic if and only if $x$ is $p$-nilpotent. Moreover, $[\omega(x)]$ is admissible if and only if $[\rho_z(x)]^p \in \text{Im}\rho_z$. As a corollary, when $[\rho_z(x)]^p$ is not contained in the image of $\rho_z$, we know $[\omega(x)]$ is not admissible.

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