Well-posedness of stochastic second grade fluids

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Abstract

The theory of turbulent Newtonian fluids turns out that the choice of the boundary condition is a relevant issue, since it can modify the behavior of the fluid by creating or avoiding a strong boundary layer. In this work we study stochastic second grade fluids filling a two-dimensional bounded domain, with the Navier-slip boundary condition (with friction). We prove the well-posedness of this problem and establish a stability result. Our stochastic model involves a multiplicative white noise and a convective term with third order derivatives, which significantly complicate the analysis.

Key words. Stochastic, second grade fluids, solvability, stability.

AMS Subject Classification. 76A05, 76D03, 76F55, 76M35

1 Introduction

The present work is devoted to the study of the stochastic incompressible fluids of second grade, which are a special class of non-Newtonian fluids. Unlike the Newtonian fluids, where only the stretching tensor appears in the characterization of the stress response to a deformation fluid, here the Cauchy stress tensor $T$ of the non-Newtonian fluids is defined by

$$T = -\pi I + \nu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,$$

where the first term $-\pi I$ is due to the incompressibility of the fluid and $A_1, A_2$ are the two first Rivlin-Ericksen tensors (cf. [35])

$$A_1(y) = \nabla y + (\nabla y)^\top$$
$$A_2(y) = \dot{A}_1(y) + A_1(y)\nabla y + (\nabla y)^\top A_1(y),$$

where $y$ denotes the velocity of the fluid, the superposed dot is the material time derivative, $\nu$ is the kinematic viscosity of the fluid and $\alpha_1, \alpha_2$ are constant material moduli. The study developed in [18] turns out that thermodynamic laws and stability principles impose $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = 0$. We set $\alpha = \alpha_1$ and assume $\alpha_1 > 0$.

It is well known that in turbulent fluids, small random perturbations can produce relevant macroscopic effects. By this reason, the incorporation of a stochastic white noise force in the Navier-Stokes equations [6] is widely recognized as an important step to understand the turbulence phenomena. In this perspective, we can find in [5] (see Lemma 2.2) a deduction of stochastic Navier-Stokes equations from fundamental principles, by showing that the stochastic Navier-Stokes equations are a real physical model. Nowadays, the stochastic Navier-Stokes equations are quite well understood, see for instance in [16], [20], [30], [36] and the references

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therein. In spite of that, there are few results in the literature about stochastic non-Newtonian fluids [17], [32], [33], [34]. In this paper we consider the stochastic second grade equations with multiplicative noise given by

$$\begin{align*}
\frac{\partial}{\partial t}(Y - \alpha \Delta Y) &= \nu \Delta Y - \text{curl}(Y - \alpha \Delta Y) \times Y - \nabla \pi + U + G(t, Y) \dot{W}_t, \\
\text{div} \ Y &= 0 \quad \text{in } O \times (0, T), \tag{1.1}
\end{align*}$$

where $U$ is a body force, $G(t, Y) \dot{W}_t$ is a multiplicative white noise and $O$ is a bounded domain of $\mathbb{R}^2$ with a boundary $\Gamma$.

The study of this system requires suitable boundary conditions on the boundary $\Gamma$ of the domain. The Dirichlet boundary condition given by

$$Y = 0 \quad \text{on } \Gamma$$

is accepted as an appropriate boundary condition and is the more usual one. Another physical relevant boundary condition considered in the literature is the Navier boundary condition

$$Y \cdot n = 0, \quad [2(n \cdot DY) + \gamma Y] \cdot \tau = 0 \quad \text{on } \Gamma, \tag{1.2}$$

where $n = (n_1, n_2)$ and $\tau = (-n_2, n_1)$ are the unit normal and tangent vectors, respectively, to the boundary $\Gamma$, $DY = \frac{\nabla Y + (\nabla Y)^\top}{2}$ is the symmetric part of the velocity gradient and $\gamma > 0$ is a friction coefficient on $\Gamma$.

The stochastic partial differential equations (1.1) with the Dirichlet boundary condition has been studied in [32] and [34]. In the former paper, the authors used tightness arguments that conjugated with the Skorohod theorem provided the existence of a weak stochastic solution, in the sense that the Brownian motion, being part of the solution, was not given in advance; while in the second one, the authors proved the existence and uniqueness of a strong stochastic solution. Let us refer the pioneer papers [31] and [13] (see also [12]), where the deterministic second grade equations with the Dirichlet boundary condition were mathematically studied for the first time, and [1] where the deterministic equations were studied with a particular Navier boundary condition (without friction, i.e. when $\gamma = 0$). The physical interpretation of these second grade equations can be found in [8], [19], [21], [23] and [24]. It is relevant to recall that the deterministic methods are based on the Faedo-Galerkin approximation method and a priori estimates. Then, compactness arguments can be used to pass to the limit of the respective approximate equations in the distributional sense. Unfortunately, for the stochastic partial differential equations a priori estimates are not enough to pass to the limit of the approximate equations, due to the lack of regularity on the time and stochastic variables. In order to obtain a strong stochastic solution we should verify that the sequence of the Galerkin approximations converges strongly in some adequate topology.

We should mention that even if the Dirichlet boundary condition is widely accepted as an appropriate boundary condition at the surface of contact between a fluid and a solid, it is also a source of many problems since it attaches fluid particles to the boundary, creating a strong boundary layer (cf. [15], [25], [26], [28]). On the other hand, the Navier boundary condition allows the slippage of the fluid on the boundary, making it possible to treat important problems as for instance the boundary layer problem, when the viscosity $\nu$ and/or the elastic response $\alpha$ tend to zero (cf. [3], [9]-[11], [14], [27], [29]). However, even if the Navier-slip boundary condition allows to solve interesting problems, technically, when comparing with the Dirichlet boundary condition, it requires a more careful mathematical analysis to show the well-posedness of system (1.1)-(1.2) as well as to establish stability properties for the solution, since the boundary terms
resulting from integrating by parts of the convective term do not vanish and should be estimated in an appropriate way.

As far as we know, the stochastic second grade fluid equations with the Navier boundary condition are studied here for the first time. To show the well-posedness, as in previous articles, we follow the Faedo-Galerkin approximation method by taking an appropriate basis. We first deduce uniform estimates for the approximate solutions that allow to pass to the limit with respect to the weak topology. In order to show that the limit process is a solution, we adapt the methods developed in [7] to study the stochastic Navier-Stokes equations. More precisely, we show that the approximate solutions already converge strongly up to a certain stopping time, therefore we establish the existence and uniqueness results for the solution of system (1.1)-(1.2), as a stochastic process with values in $H^1$. We should mention that an analogous reasoning is considered in [31] to deal with the stochastic second grade fluid equations with homogeneous Dirichlet boundary condition.

The plan of the present paper is as follows. In Section 2 we state the functional setting and introduce useful notations. In Section 3 we present some well known results and relevant lemmas related with the nonlinear term of $F_1$-$F_2$, which will be applied in the next sections. The main result concerning the existence of a strong stochastic solution is established in Section 4. Finally Section 5 is devoted to the study of the stability property.

2 Functional setting and notations

We consider the stochastic second grade fluid model in a bounded and simply connected domain $\Omega$ of $\mathbb{R}^2$ with a sufficiently regular boundary $\Gamma$

$$
\begin{aligned}
\left\{
\begin{array}{ll}
d(\nu(Y)) = (\nu \Delta Y - \text{curl}(\nu(Y)) \times Y - \nabla \pi + U) \ dt + G(t, Y) \ dW_t, \\
\text{div} \ Y = 0 \\
Y \cdot n = 0, \\
Y(0) = Y_0
\end{array}
\right.
\end{aligned}
$$

in $\Omega \times (0, T)$,

$$
\begin{aligned}
[2(n \cdot DY) + \gamma Y] \cdot \tau = 0
\end{aligned}
$$

on $\Gamma \times (0, T)$,

$$
\begin{aligned}
α
\end{aligned}
$$

in $\Omega$,

where $\nu > 0$ is a constant viscosity of the fluid, $α > 0$ is a constant material modulus, the constant $γ > 0$ is a friction coefficient of $\Gamma$, $\Delta$ and $\nabla$ respectively denote the Laplacian and the gradient, $Y = (Y_1, Y_2)$ is a 2D velocity field and

$$
\begin{aligned}
v(Y) = Y - α\Delta Y.
\end{aligned}
$$

The function $π$ represents the pressure, $U$ is a distributed mechanical force and the term

$$
\begin{aligned}
G(t, Y) \ dW_t = \sum_{k=1}^{m} G^k(t, Y) \ dW^k_t
\end{aligned}
$$

corresponds to the stochastic perturbation, where $G(t, Y) = (G^1(t, Y), \ldots, G^m(t, Y))$ has suitable growth assumptions defined below and $W_t = (W^1_t, \ldots, W^m_t)$ is a standard $\mathbb{R}^m$-valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. We assume that $\mathcal{F}_0$ contains every $P$-null subset of $\Omega$.

Let $X$ be a real Banach space endowed with the norm $\|\cdot\|_X$. We denote by $L^p(0, T; X)$ the space of $X$-valued measurable $p$–integrable functions $y$ defined on $[0, T]$ for $p \geq 1$.

For $p, r \geq 1$ let $L^p(\Omega, L^r(0, T; X))$ be the space of processes $y = y(\omega, t)$ with values in $X$ defined on $\Omega \times [0, T]$, adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$, and endowed with the norms

$$
\begin{aligned}
\|y\|_{L^p(\Omega, L^r(0, T; X))} = \left( E \left( \int_0^T \|y\|_X^r \ dt \right)^{\frac{p}{r}} \right)^{\frac{r}{p}}
\end{aligned}
$$
and
\[ \|y\|_{L^{r}(\Omega;L^{\infty}(0,T;X))} = \left( \mathbb{E} \sup_{t \in [0,T]} \|y\|_{X}^{r} \right)^{\frac{1}{r}} \] if \( r = \infty \),
where \( \mathbb{E} \) is the mathematical expectation with respect to the probability measure \( P \). As usual in the notation of processes \( y = y(\omega, t) \) we normally omit the dependence on \( \omega \in \Omega \).

In equation (2.1) the vector product \( \times \) for 2D vectors \( y = (y_1, y_2) \) and \( z = (z_1, z_2) \) is calculated as \( y \times z = (y_1, y_2, 0) \times (z_1, z_2, 0) \); the curl of the vector \( y \) is equal to \( \text{curl} y = \frac{\partial y_2}{\partial x_1} - \frac{\partial y_1}{\partial x_2} \) and the vector product of \( \text{curl} y \) with the vector \( z \) is understood as
\[ \text{curl} y \times z = (0, 0, \text{curl} y) \times (z_1, z_2, 0). \]

Given two vectors \( y, z \in \mathbb{R}^2 \), \( y \cdot z = \sum_{i=1}^{2} y_i z_i \) stands for the usual scalar product in \( \mathbb{R}^2 \) and given two matrices \( A, B \), we denote \( A \cdot B = \sum_{ij=1}^{2} A_{ij} B_{ij} \).

Let us introduce the following Hilbert spaces
\[
\begin{align*}
H(\text{curl}; \mathcal{O}) &= \{ y \in L^2(\mathcal{O}) \mid \text{curl} y \in L^2(\mathcal{O}), \ \text{div} y = 0 \ \text{in} \ \mathcal{O} \}, \\
H &= \{ y \in L^2(\mathcal{O}) \mid \text{div} y = 0 \ \text{in} \ \mathcal{O} \ \text{and} \ y \cdot n = 0 \ \text{on} \ \Gamma \}, \\
V &= \{ y \in H^1(\mathcal{O}) \mid \text{div} y = 0 \ \text{in} \ \mathcal{O} \ \text{and} \ y \cdot n = 0 \ \text{on} \ \Gamma \}, \\
W &= \{ y \in V \cap H^2(\mathcal{O}) \mid [2(n \cdot \partial y) + \gamma y] \cdot \tau = 0 \ \text{on} \ \Gamma \}, \\
\widetilde{W} &= W \cap H^3(\mathcal{O}).
\end{align*}
\]

We denote by \((\cdot, \cdot)\) the inner product in \( L^2(\mathcal{O}) \) and by \( \| \cdot \|_2 \) the associated norm. The norm in the space \( H^p(\mathcal{O}) \) is denoted by \( \| \cdot \|_{H^p} \). Let us note that \( H(\text{curl}; \mathcal{O}) \) is a subspace of \( H^1(\mathcal{O}) \). Let us denote
\[ (Dy, Dz) = \int_{\mathcal{O}} Dy \cdot Dz. \]
On the space \( V \), we consider the following inner product
\[ (y, z)_V = (\nu(y), z) = (y, z) + 2\alpha (Dy, Dz) + \alpha \gamma \int_{\Gamma} y \cdot z \]
and the corresponding norm \( \| \cdot \|_V \). We can verify that the norms \( \| \cdot \|_{H^1} \) and \( \| \cdot \|_V \) are equivalent because of the Korn inequality
\[
\|y\|_{H^1} \leq C (\|Dy\|_2 + \|y\|_2), \quad \forall y \in H^1(\mathcal{O}). \tag{2.3}
\]
Here and below, \( C \) will denote a generic positive constant that may depend only on the domain \( \mathcal{O} \), the regularity of the boundary \( \Gamma \), the physical constants \( \nu, \alpha, \gamma \) and \( K \), defined in \( (2.5) \).

Let \( B \) be a given Hilbert space with inner product \((\cdot, \cdot)_B\). For a vector
\[ h = (h^1, \ldots, h^m) \in B^m = \underbrace{B \times \ldots \times B}_{m\text{-times}}, \]
we introduce the norm
\[ \|h\|_B = \sum_{i=1}^{m} \|h_i\|_B \]
and the module of the inner product of $h$ and a fixed $v \in B$ as
\[
| (h, v)_B | = \left( \sum_{k=1}^{m} (h^k, v)_B^2 \right)^{1/2}.
\] (2.4)

Assume that $G(t, y) : [0, T] \times V \to V^m$ is Lipschitz on $y$, and satisfies a linear growth; that is, there exists a positive constant $K$ such that
\[
\|G(t, y) - G(t, z)\|_V^2 \leq K \|y - z\|_V^2,
\]
\[
\|G(t, y)\|_V \leq K (1 + \|y\|_V), \quad \forall y, z \in V, \; t \in [0, T].
\] (2.5)

3 Preliminary results

Let us introduce the Helmholtz projector $P : L^2(O) \to H$, which is the linear bounded operator defined by $Py = \tilde{y}$, where $\tilde{y} \in H$ is characterized by the Helmholtz decomposition
\[
y = \tilde{y} + \nabla \phi, \quad \phi \in H^1(O).
\]

We recall some useful inequalities, namely, the Poincaré inequality
\[
\|y\|_2 \leq C \|\nabla y\|_2 \quad \text{for all } y \in V
\]
and the Sobolev inequality
\[
\|y\|_4 \leq C \|\nabla y\|_2 \quad \text{for all } y \in V.
\]

Now, we present the first result of this section. This is a well known and very important property concerning the Navier boundary conditions (see Lemma 4.1 and Corollary 4.2 in [26]). Let $k$ be the curvature of $\Gamma$. Parameterizing $\Gamma$ by arc length $s$, the following relation holds
\[
\frac{d n}{d s} = k \tau.
\]

Lemma 3.1 Let $y \in H^2(O) \cap V$ be a vector field verifying the Navier boundary condition. Then
\[
\text{curl } y = g(y) \text{ on } \Gamma \text{ with } g(y) = (2k - \gamma) y \cdot \tau.
\] (3.1)

Proof. Let us first notice that the anti-symmetric tensor $Ay = \nabla y - (\nabla y)^\top$ can be written in the form
\[
Ay = \text{curl } y \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]
The symmetry of $Dy$ and the anti-symmetry of $Ay$ imply that
\[
(Dy) \tau \cdot n = (Ay) n \cdot \tau \quad \text{and} \quad (Ay) \tau \cdot n = - (Ay) n \cdot \tau.
\]
It follows that
\[
(\nabla y) \tau \cdot n = (Dy) n \cdot \tau - \frac{1}{2} (Ay) n \cdot \tau
\]
which is equivalent to
\[
\text{curl } y = -2(\nabla y) \tau \cdot n + 2(Dy)n \cdot \tau.
\] (3.2)
Taking the derivative of the expression $y \cdot n = 0$ in the direction of the tangent vector $\tau$, we deduce
\[ (\nabla y)\tau \cdot n = -ky \cdot \tau. \] (3.3)

The conclusion is then a consequence of (3.2) and (3.3). ■

Now, we state a formula that can be easily derived by taking integration by parts
\[ -\int_O \nabla y \cdot z = -\int_\Gamma 2(Dy)n \cdot z + \int_O 2Dy \cdot Dz, \] (3.4)
that holds for any $y \in H^2(O) \cap V$ and $z \in H^1(O)$. Using the boundary conditions, that gives the relation
\[ -\int_O \nabla y \cdot z = \gamma \int_\Gamma y \cdot z \] for any $y \in W$ and $z \in V$ (3.5)
that will be used throughout the article.

Let us consider the following modified Stokes system with Navier boundary condition
\[
\begin{aligned}
\begin{cases}
h - \alpha \Delta h + \nabla p &= f, \quad \text{div } h = 0 \quad \text{in } O, \\
h \cdot n &= 0, \quad [2(n \cdot Dh) + \gamma h] \cdot \tau = 0 \quad \text{on } \Gamma.
\end{cases}
\end{aligned}
\] (3.6)

Next, we state a lemma concerning the regularity properties of the solution of this system.

**Lemma 3.2** Suppose $f \in H^m(O)$, $m = 0, 1$. Then system (3.6) has a solution $(h, p) \in H^{m+2}(O) \times H^{m+1}(O)$, moreover the following estimates hold
\[ \|h\|_{H^2} \leq C\|f\|_2, \] (3.7)
\[ \|h\|_{H^3} \leq C\|f\|_{H^1}. \] (3.8)

**Proof.** Supposing that $f \in L^2(O)$, the existence of the solution $(h, p)$ with $h$ in $H^1(O)$ is given by the Lax-Millgram lemma. Multiplying (3.6)_1 by $h$, we derive
\[ \|h\|_2^2 + \alpha \left(2\|Dh\|_2^2 + \gamma\|h\|_{L^2(\Gamma)}^2\right) = (f, h) \leq \|f\|_2 h_2, \]
which gives
\[ \|h\|_{H^3} \leq C\|f\|_2. \] (3.9)

On the other hand, applying the operator curl to system (3.6), we derive the following system for $u = \text{curl } h$
\[
\begin{aligned}
\begin{cases}
u - \alpha \Delta u &= \text{curl } f \quad \text{in } O, \\
u &= g(h) = (2k - \gamma) h \cdot \tau \quad \text{on } \Gamma.
\end{cases}
\end{aligned}
\] (3.10)

Let us denote the extension of the unit exterior normal $n$ (and the tangent $\tau = (-n_2, n_1)$) on the whole domain $\overline{O}$ by the same notation $n$ (and $\tau$). Then the function $z = u - (2k - \gamma) h \cdot \tau$ solves the system
\[
\begin{aligned}
\begin{cases}
z - \alpha \Delta z &= \text{curl } f - (2k - \gamma) h \cdot \tau + \alpha \Delta [(2k - \gamma) h \cdot \tau] \quad \text{in } O, \\
z &= 0 \quad \text{on } \Gamma.
\end{cases}
\end{aligned}
\] (3.11)
Multiplying equation (3.11) by $z$, integrating by parts and using (3.9), we deduce
\[ \|z\|_2 + \alpha \|\nabla z\|_2 \leq C (\|f\|_2 + \|h\|_{H^1}) \leq C \|f\|_2, \]
which implies
\[ \|u\|_{H^1} \leq C (\|f\|_2 + \|h\|_{H^1}) \leq C \|f\|_2. \]  

(3.12)

In addition estimate (2.3.3.7), p. 110 of [22] for system (3.10) gives
\[ \|u\|_{H^2} \leq C \left( \|\text{curl} f\|_2 + \|(2k - \gamma) h \cdot \tau\|_{H^{2-\frac{1}{2}}(\Gamma)} \right) \leq C (\|f\|_{H^1} + \|h\|_{H^2}). \]  

(3.13)

Since $h$ solves system (3.6), then there exists a stream function $\varphi$ such that $h = \nabla \perp \varphi$, satisfying the system
\[ \begin{cases} 
\Delta \varphi = u & \text{in } \Omega, \\
\varphi = 0 & \text{on } \Gamma 
\end{cases} \]  

(3.14)

and the estimate
\[ \|\varphi\|_{H^{2+m}} \leq \|u\|_{H^m}, \quad m \in \mathbb{N}_0, \]  

(3.15)

by Theorem 2.5.1.1, p. 128 of [22].

Combining (3.12) and (3.15) with $m = 1$, we deduce
\[ \|\varphi\|_{H^3} \leq \|u\|_{H^1} \leq C \|f\|_2, \]

hence $h = \nabla \perp \varphi \in H^2$ and (3.7) hold. Moreover (3.13) and (3.15) with $m = 2$ imply
\[ \|\varphi\|_{H^4} \leq \|u\|_{H^2} \leq C (\|f\|_{H^1} + \|h\|_{H^2}). \]

Invoking (3.7), we conclude that $h = \nabla \perp \varphi \in H^3$ and (3.8) hold.

Let us recall that the space $W$ introduced in (2.2) is naturally endowed with the Sobolev norm $\|\cdot\|_{H^2}$. The next result follows directly from Lemma 5 in [2] and helps to introduce on $W$ an equivalent norm that will be useful to analyze the stability in Section 5.

**Lemma 3.3** For each $y \in W$, we have
\[ \|v(y) - P v(y)\|_2 \leq C \|y\|_{H^1}, \]  

(3.16)

\[ \|v(y) - P v(y)\|_{H^1} \leq C \|y\|_{H^2}. \]  

(3.17)

The next regularity result will be fundamental to establish the well-posedness of the velocity equation (see Propositions 6 in [3] and Lemma 2.1 in [12] for similar results).

**Lemma 3.4** Let $y \in \widetilde{W}$. Then, the following estimates hold
\[ \|y\|_{H^2} \leq C (\|P v(y)\|_2 + \|y\|_{H^1}), \]  

(3.18)

\[ \|y\|_{H^3} \leq C (\|\text{curl} v(y)\|_2 + \|y\|_{H^1}). \]  

(3.19)
**Proof.** Considering system (3.6) with \( f = \upsilon(y) \), then the pair \((y, 0)\) is obviously the solution of such system. Hence estimate (3.7) yields
\[
\|y\|_{H^2} \leq C \|\upsilon(y)\|_2 \leq C (\|\upsilon(y) - \mathbb{P}\upsilon(y)\|_2 + \|\mathbb{P}\upsilon(y)\|_2)
\]
Applying (3.16), we deduce (3.18).

Since \( \text{curl} \upsilon(y) \in L^2(\Omega) \) and \( \nabla \cdot (\text{curl} \upsilon(y)) = 0 \), there exists a unique vector-potential \( \psi \in H^1(\Omega) \) such that
\[
\begin{aligned}
\text{curl} \psi &= \text{curl} \upsilon(y), & \text{div} \psi &= 0 & \text{in} \ \Omega, \\
\psi \cdot n &= 0 & \text{on} \ \Gamma
\end{aligned}
\]
and
\[
\|\psi\|_{H^1} \leq C \|\text{curl} \upsilon(y)\|_2. \tag{3.20}
\]

It follows that \( \text{curl} (y - \alpha \Delta y - \psi) = 0 \) and there exists \( \pi \in L^2(\Omega) \), such that
\[
y - \alpha \Delta y - \psi + \nabla \pi = 0.
\]

Hence \( y \) is the solution of the Stokes system (3.6) where \( f \) is replaced by \( \psi \).

As a consequence of (3.8), we have
\[
\|y\|_{H^3} \leq C (\|\psi\|_{H^1} + \|y\|_{H^1}). \tag{3.21}
\]
Using (3.20) we obtain the claimed result (3.19). \( \blacksquare \)

In order to define the solution of equation (2.11) in the distributional sense, we introduce a trilinear functional that is well known in the context of the Navier-Stokes equations
\[
b(\phi, z, y) = (\phi \cdot \nabla z, y), \quad \forall \phi, z, y \in V.
\]

In what follows we often will use the following property
\[
b(\phi, z, y) = -b(\phi, y, z), \tag{3.22}
\]
that follows taking integration by parts, knowing that \( \phi \) is divergence free and \( (\phi \cdot n) = 0 \) on \( \Gamma \).

Straightforward computations yield the following relation
\[
(b(\phi, z, \upsilon(y) \times z, \phi) = b(\phi, z, \upsilon(y)) - b(z, \phi, \upsilon(y)) \quad \forall y \in \tilde{W}, z, \phi \in V. \tag{3.23}
\]

In the next lemma, we deduce crucial estimates of major importance to establish the well-posedness of system (2.11), as well as to prove the stability property of their solutions. We should mention that some estimates follow from an adaptation of the method considered in [4] to prove the uniqueness.

**Lemma 3.5** Let \( y, z, \phi \in \tilde{W} \). Then
\[
\begin{align*}
\|(\text{curl} \upsilon(y) \times z, \phi)\| &\leq C \|y\|_{H^3} \|z\|_{H^1} \|\phi\|_{H^3}, \\
\|(\text{curl} \upsilon(y) \times z, \phi)\| &\leq C \|y\|_{H^1} \|z\|_{H^3} \|\phi\|_{H^3}, \\
\|(\text{curl} \upsilon(y) \times z, y)\| &\leq C \|y\|_{H^1}^2 \|z\|_{H^3}.
\end{align*}
\]
Proof. \textbf{1st step. The proof of estimate (3.23).} We directly can estimate
\[
\|(\text{curl } v(y) \times z, \phi)\| \leq \| \phi \|_\infty \|z\|_2 \|\text{curl } v(y)\|_2 \leq \|\phi\|_{H^3} \|z\|_2 \|y\|_{H^3}
\]
by Sobolev’s embedding $H^3(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$. Hence we have (3.24).

\textbf{2nd step. The proof of estimate (3.27).} Equality (3.23) gives
\[
(\text{curl } v(y) \times z, \phi) = b(\phi, z, y) - b(z, \phi, y) - \alpha (b(\phi, z, \Delta y) - b(z, \phi, \Delta y)) .
\]
(3.27)

With the help of Sobolev’s embedding $H^1(\mathcal{O}) \hookrightarrow L^4(\mathcal{O})$, it is easy to see that
\[
|b(\phi, z, y) - b(z, \phi, y)| \leq C\|\phi\|_{H^1} \|z\|_{H^1} \|y\|_{H^1} .
\]
(3.28)

Integrating by parts and using the boundary conditions, we derive
\[
b(\phi, z, \Delta y) = \sum_{i,j=1}^2 \int_\mathcal{O} \phi_i \frac{\partial g(y)}{\partial x_j} \Delta y_j = \sum_{i,j,k=1}^2 \int_\mathcal{O} \phi_i \frac{\partial g(y)}{\partial x_j} \frac{\partial g(y)}{\partial x_k} (\frac{\partial g(y)}{\partial x_j} - \frac{\partial g(y)}{\partial x_k}) = \sum_{i,j,k=1}^2 \int_\mathcal{O} \phi_i \frac{\partial g(y)}{\partial x_j} \frac{\partial g(y)}{\partial x_k} (A_{jk}(y)) = \sum_{i,j,k=1}^2 \int_\Gamma \phi_i \frac{\partial g(y)}{\partial x_j} g(y) \tau_j - \sum_{i,j,k=1}^2 \int_\mathcal{O} \frac{\partial g(y)}{\partial x_k} \frac{\partial g(y)}{\partial x_j} A_{jk}(y) = \sum_{i,j,k=1}^2 \int_\mathcal{O} \frac{\partial g(y)}{\partial x_k} \frac{\partial g(y)}{\partial x_j} A_{jk}(y)
\]
(3.29)

Again, integrating by parts, it follows that
\[
I_1 = \sum_{i,j=1}^2 \int_\Gamma \phi_i \frac{\partial g(y)}{\partial x_j} g(y) \tau_j = \sum_{i,j=1}^2 \int_\mathcal{O} \phi_i \frac{\partial g(y)}{\partial x_j} g(y) \tau_j \text{ div } n + \sum_{i,j,k=1}^2 \int_\mathcal{O} \frac{\partial g(y)}{\partial x_k} (\phi_i \frac{\partial g(y)}{\partial x_j} g(y) \tau_j)n_k = b(\phi, z, \text{ div } n g(y) \tau_j) + b((n \cdot \nabla) \phi, z, n_k g(y) \tau_j) + \sum_{k=1}^2 b(\phi, \frac{\partial g(y)}{\partial x_k}, n_k g(y) \tau_j) + b(\phi, z, (n \cdot \nabla) (g(y) \tau_j)) .
\]

Then, using Sobolev’s embedding $H^2(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$, we easily derive
\[
|b(\phi, z, \Delta y)| \leq |b(\phi, z, \text{ div } n g(y) \tau_j)| + |b((n \cdot \nabla) \phi, z, g(y) \tau_j)| + |b(\phi, z, (n \cdot \nabla) (g(y) \tau_j))| + \sum_{k=1}^2 \left| b(\phi, \frac{\partial g(y)}{\partial x_k}, n_k g(y) \tau_j) + b(\phi, \frac{\partial g(y)}{\partial x_k}, A_k g(y)) \right| + \left| b(\phi, \frac{\partial g(y)}{\partial x_k}, A_k g(y)) \right|
\]
\[
\leq C\|z\|_{H^1} \|\phi\|_{H^3} \|g\|_{H^1} .
\]
(3.30)

By symmetry, it follows that
\[
|b(z, \phi, \Delta y)| \leq C\|z\|_{H^1} \|\phi\|_{H^3} \|g\|_{H^1} .
\]
(3.31)

Then (3.25) follows from (3.27 - 3.31).
3d step. The proof of estimate \((3.32)\). As in the above computations for \((3.22)\), we obtain

\[
b(z, y, \Delta y) = \sum_{i,j=1}^{2} \int_{\Gamma} \frac{\partial y_i}{\partial x_j} g(y) \tau_j - \sum_{i,j,k=1}^{2} \int_{\Omega} \frac{\partial^2 y_i}{\partial x_k \partial x_j} A_{jk}(y) = \sum_{i,j,k=1}^{2} \int_{\Omega} \frac{\partial^2 y_i}{\partial x_k \partial x_j} A_{jk}(y)
\]

where

\[
J_1 = b(z, y, \text{div } g(y) \tau) + b((n \cdot \nabla) z, y, g(y) \tau) + \sum_{k=1}^{2} b(z, \frac{\partial y_k}{\partial x_k}, n_k g(y) \tau)
\]

\[
+ b(z, y, (n \cdot \nabla) (g(y) \tau)),
\]

\[
J_2 = -\sum_{k=1}^{2} b \left( \frac{\partial y_k}{\partial x_k}, y, A_k(y) \right)
\]

and

\[
J_3 = \sum_{i,j,k=1}^{2} \int_{\Omega} \frac{\partial y_i}{\partial x_k} \left( \frac{\partial^2 y_i}{\partial x_k \partial x_j} - \frac{\partial^2 y_j}{\partial x_k \partial x_i} \right) = \sum_{i,j,k=1}^{2} \int_{\Omega} \frac{\partial y_i}{\partial x_k} \left( \frac{\partial^2 y_i}{\partial x_k \partial x_j} - \frac{\partial^2 y_j}{\partial x_k \partial x_i} \right) = \sum_{i,j,k=1}^{2} \int_{\Omega} \frac{\partial y_i}{\partial x_k} \left( \frac{\partial^2 y_k}{\partial x_i \partial x_k} \right) = 0.
\]

Therefore, we derive

\[
|b(z, y, \Delta y)| \leq |b(z, y, \text{div } g(y) \tau)| + |b((n \cdot \nabla) z, y, g(y) \tau)| + \sum_{k=1}^{2} \left| b \left( z, n_k g(y) \tau, \frac{\partial y_k}{\partial x_k} \right) \right|
\]

\[
+ \left| b \left( z, y, (n \cdot \nabla) (g(y) \tau) \right) \right| + \left| \sum_{k=1}^{2} b \left( \frac{\partial y_k}{\partial x_k}, y, A_k(y) \right) \right| \leq C \|z\|_{H^2} \|y\|_{H^1}^2,
\]

where we have used that

\[
\sum_{k=1}^{2} b \left( z, \frac{\partial y_k}{\partial x_k}, n_k g(y) \tau \right) = -\sum_{k=1}^{2} b \left( z, n_k g(y) \tau, \frac{\partial y_k}{\partial x_k} \right)
\]

by \((3.22)\).

Taking \( \phi = y \) in \((3.20)\), we have

\[
b(y, z, \Delta y) = \sum_{i,j=1}^{2} \int_{\Gamma} y_i \frac{\partial y_j}{\partial x_j} g(y) \tau_j - \sum_{i,j,k=1}^{2} b \left( \frac{\partial y_k}{\partial x_k}, z, A_k(y) \right) - \sum_{k=1}^{2} b \left( y, \frac{\partial y_k}{\partial x_k}, A_k(y) \right).
\]

Taking into account the embedding theorems \(H^2(\Omega) \hookrightarrow C(\overline{\Omega}), H^1(\Omega) \hookrightarrow L^2(\Gamma)\) and \(H^1(\Omega) \hookrightarrow L^4(\Omega)\), we have

\[
|b(y, z, \Delta y)| \leq \sum_{i,j=1}^{2} \left| \int_{\Gamma} y_i \frac{\partial y_j}{\partial x_j} g(y) \tau \right| + \sum_{k=1}^{2} \left( |b \left( \frac{\partial y_k}{\partial x_k}, z, A_k(y) \right)| + \left| b \left( y, \frac{\partial y_k}{\partial x_k}, A_k(y) \right) \right| \right)
\]

\[
\leq C \|y\|_{L^2(\Gamma)} \|\nabla z\|_{C(\overline{\Omega})} + C \|\nabla y\|_{L^2}^2 \|\nabla z\|_{\infty} + \sum_{i,j=1}^{2} \|y\|_4 \left\| \frac{\partial^2 y_i}{\partial x_i \partial x_j} \right\|_{L^4} \|\nabla y\|_2
\]

\[
\leq C \|y\|_{H^1}^2 \|z\|_{H^2}. \quad (3.33)
\]
Then (3.26) is a consequence of (3.28) and (3.32)-(3.33).

4 Existence of strong solution

The aim of the present section is to establish the existence of a strong solution for system (2.1) in the probabilistic sense.

Definition 4.1 Let

\[ U \in L^2(\Omega \times (0, T); H(\text{curl}; \mathcal{O})), \quad Y_0 \in L^2(\Omega, \overline{W}). \]

A stochastic process \( Y \in L^2(\Omega, L^\infty(0, T; \overline{W})) \) is a strong solution of (2.1), if for a.e.-\( P \) and a.e. \( t \in (0, T) \), the following equation holds

\[
\begin{align*}
(v(Y(t)), \phi) &= \int_0^t \left[ -2\nu \langle DY(s), D\phi \rangle - \nu \gamma \int_\Gamma y \cdot \phi \, dx - \langle \text{curl} \, v(Y(s)) \times Y(s), \phi \rangle \right] \, ds \\
&\quad + (v(Y(0)), \phi) + \int_0^t (U(s), \phi) \, ds + \int_0^t (G(s, Y(s)), \phi) \, dW_s
\end{align*}
\]

for all \( \phi \in V \), where the nonlinear term should be understood in the sense

\[
\langle \text{curl} \, v(Y(t)) \times Y(t), \phi \rangle = b(\phi, Y(t), v(Y(s))) - b(Y(t), \phi, v(Y(s))
\]

and the stochastic integral is defined by

\[
\int_0^t (G(s, Y(s)), \phi) \, dW_s = \sum_{k=1}^n \int_0^t (G^k(s, Y(s)), \phi) \, dW^k_s.
\]

Let us formulate our main existence and uniqueness result, which will be shown in this section.

Theorem 4.2 Assume that

\[ U \in L^p(\Omega \times (0, T); H(\text{curl}; \mathcal{O})), \quad Y_0 \in L^p(\Omega, V) \cap L^2(\Omega, \overline{W}) \quad \text{for some} \quad 4 \leq p < \infty. \]

Then there exists a unique solution \( Y \) to equation (4.1) which belongs to

\[ L^2(\Omega, L^\infty(0, T; \overline{W})) \cap L^p(\Omega, L^\infty(0, T; V)). \]

Moreover, the following estimates hold

\[
\frac{1}{2} \mathbb{E} \sup_{s \in [0, t]} \|Y(s)\|_V^2 + \mathbb{E} \int_0^t \left( 4\nu \|DY\|_2^2 + 2\nu \gamma \|Y\|_{L^2(\Gamma)}^2 \right) \, ds \leq C \left( \mathbb{E} \|Y_0\|_V^2 + \mathbb{E} \|U\|_{L^2(0, t; L^2)}^2 + 1 \right),
\]

\[
\mathbb{E} \sup_{s \in [0, t]} \|\text{curl} \, v(Y(s))\|_2^2 \leq C \left( \mathbb{E} \|\text{curl} \, v(Y_0)\|_2^2 + \mathbb{E} \|U\|_{L^2(0, t; H^1)}^2 + 1 \right).
\]
The proof of the theorem is given by Galerkin's approximation method. We consider the inner product of $\tilde{W}$ defined by

$$
(y, z)_{\tilde{W}} = (\text{curl}(y), \text{curl}(z)) + (y, z)_{V}.
$$

(4.2)

Taking into account (2.3) and (3.19) the norm $\| \cdot \|_{\tilde{W}}$ induced by this inner product is equivalent to $\| \cdot \|_{H^2}$. The injection operator $I : \tilde{W} \rightarrow V$ is a compact operator, then there exists a basis $\{e_i\} \subset \tilde{W}$ of eigenfunctions

$$
(y, e_i)_{\tilde{W}} = \lambda_i (y, e_i)_{V}, \quad \forall y \in \tilde{W}, \ i \in \mathbb{N},
$$

(4.3)

being an orthonormal basis for $V$ and the corresponding sequence $\{\lambda_i\}$ of eigenvalues verifies $\lambda_i > 0, \forall i \in \mathbb{N}$ and $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$. Let us notice that the ellipticity of equation (4.3) increases the regularity of their solutions. Hence without loss of generality we can consider $\{e_i\} \subset H^4$ (see [4]).

In this section, we consider this basis and introduce the Faedo-Galerkin approximation of system (2.4). Let $W_n = \text{span}\{e_1, \ldots, e_n\}$ and define

$$
Y_n(t) = \sum_{j=1}^{n} c_j^n(t) e_j
$$

as the solution of the stochastic differential equation

$$
\begin{aligned}
& d(v(Y_n), \phi) = ((\nu \Delta Y_n - \text{curl}(v(Y_n)) \times Y_n + U), \phi) \ dt + (G(t, Y_n), \phi) \ dW_t, \\
& Y_n(0) = Y_{n,0}, \quad \forall \phi \in W_n.
\end{aligned}
$$

(4.4)

Here $Y_{n,0}$ denotes the projection of the initial condition $Y_0$ onto the space $W_n$.

Let us notice that $\{\tilde{e}_j = \frac{1}{\sqrt{\lambda_j}} e_j\}_{j=1}^{\infty}$ is an orthonormal basis for $\tilde{W}$ and

$$
Y_{n,0} = \sum_{j=1}^{n} (Y_0, e_j)_{V} e_j = \sum_{j=1}^{n} (Y_0, \tilde{e}_j)_{\tilde{W}} \tilde{e}_j,
$$

then the Parseval’s identity gives

$$
\|Y_n(0)\|_V \leq \|Y_0\|_V \quad \text{and} \quad \|Y_n(0)\|_{\tilde{W}} \leq \|Y_0\|_{\tilde{W}}.
$$

(4.5)

Equation (4.4) defines a system of stochastic ordinary differential equations in $\mathbb{R}^n$ with locally Lipschitz nonlinearities. Hence there exists a local-in-time solution $Y_n$ as an adapted process in the space $C([0, T_n]; W_n)$. The global-in-time existence of $Y_n$ follows from uniform estimates on $n = 1, 2, \ldots$, that will be deduced in the next Lemma (a similar reasoning can be found in [1], [34]).

**Lemma 4.3** Assume that

$$
U \in L^2(\Omega \times (0, T); H(\text{curl}; \mathcal{O})), \quad Y_0 \in L^2(\Omega, \tilde{W}).
$$

Then problem (4.4) admits a unique solution $Y_n \in L^2(\Omega, L^\infty(0, T; \tilde{W}))$. Furthermore, for any $t \in [0, T]$, the following estimates hold

$$
\begin{aligned}
\frac{1}{2} \mathbb{E} \sup_{s \in [0, t]} \|Y_n(s)\|_V^2 &+ \mathbb{E} \int_{0}^{t} \left(4\nu \|DY_n\|_{L^2(\Omega)}^2 + 2\nu \gamma \|Y_n\|_{L^2(\Omega)}^2\right) ds \\
&\leq C \left(1 + \mathbb{E} \|Y_0\|_V^2 + \mathbb{E} \|U\|_{L^2(0, t; L^2)}^2\right),
\end{aligned}
$$

(4.6)
We know that
\[
\text{Step 1. Estimate in the space } C\text{ we obtain }
\phi = \sup_{t \in [0, T]} \|\text{curl } (Y_n(t))\|_2 \leq C \|\text{curl } (Y_0)\|_2^2
\]

and
\[
\mathbb{E} \sup_{s \in [0, t]} \|Y_n(s)\|_W^2 \leq C \mathbb{E} \|Y_0\|_W^2 + \mathbb{E} \|U\|_{L^2(0, t; H(\text{curl}, \mathcal{O}))}^2,
\]

where \( C \) are positive constants independent of \( n \) (and may depend on the data of our problem the domain \( \mathcal{O} \), the regularity of \( \Gamma \), the physical constants \( \nu, \alpha, \gamma \)).

**Proof.** For each \( n \in \mathbb{N} \), let us consider the sequence \( \{\tau_n^N\}_{N \in \mathbb{N}} \) of the stopping times
\[
\tau_n^N = \inf\{t \geq 0 : \|Y_n(t)\|_{H^3} \geq N\} \wedge T_n.
\]

In order to simplify the notation, let us introduce the function
\[
f(Y_n) = (\nu \Delta Y_n - \text{curl } (Y_n)) \times Y_n + U \in H^1(\mathcal{O}).
\]

Taking \( \phi = e_i \) for each \( i = 1, \ldots, n \) in equation (3.10), we obtain
\[
d(Y_n, e_i)\text{,}_V = (f(Y_n), e_i) dt + (G(t, Y_n), e_i) dW_t.
\]

**Step 1. Estimate in the space \( V \) for \( Y_n \), depending on the stopping times \( \tau_n^N \).**

The Itô formula gives
\[
d(Y_n, e_i)\text{,}_V^2 = 2(Y_n, e_i)\text{,}_V (f(Y_n), e_i) dt + 2(Y_n, e_i)\text{,}_V (G(t, Y_n), e_i) dW_t + |(G(t, Y_n), e_i)|^2 dt,
\]

where the module in the last term is defined by (2.4). Summing these equalities over \( i = 1, \ldots, n \), we obtain
\[
d\|Y_n\|_V^2 = 2(f(Y_n), Y_n) dt + 2(G(t, Y_n), Y_n) dW_t + \sum_{i=1}^n |(G(t, Y_n), e_i)|^2 dt.
\]

We know that
\[
(f(Y_n), Y_n) = -2\nu \|DY_n\|_2^2 - \nu\gamma \|Y_n\|_{L^2(\Gamma)}^2 + \alpha (b(Y_n, \Delta Y_n) - b(Y_n, \Delta Y_n))
\]
\[
- b(Y_n, Y_n, Y_n) + (U, Y_n)
\]
\[
= -2\nu \|DY_n\|_2^2 - \nu\gamma \|Y_n\|_{L^2(\Gamma)}^2 + (U, Y_n),
\]

hence
\[
d\|Y_n\|_V^2 = 2 \left(-2\nu \|DY_n\|_2^2 - \nu\gamma \|Y_n\|_{L^2(\Gamma)}^2 + (U, Y_n)\right) dt
\]
\[
+ 2(G(t, Y_n), Y_n) dW_t + \sum_{i=1}^n |(G(t, Y_n), e_i)|^2 dt,
\]

Let \( \tilde{G}_n \) be the solution of (3.8) for \( f = G(t, Y_n) \). Then
\[
(\tilde{G}_n, e_i)\text{,}_V = (G(t, Y_n), e_i) \quad \text{for } i = 1, \ldots, n
\]
which implies
\[
\sum_{i=1}^{n} |(G(t, Y_n), e_i)|^2 = \|\hat{G}_n\|_{V}^2 \leq C\|G(t, Y_n)\|_{V}^2 \leq C(1 + \|Y_n\|_{V}^2).
\] (4.12)

Here we used the fact that \(\hat{G}_n\) solves the elliptic type problem (3.6) for \(f = G(t, Y_n)\) and assumption (2.5).

Let us take \(t \in [0, T]\), the integration over the time interval \((0, s), 0 \leq s \leq \tau^n \land t\) of equality (4.11) and estimate (4.12) yield
\[
\|Y_n(s)\|_{V}^2 + \int_0^s (4\nu \|DY_n\|_{L^2(\Omega)}^2 + 2\nu\gamma \|Y_n\|_{L^2(\Gamma)}^2) \, dr \leq \|Y_n(0)\|_{V}^2 + C(1 + \int_0^s \|U\|_{V}^2 \, dr)
+ \int_0^s \|Y_n\|_{V}^2 \, dr + 2 \int_0^s (G(r, Y_n), Y_n) \, dW_r.
\] (4.13)

The Burkholder-Davis-Gundy inequality gives
\[
E \sup_{s \in [0, \tau^n \land t]} \left| \int_0^t (G(r, Y_n), Y_n) \, dW_r \right| \leq E \left( \int_0^{	au^n \land t} \|G(s, Y_n)\|_{V}^2 \, ds \right)^\frac{1}{2}
\leq E \sup_{s \in [0, \tau^n \land t]} \|Y_n(s)\|_2 \left( \int_0^{	au^n \land t} \|G(s, Y_n)\|_{V}^2 \, ds \right)^\frac{1}{2}
\leq \varepsilon E \sup_{s \in [0, \tau^n \land t]} \|Y_n(s)\|_2^2 + C_\varepsilon \int_0^{	au^n \land t} (1 + \|Y_n\|_{V}^2) \, ds.
\]

Substituting the last inequality with the chosen \(\varepsilon = \frac{1}{2}\) in (4.13) and considering (4.15), we derive
\[
\frac{1}{2} E \sup_{s \in [0, \tau^n \land t]} \|Y_n(s)\|_{V}^2 + E \int_0^{	au^n \land t} \left( 4\nu \|DY_n\|_{L^2(\Omega)}^2 + 2\nu\gamma \|Y_n\|_{L^2(\Gamma)}^2 \right) \, ds \leq E \|Y_0\|_{V}^2
+ CE \int_0^t (1 + \|U\|_{V}^2) \, ds + CE \int_0^{	au^n \land t} \|Y_n\|_{V}^2 \, ds.
\]

Hence, if we denote by \(1_{[0, \tau^n]}\) the characteristic function of the interval \([0, \tau^n]\), the function
\[
f(t) = E \sup_{s \in [0, t]} 1_{[0, \tau^n]} \|Y_n(s)\|_{V}^2
\]
fulfills Gronwall’s type inequality
\[
\frac{1}{2} f(t) \leq C \int_0^t f(s) \, ds + E \|Y_n(0)\|_{V}^2 + CE \int_0^t (1 + \|U\|_{V}^2) \, ds,
\]
which implies
\[
\frac{1}{2} E \sup_{s \in [0, \tau^n \land t]} \|Y_n(s)\|_{V}^2 + E \int_0^{	au^n \land t} \left( 4\nu \|DY_n\|_{L^2(\Omega)}^2 + 2\nu\gamma \|Y_n\|_{L^2(\Gamma)}^2 \right) \, ds
\leq C \left( 1 + E \|Y_0\|_{V}^2 + E \|U\|_{L^2(0, T; L^2)}^2 \right).
\] (4.14)
Step 2. \( L^2 \) estimate for \( \text{curl} \, v(Y_n) \), depending on the stopping times \( \tau^n \).

The deduction of this estimate is quite long. Let us first consider the solutions \( \tilde{f}_n \) and \( \tilde{G}_n \) of (3.6) for \( f = f(Y_n) \) and \( f = G(t, Y_n) \), respectively. Then the following relations hold

\[
(\tilde{f}_n, e_i)_V = (f(Y_n), e_i), \quad (\tilde{G}_n, e_i)_V = (G(t, Y_n), e_i). \tag{4.15}
\]

If we use these relations in equality (4.9), we get

\[
d(Y_n, e_i)_V = (\tilde{f}_n, e_i)_V \ dt + (\tilde{G}_n, e_i)_V \ dW_t.
\]

Multiplying the last identity by \( \lambda_i \) and using (4.3) in the resulting equation yields

\[
d(Y_n, e_i)_{\tilde{W}} = (\tilde{f}_n, e_i)_{\tilde{W}} \ dt + (\tilde{G}_n, e_i)_{\tilde{W}} \ dW_t.
\]

On the other hand, the Itô formula gives

\[
d(Y_n, e_i)^2_{\tilde{W}} = 2(Y_n, e_i)_{\tilde{W}} (\tilde{f}_n, e_i)_{\tilde{W}} \ dt + 2(Y_n, e_i)_{\tilde{W}} (\tilde{G}_n, e_i)_{\tilde{W}} \ dW_t + |(\tilde{G}_n, e_i)_{\tilde{W}}|^2 dt.
\]

Multiplying this equality by \( \frac{1}{\lambda_i} \) and summing over \( i = 1, \ldots, n \), we obtain

\[
d \| Y_n \|^2_{\tilde{W}} = 2(\tilde{f}_n, Y_n)_{\tilde{W}} \ dt + 2(\tilde{G}_n, Y_n)_{\tilde{W}} \ dW_t + \sum_{i=1}^{n} \frac{1}{\lambda_i} |(\tilde{G}_n, e_i)_{\tilde{W}}|^2 \ dt,
\]

that is

\[
d(\| \text{curl} \, v(Y_n) \|^2_{\tilde{W}} + \| Y_n \|^2_{\tilde{W}}) = 2 \left( (\text{curl} \, v(\tilde{f}_n), \text{curl} \, v(Y_n)) + (\tilde{f}_n, Y_n)_V \right) \ dt
+ 2 \left( (\text{curl} \, v(\tilde{G}_n), \text{curl} \, v(Y_n)) + (\tilde{G}_n, Y_n)_V \right) \ dW_t + \sum_{i=1}^{n} \lambda_i |(G(t, Y_n), e_i)_V|^2 \ dt
\]

by the definition of the inner product (4.3). The definition of \( \tilde{f}_n \) and \( \tilde{G}_n \) as solutions of (3.6) implies

\[
d(\| \text{curl} \, v(Y_n) \|^2_{\tilde{W}} + \| Y_n \|^2_{\tilde{W}}) = 2 \left( (\text{curl} \, f(Y_n), \text{curl} \, v(Y_n)) + (f(Y_n), Y_n)_V \right) \ dt
+ 2 \left( (\text{curl} \, G(t, Y_n), \text{curl} \, v(Y_n)) + (G(t, Y_n), Y_n)_V \right) \ dW_t + \sum_{i=1}^{n} \lambda_i |(G(t, Y_n), e_i)_V|^2 \ dt,
\]

that reduces to

\[
d(\| \text{curl} \, v(Y_n) \|^2_{\tilde{W}}) = 2 \left( (\text{curl} \, f(Y_n), \text{curl} \, v(Y_n)) \right) \ dt
+ 2 \left( (\text{curl} \, G(t, Y_n), \text{curl} \, v(Y_n)) \right) \ dW_t + \sum_{i=1}^{n} (\lambda_i - 1) |(G(t, Y_n), e_i)|^2 \ dt, \tag{4.16}
\]

taking into account equality (4.11).

Since

\[
\text{curl} \, (\text{curl} \, v(Y_n)) = (Y_n \cdot \nabla) \text{curl} \, v(Y_n) \quad \text{and} \quad ((Y_n \cdot \nabla) \text{curl} \, v(Y_n), \text{curl} \, v(Y_n)) = 0,
\]

we have

\[
(\text{curl} \, f(Y_n), \text{curl} \, v(Y_n)) = (\nu \text{curl} \, \Delta Y_n + \text{curl} \, U, \text{curl} \, v(Y_n))
= \left( -\frac{\nu}{\alpha} \text{curl} \, v(Y_n) + \frac{\nu}{\alpha} \text{curl} \, Y_n + \text{curl} \, U, \text{curl} \, v(Y_n) \right).
\]
Substituting this last relation in (4.10), we derive
\[
d \| \text{curl} \, (Y_n) \|^2 + \frac{2\nu}{\alpha} \| \text{curl} \, v \, (Y_n) \|^2 \, dt = 2 \left( \frac{\nu}{\alpha} \text{curl} \, Y_n + \text{curl} \, U, \text{curl} \, v \, (Y_n) \right) \, dt
\]
\[
+ 2 \left( (\text{curl} \, G(t, Y_n), \text{curl} \, v \, (Y_n)) \right) \, dW_t + \sum_{i=1}^n (\lambda_i - 1) ||(G(t, Y_n), e_i)||^2 \, dt. \quad (4.17)
\]

Let us take \( t \in [0, T] \). By integrating over the time interval \( (0, s) \), \( 0 \leq s \leq \tau_N^\wedge t \), taking the supremum and the expectation, we get
\[
E \sup_{s \in [0, \tau_N^\wedge t]} \| \text{curl} \, v \,(Y_n(s)) \|^2 + \frac{2\nu}{\alpha} E \int_0^{\tau_N^\wedge t} \| \text{curl} \, v \,(Y_n) \|^2 \, ds \leq E \| \text{curl} \, v \,(Y_n(0)) \|^2
\]
\[
+ 2E \int_0^{\tau_N^\wedge t} \left( \left( \frac{\nu}{\alpha} \text{curl} \, Y_n + \text{curl} \, U, \text{curl} \, v \,(Y_n) \right) \right) \, ds
\]
\[
+ 2E \sup_{s \in [0, \tau_N^\wedge t]} \left| \int_0^s (\text{curl} \, G(r, Y_n), \text{curl} \, v \,(Y_n)) \, dW_r \right|
\]
\[
+ E \int_0^{\tau_N^\wedge t} \sum_{i=1}^n |\lambda_i - 1| ||(G(s, Y_n), e_i)||^2 \, ds. \quad (4.18)
\]

Moreover
\[
2E \int_0^{\tau_N^\wedge t} \left| \left( \frac{\nu}{\alpha} \text{curl} \, Y_n + \text{curl} \, U, \text{curl} \, v \,(Y_n) \right) \right| \, ds
\]
\[
\leq E \sup_{s \in [0, \tau_N^\wedge t]} \| \text{curl} \, v \,(Y_n(s)) \|^2 \int_0^{\tau_N^\wedge t} \left( \frac{\nu}{\alpha} ||\text{curl} \, Y_n||_2 + ||\text{curl} \, U||_2 \right) \, ds
\]
\[
\leq \varepsilon E \sup_{s \in [0, \tau_N^\wedge t]} \| \text{curl} \, v \,(Y_n(s)) \|^2 + C_\varepsilon E \int_0^{\tau_N^\wedge t} (||\text{curl} \, Y_n||_2^2 + ||\text{curl} \, U||_2^2) \, ds,
\]
that is
\[
2E \int_0^{\tau_N^\wedge t} \left| \left( \frac{\nu}{\alpha} \text{curl} \, Y_n + \text{curl} \, U, \text{curl} \, v \,(Y_n) \right) \right| \, ds
\]
\[
\leq \varepsilon E \sup_{s \in [0, \tau_N^\wedge t]} \| \text{curl} \, v \,(Y_n(s)) \|^2 + C_\varepsilon E \int_0^{\tau_N^\wedge t} (||\text{curl} \, Y_n||_2^2 + ||\text{curl} \, U||_2^2) \, ds. \quad (4.19)
\]

The Burkholder-Davis-Gundy inequality and estimate (2.5) imply
\[
2E \sup_{s \in [0, \tau_N^\wedge t]} \left| \int_0^s (\text{curl} \, G(r, Y_n), \text{curl} \, v \,(Y_n)) \, dW_r \right| \leq 2E \left( \int_0^{\tau_N^\wedge t} ||(\text{curl} \, G(s, Y_n), \text{curl} \, v \,(Y_n))||^2 \, ds \right)^{\frac{1}{2}}
\]
\[
\leq 2E \sup_{s \in [0, \tau_N^\wedge t]} \| \text{curl} \, v \,(Y_n(s)) \|^2 \left( \int_0^{\tau_N^\wedge t} ||\text{curl} \, G(s, Y_n)||_2^2 \, ds \right)^{\frac{1}{2}}
\]
\[
\leq \varepsilon E \sup_{s \in [0, \tau_N^\wedge t]} \| \text{curl} \, v \,(Y_n)||_2^2 + C_\varepsilon E \int_0^{\tau_N^\wedge t} ||\text{curl} \, G(s, Y_n)||_2^2 \, ds
\]
\[
\leq \varepsilon E \sup_{s \in [0, \tau_N^\wedge t]} \| \text{curl} \, v \,(Y_n)||_2^2 + C_\varepsilon E \int_0^{\tau_N^\wedge t} \left( 1 + ||Y_n(s)||_2^2 \right) \, ds,
\]
that is

\[ 2\mathbb{E}\sup_{s \in [0, \tau^N_T \wedge T]} \left| \int_0^s (\text{curl} G(r, Y_n), \text{curl} v(Y_n)) \ dW_r \right| \]

\[ \leq \varepsilon \mathbb{E}\sup_{s \in [0, \tau^N_T \wedge T]} \|\text{curl} v(Y_n)\|_2^2 + C \varepsilon \int_0^{\tau^N_T \wedge T} \left( 1 + \|Y_n\|_V^2 \right) \ ds. \]  \hspace{1cm} (4.20)

Substituting (4.19)-(4.20) in (4.18) and choosing \( \varepsilon = \frac{1}{4} \), we obtain

\[ \frac{1}{2} \mathbb{E}\sup_{s \in [0, \tau^N_T \wedge T]} \|\text{curl} v(Y_n(s))\|_2^2 + \frac{2\nu}{\alpha} \mathbb{E}\int_0^{\tau^N_T \wedge T} \|\text{curl} v(Y_n)\|_2^2 \ ds \leq \mathbb{E}\|\text{curl} v(Y_n(0))\|_2^2 \]

\[ + C \varepsilon \int_0^{\tau^N_T \wedge T} \|\text{curl} U\|_2^2 \ ds + C \varepsilon \int_0^{\tau^N_T \wedge T} \left( 1 + \|Y_n(s)\|^2 \right) \ ds. \]  \hspace{1cm} (4.21)

**Step 3. The limit transition, as \( N \to \infty \), in estimates (4.14), (4.21).**

Since

\[ \mathbb{E}\|\text{curl} v(Y_n(0))\|_2^2 \leq C \mathbb{E}\|Y_n(0)\|^2_{H^1} \leq C \mathbb{E}\|Y_0\|^2_{H^1} \leq C \]

and

\[ \mathbb{E}\int_0^{\tau^N_T \wedge T} \left( 1 + \|Y_n(s)\|_V^2 \right) \ ds \leq C \]

by (4.20) and (4.14), we obtain

\[ \mathbb{E}\sup_{s \in [0, \tau^N_T \wedge T]} \|\text{curl} v(Y_n(s))\|_2^2 \leq C. \]

Therefore estimates (4.10), (4.11) imply

\[ \mathbb{E}\sup_{s \in [0, \tau^N_T \wedge T]} \|Y_n(s)\|^2_{H^1} \leq C, \]

where \( C \) is a constant independent of \( N \) and \( n \). Let us fix \( n \in \mathbb{N} \), writing

\[ \mathbb{E}\sup_{s \in [0, \tau^N_T \wedge T]} \|Y_n(s)\|^2_{H^1} = \mathbb{E}\left( \sup_{s \in [0, \tau^N_T \wedge T]} 1_{\{\tau^N_T < T\}} \|Y_n(s)\|^2_{H^1} \right) \]

\[ + \mathbb{E}\left( \sup_{s \in [0, \tau^N_T \wedge T]} 1_{\{\tau^N_T \geq T\}} \|Y_n(s)\|^2_{H^1} \right) \]

\[ \geq \mathbb{E}\left( \max_{s \in [0, \tau^N_T]} 1_{\{\tau^N_T < T\}} \|Y_n(s)\|^2_{H^1} \right) \geq N^2 P(\tau^N_T < T), \]  \hspace{1cm} (4.22)

we deduce that \( P(\tau^N_T < T) \to 0 \), as \( N \to \infty \). This means that \( \tau^N_T \to T \) in probability, as \( N \to \infty \). Then there exists a subsequence \( \{\tau^N_{T_k}\} \) of \( \{\tau^N_T\} \) (that may depend on \( n \)) such that

\[ \tau^N_{T_k}(\omega) \to T \quad \text{for a.e. } \omega \in \Omega \quad \text{as } k \to \infty. \]

Since \( \tau^N_{T_k} \leq T_n \leq T \), we deduce that \( T_n = T \), so \( Y_n \) is a global-in-time solution of the stochastic differential equation (4.4). On the other hand, the sequence \( \{\tau^N_T\} \) of the stopping times is
monotone on \( N \) for each fixed \( n \), then we can apply the monotone convergence theorem in order to pass to the limit in inequalities (4.14) and (4.21) as \( N \to \infty \), deducing estimates (4.6) and (4.7).

**Lemma 3.4**, we immediately derive the main estimate (4.8) of this lemma.

For each \( \theta \), assume that \( \text{Lemma 4.4} \) we improve the integrability properties for the solution \( Y_n \) of problem (4.4).

**Lemma 4.4** Assume that \( U \in L^p(\Omega \times (0, T); H), \quad Y_0 \in L^p(\Omega, V) \) for some \( 4 \leq p < \infty \).

Then the solution \( Y_n \) of problem (4.4) belongs to \( L^p(\Omega, L^\infty(0, T; V)) \) and verifies the estimate

\[
\mathbb{E} \sup_{s \in [0, t]} \| Y_n(t) \|_V^p \leq C \mathbb{E} \| Y_0 \|_V^p + C (1 + \mathbb{E} \int_0^t \| U \|_2^2 \, ds),
\]

where \( C \) is a positive constant independent of \( n \).

**Proof.** For each \( n \in \mathbb{N} \), let us define the suitable sequence \( \{ \tau_n^N \}_{N \in \mathbb{N}} \) of the stopping times

\[
\tau_n^N = \inf \{ t \geq 0 : \| Y_n(t) \|_V \geq N \} \wedge T.
\]

Applying the Itô formula for the function \( \theta(x) = x^q, \quad q \geq 1 \), to process (4.11), we have

\[
d\| Y_n \|_V^{2q} = q \| Y_n \|_V^{2q-2} \left[ - \left( 4 \nu \| D Y_n \|_2^2 + 2 \nu \| Y_n \|_2^2 \right)_t + 2 \langle U, Y_n \rangle + \sum_{i=1}^n \| (G(t, Y_n), e_i) \|_2^2 \right] \, dt
\]

\[
+ 2 q \| Y_n \|_V^{2q-2} \langle G(t, Y_n), Y_n \rangle \, dW_t + 2 q (q-1) \| Y_n \|_V^{2q-4} \| G(t, Y_n), Y_n \|_2^2 \, dt.
\]

Let us take \( t \in [0, T] \). Integrating over the time interval \( [0, s] \), \( 0 \leq s \leq \tau_n^N \wedge t \), we obtain

\[
\| Y_n(s) \|_V^{2q} \leq \| Y_n(0) \|_V^{2q} + q \int_0^s \| Y_n \|_V^{2q-2} \left( 2 \langle U, Y_n \rangle + \sum_{i=1}^n \| (G(r, Y_n), e_i) \|_2^2 \right) \, dr
\]

\[
+ 2 q \int_0^s \| Y_n \|_V^{2q-2} \langle G(r, Y_n), Y_n \rangle \, dW_r
\]

\[
+ 2 q (q-1) \int_0^s \| Y_n \|_V^{2q-4} \| (G(r, Y_n), Y_n) \|_2^2 \, dr.
\]

From estimate (4.12) we have

\[
\sum_{i=1}^n \| (G(t, Y_n), e_i) \|_2^2 \leq C(1 + \| Y_n \|_V^2).
\]

Taking the supremum on \( s \in [0, \tau_n^N \wedge t] \), the expectation in (4.24), applying Burkholder-Davis-Gundy’s and Young’s inequalities, and proceeding analogously to (4.13), we obtain

\[
\mathbb{E} \sup_{s \in [0, \tau_n^N \wedge t]} \| Y_n(s) \|_V^{2q} \leq \mathbb{E} \| Y_n(0) \|_V^{2q} + C_q \mathbb{E} \int_0^t \| U \|_2^{2q} \, ds
\]

\[
+ \frac{1}{2} \mathbb{E} \sup_{s \in [0, \tau_n^N \wedge t]} \| Y_n(s) \|_V^{2q} + C_q (1 + \mathbb{E} \int_0^{\tau_n^N \wedge t} \| Y_n \|_V^{2q} \, ds).
\]
Using Gronwall’s inequality, we deduce
\[
\mathbb{E} \sup_{s \in [0, \tau_n^N \wedge t]} \|Y_n(s)\|_{L^q}\leq \mathbb{E} \|Y_0\|_{L^q}^2 + C (1 + \mathbb{E} \int_0^t \|U\|_{L^2}^2 \, ds) \tag{4.25}
\]
for any \( q \geq 1 \) and \( t \in [0, T] \). Using the fact that \( \mathbb{E} \sup_{s \in [0, \tau_n^N \wedge T]} \|Y_n(s)\|_{L^q}^2 \leq C \) with \( C \) independent of \( n \) and \( N \), we may reasoning as in the proof of Lemma [4.3] in order to verify that for each \( n \), \( \tau_n^N \rightarrow T \) in probability, as \( N \rightarrow \infty \). Then, there exists a subsequence \( \{\tau_n^k\} \) of \( \{\tau_n^N\} \) (that may depend on \( n \)) such that \( \tau_n^k \rightarrow T \) a. e. \( \omega \in \Omega \) as \( k \rightarrow \infty \). Now, let us consider \( q = \frac{2}{p} \). Using the monotone convergence theorem, we pass to the limit in (4.25) as \( k \rightarrow \infty \), deriving estimate (4.23).

Proof of Theorem 4.2. Existence. The proof is split into four steps.

Step 1. Estimates and convergences, related with the projection operator.
Let \( P_n : \tilde{W} \rightarrow W_n \) be the orthogonal projection defined by
\[
P_n y = \sum_{j=1}^n \tilde{c}_j \tilde{e}_j \quad \text{with} \quad \tilde{c}_j = (y, \tilde{e}_j)_{\tilde{W}}, \quad \forall y \in \tilde{W},
\]
where \( \{\tilde{e}_j = \frac{1}{\sqrt{\lambda_j}} e_j\}_{j=1}^\infty \) is the orthonormal basis of \( \tilde{W} \). It is easy to check that
\[
P_n y = \sum_{j=1}^n c_j e_j \quad \text{with} \quad c_j = (y, e_j)_V, \quad \forall y \in \tilde{W}.
\]
By Parseval’s identity we have that
\[
||P_n y||_V \leq ||y||_V, \quad \forall y \in V,
\]
\[
||P_n y||_{\tilde{W}} \leq ||y||_{\tilde{W}} \quad \text{and} \quad P_n y \rightarrow y \quad \text{strongly in} \quad \tilde{W}, \quad \forall y \in \tilde{W}.
\]
Considering an arbitrary \( Z \in L^2(\Omega \times (0,T); \tilde{W}) \), we have
\[
||P_n Z||_{\tilde{W}} \leq ||Z||_{\tilde{W}} \quad \text{and} \quad P_n Z(\omega, t) \rightarrow Z(\omega, t) \quad \text{strongly in} \quad \tilde{W},
\]
which are valid for \( P \)-a. e. \( \omega \in \Omega \) and a.e. \( t \in (0,T) \). Hence Lebesgue’s dominated convergence theorem and the inequality
\[
||Z||_V \leq C ||Z||_{\tilde{W}} \quad \text{for any} \quad Z \in \tilde{W}
\]
imply
\[
P_n Z \quad \rightarrow \quad Z \quad \text{strongly in} \quad L^2(\Omega \times (0,T); \tilde{W}),
\]
\[
P_n Z \quad \rightarrow \quad Z \quad \text{strongly in} \quad L^2(\Omega \times (0,T); V). \tag{4.26}
\]

Step 2. Passing to the limit in the weak sense.
We have
\[ \mathbb{E} \sup_{t \in [0,T]} \| Y_n(t) \|_{\tilde{W}}^2 \leq C, \quad \mathbb{E} \sup_{t \in [0,T]} \| Y_n(t) \|_V^p \leq C \] (4.27)
for some constants \( C \) independent of the index \( n \), by estimates (4.8) and (4.23). Therefore there exists a suitable subsequence \( Y_n \), which is indexed by the same index \( n \), for simplicity of notations, such that
\[ Y_n \rightharpoonup Y \quad \text{*-weakly in } L^2(\Omega, L^\infty(0,T;\tilde{W})), \]
\[ Y_n \rightharpoonup Y \quad \text{*-weakly in } L^p(\Omega, L^\infty(0,T;V)). \] (4.28)
Moreover, we have
\[ P_n Y \rightarrow Y \quad \text{strongly in } L^2(\Omega \times (0,T);\tilde{W})), \]
\[ P_n Y \rightarrow Y \quad \text{strongly in } L^2(\Omega \times (0,T);V)). \] (4.29)

Let us introduce the operator \( B : \tilde{W} \times V \rightarrow \tilde{W}^* \), defined as
\[ (B(y,z), \phi) = (\text{curl} \upsilon(y) \times z, \phi) \quad \text{for any } y, \phi \in \tilde{W} \quad \text{and } z \in V, \]
and state some useful properties. Relation (3.23) gives
\[ (B(y,z), \phi) = - (B(y,\phi), z), \quad (B(y,z), z) = 0, \] (4.30)
and (3.24), (3.25) yield
\[ \| B(y,z) \|_{\tilde{W}^*} \leq C \| z \|_V \| y \|_{\tilde{W}}, \]
\[ \| B(y,z) \|_{\tilde{W}^*} \leq C \| y \|_V \| z \|_{\tilde{W}}. \] (4.31)
(4.32)
From (3.26) there exists a fixed constant \( C_1 \) such that
\[ \| B(y,y) \|_{\tilde{W}^*} \leq C_1 \| y \|_V^2, \] (4.33)
then
\[ \| B(y,y) \|_{L^2(\Omega \times (0,T);\tilde{W}^*))} \leq C_1 \| y \|_{L^2(\Omega \times (0,T);V)}^2. \] (4.34)
On the other hand, taking into account (2.5), (4.28), there exist operators \( B^*(t) \) and \( G^*(t) \), such that
\[ B(Y_n,Y_n) \rightharpoonup B^*(t) \quad \text{weakly in } L^2(\Omega \times (0,T);\tilde{W}^*)), \]
\[ G(t,Y_n) \rightharpoonup G^*(t) \quad \text{weakly in } L^2(\Omega \times (0,T);V^m). \] (4.35)
Passing on the limit \( n \rightarrow \infty \) in equation (4.4), we derive that the limit function \( Y \) satisfies the stochastic differential equation
\[ d(\upsilon(Y), \phi) = [(\nu \Delta Y + U, \phi) - (B^*(t), \phi)] \, dt + (G^*(t), \phi) \, dW_t, \quad \forall \phi \in \tilde{W}. \] (4.36)

**Step 3. Deduction of strong convergences, as \( n \rightarrow \infty \), depending on the stopping times \( \tau_M \).**

In order to prove that the limit process \( Y \) satisfy equation (4.1), we will adapt the methods in [7] (see also [34]). Let us introduce a sequence \( (\tau_M) \), \( M \in \mathbb{N} \), of stopping times defined by
\[ \tau_M(\omega) = \inf\{ t \geq 0 : \| Y(t) \|_{\tilde{W}}(\omega) \geq M \} \land T, \quad \omega \in \Omega. \]
Taking the difference of (4.33) and (4.36), we deduce
\[ d(P_nY - Y_n, e_i)_V = \left[ \nu \Delta (Y - Y_n), e_i \right] + \langle B(Y_n, Y_n) - B^*(t), e_i \rangle \ dt - \langle G(t, Y_n) - G^*(t), e_i \rangle \ dW_t, \] (4.37)
which is valid for any \( e_i \in \mathcal{W}, i = 1, ..., n. \)

By applying Itô’s formula, equation (4.37) gives
\[ d(P_nY - Y_n, e_i)_V^2 = 2 \left( P_nY - Y_n, e_i \right)_V \left[ \nu \Delta (Y - Y_n), e_i \right] + \langle B(Y_n, Y_n) - B^*(t), e_i \rangle \ dt - \langle G(t, Y_n) - G^*(t), e_i \rangle^2 \ dt, \]

and summing over the index \( i \) from 1 to \( n \), we derive
\[
\begin{align*}
&d \left( \| P_nY - Y_n \|^2 \right) + \left( 4\nu \| D(P_nY - Y_n) \|^2 + 2\nu \gamma \| P_nY - Y_n \|^2 \right) \ dt \\
= &2\nu \Delta (P_nY - Y_n), P_nY - Y_n) \ dt \\
+ &2 \langle B(Y_n, Y_n) - B^*(t), P_nY - Y_n \rangle \ dt \\
+ &\sum_{i=1}^n \langle (G(t, Y_n) - G^*(t), e_i) \rangle^2 \ dt - 2 \langle G(t, Y_n) - G^*(t), P_nY - Y_n \rangle \ dW_t.
\end{align*}
\] (4.38)

Let us notice that
\[
\begin{align*}
\langle B(Y_n, Y_n) - B^*(t), P_nY - Y_n \rangle &= \langle B(Y_n, Y_n) - B(P_nY, P_nY), P_nY - Y_n \rangle \\
&\quad + \langle B(P_nY, P_nY) - B(Y, Y), P_nY - Y_n \rangle + \langle B(Y, Y) - B^*(t), P_nY - Y_n \rangle \\
&= I_1 + I_2 + I_3.
\end{align*}
\] (4.39)

Using (4.30), we derive
\[
\begin{align*}
I_1 &= \langle B(Y_n, Y_n) - B(P_nY, P_nY), P_nY - Y_n \rangle \\
&= \langle B(Y_n, Y_n) - B(Y, Y), P_nY - Y_n \rangle + \langle B(P_nY, P_nY) - B(Y, Y), P_nY - Y_n \rangle \\
&\quad - \langle B(Y, Y) - B^*(t), P_nY - Y_n \rangle \\
&= \langle B(P_nY - Y_n, P_nY - Y_n), P_nY \rangle,
\end{align*}
\]
which along with (4.33) implies
\[ |I_1| \leq C_1 \| Y \|^2_{\mathcal{W}} \| P_nY - Y_n \|^2_V. \] (4.40)

For the term \( I_2 \), we have
\[
|I_2| = \| (B(P_nY, P_nY) - B(Y, Y), P_nY - Y_n) \| \leq \| B(P_nY, P_nY) - B(Y, Y) \|_{\mathcal{W}} \| P_nY - Y_n \|_{\mathcal{W}},
\]
and for every \( \phi \in \mathcal{W} \), it follows from (4.31) and (4.32) that
\[
\| B(P_nY, P_nY) - B(Y, Y) \|_{\mathcal{W}} \leq \| B(P_nY - Y, P_nY) \|_{\mathcal{W}} + \| B(Y, P_nY - Y) \|_{\mathcal{W}} \leq C \| Y \|_{\mathcal{W}} \| P_nY - Y \| V.
\]

and consequently, we obtain
\[ |I_2| \leq \| Y \|_{\mathcal{W}} \| P_nY - Y \| V \| P_nY - Y_n \|_{\mathcal{W}}. \] (4.41)
On the other hand, denoting by $\tilde{G}_n, \tilde{G}$ and $\tilde{G}^*$ the solutions of the Stokes system (3.6) for $f = G(t, Y_n), f = G(t, Y)$ and $f = G^*(t)$, respectively, we have

$$(G(t, Y_n) - G^*(t), e_i) = (\tilde{G}_n - \tilde{G}^*, e_i)_V, \quad i = 1, 2, \ldots, n.$$  

Then

$$\sum_{i=1}^n |(G(t, Y_n) - G^*(t), e_i)|^2 = \|P_n \tilde{G}_n - P_n \tilde{G}^*\|^2_V.$$  

The standard relation $x^2 = (x - y)^2 - 2xy + y^2$ allows to write

$$\|P_n \tilde{G}_n - P_n \tilde{G}^*\|^2_V = \|P_n \tilde{G}_n - P_n \tilde{G}\|^2_V - \|P_n \tilde{G} - P_n \tilde{G}^*\|^2_V$$

$$+ 2(P_n \tilde{G}_n - P_n \tilde{G}^*, P_n \tilde{G} - P_n \tilde{G}^*)_V.$$  

From the properties of the solutions of the Stokes system (3.6) and (2.5), we have

$$\|P_n \tilde{G}_n - P_n \tilde{G}\|^2_V \leq \|\tilde{G}_n - \tilde{G}\|^2_V \leq \|G(t, Y_n) - G(t, Y)\|^{2}_{L^2} \leq K \|Y_n - Y\|^2_V,$$

then, for the fixed constant $C_2 = 2K$, we have

$$\|P_n \tilde{G}_n - P_n \tilde{G}^*\|^2_V \leq K \|Y_n - Y\|^2_V - \|P_n \tilde{G} - P_n \tilde{G}^*\|^2_V$$

$$+ 2(P_n \tilde{G}_n - P_n \tilde{G}^*, P_n \tilde{G} - P_n \tilde{G}^*)_V$$

$$\leq C_2 \|Y_n - P_n Y\|^2_V + C \|P_n Y - Y\|^2_V - \|P_n \tilde{G} - P_n \tilde{G}^*\|^2_V$$

$$+ 2(P_n \tilde{G}_n - P_n \tilde{G}^*, P_n \tilde{G} - P_n \tilde{G}^*)_V. \quad (4.42)$$  

The positive constants $C_1$ and $C_2$ in (4.40) and (4.42), are independent of $n$ and may depend on the data: the domain $O$, the regularity of $\Gamma$, the physical constants $\nu, \alpha, \gamma, K$.

Let us notice that from the convergence results (4.26) - (4.29), (4.35), we can guess that by passing to the limit in equation (4.38), in a suitable way, as $n \to \infty$, all terms containing $P_n Y - Y$ will vanish on the right hand side of equality (4.38), according to relations (4.26), (4.11) and (4.32). But the terms with $Y_n - P_n Y$ will remain. Fortunately, these terms can be eliminated by the introduction of the auxiliary function

$$\xi(t) = e^{-C_2 t - 2C_1 \int_0^t \|Y\|^2_W ds}.$$  

Now, applying Itô’s formula in equality (1.38), we get

$$d(\xi(t)\|P_n Y - Y_n\|^2_V) + \xi(t) \left(4\nu \|D(P_n Y - Y_n)\|^2_V + 2\nu\gamma \|P_n Y - Y_n\|^2_{L^2(\Gamma)}\right) dt$$

$$= 2\nu \xi(t) \langle \Delta(P_n Y - Y), P_n Y - Y_n \rangle dt$$

$$+ 2\xi(t) \langle B(Y_n, Y_n) - B^*(t), P_n Y - Y_n \rangle dt + \xi(t) \sum_{i=1}^n |(G(t, Y_n) - G^*(t), e_i)|^2 dt$$

$$- 2\xi(t) \langle G(t, Y_n) - G^*(t), P_n Y - Y_n \rangle dW_t$$

$$- C_2 \xi(t)\|P_n Y - Y_n\|^2_V dt - 2C_1 \xi(t) \|Y\|^2_W \|P_n Y - Y_n\|^2_V dt. \quad (4.43)$$  

Integrating it over the time interval $(0, \tau_M(\omega))$, taking the expectation and applying estimates
In what follows we show that, for each $M \in \mathbb{N}$, the right hand side of this inequality goes to zero, as $n \to \infty$.

Using (4.27), (4.28) and the properties of the projection $P_n$, we have

$$|J_1| = 2\nu E \int_0^T \xi(s) |1_{[0,\tau,M]}(s)\Delta(P_n Y - Y), P_n Y - Y_n| ds \leq C\|P_n Y - Y\|_{L^2(\Omega \times (0,T), H^2)} \|P_n Y - Y_n\|_{L^2(\Omega \times (0,T), H^2)} \leq C\|P_n Y - Y\|_{L^2(\Omega \times (0,T), H^2)} \left(\|Y\|_{L^2(\Omega \times (0,T), H^2)} + \|Y_n\|_{L^2(\Omega \times (0,T), H^2)}\right),$$

which goes to zero, as $n \to \infty$, by (4.29). Taking into account estimates (4.8), (4.41) and knowing that $1_{[0,\tau,M]}(s)\|Y(s)\|_{\tilde{W}} \leq M$, $P$ - a. e. in $\Omega$, we deduce that

$$|J_2| \leq 2E \int_0^T \xi(s) I_2 ds \leq 2E \int_0^T \xi(s) |1_{[0,\tau,M]}(s)\|Y\|_{\tilde{W}} \|P_n Y - Y\|_{\tilde{W}} \left(\|Y\|_{\tilde{W}} + \|Y_n\|_{\tilde{W}}\right) ds \leq CM\|P_n Y - Y\|_{L^2(\Omega \times (0,T), \tilde{W})} \left(\|Y\|_{L^2(\Omega \times (0,T), \tilde{W})} + \|Y_n\|_{L^2(\Omega \times (0,T), \tilde{W})}\right) \leq CM\|P_n Y - Y\|_{L^2(\Omega \times (0,T), \tilde{W})},$$

which also converges to zero by (4.29).

Convergences (4.28) and (4.29) give that

$$P_n Y - Y_n \to 0 \quad \text{weakly in } L^2(\Omega \times (0,T), \tilde{W}),$$

then for any operator $A \in L^2(\Omega \times (0,T), \tilde{W}^*)$ we have

$$E \int_0^T \langle A, P_n Y - Y_n \rangle ds \to 0 \quad \text{as } n \to \infty.$$

Since the function $1_{[0,\tau,M]}(s)\xi(s)$ is bounded and independent of the space variable, we have

$$\|1_{[0,\tau,M]}(s)\xi(s) (B(Y,Y) - B^*)\|_{L^2(\Omega \times (0,T), \tilde{W}^*)} \leq C \left(\|B(Y,Y)\|_{L^2(\Omega \times (0,T), \tilde{W}^*)}^2 + \|B^*\|_{L^2(\Omega \times (0,T), \tilde{W}^*)}^2\right) \leq C;$$
by (4.29), (1.31) and (1.35). Therefore

\[ J_3 = 2\mathbb{E} \int_0^T \xi(s) J_3 \, ds \]
\[ = 2\mathbb{E} \int_0^T (1_{[0,\tau_M]}(s) \xi(s) (B(Y_s Y) - B^*(s), P_n Y - Y_n) \, ds \to 0 \quad \text{as } n \to \infty. \]

We write

\[ J_4 = \mathbb{E} \int_0^T \xi(s) \left[ C \| P_n Y - Y \|_{V^1}^2 + 2(P_n G_n - P_n G^*) (P_n G - P_n G^*)_V \right] \, ds \]
\[ = C\mathbb{E} \int_0^T 1_{[0,\tau_M]}(s) \xi(s) \| P_n Y - Y \|_{V^1}^2 \, ds \]
\[ + C\mathbb{E} \int_0^T 1_{[0,\tau_M]}(s) \xi(s) (G_n - G^*, P_n G - P_n G^*)_V \, ds. \]

Due to (1.29), we have

\[ \left| \mathbb{E} \int_0^T 1_{[0,\tau_M]}(s) \xi(s) \| P_n Y - Y \|_{V^1}^2 \, ds \right| \leq \mathbb{E} \int_0^T \| P_n Y - Y \|_{V^1}^2 \, ds \to 0, \quad \text{as } n \to \infty. \]

Now, for each stochastic process \( Z \in L^2(\Omega \times (0, T), H) \) let us denote by \( \tilde{Z} \) the solution of the modified Stokes problem (5.6). We recall that the operator

\[ A : Z \to \tilde{Z} \]

is linear and continuous operator from \( L^2(\Omega \times (0, T), V) \) into \( L^2(\Omega \times (0, T), V^1) \). Applying Proposition A.2 in [7] (see also references therein), it follows that \( A \) is continuous for the weak topology, namely if \( Z_n \to Z \) weakly in \( L^2(\Omega \times (0, T), V) \), then \( \tilde{Z}_n \to \tilde{Z} \) weakly in \( L^2(\Omega \times (0, T), V^1) \). Due to this property and the convergence result (4.35), we obtain

\[ \tilde{G}_n - \tilde{G}^* \to \tilde{G} - \tilde{G}^* \quad \text{weakly in } L^2(\Omega \times (0, T), V^m). \quad (4.44) \]

Moreover, we have \( \tilde{G} - \tilde{G}^* \in \bar{V}^m \) and

\[ P_n(\tilde{G} - \tilde{G}^*) \to \tilde{G} - \tilde{G}^* \quad \text{strongly in } L^2(\Omega \times (0, T), \bar{W}^m), \]
\[ P_n(\tilde{G} - \tilde{G}^*) \to \tilde{G} - \tilde{G}^* \quad \text{strongly in } L^2(\Omega \times (0, T), V^m). \]

Then we can verify that

\[ 1_{[0,\tau_M]}(s) \xi(s) P_n(\tilde{G} - \tilde{G}^*) \to 1_{[0,\tau_M]}(s) \xi(s)(\tilde{G} - \tilde{G}^*) \quad \text{strongly in } L^2(\Omega \times (0, T), V^m). \quad (4.45) \]

As a consequence of (4.44) and (4.45), we have

\[ \mathbb{E} \int_0^T 1_{[0,\tau_M]}(s) \xi(s)(\tilde{G}_n - \tilde{G}^*, P_n \tilde{G} - P_n \tilde{G}^*)_V \, ds \]
\[ = \mathbb{E} \int_0^T (\tilde{G}_n - \tilde{G}^*, 1_{[0,\tau_M]}(s) \xi(s) P_n(\tilde{G} - \tilde{G}^*)_V \, ds \to 0, \quad \text{as } n \to \infty. \]
Collecting all convergence results, we obtain the following strong convergences, depending on the stopping times $\tau_M$,

$$
\lim_{n \to \infty} E \left(\xi(\tau_M) || P_n Y(\tau_M) - Y_n(\tau_M) ||_V^2 \right) = 0,
$$

$$
\lim_{n \to \infty} E \int_0^{\tau_M} \xi(s) || P_n \tilde{G} - P_n \tilde{G}^* ||_V^2 ds = 0,
$$

$$
\lim_{n \to \infty} E \int_0^{\tau_M} \xi(s) \left(4\nu || P_n Y - Y_n ||_2^2 + 2\nu\gamma || P_n Y - Y_n ||_{L^2(\Gamma)}^2 \right) ds = 0,
$$

for each $M \in \mathbb{N}$. Since there exists a strictly positive constant $\mu$, such that $\mu \leq 1_{[0,\tau_M]}(s) \xi(s) \leq 1$, it follows that

$$
\lim_{n \to \infty} E \int_0^{\tau_M} || P_n Y - Y_n ||_V^2 ds = 0 \quad \text{implying} \quad \lim_{n \to \infty} E \int_0^{\tau_M} || Y - Y_n ||_V^2 ds = 0
$$

by (4.29). In addition, considering (4.26), we have

$$
E \int_0^{\tau_M} || \tilde{G} - \tilde{G}^* ||_V^2 ds = 0. \quad (4.46)
$$

Step 4. Identification of $B^*(t)$ with $B(Y,Y)$ and $G^*(t)$ with $G(t,Y)$.

Now, we are able to show that the limit function $Y$ satisfies equation (4.1). Integrating equation (4.36) on the time interval $(0, \tau_M \wedge t)$, we derive

$$
(v(Y(\tau_M \wedge t)), \phi) - (v(Y_0), \phi) = \int_0^{\tau_M \wedge t} [\nu \Delta Y + U, \phi] - (B^*(s), \phi) \right) ds
$$

$$
+ \int_0^{\tau_M \wedge t} (G^*(s), \phi) \right) dW_s \quad (4.47)
$$

for any $\phi \in \overline{W}$.

From (4.40) it follows that

$$
1_{[0,\tau_M]}(t)\tilde{G} = 1_{[0,\tau_M]}(t)\tilde{G}^* \quad \text{a. e. in } \Omega \times (0,T),
$$

which implies

$$
1_{[0,\tau_M]}(t)G(t,Y) = 1_{[0,\tau_M]}(t)G^*(t) \quad \text{a. e. in } \Omega \times (0,T) \quad (4.48)
$$

by (3.6). Since $B(Y_n, Y_n) - B(Y,Y) = B(Y_n, Y_n - Y) - B(Y_n - Y, Y)$, using (4.31)-(4.32), we have

$$
|| B(Y_n, Y_n) - B(Y,Y) ||_{\overline{W}} \leq C \left( || Y_n ||_{\overline{W}} + || Y_n ||_{\overline{W}} \right) || Y_n - Y ||_{V}.
$$

Then for any $\varphi \in L^\infty(\Omega \times (0,T); \overline{W})$, using (4.27), (4.28)

$$
| E \int_0^T 1_{[0,\tau_M]}(s) (B(Y_n, Y_n) - B(Y,Y), \varphi) ds |
$$

$$
\leq CE \int_0^T 1_{[0,\tau_M]}(s) \left( || Y_n ||_{\overline{W}} + || Y ||_{\overline{W}} \right) || Y_n - Y ||_{V} || \varphi ||_{\overline{W}} ds
$$

$$
\leq C|| \varphi ||_{L^\infty(\Omega \times (0,T); \overline{W})} E \int_0^T \left( || Y_n ||_{\overline{W}} + || Y ||_{\overline{W}} \right) || Y_n - Y ||_{V} ds
$$

$$
\leq C|| \varphi ||_{L^\infty(\Omega \times (0,T); \overline{W})} \left( E \int_0^{\tau_M} || Y_n - Y ||_{V}^2 ds \right)^{\frac{1}{2}} \rightarrow 0, \quad \text{as } n \to \infty.
$$
Taking into account (4.35) and that the space \( L^\infty(\Omega \times (0, T); \tilde{W}) \) is dense in \( L^2(\Omega \times (0, T); \tilde{W}) \), we obtain
\[
1_{[0, \tau_M]}(s)B^*(s) = 1_{[0, \tau_M]}(s)B(Y, Y) \quad \text{a. e. in } \Omega \times (0, T).
\]
By introducing identities (4.48), (4.49) in equation (4.47), it follows that
\[
(v(Y_{M \land t}), \phi) - (v(Y_0), \phi) = \int_0^{\tau_M \land t} [(v \Delta Y + U, \phi) - \langle B(Y, Y), \phi \rangle] \, ds + \int_0^{\tau_M \land t} (G(s, Y), \phi) \, dW_s.
\]
Now, reasoning as in (4.22) we have \( \tau_M \rightarrow T \) a. e. in \( \Omega \). We can pass to the limit in each term of equation (4.50) in \( L^1(\Omega \times (0, T)) \), as \( M \rightarrow \infty \), by applying the Lebesgue dominated convergence theorem and the Burkholder-Davis-Gundy inequality for the last (stochastic) term, deriving equation (4.1) a. e. in \( \Omega \times (0, T) \).

Let us notice that the estimates for \( Y_n \) in Lemmas 4.3 and 4.4 are valid also for the limit process \( Y \), due to convergence (4.28).

The uniqueness of the solution \( Y \) follows from the stability result that we will show in the next section. ■

5 Stability result for solutions

In this section we will establish a stability property for solutions of the stochastic second grade fluid model (2.1). In spite of the existence result with \( H^3 \) space regularity, the difference of two solutions can only be estimated (with respect to the initial data) in space \( H^2 \). It will be convenient to introduce the following norm on the space \( W \)
\[
\|y\|_W = \|y\|_V + \|Pv(y)\|_2, \quad y \in W.
\]
As a consequence of (3.18) and (2.3) this norm \( \| \cdot \|_W \) is equivalent to \( \| \cdot \|_{H^2} \).

**Theorem 5.1** Assume that for some \( 4 \leq p < \infty \)
\[
U_1, U_2 \in L^p(\Omega, L^p(0, T; H(\text{curl}; \mathcal{O}))), \quad Y_{1,0}, Y_{2,0} \in L^p(\Omega, V) \cap L^2(\Omega, \tilde{W})
\]
and
\[
Y_1, Y_2 \in L^2(\Omega, L^\infty(0, T; \tilde{W})) \cap L^p(\Omega, L^\infty(0, T; V))
\]
are corresponding solutions of (2.1) in the sense of the variational equality (1.1).

Then there exist strictly positive constants \( C_3 \) and \( C \), which depend only on the data (the domain \( \mathcal{O} \), the regularity of \( \Gamma \), the physical constants \( \nu, \alpha, \gamma, K \)) and satisfy the following estimate
\[
\mathbb{E} \sup_{s \in [0, t]} \xi(s) \|Y_1(s) - Y_2(s)\|_W^2 \leq C(\mathbb{E} \|Y_{1,0} - Y_{2,0}\|_W^2 + \mathbb{E} \int_0^t \xi(s) \|U_1(s) - U_2(s)\|_2^2 \, ds) \quad (5.1)
\]
with the function \( \xi \) defined as
\[
\xi(t) = e^{-C_3 t} \left( \|Y_1\|_{H^3} + \|Y_2\|_{H^3} \right),
\]

\[\]
where \( \pi = \pi_1 - \pi_2 \) and \( U = U_1 - U_2 \). Applying the operator \((I - \alpha P \Delta)^{-1}\) to equation (5.2), we deduce a stochastic differential equation for \( Y \), then with the help of Itô’s formula we obtain

\[
d ||Y||_V^2 = 2((\nu \Delta Y - \text{curl} \, v(Y) \times Y_2 - \text{curl} \, v(Y_1) \times Y + U, Y) \, dt
\]

\[
+ \tilde{G}_1 - \tilde{G}_2 \, dt + 2(G(t, Y_1) - G(t, Y_2), Y) \, dW_t,
\]

where \( \tilde{G}_i \) are the solutions of the modified Stokes problem with \( f = G(t, Y_i) \), \( i = 1, 2 \). Hence, using assumption (2.5), we have

\[
\|\tilde{G}_1 - \tilde{G}_2\|_V^2 \leq C\|G(t, Y_1) - G(t, Y_2)\|_2^2 \leq C\|Y\|_V^2.
\]

Taking into account property \( 3.29 \), estimate \( 3.24 \) and the Young inequality, we derive

\[
\|Y(t)\|_V^2 + \int_0^t \left( 4\nu \|DY\|^2_2 + 2\nu \gamma \|Y\|_{L^2(\gamma)}^2 \right) \, ds \leq \|Y_0\|_V^2 + C \int_0^t \|Y_2\|_{H^3} \|Y\|_V^2 \, ds
\]

\[
+ \int_0^t \|U\|_2^2 \, ds + C \int_0^t \|Y\|_V^2 \, ds + 2 \int_0^t (G(s, Y_1) - G(s, Y_2), Y) \, dW_s.
\]

The Itô formula also gives

\[
d \|P_v (Y)\|_2^2 = 2((\nu \Delta Y - \text{curl} \, v(Y) \times Y_2 - \text{curl} \, v(Y_1) \times Y + U, P_v (Y)) \, dt
\]

\[
+ G(t, Y_1) - G(t, Y_2) \|_2^2 \, dt + 2(G(t, Y_1) - G(t, Y_2), P_v (Y)) \, dW_t.
\]

Estimating the nonlinear term

\[
||\text{curl} \, v(Y) \times Y_2 + \text{curl} \, v(Y_1) \times Y, P_v (Y)|| \leq C_3 (\|Y_2\|_{H^3} + \|Y_1\|_{H^3}) \left( \|Y\|_V^2 + \|P_v (Y)\|_2^2 \right)
\]

and using (2.5), we deduce

\[
\|P_v (Y(t))\|_2^2 + \frac{2\nu}{\alpha} \int_0^t \|v(Y)\|_2^2 \, ds \leq \|P_v (Y_0)\|_2^2 + 2 \int_0^t \left( \frac{\nu}{\alpha} Y + U, P_v (Y) \right) \, ds
\]

\[
+ C_3 \int_0^t (\|Y_2\|_{H^3} + \|Y_1\|_{H^3}) \left( \|Y\|_V^2 + \|P_v (Y)\|_2^2 \right) \, ds
\]

\[
+ C \int_0^t \|Y\|_V^2 \, ds + 2 \int_0^t (G(s, Y_1) - G(s, Y_2), P_v (Y)) \, dW_s.
\]
Summing this inequality with (5.4), we obtain
\[
\|Y(t)\|_V^2 + \|\mathbb{P}v(Y(t))\|_2^2 \leq \|Y_0\|_V^2 + \|\mathbb{P}v(Y_0)\|_2^2 + \int_0^t \|U\|_V^2 ds + C \int_0^t (\|Y\|_V^2 + \|\mathbb{P}v(Y)\|_2^2) ds + C_3 \int_0^t (\|Y_2\|_{H^3} + \|Y_1\|_{H^3}) (\|Y\|_V^2 + \|\mathbb{P}v(Y)\|_2^2) ds \\
+ \frac{2}{C} \int_0^t (G(s, Y_1) - G(s, Y_2), Y + \mathbb{P}v(Y)) dW_s.
\]

Taking \(\xi(t) = e^{-C_3 \int_0^t (\|Y_2\|_{H^3} + \|Y_1\|_{H^3}) ds}\) and applying Itô’s formula, then we easily obtain
\[
\xi(t) \left(\|Y(t)\|_V^2 + \|\mathbb{P}v(Y(t))\|_2^2\right) \leq \|Y_0\|_V^2 + \|\mathbb{P}v(Y_0)\|_2^2 + \int_0^t \xi(s) \|U\|_V^2 ds + C \int_0^t \xi(s) (\|Y\|_V^2 + \|\mathbb{P}v(Y)\|_2^2) ds \\
+ 2 \int_0^t \xi(s) (G(s, Y_1) - G(s, Y_2), Y + \mathbb{P}v(Y)) dW_s. \tag{5.5}
\]

The Burkholder-Davis-Gundy inequality gives
\[
\mathbb{E} \sup_{s \in [0, t]} \left| \int_0^s \xi(r) (G(r, Y_1) - G(r, Y_2), Y + \mathbb{P}v(Y)) dW_r \right|^2 \\
\leq \mathbb{E} \left( \int_0^t \xi^2(s) \|Y\|_V^2 + \|\mathbb{P}v(Y)\|_2^2 ds \right)^2 \\
\leq \varepsilon \mathbb{E} \sup_{s \in [0, t]} \xi(s) (\|Y\|_V + \|\mathbb{P}v(Y)\|_2) \\
+ C_2 \mathbb{E} \int_0^t \xi(s) (\|Y\|_V + \|\mathbb{P}v(Y)\|_2) ds.
\]

Substituting this inequality with \(\varepsilon = \frac{1}{4}\) in (5.5) and taking the supremum on the time interval \([0, t]\) and the expectation, we deduce
\[
\mathbb{E} \sup_{s \in [0, t]} \xi(s) \|Y(s)\|_W^2 \leq \mathbb{E} \|Y_0\|_V^2 + \mathbb{E} \int_0^t \xi(s) \|U\|_V^2 ds + C \mathbb{E} \int_0^t \xi(s) \|Y(s)\|_V^2 ds.
\]

Hence Gronwall’s inequality yields (5.1).}

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