ON LIOUVILLE SYSTEMS AT CRITICAL PARAMETERS, PART 1:
ONE BUBBLE

CHANG-SHOU LIN AND LEI ZHANG

ABSTRACT. In this paper we consider bubbling solutions to the general Liouville system:

\[(0.1)\quad \Delta_g u_i + \sum_{j=1}^{n} a_{ij} \rho_j^k \left( \frac{h_j e^u_j}{h_i e^u_i} - 1 \right) = 0 \quad \text{in } M, \quad i = 1, \ldots, n \quad (n \geq 2)\]

where \((M, g)\) is a Riemann surface, and \(A = (a_{ij})_{n \times n}\) is a constant non-negative matrix and \(\rho_j^k \to \rho_j\) as \(k \to \infty\). Among other things we prove the following sharp estimates.

1. The location of the blowup point.
2. The convergence rate of \(\rho_j^k - \rho_j, j = 1, \ldots, n\).

These results are of fundamental importance for constructing bubbling solutions. It is interesting to compare the difference between the general Liouville system and the SU(3) Toda system on estimates (1) and (2).

1. INTRODUCTION

Let \((M, g)\) be a compact Riemann surface whose volume is normalized to be 1, \(h_1, \ldots, h_n\) be positive \(C^3\) functions on \(M\), \(\rho_1, \ldots, \rho_n\) be nonnegative constants. In this article we continue our study of the following Liouville system defined on \((M, g)\):

\[(1.1)\quad \Delta_g u_i + \sum_{j=1}^{n} \rho_j a_{ij} \left( \frac{h_j e^u_j}{\int_M h_j e^u_j dV_g} - 1 \right) = 0, \quad i \in I := \{1, \ldots, n\}\]

where \(dV_g\) is the volume form, \(A = (a_{ij})\) is a non-negative constant matrix, \(\Delta_g\) is the Laplace-Beltrami operator \((-\Delta_g \geq 0)\). When \(n = 1\) and \(a_{11} = 1\), equation (1.1) is the mean field equation of the Liouville type:

\[(1.2)\quad \Delta_g u + \rho \left( \frac{h e^u}{\int_M h e^u dV_g} - 1 \right) = 0 \quad \text{in } M.\]

Therefore, the Liouville system (1.1) is a natural extension of the classical Liouville equation, which has been extensively studied for the past three decades. Both the Liouville equation and the Liouville system are related to various fields of geometry, Physics, Chemistry and Ecology. For example in conformal geometry,
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when \( \rho = 8\pi \) and \( M \) is the sphere \( S^2 \), equation (1.2) is equivalent to the famous Nirenberg problem. For a bounded domain in \( \mathbb{R}^2 \) and \( n = 1 \), a variant of (1.2) can be derived from the mean field limit of Euler flows or spherical Onsager vortex theory, as studied by Caglioti, Lions, Marchioro and Pulvirenti \([6, 7]\), Kiessling \([28]\), Chanillo and Kiessling \([9]\) and Lin \([34]\). In classical gauge field theory, equation (1.1) is closely related to the Chern-Simons-Higgs equation for the abelian case, see \([5, 24, 25, 45]\). Various Liouville systems are also used to describe models in the theory of self-gravitating systems \([1]\), Chemotaxis \([17, 27]\), in the physics of charged particle beams \([4, 20, 29, 30]\), in the non-abelian Chern-Simons-Higgs theory \([21, 26, 45]\) and other gauge field models \([22, 23, 31]\). For recent developments of these subjects or related Liouville systems in more general settings, we refer the readers to \([2, 3, 12, 13, 14, 15, 18, 19, 32, 33, 34, 35, 36, 37, 40, 41, 42, 43, 44, 46, 47]\) and the references therein.

For any solution \( u \) of (1.2), clearly adding any constant to \( u \) gives another solution. So it is nature to assume \( u \in \tilde{H}^1(M) \), where

\[
\tilde{H}^1(M) = \left\{ u \in L^2(M) \left| |\nabla_g u| \in L^2(M) \right. \right\}.
\]

Corresponding to (1.1) we set

\[
\tilde{H}^{1, n} = \tilde{H}^1(M) \times \cdots \times \tilde{H}^1(M)
\]

to be the space for solutions. For any \( \rho = (\rho_1, \cdots, \rho_n) \), \( \rho_i > 0 (i \in I = \{1, \ldots, n\}) \), let \( \Phi_{\rho} \) be a nonlinear functional defined in \( \tilde{H}^{1, n} \) by

\[
\Phi_{\rho}(u) = \frac{1}{2} \sum_{i, j \in I} a^{ij} \int_M \nabla_g u_i \cdot \nabla_g u_j dV_g - \sum_{j \in I} \rho_j \log \int_M h_j e^{u_j} dV_g
\]

where \((a^{ij})_{n \times n}\) is the inverse of \( A = (a_{ij})_{n \times n} \). It is easy to see that equation (1.1) is the Euler-Lagrangian equation of \( \Phi_{\rho} \).

If the matrix \( A \) satisfies the following two assumptions:

\[
(H1) : \quad A \text{ is symmetric, nonnegative, irreducible and invertible.}
\]

\[
(H2) : \quad a^{ii} \leq 0, \forall i \in I, \quad a^{ij} \geq 0, \forall i \neq j, \quad \sum_{j \in I} a^{ij} \geq 0, \forall i \in I,
\]

the authors prove in \([38]\) that for \( \rho \) satisfying

\[
8\pi N \sum_{i \in I} \rho_i < \sum_{i, j \in I} a_{ij} \rho_i \rho_j < 8\pi (N + 1) \sum_{i \in I} \rho_i,
\]

there is a priori estimate for all solutions \( u \) to (1.1), and the Leray-Schauder degree \( d_{\rho} \) for equation (1.1) is

\[
d_{\rho} = \frac{1}{N!} \left( (-\chi_M + 1) \cdots (-\chi_M + N) \right) \quad \text{if (1.3) holds}
\]

where \( \chi_M \) is the Euler characteristic of \( M \). Moreover, if \( \rho^k \) tends to the hypersurface \( \{ \rho : \quad 8\pi N \sum_{i \in I} \rho_i = \sum_{i, j \in I} a_{ij} \rho_i \rho_j \} \), there exist exactly \( N \) disjoint blowup points (see \([38]\)).
The proof of the a priori bound in \cite{38} relies on the sharp estimate for a sequence of bubbling solutions to (1.1). Let \( u^k \) be the blowup solutions corresponding to \( \rho^k \) and \( B(p_t, \delta_0) \) \((t = 1, \ldots, N)\) be disjoint balls around distinct blowup points in \( M \). Then under assumptions (H1) and (H2), the behavior of \( u^k \) around any \( p_t \) is fully bubbling, that is, the maximum values of any components of \( u^k \) in any of the balls are of the same magnitude:

\[
\max_{B(p_t, \delta_0)} u^k_i = \max_{B(p_t, \delta_0)} u^k_j + O(1), \quad \forall i, j \in I.
\]

Moreover, after a suitable scaling around each blowup point \( p_t \), \( u^k \) converges to an entire solution \( \tilde{U} = (\tilde{U}_1, \ldots, \tilde{U}_n) \) of the following Liouville system:

\[
\begin{align*}
\Delta \tilde{U}_i + \sum_{j=1}^n a_{ij} e^{\tilde{U}_j} &= 0, \quad \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{\tilde{U}_i} &< \infty, \quad \tilde{U}_i \text{ is radial}, \forall i \in I.
\end{align*}
\]

One may expect the limiting entire solution to be different around each blowup point, however the authors proved that \( \tilde{U} \) is independent of blowup points, and only depends on the ratio of \( \rho^k_1 - \rho_1 \): \( \rho^k_2 - \rho_2 \): \ldots: \( \rho^k_n - \rho_n \) (see \cite{37}). Naturally it leads to the question: how to construct bubbling solutions with the help of this information?

In this paper and subsequent ones, we are devoted to study the bubbling phenomenon of Liouville systems: how to accurately estimate the bubbling solutions of (1.1) and how to construct them. These are quite challenging analytic problems. In general, blowup analysis for a system of equations is much harder than that for the single equation. One reason is that the Pohozaev identity, a balancing condition, is no longer so powerful as in the scalar case. Another reason is that there are too many entire solutions: the parameter \( \sigma = (\sigma_1, \ldots, \sigma_n) \) \((\sigma_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{u_i})\) represents the energy of the entire solution, forms a submanifold of \( n - 1 \) dimension. However, for the Liouville equation, the energy is just one number: \( \int_{\mathbb{R}^2} e^u = 8\pi \).

In this article we consider the case of one blowup point, and always assume (H1) only. Let \( \rho = (\rho_1, \ldots, \rho_n) \) and

\[
\Lambda_J(\rho) = 8\pi \sum_{i \in J} \rho_i - \sum_{i, j \in J} a_{ij} \rho_i \rho_j
\]

for any \( J \subset I := \{1, \ldots, n\} \). Define

\[
\Gamma_1 = \{ \rho; \Lambda_J(\rho) = 0 \quad \text{and} \quad \Lambda_J(\rho) > 0 \quad \text{for all} \quad \emptyset \subsetneq J \subsetneq I \}.
\]

Note that if (H2) also holds also, then \( \Lambda_J(\rho) = 0 \) implies \( \Lambda_J(\rho) > 0 \) for all \( J \subsetneq I \) (see \cite{38}). For any \( \rho \) we define \((m_1, \ldots, m_n)\) by

\[
m_i = \frac{1}{2\pi} \sum_{j \in I} a_{ij} \rho_j.
\]

The quantity \( m_i \) can be interpreted by the entire solution \( \tilde{U} \) of (1.4). In fact

\[
\rho_i = \int_{\mathbb{R}^2} e^{\tilde{U}_i} dx, \quad i \in I,
\]
and
\begin{equation}
\bar{U}_i(x) = -m_i \log |x| + O(1), \quad \text{for } |x| \text{ near infinity.}
\end{equation}

The integrability of $e^{\bar{U}_i}$ implies $m_i > 2$ for all $i$. On the other hand, $\Lambda_i(p) = 0$ can be written as $\sum_{i \in I} (m_i - 4) p_i = 0$. Thus either $\min\{m_1, \ldots, m_n\} < 4$ or $m_i = 4$ for all $i \in I$. We also note that (1.7) implies that $\Gamma_1$ is a smooth submanifold because the normal vector at $\rho \in \Gamma_1$:

$$
(\sum_{j \in I} a_{ij} \rho_j - 4\pi, \sum_{j \in I} a_{nj} \rho_j - 4\pi),
$$

does not have all its components positive.

The asymptotic behavior of $\bar{U}_i(x)$ shows that the decay rate of $e^{\bar{U}_i(x)}$ is $O(|x|^{-m})$, where

$$
m = \min\{m_1, \ldots, m_n\}.
$$

In this article we define $Q \in \Gamma_1$ with $m = 4$, i.e. $m_i = 4$ for all $i$. Thus the decay rate of $e^{\bar{U}_i}$ for $\rho = Q$ is $O(|x|^{-4})$. The difference on the decay rate for $Q$ and $\rho \neq Q$ will have great effects on bubbling analysis later.

Let $u^k = (u_1^k, \ldots, u_n^k)$ be a sequence of blow up solutions to (1.1) with $\rho = \rho^k$ such that $\rho^k \to \rho \in \Gamma_1$. The point $Q$ defined above is of particular importance, and the readers will see that in our main theorems below, the asymptotic behavior of blowup solutions, the nature of $\Lambda_i(\rho^k)$ and the location of blowup point are all significantly different depending on $\rho = Q$ or not.

Let $p$ be the blowup point of $u^k$ and

\begin{equation}
M_k = \max_{B(p, \delta)} \left( u_1^k(x) - \log \int_M h_1 e^{u_1^k} dV_g \right),
\end{equation}

\begin{equation}
\epsilon_k = e^{-\frac{1}{2}M_k}.
\end{equation}

Since $\rho^k \to \Gamma_1$, there is only one blowup point $p$. It is easy to see that $u^k$ fully blows up at $p$ (see Lemma 6.1):

\begin{equation}
\max_{B(p, \delta)} \left( u_1^k(x) - \log \int_M h_1 e^{u_1^k} dV_g \right) = M_k + O(1), \quad i \in I.
\end{equation}

Our first result is on the location of the blowup point $p$. Let $p_k \to p$ be where the maximum of $\{u_1^k, \ldots, u_n^k\}$ is attained, then we have

\textbf{Theorem 1.1.} Let $\rho^k \to \rho \in \Gamma_1$ and all $\rho_i^k - \rho_i$ have the same sign.

(1) If $\rho \neq Q$, then

\begin{equation}
\sum_{i \in I} \left( \nabla (\log h_i)(p_k) + 2\pi m \nabla_1 \gamma(p_k, p_k) \right) \rho_i = O(\epsilon_k^{m-2}).
\end{equation}

(2) If $\rho = Q$, then

\begin{equation}
\sum_{i} \left( \nabla (\log h_i)(p_k) + 8\pi \nabla_1 \gamma(p_k, p_k) \right) \rho_i = O(\epsilon_k^2 \log \epsilon_k^{-1}).
\end{equation}

where $\nabla_1$ denotes the derivative with respect to the first variable, and $\gamma(x, y)$ stands for the regular part of the Green’s function.
Our second result is about the decay rate of $\Lambda_f(\rho^k)$. To state the result, we need to define the following quantity $D_i (i \in I = \{1, \ldots, n\}):$

$$D_i = \lim_{\delta_0 \to 0} \left( \delta_0^{2-m} - \frac{m-2}{2\pi} \int_{M \setminus B(p, \delta_0)} h_i(x) e^{2\pi m(G(x,p)-\gamma(p,p))} dV_s \right)$$

provided that $m < 4$. The limit is well defined if $m < 4$, see section 6.

**Theorem 1.2.** Suppose $\rho^k \to \rho \in \Gamma_1$ and $\rho \neq Q$, if all $\rho_i^k - \rho_i$ have the same sign, then

$$\Lambda_f(\rho^k) = 8\pi^2 \sum_{i \in I_1} (c_i D_i + o(1)) e_i^{m-2},$$

where $I_1$ is a subset of $I$ where $m_i = m$ for all $i \in I_1$, $c_i$ is a constant determined in (6.14), $o(1) \to 0$ as $k \to \infty$.

If $M$ is a flat torus with fundamental cell domain $\Omega \subset \mathbb{R}^2$, then $D_i$ can be written as

$$D_i = \frac{m-2}{2\pi} \left( \int_{\mathbb{R}^2 \setminus \Omega} \frac{1}{|x-p|^m} dx - \lim_{\delta_0 \to 0} \int_{\Omega \setminus B(p, \delta_0)} \frac{H_i(x,p)}{|x-p|^m} dx \right)$$

where

$$H_i(x,p) = \frac{h_i(x)}{h_i(p)} e^{2\pi m(\gamma(x,p)-\gamma(p,p))} - 1, \quad i \in I.$$

See [8] and [15] for related discussions.

**Remark 1.1.** The assumption that all $\rho_i^k - \rho_i$ have the same sign seems due to some technical difficulties. When $n = 2$ and $\rho \neq Q$, this assumption is not needed for both Theorem 1.1 and Theorem 1.2.

**Theorem 1.3.** Suppose $\rho^k \to \rho \in \Gamma_1$ and $\rho = Q$. If all $\rho_i^k - Q_i$ have the same sign, then

$$\Lambda_f(\rho^k) = -16\pi^2 \left( \sum_{i \in I} b_i e_i^2 + o(1) \right) e_i^2 \log e_i^{-1},$$

where

$$b_i = \frac{1}{4} \left( \Delta \log h_i(p) - 2K(p) + 8\pi + \| \nabla \log h_i(p) + 8\pi \nabla_1 \gamma(p,p) \|_2 \right),$$

$c_i$ is determined in (6.14).

Important information on bubbling solutions can be observed on the two cases: $\rho \neq Q$ and $\rho = Q$. Theorem 1.2 which is on $\rho \neq Q$, has its leading term in $\Lambda_f(\rho^k)$ involved with global information of the manifold, while the leading term in Theorem 1.3 which corresponds to $\rho = Q$, only depends on the geometric information at the blowup point. Moreover, the error terms in Theorem 1.2 and Theorem 1.3 respectively also indicate the different asymptotic behaviors of blowup solutions near the singularity. All these differences in the two cases will lead to separate strategies in the construction of bubbling solutions in forthcoming works.

Since Liouville systems and Toda systems share a lot of common features, it is informative to compare our main theorems with the ones for the $SU(3)$ Toda system. First, the location of the blowup point in Theorem 1.3 is a critical point of
a combination of \( \log h_i, \rho_i \) and \( \gamma \) (\( \nabla \gamma \) vanishes if the Riemann surface has constant curvature). However for the \( SU(3) \) Toda system, the blowup point \( p \) is a critical point of both \( \log h_1 \) and \( \log h_2 \), i.e. \( p \) satisfies (see \( \text{(36)} \))

\[
\nabla h_1(p) = \nabla h_2(p) = 0.
\]

Second, for the \( SU(3) \) Toda system, the convergence rate of \( \rho^k_i - \rho_i \) is estimated to be

\[
\rho^k_i - \rho_i = (e^{\tilde{c}^k b_i} + o(1))\varepsilon^2 \log \varepsilon^{-1},
\]

where \( b_i \) is the term in \( \text{(1.14)} \). Nevertheless our result in \( \text{(1.14)} \) is again a combination of the \( b_i \)s. The comparison of the results reflects some major difference between the Toda system and our Liouville system:

1. The dimension of kernel space of the linearized operator at an entire solution is 8 for \( SU(3) \) Toda system, and is 3 for our Liouville system.
2. The set \( \Gamma = \{ (\rho_1, \ldots, \rho_n); \rho_i = \int_{\mathbb{R}^2} e^{u_i}, (u_1, \ldots, u_n) \text{ is an entire solution} \} \) is only a point for \( SU(3) \) Toda system, while for the Liouville system it is a \( (n-1) \) dimensional manifold.

As far as the blowup analysis is concerned, our Liouville system has disadvantages in both respects, as the kernel space is too small and \( \Gamma \) is too large. For a sequence of bubbling solutions, it is extremely difficult to pin-point suitable approximating solutions from \( \Gamma \), because at the beginning, the local energy of bubbling solutions could be estimated in some rough way. This rough estimate of the local energy leads to a small perturbation of global bubbling solutions. This perturbation on global solutions, albeit small, has a non-negligible effect on the approximation of blowup solutions. This difficulty is particularly evident when we study bubbling solutions with multiple blowup points in \( \text{(39)} \). Therefore our method to obtain those sharp results is different from ones in Chen-Lin \( \text{(12)} \) for the mean field equations and Lin-Wei-Zhao \( \text{(36)} \) for the \( SU(3) \) Toda system (The methods in \( \text{(12)} \) and \( \text{(36)} \) are similar).

The organization of the paper is as follows. In section two we first prove a uniqueness theorem for globally defined linearized Liouville systems. This result plays a central role for the delicate blowup analysis in sections three to five. The main idea of the proof uses a monotonicity property of solutions and we introduce a way to use maximum principles suitable for Liouville systems. In the second part of section two, we study the asymptotic behavior of global solutions to the Liouville system on \( \mathbb{R}^2 \) and obtain some Pohozaev identities. In section three and section four we obtain a sharp expansion result for blowup solutions around a blowup point. Then in section five, for local equations we use Pohozaev identity to determine the locations of blowup points. Then in section six we return to the equation on manifold and compute the leading term for \( \rho^k \rightarrow \rho \) in both situations \( \rho \neq Q \) or \( \rho = Q \) and complete the proofs of the main theorems.

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2. PRELIMINARY RESULTS ON THE LIOUVILLE SYSTEMS

In this section we prove two theorems on the Liouville systems with the matrix $A$ satisfying $(H1)$. They are important for the blowup analysis and the computation of the leading terms of $\rho \to \Gamma_1$ in this paper and $\rho \to \Gamma_N$ in the forthcoming work [39].

2.1. A uniqueness theorem for the linearized system of $n$ equations. In the first subsection we prove a uniqueness theorem for the linearized system of $n$ equations.

**Theorem 2.1.** Let $A$ be a matrix that satisfies $(H1)$, $u = (u_1, \ldots, u_n)$ be a radial solution of

\begin{equation*}
-\Delta u_i = \sum_{j=1}^{n} a_{ij} e^{u_j}, \quad i \in I = \{1, \ldots, n\}
\end{equation*}

in $\mathbb{R}^2$, $\int_{\mathbb{R}^2} e^{u_i} < \infty$.

Suppose $\phi = (\phi_1, \ldots, \phi_n)$ satisfies

\begin{equation}
-\Delta \phi_i = \sum_{j=1}^{n} a_{ij} e^{u_j} \phi_j, \quad i \in I.
\end{equation}

(1)

(2.1) \quad |\phi_i(x)| \leq C(1 + |x|^\tau), \quad x \in \mathbb{R}^2,

for some $\tau \in (0, 1)$ and

\begin{equation*}
\phi_i(0) = 0, \quad i \in I.
\end{equation*}

Then there exist $c_1, c_2 \in \mathbb{R}$ such that

\begin{equation*}
\phi_i(r) = c_1 u_i'(r) \frac{x_1}{r} + c_2 u_i'(r) \frac{x_2}{r}, \quad i \in I.
\end{equation*}

(2) If $|\phi_i(x)| \leq C$ for all $x \in \mathbb{R}^2$, then there exist $c_0, c_1, c_2 \in \mathbb{R}$ such that

\begin{equation*}
\phi_i(r) = c_0 (ru_i'(r) + 2) + c_1 u_i'(r) \frac{x_1}{r} + c_2 u_i'(r) \frac{x_2}{r}, \quad \mathbb{R}^2, \quad i \in I.
\end{equation*}

(3) If $\phi_i(x) = O(|x|^2)$ near 0 and $|\phi_i(x)| \leq C(1 + |x|)^{2-\epsilon_0}$ for some $\epsilon_0 > 0$, then $\phi_i \equiv 0$.

Before the proof of Theorem 2.1 we first establish

**Lemma 2.1.** Let $A$ and $u$ be as in Theorem 2.1 let $\Phi = (\Phi_1, \ldots, \Phi_n)$ solve

\begin{equation}
\begin{cases}
\Phi''_i(r) + \frac{1}{r} \Phi'_i(r) - \frac{1}{r} \sum_{j=1}^{n} a_{ij} e^{u_j} \Phi_j = 0, & 0 < r < \infty,

|\Phi_i(r)| \leq C r / (1 + r)^{\epsilon_0} \quad \text{for some } \epsilon_0 \in (0, 1), \quad \forall i \in I.
\end{cases}
\end{equation}

Then there exists a constant $C$ such that $\Phi_i(r) = Cu_i'(r)$ for $i \in I$. 
Proof of Lemma 2.1:

The proof is in two steps. First we show that under the assumption of $\Phi_i$ at infinity we have the following sharper decay estimate:

\[(2.3) \quad |\Phi_i(r)| \leq Cr(1+r)^{-2}, \quad 0 < r < \infty, \quad i \in I.\]

Indeed, let $\tilde{\psi}_i(r) = \Phi_i(r)/r$. By direct computation we see that $\tilde{\psi}_i = (\tilde{\psi}_1, \ldots, \tilde{\psi}_n)$ satisfies

\[(2.4) \quad \psi''_i(r) + \frac{3}{r} \psi'_i(r) + \sum_j a_{ij} e^{\psi_j} \psi_j = 0, \quad 0 < r < \infty.\]

Clearly in order to show (2.3) we only need to show $|\tilde{\psi}_i(r)| \leq Cr^{-2}$ for $r > 1$ under the assumption that $|\tilde{\psi}_i(r)| \leq Cr^{-\delta_0}$ for $i \in I$ and $r > 1$. Let

\[\tilde{\psi}_i(t) = \tilde{\psi}_i(e^t) \quad \text{and} \quad \bar{u}_i(t) = u_i(e^t) + 2t,\]

it is easy to see that $\tilde{\psi}(t) = (\tilde{\psi}_1(t), \ldots, \tilde{\psi}_n(t))$ satisfies

\[(2.5) \quad \tilde{\psi}''_i(t) + 2\tilde{\psi}'_i(t) + \sum_{j \in I} a_{ij} e^{\bar{u}_j(t)} \tilde{\psi}_j(t) = 0, \quad -\infty < t < \infty\]

and our goal is to show

\[(2.6) \quad \tilde{\psi}_i(t) = O(e^{-2l})\]

knowing $\tilde{\psi}_i = O(e^{-\delta_0})$ for $t > 0, i \in I$. Set

\[l_i = \frac{1}{2\pi} \sum_{j \in I} a_{ij} \int_R e^{\psi_j}, \quad l = \min\{l_1, \ldots, l_n\}.\]

By Lemma 2.2 below $l > 2$. Let

\[h_i(t) = -\sum_{j \in I} a_{ij} e^{\bar{u}_j(t)} \tilde{\psi}_j(t) = O(e^{(2-l-\delta_0)t}), \quad t > 0.\]

Then

\[\tilde{\psi}_i(t) = C_0 + C_1 e^{-2t} + \frac{1}{2} \int_0^t h_i(s)ds - \frac{1}{2} e^{-2t} \int_0^t e^{2s} h_i(s)ds.\]

Using the asymptotic rate of $h_i(t)$ at infinity we further have

\[\tilde{\psi}_i(t) = (C_0 + \frac{1}{2} \int_0^\infty h_i(s)ds) + C_1 e^{-2t} + O(e^{(2-l-\delta_0)t}).\]

Since $\tilde{\psi}_i(t)$ tends to 0 as $t$ goes to infinity we know $\tilde{\psi}_i(t) = O(e^{-2l})$ if $l + \delta_0 \geq 4$, in which case (2.6) is established. Otherwise we obtain $\tilde{\psi}_i(t) = O(e^{(2-l-\delta_0)t})$. In the latter case, we apply the same procedure to obtain a better decaying rate of $\tilde{\psi}_i(t)$ at infinity. After finite steps, (2.6) is established.

In the second step we complete the proof of the Lemma 2.1. By way of contradiction we suppose there is a solution $\Phi = (\Phi_1, \ldots, \Phi_n)$ that satisfies (2.2) and $\Phi$ is not a multiple of $f = (u_1(r), \ldots, u_n(r))$. Let $\tilde{\psi}_i = -u_i(r)/r$, then clearly both $\psi^0 = (\psi_1^0, \ldots, \psi_n^0)$ and $\tilde{\psi} = (\tilde{\psi}_1, \ldots, \tilde{\psi}_n)$ satisfy (2.4). We verify by direct computation that

\[\int_0^e e^{\psi_j^0}(\sigma) \sigma^2 d\sigma = -\int_0^e (e^{\psi_j^0})' \sigma^2 d\sigma = -e^{\psi_j^0(r)} r^2 + 2 \int_0^e e^{\psi_j}(\sigma) d\sigma > 0.\]
Note that the last inequality is justified by \( u_i'(r) < 0 \) for \( r > 0 \) and \( i \in I \). Also, since 
\[ e^u \leq Cr^{-2-\delta} \]
for some \( \delta > 0 \) and \( r > 1 \), 
\[ \int_0^{\infty} e^u \psi_0^i(\sigma) \sigma^3 d\sigma < \infty. \]
Based on the computation above we set 
\[ S = \left\{ t \left| \psi_i'(t) = \psi_i^0 + t(\psi_i^0 - \bar{\psi}) \right. \right. \} \]
Let \( \bar{\psi} = (\bar{\psi}_1, \ldots, \bar{\psi}_n) \) and \( \psi^0 = (\psi_1^0, \ldots, \psi_n^0) \). We first observe that \( \psi_i^0(0) > 0 \). Suppose \( \bar{\psi} \neq \psi^0 \), we can assume that \( \psi_i(0) \neq 0 \) and \( |\bar{\psi}_i(0)| < \psi_i^0(0) \) for all \( i \in I \).

From the definition of \( S \) we immediately see that \( 0 \in S \). Moreover, since \( |\bar{\psi}(r)| \leq Cr^{-2} \) near infinity, we can choose \( |t| < \delta \) with \( \delta \) small so that all \( |t| < \delta \) belong to \( S \). Another immediate observation is that \( S \) has a lower bound. Indeed, for \( T \) sufficiently negative, \( \psi_i'(0) < 0 \), which is impossible to have \( \int_0^T e^u \psi_i^T(\sigma) \sigma^3 d\sigma > 0 \) for \( r \) small.

Let \( T = \inf S \) and let \( t_m \to T^+ \). The sequence \( \psi_i^m \) obviously converges to a function \( \psi_i \), which is just \( \psi_i^0 + T(\psi_i^0 - \bar{\psi}_i) \). \( \psi = (\psi_1, \ldots, \psi_n) \) satisfies the property 
\[ \int_0^r e^u(\sigma) \psi_i(\sigma) \sigma^3 d\sigma \geq 0, \quad \forall r > 0, \quad i = 1, \ldots, n. \]
On the other hand, from the behavior of \( \psi \) and \( \psi^0 \) at infinity (both are \( O(r^{-2}) \)) we immediately observe that 
\[ \int_0^{\infty} e^u(r) \psi_i(r) r^3 dr < \infty. \]
In regard to (2.4) we have 
\[ r^3 \psi_i'(r) = -\int_0^r \sum_j a_{ij} e^u \psi_j \sigma^3 d\sigma, \quad 0 < r < \infty. \]
From (2.7) and (2.8) we see that \( \psi_i \) is non-increasing. Since we have known that \( |\psi_i(r)| \leq Cr^{-2} \) near infinity we have 
\[ \psi_i(r) \geq 0, \quad \forall r > 0, \quad i \in I. \]
It is not possible to have all \( \psi_i(0) = 0 \) because this implies \( \bar{\psi} \equiv 0 \), a contradiction to the assumption that \( \bar{\psi} \) is not a multiple of \( \psi^0 \). Therefore without loss of generality we assume \( \psi_i(0) > 0 \). Then we further claim that \( \psi_i \) is strictly decreasing for all \( i \in I \). Indeed, let \( I_1 = \{ j \in I | a_{ij} > 0 \} \), for each \( j \in I_1 \) we use (2.9) and \( \psi_i(0) > 0 \) to obtain 
\[ r^3 \psi_j'(r) \leq -\int_0^r a_{ij} e^u \psi_i \sigma^3 d\sigma < 0, \quad 0 < r < \infty. \]
Therefore for each \( j \in I_1 \), \( \psi_j \) is strictly decreasing, which immediately implies that \( \psi_j(0) > 0 \). We can further define \( I_2 = \{ i \in I | a_{ij} > 0 \text{ for some } j \in I_1 \} \). Then the same argument shows that \( \psi_i \) is strictly decreasing for each \( i \in I_2 \) as well. Since the matrix \( A = (a_{ij})_{n \times n} \) is irreducible, this process exhausts all \( i \in I \).

(2.8) yields \( \psi_i'(r) \leq -Cr^{-3} \) for \( r > 1 \) and \( i \in I \). Then by using \( \lim_{r \to \infty} \psi_i(r) = 0 \) we further have 
\[ \psi_i(r) \geq Cr^{-2}, \quad r \geq 1. \]
Then it is easy to see that for $t = T - \varepsilon$ with $\varepsilon > 0$ small, we also have
\[
\int_0^\tau e^{ui} \psi_i(\sigma) \sigma^3 d\sigma > 0, \quad \text{for all } \tau > 0,
\]
a contradiction to the definition of $T$. Lemma 2.1 is proved. □

**Proof of Theorem 2.1** We first prove the third statement. The following function plays an important role: Let $f = (f_1, \ldots, f_n) = (u'_1, \ldots, u'_n)$. Then
\[
(2.11) \quad -\Delta f_i = \sum_j a_{ij} \phi_k \phi_j - \frac{k^2}{r^2} \phi_i, \quad i = 1, \ldots, n.
\]
Let $\phi^k = (\phi^k_1, \ldots, \phi^k_n)$ be defined as
\[
\phi^k_i(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi_i(r \cos \theta, r \sin \theta) \cos kr d\theta, \quad i \in I.
\]
Then $\phi^k$ satisfies
\[
(2.12) \quad -\Delta \phi^k_i = \sum_j a_{ij} \phi^k_j - \frac{k^2}{r^2} \phi^k_i, \quad i \in I, \quad k = 2, \ldots
\]
Clearly $\phi^k_i(r) = o(r)$ near 0 and $\phi^k_i(r) = O(r^{2-\delta_0})$ at $\infty$. We claim that
\[
(2.13) \quad \phi^k_i(r) \equiv 0, \quad \forall k \geq 2, \quad \text{provided that } \phi^k_i(r) = O(r^{k-1+\tau}), \ r > 1, \ k \geq 2.
\]
Note that the growth condition in (2.13) is weaker than what is assumed in the assumption in Theorem 2.1.

The argument below also applies if $\phi$ is projected on $\sin k\theta$. First we show that $\phi^k_i = o(r^{-1})$ as $r \to \infty$. Indeed, using $\phi^k_j(x) \leq C|x|^{k-1+\tau}$ we write $\sum_j a_{ij} e^{\mu_j} \phi^k_j$ as $O(r^{k-1+\tau-2\delta_0})$ (for some $\delta_0 > 0$). Let $g(t) = \phi^k_i(\cdot \tau)$, then from (2.12) $g(t)$ satisfies
\[
g''(t) - k^2 g(t) = h(t), \quad t \in \mathbb{R}
\]
where
\[
(2.14) \quad h(t) = O(e^{(k-1+\tau-\delta_0)t}), \quad t > 0.
\]
Let $g_1(t) = e^{kt}$ and $g_2(t) = e^{-kt}$ be two fundamental solutions of the homogeneous equation, a general solution $g(t)$ is of the form
\[
g(t) = c_1 g_1(t) + c_2 g_2(t) - \frac{g_1(t)}{2k} \int_0^t g_2(s) h(s) ds + \frac{g_2(t)}{2k} \int_0^t g_1(s) h(s) ds
\]
where $c_1, c_2$ are constants. Using (2.14) in the above we obtain
\[
g(t) = c'_1 g_1(t) + c'_2 g_2(t) + O(e^{(k-1+\tau-\delta_0)t}), \quad \text{for } t > 1
\]
where $c'_1, c'_2$ are two constants. Since $g(t) = O(e^{(k-1+\tau)t})$ for $t \to \infty$, we see that $c'_1 = 0$ and therefore $g(t) = O(e^{(k-1+\tau-\delta_0)t})$ as $t \to \infty$. Equivalently
\[
(2.15) \quad \phi^k_i(r) = O(r^{k-1+\tau-\delta_0}), \quad i \in I.
\]
With (2.15) we further obtain
\[
\sum_j a_{ij} e^{\mu_j} \phi^k_j = O(r^{k-1+\tau-2\delta_0}).
\]
Consequently $\phi_k^i(x) = O(r^{k-1+\frac{\tau}{2} - 2\delta})$. Keep doing this for finite steps we obtain that $\phi_k^i$ decays faster than $r^{-1}$ at infinity. The asymptotic theory of ODE can be similarly used to show that $\phi_k^i(r) = o(r)$ as $r \to 0$.

To get a contradiction, without loss of generality, we may assume that some of $\phi_k^i$, say $\phi_k^i(r) > 0$ for some $r > 0$ and

$$
\max_{\mathbb{R}^+} \left( \frac{\phi_k^i(r)}{f_1(r)} \right) = \max_{1 \leq j \leq n} \left( \frac{\phi_k^j(r)}{f_j(r)} \right).
$$

By noting $\phi_k^i(r) = o(r)$ as $r \to 0$ and $\phi_k^i(r) = o(\frac{1}{r})$ as $r \to \infty$, $\phi_k^i / f_1(r)$ attains its maximum at some point $r_0 \in \mathbb{R}^+$. Let $w_1(r) = \phi_k^i(r) / f_1(r)$. By a direct computation, $w_1(r)$ satisfies

$$
\Delta w_1 + 2\nabla w_1 \cdot \frac{\nabla f_1}{f_1} + 1 - \frac{k^2}{r^2} w_1 = \sum_{j=2}^n a_{1j} e^{u_j} \left( \frac{w_1 f_j - \phi_j^i}{f_1} \right).
$$

Now we apply the maximum principle at $r = r_0$, and obtain

$$
\Delta w_1(r_0) \leq 0, \quad \text{and} \quad \nabla w_1(r_0) = 0.
$$

Since $k > 1$, (2.16) yields

$$
\sum_{j=2}^n a_{1j} e^{u_j} \left( \frac{w_1 f_j - \phi_j^i}{f_1} \right)(r_0) < 0
$$

because $w_1(r_0) > 0$. On the other hand, for $j \geq 2$,

$$
w_1(r_0) f_j(r_0) - \phi_j^i(r_0) = f_j(r_0) \frac{\phi_k^i(r_0)}{f_1(r_0)} - \frac{\phi_k^j(r_0)}{f_j(r_0)} \geq 0,
$$

which obviously contradicts (2.17). Therefore (2.13) is established. When $k = 1$, $\phi_1^i \equiv 0$ because by Lemma 2.1 $\phi_1^i(r) = Cu_1^i(r)$. By the assumption $\phi_k^i(x) = O(|x|^2)$ near $0$, $C = 0$. The third statement of Theorem 2.1 is established.

Again by Lemma 2.1, the first statement of Theorem 2.1 is established.

Finally, the second statement of Theorem 2.1 is an immediate consequence of Lemma 3.1 of [37]. Theorem 2.1 is established. \qed

2.2. A Pohozaev identity for global solutions.

Lemma 2.2. Let $u = (u_1, \ldots, u_n)$ be an entire, radial solution of

$$
\begin{cases}
-\Delta u_i = \sum_{j=1}^n a_{ij} e^{u_j}, & \text{in } \mathbb{R}^2, \\
f_{\mathbb{R}^2} e^{u_i} < \infty
\end{cases}
$$

where $A$ is a constant matrix that satisfies (H1). Let

$$
c_i = u_i(0) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |\eta| \left( \sum_{j=1}^n a_{ij} e^{u_j(\eta)} \right) d\eta,
$$

$$
\sigma_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{u_i}, \quad l_i = \sum_{j=1}^n a_{ij} \sigma_j, \quad l = \min\{l_1, \ldots, l_n\}
$$

...
and
\[ \sigma_{IR} = \frac{1}{2\pi} \int_{B_r} e^{u_i}. \]

Then for some \( \delta_0 > 0 \)
\begin{equation}
(2.18) \quad e^{u_i(r)} = e^{c_i} r^{-l_i} (1 + o(r^{-\delta_0})), \quad r > 1,
\end{equation}

\begin{equation}
(2.19) \quad 4 \sum_{i \in I} \sigma_{IR} = \sum_{i,j \in I} a_{ij} \sigma_{IR} \sigma_{JR} + 2 \sum_{i \in I} e^{c_i} R^{2-l_i} + O(R^{2-l_i-\delta_0}).
\end{equation}

**Proof of Lemma 2.2:**

It is well known that
\begin{equation}
(2.20) \quad u_i(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - \eta| \left( \sum_{j=1}^n a_{ij} e^{u_j(\eta)} \right) d\eta + c_i.
\end{equation}

Indeed, let \( w_i \) be the function defined by the right hand side of (2.20). Then \( w_i - u_i \) is a harmonic function. Since they both have logarithmic growth at infinity, \( w_i - u_i = c_i \). Evaluating both functions at 0 we have \( c_i = c_i \).

Clearly
\begin{equation}
(2.21) \quad u_i(x) + l_i \log |x| = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log \left( \frac{|x - \eta|}{|x|} \right) \sum_j a_{ij} e^{u_j(\eta)} d\eta + c_i.
\end{equation}

Using \( \sum_j a_{ij} e^{u_j(\eta)} = O(r^{-2-\delta_0}) \) for some \( \delta_0 > 0 \) and \( r \) large, we obtain, by elementary estimates,

\[ u_i(r) = -l_i \log r + c_i + o(r^{-\delta_0}), \]

which leads to
\begin{equation}
(2.22) \quad \sigma_i = \sigma_{IR} + \frac{e^{c_i}}{l_i - 2} R^{2-l_i} + O(R^{2-l_i-\delta_0}).
\end{equation}

We arrive at (2.19) by using (2.22) in the Pohozaev identity for \( \sigma \):
\[ 4 \sum_i \sigma_i = \sum_{i,j} a_{ij} \sigma_i \sigma_j. \]

Lemma 2.2 is established. \( \Box \).

3. First order estimates

Let \((h_1^k, \ldots, h_n^k)\) be a family of positive, \( C^3 \) functions on \( B_1 \) with a uniform bound on their positivity and \( C^3 \) norm:
\begin{equation}
(3.1) \quad \frac{1}{C} \leq h_i^k(x) \leq C, \quad \|h_i^k\|_{C^3(B_1)} \leq C, \quad x \in B_1, \quad i = 1, \ldots, n.
\end{equation}

In the next three sections we consider a sequence of locally defined, fully blown-up solutions \( u^k = (u_1^k, \ldots, u_n^k) \) and we shall derive their precise asymptotic behavior near their singularity and the precise location of their singularity. Here we abuse the notation \( u^k = (u_1^k, \ldots, u_n^k) \) and it is independent of the one used in the introduction.
Specifically we assume that $u^k$ satisfies the following equation in $B_1$, the unit ball:
\begin{equation}
- \Delta u^k_i = \sum_{j=1}^{n} a_{ij} h^k_j e^{u^k_i}, \quad i = 1, \ldots, n, \quad x \in B_1,
\end{equation}
with 0 being the only blowup point in $B_1$:
\begin{equation}
\max_k u^k_i = C(K), \quad \forall K \subset \bar{B}_1 \setminus \{0\}, \quad \text{and } \max_{\bar{B}_1} u^k_i \to \infty,
\end{equation}
with bounded oscillation on $\partial B_1$:
\begin{equation}
|u^k_i(x) - u^k_i(y)| \leq C, \forall x, y \in \partial B_1, \quad C \text{ independent of } k
\end{equation}
and uniformly bounded energy:
\begin{equation}
\int_{B_1} e^{u^k_i} \leq C, \quad C \text{ is independent of } k.
\end{equation}

Finally we assume that $u^k$ is a fully blown-up sequence, which means when re-scaled according its maximum, \{u_k\} converges to a system of $n$ equations: Let $u^k_i(0) = \max_{B_1} u^k_i$ and $\varepsilon_k = e^{-\frac{1}{4} u^k_i(0)}$, and
\begin{equation}
v^k_i(y) = u^k_i(\varepsilon_k y) - u^k_i(0), \quad y \in \Omega_k := B(0, \varepsilon_k^{-1}).
\end{equation}
Then $v^k = (v^k_1, \ldots, v^k_n)$ converges in $C^2_{\text{loc}}(\mathbb{R}^2)$ to $v = (v_1, \ldots, v_n)$, which satisfies
\begin{equation}
\begin{cases}
-\Delta v_i = \sum_{j=1}^{n} a_{ij} \tilde{h}_j(0)e^{v_j}, \quad \mathbb{R}^2, \quad i = 1, \ldots, n \\
\int_{\mathbb{R}^2} e^{v_i} < \infty, \quad i = 1, \ldots, n, \quad v_1(0) = 0,
\end{cases}
\end{equation}
where $\tilde{h}_j(0) = \lim_{k \to \infty} h^k_j(0)$.

For the rest of the paper we set
\begin{equation}
m_i := \frac{1}{2\pi} \int_{\mathbb{R}^2} \sum_{j=1}^{n} a_{ij} \tilde{h}_j(0)e^{v_j} > 2, \quad i \in I
\end{equation}
and $m = \min\{m_1, \ldots, m_n\}$. In this section we derive a first order estimate for $v^k$ in $\Omega_k$. In \cite{37} the authors prove that there is a sequence of global radial solutions $U^k = (U^k_1, \ldots, U^k_n)$ of (3.6) such that
\begin{equation}
|u^k_i(\varepsilon_k y) - U^k_i(y)| \leq C, \quad \text{for } |y| \leq r_0 \varepsilon_k^{-1}.
\end{equation}
From (3.7) we have the following spherical Harnack inequality:
\begin{equation}
|u^k_i(\varepsilon_k y) - U^k_i(\varepsilon_k y')| \leq C
\end{equation}
for all $|y| = |y'| = r \leq r_0 \varepsilon_k^{-1}$ and $C$ is a constant independent of $k, r$. (3.8) will play an essential role in the first order estimate. To improve (3.7) we introduce $\phi^k_i$ to be a harmonic function:
\begin{equation}
\begin{cases}
-\Delta \phi^k_i = 0, \quad B_1, \\
\phi^k_i = u^k_i - \frac{1}{2\pi} \int_{\partial B_1} u^k_i, \quad \text{on } \partial B_1,
\end{cases}
\end{equation}
Obviously \( \phi^k(0) = 0 \) by the mean value theorem and \( \phi^k \) is uniformly bounded on \( B_{1/2} \) because of (3.3). Later in section 6, when the results in section 3.4.5 will be used to prove the main theorems, the function \( \phi^k \) will be specified when we consider the system defined on Riemann surface.

Let \( V^k = (V^k_1, \ldots, V^k_n) \) be the radial solutions of

\[
\begin{aligned}
-\Delta V^k_i &= \sum_{j=1}^n a_{ij} h^k_j(0) e^{V^k_j}, \quad i \in I \\
V^k_i(0) &= v^k_i(0), \quad i \in I
\end{aligned}
\tag{3.10}
\]

where \( v^k_i \) is defined in (3.6). It is easy to see that any radial solution \( V \) of (3.10) exists for all \( r > 0 \) and \( e^{V^k} \in L^2(\mathbb{R}^2) \). The main result of this section is to prove that \( V^k(y) + \phi^k(\varepsilon_k y) \) is the first order approximation to \( v^k(y) \).

**Theorem 3.1.** Let \( A, u^k, h^k = (h^k_1, \ldots, h^k_n), \phi^k \) and \( v^k \) be described as above. Then for any \( \delta > 0 \), there exist \( k_0(\delta) > 1 \) and \( C \) independent of \( k \) and \( \delta \) such that for all \( k \geq k_0 \),

\[
|D^\alpha (v^k_i(y) - V^k_i(y) - \phi^k(\varepsilon_k y))| \leq \begin{cases} C \varepsilon_k (1 + |y|)^{3-\min\{m, k\}} & m \leq 3, \\
C \varepsilon_k (1 + |y|)^{m-k} & m > 3,
\end{cases} \quad |y| < \varepsilon_k^{-1}/2, \quad |\alpha| = 0, 1, 2.
\tag{3.11}
\]

**Definition 3.1.**

\[
\sigma^k_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} h^k_i(0) e^{V^k_i}, \quad m^k_i = \sum_{j=1}^n a_{ij} \sigma^k_j, \quad m^k = \min\{m^k_1, \ldots, m^k_n\}.
\]

From Theorem 3.1 it is easy to see that \( \lim_{k \to \infty} m^k = m_i \). Thus \( m^k_i \geq 2 + \delta_0 \) for some \( \delta_0 > 0 \) independent of \( k \).

To prove Theorem 3.1 we have

\[
-\Delta (v^k_i(y) - \phi^k(\varepsilon_k y)) = \sum_j a_{ij} H^k_j(\varepsilon_k y) e^{\phi^k_j(\varepsilon_k y)}, \quad \text{in } \Omega_k \ (\text{see (3.5)}
\tag{3.12}
\]

where

\[
H^k_i(\cdot) = h^k_i(\cdot) e^{\phi^k(\cdot)}.
\tag{3.13}
\]

Since \( \phi^k(0) = 0 \) we have \( H^k_i(0) = h^k_i(0) \). Also, the definition of \( \phi^k \) implies that \( v^k_i - \phi^k(\varepsilon_k y) \) is a constant on \( \partial \Omega_k \).

To estimate the error term \( w^k_i = v^k_i - \phi^k(\varepsilon_k y) - V^k_i \). We find \( w^k_i \) satisfies

\[
\begin{aligned}
\Delta w^k_i(y) + \sum_j a_{ij} H^k_j(\varepsilon_k y) e^{\xi^k_j} w^k_j &= -\sum_j a_{ij} (H^k_j(\varepsilon_k y) - H^k_j(0)) e^{V^k_j}, \\
w^k_i(0) &= 0, \quad i \in I, \quad \nabla w^k_i(0) = O(\varepsilon_k),
\end{aligned}
\tag{3.14}
\]

where \( \xi^k_i \) is defined by

\[
e^{\xi^k_i} = \int_0^1 e^{v^k_i(t)(1-t)\varepsilon_k} dt.
\tag{3.15}
Since both \( v^k \) and \( V^k \) converge to \( v \), \( w^k = o(1) \) over any compact subset of \( \mathbb{R}^2 \). The first estimate of \( w^k \) is the following

**Lemma 3.1.**

\[(3.16) \quad w^k(y) = o(1) \log(1 + |y|) + O(1), \quad \text{for } y \in \Omega_k. \]

**Proof:** By (3.8)

\[ |v^k(y) - v^k(|y|)| \leq C, \quad \forall y \in \Omega_k \]

where \( v^k(r) \) is the average of \( v^k \) on \( \partial B_r \):

\[ v^k(r) = \frac{1}{2\pi r} \int_{\partial B_r} v^k. \]

Thus we have \( e^{v^k(y)} = O(r^{-2-\delta}) \) and \( e^{v^k(y)} = O(r^{-2-\delta}) \) where \( r = |y| \) and \( \delta_0 > 0 \). Then

\[ r(\tilde{w}^k)'(r) = \frac{1}{2\pi} \left( \int_{\partial B_r} \sum a_{ij} H^k_j(\xi_k \cdot) e^{v^k} - \int_{\partial B_r} \sum a_{ij} H^k_j(0) e^{v^k} \right) \]

It is easy to use the decay rate of \( e^{v^k} \), \( e^{V^k} \) and the closeness between \( v^k \) and \( V^k \) to obtain

\[ r(\tilde{w}^k)'(r) = o(1), \quad r \geq 1. \]

Hence \( \tilde{w}^k(r) = o(1) \log r \) and (3.16) follows from this easily. Lemma 3.1 is established. \( \square \)

The following estimate is immediately implied by Lemma 3.1

\[ e^{\tilde{w}^k(y)} \leq C(1 + |y|)^{-m + o(1)} \quad \text{for } y \in \Omega_k = B(0, \varepsilon_k^{-1}). \]

Before we derive further estimate for \( w^k \) we establish a useful estimate for the Green’s function on \( \Omega_k \) with respect to the Dirichlet boundary condition:

**Lemma 3.2.** Let \( G(y, \eta) \) be the Green’s function with respect to Dirichlet boundary condition on \( \Omega_k \). For \( y \in \Omega_k \), let

\[
\begin{align*}
\Sigma_1 &= \{ \eta \in \Omega_k; \quad |\eta| < |y|/2 \} \\
\Sigma_2 &= \{ \eta \in \Omega_k; \quad |y - \eta| < |y|/2 \} \\
\Sigma_3 &= \Omega_k \setminus (\Sigma_1 \cup \Sigma_2).
\end{align*}
\]

Then in addition for \( |y| > 2 \),

\[(3.17) \quad |G(y, \eta) - G(0, \eta)| \leq \begin{cases} 
C(|\log |y| + |\log |\eta||), & \eta \in \Sigma_1, \\
C(|\log |y| + |\log |y - \eta||), & \eta \in \Sigma_2, \\
C|y|/|\eta|, & \eta \in \Sigma_3,
\end{cases} \]

**Proof:** The expression for \( G(y, \eta) \) is

\[ G(y, \eta) = -\frac{1}{2\pi} \log |y - \eta| + \frac{1}{2\pi} \log \left( \frac{|y|}{|\eta - \eta_k|} \left| \frac{\varepsilon_k^{-2} y}{|y|^2} - \eta \right| \right), \quad y, \eta \in \Omega_k. \]

In particular

\[ G(0, \eta) = -\frac{1}{2\pi} \log |\eta| + \frac{1}{2\pi} \log \varepsilon_k^{-1}, \quad \eta \in \Omega_k. \]
Therefore we write $G(y, \eta) - G(0, \eta)$ as

$$
G(y, \eta) - G(0, \eta) = \frac{1}{2\pi} \log \frac{|\eta|}{|y - \eta|} + \frac{1}{2\pi} \log \frac{|y - \eta|}{|y|}.
$$

The proof of (3.17) for $\eta \in \Sigma_1$ is obvious. For $\eta \in \Sigma_2$, (3.17) also obviously holds if either $|y|$ or $|\eta|$ is less than $\frac{7}{8} \epsilon_k^{-1}$ because in this case

$$
\log \left| \log \frac{|y - \eta|}{|y|} \right| \leq C.
$$

Consequently

$$
|G(y, \eta) - G(0, \eta)| \leq C(\log |\eta| + |\log |y - \eta|| + C)
$$

$$
\leq C(\log |\eta| + |\log |y - \eta||).
$$

Therefore for $\eta \in \Sigma_2$ we only need to consider the case when $|y|, |\eta| > \frac{7}{8} \epsilon_k^{-1}$. In this case it is immediate to observe that

$$
\log \left| \log \frac{|y - \eta|}{|y|} \right| < C, \quad \text{if } \angle \left( \frac{y}{|y|}, \frac{\eta}{|\eta|} \right) > \frac{\pi}{8}
$$

where $\angle (\cdot, \cdot)$ is the angle between two unit vectors. Thus for $\eta \in \Sigma_2$ we only consider the situation when $|y|, |\eta| > \frac{7}{8} \epsilon_k^{-1}, \angle (\frac{y}{|y|}, \frac{\eta}{|\eta|}) < \frac{\pi}{8}$. For this case we estimate $G(y, \eta) - G(0, \eta)$ as follows:

$$
|G(y, \eta) - G(0, \eta)| \leq |G(y, \eta)| + |G(0, \eta)|
$$

$$
|G(0, \eta)| \leq C \log |y|.
$$

$$
|G(y, \eta)| \leq \frac{1}{2\pi} |\log |y - \eta|| + \frac{1}{2\pi} \log \frac{8}{7} + \frac{1}{2\pi} |\log \frac{\epsilon_k^{-2} y}{|y|^2} - \eta||
$$

$$
\leq C(\log |y| + |\log |y - \eta||)
$$

where the last inequality holds because

$$
|y - \eta| < \frac{\epsilon_k^{-2} y}{|y|^2} - \eta < C|y|
$$

which implies

$$
|\log \frac{\epsilon_k^{-2} y}{|y|^2} - \eta| \leq C(\log |y - \eta|| + |y|).
$$

The second case of (3.17) (when $\eta \in \Sigma_2$) is proved.

For $\eta \in \Sigma_3$, we first consider when $|\eta| > 2|y|$. In this case

$$
|\log \frac{|\eta|}{|\eta - y|} = |\log \frac{|\eta|}{|\eta|} - \frac{y}{|\eta||y|}| \leq C \frac{|y|}{|\eta|}.
$$

For the second term, since $\eta, y \in \Omega_k$ and $|\eta| > 2|y|$, we have $|\log |\eta|| < \frac{1}{2} \epsilon_k^{-2}$, consequently

$$
|\log \frac{y}{|y|} - \frac{|\eta||\eta|}{\epsilon_k^{-2}}| \leq C|\eta||\eta| \epsilon_k^{-2} \leq C \frac{|y|}{|\eta|}.
$$
So (3.17) is proved in this case. Now we consider $\frac{|y|}{2} \leq |\eta| \leq 2|y|$ and $|\eta - y| \geq \frac{|y|}{2}$. For the first term we have

$$\left| \log \frac{|\eta|}{|y - \eta|} \right| \leq \log 4 \leq C \frac{|y|}{|\eta|}.$$ 

For the second term, we want to show

$$\left| \log \left( \frac{y}{|y|} - \frac{|y| |\eta|}{\varepsilon_k^2} \right) \right| \leq C \frac{|y|}{|\eta|},$$

If either $|y| \leq \frac{15}{16} \varepsilon_k^{-1}$ or $|\eta| \leq \frac{15}{16} \varepsilon_k^{-1}$ we have

$$\left| \frac{y}{|y|} - \frac{|y| |\eta|}{\varepsilon_k^2} \right| \geq 16,$$

therefore (3.19) obviously holds. For $\frac{15}{16} \varepsilon_k^{-1} < |y|, |\eta| \leq \varepsilon_k^{-1}$, using $|y - \eta| > \frac{1}{2} |y|$ we obtain easily

$$\left| \frac{y}{|y|} - \frac{|y| |\eta|}{\varepsilon_k^2} \right| \geq \frac{3}{8}.$$ 

Therefore (3.17) is proved in all cases. Lemma 3.2 is established. \(\Box\)

**Proof of Theorem 3.1** First we prove (3.11) for $\alpha = 0$. We consider the case $m \leq 3$, the proof for the case $m > 3$ is similar. By way of contradiction, we assume

$$\Lambda_k := \max_{y \in \Omega_k} \frac{\max_{i \in I} |w_i^k(y)|}{\varepsilon_k (1 + |y|)^{3+\delta-m}} \to \infty.$$ 

Suppose $\Lambda_k$ is attained at $y_k \in \overline{\Omega}_k$ for some $i_0 \in I$. We thus define

$$\bar{w}_i^k(y) = \frac{w_i^k(y)}{\Lambda_k \varepsilon_k (1 + |y_k|)^{3+\delta-m}}.$$ 

Here we require $\delta$ to be small so that $m - 2 - \delta > 0$ (Thus $3 - m + \delta < 1$). It follows from the definition of $\Lambda_k$ that for $y \in \Omega_k$

$$|\bar{w}_i^k(y)| = \frac{|w_i^k(y)|}{\Lambda_k \varepsilon_k (1 + |y|)^{3+\delta-m}} \leq \frac{(1 + |y|)^{3+\delta-m}}{(1 + |y_k|)^{3+\delta-m}}.$$ 

The equation for $\bar{w}_i^k$ is

$$(3.21) \quad -\Delta \bar{w}_i^k(y) = \sum_j a_{ij} h^k_j(0) e^{i_j y} \bar{w}_j^k + o(1) \frac{(1 + |y|)^{1-m}}{(1 + |y_k|)^{3+\delta-m}}, \quad y \in \Omega_k$$

for $i \in I$. Here $\xi_i^k$ is given by (3.15). $\xi_i^k$ converges to $v_i$ in $C^2_{loc}(\mathbb{R}^2)$. Besides, we also have $\bar{w}_i^k(0) = 0$ for all $i$ and $\nabla \bar{w}_i^k(0) = o(1)$. If a subsequence of $y_k$ stays bounded, then along a subsequence $\bar{w}^k = (\bar{w}_1^k, \ldots, \bar{w}_n^k)$ converges to $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_n)$ that satisfies

$$\left\{ \begin{array}{ll}
-\Delta \bar{w}_i = \sum_j a_{ij} h(j)(0) e^{i y} \bar{w}_j, & y \in \mathbb{R}^2, \quad i \in I, \\
\bar{w}_i(0) = 0, \quad \nabla \bar{w}_1(0) = 0, & |\bar{w}_i(y)| \leq C(1 + |y|)^{3+\delta-m}, \quad y \in \mathbb{R}^2.
\end{array} \right.$$
Thanks to (1) of Theorem 2.1

\[ \tilde{w}_i(x) = c_1 \frac{\partial v_i}{\partial x_1} + c_2 \frac{\partial v_i}{\partial x_2}. \]

Since \( \nabla \tilde{w}_1(0) = 0 \), we have \( c_1 = c_2 = 0 \), thus \( \tilde{w}_i \equiv 0 \) for all \( i \). On the other hand, the fact that \( \tilde{w}_k(y_k) = \pm 1 \) for some \( i_0 \in I \) implies that \( \tilde{w}_k(\tilde{y}) = \pm 1 \) where \( \tilde{y} \) is the limit of \( y_k \). This contradiction means that \( y_k \to \infty \). Next we shall show a contradiction if \( |y_k| \to \infty \). By the Green's representation formula for \( \tilde{w}_i^k \),

\[ \tilde{w}_i^k(y) = \int_{\Omega_k} G(y, \eta)(-\Delta \tilde{w}_i^k(\eta))d\eta + \tilde{w}_i^k|_{\partial \Omega_k} \]

where \( \tilde{w}_i^k|_{\partial \Omega_k} \) is the boundary value of \( \tilde{w}_i \) on \( \partial \Omega_k \) (which is a constant). From (3.20) and (3.21) we have

\[ \left| -\Delta \tilde{w}_i^k(\eta) \right| \leq \frac{C(1 + |\eta|)^{3+\delta-2m}}{(1 + |y_k|)^{3+\delta-m}} + \frac{C(1 + |\eta|)^{1-m+\delta}}{\Lambda_k(1 + |y_k|)^{3+\delta-m}}. \]

Thus for some \( i \in I \) we have

(3.22) \[ 1 = |\tilde{w}_i^k(y_k) - \tilde{w}_i^k(0)| \]

\[ \leq C \int_{\Omega_k} |G(y_k, \eta) - G(0, \eta)| \left( \frac{(1 + |\eta|)^{3+\delta-2m}}{(1 + |y_k|)^{3+\delta-m}} + \frac{(1 + |\eta|)^{1-m+\delta}}{\Lambda_k(1 + |y_k|)^{3+\delta-m}} \right) d\eta \]

where the constant on the boundary is canceled out. To compute the right hand side of the above, we decompose the \( \Omega_k \) as \( \Omega_k = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \) as in Lemma 3.2 Using (3.17) we have

\[ \int_{\Sigma_1 \cup \Sigma_2} |G(y_k, \eta) - G(0, \eta)|(1 + |\eta|)^{3+\delta-2m} d\eta = O(1)(\log |y_k|)(1 + |y_k|)^{(5+\delta-2m)}. \]

where

\[ (1 + |y_k|)^{\alpha+} = \begin{cases} (1 + |y_k|)^{\alpha}, & \alpha > 0, \\ \log(1 + |y_k|), & \alpha = 0, \\ 1, & \alpha < 0. \end{cases} \]

\[ \int_{\Sigma_3} |G(y_k, \eta) - G(0, \eta)|(1 + |\eta|)^{3+\delta-2m} d\eta = O(1)(1 + |y_k|)^{5+\delta-2m}. \]

Hence

\[ \int_{\Omega_k} |G(y_k, \eta) - G(0, \eta)| \left( \frac{(1 + |\eta|)^{3+\delta-2m}}{(1 + |y_k|)^{3+\delta-m}} d\eta = O(1)(1 + |y_k|)^{2-m}. \right. \]

Similarly we can compute the other term:

\[ \int_{\Omega_k} |G(y_k, \eta) - G(0, \eta)| \frac{(1 + |\eta|)^{1-m+\delta}}{\Lambda_k(1 + |y_k|)^{3+\delta-m}} d\eta = O(1)\Lambda_k^{-1}(\log(1 + |y_k|))^{-\frac{\delta}{m}}. \]

By the computations above we see that the right hand side of (3.22) is \( o(1) \), a contradiction to the left hand side of (3.22). Thus (3.11) is established for \( \alpha = 0 \). The estimates for \( |\alpha| = 1 \) and 2 follow easily by scaling and standard elliptic estimates. Therefore Theorem 3.1 is completely proved. \( \square \)
4. Second Order Estimates

In this section we improve the estimates in Theorem 3.1 for $m < 4$ and $m = 4$, respectively. Let $p_{i,k}$ be the maximum point of $v^k_i(\cdot) - \phi^k_i(\varepsilon_k \cdot)$. The following lemma estimates the location of $p_{i,k}$.

Lemma 4.1. $p_{i,k} = O(\varepsilon_k)$, $i \in I$.

Proof: Applying Theorem 3.1 to $v^k_i - \phi^k_i(\varepsilon_i \cdot)$ on $B_1$:

(4.1) \[ D^\alpha (v^k_i(y) - \phi^k_i(\varepsilon_k y)) = D^\alpha (V^k_i(|y|)) + O(\varepsilon_k), \quad |y| < 1, \quad |\alpha| = 0, 1, 2. \]

The equation for $V^k_i$ is

\[
(V^k_i)^\prime\prime(r) + \frac{1}{r} (V^k_i)^\prime(r) + \sum_{j=1}^{n} a_{ij} H^k_j(0) e^{V^k_j(r)} = 0, \quad r > 0.
\]

From $(V^k_i)^\prime(0) = 0$, we see $\lim_{r \to 0} (V^k_i)^\prime(r)/r = (V^k_i)^\prime\prime(0)$. Thus

(4.2) \[ (V^k_i)^\prime\prime(0) = -\frac{1}{2} \sum_{j=1}^{n} a_{ij} H^k_j(0) e^{V^k_j(0)} < -C \]

for some $C > 0$ independent of $k$. Since $p_{i,k}$ is the maximum point of $v^k_i(\cdot) - \phi^k_i(\varepsilon_k \cdot)$, we deduce from (4.1) that $(V^k_i)^\prime(p_{i,k}) = O(\varepsilon_k)$, thus from (4.2) we have $p_{i,k} = O(\varepsilon_k)$. Lemma 4.1 is established. \( \square \)

The main result in this section is to find the $\varepsilon_k$ approximation to $v^k_i(\cdot) - \phi^k_i(\varepsilon_k \cdot)$. It is most convenient to write the expansion around one of the $p_{i,k}$s. We choose $p_{1,1}$ and shall use $\Phi^k = (\Phi^k_1, \ldots, \Phi^k_n)$ to denote the projection of $v^k_i(\cdot) - \phi^k_i(\varepsilon_k \cdot)$ onto $\text{span}\{\sin \theta, \cos \theta\}$, i.e.

(4.3) \[ \Phi^k_i(r \cos \theta, r \sin \theta) = \varepsilon_k (G^k_{1,i}(r) \cos \theta + G^k_{2,i}(r) \sin \theta), \quad i \in I \]

with $G^k_{i,1}(r)$ ($i = 1, 2$) satisfying some ordinary differential equations to be specified later.

Set $v^{1,k} = (v^{1,k}_1, \ldots, v^{1,k}_n)$ as

(4.4) \[ v^{1,k}_i(\cdot) = v^k_i(\cdot + p_{1,k}) - \phi^k_i(\varepsilon_k(\cdot + p_{1,k})) \]

in

(4.5) \[ \Omega_{1,k} := \{ \eta : \eta + p_{1,k} \in \Omega_k \}. \]

Using $\nabla v^k_i(0) = O(\varepsilon_k)$ (by Theorem 3.1) and $\phi^k_i(0) = 0$ we observe that

(4.6) \[
\begin{align*}
\nabla v^{1,k}_i(0) &= v^k_i(p_{1,k}) - \phi^k_i(\varepsilon_k p_{1,k}) \\
&= v^k_i(0) + \nabla v^k_i(0) \cdot p_{1,k} + O(|p_{1,k}|^2) + O(\varepsilon_k) \\
&= v^k_i(0) + O(\varepsilon_k^2)
\end{align*}
\]
The equation that $v^{1,k}$ satisfies is (combining (3.11) and (3.12))

\[
\begin{aligned}
\Delta v^{1,k}_i + \sum_{j=1}^n a_{ij} H^{1,k}_j(y) e^{v^{1,k}_j} &= 0, & \text{in } \Omega_{1,k} \\
\nabla v^{1,k}_i(0) &= 0, & \nabla v^{1,k}_i(0) &= O(\varepsilon_k), & i = 2, \ldots, n
\end{aligned}
\]

(4.7)

where $H^{1,k} = (H^{1,k}_1, \ldots, H^{1,k}_n)$ is defined by (see (3.13)).

\[
H^{1,k}_i(\cdot) = H^k_i(\varepsilon_k \cdot + \varepsilon_k p_{1,k}) = h^k_i(\varepsilon_k \cdot + \varepsilon_k p_{1,k}) e^{\theta^i(\varepsilon_k + \varepsilon_k p_{1,k})}.
\]

Trivially

\[
H^{1,k}_i(0) = h^k_i(0) + O(\varepsilon_k^2).
\]

Step one:

Let $w^{1,k} = (w^{1,k}_1, \ldots, w^{1,k}_n)$ be the difference between $v^k$ and $V^k$:

\[
w^{1,k}_i(y) = v^{1,k}_i(y) - V^k_i(|y|), \quad y \in \Omega_{1,k}.
\]

Taking the difference between (4.7) and (3.10), we have

\[
\Delta w^{1,k}_i + \sum_j a_{ij} H^{1,k}_j(y) e^{V^k_j + w^{1,k}_j} - \sum_j a_{ij} h^k_j(0) e^{V^k_j} = 0,
\]

which is

\[
\Delta w^{1,k}_i + \sum_j a_{ij} h^k_j(0) e^{V^k_j} \left( \frac{e^{V^k_j}}{h^k_j(0)} - 1 \right) = 0.
\]

Here we observe that the oscillation of $V^k_i$ on $\partial \Omega_k$ is $O(\varepsilon_k^2)$. Indeed, recall that $\Omega_{1,k}$ is the shift of the large ball $\Omega_k$ by $p_{1,k}$. Let $y_1, y_2 \in \partial \Omega_{1,k}$, one can find $y_3$ such that $|y_3| = |y_2|$ and $|y_3 - y_1| \leq C \varepsilon_k$. Since $(V^k_i)'(r) \sim r^{-1}$ for $r > 1$ and $|y_1| \sim \varepsilon_k^{-1}$, we have

\[
V^k_i(y_1) - V^k_i(y_2) = V^k_i(y_1) - V^k_i(y_3) = O(\varepsilon_k^2).
\]

(4.10)

With (4.10) we further write the equation for $w^{1,k}_i$ as

\[
\begin{aligned}
\Delta w^{1,k}_i + \sum_j a_{ij} h^k_j(0) e^{V^k_j} w^{1,k}_j &= E^k_i, & \text{in } \Omega_{1,k} \\
\quad w^{1,k}_j(0) &= O(\varepsilon_k^2), & i \in I, & \quad w^{1,k}_j = O(\varepsilon_k^2) \text{ on } \partial \Omega_{1,k}, \\
\nabla w^{1,k}_i(0) &= 0, & \nabla w^{1,k}_i(0) &= O(\varepsilon_k), & i \in I \setminus \{1\}
\end{aligned}
\]

(4.11)
where
\[
E_i^k = -\sum_j a_{ij} h_j^k(0) e^{V_j^i} \left( \frac{H_j^{1,k}(y)}{h_j^k(0)} w_j^{1,k} - 1 - w_j^{1,k} \right).
\]

Similar to Theorem 3.1 we also have

**Lemma 4.2.** For any \( \delta > 0 \), there exists \( k_0(\delta) > 1 \) such that for some \( C > 0 \) independent of \( k \) and \( \delta \), the following estimate holds for all \( k \geq k_0 \):
\[
|w_i^{1,k}(y)| \leq \begin{cases} 
C\varepsilon_k (1 + |y|)^{3-m+\delta}, & m \leq 3, \\
C\varepsilon_k (1 + |y|)^{\delta}, & m > 3,
\end{cases} 
\]

where \( y \in \Omega_{1,k} \).

**Proof:** Using the definition of \( v_i^{1,k} \) and Theorem 3.1 we have
\[
|v_i^{1,k}(y) - V_i^k(y + p_{1,k})| \leq \begin{cases} 
C\varepsilon_k (1 + |y|)^{3-m+\delta}, & m \leq 3, \\
C\varepsilon_k (1 + |y|)^{\delta}, & m > 3.
\end{cases}
\]

On the other hand we clearly have
\[
|V_i^k(y) - V_i^k(y + p_{1,k})| \leq C\varepsilon_k (1 + |y|)^{-1}
\]
by mean value theorem and the estimate of \( \nabla V_i^k \). Lemma 4.2 is established. \( \square \)

Using Lemma 4.2 and (4.9) we now rewrite \( E_i^k \), clearly
\[
E_i^k = -\sum_j a_{ij} e^{V_j^i} (H_j^{1,k}(y) - h_j^k(0)) + (H_j^{1,k}(y) - h_j^k(0)) w_j^{1,k} + O((w_j^{1,k})^2).
\]

By Lemma 4.2 and (4.9), the last two terms are \( O(\varepsilon_k^2 (1 + |y|)^{2-m}) \) regardless whether \( m \geq 3 \) or not. Thus
\[
E_i^k = -\sum_j a_{ij} e^{V_j^i} (H_j^{1,k}(y) - h_j^k(0)) + O(\varepsilon_k^2 (1 + |y|)^{2-m})
\]
\[
= -\sum_j a_{ij} e^{V_j^i} (H_j^{1,k}(y) - H_j^{1,k}(0)) + O(\varepsilon_k^2 (1 + |y|)^{2-3m})
\]
where in the last step we used (4.9) again.

**Step Two: Estimate of the radial part of \( w_i^{1,k} \):**

Let \( g_i^{k,0} = (g_1^{k,0}, \ldots, g_{n_i}^{k,0}) \) be the radial part of \( w_i^{1,k} \):
\[
g_i^{k,0}(r) = \frac{1}{2\pi} \int_0^{2\pi} w_i^{1,k}(r \cos \theta, r \sin \theta) d\theta.
\]

Due to the radial symmetry of \( V_i^k \), \( g_i^{k,0} \) satisfies
\[
\begin{cases} 
L_i g_i^{k,0} = -\frac{\varepsilon_k^2}{4} \sum_j a_{ij} \Delta H_j^{1,k}(0) r^2 e^{V_j^i} + O(\varepsilon_k^2 (1 + r)^{\delta-m}) \\
g_i^{k,0}(0) = O(\varepsilon_k^2), & i \in I, \quad \frac{d}{dr} g_i^{k,0}(0) = 0.
\end{cases}
\]

where (for simplicity we omit \( k \) in \( L_i \))
\[
L_i g_i^{k,0} = \frac{d^2}{dr^2} g_i^{k,0} + \frac{1}{r} \frac{d}{dr} g_i^{k,0} + \sum_j a_{ij} h_j^k(0) e^{V_j^i} g_j^{k,0}.
\]
We claim that for $\delta > 0$, there exists $k_0(\delta) > 1$ such that for all $k \geq k_0$

(4.16) \[ |g^{k,0}_i(r)| \leq C\varepsilon_k^2(1 + r)^{4 - m - \delta}, \quad 0 < r < \varepsilon_k^{-1} \]

holds for some $C$ independent of $k$ and $\delta$. So, $g^{k,0}_i(r)$ can be discarded as an error term.

To prove (4.16), we first observe that $m \leq 4$ and by (4.15)

\[ |L_i g^{k,0}| \leq C\varepsilon_k^2(1 + r)^{2 - m - \delta/2}. \]

Let $f^k = (f^k_1, \ldots, f^k_n)$ be the solution of

\[
\begin{cases}
\frac{d^2}{dr^2} f^k_i + \frac{1}{r} \frac{d}{dr} f^k_i = L_i g^{k,0}, \\
f^k_i(0) = \frac{d}{dr} f^k_i(0) = 0.
\end{cases}
\]

Then elementary estimate shows

(4.17) \[ |f^k_i(r)| \leq C\varepsilon_k^2(1 + r)^{4 - m - \delta}. \]

If $m > \frac{5}{2}$, we claim

(4.18) \[ |g^{k,0}_i(r) - f^k_i(r)| \leq C\varepsilon_k^2(1 + r)^\delta, \quad 0 < r < \varepsilon_k^{-1}. \]

Indeed, let $\tilde{g}^k = g^{k,0} - f^k$, then clearly

\[
\begin{cases}
L_i \tilde{g}^k = F^k_i, \\
\tilde{g}^k_i(0) = O(\varepsilon_k^2), \quad \frac{d}{dr} \tilde{g}^k_i(0) = 0,
\end{cases}
\]

where

\[ F^k_i := -\sum_j a_{ij} h^k_f(0) e^{V^k_j} f^k_j = O(\varepsilon_k^2)(1 + r)^{4 - 2m + \delta}. \]

By considering $\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}$ as $\Delta$ in $\mathbb{R}^2$ and $\tilde{g}^k$ a solution with boundary oscillation 0 in $B(0, \varepsilon_k^{-1})$, we obtain (4.18) by the argument for Theorem 3.1. Here we note that to apply Theorem 2.1, it is essential to require $6 - 2m + \delta < 1$ (it holds if $m > 5/2$), the estimate on the Green’s function in Lemma 3.2 and the condition $\frac{d}{dr} \tilde{g}^k_i(0) = 0$. Since $m \leq 4$, $O(\varepsilon_k^2)(1 + r)^\delta$ is part of the error.

If $m \leq \frac{5}{2}$ we apply the same ideas by adding more correction functions to $g^{k,0}$. Let $N$ satisfy $2 + (2 - m)N < 1$, we add $N$ correcting functions to make the right hand side of the equation of the order $O(\varepsilon_k^2(1 + r)^{(2 - m)N + \delta})$. Note that each correction can be discarded as an error in the sense that they are smaller than the right hand side of (4.47). Using $2 + 2N - mN < 1$ and the argument in the proof of Theorem 3.1 we obtain (4.10).

**Step Three: The projection on $\sin \theta$ and $\cos \theta$**

In this step we consider the projection of $w^{1,k}$ over $\cos \theta$ and $\sin \theta$, respectively:

\[ \varepsilon_k G^k_{1,l}(r) = \frac{1}{2\pi} \int_0^{2\pi} w^{1,k}_l(r, \theta) \cos \theta d\theta, \quad \varepsilon_k G^k_{2,l}(r) = \frac{1}{2\pi} \int_0^{2\pi} w^{1,k}_l(r, \theta) \sin \theta d\theta. \]

Let

(4.19) \[ \Phi^k_i = \varepsilon_k G^k_{1,i}(r) \cos \theta + \varepsilon_k G^k_{2,i}(r) \sin \theta \]
clearly \( G_{1,j}^k \) and \( G_{2,j}^k \) solve the following linear systems: For \( 0 < r < \varepsilon_k^{-1} \) and \( t = 1, 2 \)

\[
\begin{align*}
\left( \frac{d^2}{dx^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) G_{i,j}^k + \sum_j a_{ij} h_j^k(0) e^{V_j} G_{i,j}^k &= -\sum_j a_{ij} \partial_i H_j^{1,k}(0) r e^{V_j} + O(\varepsilon_k^2(1 + r)^{2-m}). \tag{4.24}
\end{align*}
\]

Then \( \Phi^k \) solves

\[
\begin{align*}
\Delta \Phi_i^k + \sum_j a_{ij} h_j^k(0) e^{V_j} \Phi_j^k &= -\varepsilon_k \sum_j a_{ij} (\partial_1 H_j^{1,k}(0) y_1 + \partial_2 H_j^{1,k}(0) y_2) e^{V_j} + O(\varepsilon_k^2(1 + |y|)^{2-m}). \tag{4.21}
\end{align*}
\]

By the long behavior of \( w^{1,k} \) (Lemma 4.2) we have

\[
\begin{align*}
|G_{1,i}^k(r)| + |G_{2,j}^k(r)| \leq \begin{cases} 
C(1 + r)^{3-m-\delta}, & m \leq 3, \\
C(1 + r)^{\delta}, & m > 3.
\end{cases} \tag{4.22}
\end{align*}
\]

Note that \( G_{1,i}^k(0) = 0 \) and \( G_{1,i}^k(r) = O(r) \) near 0.

**Step Four: Projection of \( w^{1,k} \) onto higher frequencies**

Let

\[
g^{k,l} = (g_1^{k,l}, \ldots, g_n^{k,l})
\]

be the projection of \( w^{1,k} \) on \( \sin l/\theta \). In this step we first establish a preliminary estimate for all these projections:

**Lemma 4.3.** There exist \( l_0 \geq 3 \) and \( C > 0 \) independent of \( k, l \) such that

\[
|g_i^{k,l}(r)| \leq C\varepsilon_k^2 r^2, \quad 0 < r < \varepsilon_k^{-1}, \quad \forall l \geq l_0.
\]

**Proof:**

By (4.11), (4.14) and Lemma 4.2 \( g^{k,l} \) satisfies

\[
\begin{cases}
\left( \frac{d^2}{dx^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) g_i^{k,l} + \sum_j a_{ij} h_j^k(0) e^{V_j} g_j^{k,l} = h_{ikl}, \\
g_i^{k,l}(0) = 0.
\end{cases} \tag{4.23}
\]

where, applying the Taylor expansion of \( H_j^{1,k} \) up to the second order,

\[
|h_{ikl}(r)| \leq C\varepsilon_k^2 (1 + r)^{2-m-\delta/2}
\]

for some \( C \) independent of \( k \) and \( l \). Thus

\[
\Delta g_i^{k,l} - \frac{l^2}{r^2} g_i^{k,l} + \sum_j a_{ij} h_j^k(0) e^{V_j} g_j^{k,l} > -c_0 \varepsilon_k^2 (1 + r)^{2-m-\delta}
\]

for some \( c_0 > 0 \). Let

\[
g(r) = \frac{r^2}{4} \int_r^{\infty} \frac{c_0}{s} (1 + s)^{2-m-\delta} ds + c_0 \frac{r^{-2}}{4} \int_0^r s^3 (1 + s)^{2-m-\delta} ds + r^2.
\]
Then clearly \( g(r) > 0 \) for \( r > 0 \), \( g \) solves
\[
\begin{align*}
g'' + \frac{4}{r} g' - \frac{4}{r^2} g(r) &= -c_0 r^2(1 + r)^{-m - \delta}, \quad r > 0, \\
g(0) &= g'(0) = 0.
\end{align*}
\] (4.26)

and
\[ g(r) = r^2 + O(r^{4 - m + \delta}), \quad r > 1, \quad g(r) \leq C r^2 \log(\frac{1}{r} + 1), \quad r \leq 1. \]

Clearly by Lemma 4.2
\[ \varepsilon_k^2 g(\varepsilon_k^{-1}) > \max_{y \in \partial \Omega_k} |w_i^{1,k}(y)| \geq |g^{k,l}(\varepsilon_k^{-1})|. \] (4.27)

The reason that we include \( r^2 \) in the definition of \( g(\cdot) \) is because by Lemma 4.2 we only know \( w_i^{1,k}(x) = O(\varepsilon_k(1 + |x|)^\delta) \) for \( m > 3 \). Let \( g^k = (g^{1,k}, \ldots, g^{n,k}) = \varepsilon_k^2 (g, \ldots, g) \), then it is easy to see that for \( l_0 \) sufficiently large and \( l > l_0 \)
\[ \begin{align*}
\Delta g_i^k &= \frac{l^2}{r^2} g_i^k + \sum_j a_{ij} h_j^k(0) e^{V_i^k} g_j^k \\
&= -\varepsilon_k^2 c_0 (1 + r)^{2 - m + \delta} + \sum_j a_{ij} h_j^k(0) e^{V_i^k} g_j^k - \frac{l^2}{r^2} g_i^k \\
&\leq -\varepsilon_k^2 c_0 (1 + r)^{2 - m + \delta}.
\end{align*} \] (4.28)

To prove Lemma 4.3 it is enough to show
\[ |g_i^{k,l}(r)| \leq C g_i^k, \quad 0 < r < \varepsilon_k^{-1}, \quad l \geq l_0, \]
with \( C \) independent of \( k \) and \( l \). To this end, we shall use (4.25) and the initial value
\[ g_i^{k,l}(r) = o(r) \] near 0.

In the following we use the argument in the proof of Theorem 2.1 Let \( \psi_i^k = g_i^k - g_i^{k,l} \), by (4.27), (4.28), (4.25) and (4.30)
\[
\begin{align*}
\Delta \psi_i^k &= \frac{l^2}{r^2} \psi_i^k + \sum_j a_{ij} h_j^k(0) e^{V_i^k} \psi_j^k < 0, \quad 0 < r < \varepsilon_k^{-1}, \\
\psi_i^k(r) &= o(r) \text{ near 0}, \quad \psi_i^k(\varepsilon_k^{-1}) > 0 \text{ for all } i.
\end{align*}
\]

Our goal is to prove \( \psi_i^k(r) > 0 \) for all \( 0 < r < \varepsilon_k^{-1} \). To do this, let \( f_i^k = (f_1^k, \ldots, f_n^k) \) be the positive solution to the homogeneous system:
\[ \Delta f_i^k - \frac{1}{r^2} f_i^k + \sum_j a_{ij} h_j^k(0) e^{V_i^k} f_j^k = 0, \quad r > 0 \]
such that
\[ f_i^k(r) \sim r, \text{ near 0}, \quad f_i^k(r) \sim \frac{1}{r} \text{ near } \infty, \quad f_i^k(r) > 0 \text{ for all } r > 0. \]
If $\psi^k_i$ is not always non-negative we assume
\[ \min_{\mathbb{R}^n} \frac{\psi^k_i(r)}{f^k_i(r)} = \min_{1 \leq j \leq n} \frac{\psi^k_j(r)}{f^k_j(r)} < 0. \]

Suppose the minimum is attained at $r_k$. By the behavior of $\psi^k_i$ and $f^k_i$, we have $0 < r_k < \varepsilon^k_i$. Let $w^k_i = \psi^k_i / f^k_i$, then $w^k_i$ satisfies
\[ \Delta w^k_i + 2 \nabla w^k_i \cdot \nabla f^k_i + \frac{1 - l^2}{r^2} w^k_i = \sum_{j=2}^n a_{ij} h^k_j(0) e^j_k \left( \frac{w^k_j f^k_j - \psi^k_j}{f^k_j} \right). \]

Evaluating both sides at $r_k$, the left hand side is strictly positive while the right hand side is non-positive by the definition of $w^k_i$. This contradiction proves (4.29). Finally since $g(r) = O(r^2 \log 1/\rho)$ near 0, $g_i^{k,l}(r) = O(\varepsilon^k_i r^2)$ near 0 for $l \geq l_0 \geq 3$. Lemma 4.3 is established. \( \square \)

**Lemma 4.4.** Given $\delta > 0$, there exist $C(\delta) > 0$ independent of $k,l$ and $k_0(\delta) > 1$ such that for $l \geq 3$ and $k \geq k_0$

\[ |g_i^{k,l}(r)| \leq C \varepsilon_k^{m-2-\delta} \left( \varepsilon_k r \right)^l + C \varepsilon_k^2 r^2 (1 + r)^{2-m+\delta}, \quad r \leq \frac{1}{2} \varepsilon_k^{-1} \]
where
\[ (m-2-\delta)^+ = \begin{cases} 
2m - 2 - \delta & \text{if } m \leq 3 \\
1 - \delta & \text{if } m > 3.
\end{cases} \]

**Proof:**

By Lemma 4.3 (4.23) can be rewritten as
\[ (\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{l^2}{r^2}) g_i^{k,l} = \tilde{h}_{ijkl}, \quad g_i^{k,l}(0) = 0 \]
where
\[ |\tilde{h}_{ijkl}(r)| \leq C \varepsilon_k^2 (1 + r)^{2-m+\delta}, \quad k \geq k_0(\delta) \]
for some $C$ independent of $k$ and $l$. By standard ODE theory
\[ g_i^{k,l}(r) = c_{ijkl} r^l + c_{2kl} r^{-l} - \frac{l^2}{2l} \int_0^r s^{-l+1} h_{ijkl}(s) ds - \frac{r-l}{2l} \int_0^r s^{l+1} \tilde{h}_{ijkl}(s) ds. \]

When $r \to 0$, it is easy to see that the last two terms in (4.33) both tend to 0. Thus $c_{ijkl} = 0$. Let $I_3$ and $I_4$ represent the last two terms in (4.33), respectively. By (4.32), we have
\[ |I_3(r)| + |I_4(r)| \leq \frac{C \varepsilon_k^2}{l^2} (1 + r)^{4-m+\delta} \]
where $C$ is independent of $k$ and $l$. On the other hand, using the information of $w^{1,k}$ for $r \sim \varepsilon_k^{-1}$, we know
\[ |g_i^{k,l}(r)| \leq C \varepsilon_k^{(m-2-\delta)^+}, \quad r \sim \varepsilon_k^{-1}. \]

In regard to (4.34) and (4.33) we have
\[ |c_{ijkl} | \leq C \varepsilon_k^{(m-2-\delta)^+ + l}. \]
To this end, we need to evaluate the value of $\log z^k(\eta)$ on $\partial \Omega_{1,k}$. The Green’s representation formula for $w_i^{1,k}$ gives (see (4.11))

\[ O(\varepsilon_k^2) = w_i^{1,k}(0) = \int_{\Omega_{1,k}} G(0, \eta) \left( \sum_j a_{ij} h_j^{1,k}(0) e^{V_i^{1,k}(\eta)} - V_i^{1,k}(\eta) \right) d\eta \]

where $G(\cdot, \cdot)$ is the Green’s function on $\Omega_{1,k}$ with respect to the Dirichlet boundary condition. We observe that

\[ G(0, \eta) = -\frac{1}{2\pi} \log |\eta| + \frac{1}{2\pi} \log \varepsilon_k^{-1} + O(\varepsilon_k^2). \]

Indeed, since $\Omega_{1,k}$ is a translation of $\Omega_k$ by $p_{1,k}$ (recall $p_{1,k} = O(\varepsilon_k)$), the oscillation of $\log |\eta|$ on $\partial \Omega_{1,k}$ is $O(\varepsilon_k^2)$. Thus the oscillation of the regular part of $G(0, \eta)$ is $O(\varepsilon_k^2)$, which leads to (4.42). On the other hand let $\tilde{w}_i^{1,k}$ be the average of $w_i^{1,k}$ on $\partial \Omega_{1,k}$, using the fact that $v_i^{1,k}$ is constant on $\partial \Omega_{1,k}$ and $V_i^k$ has oscillation $O(\varepsilon_k^2)$ on $\partial \Omega_{1,k}$ we have

\[ w_i^{1,k}(y) = \tilde{w}_i^{1,k} + O(\varepsilon_k^2), \quad \forall y \in \partial \Omega_{1,k}. \]

Hence (4.31) follows immediately. Lemma 4.4 is established. □

Let $z^k = (z^k_1, \ldots, z^k_n)$ be the projection of $w^{1,k}$ to $\text{span}\{\sin l\theta, \cos l\theta, l \geq 2\}$, i.e.

\[ z^k_i = \sum_{l=2}^{\infty} \left( g^{k,j}_i(r) \sin l\theta + \tilde{g}^{k,j}_i(r) \cos l\theta \right) \]

where $g^{k,j}_i$ is the projection of $w^{1,k}_i$ on $\cos l\theta$. $\tilde{g}^{k,j}_i$ has similar estimates as that for $g^{k,j}_i$. Then Lemma 4.4 leads to

\[ |z^k_i(y)| \leq C \varepsilon_k^2 (1 + |y|)^{4-m+\delta}, \quad |y| \leq \frac{1}{2} \varepsilon_k^{-1} \]

for $m \leq 3$. However for $m > 3$, Lemma 4.4 only gives

\[ |z^k_i(y)| \leq C \varepsilon_k^{3-\delta} (1 + |y|)^2 + C \varepsilon_k^2 (1 + |y|)^{4-m+\delta}, \quad |y| \leq \frac{1}{2} \varepsilon_k^{-1}. \]

In the following we shall get rid of the first term on the right hand side of (4.38). To this end, we need to evaluate the value of $w_i^{1,k}$ on $\partial \Omega_{1,k}$.

**Lemma 4.5.** (a) If $m < 4$, then

\[ |w_i^{1,k}(y)| \leq C \varepsilon_k^{m-2} \log \varepsilon_k^{-1}, \quad y \in \partial \Omega_{1,k}. \]

(b) If $m = 4$ and $|m_k^i - 4| \leq C / \log \varepsilon_k^{-1}$ for all $i$, then

\[ |w_i^{1,k}(y)| \leq C \varepsilon_k^2 (\log \varepsilon_k)^2, \quad y \in \partial \Omega_{1,k}. \]

**Remark 4.1.** The assumption $|m_k^i - 4| \leq C / \log \varepsilon_k^{-1}$ when $m = 4$ is natural and will be justified in the proof of the main theorems in section 6. We also remark that in (4.39) we use $\varepsilon_k^{m-2} \log 1 / \varepsilon_k$ instead of the crude $\varepsilon_k^{m-\delta}$ as before.

**Proof:**

The Green’s representation formula for $w_i^{1,k}$ gives (see (4.11))

\[ O(\varepsilon_k^2) = w_i^{1,k}(0) = \int_{\Omega_{1,k}} G(0, \eta) \left( \sum_j a_{ij} h_j^{1,k}(0) e^{V_i^{1,k}(\eta)} - V_i^{1,k}(\eta) \right) d\eta \]

\[ - \int_{\partial \Omega_{1,k}} \partial_j G(0, \eta) w_i^{1,k}(\eta) dS_\eta \]

where $G(\cdot, \cdot)$ is the Green’s function on $\Omega_{1,k}$ with respect to the Dirichlet boundary condition. We observe that

\[ G(0, \eta) = -\frac{1}{2\pi} \log |\eta| + \frac{1}{2\pi} \log \varepsilon_k^{-1} + O(\varepsilon_k^2). \]

Indeed, since $\Omega_{1,k}$ is a translation of $\Omega_k$ by $p_{1,k}$ (recall $p_{1,k} = O(\varepsilon_k)$), the oscillation of $\log |\eta|$ on $\partial \Omega_{1,k}$ is $O(\varepsilon_k^2)$. Thus the oscillation of the regular part of $G(0, \eta)$ is $O(\varepsilon_k^2)$, which leads to (4.42). On the other hand let $\tilde{w}_i^{1,k}$ be the average of $w_i^{1,k}$ on $\partial \Omega_{1,k}$, using the fact that $v_i^{1,k}$ is constant on $\partial \Omega_{1,k}$ and $V_i^k$ has oscillation $O(\varepsilon_k^2)$ on $\partial \Omega_{1,k}$ we have

\[ w_i^{1,k}(y) = \tilde{w}_i^{1,k} + O(\varepsilon_k^2), \quad \forall y \in \partial \Omega_{1,k}. \]
Thus
\begin{equation}
- \int_{\partial \Omega_{1,k}} \partial \nu G(0, \eta) w_{l}^{1,k}(\eta) dS_{\eta} = \tilde{w}_{l}^{1,k} + O(\varepsilon_{k}^{2}).
\end{equation}
By using (4.42), (4.43) and (4.44) in (4.44) we have
\begin{align*}
- \tilde{w}_{l}^{1,k} + O(\varepsilon_{k}^{2}) &= \int_{\Omega_{1,k}} \left( \frac{1}{2\pi} \log \frac{\varepsilon_{k}^{-1}}{\eta} \right) \left( \sum_{j} a_{ij} h_{j}^{k}(0) e^{v_{j}^{k}} w_{j}^{1,k}(\eta) - E_{l}^{k}(\eta) \right) d\eta.
\end{align*}
To evaluate the right hand side, we divide $\Omega_{1,k}$ into a symmetric part: $D_{1} := B(0, \varepsilon_{k}^{-1} - |p_{1,k}|)$ and a nonsymmetric part: $\Omega_{1,k} \setminus D_{1}$ and use $I_{1}$ and $I_{2}$ to represent the corresponding integrals on them. If $\eta \in \Omega_{1,k} \setminus D_{1}$, it is easy to see from $|p_{1,k}| = O(\varepsilon_{k})$ that
\begin{equation*}
\log \left( \frac{\varepsilon_{k}^{-1}}{\eta} \right) = O(\varepsilon_{k}^{2}) \quad \text{and} \quad |\Omega_{1,k} \setminus D_{1}| = O(1).
\end{equation*}
Moreover for $\eta \in \Omega_{1,k} \setminus D_{1}$, by (4.13) and (4.14)
\begin{equation*}
w_{j}^{1,k}(\eta) = O(\varepsilon_{k}^{-1}), \quad E_{l}^{k}(\eta) = O(\varepsilon_{k}^{m}).
\end{equation*}
Combining these facts we have $I_{2} = O(\varepsilon_{k}^{2})$.
To evaluate $I_{1}$, since $\log(\varepsilon_{k}^{-1}/|\eta|)$ is radial, by symmetry only the projections of $w_{j}^{1,k}$ and $E_{k}^{k}$ onto 1 remain. By (4.16) and (4.14)
\begin{equation*}
|I_{1}| \leq C \log \varepsilon_{k}^{-1} \int_{D_{1}} \varepsilon_{k}^{2} (1 + |\eta|)^{2-m} d\eta.
\end{equation*}
(Note that we use the fact that $\varepsilon_{k}^{-1} \leq C(1 + |\eta|)^{-m}$ for $C$ independent of $k$.) Therefore if $m < 4$, $I_{1} = O(\varepsilon_{k}^{-m-2}) \log \varepsilon_{k}^{-1}$. If $m = 4$ and $|m_{k}^{k} - 4| \leq C \log \varepsilon_{k}^{-1}$ we have $(1 + r)^{-m} \leq C(1 + r)^{-4}$, thus by elementary computation
\begin{equation*}
|I_{1}| \leq C \varepsilon_{k}^{2} (\log \varepsilon_{k})^{2}.
\end{equation*}
Lemma 4.5 is established. \(\Box\)
By Lemma 4.5 (4.35) can be replaced by
\begin{equation*}
|g_{l}^{k}(r)| \leq C \varepsilon_{k}^{m-\delta}, \quad r \sim \varepsilon_{k}^{-1}.
\end{equation*}
Correspondingly, the estimate for $c_{1kl}^{k}$ becomes
\begin{equation*}
|c_{1kl}^{k}| \leq C \varepsilon_{k}^{m-\delta+l}.
\end{equation*}
Then it is easy to see that the first term in (4.38) can be removed.

**Step five:**

For the projection on $span\{\sin \theta, \cos \theta\}$, we write (4.20) as
\begin{equation*}
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right)G_{l}^{k} = h(r), \quad 0 < r < \varepsilon_{k}^{-1}
\end{equation*}
where $h$ is the collection of other terms. By (4.22)
\begin{equation*}
|h(r)| \leq C(1 + r)^{1-m+\delta}.
\end{equation*}
Then
\begin{equation}
G_{1,i}^k = c_{1k} + \frac{c_{2k}}{r} - \frac{r}{2} \int_r^\infty h(s)ds - \frac{r^{-1}}{2} \int_0^r s^2 h(s)ds.
\end{equation}

Since $G_{1,i}$ is bounded near 0, $c_{2k} = 0$. Using $G_{1,i}^k(\varepsilon_k^{-1}) = O(\varepsilon_k^{m-3-\delta})$ we have
\[|c_{1k}| \leq C\varepsilon_k^{m-2-\delta}.
\]

Then it is easy to see from (4.45) that
\begin{equation}
|G_{1,i}^k(r)| \leq Cr(1 + r)^{2-m+\delta}, \quad t = 1, 2, \quad \text{when } m > 3.
\end{equation}

Similarly (4.46) also holds for $G_{2,j}^k$.

Combining the results in the five steps we arrive at the following estimate without distinguishing $m < 4$ or not.

**Theorem 4.1.** Given $\delta > 0$, there exist $C(\delta) > 0$, $k_0(\delta) > 1$ such that for $|y| \leq \varepsilon_k^{-1}/2$ and $|\alpha| = 0, 1$, the following holds for all $k \geq k_0$
\begin{equation}
|D^\alpha(v_{i,j}^k(y) - V_i^k(y) - \Phi_i^k(y))| \leq C\varepsilon_k^2(1 + |y|)^{4-m-|\alpha|+\delta}.
\end{equation}

where
\[\Phi_i^k(y) = \varepsilon_k(C_{1,i}^k(r) \cos \theta + C_{2,i}^k(r) \sin \theta)
\]
with
\begin{equation}
|G_{i,j}^k(r)| \leq Cr(1 + r)^{2-m+\delta} \quad t = 1, 2.
\end{equation}

Note that the estimate for $|\alpha| = 0$ follows directly from the five steps. The estimate for $|\alpha| = 1$ follows from standard gradient estimate for elliptic equations.

Theorem 4.1 does not distinguish $m < 4$ or $m = 4$. In the following we apply Theorem 4.1 to obtain more accurate estimates for $m < 4$ and $m = 4$, respectively. Both results in the sequel (Theorem 4.2 and Theorem 4.3) play a crucial role in determining the location of maximum points of bubbling solutions.

**Theorem 4.2.** Suppose $m < 4$, then for $|y| \leq \varepsilon_k^{-1}/2$ and $i \in I$
\begin{equation}
|D^\alpha(v_{i,j}^k(y) - \phi_i^k(y) - V_i^k(y - p_{1,k}) - \Phi_i^k(y - p_{1,k}))| \leq C\varepsilon_k^2(1 + |y|)^{4-m-|\alpha|+\delta} \quad |\alpha| = 0, 1
\end{equation}

where $v_{i,j}^k, \phi_i^k, V_i^k, \Phi_i^k$ are defined by (3.5), (3.9), (3.10) and (4.3), respectively. Moreover $G_{i,j}^k(t = 1, 2, i \in I)$ satisfy
\begin{equation}
|G_{i,j}^k(r)| \leq Cr(1 + r)^{2-m} \quad 0 < r < \varepsilon_k^{-1}.
\end{equation}

**Proof:**
We use the same notations as in the proof of Proposition 4.1. First we consider the radial part of $w^{1,k}$: Recall $m < 4$. Using Proposition 4.1 and (4.15) we have

\[
\frac{d^2}{dr^2} \left( \frac{1}{r} \frac{d}{dr} \right) g_i^{k,0}
\]

\[
= - \sum_j a_{ij} h_j^k(0)e^{V_j^k} g_j^{k,0} \frac{\varepsilon^2}{4} \sum_j a_{ij} \Delta h_j^k(0) r^2 e^{V_j^k} + O(\varepsilon_k^2)(1 + r)^{2-m}
\]

\[
= O(\varepsilon_k^2)(1 + r)^{2-m}, \quad 0 < r < \varepsilon_k^{-1},
\]

and

\[
g_i^{k,0}(0) = O(\varepsilon_k^2).
\]

Multiplying $r$ on both sides of the equation and integrating, we obtain

\[
|g_i^{k,0}(r)| \leq C \varepsilon_k^2 \log(2 + r).
\]

Next we consider the projection of $w^{1,k}$ on $\sin \theta$ and $\cos \theta$. Let $\omega^k = (\omega_1^k, \ldots, \omega_n^k)$ be

\[
\omega_i^k = w_i^{1,k} - g_i^{k,0}.
\]

Then $\omega^k$ satisfies

\[
\Delta \omega_i^k + \sum_j a_{ij} h_j^k(0)e^{V_j^k} \omega_j^k = E_{1,i}^k, \quad \Omega_{1,i}
\]

where $E_{1,i}^k$ is the projection of the original right hand side on the subspace spanned by $\sin k \theta$ and $\cos ku (k = 1, 2, \ldots)$. Since $g_i^{k,0}$ is a radial function, from the asymptotic behavior of $g^{k,0}$ it is easy to see that the oscillation of it on $\partial \Omega_{1,k}$ is $O(\varepsilon_k^2)$, therefore the oscillation of $\omega_i^k$ on $\partial \Omega_{1,k}$ is $O(\varepsilon_k^2)$. Let $\bar{\omega}_i^k$ be the average of $\omega_i^k$ on $\partial \Omega_{1,k}$, then by $\omega_i^k(0) = 0$ we have

\[
0 = \int_{\Omega_{1,k}} G(0, \eta) \left( \sum_j a_{ij} h_j^k(0)e^{V_j^k} \omega_j^k(\eta) - E_{1,i}^k(\eta) \right) d\eta + \bar{\omega}_i^k + O(\varepsilon_k^2).
\]

Using (4.42) in the equation above we have $\bar{\omega}_i^k = O(\varepsilon_k^2)$, thus

\[
\omega_i^k = O(\varepsilon_k^2), \quad \text{on} \quad \partial \Omega_{1,k}.
\]

Since we have known that $\varepsilon_k G_{i,j}^k(\varepsilon_k^{-1}) = O(\varepsilon_k^m - 2)$, we can improve the estimate of $\Phi^k$ using (4.45). The estimate of $G_{i,j}^k$ now is

\[
|G_{i,j}^k(r)| \leq C r(1 + r)^{2-m}, \quad r < \varepsilon_k^{-1}
\]

and

\[
|\frac{d}{dr} G_{i,j}^k| \leq C r(1 + r)^{1-m}, \quad 0 < r < \varepsilon_k^{-1}/2
\]

which leads to

\[
|\nabla \Phi_i^k(y)| \leq C \varepsilon_k r(1 + r)^{1-m}, \quad |y| < \varepsilon_k^{-1}/2.
\]
As far as the projection of \(w^{1,k}\) on higher frequencies is concerned, since we have (4.51), for \(s^{k,l}\) we now have, instead of Lemma 4.4,

\[
|s^{k,l}_i(r)| \leq C \epsilon_k^n(\epsilon_k r)^l + \frac{C}{r^2} \epsilon_k^2 r^2(1 + r)^{2-m_k}, \quad l \geq 2.
\]

As before we let \(z^k_i = w^{1,k}_i - \Phi^k_i\), then \(z^k_i\) satisfies

\[
\begin{align*}
\Delta z^k_i &= O(\epsilon_k^2) |y|^2 (1 + |y|)^{-m_k}, \quad \Omega_{1,k} \subset \mathbb{R}^2, \\
\frac{\partial z^k_i}{\partial \nu} &= O(\epsilon_k^2), \quad \partial \Omega_{1,k}.
\end{align*}
\]

Because of (4.53), we have

\[
|z^k_i(y)| \leq C \epsilon_k^2 (1 + |y|)^{4-m_k}, \quad |y| \leq \epsilon_k^{-1}.
\]

By standard re-scaling method

\[
|\nabla z^k_i(y)| \leq C \epsilon_k^2 (1 + |y|)^{3-m_k}, \quad |y| < \epsilon_k^{-1}/2.
\]

We have established

\[
|D^\alpha (v^{1,k}_i(y) - V^k_i(y) - \Phi^k_i(y))| \leq C \epsilon_k^2 (1 + |y|)^{4-m_k - |\alpha| \log (2 + |y|)}
\]

for \(|\alpha| = 0, 1\) and \(|y| \leq \epsilon_k^{-1}/2\). Recall that \(v^{1,k}_i\) is defined in (4.4). Instead of using the coordinate around \(p_{1,k}\) we use the coordinate around the origin to obtain (4.49).

Theorem 4.3.

If \(m = 4\) and \(|m_i - 4| \leq C \log \epsilon_k^{-1}\) for all \(i \in I\), then we have, for \(|y| \leq \epsilon_k^{-1}/2\) and \(i \in I\)

\[
|D^\alpha (v^i_k(y) - \Phi^k_i(\epsilon_k y) - V^k_i(y - p_{1,k}) - \Phi^k_i(y - p_{1,k}))| \leq C \epsilon_k^2 (1 + |y|)^{-|\alpha| (\log (2 + |y|)^2)}, \quad |\alpha| = 0, 1,
\]

where \(\Phi^k\) is of the form stated in (4.3) with \(G^t_i (t = 1, 2)\) satisfying

\[
|G^t_i(r)| \leq C r (1 + r)^{-2}, \quad 0 < r < \epsilon_k^{-1}, \quad i \in I.
\]

**Proof:**

For \(m = 4\) we use (4.16) to write (4.15) as

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d^2}{dt^2} + \frac{1}{r} \frac{d}{dt} \right\} g^0_{i,k} = - \epsilon_k^2 \sum_j a_{ij} \frac{\Delta H^{1,k}_j(0)}{4} r^2 e^{V^i_j} + O(\epsilon_k^2)(1 + r)^{-4+\epsilon}, \\
g^0_{i,k}(0) = O(\epsilon_k^2), \quad \frac{d}{dt} g^0_{i,k}(0) = 0.
\end{array} \right.
\]

Note that \((1 + r)^{-m_k} \leq C (1 + r)^{-4}\). Multiplying both sides by \(r^2\) and integrating, we have

\[
g^{k,0}_i(r) = - \epsilon_k^2 \sum_j a_{ij} \frac{\Delta H^{1,k}_j(0)}{4} \int_0^r t^2 e^{V^i_j}(\log r - \log t) dt + O(\epsilon_k^2) \log (1 + r).
\]

To evaluate the integral, we use (2.21) to write

\[
H^{1,k}_j(0) e^{V^i_j(t)} = e^{\epsilon_k^2 t^{-m_k}} + O(t^{-4-\epsilon_0}), \quad t > 1
\]
for some $\varepsilon_0 > 0$. Under the assumption of Theorem 4.3, $|m^{k}_l - 4| \leq C/\log \varepsilon_{k}^{-1}$ we have $t^{-m^k} = O(t^{-4})$, thus

$$|g^{k,0}_l (r)| \leq C\varepsilon_{k}^2 (\log(2 + r))^2, \quad 0 < r < \varepsilon_{k}^{-1}.$$  

The projection on higher frequencies has the same estimates as in the case for $m < 4$. Specifically, let $\omega^k_l$ be the same as in Theorem 4.2. Then (5.1) also holds. Correspondingly (4.52), (4.53) and (4.54) still hold with $m^k = 4$ by the same proof. Theorem 4.3 is established. □

5. LOCATION OF THE BLOWUP POINTS

In this section we determine the locations of the blowup points in Theorem 4.2 and Theorem 4.3.

**Theorem 5.1.** Under the same assumptions as in Theorem 4.2

$$|\sum_i \left( \frac{\partial_t h^k_l (0)}{h^k_l (0)} + \partial_t \phi^{k}_l (0) \right) \sigma^k_i | \leq C\varepsilon_{k}^{m^k - 2}, \quad l = 1, 2,$$

where $C$ is independent of $k$. On the other hand, under the assumptions in Theorem 4.3 we have

$$|\sum_i \left( \frac{\partial_t h^k_l (0)}{h^k_l (0)} + \partial_t \phi^{k}_l (0) \right) \sigma^k_i | \leq C\varepsilon_{k}^{2 \log \varepsilon_{k}^{-1}}, \quad l = 1, 2,$$

where $\sigma^k_i$ is defined in Definition 3.1.

**Proof of Theorem 5.1.** Recall $H^{k}_l (\varepsilon_i y) = h^{k}_l (\varepsilon_i y) e^{\phi^{k}_l (\varepsilon_i y)}$ and $H^{1,k}_l$ is defined in (4.8). Let $\hat{\Omega}_k = B(0, \varepsilon_{k}^{-1}/2)$, we use the following Pohozaev identity for the equation for $v^{1,k}_l$: For $\xi \in S^1$,

$$\sum_i \int_{\hat{\Omega}_k} \partial_{\xi} H^{1,k}_l (y) e^{v^{1,k}_l (y)}$$

$$= \int_{\partial \hat{\Omega}_k} (\xi \cdot \nu) \sum_i e^{v^{1,k}_l} H^{1,k}_l + \sum_{ij} d^{ij} \left( \partial_{\xi} v^{1,k}_l \partial_{\nu} v^{1,k}_l - \frac{1}{2} \nabla v^{1,k}_l \cdot \nabla v^{1,k}_l (\xi \cdot \nu) \right)$$

According to the definition of $H^{1,k}_l$ in (4.8)

$$\partial_{\xi} H^{1,k}_l (y) = \varepsilon_i \partial_{\xi} H^{1}_l (0) + \sum_{l=1}^{2} \varepsilon_k \partial_{\xi} H^{1}_l (0) y^l + O(\varepsilon_k (1 + |y|)^2).$$

Using the expansion of $v^{1,k}_l$ in Proposition 4.1 (which holds for $m < 4$ and $m = 4$) we have

$$\int_{\hat{\Omega}_k} \partial_{\xi} H^{1,k}_l (y) e^{v^{1,k}_l (y)}$$

$$= \int_{\hat{\Omega}_k} \left( \varepsilon_i \partial_{\xi} H^{1}_l (0) + \sum_{l=1}^{2} \varepsilon_k \partial_{\xi} H^{1}_l (0) y^l + O(\varepsilon_k (1 + |y|)^2) \right)$$

$$\cdot \left( e^{v^{1}_l (y)} (1 + \Phi^k_l ) + O(\varepsilon_k^2 (1 + |y|)^4 - 2m + \varepsilon) \right) dy.$$
By symmetry we have
\[ \int_{\Omega_k} \sum_{i=1}^{2} \varepsilon_k^2 \partial_{i\xi} H_i^k(0)y^l e^{V_i^k(y)} = \int_{\Omega_k} \varepsilon_k\partial_{i\xi} H_i^k(0) e^{V_i^k(y)} \Phi_i^k(y) dy = 0. \]

Also by elementary estimates we have
\[ \int_{\Omega_k} \left( \sum_{i=1}^{2} \varepsilon_k^2 \partial_{i\xi} H_i^k(0)y^l + O(\varepsilon_k^3 (1 + |y|^2)) \right) (e^{V_i^k(y)} \Phi_i^k + O(\varepsilon_k^2 (1 + |y|)^{2m+\varepsilon})) = O(\varepsilon_k^{m-1}) \]
and
\[ \int_{\Omega_k} \varepsilon_k \partial_{i\xi} H_i^k(0) O(\varepsilon_k^2 (1 + |y|)^{4-2m+\varepsilon}) = O(\varepsilon_k^{m-1}). \]

Thus
\[ \int_{\Omega_k} \partial_{i\xi} H_i^{1,k}(y) e^{V_i^k(y)} = \int_{\Omega_k} \varepsilon_k \partial_{i\xi} H_i^k(0) e^{V_i^k(y)} + O(\varepsilon_k^3 (1 + |y|^2)) e^{V_i^k(y)}. \]

The only difference on whether \( m < 4 \) or \( m = 4 \) is on this term. For \( m < 4 \), direct computation gives
\[ \int_{\Omega_k} \partial_{i\xi} H_i^{1,k}(y) e^{V_i^k(y)} = \varepsilon_k \partial_{i\xi} H_i^k(0) \int_{\Omega_k} e^{V_i^k(y)} dy + O(\varepsilon_k^{m-1}). \]
\[ = \varepsilon_k \partial_{i\xi} H_i^k(0) \int_{\Omega_k} H_i^k(0) e^{V_i^k(y)} dy + O(\varepsilon_k^{m-1}) \]
\[ = 2\pi \varepsilon_k \partial_{i\xi} (\log H_i^k)(0) \sigma_i^k + O(\varepsilon_k^{m-1}). \]

On the other hand for \( m = 4 \), using the closeness between \( m_i^k \) and 4, we have
\[ \int_{\Omega_k} \partial_{i\xi} H_i^{1,k}(y) e^{V_i^k(y)} = 2\pi \varepsilon_k \partial_{i\xi} (\log H_i^k)(0) \sigma_i^k + O(\varepsilon_k^{3} \log \varepsilon_k^{-1}). \]

Next we estimate the right hand side of the Pohozaev identity. The key point in this part is that the only difference between Theorem 4.2 and Theorem 4.3 is on the radial part of \( w_i^{1,k} \cdot g_i^{k,0} \). The projections of \( w_i^{1,k} \) on higher frequencies \( (\omega_i^k) \) have the same estimates. i.e. \( (4.31)-(4.54) \) hold for \( m < 4 \) and \( m = 4 \). The detail is as follows: First by the decay rate of \( v_i^{1,k} \), one sees easily that
\[ \int_{\Omega_k} (\xi \cdot v) \sum_{i} e^{v_i^{1,k}} H_i^{1,k} = O(\varepsilon_k^{m-1}). \]

Before evaluating the remaining two terms we first observe that by \( (4.52) \) and \( (4.54) \) that
\[ \nabla v_i^{1,k} = \nabla V_i^k + \nabla g_i^{k,0} + O(\varepsilon_k^{m-1}), \quad y \in \partial \Omega_k. \]
Moreover, by symmetry we have
\[ \int_{\partial \Omega_k} \partial_{\xi} V_i^k \partial_{\xi} g_j^k = \int_{\partial \Omega_k} \partial_{\xi} V_i^k \partial_{\nu} g_j^k = 0. \]
Thus
\[ \sum_{i,j} \int_{\partial \Omega_k} a^{ij} \partial_{\nu} V_i^k \partial_{\xi} V_j^k \]
\[ = \sum_{i,j} \int_{\partial \Omega_k} a^{ij} \left( (\partial_{\nu} (V_i^k + g_i^{k,0}) + O(\epsilon_k^{m-1})) (\partial_{\xi} (V_j^k + g_j^{k,0}) + O(\epsilon_k^{m-1})) \right) \]
\[ = O(\epsilon_k^{m-1}). \]
Similarly in the last integral:
\[ \sum_{i,j} \int_{\partial \Omega_k} a^{ij} \nabla_{\nu} V_i^k \nabla_{\xi} V_j^k (\xi \cdot v) \]
\[ = \sum_{i,j} \int_{\partial \Omega_k} a^{ij} \left( \frac{d}{dr} (V_i^k + g_i^{k,0}) + O(\epsilon_k^{m-1})) (\frac{d}{dr} (V_j^k + g_j^{k,0}) + O(\epsilon_k^{m-1})) (\xi \cdot v) \right) \]
\[ = O(\epsilon_k^{m-1}). \]
Theorem 5.1 is established. □

6. THE LEADING TERM FOR $\rho^k \to \rho \in \Gamma_1$

In this section we complete the proofs of Theorem 1.2, Theorem 1.3 and Theorem 1.4. The notation $u^k$ refer to the one in the introduction. Let
\[ (6.1) \quad \Theta_i^k = u_i^k - \log \int_M h_i e^{u_i^k} dV_g, \quad i \in I. \]
Then we have
\[ (6.2) \quad -\Delta_{\xi} \Theta_i^k = \sum_{j=1}^n a_{ij} \rho_j^k (h_j e^{\Theta_j^k} - 1). \]
where
\[ (6.3) \quad \int_M h_i e^{\Theta_i^k} dV_g = 1, \quad Vol(M) = 1. \]
$\Theta^k = (\Theta_1^k, ..., \Theta_n^k)$ is a sequence of blow-up solutions. We use $p_k$ to denote the point where the maximum of $\Theta^k$ on $M$ is taken. Take the local coordinates around $p_k$. Then $d\sigma^2$ has the form $e^{\psi(y_1, y_2)} (dy_1^2 + dy_2^2)$ where
\[ |\nabla \psi(0)| = 0, \quad \psi(0) = 0, \quad \Delta \psi = -2K e^\psi, \]
$K$ is the Gauss curvature. In local coordinates, (6.2) becomes
\[ (6.4) \quad -\Delta \Theta_i^k = \sum_{j=1}^n a_{ij} \rho_j^k e^\psi (h_j e^{\Theta_j^k} - 1), \quad \text{in} \quad B_{\delta_0}. \]
for some $\delta_0$ small. Let $f_i^k$ solve
\begin{equation}
\begin{cases}
\Delta f_i^k = \sum_{j=1}^n a_{ij}\rho_{ij}^k e^{\psi}, & B_{\delta_0} \\
f_i^k(0) = 0, & \nabla f_i^k(0) = 0.
\end{cases}
\end{equation}
(6.5)
and $f_i^k$ has bounded oscillation on $\partial B_{\delta_0}$. Set
\[ \tilde{\Theta}_i^k = \Theta_i^k(p_k + \cdot) - f_i^k \quad \text{and} \quad \tilde{h}_i^k = \rho_i^k h_i(p_k + \cdot) e^{\psi + f_i^k}, \]
then $\tilde{\Theta}_i^k$ satisfies
\begin{equation}
- \Delta \tilde{\Theta}_i^k = \sum_{j} a_{ij}\tilde{h}_j^k e^{\Theta_j^k}, \quad B_{\delta_0}.
\end{equation}
(6.6)
Note that from the definition of $\tilde{h}_i^k$, $f_i^k$ and $\psi$, the following can be verified by direct computation:
\begin{equation}
\begin{cases}
\tilde{h}_i^k(0) = \rho_i^k h_i(p_k), & \nabla \tilde{h}_i^k(0) = \rho_i^k \nabla h_i(p_k), \\
\Delta \tilde{h}_i^k(0) = \rho_i^k \left( \Delta h_i(p_k) + h_i(p_k)(-2K(p_k) + \sum_{j=1}^n a_{ij}\rho_j^k) \right).
\end{cases}
\end{equation}
(6.7)
Let $M_k = \max_{i\in I} \max M \Theta_i^k, \varepsilon_k = e^{-\frac{1}{2}M_k}$,
\[ v_i^k(y) = \tilde{\Theta}_i^k(\varepsilon_k y) + 2\log \varepsilon_k = \tilde{\Theta}_i^k(\varepsilon_k y) - M_k, \]
then we have
\begin{equation}
\Delta v_i^k(y) + \sum_{j=1}^n a_{ij}\tilde{h}_j^k(\varepsilon_k y)e^{v_j^k(y)} = 0, \quad |y| \leq \delta_0\varepsilon_k^{-1}.
\end{equation}
(6.9)

The following lemma proves that the $\Theta_i^k$ is a fully blown up sequence.

**Lemma 6.1.** Along a subsequence $(\tilde{v}_1^k, ..., \tilde{v}_n^k)$ converge in $C^2_{loc}(\mathbb{R}^2)$ to a system of $n$ equations.

**Proof:** By way of contradiction we assume that only $l$ ($l < n$) components of $(\tilde{v}_1^k, ..., \tilde{v}_n^k)$ converge to a system of $l$ equations. Without loss of generality we assume that the first $l$ components of $(\tilde{v}_1^k, ..., \tilde{v}_n^k)$ converge to $(\tilde{v}_1, ..., \tilde{v}_l)$ that satisfies
\[ -\Delta \tilde{v}_i = \sum_{j=1}^l a_{ij}\tilde{h}_j e^{\tilde{v}_j}, \quad i = 1, ..., l, \text{ in } \mathbb{R}^2 \]
where $\tilde{h}_i = \rho_i h_i(p)$. Let
\[ \sigma_{i,v} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{h}_i e^{\tilde{v}_i}. \]
Then the entire solution $(\tilde{v}_1, ..., \tilde{v}_l)$ with finite energy satisfies
\[ \sum_{j=1}^l a_{ij}\sigma_{j,v} > 2. \]
Let $J = \{1, \ldots, l\}$, then by Theorem C in [38] $(\bar{\nu}_1, \ldots, \bar{\nu}_l)$ also satisfies
\[
\frac{1}{4\pi^2} \Lambda_J(\sigma_i) := 4 \sum_{i \in J} \sigma_{i,v} - \sum_{i,j \in J} a_{ij} \sigma_{i,v} \sigma_{j,v} = 0.
\]
By (6.3)
\[
\frac{\rho_i^k}{2\pi} \geq \frac{1}{2\pi} \int_{B(p_k, \delta)} \rho_i^k \sigma_i e^{\Theta^k} dV_k = \frac{1}{2\pi} \int_{B_8} \sigma_i e^{\Theta^k} dx.
\]
Thus by letting $k \to \infty$ we have
\[
\frac{\rho_i}{2\pi} \geq \sigma_{i,v}, \quad i = 1, \ldots, l.
\]
Let $s_i = \frac{\rho_i}{2\pi} - \sigma_{i,v}, \ i \in J$, then easy to see
\[
(6.10) \quad \frac{1}{4\pi^2} (\Lambda_J(\rho) - \Lambda_J(\sigma_i)) = -2 \sum_{i \in J} (\sum_{j \in J} a_{ij} \sigma_{j,v} - 2)s_i - \sum_{i,j} a_{ij} s_i s_j \leq 0.
\]
Since $\Lambda_J(\sigma_i) = 0$, (6.10) is a violation of the definition of $\Gamma_1$ in the introduction. Lemma 6.1 is established. □

Let $\phi^k = (\phi_1^k, \ldots, \phi_l^k)$ be the harmonic function that takes 0 at 0 and that kills the oscillation of $\tilde{\Theta}^k$ on $\partial B_{8}$. The first term in the expansion of $\tilde{\nu}^k - \phi^k(\epsilon_k \cdot)$ is $U^k(\cdot - p_{1,k})$ where $U^k = (U_1^k, \ldots, U_n^k)$ satisfies
\[
\begin{cases}
\Delta U^k_i = \sum_j a_{ij} \tilde{h}_j^k(0)e^{U^k_j} & \mathbb{R}^2, \\
U^k_i(0) = \bar{\nu}_i^k(0), & i \in I.
\end{cases}
\]
Moreover, $U_i^k + \log(\rho_i^k \tilde{h}_i(p_k))$ satisfies
\[
-\Delta(U_i^k + \log(\rho_i^k \tilde{h}_i(p_k))) = \sum_j a_{ij} e^{U^k_j + \log(\rho_i^k \tilde{h}_i(p_k))} & \mathbb{R}^2.
\]
Set
\[
(6.12) \quad \bar{\sigma}^k_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} \rho^k_i \tilde{h}_i(p_k) e^{U^k_i} dx, \quad m_i^k = \sum_{j=1}^n a_{ij} \bar{\sigma}^k_j.
\]
Correspondingly
\[
m^k = \min\{m_1^k, \ldots, m_n^k\}, \quad m = \lim_{k \to \infty} m^k.
\]
Later we shall show that $m$ is equal to the one defined in (1.6). In regard to Lemma 2.2 we set $c_i^k$ as
\[
(6.13) \quad c_i^k = U_i^k(0) + \log(\rho_i^k \tilde{h}_i(p_k)) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|\eta| \sum_{j=1}^n a_{ij} \tilde{h}_j^k(\eta)e^{U^k_j} d\eta
\]
\[
= \Theta_i^k(p_k) - M_k + \log(\rho_i^k \tilde{h}_i(p_k)) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|\eta| \sum_{j=1}^n a_{ij} \rho_j^k \tilde{h}_j(p_k)e^{U^k_j} d\eta,
\]
and
\[
(6.14) \quad c_i = \lim_{k \to \infty} c_i^k.
\]
According to Proposition 4.1, the expansion of $\tilde{v}^k_i$ can be written as
\begin{equation}
\tilde{v}^k_i(y) = U^k_i(y - p_{1,k}) + \Phi^k_i(\varepsilonKY) + \Phi^k_i(y - p_{1,k}) + O(\varepsilon^2) \Gamma(2 + |y|)^{2-m+e}
\end{equation}
on $B(0, \delta_0 \varepsilon^{-1})$, where $p_{1,k} = O(\varepsilon_k)$,
\[ \Phi^k_i(y) = \varepsilon_k(G^k_{t,i}(\rm{cos} \theta) + G^k_{t,i}(\sin \theta)) \]
such that
\[ |G^k_{t,i}(r)| \leq Cr(1 + r)^{-m^k}, \quad 0 < r < \delta_0 \varepsilon^{-1}, \quad t = 1, 2. \]

Here we note that the Green's function is of the form
\[ G(x, \eta) = -\frac{1}{2\pi} \log |x - \eta| \chi + \gamma(x, \eta) \]
where $\chi$ is a cut-off function such that $\chi \equiv 1$ in $B_{\varepsilon_1}$ for some $\varepsilon_1 > 0$ ($2\varepsilon_1$ is less than the injectivity radius of $M$) and $\chi \equiv 0$ outside $B_{2\varepsilon_1}$. In the sequel we always assume $\delta_0 < \varepsilon_1/4$.

The Green's representation formula for $\Theta^k_i$ is
\begin{equation}
\Theta^k_i(x) = \Theta^k_i + \int_M G(x, \eta) \sum_{j} a_{ij} \rho^k_i h_j e^{\Theta^k_j} dV_g.
\end{equation}

Next we claim that
\begin{equation}
|\Theta^k_i + \frac{m^k_i - 2}{2} \rho^k_i - c^k_i + \log(\rho^k_i h_i(p_k)) + 2\pi \gamma(p_k, p_k) m^k_i| \leq C_0(\delta_0) \varepsilon_k^m - 2 - \delta
\end{equation}
where $\delta > 0$ is small positive number, $\Theta^k_i$ is the average of $\Theta^k_i$ on $M$, $C_0(\cdot)$ is a positive function such that $C_0(\delta_0) \to \infty$ as $\delta_0 \to 0$. To see this, let $x = p_k$ in (6.16),
\begin{align*}
\Theta^k_i(p_k) &= \Theta^k_i - \frac{1}{2\pi} \int_{B(p_k, \delta_0)} \log |p_k - \eta| \sum_j a_{ij} \rho^k_i h_j e^{\Theta^k_j} dV_g \\
&\quad + \int_{B(p_k, \delta_0)} \gamma(p_k, \eta) \sum_j a_{ij} \rho^k_i h_j e^{\Theta^k_j} dV_g \\
&\quad + \int_{M \setminus B(p_k, \delta_0)} G(p_k, \eta) \sum_j a_{ij} \rho^k_i h_j e^{\Theta^k_j} dV_g.
\end{align*}

Observe that
\begin{equation}
\rho^k_i h_j e^{\Theta^k_j} dV_g(\eta) = \rho^k_i h_j e^{f_i^j + \omega e^{\Theta^k_j} - f_i^j} d\eta
\end{equation}
Then by the definition of $\varepsilon_k$, $\tilde{v}^k_i$, $c^k_i$ we obtain
\begin{align*}
-\frac{1}{2\pi} \int_{B(p_k, \delta_0)} \log |p_k - \eta| \sum_j a_{ij} \rho^k_i h_j e^{\Theta^k_j} dV_g \\
&= -\frac{1}{2\pi} \int_{B(0, \delta_0 \varepsilon^{-1})} (\log(\varepsilon_k \eta_1)) \sum_j a_{ij} (\tilde{h}^k_i e^{\Theta^k_j}) (\varepsilon_k \eta_1) e^{\Theta^k_i (\eta) - \phi^k_j (\varepsilon_k \eta_1)} d\eta_1 \\
&= -\frac{1}{2\pi} \int_{B(0, \delta_0 \varepsilon^{-1})} (\log \varepsilon_k + \log |\eta_1|) \sum_j a_{ij} H^1_j (\eta) e^{\phi^k_j (\eta_1)} d\eta_1
\end{align*}
where, by the same kind of notations used in section 3 and section 4,
\[
v_{i}^{1,k}(\cdot) = v_{i}^{k}(p_{1,k} + \cdot) - \phi_{i}^{k}(\varepsilon_{k}p_{1,k} + \varepsilon_{k}\cdot)
\]
and
\[(6.21)\]
\[
H_{i}^{1,k}(\cdot) = \bar{h}_{i}^{k}(\varepsilon_{k}p_{1,k} + \varepsilon_{k}\cdot)e^{\phi_{i}^{k}(\varepsilon_{k}p_{1,k} + \varepsilon_{k}\cdot)}.
\]

By (6.7) we have
\[
H_{i}^{1,k}(0) = \rho_{i}^{k}h_{i}(p_{k}) + O(\varepsilon_{k}^{2}), \quad \nabla H_{i}^{1,k}(0) = \varepsilon_{k}\rho_{i}^{k}\nabla h_{i}(p_{k}) + O(\varepsilon_{k}^{3})
\]
and
\[
H_{i}^{1,k}(\eta) = \rho_{i}^{k}h_{i}(p_{k}) + \varepsilon_{k}\rho_{i}^{k}\nabla h_{i}(p_{k})\cdot\eta + O(\varepsilon_{k}^{2}(1 + |\eta|)^{2}.
\]
On the other hand by Proposition 4.1
\[
e^{\nu_{i}^{1,k}(\eta)} = e^{U_{i}^{k}(1 + \Phi_{i}^{k}(\eta_{1}) + O(\varepsilon_{k}^{2})(1 + |\eta|)^{4-m+\delta}).
\]
Hence
\[
(6.22) \quad H_{i}^{1,k}(\eta)e^{\nu_{i}^{1,k}(\eta)}
\]
\[
= (\rho_{i}^{k}h_{i}(p_{k}) + \varepsilon_{k}\rho_{i}^{k}\nabla h_{i}(p_{k})\cdot\eta + \Phi_{i}^{k}(\eta))e^{U_{i}^{k}(\eta)} + O(\varepsilon_{k}^{2}(1 + |\eta|)^{2-m+\delta}).
\]
Using (6.22) in the evaluation of (6.20) we have
\[
-\frac{1}{2\pi} \int_{B(p_{k},\delta_{0})} \log |p_{k} - \eta| \sum_{j} a_{ij} \rho_{j}^{k} h_{j} e^{\Theta_{j}^{k}} dV_{k}
\]
\[
= \frac{M_{k}}{2} m_{k}^{i} + \Theta_{i}^{k}(p_{k}) - M_{k} - c_{k}^{i} + \log(\rho_{i}^{k}h_{i}(p_{k})) + C_{0}(\delta_{0})(\varepsilon_{k}^{m-2-\delta}).
\]
Similarly
\[
\int_{B(p_{k},\delta_{0})} \gamma(p_{k},\eta) \sum_{j} a_{ij} \rho_{j}^{k} h_{j} e^{\Theta_{j}^{k}} dV_{k} = \gamma(p_{k},p_{k})2\pi m_{k}^{i} + C_{0}(\delta_{0})(\varepsilon_{k}^{m-2-\delta}).
\]
For the final term in (6.18) by the following crude estimate established in [38]
\[
\Theta_{i}^{k}(x) = -\frac{m_{k}^{i} - 2}{2}M_{k} + O(1), \quad x \in M \setminus B(p_{k},\delta_{0}), \quad O(1) \rightarrow \infty \text{ if } \delta_{0} \rightarrow 0
\]
it is easy to see
\[
\int_{M \setminus B(p_{k},\delta_{0})} G(p_{k},\eta) \sum_{j} a_{ij} \rho_{j}^{k} h_{j} e^{\Theta_{j}^{k}} dV_{k} = C_{0}(\delta_{0})O(\varepsilon_{k}^{m-2-\delta}).
\]
Thus, back to (6.18),
\[
\Theta_{i}^{k}(p_{k}) = \bar{\Theta}_{i}^{k} + \frac{M_{k}}{2} m_{k}^{i} + \Theta_{i}^{k}(p_{k}) - M_{k} - c_{k}^{i} + \log(\rho_{i}^{k}h_{i}(p_{k}))
\]
\[
+ \gamma(p_{k},p_{k})2\pi m_{k}^{i} + O(\varepsilon_{k}^{m-2-\delta}).
\]
Then (6.17) follows.

Next for \(x \in M \setminus B(p_{k},\delta_{0})\), by (6.16) and standard estimates
\[
(6.23) \quad \Theta_{i}^{k}(x) = \bar{\Theta}_{i}^{k} + 2\pi G(x,p_{k})m_{k}^{i} + C_{0}(\delta_{0})\varepsilon_{k}^{m-2-\delta}.
\]
Consequently by (6.23) and (6.17)

\[ e^{\Theta_k(x)} = e^{\bar{\Theta}_k} + 2\pi G(x, p_k) m_k + E_\delta, \quad x \in M \setminus \overline{B(p_k, \delta_0)} \]

where

\[ |E_\delta| \leq C_0(\delta_0) e^{m_k - 2 + \delta}. \quad (6.25) \]

In the sequel we shall always use $E_\delta$ to represent a term bounded by the right hand side of (6.25).

6.1. **Proof of Theorem 1.2** In this case $m < 4$. Since $\int_M h_1 e^{\Theta_k} dV_k = 1$ we write

\[ \rho_{ia}^k = \rho_{ia}^k h_i e^{\Theta_k} dV_k + \int_{M \setminus \overline{B(p_k, \delta_0)}} \rho_{ia}^k h_i e^{\Theta_k} dV_k = \rho_{ia}^k + \rho_{ib}^k. \]

By (6.19), (6.8)

\[ \rho_{ia}^k = \int_{B(0, \delta_0)} \tilde{h}_i^k e^{\tilde{\Theta}_i} d\eta = \int_{B(0, \delta_0 \varepsilon_k^{-1})} \tilde{h}_i^k (\varepsilon_k y) e^{\psi_i(y)} dy. \]

Let

\[ I_1 = \{ i \in I; \lim_{k \to \infty} m_k^i = \lim_{k \to \infty} m_k^i \}. \]

Using the expansion of $\tilde{v}_i^k$ in (6.15) we have (since $m < 4$)

\[ \rho_{ia}^k = \int_{B(0, \delta_0 \varepsilon_k^{-1})} \tilde{h}_i^k (0) e^{U_i^k(y)} dy + o(\delta_0) e^{m_k^i - 2}; \quad i \in I_1. \]

Now for $i \notin I_1$ we have

\[ \rho_{ia}^k = \int_{B(0, \delta_0 \varepsilon_k^{-1})} \tilde{h}_i^k (0) e^{U_i^k(y)} dy + E_\delta, \quad i \notin I_1 \]

and

\[ |\rho_{ib}^k| = E_\delta, \quad i \notin I_1. \]

It is easy to see from (6.12) and (6.7) that the $m$ defined by (6.12) is the same as the one in (1.6).

Combining (6.26), (6.27), (6.28) and Lemma 2.2 we have

\[ \sum_{i} \frac{1}{2\pi} \rho_{ia}^k - \sum_{ij} a_{ij} \frac{\rho_{ia}^k \rho_{ja}^k}{2\pi} \]

\[ = 2 \sum_{i \in I_1} e^{\tilde{\varepsilon}_0^k} \delta_0^{2 - m_k^i} e^{m_k^i - 2} + o(\delta_0) e^{m_k^{*2}} + E_\delta. \]
Using (6.27), (6.28) to change from $\rho^k_0$ to $\rho^k_i$,

\begin{equation}
\frac{4}{2\pi} \sum_i \rho^k_i = \sum_i \alpha_{ij} \frac{\rho^k_j}{2\pi} + 2 \sum_{i \in I_1} e^{i\delta_0^{2-m_i^k} \varepsilon_m^k - 2} - 2 \sum_{i \in I_1} (m_i^k - 2) \frac{\rho^k_i}{2\pi} + o(\delta_0) \varepsilon_m^k - 2 + E_{\delta_0}.
\end{equation}

Using (6.24) we have

\[ \rho^k_0 = \int_{M \setminus B(p_k, \delta_0)} \rho^k_i e^{\Theta_i} dV_g \]

\[ = \varepsilon_k^{m_i^k-2} \int_{M \setminus B(p_k, \delta_0)} \frac{h_i(x) e^{\Phi_i}}{h_i(p_k)} e^{2\pi n_i^k (G(x, p_k) - \gamma(p_k, p_k))} dV_g + E_{\delta_0}, \quad i \in I_1. \]

Combining terms we have

\[ \frac{4}{2\pi} \sum_i \rho^k_i - \sum_{i,j} \alpha_{ij} \frac{\rho^k_j}{2\pi} \]

\[ = 2 \sum_{i \in I_1} e^{i\delta_0^{2-m_i^k}} \left( \delta_0^{2-m_i^k} - \frac{m_i^k - 2}{2\pi} \int_{M \setminus B(p_k, \delta_0)} \frac{h_i(x)}{h_i(p_k)} e^{2\pi n_i^k (G(x, p_k) - \gamma(p_k, p_k))} dV_g \right) + o(\delta_0) \varepsilon_m^k - 2 + E_{\delta_0}.
\]

We claim that for fixed $k$ the following limit exists:

\begin{equation}
\lim_{\delta_0 \to 0} \left( \delta_0^{2-m_i^k} - \frac{m_i^k - 2}{2\pi} \int_{M \setminus B(p_k, \delta_0)} \frac{h_i(x)}{h_i(p_k)} e^{2\pi n_i^k (G(x, p_k) - \gamma(p_k, p_k))} dV_g \right)
\end{equation}

Indeed, write the second integral as the sum of one integral over $B(p_k, \delta_1) \setminus B(p_k, \delta_0)$ and the other over $M \setminus B(p_k, \delta_1)$, where $\delta_1$ is chosen small enough so that $\delta_0 = \delta_1$,

\[ dV_g = e^{\psi} d\psi \quad \text{where} \quad \psi(x) = O(|x|^2) \text{ in } B_{\delta_1}. \]

In the local coordinate at $p_k$, it suffices to prove

\[ \lim_{\delta_0 \to 0} \left( \delta_0^{2-m_i^k} - \frac{m_i^k - 2}{2\pi} \int_{B_{\delta_0} \setminus B_{\delta_0}} e^{2\pi n_i^k (G(p_k + x, p_k) - \gamma(p_k, p_k))} \frac{h_i(p_k + x)}{h_i(p_k)} dV_g(x) \right)
\]

exists. Since

\[ G(p_k + x, p_k) = -\frac{1}{2\pi} \log |x| + \gamma(p_k + x, p_k), \]

we use (6.32) and the Taylor expansions of $h_i$ and $\gamma(p_k + x, p_k) - \gamma(p_k, p_k)$ to obtain

\[ e^{2\pi n_i^k (G(p_k + x, p_k) - \gamma(p_k, p_k))} \frac{h_i(p_k + x)}{h_i(p_k)} dV_g = |x|^{-m_i^k} (1 + \frac{2}{3} c_i |x| + O(|x|^2)) dx. \]
where \( c_1, c_2 \) are some constants. Observe that the terms with \( c_1, c_2 \) disappear in the integration and \( m^k_i < 4 - \varepsilon_1 \) for some \( \varepsilon_1 > 0 \) for all \( i \in I_1 \), thus the limits in (6.31) exists.

The following lemma states the closeness between \( \rho^k \) to \( \rho \) if \( m < 4 \). It will be used to simplify the leading terms in the statements of main theorems.

**Lemma 6.2.** Let \( \rho^k \to \rho \in \Gamma_1 \) (\( \rho \neq Q \)) such that all \( \rho^k_i - \rho_i \) have the same sign, then

\[
(6.33) \quad \rho^k_i - \rho_i = O(e_k^{m - 2 - \varepsilon}) \quad \text{and} \quad m^k_i - m_i = O(e_k^{m - 2 - \varepsilon}).
\]

**Proof of Lemma 6.2.** Let \( s^k_i = \rho^k_i - \rho_i \). Then all \( s^k_i \) are of the same sign. By (6.30) we have

\[
\sum_i (m_i - 2)s^k_i = O(e_k^{m - 2 - \varepsilon}).
\]

Thus (6.33) holds and Lemma 6.2 is established. \( \square \)

Using (6.33) in (6.30) we can rewrite the leading term as

\[
\frac{4}{2\pi} \sum_i \rho^k_i - \sum_{i,j} (\rho^k_i \rho^k_j) = 2 \sum_{i \in I_1} \varepsilon_i e_k^{m - 2} \left( \delta_0 - \frac{m - 2}{2\pi} \int_{M \setminus B(p_k, \delta_0)} \frac{h_i(x)}{h_i(p_k)} e^{2\pi m (G(x, p_k) - \gamma(p_k, p_k))} dV \right) + o(\delta_0) e_k^{m - 2} + E_{\delta_0}.
\]

Since \( \varepsilon_i \to c_i \), the \( \varepsilon_i \) can be replaced by \( c_i \) in (6.14). Theorem 1.2 is established. \( \square \)

Next we establish the following lemma regardless of \( m = 4 \) or not.

**Lemma 6.3.**

\[
(6.34) \quad \nabla \phi^k_i(0) = 2\pi m^k_i \nabla \gamma(p_k, p_k) + O(e_k^{m - 2 - \varepsilon}).
\]

where \( \nabla \gamma \) means the differentiation with respect to the first component.

**Proof of Lemma 6.3:**

First from (6.23), for \( x \in \partial B(p_k, \delta_0) \)

\[
\tilde{\Theta}^k_i(x) = \Theta^k_i(x) - f^k_i(x)
\]

\[
= \tilde{\Theta}^k_i - m^k_i \log |x - p_k| + 2\pi m^k_i \gamma(x, p_k) - f^k_i(x) + O(e_k^{m - 2 - \varepsilon}).
\]

For the first derivative, by standard estimates

\[
|D^l(\tilde{\Theta}^k_i(x))| - D^l(-m^k_i \log |x - p_k| + 2\pi m^k_i \gamma(x, p_k) - f^k_i(x))| \leq C e_k^{m - 2 - \varepsilon}, \quad l = 0, 1
\]

for \( x \in B(p_k, \delta_0) \). On the other hand, we recall that \( \tilde{v}^k_i(y) = \tilde{\Theta}^k_i(\varepsilon_k y) + 2\log \varepsilon_k \). According to Proposition 4.1

\[
|D^l(\tilde{v}^k_i(y)) - U^l_i(y - p_{1,k}) - \Phi^k_i(\varepsilon_k y) - \Phi^k_i(y - p_{1,k})| \leq C e_k^2 (1 + |y|)^{4 - m + \varepsilon - l}
\]


for $l = 0, 1$ and $|y| \leq 2\delta_0\varepsilon_k^{-1}$. Using the asymptotic behavior $U^k_l$ for $|y| \sim \varepsilon_k^{-1}$ we have

$$U^k_l(y - p_{1,k}) + \log(p^k_r h(p_k)) = -\frac{m^k}{2}k^{-l}M_k - m^k_l \log|x| + c^k_l + O(\varepsilon_k^{m^k - 2})$$

where $|x| = \varepsilon_k|y|$. Thus by (6.17) for $l = 0$ we have (6.35)

$$|\phi^k_i(x)| = 2\pi m^k \gamma(x, p_k) - \gamma(p_k, p_k) - f^k_i(x) + O(\varepsilon_k^{2+\varepsilon}), \quad x \in B(p_k, 2\delta_0) \setminus B(p_k, \delta_0/2).$$

Thus the comparison on $l = 1$ and (6.5) yield (6.34). Lemma 6.3 is established.

6.2. Proof of Theorem 1.3. In this case $m = 4$, we first give a rough estimate of $\rho^k - \rho_i$.

**Lemma 6.4.** Let $\rho^k$ tend to $Q$ such that all $p^k_i - Q_i$ have the same sign. Then

$$m^k - 4 = O(\varepsilon_k^{2+\varepsilon}), \quad \rho^k_i - \rho_i = O(\varepsilon_k^{2+\varepsilon}) \quad \forall i \in I.$$

**Proof of Lemma 6.4:** Since $\rho^k \to Q$, $m = 4$, then all $m^k_i \to 4$. Recall that $\rho^k_i = 2\pi \sigma_i^k + O(\varepsilon_k^{2+\varepsilon})$, The rest of the proof is the same as that in the proof of Lemma 6.2. Lemma 6.4 is established.

By (5.2)

$$\sum_{i} (\partial_i \log h^k_i(0) + \partial_i \phi^k_i(p_k)) \bar{\sigma}^k_i = O(\varepsilon_k^2 \log \varepsilon_k^{-1}),$$

then by (6.7)

$$\sum_{i=1}^{n} \left( \partial_i (\log h_i + \phi^k_i(p_k)) \right) \bar{\sigma}^k_i = O(\varepsilon_k^2) \log \varepsilon_k^{-1}, \quad l = 1, 2.$$

The distance from $\rho^k$ to $\Gamma_1$ can be computed as follows:

$$\rho^k_i = \int_{B(p_k, \delta_0)} p^k_i h_i e^{\Theta_i} dV_g + \int_{\partial B(p_k, \delta_0)} \rho^k_i h_i e^{\Theta_i} dV_g.$$

By (6.36) and Theorem 4.3 the second integral is $O(\varepsilon_k^2)$, this is the same as the computation for the single equation [12]. Therefore

$$\rho^k_i = \int_{B(p_k, \delta_0)} p^k_i h_i e^{\Theta_i} dV_g + O(\varepsilon_k^2)$$

$$= \int_{B(0, \delta_0)} \tilde{h}^k_i e^{\phi^k_i} e^{\Theta_i - \Theta^k_i} d\eta + O(\varepsilon_k^2)$$

$$= \int_{B(0, \delta_0) \varepsilon_k^{-1}} H^k_1(\eta) e^{\phi^k_i(\eta)} d\eta + O(\varepsilon_k^2)$$

$$= \int_{B(0, \delta_0) \varepsilon_k^{-1}} \rho^k_i h_i(p_k) e^{U^k_i(\eta)} d\eta + \int_{B(0, \delta_0) \varepsilon_k^{-1}} \frac{1}{4} \Delta H^k_1(0) |\eta|^2 e^{U^k_i(\eta)} d\eta$$

$$+ O(\varepsilon_k^2).$$
The first integral on the right hand side of the above is $2\pi \sigma_i^k + O(\varepsilon_k^2)$. To evaluate the second term on the right hand side, we first use the definition of the $H_i^{1,k}$ in (6.21), (6.36), (6.7) and (6.34) to have

$$
(6.39) \quad H_i^{1,k}(0) = \rho_i^k h_i(p_k) \left( \frac{\Delta h_i(p_k)}{h_i(p_k)} - 2K(p_k) + 8\pi \sum \nabla h_i(p_k) \cdot \nabla_1 \gamma(p_k,p_k) + 64\pi^2 |\nabla_1 \gamma(p_k,p_k)|^2 \right) \varepsilon_k^2 + O(\varepsilon_k^{4-\varepsilon}).
$$

For $e^{U_i^k}$ we use the definition of $c_i^k$ in (6.13) and (2.18) to have

$$
(6.40) \quad \rho_i^k h_i(p_k) e^{U_i^k(\eta)} = e^{\varepsilon_k |\eta| - 4} + O(|\eta|^{-4-\delta}), \quad |\eta| > 1
$$

for some $\delta_0 > 0$ independent of $k$. Using (6.39) and (6.40) in the computation of the second integral of $\rho_i^k$ we have

$$
\rho_i^k = 2\pi \sigma_i^k + 2\pi \varepsilon_k^2 \log \varepsilon_k^{-1} b_i^k e^{\varepsilon_k} + O(\varepsilon_k^2).
$$

where

$$
b_i^k = \frac{1}{4} \left( \frac{\Delta h_i(p_k)}{h_i(p_k)} - 2K(p_k) + 8\pi + 16\pi \frac{\nabla h_i(p_k)}{h_i(p_k)} \nabla_1 \gamma(p_k,p_k) + 64\pi^2 |\nabla_1 \gamma(p_k,p_k)|^2 \right)
$$

Consequently,

$$
(6.41) \quad 4 \sum_i \frac{\rho_i^k}{2\pi} - \sum_i \frac{\rho_i^k \rho_j^k}{2\pi} \frac{a_{ij}}{2\pi} = \varepsilon_k^2 \log \varepsilon_k^{-1} (\sum_i (4 - 2m_i^k) b_i^k e^{\varepsilon_k}) + O(\varepsilon_k^2).
$$

Theorem 1.3 is established. □

**Remark 6.1.** Even though there is a cut-off function in the definition of $\chi$, changing the domain of this cut-off function does not change $\nabla_1 \gamma(p_k,p_k)$.

**Proof of Theorem 1.1.**

By going through the proof of Lemma 6.3 using Theorem 1.2 for $\rho \neq Q$ or Theorem 1.3 for $\rho = Q$ instead of Proposition 4.1 one sees easily that

$$
\phi_i^k(x) = 2\pi m_i^k (\gamma(p_k + x, p_k) - \gamma(p_k, p_k)) - c_i^k - f_i^k(x) + O(\varepsilon_k^{m_i^k-2})
$$

if $\rho \neq Q$. On the other hand

$$
\phi_i^k(x) = 2\pi m_i^k (\gamma(p_k + x, p_k) - \gamma(p_k, p_k)) - c_i^k - f_i^k(x) + O(\varepsilon_k^2 \log \varepsilon_k^{-1})
$$

if $\rho = Q$. Correspondingly

$$
(6.42) \quad \nabla \phi_i^k(0) = 2\pi m_i^k \nabla_1 \gamma(p_k, p_k) + O(\varepsilon_k^{m_i^k-2})
$$
if $\rho \neq Q$ and
(6.43) \[ \nabla \phi_k^i(0) = 2\pi m_k^i \nabla \gamma(p_k, p_k) + O(\varepsilon_k^2 \log \varepsilon_k^{-1}) \]
if $\rho = Q$. Theorem 5.1 yields
\[
\left| \sum_i \left( \partial_l (\log h_k^i)(0) + \partial_l \phi_l^i(0) \right) \sigma_{k i}^l \right| \leq C \varepsilon_k^{m_k^i - 2} \quad \text{if } \rho \neq Q
\]
and
\[
\left| \sum_i \left( \partial_l (\log h_k^i)(0) + \partial_l \phi_l^i(0) \right) \sigma_{k i}^l \right| \leq C \varepsilon_k^2 \log \varepsilon_k^{-1} \quad \text{if } \rho = Q.
\]
Also the proofs of Theorem 1.2 and Theorem 1.3 give
\[
\rho_k^i - \rho_i = O(\varepsilon_k^{m_k^i - 2}) = O(\varepsilon_k^{m_k^i - 2}), \quad m < 4
\]
and
\[
\rho_k^i - \rho_i = O(\varepsilon_k^{m_k^i - 2}) \log \varepsilon_k^{-1} = O(\varepsilon_k^2 \log \varepsilon_k^{-1}), \quad m = 4.
\]
So
\[
\bar{\sigma}_k^i - \rho_i \ \frac{2\pi}{2\pi} = \begin{cases} O(\varepsilon_k^{m_k^i - 2}) , & m < 4, \\ O(\varepsilon_k^2 \log \varepsilon_k^{-1}) , & m = 4. \end{cases}
\]
By (6.42) and (6.43) we obtain (1.11) and (1.12) in Theorem 1.1. Theorem 1.1 is established \(\square\)

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DEPARTMENT OF MATHEMATICS, TAIWA N INSTITUTE OF MATHEMATICAL SCIENCES, NATIONAL TAIWAN UNIVERSITY, TAIPEI 106, TAIWAN

E-mail address: cslin@math.ntu.edu.tw

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, 358 Little Hall P.O.Box 118105, GAINESVILLE FL 32611-8105

E-mail address: leizhang@ufl.edu