The homogeneous flow of a parallelizable manifold

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Abstract

Motivated by the Hamilton’s Ricci flow, we define the homogeneous flow of a parallelizable manifold and show the long time existence and uniqueness of its solutions on $[0, \infty)$. Using this flow, we outline a simple proof of the Poincare Conjecture.

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1 Introduction

This note is the result of our efforts to justify the significance and efficiency of the generalization of the Klein’s Erlangen Program proposed in [8] based on our earlier work in arXiv. A parallel theory is proposed in [2]. Our main purpose here is to use the Poincare Conjecture (PC) as a means for this justification.

The central concept in the framework of [8] is that of a prehomogeneous geometry (phg) and its curvature. The order of a phg is the order of jets involved in its definition. The curvature is the obstruction to the local homogeneity of the phg. In this note we are interested in the simplest phg of order zero, i.e., a parallelizable manifold $(M, \varepsilon)$ where $\varepsilon$ denotes the parallelization. If the curvature $\mathcal{R}(\varepsilon)$ vanishes, $M$ becomes locally homogeneous in two ways and is called a local Lie group in [1]. If $M$ is also simply connected and $\varepsilon$ is complete, then $M$ is the homogeneous space of two global and simply transitive transformation groups which correspond to the left-right actions of a Lie group. Section 2 contains a concise exposition of this theory with more details than in [1] on certain points, also clarifying certain ambiguities in [1]. It is worth stressing here that
the theory of local Lie groups is not a simple consequence of the present global theory but has its own set of interesting and delicate geometric structures as stated in [5] which deeply inspired our work. For instance, a local Lie group in this sense does not always imbed in a global Lie group ([5]). In fact, it is shown in [1] that the opposite is true: a Lie group is a special (globalizable) local Lie group! Therefore, in the words of [6], Section 2 "reinstates the paradigm of local to global to its historical record".

In Section 3 we define the homogeneous flow (HF) of a parallelizable manifold which is inspired by the Ricci flow of Hamilton. We show that HF is weakly parabolic. Using the DeTurck trick [3], we show that HF is equivalent to a strongly parabolic flow thus establishing the existence and uniqueness of the short time solutions of HF.

Semigroups supply an important tool in the study of evolution equations. Using this idea, we show in Section 4 by a very simple argument that the short time solutions of HF extend uniquely to long time solutions on [0, ∞). The key fact is the use of the gauge group of the parallelizable manifold which is an infinite dimensional Frechet Lie group in the sense of [7] with a locally bijective exponential map. Further, this group acts simply transitively on the set of all parallelisms and its 1-parameter subgroups extend the short time solutions to [0, ∞). This fact shows that HF does not develop any finite time singularities which is a serious difficulty with the Ricci flow.

We outline in Section 5 how PC can be derived from the convergence of HF as $t \to \infty$ on a compact and simply connected 3-manifold. The level of the technical difficulty of this derivation seems to be less than the seminal result of Hamilton that the normalized Ricci flow starting with a metric with positive Ricci curvature converges to a metric with constant positive sectional curvature on a compact and simply connected 3-manifold ([4]).

In the Appendix we comment on the relation between HF and the Ricci flow on a parallelizable manifold.

Finally, it is worth stressing here that HF is defined for any phg (in particular for a Riemannian geometry as a phg of order one), the key fact being that the top principal bundle defined by the phg is parallelizable ([8]). The curvature of a Riemannian geometry as a phg vanishes if and only if the underlying metric has constant sectional curvature (which is equivalent to local homogeneity. See page 6 of [2] for a simple formula for this curvature).

2 Parallelizable manifolds and local Lie groups

Let $M$ be a smooth manifold with dim $M \geq 2$ and $U_k(M)$ (shortly $U_k$) be the universal groupoid of order $k$ on $M$. The elements of $U_k$ are the $k$-jets of local diffeomorphisms of $M$. We call an element of $U_k$ with source at $p$ and target at $q$ a $k$-arrow from $p$ to $q$ and denote it by $j_k(f)^{p,q}$. Therefore $U_0 = M \times M$ is the pair groupoid. The relevant universal groupoids in this section are $U_0$ and $U_1$. The projection homomorphism $\pi : U_1 \to U_0$ of groupoids maps a 1-arrow from $p$ to $q$ to the pair $(p, q)$. A splitting $\varepsilon : U_0 \to U_1$ is a homomorphism of
groupoids so that \( \pi \circ \varepsilon = id_{U_0} \). Thus \( \varepsilon \) assigns to any pair \( (p, q) \) a unique 1-arrow from \( p \) to \( q \) and this assignment preserves the composition and inversions of arrows. We easily check that \( \pi : U_1 \to U_0 \) admits a splitting if and only if \( M \) is parallelizable. If \( p \in (U, x^i) \) has coordinates \( x^i \) and \( q \in (V, y^i) \) has coordinates \( y^i \), then \( \varepsilon(p, q) \) has the local representation \( \varepsilon_j^i(x, y) = \varepsilon_j^i(x, y), \quad 1 \leq i, j \leq n = \dim M. \) Thus we have the coordinate formulas

\[
\begin{align*}
\varepsilon_j^i(z, y)\varepsilon_j^a(x, z) &= \varepsilon_j^i(x, y) \\
\varepsilon_j^i(x, x) &= \delta_j^i \\
\varepsilon_j^i(y, x)\varepsilon_j^a(x, y) &= \delta_j^i
\end{align*}
\]

We use summation convention in (1). In this section we fix the splitting \( \varepsilon \) once and for all and let \( (M, \varepsilon) \) denote the parallelizable manifold \( M \).

Now we consider the first order nonlinear PDE

\[
\frac{\partial f^i(x)}{\partial x^j} = \varepsilon_j^i(x, f(x))
\]

for some local diffeomorphism \( y^i = f^i(x) \). The integrability conditions of (2) are given by

\[
R_{jk}(x, y) \overset{\text{def}}{=} \left[ \frac{\partial \varepsilon_j^i(x, y)}{\partial x^3} + \frac{\partial \varepsilon_k^i(x, y)}{\partial y^3} \varepsilon_j^a(x, y) \right]_{[jk]} = 0
\]

where \([jk]\) denotes the alternation of the indices \( j, k \). We have \( R(p, q) \in \wedge^2 T_p^* \otimes T_q \). If (3) admits a solution \( f \) with \( f(p) = q \) for any \( (p, q) \in U \times V \), then clearly \( R = 0 \) on \( U \times V \). Conversely, by the well known existence and uniqueness theorem for the first order systems of PDE’s, if \( R = 0 \) on \( U \times V \), then we may assign any pair \( (p, q) \in U \times V \) as initial condition and solve (2) uniquely for some \( f \) defined on \( U \subset U \) satisfying \( f(p) = q \). Further, \( j_1(f)^{x, f(x)} \in \varepsilon(U_0) \) for all \( x \in U \) and we may choose \( U = U \) if \( U \) is simply connected. Note that \( R(p, p) = 0 \) for all \( p \in M \).

**Definition 1** \( R \) is the groupoid curvature of \((M, \varepsilon)\) and \((M, \varepsilon)\) is locally homogeneous (or a local Lie group) if \( R = 0 \) on \( M \times M \).

To justify the term local homogeneity, we assume \( R = 0 \) and let \( S \) denote the set of all local solutions of (2). Since \( \varepsilon \) is a homomorphism of groupoids, \( S \) is easily seen to be a pseudogroup. Some \( f \in S \) is determined on its domain by any of its 0-arrows \( (p, f(p)) \). Now let \( f \in S \) be defined on \( U, p \in U \) and \( C \) a path from \( p \) to some \( q \in M \). We can "analytically continue" \( f \) along \( C \) but may not be able to "reach" \( q \). We call \((M, \varepsilon)\) complete if all elements of \( S \) can be continued indefinitely along all paths in \( M \). Note that we define the completeness of \((M, \varepsilon)\) only when \( R = 0 \) (at least here). Assuming completeness, two paths from \( p \) to \( q \) may give different values at \( q \) if these paths are not homotopic. However, the standard monodromy argument shows that we get the same values at \( q \) if these
paths are homotopic. In particular, if $M$ is simply connected, we easily see that any $f \in S$ extends to a global diffeomorphism of $M$. Further, these global transformations are closed under composition and inversion and therefore they form a global transformation group of $M$ which acts simply transitively. We continue to denote this transformation group by $S$ and call $S$ globalizable (as a pseudogroup). Note that $S$ may be globalizable without $M$ being simply connected but $S$ is of course complete if it is globalizable and what we have shown above is that completeness together with simple connectedness implies globalizability. If $(M, \varepsilon)$ is complete but not globalizable, then we can lift $S$ to a pseudogroup $S'$ on the universal cover $M'$ of $M$ and globalize $S'$ on $M'$ such that the covering transformations form a discontinuous subgroup of $S'$ isomorphic to the fundamental group of $M$.

If $\mathcal{R} = 0$, there is another pseudogroup on $M$ defined as follows. Let $f(a, b, z)$ denote the unique local solution of (2) in the variable $z$ satisfying the initial condition $a \to b$. We fix some $p, q \in (U, x^i)$ and define

$$\tilde{\varepsilon}^i_j(p, q) \stackrel{def}{=} \left( \frac{\partial f^i(p, x, q)}{\partial x^j} \right)_{x=p}$$

(4)

Note that $\tilde{\varepsilon}(p, q)$ is defined for close $p, q$ unless $S$ is globalizable. We check that $\tilde{\varepsilon}$ is a local splitting of $\pi : U_1 \to U_0$. Therefore we can replace (2) by

$$\frac{\partial h^i(x)}{\partial x^j} = \tilde{\varepsilon}^i_j(x, h(x))$$

(5)

Now the local diffeomorphism $h : x \to f(p, x, q)$ satisfies $h(p) = q$ and solves (5). In particular the integrability conditions of (5) are satisfied. Thus we get a pseudogroup $\tilde{S}$ in the same way we get $S$. The only difference is that $\tilde{S}$ is locally transitive whereas $S$ is globally transitive. The elements of $S$ and $\tilde{S}$ commute whenever their compositions are defined. If $S$ globalizes, then so does $\tilde{S}$ in which case we get two global commuting transformation groups of $M$. Now it is easy to construct an abstract Lie group $G$ whose underlying manifold is $M$ and its left-right (or right-left) translations can be identified with $S$ and $\tilde{S}$. However, note that there is no such canonical identification!

Up to now we assumed $\mathcal{R} = 0$ and dealt with the "Lie group". Now we drop the assumption $\mathcal{R} = 0$ and consider the parallelizable manifold $(M, \varepsilon)$ our purpose being to construct the "Lie algebra".

We define

$$\Gamma^i_{jk}(x) \stackrel{def}{=} \left( \frac{\partial \varepsilon^i_k(x, y)}{\partial y^j} \right)_{y=x}$$

(6)

It is extremely crucial that $\Gamma^i_{jk}(x)$ need not be symmetric in $j, k$. Differentiating the third formula in (1) with respect to $x$ at $y = x$ gives

$$\left( \frac{\partial \varepsilon^i_k(x, y)}{\partial x^j} \right)_{y=x} = -\Gamma^i_{jk}(x)$$

(7)
The 1-arrow \( \varepsilon(p, q) \) induces an isomorphism \( \varepsilon(p, q)_* \) of the tangent spaces \( \varepsilon(p, q)_*: T_p \to T_q \) which extends to an isomorphism \( \varepsilon(p, q)_*: (T^m_r)_p \to (T^m_r)_q \) of the tensor spaces. A tensor field \( t \) is \( \varepsilon \)-parallel if \( \varepsilon(p, q)_*t(p) = t(q) \) for all \( p, q \in M \). Thus an \( \varepsilon \)-parallel \( t \) is globally determined by its value at any point. For instance, the tensor \( (t^i_j) \) is \( \varepsilon \)-parallel if and only if

\[
t^i_j(x) = \varepsilon^i_a(p, x)t^a_j(p, p) \varepsilon^b_j(x, p)
\]

for any fixed but arbitrary \( p \) and all \( x \). Differentiating (8) with respect to \( x \) at \( x = p \), substituting from (6), (7) and omitting \( p \) from our notation, we get

\[
\nabla_j t^i_j \overset{\text{def}}{=} \frac{\partial t^i_j}{\partial x^r} - \Gamma^i_{ra} t^a_j + \Gamma^a_{rj} t^i_a = 0
\]

The order of the indices is quite relevant in (11). If \( \tilde{\mathfrak{R}} = 0 \) is identically satisfied on \( M \), then for any initial condition \( \xi^i(p) \) at some \( p \in M \), we have a unique solution \( \xi^i(x) \) of (11) around \( p \) satisfying this initial condition. However, \( \xi^i(p) \) determines an \( \varepsilon \)-parallel vector field which is known to solve (11) on \( M \). By uniqueness, the unique solution \( \xi^i(x) \) is the restriction of an \( \varepsilon \)-parallel vector field and therefore \( \nabla t = 0 \) implies that \( t \) is \( \varepsilon \)-parallel if \( t \) is a vector field. In particular, we observe that we always have \( \tilde{\mathfrak{R}} = 0 \) on a parallelizable manifold \((M, \varepsilon)\). Let \( \mathfrak{X}(M) \) denote the Lie algebra of vector fields on \( M \) and \( \mathfrak{X}_{\varepsilon}(M) \subset \mathfrak{X}(M) \) denote the subspace of \( \varepsilon \)-invariant vector fields. We conclude that some \( \xi \in \mathfrak{X}(M) \) belongs to \( \mathfrak{X}_{\varepsilon}(M) \) if and only if it solves (11) on \( M \). Now the integrability conditions of (10) for an arbitrary tensor field \( t \) is an expression in terms of \( \tilde{\mathfrak{R}} \) well known from tensor calculus. Therefore these conditions are identically satisfied since \( \tilde{\mathfrak{R}} = 0 \) and we deduce the desired implication for any tensor by a similar reasoning.

Clearly \( \dim \mathfrak{X}_{\varepsilon}(M) = \dim M \). However \( \mathfrak{X}_{\varepsilon}(M) \) need not be a Lie algebra, i.e., the bracket of two \( \varepsilon \)-parallel vector fields need not be \( \varepsilon \)-parallel. We define

\[
\nabla_j \xi^i \overset{\text{def}}{=} \frac{\partial \xi^i}{\partial x^r} - \Gamma^i_{ar} \xi^a
\]
and extend \( \tilde{\nabla} \) to all tensor fields. Note that \( r \) is now the second index in \( \Gamma^r_{\cdot \cdot \cdot} \) in (12). Assuming \( R = 0 \), we check

\[
\left( \frac{\partial \tilde{\varepsilon}^j_i}{\partial y^k} \right)_{y=x} = \Gamma^i_{jk}
\]

(13)

So if we define \( \tilde{\Gamma}^i_{kj} \) by the LHS of (13) as in (6), we get \( \tilde{\Gamma}^i_{kj} = \Gamma^i_{jk} \). Recalling that \( \tilde{\varepsilon} \) is defined only if \( R = 0 \), it is a remarkable fact that \( \tilde{\nabla} \) is defined without the assumption \( R = 0 \).

The integrability conditions of \( \tilde{\nabla}_r \xi^i = 0 \) are given by

\[
\mathcal{R}^i_{rj,k} \overset{\text{def}}{=} \left[ \frac{\partial \tilde{\Gamma}^i_{kj}}{\partial x^r} + \Gamma^q_{kr} \Gamma^i_{aj} \right]_{[rj]} = 0
\]

(14)

**Definition 2** \( \mathcal{R} \) is the algebroid curvature of the parallelizable manifold \( (M, \varepsilon) \).

The following important proposition whose proof follows easily from definitions (like all other facts in this section, except Proposition 4 below) clarifies the geometric meaning of \( \mathcal{R} \).

**Proposition 3** \( \mathfrak{X}_\varepsilon(M) \) is a Lie algebra if and only if \( \mathcal{R} = 0 \). In this case, \( \mathfrak{X}_{\tilde{\varepsilon}}(M) \) is also a Lie algebra (\( \tilde{\varepsilon} \) is defined since \( R = 0 \) by Proposition 4) and the vector fields of \( \mathfrak{X}_\varepsilon(M) \) and \( \mathfrak{X}_{\tilde{\varepsilon}}(M) \) commute.

Equation (12) is obtained from (2) by a "linearization" process whose meaning will be clear shortly. In principle this process is the passage from a groupoid to its algebroid. This formalism can be avoided in our simple case of parallelizable manifolds but becomes indispensible for general phg's. In the same way, \( \mathcal{R} \) is obtained from \( R \) by the same linearization: substituting \( y^i = x^i + t\xi^i \) into \( R^{i}_{rj}(x, y) \) and differentiating with respect to \( t \) at \( t = 0 \) gives \( \mathcal{R}^m_{rj,a} \xi^a \). In particular, \( R = 0 \) implies \( \mathcal{R} = 0 \).

Now we have the following fundamental

**Proposition 4** \( R = 0 \iff \mathcal{R} = 0 \)

The implication \( \Rightarrow \) states that "the Lie group has a Lie algebra" whereas the nontrivial \( \Leftarrow \) asserts that "the Lie algebra has a Lie group" which is Lie's 3rd Fundamental Theorem.

Now suppose \( \mathcal{R} = 0 \) so that \( (M, \varepsilon) \) is locally homogeneous. The Lie algebra \( \mathfrak{X}_\varepsilon(M) \) integrates to the pseudogroup \( S \), i.e., \( \mathfrak{X}_\varepsilon(M) \) is the Lie algebra of the infinitesimal generators of the action of \( S \). Similarly, the Lie algebra \( \mathfrak{X}_{\tilde{\varepsilon}}(M) \) integrates to the pseudogroup \( \tilde{S} \), in analogy with the familiar fact from Lie groups that the left (right) invariant vector fields integrate to the right (left) actions. Recall, however, that there is no canonical identification even if \( S \) is globalizable.
Now we define the fundamental object

\[ T_{jk}^i \overset{\text{def}}{=} \Gamma^{ij}_{jk} - \Gamma^{ij}_{kj} \tag{15} \]

We have

\[ \nabla_r \xi^i = \tilde{\nabla}_r \xi^i + T^i_{ra} \xi^a \tag{16} \]

and (16) easily generalizes to all tensor fields. The next proposition gives the first hint that \( T \) dominates the whole theory.

**Proposition 5**

\[ \nabla_r T^i_{jk} = R^i_{jk,r} \tag{17} \]

It follows that \( R \) is determined by \( T \) and \( R = 0 \) if and only if \( T \) is \( \varepsilon \)-parallel!

To clarify the meaning of \( T \) further, let \( \xi, \eta \in \mathfrak{X}(M) \). We define the torsion bracket \( T(\xi, \eta) \in \mathfrak{X}(M) \) by

\[ T(\xi, \eta)^i \overset{\text{def}}{=} T^{i}_{ab} \xi^a \eta^b \tag{18} \]

and the Jacobi 3-form by

\[ J(\xi, \eta, \sigma) \overset{\text{def}}{=} T(\xi, T(\eta, \sigma)) + T(\eta, T(\sigma, \xi)) + T(\sigma, T(\xi, \eta)) \tag{19} \]

**Proposition 6 (The First Bianchi Identity)** Let \((M, \varepsilon)\) be parallelizable. Then

\[ \nabla_\xi T(\eta, \sigma) + \nabla_\eta T(\sigma, \xi) + \nabla_\sigma T(\eta, \sigma) = R(\eta, \sigma)(\xi) + R(\sigma, \xi)(\eta) + R(\xi, \eta)(\sigma) \tag{20} \]

In particular, if \( R = 0 \), then \( J = 0 \). In this case, \( T(\xi, \eta) = [\xi, \eta] \) for \( \xi, \eta \in \mathfrak{X}(M) \), which explains (15) to some extent.

To finish this section, we recall that tensor calculus originated from Riemannian geometry as an attempt to formalize Riemann’s ideas. We hope to have convinced the reader that tensor calculus (which we barely touched in this section) could have originated also from Lie theory....and if this had happened, the concepts of torsion and curvature would have quite different meanings today. We hope that the next section, where the above formulas will be used in an essential way, will give further support to this view.

### 3 The homogeneous flow of a parallelizable manifold

We recall the universal groupoid \( \mathcal{U}_1 \) and the subgroupoid \( \varepsilon(\mathcal{U}_0) = \varepsilon(M \times M) \subset \mathcal{U}_1 \). We fix some basepoint \( \mathfrak{e} \in M \) and consider the principal bundle \( \varepsilon(\mathfrak{e} \times M) \)
whose structure group is trivial as it is the 1-arrow of the identity map with source and target at \( \mathfrak{g} \). We fix some coordinates around \( 0 \) once and for all.

We define a geometric object on \( M \) whose components on \((U, x^i)\) are \( \varepsilon_{ij}^a(\mathfrak{g}, x) \).

Now (1) gives

\[
\varepsilon_i^a(x, y)\varepsilon_j^a(\mathfrak{g}, x) = \varepsilon_j^a(\mathfrak{g}, y)
\]

(21) asserts that \( \varepsilon(M \times M) \) consists of those 1-arrows in \( \mathcal{U}_1 \) which preserve the geometric object \( \varepsilon_{ij}^a(\mathfrak{g}, x) \). In view of (21), a change of coordinates \((U, x^i) \to (V, y^i)\) transforms \( \varepsilon_{ij}^a(\mathfrak{g}, x) \) by

\[
\frac{\partial y^i}{\partial x^a} \varepsilon_j^a(\mathfrak{g}, x) = \varepsilon_j^a(\mathfrak{g}, y)
\]

(22) shows that \( \varepsilon_{ij}^a(\mathfrak{g}, y) \) transforms only in the index \( i \) but not in the index \( j \). We call \( i \) the coordinate index and \( j \) the \( \mathbb{R}^n \) index. We also define the dual object \( \varepsilon_{ij}^x(\mathfrak{g}, x) \) with \( \varepsilon_{ij}^x(\mathfrak{g}, x) \varepsilon_{kl}^y(\mathfrak{g}, x) = \varepsilon_{kl}^y(\mathfrak{g}, x) \varepsilon_{ij}^x(\mathfrak{g}, x) = \delta_{ij}^y \) where \( i \) is the \( \mathbb{R}^n \) index and \( j \) is the coordinate index. If \( \mathfrak{g} = 0 \), it is an amusing fact to check that \( \varepsilon_{ij}^x(\mathfrak{g}, \mathfrak{g}) \) becomes the Maurer-Cartan form (see (51) in [1]).

Now differentiating (21) with respect to \( y \) at \( y = x \) and substituting from (6) gives

\[
\Gamma^i_{jk}(x) = \varepsilon_{ij}^a(x, 0) \varepsilon_k^a(\mathfrak{g}, x) - \varepsilon_{ij}^a(x, 0) \Gamma^a_{ra}(x) \varepsilon_r^a(\mathfrak{g}, x) = 0
\]

(23) shows that the RHS of (23) is independent of the choice of the base point \( \mathfrak{g} \). Now we identify \( \mathfrak{g} \) with the origin \( \mathfrak{o} \) in \( \mathbb{R}^n \) and (23) shows that we can define \( \Gamma^i_{jk}(x) \) consistently on the principal bundle \( \varepsilon(\mathfrak{o} \times M) \). This identification will be useful in Section 4. Note that the principal bundle \( \varepsilon(\mathfrak{o} \times M) \) determines the groupoid \( \varepsilon(M \times M) \) since \( \varepsilon(p, q) = \varepsilon(p, \mathfrak{o}) \circ \varepsilon(\mathfrak{o}, q) \).

We rewrite (23) as

\[
\nabla_r \varepsilon_{ij}^x(\mathfrak{o}, x) = \frac{\varepsilon_{ij}^x(\mathfrak{o}, x)}{\partial x^r} - \Gamma^i_{ra}(x) \varepsilon_r^a(\mathfrak{o}, x) = 0
\]

(24)

Similarly we have \( \nabla_r \varepsilon_{ij}^x(\mathfrak{o}, x) = 0 \) keeping in mind that we always differentiate with respect to the coordinate indices.

Now we define the geometric object \( \mathfrak{g} \) by defining its components \( g_{ij}(x) \) on \((U, x^i)\) by

\[
g_{ij}(x) \overset{\text{def}}{=} \varepsilon_{ij}^x(\mathfrak{o}, x) \varepsilon_{ij}^y(\mathfrak{o}, x) = \varepsilon_{ij}^x(\mathfrak{o}, x) \left( \delta_{ab} \varepsilon_{ij}^y(\mathfrak{o}, x) \right)
\]

(25)

where we identify \( \mathbb{R}^n \) with \( (\mathbb{R}^n)^* \) by the canonical metric \( \delta_{ij} \). Clearly \( g_{ij} \) is symmetric. It is also positive definite since the matrix \( \varepsilon_{ij}^x(\mathfrak{o}, x) \) is invertible. From (24) and (26) we deduce

\[
\nabla_r g_{ij} = 0
\]

(26)

**Definition 7** \( g \) is the canonical metric of the parallelizable manifold \((M, \varepsilon)\).
Let $t_{km}^{ij}$ be a tensor field. We define

$$t_{km}^{i(j)}(x) \overset{def}{=} \varepsilon^a_j(x, o) t_{km}^{ia}(x)$$  \hspace{1cm} (27)

Now $t_{km}^{i(j)}(x)$ does not transform in the index $j$. We say that the tensor $t_{km}^{i(j)}(x)$ is obtained from $t_{km}^{ij}(x)$ by moving the index $j$ to $o$. Similarly we can move the index, say, $k$ to $o$ using $\varepsilon^i_j(o, x)$ and this operation extends to all tensors in an obvious way. With an abuse of notation we will also move a covariant or contravariant $\mathbb{R}^n$ index $j$ to the coordinate index $(j)$ as in (33) below.

Now we define

$$\mathfrak{H}^j_i(\varepsilon) \overset{def}{=} -\varepsilon^a_j(o, x) g^{bc} \nabla_b T_{ac}^i = -g^{bc} \nabla_b T_{(j)c}^i$$

Clearly $\mathfrak{H} = 0$ implies $\mathfrak{H} = 0$. The converse will be quite relevant in Section 5.

We now assume that the splitting $\varepsilon^i_j(o, x, t)$ depends on time $t \geq 0$ smoothly and $\varepsilon(o, x, 0) = \varepsilon_0$. So for any small $t \geq 0$ and $x \in M$, $\varepsilon(o, x, t)$ assigns a 1-arrow with source at $o$ and target at $x$ and this assignment is smooth in $x, t$.

We observe that $\mathfrak{H}^j_i(\varepsilon)$ depends non-linearly on the second order derivatives of $\varepsilon$. For simplicity of notation, henceforth we omit the arguments of our functions, except those of $\varepsilon$ since the notation $\varepsilon$ does not distinguish between $\varepsilon(o, x)$ and $\varepsilon(x, o)$ which is quite crucial below.

**Definition 8** The homogeneous flow of a parallelizable manifold is the second order nonlinear evolution equation

$$\frac{d\varepsilon^i_j(o, x, t)}{dt} = \mathfrak{H}^j_i(\varepsilon)$$  \hspace{1cm} (29)

with the initial condition $\varepsilon(o, x, 0) = \varepsilon_0$.

Note that (29) stabilizes if $\mathfrak{H} = 0$.

**Proposition 9** If $M$ is compact, (29) admits a unique short time solution with any initial condition.

**Proof:** We compute the symbol of the linearization of $\mathfrak{H}$. So we set

$$\frac{d\varepsilon^i_j(o, x, t)}{dt} = h^j_i$$  \hspace{1cm} (30)

and compute the terms which depend on the second order derivatives of $h^j_i$ with respect $x$ in the expression

$$\frac{d\mathfrak{H}^j_i(\varepsilon(o, x, t))}{dt}$$  \hspace{1cm} (31)
According to (24) $\nabla_r \varepsilon^i_j(\omega, x, t) = 0$ for all $t$ where $\nabla$ is the operator defined by $\varepsilon(\omega, x, t)$ which we will write as $\varepsilon(\omega, x)$. Therefore

$$0 = \frac{d}{dt} \nabla_r \varepsilon^i_j(\omega, x)$$

$$= \frac{d}{dt} \left( \frac{\partial \varepsilon^i_j(\omega, x)}{\partial x^r} - \Gamma^i_{ra} \varepsilon^a_j(\omega, x) \right)$$

$$= \frac{\partial}{\partial x^r} \left( \frac{d \varepsilon^i_j(\omega, x)}{dt} \right) - \Gamma^i_{ra} \frac{d \varepsilon^a_j(\omega, x)}{dt} - \frac{d \Gamma^i_{ra}}{dt} \varepsilon^a_j(\omega, x)$$

$$= \frac{\partial h^i_j}{\partial x^r} - \Gamma^i_{ra} h^a_j - \frac{d \Gamma^i_{ra}}{dt} \varepsilon^a_j(\omega, x)$$

$$= \nabla_r h^i_j - \frac{d \Gamma^i_{ra}}{dt} \varepsilon^a_j(\omega, x)$$

(32)

which gives

$$\frac{d \Gamma^i_{rk}}{dt} = \varepsilon^a_k(x, \omega) \nabla_r h^i_a$$

$$= \nabla_r \left( \varepsilon^a_k(x, \omega) h^i_a \right)$$

$$= \nabla_r h^i_{(k)}$$

(33)

Note the simplicity of the variation (33) compared to the variation of the Christoffel symbols of a metric in the Ricci flow.

From (33) we deduce

$$\frac{d T^i_c}{dt} = \nabla_r h^i_{(r)} - \nabla_k h^i_{(k)}$$

(34)

Now

$$\frac{d \delta^i_j(\varepsilon)}{dt}$$

(35)

$$= - \frac{d \varepsilon^a_j(\omega, x)}{dt} g^{bc} \nabla_b T^i_{ac} - \varepsilon^a_j(\omega, x) g^{bc} \frac{d \nabla_b T^i_{ac}}{dt} - \varepsilon^a_j(\omega, x) g^{bc} \frac{d}{dt} \nabla_b T^i_{ac}$$

It is only the last term in (35) which contains second order derivatives of $h$. Further

$$\frac{d}{dt} \nabla_b T^i_{ac} = \nabla_b \left( \frac{d T^i_c}{dt} \right) + \text{lower order terms}$$

(36)

Substituting (36) into (35), the symbol is given by
\[-\varepsilon^a_j(\mathbf{o}, x)g^{bc}\nabla_b \left( \frac{dT_{ac}}{dt} \right) = -\varepsilon^a_j(\mathbf{o}, x)g^{bc}\nabla_b \left( \varepsilon^d(x, \mathbf{o})\nabla_a h_d^i - \varepsilon^a(x, \mathbf{o})\nabla_c h_d^i \right) = -\varepsilon^a_j(\mathbf{o}, x)\varepsilon^d(x, \mathbf{o})g^{bc}\nabla_b \nabla_a h_d^i + g^{bc}\nabla_b \nabla_c h_d^i \]  \hspace{1cm} (37)

and the second "elliptic" term in (37) shows that (29) is weakly parabolic.

Now we fix an arbitrary "connection" $\Gamma^i_{jk}$ and define the time-dependent vector field $W(x, t)$ by

$$W^i \overset{def}{=} g^{ab} \left( \Gamma^i_{ab} - \Gamma^i_{ab} \right)$$  \hspace{1cm} (38)

The key fact in (38) is that $\Gamma - \bar{\Gamma}$ is a tensor and $\Gamma$ does not depend on $t$. We define the second order nonlinear operator $\mathfrak{M}$ by the formula

$$\mathfrak{M}^i_j(\varepsilon) \overset{def}{=} \varepsilon^a_j(\mathbf{o}, x)\nabla_a W^i = \nabla(J)W^i$$  \hspace{1cm} (39)

We compute

$$\frac{d\mathfrak{M}^i_j}{dt} = \varepsilon^a_j(\mathbf{o}, x)g^{ab}\nabla_a \frac{d\Gamma^i_{ab}}{dt} + \ldots = \varepsilon^a_j(\mathbf{o}, x)g^{ab}\nabla_a \left( \varepsilon^d(x, \mathbf{o})\nabla_a h_d^i \right) + \ldots = \varepsilon^a_j(\mathbf{o}, x)g^{ab}\varepsilon^d(x, \mathbf{o})\nabla_a \nabla_a h_d^i + \ldots = \varepsilon^a_j(\mathbf{o}, x)g^{ab}\varepsilon^d(x, \mathbf{o})\nabla_a \nabla_a h_d^i + \ldots$$  \hspace{1cm} (40)

From (37) and (40) we conclude that the evolution equation

$$\frac{d\varepsilon^a_j(\mathbf{o}, x)}{dt} = S^a_j(\varepsilon) + \mathfrak{M}^i_j(\varepsilon)$$  \hspace{1cm} (41)

is strongly parabolic. By the well known existence theorem, we conclude that (41) admits unique short time solutions. Now let $\varepsilon(\mathbf{o}, x, t) = \varepsilon_t$ be the unique short time solution of (41) starting from $\varepsilon_0$. Let $\varphi_t$ be the unique short time solution of the ODE

$$\frac{d\varphi_t}{dt} = W$$  \hspace{1cm} (42)

so that $\varphi_t$ is a family of diffeomorphisms of $M$ with $\varphi_0 = id$. Now it is easily shown (see [4]) that $\varphi_t\varepsilon_t$ solves (29) and in fact the solutions of (29) are unique, finishing the proof.

Note that the short time solution $\varepsilon^a_j(\mathbf{o}, x, t)$ is a 1-arrow, i.e., an invertible matrix for sufficiently small $t$ and for all $x \in M$ since this is so for the initial condition $\varepsilon^a_j(\mathbf{o}, x, 0)$. It turns out, however, that the condition "sufficiently small" is redundant.
4 Gauge group

In this section we will prove

**Proposition 10** The unique short time solutions of (29) extend uniquely to long time solutions on \([0, \infty)\).

**Proof:** We first define the first order universal gauge group \(A\) of \(M\). Let \(U^p_p\) denote the set of all 1-arrows with source and target at \(p \in M\). A choice of coordinates around \(p\) identifies \(U^p_p\) with the Lie group \(GL(n, \mathbb{R})\). We define \(A \equiv \bigcup_{p \in M} U^p_p\). A local section of \(A\) is of the form \(f^i_j(x)\) in coordinates and is an invertible linear map on the tangent space at \(x\). Let \(\Gamma A\) denote the space of smooth sections of \(A\). Now \(\Gamma A\) is a group with the fiberwise composition of jets.

Now let \(M\) be a parallelizable manifold and \(E\) denote the set of all splittings on \(M\). We recall that for any \(x \in M\) the splitting \(\varepsilon\) assigns a 1-arrow from \(0 \in \mathbb{R}^n\) to \(x\) and we also write \(\varepsilon(0 \times M)\) for \(\varepsilon\). Let \(g \in \Gamma A\) and \(\varepsilon(0 \times M) \in E\). We define \(g\varepsilon \in E\) by

\[
(g\varepsilon)(0, p) \equiv g(p) \circ \varepsilon(0, p) \quad p \in M
\]

The action (43) is easily seen to be simply transitive. Hence a choice of \(\varepsilon\) gives a 1-1 correspondence between \(\Gamma A\) and \(E\). Now \(\Gamma A\) is an infinite dimensional Frechet Lie group as follows (see [7] for the technical details of this theory). Let \(\xi\) be a section of the vector bundle \(Hom(T, T) \to M\), so that \(\xi(p)\) is a linear map at \(T_p\). We note that the Lie algebra of \(U^p_p\) is canonically isomorphic to the fiber \(Hom(T_p, T_p)\) endowed with the usual bracket of matrices. Therefore the space of sections \(\Gamma Hom(T, T)\) is a Lie algebra. Note that the bracket of \(\Gamma Hom(T, T)\) is defined pointwise and does not involve differentiation. Now the Lie algebra \(\Gamma Hom(T, T)\) may be viewed (and this view can be made rigorous) as the Lie algebra of \(\Gamma A\) as follows. The value \(\xi(p)\) determines a 1-parameter subgroup \(\gamma(p, t)\) in \(U^p_p\) with \(\left.\frac{d\gamma(p, t)}{dt}\right|_{t=0} = \xi(p)\), \(\gamma(p, 0) = p\) for all \(p \in M\) and \(\gamma(p, t)\) is defined for all \(t \geq 0\). So \(\xi \in \Gamma Hom(T, T)\) determines a "1-parameter subgroup" of \(\Gamma A\) which consists of the 1-parameter subgroups at all points. The local surjectivity of the pointwise exponential maps gives the local surjectivity of the exponential map \(\exp : \Gamma Hom(T, T) \to \Gamma A\) if \(M\) is compact.

Now let \(\varepsilon(0, x, t)\) be the unique short time solution of (29) starting from \(\varepsilon(0, x, 0)\). By the above 1-1 correspondence, \(\varepsilon_0\) identifies \(\Gamma A\) with \(E\). In view of the simple transitivity of (43), for any small \(t\) there exists a unique \(g \in \Gamma A\) satisfying

\[
\varepsilon(0, x, t) = g(x, t) \circ \varepsilon(0, x, 0)
\]

By the uniqueness of the short time solutions of (29), evolving the initial splitting \(\varepsilon(0, x, 0)\) by \(t + s\) is the same as evolving \(\varepsilon(0, x, t)\) by \(s\) for small \(s, t\). It follows that \(g(x, t)\) is a 1-parameter subgroup of \(\Gamma A\) defined for small \(t\).
Therefore it extends to a 1-parameter subgroup defined for all values of \( t \geq 0 \).

We now define \( \varepsilon(t, x, t) \) by the RHS of (44) for all \( t \geq 0 \) and easily check that this gives a long time solution of (29) on \([0, \infty)\), concluding the proof. Note the interesting interpretation of the "backward" solutions.

5 Poincare Conjecture

Assume that \( M \) is a parallelizable manifold. We recall here that an orientable 3-manifold is parallelizable. Consider the following two assertions.

A1. Suppose \( \varepsilon_t \) converges to some parallelism \( \varepsilon_\infty \) as \( t \to \infty \), i.e., no singularity occurs at infinite. Then \( \mathcal{O}(\varepsilon_\infty) = 0 \).

A2. If \( \dim M = 3 \), then \( \mathcal{O} = 0 \iff \mathcal{R} = 0 \) and the hypothesis of A1 holds if \( M \) is compact and simply connected.

Proposition 11 A1 and A2 imply PC.

Proof: It suffices to show that the compact and simply connected local Lie group \((M, \varepsilon_\infty)\) of dimension three is diffeomorphic to \( S^3 \). Since \( M \) is compact, \( \varepsilon_\infty \) is easily seen to be complete (see Lemma 7.3 in [1]). Since \( M \) is also simply connected, the pseudogroup \( S \) in Section 2 globalizes to a Lie group. However \( S^3 \) is the only compact and simply connected Lie group in dimension three up to diffeomorphism.

We believe that the above approach whose main idea is stated in [1] will greatly simplify the present proof of PC.

6 Appendix. The relation of HF to the Ricci flow

Suppose we evolve the initial canonical metric given by Definition 7 according to the Ricci flow

\[
\frac{dg}{dt} = -2Ric(g)
\]  

(45)

The natural question is how (29) and (45) are related. So let \( g_t \) be the unique short time solution of (45) with \( g_0 \) being the canonical metric of \( \varepsilon_0 \) so that \( \varepsilon_0(\mathfrak{o}, M) \) is a trivialization of the \( O(n) \)-principal bundle \( P(g_0) \) determined by \( g_0 \). It is easy to see that the principal bundles \( P(g_t) \) admit trivializations for small \( t \). However, there is no canonical way of choosing these trivializations. Therefore, (45) does not imply (29). The main idea of this note is to reverse this reasoning and try to derive (45) from (29) as follows. Now (27) gives

\[
\frac{\partial g_{ij}}{\partial x^r} + \Gamma^a_{ri} g_{aj} + \Gamma^a_{rj} g_{ia} = 0
\]  

(46)

Let \( \Sigma^i_{jk} \) be the Christoffel symbols of \( g_{ij} \) so that (46) holds also for \( \Sigma^i_{jk} \). The Gauss trick of shifting the indices in (46) gives the formula
\[ \Sigma_{jk}^i = -\frac{1}{2} \left( \Gamma_{jk}^i + T_{jk}^ag_{ka}g^{ib} \right)_{(jk)} \]  

(47)

where \((jk)\) denotes the symmetrization of \(j, k\). It is natural to substitute (47) into (45) and try to derive (45) from (29). Equivalently, we may differentiate (26), substitute from (29) and try to express the resulting expression in terms of \(g\) by eliminating \(\varepsilon\). Unable to carry out this derivation, we came up with the expression (28). We now believe that (29) and (45) are independent. However, differentiation of (26) suggests that (45) is the \"symmetrization\" of (29) in some vague sense.

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