DUNKL OPERATOR AND QUANTIZATION OF $\mathbb{Z}_2$-SINGULARITY

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Abstract. Let $(X, \omega)$ be a symplectic orbifold which is locally like the quotient of a $\mathbb{Z}_2$ action on $\mathbb{R}^n$. Let $A^{(\hbar)}_X$ be a deformation quantization of $X$ constructed via the standard Fedosov method with characteristic class being $\omega$. In this paper, we construct a universal deformation of the algebra $A^{(\hbar)}_X$ parametrized by codimension 2 components of the associated inertia orbifold $\tilde{X}$. This partially confirms a conjecture of Dolgushev and Etingof (see [4]) in the case of $\mathbb{Z}_2$ orbifolds. To do so, we generalize the interpretation of Moyal star-product as a composition of symbol of pseudodifferential operators in the case where partial derivatives are replaced with Dunkl operators. The star-products we obtain can be seen as globalizations of symplectic reflection algebras ([6]).

1. Introduction

In this paper, we construct exotic deformation quantizations of symplectic orbifolds. Orbifolds provide a large class of examples of topological spaces which are obtained as quotients of manifolds by actions of compact groups. We consider a compact manifold $M$ endowed with a symplectic structure $\omega$ and with a $\mathbb{Z}_2$ action which preserves the symplectic structure. Given these data one can construct a $\mathbb{Z}_2$-invariant (associative) star-product (using Fedosov method via a $\mathbb{Z}_2$ invariant connection for instance) with the characteristic class being $\omega$. The restriction of the invariant star-product on $C^\infty(M/\mathbb{Z}_2)$ defines a deformation quantization of the orbifold $X = M/\mathbb{Z}_2$.

Let $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ denote the star algebra on $M/\mathbb{Z}_2$ with the characteristic class being $\omega$. In [4, Theorem 1.1] and [13, Theorem VII], the Hochschild cohomology of $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ was computed to be equal to the cohomology of the corresponding inertia orbifold with coefficient in $C((\hbar))$. In particular, Dolgushev and Etingof ([4]) conjectured that the deformations of the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ is unobstructed.

Let $\gamma$ be the non unital element in $\mathbb{Z}_2$ and $M^\gamma$ be the $\gamma$ fixed point subsets. The inertia orbifold $\tilde{X}$ associated to the quotient $M/\mathbb{Z}_2$ is equal to $\tilde{X} = M/\mathbb{Z}_2 \sqcup M^\gamma/\mathbb{Z}_2$. The Hochschild cohomology of $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ is equal to

$$H^2(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), C^\infty(M)^{\mathbb{Z}_2}((\hbar))) = H^2(M/\mathbb{Z}_2)((\hbar)) \bigoplus H^0(M^\gamma/\mathbb{Z}_2)((\hbar)),$$

where $M^\gamma_{\mathbb{Z}_2}$ is the union of components of $M^\gamma$ of codimension 2. The Dolgushev-Etingof conjecture implies that the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ has a deformation coming from every $\gamma$ fixed point component with codimension 2.

The aim of this paper is to prove that for every class in $H^0(M^\gamma_{\mathbb{Z}_2}/\mathbb{Z}_2)((\hbar))$, namely every codimension 2 component of the inertia orbifold $\tilde{X}$, we are able to construct a deformation of the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$. Moreover, there exists a universal
deformation of \((C^\infty(M)^{\mathbb{Z}_2}, \star)\) parametrized by \(H^0(M^2/\mathbb{Z}_2)((\hbar_2))\). This partially confirms the conjecture of Dolgushev and Etingof in the case of \(\mathbb{Z}_2\) orbifolds. Our result is not far away from the full Dolgushev-Etingof conjecture, and the detailed relations are explained in Remark 4.4. The Dolgushev-Etingof conjecture was proved by Etingof [7] when an orbifold is the cotangent bundle of a global quotient orbifold. It is the first time that we know that a large portion of this conjecture holds true for a large class of compact symplectic orbifolds.

In the case where \(M\) is \(\mathbb{R}^{2n}\), the deformations we get are formal versions of symplectic reflection algebras ([6]) and our construction can be seen as a globalization of such algebras.

To globalize star-products on \(\mathbb{R}^{2n}\), one should start with local formulas of the star-products like the Moyal product and Kontsevich star product [11]. Moyal product, which deforms the standard symplectic structure on \(\mathbb{R}^{2n}\), can be described using composition of symbols of pseudodifferential operators on \(\mathbb{R}^n\). One of the main ideas of the paper is to get a generalized Moyal product formula out of composition of symbols associated to difference-pseudodifferential operators. Following this approach, we replace partial derivatives with Dunkl operators to take into account the \(\mathbb{Z}_2\) action and define local formulas for deformation of any non-commutative Poisson structure [10] associated with \(\omega\). In this sense, we can also view our construction as globalization of difference-pseudodifferential operators of “Dunkl type”.

In Section 2, we recall general material on Dunkl operators and Dunkl pseudodifferential operators. This will allow us to construct operator-symbol product formula in Section 3: we will get two families of \(\mathbb{Z}_2\)-local bilinear operators satisfying properties summarized in Theorem 3.10. Those operators will allow us to define a \(\gamma\)-local associative star product (Proposition 3.16) generalizing the standard Moyal star product. Interesting combinatorics appears in the associativity the new star product. The proof of this main theorem is done in Section 5, using series expansions of pseudodifferential calculous and explicit computations.

Section 4 is devoted to globalization and thus to give a positive answer to Dolgushev-Etingof conjecture. The main idea there is to use Fedosov standard method on the complement of a tubular neighborhood of the \(\mathbb{Z}_2\) fixed point submanifold of codimension 2. This can be done as the star product there is locally equivalent to the Moyal product. In the neighborhood of the fixed point submanifold of codimension 2, we use our generalized Moyal product and Fedosov’s method of quantization of fixed point submanifolds. The fact that both the Moyal product and the generalized Moyal product are \(\gamma\)-local allows us to restrict the two deformations above on the intersections of the two open sets, which is diffeomorphic to the tubular neighborhood of the fixed point submanifold of codimension 2 with the fixed point submanifold removed. We are able to glue the two deformations on the intersection together to get a global deformation on \(M/\mathbb{Z}_2\) as \(\mathbb{Z}_2\) acts on the intersection freely.

Here are some remarks and questions for future directions.

(1) The fact that the group acting on \(M\) is \(\mathbb{Z}_2\) is of major importance for our construction: if \(M = \mathbb{R}^{2n}\), the \(\mathbb{Z}_2\) action stabilizes the two corresponding copies of
and thus allows us to play with (Dunkl) operators. Such an idea was also used by Etingof [7] in his construction of universal deformation of the cotangent bundle of a global quotient orbifold. To extend our results to more general orbifolds, an important question to answer is how to quantize a symplectic orbifold when such a “polarization” of the symplectic orbifold does not exist. (2) One could try to generalize our results to every \( \mathbb{Z}_2 \) invariant Poisson structure (and so deform the corresponding noncommutative Poisson structure). One would expect that with the help of the above mentioned polarization on a \( \mathbb{Z}_2 \) orbifold, we can play with the corresponding conjectural generalized Poisson sigma models to define the generalized Moyal products. (3) Another natural question is to compute Hochschild cohomology (and \( K \)-theory) of our deformed algebra. It will be interesting to develop an algebraic index theorem for our deformed algebra. We hope to extract the information of singularities from the algebraic index theorem.

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2. Dunkl Operator

In this section, we briefly review the theory of Dunkl operators, Dunkl transforms, and Dunkl pseudodifferential operators, which we will need in this paper. We will focus ourselves to a very special case in the theory of Dunkl operators. Most constructions and results we are reviewing go back to Dunkl’s original work [5]. We refer readers to [14] and [3] for the proofs of the statements in this section.

Let \( \mathbb{Z}_2 = \{1, \gamma\} \) be the group of two elements. It acts on the space \( \mathbb{R} \) by reflection. We will use \( C^\infty_c(\mathbb{R}) \) to denote the space of compactly supported smooth functions on \( \mathbb{R} \), and \( S(\mathbb{R}) \) to denote the space of Schwartz functions on \( \mathbb{R} \). For a real parameter \( k > 0 \), we consider the following differential-difference operator defined by

\[
T_k(f)(x) = \frac{df}{dx}(x) + k \frac{f(x) - f(-x)}{x}, \quad f \in C^\infty(\mathbb{R}),
\]

which is called Dunkl operator.

For the spectral of the operator \( T_k \), one considers the following equation

\[
\begin{cases}
T_k(u)(x) = -i\lambda u(x) \\
u(0) = 1
\end{cases}
\]

for \( \lambda \in \mathbb{C} \).

The above equation actually have a unique solution \( E_k(x, -i\lambda) \), called Dunkl kernel given by

\[
E_k(x, -i\lambda) = j_{k-1/2}(i\lambda x) + \frac{\lambda x}{2k+1} j_{k+1/2}(i\lambda x),
\]

where \( j_\alpha \) is the “normalized first kind Bessel function of order \( \alpha \). From the above expression, one easily see that \( E_k(x, -i\lambda) \) can be extended to a holomorphic function of variable \( x \in \mathbb{C}, \lambda \in \mathbb{C}, \Re k \geq 0 \). One can even show that for \( x, \lambda \in \mathbb{R}, \)

\[
|E_k(x, i\lambda)| \leq 1.
\]
We consider the following measure $\mu_k$ on $\mathbb{R}$ by
\[
d\mu_k(x) = \frac{|x|^{2k}}{2^{k+1/2} \Gamma(k + 1/2)} dx,
\]
with $\Gamma(x)$ the Gamma function. It is straightforward to check that the Dunkl operator $T_k$ is skew symmetric with respect to the $L^2$-norm associated to the measure $\mu_k$, i.e.
\[
\int_{\mathbb{R}} T_k(f)(x) \overline{g} d\mu_k(x) = -\int_{\mathbb{R}} f(x) \overline{T_k(g)(x)} d\mu_k(x).
\]
For $1 \leq p < \infty$, define $L^p_k(\mathbb{R})$ to be the space of measurable complex valued functions on $\mathbb{R}$ such that
\[
||f||_{p,k} = \left( \int_{\mathbb{R}} |f(x)|^p d\mu_k(x) \right)^{1/p} < \infty.
\]
For $f \in L^1_k(\mathbb{R})$, define the Dunkl transform $\mathcal{F}_k$ of $f$ by
\[
\mathcal{F}_k(f)(\lambda) = \int_{\mathbb{R}} E_k(y, -i\lambda) f(y) d\mu_k(y).
\]
When $f$ is in $\mathcal{S}(\mathbb{R})$, then
1. $\mathcal{F}_k(f) \in C^\infty(\mathbb{R})$, and $T_k \mathcal{F}_k(f) = -\mathcal{F}_k(ixf)$,
2. $\mathcal{F}_k(T_k f) = i\lambda \mathcal{F}_k(f)$,
3. the Dunkl transform leaves $\mathcal{S}(\mathbb{R})$ invariant.
4. For all $f \in L^1_k(\mathbb{R})$ such that $\mathcal{F}_k(f) \in L^1_k(\mathbb{R})$, the inverse Dunkl transform is defined to be
\[
\mathcal{F}_k^{-1}(f)(x) = \int_{\mathbb{R}} E_k(x, i\lambda) f(\lambda) d\mu_k(\lambda),
\]
5. For $f \in L^2_k(\mathbb{R})$, $||\mathcal{F}_k(f)||_{2,k} = ||f||_{2,k}$.

3. Generalized pseudodifferential operators and Moyal type formula

Pseudo-differential operators associated to Dunkl operators in the case of $\mathbb{Z}_2$ have been studied by Dachraoui [2] and Abdelkefi-Amri-Sifi [1]. Let $D(\mathbb{R})$ be the algebra of differential operators on $\mathbb{R}$. In this section, our goal is to use the idea of operatorsymbol calculus to define an associative deformation of the algebra $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2$ and also $D(\mathbb{R}) \rtimes \mathbb{Z}_2$. When one restricts such a deformation to the subalgebra\(^1\) $\text{Poly}(\mathbb{R}^2) \rtimes \mathbb{Z}_2$, we obtain a Moyal type formula for the symplectic reflection algebra introduced by Etingof-Ginzburg [6] in the case of $\mathbb{Z}_2$ action on $\mathbb{R}^2$ by reflection.

3.1. Operator product.

Definition 3.1. We say that a function $a(x, p) \in C^\infty(\mathbb{R}^2)$, a complex valued function on $\mathbb{R}^2$, belongs to the symbol class $\mathfrak{S}_m^m$ if for any $r, s \in \mathbb{N}$,
\[
|\partial_p^r \partial_x^s a(x, p)| \leq C_{m,r,s} (1 + |p|^2)^{(m-r)/2}.
\]

\(^1\)Poly(\mathbb{R}^2) denotes the algebra of polynomial functions on \mathbb{R}^2.
**Definition 3.2.** Let \( a \in \mathcal{S}_0^m \), then define \( \text{Op}_k(a) \) a linear operator on \( \mathcal{S}(\mathbb{R}) \) by

\[
\text{Op}_k(a)(f)(x) = \int_{\mathbb{R}} a(x, -ip) E_k(x, ip) \mathcal{F}_k(f)(p) d\mu_k(p).
\]

Dachraoui [2, Thm. 4.1] proves the following theorem:

**Theorem 3.3.** [2] Let \( a \in \mathcal{S}_0^m \), then the operator \( \text{Op}_k(a) \) associated to \( a \) is a linear continuous mapping from \( \mathcal{S}(\mathbb{R}) \) to itself.

**Remark 3.4.** For \( a \in \mathcal{S}_0^m \), [1, Proposition 4.1] proves that \( \text{Op}_k(a) \) defines a bounded operator on \( L_k^p(\mathbb{R}) \) for \( 1 < p < \infty \).

**Example 3.5.** For \( a(x, p) = x^i p^j \in \mathcal{S}_0^m \), \( \text{Op}_k(a) = x^i T^j_k \).

We consider the translation operator \( \hat{\gamma} : f(x) \mapsto f(-x) \). It is easily seen that \( \hat{\gamma} \) is an isometry on \( L_k^2(\mathbb{R}) \). We have the following observation:

**Lemma 3.6.** For \( a_j, b_j \in \text{Poly}(\mathbb{R}^2), j = 0, \ldots, n \), if \( \sum_j k^j (\text{Op}_k(a_j) + \text{Op}_k(b_j) \circ \hat{\gamma}) \) is the zero operator for any \( k \geq 0 \), then \( a_j = b_j = 0, j = 0, \ldots, n \).

**Proof.** As \( \text{Op}_k(\sum_j k^j a_j) + \text{Op}_k(\sum_j k^j b_j) \circ \hat{\gamma} = 0 \), then

\[
\int_{\mathbb{R}} \left( \sum_j k^j a_j(x, -ip) \right) E_k(x, ip) \mathcal{F}_k(f)(p) d\mu_k(p)
+ \int_{\mathbb{R}} \left( \sum_j k^j b_j(x, -ip) \right) E_k(x, ip) \mathcal{F}_k(\hat{\gamma}(f))(p) d\mu_k(p) = 0,
\]

for any \( f \in \mathcal{S}(\mathbb{R}) \).

We notice that \( \mathcal{F}_k(\hat{\gamma}(f))(p) = \mathcal{F}_k(f)(-p) \), then Equation (1) becomes

\[
\int_{\mathbb{R}} \sum_j k^j (a_j(x, p) E_k(x, ip) + b_j(x, -p) E_k(x, -ip)) \mathcal{F}_k(f)(p) d\mu_k(p) = 0,
\]

for any \( f \in \mathcal{S}(\mathbb{R}) \). Therefore, we conclude that

\[
\sum_j k^j (a_j(x, p) E_k(x, ip) + b_j(x, -p) E_k(x, -ip)) = 0,
\]

for any \( x, p \in \mathbb{R} \). If we consider the above equation at \( k = 0 \), then

\[
a_0(x, p) \exp(ip) + b_0(x, -p) \exp(-ip) = 0.
\]

From the above equation, we have that

\[
\partial_x a(x, p)b(x, -p) - a(x, p)\partial_x b(x, -p) = -2ipa(x, p)b(x, -p).
\]

By comparing the leading terms on both sides, we can quickly conclude that \( a_0 = b_0 = 0 \).

By induction, we conclude that \( a_j = b_j = 0 \) for \( j = 0, \ldots, n \). \( \square \)

To motivate the main result of this section, we introduce the following notion of a \( \gamma \)-local operator. (Recall that \( \gamma \) acts on \( \mathbb{R} \) by reflection.)
Definition 3.7. A linear operator $D$ on $C^\infty(\mathbb{R}^2)$ is called $\gamma$-local if for any $f \in C^\infty(\mathbb{R}^2)$, $D(f)(x, p)$ is determined completed by finitely many jets of $f$ at $(x, p)$, $(-x, p)$, $(x, -p)$, and $(-x, -p)$. In general, a $k$-linear operator $D$ on $C^\infty(\mathbb{R}^2)$ is called $\gamma$-local, if for any $f_1, \ldots, f_k \in C^\infty(\mathbb{R}^2)$, $D(f_1, \ldots, g_k)(x, p)$ is determined by finitely many jets of $f_1, \ldots, f_k$ at $(x, p)$, $(-x, -p)$, and $(-x, -p)$.

Example 3.8. Let us list some examples of $\gamma$-local operators.

1. Differential operators on $\mathbb{R}$ are $\gamma$-local.
2. The partial translation operator $\sigma_i : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ for $i = 1, 2$ with $\sigma_1(f)(x, p) := f(-x, p)$ and $\sigma_2(f)(x, p) = f(x - p)$ are $\gamma$-local.
3. The difference operators $\tilde{\partial}_x, \tilde{\partial}_p : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ with $\tilde{\partial}_x(f)(x, p) = (f(x, p) - f(-x, p))/x$ and $\tilde{\partial}_p(f)(x, p) = (f(x, p) - f(x, -p))/p$ are $\gamma$-local. We observe that $\partial_x + \tilde{\partial}_x, (\partial_p + \tilde{\partial}_p)$ is the Dunkl operator $T_1$ acting on the $x$-variable ($p$-variable), and is also $\Gamma$-local.

Proposition 3.9. The space of $\gamma$-local operators on $C^\infty(\mathbb{R}^2)$ is an associative algebra under composition.

Proof. This is a straightforward check. □

The main result of this section can be summarized into the following Theorem.

Theorem 3.10. There are 2 families of $\gamma$-local bilinear operators $C^1_{j,l}$ and $C^2_{j,l}$ on $C^\infty(\mathbb{R}^2)$ satisfying

1. For two polynomials $a_1$ and $a_2$ of degrees $(m_1, n_1)$ and $(m_2, n_2)$, $C^0_{j,l}(a_1, a_2)$ and $C^1_{j,l}(a_1, a_2)$ are again polynomials of degree $(m_1 + m_2 - j, n_1 + n_2 - j)$.
2. $C^0_{j,l}$ and $C^1_{j,l}$ vanish when $l > j$.
3. For two polynomials $a_1(x, p)$ and $a_2(x, p)$,

$$\text{Op}_k(a_1) \circ \text{Op}_k(a_2) = \sum_{j,l} k! \left( \text{Op}_k \left( C^0_{j,l}(a_1, a_2) \right) + \text{Op}_k \left( C^1_{j,l}(a_1, a_2) \right) \circ \gamma \right).$$

We observe that for any given $a_1, a_2$, the above sum is actually finite and therefore well defined.

The proof of this theorem will be given in Section 5. In the left of this section, we will provide an explicit formula for each bilinear operator $C^0_{j,l}$. In particular, when $l = 0$, $C^1_{j,0}$ vanishes and $C^0_{j,0}$ is the $j$-th component of the Moyal product,

$$C^1_{j,0}(a_1, a_2) = \frac{(-i)^j}{j!} \tilde{\partial}_p(a_1) \tilde{\partial}_p(a_2).$$

From this, we can see that the above operator-symbol calculus defines a deformation of the crossed production of $D(\mathbb{R}) \rtimes \mathbb{Z}_2$.

3.2. A coproduct structure on Poly(\mathbb{R}). We consider a coproduct structure on the algebra of polynomials of one variable, which is useful in describing the operators $C^0_{j,l}$.

Define $\Delta$ to be a linear map from Poly(\mathbb{R}) to Poly(\mathbb{R}) $\otimes_C$ Poly(\mathbb{R}) by

$$\Delta(f)(x, y) := \frac{f(x) - f(y)}{x - y}.$$
According to Proposition 3.11, (2), the operator \( \Delta \) satisfies the following properties:

1. **coassociative**, i.e.
   \[
   (\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : \text{Poly}(\mathbb{R}) \to \text{Poly}(\mathbb{R}) \otimes \text{Poly}(\mathbb{R}) \otimes \text{Poly}(\mathbb{R});
   \]

2. **Leibnitz rule**, i.e.
   \[
   \Delta(fg) = (f \otimes 1)\Delta(g) + \Delta(f)(1 \otimes g);
   \]

3. \( \Delta(f)(x, x) = f'(x) = D(f)(x) \), and \( \Delta(f)(x, -x) = (f(x) - f(-x))/2x = 1/2\hat{D}(f)(x) \), and \( T_k(f)(x) = (D + k\hat{D})(f)(x) \);

4. \( \Delta(f) \) is a symmetric function of 2 variables;

5. \( \Delta \) extends to be a linear map \( \Delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R}) \) satisfying the same properties (1)-(4), where \( \otimes \) is the complete topological tensor product.

**Remark 3.12.** According to Proposition 3.11, (2), the operator \( \Delta \) is a Hochschild cocycle of \( \text{Poly}(\mathbb{R}) \) with coefficient in \( \text{Poly}(\mathbb{R}) \otimes \text{Poly}(\mathbb{R}) \). By the Koszul complex, we can compute the Hochschild cohomology \( H^1(\text{Poly}(\mathbb{R}), \text{Poly}(\mathbb{R}) \otimes^2) \) is equal to \( \text{Poly}(\mathbb{R}) \). Under this identification, \( \Delta \) is mapped to the unit of \( \text{Poly}(\mathbb{R}) \).

**Remark 3.13.** For \( \mathbb{R}^n \), we can generalize \( \Delta \) to a cocycle \( \Delta_n : \text{Poly}(\mathbb{R}^n) \otimes^n \to \text{Poly}(\mathbb{R}^n) \otimes^2 \) by

\[
\Delta_n(f_1, \ldots, f_n)(x, y) = (f_1(x_1, \ldots, x_n) - f_2(y_1, x_2, \ldots, x_n)) \cdots (f_n(y_1, \ldots, y_{n-1}, x_n) - f_n(y_1, \ldots, y_n))/ (x_1 - y_1) \cdots (x_n - y_n),
\]

where \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \).

### 3.3. Formulas for asymptotic expansion

We will give explicit expressions for \( C^i_{j,l} \), \( i = 1, 2 \), which involves interesting combinatorics.

We start with considering the linear equation

\[
y_0 + y_1 + \cdots + y_l = j - l.
\]

Let \( P_{j-l,l} \) be the set of integer solutions to Equation (3) where \( y_0, y_l \) are nonnegative and \( y_1, \ldots, y_{l-1} \) are positive.

Let \( D(f)(x) = f'(x) \) and \( \hat{D}(f)(x) = (f(x) - f(-x))/x \).

For an element \( \nu \in P_{m,n} \), define \( D_{\nu} \) a linear operator on \( \text{Poly}(\mathbb{R}) \) by

\[
B_{\nu}(f)(x) := D^{y_0} \circ \hat{D} \circ D^{y_{n-1}} \cdots D^{y_1} \circ \hat{D} \circ D^{y_0}(f)(x).
\]

For \( n \in \mathbb{N} \), define \( n_0 \) to be the number of positive even numbers less than or equal to \( n \), and \( n_1 \) to be the number of positive odd numbers less than or equal to \( n \). Obviously, \( n = n_0 + n_1 \). Given \( \nu \in P_{m,n} \), we define \( \Lambda_0 = y_0 + \sum_{i \text{ even}} y_i \) and \( \Lambda_1 = \sum_{i \text{ odd}} y_i \). We have \( \Lambda_0 + \Lambda_1 = m \). Define \( A_{\nu} \), a linear operator on \( \text{Poly}(\mathbb{R}) \) by

\[
\Delta^{m+n}(f)(x_0, \ldots, x_{\Lambda_0+n_0+1}, -x_0, \ldots, -x_{\Lambda_1+n_1}).
\]
By the associativity of $\Delta$ (Prop. 3.11, (1)), define $\Delta^k(f) = (\Delta \otimes 1 \otimes \cdots \otimes 1) \cdots (\Delta \otimes 1)\Delta(f)$. And according to Prop. 3.11 (4), $\Delta^k(f)$ is a symmetric function of $k + 1$ variables.

In order to define $C_{j, l}^i$, which are bilinear operators on $\text{Poly}(\mathbb{R}^2)$, we lift $A_\nu$ and $B_\nu$ on $\text{Poly}(\mathbb{R}^2)$ by applying $A_\nu$ on the variable $p$ and $B_\nu$ on the variable $x$.

Now we are ready to define $C_{j, l}^i$.

I. $C_{j, l}^0$. The bilinear operator $C_{j, l}^0$ vanishes if $l$ is odd, and when $l$ is even,

$$C_{j, l}^0(a_1, a_2) := (-i)^j \sum_{\nu \in P_{j-1, l}} A_\nu(a_1)(x, p)B_\nu(a_2)(x, p).$$

II. $C_{j, l}^1$. The bilinear operator $C_{j, l}^1$ vanishes if $l$ is even, and when $l$ is odd,

$$C_{j, l}^1(a_1, a_2)(x, p) := (-i)^j \sum_{\nu \in P_{j-1, l}} A_\nu(a_1)(x, p)B_\nu(a_2)(x, -p).$$

We point out that with the expression of $C_{j, l}^i$, Theorem 3.10, (1) follows obviously by the definition of $A_\nu$ and $B_\nu$. Furthermore, one notices that if $j - l < l - 1$, then $P_{j-1, l}$ is an empty set, and therefore $C_{j, l}^i$ vanishes. This gives a stronger version of Theorem 3.10, (2).

From the above discussion, we are left to prove part (3) of Theorem 3.10. This is an interesting application of operator-symbol calculus and the detail will be in Section 5. In particular, we will explain how we obtain the operators $A_\nu$ and $B_\nu$.

3.4. A “Moyal” formula. Motivated by the result of Theorem 3.10, we introduce the following algebra.

**Definition 3.14.** Define the following product $\star$ on $C^\infty(\mathbb{R}^2) \rtimes \mathbb{C} \mathbb{Z}_2[[h_1, h_2]]$ by

1. $\star$ is $\mathbb{C}[[h_1, h_2]]$ linear;
2. For $a_1, a_2 \in C^\infty(\mathbb{R}^2)$, $a_1 \star a_2$ is defined by

$$a_1 \star a_2 = \sum_{j, l} h_1^j h_2^l (C_{j, l}^0(a_1, a_2) + C_{j, l}^1(a_1, a_2)\gamma).$$

As we have explained at the end of Section 3.1, when $h_2 = 0$, the above product $\star$ reduces back the standard Moyal product. Hence, we can view $(C^\infty(\mathbb{R}^2) \rtimes \mathbb{C} \mathbb{Z}_2[[h_1, h_2]], \star)$ as a deformation of the crossed product of the Weyl algebra $\mathbb{W}_2$ with $\mathbb{Z}_2$. Furthermore, we point out that as $C_{j, l}^i = 0$ when $l > j$, we can allow $h_2$ be a complex number in $\mathbb{C}$ rather than a formal parameter. In this way, we can also view $(C^\infty(\mathbb{R}^2) \rtimes \mathbb{C} \mathbb{Z}_2[[h_1, h_2]], \star)$ as a formal deformation quantization of the crossed product algebra $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2$ along the noncommutative Poisson structure $\pi + h_2\pi\gamma$ on $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2$ as we introduced in [10].

**Lemma 3.15.** For any $(x_0, p_0) \in \mathbb{R}^2$, and $f \in C^\infty(\mathbb{R}^2)$, and $m, n \in \mathbb{N} \cup \{0\}$, there is a polynomial $g_{m, n} \in \text{Poly}(\mathbb{R}^2)$ such that $\partial_x^i \partial_p^j f$ agrees with $\partial_x^i \partial_p^j g_{m, n}$ at $(x_0, p_0)$, $(x_0, -p_0)$, $(-x_0, p_0)$, and $(-x_0, -p_0)$ for $0 \leq i \leq m, 0 \leq j \leq n$.

**Proof.** We divide our proofs into 4 different situations according to $(x_0, p_0)$.

1. $x_0 = p_0 = 0$,
We need to solve a. For any Case (1) Define

\[ g_{m,n} = \sum_{0 \leq i \leq m, 0 \leq j \leq n} \frac{1}{i! j!} \partial_x^i \partial_p^j (f)(0,0) x^i p^j. \]

It is easy to check \( \partial_x^i \partial_p^j g_{m,n} \) agrees \( \partial_x^i \partial_p^j f \) at \( (0,0) \) for \( 0 \leq i \leq m, 0 \leq j \leq n \).

**Case (2) and (3).** The proof for these two cases are exactly same. Therefore, we will only prove Case (2). Define

\[ g_1 = \sum_{0 \leq i \leq m, 0 \leq j \leq n} \frac{1}{i! j!} \partial_x^i \partial_p^j (f)(0,0)(x - x_0)^i p^j. \]

Define \( g_{m,n} = g_1 + (x - x_0)^{m+1} g_2 \) where \( g_2 \) is some polynomial to be determined. It is easy to check that \( \partial_x^i \partial_p^{m+1} g_{m,n} \) agrees with \( \partial_x^i \partial_p^j f(x_0,0) \). We proceed to look for \( g_2 \) such that \( \partial_x^i \partial_p^j g_{m,n}(-x_0,0) \) agrees with \( \partial_x^i \partial_p^j f(-x_0,0) \). We write

\[ g_2 = \sum_{1 \leq s \leq m, 1 \leq t \leq n} 1/s! t! a_{st}(x + x_0)^s p^t. \]

We need to solve \( a_{st} \). From the requirement that \( \partial_x^i \partial_p^j g_{m,n}(-x_0,0) = \partial_x^i \partial_p^j f(-x_0,0) \), we know that

\[ \partial_x^i \partial_p^j (g_1)(-x_0,0) + \left( \begin{array}{c} i \\ k \end{array} \right) \partial_x^{-k}(x - x_0)^{m+1} \partial_x^k \partial_p^j g_2(-x_0,0) = \partial_x^i \partial_p^j f(-x_0,0). \]

If we order \( a_{st} \) lexicographically, then it is not difficult to see that the above equations for \( 1 \leq i \leq m, 1 \leq j \leq n \) define a system of linear equations for variable \( a_{st} \). We notice that in Eq. (4), the leading term is \( a_{ij} \) with coefficient \( (2x_0)^{m+1} \). When \( i \) and \( j \) vary, we have a system of linear equations whose coefficient matrix is an upper triangular matrix with a nonzero number \( (2x_0)^{m+1} \) everywhere on the main diagonal. This implies that we have a unique solution for \( a_{st} \), and therefore a solution for \( g_{m,n} \).

**Case (4).** From the proof of Case (2), we learn that we need to construct \( g \) step by step. Firstly, define \( g_0 \) to be

\[ g_0 = \sum_{0 \leq i \leq m, 0 \leq j \leq n} \frac{1}{i! j!} \partial_x^i \partial_p^j f(x_0, y_0)(x - x_0)^i (p - p_0)^j. \]

We now look for \( g_1 \) of the form \( \sum_{0 \leq i \leq m, 0 \leq j \leq n} 1/i! j! a_{ij}(x + x_0)^i (p - p_0)^j \) such that \( \partial_x^i \partial_p^j (g_0 + (x - x_0)^{m+1} g_1) \) agrees with \( \partial_x^i \partial_p^j f \) at both \((x_0, p_0)\) and \((-x_0, p_0)\) for \( 0 \leq i \leq m, 0 \leq j \leq n \).

We notice that it is always true that \( \partial_x^i \partial_p^j (g_0 + (x - x_0)^{m+1} g_1)(x_0, p_0) = \partial_x^i \partial_p^j f(x_0, p_0) \) for \( 0 \leq i \leq m, 0 \leq j \leq n \). By the same arguments as in the proof of Case (2), we can find a unique family \( a_{ij} \) such that \( \partial_x^i \partial_p^j (g_0 + (x - x_0)^{m+1} g_1)(-x_0, p_0) \) is same to \( \partial_x^i \partial_p^j (f)(-x_0, p_0) \) for \( 0 \leq i \leq m, 0 \leq j \leq n \).

We next look for \( g_2 \) of the form \( \sum_{0 \leq i \leq m, 0 \leq j \leq n} 1/i! j! b_{ij}(x - x_0)^i (p + p_0)^j \) such that \( \partial_x^i \partial_p^j (g_0 + (x - x_0)^{m+1} g_1 + (p - p_0)^{n+1} g_2) \) agrees with \( \partial_x^i \partial_p^j f \) at \((x_0, p_0), (-x_0, p_0), (x_0, -p_0)\) for \( 0 \leq i \leq m, 0 \leq j \leq n \). Again, it not difficult to check that the partial derivatives
of these two functions agree at \((x_0, p_0)\) and \((-x_0, p_0)\) no matter what \(g_2\) is like. With the above arguments, we know that there exists a unique solution for \(b_{st}\) such that the derivatives of the two functions agree at \((x_0, -p_0)\).

Continuing the above procedure, we look for \(g_3\) of the form \(\sum_{0 \leq i \leq m, 0 \leq j \leq n} 1/!j!c_{ij}(x+x_0)^i(p+p_0)^j\) such that \(\partial_x^i \partial_p^j (g_0 + (x-x_0)^{m+1}g_1 + (p-p_0)^{n+1}g_2 + (x-x_0)^{m+1}(p-p_0)^{n+1}g_3)\) agrees with \(\partial_x^i \partial_p^j f\) at \((x_0, p_0)\), \((-x_0, p_0)\), \((x_0, -p_0)\), and \((-x_0, -p_0)\) for \(0 \leq i \leq m, 0 \leq j \leq n\). Again the two functions have the same derivatives at \((x_0, p_0)\), \((-x_0, p_0)\), \((x_0, -p_0)\), and \((-x_0, -p_0)\) no matter what \(g_3\) is like. The same arguments as in the proof of Case (2) shows that there is a unique solution for \(c_{ij}\).

In summary, we have fund a function \(g_{m,n} = g_0 + (x-x_0)^{m+1}g_1 + (p-p_0)^{n+1}g_2 + (x-x_0)^{m+1}(p-p_0)^{n+1}g_3\) such that \(\partial_x^i \partial_p^j g_{m,n}\) agrees with \(\partial_x^i \partial_p^j g_{m,n}\) at \((x_0, p_0)\), \((-x_0, p_0)\), \((x_0, -p_0)\), and \((-x_0, -p_0)\) for \(0 \leq i \leq m, 0 \leq j \leq n\).

\[\square\]

**Proposition 3.16.** The product \(*\) is associative on \(C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[h_1, h_2]]\). For \(\epsilon^{i\theta} \in U(1)\), the map \(x \mapsto e^{i\theta}x, p \mapsto e^{-i\theta}p\) defines a \(U(1)\) action on the algebra \((C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[h_1, h_2]], *)\)

**Proof.** We observe that \(\text{Poly}(\mathbb{R}^2) \rtimes \mathbb{Z}_2\) is closed under \(*\). If \(a_i\) \((i = 1, 2, 3)\) are monomials of degrees \((m_i, n_i)\), then \(\sum_j k_j^i (C^0_{j_1,l_1}(a_1, a_2) + C^1_{j_1,l_1}(a_1, a_2) \gamma)\) is the degree \((m_1 + m_2 - j, n_1 + n_2 - j)\) in the expansion of \((\text{Op}_k(a_1) \circ \text{Op}_k(a_2))\). Therefore,

\[
\sum_{j_1+j_2=j} \sum_l k^l \sum_{l_1+l_2=l} \left(\begin{array}{c}
(C^0_{j_1,l_1}(a_1, a_2), a_3) + C^1_{j_1,l_1}(C^1_{j_2,l_2}(a_1, a_2), \gamma(a_3)) \\
(C^0_{j_1,l_1}(C^1_{j_2,l_2}(a_1, a_2), \gamma(a_3)) + C^1_{j_1,l_1}(C^0_{j_2,l_2}(a_1, a_2), a_3))\gamma
\end{array}\right)
\]

is the degree \((m_1 + m_2 + m_3 - j, n_1 + n_2 + n_3 - j)\) component of the expansion of \((\text{Op}_k(a_1) \circ \text{Op}_k(a_2))\).

As the composition between operators on \(S(\mathbb{R})\) is associative, by Theorem 3.10 and Lemma 3.6, we conclude that the product \(*\) on \(\text{Poly}(\mathbb{R}^2) \rtimes \mathbb{Z}_2\) is associative by comparing components with degree \((m_1 + m_2 + m_3 - j, n_1 + n_2 + n_3 - j)\) and power \(k^l\) in the expansions of \((\text{Op}_k(a_1) \circ \text{Op}_k(a_2)) \circ \text{Op}_k(a_3)\) and \(\text{Op}_k(a_1) \circ (\text{Op}_k(a_2) \circ \text{Op}_k(a_3))\).

To prove that \(*\) is associative on \(C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2\), it is sufficient to check that \(*\) is associative at every point \((x, p)\) up to any \(h^l i^l\). We notice that \(C^i_{ji}(a_1, a_2)(x, p)\) is determined by the values of \(a_1, \partial_y a_1, \ldots, \partial^i_p a_1\) at \((x, p)\) and \((x, -p)\), together with values of \(a_2, \partial_y a_2, \ldots, \partial^i_p a_2\) at \((x, p), (x, -p), (x, p), (x, -p)\). Therefore, to check \(((a_1 \ast a_2) \ast a_3)(x, p)\) agrees with \((a_1 \ast (a_2 \ast a_3))(x, p)\) up to degree \(h^l i^l\), it sufficient to check \((b_1 \ast b_2) \ast b_3(x, p)\) agrees with \(b_1 \ast (b_2 \ast b_3)(x, p)\) up to degree \(h^l i^l\) for polynomials \(b_1, b_2, b_3\) where the values of \(\partial^i_p b_i\) at \((x, p), (x, -p), (x, p), (x, -p)\) agree with the corresponding values of \(\partial^i_x a_i\) for \(i = 1, 2, 3\), \(1 \leq s, t \leq j\). Hence by the associativity of \(*\) on \(\text{Poly}(\mathbb{R}^2) \rtimes \mathbb{Z}_2\) and Lemma 3.15, we conclude that \(*\) is associative on \(C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2\).

For the action of \(t = e^{i\theta}\), we notice that \(t = e^{i\theta}\) acts on operators \(A_\nu\) and \(B_\nu\) with eigenvalues \(t^{-j}\) and \(t^j\). Therefore, one can quickly check that \(C^i_{ji}\) is a \(U(1)\) invariant bilinear operator on \(C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[h_1, h_2]]\) for any \(i, j, l\). Therefore, \(U(1)\) acts on \(C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[h_1, h_2]]\) by algebra automorphisms.

\[\square\]

**Remark 3.17.** The algebra \((C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[h_1, h_2]], *)\) is the formal version of a symplectic reflection algebra \([6]\) for \(\mathbb{Z}_2\) action on the standard symplectic vector space \(\mathbb{R}^2\).
Theorem 3.10 gives an operator interpretation of this symplectic reflection algebra and furthermore a Moyal type expansion formula.

Let \( P = (1 + \gamma)/2 \in C^\infty(\mathbb{R}^2) \times \mathbb{Z}_2 \). Consider the subspace of \( A = (C^\infty(\mathbb{R}^2) \times \mathbb{Z}_2[[h_1, h_2]], \star) \) defined by \( P \star A \star P = P \star C^\infty(\mathbb{R}^2) \times \mathbb{Z}_2[[h_1, h_2]] \star P \). In [6], Etingof and Ginzburg proved that \( P \star A \star P \) is Morita equivalent to \( A \). In particular, one can quickly check that the space \( P \star A \star P \) as a vector space is isomorphic to \( C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}[[h_1, h_2]] \). Via the natural identification,

\[
a \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}[[h_1, h_2]] \mapsto aP \in C^\infty(\mathbb{R}^2) \times \mathbb{Z}_2[[h_1, h_2]],
\]

\( C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}[[h_1, h_2]] \) is equipped with a star-product which we will again denote by \( \star \). We call this algebra Dunkl-Weyl algebra \( \mathbb{D}_2 \), which is called the spherical subalgebra by Etingof and Ginzburg [6]. By Proposition 3.16, we conclude that the Dunkl-Weyl algebra \( \mathbb{D}_2 \) is an associative algebra with a natural \( U(1) \) action. In the later applications, we many times will work with the algebra \( \mathbb{D}_2 \otimes_{\mathbb{C}} \mathbb{C}((h_1, h_2)) \), which will be denoted by \( \mathbb{D}_2((h_1, h_2)) \).

4. Quantization of \( \mathbb{Z}_2 \)-orbifold

In this section, we consider deformation quantization of \( \mathbb{Z}_2 \)-orbifolds. Let \( M \) be a symplectic manifold with a symplectic \( \mathbb{Z}_2 \) action. As \( \mathbb{Z}_2 \) is finite, we can always find a \( \mathbb{Z}_2 \) invariant symplectic connection on \( M \). Using Fedosov’s method, we can construct a \( \mathbb{Z}_2 \) invariant star-product \( \star \) on \( C^\infty(M)[[h]] \). (As our construction is local, it works more generally for an orbifold which locally is a quotient of a \( \mathbb{Z}_2 \) action.) The restriction of the invariant star-product on \( C^\infty(M)^{\mathbb{Z}_2}[[h]] \) defines a deformation quantization of the orbifold \( X = M/\mathbb{Z}_2 \). We use \( A_{M/\mathbb{Z}_2}^{(h)} \) to denote the quantized algebra on \( M/\mathbb{Z}_2 \) with the characteristic class equal to \( \omega \) (with \( A_{M/\mathbb{Z}_2}^{(h)} \), we refer to the algebra \( C^\infty(M)^{\mathbb{Z}_2} \otimes_{\mathbb{C}} \mathbb{C}((h)) \) with the extended star-product \( \star \)).

According to [4, Theorem 1.1] and [13, Theorem VII], the Hochschild cohomology of \( A_{M/\mathbb{Z}_2}^{(h)} \) is equal to the cohomology of the corresponding inertia orbifold with coefficient in \( \mathbb{C}((h)) \). In the case of \( M/\mathbb{Z}_2 \), the corresponding inertia orbifold is defined to be \( \tilde{X} := M/\mathbb{Z}_2 \sqcup M^\gamma/\mathbb{Z}_2 \), where \( M^\gamma \) is the fixed submanifold of the group element \( \gamma \) in \( \mathbb{Z}_2 \). If \( M^\gamma \) has several components maybe of different dimensions, we will take the disjoint union of all components. We use \( \ell \) to denote the codimension of \( M^\gamma \) in \( M \), and \( \ell \) is a locally constant function on \( X \). We point out that \( \mathbb{Z}_2 \) acts on \( M^\gamma \) trivially, but we will view \( M^\gamma \) as an orbifold with a global stabilizer group \( \mathbb{Z}_2 \). We have

\[
H^\bullet(A_{M/\mathbb{Z}_2}^{(h)}; A_{M/\mathbb{Z}_2}^{(h)}) = H^{\bullet - \ell}(\tilde{X}, \mathbb{C}((h))).
\]

Looking at Equation (5), we conclude that the second Hochschild cohomology of \( A_{M/\mathbb{Z}_2}^{(h)} \) is equal to a direct sum of \( H^2(M/\mathbb{Z}_2, \mathbb{C}((h))) \) and \( H^0(M^\gamma_2/\mathbb{Z}_2, \mathbb{C}((h))) \) for the components \( M^\gamma_2 \) of \( M^\gamma \) with codimension 2 (we have degree 0 cohomology on \( M^\gamma_2 \) because of the degree shifting in Equation (5)). From the experience of deformation quantization of a symplectic manifold, we know that the component \( H^2(M/\mathbb{Z}_2, \mathbb{C}((h))) \) of \( H^2(A_{M/\mathbb{Z}_2}^{(h)}, A_{M/\mathbb{Z}_2}^{(h)}) \) corresponds to isomorphism classes of \( \mathbb{Z}_2 \) invariant deformation quantizations on \( M \). In the following of this section, we construct deformations of \( A_{M/\mathbb{Z}_2}^{(h)} \) corresponding to
This gives a partial positive answer to [4, Conjecture 1] in the case of $Z_2$ orbifolds. We construct a deformation of $A_{M/Z_2}^{(\hbar)}$ in 3 steps,

1. Dunkl-Weyl algebra bundle,
2. Quantization of punctured disk bundle,
3. Global quantization.

We briefly explain the strategy before we go into the details of the construction. In the first step, we will quantize the normal bundle of the $\gamma$ fixed point submanifold with codimension 2. Quantization of normal bundle of a fixed point submanifold has been considered by Fedosov [9] and Kravchenko [12]. Here the new input is that along the fiber direction of the normal bundle, we will use the Dunkl-Weyl algebra introduced at the end of Section 3. The main result will be that with the new algebra $D_2((\hbar_1, \hbar_2))$, the construction of Fedosov [9] and Kravchenko [12] has a natural generalization and we obtain a flat connection on the associated Dunkl-Weyl algebra bundle. This first step can be viewed as a quantization of a tubular neighborhood of the $\gamma$ fixed point submanifolds.

In Step 2, we restrict the quantization we obtained in Step 1 to a punctured tubular neighborhood of the $\gamma$ fixed point submanifold with the zero section removed. We are allowed to restrict this quantization because of the locality of the product $\star$ on $D_2((\hbar_1, \hbar_2))$ discussed in Section 3, Theorem 3.10. An important property of the punctured tubular neighborhood is that the $Z_2$ action on it is free, and there is no fixed point. Therefore, quantizations of such a punctured neighborhood can be classified by Fedosov’s theory without any extra contribution from the fixed point submanifold.

In Step 3, we will extend the quantization obtained in Step 1 of the tubular neighborhood of the $\gamma$ fixed point submanifold with codimension 2 to the whole orbifold. Here the key is that with the study in Step 2, we can regularize the quantization obtained in Step 1 on the punctured tubular neighborhood. Namely, it is isomorphic to some standard quantization of the punctured tubular neighborhood using Fedosov’s construction via the characteristic classes developed by Fedosov [8] and Kravchenko [12]. We point out the above strategy is possible to be generalized by replacing the Dunkl-Weyl algebra $D_2((\hbar_1, \hbar_2))$ by the spherical subalgebra of other symplectic reflection algebras [6] if we know the product is “local”.

4.1. Dunkl-Weyl algebra bundle. We consider the collection of connected components of $M^\gamma$ which are of codimension 2, and we denote it by $M_2^\gamma$. The symplectic orthogonal space of $TM_2^\gamma$ in $TM|M_2^\gamma$ defines a normal bundle $N$ of $M_2^\gamma$ in $M$. $N$ inherits a $Z_2$ action from the $Z_2$ action on $M$. The restriction of the symplectic form $\omega$ to $N$ makes $N$ a $Z_2$ equivariant symplectic vector bundle with the symplectic structure $\omega^N$. We will fix a global $Z_2$ invariant compatible almost complex structure on $M$. (Such an almost complex structure always exists.) An invariant almost complex structure makes $N$ into a $Z_2$ equivariant hermitian line bundle. In particular, the corresponding principal bundle $P$ associated to $N$ is a principal $U(1)$ bundle. By Proposition 3.16, $U(1)$ naturally acts on the Dunkl-Weyl algebra ($D_2((\hbar_1, \hbar_2)), \star$). Therefore, we define the following Dunkl-Weyl algebra bundle over $M_2^\gamma$ by

$$\mathcal{V} := P \times_{U(1)} D_2((\hbar_1, \hbar_2)).$$
We have constructed a bundle \( \mathcal{V} \) of infinitely dimensional algebras over a symplectic manifold \( M_2^\gamma \). The hermitian connection on the principal bundle \( P \) induces a connection on the Dunkl-Weyl algebra bundle. We exhibit this connection in local coordinates. Let \( x^\nu (\nu = 1, \ldots, 2n - 2) \) be coordinates on \( M_2^\gamma \) and \( z, \bar{z} \) be coordinates along the fiber direction. The hermitian connection \( \nabla^N \) on \( N \) can be written as

\[
\nabla^N_{\partial z} \partial_z = i \Gamma_\nu (x) \partial_z, \quad \nabla^N_{\partial \bar{z}} \partial_z = -i \Gamma_\nu (x) \partial_{\bar{z}},
\]

where \( \Gamma_\nu \) is a real valued function on \( M_2^\gamma \).

The induced connection \( \partial^N \) on \( \mathcal{V} \) is defined by

\[
\partial^N \xi = dx^i \otimes \left( \frac{\partial \xi}{\partial x^i} + \frac{i}{2\hbar_1} [\Gamma, z\bar{z}, \xi]_* \right), \quad \xi \in \Gamma(\mathcal{V}),
\]

where \([ , ]_*\) is the star-commutator.

Let \( R^N_{\nu_1\nu_2} \) be the curvature tensor associated to the hermitian connection \( \nabla \). Then one can quickly compute that

\[
\partial^N \circ \partial^N (\xi) = \frac{1}{2\hbar_1} [dx^{\nu_1} \wedge dx^{\nu_2} R_{\nu_1\nu_2} z\bar{z}, \xi]_*.
\]

We remark that because \( N \) is a complex 1-dim vector bundle, \( h_2 \) does not appear in the above curvature expression although it does show up in general in the star-product.

Let \( \mathcal{W} \) be the Weyl algebra (with coefficient in \( \mathbb{C}((h_1)) \)) bundle associated to the symplectic form \( \omega^0 \) on \( M_2^\gamma \). Following Fedosov’s method [8] and Kravchenko’s modification [12], we construct a flat connection \( D \) on the associated bundle

\[
\wedge^* T^\ast M_2^\gamma \otimes \mathcal{W} \otimes \mathcal{V}.
\]

We remark that \( \mathcal{W} \) is a bundle of algebras with respect to the ring \( \mathbb{C}((h_1)) \), and \( \mathcal{V} \) is an algebra with respect to the ring \( \mathbb{C}((h_1, h_2)) \). The tensor product between \( \mathcal{W} \) and \( \mathcal{V} \) is taken over the ring \( C^\infty(M)((h_1)) \). Though our construction is essentially a repetition of the ones in [12], since the Dunkl-Weyl algebra is a new ingredient, we recall the construction of the flat connection on \( \wedge^* T^\ast M_2^\gamma \otimes \mathcal{W} \otimes \mathcal{V} \) briefly.

Let \( \nabla^T \) be a symplectic connection on \( T M_2^\gamma \) with respect to the symplectic form \( \omega_0 \). If \( \Gamma^k_{ij} \) be the Christoffel symbol associated to the connection \( \nabla^T \) on \( T M_2^\gamma \), then

\[
\partial^T \eta = d\eta + \frac{i}{2\hbar_1} [\omega_0 \Gamma^i_{jk} y^j y^k dx^i, \eta]_*, \quad \eta \in \mathcal{W}
\]

defines a connection on the Weyl algebra bundle \( \mathcal{W} \), where \( y^i, i = 1, \ldots, 2n - 2 \) are coordinates along the fiber direction of \( T M_2^\gamma \) and \([ , ]_*\) is the commutator with respect to the star-product on \( \mathcal{W} \). Accordingly, \( \partial := \partial^T \otimes 1 + 1 \otimes \partial^N \) defines a connection on the bundle \( \mathcal{W} \otimes \mathcal{V} \). It is a straightforward computation to find

\[
\partial^2 a = \frac{i}{\hbar_1} [R^T \otimes 1 + 1 \otimes R^N, a], \quad a \in \Gamma^\infty (\mathcal{W} \otimes \mathcal{V}),
\]

where \( R^T \) is the curvature form of \( \partial^T \).

Define \( \delta : \wedge^* T^\ast M_2^\gamma \otimes \mathcal{W} \otimes \mathcal{V} \rightarrow \wedge^* T^\ast M_2^\gamma \otimes \mathcal{W} \otimes \mathcal{V} \) by \( \delta(a) = \sum_{i=1}^{2n-2} dx^i \partial a/\partial y^i \).

Then the same proof as [12, Thm. 5.5] proves that there is a flat connection \( D \) on
\[ D = d + \frac{i}{\hbar} [\gamma, \cdot] = \partial + \frac{i}{\hbar} [r, \cdot], \]

where \( r \) is an element in \( T^*M_2^\gamma \otimes \mathcal{W} \otimes \mathcal{V} \). The key point in the Kravchenko’s proof of [12, Theorem 5.5] is that one has to modify the definition of the operator \( \delta \) to compensate the existence of the curvature form \( R^N \) in the expression of \( \partial^N \) because \( i/\hbar R^N \) will contribute an extra term of degree -1 in Fedosov’s iteration procedure of constructing a flat connection.

By the definition of the star-product on the Dunkl-Weyl algebra, one can easily check that for \( j \leq 2 \), \( C_j^0 \) and \( C_j^1 \) vanishes on \( z\bar{z} \). Therefore, we have for an arbitrary \( f \in \mathbb{D}_2 \), as \( f \) is \( \mathbb{Z}_2 \) invariant,

\[
[z\bar{z}, f]_\star = h_1 \left( \{ z\bar{z}, f \} + h_2 \left( \frac{(z\bar{z} + z\bar{z}) (f(z, \bar{z}) - f(-z, -\bar{z}))}{2z} - \frac{(f(z, \bar{z}) - f(z, -\bar{z})) (-z\bar{z} - \bar{z}z)}{2z} \right) \right)
\]

where \( \{ ., . \} \) is the Poisson structure \( \{ f, g \} = i(\partial_z f \partial_{\bar{z}} g - \partial_{\bar{z}} g \partial_z f) \). This shows that on the bundle \( \mathcal{V} \) with the fiberwise algebra isomorphic to \( \mathbb{D}_2((h_1, h_2)) \), the curvature of the connection \( \partial^N \) which is equal to the star-commutator of \( R^N \) with respect to the product on \( \mathbb{D}_2((h_1, h_2)) \) acts as same as the Poisson commutator associated to the restriction of the symplectic form \( \omega^N \) along the fiber direction of \( N \). With this observation, we can repeat the exactly same construction as those in Kravchenko’s proof of [12, Theorem 5.5]. And we conclude that there is a flat connection \( D \) on the bundle \( \wedge^* T^* M_2^\gamma \otimes \mathcal{W} \otimes \mathcal{V} \) whose Weyl curvature is equal to \( \omega \), the pullback of the symplectic form on \( M \) to the normal bundle \( N \).

4.2. Quantization of punctured disk bundle. With the above flat connection \( D \) on \( \mathcal{W} \otimes \mathcal{V} \), we consider flat sections with respect to the connection \( D \). The space \( \mathcal{A}_D \) of flat sections is isomorphic \( C^\infty(M_2^\gamma, \mathcal{V}) \) as a vector space. Furthermore, including the isomorphism between \( \mathbb{D}_2((h_1, h_2)) \) with \( \text{Poly}(\mathbb{R}^2)^{\mathbb{Z}_2}((h_1, h_2)) \) as vector spaces, we conclude that \( \mathcal{A}_D \) is isomorphic to the space of functions on \( M_2^\gamma \) with value in the associated bundle \( P \otimes_{U(1)} \text{Poly}(\mathbb{R}^2)^{\mathbb{Z}_2}((h_1, h_2)) \). The later space can be viewed as the space \( C^\infty(N)^{\mathbb{Z}_2}((h_1, h_2)) \) of \( \mathbb{Z}_2 \)-invariant smooth functions on \( N \). The identification between \( C^\infty(N)^{\mathbb{Z}_2}((h_1, h_2)) \) and \( \mathcal{A}_D \) equips \( C^\infty(N)^{\mathbb{Z}_2}((h_1, h_2)) \) with a new associative product, which is a deformation of the standard commutative product on \( C^\infty(N)^{\mathbb{Z}_2}((h_1, h_2)) \).

Via the exponential map with respect to some \( \mathbb{Z}_2 \) invariant metric on \( N \), we can identify a tubular neighborhood \( B_\epsilon \) of \( M_2^\gamma \) in \( M \) with an \( \epsilon \) neighborhood \( N_\epsilon \) of the zero section in \( N \) for some \( \epsilon > 0 \). As was discussed in Theorem 3.10, the star-product on \( \mathbb{D}_2((h_1, h_2)) \) is \( \gamma \)-local. Furthermore, the product on the standard Weyl algebra is also local. These locality results imply that the deformed product on \( C^\infty(N)^{\mathbb{Z}_2}((h_1, h_2)) \) is \( \gamma \)-local, and therefore, we are allowed to restrict \( (C^\infty(N)^{\mathbb{Z}_2}((h_1, h_2)), \star) \) to the \( \epsilon \) neighborhood \( N_\epsilon \) of the zero section in \( N \), which defines an associative deformation of \( \mathbb{Z}_2 \)-invariant functions on \( N_\epsilon \). Finally, pushing forward along the exponential map, we obtain a deformation of \( \mathbb{Z}_2 \)-invariant smooth functions on \( B_\epsilon \), namely \( (C^\infty(B_\epsilon)^{\mathbb{Z}_2}((h_1, h_2)), \star) \).
We next look at the space $B^*_\epsilon := B_\epsilon - M_2^\epsilon$ of punctured neighborhood, which is diffeomorphic to $N_\epsilon - M_2^\epsilon$, the punctured disk bundle via the exponential map. As the star product on the Dunkl-Weyl algebra $\mathbb{D}_2$ is $\gamma$-local, we can restrict the algebra $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star)$ to the punctured neighborhood, which is denoted by $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star)$.

We observe that as the $\mathbb{Z}_2$ action on $B^*_\epsilon$ is free, the space of $\mathbb{Z}_2$-invariant functions on $B^*_\epsilon$ can be identified with the space of functions on the quotient $B^*_\epsilon/\mathbb{Z}_2$, which is a smooth manifold. Hence the algebra $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star)$ can be viewed as a deformation quantization of the quotient $B^*_\epsilon/\mathbb{Z}_2$. On the other hand, the $\mathbb{Z}_2$-invariant symplectic form $\omega$ on $M$ restricts to define a symplectic form on the quotient $B^*_\epsilon/\mathbb{Z}_2$. As $B^*_\epsilon/\mathbb{Z}_2$ is a smooth symplectic manifold, one can apply the standard Fedosov construction of a deformation quantization on $B^*_\epsilon/\mathbb{Z}_2$, and therefore obtain an associative algebra $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star_F)$ with the characteristic class equal to $\omega$.

**Proposition 4.1.** The two algebras $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star)$ and $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star_F)$ are isomorphic.

**Proof.** We consider an intermediate algebra to relate the above two algebras. We look at the normal bundle $N$ over $M_2^\epsilon$. The restriction of the symplectic form $\omega$ to each fiber makes $N$ into a symplectic vector bundle. We consider the associated Weyl algebra bundle to $\mathcal{V}_N$ by $\mathcal{V}_N := P \times_{U(1)} \mathbb{W}_2^{\mathbb{Z}_2}$, where $\mathbb{W}_2$ is the tensor of the Weyl algebra (with coefficient in $\mathbb{C}((h_1))$) on $\mathbb{R}^2$ with $\mathbb{C}((h_2))$ and $\mathbb{W}_2^{\mathbb{Z}_2}$ is the $\mathbb{Z}_2$ invariant subalgebra of $\mathbb{W}_2$. Similar to what we have done in Section 4.1, we can construct a flat connection $D_N$ on the bundle $\wedge^\cdot T^* M_2^\epsilon \otimes \mathcal{W} \otimes \mathcal{V}_N$. The space of flat sections with respect to the flat connection $D_N$ is isomorphic to $(C^\infty(N))^{\mathbb{Z}_2}((h_1, h_2))$. Therefore we obtain an associative algebra $(C^\infty(N))^{\mathbb{Z}_2}((h_1, h_2)), \star_W)$ as a deformation quantization of $(C^\infty(N))^{\mathbb{Z}_2}((h_1, h_2))$. As the Weyl algebra $\mathbb{W}_2$ has a local product, the algebra $(C^\infty(N))^{\mathbb{Z}_2}((h_1, h_2)), \star_W)$ restricts to the punctured disk bundle $N_\epsilon - M_2^\epsilon$. And via the exponential map with respect to the $\mathbb{Z}_2$ invariant riemannian metric, $(C^\infty(N))^{\mathbb{Z}_2}((h_1, h_2)), \star_W)$ restricts to define a deformation quantization of the punctured tubular neighborhood, $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star_W)$.

Now we compare the algebra $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star_W)$ with $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star_F)$. According to [9, Sec. 5] and [12, Thm. 5.6], these two algebras are isomorphic as they have the same characteristic class $\omega$. To compare the algebra $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star)$ with $(C^\infty(B^*_\epsilon))^{\mathbb{Z}_2}((h_1, h_2)), \star_W)$, we see that the procedure to obtain these two algebras are different only at one step, where $V$ is $P \times_{U(1)} \mathbb{D}_2((h_1, h_2))$ and $\mathcal{V}_N$ is $P \times_{U(1)} \mathbb{W}_2^{\mathbb{Z}_2}$. We have pointed out that the product on $\mathbb{D}_2((h_1, h_2))$ is $\gamma$-local and the product on $\mathbb{W}_2^{\mathbb{Z}_2}$ is local. Therefore, the restrictions of $(C^\infty(N))^{\mathbb{Z}_2}((h_1, h_1)), \star)$ and $(C^\infty(N))^{\mathbb{Z}_2}((h_1, h_2)), \star_W)$ to $N_\epsilon - M_2^\epsilon$ can be constructed via the flat connections $D$ and $D_N$ on the bundle $\wedge^\cdot T^* M_2^\epsilon \otimes \mathcal{W} \otimes (P \times_{U(1)} \mathbb{D}_2((h_1, h_2))|_{D^\epsilon})$ and $\wedge^\cdot T^* M_2^\epsilon \otimes \mathcal{W} \otimes (P \times_{U(1)} \mathbb{W}_2^{\mathbb{Z}_2}|_{D^\epsilon})$, where $D^\epsilon$ is the puncture disk of radius $\epsilon$ in $\mathbb{R}^2$.

The algebras $\mathbb{D}_2((h_1, h_2))|_{D^\epsilon}$ and $\mathbb{W}_2^{\mathbb{Z}_2}|_{D^\epsilon}$ are both deformation quantization of the punctured disk $D^\epsilon/\mathbb{Z}_2$. In fact, if we look at the algebra $\mathbb{D}_2((h_1, h_2))|_{D^\epsilon}$ more carefully, we notice that the product of this algebra is an expression of $f, g \in C^\infty(D^\epsilon)^{\mathbb{Z}_2}$ of power series $h_1$ and $h_2$. In particular, if we look at $f \cdot g$ as a formal power series of $h_2$, the 0-th power term is exactly the product on $\mathbb{W}_2^{\mathbb{Z}_2}|_{D^\epsilon}$. Therefore, we can view $\mathbb{D}_2((h_1, h_2))|_{D^\epsilon}$
as a formal deformation quantization of the algebra $\mathbb{W}_2^{Z_2}|_{D^*_t}$. As we can identify $\mathbb{W}_2^{Z_2}|_{D^*_t}$ as a deformation quantization of the quotient $D^*_t/\mathbb{Z}_2$, its Hochschild cohomology of $\mathbb{W}_2^{Z_2}|_{D^*_t}$ can be computed using the result of [13]. In particular, the second Hochschild cohomology of $\mathbb{W}_2^{Z_2}|_{D^*_t}$ is equal to the degree 2 de Rham cohomology of $D^*_t/\mathbb{Z}_2$ with coefficient in $\mathbb{C}((h_1, h_2))$. As $D^*_t/\mathbb{Z}_2$ is homotopic to a circle, its degree 2 de Rham cohomology is zero. This implies that $\mathbb{D}_2((h_1, h_2))|_{D^*_t}$ must be a trivial deformation of $\mathbb{W}_2^{Z_2}|_{D^*_t}$. Furthermore, as $U(1)$ is compact, by the standard average trick, we can obtain a $U(1)$ equivariant isomorphism from $\mathbb{D}_2((h_1, h_2))|_{D^*_t}$ to $\mathbb{W}_2^{Z_2}|_{D^*_t}$.

With the above $U(1)$ equivariant isomorphism between the algebras on each fiber, we have an natural isomorphism of bundles $\mathcal{V}$ and $\mathcal{V}_F$ which accordingly identifies the connections $\partial_N$ on the corresponding bundles. Noticing that Fedosov’s construction of flat connection is canonical with respect to the choice of a symplectic connection, we can easily check that the construction of flat connections $D$ and $D_W$ are actually compatible with respect to this isomorphism of bundles. Hence, we can conclude that there is an isomorphism between $(C^\infty(B^*_t)^{Z_2}((h_1, h_2)), \star)$ and $(C^\infty(B^*_t)^{Z_2}((h_1, h_2)), \star_W)$ as flat sections of $D$ and $D_W$.

We conclude that $(C^\infty(B^*_t)^{Z_2}((h_1, h_2)), \star)$ is isomorphic to $(C^\infty(B^*_t)^{Z_2}((h_1, h_2)), \star_W)$ and therefore is isomorphic to $(C^\infty(B^*_t)^{Z_2}((h_1, h_2)), \star_F)$. \hfill \square

4.3. Global algebra. In this subsection, we construct the algebra promised at the beginning of this section, which is a deformation of $A_{M/Z_2}^{((h_1))}$.

Recall that $M_2^*$ is a disjoint union of fixed point submanifolds of $M$ which are of codimension 2. Fix a $\mathbb{Z}_2$-invariant almost complex structure on $M$, which also defines a $\mathbb{Z}_2$ invariant metric on $M$. We choose a sufficiently small $\epsilon$ such that the $\epsilon$ tubular neighborhood of each component of $M_2^*$ in $M$ does not intersect with each other. We use $B_\epsilon$ to denote the disjoint union of the $\epsilon$ tubular neighborhood of each component of $M_2^*$. Furthermore, we denote $M^-$ to be the open complement $M - M_2^*$ to the closed subset $M_2^*$. In this way, we have the orbifold as a union of two open subsets $B_\epsilon/\mathbb{Z}_2$ and $M^-/\mathbb{Z}_2$, and the intersection of these two open sets is $B^*_\epsilon/\mathbb{Z}_2$, the $\epsilon$ punctured neighborhood of $M^*$ in $M/\mathbb{Z}_2$.

We construct an algebra $\mathfrak{A}_{M/Z_2}^{((h_1, h_2))}$ as follows. On $B_\epsilon/\mathbb{Z}_2$, this algebra is isomorphic to $(C^\infty(B^*_\epsilon)^{Z_2}((h_1, h_2)), \star)$ which, via the exponential map, can be identified as the restriction to the bundle $N_\epsilon$ of the space of flat sections of the Dunkl-Weyl algebra introduced in Section 4.1. On $M^-/\mathbb{Z}_2$, the algebra $\mathfrak{A}_{M/Z_2}^{((h_1, h_2))}$ is isomorphic to the Fedosov quantization\(^2\) $A_{M^-/\mathbb{Z}_2}^{((h_1, h_2))}$ of $M^-$ with the Weyl curvature being $\omega$. By Proposition 4.1, the restriction of $(C^\infty(B^*_\epsilon)^{Z_2}((h_1, h_2)), \star)$ to $B^*_\epsilon/\mathbb{Z}_2 = M^-/\mathbb{Z}_2 \cap B^*_\epsilon/\mathbb{Z}_2$ is isomorphic to $(C^\infty(B^*_\epsilon)^{Z_2}((h_1, h_2)), \star_F)$, which is the restriction of $(A_{M^-/\mathbb{Z}_2}^{((h_1, h_2))})^{Z_2}$ to $B^*_\epsilon/\mathbb{Z}_2$. We define $\mathfrak{A}_{M/Z_2}^{((h_1, h_2))}$ to be the algebra defined by gluing $(C^\infty(B^*_\epsilon)^{Z_2}((h_1, h_2)), \star)$ and $(A_{M^-/\mathbb{Z}_2}^{((h_1, h_2))})^{Z_2}$ via the isomorphism on $B^*_\epsilon/\mathbb{Z}_2$.

We summarize the above construction into the following theorem.

**Theorem 4.2.** The algebra $\mathfrak{A}_{M/Z_2}^{((h_1, h_2))}$ is a nontrivial deformation of the algebra $A_{M/Z_2}^{((h_1))}$.

\(^2\)The algebra $A_{M^-/\mathbb{Z}_2}^{((h_1, h_2))}$ is defined to $A^{((h_1))} \otimes \mathbb{C}((h_2))$. 
Proof. We look at the product on the algebra $\mathfrak{A}^{h_1,h_2}$. If $f$ and $g$ are two elements of $C^\infty(M/\mathbb{Z}_2)$, $f \ast g$ can be written as a formal power series of $h_2$. From the construction in Section 4.1, it is not difficult to see that the $h_2^0$ component is exactly the product of the algebra $A_{M/\mathbb{Z}_2}^{h_1}$. Furthermore, from the local computation on $B_\epsilon$, we can see that as $D_2((h_1,h_2))$ is a nontrivial deformation of the invariant Weyl algebra $W^Z_2$, the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}^{h_1,h_2}$ is a nontrivial deformation. \hfill $\Box$

Remark 4.3. In the construction of the algebra $\mathfrak{A}^{((h_1,h_2))}$, we have chosen an $\epsilon$ neighborhood $B_\epsilon$ of $M^X_2$. We point out that different choices of $\epsilon$ give rise to isomorphic algebras $\mathfrak{A}^{((h_1,h_2))}$. One can easily check the $U(1)$ equivariant isomorphism from $D_2((h_1,h_2))\mid_{D^*_1}$ to $W^Z_2\mid_{D^*_1}$ constructed in Proposition 4.1 can be made to be compatible with the restriction map to $B_\epsilon$ with $\epsilon' < \epsilon$, as the operators appearing in the isomorphism are all $\gamma$-local operators. This compatibility with respect to the restriction map assures that the outcome algebra $\mathfrak{A}^{((h_1,h_2))}$ are all isomorphic for different choices of $\epsilon$.

We have used an almost complex structure and therefore a compatible riemannian metric to identify the $\epsilon$ neighborhood $B_\epsilon$ with the $\epsilon$ neighborhood $N_\epsilon$ of the zero section of $N$. Our algebra $\mathfrak{A}^{((h_1,h_2))}$ does seem to depend on the choice of almost complex structures since we are taking the normal ordering of the operator symbol calculus in Definition 3.1 and also in Definition 3.14 of the Dunkl-Weyl algebra $D_2((h_1,h_2))$. The analogous well-known phenomena is that wick and anti-wick deformation quantization of an almost Kähler manifold depends on the choices of almost complex structures. We plan to discuss this dependence of almost complex structures in the future.

Remark 4.4. We explain how far we are away from a full proof of the Dolgushev-Etingof conjecture in the case of a $\mathbb{Z}_2$ orbifold. In this section, we have constructed a deformation of the algebra $A_{M/\mathbb{Z}_2}^{((h_1))}$ along the direction of the union of all codimension 2 components in the inertia orbifold $\mathbb{M}/\mathbb{Z}_2$. Furthermore, we observe that our constructions are local with respect to every connected component $M^X_2$. Such an observation allows us to construct a deformation of $A_{M/\mathbb{Z}_2}^{((h_1))}$ for every connected component of $M^X_2$. If we associate a formal parameter $c_i$ for every component of $M^X_2$, we actually have constructed a universal deformation of $A_{M/\mathbb{Z}_2}^{((h_1))}$ parametrized by codimension 2 components in $\mathbb{M}/\mathbb{Z}_2$. This is the main part of the Dolgushev-Etingof conjecture [4].

We are not able to prove the full conjecture of Dolgushev-Etingof for $\mathbb{Z}_2$ orbifolds because in our construction, in particular the proof of Proposition 4.1, we have used crucially two extra assumptions. One is that we have assumed the characteristic class of $A_{M/\mathbb{Z}_2}^{((h_1))}$ is $\omega$, the other is that the parameter $h_2$ is formal. However, the Dolgushev-Etingof conjecture [4] does not require these two assumptions. The first assumption allows us to compare the Fedosov quantizations of the normal bundle $N$ of the fixed point submanifold $M^X_2$ and a tubular neighborhood $B_\epsilon$. If we change the characteristic class $\omega$ by a class $t$ in $H^2(M/\mathbb{Z}_2)((h_1))$, it is hard to realize the information of $t$ on the normal bundle $N$, which prevents us from comparing the corresponding Fedosov quantizations. The second assumption that $h_2$ is formal allows us apply homological arguments to show that $D_2((h_1,h_2))\mid_{D^*_1}$ is isomorphic to $W^Z_2\mid_{D^*_1}$. If we are able to prove that the constructed isomorphism in power series of $h_2$ is convergent, then we can actually allow $h_2$ to be a
number in \( \mathbb{C} \). We do not have solutions to avoid these two assumptions now, and hope to address these problems in a future publication.

We finally remark that in this section, we have been working with the globalquotient orbifold, the quotient of a symplectic manifold \( M \) by a \( \mathbb{Z}_2 \) action. Our construction does generalize for general orbifolds which is locally either diffeomorphic to \( \mathbb{R}^n \) or thequotient of \( \mathbb{R}^n \) by the linear \( \mathbb{Z}_2 \) action.

5. PROOF THEOREM 3.10

We will prove Theorem 3.10 in 2 steps. In the first step, we work with Dunkl pseudo-differential operators of the forms introduced in Definition 3.2 to derive an asymptotic expansion of the symbol of the product of two operators. The asymptotic expansion of the symbol we obtain in Step I may contain a sum of infinitely many terms in a fixed symbol class. In the second step, we will rewrite the asymptotic expansion of the symbol obtained in Step I into the expressions introduced in Section 3.3.

5.1. Step I. Let \( a, b \) be two polynomials on \( \mathbb{R}^2 \). To compute the asymptotic expansion of \( Op_k(a) \circ Op_k(b) \) we need to study the following integral

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} d\mu_k(p) d\mu_k(y) d\mu_k(p_1) \sum_{\alpha} E_k(x, ip) \frac{p^{\alpha}}{\alpha!} \partial_p^\alpha a(x, 0) E_k(y, -ip) \cdot E_k(y, ip_1) b(y, p_1) E_k(z, -ip_1).
\]

Since \( a(x, p) \) is a polynomial, we take the Taylor expansion of \( a(x, p) \) with respect to \( p \), i.e. \( a(x, p) = \sum_{\alpha} \frac{p^{\alpha}}{\alpha!} \partial_p^\alpha a(x, 0) \).

We insert the Taylor expansion into the above equation of \( Op_k(a) \circ Op_k(b) \).

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} d\mu_k(p) d\mu_k(y) d\mu_k(p_1) \sum_{\alpha} E_k(x, ip) \frac{p^{\alpha}}{\alpha!} \partial_p^\alpha a(x, 0) E_k(y, -ip) \cdot E_k(y, ip_1) b(y, p_1) E_k(z, -ip_1).
\]

Recall that when applying the variable \( y \), we have \( T_k(E_k(y, -ip)) = (-ip) E_k(y, -ip) \). Hence we can replace \( pE_k(y, -ip) \) by \( iT_k(E_k(y, -ip)) \) in the above integral, and obtain

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} d\mu_k(p) d\mu_k(y) d\mu_k(p_1) \sum_{\alpha} E_k(x, ip) \frac{p^{\alpha-1}}{\alpha!} \partial_p^\alpha a(x, 0) iT_k(E_k(y, -ip)) \cdot E_k(y, ip_1) b(y, p_1) E_k(z, -ip_1).
\]

As \( T_k \) is a skew adjoint operator on \( L^2_k(\mathbb{R}) \), we can rewrite the above equation as

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} d\mu_k(p) d\mu_k(y) d\mu_k(p_1) \sum_{\alpha} E_k(x, ip) \frac{p^{\alpha-1}}{\alpha!} \partial_p^\alpha a(x, 0) E_k(y, -ip) \cdot (-i) T_k(E_k(y, ip_1) b(y, p_1)) E_k(z, -ip_1).
\]

We apply

\[
T_k(E_k(y, ip_1) b(y, p_1)) = ip_1 E_k(y, ip_1) b(y, p_1) + E_k(y, ip_1) \partial_y b(y, p_1) + E_k(-y, ip_1) k \tilde{\partial_y} b(y, p_1)
\]
We observe the following commuting relations, and obtain
\[
\int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} d\mu_k(p)d\mu_k(y)d\mu_k(p_1) \sum_\alpha E_k(x, ip) \frac{p^{\alpha-1}_p}{\alpha!} \partial_p^\alpha a(x, 0) E_k(y, -ip)
\]
\[
(p_1 E_k(y, ip_1) b(y, p_1) - iE_k(y, ip_1) \partial_y b(y, p_1) - E_k(-y, ip_1) k \tilde{\partial}_y b(y, p_1)) E_k(z, -ip_1).
\]
Substituting \( p_1 \) by \(-p_1\), we obtain the following expression
\[
\int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} d\mu_k(p)d\mu_k(y)d\mu_k(p_1) \sum_\alpha E_k(x, ip) \frac{p^{\alpha-1}_p}{\alpha!} \partial_p^\alpha a(x, 0) E_k(y, -ip)
\]
\[
E_k(y, ip_1)((p_1 - i\partial_y)b(y, p_1) - ik\tilde{\partial}_y b(y, -p_1\tilde{\gamma})) E_k(z, -ip_1),
\]
where \( \tilde{\gamma} \) is an operator on variable \( z \) changing \( z \) to \(-z\). In summary, we have seen above that an extra variable \( p \) on \( a(x, p) \) in the integral of \( Op_k(a) \circ Op_k(b) \) is equivalent to apply \( p_1 - i\partial_y - ik\sigma_2 \tilde{\partial}\gamma \) on \( b \), where \( \sigma_2 \) is mapping \( b(x, p) \) to \( b(x, -p) \).

By induction with respect to the power \( \alpha \), we have the following expression
\[
\int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} d\mu_k(p)d\mu_k(y)d\mu_k(p_1) E_k(x, ip) a(x, p) E_k(y, -ip) E_k(y, ip_1)
\]
\[
\quad b(y, p_1) E_k(z, -ip_1)
\]
\[
= \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} d\mu_k(p)d\mu_k(y)d\mu_k(p_1) E_k(x, ip) E_k(y, -ip) E_k(y, ip_1)
\]
\[
\quad \sum_\alpha \frac{1}{\alpha!} \partial_p^\alpha a(x, 0)[p_1 - i\partial_y - ik\sigma_2 \tilde{\partial}\gamma]^{\alpha} b(y, p_1) E_k(z, -ip_1).
\]

Integrating over variable \( p \), we have the integral on the right hand side equal to
\[
\int_\mathbb{R} d\mu_k(y)d\mu_k(p) E_k(x, ip_1)
\]
\[
\sum_\alpha \frac{1}{\alpha!} \partial_p^\alpha a(x, 0)[p_1 - i\partial_y - ik\sigma_2 \tilde{\partial}\gamma]^{\alpha} b(y, p_1)|_{y=x} E_k(z, -ip_1).
\]

Therefore, we conclude that
\[
Op_k(a) \circ Op_k(b) = Op_k(\sum_\alpha \frac{1}{\alpha!} \partial_p^\alpha a(x, 0)[p_1 - i\partial_y - ik\sigma_2 \tilde{\partial}\gamma]^{\alpha} b(y, p_1)|_{y=x}).
\]

We remark that the above sum is finite as \( a \) is a polynomial.

5.2. Step II. In this step, we aim to understand the expansion formula
\[
\sum_\alpha \frac{1}{\alpha!} \partial_p^\alpha a(x, 0)[p_1 - i\partial_y - ik\sigma_2 \tilde{\partial}\gamma]^{\alpha} b(y, p_1)|_{y=x}
\]
obtained in the previous subsection.

We look at the power \([p_1 - i\partial_y - ik\sigma_2 \tilde{\partial}\gamma]^{\alpha}\). Define \( A = -i\partial_y \), and \( B = -ik\sigma_2 \tilde{\partial}\gamma \).

We observe the following commuting relations,
\[
Ap_1 = p_1 A, \quad p_1 B = -Bp_1.
\]
With these relations, we write \[ [p_1 - i \partial_y - ik \sigma_2 \hat{\partial}_y \hat{\gamma}]^\alpha \] as
\[
\sum_{\nu \in P_{m,n}} c_\nu (-i)^{m+n} k^n p_1^{\alpha-m-n} B_\nu \sigma_2^n \hat{\gamma}^n,
\]
where \( P_{m,n} \) is the set of solutions to Eq. (3) introduced in Section 3.3, and \( B_\nu \) is the operator introduced in Section 3.3 as compositions of \( \partial_y \) and \( \hat{\partial}_y \), and \( c_\nu \) is number determined by \( \nu \in P_{m,n} \).

We study the number \( c_\nu \) more carefully. \( c_\nu \) is the number of the term
\[
(-i)^{m+n} k^n p_1^{\alpha-m-n} B_\nu \sigma_2^n \hat{\gamma}^n
\]
appearing in the expansion \([p_1 - i \partial_y - ik \sigma_2 \hat{\partial}_y \hat{\gamma}]^\alpha \). When we write out the expansion of \([p_1 - i \partial_y - ik \sigma_2 \hat{\partial}_y \hat{\gamma}]^\alpha \), it is a sum of monomials of the form \( p_1^{x_0} \partial_y^{\nu_0} \hat{\partial}_y p_1^{x_1} \partial_y^{\nu_1} \hat{\partial}_y \cdots \hat{\partial}_y p_1^{x_n} \partial_y^{\nu_n} \sigma_2^n \hat{\gamma}^n \), where \( \nu = (\nu_0, \ldots, \nu_n) \) is a fixed element of \( P_{m,n} \), and \( x_0, \ldots, x_n \) are nonnegative integers with \( x_0 + \cdots + x_n = \alpha - m - n \). We remark that as \( p_1 \) commutes with \( \partial_y \), we do not need to count the relative positions between \( \partial_y \) and \( p_1 \). Therefore, totally there are
\[
\prod_{j=0}^{n} \begin{pmatrix} y_j + x_j \\ x_j \end{pmatrix}
\]
number of the term \( p_1^{x_0} \partial_y^{\nu_0} \hat{\partial}_y p_1^{x_1} \partial_y^{\nu_1} \hat{\partial}_y \cdots \hat{\partial}_y p_1^{x_n} \partial_y^{\nu_n} \sigma_2^n \hat{\gamma}^n \) in the power \([p_1 - i \partial_y - ik \sigma_2 \hat{\partial}_y \hat{\gamma}]^\alpha \).

Furthermore, as \( \sigma_2 \) changes the sign of \( p_1 \), when we move \( \sigma_2 \) to the right end, we need to count the change of signs. Therefore, in front of the term \( p_1^{x_0} \partial_y^{\nu_0} \hat{\partial}_y p_1^{x_1} \partial_y^{\nu_1} \hat{\partial}_y \cdots \hat{\partial}_y p_1^{x_n} \partial_y^{\nu_n} \sigma_2^n \hat{\gamma}^n \), there should be a sign
\[
(-1)^{x_1 + 2x_2 + \cdots + nx_n}.
\]

In summary, for \( \nu \in P_{m,n} \), \( c_\nu \) is equal to
\[
\sum_{x_0 + \cdots + x_n = \alpha - m - n} \prod_{j=0}^{n} \begin{pmatrix} y_j + x_j \\ x_j \end{pmatrix} (-1)^{x_j}.
\]

Separating \( j \) from even to odd, we have \( c_\nu \) equal to
\[
\sum_{x_0 + \cdots + x_n = \alpha - m - n} \left[ \prod_{j \text{ is even}} \begin{pmatrix} y_j + x_j \\ x_j \end{pmatrix} \right] \left[ \prod_{j \text{ is odd}} \begin{pmatrix} y_j + x_j \\ x_j \end{pmatrix} (-1)^{x_j} \right]
\]
\[
= \sum_{s+t = \alpha - m - n} \sum_{x_0 + \cdots + x_{\text{even}} = s} \prod_{j \text{ is even}} \begin{pmatrix} y_j + x_j \\ x_j \end{pmatrix} \left[ \sum_{x_1 + \cdots + x_{\text{odd}} = t} \prod_{j \text{ odd}} \begin{pmatrix} y_j + x_j \\ x_j \end{pmatrix} (-1)^{x_j} \right]
\]

To evaluate the above number, we introduce the generating function \( 1/(1 - x)^{s+1} \). The Taylor expansion of \( 1/(1 - x)^{1+s} \) and \( 1/(1 + x)^{1+s} \) at 0 with \( |x| < 1 \) is
\[
\frac{1}{(1 - x)^{1+s}} = \sum_{k=0}^{\infty} \binom{s+k}{k} x^k,
\]
\[
\frac{1}{(1 + x)^{1+s}} = \sum_{k=0}^{\infty} \binom{s+k}{k} (-1)^k x^k.
\]
In summary, if we denote \( \Lambda_0 = \nu_0 + \sum_{\text{even }i} \nu_i \) and \( \Lambda_1 = \sum_{\text{odd }i} \nu_i \), then \( c_\nu \) is the coefficient of the term \( x^{a-m-n} \) of the Taylor expansion of the following function

\[
\frac{1}{(1-x)^{\Lambda_0+\nu_0+1}(1+x)^{\Lambda_1+\nu_1}},
\]

where \( n_0 \) (and \( n_1 \)) is the number of positive even (odd) numbers less than or equal to \( n \).

We next consider the expansion

\[
\sum_\alpha \frac{1}{\alpha!} \partial^\alpha_p a(x,0) [p_1 - i \partial_y - ik \sigma_2 \partial_y \hat{\gamma}]^\alpha b(y,p_1)|_{y=x}.
\]

By inserting the expansion of the power \([p_1 - i \partial_y - ik \sigma_2 \partial_y \hat{\gamma}]^\alpha\), we have the above expansion equal to

\[
\sum_\alpha \sum_{m,n,\nu} \frac{(-i)^{m+n} k^n}{\alpha!} \partial^\alpha_p a(x,0) c_\nu p_1^{a-m-n} B_\nu \sigma_2^n(b) \hat{\gamma}^n
\]

\[
= \sum_{m,n} (-i)^{m+n} k^n \left( \sum_{\nu=0}^{\rho_m+n} \sum_\alpha \frac{c_\nu}{p_1^{m+n}} \frac{p_1^\alpha}{\alpha!} \partial^\alpha_p a(x,0) \right) B_\nu \sigma_2^n(b) \hat{\gamma}.
\]

We notice that the terms \( B_\nu \sigma_2^n(b) \hat{\gamma}^k \) are independent of \( \alpha \) and \( \nu \), then we are left to deal with \( \frac{c_\nu}{p_1^{m+n}} \frac{p_1^\alpha}{\alpha!} \partial^\alpha_p a(x,0) \) for the sum over \( \alpha \).

Considering the above interpretation of \( c_\nu \), if we introduce an auxiliary variable \( t \in \mathbb{C} - \{0\} \), then we have \( \sum_\alpha \frac{c_\nu}{p_1^{m+n}} \frac{p_1^\alpha}{\alpha!} \partial^\alpha_p a(x,0) \) equal to the \( t^0 \) term of the product between

\[
\frac{t^{m+n}}{(1-t p_1)^{\Lambda_1+\nu_1+1}(1+t p_1)^{\Lambda_0+\nu_0+1}}
\]

and \( a(x,1/t) \) for \( |t p_1| < 1 \). We remark that by \( 1/(1-t p_1)^{\Lambda_0+\nu_0+1}(1+t p_1)^{\Lambda_1+\nu_1} \) really mean the Taylor expansion with respect to variable \( t p_1 \) as \( |t p_1| < 1 \), and by \( a(x,1/t) \) we mean the Taylor expansion of \( a \) with respect to the variable \( 1/t \). As \( a \) is assumed to be a polynomial, its Taylor expansion with respect to variable \( 1/t \) only has finitely many terms, and \( a(x,1/t) \) is an element in \( \mathbb{C}[x]( (t) ) \). This assures the product between \( a(x,1/t) \) and \( t^{m+n}/(1-t p_1)^{\Lambda_0+\nu_0+1}(1+t p_1)^{\Lambda_1+\nu_1} \) well defined, as a product of two Laurent series of variable \( t \). In conclusion, we conclude that \( \sum_\alpha \frac{c_\nu}{p_1^{m+n}} \frac{p_1^\alpha}{\alpha!} \partial^\alpha_p a(x,0) \) is equal to the \( t^0 \) component of the product \( t^{m+n} a(x,1/t) 1/(1-t p_1)^{\Lambda_0+\nu_0+1}(1+t p_1)^{\Lambda_1+\nu_1} \).

Now we relate the above explanation of \( \sum_\alpha \frac{c_\nu}{p_1^{m+n}} \frac{p_1^\alpha}{\alpha!} \partial^\alpha_p a(x,0) \) with the operator \( \Delta \) in Proposition 3.11. Let \( f \) be a polynomial of one variable. Then \( \Delta(f)(q_1,q_2) = (f(q_1) - f(q_2))/(q_1 - q_2) \) is equal to the \( t^0 \) component of the product between \( f(1/t) \) and \( t/(1-t q_1)(1-t q_2) \) for \( |t q_i| < 1 \) and \( |t q_i| < 1 \). In this identification we have viewed \( 1/(1-t q_1)(1-t q_2) \) as a Taylor series of variables \( t, q_1, q_2 \). Now applying the same trick, we can identify \( \Delta^2(f)(q_1,q_2,q_3) \) as the \( t^0 \) component of the product \( f(1/t)t^2/(1-t q_1)(1-t q_2)(1-t q_3) \) for \( |t q_i| < 1 \), \( i = 1,2,3 \). Extending this procedure, we have that in general, for \( k \in \mathbb{N} \), \( \Delta^k(f)(q_1,\ldots,q_k) \) is the \( t^0 \) component of the product \( f(1/t)t^k/(1-t q_1) \cdots (1-t q_k) \) for \( |t q_i| < 1 \), \( i = 1,\ldots,k \). Finally, comparing this interpretation of \( \Delta^k(f) \), we conclude that
\[ \sum_{\alpha} \frac{c_\alpha}{p_1^\alpha} \partial^{\alpha}_p a(x,0) \] is equal to \( \Delta^{m+n}(a) \) evaluating at
\[ \left( x, p_1 \right) \times \cdots \times \left( x, p_1 \right) \times \left( x, -p_1 \right) \times \cdots \times \left( x, -p_1 \right). \]

This is exactly the expression of the operator \( A_{\nu} \), introduced in Section 3.3, on the function \( a \) with respect to the variable \( p_1 \).

In summary, from the above expression about the symbol of the operator \( Op_k(a) \circ Op_k(b) \), we can quickly check property (1)-(3) in Theorem 3.10.

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