Cache-Aided Interference Channels

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Abstract

Over the past decade, the bulk of wireless traffic has shifted from speech to content. This shift creates the opportunity to cache part of the content in memories closer to the end users, for example in base stations. Most of the prior literature focuses on the reduction of load in the backhaul and core networks due to caching, i.e., on the benefits caching offers for the wireline communication link between the origin server and the caches. In this paper, we are instead interested in the benefits caching can offer for the wireless communication link between the caches and the end users.

To quantify the gains of caching for this wireless link, we consider an interference channel in which each transmitter is equipped with an isolated cache memory. Communication takes place in two phases, a content placement phase followed by a content delivery phase. The objective is to design both the placement and the delivery phases to maximize the rate in the delivery phase in response to any possible user demands. Focusing on the three-user case, we show that through careful joint design of these phases, we can reap three distinct benefits from caching: a load balancing gain, an interference cancellation gain, and an interference alignment gain. In our proposed scheme, load balancing is achieved through a specific file splitting and placement, producing a particular pattern of content overlap at the caches. This overlap allows to implement interference cancellation. Further, it allows us to create several virtual transmitters, each transmitting a part of the requested content, which increases interference-alignment possibilities.

Index Terms

Cache-Aided Multiuser Systems, Interference Channels, Coded Caching, Interference Alignment, Cache-Aided Interference Management

I. INTRODUCTION

The traditional information-theoretic analysis of communication networks assumes that each transmitter has access to an independent message. This assumption is appropriate if the messages are generated locally, such as speech in a telephone call. However, over the last decade or so, the bulk of traffic carried especially in cellular networks has shifted from locally generated speech to centrally generated content [1]. Since content is usually created well ahead of transmission time, it can be made available in several places in the network, either in caches or other storage elements. This shift from speech to content thus renders questionable the classical assumption of independent messages in the network analysis.

Thus, we require a different formulation more appropriate for content distribution scenarios as follows. A number of messages, each corresponding to one piece of content, are independently generated. During a content placement phase, each transmitter can store an arbitrary function of these messages up to a storage memory limit. During a subsequent content delivery phase, each receiver selects one of the messages, and the transmitters’ aim is to satisfy the receivers’ message demands with the fewest number of channel uses. Unlike in the traditional problem formulation, the goal here is to optimally design both the delivery and the placement phases. It is worth emphasizing that the content placement is performed without prior knowledge of the receivers’ demands nor the channel gains. Thus, we need to construct a content placement that is simultaneously suitable for all possible demands and channel conditions.

We formalize this approach in the context of communication over a $K$-user Gaussian interference channel as depicted in Fig. [1] We then focus on the three-user case. For this setting we provide a content placement scheme and a delivery scheme that reaps the following three distinct caching gains.

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• **Load Balancing:** Judicious content placement allows to balance the load among the transmitters in the delivery phase, thereby avoiding bottlenecks.

• **Interference Cancellation:** If in the delivery phase a piece of requested content is available at more than one transmitter, its interference can be cancelled through transmit zero forcing at some of the unintended receivers.

• **Interference Alignment:** It is known that in multi-user communication networks, interference alignment can increase communication rates [2]–[4]. We show here that this alignment gain can be increased through proper content placement.

The challenge when combining transmit zero forcing with interference alignment is that the channel coefficients in the equivalent zero-forced channel are not independent. We show how to address this challenge by adding several precoding factors at each transmitter and by exploiting the algebraic structure of the mapping between the equivalent zero-forced channel coefficients and the original channel coefficients.

### A. Related Work

Several works have investigated wireless networks with caches. Some papers focus on optimizing the content placement to increase local content delivery. In [5], a cache-aided small-cell system is considered, and the cache placement is formulated as maximizing the weighted-sum of the probability of local delivery. The wireless channel is modeled by a connectivity graph. [6] considers a related model but the content placement is optimized without a-priori knowledge of file popularities, which are learned from the demand history.

Other papers consider the wireless delivery as well. The load-balancing benefits of caching in ad-hoc wireless networks is investigated in [7]. In [8], content placement is optimized assuming a fixed capacity at each small cell, which is justified for orthogonal wireless multiple-access. [9] focuses on device-to-device wireless networks, in which devices are equipped with a limited cache memory and may opportunistically download the requested file from caches of neighboring devices. In [10], caching the same content at the base stations is suggested to improve the overall throughput by inducing cooperation among transmitters, and in [11] the availability of common files at different devices is exploited to match transmitters with receivers such that treating interference as noise is approximately optimal.

An information-theoretic framework for the analysis of cache-aided communication was introduced in [12] in the context of broadcast channels. Unlike here, the caches there are placed at the receivers. It is shown in [12] that in this setting the availability of caches allows the exploitation of a coded multicasting gain.

### B. Organization

The remainder of this paper is organized as follows. Section II introduces the problem formulation. Section III presents the main result and an outline of the proof. Section IV is dedicated to the formal proof.
II. PROBLEM FORMULATION

We now formally introduce the problem of communication over a cache-aided interference channel. We consider a $K$-user Gaussian interference channel with time-invariant channel coefficients $h_{jk} \in \mathbb{C}$ between transmitter $k$ and receiver $j$. We assume availability of full channel state information, i.e., all transmitters and receivers know all channel gains. The channel input is corrupted by standard complex Gaussian additive noise at all the receivers.

Unlike the standard interference channel, in which there are $K$ messages, in the caching setting introduced here there are $N$ messages called files in the following and denoted by $W_1, W_2, \ldots, W_N$. Each file is chosen i.i.d. uniformly at random from $\mathbb{F} \triangleq \{1, 2, \ldots, 2^F\}$, where $F$ is the file size in bits. Each transmitter has a local cache able to store $MF$ bits. Thus, each transmitter can store the equivalent of $M$ entire files in its cache. It will be convenient to define the normalized cache size

$$\mu \triangleq \frac{M}{N}$$

as the fraction of the library of files that can be stored locally in each transmitter’s cache.

Communication over the interference channel proceeds in two phases, a placement phase followed by a delivery phase. During the placement phase, each transmitter is given access to the complete library of files and can fill its cache as an arbitrary function of those files. During the subsequent delivery phase, each receiver $k$ requests one file $d_k$ out of the $N$ total files. We denote by $d \triangleq (d_k)_{k=1}^K \in [N]^K$ the vector of demands. The transmitters are informed of all $K$ demands $d$. Knowing these demands and having access to only their local cache, the encoder at each transmitter outputs a codeword that is sent over the interference channel. We impose an average power constraint of $P$ on those codewords. Each receiver $j$ decodes an estimate $\hat{W}_j$ of its requested file $W_{d_j}$.

Formally, each transmitter $k \in [K]$ consists of a caching function

$$\phi_k : [2^F]^N \rightarrow [2^{FM}]$$

mapping the files $W_1, W_2, \ldots, W_N$ to its local cache content

$$V_k \triangleq \phi_k(W_1, W_2, \ldots, W_N)$$

during the placement phase. Each transmitter $k \in [K]$ further consists of an encoding function

$$\psi_k : [2^{FM}] \times [N]^K \times \mathbb{C}^K \rightarrow \mathbb{C}^T$$

During the delivery phase, transmitter $k$ uses the encoder $\psi_k$ to map its cache content $V_k$, the now available receiver demands $d$, and the channel coefficients $H = (h_{jk})_{j,k}$ to the channel inputs

$$(X_k[t])_{t=1}^T \triangleq \psi_k(V_k, d, H),$$

where $T$ is the block length of the channel code. We impose an average power constraint of $P$ on each codeword $(X_k[t])_{t=1}^T$.

Each receiver $j \in [K]$ consists of a decoding function

$$\eta_j : \mathbb{C}^T \times [N]^K \times \mathbb{C}^K \rightarrow [2^F].$$

Again during the delivery phase, receiver $j$ uses the decoder $\eta_j$ to map the channel outputs $(Y_j[t])_{t=1}^T$, the receiver demands $d$, and the channel coefficients $H$ to the estimate

$$\hat{W}_j \triangleq \eta_j((Y_j[t])_{t=1}^T, d, H)$$

of the requested file $W_j$. 
Together, the caching, encoding, and decoding functions define a coding scheme. The sum rate of this coding scheme is \(KF/T\), and its probability of error is

\[
\max_{d \in [N]^K} \max_{k \in [K]} \mathbb{P}(\hat{W}_{dk} \neq W_{dk}),
\]

where the first maximization is over all demands \(d\) and the second maximization is over all receivers \(k\). We say that a rate \(R(\mu, P)\) is achievable, if for fixed normalized cache size \(\mu\) and power constraint \(P\), there exists a sequence of coding schemes, indexed by the file size \(F\), each of rate at least \(R(\mu, P)\) and with vanishing probability of error as \(F \to \infty\). The capacity \(C(\mu, P)\) is the supremum of all achievable rates.

The sum degrees of freedom of this system is defined as

\[\text{DoF}(\mu) \triangleq \liminf_{P \to \infty} \frac{C(\mu, P)}{\log P}\]

and describes the high-SNR behavior of capacity. Our goal in this paper is to characterize the tradeoff between the normalized cache size \(\mu\) and the sum degrees of freedom \(\text{DoF}(\mu)\).

**Remark 1 (Side Information):** A crucial component of this problem setting is that side information about the receiver demands and channel gains is only available during the delivery phase but not during the earlier placement phase. As a consequence, the cache contents at each transmitter have to be chosen without knowledge of the future receiver demands and channel gains.

Operationally, this lack of side information is due to different times during which the two phases occur. The placement phase occurs during a time of low network load, say late at night, when idle network resource can be used to push content from origin servers onto the caches. The delivery phase, on the other hand, occurs during a later time, say the next morning or evening, when users are trying to download or stream content onto their mobile devices.

**Remark 2 (Domain of \(\text{DoF}(\mu)\)):** Observe that for \(\mu < 1/K\) the collection of all caches can hold less than \(KN\mu F < NF\) bits. Hence, it is easily seen by a cut-set argument that the probability of error can not be vanishing in this case (intuitively, this is because some file bits are necessarily not cached anywhere in the system). Thus, \(\text{DoF}(\mu)\) is not defined for \(\mu < 1/K\). This formulation, in which each file of the content library should be available at some caches, allows us to isolate the gain of caching over the wireless channel, i.e., between transmitters and receivers, from the gain of caching over the backhaul and core networks, i.e., between the original server and caches.

On the other hand, for \(\mu \geq 1\), each transmitter can cache the entire library of files and further increasing \(\mu\) offers no additional benefit. Thus, \(\text{DoF}(\mu)\) is constant for \(\mu \geq 1\). The interesting domain of the tradeoff between cache size and degrees of freedom is therefore \(1/K \leq \mu \leq 1\).

**Remark 3 (Convexity of \(1/\text{DoF}(\mu)\)):** Instead of working directly with \(\text{DoF}(\mu)\), we will instead express our results in terms of the reciprocal \(1/\text{DoF}(\mu)\). This is because \(1/\text{DoF}(\mu)\) is a convex function of \(\mu\). Convexity can be shown by a memory-sharing argument (see Appendix [A]). For future reference, we summarize this fact in the following lemma.

**Lemma 1.** The reciprocal degrees of freedom \(1/\text{DoF}(\mu)\) is a convex function of the normalized cache size \(\mu\).

### III. Main Results

In the remainder of this paper, we focus on the three-user interference channel, i.e., \(K = 3\). We present an achievable scheme yielding a lower bound on \(\text{DoF}(\mu)\). As is explained in Section [II] (see Lemma [I]), due to the convexity of \(1/\text{DoF}(\mu)\), we express all results in terms of the reciprocal sum degrees of freedom \(1/\text{DoF}(\mu)\).
Theorem 2. For $N \in \mathbb{N}$ files and three users, where each transmitter has a cache size of $N\mu$, 
\[
1/\text{DoF}(\mu) \leq \begin{cases} 
13/18 - \mu/2, & \text{for } 1/3 \leq \mu \leq 2/3 \\
1/2 - \mu/6, & \text{for } 2/3 \leq \mu \leq 1.
\end{cases}
\]
for almost all channel gains $H \in \mathbb{C}^{3 \times 3}$.

This upper bound on $1/\text{DoF}(\mu)$ is depicted in Fig. 2.

![Graph showing the upper bound on $1/\text{DoF}(\mu)$ for $K = 3$-user case.](image_url)

Fig. 2. Upper bound on the tradeoff between normalized cache size $\mu$ and reciprocal sum degrees of freedom $1/\text{DoF}(\mu)$ for the $K = 3$-user case.

We next outline the proof of achievability of this upper bound. We focus on the corner points at $\mu = 1/3$ (Section III-A), $\mu = 2/3$ (Section III-C), and $\mu = 1$ (Section III-A). Convexity of $1/\text{DoF}(\mu)$ as a function of $\mu$ guarantees that any point on the line connecting two achievable $(\mu, 1/\text{DoF}(\mu))$ points is also achievable. The details of the proofs are provided in Section IV.

As we will see in the construction of the achievable schemes for the three corner points, the availability of caches enables three distinct gains compared to the conventional interference channel: load balancing, interference cancellation, and increased interference alignment. Our proposed scheme exploits all these three gains.

A. Corner Point at $\mu = 1$

This is the most straightforward case. Since $\mu = 1$, we have $M = N$, so that in the placement phase each transmitter can cache the entire content library $(W_1, W_2, \ldots, W_N)$. Without loss of generality, let us assume that in the delivery phase receiver one requests file $A$, receiver two requests file $B$, and receiver three requests file $C$. In other words, $W_{d1} = A$, $W_{d2} = B$, and $W_{d3} = C$. From the preceding placement phase, each transmitter has local access to all the three files $A$, $B$, and $C$. Therefore, in the delivery phase, the three transmitters can cooperatively transmit these files using multiple-antenna broadcast techniques, such as zero-forcing or other advanced schemes [13], to achieve a sum DoF of 3.

We point out that this cooperation between transmitters is achieved without any backhaul collaboration during the delivery phase. This corner case therefore reveals that one of the gains of cache-aided communication over the interference channel is to enable cooperation among the transmitters without any additional backhaul load. This cooperation is used to perform interference cancellation in the form of transmitter zero-forcing.

Here, it is worth mentioning that even though the theoretical derivation of the zero-forcing scheme is rather straightforward, there are many system-level challenges to be addressed to realize this gain in practice.
B. Corner Point at $\mu = 1/3$

We next focus on the other extreme corner point $\mu = 1/3$. In this case, each transmitter has space to cache one third of the content library. In other words, the transmitters collectively have just enough memory to store the entire content.

In the placement phase, we split each file into three nonoverlapping subfiles of equal size. Each transmitter caches a unique subfile of each file. Formally, each file $W_n$ is split into three equal-sized subfiles $W_{n,1}$, $W_{n,2}$, and $W_{n,3}$, with $W_n = (W_{n,1}, W_{n,2}, W_{n,3})$. Then transmitter $k$ caches $W_{n,k}$ for all $n \in [N]$.

For the delivery phase, let us again assume without loss of generality that receivers one, two, and three request files $A$, $B$, and $C$, respectively. From the preceding placement phase, transmitter one has access to subfiles $A_1$, $B_1$, and $C_1$, transmitter two has access to subfiles $A_2$, $B_2$, and $C_2$, transmitter three has access to subfiles $A_3$, $B_3$, and $C_3$. In the delivery phase, receiver one needs to receive $A_1$, $A_2$, and $A_3$ from transmitters one, two, and three, respectively. Similarly, receiver two needs to receive $B_1$, $B_2$, and $B_3$ from transmitters one, two, and three, respectively. Finally, receiver three needs to receive $C_1$, $C_2$, and $C_3$ from transmitters one, two, and three, respectively. We can recognize this as an X-channel message setting with three transmitters and three receivers. For this type of X-channel the sum DoF is equal to 9/5 achieved by interference alignment [3], [4]. By applying this delivery scheme in our case, we achieve a sum DoF of 9/5 for the cache-aided interference channel.

Remark 4 (Alternative Suboptimal Approach): To appreciate the advantage of the proposed scheme just described, let us consider an alternative placement approach in which files are not split into subfiles. That is, each transmitter caches $N/3$ distinct whole files, rather than 1/3 of each file. Let us assume that in the delivery phase receivers one, two, and three request files $A$, $B$, and $C$, respectively. For this content placement, various cases can happen in the delivery phase: either all three requested files $A$, $B$, and $C$ are stored at the same transmitter, or two of the requested files, say $A$ and $B$, are stored at one transmitter and the remaining file $C$ at another transmitter, or each requested file is stored at exactly one distinct transmitter.

Consider first an extreme case, in which the three files are all stored at the same transmitter. Observe that, even for large number of files $N$ in the content library, this case happens for a fraction $1/N$ of all possible $N^3$ receiver requests. In this case, the transmitter holding the three requested files becomes a bottleneck, limiting the sum DoF of the system to 1. Note that this is less than the sum DoF of 9/5 achieved by our proposed scheme described earlier.

Consider then the other extreme case, in which each transmitter stores exactly one of the requested files. This case happens for a fraction $2/N$ of all possible $N^3$ receiver requests. The system then forms a standard interference channel, for which the sum DoF is equal to $3/N$ which is achieved by interference alignment. Thus, even in the case of balanced user demands, the sum DoF of this alternative approach is less than the sum DoF of 9/5 achieved by our proposed scheme described earlier.

From the above discussion, we see that the proposed cache-aided communication scheme offers two advantages. First, it avoids creating bottlenecks by balancing the transmitter load for all possible receiver requests. Thus, caching enables a load-balancing gain. Second, it increases interference-alignment opportunities since each requested file can be partially delivered from all transmitters. Thus, caching further enables an increased interference-alignment gain.

C. Corner Point at $\mu = 2/3$

This is the most interesting corner point. In this case, the aggregated cache size is twice of the content size, allowing us to exploit all three gains that we observed for $\mu = 1$ and $\mu = 1/3$, i.e., the gains of load balancing, interference cancellation, and increased interference alignment.

The proposed placement phase operates as follows. We split each file into three subfiles of equal size. We label each subfile by a subset of $\{1, 2, 3\}$ of cardinality two. For example, file $A$ is split into the
subfiles $A = (A_{12}, A_{13}, A_{23})$. The subfile $A_{12}$ is cached at transmitters one and two. Similarly, subfile $A_{13}$ is cached at transmitters one and three, and subfile $A_{23}$ is cached at transmitters two and three. We follow the same content placement approach for all $N$ files. This is depicted in Fig. [3] for the three files $A$, $B$, and $C$. This placement approach was initially presented in [12] for a different scenario where the channel is a shared bottleneck link and the cache memories are located at the receiver side.

Consider next the delivery phase. Assume without loss of generality that receivers one, two, and three request files $A$, $B$, and $C$, respectively. We focus on one of them subfile of $A$, say $A_{12}$. Recall that, from the placement phase, $A_{12}$ is available at transmitters one and two. This allows transmitters one and two to cooperate on transmitting $A_{12}$ in order to achieve both interference cancellation and interference alignment. We start with the interference cancellation step. Observe that the interference of $A_{12}$ can be zero-forced at one of the unintended receivers two or three (for which file $A$ constitutes interference). To preserve symmetry, we further sub-split $A_{12}$ into two parts of equal size, denoted by $A_{12}^2$ and $A_{12}^3$. Transmitters two and three cooperate to zero-force the interference caused by $A_{12}^2$ at receiver two and to zero-force the interference caused by $A_{12}^3$ at receiver three. Similar sub-splitting is used for all sub-files of $A$, $B$, and $C$ as depicted in Fig. [4].

In general, for file $W_{d_j}$ requested by receiver $j$, the subfile $W_{d_j,S}$ (cached at all transmitters in $S \subseteq \{1, 2, 3\}$ with $|S| = 2$) is sub-split into two equal-sized parts $W_{d_j, \tau}^S$, $\tau \in \{1, 2, 3\} \setminus \{j\}$. The two transmitters in $S$ cooperatively zero-force the interference caused by $W_{d_j, \tau}^S$ at receiver $\tau$.

Achieving this zero-forcing is straightforward. For example, to zero-force file part $A_{12}^2$ at receiver two, transmitters one and two send $A_{12}^2$ in the direction of $(h_{22}, -h_{21})^T$, which is orthogonal to the matrix $(h_{21}, h_{22})^T$ of channels from those two transmitters to receiver two. Put differently, transmitter one precodes the transmission of $A_{12}^2$ with the coefficient $h_{22}$ and transmitter two precodes the transmission of $A_{12}^2$ with the coefficient $-h_{21}$ in order to cancel it out at receiver two. This precoding transforms the interference channel experienced by file part $A_{12}^2$ into an equivalent precoded channel with coefficients $h_{11}h_{22} - h_{12}h_{21}$, $0$, and $h_{31}h_{22} - h_{32}h_{21}$ to receivers one, two, and three, respectively. We can thus see file part $A_{12}^3$ as being sent from a single virtual transmitter with these equivalent channel gains. Performing similar zero-forcing for all 18 distinct file parts transforms the original interference channel into an equivalent precoded channel with 18 virtual transmitters and three receivers. Each virtual transmitter has access to one of the file parts shown in Fig. [4].

![Fig. 3. Content placement for normalized cache size $\mu = 2/3$. For simplicity, only three files, labeled $A$, $B$, and $C$, are shown. Each file is split into three subfiles of equal size, e.g., $A = (A_{12}, A_{13}, A_{23})$, with the subscript indicating at which transmitters this subfile is to be cached.](image-url)
We next explain the interference alignment step. Note that receiver one is interested in the six file parts $A_{12}, A_{13}, A_{23}, A_{23}, A_{23},$ and $A_{23}$. At the same time, it receives interference from $B_{12}, B_{13}, B_{23}, B_{23}, B_{23},$ and $B_{23}$. The remaining six interference terms are zero-forced at receiver one. We would like to align the six interfering terms at this receiver as depicted in Fig. 5. If each file part carries $1/7$ of one DoF, then out of one DoF available at receiver one, the aligned interference contributions together use up $1/7$ of one DoF, and the desired file parts take up $6/7$ of one DoF. Therefore, we can achieve a per-user DoF of $6/7$ corresponding to a sum DoF of $18/7$.

To implement this interference alignment, we use asymptotic interference alignment (see [14]) for time-invariant channels (proposed in [4] under the name real interference alignment) over the equivalent channel with 18 virtual inputs and three outputs. We face one major challenge that precludes a direct application of these techniques: the channel coefficients of the equivalent channel are not linearly independent functions of the original channel coefficients. In fact, the 36 nonzero coefficients of the equivalent channel are
functions of the only nine coefficients of the original channel. Thus, the conditions needed for real interference alignment do not hold.

For example, recall that the equivalent channel coefficient from virtual transmitter for file part \(A_{12}^1\) to receiver one is \(h_{11} h_{22} - h_{12} h_{21}\). At the same time, the equivalent channel coefficient from virtual transmitter for file part \(B_{12}^1\) to receiver two is \(-(h_{11} h_{22} - h_{12} h_{21})\). Therefore, channel coefficients in the equivalent channel are not linearly independent functions of the original channel coefficients. Nevertheless, by applying a unique scaling prefactor to each equivalent channel input together with a more careful analysis of decodeability, we show that real interference alignment is still possible. The analysis is based on representing the channel coefficients of the equivalent channel as the entries of the adjugate of the channel matrix and on showing that the adjugate operator is a diffeomorphism. The details of these arguments are presented in the next section.

Remark 5: Consider an X-Channel, with \(K_t\) single-antenna transmitters and \(K_r\) single-antenna receivers, where each transmitter has an independent message for each receiver. The sum DoF of this channel is \(\frac{K_t K_r}{K_t + K_r - 1}\) [2]–[4]. Note the as the number of transmitters \(K_t\) increases, the sum DoF approaches \(K_r\). In other words, each receiver is able to attain a per-user DoF approaching one. The reason is that, as the number of transmitters increases, the gain of interference alignment increases.

In the cache-aided interference channel with \(K\) users, when the same piece of content is available at several transmitters, those transmitters collectively form a virtual transmitter for that particular piece of content. Since we can form one such virtual transmitter for every subset of appropriate size of the \(K\) transmitters, this leads to a large number of virtual transmitters. This, in turn, improves the gain of interference alignment.

As pointed out earlier, this large number of virtual transmitters communicate over an equivalent channel created from a fixed set of only \(K^2\) channel coefficients. The difficulty in extending the results presented here to more than \(K = 3\) users is in showing that, even with this linear dependence of the equivalent channel gains, interference alignment is still feasible.

IV. PROOF OF ACHIEVABILITY FOR \(\mu = 2/3\)

A. Preliminaries

To be self contained, we start by reviewing some results on real interference alignment from [4] and its extension to complex channels from [15]. For a vector \(g \in \mathbb{C}^I\), the multiplicative Diophantine exponent \(\omega(g)\) is defined as the supremum of all \(\zeta\) satisfying

\[
|p + \sum_{i=1}^{I-1} g_i q_i| \leq \left(\prod_{i=1}^{I-1} \max\{1, |q_i|\}\right)^{-\zeta/(I-1)}
\]

for infinitely many \((p, q_1, \ldots, q_{I-1}) \in \mathbb{Z}^I\). We have the following result from [16] for the multiplicative Diophantine exponent of a vector function.

Theorem 3. Let \(f = (f_1, \ldots, f_{I-1})\) be a map from an open set \(\mathcal{H} \subset \mathbb{C}^d\) to \(\mathbb{C}^{I-1}\) with each \(f_i\) being analytic. If \(1, f_1, \ldots, f_{I-1}\) are linearly independent functions over \(\mathbb{R}\), then \(\omega(f(h)) = I/2 - 1 \) for almost all \(h \in \mathcal{H}\).

The key implication of Theorem 3 is that, if we choose \(\zeta = I/2 - 1 + \varepsilon\) with \(\varepsilon > 0\), then for almost all \(h \in \mathcal{H}\), the inequality

\[
|p + \sum_{i=1}^{I-1} f_i(h) q_i| \geq \left(\prod_{i=1}^{I-1} \max\{1, |q_i|\}\right)^{-\zeta/(I-1)}
\]

holds for all but at most finitely many values of \((p, q_1, \ldots, q_{I-1}) \in \mathbb{Z}^I\).
This result from number theory allows us to lower bound the minimum distance for a particular class of signal constellations used in real interference alignment described next. The presentation here follows [15]. For a natural number $Q$, define

$$Z_Q \triangleq \mathbb{Z} \cap [-Q, Q]$$

(1)

as the set of the $2Q + 1$ integers in the interval $[-Q, Q]$. Consider the signal constellation

$$\mathcal{C} \triangleq \Gamma \cdot \left( Z_Q + \sum_{i=1}^{I-1} f_i(h)Z_Q \right),$$

(2)

where

$$Q \triangleq \frac{P(1-\varepsilon)}{(I+2\varepsilon)},$$

$$\Gamma \triangleq c_1 \frac{P(I-2+4\varepsilon)/(2(I+2\varepsilon))}{},$$

for some positive constants $c_1, \varepsilon$ and for per-symbol power $P$. In addition, assume that $1, f_1, \ldots, f_{I-1}$ are linearly independent over $\mathbb{R}$ and that each $f_i$ is analytic.

Then, from Theorem 3 and by setting $\zeta = I/2 - 1 + \varepsilon$, the minimum distance of the constellation $\mathcal{C}$ is greater than

$$c_2 \Gamma Q^{-\zeta} = c_1 c_2 P^{\varepsilon/2},$$

for $P$ large enough. Here, $c_2$ is some constant depending only on $f$ and $h$ (the constant $c_2$ does not depend on $P$ and accounts for the finite number of points for which Theorem 3 does not hold), and $c_2$ is positive for almost all $h$. Furthermore, the points in the constellation $\mathcal{C}$ have power at most

$$c_3 \Gamma^2 Q^2 = c_3 c_1^2 P$$

for some positive constant $c_3$ depending only on $f$ and $h$. Therefore, the minimum distance of the constellation grows with $P$, meaning that for high signal-to-noise ratios the average probability of error is arbitrarily small. In addition, by choosing $c_1$, we can control the power of the constellation. Note that each term $Z_Q$ in the summation (2) carries $\log(2Q + 1)$ bits of information, corresponding to a DoF of

$$\lim_{P \to \infty} \frac{\log(2Q + 1)}{\log(P)} = \frac{1 - \varepsilon}{I + 2\varepsilon},$$

which can be made arbitrarily close to $1/I$ by choosing $\varepsilon$ small enough.

Real interference alignment uses this class of constellations. As will be explained in the next section, the alignment is performed by choosing the map $f$ as monomials of the channel coefficients. In particular, the following type of maps will be important. Fix a value of $J$ and $L \in \mathbb{N}$. Consider $J$ integers $\ell_1, \ell_2, \ldots, \ell_J$ all between 1 and $L$, and consider the monomial mapping the complex vector $u = (u_1, u_2, \ldots, u_J)$ into $u_1^{\ell_1} u_2^{\ell_2} \ldots u_J^{\ell_J}$. Note that each such monomial is an analytic function of $u$. Define the collection of all $L^J$ such monomials as $\mathcal{T}_L$, and denote by

$$\mathcal{T}_L(u) \triangleq (u_1^{\ell_1} u_2^{\ell_2} \ldots u_J^{\ell_J}, 1 \leq \ell_1, \ell_2, \ldots, \ell_J \leq L)$$

(3)

the complex-valued vector of length $L^J$ arising from evaluating each of these monomials at the point $u \in \mathbb{C}^J$. 


B. Achievable Scheme

Consider the content placement as described in Section III-C. Assume without loss of generality that receivers one, two, and three request files $A$, $B$, and $C$, respectively. Recall that each file is split into six subfiles. For example, file $A$ is split into the six parts $A^r_{kk}$ with $k, \bar{k} \in [3], \tau \in [3] \setminus \{1\}$ and $k < \bar{k}$. This file part is to be stored at transmitters $k$ and $\bar{k}$ and precoded to be zero forced at receiver $\tau$. Our target is to align all the uncanceled interference at each receiver and to achieve a per-user DoF of $6/7$.

Consider a file part available at transmitters $k$ and $\bar{k}$ with $k < \bar{k}$. To zero-force the interference caused by this file part at receiver $\tau$, we use the scaling factors $h^r_{\tau k}$ at transmitter $k$ and $-h^r_{\tau \bar{k}}$ at transmitter $\bar{k}$. The equivalent channel coefficient from the virtual transmitter for this file part to receiver $j$ is equal to $h_{jk}h^r_{\tau k} - h_{j\bar{k}}h^r_{\tau \bar{k}}$. Clearly, if $j = \tau$, the equivalent channel coefficient is zero. Moreover, if $j \neq \tau$, then the equivalent channel coefficient $h_{jk}h^r_{\tau k} - h_{j\bar{k}}h^r_{\tau \bar{k}}$ is equal to plus or minus the entry $g_{\xi}g_{\xi}$ of the adjugate matrix $G$ of the channel matrix $H$, with $\xi = [3] \setminus \{k, \bar{k}\}$ and $\xi = [3] \setminus \{\tau, j\}$. (See Appendix B for a definition of the adjugate.)

This construction results in an equivalent channel with 18 virtual transmitters and three receivers. Each receiver is interested in the file parts of six virtual transmitters. From the 12 remaining virtual transmitters, six are already zero-forced at that receiver, and six are creating interference. Specifically:

- At receiver one, the desired data streams are $A^r_{1k}, A^r_{12}, A^r_{13}, A^r_{1\bar{k}}, A^r_{23}$, and $A^r_{2\bar{k}}$. The interfering data streams are $B^r_{1k}, B^r_{12}, B^r_{13}, B^r_{1\bar{k}}, B^r_{23}$, and $B^r_{2\bar{k}}$.
- At receiver two, the desired data streams are $B^r_{1k}, B^r_{12}, B^r_{13}, B^r_{1\bar{k}}, B^r_{23}$, and $B^r_{2\bar{k}}$. The interfering data streams are $A^r_{1k}, A^r_{12}, A^r_{13}, A^r_{1\bar{k}}, A^r_{23}$, and $A^r_{2\bar{k}}$.
- At receiver three, the desired data streams are $C^r_{1k}, C^r_{12}, C^r_{13}, C^r_{1\bar{k}}, C^r_{23}$, and $C^r_{2\bar{k}}$. The interfering data streams are $A^r_{1k}, A^r_{12}, A^r_{13}, A^r_{1\bar{k}}, A^r_{23}$, and $A^r_{2\bar{k}}$.

At each receiver, we aim to align all six interference contributions using real interference alignment. For reasons that will be explained later, at each virtual transmitter, we introduce an additional scaling factor. In particular, at the virtual transmitter for file part $A^r_{kk}$, we use the scalar scaling factor $\alpha^r_{kk}$. Similarly, at the virtual transmitter for file part $B^r_{kk}$, we use the scaling factor $\beta^r_{kk}$, and at the virtual transmitter for file part $C^r_{kk}$, we use the scaling factor $\gamma^r_{kk}$. Each scaling factor is a complex number to be chosen later.

We are now ready to describe the interference-alignment solution. At receiver one, the six sources of interference are $B^r_{1k}, B^r_{12}, B^r_{13}, B^r_{1\bar{k}}, B^r_{23}$, and $B^r_{2\bar{k}}$, received respectively through links with equivalent channel coefficients $-\beta^r_{12}g_{23}, \gamma^r_{12}g_{33}, \beta^r_{13}g_{23}, -\gamma^r_{13}g_{23}, -\beta^r_{23}g_{12}$, and $\gamma^r_{23}g_{13}$. The goal is to align all these six interfering data streams at receiver one. Fix natural numbers $L$ and $Q$. Each data stream for the six interfering file parts is formed as a superposition of $L^6$ substreams. Each such substream is modulated over the integer constellation $\mathbb{Z}_Q$ as defined in (1) and scaled with a unique monomial from the set $T_L(u^{(1)})$ with

$$u^{(1)} \triangleq \{\beta^r_{12}g_{32}, \gamma^r_{12}g_{33}, \beta^r_{13}g_{23}, -\gamma^r_{13}g_{23}, -\beta^r_{23}g_{12}, \gamma^r_{23}g_{13}\}$$

and with $T_L(\cdot)$ as defined in (3). The $L^6$ modulated substreams are added up to form the data stream for that file part. Thus, each of these six interfering data streams consists of points from the signal constellation

$$\sum_{v \in T_L(u^{(1)})} v \mathbb{Z}_Q,$$

where the sum is over all components of the vector $T_L(u^{(1)})$.

Similarly, the interfering data streams $A^r_{1k}, A^r_{12}, A^r_{13}, A^r_{1\bar{k}}, A^r_{23}$, and $A^r_{2\bar{k}}$ at receiver two are received respectively through the equivalent channel coefficients $\alpha^r_{12}g_{31}, -\gamma^r_{12}g_{33}, -\alpha^r_{13}g_{21}, \gamma^r_{13}g_{23}, \alpha^r_{23}g_{11}$, and $-\gamma^r_{23}g_{13}$. Again, each of these six interfering data stream is formed as a superposition of $L^6$ substreams resulting in the signal constellation

$$\sum_{v \in T_L(u^{(2)})} v \mathbb{Z}_Q.$$
with
\[ u^{(2)}(2) \triangleq \left\{ \alpha_{12}^{(2)} g_{3}, \beta_{12}^{(2)} g_{3}, \alpha_{13}^{(2)} g_{21}, \beta_{13}^{(2)} g_{21}, \alpha_{23}^{(2)} g_{11}, \beta_{23}^{(2)} g_{11} \right\}. \]

Finally, the interfering data streams \( A_{12}^{2}, B_{12}^{2}, A_{13}^{2}, B_{13}^{2}, A_{23}^{2} \), and \( B_{23}^{2} \) at receiver three are received respectively through the equivalent channel coefficients \(-\alpha_{12}^{(2)} g_{3}, \beta_{12}^{(2)} g_{3}, \alpha_{13}^{(2)} g_{21}, -\beta_{13}^{(2)} g_{21}, -\alpha_{23}^{(2)} g_{11}, \beta_{23}^{(2)} g_{11}\). Again, each of these six interfering data stream is formed as a superposition of \( L^6 \) substreams resulting in the signal constellation
\[ \sum_{\nu \in T_{L}(u^{(2)})} v Z_{Q} \]
with
\[ u^{(3)}(3) \triangleq \left\{ \alpha_{12}^{(2)} g_{3}, \beta_{12}^{(2)} g_{3}, \alpha_{13}^{(2)} g_{21}, \beta_{13}^{(2)} g_{21}, \alpha_{23}^{(2)} g_{11}, \beta_{23}^{(2)} g_{11} \right\}. \]

To satisfy the power constraint at each transmitter, we further scale the superposition of all signals at each transmitter by the positive factor \( \Gamma \).

We now shift focus to the receivers. In particular, we consider receiver one. By symmetry, analogous arguments are valid for the other receivers as well. We start with the analysis of the desired file parts of file \( A \). At receiver one, the data streams for \( A_{12}^{2}, A_{13}^{2}, A_{23}^{2} \) are observed with channel coefficients \( \alpha_{12}^{(2)} g_{3}, -\alpha_{13}^{(2)} g_{3}, \) and \( \alpha_{23}^{(2)} g_{11} \). Recall that these three data streams were chosen from the signal constellation
\[ \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} \]
Therefore, the received signal constellation for \( A_{12}^{2}, A_{13}^{2}, \) and \( A_{23}^{2} \) is
\[ C_{1} \triangleq \Gamma \cdot \left( \alpha_{12}^{(2)} g_{3} \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} - \alpha_{13}^{(2)} g_{21} \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} + \alpha_{23}^{(2)} g_{11} \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} \right). \]

The remaining three desired data streams for \( A_{12}^{3}, A_{13}^{3}, \) and \( A_{23}^{3} \) at receiver one are observed with channel coefficients \(-\alpha_{12}^{(3)} g_{3}, \alpha_{13}^{(3)} g_{22}, \) and \(-\alpha_{23}^{(3)} g_{12}\). These three data streams were chosen from the signal constellation
\[ \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} \]
Therefore, the received signal constellation for \( A_{12}^{3}, A_{13}^{3}, \) and \( A_{23}^{3} \) is
\[ C_{2} \triangleq \Gamma \cdot \left( -\alpha_{12}^{(3)} g_{3} \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} + \alpha_{13}^{(3)} g_{22} \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} - \alpha_{23}^{(3)} g_{12} \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} \right). \]

Consider next the six interfering data streams for \( B_{12}^{2}, C_{12}^{2}, B_{13}^{2}, C_{13}^{2}, B_{23}^{2}, \) and \( C_{23}^{2} \) at receiver one. These interfering data streams are observed with channel coefficients \(-\beta_{12}^{(3)} g_{3}, \gamma_{12}^{(3)} g_{3}, \beta_{13}^{(3)} g_{22}, -\gamma_{13}^{(3)} g_{22}, -\gamma_{13}^{(3)} g_{23}, \) and \( \gamma_{13}^{(3)} g_{23} \). Each of these data streams was selected from the same signal constellation
\[ \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} \]
Thus, the received constellation for the interference is equal to
\[ C_{3} \triangleq \Gamma \cdot \left( -\beta_{12}^{(3)} g_{3} \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} + \beta_{13}^{(3)} g_{22} \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} - \gamma_{13}^{(3)} g_{23} \sum_{\nu \in T_{L}(u^{(3)})} v Z_{Q} \right) \]
\[ \left. + \sum_{\nu \in T_{L+1}(u^{(1)})} v Z_{Q} \right) \]
\[ \left. \left( \sum_{\nu \in T_{L+1}(u^{(1)})} v Z_{Q} \right) \right). \]
Here (a) follows from the definition of $\mathcal{T}_L(\cdot)$ in (3) and the definition of $u(1)$ in (4). This equation is a key step in the derivation of the alignment scheme: It states that the received interference constellations have collapsed into a constellation of cardinality smaller than the multiplication of the six transmitted constellation cardinalities. Put differently, interference is aligned.

The combined received constellation is $C_1 + C_2 + C_3$. We next argue that receiver one can recover all six desired data streams plus one data stream of aligned interference by appealing to Theorem 3. To use this theorem, we need to show that the monomials forming the received constellation are linearly independent as a function of the adjugate $G \in \mathbb{C}^{3 \times 3}$ of the channel matrix $H \in \mathbb{C}^{3 \times 3}$ and as a function of the scaling factors $\alpha_{kk}^{(r)}$, $\beta_{kk}^{(r)}$, and $\gamma_{kk}^{(r)}$. We then express the result as a function of the channel gains $H$. Linear independence can be proven based on the following observations:

- It is easy to see that (i) the monomials forming the constellation $C_1$ have terms that are functions of $\alpha_{12}^{(2)}$, $\beta_{12}^{(1)}$, $\alpha_{13}^{(2)}$, $\beta_{13}^{(1)}$, $\alpha_{23}^{(2)}$, and $\beta_{23}^{(1)}$, (ii) the monomials forming the constellation $C_2$ have terms that are functions of $\alpha_{12}^{(3)}$, $\gamma_{12}^{(1)}$, $\alpha_{13}^{(3)}$, $\gamma_{13}^{(1)}$, $\alpha_{23}^{(3)}$, $\gamma_{23}^{(1)}$, and (iii) the monomials forming constellation $C_3$, have terms that are functions of $\beta_{12}^{(3)}$, $\gamma_{12}^{(2)}$, $\beta_{13}^{(3)}$, $\gamma_{13}^{(2)}$, $\beta_{23}^{(3)}$, and $\gamma_{23}^{(2)}$. Since there is no overlap between these coefficients, we have linear independence across the monomials forming different constellations $C_1$, $C_2$, and $C_3$.

- It remains to prove that the monomials within each constellation $C_i$ are linearly independent functions. The monomials forming the constellation $C_1$ are $\alpha_{12}^{(2)} g_{33} \cdot \mathcal{T}_L(u(3))$, $\alpha_{13}^{(2)} g_{23} \cdot \mathcal{T}_L(u(3))$, and $\alpha_{23}^{(2)} g_{13} \cdot \mathcal{T}_L(u(3))$, where $g_{33}$, $g_{23}$, and $g_{13}$ have no contribution in $\mathcal{T}_L(u(3))$. Then it is easy to see that these monomials are linearly independent.

Similar arguments hold for monomials forming $C_2$. Finally, the monomials forming $C_3$ are a subset of $\mathcal{T}_{L+1}(u(1))$, and a similar argument can be applied to prove linear independence.

Given the linear independence of the monomials and assuming properly chosen $I$, $Q$, and $\Gamma$, Theorem 5 guarantees decodeability of the combined received constellation $C_1 + C_2 + C_3$ for almost all values of the adjugate $G$ and the scaling factors $\alpha_{kk}^{(r)}$, $\beta_{kk}^{(r)}$, and $\gamma_{kk}^{(r)}$. This implies that there exists a fixed choice of $\alpha_{kk}^{(r)}$, $\beta_{kk}^{(r)}$, $\gamma_{kk}^{(r)}$ such that for these scaling factors decodeability is guaranteed for all adjugate matrices $G$ outside a set $B \subset \mathbb{C}^{3 \times 3}$ of measure zero. We show in Lemma 4 in Appendix B that the inverse adjugate mapping preserves sets of measure zero. In other words, the set of channel matrices $H \in \mathbb{C}^{3 \times 3}$ for which the adjugate matrix $G$ is in $B$ has measure zero. Thus, for the fixed choice of scaling factors, decodeability is guaranteed for almost all channel matrices $H$.

Let

\[
I \triangleq 6L^6 + (L + 1)^6,
\]

\[
Q \triangleq \frac{1}{6} P^{(1-e)/(I+2e)},
\]

\[
\Gamma \triangleq c_1 P^{(I-2+4e)/(2(I+2e))}.
\]

The average transmit power of each signal constellation is then upper bounded by

\[
c_3 \Gamma^2 Q^2 = \frac{c_3 c_1^2}{36} P,
\]

and we can choose $c_1$ such that the power constraint at each transmitter is satisfied. The minimum distance $\Delta$ of the received constellation is at least

\[
\Delta \geq c_2 \frac{\Gamma}{(6Q)^{1/2}} = c_1 c_2 P^{e/2},
\]
which grows with $P$ for any positive $\varepsilon$. Therefore, the probability of error goes to zero as $P$ increases. By appealing to Fano’s inequality (see [4] for the details), this implies that there exist block codes over this modulated channel that achieve a per-user DoF of

$$\lim_{P \to \infty} \frac{1 - \exp(-\Delta^2/8)}{\log(P)} (\log|C_1| + \log|C_2|) = 6L^6 \frac{1 - \varepsilon}{1 + 2\varepsilon} = \frac{6(1 - \varepsilon)}{6 + (1 + 1/L)^6 + 2\varepsilon/L^6},$$

which approaches $6/7$ as $L \to \infty$ followed by $\varepsilon \to 0$. Thus, by choosing $L$ large enough and $\varepsilon$ small enough, we achieve a sum DoF arbitrarily close to $18/7$, as needed to be shown.

V. CONCLUSIONS AND DISCUSSION

This paper introduced the problem of communication over the interference channel aided by caches at the transmitters. It proposed and analyzed a communication scheme for the case with 3 transmitters and receivers. This communication scheme combines both transmitter zero forcing with interference alignment. The analysis of this combination of techniques is non-trivial, since the transmit zero forcing introduces dependence among the effective (i.e., including the zero-forcing operation) channel coefficients that have to be carefully handled while performing the interference alignment.

Since the conference version of this paper was presented at ISIT in June 2015, several follow-up papers have extended the setting and results presented here. [17] extended the joint zero-forcing and interference-alignment scheme of this present paper to the case of arbitrary number of receivers but, critically, still with only three transmitters. [18] added a communication constraint on the backhaul link from the content library to the transmitters during the delivery phase. This backhaul constraint allows to meaningfully study the case where the transmitter memory $\mu$ is less than $1/K$ so that the transmitters jointly cannot cache the entire library. [19] considered cache-aided cellular systems with backhaul links and proposed a graph-based approach to identify zero-forcing opportunities. [20] studied two-user cache aided systems with backhaul communication, where contents have different latency requirements. Finally, [21]–[23] analyzed the problem of communication over the interference channel with caches at both the transmitters and receivers.

When specialized to the setting considered in this paper (i.e., equal number of transmitters and receivers $K$ and caches only at the transmitters), [21] shows that a communication scheme using only transmitter zero-forcing achieves degrees of freedom within a constant factor of the best when restricted to the class of linear, one-shot coding schemes. With the same specialization, [23] shows that a communication scheme using only transmitter interference alignment achieves degrees of freedom with a constant factor of optimal (without any restrictions on the class of schemes).

Recall that in this paper we propose a scheme combining both transmitter zero forcing and interference alignment. Fig. 6 compares the inverse degrees of freedom of this scheme with the ones of [21] (using only zero-forcing) and [23] (using only interference alignment). The figure indicates that for small cache sizes, interference-alignment provides the main performance gain, whereas for large cache sizes zero forcing provides the main performance gain. However, for moderate cache sizes, both gains need to be exploited for optimal operation. How to achieve both these gains jointly for scenarios with more than 3 transmitters is still an open problem.

APPENDIX A

PROOF OF LEMMA 1

Consider two sequences of coding schemes, each indexed by file size $F$ and for a fixed power constraint $P$. Assume the scheme $i \in \{1, 2\}$ uses normalized cache size $\mu_i$ and achieves rate $R_i$. Denote by $\varepsilon_i(F)$ the probability of error of scheme $i$ as a function of file size $F$. By definition of achievability, we have that $\lim_{F \to \infty} \varepsilon_i(F) = 0$. 

Fig. 6. Comparison of the inverse sum degrees of freedom of the scheme proposed in this paper with the ones of follow-up work by Naderializadeh et al. [21] and by Hachem et al. [23].

Fix $\alpha_1 \in (0, 1)$ and set $\alpha_2 \triangleq 1 - \alpha_1$.

Consider normalized cache size

$$\mu \triangleq \alpha_1 \mu_1 + \alpha_2 \mu_2.$$  

Split each file into two parts of size $\alpha_1 F$ and $\alpha_2 F$, and partition the memory at each cache into two parts of size $\alpha_1 \mu_1$ and $\alpha_2 \mu_2$. We will use the coding scheme $i$ using part $i$ of the memory for file parts $i$. Note that, with respect to the reduced file size, coding scheme $i$ deals with a normalized cache size of $\mu_i$ for a file size of $\alpha_i F$. Hence, the coding scheme $i$ applied to these subfiles achieves a rate of $R_i$ with probability of error $\varepsilon_i(\alpha_i F)$.

The probability of error $\varepsilon(F)$ of the combined scheme satisfies

$$\varepsilon(F) \leq \varepsilon_1(\alpha_1 F) + \varepsilon_2(\alpha_2 F) \to 0$$

as $F \to \infty$ since both $\alpha_1$ and $\alpha_2$ are strictly positive constants. To transmit $\alpha_i F$ file bits at rate $R_i$, scheme $i$ uses $\alpha_i F / R_i$ channel slots. Hence, the total number of channel uses to transmit all $F$ file bits is

$$\alpha_1 F / R_1 + \alpha_2 F / R_2.$$  

The rate of the combined scheme is thus

$$R = \frac{F}{\alpha_1 F / R_1 + \alpha_2 F / R_2},$$

or, put differently,

$$1/R = \alpha_1 / R_1 + \alpha_2 / R_2.$$
This implies that, for fixed power constraint $P$, the set $\mathcal{A}(P)$ of all achievable memory–reciprocal-rate pairs $(\mu, 1/R(\mu, P))$ is convex. Hence,

$$\frac{1}{\text{DoF}(\alpha_1 \mu_1 + \alpha_2 \mu_2)} = \limsup_{P \to \infty} \frac{\log P}{C(\alpha_1 \mu_1 + \alpha_2 \mu_2, P)}$$

$$= \limsup_{P \to \infty} \log P \inf \left\{ \frac{1}{R} : (\alpha_1 \mu_1 + \alpha_2 \mu_2, 1/R) \in \mathcal{A}(P) \right\}$$

$$\leq \limsup_{P \to \infty} \log P \left( \alpha_1 \inf \left\{ \frac{1}{R_1} : (\mu_1, 1/R_1) \in \mathcal{A}(P) \right\} + \alpha_2 \inf \left\{ \frac{1}{R_2} : (\mu_2, 1/R_2) \in \mathcal{A}(P) \right\} \right)$$

$$\leq \alpha_1 \limsup_{P \to \infty} \log P \inf \left\{ \frac{1}{R_1} : (\mu_1, 1/R_1) \in \mathcal{A}(P) \right\} + \alpha_2 \limsup_{P \to \infty} \log P \inf \left\{ \frac{1}{R_2} : (\mu_2, 1/R_2) \in \mathcal{A}(P) \right\}$$

$$= \frac{\alpha_1}{\text{DoF}(\mu_1)} + \frac{\alpha_2}{\text{DoF}(\mu_2)},$$

where the first inequality follows from the convexity of the set $\mathcal{A}(P)$. Therefore $1/\text{DoF}(\mu)$ is convex. ■

**APPENDIX B**

**THE ADJUGATE AND SETS OF MEASURE ZERO**

First, we recall the definition of the adjugate.

**Definition.** The adjugate of a matrix $A \in \mathbb{C}^{K \times K}$, denoted by $\text{adj}(A) \in \mathbb{C}^{K \times K}$, is defined as follows. The entry $(i, j)$ of $\text{adj}(A)$ is equal to $(-1)^{i+j} \det(M_{i,j})$, where $M_{i,j} \in \mathbb{C}^{(K-1) \times (K-1)}$ results from eliminating row $i$ and column $j$ of the matrix $A$.

For a subset $\mathcal{A} \subset \mathbb{C}^{K \times K}$, denote by $\lambda(\mathcal{A})$ its Lebesgue measure.

**Lemma 4.** Let $\mathcal{B} \subset \mathbb{C}^{K \times K}$ be a set of (not necessarily invertible) matrices. If $\mathcal{B}$ has measure $\lambda(\mathcal{B}) = 0$, then the preimage $\text{adj}^{-1}(\mathcal{B})$ of $\mathcal{B}$ under $\text{adj}(\cdot)$ has also measure $\lambda(\text{adj}^{-1}(\mathcal{B})) = 0$.

**Proof:** Let $\mathcal{GL} \subset \mathbb{C}^{K \times K}$ denote the set of $K \times K$ invertible matrices. $\mathcal{GL}$ is an open set since $\mathcal{GL} = \text{det}^{-1}(\mathbb{C} \setminus \{0\})$, i.e., it is the preimage of an open set under a continuous (in fact polynomial) function. Furthermore, since $\mathcal{GL}^c$ are the roots of this polynomial function, $\lambda(\mathcal{GL}^c) = 0$.

Consider next the restriction of $\text{adj}(\cdot)$ to the invertible matrices. While $\text{adj}(\cdot)$ is not an invertible function on $\mathbb{C}^{K \times K}$, it is invertible on $\mathcal{GL}$. A short computation shows that, for $B \in \mathcal{GL}$, the inverse function of $\text{adj}(\cdot)$ is

$$\text{adj}^{-1}(B) = \frac{\text{adj}(B)}{(\det(B))^{(K-2)/(K-1)}}. \quad (5)$$

From (5), it follows that $\text{adj}(\cdot)$ is a bijection from $\mathcal{GL}$ to $\mathcal{GL}$. Moreover, (5) also implies that, seen as a function from $\mathbb{R}^{2K \times 2K}$ to $\mathbb{R}^{2K \times 2K}$, $\text{adj}^{-1}(B)$ is continuously differentiable on $\mathcal{GL}$.

Since $\mathcal{GL}$ is open and $\text{adj}^{-1}(\cdot)$ is continuously differentiable, it follows by [24, Lemma 18.1] that if $\mathcal{B} \subset \mathcal{GL}$ has measure $\lambda(\mathcal{B}) = 0$, then its image under $\text{adj}^{-1}(\cdot)$ has also measure $\lambda(\text{adj}^{-1}(\mathcal{B})) = 0$.

Consider now a subset $\mathcal{B} \subset \mathbb{C}^{K \times K}$ of (not necessarily invertible) matrices. While the function $\text{adj}(\cdot)$ is not invertible on $\mathbb{C}^{K \times K}$, we may nevertheless consider the preimage of $\mathcal{B}$ under $\text{adj}(\cdot)$. With slight abuse of notation, we denote this preimage by $\text{adj}^{-1}(\mathcal{B})$. If $\lambda(\mathcal{B}) = 0$, we then have

$$\lambda(\text{adj}^{-1}(\mathcal{B})) \leq \lambda(\text{adj}^{-1}(\mathcal{B}) \cap \mathcal{GL}) + \lambda(\text{adj}^{-1}(\mathcal{B}) \cap \mathcal{GL}^c)$$

$$\leq \lambda(\text{adj}^{-1}(\mathcal{B} \cap \mathcal{GL})) + \lambda(\mathcal{GL}^c)$$

$$= 0,$$

as needed to be shown. ■
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