Non-existence of genuine (compact) quantum symmetries of compact, connected smooth manifolds

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Abstract

Suppose that a compact quantum group $Q$ acts faithfully on a smooth, compact, connected manifold $M$, i.e. has a $C^*$ (co)-action $\alpha$ on $C(M)$, such that $\alpha(C^\infty(M)) \subseteq C^\infty(M, Q)$ and the linear span of $\alpha(C^\infty(M))(1 \otimes Q)$ is dense in $C^\infty(M, Q)$ with respect to the Frechet topology. It was conjectured by the author quite a few years ago that $Q$ must be commutative as a $C^*$ algebra i.e. $Q \cong C(G)$ for some compact group $G$ acting smoothly on $M$. The goal of this paper is to prove the truth of this conjecture. A remarkable aspect of the proof is the use of probabilistic techniques involving Brownian stopping time.

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1 Introduction

In this article, we settle a conjecture about quantum group actions on classical spaces, which was made by the author in [16] quite a few years ago and which has been proved in certain cases by him and others over the recent years. Let us give some background before stating it.

Quantum groups have their origin in both physics and mathematics, as generalized symmetry objects of possibly noncommutative spaces. Following pioneering works by Drinfeld [9] Jimbo [21], Faddeev-Reshetikhin-Takhtajan [12] and others (see, e.g. [30]) in the algebraic framework and later Woronowicz [34], Podles [27], Vaes-Kustermans [23] and others in the analytic setting, there is by now a huge and impressive literature on quantum groups. In [25], Manin studied quantum symmetry in terms of certain universal Hopf algebras. In the analytic framework of compact quantum groups a la Woronowicz, Wang, Banica, Bichon, Collins (see, e.g. [1], [6], [32]) and many other mathematicians formulated and studied quantum analogues of permutation and automorphism groups for finite sets, graphs, matrix algebras etc. This motivated the more recent theory of quantum isometry groups [15] by the author of the present article in the context of Connes’ noncommutative geometry (c.f. [8]), which was developed further by many others including Bhowmick, Skalski, Banica, Soltan, De-Commer, Thibault, just to name a few (see, e.g. [5], [31] etc. and the references therein).

In this context, it is important to study quantum symmetries of classical spaces. One may hope that there are many more genuine quantum symmetries of a given classical space than classical group symmetries which will help one understand the space better. By ‘genuine’ we mean that the underlying algebra structure of the quantum group is noncommutative. In this context, one may mention Wang’s discovery of infinite dimensional quantum permutation group $S_n^+$ of a finite set with $n$ points where $n \geq 4$ and the discussion on ‘hidden symmetry in algebraic geometry’ in Chapter 13 of [25]. It follows from Wang’s work that any disconnected space with 4 or more homoemorphic components will admit a faithful quantum symmetry given by a suitable quantum permutation group. It is more interesting to look for nontrivial and interesting examples of (faithful) (co)-actions of genuine quantum groups on connected classical topological spaces as well as connected algebraic varieties. Indeed, several such examples are known by now, which include:

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(i) (co)action of $S_n^+$ on the connected compact space formed by topologically gluing $n$ copies of a given compact connected space [18];

(ii) (co)action of the group $C^*$ algebra $C^*(S_3)$ of the group of permutations of 3 objects on the coordinate ring of the variety \( \{xy = 0\} \) as in [11];

(iii) algebraic co-action of Hopf-algebras corresponding to genuine non-compact quantum groups on commutative domains associated with affine varieties as in [33] (Example 2.20).

However, one striking observation is that in each of the above examples, either the underlying space is not a smooth manifold ((i), (ii)) or the quantum group is not of compact type (in iii). There seems to be a natural obstacle to construct genuine compact quantum group action on a compact connected smooth manifold, at least when the action is assumed to be smooth in a natural sense. Motivated by the fact that a topological action $\beta$ of compact group $G$ on a smooth manifold $M$ is smooth in the sense that each $\beta_g$ is a smooth map (diffeomorphism) if and only if it is isometric w.r.t. some Riemannian structure on the manifold, the first author of this paper and some of his collaborators and students tried to compute quantum isometry groups for several classical (compact) Riemannian manifolds including the spheres and the tori. Quite remarkably, in each of these cases, the quantum isometry group turned out to be the same as $C(G)$ where $G$ is the corresponding isometry group. On the other hand, Banica et al ([2]) ruled out the possibility of (faithful) isometric actions of a large class of compact quantum groups including $S_n^+$ on a connected compact Riemannian manifold. All these led the first author of the present paper to make the following conjecture in [16], where he also gave some supporting evidence to this conjecture considering certain class of homogeneous spaces.

**Conjecture I:** It is not possible to have smooth faithful action of a genuine compact quantum group on $C(M)$ when $M$ is a compact connected smooth manifold.

There have been several results, both in the algebraic and analytic set-up, which point towards the truth of this conjecture. For example, it is verified in [13] under the additional condition that the action is isometric in the sense of [15] for some Riemannian metric on the manifold. In [11], Etingof and Walton obtained a somewhat similar result in the purely algebraic set-up by proving that there cannot be any finite dimensional Hopf algebra having inner faithful action on a commutative domain. However, their proof does not seem to extend to the infinite dimension as it crucially depends on the semisimplicity and finite dimensionality of the Hopf algebra. We should also mention the proof by A. L. Chirvasitu ([7]) of non-existence of genuine quantum isometry in the metric space set-up (see [29], [1], [17] etc.) for the geodesic metric of a negatively curved, compact connected Riemannian manifold.

In the present article, we settle the above conjecture in the affirmative. In fact, in a preprint written with two other collaborators, the author of the present paper posted a claim of the proof of this fact on the archive quite a few years ago but it contained a crucial gap. The idea was to emulate the classical averaging trick to construct a Riemannian metric for which the given smooth CQG action is isometric. However, the idea did not work mainly because we could not prove that the candidate of the Laplacian associated to the averaged metric was a second order differential operator. In the present article, we circumvent the difficulties using techniques of stopping time from the theory of probability. In fact, we follow the classical line of proving locality of the infinitesimal generator of the heat semigroup using stopping time of Brownian motion on manifolds.

**Remark 1.1** In some sense, our results indicate that one cannot possibly have a genuine ‘hidden quantum symmetry’ in the sense of Manin (Chapter 13 of [25]) for smooth connected varieties coming from compact type Hopf algebras; i.e. one must look for such quantum symmetries given by Hopf algebras of non-compact type only. From a physical point of view, it follows that for a classical mechanical system with phase-space modeled by a compact connected manifold, the generalized notion of symmetries in terms of (compact) quantum groups coincides with the
2 Preliminaries

2.1 Notational convention

We will mostly follow the notation and terminology of [13], some of which we briefly recall here. All the Hilbert spaces are over \( \mathbb{C} \) unless mentioned otherwise. For a complex \(*\)-algebra \( \mathcal{C} \), let \( \mathcal{C}_{a.a.} = \{ c \in \mathcal{C} : c^* = c \} \). We shall denote the \( C^* \) algebra of bounded operators on a Hilbert space \( \mathcal{H} \) by \( \mathcal{B}(\mathcal{H}) \) and the \( C^* \) algebra of compact operators on \( \mathcal{H} \) by \( \mathcal{B}_0(\mathcal{H}) \). \( \text{Sp}, \overline{\text{Sp}} \) stand for the linear span and closed linear span of elements of a vector space respectively, whereas \( \text{Im}(A) \) denotes the image of a linear map.

We will deviate from the convention of [13] in one context: we’ll use the same symbol \( \otimes \) for any kind of topological tensor product, namely minimal \( C^* \) tensor product, projective tensor product of locally convex spaces as well as tensor product of Hilbert spaces and Hilbert modules. However, \( \otimes_{\text{alg}} \) will be used for algebraic tensor product of vector spaces, algebras or modules. A scalar valued inner product of Hilbert spaces will be denoted by \( \langle \cdot, \cdot \rangle \) and some (non-scalar) \(*\)-algebra valued inner product of Hilbert modules over locally convex \(*\)-algebras will be denoted by \( \langle \cdot, \cdot \rangle \). For a Hilbert \( \mathcal{A} \)-module \( E \) where \( \mathcal{A} \) is a \( C^* \) algebra, we denote the \( C^* \)-algebra of adjointable right \( \mathcal{A} \)-linear maps by \( \mathcal{L}(E) \). In particular, we’ll consider the trivial Hilbert modules of the form \( \mathcal{H} \otimes \mathcal{A} \).

Throughout the paper, let \( M \) be a compact smooth manifold. Let us also fix an embedding of \( M \) in some \( \mathbb{R}^n \) and let \( x_1, \ldots, x_n \) denote the restriction of the canonical coordinate functions of \( \mathbb{R}^n \) to \( M \).

2.2 Compact quantum groups and their actions

We recall from [13] and the references therein, including [24], [34], some basic facts about compact quantum groups and their actions. A compact quantum group (CQG for short) is a unital \( C^* \)-algebra \( \mathcal{Q} \) with a coassociative coproduct (see [24]) \( \Delta \) from \( \mathcal{Q} \) to \( \mathcal{Q} \otimes \mathcal{Q} \) such that each of the linear spans of \( \Delta(\mathcal{Q})(\mathcal{Q} \otimes 1) \) and that of \( \Delta(\mathcal{Q})(1 \otimes \mathcal{Q}) \) is norm-dense in \( \mathcal{Q} \otimes \mathcal{Q} \). From this condition, one can obtain a canonical dense unital \(*\)-algebra of adjointable right \( \mathcal{A} \)-linear maps by \( \mathcal{L}(E) \). In particular, we’ll consider the trivial Hilbert modules of the form \( \mathcal{H} \otimes \mathcal{A} \).

It is known that there is a unique state \( h \) on a CQG \( \mathcal{Q} \) (called the Haar state) which is bi-invariant in the sense that \( (\text{id} \otimes h) \circ \Delta(a) = (h \otimes \text{id}) \circ \Delta(a) = h(a)1 \) for all \( a \in \mathcal{Q} \). The Haar state need not be faithful in general, though it is always faithful on \( \mathcal{Q}_0 \) at least. The image of \( \mathcal{Q} \) in the GNS representation of \( h \) in the GNS Hilbert space \( L^2(\mathcal{H}, h) \) is denoted by \( \mathcal{Q}_r \) and it is called the reduced CQG corresponding to \( \mathcal{Q} \).

A unitary representation of a CQG \( (\mathcal{Q}, \Delta) \) on a Hilbert space \( \mathcal{H} \) is a unitary \( U \in \mathcal{L}(\mathcal{H} \otimes \mathcal{Q}) \) such that the \( \mathcal{C} \)-linear map \( V \) from \( \mathcal{H} \) to the Hilbert module \( \mathcal{H} \otimes \mathcal{Q} \) given by \( V(\xi) = U(\xi \otimes 1) \) satisfies \( (V \otimes \text{id})V = (\text{id} \otimes \Delta)V \). Here, the map \( (V \otimes \text{id}) \) denotes the extension of \( V \otimes \text{id} \) to the completed tensor product \( \mathcal{H} \otimes \mathcal{Q} \) which exists as \( V \) is an isometry.

For a Hopf algebra \( H \) with the coproduct \( \Delta \), we write \( \Delta(q) = q(1) \otimes q(2) \) suppressing the summation notation (Sweedler’s notation). For an algebra (other than \( H \) itself) or module \( \mathcal{A} \) and a \( \mathcal{C} \)-linear map \( \Gamma : \mathcal{A} \rightarrow \mathcal{A} \otimes_{\text{alg}} H \) (typically a comodule map or a coaction) we shall also use an analogue of Sweedler’s notation and write \( \Gamma(a) = a(0) \otimes a(1) \).

Definition 2.1 A unital \(*\)-homomorphism \( \alpha : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{Q} \), where \( \mathcal{C} \) is a unital \( C^* \)-algebra and \( \mathcal{Q} \) is a CQG, is said to be an action of \( \mathcal{Q} \) on \( \mathcal{C} \) if

1. \( (\alpha \otimes \text{id}) \alpha = (\text{id} \otimes \Delta) \alpha \) (co-associativity).
2. \( \text{Sp} \alpha(\mathcal{C})(1 \otimes \mathcal{Q}) \) is norm-dense in \( \mathcal{C} \otimes \mathcal{Q} \).
Given an action $\alpha$ of a CQG $Q$ on $C$, there exists a norm-dense unital $*$-subalgebra of $C$ over which $\alpha$ restricts to an algebraic co-action of the Hopf algebra $Q_0$.

An action $\alpha$ of $Q$ on $C$ induces an action (say $\alpha_r$) of the corresponding reduced CQG $Q_r$ and the original action is faithful if and only if the action of $Q_r$ is so. We say that the action of $\alpha$ can be implemented by unitary representation if we can find a Hilbert space $H$ such that $A \subseteq B(H)$ and a unitary representation $U$ on $H$ such that $\alpha(a) = U(a \otimes 1)U^{-1}$ for all $a \in C$. It is easy to see that any unitarily implemented action is injective. In fact, as the unitary representation $\alpha$ is faithful if and only if the action of $Q$ is faithful. By invariance, the map $a \otimes q \mapsto \alpha_r(a)(1 \otimes q)$, $a \in C$, $q \in Q_r$ extends to a unitary representation of $Q_r$ on the GNS space $L^2(C,\overline{\alpha})$. This unitary representation implements $\alpha_r$.

In the special case $C = C(X)$ where $X$ is a compact Hausdorff space, the above invariant state will correspond to a faithful Borel measure, say $\mu$, so that the injective reduced action is implemented by a unitary representation in $L^2(X,\mu)$.

2.3 Riemannian metric from a nondegenerate, conditionally positive definite, local operator

Definition 2.2 Consider a linear map $L$ from $C^\infty(M)$ to $C(M)$ satisfying $L(1) = 0$. We say that $L$ is

(i) real, if $L(f) = \overline{L(f)}$ for all $f \in C^\infty(M)$;

(ii) local, if for any $x \in M$ and any $f \in C^\infty(M)$ such that $f(y) = 0$ for all $y$ in an open neighbourhood of $x$, we must have $L(f)(x) = 0$;

(iii) conditionally positive definite if $L$ is real and for any $f_1, \ldots, f_k \in C^\infty(M)$, $k \geq 1$ and $x \in M$, the $k \times k$ matrix $((k_L(f_i, f_j))(x))$ is nonnegative definite, where $k_L(f, g) := L(f)g - L(g)f$;

(iv) non-degenerate, if for every point $x \in M$ and for some (hence any) choice of local coordinates $f_1, \ldots, f_m$ around $x$, $((k_L(f_i, f_j))(x))$ is invertible.

The following is well-known, but we give a complete proof as we could not locate a precise reference of the result stated in this form.

Proposition 2.3 Let $L$ be a non-degenerate, local, conditionally positive definite $L$ with $L(1) = 0$ as above. Furthermore, assume that for every $x \in M$, there is a set of $m$ (dimension of $M$) smooth real-valued functions $\xi_1, \ldots, \xi_m$ such that they give a local coordinate system around $x$ and $k_L(\xi_i, \xi_j) \in C^\infty(M)$ for $i, j = 1, \ldots, m$. Then there is a unique Riemannian structure $\langle \cdot, \cdot \rangle$ on $M$ such that $\langle df, dg \rangle_x = k_L(f, g)(x)$ for all real valued $C^\infty$ functions $f, g$.

Proof

It is easy to see that $k_L(f, g) = k_L(g, f)$ for $f, g$ real and also $k_L(f, g)(x) = 0$ if $f$ (or $g$) is zero on an open neighborhood of $x$. Moreover, as $L(1) = 0$,

$$k_L(f, 1) = k_L(1, f) = 0$$

for any $f \in C^\infty(M)$.

Now, fix any $x \in M$. Also, fix any positive integer $k$ and smooth real functions $f_1, \ldots, f_k$ such that $f_i(x) = 0$ for each $i$. Consider the linear map $M_k(C^\infty(M)) \ni G = (g_{ij}) \rightarrow ((b_{ij}(x))) \in M_k(\mathbb{C})$ where $b_{ij} = L(f_if_jg_{ij})$. By the condition (iii) of Definition 2.2, we have for complex
numbers $c_1, \ldots, c_k$ and for $G = H^*H$, where $H = ((h_{ij})) \in M_k(C^\infty(M))$:

$$
\sum_{ij} c_i c_j \mathcal{L}(f_i f_j g_{ij})(x)
= \sum_{i,j,p=1}^k c_i c_j \mathcal{L}(f_i f_j \tilde{h}_{jp})(x)
\geq \sum_{i,j,p=1}^k (f_i(x) \tilde{h}_{jp}(x) \mathcal{L}(f_j h_{jp})(x) + \mathcal{L}(\tilde{h}_{jp} f_i)(x) f_j(x) h_{jp}(x)) = 0,
$$

which proves that this is a positive linear map. As $M_k(C^\infty(M))$ is a unital *-subalgebra of $M_k(C(M))$ which is unital and closed under the holomorphic functional calculus, any positive linear map $\psi$ on $M_k(C^\infty(M))$ extends uniquely to $M_k(C(M))$ as a positive linear (hence bounded with the norm $\psi(1)$, as $C(M)$ is unital) map. This gives us $\|((\mathcal{L}(f_i f_j g_{ij})(x)))\| \leq \|((g_{ij}))\| \|((\mathcal{L}(f_i f_j)(x)))\|$ for all $((g_{ij})) \in M_k(C^\infty(M))$. But as $\mathcal{L}$ is local, $\mathcal{L}(f_i f_j g_{ij})(x)$ depends only on the values of $((f_i f_j g_{ij}))$ in an arbitrarily small open neighbourhood of $x$. If $g_{ij}(x) = 0$ for all $i,j$, then for any $\epsilon > 0$ we can choose open neighbourhoods $V, W$ (say) of $x$ such that $V \subset W$ and $\|(g_{ij}(y))\| \leq \epsilon$ for all $y \in W$. Let $\chi$ be a smooth function supported in $W$ with $0 \leq \chi \leq 1$ and $\chi|_V \equiv 1$. Then $G_1 = \chi((g_{ij}))$ satisfies $G_1(y) = ((g_{ij}(y)))$ for all $y \in V$ and $\|G_1(y)\| \leq \epsilon$ for all $y \in M$. Thus, $\|((\mathcal{L}(f_i f_j g_{ij})(x)))\| = \|\mathcal{L}(f_i f_j G_1(x))\| \leq \epsilon \|((\mathcal{L}(f_i f_j)(x)))\|$. As $\epsilon$ is arbitrary, this proves $\mathcal{L}(f_i f_j g_{ij})(x) = 0$. It also follows that

$$
k_L(f_i f_j g_{ij})(x) = 0 \quad (2)
$$

if $f_i, g_i$ real valued smooth functions with $f_i(x) = g_i(x) = 0$.

Next, choose a local coordinate $(U, \xi_1, \ldots, \xi_m)$ (say) around $x$ such that $\xi_i(x) = 0$ for each $i$. Choose another open neighbourhood $V$ of $x$ such that $V \subset U$ and a smooth positive function $\chi$ supported in $U$ such that $\chi|_V \equiv 1$. Now, given real valued smooth $f$ we can write $f = f(x)1 + \sum_i \partial_i(f)(x) \xi_i + R_f$ on $U$, where $\partial_i(f)(x)$ denotes the partial derivative if $f$ w.r.t. the coordinate $\xi_i$ at $x$ and $R_f$ is defined in $U$. Using the local Taylor expansion of $f$ around $x$ we can write $R_f = \sum_i \xi_i h_i$ where $h_i$ are smooth functions defined on $U$ with $h_i(x) = 0$. Writing $\phi = \chi \phi$ for any smooth function defined at least $U$ (so that $\phi \in C^\infty(M)$), we get $\tilde{f} = f(x)1 + \sum_i \partial_i(\phi)(x) \xi_i + \tilde{R}_f$. As $\tilde{f} = f, \tilde{g} = g$ on $V$, we have $k_L(f, g)(x) = k_L(\tilde{f}, \tilde{g})(x)$. It also follows from (2) proved before that $k_L(h_i \tilde{\xi}_i, \tilde{h}_j \tilde{\xi}_j)(x) = 0$, hence also $k_L(\tilde{R}_f, \tilde{R}_f)(x) = 0$. By positive definiteness of $k_L$, we have $|k_L(\phi, \tilde{R}_f)(x)|^2 \leq k_L(\phi, \phi)(x) k_L(\tilde{R}_f, \tilde{R}_f)(x) = 0$. Using this as well as (1), we get

$$
k_L(f, g)(x) = \sum_{i,j} \partial_i(f)(x) \partial_j(g)(x) k_L(\xi_i, \xi_j)(x). \quad (3)
$$

Define a bilinear form $<\cdot, \cdot>_x$ on $T^*_x M$ by setting $<d\tilde{t}l dx_i | x, d\tilde{t}l_j | x > = |x = g_{ij}(x)$ on the basis $d\tilde{t}l_i | x, i = 1, \ldots, m$, where $g_{ij} = k_L(\xi_i, \xi_j)$. To see well-definedness, i.e. independence of choice of coordinates, it suffices to note that for another set of local coordinates $(\eta_1, \ldots, \eta_m)$ around $x$, we have by (3)

$$
k_L(\tilde{\eta}_i, \tilde{\eta}_j)(x) = \sum_{kl} \frac{\partial \eta_i}{\partial \xi_k}(x) \frac{\partial \eta_j}{\partial \xi_l}(x) g_{kl}(x).
$$

This also shows that $k_L(\tilde{\eta}_i, \tilde{\eta}_j)(\cdot)$ is smooth around $x$ if and only if $g_{ij}$ are so and $((k_L(\tilde{\eta}_i, \tilde{\eta}_j)(x)))$ is invertible if and only if $((g_{ij}(x)))$ is invertible. By hypothesis we can choose such $\xi_i, i = 1, \ldots, m$ satisfying such local smoothness and invertibility conditions. Hence the symmetric bilinear form defined by us is indeed a Riemannian inner product.

Finally, it also follows from (3) that $<df, dg>_x = |x = k_L(f, g)(x)$ for all real smooth functions $f, g$ and this uniquely determines the metric. □
2.4 Martingales and Brownian flow on manifolds

We will need some standard results about the Brownian motion on a compact Riemannian manifold which we briefly summarize here. For the definition, construction and properties of this stochastic process, we refer to [10], [20], [28] and the references therein. Let us consider the Riemannian structure on $M$ inherited from the Euclidean Riemannian structure of $\mathbb{R}^n$ and follow the construction of [20], page 11, Subsect. 1.4, namely define $X_t$ to be the unique solution of the stochastic differential equation $dX_t = \sum_{i=1}^n X_t P_i(X_t) \circ dW_t(t)$, $X_0 \in M$, in the notation of [20]. Here, $P_i(x)$ denotes the projection of the $i$-th coordinate unit vector of $\mathbb{R}^n$ on the tangent space $T_x M$ and $(W_1(t), \ldots, W_n(t))$ denotes the standard Brownian motion of $\mathbb{R}^n$ starting at the origin. In this picture, $X_t$ is a process on the sample space $(\Omega, \mathcal{F}, P)$ (say) of the standard $n$-dimensional Brownian motion. Let $X_t(x, \omega)$ be the process ‘starting at $x$’, i.e. the solution with $X_0 = x$. Let $\mathcal{L} = \sum_i P_i^2$ be the Laplacian on $M$. The it is known that the Markov semigroup (‘heat semigroup’) given by $T_t(f)(x) := \mathbb{E}_P(f(X_t(x, \cdot)))$ has $\mathcal{L}$ as the infinitesimal generator. We also need the following fact, which can be seen from [10], Prop. 4C, Chap. I:

Proposition 2.4 For almost all $\omega$ in the sample space, the following hold:
(i) The random map $\gamma_t(\omega)$ given by $x \mapsto X_t(x, \omega)$ is a diffeomorphism for every $t$,
(ii) $(x, t) \mapsto X_t(x, \omega)$ is continuous.
(iii) $X_{t+s}(x, \omega) = X_t(X_s(x, \omega), \omega)$.

We will need the concept of stopping time (or stop time) and a version of Doob’s Optional Sampling Theorem suitable for us. Let us briefly recall here the basics (see also [28] and the reference therein). We assume the usual hypotheses such as right continuity of the filtrations considered.

Definition 2.5 A stopping time adapted to a filtered probability space $(\Sigma, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)$, is a random variable $\tau : \Sigma R_+$ satisfying $\{\omega \in \Sigma : \tau(\omega) \leq t\} \in \mathcal{G}_t$ for all $t \in R_+$.

Let $Z$ be a separable Banach space. We restrict our attention to separable Banach space valued random variables to avoid measure-theoretic difficulties. For example, the notion of Bochner or strong measurability and weak measurability coincide for separable Banach-space valued random variables. We refer the reader to the lecture note by Pisier [20] for some more details of Banach space valued measurable functions related topics.

A family $(M_t)_{t \geq 0}$ of $Z$-valued random variables on the above filtered probability space is called a $(\mathcal{G}_t)$-martingale (simply martingale if the filtration is understood) if $\mathbb{E}(\|M_t\|) < \infty$ for each $t$, $M_t$ is adapted to $\mathcal{G}_t$ in the sense that $M_t$ is measurable w.r.t. $(\Sigma, \mathcal{G}_t)$ and $\mathbb{E}(M_t|\mathcal{G}_s) = M_s$ (almost surely) for all $0 \leq s \leq t < \infty$, where $\mathbb{E}(\cdot|\mathcal{G}_s)$ denotes the conditional expectation with respect to $\mathcal{G}_s$.

Clearly, a Banach space valued family of random variable $M_t$ is martingale if and only if for every bounded linear functional $\phi$ on $Z$, the complex valued process $\phi(M_t)$ is a martingale in the usual classical sense. Adapting the proof of the classical Optional Sampling Theorem, we get the following version of Theorem 18 of Chapter I, page 10 of [28]:

Proposition 2.6 Let $(M_t)$ be a $Z$-valued right continuous (i.e. for almost all $\omega$, $t \mapsto M_t(\omega)$ is right continuous) martingale as above. Then for any bounded stopping time, the process $M_{\tau + \mathcal{M}}$ is a martingale, where $a \wedge b := \min(a, b)$.

Proof:
Let $\tau \leq t_0$ for some constant $t_0 > 0$ almost surely. It is enough to prove that $\phi(M_{\tau + \mathcal{M}})$ is a scalar-valued martingale for each bounded linear functional $\phi$ on $Z$. But this follows by applying Theorem 18 of Chapter I of [28] to the scalar-valued martingale $\phi(M_{t \wedge t_0})$, or applying Problem 3.23 (part (i)) of Chapter I, page 20 of [22] to $\phi(M_t)$. □
3 Main results

Throughout this section, let $M$ be a compact smooth manifold of dimension $m$ and $Q$ be a CQG with a faithful action $\alpha$ on $C(M)$.

3.1 Smooth action

We refer to [13] for a detailed discussion on the natural Fréchet topology of $C^\infty(M)$ as well as the space of $\mathcal{B}$-valued smooth functions $C^\infty(M, \mathcal{B})$ for any Banach space $\mathcal{B}$. Indeed, by the nuclearity of $C^\infty(M)$ as a locally convex space, $C^\infty(M, \mathcal{B})$ is the unique topological tensor product of $C^\infty(M)$ and $\mathcal{Q}$ in the category of locally convex spaces. This allows us to define $T \otimes \text{id}$ from $C^\infty(M, \mathcal{B})$ for any Fréchet continuous linear map $T$ from $C^\infty(M)$ to $C^\infty(M)$ (or, more generally, to some other locally convex space). For we also recall from [13] the space $\Omega^1(M) \equiv \Omega^1(C^\infty(M))$ of smooth one-forms and $\Omega^1(M, \mathcal{B})$ of the space of smooth $\mathcal{B}$-valued one-forms, as well as the natural extension of the differential map $d$ to a Fréchet continuous map from $C^\infty(M, \mathcal{B})$ to $\Omega^1(M, \mathcal{B})$. In fact, for $F \in C^\infty(M, \mathcal{B})$, the element $dF \in \Omega^1(M, \mathcal{B})$ is the unique element satisfying $(\text{id} \otimes \eta)(dF(m)) = (dF_\xi)(m)$, for every continuous linear functional $\xi$ on $\mathcal{B}$, where $m \in M$, $dF(m) \in T^*_mM \otimes_{\text{alg}} \mathcal{B}$ and $F_\xi \in C^\infty(M)$ given by $F_\xi(x) := \xi(F(x)) \forall x \in M$.

We now define a smooth action following [13].

**Definition 3.1** In case $C = C(M)$, where $M$ is a smooth compact manifold, we say that an action $\alpha$ of a CQG $Q$ on $C(M)$ is smooth if $\alpha$ maps $C^\infty(M)$ into $C^\infty(M, Q)$ and $\text{Sp} \alpha(C)(1 \otimes Q)$ is dense in $C^\infty(M, Q)$ in the Fréchet topology. We say that the action is faithful if the algebra generated by elements of the form $\alpha(f)(p) \equiv \text{ev}_p(\alpha(f))$, where $f \in C(M), p \in M$ is norm-dense in $C(M, Q)$.

**Remark 3.2** In case $Q = C(G)$ where $G$ is a compact group acting on $M$, say by $\alpha_g : x \mapsto gx$, the smoothness of the induced action $\alpha$ given by $\alpha(f)(x, g) = f(gx)$ on $C(M)$ in the sense of the above definition means the smoothness of the map $M \ni x \mapsto gx$ for each $g$.

It has been proved in [13], following arguments of [27, 3] etc. that given any smooth action $\alpha$ of $Q$ on $C(M)$ there is a Fréchet dense unital $*$-subalgebra $C_0$ of $C^\infty(M)$ on which $\alpha$ restricts to an algebraic co-action of $Q_0$. It also follows (see [14], Corollary 3.3) that for any smooth action $\alpha$, the corresponding reduced action $\alpha_r$ is injective and hence it is unitarily implemented.

Suppose that $M$ has a Riemannian structure with the corresponding $C^\infty(M)$ valued inner product $\langle \cdot, \cdot \rangle$ on $\Omega^1(C^\infty(M))$ as in [13] using the Riemannian structure, determined by $\langle \omega_1 \otimes q_1, \omega_2 \otimes q_2 \rangle(x) = \langle \omega_1, \omega_2 \rangle_x q_1^*q_2$, where $\langle \cdot, \cdot \rangle_x$ denotes the inner product coming from the Riemannian structure on the complexified cotangent space at $x$. If $\mathcal{L}$ is the Laplacian corresponding to the Riemannian structure and $k_\mathcal{L}(f, g) := \mathcal{L}(\mathcal{T}g) - \mathcal{L}(\mathcal{T})g - \mathcal{T}\mathcal{L}(g)$ for $f, g \in C^\infty(M)$, we have $\langle df \otimes q, dg \otimes q' \rangle = k_\mathcal{L}(f, g)q^*q'$.

**Definition 3.3** A smooth action $\alpha$ on $M$ is said to preserve the inner product if

$$\langle \langle df, dg \rangle \rangle = \alpha(\langle df, dg \rangle)$$

for all $f, g \in C^\infty(M)$.

It is easy to see, by the Fréchet continuity of the maps involved that it is enough to have $\alpha$ for $f, g \in C_0$.

3.2 Averaging of the Riemannian metric

Let $M$ be as before and let $\alpha$ be a faithful smooth action of $Q$ on $C(M)$. Replacing $Q$ by $Q_r$ we can assume without loss of generality that $Q$ has faithful Haar state and $\alpha = \alpha_r$. It
is also known (see [19]) that $Q_r$ is of Kac type, hence $h$ is tracial and $\kappa$ is norm-bounded on $Q = Q_r$. Let $Q_0$ be the canonical dense Hopf *-algebra for $Q$ and $C_0$ be a Frechet-dense unital *-subalgebra of $C^\infty(M)$ on which $\alpha$ is algebraic. Moreover, as explained in Subsection 2.2, choose some faithful $\alpha$-invariant Borel measure $\mu$ on $M$ and the corresponding unitary representation $U$ on $L^2(M,\mu)$ implementing $\alpha$, i.e. $\alpha(f) = U(f \otimes 1)U^{-1}$, where $f \in C(M)$ is viewed as a multiplication operator on $L^2(M,\mu)$. Let $L^2(Q)$ be the GNS space of the Haar state $h$ and identify $L(H \otimes Q)$ (for any Hilbert space $H$) as a subalgebra of $B(H \otimes L^2(Q))$. The vector state $\langle 1,1 \rangle$ on $B(L^2(Q))$ extends $h$ and we continue to denote it by $h$.

Denote by $M_F$ and $M_f$ the operators of left multiplication by $F$ (respectively $f$) on the Hilbert $Q$-module $L^2(M,\mu) \otimes Q$ (respectively $L^2(M,\mu)$), most often we may write simply $F$ or $f$ for $M_F$ or $M_f$ respectively by making slight abuse of notation.

**Lemma 3.4** For $F \in C_0 \otimes_{alg} Q_0 \subset C^\infty(M, Q)$, we have 

$$(\text{id} \otimes h)(U^{-1}M FU) = M_F,$$

where $F^g = (\text{id} \otimes h)(U^{-1}(F)) \in C_0$.

**Proof:**

It is sufficient to prove the lemma for $F = f \otimes q$, where $f \in C_0, q \in Q_0$. Using Sweedler’s notation and the trace property of $h$, we have for $g \in C_0$:

$$(\text{id} \otimes h)(U^{-1}M FU)g = (\text{id} \otimes h)(U^{-1}M FU(g \otimes 1)) = (\text{id} \otimes h)(U^{-1}(fg_0) \otimes q(g_1)) = f_0 g_0 h(\kappa f_1 q g_2 \kappa(g_1)) = f_0 g_0 h(\kappa f_1 q g_2 \kappa g_1)g = F^g.$$ 

□

**Corollary 3.5** The map $F \mapsto \Psi(F) := (\text{id} \otimes h)(U^{-1}M FU)$ extends to a unital completely positive map from $C(M, Q)$ to $C(M)$. In particular, $\text{ev}_p \circ (\text{id} \otimes h)(U^{-1} \cdot U)$ extends to a well-defined state on $C(M)$. Moreover, $\Psi$ is $Q$-invariant in the sense that

$$(\Psi \otimes \text{id}) \circ (\text{id} \otimes \Delta) = \alpha(\Psi(\cdot)). \quad (5)$$

**Proof:**

The map is clearly norm-bounded and completely positive by the formula that defines it. It also follows from Lemma 3.4 that it maps the dense subspace $C_0 \otimes_{alg} Q_0$ into $C(M)$. By norm-continuity, the image of the map must be contained in $C(M)$.

To prove the invariance, it is enough to prove (5) for $F = f \otimes q$, where $f \in C_0, q \in Q_0$. To this end, note that as $\kappa^2 = \text{id}$, we have $q_1 q_2 q_1 q_2 = q_2 q_1 q_1 q_2 = \epsilon(q)1$ for all $q \in Q_0$. Moreover, we have

$$(\kappa \otimes \Delta) \circ (\kappa \otimes \kappa) = \sigma \circ (\kappa \otimes \kappa) \circ \Delta$$

where $\sigma$ denotes flip, we have

$h(\kappa f_1 q, k g_1) = h(\kappa f_1 q, g_1 k) = h(\kappa f_1 q, g_1 k) = h(\kappa f_1 q, k g_1) = h(\kappa f_1 q, k g_1) = h(\kappa f_1 q, k g_1)$

for all $q \in Q_0$. Now, the left hand side of (5) for $F = f \otimes q$ equals $f_0 \otimes h(\kappa f_1 q g_2)$. By (5), $\kappa^2 = \text{id}$ as well as the identity $\Delta \circ \kappa = \sigma \circ (\kappa \otimes \kappa) \circ \Delta$ where $\sigma$ denotes flip, we have

$h(\kappa f_1 q, k g_1) = h(\kappa f_1 q, k g_1)$

for all $q \in Q_0$. Now, the left hand side of (5) for $F = f \otimes q$ equals $f_0 \otimes h(\kappa f_1 q g_2)$. By (5), $\kappa^2 = \text{id}$ as well as the identity $\Delta \circ \kappa = \sigma \circ (\kappa \otimes \kappa) \circ \Delta$ where $\sigma$ denotes flip, we have

$h(\kappa f_1 q, k g_1) = h(\kappa f_1 q, k g_1)$

for all $q \in Q_0$. Now, the left hand side of (5) for $F = f \otimes q$ equals $f_0 \otimes h(\kappa f_1 q g_2)$.
It follows that
\[ f_{(0)} \otimes h(\kappa(f_{(1)}))q_{(2)} = f_{(0)} \otimes f_{(1)}h(\kappa(f_{(1)})q) = f_{(0)}f_{(1)}h(\kappa(f_{(1)}))q = \alpha(f_{(0)}h(\kappa(f_{(1)}))q) = \alpha(\Psi(f \otimes q)), \]
which is the right hand side of (5).

\[ \Box \]

Choose and fix any Riemannian structure on \( M \) and write \( \mathcal{L} \) for the Laplacian on \( M \) and let
\[ \hat{\mathcal{L}}(f) = (\text{id} \otimes h)(U^{-1}(\mathcal{L} \otimes \text{id})(\alpha(f))U) \]
for \( f \in C^\infty(M) \). Here we have identified scalar or \( \mathcal{Q} \)-valued functions with the corresponding left multiplication operators in appropriate Hilbert spaces or Hilbert modules, as understood from the context. By Lemma 3.4 and Corollary 3.5, \( \hat{\mathcal{L}}(f) \in C(M) \). As \( \alpha \) is Frechet continuous, it is clear that \( \hat{\mathcal{L}} \) is continuous w.r.t. the Frechet topology on \( C^\infty(M) \) and the norm topology on \( C(M) \). We also observe that for \( f \in C_0, \hat{\mathcal{L}}(f) = ((\mathcal{L} \otimes \text{id})(\alpha(f)))^2 = f_{(0)} = \Psi(\mathcal{L}(f_{(0)}) \otimes f_{(1)}). \)
We now claim that

**Theorem 3.6** \( \hat{\mathcal{L}} \) satisfies the hypotheses of Proposition 2.3.

**Proof:**
As before, we’ll throughout make the identifications with functions (scalar or \( \mathcal{C}^\ast \) algebra valued) and operators of left multiplication by them on appropriate Hilbert spaces or modules. Clearly, \( \hat{\mathcal{L}}(1) = 0 \). Clearly, as \( \Psi(F^*) = \Psi(F)^* \), we have \( \hat{\mathcal{L}}(\overline{f}) = \overline{\hat{\mathcal{L}}(f)} \) for all \( f \in C^\infty(M) \). We now prove that \( \hat{\mathcal{L}} \) is conditionally completely positive. It follows from the Corollary 3.5 and the following observation:
\[ k_{\mathcal{L}}(f, g) = \Psi(\langle \langle da(f), da(g) \rangle \rangle) \equiv (\text{id} \otimes h)(U^{-1}(\langle \langle da(f), da(g) \rangle \rangle)U), \tag{7} \]
where \( \langle \langle \cdot, \cdot \rangle \rangle \) is the \( C^\infty(M, \mathcal{Q}) \) valued inner product on \( \Omega^1(M, \mathcal{Q}) \) mentioned before. Note that \( \langle \langle da(f), da(g) \rangle \rangle = (\mathcal{L} \otimes \text{id})(\alpha(\overline{f}g)) - (\mathcal{L} \otimes \text{id})(\alpha(\overline{f}))\alpha(g) - \alpha(\overline{f})(\mathcal{L} \otimes \text{id})(\alpha(g)). \)
To prove (7), it suffices to observe that by Lemma 3.4 with \( F = (\mathcal{L} \otimes \text{id})(\alpha(f)) \), we have for \( f, g \in C_0 \otimes_{\text{alg}} \mathcal{Q}_0: \)
\[ \hat{\mathcal{L}}(f)g = (\text{id} \otimes h)(U^{-1}(\mathcal{L} \otimes \text{id})(\alpha(f))U(g \otimes 1)) = (\text{id} \otimes h)(U^{-1}(\mathcal{L} \otimes \text{id})(\alpha(f))\alpha(g)U). \]
By continuity of \( \hat{\mathcal{L}} \) and \( \mathcal{L} \otimes \text{id} \), the above equation extends to all \( f, g \in C^\infty(M) \).

We now come to the proof of locality.
Let us consider the Brownian motion \( (X_t) \) corresponding to the Riemannian structure given by \( \mathcal{L} \) and let \( \gamma_t \) be the random flow of automorphism as in the Proposition 2.3. For a Banach space \( E \) let \( L^\infty(\Omega, E) \) be the Banach space of \( E \)-valued essentially bounded measurable functions, to be viewed as \( E \)-valued random variables. Let \( j_t : C(M, \mathcal{Q}) \to L^\infty(\Omega, C(M, \mathcal{Q})) \) be the \(*\)-homomorphism given by
\[ j_t(F)(\omega)(x) = j_t(F)(x, \omega) = F(X_t(x, \omega)). \]
That is, \( j_t(F)(\omega) = F \circ \gamma_t(\omega) \). Let \( \mathcal{E} \) be the \( \mathcal{C}^\ast \) subalgebra of \( \mathcal{L}(L^2(M, \mu) \otimes \mathcal{Q}) \) given by \( \mathcal{E} = \{ U^{-1}MFU : F \in C(M, \mathcal{Q}) \} \equiv U^{-1}C(M, \mathcal{Q})U \). We note that \( \mathcal{E} \) is separable. Indeed, as
$M$ is compact and $Q$ acts faithfully on the separable $C^*$ algebra $C(M)$, $Q$ is separable too. This implies $C(M,Q) = (C(M) \otimes Q$ and hence $E \cong C(M,Q)$ is separable. Moreover, $E$ contains $C(M) \otimes 1 = U^{-1} \alpha(C(M))U$.

Using the identification of $C(M,Q)$ as left multiplication operators, we have an embedding of $L^\infty(\Omega,C(M,Q))$ in $L^\infty(\Omega,E)$. Using this, define $J_t : C(M,Q) \to L^\infty(\Omega,E)$ by

$$J_t(F)(\omega) = U^{-1} j_t(F)(\omega)U.$$

Clearly, $J_t$ is a unital $*$-homomorphism. We also have a natural embedding $L^\infty(\Omega,E) \subseteq \mathcal{B}(L^2(\Omega) \otimes L^2(M,M) \otimes L^2(\Omega,h))$ and in this picture, we an write $J_t(F) = \tilde{U}^{-1} j_t(F)\tilde{U}$, where $\tilde{U} = I_{L^2(\Omega)} \otimes U$.

Let $T_t$ be Markov semigroup (heat semigroup) generated by $L$, which is given by the formula

$$T_t(f)(x) = E(f(X_t(x,\cdot))) \text{ for all } t \geq 0.\quad (\text{ii})$$

As $T_t$ is a $C_0$ semigroup of completely positive maps from $C(M)$ to $C(M)$, the map $\tilde{T}_t := T_t \otimes id : C(M,Q) \to C(M,Q)$. In fact, we have $\tilde{T}_t(F)(x) = E(F(X_t(x,\cdot))) \text{ for } F \in C(M,Q)$.

Let $E_s$ denote the conditional expectation w.r.t. the sub $\sigma$-algebra generated by $\{X^{-1}_s(B), B \in \mathcal{B}_E, 0 \leq s \leq \}$, where $\mathcal{B}_E$ denotes the usual Borel $\sigma$-algebra of the Banach space $E$. We have

$$E_s \circ j_{s+t} = j_s \circ \tilde{T}_t\quad (8)$$

for $s,t \geq 0$, which follows from the Markov property of the Brownian motion, e.g., as given by (iii) of Proposition 2.4. Let $\bar{\mathcal{L}} = \mathcal{L} \otimes id$ on $C^\infty(M,Q)$. Clearly, for all $F \in C^\infty(M,Q)$, we have

$$\frac{d}{dt} \tilde{T}_t(F) = \bar{\mathcal{L}} \circ \tilde{T}_t(F) = \tilde{T}_t \circ \bar{\mathcal{L}}(F).\quad (9)$$

Indeed, to verify this, we should at first note that $\tilde{T}_t$ and $\bar{\mathcal{L}}$ commute as $T_t$ and $\mathcal{L}$ do and they are continuous in appropriate topologies. Furthermore, we have $\tilde{T}_t(F) - F = \int_0^t \tilde{T}_s \circ \bar{\mathcal{L}}(F)ds$, first verifying on the Frechet dense subspace $C_0 \otimes_{\text{alg}} Q_0$ and then extending to the whole of $C^\infty(M,Q)$ by continuity of the maps involved. From this, (9) follows immediately.

Define a unital $*$-homomorphism $\Pi_t : C(M) \to L^\infty(\Omega,E)$ by

$$\Pi_t(f) = J_t(\alpha(f)).$$

For $f \in C^\infty(M)$, define

$$M^f_t = \Pi_t(f) - \int_0^t J_s(\bar{\mathcal{L}}(\alpha(f)))ds.$$

Note that by the continuity of the Brownian flow, the integrand on the right hand side is continuous in $s$ for almost all $\omega$ and hence convergent absolutely in norm of $E$. We claim that this is an $E$- valued martingale w.r.t. the filtration of the Brownian motion. To this end, note that for $0 \leq u \leq s \leq t$, $E_{u,j_s}(\bar{\mathcal{L}}(\alpha(f))) = j_s(T_{s-u} \circ \bar{\mathcal{L}}(\alpha(f))) = \frac{d}{ds}j_u(T_{s-u}(\alpha(f)))$, using (8) and (9). On the other hand, $E_{u,j_s}(F) = j_s(F)$ for $s \leq u$ and $F \in C^\infty(M)$ by definition of the filtration. Hence we have (for almost all $\omega \in \Omega$):

$$E_u(M^f_t)(\omega) = E_u(\Pi_t(f))(\omega) + U^{-1} \left( \int_0^u j_s(\bar{\mathcal{L}}(\alpha(f)))(\omega)ds - \int_u^t \frac{d}{ds}j_u(T_{s-u}(\alpha(f)))(\omega)ds \right) U.\quad (10)$$

But observe that

$$\int_0^u j_s(\bar{\mathcal{L}}(\alpha(f)))ds + \int_u^t \frac{d}{ds}j_u(T_{s-u}(\alpha(f)))(\omega)ds = \int_0^u j_s(\bar{\mathcal{L}}(\alpha(f)))ds + j_u(T_{t-u}(\alpha(f)) - j_u(\alpha(f))).\quad (11)$$

As $j_s(\cdot)$ is measurable w.r.t. the $\sigma$-algebra of $X_v, v \leq s$, we have $E_u \circ j_s = j_s$ for all $s$. Moreover, $E_u \circ j_t = j_t \circ \tilde{T}_t$ - hence $E_u(\Pi_t(f))(\omega) = U^{-1} j_u T_{t-u}(\alpha(f))(\omega)U$. We also have
\[ j_t \circ T_{t-u} = E_t \circ j_t = j_t. \]

Combining the above observations with (11), the right hand side of (10) reduces to

\[ \Pi_u(f)(\omega) - U^{-1} \left( \int_0^u j_s(\mathcal{L}(\alpha(f)))(\omega)ds \right) U = M_u(f)(\omega). \]

In the above, we could interchange \( E_u \) with the integral by appropriate continuity of the maps involved.

Now, let \( Y_t(x) = \Pi_t(x_1) \). Observe that \( Y_t(0) = y_t \otimes 1 \). To show the locality of \( \mathcal{L} \) at a point \( p = (p_1, \ldots, p_n) \) of \( M \subset \mathbb{R}^n \), consider \( f = \phi(x_1, \ldots, x_n) \), where \( \phi \) is a smooth function on \( \mathbb{R}^n \), and assume that \( f \) is zero on a neighbourhood of \( p \). Choose small enough \( \epsilon_0 > 0 \) such that \( \phi(y_1, \ldots, y_n) = 0 \) whenever \( |y_i - p_i| \leq \epsilon_0 \) for all \( i \) and \( y = (y_1, \ldots, y_n) \in M \). It is clear from the continuity properties of the Brownian flow (see Proposition 2.4) that \( t \mapsto J_t(\omega) \) is norm continuous for almost all \( \omega \) and fixed \( F \in C^\infty(M, \mathbb{Q}) \). Let \( F = \mathcal{L}(\alpha(f)) \) and let \( \tau'_{\epsilon}(\omega) (\epsilon > 0) \) be the infimum of \( t \geq 0 \) (which is defined to be \( +\infty \) if no such \( t \) exists) for which \( \| J_t(\omega) - J_0(\omega) \| > \epsilon \). It is clearly a stopping time. Observe that, as \( \Pi_t \) is a homomorphism, \( \Pi_t(f) = \phi(\Pi_t(x_1), \ldots, \Pi_t(x_n)) = \phi(Y_t(1), \ldots, Y_{n}(t)) \). Furthermore, consider another stopping time \( \tau''_\epsilon(\omega) \) to be the infimum of all \( t \geq 0 \) for which \( \| Y_t(\omega) - y_t \otimes 1 \| > \epsilon \) for some \( i \). Finally, let \( \tau_{\epsilon} = \min(\tau'_{\epsilon}, \tau''_\epsilon, 1) \), which is a bounded stopping time. Applying Proposition 2.6 to the (continuous) martingale \( M^f_t \), we conclude that \( M^f_{\tau_\epsilon} \) is a martingale too, hence in particular, \( E(M^f_{\tau_\epsilon}) = M^f_0 = f \otimes 1 \). In other words,

\[
E(\Pi_{\tau_{\epsilon}}(f)) - f \otimes 1 = E(\int_0^{\tau_{\epsilon}} \tilde{U}^{-1} j_s(\mathcal{L}(\alpha(f)))\tilde{U} ds) = E(\int_0^{\tau_{\epsilon}} J_s(F)ds).
\]

By definition of \( \tau_{\epsilon} \) and continuity of the Brownian flow, it is clear that \( \| J_s(\omega) - J_0(\omega) \| \leq \epsilon \) for all \( s \leq \tau_{\epsilon} \). Hence we have \( \int_0^{\tau_{\epsilon}} \| J_s(\omega) - J_0(\omega) \| ds \leq \tau_{\epsilon} \cdot \epsilon \). It follows that

\[
\left\| E(\int_0^{\tau_{\epsilon}} J_s(F)ds) - E(\tau_{\epsilon})J_0(F) \right\|
\leq \epsilon E(\tau_{\epsilon}),
\]

hence

\[
\lim_{\epsilon \to 0+} \frac{E(\Pi_{\tau_{\epsilon}}(f)) - f \otimes 1}{E(\tau_{\epsilon})} = \lim_{\epsilon \to 0+} \frac{E(\int_0^{\tau_{\epsilon}} J_s(F)ds)}{E(\tau_{\epsilon})} = E(J_0(F)) = U^{-1}\mathcal{L}(\alpha(f))U. \tag{12}
\]

For a fixed \( t \) and \( \omega \), let us denote by \( B_{t,\omega} \) the commutative unital C*-algebra generated by \( \{ \Pi_t(\omega), g \otimes 1, f, g \in C^\infty(M) \} \), in \( \mathcal{L}(L^2(M, \mu \otimes \mathbb{Q})) \). Clearly, \( B_{0,\omega} = C(M) \otimes 1 \) is a common C*-subalgebra of all \( B_{t,\omega} \). Let \( S \) be the (convex, weak * compact) set of states \( \zeta \) on \( B_{t,\omega} \) which extends \( ev_p \) on \( C(M) \otimes 1 \cong C(M) \), i.e. \( \zeta(g \otimes 1) = ev_p(g \otimes 1) = g(p) \forall g \). By standard arguments we can prove that every extreme point of \( S \) is also an extreme point of the set of all states on \( B_{t,\omega} \), i.e. a *-homomorphism. Indeed, if an extreme point \( \zeta \) of \( S \) can be written as \( \zeta = q_{\zeta_1} + (1 - q)\zeta_2 \), where \( 0 < q < 1 \) and \( \zeta_1, \zeta_2 \) are states on \( B_{t,\omega} \), we have \( ev_p = q_{\zeta'_1} + (1 - q)\zeta'_2 \), where \( \zeta'_i \) denotes the restriction of \( \zeta_i \) to \( C(M) \otimes 1 \). As \( ev_p \) is a pure state of \( C(M) \otimes 1 \), this implies \( \zeta'_i = ev_p \) for \( i = 1, 2 \), i.e. \( \zeta_i \in S \). Then, by the extremality of \( \zeta \) in \( S \), \( \zeta_i = \zeta \) for \( i = 1, 2 \). Hence \( \zeta \) is a pure state of \( B_{t,\omega} \), i.e. *-homomorphism and we have \( \zeta(\Pi_t(\omega)) = \phi(\zeta(Y_1(t), \ldots, Y_{n}(t))) \).

Now, recall from Corollary 3.3 that \( (id \otimes h)(B_{t,\omega}) \subseteq C(M) \), so \( \eta := (ev_p \otimes h) \) is a well-defined state on \( B_{t,\omega} \) and it is also an element of \( S \). Moreover, \( \mathcal{L}(f)(p) = \lim_{\epsilon \to 0+} \frac{E_{(\eta(\Pi_{\tau_{\epsilon}}(f)))}}{E_{(\tau_{\epsilon})}} \), as \( f(p) = 0 \). We claim that \( \zeta(\Pi_{\tau_{\epsilon}}(f)) = 0 \) for all sufficiently small \( \epsilon \) and for \( \zeta \in S \). It is enough
to prove it when \( \zeta \) is an extreme point, i.e. \( * \)-homomorphism. For any such \( \zeta \) (extremal) we get by continuity of the Brownian flow \( |\zeta(Y_i(\tau_i) - y_i)\| \leq \epsilon \forall i \). As \( \zeta(y_i \oplus x) = y_i \) by definition of \( S \), the tuple \((\zeta(Y_1(\tau_1)), \ldots, \zeta(Y_n(\tau_n))) \in \mathbb{R}^{n}\) is contained in an \( n \)-cube of sidelength \( \epsilon \) around \((y_1, \ldots, y_n)\). Moreover, as \( \zeta \circ \Pi_\epsilon \) is a character of \( C(M) \), there is some point \( v = (v_1, \ldots, v_n) \in M \subset \mathbb{R}^{n} \) such that \( \zeta \circ \Pi_\epsilon(f) = f(v) \) for all \( f \in C(M) \). In particular, \((\zeta(Y_1(\epsilon)), \ldots, \zeta(Y_n(\epsilon))) = (v_1, \ldots, v_n) \in M \). Thus, \( \zeta((\Pi_\epsilon(f)) = \phi(\zeta(Y_1(\epsilon)), \ldots, \zeta(Y_n(\epsilon))) = 0 \) for all \( \epsilon < \epsilon_0 \), proving our claim. In particular, we have \( \eta(\Pi_\epsilon(f)) = 0 \) for all sufficiently small \( \epsilon \), hence \( \hat{\mathcal{L}}(f)(p) = 0 \).

Finally, we come to the proof of non-degeneracy. It is enough to prove that for \( x \in M \) and some real valued \( f_1, \ldots, f_m \) from \( C_0 \) such that they form a set of local coordinates around \( x \) (such a choice can be made by the Frechet density of \( C_0 \) in \( C^\infty(M) \)), the matrix \((k_{\mathcal{L}}(f_i, f_j)(x))\) is invertible. To this end, let \( c_1, \ldots, c_m \) be real numbers such that \( \sum_{ij} c_i c_j k_{\mathcal{L}}(f_i, f_j)(x) = 0 \). From \( \mathcal{L} \) and the faithfulness of \( h \) on \( Q_0 \), we get \( \sum_{ij} c_i c_j (\alpha \otimes \kappa) \circ \alpha(k_{\mathcal{L}}(f_i, f_j))(x) = 0 \). Applying the counit \( \epsilon \), it follows that \( \sum_{ij} c_i c_j k_{\mathcal{L}}(f_i, f_j)(x) = 0 \). But

\[
\sum_{ij} c_i c_j k_{\mathcal{L}}(f_i(0), f_j(0))(x) = 0.
\]

where \( < \cdot, \cdot >_x \) is the Riemannian inner product at \( x \). Hence we have got \( \sum_{ij} k_{\mathcal{L}}(f_i, f_j)(x) = 0 \).

As \( \mathcal{L} \) is nondegenerate, we conclude that \( c_i = 0 \) for all \( i \), proving the nondegeneracy of \( \hat{\mathcal{L}} \). Finally, for the above choice of \( f_i \)’s it is clear that \( \hat{\mathcal{L}}(f_i) \in C_0 \), hence \( k_{\mathcal{L}}(f_i, f_j) \in C^\infty(M) \) too. This completes the proof.

\[ \square \]

**Corollary 3.7** Any smooth action on a compact Riemannian manifold preserves some Riemannian metric on \( M \).

**Proof:**

By Proposition 3.3 and Theorem 3.6 \( \hat{\mathcal{L}} \) induces a Riemannian metric given by \( <df, dg> = \hat{\mathcal{L}}(fg) - \hat{\mathcal{L}}(f)g - f\hat{\mathcal{L}}(g) \) for all real valued smooth \( f, g \). We claim that \( \alpha \) preserves this Riemannian inner product. As \( \alpha \) is smooth, it is enough to prove \( <df_0, dg_0> = \alpha(<df_0, dg_0>) \) for all real valued elements \( f, g \in C_0 \). For this, it is enough to prove that \( (\hat{\mathcal{L}} \otimes \text{id})(\alpha(f)) = \alpha(\hat{\mathcal{L}}(f)) \) for all \( f \in C_0 \) (hence for all \( f \in C^\infty(M) \) by appropriate continuity of \( \alpha \) and \( \hat{\mathcal{L}} \)). Once we prove this, the argument of Lemma 4.3 of [13] can be applied verbatim. There \( \mathcal{L} \) is the Laplacian of a Riemannian structure but that has no role in the proof; the algebraic calculation requires only that \( \mathcal{L} \) commutes with \( \alpha \).

Now, \( \hat{\mathcal{L}}(f) = \Psi(F) \) where \( F = \mathcal{L}(f(0)) \otimes f(1) \) and we have by (15) the following

\[
\begin{align*}
(\hat{\mathcal{L}} \otimes \text{id})(\alpha(f)) &= (\Psi \otimes \text{id})(\mathcal{L}(f(0)) \otimes f(1)) \\
&= (\Psi \otimes \text{id})(\mathcal{L}(f(0)) \otimes f(1)) \\
&= \alpha(\Psi(\mathcal{L}(f(0)) \otimes f(1))) \\
&= \alpha(\hat{\mathcal{L}}(f)).
\end{align*}
\]

12
From Theorem 3.5 of [13], we conclude the following, which can be called ‘commutativity of partial derivatives up to the first order’:

**Corollary 3.8** For any point \( x \in M \) and local coordinates \((x_1, \ldots, x_m)\) around \( x \), the algebra \( Q_x \) generated by \( \alpha(f)(x), \frac{\partial}{\partial x_i}\alpha(g)(x) \), where \( f, g \in C^\infty(M) \) and \( i = 1, \ldots, m \), is commutative.

### 3.3 Proof of the conjecture

Let \( \alpha \) be a smooth action as in the previous subsection. We have already seen commutativity of partial derivatives up to the first order. We want to prove similar commutativity for higher order partial derivatives. This involves lift to the cotangent bundle.

**Lemma 3.9** For any point \( x \in M \) and local coordinates \((x_1, \ldots, x_m)\) around \( x \), the algebra generated by \( \alpha(f)(x), \frac{\partial}{\partial x_i} \alpha(g)(x) \), where \( f, g \in C^\infty(M) \), \( k \geq 1 \) and \( i_j \in \{1, \ldots, m\} \), is commutative.

**Proof:**

We need an analogue of Theorem 3.4 of [13], to lift the given action to a smooth action on the sphere bundle of the cotangent space. As the constructions and arguments in [13] go through almost verbatim, we just sketch the main line of arguments very briefly.

First, we choose a Riemannian metric \(<\cdot, \cdot>\) by Corollary 3.7 which is preserved by the action. Consider the compact smooth manifold \( S \) given by:

\[
S = \{(x, \omega) : x \in M, \omega \in T_x^* M <\omega, \omega>_x = 1\}.
\]

Let \( \tilde{\pi} : \tilde{S} \to M \) be the projection \( \tilde{\pi}(x, \omega) = x \), which extends \( \pi : S \to M \). In analogy with the construction of Subsection 3.3 of [13], we define \( \tilde{\theta}_x \in C^\infty(\tilde{S}) \), where \( \xi \in \Omega^1(C^\infty(M)) \), by \( \tilde{\theta}_x(x, \omega) = <\omega, \xi(x)>_x \). For any local coordinate chart \((U, (x_1, \ldots, x_m))\) for \( M \) and \( U\)-orthonormal one-forms \( \omega_1, \ldots, \omega_m \) in the sense of [13], i.e. \( \omega_i(y), \ldots, \omega_m(y) \) is an orthonormal basis of \( T_y M \) for all \( y \in U \), we define \( T^U_{\omega} = \theta_{\omega'} \). Similarly, define \( T^U_{\omega} \in C^\infty(\tilde{S}, \mathbb{Q}) \) by

\[
T^U_{\omega}(x, \omega) = <\omega \otimes 1, d\alpha_{(1)}(\omega')>(x) >,\n\]

where \( d\alpha_{(1)} \) is the lift of \( \alpha \) to the module of one-forms as in [13] and \(<\cdot, \cdot, \cdot>\) denotes the \( \mathbb{Q}\)-valued inner product of \( T^*_x M \otimes \mathbb{Q} \). Then, following the arguments of Subsection 3.3 of [13], we can prove that there exists a faithful smooth action \( \beta \) (say) of \( \mathbb{Q} \) on \( S \). The action is determined by

\[
\beta((f \circ \pi)t^U_{\omega}) = (\alpha(f) \circ \pi)T^U_{\omega},
\]

for any \( f \in C^\infty_c(M) \) supported in a coordinate chart \( U \) and where \( \pi : S \to M \) is the bundle projection \( \pi(x, \omega) = x \). Equivalently, we have \( \beta(\theta_g)(x, \omega) = <\omega \otimes 1, d\alpha(f)(x) >. \) Applying Corollary 3.8 to \( \beta \), we conclude that for any \( e \in S \), the algebra \( (Q_{c_e}^\infty, \text{say}) \) generated by \( (X \otimes \text{id})(\beta(F))(e), \beta(G)(e) \) where \( F, G \) are smooth functions on \( S, X \) is any smooth vector field on \( S \), is commutative. Let us extend \( \beta \) further to \( \tilde{S} := \{(x, \omega) : x \in M, \omega \in T^*_x M, \omega \neq 0\} \). Clearly, \( \tilde{S} \) is diffeomorphic to \( S \times \mathbb{R}^\times \), where \( \mathbb{R}^\times = \mathbb{R} \setminus \{0\} \) and the diffeomorphism \( \psi : \mathbb{R}^\times \times S \to \tilde{S} \) (say) is given by \( \psi((x, \omega), r) = (x, r\omega) \). This induces the isomorphism \( C^\infty_c(\tilde{S}) \cong C^\infty_c(\mathbb{R}^\times) \otimes C(S) \). In what follows, we will interchangeably use the two equivalent descriptions of \( \tilde{S} \) explained above, without explicitly mentioning the diffeomorphism \( \psi \).

Define \( \tilde{\beta} : C(\tilde{S}) \to C(\tilde{S}, \mathbb{Q}) \) by \( \tilde{\beta}(\tilde{F})(x, \omega, r) = \beta(\tilde{F})(x, \omega), \) where \( \tilde{F} \in C(S) \) is given by \( \tilde{F}(x, \omega) = \tilde{F}(x, \omega),r \).

From the definition it is clear that \( \tilde{\beta} \) maps \( C^\infty_c(\tilde{S}) \) into \( C^\infty_c(\tilde{S}, \mathbb{Q}) \) and for \( \tilde{e} = (e, r) \in \tilde{S} \) (\( e \in S \)) any smooth vector field \( Y \) on \( \tilde{S} \) and smooth compactly supported function \( \tilde{F} \) on \( \tilde{S}, (Y \otimes \text{id})(\beta(F))(\tilde{e}) \) belongs to \( Q^1_{c_{\mathbb{Q}}} \). Indeed, it is enough to check this for \( \tilde{F} \) of the form
\[ \tilde{F}(x, \omega, r) = F(x, \omega)g(r), \quad F \in C^\infty(S), \quad g \in C^\infty(\mathbb{R}^\infty). \] For such \( \tilde{F} \), we have \( \tilde{\beta}(\tilde{F})(x, \omega, r) = \beta(F)(x, \omega)g(r) \). Moreover, any smooth vector field \( Y \) on \( \tilde{S} \) can be written (locally) as \( \phi_1 X + \phi_2 \tilde{X} \) where \( \phi_1, \phi_2 \) are smooth functions on \( \tilde{S} \) and \( X \) is a vector field in the direction of \( S \). Thus, \( \langle Y \circ \text{id} \rangle(\beta(F))(\tilde{\epsilon}) = g(r)\phi_1(\tilde{\epsilon})(X \circ \text{id})\langle \beta(F)\rangle(\tilde{\epsilon}) + g'(r)\phi_2(\tilde{\epsilon})\beta(F)(\tilde{\epsilon}) \in \mathbb{Q}^1. \)

For a set of local coordinates \( (U, (x_1, \ldots, x_m)) \) for \( M \) and \( U \)-orthonormal one-forms \( \omega'_1, \ldots, \omega'_m \) as before, define \( \tilde{t}^U_j : \tilde{S} \to \mathbb{R} \) by \( \tilde{t}^U_j(e, r) = rt^U_j(e) \). It is clear from the definition of \( \tilde{t}^U_j \) that \( \sum_{j=1}^m (\tilde{t}^U_j)^2 = r^2 \) for all \( (e, r) \) with \( \pi(e) \in U \). Moreover, \( (x_1, \ldots, x_m, \tilde{t}^U_1, \ldots, \tilde{t}^U_m) \) is a set of local coordinates for \( \tilde{S} \) on the neighbourhood \( \tilde{\pi}^{-1}(U) \cong \mathbb{R}^{-1} \times \mathbb{R}^\infty \). Let us write \( y_i \) for \( \tilde{t}^U_i \) and define \( \tilde{\vartheta}_j : \tilde{S} \to \mathbb{R} (\xi \in \Omega^1(C^\infty(M))) \) by \( \tilde{\vartheta}_j((x, \omega), r) = \langle r, \omega, \xi \rangle = \langle r\theta(x, \omega) \rangle \).

Fix a point \( \tilde{e}_0 = (x_0^0, \ldots, x_0^m, y_0^1, \ldots, y_0^m) \) of \( \tilde{S} \), a coordinate neighbourhood \( (U, (x_1, \ldots, x_m)) \) of \( M \) around \( (x_0^1, \ldots, x_0^m) \), a set of \( U \)-orthonormal one-forms \( \omega'_1, \ldots, \omega'_m \) and functions \( f \in C^\infty(M) \) and \( g \in C^\infty_c(\mathbb{R}^m) \) such that \( g = 1 \) in an open neighbourhood of \( r_0 := (y_0^1)^2 + \ldots + (y_0^m)^2 \). Consider \( \tilde{F} = g\tilde{\vartheta}_g \in C^\infty(\tilde{S}) \). Clearly, on a sufficiently small neighbourhood of \( \tilde{e}_0 \), we have (using the orthonormality of \( \omega'_j(x) \)’s)

\[
\tilde{\beta}(\tilde{F})(x, \omega, r) = \left< \omega \odot 1, d\alpha(f) \right> = \sum_j \left< \omega_j \odot 1, d\alpha(f) \right> (x).
\]

In other words, writing \( x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \), we have

\[
\tilde{\beta}(\tilde{F})(x, y) = \sum_j y_j \eta_j(x),
\]

where \( \eta_j(x) = \left< \omega_j, d\alpha(f) \right> (x) \). This implies,

\[
\frac{\partial}{\partial x_i} \tilde{\beta}(\tilde{F})(x, y) = \sum_j y_j \left( \frac{\partial}{\partial x_i} \eta_j \right)(x).
\]

We have already seen that the left hand side of the above belongs to \( \mathbb{Q}_c^1 \). Therefore, fixing \( x = x_0^0 := (x_0^1, \ldots, x_0^m) \) we get \( \sum_j y_j C_j \in \mathbb{Q}^1_{(x_0, y)} \), where \( C_j = (\frac{\partial}{\partial x_i} \eta_j)(x_0) \), for all \( y \) in an open neighbourhood of \( \mathbb{R}^m \). As \( \mathbb{Q}^1_{(x_0, y)} \) is a commutative algebra by Corollary 4.3, we have

\[
\sum_{j \leq k \leq 1} y_j y_k [C_j, C_k] = 0.
\]

Using the fact that \( \{ y_j y_k, j \leq k \} \) are linearly independent (they are the coordinates for an \( m \)-dimensional open neighbourhood) we conclude \( [C_j, C_k] = 0 \). Moreover, for any \( \phi \in C^\infty(M) \), we have \( \alpha(\phi)(x) = \beta(\phi \circ \pi)(x, \omega) \) for any \( \omega \), hence \( \alpha(\phi)(x) \in \mathbb{Q}^1_{(x_0, \omega)} \). It follows that \( \alpha(\phi)(x_0) \) commutes with \( \sum_j y_j C_j \), and using the linear independence of the \( y_i \), we get \( \{ \alpha(\phi)(x_0), C_j \} = 0 \) for all \( j \). Similarly, \( \langle \omega_j \odot 1, d(\alpha(f)) \rangle = (x_0) \in \mathbb{Q}^1_{(x_0, y)} \) and this helps us conclude the commutativity between \( \langle \omega_j \odot 1, d(\alpha(f)) \rangle (x_0) \) and \( C_k \) for any \( j, k = 1, \ldots, m \). In other words, we have proved the commutativity of the algebra generated by \( \alpha(f_1)(x_0), \langle \omega_j \odot 1, d(\alpha(f_2)) \rangle (x_0) \), \( f_p \in C^\infty(M), i, j, k = 1, \ldots, m \). Writing \( \frac{\partial}{\partial x_i} \) in terms of \( \omega_j \)’s (see also the arguments in the beginning of Theorem 4.6.2 of [13]) we can see that this is same as the algebra generated by \( \alpha(f)(x), \frac{\partial}{\partial x_i} \alpha(g)(x), \frac{\partial}{\partial x_j} \alpha(e)(x) \), where \( f, g, \phi \in C^\infty(M), i, j, k = 1, \ldots, m \).

We can go on like this and set up an induction hypothesis that the algebra \( \mathbb{B}^1_{(x)} \) (say) generated by \( \alpha(f)(x), \frac{\partial}{\partial x_i}, \ldots, \frac{\partial}{\partial x_k} \alpha(g)(x) \), where \( f, g \in C^\infty(M), 1 \leq k \leq l \) is commutative.
for any smooth action $\alpha$ on a compact smooth manifold $M$. Using the induction hypothesis (for $l$) for $\beta$ on $S$, we see that $\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_l} \beta(\tilde{F})(x,y)$ belongs to a commutative algebra $B^l_{\beta}(e)$. Proceeding as before, we conclude the commutativity of $B^l_{\alpha+1}$. □

**Theorem 3.10** Let $\alpha$ be a smooth faithful action of a CQG $Q$ on a compact connected smooth manifold $M$. Then $Q$ must be classical, i.e. isomorphic with $C(G)$ for a compact group $G$ acting smoothly on $M$.

**Proof**

Note that in the proof of Theorem 5.3 of [13], the isometry condition, i.e. commutation with the Laplacian, was used only to get commutativity of all order partial derivatives of the action. However, we have already proved this commutativity in Lemma 3.9. This allows the proof of Theorem 5.3 of [13] to be carried through more or less verbatim. Let us sketch it briefly.

Given the smooth action $\alpha$ of $Q$ on $M$, we choose a Riemannian metric by Corollary 3.7 which is preserved by the action. This implies the commutativity of $Q_{\alpha}$. Using this, we can proceed along the lines of [13] to lift the given action to $O(M)$. Now, by Lemma 3.9, we do have the commutativity of partial derivatives of all orders for the lifted action $\Phi$ needed in steps (i) and (iv) of the proof of Theorem 5.3 of [13] and the rest of the arguments of Theorem 5.3 of [13] will go through. □

**Remark 3.11** Observe that in the proof of Lemma 5.1 of [13], only commutativity of partial derivatives up to the second order is necessary. This means it is actually sufficient to state and prove Lemma 3.9 for commutativity up to second order.

As an application, we can generalize the results obtained by Chirvasitu in [4] for some other class of Riemannian manifolds. More precisely,

**Corollary 3.12** Let $M$ be any compact connected Riemannian manifold so that the metric space $(M,d)$ (where $d$ is the Riemannian geodesic distance) satisfies the hypotheses of Corollary 4.9 of [17]. Then the quantum isometry group $QISO(M,d)$ in the sense of [17] coincides with $C(ISO(M,d))$.

**Proof:**

It follows from the proof of existence of $QISO(M,d)$ in [17] that the action of $QISO(M,d)$ on $C(M)$ is affine w.r.t. the coordinate functions coming from any embedding $M \subseteq \mathbb{R}^N$ satisfying the conditions of Corollary 4.9 of [17]. But this means the action is smooth in our sense, hence by Theorem 3.10 we complete the proof. □

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