Abstract. In this paper, we prove variational principles between metric mean dimension and rate distortion function for countable discrete amenable group actions which extend recently results by Lindenstrauss and Tsukamoto.

1. Introduction

Entropy is the most successful invariant in dynamical systems which measures the complexity or uncertainty of systems and it has close relationship with information theory, dimension theory, fractal geometry and many other aspects in mathematics.

Due to the value of the entropy, dynamical systems can be divided into three classes: 1. systems with zero entropy; 2. systems with finite and positive entropy; 3. systems with infinite entropy. For zero entropy case, to give the quantitative measure of randomness or disorder, various of entropy type invariants were introduced: sequence entropy (Kushnirenko [21] and Goodman [10]), scaled entropy (Vershik [34, 35, 36]), entropy dimension (Carvalho [2], Ferenczi-Park [9] and Dou-Huang-Park [6, 7]) and so on. The studies on these invariants rely on the detailed analysis to the entropy-related quantities or functions. For infinite entropy case, the Gromov-Lindenstrauss-Weiss mean dimension is proved to be a meaningful quantity. The concept of mean dimension was first introduced by Gromov [11] in 1999 and then Lindenstrauss and Weiss [29] defined a metric version which is called metric mean dimension. These definitions of mean dimension can be viewed as analogies of the concepts of dimension in dynamical systems. Mean dimension can be applied to solve imbedding problems in dynamical systems (see for example, [12, 13, 15, 25, 27]) and also supplies interesting quantities when characterizing large dynamics ([31, 32, 33]). In fact, from the definition, one may see easily that metric mean dimension is also an entropy-related quantity.

In the study of dynamical system and ergodic theory, people are always interested with the relationships between topological concepts and measure-theoretic ones. For entropy, there exists a variational principle which says that topological entropy is the supreme of measure-theoretic entropy over all invariant Borel probability measures. A natural question follows is that whether there exist variational principles for other entropy-related invariants?

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For zero entropy case, it is shown that the traditional variational principle does not hold for both sequence entropy ([10]) and entropy dimension ([1]).

For infinite entropy case, people have been seeking variational principle of mean dimension for almost twenty years since Gromov-Lindenstrauss-Weiss’s mean dimension theory was established. In [17], Kawabata and Dembo applied the rate-distortion function in information theory to investigate the dimension of fractal sets and established connections between dimensions and rate-distortion functions. Motivated by their work, recently Lindemstrauss and Tsukamoto [28] proved variational principles for metric mean dimensions. In the following let us give a brief review of their results.

Let \((X, d, T)\) be a TDS, where \(X\) is a compact metric space with metric \(d\) and \(T\) a continuous onto map from \(X\) to itself. Denote by \(M(X, T)\) the collection of \(T\)-invariant Borel probability measure on \(X\). Let \(\overline{\text{mdim}}_M(X, d)\) and \(\underline{\text{mdim}}_M(X, d)\) be the upper and the lower metric mean dimension of TDS \((X, d, T)\) respectively. Let \(R_\mu(\cdot), R_{\mu,p}(\cdot)\) and \(R_{\mu,\infty}(\cdot)\) be the \(L^1\), \(L^p\) \((p > 1)\) and \(L^\infty\) rate-distortion function of \((X, d, T)\) with respect to \(\mu \in M(X, T)\) respectively. For the definitions one may refer to [28] and we will also give the detailed definitions for amenable group actions in section 3.

Recall that the compact metric space \((X, d)\) is said to have tame growth of covering numbers if for every \(\delta > 0\) it holds that

\[
\lim_{\varepsilon \to 0} \varepsilon^\delta \log \#(X, d, \varepsilon) = 0.
\]

Lindemstrauss and Tsukamoto’s variational principles are the following:

**Theorem 1.1** (\(L^1\) and \(L^p\) \((p > 1)\) variational principles, Theorem 1.5 and Corollary 1.10 of [28]). Let \((X, d, T)\) be a TDS and \((X, d)\) has tame growth of covering numbers, then

\[
\overline{\text{mdim}}_M(X, d) = \limsup_{\varepsilon \to 0} \sup_{\mu \in M(X, T)} \frac{R_\mu(\varepsilon)}{|\log \varepsilon|} = \limsup_{\varepsilon \to 0} \sup_{\mu \in M(X, T)} \frac{R_{\mu,p}(\varepsilon)}{|\log \varepsilon|},
\]

\[
\underline{\text{mdim}}_M(X, d) = \liminf_{\varepsilon \to 0} \sup_{\mu \in M(X, T)} \frac{R_\mu(\varepsilon)}{|\log \varepsilon|} = \liminf_{\varepsilon \to 0} \sup_{\mu \in M(X, T)} \frac{R_{\mu,p}(\varepsilon)}{|\log \varepsilon|}.
\]

**Theorem 1.2** (\(L^\infty\) variational principles, Theorem 1.9 of [28]). Let \((X, d, T)\) be a TDS, then

\[
\overline{\text{mdim}}_M(X, d) = \limsup_{\varepsilon \to 0} \sup_{\mu \in M(X, T)} \frac{R_{\mu,\infty}(\varepsilon)}{|\log \varepsilon|},
\]

\[
\underline{\text{mdim}}_M(X, d) = \liminf_{\varepsilon \to 0} \sup_{\mu \in M(X, T)} \frac{R_{\mu,\infty}(\varepsilon)}{|\log \varepsilon|}.
\]

Since many classic results include SMB theorem and variational principle for entropy have been generalized to actions by more larger class of groups beyond \(\mathbb{Z}\) or \(\mathbb{Z}^d\), it is natural to ask whether the above variational principles still hold for such groups. In this paper we will work in the frame of countable discrete amenable group actions and establish the corresponding variational principles for amenable metric mean dimension. For the proofs we will follow Lindenstrauss and Tsukamoto’s steps. But there are
additional difficulties for amenable group actions: when we construct the related invariant measures, we need some further tiling or quasi-tiling result for amenable groups (Lemma 2.4) to produce some specific Følner sequence (Lemma 2.6). To avoid complicated technical details, we employ the recent finite tiling result on amenable groups (Downarowicz et. [8]).

We would like to mention here that after Gromov-Lindenstrauss-Weiss’s foundation works on mean dimension, there are sequences of articles on the theme for amenable mean dimensions. See, for example [3, 4, 5, 19, 20, 23]. There are also works for sofic group actions beyond amenable group actions [16, 22, 24]. We are also interested that whether there exist variational principles for sofic mean dimensions.

The paper is organized as follows. In section 2, we will briefly recall the preliminaries for countable discrete amenable group including its tiling or quasi-tiling theory. And then prove our Lemma 2.4 and 2.6. In section 3, we will introduce concepts and some properties for amenable metric mean dimensions, mutual information and amenable \((L^1)\) rate-distortion function. Especially we will show the definition of rate-distortion function is independent of the choice of the Følner sequences as well. Then in section 4 we will prove our \((L^1)\) variational principles for amenable metric mean dimensions (Theorem 4.1). In section 5, we will consider \(L^\infty\) and \(L^p\) \((p > 1)\) rate distortion functions and formulate the corresponding \(L^\infty\) and \(L^p\) \((p > 1)\) variational principles. Since the proof is parallel to the \(L^1\) variational principles, we leave it to Appendix A.

2. Amenable groups and preliminary tiling lemmas

Recall that a group \(G\) is said to be amenable if there always exists an invariant Borel probability measure when it acts to any compact metric space. In the case \(G\) is a countable discrete group, amenability is equivalent to the existence of a Følner sequence: a sequence of finite subsets \(\{F_n\}\) of \(G\) such that

\[
\lim_{n \to +\infty} \frac{|F_n \triangle gF_n|}{|F_n|} = 0, \text{ for all } g \in G.
\]

From now on, we always assume the group \(G\) to be a countable discrete amenable group.

Denote by \(F(G)\) the collection of nonempty finite subsets of \(G\). Let \(A, K \in F(G)\) and \(\delta > 0\). The set \(A\) is said to be \((K, \delta)\)-invariant if

\[
\frac{|KA \triangle A|}{|A|} < \delta.
\]

Another equivalent condition for the sequence of finite subsets \(\{F_n\}\) of \(G\) to be a Følner sequence is that \(\{F_n\}\) becomes more and more invariant, i.e. for any \(\delta > 0\) and any finite subset \(K\) of \(G\), \(F_n\) is \((K, \delta)\)-invariant for sufficiently large \(n\). One may refer to Ornstein and Weiss [30] for more details on amenable groups, or Kerr and Li [18] for reference.

When considering amenable group actions in ergodic theory and dynamical systems, some kinds of “tiling properties” are strongly involved in most of situations. Not as good as the groups \(\mathbb{Z}\) or \(\mathbb{Z}^d\), in general it is still not known whether there always exist
tiling Følner sets for all general amenable groups. Ornstein and Weiss developed their quasi-tiling theory allowing some errors for the needed tiling properties and then many results can be extended to all general amenable groups. Recently, Downarowicz etc [8] proved a finite tiling result for general amenable groups. With the help of their result, some of the proofs obtained from the quasi-tiling techniques can be simplified.

In the next let us recall the finite tiling result of Downarowicz etc [8].

We call \( T \subset F(G) \) a tiling if \( T \) forms a partition of \( G \). An element in a tiling \( T \) is called a \( T \)-tile or tile. A tiling \( T \) is said to be finite if there exists a finite collection \( S = S(T) = \{S_1, S_2, \ldots, S_k\} \) of \( F(G) \), which is called the shapes of \( T \), such that each element in \( T \) is a translation of some set in \( S \). For convenience, we always assume that the shapes \( S \) has minimal cardinality, i.e. any set in \( S \) cannot be a translation of others. Moreover, through some suitable translation, we can assume each set in \( S \) contain \( e_G \).

Let \( S \) be a shape of a finite tiling \( T \), the center of shape \( S \) is the set \( C(S) = \{c \in G : Sc \in T\} \). For convenience, we need \( C(S) \) to be nonempty for each shape \( S \). We also require the centers \( C(S) \)'s satisfy that \( Sc \)'s are disjoint for \( c \in C(S) \) and \( S \in S \).

For a tiling \( T \) with shapes \( S \), we can define a subshift \( X_T \) of \( (S \cup \{0\})^G \) by

\[
X_T = \bigcup_{g \in G} \{gx\},
\]

where \( x = (x_g)_{g \in G} \) is defined by

\[
x_g = \begin{cases} 
S, & \text{if } g \in C(S), \\
0, & \text{otherwise},
\end{cases}
\]
i.e., \( x \) is a transitive point of the subshift \( X_T \). We recall here that the shift action is defined by \( (hx)_g = x_{gh} \) for \( g, h \in G \).

Let \( T \) be a finite tiling of a countable discrete amenable group \( G \). Denote by \( h(T) = h_{\text{top}}(X_T, G) \), the topological entropy of the associated subshift \( (X_T, G) \). The following is Theorem 5.2 of [8] by Downarowicz etc. Recall that a sequence of tiles \( (T_k)_{k \geq 1} \) is said to be congruent if for each \( k \geq 1 \), every tile of \( T_{k+1} \) equals a union of tiles of \( T_k \).

**Theorem 2.1.** Let \( G \) be a countable discrete amenable group. Fix a converging to zero sequence \( \varepsilon_k > 0 \) and a sequence \( K_k \) of finite subsets of \( G \). There exists a congruent sequence of finite tilings \( T_k \) of \( G \) such that the shapes of \( T_k \) are \((K_k, \varepsilon_k)\)-invariant and \( h(T_k) = 0 \) for each \( k \).

In the present paper, we just need to use the following extract which is taken from Theorem 4.3 of Downarowicz etc [8], a weaker version of the above theorem.

**Theorem 2.2.** For any \( \varepsilon > 0 \) and \( K \in F(G) \). There exists a finite tiling \( T \) of \( G \), such that every shape of \( T \) is \((K, \varepsilon)\)-invariant.
Recall that a Følner sequence \( \{F_n\} \) in \( G \) is said to be tempered if there exists a constant \( C \) which is independent of \( n \) such that
\[
| \bigcup_{k<n} F_k^{-1} F_n | \leq C |F_n|, \quad \text{for any } n.
\]
(2.1)

Note that every Følner sequence \( F_n \) has a tempered subsequence and in particular, every amenable group has a tempered Følner sequence (see Proposition 1.4 of Lindenstrauss [26]).

The following is the pointwise ergodic theorem for amenable group actions (Theorem 1.2 of Lindenstrauss [26], see also Weiss [37]).

**Theorem 2.3** (Pointwise Ergodic Theorem). Let \( (X, G, \mu) \) be an ergodic \( G \)-system, \( \{F_n\} \) be a tempered Følner sequence in \( G \) and \( f \in L^1(X, B, \mu) \). Then
\[
\lim_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) = \int_X f(x) \, d\mu,
\]
almost everywhere and in \( L^1 \).

Let \( T \) be a tiling of \( G \) with shapes \( S = \{T_1, \ldots, T_l\} \). For \( F \in F(G) \), \( 1 \leq j \leq l \), denote by
\[
\rho_T(T_j, F) = \frac{1}{|F|} \# \{ c \in G : T_j c \subset T_j c \in T \text{ and } T_j c \subset F \}|T_j|,
\]
the density or the portion of tiles of \( T \) with shape \( T_j \) that completely contained in \( F \). It is easy to note that \( \sum_{j=1}^l \rho_T(T_j, F) \leq 1 \).

**Lemma 2.4.** Let \( \{F_n\} \) be any tempered Følner sequence in \( G \). For any \( K \in F(G) \) and \( 0 < \varepsilon < \frac{1}{2} \), there exists a finite tiling \( T = T(K, \varepsilon) \) of \( G \) such that
\begin{enumerate}
  \item \( T \) has shapes \( T_1, T_2, \ldots, T_l \) and each shape is \( (K, \varepsilon) \)-invariant;
  \item for sufficiently \( n \in \mathbb{N} \), for \( 1 \leq j \leq l \), there exists \( \tilde{F}_n \subset F_n \) with \( |	ilde{F}_n| > (1-\varepsilon)|F_n| \)
\end{enumerate}

such that
\[
\left| \frac{1}{|F_n|} \sum_{g \in F_n} 1_{C_j g^{-1} \cap F_n}(h) - \frac{\rho_T(T_j, F_n)}{|T_j|} \right| < \varepsilon \frac{\rho_T(T_j, F_n)}{|T_j|},
\]
for all \( h \in \tilde{F}_n \), where \( C_j = S(T_j) \) is the center of the shape \( T_j \).

**Proof.** Let \( T' \) be a finite tiling with shapes \( T_1, T_2, \ldots, T_l \) and each shape \( T_j \) is \( (K, \varepsilon) \)-invariant due to Theorem 2.2. Let \( (X_{T'}, G) \) be the associated subshift. Let \( \mu \) be a \( G \)-invariant ergodic measure of \( (X_{T'}, G) \). For each \( j = 1, 2, \ldots, l \), set
\[
f_j(x) = \begin{cases} 
 1, & \text{if } x_{cG} = T_j, \\
 0, & \text{otherwise},
\end{cases} \quad \text{for } x = (x_g)_{g \in G} \in X_{T'}.
\]

By the pointwise ergodic theorem, for \( \mu \)-a.e. \( x \in X_{T'} \),
\[
\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} f_j(gx) = \int f_j(x) \, d\mu.
\]
Denote the above limit by $t_j$. Note that when $t_j \neq 0$, for sufficiently invariant $F \in F(G)$, $\frac{\rho_T(T_j, F_n)}{|T_j|} \neq 0$. Then for sufficiently large $N_0 \in \mathbb{N}$,

$$\mu\left(\{x \in X_{\mathcal{T}'} : \text{for any } n > N_0, \frac{1}{|F_n|} \sum_{g \in F_n} f_j(gx) - t_j < \frac{\varepsilon \rho_T(T_j, F_n)}{3 |T_j|}\}\right) > 1 - \varepsilon.$$ 

Denote by $X_0$ the set in the left-hand side of the above inequality. Applying the pointwise ergodic theorem again, there exists an $N_1 \in \mathbb{N}$ which is greater than $N_0$, such that for any $n > N_1$, it holds for $\mu$-a.e. $x \in X_{\mathcal{T}'}$ that

$$\mu\left(\{x \in X_{\mathcal{T}'} : \text{for any } n > N_1, \frac{1}{|F_n|} \sum_{g \in F_n} 1_{X_0}(gx) > 1 - \varepsilon\}\right) > 1 - \varepsilon.$$ 

Denote by $X_1$ the set in the left-hand side of the above inequality. Now we choose an $x$ from the intersection of the sets $X_0$ and $X_1$ and then let $\mathcal{T}$ be the finite tiling generated by $x$. We assume that $\mathcal{T}$ still has shapes $T_1, T_2, \ldots, T_l$ (if $t_j = 0$ for some $j$, we can delete the corresponding $T_j$ from the shape set).

Since the tiling $\mathcal{T}$ is generated by $x$, there exists $N_2 > N_1 \in \mathbb{N}$ such that whenever $n > N_2$, it holds that

$$\left|\frac{1}{|F_n|} \sum_{g \in F_n} f_j(gx) - \frac{\rho_T(T_j, F_n)}{|T_j|}\right| < \frac{\varepsilon \rho_T(T_j, F_n)}{3 |T_j|}.$$ 

Since $x \in X_0 \cap X_1$, when $n$ is larger than $N_1$, we have that

$$\left|\frac{1}{|F_n|} \sum_{g \in F_n} f_j(gx) - t_j\right| < \frac{\varepsilon \rho_T(T_j, F_n)}{3 |T_j|}$$

and

$$\frac{1}{|F_n|} \sum_{g \in F_n} 1_{X_0}(gx) > 1 - \varepsilon.$$ 

Joint [2.2] and [2.3] together, for every $n > N_2$, it holds that

$$\left|\frac{\rho_T(T_j, F_n)}{|T_j|} - t_j\right| < \frac{2\varepsilon \rho_T(T_j, F_n)}{3 |T_j|}.$$ 

Let $\tilde{F}_n = \{h \in F_n : hx \in X_0\}$. Then by [2.4], $|\tilde{F}_n| > (1 - \varepsilon)|F_n|$.

For each $h \in \tilde{F}_n$, since $hx \in X_0$ and $n > N$, it holds that

$$\left|\frac{1}{|F_n|} \sum_{g \in F_n} f_j(ghx) - t_j\right| < \frac{\varepsilon \rho_T(T_j, F_n)}{3 |T_j|},$$

i.e.

$$\left|\frac{1}{|F_n|} \#\{g \in F_n : (hx)_g = x_{gh} = T_j\} - t_j\right| < \frac{\varepsilon \rho_T(T_j, F_n)}{3 |T_j|}.$$
Note that $x_{gh} = T_j$ if and only if $gh \in C_j$, i.e. $h \in C_j g^{-1}$. Hence
\[
\left| \frac{1}{|F_n|} \sum_{g \in F_n} 1_{C_j g^{-1} \cap F_n}(h) - t_j \right| < \varepsilon \frac{\rho_T(T_j, F_n)}{|T_j|},
\]
for all $h \in \tilde{F}_n$.

Then whenever $n > N_2$, we have
\[
\left| \frac{1}{|F_n|} \sum_{g \in F_n} 1_{C_j g^{-1} \cap F_n}(h) - \rho_T(T_j, F_n) \right| < \varepsilon \frac{\rho_T(T_j, F_n)}{|T_j|},
\]
for all $h \in \tilde{F}_n$.

\[\blacksquare\]

**Remark 2.5.** From the proof of Lemma 2.4, we can see that the shapes $T_1, T_2, \ldots, T_l$ do not depend on the given Følner sequence $\{F_n\}$, although the tiling $\mathcal{T}$ itself does depend on $\{F_n\}$.

With the help of Lemma 2.4, we can construct a specific Følner sequence of $G$, which plays a crucial role for proving the variational principles.

**Lemma 2.6.** Let $\{H_n\}$ be any tempered Følner sequence of $G$. There exists a Følner sequence of $G$ (independent of $\{H_n\}$), denoted by $\{F_n\}$, such that for any $K \in F(G)$ that contains $e_G$ and $0 < \varepsilon < \frac{1}{2}$, there is a finite tiling $\mathcal{T}$ of $G$ satisfying the following:

1. $\mathcal{T}$ has shapes $\{F_{m_1}, \ldots, F_{m_l}\}$ consisted with Følner sets in $\{F_n\}$ and each Følner set is $(K, \varepsilon)$-invariant;
2. let $C_j$ be the center of the shape $F_{m_j}$ for each $1 \leq j \leq l$, then the family of sets $\{C_j g^{-1}\}_{g \in H_n}$ covers a subset $\tilde{H}_n \subseteq H_n$ with $|\tilde{H}_n| > (1 - \varepsilon)|H_n|$ at most $(1 + \varepsilon)\rho_T(F_{m_j}, H_n)|F_{m_j}|^{-1}|H_n|$ many times, whenever $n$ is sufficiently invariant.

**Proof.** Let $(\varepsilon_n)$ be a sequence of real numbers decreasing to 0 and let $\{K_n\}$ be a sequence of finite subsets of $G$ such that

1. $\{e_G\} \subset K_1 \subset K_2 \subset \cdots$ and $\lim_{n \to \infty} K_n = G$;
2. $K_n$ becomes more and more invariant as $n \to \infty$ (in fact $\{K_n\}$ is also a Følner sequence).

Then we collect the shapes of tiling $\mathcal{T}'(K_n, \varepsilon_n)$ associated with each pair $(K_n, \varepsilon_n)$ due to Theorem 2.2 to form a sequence of finite subsets of $G$ and denote this sequence by $\{F_n\}$. Since the shapes become more and more invariant as $n \to \infty$, $\{F_n\}$ is a Følner sequence of $G$.

Then for any $K \in F(G)$ and $\epsilon > 0$, let $K_n \supset K$ and $\varepsilon_n < \varepsilon$. We then take the finite tiling $\mathcal{T}'$ to be $\mathcal{T}' = \mathcal{T}(K_n, \varepsilon_n)$ as in Lemma 2.4. Then every shape of $\mathcal{T}'$ is taken from the Følner sequence $\{F_n\}$ and $(K_n, \varepsilon_n)$-invariant (hence $(K, \varepsilon)$-invariant since $e_G \in K$).

Moreover, by Lemma 2.4 we can use the tiling $\mathcal{T}'$ to form the required tiling $\mathcal{T}$. Then for any sufficiently large $n \in \mathbb{N}$, for $1 \leq j \leq l$, there exists $\tilde{H}_n \subset H_n$ with
\[ |\tilde{H}_n| > (1 - \varepsilon)|H_n| \text{ such that} \]
\[
\left| \frac{1}{|H_n|} \sum_{g \in H_n} 1_{C_j g^{-1} \cap H_n}(h) - \frac{\rho_T(F_{m_j}, H_n)}{|F_{m_j}|} \right| < \varepsilon \frac{\rho_T(F_{m_j}, H_n)}{|F_{m_j}|}, \text{ for any } h \in \tilde{H}_n. \]

Hence for any \( h \in \tilde{H}_n \),
\[
\frac{1}{|H_n|} \sum_{g \in H_n} 1_{C_j g^{-1} \cap H_n}(h) < (1 + \varepsilon) \frac{\rho_T(F_{m_j}, H_n)}{|F_{m_j}|}.
\]

This shows that the set \( \tilde{H}_n \) is covered by the family of sets \( \{C_j g^{-1}\}_{g \in H_n} \) at most \((1 + \varepsilon)\rho_T(F_{m_j}, H_n)|F_{m_j}|\)-many times. \( \square \)

**Remark 2.7.** Since the construction of the Følner sequence \( \{F_n\} \) is independent on the given tempered Følner sequence \( \{H_n\} \), we can make \( \{H_n\} \) to be a tempered subsequence of \( \{F_n\} \). It would be more convenient if we can choose \( \{H_n\} \) just to be the whole \( \{F_n\} \), but we don’t know whether we can make the whole Følner sequence \( \{F_n\} \) tempered.

### 3. Mean dimension, mutual information and rate distortion function

#### 3.1. Topological mean dimension and metric mean dimension

Let \( \mathcal{X} \) be a compact metrizable space and \( \alpha = \{U_1, U_2, \ldots, U_k\} \) be a finite open cover of \( \mathcal{X} \). The **order** of \( \alpha \) is defined by
\[
\text{ord}(\alpha) = \max_{x \in \mathcal{X}} \sum_{i=1}^k 1_{U_i}(x) - 1.
\]
Denote by
\[
D(\alpha) = \min_\beta \text{ord}(\beta),
\]
where \( \beta \) is taken over all finite open covers of \( \mathcal{X} \) with \( \beta \succ \alpha \).

The **topological dimension** of \( \mathcal{X} \) is then defined by
\[
\dim \mathcal{X} = \sup_\alpha D(\alpha),
\]
where \( \alpha \) runs over all finite open covers of \( \mathcal{X} \).

Let \( (\mathcal{X}, G) \) be a \( G \)-system, where \( G \) is a countable discrete amenable group. For \( F \in F(G) \) and a finite open cover \( \alpha \) of \( \mathcal{X} \), denote by \( \alpha_F = \bigvee_{g \in F} g^{-1} \alpha \). Then we can define
\[
D(\alpha, G) = \lim_{n \to \infty} \frac{D(\alpha_{F_n})}{|F_n|},
\]
where \( \{F_n\} \) is a Følner sequence of \( G \). It is shown that this limit exists and is independent on the choice of the Følner sequence. The **mean topological dimension** \( \text{mdim}(\mathcal{X}, G) \) of \( (\mathcal{X}, G) \) is defined by
\[
\text{mdim}(\mathcal{X}, G) = \sup_\alpha D(\alpha, G),
\]
where \( \alpha \) runs over all finite open covers of \( \mathcal{X} \).
Let \((\mathcal{X}, G)\) be a \(G\)-system with metric \(d\). For \(F \in \mathcal{F}(G)\), define metrics \(d_F\) and \(\bar{d}_F\) on \(\mathcal{X}\) by
\[
d_F(x, y) = \max_{g \in F} d(gx, gy)
\]
and
\[
\bar{d}_F(x, y) = \frac{1}{|F|} \sum_{g \in F} d(gx, gy), \quad x, y \in \mathcal{X}.
\]
We note here that we also use \(\bar{d}_F\) to denote the metric on \(\mathcal{X}^F\) defined by
\[
\bar{d}_F((x_g)_{g \in F}, (y_g)_{g \in F}) = \frac{1}{|F|} \sum_{g \in F} d(x_g, y_g),
\]
for \((x_g)_{g \in F}, (y_g)_{g \in F} \in \mathcal{X}^F\).

For any \(\varepsilon > 0\), let \(#(\mathcal{X}, d, \varepsilon)\) be the minimal cardinality of open cover \(U\) of \(\mathcal{X}\) with \(\text{diam}(U, d) < \varepsilon\). Then we define
\[
S(\mathcal{X}, G, d, \varepsilon) = \lim_{n \to \infty} \frac{1}{|F_n|} \log \#(\mathcal{X}, d_{F_n}, \varepsilon).
\]
This limit always exists and does not depend on the choice of the Følner sequence \(\{F_n\}\). Note that \(h_{\text{top}}(\mathcal{X}, G)\), the topological entropy of the system \((\mathcal{X}, G)\), equals \(\lim_{\varepsilon \to 0} S(\mathcal{X}, G, d, \varepsilon)\) for any metric \(d\) which is compatible with the topology of \(\mathcal{X}\).

The upper and lower metric mean dimension is then defined by
\[
\overline{\text{mdim}}_M(\mathcal{X}, G, d) = \limsup_{\varepsilon \to 0} \frac{S(\mathcal{X}, G, d, \varepsilon)}{|\log \varepsilon|},
\]
\[
\underline{\text{mdim}}_M(\mathcal{X}, G, d) = \liminf_{\varepsilon \to 0} \frac{S(\mathcal{X}, G, d, \varepsilon)}{|\log \varepsilon|}.
\]
When the limits agree, the common value is denoted by \(\text{mdim}_M(\mathcal{X}, G, d)\).

Replacing \(d_F\) by \(\bar{d}_F\) in the definition of \(S(\mathcal{X}, G, d, \varepsilon)\), define
\[
\tilde{S}(\mathcal{X}, G, d, \varepsilon) = \lim_{n \to \infty} \frac{1}{|F_n|} \log \#(\mathcal{X}, \bar{d}_{F_n}, \varepsilon).
\]
This limit also exists and does not depend on the choice of the Følner sequence \(\{F_n\}\).

It is easy to see that
\[
\tilde{S}(\mathcal{X}, G, d, \varepsilon) \leq S(\mathcal{X}, G, d, \varepsilon).
\]
Recall that the compact metric space \((\mathcal{X}, d)\) is said to have tame growth of covering numbers if for every \(\delta > 0\) it holds that
\[
\lim_{\varepsilon \to 0} \varepsilon^\delta \log \#(\mathcal{X}, d, \varepsilon) = 0.
\]

**Proposition 3.1.** If \((\mathcal{X}, d)\) has tame growth of covering numbers, then
\[
\overline{\text{mdim}}_M(\mathcal{X}, G, d) = \limsup_{\varepsilon \to 0} \frac{\tilde{S}(\mathcal{X}, G, d, \varepsilon)}{|\log \varepsilon|},
\]
\[
\underline{\text{mdim}}_M(\mathcal{X}, G, d) = \liminf_{\varepsilon \to 0} \frac{\tilde{S}(\mathcal{X}, G, d, \varepsilon)}{|\log \varepsilon|}.
\]
**Proof.** We first prove the case of \( \overline{\dim}_M(\mathcal{X}, G, d) \). Since \( \tilde{S}(\mathcal{X}, G, d, \varepsilon) \leq S(\mathcal{X}, G, d, \varepsilon) \), it obviously holds that

\[
\overline{\dim}_M(\mathcal{X}, G, d) = \limsup_{\varepsilon \to 0} \frac{S(\mathcal{X}, G, d, \varepsilon)}{|\log \varepsilon|} \geq \limsup_{\varepsilon \to 0} \frac{\tilde{S}(\mathcal{X}, G, d, \varepsilon)}{|\log \varepsilon|}.
\]

Let \( M = \#(\mathcal{X}, d, \varepsilon) \) and choose an open cover \( \mathcal{W} = \{ W_1, \ldots, W_M \} \) of \( \mathcal{X} \) with \( \text{diam}(W_m, d) < \varepsilon \) for every \( 1 \leq m \leq M \). Respectively, for \( F \in F(G) \), let \( N = \#(\mathcal{X}, d_F, \varepsilon) \) and choose an open cover \( \mathcal{U} = \{ U_1, \ldots, U_N \} \) of \( \mathcal{X} \) with \( \text{diam}(U_i, d_F) < \varepsilon \) for every \( 1 \leq i \leq N \).

Now for each \( 1 \leq i \leq N \) choose a point \( p_i \in U_i \). Then \( d_F(x, p_i) < \varepsilon \) for every \( x \in U_i \). Hence for \( L \geq 1 \),

\[
|\{ g \in F : d(gx, gp_i) \geq L \varepsilon \}| < \frac{|F|}{L},
\]

which follows that

\[
U_i \subset \bigcup_{A \subset F \text{ with } |A| < \frac{|F|}{L}} B_L(p_i, d_{F \setminus A}).
\]

For \( A \subset F \), since \( \bigvee_{g \in A} g^{-1} \mathcal{W} \) is a cover of \( X \), it holds that

\[
B_L(p_i, d_{F \setminus A}) = \bigcup_{g \in A, 1 \leq m \leq M} (g^{-1}W_m \cap B_L(p_i, d_{F \setminus A})).
\]

Noticing that

\[
\text{diam}(g^{-1}W_m \cap B_L(p_i, d_{F \setminus A}), d_F) < 2L \varepsilon;
\]

we have

\[
\#(B_L(p_i, d_{F \setminus A}), d_F, 2L \varepsilon) \leq M^{|A|} \leq M^{|F|}.
\]

Since there are \( N \) choices of \( U_i \) and totally \( 2^{|F|} \) choices of \( A \subset F \), it holds that

\[
\#(\mathcal{X}, d_F, 2L \varepsilon) \leq 2^{|F|} M^{|F|} N.
\]

Thus

\[
\frac{1}{|F|} \log \#(\mathcal{X}, d_F, 2L \varepsilon) \leq \log 2 + \frac{1}{L} \log \#(\mathcal{X}, d, \varepsilon) + \frac{1}{|F|} \log \#(\mathcal{X}, d_F, \varepsilon).
\]

Now take \( 0 < \delta < 1 \) and let \( L = (1/\varepsilon)\delta \) in the above inequality. We have

\[
\frac{1}{|F|} \log \#(\mathcal{X}, d_F, 2\varepsilon^{1-\delta}) \leq \log 2 + \varepsilon^\delta \log \#(\mathcal{X}, d, \varepsilon) + \frac{1}{|F|} \log \#(\mathcal{X}, d_F, \varepsilon).
\]

Letting \( F = F_n \) with \( n \to \infty \) in any Følner sequence \( \{ F_n \} \),

\[
S(\mathcal{X}, G, d, 2\varepsilon^{1-\delta}) \leq \log 2 + \varepsilon^\delta \log \#(\mathcal{X}, d, \varepsilon) + \tilde{S}(\mathcal{X}, G, d, \varepsilon).
\]

Applying the condition of the tame growth of covering numbers and then letting \( \delta \to 0 \), it follows that

\[
\overline{\dim}_M(\mathcal{X}, G, d) \leq \limsup_{\varepsilon \to 0} \frac{\tilde{S}(\mathcal{X}, G, d, \varepsilon)}{|\log \varepsilon|}.
\]

The case of \( \underline{\dim}_M(\mathcal{X}, G, d) \) is similar. \( \square \)
3.2. Mutual information. Mutual information is an important concept in information theory via entropy.

In the following we will introduce its definition and collect some of its basic properties from [28]. Now let \((\Omega, \mathbb{P})\) be a probability space. Let \(\mathcal{X}, \mathcal{Y}\) be two measurable spaces and let \(X : \Omega \mapsto \mathcal{X}\) and \(Y : \Omega \mapsto \mathcal{Y}\) be two measurable maps. \(I(X;Y)\), the mutual information of \(X\) and \(Y\) is defined by the following:

\[
I(X;Y) := \sup_{\mathcal{P}, \mathcal{Q}} \sum_{P \in \mathcal{P}, Q \in \mathcal{Q}} \mathbb{P}((X,Y) \in P \times Q) \log \frac{\mathbb{P}(X,Y) \in P \times Q}{\mathbb{P}(X \in P) \mathbb{P}(Y \in Q)},
\]

where \(\mathcal{P}\) and \(\mathcal{Q}\) run over all finite measurable partitions of \(\mathcal{X}\) and \(\mathcal{Y}\) respectively and with the convention we set that \(0 \log \frac{0}{a} = 0\) for all \(a \geq 0\).

It is easy to see that \(I(X;Y) = I(Y;X) \geq 0\) for any measurable maps \(X\) and \(Y\).

The mutual information has the following properties.

**Proposition 3.2.** Let \(\mathcal{X}, \mathcal{Y}, \mathcal{Z}\) be measurable spaces, \(X, Y, Z\) be measurable maps from \(\Omega\) to \(\mathcal{X}, \mathcal{Y}, \mathcal{Z}\) respectively, and \(f : \mathcal{Y} \mapsto \mathcal{Z}\) be a measurable map.

1. (Data-processing inequality). 
   \[
   I(X; f(Y)) \leq I(X; Y).
   \]
   If in addition \(\mathcal{X}, \mathcal{Y}\) and \(\mathcal{Z}\) are finite sets, then the following (2)-(6) holds.

2. \(I(X; Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y)\).

3. Let \((X_n, Y_n) : \Omega \mapsto \mathcal{X} \times \mathcal{Y}\) be a sequence of measurable maps converging to \((X,Y)\) in law, then \(I(X_n; Y_n)\) converges to \(I(X; Y)\).

4. (Fano’s inequality). Let \(P_e = \mathbb{P}(X \neq f(Y))\), then
   \[
   H(X|Y) \leq H(P_e) + P_e \log |\mathcal{X}|.
   \]

5. (Subadditivity). If \(X\) and \(Z\) are conditionally independent given \(Y\), i.e. for every \(y \in \mathcal{Y}\) with \(\mathbb{P}(Y = y) \neq 0\) and for every \(x \in \mathcal{X}\) and \(y \in \mathcal{Y}\),
   \[
   \mathbb{P}(X = x, Z = z|Y = y) = \mathbb{P}(X = x|Y = y) \mathbb{P}(Z = z|Y = y),
   \]
   then
   \[
   I(Y; X, Z) \leq I(Y; X) + I(Y; Z).
   \]

6. (Superadditivity). If \(X\) and \(Z\) are independent, then
   \[
   I(Y; X, Z) \geq I(Y; X) + I(Y; Z).
   \]

Let \((X,d)\) be a compact metric space and \(\varepsilon > 0\). A subset \(S \subset \mathcal{X}\) is said to be \(\varepsilon\)-separated if \(d(x,y) \geq \varepsilon\) for any two distinct points \(x, y \in S\). The following lemma is Corollary 2.5 of [28], which is a corollary of Fano’s inequality.

**Lemma 3.3.** Let \((X,d)\) be a compact metric space. Let \(\varepsilon > 0\) and \(D > 2\). Suppose \(S \subset \mathcal{X}\) is a \(2D\varepsilon\)-separated set. Let \(X\) and \(Y\) be measurable maps from \(\Omega\) to \(\mathcal{X}\) such that \(X\) is uniformly distributed over \(S\) and \(\mathbb{E}(d(X,Y)) < \varepsilon\). Then

\[
I(X;Y) \geq (1 - \frac{1}{D}) \log |S| - H(\frac{1}{D}).
\]
Let $\mathcal{X}, \mathcal{Y}$ be finite and let $X$ and $Y$ be measurable maps from $\Omega$ to $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let $\mu(x) = \mathbb{P}(X = x), \nu(y|x) = \mathbb{P}(Y = y|X = x)$, then $\mu(x)\nu(y|x)$ determines the distribution of $(X, Y)$ and hence the mutual information $I(X; Y)$. So sometimes we use $I(\mu, \nu)$ to instead $I(X; Y)$. The following proposition shows the concavity and convexity of mutual information.

**Proposition 3.4.**

1. Suppose for each $x \in \mathcal{X}$ we are given a probability mass function $\nu(\cdot|x)$ on $\mathcal{Y}$. Let $\mu_1$ and $\mu_2$ be two probability mass function on $\mathcal{X}$. Then
   $$I((1-t)\mu_1 + t\mu_2, \nu) \geq (1-t)I(\mu_1, \nu) + tI(\mu_2, \nu) \quad (0 \leq t \leq 1).$$

2. Suppose for each $x \in \mathcal{X}$ we are given two probability mass functions $\nu_1(\cdot|x)$ and $\nu_2(\cdot|x)$ on $\mathcal{Y}$. Let $\mu$ be a probability mass function on $\mathcal{X}$. Then
   $$I(\mu, (1-t)\nu_1 + t\nu_2) \leq (1-t)I(\mu, \nu_1) + tI(\mu, \nu_2) \quad (0 \leq t \leq 1).$$

### 3.3. Rate distortion functions.

Now we introduce rate distortion functions for dynamical systems.

Let $(\mathcal{X}, G)$ be a $G$-system with metric $d$. Denote by $M(\mathcal{X}, G)$ the collection of $G$-invariant probability measures of $\mathcal{X}$.

Let $\varepsilon > 0$ and $\mu \in M(\mathcal{X}, G)$. For $F \in F(G)$, let $X : \Omega \mapsto \mathcal{X}$ and $Y_g : \Omega \mapsto \mathcal{X}, g \in F$ be random variables defined on some probability space $(\Omega, \mathbb{P})$. Assume the law of $X$ is given by $\mu$. We say $X$ and $Y = (Y_g)_{g \in F}$ are $(F, \varepsilon)$-close (or $(F, L^1, \varepsilon)$-close) if

$$\mathbb{E}\left(\frac{1}{|F|} \sum_{g \in F} d(gX, Y_g)\right) < \varepsilon. \quad (3.2)$$

(3.2) is also called the $(L^1)$ distortion condition.

Denote by

$$R_\mu(\varepsilon, F) = \inf_{X, Y \text{ are } (F, \varepsilon)\text{-close}} I(X; Y).$$

The $(L^1)$ rate distortion function $R_\mu(\varepsilon)$ is then defined by

$$R_\mu(\varepsilon) = \lim_{n \to \infty} \frac{R_\mu(\varepsilon, F_n)}{|F_n|},$$

where $\{F_n\}$ is a Følner sequence in $G$. Using quasi-tiling technique, it is not hard to show that the above limit does exist and is independent of the specific Følner sequence $\{F_n\}$.

In section 5 we will consider $L^\infty$ and $L^p \ (p \geq 1)$ rate distortion functions.

**Remark 3.5.** Similar to Remark 2.3 of Lindenstrauss and Tsukamoto [28], in the definition of $R_\mu(\varepsilon)$, the random variable $Y$ can be assumed to take only finitely many values.
4. L¹ VARIATIONAL PRINCIPLE

In this section, we will prove the following L¹ variational principle between metric mean dimension and the L¹ rate distortion function.

**Theorem 4.1.** If \((\mathcal{X}, d)\) has tame growth of covering numbers, then

\[
\overline{\text{mdim}}_M(\mathcal{X}, G, d) = \limsup_{\varepsilon \to 0} \sup_{\mu \in M(\mathcal{X}, G)} \frac{R_\mu(\varepsilon)}{|\log \varepsilon|},
\]

\[
\underline{\text{mdim}}_M(\mathcal{X}, G, d) = \liminf_{\varepsilon \to 0} \sup_{\mu \in M(\mathcal{X}, G)} \frac{R_\mu(\varepsilon)}{|\log \varepsilon|}.
\]

4.1. The lower bound.

**Lemma 4.2.** For \(\varepsilon > 0\) and \(\mu \in M(\mathcal{X}, G)\), we have

\[
R_\mu(\varepsilon) \leq \tilde{S}(\mathcal{X}, G, d, \varepsilon).
\]

*Proof.* Let \(\{F_n\}\) be a the Følner sequence in \(G\). For \(n > 0\), denote by \(M = \#(\mathcal{X}, \tilde{d}_{F_n}, \varepsilon)\) and let \(\{U_1, \ldots, U_M\}\) be an open cover of \(\mathcal{X}\) with \(\text{diam}(U_m, \tilde{d}_{F_n}) < \varepsilon\) for each \(1 \leq m \leq M\). Choose a point \(p_m \in U_m\) for each \(m\). For any \(x \in \mathcal{X}\), let \(m\) be the smallest number satisfying \(x \in U_m\). Then by setting \(f(x) = p_m\) we can define a map \(f : \mathcal{X} \to \{p_1, \ldots, p_M\}\) and hence \(\tilde{d}_{F_n}(x, f(x)) < \varepsilon\). Let \(X\) be a random variable obeying \(\mu\) and let \(Y = (gX)_{g \in F_n}\). Then

\[
\mathbb{E}\left(\frac{1}{|F_n|} \sum_{g \in F_n} d(gX, gf(X))\right) = \mathbb{E} \tilde{d}_{F_n}(X, f(X)) < \varepsilon.
\]

Hence

\[
I(X; Y) \leq H(Y) \leq \log M = \log \#(\mathcal{X}, \tilde{d}_{F_n}, \varepsilon).
\]

Dividing by \(|F_n|\) and letting \(n \to \infty\), we have

\[
R_\mu(\varepsilon) \leq \tilde{S}(\mathcal{X}, G, d, \varepsilon).
\]

\[\square\]

Since \(\tilde{S}(\mathcal{X}, G, d, \varepsilon) \leq S(\mathcal{X}, G, d, \varepsilon)\), we have

**Proposition 4.3.**

\[
\overline{\text{mdim}}_M(\mathcal{X}, G, d) \geq \limsup_{\varepsilon \to 0} \sup_{\mu \in M(\mathcal{X}, G)} \frac{R_\mu(\varepsilon)}{|\log \varepsilon|},
\]

\[
\underline{\text{mdim}}_M(\mathcal{X}, G, d) \geq \liminf_{\varepsilon \to 0} \sup_{\mu \in M(\mathcal{X}, G)} \frac{R_\mu(\varepsilon)}{|\log \varepsilon|}.
\]
4.2. The upper bound.

Proposition 4.4. For any $\varepsilon > 0$ and $D > 2$ there exists $\mu \in M(\mathcal{X}, G)$ such that

$$R_\mu(\varepsilon) \geq (1 - \frac{1}{D}) \tilde{S}(\mathcal{X}, G, d, (16D + 4)\varepsilon).$$

Proof. Let $\{F_n\}$ be the Følner sequence in $G$ constructed as in Lemma 2.6.

For each $F_n$ we choose $S_n$ to be a maximal $(8D + 2)\varepsilon$-separated set of $\mathcal{X}$ with respect to the metric $\tilde{d}_{F_n}$. Then

$$|S_n| \geq \#(\mathcal{X}, \tilde{d}_{F_n}, (16D + 4)\varepsilon).$$

Define

$$\nu_n = \frac{1}{|S_n|} \sum_{x \in S_n} \delta_x$$

and

$$\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} \nu_n \circ g^{-1}.$$

Let $\{F_n\}$ be a tempered subsequence of $\{F_n\}$. Then choose a convergence subsequence of $\{\mu_n\}_{i=1}^\infty$ in the weak* topology and assume it converges to $\mu$. For simplicity, we still denote this subsequence by $\{\mu_n\}_{i=1}^\infty$. Then $\mu \in M(\mathcal{X}, G)$ and we will show that it satisfies the inequality (4.1).

Let $\mathcal{P} = \{P_1, \ldots, P_M\}$ be a measurable partition of $\mathcal{X}$ with $\text{diam}(P_m, d) < \varepsilon$ and $\mu(\partial P_m) = 0$ for each $1 \leq m \leq M$.

Assign each $P_m$ a point $p_m \in P_m$ and set $A = \{p_1, \ldots, p_M\}$. Denote by $\mathcal{P}(x) = p_m$ for $x \in P_m$. Then

$$d(x, \mathcal{P}(x)) < \varepsilon, \text{ for any } x \in \mathcal{X}.$$

Let $\mathcal{P}^F(x) = (\mathcal{P}(gx))_{g \in F}$ for $F \in F(G)$. Recall that we also use $\tilde{d}_F$ to denote the metric on $\mathcal{X}^F$ for $F \in F(G)$ (see (3.1) for the definition). By (4.3), we have $\tilde{d}_{F_n}((gx)_{g \in F_n}, \mathcal{P}^{F_n}(x)) < \varepsilon$ for any $x \in \mathcal{X}$. For any two distinct points $x, y \in S_n$, we have

$$\tilde{d}_{F_n}(\mathcal{P}^{F_n}(x), \mathcal{P}^{F_n}(y)) \geq \tilde{d}_{F_n}(x, y) - \tilde{d}_{F_n}((gx)_{g \in F_n}, \mathcal{P}^{F_n}(x)) - \tilde{d}_{F_n}((gy)_{g \in F_n}, \mathcal{P}^{F_n}(y)) > (8D + 2)\varepsilon - 2\varepsilon = 8D\varepsilon.$$

Hence the set

$$\mathcal{P}^{F_n}(S_n) = \{\mathcal{P}^{F_n}(x)|x \in S_n\}$$

is a $8D\varepsilon$-separated set of $\mathcal{X}^{F_n}$ with respect to the metric $\tilde{d}_{F_n}$. Moreover, since $\nu_{F_n}$ is the uniform distribution over $S_n$, the push-forward measure $\mathcal{P}^{F_n}_{\nu_{F_n}}$ is also the uniform distribution measure over $\mathcal{P}^{F_n}(S_n)$. Note that $|\mathcal{P}^{F_n}(S_n)| = |S_n|$. 

Let $X : \Omega \mapsto \mathcal{X}$ be a random variable defined on some probability space $(\Omega, \mathcal{P})$ such that the law of $X$ is given by $\mu$. For $F \in F(G)$, let $Y_{F,g} : \Omega \mapsto \mathcal{X}$ ($g \in F$) be random
variables defined on the same probability space \((\Omega, \mathbb{P})\) such that \(Y_F = (Y_{F,g})_{g \in F}\) and \(X\) are \((F, L^1, \varepsilon)\)-close, i.e.

\[
(4.4) \quad \mathbb{E}\left( \frac{1}{|F|} \sum_{g \in F} d(gX, Y_{F,g}) \right) < \varepsilon.
\]

We can assume the distribution of \(Y_F\) is supported on a finite set \(Y_F \subset X^F\) (by (2) of Remark 3.5). By (1) of Proposition 3.2, the Data-processing inequality,

\[
I(X; Y_F) \geq I(P_F(X); Y_F).
\]

Let \(\tau_F\) be the law of \((P_F(X), Y_F)\), which is a probability measure on \(A^F \times Y_F\). It follows that

\[
\int_{A^F \times Y_F} \bar{d}_F(x,y) d\tau_F(x,y) = \mathbb{E}\left( \frac{1}{|F|} \sum_{g \in F} d(P(gX), Y_{F,g}) \right) 
\leq \mathbb{E}\left( \frac{1}{|F|} \sum_{g \in F} d(P(gX), gX) \right) + \mathbb{E}\left( \frac{1}{|F|} \sum_{g \in F} d(gX, Y_{F,g}) \right) < 2\varepsilon \quad \text{(by (4.3) and (4.4))}.
\]

For each \(n \geq 1\), we consider the couplings of \((\mathcal{P}_F^* \mu_n, \mathcal{P}_F^* \mu)\) (i.e. a probability measure on \(A^F \times A^F\) whose first and second marginals are \(\mathcal{P}_F^* \mu_n\) and \(\mathcal{P}_F^* \mu\) respectively). We choose a probability measure \(\pi_{F,n}\) that minimizes the following integral

\[
\int_{A^F \times A^F} \bar{d}_F(x,y) d\pi(x,y)
\]

among all such couplings \(\pi\). Similar to Claim 3.6 of [28], the sequence \(\pi_{F,n}\) converges to \((\mathcal{P}_F^* \times \mathcal{P}_F^*)^* \mu\) in the weak* topology.

Since both the second marginal of \(\pi_{F,n}\) and the first marginal of \(\tau_F\) are equal to the measure \(\mathcal{P}_F^* \mu\), we can compose them to produce a coupling \(\tau_{F,n}\) of \((\mathcal{P}_F^* \mu_n, \text{Law}(Y))\) by the following way:

\[
\tau_{F,n}(x,y) = \sum_{x' \in A^F} \pi_{F,n}(x,x') \mathbb{P}(Y = y|\mathcal{P}(X) = x'), \quad (x \in A^F, y \in Y_F).
\]

We note here that the sequence \(\tau_{F,n}\) converges to \(\tau_F\) in the weak* topology and hence by (4.5),

\[
(4.6) \quad \mathbb{E}_{\tau_{F,n}}(\bar{d}_F(x,y)) := \int_{A^F \times Y_F} \bar{d}_F(x,y) d\tau_{F,n}(x,y) < 2\varepsilon
\]

for all sufficiently large \(n_i\).

For \(x \in \bigcup_{g \in F_n} \mathcal{P}(gS_n)\) and \(y \in X^F\), define a conditional probability mass function \(\tau_{F,n}(y|x)\) by

\[
\tau_{F,n}(y|x) = \frac{\tau_{F,n}(x,y)}{\mathcal{P}_F^* \mu_n(x)}.
\]

Recall that our Følner sequence \(\{F_n\}\) is constructed as in Lemma 2.6. Then for any \(K \in F(G)\) with \(\varepsilon_G \in K\) and \(0 < \varepsilon_1 < \min\{\frac{1}{2}, \frac{\varepsilon}{diam(X,d)}\}\), by Lemma 2.6 (here we choose
\{H_n\} to be the tempered Følner sequence \(\{F_n\}\), there exists \(\mathcal{T}\), a finite tiling of \(G\), satisfying the following two conditions:

1. \(\mathcal{T}\) has shapes \(\{F_{m_1}, \ldots, F_{m_l}\}\) consisted with Følner sets in \(\{F_n\}\) and each \(F_{m_i}\) is \((K, \varepsilon_1)\)-invariant;
2. for sufficiently large \(i\), for each \(1 \leq j \leq l\), the family of sets \(\{C_j g^{-1}\}_{g \in F_{n_i}}\) covers a subset \(\tilde{F}_n \subset F_{n_i}\) with \(|\tilde{F}_n| > (1 - \varepsilon_1)|F_{n_i}|\) at most \((1 + \varepsilon_1)\rho_{\mathcal{T}}(F_{m_j}, F_{n_i})|F_{n_i}|/|F_{m_j}|\) many times, where \(C_j\) is the center of the shape \(F_{m_j}\).

Note that \(\mathcal{T} = \{F_{m_j} c : c \in C_j, 1 \leq j \leq l\}\) and

\[
G = \prod_{j=1}^l \prod_{c \in C_j} F_{m_j} c.
\]

For \(g \in F_{n_i}\), denote by

\[
R_g = F_{n_i} \setminus \left( \prod_{j=1}^l \prod_{c \in C_j, F_{m_j} c \subset F_{n_i}, g c^{-1} \in \tilde{F}_n} F_{m_j} c^{-1} \right).
\]

Fix a point \(a \in \mathcal{X}\). For \(x = (x_g)_{g \in F_{n_i}} \in \mathcal{P}^{F_{n_i}}(S_{n_i})\) and \(g \in F_{n_i}\), we define probability mass functions \(\sigma_{F_{n_i}, g}(\cdot|x)\) on \(\mathcal{X}^{F_{n_i}}\) as the following: for \(y = (y_g)_{g \in F_{n_i}} \in \mathcal{X}^{F_{n_i}}\),

\[
\sigma_{F_{n_i}, g}(y|x) = \prod_{j=1}^l \prod_{c \in C_j, F_{m_j} c \subset F_{n_i}, g c^{-1} \in \tilde{F}_n} \tau_{F_{m_j}, n_i}(y_{F_{m_j} c^{-1}}|x_{F_{m_j} c^{-1}}) \cdot \prod_{h \in R_g} \delta_a(y_h).
\]

Here we note that

\[
y_{F_{m_j} c^{-1}} = (y_h)_{h \in F_{m_j} c^{-1}} \in \mathcal{X}^{F_{m_j} c^{-1}}
\]

and

\[
x_{F_{m_j} c^{-1}} = (x_h)_{h \in F_{m_j} c^{-1}} \in \mathcal{P}^{F_{m_j} c^{-1}}(S_{n_i}) \subset \mathcal{X}^{F_{m_j} c^{-1}}.
\]

Then we set

\[
\sigma_{F_{n_i}}(y|x) = \frac{1}{|F_{n_i}|} \sum_{g \in F_{n_i}} \sigma_{F_{n_i}, g}(y|x).
\]

**Claim 4.5.** For sufficiently large \(n_i\), there exists some \(1 \leq j \leq l\) such that

\[
(1 - \varepsilon_1) \frac{1}{|F_{n_i}|} I(\mathcal{P}^{F_{n_i}} \nu_{n_i}, \sigma_{F_{n_i}}) \leq \frac{1}{|F_{m_j}|} I(\mathcal{P}^{F_{m_j}} \mu_{n_i}, \tau_{F_{m_j}, n_i}).
\]

**Proof of Claim 4.5.** By (2) of Proposition 3.4, the convexity of mutual information,

\[
I(\mathcal{P}^{F_{n_i}} \nu_{n_i}, \sigma_{F_{n_i}}) \leq \frac{1}{|F_{n_i}|} \sum_{g \in F_{n_i}} I(\mathcal{P}^{F_{n_i}} \nu_{n_i}, \sigma_{F_{n_i}, g}).
\]
By (5) of Proposition 3.2, the subadditivity of mutual information, together with (4.7), we have

\[(4.10) \quad I(\mathcal{P}_F^{F_{n_i}}\nu_{n_i}, \sigma_{F_{n_i}}) \leq \sum_{j=1}^{l} \sum_{c \in C_j, F_{m_j} \subset F_{n_i}, g, cg^{-1} \in \tilde{F}_{n_i}} I \left( \mathcal{P}_F^{F_{m_j}} \left( (cg^{-1}) \nu_{n_i} \right), \tau_{F_{m_j}, n_i} \right). \]

Joint (4.9) and (4.10) together,

\[I(\mathcal{P}_F^{F_{n_i}}\nu_{n_i}, \sigma_{F_{n_i}}) \leq \frac{1}{|F_{n_i}|} \sum_{j=1}^{l} \sum_{g \in F_{n_i}} I \left( \mathcal{P}_F^{F_{m_j}} \left( (cg^{-1}) \nu_{n_i} \right), \tau_{F_{m_j}, n_i} \right) = \frac{1}{|F_{n_i}|} \sum_{j=1}^{l} \sum_{g \in F_{n_i}} I \left( \mathcal{P}_F^{F_{m_j}} \left( (cg^{-1}) \nu_{n_i} \right), \tau_{F_{m_j}, n_i} \right). \]

For convenience, denote by \( t_j = \rho \tau(F_{m_j}, F_{n_i}) \), then we have

\[I(\mathcal{P}_F^{F_{n_i}}\nu_{n_i}, \sigma_{F_{n_i}}) \leq \frac{1}{|F_{n_i}|} \sum_{j=1}^{l} \sum_{h \in F_{n_i}} (1 + \varepsilon_1) t_j \frac{|F_{n_i}|}{|F_{m_j}|} I \left( \mathcal{P}_F^{F_{m_j}} (h \nu_{n_i}), \tau_{F_{m_j}, n_i} \right) \]

(by condition (C2))

\[\leq \sum_{j=1}^{l} (1 + \varepsilon_1) t_j \frac{|F_{n_i}|}{|F_{m_j}|} \frac{1}{|F_{n_i}|} \sum_{h \in F_{n_i}} I \left( \mathcal{P}_F^{F_{m_j}} (h \nu_{n_i}), \tau_{F_{m_j}, n_i} \right) \]

\[\leq \sum_{j=1}^{l} (1 + \varepsilon_1) t_j \frac{|F_{n_i}|}{|F_{m_j}|} I \left( \mathcal{P}_F^{F_{m_j}} \left( \frac{1}{|F_{n_i}|} \sum_{h \in F_{n_i}} h \nu_{n_i} \right), \tau_{F_{m_j}, n_i} \right) \]

(by (1) of Proposition 3.4, the concavity of mutual information)

\[= (1 + \varepsilon_1) |F_{n_i}| \sum_{j=1}^{l} t_j \frac{1}{|F_{m_j}|} I \left( \mathcal{P}_F^{F_{m_j}} (\mu_{n_i}), \tau_{F_{m_j}, n_i} \right), \]

i.e.

\[(1 - \varepsilon_1) \frac{1}{|F_{n_i}|} I \left( \mathcal{P}_F^{F_{n_i}}\nu_{n_i}, \sigma_{F_{n_i}} \right) \leq \sum_{j=1}^{l} t_j \frac{1}{|F_{m_j}|} I \left( \mathcal{P}_F^{F_{m_j}} (\mu_{n_i}), \tau_{F_{m_j}, n_i} \right). \]

Noticing that \( \sum_{l=1}^{l} t_j \leq 1 \), there must exists some \( 1 \leq j \leq l \) such that

\[(1 - \varepsilon_1) \frac{1}{|F_{n_i}|} I \left( \mathcal{P}_F^{F_{n_i}}\nu_{n_i}, \sigma_{F_{n_i}} \right) \leq \frac{1}{|F_{m_j}|} I \left( \mathcal{P}_F^{F_{m_j}} (\mu_{n_i}), \tau_{F_{m_j}, n_i} \right). \]

This finishes the proof of Claim 4.5 □

Denote by \( \mathbb{E}_{\mathcal{P}_F^{F_{n_i}}\nu_{n_i}, \sigma_{F_{n_i}}} \left( d_{F_{n_i}}(x, y) \right) \) the expected value of \( d_{F_{n_i}}(x, y) \) \( (x, y \in \mathcal{X}^{F_{n_i}}) \) with respect to the probability measure \( \mathcal{P}_F^{F_{n_i}}\nu_{n_i}(x)\sigma_{F_{n_i}}(y|x) \).
**Claim 4.6.** For sufficiently large $n_i$,

$$
\mathbb{E}_{P_{n_i}^{F_{n_i}}} (\bar{d}_{F_{n_i}}(x, y)) < 4\varepsilon.
$$

and

$$
I(P_{n_i}^{F_{n_i}}, \sigma_{F_{n_i}}) \geq (1 - \frac{1}{D}) \log |S_{n_i}| - H(\frac{1}{D}).
$$

**Proof of Claim 4.6.** By (4.7) and (4.8), the definitions of probability mass functions $\sigma_{F_{n_i}, g}(|x|) (g \in F_{n_i})$ and $\sigma_{F_{n_i}}(|x|)$, we have

$$
\mathbb{E}_{P_{n_i}^{F_{n_i}}} (\bar{d}_{F_{n_i}}(x, y)) = \frac{1}{|F_{n_i}|} \sum_{g \in F_{n_i}} \mathbb{E}_{P_{n_i}^{F_{n_i}}, \sigma_{F_{n_i}, g}} (\bar{d}_{F_{n_i}}(x, y))
$$

and

$$
|F_{n_i}| \mathbb{E}_{P_{n_i}^{F_{n_i}}} (\bar{d}_{F_{n_i}}(x, y))
$$

$$
\leq \sum_{j=1}^{t} \sum_{c \in C_j} \sum_{g \in F_{n_i}} |F_{m_j}| \mathbb{E}_{P_{n_i}^{F_{n_i}}} (\bar{d}_{F_{m_j}}(x', y'))
$$

$$
+ \varepsilon_1 |F_{n_i}| \text{diam}(\mathcal{X}, d),
$$

where $x, y$ are random points in $\mathcal{X}^{F_{n_i}}$ and $x', y'$ appear in $\bar{d}_{F_{m_j}}(x', y')$ are in $\mathcal{X}^{F_{m_j}}$.

When $F_{n_i}$ is sufficiently invariant, $|R_g| < \varepsilon_1 |F_{n_i}|$. Hence

$$
\mathbb{E}_{P_{n_i}^{F_{n_i}}} (\bar{d}_{F_{n_i}}(x, y))
$$

$$
\leq \frac{1}{|F_{n_i}|} \sum_{j=1}^{t} \sum_{c \in C_j} \sum_{g \in F_{n_i}} |F_{m_j}| \mathbb{E}_{P_{n_i}^{F_{n_i}}} (\bar{d}_{F_{m_j}}(x', y'))
$$

$$
+ \varepsilon_1 \text{diam}(\mathcal{X}, d)
$$

$$
\leq \frac{1}{|F_{n_i}|} \sum_{j=1}^{t} \sum_{c \in C_j} \sum_{g \in F_{n_i}} (1 + \varepsilon_1) t_j \mathbb{E}_{P_{n_i}^{F_{m_j}}} (\bar{d}_{F_{m_j}}(x', y')) + \varepsilon_1 \text{diam}(\mathcal{X}, d)
$$

(by condition (C2) and recall here $t_j = \rho_T(F_{m_j}, F_{n_i})$)

$$
\leq \sum_{j=1}^{t} (1 + \varepsilon_1) t_j \mathbb{E}_{P_{n_i}^{F_{m_j}}} (\bar{d}_{F_{m_j}}(x', y')) + \varepsilon_1 \text{diam}(\mathcal{X}, d)
$$

$$
= \sum_{j=1}^{t} (1 + \varepsilon_1) t_j \int_{\mathcal{X}^{F_{m_j}} \times \mathcal{Y}_{F_{m_j}}} \bar{d}_{F_{m_j}}(x', y') d\tau_{F_{m_j}, n_i}(x, y) + \varepsilon_1 \text{diam}(\mathcal{X}, d).\]
Recall that $0 < \varepsilon_1 < \min\{\frac{1}{2}, \frac{\varepsilon}{\text{diam}(X,d)}\}$ and $\sum_{j=1}^{l} t_j \leq 1$. By (4.6), for sufficiently large $n_i$, we have

$$
\mathbb{E}_{\mathcal{P}_{\nu_{n_i}, \sigma_{F_{n_i}}}^{F_{n_i}}} (d_{F_{n_i}}(x,y)) < (1 + \frac{1}{2})2\varepsilon + \varepsilon = 4\varepsilon.
$$

Since $\mathcal{P}_{\nu_{n_i}}^{F_{n_i}}$ is uniformly distributed over $\mathcal{P}_{\nu_{n_i}}^{F_{n_i}}(S_{n_i})$ and $\mathcal{P}_{\nu_{n_i}}^{F_{n_i}}(S_{n_i})$ is a $(8D\varepsilon)$-separated set of cardinality $|S_{n_i}|$, by Lemma 3.3 for sufficiently large $n_i$,

$$
I(\mathcal{P}_{\nu_{n_i}, \sigma_{F_{n_i}}}^{F_{n_i}}) \geq (1 - \frac{1}{D})\log |S_{n_i}| - H(\frac{1}{D}).
$$

This finishes the proof of Claim 4.6. \hfill \Box

Now we proceed with the proof of Lemma 4.4.

For any $K \in F(G)$ with $\epsilon_G \in K$ and $0 < \varepsilon_1 < \min\{\frac{1}{2}, \frac{\varepsilon}{\text{diam}(X,d)}\}$, for sufficiently large $n_i$, there exists a $1 \leq j \leq l$ (here $j$ depends on $n_i$, whereas $l$ depends on $K$ and $\varepsilon_1$ but does not depend on $n_i$),

$$
\frac{1}{|F_{m_j}|} I(\mathcal{P}_{\nu_{n_i}, \sigma_{F_{n_i}}}^{F_{m_j}}) \geq (1 - \varepsilon_1) \frac{1}{|F_{m_j}|} I(\mathcal{P}_{\nu_{n_i}, \sigma_{F_{n_i}}}^{F_{m_j}}) \text{ by Claim 4.5}
$$

$$
\geq (1 - \varepsilon_1) \left( (1 - \frac{1}{D})\log |S_{n_i}| - \frac{H(\frac{1}{D})}{|F_{n_i}|} \right) \text{ by Claim 4.6}
$$

$$
\geq (1 - \varepsilon_1) \left( (1 - \frac{1}{D})\log |S_{n_i}| - \frac{H(\frac{1}{D})}{|F_{n_i}|} \right) \text{ by (4.2)).}
$$

By choosing some subsequence of $\{n_i\}$ (we still denote it by $\{n_i\}$), for some $1 \leq j \leq l$, the probability measures $\tau_{F_{m_j}, n_i}$ converge to $\tau_{F_{m_j}, \nu_{n_i}} = \text{Law}(\mathcal{P}_{\nu_{n_i}}^{F_{m_j}}(X), Y_{F_{m_j}})$ in the weak* topology. Let $n_i \to \infty$. By (3) of Proposition 3.2

$$
\frac{1}{|F_{m_j}|} I(\mathcal{P}_{\nu_{n_i}, \sigma_{F_{n_i}}}^{F_{m_j}}(X); Y_{F_{m_j}}) \geq (1 - \varepsilon_1)(1 - \frac{1}{D})\tilde{S}(X, G, d, (16D + 4)\varepsilon).
$$

By (1) of Proposition 3.2 the data-processing inequality,

$$
\frac{1}{|F_{m_j}|} I(X; Y_{F_{m_j}}) \geq (1 - \varepsilon_1)(1 - \frac{1}{D})\tilde{S}(X, G, d, (20D + 4)\varepsilon).
$$

Let $K = K_n \in F(G)$ be chosen from some Følner sequence $\{K_n\}$ in $G$, for example, we can let $\{K_n\}$ be constructed as in the proof of Lemma 2.6 (a Følner sequence with $\{\epsilon_G\} \subset K_1 \subset K_2 \subset \cdots$ and $\lim_{n \to \infty} K_n = G$). Then let $n \to \infty$ for $K_n$ to make $K$ sufficiently invariant and $\varepsilon_1 \to 0$. Hence $m_j \to \infty$. Noticing that $R_{\mu}(\varepsilon)$ is independent of the selection of the Følner sequence $\{F_n\}$, it follows that

$$
R_{\mu}(\varepsilon) \geq (1 - \frac{1}{D})\tilde{S}(X, G, d, (16D + 4)\varepsilon).
$$

This completes the proof of Proposition 4.4. \hfill \Box
Proof of Theorem 4.1

For $D > 2$,

\[
\limsup_{\varepsilon \to 0} \sup_{\mu \in M(X, G)} \frac{R_\mu(\varepsilon)}{\log \varepsilon} \geq \limsup_{\varepsilon \to 0} \frac{(1 - \frac{1}{D})\tilde{S}(X, G, d, (16D + 4)\varepsilon)}{\log \varepsilon}
\]

(by Proposition 4.4)

\[
= (1 - \frac{1}{D}) \limsup_{\varepsilon \to 0} \frac{\tilde{S}(X, G, d, (16D + 4)\varepsilon)}{\log(16D + 4)\varepsilon}
\]

\[
= (1 - \frac{1}{D}) \text{mdim}_M(X, G, d) \quad \text{(by Proposition 3.1)}.
\]

Let $D \to \infty$, we have

\[
\text{mdim}_M(X, G, d) \leq \limsup_{\varepsilon \to 0} \sup_{\mu \in M(X, G)} \frac{R_\mu(\varepsilon)}{\log \varepsilon}.
\]

And similarly,

\[
\text{mdim}_M(X, G, d) \leq \liminf_{\varepsilon \to 0} \sup_{\mu \in M(X, G)} \frac{R_\mu(\varepsilon)}{\log \varepsilon}.
\]

Joint with Proposition 4.3, we obtain

\[
\text{mdim}_M(X, G, d) = \limsup_{\varepsilon \to 0} \sup_{\mu \in M(X, G)} \frac{R_\mu(\varepsilon)}{\log \varepsilon},
\]

\[
\text{mdim}_M(X, G, d) = \liminf_{\varepsilon \to 0} \sup_{\mu \in M(X, G)} \frac{R_\mu(\varepsilon)}{\log \varepsilon}.
\]

□

5. \(L^\infty\) and \(L^p\) \((p \geq 1)\) VARIATIONAL PRINCIPLES

Modifying the distortion condition (3.2), we can also define \(L^\infty\) and \(L^p\) \((p \geq 1)\) rate distortion functions. Similarly, we have \(L^\infty\) and \(L^p\) \((p \geq 1)\) variational principles between metric mean dimensions and the corresponding rate distortion functions.

Let \((X, G)\) be a \(G\)-system with metric \(d\). We define the \(L^\infty\) rate distortion function of \((X, G)\) in the following way.

Let \(\varepsilon > 0\) and \(\mu \in M(X, G)\). For \(F \in F(G)\), let \(X : \Omega \mapsto X\) and \(Y_g : \Omega \mapsto X, g \in F\) be random variables defined on some probability space \((\Omega, \mathbb{P})\). Assume \(\mu = \text{Law}(X)\). We say \(X\) and \(Y = (Y_g)_{g \in F}\) are \((F, L^\infty, \varepsilon, \alpha)\)-close for \(\alpha > 0\) if

\[
\mathbb{E}\left(\frac{1}{|F|} \#\{g \in F : d(gX, Y_g) \geq \varepsilon\}\right) < \alpha.
\]

Denote by

\[
R_{\mu, \infty}(\varepsilon, \alpha, F) = \inf_{X, Y \text{ are } (F, L^\infty, \varepsilon, \alpha)\text{-close}} I(X; Y)
\]
and

\[ R_{\mu,\infty}(\varepsilon, \alpha) = \lim_{n \to \infty} \frac{R_{\mu,\infty}(\varepsilon, \alpha, F_n)}{|F_n|}, \]

where \( \{F_n\} \) is a Følner sequence in \( G \). It is not hard to show that the above limit does exist and is independent of the choice of the Følner sequence \( \{F_n\} \). The \( L^\infty \) rate distortion function \( R_{\mu,\infty}(\varepsilon) \) is then defined by

\[ R_{\mu,\infty}(\varepsilon) = \lim_{\alpha \to 0} R_{\mu,\infty}(\varepsilon, \alpha). \]

Fix \( 1 \leq p < \infty \). Let \( F \in F(G), X : \Omega \mapsto \mathcal{X} \) and \( Y_g : \Omega \mapsto \mathcal{X}, g \in F \) be given as previous. We say \( X \) and \( Y = (Y_g)_{g \in F} \) are \((F, L^p, \varepsilon)\)-close if

\[ \mathbb{E}\left( \frac{1}{|F|} \sum_{g \in F} d(gX, Y_g)^p \right) < \varepsilon^p. \]

Denote by

\[ R_{\mu,p}(\varepsilon, F) = \inf_{X,Y \text{ are \((F, L^p, \varepsilon)\)-close}} I(X;Y). \]

The \( L^p \) rate distortion function \( R_{\mu,p}(\varepsilon) \) is then defined by

\[ R_{\mu,p}(\varepsilon) = \lim_{n \to \infty} \frac{R_{\mu,p}(\varepsilon, F_n)}{|F_n|}, \]

where \( \{F_n\} \) is a Følner sequence in \( G \). It is not hard to show that the above limit also exists and is independent of the choice of the Følner sequence \( \{F_n\} \). When \( p = 1 \), the \( L^1 \) rate distortion function \( R_{\mu,1}(\varepsilon) \) coincide with the rate distortion function \( R_{\mu}(\varepsilon) \) defined in Section 3.

**Remark 5.1.** Also similar to Remark 2.3 of Lindenstrauss and Tsukamoto [28], in the definitions of \( R_{\mu,p}(\varepsilon) \) and \( R_{\mu,\infty}(\varepsilon, \alpha) \), the random variable \( Y \) can be also assumed to take only finitely many values.

The following theorem is the \( L^\infty \) variational principles for metric mean dimension. The proof is similar with that of the \( L^1 \) variational principle (Theorem 4.1). Since the \( \bar{d}_F \) metric and \( \tilde{S}(\mathcal{X}, G, d, \varepsilon) \) are not involved, the proof is simpler than that of Theorem 4.1 (but it is still complicated). We will put the proof in Appendix A.

**Theorem 5.2.**

\[
\overline{\operatorname{mdim}}_M(\mathcal{X}, G, d) = \limsup_{\varepsilon \to 0} \sup_{\mu \in M(\mathcal{X}, G)} \frac{R_{\mu,\infty}(\varepsilon)}{|\log \varepsilon|},
\]

\[
\underline{\operatorname{mdim}}_M(\mathcal{X}, G, d) = \liminf_{\varepsilon \to 0} \sup_{\mu \in M(\mathcal{X}, G)} \frac{R_{\mu,\infty}(\varepsilon)}{|\log \varepsilon|}.
\]

**Proof.** See Appendix A. \( \square \)

We also note that the space \((\mathcal{X}, d)\) need not have tame growth of covering numbers in the \( L^\infty \) variational principles. Applying the \( L^1 \) and \( L^\infty \) variational principles, we can obtain the following \( L^p \ (p \geq 1) \) variational principles under the condition that \((\mathcal{X}, d)\) has tame growth of covering numbers.
Theorem 5.3. If \((\mathcal{X}, d)\) has tame growth of covering numbers, then for any \(p \geq 1\),
\[
\text{mdim}_M(\mathcal{X}, G, d) = \limsup_{\varepsilon \to 0} \frac{\sup_{\mu \in M(\mathcal{X}, G)} R_{\mu, p}(\varepsilon)}{|\log \varepsilon|},
\]
\[
\text{mdim}_M(\mathcal{X}, G, d) = \liminf_{\varepsilon \to 0} \frac{\sup_{\mu \in M(\mathcal{X}, G)} R_{\mu, p}(\varepsilon)}{|\log \varepsilon|}.
\]

Proof. Let \(p \geq 1, \alpha > 0, \varepsilon > 0\) and \(\mu \in M(\mathcal{X}, G)\). For \(F \in F(G)\), let \(X : \Omega \mapsto \mathcal{X}\) and \(Y_g : \Omega \mapsto X, g \in F\) be random variables as in the definition of the rate distortion functions.

If \(X\) and \(Y = (Y_g)_{g \in F}\) are \((F, \varepsilon)\)-close, then by the Hölder inequality, it holds that
\[
\mathbb{E}\left(\frac{1}{|F|} \sum_{g \in F} d(gX, Y_g)\right) < \left(\mathbb{E}\left(\frac{1}{|F|} \sum_{g \in F} d(gX, Y_g)^p\right)\right)^{\frac{1}{p}} < \varepsilon,
\]
i.e. \(X\) and \(Y = (Y_g)_{g \in F}\) are \((F, \varepsilon)\)-close. And hence by the definition of the rate distortion functions,
\(R_{\mu}(\varepsilon) \leq R_{\mu, p}(\varepsilon)\).

If \(X\) and \(Y = (Y_g)_{g \in F}\) are \((F, L^\infty, \varepsilon, \alpha)\)-close for \(\alpha > 0\), i.e.
\[
\mathbb{E}\left(\frac{1}{|F|} \#\{g \in F : d(gX, Y_g) \geq \varepsilon\}\right) < \alpha,
\]
then
\[
\frac{1}{|F|} \sum_{g \in F} d(gX, Y_g)^p \leq \varepsilon^p + \frac{1}{|F|} \sum_{g \in F, d(gX, Y_g) \geq \varepsilon} d(gX, Y_g)^p
\]
\[
\leq \varepsilon^p + \frac{1}{|F|} \#\{g \in F : d(gX, Y_g) \geq \varepsilon\} \cdot (\text{diam}(\mathcal{X}, d))^p.
\]
And hence
\[
\mathbb{E}\left(\frac{1}{|F|} \sum_{g \in F} d(gX, Y_g)^p\right) < \varepsilon^p + \alpha (\text{diam}(\mathcal{X}, d))^p.
\]
Then it follows that for any \(\varepsilon' > \varepsilon\), when \(\alpha\) is sufficiently small,
\[
\left(\mathbb{E}\left(\frac{1}{|F|} \sum_{g \in F} d(gX, Y_g)^p\right)\right)^{\frac{1}{p}} < \varepsilon'.
\]
Hence
\(R_{\mu}(\varepsilon') \leq R_{\mu, \infty}(\varepsilon), \text{ for any } \varepsilon' > \varepsilon\).

The conclusion then follows by Theorem 4.1 and Theorem 5.2. \(\square\)

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Appendix A. Proof of Theorem 5.2

The following Lemma is a small modification of Lemma 2.6 of [28] and we omit the proof.

Lemma A.1. Let \((\mathcal{X}, d)\) be a compact metric space with a finite subset \(A\). Let \(F \in F(G), \varepsilon > 0\) and \(\alpha \leq \frac{1}{2}\). Suppose \(S \subset A^F\) is a \(2\varepsilon\)-separated set with respect to the metric \(d_F((x_g)_{g \in F}, (y_g)_{g \in F})\). Let \(X = (X_g)_{g \in F}\) and \(y = (Y_g)_{g \in F}\) be measurable maps from \(\Omega\) to \(\mathcal{X}^F\) such that \(X\) is uniformly distributed over \(S\) and

\[
\mathbb{E}(\#\{g \in F : d(X_g, Y_g) \geq \varepsilon\}) < \alpha|F|.
\]

Then

\[
I(X; Y) \geq \log |S| - |F|H(\alpha) - \alpha|F| \log |A|.
\]

Lemma A.2. For \(\varepsilon > 0\) and \(\mu \in M(\mathcal{X}, G)\), we have

\[
R_{\mu, \infty}(\varepsilon) \leq S(\mathcal{X}, G, d, \varepsilon).
\]

Proof. Let \(\{F_n\}\) be a Følner sequence in \(G\). For \(n > 0\), denote by \(M = \#(\mathcal{X}, d_{F_n}, \varepsilon)\) and let \(\{U_1, \ldots, U_M\}\) be an open cover of \(\mathcal{X}\) with \(\text{diam}(U_m, d_{F_n}) < \varepsilon\) for each \(1 \leq m \leq M\). Choose a point \(p_m \in U_m\) for each \(m\). For any \(x \in \mathcal{X}\), let \(m\) be the smallest number satisfying \(x \in U_{m}\). Then by setting \(f(x) = p_m\) we can define a map \(f : \mathcal{X} \to \{p_1, \ldots, p_M\}\) and hence \(d_{F_n}(x, f(x)) < \varepsilon\). Let \(X\) be a random variable with \(\text{Law}(X) = \mu\). Then \(d_{F_n}(X, f(X)) < \varepsilon\) almost surely. Hence

\[
\mathbb{E}\left(\frac{1}{|F|}\#\{g \in F : d(gX, gf(X)) \geq \varepsilon\}\right) = 0.
\]

Let \(Y = (gf(X))_{g \in F_n}\). Obviously \(X\) and \(Y\) are \((F, L^{\infty}, \varepsilon, \alpha)\)-close for any \(\alpha > 0\). Hence

\[
I(X; Y) \leq H(Y) \leq \log M = \log \#(\mathcal{X}, d_{F_n}, \varepsilon).
\]

Dividing by \(|F_n|\) and letting \(n \to \infty\), we have

\[
R_{\mu, \infty}(\varepsilon) \leq S(\mathcal{X}, G, d, \varepsilon).
\]

Proposition A.3. For any \(\varepsilon > 0\) there exists \(\mu \in M(\mathcal{X}, G)\) such that

\[
R_{\mu, \infty}(\varepsilon) \geq S(\mathcal{X}, G, d, 12\varepsilon).
\]

Proof. Let \(\{F_n\}\) be the Følner sequence in \(G\) constructed as in Lemma 2.6

For each \(F_n\) we choose \(S_n\) to be a maximal \(6\varepsilon\)-separated set of \(\mathcal{X}\) with respect to the metric \(d_{F_n}\). Then

\[
|S_n| \geq \#(\mathcal{X}, d_{F_n}, 12\varepsilon).
\]

Define

\[
\nu_n = \frac{1}{|S_n|} \sum_{x \in S_n} \delta_x
\]
and
\[ \mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} \nu_n \circ g^{-1}. \]

As in the proof of Proposition 4.4, we first choose a tempered subsequence \( \{F_n\} \) of \( \{F_n\} \), then choose a convergence subsequence of \( \{\mu_n\}_{n=1}^\infty \) in the weak* topology and assume it converges to \( \mu \). Hence \( \mu \in M(X,G) \) and we will show it satisfies the inequality (A.1). For simplicity, we still denote this subsequence by \( \{\mu_n\}_{n=1}^\infty \).

Let \( \mathcal{P} = \{P_1, \ldots, P_M\} \) be a measurable partition of \( X \) with \( \text{diam}(P_m, d) < \varepsilon \) and \( \mu(\partial P_m) = 0 \) for each \( 1 \leq m \leq M \).

Assign each \( P_m \) a point \( p_m \in P_m \) and set \( A = \{p_1, \ldots, p_M\} \). Denote by \( \mathcal{P}(x) = p_m \) for \( x \in P_m \). Then
\[ \text{(A.3)} \quad d(x, \mathcal{P}(x)) < \varepsilon. \]

Let \( \mathcal{P}^F(x) = (\mathcal{P}(gx))_{g \in F} \) for \( F \in F(G) \). Recall that we also use \( d_F \) to denote the metric on \( X^F \) for \( F \in F(G) \) (see (3.1) for the definition). By (A.3), we have
\[ d_{F_n}(\mathcal{P}^{F_n}(x), \mathcal{P}^{F_n}(y)) < \varepsilon \]
for any \( x, y \in X \). For any two distinct points \( x, y \in S_n \), we have
\[ d_{F_n}(\mathcal{P}^{F_n}(x), \mathcal{P}^{F_n}(y)) > 6\varepsilon - 2\varepsilon = 4\varepsilon. \]

Hence the set
\[ \mathcal{P}^{F_n}(S_n) = \{\mathcal{P}^{F_n}(x) \mid x \in S_n\} \subset A^{F_n} \]
is a \( 4\varepsilon \)-separated set of \( X^{F_n} \) with respect to the metric \( d_{F_n} \). Moreover, since \( \nu_{F_n} \) is the uniform distribution over \( S_n \), the push-forward measure \( \mathcal{P}^{F_n}_*\nu_{F_n} \) is also the uniform distribution measure over \( \mathcal{P}^{F_n}(S_n) \). Note that \( |\mathcal{P}^{F_n}(S_n)| = |S_n| \).

Let \( 0 < \alpha < \frac{1}{4} \), let \( X : \Omega \rightarrow X \) be a random variable defined on some probability space \( (\Omega, \mathbb{P}) \) such that the law of \( X \) is given by \( \mu \). For \( F \in F(G) \), let \( Y_{F,g} : \Omega \rightarrow X \) \( (g \in F) \) be random variables defined on the same probability space \( (\Omega, \mathbb{P}) \) such that \( Y_F = (Y_{F,g})_{g \in F} \) and \( X \) are \( (F, L^\infty, \varepsilon, \alpha) \)-close, i.e.
\[ \text{(A.4)} \quad \mathbb{E}\left( \frac{1}{|F|} \# \{ g \in F : d(gX, Y_g) \geq \varepsilon \} \right) < \alpha. \]

We can assume the distribution of \( Y_F \) is supported on a finite set \( \mathcal{Y}_F \subset X^F \) (by (2) of Remark 5.1). By the Data-processing inequality,
\[ I(X; Y_F) \geq I(\mathcal{P}^F(X); Y_F). \]
Let \( \tau_F = \text{Law}(\mathcal{P}^F(X), Y_F) \) be the law of \( (\mathcal{P}^F(X), Y_F) \), which is supported on \( A^F \times \mathcal{Y}_F \).

Since \( d(gX, \mathcal{P}(gX)) < \varepsilon \), it follows that
\[ \{ g \in F : d(\mathcal{P}(gX), Y_{F,g}) \geq 2\varepsilon \} \subset \{ g \in F : d(gX, Y_{F,g}) \geq \varepsilon \}. \]
Thus
\[
\mathbb{E}_{\tau_F} f_F(x, y) := \int_{A^F \times Y} f_F(x, y) d\tau_F(x, y)
= \mathbb{E} \left( \# \{ g \in F : d(\mathcal{P}(gX), Y_{F,g}) \geq 2\varepsilon \} \right)
\leq \alpha |F|,
\]
where we denote by \( f_F(x, y) = \# \{ g \in F : d(x_g, y_g) \geq 2\varepsilon \} \) for \( x = (x_g)_{g \in F} \in A^F \) and \( y = (y_g)_{g \in F} \in Y_F \). For each \( n \geq 1 \), we consider the couplings of \( (\mathcal{P}_F^F \mu_n, \mathcal{P}_F^F \mu) \). Choose a probability measure \( \pi_{F,n} \) that minimizes the following integral
\[
\int_{A^F \times A^F} \tilde{d}_F(x, y) d\pi(x, y)
\]
among all such couplings \( \pi \). Also similar to Claim 3.6 of \([28]\), the sequence \( \pi_{F,n} \) converges to \( (\mathcal{P}_F^F \times \mathcal{P}_F^F)_* \mu \) in the weak* topology.

Compose \( \pi_{F,n} \) and \( \tau_F \) to produce a coupling \( \tau_{F,n} \) of \( (\mathcal{P}_F^F \mu_n, \text{Law}(Y)) \) by the following way:
\[
\tau_{F,n}(x, y) = \sum_{x' \in A^F} \pi_{F,n}(x, x') \mathbb{P}(Y = y | \mathcal{P}_F^F(X) = x'), \quad (x \in A^F, y \in Y_F).
\]
We note here that the sequence \( \tau_{F,n} \) converges to \( \tau_F \) in the weak* topology and hence by (A.5),
\[
\mathbb{E}_{\tau_{F,n}} f_F(x, y) = \int_{A^F \times Y} f_F(x, y) d\tau_{F,n}(x, y) < \alpha |F|
\]
for all sufficiently large \( n_i \).

Similar to the proof of Proposition 4.4, for \( x \in \bigcup_{g \in F_n} \mathcal{P}_F^F(gS_g) \) and \( y \in \mathcal{X}_F \), we define a conditional probability mass function \( \tau_{F,n}(y|x) \) by
\[
\tau_{F,n}(y|x) = \frac{\tau_{F,n}(x, y)}{\mathcal{P}_F^F \mu_n(x)}.
\]

For any \( K \in F(G) \) with \( e_G \in K \) and \( 0 < \varepsilon_1 < \alpha \), as in Proposition 4.4, by Lemma 2.6, there exists \( T \), a finite tiling of \( G \), satisfying conditions (C1) and (C2) in Proposition 4.4

(C1) \( T \) has shapes \( \{ F_{m_1}, \ldots, F_{m_l} \} \) consisted with Følner sets in \( \{ F_n \} \) and each \( F_{m_i} \) is \((K, \varepsilon_1)\)-invariant;

(C2) for sufficiently large \( i \) (hence \( F_{n_i} \in F(G) \) is sufficiently invariant), for each \( 1 \leq j \leq l \), the family of sets \( \{ C_j g^{-1} \}_{g \in F_{n_i}} \) covers a subset \( \tilde{F}_{n_i} \subset F_{n_i} \) with \( |\tilde{F}_{n_i}| > (1 - \varepsilon_1) |F_{n_i}| \) at most \( (1 + \varepsilon_1)^{\rho_F(F_{m_i}, F_{m_i}) |F_{n_i}|} \)-many times, where \( C_j \) is the center of the shape \( F_{m_j} \).

For \( x = (x_g)_{g \in F_{n_i}} \in \mathcal{P}_F^{F_{n_i}}(S_{n_i}) \) and \( g \in F_{n_i} \), we define probability mass functions \( \sigma_{F_{n_i},g}(\cdot|x) \) and \( \sigma_{F_{n_i}}(\cdot|x) \) on \( \mathcal{X}_F^{F_{n_i}} \) as exactly as (4.7) and (4.8) respectively. For \( y = \)
\[(y_g)_{g \in F_{n_i}} \in \mathcal{X}^{F_{n_i}} \text{ and } g \in F_{n_i}, \]

\[(A.7) \quad \sigma_{F_{n_i}, g}(y|x) = \prod_{j=1}^{l} \prod_{c \in C_j, F_{m_j}c \subset F_{n_i}, g \in g^{-1} \in \bar{F}_{n_i}} \tau_{F_{m_j}, n_i}(y_{F_{m_j}cg^{-1} | x_{F_{m_j}cg^{-1}}}) \cdot \prod_{h \in R_g} \delta_a(y_h) \]

and

\[(A.8) \quad \sigma_{F_{n_i}}(y|x) = \frac{1}{|F_{n_i}|} \sum_{g \in F_{n_i}} \sigma_{F_{n_i}, g}(y|x). \]

Here we recall that

\[y_{F_{m_j}cg^{-1}} = (y_h)_{h \in F_{m_j}cg^{-1} \in \mathcal{X}^{F_{m_j}cg^{-1}}}, \]

\[x_{F_{m_j}cg^{-1}} = (x_h)_{h \in F_{m_j}cg^{-1} \in \mathcal{P}^{F_{m_j}cg^{-1}}(S_{n_i})} \]

and

\[R_g = F_{n_i} \setminus \left( \prod_{i=1}^{m} \prod_{c \in C_j, F_{m_j}c \subset F_{n_i}, g \in g^{-1} \in \bar{F}_{n_i}} F_{m_j}cg^{-1} \right). \]

Then as exactly as Claim 4.5, when \(n_i\) is large enough, there exists some \(1 \leq j \leq l\) such that

\[(A.9) \quad (1 - \varepsilon_1) \frac{1}{|F_{n_i}|} I\left(P_{n_i}^{F_{n_i}} \nu_{n_i}, \sigma_{F_{n_i}}\right) \leq \frac{1}{|F_{m_j}|} I\left(P_{m_j}^{F_{m_j}}(\mu_{n_i}), \tau_{F_{m_j}, n_i}\right). \]

Denote by \(E_{P_{n_i}^{F_{n_i}} \nu_{n_i}, \sigma_{F_{n_i}}} f_{F_{n_i}}(x, y)\) the expected value of the function \(f_{F_{n_i}}(x, y)\) with respect to the probability measure \(P_{n_i}^{F_{n_i}} \nu_{n_i}(x)\sigma_{F_{n_i}}(y|x)\).

**Claim A.4.** For sufficiently large \(n_i\),

\[E_{P_{n_i}^{F_{n_i}} \nu_{n_i}, \sigma_{F_{n_i}}} f_{F_{n_i}}(x, y) < 3\alpha |F_{n_i}|. \]

**Proof of Claim A.4.** By (A.7) and (A.8), the definitions of probability mass functions \(\sigma_{F_{n_i}, g}(\cdot|x) (g \in F_{n_i})\) and \(\sigma_{F_{n_i}}(\cdot|x)\), we have

\[E_{P_{n_i}^{F_{n_i}} \nu_{n_i}, \sigma_{F_{n_i}}} f_{F_{n_i}}(x, y) = \frac{1}{|F_{n_i}|} \sum_{g \in F_{n_i}} E_{P_{n_i}^{F_{n_i}} \nu_{n_i}, \sigma_{F_{n_i}, g}} f_{F_{n_i}}(x, y) \]

and

\[E_{P_{n_i}^{F_{n_i}} \nu_{n_i}, \sigma_{F_{n_i}, g}} f_{F_{n_i}}(x, y) \leq \sum_{j=1}^{l} \sum_{c \in C_j, F_{m_j}c \subset F_{n_i}, g \in g^{-1} \in \bar{F}_{n_i}} E_{P_{n_i}^{F_{m_j}}((g^{-1})_c \nu_{n_i}), \tau_{F_{m_j}, n_i}} f_{F_{m_j}}(x', y') + |R_g|, \]

where \(x, y\) are random points in \(\mathcal{X}^{F_{n_i}}\) and \(x', y'\) appear in \(f_{F_{m_j}}(x', y')\) are in \(\mathcal{X}^{F_{m_j}}\).
When $F_{n_i}$ is sufficiently invariant, $|R_g| < \varepsilon_1 |F_{n_i}|$. Hence

$$
\mathbb{E}_{P_{F_{n_i}}^\nu_{n_i}, \sigma_{F_{n_i}}} f_{F_{n_i}}(x, y)
\leq \frac{1}{|F_{n_i}|} \sum_{j=1}^l \sum_{g \in F_{n_i}} \sum_{C_j, F_{n_j} \subset F_{n_i}} \mathbb{E}_{P_{F_{n_j}}^\nu_{n_j}, \tau_{F_{n_j}, n_i}} f_{F_{n_j}}(x', y') + \varepsilon_1 |F_{n_i}|
$$

(by condition (C2) and recall here $t_j = \rho_T(F_{m_j}, F_{n_i})$)

$$
\leq \sum_{j=1}^l (1 + \varepsilon_1) t_j \frac{|F_{n_i}|}{F_{m_j}} \mathbb{E}_{P_{F_{m_j}}^{\mu_{n_j}, \tau_{F_{m_j}, n_i}}} f_{F_{m_j}}(x', y') + \varepsilon_1 |F_{n_i}|
$$

$$
= \sum_{j=1}^l (1 + \varepsilon_1) t_j \frac{|F_{n_i}|}{F_{m_j}} \int f_{F_{m_j}}(x', y') d\tau_{F_{m_j}, n_i}(x, y) + \varepsilon_1 |F_{n_i}|
$$

Recall that $0 < \varepsilon_1 < \alpha < \frac{1}{4}$ and $\sum_{j=1}^l t_j \leq 1$. By (A.6), for sufficiently large $n_i$, we have

$$
\mathbb{E}_{r_{F_{m_j}, n_i}} f_{F_{m_j}}(x', y') < \alpha |F_{m_j}|, \text{ for each } 1 \leq j \leq l.
$$

Hence for sufficiently large $n_i$,

$$
\mathbb{E}_{P_{F_{n_i}}^\nu_{n_i}, \sigma_{F_{n_i}}} f_{F_{n_i}}(x, y) < \left( (1 + \varepsilon_1)\alpha + \varepsilon_1 \right) |F_{n_i}|
\leq 3\alpha |F_{n_i}|.
$$

This finishes the proof of Claim A.4. \qed

Note that the set $P_{F_{n_i}}(S_{n_i}) = \{P_{F_{n_i}}(x) | x \in S_{n_i} \} \subset A^{F_{n_i}}$, $|P_{F_{n_i}}(S_{n_i})| = |S_{n_i}|$ is a 4$\varepsilon$-separated set of $A^{F_{n_i}}$ with respect to the metric $d_{F_{n_i}}$ and the push-forward measure $P_{F_{n_i}}^\nu_{n_i}$ is the uniform distribution measure over $P_{F_{n_i}}(S_{n_i})$. By Claim A.4 and Lemma A.1, for sufficiently large $n_i$,

$$(A.10) \quad \frac{1}{|F_{n_i}|} I(P_{F_{n_i}}^\nu_{n_i}, \sigma_{F_{n_i}}) \geq \frac{1}{|F_{n_i}|} \log |S_{n_i}| - 3\alpha \log M - H(3\alpha).$$
It follows from (A.2), (A.9) and Claim A.4 that for sufficiently large $n_i$, there exists $1 \leq j \leq l$ ($j$ depends on $n_i$ and $l$ is independent on $n_i$) such that

$$\frac{1}{|F_{m_j}|} I(\mathcal{P}_{\tau_{F_{m_j},n_i}}^{F_{m_j}}(\mu_{n_i}), \tau_{F_{m_j},n_i}) \geq (1 - \varepsilon_1) \left( \frac{1}{|F_{n_i}|} \log |\mathcal{X}, d_{F_{n_i}}, 12 \varepsilon| - 3\alpha \log M - H(3\alpha) \right).$$

By choosing some subsequence of $\{n_i\}$ (we still denote it by $\{n_i\}$), for some $1 \leq j \leq l$, the probability measures $\tau_{F_{m_j},n_i}$ converge to $\tau_{F_{m_j}} = \text{Law}(\mathcal{P}_{F_{m_j}}^{F_{m_j}}(X), Y_{F_{m_j}})$ in the weak* topology. Let $n_i \to \infty$. By (3) of Proposition 3.2,

$$\frac{1}{|F_{m_j}|} I(\mathcal{P}_{\tau_{F_{m_j}}}(X); Y_{F_{m_j}}) \geq (1 - \varepsilon_1) \left( S(\mathcal{X}, G, d, 12 \varepsilon) - 3\alpha \log M - H(3\alpha) \right).$$

By (1) of Proposition 3.2 the data-processing inequality,

$$\frac{1}{|F_{m_j}|} I(X; Y_{F_{m_j}}) \geq (1 - \varepsilon_1) \left( S(\mathcal{X}, G, d, 12 \varepsilon) - 3\alpha \log M - H(3\alpha) \right).$$

Choose $K \in F(G)$ to be more and more invariant as in the proof of Proposition 4.4 and let $\varepsilon_1$ tend to 0 to force $m_j \to \infty$. Noticing that $R_{\mu, \infty}(\varepsilon, \alpha)$ is independent of the selection of the Følner sequence, it follows that

$$R_{\mu, \infty}(\varepsilon, \alpha) \geq S(\mathcal{X}, G, d, 12 \varepsilon) - 3\alpha \log M - H(3\alpha).$$

Letting $\alpha \to 0$, we have

$$R_{\mu, \infty}(\varepsilon) \geq S(\mathcal{X}, G, d, 12 \varepsilon).$$

This completes the proof of Proposition A.3. \hfill \Box

Theorem 5.2 then follows from Lemma A.2 and Proposition A.3.

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