QUANTUM DOUBLE FOR QUASI-HOPF ALGEBRAS

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Abstract We introduce a quantum double quasitriangular quasi-Hopf algebra $D(H)$ associated to any quasi-Hopf algebra $H$. The algebra structure is a cocycle double cross product. We use categorical reconstruction methods. As an example, we recover the quasi-Hopf algebra of Dijkgraaf, Pasquier and Roche as the quantum double $D^\phi(G)$ associated to a finite group $G$ and group 3-cocycle $\phi$. Keywords: quantum double – quasi-Hopf algebra – finite group – cocycle – category – reconstruction.

1 Introduction

The quantum double\cite{1} of V.G. Drinfeld is one of the most important of quantum group constructions. It associates to a Hopf algebra $H$ a quasitriangular one. The quasitriangular structure leads to a braiding in the category of representations and many ensuing applications.

In this note we introduce the corresponding construction for quasi-Hopf algebras. Quasi-Hopf algebras have coproducts which are coassociative only up to a 3-cocycle $\phi \in H \otimes H \otimes H$\cite{2}. This greater freedom allows, for example, the simplest formulation of quantum groups $U_q(g)$ as ordinary enveloping algebras $U(g)$ equipped with a quasi-Hopf structure obtained from the Knizhnik-Zamolodchikov equations\cite{3}. This is the form in which quantum groups ‘naturally arise’ in conformal field theory and in the theory of Vassilyev invariants, for example.

At first, it would appear difficult to define directly the corresponding quantum double because, in Drinfeld’s construction, the quantum double $D(H)$ contains $H, H^{\text{op}}$ in a symmetrical way; if $H$ is quasi-coassociative then $H^{\text{op}}$ is quasi-associative and one might expect the double to be some kind of hybrid object. We will see that this is not necessary. Instead, we use a
categorical formulation of the quantum double which must necessarily give us as the double an ordinary quasi-Hopf algebra.

The categorical picture of the usual quantum double was also provided by V.G. Drinfeld, in terms of the ‘center’ or ‘double’ $Z(C)$ of a monoidal category. The same (slightly more general) construction was introduced at the same time in $\mathcal{Z}(\mathcal{C})$ as a generalised ‘Pontryagin dual’ $\mathcal{C}^\circ$ of a monoidal category. Drinfeld observed that this category is braided and is $D(H)_M$ when $C = H_M$, the modules over a Hopf algebra. We recall the basic facts in the following Preliminaries section. In Section 2, we compute the double category when $H$ is the category of modules over a quasi-Hopf algebra $H$, $\phi$. We can then use the Tannaka-Krein reconstruction in the generalised form cf.[5] to define $D(H)$ for quasi-Hopf algebras, in Section 3. We cover in particular the example when $H = k(G)$ the algebra of functions on a finite group $G$, equipped with a group cocycle $\phi \in Z^3(G)$. The double in this case recovers the quasi-Hopf algebra introduced by direct means in [3] in connection with a ‘toy model’ of conformal field theory.

Preliminaries

We work over a general field $k$ or, with suitable care, over a commutative ring. Following Drinfeld, a quasi-Hopf algebra means $(H, \Delta, \epsilon, \phi, S, \alpha, \beta)$ where $H$ is a unital algebra, $\Delta : H \to H \otimes H$ and $\epsilon : H \to k$ are algebra homomorphisms obeying

$$(\text{id} \otimes \Delta) \circ \Delta = \phi((\Delta \otimes \text{id}) \circ \Delta) \phi^{-1}, \quad (\text{id} \otimes \epsilon) \circ \Delta = \text{id} = (\text{id} \otimes \epsilon) \circ \Delta,$$

where $\phi \in H^{\otimes 3}$ is invertible and a 3-cocycle in the sense

$$(1 \otimes \phi)((\text{id} \otimes \Delta \otimes \text{id})\phi)(\phi \otimes 1) = ((\text{id} \otimes \text{id} \otimes \Delta)\phi)((\Delta \otimes \text{id} \otimes \text{id})\phi)$$

and $(\text{id} \otimes \epsilon \otimes \text{id})\phi = 1 \otimes 1$. This part defines a quasi-bialgebra. In addition, we require $S : H \to H$ and $\alpha, \beta \in H$ such that

$$\sum (Sh_{(1)})\alpha h_{(2)} = \epsilon(h)\alpha, \quad \sum h_{(1)} \beta Sh_{(2)} = \epsilon(h)\beta, \quad \forall h \in H$$

$$\sum \phi^{(1)} \beta S\phi^{(2)} \alpha \phi^{(3)} = 1, \quad \sum (S\phi^{-1})\alpha \phi^{-2} \beta S\phi^{-3} = 1.$$  \hspace{1cm} (3)

We use the notation $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ and $\phi = \sum \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)}$. Similarly for $\phi^{-1}$.

A quasi-bialgebra or quasi-Hopf algebra $H$ is quasitriangular if there is an invertible element $R \in H \otimes H$ such that

$$(\Delta \otimes \text{id})R = \phi_{312} R_{132} \phi^{-1}_{132} R_{23} \phi, \quad (\text{id} \otimes \Delta)R = \phi^{-1}_{231} R_{132} \phi_{231} R_{12} \phi^{-1}$$

in another standard notation. Explicitly, $\phi_{213} = \sum \phi^{(2)} \otimes \phi^{(1)} \otimes \phi^{(3)}$, etc.
A monoidal category means a category $\mathcal{C}$ with objects $V, W, Z$ etc., a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ equipped with an associativity natural transformation consisting of functorial isomorphisms $\Phi_{V,W,Z} : (V \otimes W) \otimes Z \to V \otimes (W \otimes Z)$ obeying a pentagon identity. There is also a compatible unit object and associated functorial isomorphisms. A braided category is a monoidal category equipped with a commutativity natural transformation consisting of functorial isomorphisms $\Psi_{V,W} : V \otimes W \to W \otimes V$ compatible with the unit and associativity structures in a natural way. We generally suppress writing the tensor product by identity morphisms, as well as isomorphisms associated with the identity object.

A representation of a monoidal category $\mathcal{C}$ in itself is an object $V$ of $\mathcal{C}$ and a natural equivalence $\lambda_V \in \text{Nat}(V \otimes \text{id}, \text{id} \otimes V)$, i.e. functorial isomorphisms $\lambda_{V,W} : V \otimes W \to W \otimes V$ (functorial in $W$), such that

$$\lambda_{V,1} = \text{id}, \quad \lambda_{V,Z} \circ \Phi_{W,V,Z} \circ \lambda_{V,W} = \Phi_{W,Z,V} \circ \lambda_{V,W} \otimes Z \circ \Phi_{W,Z,Z} \circ \lambda_{W,Z}, \quad \forall W, Z \in \mathcal{C}. \quad (5)$$

The ‘Pontryagin dual’ monoidal category $\mathcal{C}^\circ$, or ‘double’, has as objects such representations. Morphisms $\phi : (V, \lambda_V) \to (W, \lambda_W)$ are morphisms $\phi : V \to W$ of $\mathcal{C}$ obeying

$$(\text{id} \otimes \phi) \circ \lambda_{V,Z} = \lambda_{W,Z} \circ (\phi \otimes \text{id}). \quad (6)$$

The monoidal structure $(V, \lambda_V) \otimes (W, \lambda_W)$ consists of $V \otimes W$ and the natural transformation

$$\lambda_{V \otimes W,Z} = \Phi_{Z,V,W} \circ \lambda_{V,Z} \circ \Phi_{V,Z,W}^{-1} \circ \lambda_{W,Z} \circ \Phi_{V,W,Z}, \quad \forall Z \in \mathcal{C}. \quad (7)$$

The associator $\Phi_{V,W,Z}$ is the underlying one for the category $\mathcal{C}$, viewed as functorial isomorphisms in $\mathcal{C}^\circ$. The construction also works more generally for representations in another monoidal category. In the present case, there is a braiding

$$\Psi_{(V, \lambda_V), (W, \lambda_W)} = \lambda_{V,W} \quad (8)$$

due to $\phi$. Other notations for this category are $D(\mathcal{C})$ or $Z(\mathcal{C})$. Further results are in $\cite{10}$.

If $H$ is a quasi-bialgebra or quasi-Hopf algebra, we denote by $H \mathcal{M}$ its category of modules. This forms a monoidal category with tensor product defined via $\Delta$ and with associativity transformation

$$\Phi_{V,W,Z}((v \otimes w) \otimes z) = \sum \phi^{(1)} \triangleright v \otimes (\phi^{(2)} \triangleright w \otimes \phi^{(3)} \triangleright z), \quad \forall v \in V, w \in W, z \in Z \quad (9)$$

and (in the quasitriangular case) braiding defined by

$$\Psi_{V,W}(v \otimes w) = \sum \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v, \quad \forall v \in V, w \in W. \quad (10)$$

The forgetful functor is multiplicative but not monoidal unless $H$ is twisting equivalent to an ordinary Hopf algebra. Further details are in $\cite{11}$. 

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2 Double category of modules over a quasi-Hopf algebra

Let $H, \phi$ be a quasi-Hopf algebra with bijective antipode and $\mathcal{C} = _H\mathcal{M}$. We denote the action of $H$ on a module $V$ by $\triangleright$.

Lemma 2.1 Let $V \in \mathcal{C}$. Natural transformations $\text{Nat}(V \otimes \text{id}, \text{id} \otimes V)$ are in 1-1 correspondence with linear maps $V \to H \otimes V$, denoted $v \mapsto \sum v^{(1)} \otimes v^{(2)}$, such that

$$\sum h_{(1)} v^{(1)} \otimes h_{(2)} \triangleright v^{(2)} = \sum (h_{(1)} \triangleright v)^{(1)} h_{(2)} \otimes (h_{(1)} \triangleright v)^{(2)}.$$ 

Proof Let $H_L$ denote $H \in _H\mathcal{M}$ by the left regular representation. If $\lambda_V$ is a natural transformation, the corresponding linear map $\beta : V \to H \otimes V$ is

$$\beta(v) = \lambda_{V,H_L}(v \otimes 1).$$

For and $W \in \mathcal{C}$ and $w \in W$ we consider the morphism $i_w : H_L \to W$ defined by $h \mapsto h \triangleright w$. Then naturality of $\lambda_V$ implies that $\lambda_{V,W}(v \otimes w) = \lambda_{V,W}(v \otimes i_w,1) = v^{(1)} \triangleright w \otimes v^{(2)}$. In particular, for any $h \in H$ we consider right-multiplication $R_h : H_L \to H_L$ as a morphism and hence $\lambda_{V,H_L}(v \otimes h) = \lambda_{V,H_L}(v \otimes R_h,1) = v^{(1)} h \otimes v^{(2)}$. Then the assumption that each $\lambda_{V,H_L}$ are morphisms implies that

$$(h_{(1)} \triangleright v)^{(1)} h_{(2)} \otimes (h_{(1)} \triangleright v)^{(2)} = \lambda_{V,H_L}(h_{(1)} \triangleright v \otimes h_{(2)},1) = h_{(1)} v^{(1)} \otimes h_{(2)} \triangleright v^{(2)},$$

which is the condition stated. Conversely, given $\beta$ obeying this condition, we define $\lambda_{V,W}(v \otimes w) = v^{(1)} \triangleright w \otimes v^{(2)}$ and verify easily that this is a natural transformation. □

Proposition 2.2 Representations of $_H\mathcal{M}$ in itself are in 1-1 correspondence with pairs $(V, \beta_V)$ where $V$ is an $H$-module and $\beta_V : V \to H \otimes V$ obeys the condition in Lemma 2.1 and

$$\phi^{(1)} v^{(1)} \otimes (\phi^{(2)} \triangleright v^{(2)})^{(1)} \phi^{(3)} \otimes (\phi^{(2)} \triangleright v^{(2)})^{(2)} = \phi \left( (\phi^{(1)} \triangleright v)^{(1)} (\phi^{(1)} \triangleright v)^{(1)} \phi^{(2)} \otimes (\phi^{(1)} \triangleright v)^{(1)} (\phi^{(1)} \triangleright v)^{(2)} \phi^{(3)} \otimes (\phi^{(1)} \triangleright v)^{(2)} \right),$$

where $\phi$ acts on $H_L \otimes H_L \otimes V$. We also require $(\epsilon \otimes \text{id}) \circ \beta_V = \text{id}$. The category $(_H\mathcal{M})^0$ consists of such objects and morphisms which intertwine the $H$ action and the corresponding $\beta$. The category has monoidal product $(V, \beta_V) \otimes (W, \beta_W)$ built on $V \otimes W$ as an $H$-module and

$$\beta_{V \otimes W} = \phi \left( (\phi^{-1} \phi^{(1)} \triangleright v)^{(1)} \phi^{-1} (\phi^{(2)} \triangleright v)^{(1)} \phi^{(3)} \otimes (\phi^{-1} \phi^{(1)} \triangleright v)^{(2)} \phi^{(3)} \otimes (\phi^{-1} \phi^{(1)} \triangleright v)^{(3)} \phi^{(2)} \triangleright w \otimes v^{(2)} \right),$$

where $\phi$ acts on $H_L \otimes V \otimes W$. The associativity isomorphisms are given by the same formula (4) as for $_H\mathcal{M}$. The category is braided, with

$$\Psi_{V,W}(v \otimes w) = v^{(1)} \triangleright w \otimes v^{(2)}.$$
Proof We write out (5) and (7) using the identifications in Lemma 2.1. In the converse direction, the $\lambda_{V,W}$ defined by $\beta_V$ are isomorphisms since $H$ using the assumed inverse antipode (this corresponds to existence of left duals by [3]).  

These steps are similar to the computation of the double category for ordinary Hopf algebras, with the additional presence of $\phi$ in our quasi-Hopf algebra case. Whereas one could also come to such a category of ‘crossed modules’ as a generalisation of Whitehead’s crossed $G$-sets [12], one really needs the above double approach in the quasi-Hopf algebra case, in order to place the $\phi$ correctly.

3 Double of a quasiHopf algebra

We are now in a position to define the double quasi-Hopf algebra $D(H)$ as that obtained by Tannaka-Krein reconstruction from the category constructed in the preceding section as the automorphisms of the forgetful functor.

Corollary 3.1 If $(H, \phi)$ is a finite-dimensional quasi-Hopf algebra, it has a quantum double $D(H)$ uniquely defined, up to isomorphism, as a quasitriangular quasiHopf algebra such that its category of representations is $(\mathcal{H}_\mathcal{M})^\circ$ in Proposition 2.2. In particular, it may be built on the vector space $H^* \otimes H$ with $H$ as a sub-quasiHopf algebra.

Proof The forgetful functor from $(\mathcal{H}_\mathcal{M})^\circ$ to $Vec$ is multiplicative and hence we can use the reconstruction theorem [5]. This builds $D(H)$ such that we can identify $D(H)_\mathcal{M} = (\mathcal{H}_\mathcal{M})^\circ$ as braided monoidal categories. It is clear from the characterisation of the latter in Proposition 2.2 that $D(H)$ may be built on $H^* \otimes H$ as a vector space with a certain product and coproduct: the action of $(f \otimes h) \in D(H)$ is

$$(f \otimes h) \triangleright v = \langle f, (h \triangleright v)^{(1)} \rangle (h \triangleright v)^{(2)}$$

(11)

where $(V, \triangleright, \beta)$ is the corresponding object of $(\mathcal{H}_\mathcal{M})^\circ$ and $\langle \cdot, \cdot \rangle$ is the evaluation pairing. Moreover, the forgetful functor factors through the monoidal functor $(\mathcal{H}_\mathcal{M})^\circ \rightarrow \mathcal{H}_\mathcal{M}$, which corresponds to an inclusion $H \subset D(H)$ as $h \equiv 1 \otimes h$. Note that the form of $\Phi$ gives immediately

$$\Phi_{D(H)} = \phi$$

(12)

under this inclusion. Likewise, the form of $\Psi$ in Proposition 2.2 and comparison with (10) gives immediately

$$\mathcal{R} = \sum_a (f^a \otimes 1) \otimes (1 \otimes e_a),$$

(13)
where \( \{e_a\} \) is a basis of \( H \) and \( \{f^a\} \) is a dual basis. The reconstructed product and coproduct are more complex, although reducing to those for the usual quantum double \( H^{\mathrm{op} \circ \Delta}H \) when \( \phi = 1. \]

If one wants explicit formulae for the product and coproduct of \( D(H) \), they are immediately obtained from the formulae in Lemma 2.1 and Proposition 2.2. We can write the former as

\[
(\phi^{-(1)} h \bowtie v) (1) \phi^{-(2)} \beta S \phi^{-(3)} \otimes (\phi^{-(1)} h \bowtie v)^{(2)} = h_{(1)(1)} (\phi^{-(1)} h \bowtie v)^{(1)} \phi^{-(2)} \beta S (\phi^{-(3)} h_{(2)}) \otimes h_{(1)(2)} \phi^{-(1)} h^{(1)} \bowtie v^{(2)}
\]

in view of the quasi-coassociativity and antipode axioms for \( H \). Applying \( \langle f, \cdot \rangle \) for \( f \in H^* \) to the first factor, and making use of (11), we eliminate \( v \) and obtain the relations

\[
f_{(1)} \cdot \phi^{-(1)} \cdot h \langle f_{(2)}, \phi^{-(2)} \beta S \phi^{-(3)} \rangle = \langle f_{(1)}, h_{(1)(1)} \rangle \langle f_{(3)}, \phi^{-(2)} \beta S (\phi^{-(3)} h_{(2)}) \rangle h_{(1)(2)} \cdot f_{(2)} \cdot \phi^{-(1)}
\]  

for all \( h \in H \) and \( f \in H^* \). Here we write \( h_{(1)(1)} = 1 \otimes h_{(1)(1)} \) and \( f_{(1)} = f_{(1)} \otimes 1 \), etc., and \( \cdot \) is the product in \( D(H) \). Moreover, \( f_{(1)} \otimes f_{(2)} \) etc., denotes the coassociative coproduct of \( H^* \) dual to the product of \( H \). Similarly, applying \( \langle g, \cdot \rangle \otimes \langle f, \cdot \rangle \) to the quasi-coaction condition for \( \beta_V \) in Proposition 2.2, using (11) and cancelling \( h \)

\[
\langle g_{(1)}, \phi^{(1)} \rangle \langle f_{(2)}, \phi^{(3)} \rangle f_{(1)} \cdot \phi^{(2)} \cdot g_{(2)} = \langle g_{(1)}, \phi^{(1)} \rangle \langle f_{(1)}, \phi^{(2)} \rangle \langle g_{(3)}, \phi^{(3)} \rangle \langle f_{(3)}, \phi^{(3)} \rangle \langle g_{(2)}, f_{(2)} \rangle \cdot \phi^{(1)}
\]  

for all \( f, g \in H^* \). Here \( \phi' \) denotes a second copy of \( \phi \) and \( g_{(2)} f_{(2)} \) is multiplied in the (non-associative) product on \( H^* \) dual to the coproduct of \( H \). Finally, applying \( \langle f, \cdot \rangle \) to the formula for \( \beta_V \otimes \beta_W \) in Proposition 2.2 and cancelling \( v, w \) gives

\[
\Delta_{D(H)} f = \langle f_{(1)}, \phi^{(1)} \rangle \langle f_{(3)}, \phi^{(2)} \rangle \phi^{(3)} \phi^{(2)} \cdot f_{(2)} \cdot \phi^{-(1)} \otimes \phi^{(3)} \phi^{-(3)} \cdot f_{(4)} \cdot \phi^{(2)}.
\]  

for \( f \in H^* \). We already know that \( \Delta_{D(H)} h = \Delta h \) for \( h \in H \). These more explicit formulae (12)–(16) for \( D(H) \) correspond directly to the characterisation of its representations in Lemma 2.1 and Proposition 2.2.

To illustrate this theory, we content ourselves with the simplest case, which is, however, the case relevant for conformal field theory so far. Thus, let \( G \) be a finite group and \( \phi \in Z^3(G) \), i.e. a 3-cocycle in the sense

\[
\phi(y, s, t) \phi(x, y, s, t) = \phi(x, y, s) \phi(x, y, s, t), \quad \phi(x, e, y) = 1
\]

for all \( x, y, s, t \in G \) and \( e \) the group identity element. Let \( k(G) \) be the Hopf algebra of functions on \( G \) with coproduct \( (\Delta f)(x, y) = f(xy) \). We view it as a quasi-Hopf algebra \( k^\phi(G) \) with \( \phi \in k(G)^{\otimes 3} \). We also make use of \( kG \), the group algebra of \( G \).

\[\text{We have added them here at the request of the referee. We note that in the meantime the recent preprint} \]

‘Doubles of quasi-quantum groups’ by F. Hauser and F. Nill, following up the preprint version of the present paper, provides some further explicit formulae for the above \( D(H) \).
Proposition 3.2 An object of the double category $(\mathcal{M}^\circ)_G$ is a $G$-graded vector space $V$ and a right cocycle action $\triangleleft$ of $kG$ on $V$ which is compatible with the grading, in the sense

$$(v \triangleleft x) \triangleleft y = \frac{\phi(x, y, y^{-1}x^{-1}y^{-1}|v||x||y)\phi(|v||x||y)}{\phi(x, x^{-1}|v||x||y)} v \triangleleft y, \quad v \triangleleft 1 = v, \quad |v \triangleleft x| = x^{-1}|v||x$$

for all $v \in V$ homogeneous of degree $|v|$ and $x, y \in G$. Morphisms in the category are linear maps preserving the right action and the grading. The tensor product grading and action are given by

$$|v \otimes w| = |v||w|, \quad (v \otimes w) \triangleleft x = \frac{\phi(x, x^{-1}|v||x||x^{-1}y|v)\phi(|v||x||y)}{\phi(|v||x||y)} v \triangleleft x \otimes w \triangleleft x,$$

for homogeneous $v, w$. We call this the category of cocycle crossed $G$-modules. It is braided, with

$$\Psi_{V \otimes W}(v \otimes w) = w \triangleleft |v||v \otimes v|.$$

**Proof** If $V$ is a $k(G)$-module, the projection operators given by the left action of Kronecker delta-functions $\delta_x$ provide a decomposition of $V$ into subspaces of degree $x$. Conversely, if $V$ is $G$-graded, it becomes a $k(G)$-module by $f \triangleright v = f(|v|)v$ on homogeneous elements. Next, we write $\beta_V$ in Proposition 2.2 as a right ‘quasi-action’ by $kG$ according to the correspondence $v \triangleleft x = \sum v^{(1)}(x)v^{(2)}$. The condition in Proposition 2.2 in these terms comes out as stated. It is easy to see that $\chi : G \times G \to k(G)$ defined by

$$\chi(x, y)(s) = \frac{\phi(x, y, y^{-1}x^{-1}s|xy)\phi(s, x, y)}{\phi(x, x^{-1}|x, y)}$$

is a 2-cocycle in $Z^2_{Ad}(G, k(G))$ with values in $k(G)$ as a left module induced by the group adjoint action. This is the sense in which $\triangleleft$ is a cocycle action. The remaining structure easily computes in this case as stated. □

This category generalises J.H.C. Whitehead’s notion of crossed $G$-sets to the case of a non-trivial 3-cocycle $\phi$. It is easy to identify the algebra, $D^\phi(G)$, say, with the representations of which this category can be identified. Namely, cocycle representations are naturally identified with modules of the corresponding cocycle cross product algebra.

**Proposition 3.3** The quasi-Hopf algebra double in Corollary 3.1 reduces in this example to the quasi-Hopf algebra $D^\phi(G) = kG^{\otimes 3}, \chi k(G)$ in [2]. Explicitly,

$$(x \otimes \delta_a) \cdot (y \otimes \delta_b) = yx \otimes \delta_b \delta_y s^{-1} \cdot \chi(y, x)(t)$$

$$\Delta(x \otimes \delta_a) = \sum_{ab=s} \frac{\phi(x, a^{-1}y^{-1}x^{-1}b, \phi(a, b, x) \cdot x \otimes \delta_a \otimes x \otimes \delta_b, \quad \epsilon(x \otimes \delta_a) = \delta_{s, e}}$$

and $\phi \in D^\phi(G)^{\otimes 3}$ by the standard inclusion. There is a quasitriangular structure $R = \sum (\delta_x \otimes 1) \otimes (1 \otimes x)$. 7
Proof This is a special case of Section 3. However, it is also easy enough to see directly in the present example. We view $\chi_{21}(x, y) = \chi(y, x)$ as a right-handed 2-cocycle on $G^{\text{op}}$ with values in $k(G)$ viewed as a right $G^{\text{op}}$-module (by the adjoint action). The 2-cocycle property means $\chi_{21}(x, y) \circ s \chi_{21}(xy, s) = \chi_{21}(y, s) \chi_{21}(x, ys)$. Hence by a standard construction for cocycle cross product algebras by group cocycle-actions\[13\], we have an algebra $kG^{\text{op}} \ltimes k(G)$ with product $(x \otimes f)(y \otimes h) = yx \otimes \chi(y, x)(f \circ y)h$ for $f, h \in k(G)$. This has the form shown on delta-functions. Moreover, the category in Proposition 4.1 can be identified with its left modules in the obvious way. If $V$ is an object of the category, the corresponding left module is $(x \otimes f) \triangleright v = v \triangleleft x f(|v|)$ on homogeneous elements. Given this identification, the tensor product of objects in Proposition 4.1 corresponds to the map coproduct shown. □

This is the content of our more formal Tannaka-Krein arguments in the finite groups setting. It is clear that the result is a special case of the general formulae (12)–(16) of $D(H)$. It should also be clear that this double quasi-Hopf algebra construction has an immediate generalisations to the case of an infinite-dimensional quasi-Hopf algebra and a dual quasi-Hopf algebra paired with it. (And we do not really require an antipode, provided the pairing is convolution-invertible in the usual sense.) Alternatively, one may work over $\mathbb{C}[[\hbar]]$ and take suitable topological duals as in the usual quantum double construction. In such a setting, an interesting project for further work would be to apply the double quasi-Hopf algebra construction to $H = (Ug, \phi)$, where $g$ is a complex semisimple Lie algebra and $\phi$ is Drinfeld’s 3-cocycle obtained by solving the KZ equation\[2\]. For example, when $g = su_2$, this provides in principle a quasi-Hopf algebra approach to the q-Lorentz group.

References

[1] V.G. Drinfeld. Quantum groups. In A. Gleason, editor, Proceedings of the ICM, pages 798–820, Rhode Island, 1987. AMS.

[2] V.G. Drinfeld. QuasiHopf algebras. Leningrad Math. J., 1:1419–1457, 1990.

[3] S. Majid. Representations, duals and quantum doubles of monoidal categories. Suppl. Rend. Circ. Mat. Palermo, Ser. II, 26:197–206, 1991.

[4] V.G. Drinfeld. Private communication, February, 1990.

[5] S. Majid. Tannaka-Krein theorem for quasiHopf algebras and other results. Contemp. Maths, 134:219–232, 1992.
[6] V. Pasquier R. Dijkgraaf and P. Roche. Quasi-quantum groups related to orbifold models. In Proc. Modern Quantum Field Theory, Tata Institute, Bombay, 1990, pages 375–383.

[7] R. Dijkgraaf and E. Witten. Topological gauge theories and group cohomology. Commun. Math. Phys, 129:393–429, 1990.

[8] S. Mac Lane. Categories for the Working Mathematician. Springer, 1974. GTM vol. 5.

[9] A. Joyal and R. Street. Braided monoidal categories. Mathematics Reports 86008, Macquarie University, 1986.

[10] S. Majid. Braided groups and duals of monoidal categories. Canad. Math. Soc. Conf. Proc., 13:329–343, 1992.

[11] S. Majid. Foundations of Quantum Group Theory. Cambridge University Press, 1995.

[12] J.H.C. Whitehead. Combinatorial homotopy, II. Bull. Amer. Math. Soc., 55:453–496, 1949.

[13] M.E. Sweedler. Cohomology of algebras over Hopf algebras. Ann. Math., pages 205–239, 1968.