Towards the Lax formulation of $SU(2)$ principal models with nonconstant metric

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Abstract

The equations that define the Lax pairs for generalized principal chiral models can be solved for any constant nondegenerate bilinear form on $su(2)$. The solution is dependent on one free variable that can serve as the spectral parameter. Necessary conditions for the nonconstant metric on $SU(2)$ that define the integrable models are given.

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1 Formulation of generalized principal chiral models

Principal chiral models are important example of relativistically invariant field theory. They are given by the action

\[ I[g] = - \int d^2 x \eta^{\mu \nu} L(A_\mu, A_\nu), \]  

(1)

where

\[ A_\mu := -i(g^{-1} \partial_\mu g) \in \mathcal{L}(G), \]  

(2)

\[ g : \mathbb{R}^2 \to G, \ \mu, \nu \in \{0, 1\}, \ \eta := \text{diag}(1, -1) \] and \( L \) is the Killing form on the corresponding Lie algebra \( \mathcal{L}(G) \). They are integrable by the inverse scattering method. Their Lax pair formulation was given in [1]. (For Lax formulation of related \( O(N) \)-sigma models see [2], [3].)

An immediate generalization of the principal chiral models is obtained when in the action \( (1) \) one considers a general bilinear form instead of the Killing \( (4) \). Such a type of models were introduced e.g. as a quasiclassical limit of the Baxter quantum XYZ model [5]. Next step of generalization \( (3) \) is introducing \( G \)-dependent symmetric bilinear forms into the action. In the coordinate dependent version the generalized principal chiral are defined by the action

\[ I[g] = \int d^2 x L_{ab}(g) \eta^{\mu \nu} (g^{-1} \partial_\mu g)^a (g^{-1} \partial_\nu g)^b, \]  

(3)

where \( L_{ab}(g) \) is matrix \( \text{dim}G \times \text{dim}G \) defined by the \( G \)-dependent bilinear form \( L(g) \) as

\[ L_{ab}(g) := L(g)(t_a \otimes t_b), \]  

(4)

and \( t_j \) are elements of a basis in the Lie algebra of the left-invariant fields. It is useful to consider the bilinear form \( L(g) \) as a metric on the group manifold. Lie products of elements of the basis define the structure coefficients

\[ [t_a, t_b] = i f_{ac} t_c \]  

(5)

and in the same basis we define the coordinates of the field \( A \)

\[ i A_\nu = (g^{-1} \partial_\nu g) = (g^{-1} \partial_\nu g)^b t_b = i A^b_\nu t_b. \]  

(6)
Varying the action \( \{ 3 \} \) w.r.t. \( \eta := g^{-1} \delta g \) we obtain equations of motion for the generalized principal chiral models

\[
\partial_\mu A^{\mu,a} + \Gamma^a_{bc} A^b_\mu A^{\mu,c} = 0 \tag{7}
\]

where

\[
\Gamma^a_{bc} := S^a_{bc} + \gamma^a_{bc}, \tag{8}
\]

and \( S^a_{bc} \) is the so called flat connection given by the structure coefficients

\[
S^a_{bc} := -\frac{1}{2} (F^a_{bc} + F^a_{cb}), \quad F^a_{bc} := (L^{-1})^{ap} f_{pbq} L^{q}_{c} \tag{9}
\]

while \( \gamma^a_{bc} \) is the Christoffel symbol for the metric \( L_{ab}(g) \) on the group manifold

\[
\gamma^a_{bc} := \frac{1}{2} (L^{-1})^{ad} (U_b L_{cd} + U_c L_{bd} - U_d L_{bc}) \tag{10}
\]

The vector fields \( U_a \) in (10) are defined in the local group coordinates \( \theta_i \) as

\[
U_a := U^i_a (\theta) \frac{\partial}{\partial \theta^i} \tag{11}
\]

where the matrix \( U \) is inverse to the matrix \( V \) of vielbein coordinates

\[
U^i_a (\theta) := (V^{-1})^i_a (\theta), \quad V^a_i (\theta) := -i (g^{-1} \frac{\partial g}{\partial \theta^i})^a. \tag{12}
\]

Note that the connection (8) is symmetric in the lower indices

\[
\Gamma^a_{bc} = \Gamma^a_{cb}. \tag{13}
\]

2 The Lax pairs

In the paper \([4]\), ansatz for the Lax formulation of the generalized chiral models was taken in the form

\[
[i \partial_0 + P_{ab} A^b_0 t_a + Q_{ab} A^b_1 t_a, \ i \partial_1 + P_{ab} A^b_1 t_a + Q_{ab} A^b_0 t_a] = 0 \tag{14}
\]

where \( P, Q \) are two auxiliary \( \text{dim} G \times \text{dim} G \) matrices. The ansatz (14) is a generalization of the Lax pair for \( L \) equal to the Killing form \( L_{ab} = Tr(t_a t_b) \) (where \( Q \) and \( P \) are multiples of the unit matrix) and for the anisotropic \( SU(2) \) model where \( L_{ab} := L_{a} \delta_{ab} \) (no summation)
with \( L_a = \text{const.}, \) \( L_1 = L_2 \neq L_3. \) A less general ansatz used in \[3\] leads just to the symmetric spaces, i.e. \( L_{ab} := \delta_{ab} \) for \( SU(2). \) Necessary conditions that the operators in \[4\] form the Lax pair for the equations of motion \( \Xi \) are

\[
P_{ab} f_{pq}^b + (P_{bp} P_{cq} - Q_{bp} Q_{cq}) f_{bc}^a = 0, \tag{15}
\]

\[
\frac{1}{2} f_{cd}^a (P_{cp} Q_{dq} + P_{cq} Q_{dp}) = Q_{ab} \Gamma_{pq}^b. \tag{16}
\]

If \( Q \) is invertible, that we shall assume in the following, then these conditions are also sufficient. Note that the first condition is independent of the bilinear form \( L \) so that one can start with solving the equation \( \Xi \) and then look for the bilinear forms \( L \) that admit solution of the equation \( \Omega. \)

### 2.1 Solution of the equation \( \Xi \)

The structure coefficients for \( su(2) \) can be chosen in terms of the totally antisymetric Levi-Civita tensor

\[
f_{ab}^c = \epsilon_{abc}. \tag{17}
\]

In this case the equation \( \Xi \) can be rewritten to the form

\[
(\text{Adj } Q)_{ab} = (\text{Adj } P)_{ab} + P_{ba} \tag{18}
\]

and solved by

\[
Q = \pm (P^T + \text{Adj } P)^{-1} / \sqrt{\text{det}(P^T + \text{Adj } P)}. \tag{19}
\]

where \( P^T \) is the transpose of \( P \) and elements \( (\text{Adj } N)_{ab} \) of the adjoint matrix to \( N \) are obtained as determinants of matrix \( N \) with dropped \( b \)-th row and \( a \)-th column multiplied by \( (-)^{a+b}. \) If we assume that \( Q \) is invertible then the solution \( \Xi \) is unique up to the sign.

Inserting \( \Xi \) into the conditions \( \Omega \) that remain to be solved we obtain rather complicated set of of \( P \) equations of the form

\[
G_{pq}^b(P) := \frac{1}{2} R_{ba} \epsilon_{cda} (P_{cp} R_{dq}^{-1} + P_{cq} R_{dp}^{-1}) = \Gamma_{pq}^b(L). \tag{20}
\]

where \( R = P^T + \text{Adj } P. \) As we shall see, the fact that the left–hand side is expressed only in terms of elements of the matrix \( P \) while the right–hand side only in terms of elements of the metric \( L \) imposes restrictions on the metric for which the above given form of Lax pair exists. On the other hand, solvability of these equations for \( L \) imposes conditions for \( P \) independent of \( L. \)
2.2 Conditions for the metric

As the matrix $L_{ab}$ is symmetric we can diagonalize it by orthogonal transformations and the structure coefficients \( (17) \) remain invariant. It means that for the $SU(2)$ models we can assume without loss of generality that $L$ is diagonal

$$L = \text{diag}(L_1(g), L_2(g), L_3(g)). \quad (21)$$

The right-hand sides of (20), i.e. elements of the connection $\Gamma(L)$ have rather special and simple form in this case.

$$\Gamma^a_{bc} = \Gamma^a_{cb} = \frac{L_b - L_c}{2L_a}, \text{ for } a \neq b, a \neq c, c \neq b. \quad (22)$$

$$\Gamma^a_{bb} = -\frac{U_b L_b}{2L_a}, \text{ for } a \neq b, \text{ no sums.} \quad (23)$$

$$\Gamma^a_{ab} = \Gamma^a_{ba} = \frac{U_b L_a}{2L_b}, \text{ no sums} \quad (24)$$

On the other hand, the left-hand sides of (20) satisfy two important identities, namely

$$G^a_{ab} = 0, \quad G^a_{bb} = 0 \quad (25)$$

that hold for arbitrary $P$. Indeed, using the antisymmetry of Levi-Civita symbol $\epsilon$ and (20) one gets

$$G^b_{bq} = \frac{1}{2} R_{ba} \epsilon_{cda} P_{cb} R_{dq}^{-1} = \frac{1}{2} (\text{Adj } P)_{ba} \epsilon_{cda} P_{cb} R_{dq}^{-1} = \delta_{ca} \epsilon_{cda} R_{dq}^{-1} \det P = 0$$

Proof of the second identity in (25) can be done e.g. by expressing both inverse and adjoint matrix via Levi-Civita symbol $\epsilon$ but it is very tedious. The easiest way is to check it by computer.

From (25b), (20) and (23) we get

$$U_1(L_1 - L_2 - L_3) = 0 \text{ and cyclic permutations of } (1, 2, 3) \quad (26)$$

wherefrom we find that

$$L_1(g) = f_2(g) + f_3(g) \text{ and cyclic permutations of } (1, 2, 3) \quad (27)$$

where the functions $f_1, f_2, f_3$ are invariant with respect to the fields $U_1, U_2, U_3$, respectively. From (25a), (20) and (24) we get

$$\sum_{b=1}^{3} U_a \log L_b = 0 \forall a \Rightarrow \det L = L_1 L_2 L_3 = \text{const.} \quad (28)$$

because the the fields $U_1, U_2, U_3$ form a basis in $T_g G$. 

5
2.3 Conditions for the matrix $P$

Comparing (24) and (23) we immediately see that the following identities hold for arbitrary diagonal metric and $a \neq b$

$$L_a \Gamma^a_{bb} + L_b \Gamma^b_{ba} = 0, \text{ no sums} \quad (29)$$

Using the equation (20), we can replace $\Gamma^a_{bc}$ by $G^a_{bc}$ and obtain a set of six linear equations for $L_j$. Their solvability then yields three algebraic equations for elements of the matrix $P$

$$G^a_{bb}G^b_{aa} = G^b_{ba}G^a_{ba}, \text{ no sums.} \quad (30)$$

Another set of conditions for $G^a_{bc}$ can be obtained from (20) for $a \neq b, a \neq c, c \neq b$ because in these cases we get three linear equations for $L_j$

$$2L_a G^a_{cb} = L_b - L_c \text{ no sums} \quad (31)$$

due to (22). The solvability condition for these equations reads

$$G^1_{23} G^2_{13} G^3_{12} + G^1_{23} + G^2_{13} + G^3_{12} = 0. \quad (32)$$

Note that the equations (30) and (32) are independent of $L$. Unfortunately, they are highly nonlinear in elements of $P$ and it seems impossible to solve them without an ansatz.

3 Solutions for general constant metric

As it was mentioned above, without loss of generality we can assume that the metric $L$ is diagonal. We shall prove that we can satisfy the conditions (15), (16) for any constant diagonal $L$ by matrices $P, Q$ containing one free (“spectral”) parameter. It means that any generalized principal $SU(2)$ model with constant metric has a Lax pair.

3.1 Diagonal ansatz

The most natural extensions of the results obtained e.g. in [1] and [4] is the diagonal ansatz for $P$

$$P = \text{diag}(P_1, P_2, P_3). \quad (33)$$
It is rather easy to check that in this case the conditions (32) and (30) for the matrix $P$ are satisfied identically.

Inserting (33) into (19) one can immediately see that the matrix $Q$ is diagonal as well, namely $Q = \text{diag}(Q_1, Q_2, Q_3)$ where

$$Q_1 = \pm \sqrt{\frac{R_{22}R_{33}}{R_{11}}}, \quad Q_2 = \frac{R_{33}}{Q_1}, \quad Q_3 = \frac{R_{22}}{Q_1}.$$ (34)

and

$$R_{11} = P_2P_3 + P_1, \quad R_{22} = P_3P_1 + P_2, \quad R_{33} = P_1P_2 + P_3.$$ (35)

Inserting (33), (34) and (35) into the left-hand side of (20) for $p = q$ and $p = b$ we get zero and using (24), (23) we find that the metric must be constant in this case.

The equations (20) for diagonal $P$ reduce to three nonlinear non-homogeneous equations for $P_j$

$$E_1 := P_1P_2[\sigma_3(P_3^2 - 1) + P_2^2 - P_1^2] + P_3[(P_1^2 + P_2^2)\sigma_3 + P_2^2 - P_1^2] = 0, \quad (36)$$

$$E_2 := P_2P_3[\sigma_1(P_1^2 - 1) + P_3^2 - P_2^2] + P_1[(P_2^2 + P_3^2)\sigma_1 + P_3^2 - P_2^2] = 0, \quad (37)$$

$$E_3 := P_3P_1[\sigma_2(P_2^2 - 1) + P_1^2 - P_3^2] + P_2[(P_3^2 + P_1^2)\sigma_2 + P_1^2 - P_3^2] = 0. \quad (38)$$

These equations, where

$$\sigma_1 = \frac{L_2 - L_3}{L_1}, \quad \sigma_2 = \frac{L_3 - L_1}{L_2}, \quad \sigma_3 = \frac{L_1 - L_2}{L_3},$$

are not independent because the following relation holds identically

$$E_1P_1L_1(-L_1 + L_2 + L_3) + E_2P_2L_2(-L_2 + L_3 + L_1) + E_3P_3L_3(-L_3 + L_1 + L_2) = 0$$ (39)

and that’s why the variety of solutions of (36)–(38) has the dimension one. The solution curves can be written as

$$P_1 = \kappa_1 \frac{\sqrt{\mu + L_2} \sqrt{\mu + L_3}}{\sqrt{L_2} \sqrt{L_3}}, \quad P_2 = \kappa_2 \frac{\sqrt{\mu + L_3} \sqrt{\mu + L_1}}{\sqrt{L_3} \sqrt{L_1}},$$

$$P_3 = \kappa_3 \frac{\sqrt{\mu + L_1} \sqrt{\mu + L_2}}{\sqrt{L_1} \sqrt{L_2}}.$$ (40)

where $\mu$ is a free parameter and

$$\kappa_1^2 = \kappa_2^2 = 1, \quad \kappa_3 = \kappa_1 \kappa_2.$$
Inserting (40) into (34) we get

\[ Q_1 = \omega_1 \frac{\sqrt{\mu^2 + L_1^2}}{\sqrt{L_2 L_3}}, \quad Q_2 = \omega_2 \frac{\sqrt{\mu^2 + L_2^2}}{\sqrt{L_3 L_1}}, \quad Q_3 = \omega_3 \frac{\sqrt{\mu^2 + L_3^2}}{\sqrt{L_1 L_2}}, \]

where \( \omega_1 = \kappa_2 \omega_3, \quad \omega_2 = \kappa_1 \omega_3, \quad \omega_3^2 = 1. \)

The formulas (33), (34), (40), and (41) yield the solution of the equations (15), (16) for

\[ L_{ab} = L_a \delta_{ab}, \quad f_{abc} = \epsilon_{abc} \]

and up to an eventual orthogonal transformation of the algebra basis, they define the Lax pair for the generalized principal SU(2) chiral model with the constant anisotropic metric. It is easy to check that for \( L_1 = L_2 = L_3 \) and \( L_1 = L_2 \neq L_3 \), the Lax pairs coincide with the previously known cases [4].

### 3.2 Block–diagonal ansatz

Our next goal is to find model with the \( G \)-dependent metric. As it follows from the preceding subsection, the only possibility is the non–diagonal matrix \( P \). On the other hand the calculations with the general matrix seem hopelessly complicated so that we can try the block–diagonal form

\[ P = \begin{pmatrix} p_1 & b_1 & 0 \\ b_2 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}. \]

However, as we shall see, this ansatz leads again to the constant metric.

Inserting (43) into (19) we find that the matrix \( Q \) has the same block–diagonal form as \( P \). The conditions (30) are satisfied identically for block–diagonal \( P \) while the equation (32) now reads

\[ 0 = p_3 (b_1 p_1 + b_2 p_2) \left( b_1^2 - b_2^2 - p_1^2 + p_2^2 \right) \times \left( 2 p_1 p_2 - 2 b_1 b_2 + p_3 \left( 1 + b_1^2 + b_2^2 + p_1^2 + p_2^2 - p_3^2 \right) \right). \]

On the other hand, from (24), (24) and (23) we find that

\[ U_3 L_k = (-)^k L_k p_3 (b_1 p_1 + b_2 p_2) \times \]
\[(2 p_1 p_2 - 2 b_1 b_2 + p_3(1 + b_1^2 + b_2^2 + p_1^2 + p_2^2) - p_3^2)) \]  
for \(k = 1, 2\) and \(U_a L_b = 0\) for all other combinations of indices. Comparing (44) and (45) we can see that the Lax pair for nonconstant metric can exist only if

\[b_1^2 - b_2^2 = p_1^2 - p_2^2\]  
(46)

Unfortunately in this case \(G_{12}^3 = 0\) so that from (20) and (22) we get \(L_1 = L_2\) and from (45) we find that the metric must be constant.

4 Conclusions

The ansatz of the form (14) for the Lax pair formulation of the generalized principal chiral model (3) implies rather complicated set of equations (15), (16) for elements of the matrices \(P, Q\) and metric \(L\).

For the group \(SU(2)\) the matrix \(Q\) can be solved in terms of \(P\) and one can derive admissible group dependence of the metric \(L\) under which the the Lax pair for the generalized principal chiral model may exist.

It seems that there is no other way to solve the equations (15), (16) but using an ansatz. Using the diagonal form of the matrix \(P\) we have found the explicit form of the Lax pair with the spectral parameter for the general anisotropic \(SU(2)\) model with constant metric and then we have proved that the Lax pair for the nonconstant metric (if it exists in the form (14)) requires a more general form of the matrix \(P\) than the block–diagonal (13). Unfortunately, all more general forms that were tried (Jordan form and others) did not simplify the equations for \(P\) and \(L\) to a form that would be solvable.

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