INHOMOGENEOUS PARABOLIC NEUMANN PROBLEMS

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Abstract. We study second order parabolic equations on Lipschitz domains subject to inhomogeneous Neumann (or, more generally, Robin) boundary conditions. We prove existence and uniqueness of weak solutions and their continuity up to the boundary of the parabolic cylinder. Under natural assumptions on the coefficients and the inhomogeneity we can also prove convergence to an equilibrium or asymptotic almost periodicity.

1. Introduction

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$. Our model problem is the heat equation

$$
\begin{cases}
 u_t(t, x) - \Delta u(t, x) = f(t, x), & t > 0, \ x \in \Omega \\
 \frac{\partial u(t, z)}{\partial \nu} = g(t, z), & t > 0, \ z \in \partial \Omega \\
u(0, x) = u_0(x), & x \in \Omega
\end{cases}
$$

subject to inhomogeneous Neumann boundary conditions. The above problem has a unique weak solution in an $L^2$-sense if $f$, $g$ and $u_0$ are square-integrable. We are interested in its regularity at the boundary and its asymptotic behavior. Such problems appear in a natural way in control theory [5, 6] or thermal imaging [7].

More precisely, we show the following: if $u_0$ is continuous and $f$ and $g$ satisfy some integrability conditions, then the solution $u$ is continuous up to the boundary of the parabolic cylinder; if $f$ and $g$ converge to zero in a time-averaged sense, then $u$ converges to zero uniformly on $\Omega$; finally, if $f$ and $g$ are almost periodic functions, then $u$ is asymptotically almost periodic with essentially the same frequencies.

Even though the heat equation will be our model case, we will admit general strongly elliptic operators subject to Robin boundary conditions in all of our results. For homogeneous boundary conditions, i.e., if $g = 0$, these problem are well understood and can be studied by semigroup methods. Inhomogeneous boundary conditions, however, are more delicate. For smooth data, some existence and regularity results can be found in [13, Theorem 5.18] or [10]. Existence of weak solution is shown in [10, §4.15.3]. Regularity theory in $L^p$-spaces for the inhomogeneous elliptic Neumann problem can be found for example in [13, 16] and for the parabolic Neumann problem in [23], both with a different emphasis. Asymptotic almost periodicity has been studied in [1] for the inhomogeneous Dirichlet problem.

In order to study the asymptotic behavior we want to follow a semigroup approach by considering the equation as an abstract Cauchy problem in a suitable space, which is adapted to the boundary data. To this end one could use spaces of distributions that contain functionals arising from boundary integrals, a strategy...
which has been pursued with negative exponent Sobolev spaces \([14]\) and Sobolev-Morrey spaces \([13]\). This approach, however, has the disadvantage that a priori the solutions no more regular than generic elements of these spaces, whereas it would be favorable to have continuous functions as solutions. The parabolic structure of the equation does not immediately help because a gain in regularity is not obvious in presence of the inhomogeneities. The regularity matters in particular in the limits \(t \to 0\) and \(t \to \infty\) since semigroup methods provide us typically with convergence in the norm of the underlying space.

In view of these considerations we aim towards results in the space \(C(\overline{\Omega})\). Existence is however much more convenient in \(L^2(\Omega)\), which is why we will start out by considering \(L^2\)-solution. By using \(C(\overline{\Omega})\) we are able to obtain uniform convergence of \(u\) on \(\overline{\Omega}\) as \(t \to 0\) and as \(t \to \infty\), or more generally asymptotic almost periodicity. This seems to be completely new for Neumann boundary conditions and is our main result.

Our strategy is the following. When formulating the initial-boundary value problem as an abstract Cauchy problem on \(L^2(\Omega)\) or \(C(\overline{\Omega})\), we switch to a product space. More precisely, we regard the inhomogeneous heat equation as an inhomogeneous abstract Cauchy problem for the operator \(A\) given by \(A(u,0) = (\Delta u, -\partial u/\partial \nu)\) in the space \(L^2(\Omega) \times L^2(\partial \Omega)\). This operator \(A\) is not densely defined and hence not the generator of a strongly continuous semigroup. In fact, it turns out that \(A\) does not even satisfy the Hille-Yosida estimates. Still, the operator is resolvent positive and hence generates a once integrated semigroup. This implies existence and uniqueness of solutions for regular right hand sides \(f\) and \(g\) and gives information about the asymptotic behavior of solutions. These results can be extended to a larger class of less regular right hand sides once we obtain suitable a priori estimates.

The idea to consider a non-densely defined operator \(A\) on a product space in order to treat inhomogeneous boundary conditions has first been used by Arendt for the study of the heat equation with inhomogeneous Dirichlet boundary conditions \([1]\). Here we copy the skeleton of his proofs. The details are however quite different, the main aspects being the following:

1. We restrict ourselves to Lipschitz domains, which is the usual framework for Neumann problems, whereas one of Arendt’s main points are the optimal boundary regularity assumptions.
2. Our a priori estimate needs more sophisticated methods, whereas for the Dirichlet problem it is a consequence of the parabolic maximum principle.
3. The Neumann problem has a smoothing effect with respect to the boundary conditions, which allows us to obtain continuous solutions even for non-smooth functions \(g\), whereas for Dirichlet problems the boundary has to be continuous. The latter fact is reflected in various places. It explains for example why for the Neumann problem the solution is asymptotically almost periodic in the sense of Bohr even if the right hand side is almost periodic only in the sense of Stepanoff, whereas for the Dirichlet problem this cannot hold.

The article is organized as follows. In Section 2 we introduce the initial-boundary value problem. We show existence and uniqueness of solutions and discuss the relationship between three different notions of solutions. Section 3 contains results and pointwise estimates for the solutions as well as their continuity. The most technical part of this section is however postponed to Appendix A in the hope that this improves the readability of the article as a whole. In Section 4 we study the convergence of solutions. More precisely, we give natural sufficient conditions for the solution to be bounded or to converge to a constant function. Finally, in Section 5 we show that for asymptotically almost periodic right hand sides in the
Throughout the article we will always refer to the inhomogeneous Robin problem

\[ u \in L^q(\Omega), \quad b_j, c_i \in L^q(\Omega), \quad d \in L^2(\Omega) \quad \text{and} \quad \beta \in L^{q-1}(\partial\Omega) \]

be given, where \( q > N \) is arbitrary, and assume that there exists \( \mu > 0 \) such that

\[
\sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \geq \mu |\xi|^2 \quad \text{for all} \ \xi \in \mathbb{R}^N.
\]  

(2.1)

Throughout the article we will always refer to the inhomogeneous Robin problem

\[
(P_{u_0, f, g}) \quad \begin{cases}
\begin{aligned}
& u_t(t, x) - Au(t, x) = f(t, x), \quad t > 0, \ x \in \Omega \\
& \partial u(t, z) + \beta u(t, z) = g(t, z), \quad t > 0, \ z \in \partial\Omega \ \\
& u(0, x) = u_0(x), \quad x \in \Omega,
\end{aligned}
\end{cases}
\]

(2.2)

with given \( u_0 \in L^2(\Omega) \), \( f \in L^2(0, T; L^2(\Omega)) \) and \( g \in L^2(0, T; L^2(\partial\Omega)) \). Here, at least on a formal level,

\[
Au := \sum_{j=1}^{N} D_j \left( \sum_{i=1}^{N} a_{ij} D_i u + b_j u \right) - \left( \sum_{i=1}^{N} c_i D_i u + d u \right)
\]

\[
\frac{\partial u}{\partial \nu_A} := \sum_{j=1}^{N} \left( \sum_{i=1}^{N} a_{ij} D_i u + b_j u \right) \nu_j,
\]

where \( \nu = (\nu_j)_{j=1}^{N} \) denotes the outer unit normal of \( \Omega \) at the boundary \( \partial\Omega \). It is convenient to introduce also the bilinear forms

\[
a_0(u, v) := \int_{\Omega} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} a_{ij} D_i u + b_j u \right) D_j v + \int_{\Omega} \left( \sum_{i=1}^{N} c_i D_i u + d u \right) v
\]

(2.3)

\[
a_{\beta}(u, v) := a_0(u, v) + \int_{\partial\Omega} \beta u v
\]

(2.4)

for \( u \) and \( v \) in \( H^1(\Omega) \), where \( H^1(\Omega) \) refers to the Sobolev space of all functions in \( L^2(\Omega) \) whose first derivative also lie in \( L^2(\Omega) \).

We introduce and compare various notions for a solution of \( (P_{u_0, f, g}) \), which are based on the observation that on a formal level the divergence theorem gives

\[
a_0(u, v) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu_A} v - \int_{\Omega} Au \ v
\]

(2.5)

for all \( v \in H^1(\Omega) \). A weak solution is now defined by testing against a smooth function and formally integrating by parts.

**Definition 2.1.** We say that a function \( u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \) is a weak solution of \( (P_{u_0, f, g}) \) on \([0, T]\) for some \( T > 0 \) if

\[
-\int_{0}^{T} \int_{\Omega} u(s) \psi_t(s) + \int_{0}^{T} a_{\beta}(u(s), \psi(s)) = \int_{0}^{T} \int_{\Omega} f(s) \psi(s) + \int_{0}^{T} \int_{\partial\Omega} g(s) \psi(s)
\]

(2.6)

for all \( \psi \in H^1(0, T; H^1(\Omega)) \) that satisfy \( \psi(T) = 0 \).
We say that a function \( u : [0, \infty) \to L^2(\Omega) \) is a weak solution of \((P_{a_0, f, g})\) on [0, \infty) if for every \( T > 0 \) its restriction to \([0, T]\) is a weak solution on \([0, T]\).

In order to give two further definitions of a solution, we first introduce the \( L^2 \)-realization \( A_2 \) of \( A \) with Robin boundary conditions, which is also based on (2.5).

**Definition 2.2.**
(a) Let \( u \in H^1(\Omega) \). We say that \( Au \in L^2(\Omega) \) if there exists a (necessarily unique) function \( f \in L^2(\Omega) \) satisfying \( a_0(u, \eta) = -\int_{\Omega} f \eta \) for all \( \eta \in H^1_0(\Omega) \). In this case we define \( Au := f \).
(b) Let \( u \in H^1(\Omega) \) satisfy \( Au \in L^2(\Omega) \). We say that \( \frac{\partial u}{\partial \nu_{A'}} \in L^2(\Omega) \) if there exists a (necessarily unique) function \( g \in L^2(\partial\Omega) \) satisfying \( a_0(u, \eta) = \int_{\partial\Omega} g \eta - \int_{\Omega} Au \eta \) for all \( \eta \in H^1(\Omega) \). In this case we define \( \frac{\partial u}{\partial \nu_{A'}} := g \).
(c) We define the operator \( A_2 \) on the space \( L^2(\Omega) \times L^2(\partial\Omega) \) by

\[
D(A_2) := \left\{ (u, 0) : u \in H^1(\Omega), \; \begin{pmatrix} u \\ \frac{\partial u}{\partial \nu_{A'}} \end{pmatrix} \in L^2(\partial\Omega) \right\}
\]

\[
A_2(u, 0) := \left( Au, \frac{\partial u}{\partial \nu_{A'}} - \beta u|_{\partial\Omega} \right).
\]

**Remark 2.3.** It is easily checked that \((u, 0) \in D(A_2)\) with \(-A_2(u, 0) = (f, g)\) if and only if

\[
a_\beta(u, v) = \int_{\Omega} f v + \int_{\partial\Omega} g v
\]

for all \( v \in H^1(\Omega) \).

It is an exercise in applying Hölder’s inequality, the Sobolev embedding theorems and Young’s inequality to prove that there exists \( \omega \geq 0 \) such that

\[
a_\beta(u, u) \geq \frac{\mu}{2} \int_{\Omega} |\nabla u|^2 - \omega \int_{\Omega} |u|^2
\]

(2.7)

for all \( u \in H^1(\Omega) \). We leave the verification to the reader.

Next we collect a few facts about \( A_2 \).

**Lemma 2.4.** The operator \( A_2 \) is resolvent positive. More precisely, the operator \( \lambda - A_2 : D(A_2) \to L^2(\Omega) \times L^2(\partial\Omega) \) is invertible for all \( \lambda > \omega \), where \( \omega \) is as in (2.7), and if \( A_2(u, 0) = (f, g) \) with non-negative functions \( f \in L^2(\Omega) \) and \( g \in L^2(\partial\Omega) \), then \( u \geq 0 \) almost everywhere. Moreover, if \( D(A_2) \) is equipped with the graph norm, then \( D(A_2) \) is continuously embedded into \( H^1(\Omega) \times \{0\} \).

**Proof.** Let \( \omega \) be as in (2.7) and fix \( \lambda > \omega \). Then

\[
\lambda \int_{\Omega} |u|^2 + a_\beta(u, u) \geq \alpha \|u\|^2_{H^1(\Omega)}
\]

(2.8)

for all \( u \in H^1(\Omega) \) with \( \alpha := \min\{\lambda - \omega, \frac{\mu}{2}\} > 0 \). Hence by the Lax-Milgram theorem \([12, \S 5.8]\) for every \( f \in L^2(\Omega) \) and \( g \in L^2(\partial\Omega) \) there exists a unique function \( u \in H^1(\Omega) \) such that

\[
\lambda \int_{\Omega} u v + a_\beta(u, v) = \int_{\Omega} f v + \int_{\partial\Omega} g v
\]

(2.9)

for all \( v \in H^1(\Omega) \). By Remark 2.3 this means precisely that there is a unique function \( u \in H^1(\Omega) \) with \((u, 0) \in D(A_2)\) and

\[
(\lambda - A_2)(u, 0) = (\lambda u, 0) - A_2(u, 0) = (f, g).
\]
Theorem 2.6. Let either

(a) Let \( I \) so it suffices to consider the case

\[ L(u) \]

(b) Every classical

\[ u \]

We say that a function \( u \)

\[ D_j v = D_j u \mathbf{1}_{\{u > 0\}} \quad \text{and} \quad v|_{\partial \Omega} = u|_{\partial \Omega} \mathbf{1}_{\{u|_{\partial \Omega} > 0\}} \]

and hence

\[
0 \geq \int_{\Omega} fv + \int_{\partial \Omega} gv = \lambda \int_{\Omega} uv + a_\beta(u, v) = \lambda \int_{\Omega} |v|^2 + a_\beta(v, v) \geq 0
\]

by (2.9). By (2.8) this shows that \( u \leq 0 \) almost everywhere. We have shown that the resolvent \((\lambda - A_2)^{-1}\) is a positive operator. Since every positive operator is continuous \([3]\) we deduce that \( \lambda - A_2 \) is in fact invertible.

In particular we have proved that \( A_2 \) is closed. Hence \( D(A_2) \) is a Banach space for the graph norm of \( A_2 \), and by definition of \( A_2 \) we have \( D(A_2) \subset H^1(\Omega) \times \{0\} \).

Since both of these spaces are continuously embedded into \( L^2(\Omega) \times L^2(\partial \Omega) \), we deduce from the closed graph theorem that \( D(A_2) \) is continuously embedded into \( H^1(\Omega) \times \{0\} \).

We always equip \( D(A_2) \) with the graph norm.

Now we can define mild and classical solutions of \((P_{u_0, f, g})\). The definition of a classical solution is obtained by writing \((P_{u_0, f, g})\) in terms of \( A_2 \) in a straightforward way, assuming smoothness in the time variable. The definition of a mild solution is similar, but uses an integrated form of the equation. These two notions are the most common ones in the study of abstract Cauchy problems.

Definition 2.5. Let \( I = [0, T] \) for some \( T > 0 \), or let \( I = [0, \infty) \).

(a) We say that a function \( u \) is a classical \( L^2 \)-solution of \((P_{u_0, f, g})\) on \( I \) if \( u \) is in \( C^1(I; L^2(\Omega)) \), we have \( u(0) = u_0 \), the mapping \( t \mapsto (u(t), 0) \) is in \( C(I; D(A_2)) \) and the relation

\[
(u_t(t), 0) - A_2(u(t), 0) = (f(t), 0) \tag{2.10}
\]

holds for all \( t \in I \).

(b) We say that a function \( u \) is a mild \( L^2 \)-solution of \((P_{u_0, f, g})\) on \( I \) if \( u \) is in \( C(I; L^2(\Omega)) \), \((\int_0^t u(s), 0) \in D(A_2) \) for all \( t \geq 0 \) and

\[
(u(t) - u_0, 0) - A_2\left(\int_0^t u(s), 0\right) = \left(\int_0^t f(s), \int_0^t g(s)\right) \tag{2.11}
\]

for all \( t \geq 0 \).

It will turn out later that weak solutions and mild \( L^2 \)-solutions are in fact the same. Let us start with an easy relationship between the three notions of a solution.

Theorem 2.6. Let either \( I = [0, T] \) with \( T > 0 \) or \( I = [0, \infty) \).

(a) Every classical \( L^2 \)-solution of \((P_{u_0, f, g})\) on \( I \) is a weak solution on \( I \).

(b) Every weak solution of \((P_{u_0, f, g})\) on \( I \) is a mild \( L^2 \)-solution on \( I \).

Proof. All three definitions depend only on the behavior of \( u \) on bounded intervals, so it suffices to consider the case \( I = [0, T] \).

(a) Let \( u \) be a classical \( L^2 \)-solution. Then \( u \in C([0, T]; H^1(\Omega)) \) by Lemma 2.4 which shows that \( u \) has the regularity requested in Definition 2.4. Let \( \psi \) be in \( \mathcal{H}^1(0, T; H^1(\Omega)) \) and satisfy \( \psi(T) = 0 \). From (2.10) and Remark 2.3 we obtain that

\[
\int_{\Omega} u_t(t) \psi(t) + a_\beta(u(t), \psi(t)) = \int_{\Omega} f(t) \psi(t) + \int_{\partial \Omega} g(t) \psi(t)
\]

for all \( t \in [0, T] \). Integrating over \([0, T]\) and integrating the first summand by parts this gives (2.6).
(b) Let $u$ be a weak solution. Fix functions $\varphi \in H^1(0, T)$ and $\eta \in H^1(\Omega)$, where $\varphi(T) = 0$. Define $\psi(t) := \varphi(t) \cdot \eta$. Then $\psi \in H^1(0, T; H^1(\Omega))$ with $\psi(T) = 0$ and hence

$$-\int_0^T \left( \int_\Omega u(s) \right) \varphi(t) \eta = \left( \int_\Omega u_0 \eta \right) \varphi(0) + \int_0^T \left( -a_\beta(u(s), \eta) + \int_\Omega f(s) \eta + \int_{\partial\Omega} g(s) \eta \right) \varphi(s)$$

by (2.6). Hence $t \mapsto \int_\Omega u(t) \eta$ is weakly differentiable for all $\eta \in H^1(\Omega)$ with weak derivative

$$\frac{d}{dt} \int_\Omega u(t) \eta = -a_\beta(u(s), \eta) + \int_\Omega f(s) \eta + \int_{\partial\Omega} g(t) \eta.$$ 

and initial value $\int_\Omega u(0) \eta = \int_\Omega u_0 \eta$, hence $u(0) = u_0$. We deduce that

$$\int_\Omega u(t) \eta = \int_\Omega u_0 \eta + \int_0^t \left( -a_\beta(u(s), \eta) + \int_\Omega f(s) \eta + \int_{\partial\Omega} g(s) \eta \right)$$

for all $t \in [0, T]$ and all $\eta \in H^1(\Omega)$. Since $u \in L^2(0, T; H^1(\Omega))$ and $v \mapsto a_\beta(v, \eta)$ is a continuous linear functional on $H^1(\Omega)$, this implies that

$$\int_\Omega (u(t) - u_0) \eta + a_\beta \left( \int_0^t u(s), \eta \right) = \int_\Omega \left( \int_0^t f(s) \right) \eta + \int_{\partial\Omega} \left( \int_0^t g(s) \right) \eta$$

for all $\eta \in H^1(\Omega)$. Hence by Remark 2.3, the function $u$ is a weak solution. □

We want to establish the existence of a weak solution via the theory of resolvent positive operators. Since $L^2(\Omega) \times L^2(\partial\Omega)$ is a Banach lattice with order continuous norm, the resolvent positive operator $A_2$ generates a once integrated semigroup [2, Theorem 3.11.7]. This yields the following existence, uniqueness and comparison results for $L^2$-solutions.

**Proposition 2.7.** Let $u_0 \in L^2(\Omega)$, $f \in L^2(0, T; L^2(\Omega))$ and $g \in L^2(0, T; L^2(\partial\Omega))$ for some $T > 0$.

(a) Problem $(P_{u_0, f, g})$ has at most one mild $L^2$-solution.

(b) Assume that $u_0 \in L^2(\Omega)$ satisfies $Au_0 \in L^2(\partial\Omega)$ and that $f \in C^2([0, T]; L^2(\Omega))$ and $g \in C^2([0, T]; L^2(\partial\Omega))$. If $u_0 + \beta u_0 = g(0)$ holds and $v := Au_0 + f(0) \in L^2(\Omega)$ satisfies $Av \in L^2(\Omega)$ and $\frac{\partial}{\partial \nu} u_0 \in L^2(\partial\Omega)$, then $(P_{u_0, f, g})$ has a classical $L^2$-solution.

(c) Assume that $u_0 \geq 0$, $f(t) \geq 0$ and $g(t) \geq 0$ almost everywhere for almost every $t \in (0, T)$. If $u$ is a mild $L^2$-solution of $(P_{u_0, f, g})$, then $u(t) \geq 0$ almost everywhere for every $t \in (0, T)$.

**Proof.** By Definition 2.5 a function $u$ is a mild (resp.: classical) $L^2$-solution of $(P_{u_0, f, g})$ if and only if the mapping $t \mapsto (u(t), 0)$ is a mild (resp.: classical) solution of the abstract Cauchy problem associated with $A_2$ with inhomogeneity $(f, g)$, confer [2, §3.1]. Hence part (a) follows from [2, Theorem 3.11.11]. This implies in particular that $u = 0$ is the unique mild $L^2$-solution if $u_0 = 0$, $f = 0$ and $g = 0$, so part (b) follows from the linearity of the equation. Finally, the conditions on $u_0$ in part (a) can be rephrased by saying that

$$(u_0, 0) \in D(A_2) \quad \text{and} \quad A_2(u_0, 0) + (f(0), g(0)) \in D(A_2).$$

Hence the existence of a classical $L^2$-solutions follows from [2, Corollary 3.2.11]. □

We want to show that for all square-integrable functions $u_0$, $f$ and $g$ we have a unique weak solution. As a first step we prove a bound for classical $L^2$-solutions in the norm of $C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$. 
Lemma 2.8. If $u$ is a classical $L^2$-solution of $(P_{u_0,f,g})$ on $[0,T]$ for some $T > 0$, then
\[
\sup_{0 \leq t \leq T} \int_{\Omega} |u(t)|^2 + \int_0^T \int_{\Omega} |\nabla u|^2 \leq c \int_{\Omega} |u_0|^2 + c \int_0^T \int_{\Omega} |f(t)|^2 + c \int_0^T \int_{\partial\Omega} |g(t)|^2
\]
for a constant $c \geq 0$ that depends only on $T, \Omega$ and the values $\mu$ and $\omega$ in \ref{2.12}.

Proof. Let $t \in [0,T]$ be arbitrary. Then
\[
\frac{1}{2} \int_{\Omega} |u(t)|^2 - \frac{1}{2} \int_{\Omega} |u_0|^2 = \frac{1}{2} \int_0^t \frac{d}{ds} \int_{\Omega} |u(s)|^2 = \int_0^t \int_{\Omega} u(s) u_t(s)
\]
\[
= \int_0^t \int_{\Omega} \partial_t u(s) [Au(s) + f(s)]
\]
\[
= \int_0^t \int_{\partial\Omega} \frac{\partial u}{\partial n} u(s) - \int_0^t a_0(u(s),u(s)) + \int_0^t \int_{\Omega} f(s) u(s) = \int_0^t \int_{\Omega} f(s) u(s) + \int_0^t \int_{\partial\Omega} g(s) u(s) - \int_0^t \int_{\Omega} a_0(u(s),u(s))
\]
\[
\leq \frac{1}{2} \int_0^t \int_{\Omega} |f(s)|^2 + \frac{1}{4c} \int_0^t \int_{\partial\Omega} |g(s)|^2
\]
\[
- (\frac{\mu}{2} - \varepsilon c_1^2) \int_0^t \int_{\Omega} \nabla u(s)^2 + (\omega + \varepsilon c_1^2) \int_0^t \int_{\Omega} |u(s)|^2,
\]
where we have used Young’s inequality and \ref{2.12}. Here $c_1 \geq 0$ is the norm of the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$. We pick $\varepsilon := \frac{\mu}{4c}$ and vary over $t$ to deduce that
\[
\sup_{0 \leq s \leq t} \int_{\Omega} |u(s)|^2 + \int_0^t \int_{\Omega} |\nabla u(s)|^2
\]
\[
\leq c_2 \int_{\Omega} |u_0|^2 + c_2 \int_0^t \int_{\Omega} |f(s)|^2 + c_2 \int_0^t \int_{\partial\Omega} |g(s)|^2 + c_2 \int_0^t \int_{\Omega} |u(s)|^2
\]
\[
\leq c_2 \int_{\Omega} |u_0|^2 + c_2 \int_0^t \int_{\Omega} |f(s)|^2 + c_2 \int_0^t \int_{\partial\Omega} |g(s)|^2 + t c_2 \sup_{0 \leq s \leq t} \int_{\Omega} |u(s)|^2
\]
for all $t \in [0,T]$ with a constant $c_2 \geq 0$ that depends only on $c_1, \mu$ and $\omega$. This shows that with $t_0 := \frac{1}{2c_2}$ we have
\[
\sup_{0 \leq s \leq t} \int_{\Omega} |u(s)|^2 + \int_0^t \int_{\Omega} |\nabla u(s)|^2
\]
\[
\leq 2c_2 \int_{\Omega} |u_0|^2 + 2c_2 \int_0^t \int_{\Omega} |f(s)|^2 + 2c_2 \int_0^t \int_{\partial\Omega} |g(s)|^2
\]
for all $t \in [0,t_0]$. We split $[0,T]$ into finitely many intervals of length at most $s_0$ and apply the last inequality successively on these intervals. This gives \ref{2.12}. $\square$

We also collect some results about the homogeneous problem $(P_{u_0,0,0})$ for later use. To this end we introduce the generator $A_{2,h}$ for the homogeneous problem, which is the part of $A_2$ in $L^2(\Omega) \times \{0\}$. All of the following results stem from semigroup theory.

Proposition 2.9. The operator $A_{2,h}$ given by
\[
D(A_{2,h}) = \left\{ u \in H^1(\Omega) : Au \in L^2(\Omega), \left[ \frac{\partial u}{\partial n} + \beta u \right] = 0 \right\}
\]
\[
A_{2,h} u = Au
\]
is the generator of an analytic $C_0$-semigroup $(T_{2,h}(t))_{t \geq 0}$ on $L^2(\Omega)$. Given $u_0 \in L^2(\Omega)$, the function $u$ defined by $u(t) := T_{2,h}(t)u_0$ is the unique mild $L^2$-solution of $(P_{u_0,0,0})$, and we have the following properties:

(i) There exist $M \geq 0$ and $\omega \in \mathbb{R}$ depending only on $N$, $\Omega$ and the coefficients of the equation such that $\|u(t)\|_{L^\infty(\Omega)} \leq M e^{\omega t}\|u_0\|_{L^\infty(\Omega)}$ for all $t \geq 0$.

(ii) For every $t > 0$ we have $u(t) \in C(\Omega)$.

(iii) If $u_0 \in C(\overline{\Omega})$, then $u \in C([0,\infty);C(\overline{\Omega}))$ for all $T > 0$.

**Proof.** The operator $-A_{2,h}$ is associated with the bounded, $L^2(\Omega)$-elliptic bilinear form $a_2: H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined in (2.1). Hence $A_{2,h}$ generates an analytic $C_0$-semigroup on $L^2(\Omega)$, see [1, Proposition XVII.6.3]. By construction a function $u$ is a mild solution for the abstract Cauchy problem associated with $A_{2,h}$ if and only if it is a mild $L^2$-solution of $(P_{u_0,0,0})$, which proves the assertion about the mild $L^2$-solutions [2, Theorem 3.1.12]. Property (i) follows from [3, Proposition 7.1]. Properties (ii) and (iii) have been proved in [22, Theorem 4.3] for bounded coefficients. The same arguments work here, but compare also [20, 21], where unbounded (and nonlinear) coefficients are considered. \hfill $\square$

The following is our main existence theorem.

**Theorem 2.10.** Let $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ and $g \in L^2(0,T;L^2(\partial \Omega))$ be given, where $T > 0$ is arbitrary. Then there exists a weak solution $u$ of $(P_{u_0,f,g})$ on $[0,T]$, which is unique even within the class of mild $L^2$-solutions.

**Proof.** Pick sequences $(f_n) \subset C^2([0,T];L^2(\Omega))$ and $(g_n) \subset C^2([0,T];L^2(\partial \Omega))$ that satisfy $f_n(0) = 0$, $g_n(0) = 0$, $f_n \to f$ in $L^2(0,T;L^2(\Omega))$ and $g_n \to g$ in $L^2(0,T;L^2(\partial \Omega))$. Since $A_{2,h}$ is the generator of a $C_0$-semigroup, there exists a sequence $(u_{n,0}) \subset D(A^2_{2,h})$ satisfying $u_{n,0} \to u_0$ in $L^2(\Omega)$, see [11, Proposition II.1.8]. By Proposition 2.8 there exists a classical $L^2$-solutions $u_n$ of $(P_{u_0,f_n,g_n})$.

By Lemma 2.8 the sequence $(u_n)$ is Cauchy in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Denote its limit by $u$. Using that $u_n$ is a weak solution of $(P_{u_0,f_n,g_n})$ by Theorem 2.6 we can pass in (2.6) to the limit and obtain that $u$ is a weak solution of $(P_{u_0,f,g})$. Uniqueness has already been asserted in Proposition 2.7. \hfill $\square$

Since being a solution is a local concept, we obtain the following corollary.

**Corollary 2.11.** For given functions $u_0 \in L^2(\Omega)$, $f \in L^2_{\text{loc}}([0,\infty);L^2(\Omega))$ and $g \in L^2_{\text{loc}}([0,\infty);L^2(\partial \Omega))$, equation $(P_{u_0,f,g})$ has a weak solution on $[0,\infty)$, which is unique even in the class of mild solutions.

We deduce the following from Theorem 2.6 and Theorem 2.10 or Corollary 2.11 respectively.

**Corollary 2.12.** For problem $(P_{u_0,f,g})$ the notions of weak and mild solutions coincide.

We have seen that Problem $(P_{u_0,f,g})$ admits unique weak solutions. One might expect that this implies that $A_2$ is the generator of a strongly continuous semigroup. Obviously, this is not true since $A_2$ is not densely defined. Even worse, the operator does not even satisfy Hille-Yosida estimates as the following example shows.

**Example 2.13.** Set $\Omega = (0,1)$ and consider the Laplace operator with Neumann boundary conditions, i.e., $A_2(u,0) := (u'',(u'(0),-u'(1)))$ on $L^2(0,1) \times \mathbb{R}^2$. For $\lambda > 0$ we can explicitly calculate that $u_\lambda := (\lambda - A)^{-1}(0,(0,1))$
is given by the formula
\[ u_\lambda(x) = \frac{\exp(\sqrt{\lambda}x) + \exp(-\sqrt{\lambda}x)}{\sqrt{\lambda}(\exp(\sqrt{\lambda}) - \exp(-\sqrt{\lambda}))}, \]
from which we obtain after some calculations that
\[ \|u_\lambda\|_{L^2(\Omega)} \sim \frac{1}{\sqrt{\lambda}} \lambda^{-\frac{1}{4}}, \]
as \( \lambda \to \infty \). Hence \( \|\lambda R(\lambda, A)\| \sim c \lambda^{\frac{1}{8}} \) as \( \lambda \to \infty \) for some constant \( c > 0 \), which shows that \( A_2 \) is not a Hille-Yosida operator in the sense of [2, §3.5]. This was already clear since every Hille-Yosida operator on reflexive space is densely defined [2, Proposition 3.3.8].

3. Regularity

The goal of this section is to show that for \( u_0 \in C(\overline{\Omega}) \) the weak solution of \((P_{u_0,f,g})\) is continuous on the parabolic cylinder \([0, \infty) \times \Omega\), so in particular continuous up to the boundary. The main tool is the following pointwise a priori estimate, which we will use also for the study of the asymptotic behavior.

**Proposition 3.1.** Fix \( T > 0 \). Let \( r_1, r_2, q_1, q_2 \in [2, \infty) \) satisfy
\[
\frac{1}{r_1} + \frac{N}{2q_1} < 1 \quad \text{and} \quad \frac{1}{r_2} + \frac{N-1}{2q_2} < \frac{1}{2}.\tag{3.1}
\]
Let \( u_0 \in L^2(\Omega) \), \( f \in L^{r_1}(0, T; L^{q_1}(\Omega)) \) and \( g \in L^{r_2}(0, T; L^{q_2}(\partial \Omega)) \) be given and denote by \( u \) the weak solution of \((P_{u_0,f,g})\). Then
\[
\|u\|_{L^{\infty}(0, T; L^{\infty}(\Omega))} \leq c \|u_0\|_{L^2(\Omega)}^2 + c \|f\|_{L^{r_1}(0, T; L^{q_1}(\Omega))}^2 + c \|g\|_{L^{r_2}(0, T; L^{q_2}(\partial \Omega))}^2,\tag{3.2}
\]
where \( c \) depends only on \( T, N, \Omega, r_1, q_1, r_2, q_2 \) and the coefficients of the equation.

If we have \( u_0 = 0 \), then we obtain the global estimate
\[
\|u\|_{L^{\infty}(0, T; L^{\infty}(\Omega))} \leq c \|f\|_{L^{r_1}(0, T; L^{q_1}(\Omega))} + c \|g\|_{L^{r_2}(0, T; L^{q_2}(\partial \Omega))}.\tag{3.3}
\]

The proof of Proposition 3.1 is lengthy and technical. We postpone it to Appendix A in order not to interrupt the train of thought. We will use mainly the following consequence of Proposition 3.1 which arises from combining it with Proposition 2.9.

**Theorem 3.2.** Let \( T > 0 \) be arbitrary, let \( f \) and \( g \) satisfy the conditions of Proposition 3.1 and let \( u_0 \in L^\infty(\Omega) \) be given. Then the weak solution \( u \) of \((P_{u_0,f,g})\) satisfies
\[
\|u\|_{L^{\infty}(0, T; L^{\infty}(\Omega))} \leq c \|u_0\|_{L^2(\Omega)} + c \|f\|_{L^{r_1}(0, T; L^{q_1}(\Omega))} + c \|g\|_{L^{r_2}(0, T; L^{q_2}(\partial \Omega))},\tag{3.4}
\]
where \( c \) depends on the same parameters as in Proposition 3.1.

**Proof.** By linearity we have \( u(t) = T_{2,h}(t)u_0 + v(t) \), where \((T_{2,h}(t))_{t \geq 0}\) has been introduced in Proposition 2.9 and \( v \) is the weak solution of \((P_{0,f,g})\). Hence we deduce from (3.3) and Proposition 2.9 that
\[
\|u\|_{L^{\infty}(0, T; L^{\infty}(\Omega))} \leq 2 \sup_{0 \leq s \leq T} \|T_{2,h}(t)u_0\|_{L^{\infty}(\Omega)} + 2\|v\|_{L^{\infty}(0, T; L^{\infty}(\Omega))}^2 \\
\leq 2M^2c^{2|\omega|T}\|u_0\|_{L^{\infty}(\Omega)} + 2c\|f\|_{L^{r_1}(0, T; L^{q_1}(\Omega))}^2 \\
+ 2c\|g\|_{L^{r_2}(0, T; L^{q_2}(\partial \Omega))}^2.
\]
In addition, by Lemma 2.8 and Hölder’s inequality we have
\[
\|v(s)\|_{L^{r_2}(\Omega)} \leq c \|u_0\|_{L^{\infty}(\Omega)}^2 + c \|f\|_{L^{r_1}(0, T; L^{q_1}(\Omega))}^2 + c \|g\|_{L^{r_2}(0, T; L^{q_2}(\partial \Omega))}^2.
\]
for all \( s \in [0, T] \), where we note that by the proof of Theorem 3.3 the lemma is valid for all weak solutions, not only classical solutions. Combining these two estimates we have proved (3.4).

We use (3.4) to deduce continuity of the solution up to the boundary of the parabolic cylinder, which is our main regularity result.

**Theorem 3.3.** Let \( T > 0 \) be arbitrary, let \( f \) and \( g \) satisfy the conditions of Proposition 3.1 and let \( u_0 \in C(\Omega) \) be given. Then the weak solution \( u \) of \((P_{u_0, f, g})\) is in \( C([0, T]; C(\Omega)) \). So in particular \( u(t) \to u_0 \) uniformly on \( \Omega \) as \( t \to 0 \).

**Proof.** Let \( A_X \) denote the realization of \( A \) in \( X := L^q(\Omega) \times L^{q_2}(\partial\Omega) \) with the same boundary conditions as \( A_2 \), i.e.,

\[
D(A_X) := \left\{(u, 0) \in D(A_2) : Au \in L^q(\Omega), \frac{\partial u}{\partial \nu_A} \in L^{q_2}(\partial\Omega)\right\}
\]

\[
A_X(u, 0) := \left(\frac{\partial u}{\partial \nu_A} - \beta u|_{\partial\Omega}\right).
\]

Thus \((u, 0) \in D(A_X)\) if and only if there exist \( f \in L^q(\Omega) \) and \( g \in L^{q_2}(\partial\Omega) \) such that \( u \) solves

\[
\begin{cases}
Au = f & \text{on } \Omega \\
\frac{\partial u}{\partial \nu_A} + \beta u = g & \text{on } \partial\Omega
\end{cases}
\]

in the weak sense. Since by (3.1) we have in particular \( q_1 > \frac{n}{2} \) and \( q_2 > \frac{n-1}{2} \), elliptic regularity theory shows that in this case \( u \in C(\Omega) \), compare [22, Theorem 3.14] for bounded coefficients or [24, Example 4.2.7] for the general case. Hence \( D(A_X) \subset C(\Omega) \times \{0\} \) and in particular \( D(A_X) \subset X \). Hence \( A_X \) is the part of the resolvent positive operator \( A_2 \) in \( X \), and hence is resolvent positive. Thus \( A_X \) generates a once integrated semigroup on \( X \) by [2, Theorem 3.11.7].

Pick sequences \((f_n) \subset C^2([0, T]; L^\infty(\Omega))\) and \((g_n) \subset C^2([0, T]; L^\infty(\partial\Omega))\) that satisfy \( f_n(0) = 0, g_n(0) = 0, f_n \to f \) in \( L^r(0, T; L^q(\Omega)) \) and \( g_n \to g \) in \( L^r(0, T; L^{q_2}(\partial\Omega)) \), and let \( v_n \) denote the weak solution of \((P_{0, f_n, g_n})\).

By [2, Corollary 3.2.11] the abstract Cauchy problem

\[
\begin{aligned}
W_n(t) &= A_X W_n(t) + (f_n(t), g_n(t)) \\
W(0) &= (0, 0)
\end{aligned}
\]

has a unique solution \( W_n = (w_n, 0) \in C^1([0, T]; X) \cap C([0, T]; D(A_X)) \), and in particular we have \( w_n \in C([0, T]; C(\Omega)) \); we could call \( w_n \) a classical \( X \)-solution of \((P_{0, f_n, g_n})\) in analogy to Definition 2.3. The function \( w_n \) is in particular a classical \( L^2 \)-solution of \((2.3)\), hence \( w_n = v_n \) by uniqueness. We have shown that \( v_n \in C([0, T]; C(\Omega)) \).

Now, since by Theorem 3.2 we have \( v_n \to v \) uniformly on \([0, T] \times \Omega\), where \( v \) denotes the weak solution of \((P_{0, f, g})\), we deduce that \( v \in C([0, T]; C(\Omega)) \). Hence, since \( u(t) = T_{2, h}(t)u_0 + v(t) \) with \((T_{2, h}(t))_{t \geq 0} \) defined in Proposition 2.9, continuity of \( u \) follows from Proposition 2.9.

**Remark 3.4.** If in Theorem 3.3 we only have \( u_0 \in L^2(\Omega) \) instead of \( u_0 \in C(\Omega) \), we still obtain that \( u_{|_{[t_0, T]} \in C([t_0, T]; C(\Omega)) \) for all \( t_0 \in (0, T) \). In fact, this can be seen easily from the proof since by Proposition 2.9 \( t \mapsto T_{2, h}(t)u_0 \) is continuous from \([t_0, \infty) \) to \( C(\Omega) \) for every \( t_0 > 0 \).

In particular, \( u_0 \in C(\Omega) \) is a necessary condition for the convergence \( u(t) \to u_0 \) as \( t \to 0 \) to be uniform on \( \Omega \). Theorem 3.3 shows that it is also sufficient if \( f \) and \( g \) do not behave too badly.
We close this section by a comparison with the situation for Dirichlet boundary conditions.

**Remark 3.5.** For the Dirichlet initial-boundary value problem studied in [1] one has to work with a realization $A_{c,D}$ of $A$ with Dirichlet boundary conditions in a space of continuous functions because $L^p$-regularity conditions on the boundary do not suffice in order to obtain continuous solutions, which contrasts the situation in Theorem 3.3 for Neumann boundary data. This leads to a minor difficulty. More precisely, since $C(\partial \Omega)$ does not have order continuous norm, it is not immediately clear that $A_{c,D}$ is the generator of a once integrated semigroup. In fact, this is even false since if $A_{c,D}$ were the generator of a once integrated semigroup, then by [2, Corollary 3.2.11] there would exist a mild solution of the corresponding abstract Cauchy problem

\[
\begin{cases}
  u_t(t) = \Delta u(t) \\
  u(t)|_{\partial \Omega} = \varphi(t) \\
  u(0) = u_0
\end{cases}
\]

regardless of any compatibility assumptions between $\varphi \in C^1([0, \infty) ; C(\partial \Omega))$ and $u_0 \in C(\Omega)$. This contradicts the simple observation that the existence of a mild solution enforces the condition $\varphi(0) = u_0|_{\partial \Omega}$, see [1, Proposition 3.2]. Still, $A_{c,D}$ generates a twice integrated semigroup [2, Theorem 3.11.5], which is sufficient for the results in [1].

The situation is different for Neumann boundary conditions, as we can already expect from the fact that no compatibility condition appears in Theorem 3.3. In fact, we have a once integrated semigroup in that case. In order to see this, consider the realization $A_c$ in $C(\Omega) \times C(\partial \Omega)$ of $A$ with Robin boundary conditions and set $Z := C(\Omega) \times \{0\}$. The space $Z$ is invariant under the resolvent of $A_c$ and the part of $A_c$ in $Z$ is the generator of a strongly continuous semigroup [22, Theorem 4.3]. Hence by [2, Theorem 3.10.4] the operator $A_c$ generates a once integrated semigroup on $C(\Omega) \times C(\partial \Omega)$.

It can be seen from Example 2.13 that the operator $A_c$ fails to be a Hille-Yosida operator. In this respect, the situation is the same as for Dirichlet boundary conditions [1, Remark 2.5 b)].

### 4. Convergence

In this section we study boundedness of the solution $u$ of $(P_{u_0,f,g})$ as $t \to \infty$. We are not interested in (exponential) blow-up or decay, but want to consider the border case only. Inspired by our model case, i.e., $A = \Delta$ and $\beta = 0$, a natural condition that helps with this issue is to assume conservation of total energy, i.e.,

\[
\int_\Omega u(t) = \int_\Omega u_0 + \int_0^t \int_\Omega f(s) + \int_0^t \int_{\partial \Omega} g(s)
\]

for all $t > 0$. We restrict ourselves to this situation, which can be characterized as follows.

**Proposition 4.1.** The following assertions are equivalent:

(i) for every $T > 0$, $f \in L^2(0,T;L^2(\Omega))$, $g \in L^2(0,T;L^2(\partial \Omega))$ and $u_0 \in L^2(\Omega)$ relation (4.1) holds for all $t \in [0,T]$, where $u$ is the weak solution of $(P_{u_0,f,g})$;

(ii) for every $u_0 \in L^2(\Omega)$ we have $\int_\Omega u(t) = \int_\Omega u_0$ for all $t > 0$, where $u$ is the weak solution of $(P_{u_0,0,0})$;
(iii) the relation
\[
\begin{cases}
\sum_{i=1}^N c_i = d & \text{on } \Omega \\
\sum_{i=1}^N c_i \nu_i = -\beta & \text{on } \partial \Omega
\end{cases}
\] (4.2)
holds in the weak sense, i.e.,
\[
\sum_{i=1}^N \int_{\Omega} c_i D_i \eta + \int_{\Omega} d\eta + \int_{\partial \Omega} \beta \eta = 0 \quad \text{for all } \eta \in H^1(\Omega).
\]

Proof. Assume (iii) and let \( u \) be the weak solution of \((P_{u_0,f,g})\), which is a mild \( L^2 \)-solution by Theorem 2.6. By Remark 2.3 we have
\[
a_\beta \left( \int_0^t u(s), v \right) = \int_0^t \int_{\Omega} f(s) v + \int_0^t \int_{\partial \Omega} g(s) v - \int_{\Omega} \left( u(t) - u_0 \right) v \quad (4.3)
\]
for all \( v \in H^1(\Omega) \). Picking \( v := 1_\Omega \) and using that by (4.2) we have \( a_\beta(\eta, 1_\Omega) = 0 \) for all \( \eta \in H^1(\Omega) \) this gives (4.4).

It is trivial that (i) implies (ii). So now assume that (ii) holds, i.e., \( \int_{\Omega} T_{2,h}(t) u_0 = \int_{\Omega} u_0 \) for all \( t \geq 0 \) and all \( u_0 \in L^2(\Omega) \), where \((T_{2,h}(t))_{t \geq 0}\) is defined in Proposition 2.9. Then \( 1_\Omega \) is a fixed point of the adjoint semigroup \((T_{2,h}^*(t))_{t \geq 0}\), which implies \( A_{2,h}^* 1_\Omega = 0 \), i.e., \( a_\beta(\eta, 1_\Omega) = 0 \) for all \( \eta \in H^1(\Omega) \). This is (1.2).

We aim towards a bound of the solution of \((P_{u_0,f,g})\) in \( L^\infty(0, \infty; L^\infty(\Omega)) \). As a first step, we consider this problem only for the homogeneous problem \((P_{u_0,0,0})\), as we describe in the following lemma.

**Lemma 4.2.** Under condition (1.2) we have \( \|u\|_{L^\infty(0, \infty; L^\infty(\Omega))} \leq \|u_0\|_{L^\infty(\Omega)} \) for the weak solution \( u \) of \((P_{u_0,0,0})\) if and only if
\[
\begin{cases}
\sum_{j=1}^N b_j = d & \text{on } \Omega \\
\sum_{j=1}^N b_j \nu_j = -\beta & \text{on } \partial \Omega
\end{cases}
\] (4.4)
in the weak sense.

Proof. Relation (4.4) is equivalent to \( a_\beta(1_\Omega, \eta) = 0 \) for all \( \eta \in H^1(\Omega) \), i.e., \( A_{2,h} 1_\Omega = 0 \). Hence (4.4) is equivalent to \( 1_\Omega \) being a fixed point of \((T_{2,h}(t))_{t \geq 0}\), where \((T_{2,h}(t))_{t \geq 0}\) is defined in Proposition 2.9.

Since \((T_{2,h}(t))_{t \geq 0}\) is positive, \( T_{2,h}(t) 1_\Omega = 1_\Omega \) for all \( t \geq 0 \) implies that the semigroup is contractive with respect to the norm of \( L^\infty(\Omega) \), which is precisely the bound for \( u \). On the other hand, if \((T_{2,h}(t))_{t \geq 0}\) is \( L^\infty(\Omega) \)-contractive and \( \int_{\Omega} T_{2,h}(t) u_0 = \int_{\Omega} u_0 \) for all \( t \geq 0 \), which is satisfied by Proposition 4.1 then \( 1_\Omega \) is a fixed point of \((T_{2,h}(t))_{t \geq 0}\).

We will see in Corollary 4.3 that (1.3) implies that also the inhomogeneous problem \((P_{u_0,f,g})\) has bounded solutions if we assume in addition that \( \int_{\Omega} f(t) + \int_{\partial \Omega} g(t) = 0 \) for all \( t \geq 0 \) and the functions \( f \) and \( g \) are not too irregular. The first step in this direction is an \( L^2 \)-bound on bounded time intervals, Proposition 4.4 for which we need the following lemma.

**Lemma 4.3.** If (1.2) and (1.4) hold, then \( a_\beta(v, v) \geq \mu \int_{\Omega} |\nabla v|^2 \) for all \( v \in H^1(\Omega) \).
we obtain that

\[ a_\beta(v,v) \geq \mu \int_\Omega |\nabla v|^2 + \frac{1}{2} \sum_{j=1}^N \int_\Omega b_j D_j(v^2) + \frac{1}{2} \sum_{i=1}^N \int_\Omega c_i D_i(v^2) + \int_\Omega dv^2 + \int_{\partial \Omega} \beta v^2 \]

where in the second step we used the weak formulations of (4.2) and (4.4) with \( u \) without loss of generality that \( c \) inequality and the Sobolev embedding theorems there exists \( c_\eta \) for all \( v \in H^1(\Omega) \).

Prove. By continuity of \( a_\beta \) it suffices to prove the estimate for all \( v \in H^1(\Omega) \cap L^\infty(\Omega) \). For such \( v \) we have by (2.1) and the chain rule that

\[ a_\beta(v,v) \geq \mu \int_\Omega |\nabla v|^2, \]

\( \eta \) for such \( u \) was assumed to be connected throughout the article. Hence by Poincaré’s inequality and the Sobolev embedding theorems there exists \( c_1 \geq 0 \) depending only on \( \mu \) and \( \Omega \) such that

\[ \int_\Omega |u(t)|^2 \leq e^{-\tau t} \int_\Omega |u_0|^2 + c \int_0^t e^{(s-t)/\tau} \left( \int_\Omega |f(s)|^2 + \int_{\partial \Omega} |g(s)|^2 \right) ds \]

for all \( t \in [0, T] \).

Proof. Since \( u \) can be approximated by classical \( L^2 \)-solutions of equations with right hand sides close to \( f \) and \( g \), compare the proof of Theorem 2.10, we can assume without loss of generality that \( u \) is a classical \( L^2 \)-solution of \( (P_{u_0,f,g}) \).

By (1.5) and Proposition 4.4 we have \( \int_\Omega u(t) = \int_{\partial \Omega} u_0 = 0 \) for all \( t \in [0, T] \). Recall that \( \Omega \) was assumed to be connected throughout the article. Hence by Poincaré’s inequality and the Sobolev embedding theorems there exists \( c_1 \geq 0 \) depending only on \( \Omega \) such that

\[ \int_\Omega |u(t)|^2 + \int_{\partial \Omega} |u(t)|^2 \leq c_1 \int_\Omega |\nabla u(t)|^2 \]

for all \( t \geq 0 \). Using Remark 2.3, Lemma 4.3, Young’s inequality and estimate 4.7 we obtain that

\[ \frac{d}{dt} \frac{1}{2} \int_\Omega |u(t)|^2 = \int_\Omega u(t) u_t(t) = \int_\Omega u(t) (Au(t) + f(t)) \]

\[ \leq \int_\Omega f(t) u(t) + \int_{\partial \Omega} g(t) u(t) - a_\beta(u(t), u(t)) \]

\[ \leq c_2 \left( \int_\Omega |f(t)|^2 + \int_{\partial \Omega} |g(t)|^2 \right) + \frac{\mu}{2c_1} \left( \int_\Omega |u(t)|^2 + \int_{\partial \Omega} |u(t)|^2 \right) \]

\[ \leq c_2 \left( \int_\Omega |f(t)|^2 + \int_{\partial \Omega} |g(t)|^2 \right) - \frac{\mu}{2} \int_\Omega |\nabla u(t)|^2. \]
with \( c_2 := \frac{2r}{2} \). Define \( \tau := \frac{2r}{2} \). Then by (4.7) and the above inequality
\[
\frac{1}{2} \int_0^t |u(t)|^2 - e^{-t/\tau} \frac{1}{2} \int_0^t |u_0|^2 = \int_0^t \frac{d}{ds} (e^{(s-t)/\tau} \frac{1}{2} \int_0^s |u(s)|^2) \\
\leq \frac{1}{2} \int_0^t e^{(s-t)/\tau} \int_0^s |u(s)|^2 \\
+ \frac{1}{2} \int_0^t e^{(s-t)/\tau} \left( c_2 \int_0^s |f(s)|^2 + c_2 \int_0^s |g(s)|^2 \right) \\
\leq c_2 \int_0^t e^{(s-t)/\tau} \left( \int_0^s |f(s)|^2 + \int_0^s |g(s)|^2 \right),
\]
where in the last step we have used that \( \frac{2r}{2} = \frac{2}{2} \).

We want to find a condition on \( f \) and \( g \) which ensures that the right hand side of (4.7) remains bounded as \( t \to \infty \). To this end we introduce some function spaces.

**Definition 4.5.** Let \( r_1 \) and \( q_1 \) be in \([1, \infty)\), and let \( T > 0 \). For a strongly measurable function \( f: (0, \infty) \to L^{q_1}(\Omega) \) we define
\[
R_{f,T}^{r_1,q_1}(t) := \|f\|_{L^{r_1,q_1}(t, t+T; L^{q_1}(\Omega))} = \left( \int_0^T \|f(s)\|_{L^{r_1,q_1}(\Omega)}^{r_1} \right)^{\frac{1}{r_1}}
\]
and introduce the spaces
\[
L_{r_1,q_1}(\Omega) := \{ f: (0, \infty) \to L^{q_1}(\Omega) \mid R_{f,T}^{r_1,q_1} \in L^\infty(0, \infty) \}
\]
and
\[
L_{r_1,q_1}^T(\Omega) := \{ f \in L_{r_1,q_1}(\Omega) \mid \lim_{t \to \infty} R_{f,T}^{r_1,q_1}(t) = 0 \}
\]
of uniformly mean integrable functions, where we identify functions that coincide almost everywhere. Similarly, for \( r_2 \) and \( q_2 \) in \([1, \infty)\) and \( g: (0, \infty) \to L^{q_2}(\partial \Omega) \) we set
\[
R_{g,T}^{r_2,q_2}(t) := \|g\|_{L^{r_2,q_2}(t, t+T; L^{q_2}(\partial \Omega))},
\]
\[
L_{r_2,q_2}(\partial \Omega) := \{ g: (0, \infty) \to L^{q_2}(\partial \Omega) \mid R_{g,T}^{r_2,q_2} \in L^\infty(0, \infty) \},
\]
\[
L_{r_2,q_2}^T(\partial \Omega) := \{ g \in L_{r_2,q_2}(\partial \Omega) \mid \lim_{t \to \infty} R_{g,T}^{r_2,q_2}(t) = 0 \}.
\]

Let us collect a few properties of the spaces introduced in Definition 4.5.

**Lemma 4.6.** Let \( r_1 \) and \( q_1 \) be in \([1, \infty)\). Then
(a) for every \( T > 0 \), the expression \( \|f\|_{L_{r_1,q_1}(\Omega)} := \sup_{t \geq 0} R_{f,T}^{r_1,q_1}(t) \) defines a complete norm on \( L_{r_1,q_1}(\Omega) \);
(b) the norms in (b) are pairwise equivalent for different values of \( T \);
(c) for every \( f \in L_{r_1,q_1}(\Omega) \) and every \( T > 0 \) the function \( R_{f,T}^{r_1,q_1} \) is continuous on \([0, \infty)\);
(d) the space \( L_{r_1,q_1}(\Omega) \) is a closed subspace of \( L_{r_1,q_1}^T(\Omega) \);
(e) if \( 1 \leq r_1' \leq r_1 \) and \( 1 \leq q_1' \leq q_1 \), then
\[
L_{r_1,q_1}(\Omega) \subset L_{r_1',q_1'}(\Omega) \quad \text{and} \quad L_{r_1,q_1}(\Omega) \subset L_{r_1',q_1'}(\Omega)
\]
with continuous embeddings;
(f) we have \( L^{\infty}(0, \infty; L^{q_1}(\Omega)) \subset L_{r_1,q_1}(\Omega) \) and \( C_0([0, \infty); L^{q_1}(\Omega)) \subset L_{r_1,q_1}(\Omega) \) with continuous embeddings;
(g) for \( f \in L_{r_1,q_1}(\Omega) \) and every non-increasing function \( h \in L^1(0, \infty) \cap L^\infty(0, \infty) \) we have
\[
\int_0^t h(t-s) \|f(s)\|_{L^{r_1,q_1}(\Omega)} \, ds \leq (\|h\|_{L^\infty(0, \infty)} + \frac{2}{r_1} \|h\|_{L^1(0, \infty)}) \|R_{f,T}^{r_1,q_1}\|_{L^\infty(0, \infty)}
\]
for all \( T > 0 \) and \( t \geq 0 \);
(h) for \( f \in L^{r_1,q_1}_{m,0}(\Omega) \) and every non-increasing function \( h \in L^1(0,\infty) \cap L^\infty(0,\infty) \) we have
\[
\lim_{t \to 0} \int_0^t h(t-s) \|f(s)\|_{L^{r_1}_{m,0}(\Omega)}^2 \, ds = 0.
\]
Analogous assertions hold for the spaces \( L^{r_2,q_2}_{m,0}(\partial \Omega) \) and \( L^{r_2,q_2}_{m,0}(\partial \Omega) \) with \( r_2, q_2 \in [1,\infty) \).

Part (i) justifies that we suppress the dependence on \( T \) in the notation for \( L^{r_1,q_1}_{m,0}(\Omega) \) and its norm.

**Proof.** Part (a) is routinely checked and we leave the verification to the reader.

Now let \( T > 0 \) and \( T' > 0 \) be given and pick a natural number \( n \geq T' \). Then by Hölder’s inequality
\[
R^{r_1,q_1}_{f,T'}(t) \leq R^{r_1,q_1}_{f,T}(t) = \left( \sum_{k=0}^{n-1} R^{r_1,q_1}_{f,T}(t+kT)^{r_1} \right)^{\frac{1}{r_1}}
\]
\[
\leq \sum_{k=0}^{n-1} R^{r_1,q_1}_{f,T}(t+kT) \leq n \sup_{s \geq 0} R^{r_1,q_1}_{f,T}(s)
\]
for all \( t \geq 0 \), which implies (i).

By the reverse triangle inequality we have
\[
|R^{r_1,q_1}_{f,T}(t+h) - R^{r_1,q_1}_{f,T}(t)| \leq \left( \int_0^\infty \|f(s)\|_{L^{r_1}_{m,0}(\Omega)}^2 \mathbf{1}_{[t+1,T+h]}(s) - \mathbf{1}_{[t,T]}(s) \right)^{\frac{1}{2}}.
\]
Since moreover \( \mathbf{1}_{[t+1,T+h]} \to \mathbf{1}_{[t,T]} \) almost everywhere as \( h \to 0 \), part (ii) follows from the dominated convergence theorem, where as dominating function we may take \( \|f\|_{L^{r_1}_{m,0}(\Omega)}^2 \mathbf{1}_{(0,T+2T)} \in L^1(0,\infty) \).

By (ii) and the definition of the norm the mapping \( f \mapsto R^{r_1,q_1}_{f,T} \) is Lipschitz continuous from \( L^{r_1,q_1}_{m,0}(\Omega) \) to \( C_0([0,\infty)) \) for every \( T > 0 \). Hence the preimage of \( C_0([0,\infty)) \) under this function is closed, which proves (iii).

For \( 1 \leq r'_1 \leq r_1 \) and \( 1 \leq q'_1 \leq q_1 \) we obtain from Hölder’s inequality that
\[
R^{r_1,q_1}_{f,T}(t) \leq T^{\frac{r_1-r'_1}{r'_1}} \left| \Omega \right|^{\frac{q_1-q'_1}{q'_1}} R^{r'_1,q'_1}_{f,T}
\]
for all \( t \geq 0 \). This implies (iv), and (i) is proved similarly.

For (v) let \( f \in L^{r_1,q_1}_{m,0}(\Omega) \), \( T > 0 \) and \( T' > 0 \) be fixed and define \( n_1 \in \mathbb{N} \) by \( (n_1-1)T < t < n_1T \). Let \( h \in L^1(0,\infty) \cap L^\infty(0,\infty) \) be non-increasing and assume without loss of generality that \( h(0) = \|h\|_{L^\infty(0,\infty)} \). Then for \( t \leq T \) the estimate in (v) is trivial, we may assume that \( t > T \), i.e., \( n_1 \geq 2 \). Then
\[
\frac{1}{n_1} \sum_{k=0}^{n_1-1} h\left( \frac{(n_1-k)t}{n_1} \right) \leq \frac{1}{n_1} \sum_{k=0}^{n_1-1} \int_{\frac{(n_1-k+1)t}{n_1}}^{\frac{(n_1-k)t}{n_1}} h(s) \, ds \leq \frac{2}{T} \int_0^t h(s) \, ds.
\]
Moreover,
\[
\int_0^t h(t-s) \|f(s)\|_{L^{r_1}_{m,0}(\Omega)}^2 \, ds \leq \sum_{k=1}^{n_1} h\left( \frac{k-1}{n_1} \right) \left( \frac{k}{n_1} \right)^{r_1} \|f(s)\|_{L^{r_1}_{m,0}(\Omega)}^2 \, ds
\]
\[
\leq \sum_{k=1}^{n_1} h\left( \frac{(n_1-k)t}{n_1} \right) \left( R^{r_1,q_1}_{f,T}\left( \frac{(k-1)t}{n_1} \right) \right)^{r_1}.
\]
(4.8)
The estimate in (vi) is an immediate consequence of (4.8) and (4.9).

Now assume in addition that \( f \in L^{r_1,q_1}_{m,0}(\Omega) \). Let \( \varepsilon > 0 \) be given and pick \( k_1 \in \mathbb{N} \) so large that \( R^{r_1,q_1}_{f,T}(s)^{r_1} \leq \varepsilon \) for all \( s \geq k_1T \). Let \( k_2 \in \mathbb{N} \) be so large that \( h(s) \leq \frac{\varepsilon}{k_1T} \).
for all \( s \geq k_2 T \), set \( k_0 := \max\{4k_1, 2k_2\} \) and define \( t_0 := k_0 T \). Let \( t \geq t_0 \) be fixed, so \( n_t \geq k_0 \). Then for \( k \leq 2k_1 \) we have

\[
\frac{(n_t - k) t}{n_t} = \left(1 - \frac{k}{n_t}\right) t \geq \left(1 - \frac{2k_1}{k_0}\right) t \geq \frac{t}{2} \geq 2T,
\]

whereas for \( k \geq 2k_1 + 1 \) we have

\[
\frac{(k - 1) t}{n_t} \geq \frac{2k_1 t}{2(n_t - 1)} \geq k_1 T.
\]

Hence from (4.8) and the definitions of \( k_1 \) and \( k_2 \) we obtain for \( t \geq k_0 T \) that

\[
\sum_{k=1}^{n_t} h\left(\frac{(n_t-k)t}{n_t}\right) \left(R^r_{1}, q_i \left(\frac{(k-1)t}{n_t}\right)\right)^{r_i} \leq \frac{\varepsilon}{2k_1} \sum_{k=1}^{2k_1} \left(R^r_{1}, q_i \left(\frac{(k-1)t}{n_t}\right)\right)^{r_i} + \varepsilon \sum_{k=2k_1+1}^{n_t} h\left(\frac{(n_t-k)t}{n_t}\right) \leq \varepsilon \left(\|R^r_{1}, q_i\|^{r_i}_{\infty} + h(0) + \|h\|_{L^1(0,\infty)}\right).
\]

We have shown that

\[
\lim_{t \to 0} \sum_{k=1}^{n_t} h\left(\frac{(n_t-k)t}{n_t}\right) \left(R^r_{1}, q_i \left(\frac{(k-1)t}{n_t}\right)\right)^{r_i} = 0,
\]

which by (4.9) implies (4.8).

We can now formulate our criterion for boundedness and convergence of solutions of \((P_{u_0, f, g})\), which together with its corollary is the main result of this section.

**Theorem 4.7.** If (4.2) and (4.4) hold, then for all \( u_0 \in L^2(\Omega) \), \( f \in L^2_{m, 2}(\Omega) \) and \( g \in L^2_{m, 2}(\partial \Omega) \) that satisfy (4.5) the weak solution \( u \) of \((P_{u_0, f, g})\) is bounded in \( L^2(\Omega) \), and more precisely

\[
\int_\Omega |u(t)|^2 dx \leq c \int_\Omega |u_0|^2 dx + c \|f\|_{L^2_{m, 2}(\Omega)}^2 + c \|g\|_{L^2_{m, 2}(\partial \Omega)}^2
\]

for all \( t \geq 0 \) with a constant \( c \geq 0 \) that depends only on \( \Omega \) and the coefficients. If even \( f \in L^2_{m, 0}(\Omega) \) and \( g \in L^2_{m, 0}(\partial \Omega) \), then \( \lim_{t \to \infty} u(t) = \frac{1}{|\Omega|} \int_\Omega u_0 \) in \( L^2(\Omega) \).

**Proof.** Write \( u_0 = \hat{u}_0 + k \) with \( k := \frac{1}{|\Omega|} \int_\Omega u_0 \). Then \( u(t) = \hat{u}(t) + k \) by Lemma 4.2, where \( \hat{u} \) denotes the weak solution of \((P_{\hat{u}_0, f, g})\). Proposition 4.4 and part (ii) of Lemma 4.6 applied with \( h(r) := e^{-r^\gamma} \) show that

\[
\int_\Omega |\hat{u}(t)|^2 dx \leq c \int_\Omega |\hat{u}_0|^2 dx + c \|f\|_{L^2_{m, 2}(\Omega)}^2 + c \|g\|_{L^2_{m, 2}(\partial \Omega)}^2,
\]

whereas part (i) shows that \( \lim_{t \to \infty} \hat{u}(t) = 0 \) in \( L^2(\Omega) \) if \( f \in L^2_{m, 0}(\Omega) \) and \( g \in L^2_{m, 2}(\partial \Omega) \).

Under slightly stronger assumptions on \( u_0 \), \( f \) and \( g \) we obtain even uniform boundedness and uniform convergence.

**Corollary 4.8.** Let \( r_1 \), \( q_1 \), \( r_2 \) and \( q_2 \) be numbers in \([2, \infty)\) that satisfy (3.1). If (4.2) and (4.4) hold, then for all \( u_0 \in L^\infty(\Omega) \), \( f \in L^r_{m, q_1}(\Omega) \) and \( g \in L^r_{m, q_2}(\partial \Omega) \) which satisfy (4.5) the weak solution \( u \) of \((P_{u_0, f, g})\) is bounded in \( L^\infty(\Omega) \), and more precisely

\[
\|u(t)\|_{L^\infty(\Omega)}^2 \leq c \|u_0\|_{L^\infty(\Omega)}^2 + c \|f\|_{L^r_{m, q_1}(\Omega)}^2 + c \|g\|_{L^r_{m, q_2}(\partial \Omega)}^2
\]

for all \( t \geq 0 \). If even \( f \in L^r_{m, 0}(\Omega) \) and \( g \in L^r_{m, 0}(\partial \Omega) \), then \( \lim_{t \to \infty} u(t) = \frac{1}{|\Omega|} \int_\Omega u_0 \) in \( L^\infty(\Omega) \).
Proof. By Theorem 4.7 and part (vi) of Lemma 4.6 we have
\[ \|u(t)\|_{L^2(\Omega)}^2 \leq c \|\hat{u}\|_{L^\infty(\Omega)}^2 + c \|f\|_{L^2,\Omega}^2 + c \|\hat{g}\|_{L^2,\Omega}^2. \]

On the other hand, inequality 3.2, applied to the interval \([t-2,t]\) shows that
\[ \|u(t)\|_{L^\infty(\Omega)}^2 \leq 2e \sup_{s \geq t-2} \|u(s)\|_{L^2(\Omega)}^2 + c (R_{f,2}^*)^2 + c (R_{g,2}^*)^2 \]for every \( t \geq 2 \). Using in addition Theorem 3.2 to bound \( u \) on \([0,2]\), we have shown 3.10.

Let now \( f \in L^{r_1,0}_{m,0}(\Omega) \subset L^{2,2}_{m,0}(\Omega) \) and \( g \in L^{r_2,0}_{m,0}(\partial\Omega) \subset L^{2,2}_{m,0}(\partial\Omega) \), see Lemma 4.6. Write \( u(t) = \tilde{u}(t) + k \) with \( k := \frac{1}{|\Omega|} \int_\Omega u_0 \) as in the proof of Theorem 4.7. Then \( \lim_{t \to \infty} \|\tilde{u}(t)\|_{L^2(\Omega)} = 0 \) by Theorem 4.7. Using the definitions of \( L^{r_1,0}_{m,0}(\Omega) \) and \( L^{r_2,0}_{m,0}(\partial\Omega) \), this gives \( \lim_{t \to \infty} \|\tilde{u}(t)\|_{L^\infty(\Omega)} = 0 \) by 4.11 applied to \( \tilde{u} \). The additional claim is proved.

Remark 4.9. Remark 4.4 shows that if in the situation of Corollary 4.8 we only have \( u_0 \in L^2(\Omega) \) instead of \( u_0 \in L^\infty(\Omega) \), the assertions remain valid apart from the exception that we will not be bounded in \( L^\infty(\Omega) \) as \( t \to 0 \), i.e., estimate 4.11 holds only for \( t \geq t_0 > 0 \) with a constant \( c \geq 0 \) that depends in addition on \( t_0 \).

5. Periodicity

We are going to study the periodic behavior of solutions of \((P_{u_0,0})\) under periodicity assumptions on \( f \) and \( g \). This relies on spectral theory, which is why in this section we assume our Banach spaces to be complex. Thus \( u_0 \), \( f \) and \( g \) are complex-valued functions, and hence also the solution \( u \) will be complex-valued. For the theory developed in the other sections this makes no difference since we can always treat the real and imaginary part separately as long as the coefficients of the equation are real-valued, which we still assume. Thus we will neglect this detail in the notation and reuse the symbols for the real spaces for their complex counterparts.

We start this section with a short summary on almost periodic functions in the sense of Harald Bohr, i.e., uniformly almost periodic functions. For further details and proofs we refer to [2, §4.5–4.7] or [3].

Definition 5.1. Let \( X \) be a complex Banach space. A function \( f: [0, \infty) \to X \) is called \( \tau \)-periodic (for some \( \tau > 0 \)) if \( f(t + \tau) = f(t) \) for all \( t \geq 0 \). Set \( e^{i\theta t} \) for \( \theta \in \mathbb{R} \) and \( t \geq 0 \). The members of the space
\[ \text{AP}([0, \infty); X) := \overline{\text{span}} \{e^{i\theta x} : \theta \in \mathbb{R}, x \in X\}, \]
are called uniformly almost periodic functions, where the closure is taken in the space of bounded, uniformly continuous functions \( \text{BUC}([0, \infty); X) \), which is a Banach space for the uniform norm. The direct topological sum
\[ \text{AAP}([0, \infty); X) := \text{AP}([0, \infty); X) \oplus C_0([0, \infty); X) \subset \text{BUC}([0, \infty); X) \]
called the space of uniformly asymptotically almost periodic functions. For all \( f \in \text{AAP}([0, \infty); X) \) and \( \eta \in \mathbb{R} \) the Cesàro limit
\[ C_\eta f := \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-i\eta s} f(s) \, ds \]
exist in \( X \). We let
\[ \text{Freq}(f) := \{ \eta \in \mathbb{R} : C_\eta f \neq 0 \} \]
de note the set of frequencies of \( f \). For \( f \in \text{AAP}([0, \infty); X) \) the set \( \text{Freq}(f) \) is countable. The function \( f \) can be decomposed into its frequencies in the sense that
\[ f \in \overline{\text{span}} \{e^{i\theta x} : \theta \in \text{Freq}(f), x \in X\} \oplus C_0([0, \infty); X). \]
In particular, $f \in C_0([0, \infty); X)$ if and only if $\text{Freq}(f) = \emptyset$. Moreover, $\text{Freq}(f) \subset \mathbb{Z}$ if and only there exists a $\tau$-periodic function $g$ such that $f - g \in C_0([0, \infty); X)$.

We show that for uniformly asymptotically almost periodic data, the solution is uniformly asymptotically almost periodic with essentially the same frequencies. In fact, this is a general phenomenon for mild solutions of abstract Cauchy problems and we merely have to check the assumptions of [2, Corollary 5.6.9]. We are going to improve this result later, which is why we call this preliminary result a lemma.

**Lemma 5.2.** Assume (1.2) and (4.3) and let $u_0 \in L^2(\Omega)$, $f \in \text{AAP}([0, \infty); L^2(\Omega))$ and $g \in \text{AAP}([0, \infty); L^2(\partial\Omega))$ satisfy (1.3). Then the weak solution $u$ of $(P_{u_0, f, g})$ is in $\text{AAP}([0, \infty); L^2(\Omega))$.

**Proof.** Define $\bar{u}_0(t) := u(t + h)$, $f_h(t) := f(t + h)$ and $g_h(t) := g(t + h)$ for $h \geq 0$ and $t \geq 0$. Then by uniform continuity of $f$ for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f_h - f\|_{L^2(\Omega)} \leq \varepsilon$ holds whenever $0 \leq h < \delta$, see part (b) of Lemma 4.6. A similar assertion holds for $g$. Applying Theorem 4.7 to $u$ and $u_0 - u$, which is the weak solution of $(P_{u_0(h) - u(0), f_h - f, g_h - g})$, and using in addition that $u$ is continuous by Definition 2.4, we thus obtain that $u \in \text{BUC}([0, \infty); L^2(\Omega))$.

Let $A_2$ be as in Definition 2.2. By Lemma 2.3 the operator $A_2$ generates a once integrated semigroup $(S(t))_{t \geq 0}$ on $L^2(\Omega) \times L^2(\partial\Omega)$, see [2, Theorem 3.11.7], which by [2, Lemma 3.2.9] satisfies $S(t)(v, 0) = \left( \int_0^t T_{2,h(s)} v, 0 \right)$ for all $v \in L^2(\Omega)$, where $(T_{2,h}(t))_{t \geq 0}$ is defined in Proposition 2.9. By Proposition 4.1 the closed subspace

$$X_0 := \left\{ (v, 0) : v \in L^2(\Omega), \int_\Omega v = 0 \right\}$$

of $L^2(\Omega) \times L^2(\partial\Omega)$ is invariant under the action of $(S(t))_{t \geq 0}$, which by Definition 3.2.1 implies that $X_0$ is invariant under the resolvent of $A_2$. Hence for the part $A_2|_{X_0}$ of $A_2$ in $X_0$ we have $\sigma(A_2|_{X_0}) \subset \sigma(A_2)$ and in particular $\rho(A_2|_{X_0}) \neq \emptyset$. We obtain from Lemma 2.4 and the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ that $A_2|_{X_0}$ has compact resolvent.

We now show that $\sigma(A_2|_{X_0}) \cap i\mathbb{R} = \emptyset$. Assume to the contrary that there exists $\eta \in \mathbb{R}$ such that $i\eta \in \sigma_p(A_2|_{X_0}) = \sigma(A_2|_{X_0})$. Then there exists $0 \neq v_0 \in L^2(\Omega)$ satisfying $\int_\Omega v_0 = 0$ and $A_2(v_0, 0) = (i\eta v_0, 0)$. Then $v(t) := e^{i\eta t}v_0$ defines a classical $L^2$-solution of $(P_{v_0, 0, 0})$. This contradicts Proposition 4.1 because $\|v(t)\|_{L^2(\Omega)} \not\to 0$ as $t \to \infty$.

Write $u_0 = \hat{u}_0 + k$ with $k := \frac{1}{\|\cdot\|_{L^2(\Omega)}} \int_\Omega u_0$. Then $u(t) = \hat{u}(t) + k$ by Lemma 4.2 where $\hat{u}$ is the weak (and hence mild) solution of $(P_{u_0, f, g})$. Since in addition $\int_\Omega u(t) = 0$ for all $t \geq 0$ by Proposition 4.1 we deduce that $(u, 0)$ is a mild solution of the abstract Cauchy problem associated with $A_2|_{X_0}$ for the inhomogeneity $(f, g)$. Since $\hat{u} \in \text{BUC}([0, \infty); L^2(\Omega))$ we now obtain from [2, Corollary 5.6.9] that $\hat{u} \in \text{AAP}([0, \infty); L^2(\Omega))$, which shows $u \in \text{AAP}([0, \infty); L^2(\Omega))$. 

Via an approximation argument we can relax the assumptions of Lemma 5.2. For this we introduce Stepanoff almost periodic functions. We omit the proofs of the implicit statements about this class of functions, which are similar to the ones for uniformly almost periodic functions. The interested reader may consult [4, §99] and [24] for the scalar-valued case.

**Definition 5.3.** Let $X$ be a complex Banach space. For $r \in [1, \infty)$ the members of the space

$$\text{AP}^r([0, \infty); X) := \overline{\text{span}} \{ e^{i\eta x} : \eta \in \mathbb{R}, x \in X \},$$
are called Stepanoff almost periodic functions (to the exponent $r$), where the closure is taken with respect to the norm

$$
\|f\|_{L^r_m(X)} := \sup_{t \geq 0} \left( \int_0^{t+1} \|f(s)\|^r_X \right)^{1/r}.
$$

The space of Stepanoff asymptotically almost periodic functions is defined as

$$
\text{AAP}^r([0, \infty); X) := \text{AP}^r([0, \infty); X) \oplus L_{r,0}^r(X),
$$

where we set $L_{r,0}^r(X) := \{ f \in L_r^r(X) : \lim_{t \to \infty} \int_t^{t+1} \|f(s)\|^r \to 0 \}$. The Cesàro limit

$$
C_\eta := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(s)
$$

exists for all $\eta \in \mathbb{R}$ and $f \in \text{AAP}^r([0, \infty); X)$. We define the set of frequencies of $f$ as

$$
\text{Freq}(f) := \{ \eta \in \mathbb{R} : C_\eta f \neq 0 \}
$$

and remark that $\text{Freq}(f) \subseteq \frac{2\pi}{r} \mathbb{Z}$ if and only there exists a $\tau$-periodic function $g$ such that $f - g \in L_{r,0}^r(X)$.

Now improve the statement of Lemma 5.2 by showing that for Stepanoff asymptotically almost periodic data we obtain uniformly asymptotically almost periodic solutions with a precise description of their frequencies. We start with the result in the $L^2$-framework.

**Theorem 5.4.** Assume that (4.2) and (4.3) hold. We assume that $u_0 \in L^2(\Omega)$, $f \in \text{AAP}^2([0, \infty); L^2(\Omega))$ and $g \in \text{AAP}^2([0, \infty); L^2(\partial\Omega))$ satisfy (4.5). Then the weak solution $u$ of $(P_{u_0,f,g})$ is in $\text{AAP}([0, \infty); L^2(\Omega))$. For $\eta \neq 0$ we have $\eta \in \text{Freq}(u)$ if and only if $\eta \in \text{Freq}(f) \cup \text{Freq}(g)$. Moreover, $0 \in \text{Freq}(u)$ if and only if $0 \in \text{Freq}(f) \cup \text{Freq}(g)$ or $f_{\eta} u_0 \neq 0$.

**Proof.** Write $f = f_P + f_C$ with $f_P \in \text{AP}([0, \infty); L^2(\Omega))$ and $f_C \in L^2_{m,0}(L^2(\Omega))$, $g = g_P + g_C$ with $g_P \in \text{AP}([0, \infty); L^2(\partial\Omega))$ and $g_C \in L^2_{m,0}(L^2(\partial\Omega))$ and $u_0 = \hat{u}_0 + k$ with $k := \frac{1}{m} \int_0^r \eta u_0$. Then $u = u_P + u_C + k$ by Lemma 5.2 where $u_P$ denotes the solution of $(P_{u_0,f_P,g_P})$ and $u_C$ is the solution of $(P_{u_0,f_C,g_C})$.

Pick $f_n \in \text{span}\{v \in \text{AAP}^2([0, \infty); L^2(\Omega)) \}$ and $g_n \in \text{span}\{v \in \text{AAP}^2([0, \infty); L^2(\partial\Omega)) \}$ such that $f_n \to f$ in the norm of $L^2_{m,0}(L^2(\Omega)) = L^2_{m,0}(\Omega)$ and $g_n \to g$ in the norm of $L^2_{m,0}(L^2(\partial\Omega)) = L^2_{m,0}(\partial\Omega)$. Let $u_n$ denote the weak solution of $(P_{u_0,f_n,g_n})$. Then $u_n \to u_P$ in $L^\infty([0, \infty); L^2(\Omega))$ by Theorem 5.4 and $u_n \to u_C$ in $L^2_{m,0}(\Omega)$ by Lemma 5.2. Hence $u_n \to u$ in $L^2_{m,0}(\Omega)$. Since $(u_0, 0)$ is a mild solution of the abstract Cauchy problem associated with $A_\eta$ and $A_\eta$ is the homogeneity of $(f_n, g_n)$, see the proof of Lemma 5.2 we obtain from [2, Proposition 5.6.7] that $C_\eta u_n = (i\eta - A_\eta)^{-1}(C_\eta f_n, C_\eta g_n)$ for all $\eta \in \mathbb{R}$. Passing to the limit we have the relation $C_\eta u_P = (i\eta - A_\eta)^{-1}(C_\eta f, C_\eta g)$. Thus $\text{Freq}(u_P) = \text{Freq}(f) \cup \text{Freq}(g)$.

Since $u_C \in C_\eta([0, \infty); L^2(\Omega))$ by Theorem 4.3 and $u_P(t) \perp k$ for all $t \geq 0$ by Proposition 4.1, we deduce that $u \in \text{AAP}([0, \infty); L^2(\Omega))$ and

$$
\text{Freq}(u) = \text{Freq}(u_P) + \text{Freq}(k) = \text{Freq}(f) \cup \text{Freq}(g) \cup \text{Freq}(k),
$$

which is a different way to write down the description of $\text{Freq}(u)$. \qed

We can also obtain an analogue of Theorem 5.4 in the more regular setting of continuous solutions.

**Theorem 5.5.** Let $r_1, q_1, r_2$ and $q_2$ be numbers in $[2, \infty)$ that satisfy relation (4.3). Assume that (4.2) and (4.3) hold and let $u_0 \in L^\infty(\Omega)$, $f \in \text{AAP}^{r_1}([0, \infty); L^{q_1}(\Omega))$ and $g \in \text{AAP}^{r_2}([0, \infty); L^{q_2}(\partial\Omega))$ satisfy (4.5). Then the weak solution $u$ of $(P_{u_0,f,g})$ is in $\text{AAP}([0, \infty); L^\infty(\Omega))$. For $\eta \neq 0$ we have $\eta \in \text{Freq}(u)$ if and only if $\eta \in \text{Freq}(u)$.
We have to use Corollary 4.8 instead of Theorem 4.7 and the realization of \( u \) which contradicts that
\[
\text{another motivation to give the details is that relevant parts in [17] contain some}
\]
\( \tau \)
\( \text{their} \)
\( L \)
\( u \)
\( \text{such that the weak solution} \)
\( u \)
\( \text{is easily proved by induction.} \)
\[
\text{we need, however, the following improvements over [17]:}
\]
\( \text{(i) the presence of the inhomogeneity} \)
\( g \)
\( \text{in} \)
\( (P_{w_0,f,g}) \)
\( \text{makes it necessary to keep} \)
\( \text{track of the measure of the sublevel sets of} \)
\( u_{|\partial\Omega}; \)
\( \text{(ii) we need a precise dependence of the constants on} \)
\( f \)
\( g \)
\( \text{more precisely,} \)
\( \text{these quantities have to enter linearly into the right hand side.} \)
\( \text{this is not} \)
\( \text{obvious from the proofs in [17], but can be asserted after some small modifications;}
\]
\( \text{(iii) we need an estimate that is local in time but global in space, whereas the} \)
\( \text{results in [17] are either global in both variables or local.} \)
\( \text{this requires only} \)
\( \text{trivial modifications.} \)
\[
\text{Another motivation to give the details is that relevant parts in [17] contain some misprints. For example, the relations between} \)
\( n, \hat{\tau} \)
\( \hat{q} \)
\( \text{in the proof of [17, Theorem III.7.1]} \)
\( \text{are faulty, as can be seen by taking} \)
\( n = 2, \hat{r} = q = 4 \)
\( \text{and} \)
\( \kappa = 1/2. \)
\[
\text{A more subtle mistake is the claim that the constant in [17, (II.6.11)] does not depend on} \)
\( \tau_0 \)
\( \theta \)
\( \text{this is wrong, which renders the seemingly precise elaboration of the dependence on} \)
\( \tau_0 \)
\( \text{and} \)
\( \theta \)
\( \text{useless. more precisely, a closer look at the proof} \)
\( \text{exhibits that the explicit constant given in [17, (II.6.25)] still contains} \)
\( \theta = \tau_0\theta_0^{-2}. \)
\[
\text{in fact, if estimate [17, (II.6.11)] was true, then applying it to the solution} \)
\( u \)
\( \text{of the heat equation with initial datum} \)
\( u_0 \)
\( \in L^2(\mathbb{R}^N) \)
\( L^\infty(\mathbb{R}^N) \)
\( \text{like in [17, §III.8] we could deduce that given a ball} \)
\( B \)
\( \subset \mathbb{R}^N \)
\( \text{we have} \)
\[
\sup_{\frac{T}{4} \leq t \leq T} \|u(t)\|_{L^\infty(B)}^2 \leq c\|u_0\|_{L^2(\mathbb{R}^N)}^2
\]
\( \text{for all} \)
\( T > 0 \)
\( \text{with a constant} \)
\( c \geq 0 \)
\( \text{that depends only on the radius of the ball, which contradicts that} \)
\( u(t) \rightarrow u_0 \)
\( \text{in} \)
\( L^2(\mathbb{R}^N). \)
\[
\text{for these reasons, we give a complete proof of Proposition 5.1.} \text{The only part of the argument that we copy from [17] without change is the following lemma, which is easily proved by induction.}
\]
\[
\text{Appendix A. Pointwise estimates via De Giorgi’s techniques}
\]
\text{In this section we prove Proposition 5.1. The proof is similar to what can be found in [17, §III.7–8], which in turn is a refined version of De Giorgi’s famous technique. We need, however, the following improvements over [17]:}
\text{(i) the presence of the inhomogeneity} \)
\( g \)
\( \text{in} \)
\( (P_{w_0,f,g}) \)
\( \text{makes it necessary to keep} \)
\( \text{track of the measure of the sublevel sets of} \)
\( u_{|\partial\Omega}; \)
\text{(ii) we need a precise dependence of the constants on} \)
\( f \)
\( g \)
\text{more precisely,}
\text{these quantities have to enter linearly into the right hand side.} \text{this is not}
\text{obvious from the proofs in [17], but can be asserted after some small modifications;}
\text{(iii) we need an estimate that is local in time but global in space, whereas the}
\text{results in [17] are either global in both variables or local.} \text{this requires only}
\text{trivial modifications.}
\[
\text{Another motivation to give the details is that relevant parts in [17] contain some misprints. For example, the relations between} \)
\( n, \hat{\tau} \)
\( \hat{q} \)
\( \text{in the proof of [17, Theorem III.7.1]} \)
\( \text{are faulty, as can be seen by taking} \)
\( n = 2, \hat{r} = q = 4 \)
\( \text{and} \)
\( \kappa = 1/2. \)
\[
\text{A more subtle mistake is the claim that the constant in [17, (II.6.11)] does not depend on} \)
\( \tau_0 \)
\( \theta \)
\( \text{this is wrong, which renders the seemingly precise elaboration of the dependence on} \)
\( \tau_0 \)
\( \theta \)
\( \text{useless. more precisely, a closer look at the proof} \)
\( \text{exhibits that the explicit constant given in [17, (II.6.25)] still contains} \)
\( \theta = \tau_0\theta_0^{-2}. \)
\[
\text{in fact, if estimate [17, (II.6.11)] was true, then applying it to the solution} \)
\( u \)
\( \text{of the heat equation with initial datum} \)
\( u_0 \)
\( \in L^2(\mathbb{R}^N) \)
\( L^\infty(\mathbb{R}^N) \)
\( \text{like in [17, §III.8] we could deduce that given a ball} \)
\( B \)
\( \subset \mathbb{R}^N \)
\( \text{we have} \)
\[
\sup_{\frac{T}{4} \leq t \leq T} \|u(t)\|_{L^\infty(B)}^2 \leq c\|u_0\|_{L^2(\mathbb{R}^N)}^2
\]
\( \text{for all} \)
\( T > 0 \)
\( \text{with a constant} \)
\( c \geq 0 \)
\( \text{that depends only on the radius of the ball, which contradicts that} \)
\( u(t) \rightarrow u_0 \)
\( \text{in} \)
\( L^2(\mathbb{R}^N). \)
\[
\text{for these reasons, we give a complete proof of Proposition 5.1. The only part of the argument that we copy from [17] without change is the following lemma, which is easily proved by induction.}
Lemma A.1 ([17, Lemma II.5.7]). Let \((y_n)_{n \in \mathbb{N}_0}\) and \((z_n)_{n \in \mathbb{N}_0}\) be sequences of non-negative real numbers such that
\[
y_{n+1} \leq cb^N\left(y_1^+ + z_1^+ + y_n^+ + z_n^+ + r_n\right)\quad \text{and} \quad z_{n+1} \leq cb^N(z_n + r_n^+ + y_n)
\]
for all \(n \in \mathbb{N}_0\) with positive constants \(c, b, \varepsilon\) and \(\delta\), where \(b \geq 1\). Define
\[
d := \min\{\delta, \frac{\varepsilon}{1+b}\} \quad \text{and} \quad \lambda := \min\{(2c)^{-\frac{1}{2}}b^{-\frac{1}{2}}, (2c)^{-\frac{1}{2}}b^{-\frac{1}{2}}\}
\]
and assume that
\[
y_0 \leq \lambda \quad \text{and} \quad z_0 \leq \lambda^+.
\]
Then
\[
y_n \leq \lambda b^{-\frac{\varepsilon}{2}} \quad \text{and} \quad z_n \leq (\lambda b^{-\frac{\varepsilon}{2}})^{\frac{1}{1+b}}
\]
for all \(n \in \mathbb{N}_0\).

We imitate the notation of [17] to a certain degree. More precisely, let \(\Omega \subset \mathbb{R}^N\) be a bounded Lipschitz domain and \(T > 0\). It will be convenient to work with functions defined for negative times, so we will always assume that \(u \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H^1(\Omega))\). In that case we write
\[
\|u\|_{Q(\tau)}^2 := \sup_{-\tau \leq t \leq 0} \int_\Omega |u(t)|^2 + \int_{-\tau}^0 \int_\Omega |\nabla u(t)|^2.
\]
and for \(k \geq 0\) we define
\[
u^{(k)}(t) := (u(t) - k)^+.
\]
In what follows we will frequently need that for \(r_1 \in [2, \infty], q_1 \in [2, \frac{2N}{N-2}], r_2 \in [2, \infty]\) and \(q_2 \in [2, \frac{2(N-1)}{N-2}]\) satisfying
\[
\frac{1}{r_1} + \frac{N}{2q_1} = \frac{N}{4} \quad \text{and} \quad \frac{1}{r_2} + \frac{N - 1}{2q_2} = \frac{N}{4},
\]
we have
\[
\|u\|_{L^{r_1}(-\tau, 0; L^{q_1}(\Omega))} + \|u\|_{L^{r_2}(-\tau, 0; L^{q_2}(\partial\Omega))} \leq c\|u\|_{Q(\tau)}, \tag{A.1}
\]
where \(c \geq 0\) depends only on \(\Omega, r_1, q_1, r_2\) and \(q_2\). This anisotropic Sobolev inequality follows from the multiplicative Sobolev inequalities on \(\Omega\), see [17, §II.3].

We start with a modified version of [17, Theorem II.6.2].

Theorem A.2. Let \(\Omega \subset \mathbb{R}^N\) be a bounded Lipschitz domain, \(N \geq 2\). Fix \(T > 0\) and \(u \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H^1(\Omega))\). Let \(r_1, r_2, q_1, q_2, \ell \in [2, \frac{2N}{N-2}]\), \(r_2, \ell \in [2, \infty]\) and \(q_2, \ell \in [2, \frac{2(N-1)}{N-2}]\) satisfy
\[
\frac{1}{r_{1, \ell}} + \frac{N}{2q_{1, \ell}} = \frac{N}{4}, \quad (1 \leq \ell \leq L_1) \quad \text{and} \quad \frac{1}{r_{2, \ell}} + \frac{N - 1}{2q_{2, \ell}} = \frac{N}{4}, \quad (1 \leq \ell \leq L_2). \tag{A.2}
\]
Assume that there exist \(\hat{k} \geq 0\), \(\gamma \geq 0\) and numbers \(\kappa_{1, \ell} > 0\) and \(\kappa_{2, \ell} > 0\) such that
for all \(\tau \in (0, T], \sigma \in (0, \frac{1}{4})\) and \(k \geq \hat{k}\) we have
\[
\|u^{(k)}\|_{Q((1-\sigma)\tau)}^2 \leq \frac{\gamma}{\sigma^2} \int_{-\tau}^0 \int_\Omega |u^{(k)}(t)|^2 + \gamma k^2 \sum_{\ell = 1}^{L_1} \left(\int_{-\tau}^0 |A_{k}(t)|^{q_{1, \ell}} \right)^{2/(1+q_{1, \ell})} \tag{A.3}
\]
\[
+ \gamma k^2 \sum_{\ell = 1}^{L_2} \left(\int_{-\tau}^0 |B_{k}(t)|^{q_{2, \ell}} \right)^{2/(1+q_{2, \ell})}.
\]
\[
\text{Then}
\]
\[
\text{ess sup}_{(t,x) \in [-\frac{T}{4}, 0] \times \Omega} u(t, x) \leq c \left(\int_{-T}^0 \int_\Omega |u(t)|^2 + \hat{k}^2\right)^{\frac{1}{2}}, \tag{A.4}
\]
where the constant \(c \geq 0\) is independent of \(u\) and \(\hat{k}\).
Proof. In the proof the constants $c, c_0, c_1$ and $c_2$ never depend on $u$ and $\hat{k}$. Moreover, $c$ is a generic constant in the sense that it may change its numeric value between occurrences.

Since $|A_k(t)| \leq |\Omega|$ and $|B_k(t)| \leq |\partial \Omega|$ estimate $[\text{A.3}]$ remains valid if we replace all the $\kappa_{1,\ell}$ and $\kappa_{2,\ell}$ by their least member

$$\kappa := \min \{ \kappa_{1,1}, \ldots, \kappa_{1,L_1}, \kappa_{1,L_1}, \kappa_{2,1}, \kappa_{2,1} \} > 0$$

provided we replace $\gamma$ by a larger constant $\gamma'$ that depends on $\kappa_{1,\ell}$, $\kappa_{2,\ell}$, $r_{1,\ell}$, $q_{1,\ell}$ $r_{2,\ell}$, $q_{2,\ell}$, $T$, $\gamma$, $|\Omega|$ and $|\partial \Omega|$. Thus we may assume without loss of generality that $\kappa_{1,\ell} = \kappa$ for all $1 \leq \ell \leq L_1$ and $\kappa_{2,\ell} = \kappa$ for all $1 \leq \ell \leq L_2$.

Let $M \geq \hat{k}$ be arbitrary and define

$$\tau_n := (1 + 2^{-(n+1)})^\ell \in \left[ \frac{\tau_n}{2}, T \right],$$

$$k_n := (2 - 2^{-n})M \geq \hat{k},$$

$$y_n := \frac{1}{M^2} \int_{-\tau_{n}}^{0} \int_{\Omega} |u_k(t)|^2, $$

$$z_n := \sum_{\ell=1}^{L_2} \left( \int_{-\tau_{n}}^{0} |A_{k_n}(t)|^{\frac{r_{1,\ell}}{r_{1,\ell}}} \right)^\frac{q_{1,\ell}}{q_{1,\ell}} + \sum_{\ell=1}^{L_2} \left( \int_{-\tau_{n}}^{0} |B_{k_n}(t)|^{\frac{r_{2,\ell}}{r_{2,\ell}}} \right)^\frac{q_{2,\ell}}{q_{2,\ell}}$$

for all $n \in \mathbb{N}_0$.

To this end, let $n \in \mathbb{N}_0$ be fixed. From $[\text{A.1}]$ and the trivial estimate

$$|u_k(t)|^2 \geq (k_{n+1} - k_n)^2 \mathbb{I}_{A_{k_{n+1}}(t)}$$

we obtain that

$$M^2 y_{n+1} \leq c \left( \int_{-\tau_{n+1}}^{0} |A_{k_{n+1}}(t)| \right)^\frac{q_{1,\ell}}{q_{1,\ell}} \|u_k(t)\|_2^2 \|A_{k_{n+1}}(t)\|^2 \Omega(\tau_{n+1})$$

$$\leq c \left( (k_{n+1} - k_n)^{-2} M^2 y_n \right)^\frac{q_{1,\ell}}{q_{1,\ell}} \|u_k(t)\|_2^2 \|A_{k_{n+1}}(t)\|^2 \Omega(\tau_{n+1})$$

$$\leq c 2^{2(n+1)} y_n \|u_k(t)\|_2^2 \|A_{k_{n+1}}(t)\|^2 \Omega(\tau_{n+1}).$$

Similarly,

$$2^{-2(n+1)} M^2 z_{n+1} = (k_{n+1} - k_n)^2 \mathbb{I}_{A_{k_{n+1}}(t)}$$

$$\leq \sum_{\ell=1}^{L_2} \left( \int_{-\tau_{n+1}}^{0} \int_{\Omega} |u_k(t)|^{q_{1,\ell}} \right)^\frac{1}{q_{1,\ell}} \left( \int_{-\tau_{n+1}}^{0} \int_{\Omega} |u_k(t)|^{q_{2,\ell}} \right)^\frac{1}{q_{2,\ell}} \|u_k(t)\|_2^2 \left( \int_{-\tau_{n+1}}^{0} \int_{\partial \Omega} |u_k(t)|^{r_{1,\ell}} \right)^\frac{1}{r_{1,\ell}} \|u_k(t)\|_2^2 \left( \int_{-\tau_{n+1}}^{0} \int_{\partial \Omega} |u_k(t)|^{r_{2,\ell}} \right)^\frac{1}{r_{2,\ell}}$$

$$\leq c \|u_k(t)\|_2^2 \Omega(\tau_{n+1})$$

Moreover, from $[\text{A.3}]$ applied with $\tau = \tau_n$ and $\sigma = 1 - \frac{\gamma \tau_n}{2} \geq 2^{-(n+3)}$ we get that

$$\|u_k(t)\|_2^2 \Omega(\tau_{n+1}) \leq \|u_k(t)\|_2^2 \Omega(\tau_{n+1}) \leq \frac{\gamma}{\sigma \tau_n} M^2 y_n + \frac{\gamma k_n^{2+\kappa}}{\sigma}$$

$$\leq \gamma M^2 2^{n+4} \left( T^{-1} + 1 \right) (y_n + \mathbb{I}_{A_{k_{n+1}}(t)})$$

Combining $[\text{A.5}]$, $[\text{A.6}]$ and $[\text{A.7}]$ we obtain with $\delta := \frac{2}{N+2}$ that

$$\begin{cases}
  y_{n+1} \leq c_0 2^{3n} (y_n + \mathbb{I}_{A_{k_{n+1}}(t)}) \\
  z_{n+1} \leq c_0 2^{3n} (y_n + \mathbb{I}_{A_{k_{n+1}}(t)})
\end{cases}$$

for all $n \in \mathbb{N}_0$. 
Next we want to estimate \( y_0 \) and \( z_0 \) for large \( M \). On the one hand, we have
\[
y_0 \leq \frac{1}{M^2} \int_{-T}^{0} \int_{\Omega} |u(t)|^2. \tag{A.9}
\]
On the other hand, similarly to (A.9) and (A.10), we have
\[
(M - \hat{k})^2 z_0 \leq \sum_{\ell = 1}^{L_1} \left( \int_{-\tau_0}^{0} \int_{\Omega} |u(\bar{k})(t)|^2 \frac{T}{T-t} \right)^{\frac{1}{2}}
+ \sum_{\ell = 1}^{L_2} \left( \int_{-\tau_0}^{0} \int_{\partial\Omega} |u(\bar{k})(t)|^2 \frac{T}{T-t} \right)^{\frac{1}{2}}
\leq c_1 \|u(\bar{k})\|_{Q(\tau_0)}
\leq \frac{4\gamma k}{T} \int_{-T}^{0} \int_{\Omega} \left| u(\bar{k})(t) \right|^2 + \gamma \hat{k}^2 (T |\Omega|)^{\frac{2(1+\gamma)}{\gamma}} + \gamma \hat{k}^2 (T |\partial\Omega|)^{\frac{2(1+\gamma)}{\gamma}}
\]
so that
\[
z_0 \leq \frac{c_1}{(M - \hat{k})^2} \left( \int_{-T}^{0} \int_{\Omega} |u(t)|^2 + \hat{k}^2 \right) \tag{A.10}
\]
for all \( M \geq \hat{k} \). Define \( d := \min\{\delta, \frac{1}{T}\} \) and
\[
\lambda := \min\{(2c_0)^{-\frac{1}{2}} - \frac{1}{\tau}, (2c_0)^{-\frac{1}{2}} - \frac{1}{\tau}\}.
\]
Then for
\[
M := \max\left\{ \lambda^{-\frac{1}{2}} \left( \int_{-T}^{0} \int_{\Omega} |u(t)|^2 \right)^{\frac{1}{4}}, \hat{k} + \lambda^{-\frac{1}{2}} \int_{-T}^{0} \int_{\Omega} |u(t)|^2 + \hat{k}^2 \right\}
\leq c_2 \left( \int_{-T}^{0} \int_{\Omega} |u(t)|^2 + \hat{k}^2 \right)^{\frac{1}{4}}
\]
we obtain from (A.9) and (A.10) that
\[
\begin{align*}
y_0 & \leq \lambda \\
z_0 & \leq \lambda^{\frac{1}{4}}. \tag{A.12}
\end{align*}
\]

Estimates (A.8) and (A.12) show in view of Lemma A.1 that \( z_n \to 0 \) as \( n \to \infty \), which implies that \( u(t) \leq \lim_{n \to \infty} k_n = 2M \) almost everywhere on \( \Omega \) for almost every \( t \in \{k_n \in \mathbb{N} \mid \tau_n, 0 = [-\frac{T}{2}, 0] \) if we define \( M \) as in (A.11). This is (A.13). \( \square \)

Theorem A.2 is a local estimate in time therefore allows us to estimate the solution of \( (P_{u,v}) \) independently of the initial value \( u_0 \). The price is that we obtain estimates only away from \( t = 0 \). We also need the following modification of Theorem A.2 that gives good estimates for small \( t \).

**Corollary A.3.** In the situation of Theorem A.2, assume that instead of (A.3) we even have
\[
\|u(k)\|_{Q_1(\tau)}^2 \leq \gamma \int_{-T}^{0} \int_{\Omega} |u(k)(t)|^2 + \gamma \hat{k}^2 \sum_{\ell = 1}^{L_1} \left( \int_{-T}^{0} |A_k(t)|^{\frac{2(1+\gamma)}{\gamma}} \right)^{\frac{1}{2}}
+ \gamma \hat{k}^2 \sum_{\ell = 1}^{L_2} \left( \int_{-T}^{0} |B_k(t)|^{\frac{2(1+\gamma)}{1+\gamma}} \right)^{\frac{1}{2}}
\]
for all \( k \geq \hat{k} \). Then
\[
\text{ess sup}_{t \in [-T,0], x \in \Omega} u(t,x) \leq c \left( \int_{-T}^{0} \int_{\Omega} |u(t)|^2 + \hat{k}^2 \right)^{\frac{1}{4}}
\]
for all $t \in [-T, 0]$, where the constant $c \geq 0$ is independent of $u$ and $\hat{k}$.

**Proof.** The proof is very similar to the one of Theorem A.2. In fact, we only have to notice that after changing the definition of $\tau_n$ to $\tau_n := T$ for all $n \in \mathbb{N}$ the rest of the proof carries over verbatim with the mere exception that this time we have $\bigcap_{n \in \mathbb{N}} [-\tau_n, 0] = [-T, 0]$, which gives the result.

Before we can check that Theorem A.2 applies to the solutions of $(P_{uo,f,g})$, we have to supply the following tool for the calculations.

**Lemma A.4.** Let $T > 0$, $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ and $k \geq 0$. Define $u^{(k)}(t) := (u(t) - k)^+$ for $t \geq 0$. Then $u^{(k)} \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ with derivative $(u^{(k)})_t(t) = u_t(t) 1_{\{u(t) > k\}}$ and $\nabla u^{(k)}(t) = \nabla u(t) 1_{\{u(t) > k\}}$. Moreover, $u^{(k)}(t)|_{\partial \Omega} = (u|_{\partial \Omega}(t) - k)^+$.

**Proof.** After identifying $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ with $H^1((0, T) \times \Omega)$ up to equivalent norms in the obvious way, the formulas for the derivatives follow from the chain rule for weakly differentiable functions, see for example [12, Theorem 7.8]. The assertion about the trace is true for continuous functions and thus by approximation for all functions under consideration.

We now prove Proposition 3.1 for classical $L^2$-solutions. Basically, we will check that every solution of $(P_{uo,f,g})$ satisfies $A.3$.

**Lemma A.5.** Proposition 3.1 holds if in addition we assume that $u$ is a classical $L^2$-solution and $T \leq T_0$, where $T_0 > 0$ depends only on $N$, $\Omega$, $r_1$, $q_1$, $r_2$, $q_2$ and the coefficients of the equation.

**Proof.** After a linear substitution in the time variable we may consider problem $(P_{uo,f,g})$ on $[-T, 0]$ instead of $[0, T]$, the initial value now being $u_0 = u(-T)$. We check the conditions of Theorem A.2 with

$$\hat{k}^2 := \|f\|_{L^2(-T, 0; L^2(\Omega))}^2 + \|g\|_{L^2(-T, 0; L^2(\Omega))}^2.$$  \hfill (A.13)

Fix $0 < \tau \leq T$ and let $\zeta$ be a function in $H^1(-\tau, 0)$ satisfying $0 \leq \zeta(t) \leq 1$ for all $t \in [-\tau, 0]$. Assume either that $\zeta(-\tau) = 0$ or that $\tau = T$ and $u^{(k)}(-T) = 0$. Then for $t \in [-\tau, 0]$ we have

$$\zeta(t)^2 \cdot \frac{1}{2} \int_\Omega |u^{(k)}(t)|^2 = \int_{-\tau}^{t} \frac{d}{ds} \left( \zeta(s)^2 \cdot \frac{1}{2} \int_\Omega |u^{(k)}(s)|^2 \right)$$

$$= \int_{-\tau}^{t} \zeta(s) \zeta'(s) \int_\Omega |u^{(k)}(s)|^2 + \int_{-\tau}^{t} \zeta(s)^2 \int_\Omega u^{(k)}(s) u_t^{(k)}(s).$$

From Lemma A.4 and the fact that $u$ is a classical $L^2$-solution of $(P_{uo,f,g})$ we obtain that for all $s \in [-\tau, 0]$ we have

$$\int_\Omega u_t^{(k)}(s) u^{(k)}(s) = \int_\Omega u_t(s) u^{(k)}(s) = \int_\Omega (Au(s) + f(s)) u^{(k)}(s)$$

$$= \int_\Omega f(s) u^{(k)}(s) + \int_{\partial \Omega} g(s) u^{(k)}(s) - a_\beta(u(s), u^{(k)}(s)).$$

\hfill (A.15)
We now estimate the right hand side of (A.15). From Lemma (A.4) (2.7) and Young’s inequality we obtain that
\[
\tau_a \beta(u(s), u^k(s))
\]
\[
= a \beta(u^k(s), u^k(s)) + \sum_{j=1}^N b_j kD_j u^k(s) + \int_\Omega d \beta k u^k(s)
\]
\[
\geq \frac{\mu}{2} \int_\Omega |\nabla u^k(s)|^2 - \omega \int_\Omega |u^k(s)|^2 - k^2 \mu \sum_{j=1}^N \int_{A_k(s)} |b_j|^2 - \frac{\mu}{4} \int_\Omega |\nabla u^k(s)|^2
\]
\[
- \int_{A_k(s)} |df| (|u^k(s)|^2 + k^2) - \int_{B_k(s)} |\beta| (|u^k(s)|^2 + k^2).
\]
Using (A.15) and again Young’s inequality this gives
\[
\int_\Omega u^k(s) (u^k(s)) \leq \frac{\mu}{4} \int_\Omega |\nabla u^k(s)|^2 + \int_{A_k(s)} \left( \frac{k}{2} |f(s)| + D_0 \right) (|u^k(s)|^2 + k^2)
\]
\[
+ \int_{B_k(s)} \left( \frac{k}{2} |g(s)| + |\beta| \right) (|u^k(s)|^2 + k^2) \tag{A.16}
\]
with
\[
D_0 := \omega + \frac{1}{\mu} \sum_{j=1}^N |b_j|^2 + |df| \in L^\frac{2}{r}(\Omega),
\]
where \( q > N \). Plugging (A.16) into (A.14) and varying over \( t \) we arrive at the estimate
\[
\min \left\{ \frac{\mu}{\omega}, \frac{4}{\omega} \right\} ||u^k||^2_{Q(t)}
\]
\[
\leq \sup_{-\tau \leq t \leq 0} \left( \zeta(t)^2 + \frac{1}{2} \int_\Omega |u^k(t)|^2 \right) + \frac{\mu}{4} \int_{-\tau}^0 \zeta(s)^2 \int_\Omega |\nabla u^k|^2
\]
\[
\leq ||\zeta||_{L^\infty(-\tau,0)} \int_{-\tau}^0 \int_\Omega |u^k(s)|^2
\]
\[
+ \int_{-\tau}^0 \int_{A_k(s)} \left( \frac{k}{2} |f(s)| + D_0 \right) (|\zeta|^2 |u^k(s)|^2 + k^2)
\]
\[
+ \int_{-\tau}^0 \int_{B_k(s)} \left( \frac{k}{2} |g(s)| + |\beta| \right) (|\zeta|^2 |u^k(s)|^2 + k^2) \tag{A.17}
\]
We estimate the right hand side of (A.17). Define \( \kappa_1 > 0 \) and \( \kappa_2 > 0 \) by
\[
\frac{1}{r_1} + \frac{N}{2q_1} = 1 - \frac{\kappa_1 N}{2} \quad \text{and} \quad \frac{1}{r_2} + \frac{N - 1}{2q_2} = 1 - \frac{\kappa_2 N}{2} \tag{A.18}
\]
With \( \tilde{r}_1 := \frac{2^q_1}{r_1} \) and \( \tilde{q}_1 := \frac{2q_1}{q_1 - 1} \) we obtain from Hölder’s inequality that
\[
\int_{-\tau}^0 \int_{A_k(s)} |f(s)| \cdot |\zeta(s)|^2 |u^k(s)|^2
\]
\[
\leq ||f||_{L^{\tilde{r}_1}(-\tau,0;L^{\tilde{q}_1}(\Omega))} ||\zeta u^k||_{L^{r_1}(-\tau,0;L^{q_1}(\Omega))}^2 \leq \tilde{k} ||u^k||_{L^{r_1}(-\tau,0;L^{q_1}(\Omega))}^2
\]
The last factor tends to zero as \( \tau \to 0 \). Since moreover \( \frac{r_1}{2(q_1 + \kappa_1 s)} + \frac{N}{2(q_1 + \kappa_1 s)} = \frac{q_1}{r_1} \) by (A.18), we deduce from (A.17) that
\[
\int_{-\tau}^0 \int_{A_k(s)} |f(s)| \cdot |\zeta(s)|^2 |u^k(s)|^2 \leq \frac{k}{8} \min \left\{ \frac{\mu}{\omega}, \frac{4}{\omega} \right\} ||u^k||^2_{Q(t)}
\]
if \( \tau \) is sufficiently small, say \( \tau \leq T_0 \), where \( T_0 \) depends on \( \mu, N, \Omega, \kappa_1, r_1, q_1 \). Similarly, since \( \frac{2q}{q-2} < \frac{2N}{N-2} \) we obtain that

\[
\int_{-\tau}^{0} \int_{A_k(s)} D_0 \cdot \zeta(s)^2 |u^{(k)}(s)|^2 \leq \|D_0\|_{L^{\frac{0}{2}}(\Omega)} \|\zeta u^{(k)}\|_{L^2(-\tau, 0, L^{\frac{2q}{q-2}}(\Omega))}^2
\leq \frac{1}{8} \min\{\frac{1}{2}, \frac{\mu}{\tau}\} \|\zeta u^{(k)}\|_{Q(\tau)}^2
\]

for \( \tau \leq T_0 \) with some possibly smaller \( T_0 > 0 \) that depends in addition on \( D_0 \) and \( q \).

Analogously, with \( \bar{r}_2 := \frac{2q}{r_2 - 1} \) and \( \bar{q}_2 := \frac{2q_2}{q_2 - 1} \) we have

\[
\int_{-\tau}^{0} \int_{B_k(s)} |g(s)| \cdot \zeta(s)^2 |u^{(k)}(s)|^2
\leq \frac{k}{8} \|\zeta u^{(k)}\|_{L^{(1+\tau)}(\Omega)} \|\zeta u^{(k)}\|_{L^2(-\tau, 0, L^{\frac{2q_2}{q_2-1}}(\Omega))}^2
\leq \frac{1}{8} \min\{\frac{1}{2}, \frac{\mu}{\tau}\} \|\zeta u^{(k)}\|_{Q(\tau)}^2
\]

and since \( \frac{2q-1}{q-2} < \frac{2(N-1)}{N-2} \) also

\[
\int_{-\tau}^{0} \int_{B_k(s)} |\beta| \cdot \zeta(s)^2 |u^{(k)}(s)|^2 \leq \|\beta\|_{L^{\frac{1}{(1+\tau)}(\Omega)}} \|\zeta u^{(k)}\|_{L^2(-\tau, 0, L^{\frac{2q_2}{q_2-1}}(\Omega))}^2
\leq \frac{1}{8} \min\{\frac{1}{2}, \frac{\mu}{\tau}\} \|\zeta u^{(k)}\|_{Q(\tau)}^2
\]

for \( \tau \leq T_0 \), where this new \( T_0 \) depends also on \( r_2, q_2, \kappa_2 \) and \( \beta \).

Combining the latter estimates with (A.17) we obtain that

\[
\|\zeta u^{(k)}\|_{Q(\tau)}^2 \leq c_\mu \|\zeta\|_{L^\infty(-\tau, 0)} \int_{-\tau}^{0} \int_{\Omega} |u^{(k)}(s)|^2 + c_\mu k^2 \int_{-\tau}^{0} \int_{A_k(s)} \left( \frac{1}{k} |f(s)| + D_0 \right)
\]

\[
+ c_\mu k^2 \int_{-\tau}^{0} \int_{B_k(s)} \left( \frac{1}{k} |g(s)| + |\beta| \right) \tag{A.19}
\]

if \( \tau \leq T_0 \) and \( k \geq \bar{k} \), where \( c_\mu \) depends only on \( \mu \).

Now we estimate for \( k \geq \bar{k} \)

\[
\int_{-\tau}^{0} \int_{A_k(s)} \frac{1}{k} |f(s)| \leq \frac{1}{k} \|f\|_{L^{\frac{1}{1+\tau}}(-\tau, 0, L^{\frac{2q_1}{q_1-1}}(\Omega))} \|\tau A_k\|_{L^{\frac{2q_1}{q_1-1}}(-\tau, 0, L^{\frac{2q_1}{q_1-1}}(\Omega))}
\leq \|\tau A_k\|_{L^{\frac{2q_1}{q_1-1}}(-\tau, 0, L^{\frac{2q_1}{q_1-1}}(\Omega))}
\leq \|\tau A_k\|_{L^{(1+\kappa_1)}(-\tau, 0, L^{\frac{2q_1}{q_1-1}}(\Omega))}
\]

with \( \kappa_{1,1} := \kappa_1, r_{1,1} := 2(1 + \kappa_1) \frac{r_1}{r_1 - 1} \) and \( q_{1,1} := 2(1 + \kappa_1) \frac{q_1}{q_1 - 1} \) and similarly

\[
\int_{-\tau}^{0} \int_{A_k(s)} D_0 \leq \|D_0\|_{L^{\frac{0}{2}}(\Omega)} \|\tau A_k\|_{L^{\frac{2q_2}{q_2-1}}(-\tau, 0, L^{\frac{2q_2}{q_2-1}}(\Omega))}
\leq \|D_0\|_{L^{\frac{0}{2}}(\Omega)} \|\tau A_k\|_{L^{(1+\kappa_2)}(-\tau, 0, L^{\frac{2q_2}{q_2-1}}(\Omega))}
\]

with \( \kappa_{1,2} := \frac{2(q-N)+(q-2)N}{qN}, r_{1,2} := 2(1 + \kappa_{1,2}) \) and \( q_{1,2} := 2(1 + \kappa_{1,2}) \frac{q}{q-2} \). Analogously,

\[
\int_{-\tau}^{0} \int_{B_k(s)} \frac{1}{k} |g(s)| \leq \|\tau B_k\|_{L^{\frac{2(1+\kappa_{2,1})}{q_2-2}}(-\tau, 0, L^{\frac{2q_2}{q_2-1}}(\Omega))}
\]
with \( \kappa_{2,1} := \kappa_2 \), \( r_{2,1} := 2(1 + \kappa_{2,1}) \frac{q_2}{r_2 - 1} \) and \( q_{2,1} := 2(1 + \kappa_{2,1}) \frac{q_2}{q_2 - 1} \) and
\[
\int_{-\tau}^{0} \int_{B_k(s)} |\beta| \leq \| \beta \|_{L^{q_2 - 1}(\Omega)} \| f \|_{L^{2(1+\kappa_{2,2})}(\tau, 0; L^{q_2}(\Omega))}
\]
with \( \kappa_{2,2} := \frac{N(q-N)+2(N-1)}{q-1} \), \( r_{2,2} := 2(1+\kappa_{2,2}) \) and \( q_{2,2} := 2(1+\kappa_{2,2}) \frac{q_2}{q_2 - 1} \). Thus (A.19) yields
\[
\| \zeta u^{(k)} \|_{L^{2(q(1-\sigma))}(\Omega)}^2 \leq c\mu \| \zeta \|_{L^{\infty}(\tau,0)} \int_0^\tau \int_\Omega |u^{(k)}(s)|^2 + c\kappa^2 \sum_{\ell=1}^2 \left( \int_{-\tau}^0 \| A_k(s) \|_{\tau,\ell}^{(1,\kappa_{1,\ell})} \| \zeta u^{(k)} \|_{\tau,\ell} \right)^{2(1+\kappa_{1,\ell})} \frac{1}{\tau,\ell}
\]
Moreover, (A.18) implies that the parameters \( r_{i,\ell} \) and \( q_{i,\ell} \) satisfy (A.2) for \( i = 1, 2 \) and \( \ell = 1, 2 \) as elementary calculations show.

If we pick \( \zeta(t) := \frac{t}{\tau} \) for \( t \in [-\tau, -1-\sigma] \) and \( \zeta(t) := 1 \) for \( t \in [-(1-\sigma)\tau, 0) \) with some given \( \sigma \in (0, \frac{1}{2}) \), we have
\[
\| u^{(k)} \|_{L^{2(q(1-\sigma))}(\Omega)}^2 \leq \| \zeta u^{(k)} \|_{L^{2(q(\sigma))}(\Omega)}^2
\]
and \( \| \zeta \|_{L^{\infty}(-\tau,0)} \leq \frac{1}{\tau} \) if \( T \leq T_0 \), where \( c \) depends only on \( \mu, D_0 \) and \( \beta \). Thus (A.20) implies (A.3). Hence by Theorem A.2 applied to \( u \) and \( -u \), the latter being a classical solution of \((P_{u_0, -f, g})\), we obtain (A.2).

If in addition \( u(-T) = 0 \), then we can set \( \tau := T \) and choose \( \zeta(t) := 1 \) for all \( t \in [-T, 0] \). Now using Corollary A.3 instead of Theorem A.2 we obtain (A.3) from (A.20) like above.

We finally make the step from classical \( L^2 \)-solutions to weak solutions and drop the assumption that \( T \) be small enough, thus proving Proposition 3.1.

**Proof of Proposition 3.1** Let \( u \) be the weak solution of \((P_{u_0, f, g})\). Pick a sequence \((u_{n,n}) \in D(A_{2,\ell,s})\) that satisfies \( u_{n,n} \to u_0 \) in \( L^2(\Omega) \), which exists since by Proposition 2.4 the operator \( A_{2,\ell,s} \) is a generator of a strongly continuous semigroup and hence densely defined. Pick sequences \((f_n) \) and \((g_n) \in C^0([0,T]; L^p(\Omega)) \) and \( C^2([0,T]; L^p(\Omega)) \) respectively, that satisfy \( f_n \to f \) in \( L^p(0,T; L^p(\Omega)) \) and \( g_n \to g \) in \( C^0(\Omega) \), while \( f_n(0) = 0 \) and \( g_n(0) = 0 \) for all \( n \in \mathbb{N} \). Then problem \((P_{u_0,f_n,g_n})\) has a unique classical \( L^2 \)-solution \( u_n \) by Proposition 2.4 and as in the proof of Theorem 2.10 we see that \( u_n \to u \) in \( C(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \).

Pick \( T_0 > 0 \) as in Lemma A.3. Shrinking \( T_0 \), if necessary, we can assume that \( T_0 \leq T \). Let \( I \subset \left[ \frac{T}{2}, T_0 \right] \) be an interval of length at most \( \frac{T}{4} \). Applying (A.2) for the classical \( L^2 \)-solutions \( u_{n,n} \) and \( u_{n,m} \) on \( I \), which is allowed by Lemma A.5 we obtain that
\[
\| u_n \|_{L^\infty(I; L^\infty(\Omega))} \leq c \int_0^T \int_\Omega |u_n(s)|^2 + c \| f_n \|_{L^2(0,T; L^2(\Omega))}^2 + c \| g_n \|_{L^2(0,T; L^2(\Omega))}^2 \qquad \text{(A.21)}
\]
and that \((u_n)\) is a Cauchy sequence in \( L^\infty(I; L^\infty(\Omega)) \). Hence \( u_n \to u \) in \( L^\infty(I; L^\infty(\Omega)) \) and passing to the limit in (A.21) we have
\[
\| u \|_{L^\infty(I; L^\infty(\Omega))} \leq c \int_0^T \int_\Omega |u(s)|^2 + c \| f \|_{L^2(0,T; L^2(\Omega))}^2 + c \| g \|_{L^2(0,T; L^2(\Omega))}^2 \quad \text{(A.22)}
\]
Covering \( \left[ \frac{T}{2}, T \right] \) by finitely many intervals of length at most \( \frac{T}{2} \) and using (A.22) for each of these intervals we obtain (3.2).
If in addition $u_0 = 0$, then we can pick $u_{0,n} := 0$ and the same strategy as above yields that

$$\|u\|_{L^\infty(0,T_0,L^\infty(\Omega))} \leq c \int_0^T \int_\Omega |u(s)|^2 + c\|f\|_{L^{r_1}(0,T;L^{q_1}(\Omega))}^2 + c\|g\|_{L^{r_2}(0,T;L^{q_2}(\Omega))}^2.$$  

Using in addition (3.2) to estimate $\|u\|_{L^\infty(I,L^\infty(\Omega))}$ for finitely many intervals $I$ of length $\frac{T_0}{2}$ that cover $[T_0, T]$, we have proved also (3.3).

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