Smoothed Analysis of Trie Height
by Star-like PFAs

Stefan Eckhardt¹, Sven Kosub², and Johannes Nowak¹

¹ Fakultät für Informatik, Technische Universität München, Boltzmannstraße 3, D-85748 Garching, Germany
{eckhardt,nowak}@in.tum.de
² Fachbereich Informatik und Informationswissenschaft, Universität Konstanz, Box D 67, D-78457 Konstanz, Germany
kosub@inf.uni-konstanz.de

Abstract. Tries are general purpose data structures for information retrieval. The most significant parameter of a trie is its height $H$ which equals the length of the longest common prefix of any two strings in the set $A$ over which the trie is built. Analytical investigations of random tries suggest that $\mathbb{E}[H] \in O(\log(|A|))$, although $H$ is unbounded in the worst case. Moreover, sharp results on the distribution function of $H$ are known for many different random string sources. But because of the inherent weakness of the modeling behind average-case analysis—analyses being dominated by random data—these results can utterly explain the fact that in many practical situations the trie height is logarithmic. We propose a new semi-random string model and perform a smoothed analysis in order to give a mathematically more rigorous explanation for the practical findings. The perturbation functions which we consider are based on probabilistic finite automata (PFA) and we show that the transition probabilities of the representing PFA completely characterize the asymptotic growth of the smoothed trie height. Our main result is of dichotomous nature—logarithmic or unbounded—and is certainly not surprising at first glance, but we also give quantitative upper and lower bounds, which are derived using multivariate generating function in order to express the computations of the perturbing PFA. A direct consequence is the logarithmic trie height for edit perturbations (i.e., random insertions, deletions and substitutions).

1 Introduction

Motivation. Tries are very simple general purpose data structures for information retrieval. This explains why many parameters of tries, such as height, path length or size have been and are still subject to extensive average-case analysis under various random string models. Though almost all investigations of trie height using analytical methods suggest the height of a random trie to be logarithmic in the number of strings, it is not immediately clear that these results can utterly explain the fact that in many practical settings the height is in fact logarithmic in the number of strings and thus far from its worst case. This holds particularly
in the case of non-random data. Nilsson and Tikkanen [10] have experimentally investigated the height of PATRICIA trees, or path-compressed tries, and other search structures. There, the height of a PATRICIA tree, built over a set of 50,000 unique random uniform strings was 16 on average and 20 at most. For non-random data consisting of 19,461 strings from geometric data, of 16,542 ASCII character strings from a book, and of 38,367 strings from Internet routing tables, the height of a path-compressed trie, built over these data sets, was on average 21, 20, and 18, respectively, and at most 30, 41 and 24, respectively. These findings suggest that worst-case inputs, i.e., sets for which the height of the respective trie is unbounded, are isolated peaks in the input space and even small deviations from worst-case inputs yield logarithmic trie height. In this work we try to give an analytical explanation of these findings.

The previous average-case approaches typically suffer from two drawbacks: such analyses are usually dominated by a high proportion of purely random inputs, even if the random inputs are produced by very sophisticated random string models such as the recently introduced symbolic dynamical systems [20]; moreover, even those results that give sharper bound on the higher moments of the distribution function of $H_S$ cannot explain the behavior of a trie on an input that is very close to worst-case. Smoothed analysis, introduced by Spielman and Teng in their seminal paper [16] in order to explain the good practical performance of the simplex algorithm which is opposed to its bad worst-case behavior, gives a mathematical framework to better understand such findings: one is not interested in finding a probability distribution which models the typical input space more accurately. Rather, one aims at answering the following kind of question: are worst-case inputs “isolated peaks” or “plateaus”? To this end, the smoothed complexity of an algorithm—or more generally of a random variable—is defined as the maximum over all inputs of the expected running time of the algorithm under slight random perturbations of the respective input. In order to perform a meaningful smoothed analysis, one must find an adequate perturbation function, i.e., one which resembles those random influences which real world inputs are typically subject to.

In order to perform a meaningful smoothed analysis of the most significant parameter of a trie, namely its height, we present a new semi-random model for strings: the set of input strings is chosen in advance by an adversary and then strings are randomly perturbed independently using the same perturbation functions. The adversary has full information on the parameters of the perturbation function, but has no control over the random perturbations and the parameters, once the input set is chosen. This model fits into the framework of smoothed analysis. (A somewhat stronger model for semi-random sources was considered by Santha and Vazirani in [15], though it was not in the context of tries but in the context of random and quasi-random number generators: there, the adversary had (limited) control over each of the biases in a sequence biased coin flips and full knowledge over the previous history.) The class of string perturbation functions which we consider can be represented by (Mealy-type) probabilistic finite automata (PFAs). PFAs are a standard tool for modeling unreliable deter-
ministic systems and they provide a compact representation for a very natural class of string perturbation functions, namely random edit perturbations, which occur in those settings and thus resembles some of the typical random influences that strings are exposed to. To the best of our knowledge, we are the first to perform a smoothed analysis of trie parameters.

Results. The main technical contribution of this paper is a characterization of the smoothed trie height depending on the probabilistic automaton underlying the perturbation function. For a star-like perturbation automaton, it is logarithmic if and only if certain conditions for the automaton’s transitions hold; if the conditions do not hold then the height is unbounded (see Theorem 5). The logarithmic/unbounded-height dichotomy is certainly not surprising, but the conditions are very easy to check. So, the theorem can be applied to rather complex perturbation models for which an ad-hoc analysis appears quite involved.

In order to derive the result, it turns out that we must bound the coincidence probability of length \(k\) by an exponentially decreasing term in \(k\). To do so, we use multivariate rational generating functions to express the computations of the perturbing PFA. This approach, which is called the weighted words model (cf. [7]), seems to fit best the requirements of our analysis. A direct consequence of the theorem is a proof of the logarithmic smoothed trie height for random edit perturbations (i.e., insertions, deletions, substitutions). We should note that not all plausible string perturbation functions can be modeled by star-like automata, e.g., transpositions.

Due to the page limit, all technical proofs of this paper are omitted. Instead, they can be found in the full paper [6] (or in the appendix).

2 Preliminaries

Let \(\mathbb{N} = \{0, 1, 2, \ldots\}\) and \(\mathbb{N}_+ = \{1, 2, \ldots\}\). Let \(A\) denote the finite alphabet. The elements of \(A\) are called the symbols of the alphabet. For \(m \in \mathbb{N}\), the finite sequence \(s = (a_1, \ldots, a_m)\) of symbols \(a_i \in A\) is called a finite string over \(A\) of length \(m\), denoted by \(|s|\). If \(m = 0\) then the string is called the empty string and is denoted by \(\epsilon\). An infinite sequence \(s = (a_1, a_2, \ldots)\) of symbols such that for \(i \in \mathbb{N}_+\) it holds that \(a_i \in A\) is called and infinite string. In this case, we set \(|s| = +\infty\). A string \(s = (a)\) of length one will by abbreviated by \(a\). For a finite string \(s = (a_1, \ldots, a_m)\) of length \(m\) and \(i \in \{1, \ldots, m\}\) we access the \(i\)-th element \(a_i\) by \(s[i]\). Also, for an infinite string \(s\) we access the \(i\)-th element for \(i \in \mathbb{N}\) by \(s[i]\) and for every string \(s\) it holds the \(s[0] = \epsilon\). For a finite string \(s\) and \(i, j \in \{1, \ldots, |s|\}\) satisfying \(i < j\), the subsequence \((a_i, \ldots, a_j)\) is called a substring of \(s\) and is accessed by \(s[i \ldots j]\). Here, for \(i, j \in \{1, \ldots, m\}\) satisfying \(i > j\) we define \(s[i \ldots j] = \epsilon\) as the access to the empty string. If \(s\) is infinite, we access the infinite substring starting at the \(i\)-th position of \(s\) by \(s[i \ldots]\). For a symbol \(a \in A\) and a string \(s\) over the same alphabet we denote by \(|s|_a\) the number of occurrences of the symbol \(a\) in \(s\). For a finite string \(s\), it clearly holds that \(|s| = \sum_{a \in A} |s|_a\). For a natural number \(m\) we denote by \(A^m\) the set of all
strings over $A$ that have length exactly $m$ and by $A^{\leq m}$ the set of all strings that have length at most $m$. Let $A^\infty$ denote the set of all infinite strings over $A$, let $A^{<\infty}$ denote the set of all finite strings over $A$, and let $A^{\leq \infty}$ denote the set of all finite and infinite strings over $A$. A string $s$ is a prefix of a string $t$, if $|s| \leq |t|$ and for all indices $i \in \{1, \ldots, |s|\}$ it holds that $s[i] = t[i]$. We write $s \preceq t$ in this case. A prefix $s$ of $t$ is a proper prefix, if $|s| < |t|$. We write $s \subset t$ in this case. Note that for the empty string $\epsilon$ only $\epsilon \subset t$ for every non-empty string $t$.

3 Towards Smoothed Trie Height

3.1 Related Studies: The Height of Random Tries

Let $A = \{a_1, \ldots, a_r\}$ be a finite alphabet of cardinality $r \geq 2$ and let $A \subseteq A^\infty$ be a set of $|A|$ distinct strings. Tries were first introduced and analyzed by Fredkin [8] and Knuth [9]. For the analysis of random tries, i.e., tries built over a set of random strings, the $n$-dimensional product space $\Omega = A^\infty \times \cdots \times A^\infty$ together with some joint probability function $\mu : \Omega \rightarrow [0,1]$ constitutes the probability space. Let $A$ be a finite alphabet of cardinality $r$ and let $A^{\leq \infty}$ denote the set of all finite and infinite strings over $A$. A string $s$ is a prefix of a string $t$, if $|s| \leq |t|$ and for all indices $i \in \{1, \ldots, |s|\}$ it holds that $s[i] = t[i]$. We write $s \preceq t$ in this case. A prefix $s$ of $t$ is a proper prefix, if $|s| < |t|$. We write $s \subset t$ in this case. Note that for the empty string $\epsilon$ only $\epsilon \subset t$ for every non-empty string $t$.

Markovian sources in the literature are

3.1 Related Studies: The Height of Random Tries

Let $A = \{a_1, \ldots, a_r\}$ be a finite alphabet of cardinality $r \geq 2$ and let $A \subseteq A^\infty$ be a set of $|A|$ distinct strings. Tries were first introduced and analyzed by Fredkin [8] and Knuth [9]. For the analysis of random tries, i.e., tries built over a set of random strings, the $n$-dimensional product space $\Omega = A^\infty \times \cdots \times A^\infty$ together with some joint probability function $\mu : \Omega \rightarrow [0,1]$ constitutes the probability space. For an $r$-ary trie built over the set $A$ it holds that the height of the trie $H_A = \max_{s,t \in A} \text{lcp}(s,t)$, where $\text{lcp} : A^{\leq \infty} \times A^{\leq \infty} \rightarrow \mathbb{N}_+$ measures the length of the longest common prefix of two strings. To analyze its behavior, $H_A$ is viewed as a random variable over the above sample space $\Omega$. Clearly, in the worst case $H_A$ is unbounded for standard tries. By choosing some joint probability function, one can analyze the expected value of $H_A$ and other asymptotic properties, e.g., its asymptotic density. This has been done for various kinds of probability density functions, where in general the $n$ strings in the set $A$ are assumed to be independent and identically distributed. Thus, the joint density function is completely characterized by the density function $\tilde{\mu} : A^\infty \rightarrow [0,1]$ for one random string. Let $Z$ be a random variable that takes values from $A$. Then the one-sided infinite sequence $\{Z_i\}_{i=1}^\infty$ can be considered a random string over $A$.

The oldest model is the memory-less random source, were each symbol corresponds to a possible outcome of a Bernoulli trial [9]. This means, we are given a parameter vector $p = (p_1, \ldots, p_r) \in (0,1)^r$ and for all $i \in \mathbb{N}_+$ it holds that $\mathbf{P}\{Z_i = a_j\} = p_j$. Another model for random strings that is discussed intensively in the literature are Markovian sources [17, 1]: a string can be considered the outcome of transitions of a finite and ergodic Markov chain with state space $A$ which has reached its stationary distribution. These two models can be subsumed under a the wider class of random strings which satisfy the mixing property. Pittel [12, 13] considered the growth of different types of random trees under the assumption that the underlying random process $\{Z_i\}_{i=1}^\infty$ satisfies the mixing property: the sequence $\{Z_i\}_{i \geq 1}$ satisfies the mixing property, if there exists $n_0 \in \mathbb{N}$ and positive constants $c_1, c_2$ such that for all $1 \leq m \leq m + n_0 \leq n$ and $A \in \mathcal{F}_m^n$ and $B \in \mathcal{F}_{m+n_0}^{n}$ it holds that $c_1 \cdot \mathbf{P}\{A\} \mathbf{P}\{B\} \leq \mathbf{P}\{A \cap B\} \leq c_2 \cdot \mathbf{P}\{A\} \mathbf{P}\{B\}$, where for $1 \leq k \leq l$, $\mathcal{F}_k$ denotes the $\sigma$-field generated by the subsequence $\{Z_i\}_{i=k}^l$. Under this assumption the following limit—the Rényi entropy of second order—
exists\(^3\)

\[ h = \lim_{n \to \infty} -\frac{\ln \sum_{\alpha \in A^n} P\{Z_1^n = \alpha\}^2}{2n}, \] (1)

where \(Z_1^n = (Z_1, \ldots, Z_n)\), and the height \(H_{\text{MM}}(n)\) of a random trie built over a set of \(n\) independent strings produced by a mixing source satisfies

\[ H_{\text{MM}}(n) \overset{\text{w.h.p.}}{\to} (\ln n)/h. \] (2)

Devroye [3–5] has introduced the density model, where each string can be considered the fractional binary expansion of a random variable from \([0, 1)\) and all \(n\) random variables are assumed to be independent having identical density. Particularly, it was shown that the height \(H_{\text{DM}}(n)\) of a random trie under the density model satisfies

\[ -1 \leq \liminf_{n \to \infty} \mathbb{E}[H_{\text{DM}}(n)] - \frac{\ln \alpha + e}{\ln 2} \leq \limsup_{n \to \infty} \mathbb{E}[H_{\text{DM}}(n)] - \frac{\ln \alpha + e}{\ln 2} \leq 1, \] (3)

if \(\int f^2(x)dx < \infty\), and is unbounded, otherwise. Here, \(\alpha = \frac{n^2 \int f^4(x)dx}{2}\) and \(e = 2.718\ldots\) is Euler’s constant. Note that this model for random strings accounts for unlimited dependency between symbols. Another model, that allows for unlimited dependency are symbolic dynamical systems which were introduced by Vallée [20] as a very general model for random strings. Clément, Vallée and Flajolet [2] have analyzed the height of random tries under this model for random strings.

### 3.2 Smoothed Trie Height

Depending on the real world application in which the tries are used, the previous analyses of trie height and other trie parameters give satisfactionary explanations of their good practical performance, which is opposed to their bad worst-case behavior: if successive data items are independent then the analyses with respect to the memory-less random source provide a sound mathematical explanation for the practical findings. If, on the other hand, data items are not independent, then there are many situations in which the analyses with respect to the Markovian source give adequate answers. Nevertheless, none of the results on the height of random tries can be accounted for a thorough explanation of the practical findings: this is particularly the case in situations where tries are built over natural languages or biological data like DNA or protein sequences. Those analyses which use random string models suffer from the following two drawbacks of average-case analyses: first, it is unclear to which amount the analyses are dominated by purely random inputs; second, even the w.h.p. results and relatively exact knowledge of the distribution function of the height cannot explain the behavior of tries on nearly-worst-case inputs. To answer these kind of questions, it seems appropriate to perform a smoothed analysis and to model a

\(^3\) originally referred to as \(h_3\) in [12, 13], but we drop the subscript
string by means of a semi-random model, where non-random inputs are subject to slight random perturbations. We initiate this line of research by performing a smoothed analysis of the most crucial parameter of a trie, i.e., its height. Having motivated the need of a smoothed analysis of trie parameters, we now turn to the formal definition of the smoothed trie height $H(S, n, X)$. Here, $S$ and $X$ denote the input set and the string perturbation function, respectively, and $n$ is the number of strings that are stored in the trie.

**Definition 1.** Let $A$ be a finite alphabet and let $S \subseteq A^\infty$ be some non-empty set of infinite strings over $A$. Given a perturbation function $X : A^\infty \rightarrow A^\leq \infty$ the smoothed trie height for $n$ strings over the set $S$ under the perturbation function $X$, denoted by $H(S, n, X)$, is defined by

$$H(S, n, x) = \max_{A \subseteq S} \max_{\|A\| = n} E \left[ \max_{s, t \in A} \text{lcp}(X(s), X(t)) \right].$$

Note that we assume that strings are perturbed independently. For our smoothed analysis, the input set $S$ can either be arbitrary, i.e., the above product space over all infinite strings from $A$, or restricted. We consider only the first variants, where the inputs are unconstrained.

### 3.3 Perturbations by Probabilistic Finite Automata

In this subsection we present our perturbation model which is based on probabilistic finite automata.

*(Mealy-type) Probabilistic Finite Automata.* A probabilistic finite automaton [11, 14] is a standard way to model an unreliable deterministic system or a communication channel. We suggest to consider random perturbation functions representable by probabilistic automata. It is not our aim to develop a general theory of automata-based perturbation functions. Instead, we use probabilistic finite automata as a compact, but nevertheless fairly general representation for string perturbation functions. We will define the probabilistic finite automata in a slightly non-standard way by separating input states from output states. This provides an easy way to describe automata computing non-length-respecting input-output relations.

A (Mealy-type) probabilistic finite automaton (PFA) over a finite alphabet $A$ is a tuple $X = (R, W, \mu_R, \mu_W, \sigma)$ where:

- $R$ is a non-empty, finite set of *input states*.
- $W$ is a non-empty, finite set of *output states*.
- $\mu_R : R \times A \times (R \cup W) \rightarrow [0, 1]$ is the *transition probability function for input states* satisfying

$$\forall q \in R \forall a \in A \sum_{p \in R \cup W} \mu_R(q, a, p) = 1.$$
The semantics of the function $\mu_R$ is: if the PFA $X$ is in input state $q$ and the symbol $a$ is read, move into state $p$ with probability $\mu_R(q,a,p)$. Note that possibly $\mu_R(q,a,q) > 0$.

$\mu_W : W \times A \times (R \times W) \to [0,1]$ is the transition probability function for output states satisfying

$$(\forall q \in W)(\forall a \in A) \sum_{p \in R \cup W} \mu_W(q,a,p) = 1.$$  

The semantics of the function $\mu_W$ is: if the PFA $X$ is in output state $q$, with probability $\mu_W(q,a,p)$, write the symbol $a$ and move into state $p$. Note that possibly $\mu_W(q,a,q) > 0$.

$\sigma : R \cup W \to [0,1]$ is the initial probability distribution, i.e., $\sigma$ satisfies $\sum_{q \in R \cup W} \sigma(q) = 1$.

We will identify with a PFA $X$ over the alphabet $A$ a random mapping $X : A^{\leq \infty} \to A^{\leq \infty}$, mapping finite of infinite strings to finite or infinite strings. A computation of a PFA $X$ on an input symbol $a \in A$ starts in some input state and stops when $X$ moves into an input state, again. The (possibly empty) output of the computation is composed by concatenating all output symbols of transitions leaving output states along which $X$ moved during the computation. A computation of $X$ on an input string $t \in A^{\leq \infty}$ is composed by the concatenation of the computations on the successive symbols of the string $t$, where the computation of $X$ on the symbol $t[i]$ starts in that input state in which the computation of $X$ on the symbol $t[i]$ stopped. The computation stops when $X$ reaches an input state and there is no more input symbol left to read. If $t$ is infinite, the computation never stops. The output of the computation is composed by concatenating all outputs of the computations on the individual symbols $t[1], t[2], \ldots$. A computation of $X$ is said to have output length $m$ if the output has length $m$ and is said to have input length $l$ if it has read $l$ symbols of the input.

**Edit Perturbations of Binary Strings.** Edit operations, i.e., substituting, deleting or inserting symbols, are among the most fundamental operations for locally manipulating strings. Therefore, a smoothed analysis with respect to perturbation functions that resemble these operations provide a better understanding of the good practical performance of tries. We say that a perturbation function on strings is an edit perturbation if it perturbs the input by randomly substituting, inserting or deleting symbols. Let $p, q \in (0, 1)$. The perturbation function $\text{SUB}_p$ substitutes each symbol in the input string with its opposite symbol independently with probability $p$; the perturbation function $\text{INS}_{pq}$ inserts before each symbol in the input string a number of $k$ symbols $a_1, \ldots, a_k$, where for $i \in \{1, \ldots, k\}$, $a_i$ equals 0 with probability $q$ and 1 with probability $1 - q$. The number of inserted symbols is geometrically distributed with parameter $p$. Finally, the perturbation function $\text{DEL}_p$ reads the input string and deletes each symbols independently with probability $p$. 

Analyzing the smoothed trie height under each of the edit perturbations of binary strings has been the starting point of our research in this field. It can be shown that the smoothed trie height under SUB$_p$ and INS$_pq$ is logarithmic and it is immediate that this does not hold for the function DEL$_p$ because the input string 111... is mapped to the output string 11... deterministically. For the convex combination of the edit perturbation matters are less trivial: let $p_S, p_I, q_I, p_D \in (0, 1)$ be the respective parameters for the edit perturbations and let $v = (v_S, v_I, v_D) \in [0, 1]^3$ be such that $v_S + v_I + v_D = 1$ be the parameter vector for the convex combination. We say that a perturbation function $Y : \mathcal{A}^{\leq \infty} \rightarrow \mathcal{A}^{\leq \infty}$ is the convex combination of the binary edit perturbations, if $Y$ can be represented by the PFA depicted in Figure 1.

Star-like Perturbation Functions. All of the perturbations considered in the last section have in common that there is exactly one input state and that the computations on the individual symbols never move between distinct output states. We now formally define a class of perturbation functions which are characterized by exactly these properties. Since their representation is a directed star graph with multi-edges and loops, where the unique input state is the center vertex, the set of output states is the set of terminal vertices, and the transitions having
strictly positive probability gives the set edges, we call those PFAs and their respective perturbation functions star-like.

**Definition 2.** Let \( \mathcal{A} \) be finite a alphabet and let \( X = (R, W, \mu_R, \mu_W, \sigma) \) be a PFA over \( \mathcal{A} \). \( X \) is said to be star-like if the following hold:

1. \( \|R\| = 1 \), i.e., \( R = \{ s \} \).
2. The function \( \mu_W \) is such that
   \[
   (\forall q, q' \in W, q \neq q') (\forall a \in \mathcal{A}) \mu_W(q, a, q') = 0,
   \]
   i.e., the graph induced by the set \( W \) and edge set \( \{(q, q') : (\exists a \in \mathcal{A}) \mu_W(q, a, q') > 0\} \) consists of a number of connected components each of which is a single vertex.
3. For all \( q \in W \) it holds that \( \sum_{a \in \mathcal{A}} \mu_W(q, a, q) < 1 \), i.e., the probability that \( X \) loops at \( q \) is strictly less than one.

Further, we consider a strict subclass of the star-like perturbation functions, namely the class of those perturbation functions which are such that for each symbol \( a \in \mathcal{A} \), there is exactly one output state, say \( q_a \), that can be reached from \( s \) with positive probability when reading \( a \). If additional to this the perturbation functions are non-deleting, i.e., there are no loops at \( s \), then we say that they are read-deterministic perturbation functions. Otherwise, i.e., there are symbols \( a \) which are deleted with positive probability, we say that the perturbation functions are read-semi-deterministic. It is easy to verify that all edit perturbations are star-like perturbation functions and further that the functions INS\(_{pq}\) and SUB\(_p\) are read-deterministic and the function DEL\(_p\) is read-semi-deterministic.

**Definition 3.** Let \( \mathcal{A} \) be a finite alphabet and let \( X = (\{s\}, W, \mu_R, \mu_W, \sigma) \) be a star-like PFA over \( \mathcal{A} \). \( X \) is said to be read-semi-deterministic, if for all \( a \in \mathcal{A} \), there exist a constant \( p_a \in [0, 1] \) and exactly one output state \( q_a \) such that \( \mu_R(s, a, s) = p_a \) and \( \mu_R(s, a, q_a) = 1 - p_a \). Further, \( X \) is said to be read-deterministic, if for all \( a \in \mathcal{A} \), \( p_a = 0 \), i.e., \( \mu_R \) has no loops at \( s \).

### 3.4 Comparison to Previous Random String Models

In this subsection we compare our semi-random string model to purely random string models. One property that the sequences from most random sources possess is the mixing property, which as we mentioned implies that Rényi’s Entropy of second order, i.e., the limit (1), exists. We show that these assumptions do not hold in general for sequences which result from the perturbation of a non-random input sequence by means of a star-like perturbation function. To this end, let \( X \) be a read-semi-deterministic PFA such that for two distinct symbols \( a \) and \( b \) it holds that \( Q_a \neq Q_b \), where for \( i \in \{a, b\} \),

\[
Q_i = (\mu_W(q_i, a_1, q_i) + \mu_W(q_i, a_1, s), \ldots, \mu_W(q_i, a_r, q_i) + \mu_W(q_i, a_r, s))
\]

It is easy to verify that the output of the pairs \((aaa\ldots, X)\) and \((bbb\ldots, X)\), respectively, have the same probability distributions as memory-less random
sources with parameter vectors $Q_a$ and $Q_b$, respectively. Then a standard calculation (cf. [19]) gives the following: For $i \in \{a, b\}$ the limit depends on the input string:

$$\lim_{n \to \infty} -\ln \sum_{\alpha \in A^n} P\{X(iii \ldots)[1 \ldots n] = \alpha\}^2 = Q_i$$

The enables us to give lower bounds on the smoothed trie height.

**Proposition 4.** Let $X = (\{s\}, W, \mu_R, \mu_W, \sigma)$ be a read-semi-deterministic PFA over a finite alphabet $A$ in canonical form (for a definition see below) and let $P = \max_{a \in A} \sum_{b \in A}(\mu_W(q_a, b, q) + \mu_W(q_a, b, s))^2$. Then for all $\varepsilon > 0$,

$$H(A^\infty, n, X) \geq 2(1 - \varepsilon) \log_{1/P} n - o(1).$$

**4 Main Result: Smoothed Trie Height under Star-like Perturbation Functions**

**A Dichotomous Result.** In this section we present the main result of this work. Let $X$ be star-like and let $X = (\{s\}, W, \mu_R, \mu_W, \sigma)$ be the representing PFA. To ease the analysis we assume that perturbations start in the input state $s$ with probability one, i.e., that $\sigma(s) = 1$ and for all $q \in W$, $\sigma(q) = 0$ holds, and we say that such a perturbation function is in canonical form. The following dichotomous result for star-like perturbation functions over arbitrary input sets can be proven.

**Theorem 5.** Let $X$ be a star-like string perturbation function over a finite alphabet $A$ in canonical form, represented by the PFA $X = (\{s\}, W, \mu_R, \mu_W, \sigma)$ such that for all $a \in A$ it holds that $\mu_R(s, a, s) < 1$. Then the following statements are equivalent.

1. $(\forall a, b \in A) \mu_R(s, a, s) + \sum_{q \in W} \mu_R(s, a, q) \cdot (\mu_W(q, b, q) + \mu_W(q, b, s)) < 1$
2. $H(A^\infty, n, X) \in O(\log n)$.

Before we discuss the meaning of the above theorem, we note that it directly yields the following corollary concerning the smoothed trie height under convex combinations of edit perturbations of arbitrary binary strings.

**Corollary 6.** Let $p_S, p_I, q_I, p_D \in (0, 1)$ and let $v = (v_S, v_I, v_D) \in [0, 1]^3$ be such that $v_S + v_I + v_D = 1$ and let $Y$ be string perturbation function which is computed by the PFA depicted in Figure 1. Then, $H(\{0, 1\}^\infty, n, Y) \in O(\log n)$ if and only if $v_D < 1$. In other words, the smoothed trie height is logarithmic if and only if the convex combination of edit perturbations does not collapse to deletions.

In general, statement (1) of the theorem gives a set of necessary and sufficient conditions such that the smoothed trie height $H(A^\infty, n, X)$ is logarithmic in $n$ if those conditions are satisfied and unbounded, otherwise. These conditions are especially appealing, because they can be verified easily and efficiently by looking at the transition probability function of the representing PFA. For
general star-like perturbation functions the verification can be done algorithmically in time $O(\|A\|^2 \cdot \|W\|)$. Note that the additional constraint regarding the deletion probabilities, i.e., that for all $a \in A$ it holds that $\mu_R(s,a,s) < 1$ cannot be dropped: let $a \in A$ be such that $\mu_R(s,a,s) = 1$ and let $t = aaa\ldots$. Then $X(a) = \epsilon$ with probability one and it becomes obsolete to speak of smoothed trie height in this particular case.

Quantitative Analyses. When performing a smoothed analysis it is usual to quantify the influence of the parameters of the perturbation function on the smoothed complexity of a problem. We can give the following quantitative result on the smoothed trie height. Let $X(\{s\}, W, \mu_R, \mu_W, \sigma)$ be a star-like PFA over the finite alphabet $A$ in canonical form. For $q \in W$ the return probability from state $q$ is defined as

$$\eta_q \overset{\text{def}}{=} \sum_{a \in A} \mu_W(q,a,s).$$

Also, for the sake of exposition, define for $a \in A$ and $q \in W$

$$\rho_{a,q} \overset{\text{def}}{=} \mu_R(s,a,q) \quad \text{and} \quad \rho_a \overset{\text{def}}{=} \mu_R(s,a,s).$$

**Theorem 7.** Let $X$ be a star-like string perturbation function over a finite alphabet $A$ in canonical form, represented by the PFA $X = (\{s\}, W, \mu_R, \mu_W, \sigma)$ such that for all $a \in A$ it holds that $p_a < 1$ and such that

$$(\forall a, b \in A) \mu_R(s,a,s) + \sum_{q \in W} \mu_R(s,a,q) \cdot (\mu_W(q,b,q) + \mu_W(q,b,s)) < 1,$$

where we denote the maximum term by $\delta$. Let $\gamma = 1/\tilde{z}$, where $\tilde{z}$ is the pole of minimum modulus of the function

$$\tilde{Z}_X(z) = \prod_{i=1}^r \left(1 - \delta \cdot \rho_a - \sum_{j=1}^n \delta \cdot \rho_{a,j} \cdot \eta_{q_j} \cdot z \right)^{-1} \cdot \frac{1}{z}.$$

Then, for $n$ sufficiently large and for all $\varepsilon > 0$ it holds that

$$H(A^\infty, n, X) \leq 2 \cdot \lceil (1 + \varepsilon) \log_1/\gamma \cdot n \rceil + o(1).$$

Note that for the case of read-semi-deterministic perturbation functions we also get a lower bounds from Proposition 4. Unfortunately, this lower bound does not match our quantitative upper bound. Non-matching upper and lower bounds can also be found in the dichotomous result of Devroye [4].

5 Conclusions

There are two main open problems posed by this paper: the first concerns the extension of our perturbation functions to more general string perturbation functions which can be represented by PFAs. Clearly, general PFAs which can model
real-world string sources such as sensors more appropriately are one possible extension. We are particularly interested in probabilistic push-down automata because they provide a way to model random transpositions, which occur quite frequently in non-random data such as DNA sequences. The second open problem concerns the smoothed analysis of other parameters and related data structures under our model. Particularly, we actually try to analyze the smoothed trie height of suffix trees. There, it is believed that the mixing condition is a necessary ingredient to prove logarithmic smoothed trie height (cf. [18]). Since our model does not satisfy the mixing condition, a positive result would give new insights in the practical performance of such data structures.

References

1. A. Apostolico, W. Szpankowski. Self-alignments in words and their applications. *Journal of Algorithms*, 13(3):446–467, 1992.
2. J. Clément, P. Flajolet, B. Vallée. Dynamical sources in information theory: A general analysis of trie structures. *Algorithmica*, 29(1-2):307–369, 2001.
3. L. Devroye. A note on the average depth of tries. *Computing*, 28:367–371, 1982.
4. L. Devroye. A probabilistic analysis of the height of tries and the complexity of triesort. *Acta Informatica*, 21(3):229–237, 1984.
5. L. Devroye. A study of trie-like structures under the density model. *Annals of Applied Probability*, 2(2):402–434, 1992.
6. S. Eckhardt, S. Kosub, J. Nowak. *Smoothed Analysis of Trie Height*. Technical Report TUM-I0715, Institut für Informatik, Technische Universität München, 2007.
7. P. Flajolet, R. Sedgewick. *Analytic Combinatorics*. Web edition, 9th edition, 2007.
8. E. Fredkin. Trie memory. *Communication of the ACM*, 3:490–500, 1960.
9. D. Knuth. *The Art of Computer Programming*, volume Vol. 3: Sorting and Searching. Addison-Wesley Publishing Co., Reading, MA, 1997.
10. S. Nilsson, M. Tikkanen. An experimental study of compression methods for dynamic tries. *Algorithmica*, 33(1):19–33, 2002.
11. A. Paz. *Introduction to Probabilistic Automata*. Academic Press, 1971.
12. B. Pittel. Asymptotical growth of a class of random trees. *Annals of Probability*, 13(2):414–427, 1985.
13. B. Pittel. Paths in a random digital tree: Limiting distributions. *Advances in Applied Probability*, 18(1):139–155, 1986.
14. M. Rabin. Probabilistic automata. *Information and Control*, 6(3):230–245, 1963.
15. M. Santha, U. Vazirani. Generating quasi-random sequences from semi-random sources. *Journal of Computer and System Sciences*, 33(1):75–87, 198.
16. D. Spielman, S.-H. Teng. Smoothed analysis of algorithms: why the simplex algorithm usually takes polynomial time. *Journal of the ACM*, 51(3):385–463, 2004.
17. W. Szpankowski. On the height of digital trees and related problems. *Algorithmica*, 6:256–277, 1991.
18. W. Szpankowski. A generalized suffix tree and its (un)expected asymptotic behaviors. *SIAM Journal on Computing*, 22(6):1176–1198, 1993.
19. W. Szpankowski. *Average Case Analysis of Algorithms on Sequences*. John Wiley, New York, NY, 2001.
20. B. Vallée. Dynamical sources in information theory: Fundamental intervals and word prefixes. *Algorithmica*, 29(1-2):269–306, 2001.
A Proof of Proposition 4

Proof. Let $X = \{s\}, W, \mu_R, \mu_W, \sigma$ be a read-semi-deterministic PFA over a finite alphabet $A = \{a_1, \ldots, a_r\}$ in canonical form. Let $A$ be a set of $n$ infinite strings each of which starts with $2n$ repetitions the symbol $a \in A$ such that

$$P = \max_{a \in A} \left( \frac{\sum_{b \in A} (\mu_W(q_a, b, q_a) + \mu_W(q_a, b, s))}{2^{2n}} \right)$$

is maximal. It holds that

$$H(A, n, X) \geq \mathbb{E}\left[ \max_{s, t \in A} \text{lcp}(X(s), X(t)) \right].$$

Let $k = 2[\log_1/p \ n]$. Now, for each string $s \in A$ the probability that $|X(s[1 \ldots 2n])| \geq k$,

i.e., the computations of $X$ on the prefix of $s$ of input length $2n$ has length at least $k$ satisfies

$$P\{|X(s[1 \ldots 2n])| \geq k\} = 1 - P\{|X(s[1 \ldots 2n])| < k\} \geq 1 - \sum_{i=0}^{k-1} \binom{2n}{i} \cdot (p_a)^{2n-i} \geq 1 - k \cdot \binom{2n}{k} \cdot (p_a)^{2n-k} = 1 - o((p_a)^n).$$

Now, with probability $1 - o((p_a)^n)$ the prefix of length $k$ of output of the computation of $X$ on $s$ has the same distribution as the prefix of an string that is written by a Memory-less random source with parameter vector $p \in (0, 1)^r$, where for $i \in \{1, \ldots, r\}, p_i = \mu_W(q_a, a_i, q_a) + \mu_W(q_a, a_i, s)$. For such a source and two random strings $s', s''$ it holds for every $k \in \mathbb{N}_+$ that

$$P\{|\text{lcp}(s', s'')| \geq k\} = P_k.$$

Let $A = \{s_1, \ldots, s_n\}$ and $k = 2(1 - \varepsilon) \log_1/p \ n$ for $\varepsilon > 0$ and for $i, j \in \{1, \ldots, r\}$ let $C_{ij} = \text{lcp}(s_i, s_j)$. From the preceeding,

$$P\{C_{ij} \geq k\} = P_k \cdot (1 - o((p_a)^n))^2.$$

This holds particularly, because $k$ is exponentially smaller than $n$, i.e., $k = 2[\log_1/p \ n]$. Let $H$ be the height of a trie which is build over the set $A$. Using the Second Moment Method, the following claim can be shown.
Claim. [see Section 4.2.3 in [19]] Under the above conditions, for any \( \varepsilon > 0 \) it holds that
\[
P\left\{ H > 2(1 - \varepsilon) \log_{1/P} n \right\} = 1 - O(1/n^\varepsilon).
\]
The above claim implies that for every \( \varepsilon > 0 \)
\[
H(A^n, n, X) \geq E\max_{s, t \in A} lcp(X(s), X(t)) = E[H] \geq 2(1 - \varepsilon) \log_{1/P} n - o(1).
\]
This proves the Theorem

B Overview on the Proofs of Theorem 5 and Theorem 7

In this section we give an overview on the proofs of Theorem 5 Theorem 7. The details of the proofs are given in the subsequent sections.

First, we show that \( H(S, n, X) \) grows at most as \( 2 \log_{1/\gamma} n \), if the coincidence probability of length \( m \) of two independent perturbations of the same string \( s \in S \), i.e., \( P\{lcp(X(s), X(s)) \geq m\} \), can be bounded from above by \( \gamma^m \) for some \( \gamma < 1 \). The following lemma holds for arbitrary string perturbation functions.

Its formal proof can be found in Section C.

Lemma 8. Let \( A \) be a finite alphabet and let \( m_0 \in \mathbb{N} \) and \( \gamma \in \mathbb{R} \) satisfying \( 0 < \gamma < 1 \). Let \( X : A^\infty \rightarrow A^{\leq \infty} \) be a perturbation function and let \( S \in A^\infty \) be a non-empty set of infinite strings. Let \( n > \gamma^{-m_0}/2 \). If there is a polynomial \( \Pi(z) \) of fixed degree \( d \in \mathbb{N} \), such that for all \( s \in S \) and all \( m \geq m_0 \) it holds that the coincidence probability of two independent perturbations of \( s \) satisfies
\[
P\{lcp(X(s), X(s)) \geq m\} \leq \Pi(m) \cdot \gamma^m,
\]
then for all \( \varepsilon > 0 \) it holds that \( H(S, n, X) \leq 2 \cdot \left[ (1 + \varepsilon) \log_{1/\gamma} n \right] + o(1) \).

Proof (Proof of Theorem 5). Let \( X = (\{s\}, W, \mu_R, \mu_W, \sigma) \) be a star-like PFA over the alphabet \( A = \{a_1, \ldots, a_r\} \) in canonical form. In order to prove the equivalence of the two statements, we claim that \( (2) \Rightarrow (1) \) and that \( (1) \Rightarrow (2) \). Then, the theorem follows. The first claim, i.e., that \( (2) \Rightarrow (1) \), can easily be established by contraposition.

Claim. In the setting of Theorem 5, it holds that \( (2) \Rightarrow (1) \).

Proof. We prove the claim by contraposition: to this end assume that \( (1) \) does not hold, i.e., there are symbols \( a, b \in A \) such that
\[
\mu_R(s, a, s) + \sum_{q \in W} \mu_R(s, a, q) \cdot (\mu_W(q, b, q) + \mu_W(q, b, s)) = 1.
\]
Thus \( P\{b \sqsubseteq X(a)\} = 1 - \mu_R(s, a, s) \). Let \( t = aaa \ldots \) and let \( s = bbb \ldots \). Then \( X \) maps \( t \) to \( s \) with probability one. Therefore, \( H(A^\infty, n, X) \) is unbounded. The claim follows.
The second claim is less easy to prove: in order to show that \((1) \Rightarrow (2)\), we prove that \((1)\) is a sufficient condition such that the tail-bound (Lemma 8) can be applied. Particularly, we show that under the assumption that \((1)\), for arbitrary \(t \in \mathcal{A}^\infty\) and \(m \in \mathbb{N}_+\) sufficiently large there are suitable positive constants \(u, v, \gamma\) satisfying \(0 < \gamma < 1\) such that

\[
P\{\text{lcp}(X(t), X(t) \geq m) = \sum_{\alpha \in \mathcal{A}^m} P\{\alpha \sqsubseteq X(t)\}^2 \leq (u \cdot m + v) \cdot \gamma^m.
\]

To this end, for \(\alpha, t \in \mathcal{A}^\infty\) let \(\mu_X(\alpha, t)\) be the probability that a computation of \(X\) on \(t\) that has input length \(|t|\) has the prefix \(\alpha\). Then for \(t \in \mathcal{A}^\infty\) and \(\alpha \in \mathcal{A}^\infty\) we have the following identity

\[
P\{\alpha \sqsubseteq X(t)\} = \sum_{l=1}^{\infty} \mu_X(\alpha, t[1..l])
\]

and thus for \(m \in \mathbb{N}_+\) we have

\[
P\{\text{lcp}(X(t), X(t) \geq m) = \sum_{\alpha \in \mathcal{A}^m} \left( \sum_{l=1}^{\infty} \mu_X(\alpha, t[1..l]) \right)^2.
\]

Next we split the right-hand side of the above equation into two suitable parts by an application of Cauchy's Inequality: let \(d \in \mathbb{R}_+\) be a constant to be defined in a moment. Then

\[
\sum_{\alpha \in \mathcal{A}^m} \left( \sum_{l=1}^{\infty} \mu_X(\alpha, t[1..l]) \right)^2 = \sum_{\alpha \in \mathcal{A}^m} \left( \sum_{l=1}^{[d \cdot m]} \mu_X(\alpha, t[1..l]) + \sum_{l=[d \cdot m] + 1}^{\infty} \mu_X(\alpha, t[1..l]) \right)^2 \leq 2 \cdot \sum_{\alpha \in \mathcal{A}^m} \left( \sum_{l=1}^{[d \cdot m]} \mu_X(\alpha, t[1..l]) \right)^2 + 2 \cdot \sum_{\alpha \in \mathcal{A}^m} \left( \sum_{l=[d \cdot m] + 1}^{\infty} \mu_X(\alpha, t[1..l]) \right)^2.
\]

Then, we prove an exponentially decreasing upper bound on each of the two addends in (5) under the assumption that \((1)\). To this end, we define for \(m \in \mathbb{N}_+, d \in \mathbb{R}_+\) and \(t \in \mathcal{A}^\infty:\)

\[
\Phi(t, m, d) \overset{\text{def}}{=} 2 \cdot \sum_{\alpha \in \mathcal{A}^m} \left( \sum_{l=1}^{[d \cdot m]} \mu_X(\alpha, t[1..l]) \right)^2,
\]

\[
\Psi(t, m, d) \overset{\text{def}}{=} 2 \cdot \sum_{\alpha \in \mathcal{A}^m} \left( \sum_{l=[d \cdot m] + 1}^{\infty} \mu_X(\alpha, t[1..l]) \right)^2.
\]
Claim. Let \( d \in \mathbb{R}_+ \) be fixed and let \( \gamma = 1/\tilde{z} \), where \( \tilde{z} \) is the pole of minimum modulus of the function

\[
\tilde{Z}_X(z) = \prod_{i=1}^{r} \left( 1 - \delta \cdot \rho_{ai} - \sum_{j=1}^{v} \frac{\delta \cdot \rho_{ai} \cdot q_{ij} \cdot \eta_{ij} \cdot z}{1 - (1 - \eta_{ij}) \cdot z} \right)^{-1}.
\]

Under the assumption that (1), there is polynomial \( \Pi(z) \) of fixed degree \( \leq r \) such that

\[
\Phi(t, m, d) \leq \Pi(m) \cdot \gamma^m.
\]

Claim. Let \( d \in \mathbb{R}_+ \) be fixed. For a star-like perturbation function \( X \) as in the setting of Theorem 5, there exist constants \( c, \gamma_2 \in \mathbb{R} \) satisfying \( \gamma_2 < 1 \) and such that

\[
\Psi(t, m, d) \leq c \cdot \gamma_2^m.
\]

The detailed proofs of the two claims can be found in Section D. We now fix \( d \). The above directly yields

\[
\mathbb{P}\{\text{lcp}(X(t), X(t)) \geq m\} \leq \Psi(t, m, d) + \Phi(t, m, d) \leq (\Pi(m) + c) \cdot \tilde{\gamma}^m
\]

for \( \tilde{\gamma} = \max\{\gamma_1, \gamma_2\} < 1 \). Thus we may apply the tail-bound. Together this shows the sought-after claim.

Claim. In the setting of Theorem 5, it holds that (1) \( \Rightarrow \) (2).

This proves Theorem 5.

Note that it can also be shown (see Appendix D) that for \( d \) sufficiently large,

\[
\lim_{m \to \infty} \Phi(t, m, d)/\Psi(t, m, d) = 0
\]

which implies Theorem 7.

C Proof of Lemma 8

Proof. Let \( S \) be a non-empty set of infinite strings over a finite alphabet \( \mathcal{A} \). Let \( \varepsilon > 0 \) and let \( k \in \mathbb{N}_+ \) be arbitrary. Then

\[
H(S, n, X) = \max_{A \subseteq \mathcal{A}^n} \mathbb{E} \left[ \max_{s, t \in A} \text{lcp}(X(s), X(t)) \right] = \max_{|A| = n} \sum_{A \subseteq \mathcal{A}^n} \mathbb{P} \left\{ \max_{s, t \in A} \text{lcp}(X(s), X(t)) \geq i \right\}
\]

\[
\leq \sum_{i=1}^{\infty} \max_{|A| = n} \mathbb{P} \left\{ \max_{s, t \in A} \text{lcp}(X(s), X(t)) \geq i \right\}
\]

\[
\leq k + \sum_{i=k+1}^{\infty} \max_{|A| = n} \mathbb{P} \left\{ \max_{s, t \in A} \text{lcp}(X(s), X(t)) \geq i \right\}
\]

\[
\leq k + n^2 \cdot \sum_{i=k+1}^{\infty} \max_{s, t \in S} \mathbb{P} \{ \text{lcp}(X(s), X(t)) \geq i \}.
\]
Inequality (7) follows from Boole’s Inequality and (6) holds, because the in sum of probabilities each addend of the first $k$ addends can by bounded by one. Now we expand each addend of the right-hand side and apply Cauchy’s Inequality in its standard form:

$$\max_{s,t \in S} \mathbb{P}\{\text{lcp}(X(s), X(t)) \geq i\} = \max_{s,t \in S} \sum_{\alpha \in A^i} \mathbb{P}\{\alpha \subseteq X(s)\} \cdot \mathbb{P}\{\alpha \subseteq X(t)\}$$

$$\leq \max_{s,t \in S} \left( \sum_{\alpha \in A^i} \mathbb{P}\{\alpha \subseteq X(s)\} \right)^2 \cdot \left( \sum_{\alpha \in A^i} \mathbb{P}\{\alpha \subseteq X(t)\} \right)^2$$

$$\leq \max_{s \in S} \sum_{\alpha \in A^i} \mathbb{P}\{\alpha \subseteq X(s)\}^2$$

$$= \max_{s \in S} \mathbb{P}\{\text{lcp}(X(s), X(s)) \geq i\}.$$

Now, we have that for all $k \in \mathbb{N}_+$ it holds that

$$H(S, n, X) \leq k + n^2 \cdot \sum_{i=k+1}^{\infty} \max_{s \in S} \mathbb{P}\{\text{lcp}(X(s), X(s)) \geq i\}.$$ 

Let $d \in \mathbb{N}_+$ and let $\Pi(z)$ be a polynomial of degree $d$ such that the assumption of the theorem holds. Set $k = 2 \cdot \lceil (1 + \varepsilon \log_{1/\gamma} n) \rceil \geq m_0$. Then

$$H(S, n, X) \leq k + n^2 \cdot \sum_{i=k+1}^{\infty} \max_{s \in S} \mathbb{P}\{\text{lcp}(X(s), X(s)) \geq i\}$$

$$\leq 2 \cdot \lceil (1 + \varepsilon \log_{1/\gamma} n) \rceil + \sum_{i=2 \cdot \lceil (1 + \varepsilon \log_{1/\gamma} n) \rceil + 1}^{\infty} \Pi(i) \cdot n^2 \cdot \gamma^i$$

It is easy to see that the latter term is in $o(1)$:

$$\sum_{i=2 \cdot \lceil (1 + \varepsilon \log_{1/\gamma} n) \rceil + 1}^{\infty} \Pi(i) \cdot n^2 \cdot \gamma^i$$

$$= \sum_{i=1}^{\infty} \Pi(2 \cdot \lceil (1 + \varepsilon \log_{1/\gamma} n) \rceil + i) \cdot n^2 \cdot \gamma^{2 \cdot \lceil (1 + \varepsilon \log_{1/\gamma} n) \rceil} \cdot \gamma^i$$

$$\leq \sum_{i=1}^{\infty} \Pi(2 \cdot \lceil (1 + \varepsilon \log_{1/\gamma} n) \rceil + i) \cdot n^2 \cdot n^{\omega} \cdot \gamma^i$$

$$= n^{-2 \varepsilon} \cdot \sum_{i=1}^{\infty} \Pi(2 \cdot \lceil (1 + \varepsilon \log_{1/\gamma} n) \rceil + i) \cdot \gamma^i \in o(1).$$

This proves the lemma.

**D Detailed Proof of Theorem 5**

Before we actually start with proving the first claim, we first show how to express the term $\sum_{\alpha \in A^m} \mu_X(\alpha, t[1 \ldots l])$ subject to the transition probabilities of $X$. 
Then we establish Claim B in Section D.2. Afterwards, we turn to the proof of Claim B in Section D.3. This then proves Theorem 5.

D.1 Prerequisites: computations of star-like PFAs

In this section, we prove the following Lemma which will be one of the important ingredients in the proofs of Claims B and B. In particular, the lemma gives a (nearly exact) expression of the term $\sum_{\alpha \in A} \mu_X(\alpha, t[1...l])$, i.e., the probability that a computation of input length $l$ on the prefix of $t$ has output length at least $m$, subject to the transition probabilities of $X$.

**Lemma 9.** Let $X = (\{s\}, W, \mu_R, \mu_W, \sigma)$ be a star-like PFA over the finite alphabet $A$ in canonical form and let $t \in A^\infty$ and $l \in \mathbb{N}_+$. Let $f: \mathbb{N} \to \{0,1\}$ the following defined function: for $x \in \mathbb{N}$,

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise.} \end{cases}$$

The function $f$ is used to indicate deleted symbols in the computations of $X$. Then

$$\sum_{\alpha \in A^m} \mu_X(\alpha, t[1...l]) \leq \frac{1}{\eta} \cdot \sum_{m_1+\ldots+m_l=m, i=1}^l \prod_{i=1}^l \left( f(m_i) \cdot \rho_i + (1-f(m_i)) \cdot \sum_{q \in W} \rho_{i,q} \eta_q (1-\eta_q)^{-m_i-1} \right),$$

where $\eta = \min_{q \in W} \eta_q$ denotes the minimum return probability.

**Proof.** Recall that the term which we seek to bound is the probability that the computation of $X$ on $t$ of input length $l$ has output length at least $m$. Since $X$ is star-like and given in canonical form, each computation of $X$ on $t$ starts in the input state and then moves into some output state, from which it writes the output, before it moves into the input state again, where it reads the next symbol of the input and continues the computations as described above. Thus, each computation can be decomposed into the computations on the successive individual symbols of $t$. For a computation of input length $l$ and output length $m$, there are $\binom{m+l-1}{l-1}$ possibilities to concatenate $l$ such computations on individual symbols such that they give a computation of output length $m$: this equals the number of decompositions of $m$ into $l$ non-negative addends. Note, that addends might be equal to zero, because computations might have output length zero. The computations on the first $l-1$ input symbols must return into the input state, whereas the computation on the $l$-th and last input symbol may either loop at its output state or return back into the input state after having written the $m$-th and last symbol of the output.
Now, consider a fixed decomposition $m_1 + \ldots + m_l = m$ into possibly empty computations. For $i \in \{0, \ldots, l\}$, if $m_i = 0$ then the probability that the computation of $X$ on the symbol $t[i]$ has output length zero is
\[
P\{X(t[i]) = \epsilon\} = \rho_{t[i]}.
\] (8)
For $i \in \{1, \ldots, l-1\}$, if $m_i > 0$ then the probability that the computation of $X$ on the symbol $t[i]$ has output length exactly $m_i$ is equal to
\[
\sum_{\alpha \in A^{m_i}} P\{X(t[i]) = \alpha\} = \sum_{q \in W} \rho_{a,q} \cdot \eta_q \cdot (1 - \eta_q)^{m_i-1}
\] (9)
and the probability that the computation of $X$ on the symbol $t[l]$ has output length at least $m_l > 0$ is equal to
\[
\sum_{\alpha \in A^{m_l}} P\{\alpha \sqsubseteq X(t[l])\} = \sum_{q \in W} \rho_{a,q} \cdot (1 - \eta_q)^{m_l-1}.
\]
Let $\tilde{\eta} = \min_{q \in W} \eta_q$ be the minimum return probability. The term (10) can be bounded as
\[
\sum_{q \in W} \rho_{a,q} \cdot (1 - \eta_q)^{m_i-1} \leq \frac{1}{\tilde{\eta}} \cdot \sum_{q \in W} \rho_{a,q} \cdot \eta_q \cdot (1 - \eta_q)^{m_i-1}.
\] (10)
Now, using the indicator function $f$ to choose the correct term for $i \in \{1, \ldots, l\}$, i.e., the term (8) if $m_i = 0$ and the term (9) if $m_i > 0$ and $i < l$ or the term (10) if $m_i > 0$ and $i = l$, the probability that the computation of $X$ on $t$ of input length $l$ that can be decomposed as $m_1 + \ldots + m_l = m$ has length at least $m$ is can be bounded by the product
\[
\frac{1}{\tilde{\eta}} \cdot \prod_{i=1}^{l} \left( f(m_i) \cdot \rho_{t[i]} + (1 - f(m_i)) \cdot \sum_{q \in W} \rho_{a,q} \cdot \eta_q \cdot (1 - \eta_q)^{m_i-1} \right).
\]
Summing over all possible decompositions, we get
\[
\sum_{\alpha \in A^m} \mu_X(\alpha, t[1..l]) \leq \frac{1}{\tilde{\eta}} \cdot \sum_{m_1 + \ldots + m_l = m} \prod_{i=1}^{l} \left( f(m_i) \cdot \rho_{t[i]} + (1 - f(m_i)) \cdot \sum_{q \in W} \rho_{a,q} \cdot \eta_q (1 - \eta_q)^{m_i-1} \right)
\]
which proves the lemma.

**D.2 Bounding $\Phi(t, m, d)$**

In order to prove an exponentially decreasing upper bound on
\[
\Phi(t, m, d) = 2 \cdot \sum_{\alpha \in A^m} \left( \sum_{l=1}^{\lfloor d/m \rfloor} \mu_X(\alpha, t[1..l]) \right)^2
\]
for fixed $d \in \mathbb{R}_+$ an thereby prove Claim B, we first apply Cauchy’s inequality and then bound by counting over all possible $l \in \mathbb{N}_+$:

\[
2 \cdot \sum_{\alpha \in A^m} \left( \sum_{l=1}^{[d \cdot m]} \mu_X(a, t[1\ldots l]) \right)^2 \leq 2[d \cdot m] \cdot \sum_{\alpha \in A^m} \sum_{l=1}^{[d \cdot m]} \mu_X(a, t[1\ldots l])^2 \leq 2[d \cdot m] \cdot \sum_{l=1}^{\infty} \sum_{\alpha \in A^m} \mu_X(a, t[1\ldots l])^2 \tag{11}
\]

Here, exchanging the two sums does not change the value of the expression.

Bounding $\sum_{\alpha \in A^m} \mu_X(a, t[1\ldots l])^2$. To proceed, we use the Conditions given by statement (1) of Theorem 5 which as we will prove in Lemma 11 imply the existence of a constant $\delta < 1$ such that for all $l, m \in \mathbb{N}_+$ and $t \in A^\infty$ the $l$-th addend of the outer sum of (11) can be bounded by $\sum_{\alpha \in A^m} \mu_X(a, t[1\ldots l])$.

Before we proceed to the lemma, we state the following Proposition which is a direct consequence of the definition of $\mu_X(a, t[k\ldots l])$.

**Proposition 10.** Let $j, k, l \in \mathbb{N}_+$ satisfying $k \leq j < l$ and $\alpha \in A^m$. Then

\[
\mu_X(a, t[k\ldots l]) = \sum_{i=0}^{m} P\{X(a) = \alpha[0\ldots i]\} \cdot \mu_X(\alpha[i+1\ldots m], t[j+1\ldots l]).
\]

**Lemma 11.** Let $X = (\{s\}, W, \mu_R, \mu_W, \sigma)$ be a star-like PFA in canonical form over the finite alphabet $A$. Let

\[
\delta = \max_{a, b \in A} \left( \rho_a + \sum_{q \in W} \rho_{a,q} \cdot (\mu_W(q, b, q) + \mu_W(q, b, s)) \right). \tag{12}
\]

Then for all infinite strings $t \in A^\infty$ and all $k, l, m \in \mathbb{N}_+$ it holds that

\[
\sum_{\alpha \in A^m} \mu(a, t[k\ldots l])^2 \leq \delta^{l-k} \cdot \sum_{\alpha \in A^m} \mu(a, t[k\ldots l]).
\]

**Proof (Proof of Lemma 11).** Let $X$ be a star-like PFA and let $t \in A^\infty$ be an arbitrary input string. For $a \in A$ and $\alpha \in A^{\leq \infty}$ satisfying $|\alpha| \geq 1$ we have

\[
\sum_{i=0}^{|\alpha|} P\{X(a) = \alpha[0\ldots i]\} = P\{X(a) = \varepsilon\} + P\{X(a) = \alpha[1]\} + \ldots \leq P\{X(a) = \varepsilon\} + P\{\alpha[1] \subseteq X(a)\} \leq \delta. \tag{13}
\]

We prove the lemma by induction on the length $\ell = l - k + 1$ of the part of $t$ which is read. First note that for $l < k$ the left and the right hand side of Inequality (13) are equal to zero. This holds particularly, because $X$ is given in
canonical form. Therefore we may without loss of generality assume that \( l \geq k \) holds.

**Induction basis:** for \( \ell = 1 \) it holds that

\[
\sum_{\alpha \in \mathcal{A}^m} \mu_X(\alpha, t[l])^2 \leq \sum_{\alpha \in \mathcal{A}^m} \mu_X(\alpha, t[l]),
\]

because probabilities are less than one.

**Induction step:** assume that (13) holds for \( l - k \leq \ell - 2 \). By Proposition 10

we get

\[
\sum_{\alpha \in \mathcal{A}^m} \mu_X(\alpha, t[k...l])^2 = \sum_{\alpha \in \mathcal{A}^m} \left( \sum_{i=0}^{m} \Pr \{ X(t[k]) = \alpha[0...i] \} \cdot \mu_X(\alpha[i+1...m], t[k+1...l]) \right)^2
\]

We apply Jensen’s Inequality: let \( x_0, \ldots, x_m, y_0, \ldots, y_m \in \mathbb{R}^+ \). Then

\[
\left( \sum_{i=1}^{m} x_i y_i \right)^2 = \left( \sum_{i=0}^{m} x_i \right)^2 \cdot \left( \frac{\sum_{i=0}^{m} x_i y_i}{\sum_{i=0}^{m} x_i} \right)^2 \leq \left( \sum_{i=0}^{m} x_i \right) \cdot \left( \sum_{i=0}^{m} x_i y_i^2 \right)
\]

For \( i \in \{0, \ldots, m\} \) we set

\[
x_i = \Pr \{ X(t[k]) = \alpha[0...i] \}
\]

and

\[
y_i = \mu_X(\alpha[i+1...m], t[k+1...l])
\]

in Inequality (14). Additionally we know from Inequality (13) that

\[
\sum_{i=0}^{m} x_i = \sum_{i=0}^{m} \Pr \{ X(t[k]) = \alpha[0...i] \} \leq \delta.
\]

Together, we get that

\[
\left( \sum_{i=0}^{m} x_i y_i \right)^2 \leq \left( \sum_{i=0}^{m} x_i \right) \cdot \left( \sum_{i=0}^{m} x_i y_i^2 \right) \leq \delta \cdot \left( \sum_{i=0}^{m} x_i y_i^2 \right)
\]

which after re-translating gives

\[
\sum_{\alpha \in \mathcal{A}^m} \left( \sum_{i=0}^{m} \Pr \{ X(t[k]) = \alpha[0...i] \} \cdot \mu_X(\alpha[i+1...m], t[k+1...l]) \right)^2
\]

\[
\leq \sum_{\alpha \in \mathcal{A}^m} \delta \cdot \sum_{i=0}^{m} \Pr \{ X(t[k]) = \alpha[0...i] \} \cdot \mu_X(\alpha[i+1...m], t[k+1...l])^2.
\]
Using this we proceed as follows:

\[
\sum_{\alpha \in \mathcal{A}^m} \delta \cdot \sum_{i=0}^{m} \mathbb{P}\{X(t|k) = \alpha[0\ldots i]\} \mu_X(\alpha[i + 1\ldots m], t[k + 1\ldots l])^2
\]

\[
= \delta \cdot \sum_{i=0}^{m} \left( \sum_{\alpha_1 \in \mathcal{A}^i} \mathbb{P}\{X(t|k) = \alpha_1\} \right) \cdot \left( \sum_{\alpha_2 \in \mathcal{A}^{m-i}} \mu_X(\alpha_2, t[i + 1\ldots l])^2 \right)
\]

\[
\leq \delta \cdot \sum_{i=0}^{m} \left( \sum_{\alpha_1 \in \mathcal{A}^i} \mathbb{P}\{X(t|k) = \alpha_1\} \right) \cdot \left( \delta^{i-k-1} \sum_{\alpha_2 \in \mathcal{A}^{m-i}} \mu_X(\alpha_2, t[k + 1\ldots j]) \right)
\]

\[
= \delta^{i-k} \cdot \sum_{\alpha \in \mathcal{A}^m} \mu_X(\alpha, t[k\ldots l]).
\]

Here, Inequality (15) follows from the induction hypothesis. Altogether, we have shown

\[
\sum_{\alpha \in \mathcal{A}^m} \mu_X(\alpha, t[k\ldots l])^2 \leq \delta^{i-k} \cdot \sum_{\alpha \in \mathcal{A}^m} \mu_X(\alpha, t[k\ldots l]).
\]

This proves the lemma.

Lemma 11 tells us that the Conditions given by Statement (1) of Theorem 5 allows us to bound \(\Phi(t, m, d)\) subject to the probability mass which is induced by the perturbation function \(X\) on input \(t \in \mathcal{A}^\infty\) multiplied by a factor of \(\delta < 0\) for every input symbol which is read in the respective term. This is, using Inequality (11) from the beginning of this section and the lemma, we can bound \(\Phi(t, m, d)\) as

\[
\Phi(t, m, d) \leq 2[d \cdot m] \sum_{l=1}^{\infty} \sum_{\alpha \in \mathcal{A}^m} \mu_X(\alpha, t[1\ldots l])^2 \leq \frac{2[d \cdot m]}{\delta} \sum_{l=1}^{\infty} \sum_{\alpha \in \mathcal{A}^m} \delta^l \mu_X(\alpha, t[1\ldots l]).
\]

Now, we can expand (and bound) each addend of the last sum according to Lemma 9 as

\[
\sum_{\alpha \in \mathcal{A}^m} \delta^l \cdot \mu_X(\alpha, t[1\ldots l])
\]

\[
\leq \frac{1}{\tilde{\eta} m_1 + \ldots + m_i + 1} \prod_{i=m_{i+1}=1}^{m_{i+1} \ldots m_{i+1} = m_{i+1}} \left( f(m_i) \cdot \delta \rho_{t|j} + (1 - f(m_i)) \cdot \sum_{q \in W} \delta \rho_{t|j, q} \eta_q (1 - \eta_q)^{m_{i+1} - 1} \right)
\]

where for \(q \in W\), \(\eta_q\) was defined as the return probability from state \(q\). The minimum such probability was \(\tilde{\eta} = \min_{q \in W} \eta_q\) and \(f : \mathbb{N} \to \{0, 1\}\) was defined to be the indicator function of deleted letters.
Valid expressions Call each non-zero addend in the above sum a valid expression of $X$ on $t$. A valid expression is said to be of input length $l$ if it corresponds to a set of computations of input length $l$ and is said to be of output length $m$ if its corresponding set of computations has output length $m$. Let $W = \{q_1, \ldots, q_v\}$. Each valid expression is a product over the set of variables $\{\delta, \rho_a, \rho_{a,q}, \rho_{a,q_1}, \eta_{q_1}, \ldots, \eta_{q_v}\}$. The products have a regular structure which we exploit in order to bound the term (16): to this end, let $\mathcal{B}$ be the following alphabet, where we interpret variables as letters (we intentionally use the term 'letter' for an element of the alphabet and 'word' for a sequence of letters in order to avoid confusion)

$$
\mathcal{B} = \text{def} \{\delta\} \cup \{\eta_q \; : \; q \in W\} \cup \{(1 - \eta_q) \; : \; q \in W\} \cup \{\rho_{a,q} \; : \; q \in W \text{ and } a \in A\} \cup \{\rho_a \; : \; a \in A\}.
$$

Then, each valid expression in (16) is readily identifiable with a word $w$ over the alphabet $\mathcal{B}$: e.g., the word

$$
\delta \rho_a \delta \rho_b \delta \rho_{a,q} (1 - \eta_q) (1 - \eta_q) (1 - \eta_q) \eta_q
$$

is corresponds to a valid expression of $X$ on $t = aba\ldots$ of input length 3 and output length 4 and thus to a set of computations of $X$ on $t$, where each computations deletes the first two letters and then moves into state $q$ after having read the letter $a$, whereupon it loops three times at $q$ and then moves back to the input state again. Clearly, not all words over $\mathcal{B}$ are valid expressions of $X$ on $t$. Call a word valid if it does. Let $W_{X}^{(t,l,m)}$ be the set of all valid words over $\mathcal{B}$ that have input length $l$ and output length $m$, i.e., corresponding to a valid expression of $X$ on $t$ that has the respective input and output lengths. I.e.,

$$
W_{X}^{(t,l,m)} = \text{def} \{w \in \mathcal{B}^{<\infty} : w \text{ is a valid word having input length } l \text{ and output length } m\}.
$$

In order to evaluate the term (16) using the framework of valid expressions, we follow the weighted words model: we define the weight $\pi(w)$ of a word $w \in \mathcal{B}^{<\infty}$ as the product of all letters which constitute $w$, where the multiplicity of a letter in the product equals the number of times it occurs in the word $w$:

$$
\pi(w) = \text{def} \sum_{x \in \mathcal{B}} x^{|w|_x}.
$$

E.g., for

$$
w' = \delta \rho_a \delta \rho_b \delta \rho_{a,q} (1 - \eta_q) (1 - \eta_q) (1 - \eta_q) \eta_q
$$

the example word from above, we have that

$$
\pi(w') = \delta^3 \cdot \rho_a \cdot \rho_b \cdot \rho_{a,q} \cdot (1 - \eta_q)^3 \cdot \eta_q,
$$

as $|w'|_\delta = |w'|(1-\eta_q) = 3$ and $|w'|_{\rho_a} = |w'|_{\rho_b} = |w'|_{\rho_{a,q}} = |w'|_{\eta_q} = 1$. Also, the weight of a set is then defined as the weight of all elements in the set.
Clearly, the weight of a valid word equals the value of its corresponding valid expression. It is easy to see that for \( l, l' \in \mathbb{N}^+ \) satisfying \( l \neq l' \) it holds that \( W_X^{(t,l,m)} \cap W_X^{(t,l',m)} = \emptyset \). Thus

\[
\sum_{m_1 + \ldots + m_l = m} \prod_{i=1}^l \left( f(m_i) \cdot \delta \rho_{t[i]}(1 - f(m_i)) \cdot \sum_{q \in W} \delta \rho_{t[i],q} \eta_q (1 - \eta_q)^{m_i - 1} \right) = \pi(W_X^{(t,l,m)})
\]

and therefore

**Proposition 12.**

\[
\sum_{l=1}^{\infty} \sum_{\alpha \in A^m} \delta^l \cdot \mu_X(\alpha, t[1...l]) \leq \frac{1}{\eta \delta} \cdot \sum_{l=1}^{\infty} \pi(W_X^{(t,l,m)}).
\]

Let \( W_X^{(t,l)} \supset W_X^{(t,l,m)} \) be the set of all valid words having input length \( l \), but arbitrary output length. The set \( W_X^{(t,l)} \) is a regular language: let \(|\cdot\)| denote the choice operator and \( \ast \) denote the sequence operator for a possibly zero number of repetitions of the respective letter. The regular specification is as follows:

\[
W_X^{(t,l)} = \{ w \in B^{\leq \infty} : w := W_1 W_2 \ldots W_l \\
W_1 := \delta \rho_{t[1]} \mid \delta \rho_{t[1],q_1} \eta_{q_1} (1 - \eta_{q_1}) \ast \ldots \mid \delta \rho_{t[l],q_l} \eta_{q_l} (1 - \eta_{q_l}) \ast \\
\vdots \\
W_l := \delta \rho_{t[l]} \mid \delta \rho_{t[l],q_1} \eta_{q_1} (1 - \eta_{q_1}) \ast \ldots \mid \delta \rho_{t[l],q_l} \eta_{q_l} (1 - \eta_{q_l}) \ast \}
\]

Here, we use \( W_1, \ldots, W_l \) as placeholder for the below defined regular expressions. Clearly, for all words \( w \in W_X^{(t,l)} \), it holds that \( \sum_{i=1}^l |w|_{\delta} = l \). Now, we can formally define

\[
W_X^{(t,l,m)} = \{ w \in W_X^{(t,l)} : \sum_{i=1}^l (|w|_{q_{q_i}} + |w|_{(1 - \eta_{q_i})}) = m \}.
\]

**Embedding valid words** In order to evaluate the sum over all valid words of \( X \) on \( t \) of output length \( m \), we first construct a family of structurally simpler languages such for each set \( W_X^{(t,l,m)} \) there exists a corresponding set in the structurally simpler family having the same weight and such that the set in the family are still disjoint. Thus, the sum over the weights of all such new sets equals weight of valid words of \( X \) on \( t \) of output length \( m \). Still, the sum over the weights of these new sets depends on the structure of the input string \( t \) which is unknown. Thus, in order to get rid of this dependence on \( t \), we do not evaluate the sum over the weights of all new sets exactly, but we over-count slightly. This over-counting can once again be best expressed by constructing a structurally even simpler language which contains each word that we need to account for (ans
some more words). To this end, let \( Y_{X}^{(t,i,m)} \) be the new set of words. For a fixed input string \( t \in A^{\infty} \) and a prefix \( t[1 \ldots l] \) let for \( i \in \{1, \ldots, r\} \),

\[
l_i = |t[1 \ldots l]|_{a_i}
\]

be the number of occurrences of the symbol \( a_i \) in the prefix \( t[1 \ldots l] \). For each such prefix we give a canonical input string \( t' \) such that the corresponding set of valid words is structurally simpler and such that there is a function \( g : B^{< \infty} \rightarrow B^{< \infty} \) that describes a bijection from \( W_{X}^{(t,i,m)} \) to \( W_{X}^{(t',l,m)} \), from which it follows that

\[
\pi \left(Y_{X}^{(t,i,m)}\right) = \pi \left(W_{X}^{(t',l,m)}\right).
\]

The string \( t' \) is defined as

\[
t' = a_1 \ldots a_1 \ldots a_r \ldots a_r\]

\( l_1 \) times \( l_r \) times

The corresponding sets \( W_{X}^{(t',l,m)} \) is then such that for each word \( w \in W_{X}^{(t,i,m)} \) there is a word \( w' \in W_{X}^{(t',l,m)} \) that is composed of exactly the same set of sub-words, but in different ordering: Consider the decomposition of \( w \) as

\[
w = W_1 \ldots W_l,
\]

where the \( i \)-th sub-word \( W_i \) for \( i \in \{1, \ldots, l\} \) corresponded to the symbol \( t[i] \).

Now, for \( w' \) with the decomposition as

\[
w' = W'_1 \ldots W'_l
\]

it holds that the the first \( l_1 \) sub-words \( W'_1, \ldots, W'_l \) correspond to the symbol \( a_1 \), the next \( l_2 \) sub-words \( W'_1, \ldots, W'_{l_1+l_2} \) correspond to the symbol \( a_2 \), and so on. Thus, the function \( g \) is a permutation of sub-words: assume w.l.o.g. that the \( i \)-th subword \( W_i \) corresponds to the symbols \( t[i] = a_j \). Then \( W_i \) is mapped to the position \( \sum_{k=1}^{i-1} l_k + |t[1 \ldots i]|_{a_j} \) in \( w' \). Such a permutation is clearly weight-preserving. Now, define for a string \( t \) the set \( Y_{X}^{(t,l,m)} \) as

\[
Y_{X}^{(t,l,m)} = \text{def } Y_{X}^{(t',l,m)}
\]

where \( t' \) is the canonical input string corresponding to \( t[1 \ldots l] \). Clearly, for \( l, l' \in \mathbb{N}_+ \) is still holds that \( Y_{X}^{(t,l,m)} \cap Y_{X}^{(t',l,m)} = \emptyset \).

**Proposition 13.**

\[
\pi \left(W_{X}^{(t,l,m)}\right) = \pi \left(Y_{X}^{(t',l,m)}\right)
\]

Now, we get rid of the dependency on the input string \( t \): let \( Y_{X} \) be the following regular language over the alphabet \( B \), where again \( Y_1, \ldots, Y_l \) are placeholder for regular expressions:

\[
Y_X = \{w \in B^{\leq \infty} \mid w := Y_1 Y_2 \ldots Y_r \}
\]

\[
Y_1 := (\delta \rho_{a_1} \mid \delta \rho_{a_1,q_1} \eta_{q_1} (1 - \eta_{q_1})^* \mid \ldots | \delta \rho_{a_1,q_r} \eta_{q_r} (1 - \eta_{q_r})^*)^*;
\]

\[
Y_r := (\delta \rho_{a_r} \mid \delta \rho_{a_r,q_1} \eta_{q_1} (1 - \eta_{q_1})^* \mid \ldots | \delta \rho_{a_r,q_r} \eta_{q_r} (1 - \eta_{q_r})^*)^* \}
\]
Clearly, $\mathcal{Y}^{(t,l,m)} \subset \mathcal{Y}_X$. We have,

$$\mathcal{Y}^{(t,l,m)}_X = \{ w \in \mathcal{Y}_X : \text{there exits } w' \in \mathcal{Y}^{(t,l,m)} \text{ such that } w = g(w') \} = \{ w \in \mathcal{Y}_X : \sum_{i=1}^v (|w|_{\eta_q} + |w|_{(1-\eta_q)}) = m \text{ and } \sum_{i=1}^v |w|_l = l \}.$$ 

Now, define

$$\mathcal{Y}^{(m)}_X = \text{def} \{ w \in \mathcal{Y}_X : \sum_{i=1}^v (|w|_{\eta_q} + |w|_{(1-\eta_q)}) = m \}.$$ 

Clearly, for $m \in \mathbb{N}_+$ it holds that $\mathcal{Y}^{(t,l,m)}_X \subset \mathcal{Y}^{(m)}_X$ and thus we have

$$\sum_{l=1}^\infty \pi \left( \mathcal{Y}^{(t,l,m)}_X \right) \leq \pi \left( \mathcal{Y}^{(m)}_X \right). \quad (17)$$

Altogether, we have established the following relation between the sum over all terms (16) over all $l \in \mathbb{N}_+$ and the weight of the set $\mathcal{Y}^{(m)}_X$:

**Lemma 14.** For $m \in \mathbb{N}_+$ and $t \in A^\infty$,

$$\sum_{l=1}^\infty \sum_{\alpha \in A^m} \delta^l \cdot \mu_X(\alpha, t[1 \ldots l]) \leq 1/(\bar{\eta} \cdot \delta) \cdot \pi \left( \mathcal{Y}^{(m)}_X \right).$$

**Proof.** The lemma is easy to prove:

$$\sum_{l=1}^\infty \sum_{\alpha \in A^m} \delta^l \cdot \mu_X(\alpha, t[1 \ldots l]) = 1/(\bar{\eta} \cdot \delta) \cdot \sum_{l=1}^\infty \pi \left( \mathcal{W}^{(t,l,m)}_X \right) \quad \text{by Proposition 12}$$

$$= 1/(\bar{\eta} \cdot \delta) \cdot \sum_{l=1}^\infty \pi \left( \mathcal{Y}^{(t,l,m)}_X \right) \quad \text{by Proposition 13}$$

$$\leq 1/(\bar{\eta} \cdot \delta) \cdot \pi \left( \mathcal{Y}^{(m)}_X \right) \quad \text{by (17)}$$

This proves the lemma.

*A crude bound on $\pi \left( \mathcal{Y}^{(m)}_X \right)$ using the saddle point method* In order to get a bound on the term $\pi \left( \mathcal{Y}^{(m)}_X \right)$, we proceed as follows: after having given the regular specification of the set $\mathcal{Y}_X$, we translate this specification into the language of generating functions, where we use the variable $z$ to mark the length, i.e., the number $\sum_{i=1}^v (|w|_{\eta_q} + |w|_{(1-\eta_q)})$ for a word $w \in \mathcal{Y}_X$. Also, we symbolically use the letters as variables. A regular specification for a set of combinatorial objects translates into a *invariably positive rational* generating function, where we have the following relationship between the operators of the regular description and the algebraic operators: let $a, b \in \mathcal{B}$. Then union, i.e., $'a | b'$, corresponds to $'a + b'$,
combinatorial product, i.e., 'a \ b', corresponds to 'a \cdot b' and sequence building, i.e., 'a*', corresponds to 1/(1 - a) (where a \neq \varepsilon). Thus, the regular specification of the language \mathcal{Y}_X readily lends itself to the following ordinary multivariate generating function.

**Lemma 15.** The ordinary multivariate generating function corresponding to the language \mathcal{Y}_X is

\[
Z_X(z, \delta, \eta, \rho) = \prod_{i=1}^{r} \left( 1 - \delta_i \cdot \rho_{a_i} - \sum_{j=1}^{v} \frac{\delta_i \cdot \rho_{a_i, q_j} \cdot \eta_{q_j} \cdot z}{1 - (1 - \eta_{q_j}) \cdot z} \right)^{-1}
\]

where \eta = (\eta_{q_1}, \ldots, \eta_{q_k}, (1 - \eta_{q_1}), \ldots, (1 - \eta_{q_k})) and \rho = (\rho_{q_1}, \ldots, \rho_{q_k}, \rho_{q_1, q_k}, \ldots, \rho_{q_k, q_k}). Here the variable z marks the number \sum_{i=1}^{v} (|w|_{a_i} + |w|_{1 - \eta_{q_i}}) and the other variables mark the number of occurrences of the respective letters.

**Proof.** In order to make the proof more readable, we mark the number of occurrences of a letter by a variable with the corresponding latin symbol.

- \delta is marked by \delta, d.
- For \( i \in \{1, \ldots, r \} \), \( \rho_{a_i} \) is marked by \( r_{a_i} \).
- For \( i \in \{1, \ldots, r \} \) and \( j \in \{1, \ldots, v \} \), \( \rho_{a_i, q_j} \) is marked by \( r_{a_i, q_j} \).
- For \( j \in \{1, \ldots, v \} \), \( \eta_{q_j} \) is marked by \( e_{q_j} \) and \( \eta_{q_j} (1 - \eta_{q_j}) \) is marked by \( (1 - e_{q_j}) \).

Also, z marks the number \sum_{i=1}^{v} (|w|_{a_i} + |w|_{1 - \eta_{q_i}}). Consider the \( i \)-th addend in the product for \( i \in \{1, \ldots, r \} \). The set of words \{\delta_i \rho_{a_i}\} is generated by the multivariate generating function (MGF)

\[
g_i(d, r_{a_i}) = d \cdot r_{a_i}.
\]

For \( j \in \{1, \ldots, v \} \), the set of words

\[
\{\delta_i \rho_{a_i, q_j} \eta_{q_j}, \delta_i \rho_{a_i, q_j} (1 - \eta_{q_j}), \delta_i \rho_{a_i, q_j} \eta_{q_j} (1 - \eta_{q_j}) (1 - \eta_{q_j}), \ldots\}
\]

is generated by the MGF

\[
f_{ij}(z, d, r_{a_i, q_j}, e_{q_j}, (1 - e_{q_j})) = \frac{d \cdot r_{a_i, q_j} \cdot e_{q_j} \cdot z}{1 - (1 - e_{q_j}) \cdot z}
\]

Now the words corresponding to state \( q_j \) are generated by the regular expression

\[
(\delta_i \rho_{a_i} | \delta_i \rho_{a_i, q_j} \eta_{q_j} (1 - \eta_{q_j})^* | \ldots | \delta_i \rho_{a_i, q_k} \eta_{q_k} (1 - \eta_{q_k})^*)^* \tag{18}
\]

are generated by the function

\[
f_i(z, d, r_{a_i}, r_{a_i, q_1}, \ldots, r_{a_i, q_k}, e_{q_1}, \ldots, e_{q_k}, (1 - e_{q_1}), \ldots, (1 - e_{q_k}))
\]

\[
= \left( 1 - g_i(d, r_{a_i}) - \sum_{j=1}^{v} f_{ij}(z, d, r_{a_i, q_j}, e_{q_j}, (1 - e_{q_j})) \right)^{-1}
\]

\[
= \left( 1 - d \cdot r_{a_i} - \sum_{j=1}^{v} \frac{d \cdot r_{a_i, q_j} \cdot e_{q_j} \cdot z}{1 - (1 - e_{q_j}) \cdot z} \right)^{-1}.
\]
Now, the set $\mathcal{Y}_X$, which is defined by a regular expression that is the concatenation of the regular expression (18) for state $q_j$ for $j \in \{1, \ldots, v\}$ is generated the the product over the corresponding MGF’s. Resubstituting the respective variables proves the Lemma.

In order to evaluate $\pi(\mathcal{Y}_X^{(m)})$, we follow the weighted words model: this is, the former variables are treated as parameters in the new generating function. The respective function is then

$$\hat{Z}_X(z) = \prod_{i=1}^{r} \left( 1 - \delta \cdot \rho_{a_i} - \sum_{j=1}^{v} \frac{\delta \cdot \rho_{a_i} \cdot q_j \cdot \eta_{q_j} \cdot z}{1 - (1 - \eta_{q_j}) \cdot z} \right)^{-1}.$$ 

Now, there are (at least) two ways to proceed in order to derive the weight $\pi(\mathcal{Y}_X^{(m)})$: since the function $\hat{Z}_X(z)$ is a rational function, it lends itself to a partial fraction decomposition. Then, one can easily translate this form back into a formal power series $A(z) = \sum_{i=1}^{\infty} a_i z^i$ and $\pi(\mathcal{Y}_X^{(m)})$ equals the coefficient at $z^m$ of this power series, i.e.,

$$\pi(\mathcal{Y}_X^{(m)}) = [z^m] A(z) = a_m.$$

Since a partial fraction decomposition of the function $\hat{Z}_X(z)$ is quite involved, we do not follow this vein here; instead, we use the following Theorem on the expansion of rational functions

**Theorem 16 (Expansion of rational functions).** [Theorem IV in [7]] If $f(z)$ is a rational function that is analytic at zero and has poles at $z_1 \leq z_2 \leq \ldots \leq z_k$ then its coefficients are a sum of exponential polynomials: there exist $k$ polynomials $\Pi_1(z), \ldots, \Pi_k(z)$ such that for $m$ larger than some fixed $m_0$,

$$[z^m] f(z) = \sum_{j=1}^{k} \Pi_j(m) \cdot \left( \frac{1}{z_j} \right)^m.$$

Furthermore, the polynomial $\Pi_j$ has degree equal to the order of the pole at $z_j$ minus one.

By construction of the regular language $\mathcal{Y}_X$, all poles of $\hat{Z}$ are of order at most $r$, where $r$ is the cardinality of $A$. Let $\tilde{z}_1, \ldots, \tilde{z}_{r'}$, where $r' \in \{1, \ldots, r\}$ be these poles (which have not yet been specified) and let $\tilde{z}_1$ the pole of smallest modulus. Then according to the above theorem we have that

$$[z^m] \hat{Z}_X = \pi(\mathcal{Y}_X^{(m)}) = \sum_{i=1}^{r'} \Pi_i(m) \cdot \left( \frac{1}{\tilde{z}_i} \right)^m \leq \left( \frac{1}{\tilde{z}_1} \right)^m \cdot \sum_{i=1}^{r'} \Pi_i(m), \quad (19)$$

where for $i \in \{1, \ldots, r'\}$ is a polynomial of degree at most equal to the order of the pole at $z_i$ minus one. Now, we are in a position to prove the exponentially decreasing upper bound on $\Phi(t, m, d)$. 
Proof (Proof of Claim B). Recapitulating the previous calculation, we have

\[ \Phi(t, m, d) = 2 \cdot \sum_{\alpha \in A^m} \left( \sum_{l=1}^{[d \cdot m]} \mu_X(\alpha, t[1...l]) \right)^2 \]  
(Def.)

\[ \leq 2[d \cdot m] \sum_{l=1}^{\infty} \sum_{\alpha \in A^m} \mu_X(\alpha, t[1...l])^2 \]  
(Ineq. (11))

\[ \leq \frac{2[d \cdot m]}{\tilde{\eta} \delta} \sum_{l=1}^{\infty} \sum_{\alpha \in A^m} \sum_{\sigma = 1}^{[l \cdot m]} \mu_X(\alpha, t[1...l]) \]  
(Lem. 11)

\[ \leq \frac{2[d \cdot m]}{\tilde{\eta} \delta} \sum_{l=1}^{\infty} \pi \left( Y^\alpha X \right) \]  
(Lem. 14)

\[ \leq \frac{2[d \cdot m]}{\tilde{\eta} \delta} \sum_{l=1}^{\infty} \sum_{\alpha \in A^m} \sum_{l'=1}^{r'} \Pi_i(m) \cdot \left( \frac{1}{\tilde{z}_1} \right)^m \]  
(Eq. (19) and Thm. 16),

where \( \tilde{z}_1 \) is the pole of minimum modulus of the function \( \tilde{Z}_X(z) \). Now, since \([d \cdot m] < (d + 1) \cdot m\) the Claim follows with \( \Pi(m) = \frac{2(d+1)}{\tilde{\eta} \delta} \cdot \sum_{i=1}^{r'} \Pi_i(m) \).

D.3 Bounding \( \Psi(t, m, d) \)

In this section, we derive the exponentially decreasing upper bound on the term \( \Psi(t, m, d) \) for fixed \( d \in \mathbb{R}_+ \). Remember that we fixed

\[ d =_{\text{def}} \min\{d' \in \mathbb{R} : (\sqrt{e} \cdot p_{\text{max}} < 1) \land (d' \cdot (p_{\text{max}})^{d'} < \left( \frac{p_{\text{max}}^2}{3} \right))\}, \]

where \( p_{\text{max}} = \max_{\alpha \in A} \rho_\alpha \) was the maximum deletion probability. Set

\[ \gamma_B =_{\text{def}} e \cdot (d + 1) \cdot (p_{\text{max}})^{d-1} \]

and

\[ c_B =_{\text{def}} \frac{d+1}{\tilde{\eta} \cdot (1 - \sqrt[3]{e} \cdot p_{\text{max}})}. \]

Here, \( e \approx 2.71... \) is the base of the natural logarithm and \( \tilde{\eta} = \min_{q \in W} \eta_q \). We prove Claim B by showing that for the above choice of constants it holds that

\[ \Psi(t, m, d) \leq c_B \cdot (\gamma_B)^m. \]

The choice of \( d \) gives that \( \gamma_B < 1 \). This justifies the choice.
Proof (Proof of Claim B). Let $d$ and $p_{\text{max}}$ be defined as above. We start as follows:

\[
\Psi(t, m, d) = 2 \cdot \sum_{\alpha \in \mathcal{A}^m} \left( \sum_{l=\lfloor d \cdot m \rfloor + 1}^{\infty} \mu(\alpha, t[1 \ldots l]) \right)^2 \leq \sum_{\alpha \in \mathcal{A}^m} \sum_{l=\lfloor d \cdot m \rfloor + 1}^{\infty} \mu(\alpha, t[1 \ldots l]) = \sum_{l=\lfloor d \cdot m \rfloor + 1}^{\infty} \sum_{\alpha \in \mathcal{A}^m} \mu(\alpha, t[1 \ldots l]).
\]

This holds particularly, because we deal with probabilities, i.e., quantities less than one. Thus, we have bounded $\Psi(t, m, d)$ by the that part of the probability mass induced by $X$ on input $t$ which corresponds to the cases in which $X$ has read a relatively long prefix of $t$. Next, we consider the expansion of $\sum_{\alpha \in \mathcal{A}^m} \mu(\alpha, t[1 \ldots l])$ for a fixed $l \geq \lfloor d \cdot m \rfloor + 1$ due to Lemma 9:

\[
\sum_{\alpha \in \mathcal{A}^m} \mu(\alpha, t[1 \ldots l]) \leq \frac{1}{\tilde{\eta}} \cdot \frac{\sum_{m_1 + \ldots + m_i = m} \prod_{i=1}^{l} \left( f(m_i) \cdot \rho_{t[i]} + (1-f(m_i)) \cdot \sum_{q \in W} \rho_{t[i],q} \eta_q (1-\eta_q)^{m_i-1} \right)}{(m+l-1) \cdot (p_{\text{max}})^{l-m}}, \tag{20}
\]

Inequality (20) follows from the fact that for $l \geq \lfloor d \cdot m \rfloor + 1$ in every decomposition $m = m_1 + \ldots + m_i$ of $m$ into $l$ non-negative addends, there are at least $l - m$ indices $i$, where $1 \leq i \leq l$ such that $m_i = 0$. For each such $m_i$, it holds that $f(m_i) = 1$ and thus a factor of $\rho_{t[i]} \leq p_{\text{max}}$ is “added” in the product. Also, there are at most $\binom{m+l-1}{l-1}$ such decompositions. Using Stirling’s Approximation for the Binomial Coefficient and the Fact that $(n+1/n)^n < e$ we may further bound as follows:

\[
\sum_{l=\lfloor d \cdot m \rfloor + 1}^{\infty} \sum_{\alpha \in \mathcal{A}^m} \mu(\alpha, t[1 \ldots l]) \leq \frac{1}{\tilde{\eta}} \cdot \frac{\sum_{l=\lfloor d \cdot m \rfloor + 1}^{\infty} \left( m + l - 1 \right) \cdot (p_{\text{max}})^{l-m}}{l-1} \leq \frac{1}{\tilde{\eta}} \cdot (p_{\text{max}})^{(d-1)m} \sum_{l=0}^{\infty} \left( \frac{[d \cdot m] + m + l}{m} \right) \cdot (p_{\text{max}})^l \leq \frac{d \sqrt[3]{e}}{\tilde{\eta}} \cdot (e(d + 1)(p_{\text{max}})^{(d-1)m} \sum_{l=0}^{\infty} \left( d \sqrt[3]{e} \cdot p_{\text{max}} \right)^l \leq \frac{d \sqrt[3]{e}}{\tilde{\eta}} \cdot (e(d + 1)(p_{\text{max}})^{(d-1)m} \gamma_B)^m \tag{21}
\]

$= c_B \cdot (\gamma_B)^m$. 


Here, (21) holds, because \( d\sqrt{e} \cdot p_{\text{max}} < 1 \) by our choice of \( d \). Hence, Claim B follows.

Remark 17. Note that \( \gamma_B \) can be made arbitrarily small, as \( \lim_{d \to \infty} d \cdot (p_{\text{max}})^d = 0 \). Our choice of \( d \) being minimal such that the exponentially decreasing upper bound on \( \Psi(t, m, d) \) can be shown can thus be improved such that for \( d \) sufficiently large,

\[
\lim_{m \to \infty} \frac{\Phi(t, m, d)}{\Psi(t, m, d)} = 0
\]

and therefore the base of the logarithm for the smoothed trie height depends only on the upper bound on \( \Phi(t, m, d) \).

In smoothed analysis it is usual to quantify the influence of the perturbation function on the smoothed complexity. Here, the respective quality is the trie height. So far, we have ignored the quantitative influence of the perturbation function and have only given a qualitative result. Note that by Remark 17 immediately implies Theorem 7.