Torus embeddings and
algebraic intersection complexes

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Introduction

In [GM], Goreskey and MacPherson defined and constructed intersection complexes
for topological pseudomanifolds. The complexes are defined in the derived category
of sheaves of modules over a constant ring sheaf. Since analytic spaces are of this
category, any algebraic variety defined over $\mathbb{C}$ has an intersection complex for each
perversity.

The purpose of this paper is to give an algebraic description of the intersection
complex of a toric variety. Namely, we describe it as a finite complex of coherent
sheaves whose coboundary map is a differential operator of order one.

Let $Z_h$ be the complete toric variety associated to a complete fan $\Delta$. For each
$\sigma \in \Delta$, let $X(\sigma)_h$ be the associated closed subvariety of $Z_h$. For each perversity $\mathbf{p}$,

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we construct a bicomplex \((\text{ic}_p(Z_h)^{\bullet \bullet}, d_1, d_2)\) with the following properties.

(1) \(\text{ic}_p(Z_h)^{i,j} = \{0\}\) for \((i, j) \not\in [0, r] \times [-r, 0]\).

(2) Each \(\text{ic}_p(Z_h)^{i,j}\) is a direct sum for \(\sigma \in \Delta\) of free \(\mathcal{O}_{X(\sigma)_h}\)-modules of finite rank.

(3) \(d_1\) is an \(\mathcal{O}_{Z_h}\)-homomorphism and \(d_2\) is a differential operator of order one.

(4) The associated single complex \(\text{ic}_p(Z_h)^\bullet\) is quasi-isomorphic to the \(r\)-times dimension shifts to the right of the intersection complex defined in [GM2]. In other words, our complex belongs to “Beilinson-Bernstein-Deligne-Gabber scheme” (cf. [GM2, 2.3,(d)] and [BBD, 2.1]).

In §1, we introduce an abelian category \(\text{GM}(A(\sigma))\) of finitely generated graded \(A(\sigma)\)-modules, where \(A(\sigma)\) is the exterior algebra of the \(\mathbb{Q}\)-vector space \(N(\sigma)_\mathbb{Q}\) defined by a cone \(\sigma\).

In §2, we define an additive category \(\text{GEM}(\Delta)\) for a finite fan \(\Delta\). Each object \(L\) of this category is a collection \((L(\sigma) ; \sigma \in \Delta)\) of \(L(\sigma) \in \text{GM}(A(\sigma))\). We define a dualizing functor \(D\) on the category \(\text{CGEM}(\Delta)\) of finite complexes in \(\text{GEM}(\Delta)\).

A perversity \(p\) on \(\Delta\) is defined to be a \(\mathbb{Z}\)-valued map on \(\Delta \setminus \{0\}\). The intersection complex \(\text{ic}_p(\Delta)^\bullet\) as an object of \(\text{CGEM}(\Delta)\) is defined and constructed in §2.

In §3, we work on the toric variety \(Z(\Delta)\) associated to the fan \(\Delta\). For each \(L^\bullet \in \text{CGEM}(\Delta)\), we define a finite bicomplex \(\Lambda_{Z(\Delta)}(L)^{\bullet \bullet}\) of coherent \(\mathcal{O}_{Z(\Delta)}\)-modules whose second coboundary map is a differential operator of order one.

We consider the normal analytic space \(Z_h := Z(\Delta)_h\) associated to the toric variety \(Z(\Delta)\) in §4. The bicomplex \(\text{ic}_p(Z_h)^{\bullet \bullet}\) stated above is defined to be the bicomplex \(\Lambda_{Z(\Delta)_h}(\text{ic}_p(\Delta))^{\bullet \bullet}\) on this analytic space.

When \(\Delta\) is a complete fan, we show that the intersection cohomologies are described in terms of the complex \(\text{ic}_p(\Delta)^\bullet\) in \(\text{CGEM}(\Delta)\).

The middle perversity \(m\) is defined by \(m(\sigma) := 0\) for all \(\sigma \in \Delta \setminus \{0\}\). In [E], we will discuss on \(\text{ic}_m(\Delta)^\bullet\) and prove the decomposition theorem for a barycentric
subdivision of the fan.

**Notation**

We denote by $\mathbb{Z}$ the ring of rational integers and by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the fields of rational numbers, real numbers and complex numbers, respectively.

For a free $\mathbb{Z}$-module $F$ of finite rank, we denote $F_\mathbb{Q} := F \otimes \mathbb{Q}$ and $F_\mathbb{R} := F \otimes \mathbb{R}$.

We denote a complex $E^\bullet$ of modules or of sheaves of modules simply by $E^\bullet$ when the substitution of the dot by an integer is not suitable. In particular, we prefer to write $G(E)^\bullet$ rather than $G(E^\bullet)$, if $G$ is a functor between categories of complexes. However, we left the dot in representing the cohomologies $H^i(E^\bullet)$.

For a complex $(E^\bullet, d_E)$ and a integer $n$, the complex $(E[n]^\bullet, d_{E[n]}^\bullet)$ is defined by $E[n]^i := E^{i+n}$ and $d_{E[n]}^i := (-1)^n d_E^{i+n}$ for $i \in \mathbb{Z}$.

By a bicomplex $(E^{\bullet, \bullet}, d_1, d_2)$, we mean a naive double complex $[D2, 0.4]$, i.e., a double complex satisfying $d_1 \cdot d_2 = d_2 \cdot d_1$. The associated single complex $(E^\bullet, d)$ is defined by $E^k := \bigoplus_{i+j=k} E^{i,j}$ for every $k \in \mathbb{Z}$ and $d(x) := d_1^{i,j}(x) + (-1)^i d_2^{i,j}(x)$ for $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ and $x \in E^{i,j}$. In general, we follow $[D2]$ for the sign configuration of complexes.

By a $d$-complex and a $\partial$-complex, we mean complexes in an additive category whose coboundary maps are denoted by $d$ and $\partial$, respectively. Some important bicomplexes in this paper has $d_1 = d$ and $d_2 = \partial$.

1 The exterior algebras and modules

Let $r$ be a non-negative integer, and let $N$ and $M$ be mutually dual free $\mathbb{Z}$-modules of rank $r$ with the pairing $\langle , \rangle : M \times N \rightarrow \mathbb{Z}$. This pairing is extended $\mathbb{R}$-bilinearly to $\langle , \rangle : M_\mathbb{R} \times N_\mathbb{R} \rightarrow \mathbb{R}$.

By a cone in $N_\mathbb{R}$, we mean a strongly convex rational polyhedral cone $[O1]$. 

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other hand, we set \( \sigma \) so that \( N/N \) complement of each other with respect to the pairing. These notations are defined of \( N \) and \( M \) dimensional of zero and the homogeneous elements of degree \( j \) for each \( i \in A \). Then \( N/N \) be such a \( Q \) for each \( i \in A \). Then \( N/N \) \( \mathbb{Q} \) for a cone \( \sigma \) in \( R_{\mathbb{Q}} \), we define \( N(\sigma) := N \cap (\sigma + (-\sigma)) \cong \mathbb{Z}^{\sigma^*} \) and \( N[\sigma] := N/N(\sigma) \cong \mathbb{Z}^{(-\sigma)} \). Hence, \( N(\sigma)_{\mathbb{R}} \) is the real subspace \( \sigma + (-\sigma) \) of \( N_{\mathbb{R}} \). On the other hand, we set \( \sigma^\perp := \{ x \in M_{\mathbb{R}} \mid \langle x, a \rangle = 0, \forall a \in \sigma \} \), \( M[\sigma] := M \cap \sigma^\perp \) and \( M(\sigma) := M/M[\sigma] \). Then \( N(\sigma)_{\mathbb{R}} \subset N_{\mathbb{R}} \) and \( M[\sigma]_{\mathbb{R}} \subset M_{\mathbb{R}} \) are orthogonal complement of each other with respect to the pairing. These notations are defined so that \( N(\sigma) \) and \( M(\sigma) \) as well as \( N[\sigma] \) and \( M[\sigma] \) are mutually dual, respectively.

In this article, we often treat finite dimensional graded \( \mathbb{Q} \)-vector spaces. Let \( V \) be such a \( \mathbb{Q} \)-vector space. Then, we denote by \( V_j \) the vector subspace consisting of zero and the homogeneous elements of degree \( j \) for each \( j \in \mathbb{Z} \). For two finite dimensional \( \mathbb{Q} \)-vector spaces \( V \) and \( W \), we define the grading of \( V \otimes_{\mathbb{Q}} W \) by

\[
(V \otimes_{\mathbb{Q}} W)_k := \bigoplus_{i+j=k} V_i \otimes_{\mathbb{Q}} W_j
\]

for each \( k \in \mathbb{Z} \). We identify \( V \otimes_{\mathbb{Q}} W \) with \( W \otimes_{\mathbb{Q}} V \) by the identifications

\[
a_i \otimes b_j = (-1)^{ij} b_j \otimes a_i
\]

for \( a_i \in V_i, b_j \in W_j \) for all \( i, j \in \mathbb{Z} \).

We denote by \( A(M_{\mathbb{Q}}) \) the exterior algebra \( \Lambda^* M_{\mathbb{Q}} \) over the rational number field \( \mathbb{Q} \). Then \( A(M_{\mathbb{Q}}) \) is a graded \( \mathbb{Q} \)-algebra with \( A(M_{\mathbb{Q}})_i := \Lambda^i M_{\mathbb{Q}} \) for each \( i \in \mathbb{Z} \).

The algebra \( A(N_{\mathbb{Q}}) := \Lambda^* N_{\mathbb{Q}} \) is more important in our theory. The grading of \( A(N_{\mathbb{Q}}) \) is defined negatively, i.e., \( A(N_{\mathbb{Q}})_i := \Lambda^{-i} N_{\mathbb{Q}} \) for each \( i \). For a cone \( \sigma \) in \( N_\mathbb{R} \), \( A(N(\sigma)_{\mathbb{Q}}) := \Lambda^* N(\sigma)_{\mathbb{Q}} \) is a graded subalgebra of \( A(N_{\mathbb{Q}}) \).

In order to simplify the notation, we set \( A := A(N_{\mathbb{Q}}) \) and \( A^* := A(M_{\mathbb{Q}}) \). For a cone \( \sigma \) in \( N_\mathbb{R} \), we set \( A(\sigma) := A(N(\sigma)_{\mathbb{Q}}) \subset A \) and \( A^*[\sigma] := A(M[\sigma]_{\mathbb{Q}}) \subset A^* \).

Let \( C = \bigoplus_{p \in \mathbb{Z}} C_p \) be a graded \( \mathbb{Q} \)-subalgebra of \( A \) or \( A^* \). For a graded left \( C \)-module \( V = \bigoplus_{q \in \mathbb{Z}} V_q \), we define the associated graded right \( C \)-module structure on...
\[ x a := \sum_{q \in \mathbb{Z}} (-1)^{pq} ax_q \] (1.3)

for \( a \in C_p \) and \( x = \sum_{q \in \mathbb{Z}} x_q \in V \). Conversely, if \( V \) is a graded right \( C \)-module, then the associated graded left \( C \)-module structure is defined similarly. In both cases, we see easily that \( a(xb) = (ax)b \) for \( a, b \in C \) and \( x \in V \). Hence \( V \) is a two-sided \( C \)-module and we call it simply a \( C \)-module. The following lemma is checked easily.

**Lemma 1.1** Let \( V, V' \) be graded \( C \)-modules for \( C \) as above. If \( W \) is a homogeneous left or right \( C \)-submodule of \( V \), then it is a two-sided \( C \)-submodule of \( V \). If \( f : V \to V' \) is a homogeneous homomorphism as left or right \( C \)-modules of degree zero, then it is a homomorphism of two-sided \( C \)-modules.

Let \( \text{GM}(C) \) be the abelian category of finitely generated graded \( C \)-modules where the morphisms are defined to be homogeneous homomorphisms of degree zero. By definition, every object in \( \text{GM}(C) \) is finite-dimensional as a \( \mathbb{Q} \)-vector space.

Since \( M_\mathbb{Q} \) is the dual \( \mathbb{Q} \)-vector space of \( N_\mathbb{Q} \), each homogeneous element \( a \in A_p^* \) for any \( p \in \mathbb{Z} \) induces a homogeneous \( \mathbb{Q} \)-linear map \( i(a) : A \to A \) (cf. [G, 5.14]) which is usually called the right interior product. Here note that the degree of the interior product \( i(a) \) is \( p \), since the indices of \( A \) are given negatively. It is known that this operation induces a graded right \( A^* \)-module structure on \( A \) (cf. [G, (5.50)]). With the associated left \( A^* \)-module structure, we regard \( A \) as a two-sided \( A^* \)-module.

**Lemma 1.2** Let \( \sigma \) be a cone in \( N_\mathbb{R} \). Then the left operations of \( A(\sigma) \) and \( A^*[\sigma] \) on \( A \) commute with each other. This is also true for the right operations.

**Proof.** For \( a \in A^*[\sigma]_p, b \in A(\sigma)_q \) and \( x \in A_s \), we prove the equalities

\[ b(ax) = a(bx) \] (1.4)

\[ (xa)b = (xb)a \] (1.5)

\[ (bx)a = (-1)^{pq} b(xa) \] (1.6)
Since $N(\sigma)_Q$ is the orthogonal complement of $M[\sigma]_Q$ and $(bx)a = i(a)(b \wedge x)$, the equality (1.6) is equal to $[Q, (5.58)]$. By this equality, we have $b(ax) = (-1)^p b(xa) = (-1)^{p+q} b(xa) = (xa)b = (-1)^q b(xa) = (xb)a$. These are the equalities (1.4) and (1.5).

Let $\sigma$ be a cone in $N_R$ and let $V$ be a graded $A(\sigma)$-module. Then the above lemma implies that $V_A := V \otimes_{A(\sigma)} A$ has a structure of $A^*[\sigma]$-module such that $a(u \otimes x) = u \otimes ax$ for $u \in V$, $x \in A$ and $a \in A^*[\sigma]$.

**Lemma 1.3** Let $V$ be a finitely generated graded $A(\sigma)$-module with a homogeneous $Q$-basis $\{u_1, \ldots, u_s\}$ and let $\{x_1, \ldots, x_r\}$ be a $Q$-basis of $N_Q$ such that $N(\sigma)_Q$ is generated by $\{x_1, \ldots, x_k\}$ for $k := r_\sigma$. Then $V_A$ is a free $A^*[\sigma]$-module with the basis $\{u'_1, \ldots, u'_s\}$, where $u'_i := u_i \otimes (x_{k+1} \wedge \cdots \wedge x_r)$ for each $i$.

**Proof.** Let $E$ be the subspace $Qx_{k+1} + \cdots + Qx_r$ of $N_Q$. Since the operation of $A^*[\sigma]$ on $A$ is defined by the interior products, we have $A(E) = A^*[\sigma](x_{k+1} \wedge \cdots \wedge x_r)$ in $A$. Hence $A = A(\sigma) \otimes_Q A(E)$ is equal to $(A(\sigma) \otimes_Q A^*[\sigma])(x_{k+1} \wedge \cdots \wedge x_r)$. Hence

$$V_A = V \otimes_{A(\sigma)} A \simeq V \otimes_Q A^*[\sigma](x_{k+1} \wedge \cdots \wedge x_r) \quad (1.7)$$

as $A^*[\sigma]$-modules and we get the lemma. q.e.d.

Let $\sigma$ be a cone of $N_R$. We set

$$\det(\sigma) := \bigwedge^{r_\sigma} N(\sigma) \simeq \mathbb{Z} \quad (1.8)$$

and $\det(\sigma)_Q := \det(\sigma) \otimes Q$. We denote by $\text{Det}(\sigma)_Q$ the graded $Q$-vector space defined by $(\text{Det}(\sigma)_Q)_{-r_\sigma} := \det(\sigma)_Q$ and $(\text{Det}(\sigma)_Q)_{j} := \{0\}$ for $j \neq -r_\sigma$.

For a finitely generated graded $A(\sigma)$-module $V$, we define a graded $A(\sigma)$-module $D_\sigma(V)$ as follows.

We define graded $Q$-vector spaces $D^\left(\sigma\right)_\sigma(V)$ and $D^\right(\sigma\right)_\sigma(V)$ by

$$D^\left(\sigma\right)_\sigma(V) = D^\right(\sigma\right)_\sigma(V) := \text{Hom}_Q(V, \text{Det}(\sigma)_Q) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_Q(V_{-r_\sigma - i}, \text{Det}(\sigma)_Q) \quad (1.9)$$
For $x \in V$ and $y \in D^\text{left}_\sigma(V)$ (resp. $z \in D^\text{right}_\sigma(V)$) we denote the operation by $(y, x)$ (resp. $(x, z)$). The right $A(\sigma)$-module structure of $D^\text{left}_\sigma(V)$ (resp. the left $A(\sigma)$-module structure of $D^\text{right}_\sigma(V)$) is defined by}

$$(ya, x) := (y, ax) \quad (\text{resp. } (x, az) := (xa, z)) \quad (1.10)$$

for $a \in A(\sigma)$. There exists a unique homogeneous isomorphism $\varphi : D^\text{left}_\sigma(V) \to D^\text{right}_\sigma(V)$ degree zero such that $(y, x) = (-1)^{pq}(x, \varphi(y))$ for homogeneous elements $x \in V_p$ and $y \in D^\text{left}_\sigma(V)_q$ for all $p, q \in \mathbb{Z}$. We define $D_\sigma(V)$ to be the identification of $D^\text{left}_\sigma(V)$ and $D^\text{right}_\sigma(V)$ by the isomorphism $\varphi$. Namely, we have $(y, x) = (-1)^{pq}(x, y)$ for $x \in V_p$ and $y \in D_\sigma(V)_q$. Here note that $(y, x) = 0$ if $p + q \neq -r_\sigma$. It is easy to see that the induced left and right $A(\sigma)$-module structures on $D_\sigma(V)$ have the compatibility $(1.3)$. By definition, we have

$$\dim_{\mathbb{Q}} D_\sigma(V)_j = \dim_{\mathbb{Q}} V_{-r_\sigma-j} \quad (1.11)$$

for every $j \in \mathbb{Z}$.

For $V \in \text{GM}(A(\sigma))$, we define an $A(\sigma)$-homomorphism $\iota : V \to D_\sigma(D_\sigma(V))$ by $(y, \iota(x)) := (y, x)$ for $x \in V$ and $y \in D_\sigma(V)$. It is easy to see that the symmetric equality $(\iota(x), y) = (x, y)$ holds. Since $V$ is a finite dimensional $\mathbb{Q}$-vector space, the pairings are perfect and $\iota$ is an isomorphism. We call $\iota$ the canoniacal isomorphism.

For a homomorphism $f : V \to W$ in $\text{GM}(A(\sigma))$, we define $D_\sigma(f)$ to be the natural induced homomorphism $D_\sigma(W) \to D_\sigma(V)$ of $A(\sigma)$-modules, i.e., the equality $(y, f(x)) = (D_\sigma(f)(y), x)$ for $x \in V$ and $y \in D_\sigma(W)$. It is clear by definition that the correspondence $V \mapsto D_\sigma(V)$ is a contravariant exact functor from $\text{GM}(A(\sigma))$ to itself.

If $\pi$ is of dimension $r$, then $A(\pi) = A$ and $D_\pi$ is a functor from $\text{GM}(A)$ to itself. We denote this functor by $D_N$ which does not depend on the choice of $\pi$.

Let $\sigma$ and $\rho$ be cones in $N_\mathbb{R}$ with $\sigma \prec \rho$. 7
For $V$ in $\text{GM}(A(\sigma))$, we denote by $V_{A(\rho)}$ the graded $A(\rho)$-module $V \otimes_{A(\sigma)} A(\rho) = A(\rho) \otimes_{A(\sigma)} V$, where we identify $x \otimes a$ with $(-1)^{pq}a \otimes x$ for $x \in V_p$ and $a \in A(\rho)_q$. For a morphism $f : V \to V'$ in $\text{GM}(A(\sigma))$, we denote $f_{A(\rho)} := f \otimes 1_{A(\rho)} : V_{A(\rho)} \to V'_{A(\rho)}$. The correspondence $V \mapsto V_{A(\rho)}$ is a covariant functor from $\text{GM}(A(\sigma))$ to $\text{GM}(A(\rho))$. Let $H$ be a linear subspace of $N(\rho)_Q$ such that $N(\rho)_Q = N(\sigma)_Q \oplus H$. Then $A(\rho) = A(\sigma) \otimes_Q A(H)$ and $V_{A(\rho)} = V \otimes_Q A(H)$ for any $V$ in $\text{GM}(A(\sigma))$. This implies that the functor is exact. Similarly, the functor from $\text{GM}(A(\sigma))$ to $\text{GM}(A)$ defined by $V \mapsto V_A$ is exact.

For $V \in \text{GM}(A(\sigma))$, we define an $A(\sigma)$-homomorphism
\begin{equation}
\varphi : D_\sigma(V) \to D_\rho(V_{A(\rho)})
\end{equation}
by $(\varphi(y), xa) := \phi_\rho((y, xa))$ for $x \in V$, $y \in D_\sigma(V)$ and $a \in A(\rho)$, where $\phi_\rho$ is the homogeneous projection $A(\rho) \to A(\rho)_{-r_\rho} = \text{Det}(\rho)_Q$.

**Lemma 1.4** Let $\sigma, \rho$ be cones in $N_R$ with $\sigma \prec \rho$. Then the homomorphism \((1.12)\) induces $A(\rho)$-isomorphism $D_\rho(V_{A(\rho)}) \simeq D_\sigma(V)_{A(\rho)}$ for every $V$ in $\text{GM}(A(\sigma))$.

**Proof.** Consider the case $V = A(\sigma)$. Let $\phi_\sigma : A(\sigma) \to \text{Det}(\sigma)_Q = A(\sigma)_{-r_\sigma}$ be the homogeneous projection. For $u \in A(\sigma)$, the corresponding element $\phi_\sigma u$ of $D_\sigma(A(\sigma))$ is given by $(\phi_\sigma u, x) := \phi_\sigma(u \wedge x)$ for $x \in A(\sigma)$. Hence $D_\sigma(A(\sigma))$ is a free $A(\sigma)$-module generated by $\phi_\sigma$. Similarly, $D_\rho(A(\rho))$ is equal to $\phi_\rho A(\rho)$.

By the definiton of $\varphi$ in \((1.12)\), we have $\varphi(\phi_\sigma) = \phi_\rho$. Hence $\varphi$ induces an isomorphism $D_\sigma(A(\sigma))_{A(\rho)} \simeq D_\rho(A(\rho))$.

For general $V$, we take an exact sequence
\begin{equation}
\bigoplus_{i=1}^m A(\sigma)x_i \xrightarrow{f} \bigoplus_{j=1}^n A(\sigma)y_j \longrightarrow V \longrightarrow 0
\end{equation}
of graded left $A(\sigma)$-modules, where $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ are homogeneous bases. Then by the exactness of the functors, we get exact sequences
\begin{equation}
0 \longrightarrow D_\sigma(V)_{A(\rho)} \longrightarrow \bigoplus_{j=1}^n y_j^* A(\rho) \xrightarrow{f^*} \bigoplus_{i=1}^m x_i^* A(\rho)
\end{equation}
and
\[ 0 \rightarrow D_\rho(V_{A(\rho)}) \rightarrow \bigoplus_{j=1}^{n} y_j^* A(\rho) \xrightarrow{i_f} \bigoplus_{i=1}^{m} x_i^* A(\rho) \] (1.15)
of graded right \( A(\rho) \)-modules. Hence we get the required isomorphism. \( \text{q.e.d.} \)

Let \( V^\bullet \) be a finite \( d \)-complex of graded \( A(\sigma) \)-modules in \( \text{GM}(A(\sigma)) \) with \( d = (d^i_V : i \in \mathbb{Z}) \). We define the complex \( D_\sigma(V)^\bullet \) by \( D_\sigma(V)^i := D_\sigma(V^{-i}) \) for \( i \in \mathbb{Z} \). The coboundary map \( d = (d^i_{D_\sigma(V)}) \) is defined by
\[ d^i_{D_\sigma(V)} := (-1)^{i+1} D_\sigma(d^{-i-1}_V) : D_\sigma(V)^i \rightarrow D_\sigma(V)^{i+1} \] (1.16)
for each \( i \in \mathbb{Z} \) (cf. [D2, 1.1.5]). Note that we have the principle to put the dot sign of complexes at the right end.

Since \( D_\sigma \) is an exact functor, the cohomology group \( H^p(D_\sigma(V)^\bullet) \) is isomorphic to \( D_\sigma(H^{-p}(V^\bullet)) \) as a graded \( \mathbb{Q} \)-vector space. By the equality (1.11), we get the following lemma.

**Lemma 1.5** Let \( \sigma \subset N_R \) be a cone and let \( V^\bullet \) be a finite \( d \)-complex in \( \text{GM}(A(\sigma)) \). Then
\[ \dim_{\mathbb{Q}} H^p(D_\sigma(V)^\bullet)_q = \dim_{\mathbb{Q}} H^{-p}(V^\bullet)_{-r_\sigma - q} \] (1.17)
for any integers \( p, q \). In particular, we have
\[ \dim_{\mathbb{Q}} H^p(D_N(V)^\bullet)_q = \dim_{\mathbb{Q}} H^{-p}(V^\bullet)_{-r - q} \] (1.18)
if \( V^\bullet \) is a finite \( d \)-complex in \( \text{GM}(A) \).

For a homomorphism \( f : V^\bullet \rightarrow W^\bullet \) in \( \text{CGM}(A(\sigma)) \), the homomorphism
\[ D_\sigma(f) : D_\sigma(W)^\bullet \rightarrow D_\sigma(V)^\bullet \] (1.19)
is defined as the collection
\[ \{ D_\sigma(f)^i = D_\sigma(f^{-i}) ; i \in \mathbb{Z} \} . \] (1.20)
For a homogeneous $\mathbb{Q}$-subalgebra $C$ of $A$, we denote by $\text{CGM}(C)$ the category of finite $d$-complexes in $\text{GM}(C)$. It is easy to see that $D_\sigma$ is a contravariant exact functor of the abelian category $\text{CGM}(A(\sigma))$ to itself.

Let $V^\bullet$ be an object of $\text{CGM}(C)$. Hence each $V_i = \bigoplus_{j \in \mathbb{Z}} V^i_j$ is in $\text{GM}(C)$. For each $i, j \in \mathbb{Z}$, we denote by $d^{i,j} : V^i_j \to V^{i+1}_j$ the homogeneous component of $d^i$ of degree $j$. For each integer $k$, the \textit{gradual truncation} below $(\text{gt} \leq k V^\bullet)$ is the homogeneous subcomplex of $V^\bullet$ defined by

$$
(\text{gt} \leq k V^\bullet)_j^i := \begin{cases} 
V^i_j & \text{if } i + j < k \\
\text{Ker } d^{i,j} & \text{if } i + j = k \\
\{0\} & \text{if } i + j > k 
\end{cases}
$$

and the gradual truncation above $(\text{gt} \geq k V^\bullet)$ is the homogeneous quotient complex of $V^\bullet$ defined by

$$
(\text{gt} \geq k V^\bullet)_j^i := \begin{cases} 
\{0\} & \text{if } i + j < k \\
\text{Coker } d^{-1,j} & \text{if } i + j = k \\
V^i_j & \text{if } i + j > k 
\end{cases}
$$

Since $C$ is graded negatively, $(\text{gt} \leq k V^\bullet)$ and $(\text{gt} \geq k V^\bullet)$ are $d$-complexes in $\text{GM}(C)$. Hence, these are covariant functors from $\text{CGM}(C)$ to itself.

The variant gradual truncations $(\tilde{\text{gt}} \leq k V^\bullet)$ and $(\tilde{\text{gt}} \geq k V^\bullet)$ are defined by

$$
(\tilde{\text{gt}} \leq k V^\bullet)_j^i := \begin{cases} 
V^i_j & \text{if } i + j \leq k \\
\text{Im } d^{-1,j} & \text{if } i + j = k + 1 \\
\{0\} & \text{if } i + j > k + 1 
\end{cases}
$$

and

$$
(\tilde{\text{gt}} \geq k V^\bullet)_j^i := \begin{cases} 
\{0\} & \text{if } i + j < k - 1 \\
\text{Im } d^{i,j} & \text{if } i + j = k - 1 \\
V^i_j & \text{if } i + j \geq k 
\end{cases}
$$

respectively.
It is easy to see that \((\tilde{g}_t \leq k V)^*\) is quasi-isomorphic to \((g_t \leq k V)^*\), while \((\tilde{g}_t \geq k V)^*\) is quasi-isomorphic to \((g_t \geq k V)^*\) (cf. [GM2, p.93]).

It is clear that

\[
H^p((\text{gt} \leq k V)^*)_q = \begin{cases} 
H^p(V^*)_q & \text{for } p + q \leq k \\
0 & \text{for } p + q > k
\end{cases}
\]  \hspace{1cm} (1.25)

and

\[
H^p((\text{gt} \geq k V)^*)_q = \begin{cases} 
0 & \text{for } p + q < k \\
H^p(V^*)_q & \text{for } p + q \geq k
\end{cases}
\]  \hspace{1cm} (1.26)

An object \(V^*\) in \(\text{CGM}(C)\) is said to be acyclic if \(H^p(V^*) = \{0\}\) for all integers \(p\).

**Lemma 1.6** Let \(\sigma\) be a cone of \(N_\mathbb{R}\). Let \(V^*\) be in \(\text{CGM}(A(\sigma))\) and \(k\) be an integer. Then the \(d\)-complex \((\text{gt} \leq k D_\sigma(V))^*\) is acyclic if and only if \((\text{gt} \geq -r_\sigma - k V)^*\) is. Similarly, the \(d\)-complex \((\text{gt} \geq k D_\sigma(V))^*\) is acyclic if and only if \((\text{gt} \leq -r_\sigma - k V)^*\) is.

If \(V^*\) is in \(\text{CGM}(A)\), then these assertions with \(D_\sigma\) replaced by \(D_\mathbb{N}\) and \(r_\sigma\) replaced by \(r\) hold.

**Proof.** By (1.25), \((\text{gt} \leq k D_\sigma(V))^*\) is acyclic if and only if \(H^p(D_\sigma(V)^*)_q = \{0\}\) for \(p + q \leq k\). By Lemma 1.5, this is equivalent to the condition

\[
\dim Q H^p(V^*)_q = \dim Q H^{-p}(D_\sigma(V)^*)_{-r_\sigma - q} = 0
\]  \hspace{1cm} (1.27)

for \(p + q \geq -r_\sigma - k\). This condition means that \((\text{gt} \geq -r_\sigma - k V)^*\) is acyclic. The second assertion is similarly proved.

The last assertion is obtained by taking \(\sigma\) with \(r_\sigma = r\). \(q.e.d.\)

### 2 The graded exterior modules on a fan

Let \(\Delta\) be a finite fan of \(N_\mathbb{R}\) [OT 1.1]. We introduce an additive category \(\text{GEM}(\Delta)\) which contains \(\text{GM}(A(\sigma))\) as full subcategories for all \(\sigma \in \Delta\).
Let $V$ be in $\text{GM}(A(\sigma))$ and $W$ in $\text{GM}(A(\rho))$. If $\sigma \prec \rho$, then $A(\sigma) \subset A(\rho)$ and $W$ has an induced structure of graded $A(\sigma)$-module.

A morphism $f : V \to W$ in $\text{GEM}(\Delta)$ is defined to be a homogeneous $A(\sigma)$-homomorphism of degree zero. If $\sigma$ is not a face of $\rho$, we allow only the zero map as a morphism even if $A(\sigma)$ happens to be contained in $A(\rho)$.

Consequently, the additive category $\text{GEM}(\Delta)$ of graded exterior modules on $\Delta$ is defined as follows.

A graded exterior module $L$ on $\Delta$ is a collection $(L(\sigma) : \sigma \in \Delta)$ of objects $L(\sigma)$ in $\text{GM}(A(\sigma))$ for $\sigma \in \Delta$. A homomorphism $f : L \to K$ of graded exterior modules on $\Delta$ is a collection $f = (f(\sigma/\rho))$ of morphisms

$$f(\sigma/\rho) : L(\sigma) \longrightarrow K(\rho)$$

in $\text{GM}(A(\sigma))$ for all pairs $(\sigma, \rho)$ of cones in $\Delta$ with $\sigma \prec \rho$. For $f : L \to K$ and $g : K \to J$, the composite $(g \cdot f) : L \to J$ is defined by

$$(g \cdot f)(\sigma/\rho) := \sum_{\tau \in F[\sigma, \rho]} g(\tau/\rho) \cdot f(\sigma/\tau)$$

for $\sigma, \rho$ with $\sigma \prec \rho$, where $F[\sigma, \rho]$ is the set of the faces $\tau$ of $\rho$ with $\sigma \prec \tau$.

The direct sum of finite objects in $\text{GEM}(\Delta)$ is defined naturally. An object $V$ of $\text{GM}(A(\sigma))$ is also regarded as an object of $\text{GEM}(\Delta)$ by defining $V(\sigma) := V$ and $V(\rho) := \{0\}$ for $\rho \neq \sigma$. In this sense, we may write $L = \bigoplus_{\sigma \in \Delta} L(\sigma)$.

A homomorphism $f : L \to K$ is said to be unmixed if $f(\sigma/\rho) = 0$ for any $\sigma, \rho$ with $\sigma \neq \rho$. If $f$ is unmixed, $\text{Ker} f$, $\text{Coker} f$ and $\text{Im} f$ are defined naturally as an object of $\text{GEM}(\Delta)$. We denote by $\text{UGEM}(\Delta)$ the category of the objects of $\text{GEM}(\Delta)$ with the class of homomorphisms restricted to unmixed ones. It is easy to see that $\text{UGEM}(\Delta)$ is an abelian category.

Let $f : L \to K$ be a homomorphism in $\text{GEM}(\Delta)$. We say that $L$ is a submodule or a subobject of $K$, if $f$ is unmixed and $f(\sigma/\sigma) : L(\sigma) \to K(\sigma)$ is an inclusion.
map for every \( \sigma \in \Delta \). If \( L \) is a submodule of \( K \), then we define an object \( K/L \) in \( \text{GEM}(\Delta) \) by \( (K/L)(\sigma) := K(\sigma)/L(\sigma) \) for \( \sigma \in \Delta \). Namely, we have a short exact sequence
\[
0 \longrightarrow L \longrightarrow K \longrightarrow K/L \longrightarrow 0 \quad (2.3)
\]
in \( \text{UGEM}(\Delta) \).

We denote \( \hat{\Delta} := \Delta \cup \{ \alpha \} \) and call it an augmented fan where \( \alpha \) is an imaginary cone. We define \( A(\alpha) := A \). The category \( \text{GEM}(\hat{\Delta}) \) is defined similarly by supposing \( \sigma \prec \alpha \) for all \( \sigma \in \hat{\Delta} \). An object \( L \) of \( \text{GEM}(\Delta) \) is also regarded as that of \( \text{GEM}(\hat{\Delta}) \) by setting \( L(\alpha) := \{0\} \).

For each \( \rho \in \hat{\Delta} \), an additive covariant functor
\[
i_{\rho}^* : \text{GEM}(\hat{\Delta}) \longrightarrow \text{GM}(A(\rho)) \quad (2.4)
\]
is defined by
\[
i_{\rho}^*(L) := \bigoplus_{\sigma \in F(\rho)} L(\sigma)_{A(\rho)},
\]
where \( F(\rho) \) is the set of faces of \( \rho \) and we suppose \( F(\rho) = \hat{\Delta} \) if \( \rho = \alpha \). Recall that \( L(\sigma)_{A(\rho)} \) is the graded \( A(\rho) \)-module \( L(\sigma) \otimes_{A(\alpha)} A(\rho) \) for each \( \sigma \). We usually denote by \( \Gamma \) the functor \( i_{\alpha}^* \).

For a homomorphism \( f : L \rightarrow K \) in \( \text{GEM}(\hat{\Delta}) \), the \((\sigma, \tau)\)-component of the homomorphism
\[
i_{\rho}^*(f) : \quad i_{\rho}^*(L) \quad \longrightarrow \quad i_{\rho}^*(K)
\]
\[
\begin{array}{c}
\bigoplus_{\sigma \in F(\rho)} L(\sigma)_{A(\rho)} \\
\bigoplus_{\tau \in F(\rho)} K(\tau)_{A(\rho)}
\end{array}
\]
is defined to be \( f(\sigma/\tau)_{A(\rho)} \) if \( \sigma \prec \tau \) and zero otherwise.

For each \( \rho \in \Delta \), the additive covariant functor
\[
i_{\rho}^! : \text{GEM}(\hat{\Delta}) \longrightarrow \text{GM}(A(\rho)) \quad (2.7)
\]
is defined by $i^i(\rho) := L(\rho)$. For $f : L \to K$, the homomorphism $i^i(f)$ is defined to be $f(\rho/\rho)$.

For cones $\sigma, \tau$ with $\sigma \prec \tau$ and $r_\tau = r_\sigma + 1$, we define the *incidence isomorphism* $q'_{\sigma/\tau} : \det(\sigma) \to \det(\tau)$ of free $\mathbb{Z}$-modules of rank one as follows.

By the condition, $N(\tau)/N(\sigma)$ is a free $\mathbb{Z}$-module of rank one. We take $a \in N(\tau) \cap \tau$ such that the class of $a$ in $N(\tau)/N(\sigma)$ is a generator. Then we define $q'_{\sigma/\tau}(w) := a \wedge w$ for $w \in \det(\sigma)$.

For cones $\sigma, \rho$ with $\sigma \prec \rho$ and $r_\rho = r_\sigma + 2$, there exists exactly two cones $\tau$ with $\sigma \prec \tau \prec \rho$ and $r_\tau = r_\sigma + 1$. Let these cones be $\tau_1, \tau_2$. Then the equality

$$q'_{\sigma/\tau_1} \cdot q'_{\tau_1/\rho} + q'_{\sigma/\tau_2} \cdot q'_{\tau_2/\rho} = 0 \quad (2.8)$$

holds (cf. [I1, Lem.1.4]).

For a subset $\Phi \subset \Delta$ and an integer $i$, we set

$$\Phi(i) := \{ \sigma \in \Phi ; r_\sigma = i \} . \quad (2.9)$$

A subset $\Phi$ of $\Delta$ is said to be *locally star closed* if $\sigma, \rho \in \Phi, \tau \in \Delta$ and $\sigma \prec \tau \prec \rho$ imply $\tau \in \Phi$.

For a locally star closed subset $\Phi$ of $\Delta$, we define a complex $E(\Phi, \mathbb{Z})^\bullet$ of free $\mathbb{Z}$-modules as follows.

For each integer $i$, we set

$$E(\Phi, \mathbb{Z})^i := \bigoplus_{\sigma \in \Phi(i)} \det(\sigma) . \quad (2.10)$$

For $\sigma \in \Phi(i)$ and $\tau \in \Phi(i + 1)$, the $(\sigma, \tau)$-component of the coboundary map

$$d^i : E(\Phi, \mathbb{Z})^i \to E(\Phi, \mathbb{Z})^{i+1} \quad (2.11)$$

is defined to be $q'_{\sigma/\tau}$. The equality $d^{i+1} \cdot d^i = 0$ follows from (2.8) for every $i$.

We say that a locally star closed subset $\Phi \subset \Delta$ is 1-*complete* if, for each $\sigma \in \Phi(r - 1)$, there exist exactly two $\tau$’s in $\Phi(r)$ with $\sigma \prec \tau$. 

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If the finite fan $\Delta$ is complete [O1, Thm.1.11], then $\Delta(\sigma \prec) := \{\rho \in \Delta \mid \sigma \prec \rho\}$ is 1-complete for every $\sigma \in \Delta$.

If $\Phi$ is 1-complete, then we can define an augmented complex $E(\hat{\Phi}, \mathbb{Z})^\bullet$ for $\hat{\Phi} := \Phi \cup \{\alpha\}$ by defining $r_\alpha := r + 1$, $\det(\alpha) := \bigwedge^r N$ and $q_{\tau/\alpha} := \text{id}$ for every $\tau \in \Phi(r)$ with respect to the identification $\det(\tau) = \det(\alpha)$. In particular, $E(\hat{\Phi}, \mathbb{Z})^i = E(\Phi, \mathbb{Z})^i$ for $i \neq r + 1$ and $E(\hat{\Phi}, \mathbb{Z})^{r+1} = \det(\alpha)$.

When $\Delta$ is complete, $E(\hat{\Delta}(\sigma \prec), \mathbb{Z})^\bullet$ is acyclic for every $\sigma \in \Delta$. Actually, $H^i(E(\Delta(\sigma \prec), \mathbb{Z})^\bullet)$ is equal to the $(i - r_{\sigma} - 1)$-th reduced cohomology group of an $(r - r_{\sigma} - 1)$-dimensional sphere, and hence it vanishes if $i \neq r$. The $r$-th cohomology is killed by $E(\hat{\Phi}, \mathbb{Z})^{r+1} = \det(\alpha)$.

We denote by $\text{CGEM}(\Delta)$ and $\text{CGEM}(\hat{\Delta})$ the category of finite $d$-complexes in $\text{GEM}(\Delta)$ and $\text{GEM}(\hat{\Delta})$, respectively.

Let $(L^\bullet, d_L)$ be an object of $\text{CGEM}(\Delta)$. Then, for each $\rho \in \Delta$, we get an object $(L(\rho)^\bullet, d_L(\rho/\rho))$ of $\text{CGM}(A(\rho))$ which we denote simply $L(\rho)^\bullet$.

For $\rho, \mu \in \Delta$ with $\rho \prec \mu$, we set $F[\rho, \mu] := \{\sigma \in F(\mu) \mid \rho \prec \sigma\}$. Then the equality $d_L \cdot d_L = 0$ implies that

$$\sum_{\sigma \in F[\rho, \mu]} d_L^{i+1}(\sigma/\mu) \cdot d_L^i(\rho/\sigma) = 0 \quad (2.12)$$

for each integer $i$. In particular, if $r_\mu = r_\rho + 1$, then the collection $(d_L^i(\rho/\mu) \ ; \ i \in \mathbb{Z})$ defines a homomorphism of complexes $d_L(\rho/\mu) : L(\rho)^\bullet \to L(\mu)[1]^{\bullet}$ since then $F[\rho, \mu] = \{\rho, \mu\}$ and the equality (2.12) imply the commutativity of the diagram

$$\begin{array}{ccc}
L(\rho)^i & \xrightarrow{d_L^i(\rho/\rho)} & L^{i+1}(\rho) \\
\downarrow d_L^i(\rho/\mu) & & \downarrow d_L^{i+1}(\rho/\mu) \\
L(\mu)[1]^i & \xrightarrow{d_L^i(\mu/\mu)} & L(\mu)[1]^{i+1}
\end{array},$$

where $d_L^i(\mu/\mu) = -d_L^{i+1}(\mu/\mu)$. 

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Conversely, assume that complexes \( L(\rho)^{\bullet} \in \text{CGM}(A(\rho)) \) for \( \rho \in \Delta \) and homomorphisms

\[
d^i_L(\sigma/\tau) : L(\sigma)^i \to L(\tau)^{i+1}
\]

for \( \sigma, \tau \in \Delta \) with \( \sigma \prec \tau \) and \( i \in \mathbb{Z} \) are given. If they satisfy (2.12) for all \((\rho, \mu)\) and \( i \in \mathbb{Z} \), then we get a complex \((L^{\bullet}, d_L)\) in \( \text{CGEM}(\Delta) \).

An object \( L^{\bullet} \) in \( \text{CGEM}(\Delta) \) is said to be shallow if \( d_L(\sigma/\rho) = 0 \) for any \( \rho, \sigma \) with \( r_\rho - r_\sigma \geq 2 \).

In order to define a shallow object \( L^{\bullet} \) of \( \text{CGEM}(\Delta) \), it is sufficient to give the following data (1), (2) and check the condition (3).

1. A complex \( L(\sigma)^{\bullet} \in \text{CGM}(A(\sigma)) \) for each \( \sigma \in \Delta \).

2. A homomorphism \( d(\sigma/\tau) : L(\sigma)^{\bullet} \to L(\tau)[1]^{\bullet} \) for each pair \((\sigma, \tau)\) of cones in \( \Delta \) with \( \sigma \prec \tau \) and \( r_\tau = r_\sigma + 1 \).

3. The equality

\[
d(\tau_1/\rho)^{i+1} \cdot d(\sigma/\tau_1)^i + d(\tau_2/\rho)^{i+1} \cdot d(\sigma/\tau_2)^i = 0
\]

holds for all \( i \in \mathbb{Z} \) and all pairs \((\sigma, \rho)\) with \( \sigma \prec \rho \) and \( r_\rho = r_\sigma + 2 \), where \( \tau_1, \tau_2 \) are the dual cones with \( \sigma \prec \tau_i \prec \rho \) and \( r_{\tau_i} = r_\sigma + 1 \).

For an object \( L^{\bullet} \) in \( \text{CGEM}(\Delta) \), we define a shallow object \( \tilde{L}^{\bullet} \) as follows.

For each \( \rho \in \Delta \) and \( i \in \mathbb{Z} \), we set

\[
\tilde{L}(\rho)^i := \bigoplus_{\sigma \in F(\rho)} \bigoplus_{\eta \in F(\sigma)} \det(\rho) \otimes \det(\sigma)^* \otimes L(\eta)^{-r_\rho + r_\sigma + i}
\]

where \( \det(\sigma)^* := \text{Hom}_{\mathbb{Z}}(\det(\sigma), \mathbb{Z}) \). For

\[
\tilde{L}(\rho)^{i+1} := \bigoplus_{\tau \in F(\rho)} \bigoplus_{\zeta \in F(\tau)} \det(\rho) \otimes \det(\tau)^* \otimes L(\zeta)^{-r_\rho + r_\tau + i+1}
\]

the \((\sigma, \eta), (\tau, \zeta)\)-component of the coboundary map \( d(\rho/\rho)^i : \tilde{L}(\rho)^i \to \tilde{L}(\rho)^{i+1} \) is defined to be zero map except for the cases (a) \( \sigma = \tau \) and \( \eta \prec \zeta \), or (b) \( \tau \prec \sigma \), \( r_\sigma = r_\tau + 1 \) and \( \eta = \zeta \). In case (a), the component is defined to be \((-1)^{r_\rho - r_\sigma} \text{id} \otimes \)
\( d(\eta/\zeta)^{-r_\rho+r_\sigma+i} \), and in case (b), it is defined to be \((-1)^{r_\rho-r_\sigma-1}\det(\rho) \otimes (q_{\tau/\sigma}')^* \otimes \text{id} \), where \text{id}'s are identity maps of the corresponding parts.

For \( \rho \prec \mu \) with \( r_\mu = r_\rho + 1 \), the homomorphism \( d(\rho/\mu) : \tilde{L}(\rho)^* \rightarrow \tilde{L}(\mu)[1]^* \) is defined to be the tensor product of \( q_{\rho/\mu}' \) and the natural inclusion map.

It is easy to check that \( \tilde{L}(\rho)^* \) is actually a complex and \( d(\rho/\mu) \) is a homomorphism of complexes. The condition (3) in the definition of shallow complexes is satisfied by the equality (2.8).

We define a homomorphism \( f_L : L^* \rightarrow \tilde{L}^* \) as follows.

For \( \tau, \rho \in \Delta \) with \( \tau \prec \rho \), the \( A(\tau) \)-homomorphism \( f_L(\tau/\rho)^i : L(\tau)^i \rightarrow \tilde{L}(\rho)^i \) is the inclusion map to the component \( \det(\rho) \otimes \det(\sigma)^* \otimes L(\eta)^{-r_\rho+r_\sigma+i} \) for \( \sigma = \rho \) and \( \eta = \tau \) in the description (2.16). The compatibility with the coboundary maps is checked easily.

A homomorphism \( f : L^* \rightarrow K^* \) of \( d \)-complexes of graded exterior modules on \( \Delta \) is said to be a quasi-isomorphism if \( f(\sigma/\sigma) : L(\sigma)^* \rightarrow K(\sigma)^* \) is a quasi-isomorphism of \( d \)-complexes of \( A(\sigma) \)-modules for every \( \sigma \in \Delta \).

The following proposition shows that any object in \( \text{CGEM}(\Delta) \) is quasi-isomorphic to a shallow one.

**Proposition 2.1** For any \( L^* \) in \( \text{CGEM}(\Delta) \), the homomorphism \( f_L : L^* \rightarrow \tilde{L}^* \) is quasi-isomorphic.
\[ F^k(V)^\bullet / F^{k+1}(V)^\bullet \] to a direct sum

\[ \bigoplus_{\eta \in F(\rho)(k)} V^\bullet_{k,\eta}, \tag{2.19} \]

where \( V^\bullet_{k,\eta} \) is the part consisting of the components related to \( \eta \). Then we see that \( V^\bullet_{k,\eta} \) is isomorphic to the associated single complex of the bicomplex

\[ \det(\rho) \otimes \text{Hom}(E(F[\eta, \rho], \mathbf{Z}), \mathbf{Z})[-r_\rho] \otimes \bigotimes_{\eta \in F(\rho)(k)} L(\eta)^\bullet_{A(\rho)}, \tag{2.20} \]

where we denote by \( \text{Hom}(E, \mathbf{Z})^\bullet \) the complex defined by

\[ \text{Hom}(E, \mathbf{Z})^i := \text{Hom}(E^{-i}, \mathbf{Z}) \tag{2.21} \]

and \( d^i := (-1)^{i+1}(d_E^{-i-1})^\ast \) for \( i \in \mathbf{Z} \). If \( k < r_\rho \), then \( \eta \neq \rho \) and \( E(F[\eta, \rho], \mathbf{Z})^\bullet \) is acyclic, and hence so is \( V^\bullet_{k,\eta} \). Hence \( V^\bullet \) is quasi-isomorphic to the subcomplex \( F^{r_\rho}(V)^\bullet \). Since \( f_L(\rho/\rho) \) is an isomorphism from \( L(\rho)^\bullet \) onto \( F^{r_\rho}(V)^\bullet \), it is a quasi-isomorphism to \( V^\bullet = \tilde{L}(\rho)^\bullet \).

q.e.d.

For a complex \( L^\bullet \) in \( \text{CGEM}(\Delta) \), we define a shallow complex \( D(L)^\bullet \) in \( \text{CGEM}(\Delta) \) as follows.

For each \( \rho \in \Delta \), we set

\[ D(L)(\rho)^\bullet := \det(\rho) \otimes D_\rho(i^*_\rho L)[{-r_\rho}]^\bullet. \tag{2.22} \]

Let \( \rho, \mu \in \Delta \) satisfy \( \rho < \mu \) and \( r_\mu = r_\rho + 1 \). By the definition of \( i^*_\mu \) and \( i^*_\rho \), we see that \( (i^*_\rho L)^{A(\mu)} \) is a direct summand of \( i^*_\mu L^i \). Let \( p(\mu/\rho)^i : i^*_\mu L^i \to (i^*_\rho L)^{A(\mu)} \) be the natural projection map. We see that \( \{p(\mu/\rho)^i : i \in \mathbf{Z} \} \) defines a homomorphism \( p(\mu/\rho) : i^*_\mu L^\bullet \to (i^*_\rho L)^{A(\mu)} \). Hence we get a homomorphism

\[ D_\mu(p(\mu/\rho)) : D_\mu((i^*_\rho L)^{A(\mu)})^\bullet \to D_\mu(i^*_\mu L)^\bullet. \tag{2.23} \]

Since

\[ D_\mu((i^*_\rho L)^{A(\mu)})^\bullet = D_\rho(i^*_\rho L)^{A(\mu)}^\bullet \tag{2.24} \]
by Lemma 1.4, we get a homomorphism

\[ i(\rho/\mu) : D_{\rho}(i_{\rho}^*L)^{\bullet} \longrightarrow D_{\mu}(i_{\mu}^*L)^{\bullet} \]  \hspace{1cm} (2.25)

as the composite of the inclusion \( D_{\rho}(i_{\rho}^*L)^{\bullet} \to D_{\rho}(i_{\rho}^*L)_{A(\rho)}^{\bullet} \) and \((2.23)\). We define

\[ d_{D(L)}(\rho/\mu) : D(L)(\rho)^{\bullet} \longrightarrow D(L)(\mu)[1]^{\bullet} \]  \hspace{1cm} (2.26)

to be \( q_{\rho/\mu} \otimes i(\rho/\mu)[-r_{\rho}] \). By the equality \((2.8)\), the condition (3) of the construction of shallow complexes is satisfied and we get a complex \( D(L)^{\bullet} \).

**Lemma 2.2** Let \( L^{\bullet} \) be an object of \( CGEM(\Delta) \). Then, there exists a quasi-isomorphism \( \varphi : L^{\bullet} \to D(D(L))^{\bullet} \).

**Proof.** We prove that \( D(D(L))^{\bullet} \) is isomorphic to \( \tilde{L}^{\bullet} \). Then the lemma follows from Proposition 2.1.

Let \( \rho \) be an element of \( \Delta \). For each integer \( i \),

\[
\begin{align*}
D(D(L))(\rho)^i &= \bigoplus_{\sigma \in F(\rho)} \det(\rho) \otimes D_{\rho}(D(L)(\sigma)_{A(\rho)}^{-r_{\sigma}+r_{\rho}+i}) \\
&= \bigoplus_{\sigma \in F(\rho)} \bigoplus_{\eta \in F(\sigma)} \det(\rho) \otimes D_{\rho}(\det(\sigma) \otimes D_{\sigma}(L(\eta)_{A(\sigma)}^{-r_{\sigma}+r_{\sigma}+i})_{A(\rho)}) \\
&= \bigoplus_{\sigma \in F(\rho)} \bigoplus_{\eta \in F(\sigma)} \det(\rho) \otimes \det(\sigma)^* \otimes D_{\rho}(D_{\sigma}(L(\eta)_{A(\sigma)}^{-r_{\sigma}+r_{\sigma}+i})_{A(\rho)}) .
\end{align*}
\]

By Lemma 1.4, we have a natural isomorphism

\[
D_{\rho}(D_{\sigma}(L(\eta)_{A(\sigma)}^{-r_{\sigma}+r_{\sigma}+i})_{A(\rho)}) \simeq D_{\rho}(D_{\sigma}(L(\eta)_{A(\sigma)}^{-r_{\sigma}+r_{\sigma}+i})) . \]  \hspace{1cm} (2.27)

We identify the last \( A(\rho) \)-module with \( L(\eta)_{A(\rho)}^{-r_{\rho}+r_{\sigma}+i} \) by the canonical isomorphism for all \((\sigma, \eta)\). we know \( D(D(L))(\rho)^i = \tilde{L}(\rho)^i \) for every \( \rho \in \Delta \) and \( i \in \mathbb{Z} \), however the coboundary map is not equal to that of \( \tilde{L}^{\bullet} \).
We consider the descriptions (2.16) and (2.17) with replacing \( \tilde{L}(\rho)^i \) and \( \tilde{L}(\rho)^{i+1} \) by \( \mathbf{D}(\mathbf{D}(L))(\rho)^i \) and \( \mathbf{D}(\mathbf{D}(L))(\rho)^{i+1} \), respectively. We can check that the \(((\sigma, \eta), (\tau, \zeta))\)-component of the coboundary map \( d(\rho/\rho)^i : \tilde{\mathbf{D}}(\mathbf{D}(L))(\rho)^i \to \mathbf{D}(\mathbf{D}(L))(\rho)^{i+1} \) is the zero map except for the cases (a) \( \sigma = \tau \) and \( \eta \prec \zeta \), or (b) \( \tau \prec \sigma, r_\sigma = r_\tau + 1 \) and \( \eta = \zeta \). In case (a), the component is calculated to be \((-1)^{r_\rho + 1} \otimes d(\eta/\zeta)^{-r_\rho + r_\sigma + i}\), and in case (b), it is \((-1)^{i+1} \det(\rho)^{-1} \otimes (q_{\tau/\sigma})^* \otimes \text{id}\).

For each \( \rho \) and \( i \), we define an isomorphism

\[ \varphi(\rho)^i : \tilde{L}(\rho)^i \longrightarrow \mathbf{D}(\mathbf{D}(L))(\rho)^i \]  

by defining its restriction to the component

\[ \det(\rho) \otimes \det(\sigma)^* \otimes \mathbf{L}(\mathbf{L})(\rho)^i \]

\[ \longrightarrow (-1)^{(r_\rho + 1)(r_\sigma + 1)} \times \text{id} \text{ to the same component of } \mathbf{D}(\mathbf{D}(L))(\rho)^i. \]

Then we see that the collection \( \{\varphi(\rho)^i\} \) defines an unmixed isomorphism \( \tilde{L}^\bullet \to \mathbf{D}(\mathbf{D}(L))^\bullet \).

q.e.d.

We can define a similar functor

\[ \tilde{\mathbf{D}} : \text{CGEM}(\hat{\Delta}) \longrightarrow \text{CGEM}(\hat{\Delta}) \]

for an augmented 1-complete fan \( \hat{\Delta} \) by the convention that \( r_\alpha = r + 1 \), \( F(\alpha) = \hat{\Delta} \)
and \( q_{\tau/\alpha} := 1 \det N \) for \( \tau \in \Delta(r) \). When \( L^\bullet \) is an object of \( \text{CGEM}(\Delta) \), \( \tilde{\mathbf{D}}(L)^\bullet \) is in \( \text{CGEM}(\hat{\Delta}) \). Since \( \det(\alpha) = \det N \), we get an exact sequence

\[ 0 \rightarrow \tilde{\mathbf{D}}(L)(\alpha)^\bullet \longrightarrow \tilde{\mathbf{D}}(L)^\bullet \longrightarrow \mathbf{D}(L)^\bullet \longrightarrow 0 \]

in \( \text{CGEM}(\hat{\Delta}) \) as well as an exact sequence

\[ 0 \rightarrow \det N \otimes \mathbf{D}_N(\Gamma(L))[-r - 1]^\bullet \longrightarrow \Gamma(\tilde{\mathbf{D}}(L))^\bullet \longrightarrow \Gamma(\mathbf{D}(L))^\bullet \longrightarrow 0 \]

in \( \text{CGM}(A) \) by applying \( \Gamma \).
Lemma 2.3 Assume that $\Delta$ is a complete fan. Then, for any $L^\bullet$ in CGEM$(\Delta)$, the complex of $A$-modules $\Gamma((\hat{D}(L)))^\bullet$ is acyclic.

Proof. For each $i \in \mathbb{Z}$, we have

$$\Gamma(\hat{D}(L))^i = \bigoplus_{\rho \in \Delta} \bigoplus_{\sigma \in F(\rho)} \det(\rho) \otimes D_\rho(L(\sigma)_{A(\rho)}^{-r_\rho-i})_A . \quad (2.33)$$

For

$$\Gamma(\hat{D}(L))^{i+1} = \bigoplus_{\mu \in \Delta} \bigoplus_{\tau \in F(\mu)} \det(\mu) \otimes D_\mu(L(\tau)_{A(\mu)}^{-r_\mu-i-1})_A , \quad (2.34)$$

the $((\rho, \sigma), (\mu, \tau))$-component of the coboundary map is nonzero only for (a) $\rho = \mu$ and $\tau \prec \sigma$, or (b) $\rho \prec \mu$, $r_\mu = r_\rho + 1$ and $\sigma = \tau$. In case (a), the component is $(-1)^{i+1} \text{id} \otimes D_\rho(d(\tau/\sigma)_{A(\rho)}^{-r_\rho-i})_A$, and in case (b), it is $q'_{\rho/\mu} \otimes \text{id}$, where id’s are the identities of the corresponding parts, respectively.

For $V^\bullet := \Gamma(\hat{D}(L))^\bullet$, we introduce a deceasing filtration $\{F^k\}$ as follow. For each integer $i$, we define $F^k(V)^i$ to be the direct sum of the components $\det(\rho) \otimes D_\rho(L(\sigma)_{A(\rho)}^{-r_\rho-i})_A$ of $\Gamma(\hat{D}(L))^i$ in the description $(2.33)$ for all pairs $(\rho, \sigma)$ with $r_\sigma \geq k$. Then $F^k(V)^\bullet$ is a subcomplex of $V^\bullet$ for every $k \in \mathbb{Z}$, and $F^k(V)^\bullet/F^{k+1}(V)^\bullet$ is a direct sum of complexes $\bigoplus_{\sigma \in \Delta(k)} V^\bullet_{\sigma}$, where $V^\bullet_{\sigma}$ is the part related to each $\sigma$. We see that $V^\bullet_{\sigma}$ is isomorphic to the associated single complex of the bicomplex

$$E(F[\sigma, \alpha], \mathbb{Z})^\bullet \otimes D_\sigma(L(\sigma))^\bullet . \quad (2.35)$$

Since $\Delta$ is complete, $E(F[\sigma, \alpha], \mathbb{Z})^\bullet$ is acyclic for every $\sigma \in \Delta$. Hence $F^k(V)^\bullet/F^{k+1}(V)^\bullet$ is also acyclic for every $k$. Since $F^0(V)^\bullet = V^\bullet$ and $F^{r+1}(V)^\bullet = \{0\}$, $V^\bullet$ is acyclic.

$q.e.d.$

By the long exact sequence obtained from the exact sequence $(2.32)$, we get the following corollary.

Corollary 2.4 If $\Delta$ is complete, then there exists an isomorphism

$$H^{-p}(D_N(\Gamma(L))^\bullet) \simeq H^{r-p}(\Gamma(D(L))^\bullet) \quad (2.36)$$
in GM(A) for each integer $p$.

The following proposition is a consequence of Lemma 1.3 and Corollary 2.4.

**Proposition 2.5** Assume that $\Delta$ is complete. For any integer $p, q$, the equality

$$\dim Q H^p(\Gamma(L)^\bullet)_q = \dim Q H^{r-p}(\Gamma(D(L))_q^{\bullet})_{-r-q}$$

(2.37)

holds.

A finite fan $\Delta$ of $N_R$ is said to be lifted complete if there exists a rational line $\ell$ of $N_R$ going through the origin with the following property.

Let $\bar{\sigma}$ be the image of $\sigma$ in the quotient $N_R/\ell$ for each $\sigma \in \Delta$. Then (1) $\dim \bar{\sigma} = r_\sigma$ for every $\sigma \in \Delta$, (2) $\bar{\sigma} \neq \bar{\tau}$ for any distinct $\sigma, \tau \in \Delta$ and (3) $\Delta_\ell := \{ \bar{\sigma} ; \sigma \in \Delta \}$ is a complete fan of $N_R/\ell$.

For a lifted complete fan, the associated toric variety has an action of the multiplicative algebraic group $G_m$, and it has a complete toric variety of dimension $r - 1$ as the geometric quotient in the sense of Mumford’s geometric invariant theory.

Let $\Delta$ be a lifted complete fan with respect to $\ell$ and let $\ell^+$ be one of the one-dimensional cones contained in $\ell$. By the property (1), $\dim (\tau + \ell^+) = r_\tau + 1$ for every $\tau \in \Delta$. The oriented lifted complete fan $\tilde{\Delta}$ is defined to be $\Delta \cup \{ \beta \}$ where $\beta$ is an imaginary cone of dimension $r$. We suppose $\sigma \prec \beta$ for every $\sigma \in \Delta$. We define $\det(\beta) := \det N$ and $q_{\tau/\beta} : = q_{\tau'/\tau'}$ for $\tau \in \Delta(r-1)$, where $\tau' := \tau + \ell^+$.

For any star closed subset $\Phi \subset \tilde{\Delta}$, the complex $E(\Phi, Z)^\bullet$ is defined similarly as in the previous case. The complex $E(\tilde{\Delta}(\sigma \prec), Z)^\bullet$ is acyclic for every $\sigma \in \Delta$ since it is isomorphic to the augmented complex $E(\tilde{\Delta}_\ell, Z)^\bullet$.

Each lifted complete fan has two orientations according to the choice of $\ell^+$, but it does not depend on the choice of the line $\ell$.

Here we give three typical examples of oriented lifted complete fans.
(1) Let \( C \subset N_R \) be a closed convex cone of dimension \( r \) which may not be strongly convex and which is not equal to \( N_R \). Let \( \partial C \) be the boundary set of \( C \). Then a finite fan \( \Delta \) with the support \( \partial C \) is lifted complete. Any \( \ell \) which intersects the interior of \( C \) satisfies the condition. As the natural orientation, we take \( \ell^+ := \ell \cap C \).

(2) Let \( \Phi \) be a simplicial complete fan of \( N_R \) and let \( \gamma \) be a one-dimensional cone in \( \Phi \). Set
\[
\Delta := \{ \sigma \in \Phi ; \gamma \nleq \sigma, \sigma + \gamma \in \Phi \} .
\] Then \( \Delta \) is a lifted complete fan and \( \ell^+ := \gamma \) defines an orientation (cf. [O2]).

(3) Let \( N' \) be a free \( \mathbb{Z} \)-module of rank \( r - 1 \), \( \Phi \) a finite complete fan of \( N'_R \) and \( h \) a real-valued continuous function on \( N'_R \) which is linear on each cone \( \sigma \in \Phi \) and has rational values on \( N'_Q \). Then the fan \( \Delta = \{ \sigma' ; \sigma \in \Phi \} \) of \( N_R := N'_R \oplus R \) is lifted complete, where \( \sigma' := \{(x, h(x)) \mid x \in \sigma \} \) for each \( \sigma \in \Phi \). We take \( \ell^+ := \{0\} \times R_0 \) as the orientation. This type of fan is treated in [O3].

Let \( \tilde{\Delta} = \Delta \cup \{\beta\} \) be an oriented lifted complete fan. The category \( \text{GEM}(\tilde{\Delta}) \) is defined by setting \( A(\beta) := A \). Then the functors
\[
i^\beta : \text{GEM}(\tilde{\Delta}) \longrightarrow \text{GM}(A)
\]
and
\[
\hat{D} : \text{GEM}(\tilde{\Delta}) \longrightarrow \text{GEM}(\tilde{\Delta})
\]
are defined similarly as \( i^*_\alpha \) and \( \hat{D} \) in the case of augmented 1-complete fans, respectively. We denote also by \( \Gamma \) the functor \( i^*_\beta \).

We omit the proofs of the following results, since they are similar to those of the corresponding results for an augmented complete fan.

**Lemma 2.6** Let \( \tilde{\Delta} = \Delta \cup \{\beta\} \) be an oriented lifted complete fan. Then, for any \( L^\bullet \) in \( \text{CGEM}(\Delta) \), the complex of \( A \)-modules \( \Gamma(\hat{D}(L))^\bullet \) is acyclic, and there exists an exact sequence
\[
0 \longrightarrow \text{det} N \otimes D(\Gamma(L))^\bullet[-r] \longrightarrow \Gamma(\hat{D}(L))^\bullet \longrightarrow \Gamma(D(L))^\bullet \longrightarrow 0 .
\]
Corollary 2.7 Let \( \Delta \cup \{ \beta \} \) be an oriented lifted complete fan. Then for any \( L^\bullet \in \text{CGEM}(\Delta) \), there exists an isomorphism
\[
H^{-p}(D_N(\Gamma(L))^\bullet) \simeq H^{r-p-1}(\Gamma(D(L))^\bullet)
\]
(2.42)
in \( \text{GM}(A) \) for each integer \( p \).

Proposition 2.8 Let \( \Delta \cup \{ \beta \} \) be an oriented lifted complete fan. For any integer \( p, q \), the equality
\[
\dim H^p(\Gamma(L)^\bullet)q = \dim H^{r-p-1}(\Gamma(D(L))^\bullet)_{-r-q}
\]
(2.43)
holds.

Let \( \Delta \) be a finite fan of \( N_\mathbb{R} \) and let \( \Phi \) be a subfan of \( \Delta \). For \( K^\bullet \) in \( \text{CGEM}(\Phi) \), we denote by the same symbol \( K^\bullet \) the trivial extension to \( \Delta \), i.e., we define \( K(\sigma)^\bullet := \{0\} \) for \( \sigma \in \Delta \setminus \Phi \).

For \( L^\bullet \in \text{CGEM}(\Delta) \) and \( \sigma \in \Delta \), we set
\[
i^*_\sigma(L)^\bullet := i^*_\sigma(L|F(\sigma)\setminus\{\sigma\})^\bullet,
\]
(2.44)
where \( (L|F(\sigma)\setminus\{\sigma\})^\bullet \) is the restriction of \( L^\bullet \) to \( F(\sigma) \setminus \{\sigma\} \).

We see the case \( \Phi = \Delta \setminus \{\pi\} \) for a maximal element \( \pi \in \Delta \). Let \( L^\bullet \in \text{CGEM}(\Delta) \). Since
\[
i^0_\pi(L)^i = \bigoplus_{\sigma \in F(\pi) \setminus \{\pi\}} L(\sigma)^i_{A(\pi)}
\]
(2.45)
and \( i^*_\pi(L)^i = L(\pi)^i \oplus i^0_\pi(L)^i \) for each \( i \in \mathbb{Z} \), there exists an exact sequence
\[
0 \to L(\pi)^\bullet \longrightarrow i^*_\pi(L)^\bullet \longrightarrow i^0_\pi(L)^\bullet \to 0
\]
(2.46)
in \( \text{CGM}(A(\pi)) \). In other words, \( i^*_\pi(L)^\bullet \) is equal to the mapping cone of the homomorphism
\[
\phi : i^0_\pi(L)[-1]^\bullet \longrightarrow L(\pi)^\bullet
\]
(2.47)
whose component for each $\sigma \in F(\pi) \setminus \{\pi\}$ is $d(\sigma/\pi)_{A(\pi)}$.

The extension of $(L|\Phi)^{\bullet}$ to $L^{\bullet}$ is determined by the above homomorphism $\phi$. Actually, if $(L|\Phi)^{\bullet} \in \text{CGEM}(\Phi)$, $L(\pi)^{\bullet}$ and the homomorphism (2.47) is given, then we get the extension $L^{\bullet}$.

We define a functor $j^{\Phi}_{!} : \text{CGEM}(\Phi) \to \text{CGEM}(\Delta)$ as follows.

For $K^{\bullet}$ in $\text{CGEM}(\Phi)$, we define $j^{\Phi}_{!}(K)^{\bullet}$ by

$$j^{\Phi}_{!}(K)(\sigma)^{\bullet} := K(\sigma)^{\bullet}$$  \hspace{1cm} (2.48)

for $\sigma \in \Phi$ and

$$j^{\Phi}_{!}(K)(\pi)^{\bullet} := i_{\pi}^{\circ}(K)[-1]^{\bullet} = i_{\pi}^{\circ}(K)[-1]^{\bullet}.$$  \hspace{1cm} (2.49)

In particular,

$$j^{\Phi}_{!}(K)(\pi)^{i+1} = \bigoplus_{\sigma \in F(\pi) \setminus \{\pi\}} K(\sigma)^{i}_{A(\pi)}.$$  \hspace{1cm} (2.50)

We define the extension $j^{\Phi}_{!}(K)^{\bullet} \in \text{CGEM}(\Delta)$ of $K^{\bullet}$ by the identity map

$$i_{\pi}^{\circ}(K)[-1]^{\bullet} \longrightarrow j^{\Phi}_{!}(K)(\pi)^{\bullet}.$$  \hspace{1cm} (2.51)

Let $(L^{\bullet}, d_{L})$ and $(K^{\bullet}, d_{K})$ be $d$-complexes in $\text{CGEM}(\Delta)$. By an unmixed homomorphism $f : L^{\bullet} \to K^{\bullet}$, we means a collection $\{f^{i} : i \in \mathbb{Z}\}$ of unmixed homomorphisms $f^{i} : L^{i} \to K^{i}$ such that $d_{K}^{i} \cdot f^{i} = f^{i+1} \cdot d_{L}^{i}$ for every $i \in \mathbb{Z}$. Note that $d_{L}^{i}$ and $d_{K}^{i}$ are not necessary unmixed.

Let $L^{\bullet}$ be an object of $\text{CGEM}(\Delta)$. Then there exists a unique unmixed homomorphism $j^{\Phi}_{!}(L|\Phi)^{\bullet} \to L^{\bullet}$ which is the identity map on $\Phi$. The homomorphism

$$j^{\Phi}_{!}(L|\Phi)(\pi)^{\bullet} = i_{\pi}^{\circ}(L)[-1]^{\bullet} \longrightarrow L(\pi)^{\bullet}$$  \hspace{1cm} (2.52)

is defined to be (2.47).

We can define a functor $j^{\Phi}_{!} : \text{CGEM}(\Phi) \to \text{CGEM}(\Delta)$ for general subfan $\Phi \subset \Delta$ as the composite of the above functors. However, we will not use the general case.
Let $\pi$ be a maximal element of $\Delta$. For $L^\bullet$ in CGEM($\Delta$) and an integer $k$, we define $gt^\geq_k L^\bullet$ by

$$
(gt^\geq_k L)(\sigma)^\bullet = \begin{cases} 
L(\sigma)^\bullet & \text{if } \sigma \neq \pi \\
geq^\geq_k L(\pi)^\bullet & \text{if } \sigma = \pi .
\end{cases}
$$

(2.53)

There exists a natural unmixed homomorphism $L^\bullet \rightarrow gt^\geq_k L^\bullet$ such that the components for $\sigma \neq \pi$ are the identity maps and the component for $\pi$ is the natural surjection $L^\bullet(\pi) \rightarrow gt^\geq_k L(\pi)^\bullet$. This homomorphism is a quasi-isomorphism if and only if $gt^\leq_{k-1} (L^\bullet(\pi))$ is acyclic.

Two objects $L^\bullet, K^\bullet$ in CGEM($\Delta$) are said to be quasi-isomorphic if there exist a finite sequence $L^\bullet_0, L^\bullet_1, \cdots, L^\bullet_{2k}$ of objects in CGEM($\Delta$) with $L^\bullet = L^\bullet_0$ and $L^\bullet_{2k} = K^\bullet$ and quasi-isomorphisms $L^\bullet_{2i-2} \rightarrow L^\bullet_{2i-1}$ and $L^\bullet_{2i} \rightarrow L^\bullet_{2i-1}$ for $i = 1, \cdots, k$. Some lemmas on this definition are given at the end this section.

A map $p : \Delta \setminus \{0\} \rightarrow \mathbb{Z}$ is called a perversity on a finite fan $\Delta$. We prove the following theorem.

**Theorem 2.9** Let $p$ be a perversity on $\Delta$. Then there exists a finite $d$-complex $ic_p(\Delta)^\bullet$ of graded exterior modules on $\Delta$ satisfying the following conditions.

1. $H^i(ic_p(\Delta)(0)^\bullet) = \mathbb{Q}$ and $H^i(ic_p(\Delta)(0)^\bullet) = \{0\}$ for $i \neq 0$.
2. For $\sigma \in \Delta \setminus \{0\}$ and $i, j \in \mathbb{Z}$ with $i + j \leq p(\sigma)$, we have $H^i(ic_p(\Delta)(\sigma)^\bullet) = \{0\}$.
3. For $\sigma \in \Delta \setminus \{0\}$ and $i, j \in \mathbb{Z}$ with $i + j \geq p(\sigma)$, we have $H^i(i^*_\sigma(ic_p(\Delta))^\bullet) = \{0\}$.

Furthermore, if $L^\bullet$ is another finite $d$-complex satisfying the above conditions, then $L^\bullet$ is quasi-isomorphic to $ic_p(\Delta)^\bullet$.

**Proof.** We prove the theorem by induction on the number of cones in $\Delta$. If $\Delta = \{0\}$, then we set $ic_p(\Delta)(0)^0 = \mathbb{Q}$ and $ic_p(\Delta)(0)^i = \{0\}$ for $i \neq 0$. Then (1) and the last assertion are clearly satisfied.
Assume that $\Delta \neq \{0\}$. Let $\pi \in \Delta$ be a cone of maximal dimension. We assume that the $d$-complex $ic_\Phi(\Phi)^\bullet$ exists for $\Phi = \Delta \setminus \{\pi\}$.

Let $L^\bullet := j_\Phi \cdot ic_\Phi(\Phi)^\bullet$. We define

$$ic_\Phi(\Delta)^\bullet := gt^{\geq p(\pi)+1}_\pi L^\bullet.$$  \hfill (2.54)

Since

$$ic_\Phi(\Delta)(\pi)^\bullet = gt^{\geq p(\pi)+1}_\pi L(\pi)^\bullet,$$  \hfill (2.55)

the truncation $gt_{\leq p(\pi)}(ic_\Phi(\Delta)(\pi))^\bullet$ is the zero complex.

On the other hand, $i_\pi^\bullet ic_\Phi(\Delta)^\bullet$ is equal to the mapping cone

$$(L(\pi)[1] \oplus gt^{\geq p(\pi)+1}_\pi L(\pi))^\bullet$$  \hfill (2.56)

of the natural surjection

$$L(\pi)^\bullet \longrightarrow gt^{\geq p(\pi)+1}_\pi L(\pi)^\bullet.$$  \hfill (2.57)

There exists an exact sequence

$$0 \longrightarrow gt_{\leq p(\pi)} L(\pi)^\bullet \longrightarrow L(\pi)^\bullet \longrightarrow gt^{\geq p(\pi)+1}_\pi L(\pi)^\bullet \longrightarrow 0$$  \hfill (2.58)

of $d$-complexes in $GM(A(\pi))$. Hence $i_\pi^\bullet ic_\Phi(\Delta)^\bullet$ is quasi-isomorphic to

$$(gt_{\leq p(\pi)} L(\pi))[1]^\bullet = gt_{\leq p(\pi)-1} (L(\pi)[1])^\bullet.$$  \hfill (2.59)

and to $gt_{\leq p(\pi)-1}(L(\pi)[1])^\bullet$. Hence $gt^{\geq p(\pi)} i_\pi^\bullet ic_\Phi(\Delta)^\bullet$ is acyclic. This is equivalent to the condition (3).

For the last assertion, it is sufficient to prove the following lemma.

**Lemma 2.10** Let $\pi \neq 0$ be a maximal element of $\Delta$ and let $\Phi := \Delta \setminus \{\pi\}$. Let $L^\bullet, K^\bullet$ be objects of $CGEM(\Delta)$ which satisfy the conditions of Theorem 2.9, and assume that there exists a quasi-isomorphism $(L|\Phi)^\bullet \rightarrow (K|\Phi)^\bullet$. Then, $L^\bullet$ and $K^\bullet$ is connected by a finite sequence of unmixed quasi-isomorphisms.
Proof. By the condition (2), the natural homomorphism
\[ L^\bullet \to \operatorname{gt}^{\geq p(\sigma)+1}_\pi L^\bullet \] (2.60)
is a quasi-isomorphism. Since \( i^*_\pi(L)^\bullet \) is the mapping cone of the homomorphism \( j^\Phi_i(L|\Phi)^\bullet \to L(\pi)^\bullet \), the condition (3) implies the homomorphism
\[ \operatorname{gt}^{\geq p(\sigma)+1}_\pi (j^\Phi_i(L|\Phi))^\bullet \to \operatorname{gt}^{\geq p(\sigma)+1}_\pi L^\bullet \] (2.61)
is also a quasi-isomorphism.

There are similar quasi-isomorphisms for \( K^\bullet \). We are done since the quasi-isomorphism \( (L|\Phi)^\bullet \to (K|\Phi)^\bullet \) induces a quasi-isomorphism
\[ \operatorname{gt}^{\geq p(\sigma)+1}_\pi (j^\Phi_i(L|\Phi))^\bullet \to \operatorname{gt}^{\geq p(\sigma)+1}_\pi (j^\Phi_i(K|\Phi))^\bullet \] (2.62)
=q.e.d.

Thus we complete the proof of Theorem 2.3.

In the rest of this paper, we denote by \( \operatorname{ic}_p(\Delta)^\bullet \) the \( d \)-complex constructed in the proof of the above theorem, and we call it the intersection complex of the fan \( \Delta \) with the perversity \( p \). It does not depend on the choice of the order of the induction, and is uniquely determined by \( \Delta \) and \( p \). If \( \Phi \) is a subfan of \( \Delta \), then the construction implies that \( \operatorname{ic}_p(\Phi)^\bullet \) is isomorphic to the restriction of \( \operatorname{ic}_p(\Delta)^\bullet \) to \( \Phi \). In particular, \( \operatorname{ic}_p(\Delta)(\sigma)^\bullet = \operatorname{ic}_p(F(\sigma))(\sigma)^\bullet \). Hence the complex \( \operatorname{ic}_p(\Delta)(\sigma)^\bullet \) of \( A(\sigma) \)-modules depends only on \( p \) and \( \sigma \).

**Proposition 2.11** Let \( \sigma \) be a nonzero cone of \( \Delta \) and let \( p \) be a perversity on \( \Delta \). Then (1) \( \operatorname{ic}_p(\Delta)(\sigma)^j_i = \{0\} \) unless \( (i,j) \in [1,r_{\sigma}] \times [-r_{\sigma},0] \), (2) \( i^*_\sigma(\operatorname{ic}_p(\Delta))^j_i = \{0\} \) unless \( (i,j) \in [0,r_{\sigma}-1] \times [-r_{\sigma},0] \) and (3) \( i^*_\sigma(\operatorname{ic}_p(\Delta))^j_i = \{0\} \) unless \( (i,j) \in [0,r_{\sigma}] \times [-r_{\sigma},0] \). Furthermore, \( \Gamma(\operatorname{ic}_p(\Delta))^j_i = \{0\} \) unless \( (i,j) \in [0,r] \times [-r,0] \).

**Proof.** We prove the proposition by induction on \( r_{\sigma} \). Note that \( \operatorname{ic}_p(\Delta)(0)^j_i = \{0\} \) unless \( (i,j) = (0,0) \) by the construction of \( \operatorname{ic}_p(\Delta)^\bullet \).
Let \( \eta, \sigma \) be cones in \( \Delta \) with \( \eta \prec \sigma \). Recall that, if we take an \((r_\sigma - r_\eta)\)-dimensional linear subspace \( H \) of \( N(\sigma)q \) such that \( N(\sigma)q = N(\eta)q \oplus H \), then \( V_{A(\sigma)} = V \otimes_q A(H) \) for \( V \in GM(A(\eta)) \). Hence, if \( V_j = \{0\} \) unless \( j \in [a, b] \) for integers \( a, b \) with \( a \leq b \), then \( (V_{A(\sigma)})_j = \{0\} \) unless \( j \in [a - (r_\sigma - r_\eta), b] \).

Since

\[
i^\sigma_\sigma(i_{Cp}(\Delta))^i = \bigoplus_{\eta \in F(\sigma) \setminus \{\sigma\}} i_{Cp}(\Delta)(\eta)^i_{A(\sigma)}, \tag{2.63}\]

(2) is a consequence of (1) for \( \eta \in F(\sigma) \setminus \{\sigma\} \) which are true by the assumption of the induction. Since

\[
i_{Cp}(\Delta)(\sigma)^i = g_{t \geq p(\sigma) + 1}(i^\sigma_{Cp}(\Delta)[-1])^i, \tag{2.64}\]

(1) follows from (2). Since

\[
i^\sigma_{Cp}(\Delta))_i = i^\sigma_{Cp}(\Delta))^i \oplus i_{Cp}(\Delta)(\sigma)^i \tag{2.65}\]

for every \( i \in \mathbb{Z} \), (3) follows from (1) and (2).

Since

\[
\Gamma(i_{Cp}(\Delta)^i = \bigoplus_{\sigma \in \Delta} i_{Cp}(\Delta)(\sigma)^i \tag{2.66}\]

for every \( i \), the last assertion is a consequence of (1) for all \( \sigma \in \Delta \setminus \{0\} \). q.e.d.

**Corollary 2.12** For any perversity \( p \) on \( \Delta \), \( D(i_{Cp}(\Delta))^* \) is quasi-isomorphic to \( ic_{-p}(\Delta)^* \).

**Proof.** It is sufficient to show that \( D(i_{Cp}(\Delta))^* \) satisfies the conditions of the theorem for the perversity \(-p\). (1) is satisfied since \( D(i_{Cp}(\Delta))(0)^i = Q \) and \( D(i_{Cp}(\Delta))(0)^i = \{0\} \) for \( i \neq 0 \) by the definition of \( D \).

We check the conditions (2) and (3) for each \( \sigma \) in \( \Delta \setminus \{0\} \).

Since \( D(i_{Cp}(\Delta))(\sigma)^* = \text{det}(\sigma) \otimes D_\sigma(i^\sigma_{Cp}(\Delta))[-r_\sigma]^* \), the condition (3) of the theorem and Lemma [1,3] imply that \( g_{t \leq -p(\sigma)}(D(i_{Cp}(\Delta))(\sigma))^* \) is acyclic, i.e., (2) for \(-p\).
By Lemma 2.2, ic$^p(\Delta)(\sigma)^\bullet$ is quasi-isomorphic to $D(D(ic^p(\Delta))(\sigma))^\bullet$. Hence $gt_{\leq p(\sigma)}(D(D(ic^p(\Delta))(\sigma))^\bullet)$ is acyclic by (2) for $ic^p(\Delta)(\sigma)^\bullet$. Since
\[
det(\sigma) \otimes D_\sigma(i_\sigma^* D(ic^p(\Delta)))^\bullet = D(D(ic^p(\Delta))(\sigma)[r_\sigma]^\bullet, \tag{2.67}
\]
Lemma 1.6 implies that $gt_{\geq -p(\sigma)}(i_\sigma^* D(ic^p(\Delta)))^\bullet$ is acyclic, i.e., (3) for $-p^\bullet$. q.e.d.

Two homomorphisms $f, g : L^\bullet \to K^\bullet$ in CGEM(\Delta) are said to be homotopic if there exists a collection of homomorphisms \{u_i : L_i \to K_{i-1}^\bullet ; i \in \mathbb{Z}\} in GEM(\Delta) such that
\[
f^i - g^i = d_{K}^{i-1} \cdot u^i + u^{i+1} \cdot d_L^i \tag{2.68}
\]
for every $i \in \mathbb{Z}$. If $f$ and $g$ are homotopic, then $f(\sigma), g(\sigma) : L(\sigma)^\bullet \to K(\sigma)^\bullet$ and $i_\sigma^* (f), i_\sigma^* (g) : i_\sigma^* L^\bullet \to i_\sigma^* K^\bullet$ for $\sigma \in \Delta$ as well as $\Gamma(f), \Gamma(g) : \Gamma(L)^\bullet \to \Gamma(K)^\bullet$ are homotopic as complexes in abelian categories. Actually, it is sufficient to take \{u^i(\sigma/\sigma)\}, \{i_\sigma^* (u^i)\} and \{\Gamma(u^i)\}, respectively. In particular, the maps of the cohomologies induced by the two homomorphisms of the complexes are respectively equal.

We give here some elementary lemmas on the quasi-isomorphism property in CGEM(\Delta).

**Lemma 2.13** Let $f_1 : L_1^\bullet \to L_2^\bullet$ be a quasi-isomorphism and $f_2 : L_1^\bullet \to L_3^\bullet$ a homomorphism in CGEM(\Delta). Then there exist $L_4^\bullet$ in CGEM(\Delta), a quasi-isomorphism $g_1 : L_3^\bullet \to L_4^\bullet$ and a homomorphism $g_2 : L_2^\bullet \to L_4^\bullet$ such that the homomorphisms $g_2 \cdot f_1$ and $g_1 \cdot f_2$ are homotopic. If $f_2$ is a quasi-isomorphism, then so is $g_2$.

**Lemma 2.14** Let $g_1 : L_3^\bullet \to L_4^\bullet$ be a quasi-isomorphism and $g_2 : L_2^\bullet \to L_4^\bullet$ a homomorphism in CGEM(\Delta). Then there exist $L_1^\bullet$ in CGEM(\Delta), a quasi-isomorphism $f_1 : L_1^\bullet \to L_2^\bullet$ and a homomorphism $f_2 : L_1^\bullet \to L_3^\bullet$ such that the homomorphisms $g_2 \cdot f_1$ and $g_1 \cdot f_2$ are homotopic. If $g_2$ is a quasi-isomorphism, then so is $f_2$. 30
We get these lemmas by setting $L^4_1$ and $L^4_2$ the mapping cones of the homomorphisms $L^1_1 \rightarrow L^2_1 \oplus L_3$ and $L^2_1 \oplus L_3 \rightarrow L^4_1$, respectively.

By applying these lemmas, we get the following equivalent conditions.

**Lemma 2.15** For $L^\bullet, K^\bullet$ in CGEM$(\Delta)$, the following conditions are equivalent.

1. $L^\bullet$ is quasi-isomorphic to $K^\bullet$.
2. There exists $J^\bullet$ in CGEM$(\Delta)$ and quasi-isomorphisms $L^\bullet \rightarrow J^\bullet$ and $K^\bullet \rightarrow J^\bullet$.
3. There exists $I^\bullet$ in CGEM$(\Delta)$ and quasi-isomorphisms $I^\bullet \rightarrow L^\bullet$ and $I^\bullet \rightarrow K^\bullet$.

**Lemma 2.16** Let $f_1 : L^1_1 \rightarrow K^1_1$ be a homomorphism in CGEM$(\Delta)$. Assume that $L^2_2, K^2_2$ in CGEM$(\Delta)$ are quasi-isomorphic to $L^1_1$ and $K^1_1$, respectively. Then there exist $K^3_3$ in CGEM$(\Delta)$, a quasi-isomorphism $K^2_2 \rightarrow K^3_3$ and a homomorphism $f_2 : L^2_2 \rightarrow K^3_3$ such that the diagrams

$$
\begin{align*}
\text{H}^i(\Gamma(L^1_1^\bullet)) & \rightarrow \text{H}^i(\Gamma(K^1_1^\bullet)) \\
\text{H}^i(\Gamma(L^2_1^\bullet)) & \rightarrow \text{H}^i(\Gamma(K^3_3^\bullet))
\end{align*}
$$

(2.69)

of the cohomologies are commutative for all $i \in \mathbb{Z}$.

There exist also $L^3_3$ in CGEM$(\Delta)$, a quasi-isomorphism $L^3_3 \rightarrow L^2_2$ and a homomorphism $f_3 : L^3_3 \rightarrow K^2_2$ with the similar compatibility with $f_1$.

**Proof.** By Lemma 2.15, there exists $L^0_0$ and quasi-isomorphisms $g_1 : L^0_0 \rightarrow L^1_1$ and $g_2 : L^0_0 \rightarrow L_2$. By applying Lemma 2.14 for $f_1 \cdot g_1$ and $g_2$, we get $K^0_0$ with a quasi-isomorphism $K^1_1 \rightarrow K^0_0$ and a homomorphism $h_0 : L^2_2 \rightarrow K^0_0$ with the compatibility condition. Since $K^0_0$ is quasi-isomorphic to $K_2$, there exists $K^3_3$ and quasi-isomorphisms $h_1 : K^0_0 \rightarrow K^3_3$ and $K^2_2 \rightarrow K^3_3$ by Lemma 2.15. It is sufficient to set $f_3 := h_1 \cdot h_0$.

The second assertion is proved similarly. q.e.d.
3 The algebraic theory on toric varieties

In this section, we construct a functor from the category of graded exterior modules to that of complexes on the toric variety associated to the fan. For a finite fan $\Delta$, we denote by $Z(\Delta)$ the associated toric variety defined over $\mathbb{Q}$ (cf. [O1]).

We start with the case of an affine toric variety. Assume that $\Delta$ is $F(\pi)$, i.e. the set of all faces of a cone $\pi$ in $\mathbb{N}_R$.

We denote by $\mathbb{Q}[M]$ the group ring $\bigoplus_{m \in M} \mathbb{Q}e(m)$ defined by $e(m)e(m') = e(m + m')$ for $m, m' \in M$ and $e(0) = 1$. This $\mathbb{Q}$-algebra has a grading in the free $\mathbb{Z}$-module $M$. For a subset $U$ of $M$, we denote $\mathbb{Q}[U] := \bigoplus_{m \in U} \mathbb{Q}e(m)$. Note that $1 \not\in \mathbb{Q}[U]$ if $0 \not\in U$.

For the subsemigroup $M \cap \pi^\vee \subset M$, we denote by $S(\pi)$ the $M$-graded $\mathbb{Q}$-subalgebra $\mathbb{Q}[M \cap \pi^\vee]$ of $\mathbb{Q}[M]$. Then the affine toric variety $Z(F(\pi))$ is equal to $\text{Spec } S(\pi)$. The algebraic torus $T_N$ is equal to $\text{Spec } \mathbb{Q}[M]$ and the reduced complement $Z(F(\pi)) \setminus T_N$ is defined by the ideal $J(\pi) := \mathbb{Q}[M \cap \text{int } \pi^\vee]$.

The logarithmic de Rham complex $\Omega_{S(\pi)}(\log J(\pi))^\bullet$ is defined as follows.

We set
\[
\Omega_{S(\pi)}(\log J(\pi))^1 := S(\pi) \otimes M
\]
and
\[
\Omega_{S(\pi)}(\log J(\pi))^i := \bigwedge_i \Omega_{S(\pi)}(\log J(\pi))^1 = S(\pi) \otimes \bigwedge_i M
\]
for $0 \leq i \leq r$, where the exterior powers are taken as an $S(\pi)$-module and as a $\mathbb{Z}$-module, respectively. These are clearly free $S(\pi)$-modules. By the notation $A^* = A(M_\mathbb{Q}) = \wedge^* M_\mathbb{Q}$, the direct sum $\bigoplus_{i=0}^r \Omega_{S(\pi)}(\log J(\pi))^i$ is equal to the $M$-graded free $S(\pi)$-module
\[
S(\pi) \otimes_\mathbb{Q} A^* = \bigoplus_{m \in M \cap \pi^\vee} \mathbb{Q}e(m) \otimes_\mathbb{Q} A^*.
\]

The $\mathbb{Q}$-endomorphism $\partial$ of this $S(\pi)$-module is defined to be the $M$-homogeneous morphism such that the the restriction to the component $\mathbb{Q}e(m) \otimes_\mathbb{Q} A^*$ is $1 \otimes d_m$ for
each $m \in M \cap \pi^\vee$, where $d_m$ is the left multiplication of $m$. Since $\partial(\Omega_S(\pi)(\log J(\pi))^i) \subset \Omega_S(\pi)(\log J(\pi))^{i+1}$ for each $i$, $\Omega_S(\pi)(\log J(\pi))^\bullet$ is a $\partial$-complex of $M$-graded $Q$-vector spaces.

For each face $\sigma$ of $\pi$, $\pi^\vee \cap \sigma^\perp$ is a face of the dual cone $\pi^\vee \subset M_R$. Furthermore, it is known that the correspondence $\sigma \mapsto \pi^\vee \cap \sigma^\perp$ defines a bijection from $F(\pi)$ to $F(\pi^\vee)$ [O1, Prop.A.6].

For each $\sigma \in F(\pi)$, we denote by $P(\pi; \sigma)$ the $M$-homogeneous prime ideal
\[
Q[M \cap (\pi^\vee \setminus \sigma^\perp)] = \bigoplus_{m \in M \cap (\pi^\vee \setminus \sigma^\perp)} Qe(m)
\] (3.4)
of $S(\pi)$. The $M$-homogeneous quotient ring $S(\pi)/P(\pi; \sigma)$ is denoted by $S(\pi; \sigma)$. We denote the image of $e(m)$ in $S(\pi; \sigma)$ for $m \in M \cap \pi^\vee \cap \sigma^\perp$ also by $e(m)$. Since $M[\sigma] = M \cap \sigma^\perp$, we have a description $S(\pi; \sigma) = \bigoplus_{m \in M[\sigma] \cap \pi^\vee} Qe(m).

Let $J(\pi; \sigma)$ be the ideal $Q[M[\sigma] \cap \text{rel.int}(\pi^\vee \cap \sigma^\perp)]$ of $S(\pi; \sigma)$. The $\partial$-complex $\Omega_{S(\pi; \sigma)}(\log J(\pi; \sigma))^\bullet$ is defined to be $S(\pi; \sigma) \otimes_Q A^*[\sigma]$ with the $M$-homogeneous $Q$-homomorphism $\partial$ defined similarly as above, where $A^*[\sigma] = \wedge^\bullet M[\sigma]_Q$. Note that $\Omega_{S(\pi; 0)}(\log J(\pi; 0))^\bullet$ is equal to $\Omega_{S(\pi)}(\log J(\pi))^\bullet$.

We denote by $\text{Coh}(S(\pi))$ the category:

**object:** A finitely generated $M$-graded $S(\pi)$-module.

**morphism:** An $M$-homogeneous $S(\pi)$-homomorphism of $M$-degree zero.

A $Q$-homomorphism $\delta : F \to G$ of $S(\pi)$-modules is said to be a **differential operator of order one** if the map $(\delta \cdot f - f \cdot \delta) : F \to G$ defined by $(\delta \cdot f - f \cdot \delta)(x) := \delta(f(x)) - f(\delta(x))$ is an $S(\pi)$-homomorphism for every $f \in S(\pi)$. We denote by $\text{CCohDiff}(S(\pi))$ the category:

**object:** A finite $\partial$-complex $F^\bullet$ such that $F^i$’s are in $\text{Coh}(S(\pi))$ and $\partial$ is $M$-homogeneous of $M$-degree zero and is a differential operator of order one.
morphism: An $M$-homogeneous $S(\pi)$-homomorphism of $M$-degree zero.

We construct a functor $\Lambda_{S(\pi)}$ from the category $\text{GEM}(F(\pi))$ of graded exterior modules on $F(\pi)$ to this category $\text{CCohDiff}(S(\pi))$.

Let $\rho$ be a cone in $F(\pi)$. Recall that an object $V$ of $\text{GM}(A(\rho))$ is a finitely generated graded $A(\rho)$-module and the $A$-module $V_A$ has a structure of a free $A^*[\rho]$-module (cf. Lemma 1.3). Each $m \in M[\rho] = M \cap \rho^\perp$ is a homogeneous element of $A^*[\rho]$ of degree one. We denote by $d_m$ the left operation of $m$ on $V_A$. Then $d_m^2 = 0$ since $m \wedge m = 0$. For each $m \in M[\rho]$, we denote by $V_A(m)^\bullet$ the $\partial$-complex defined by $V_A(m)^i := (V_A)_i$ for each $i \in \mathbb{Z}$ and $\partial := d_m$.

We set $\Lambda^\rho_{S(\pi)}(V)^i := S(\pi; \rho) \otimes \mathbb{Q}(V_A)_i$ for each integer $i$. We define the $\partial$-complex $\Lambda^\rho_{S(\pi)}(V)^\bullet$ by

$$
\Lambda^\rho_{S(\pi)}(V)^\bullet := S(\pi; \rho) \otimes \mathbb{Q} V_A = \bigoplus_{m \in M[\rho] \cap \pi^\vee} \mathbb{Q}e(m) \otimes \mathbb{Q} V_A(m)^\bullet, 
$$

i.e., the its $m$-component of $\partial$ is $1_{\mathbb{Q}e(m)} \otimes d_m$ for every $m \in M[\rho] \cap \pi^\vee$.

In order to check that $\partial$ is a differential operator of order one, it is sufficient to show the $S(\pi; \rho)$-linearly $\partial \cdot e(m_0) - e(m_0) \cdot \partial$ for $m_0 \in M \cap \pi^\vee$. Since $P(\pi; \rho)$ is the annihilator of $S(\pi; \rho)$, $\partial \cdot e(m_0) - e(m_0) \cdot \partial = 0$ if $m_0 \notin M[\rho]$. Assume $m_0, m_1 \in M[\rho] \cap \pi^\vee$ and $x \in V_A$. For any $m \in M[\rho] \cap \pi^\vee$, we have

$$(\partial \cdot e(m_0) - e(m_0) \cdot \partial)(e(m)e(m_1) \otimes x)$$

$$= \partial(e(m + m_0 + m_1) \otimes x) - e(m_0)\partial(e(m + m_1) \otimes x)$$

$$= e(m + m_0 + m_1) \otimes (m + m_0 + m_1)x - e(m + m_0 + m_1) \otimes (m + m_1)x$$

$$= e(m + m_0 + m_1) \otimes m_0x$$

$$= e(m + m_0 + m_1) \otimes (m_0 + m_1)x - e(m + m_0 + m_1) \otimes m_1x$$

$$= e(m)(\partial \cdot e(m_0) - e(m_0) \cdot \partial)(e(m_1) \otimes x).$$

Hence $\partial \cdot e(m_0) - e(m_0) \cdot \partial$ is an $S(\pi; \rho)$-homomorphism.
**Proposition 3.1** Let $\rho$ be in $F(\pi)$ and $V$ in $GM(A(\rho))$. Then $\Lambda^\rho_{\pi}(V)^\bullet$ is isomorphic to a finite direct sum of subcomplexes which are isomorphic to dimension shifts of $\Omega_{\pi;\rho}(\log J(\rho))^\bullet$.

**Proof.** By Lemma 1.3, $V_A$ is a free $A^*\rho$-module. Let $\{x_1, \cdots, x_s\}$ be a homogeneous basis. By the definition of $\partial$, the decomposition

$$\Lambda^\rho_{\pi}(V)^\bullet = \bigoplus_{i=1}^s S(\pi; \rho) \otimes Q A^*\rho x_i$$

(3.6)

is a direct sum of subcomplexes. Furthermore, there exists an isomorphism

$$\Omega_{\pi;\rho}(\log J(\rho))[-\deg x_i]^{\bullet} \rightarrow (S(\pi; \rho) \otimes Q A^*\rho x_i)^{\bullet}$$

(3.7)

for each $i$. $\quad q.e.d.$

For a homomorphism $f : V \rightarrow W$ in $GM(A(\rho))$, the $S(\pi)$-homomorphism $\Lambda^\rho_{\pi}(f) : \Lambda^\rho_{\pi}(V)^\bullet \rightarrow \Lambda^\rho_{\pi}(W)^\bullet$ of $\partial$-complexes is defined by $\Lambda^\rho_{\pi}(f) = 1_{S(\pi)} \otimes f_A$. Since $f_A$ is an $A^*\rho$-homomorphism, $\Lambda^\rho_{\pi}(f)$ commutes with the coboundry maps of $\Lambda^\rho_{\pi}(V)^\bullet$ and $\Lambda^\rho_{\pi}(W)^\bullet$.

**Proposition 3.2** Let $\sigma, \rho$ be cones in $F(\pi)$ with $\sigma \prec \rho$. For $V$ in $GM(A(\sigma))$, there exists a natural isomorphism

$$\Lambda^\sigma_{\pi}(V)^\bullet \otimes_{S(\pi;\sigma)} S(\pi; \rho) \simeq \Lambda^\rho_{\pi}(V_A(\rho))^\bullet$$

(3.8)

of $M$-graded $\partial$-complexes.

**Proof.** $\Lambda^\sigma_{\pi}(V)^\bullet \otimes_{S(\pi;\sigma)} S(\pi; \rho)$ is actually an $M$-graded $\partial$-complex since it is the quotient of $\Lambda^\sigma_{\pi}(V)^\bullet$ by the $M$-homogeneous subcomplex $P(\pi; \rho)\Lambda^\sigma_{\pi}(V)^\bullet$. Both sides are equal to $S(\pi; \rho) \otimes Q V_A$ by the identification $(V_A(\rho))_A = V_A$. The differential operators commute with the identification since the $m$-components of them for $m \in M(\rho) \cap \pi^\vee$ are both the operation $d_m$. $\quad q.e.d.$

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For $L$ in $\text{GEM}(F(\pi))$, we define the $\partial$-complex $\Lambda_{S(\pi)}(L)^\bullet$ in $\text{CCohDiff}(S(\pi))$ by

$$
\Lambda_{S(\pi)}(L)^\bullet := \bigoplus_{\sigma \in F(\pi)} \Lambda^\rho_{S(\pi)}(L(\rho))^\bullet, \quad (3.9)
$$

where the coboundary map $\partial$ is also defined as the direct sum.

Let $f : L \to K$ be a morphism in $\text{GEM}(F(\pi))$. We define an $M$-homogeneous $S(\pi)$-homomorphism $\Lambda_{S(\pi)}(f) : \Lambda_{S(\pi)}(L)^\bullet \to \Lambda_{S(\pi)}(K)^\bullet$ of $\partial$-complexes as follows.

For $\sigma, \rho \in F(\pi)$, the $(\sigma, \rho)$-component of the morphism $\Lambda_{S(\pi)}(f) : \Lambda_{S(\pi)}(L)^\bullet \to \Lambda_{S(\pi)}(K)^\bullet$ is defined to be the composite of the natural surjection

$$
\Lambda^\sigma_{S(\pi)}(L(\sigma))^\bullet \to \bigoplus_{\rho \in F(\pi)} \Lambda^\rho_{S(\pi)}(K(\rho))^\bullet
$$

and

$$
1_{S(\pi;\rho)} \otimes f(\sigma/\rho)_A : \Lambda^\rho_{S(\pi)}(L(\sigma)_{A(\rho)})^\bullet \to \Lambda^\rho_{S(\pi)}(K(\rho))^\bullet
$$

if $\sigma \prec \rho$ and is defined to be the zero map otherwise. Then $\Lambda_{S(\pi)}$ is a covariant functor from $\text{GEM}(\Delta)$ to $\text{CCohDiff}(S(\pi))$.

Let $k$ be a field. For a topological space $X$, we denote by $k_X$ the sheaf of rings with the constant stalk $k$. A $k_X$-module on $X$ is called a $k$-sheaf and a $k_X$-homomorphism of $k_X$-modules is called simply a $k$-homomorphism. In this paper, we treat only the cases $k = \mathbb{Q}$ and $k = \mathbb{C}$.

Let $\mathcal{O}_X$ be a sheaf of commutative $k$-algebras. A $k$-homomorphism $f : F \to G$ of $\mathcal{O}_X$-modules $F, G$ is said to be a differential operator of order one if the $k$-homomorphism $f(U) : F(U) \to G(U)$ is a differential operator of order one of $\mathcal{O}_X(U)$-modules for every open subset $U \subset X$.

For a finitely generated $S(\pi)$-module $E$, we denote by $\mathcal{F}(E)$ the associated coherent sheaf on the affine scheme $\text{Spec } S(\pi)$. If $d : E \to G$ is a differential operator of
order one of finitely generated $S(\pi)$-modules, then it defines a differential operator $F(d) : F(E) \to F(G)$ of order one of $\mathcal{O}_{Z(F(\pi))}$-modules on the affine toric variety $Z(F(\pi))$. If $(E^\bullet, \partial = (\partial_E))$ is a $\partial$-complex such that each $E^i$ is a finitely generated $S(\pi)$-module and $\partial$ is a differential operator of order one, then we denote by $F(E)^\bullet$ the $\partial$-complex on $Z(F(\pi))$ with the coboundary map $\partial = (F(\partial_E))$.

Let $\mu$ be an element of $F(\pi)$. Take an element $m \in M \cap \text{rel. int}(\pi^\vee \cap \mu^\perp)$. Since $\mu^\vee = \pi^\vee + \mathbf{R}(-m)$ [\text{Cor.A.7}], $S(\mu) := \mathbf{Q}[M \cap \mu^\vee]$ is equal to the localization $S(\pi)[\mathbf{e}(m)^{-1}]$. For any $\rho \in F(\pi)$, we see easily that $S(\pi; \rho) \otimes_{S(\pi)} S(\mu)$ is equal to $S(\mu; \rho)$ if $\rho \prec \mu$ and is $\{0\}$ otherwise.

**Lemma 3.3** Let $\mu$ be an element of $F(\pi)$. For $\rho \in F(\mu)$ and $V$ in $\text{GM}(A(\rho))$, the localization $\Lambda^\rho_{S(\pi)}(V)^\bullet \otimes_{S(\pi)} S(\mu)$ is equal to $\Lambda^\rho_{S(\mu)}(V)^\bullet$.

For $L$ in $\text{GEM}(F(\pi))$, the localization $\Lambda_{S(\pi)}(L)^\bullet \otimes_{S(\pi)} S(\mu)$ of $\partial$-complex is equal to $\Lambda_{S(\mu)}(L|F(\mu))^\bullet$, where $L|F(\mu)$ is the restriction of $L$ to $F(\mu)$.

**Proof.** The first equality is clear as $S(\mu)$-modules. The coboundary maps $\partial$ are compatible with the inclusion $\Lambda^\rho_{S(\pi)}(V)^\bullet \subset \Lambda^\rho_{S(\mu)}(V)^\bullet$ since they are defined as the direct sums of $\mathbf{Q}\mathbf{e}(m) \otimes_{\mathbf{Q}} V_A(m)^\bullet$’s. Since differential operators are extended to localizations uniquely, they are equal as $\partial$-complexes.

Since $\Lambda_{S(\pi)}(L)^\bullet$ and $\Lambda_{S(\mu)}(L)^\bullet$ are defined as the direct sums of $\Lambda^\rho_{S(\pi)}(L(\rho))^\bullet$ for $\rho \in F(\pi)$ and $\Lambda^\rho_{S(\mu)}(L(\rho))^\bullet$ for $\rho \in F(\mu)$, respectively, the second assertion follows from the first. \hspace{1cm} q.e.d.

Let $\Delta$ be a finite fan of $N_{\mathbf{R}}$. The toric variety $Z(\Delta)$ has the affine open covering $\bigcup_{\pi \in \Delta} Z(F(\pi))$. The reduced divisor $D(\Delta) := Z(\Delta) \setminus T_N$ is defined by the ideal $J(\pi) \subset S(\pi)$ on each affine open subscheme $Z(F(\pi))$.

The logarithmic de Rham complex $\Omega_{Z(\Delta)}(\log D(\Delta))^\bullet$ is defined by

$$\Omega_{Z(\Delta)}(\log D(\Delta))^\bullet := \mathcal{O}_{Z(\Delta)} \otimes_{\mathbf{Q}} A^*.$$ (3.13)
We define the coboundary map \( \partial = (\partial^i : i \in \mathbb{Z}) \)

\[
\partial^i : \Omega_{Z(D)}(\log D)^i \to \Omega_{Z(D)}(\log D)^{i+1}
\]

so that the restriction to \( Z(F(\pi)) \) is equal to \( \partial \) of \( \mathcal{F}(\Omega_{S(\pi)}(\log J(\pi)))^\bullet \) for each \( \pi \in \Delta \).

Although it is common to write a logarithmic de Rham complex as \( \Omega_X(\log D) \), we put the dot at the right end as \( \Omega_X(\log D)^\bullet \) for the compatibility with the other notation in this paper.

For each \( \sigma \in F(\rho) \), the subscheme \( \text{Spec } S(\rho; \sigma) \) of \( Z(F(\rho)) \) is denoted by \( X(\rho; \sigma) \).

In particular, \( X(\rho; \rho) \) is the algebraic torus \( T_N[\rho] := \text{Spec } \mathbb{Q}[M[\rho]] \) of dimension \( r - r_\rho \).

In order to simplify the notation, we set \( T := T_N \) and \( T[\rho] := T_N[\rho] \) for each \( \rho \in \Delta \).

Then the toric variety \( Z(\Delta) \) is decomposed as the disjoint union

\[
\bigcup_{\rho \in \Delta} T[\rho]
\]

of \( T \)-orbits \([11], \text{Prop.1.6}\).

For each \( \sigma \in \Delta \), we denote by \( X(\Delta; \sigma) \) or simply \( X(\sigma) \) the union of \( X(\rho; \sigma) \) for \( \rho \in \Delta \) with \( \sigma \prec \rho \). \( X(\sigma) \) is a \( T \)-invariant irreducible closed subvariety of \( Z(\Delta) \).

For each \( \sigma \in \Delta \), let \( N[\sigma] := N/N(\sigma) \). For each \( \rho \in \Delta \) with \( \sigma \prec \rho \), we denote by \( \rho[\sigma] \) the image of \( \rho \) in \( N[\sigma]_R := N_R/N(\sigma)_R \). Then \( \Delta[\sigma] := \{ \rho[\sigma] : \rho \in \Delta, \sigma \prec \rho \} \) is a fan of \( N[\sigma]_R \). It is known that \( X(\sigma) \subset Z(\Delta) \) is equal to the toric variety \( Z(\Delta[\sigma]) \) with the torus \( T[\sigma] \) \([11], \text{Cor.1.7}\). The reduced complement \( X(\sigma) \setminus T[\sigma] \) is denoted by \( D(\Delta; \sigma) \) or \( D(\sigma) \).

We define

\[
\Omega_{X(\sigma)}(\log D(\sigma))^\bullet := \mathcal{O}_{X(\sigma)} \otimes_{\mathbb{Q}} A^*[\sigma]
\]

and \( \partial \) of it is defined so that the restriction to the affine open subscheme \( Z(F(\rho)) \) is equal to that of \( \mathcal{F}(\Omega_{S(\rho; \sigma)}(\log J(\rho; \sigma)))^\bullet \) for every \( \rho \in \Delta \) with \( \sigma \prec \rho \).
For each $V \in \text{GM}(A(\sigma))$, we define the $\partial$-complex $\Lambda_{Z(\Delta)}^\sigma(V)^\bullet$ on $Z(\Delta)$ by

$$
\Lambda_{Z(\Delta)}^\sigma(V)^\bullet := \mathcal{O}_{X(\sigma)} \otimes_\mathbb{Q} V_A.
$$

The coboundary map $\partial$ is defined so that the restriction to each open set $Z(F(\rho))$ is equal to that of $\mathcal{F}(\Lambda_{S(\rho)}^\sigma(V))^\bullet$ for every $\rho \in \Delta$ with $\sigma \prec \rho$.

Note that $M$-gradings of $\Lambda_{S(\rho)}^\sigma(V)^\bullet$ for $\rho \in \Delta$ induce a natural $T$-action on the $\partial$-complex $\Lambda_{Z(\Delta)}^\sigma(V)^\bullet$.

The following propositions follow from Propositions 3.1 and 3.2, respectively.

**Proposition 3.4** Let $\rho$ be in $\Delta$ and $V$ in $\text{GM}(A(\rho))$. Then $\Lambda_{Z(\Delta)}^\rho(V)^\bullet$ is isomorphic to a finite direct sum of subcomplexes which are isomorphic to dimension shifts of $\Omega_{X(\rho)}(\log D(\rho))^\bullet$.

**Proposition 3.5** Let $\sigma, \rho$ be cones in $\Delta$ with $\sigma \prec \rho$. For $V$ in $\text{GM}(A(\sigma))$, there exists a natural $T$-equivariant isomorphism

$$
\Lambda_{Z(\Delta)}^\sigma(V)^\bullet \otimes_{\mathcal{O}_{X(\sigma)}} \mathcal{O}_{X(\rho)} \simeq \Lambda_{Z(\Delta)}^\rho(V_{A(\rho)})^\bullet
$$

of $\partial$-complexes.

Let $V^\bullet$ be in $\text{CGM}(A(\sigma))$. Then the bicomplex $\Lambda_{Z(\Delta)}^\sigma(V)^{\bullet,\bullet}$ is defined by

$$
\Lambda_{Z(\Delta)}^\sigma(V)^{i,j} := \Lambda_{Z(\Delta)}^\sigma(V^i)^j
$$

for $i, j \in \mathbb{Z}$ and $d_1 := d$ and $d_2 := \partial$. We denote by $\Lambda_{Z(\Delta)}^\sigma(V)^\bullet$ the associated single complex and by $\delta$ the coboundary map.

For each object $L$ of $\text{GEM}(\Delta)$, we set

$$
\Lambda_{Z(\Delta)}^\sigma(L)^\bullet := \bigoplus_{\sigma \in \Delta} \Lambda_{Z(\Delta)}^\sigma(L(\sigma))^\bullet.
$$
For a morphism $f : L \to K$ in $\text{GEM}(\Delta)$, the $T$-equivariant homomorphism

$$\Lambda_{Z(\Delta)}(f) : \Lambda_{Z(\Delta)}(L)\bullet \longrightarrow \Lambda_{Z(\Delta)}(K)\bullet$$

(3.21)

is defined naturally. Then $\Lambda_{Z(\Delta)}$ is a covariant functor from $\text{GEM}(\Delta)$ to the category $\text{CCohDiff}(Z(\Delta))$ which is defined naturally as the globalization of $\text{CCohDiff}(S(\pi))$.

Let $L\bullet$ be a $d$-complex in $\text{CGEM}(\Delta)$. Then the bicomplex $\Lambda_{Z(\Delta)}(L)^{\bullet,\bullet}$ is defined by

$$\Lambda_{Z(\Delta)}(L)^{i,j} := \Lambda_{Z(\Delta)}(L_i)^j$$

(3.22)

for $i, j \in \mathbb{Z}$. Note that $d_1 := d$ of this bicomplex is a $\mathcal{O}_{Z(\Delta)}$-homomorphism and $d_2 := \partial$ is a differential operator of order one. If there is no danger of confusion, we denote by $\Lambda_{Z(\Delta)}(L)^{\bullet}$ the associated single complex. The coboundary map, which we denote by $\delta$, is a differential operator of order one. For each integer $j$, we denote by $\Lambda_{Z(\Delta)}(L)^{j,\bullet}$ the $d$-complex $\Lambda_{Z(\Delta)}(L)^{\bullet,j}$. Then $\Lambda_{Z(\Delta)}(L)^{j,\bullet}$ is a finite $d$-complex in the category of coherent $\mathcal{O}_{Z}$-modules.

If the fan $\Delta$ is complete, then the toric variety $Z(\Delta)$ is complete. For the functor $\Gamma : \text{GEM}(\Delta) \to \text{GM}(A)$ defined in Section 1, we get the following lemma.

**Lemma 3.6** Assume that $\Delta$ is a complete fan. Let $L\bullet$ be an object of $\text{CGEM}(\Delta)$.

For any integers $p, q$, we have an isomorphism

$$H^p(Z(\Delta), \Lambda_{Z(\Delta)}(L)^{\bullet}) \simeq H^p(\Gamma(L)^{\bullet})_q$$

(3.23)

of finite dimensional $\mathbb{Q}$-vector spaces, where the lefthand side is the hypercohomology group of the complex of coherent sheaves on $Z(\Delta)$.

**Proof.** For each $\sigma \in \Delta$, we have $H^0(X(\sigma), \mathcal{O}_{X(\sigma)}) = \mathbb{Q}$ and $H^p(X(\sigma), \mathcal{O}_{X(\sigma)}) = \{0\}$ for $p > 0$, since $X(\sigma)$ is a complete toric variety [O1, Cor.2.8]. Since $\Lambda_{Z(L)}^p = \Lambda_{Z(L)}^{p,q}$ is a direct sum for $\sigma \in \Delta$ of free $\mathcal{O}_{X(\sigma)}$-modules for every $p \in \mathbb{Z}$, the
hypercohomology $H^p(Z(\Delta), \Lambda^\ast Z(L)_{q})$ is equal to the $p$-th cohomology of the complex $\Gamma(Z, \Lambda^\ast Z(L)_{q})$ of $\mathbb{Q}$-vector spaces. For any $\sigma \in \Delta$ and $p, q \in \mathbb{Z}$, we have

$$\Gamma(Z, \Lambda^\ast Z(L(\sigma))_q)^p = (L(\sigma)_\Delta^p)_q.$$  \hspace{1cm} (3.24)

Hence $\Gamma(Z, \Lambda^\ast Z(L)_{q})$ is isomorphic to $\Gamma(L)_{q}^\ast$ as a complex of $\mathbb{Q}$-vector spaces. \hspace{1cm} q.e.d.

4 The analytic theory on toric varieties

For any $\mathbb{Q}$-algebra $B$ and any $\mathbb{Q}$-scheme $X$, we denote by $B_C$ and $X_C$ the scalar extensions $B \otimes_\mathbb{Q} \mathbb{C}$ and $X \times_{\text{Spec} \mathbb{Q}} \text{Spec} \mathbb{C}$, respectively.

When $X$ is of finite type over $\mathbb{Q}$, we denote by $X_h$ the analytic space associated to the algebraic $\mathbb{C}$-scheme $X_C$ [H, Chap.1, §6]. For a coherent sheaf $F$ on $X$, the pulled-back coherent sheaf on $X_C$ and the associated analytic coherent sheaf on $X_h$ are denoted by $F_C$ and $F_h$, respectively. Let $f : F \to G$ be a differential operator of order one over $\mathbb{Q}$. Then it is easy to see that the $\mathbb{C}$-homomorphism $f_C : F_C \to G_C$ on $X_C$ obtained by scalar extension is a differential operator of order one over $\mathbb{C}$. By [EGA4, 16.8], the differential operator $f_C : F_C \to G_C$ is decomposed uniquely to $u \cdot d^1_{X_C}$ where $d^1_{X_C} : F \to \mathcal{P}^1_{X_C}(F)$ is a canonical $\mathbb{C}$-homomorphism [EGA4, 16.7.5] and $u : \mathcal{P}^1_{X_C}(F) \to G$ is a $\mathcal{O}_{X_C}$-homomorphism. For the definition of $\mathcal{P}^1_{X_C}(F)$, see [EGA4]. Since these homomorphisms are canonically pulled-back to $X_h$, we get a differential operator $f_h : F_h \to G_h$ of order one of the analytic coherent sheaves on $X_h$.

Let $\rho$ be a cone in $N_\mathbb{R}$. By the notation of the scalar extensions,

$$S(\rho)_{\mathbb{C}} = S(\rho) \otimes_\mathbb{Q} \mathbb{C} = \mathbb{C}[M \cap \rho^\vee] \hspace{1cm} (4.1)$$

and

$$S(\rho; \sigma)_{\mathbb{C}} = S(\rho; \sigma) \otimes_\mathbb{Q} \mathbb{C} = \mathbb{C}[M[\sigma] \cap \rho^\vee] \hspace{1cm} (4.2)$$

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for each \( \sigma \in F(\rho) \). Since \( S(\rho; \sigma) = S(\rho)/P(\rho; \sigma) \), \( S(\rho; \sigma)_C \) is the quotient of \( S(\rho)_C \) by the prime ideal \( P(\rho; \sigma)_C = C[M \cap (\rho^\vee \setminus \sigma^\perp)] \).

We fix a finite fan \( \Delta \) of \( N \mathbb{R} \) in this section.

We denote simply by \( Z \) the toric variety \( Z(\Delta) \). Since \( Z \) is normal, so are the toric variety \( Z_C = Z(\Delta)_C \) over \( C \) and the analytic space \( Z_h = Z(\Delta)_h \). For each \( \sigma \in \Delta \), a \( T \)-invariant irreducible closed subvariety \( X(\sigma) \) of \( Z \) was defined in Section 3. Hence \( X(\sigma)_C \) and \( X(\sigma)_h \) are irreducible closed subvarieties of \( Z_C \) and \( Z_h \), respectively.

Let \( n \) be an element of \( N \). The group homomorphism \( M \to \mathbb{Z} \) defined by \( m \mapsto \langle m, n \rangle \) induces a homomorphism of group rings \( C[M] \to C[t, t^{-1}] \) and the associated morphism \( \lambda_n : \text{Spec} C[t, t^{-1}] \to T_C = \text{Spec} C[M] \), where \( t \) is the monomial corresponding to \( 1 \in \mathbb{Z} \). We call \( \lambda_n \) the one-parameter subgroup associated to \( n \) [Ol1, 1.2]. We denote also by \( \lambda_n \) the associated map \( \mathbb{C}^* \to T_h \) of complex Lie groups. If \( n \in N \cap \rho \) for a cone \( \rho \in \Delta \), then \( C[M \cap \rho^\vee] \) is mapped to \( C[t] \). Hence the one-parameter subgroup is extended to a regular map \( \lambda_n : \mathbb{C} \to Z(F(\rho))_C \subset Z_C \), uniquely.

For each \( m \in M \), the monomial \( e(m) \in C[M] \) is regarded as a character \( T_h \to \mathbb{C}^* \). The composite \( e(m) \cdot \lambda_n : \mathbb{C}^* \to \mathbb{C}^* \) of \( \lambda_n \) and \( e(m) \) is equal to the map \( t \mapsto t^{(m, n)} \).

Since \( Z_C \) is a toric variety, the group \( T_h \) acts on \( Z_h \) analytically. For \( a \in T_h \) and \( x \in Z_h \), we denote by \( ax \) the corresponding point of \( Z_h \) by the action.

Let \( f \) be a complex analytic function on an open subset \( U \) of \( Z_h \). For each \( n \in N \), the derivation \( \partial_n f \) of \( f \) is defined by

\[
\partial_n f(x) := \left. \frac{d}{dt} \right|_{t=1} f(\lambda_n(t)x) = \lim_{t \to 1} \frac{f(x) - f(\lambda_n(t)x)}{1 - t} \quad (4.3)
\]

for \( x \in U \). Since \( f(\lambda_n(t)x) \) is analytic in the variables \( x, t \), the function \( \partial_n f \) is analytic on \( U \). Hence we get a \( \mathbb{C} \)-derivation \( \partial_h : \mathcal{O}_{Z_h} \to \mathcal{O}_{Z_h} \) of the structure sheaf.

For each \( \sigma \in \Delta \), we denote by \( \mathcal{P}(\sigma)_h \) the ideal sheaf of \( \mathcal{O}_{Z_h} \) defining \( X(\sigma)_h \subset Z_h \). If \( f \) is in \( \mathcal{P}(\sigma)_h(U) \), then so is \( \partial_n f \) since \( f \) is zero on \( X(\sigma) \cap U \) and \( X(\sigma)_h \) is closed.
Lemma 4.1 Let \( \sigma \) be an element of \( \Delta \) and let \( y \) be a point in \( X(\sigma)_h \). If \( n \) is in \( N \cap \text{rel. int } \sigma \), then the endomorphism \( (\partial_n)_y \) of the stalk \( (\mathcal{P}(\sigma)_h)_y \) is an automorphism as a \( \mathbb{C} \)-vector space.

Proof. Let \( U \subset Z_h \) be an open neighborhood of \( y \) such that \( \lambda_n(t)U \subset U \) for every \( t \) with \( |t| \leq 1 \). For \( g \in \mathcal{P}(\sigma)_h(U) \), we define
\[
f(x) := \int_0^1 \frac{g(\lambda_n(s)x)}{s} ds ,
\]
where the integration is taken on the real interval \([0, 1]\). Since \( \lambda_n(0)x \) is in \( X(\sigma)_h \), the analytic function \( g(\lambda_n(s)x) \) on \( U \times \{s ; |s| \leq 1\} \) has zero at the divisor \( (s = 0) \). Hence \( g(\lambda_n(s)x)/s \) is an analytic function. Hence the integral \( f \) is an analytic function on \( U \). By the definition, we have
\[
f(\lambda_n(t)x) = \int_0^1 \frac{g(\lambda_n(st)x)}{s} ds = \int_0^t \frac{g(\lambda_n(u)x)}{u} du .
\]
Hence \( \partial_n f = g \). This implies that \( \partial_n : \mathcal{P}(\sigma)_h(U) \to \mathcal{P}(\sigma)_h(U) \) is surjective.

For \( f \in \mathcal{P}(\sigma)_h(U) \), suppose that \( \partial_n f = 0 \). Then \( (d/dt)f(\lambda_n(t)x) = 0 \) for \( t \in [0, 1] \) and we have \( f(x) = f(\lambda_n(0)x) = 0 \). Since \( \partial_n \) is \( \mathbb{C} \)-linear, it is also injective.

Since \( U \)'s with this property form a fundamental system of neighborhood of \( y \), \( \partial_n : \mathcal{P}(\sigma)_h \to \mathcal{P}(\sigma)_h \) is isomorphic at the stalk of \( y \). q.e.d.

Let \( D = D(\Delta) \) be the complementary reduced divisor of \( T \) in \( Z \). We set
\[
\Omega_{Z_h}(\log D_h)^1 := \Omega_Z(\log D)^1_h = \mathcal{O}_{Z_h} \otimes_{\mathbb{Z}} M .
\]
We define the \( \mathbb{C} \)-derivation \( \partial : \mathcal{O}_{Z_h} \to \Omega_{Z_h}(\log D_h)^1 \) as follows.

Let \( n, n' \) be elements of \( N \). The equality \( \lambda_{n+n'}(t) = \lambda_n(t)\lambda_{n'}(t) \) holds for \( t \in \mathbb{C}^* \). For an analytic function \( f \) on an open subset \( U \) of \( Z_h \), we have
\[
\partial_{n+n'}f(x)
\]
\[ \lim_{t \to 1} \frac{f(x) - f(\lambda_n(t)x)}{1 - t} = \nabla_n f(x) + \nabla_n f(t) \]

for any \( x \in U \). Hence the map \( n \mapsto \nabla_n f \in \mathcal{O}_{\mathbb{Z}}(U) \) is a homomorphism, i.e., an element of \( \text{Hom}_{\mathbb{Z}}(N, \mathcal{O}_{\mathbb{Z}}(U)) \). We define \( \nabla f \) to be the corresponding element of \( \Omega_{\mathbb{Z}}(\log D_{\mathbb{Z}})(U)^1 = \mathcal{O}_{\mathbb{Z}}(U) \otimes_{\mathbb{Z}} M \).

Let \( m \) and \( n \) be elements of \( M \) and \( N \), respectively. Since the character \( e(m) : T_h \to C^* \) is a homomorphism,

\[ e(m)(\lambda_n(t)x) = e(m)(\lambda_n(t)) \cdot e(m)(x) = t^{\langle m, n \rangle} \cdot e(m)(x). \] (4.7)

Hence \( \nabla_n e(m) = \langle m, n \rangle e(m) \) for every \( n \in N \). This implies \( \nabla e(m) = e(m) \otimes m \).

Hence we denote the global section \( 1 \otimes m \) of \( \Omega_{\mathbb{Z}}(\log D_{\mathbb{Z}})^1 \) by \( de(m) / e(m) \) for every \( m \in M \), however it is common to write it by \( d e(m) / e(m) \).

Let \( \{m_1, \cdots, m_r\} \) be a \( \mathbb{Z} \)-basis of \( M \) and let \( \{n_1, \cdots, n_r\} \) be the dual basis of \( N \).

If we write \( \nabla f = \sum a_i \otimes m_i \), we have \( a_i = \langle \nabla f, n_i \rangle = \nabla_n f \) for each \( i \). Hence

\[ \nabla f = \sum_{i=1}^{r} \nabla_n f \frac{de(m_i)}{e(m_i)}. \] (4.8)

In particular \( \nabla \) is a \( C \)-derivation.

Since \( e(m) \)'s form a \( C \)-basis of the coordinate ring of each affine toric variety, we know that this \( C \)-derivation \( \nabla \) is compatible with the algebraic \( C \)-derivation \( \partial : \mathcal{O}_{\mathbb{C}} \to \Omega_{\mathbb{C}}(\log D_{\mathbb{C}})^1 \).

For each \( 0 \leq i \leq r \) we set

\[ \Omega_{\mathbb{Z}}(\log D_{\mathbb{Z}})^i := \bigwedge^i \Omega_{\mathbb{Z}}(\log D_{\mathbb{Z}})^1 = \mathcal{O}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \bigwedge^i M. \] (4.9)

Since \( \Omega_{\mathbb{Z}}(\log D_{\mathbb{Z}})^1 \) is a free \( \mathcal{O}_{\mathbb{Z}} \)-module of rank \( r \), \( \Omega_{\mathbb{Z}}(\log D_{\mathbb{Z}})^i \) is free of rank \( r, Ci \).

For \( 0 < i < r \), we define a pairing

\[ \mathcal{O}_{\mathbb{Z}}(U) \times \bigwedge^i M \longrightarrow \Omega_{\mathbb{Z}}(\log D_{\mathbb{Z}})^{i+1}(U) \] (4.10)
by \((f, w) \mapsto \partial f \wedge w\) which induces a \(\mathbf{C}\)-homomorphism
\[
\partial : \Omega_{Z_h}(\log D_h)^i \to \Omega_{Z_h}(\log D_h)^{i+1}
\] of sheaves which we denote also by \(\partial\). We check easily that \(\partial \cdot \partial = 0\), and we get a \(\partial\)-complex \(\Omega_{Z_h}(\log D_h)^\bullet\) which we call the logarithmic de Rham complex on \(Z_h\). For any \(\sigma \in \Delta\), we see easily by the description (4.8) of \(\partial f\) that \(P(\sigma)_h \Omega_{Z_h}(\log D_h)^\bullet\) is a subcomplex of \(\Omega_{Z_h}(\log D_h)^\bullet\).

For each \(\sigma \in \Delta\), we denote by \(\bar{I}_\sigma\) the closed immersion \(X(\sigma)_h \to Z_h\) and by \(I_\sigma\) the immersion \(T[\sigma]_h \to Z_h\). For a \(\mathbf{C}\)-sheaf \(F\) on \(Z_h\), we denote by \(\bar{I}_\sigma^* F\) and \(I_\sigma^* F\) the pull-back of \(F\) to \(X(\sigma)_h\) and \(T[\sigma]_h\), respectively. Note that, even if \(F\) is an \(\mathcal{O}_{Z_h}\)-module, the pull-back is taken as a \(\mathbf{C}\)-sheaf.

**Lemma 4.2** For every \(\sigma \in \Delta\), the \(\partial\)-complex \(\bar{I}_\sigma^*(\mathcal{P}(\sigma)_h \Omega_{Z_h}(\log D_h))^{\bullet}\) on \(X(\sigma)_h\) is homotopically equivalent to the zero complex.

**Proof.** We have to show that the identity map of this \(\partial\)-complex is homotopic to the zero map. Take an element \(n_0\) in \(N \cap \text{rel. int } \sigma\). Let \(h\) be the \(\mathcal{O}_{Z_h}\)-homomorphism of degree \(-1\) of the graded \(\mathcal{O}_{Z_h}\)-module \(\Omega_{Z_h}(\log D_h)^\bullet\) induced by the right interior product \(i(n_0) : \wedge^\bullet M \to \wedge^\bullet M\). Then the \(\mathbf{C}\)-homomorphism \(h \cdot \partial + \partial \cdot h\) on \(\Omega_{Z_h}(\log D_h)^\bullet = \mathcal{O}_{Z_h} \otimes \wedge^\bullet M\) is equal to \(\partial_{n_0} \otimes 1\) (cf. [D2, 13.4]). By Lemma 4.1, this induces an automorphism of \(\bar{I}_\sigma^*(\mathcal{P}(\sigma)_h \Omega_{Z_h}(\log D_h))^{\bullet}\) as a \(\mathbf{C}\)-sheaf. Let \(u\) be the inverse isomorphism of \(\bar{I}_\sigma^*(h \cdot \partial + \partial \cdot h)\). Since

\[
(h \cdot \partial + \partial \cdot h) \cdot \partial = \partial \cdot (h \cdot \partial + \partial \cdot h) = \partial \cdot h \cdot \partial ,
\]
we have \(\bar{I}_\sigma^* \partial \cdot u = u \cdot \bar{I}_\sigma^* \partial = u \cdot I_\sigma^*(\partial \cdot h \cdot \partial) \cdot u\). Set \(h' := u \cdot \bar{I}_\sigma^* h\). Then we have

\[
h' \cdot \bar{I}_\sigma^* \partial + \bar{I}_\sigma^* \partial \cdot h' = u \cdot I_\sigma^*(\partial \cdot h + \partial \cdot h) = 1.
\] 

q.e.d.
Let $\sigma$ be an element of $\Delta$. We set $N[\sigma] := N/(N \cap (\sigma + (-\sigma)))$. This is naturally the dual $\mathbb{Z}$-module of $M[\sigma] = M \cap \sigma^\perp$. It is known that the closed subscheme $X(\sigma)$ of $Z$ is naturally identified with the toric variety $Z(\Delta[\sigma])$ with the torus $T[\sigma]$ \cite[Cor.1.7]{O1}. Furthermore, the action of $T[\sigma]$ on $Z(\Delta[\sigma])$ is equivariant with that of $T$ with respect to the natural surjection $T \to T[\sigma]$.

We set $D(\sigma)_h := X(\sigma)_h \setminus T[\sigma]_h$ and

$$\Omega_{X(\sigma)_h}(\log D(\sigma)_h)^\bullet := \mathcal{O}_{X(\sigma)_h} \otimes_{\mathbb{Z}} \bigwedge^\bullet M[\sigma]. \quad (4.14)$$

Then $\Omega_{X(\sigma)_h}(\log D(\sigma)_h)^\bullet$ has the $\partial$-complex structure so that it is identified with the logarithmic de Rham complex of $Z(\Delta[\sigma])_h$ similarly as $\Omega_{Z_h}(\log D_h)^\bullet$ for $Z_h = Z(\Delta)_h$.

**Lemma 4.3** Let $\sigma$ be an element of $\Delta$. For $\rho \in \Delta$ with $\sigma < \rho$, the $\partial$-complex $\overline{I}_\rho^*(\mathcal{P}(\rho)_h \Omega_{X(\sigma)_h}(\log D(\sigma)_h))_h^\bullet$ of $\mathcal{C}$-sheaves on $X(\rho)_h$ is homotopically equivalent to the zero complex.

**Proof.** For the toric variety $X(\sigma)_h = Z(\Delta[\sigma])_h$, $X(\rho)_h$ is the closed subvariety associated to $\rho[\sigma] \in \Delta[\sigma]$. The closed subvariety $X(\rho)_h$ of $X(\sigma)_h$ is defined by the image of $\mathcal{P}(\rho)_h$ in $\mathcal{O}_{X(\sigma)_h}$. Hence, this is a consequence of Lemma 4.2 applied to the toric variety $Z(\Delta[\sigma])_h$. q.e.d.

Let $\rho$ be an element of $\Delta$ and $V$ an object of $\text{GM}(A(\rho))$. Then the $\partial$-complex $\Lambda^\rho_Z(V)^\bullet$ defined in Section 3 induces a $\partial$-complex $\Lambda^\rho_{Z_c}(V)^\bullet$ and its analytic version $\Lambda^\rho_{Z_h}(V)^\bullet$. By Proposition 3.1, $\Lambda^\rho_{Z_h}(V)^\bullet$ is isomorphic to a finite direct sum of dimension shifts of $\Omega_{X(\rho)_h}(\log D(\rho)_h)^\bullet$. In particular, we get the following corollary by Lemma 4.3.

**Corollary 4.4** Let $\sigma$ be an element of $\Delta$ and $V$ in $\text{GM}(A(\sigma))$. For $\rho \in \Delta$ with $\sigma < \rho$, the $\partial$-complex $\overline{I}_\rho^*(\mathcal{P}(\rho)_h \Lambda^\rho_{Z_h}(V))_h^\bullet$ on $X(\rho)_h$ is homotopically equivalent to the zero complex.
We recall some general notation of derived categories (cf. [V]).

Let \( X \) be a locally compact topological space and let \( A(C_X) \) be the abelian category of the \( C \)-sheaves on \( X \). We denote by \( C^+(C_X) \) and \( D^+(C_X) \) the category of the complexes bounded below in \( A(C_X) \) and the derived category of it, respectively. For a continuous map \( f : X \to Y \), let \( f^* : A(C_Y) \to A(C_X) \) be the pull-back functor which is exact. We denote also by \( f^* \) the induced functors \( C^+(C_Y) \to C^+(C_X) \) and \( D^+(C_Y) \to D^+(C_X) \).

If the direct image functor with proper support \( f_! : A(C_X) \to A(C_Y) \) is of finite cohomological dimension, the functor \( f_! : D^+(C_Y) \to D^+(C_X) \) is defined [V, 2.2]. Note that this condition is satisfied for any regular morphisms of finite dimensional analytic spaces.

For \( V^\bullet \in \text{CGM}(A(\sigma)) \), the bicomplex \( \Lambda^\sigma_{Z_h}(V)^{\bullet \bullet} \) is defined as the analytic version of \( \Lambda^\sigma_Z(V)^{\bullet \bullet} \). The associated single complex is denoted by \( \Lambda^\sigma_{Z_h}(V)^\bullet \).

For \( L^\bullet \in \text{CGEM}(\Delta) \), we get the bicomplex \( \Lambda_{Z_h}(L)^{\bullet \bullet} \) and its associated single complex \( \Lambda_{Z_h}(L)^\bullet \), similarly. We denote by \( \delta \) the coboundary map of \( \Lambda_{Z_h}(L)^\bullet \).

**Proposition 4.5** For \( \rho \in \Delta \) and a \( d \)-complex \( L^\bullet \) in \( \text{CGEM}(\Delta) \), there exist a quasi-isomorphism

\[
I^\rho_\pi \Lambda_{Z_h}(L)^\bullet \simeq I^\rho_\pi \Lambda^\rho_{Z_h}(i^\rho_\pi L)^\bullet
\]

as \( \delta \)-complexes of \( C \)-sheaves on \( T[\rho]_{\text{h}} \) and an isomorphism

\[
I^\rho_\pi \Lambda_{Z_h}(L)^\bullet \simeq I^\rho_\pi \Lambda^\rho_{Z_h}(i^\rho_\pi L)^\bullet
\]

in the derived category \( D^+(C_{T[\rho]_{\text{h}}}) \).

**Proof.** Let \( \sigma \) be an element of \( F(\pi) \) with \( \sigma \prec \rho \). For \( V \) in \( \text{GM}(A(\sigma)) \), there exists an exact sequence

\[
0 \to \mathcal{P}(\rho)_{\text{h}} \Lambda^\sigma_{Z_h}(V)^\bullet \to \Lambda^\sigma_{Z_h}(V)^\bullet \to \Lambda^\rho_{Z_h}(V_{A(\rho)})^\bullet \to 0.
\]

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In particular, the homomorphism $I^*_\rho \lambda(V)$ of $\partial$-complexes is a quasi-isomorphism by Corollary 4.4.

For each $L^i$, we get a quasi-isomorphism

$$I^*_\rho \lambda(L^i) : I^*_\rho \Lambda^*_A \Lambda(V(L^i))^\bullet \rightarrow I^*_\rho \Lambda^*_A \Lambda(V(i^*_\rho L)^i)^\bullet$$

as a collection of quasi-isomorphisms $I^*_\rho \lambda(L^i(\sigma))$ for $\sigma \in F(\rho)$.

We get the first quasi-isomorphism as the collection of $I^*_\rho \lambda(L^i)$’s for $i \in \mathbb{Z}$.

Since $i^*_\rho L^i = L(\rho)^i$, it is enough to show that $\bar{I}^*_\rho \Lambda^*_A \Lambda(V(L^i(\sigma)))$ are quasi-isomorphic to the zero complex for all $\sigma \in F(\rho) \setminus \{\rho\}$ and $i \in \mathbb{Z}$. By proposition 3.1, $\Lambda^*_A \Lambda(V(L^i(\sigma)))$ is isomorphic to the finite direct sum of dimension shifts of the logarithmic de Rham complex $\Omega_{X(\sigma)h}(\log D(\sigma)h)^i$ on $X(\sigma)h$. By [2], Prop.1.2, the logarithmic de Rham complex is quasi-isomorphic to the direct image $Rj_* C_{T[\sigma]}$ for the open immersion $j : T[\sigma]h \rightarrow X(\sigma)h$. Since $T[\sigma]h \cap X(\rho)h = \emptyset$, $\bar{I}^*_\rho Rj_* C_{T[\sigma]h}$ is equivalent to zero. Hence $\bar{I}^*_\rho \Lambda^*_A \Lambda(V(L^i(\sigma)))$ is also zero in the derived category.

q.e.d.

For any $\mathbb{Q}$-vector space $W$, we denote by $\bar{W}$ the scalar extension $W \otimes_{\mathbb{Q}} \mathbb{C}$. If $W$ is graded, so is $\bar{W}$.

Let $\rho$ be an element of $\Delta$. For $V$ in $\text{GM}(A(\rho))$, we denote by $\bar{V}_{T[\rho]h}$ the constant sheaf on $T[\rho]h$ with the stalk $\bar{V}$. We regard it as a $\partial$-complex of $\mathbb{C}$-sheaves by setting $\bar{V}^i_{T[\rho]h} := (\bar{V}_i)_{T[\rho]h}$ and $\partial = 0$.

For $N[\rho] := N/N(\rho)$, we set $\det[\rho]_\mathbb{Q} := \det N[\rho]_\mathbb{Q}$. We denote by $\text{Det}[\rho]$ the graded $\mathbb{Z}$-module defined by $(\text{Det}[\rho])_{-r+r^\rho} := \det[\rho]$ and $(\text{Det}[\rho])_i := \{0\}$ for $i \neq -r + r^\rho$. Here note that rank $N[\rho] = r - r^\rho$.

We define a homomorphism $(\bar{V} \otimes \text{Det}[\rho])^*_T[\rho]h \rightarrow I^*_\rho (\Lambda^*_A \Lambda(V))^\bullet$ as follows.

We take a $\mathbb{Q}$-linear subspace $H$ of $N_\mathbb{Q}$ such that $N_\mathbb{Q} = N(\rho)_\mathbb{Q} \oplus H$. Then we have a natural isomorphisms $N[\rho]_\mathbb{Q} = N_\mathbb{Q}/N(\rho)_\mathbb{Q} \simeq H$ and $\det[\rho]_\mathbb{Q} = \det N[\rho]_\mathbb{Q} \simeq \det H \subset A_{-r+r^\rho}$. We denote by $\text{Det} H$ the graded $\mathbb{Q}$-vector space defined by $(\text{Det} H)_{-r+r^\rho} := \det H$ and $(\text{Det} H)_i := \{0\}$ for $i \neq -r + r^\rho$. By Lemma 1.3, $V_A = V \otimes_{A(\rho)} A$ is equal to the free $A^*[\rho]$-module $(V \otimes_{\mathbb{Q}} \text{Det} H) \otimes_{\mathbb{Q}} A^*[\rho]$. 

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The restriction of $\partial$ of $\Lambda^0_{\mathbb{Z}}(V)^\bullet = \mathcal{O}_Z \otimes_{\mathbb{Q}} V_A$ to the constant subsheaf $\mathcal{Q}e(0) \otimes_{\mathbb{Q}} V_A$ is zero. Hence the composite of the natural homomorphisms

$$\bar{V} \otimes \text{Det}[\rho] \longrightarrow \bar{V} \otimes_{\mathbb{Q}} \text{Det} H \longrightarrow \bar{V}_A \longrightarrow \mathcal{C}e(0) \otimes_{\mathbb{C}} \bar{V}_A$$

defines a homomorphism $(\bar{V} \otimes \text{Det}[\rho])_{T[\rho]} \longrightarrow \Lambda^0_{\mathbb{Z}}(V)|_{T[\rho]}$ of $\partial$-complexes of $\mathbb{C}$-sheaves on the algebraic torus $T[\rho]$. We denote by $\phi^H$ the associated homomorphism

$$(\bar{V} \otimes \text{Det}[\rho])_{T[\rho]} \longrightarrow I_\rho^*(\Lambda^0_{\mathbb{Z}}(V))$$

on the smooth analytic space $T[\rho]_h$.

**Proposition 4.6** Let $\rho \in \Delta$ and let $V$ be an object of $\text{GM}(A(\rho))$. Then the homomorphism $\phi^H_V$ is a quasi-isomorphism for any $H \subset N_\mathbb{Q}$ with $N_\mathbb{Q} = N(\rho)_{\mathbb{Q}} \oplus H$. For another $K \subset N_\mathbb{Q}$ with $N_\mathbb{Q} = N(\rho)_{\mathbb{Q}} \oplus K$, the homomorphisms $\phi^H_V$ and $\phi^K_V$ are locally homotopic.

**Proof.** Since $V$ is of finite dimension, we prove the proposition by induction on the dimension of $V$. If $V = \{0\}$, then the $\partial$-complexes are trivial and the assertion is clear. Assume $\dim V \geq 1$. Let $k$ be the maximal integer with $V_k \neq \{0\}$. We take a vector subspace $V'_k \subset V_k$ of codimension one. By setting $V'_i := V_i$ for $i \neq k$, we get a homogeneous subspace $V' \subset V$ of codimension one. Since $A(\rho)$ is graded negatively, $V'$ is an object of $\text{GM}(A(\rho))$. We get an exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow \mathbb{Q}(-k) \longrightarrow 0$$

of graded $A(\rho)$-modules, where $Q(a)$ is the graded $\mathbb{Q}$-vector space defined by $Q(a)_{i} := \{0\}$ for $i \neq -a$. Set $c := r - r_\rho$. Then we get a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & (\bar{V}' \otimes \text{Det}[\rho])_{T[\rho]}^\bullet & \rightarrow & (\bar{V} \otimes \text{Det}[\rho])_{T[\rho]}^\bullet & \rightarrow & (C(-k) \otimes \text{Det}[\rho])_{T[\rho]}^\bullet & \rightarrow & 0 \\
\downarrow & & \phi^H_{V'} & & \phi^H_{V} & & \phi^H_{Q(-k)} & & \\
0 & \rightarrow & I_\rho^*(\Lambda^0_{\mathbb{Z}}(V'))^\bullet & \rightarrow & I_\rho^*(\Lambda^0_{\mathbb{Z}}(V))^\bullet & \rightarrow & I_\rho^*(\Lambda^0_{\mathbb{Z}}(\mathbb{Q}(-k)))^\bullet & \rightarrow & 0
\end{array}
$$

(4.22)
By the induction assumption, $\phi^H_V$ is a quasi-isomorphism of $\partial$-complexes of $\mathbf{C}$-sheaves. On the other hand, $I^*_\rho(\Lambda^p_{Z_h}(Q(-c)))^\bullet$ is equal to the analytic de Rham complex on the complex manifold $T[\rho]_h$. Since $\phi^H_{Q(-k)}$ is the dimension shift of the natural homomorphism $\mathbf{C}_{T[\rho]_h} \rightarrow \Omega^\bullet_{T[\rho]_h}$, this is a quasi-isomorphism, by the complex analytic version of the Poincaré Lemma. Hence $\phi^H_V$ is also a quasi-isomorphism.

Clearly, $\phi^H_{Q(-k)}$ does not depend on the choice of $H$. Hence $\phi^H_{Q(-k)} - \phi^K_{Q(-k)} = 0$. By assumption, $\phi^H_V - \phi^K_V$ is locally homotopic to zero. Hence, we know that $\phi^H_V - \phi^K_V$ gives zero maps on the cohomology sheaves. Since $\bar{V}_{T[\rho]_h}$ is a locally free $\mathbf{C}$-sheaf, $\phi^H_V - \phi^K_V$ is locally homotopic to zero. q.e.d.

Recall that $Z_h$ has the decomposition $\bigcup_{\sigma \in \Delta} T[\sigma]_h$ into $T_h$-orbits. For each integer $0 \leq i \leq r$, we define

$$Z^i_h = Z^{2i+1}_h =: \bigcup_{\sigma \in \Delta \atop r_\sigma \geq r-i} T[\sigma]_h.$$ (4.23)

Then we have a filtration

$$Z_h = Z^{2r}_h \supset Z^{2r-1}_h \supset \cdots \supset Z^2_h \supset Z^1_h \supset Z^0_h$$ (4.24)

of $Z_h$ satisfying the conditions of the topological stratification in [GM2, 1.1].

The intersection complex of a stratified space is defined for a sequence of integers $(p(2), p(3), p(4), \cdots)$ which is called a perversity [GM2, 2.0]. Since $Z_h$ is a complex analytic space of dimension $r$, only $p(i)$’s for even $i$ less than or equal to $2r$ are relevant for the intersection complex.

Let $p = (p(2), p(4), p(6), \cdots, p(2r))$ be a sequence of integers with $p(2) = 0$ and $p(2i) \leq p(2i+2) \leq p(2i)+2$ for $i = 1, \cdots, r-1$ as a perversity for $Z_h$. We denote by the same symbol $p$ the perversity on $\Delta$ defined by $p(\sigma) := p(2r_\sigma) - r_\sigma + 1$ for $\sigma \in \Delta \setminus \{0\}$. Note that, for the middle perversity $m := (0, 1, 2, \cdots, r-1)$, we have $m(\sigma) = 0$ for all $\sigma \in \Delta \setminus \{0\}$.

We consider the $\delta$-complex $\Lambda_Z(\text{ic}_p(\Delta))^\bullet$ which is the associated single complex of the bicomplex $\Lambda_Z(\text{ic}_p(\Delta))^\bullet$. 50
By [GM2, 3.3, AX1], the intersection complex $\text{IC}_p(Z_h)^\bullet$ in the derived category $D^b(C_{Z_h})$ of bounded complexes of $C$-sheaves is characterized by the following properties.

For the convenience of our use, we set $F := \text{IC}_p(Z_h)[-r]^\bullet$. Note that $n$ in [GM2, 3.3] is $2r$ in our case.

(a) The restriction of $F$ to $T_h$ is quasi-isomorphic to $C_{T_h}[-r]$. 

(b) $H^i(F)$ is a trivial sheaf for $i < -r$. 

(c) For any $\sigma \in \Delta \setminus \{0\}$, $H^i(F_x) = \{0\}$ for $i \geq p(2r_\sigma) - r + 1$. 

(d) For any $\sigma \in \Delta \setminus \{0\}$, $H^i((I_\sigma^r F)_x) = \{0\}$ for $i \leq p(2r_\sigma) - r + 1$.

**Theorem 4.7** Let $p = (p(2), p(4), p(6), \ldots, p(2r))$ be a perversity. The complex of $C$-sheaves $\Lambda_{Z_h}(\text{ic}_p(\Delta))^\bullet$ is isomorphic to $\text{IC}_p(Z_h)[-r]^\bullet$.

**Proof.** We set $L^\bullet := \Lambda_{Z_h}(\text{ic}_p(\Delta))^\bullet$. It is sufficient to show that $L^\bullet$ satisfies the above properties. Let $\sigma$ be in $\Delta \setminus \{0\}$. Then the stratum $T[\sigma]_h$ is of dimension $2r - 2r_\sigma$ in the real dimension. By Propositions 4.5 and 4.6, we have an isomorphism

$$H^i(L^\bullet_x) \simeq H^{i+r-r_\sigma}(i^\star_\sigma \text{ic}_p(\Delta)^\bullet) \otimes \mathbb{Q} C$$

(4.25)

for every point $x \in T[\sigma]_h$. By the condition (3) of Theorem 2.9, the cohomology $H^i(i^\star_\sigma \text{ic}_p(\Delta)^\bullet)$ vanishes for $i \geq p(\sigma) = p(2r_\sigma) - r_\sigma + 1$. Hence $H^i(L^\bullet_x) = 0$ for $i \leq p(2r_\sigma) - r + 1$. By Propositions 4.5 and 4.6, we also have

$$H^i((I_\sigma^r L)^\bullet_x) \simeq H^{i+r-r_\sigma}(i^\star_\sigma \text{ic}_p(\Delta)^\bullet) \otimes \mathbb{Q} C$$

(4.26)

for all $x \in T[\sigma]_h$. The condition (2) of Theorem 2.9 implies $H^i(i^\star_\sigma \text{ic}_p(\Delta)^\bullet) = 0$ for $i \leq p(\sigma) = p(2r_\sigma) - (r_\sigma)+1$. Hence $H^i((I_\sigma^r L)^\bullet_x) = 0$ for $i \leq p(2r_\sigma) - r + 1$. Hence, by the Axioms [AX1] of [GM2, 3.3], $L^\bullet$ is the intersection complex with the perversity $p$.

q.e.d.
Proposition 4.8 Assume that $\Delta$ is a complete fan. Let $L^\bullet$ be an object of CGEM($\Delta$). Then there exists a natural isomorphism
\[
H^k(\Lambda Z_h(L)^\bullet) \simeq \bigoplus_{p+q=k} H^p(\Gamma(L)^\bullet)_q \otimes_\mathbb{Q} \mathbb{C}
\] (4.27)
for every integer $k$.

Proof. Since any $X(\sigma)$ is a complete toric variety, $H^i(X(\sigma), \mathcal{O}_{X(\sigma)}) = 0$ for every $i > 0$ [O1, Cor.2.8]. By [GAGA], we have
\[
H^i(X(\sigma)_h, \mathcal{O}_{X(\sigma)_h}) = \begin{cases} 
\mathbb{C} & \text{if } i = 0 \\
\{0\} & \text{if } i > 0 
\end{cases}
\] (4.28)

Since $\Lambda^\sigma_{Z_h}(L^i(\sigma))$ is a free $O_{X(\sigma)_h}$-module for any $\sigma$ and $i$, we have an isomorphism
\[
H^k(\Lambda Z_h(L)^\bullet) = H^k(\Gamma(Z_h, \Lambda Z_h(L))^\bullet),
\] (4.29)
where $\Gamma(Z_h, \Lambda Z_h(L))^\bullet$ is the single complex of $\mathbb{C}$-vector spaces associated to the bicomplex $\Gamma(Z_h, \Lambda Z_h(L))^{\bullet\bullet}$. Since the global sections are locally $M$-homogeneous of degree $0 \in M$, $d_2 = \partial$ of the last bicomplex is zero. Hence the spectral sequence
\[
E_1^{p,q} := H^p(\Gamma(Z_h, \Lambda Z_h(L))^\bullet)_q \Rightarrow H^{p+q}(\Gamma(Z_h, \Lambda Z_h(L))^\bullet)
\] (4.30)
degenerates at $E_1$-terms. Consequently, the cohomology group $H^k(\Gamma(Z_h, \Lambda Z_h(L))^\bullet)$ is equal to the direct sum
\[
\bigoplus_{q \in \mathbb{Z}} H^{k-q}(\Gamma(Z_h, \Lambda Z_h(L))^\bullet)_q = \bigoplus_{q \in \mathbb{Z}} H^{k-q}(\Gamma(L)^\bullet)_q \otimes_\mathbb{Q} \mathbb{C}.
\] (4.31)
q.e.d.

5 The Serre duality

In this section, we again fix a finite fan $\Delta$ of $N_\mathbb{R}$ and let $Z := Z(\Delta)$ be the associated toric variety defined over $\mathbb{Q}$.
Let \( p \) be a perversity on \( \Delta \). Then \( \Lambda_Z(\text{ic}_p(\Delta))^{\bullet \bullet} \) is a bicomplex of coherent \( \mathcal{O}_Z \)-modules such that \( d_1 = d \) is a \( \mathcal{O}_Z \)-homomorphism and \( d_2 = \partial \) is a differential operator of order one. Note that \( \Lambda_Z(\text{ic}_p(\Delta))^{i,j} = \{0\} \) unless \( 0 \leq i \leq r \) and \( -r \leq j \leq 0 \) by Proposition \[2.11\].

For each integer \( j \), we set

\[
\Omega_j(p; Z) := \Lambda_Z(\text{ic}_p(\Delta))^{*, -j} .
\]

(5.1)

Clearly, \( \Omega_j(p; Z) \) is the zero complex unless \( 0 \leq j \leq r \). Since \( d \) is an \( \mathcal{O}_Z \)-homomorphism, \( \Omega_j(p; Z) \) is a finite complex in the category of coherent \( \mathcal{O}_Z \)-modules.

For the top perversity \( t \), we have

\[
\Omega_j(t; Z)^i = \bigoplus_{\sigma \in \Delta(i)} \det(\sigma) \otimes \bigwedge^j N[\sigma] \otimes \mathcal{O}_{X(\sigma)} ,
\]

(5.2)

where \( N[\sigma] := N/N(\sigma) \). For \( \sigma \in \Delta(i) \) and \( \tau \in \Delta(i + 1) \), the \((\sigma, \tau)\)-component of the coboundary map is the tensor product of \( q'_{\sigma/\tau} : \det(\sigma) \to \det(\tau) \) and the natural surjection

\[
\bigwedge^j N[\sigma] \otimes \mathcal{O}_{X(\sigma)} \to \bigwedge^j N[\tau] \otimes \mathcal{O}_{X(\tau)}
\]

(5.3)

if \( \sigma \prec \tau \) and is the zero map otherwise.

For finite complexes \( B^\bullet, C^\bullet \) of coherent sheaves on \( Z \), we denote by \( \mathcal{H}om_{\mathcal{O}_Z}(C, B)^{\bullet \bullet} \) the bicomplex with the component

\[
\mathcal{H}om_{\mathcal{O}_Z}(C, B)^{i,j} := \mathcal{H}om_{\mathcal{O}_Z}(C_j, B_i)
\]

(5.4)

for each pair \((i,j)\) of integers. We denote by \( \mathcal{H}om_{\mathcal{O}_Z}(C, B)^\bullet \) the associated single complex. The coboundary map

\[
d^k : \mathcal{H}om_{\mathcal{O}_Z}(C, B)^k \longrightarrow \mathcal{H}om_{\mathcal{O}_Z}(C, B)^{k+1}
\]

(5.5)

is determined as follows. For an integer \( i \), the restriction of \( d^k \) to the component \( \mathcal{H}om_{\mathcal{O}_Z}(C^i, B^{i+k}) \) is the sum of

\[
(d_B^{i+k})_* : \mathcal{H}om_{\mathcal{O}_Z}(C^i, B^{i+k}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Z}(C^i, B^{i+k+1})
\]

(5.6)

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\( (−1)^{k+1}(d_C^{i+1})^*\mathcal{H}om_{O_Z}(C^i, B^{i+k}) \longrightarrow \mathcal{H}om_{O_Z}(C^{i-1}, B^{i+k}) \) \hspace{1cm} (5.7)

(cf. [D2, 1.1.10,(ii)])

**Theorem 5.1** Let \( L^* \) be an object of CGEM(\( \Delta \)). For each integer \( q \), there exists

\[ \mathcal{H}om_{O_Z}(\Lambda_{Z}(L)_q, \Omega(t; Z) \otimes \text{det } N)^* \simeq \Lambda_{Z}(D(L))^*_{r-q}. \] \hspace{1cm} (5.8)

**Proof.** For each pair \((i, j)\) of integers, we have

\[ \mathcal{H}om_{O_Z}(\Lambda_{Z}(L)_q, \Omega(t; Z) \otimes \text{det } N)^{i,j} \]

\[ = \bigoplus_{(\sigma, \rho)} \text{Hom}_Q((L(\sigma)^{−j})_q, \text{det}(\rho) \otimes \text{det } N) \otimes Q \mathcal{O}_{X(\rho)}, \] \hspace{1cm} (5.10)

where the sum is taken over all pairs \((\sigma, \rho)\) with \( \sigma \in \Delta, \rho \in \Delta(i) \) and \( \sigma \prec \rho \). We identify \( \text{Hom}_Q((L(\sigma)^{−j})_q, \text{det } N_Q) \) with \( D_N((L(\sigma)^{−j})−r−q) \) through the set of right operations \( D_N^{\text{right}}((L(\sigma)^{−j})−r−q) \). Hence

\[ \mathcal{H}om_{O_Z}(\Lambda_{Z}(L)_q, \Omega(t; Z) \otimes \text{det } N)^k \]

\[ = \bigoplus_{\rho \in \Delta} \text{det}(\rho) \otimes D_N((i^*_\rho L^{−r\rho+k})_A)_{−r−q} \otimes Q \mathcal{O}_{X(\rho)} \] \hspace{1cm} (5.12)

\[ = \bigoplus_{\rho \in \Delta} \text{det}(\rho) \otimes (D_{\rho}(i^*_\rho L^{−r\rho+k})_A)_{−r−q} \otimes Q \mathcal{O}_{X(\rho)} \] \hspace{1cm} (5.13)

\[ = \Lambda_{Z}(D(L))_{r-q}^k. \] \hspace{1cm} (5.14)

The equality of the coboundary maps is also checked. Namely, for \( \rho, \mu \in \Delta \), the

\((\rho, \mu)\)-component of \( d^k \)'s are nonzero only for (a) \( \rho < \mu \) and \( r_\mu = r_\rho + 1 \), or (b) \( \rho = \mu \). In case (a), they are both equal to the tensor product of \( q^{r_{\rho/\mu}} \), inclusion map

\[ D_N((i^*_\rho L^{−r_\rho+k})_A)_{−r−q} \rightarrow D_N((i^*_\mu L^{−r_\mu+k})_A)_{−r−q} \] \hspace{1cm} (5.15)

and the natural surjection \( \mathcal{O}_{X(\rho)} \rightarrow \mathcal{O}_{X(\mu)} \) in the description \( (5.12) \). In case (b), they are both equal to \( (−1)^{k+1} \text{id} \otimes D_N(i^*_\rho (d_L)^{k-1}) \otimes \text{id}. \) q.e.d.
Let $S$ be a scheme of finite type over a field and let $D^+_{\text{coh}}(S)$ be the derived category of complexes bounded below of $\mathcal{O}_S$-modules with coherent cohomologies. The Grothendieck theory of residues and duality \cite{RD} says that there exists an object of $D^+_{\text{coh}}(S)$ which is called the dualizing complex, and the Serre duality theorem for nonsingular projective varieties is generalized to $S$ by using the the dualizing complex in place of the canonical invertible sheaf.

For the toric variety $Z$, the dualizing complex which is denoted by $C^*(Z, \Omega_0^\vee)$ in \cite[§5]{I2} is described as follows. For the compatibility with the notation of this paper, we write it by $C(Z, \Omega_0^\vee)^*$. For each integer $-r \leq i \leq 0$, we set

$$C^i(Z, \Omega_0^\vee) := \bigoplus_{\sigma \in \Delta(r+i)} \mathcal{O}_{X(\sigma)} \otimes \det M[\sigma] = \bigoplus_{\sigma \in \Delta(r+i)} \Omega_{X(\sigma)}(\log D(\sigma))^{-i}. \quad (5.16)$$

For $\sigma, \tau \in \Delta$, the subvariety $X(\tau)$ of $Z$ is contained in $X(\sigma)$ if and only if $\sigma \prec \tau$. When $\sigma \prec \tau$, let $\varphi_{\sigma/\tau} : \mathcal{O}_{X(\sigma)} \to \mathcal{O}_{X(\tau)}$ be the natural surjection. The component of the coboundary map

$$C^i(Z, \Omega_0^\vee) \xrightarrow{d} C^{i+1}(Z, \Omega_0^\vee)$$

for $\sigma \in \Delta(r+i)$ and $\tau \in \Delta(r+i+1)$ is defined to be $\varphi_{\sigma/\tau} \otimes q_{\sigma/\tau}$ if $\sigma \prec \tau$ and the zero map otherwise, where $\varphi_{\sigma/\tau}$ is the natural surjection $\mathcal{O}_{X(\sigma)} \to \mathcal{O}_{X(\tau)}$. For the definition of $q_{\sigma/\tau}$, see \cite[§1]{I2}. With respect to the identifications $\det M[\sigma] \otimes \det N = \det(\sigma)$ and $\det M[\tau] \otimes \det N = \det(\tau)$, the isomorphism $q_{\sigma/\tau}'$ defined in §2 is equal to $q_{\sigma/\tau} \otimes 1_{\det N}$.

As a special case of \cite[Thm.5.4]{I2}, the $d$-complex $C(Z, \Omega_0^\vee)^*$ is quasi-isomorphic to the residual complex $f_Z^*\mathbb{Q}$ \cite[VI,§3]{RD} for the structure morphism $f_Z : Z \to \text{Spec} \mathbb{Q}$, i.e., it is a global dualizing complex of the $\mathbb{Q}$-scheme $Z$. 

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For a finite complex $B^\bullet$ of $\mathcal{O}_Z$-modules with coherent cohomology sheaves, the object $R\mathcal{H}om(B, f_Z^* \mathbb{Q})^\bullet$ in the derived category $D^{+}_{\text{coh}}(Z)$ is called the Grothendieck dual of $B^\bullet$.

It is easy to see that $\Lambda_Z(\text{ic}_t(\Delta))^\bullet_0$ is isomorphic to $(\mathcal{C}(Z, \Omega^\vee_0)) \otimes (\det N)[-r]^\bullet$.

**Corollary 5.2** For each integer $0 \leq j \leq r$, the Grothendieck dual of the $d$-complex $\Lambda_Z(L)_{-j}$ is quasi-isomorphic to $\Lambda_Z(D(L))_{j-r}[r]^\bullet$.

**Proof.** Since $\Lambda_Z(L)_{-j}$ is a direct sum of free $\mathcal{O}_{X(\sigma)}$-modules for $\sigma \in \Delta$, the dual $R\mathcal{H}om(\Lambda_Z(L)_{-j}, \mathcal{C}(Z, \Omega_0^\vee))^\bullet$ in the derived category $D^{+}_{\text{coh}}(Z)$ is represented by the complex $\mathcal{H}om(\Lambda_Z(L)_{-j}, \mathcal{C}(Z, \Omega_0^\vee))^\bullet$ by [2, Lem.3.6]. Hence we get the corollary by Theorem 5.1.

The following corollary follows from Corollaries 2.12 and 5.2.

**Corollary 5.3** For each integer $0 \leq j \leq r$, the Grothendieck dual of the $d$-complex $\Omega_j(p; Z)$ is quasi-isomorphic to $\Omega_{r-j}(-p; Z)[r]$.

We assume that $\Delta$ is a complete fan. Then $Z$ is a complete variety.

Let $F^\bullet$ be a finite complex of coherent $\mathcal{O}_Z$-modules. By the Grothendieck duality theorem [Rd, VI, Thm.3.3] applied for the proper morphism $f_Z : Z \rightarrow \text{Spec} \mathbb{Q}$, we have a natural isomorphism

$$H^i(Z, R\mathcal{H}om(F, \mathcal{C}(Z, \Omega_0^\vee))^\bullet) \simeq \text{Hom}(H^{-i}(Z, F^\bullet), \mathbb{Q}). \quad (5.18)$$

**Theorem 5.4** Assume that $\Delta$ is a complete fan. Then the equality

$$\dim_{\mathbb{Q}} H^i(Z, \Omega_j(p; Z)) = \dim_{\mathbb{Q}} H^{-i}(Z, \Omega_{r-j}(-p; Z)) \quad (5.19)$$

holds for any integers $i, j$. 

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Proof. By Corollary 5.2, this is a consequence of the Grothendieck-Serre duality theorem.

We can also prove the equality directly as follows. We have equalities

\[ \dim_Q H^i(Z, \Omega_j(p; Z)^*) = \dim_Q H^i(\Gamma(ic_p(\Delta)^*))_{-j} \quad (5.20) \]

and

\[ \dim_Q H^{-i}(Z, \Omega_{r-j}(p; Z)^*) = \dim_Q H^{-i}(\Gamma(ic_{-p}(\Delta)^*))_{-r+j} \quad (5.21) \]

by Lemma 3.6. Since $ic_p(\Delta)^*$ is quasi-isomorphic to $D(ic_{-p}(\Delta))^*$ by Corollary 2.12, we get the equality by Proposition 2.5.

q.e.d.

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