Higher (Odd) Dimensional Quantum Hall Effect
and
Extended Dimensional Hierarchy

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Abstract

We demonstrate dimensional ladder of higher dimensional quantum Hall effects by exploiting quantum Hall effects on arbitrary odd dimensional spheres. Non-relativistic and relativistic Landau models are analyzed on $S^{2k-1}$ in the $SO(2k-1)$ monopole background. The total sub-band degeneracy of the odd dimensional lowest Landau level is shown to be equal to the winding number from the base-manifold $S^{2k-1}$ to the one-dimension higher $SO(2k)$ gauge group. Based on the chiral Hopf maps, we clarify the underlying quantum Nambu geometry for odd dimensional quantum Hall effect and the resulting quantum geometry is naturally embedded also in one-dimension higher quantum geometry. An origin of such dimensional ladder connecting even and odd dimensional quantum Hall effects is illuminated from a viewpoint of the spectral flow of Atiyah-Patodi-Singer index theorem in differential topology. We also present a BF topological field theory as an effective field theory in which membranes with different dimensions undergo non-trivial linking in odd dimensional space. Finally, an extended version of the dimensional hierarchy for higher dimensional quantum Hall liquids is proposed, and its relationship to quantum anomaly and D-brane physics is discussed.
Table 1: The periodic table for topological insulator [5].

| Class | Charge | Z₁ | Z₂ | Z₃ | Z₄ | Z₅ | Z₆ | Z₇ | Z₈ |
|-------|--------|----|----|----|----|----|----|----|----|
| A-Z   | o      | o  | o  | o  | Z  | o  | Z  | o  | Z  |
| AIII  | o      | o  | 1  | Z  | o  | Z  | o  | Z  | o  |
| AI    | +1     | o  | o  | o  | o  | Z  | o  | Z  | Z  |
| BDI   | +1     | +1 | 1  | Z  | o  | o  | o  | Z  | Z  |
| D     | o      | +1 | o  | Z₂ | Z  | o  | o  | Z  | Z  |
| DIII  | −1     | +1 | 1  | Z₂ | Z₂ | Z  | o  | o  | Z  |
| AII   | −1     | o  | o  | o  | Z₂ | Z₂ | Z  | o  | Z  |
| CII   | −1     | −1 | 1  | Z  | o  | Z₂ | Z₂ | Z  | o  |
| CI    | +1     | −1 | 1  | o  | Z  | o  | Z₂ | Z₂ | Z  |

1 Introduction

Topological insulators [1, 2] have attracted a great deal of attention in contemporary physics. Quantum Hall effect [3, 4] is a well investigated topological insulator fitted in the 2D A-class of the periodic table (Table 1) [5, 6, 7]. Other A-class topological insulators which appear in arbitrary even dimensions can be regarded as higher dimensional analogues of the 2D quantum Hall effect [8, 9]. Zhang and Hu launched the study of the higher dimensional quantum Hall effect by their four-dimensional system [10] and it has been further generalized by many researchers, for instance, on complex projective manifolds [11]. However most works have focused on even dimensional quantum Hall effects, and studies about odd dimensional quantum Hall effects are very few [14, 15]. As the even dimensional quantum Hall effect is a manifestation of the A-class topological insulator [8], AIII class is naturally regarded as odd dimensional counterpart of the quantum Hall effects [15]. A-class and AIII-class are special topological insulators in the sense that the topological invariant $Z$ regularly appear in even and odd spacial dimensions in the periodic table of topological insulators (Table 1). Indeed, intimate relations are known to exist between A and AIII classes: AIII-class topological insulators can be induced by dimensional reduction from one-dimension higher A-class topological insulators [5].

Interestingly, A and AII-class topological insulators accommodate quantum Nambu geometry [16, 17, 18] as a consequence of the level projection [19, 20, 21]. In string theory, the quantum Nambu geometry is considered to be relevant to the geometry of M-theory. For instance, the fuzzy three-sphere [22, 23, 24, 25] appears in a geometry of M2-M5 boundstate [26], and the quantum Nambu bracket plays a crucial role in the BLG formulation of M theory [27, 28, 29]. It is rather fascinating that such exotic geometry emerges in the context of topological insulators [5]. Also, the

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1 There are so many works up to the present. See Refs. [12, 13] as reviews and references therein at that time.
2 See Ref. [30] as a recent good review.
3 Non-commutative geometries for AII class topological insulator and topological semi-metal are also reported in Refs. [31, 32, 33] and Ref. [34].
non-commutative geometry has been practically applied to investigations of fractional quantum Hall effect (see Ref. [35] for instance). However, odd dimensional quantum Nambu geometry is known to suffer from several pathological properties such as difficulties in quantization from the Nambu-Poisson bracket to commutator and violation of fundamental identity, both of which even dimensional Nambu geometry does not sustain [17]. A natural way to resolve such difficulties is to realize odd dimensional Nambu geometry as a subspace of the consistent even dimensional Nambu geometry. Based on the idea, fuzzy three-sphere is constructed as a narrow strip around the equator of one-dimension higher fuzzy four-sphere [36, 37]. See also earlier works [38, 39].

In this paper, we construct higher odd dimensional quantum Hall effect. In Ref. [8], we investigated even dimensional quantum Hall effect and unveiled the underlying quantum Nambu geometry. Also we demonstrated exact correspondence between the zero-mode Landau level degeneracy and Chern number of non-Abelian monopole bundle configuration, guaranteed by the Atiyah-Singer index theorem [40]. In Ref. [15], we investigated three dimensional quantum Hall effect introduced by Nair and Daemi [14] and established its relation to fuzzy three-sphere. The Nair and Daemi’s setup is a natural three dimensional counterpart of the Haldane’s quantum Hall effect on $S^2$ [4] and Zhang and Hu’s on $S^4$. We combine our two preceding works [8, 15] to explore an entire picture of the higher dimensional quantum Hall effects. We start from odd-dimensional Landau problem and clarify the associated non-commutative geometry. Though our starting point is completely odd-dimension, what is interesting is that the odd dimensional Landau model “knows” one-dimension higher lowest Landau level structure. This is the heart of this paper. As we will discuss in detail, there is a dimensional ladder between even and odd dimensions. Such ladder is an analogue of the celebrated dimensional ladder of anomalies in relativistic quantum field theory [41, 42, 43, 44, 45, 46] and, in differential topology, the one known as the spectral flow of the Atiyah-Patodi-Singer index theorem [47, 48, 49]. The existence of the dimensional ladder is also compatible with the ideas of the dimensional relation between A and AIII topological insulators and that of non-commutative geometry stated above.

This paper is organized as follows. In Sec.2, we introduce the chiral Hopf maps based on the original Hopf maps as mathematical background of odd dimensional quantum Hall effect. Sec.3 discusses the non-relativistic Landau model on $S^{2k-1}$, and in Sec.4 the corresponding quantum Nambu geometry is elucidated. The relativistic Landau problem on odd dimensional sphere is exploited in Sec.5. We demonstrate equality between the total dimension of sub-bands of the lowest Landau level in odd dimension and that of even dimension. Sec.6 provides an explanation of the dimensional ladder based on the spectral flow argument of Atiyah-Patodi-Singer. As effective field theory for the odd dimensional quantum Hall effect, BF topological field theory is presented in Sec.7. In Sec.8 we discuss physical implications of the dimensional ladder in the higher dimensional quantum Hall effect. Sec.9 is devoted to summary and discussions.
2 Mathematical Background: The Chiral Hopf Maps

The quantum Hall effect on even dimensional sphere is constructed based on the Hopf maps \[9\]:
\[S^{4k-1} \xrightarrow{S^{2k-1}} S^{2k}. \tag{1}\]

\((k = 1, 2 \text{ and } 4 \text{ respectively correspond to the 1st, 2nd and 3rd Hopf maps.})\) By imposing one more condition on the total manifold, we introduce a reduced Hopf map which we shall call the chiral Hopf map:
\[S^{2k-1}_L \otimes S^{2k-1}_R \xrightarrow{S^{2k-1}_D} S^{2k-1}. \tag{2}\]

We take the \(SO(2k)\) gamma matrices \(\left(2^k \times 2^k\right)\) matrices as
\[\Gamma_{\mu} = \begin{pmatrix} 0 & i\bar{\gamma}_{\mu} \\ -i\gamma_{\mu} & 0 \end{pmatrix} \quad (\mu = 1, 2, \cdots, 2k) \tag{3}\]

where
\[\gamma_i = \bar{\gamma}_i \quad (i = 1, 2, \cdots, 2k - 1), \quad \gamma_{2k} = -\bar{\gamma}_{2k} = i1_{2k-1} \tag{4}\]
with the \(SO(2k-1)\) gamma matrices \(\gamma_i^4\).

\(\{\gamma_i, \gamma_j\} = 2\delta_{ij}. \tag{5}\)

\(\Gamma_{\mu} \tag{3}\) can be expressed as
\[\Gamma_i = \begin{pmatrix} 0 & i\bar{\gamma}_i \\ -i\gamma_i & 0 \end{pmatrix} \quad (i = 1, 2, \cdots, 2k - 1), \quad \Gamma_{2k} = \begin{pmatrix} 0 & 1_{2k-1} \\ 1_{2k-1} & 0 \end{pmatrix}. \tag{6}\]

The corresponding \(SO(2k)\) generators are derived as
\[\Sigma_{\mu\nu} = -i\frac{1}{4}[\Gamma_{\mu}, \Gamma_{\nu}] = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar{\sigma}_{\mu\nu} \end{pmatrix}, \tag{7}\]

where each set of \(\sigma_{\mu\nu}\) and \(\bar{\sigma}_{\mu\nu}\) composes the \(SO(2k)\) algebra of the Weyl representation:
\[\sigma_{\mu\nu} = -i\frac{1}{4}[\bar{\gamma}_{\mu}\gamma_{\nu} - \bar{\gamma}_{\nu}\gamma_{\mu}], \quad \bar{\sigma}_{\mu\nu} = -i\frac{1}{4}[\gamma_{\mu}\bar{\gamma}_{\nu} - \gamma_{\nu}\bar{\gamma}_{\mu}], \tag{8}\]

with
\[\sigma_{ij} = \bar{\sigma}_{ij} = -i\frac{1}{4}[\gamma^i, \gamma^j], \quad \sigma_{i,2k} = -\bar{\sigma}_{i,2k} = \frac{1}{2}\gamma^i. \tag{9}\]

\(\sigma_{ij} (i, j = 1, 2, \cdots 2k - 1)\) are \(SO(2k - 1)\) generators. The chiral matrix \(\Gamma_{2k+1}\) is given by
\[\Gamma_{2k+1} = (-i)^k \Gamma_1 \Gamma_2 \cdots \Gamma_{2k} = \begin{pmatrix} 1_{2k-1} & 0 \\ 0 & -1_{2k-1} \end{pmatrix}. \tag{10}\]

\(^5\text{For } k = 1, \gamma_{i=1} = 1, \text{ for } k = 2, \gamma_i = \sigma_i \ (i = 1, 2, 3) \text{ and for } k = 4, \gamma_i = -i\lambda_i \ (i = 1, 2, \cdots, 7) \text{ with real antisymmetric matrices } \lambda_i \text{ (see Ref.\[12\] for details).} \)
To represent the total manifold $S^2_{L} \otimes S^2_{R}$, we introduce a spinor

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix},$$

subject to

$$\Psi \Psi^\dagger = 1,$$

$$\Psi \Gamma_{2k+1} \Psi^\dagger = 0,$$

or

$$\Psi_L \Psi_L^\dagger + \Psi_R \Psi_R^\dagger = \frac{1}{2}.$$

Using Eq. (11), we realize the chiral Hopf map (2) as

$$\Psi \rightarrow x_\mu = \Psi \Gamma_\mu \Psi = -i \Psi_R \gamma_\mu \Psi_L + i \Psi_L \gamma_\mu \Psi_R \quad (\mu = 1, 2, \ldots, 2k),$$

where $x_\mu$ satisfy

$$\sum_{\mu=1}^{2k} x_\mu x_\mu = (\Psi \Psi^\dagger)^2 - (\Psi \Gamma_{2k+1} \Psi^\dagger)^2 = 4(\Psi_L \Psi_L^\dagger)^2(\Psi_R \Psi_R^\dagger)^2 = 1.$$

$x_\mu$ thus denote coordinates of $S^{2k-1}$. Inverting the map (14), $\Psi$ can be expressed as

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \frac{1}{2\sqrt{1+x_{2k}}^2} \begin{pmatrix} (1 + x_{2k} + ix_i \gamma_i) \\ (1 + x_{2k} - ix_i \gamma_i) \end{pmatrix}.$$

Notice that we can obtain the chiral Hopf spinor (17) also from the original Hopf spinor

$$\Psi_{ori} = \frac{1}{\sqrt{2(1+x_{2k+1})}} \begin{pmatrix} (1 + x_{2k+1}) \psi \\ (x_{2k} - ix_i \gamma_i) \psi \end{pmatrix} \quad (\psi \psi^\dagger = 1),$$

by focusing on the equator of $S^{2k}$

$$x_{2k+1} = 0$$

and choosing a gauge

$$\psi = \frac{1}{\sqrt{2(1+x_{2k})}} (1 + x_{2k} + ix_i \gamma_i).$$

In this sense, the chiral Hopf spinor is equivalent to the original Hopf spinor on the equator of $S^{2k}$.

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"Eq. (14) suggests that $\Psi$ is a coherent state that satisfies

$$\sum_{\mu=1}^{2k} x_\mu \Gamma_\mu \Psi = \Psi.$$"

Indeed with the use of Eq. (6) and (17), one may readily verify Eq. (16).
The chiral Hopf map naturally induces a product of two Hopf maps:

\[ S^{2k-1}_L \otimes S^{2k-1}_R \rightarrow S^{k-1}_L \otimes S^{k-1}_R \rightarrow S^k_L \otimes S^k_R. \]  \tag{21}

When \( k = 2, 4 \) and \( 8 \), \( L \) and \( R \) respectively correspond to 1st, 2nd and 3rd Hopf maps. They are explicitly given by

\[ \Psi_L \rightarrow x^i_L = \Psi^\dagger_L \gamma_i \Psi_L \quad (i = 1, 2, \cdots, 2k - 1), \]
\[ \Psi_R \rightarrow x^i_R = \Psi^\dagger_R \gamma_i \Psi_R \quad (i = 1, 2, \cdots, 2k - 1). \]  \tag{22}

From \( \Psi_L \) and \( \Psi_R \), we can derive two gauge fields for \( L \) and \( R \):

\[ A_L = -i \Psi^\dagger_L d\Psi = -\frac{1}{2(1 + x^{2k})} \sigma_{i\mu} x^{x_\mu} dx_i + \frac{1}{4(1 + x^{2k})} \gamma_i dx_i, \]
\[ A_R = -i \Psi^\dagger_L d\Psi = -\frac{1}{2(1 + x^{2k})} \sigma_{i\mu} x^{x_\mu} dx_i - \frac{1}{4(1 + x^{2k})} \gamma_i dx_i, \]  \tag{23}

and then the total gauge field is given by

\[ A = -i \Psi^\dagger d\Psi = A_L + A_R = A_\mu dx_\mu \]  \tag{24}

with

\[ A_i = -\frac{1}{1 + x^{2k}} \sigma_{ij} x^j, \quad A_{2k} = 0. \]  \tag{25}

Notice that both \( A_L \) and \( A_R \) depend on the \( SO(2k) \) generators but their sum amounts to \( A_\mu \) \( \tag{24} \) that depends on the \( SO(2k - 1) \) generators only, and \( A \) represents the \( SO(2k - 1) \) monopole gauge field.\footnote{The \( SO(2k-1) \) gauge field \( \tag{25} \) exhibits singularity at the south pole on \( S^{2k-1} \). The \( SO(2k-1) \) gauge field with singularity at the north pole

\[ A' = -\frac{1}{1 - x^{2k}} \sigma_{ij} x^j dx_i \]  \tag{26}

is related to \( A \) \( \tag{25} \) by the following gauge transformation

\[ A' = g^\dagger A g - ig^\dagger dg \]  \tag{27}

with

\[ g = \frac{1}{\sqrt{1 - x^{2k}}} i x_i \gamma_i = -g^\dagger = -g^{-1}. \]  \tag{28}

Notice

\[ A = -\frac{i}{2} (1 - x^{2k}) dg g^\dagger, \quad A' = -\frac{i}{2} (1 + x^{2k}) g^\dagger dg. \]  \tag{29}

The field strengths \( F = dA + iA^2 \) and \( F' = dA' + iA'^2 \) are related by \( F' = g^\dagger F g \).}
While the above derivation of $SO(2k - 1)$ monopole gauge field is apparently parallel to that of $SO(2k)$ gauge field from the Hopf map \cite{9}, there is a problem, because there does not exist homotopy that guarantees the quantization of the $SO(2k - 1)$ monopole charge:

$$\pi_{2k-2}(SO(2k-1)) = 0.$$  \hspace{1cm} (32)

This problem is resolved when the present system is embedded in one-dimension higher system in which the monopole gauge field is $SO(2k)$ and then the charge is quantized:

$$\pi_{2k-1}(SO(2k)) = \mathbb{Z}.$$  \hspace{1cm} (33)

We will discuss the embedding later, but for the moment, we just suppose quantization of the $SO(2k - 1)$ monopole charge to be quantized.

## 3 Non-Relativistic Landau Model on $S^{2k-1}$

Our next task is to analyze the non-relativistic Landau model on $S^{2k-1}$ in the $SO(2k - 1)$ monopole background.

$$H = -\frac{1}{2M} \sum_{\mu=1}^{2k} D_\mu^2 \bigg|_{r=1} = \frac{1}{2M} \sum_{\mu<\nu=1}^{2k} \Lambda_{\mu\nu}^2,$$  \hspace{1cm} (35)

where $\Lambda_{\mu\nu}$ are the $SO(2k)$ covariant angular momentum

$$\Lambda_{\mu\nu} = -ix_\mu D_\nu + ix_\nu D_\mu$$  \hspace{1cm} (36)

with

$$D_\mu = \partial_\mu + iA_\mu.$$  \hspace{1cm} (37)

The total angular momentum is constructed as

$$L_{\mu\nu} = \Lambda_{\mu\nu} + F_{\mu\nu}.$$  \hspace{1cm} (38)

In detail, they are

$$L_{ij} = L_{ij}^0 + \sigma_{ij}, \quad L_{i,2k} = L_{i,2k}^0 - \frac{1}{1 + x_{2k}{ij}^2} \sigma_{ij} x_j$$  \hspace{1cm} (39)

with

$$L_{ij}^0 = -ix_i \frac{\partial}{\partial x_j} + ix_j \frac{\partial}{\partial x_i}.$$  \hspace{1cm} (40)

It is easy to check that $L_{\mu\nu}$ satisfy the $SO(2k)$ algebra, and the $SO(2k)$ Casimir $L_{\mu\nu}^2$ is given by

$$L_{\mu\nu}^2 = \Lambda_{\mu\nu}^2 + F_{\mu\nu}^2,$$  \hspace{1cm} (41)

\begin{footnote}{In $2kD$, the square of the covariant derivative is given by
\begin{equation}
\sum_{\mu=1}^{2k} D_\mu^2 = \frac{\partial^2}{\partial r^2} + (2k - 1) \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r} \sum_{\mu<\nu} \Lambda_{\mu\nu}^2.
\end{equation}
\end{footnote}
and the $SO(2k)$ Landau Hamiltonian (35) can be rewritten as

$$H = \frac{1}{2M} \left( \sum_{\mu < \nu = 1}^{2k} L_{\mu\nu}^2 - \sum_{\mu < \nu = 1}^{2k} F_{\mu\nu}^2 \right) = \frac{1}{2M} (C_{SO(2k)} - C_{SO(2k-1)}).$$

(42)

Here we generalize the previous $SO(4)$ analysis [15] to $SO(2k)$ ($k \geq 2$) and specify the irreducible representation as

$$[l_1, l_2, \cdots, l_{k-1}, l_k] = [n + \frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}, s] \equiv [n, \frac{I}{2}, s]_{SO(2k)},$$

(43)

where $n (= 0, 1, 2, \cdots)$ denotes the Landau level index, and $s$ which we call the chirality parameter takes the following values,

$$s = \frac{I}{2}, \frac{I}{2} - 1, \cdots, -\frac{I}{2} + 1, -\frac{I}{2}.$$

(44)

Notice that the minimum of the absolute value of the chirality parameter takes different values depending on the parity of $I$: For even $I$, while for odd $I/2$. The $SO(2k)$ Casimir is evaluated as

$$C[n, \frac{I}{2}, \pm s]_{SO(2k)} = \sum_{i=1}^{k} l_i(l_i + 2k - 2i) = n(n + 2k - 2) + \frac{1}{2} I(2n + k^2 - k) + \frac{1}{4} I^2(k - 1) + s^2.$$  

(45)

For the $SO(2k-1)$ monopole gauge group, the representation is given by the fully symmetric representation

$$[l_1, l_2, \cdots, l_{k-1}] = \left[ \frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2} \right] \equiv [0, \frac{I}{2}]_{SO(2k-1)},$$

(46)

and the corresponding $SO(2k-1)$ Casimir eigenvalue becomes

$$C[0, \frac{I}{2}]_{SO(2k-1)} = \sum_{i=1}^{k-1} l_i(l_i + 2k - 1 - 2i) = \frac{1}{4} I^2(k - 1) + \frac{1}{2} I(k - 1)^2.$$  

(47)

Consequently, the energy eigenvalues (42) for $k \geq 2$ are derived as

$$E_n(s) = \frac{1}{2M} \left( n(n + 2k - 2) + I(n + \frac{k-1}{2}) + s^2 \right) = E_n(-s).$$  

(49)

Notice that the energy (49) depends not only on the Landau level index $n$ but also on the chirality parameter $s$. In particular the "lowest Landau level" ($n = 0$) is split to the sub-bands

$$E_{n=0}(s) = \frac{1}{2M} \left( \frac{I}{2}(k - 1) + s^2 \right).$$

(50)

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9 One can find the Casimir eigenvalues and the dimensions of the corresponding representations in Ref.[50].

10 Recovering the radius of sphere $r$ in the $SO(2k)$ Landau Hamiltonian (35) with $M \to Mr^2$, we take the planar limit $I, r \to \infty$ with fixed $I/r^2$ and $s$. The $SO(2k)$ Landau levels (49) are reduced to

$$E_n \to \frac{I}{2M}(n + \frac{k-1}{2}).$$

(48)

These are equal to the planar limit of the Landau levels of the $SO(2k - 1)$ Landau model [8].
For $k = 1$, the energy eigenvalues are given by (see Appendix A for details)

\[
SO(2) : \quad E_n = \frac{1}{2M} n^2 \quad \text{(for even } I),
\]

\[
E_n = \frac{1}{2M} (n + \frac{1}{2})^2 \quad \text{(for odd } I).
\]

(51)

The corresponding degeneracy of the sub-band of the Landau level \([49]\) is derived as

\[
d[n, \frac{I}{2}, s]_{SO(2k)} = \frac{(2n + I + 2k - 2)^2 - 4s^2}{4(k - 1)^2} \cdot \prod_{2 \leq i \leq k-1} \frac{(n + I + 2k - i - 1)(n + i - 1)}{(2k - i - 1)(i - 1)}
\]

\[
\cdot \prod_{2 \leq i < j \leq k-1} \frac{I + 2k - i - j}{2k - i - j} \cdot \prod_{2 \leq i \leq k-1} \frac{(I + 2k - 2i)^2 - 4s^2}{4(k - i)^2},
\]

(52)

and for the lowest Landau level \([50]\), the sub-band has the following degeneracy

\[
d[0, \frac{I}{2}, s]_{SO(2k)} = d[0, \frac{I}{2}, -s]_{SO(2k)} = \prod_{j=1}^{k-1} \frac{(I + 2j)^2 - 4s^2}{(2j)^2} \times \prod_{l=1}^{k-2} \prod_{i=1}^{k-l-1} \frac{I + 2l + i}{2l + i}.
\]

(53)

For instance\([11]\),

\[
d[0, \frac{I}{2}, s]_{SO(2)} \equiv 1 \sim I^0,
\]

\[
d[0, \frac{I}{2}, s]_{SO(4)} = \frac{1}{4} ((I + 2)^2 - 4s^2) \sim I^2,
\]

\[
d[0, \frac{I}{2}, s]_{SO(6)} = \frac{1}{192} ((I + 2)^2 - 4s^2)(I + 3)((I + 4)^2 - 4s^2) \sim I^5,
\]

\[
d[0, \frac{I}{2}, s]_{SO(8)} = \frac{1}{138240} ((I + 2)^2 - 4s^2)(I + 3)((I + 4)^2 - 4s^2)(I + 4)(I + 5)((I + 6)^2 - 4s^2) \sim I^9.
\]

(55)

Here $\sim$ means the limit $I \to \infty$ with finite $s$. The chiral Hopf spinor \([17]\) is the lowest Landau level eigenstate of the Landau Hamiltonian \([35]\) for $\frac{I}{2} = |s| = \frac{1}{2}$. Taking the total antisymmetric combination of $N$ chiral Hopf spinors, we can readily construct a Laughlin like $N$-body wavefunction or Laughlin-Halperin like wave-function as in Ref.\([15]\).

From Eq.\([53]\), one may find a hierarchical relation in the degeneracies between different dimensions ($2k - 1$ and $2k - 3$):

\[
d[0, \frac{I}{2}, s]_{SO(2k)} = \frac{(k - 1)!}{(2k - 3)!} \frac{(I + 2k - 2)^2 - 4s^2}{(2k - 2)^2} \frac{(I + 2k - 3)!}{(I + k - 1)!} \cdot d[0, \frac{I}{2}, s]_{SO(2k-2)}
\]

\[
\sim I^k \cdot d[0, \frac{I}{2}, s]_{SO(2k-2)}.
\]

(56)

\[^{11}\text{In particular,}
\]

\[
d[0, 0, 0]_{SO(2k)} = 1, \quad d[0, \frac{1}{2}, \pm \frac{1}{2}]_{SO(2k)} = 2^{k-1}.
\]

(54)

The second relation stands for the well known formula of the dimension of the $SO(2k)$ fundamental spinor.
The total dimension of the lowest Landau level can be read off from Eq.(53):

\[
\sum_{s=-\frac{l}{2}}^{\frac{l}{2}} d[0, \frac{I}{2}, s]_{SO(2k)} = \prod_{l=1}^{k} \prod_{i=1}^{l} \frac{I + l + i - 1}{l + i - 1}.
\] (57)

For instance,

\[
\sum_{s=-\frac{l}{2}}^{\frac{l}{2}} d[0, \frac{I}{2}, s]_{SO(2)} = I + 1,
\]

\[
\sum_{s=-\frac{l}{2}}^{\frac{l}{2}} d[0, \frac{I}{2}, s]_{SO(4)} = \frac{1}{6}(I + 1)(I + 2)(I + 3),
\]

\[
\sum_{s=-\frac{l}{2}}^{\frac{l}{2}} d[0, \frac{I}{2}, s]_{SO(6)} = \frac{1}{360}(I + 1)(I + 2)(I + 3)^2(I + 4)(I + 5),
\]

\[
\sum_{s=-\frac{l}{2}}^{\frac{l}{2}} d[0, \frac{I}{2}, s]_{SO(8)} = \frac{1}{302400}(I + 1)(I + 2)(I + 3)^2(I + 4)^2(I + 5)^2(I + 6)(I + 7).
\] (58)

Interestingly, these quantities correspond to the winding numbers from \( S^{2k-1} \) to \( SO(2k) \) gauge group (rather than the original \( SO(2k-1) \) monopole gauge group):

\[
\pi_{2k-1}(SO(2k)) \simeq \mathbb{Z},
\] (59)

which are represented as

\[
\nu_k = \frac{1}{(2k-1)!(2i)^{k-1}A(S^{2k-1})} \int_{S^{2k-1}} \text{tr}(-ig^\dagger dg)^{2k-1} = (-i)^{k-1} \frac{1}{(2\pi)^k} \frac{(k-1)!}{(2k-1)!} \int_{S^{2k-1}} \text{tr}(-ig^\dagger dg)^{2k-1}.
\] (60)

Here \( A(S^{2k-1}) = \frac{2\pi^k}{(k-1)!} \) is the area of \( S^{2k-1} \), and \( g \) denotes the \( SO(2k) \) group element. When \( g \) is the \( SO(2k) \) element of the fully symmetric representation \([0, \frac{I}{2}, \frac{I}{2}]_{SO(2k)}\), the winding number is explicitly evaluated as (see Appendix B)

\[
\nu_k(I) = \frac{1}{2k-1} \frac{(I + 2k - 4)!!}{(2k - 3)!!I!!} \sum_{s=-\frac{l}{2}}^{\frac{l}{2}} (2s)^2 d[0, \frac{I}{2}, s]_{SO(2k-2)} \quad (k \geq 2),
\] (61)
and then\textsuperscript{12}

\[ \nu_1(I) = \frac{1}{2\pi} \int_{S^1} \text{tr}(-ig^\dagger dg) = I, \]

\[ \nu_2(I) = -i \frac{1}{24\pi^2} \int_{S^3} \text{tr}(-ig^\dagger dg)^3 = \frac{1}{6} I(I+1)(I+2), \]

\[ \nu_3(I) = -\frac{1}{480\pi^3} \int_{S^5} \text{tr}(-ig^\dagger dg)^5 = \frac{1}{360} I(I+1)(I+2)^2(I+3)(I+4), \]

\[ \nu_4(I) = i \frac{1}{13440\pi^4} \int_{S^7} \text{tr}(-ig^\dagger dg)^7 = \frac{1}{302400} I(I+1)(I+2)^2(I+3)^2(I+4)^2(I+5)(I+6), \] \hspace{1cm} (62)

where we used Eq. (55) to evaluate \[ \sum_{s=-\frac{l}{2}}^{\frac{l}{2}} (2s)^2 d[0, \frac{l}{2}, s]_{SO(2k-2)}. \] Comparison between Eq. (58) and (62) manifests the exact equality between the total number of the \((2k-1)D\) lowest Landau level states for \(I\) and the \(SO(2k)\) winding number for \(I+1\):

\[ \sum_{s=-\frac{l}{2}}^{\frac{l}{2}} d[0, \frac{l}{2}, s]_{SO(2k)} = \nu_k(I+1). \] \hspace{1cm} (63)

The origins of the \(SO(2k)\) gauge group and the relation (63) will be unveiled in the following sections both from non-commutative geometry and differential topology point of view.

### 4 Non-Commutative Geometry

We now discuss non-commutative geometry associated with the odd dimensional Landau model. Quantization of the chiral Hopf maps naturally realizes the geometry of odd dimensional fuzzy spheres.

#### 4.1 Quantum Nambu geometry

In the lowest Landau level the complex conjugate is replaced with derivative,

\[ \Psi^* \to \frac{\partial}{\partial \Psi}, \] \hspace{1cm} (64)

and the coordinates on \(S^{2k-1}\) introduced by the chiral Hopf map (14) are transformed to the operators

\[ X_\mu = \alpha \Psi^I \Gamma_\mu \frac{\partial}{\partial \Psi}, \] \hspace{1cm} (65)

or

\[ X_\mu = i\alpha \Psi^I L_\mu \frac{\partial}{\partial \Psi_R} - i\alpha \Psi^I R_\mu \frac{\partial}{\partial \Psi_L}, \] \hspace{1cm} (66)

where \(\Gamma_\mu\) are the \(SO(2k)\) gamma matrices \(\{3\}\), and \(\alpha\) denotes the parameter of the non-commutative scale;

\[ \alpha = \frac{2r}{I}. \] \hspace{1cm} (67)

\textsuperscript{12}For \(\nu_1(I)\), we used \(g = (\frac{1}{\sqrt{1-x^2}}(x_2 - ix_1))^I = (-i)^I e^{i\phi}.\)
(We recovered the radius of sphere $r$.) The commutation relation of $X_{\mu}$ yields

$$[X_{\mu}, X_{\nu}] = 4i\alpha (X_{\mu}^{L} + X_{\mu}^{R}),$$

(72)

where

$$X_{\mu}^{L} \equiv \alpha \Psi_{L} \frac{\partial}{\partial \Psi_{L}} \frac{\partial}{\partial \Psi_{L}} \frac{\partial}{\partial \Psi_{R}} \frac{\partial}{\partial \Psi_{R}}.$$

(73)

Notice that only among $X_{\mu}$, their commutators do not close but introduce the new operators, $X_{\mu}^{L}$ and $X_{\mu}^{R}$, that respectively satisfy the independent $SO(2k)$ algebra:

$$[X_{\mu}^{L}, X_{\mu}^{L}] = i\alpha \delta_{\mu\rho} X_{\mu\rho}^{L} - i\alpha \delta_{\mu\sigma} X_{\mu\sigma}^{L} + i\alpha \delta_{\nu\rho} X_{\nu\rho}^{L} - i\alpha \delta_{\nu\sigma} X_{\nu\sigma}^{L},$$

$$[X_{\mu}^{R}, X_{\mu}^{R}] = i\alpha \delta_{\mu\rho} X_{\mu\rho}^{R} - i\alpha \delta_{\mu\sigma} X_{\nu\rho}^{R} + i\alpha \delta_{\nu\rho} X_{\nu\rho}^{R} - i\alpha \delta_{\nu\sigma} X_{\nu\sigma}^{R},$$

$$[X_{\mu}^{L}, X_{\mu}^{R}] = 0.$$

(74)

Furthermore, with

$$X_{\mu\nu} \equiv X_{\mu\nu}^{L} + X_{\mu\nu}^{R},$$

(75)

we have

$$[X_{\mu}, X_{\nu}] = 4i\alpha X_{\mu\nu},$$

$$[X_{\mu\nu}, X_{\rho}] = i\delta_{\mu\rho} X_{\nu\nu} - i\delta_{\nu\rho} X_{\mu\mu},$$

$$[X_{\mu\nu}, X_{\rho\sigma}] = i\delta_{\mu\rho} X_{\nu\sigma} - i\delta_{\nu\rho} X_{\mu\sigma} - i\delta_{\nu\sigma} X_{\mu\nu}. $$

(76)

With the identification $X_{\mu,2k+1} = -X_{2k+1,\mu} = \frac{1}{4} X_{\mu}$, Eq. (76) is given by the closed $SO(2k+1)$ algebra of $X_{ab} (a, b = 1, 2, \cdots, 2k + 1)$ determining the non-commutative structure of $S^{2k-1}_{F}$.

Instead of using the usual commutator formalism, we can describe $S^{2k-1}_{F}$ in quantum Nambu formalism where $X_{\mu}$ satisfy a concise algebra. The quantum Nambu bracket or $n$-bracket is defined by

$$[O_{a1}, O_{a2}, \cdots, O_{an}] \equiv O_{a1} O_{a2} \cdots O_{an};$$

(77)

Expanding the $SO(2k)$ generators by the Spin$(2k)$ generators, $t^{A}$ ($A = 1, 2, \cdots, k(2k - 1)$),

$$\sigma_{\mu\nu}^{+} = \sum_{A=1}^{k(2k-1)} \eta^{A} \tau_{A}, \quad \sigma_{\mu\nu}^{-} = \sum_{A=1}^{k(2k-1)} \eta^{A} \tau_{A},$$

(68)

we can express Eq. (72) as

$$[X_{\mu}, X_{\nu}] = 4i\alpha \sum_{A=1}^{k(2k-1)} (\eta_{\mu\nu}^{A} X_{A}^{L} + \eta_{\mu\nu}^{A} X_{A}^{R})$$

(69)

with

$$X_{A}^{L} \equiv \alpha \Psi_{L} \frac{\partial}{\partial \Psi_{L}} \frac{\partial}{\partial \Psi_{R}} \frac{\partial}{\partial \Psi_{R}} \frac{\partial}{\partial \Psi_{R}} \frac{\partial}{\partial \Psi_{R}} \frac{\partial}{\partial \Psi_{R}},$$

(70)

For $SO(4)$, the expansion coefficients $\eta^{A}_{\mu\nu}$ and $\eta^{A}_{\mu\nu}$ are self- and anti-self dual 'tHooft symbols. For $SO(2k)$, $\sigma_{\mu\nu}^{+}$ and $\sigma_{\mu\nu}^{-}$ satisfy the generalized self- and anti-self dual relations respectively:

$$\sigma_{\mu_{1}\mu_{2}}^{\pm} = \pm \frac{k(2k-1)}{(2k-2)!} \varepsilon_{\mu_{1}\mu_{2} \cdots \mu_{2k-1} \mu_{2k}} \sigma_{\mu_{3} \mu_{4}}^{\pm} \cdots \sigma_{\mu_{2k-1} \mu_{2k}}^{\pm}.$$

(71)
where the right-hand side represents the totally antisymmetric combination of the \( n \) quantities. For instance,

\[
\begin{align*}
[O_{a_1}, O_{a_2}] &= O_{a_1} O_{a_2} - O_{a_2} O_{a_1}, \\
[O_{a_1}, O_{a_2}, O_{a_3}] &= O_{a_1} O_{a_2} O_{a_3} - O_{a_1} O_{a_3} O_{a_2} - O_{a_2} O_{a_1} O_{a_3} + O_{a_2} O_{a_3} O_{a_1} + O_{a_3} O_{a_1} O_{a_2} - O_{a_3} O_{a_2} O_{a_1}.
\end{align*}
\]

(78)

Unlike the case of even D fuzzy spheres [8], the quantized coordinates [65] do not satisfy a closed algebra with the usual definition [77] [15]:

\[
[O_{\mu_1}, O_{\mu_2}, \cdots, O_{\mu_{2k-1}}] = O_{[\mu_1, O_{\mu_2}, \cdots, O_{\mu_{2k-1}}]},
\]

(79)

We need to introduce the chiral bracket defined by [36, 37]

\[
[O_{\mu_1}, O_{\mu_2}, \cdots, O_{\mu_{2k-1}}]_\chi \equiv [O_{\mu_1} O_{\mu_2} \cdots O_{\mu_{2k-1}}, \Gamma_{2k+1}],
\]

(80)

where \( \Gamma_{2k+1} \) denotes the chiral matrix of the \( SO(2k) \) group. The use of the chiral bracket is favorable in the sense that it avoids problems inherent to the usual definition of the odd dimensional bracket: As briefly mentioned in Sec[14] there exist difficulties in quantization and fundamental identity in the odd D Nambu bracket [17], while such difficulties do not arise in even dimensions. Since the chiral bracket [60] is essentially \( 2k \)-bracket, it can evade such problems [36, 37]. It is easy to see that the quantized coordinates \( X_\mu \) [65] satisfy a closed algebra with the use of the chiral bracket:

\[
[X_{\mu_1}, X_{\mu_2}, \cdots, X_{\mu_{2k-1}}]_\chi = -i^k C(k, I) \alpha^{2k-2} \epsilon_{\mu_1 \mu_2 \cdots \mu_{2k}} X_{\mu_{2k}}
\]

(81)

where

\[
C(k, I) = \frac{(2k)!!(I + 2k - 2)!!}{2^{2k-1}I!!}.
\]

(82)

Eq. (81) represents the algebraic relation of \( S_F^{2k-1} \) in quantum Nambu formalism, and it can also be derived by “dimensional reduction” from the quantum Nambu algebra of \( S^2_F \) [8]

\[
[X_{a_1}, X_{a_2}, \cdots, X_{a_{2k}}] = i^k C(k, I) \alpha^{2k-1} \epsilon_{a_1 a_2 \cdots a_{2k+1}} X_{a_{2k+1}},
\]

(83)

with identification \( \Gamma_{2k+1} = \frac{1}{\alpha} X_{2k+1} \). In low dimensions, we have

\[
\begin{align*}
S^1_F & \colon [X_{\mu}]_\chi = -i \epsilon_{\mu \nu} X_{\nu}, \\
S^2_F & \colon [X_{\mu_1}, X_{\mu_2}, X_{\mu_3}]_\chi = \alpha^2 (I + 2) \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} X_{\mu_4}, \\
S^3_F & \colon [X_{\mu_1}, X_{\mu_2}, X_{\mu_3}, X_{\mu_4}]_\chi = i^3 \frac{3}{2} \alpha^4 (I + 2)(I + 4) \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} X_{\mu_5}, \\
S^4_F & \colon [X_{\mu_1}, X_{\mu_2}, X_{\mu_3}, X_{\mu_4}, X_{\mu_5}]_\chi = -3 \alpha^6 (I + 2)(I + 4)(I + 6) \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} X_{\mu_5}.
\end{align*}
\]

(84)

Square of radius of \( S^{2k-1}_F \) is derived as [24, 37]

\[
\sum_{\mu=1}^{2k} X_\mu X_\mu = \frac{1}{2} \alpha^2 (I + 1)(I + 2k - 1) \mathbf{1}_{2d[0, 1 \frac{1}{2}]}(SO(2k))^{-1}.
\]

(85)
4.2 Dimensional ladder

In the above, we introduced the chiral Nambu bracket by “embedding” it in one-dimension higher even-bracket. Since the basic philosophy of non-commutative geometry is to realize quantum geometry by associated algebra, the algebraic embedding implies a geometric embedding of fuzzy odd-sphere in one-dimension higher non-commutative space. Here we discuss the geometric structure of arbitrary odd dimensional fuzzy spheres in the context of the Landau models, which naturally generalizes the preceding study of fuzzy three-sphere [37].

Recall that the lowest Landau level Hilbert space of $2kD$ Landau model is spanned by the $SO(2k+1)$ fully symmetric representation $|0, I_2⟩_{SO(2k+1)}$ with dimension [8]

$$d[n = 0, I_2]_{SO(2k+1)} = \prod_{l=1}^{k} \prod_{i=1}^{l} \frac{I + l + i - 1}{l + i - 1},$$

which is equal to the total dimension of the lowest Landau level of the $(2k-1)D$ Landau model [57]:

$$d[0, I_2]_{SO(2k+1)} = \sum_{s=-\frac{I}{2}}^{\frac{I}{2}} d[0, I_2, s]_{SO(2k)}.$$  \hspace{1cm} (87)

This implies a close relation between the $2k$ and $(2k-1)D$ lowest Landau levels. Meanwhile from Eq.(53), we obtain

$$d[0, I_2, \pm I_2]_{SO(2k)} = d[0, I_2]_{SO(2k-1)}$$  \hspace{1cm} (89)

indicating another relationship between $(2k-1)$ and $(2k-2)D$. Eq.(87) and (89) suggest the dimensional ladder of the lowest Landau levels

$$2kD LLL \xrightarrow{\text{Eq.}(87)} (2k-1)D LLL \xrightarrow{\text{Eq.}(89)} (2k-2)D LLL.$$  \hspace{1cm} (90)

We first discuss a geometric interpretation of the relation (87) between $2k$ and $(2k-1)D$. In the $(2k-1)D$ Landau model, the lowest Landau level eigenstates carry the internal chiral parameter $s$, which we will regard as a virtual $x_{2k+1}$-axis. Justification of this postulation is that, for the lowest Landau level eigenstates, $s$ denotes the eigenvalues of the $(2k+1)$th coordinate of $S^{2k}_F$, \hspace{1cm} (91)

$$\frac{1}{2\alpha} X_{2k+1} = \frac{1}{2} \Psi^T \gamma_{2k+1} \frac{\partial}{\partial \Psi} = \frac{1}{2} \Psi^T_L \frac{\partial}{\partial \Psi_L} - \frac{1}{2} \Psi^T_R \frac{\partial}{\partial \Psi_R}.$$

Geometrically, $s$ specifies the position of the hyper-latitude on the virtual $S^{2k}_F$ (Fig. 1), and such hyper-latitude carries $d[0, I_2, s]_{SO(2k)}$ degrees of freedom that represents the sub-band degeneracy of the lowest Landau level of the $(2k-1)D$ Landau model. The sub-band degrees of freedom are stacked as the hyper-latitudes along the $(2k+1)$th axis and in total they constitute the $S^{2k}_F$. Each hyper-latitude corresponds to the lowest Landau level sub-band or the odd dimensional fuzzy sphere $S^{2k-1}_F$. As the magnitude of $s$ increases, $d[0, I_2, s]_{SO(2k)}$ decreases. This coincides with our

\hspace{1cm} \footnote{In particular for the spinor representation, we have

$$d[0, I_2]_{SO(2k)} = 2^{k-1} = d[0, I_2]_{SO(2k-1)}.$$  \hspace{1cm} (88)

We first discuss a geometric interpretation of the relation (87) between $2k$ and $(2k-1)D$. In the $(2k-1)D$ Landau model, the lowest Landau level eigenstates carry the internal chiral parameter $s$, which we will regard as a virtual $x_{2k+1}$-axis. Justification of this postulation is that, for the lowest Landau level eigenstates, $s$ denotes the eigenvalues of the $(2k+1)$th coordinate of $S^{2k}_F$, \hspace{1cm} (91)

$$\frac{1}{2\alpha} X_{2k+1} = \frac{1}{2} \Psi^T \gamma_{2k+1} \frac{\partial}{\partial \Psi} = \frac{1}{2} \Psi^T_L \frac{\partial}{\partial \Psi_L} - \frac{1}{2} \Psi^T_R \frac{\partial}{\partial \Psi_R}.$$

Geometrically, $s$ specifies the position of the hyper-latitude on the virtual $S^{2k}_F$ (Fig. 1), and such hyper-latitude carries $d[0, I_2, s]_{SO(2k)}$ degrees of freedom that represents the sub-band degeneracy of the lowest Landau level of the $(2k-1)D$ Landau model. The sub-band degrees of freedom are stacked as the hyper-latitudes along the $(2k+1)$th axis and in total they constitute the $S^{2k}_F$. Each hyper-latitude corresponds to the lowest Landau level sub-band or the odd dimensional fuzzy sphere $S^{2k-1}_F$. As the magnitude of $s$ increases, $d[0, I_2, s]_{SO(2k)}$ decreases. This coincides with our
intuition that the hyper-latitude $S^{2k-1}_F$ farther from the equator becomes smaller to accommodate fewer degrees of freedom. The number of degrees of freedom on each hyper-latitude is given by

$$d[0, \frac{I}{2}, s]_{SO(2k)} \sim I^\frac{1}{2}(k+2)(k-1) \quad (s << I),$$

and the number of the latitudes is $\sim I$. Consequently, the total number of degrees of freedom of $S^{2k}_F$ is roughly evaluated as

$$I^\frac{1}{2}(k+2)(k-1) \times I = I^\frac{1}{2}k(k+1),$$

which essentially reproduces Eq.(93) as

$$\sum_{s=-\frac{I}{2}}^{\frac{I}{2}} d[0, \frac{I}{2}, s]_{SO(2k)} \sim I^\frac{1}{2}(k+2)(k-1) \times I = I^\frac{1}{2}k(k+1) \sim d[0, \frac{I}{2}]_{SO(2k+1)}.$$  \hfill (95)

Thus the dimensional ladder between the $2k$ and $(2k-1)$D Landau models [37] can be understood in the context of non-commutative geometry. Interestingly, we have not a priori assumed the embedding of the odd dimensional Landau model in one-dimension higher space, but as the analysis proceeds, the embedding structure has naturally emerged. The $SO(2k)$ gauge group in the winding number (59) now finds its origin too: Since $SO(2k)$ is the gauge group of the $2k$D Landau model [8], its appearance is accounted for by the embedding of the $(2k-1)$D Landau model in the $2k$D Landau model.

15In a precise sense [37], $(2k-1)$D fuzzy sphere is realized as a superposition of two latitudes, $s = 1/2$ and $s = -1/2$, nearest to the equator of fuzzy $S^{2k}$ for odd $I$ (the red circles in the right of Fig. 1).

16Considering that the number of the hyper-latitudes on $S^{2k}_F$ is $I + 1 = \deg(S^2_F)$ and $S^{2k-1}_F$ corresponds to each hyper-latitude, we can restate Eq. (93) as

$$\deg(S^{2k-1}_F) \sim \deg(S^{2k}_F)/\deg(S^2_F).$$  \hfill (94)
A simple geometric picture also explains the relation between \((2k - 1)\) and \((2k - 2)\)D. Even dimensional fuzzy sphere \(S^2_F\) is locally equivalent to \(S^{2k-2}_F\) fibration over \(S^{2k}\):

\[
S^2_F \sim S^{2k} \otimes S^{2k-2}_F,
\]

and then the number of the degrees of freedom of the latitude around the north-pole (or the south-pole), \(d[0, \frac{I}{2}, \pm \frac{I}{2}]\), should coincide with that of \(S^{2k-2}_F\), \(d[0, \frac{I}{2}]\) of \(SO(2k-2)\). Eq. (89) reflects such geometry (see Fig. 1).

We thus have seen the following dimensional ladder: From the point of view of the group theory,

\[
SO(2k + 1) \rightarrow SO(2k) \rightarrow SO(2k - 1),
\]

from the non-commutative geometry,

\[
S^{2k}_F \rightarrow S^{2k-1}_F \rightarrow S^{2k-2}_F,
\]

and from the lowest Landau level,

\[
2k\text{D LLL} \rightarrow (2k - 1)\text{D LLL} \rightarrow (2k - 2)\text{D LLL}.
\]

In particular for \(k = 2\), we have

\[
SO(5) \rightarrow SO(4) \simeq SU(2)_L \otimes SU(2)_R \rightarrow SU(2)_{L/R},
\]

\[
S^4_F \rightarrow S^3_F \rightarrow S^2_F,
\]

\[
4\text{D LLL} \rightarrow 3\text{D LLL} \rightarrow 2\text{D LLL}.
\]

Eq. (100) may remind the readers of the dimensional ladder of anomaly associated with relativistic fermion on \(S^4\) in \(SU(2)\) instanton background [51, 52]. In these literatures, instantons with gauge group of the fundamental representation \((I = 1)\) and non-trivial Potryagin index are adopted, while in the present work we deal with gauge bundle with unit Pontryagin index and gauge group of higher representation \((I \geq 1)\). We will revisit the dimensional ladder in Sec. 6 but before that, we exploit the relativistic Landau models on odd dimensional spheres.

5 Relativistic Landau Models on \(S^{2k-1}\)

5.1 Dirac-Landau models

Here, we consider Dirac-Landau models on \(S^{2k-1}\). The \(SO(2k)\) gamma matrices are taken as

\[
\Gamma^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix} \quad (i = 1, 2, \ldots, 2k - 1),
\]

\[
\Gamma^{2k} = \begin{pmatrix} 0 & 1_{2^{k-1}} \\ 1_{2^{k-1}} & 0 \end{pmatrix},
\]

\[
\text{Eq. (101)}
\]

The instanton Chern number is generally given by the product of the Pontryagin index \(N\) and the gauge group representation part [39]:

\[
e_{2} = N \cdot \frac{1}{6} I(I + 1)(I + 2).
\]

In Refs. [51, 52] \(I = 1\) and \(N \geq 1\), while in the present paper \(N = 1\) and \(I \geq 1\).
where $\gamma^i$ denote $SO(2k - 1)$ gamma matrices. The chirality matrix is given by

$$\Gamma^{2k+1} = \begin{pmatrix} 0 & i1_{2k-1} \\ -i1_{2k-1} & 0 \end{pmatrix}. \quad (103)$$

The Dirac-Landau operator on $S^{2k-1}$ is constructed as

$$-i\mathcal{D} = -i \sum_{i=1}^{2k-1} \gamma^i e_i \alpha D_\alpha = \begin{pmatrix} -i\mathcal{D}_W & 0 \\ 0 & i\mathcal{D}_W \end{pmatrix}, \quad (104)$$

where $-i\mathcal{D}_W$ denotes the Weyl-Landau operator \[101\], and the massive Dirac-Landau Hamiltonian is given by

$$\mathcal{H} = -i\mathcal{D} + M \Gamma^{2k} = \begin{pmatrix} -i\mathcal{D}_W & M \\ M & i\mathcal{D}_W \end{pmatrix}. \quad (105)$$

$\mathcal{H}$ is a Dirac-Landau Hamiltonian of the AIII class topological insulator that respects the chiral symmetry,

$$\{\mathcal{H}, \Gamma^{2k+1}\} = 0, \quad (106)$$

and its spectrum is symmetric with respect to the zero-energy:

$$\pm \Lambda = \pm \sqrt{\lambda^2 + M^2}, \quad (107)$$

where $\lambda^2$ denote the eigenvalue of $(-i\mathcal{D})^2$. From the results of Sec. 5.2 one can readily obtain the eigenvalues and the corresponding degeneracies of the Dirac-Landau operator. The eigenvalues are the same as of the Weyl-Landau operator \[117\]:

$$(-i\mathcal{D})^2 = \lambda_n^+(s)^2, \quad \lambda_n^-(s)^2. \quad (108)$$

For each of $\lambda_n^\pm(s)^2$, the degeneracy is equally given by

$$s^2 > 0 : \quad d[n, I/2 \pm 1/2, s]_{SO(2k)},$$

$$s^2 = 0 : \quad d[n, I/2 \pm 1/2, 0]_{SO(2k)}. \quad (109)$$

For the relativistic Landau models on $S^1$, we give a detailed analysis in Appendix A.

### 5.2 Weyl-Landau models

We analyze the Weyl-Landau operator on $S^{2k-1}$:

$$-i\mathcal{D}_W = -i \sum_{i=1}^{2k-1} \gamma^i e_i \alpha D_\alpha = \sum_{i=1}^{2k-1} \gamma^i (\partial_\alpha + \omega_\alpha + \mathcal{A}_\alpha), \quad (110)$$

where $\omega_\alpha$ represent the $SO(2k - 1)$ spin connection and $\mathcal{A}_\alpha$ ($\alpha = x_1, x_2, \cdots, x_{2k-1}$) denote the $SO(2k-1)$ gauge field on $S^{2k-1}$ that corresponds to $A_\mu$ \[25\] in the Cartesian coordinate. Following
Dolan’s work [10], we investigate the eigenvalue problem of the Weyl-Landau operator (110). On the coset \( M \cong G/H \), the square of the Dirac (Weyl) operator is given by

\[
(-i\mathbb{D})^2 = C(G) - C(H) + \frac{1}{8} \mathcal{R}_\mathcal{M},
\]

where \( C \) denotes quadratic Casimir of the corresponding group, and \( \mathcal{R}_\mathcal{M} \) is the Ricci scalar of \( \mathcal{M} \). For the present case \( S^{2k-1} \simeq SO(2k)/SO(2k-1) \), Eq. (111) becomes

\[
(-i\mathbb{D}_W)^2 = C_{SO(2k)} - C_{SO(2k-1)} + \frac{1}{8} \mathcal{R}_{S^{2k-1}}
\]

where

\[
C[n, J, s]_{SO(2k)} = J^2(k-1) + J(2n + k^2 - k) + n(n + 2k - 2) + s^2,
\]

\[
C[0, \frac{I}{2}]_{SO(2k-1)} = \frac{I^2}{4}(k-1) + \frac{I}{2}(k-1)^2,
\]

\[
\mathcal{R}_{S^{2k-1}} = d(d-1)|_{d=2k-1} = 2(k-1)(2k-1).
\]

By substituting these quantities to the formula (111) with

\[
J = \frac{I}{2} \pm \frac{1}{2},
\]

we have

\[
\lambda_n(s)^2 = \lambda_n(-s)^2 = \begin{pmatrix}
(n+k)(n+k-1) + I(n+k-1) + s^2 & 0 \\
0 & n(n+2k-3) + In + s^2
\end{pmatrix}
\]

\[
\equiv \begin{pmatrix}
\lambda_n^+(s)^2 & 0 \\
0 & \lambda_n^-(s)^2
\end{pmatrix}.
\]

\[\text{For the fundamental representation} \ (I = 1), \text{the } SO(2k-1) \text{ Casimir is given by}
\]

\[
\sum_{i<j=1}^{2k-1} \sigma_{ij}^2 = \frac{1}{4}(k-1)(2k-1) = \frac{1}{8} \mathcal{R}_{S^{2k-1}}.
\]

\[\text{In the thermodynamic limit } I, R \to \infty \text{ with fixed } s \text{ and } \omega \equiv I/R^2, \text{ Eq. (111) is reduced to}
\]

\[
\lambda_n^+(s)^2 \simeq I(n+k-1), \quad \lambda_n^-(s)^2 \simeq In.
\]
For instance

\[
SO(4) \ (k = 2) : \begin{pmatrix}
\lambda_0^+(s)^2 & 0 \\
0 & \lambda_0^-(s)^2
\end{pmatrix} = \begin{pmatrix}
(n + 2)(n + 1) + I(n + 1) + s^2 & 0 \\
0 & n(n + 1) + In + s^2
\end{pmatrix},
\]

\[
SO(6) \ (k = 3) : \begin{pmatrix}
\lambda_n^+(s)^2 & 0 \\
0 & \lambda_n^-(s)^2
\end{pmatrix} = \begin{pmatrix}
(n + 3)(n + 2) + I(n + 2) + s^2 & 0 \\
0 & n(n + 3) + In + s^2
\end{pmatrix},
\]

\[
SO(8) \ (k = 4) : \begin{pmatrix}
\lambda_n^+(s)^2 & 0 \\
0 & \lambda_n^-(s)^2
\end{pmatrix} = \begin{pmatrix}
(n + 4)(n + 3) + I(n + 3) + s^2 & 0 \\
0 & n(n + 5) + In + s^2
\end{pmatrix}.
\]

(121)

For the eigenvalues \(\lambda_n^\pm(s)^2\), the corresponding degeneracies are given by

\[
2(1 - \delta_{s,0}) \ d[n, \frac{I}{2} \pm \frac{1}{2}, s]_{SO(2k)} + \delta_{s,0} \ d[n, \frac{I}{2} \pm \frac{1}{2}, 0]_{SO(2k)},
\]

(122)

with \(d_{SO(2k)}\). The coefficient 2 of the first term comes from the contribution of \(+s\) and \(-s\).

Depending on the parity of \(I\), the system exhibits different behaviors. For odd \(I\), \(s\) can take \(s = 0\), and the representations, \([n, \frac{I}{2} + \frac{1}{2}, 0]_{SO(2k)}\) and \([n, \frac{I}{2} - \frac{1}{2}, 0]_{SO(2k)}\), respectively yield the eigenvalues

\[
\lambda^+_n(0)^2 = (n + k + 1)(n + k - 1),
\]

\[
\lambda^-_n(0)^2 = n(n + 2k - 3 + I).
\]

(123)

Thus zero-modes are realized as \(\lambda_n^\pm(0) = 0\) for the representation \([0, \frac{I}{2} - \frac{1}{2}, 0]_{SO(2k)}\). Meanwhile for even \(I\), the representations, \([n, \frac{I}{2} + \frac{1}{2}, \pm\frac{1}{2}]_{SO(2k)}\) and \([n, \frac{I}{2} - \frac{1}{2}, \pm\frac{1}{2}]_{SO(2k)}\), respectively yield the eigenvalues

\[
\lambda_n^+(s = \pm 1/2)^2 = (n + k - \frac{1}{2})^2 + I(n + k - 1),
\]

\[
\lambda_n^-(s = \pm 1/2)^2 = n(n + 2k - 3 + I) + \frac{1}{4}.
\]

(124)

The corresponding degeneracies are summarized in Appendix C. Notice that the minimum value is not zero, but \(\lambda_{n=0}^-(s = \pm 1/2)^2 = \frac{1}{4}\).

\[\text{For } SO(4), \text{ the eigenvalues } \lambda_n^\pm(s)^2 \text{ of } (-i\partial_t)^2 \text{ satisfy the special relation:}
\]

\[
\lambda_n^+(s)^2 = \lambda_{n+1}^-(s)^2 = (n + 2)(n + 1) + I(n + 1) + s^2 \quad (n = 0, 1, 2, \cdots),
\]

\[
\lambda_0^-(s)^2 = s^2,
\]

(118)

with the degeneracies

\[
d[n, \frac{I}{2} + \frac{1}{2}, s]_{SO(4)} = \frac{1}{4}(2n + I + 3)^2 - s^2 \quad (n = 0, 1, 2, \cdots), \quad d[n, \frac{I}{2} - \frac{1}{2}, s]_{SO(4)} = \frac{1}{4}(I + 1)^2 - s^2.
\]

(119)

These results coincide with Eq.(18) of Nair and Daemi \(\text{[14]}\) by the replacement:

\[
(J, q, \mu, T, S) \rightarrow (\frac{I}{2} \pm \frac{1}{2}, n, \frac{I}{2} \pm \frac{1}{2}, \frac{\lambda}{2}, I, \frac{1}{2}).
\]

(120)
In the limit $I \to 0$, the former representation $[n, \frac{1}{2} + \frac{1}{2}, \pm \frac{1}{2}]_{\text{SO}(2k)}|_{I=0} = [n, \frac{1}{2}, \pm \frac{1}{2}]_{\text{SO}(2k)}$ gives the spectrum of the free spinor particle on $S^{2k-1}$

$$\pm \lambda_n^+(\pm 1/2) \xrightarrow{I \to 0} -i \nabla = \pm (n + k - \frac{1}{2}),$$

(126)

while the latter representation $[n, \frac{1}{2} - \frac{1}{2}, \pm \frac{1}{2}]_{\text{SO}(2k)}$ is not well defined, and then $\pm \lambda_n^-(s = \pm 1/2)$ do not have their counterparts in the spectrum of the free Dirac operator.

### 5.3 Total degeneracy of 0th Landau level revisited

For $n = 0$, $\lambda_n^-(s)^2$ takes a very simple form:

$$\lambda_{n=0}^-(s)^2 = s^2,$$

(127)

with degeneracy

$$2(1 - \delta_{s,0}) \left[ d[0, \frac{I}{2} - \frac{1}{2}, s]_{\text{SO}(2k)} + \delta_{s,0} \left[ d[0, \frac{I}{2} - \frac{1}{2}, 0]_{\text{SO}(2k)} \right] \right] = \sum_{s = -\frac{I}{4} + \frac{1}{2}}^{\frac{I}{4} + \frac{1}{2}} \left[ d[0, \frac{I}{2} - \frac{1}{2}, s]_{\text{SO}(2k)} \right].$$

(128)

Then we have

$$\text{total dimension of the 0th Landau level}$$

$$= 2 \sum_{s > 0} d[0, \frac{I}{2} - \frac{1}{2}, s]_{\text{SO}(2k)} + d[0, \frac{I}{2} - \frac{1}{2}, 0]_{\text{SO}(2k)} = \sum_{s = -\frac{I}{4} + \frac{1}{2}}^{\frac{I}{4} + \frac{1}{2}} d[0, \frac{I}{2} - \frac{1}{2}, s]_{\text{SO}(2k)}$$

$$= \nu_k(I),$$

(130)

where Eq. (123) was used in the last equation. Zero-mode contribution ( ) on the second equation exists when $I$ is odd. We will explain a differential topology origin behind this relation. First notice, since $d[0, \frac{I}{2} - \frac{1}{2}]_{\text{SO}(2k+1)}$ denotes the 0-mode degeneracy of the 2kD relativistic Landau model, i.e., the index of the Dirac-Landau operator on $S^{2k}$, the dimensional ladder between 2k and $(2k - 1)D$ can be rewritten as

$$\sum_{s = -\frac{I}{4} + \frac{1}{2}}^{\frac{I}{4} + \frac{1}{2}} d[0, \frac{I}{2} - \frac{1}{2}, s]_{\text{SO}(2k)} = \text{Ind}[\mathcal{D}_{\text{SO}(2k+1)}].$$

(131)

\footnote{For each of the eigenvalues $[126]$, $(n + k - \frac{1}{2})$ and $-(n + k - \frac{1}{2})$, the corresponding degeneracy is equally given by

$$d[n, 1/2, 1/2]_{\text{SO}(2k)} = 2^{k-1} \binom{n + 2k - 2}{n}$$

(125)

which reproduces the known results of the free Dirac operator on $S^{2k-1}$ (see Ref. [54] for instance).}
Second, we find that the Atiyah-Singer index theorem tells us

\[
\text{Ind}[-i\mathcal{D}_{SO(2k+1)}] = c_k(I),
\]

where \(c_k\) denotes the \(k\)th Chern number of the \(SO(2k)\) monopole charge. For the \(2k\)D Landau models, Eq. (135) can be verified explicitly [8]. Third, from the fibre-bundle theory, the non-trivial gauge topology on \(2k\)-sphere is evaluated by the winding number (60) from the transition function on the equator \(S^{2k-1}\):

\[
c_k(I) = \nu_k(I).
\]

See Appendix D about a general proof of Eq. (136). In the present case, we can explicitly verify Eq. (136) as follows. Using the formula of \(d[0, I, s]_{SO(2k-2)}\) (53), we perform the summation of the winding number (61) to have

\[
\nu_k(I) = \frac{1}{2^{k-1}} \frac{(I + 2k - 4)!}{(2k - 3)!} \sum_{s=-\frac{2}{k}} \frac{(2s)^2 d[0, I, s]_{SO(2k-2)}}{l \prod_{i=1}^l \frac{I + l + i - 2}{l + i - 2}} = d[0, I - \frac{1}{2}]_{SO(2k+1)},
\]

where \(d[0, I, s]_{SO(2k+1)}\) is defined by Eq. (56). In Ref. [3], we showed \(d[0, I - \frac{1}{2}]_{SO(2k+1)} = c_k(I)\), and then

\[
c_k(I) = d[0, I - \frac{1}{2}]_{SO(2k+1)} = \nu_k(I).
\]

Finally, from Eq. (131), (135) and (136), we obtain the sequence

\[
\text{total dim. of odd D LLL} \quad \Rightarrow \quad \text{dim. ladder} \quad \Rightarrow \quad \text{Ind}[-i\mathcal{D}]_{\text{even D}} \quad \Rightarrow \quad \text{AS index theorem} \quad \Rightarrow \quad c_fibre-bundle theory \quad \Rightarrow \quad \nu,
\]

which verifies Eq. (130) after all. However, there is one speck in the above discussions. As for the dimensional ladder (131), we utilized specific properties of the Landau models, while the other processes of the sequence depend on general arguments of differential topology, such as Atiyah-Singer index theorem and fibre-bundle theory. We will reconsider the dimensional ladder and show that the dimensional ladder can also be understood from a differential topology point of view. In Sec 6, the total dimension of the 0th Landau level is shown to be equal to the spectral flow:

\[
\sum_s d[0, I, s]_{SO(2k)} = -\text{spectral flow}.
\]

\[\]On general spin manifold without boundary, the Atiyah-Singer theorem is given by

\[
\text{Ind}(-i\mathcal{D}) = \int_M \hat{A}(R) \cdot \text{ch}(F),
\]

where \(\text{ch}(F)\) denotes the Chern number density and \(\hat{A}(R)\) represents the Dirac-genus given by the polynomials of the Pontryagin class:

\[
\hat{A}(R) = 1 - \frac{1}{24} p_1(R) + \frac{1}{5760} (7 p_1(R)^2 - 4 p_2(R)) + \cdots.
\]

For spheres, the Pontryagin class is identically zero, and so the Dirac-genus becomes trivial. Consequently for \(S^{2k}\), the index of the Dirac operator is determined solely by the gauge bundle topology:

\[
\text{Ind}(-i\mathcal{D}) = \int_{S^{2k}} \text{ch}(F) = c_k.
\]
With Eq.(140) and the spectral flow argument of Atiyah-Patodi-Singer \[47, 48, 49, 46\]

\[-\text{spectral flow} = \text{Ind}(-iD), \quad (141)\]

every process of the sequence (139) can be restated in the language of differential topology:

\[
\text{total dim. of odd D LLL} \xrightarrow{\text{APS spectral flow}} \text{Ind}(-iD)_{\text{even D}} \xrightarrow{\text{AS index theorem}} c \xrightarrow{\text{fibre-bundle theory}} \nu. \quad (142)
\]

6 Ladder of Dimensions: Spectral Flow of Atiyah-Patodi-Singer

Let us discuss relations between the spectral flow argument of Atiyah-Patodi-Singer and the dimensional ladder between the $2k$ and $(2k-1)$D Landau models:

\[
d[0, \frac{I}{2}]_{SO(2k+1)} = \sum_{s=-\frac{1}{2}}^{\frac{1}{2}} d[0, \frac{I}{2}, s]_{SO(2k)}. \quad (143)
\]

We demonstrate how Eq.(143) works in 1D and 2D relativistic Landau models and clarify relations to the spectral flow argument. For this purpose, we apply the Atiyah’s discussion \[55\] originally developed on torus $S^1 \otimes S^1$ is applied to fuzzy two-sphere\[23\]. In this section, we use the terminologies “lowest Landau level” and “0th Landau level” interchangeably.

6.1 2D and 1D relativistic Landau models

The massive Dirac-Landau operator on two-sphere is given by

\[
-\sigma_x(\partial_{\theta} - \frac{1}{2} \tan \frac{\theta}{2}) - i\sigma_y(\partial_{\phi} - iA_{\phi}) + M\sigma_z, \quad (144)
\]

where $\theta$ and $\phi$ represent the standard polar coordinates, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$, and $A_{\phi}$ denotes the monopole gauge field

\[
A = A_{\phi}d\phi = \frac{I}{2}(1 - \cos \theta)d\phi. \quad (145)
\]

We omit the spin connection part to deduce a massive 1D Dirac-Landau operator from Eq.(141):

\[
-\mathcal{D}_{S^1} + M\sigma_z = -i\sigma_y(\partial_{\phi} - iA_{\phi}) + M\sigma_z = \begin{pmatrix} M & -\partial_{\phi} + iA_{\phi} \\ \partial_{\phi} - iA_{\phi} & -M \end{pmatrix}
\]

that respects the chiral symmetry:

\[
\{\sigma_x, -i\mathcal{D}_{S^1} + M\sigma_z\} = 0. \quad (146)
\]

\[23\] Also see Refs.\[56, 57\] for relations to quantum anomaly.
\(-i\mathcal{D}_S + M\sigma_z\) denotes a 1D chiral topological insulator Hamiltonian. The 1D Dirac-Landau operator \(-i\mathcal{D}_S\)

\[-i\mathcal{D}_S = -i\sigma_y D_{S^1} = \begin{pmatrix} 0 & -D_{S^1} \\ D_{S^1} & 0 \end{pmatrix}\]  

(147)

consists of two copies of the 1D Weyl-Landau operator

\[-iD_{S^1} = -i\partial_\phi - A^{(\theta)}_\phi,\]  

(148)

with

\[A^{(\theta)}_\phi = \frac{I}{2}(1 - \cos \theta).\]  

(149)

Notice that in the 1D Weyl-Landau operator (148), \(\theta\) is no longer a coordinate but denotes an external parameter to tune the magnitude of the gauge field. While magnetic field is constant on the surface of the two-sphere, Eq. (148) describes the 1D Weyl operator with the magnetic flux at the center of \(S^1\) (Fig. 2). The magnitude of the magnetic flux is given by

\[\Phi^{(\theta)} = \oint d\phi A^{(\theta)}_\phi = 2\pi A^{(\theta)}_\phi = \pi I(1 - \cos \theta),\]  

(150)

which monotonically increases from 0 to \(2\pi I\) as the parameter \(\theta\) changes from \(\theta = 0\) to \(\pi\).

\[\textbf{Figure 2: 1D Landau models as latitudes of the 2D Landau model.}\]

### 6.1.1 Atiyah-Singer index theorem

There are not a few literatures about the non-relativistic/relativistic Landau problem on \(S^2\) (see Ref. [58] for instance). In the non-relativistic case, the lowest Landau level degeneracy is given by

\[d_{2D}^{\text{LLL}}(I, \frac{I}{2}) = d[0, \frac{I}{2}]_{SO(3)} = I + 1,\]  

(151)

\[\text{Without the gauge field, Eq. (146) is reduced to the Jackiw-Rebbi Hamiltonian used for the low energy description of polyacetylene.}^\text{[59]}\]
and the lowest Landau level basis states are the eigenstates of $X_z = \frac{2}{I} L_z$ with the eigenvalues

$$X_z = 1 - \frac{2}{I} m \quad (m = 0, 1, 2, \cdots, I).$$

(152)

In other words, the lowest Landau level basis states correspond to the latitudes with the azimuthal angles $\theta_m$:

$$\cos \theta_m = 1 - \frac{2}{I} m.$$  

(153)

The zero-mode degeneracy of the Dirac-Landau operator is obtained from the lowest Landau level degeneracy (151) with the replacement $\frac{I}{2} \rightarrow \frac{I}{2} - \frac{1}{2}$, and the index of the Dirac-Landau operator is given by

$$\text{Ind}(-iD_{S^2}) = d_{\text{LLL}}(\frac{I}{2} - \frac{1}{2}) = I.$$  

(154)

Meanwhile, the 1st Chern number of the monopole bundle is obtained as

$$c_1(g) = \frac{1}{2\pi} \int_{S^2} F = I.$$  

(155)

Their equality manifests the Atiyah-Singer index theorem:

$$\text{Ind}(-iD_{S^2}) = c_1.$$  

(156)

6.1.2 Spectral flow argument

Next let us consider the 1D Weyl-Landau model. With the magnetic flux $\Phi(\theta)$ (150), the 1D Weyl-Landau operator is represented as

$$-iD_{S^1} = -i(\partial_{\phi} - i\frac{1}{2\pi}\Phi(\theta)) = L_\phi,$$  

(157)

which is simply the covariant derivative in $\phi$-direction or the covariant angular momentum, $L_\phi$. Intuitively, the flux insertion brings the counter-propagating angular momentum along $\phi$ direction (Fig 3). Since $\theta$ is a continuum variable, the magnitude of the flux $\Phi(\theta) = \pi I(1 - \cos \theta)$, is not generally quantized unlike the monopole charge. However, with respect to the latitudes (153), the magnetic flux (150) is quantized as a multiple of unit flux:

$$\Phi(\theta_m) = 2\pi m \quad (m = 0, 1, 2, \cdots, I).$$  

(158)

Figure 3: The flux insertion reduces the particle angular momentum due to the associated counter-propagating gauge field.
Thus by embedding, the magnetic charges of the odd D Landau models become quantized. We solve the eigenvalue problem of the Weyl-Landau operator,
\[-iD_S\psi_n(\phi) = \lambda_n^{(\theta)} \psi_n(\phi).\] (159)

Due to the periodic boundary condition, the eigenfunctions are obtained as
\[\psi_n(\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi} \quad (n = 0, \pm 1, \pm 2, \cdots),\] (160)
with the eigenvalues
\[\lambda_n^{(\theta)} = n - \frac{1}{2\pi} \Phi^{(\theta)} = n - \frac{I}{2}(1 - \cos \theta).\] (161)

The 1D Weyl-Landau model on the latitude\(^{158}\) takes the energy eigenvalues
\[\lambda_n^{(\theta_m)} = n - m.\] (162)

When \(\theta\) changes from 0 to \(\pi\) (or the magnetic flux \(\Phi^{(\theta)}\) increases adiabatically from 0 to \(2\pi I\)), the spectrum evolves from \(n\) to \((n - I)\) according to Eq.(161) (Fig.4). The flux insertion

![Figure 4](image)

Figure 4: Evolving eigenvalues\(^{161}\) during the process from \(\theta = 0\) to \(\theta = \pi\) with respect to \(I/2 = 1/2, 1, 3/2\) from left to right. The eigenvalues cross the zero-energy (the red horizontal line) \(I\) times at the points indicated by the red filled circles, the number of which is the definition of the spectral flow. Since each Landau level reduces its energy by \(I\) during the process, the spectral flow is equal to the descent of energy level (the length of the purple down-arrow).

brings the counter-propagating gauge field during the process, and the particle on \(S^1\) reduces its angular momentum and travels down from the \(n\)th circle to the \((n - I)\)th inner circle. Since the energy transition of 1D Landau model is equal to the number of the points at which the evolving eigenvalues cross the zero-energy horizontal line (Fig.4), we find that \(I\) is equal to (the minus of) the spectral flow\(^{25}\)
\[\text{spectral flow} = I.\] (163)

\(^{25}\)The sign of the spectral flow is taken so that each positive/negative eigenvalue that becomes negative/positive contributes -1/+1 to the spectral flow counting.
Recall that \( I \) was the index of the 2D Dirac-Landau operator (154). Then we have

\[
\text{Ind}(-i\mathcal{D}) = - \text{spectral flow},
\]

which manifests the spectral flow argument of Atiyah-Patodi-Singer.

### 6.2 Dimensional ladder of non-commutative geometry

In the above, we have seen the validity of the spectral flow argument in the context of the low dimensional Landau models, while in the following we will show that the spectral flow argument generally guarantees the dimensional ladder of the Landau models (143). The spectral flow indicates the number of points at which the evolving eigenstates cross the zero-energy. In view of the 1D Landau models, each crossing point corresponds to the lowest Landau level of the Landau model with different magnetic flux. Concretely, the \( m \)th crossing point corresponds to the lowest Landau level of the Weyl-Landau model (157) with flux \( \Phi^{(\theta_m)} = 2\pi m \) (Fig.1). Since the spectral flow indicates the number of the crossing points, the spectral flow is equal to the total number of the lowest Landau level states of \( I \) 1D Weyl-Landau models with \( \Phi^{(\theta_m)} = 2\pi m \) (\( m = 1, 2, \cdots, I \)):

\[
- \text{spectral flow} = \sum_{m=1}^{I} d_{1\text{D LLL}}^{(I)}(m),
\]

where \( d_{1\text{D LLL}}^{(I)}(m) \) denotes the lowest Landau level degeneracy of the Weyl-Landau model with \( \Phi^{(\theta_m)} \), and in the present case, obviously \( d_{1\text{D LLL}}^{(I)}(m) = 1 \). Besides, the sum of the lowest Landau level states of the \( I \) Weyl-Landau models can be given by the sum of the states in different energy levels of a single Weyl-Landau model. Notice that the lowest Landau level of the Weyl-Landau model for \( m \) and \( s = m - \frac{I}{2} - \frac{1}{2} \)th level of the Weyl-Landau model with \( \Phi^{(\theta_{m=I/2-1/2})} = \pi(I-1) \) are on the same evolving eigenvalue curve (see Fig.5):

\[
d_{1\text{D LLL}}^{(I)}(m) = d_{s=m-\frac{I}{2}-\frac{1}{2}}^{1\text{D}}(\frac{I}{2} - \frac{1}{2}),
\]

where \( d_{s}^{1\text{D}}(\frac{I}{2} - \frac{1}{2}) \) denotes the \( s \)th level degeneracy of the Weyl-Landau model with flux \( \Phi/(2\pi) = \frac{I}{2} - \frac{1}{2} \). Hence, the spectral flow (165) can be expressed by the sum of the level degeneracies of the single Weyl-Landau model with \( \Phi/(2\pi) = \frac{I}{2} - \frac{1}{2} \):

\[
- \text{spectral flow} = \sum_{s=-\frac{I}{2}+\frac{1}{2}}^{\frac{I}{2}-\frac{1}{2}} d_{s}^{1\text{D}}(\frac{I}{2} - \frac{1}{2}).
\]

This relation is naturally generalized to higher dimensions as

\[
- \text{spectral flow} = \sum_{s=-\frac{I}{2}+\frac{1}{2}}^{\frac{I}{2}-\frac{1}{2}} d_{s}^{(2k-1)\text{D}}(\frac{I}{2} - \frac{1}{2}) = \sum_{s=-\frac{I}{2}+\frac{1}{2}}^{\frac{I}{2}-\frac{1}{2}} d_{[0, \frac{I}{2} - \frac{1}{2}, s]}^{ \text{SO}(2k)},
\]

which is the sought formula (140) that relates the odd D Landau level degeneracies and the spectral flow. Besides, the spectral flow argument tells

\[
\text{Ind}(-i\mathcal{D}) = - \text{spectral flow},
\]
Figure 5: Evolving eigenvalues of the 1D Weyl-Landau operator for $I = 3$. The intersections (blue circles) between the blue vertical line and the energy curves represent the $s(= -1, 0, 1)$th levels of the Weyl-Landau operator with $\Phi^{(\theta_m - \frac{1}{2}, \frac{1}{2})} = \pi(I - 1)$. The intersections (red filled circles) between the zero-energy horizontal line and the energy curves indicate the lowest Landau levels of the Weyl-Landau models with $\Phi^{(\theta_m = 1, 2, \ldots, I)} = 2\pi, 4\pi, \ldots, 2\pi I$. The blue circle and the red filled one on same eigenvalue curve are related by $s = m - \frac{I}{2} - \frac{1}{2}$.

and we eventually have

$$d[0, \frac{I}{2} - \frac{1}{2}]_{SO(2k+1)} = \text{Ind}(-iD) = -\text{spectral flow} = \sum_{s=-\frac{I}{2} + \frac{1}{2}}^{\frac{I}{2} - \frac{1}{2}} d[0, \frac{I}{2} - \frac{1}{2}, s]_{SO(2k)}. \quad (170)$$

This is the very formula of the dimensional ladder [143]. As discussed above, the spectral flow argument of Atiyah-Patodi-Singer guarantees the validity of the dimensional ladder of the Landau models. In other words, the dimensional ladder of non-commutative geometry can thus find its origin in the differential topology.

We recapitulate the above discussions with the 1D and 2D Landau models. From the two-sphere point of view, $\theta$ is the azimuthal angle, and the change of $\theta$ from 0 to $\pi$ corresponds to a sweep on $S^2$ from north pole to the south pole transversing $I$ latitudes, which is the dimension of the 2D relativistic lowest Landau level, $d^{2D}_{\text{LLL}}(\frac{I}{2} - \frac{1}{2})$. During the sweep, the magnetic flux changes from 0 to $2\pi I$, and then each energy level of the 1D Weyl-Landau model decreases by $I$ that is the spectral flow. The spectral flow is equal to the total dimension of the corresponding 1D Landau levels. We can thus restate Eq. (170) as

$$2\text{D LLL dim.} \Rightarrow \text{sweep on (fuzzy) two-sphere} \Rightarrow \text{spectral flow} \Rightarrow \text{total dim. of 1D LLs.} \quad (171)$$

Fig.6 shows (171) pictorially.
7 BF Effective Field Theory

There arise $(2k - 2)$-branes as fundamental excitations in $2k$D quantum Hall effect, and a $(4k - 1)$D tensor Chern-Simons action is proposed for the description of such membranes [8]:

$$S = -e^{2k-2} \int_{4k-1} C_{2k-1} J_{2k} + \frac{\kappa}{2} \int_{4k-1} C_{2k-1} G_{2k},$$

(172)

where $G_{2k} = dC_{2k-1}$, $J_{2k}$ denotes the $(2k - 2)$-brane current, and $\kappa$ stands for the Chern–Simons coupling constant,

$$\kappa = \frac{e^{2k-2}}{2\pi} \frac{1}{m^k}. \quad (173)$$

$m$ represents an odd integer to specify the filling factor by $\nu_{2k} = \frac{1}{m^k}$. The tensor Chern–Simons action induces the tensor flux attachment to membrane and accounts for non-trivial linking of $(2k - 2)$-brane currents [60]. Let us consider similar field theory formulation for the odd D quantum Hall effect. As the $(4k - 1)$D tensor Chern–Simons action was used for the description of the $2k$D quantum Hall effect, one may be tempted to adopt $(4k - 3)$D tensor Chern-Simons action for the present odd D case, but the $(4k - 3)$D tensor Chern–Simons action identically vanishes: In $(4k - 3)$D, the tensor Chern–Simons field has to be even rank, but the Chern–Simon form of even rank tensor becomes

$$C_{2k-2} \wedge dC_{2k-2} = -dC_{2k-2} \wedge C_{2k-2} = -C_{2k-2} \wedge dC_{2k-2} = 0$$

(174)

on closed manifolds. The use of $(4k - 3)$D tensor Chern-Simons action is thus in vain.

Considering that the odd D quantum Hall effect is embedded in one-dimension higher quantum Hall effect, a natural idea to construct effective field theory for the odd D quantum Hall effect will be to use one-dimension reduction of the tensor Chern–Simons theory for the even D. Such theory is a tensor type BF theory in $(4k - 2)$D that we will pursue in this section. $(4k - 2)$D space–time

---

26 $\nu_{2k}$ has nothing to do with the winding number in Sec. 4.
The original space-time D  

| The emergent space-time D |
|---------------------------|
| 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 | 18 | ... |
| 2  | 6  | 10 | 14 | 18 | 22 | 26 | 30 | 34 | ... |

Table 2: The emergent (4k − 2)D space-time for the (2k − 1)D quantum Hall effect.

is the emergent space–time for (2k − 1)D quantum Hall effect as (4k − 1)D for 2kD quantum Hall effect (Table 2). BF theory is a topological field theory defined in arbitrary dimensions [61, 62, 63] and describes fractional statistics of extended objects. For instance in (3 + 1)D, BF theory is used to describe the linking and fractional statistics [64, 65, 66, 67] between particle and vortex-string coupled to 2-rank antisymmetric tensor field [68]. We will make use of such properties of BF theory in (4k − 2)D space–time.

7.1 BF theory as a reduced Chern-Simons theory

Accompanied with one dimension reduction of the (4k − 1)D Chern-Simons action, the (2k − 1) tensor Chern-Simons field and (2k − 2)-brane current \( J_{2k} \) in Eq. (172) are decomposed as

\[
\begin{align*}
C_{2k-1} &= C_{2k-1} + C_{2k-2} x^{4k-2}, \\
J_{2k} &= J_{2k-1} \wedge x^{4k-2} + J_{2k}.
\end{align*}
\]

(175)

In component representation, Eq. (7.1) is expressed as

\[
\begin{align*}
C_{a1a2\cdots a_{2k-1}} &\rightarrow C_{\mu_1\mu_2\cdots \mu_{2k-1}}, \\
J_{a1a2\cdots a_{2k-1}} &\rightarrow J_{\mu_1\mu_2\cdots \mu_{2k-1}}, \\
J_{\mu_1\mu_2\cdots \mu_{2k-2}} &\rightarrow J_{\mu_1\mu_2\cdots \mu_{2k-2},4k-2}.
\end{align*}
\]

(176)

with \( a_1, a_2, \cdots, a_{2k-1} = 0, 1, 2, \cdots, 4k-2 \) and \( \mu_1, \mu_2, \cdots, \mu_{2k-1} = 0, 1, 2, \cdots, 4k-3 \). Here, \( C_{2k-1} \) and \( C_{2k-2} \) are taken to be independent on the compactified space coordinate \( x^{4k-2} \). With \( S^1 \)-
compactification with radius $R$, we derive the reduced Chern-Simons action in $(4k - 2)D$ as:

$$S_{(4k-2)D} = -2\pi R \cdot \left( e_{2k-2} \int_{4k-2} \left( C_{2k-1} J_{2k-1} + C_{2k-2} J_{2k} \right) - \frac{\kappa}{2} \int_{4k-2} \left( C_{2k-1} G_{2k-1} + C_{2k-2} G_{2k} \right) \right)$$

$$= -2\pi R \cdot \left( e_{2k-2} \int_{4k-2} \left( C_{2k-1} J_{2k-1} + C_{2k-2} J_{2k} \right) - \kappa \int_{4k-2} C_{2k-1} dC_{2k-2} + \text{(surface term)} \right),$$

(181)

where

$$G_{2k-1} = dC_{2k-2}, \quad G_{2k} = dC_{2k-1}. \quad (182)$$

The two tensor gauge fields, $C_{2k-1}$ and $C_{2k-2}$, are coupled to different membranes with dimensions $(2k - 2)$ and $(2k - 3)$, and the membrane currents are given by $J_{2k-1}$ and $J_{2k}$. Thus there exist two kinds of membranes in odd dimensional quantum Hall effect. However, since both $(2k - 2)$ and $(2k - 3)$-branes originate from $(2k - 2)$-brane in one-dimension higher space, their coupling constants to the Chern-Simons fields are equally given by $(2\pi R)e_{2k-2}$. From Eq.(181), the equations of motion are derived as

$$J_{2k-1} = \frac{\kappa}{e_{2k-2}} G_{2k-1}, \quad J_{2k} = -\frac{\kappa}{e_{2k-2}} G_{2k}. \quad (183)$$

As component equations, Eq.(183) becomes

$$j_{i_1 \cdots i_{2k-3} 0} = \frac{\kappa}{e_{2k-2}} B_{i_1 \cdots i_{2k-3}}, \quad j_{i_1 \cdots i_{2k-2} 0} = \frac{\kappa}{e_{2k-2}} B_{i_1 \cdots i_{2k-2}}, \quad (184)$$

In component representation, the decomposition (7.1) is given as follows. With

$$C_{2k-1} = \frac{1}{(2k-1)!} C_{\alpha_{1} \alpha_{2} \cdots \alpha_{2k-1}} dx^{\alpha_{1}} dx^{\alpha_{2}} \cdots dx^{\alpha_{2k-1}},$$

$$J_{2k} = \frac{1}{(2k-1)! (2k)!} B_{\alpha_{1} \alpha_{2} \cdots \alpha_{2k-1}} \mu_{1} \mu_{2} \cdots \mu_{2k-1} dx_{\alpha_{1}} dx_{\alpha_{2}} \cdots dx_{\alpha_{2k-1}} dx_{\mu_{1}} dx_{\mu_{2}} \cdots dx_{\mu_{2k-1}}, \quad (177)$$

we have

$$C_{2k-1} = \frac{1}{(2k-1)!} C_{\mu_{1} \mu_{2} \cdots \mu_{2k-1}} dx^{\mu_{1}} dx^{\mu_{2}} \cdots dx^{\mu_{2k-1}},$$

$$C_{2k-2} = \frac{1}{(2k-2)!} C_{\mu_{1} \mu_{2} \cdots \mu_{2k-2} \mu_{2k}} dx^{\mu_{1}} dx^{\mu_{2}} \cdots dx^{\mu_{2k-2}} \cdot dx_{\mu_{2k}}, \quad (178)$$

and

$$J_{2k-1} = \frac{1}{((2k-1)!)^{2}} \epsilon^{\mu_{1} \mu_{2} \cdots \mu_{4k-2}} J_{\mu_{1} \mu_{2} \cdots \mu_{2k-1} \mu_{2k}} dx_{\mu_{1}} dx_{\mu_{2}} \cdots dx_{\mu_{4k-2}}, \quad (179)$$

The $(4k - 2)D$ reduced Chern-Simons action (181) is expressed as

$$S_{(4k-2)D} = \frac{2\pi R}{(2k-1)!} \int d^{4k-2}x \left( -e_{2k-2} C_{\mu_{1} \mu_{2} \cdots \mu_{2k-1}} J^{\mu_{1} \mu_{2} \cdots \mu_{2k-1}} - e_{2k-2} (2k-1) C_{\mu_{1} \mu_{2} \cdots \mu_{2k-2}} J^{\mu_{1} \mu_{2} \cdots \mu_{2k-2}} \right) + \text{(surface-term)} \quad (180)$$
Table 3: Two non-overlapping \((2k-2)\) and \((2k-3)\)-branes in \((4k-2)\)D space-time. From the co-dimension 2, the \((2k-2)\) and \((2k-3)\)-branes are regarded as two point particles.

$$J_{i_1 \cdots i_{2k-1}} = -\frac{1}{(2k-2)!} \frac{\kappa}{e_{2k-2}} \epsilon_{i_1 \cdots i_{4k-3}} E_{i_{2k-1} \cdots i_{4k-3}}^1, \quad J_{i_1 \cdots i_{2k-2}} = -\frac{1}{(2k-1)!} \frac{\kappa}{e_{2k-2}} \epsilon_{i_1 \cdots i_{4k-3}} E_{i_{2k-1} \cdots i_{4k-3}}^2.$$  

(185)

Here, \(B_{i_1 \cdots i_{2k-2}} = -\frac{1}{(2k-1)!} \epsilon_{i_1 \cdots i_{4k-3}} G_{i_{2k-1} \cdots i_{4k-3}}, \quad B_{i_1 \cdots i_{2k-3}} = -\frac{1}{(2k)!} \epsilon_{i_1 \cdots i_{4k-3}} G_{i_{2k-1} \cdots i_{4k-3}}, \quad E_{i_1 \cdots i_{2k-2}} \equiv G_{i_1 \cdots i_{2k-2}0} \) and \(E_{i_1 \cdots i_{2k-2}} \equiv G_{i_1 \cdots i_{2k-1}}\). Eq. (184) signifies a generalized flux attachment to membranes, and similarly Eq. (185) a generalized Hall effect. Either \((2k-2)\)-brane or \((2k-3)\)-brane current does not solely satisfy the Hall effect, \(E_{i_1 \cdots i_{2k-1}} J_{i_1 \cdots i_{2k-1}} \neq 0\), \(E_{i_1 \cdots i_{2k-2}} J_{i_1 \cdots i_{2k-2}} \neq 0\), but their sum does:

$$E_{i_1 \cdots i_{2k-1}} J_{i_1 \cdots i_{2k-1}} + (2k-1) E_{i_1 \cdots i_{2k-2}} J_{i_1 \cdots i_{2k-2}} = 0.$$  

(186)

These properties reflect the fact that the present BF action originates from a single Chern-Simons action in one dimension higher space.

### 7.2 Effective action for membranes and linking number

The present BF theory in \((4k-2)\)D space-time accommodates the \((2k-2)\) and \((2k-3)\)-branes with unoccupied two-dimensional space (Table 3). In such two dimensions, \((2k-2)\) and \((2k-3)\)-branes are regarded as point-like objects, and the braiding of these objects is well-defined. In this sense, the membranes obey a generalized fractional statistics. With the use of Eq. (184), we can express the tensor fields by the associated membrane currents, and an effective membrane-membrane interaction can be derived from Eq. (185) as

$$S_{\text{eff}} = 2\pi R \cdot \frac{e_{2k-2}^2}{\kappa} \int d^{4k-2}x \int d^{4k-2}y \epsilon_{\mu_1 \mu_2 \cdots \mu_{4k-2}} J_{\mu_1 \cdots \mu_{2k-1}}(x) \partial_{\mu_{2k}} \left( \frac{1}{|x-y|^2} \right) x, y J_{\mu_{2k+1} \cdots \mu_{4k-2}}(y).$$  

(187)

In the thin membrane limit (see Appendix E for details), the integral part becomes the linking number between the membrane trajectories,

$$L(M_{2k-2} \times S^1, M_{2k-3} \times S^1) = \frac{1}{2(2k-1)!} \frac{1}{\pi^{2k-1}} \int_{M_{2k-2} \times S^1} d\mu_{\mu_1 \cdots \mu_{2k-1}} \times \int_{M_{2k-3} \times S^1} d\mu_{\mu_{2k} \cdots \mu_{4k-3}} \cdot \frac{1}{|x-y|^{4k-2}} \epsilon_{\mu_1 \mu_2 \cdots \mu_{4k-2}}(x_{\mu_{4k-2}} - y_{\mu_{4k-2}}),$$  

(188)

and the effective action (187) is reduced to

$$S_{\text{eff}} = 2\pi R \cdot \frac{e_{2k-2}^2}{\kappa} \cdot L(M_{2k-2} \times S^1, M_{2k-3} \times S^1).$$  

(189)
The integral part of Eq. (188) denotes the area of $S^{4k-3}$ with coordinates $\frac{x^\mu - y^\mu}{|x-y|}$ (Appendix E), and so the linking number (188) is equivalent to the winding number accompanied with

$$(M^{2k-2} \times S^1) \times (M^{2k-3} \times S^1) \rightarrow S^{n-1}.$$  \hspace{1cm} (190)

When the membranes are spherical, we have

$$(S^{2k-2} \times S^1) \times (S^{2k-3} \times S^1) \rightarrow S^{4k-3},$$  \hspace{1cm} (191)

indicating that the linking number is integer.

In the case that quantum Hall effect contains internal degrees of freedom [15], the BF theory is simply generalized to include $K$ matrix:

$$S^{(4k-2)D} = -2\pi R \cdot \left( e_{2k-2} \int_{4k-2} (C_\alpha^{\alpha} G_{2k-1}^\alpha F_{2k-1} + C_\alpha^{\beta} \int_{2k-2} (C_{2k-1}^\alpha G_{2k-1}^\beta + C_{2k-2}^\alpha G_{2k-1}^\beta) \right),$$

and the effective action of the membrane-membrane interaction is derived as

$$S_{\text{eff}} = 2\pi R \cdot e_{2k-2} \left( \frac{1}{K} \right) L^{\alpha\beta},$$

with

$$L^{\alpha\beta} \equiv l(M_{2k-2}^{\alpha} \times S^1, M_{2k-3}^{\beta} \times S^1).$$ \hspace{1cm} (194)

\section{8 Extended Dimensional Hierarchy}

In 2kD quantum Hall effect, the filling factor exhibits a hierarchical structure referred to as the dimensional hierarchy [8][9]:

$$\nu_{2k} = \frac{1}{m_2^{2k(k+1)}} = \frac{1}{m_1 m_2 \cdots m_k}$$ \hspace{1cm} (195)

or

$$\nu_{2k} = \frac{1}{m_k} \nu_{2k-2}.$$ \hspace{1cm} (196)

According to the Haldane-Halperin hierarchy [4][69], in the usual 2D quantum Hall liquid, hierarchical nature of the filling factor is a reflection of new formations of incompressible liquids by quasiparticle condensation. Therefore, the dimensional hierarchy suggests “dimensional condensation” of membrane-like excitations: Low dimensional membranes condense to form incompressible liquid in two-dimension higher space, and the resulting higher D liquids also behave as membranes and condense again to form a new incompressible liquid in even higher dimensional space. Such condensation process repeatedly occurs from 0 to 2kD, and consequently the filling factor bears a hierarchical structure in dimensions (195). The dimensional hierarchy can be understood by simple dimension counting. In 2kD quantum Hall effect there are $(2k-2)$-branes occupying an area $\ell_B^{2k}$ ($\ell_B = \sqrt{T/R}$), and the number of $(2k-2)$-branes on 2k-sphere is evaluated as

$$R^{2k}_{\ell_B^{2k}} = R^k.$$ \hspace{1cm} (197)
Since the membrane condensation repeatedly occurs from 0-branes to $(2k - 2)$-branes, the total number of 0-branes on $S^{2k}$ is

$$d_{2k}(I) \equiv I \cdot I^2 \cdots I^k = I^{\frac{k}{2}(k+1)}. \quad (198)$$

The filling factor is scaled inversely in terms of the magnetic field (or the monopole charge), and then the scaling of the filling factor becomes Eq. (195). One may also see that Eq. (198) can be obtained from the lowest Landau level degeneracy of the Dirac-Landau operator, i.e., the Dirac operator index:

$$\text{Ind } (-iP)_{SO(2k+1)} = d_{SO(2k+1)}[0, \frac{I}{2} - \frac{1}{2}] = \prod_{j=1}^{k} \prod_{i=1}^{l} \frac{I + j - 2}{l + i - 1} \sim I^{\frac{k}{2}(k+1)}. \quad (199)$$

Let us consider whether such dimensional hierarchy will occur in the odd $D$ quantum Hall effects. Membranes on $S^{2k-1}$ are supposed to occupy the area $\ell_B^{2k-1}$, and similar dimension counting suggests that the number of the membranes would be

$$\frac{R^{2k-1}_{\ell_B^{2k-1}}}{\ell_B^{2k-1}} = I^{k-\frac{1}{2}}, \quad (200)$$

and the number of 0-branes would also be

$$I^{\frac{1}{2}} \cdot I^{\frac{3}{2}} \cdots I^{k-\frac{3}{2}} = I^{\frac{k}{2}}, \quad (201)$$

which however does not coincide with the sub-band degeneracy of the $(2k - 1)D$ lowest Landau level:

$$d_{SO(2k)}[0, \frac{I}{2} - \frac{1}{2}, s] = \prod_{j=1}^{k-1} \frac{(I + 2j - 1)^2 - 4s^2}{(2j)^2} \prod_{l=1}^{k-2} \prod_{j=1}^{l-1} \frac{I + 2l + j - 1}{2l + j} \sim d_{2k-1}(I) \equiv I^{\frac{1}{2}(k-1)(k+2)} \quad (s << I). \quad (202)$$

The understanding of the scaling based on the simple dimensional analysis thus fails in the case of the odd $D$ quantum Hall effects. However, Eq. (202) suggests the alternative scaling of the filling factor

$$\nu_{2k-1} = \frac{1}{m^{\frac{1}{2}(k-1)(k+2)}}, \quad (203)$$

which exhibits a hierarchical structure similar to the even $D$ case (195):

$$\frac{1}{m^{\frac{1}{2}(k-1)(k+2)}} = \frac{1}{m^2} \frac{1}{m^3} \cdots \frac{1}{m^k}, \quad (204)$$

or

$$\nu_{2k-1} = \frac{1}{m^k} \nu_{2k-3}. \quad (205)$$

Therefore, the condensation mechanism is expected to work also in the odd $D$ quantum Hall effect. Considering that $(2k - 1)D$ quantum Hall effect is realized as a “subspace” of $2kD$ quantum Hall effect, it is no wonder if similar mechanism may operate in odd $D$ case.
From Eq. (195) and (203), we find the relationship between even D and odd D:

\[ \nu_{2k} = \frac{1}{m} \cdot \nu_{2k-1}, \]  

(206)

or

\[ I \cdot d_{2k-1}(I) = d_{2k}(I). \]  

(207)

Obviously, Eq. (207) is a manifestation of the dimensional ladder between \( (2k-1) \) and \( 2kD \) discussed in Sec 4.2. From Eq. (196) and (206), another dimensional ladder between \( (2k+1) \) and \( 2kD \) is inferred as

\[ \nu_{2k+1} = \frac{1}{m^k} \nu_{2k}, \]  

(208)

or

\[ I^k \cdot d_{2k}(I) = d_{2k+1}(I). \]  

(209)

This relation implies that \( I^k \) \( 2k \)-branes or \( 2kD \) quantum Hall liquids condense to form one-dimension higher \( (2k+1)D \) quantum Hall liquid. Fig. 7 summarizes mutual relations among the higher D quantum Hall effects that we call the extended dimensional hierarchy.

![The original dimensional hierarchy in even D.](image)

**Figure 7:** The extended dimensional hierarchy. The quantum Hall liquids in different dimensions are related by the web. See also the topological periodic table (Table 1) of A and AIII-class.

We add some comments on the relations to anomaly in relativistic quantum field theory and D-brane physics in string theory. The original dimensional hierarchy [196] was only for even D quantum Hall effects, which may correspond to the relationship between chiral anomaly and gauge anomaly in even dimensions [41, 42, 43, 44]. We have pointed out a new relation in dimensions of the quantum Hall effects (206) – the dimensional ladder between even and odd dimensions. This ladder is associated with the relationship between even D chiral anomaly and odd D parity anomaly [46]. Therefore, the extended dimensional hierarchy including the original hierarchy and ladder will be a counterpart of more general relationships among anomalies in arbitrary dimensions, such as Stora–Zumino chain of descent equations [41, 42, 43] or the family index theorem in differential topology [45, 46]. On string theory side, D-brane democracy [70]...
or D-brane descent/ascent relation \[71\] tells that any D-brane is comprised of other D-branes in different dimensions. Obviously, such D-brane mechanism is quite analogous to the physics of the dimensional condensation of quantum Hall liquids.

9 Summary

In this work, we developed a formulation of the odd D quantum Hall effect in the \(SO(2k - 1)\) monopole background. Based on the original Hopf maps, the chiral Hopf maps for the odd D Landau models were introduced, and their quantization was found to give a physical realization of the quantum Nambu geometry. We saw that the peculiar properties of the odd D quantum Hall effect can be naturally understood, provided it is embedded in one-dimension higher even D quantum Hall effect. We demonstrated the dimensional ladder between even and odd D quantum Hall effects through the analysis of the non-relativistic/relativistic Landau models by focusing on their lowest Landau level structures. It was shown that the spectral flow argument of the Atiyah–Patodi–Singer index theorem is the essential mathematics behind the dimensional ladder. From dimensional reduction of the tensor Chern–Simons theory, we constructed BF topological field theory in emergent \((4k - 2)D\) space–time for \((2k - 1)D\) quantum Hall effect. In the BF field theory description, \((2k - 2)\) and \((2k - 3)\)-branes obey a generalized fractional statistics and bring a generalized Hall effect and flux attachment in higher dimensions. Finally, we proposed an extended version of the dimensional hierarchy that gives an overall picture about mutual relations among higher D quantum Hall effects.

The extended dimensional hierarchy is closely related to anomaly in relativistic quantum field theory and D-brane physics in string theory. We summarize the corresponding concepts in Table 4. It may be worthwhile to pursue further consequences behind such analogies in apparently different theories. From topological insulator point of view, the extended hierarchy is only about the relations between A and AIII classes. The periodic topological table suggests wider relations among topological insulators with different symmetries. It may be interesting to see how the extended dimensional hierarchy can further be “extended” to include other topological classes.

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| Fibre-bundle | Diff. topol. | Anomaly | NCG | QHE |
|--------------|-------------|---------|-----|-----|
| 2kD          | Chern number | Dirac index | Axial anomaly | Fuzzy even-sphere | LLL degeneracy |
| (2k – 1)D    | Winding number | Spectral flow | Parity anomaly | Fuzzy odd-sphere | Total dim. of LLL |

Table 4: The dimensional ladder appears in apparently different theories.
A Landau Models on $S^1$

Here we consider Landau models on $S^1$ \[56, 57\]. Unlike other odd dimensional spheres, there does not exist the spin connection for $S^1$.

A.1 Non-relativistic Landau model

We take $U(1)$ gauge field as

$$A_x = -\frac{I}{2r^2}y, \quad A_y = \frac{I}{2r^2}x \quad (r^2 = x^2 + y^2), \quad (210)$$

which yields the magnetic flux at the origin:

$$B = \partial_x A_y - \partial_y A_x = I\pi \delta(x, y). \quad (211)$$

Notice that a charged particle living on $S^1$ does not classically interact with the magnetic flux at the origin, but does quantum mechanically through the gauge potential. The Landau Hamiltonian on $S^1$ is given by

$$H = \frac{1}{2Mr^2} \Lambda^2, \quad (212)$$

where

$$\Lambda = -i x (\partial_y + i A_y) + i y (\partial_x + i A_x) = -i \frac{\partial}{\partial \theta} + i \frac{I}{2}. \quad (213)$$

We can readily solve the eigenvalue problem of the Landau Hamiltonian. Due to the periodic boundary condition on $S^1$

$$\psi(\theta = 2\pi) = \psi(\theta = 0), \quad (214)$$

the solutions are classified to two cases, even $I$ and odd $I$. For even $I$, we have

$$E_n = \frac{1}{2Mr^2} n^2, \quad (215a)$$

$$\psi_{\pm n}(\theta) = \frac{1}{\sqrt{2\pi}} e^{i(\pm n - \frac{I}{2}) \theta}, \quad (215b)$$

while for odd $I$,

$$E'_n = \frac{1}{2Mr^2} \left( n + \frac{1}{2} \right)^2, \quad (216a)$$

$$\psi'_{\pm(n+\frac{1}{2})}(\theta) = \frac{1}{\sqrt{2\pi}} e^{i(\pm (n+\frac{1}{2}) - \frac{I}{2}) \theta}, \quad (216b)$$
where \( n = 0, 1, 2, \cdots \). As mentioned in Sec.3, for even \( I \) the lowest Landau level \( (n = 0) \) is non-degenerate while for odd \( I \) doubly degenerate. The energy spectrum for even \( I \) \((215a)\) is identical to that of the free particle on \( S^1 \), but the spectrum for odd \( I \) \((216a)\) exists only in the presence of the magnetic flux, and is peculiar to quantum mechanics. Also notice that by the specialty of \( S^1 \), the magnitude of \( I \) does not affect the Landau level spacing and the Landau level degeneracy. Each of the non-zero energy levels are two-fold degenerate corresponding to the left and right moving modes on the circle.

### A.2 Relativistic Landau model

On the equator of \( S^2 \) \((\theta = \pi/2)\), the 1D Dirac-Landau operator \((147)\) is given by (we have changed the basis)

\[
- iD_{S^1} = -i\sigma_z(\partial_\phi + iI/2) = \begin{pmatrix} -i\partial_\phi + iI/2 & 0 \\ 0 & i\partial_\phi - iI/2 \end{pmatrix} = \begin{pmatrix} -iD_W & 0 \\ 0 & iD_W \end{pmatrix},
\]

where

\[
- iD_W = -i\partial_\phi + iI/2.
\]

The Weyl-Landau operator is the covariant angular momentum itself \((213)\), and the \( SO(2) \) Landau Hamiltonian \((212)\) can be written as \( H = \frac{1}{2M}(-iD_W)^2 \). For even \( I \), the eigenvalues of the Dirac operator \((217)\) are given by

\[
\pm \lambda_n = \pm n \quad (n = 0, 1, 2, \cdots),
\]

and the corresponding eigenstates are

\[
\begin{pmatrix} \psi_{\pm n} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \psi_{\mp n} \end{pmatrix},
\]

with \( \psi_{\pm n} \) \((215b)\). Meanwhile for odd \( I \), the eigenvalues are given by

\[
\pm \lambda_n = \pm (n + 1/2) \quad (n = 0, 1, 2, \cdots),
\]

and their eigenstates are

\[
\begin{pmatrix} \psi_{\pm(n+1/2)}' \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \psi_{\mp(n+1/2)}' \end{pmatrix},
\]

with \( \psi_{\pm n} \) \((216b)\).

The massive Dirac operator is constructed as

\[
- iD_{S^1} + \sigma_x M = \begin{pmatrix} -iD_W & M \\ M & iD_W \end{pmatrix},
\]

which respects the chiral symmetry

\[
\{-iD_{S^1} + \sigma_x M, \sigma_y\} = 0,
\]
where $\sigma_y$ denotes the chiral matrix. The eigenvalues are readily obtained as

$$\pm \Lambda_n = \pm \sqrt{\lambda_n^2 + M^2} \geq M.$$  \hfill (225)

For $|\Lambda_n| > M$, every Landau level is doubly degenerate (Fig.8), and the corresponding eigenstates are

$$+ \Lambda_n (>|M|) : \frac{1}{\sqrt{2\Lambda_n}} \begin{pmatrix} \Lambda_n + \lambda_n \\ M \end{pmatrix} e^{i(\lambda_n - \frac{1}{2})\phi}, \quad - \Lambda_n (<-M) : \frac{1}{\sqrt{2\Lambda_n}} \begin{pmatrix} M \\ \Lambda_n + \lambda_n \end{pmatrix} e^{-i(\lambda_n + \frac{1}{2})\phi}. \hfill (226)$$

For even $I$, non-degenerate energy levels exist for $\pm \Lambda_{n=0} = \pm M$ and the eigenstates are

$$+ \Lambda_{n=0} (s = 0) = +M : \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\frac{1}{2}\phi}, \quad - \Lambda_{n=0} (s = 0) = -M : \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\frac{1}{2}\phi}. \hfill (227)$$

**B Calculation of the Winding Number**

Two patches are needed to represent the $SO(2k)$ monopole gauge field on $S^{2k}$. The gauge field regular except for the south/north pole is given by

$$A_N = -\frac{1}{2(1 + x_{2k+1})} \sigma_{\mu\nu} x_\nu dx_\mu = -i\frac{1}{2}(1 - x_{2k+1}) g^I dg,$$

$$A_S = -\frac{1}{2(1 - x_{2k+1})} \bar{\sigma}_{\mu\nu} x_\nu dx_\mu = -i\frac{1}{2}(1 + x_{2k+1}) gdg^I, \hfill (228)$$

where $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are the $SO(2k)$ generators in the Weyl representation. $g$ denotes the $SO(2k)$ gauge group element

$$g = \frac{1}{\sqrt{1 - x_{2k+1}^2}}(x_{2k} - i\gamma_i x_i), \hfill (229)$$
which acts as the transition function for the \( SO(2k) \) monopole gauge bundle. Obviously, \( A_N \) and \( A_S \) are related as

\[
A_N = g^\dagger A_S g - ig^\dagger dg. \tag{230}
\]

On the equator of \( S^{2k} \), \( g \) is given by

\[
g = x_{2k} - i\gamma_i x_i \in SO(2k), \tag{231}
\]

with

\[
x_i^2 + x_{2k}^2 = 1 : S^{2k-1}. \tag{232}
\]

\( g \) specifies a map from \( S^{2k-1} \) to \( SO(2k) \) and the associated winding number is

\[
\nu_k = \frac{1}{(2k-1)!/(2i)^k-1} \int_{S^{2k-1}} tr(-ig^\dagger dg)^{2k-1}. \tag{233}
\]

We explicitly calculate Eq.\(^{(233)}\) as follows. Around the north-pole of \( S^{2k-1} \) \((x_{2k} \simeq 1)\), \( g \) behaves as

\[
-ig^\dagger dg \simeq -\gamma_i dx_i, \tag{234}
\]

and from the \( SO(2k) \) symmetry, the integration part is represented as

\[
\int_{S^{2k-1}} tr(-ig^\dagger dg)^{2k-1} \sim -\int_{S^{2k-1}} tr(\gamma_i dx_i)^{2k-1} = -\int_{S^{2k-1}} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2k-1} tr(\gamma_1 \gamma_2 \cdots \gamma_{2k-1})
\]

\[
= A(S^{2k-1}) tr(\gamma_1 \gamma_2 \cdots \gamma_{2k-1}), \tag{235}
\]

where the direction of the wedge product is taken in the inward of \( S^{2k-1} \); \( \int_{S^{2k-1}} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2k-1} = -A(S^{2k-1}) \). Eq.\(^{(233)}\) now becomes

\[
\nu_k = \frac{1}{(2i)^{k-1}(2k-1)!} tr(\gamma_1 \gamma_2 \cdots \gamma_{2k-1}), \tag{236}
\]

and then the calculation of the winding number is boiled down to taking the trace of the product of the \( SO(2k-1) \) gamma matrices. The \( SO(2k-1) \) gamma matrices in the totally symmetric representation \([0, \frac{I}{2}]_{SO(2k-1)}\) satisfies

\[
\gamma_{i_{2k-1}} = (-i)^{k-1} \frac{I!!}{(2k-2)!!(I+2k-4)!!} \gamma_{i_1 i_2 \cdots i_{2k-1}} \gamma_1 \gamma_2 \cdots \gamma_{2k-2} \quad (k \geq 2), \tag{237}
\]

and hence the trace is reduced to

\[
tr(\gamma_1 \gamma_2 \cdots \gamma_{2k-1}) = (2k-1) \ tr(\gamma_1 \gamma_2 \cdots \gamma_{2k-2} \gamma_{2k-1})
\]

\[
= i^{k-1} \frac{(2k-1)!!(I+2k-4)!!}{(2k-3)!!I!!} tr(\gamma_{2k-1}^2). \tag{238}
\]

Since the \( SO(2k-1) \) chiral matrix is given by the following diagonal form

\[
\gamma_{2k-1} = \text{diag}(I_{[0, \frac{I}{2}, \cdots \frac{I}{2}]_{SO(2k-2)}}, (-I-2)_{d[0, \frac{I}{2}, \cdots \frac{I}{2}]_{SO(2k-2)}}, \cdots, (2s)_{d[0, \frac{I}{2}, \cdots \frac{I}{2}]_{SO(2k-2)}}, \cdots, -I_{d[0, \frac{I}{2}, \cdots \frac{I}{2}]_{SO(2k-2)}}), \tag{239}
\]
with $d[0, \frac{I}{2}, s]_{SO(2k-2)} \equiv (I, I, \cdots, I)$, Eq. (238) can be expressed as

$$\text{tr}(\gamma_1 \gamma_2 \cdots \gamma_{2k-1}) = i^{k-1} \frac{(2k-1)!(I + 2k - 4)!!}{(2k - 3)!!I!!} \sum_{s = -\frac{k}{2}}^{\frac{k}{2}} (2s)^2 d[0, \frac{I}{2}, s]_{SO(2k-2)}.$$ (240)

and the winding number (233) is eventually derived as

$$\nu_k = \frac{1}{2k-1} \frac{(I + 2k - 4)!!}{(2k - 3)!!I!!} \sum_{s = -\frac{k}{2}}^{\frac{k}{2}} (2s)^2 d[0, \frac{I}{2}, s]_{SO(2k-2)} \quad (k \geq 2).$$ (241)

C Weyl-Landau Problem on $S^{2k-1}$

Several properties of the eigenstates of the Weyl-Landau operator (110) are summarized here.

C.1 Odd $I$

The minimum eigenvalue of the square of the Weyl-Landau operator is zero

$$\lambda_{n=0}(s = 0) = 0,$$ (242)

for the representation

$$[n = 0, \frac{I}{2} - \frac{1}{2}, s = 0]_{SO(2k)}.$$ (243)

The number of the zero-modes is

$$d[0, \frac{I}{2} - \frac{1}{2}, 0]_{SO(2k)} = \left(\frac{(I + 2k - 3)!!}{(I - 1)!!(2k - 2)!!}\right)^2 \times \prod_{l=1}^{k-2} \frac{(2l)!(I + l + k - 2)!}{(l + k - 1)!(I + 2l - 1)!} \sim I^{\frac{1}{2}(k+2)(k-1)}.$$ (244)

For instance,

$$d[0, \frac{I}{2} - \frac{1}{2}, 0]_{SO(2)} = 1 = I^0,$$

$$d[0, \frac{I}{2} - \frac{1}{2}, 0]_{SO(4)} = \frac{1}{4} (I + 1)^2 \sim I^2,$$

$$d[0, \frac{I}{2} - \frac{1}{2}, 0]_{SO(6)} = \frac{1}{192} (I + 1)^2 (I + 2)(I + 3)^2 \sim I^5,$$

$$d[0, \frac{I}{2} - \frac{1}{2}, 0]_{SO(8)} = \frac{1}{138240} (I + 1)^2 (I + 2)(I + 3)^2 (I + 4)(I + 5)^2 \sim I^9.$$ (245)

C.2 Even $I$ ($I \neq 0$)

The minimum eigenvalue of the square of the Weyl-Landau operator is given by

$$\lambda_{n=0}(s = \frac{1}{2})^2 = \lambda_{n=0}(s = -\frac{1}{2})^2 = \left(\frac{1}{2}\right)^2.$$ (246)
for the representations,
\[
[n = 0, \frac{I}{2} - \frac{1}{2}, s = \frac{1}{2}]_{SO(2k)}, \quad [n = 0, \frac{I}{2} - \frac{1}{2}, s = -\frac{1}{2}]_{SO(2k)},
\]  
(247)

The corresponding degeneracy is
\[
2 \left[ d[0, \frac{I}{2} - \frac{1}{2}, \frac{1}{2}]_{SO(2k)} \right] = 2 \left[ d[0, \frac{I}{2} - \frac{1}{2}, -\frac{1}{2}]_{SO(2k)} \right] = 2 \prod_{j=0}^{k-2} \frac{(I + 2j)(I + 2j + 2)}{(2j + 2)^2} \times \prod_{l=1}^{k-2} \prod_{j=1}^{l-1} \frac{I + 2l + j - 1}{2l + j} \sim I^{\frac{1}{2}(k+2)(k-1)}. 
\]  
(248)

For instance,
\[
d[0, \frac{I}{2} - \frac{1}{2}, \frac{1}{2}]_{SO(2)} = 1 - \delta_{I,0} \sim I^0, \\
d[0, \frac{I}{2} - \frac{1}{2}, \frac{1}{2}]_{SO(4)} = \frac{1}{4} I(I + 2) \sim I^2, \\
d[0, \frac{I}{2} - \frac{1}{2}, \frac{1}{2}]_{SO(6)} = \frac{1}{192} I(I + 2)^3(I + 4) \sim I^5, \\
d[0, \frac{I}{2} - \frac{1}{2}, \frac{1}{2}]_{SO(6)} = \frac{1}{138240} I(I + 2)^3(I + 3)(I + 4)^3(I + 6) \sim I^9. 
\]  
(249)

Though these states are non-zero modes, in the absence of the external gauge field, these states vanish just like the zero-modes of even-sphere case.\(^{28}\) (See also Appendix A for \(S^1\).)

### D Chern Number and Winding Number

We demonstrate equality between the Chern number and the winding number associated with monopole bundle on \(2k\)-sphere. Two patches that cover the north and south hemispheres are introduced to represent non-singular gauge field on the base-manifold \(S^{2k}\), and on their overlapping region, say the equator \(S^{2k-1}\), the transition function \(g\) is defined so as to satisfy
\[
A_N = g^\dagger A_S g - ig^\dagger dg. 
\]  
(251)

Using \(g\), we will express the \(k\)th Chern number
\[
c_k = \frac{1}{(2k)! \pi^{k}} \int_{S^{2k}} \text{tr} F^k. 
\]  
(252)

\(^{28}\)In low dimensions, the numbers of zero-modes of the \(2k\)D relativistic Landau models are given by
\[
\text{Ind}(-i\mathcal{D})_{SO(3)} = I, \\
\text{Ind}(-i\mathcal{D})_{SO(5)} = \frac{1}{6} I(I + 1)(I + 2), \\
\text{Ind}(-i\mathcal{D})_{SO(7)} = \frac{1}{360} I(I + 1)(I + 2)^2(I + 3)(I + 4). 
\]  
(250)
Since $\text{tr}F^k$ is given by total derivative of the Chern–Simons form:

$$\text{tr}F^k = dL_{CS}^{2k-1},$$

(253)

with

$$L_{CS}^{2k-1}[A] = k \int_0^1 dt \text{ tr}(A(tdA + it^2A^2)^{k-1}),$$

(254)

the Chern number can be rewritten as

$$c_k = \frac{1}{(2k)!\pi^{k}} \int_{D^{2k}} \text{tr}F^k + \int_{D^{2k}} \text{tr}F^k = \frac{1}{(2k)!\pi^{k}} \int_{S^{2k-1}} (L_{CS}^{2k-1}[A_N] - L_{CS}^{2k-1}[A_S])$$

$$= \frac{1}{(2k)!\pi^{k}} \int_{S^{2k-1}} L_{CS}^{2k-1}[-ig^\dagger dg].$$

(255)

In the last equation, we used $L_{CS}^{2k-1}[A_N] - L_{CS}^{2k-1}[A_S] = L_{CS}^{2k-1}[-ig^\dagger dg] + \text{ (total derivative term)}$, and $L_{CS}[-ig^\dagger dg]$ is the pure gauge Chern-Simons action:

$$L_{CS}[A = -ig^\dagger dg] = (-i)^{k-1} \frac{k!(k-1)!}{(2k-1)!} \text{ tr}(-ig^\dagger dg)^{2k-1}.$$  

(258)

Consequently, Eq. (255) is represented as

$$\nu_k = (-i)^{k-1} \frac{1}{(2\pi)^{k}} \frac{(k-1)!}{(2k-1)!} \int_{S^{2k-1}} \text{ tr}(-ig^\dagger dg)^{2k-1},$$

(259)

which is the winding number from $S^{2k-1}$ to the transition function $g$. Thus it was shown that the Chern number (255) is equal to the winding number (259) in general:

$$c_k = \nu_k.$$  

(260)

### E BF Theory and Linking Number

In 3D, the Gauss linking number is defined for 0D objects with world lines $V_1$ and $U_1$:

$$L(V_1, U_1) = -\frac{1}{4\pi} \int_{V_1} dx_\mu \int_{U_1} dy_\nu \epsilon_{\mu\nu\rho} \frac{1}{|x-y|}$$

$$= \frac{1}{4\pi} \int_{V_1} dx_\mu \int_{U_1} dy_\nu \epsilon_{\mu\nu\rho} \frac{x_\rho - y_\rho}{|x-y|^3}. $$

(261)

For $k = 2$, the 3D Chern-Simons form is given by

$$L_{CS}^3 = \text{ tr}(AdA + \frac{2}{3} iA^3),$$

(256)

and we have

$$L_{CS}^3[A_N] - L_{CS}^3[A_S] = \frac{1}{3} \text{ tr}(g^\dagger dg)^3 + id(\text{ tr}A_s gdg^\dagger) = L_{CS}^3[-ig^\dagger dg] + id(\text{ tr}A_s gdg^\dagger).$$

(257)
In \( n \)D, \((p - 1)\)D and \((n - p - 2)\)D membranes undergo non-trivial linking.\(^{30}\)

\[
L(V_p, U_{n-p-1}) = \frac{1}{p!(n-p-1)!} \int_{V_p} dx_{\mu_1 \mu_2 \cdots \mu_p} \int_{U_{n-p-1}} dy_{\mu_{p+1} \cdots \mu_{n-1}} \epsilon_{\mu_1 \mu_2 \cdots \mu_n} \partial^n_{x y} \left( \frac{1}{\partial^2} \right), \tag{262}
\]

where \( V_p \) and \( U_{n-p-1} \) respectively represent the world volumes of the \((p - 1)\) and \((n - p - 2)\)-membranes and \( \partial^n_{x y} \) denotes the Green function in \( n \)D:

\[
\frac{1}{\partial^2} = -\frac{1}{(n-2)A(S^{n-1})} \frac{1}{r^{n-2}} \quad (n \geq 3), \tag{263}
\]

which satisfies

\[
\partial^2 \frac{1}{\partial^2} = \delta^{(n)}(r). \tag{264}
\]

Here, \( A(S^n) \) represents the area of \( S^n \).\(^{31}\)

Eq.\,(262) can be rewritten as

\[
L(V_p, U_{n-p-1}) = -\frac{\Gamma(\frac{n}{2})}{2(n-2)\pi^\frac{n}{2} p!(n-p-1)!} \int_{V_p} dx_{\mu_1 \cdots \mu_p} \int_{U_{n-p-1}} dy_{\mu_{p+1} \cdots \mu_{n-1}} \epsilon_{\mu_1 \mu_2 \cdots \mu_n} \partial^n_{x y} \frac{1}{|x-y|^{n-2}}
\]

\[
= \frac{\Gamma(\frac{n}{2})}{2\pi^\frac{n}{2} p!(n-p)!} \int_{V_p} dx_{\mu_1 \cdots \mu_p} \int_{U_{n-p-1}} dy_{\mu_{p+1} \cdots \mu_{n-1}} \frac{1}{|x-y|^n} \epsilon_{\mu_1 \mu_2 \cdots \mu_n} (x_{\mu_n} - y_{\mu_n}). \tag{267}
\]

This expression is mentioned in Ref.\,[64].

By integrating out the BF fields, we will derive the linking number \,(267) from the BF action

\[
S_{BF} = \frac{1}{\theta p!(n-p)!} \int d^n x \epsilon^{\mu_1 \mu_2 \cdots \mu_n} B_{\mu_1 \cdots \mu_p} F_{\mu_{p+1} \cdots \mu_n}
\]

\[
- \frac{1}{p!} \int d^n x J_{\mu_1 \cdots \mu_p} B^{\mu_1 \cdots \mu_p} - \frac{1}{(n-p-1)!} \int d^n x I_{\mu_1 \cdots \mu_{n-p-1}} A^{\mu_1 \cdots \mu_{n-p-1}}, \tag{268}
\]

where

\[
F = dA, \quad H = dB. \tag{269}
\]

Equations of motion are derived as

\[
J_{\mu_1 \mu_2 \cdots \mu_p} = \frac{1}{\theta(n-p)!} \epsilon_{\mu_1 \mu_2 \cdots \mu_n} F^{\mu_{p+1} \cdots \mu_n},
\]

\[
I_{\mu_1 \mu_2 \cdots \mu_{n-p-1}} = (-1)^{(n+1)(p+1)} \frac{1}{\theta(p+1)!} \epsilon_{\mu_1 \mu_2 \cdots \mu_n} H^{\mu_{n-p} \mu_{n-p+1} \cdots \mu_n}, \tag{270}
\]

\(^{30}\)Eq.\,(158) is obtained from Eq.\,(262) for \( n = 4k - 2 \) and \( p = 2k - 1 \).

\(^{31}\)Eq.\,(266) gives

\[
A(S^{2k}) = \frac{2^{k+1} \pi^k}{(2k-1)!!}, \quad A(S^{2k-1}) = \frac{2^k \pi^{k-1}}{(k-1)!}. \tag{265}
\]

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which suggest the generalized current conservation laws
\[
\partial_{\mu_i} J^{\mu_1 \cdots \mu_p} = 0 \quad (i = 1, 2, \cdots, p),
\]
\[
\partial_{\mu_i} I^{\mu_1 \cdots \mu_{n-p-1}} = 0 \quad (i = 1, 2 \cdots, n - p - 1). \quad (271)
\]
From Eq. (270), we have
\[
F^{\mu_1 \mu_2 \cdots \mu_{n-p}} = (-1)^{(n+1)p} \frac{1}{p!} \epsilon^{\mu_1 \mu_2 \cdots \mu_n} J_{\mu_{n-p+1} \cdots \mu_n},
\]
\[
H^{\mu_1 \mu_2 \cdots \mu_{p+1}} = (-1)^{p+1} \frac{1}{(n-p-1)!} \epsilon^{\mu_1 \cdots \mu_n} I_{\mu_{p+2} \cdots \mu_n}, \quad (272)
\]
and in the Coulomb like gauge, \( \partial_\mu A^{\mu_1 \cdots \mu_{n-p-1}} = 0 \) and \( \partial_\mu B^{\mu_1 \cdots \mu_p} = 0 \), \( A \) and \( B \) are given by
\[
A^{\mu_1 \cdots \mu_{n-p-1}} = -(-1)^{(n+1)} \frac{1}{p!} \epsilon^{\mu_1 \mu_2 \cdots \mu_n} \partial_{\mu_{n-p}} \frac{1}{1} \partial^2 J_{\mu_{p+2} \cdots \mu_n},
\]
\[
B^{\mu_1 \cdots \mu_p} = -\theta \frac{1}{(n-p-1)!} \epsilon^{\mu_1 \cdots \mu_n} \partial_{\mu_{p+1}} \frac{1}{1} \partial^2 I_{\mu_{p+2} \cdots \mu_n}. \quad (273)
\]
Substituting these expressions to the BF action [208], we obtain an effective membrane current-current interaction
\[
S_{\text{eff}} = \theta \frac{1}{p!(n-p-1)!} \int d^m x \int d^m y \; \epsilon^{\mu_1 \mu_2 \cdots \mu_n} J_{\mu_1 \cdots \mu_p}(x) \partial_{\mu_{p+1}} \left( \frac{1}{\partial^2} \right)_{x,y} I_{\mu_{p+2} \cdots \mu_n}(y). \quad (274)
\]
In the thin membrane limit
\[
J^{\mu_1 \cdots \mu_p}(x) = \int d^p \sigma \; \frac{\partial(y^{\mu_1}, y^{\mu_2}, \cdots, y^{\mu_p})}{\partial(\sigma^0, \sigma^1, \cdots, \sigma^{p-1})} \delta^{(n)}(x - y(\sigma)),
\]
\[
I^{\mu_1 \cdots \mu_{n-p-1}}(x) = \int d^p \sigma \; \frac{\partial(y^{\mu_1}, y^{\mu_2}, \cdots, y^{\mu_{n-p-1}})}{\partial(\sigma^0, \sigma^1, \cdots, \sigma^{n-p-2})} \delta^{(n)}(x - y'(\sigma')), \quad (275)
\]
Eq. (274) is reduced to
\[
S_{\text{eff}} = \theta \frac{1}{p!(n-p-1)!} \int_{V_p} dx^{\mu_1 \cdots \mu_p} \int_{U_{n-p-1}} dy^{\mu_{p+2} \cdots \mu_n} \epsilon^{\mu_1 \mu_2 \cdots \mu_n} \partial_{\mu_{p+1}} \left( \frac{1}{\partial^2} \right)_{x,y}, \quad (276)
\]
where
\[
dx^{\mu_1 \cdots \mu_p} \equiv d\sigma^0 d\sigma^1 \cdots d\sigma^{p-1} \frac{\partial(x^{\mu_1}, x^{\mu_2}, \cdots, x^{\mu_p})}{\partial(\sigma^0, \sigma^1, \cdots, \sigma^{p-1})},
\]
\[
dy^{\mu_1 \cdots \mu_{n-p-1}} \equiv d\sigma^0 d\sigma^1 \cdots d\sigma^{n-p-2} \frac{\partial(y^{\mu_1}, y^{\mu_2}, \cdots, y^{\mu_{n-p-1}})}{\partial(\sigma^0, \sigma^1, \cdots, \sigma^{n-p-2})}. \quad (277)
\]
In Euclidean space-time, we have
\[
S_{\text{eff}} = \theta \cdot L(V_p, U_{n-p-1}), \quad (278)
\]
where
\[
L(V_p, U_{n-p-1}) = \frac{1}{p!(n-p-1)!} A(S^{n-1}) \int_{V_p} dx^{\mu_1 \cdots \mu_p} \int_{U_{n-p-1}} dy^{\mu_{p+1} \cdots \mu_{n-1}} \epsilon_{\mu_1 \mu_2 \cdots \mu_n} \frac{x_{\mu_n}(\sigma) - y_{\mu_n}(\sigma')}{|x(\sigma) - y(\sigma')|^n}. \quad (279)
\]
which is exactly equal to the linking number in higher dimensions \(267\). With
\[
z_{\mu}(\sigma, \sigma') \equiv \frac{x_{\mu}(\sigma) - y_{\mu}(\sigma')}{|x(\sigma) - y(\sigma')|} \quad \left(\sum_{\mu=1}^{n} z_{\mu}z_{\mu} = 1 : S^{n-1}\right) ,
\]
the linking number \(279\) can be expressed as
\[
L = \frac{1}{A(S^{n-1})} \cdot \frac{1}{(n-1)!} \int dz^{\mu_1} ... dz^{\mu_{n-1}} \epsilon_{\mu_1 ... \mu_n} z_{\mu_n},
\]
where
\[
dz^{\mu_1 ... \mu_n} \equiv d\sigma^{0} ... d\sigma^{p-1}d\sigma^{0} ... d\sigma^{n-p-1} \frac{\partial(z^{\mu_1}, z^{\mu_2}, z^{\mu_3}, ..., z^{\mu_n})}{\partial(\sigma^{0}, ..., \sigma^{p-1}, \sigma^{0}, ..., \sigma^{n-p-1})}.
\]
To derive Eq.\((281)\), we used the following formula about the determinant
\[
\frac{1}{p!(n-p)!} \epsilon_{\mu_1 \mu_2 ... \mu_n} \frac{\partial(z_{\mu_1}, z_{\mu_2}, ..., z_{\mu_p})}{\partial(\sigma_1, \sigma_2, ..., \sigma_p)} \frac{\partial(z_{\mu_{p+1}}, z_{\mu_{p+2}}, ..., z_{\mu_n})}{\partial(\sigma_{p+1}, \sigma_{p+2}, ..., \sigma_n)} = \frac{1}{n!} \epsilon_{\mu_1 \mu_2 ... \mu_n} \frac{\partial(z_{\mu_1}, z_{\mu_2}, ..., z_{\mu_n})}{\partial(\sigma_1, \sigma_2, ..., \sigma_n)} \quad (282)
\]
The integral of Eq.\((281)\) gives the area of \(S^{n-1}\), and then the linking number is equivalent to the winding number that takes integer values associated with the map
\[
V_p \times U_{n-p-1} \longrightarrow S^{n-1}.
\]

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