Non-local Symmetries of Nonlinear Field Equations: an Algebraic Approach

M. Leo, R.A. Leo, G. Soliani and P. Tempesta*
Dipartimento di Fisica dell’Università di Lecce, 73100 Lecce, Italy.

Abstract

An algebraic method is devised to look for non-local symmetries of the pseudopotential type of nonlinear field equations. The method is based on the use of an infinite-dimensional subalgebra of the prolongation algebra $L$ associated with the equations under consideration. Our approach, which is applied by way of example to the Dym and the Korteweg-de Vries equations, allows us to obtain a general formula for the infinitesimal operator of the non-local symmetries expressed in terms of elements of $L$. The method could be exploited to investigate the symmetry properties of other nonlinear field equations possessing nontrivial prolongations.

1 Introduction

Local symmetries of differential equations (DEs) are defined by infinitesimal operators which generally are functions of the independent and the dependent variables (fields) involved in the equations under consideration. On the contrary, non-local symmetries are characterized by infinitesimal operators depending on the global behavior of the fields, expressed for instance by their integrals [1,2].

*Address for correspondence: Dott. P. Tempesta, Dipartimento di Fisica, Università degli Studi di Lecce, Via per Arnesano, 73100-Lecce (Italy). E-mail: Tempesta@le.infn.it.
The study of non-local symmetries can be performed following a procedure which relates these symmetries to the local symmetries of certain auxiliary systems of equations connected with the original DEs. An interesting situation is that where these equations can be expressed as conservation laws. Indeed, in this case it is possible to introduce new dependent variables, called potentials, which can be defined by quadratures. Hence, with a given original system of DEs $\Delta(x,u)$, where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, one can associate a system $\Theta(x,u,y)$ (see later) which depends on the set of potential variables $y = \{y_i\}$ as well, with the property that any solution $(u(x), y(x))$ of $\Theta$ is also a solution of $\Delta$. Vice versa, for any solution $u(x)$ of $\Delta$ there is a potential $y(x)$ such that the pair $(u(x), y(x))$ is a solution of $\Theta$ [2]. Of course, any symmetry group $S_\Theta$ of $\Theta$ will be also a symmetry group of $\Delta$.

More generally, any local group of transformations of $\Theta$ defines a non-local group of transformations of $\Delta$, provided that the generators of the infinitesimal symmetry transformations (of the independent and dependent variables) are explicit functions of the potential. Then, the non-local case can be treated using the same algorithmic procedures valid for the local symmetries.

The potential symmetries can also be exploited to carry out the symmetry reduction of a given system of differential equations. The case of partial differential equations has been dealt with by Bluman and Reid [3]. Other kinds of non-local symmetries have been studied in [4-6]. Recently, Guthrie and Hickman [7] have derived new algebraic structures for the bi-Hamiltonian version of the Korteweg-de Vries (KdV) equation. Since the inverse recurrence operator for this equation, because of its bi-Hamiltonian origin, is still a recurrence operator, it can generate three new families of generalized symmetries, depending on non-local variables. These symmetries can be interpreted as isovectors of a prolonged system of the KdV equation, which are found starting from an infinite-dimensional realization of the Estabrook-Wahlquist (EW) prolongation algebra.

A remarkable generalization of the concept of non-local symmetry, i.e. the Lie-Bäcklund (non-local) symmetry, has been devised by Edelen [8] and Krasil’shchik and Vinogradov [9], where the non-locality character is carried out by variables of the pseudopotential type. In this context, Galas has rederived the one-soliton solutions for the KdV and the Dym equations, and the AKNS system [10]. The non-local Lie-Bäcklund technique presents the
benefit to join up the EW prolongation method [11], which yields the set of pseudopotentials, to the symmetry reduction approach [1], giving the tools for analyzing field equations via the group theory. Other interesting results on the theory of non-local symmetries have been achieved in [12] and [13].

In this paper we outline a procedure to obtain non-local symmetries of the pseudopotential type of a partial differential equation admitting an incomplete prolongation algebra $L$ (in the sense that not all of the commutators of $L$ are known). Equations enjoying this property are important both from a mathematical point of view and in physical applications. Our method consists essentially in looking for an infinite-dimensional subalgebra $L$ of $L$ containing as a special case a finite-dimensional subalgebra $L_o$ of $L$. This procedure is suggested by the fact that in the framework of $L_o$ only trivial non-local symmetries can be found. The infinite dimensionality of the subalgebra seems to be crucial for the determination of nontrivial non-local symmetries. Within our approach, which is applied to the Dym and the KdV equations, we obtain a general formula for the infinitesimal operator of the non-local symmetries expressed in terms of elements of $L$. In theory, other equations (or systems of equations) endowed with nontrivial prolongations could be treated in a similar way.

The paper is organized as follows. Section 2 is devoted to some preliminaries on potential and pseudopotential symmetries. In Sections 3 and 4 we consider the Dym and the KdV equations, respectively. The prolongation algebras allowed by these equations are incomplete. We apply the algebraic technique described above to yield the generators of the non-local symmetries (of the pseudopotential type) together with examples of interesting solutions of the equations under study. Some of these solutions are well-known and do not represent, of course, the main goal of the paper, which is based conversely on a unifying (algebraic) tool to find non-local symmetries. Finally, Section 5 contains some concluding remarks, while in the Appendices A and B details of the calculations are reported.

2 Potential and pseudopotential symmetries

Let us deal with a system of nonlinear field equations

$$
\Delta(x, u) \equiv u_t + K(x, t, u, u, u_x, ..., u_{x...x}) = 0,
$$

(2.1)
with \((x,t) \in \mathbb{R}^2, u = (u_1, ..., u_n) \in \mathbb{R}^n\) and \(u_t = D_t(u), u_x = D_x(u)\), and so on, where \(D_t\) and \(D_x\) stand for the total derivatives with respect to \(t\) and \(x\).

Let us suppose that the system (2.1) admits conservation laws of the type
\[
\frac{\partial}{\partial t} F_i(x,t,u,u_x,...,u_{x...x}) = \frac{\partial}{\partial x} G_i(x,t,u,u_x,...,u_{x...x}).
\]  

Equations (2.2) allow us to introduce a set of potentials \(y = \{y_i\}\) such that
\[
y_{i,x} = F_i, \quad y_{i,t} = G_i. \tag{2.3a, b}
\]

Then one can consider non-local Lie-Bäcklund operators for the system (2.1) of the form
\[
V = Q_j(x,t,u,u_x,...,u_{x...x},y) \frac{\partial}{\partial u_j}. \tag{2.4}
\]

The non-local character of \(V\) is due to the fact that it depends on the variables \(y_i\), which are defined by quadratures.

The Lie-Bäcklund non-local symmetries corresponding to \(V\) are equivalent to the Lie-point symmetries of the system \(\Theta(x,u,y)\) constituted by Eq. (2.1) and by the set of conservation laws (2.2). These symmetries are generated by vector fields of the form
\[
V_\Theta = \xi(x,t,u,y) \frac{\partial}{\partial x} + \tau(x,t,u,y) \frac{\partial}{\partial t} + \phi_j(x,t,u,y) \frac{\partial}{\partial u_j} + \pi_i(x,t,u,y) \frac{\partial}{\partial y_i}. \tag{2.5}
\]

where the coefficients depend explicitly on the potential variables.

Non-local symmetries can be extended to the case in which the (non-local) symmetry operators depend on pseudopotential variables [10]. In opposition to the potential variables, pseudopotential variables can be defined by the set of implicit equations
\[
y_{i,x} = F_i(x,t,u,u_x,...,u_{x...x},y), \quad y_{i,t} = G_i(x,t,u,u_x,...,u_{x...x},y), \tag{2.6}
\]

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where the functional dependence of $F_i$ and $G_i$ includes the pseudopotential variables also. The compatibility condition $y_{i,xt} = y_{i,tx}$ reproduces the original equations of the type (2.1). Equations (2.6) allow us to obtain, in theory, the prolongation algebra associated with the differential equations under consideration and the related spectral problem [12].

In the following we shall display two case studies concerning the determination of non-local symmetries based on the use of pseudopotentials.

### 3 Non-local symmetries of the Dym equation

The EW prolongation technique applied to the Dym equation

$$u_t = u^3 u_{xxx}$$

(3.1)

gives

$$y_{i,x} = F_i = \frac{A_i}{u^2} + B_i,$$

(3.2a)

$$y_{i,t} = G_i = -2A_i u_{xx} - 2C_i u_x - \frac{2}{u} [A, C]_i + 2u [B, C]_i,$$

(3.2b)

where $A_i, B_i, C_i$ are functions depending on the pseudopotential variables only, and $[A, C]_i, [B, C]_i$ are Lie brackets defined by

$$[A, C]_i = A_j \frac{\partial}{\partial y_j} C_i - C_j \frac{\partial}{\partial y_j} A_i,$$

and so on.

Hereafter, for simplicity we shall omit the index $i$ and adopt an operator formalism, in the sense that we shall define the operators

$$\hat{F} = F_j \frac{\partial}{\partial y_j}, \hat{G} = G_j \frac{\partial}{\partial y_j}, \hat{A} = A_j \frac{\partial}{\partial y_j}, \ldots,$$

etc.. Then the Lie brackets are transformed into commutators, viz.

$$[\hat{A}, \hat{C}] \equiv [A, C]_j \frac{\partial}{\partial y_j}, \ldots.$$  Furthermore, to avoid redundance of symbols, we shall continue to use $F$ instead of $\hat{F}$, and so on.
From the compatibility condition $y_{xt} = y_{tx}$ we have that the operators $A, B, C$ obey the commutation rules

$$[A, B] = C, \quad (3.3a)$$
$$[A, [A, C]] = 0, \quad (3.3b)$$
$$[B, [B, C]] = 0. \quad (3.3c)$$

Equations (3.3a)-(3.3c) define an incomplete Lie algebra. In order to investigate the existence of non-local symmetries (of the pseudopotential type) of Eq. (3.1), let us deal with the infinitesimal transformation

$$\tilde{u} = u + \varepsilon \varphi(u, u_x, y), \quad (3.4)$$

where $\varphi$ is a function to be found and $\varepsilon$ is a real parameter.

By imposing (3.4), Eq. (3.1) yields

$$\varphi_t = 3u^2 \varphi u_{xxx} + u^3 \varphi_{xxx}, \quad (3.5)$$

where $\varphi_t = D_t \varphi$ and $\varphi_x = D_x \varphi$.

At this stage it is convenient to adopt a procedure which consists in the use of the prolongation algebra (3.3a)-(3.3c) without employing any specific representation for the vector fields $A, B, C$.

By expliciting (3.5) and equating to zero the coefficients of the derivatives of $u$ regarded as independent functions, we get

$$\varphi = (u_x - uB)\varphi_o(y), \quad (3.6)$$

where $\varphi_o$ is a function of $y$ satisfying the constraints

$$A\varphi_o = 0, \quad (3.7a)$$
$$A^2 B \varphi_o = 0, \quad (3.7b)$$
$$B^3 \varphi_o = 0, \quad (3.7c)$$
$$BCB \varphi_o = 0, \quad (3.7d)$$
$$2[A, C]B \varphi_o + A^2 B^2 \varphi_o + ABAB \varphi_o = 0. \quad (3.7e)$$
Equations (3.7a) and (3.7b) entail

$$[A, C]\varphi_o = 0. \quad (3.8)$$

Now let us introduce an infinitesimal transformation for the pseudopotential $y$ analogous to (3.4):

$$\tilde{y} = y + \varepsilon \eta(u, y), \quad (3.9)$$

where $\eta(u, y)$ is a vector field to be determined.

We observe that both $\eta$ and $\varphi$ could have a functional dependence more complicated than that made here; indeed, the choice expressed by Eqs. (3.4) and (3.9) corresponds to a minimal assumption. We notice also that if we take $\varphi$ independent from $u_x$, only the identity arises as a symmetry.

Combining together (3.9) and (3.2a) we obtain

$$\tilde{y}_x = y_x + \varepsilon \eta_x(u, y) = F(y + \varepsilon \eta, u + \varepsilon \varphi) = F + \varepsilon (F_y \eta + F_u \varphi).$$

Hence

$$u_x \eta_u + [F, \eta] - \varphi F_u = 0,$$

namely

$$u_x \eta_u + \frac{1}{u^2} [A, \eta] + [B, \eta] + \frac{2}{u^3} u_x \varphi_o A - 2 \frac{2}{u^2} (B \varphi_o) A = 0. \quad (3.10)$$

Taking equal to zero the coefficients of the independent functions of $u$ and their partial derivatives appearing in (3.10), we find

$$\eta = \frac{1}{u^2} \varphi_o A + \eta_o, \quad (3.11)$$

$\eta_o = \eta_o(y)$ being a vector field of integration, and

$$[A, \varphi_o A] = 0, \quad (3.12a)$$

$$[A, \eta_o] + [B, \varphi_o A] - 2 (B \varphi_o) A = 0, \quad (3.12b)$$

$$[B, \eta_o] = 0. \quad (3.12c)$$

In a similar manner, starting from (3.2b) and taking account of (3.9), we obtain (at the first order in $\varepsilon$):
\[
u^3 u_{xxx} \eta_u + [G, \eta] - \frac{2}{u^2} \varphi [A, C] - 2 \varphi [B, C] + 2 \varphi_x C + 2 \varphi_{xx} A = 0. \quad (3.13)\]

Equation (3.13) can be explicited to give

\begin{align*}
(AB\varphi_o)A - [C, \varphi_o A] - \varphi_o A, C] &= 0, \\
[C, \eta_o] + \varphi_o [B, C] + (B^2 \varphi_o) A &= 0, \\
[[A, C], \varphi_o A] &= 0, \\
[[B, C], \eta_o] + (B\varphi_o) [B, C] - (B^2 \varphi_o) C &= 0, \\
[[B, C], \varphi_o A] - [[A, C], \eta_o] + (B\varphi_o) [A, C] - A(B\varphi_o)C - (AB^2 \varphi_o) A \\
&- (BAB\varphi_o) A = 0. \quad (3.14e)
\end{align*}

Finally, the infinitesimal generator of the non–local symmetries of the pseudopotential type for the Dym equation is (see (3.6) and (3.11))

\[
V_{NL} = \varphi \partial_u + \eta = (u_x \varphi_o - uB\varphi_o) \partial_u + \frac{1}{u^2} \varphi_o A + \eta_o. \quad (3.15)
\]

The problem of the determination of the non-local symmetries of the Dym equation is led to find suitable representations of the incomplete prolongation algebra (3.3a)-(3.3c) and the constraints associated with this equation.

In this context, we shall show below that by choosing a finite-dimensional representation of the prolongation algebra (3.3a)-(3.3c) of the \(\mathfrak{sl}(2,R)\) type, we arrive at trivial transformations only.

To this aim, let us set

\begin{align*}
[A, C] &= b_1 A + b_2 B + b_3 C, \quad (3.16a) \\
[B, C] &= c_1 A + c_2 B + c_3 C, \quad (3.16b) \\
[A, B] &= C, \quad (3.16c)
\end{align*}

where \(b_j, c_j\) are constants.

Then, from (3.3a) and (3.3c) we find \(b_2 = b_3 = 0\) and \(c_1 = c_3 = 0\), respectively. Thus, putting \(b_1 = -2\lambda, b_2 = 2\lambda\) Eqs. (3.16a)-(3.16c) become

\[
[A, C] = -2\lambda A, \quad (3.17a)
\]

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\[ [B, C] = 2\lambda B, \quad (3.17b) \]
\[ [A, B] = C. \quad (3.17c) \]

On the other hand, Eq. (3.14d) gives
\[ 2\lambda(B\phi_o)B = (B^2\phi_o)C, \quad (3.18) \]
while from (3.14e):
\[ (2\lambda(B\phi_o) - (AB^2\phi_o) - (BA(B\phi_o))A = (AB\phi_o)C. \quad (3.19) \]

Applying (3.19) to \( \phi_o \) yields
\[ (AB\phi_o)(C\phi_o) = 0, \]
from which
\[ C\phi_o = 0. \quad (3.20) \]

As a consequence, the constraint (3.18) provides \( B\phi_o = 0 \). To conclude, the quotient algebra (3.17a)-(3.17c) leads to the relations
\[ A\phi_o = 0, B\phi_o = 0, C\phi_o = 0, \quad (3.21) \]
which tell us that \( \phi_o = \text{const.} \).

3.1 The method of the ”extended” algebra

We have seen that within a finite-dimensional subalgebra (such as \( sl(2, R) \)) of the prolongation algebra \( L \) defined by (3.3a)-(3.3c), we have not been able to find nontrivial non-local symmetries. Then, we have exploited an infinite-dimensional subalgebra of \( L \). As we show below, this approach succeeds. To this aim, let us introduce the operators
\[ A = A_0 + \varphi_0 A_1, \quad (3.22) \]
\[ B = B_0, \quad (3.23) \]
\[ C = C_0 - (B_0 \varphi_0) A_1, \]  
\text{(3.24)}

into the commutation relations (3.3a)-(3.3c), where

\[ [A_0, B_0] = C_0, [A_0, C_0] = -2\lambda A_0, [B_0, C_0] = 2\lambda B_0 \]  
\text{(3.25)}

and \( A_1 \) denotes an operator obeying the commutation rules

\[ [A_1, A_0] = 0, [A_1, B_0] = 0, [A_1, C_0] = 0. \]  
\text{(3.26)}

Furthermore, we assume that

\[ A_1 \varphi_0 = 0, \]  
\text{(3.27)}

and

\[ A_0 \varphi_0 = A_0^2 B_0 \varphi_0 = B_0^3 \varphi_0 = 0. \]  
\text{(3.28)}

(We observe that the constraints (3.28) are really special cases of (3.7a), (3.7b) and (3.7c), respectively).

By virtue of (3.22)-(3.28), we can prove directly that the commutators

\[ [A, B_0] = C_0 - (B_0 \varphi_0) A_1, \]  
\text{(3.29a)}

\[ [A, C] = -2\lambda A_0 - \{(A_0 B_0 \varphi_0) + (C_0 \varphi_0)\} A_1, \]  
\text{(3.29b)}

\[ [B_0, C] = 2\lambda B_0 - (B_0^2 \varphi_0) A_1, \]  
\text{(3.29c)}

realize the prolongation algebra \( \mathcal{L} \) expressed by (3.3a)-(3.3c).

When \( A_1 = 0 \), we obtain \( A = A_0, C = C_0, \) and the commutation rules (3.29a)-(3.29c) reproduce just the \( sl(2, R) \) algebra (3.25). In some sense, the algebra defined by Eqs. (3.29a)-(3.29c) plays the role of an ”extended” algebra, \( \mathcal{L}_E \), relatively to the \( sl(2, R) \) algebra (3.25). Since \( A_1 \) is multiplied by an arbitrary function, it turns out that \( \mathcal{L}_E \) is an infinite-dimensional subalgebra of \( \mathcal{L} \) (satisfying all the constraints involved by the theory of non-local symmetries). In the following, we shall see that the use of \( \mathcal{L}_E \) instead of \( sl(2, R) \) enables us to obtain nontrivial non-local symmetries.
Now let us demand that

$$[A_o, \eta_o] = 0, [B_o, \eta_o] = 0, [C_o, \eta_o] = 0. \quad (3.30)$$

A possible bidimensional realization of the algebra (3.25) is

$$A_o = \lambda \partial_{y_1}, \quad (3.31a)$$
$$B_o = -y_1^2 \partial_{y_1} + y_1 \partial_{y_2}, \quad (3.31b)$$
$$C_o = -2\lambda y_1 \partial_{y_1} + \lambda \partial_{y_2}. \quad (3.31c)$$

From (3.12b) we find

$$[A_1, \eta_o] = \frac{\eta_o \varphi_o}{\varphi_o} A_1 + \frac{B_o \varphi_o}{\varphi_o} A_o + C_o, \quad (3.32)$$

while (3.14b) gives

$$[A_1, \eta_o] = \frac{(\eta_o B_o \varphi_o)}{(B_o \varphi_o)} A_1 + \frac{2\lambda \varphi_o B_o}{(B_o \varphi_o)} + \frac{(B_o^2 \varphi_o) A_o}{(B_o \varphi_o)}. \quad (3.33)$$

Furthermore, from (3.14d) we deduce

$$[A_1, \eta_o] = \frac{(\eta_o B_o^2 \varphi_o)}{(B_o^2 \varphi_o)} A_1 + \frac{2\lambda \varphi_o B_o}{(B_o^2 \varphi_o)} A_o - C_o. \quad (3.34)$$

By comparing (3.32), (3.33) and (3.34) we have

$$\frac{(\eta_o \varphi_o)}{\varphi_o} = \frac{(\eta_o B_o \varphi_o)}{(B_o \varphi_o)} = \frac{(\eta_o B_o^2 \varphi_o)}{(B_o^2 \varphi_o)}, \quad (3.35)$$
$$\frac{(B_o \varphi_o)}{\varphi_o} A_o + C_o = \frac{(B_o \varphi_o)}{(B_o \varphi_o)} A_o = \frac{2\lambda \varphi_o B_o}{(B_o \varphi_o)} A_o + C_o. \quad (3.36)$$

Now we are ready to derive $\varphi_o, A_1,$ and $\eta_o.$ In doing so, let us suppose that these quantities can be expressed as

$$\varphi_o = \varphi_o(y) \equiv \varphi_o(y_1, y_2, y_3), \quad (3.37)$$
$$A_1 = a_1(y) \partial_{y_1} + a_2(y) \partial_{y_2} + a_3(y) \partial_{y_3}, \quad (3.38)$$
$$\eta_o = f_1(y) \partial_{y_1} + f_2(y) \partial_{y_2} + f_3(y) \partial_{y_3}, \quad (3.39)$$

where $a_j, f_j$ are functions of $y \equiv (y_1, y_2, y_3)$ to be determined.
From (3.36) we deduce

$$B_o \varphi_o = 2 \varphi_o y_1,$$  
(3.40)

which yields

$$\varphi_o = \phi(y_3) e^{2y_3},$$  
(3.41)

$\phi(y_3)$ being a function of integration depending on $y_3$ only.

Then, taking account of (3.40), it turns out that (3.35) and (3.36) are identically satisfied.

On the other hand, since $A_1 \varphi_o = 0$ we get (see (3.41) and (3.38)):

$$2a_2(y) \phi(y_3) + a_3(y) \phi_{y_3}(y_3) = 0.$$  
(3.42)

Then, resorting to the commutation rule $[A_1, B_o] = 0$, we can write

$$[B_o, A_1] = (-y_1^2 \partial_{y_1} + y_1 \partial_{y_2}) A_1 \varphi_o + A_1 (y_1^2 \partial_{y_1} - y_1 \partial_{y_2}) \varphi_o = 0,$$  
(3.43)

which gives $a_1(y) = 0$.

Furthermore, from $[A_1, A_o] = 0$ we infer that the functions $a_2$ and $a_3$ appearing in (3.42) are independent from the pseudopotential variable $y_1$, i.e. $a_2 = a_2(y_2, y_3), a_3 = a_3(y_2, y_3)$.

Finally, the commutation relation $[A_1, C_o] = 0$ yields

$$A_1 = a_2(y_3) \partial_{y_2} + a_3(y_3) \partial_{y_3},$$  
(3.44)

where $a_2$ and $a_3$ are functions of the pseudopotential variable $y_3$ only.

At this point let us consider the operator $\eta_o$, which obeys the commutation rules $[A_o, \eta_o] = 0, [B_o, \eta_o] = 0, [C_o, \eta_o] = 0$. The role of $\eta_o$ is formally analogous to that played by $A_1$. Therefore, we easily get

$$\eta_o = f_2(y_3) \partial_{y_2} + f_3(y_3) \partial_{y_3}.$$  
(3.45)

Then, we can exploit the commutator $[A_1, \eta_o]$ expressed by (3.32). Indeed, substituting (3.44) and (3.45) into (3.32) and equating the coefficients of $\partial_{y_2}$ and $\partial_{y_3}$ to zero, we are led to the relations

$$f_3 a_{2y_3} - a_3 f_{2y_3} + 2f_2 a_2 + f_3 a_2 \frac{\phi_{y_3}}{\phi} + \lambda = 0,$$  
(3.46)
\[ -a_3 f_{3y_3} + a_{3y_3} f_3 + 2f_2 a_3 + f_3 a_3 \frac{\phi_{y_3}}{\phi} = 0. \]  
(3.47)

We remark that the same relations come from (3.14e).

Equations (3.42), (3.46) and (3.47) represent an overdetermined system with unknowns \(a_2, a_3, f_2, f_3,\) and \(\phi\). The knowledge of these quantities provides the function \(\varphi_o\) and the operators \(A_1, \eta_o\).

By using the realization (3.31a)-(3.31c), (3.44) and (3.45), the generator (3.15) takes the form

\[ V_{NL} = -\phi(y_3)e^{2y_2} \partial_x - 2u \phi(y_3)y_1 e^{2y_2} \partial_u + \phi(y_3)e^{2y_2} y_1^2 \partial_{y_1} + f_2(y_3) - y_1 \phi(y_3)e^{2y_2} \partial_{y_2} + f_3(y_3) \partial_{y_3}, \]  
(3.48)

Thus, the corresponding group transformations arise from the differential equations

\[
\frac{d \tilde{x}}{d \varepsilon} = -\phi(\tilde{y}_3)e^{2\tilde{y}_2}, \\
\frac{d \tilde{u}}{d \varepsilon} = -2 \tilde{u} \phi(\tilde{y}_3) \tilde{y}_1 e^{2\tilde{y}_2}, \\
\frac{d \tilde{y}_1}{d \varepsilon} = \phi(\tilde{y}_3) \tilde{y}_1^2 e^{2\tilde{y}_2}, \\
\frac{d \tilde{y}_2}{d \varepsilon} = -\phi(\tilde{y}_3) \tilde{y}_1 e^{2\tilde{y}_2} + f_2(\tilde{y}_3), \\
\frac{d \tilde{y}_3}{d \varepsilon} = f_3(\tilde{y}_3),
\]  
(3.49a-d)

where \(\tilde{t} = t, \varepsilon\) is the group parameter, and the boundary conditions

\[
\tilde{x}|_{\varepsilon=0} = x, \tilde{u}|_{\varepsilon=0} = u, \tilde{y}_1|_{\varepsilon=0} = y_1, \tilde{y}_2|_{\varepsilon=0} = y_2, \tilde{y}_3|_{\varepsilon=0} = y_3,
\]  
(3.50)

are considered.

Now, in order to illustrate how our method works, let us consider the trivial solution \(u = -1\) to Eq. (3.1). Consequently, the prolongation equations (3.2a) and (3.2b) provide
\[ y_{1x} = \lambda - y_1^2, \quad (3.51a) \]
\[ y_{1t} = -4\lambda^2 + 4\lambda y_1^2, \quad (3.51b) \]
\[ y_{2x} = y_1 + a_2 \phi(y_3) e^{2y_2}, \quad (3.51c) \]
\[ y_{2t} = -8\lambda a_2 \phi(y_3) e^{2y_2} - 4\lambda y_1 + 4a_2 y_1^2 \phi(y_3) e^{2y_2}, \quad (3.51d) \]
\[ y_{3x} = a_3 \phi(y_3) e^{2y_2}, \quad (3.51e) \]
\[ y_{3t} = -8\lambda a_3 \phi(y_3) e^{2y_2} + 4a_3 y_1^2 \phi(y_3) e^{2y_2}, \quad (3.51f) \]

where here \( y_1, y_2, y_3 \) have not to be regarded as vector fields, but functions of \((x, t)\).

After some manipulations, from the equations (3.49a)-(3.49e) and (3.51a)-(3.51f) we obtain

\[ \tilde{u} = u \frac{y_1^2}{\gamma^2} \quad (3.52) \]

where \( y_1 \) can be derived by solving the pair of Riccati equations (3.51a) and (3.51b). By choosing for example \( \lambda > 0 \), we obtain

\[ y_1 = \sqrt{\lambda} \frac{e^{\sqrt{\lambda} \xi} - ae^{-\sqrt{\lambda} \xi}}{e^{\sqrt{\lambda} \xi} + ae^{-\sqrt{\lambda} \xi}}, \quad (3.53) \]

where \( a \) is a constant of integration, and \( \xi = x - 4\lambda t \).

By scrutinizing the remaining equations involving the prolongation variables, i.e. (3.51c)-(3.51f), we have

\[ \tilde{y}_1 = y_1 \frac{1 - 2\varepsilon \chi \lambda a(x - 12\lambda t) + \frac{\varepsilon \sqrt{\lambda}}{2}(e^{2\sqrt{\lambda} \xi} - a^2 e^{-2\sqrt{\lambda} \xi})}{1 - 2\varepsilon \chi \lambda a(x - 12\lambda t) - \frac{\varepsilon \sqrt{\lambda}}{2}(e^{2\sqrt{\lambda} \xi} - a^2 e^{-2\sqrt{\lambda} \xi})}. \quad (3.54) \]

In order to look for interesting solutions to Eq. (3.1) (starting from the trivial solution \( u = -1 \)), we have to use Eq. (3.52) and write the variables...
$x, t$ in terms of $\tilde{x}, \tilde{t}$. In doing so, since $t = \tilde{t}$, it is sufficient to consider Eqs. (3.49a) and (3.49c). These yield

$$\tilde{x} = x + \frac{1}{y_1} - \frac{1}{y_1}. \quad (3.55)$$

Then, by using (3.54) and (3.53) we have ($t = \tilde{t}$)

$$\tilde{x} = x - \frac{\varepsilon \chi (e^{\sqrt{\lambda} \xi} + ae^{-\sqrt{\lambda} \xi})^2}{1 - 2\varepsilon \chi \lambda a (x - 12\lambda t) + \frac{\varepsilon \chi \sqrt{\lambda}}{2} (e^{2\sqrt{\lambda} \xi} - a^2 e^{-2\sqrt{\lambda} \xi})}. \quad (3.56)$$

From (3.56) we can derive, formally, $x$ as a function of $\tilde{x}$ and $\tilde{t}$:

$$x = \tau(\tilde{x}, \tilde{t}). \quad (3.57)$$

where $\xi = \tau(\tilde{x}, \tilde{t}) - 4\lambda \tilde{t}$.

We remark that for $a = 0$, $\chi = -1$ and $\varepsilon < 0$, Eq. (3.57) produces the (formal) solitary wave solution

$$\tilde{u} = \sec h^2 \sqrt{\lambda} (\xi + \delta) - 1, \quad (3.58)$$

where $\delta$ is a constant defined by $\frac{\varepsilon \chi \sqrt{\lambda}}{2} = e^{2\sqrt{\lambda} \delta}$. This solution corresponds to that found in [10]. By choosing in the Riccati equations (3.51a) and (3.51b) $\lambda < 0$, a procedure similar to that employed to derive formula (3.57) leads to the solution (A3) reported in Appendix A.

To conclude this Section, we observe that the solutions (3.57) and (A3) are a consequence of the choice $u = -1$ (i.e. the trivial solution of the Dym equation) in the Bäcklund transformation (3.52). Of course, in theory other choices should be possible and, correspondingly, other solutions should be derived.
4 Non-local symmetries of the Korteweg-de Vries equation

Another interesting case which constitutes a good laboratory for checking the validity of our approach to non-local symmetries with pseudopotentials, is given by the KdV equation

\[ u_t + 6uu_x + u_{xxx} = 0. \]  \hspace{1cm} (4.1)

The pseudopotential for Eq. (4.1) is defined by

\[ y_x = F = \frac{1}{2} u^2 A + uB + C, \]  \hspace{1cm} (4.2a)

\[ y_t = G = (-uu_{xx} + \frac{1}{2} u_x^2 - 2u^3)A - (u_{xx} + 3u^2)B - \frac{1}{2} u^2 [B, D] - u[C, D] + u_x D + E, \]  \hspace{1cm} (4.2b)

where \( A, B, C, D, E \) are vector fields (depending on \( y \equiv \{y_j\} \) only) satisfying the commutation relations

\[ [A, B] = [A, C] = [A, D] = 0, \]  \hspace{1cm} (4.3a)

\[ [A, [B, D]] = 0, \]  \hspace{1cm} (4.3b)

\[ [B, [B, D]] = [A, [D, C]] = 0, \]  \hspace{1cm} (4.3c)

\[ [A, E] = 3[C, [B, D]] + 6D, \]  \hspace{1cm} (4.3d)

\[ [B, E] = [C, [C, D]], \]  \hspace{1cm} (4.3e)

\[ [C, E] = 0, \]  \hspace{1cm} (4.3f)

\[ [C, [B, D]] = [B, [C, D]], \]  \hspace{1cm} (4.3g)
\[ D = [C, B]. \] (4.3h)

The prolongation algebra (4.3a)-(4.3h) is incomplete.

Now let us carry out the infinitesimal transformation

\[ \tilde{u} = u + \varepsilon \varphi(u, y). \] (4.4)

Then, Eq. (4.1) provides

\[ \varphi_t + 6u\varphi_x + 6\varphi u_x + \varphi_{xxx} = 0. \] (4.5)

Equation (4.5) can be managed to get the following set of constraints:

\[ A\varphi = 0, \] (4.6a)

\[ B^2\varphi = 0, \] (4.6b)

\[ CB\varphi = -2\varphi, \] (4.6c)

\[ -3B\varphi - \frac{1}{2}[B, D]\varphi + 6B\varphi + B^2C\varphi + BCB\varphi = 0, \] (4.6d)

\[ -[C, D]\varphi + 6C\varphi + BCB\varphi + C^2B\varphi = 0, \] (4.6e)

\[ E\varphi + C^3\varphi = 0, \] (4.6f)

where \( \varphi = \varphi(y) \), i.e. \( \varphi \) depends on the pseudopotential only.

Now let us perform the infinitesimal transformation for the pseudopotential \( y : \)

\[ \tilde{y} = y + \varepsilon \eta(u, y). \] (4.7)

By virtue of (4.7), Eq. (4.2a) yields

\[ u_x\eta_u + [F, \eta] - \varphi F_u = 0. \] (4.8)

Equation (4.8) tells us that \( \eta_u = 0 \), i.e. \( \eta = \eta(y) \). Moreover, the following conditions

\[ [B, \eta] = 0, \] (4.9a)
\[ [C, \eta] - \varphi B = 0, \]  
\[ (4.9b) \]
\[ [D, \eta] + (B\varphi)B = 0, \]  
\[ (4.9c) \]
\[ [[B, D], \eta] = 0, \]  
\[ (4.9d) \]
\[-[[C, D], \eta] + 6\varphi B + \varphi[B, D] - (B\varphi)D + (BC\varphi)B + (CB\varphi)B = 0, \]  
\[ (4.9e) \]
\[ [E, \eta] + \varphi[C, D] - (C\varphi)D + (C^2\varphi)B = 0, \]  
\[ (4.9f) \]

hold.

Then, our task is to find the infinitesimal generator of the non-local symmetries for Eq. (4.1), which reads

\[ V_{NL} = \varphi \partial_u + \eta. \]  
\[ (4.10) \]

To this aim, first we look for a finite-dimensional representation of the prolongation algebra (4.3a)-(4.3h). Assuming \( A = 0 \), a representation of this kind is

\[ [C_o, B_o] = D_o, \]  
\[ (4.11a) \]
\[ [B_o, D_o] = -2B_o, \]  
\[ (4.11b) \]
\[ [C_o, D_o] = 4\lambda B_o + 2C_o, \]  
\[ (4.11c) \]
\[ E_o = -4\lambda C_o, \]  
\[ (4.11d) \]

where \( \lambda \) is an arbitrary parameter.

We have seen that limiting ourselves to starting from the closed algebra (4.11a)-(4.11d), namely taking \( A = 0, B = B_o, C = C_o, E = E_o \), only trivial non-local symmetries arise. Thus, as a possible way out we employ a procedure similar to that applied in Section 3 for the case of Dym equation. In other words, we search a realization of the prolongation algebra (4.3a)-(4.3h) such that

\[ A = 0, B = B_o, \]  
\[ (4.12) \]

and

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\[ C = C_o + \mu C_1, \quad (4.13) \]

where \( \mu = \mu(y) \) is a function of the pseudopotential to be determined, \( B_o, C_o, D_o \) satisfy the relations (4.11a)-(4.11c), and the operator \( C_1 \neq 0 \) is supposed to obey the commutation rules

\[ [C_1, B_o] = [C_1, C_o] = [C_1, D_o] = 0. \quad (4.14) \]

For simplicity, we take \( E_1 = C_1 \).

Now let us introduce the operator

\[ E = E_o + \nu E_1, \quad (4.15) \]

where \( E_o \) is given by (4.11d) and \( \nu \) is a function depending on the pseudopotential. Let us assume also that the relations

\[ [E_1, B_o] = [E_1, C_o] = [E_1, D_o] = 0, \quad (4.16) \]

\[ E_1 \varphi = E_1 \mu = 0, \quad (4.17) \]

are valid.

By dealing with the commutation rule (4.9a), it is natural to put

\[ \eta = \gamma \mu B_o + \eta_1, \quad (4.18) \]

where \( \gamma \) is a constant, and the operator \( \eta_1 \) is chosen in such a way that

\[ [\eta_1, B_o] = [\eta_1, C_o] = [\eta_1, D_o] = 0. \quad (4.19) \]

In order to obtain the infinitesimal generator of the non-local symmetries, we need to know the field \( \varphi \) and the operator \( \eta \) (see (4.10)). To this aim, first we shall exploit a specific realization of the algebra (4.11a)-(4.11c). Precisely, let us consider

\[ B_o = -\partial_{y_1}, \quad (4.20a) \]

\[ C_o = (\lambda - y_1^2)\partial_{y_1} + y_1\partial_{y_2}, \quad (4.20b) \]
\[ D_o = -2y_1 \partial_{y_1} + \partial_{y_2}. \]  

(4.20c)

Second, we make the hypothesis that the field \( \varphi(y) \) and the functions \( \mu(y) \) and \( \nu(y) \) depend on a pseudopotential vector having at least three components, say \( y \equiv (y_1, y_2, y_3) \). Consequently, in this context it is reliable to suppose that the operators \( C_1 \) and \( \eta_1 \), present in (4.13) and (4.18), respectively, take the form

\[ C_1 = \varphi_1(y) \partial_{y_1} + \varphi_2(y) \partial_{y_2} + \varphi_3(y) \partial_{y_3}, \]  

(4.21)

and

\[ \eta_1 = X_1(y) \partial_{y_1} + X_2(y) \partial_{y_2} + X_3(y) \partial_{y_3}, \]  

(4.22)

where \( \varphi_j(y) \) and \( X(y) \) are functions to be determined.

Starting from the prolongation algebra (4.3a)-(4.3h), and keeping in mind the previous assumptions and the algebraic constraints (4.6a)-(4.6f) and (4.9a)-(4.9f), in analogy with the case of the Dym equation, we can deduce another set of constraints which we omit for brevity.

By using all these relations, after some manipulations we have obtained the following results:

\[ \mu = \mu_1(y_3) e^{2y_2}, \]  

(4.23)

\[ \varphi = -\alpha \mu_1(y_3) y_1 e^{2y_2}, \]  

(4.24)

\[ \nu = 4\mu_1(y_3)y_1^2 e^{2y_2} + \nu_1(y_2, y_3), \]  

(4.25)

\[ C_1 = \varphi_2(y_3) \partial_{y_2} + \varphi_3(y_3) \partial_{y_3}, \]  

(4.26)

\[ \eta_1 = X_2(y_3) \partial_{y_2} + X_3(y_3) \partial_{y_3}, \]  

(4.27)

where the functions \( \mu_1 = \mu_1(y_3), \nu_1 = \nu_1(y_2, y_3), \varphi_2 = \varphi_2(y_3), \varphi_3 = \varphi_3(y_3), X_2 = X_2(y_3), X_3 = X_3(y_3) \) fulfill the system of linear differential equations

\[ 2\varphi_2 \mu_1 + \varphi_3 \mu_1 y_3 = 0, \]  

(4.28a)

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\[- \varphi_{2y_3} X_3 + \varphi_3 X_{2y_3} = \frac{\alpha}{4} + 2 \varphi_2 X_2 + \varphi_2 X_3 \frac{1}{\mu_1} \mu_{1y_3}, \quad (4.28b)\]
\[\varphi_3 X_{3y_3} - \varphi_{3y_3} X_3 = \varphi_3 X_3 \frac{1}{\mu_1} \mu_{1y_3} + 2 \varphi_3 X_2, \quad (4.28c)\]
\[X_2 \nu_{1y_2} + X_3 \nu_{1y_3} = 2 X_2 \nu_1 + X_3 \nu_1 \frac{1}{\mu_1} \mu_{1y_3}, \quad (4.28d)\]
\[16 \lambda \mu_1 e^{2y_2} + \nu_{1y_2} = 0, \quad (4.28e)\]
\[\phi_2 \nu_{1y_2} + \phi_3 \nu_{1y_3} = 0. \quad (4.28f)\]

This system can be handled by means of the procedure used for the Dym equation.

In doing so, let us explicit the infinitesimal generator (4.10), which reads

\[V_{NL} = -\alpha y_1 \mu_1 (y_3) e^{2y_2} \partial_u + \frac{\alpha}{4} \mu_1 (y_3) e^{2y_2} \partial_{y_1} + X_2 (y_3) \partial_{y_2} + X_3 (y_3) \partial_{y_1} \quad (4.29)\]

Then, from the the group transformations corresponding to the operator (4.29) we easily find

\[\tilde{u} = -2(y_1^2 - y_1^2) + u. \quad (4.30)\]

In our scheme, if we choose, for instance, \(u = 0\), the prolongation equations (4.2a) and (4.2b) become

\[y_{1x} = \lambda - y_1^2, \quad (4.31a)\]
\[y_{1t} = -4 \lambda^2 + 4 \lambda y_1^2, \quad (4.31b)\]
\[y_{2x} = y_1 + \varphi_2 (y_3) \mu_1 (y_3) e^{2y_2}, \quad (4.31c)\]
\[y_{2t} = -4 \lambda y_1 + (4 y_1^2 - 8 \lambda) \varphi_2 (y_3) \mu_1 (y_3) e^{2y_2}, \quad (4.31d)\]
\[y_{3x} = \varphi_3 (y_3) \mu_1 (y_3) e^{2y_2}, \quad (4.31e)\]

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Then, for $\lambda > 0$, from Eqs. (4.31a) and (4.31b) we obtain

$$y_1 = \sqrt{\lambda} \frac{e^{\sqrt{\lambda} \xi} - ae^{-\sqrt{\lambda} \xi}}{e^{\sqrt{\lambda} \xi} + ae^{-\sqrt{\lambda} \xi}},$$  \hspace{1cm} (4.32)

where $\xi = x - 4\lambda t$ and $a$ is a constant.

By integrating Eqs. (4.31c)-(4.31f), from Eq.(4.30) we get (for $u = 0$)

$$\tilde{u} = \frac{2\lambda(e^{\sqrt{\lambda} \xi} - ae^{-\sqrt{\lambda} \xi})^2}{(e^{\sqrt{\lambda} \xi} + ae^{-\sqrt{\lambda} \xi})^2} - 2\{\sqrt{\lambda}(e^{\sqrt{\lambda} \xi} - ae^{-\sqrt{\lambda} \xi}) \frac{(e^{\sqrt{\lambda} \xi} + ae^{-\sqrt{\lambda} \xi})}{(e^{\sqrt{\lambda} \xi} + ae^{-\sqrt{\lambda} \xi})^2} \} + \frac{\alpha \xi}{4 - \varepsilon \alpha (2a(x - 12\lambda t) + \frac{1}{2\sqrt{\lambda}}(e^{2\sqrt{\lambda} \xi} - a^2e^{-2\sqrt{\lambda} \xi}))^2}. \hspace{1cm} (4.33)$$

We notice that since $\tilde{x} = x$, $\tilde{t} = t$, this solution is explicit and contains the well-known soliton solution [10]

$$\tilde{u} = 2\lambda \sec^2(\sqrt{\lambda} \xi + \frac{1}{2} \ln \left| \varepsilon \right|), \hspace{1cm} (4.34)$$

which emerges for $a = 0$ and $\alpha = -1$.

Another solution to the KdV equation can be obtained in correspondence of the choice $\lambda < 0$. This is quoted in Appendix B.

## 5 Conclusions

We have developed a procedure to obtain non-local symmetries of the pseudopotential type of nonlinear field equations whose prolongation algebra $L$ is incomplete. We have considered two case studies: the Dym and the KdV equation, respectively. For both equations, first we have found a finite-dimensional subalgebra (quotient algebra) $L_o$ of the related (incomplete) prolongation algebra $L$. Then, we have tried to use $L_o$ to look for non-local symmetries.
Unfortunately, through $\mathcal{L}_o$ only trivial symmetries emerge. Consequently, we have "extended" the subalgebra $\mathcal{L}_o$ by introducing new operators to be determined by the requirement that the commutation relations defining $L$ and the constraints involved by the infinitesimal transformations for the non-local symmetries are satisfied. The determination of these new operators is crucial, since they appear in the generator of the non-local symmetries. For the two equations under investigation this task has been successful. From the generator of the non-local symmetries, expressed in terms of pseudopotential variables, one can write the corresponding group transformations which enable us to yield exact solutions of the Dym and the KdV equations. Some of these solutions are well-known. Notwithstanding, they serve as paradigms to probe and to illustrate the potentiality of our algebraic approach.

As we can argue from the results achieved on the above-mentioned applications, our method could be exploited to treat other nonlinear field equations admitting nontrivial prolongations. But some aspects of the method remain to be elucidated, and only the accumulation of cases could indicate the appropriate way of implementation. To be precise, for instance we remark that a basic role is played by the realization of the "extended" algebra in terms of vector fields depending on pseudopotential variables. In our calculations, we have chosen simple but nonlinear realizations (of the polynomial type). Different algebraic realizations (say, polynomial realizations in higher dimensions or realizations which are not of the polynomial type) might produce different non-local symmetries and, correspondingly, different solutions to the original equations. Anyway, this question is to be explored. Another interesting attempt which deserves to be made is the use of an infinite-dimensional realization (of the prolongation algebra $L$) of the Kac-Moody type. Could this kind of realization have a significant role in the search of non-local symmetries of the pseudopotential type? Finally, we point out that an important problem is the extension of our approach to nonlinear field equations in more than 1+1 dimensions. However, due to the fact that at present only a few applications in higher dimensions have been carried out within the prolongation scheme [14], this programme is strictly connected with a possible revival of interest in the extension of the prolongation studies.
Here we sketch the calculation to find the solution to the Dym equation (3.1) corresponding to the choice \( \lambda < 0 \) in the Riccati equations (3.51a) and (3.51b). These give

\[
y_1 = \sqrt{|\lambda|} \frac{\cos \sqrt{|\lambda|} \theta - b \sin \sqrt{|\lambda|} \theta}{\sin \sqrt{|\lambda|} \theta + b \cos \sqrt{|\lambda|} \theta},
\]

(A1)

where \( \theta = x + 4 \mid \lambda \mid t \) and \( b \) is a constant.

This formula is the analogous of (3.53). Carrying out the same type of calculations leading to (3.56), we obtain the transformation

\[
\tilde{x} = x - [\varepsilon \chi (\sin \sqrt{|\lambda|} \theta + b \cos \sqrt{|\lambda|} \theta)]^2 \times \\
\{1 + \frac{\varepsilon \chi |\lambda|}{4} [2(b^2 + 1)(x + 12 \mid \lambda \mid t) + \frac{(b^2-1) \sin 2\sqrt{|\lambda|} \theta - 2b \cos 2\sqrt{|\lambda|} \theta}{\sqrt{|\lambda|}}] \\
+ \varepsilon \chi \mid \lambda \mid (\cos \sqrt{|\lambda|} \theta - b \sin \sqrt{|\lambda|} \theta)(\sin \sqrt{|\lambda|} \theta + b \cos \sqrt{|\lambda|} \theta)\}^{-1},
\]

(A2)

where \( \chi = \pm 1 \).

If \( x = \gamma(\tilde{x}, \tilde{t}) \) indicates formally the inverse of the expression (A2), keeping in mind (A1) from (3.52) we have

\[
\tilde{u} = -\{1 + \frac{\varepsilon \chi |\lambda|}{4} [2(b^2 + 1)(\gamma(\tilde{x}, \tilde{t}) + 12 \mid \lambda \mid \tilde{t}) + \frac{(b^2-1) \sin 2\sqrt{|\lambda|} \tilde{\theta} - 2b \cos 2\sqrt{|\lambda|} \tilde{\theta}}{\sqrt{|\lambda|}}] \\
+ \varepsilon \chi \mid \lambda \mid (\cos \sqrt{|\lambda|} \tilde{\theta} - b \sin \sqrt{|\lambda|} \tilde{\theta})(\sin \sqrt{|\lambda|} \tilde{\theta} + b \cos \sqrt{|\lambda|} \tilde{\theta})\}^{-1},
\]

(A3)

where \( \tilde{\theta} = \gamma(\tilde{x}, \tilde{t}) + 4 \mid \lambda \mid \tilde{t} \).
7 Appendix B

By choosing $\lambda < 0$, Eqs. (4.31a) and (4.31b) give rise to the solution expressed by (A1). Following the same procedure adopted to find the solution (4.33), we obtain

$$\tilde{u} = [-4\varepsilon\alpha\sqrt{|\lambda|} \cos \sqrt{|\lambda|}\theta - b \sin \sqrt{|\lambda|}\theta] \left( \sin \sqrt{|\lambda|}\theta + b \cos \sqrt{|\lambda|}\theta \right) \times \left\{ 4 - \varepsilon\alpha^2 \left[ \frac{2(1 + b^2)\theta}{2} \right] + \frac{(b^2 - 1) \sin 2 \sqrt{|\lambda|}\theta - 2b \cos 2 \sqrt{|\lambda|}\theta}{\sqrt{|\lambda|}} \right\}^{-1} - 2\varepsilon^2 \alpha^2 (\sin \sqrt{|\lambda|}\theta + b \cos \sqrt{|\lambda|}\theta)^4 \times \left\{ 4 - \varepsilon\alpha^2 \left[ \frac{2(1 + b^2)(x + 4|\lambda|t)}{2} \right] + \frac{(b^2 - 1) \sin 2 \sqrt{|\lambda|}\theta - 2b \cos 2 \sqrt{|\lambda|}\theta}{\sqrt{|\lambda|}} \right\}^{-2};$$

with $\theta = x + 4|\lambda|t$, $\alpha \in \mathbb{R}$.

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