BOOK:
M O D E L S  o f  s c i e n c e  d y n a m i c s -
encounters between complexity
theory and information sciences

CHAPTER 3

Knowledge epidemics
and population dynamics models
for describing idea diffusion

N I K O L A Y  K .  V I T A N O V  a n d  M A R C E L  R .  A U S L O O S
Abstract

The diffusion of ideas is often closely connected to the creation and diffusion of knowledge and to the technological evolution of society. Because of this, knowledge creation, exchange and its subsequent transformation into innovations for improved welfare and economic growth is briefly described from a historical point of view. Next, three approaches are discussed for modeling the diffusion of ideas in the areas of science and technology, through (i) deterministic, (ii) stochastic, and (iii) statistical approaches. These are illustrated through their corresponding population dynamics and epidemic models relative to the spreading of ideas, knowledge and innovations.

The deterministic dynamical models are considered to be appropriate for analyzing the evolution of large and small societal, scientific and technological systems when the influence of fluctuations is insignificant. Stochastic models are appropriate when the system of interest is small but when the fluctuations become significant for its evolution. Finally statistical approaches and models based on the laws and distributions of Lotka, Bradford, Yule, Zipf-Mandelbrot, and others, provide much useful information for the analysis of the evolution of systems in which development is closely connected to the process of idea diffusion.
| **10 IMPORTANT QUESTIONS RAISED IN THIS CHAPTER** | **AND THEIR ANSWERS in the form of guidance** |
|--------------------------------------------------|-----------------------------------------------|
| 1. What is the connection between knowledge and capital? | Knowledge is often considered as a form of human capital |
| 2. What happens in the case of knowledge diffusion? | Knowledge is transferred when the subjects interact |
| 3. Should quantitative research be supplemented by qualitative research? | Yes, surely supplemented coordinated joint aims are useful |
| 4. Who are the pioneers of scientometrics? | Alfred Lotka and Derek Price |
| 5. What is the relation between epidemic models and population dynamics models? | Epidemic models are a particular case of population dynamics models |
| 6. What has to be done if fluctuations strongly influence the system evolution? | Switch from deterministic to stochastic models and think |
| 7. Why are discrete models useful? | Often data is collected for some period of time. Thus, such data is best described by discrete models |
| 8. Around which statistical law are grouped all statistical tools described in the chapter? | Around Lotka law |
| 9. Are all possibly relevant models, presented in this chapter? | NO! Only an appropriate selection. For more models, consult the literature or ask a specialist |
| 10. What is the strategy followed by the authors of the chapter? | Proceed from simple to more complicated models and from deterministic to stochastic models supplemented by statistical tools |

Table 1: Several questions and answers that should guide and supply useful and important information for the reader.

1 Knowledge, capital, science research, and ideas diffusion

1.1 Knowledge and capital

Knowledge can be defined as a dynamic framework connected to cognitive structures from which information can be sorted, processed and understood [1]. Along economics lines of thought [2][3][4], knowledge can be treated as one of the "production factors", i.e., one of the main causes of wealth in modern capitalistic societies.
| Models described in this chapter | Are useful for |
|----------------------------------|----------------|
| Science landscapes               | Evaluation of research strategies. Decisions about personal development and promotion |
| Verhulst Logistic curve          | Description of a large class of growth processes |
| Broadcasting model of technology diffusion | Understanding the influence of mass media on technology diffusion |
| Word-of-mouth model              | Understanding the influence of interpersonal contacts on technology diffusion |
| Mixed information source model   | Understanding the influence of both mass media and interpersonal contacts on technology diffusion |
| Lotka-Volterra model of innovation diffusion with time lag | Understanding the influence of the time lag between hearing about innovation and its adoption |
| Price model of knowledge growth with time lag | Modeling the growth of discoveries, inventions, and scientific laws |
| SIR models of scientific epidemics | Modeling the epidemic stage of scientific idea spreading |
| SEIR models of scientific epidemics | Extends the SIR model by specifically adding the role of a class of scientists exposed to some scientific idea |

Table 2: List of models described in the chapter with comments on their usefulness.

According to Marshall [5] a "capital" is a collection of goods external to the economic agent that can be sold for money and from which an income can be derived. Often, knowledge is parametrized as such a "human capital" [6, 7, 8, 9, 10]. Walsh [11] was one pioneer in treating human knowledge as if it was a "capital", in the economic sense; he made an attempt to find measures for this form of "capital". Bourdieu [12], Coleman [13], Putnam [14], Becker and collaborators have further implanted the concept of such a "human capital" in economic theory [15, 16, 17].

However, the concept of knowledge as a form of capital is an oversimplification. This global-like concept does not account for many properties of knowledge strictly connected to the individual, such as the possibility for different learning paths or different views, multiple levels of interpretation,
Models described in this chapter are useful for

| Models described in this chapter                                      | Are useful for                                                                 |
|--------------------------------------------------------------------|-------------------------------------------------------------------------------|
| Discrete model for the change in the number of authors in a scientific field | Modeling and forecasting the evolution in the number of authors and papers in a scientific field |
| Daley model                                                        | Modeling the evolution of a population of papers in a scientific field         |
| Coupled discrete model for populations of scientists and papers    | Modeling and forecasting the joint evolution of population of scientists and papers in a research field |
| Goffman-Newill model for the joint evolution of one scientific field and one of its sub-fields | Epidemic model for the increase of number of scientists from a research field who start work in a sub-field of the scientific field. The model also describes the increase in the number of papers in the research sub-field |
| Bruckner-Ebeling-Scharnhorst model for the evolution of n scientific fields | Understanding the joint evolution of scientific fields in presence of migration of scientists from one field to another field |

Table 3: List of models described in the chapter with comments on their usefulness (Continuing Table 2).

and different preferences [18]. In fact, knowledge develops in a quite complex social context, within possibly different frameworks or time scales, and involves "tacit dimensions" (beside the basic space and time dimensions) requiring coding and decoding [4].

FOR POLICY-MAKERS
Take away box Nr.1: Knowledge is much more than a form of capital: it is a dynamic framework connected to cognitive structures from which information can be sorted, processed and understood.

1.2 Growth and exchange of knowledge

Science policy-makers and scholars have for many decades wished to develop quantitative methods for describing and predicting the initiation and growth of science research [19] [20] [21]. Thus, scientometrics has become one of
Models described in this chapter are useful for:

| Models described in this chapter | ARE USEFUL FOR |
|----------------------------------|----------------|
| SI model for the probability of intellectual infection | Modeling the spread of intellectual infection along a scientific network |
| SEI model for the probability of intellectual infection | Modeling the spread of intellectual infection along a scientific network in the presence of a class of scientists exposed to the intellectual infection |
| Stochastic evolution model | Modeling the number of scientists in a research subfield as a stochastic variable described by a master equation |
| Stochastic model of scientific productivity | Modeling the influence of fluctuations in scientific productivity through differential equations for the dynamics of a scientific community |
| Model of competition between ideologies | Understanding the competition between ideologies with possible migration of believers |
| Reproduction-transport model | Modeling the change of research field as a migration process |

Table 4: List of models described in the chapter with comments on their usefulness (Continuation of Table 2).

the core research activities in view of constructing science and technology indicators [22].

The accumulation of the knowledge in a country’s population arises either from acquiring knowledge from abroad or from internal engines [23, 24, 25, 26]. The main engines for the production of new knowledge in a country are usually: the public research institutes, the universities and training institutes, the firms, and the individuals [27]. The users of the knowledge are firms, governments, public institutions (such as the national education, health, or security institutions), social organizations, and any concerned individual. The knowledge is transferred from producers to the users by dissemination that is realized by some flow or diffusion of process [28], sometimes involving physical migration.

Knowledge typically appears at first as purely tacit: *a person "has" an idea* [29, 30]. This tacit knowledge must be codified for further use; after codification, knowledge can be stored in different ways, as in textbooks or digital carriers. It can be transferred from one system to another. In addition to knowledge creation, a system can gain knowledge by knowledge exchange and/or trade.
Table 5: List of laws discussed in the chapter with a few words on their usefulness (Continuation of Table 2).

| LAWS DESCRIBED IN THIS CHAPTER | ARE USEFUL FOR |
|--------------------------------|----------------|
| Lotka law                      | Describing the number distribution of scientists with respect to the number of papers they wrote |
| Pareto distribution            | Writing a continuous version of Lotka law |
| Zipf law and Zipf-Mandelbrot law| Ranking scientists by the number of papers they wrote |
| Bradford law                   | Reflecting the fact that a large number of relevant articles are concentrated in a small number of journals |

In knowledge diffusion, the knowledge is transferred while subjects interact \[31\] \[32\] \[33\]. Pioneering studies on knowledge diffusion investigated the patterns through which new technologies are spread in social systems \[34\] \[35\]. The gain of knowledge due to knowledge diffusion is one of the keys or leads to innovative products and innovations \[36\] \[37\].

FOR POLICY-MAKERS
Take away box Nr.2:
An innovative product or a process is new for the group of people who are likely to use it. Innovation is an innovative product or process that has passed the barrier of user adoption. Because of the rejection by the market, many innovative products and processes never become an innovation.

In science, the diffusion of knowledge is mainly connected to the transfer of scientific information by publications. It is accepted that the results of some research become completely scientific when they are published \[38\]. Such a diffusion can also take place at scientific meetings and through oral or other exchanges, sometimes without formal publication of exchanged ideas \[39\].
FOR POLICY-MAKERS
Take away box Nr.3:
Scientific communication has specific features. For example, citations are very important in the communication process as they place corresponding research and researchers, mentioned in the scientific literature, in a way similar to the kinship links that tie persons within a tribe. Informal exchanges happening in the process of common work at the time of meetings, workshops, or conferences may accelerate the transfer of scientific information, whence the growth of knowledge.

2 Qualitative research. Historical remarks.

2.1 Science landscapes

Understanding the diffusion of knowledge requires research complementary to mathematical investigations. For example, mathematics cannot indicate why the exposure to ideas leads to intellectual epidemics. Yet, mathematics can provide information on the intensity or the duration of some intellectual epidemics.

Qualitative research is all about exploring issues, understanding phenomena, and answering questions without much mathematics. Qualitative research involves empirical research through which the researcher explores relationships using a textual methodology rather than quantitative data. Problems and results in the field of qualitative research on knowledge epidemics will not be discussed in detail here. However, through one example it can be shown how mathematics can create the basis for qualitative research and decision making. This example is connected to the science landscape concepts outlined here below.

The idea of science landscapes has some similarity with the work of Wright in biology who proposed that the fitness landscape evolution can be treated as an optimization process based on the roles of mutation, inbreeding, crossbreeding, and selection. The science landscape idea was developed by Small, as well as by Noyons and van Raan. In this framework, Scharnhorst proposed an approach for the analysis of scientific landscapes, named “geometrically oriented evolution theory”.

7
FOR POLICY-MAKERS
Take away box Nr.4:
The concept of science landscape is rather simple: Describe the corresponding field of science or technology through a function of parameters such as height, weight, size, technical data, etc. Then a virtual knowledge landscape can be constructed from empirical data in order to visualize and understand innovation and to optimize various processes in science and technology.

As an illustration at this level, consider that a mathematical example of a technological landscape can be given by a function $C = C(S,v)$, where $C$ is the cost for developing a new airplane, and where $S$ and $v$ represent the size and velocity of the airplane.

Consider two examples concerning the use of science landscapes for evaluation purposes:

(1) Science landscape approach as a method for evaluating national research strategies
For example, national science systems can be considered as made of researchers who compete for scientific results, and subsidies, following optimal research strategies. The efforts of every country become visible, comparable and measurable by means of appropriate functions or landscapes: e.g., the number of publications. The aggregate research strategies of a country can thereby be represented by the distribution of publications in the various scientific disciplines. In so doing, within a two-dimensional space, different countries correspond to different landscapes. Various political discussions can follow and evolution strategies can be invented thereafter.

Notice that the dynamics of self-organized structures in complex systems can be understood as the result of a search for optimal solutions to a certain problem. Therefore, such a comment shows how rather strict mathematical approaches, not disregarding simulation methods, can be congruent to qualitative questions.

(2) Scientific citations as landscapes for individual evaluation
Scientific citations can serve for constructing landscapes. Indeed, citations have a key position in the retrieval and valuation of information in scientific communication systems [45, 47, 48]. This position is based on the objective

\[ E.g., \text{take the scientific disciplines and the number of publications as axes} \]
nature of the citations as components of a global information system, as represented by the Science Citation Index. A landscape function based on citations can be defined in various ways. It can take into account self-citations [49, 50, 51, 52], or time-dependent quantitative measures [53, 54, 55].

FOR POLICY-MAKERS
Take away box Nr.5:
Citation landscapes become important elements of a science policy (e.g., in personnel management decisions), thereby influencing individual scientific careers, evaluation of research institutes, and investment strategies.

2.2 Lotka and Price: pioneers of scientometrics

Alfred Lotka, one of the modern founders of population dynamics studies, was also an excellent statistician. He discovered a distribution for the number of authors $n_r$ as a function of the number of published papers $r$, i.e., $n_r = n_1/r^2$.

However, Derek Price, a physicist, set the mathematical basis in the field of measuring scientific research in recent times [58, 59, 60]. He proposed a model of scientific growth connecting science and time. In the first version of the model, the size of science was measured by the number of journals founded in the course of a number of years. Later, instead of the number of journals, the number of published papers was used as the measure of scientific growth. Price and other authors [59, 60, 61] considered also different indicators of scientific growth, such as the number of authors, funds, dissertation production, citations, or the number of scientific books.

In addition to the deterministic approach initiated by Price, the statistical approach to the study of scientific information developed rapidly and nowadays is still an important tool in scientometrics [62, 63]. More discussion on the statistical approach will be given in section 6 of this chapter.
FOR POLICY-MAKERS
Take away box Nr.6:
Price distinguished three stages in the growth of knowledge: (a) a preliminary phase with small increments; (b) a phase of exponential growth; (c) a saturation stage. The stage (c) must be reached sooner or later after the new ideas and opportunities are exhausted; the growth slows down until a new trend emerges and gives rise to a new growth stage. According to Price, the curve of this growth is a S-shaped logistic curve.

2.3 Population dynamics and epidemic models of the diffusion of knowledge

![Diagram of Population dynamics models, Lotka-Volterra models, and Epidemic models]

Figure 1: Relation among epidemic models, Lotka-Volterra models, and population dynamics models.

Population dynamics is the branch of life sciences that studies short- and long-term changes in the size and age composition of populations, and how the biological and environmental processes influence those changes. In the past, most models for biological population dynamics have been of interest only in mathematical biology [64, 65]. Today, these models are adapted and applied in many more areas of science [66, 67]. Here below, models of knowledge dynamics will be of interest as bases of epidemic models. Such models are nowadays used because some stages of idea spreading processes within a population (e.g., of scientists), possess properties like those of epidemics.

The mathematical modeling of epidemic processes has attracted much attention since the spread of infectious diseases has always been of great concern and considered to be a threat to public health [68, 69, 70]. In the history of science and society, many examples of ideas spreading seem to
occur in a way similar to the spread of epidemics. Examples of the former field pertain to the ideas of Newton on mechanics and the passion for “High Critical Temperature Superconductivity” at the end of the twentieth century. Examples of the latter field are the spreading of ideas from Moses or Buddha [71], or discussions based on the Kermack-McKendrick model [72] for the epidemic stages of revolutions or drug spreading [73].

Epidemic models belong to a more general class of Lotka-Volterra models used in research on systems in the fields of biological population dynamics, social dynamics, and economics. The models can also be used for describing processes connected to the spread of knowledge, ideas and innovations (see Fig. 1). Two examples are the model of innovation in established organizations [74] and the Lotka-Volterra model for forecasting emerging technologies and the growth of knowledge [36]. In social dynamics, the Lanchester model of war between two armies can be mentioned, a model which in the case of reinforcements coincides with the Lotka-Volterra-Gause model for competition between two species [75]. Solomon and Richmond [76, 77] applied a Lotka-Volterra model to financial markets, while the model for the trap of extinction can be applied to economic subjects [78]. Applications to chaotic pairwise competition among political parties [79] could also be mentioned.

To start the discussion of population dynamics models as applied to the growth of scientific knowledge with special emphasis on epidemic models, two kinds of models can be discussed (Fig. 2): (1) deterministic models, see Sec. 3, appropriate for large and small populations where the fluctuations are not drastically important, (2) stochastic models, see Sec. 4, appropriate for small populations. In the latter case the intrinsic randomness appears much more relevant than in the former case. Stochastic models for large populations will not be discussed. The reason for this is that such models usually consist of many stochastic differential equations, whence their evolution can be investigated only numerically.

Finally, let us mention that the knowledge diffusion is closely connected to the structure and properties of the social network where the diffusion happens. This is a new and very promising research area. For example, a combination can be made between the theory of information diffusion and the theory of complex networks [80]. For more information about the relation between networks and knowledge, see the following chapters of the book.
3 Deterministic models

Below, 13 selected deterministic models (see Fig. 3) are discussed. The emphasis is on models that can be used for describing the epidemic stage of the diffusion of ideas, knowledge, and technologies.

3.1 Logistic curve and its generalizations

In a number of cases, the natural growth of autonomous systems in competition can be described by the logistic equation and the logistic curve (S-curve) [81]. In order to describe trajectories of growth or decline in socio-technical systems, one generally applies a three-parameter logistic curve:

\[
N(t) = \frac{K}{1 + \exp[-\alpha t - \beta]} \tag{1}
\]

where \(N(t)\) is the number of units in the species or growing variable to study; \(K\) is the asymptotic limit of growth; \(\alpha\) is the growth rate which specifies the "width" of the S-curve for \(N(t)\); and \(\beta\) specifies the time \(t_m\) when the curve
reaches the midpoint of the growth trajectory, such that \( N(t_m) = 0.5 K \). The three parameters, \( K \), \( \alpha \), and \( \beta \), are usually obtained after fitting some data \[82\]. It is well known that many cases of epidemic growth can be described by parts of an appropriate S-curve. As an example, recall that the S-curve was also used for describing technological substitution \[34, 83, 84\], ca. 60 years ago.

However, different interaction schemes can generate different growth patterns for whatever system species are under consideration \[85\]. Not every interaction scheme leads to a logistic growth \[86\]. The evolution of systems in such regimes may be described by more complex curves, such as a combination of two or more simple three-parameter functions \[81, 87\].
3.2 Simple epidemic and Lotka-Volterra models of technology diffusion

As recalled here above, the simplest epidemic models could be used for describing technology diffusion, like considering two populations/species: adopters and non-adopters of some technology. Such models can be put into two basic classes: either broadcasting (Fig. 4) or word-of-mouth models (Fig. 5). In the broadcasting models, the source of knowledge about the existence and/or characteristics of the new technology is external and reaches all possible adopters in the same way. In the word-of-mouth models, the knowledge is diffused by means of personal interactions.

(1) The broadcasting model (Fig. 4)
Let us consider a population of $K$ potential adopters of the new technology and let each adopter switch to the new technology as soon as he/she hears about its existence (immediate infection through broadcasting). The probability that at time $t$ a new subject will adopt the new technology is characterized by a coefficient of diffusion $\kappa(t)$ which might or might not be a function of the number of previous adopters. In the broadcasting model $\kappa(t) = a$ with $(0 < a < 1)$; this is considered to be a measure of the infection probability.

Let $N(t)$ be the number of adopters at time $t$. The increase in adopters for each period is equal to the probability of being infected, multiplied by the current population of non-adopters [88]. The rate of diffusion at time $t$ is

$$\frac{dN}{dt} = a[K - N(t)].$$

(2) The integration of (2) leads to the number of adopters: i.e.,

$$N(t) = K[1 - \exp(-at)].$$

$N(t)$ is described by a decaying exponential curve.

(2) Word-of-mouth model (Fig. 5)
In many cases, however, the technology adoption timing is at least an order of magnitude slower than the time it takes for information spreading [89]. This requires another modelization than in (1): the word-of-mouth diffusion model. Its basic assumption is that knowledge diffuses by means of face-to-face interactions. Then the probability of receiving the relevant knowledge needed to adopt the new technology is a positive function of current users
Figure 4: Schematic representation of a broadcasting model of technology diffusion. The number of adopters of technology increases by mass media influence.

Let the coefficient of diffusion $\kappa(t)$ be $bN(t)$ with $b > 0$. The rate of diffusion at time $t$ is

$$\frac{dN}{dt} = b N(t) [K - N(t)].$$

Then

$$N(t) = \frac{K}{1 + \left(\frac{K - N_0}{N_0}\right)e^{-bK(t-t_0)}}$$

where $N_0 = N(t = t_0)$. $N(t)$ is described by an S-shaped curve.

A constraint exists in the word-of-mouth model: it explains the diffusion of an innovation not from the date of its invention but from the date when some number, $N(t) > 0$, of early users have begun using it.
Figure 5: Schematic representation of a word-of-mouth model of technology diffusion. The number of adopters of technology increases by interpersonal interactions.

(3) Mixed information source model (Fig. 6)
In the mixed information source model, existing non-adopters are subject to two sources of information (Fig. 6). The coefficient of diffusion is supposed to look like $a + bN(t)$. The model evolution equation becomes

$$\frac{dN}{dt} = (a + bN(t)) [K - N(t)].$$ (6)

The result of Eq.(6) is a (generalized) logistic curve whose shape is determined by $a$ and $b$ [88].

(4) Time lag Lotka-Volterra model of innovation diffusion (Fig. 7)
Let it be again assumed that the diffusion of innovation in a society is accounted for by a combination of two processes: a mass-mediated process
and a process connected to interpersonal (word-of-mouth) contacts. Let $N(t)$ be the number of potential adopters. Some of the potential adopters adopt the innovation and become real adopters. The equation for the the rate of growth of the real adopters $n(t)$, in absence of time lag, is

$$\frac{dn(t)}{dt} = \alpha [N(t) - n(t)] + \beta n(t)[N(t) - n(t)] - \mu n(t),$$

where $\alpha$ denotes the degree of external influence such as mass media, $\beta$ accounts for the degree of internal influence by interpersonal contact between adopters and the remaining population; $\mu$ is a parameter characterizing the
Figure 7: Schematic representation of a Lotka-Volterra model with time lag. The model accounts for the time lag between hearing about innovation and its adoption.

decline in the number of adopters because of technology rejection for whatever reason.

A basic limitation in most models of innovation diffusion has been the assumption of instantaneous acceptance of the new innovation by a potential adopter [88, 90]. Often, in reality, there is a finite time lag between the moment when a potential adopter hears about a new innovation and the time of adoption. Such time lags usually are continuously distributed [91, 92].

The time lag between the knowledge about the innovation and its adoption can be captured by a distributed time lag approach in which the effects of time delays are expressed as a weighted response over a finite time interval through appropriately chosen memory kernels [93] (see Fig. 7). Whence
Eq. (7) becomes

\[
\frac{dn(t)}{dt} = \alpha \int_0^t d\tau K_1^*(t-\tau) [N(\tau) - n(\tau)] + \\
\beta \int_0^t d\tau K_2^*(t-\tau)n(\tau)[N(\tau) - n(\tau)] - \mu \int_0^t d\tau K_3^*(t-\tau)n(\tau).
\]

(8)

Eq. (8) reduces to Eq. (7) when the memory kernels \( K_i^*(t) \) \( (i = 1, 2, 3) \) are replaced by delta functions.

Two generic types of kernels are usually considered [92]:

\[
K^*(t) = \nu e^{-\nu t},
\]

(9)

\[
K^*(t) = \nu^2 t e^{-\nu t},
\]

(10)
in which \( \nu^{-1} \) is some characteristic time scale of the system.

The number of potential adopters \( N(t) \) changes over time. Several possible functional forms of \( N(t) \) are used [94]:

\[
N(t) = N_0(1 + at); \quad N_0 > 0, a > 0
\]

(11)

\[
N(t) = N_0 \exp(gt); \quad N_0 > 0, g > 0
\]

(12)

\[
N(t) = \frac{b}{1 + d \exp(-ct)}; \quad b > 0, d > 0, c > 0
\]

(13)

\[
N(t) = b - q \exp(-rt); \quad b > 0, q > 0, r > 0.
\]

(14)

Eq. (12) represents an approximation for short- and medium-term forecasting since for \( t \) large, \( N(t) \) grows without bound, as in Keynes [95]. Eqs. (13) and (14) are useful in long-term forecasting as \( N(t) \) has an upper limit. Such forms for \( N(t) \) are valid within a deterministic framework.

However, a stochastic framework (see below) is more appropriate when the carrying capacity \( N(t) \) is governed by some stochastic process, as when the influence of socioeconomic and natural factors are subject to "random" or hardly explainable fluctuations. In such systems, \( N(t) \) can be time-dependent: for example, \( N(t) \sim N_0(1 + \epsilon \cos(\omega t)) \) where \( \epsilon \ll 1 \) and the periodicity takes into account the influence of some (strong) cyclic economic factors. In presence of a strong stochastic component, \( N(t) \) can be stochastic: \( N(t) = N_0 + \xi(t) \), where the noisy component is \( \xi(t) \) and \( N_0 \) is the average value of the so-called carrying capacity [96].
FOR POLICY-MAKERS
Take away box Nr.7:
Time lags between observations and decisions lead to complicated dynamics. Perform some preliminary careful analysis of system behavior based on time lags before making a decision.

3.3 Price model of knowledge growth. Cycles of growth of knowledge

Figure 8: Diagram of relationships between Price model and its modifications. The presence of time lags can lead to much complication in the evolution dynamics of a scientific field.
The Price evolution model of scientific growth ignited intensive research \[97, 98\] (see Fig. 8). This model is in fact a dialectical addition to Kuhn’s idea \[99\] about the revolutionary nature of science processes: after some period of evolutionary growth, a scientific revolution occurs. Price considered the exponential growth as a disease that retards the growth of stable science, producing narrower and less flexible specialists.

**FOR POLICY-MAKERS**
Take-away box Nr.8

An interesting result of the research of Price can be read as follows: if a government wants to double the usefulness of science, it has to multiply by about eight the gross number of workers and the total expenditure of manpower and national income.

The unreserved application of the Price model faces several difficulties:

- many scientific products which seem to be new are not really new;
- creativity and innovation can be confused \[100, 101\];
- creative papers with new ideas and results have the same importance as trivial duplications \[102\];
- two things are omitted:
  - quality (whatever that means, but it is an economic notion) of research;
  - the cost or measure of complexity.

In answer to this, Price formulated the hypothesis that one should be studying only the growth of important discoveries, inventions, and scientific laws, rather than both important and trivial things. In so doing, one might expect that any of such studied growth will follow the same pattern.

A generalized version of the Price model for the growth of a scientific field \[103, 104\] is based on the following assumptions: (a) the growth is measured by the number of important publications appearing at a given time; (b) the growth has a continuous character, though a finite time period \( T = \text{const} \) is needed to build up a result of the fundamental character; (c) the interactions
between various scientific fields are neglected. If, in addition, the number of scientists publishing results in this field is constant, then the rate of scientific growth is proportional to the number of important publications at time $t$ minus the time period $T$ required to build up a fundamental result. The model equation is

$$\frac{dx}{dt} = \alpha x(t - T),$$

(15)

where $\alpha$ is a constant. The initial condition $x(t) = \phi(t)$ is defined on the interval $[-T, 0]$.

Let the population of scientists be varying and consider the evolution of the average number of papers per scientist. In general, instead of the linear right-hand side Eq.(15), a non-linear model can be used:

$$\frac{dx}{dt} = f(x(t - T), x(t)),$$

(16)

where $f(t - T)$ is a homogeneous function of degree one. The simplest form of such a function is a linear function. Let $n(t)$ represent the rate of growth of the population of scientists and write $L(t) = \exp[n(t) t]$. For simplicity, let the population of scientists grow at the constant rate $n = \frac{1}{T} \frac{dL}{dt}$ and let $z = x/L$. Then the evolution of the number of papers written by a scientist has the form

$$\frac{dz}{dt} = \alpha z(t - T) - nz(t).$$

(17)

If $n = 0$ and $T = 0$, the Price model of exponential growth is recovered. Eq.(17) is linear, but a cyclic behavior may appear because of the feedback between the delayed and non-delayed terms.

3.4 Models based on three or four populations. Discrete models.

(1) SIR (Susceptible-Infected-Removed) model (Fig.9)

In 1927, Kermack and McKendrick [72] created a model in which they considered a fixed population with only three compartments: $S(t)$, the susceptibles; $I(t)$, the infected; $R(t)$, the recovered, or removed.

Following this idea, Goffman and Newill [71, 105] considered the stages of fast growth of scientific research in a scientific field as ”intellectual epidemics” and developed the corresponding scientific research epidemic stage based on
Figure 9: SIR (susceptibles $S$, infectives $I$, recovered $R$) model of intellectual infection with influxes of susceptibles and infectives to the corresponding scientific ideas.

three classes of population: (i) the susceptibles $S$ who can become infectives when in contact with infectious material (the ideas); (ii) the infectives $I$ who host the infectious material; and (iii) the recovered $R$ who are removed from the epidemics for different reasons (Fig. 9).

The epidemic stage is controlled by the system of differential equations

$$ \frac{dS}{dt} = -\beta SI - \delta S + \mu, $$  \hspace{1cm} (18)  
$$ \frac{dI}{dt} = \beta SI - \gamma I + \nu, $$ \hspace{1cm} (19)  
$$ \frac{dR}{dt} = \delta S + \gamma I $$ \hspace{1cm} (20)  

where $\mu$ and $\nu$ are the rates at which the new supply of susceptibles and infectives enter the population. A necessary condition for the process to
enter the epidemic state is $\frac{dI}{dt} > 0$. Then

$$S > \frac{\gamma - \nu}{\beta} = \rho$$

is the threshold density of susceptibles, i.e., no epidemics can develop from time $t_0$ unless $S_0$, the number of susceptibles at that time, exceeds the threshold $\rho$: the epidemic state cannot be maintained over some time interval unless the number of susceptibles is larger than $\rho$ through that interval of time. As $I$ increases, $\nu/I$ converges to 0 and $\rho$ converges rapidly to $\gamma/\beta$.

In [71], Goffman evaluated the rate of change of infectives $\Delta I/\Delta t$. From the system equations, it is difficult to determine $I(t)$. Yet in the epidemic stage, the behaviour of $I(t)$ is exponential. For small $t$ close to $t_0$, $I(t)$ can be expanded into a power series: $I(t) = C_0 + C_1 t + C_2 t^2 + \ldots C_n t^n + \ldots$ such that the approximate rate of $\Delta I/\Delta t$ can be obtained. On the basis of this rate and the raw data, the development and peak of some research activity can be predicted, - under the assumption that the research is in an epidemic stage.

(2) SEIR model for the spreading of scientific ideas (Fig. 10)
The SIR epidemic models can be further refined by introducing a fourth class, $E$, i.e., persons exposed to the corresponding scientific ideas (Fig. 10). Such models are discussed in [106, 107]; they belong to the class of so-called SEIR epidemic models. One typical model goes as follows

$$\frac{dS}{dt} = \lambda N - \frac{\beta SI}{N}; \quad \frac{dE}{dt} = \frac{\beta SI}{N} - \kappa E - \frac{\rho EI}{N};$$

$$\frac{dI}{dt} = \kappa E + \frac{\rho EI}{N} - \gamma I; \quad \frac{dR}{dt} = \gamma I$$

where $S(t)$ is the size of the susceptible population at time $t$, $E(t)$ is the size of the exposed class, $I(t)$ is the size of the infected class. These individuals have adopted the new scientific idea in their publications. Finally, $R(t)$ is the size of the population of recovered scientists, i.e., those who no longer publish on the topic. The size of the entire population is: $N = S + E + I + R$. An exit term is assumed to be very small, and because of this, $t$ is included in the recovered class. $N$ grows exponentially with rate $\lambda$. The parameters of the model are: $\beta$, the probability and effectiveness of a contact with an adopter; $1/\kappa$, the standard latency time, (in other words, the average duration of time after one has been exposed but before one includes the new idea in one’s own
Figure 10: SEIR model of intellectual infection with influxes of susceptibles and infectives to the corresponding scientific ideas, thus extending the SIR model by including a class of scientists exposed \((E)\) to the specific scientific ideas.

\(1/\gamma\), the duration of the infectious period, thus how long one publishes on the topic and teaches others; \(\rho\), the probability that an exposed person has multiple effective contacts with other adopters.

This simple model can incorporate a wide range of behaviors. For many values of the parameters \(\lambda, \beta, \kappa, \gamma\) and \(\rho\), the infected class grows as a logistic curve. For large values of the contact rate \(\beta\) or recruitment \(\lambda\), \(I(t)\) grows nearly linearly, as indeed has been found empirically for some research fields [106].
FOR POLICY-MAKERS
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Epidemic models are the best suited for describing the expansion stage of a process growth.

(3) SI discrete model for the change in the number of authors in a scientific field (Fig. 11)
With the goal of predicting the spreading out of scientific objects (such as theories or methods), Nowakowska [108] discussed several epidemic discrete models for predicting changes in the number of publications and authors in a given scientific field. With respect to the publications, the main assumption

Figure 11: Schema of a discrete SI evolution model of the number of authors of scientific papers. The model takes into account that several scientists stop their work in a scientific field; it can be due to different reasons as for example death or losing interest in particular questions.
of the models is that the number of publications in the next period of time (say, one year) will depend: (i) on the number of papers which recently appeared, and (ii) on the degree at which the subject has been exhausted. The numbers of publications appearing in successive periods of time should first increase, then would reach a maximum, and as the problem becomes more and more exhausted, the number of publications would decrease.

Let it be assumed (Fig. 11) that if at a certain moment \( t \) the epidemics state is \((x_t, y_t)\) \( (x_t \) is the number of infectives (authors who write papers on the corresponding research problems) \), \( y_t \) is the number of susceptibles\), then for a sufficiently short time interval \( \Delta t \), one may expect that the number of infectives \( x_{t+\Delta t} \) will be equal to \( x_t - ax_t \Delta t + bx_t y_t \Delta t \), while the number of susceptibles \( y_{t+\Delta t} \) will be equal to \( y_t - bx_t y_t \Delta t; \) \( a \) and \( b \) being appropriate constants. Let the expected number of individuals who either die or recover, during the interval \((t, t + \Delta t)\), be \( ax_t \Delta t \), and let \( bx_t y_t \Delta t \) be the expected number of new infections. The equations of this model are:

\[
\begin{align*}
    x_{t+\Delta t} &= ax_t - ax_t \Delta t + bx_t y_t \Delta t \\
    y_{t+\Delta t} &= y_t - bx_t y_t \Delta t.
\end{align*}
\] (24) (25)

Note here that such discrete models are useful for the analysis of realistic situations where the values of the quantities are available at selected moments (every month, every year, etc.).

(4) Daley discrete model for the population of papers (Fig. 12)

Daley [109] investigated the spread of news as follows: individuals who have not heard the news are susceptible and those who heard the news are infective. Recovery is not possible, as it is assumed that the individuals have perfect memory and never forget. The Daley model can be applied also to the population of papers [108] (see Fig. 12). For \( \Delta t = 1 \) (year), the Daley model equation reads

\[
x_{t+1} = bx_t \left( N - \sum_{i=1}^{t} x_i \right)
\] (26)

where \( x_1, x_2, \ldots \) are the numbers of papers on the subject which appear in successive periods of time, \( b \) and \( N \) being parameters. The expected number \( x_{t+1} \) of papers in year \( t + 1 \) is proportional to the number \( x_t \) of papers which appeared in year \( t \), and to the number \( N - x_1 - x_2 \ldots - x_t = N - \sum_{i=1}^{t} x_i \). \( N \) is the number of papers which have to appear in order to exhaust the
Daley model for population of papers in a research field

Figure 12: Daley model for evolution of population of papers on problems in a scientific field. The exhausting of the scientific field is taken into account.

problem: the problem under consideration may be partitioned into $N$ subproblems, such that solving any of them is worth a separate publication; these subproblems are solved successively by the scientists. The $b$ and $N$ parameters may be estimated by the method of least squares, e.g. from a given empirical histogram. A parameter characterizing the initial growth dynamics in the number of publications can also be introduced: $\tau = bN$. Therefore, Eq.\( (26) \) can be used for short-time prediction, even when the corresponding research field is in the epidemic stage of its evolution.

(5) Discrete model coupling the populations of scientists and papers (Fig. 13)

A discrete model coupling the populations of scientists and papers can be considered (Fig. 13); it depends on four parameters: $N$, $a$, $b$ and $c$. $N$ as above denotes the number of sub-problems of the given problem; $a$ is the probability that a scientist working on the subject in a given year abandons research on the subject for whatever reasons; $b$ is the probability of obtaining
Figure 13: Discrete model for the joint evolution of populations of scientists and papers. The attractiveness of the field, the exhaustion of the field, and the possibility for declining interest for working in the scientific field are taken into account through adequate rate parameters.

A solution to a given subproblem by one scientist during one year of research; $c$ denotes the coefficient of attractiveness of the subject. The basic variables of the model are: $u_t$, the number of scientists working on the subject in year $t$, and $x_t$, the number of publications on the subject which appear in year $t$.

The model equations are

$$u_{t+1} = (1 - a)u_t + cx_t$$  \hspace{1cm} (27)

$$x_{t+1} = \left[1 - (1 - b)^{u_t}\right] \left(N - \sum_{i=1}^{t} x_i\right).$$  \hspace{1cm} (28)

The equation for the number $u_{t+1}$ of scientists working on the subject in year $t+1$ tells that in year $t+1$, the expected number of scientists working on the
subject will be the number of scientists working on the subject in year $t$, $u_t$, minus the expected number of scientists who stopped working on the subject, $au_t$, plus the expected number of scientists, $cx_t$, who became attracted to the problem by reading papers which appeared in year $t$. The equation expressing the number of publications in year $t + 1$ tells us that $x_{t+1}$ equals the number of subproblems that were solved in the year $t$. The probability that a given subproblem will be solved in year $t$ by a given scientist equals $b$. Then the probability of the opposite event, i.e. a given scientist will not solve a particular problem, equals $1 - b$. As there are $u_t$ scientists working on the subject in year $t$, the probability that a given subproblem will not be solved by any of them is $(1 - b)^{u_t}$. Consequently, the probability that a given subproblem will be solved in year $t$ (by any of the $u_t$ scientists working on the subject) is equal to $1 - (1 - b)^{u_t}$. Next, in year $t$ there remained $N - \sum_{i=1}^{t} x_i$ subproblems to be solved. The expected number of subproblems solved in year $t$ is equal to the product which gives the right-hand side of Eq.(28).

N.B. It is assumed, that the waiting time for publishing of the paper is one year. A more realistic picture would be to assume that the unit of time is not one year, but two years, or that the publication has some other time delay.

\begin{center}
\small
\begin{tabular}{|l|}
\hline
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\hline
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\hline
In many cases, the data is available as one value per week, or one value per month, or one value per three months, etc. For modeling and subsequent short-range forecasting, so-called discrete (time) models are thus very appropriate.
\hline
\end{tabular}
\end{center}

3.5 Continuous models of the joint evolution of scientific sub-systems

(1) Coupled continuous model for the populations of scientists and papers: Goffman-Newill model

The Goffman-Newill model [105] (Fig. 14) is based on the idea that the spreading process within a population can be studied on the basis of the literature produced by the members of that population. There is a transfer of infectious materials (ideas) between humans by means of an intermediate
Figure 14: Schema of Goffman-Newill model for the evolution of a scientific field. Scientists are attracted to a sub-field after being intellectually infected by papers from the sub-field.

host (a written article). Let a scientific field be $F$ and $SF$ a sub-field of $F$. Let the number of scientists writing papers in the field $F$ at $t_0$ be $N_0$ and the number of scientists writing papers in $SF$ at $t_0$ (the number of infectives) be $I_0$. Thus, $S_0 = N_0 - I_0$ is the number of susceptibles; there is no removal at $t_0$, but there is removal $R(t)$ at later times $t$. The number of papers produced on $F$ at $t_0$ is $N'_0$ and the number of papers produced in $SF$ at this time is $I'_0$. The process of intellectual infection is as follows: (a) a member of $F$ is infected by a paper from $I'$; (b) after some latency period, this infected member produces 'infected' papers in $N'$, i.e. the infected member produces a paper in the subfield $SF$ citing a paper from $I'$; (c) this 'infected' paper may infect other scientists from $F$ and its sub-fields, such that the intellectual infection spreads from $SF$ to the other sub-fields of $F$.

Let $\beta$ be the rate at which the susceptibles from class $S$ become 'intel-
lectually infected’ from class $I$. Let $\beta'$ be the rate at which the papers in $SF$ are cited by members of $N$ who are producing papers in $SF$. As the infection process develops, some susceptibles and infectives are removed, i.e. some scientists are no longer active, and some papers are not cited anymore. Let $\gamma$ and $\gamma'$ be the rates of removal of infectives from the populations $I$ and $I'$ respectively, and $\delta$ and $\delta'$ be the rates of removal from the populations of susceptibles $S$ and $S'$. In addition, there can be a supply of infectives and susceptibles in $N$ and $N'$. Let the rates of introduction of new susceptibles be $\mu$ and $\mu'$, i.e. the rates at which the new authors and new papers are introduced in $F$, and let the rates of introduction of new infectives be $\upsilon$ and $\upsilon'$, i.e. the rates at which new authors and new papers are introduced in $SF$. In addition, within a short time interval a susceptible can remain susceptible or can become an infective or be removed; the infective can remain an infective or can become a removal; and the removal remains a removed. The immunes remain immune and do not return to the population of susceptibles. If, in addition, the populations are homogeneously mixed, the system of model equations reads

$$
\frac{dS}{dt} = -\beta SI' - \delta S + \mu; \quad \frac{dI}{dt} = \beta SI' - \gamma I + \upsilon
$$

(29)

$$
\frac{dR}{dt} = \gamma I + \delta S; \quad \frac{dS'}{dt} = -\beta' S'I - \delta S' + \mu'
$$

(30)

$$
\frac{dI'}{dt} = \beta' S'I - \gamma' I' + \upsilon'; \quad \frac{dR'}{dt} = \gamma' I' + \delta' S'
$$

(31)

The conditions for development of an epidemic are as follows. If as an initial condition at $t_0$, a single infective is introduced into the populations $N_0$ and $N'_0$, then for an epidemic to develop, the change of the number of infectives must be positive in both populations. Then, for $\rho = \frac{\gamma - \upsilon}{\beta}$ and $\rho' = \frac{\gamma' - \upsilon'}{\beta'}$, the threshold for the epidemic arises from the conditions $\beta SI' > \gamma I - \upsilon$ and $\beta' S'I' > \gamma' I' - \upsilon'$, such that the threshold is

$$
S_0 S'_0 > \rho \rho'.
$$

(32)

The development of epidemics is given by the equation $\frac{dI}{dt} = D(t)$. The peaks of the epidemic occur at time points where $\frac{d^2 I}{dt^2} = 0$, while the epidemic’s size is given by $I(t \rightarrow \infty)$.

(2) Bruckner-Ebeling-Scharnhorst model for the growth of $n$ subfields in a scientific field
The evolution of growth processes in a system of scientific fields can be modeled by complex continuous evolution models. One of them, the Bruckner-Ebeling-Scharnhorst approach [110] (Fig. 15), is closely related to several generalizations of Eigen’s theory of prebiotic evolution and is briefly discussed here (see also [111]). In 1912, Lotka [112] published the idea of describing biological epidemic processes, like malaria, as well as chemical oscillations, with the help of a set of differential equations. These equations, known as Lotka-Volterra equations [113, 114], are used to describe a coupled growth process of populations. However, they do not reflect several essential properties of evolutionary processes such as the creation of new structural elements. Because of this, one has to consider a more general set of equations for the change in the number $x_i$ of the scientists from the $i$-th scientific subfield (a Fisher-Eigen-Schuster kind of model), i.e.,

$$\frac{dx_i}{dt} = (A_i - D_i)x_i + \sum_{j=1; j \neq i}^{n} (A_{ij}x_j - A_{ji}x_i) + \sum_{j=1; j \neq i}^{n} B_{ij}x_i x_j - k_0 x_i,$$
$i,j = 1, \ldots, n.$ (33)

The model based on Eq. (33) describes the coupled growth of $n$ subfields, of a scientific discipline. Three fundamental processes of evolution are included in Eq. (33): (a) self-reproduction: students and young scientists join the field and start working on corresponding problems. Their choice is influenced mainly by the education process as well as by individual interests and by existing scientific schools; (b) decline: scientists are active in science for a limited number of years. For different reasons (for example, retirement) they stop working and leave the system; (c) field mobility: individuals turn to other fields of research for various reasons or maybe open up new ones themselves.

The reasoning to obtain Eq. (33) goes as follows. The general form of the law for growth of the $i$-th subfield is supposed to be

$$\frac{dx_i}{dt} = f_i(\vec{x}), \quad \vec{x} = (x_1, \ldots, x_n).$$ (34)

By separation, $f_i = w_i x_i$, one obtains the replicator equation

$$\frac{dx_i}{dt} = w_i x_i, \quad i = 1, 2, \ldots, n.$$ (35)

Notice that when $w_i = \text{const}$, the fields are uncoupled, i.e., there is an exponential growth in science. Otherwise, $w_i$ itself is a function of $x$ and of various parameters, but can be separated into three terms according to the above model assumptions, i.e.,

$$w_i = A_i - D_i + \sum_{j=1, j\neq i}^{n} \left( A_{ij} \frac{x_j}{x_i} - A_{ij} \right).$$ (36)

Eq. (33) is thus obtained from Eq. (35) and Eq. (36) for $B_{ij} = 0$, $k_0 = 0$. To adapt this model to real growth processes, it can be assumed that the coefficients $A_i$, $D_i$, and $A_{ij}$ themselves are functions of $x_i$:

$$A_i = A_i^0 + A_i^1 x_i + \ldots; \quad D_i = D_i^0 + D_i^1 x_i + \ldots; \quad A_{ij} = A_{ij}^0 + A_{ij}^1 x_j + \ldots.$$ (37)

Each of the three fundamental processes of change is represented in Eq. (33) with a linear and a quadratic term only. For example, the terms $A_i^1$ and $D_i^1$ account for cooperative effects in self-reproduction and decline processes.
respectively, while $D_0^i$ accounts for a decline, because of aging. The contributions $A_{ij}^0$ assume a linear type of field mobility behavior for scientists analogous to a diffusion process. On the other hand, the terms $A_{ij}^1$ represent a directed process of exchange of scientists between fields. The best way to obtain these parameters is to estimate them for specific data bases using the method of least squares.

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The Bruckner-Ebeling-Scharnhorst model does not belong to the class of epidemic models which are best applicable only for describing the expansion stage of a process. The Bruckner-Ebeling-Scharnhorst model is an evolution model: it describes all stages of the evolution of a system.

4 Small-size scientific and technological systems. Stochastic models (Fig. 16)

![Stochastic models diagram](image)

Figure 16: Hierarchy of stochastic models discussed in this chapter.
The movement of large bodies in mechanics is governed by deterministic laws. When the body contains a small number of molecules and atoms, stochastic effects such as the Brownian motion become important. In the area of scientific systems, the fluctuations become very important when the number of scientists in a certain research subfield is small. This is typical for new research fields with only a few researching scientists.

Several examples of stochastic models for the description of the diffusion of ideas or technology and the evolution of science are: (a) the model of evolution of scientific disciplines with an example pertaining to the case of elementary particles physics [115]; (b) stochastic models for the aging of scientific literature [116]; (c) stochastic models of the Hirsch index [55] and of instabilities in evolutionary systems [117]; (d) models of implementation of technological innovations [118], etc. [119]. In the following, see Fig. 16, two probabilistic and two stochastic models are discussed. Some attention is devoted to the master equation approach as well.

### 4.1 Probabilistic SI and SEI models

Epidemiological models of differential-equation-based compartmental type have been found to be limited in their capacity to capture heterogeneities at the individual level and in the interaction between individual epidemiological units [120]. This is one of the reasons to switch from models in which the number of individuals are in given known states to models involving probabilities. One such model [121] captures the diffusion of topics over a network of connections between scientific disciplines, as assigned by the ISI Web of Science’s classification in terms of Subject Categories (SCs). Each SC is considered as a node of a network along with all its directed and weighted connections to other nodes or SCs [121, 122]. As with epidemic models, nodes can be characterized in a medical way. SCs that are susceptible (S) are either not aware of a particular research topic or, if aware, may not be ready to adopt it. Incubating SCs (E) are those that are aware of a certain topic and have moved to do some research on problems connected with this topic. Infected SCs (I) are actively working and publishing in a particular research topic.

Two probabilistic models, i.e., (i) the Susceptible-Exposed-Infected (SEI) model (Fig. 17) and (ii) a simpler Susceptible-Infected (SI) model (Fig. 18), are thereby only discussed.

(1) **Susceptible-Exposed-Infected (SEI) model**
The SEI model equations for the evolution of the node state probabilities are given by [121]:

\[
\frac{dS_i(t)}{dt} = - \sum_j A_{ji} I_j(t) S_i(t), \quad (38)
\]

\[
\frac{dE_i(t)}{dt} = \sum_j A_{ji} I_j(t) S_i(t) - \gamma E_i(t), \quad (39)
\]

\[
\frac{dI_i(t)}{dt} = \gamma E_i(t), \quad (40)
\]

where \( 0 \leq I_i(t) \leq 1 \) denotes the probability of node \( i \) being infected at time \( t \) (likewise for \( S_i(t) \) and \( E_i(t) \)). The directed and weighted contact network...
is represented by $A_{ij} = r \Gamma_{ij}$ with $\Gamma_{ij} = (w_{ij})_{i,j=1,...,N}$ denoting the adjacency matrix that includes weighted links; $r$ is the transmission rate per contact and $1/\gamma$ is the average incubation or latent period.

This set of equations states that an increase in the probability $E_i$ of a node $i$ being exposed to an infection is directly proportional to the probability $S_i$ of node $i$ being susceptible and the probability $I_j$ of neighbouring nodes $j$ being infected. The number of such contacts and the per-contact rate of transmission are incorporated in $A_{ij}$. Likewise, $E_i$ decreases if exposed/infected nodes become infected after an average incubation time $1/\gamma$. The number of infected SCs at time $t$, according to the model, can be estimated as $I(t) = \sum_i I_i(t)$. Since $S_i(t) + E_i(t) + I_i(t) = 1$, for each $t > 0$, Eqs. (38) - (40) are readily understood, in view of Eq. (39).

(2) Susceptible-Infected (SI) model

The above SEI model can be simplified to an SI model when the possibility of an exposed period is excluded, i.e., if $\frac{dE_i(t)}{dt} = 0$. The equations for this

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{probabilistic_s_i_model.png}
\caption{Schema of the probabilistic SI model for epidemics in a network connecting scientific disciplines.}
\end{figure}
simpler SEI model are reduced to

\[ \frac{dS_i(t)}{dt} = - \sum_j A_{ji} I_j(t) S_i(t); \quad \frac{dI_i(t)}{dt} = \sum_j A_{ji} I_j(t) S_i(t), \]  

(41)

where the probability \( I_i \) of a node \( i \) being infected and infectious only depends on the probability \( S_i \) of the node \( i \) being susceptible. The comparison of both models with available data shows [121] that while the agreement at the population level is usually much better for the SEI model, for the same pair of parameters, the agreement at the individual level is better when the simpler SI model is used.

4.2 Master equation approach

(1) Stochastic evolution model with self-reproduction, decline, and field mobility

There exists a high correlation between field mobility processes and the emergence of new fields [110]. This can be accounted for by a stochastic model (see Fig. 19), in which the system at time \( t \) is characterized by a set of integers \( N_1, N_2, ..., N_i, ..., N_n \), with \( N_i \) being, e.g., the number of scientists working in the subfield \( i \), which is considered now as a stochastic variable. The three fundamental types of scientific change mentioned in the discussion of the Bruckner-Ebeling-Scharnhorst model (see above) here correspond to three elementary stochastic processes with three different transition probabilities:

- **(a)** For self-reproduction, the transition probability is given by
  \[ W(N_i + 1 \mid N_i) = A^0_i N_i = A^0_i N_i + A^1_i N_i(N_i - 1); \]

- **(b)** The transition probability for decline is
  \[ W(N_i - 1 \mid N_i) = D^0_i N_i + D^1_i N_i(N_i - 1); \]

- **(c)** The transition probability for field mobility is
  \[ W(N_i + 1, N_j - 1 \mid N_i N_j) = A^0_{ij} N_j + A^1_{ij} N_i N_j. \]

The probability density \( P(N_1, ..., N_i, N_j, ..., t) \) is given by the so-called master equation

\[ \frac{\partial P}{\partial t} = WP \]  

(42)

which can be solved analytically only in some very special cases [123].
(2) The master equation as a model of scientific productivity
The productivity factor is a very important ingredient in mathematically simulating a scientific community evolution. One way to model such an evolution is through a dynamic equation which takes into account the stochastic fluctuations of scientific community members productivity \[127\] (Fig. 20). The main processes of scientific community evolution accounted for by this model are, beside the biological constraints (like the self-reproduction, aging of scientists, and death), their departure from the field due to mobility or abandon of research activities. Call \(a\) the age of an individual and let a scientific productivity index \(\xi\) be in incorporated into the individual state space; both \(a\) and \(\xi\) are being considered to be continuous variables with values in \([0, \infty]\). The scientific community dynamics is described by a number
density function \( n(a, \xi, t) \), another form of scientific landscape, which specifies the age and productivity structure of the scientific community at time \( t \). For example, the number of individuals with age in \([a_1, a_2]\) and scientific productivity in \([\xi_1, \xi_2]\) at time \( t \) is given by the integral
\[
\int_{a_1}^{a_2} \int_{\xi_1}^{\xi_2} da d\xi \ n(a, \xi, t).
\]

A master equation for this function \( n(a, \xi, t) \) can be derived [127]:

\[
\left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) n(a, \xi, t) = -[J(a, \xi, t) + w(a, \xi, t)] n(a, \xi, t) + \\
\int_{-\infty}^{\xi} d\xi' \ \chi(a, \xi - \xi', t) \ n(a, \xi - \xi', t),
\]

where \( w(a, \xi, t) \) denotes the departure rate of community members. If \( x(t) \) is a random process describing the scientific productivity variation and if
$p_a(x,t \mid y,\tau)$ (with $\tau < t$) is the transition probability density corresponding to such a process, then

$$
\chi(a, \xi, \xi', t) = \lim_{\Delta t \to 0} \frac{p_a(\xi + \xi' + \Delta t | \xi, t)}{\Delta t}.
$$

(44)

The transition rate, at time $t$ from the productivity level $\xi$, $J(a, \xi, t)$ is by definition:

$$
J(a, \xi, t) = \int_{-\infty}^{\infty} d\xi' \chi(a, \xi, \xi', t). 
$$

The increment $\xi'$ may be positive or negative. The balance equation for $n(a, \xi, t)$ reads as follows

$$
n(a + \Delta a, \xi, t + \Delta t) = n(a, \xi, t) - J(a, \xi, t) n(a, \xi, t) \Delta t +
$$

$$
\left[ \int_{-\infty}^{\xi} \chi(a, \xi - \xi', t) n(a, \xi - \xi', t) d\xi' \right] \Delta t - w(a, \xi, t) n(a, \xi, t) \Delta t. 
$$

(45)

The term on the right-hand side, $[1 - J(a, \xi, t)\Delta t]n(a, \xi, t)$, describes the proportion of individuals whose scientific productivity does not change in $[t, t + \Delta t]$; the integral term describes the individuals whose scientific productivity becomes equal to $\xi$ because of increasing or decreasing in $[t, t + \Delta t]$; the last term corresponds to the departure of individuals due to stopping research activities or death. After expanding $n(a + \Delta t, \xi, t + \Delta t)$ around $a$ and $t$, keeping terms up to the first order in $\Delta t$, one obtains the master equation Eq.(43).

As the master equation is difficult to handle for an elaborate analysis, it is often reduced to an approximated equation similar to the well-known Fokker-Planck equation [124, 125, 126]. The approximation goes as follows. Let

$$
\mu_k(a, \xi, t) = \int_{-\xi}^{\infty} d\xi' (\xi')^k \chi(a, \xi, \xi', t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} < (\xi')^k >; \; k = 1, 2, \ldots,
$$

(46)

where the brackets denote the average with respect to the conditional probability density $p_a(\xi + \xi', t + \Delta t \mid \xi, t)$. In addition, the following assumptions are made: (i) $\mu_1, \mu_2 < \infty$; $\mu_k = 0$ for $k > 3$; (ii) $n(a, \xi, t)$ and $\chi(a, \xi, \xi', t)$ are analytic in $\xi$ for all $a$, $t$ and $\xi'$. The additional assumption $\mu_k = 0$ for $k > 3$ demands the productivity to be continuous in the sense that as $\Delta t \to 0$, the probability of large fluctuations $|\xi'|$ must decrease so quickly that $<|\xi'|^3> \to 0$ more quickly than $\Delta t$.

When the above assumptions hold, the function $n$ satisfies the equation

$$
\left( \frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) n = -\frac{\partial(\mu_1 n)}{\partial \xi} + \frac{1}{2} \frac{\partial^2(\mu_2 n)}{\partial \xi^2} - wn. 
$$

(47)
If \( w = 0 \), Eq. (47) is converted to the well known Fokker-Planck equation. Eq. (47) describes the scientific community evolution through a drift along the age component and a drift and diffusion with respect to the productivity component. The diffusion term characterized by the diffusivity \( \mu^2 \) takes into account the stochastic fluctuations of scientific productivity conditioned by internal factors (such as individual abilities, labour motivations, etc.) and external factors (such as labor organization, stimulation system, etc.). The initial and boundary conditions for Eq. (47) are: (a) \( n(a, \xi, 0) = n^0(a, \xi) \), where \( n^0(a, \xi) \) is a known function defining the community age and productivity distribution at time \( t = 0 \); and (b) \( n(0, \xi, t) = \nu(\xi, t) \) where the function \( \nu(\xi, t) \) represents the intensity of input flow of new members at age \( a = 0 \) being set \( \nu(\xi, 0) = n^0(0, \xi) \). In addition, \( n(a, \xi, t) \to 0 \) as \( a \to \infty \).

The general solution of equation Eq. (47) with the above initial condition (a) and boundary condition (b) is still a difficult task. However, for many practical applications, a knowledge of first and second moments of distribution function \( n(a, \xi, t) \) is sufficient. Eq. (47) can be solved numerically or can be reduced to a system of ordinary differential equations [127].

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FOR POLICY-MAKERS

Take away box Nr.12:

In deterministic cases, the system is robust against fluctuations: it follows some trajectory and the fluctuations are too weak to change it. When the fluctuations are important, then different trajectories for the evolution of the system become possible. To each trajectory, a probability can be assigned. This probability reflects the chance that the system will follow the corresponding trajectory. The collection of the probabilities leads to a probability distribution which can be calculated, in many evolutionary cases, on the basis of the master equation approach.

Finally, two additional problems that can be treated by the master equation approach can be mentioned:

- Age-dependent models where the birth and death rates connected to the selection are age-dependent [128] [129]
- The problem of new species in evolving networks [111]. On the basis of a stochastic treatment of the problem, the notion of ‘innovation’ can
be introduced in a broad sense as a disturbance and/or an instability of a corresponding social, technological, or scientific system. The fate of a small number of individuals of a new species in a biological system can be thought to be mathematically equivalent to some extent to the fate of a new idea, a new technology, or a new model of behavior. The evolution of the new species can be studied on evolving networks, where some nodes can disappear and new nodes can be introduced. This evolution of the network can change significantly the dynamic behavior of the entire system of interacting species itself. Some of the species can vanish in a finite time. This feature can be captured effectively by the master equation approach.

5 Space-time models. Competition of ideas. Ideological struggle

A further level of complication is to include spatial variables explicitly in the above models describing the diffusion of ideas. At this stage of globalization of economies, with several of its concomitant features, like idea, knowledge, and technology diffusion, to consider the spatial aspect is clearly a must. A large amount of research on the spatial aspects of diffusion of populations is already available. As examples of early work, papers by Kerner [130], Allen [131], Okubo [132], and Willson and de Roos [133] can be pointed out. From the point of view of diffusion of ideas and scientists, the previously discussed continuous model of research mobility [110] has to be singled out. Moreover, the model presented below is closely connected to the space-time models of migration of populations developed by Vitanov and co-authors [134, 135]. In addition, a reproduction-transport equation model (see Fig. 21) can be discussed.

5.1 Model of competition between ideologies

The diffusion of ideas is necessarily accompanied by competition processes. One model of competition between systems of ideas (ideologies) goes as follows (Fig. 22). Let a population of $N$ individuals occupy a two-dimensional plane. Suppose that there exists a set of ideas or ideologies $P = \{P_0, P_1, \ldots, P_n\}$ and let $N_i$ members of the population be followers of the $P_i$ ideology. The members $N_0$ of the class $P_0$ are not supporters of any ideology; in some
sense, they have their own individual one and do not wish to be considered associated with another one, global or not. In such a way, the population is divided in $n + 1$ sub-populations of followers of different ideologies. The total population is: $N = N_0 + N_1 + \ldots N_n$. Let a small region $\Delta S = \Delta x \Delta y$ be selected in the plane. In this region there are $\Delta N_i$ individuals holding the $i$-th ideology, $i = 0, 1, \ldots, n$. If $\Delta S$ is sufficiently small, the density of the $i$-th population can be defined as $\rho_i(x, y, t) = \frac{\Delta N_i}{\Delta S}$.

Allow the members of the $i$-th population to move through the borders of the area $\Delta S$. Let $\vec{j}_i(x, y, t)$ be the current of this movement. Then $(\vec{j}_i \cdot \vec{n}) \delta l$ is the net number of members of the $i$-th population/ideology, crossing a small line $\delta l$ with normal vector $\vec{n}$. Let the changes be summarized by the function $C_i(x, y, t)$. The total change of the number of members of the $i$-th population is

$$\frac{\partial \rho_i}{\partial t} + \text{div} \vec{j}_i = C_i.$$  \hspace{1cm} (48)
The first term in Eq. (48) describes the net rate of increase of the density of the $i$-th population. The second term describes the net rate of immigration into the area. The r.h.s. of Eq. (48) describes the net rate of increase exclusive of immigration.

Let us now specify $\vec{j}_i$ and $C_i$: $\vec{j}_i$ is assumed to be made of a non-diffusion part $\vec{j}_i^{(1)}$ and a diffusion part $\vec{j}_i^{(2)}$ where $\vec{j}_i^{(2)}$ is assumed to have the general form of a linear multicomponent diffusion [130] in terms of a diffusion coefficient $D_{ik}$

$$\vec{j}_i = \vec{j}_i^{(1)} + \vec{j}_i^{(2)} = \vec{j}_i^{(1)} - \sum_{k=0}^{n} D_{ik}(\rho_i, \rho_k, x, y, t) \nabla \rho_k.$$  \hspace{1cm} (49)$$

Let some of the followers of the ideology $P_i$ be capable of and interested in changing ideology: i.e., they can convert from the ideology $P_i$ to the ideology $P_j$. It can be assumed that the following processes can happen with respect to the members of the subpopulations of the property holders: (a) deaths: described by a term $r_i\rho_i$. It is assumed that the number of
deaths in the $i$-th population is proportional to its population density. In general $r_i = r_i(\rho_\nu, x, y, t; p_\mu)$, where $\rho_\nu$ stands for $(\rho_0, \rho_1, \ldots, \rho_N)$ and $p_\mu$ stands for $(p_1, \ldots, p_M)$ containing parameters of the environment; (b) non-contact conversion: in this class are included all conversions exclusive of the conversions by interpersonal contact between the members of whatever populations. A reason for non-contact conversion can be the existence of different kinds of mass communication media which make propaganda for whatever ideologies. As a result, members of each population can change ideology. For the $i$-th population, the change in the number of members is: $\sum_{j=0}^n f_{ij} \rho_j$, $f_{ii} = 0$. In general, $f_{ij} = f_{ij}(\rho_\nu, x, y, t; p_\mu)$; (c) contact conversion: it is assumed that there can be interpersonal contacts among the population members. The contacts happen between members in groups consisting of two members (binary contacts), three members (ternary contacts), four members, etc. As a result of the contacts, members of each population can change their ideology. For binary contacts, let it be assumed that the change of ideology probability for a member of the $j$-th population is proportional to the possible number of contacts, i.e., to the density of the $i$-th population. Then the total number of "conversions" from $P_j$ to $P_i$ is $a_{ij} \rho_i \rho_j$, where $a_{ij}$ is a parameter. In order to have a ternary contact, one must have a group of three members. The most simple is to assume that such a group exists with a probability proportional to the corresponding densities of the concerned populations. In a ternary contact between members of the $i$-th, $j$-th, and $k$-th population, members of the $j$-th and $k$-th populations can change their ideology according to $P_i = b_{ijk} \rho_i \rho_j \rho_k$, where $b_{ijk}$ is a parameter. In general, $a_{ij} = a_{ij}(\rho_\nu, x, y, t; p_\mu)$; $b_{ijk} = b_{ijk}(\rho_\nu, x, y, t; p_\mu)$; etc.

On the basis of the above, the $C_i$ term looks as follows (for more research of these types of population models see [136, 137, 138]):

$$C_i = r_i \rho_i + \sum_{j=0}^n f_{ij} \rho_j + \sum_{j=0}^n a_{ij} \rho_i \rho_j + \sum_{j,k=0}^n b_{ijk} \rho_i \rho_j \rho_k + \ldots, \quad (50)$$

and the model system of equations becomes

$$\frac{\partial \rho_i}{\partial t} + \text{div} \vec{j_i}^{(1)} - \sum_{j=0}^n \text{div}(D_{ij} \nabla \rho_j) = r_i \rho_i + \sum_{j=0}^n f_{ij} \rho_j + \sum_{j=0}^n a_{ij} \rho_i \rho_j + \sum_{j,k=0}^n b_{ijk} \rho_i \rho_j \rho_k + \ldots \quad (51)$$

The density of the entire population is $\rho = \sum_{i=0}^n \rho_i$. It can be assumed
that it changes in time according to the Verhulst law (but see the note after Eq.(56)!) 
\[ \frac{\partial \rho}{\partial t} = r \rho \left( 1 - \frac{\rho}{C} \right) \] (52)
where \( C(\rho_\nu, x, y, t; p_\mu) \) is the so-called carrying capacity of the environment [96] and \( r(\rho_\nu, x, y, t; p_\mu) \) is a positive or negative growth rate. When pertinent sociological data are available, the same type of equation could hold for any \( i \)-th population with a given \( r_i \).

First, consider the case in which the current \( \vec{j}^{(i)} \) is negligible, i.e., \( \vec{j}^{(i)} \approx 0 \) (no diffusion approximation). In addition, consider only the case when all parameters are constants. The model system of equations becomes
\[ \frac{\partial \rho_i}{\partial t} - D_{ij} \sum_{j=0}^{n} \Delta \rho_j = r_i \rho_i + \sum_{j=0}^{n} f_{ij} \rho_j + \sum_{j=0}^{n} a_{ij} \rho_i \rho_j + \sum_{j,k=0}^{n} b_{ijk} \rho_i \rho_j \rho_k + \ldots, \] (53)
for
\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad i = 0, 1, 2, \ldots, n. \] (54)

Let plane-averaged quantities and fluctuations (linear or nonlinear) be enough relevant. Let \( q(x, y, t) \) be a quantity defined in an area \( S \). By definition, a plane-averaged quantity is \( \bar{q} = \frac{1}{S} \int \int_S dx \, dy \, q(x, y, t) \). Call the fluctuations \( Q(x, y, t) \) such that \( q(x, y, t) = \bar{q}(t) + Q(x, y, t) \). If the territory is large and within the stationary approximation, \( S \) can be assumed to be large enough such that each plane-averaged combination of fluctuations vanishes, such that \( \bar{Q}_i = \bar{Q}_j = \bar{Q}_j Q_k = \ldots = 0 \). In addition to \( S \) being large and \( \int \int_S dx \, dy \, \Delta Q_k \) assumed to be finite, it can be also assumed that \( \Delta Q_k = \frac{1}{S} \int \int_S dx \, dy \, \Delta Q_k \to 0 \).

On the basis of the above (reasonable) assumptions, it is possible to separate the dynamics of the averaged quantities from the dynamics of fluctuations. As a result of the plane-average of Eq.(53), the following equations for the dynamics of the plane-averaged densities are obtained
\[ \bar{p}_0 = \bar{p} - \sum_{i=1}^{n} \bar{p}_i; \quad \frac{d\bar{p}}{dt} = r \bar{p} \left( 1 - \frac{\bar{p}}{C} \right) \] (55)
\[
\frac{d\rho_i}{dt} = r_i \bar{\rho}_i + \sum_{j=0}^{n} f_{ij} \bar{\rho}_j + \sum_{j=0}^{n} a_{ij} \bar{\rho}_i \bar{\rho}_j + \sum_{j,k=0}^{n} b_{ijk} \bar{\rho}_i \bar{\rho}_j \bar{\rho}_k + \ldots 
\] (56)

Instead of (55) we can write an equation for \(\bar{\rho}_0\) from the kind of (56). Then the total population density \(\bar{\rho}\) will not follow the Verhulst law.

Equations (55) and (56) represent the model of ideological struggle proposed by Vitanov, Dimitrova and Ausloos [139]. There is one important difference between the Lotka-Volterra models [112, 114], often used for describing prey-predator systems, and the above model of ideological struggle. The originality resides in the generalization of usual prey-predator models to the case in which a prey (or predator) changes its state and becomes a member of the predator pack (or prey band), due to some interaction with its environment or with some other prey or predator. Indeed, it can be hard for rabbits and foxes to do so, but it can be often the case in a society: a member of one population can drop his/her ideology and can convert to another one.

In order to show the relevance of such extra conditions on an evolution of populations, consider a huge (mathematical) approximation, - it might be a drastic one in particular in a country with a strictly growing total population. (Recall that the growth rate \(r\) could be positive or negative or time-dependent). Let \(r \geq 0\) and let the maximum possible population of the country be \(C\). Consider more convenient notations by setting \(\bar{\rho} = N\); \(\bar{\rho}_0 = N_0\); \(\bar{\rho}_i = N_i\) and assume that the binary contact conversion is much stronger than the ternary, etc. conversions. The system equations become

\[
N = N_0 + \sum_{i=1}^{n} N_i; \quad \frac{dN}{dt} = rN \left(1 - \frac{N}{C}\right) \quad (57)
\]

\[
\frac{dN_i}{dt} = r_i N_i + \sum_{j=0}^{n} f_{ij} N_j + \sum_{j=0}^{n} b_{ij} N_i N_j. \quad (58)
\]

Reduce the discussion of Eqs. (57) and (58) to a society in which there is the spreading of only one ideology; therefore, the population of the country is divided into two groups: \(N_1\), followers of the "invading" ideology and \(N_0\), people who are at first "indifferent" to this ideology. Let only the non-contact conversion scheme exist, as possibly moving the ideology-free population toward the single ideology; thus \(f_{10}\) is finite, but \(b_{10} = 0\). Let the initial conditions be \(N(t = 0) = N(0)\) and \(N_1(t = 0) = N_1(0)\). The solution of the
system of model equations is

\[ N(t) = \frac{CN(0)}{N(0) + (C - N(0))e^{-rt}}, \]

(59)

like the Verhulst law, but

\[ N_1(t) = e^{-(f_{10}-r_1)t}\left\{ N_1(0) + \frac{Cf_{10}}{r}\left[ \Phi\left( -\frac{C - N(0)}{N(0)}, 1, -\frac{f_{10} - r_1}{r}\right) - e^{(f_{10}-r_1)t}\Phi\left( -\frac{C - N(0)}{N(0)}e^{rt}, 1, -\frac{f_{10} - r_1}{r}\right) \right] \right\} \]

(60)

with

\[ N_0(t) = N(t) - N_1(t) \]

(61)

in which \( \Phi \) is the special function \( \Phi(z, a, v) = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)!} ; \quad |z| < 1 \).

The obtained solution describes an evolution in which the total population \( N \) reaches asymptotically the carrying capacity \( C \) of the environment. The number of adepts of the ideology reaches an equilibrium value which corresponds to the fixed point \( \hat{N}_1 = C f_{10} / (f_{10} - r_1) \) of the model equation for \( \frac{dN_1}{dt} \). The number of people who are not followers of the ideology asymptotically tends to \( N_0 = C - \hat{N}_1 \). Let \( C = 1, f_{10} = 0.03, \) and \( r_1 = -0.02, \) then \( \hat{N}_1 = 0.6, \) which means that the evolution of the system leads to an asymptotic state in which 60 % of the population are followers of the ideology and 40 % are not.

Other more complex cases with several competing ideologies can be discussed, observing steady states or/and cycles (with different values of the time intervals for each growth or/and decay), chaotic behaviors, etc. [139]. In particular, it can be shown that accepting a slight change in the conditions of the environment can prevent the extinction of some ideology. After almost collapsing, some ideology can spread again and can affect a significant part of the country’s population. Two kinds of such resurrection effects have been found and described as phoenix effects in the case of two competing ideologies. In the phoenix effect of the first kind, the equilibrium state connected to the extinction of the second ideology exists but is unstable. In the phoenix effect of the so-called second kind, the equilibrium state connected to extinction of the second ideology vanishes. In fine, the above model seems powerful enough to discuss many realistic cases. The number of control parameters seems huge, but that is the case for many competing epidemics in
complex systems. However, it was observed that the values of parameters can be monitored when enough data is available, including the time scales [139].

FOR POLICY-MAKERS
Take away box Nr.13:
Space-time models are very appropriate for modeling migration processes such as the spatial migration of scientists, besides the diffusion of ideas through competition without strictly physical motion.

5.2 Continuous model of evolution of scientific subfields. Reproduction-transport equation

The change of subject of a scientist can be considered as a migration process [110,140]. Let research problems be represented by sequences of signal words or macro-terms $P_i = (m_{i1}, m_{i2}, \ldots, m_{ik}, \ldots, m_{in})$ which are registered according to the frequency of their appearance, joint appearance, etc., respectively, in the texts. Each point of the problem space, described by a vector $\vec{q}$, corresponds to a research problem, with the problem space consisting of all scientific problems (no matter whether they are under investigation or not). The scientists distribute themselves over the space of scientific problems with density $x(\vec{q}, t)$. Thus, there is a number $x(\vec{q}, t)d\vec{q}$ working at time $t$ in the element $d\vec{q}$. The field mobility processes correspond to a density change of scientists in the problem space: instead of working on problem $\vec{q}$, a scientist may begin to work on problem $\vec{q}'$. As a result, $x(\vec{q}, t)$ decreases and $x(\vec{q}', t)$ increases. This movement of scientists (see also Fig. 23) can be described by means of the following reproduction-transport-equation:

$$\frac{\partial x(\vec{q}, t)}{\partial t} = x(\vec{q}, t) w(\vec{q} | x) + \frac{\partial}{\partial \vec{q}} \left( f(\vec{q}, x) + D(\vec{q}) \frac{\partial x(\vec{q}, t)}{\partial \vec{q}} \right). \quad (62)$$

In Eq. (62), self-reproduction and decline are represented by the term $w(\vec{q} | x) x(\vec{q}, t)$. For the reproduction rate function $w(\vec{q} | x)$, one can write

$$w(\vec{q} | x) = a(\vec{q}) + \int d\vec{q}' b(\vec{q}, \vec{q}') x(\vec{q}', t). \quad (63)$$

51
Figure 23: Schema of the reproduction-transport equation model of joint evolution of scientific fields.

The local value of $a(\vec{q})$ is an expression of the rate at which the number of scientists on field $\vec{q}$ is modified through self-reproduction and/or decline, while $b(\vec{q}, \vec{q}')$ describes the influence exerted on the field $\vec{q}$ by the neighbouring field $\vec{q}'$. The field mobility is modeled by means of the term $\frac{\partial}{\partial \vec{q}} \left( f(\vec{q}, x) + D(\vec{q}) \frac{\partial}{\partial \vec{q}} x(\vec{q}, t) \right)$. In most cases, Eq.(62) can only be solved numerically. For more details on the model, see [110].
6 Statistical approaches to the diffusion of knowledge

Solomon and Richmond [76, 77] have shown that the systems of generalized Lotka-Volterra equations are closely connected to the Pareto-Zipf probability distribution. Since such a distribution arises among other distributions and laws connected to the description of the diffusion of knowledge, it is of interest to discuss briefly the diffusion of knowledge within statistical approach studies. Lotka was its pioneer; a large amount of research has followed. Just as examples, one can mention the work of Yablonsky and Haitun on the Lotka law for the distribution of scientific productivity and its connection with the Yule distribution [141, 142, 143], where the non-Gaussian nature of the scientific activities is emphasized. Interesting applications of the Zipf law are also presented in [144]. The connection to the non-Gaussian distributions concepts of self-similarity and fractality have been applied to the scientific...
system in [145] and [146]. Several tools for appropriate statistical analysis are hereby discussed. At the center of the discussion Lotka law shall receive some special attention (see Fig. 24) 2.

As part of this discussion on the statistical approach, the analysis of the productivity of scientists can be considered. The information connected to new ideas is thought to be often codified in scientific papers. Thus, the statistical aspects of scientific productivity is of practical importance. For example, the Lotka law reflects the distribution of publications over the set of authors considered as the information sources. Bradford law describes the distribution of papers on a given topic over the set of journals publishing these papers and ranked according to the order in the decrease of the number of papers on a given topic in each journal. These laws have a non-Gaussian nature and, because of this, possess specific features such as a concentration and dispersal effect [141]: for example, it is found that there is a small number of highly productive scientists who write most of the papers on a given topic and, on the other hand, a large number of scientists with low productivity.

In order to give an example of the connection between the deterministic and statistical approaches, remember that the Goffman-Newill model, discussed here above, presents a connection between the number of scientists working in a research area and the number of relevant publications. In [106], it was found that the number of new publications scale as a simple power law with the corresponding number of new authors: $\Delta P = C(\Delta T)^\alpha$ where $\Delta P$ and $\Delta T$ are the new publications and the new authors over some time period (for an example one year). $C$ is a normalization constant, and $\alpha$ is a scaling exponent. It has been demonstrated [106] that the latter relationship provides a very good fit to data for six different research fields, but with different values of the scaling exponent $\alpha$. For $\alpha > 1$, a field would grow by showing an increase in the number of publications per capita, i.e., in such a research field, the individual productivity increases as the field attracts new scientists. A field with $\alpha < 1$ has a per capita decrease in productivity. This can be a warning signal for a dying subject matter. It would be interesting to observe whether the exponent $\alpha$ is time-dependent, as is the case in related characterizing scaling exponents of financial markets [148] or in meteorology [149]. Policy control can thus be implemented for shaking $\alpha$, 2 Let us mention a curious and interesting fact connected to statistical indicators. Very interesting is the conclusion in [147] that the scale-independent indicators show that in the fast growing innovation system of China, research institutions financed by the government play a more important role than the enterprises.
thus the field mobility.

For Policy-makers
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There exist two different kinds of statistical approaches for the analysis of scientific productivity: (i) the frequency approach and (ii) the rank approach. The frequency approach is based on the direct statistical counting of the number of corresponding information sources, such as scientists or journals. The rank approach is based on a ranking of the sources with respect to their productivity. The frequency and the rank approaches represent different and complementary reflections of the same law and form.

6.1 Lotka law. Distributions of Pareto and Yule.

Pareto [150] formulated the 80/20 rule: it can be expected that 20% of people will have 80% of the wealth. Or it can be expected that 80% of the citations refer to a core of 20% of the titles in journals. The idea of the rule of Pareto is very close to the research of Lotka who noticed the following dependence for the number of scientists $n_k$ who wrote $k$ papers

$$n_k = \frac{n_1}{k^2}; \quad k = 1, 2, \ldots, k_{max}. \quad (64)$$

In Eq. (64), $n_1$ is the number of scientists who wrote just one paper and $k_{max}$ is the maximal productivity of a scientist.

$$\sum_{k=1}^{k_{max}} n_k = n_1 \sum_{k=1}^{k_{max}} \frac{1}{k^2} = N \quad (65)$$

where $N$ is the total number of scientists. If we assume that $k_{max} \to \infty$ and take into account the fact that $\sum_{k=1}^{\infty} 1/k^2 = \pi/6$, we obtain a limiting value for the portion of scientists with the minimal productivity (single paper authors) in the given population of authors: $P_1 = n_1/N \approx 0.6$. Then, if the left and the right hand sides of Eq. (64) are divided by $N$, the frequency expression for the productivity distribution is: $p_1 = 0.6/k^2; \sum_{k=1}^{\infty} p_k = 1$. Eq. (64) is called Lotka law, or the law of inverse squares: the number of scientists who wrote a given number of papers is inversely proportional to the square of this number of papers.
It must be noted that, like many other statistical regularities, Lotka law is valid only on the average since the exponent in the denominator of Eq.(64) is not necessarily equal to two. Thus, Lotka law should be considered as the most typical among a more general family of distributions:

\[ n_k = \frac{n_1}{k^{1+\alpha}}; \quad p_1 = \frac{p_1}{k^{1+\alpha}} \quad \text{(66)} \]

where \( \alpha \) is the characteristic exponent of the distribution, \( n_1 \) is the normalizing coefficient which is determined as follows:

\[ p_1 = n_1 N = \left( \sum_{k=1}^{k_{max}} \frac{1}{1 + k^\alpha} \right)^{-1} \quad \text{(67)} \]

Then the distribution of scientific output, Eq.(66), is determined by three parameters: the proportion of scientists with the minimal productivity \( p_1 \), the maximal productivity of a scientist \( k_{max} \), and the characteristic exponent \( \alpha \). If one of these parameters is fixed, it is possible to study the dependence between two others. Let us fix \( k_{max} \) in Eq.(67). Then, we obtain the proportion of ”single paper authors” \( p_1 \) as a function of \( \alpha \): \( p_1(\alpha) \). When Eq.(67) is differentiated with respect to \( \alpha \), one can show that the corresponding derivative is positive for any \( \alpha : dp_1(\alpha)/d\alpha > 0 \). On the basis of a similar analysis of the portion of scientists with a larger productivity \( p_k(\alpha) \) as a function of \( \alpha \), we arrive at the conclusion: the increase of \( \alpha \) is accompanied by the increase of low-productivity scientists. This means that when the total number of scientists is preserved the portion of highly productive scientists will decrease.

Let us show that the Lotka law is an asymptotic expression for the Yule distribution. In order to obtain the Yule distribution, one considers the process of formation of a collection of publications as a Markov-type stochastic process. In addition, it is assumed that the probability of writing a new paper depends on the number of papers that have been already written by the scientist at time \( t \): the probability of the transition into a new state on the interval \( [t, t + \Delta t] \) should be a function of the state in which the system is at time \( t \). Moreover, the probability of publishing a new paper during a time interval \( \Delta t \), \( p(x \rightarrow x + 1, \Delta t) \) is assumed to be proportional to the number \( x \) of papers that have been written by the scientists, introducing an intensity coefficient \( \lambda \): \( p(x \rightarrow x + 1, \Delta t) \propto \lambda x \Delta t \). After solving the corresponding system of differential equations for this process, the following expression (the
Yule distribution) for the probability $p(x/t)$ of a scientist writing $x$ papers during a time $t$ is obtained [141]:

$$p(x/t) = \exp(-\lambda t)(1 - \exp(-\lambda t))^{x-1}, x = 1, 2 \ldots$$  \hspace{1cm} (68)

The mean value of the Yule distribution is $x_t = \exp(\lambda t)$. Let us take into account the fact that every scientist works on a given subject during a certain finite random time interval $[0, t]$ which depends on the scientist’s creative potential, the conditions for work, etc. With the simplest assumption that the probability of discontinuing work on a given subject is constant at any time, one obtains an exponential distribution for the time of work of any author in the scientific field under study: $p(t) = \mu \exp(-\mu t)$, where $\mu$ is the distribution parameter. The time parameter $t$ which characterizes the productivity distribution, Eq.(68), is a random number. Then in order to obtain the final distribution of scientific output observed in the experiment over sufficiently large time intervals, Eq.(68) should be averaged with respect to this parameter $t$ which is distributed according to the exponential law:

$$p(x) = \int_0^\infty dt \, p(x/t)p(t) = \int_0^\infty dt \, \exp(-\lambda t)(1 - \exp(-\lambda t))\mu \exp(-\mu t).$$  \hspace{1cm} (69)

After integrating Eq.(69), the distribution of scientific output reads

$$p(x) = \frac{\mu}{\lambda} B \left( x, \frac{\mu}{\lambda} + 1 \right) = \alpha B(x, \alpha + 1), x = 1, 2, \ldots$$  \hspace{1cm} (70)

where $B(x, \alpha + 1) = \Gamma(x)\Gamma(\alpha x + 1)/\Gamma(x + \alpha + 1)$ is a Beta-function, $\Gamma(x) \approx (x-1)!$ is a Gamma-function, and $\alpha = \mu/\lambda$ is the characteristic exponent. For instance, if $\alpha \approx 1$ then $p(x) = 1/[x(x+1)]$. Let us assume that $x \to \infty$ and apply the Stirling formula. Thus, the asymptotics of the Yule distribution Eq.(70) is like Lotka law Eq.(66) (up to a normalizing constant): $p(x) \propto \Gamma(\alpha + 1)\alpha/\lambda^{1+\alpha}/x^{1+\alpha}$.

### 6.2 Pareto distribution, Zipf-Mandelbrot and Bradford laws

For large enough values of the total number of scientists and the total number of publications, we can make the transition from discrete to continuous representation of the corresponding variables and laws. The continuous analog of Lotka law, Eq. (66), is the Pareto distribution

$$p(x) = \frac{\alpha}{x_0} \left( \frac{x_0}{x} \right)^{\alpha+1}; \quad x \geq x_0; \quad \alpha > 0$$  \hspace{1cm} (71)
which describes the distribution density for a number of scientists with \( x \) papers; \( x_0 \) is the minimal productivity \( x_0 << x << \infty \), a continuous quantity.

Zipf law is connected to the principle of least effort \[151\]: a person will try to solve his problems in such a way as to minimize the total work that he must do in the solution process. For example, to express with many words what can be expressed with a few is meaningless. Thus, it is important to summarize an article using a small number of meaningful words. Bradford law for the scattering of articles over different journals is connected to the success-breeds-success (SBS) principle \[152\]: success in the past increases chances for some success in the future. For example, a journal that has been frequently consulted for some purpose is more likely to be read again, rather than one of previously infrequent use.

In order to obtain the law of Zipf-Mandelbrot, we start from the following version of Lotka law: \( n_x = C/(1 + x)^{1+\alpha} \), where \( x \) is the scientist’s productivity, \( \alpha \) is a characteristic exponent, \( C \) is a constant which in most cases is equal to the number of authors with the minimal productivity \( x = 1 \), i.e., to \( n_1 \). On the basis of this formula, the number of scientists \( r \) who are characterized by productivity \( x_r < x < k_{\text{max}} \) (\( k_{\text{max}} \) is the maximal productivity of a scientist) reads

\[
 r = \sum_{x=x_r}^{k_{\text{max}}} n_r \approx C \int_{x_r}^{k_{\text{max}}} \frac{dx}{x^{1+\alpha}} = \frac{C}{\alpha} \left( \frac{1}{x_r^{\alpha}} - \frac{1}{k_{\text{max}}^{\alpha}} \right). \tag{72}
\]

Depending on the value of \( x_r \), \( r \) can have values 1, 2, 3, \ldots and in such a way the scientists can be ranked. If all scientists of a scientific community working on the same topic are ranked in the order of the decrease of their productivity, the place of a scientist who has written \( x_r \) papers will be determined by his/her rank \( r \). When the productivity of a scientist \( x_r \) is found from Eq.(72) as a function of rank \( r \), the relationship

\[
x_r = \left( \frac{A}{r + B} \right)^\gamma; \quad A = (C/\alpha)^{1/\alpha}; \quad B = C/(\alpha k_{\text{max}}^{\alpha}); \quad \gamma = 1/\alpha. \tag{73}
\]

This is the rank law of Zipf-Mandelbrot, which generalizes Zipf law: \( f(r) = cr^{-\beta}; r = 1, 2, 3, \ldots \), where \( c \) and \( \beta \) are parameters. Zipf law was discovered by counting words in books. If words in a book are ranked in decreasing order according to their number of occurrences, then Zipf law states that the number of occurrences of a word is inversely proportional to its rank \( r \).

Assuming that in Lotka law the exponent takes the value \( \alpha = 1 \) and that in most cases \( C = n_1 \), one has \( x_r = n_1/(r + a) \), where \( a = n_1/k_{\text{max}}, \ r \geq 0 \).
Integration of the last relationship yields the total productivity $R(n)$ of all scientists, beginning with the one with the greatest productivity $k_{\text{max}}$ and ending with the scientist whose productivity corresponds to the rank $n$ (the scientists are ranked in the order of diminishing productivity; the rank is assumed to be a continuous-like variable):

$$R(n) = n_1 \ln \left( \frac{n}{a} + 1 \right). \quad (74)$$

This is Bradford law. According to this law, for a given topic, a large number of relevant articles will be concentrated in a small number of journals. The remaining articles will be dispersed over a large number of journals. Thus, if scientific journals are arranged in order of decreasing published articles on a given subject, they may be split to a core of journals more particularly devoted to the subject and a shell consisting of sub-shells of journals containing the same numbers of articles as the core. Then the number of journals from the core zone and succeeding sub-shells will follow the relationship $1 : n : n^2 : \ldots$.

**FOR POLICY-MAKERS**

Take away box Nr.15:

The Zipf-Pareto law, in the case of the distribution of scientists with respect to their productivity, indicates that one can always single out a small number of productive scientists who wrote the greatest number of papers on a given subject, and a large number of scientists with low productivity. The same applies also to scientific contacts, citation networks, etc. This specific feature (so-called hierarchical stratification) of the Zipf-Pareto law reflects a basic mechanism in the formation of stable complex systems. This can/must be taken into account in the process of planning and the organization of science.

7 Concluding remarks

Knowledge has a complex nature. It can be created. It can lead to innovations and new technologies, and on this base, knowledge supports the advance and economic growth of societies. Knowledge can be collected. Knowledge can be spread. Diffusion of ideas is closely connected to the collection and
spreading of knowledge. Some stages of the diffusion of ideas can be described by epidemic models of scientific and technological systems. Most of the models described here are deterministic, but if the internal and external fluctuations are strong, then different kinds of models can be applied taking into account stochastic features.

Much information about properties and stability of the knowledge systems can be obtained by the statistical approach on the basis of distributions connected to the Lotka-Volterra models of diffusion of knowledge. Interestingly, new terms occur in the usual evolution equations because of the variability and flexibility in the opinions of actors, due to media contacts or interpersonal contacts, when exchanging ideas.

The inclusion of spatial variables in the models leads to new research topics, such as questions on the spreading of systems of ideas and competition among ideas in different areas/countries.

In conclusion, the epidemiological perspective renders a piece of mosaic to a better understanding of the dynamics of diffusion of ideas in science, technology, and society, which should be one of the main future tasks of the science of science [153].

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References

[1] J. R. L. Howells (2002). Tacit knowledge, innovation and economic geography. Urban Studies 39, 871 – 884.

[2] R. J. Barro, X. Sala-i-Martin (2004). Economic Growth, MIT Press, Cambridge, MA.

[3] L. Leydesdorff (2006). The knowledge-based economy – modeled, measured, simulated. Boca Raton, FL.

[4] W. Dolsma (2008). Knowledge economies. Organization, location and innovation. Routledge, London.
[5] A. Marshall (1920). Principles of economics, 8\textsuperscript{ed}. McMillan, London.

[6] D. Romer (1996). Advanced Macroeconomics. McGraw-Hill, New York.

[7] P. M. Romer (1994). The origins of endogeneous growth. \textit{Journal of Economic Perspectives} \textbf{8}, 3 – 22.

[8] P. M. Romer (1994). New goods, old theory, and the welfare costs of trade restrictions. \textit{Journal of Development Economics} \textbf{43}, 5 – 38.

[9] P. M. Romer (2002). When should we use intellectual property rights? \textit{American Economic Review} \textbf{92}, 213 – 216.

[10] A. B. Jaffe, M. Trajtenberg (2002). Patents, citations and innovations: A window on the knowledge economy, MIT Press, Cambridge, Ma.

[11] J. R. Walsh (1935). Capital concept applied to man. \textit{Quarterly Journal of Economics} \textbf{49}, 255 – 285.

[12] P. Bourdieu (1986). Forms of capital. pp. 241 – 258 in J. G. Richardson (Ed.) Handbook of theory and research for the sociology of education. Greenwood, New York.

[13] J. C. Coleman (1988). Social capital: The creation of human capital. \textit{American Journal of Sociology} \textbf{94}, Supplement, S95 – S120.

[14] R. D. Putnam (1993). Making democracy work: Civic transitions in modern Italy. Princeton University Press, Princeton, NJ.

[15] G. S. Becker, K. M. Murphy (1988). A theory of rational addiction. \textit{Journal of Political Economy} \textbf{96}, 675 – 700.

[16] G. S. Becker (1996). Accounting for tastes. Harvard University Press, Cambridge, MA.

[17] J. E. Stiglitz (1987). Learning to learn, localized learning and technological progress. p.p. 125 – 153 in P. Dasgupta , P. Stoneman (Eds.) Economic policy and technological performance, Cambridge University Press, Cambridge.

[18] J. B. Davis (2003). The theory of the individual in economics: Identity and value. Routledge, London.
[19] D. J. de Solla Price (1951). Quantitative measures of the development of science. p.p. 413 – 421 in: Actes du VIème Congres International d’Histoire des Sciences (Amsterdam). Hermann & Cie, Paris.

[20] D. J. de Solla Price (1971). Principles for projecting funding of academic science in the 1970s. Science Studies 1, 85 – 94.

[21] D. Foray (2004). The Economics of Knowledge. MIT Press, Cambridge, MA.

[22] A. F. J. van Raan (1997). Scientometrics: State of art. Scientometrics 38, 205 – 218.

[23] I. Nonaka (1994). A dynamic theory of organizational knowledge creation. Organizational Science 5, 14 – 37.

[24] I. Nonaka, N. Konno (1998). The concept of ”Ba”: Building a foundation for knowledge creation. California Management Review 40, 40 – 54.

[25] I. Nonaka, H. Takeuchi (1995). The knowledge creating company: How Japanese companies create the dynamics of innovation. Oxford University Press, New York.

[26] S. Bernius (2010). The impact of open access on the management of scientific knowledge. Online Information Review 34, 583 – 603.

[27] C. Dahlman (2009). Different innovation strategies, different results: Brazil, Russia, India, China and Korea (the BRICKs). pp. 131 – 168 in V. Chandra, D. Ercal, P. C. Padoan, C. A. Primo Braga (Eds.). Innovation and Growth. Chasing a Moving Frontier. OECD and the International Bank for Reconstruction and Development/The World Bank, Washington, DC.

[28] C. Dahlman, D. Zhihua Zeng, S. Wang (2007). Enhancing China’s competitiveness through life long life learning. The World Bank, Washington, DC.

[29] P. P. Saviotti (1999). Knowledge, information and organisational structures, pp. 120 –139 in P.L. Robertson (ed.), Authority and Control in Modern Industry, Routledge, London.
[30] R. Cowan, D. Foray (1997), The economics of codification and the diffusion of knowledge. *Industrial and Corporate Change* **6**, 595 – 622.

[31] A. B. Jaffe (1986). Technological opportunity and spillovers of R & D: evidence from firms patents, profits, and market value. *American Economic Review* **76**, 984 – 1001.

[32] C. Antonelli (1996). Localized knowledge percolation processes and information networks. *Journal of Evolutionary Economics* **6**, 281 – 295.

[33] P. Morone, R. Taylor (2010). Knowledge diffusion and innovation. Edward Elgar Publishing Inc., Northampton, MA.

[34] E. Rogers (1962). Diffusion of Innovations, Free Press, New York.

[35] E. Casetti, R. Semple (1969). Concerning the testing of spatial diffusion hypotheses. *Geographical Analysis*, **1**, 254 – 259.

[36] D. Kucharavy, E. Schenk, R. de Guio. (2009). Long-run forecasting of emerging technologies with logistic models and growth of knowledge. pp. 277 – 284 in Proceedings of the 19th CIRP Design Conference - Competitive Design, Cranfield, UK.

[37] W. Ebeling, A. Scharnhorst (1985). Selforganization models for field mobility of physicists. *Czechoslovak Journal of Physics* **36**, 43 – 46.

[38] J. M. Ziman (1969). Information, communication, knowledge. *Nature* **224**, 318 – 324.

[39] For example, at Gordon Research Conferences, it is forbidden to take written notes and to quote participant interventions later.

[40] A. Bryman (1988). Quantity and quality in social research. Unwin Hyman, London.

[41] S. Wright (1932). The roles of mutation, inbreeding, crossbreeding and selection in evolution. Proceedings of the Sixth International Congress on Genetics **1**, 356 – 366.

[42] H. Small (1997). Update on science mapping: Creating large document spaces. *Scientometrics* **38**, 275 – 293.
[43] H. Small (1998). A general framework for creating large-scale maps of science in two or three dimensions: The SciViz system. *Scientometrics* **41**, 125 – 133.

[44] E. C. M. Noyons, A.F.J. Van Raan (1998). Advanced mapping of science and technology. *Scientometrics* **41**, 61 – 67.

[45] A. Scharnhorst (1998). Citation networks, science landscapes and evolutionary strategies. *Scientometrics* **43**, 95 – 106.

[46] A. Scharnhorst (2001). Constructing knowledge landscapes within the framework of geometrically oriented evolutionary theories. pp. 505 - 515 from M. Matthies, H. Malchow and J. Kriz (Eds.). Integrative systems approaches to natural and social dynamics, Springer , Berlin.

[47] L. Egghe (1998). Mathematical theory of citation. *Scientometrics* **43**, 57 – 62.

[48] L. Egghe, R. Rousseau (1990). Introduction to Informetrics. Quantitative Methods in Library, Documentation and Information Science, Elsevier, Amsterdam.

[49] I. Hellsten, R. Lambiotte, A. Scharnhorst, M. Ausloos (2006). A journey through the landscape of physics and beyond - the self-citation patterns of Werner Ebeling”. in: Irreversible Prozesse und Selbstorganisation. T. Poeschel, H. Malchow and L. Schimansky Geier, Eds. (Logos Verlag, Berlin) pp. 375-384

[50] I. Hellsten, R. Lambiotte, A. Scharnhorst, and M. Ausloos (2007). Self-citations, co-authorships and keywords: A new method for detecting scientists field mobility? *Scientometrics* **72**, 469 – 486.

[51] I. Hellsten, R. Lambiotte, A. Scharnhorst, M. Ausloos (2007). Self-citations networks as traces of scientific careers. In Proceedings of the ISSI 2007, 11th International Conf. of the Intern. Society for Scientometrics and Informetrics, CSIC, Madrid, Spain, June 25-27, 2007. Ed. by D. Torres-Salinas; H. Moed, Vol. 1, pp. 361 – 367

[52] M. Ausloos, R. Lambiotte, A. Scharnhorst, I. Hellsten (2008). Andrzej Pekalski networks of scientific interests with internal degrees of freedom
through self-citation analysis, *International Journal of Modern Physics C* 19, 371–384

[53] J. E. Hirsch (2005). An index to quantify an individual’s scientific research output. *Proceedings of the National Academy of Sciences USA* 102, 16569–16572.

[54] J. M. Soler (2007). A Rational Indicator of Scientific Creativity. *Journal of Informetrics* 1, 123–130.

[55] Q. L. Burrell (2007). Hirsch’s h-index: A stochastic model. *Journal of Informetrics* 1, 16–25.

[56] H. Small (2006) Tracking and predicting growth areas in science. *Scientometrics* 68, 595–610.

[57] A. J. Lotka (1926). The frequency distribution of scientific productivity. *Journal of the Washington Academy of Sciences* 16, 317–324.

[58] D. J. de Solla Price (1963). Little Science, Big Science. Columbia University Press, New York.

[59] D. J. de Solla Price, S. Gürsey (1975). Some statistical results for the numbers of authors in the states of the United States and the nations of the world. pp. 26–34 in: Who is Publishing in Science 1975 Annual. Institute for Scientific Information, Philadelphia.

[60] D. J. de Solla Price (1978). Science since Babylon. Yale University Press, New Haven.

[61] G. N. Gilbert (1978). Measuring the growth of science: A review of indicators of scientific growth. *Scientometrics* 1, 9–34.

[62] K. H. Chung, R. A. K. Cox (1990). Patterns of productivity in the finance literature: A study of the bibliometric distributions. *Journal of Finance* 45, 301–309.

[63] T. Kealey (2000). More is less. Economists and governments lag decades behind Derek Price’s thinking. *Nature* 405, 279–279.

[64] J. D. Murray (2002). Mathematical Biology, Springer, Berlin.
[65] L. Edelstein-Keshet (1988). Mathematical models in biology. McGraw Hill, New York.

[66] K. Dietz (1967). Epidemics and Rumours: A Survey. Journal of the Royal Statistical Society. Series A (General) 130, 505 – 528.

[67] S. C. Dodd (1958). Formulas for spreading opinions. Public Opinion Quarterly 22, 537 – 554.

[68] R. M. Anderson, R. M. May, (1982). Population biology of infections disease. Springer, Berlin.

[69] F. Brauer, C. Castillo-Chavez (2001). Mathematical models in population biology and epidemiology. Texts in Applied Mathematics 40, Springer, New York.

[70] Z. Ma, J. Li. (Eds.) (2009). Dynamical modeling and analysis of epidemics. World Scientific, Singapore.

[71] W. Goffman (1966). Mathematical approach to the spreading of scientific ideas -the history of mast cell research. Nature 212, 449 – 452.

[72] W. O. Kermack, A. G. McKendrick (1927). A contribution to the mathematical theory of epidemics. Proceedings of the Royal Society, London, Ser. A 115, 700 – 721.

[73] J. M. Epstein (1997). Nonlinear Dynamics, Mathematical Biology, and Social Science. Addison-Wesley, Reading, MA.

[74] A. Castiaux (2007). Radical innovation in established organizations: Being a knowledge predator. Journal of Engineering and Technology Management 24, 36 – 52.

[75] G. F. Gause (1935). The Struggle for Existence. Williams and Wilkins, Baltimore

[76] S. Solomon, P. Richmond (2001). Power laws of wealth, market order volumes and market returns. Physica A 299, 188 – 197.

[77] S. Solomon, P. Richmond (2002). Stable power laws in variable economics. Lotka - Volterra implies Pareto - Zipf. The European Physical Journal B 27, 257 – 261.
[78] N. K. Vitanov, Z. I. Dimitrova, H. Kantz (2006). On the trap of extinction and its elimination. *Physics Letters A* **349**, 350 – 355.

[79] Z. I. Dimitrova, N. K. Vitanov (2004). Chaotic pairwise competition. *Theoretical Population Biology* **66**, 1 – 12.

[80] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D.-U. Hwang (2006). Complex networks: Structure and dynamics. *Physics Reports* **424**, 176 – 308.

[81] P. S. Meyer (1994). Bi-logistic growth. *Technological Forecasting and Social Change* **47**, 89 – 102.

[82] N. Meade, T. Islam (1995). Forecasting with growth curves: An empirical comparison. *International Journal of Forecasting* **11**, 199 – 215.

[83] E. Mansfield (1961). Technical change and the rate of imitation. *Econometrica* **29**, 741 – 766.

[84] T. Modis (2007). Strengths and weaknesses of S-curves, *Technological Forecasting and Social Change* **74**, 866 – 872.

[85] T. Modis (2003). A scientific approach to managing competition. *The Industrial Physicist* **9**, 24 – 27.

[86] M. Ausloos (2010). On religion and language evolutions seen through mathematical and agent based models, in Proceedings of the First Interdisciplinary CHESS Interactions Conference, C. Rangacharyulu and E. Haven, Eds., World Scientific, Singapore, pp. 157-182.

[87] P. S. Meyer, J. W. Yung, J. H. Ausubel (1999). A primer on logistic growth and substitution: The mathematics of the Loglet Lab software. *Technological Forecasting and Social Change* **61**, 247 – 271.

[88] V. Mahajan, R.A. Peterson (1985), Models for innovation diffusion, Sage Publications, Beverly Hills, CA.

[89] P. A. Geroski (2000). Models of technology diffusion. *Research Policy*, **29**, 603 – 625.

[90] D. J. Bartholomew (1982). Stochastic Models for Social Processes, Wiley, New York.
[91] R. M. May (1974). Stability and complexity in model ecosystems. Princeton University Press, Princeton, N. J..

[92] V. B. Lal, Karmeshu, S. Kaicker (1988). Modeling innovation diffusion with distributed time lag. *Technological Forecasting and Social Change* **34**, 103 – 113.

[93] Karmeshu (1982). Time lag in a diffusion model of information. *Mathematical modeling* **3**, 137 – 141.

[94] M. N. Sharif, K. Ramanathan (1981). Binomial innovation diffusion models with dynamic potential adopter population. *Technological Forecasting and Social Change* **20**, 63 – 87.

[95] J.M. Keynes (1930). A treatise on money. Harcourt, Brace and Co.

[96] E. P. Odum (1959). Fundamentals of Ecology, W.B. Saunders, Philadelphia.

[97] A. Fernandez - Camo, M. Toralbo, M. Vallejo (2004). Reconsidering Price’s model of scientific growth: An overview. *Scientometrics* **61**, 301 – 321.

[98] M. Szydlowski, A. Krawiez (2001). Scientific cycle model with delay. *Scientometrics* **52**, 83 – 95.

[99] T. T. Kuhn (1962). The structure of scientific revolutions. Chicago University Press, Chicago.

[100] P.E. Plesk (1997) Creativity, Innovation and Quality. ASQ Quality Press, Milwaukee.

[101] T.M. Amabile, R. Conti, H. Coon, J. Lazenby, M. Herron (1996). Assessing the work environment for creativity. *Academy of Management Review* **39**, 1154 – 1184.

[102] I. M. Beck (1984). A method of measurement of scientific production. *Science of Science* **4**, 183 – 195.

[103] M. Szydlowski, A. Krawiez (2009). Growth cycles of knowledge. *Scientometrics, 78*, 99 – 111.
[104] D. J. de Solla Price (1956). The exponential curve of science. *Discovery* 17, 240 – 243.

[105] W. Goffman, V. A. Newill. (1964) Generalization of epidemic theory. An application to the transmission of ideas. *Nature* 204, 225 – 228.

[106] L. M. A. Bettencourt, D. I. Kaiser, J. Kaur, C. Castillo-Chavez, D. E. Wojick. (2008). Population modeling of the emergence and development of scientific fields. *Scientometrics* 75, 495 – 518.

[107] L. M. A. Bettencourt, A. Cintron-Arias, D. I. Kaiser, C. Castillo-Chavez. (2006). The power of a good idea: Quantitative modeling of the spread of ideas from epidemiological models. *Physica A* 364, 513 – 536.

[108] M. Nowakowska (1973). Epidemical spread of scientific objects: An attempt of empirical approach to some problems of meta-science. *Theory and Decision* 3, 262 – 297.

[109] D. J. Daley (1967). Concerning the spread of news in a population of individuals who never forget. *Bulletin of Mathematical Biology* 29, 373 – 376.

[110] E. Bruckner, W. Ebeling, A. Scharnhorst. (1990). The application of evolution models in scientometrics. *Scientometrics* 18, 21 – 41.

[111] W. Ebeling, R. Feistel, I. Hartmann-Sonntag, L. Schimansky-Geier, A. Scharnhorst (2006). New species in evolving networks-stochastic theory of sensitive networks and applications on the metaphorical level. *BioSystems* 85, 65 - 71.

[112] A. J. Lotka (1912). Ein Fall von Autokatakinese mit oscillatorischem Verlauf. *Zeitschrift für Physikalische Chemie* 80, 159 - 164.

[113] A. J. Lotka (1925). Elements of physical biology. Williams and Wilkins, Baltimore.

[114] V. Volterra (1931). Variations and fluctuations of a number of individuals in animal species living together. pp. 409 - 448 from R. N. Chapman. *Animal Ecology*. McGraw Hill, New York.
[115] S. M. Kot (1987). The stochastic model for evolution of scientific disciplines. *Scientometrics* 12, 197 – 205.

[116] W. Glänzel, U. Schoepflin (1994). A stochastic model for aging of the scientific literature. *Scientometrics* 30, 49 – 64.

[117] E. Bruckner, W. Ebeling, A. Scharnhorst (1989). Stochastic dynamics of instabilities in evolutionary systems. *System Dynamics Review* 5, 176 – 191.

[118] E. Bruckner, W. Ebeling, M. A., Jimenez Montano, A. Scharnhorst (1996). Nonlinear stochastic effects of substitution - an evolutionary approach. *Journal of Evolutionary Economics* 6, 1 – 30.

[119] T. Braun, W. Glänzel, A. Schubert (1985). Scientometric indicators. A 32-country comparative evaluation of publishing performance and citation impact. World Scientific, Singapore.

[120] C. Chen, D. Hicks. (2004). Tracing knowledge diffusion. *Scientometrics*, 59, 199 – 211.

[121] I. Z. Kiss, M. Broom, P. G. Graze, I. Rafols (2000). Can epidemic models describe the diffusion of topics across disciplines. *Journal of Infometrics* 4, 74 – 82.

[122] I. Z. Kiss, D. M. Green, R. R. Kao, (2005). Disease contact tracing in random and clustered networks. *Proceedings of the Royal Society B*, 272, 1407 – 1414.

[123] N. G. VanKampen. (2007). Stochastic processes in physics and chemistry. Elsevier, Amsterdam.

[124] H. Risken. (1984) *The Fokker-Planck Equation* (Springer-Verlag, Berlin)

[125] P. Hänggi, H. Thomas (1982). Stochastic processes: Time evolution, symmetries and linear response. *Physics Reports* 88, 207 – 319.

[126] C. W. Gardiner. (1983). *Handbook of Stochastic Methods*, (Springer-Verlag, Berlin).
[127] A. K. Romanov, A. I. Terekhov (1997). The mathematical model of productivity - and age-structured scientific community evolution. *Scientometrics* 39, 3 – 17.

[128] W. Ebeling, A. Engel, V. G. Mazenko (1986). Modeling of selection processes with age-dependent birth and death rates. *BioSystems* 19, 213 – 221.

[129] W. Ebeling, A. Engel, R. Feistel (1990). Physik der Evolutionsprozesse. Akademie - Verlag, Berlin.

[130] E. H. Kerner (1959). Further considerations on the statistical mechanics of biological associations. *Bulletin of Mathematical Biophysics* 21, 217 – 253.

[131] J. C. Allen (1975). Mathematical model of species interactions in time and space. *American Naturalist* 109, 319 – 342.

[132] A. Okubo (1980). Diffusion and ecological problems: Mathematical models. Springer, Berlin.

[133] W. G. Willson, A. M. de Roos (1993). Spatial instabilities within the diffusive Lotka-Volterra system: Individual-based simulation results. *Theoretical Population Biology* 43, 91 – 127.

[134] N. K. Vitanov, I. P. Jordanov, Z. I. Dimitrova. (2009). On nonlinear population waves. *Applied Mathematics and Computation* 215, 2950 – 2964.

[135] N. K. Vitanov, I. P. Jordanov, Z. I. Dimitrova. (2009). On nonlinear dynamics of interacting populations: Coupled kink waves in a system of two populations. *Communications in Nonlinear Science and Numerical Simulation* 14, 2379 - 2388.

[136] Z. I. Dimitrova, N. K. Vitanov (2000). Influence of adaptation on the nonlinear dynamics of a system of competing populations. *Physics Letters A* 272, 368 – 380.

[137] Z. I. Dimitrova, N. K. Vitanov (2001). Adaptation and its impact on the dynamics of a system of three competing populations. *Physica A* 300, 91 – 115.
[138] Z. I. Dimitrova, N. K. Vitanov (2001). Dynamical consequences of adaptation of the growth rates in a system of three competing populations. *Journal of Physics A: Mathematical and General* 34, 7459 – 7473.

[139] N. K. Vitanov, Z. I. Dimitrova, M. Ausloos (2010). Verhulst-Lotka-Volterra (VLV) model of ideological struggle. *Physica A* 389, 4970 – 4980.

[140] W. Ebeling, A. Scharnhorst (2000). Evolutionary models of innovation dynamics, pp. 43 – 56 in D. Helbing, H. J. Herrmann, M. Schreckenberg, D. E. Wolf (Eds.) Traffic and granular flow’ 99. Social, traffic and granular dynamics. Springer, Berlin.

[141] A. I. Yablonsky. (1980). On fundamental regularities of the distribution of scientific productivity. *Scientometrics* 2, 3 – 34.

[142] A. I. Yablonsky (1985). Stable non-Gaussian distributions in scientometrics. *Scientometrics* 7, 459 – 470.

[143] S. D. Haitun (1982). Stationary scientometric distributions. Part II. Non-Gaussian nature of scientific activities. *Scientometrics* 4, 89 – 104.

[144] W. Li (2002). Zipf’s law everywhere. *Glottometrics* 5, 15 – 41.

[145] J. S. Katz (1999). The self-similar science system. *Research Policy* 28, 501 – 517.

[146] A. F. J. van Raan (2000). On growth, ageing and fractal differentiation of science. *Scientometrics* 47, 347 – 362.

[147] X. Gao, J. Guan (2009). A scale-independent analysis of the performance of the Chinese innovation system. *Journal of Infometrics* 3, 321 – 331.

[148] N. Vandewalle, M. Ausloos (1997). Coherent and random sequences in financial fluctuations. *Physica A* 246, 454 – 459.

[149] K. Ivanova, M. Ausloos (1999). Application of the Detrended Fluctuation Analysis (DFA) method for describing cloud breaking. *Physica A* 274, 349 – 354
[150] Y. S. Chen, P. P. Cheng, Y. Tong (1993). Theoretical foundation of the 80/20 rule. *Scientometrics* **28**, 283 – 204.

[151] G. K. Zipf (1949). Human behaviour and the principle of least effort, Addison-Wesley, Reading, MA.

[152] D. de Solla Price (1976). A general theory of bibliometric and other cumulative advantage processes. *Journal of the American Society for Information Science* **27**, 292 – 306.

[153] R. Wagner-Döbler, J. Berg (1994). Regularity and irregularity in the development of scientific disciplines: the case of mathematical logics. *Scientometrics* **30**, 303 – 319.