Surface charges for gravity and electromagnetism in the first order formalism

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Abstract

A new derivation of surface charges for 3 + 1 gravity coupled to electromagnetism is obtained. Gravity theory is written in the tetrad-connection variables. The general derivation starts from the Lagrangian, and uses the covariant symplectic formalism in the language of forms. For gauge theories, surface charges disentangle physical from gauge symmetries through the use of Noether identities and the exactness symmetry condition. The surface charges are quasilocal, explicitly coordinate independent, gauge invariant and background independent. For a black hole family solution, the surface charge conservation implies the first law of black hole mechanics. As a check, we show the first law for an electrically charged, rotating black hole with an asymptotically constant curvature (the Kerr–Newman (anti-)de Sitter family). The charges, including the would-be mass term appearing in the first law, are quasilocal. No reference to the asymptotic structure of the spacetime nor the boundary conditions is required and therefore topological terms do not play a rôle. Finally, surface charge formulae for Lovelock gravity coupled to electromagnetism are exhibited, generalizing the one derived in a recent work by Barnich et al Proc. Workshop ‘About Various Kinds of Interactions’ in honour of Philippe Spindel (4–5 June 2015, Mons, Belgium) C15-06-04 (2016 (arXiv:1611.01777 [gr-qc])). The two different symplectic methods to define surface charges are compared and shown equivalent.

Keywords: surface charges, symplectic formalism, gauge symmetries
1. Introduction

To find the quantum degrees of freedom responsible for black hole entropy remains one of the main questions that fuels the research of a quantum theory of gravity. The semiclassical analysis, the study of quantum field theory on fixed black hole spacetimes, ensures that the entropy is proportional to one-fourth of the horizon area (in units where $G = c = 1$).

Because the entropy corresponds to the area of the horizon, the expectation is that the microscopic degrees of freedom responsible for the entropy are localized around the horizon itself. At least two prevailing approaches construct quantum horizon models dwelling on this latter view.

The first uses symplectic methods to build a Chern–Simons description of the horizon [1–4] in the context of loop quantum gravity. The second uses tools [5, 6] developed in the context of holography, specifically asymptotic symmetries and associated charges, to describe the black hole horizon (some examples are [7–11]). Both approaches start from a classical description of the black hole horizon through boundary conditions that leave some freedom for the variables at the boundary. In both cases, one may wonder if in that freedom there are true degrees of freedom. Surprisingly, in certain contexts they have been called would-be gauge degrees of freedom [12, 13], as some of them would be degrees of freedom coming from the partial freezing of the gauge symmetries at the boundary. That seems artificial—as they are strongly dependent on the particular choice of boundary conditions.

With this context in mind, the present work is a first step in exploring, from a different perspective, the common basis upon which both the above approaches stand. That is, the study of physical and gauge symmetries of the symplectic structure when boundary conditions are imposed. To do so, we focus on surface charges, motivated in section 1.1, as the main quantities encoding the physical information related with symmetries, in the context of gauge theories.

As a result, we explicitly rederive the general expression of surface charges from the covariant symplectic method in the language of forms, in section 2. Then, to describe gravity, we choose to depart from the usual metric approach, by using the tetradic-connection formalism. We also deal directly with the more general case of a gravity theory (with cosmological constant) coupled to electromagnetism in four dimensions. This allows us to obtain, as a second important result, compact formulae for the surface charges that are gauge invariant, and do not require reference to a coordinate system—see equation (5.5). A valuable conclusion of this work is an explicit exhibition of surface charge formulae, with the tight requirement that they satisfy a conservation law. This paves the way to disentangling physical from gauge symmetries, in the context of asymptotic symmetries and boundary conditions previously discussed.

The content of these notes goes as follow: in section 2, we rederive surface charges for a general gauge theory. In sections 3–5, we progressively establish the explicit formulae of the surface charges for the theory of gravity coupled to electromagnetism. In section 3.1, we show that boundary terms in the Lagrangian have no effect on surface charges. In section 5.1, we perform a test of the reliability of the formalism, by recovering the standard first law of black hole mechanics in a quasilocal way. In section 6, we study the generalization to an arbitrary dimension. We compute the surface charge formula for Lovelock gravity coupled to electromagnetism. Finally, in the appendix, we further present a general comparison of the canonical covariant symplectic approach with other covariant techniques inherited from the BRST formalism [15].
1.1. Why surface charges?

It is not widely enough known that the first Noether theorem does not apply to gauge symmetries. The reason is that the would-be Noether current is trivially conserved—i.e., it is conserved without requiring the equations of motion (for more discussion see [14]). As we will show, this is in fact a consequence of the second Noether theorem. Thus, the would-be Noether charge is ambiguous, as it has the arbitrary gauge parameter on it. For example: for the electromagnetic field the ‘gauge current’ for the gauge symmetry, \( \delta \lambda A = d \lambda \), is \( J_\lambda = d(\lambda \ast F) \); and the ‘gauge charge’, \( Q_\lambda = \oint \lambda \ast F \), depends explicitly on the arbitrary gauge parameter.

A common way to cure this lack of meaning for the would-be Noether current is to assume extra structure for the fields and gauge parameters in an asymptotic spacetime region. With this, the charge computed out of the current may acquire a physical meaning. However, ambiguities related to the action boundary term may still affect the value of the current and extra input, as differentiability of the action may be required to have a well-defined charge.

In the case of gravity, the asymptotic structure of flat, de Sitter and anti-de Sitter spacetimes are drastically different. The boundary term in the action to guarantee differentiability of the action changes for each case. This fact makes the definition of asymptotic charges problematic. A more general approach, not resting on the particular asymptotic structure of the spacetime, is certainly desirable.

The quantities known as surface charges provide the necessary generalization [6, 18]. They are conserved quantities for a physical symmetry in the context of gauge theories.

In the next section, we show how to compute surface charges in general, from canonical symplectic methods. The computation relies on the Lagrangian, but is independent of the ambiguities in the boundary terms that one may add to it.

An interesting property of surface charges is that they are quasilocal. It is not necessary to use an asymptotic spacetime structure to define them. On the other hand, one may perfectly compute them on asymptotic regions, too.

It is worth noting that surface charges are a particular case of a generalization of conserved currents. In [24], it is explained that Noether’s first theorem can be rephrased as a cohomology in the context of the BRST symmetry. Thereafter, it is proven that higher order conserved currents are in correspondence with a generalization of ‘global symmetries’. This is a generalization of Noether’s first theorem. The usual Noether current is the first of these currents; the surface charges are built out of the second. To understand the rest of the currents from a canonical symplectic perspective is an interesting problem, left for a future work.

2. Surface charges for gauge theories

In gauge theories, field transformations due to gauge and rigid symmetries are entangled. This can muddy the definition of physical quantities like charges. In the frame of covariant symplectic methods, we can start studying both gauge and rigid symmetries on the same foot, and then assert the difference at a crucial step. We start by considering the Noether procedure for infinitesimal symmetry transformations in the language of forms. We specify this, first to the case of gauge symmetries, and then to the case where diffeomorphism is one of the gauge symmetries. In the end, we will define and assume the existence of exact transformations, to produce physically sensible results. We follow, in general lines, the canonical covariant symplectic approach [16, 17], but having in mind the invariant symplectic approach used in [6, 18], where surface charges are defined. Another useful reference is section 3 in [19], or [23], where a close general treatment is performed.
Consider a Lagrangian form \( L[\Phi] \) for a collection of fields \( \Phi \). The arbitrary variation is
\[
\delta L = E(\Phi)\delta \Phi + d\Theta(\delta \Phi),
\]
with \( E(\Phi) = 0 \) being the equations of motion, and \( \Theta(\delta \Phi) \) a boundary term. The Lagrangian has a symmetry if for certain infinitesimal variations over the configuration space it becomes at most an exact form:
\[
\delta \epsilon L = dM\epsilon;
\]
we denote by \( \epsilon \) the collection of parameters that generate the infinitesimal symmetry, and by \( \delta \epsilon \) the infinitesimal transformation generated over any quantity. In the case \( M\epsilon \neq 0 \), the usual notion of symmetry for the action \( S[\Phi] = \int_M L[\Phi] \) is recovered by choosing a vanishing of the symmetry parameters at a neighborhood of the boundary of the manifold. This can only ever be done for gauge symmetries.

The fields transform under a symmetry as \( \delta \epsilon \Phi \); therefore,
\[
dM\epsilon = E(\Phi)\delta \epsilon \Phi + d\Theta(\delta \epsilon \Phi).
\] (2.3)

Now, let us assume that the transformation \( \delta \epsilon \Phi \) is linear in the symmetry parameters \( \epsilon \). This assumption allows us to make a crucial step. We can remove the derivatives over all symmetry parameters, and formally decompose
\[
E(\Phi)\delta \epsilon \Phi = dS\epsilon - N\epsilon,
\] (2.4)
such that in \( N\epsilon \) the symmetry parameters appear only as factors. We will use a hat to remind us that the equations hold on-shell, for instance \( S\hat{\epsilon} = 0 \) or \( N\hat{\epsilon} = 0 \). Using the new expression for \( E(\Phi)\delta \epsilon \Phi \), we obtain
\[
d[\Theta(\delta \epsilon \Phi) - M\epsilon + S\epsilon] = N\epsilon.
\] (2.5)

Now, we restrict ourselves to gauge symmetries. For these, the very structure of the last equation implies
\[
N\epsilon = 0,
\] (2.6)
these are called Noether identities, and there is one of them for each independent gauge parameter. These are the usual constraints of the theory, due to the redundancy of using gauge variables and are the reason why the first Noether theorem does not apply for gauge symmetries.

Thence, it is natural to define the form
\[
J\epsilon \equiv \Theta(\delta \epsilon \Phi) - M\epsilon + S\epsilon,
\] (2.7)
which by virtue of the Noether identities satisfies
\[
dJ\epsilon = 0.
\] (2.8)

Note that the statement is off-shell, and thus \( J\epsilon \) is not a current. However this quantity reduces on-shell to what is usually called the Noether current \( J\epsilon \equiv \Theta(\delta \epsilon \Phi) - M\epsilon \). As far as \( \delta \epsilon \) generates a gauge symmetry, this current is trivial—as its off-shell conservation law suggests. However, with two more ingredients, this current generates non-trivial and finite charges. These extra assumptions are that \( \delta \epsilon \) is an exact symmetry of the fields, i.e. \( \delta \epsilon \Phi = 0 \) (further discussed in the paragraph before (2.21)), and that the boundary term in the Lagrangian is consistent with the boundary conditions \([20]\). This is the standard Noether procedure, which for gauge theories needs to be supplemented with extra information.

\footnote{\( N\epsilon \) can be factorized by the arbitrary parameters \( \epsilon \), and at the same time is equal to an exact form—this implies that \( N\epsilon \) vanishes. Proof: integrate (2.5) and choose the parameters to vanish at a neighborhood of the boundary.
However, there is an alternative. We can follow a quasilocal approach that does not make use of the asymptotic structure. The cost is reliance on a linearized theory. This is shown in the following.

The Poincaré lemma ensures that a closed form is locally exact, that is, there exists $\tilde{Q}_\epsilon$ such that

$$J_\epsilon = d\tilde{Q}_\epsilon.$$  

(2.9)

Now, consider an off-shell variation

$$\delta \Theta(\delta_1 \Phi) - \delta M_\epsilon + \delta S_\epsilon = d\delta \tilde{Q}_\epsilon.$$  

(2.10)

We assume that $\delta d = d\delta$. The double hat will be used to remind the reader that, along with the equations of motion, the linearized equations of motion also hold. For instance, $\delta S_\epsilon$ does not vanish on-shell. We need the extra assumption that $\delta S$ satisfies the linearized equations of motion, $\delta S_\epsilon \tilde{\sim} 0$.

The presymplectic structure density is an antisymmetrized double variation of the fields on the phase space, defined by

$$\Omega(\delta_1, \delta_2) \tilde{\triangleleft} \delta_1 \Theta(\delta_2 \Phi) - \delta_2 \Theta(\delta_1 \Phi) - \Theta([\delta_1, \delta_2] \Phi),$$  

(2.11)

where the boundary term in the action $\Theta(\delta \Phi)$ is also referred to as the presymplectic potential density. The variations, $\delta \Phi$, are assumed to satisfy the linearized equation of motion. Note that $\Omega(\delta_1, \delta_2)$ is a double variation in the phase space, and a $(D-1)$-form in spacetime.

The double variation can also be understood as a two-form in the phase space. The last term, $\Theta([\delta_1, \delta_2] \Phi)$, should be considered because variations of fields on the phase space do not commute in general [21]. The prefix in presymplectic stands for the fact that variations $\delta_{1,2}$ on the fields can also be gauge symmetry transformations. We need it because, using gauge variables, there is not a systematic way to disentangle the gauge redundancy from the phase space. In this sense, the phase space is degenerated. It contains gauge orbits—i.e. family of points identified through gauge transformations. In other words, if $\mathcal{M}$ is the manifold where $L[\Phi]$ is defined, then, on-shell, $\int_{\partial \mathcal{M}} \Omega(\delta_1, \delta_2)$ has degenerated directions [16]—precisely those associated with infinitesimal gauge transformations.

Considering the presymplectic structure density evaluated in a gauge variation, $\Omega(\delta, \delta_\epsilon)$, we rewrite (2.10) as

$$\Omega(\delta, \delta_\epsilon) = -\delta_\epsilon \Theta(\delta \Phi) - \Theta([\delta, \delta_\epsilon] \Phi) + \delta M_\epsilon - \delta S_\epsilon + d\delta \tilde{Q}_\epsilon.$$  

(2.12)

To go further, let us assume that $\epsilon$ contains diffeomorphisms. More precisely, suppose the collection of gauge parameters can be split as $\epsilon = (\xi, \lambda)$, i.e. $\delta_\epsilon = \delta_\xi + \delta_\lambda$, such that $\xi$ is a vector field generating infinitesimal diffeomorphisms, and $\delta_\lambda$ denotes the remaining infinitesimal gauge symmetry transformations. For a form $\omega$ that is invariant under $\delta_\lambda$, the infinitesimal diffeomorphism transformations are generated through a Lie derivative

$$\delta_\xi \omega = \mathcal{L}_\xi \omega = d(\xi \omega) + \xi.d\omega,$$  

(2.13)

where $\mathcal{L}$ denotes the interior product over forms. For a vector field $\xi = \xi^\mu \partial_\mu$ and a one-form $\omega = \omega^\mu dx_\mu$, both expressed in coordinate components, the interior product is $\xi \omega = \xi^\mu \omega_\mu$.

$^4$Equations (2.8) and (2.9) suggest that non-trivial on-shell currents are those which satisfy a conservation law in the whole spacetime, but for which the Poincaré lemma cannot be extended to the whole spacetime, i.e. they correspond to the equivalence classes of closed forms which are not exact—the de Rham cohomology. We refer to [24] for a rephrasing of Noether theorems using the cohomology of the BRST symmetry.

$^5$If the form is a gauge variable, an ambiguity arises for the Lie derivative, and the Cartan formula (2.13) has to be corrected. We address this point in the examples.
The interior product distributes over the wedge product of forms exactly as the exterior derivative does. The exterior derivative, $\mathbf{d}$, and the interior product, $\mathbf{\iota}$, act only on the term on the immediate right of the symbol, unless explicit parentheses are drawn.

We assume the Lagrangian and the presymplectic potential density are left invariant under the transformation generated by $\lambda$. As a top form in the manifold, the Lagrangian satisfies $\delta_\epsilon L = \delta_\epsilon \mathbf{d}L = \mathbf{d}(\delta_\epsilon L)$; thus, $M_\epsilon = \xi_\epsilon L$. Here and in the following, we assume $\delta\xi = 0$. Thence, we have

$$\delta M_\epsilon = \xi_\epsilon (E_\delta \Phi) + \xi_\epsilon \mathbf{d}\Theta(\delta\Phi).$$  

On the other hand

$$\delta_\epsilon \Theta(\delta\Phi) = \delta_\xi \Theta(\delta\Phi) = d\xi_\epsilon \Theta(\delta\Phi) + \xi_\epsilon \mathbf{d}\Theta(\delta\Phi),$$

and

$$\Omega(\delta, \delta_\epsilon) = -d\xi_\epsilon \Theta(\delta\Phi) - \Theta([\delta, \delta_\epsilon]\Phi) + \xi_\epsilon (E_\delta \Phi) - \delta S_\epsilon + d\delta_\epsilon \bar{Q}.$$  

After explicitly using the equations of motion and the linearized equations of motion, we obtain a simple expression for the presymplectic structure density:

$$\Omega(\delta, \delta_\epsilon) \equiv d(\delta_\epsilon \bar{Q}_\epsilon - \xi_\epsilon \Theta(\delta\Phi)) - \Theta([\delta, \delta_\epsilon]\Phi).$$  

In the case that the parameters $\delta\epsilon \neq 0$ are extended non-trivially on the phase space, the last term does not vanish (we still assume $\delta\xi = 0$ but $\delta\lambda \neq 0$). In the examples, it is going to be the case when gauge parameters get fixed to encode exact symmetries. Analogous to the decomposition $E_\delta \epsilon \Phi = dS_\epsilon - N_\epsilon$, the term $\Theta([\delta, \delta_\epsilon]\Phi)$ can be decomposed as

$$\Theta([\delta, \delta_\epsilon]\Phi) = dB_{\delta\epsilon} + C_{\delta\epsilon},$$

such that in $C_{\delta\epsilon}$, the varied parameters appear as factors. A similar argument to the one used to prove the off-shell Noether identity, $N_\epsilon = 0$, proves that on-shell $C_{\delta\epsilon} \equiv 0$.

We define a $(D-2)$-form in spacetime and first variation in phase space by

$$k_\epsilon \equiv \delta_\epsilon \bar{Q}_\epsilon - \xi_\epsilon \Theta(\delta\Phi) - B_{\delta\epsilon}.$$  

As an abuse of nomenclature we may refer to this quantity as the would-be surface charge integrand.

In the case that $\epsilon$ represents a gauge symmetry, we can choose $\delta\epsilon = 0$ such that $\Theta([\delta, \delta_\epsilon]\Phi) = 0$. Thus, equation (2.17) tells us the standard result: the presymplectic structure density for a gauge transformation is trivial—i.e. it is an exact form in spacetime.

$$\Omega(\delta, \delta_\epsilon) \equiv dk_\epsilon.$$  

6 Note that if $\lambda$ is a gauge transformation of a Chern–Simons theory, the Lagrangian is not invariant. The generalization is straightforward, but we refer to [22] for a discussion of this case.

7 In (2.17) the gauge parameters $\epsilon$ and $\delta\epsilon$ are disentangled; thus, fixing $\epsilon = 0$, and integrating over any $(D-1)$-surface, the arbitrariness of $\delta\epsilon$ implies $C_{\delta\epsilon} = 0$.

8 The presymplectic structure for an action $S[\Phi] = \int_{\partial\mathcal{M}} L$ is obtained by

$$(\delta_1 \delta_2 - \delta_2 \delta_1)S[\Phi] \equiv \int_{\partial\mathcal{M}} \left[ \Omega(\delta_1, \delta_2) + \Theta([\delta_1, \delta_2]\Phi) \right];$$

therefore, it is defined up to an exact form in spacetime.
A gauge symmetry is a degenerate direction in the presymplectic structure. Once integrated, the last expression becomes an arbitrary boundary term that, in particular, can be chosen to vanish.

Now, in the particular case that $\epsilon$ generates an exact symmetry, the presymplectic structure density vanishes. We call the condition where particular parameters $\bar{\epsilon}$ solve the equation $\delta \bar{\epsilon} \Phi = 0$ (Killing fields for the metric, for instance) exact symmetry. Thus, because the presymplectic structure density is linear in the infinitesimal transformations, we have $\Omega(\delta, \delta \bar{\epsilon}) \equiv 0$. Therefore, for exact symmetries

$$\text{d} k \bar{\epsilon} \equiv 0.$$  \hfill (2.21)

The establishment of this equation is the main goal of this section. This equation is a second conservation law of one degree less than $\text{d} J \epsilon = 0$. It has a true physical meaning because it requires the use of the equations of motion besides the property $\delta \bar{\epsilon} \Phi = 0$. As commented in section 1.1, $k \bar{\epsilon}$ is a current in the context of a generalized Noether theorem [24].

Therefore, we define the surface charge by the integral

$$\delta Q \bar{\epsilon} \equiv \oint k \bar{\epsilon}.$$  \hfill (2.22)

Note that it is called surface because it is naturally defined on a $(D - 2)$-manifold which is a surface in four dimensions. More importantly, it is called a charge because it is conserved. This happens only because the exactness of the symmetry guarantees the conservation law (2.21). This makes the integral independent of the closed surface on which the integration is performed. On the other hand, one could compute non-vanishing quantities $\delta Q \bar{\epsilon}$ for gauge symmetries $\delta \epsilon$, but these quantities are not charges. Following the notation proposed in [18], we use $\delta$ to denote quantities that are not necessarily integrable on the phase space. In other words, the function $Q \bar{\epsilon}$, such that its variation on the phase space satisfies $\delta Q \bar{\epsilon} = \delta Q \bar{\epsilon}$, may not exist. A sufficient condition for its existence is $\delta(\delta Q \bar{\epsilon}) = 0$; this is the condition of integrability for the surface charge to become a finite charge.

As explained before, a gauge symmetry produces a trivial Noether current, in the sense that it is conserved even off-shell (2.8). Thus, to use a physical symmetry or the equations of motion or nothing, in order to prove the conservation of $J \epsilon$ does not make any difference. However, if the gauge symmetry can be made exact for a certain choice of the gauge parameters $\epsilon$, it will produce a second necessarily on-shell conservation law for $k \bar{\epsilon}$. Let us remark that choosing the gauge parameters means that we are no longer dealing with a gauge symmetry. However, in the derivation of $\text{d} k \bar{\epsilon} \equiv 0$ we make intensive use of the presence of a gauge symmetry.

In the four-dimensional examples worked out in section 5.1, $k \epsilon$ is a closed two-form in spacetime that can be used to relate quantities defined on two arbitrary disconnected closed two-surfaces which are the boundaries of a given three-volume. The integration of $k \epsilon$ on a closed two-surface is trivial if the surface is contractible to a point. In the black hole example, $k \epsilon$ will be integrated over spheres enclosing the singularity. Note that, as the argument strongly depends on differentiability of fields, this should be guaranteed in the three-volume as well as in its boundary; bulk singularities and spikes in boundaries have to be treated carefully.

Two remarks regarding possible ambiguities are in order. First, note that there is an ambiguity in the definition of $\Theta(\delta \Phi) \rightarrow \Theta(\delta \Phi) + \text{d} Y(\delta \Phi)$, which percolates to an arbitrary exact form in the presymplectic structure density. However, for exact symmetries it simply vanishes.

9 This is exactly what happens in the Hamiltonian approach for asymptotic charges: the parameters generating gauge symmetry are boiled down to exact symmetries in the asymptotic region.
and does not have any effect in the definition of $k_\epsilon$. Second, another ambiguity could arise because $k_\epsilon \rightarrow k_\epsilon + d\alpha$ does not change the equation $dk_\epsilon \overset{\approx}{=} 0$. This ambiguity is harmless as far as $k_\epsilon$ is used only integrated over closed surfaces.

### 3. General relativity

In this section, we consider the action for gravity in four dimensions in the first order formalism. This formalism is fundamental, in the sense that it is suitable to include fermionic fields. At the same time, the metricity and parallelism properties of spacetime can easily be disentangled [25].

The language of forms allows us to write variables without doing explicit references to coordinates. We consider as independent variables the tetrad and the Lorentz connection, $(e^I, \omega^{IJ})$; both are one-forms. The curvature two-form reads $R^{IJ} = d\omega^{IJ} + \omega^I_K \wedge \omega^{KL}$. Besides the standard Einstein–Hilbert term, we consider a cosmological constant and a topological Euler term, all arranged in the well-known McDowell–Mansouri action [20, 26, 27]. In the following, we will suppress the indexes and the wedge product, when possible, to keep the notation compact.

Therefore, the action for gravity simple reads

$$S[e, \omega] = \kappa \int_M \bar{R} \ast \bar{R},$$

with the barred curvature given by

$$\bar{R}^{IJ} \equiv R^{IJ} \pm \frac{1}{\ell^2} \epsilon^{IJKL} e^J \bar{R}^{KL},$$

note that, as before, the wedge product between forms is understood. The $\ast$ stands for the dual of the internal group—in this case the Lorentz group—for instance $\ast R^{IJ} = \frac{1}{2} \epsilon^{IJKL} R^{KL}$. The $\pm$ stands for both possible signs of the cosmological constant. The treatment is the same when we consider both at once. The overall constant $\kappa$ has no effect in the following, but we fix it to $\kappa = \pm \frac{\ell^2}{32\pi G}$, to make contact with standard approaches. We also choose the units to set the Newton constant $G = 1$.

The dependence on the cosmological constant can be consistently removed at the end of the calculation by considering the limit $\ell \rightarrow \infty$. Note that the Euler term is multiplied by $\ell^2$; thus, the limit cannot be taken at this stage. In fact, the Euler term can be thought as providing a regulator for the Einstein–Hilbert plus cosmological constant action and for the finite Noether charges derived from it [20]. However, as far as we consider exact symmetries the present quasilocal approach is insensitive to it. This is made explicit at the end of this section.

The variation of the Lagrangian is

$$\delta L = E_i \delta e^i + E_\omega \delta \omega + d\Theta;$$

if we get rid of the term $\Theta$ by imposing boundary condition, as we will discuss in a moment, the variational principle implies the equations of motion (putting back the indexes, $E_e \rightarrow E_I$ and $E_\omega \rightarrow E_{IJ}$)

$$E_I = \mp \frac{2\kappa}{\ell^2} \epsilon^{IJKL} e^J \bar{R}^{KL} = 0,$$

$$E_{IJ} = -\kappa \epsilon^{IJKL} d_{\omega} \bar{R}^{KL} = \mp \frac{2\kappa}{\ell^2} \epsilon^{IJKL} d_{\omega} e^K e^L = 0,$$

E Frodden and D Hidalgo

Class. Quantum Grav. 35 (2018) 035002
where we use $d_\omega$ to denote the covariant exterior derivative, for instance the Bianchi identity reads
\[ d_\omega R^{IJ} = d R^{IJ} + \omega^I_K R^{JK} - \omega^J_K R^{IK} = 0,\]
and the torsion
\[ T^I \equiv d_\omega e^I = de^I + \omega^J_I e^J.\]
The second equation is equivalent to setting the torsion equal to zero: $d_\omega e^I = 0$. Because this is an algebraic equation for the Lorentz connection, $\omega$, it can be solved in terms of the tetrad, $\omega(e)$. The replacement of $\omega(e)$ in the first equation produces the usual Einstein equation with cosmological constant written in forms.

As suggested before, to have a well-posed variational principle the term
\[ \Theta = 2 \kappa \delta \omega \ast \bar{R} \]  
(3.6)
must vanish at the boundary of the spacetime $\mathcal{M}$. If we allow for an arbitrary $\delta \omega$, the following boundary condition is required:
\[ \Theta|_{\partial \mathcal{M}} = 0 \rightarrow \bar{R}|_{\partial \mathcal{M}} = 0. \]  
(3.7)
Note that this condition requires an a priori knowledge of the boundary of the spacetime. In other words, we are reducing the solution space such that the previous equation can be satisfied. The family of spacetimes with this property are named locally asymptotic (anti-)de Sitter spacetimes. On the other hand, the standard assumption $\delta \omega|_{\partial \mathcal{M}} = 0$ is more relaxed, because $\omega$ is a connection, and we would need to fix the gauge in the boundary too.

The approach we follow to define the surface charges is quasilocal. It is insensitive to the chosen prescription for the boundary term. The only requirement is that a well-posed variational principle exists in order to obtain the equations of motion.

The gauge symmetries of the action are general diffeomorphisms and local Lorentz transformations. The infinitesimal transformations of the fields by the local Lorentz group is
\[ \delta \lambda e^I = \lambda^I_J e^J \]
\[ \delta \lambda \omega^I_J = - (d_\omega \lambda)^I_J = -d \lambda^I_J - \omega^I_K \lambda^K_J + \omega^J_K \lambda^I_K, \]  
(3.9)
where $\lambda^J_K = -\lambda^K_J$ are the parameters of the infinitesimal Lorentz transformation $\Lambda^I_J \approx \delta^I_J + \lambda^I_J$. Remember that it is a gauge symmetry—that is, the group elements take different values at different points of the manifold.

The infinitesimal transformations of the fields due to diffeomorphisms are normally assumed to be generated by an arbitrary vector field $\xi$ through a Lie derivative
\[ \tilde{\delta} \xi e = \mathcal{L}_\xi e = d(\xi \cdot e) + \xi. (de) \]  
(3.10)
\[ \tilde{\delta} \xi \omega = \mathcal{L}_\xi \omega = d(\xi \cdot \omega) + \xi. (d\omega), \]  
(3.11)
where, in the second equality, we use the Cartan formula. However, note that—due to the presence of exterior derivatives—they are not homogeneous under local Lorentz transformation. The intuitive interpretation of $\delta \xi e$ and $\delta \xi \omega$ as infinitesimal variation require them to be homogeneous under the action of the local Lorentz group. More precisely, if we attach ourselves to the intuitive idea of variations as comparison of fields in a neighbourhood, $\delta e \approx e' - e$, we

\[\text{Note that the imposed boundary condition has a symmetry larger than the local Lorentz group. It is invariant under the (anti-)de Sitter group (which contains the Lorentz group SO(3,1)). Note also that for asymptotically flat spacetimes, a well-defined action principle needs a boundary term different from the Euler one. In [28], it is shown that } \omega^I_J \ast (e e^I e^J) \text{ corresponds to the Hawking–Gibbons term in the first order formalism, and thus it allows a well-defined variational principle, as well as asymptotic Hamiltonians. However, as explained in section 3.1, boundary terms in the action do not contribute in the surface charge approach that we follow.}\]
expect them to have a covariant transformation under the local Lorentz group. This criterion is not satisfied by the infinitesimal diffeomorphism transformation presented before, and we therefore correct (3.10) and (3.11) by eliminating the non-homogeneous part. This can be done by adding an infinitesimal Lorentz transformation with a parameter $\xi, \omega$. For a recent discussion see [29]. This corrects the non-homogeneous part of both transformations at once, and we get

$$
\delta_\xi e = L^\xi e + \delta_{\xi, \omega} e = d_\omega(\xi, e) + \xi, d_\omega e) \tag{3.12}
$$

$$
\delta_\xi \omega = L^\xi \omega + \delta_{\xi, \omega} \omega = \xi, R. \tag{3.13}
$$

Another way to think about this is that in the transformation of the tetrad, the exterior derivative $d$ is promoted to a covariant exterior derivative $d_\omega$, while in the transformation of the Lorentz connection, because of the identity $d(\xi, \omega) + \xi, d_\omega = d_\omega(\xi, \omega) + \xi, R$, the ill-transforming part, $d_\omega(\xi, \omega)$, is subtracted.

Therefore, the general infinitesimal gauge transformations, involving diffeomorphisms and local Lorentz transformations, with parameters $\epsilon = (\xi, \lambda)$, which are themselves homogeneous, are

$$
\delta_\epsilon e = L^\xi e + \delta_{\xi, \omega}(\xi, e) + \xi, d_\omega e) + \lambda e \tag{3.14}
$$

$$
\delta_\epsilon \omega = L^\xi \omega + \delta_{\xi, \omega}(\xi, \omega) = \xi, R - d_\omega \lambda. \tag{3.15}
$$

Now, we follow the procedure detailed in section 2 to obtain the surface charges for general relativity. The Lagrangian transforms as a total derivative, $\delta L = L^\xi L = d(\xi, L)$. Under a symmetry transformation

$$
d(\xi, L - \Theta(\delta_\omega)) = E_\omega \delta_\epsilon e + E_\omega \delta_\epsilon \omega, \tag{3.16}
$$

using explicitly the symmetry transformation on the variables we can mimic (2.4)

$$
E_\epsilon \delta_\epsilon e + E_\omega \delta_\epsilon \omega = d[E_\omega \lambda - E_\epsilon \xi, e] + (E_\epsilon e - d_\omega E_\epsilon) \lambda + d_\omega E_\epsilon \xi, e + E_\epsilon \xi, d_\omega e + E_\omega \xi, R \tag{3.17}
$$

$$
= d[E_\omega \lambda - E_\epsilon \xi, e]. \tag{3.18}
$$

In the first line, we have used exact forms, to have the gauge symmetry parameters either inside an exact form or as a factor of a term. In the second line, we used the Noether identities: $d_\omega e - E_\epsilon e = 0$ and $d_\omega E_\epsilon \xi, e + E_\epsilon \xi, d_\omega e + E_\omega \xi, R = 0$, which are a rewriting of the standard Bianchi identity. Then, we define

$$
J_\epsilon \equiv \Theta(\delta_\omega) - \xi, L - E_\omega \xi, e + E_\omega \lambda, \tag{3.19}
$$

that trivially satisfies $dJ_\epsilon = 0$. Explicit computation results in an exact three-form that depends just on the gauge parameter of the Lorentz symmetry $\lambda$

$$
J_\epsilon = J_\lambda = -\kappa \ d(2\lambda \times \bar{R}) \tag{3.20}
$$

Despite its trivial conservation, one may try to use this current to define global charges associated with exact symmetries at the asymptotic boundary. The result is non-trivial, as shown in [20]. It is in this context that the Euler density becomes crucial, to accomplish the boundary condition $\bar{R}|_{\partial M} = 0$. This regularizes the symplectic structure, such that there is no leaking

---

11 Still other prescriptions for the infinitesimal transformations are possible. For instance the ones recently introduced in [30] differ from (3.14) and (3.15) by a term depending on the equation of motion. Terms of this kind are known as trivial symmetries, because they are present in any theory (section 3.1.5 in [31]).
on the boundary for locally asymptotic (anti-)de Sitter spacetimes. Thus, global finite charges can be asymptotically computed through that method. But here we aim for a quasilocal definition of charges. Following the prescription of section 2, we make a step further, performing an arbitrary variation of \( J_\lambda \), and computing each term of (2.16). To compute the \( B_{\delta\epsilon} \) contribution note that

\[
\Theta([\delta, \delta_\epsilon]) = 2\kappa [\delta, \delta_\epsilon] \omega \star \bar{R} \\
= -2\kappa d_\epsilon \delta_\lambda [\lambda + \xi \delta \omega] \star \bar{R} \\
= -2\kappa d(\delta \lambda + \xi \delta \omega) \star \bar{R} + 2\kappa (\delta \lambda + \xi \delta \omega) \star d_\epsilon \bar{R},
\]

(3.21)

where we used \([\delta, \delta_\epsilon] \omega = \delta (\delta \lambda + \xi \delta \omega) \omega\). As it was shown in general, (2.18), the last term in the third line, corresponding to \( C_{\delta\epsilon} \), vanishes on-shell.

Thus, from (3.6), (3.20) and (3.21) we obtain the surface charge integrand for general relativity

\[
k_\epsilon = -2\kappa (\lambda \star \delta \bar{R} - \delta \omega \star \xi \bar{R}).
\]

(3.22)

Now, if the exact symmetry condition is satisfied, we have \( dk_\epsilon = 0 \), and we can define surface charges \( \delta Q_\epsilon = \oint k_\epsilon \). In the example, we will be able to integrate the varied quantities on the phase space to find \( Q_\epsilon \). The charges are varied on the phase space, through a family of solutions. The study of the phase space for a family of solutions can be done explicitly, for instance, when considering the variation of the integration constants that appear in a solution.

For completeness, we write down the presymplectic structure density for pure gravity

\[
\Omega(\delta_1, \delta_2) \cong 2\kappa (\delta_2 \omega \star \delta_1 \bar{R} - \delta_1 \omega \star \delta_2 \bar{R})
\]

(3.23)

\[
\cong -\frac{1}{8\pi} \delta [\omega^I J \delta_2 \Sigma_M] + 2\kappa d(\delta \omega^I \star \delta_2 \omega_M),
\]

(3.24)

where we used that \( \delta \bar{R} = d_\epsilon \delta \omega \), the value of \( \kappa \), and the definition \( \Sigma_M \equiv \frac{1}{2} \varepsilon_{ijkl} e^k \wedge e^l \). The first term is the conjugate pair of gravity variables \( (\omega^I, \Sigma_M) \). The second term, consequent of the Euler term in the action, is an exact form, and disappears when the density is integrated on a smooth boundary of a manifold, \( \partial \mathcal{M} \). Similarly, the Euler contribution to the surface charge will not have any effect, because for exact symmetries it becomes an exact form. Explicitly, the contribution of the Euler term to (3.22) is

\[
k_{\text{Euler}} = -2\kappa [\lambda \star \delta \bar{R} - \delta \omega \star \xi \bar{R}] = -2\kappa [d(\lambda \star \delta \omega) + \delta \lambda \omega \star \delta \omega].
\]

(3.25)

This confirms that the procedure is not affected by the action boundary terms.

Recall that to guarantee that \( k_\epsilon \) is closed, we need an exact symmetry such that \( \Omega(\delta, \delta_\epsilon) \cong 0 \). Therefore, we have to solve the parameters \( \epsilon = (\xi, \lambda) \) such that

\[
\delta_\epsilon e = 0,
\]

(3.26)

\[
\delta_\epsilon \omega = 0.
\]

(3.27)

In the following, \( \epsilon = (\xi, \lambda) \) are solutions of the previous equation. The condition \( \delta_\epsilon e = 0 \) imposes a general relation between \( \lambda \) and \( \xi \). Exact symmetries are on-shell; thus, we use \( d_\epsilon e = 0 \), and solve \( \lambda \) from (3.26)

---

12 Note that in [20] the integrand to define the charge is \( \xi \omega_M \star \bar{R}^M \), which is gauge dependent; thus, an explicit gauge fixing at the boundary is required, such that it also respects the gauge dependent asymptotic symmetry condition \( \int \varepsilon^I |_{\partial \mathcal{M}} = 0 \). That is equivalent to our expression where we can use the integrand \( \lambda \omega \star \bar{R}^M \), and can fix \( \lambda^M \) by the exactness condition expressed below in (3.28). However, these expressions are explicitly Lorentz invariant.
\[ \lambda^I = e^I e_{\alpha} (\xi \cdot e^\alpha) = e^I e^\mu \nabla_{[\mu} \xi_{\nu]} , \]

where \( e^I e_{\alpha} \) is the interior product such that \( e^I e_{\alpha} e^J = \delta^I_J \); in coordinates components, it is \( e^I e^\mu \partial_{\mu} \). In the second equality, we exhibit the solution in components with \( \nabla_\mu \), the spacetime covariant derivative. This relation is a sufficient condition that gauge parameters should fulfil, to encode an exact symmetry. Note that the Killing equation, \( \mathcal{L}_e g = 0 \) with the metric \( g = e_I \otimes e^I \), is a direct consequence of \( \delta_e e = 0 \). The Killing equation in coordinate components, \( \nabla_\mu (\mu \xi_{\nu}) = 0 \), can also be seen directly in the rightest expression for \( \lambda \); it is encoded in the fact that \( \lambda^I \) is antisymmetric. On the other hand, because \( \omega \equiv \omega(e) \), the condition \( \delta_e \omega \equiv 0 \) holds trivially.

Therefore, we have obtained the expression of for the surface charges in the tetradic first order formalism (integration of (3.22)). We have also shown that the exact symmetries condition for the tetrad is the Killing equation in this language.

As a final remark, note that there is a second and straight way to obtain the same result. Consider the following expression for the \( \delta \omega \Xi \) term does not affect the surface charge. A similar computation for the Pontryagin yields the Lagrangian which are gauge quasi-invariant forms [25]. The expression (3.25) shows that the antisymmetric. On the other hand, because \( \omega = \omega(e) \), the condition \( \delta_e \omega \equiv 0 \) holds trivially.

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Finally, in $D = 4$, we may also be interested in using the Holst term density, $e_I e_J R^{IJ}$, inside the gravity action. This is not a topological term by itself, but a part of the Nieh–Yan, and it does not affect the equations of motion, either. To deal with it, note that $e_I e_J R^{IJ} = T^I T_I - d(e^I T_I)$. The second term has already been treated; thus, it is enough to keep track of $T^I T_I$ in the computation of surface charge potential. This term also does not produce any changes, because at the level of the presymplectic structure density, $\Omega(\delta, \delta)$, the contributions are already all proportional to the torsion $T$, and therefore vanish on-shell.

Thus, boundary terms—and, in particular, topological terms—do not affect the surface charges. Note that this is already explicit for surface charges computed through the contracting homotopy operator (3.30), because it depends only on $S_\epsilon$, and not on the Lagrangian. In this sense, here, we have stressed what is already indirectly known due to the fact that surface charges obtained through both methods are equivalent (appendix).

4. Electromagnetism

Before dealing with the more general case of general relativity coupled to electromagnetism, we briefly review the pure electromagnetic theory. Because diffeomorphisms are not a gauge symmetry here, the procedure is simpler. The variable is the connection one-form $A = A_\mu dx^\mu$. The field strength two-form is $F = dA$, and it possesses the $U(1)$ gauge symmetry, $A \rightarrow A + d\alpha$.

The action is

$$S[A] = \alpha \int_M F \ast F,$$

with $\alpha = -1/8\pi$, and where the Hodge dual $\ast$, acting on the field strength in coordinate components or in tetrad components respectively, is $\ast F = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} dx^\mu \land dx^\nu = \frac{1}{2} \varepsilon_{\mu\nu\rho} F^{\rho} e^\mu \land e^\nu$, with $e = \det(e^\mu_\nu)$ and $F^{IJ} = e^{I\mu} e^{J\nu} F_{\mu\nu}$.

The variation of the Lagrangian is

$$\delta L(A) = E_\lambda \delta A + d(\Theta(\delta A))$$

$$= -2\alpha (d \ast F) \delta A + d(2\alpha \delta A \ast F),$$

where $\delta \ast = \ast \delta$ because $e^I$ is not a dynamical field for this theory. As before, the boundary term should vanish. An option is to fix the connection at the boundary, and consequently the gauge symmetry. Another option that does not restrict the connection is to assume a vanishing field strength at the boundary, $F|_{\partial M} = 0$. The infinitesimal symmetries gauge transformation is $\delta_\lambda A = -d\lambda$, and applying it to the variation of the Lagrangian, $E_\lambda(-d\lambda) = d(E_\lambda \lambda) - dE_\lambda \lambda$ we get the Noether identity, which is the trivial equation $-dE_\lambda = 2\alpha d(d \ast F) = 0$. The Lagrangian is invariant under the gauge transformation; therefore, we have

$$J_\lambda = E_\lambda \lambda + \Theta(\delta A) = -2\alpha d(\lambda \ast F).$$

Note the similarity with the corresponding equation for general relativity (3.20).

The presymplectic structure density is $\Omega(\delta_1, \delta_2) \cong -4\alpha \delta_1 A \ast d\delta_2 A$. And for a gauge symmetry, $\Omega(\delta, \delta_\lambda) \cong d\lambda$, we get the surface charge integrand

$$k_\lambda = -2\alpha \lambda \ast \delta A.$$

The exact symmetry condition $\delta_\lambda A = 0$ is solved for $\lambda = \lambda_0 = \text{constant}$, such that the gauge symmetry turns into a rigid symmetry. Note that, here, the exact condition is independent of
the fields, and admits a general solution. Thus, we have $d \lambda_0 = 0$, which can be integrated in a three-surface $\Sigma$ enclosed by a two-surface $S$, to define

$$\delta Q_{\lambda_0} = \frac{\lambda_0}{4\pi} \oint_S \ast \delta F,$$

where we have restored the value of $\alpha$. For simplicity, the parameter $\lambda_0$ is chosen to be a constant in the phase space; the variation can then be trivially removed by an integration on phase space. We set the integration constant to zero. Thus, we obtain the definition of the electric charge, enclosed by the surface $S$

$$Q_{\lambda_0} = \frac{\lambda_0}{4\pi} \oint_S \ast F;$$

the conservation $d \lambda_0 = 0$ ensures that for any other surface, $S'$, obtained by a continuous deformation of $S$, the electric charge is the same. If there is no source, $S$ can be contracted to a point, and all charges are zero. This finishes the analysis of surface charges for electromagnetism.

For completeness, we derive the Noether current for a background spacetime with rigid symmetries. Note that the spacetime may not be a solution of the Einstein equation. The rigid symmetries are controlled by a Killing field $\xi$ such that $\delta_\epsilon \mathcal{E} = 0$, where $\epsilon = (\xi, \lambda^I)$ and $\lambda^I$ is given by (3.28). The connection $A$ suffers from the same ambiguity of any gauge variable, such that a Lie derivative on it, $\mathcal{L}_\xi A = \xi \cdot F + d\xi \cdot A$, may be changed by an arbitrary gauge transformation, while keeping the same information. Repeating the argument that led us to (3.13), the symmetry infinitesimal transformation, which is also gauge invariant, is

$$\delta_\xi A = \xi \cdot F;$$

this transformation is sometimes called ‘improved.’ Note that, while being assumed as a physical transformation, it is not an exact symmetry transformation. Because $\xi$ is not an arbitrary parameter, but a fixed Killing field, there is no Noether identity associated with it. The current $J_\xi \equiv \Theta(\delta_\xi A) - \xi \cdot L$, is a true Noether current, because it is conserved just on-shell

$$d(\Theta(\delta_\xi A - \xi \cdot L) = -E_A \delta_\xi A \doteq 0,$$

explicitly

$$J_\xi = \alpha (\xi \cdot F \ast F - F \xi \cdot \ast F),$$

which is the dual of the standard four-current of electromagnetism written in forms, $\ast J_\xi = j = j_\nu d\nu = \lambda^\mu T^{EM}_{\mu \nu} d\nu$, with $T^{EM}_{\mu \nu}$ the electromagnetic stress-energy tensor. Note that, again, $\delta_\xi A \neq \mathcal{L}_\xi A$. Instead, we used an infinitesimal transformation that is covariant (actually invariant) under the gauge transformation. This subtlety has produced debate in the literature, and the transformation (4.8) has been settled as the right one, because it produces a gauge invariant current (see, for instance, section 2.3 in the recent review [33]). However, the Noether current belongs to an equivalence class of currents related by exact forms (and possibly terms vanishing on-shell). The gauge ambiguity in the chosen transformation for the symmetry (4.8) just changes the representative current of the class. To define a charge, the equation, $dJ_\xi = 0$, should be integrated on a four-dimensional manifold which is splittable in two pieces. This procedure is insensitive to the ambiguity, and therefore to the Noether charge.
5. General relativity and electromagnetism

Using the results of the previous sections, the extension to the coupled theory is easy. Here, we use * for the Hodge dual and ⋆ for the group dual. We also make explicit some indexes to differentiate the U(1) infinitesimal gauge parameter, λ, from the SO(3, 1) infinitesimal gauge parameter, λ'.

The Lagrangian and its variation are given by
\[ L = \kappa R_{ij} \ast \tilde{R}^{ij} + \alpha F \ast F \]  
\[ \delta L = E_i^\mu \delta e + E_i \omega + E_i \delta A + d \Theta(\tilde{\omega}, \delta A), \]  
where \( E_i^\mu \delta e = (E_i + M_I) \delta e^i \). We use the notation \( e^i_{\mu;\nu} = e^i_{\mu} \partial_{\nu} \). \( E_i^\mu \) is the sum of the equation of motion of pure gravity (3.4) plus the contribution due to the electromagnetic stress tensor written as a form \( M_I \equiv \alpha(e_{\mu;i}F \ast F - Fe_{\mu;i} \ast F) \sim \alpha e^i_{\mu}T^{EM}_{\mu\nu} \ast dx^\nu \). The boundary term reads
\[ \Theta(\tilde{\omega}, \delta A) = 2\kappa \tilde{\omega}_{ij} \ast \delta R^{ij} + 2\alpha \delta A \ast F. \]

The infinitesimal gauge symmetry transformations are controlled by the parameters \( \epsilon = (\xi, \lambda'^{ij}, \lambda) \), corresponding to diffeomorphisms, Lorentz local symmetry and U(1) local symmetry respectively. The transformations are the same for the gravity fields, (3.14) and (3.15). For the electromagnetic field, we need to consider that it also transforms by diffeomorphisms. As discussed in the previous section, the improved version is \( \delta A = \xi \ast F - d\lambda \).

Following the procedure of section 2 we obtain
\[ J_e = -2 \frac{\kappa \lambda'^{ij} \ast \tilde{R}^{ij} + \alpha \lambda \ast F}{\kappa \lambda'^{ij} \ast \tilde{R}^{ij} + \alpha \lambda \ast F}, \]
which is simply the sum of the gravity and electromagnetic off-shell contributions found previously, (3.20) and (4.4). While expected, this is non-trivial, because there is a non-obvious off-shell cancellation among the terms proportional to \( \xi \) appearing in \( \Theta(\tilde{\omega}, \delta A), \xi \ast L, \) and \( S_e \).

The full surface charge integrand is
\[ k_e = -2\kappa \left( \lambda'^{ij} \ast \tilde{R}^{ij} - \delta \omega_{ij} \ast \xi \tilde{R}^{ij} \right) - 2\alpha \left( \lambda \delta e - F - \delta A \ast F \right). \]

This is the sum of equations (3.22) and (4.5), plus a contribution due to diffeomorphism transformation of the electromagnetic field. Now, to ensure that \( dk_e = 0 \), we need the exactness of the symmetries \( \delta e = 0 \), which is already solved by (3.28), but we also need
\[ \delta A = \xi \ast F - d\lambda = 0. \]
from this equation, \( \lambda \) can be solved in general. In coordinates components, it is equivalent to solve \( \lambda \) from \( \frac{1}{\zeta} F_{\mu\nu} = \partial_{\mu}\lambda \). It is, in fact, the equation for the electric potential of an electromagnetic field projected with \( \xi \). We note that \( \lambda_0 \) = constant is a solution of the homogeneous equation; therefore, if \( \lambda \) is a solution of the inhomogeneous one, we have \( \lambda = \lambda = \lambda_0 \). The \( \lambda_0 \) plays exactly the same rôle as in pure electromagnetism, and therefore implies the conservation of the electric charge.

Note that, in the surface charges formalism, the definition of the electric charge and the charges due to spacetime Killing symmetries are on the same foot.

Before discussing an example, let us remark upon the linearity property of the surface charge integrands. In the general derivation of section 2, we used the assumption \( \delta \xi = 0 \). However, the obtained formula (5.5) is explicitly linear, in the vector field generating diffeomorphism and in all the gauge parameters, i.e.
\[ \alpha_1 k_1 + \alpha_2 k_2 = k_{\alpha_1 \epsilon_1 + \alpha_2 \epsilon_2}, \]  

(5.7)

where \( \alpha_{1,2} \) can be arbitrary functions on the phase space. Thus, if \( k_1 \) and \( k_2 \) are closed forms for exact symmetries generated by \( \epsilon_1 \) and \( \epsilon_2 \), then \( k_1 = k_{\alpha_1 \epsilon_1 + \alpha_2 \epsilon_2} \) is also a closed form for the exact symmetry generated by \( \epsilon_3 \), with the precise identification \( \epsilon_3 = \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 \). This fact is exploited in what follows.

### 5.1. Charged and rotating black hole

As an example, we apply the result to a particular family of black holes. We show that surface charges are compatible with those obtained through the standard asymptotic analysis. Then, we show how the quasilocal nature of surface charges makes it possible to have a first law of black hole mechanics without relying on the asymptotic structure of spacetime [34]. Note that this quasilocal perspective is the best that can be done when the black hole is embedded in an asymptotically de Sitter spacetime.

We consider a black hole solution family which is electrically charged, rotating, and satisfies the asymptotically constant curvature boundary conditions (3.7). It is known as the (anti-) de Sitter Kerr–Newman family. A possible tetrad and electromagnetic potential describing the solution are

\[
e^0 = \frac{\sqrt{\Delta r}}{\lambda} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right), \quad e^1 = \frac{\rho}{\sqrt{\Delta r}} dr,
\]

\[
e^2 = -\frac{\rho}{\sqrt{\Delta \theta}} d\theta, \quad e^3 = \frac{\sqrt{\Delta \theta} \sin \theta}{\rho} \left( adt - \frac{a^2 + r^2}{\Xi} d\phi \right),
\]

\[
A = -\frac{qr}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right),
\]

(5.8)

(5.9)

with \( \Delta r = (a^2 + r^2) \left( 1 + \frac{\rho^2}{\Xi} \right) - 2mr + q^2 \), \( \Delta \theta = 1 + \frac{\rho^2}{\Xi} \cos^2 \theta \), \( \rho^2 = r^2 + a^2 \cos^2 \theta \), and \( \Xi = 1 + \frac{\rho^2}{\Xi} \). The upper sign is reserved for the anti-de Sitter family; the lower one, for the de Sitter one. We stress that it is possible to use another set of variables related by a gauge transformation, but, as the procedure is explicitly gauge invariant, it will not have any impact on the results. In particular, to rotate \( e^l \) by an arbitrary Lorentz transformation, or to add a term of the form \( d\lambda A \), has no effect. From the equation \( \partial_\lambda e^l = 0 \), we solve the connection, and compute \( \delta \omega^l, \delta R^l, \delta A \), and \( \delta \star F \). At this level, we have reduced the phase space to the particular family solution spanned by the parameters \( (m, a, q) \); thus, the variation \( \delta \) acts only on functions of those parameters.

In the metric formalisms, \( \partial_1 \) and \( \partial_3 \) are two independent Killing fields. Through the solution of the exactness conditions for \( e^l \), (3.28), we get \( \lambda^l_1 \) and \( \lambda^l_2 \) respectively. Similarly, through the exactness conditions on \( A \), (5.6), we obtain the corresponding \( \lambda_1 \) and \( \lambda_2 \). Now we have the ingredients to compute surface charges. Plugging all these quantities into (5.5), we get the associated integrands \( k_1 \) and \( k_2 \)—one for each symmetry. The spacetime described by \( e^l \) has non-contractible spheres due to the singularity. The integration can be performed over any two-surface enclosing the singularity. The surface charges associated with the exact symmetries generated by \( \epsilon_1 = (\partial_r, \lambda^r_1, \lambda_1) \) and \( \epsilon_2 = (\partial_\phi, \lambda^\phi_1, \lambda_2) \) are

\[
\delta Q_t = \oint k_i \delta \frac{\delta m}{\Xi} \pm \frac{3am^2a}{r^2 \Xi^2},
\]

(5.10)
\[ \delta Q_\phi = \oint k_\phi = - \frac{a \delta m}{2 \ell^2} + \left( \frac{3}{2 \ell^2} - \frac{4}{3 \ell^2} \right) m \delta a. \]  

(5.11)

The exactness condition \( \delta \epsilon A = 0 \) has a further independent solution for a constant \( \lambda_0 \) such that \( \delta \lambda_0 A = - d \lambda_0 = 0 \). The corresponding exact symmetry parameter is \( \epsilon_\lambda = (0, 0, \lambda_0) \) and the surface charge is

\[ \delta Q_{\lambda_0} = \oint k_{\lambda_0} = \frac{\lambda_0}{4 \pi} \oint \delta \ast F = - \lambda_0 \left( \frac{\delta q}{\Xi} \pm \frac{2 a q \delta a}{\ell^2 \Xi^2} \right). \]  

(5.12)

To proceed now we have two strategies: to fit the scheme in the results from the asymptotic picture or to insist with a quasilocal approach. We sketch both.

5.1.1. Asymptotic strategy. In order to fit with the asymptotic picture, we can exploit the linearity of each surface charge, (5.7), and adjust the freedom of the gauge parameters in the phase space to obtain the standard integrated charges (see for instance [36]):

\[ M \equiv Q_\xi = \partial_t \mp \left( \frac{a}{\ell^2} \right) \partial_\phi = \frac{m}{\Xi^2}. \]  

(5.13)

\[ J \equiv Q_{-\phi} = \frac{a m}{\Xi^2}. \]  

(5.14)

\[ Q \equiv Q_{\lambda_0 = -1} = \frac{q}{\Xi}. \]  

(5.15)

The surface charge associated with \( \partial_t \) is not integrable. However, the linearity property allows us to choose a different combination of the symmetry parameter \( \xi \equiv \partial_t \mp \frac{a}{\ell^2} \partial_\phi \), that in fact produces an integrable charge. Note that \( \xi \) is phase space dependent: \( \delta \xi \neq 0 \).

The charges satisfy the equation known as the black hole fundamental thermodynamics relation:

\[ M^2 = \frac{S}{4 \pi} \left( 1 \pm \frac{S}{\pi \ell^2} \right)^2 + J^2 \left( \frac{\pi}{S} \pm \frac{1}{\ell^2} \right) + \frac{Q^2}{2} \left( 1 \pm \frac{S}{\pi \ell^2} \pm \frac{\pi Q^2}{2S} \right), \]  

(5.16)

which can be obtained by rewriting the condition \( \Delta_r = 0 \) in terms of the integrated charges plus \( S \equiv A/4 \), with the area of the horizon \( A = 4 \pi (r_+^2 + a^2) \). The horizon is at \( r = r_+ \), with \( r_+ \) being the largest solution of \( \Delta_r = 0 \). From the last equation it follows

\[ \delta M = T \delta S + \Omega \delta J + \Phi \delta Q, \]  

(5.17)

where the parametrization of the phase space is done with the integrable charges \( S, J, \) and \( Q \) such that \( M = M(S, J, Q) \). Then the quantities \( T, \Omega \) and \( \Phi \) have the usual physical interpretation: \( T \equiv \frac{\partial M}{\partial S} \) coincides with the Hawking temperature, \( \Omega \equiv \frac{\partial M}{\partial J} \) is the horizon angular velocity, and \( \Phi \equiv \frac{\partial M}{\partial Q} \) the electric potential at the horizon.

The drawback of this line of logic is that it relies on previous results. Ultimately, it relies on a choice of asymptotic tailing of the field components, which admits an asymptotic time symmetry, and allows us to make sense of a general asymptotic mass definition. In practice, we fixed the gauge parameters to obtain a known mass expression obtained with the asymptotic method—that, certainly for the anti-de Sitter case, relies on an asymptotic analysis. However, in the cases of asymptotically de Sitter spacetimes, there is no notion of time symmetry in the asymptotic region, and no physical argument to define a standard mass\(^{13}\); we just kept the \( \pm \) in

\(^{13}\) Remember that the boundaries of asymptotically de Sitter spacetimes are two disconnected three-dimensional spacelike regions, one for the infinite past and one for the infinite future, and therefore none of them have a notion of time symmetry.
the formulae because it is consistent. Thus, given the quasilocal construction just developed, a pertinent question is: is there a way to derive the first law of black hole mechanics based solely on quasilocal data?

5.1.2. Quasilocal strategy. To use the area of the black hole horizon as a starting point is a possibility. The area of the horizon is a well-defined quasilocal quantity, and is also a finite function of the parameters of the solution. The variation of $A(m, a, q)$ on the phase space can be expressed as a combination of all the surface charges

$$\delta A = \oint k_\epsilon = \alpha(m,a,q)\delta Q_\epsilon + \beta(m,a,q)\delta Q_\phi + \gamma(m,a,q)\delta Q_\lambda \tag{5.18}$$

$$= \alpha'(m,a,q)\oint k_\xi + \beta'(m,a,q)\oint k_{-\phi} + \gamma'(m,a,q)\oint k_{\lambda^\omega=-1} \tag{5.19}$$

$$= \frac{4}{T} \delta M - \frac{4\Omega}{T} \delta J - \frac{4\Phi}{T} \delta Q \tag{5.20}$$

we have expressed the freedom of the gauge parameter on the phase space explicitly. On the second line, we have expanded in a linear combination of integrable quantities. The problem reduces to that of finding the coefficient accompanying the integrated charges. Certainly, we already know that the result expressed in the third line is a rearrangement of the first law presented just before. However, we stress the difference in the logic: in this approach, the mass appears as an integrable charge computed quasilocally without the need of any asymptotic structure or physical interpretation to define it. This quantity coincides with the mass obtained by an asymptotic definition when such definition is to hand; but it is more general, because it requires just a quasilocal description of the spacetime.

Note that the two closed two-surfaces where the integration of $k_\epsilon$ is performed, besides enclosing the singularity, are arbitrary. For a matter of physical interpretation, that of $\oint k_\epsilon$ can be chosen to be a section of the horizon, thus being associated with the area, while for each of the other integrals it can be chosen at convenience, producing for each the same value of the charges. This freedom, plus the gauge invariance of $k_\epsilon$ can be exploited to compute the quantities easily. For instance, when a bifurcated horizon is to hand, the pullback of a particular combination of the Killing fields vanishes on it, and the surface charge formula simplifies considerably.

Summarizing, from this second perspective the first law of black hole mechanics is a consequence of the expansion of $\delta A = \oint k_\epsilon$ into independent integrable quantities. One for each independent exact symmetry $\epsilon_i$. To accomplish integrability the symmetry parameters should satisfy the condition $\delta \oint k_\epsilon = 0$ in each case, where the variation $\delta$ becomes an exterior derivative on the reduced phase space. Certainly, to have a true first law, much more should be said—and has been said—regarding the physical interpretation of each term; but the stress here is that the quantity sometimes playing the rôle of the mass can be relegated, and be indirectly defined, particularly when the asymptotic time translation symmetry is not present, or is difficult to identify\textsuperscript{14}. To decide the true thermodynamic value of the quasilocal first law relation obtained, we would need to figure out thermodynamics processes that allowed us to change the value of the integrated charges. That is, a physical exchange of the amount of charges to flow in a description outside the reduced phase space, even when the usual far

\textsuperscript{14} For instance, this is the strategy used in [35], where the embedding of a charged, rotating black hole in a magnetic field makes subtle the selection of a preferred asymptotic time-like Killing vector field to define the spacetime mass.
away of the black hole notion is not available. We leave this interesting question for future discussions.

Now, we give another step to generalize the formulae found for the surface charges.

6. General relativity and electromagnetism in a D-world

In this section, we further extend the applicability of surface charges, by exhibiting the fundamental formulae for a D-dimensional manifold. We also show that for a particular case there is an explicit equivalence of the formulae of this approach with the recent ones worked in [15] with a different method.

For a spacetime of arbitrary dimension, the Lagrangian of general relativity coupled to electromagnetism, with an a priori vanishing torsion, admits a generalization

\[ S[e, \omega, A] = \int_M \left( \sum_{p=0}^{[D/2]} L_p^D + \alpha F \ast F \right), \tag{6.1} \]

where \( L_p^D \) is a D-form given by

\[ L_p^D = \alpha_p e_{a_1 \cdots a_0} R^{a_1 a_2} \cdots R^{a_{2p-1} a_0} e^{a_{2p+1}} \cdots e^{a_0}, \tag{6.2} \]

the indexes are Lorentz and run as \( a_1, \cdots, a_D = 1, \cdots, D \). The gravity part is known as the Lovelock action, and for \( D \neq 4 \), the \( \alpha_p \) are arbitrary coefficients\(^1\). If torsion is allowed more terms should be included in the action. We do not consider this further generalization with torsion, because we do not know a compact way to treat all of them at once [38].

To get the surface charge, we can use either the Noether approach, detailed in section 2, or the contracting homotopy operator (3.30). The gravity contribution to the surface charge integrand obtained with the Noether approach (2.19) is

\[ k^{GR}_{\epsilon} = - \sum_{p=1}^{[D/2]} \alpha_p e_{a_1 \cdots a_0} \left( \lambda^{a_1 a_2} \delta - \delta \omega^{a_1 a_2} \xi \right) \left( R^{a_3 a_4} \cdots R^{a_{2p-1} a_0} e^{a_{2p+1}} \cdots e^{a_0} \right), \tag{6.3} \]

where \( \delta \) and \( \xi \) act on the forms at the right. The formula coincides with the one obtained in [39]. Operating with them, we note that each \( p \)-term of the sum can be rewritten as

\[ (D - 2p)e \left\{ R \cdots R \left( \lambda \delta e - \delta \omega \xi \right) e \cdots e \right\} + \left( p - 1 \right)e \left\{ d \left( \lambda \delta \omega R \cdots Re \cdots e \right) + \delta \omega \delta \omega R \cdots Re \cdots e \right\}, \tag{6.4} \]

where the indexes are implicit, to avoid cluttering. The terms in the second curly brackets are the generalization of (3.25) to more dimensions. They are all exact forms plus a term proportional to the exactness condition. Therefore, they do not contribute to the surface charges. This property defines an equivalence relation among the surface charge integrands. In particular, we can use the first terms of the previous equation to define

\[ k^{GR}_{\epsilon} = - \sum_{p=1}^{[D/2]} \alpha_p e_{a_1 \cdots a_0} \left( \lambda^{a_1 a_2} \delta e^{a_3} - \delta \omega^{a_1 a_2} \xi \right) R^{a_3 a_4} \cdots R^{a_{2p-1} a_0} e^{a_{2p+1}} \cdots e^{a_0}, \tag{6.5} \]

\(^1\) For odd dimensions, the Lovelock Lagrangian can be written as a Chern–Simons action for a particular fixation of the parameters [37].
this would produce exactly the same surface charge as \( k^{GR}_e \). Consequently, both belong to the same equivalence class, \( k^{GR}_e \sim k^{GR}_e' \). Remarkably, the last expression is exactly the surface charge integrand computed directly with the contracting homotopy operator (3.30)

\[
k_e^{GR} = I_{\delta e, \delta \omega} S_e.
\]

(6.6)

Now, we choose to keep just the Einstein–Hilbert term in an arbitrary dimension, i.e. to keep only the \( p = 1 \) term in the Lovelock action. Note that the cosmological constant term never contributes. The Einstein–Hilbert gravity contribution to the surface charge integrand (6.5) in \( D \) dimensions is

\[
k_e^{EH} = -\alpha \left\{ \lambda^{a_1 a_2 \cdots a_D} \delta (e^{a_3} \cdots e^{a_D}) - \delta \omega^{a_1 a_2 \cdots a_D} \xi_{a_3} \cdots \xi_{a_D} \right\},
\]

(6.7)

notably, it coincides with the expression derived in [15]\(^{16}\).

This result tells us that the surface charges defined through the conventional symplectic method and those defined through the homotopy operator are equivalent. A formal proof of the last statement is done in the appendix. Furthermore, in [15], it is shown that the surface charges integrand formula, obtained for the tetrad-connection variables, is equivalent to the one written in pure metric variables (see, for instance, [18]). Therefore, the formulae shown here (6.5) are the natural generalization to an arbitrary dimension when all the Lovelock terms are considered either using tetrad-connection or metric variables.

The electromagnetic contribution computed with the Noether approach is direct. It is the trivial generalization to \( D \) dimensions of the one obtained in (5.5)

\[
k_e^{EM} = -2\alpha \left( \lambda \delta * F - \delta A \xi * F \right),
\]

(6.8)

it is naturally a \( (D - 2) \)-form.

Therefore, the total surface charge for gravity coupled to electromagnetism in a \( D \) world is

\[
k_e = k_e^{GR} + k_e^{EM}.
\]

(6.9)

The ensemble of these formulae allows us to define the surface charges associated for a large group of theories. For a given dimension \( D \), one can pick any particular combination of Lovelock terms, and couple it (or not) to electromagnetism. If such a theory has a well-defined family of solutions with exact symmetries, then it is possible to define surface charges for them. The next step will be to integrate those surface charges to have finite charges. This can be accomplished by solving for condition \( \delta \mathcal{H}_0 = 0 \), with the help of the remaining freedom in the parameters \( e \). Those charges are the true physical quantities with which the phase space solution should be described.

7. Discussion

Covariant symplectic methods constitute a powerful tool to deal with physical symmetries in gauge theories. The approach is old and widespread; however, several subtleties are usually disregarded (the triviality of a gauge current or the \( \Theta(\{\delta_1, \delta_2\}) \) term, for example). One of the aims of these notes is to fill a key gap by obtaining the formulae of the surface charges from the usual canonical symplectic approach (2.20). Another aim is to show the relation with the so called invariant symplectic approach based on the contracting homotopy operators [18].

\(^{16}\) To do the comparison all quantities should be expanded in components and it should be noted the different prescription for the Lorentz gauge parameter \( \delta_\lambda = \mathcal{L}_{\lambda} + \delta_\lambda = \mathcal{L}_{\lambda} + \delta_\lambda + \xi_{\omega} \) thus, \( \lambda^{a_1 a_2 \cdots a_D} = \lambda^{a_1 a_2 \cdots a_D} - \xi_{\omega a_3} \cdots \xi_{\omega a_D} \). As explained before, both prescriptions are equivalent, but the one we use produces manifestly Lorentz invariant formulae.
We do so because a part of the community is unaware of the powerful results related to the surface charges—and, specifically, unaware of the way they help to solve the problems of using Noether currents in gauge theories.

It is concluded that both symplectic approaches to define surface charges are equivalent, as far as the assumptions to build the charges are respected. This is shown in the appendix for the general case, and was checked in section 6 for the theory of gravity in an arbitrary dimension. In this regard, the moral from both approaches is that, for gauge theories, physical symmetries are better understood at the level of the symplectic structure, not at the level of Noether currents. It is at the symplectic structure level that the conservation of surface charges can be established.

In the same vein, we foment the use of surface charges in gravity by deriving their explicit formulae for the first order formulation of general relativity coupled to electromagnetism, based on tetrad and connection variables: (5.5). This is one of our main results. We further extend this result to arbitrary dimensions: (6.3). The elegance of the language is expressed in the simplicity of the formulae obtained. No reference to coordinates is required. On the other hand, the translation to components depending on coordinates is straightforward, as was stated in the comparison of (6.7) with [15].

In section 5.1 we applied the formalism to the $3+1$ solution family of electrically charged, rotating black holes of asymptotically constant curvature. We also exploited the quasilocal nature of the surface charges, to present an alternative way to recover the first law of black hole mechanics.

Another interesting application of the machinery just presented is to the so-called asymptotic symmetries. Let us briefly describe the program to set the questions properly. The phase space of general relativity can be explored by perturbing the fields around an arbitrary solution. A whole research program is built on ways to perform field perturbations around solutions such that they describe a large class of spacetimes. For instance, field perturbations are usually encoded in the so-called boundary conditions: particular tailing for the fields far away from sources (e.g. the definition of asymptotically flat, de Sitter, or anti-de Sitter spacetimes). There is an interesting game in the way those boundary conditions are specified such that they reflect one or other physical situation (for example, in asymptotically flat spacetimes, to allow—or not—radiation at future null infinity). Given any particular solution, the same strategy can be applied in a specific spacetime region. Thus, by studying the symmetries of this enlarged family of spacetimes, it is possible to define a larger group of symmetries than that of the starting unperturbed solution. The surface charge formulae are quasilocal, and thus they can be used for those asymptotic symmetries producing non-vanishing quantities. In fact, non-vanishing surface charges could be computed even for gauge transformations which are not physical symmetries, in the sense that they do not respect the exactness condition. Then, the main question is to decide if the so-called asymptotic symmetries have a physical meaning—in other words, whether different values of the surface charges associated with asymptotic symmetries correspond to physically different solutions. It could happen that the enlarged
group of symmetries have a lot of gauge redundancy, and in consequence their surface charges belong to the same equivalence class. Another possibility is that they are truly physical. In this case, the phase space of spacetime solutions would have a richer structure, in which long-time considerations could be disregarded.

Assuming the latter possibility, we have for instance [11]. It is pointed out that an infinite number of charges associated to asymptotic symmetries, defined on the null region of an asymptotically flat spacetime, are related to an infinite number of charges defined for near-horizon symmetries. For the charges defined in the family of black hole solutions studied in this note—the (anti-)de Sitter Kerr–Newman family—the statement is trivial, because of the surface charge conservation (note that the asymptotically flat family is treated in exactly the same way). In this regard, and assuming that the asymptotic symmetries are exact, it would be interesting to find a systematic way to perturb this solution family, and thereby to define a larger family of solutions, such that a larger group of exact symmetries can be defined for it in the whole spacetime. Then, the associated surface charges could be computed at once, either on closed two-surfaces at the asymptotic region, or at the near horizon region (actually, on any two-surface enclosing the singularity). In this program, part of the machinery is already to hand; what is missing is the detailed description of such a family of perturbed solutions and its exact symmetries.

To study the surface charges associated with the asymptotic symmetries in the tetradic and connection variables is one of the future directions of this work. The outcome should be a better understanding of the physical relevance of those constructions. If the outcome is positive—that is, if there exist such a description of perturbed solutions admitting an infinite number of exact symmetries and related charges—the expectation is that the extra symmetries are generic for all spacetimes, regardless of their particular asymptotic structure—and, in fact, a quasilocal property.

To decide the value of the previous ideas, the covariant symplectic formalism and the surface charges expressed with tetradic and connection variables offers a solid starting point.

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Appendix. Comparison of surface charge definitions

To make contact with other approaches, in this section, we introduce a different definition for surface charges, used in [18], and further import the comparison with the prescription presented in section 2. The key of this different definition is its direct use of $S_\epsilon$, introduced in (2.4)—that is, the particular equation of motion combined with the gauge parameters that result from the use of Noether identities. In other words, the only term appearing in the trivially conserved current, $J_\epsilon = \Theta(\delta, \Phi) - \xi L + S_\epsilon$, that does not depend directly on the Lagrangian boundary term. The surface charge integrand is expressed as
\[ k' \equiv I_{\delta \Phi} S, \quad (A.1) \]

where \( I_{\delta \Phi} \) is called the homotopy operator. The homotopy operator is an efficient way to get a sensible \((p-1)\)-form from an exact \(p\)-form. In particular it can be used to select the boundary term in the Lagrangian variation

\[ \delta L = E \delta \Phi + d [\Theta'(\delta \Phi) + dY] = E \delta \Phi + d [I_{\delta \Phi} L], \quad (A.2) \]

At risk of keeping the discussion rather abstract, while brief, we just pick up the properties that allow us to understand the comparison (see [18] for a detailed definition of the homotopy operator). The defining property of the homotopy operator is its relation with a variation of fields in the configuration space

\[ \delta' \equiv d I_{\delta \Phi} + I_{\delta \Phi} d, \quad (A.3) \]

where \( d \) is the exterior derivative. In fact, the homotopy operator provides a prescription to define a variation on the phase space. Therefore we called it \( \delta' \) to distinguish it from our treatment. Note the analogy with the expression of the spacetime Lie derivative (2.13).

With this property, we can already prove

\[ dk' = -I_{\delta \Phi} dS + \delta' S, \quad (A.4) \]

\[ = -I_{\delta \Phi} [E \delta_1 \Phi] + \delta' S, \quad (A.5) \]

\[ = -I_{\delta \Phi} [E \delta_1 \Phi] - (-1)^{pE} E I_{\delta \Phi} [\delta_2 \Phi] + \delta' S, \quad (A.6) \]

where we have used the Noether identities \( E \delta_1 \Phi = dS_c - N_c = dS_c \), \( pE \) is the form degree of \( E \), i.e. \( I_{\delta \Phi} E = (-1)^{pE} E I_{\delta \Phi} \). Therefore, it is shown that \( k' \) is closed if the equation of motion, the linearized equation of motion, and the exactness condition hold, i.e. \( E = 0 \), \( \delta E = 0 \), and \( \delta \Phi = 0 \). These conditions are exactly the ones required for the surface charge integrand defined in (2.21) to be closed. In the previous calculation we made use of the so-called invariant presymplectic structure density

\[ \Omega'(\delta_1, \delta_2) \equiv I_{\delta_1 \Phi} (E \delta_2 \Phi). \quad (A.7) \]

It differs from the presymplectic structure density introduced before

\[ \Omega(\delta_1, \delta_2) = \delta_1 \Theta(\delta_2 \Phi) - \delta_2 \Theta(\delta_1 \Phi) - \Theta([\delta_1, \delta_2] \Phi). \quad (A.8) \]

Both prescription are in general inequivalent, as is shown in the following.

The boundary term \( \Theta(\delta \Phi) \) has an intrinsic ambiguity that can be selected with the homotopy operator (A.2), we use it to fix the ambiguity of the presymplectic structure density

\[ \Omega(\delta_1, \delta_2) = \delta'_1 (I_{\delta_1 \Phi} L) - \delta'_2 (I_{\delta_2 \Phi} L). \quad (A.9) \]

The use of \( \delta'_1, \delta_2 \), as defined by (A.3), ensures linearity in the variations, so that it is unnecessary to introduce the commutator term. Although we have selected the boundary term, there is still another intrinsic ambiguity if the Langrangian is allowed to change by an exact form, \( L \to L + d\alpha \); it is in this sense that this prescription for the symplectic structure density is not invariant. The comparison of both presymplectic structure densities goes as

\[ \Omega'(\delta_1, \delta_2) = I_{\delta_1 \Phi} (E \delta_2 \Phi) \]

\[ = I_{\delta_1 \Phi} (\delta_2 L - dI_{\delta_2 \Phi} L) \quad (A.10) \]

\[ = I_{\delta_1 \Phi} (\delta_2 L) - dI_{\delta_2 \Phi} L \quad (A.11) \]
\[= \delta_1^2 I_\delta_1 \phi L + \delta_2^2 I_\delta_2 \phi L - d (I_\delta_1 \phi I_\delta_2 \phi L) \quad (A.12)\]

\[= \Omega(\delta_1, \delta_2) - d \tilde{E}_{1,2} \quad (A.13)\]

where we have used the fact that the homotopy operator satisfies
\[I_\delta_1 \delta_2^2 = \delta_2^2 I_\delta_1,\]
and \(\tilde{E}_{1,2} \equiv I_\delta_1 \phi I_\delta_2 \phi L\). Thus, in the case we have exact symmetries, \(\tilde{E}\) vanishes, and there is a match in both prescriptions. This is the generalization of what was described in the computation of section 6.

It is worth to point out the differences in the prescription: \(k'\) and \(\Omega'(\delta_1, \delta_2)\) depend directly on the equation of motions and it is insensitive to the intrinsic ambiguities of the variational principle. On the other hand, \(k\) and \(\Omega(\delta_1, \delta_2)\) can be computed from standard procedures without introducing the homotopy operator. As a final remark, we note that in (3.30) we exhibited an explicit formula for the homotopy operator written for a gravity theory in tetrad-connection variables.

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