By using the canonical and symplectic approaches an (nonstandard) alternative action describing linearized gravity is studied. We identify the complete set of Dirac’s constraints, the counting of physical degrees of freedom is performed and the Dirac brackets are constructed. Furthermore, the symplectic analysis is developed which includes the complete set of Faddeev-Jackiw constraints and a symplectic tensor; from that symplectic matrix we show that the generalized Faddeev-Jackiw brackets and the Dirac ones coincide to each other. With all these results at hand, we prove that the number of physical degrees of freedom are eight, thus, we conclude that the theory does not describe the dynamics of linearized gravity. In addition, we also develop the symplectic analysis of standard linearized gravity and we compare the results for both standard and nonstandard theories. Finally we present some remarks and conclusions.

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To the memory of Filadelfo Gayosso Ríos†...
because the nonlinearity of the gravitational field is manifested in the constraints, so far the quantization of the theory has not been performed in a full and satisfactory way \[5-8\]. In this respect, in order to obtain new insights in classical or quantum analysis of gravity, it is common to study Einstein’s modified theories with the goal to provide new ideas for understanding the symmetries of the gravitational field, examples of such theories are the so-called linearized gravity and massive gravity theories. In fact, these theories are good laboratories for testing classical and quantum ideas of gravity, for instance, with the detection of gravitational waves made by LIGO/VIRGO laser interferometers \[4\], all our actual vision of the universe can change because that discovery could give us a new insight to explore the warper side of the Universe, and it could be an important starting point for testing new classical or quantum gravity proposals by means of new actions and its symmetries. In this respect, it is well-known that linearized gravity is a gauge theory that describes the propagation of a helicity-2 particle (massless graviton); from the Hamiltonian point of view, the theory presents only a set of first class constraints (then it is possible fixing the gauge) and the number of physical degrees of freedom is two in four dimensions. Nevertheless, within the quantum context, it is well-known that the usual scheme of quantization applied to the theory lead to infinities that cannot be eliminated by means of regularization and renormalization procedures \[10\]. Furthermore, one of the alternatives to possibly explain unresolved problems in cosmology such as the problem of acceleration of the Universe is performing a modification of linearized gravity. In this regard, the most natural modification is promoting the helicity-two theory to one of a massive spin-two, this theory is the so-called massive gravity \[11\]. In this manner, any alternative or modified theory of gravity is an important achievement and it is mandatory to investigate the new proposal and their symmetries in a full detail.

With these motivations, in this paper we study an alternative theory for linearized gravity that was proposed in \[12, 13\]; the model consists in an action whose dynamical variables are not the perturbation of the metric but an electric and magnetic-like fields. Moreover, the analysis reported in those works was performed ignoring that the action corresponds to a singular system, therefore the principal symmetries of the theory were not reported. In order to analyze a singular theory, there are two approaches to obtain in a systematic way the symmetries and observables of any physical theory: Dirac’s formalism \[12\] and the Faddeev-Jackiw [FJ] method \[15\]. The former allows us to know the complete set of constraints of the theory, namely, first class constraints and second class ones. As a consequence, the physical degrees of freedom can be exactly counted and the relevant symmetries can be obtained. In fact, first class constraints are the generators of gauge symmetry, and the second class constraints are used for constructing the fundamental Dirac’s brackets, which are useful for identifying observables. On the other hand, the FJ method provides a symplectic description for singular systems, where the basic feature of this approach is the construction of a symplectic tensor; from the symplectic tensor it is possible to obtain relevant physical information such as, the degrees of freedom, the gauge symmetry and the quantization brackets (the so-called generalized FJ brackets) can also be obtained. Moreover, in this framework is not necessary to classify the constraints into first and second class ones; this fact makes the FJ method more eco-
nomical than the Dirac scheme. Hence, in this paper by using the Dirac and FJ approaches the action reported in \cite{12, 13} is analyzed. We report in our analysis the complete set of constraints and the Dirac brackets are constructed, also we report that the proposed theory is neither Lorentz nor gauge invariant, thus, these results modify the number of degrees of freedom, which we obtain eight physical degrees. In addition, by using the FJ method we report the complete set of FJ constraints, then the generalized FJ brackets are constructed and the counting of physical degrees of freedom is performed; the generalized brackets and the Dirac ones coincide to each other. Finally, we have added at the end of the paper the symplectic analysis of standard linearized gravity which is absent in the literature. We compare the results of both standard and nonstandard theories and we comment their differences.

The paper is organized as follows. In the Sec.I the Hamiltonian analysis is performed. We identify the full set of constraints which are all of second class, there are not first class constraints, therefore we conclude that the action under study is not a gauge theory. Moreover, by eliminating the second class constraints we calculate the fundamental Dirac’s brackets and the counting of physical degrees of freedom is carried out. In Sec.II the symplectic analysis is developed, we obtain the full set of FJ constraints, then a symplectic tensor is constructed and the fundamental FJ brackets are identified. In addition, in the Sec. III the symplectic analysis of standard linearized gravity is performed and we compare the results of this section with those obtained previously. Finally, we present a summary and conclusions.

II. HAMILTONIAN ANALYSIS

It is well-known that the cornerstone of linearized gravity is based in considering a perturbation of the fundamental metric around the Minkowski spacetime, say

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta},$$

(1)

where the $h_{\alpha\beta}$ represents a small deviation of the fundamental metric and the background space-time is Minkowskian with metric $\eta_{\alpha\beta} = (-1, 1, 1, 1)$, here Greek indices run from 0 to 3. By introducing the metric (1) into the Einstein-Hilbert action given by

$$S[g_{\mu\nu}] = \int \sqrt{-g} R dx^4,$$

(2)

where $g$ is the metric and $R$ the Ricci scalar, and just keeping free fields for $h_{\alpha\beta}$, then the Lagrangian for standard linearized gravity is obtained \cite{10}

$$L = \frac{1}{4} \partial_\lambda h_{\mu\nu} \partial^\lambda h^\mu\nu - \frac{1}{4} \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} + \frac{1}{2} \partial_\lambda h^{\mu} \partial^\mu h^\nu_{\quad \nu} - \frac{1}{2} \partial_\lambda h_{\mu}^{\lambda} \partial_{\nu} h^{\nu\mu}.$$

(3)

The Lagrangian given in (3) has been studied in the literature (see \cite{10} and cites there in). In fact, the theory describes the propagation of a massless particle (the graviton) with two physical degrees of freedom in four dimensions, and from the Hamiltonian point of view, the theory is a gauge theory; there are only first class constraints, then in order to calculate the Dirac brackets the gauge is fixed.
and the constraints are now converted into second class constraints. In this respect, we will see along this paper that these symmetries will not be present in the theory proposed in \cite{12, 13}. Hence, a new Lagrangian for linearized gravity was proposed in \cite{12, 13}. In fact, by considering the tensor field
\[
K_{\alpha\beta\gamma\delta} = \frac{1}{2} \left[ \partial_\alpha \partial_\gamma h_{\beta\delta} - \partial_\beta \partial_\gamma h_{\alpha\delta} + \partial_\beta \partial_\delta h_{\alpha\gamma} - \partial_\alpha \partial_\delta h_{\beta\gamma} \right],
\]
which satisfies
\[
\begin{align*}
K_{\alpha\beta\gamma\delta} &= -K_{\beta\alpha\gamma\delta} = -K_{\alpha\beta\delta\gamma} = K_{\gamma\delta\alpha\beta}, \\
K_{\alpha\beta\gamma\delta} + K_{\alpha\delta\beta\gamma} + K_{\alpha\gamma\beta\delta} &= 0, \\
\partial_\alpha K_{\beta\gamma\delta\epsilon} + \partial_\epsilon K_{\beta\gamma\alpha\delta} + \partial_\delta K_{\beta\gamma\epsilon\alpha} &= 0,
\end{align*}
\]
where the second and third relations can be identified with the Ricci and Bianchi identities respectively. Moreover, it can be showed that the linearized Einstein vacuum field equations are given by (see \cite{12, 13, 16} for full details)
\[
K_{\alpha\beta} = 0,
\]
and this implies that all the components $K_{\alpha\beta\gamma\delta}$ can be expressed in terms of the fields $E_{ij}$ and $B_{ij}$ defined by
\[
E_{ij} = K_{00ij}, \quad B_{ij} = -K^{*}_{0ij},
\]
where $K^{*}_{\alpha\beta\gamma\delta} = \frac{1}{2} K_{\alpha\beta}^{\rho\sigma} \epsilon_{\rho\sigma\gamma\delta}$ is the dual of $K_{\alpha\beta\gamma\delta}$ and the fields $E_{ij}$, $B_{ij}$ have vanishing trace. Then, in \cite{12, 13} the following action for describing the linearized Einstein vacuum equations is proposed
\[
S[E, B] = \int \left[ B^{ij} \left( \frac{1}{c} \partial_t E_{ij} - \epsilon_{i}^{kl} \partial_k B_{ij} \right) - E^{ij} \left( \frac{1}{c} \partial_t B_{ij} + \epsilon_{i}^{kl} \partial_k E_{ij} \right) \right] dx^4,
\]
where $i, j, k = 1, 2, 3$. The equations of motion obtained from the action (8) are given by
\[
\frac{1}{c} \partial_t E_{ij} = \epsilon_{i}^{kl} \partial_k B_{ij}, \quad \frac{1}{c} \partial_t B_{ij} = -\epsilon_{i}^{kl} \partial_k E_{ij},
\]
and the fields $E_{ij}$ and $B_{ij}$ satisfy the following constraints
\[
\partial_i E^{ij} = 0, \quad \partial_i B^{ij} = 0,
\]
equations (9) and (10) are the equations of motion of linearized gravity given in non-covariant way. As it was commented above, the alternative action (8) was studied in \cite{12, 13, 16}, however we shall show that the action is a singular system and this fact was ignored in those works. It is important to note that the action (8) reproduces only the equations (9); the equations (10) are obtained from the equations (5) and they were added by hand in \cite{12, 13, 16} in order to obtain a set of equations similar to Maxwell’s theory, however, we will observe that these facts yield different symmetries of those known for linearized gravity. In this manner, in order to perform a complete study of the action (8) we will use the Dirac formulation for constrained systems. It is important to comment
that the action is neither Lorentz invariant nor gauge theory; these facts will be reflected in the Hamiltonian analysis, in particular in the number of physical degrees of freedom of the theory.

We can observe that the matrix elements of the Hessian given by
\[
\frac{\partial^2 L}{\partial (\partial_t E_{ij}) \partial (\partial_t E_{kl})}, \quad \frac{\partial^2 L}{\partial (\partial_t E_{ij}) \partial (\partial_t B_{kl})}, \quad \frac{\partial^2 L}{\partial (\partial_t B_{ij}) \partial (\partial_t B_{kl})},
\]
are identically zero, hence the system is singular and we expect primary constraints. In order to identify the primary constraints, the canonical formalism calls for the definition of the momenta canonically conjugate to \((E_{ij}, B_{ij})\) are given by
\[
P_{ij} = \frac{\delta L}{\delta \dot{E}_{ij}}, \quad \Pi_{ij} = \frac{\delta L}{\delta \dot{B}_{ij}}.
\]

In this manner, the fundamental Poisson brackets are
\[
\{ E_{ij}(x), P^{kl}(y) \} = \frac{1}{2} \left( \delta^k_i \delta^l_j + \delta^k_j \delta^l_i \right) \delta^3(x - y),
\]
\[
\{ B_{ij}(x), \Pi^{kl}(y) \} = \frac{1}{2} \left( \delta^k_i \delta^l_j + \delta^k_j \delta^l_i \right) \delta^3(x - y).
\]

From the definition of the momenta, we identify the following 10 primary constraints
\[
\chi^{ij} : P^{ij} - \frac{1}{c} B^{ij} \approx 0, \quad \bar{\chi}^{ij} : \Pi^{ij} + \frac{1}{c} E^{ij} \approx 0,
\]
these primary constraints are of second class and their evolution in time will fix the Lagrange multipliers; for this theory there are not more constraints. It is important to note that there are not first class constraints and therefore the theory under study is not a gauge theory. In fact, this result does not agree with the gauge invariance that is present in the standard Lagrangian of linearized gravity.

On the other hand, because of there are second class constraints, we will construct the Dirac brackets from the following matrix whose entries are given by the Poisson brackets between the second class constraints
\[
C_{\alpha\beta} = \begin{pmatrix}
0 & -\frac{1}{c} \left( \eta^{ik} \eta^{jl} + \eta^{il} \eta^{jk} \right) \\
\frac{c}{4} \left( \eta_{ik} \eta_{jl} + \eta_{il} \eta_{jk} \right) & 0
\end{pmatrix} \delta^3(x - y),
\]
its inverse is given by
\[
C^{-1}_{\alpha\beta} = \begin{pmatrix}
0 & \frac{c}{4} \left( \eta_{ik} \eta_{jl} + \eta_{il} \eta_{jk} \right) \\
-\frac{c}{4} \left( \eta^{ik} \eta^{jl} + \eta^{il} \eta^{jk} \right) & 0
\end{pmatrix} \delta^3(x - y).
\]

Furthermore, the Dirac brackets between two functionals, say \(A, B\) are expressed by
\[
\{ A(x), B(y) \}_D = \{ A(x), B(y) \} - \int du \int dv \left\{ A(x), \chi_\alpha(u) \right\} C^{-1}_{\alpha\beta} \{ \chi_\beta(v), B(y) \},
\]
where \(\{ A(x), B(y) \}\) is the usual Poisson bracket between the functionals \((A, B)\) and \((\chi_\alpha, \chi_\beta)\) represent the set of second class constraints. Hence, we obtain the following Dirac’s brackets of the theory

\[
\{ A(x), B(y) \}_D = \{ A(x), B(y) \} - \int du \int dv \left\{ A(x), \chi_\alpha(u) \right\} \frac{c}{4} \left( \eta_{ik} \eta_{jl} + \eta_{il} \eta_{jk} \right) \delta^3(x - y).
\]
\{E_{ij}(x), P^{kl}(y)\}_D = \frac{1}{4} \left( \delta^k_i \delta^l_j + \delta^k_j \delta^l_i \right) \delta^3(x - y), \\
\{B_{ij}(x), \Pi^{kl}(y)\}_D = \frac{1}{4} \left( \delta^k_i \delta^l_j + \delta^k_j \delta^l_i \right) \delta^3(x - y).

With these results at hand, we are able to calculate the physical degrees of freedom as follows; there are 20 canonical variables and 10 second class constraints, thus, there are five physical degrees of freedom. Nonetheless, we need to take into the account the equations (10). Hence, the constraints (12) satisfy the following reducibility conditions
\begin{align*}
\partial_i \chi^{ij} &= 0, \\
\partial_i \bar{\chi}^{ij} &= 0,
\end{align*}
which imply that there are [10-6]=4 second class constraints, therefore, the physical degrees of freedom are eight. It is important to remark that our results indicate that in spite of the action \(L\) yields linearized Einstein’s equations of motion, it does not describe the dynamics of linearized gravity at all. In fact, it is well-known that standard linearized gravity is both gauge invariant and describes the propagation of a massless particle with two degrees of freedom. In this manner, we have found strong differences between the action \(L\) and the standard theory for describing linearized gravity.

We finish this section with some extra comments. The Hamiltonian of the theory is given by
\begin{equation}
H = \int \left[ c \epsilon^{kli} B_{ij} \partial_k P_{lj} - c \epsilon^{kli} E_{ij} \partial_k \Pi_{lj} \right] dx^3. \tag{14}
\end{equation}
It is straightforward to prove that the Hamiltonian is of first class. In fact, we have
\begin{align*}
\{H, \chi^{qr}\} &= \frac{1}{2} c \epsilon^{kq}_i \partial_k \chi^{ir} + \frac{1}{2} c \epsilon^{kr}_i \partial_k \bar{\chi}^{iq} \approx 0, \\
\{H, \bar{\chi}^{qr}\} &= \frac{1}{2} c \epsilon^{kq}_i \partial_k \bar{\chi}^{ir} + \frac{1}{2} c \epsilon^{kr}_i \partial_k \chi^{iq} \approx 0. \tag{15}
\end{align*}
In this manner, the Hamiltonian found in [12, 13, 16] is not equivalent to that found in (14). In these works was ignored that the theory under study is singular, thus, we can not talk neither first class constraints nor second class constraints. However, we have showed that the Hamiltonian (14) is of first class and therefore within Dirac’s terminology it is an observable. All results found in this section extend and complete the work reported in [12, 13, 16].

III. SYMPLECTIC ANALYSIS.

Now we will reproduce the results obtained within the Dirac scheme by using the FJ analysis. We start with the Lagrangian \(L\) rewritten as
\begin{equation}
\mathcal{L} = \frac{2}{c} B^{ij} \dot{E}_{ij} - \frac{2}{c} E^{ij} \dot{B}_{ij} - 2 \epsilon^{kli} B^{ij} \partial_k B_{lj} - 2 \epsilon^{kli} E^{ij} \partial_k E_{lj}, \tag{16}
\end{equation}
note that we have included an additional factor of 2. This overall factor will not affect the Euler-
Lagrange equations of motion in any way. Hence, from the Lagrangian (10) we obtain the following
symplectic Lagrangian
\[ (0) \mathcal{L} = 2 P^{ij} \dot{E}_{ij} + 2 \Pi^{ij} \dot{B}_{ij} - V, \] (17)
where \( V = 2 \epsilon^{ij} B_{ij} \partial_k P_{ij} - 2 \epsilon^{ij} E_{ij} \partial_k \Pi_{ij} \) is identified as the symplectic potential. Furthermore,
in the FJ framework, the Euler-Lagrange equations of motion are given by \[ \frac{\partial}{\partial \xi^b} \frac{f_{ab}^{(0)} \dot{\xi}^b}{(18)} {\delta V^{(0)}(\xi)}{\delta \xi^a}, \]
where the symplectic matrix \( f_{ab}^{(0)} \) takes the form
\[ f_{ab}^{(0)}(x, y) = \frac{\delta a_b(y)}{\delta \xi^a(x)} - \frac{\delta a_a(x)}{\delta \xi^b(y)}, \] (19)
with \( \xi^{(0)} = (E_{ij}, P^{ij}, B_{ij}, \Pi^{ij}) \) and \( a^{(0)} = (2 P^{ij}, 0, 2 \Pi^{ij}, 0) \) representing a set of symplectic variables. The matrix (19) is not singular and this implies that there are not FJ constraints. However, as it was commented above we need to take into account the conditions (10), which implies that \( \partial_i P^{ij} = 0, \partial_i \Pi^{ij} = 0 \) and they will be considered as constraints. Moreover, we can see that \( \partial_j \partial_i P^{ij} = 0 \) and \( \partial_j \partial_i \Pi^{ij} = 0 \) which correspond to. All this information must be added to the symplectic Lagrangian by using Lagrange multipliers, namely, \( \gamma_i \) and \( \sigma_i \), thus we obtain
\[ (1) \mathcal{L} = 2 P^{ij} \dot{E}_{ij} + 2 \Pi^{ij} \dot{B}_{ij} - [2 \partial_i P^{ij} - \rho^j] \dot{\gamma}_j - [2 \partial_i \Pi^{ij} - \sigma^j] \dot{\sigma}_j - V. \] (20)
Because of the reducibility conditions, we have added the Lagrange multiplier of the Lagrange multiplier, \( \rho \) and \( \sigma \), and with this fact under consideration, we can obtain a symplectic tensor \[ \mathcal{F}^{(1)} = \left( \begin{array}{ccccccc}
0 & -\delta_{ij} & 0 & 0 & 0 & 0 & 0 \\
\delta_{ij} & 0 & 0 & 0 & -\delta_{ij} \partial_k + \delta_{kl} \partial_l & 0 & 0 \\
0 & 0 & 0 & -\delta_{ij} & 0 & 0 & 0 \\
0 & 0 & \delta_{ij} & 0 & 0 & -\delta_{ij} \partial_k + \delta_{kl} \partial_l & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\delta_{ij} \\
0 & 0 & 0 & \delta_{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta_{ij} & 0 & 0
\end{array} \right) \right) \delta^3(x - y),
\] (20)
where we can observe that is not a singular matrix, therefore it is a symplectic tensor. The inverse of $f^{(1)}_{ij}$ is given by

$$
(1)^{-1} f_{ij} = \begin{pmatrix}
0 & \frac{1}{4}(\delta_k^i \delta_j^l + \delta_l^i \delta_k^j) & 0 & 0 & 0 & 0 \\
-\frac{1}{4}(\delta_k^i \delta_l^j + \delta_l^i \delta_k^j) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4}(\delta_k^i \delta_l^j + \delta_l^i \delta_k^j) & 0 & 0 \\
0 & 0 & -\frac{1}{4}(\delta_k^i \delta_l^j + \delta_l^i \delta_k^j) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\delta_j^i & 0 \\
0 & 0 & 0 & -\frac{1}{2}(\delta^k_i \partial_j + \delta^k_j \partial_i) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_j^i & 0 & 0 \\
0 & \delta_j^i & 0 & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix}
$$

where we can identify the following generalized FJ brackets by means of

$$
\{\xi^{(1)}_i(x), \xi^{(1)}_j(y)\}_{FJ} \equiv (f^{(1)}_{ij})^{-1},
$$

hence, the following FJ brackets arise

$$
\begin{align*}
\{E_{ij}, P^{kl}\}_{FJ} &= \frac{1}{4} (\delta^k_i \delta^l_j + \delta^k_j \delta^l_i) \delta^3(x - y), \\
\{B_{ij}, \Pi^{kl}\}_{FJ} &= \frac{1}{4} (\delta^k_i \delta^l_j + \delta^k_j \delta^l_i) \delta^3(x - y),
\end{align*}
$$

and hence Dirac’s brackets and the FJ ones coincide to each other. Furthermore, we have commented above that in the FJ framework there is not a classification between the constraints in first class and second class; in the FJ scheme the counting of degrees of freedom is carried out as follows; there are 20 symplectic variables given by $(E_{ij}, P^{ij}, B_{ij}, \Pi^{ij})$ and there are 6 constraints and 2 reducibility conditions, hence, there are 4 independent constraints and at the end one obtains eight physical degrees of freedom, such as it was obtained within the Dirac formalism.
IV. SYMPLECTIC ANALYSIS OF STANDARD LINEARIZED GRAVITY

Now, we will develop the symplectic analysis of the action (3), and then we will compare the final results with those obtained in previous sections. We perform first the 3+1 decomposition

\[
\mathcal{L} = \tilde{h}_{ij} \left[ \frac{1}{4} h_{kl} \left( n^{ij} - \eta^{ij} n^{kl} \right) + \frac{1}{2} \left( \partial^j h^{0i} + \partial^i h^{0j} \right) - \eta^{ij} \partial^k h^0_k \right] - \frac{1}{2} \partial_i h_{0j} \partial^j h^0_i - \frac{1}{4} \partial_i h_{jk} \partial^j h^k_i \\
+ \frac{1}{2} \partial_i h^0_j \partial^j h^0_i + \frac{1}{2} \partial_i h^0_j \partial^j h^0_k - \frac{1}{2} \partial_i h^{ij} \partial_j h^0_k - \frac{1}{2} \partial_i h^{jk} \partial_j h^0_k + \frac{1}{2} \partial_i h_{j0} \partial^j h^0_i + \frac{1}{2} g_{ij} \partial^j h^k_i. \tag{24}
\]

By introducing the momenta given by

\[
\Pi^{mn} = \frac{1}{2} h_{ij} \left( \eta^{mi} \eta^{nj} - \eta^{ij} \eta^{mn} \right) + \frac{1}{2} \left( \partial^m h^{0n} + \partial^n h^{0m} \right) - \eta^{mn} \partial^k h^0_k, \tag{25}
\]

the Lagrangian acquires the following symplectic form

\[
\mathcal{L}^{(0)} = \tilde{h}_{ij} \Pi^{ij} - \mathcal{V}^{(0)}, \tag{26}
\]

where \( \mathcal{V}^{(0)} \) is the symplectic potential

\[
\mathcal{V}^{(0)} = \Pi_{ij} \Pi^{ij} - \frac{1}{4} n_{kl} n^{ij} \Pi^{ij} - 2 \partial_j h^0_i \Pi^{ij} - \frac{1}{4} \partial_i h_{jk} \partial^k h^0_j - \frac{1}{2} \partial_i h^{ij} \partial_j h^0_k - \frac{1}{4} \partial_i h^{jk} \partial_j h^0_k \\
+ \frac{1}{2} \partial_i h^{ij} \partial_j h^0_i + \frac{1}{2} \partial_i h^{ij} \partial_j h^0_k - \frac{1}{2} \partial_i h_{j0} \partial^j h^0_i + \frac{1}{2} \partial_i h_{jk} \partial^j h^k_i. \tag{27}
\]

On the other hand, from the symplectic Lagrangian (24) it is possible to identify the following symplectic variables and 1-forms respectively

\[
\xi^{(0)} = \left( h_{00}, h_{0i}, h_{ij}, \Pi^{ij} \right), \tag{28}
\]

\[
a^{(0)} = (0, 0, \Pi^{ij}, 0), \tag{29}
\]

thus, the symplectic matrix (13) takes the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 \left( \delta^i_l \delta^m_j + \delta^i_l \delta^m_j \right) & -\frac{1}{2} \left( \delta^i_l \delta^m_j + \delta^i_l \delta^m_j \right) & \delta^i_l (x - y)
\end{pmatrix}. \tag{30}
\]

We observe that the symplectic matrix is singular and this means that there are constraints. It is important to note the difference between the alternative action analyzed in previous sections and the standard action. In fact, in the former the symplectic matrix was not singular, and the constraints where added by hand. In the latter, there are constraints, which will be different with respect to the nonstandard theory. In order to identify the FJ constraints, we calculate the zero-modes of the
symplectic matrix. These modes are given by

\[ v^{(0)}_1 = (v^{h_{00}}, 0, 0, 0, 0), \]
\[ v^{(0)}_2 = (0, v^{h_{00}}, 0, 0, 0), \]

where \( v^{h_{00}}, v^{h_{01}} \) are arbitrary functions. Thus, the constraints will be obtained from the contraction of the null vectors and the variation of the symplectic potential, this is,

\[ \delta \mathbf{V}^{(0)}(\xi^{(0)}) = 0, \]

In this manner, the following FJ constraints arise

\[ \Omega^{(0)}_1 = \int d^3x \; v^{(0)}_1 \frac{\delta}{\delta \xi^{(0)}_j} \int d^3y \; V^{(0)}(\xi^{(0)}) = \int d^3x \; v^{h_{00}} \frac{\delta}{\delta h_{00}} \int d^3y V^{(0)} \Rightarrow \]
\[ \Omega^{(0)}_1 = \frac{1}{2} \int d^3x \; v^{h_{00}} \left( \nabla^2 h^j_j - \partial_i \partial_j h^{ij} \right) = 0 \]

\[ \Omega^{(0)}_2 = \int d^3x \; v^{h_{00}} \partial_i \Pi^{ij} = 0, \]

Furthermore, in order to determine if there are more constraints, we demand consistency conditions, thus we construct the following system

\[ \bar{f}_{ij} \xi^{(0)ij} = Z_i, \]

where

\[ \bar{f}_{ij} = \begin{pmatrix} f^{(0)}_{ij} \\ \frac{\delta \Omega^{(0)}_1}{\delta \xi^{(0)j}} \\ \frac{\delta \Omega^{(0)}_2}{\delta \xi^{(0)j}} \end{pmatrix}, \]

with

\[ Z_i = \begin{pmatrix} \frac{\delta V^{(0)}}{\delta \xi^{(0)i}} \\ 0 \\ 0 \end{pmatrix} \]
In this manner, the matrix \( \tilde{f}_{ij} \) takes the following form

\[
\tilde{f}_{ij} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} (\delta^i_d \delta^m_j + \delta^m_i \delta^i_j) \\
0 & 0 & \frac{1}{2} (\delta^i_d \delta^i_m + \delta^j_d \delta^i_m) & 0 \\
0 & 0 & \frac{1}{2} (\nabla^2 \eta^l m - \partial^i \partial^m) & 0 \\
0 & 0 & 0 & \frac{1}{2} (\partial_m \delta^i_l + \partial_l \delta^i_m)
\end{pmatrix}
\]

and

\[
Z_i = \begin{pmatrix}
\frac{1}{2} (\nabla^2 h^j_l - \partial_j \partial_l h^j_l) \\
\partial_j \Pi^i j \\
\frac{1}{2} [\eta^{ij} \nabla^2 h^0_k - \partial^i \partial^j h^0_k] + \frac{1}{2} [\eta^{ij} \nabla^2 h^k_j - \partial^i \partial^j h^k_j] - \frac{1}{2} [\nabla^2 h^j_l + \eta^{ij} \partial^l \partial^k h^l_k] \\
2 \Pi_{jm} - \eta_{jm} \Pi^j_j - \partial_m h^0_l - \partial_l h^0_j \\
\partial_m \Pi^i j & 0 \\
0 & 0
\end{pmatrix}
\]

We can observe that the matrix (34) is singular, therefore there are zero-modes. The zero-modes of that matrix are given by

\[
\bar{v}_1 = \left(0, 0, 0, (\nabla^2 \eta^l m - \partial^l \partial^m), (\delta^i_d \delta^i_m + \delta^j_d \delta^i_m), 0\right), \\
\bar{v}_2 = (0, 0, (\partial_m \delta^i_l + \partial_l \delta^i_m), 0, 0, (\delta^i_d \delta^i_j + \delta^m_i \delta^i_j)),
\]

(35)

thus, in order to determine if there are more FJ constraints, we calculate the contraction of the null vectors (35) with \( Z_i \) (34), and from that contraction we can observe that there are no more FJ constraints because the result is a combination of constraints

\[
\bar{v}_1 \ i_{z_l} = 2 \partial_l \partial_m \Pi^l m = 2 \partial_l \Omega^{(l)}_2 = 0, \\
\bar{v}_2 \ i_{z_l} = \partial^i \left(\nabla^2 h^i_j - \partial_j \partial_i h^i_j\right) = 2 \partial^i \Omega^{(l)}_1 = 0.
\]

(36)  (37)
Furthermore, we will add the information of the FJ constraints to the action via Lagrange multipliers, namely $\alpha$ and $\beta$, thus we construct a new symplectic Lagrangian

$$
\mathcal{L}^{(1)} = \dot{h}_{ij} \Pi^{ij} - \mathcal{V}^{(0)} |_{\Omega^{(0)}_1, \Omega^{(0)}_2 = 0} - \Omega^{(0)}_1 \dot{\alpha} - \Omega^{(0)}_2 \dot{\beta},
$$

$$
= \dot{h}_{ij} \Pi^{ij} - \frac{1}{2} \left( \nabla^2 h^j_j - \partial_i \partial_j h^{ij} \right) \dot{\alpha} - \partial_j \Pi^{ij} \dot{\beta} - \Pi^{ij} \Pi_{ij} + \frac{1}{2} \eta_{ij} \eta_{k\ell} \Pi^{ij} \Pi^{k\ell} - \frac{1}{4} \partial_i h_{jk} \partial^i h^{jk} + \frac{1}{4} \partial^i h^{ij} \partial_i h^{jk} - \frac{1}{2} \partial_i h^{ij} \partial_j h^{ik} + \frac{1}{2} \partial_i h_{jk} \partial^j h^{ik}. \tag{38}
$$

In this manner, from (38) we identify the following new set of symplectic variables

$$
\xi^{(1)} = (h_{ij}, \Pi^{ij}, \alpha, \beta_i),
$$

$$
a^{(1)} = (\Pi^{ij}, 0, -\frac{1}{2} \left( \nabla^2 h^j_j - \partial_i \partial_j h^{ij} \right), -\partial_j \Pi^{ij}). \tag{39}
$$

with these variables we calculate the new symplectic matrix

$$
f_{ij}^{(1)} = \begin{pmatrix}
0 & -\frac{1}{2} (\delta^i \delta^m + \delta^m \delta^i) & \frac{1}{2} \left( \nabla^2 \delta^m - \partial^i \partial^m \right) & 0 \\
-\frac{1}{2} (\nabla^2 \delta^i - \partial^i \partial^m) & 0 & 0 & \frac{1}{2} (\partial_m \delta^i + \partial_i \delta^m) \\
\frac{1}{2} \left( \delta^i \delta^m + \delta^m \delta^i \right) & 0 & 0 & \frac{1}{2} \left( \nabla^2 \delta^m - \partial^i \partial^m \right) \\
0 & -\frac{1}{2} (\delta^i \delta^m + \delta^m \delta^i) & 0 & 0 \\
\end{pmatrix} \delta^3(x - y). \tag{40}
$$

We can observe that the symplectic matrix \( f^{(1)}_{ij} \) is singular, however, we have showed that there are no more constraints. Thus, this result indicate that linearized gravity is a gauge theory, as expected.

In order to obtain a symplectic tensor, we need to fixing the gauge. We will use first the temporal gauge, we consider that $h_{00} = 0$ and $h_{0i} = 0$, this implies that $\dot{a} = 0$, $\dot{\beta} = 0$, and we will add the Lagrange multipliers $\Sigma, \Gamma^i$, enforcing this gauge choice. Thus, we obtain the following symplectic Lagrangian

$$
\mathcal{L}^{(2)} = \dot{h}_{ij} \Pi^{ij} + \left[ \Sigma - \frac{1}{2} \left( \nabla^2 h^j_j - \partial_i \partial_j h^{ij} \right) \right] \dot{\alpha} + \left[ \Gamma^i - \partial_i \Pi^{ij} \right] \dot{\beta} - \Pi^{ij} \Pi_{ij} + \frac{1}{2} \eta_{ij} \eta_{k\ell} \Pi^{ij} \Pi^{k\ell} - \frac{1}{4} \partial_i h_{jk} \partial^i h^{jk} + \frac{1}{4} \partial^i h^{ij} \partial_i h^{jk} - \frac{1}{2} \partial_i h^{ij} \partial_j h^{ik} + \frac{1}{2} \partial_i h_{jk} \partial^j h^{ik}, \tag{42}
$$

where we identify the following symplectic variables and the following 1-forms respectively

$$
\xi^{(2)} = (h_{ij}, \Pi^{ij}, \alpha, \beta_i, \Sigma, \Gamma),
$$

$$
a^{(2)} = (\Pi^{ij}, 0, \left[ \Sigma - \frac{1}{2} \left( \nabla^2 h^j_j - \partial_i \partial_j h^{ij} \right) \right], \Gamma^i - \partial_j \Pi^{ij}, 0, 0). \tag{43}
$$
with these variables we calculate the corresponding symplectic matrix, given by
\[
\begin{pmatrix}
0 & -\frac{1}{2} (\delta^i_j \delta^m_n + \delta^m_i \delta^m_n) & \frac{1}{2} \left( \nabla^2 \eta^{ij} - \partial^i \partial^j \right) & 0 & 0 & 0 \\
\frac{1}{2} (\delta^i_j \delta^m_n + \delta^m_i \delta^m_n) & 0 & 0 & \frac{1}{2} (\partial_m \delta^i_j + \partial_n \delta^m_n) & 0 & 0 \\
\frac{1}{2} (\nabla^2 \eta^{ij} - \partial^i \partial^j) & 0 & 0 & 0 & -1 & 0 \\
0 & -\frac{1}{2} (\partial_i \delta^m + \partial_j \delta^m_j) & 0 & 0 & 0 & -\delta^j_i \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \delta^j_i & 0 & 0 \\
\end{pmatrix}
\]
we observe that $f^{(2)}_{ij}$ is not singular, and therefore it is invertible. The inverse is given by
\[
\begin{pmatrix}
0 & -\frac{1}{2} (\delta^i_j \delta^m_n + \delta^m_i \delta^m_n) & -\frac{1}{2} (\partial_m \delta^i_j + \partial_n \delta^m_n) & 0 & 0 & 0 \\
-\frac{1}{2} (\delta^i_j \delta^m_n + \delta^m_i \delta^m_n) & 0 & 0 & \frac{1}{2} (\nabla^2 \eta^{ij} - \partial^i \partial^j) & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \delta^j_i & 0 \\
0 & 0 & 0 & -\delta^j_i & 0 & 0 \\
-\frac{1}{2} (\partial^i \partial_i + \partial^j \partial_j) & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
thus, it is possible to identify the following generalized FJ brackets
\[
\{ \xi^{(2)}_i (x), \xi^{(2)}_j (y) \} = \left( f^{(2)}_{ij} \right)^{-1}
\]
where the relevant brackets are given by
\[
\{ h_{ij} (x), \Pi^{kl} (y) \} = \frac{1}{2} (\delta^i_j \delta^k_l + \delta^i_l \delta^k_j) \delta^3 (x - y).
\]
Now, it is well-known that in Dirac’s terminology, to work with the temporal gauge implies to convert the primary first class constraints into second class ones. However, there are a remnant of first class constraints. In this manner, if we wish to convert all first class constraints into second class ones, we need to fix a different gauge; namely the Coulomb gauge. In fact, in \cite{10} it was performed the Dirac analysis by using the Coulomb-like gauge, hence, we will reproduce all these results by means a different approach. First, we will add to the Lagrangian \cite{22} the following Coulomb-like gauge $\partial^i h_{ij} = 0$, the momentum gauge $\Pi^i_i = 0$, and the consequent Lagrange multipliers enforcing these gauge conditions, namely, $\Lambda^i, \Upsilon$
\[
\mathcal{L}^{(3)} = \dot{h}_{ij} \Pi^{ij} + \left[ -\frac{1}{2} \left( \nabla^2 h_{ij} - \partial_i \partial_j h_{ij} \right) \right] \dot{\alpha} + \left[ \Lambda^i - \partial_j \Pi^{ij} \right] \dot{\beta}_i - \Pi^i_i \Pi^{ij} + \frac{1}{2} \eta_{ij} \Pi^{ij} \Pi^{kl} \Pi_{ij} - \Pi^i_i \Upsilon - \partial^i h_{ij} \Lambda^i - \frac{1}{4} \partial_i h_{jk} \partial^j h^k_k + \frac{1}{4} \partial^i h_{ij} \partial_j h^k_k - \frac{1}{2} \partial_i h_{ij} \partial_j h^k_k + \frac{1}{2} \partial_i h_{jk} \partial^j h^i_k,
\]

from this symplectic Lagrangian, we identify the following symplectic variables and the following 1-forms respectively

\[ \xi^{(3)} = (h_{ij}, \Pi^{ij}, \alpha, \beta_i, \Lambda^i, \Upsilon, \Sigma, \Gamma^i), \]  
\[ \alpha^{(3)} = (\Pi^{ij}, 0, \left[ \sum - \frac{1}{2} \left( \nabla^2 h^{ij} - \partial_i \partial_j h^{ij} \right) \right], \Gamma^i - \partial_j \Pi^{ij}, -\partial^i h_{ij}, -\Pi^i, 0, 0), \]  

thus, the symplectic matrix has the form

\[
\begin{pmatrix}
0 & -\frac{1}{2} \left( \delta_i^j \delta^m_m + \delta^m_i \delta_j^m \right) & -\frac{1}{2} \left( \nabla^2 \eta^{im} - \partial_i \partial^m \right) & 0 & -\frac{1}{2} \left( \delta_i^j \partial^i + \delta^i_j \partial^i \right) & 0 & 0 & 0 \\
\frac{1}{2} \left( \delta_i^j \delta^m_m + \delta^m_i \delta_j^m \right) & 0 & 0 & -\frac{1}{2} \left( \partial_m \delta_i^j + \partial_j \delta_i^m \right) & 0 & -\eta_{ij} & 0 & 0 \\
\frac{1}{2} \left( \nabla^2 \eta^{ij} - \partial^i \partial^j \right) & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & \frac{1}{2} \left( \partial_i \delta^j_m + \partial_j \delta^m_i \right) & 0 & 0 & 0 & 0 & 0 & -\delta_j^i \\
\frac{1}{2} \left( \delta_i^j \partial^i + \delta^i_j \partial^i \right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \eta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_j^i & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\delta^3(x - y),
\]  

we realize that the above matrix is not singular, and its inverse is given by

\[
\begin{pmatrix}
0 & A & \left( \eta_{ij} - \frac{\partial_j i}{\sqrt{x^2}} \right) & 0 & -B & 0 & 0 & 0 \\
-A & 0 & 0 & -C & 0 & \frac{1}{\sqrt{x^2}} \left( \eta^{ij} - \frac{\partial^i \partial^j}{\sqrt{x^2}} \right) & 0 & 0 \\
-(\eta_{ij} - \frac{\partial_j i}{\sqrt{x^2}}) \frac{1}{\sqrt{x^2}} & 0 & 0 & \frac{\partial_j i}{\sqrt{x^2}} & 0 & -\frac{1}{\sqrt{x^2}} \sqrt{x^2} & 1 & 0 \\
0 & C & -\frac{\partial_j i}{\sqrt{x^2}} & 0 & (\delta^j_i - \frac{\partial^i \partial^j}{\sqrt{x^2}}) \frac{1}{\sqrt{x^2}} & 0 & 0 & \frac{\partial_j i}{\sqrt{x^2}} \\
B & 0 & 0 & -(\delta^j_i - \frac{\partial^i \partial^j}{\sqrt{x^2}}) \frac{1}{\sqrt{x^2}} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{x^2}} \left( \eta^{ij} - \frac{\partial^i \partial^j}{\sqrt{x^2}} \right) & \frac{1}{\sqrt{x^2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\partial_j i}{\sqrt{x^2}} & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\delta^3(x - y),
\]
where we have defined
\[
A = \frac{1}{2} \left( \delta^i_\ell \delta^j_k + \delta^k_\ell \delta^j_i \right) - \left[ \delta^i_\ell \partial_j \partial^k + \delta^k_\ell \partial_j \partial^i + \delta^j_\ell \partial_i \partial^k + \delta^j_k \partial_i \partial^\ell \right] \frac{1}{2 \sqrt{\nu^2}} - \frac{1}{2 \sqrt{\nu^2}} \eta^{kl} \eta_{ij} + \frac{1}{2} \left( \eta_{ij} \partial^k \partial^l + \eta^{kl} \partial_i \partial_j \right) \frac{1}{2 \sqrt{\nu^2}}
+ \frac{\partial_i \partial_j \partial^k \partial^l}{2 \sqrt{\nu^2}},
\]
\[
B = \frac{\partial^k \partial_i \partial_j}{\sqrt{\nu^4}} - \left[ (\delta^k_\ell \partial_j + \delta^k_\ell \partial_i) \frac{1}{\sqrt{\nu^2}} \right],
\]
\[
C = \left[ \eta^{kl} \partial_i + \frac{\partial_i \partial^k \partial^l}{\sqrt{\nu^2}} \right] \frac{1}{2 \sqrt{\nu^2}} - \left[ \delta^l_\ell \partial_i \partial^k + \delta^k_\ell \partial_i \partial^l \right] \frac{1}{2 \sqrt{\nu^2}}.
\]

In this manner, we can identify the nontrivial FJ brackets by means of
\[
\{ \xi_i^{(3)}(x), \xi_j^{(3)}(y) \}_FJ \equiv \left( f_{ij}^{(3)} \right)^{-1},
\]
thus, we find
\[
\{ h_{ij}(x), \Pi^{kl}(y) \} = \frac{1}{2} \left( \delta^i_\ell \delta^j_k + \delta^k_\ell \delta^j_i \right) - \frac{1}{2} \left[ \delta^i_\ell \partial_j \partial^k + \delta^k_\ell \partial_j \partial^i + \delta^j_\ell \partial_i \partial^k + \delta^j_k \partial_i \partial^\ell \right] \frac{1}{\sqrt{\nu^2}} - \frac{1}{2 \sqrt{\nu^2}} \eta^{kl} \eta_{ij}
+ \frac{1}{2} \left( \eta_{ij} \partial^k \partial^l + (\eta_{kl} \partial_i \partial_j) \frac{1}{\sqrt{\nu^2}} + \frac{\partial_i \partial_j \partial^k \partial^l}{2 \sqrt{\nu^4}} \right) \delta^\delta(x - y),
\]
these brackets reproduce exactly those obtained in [10] by using the Dirac method, and we have reproduced the same brackets by means of a different way.

On the other hand, we can observe the differences between the nonstandard theory and the standard one; the standard linearized theory is a gauge theory, while the nonstandard theory is not. Furthermore, for both theories the generalized brackets between the fields are different, in addition, we would like to comment that our results are absent in the literature and they are an extension of those reported in [10].

V. CONCLUSIONS

In this paper, the canonical and symplectic analysis for an alternative action describing gravity were performed. With respect to Dirac’s formalism, we found the constraints of the theory, which turned out to be of second class, then the Dirac brackets were constructed. In addition, we found that the Hamiltonian is of first class and therefore it correspond to an observable; these facts were not reported in [12, 13]. Furthermore, the results were reproduced using the FJ formalism: We constructed a symplectic tensor, then the generalized FJ brackets were found, we showed that Dirac’s and FJ brackets coincide to each other. In this manner, we have confirmed by an alternative way that the action is not a gauge theory and its physical degrees of freedom are eight. In order to complete our analysis, we added the symplectic analysis of standard linearized gravity.

We finish this paper with some points to remark. If we define a system in terms of its equations of motion, then an infinity number of Hamiltonian structures can be defined for the same system [18]. This fact present a problem for singular systems because one could propose several actions yielding the same equations of motion, but the symmetries between these different actions could be
different just as it was showed in the present analysis. Thus, in order to study any new proposal for a physical system, it is mandatory to put attention in the symmetries beyond its equations of motion [19].

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