COLLAPSIBILITY TO A SUBCOMPLEX OF A GIVEN DIMENSION IS NP-COMPLETE

GIOVANNI PAOLINI

Abstract. In this paper we extend the works of Tancer and of Malgouyres and Francés, showing that \((d, k)\)-Collapsibility is NP-complete for \(d \geq k + 2\) except \((2, 0)\). By \((d, k)\)-Collapsibility we mean the following problem: determine whether a given \(d\)-dimensional simplicial complex can be collapsed to some \(k\)-dimensional subcomplex. The question of establishing the complexity status of \((d, k)\)-Collapsibility was asked by Tancer, who proved NP-completeness of \((d, 0)\) and \((d, 1)\)-Collapsibility (for \(d \geq 3\)). Our extended result, together with the known polynomial-time algorithms for \((2, 0)\) and \(d = k + 1\), answers the question completely.

1. Introduction

Discrete Morse theory is a powerful combinatorial tool which allows to explicitly simplify cell complexes while preserving their homotopy type [For98, Cha00, BW02, Koz07]. This is obtained through a sequence of “elementary collapses” of pairs of cells. Such process might decrease the dimension of the starting complex, or sometimes even leave a single point (in which case we say that the starting complex was collapsible).

The problem of algorithmically determine collapsibility, or find “good” sequences of elementary collapses, has been studied extensively [EG96, JP06, MF08, BL14, BLP16, Tan16]. Such problems proved to be computationally hard even for low dimensional simplicial complexes. For 2-dimensional complexes there exists a polynomial-time algorithm to check collapsibility [JP06, MF08], but finding the minimum number of “critical” triangles (without which the remaining complex would be collapsible) is already NP-hard [EG96]. In dimension \(d \geq 3\), collapsibility to some 1-dimensional subcomplex [MF08] or even to a single point [Tan16] were proved to be NP-complete.

In [Tan16], Tancer also introduced the general \((d, k)\)-Collapsibility problem: determine whether a \(d\)-dimensional simplicial complex can be collapsed to some \(k\)-dimensional subcomplex. He showed that \((d, k)\)-Collapsibility is NP-complete for \(k \in \{0, 1\}\) and \(d \geq 3\), extending the result of Malgouyres and Francés about NP-completeness of \((3, 1)\)-Collapsibility [MF08]. Tancer also pointed out that the codimension 1 case \((d = k + 1)\) is polynomial-time solvable as is the \((2, 0)\) case. He left open the question of determining the complexity status of \((d, k)\)-Collapsibility in general.

In this short paper we extend Tancer’s work, and prove that \((d, k)\)-Collapsibility is NP-complete in all the remaining cases.

Theorem 3.2. The \((d, k)\)-Collapsibility problem is NP-complete for \(d \geq k + 2\), except for the case \((2, 0)\).
To do so, we prove that \((d, k)\)-Collapsibility admits a polynomial-time reduction to \((d + 1, k + 1)\)-Collapsibility (Theorem 3.1). Then the main result follows by induction on \(k\). The base cases of the induction are given by NP-completeness of \((3, 1)\)-Collapsibility (for codimension 2) and of \((d, 0)\)-Collapsibility (for codimension \(d \geq 3\)).

2. Collapsibility and discrete Morse theory

We refer to [Hat02] for the definition and the basic properties of simplicial complexes, and to [Koz07] for the definition of elementary collapses. The simplicial complexes we consider do not contain the empty simplex, unless otherwise stated. Our focus is the following decision problem.

**Problem 2.1:** \((d, k)\)-Collapsibility.

*Parameters:* Non-negative integers \(d > k\).

*Instance:* A finite \(d\)-dimensional simplicial complex \(X\).

*Question:* Can \(X\) be collapsed to some \(k\)-dimensional subcomplex?

We are now going to recall a few definitions of discrete Morse theory [For98, Cha00, Koz07], so that we can state the \((d, k)\)-Collapsibility problem in terms of acyclic matchings.

Given a simplicial complex \(X\), its *Hasse diagram* \(H(X)\) is a directed graph in which the set of nodes is the set of simplexes of \(X\), and an arc goes from \(\sigma\) to \(\tau\) if and only if \(\tau\) is a face of \(\sigma\) and \(\dim(\sigma) = \dim(\tau) + 1\). A *matching* \(\mathcal{M}\) on \(X\) is a set of arcs of \(H(X)\) such that every node of \(H(X)\) (i.e. simplex of \(X\)) is contained in at most one arc in \(\mathcal{M}\). Given a matching \(\mathcal{M}\) on \(X\), we say that a simplex \(\sigma \in X\) is *critical* if it doesn’t belong to any arc in \(\mathcal{M}\). Finally we say that a matching \(\mathcal{M}\) on \(X\) is *acyclic* if the graph \(H(X)^{\mathcal{M}}\), obtained from \(H(X)\) by reversing the direction of each arc in \(\mathcal{M}\), does not contain directed cycles.

By standard facts of discrete Morse theory (see for instance [Koz07], Section 11.2), “collapsibility to some \(k\)-dimensional subcomplex” is equivalent to “existence of an acyclic matching such that the critical cells form a \(k\)-dimensional subcomplex”.

To simplify the proof of Theorem 3.1 we quote the following useful lemma from [Koz07], adapting it to our notation.

**Theorem 2.3** (Patchwork theorem, [Koz07]). Let \(P\) be a poset. Let \(\varphi: X \to P\) be an order-preserving map (where the order on \(X\) is given by inclusion), and assume
to have acyclic matchings on subposets $\varphi^{-1}(p)$ for all $p \in P$. Then the union of these matchings is itself an acyclic matching on $X$.

Notice that the subposets $\varphi^{-1}(p)$ are not subcomplexes of $X$ in general, but still they have a well-defined Hasse diagram (the induced subgraph of $H(X)$). Thus all the previous definitions (matching, critical simplex, acyclic matching) apply also to each subposet.

3. Main result

**Theorem 3.1.** Let $d > k \geq 0$. Then there is a polynomial-time reduction from ($d, k$)-Collapsibility to ($d + 1, k + 1$)-Collapsibility.

**Proof.** Let $X$ be an instance of ($d, k$)-Collapsibility, i.e. a $d$-dimensional simplicial complex. Let $V = \{v_1, \ldots, v_r\}$ be the vertex set of $X$. Construct an instance $X'$ of ($d + 1, k + 1$)-Collapsibility, i.e. a ($d + 1$)-dimensional complex, as follows. Let $n \geq 1$ be the number of simplices in $X$. Roughly speaking, $X'$ is obtained from $X$ by attaching $n + 1$ cones of $X$ to $X$. More formally, introduce new vertices $w_1, \ldots, w_{n+1}$ and define $X'$ as the simplicial complex on the vertex set $V' = \{v_1, \ldots, v_r, w_1, \ldots, w_{n+1}\}$ given by

$$X' = X \cup \left\{ \sigma \cup \{w_i\} \mid \sigma \in X, \ i = 1, \ldots, n + 1 \right\}.$$  

Then $X'$ has $n(n + 2)$ simplices. We are going to prove that $X$ is a yes-instance of ($d, k$)-Collapsibility if and only if $X'$ is a yes-instance of ($d + 1, k + 1$)-Collapsibility.

Suppose that $X$ is a yes-instance of ($d, k$)-Collapsibility. Then there exists an acyclic matching $\mathcal{M}$ on $X$ such that all critical simplices have dimension $\leq k$. Construct a matching $\mathcal{M}'$ on $X'$ as follows:

$$\mathcal{M}' = \left\{ \sigma \cup \{w_i\} \rightarrow \sigma \mid \sigma \in X \right\} \cup \left\{ \sigma \cup \{w_i\} \rightarrow \tau \cup \{w_i\} \mid (\sigma \rightarrow \tau) \in \mathcal{M}, \ i = 2, \ldots, n + 1 \right\}.$$  

This matching corresponds to collapsing the first cone together with $X$ (only the vertex $w_1$ remains), and every other "base-less" cone by itself (as a copy of $X$). To prove that $\mathcal{M}'$ is acyclic, consider the set $P = \{w_1, \ldots, w_{n+1}\}$ with the partial order $w_i < w_j$ if and only if $i = 1$ and $j > 1$.

Let $\varphi: X' \rightarrow P$ be the order-preserving map given by

$$\varphi(\sigma) = \begin{cases} 
  w_j & \text{if $\sigma$ contains $w_j$ for some $j \geq 2$;} \\
  w_1 & \text{otherwise.}
\end{cases}$$

Then $\mathcal{M}'$ is a union of matchings $\mathcal{M}'_j$ on each fiber $\varphi^{-1}(w_j)$. The matching $\mathcal{M}'_1$ is acyclic on $\varphi^{-1}(w_1)$, since the arcs of $\mathcal{M}'_1$ define a cut of the Hasse diagram of $\varphi^{-1}(w_1)$. The Hasse diagram of each $\varphi^{-1}(w_j)$ for $j \geq 2$ is isomorphic to $H(X \cup \{\emptyset\})$, and the matching $\mathcal{M}'_j$ maps to $\mathcal{M}$ via this isomorphism. Since $\mathcal{M}$ is acyclic on $H(X \cup \{\emptyset\})$, each $\mathcal{M}'_j$ is also acyclic on $\varphi^{-1}(w_j)$. By the Patchwork theorem (Theorem 2.3), $\mathcal{M}'$ is acyclic on $X'$.

The set of critical simplices of $\mathcal{M}'$ is

$$\text{Cr}(X', \mathcal{M}') = \{w_1\} \cup \left\{ \sigma \cup \{w_i\} \mid \sigma \in \text{Cr}(X, \mathcal{M}) \cup \{\emptyset\}, \ i = 2, \ldots, n + 1 \right\}.$$
In particular, all critical simplices have dimension \( \leq k + 1 \). Therefore \( X' \) is a yes-instance of \((d + 1, k + 1)\)-\textsc{Collapsibility}.

Conversely, suppose now that \( X' \) is a yes-instance of \((d + 1, k + 1)\)-\textsc{Collapsibility}. Let \( M' \) be an acyclic matching on \( X' \) such that all critical simplices have dimension \( \leq k + 1 \). Since \( X \) contains \( n \) simplices, and there are \( n + 1 \) cones, there must exist an index \( j \in \{1, \ldots, n + 1\} \) such that
\[
\left( \sigma \cup \{w_j\} \rightarrow \tau \right) \notin M' \quad \forall \sigma \in X.
\]
In other words, the matching on the \( j \)-th cone cannot mix simplices containing \( w_j \) and simplices not containing \( w_j \). Then we can construct a matching \( M \) on \( X \) as follows:
\[
M = \left\{ \sigma \rightarrow \tau \mid \sigma, \tau \in X \text{ satisfying } \left( \sigma \cup \{w_j\} \rightarrow \tau \cup \{w_j\} \right) \in M' \right\}.
\]
Notice that if there is some 0-dimensional \( \sigma \in X \) such that \( (\sigma \cup \{w_j\} \rightarrow \{w_j\}) \in M' \), then \( \sigma \) is critical with respect to \( M \) (it would be matched with \( \tau = \emptyset \) which doesn’t exist in \( X \)). The Hasse diagram of \( X \) injects into the Hasse diagram of the \( j \)-th cone via the map
\[
\iota: \sigma \mapsto \sigma \cup \{w_j\},
\]
and by construction arcs of \( M \) map to arcs of \( M' \). Since \( M' \) is acyclic, \( M \) is also acyclic. The set of critical simplices of \( M \) is
\[
\text{Cr}(X, M) = \left\{ \sigma \in X \mid \sigma \cup \{w_j\} \in \text{Cr}(X', M') \text{ or } \left( \sigma \cup \{w_j\} \rightarrow \{w_j\} \right) \in M' \right\}.
\]
In the first case \( \sigma \cup \{w_j\} \) has dimension \( \leq k + 1 \), and in the second case \( \sigma \) is 0-dimensional. In particular, all critical simplices have dimension \( \leq k \). Therefore \( X \) is a yes-instance of \((d, k)\)-\textsc{Collapsibility}. \( \square \)

The \((d, k)\)-\textsc{Collapsibility} problem admits a polynomial-time solution when \( d = k + 1 \) and also for the case \((2, 0)\) [JP06, MF08, Tan16]. Malgouyres and Francés [MF08] proved that \((3, 1)\)-\textsc{Collapsibility} is NP-complete, and Tancer extended this result to \((d, k)\)-\textsc{Collapsibility} for \( k \in \{0, 1\} \) and for all \( d \geq 3 \). Using this as the base step and Theorem 3.1 as the induction step, we obtain the following result.

**Theorem 3.2.** The \((d, k)\)-\textsc{Collapsibility} problem is NP-complete for \( d \geq k + 2 \), except for the case \((2, 0)\). \( \square \)

## 4. Acknowledgements

I would like to thank my father, Maurizio Paolini, for giving useful comments and suggesting corrections. I would also like to thank Luca Ghidelli, for checking the proof carefully and for being my best man.

## References

[BL14] B. Benedetti and F. H. Lutz, *Random discrete Morse theory and a new library of triangulations*, Experimental Mathematics **23** (2014), no. 1, 66–94.

[BLPS16] B. A. Burton, T. Lewiner, J. Paixão, and J. Spreer, *Parameterized complexity of discrete Morse theory*, ACM Transactions on Mathematical Software (TOMS) **42** (2016), no. 1, 6.

[BW02] E. Batzies and V. Welker, *Discrete Morse theory for cellular resolutions*, Journal fur die Reine und Angewandte Mathematik (2002), 147–168.
[Cha00] M. K. Chari, *On discrete Morse functions and combinatorial decompositions*, Discrete Mathematics 217 (2000), no. 1, 101–113.

[EG96] Ö. Eğecioğlu and T. F. Gonzalez, *A computationally intractable problem on simplicial complexes*, Computational Geometry 6 (1996), no. 2, 85–98.

[For98] R. Forman, *Morse theory for cell complexes*, Advances in mathematics 134 (1998), no. 1, 90–145.

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[JP06] M. Joswig and M. E. Pfetsch, *Computing optimal Morse matchings*, SIAM Journal on Discrete Mathematics 20 (2006), no. 1, 11–25.

[Koz07] D. Kozlov, *Combinatorial algebraic topology*, vol. 21, Springer Science & Business Media, 2007.

[MF08] R. Malgouyres and A. R. Francés, *Determining whether a simplicial 3-complex collapses to a 1-complex is NP-complete*, International Conference on Discrete Geometry for Computer Imagery, Springer, 2008, pp. 177–188.

[Tan16] M. Tancer, *Recognition of collapsible complexes is NP-complete*, Discrete & Computational Geometry 55 (2016), no. 1, 21–38.