Correlations in the $n \to 0$ limit of the dense $O(n)$ loop model

V S Poghosyan$^1$ and V B Priezzhev$^2$

$^1$ Institute for Informatics and Automation Problems, NAS of Armenia, 0014 Yerevan, Armenia
$^2$ Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia

E-mail: vpoghos@theor.jinr.ru and priezzvb@theor.jinr.ru

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Abstract

The two-dimensional dense $O(n)$ loop model for $n = 1$ is equivalent to the bond percolation and for $n = 0$ to the dense polymers or spanning trees. We consider the boundary correlations on the half space and calculate the probability $P_b$ that a cluster of bonds has a single common point with the boundary. In the limit $n \to 0$, we find an analytical expression for $P_b$ using the generalized Kirchhoff theorem.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The dense $O(n)$ loop model [1] is defined by drawing two lines in each elementary cell of the square lattice. Two possible states of the cell are shown in figure 1. The lines on the whole lattice with appropriate boundary conditions form a system of closed loops with the Boltzmann weight $n$ ascribed to each loop.

Increased interest in the $O(n)$ model, especially the case $n = 1$, was triggered by remarkable observations concerning the ground state of the antiferromagnetic XXZ quantum chain and, equivalently, to properties of the transfer matrix of the dense loop model taken in the Hamiltonian limit of extreme spatial anisotropy [2–7]. It was noted that the elements of the ground state of the loop Hamiltonian are equal to the cardinality of different subsets of configurations of the fully packed loop (FPL) model which, in turn, are connected with classes of alternating sign matrices (ASM).

The states of the $O(n)$ model can be defined in terms of the connectivity condition for points of intersection between loops and a horizontal line cutting the loops at these points. Two points are connected by a link if there exists a line between them via the half space above the cut. For instance, by imposing periodic boundary conditions in the horizontal direction for the lattice in figure 2 and specifying the boundary conditions at the upper edge, we obtain
three minimal links between points 1 and 6, 2 and 3, 4 and 5 which are points of intersection between loops and the bottom line of the lattice belonging to upper half space. A typical configuration of links for a larger lattice is given in figure 3.

Several interesting conjectures have been obtained about the probability distribution of configurations of links of the dense $O(1)$ loop model. To keep within the scope of this paper, we mention just one conjecture about the number of configurations containing the minimal link at a given position [7]

$$\sum \ldots \sim \ldots = \frac{3}{8} \frac{N^2}{N^2 - 1} A_{HT}(N)$$

(1.1)

for the even periodic system and

$$\sum \ldots \sim \ldots = \frac{3}{8} \frac{N^2 - 1}{N^2} A_{HT}(N)$$

(1.2)

for the odd periodic system. Here $A_{HT}(N)$ is the number of $N \times N$ half turn invariant ASM [9].

Since the total number of periodic link configurations is $A_{HT}(N)$, the probability of minimal link $\text{Prob}(\ldots \sim \ldots)$ tends to $3/8$ in the limit $N \to \infty$. Due to the simple bijection
between the loop configurations on the square lattice and the bond configurations on its sublattices, the dense $O(1)$ model is equivalent to the bond percolation. Then, the constant $3/8$ has a simple interpretation as the probability of a certain class of percolation clusters (see section 2). Despite its simplicity, an analytical derivation of the probability of minimal links as well as of other link configurations remains an open problem.

In this work, we consider the problem of evaluation of $\text{Prob}(\ldots \rightarrow \ldots)$ in the dense $O(n)$ model for the case $n = 0$. In the limit $n \to 0$, the bulk loops disappear and the system of lines is converted into the model of dense polymers [10]. The bijection, giving the bond percolation for $n = 1$ on a sublattice, leads in the limit $n \to 0$ to the spanning trees on the same sublattice. A good deal is known about correlation functions of the spanning tree model due to the Kirchhoff theorem [11]. However, we will see in subsequent sections that the evaluation of $\text{Prob}(\ldots \rightarrow \ldots)$ involves non-local diagrams which require a generalization of the Kirchhoff determinant formula. Similar correlations appear in the Abelian sandpile theory [12–16], in the problem of loop-erased random walk [17, 18], and in the problem of ‘watermelon’ embedded in the spanning tree [19, 20]. Below, we use the generalized determinant formula to find $\text{Prob}(\ldots \rightarrow \ldots)$ and compare the result obtained for $n = 0$ with that conjectured for $n = 1$.

### 2. Link structure and boundary clusters

The square lattice of sites with integer coordinates $(n, m)$ can be divided into two sublattices: black and white. For sites of the black sublattice $n + m$ is even, while for sites of the white one $n + m$ is odd. The bijection between loop and bond configurations mentioned in the introduction is shown in figure 4. The neighboring sites of each sublattice are connected by a bond if it does not intersect the lines of the elementary cell. Each connected cluster of bonds in the bulk of the lattice is situated inside a loop. Each bulk cluster on the black sublattice is surrounded by a connected cluster of bonds on the white sublattice and vice versa.

The horizontal line cutting the loops cuts bond clusters inside them and produces boundary clusters surrounded by pieces of loop. Each piece has two points of intersection with the
horizontal line and corresponds to the link between these points. The clusters of bonds corresponding to the loop configuration in figure 2 are shown in figure 5.

Since we are interested in the limiting case of large lattice $N \to \infty$, we take the infinite horizontal line and neglect the left and right boundary conditions. We put the origin of the lattice at a selected black site on the horizontal line and consider the boundary clusters of bonds containing the origin. The boundary clusters can be classified into four types shown schematically in figure 6. The cluster of type (a) consists of a single vertex coinciding with the origin. The type (b) represents clusters containing no points of the horizontal line besides

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure5}
\caption{The bond configuration corresponding to the loop configuration from figure 2.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{Four types of boundary clusters.}
\end{figure}
Table 1. The dense $O(1)$ loop model: values for $N = \infty$ are obtained from the extrapolation (see figure 7).

| $N$ | $P_a$ | $P_a + P_b$ | $P_c$ |
|-----|-------|-------------|-------|
| 3   | 0.093 747 14 | 0.531 312 73 | 0.187 474 00 |
| 5   | 0.167 509 25 | 0.442 350 40 | 0.195 035 58 |
| 7   | 0.212 467 68 | 0.411 945 24 | 0.187 785 62 |
| 9   | 0.241 849 03 | 0.398 231 48 | 0.179 956 54 |
| 11  | 0.262 378 08 | 0.390 965 27 | 0.173 345 48 |
| 13  | 0.277 454 83 | 0.386 675 26 | 0.167 990 24 |
| 15  | 0.289 009 01 | 0.383 820 77 | 0.163 609 72 |
| 17  | 0.298 121 03 | 0.381 882 81 | 0.159 981 31 |
| 19  | 0.305 524 63 | 0.380 554 39 | 0.156 946 91 |
| 21  | 0.311 466 70 | 0.379 755 36 | 0.154 289 28 |
| 23  | 0.316 652 57 | 0.378 888 34 | 0.152 521 64 |
| 25  | 0.321 187 28 | 0.378 105 30 | 0.150 389 40 |
| 27  | 0.324 754 14 | 0.377 867 65 | 0.148 676 57 |
| 29  | 0.328 150 29 | 0.377 441 92 | 0.147 226 38 |
| 31  | 0.330 867 48 | 0.377 180 18 | 0.146 015 43 |
| 33  | 0.333 417 52 | 0.376 939 01 | 0.144 784 32 |
| 35  | 0.335 712 70 | 0.376 644 02 | 0.143 811 27 |
| 37  | 0.337 711 67 | 0.376 514 94 | 0.142 866 61 |
| 39  | 0.339 558 03 | 0.376 425 72 | 0.142 023 44 |
| 41  | 0.341 236 56 | 0.376 217 96 | 0.141 312 14 |

$\infty$ 0.374 97 $\approx 3/8$ 0.374 99 $\approx 3/8$ 0.125 06 0.125 06 $\approx 1/8$

The clusters of type (c) contain the origin and one or more boundary vertices left or right from the origin. The clusters of type (d) contain the origin and an arbitrary number of boundary vertices left and right from the origin.

The minimal link probability $\text{Prob}(\ldots \leadsto \ldots)$ is the sum of probabilities $P_a$ of the clusters of type (a) and $P_b$ of the clusters of type (b) because the minimal link corresponds to the piece of loop surrounding the origin $(0,0)$ together with the cluster of bonds attached to the origin.

In the $O(1)$ model, the probability $P_a$ is elementary because the isolated site $(0,0)$ of the black sublattice corresponds to two bonds on the dual white sublattice: one bond connecting sites $(-1,0)$ and $(0,1)$ and another connecting sites $(0,1)$ and $(1,0)$. For the bond percolation, where the bond probability is $1/2$, $P_a = 1/4$. The conjectured value of $\text{Prob}(\ldots \leadsto \ldots)$ is $3/8$, then $P_b = 1/8$.

Considering the system of links as a height profile [6], one can note that the minimal links correspond to peaks of the profile and the local link configurations produced by clusters of type (d) correspond to valleys of the profile. The numbers of peaks and valleys in the periodic system coincide, so we have $P_d = 3/8$. Using the identity

$$P_a + P_b + 2P_c + P_d = 1$$

we conclude that $P_c = 1/8$. The symmetry $P_b = P_c$ is wonderful because the bond clusters of type (b) and (c) do not obey any visible symmetry in the percolation theory.

In table 1, we show the results of the enumeration of clusters of type (a), (b), (c) and (d). We considered the percolation problem on the upper-left half of the square of size $N \times N$ crossed by the diagonal line, so that the number of lattice points on the vertical, horizontal and diagonal edges of the obtained triangle is $N$. The origin is put into the central site on the diagonal. To provide a symmetry with respect to the origin, we take $N$ as odd. Despite the difference between our triangle geometry and the strip geometry used in [7], we see from
Figure 7. The Monte Carlo simulation result. The number of samples is $10^6$; the $N \times N$ upper-left half of the square with $N = 3, 5, 7, \ldots, 41$ is considered.

figure 7 that the convergence of $P_a + P_b$ to $3/8$ is of an order $1/N^2$ as is expected from the conjectures (1.1) and (1.2).

Our aim in this work is the calculation of probabilities $P_a$, $P_b$, $P_c$, $P_d$ for the free-fermion $O(0)$ model. Elimination of loops in the limit $n \to 0$ converts the wavy lines of the $O(n)$ model into dense polymers [10] and the bond clusters on the black and white sublattices into spanning trees on these sublattices. A spanning tree on the black (white) sublattice is a connected cluster containing all black (white) sites. Due to the cut, the connected spanning tree splits into separated components attached to the horizontal line. The classification of the components (figure 6) is the same as in the percolation model. For definiteness, we consider black components and calculate their statistics via probabilities of white components surrounding black clusters. Normally, one selects an arbitrary site of the tree as a root. Then all bonds of the tree become oriented toward the root. It is convenient to choose the root of tree on the white sublattice at a site below the horizontal line that cuts the lattice. Then a white cluster having $k$ sites on the cut can be oriented by $k$ different ways because every boundary site can be connected with the root by a path along the tree. Evaluation of $P_b$ is almost as elementary as that for the $O(1)$ case. Indeed, we have to find the probabilities of three configurations of two oriented bonds on the dual white sublattice.

(1a) The bonds from $(-1, 0)$ to $(0, 1)$ and from $(0, 1)$ to $(1, 0)$;
(2a) the bonds from $(1, 0)$ to $(0, 1)$ and from $(0, 1)$ to $(-1, 0)$;
(3a) the bonds from $(-1, 0)$ to $(0, 1)$ and from $(1, 0)$ to $(0, 1)$.

The probability $P_a$ is the sum of these probabilities, $P_a = P_{1a} + P_{2a} + P_{3a}$.

Evaluation of $P_b$ is a more delicate problem. The cluster of type (b) on the black sublattice is surrounded by bonds of a cluster on the white sublattice. The bonds are directed and their sequence gives continuous paths on the white sublattice of three different forms.

(1b) The self-avoiding path of an arbitrary length which starts at the site $(-1, 0)$ and stops at $(1, 0)$.
(2b) The self-avoiding path starts at the site $(1, 0)$ and stops at $(-1, 0)$. 


Figure 8. The upper-left half-plane. Open stars are roots of the spanning trees on the white sublattice.

(3b) Two self-avoiding paths start at the sites \((-1, 0)\) and \((1, 0)\), meet at a white site \(s_0\) and then the single self-avoiding path continues from \(s_0\) up to the end at a white site on the cut. In particular cases, the site \(s_0\) can coincide with sites \((-1, 0)\) and \((1, 0)\).

We denote the probabilities of these situations by \(P_{1b}, P_{2b}, P_{3b}\), and their sum by \(P_b = P_{1b} + P_{2b} + P_{3b} - P_a\).

In a similar way, the probabilities \(P_c\) and \(P_d\) can be defined. Taking into account that \(P_c = P_c(\text{left}) = P_c(\text{right})\) due to the symmetry, we can define \(P_c(\text{left})\) via probability of paths on the white sublattice which separate the origin \((0, 0)\) from the boundary vertices with positive even coordinates \((2n, 0), n \geq 1\). However, we do not need a separate evaluation of \(P_c\) because it can be determined from the relation \(P_d + P_b = P_d\) and condition (2.1).

While the probabilities constituting \(P_a\) are local correlation functions and can be calculated within a usual approach of the perturbed Laplacian, the probabilities in \(P_b\) are essentially non-local and their calculation needs a special technique. Before formulating these technical tools, we chose more convenient coordinates for our problem.

Both black and white sublattices are square lattices with the elementary cell \(\sqrt{2} \times \sqrt{2}\) that are turned by \(\pi/4\) with respect to the basic lattice. We chose the standard orientation for the sublattices and put the step of these square lattices to be equal to 1. In new coordinates, the former cut is the diagonal line crossing the origin. The sites of the black sublattice belonging to the cut have coordinates \((0, 0), (1, 1), (-1, -1), (2, 2), (-2, -2), \ldots\). The coordinates of the white sublattice on the cut are half-integers (see figure 8). To avoid the complication of notations, we shift the origin of the white sublattice to the point \((0, 0)'\) shown in figure 8.

After the transformation, the spanning trees on the sublattices belong to the upper-left half-space with respect to the diagonal cut. To describe the structure of roots of the obtained spanning trees, we use the triangular geometry of the upper-left half of a square exploited above in the numerical simulations of the percolation \(O(0)\)-model. Starting with the square of finite size \(N \times N\) with closed boundary conditions, we put a single root at the right lower
corner of the square. Then, all branches of the one-component spanning tree on the square are oriented toward the root. The diagonal cuts the spanning tree and creates a forest of trees on the upper-left half of a square, each having an individual root. We can put all the roots at the line below the cut (the line crossing the point \((0,0)\) since these roots are connected with the single root of the tree on the whole \(N \times N\) lattice. Tending \(N\) to infinity, we obtain the structure of roots in our problem.

### 3. Laplacian and generalized Kirchhoff theorem

Consider the graph \(G = (V, E)\) with vertex set \(V\) and set of bonds \(E\). The vertices are sites of the square lattice and an additional point which is the root \(*\): \(V \equiv \{s_{x,y}, (x, y) \in \mathbb{Z}^2, |x| \leq M, |y| \leq N\} \cup \{\star\}\). The bonds of \(E\) connect neighboring sites of the lattice. We set the condition that all right and bottom boundary vertices \(\{s_{x,y}\}\) and \(\{s_{-M,y}\}\) are connected with the root \(*\). The graph \(G\) represents the finite square lattice of size \((2N + 1) \times (2M + 1)\). We consider also the left-upper half plane \(V_+ = \{s_{x,y}, x \leq y\}\) with open boundary conditions at the diagonal sites \(V_0 = \{s_{x,y}, x = y\}\).

We construct the spanning tree by using the arrow representation. To this end, we attach to each vertex \(i \in V \setminus \{\star\}\) an arrow directed along one of bonds \((i, i') \in E\) incident to it. Each arrow defines a directed bond \((i \rightarrow i')\) and each configuration of arrows defines a spanning directed graph with set of bonds \(\{(i \rightarrow i') : i \in V \setminus \{\star\}, i' \in V, (i, i') \in E\}\). A sequence of directed bonds \((i_1, i_2), (i_2, i_3), (i_3, i_4), \ldots, (i_{m-1}, i_m)\) is called the path of length \(m\) from the site \(i_1\) to the site \(i_m\). This path forms a loop if \(i_m = i_1\). No arrows are attached to the root \(*\), so that it is a sink of directed paths. The spanning tree is a spanning digraph without any loops. Our aim is to construct spanning trees with a prescribed configuration of paths between fixed vertices. These configurations will be investigated by means of the determinant expansion of the Laplace matrix \([12–14]\).

Let the vertices of the set \(V\) be labeled in arbitrary order from 1 to \(n = |V \setminus \{\star\}| = (2M + 1)(2N + 1)\). Then Laplacian \(\Delta\) of size \(n \times n\) has the elements

\[
\Delta_{ij} = \begin{cases} 
  z_i & \text{if } i = j, \\
  -1 & \text{if } i, j \text{ are nearest neighbors}, \\
  0 & \text{otherwise},
\end{cases} 
\]

(3.1)

where \(z_i\) is the degree of vertex \(i \in V \setminus \{\star\}\). The determinant of \(\Delta\) is a sum over all permutations \(\sigma\) of the set \(\{1, 2, \ldots, n\}\):

\[
\det \Delta = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Delta_{1,\sigma(1)} \Delta_{2,\sigma(2)} \cdots \Delta_{n,\sigma(n)},
\]

(3.2)

where \(S_n\) is the symmetric group and \(\text{sgn}(\sigma) = \pm 1\) is the signature of permutation \(\sigma\). In general, each permutation \(\sigma \in S_n\) can be factorized into a composition of disjoint cyclic permutations, \(c_1, c_2, \ldots, c_k\). This representation partitions the set of vertices \(V \setminus \{\star\}\) into non-empty disjoint subsets which are orbits \(O_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,l_i}\} \subset V\) of the corresponding cycles \(c_i, i = 1, \ldots, k\), at that \(\bigcup_{i=1}^{k} O_i = V \setminus \{\star\}\) and \(\sum_{i=1}^{k} l_i = n\), where \(l_i\) is the length of cycle \(c_i\). Orbits consisting of just one element are the fixed points \(S_{fp}(\sigma)\) of the permutation.

A cycle \(c_i\) of length \(|c_i| = l_i \geq 2\) is called a proper cycle. The proper cycles are of even length only, hence, the number of proper cycles \(p\) defines the signature of the permutation \(\sigma\), that is \(\text{sgn}(\sigma) = (-1)^p\). Thus (3.2) can be written as follows:

\[
\det \Delta = \prod_{i=1}^{n} z_i + \sum_{p=1}^{[n/2]} (-1)^p \sum_{\sigma = \{c_1, \ldots, c_k\}} \prod_{i=1}^{p} \Delta_{c_i(v_i), c_i(v_i)} \cdots \Delta_{c_i^{l_i-1}(v_i), v_i} \prod_{j \in S_{fp}(\sigma)} z_j,
\]

(3.3)
where \( c^k_i \) is the \( k \)-fold composition of the cyclic permutation \( c_i \) of even length \( \{i, v_i \in \mathcal{O}_r(\sigma) \} \), so that \( c_i^{k-1}(v_i) \neq c_i^k(v_i) \) and \( c_i^k(v_i) = v_i \). The term \( \prod_{i=1}^m z_i \) equals to the number of all spanning digraphs having the root \(*\). Each of the other terms on the right-hand side of (3.3) having a non-zero set of fixed points equals up to a sign to \( \prod_{j \in \mathcal{O}_r(\sigma)} z_j \), because all non-diagonal elements equal to \(-1\). That product represents the number of distinct spanning digraphs which have in common the specified cycles \( c_1, \ldots, c_p \), and differ in the oriented edges outgoing from vertices \( j \in \mathcal{O}_r(\sigma) \). The expansion (3.3) can be interpreted in the form of the inclusion–exclusion principle [21]. Let \( c_1, c_2, \ldots, c_m \) be the list of all possible proper cycles. We define \( A_i, i = 1, 2, \ldots, m \) as the set of all spanning digraphs containing the particular cycle \( c_i \) and \( A_0 \) is the set of all spanning digraphs. Let \( A_{ST} \) be the set of spanning trees i.e. the set of spanning digraphs containing no cycles. Then we can write down (3.3) in the form

\[
\det \Delta = |A_{ST}| = |A_0| - \sum_{i=1}^m |A_i| + \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \cdots + (-1)^m |A_1 \cap \cdots \cap A_m|, \tag{3.4}
\]

where \( |A| \) is the cardinality of the set \( A \). Equation (3.4) is the Kirchhoff theorem for the number of spanning trees of a given graph [21].

Now we modify the Laplace matrix changing some non-diagonal elements:

\[
\Delta'_{ij} = \begin{cases} 
z_i & \text{if } i = j, 
-1 & \text{if } i, j \text{ are nearest neighbors}, 
-\varepsilon & \text{if } (i, j) \in B \equiv \{(i_1, j_1), (i_2, j_2), \ldots (i_r, j_r)\}, 
0 & \text{otherwise}, 
\end{cases} \tag{3.5}
\]

where \( B \) is some arbitrarily chosen set of \( r \) directed bonds (bridges) that are not necessarily nearest neighbors on the square lattice. The determinant expansion (3.3) with \( \Delta' \) generalizes the Kirchhoff theorem [13]. The product \( \Delta'_{i_1, j_1} \cdots \Delta'_{i_r, j_r} = (-\varepsilon)^r \) survives in the limit \( \lim_{\varepsilon \to \infty} \det \Delta'/\varepsilon^r \). Permutations corresponding to these terms contain cycles with directed bonds \( B \). If the configuration of bonds \( B \) excludes cycles containing more than one bond from the set \( B \), then the expression \( \lim_{\varepsilon \to \infty} \det \Delta'/(-\varepsilon)^r \) gives the number of configurations with the following features: (i) each configuration is the \((r + 1)\)-component spanning graph; (ii) the \( r \)th component consists of the path from site \( j_k \) to site \( i_k \), \( 1 \leq k \leq r \), and branches of the spanning tree attached to these paths; (iii) the \((r + 1)\)th component is the spanning tree containing the root \(*\). The ratio of numbers of such configurations and all one-component spanning trees is

\[
\lim_{\varepsilon \to \infty} \frac{\det \Delta'}{(-\varepsilon)^r \det \Delta} = \lim_{\varepsilon \to \infty} \frac{\det (I + \delta G)}{(-\varepsilon)^r}, \tag{3.6}
\]

where \( \delta = \Delta' - \Delta \) is the defect matrix and \( G = \Delta^{-1} \) is the Green function on the plane, which has an explicit integral representation in the thermodynamical limit \( M, N \to \infty \) [22] (see the appendix).

If \( B \) includes \( m \) bonds \((i \to i')\) between nearest neighbors on the square lattice, the diagonal elements \( \Delta'_{i,j} \) are changed from \( z_i \) to \( \varepsilon \) and non-diagonal ones \( \Delta'_{i',j} \) from \(-1\) to \(-\varepsilon\). In this case, the limit \( \lim_{\varepsilon \to \infty} \det \Delta'/\varepsilon^m \) enumerates the spanning trees with an obligatory presence of bonds \((i \to i')\).

4. Cluster probabilities

Given the above preparations, we are ready to calculate the probabilities \( P_a = P_{1a} + P_{2a} + P_{3a} \) and \( P_b = P_{1b} + P_{2b} + P_{3b} - P_{ab} \).
4.1. The probability $P_{1a}$

The two-bond configuration corresponding to $P_{1a}$ is shown in figure 9. We denote by $B$ the non-zero submatrix of the defect matrix $\delta$. For the 1a case, matrix $B$ is

$$B((1 \rightarrow 3, 3 \rightarrow 2)) = \begin{pmatrix} \varepsilon & 0 & -\varepsilon \\ 0 & 0 & 0 \\ 0 & -\varepsilon & \varepsilon \end{pmatrix}. \quad (4.1)$$

The probability $P_{1a}$ follows from (3.6)

$$P_{1a} = \lim_{\varepsilon \to \infty} \frac{\det(I + BG^{op})}{(-\varepsilon)^2}. \quad (4.2)$$

where we changed $\delta$ by its non-zero finite submatrix $B$ and the bulk Green function $G$ by the Green function $G^{op}$ for the open diagonal boundary conditions

$$G_{(p_1,q_1),(p_2,q_2)}^{op} = G_{p_2-p_1,q_2-q_1} - G_{p_2-q_1,q_2-p_1}, \quad (4.3)$$

where $(p_1, q_1)$ and $(p_2, q_2)$ are points on the white sublattice with the primed coordinates (see figure 8). This Green function is obtained as a solution of the discrete Poisson equation

$$\Delta G^{op} = I. \quad (4.4)$$

on the upper-left half-plane with the additional condition that the Green function vanishes on the diagonal line $p = q$ (open boundary). The boundary Green function (4.3) can be easily obtained due the principle of reflection symmetry [8].

Using exact values (A.6) for the Green functions at short distances given in the appendix, we obtain

$$P_{1a} = \frac{16\pi - 16 - 3\pi^2}{6\pi^2}. \quad (4.5)$$

For the case 3a we have

$$B((1 \rightarrow 3, 2 \rightarrow 3)) = \begin{pmatrix} \varepsilon & 0 & -\varepsilon \\ 0 & \varepsilon & -\varepsilon \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.6)$$
and
\[
P_{3a} = \frac{2(7\pi - 12 - \pi^2)}{\pi^2}. \tag{4.7}
\]
The resulting probability \( P_a \) is
\[
P_a = P_{3a} + 2P_{1a} = \frac{(4 - \pi)(9\pi - 22)}{3\pi^2} = 0.181903 \ldots \tag{4.8}
\]
instead of \( P_a = 1/4 \) for the dense \( O(1) \) loop model.

4.2. The probability \( P_b \)

By definition, the probability \( P_{1b} \) is the probability that the path in a spanning tree starting at site \((-1, 0)\) of the original white sublattice stops at site \((1, 0)\) and then goes to one of two neighboring roots.

The configuration of this path on the rotated sublattice is shown in figure 10, where 1 on the original (non-rotated) lattice is the site \((-1, 0)\) and 2 is the site \((1, 0)\). On the new lattice their coordinates are correspondingly \((-1, 0)\) and \((0, 1)\). Two bold arrows denote two possible connections of site 2 with the roots. To generate the path from 1 to 2, we introduce the bridge \((2 \to 1)\) with the weight \(-\epsilon\) (see figure 10) and tend \(\epsilon\) to infinity. Then the defect matrix \( B \) consists of the single element \((2 \to 1)\) and the formula (3.6) is reduced to
\[
P_{1b} = \lim_{\epsilon \to \infty} \frac{\det(I + BG^{op})}{-\epsilon} = 2(G_{1,1} - G_{2,0}). \tag{4.9}
\]
Using the table (A.6) we obtain
\[
P_{1b} = \frac{2(\pi - 3)}{\pi}. \tag{4.10}
\]
By the symmetry, we have \( P_{1b} = P_{2b} \).

The crucial point of our calculations is the probability \( P_{3b} \). The configuration of paths on the white sublattice corresponding to the \(3b\) case after the transformation obtains the form depicted in figure 11, where point 1 is the transformed position of the site \((-1, 0)\), point 2 is that of the site \((1, 0)\) and point 4 is the end of a path from the connection point \(s_0\).

In principle, it is possible to define a system of auxiliary bridges giving all three paths on figure 11; however, in this case we would obtain a product of three Green functions which should be summed up over all positions of the point \(s_0\). We choose a different method here, avoiding these tremendous calculations.
First, we define the path from point 1 to point 4 inserting the auxiliary bond ($4 \rightarrow 1$). Then, we fix an outgoing bond from point 2 in one of two possible directions (diagram I shown in figure 12). The path from point 2 can join the path from point 1 to 4, thereby forming a configuration of the type 3b. Otherwise, the path from 2 can finish at one of the white sites on the boundary between points 2 and 4. The latter case, which is shown in diagram II (figure 13)
must be excluded from all configurations in diagram I. Thus, instead of summation over two-dimensional positions of the connection point $s_0$, we obtain two one-dimensional summations over positions of points 3 and 4, and reduce by 1 the number of Green functions involved in the calculations.

Consider now the Green function $G_{(p,q),(k,k+1)}$ between an arbitrary site $(p, q), q > p$ on the left-upper half part of the square lattice and a site $(k, k + 1)$ on the diagonal boundary. The summation over the whole boundary gives

$$
\sum_{k=-\infty}^{+\infty} G_{(p,q),(k,k+1)} = \sum_{k=-\infty}^{+\infty} (G_{k-p, k+1-q} - G_{k-q, k+1-p})
$$

(4.11)

$$
= \frac{1}{8\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} \frac{e^{i(k\alpha + \beta)} e^{i\beta} (e^{-i(qx + q\beta)} - e^{-i(qx + p\beta)})}{2 - \cos \alpha - \cos \beta} d\alpha d\beta
$$

$$
= \frac{1}{4\pi} \int_{-\pi}^{\pi} i e^{-i\alpha} \sin[(q - p)\alpha] \frac{d\alpha}{1 - \cos \alpha}
$$

(4.12)

With $p = 0$ and $q = 1$ we have

$$
D_k \equiv G_{(0,1),(k,k+1)} = G_{k,k} - G_{k-1,k+1}
$$

(4.13)

and

$$
\sum_{k=-\infty}^{+\infty} D_k = \frac{1}{2}
$$

(4.14)

Due to the symmetry relation

$$
D_{-k} = D_k
$$

(4.15)
we can find the sum
\[
\sum_{k=1}^{+\infty} D_k = \frac{1}{4} - \frac{1}{2} (G_{0,0} - G_{1,1}) = \frac{1}{4} - \frac{1}{2\pi}.
\] (4.16)

We denote by \( R_k \) the probability of diagram I for distance \( k \) between points 1 and 4 (see figure 12). The defect matrices \( B \) containing the auxiliary bond between points 1 and 4, and selected outgoing bonds from point 2 are
\[
B = \begin{pmatrix}
0 & 0 & 0 & -\varepsilon & 0 & 0 \\
0 & -2\varepsilon & 0 & 0 & -\varepsilon & -\varepsilon \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\] (4.17)

where the 5th and 6th rows and columns correspond to the two neighboring vertices of point 2. The calculation of the determinant gives
\[
R_k = \frac{4(\pi - 3)}{\pi} D_{k-1} + \frac{2(\pi - 2)}{\pi} D_k.
\] (4.18)

The probability \( P_I \) is the sum
\[
P_I = \sum_{k=2}^{+\infty} R_k.
\] (4.19)

From the identity
\[
\sum_{k=-\infty}^{+\infty} R_k = 2 \sum_{k=2}^{+\infty} R_k + \frac{4(\pi - 3)}{\pi} D_0 + \frac{2(\pi - 2)}{\pi} (D_{-1} + D_0 + D_1)
\] (4.20)

we have
\[
P_I = \frac{6\pi - 8 - \pi^2}{2\pi^2}.
\] (4.21)

The probability \( P_{II} \) of diagram II (see figure 13) is a double sum of double bridges with distance \( s \) between points 1 and 4 and distance \( k \) between points 1 and 3. The corresponding \( B \) matrix reads
\[
B = \begin{pmatrix}
0 & 0 & 0 & -\varepsilon \\
0 & 0 & -\varepsilon & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\] (4.22)

which gives the probability
\[
P_{II} = 4 \lim_{\varepsilon \to \infty} \frac{\det(I + BG^{\text{op}})}{\varepsilon^2} = 4 \sum_{2 \leq k < s} (D_{k-1} D_s - D_k D_{s-1})
\] (4.23)

\[
= 4 \sum_{2 \leq k \leq s} D_k (D_{k-1} - D_k) + 4 \sum_{k=2}^{+\infty} \sum_{s=k}^{+\infty} D_k (D_s - D_{s-1})
\] (4.24)

\[
= 4 \sum_{s=2}^{+\infty} D_s (D_1 - D_s) - 4 \sum_{k=2}^{+\infty} D_k D_{k-1}
\] (4.25)

\[
= 4D_1 \sum_{s=2}^{+\infty} D_s - 4 \sum_{k=2}^{+\infty} D_k (D_k + D_{k-1}).
\] (4.26)
or, in a more convenient form,
\[ P_{II} = 4D_1 \sum_{k=1}^{+\infty} D_k - 4 \sum_{k=1}^{+\infty} D_k(D_k + D_{k+1}). \] (4.27)

One can show that
\[ \sum_{k=-\infty}^{+\infty} D_k(D_k + D_{k+1}) = 2 \sum_{k=1}^{+\infty} D_k(D_k + D_{k+1}) + D_0(D_0 + 2D_1) \] (4.28)
so
\[ P_{II} = 4D_1 \sum_{k=1}^{+\infty} D_k + 2D_0(D_0 + 2D_1) - 2 \sum_{k=-\infty}^{+\infty} D_k(D_k + D_{k+1}). \] (4.29)

We denote the latter sum by \( Q \):
\[ Q = \sum_{k=-\infty}^{+\infty} D_k(D_k + D_{k+1}). \] (4.30)

Using the integral representation of the Green function and definition of \( D_k \), after some manipulations we can show that
\[ Q = \frac{1}{16\pi^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{(\sin \alpha_1 - \sin \beta_1)(\sin \alpha_1 - \sin(2\alpha_1 + 2\alpha_2 + \beta_1))}{(2 - \cos \alpha_1 - \cos \beta_1)(2 - \cos \alpha_2 - \cos(\alpha_1 + \beta_1 + \alpha_2))} \, d\alpha_1 \, d\beta_1 \, d\alpha_2. \] (4.31)

By the substitution
\[ t_1 = \tan \frac{\alpha_1}{2}, \quad t_2 = \tan \frac{\beta_1}{2}, \quad t_3 = \tan \frac{\alpha_2}{2}, \] (4.32)
\[ \sin \alpha_1 = \frac{2t_1}{1 + t_1^2}, \quad \sin \beta_1 = \frac{2t_2}{1 + t_2^2}, \quad \sin \alpha_2 = \frac{2t_3}{1 + t_3^2}, \] (4.33)
\[ \cos \alpha_1 = \frac{1 - t_1^2}{1 + t_1^2}, \quad \cos \beta_1 = \frac{1 - t_2^2}{1 + t_2^2}, \quad \cos \alpha_2 = \frac{1 - t_3^2}{1 + t_3^2}, \] (4.34)
\[ d\alpha_1 \, d\beta_1 \, d\alpha_2 = \frac{2}{1 + t_1^2} \frac{2}{1 + t_2^2} \frac{2}{1 + t_3^2} \, dt_1 \, dt_2 \, dt_3 \] (4.35)
and symmetrization by permutation \( t_1 \leftrightarrow t_2 \), we come to the triple integral of a rational function
\[ Q = \frac{1}{\pi^3} \int_{-\infty}^{\infty} T_1(t_1,t_2,t_3)T_2(t_1,t_2,t_3) \, dt_1 \, dt_2 \, dt_3, \] (4.36)

where
\[ T_1(t_1,t_2,t_3) = \frac{(t_1 - t_2)^2(t_1t_2 - 1)(t_1t_2 + t_1t_3 + t_2t_3 - 1)(t_1t_2t_3 - t_1 - t_2 - t_3)}{(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)(t_1^2 + t_2^2 + 2t_1t_2t_3)}, \] (4.37)
\[ T_2(t_1,t_2,t_3) = \frac{2t_1t_2t_3 - 2t_3 + (t_1 + t_2)(t_1^2 - 1)}{(1 + t_1^2 + t_2^2 + t_3^2 - 2t_1t_2t_3 + (1 + t_1^2 + t_2^2)t_3^2)}. \] (4.38)

After integration over \( t_3 \) and several manipulations we have
\[ Q = \frac{2}{\pi^3} \int_{0}^{+\infty} \int_{-t_1}^{+t_1} \frac{(t_1 - t_2)^2 \left(1 + t_1^2 + t_2^2 + t_1t_2t_3 - (t_1 + t_2)\sqrt{(1 + t_1^2)(1 + t_2^2)}\right)}{(1 + t_1^2)(1 + t_2^2)(t_1^2 + t_2^2 + 2t_1t_2t_3)} \, dr_2 \, dr_1. \] (4.39)
The integration over $t_2$ gives
\begin{equation}
Q = \frac{1}{\pi^2} \int_0^{+\infty} \left( \frac{8t_1^3 \arctan \left( \frac{1 + 2t_2^2}{1 + 2t_1^2} \right)}{(1 + t_1^2)^2 \sqrt{1 + 2t_2^2}} + \frac{4(1 - t_1^2) \arctan(t_1)}{(1 + t_1^2)^2} - \frac{2(\pi - 2)t_1^2}{(1 + t_1^2)^2} \right) \, dt_1,
\end{equation}
from which we obtain
\begin{equation}
Q = \frac{4 - \pi}{2\pi}.
\end{equation}
As a result, we have
\begin{equation}
P_{\Pi} = \frac{2\pi^2 - 5\pi - 4}{\pi^2},
\end{equation}
and
\begin{equation}
P_{3b} = 2 \times P_1 - 2 \times P_{\Pi} = \frac{16 - 5\pi}{\pi}.
\end{equation}
The cumulated result is
\begin{equation}
P_{1b} + P_{2b} + P_{3b} = \frac{4 - \pi}{\pi} = 0.27323\ldots.
\end{equation}
and we obtain the probability of clusters $P_b$ in the dense $O(0)$ loop model:
\begin{equation}
P_b = P_{1b} + P_{2b} + P_{3b} - P_a = \frac{2(4 - \pi)(11 - 3\pi)}{3\pi^2} = 0.091336465\ldots
\end{equation}
The probabilities $P_c$ and $P_d$ can be easily determined from (2.1) and the condition $P_a + P_b = P_d$.

Using the one-to-one correspondence between spanning trees and loop erased random walk (the Wilson algorithm, see [23–27] for details), we can generate equally distributed spanning trees and check the obtained analytical results. Figure 14 with number of samples $10^8$ shows that the numerical value coincides with the analytical result (4.44) with high precision.

The calculation of the minimal link probability for $n = 0$ is the first step in a program of investigations of more general link correlation functions. If realized, this program would help to establish a correspondence between lattice calculations and operators of the logarithmic conformal field theory (LCFT) with the central charge $c = -2$. Then, the operators of the LCFT corresponding to link variables in the dense $O(1)$ loop model with the central charge $c = 0$ can be constructed by an analogy with the $O(0)$ model.
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Appendix. Green functions

The explicit form of the translationally invariant Green function on the plane is [22]

\[ G_{(p,q_1),(p,q_2)} \equiv G(r_2 - r_1) \equiv G_{0,0} + g_{p,q}, \quad r_2 - r_1 \equiv \vec{r} \equiv (p, q) \]  

(A.1)

with \( G_{0,0} \) an irrelevant infinite constant. The (finite) numbers \( g_{p,q} \) are given explicitly by

\[ g_{p,q} = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} \frac{e^{ipr+iq\beta} - 1}{2 - \cos \alpha - \cos \beta} \, d\beta. \]  

(A.2)

Let us mention the symmetry properties of this function

\[ g_{p,q} = g_{q,p} = g_{-p,q} = g_{p,-q}. \]  

(A.3)

After the integration over \( \alpha \), it can be expressed in a more convenient form for actual calculations,

\[ g_{p,q} = \frac{1}{4\pi} \int_{-\pi}^{\pi} t^r \frac{e^{iq\beta} - 1}{\sqrt{r^2 - 1}} \, d\beta, \]  

(A.4)

where \( t = y - \sqrt{y^2 - 1}, y = 2 - \cos \beta \) (see [14]). For \( r^2 = p^2 + q^2 \gg 1 \), it has a behavior [19]

\[ g_{p,q} = -\frac{1}{2\pi} \left( \log r + \gamma + \frac{3}{2} \log 2 \right) + \frac{\cos(4\varphi)}{24\pi r^2} + \frac{18\cos(4\varphi) + 25\cos(8\varphi)}{480\pi r^4} + \cdots, \]  

(A.5)

where \((p, q) = (r \cos \varphi, r \sin \varphi)\), and with \( \gamma = 0.57721\ldots \) the Euler constant. Let us also write down some values of the Green functions for particular points \((p, q)\), which are needed for calculations:

\[ g_{0,1} = -\frac{1}{4}, \quad g_{0,2} = -1 + \frac{2}{\pi}, \quad g_{0,3} = -\frac{17}{4} + \frac{12}{\pi}, \]

\[ g_{1,1} = -\frac{1}{\pi}, \quad g_{1,2} = \frac{1}{4} - \frac{2}{\pi}, \quad g_{1,3} = 2 - \frac{23}{3\pi}, \]

\[ g_{2,2} = -\frac{4}{3\pi}, \quad g_{2,3} = -\frac{2}{3\pi} - \frac{23}{15\pi}. \]  

(A.6)

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