GERMS OF DE RHAM COHOMOLOGY CLASSES WHICH VANISH AT THE GENERIC POINT

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Abstract. We show that hypergeometric differential equations, unitary and Gauß-Manin connections give rise to de Rham cohomology sheaves which do not admit a Bloch-Ogus resolution \cite{1}. The latter is in contrast to Panin’s theorem \cite{8} asserting that corresponding étale cohomology sheaves do fulfill Bloch-Ogus theory.

Germes de classes de cohomologie de de Rham qui s’annulent au point générique

Résumé. Nous montrons que les systèmes d’équations hypergéométriques, les connexions unitaires et de Gauß-Manin donnent lieu à des faisceaux de cohomologie de de Rham qui n’ont pas de résolution de Bloch-Ogus \cite{1}. Ce dernier exemple contraste avec le théorème de Panin \cite{8} affirmant que des faisceaux semblables en cohomologie étale vérifient la théorie de Bloch-Ogus.

Version abrégée en français. Soit \( (E, \nabla) \) une connexion plate sur une variété lisse \( S \) sur un corps \( k \) algébriquement clos en caractère 0. La restriction des faisceaux de cohomologie \( \mathcal{H}^i_{\text{DR}}(E, \nabla) \) à leur valeur au point générique de \( S \) est trivialement injective pour \( i = 0, 1 \). Afin de montrer que pour les exemples de \( (E, \nabla) \) évoqués plus haut, cela n’est plus nécessairement le cas pour \( i = 2 \), nous forçons l’existence de germes de sections de la façon suivante (voir \cite{5}). On remplace \( S \) par l’éclatement d’un point. Cela introduit un diviseur exceptionnel sur lequel la connexion est triviale, et donc acquiert des sections. Par le morphisme de Gysin, ces sections fournissent des sections non nulles dans \( H^2_{\text{DR}}(S, (E, \nabla)) \) qui en particulier s’annulent au point générique de \( S \). Que ces sections ne s’annulent pas dans le germe du faisceau en un point du diviseur exceptionnel provient d’une hypothèse convenable de résidus dans le cas hypergéométrique, de généricité dans le cas unitaire, et de la théorie de Hodge dans le cas de Gauß-Manin pour une famille à forte variation.

Let \( S \) be a smooth algebraic variety defined over an algebraically closed field \( k \). Bloch-Ogus theory \cite{1} provides a acyclic resolution of the Zariski sheaves of étale cohomology \( \mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n(j)) \) if \( \text{char } k = 0 \) or if \( (\text{char } k, n) = 1 \), of de Rham cohomology \( \mathcal{H}^i_{\text{DR}} \) if \( \text{char } k = 0 \), and of Betti cohomology \( \mathcal{H}^i_{\text{B}} \) if \( k = \mathbb{C} \). Here \( \mathcal{H}^i \) denotes the Zariski sheaf associated to the presheaf \( U \mapsto H^i(U) \). The first level of the resolution says that the restriction map to the generic point \( i_\eta : \eta = \text{Spec } k(S) \to S \)

\[
\mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n(j)) \to i_\eta^*H^i_{\text{ét}}(\eta, \mathbb{Z}/n(j))
\]

is injective (and similarly for de Rham and Betti cohomology).

Bloch-Ogus theory extends in an obvious way to the sheaves of cohomology \( \mathcal{H}^i(L) \) with values in a local system of complex vector spaces \( L \) of finite monodromy if \( k = \mathbb{C} \), or equivalently \cite{5} to the de Rham cohomology sheaves \( \mathcal{H}^i_{\text{DR}}((E, \nabla)) \) of a locally free sheaf \( E \)

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with a flat connection $\nabla$, the monodromy of which is finite with respect to one embedding $k \subset \mathbb{C}$ (and hence to all).

A remarkable generalization of the Bloch-Ogus theory for étale cohomology has been given by I. Panin \[8\]. Let $f : X \to S$ be a projective smooth morphism and let $L$ be a local system of free $\mathbb{Z}/n$-modules of finite rank (where $n$ is prime to char $k$ if char $k > 0$). Then the Zariski sheaves $\mathcal{H}^i_{\text{ét}}(f, L(j))$ associated to the presheaves

$$U \mapsto H^i_{\text{ét}}(f^{-1}(U), L(j))$$

have a Bloch-Ogus acyclic resolution on $S$. In particular the restriction to the generic point

$$\mathcal{H}^i_{\text{ét}}(f, L(j)) \to i_\eta^* H^i_{\text{ét}}(f^{-1}(\eta), L(j))$$

is injective, as in the classical case "$f = \text{identity}$ and $L = \mathbb{Z}/n"$. Panin’s proof strongly relies on the finiteness of the local systems involved.

This raises the question of a similar theorem for the de Rham cohomology in characteristic zero. In this note we give negative examples:

0.1. Bundles $E$ with a flat connection $\nabla$ for which

\[(0.1) \quad \mathcal{H}^2_{\text{DR}}((E, \nabla)) \to i_\eta^* H^2_{\text{DR}}(\eta, (E, \nabla))\]

is not injective, or equivalently (over $\mathbb{C}$), local systems $L$ of complex vector spaces for which $\mathcal{H}^2(L) \to i_\eta^* H^2(\eta, L)$ is not injective (see \[2.1\] and \[3.1\]).

0.2. Smooth projective morphisms $f : X \to S$ for which

\[(0.2) \quad \mathcal{H}^4_{\text{DR}}(f) \to i_\eta^* H^4_{\text{DR}}(f^{-1}(\eta))\]

is not injective (see \[1.3\] and \[1.4\]). Here $\mathcal{H}^i_{\text{DR}}(f)$ denotes the Zariski sheaf associated to $U \mapsto H^i_{\text{DR}}(f^{-1}(U))$. Or equivalently, over $\mathbb{C}$,

$$\mathcal{H}^1_B(f) \to i_\eta^* H^1_B(f^{-1}(\eta)) = i_\eta^* \lim_{U \subset S} H^1_B(f^{-1}(U))$$

is not injective, with a similar notation for Betti cohomology.

Deligne’s theorem \[3\], saying that $R^i f_* \mathbb{C} = \bigoplus j R^j f_* \mathbb{C}[-j]$ over $k = \mathbb{C}$, together with \[6\] imply that the injectivity of

$$\mathcal{H}^i_{\text{DR}}(f) \to i_\eta^* H^i_{\text{DR}}(f^{-1}(\eta))$$

is equivalent to the injectivity

$$\mathcal{H}^{i-j}_{\text{DR}}((R^j f_* \Omega^*_X/S, \nabla)) \to i_\eta^* H^{i-j}_{\text{DR}}((\eta, R^j f_* \Omega^*_X/S, \nabla))$$

for all $j$, where $\nabla$ is the Gauß-Manin connection. Thus in the second example we will verify that this morphism is not injective, for $i - j = j = 2$.

In all examples given the method to construct a germ of a de Rham section is as follows.

0.3. Construction of a section. Let $(E', \nabla')$ be defined over a smooth variety $S'$ of dimension at least two, and let $\delta : S \to S'$ be the blow up of a point $p \in S'$, with exceptional divisor $F$. Let $(E, \nabla) = \delta^*(E', \nabla')$ be the pullback connection. Then the restriction map

$$H^1_{\text{DR}}(S, (E, \nabla)) = H^1_{\text{DR}}(S', (E', \nabla')) \to H^1_{\text{DR}}(S' - p, (E', \nabla')) = H^1_{\text{DR}}(S - F, (E, \nabla))$$

is an isomorphism. Hence the Gysin map

$$i_F : H^0_{\text{DR}, F}(S, (E, \nabla)) = H^0_{\text{DR}}(F, (E, \nabla))|_F = k^{\text{rank} E} \to H^2_{\text{DR}}(S, (E, \nabla))$$

is injective, and any section $i_F(\sigma), \sigma \in k^{\text{rank} E}$, vanishes at the generic point $\eta$ of $S$. 
To show that the maps (0.1) and (0.2) need not be injective, we will show that for certain $\sigma$ the image $i_\sigma(\sigma)$ is non-zero in the stalk $\mathcal{H}_{\text{DR}}^q((E, \nabla))$ for all $q \in F$. The latter is equivalent to saying that for any divisor $C \subset S$, with $F \not\subset C$

\begin{equation}
(0.3) \quad i_\sigma(\sigma) \not\in i_C\mathcal{H}_{\text{DR},C}(S, (E, \nabla)).
\end{equation}

For any smooth dense open subscheme $C_0$ of $C$ and for $S_0 = S - (C - C_0)$ one has

\begin{equation}
(0.4) \quad \mathcal{H}_{\text{DR}}^2(S_0, (E, \nabla)) = \mathcal{H}_{\text{DR},C_0}^2(S_0, (E, \nabla)) = \mathcal{H}_{\text{DR}}^0(C_0, (E, \nabla)|_{C_0}) = \mathcal{H}_{\text{DR}}^0(\tilde{C}, \nu^*(E, \nabla))
\end{equation}

where $\nu : \tilde{C} \to S$ is the normalization of $C \subset S$. One way to think of this is analytically. Let $L$ be the kernel of $\nabla$, let $\lambda : S_0 \to S$, $j_0 : S_0 - C_0 \to S_0$ and $j = \lambda \circ j_0$. Then

\begin{equation}
\mathcal{H}_{\text{C}}^2(S, L) = \mathcal{H}_{\text{DR}}^0(S, R^1j_*L) \quad \text{and}
\end{equation}

\begin{equation}
\mathcal{H}_{\text{C}}^2(S_0, L) = \mathcal{H}_{\text{DR}}^0(S_0, \lambda_*R^1j_{0*}L) = \mathcal{H}_{\text{DR}}^0(S_0, R^1j_{0*}L) = \mathcal{H}_{\text{DR}}^0(C_0, L|_{C_0}).
\end{equation}

As $S - S_0$ has codimension $\geq 2$ in $S$, $R^i\lambda_*L = 0$ for $i = 1, 2$. Thus by the Leray spectral sequence for $j = \lambda \circ j_0$ the restriction map $R^1j_*L \to \lambda_*R^1j_{0*}L$ is an isomorphism. Since $\mathcal{H}_{\text{DR}}^0(C_0, L|_{C_0}) = \mathcal{H}_{\text{DR}}^0(C, \nu^*L)$ this concludes the proof of (0.4). In other words, we do as if $C$ was smooth.

This way to force geometrically the existence of sections which have nothing to do with the connection was used by the first author in [3] for example [2,1]. We thank I. Panin for explaining us the proof of his manuscript [3].

1. Gauss–Manin systems

Let $\varphi : Y \to B$ be a semi-stable family of curves of genus $g \geq 1$ with $B$ a smooth projective curve and $Y$ a smooth projective surface, defined over an algebraically closed field $k$ of characteristic zero. We assume throughout this section

**Assumption 1.1.** $\varphi_*\omega_{Y/B}$ is ample.

Let $B_0$ be the open subscheme of $B$ with $Y_0 = \varphi^{-1}(B_0)$ smooth over $B_0$, and as in (0.3) let $\delta : S \to S' = B_0 \times B_0$ be the blow up of a point $p = (b_1, b_2) \in B_0 \times B_0$ with exceptional divisor $F$. Let

\begin{equation}
\begin{split}
f : X = (Y_0 \times Y_0) \times_{(B_0 \times B_0)} S &\longrightarrow S
\end{split}
\end{equation}

be the pullback family. We consider the Gauss–Manin connection $(R^2f_*\Omega^\bullet_{X/S}, \nabla)$. On the de Rham cohomology

\begin{equation}
\mathcal{H}_{\text{DR}}^0(F, (R^2f_*\Omega^\bullet_{X/S}, \nabla)|_F) = \mathcal{H}_{\text{DR}}^2(Y_{b_1} \times Y_{b_2})
\end{equation}

one has the $F$-filtration $F^0 \supset F^1 \supset F^2$ which defines a pure Hodge-structure after base extension from $k$ to $\mathbb{C}$.

**Claim 1.2.** For $\sigma \in \{F^0 - F^1\} \mathcal{H}_{\text{DR}}^2(Y_{b_1} \times Y_{b_2})$ and for all $q \in F$ the image $i_\sigma(\sigma)$ is non-zero in the stalk $\mathcal{H}_{\text{DR}}^2((R^2f_*\Omega^\bullet_{X/S}, \nabla))_q$.

**Proof.** Let $C \subset S$ be a reduced curve with $F \not\subset C$, let $\nu : \tilde{C} \to S$ be the normalization of $C$, let $\delta : \tilde{S} \to B \times B$ be the blow up of $p$, and let $n : \Gamma \to \tilde{S}$ be the normalization of the closure of $C$ in $\tilde{S}$. Let us denote by

\begin{equation}
\begin{split}
\tilde{h} : X_{\Gamma} = (Y \times Y) \times_{(B \times B)} \Gamma &\longrightarrow \Gamma, \\
h : X_{\tilde{C}} = (Y_0 \times Y_0) \times_{(B_0 \times B_0)} \tilde{C} &\longrightarrow \tilde{C}, \\
X_F = X \times_S F = \mathbb{P}^1 \times Y_{b_1} \times Y_{b_2} &\longrightarrow \mathbb{P}^1
\end{split}
\end{equation}

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the induced families of surfaces. The Gysin map \( i_C : H^2_{\text{DR}}(X_C) \to H^4_{\text{DR}}(X) \), followed by
the restriction map
\[
\rho_F : H^4_{\text{DR}}(X) \longrightarrow H^4_{\text{DR}}(X_F) = H^4_{\text{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2})
\]
equals the restriction map
\[
\rho_{\nu^{-1}(F)} : H^2_{\text{DR}}(X_C) \longrightarrow \bigoplus_{c \in \nu^{-1}(F)} H^2_{\text{DR}}(X_C \times C) = \bigoplus_{c \in \nu^{-1}(F)} H^2_{\text{DR}}((\nu(c)) \times Y_{b_1} \times Y_{b_2})
\]
followed by the sum of the Gysin maps
\[
\bigoplus_{c \in \nu^{-1}(F)} H^2_{\text{DR}}((\nu(c)) \times Y_{b_1} \times Y_{b_2}) \longrightarrow H^4_{\text{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2}).
\]
On the other hand, \( \rho_{\nu^{-1}(F)} \) factors through the surjective map
\[
\rho : H^2_{\text{DR}}(X_C) \longrightarrow H^0_{\text{DR}}(\tilde{C}, (R^2 h_* \Omega^*_X/\tilde{C}, \nabla)) = H^2_{\text{DR}, C}(S, (R^2 f_* \Omega^*_{X/S}, \nabla)),
\]
and \( \rho_F \circ i_C = \rho_F' \circ i_C' \circ \rho \) where \( \rho_F' \) is the restriction map
\[
\rho_F' : H^2_{\text{DR}}(S, (R^2 f_* \Omega^*_{X/S}, \nabla)) \longrightarrow H^2_{\text{DR}}(F, (R^2 f_* \Omega^*_{X/S}, \nabla)|_F) = H^4_{\text{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2}),
\]
and
\[
i_C : H^2_{\text{DR}, C}(S, (R^2 f_* \Omega^*_{X/S}, \nabla)) \longrightarrow H^2_{\text{DR}}(S, (R^2 f_* \Omega^*_{X/S}, \nabla))
\]
as in \( \text{(1.3)} \). By definition, \( \tilde{h} \) is a semi-stable family of surfaces, with singular fibres \( Z = \tilde{h}^{-1}(\infty) \) for \( \infty = n^{-1}(\tilde{S} - S) = \Gamma - \tilde{C} \). Hence \( R^2 \tilde{h}_* \Omega^*_{X/T}(\log Z) \) is the Deligne extension \( \text{[2]} \) of its restriction to \( \tilde{C} \). Thus
\[
H^0_{\text{DR}}(\tilde{C}, (R^2 h_* \Omega^*_X/\tilde{C}, \nabla)) = H^0(\Gamma, \Omega^*_X(\log \infty) \otimes R^2 \tilde{h}_* \Omega^*_{X/T}(\log Z)),
\]
and
\[
\rho(\{F^0/F^1\} H^2_{\text{DR}}(X_C)) = \rho(H^2(X_C, O_{X_C}) \subset H^0(\Gamma, R^2 \tilde{h}_* O_{X_T}).
\]
The sheaf
\[
R^2 \tilde{h}_* O_{X_T} = (\delta \circ n)^* (pr^* R^1 \varphi_* O_Y \otimes pr^* R^1 \varphi_* O_Y)
\]
is dual to \( (\delta \circ n)^* (pr^* \varphi_* \omega_{Y/B} \otimes pr^* \varphi_* \omega_{Y/B}) \). Since \( \delta \circ n \) is finite, the latter is ample and
\( H^0(\Gamma, R^2 \tilde{h}_* O_{X_T}) = 0 \). Thus, since the restriction and Gysin maps are morphisms of mixed
Hodge structures \( \text{[4]} \),
\[
(\text{1.1}) \quad \text{im}(\rho_F \circ i_C) = \text{im}(\rho_F' \circ i_C') \subset F^3 H^4_{\text{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2}).
\]
On the other hand, \( \rho_F' \circ i_F \) is the multiplication by \((-1) = \text{deg}(O_F(F))\). Hence for
\( \sigma \in \{F^0 - F^1\} H^2_{\text{DR}}(Y_{b_1} \times Y_{b_2}) \) one has
\[
(\text{1.2}) \quad \rho_F' \circ i_F(\sigma) \subset \{F^2 - F^3\} H^4_{\text{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2}).
\]
(\text{1.1}) and (\text{1.2}) imply \( i_F(\sigma) \notin i_C H^2_{\text{DR}, C}(S, (E, \nabla)) \), and, as explained in \( \text{[3]} \), this proves the
claim \( \text{[2]} \). \( \square \)

Of course there are lots of families of semi-stable curves \( \varphi : Y \to B \) which satisfy the
assumption \( \text{[1]} \). The most elementary one is:

**Example 1.3.** Choose \( \varphi : Y \to B \) to be a semi-stable non-isotrivial family of elliptic
curves.

In fact, some power of the sheaf \( \varphi_* \omega_{Y/B} \) is the pullback of an ample invertible sheaf on
the moduli space \( \tilde{M}_1 = \mathbb{P}^1 \) of stable elliptic curves.
Example 1.4. There exist smooth families of curves \( \varphi : Y \to B \) of genus \( g \geq 3 \) over a projective curve \( B \) with \( \varphi_*\omega_{Y/B} \) ample, and hence there exist smooth families of surfaces \( f : X \to S \) with \( S \) projective, for which the map (1.2) is not injective.

Proof. Let \( M_{g,3} \) and \( A_g \) be the moduli spaces of curves of genus \( g \) with level 3 structure and of \( g \)-dimensional principally polarized abelian varieties, respectively. For \( g \geq 3 \) the image of \( M_{g,3} \) in the Baily-Borel compactification of \( A_g \) is a projective manifold whose boundary has codimension larger than or equal to two. Hence \( M_{g,3} \) has a projective compactification with the same property. Taking hyperplane intersections one obtains a smooth projective curve \( B \) in \( M_{g,3} \), and thereby a smooth family of curves \( \varphi : Y \to B \).

In order to show that for \( B \) in general position, \( \varphi_*\omega_{Y/B} \) is ample, we may assume that \( k = \mathbb{C} \).

The monodromy representation of the fundamental group of \( M_{g,3} \) is irreducible, and for \( B \) in general position the fundamental group of \( B \) maps surjectively to the one of \( M_{g,3} \).

On the other hand, by [7], 4.10, the sheaf \( \varphi_*\omega_{Y/B} \) is the direct sum of an ample vector bundle and a vector bundle, flat with respect to the Gauß-Manin connection. The irreducibility of the monodromy representation implies that the latter is trivial. \( \square \)

2. Hypergeometric equations

As in (1.3) let \( S' = \mathbb{P}^2 - D \), where \( D \) is the union of three lines \( H_1, H_2 \) and \( H_3 \) in general position.

Example 2.1. Choose \( a_1, a_2, a_3 \in \mathbb{C} \) such that

i) the elements \( 1, a_i, a_j \in \mathbb{C} \) are \( \mathbb{Q} \)-linearly independent, for \( 1 \leq i < j \leq 3 \),
ii) \( a_1 + a_2 + a_3 = 0; \)

for example, \( a_1 = \sqrt{2}, a_2 = \sqrt{3} \).

Let \( \omega \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(\log D)) \) be the unique form with \( \text{res}_{H_i} \omega = a_i \) for \( i = 1, 2, 3 \), and \( (E', \nabla') = (\mathcal{O}_{S'}, d + \omega) \). As in (0.3) consider the blow up \( \delta : \bar{S} \to S' \) with exceptional divisor \( F \) and the pullback \( (E, \nabla) \). We take a section \( 0 \neq \sigma \in k = H^0(F, (E, \nabla)|_F) \) and regard its image \( i_F(\sigma) \) under the Gysin map.

Claim 2.2. For all reduced curves \( C \subset S \) not containing \( F \) one has \( H^2_{\text{DR},C}(S, (E, \nabla)) = 0 \). In particular, \( 0 \neq i_F(\sigma) \in H^2_{\text{DR}}((E, \nabla)|_F) \) for all \( \sigma \).

Proof. Let \( C_0 \) be the smooth locus of \( C \). As in (1.4)

\[
H^2_{\text{DR},C}(S, (E, \nabla)) = H^2_{\text{DR}}(C_0, (E, \nabla)|_{C_0}),
\]

and since \( (E, \nabla) \) has rank one, the claim 2.2 is equivalent to \( (E, \nabla)|_{C_0} \neq (\mathcal{O}_{C_0}, d) \).

Let \( \delta : \bar{S} \to \mathbb{P}^2 \) be the blow up of \( p \), and let \( n : \Gamma \to \bar{S} \) be the normalization of the closure of \( C \) in \( \bar{S} \). For \( \infty = n^{-1}\delta^{-1}(D) \), and for some \( m_i \in \mathbb{N} \) sufficiently large, one has

\[
H^0_{\text{DR}}(C_0, (E, \nabla)|_{C_0}) = H^0(\Gamma, \Omega^*_{\bar{S}}(\log \infty) \otimes (\delta \circ n)^* (\mathcal{O}_{\mathbb{P}^2}(\sum m_i H_i), d + \omega)).
\]

The residues of \( (\delta \circ n)^*(\mathcal{O}_{\mathbb{P}^2}(\sum m_i H_i), d + \omega) \) along \( x \in \Gamma \) are in

\[
(\mathbb{N} - \{0\}) \cdot (a_i - m_i) \quad \text{for} \quad \delta(n(x)) \in H_i - \bigcup_{j \neq i} H_j,
(\mathbb{N} - \{0\}) \cdot (a_i - m_i) + (\mathbb{N} - \{0\}) \cdot (a_j - m_j) \quad \text{for} \quad \delta(n(x)) \in H_i \cap H_j.
\]

Indeed, the residue of \( (\mathcal{O}_{\mathbb{P}^2}(\sum m_i H_i), d + \omega) \) along \( H_i \) is \( a_i - m_i \), and \( \Gamma \) lies on a surface \( X \) obtained by a sequence of blow ups \( X \to S \). As well known, if \( z \) is a smooth point on a variety \( Z \) with local coordinates \( x_1, \ldots, x_n \), and \( (E, \nabla) \) a differential equation defined
around \( z \) by \( \omega = \sum_{i=1}^{n} b_i \frac{dx_i}{x_i} + \eta \) for a regular form \( \eta \) on \( Z \), then the pullback differential equation on the blow up of \( z \) has residue \( \sum_{i=1}^{n} b_i \) along the exceptional locus.

By the assumption i) in [2,3], the residues of \( (\delta \circ n)^\ast(\mathcal{O}_{\mathbb{Q}}(\sum m_i H_i), d + \omega) \) can not be in \( \mathbb{Q} \), and a fortiori \( (E, \nabla)|_{C_0} \) can not be trivial. \( \Box \)

### 3. Unitary rank one sheaves

**Example 3.1.** Let \( B_1 \) and \( B_2 \) be two non-isogeneous elliptic curves, defined over \( k = \mathbb{C} \), let \( L_i \in \text{Pic}^0(B_i) \) be non-torsion, and let \( \nabla_i \) be the unique unitary connection on \( L_i \). Using the notations introduced in 0.3, we choose

\[
\omega = \sum_{i=1}^{n} b_i \frac{dx_i}{x_i} + \eta
\]

Then for the pullback on the blow up of \( z \) around \( E \), the notations introduced in 0.3, we choose

\[
\text{Claim 3.2.} \quad H^0_{\text{DR}}(F, (E, \nabla)|_{E'}) = 0.
\]

**Proof.** Let again \( C_0 \) denote the smooth locus of \( C \) and let \( n: \Gamma \to S \) be the normalization. By (1.4)

\[
H^2_{\text{DR}, C}(S, (E, \nabla)) = H^0_{\text{DR}}(C_0, (E, \nabla)|_{C_0}) = H^0_{\text{DR}}(\Gamma, n^\ast(E, \nabla)) \subset \bigoplus_j H^0(\Gamma_j, p_{j,1}^\ast L_1 \otimes p_{j,2}^\ast L_2)
\]

where \( \Gamma_j \) are the irreducible components of \( \Gamma \), and where \( p_{j,i} \) denotes the restriction of \( pr_i \circ \delta \circ n: \Gamma \to B_i \) to \( \Gamma_j \). If \( p_{j,i} \) is dominant, the image \( B'_i \) of

\[
p_{j,i}^\ast: \text{Pic}^0(B_i) \to \text{Pic}^0(\Gamma_j)
\]

is isogeneous to \( B_i \) and it is the Zariski closure of the subgroup generated by \( p_{j,i}^\ast L_i \). Hence if one of the projections, say \( p_{j,1} \), maps \( \Gamma_j \) to a point \( p_{j,1}^\ast L_1 \otimes p_{j,2}^\ast L_2 = p_{j,2}^\ast L_2 \) has no global section.

If both, \( p_{j,1} \) and \( p_{j,2} \) are dominant, the two elliptic curves \( B'_1 \) and \( B'_2 \) are not isogeneous, hence \( B'_1 \cap B'_2 \) is finite, and \( H^0(\Gamma_j, p_{j,1}^\ast L_1 \otimes p_{j,2}^\ast L_2) = 0 \). \( \Box \)

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