COLLISION ORBITS FOR A HILL’S TYPE PROBLEM

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ABSTRACT. We study the planar problem of two satellites attracted by a center of force. Assuming that the center of mass of the two-satellite system is on a circular orbit around the center of force and using Levi-Civita regularization we prove the existence of an almost periodic orbit with an infinite number of collision between the satellites.

Key Words: Hill’s Problem, Regularization, Collisions.

1. Introduction

Consider the planar problem of three bodies of masses $m_0,m_1$ and $m_2$ in the case where $m_1$ and $m_2$ are much smaller than $m_0$. The mutual attraction of the two small bodies can be usually neglected and the problem reduces in a fair approximation to two independent two-body problems. However if the distance between the two small bodies is small their mutual attraction can no longer be ignored. This is known as Hill’s problem [1]. The derivation of Hill’s equations usually found in the literature assumes a hierarchy of masses for the three bodies:

$$m_0 >> m_1 >> m_2,$$

and proceeds in two steps: First the limit $m_2 \to 0$ is taken, which gives the restricted three-body problem; then the limit $m_1 \to 0$ is taken.

A number of problems in celestial mechanics can be approximated by Hill’s equations. Examples are: the Sun-Earth-Moon problem and the interaction between satellites on nearby orbits. The nearby orbits problem was our motivation for this work. From the data obtained during the passage of Voyager near Saturn, it was found that the two satellites Epimetheus and Janus around Saturn have almost circular orbits, with radii 151,422 Km and 151,472 Km, and periods 16.664 and 16.672. At each close encounter, they interchange orbits. To explain this behaviour some models have been proposed, among them Scaling Techniques [CH] and Assymptotic Expansions [HP]. A common feature of the proposed models is an asymptotic expansion in the variable that represents the distance between the two satellites and truncation of higher order terms. In other words, such models assume the hypothesis that there is a smaller bound for the distance among the satellites. In this paper we show that this hypothesis can not be assumed without further assumptions. Using Levi-Civita regularization we show the existence of an almost periodic orbit with an infinite number of collisions among the satellites. Therefore, it’s not true in general that such lower bound exist.

We take a different approach to Hill’s problem. First, instead of taking the limits $m_1,m_2 \to 0$ we fix the body of mass $m_0$ at the origin and assume $m_0 = 1$. We take $m_1,m_2 << 1$ but no hierarchy for the masses $m_1$ and $m_2$ will be assumed. Second,
we assume that the center of mass of the two-satellite system is on a circular orbit around the center of force. This is equivalent to consider the circular Hill problem \[1\]. The collision between the satellites is them regularized using the canonical form of the Levi-Civita regularization \[SS\]. The organization of the paper is as follows:

In section 2 we develop our model for the Hill’s Problem. The model is regularized and contrary to the usual study of Hill’s problem we do not use a rotating system of coordinates (synodical). We show that the potential is symmetric with respect to reflections about the origin in the regularized physical plane. This will be crucial to prove existence of the collision orbit.

In section 3 we study the collinear equilibria predicted by our model. It’s an interesting feature of the model that the regularized equations allows to find the equilibria points as the roots of a polynomial of degree 4. These equations are solved and the critical colinear points are found to be unstable.

In section 4 we use a continuity argument to prove the existence of an almost periodic orbit with an infinite number of collisions. The basic idea is to use the reciprocal of the distance of the center of mass of the two-satellite system to the center of force as a perturbation parameter \(\epsilon\). For \(\epsilon = 0\), the binary made by the satellites will be at an infinite distance of the center of force and the problem is integrable. Since equations are regularized, the Kepler equations for the binary are now represented by a resonant harmonic oscillator (RHO). For small \(\epsilon\), solutions of the perturbed problem are close to the solutions of the RHO. But solutions of the RHO that pass trough the origin are segments of straight lines. Perturbing these solutions we expect that at least one of the perturbed solutions will preserve the feature of passing through the origin more then one time. This is the content of our main lemma (5.2). The symmetry of the Hamiltonian with respect to the origin guarantees the existence of the collision orbit.

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2. The Model

The Hamiltonian \(\bar{H}\) of the planar problem of two bodies of mass \(m_1\) and \(m_2\) attracted by a center of force of mass \(m_0\) at the origin is

\[\bar{H} = \frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_2} - G \frac{m_0 m_1}{|\bar{q}_1|} - G \frac{m_0 m_2}{|\bar{q}_2|} - G \frac{m_1 m_2}{|\bar{q}_1 - \bar{q}_2|}.\]

where \(\bar{q}_1\) and \(\bar{q}_2\) are the coordinates of the bodies of masses \(m_1\) and \(m_2\) respectively, \(\bar{p}_1\) and \(\bar{p}_2\) their conjugate momenta and \(G\) is the gravitational constant. We choose units such that \(G = 1\) and set \(m_0 = 1\). This Hamiltonian represents a Hill problem when \(m_1, m_2 << 1\). We introduce a proportionality factor \(\lambda \in (0, \infty)\) such that \(m_2 = \lambda m_1\). The Hamiltonian becomes

\[\bar{H} = \frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2\lambda m_1} - \frac{m_1}{|\bar{q}_1|} - \frac{\lambda m_1}{|\bar{q}_2|} - \frac{\lambda m_2^2}{|\bar{q}_1 - \bar{q}_2|}.\]

Let \(\bar{w} = d\bar{q}_1 \wedge d\bar{p}_1 + d\bar{q}_2 \wedge d\bar{p}_2\) denote the standard symplectic 2-form. Let \(X_{\bar{H}}\) be the Hamiltonian vector field generated by \(\bar{H}\). Consider the fiber scaling given by

\[\Phi(\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2) = (q_1, q_2, m_1 p_1, m_1 p_2).\]
Under this scaling we have
\[ \bar{H} = m_1 \left\{ \frac{p_1^2}{2} + \frac{p_2^2}{2\lambda} - \frac{\lambda}{|q_1| - |q_2|} - \frac{\lambda m_1}{|q_1 - q_2|} \right\} ; \]
and
\[ \bar{w} = m_1 (dq_1 \wedge dp_1 + dq_2 \wedge dp_2) . \]
Dividing Hamilton's equations \( i_{\bar{X}_H} \bar{w} = d\bar{H} \) by \( m_1 \) we see that it suffices to study the Hamiltonian flow given by the Hamiltonian
\[ H = \frac{p_1^2}{2} + \frac{p_2^2}{2\lambda} - \frac{1}{|q_1| - |q_2|} - \frac{\lambda m_1}{|q_1 - q_2|} \]
with standard symplectic 2-form \( w = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 \).

We introduce Jacobi variables \( \rho \) and \( r \) by
\[
\begin{cases} 
q_1 = \rho - \frac{\lambda}{1 + \lambda} r , \\
q_2 = \rho + \frac{1}{1 + \lambda} r . 
\end{cases}
\]

Here \( \rho \) represents the position of the center of mass of the two satellites and \( r \) represents their relative position vector. The Hamiltonian \( (1) \) becomes
\[ H = \frac{p_\rho^2}{2 (1 + \lambda)} + \frac{p_r^2}{2\Gamma} - \frac{1}{|\rho - \frac{\lambda}{1 + \lambda} r|} - \frac{\lambda m_1}{|\rho + \frac{1}{1 + \lambda} r|} \]
where \( p_\rho \) and \( p_r \) are the momenta canonically conjugate to \( \rho \) and \( r \) respectively and \( \Gamma = \frac{1}{1 + \lambda} \). Assuming \( \frac{|r|}{|\rho|} < \frac{1 + \lambda}{\lambda} \) we have the convergent expansions \( [\lambda] \)
\[ \frac{1}{|\rho| \pm \frac{1}{1 + \lambda} r} = \frac{1}{|\rho|} \sum_{n=0}^{\infty} P_n(\cos \theta) \left( \mp \frac{\lambda}{1 + \lambda |\rho|} \right)^n , \]
where \( P_n(x) \) is the \( n \)-th Legendre polynomial and \( \theta \) is the positively oriented angle between \( \rho \) and \( r \). Hamiltonian \( (2) \) can be written as
\[ H = \left( \frac{p_\rho^2}{2 \lambda} - \frac{\lambda}{|\rho|} \right) + \left( \frac{p_r^2}{2 \Gamma} - \frac{\lambda m_1}{|r|} \right) - \frac{1}{|\rho|} \sum_{n=1}^{\infty} P_n(\cos \theta) \left( \frac{|r|}{|\rho|} \right)^n \Lambda_n , \]
where
\[ \Lambda_n = \Gamma^n (1 + (-1)^n \lambda) , \]
and \( \tilde{\lambda} = 1 + \lambda \).

The first parenthesis term of \( (1) \) represents the Kepler problem described by the center of mass around the center of force and the second parenthesis term of \( (2) \) represents the Kepler problem of the two satellites around their center of mass. At this point we make the principal assumption of this work, namely, we assume that \( \rho = (\rho_x, \rho_y) \), the vector representing the position of the center of mass of the two satellites, describes a circular keplerian orbit of radius \( |\rho_0| \) around the center of force, i.e. \( \rho \) is a circular solution of \( \ddot{\rho} = -\frac{\lambda}{|\rho|} \rho \) yielding
\[ \rho = |\rho_0| (\cos(\omega t), \sin(\omega t)) , \]
where \( \omega = |\rho_0|^{-\frac{3}{2}} \). This is equivalent to consider the Circular Hill's Problem. By the second law of Kepler the energy of the center of mass is given by \( E_{cm} = -\frac{\lambda}{|\rho_0|} \),
and Hamiltonian (3) becomes

\[ H = -\frac{\dot{\lambda}}{|\rho_0|} + \left( \frac{p_r^2}{2T} - \frac{\lambda m_1}{|r|} \right) - \frac{1}{|\rho_0|} \sum_{n=1}^{\infty} P_n(\cos \theta) \left( \frac{|r|}{|\rho_0|} \right)^n \Lambda_n. \tag{5} \]

We remark that this Hamiltonian is time dependent since the angle \( \theta \) depends explicitly on time. For future reference we write

\[ \cos(\theta) = \frac{r_x \rho_x + r_y \rho_y}{|r| |\rho|} = \frac{r_x \cos(\omega t) + r_y \sin(\omega t)}{|r|}. \tag{6} \]

Since energy of system (3) is not preserved we extend phase space from \( \mathbb{R}^4 \) to \( \mathbb{R}^6 \) by including the canonically conjugated pair \((E, t)\). Our new Hamiltonian system is given by

\[ \begin{align*}
\bar{\mathcal{H}} = -E - \frac{\dot{\lambda}}{|\rho_0|} + \left( \frac{p_r^2}{2T} - \frac{\lambda m_1}{|r|} \right) - \frac{1}{|\rho_0|} \sum_{n=1}^{\infty} P_n(\cos \theta) \left( \frac{|r|}{|\rho_0|} \right)^n \Lambda_n, \\
w = du \wedge dp_u + dv \wedge dp_v + dE \wedge dt,
\end{align*} \tag{7} \]

where we must restrict our attention to the level set \( \bar{\mathcal{H}} = 0 \). Denoting the new time by \( f \) it follows from Hamilton’s equation \( \frac{df}{dt} = 1 \). By choice we identify \( f \) and \( t \).

3. Regularization

We regularize the collision between the two satellites. Writing \( \rho = (\rho_x, \rho_y) \) and \( r = (r_x, r_y) \) we write the Levi-Civita transformation \[ \begin{align*}
r_x &= u^2 - v^2, \\
r_y &= 2uv, \\
\rho_x &= u^2 - z^2, \\
\rho_y &= 2wz.
\end{align*} \tag{8} \]

Identifying \( r \) and \( \rho \) with the complex vectors \( r_x + ir_y \) and \( \rho_x + i\rho_y \) respectively, we have that transformation (8) can be written as \( r = (u + iv)^2 \) and \( \rho = (w + iz)^2 \). This transformation halves angles and therefore takes the angle \( \theta \) between \( \rho \) and \( r \) to its half. Writing \( \xi = (u, v) \) and \( \gamma = (w, z) \) the transformation takes \( |r| \) and \( |\rho| \) to \( |\xi|^2 \) and \( |\gamma|^2 \) respectively. Observe that under this transformation the curve \( \rho(t) = |\rho_0| (\cos(\omega t), \sin(\omega t)) \) becomes \( \gamma(t) = |\rho_0|^2 (\cos(\omega t/2), \sin(\omega t/2)) \).

Considering the lift of (8) to the cotangent bundle, Hamiltonian (3) becomes

\[ \bar{\mathcal{H}} = -E - \frac{\dot{\lambda}}{|\gamma_0|^2} + \frac{1}{|\xi|^2} \left( \frac{p_r^2}{8T} - \lambda m_1 \right) - \frac{1}{|\gamma_0|^2} \sum_{n=1}^{\infty} P_n(\cos(\theta/2)) \left( \frac{|\xi|}{|\gamma_0|} \right)^{2n} \Lambda_n. \tag{9} \]

Observe that \( w = |\rho_0|^{-\frac{1}{2}} = |\gamma_0|^{-\frac{1}{2}} \) Regularization is achieved doing the time reparametrization given by

\[ \frac{dt}{ds} = |\xi|^2 \]

where \( s \) denotes the new independent variable. This reparametrization can be performed considering the Hamiltonian

\[ \mathcal{H} = |\xi|^2 \bar{\mathcal{H}} \tag{10} \]

in extended phase space with symplectic 2-form given by

\[ w = du \wedge dp_u + dv \wedge dp_v + dt \wedge dE. \tag{11} \]
Since the hypersurfaces \( \{ \bar{H} = 0 \} \) and \( \{ H = 0 \} \) are equal it follows that the Hamiltonian flow of (10) at the level set \( \bar{H} = 0 \) is a reparametrization of the Hamiltonian flow of (10) at the level set \( H = 0 \). (9) and (10) yields
\[
H = -\lambda m_1 + \frac{p_2^2}{8\Gamma} - \left( \frac{\bar{\lambda}}{\gamma_0^2} + E \right) |\xi|^2 - \frac{|\xi|^2}{\gamma_0^2} \sum_{n=1}^{\infty} P_n (\cos(\theta/2)) \left( \frac{|\xi|^{2n}}{\gamma_0^{2n}} \right) \Lambda_n.
\]
We are interested on the flow of \( H \) at the level 0. We can eliminate the constant \( -\lambda m_1 \) of the Hamiltonian by considering the level \( \lambda m_1 \) instead. The square of the reciprocal of the radius of the center of mass will be treated as a perturbation parameter. Writing \( \epsilon = \frac{1}{|\gamma_0|^2} \) and doing the symplectic scaling \( p_\xi \rightarrow 2p_\xi, \xi \rightarrow \xi/2 \) we have
\[
H_{\lambda m_1} = \frac{p_2^2}{2\Gamma} - \frac{1}{4} (\bar{\lambda} \epsilon + E) |\xi|^2 - \epsilon^2 \sum_{n=1}^{\infty} \epsilon^{n-1} P_n (\cos(\theta/2)) \left( \frac{|\xi|^2}{4} \right)^{n+1} \Lambda_n,
\]
where the subscript \( \lambda m_1 \) is a reminder that we must consider the level set \( H = \lambda m_1 \).

**Lemma 3.1.** Hamiltonian (12) is invariant with respect to the symmetry \( S : \mathbb{R}^6 \rightarrow \mathbb{R}^6 \) given by \( S(\xi, p_\xi, E, t) = (-\xi, p_\xi, E, t) \), i.e. \( H(\xi, p_\xi, E, t) = H(-\xi, p_\xi, E, t) \).

**Proof.** Since \( |\xi| \) is invariant under \( S \) it suffices to show that \( \cos(\theta/2) \) is also invariant. But from (6) and (8)
\[
\cos(\theta/2) = \left( \frac{u^2 - v^2}{|\xi|^2} \right) \cos(u/2) + 2uv \sin(u/2)
\]
that is clearly invariant under \( S \). \( \Box \)

The equations of motion of (12) can be written as
\[
\begin{align*}
\ddot{\xi} &= -\frac{\bar{\lambda} \epsilon + E}{2\Gamma} \xi + \epsilon^2 \nabla V; \\
\dot{E} &= -\epsilon^2 \frac{\partial V}{\partial t}; \\
\dot{t} &= \frac{|\xi|^2}{4},
\end{align*}
\]
where
\[
V = \sum_{n=1}^{\infty} \epsilon^{n-1} P_n (\cos(\theta/2)) \left( \frac{|\xi|^2}{4} \right)^{n+1} \Lambda_n.
\]
From the second equation of (13) we have that \( E(s) = E(0) + \mathcal{O}(\epsilon^2) \), and we write for future reference that
\[
\ddot{\xi} = -\frac{E_0}{2\Gamma} \xi - \frac{\bar{\lambda}}{2\Gamma} \xi + \mathcal{O}(\epsilon^2).
\]

### 4. Euler’s Critical Points

Euler’s critical points are relative equilibria of the Hamiltonian system representing colinear configurations. In what follows we compute the Euler’s critical points for our model: In a colinear configuration the two masses will be moving forming a straight line with the center of force. Therefore we must have \( \theta = 0 \) or \( \theta = \pi \) and \( \dot{\theta} = 0 \). The first case represents the case where \( m_1 \) is between the center of force and \( m_2 \) and the second case represents the case where \( m_2 \) is in between. Since
m_2 = \lambda m_1 we need to work only with the case \theta = 0. We introduce polar coordinates \((l, \phi)\) in the plane \((u, v)\), where \(l\) is the radius and \(\phi\) is the angle. Observe that \(\phi = \pi - \frac{u + l^2}{2}\). Let \(p_l\) and \(p_\phi\) be the conjugate momenta to \(l\) and \(\phi\) respectively.

Hamiltonian (12) becomes

\[
\mathcal{H}_{\lambda m_1} = \frac{p_l^2}{2\Gamma} + \frac{p_\phi^2}{2\Gamma l^2} - \frac{1}{4} (\lambda \epsilon + E) l^2 - \epsilon^2 \sum_{n=1}^{\infty} \frac{e^n}{n} P_n (\cos(\theta/2)) \left( \frac{l^2}{4} \right)^{n+1} \Lambda_n,
\]

Hamilton’s equations are

\[
\begin{align*}
\dot{l} &= \frac{p_l}{\Gamma}, \\
\dot{p}_l &= \frac{1}{2} (\lambda \epsilon + E) l + \frac{p_\phi^2}{2\Gamma l^2} + 2 \epsilon^2 l \sum_{n=1}^{\infty} \frac{e^n P_n (\cos(\theta/2))}{n} \left( \frac{l^2}{4} \right)^n \Lambda_n, \\
\dot{\phi} &= \frac{p_\phi}{\epsilon^2}, \\
\dot{p}_\phi &= \epsilon^2 \sum_{n=1}^{\infty} \frac{e^n}{n} \left( \frac{l^2}{4} \right)^{n+1} D_x P_n (\cos(\theta/2)) \sin(\theta/2) \Lambda_n, \\
\dot{E} &= -\frac{\epsilon^2}{4} \sum_{n=1}^{\infty} \frac{e^n}{n} \left( \frac{l^2}{4} \right)^{n+1} D_x P_n (\cos(\theta/2)) \sin(\theta/2) \Lambda_n, \\
l &= \frac{l^2}{4};
\end{align*}
\]

where \(\theta/2 = \pi - \phi + \frac{u + l^2}{2}\). We look for Euler’s critical points of (16). The condition \(\dot{\theta} = 0\) implies that \(\phi = wt/2\). Using the second and last equations of (16) it follows that \(p_\phi = \frac{u + l^2}{2}\). This is the value the angular momentum \(p_\phi\) must have in order to keep the colinear shape of the configuration. Since we are looking for colinear critical points in phase space (and not in extended phase space!) we must find a point for which \(l = 0\), \(\dot{p}_l = 0\), \(\dot{p}_\phi = 0\) and \(\dot{E} = 0\). For \(\theta = 0\) i.e., for \(\phi = wt/2\) it follows from (16) that the last two equalities are satisfied. We write \(E_0 = E(0)\).

The first equality will be satisfied setting \(p_l = 0\). It remains to find \(l\) for which \(\dot{p}_l = 0\). For \(\theta = 0\) we have that \(P_n (\cos(\theta/2)) = P_n (1) = 1\). Recalling that \(\epsilon = \frac{\pi}{2}\) it follows that

\[
\dot{p}_l = -\frac{\partial V}{\partial l},
\]

where the potential function \(V\) is written as

\[
V(l) = -\frac{1}{4} (\lambda \epsilon + E_0) l^2 + \frac{\epsilon^2 \Gamma l^6}{16} - \epsilon^2 \left\{ \sum_{n=1}^{\infty} \left( \epsilon \frac{l^2}{4} \Gamma \right)^n + \lambda \sum_{n=1}^{\infty} (-\epsilon \frac{l^2}{4} \Gamma)^n \right\}
\]

Suming the series we obtain

\[
V(l) = -\frac{1}{4} (\lambda \epsilon + E_0) l^2 + \frac{\epsilon^2 \Gamma l^6}{16} - \frac{\epsilon^2 l^4 \Gamma}{4} \left( \frac{4\lambda + \epsilon l^2 \Gamma (1 - \lambda)}{16 - \epsilon^2 l^4 \Gamma^2} \right)
\]

We want to find non-zero solutions of \(\frac{\partial V}{\partial l} = 0\). Making \(u = l^2\) we only need to find non-zero solutions of \(\frac{\partial V}{\partial u} = 0\). After some straightforward computations we see that the zeros of this equation is given by the zeros of a polynomial of degree 4 in \(u\). Therefore we have explicit analytic solutions for the Inner Euler’s Critical points. The explicit solutions can be computed in an algebraic manipulator software but are very messy to be of any use. The critical points can be numerically computed using a Newton algorithm. We computed the values for different values of \(E_0\). For
all of the critical points computed we have that $\frac{d^2V}{du^2} < 0$ indicating that the critical points are unstable.

## 5. Collision Orbits

Hamiltonian (12) is symmetric with respect to reflections about the origin. This symmetry will be used to prove existence of an almost periodic orbit that passes through the origin twice. Since the origin in the regularized plane represents a collision in the physical plane, this orbit represents a periodic orbit with an infinite number of collisions. The proof relies on the fact that for $\epsilon = 0$ and $E_0 < 0$, (12) represents a resonant harmonic oscillator. Therefore, the projections on the $(u,v)$ plane of trajectories leaving the origin (ejection trajectories) are segments of straight lines. We expect that when $\epsilon$ is small, some trajectories of the perturbed system will preserve the feature of passing through the origin at least two times. The symmetry with respect to the origin will then imply that this orbit will cross the origin an infinite number of times.

We set $m_1 = \mu$ and assume that $m_1, \lambda m_1 < \ll 1$ and $E_0 < 0$. We write $\mu = O(\epsilon)$.

More explicitly we will assume that there is a constant $\tilde{\mu}$ such that $\mu(\epsilon) = \epsilon \tilde{\mu}$. We will prove the following theorem

**Theorem 5.1.** If $\epsilon$ is small enough, there is an almost periodic orbit of (12) that passes through the origin.

**Proof.** For easy reference we write Hamiltonian (12)

$$H_{\epsilon, \kappa} = p_\xi^2 - \frac{1}{4} (\lambda \epsilon + E) |\xi|^2 - \epsilon^2 V,$$

where $\kappa = \lambda \mu$. We restrict our study to solutions that at time $s = 0$ leave the origin.

**Definition** We call ejection solutions, solutions of the Hamiltonian system with Hamiltonian (20) and symplectic 2-form (11) that at time $s = 0$ leave the origin, i.e., solutions with $\xi(0) = (0, 0)$.

**Remark:** The set of ejection trajectories is parametrized by a circle. In fact, for these trajectories (20) implies that $|p_\xi(0)|^2 = \epsilon \kappa$ and we can write $p_u(0) = \sqrt{\epsilon \kappa} \cos(\alpha)$ and $p_v(0) = \sqrt{\epsilon \kappa} \sin(\alpha)$ for $\alpha \in [0, 2\pi)$.

For a small $\epsilon$ and $E_0 < 0$ solutions of (20) will be close to the solutions of a resonant harmonic oscillator with period and amplitude given by

$$T_0 = \frac{2\pi \sqrt{2T}}{|E_0|},$$

$$A = 2 \sqrt{\frac{\epsilon \kappa}{|E_0|}},$$

respectively. Let $\eta$ denote an initial condition for an ejection trajectory, i.e. $\eta = (0, 0, p_u, p_v, E(0), t(0))$. Fixing $E(0)$ and $t(0)$ we have that $\eta$ is uniquely determined by $\alpha$. We write the set of ejection trajectories at time $s$ and parameter $\epsilon$ as

$$\phi^s_{\epsilon}(\alpha) = (u^s_{\epsilon}(\alpha), v^s_{\epsilon}(\alpha), p_u^s_{\epsilon}(\alpha), p_v^s_{\epsilon}(\alpha), E^s_{\epsilon}(\alpha), t^s_{\epsilon}(\alpha)).$$

Then $\phi^0_{\epsilon}(\alpha) = \eta = (0, 0, \sqrt{\epsilon \kappa} \cos(\alpha), \sqrt{\epsilon \kappa} \sin(\alpha), E(0), t(0))$.

The following lemma shows that, for $\epsilon$ small enough, an ejection trajectory with angle $\alpha$ pointing to the right half plane will cross the $v$ axis transversally at some time $\tau = \tau(\alpha)$. 

Lemma 5.2. Let $E_0 < 0$. Let $\delta$ be a positive real number with $\delta << 1$. Let $\alpha \in I$, where $I = \left[-\frac{T}{2} + \delta, \frac{T}{2} - \delta\right]$. If $\epsilon$ is small enough, there exist times $\tau = \tau(\epsilon, \alpha)$ such that

$$
\frac{1}{4} T_0 < \tau < \frac{3}{4} T_0,
$$

and $u^*_\tau(\alpha) = 0$. Moreover $v^*_\tau(\alpha)$ is a continuous function of $\alpha$.

Proof. We first prove existence of $\tau$. Let $m = \min_{\alpha \in I} \{|\cos(\alpha)|\} = |\cos(\frac{\pi}{2} - \delta)|$. We write

$$
(22) \quad \phi^s_\epsilon = \phi^s_{osc} + \epsilon \psi^s
$$

where $\phi^s_{osc}$ is the harmonic oscillator flow given by

$$
\phi^s_{osc}(\alpha) = (A \cos(\alpha) \sin(\omega s), A \sin(\alpha) \sin(\omega s), -A \omega \cos(\alpha) \cos(\omega s), -A \omega \sin(\alpha) \cos(\omega s), E(0), t(0)),
$$

where $\omega = \sqrt{\frac{|E_0|}{2T}}$. From (24) $\psi^s_\epsilon$ satisfies

$$
\ddot{\psi}^s_\epsilon = -\frac{E_0}{2} \psi^s_\epsilon - \frac{\phi^s_\epsilon}{2} + O(\epsilon).
$$

Let $\frac{1}{4} = \frac{1}{4} T_0$ and $\frac{3}{4} = \frac{3}{4} T_0$. Then

$$
(23) \quad u^s_\tau(\alpha) = \sqrt{\epsilon \kappa} \cos(\alpha) + \epsilon \psi^s_\tau(\alpha);
$$

where $\psi^s_\tau$ is the $u$ component of $\psi^s$. Let $J = [0, \bar{\epsilon}]$ where $\bar{\epsilon}$ is a positive small number. Let $M = \max_{\epsilon \in J} \left(\max_{\alpha \in I} |u^s_\tau(\alpha)|\right)$. We can find $\epsilon$ small enough such that

$$
(24) \quad \epsilon \sqrt{\kappa} m > \epsilon M,
$$

this implies that

$$
|\sqrt{\epsilon \kappa} \cos(\alpha)| > \epsilon |\psi^s_\tau(\alpha)|.
$$

Therefore for $\epsilon$ satisfying (24), (23) implies that $u^s_\tau$ has the same sign as $\sqrt{\epsilon \kappa} \cos(\alpha)$, i.e., $u^s_\tau$ is positive. Analogously we show that $u^s_\tau$ is negative. Therefore by continuity of the flow, there exists times $\tau$, for which $u^s_\tau(\alpha) = 0$ and satisfying (24). To prove continuity it suffices to prove that at time $s = \tau$ the projection of the flow on the $(u, v)$ plane intersects the $v$ axis transversally, i.e., it suffices to show that $\dot{u}_\tau^s(\alpha) \neq 0$.

But

$$
\dot{u}_\tau^s(\alpha) = -A \omega \sin(\alpha) \cos(\omega \tau) + \epsilon \psi^s_\tau.
$$

For $\epsilon$ small enough it follows from (24), using the same argument as in the existence part, that $\dot{u}_\tau^s(\alpha) \neq 0$.

For notational simplicity we will write in the next lemma $u^s = u(s)$ and $v^s = v(s)$.

Lemma 5.3. Let $E_0 < 0$. Then for $\epsilon$ small enough there exists $\alpha \in I$ such that $v(\tau_u(\alpha)) = 0$.

Proof. Let $\tau$ as in lemma (5.2). By Taylor’s Theorem it follows that there exists $c$, $0 < c < \tau$ such that $\xi(\tau) = \xi(0) + \dot{\xi}(0) \tau + \frac{\ddot{\xi}(0)}{2} \tau^2 + \frac{\dddot{\xi}(c)}{6} \tau^3$, that we write as

$$
u(\tau) = u(0) + \dot{u}(0) \tau + \frac{\ddot{u}(0)}{2} \tau^2 + \frac{\dddot{u}(c)}{6} \tau^3,$$
\[
v(\tau) = v(0) + \dot{v}(0) \tau + \frac{\ddot{v}(0)}{2} \tau^2 + \frac{\dddot{v}(c)}{6} \tau^3.
\]

For an ejection trajectory we have that \( u(0) = v(0) = 0 \) and equations (13) imply that \( \ddot{u}(0) = \ddot{v}(0) = 0 \). Therefore
\[
\begin{align*}
u(\tau) &= \sqrt{\epsilon \kappa} \cos(\alpha) \tau + \frac{\dddot{u}(c)}{6} \tau^3, \\
u(\tau) &= \sqrt{\epsilon \kappa} \sin(\alpha) \tau + \frac{\dddot{v}(c)}{6} \tau^3,
\end{align*}
\]
with \( 0 < c < \tau \). \( \tau \) satisfies (21) and by definition \( u(\tau(\alpha)) = 0 \). Solving the first equation for \( \tau \) and substituting on the second equation we obtain
\[
v(\tau_u(\alpha)) = \frac{\sqrt{\epsilon \kappa}}{\dddot{u}(c)} \sin(\alpha) \dddot{u}(c) - \cos(\alpha) \dddot{v}(c).
\]

Now we estimate \( |\dddot{u}(c) - \dddot{v}(c)| \). From the Taylor expansions we have that
\[
|\dddot{u}(c) - \dddot{v}(c)| < \frac{6}{\tau^2} |\sqrt{\epsilon \kappa} \cos(\alpha) - \sin(\alpha)| + \frac{6}{\tau^3} |u(\tau) - v(\tau)|.
\]

Using (21) we obtain
\[
|\dddot{u}(c) - \dddot{v}(c)| < \frac{6 \sqrt{\epsilon \kappa}}{\pi^2} |E_0| + \frac{6 |E_0|^{\frac{3}{2}}}{\pi^2 \sqrt{2}} |v(\tau)|.
\]

But, from (22) we have that \( v(\tau) = \frac{\sqrt{\epsilon \kappa}}{|E_0|} \sin(\alpha) \cos(\alpha) + O(\epsilon^3) \) giving that
\[
|\dddot{u}(c) - \dddot{v}(c)| < \frac{6 \sqrt{\epsilon \kappa}}{\pi^2} \frac{|E_0|}{|E_0|^{\frac{3}{2}}} + \frac{6 |E_0|^{\frac{3}{2}}}{\pi^2 \sqrt{2}} |E_0|^{\frac{3}{2}} + O(\epsilon^3).
\]

Therefore there exists a constant \( K \) such that
\[
|\dddot{u}(c) - \dddot{v}(c)| < K |E_0| \sqrt{\epsilon} + O(\epsilon^3).
\]

We can then write
\[
\dddot{v}(c) = \dddot{u}(c) + O(\epsilon^{\frac{3}{2}}),
\]
and (22) becomes
\[
v(\tau_u(\alpha)) = \frac{\sqrt{\epsilon \kappa}}{|E_0|} \left( \sin(\alpha) - \cos(\alpha) + O(\epsilon^{\frac{3}{2}}) \right).
\]

Thus for \( \epsilon \) small enough we can find \( \alpha_1, \alpha_2 \in I \) such that \( v(\tau(\alpha_1)) > 0 \) and \( v(\tau(\alpha_2)) < 0 \). By the continuity part of lemma (5.2) there is \( \alpha \in I \), such that \( v(\tau(\alpha)) = 0 \).

Therefore we proved that for \( \epsilon \) small enough there is an angle \( \alpha \) such that the ejection trajectory with this angle will pass through the origin in some future time \( \tau(\alpha) \). But by lemma (3.1) the system is symmetric with reflections about the origin. Therefore, the orbit will pass an infinite number of times around the origin proving the theorem.
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