A Family of Quasi-solvable Quantum Many-body Systems

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Abstract

We construct a family of quasi-solvable quantum many-body systems by an algebraic method. The models contain up to two-body interactions and have permutation symmetry. We classify these models under the consideration of invariance property. It turns out that this family includes the rational, hyperbolic (trigonometric) and elliptic Inozemtsev models as particular cases.

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I. INTRODUCTION

New findings of solvable or integrable models have stimulated development of new and wide research directions and ideas in both physics and mathematics. The discovery of quasi-solvability in quantum mechanics [1] is a typical example. By quasi-solvability we mean that a part of the spectra can be solved, at least, algebraically\(^1\). One of the most successful approach to construct a quasi-solvable model is the algebraic method introduced by Turbiner in 1988 [2], in which a family of quasi-solvable one-body models was constructed by the \(\mathfrak{sl}(2)\) generators on a polynomial space. This family was later completely classified under the consideration of the \(GL(2,\mathbb{R})\) invariance of the models [3, 4]. Recently, this family have been paid much attention to in the context of \(\mathcal{N}\)-fold supersymmetry [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Several attempts were made to construct quasi-solvable many-body models by naive extension to higher-rank algebras. Especially, construction of two-body problems by the rank 2 algebras was extensively investigated [4, 17, 18, 19, 20, 21, 22]. These approaches however led to Schrödinger operators in curved space in general and could hardly apply to \(M\)-body (\(M > 2\)) problems.

In 1995, a significant progress was made in Ref. [23], where the exact solvability of the rational and trigonometric \(A\) type Calogero–Sutherland (CS) models [24, 25, 26] for any finite number of particles were shown by a similar algebraic method. The key ingredient is the introduction of the elementary symmetric polynomials which reflect the permutation symmetry of the original models. The algebra for the \(M\)-body system is \(\mathfrak{sl}(M+1)\). This idea was further applied to show the exact solvability of the rational and trigonometric \(A\) and \(BC\) type CS models and their supersymmetric generalizations [27], and to show the quasi-exact solvability of various deformed CS models [28, 29]. Therefore, one can say the approach starting from Ref. [23] is, up to now, the most successful in investigating quasi-solvable quantum many-body problems. However, one has not yet known all the models that can be obtained by this approach. In other words, we have not obtained the classification of these \(\mathfrak{sl}(M+1)\) \(M\)-body models like that of the \(\mathfrak{sl}(2)\) one-body models. Recently, this classification problem was partly accessed in Ref. [30] though, as was stressed by the authors themselves, the results depend on the specific ansatz and thus are incomplete. In this Letter, we will show the complete classification of the quantum many-body systems with up to two-body interactions which can be constructed by the \(\mathfrak{sl}(M+1)\) method.

II. CONSTRUCTION OF THE MODELS

Consider an \(M\)-body quantum Hamiltonian,

\[
H_N = -\frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + V(q_1, \ldots, q_M),
\]

which possesses permutation symmetry, that is,

\[
V(\ldots, q_i, \ldots, q_j, \ldots) = V(\ldots, q_j, \ldots, q_i, \ldots),
\]

\(^1\) The term quasi-exact solvability has been widely used in this meaning. However, we keep it to express the case where the state vectors corresponding to the solvable spectra are normalizable. Importance of this distinction is explained in Refs. [12, 15].
for $\forall i \neq j$. To algebraize the Hamiltonian $\Pi$, we will proceed the following three steps. At first, we make a gauge transformation on the Hamiltonian $\Pi$:

$$\tilde{H}_N = e^{W(q)} H_N e^{-W(q)}.$$  

(3)

The function $W(q)$ is to be determined later and plays the role of the superpotential when the system Eq. $\Pi$ is supersymmetric. As in Eq. (3), we will hereafter attach tildes to both operators and vector spaces to indicate that they are quantities gauge-transformed from the original ones. In the next, we change the variables $q_i$ to $h_i$ by a function $h$ of a single variable; $h_i = h(q_i)$. Note that the way of changing of the variables preserves the permutation symmetry. The third step is the introduction of elementary symmetric polynomials of $h_i$ defined by,

$$\sigma_k(h) = \sum_{i_1 < \cdots < i_k} h_{i_1} \cdots h_{i_k} \quad (k = 1, \ldots, M),$$  

(4)

from which we further change the variables to $\sigma_i$. Then, we choose a set of components of the $\mathcal{N}$-fold supercharges in terms of the above variables $\sigma_i$ as follows:

$$\tilde{P}^{(i)}_N = \frac{\partial^\mathcal{N}}{\partial \sigma_{i_1} \cdots \partial \sigma_{i_N}} \quad (1 \leq i_1 \leq \cdots \leq i_N \leq M),$$  

(5)

where $\{i\}$ is an abbreviation of the set $\{i_1, \ldots, i_N\}$. Using these $\mathcal{N}$-fold supercharges, we define the vector space $\tilde{\mathcal{Y}}_N \equiv \bigcap_{\{i\}} \ker \tilde{P}^{(i)}_N$, which now becomes,

$$\tilde{\mathcal{Y}}_N = \text{span} \left\{ \sigma_1^{n_1} \cdots \sigma_M^{n_M} : 0 \leq \sum_{i=1}^M n_i \leq \mathcal{N} - 1 \right\}.$$  

(6)

For given $M$ and $\mathcal{N}$, the dimension of the vector space (6) becomes,

$$\dim \tilde{\mathcal{Y}}_N = \sum_{n=0}^{\mathcal{N}-1} \frac{(n + M - 1)!}{n! (M - 1)!} = \frac{(\mathcal{N} + M - 1)!}{(\mathcal{N} - 1)! M!}.$$  

(7)

We will construct the system (3) to be quasi-solvable so that the solvable subspace is given by just Eq. (6). This can be achieved by imposing the following quasi-solvability condition $[12, 13, 16],

$$\tilde{P}^{(i)}_N \tilde{H}_N \tilde{\mathcal{Y}}_N = 0 \quad \text{for} \quad \forall \{i\}.$$  

(8)

The general solution of Eq. (8) can be obtained in completely the same way as shown in Refs. [13, 16]. As in the case of the one-body models, it is sufficient to find differential operators up to the second-order as solutions for $\tilde{H}_N$ since we are constructing a Schrödinger operator in the original variables $q_i$. It turns out that the general solution which contains up to the second derivatives takes the following form,

$$\tilde{H}_N = - \sum_{\kappa,\lambda,\mu,\nu=0}^M A_{\kappa,\lambda,\mu,\nu} E_{\kappa,\lambda} E_{\mu,\nu} + \sum_{\kappa,\lambda=0}^M B_{\kappa,\lambda} E_{\kappa,\lambda} - C,$$  

(9)
where $A_{\kappa \lambda \mu \nu}$, $B_{\kappa \lambda}$, $C$ are arbitrary constants, and $E_{\kappa \lambda}$ are the first-order differential operators which constitute the Lie algebra $\mathfrak{sl}(M + 1)$:

$$E_{0i} = \frac{\partial}{\partial \sigma_i}, \quad E_{ij} = \sigma_i \frac{\partial}{\partial \sigma_j},$$

$$E_{i0} = \sigma_i E_{00} = \sigma_i \left( N - 1 - \sum_{k=1}^{M} \sigma_k \frac{\partial}{\partial \sigma_k} \right).$$

If we explicitly express the general solution (9) in terms of $\sigma_i$, we obtain the following expression,

$$\tilde{H}_N = -\sum_{k,l=1}^{M} \left[ A_0(\sigma) \sigma_k \sigma_l - A_k(\sigma) \sigma_l + A_{kl}(\sigma) \right] \frac{\partial^2}{\partial \sigma_k \partial \sigma_l}$$

$$+ \sum_{k=1}^{M} \left[ B_0(\sigma) \sigma_k - B_k(\sigma) \right] \frac{\partial}{\partial \sigma_k} - C(\sigma),$$

where $A_\kappa$, $A_{kl}$, $B_\kappa$ and $C$ are second-degree polynomials of several variables.

One of the most difficult problems one would come across in the algebraic approach to the quasi-solvable quantum many-body systems is to solve the canonical-form condition:

$$H_N = e^{-W(q)} \tilde{H}_N e^{W(q)} = -\frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + V(q).$$

If the Hamiltonian (11) is gauge-transformed back to the original one, it in general does not take the canonical form of the Schrödinger operator like Eq. (11) and one can hardly solve, for arbitrary $M$, the conditions under which a gauge-transform of Eq. (11) could be cast in the Schrödinger form. This difficulty can, however, be partly overcome by the following observation. Suppose we can solve the canonical-form condition for an $M$ and obtain a quasi-solvable $M$-body Hamiltonian constructed from the $\mathfrak{sl}(M + 1)$ generators, which would have the following form:

$$H_N = -\frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + \sum_{i=1}^{M} V_1(q_i) + \sum_{i<j}^{M} V_2(q_i, q_j) + \cdots + V_M(q_1, \ldots, q_M).$$

Then, we can get a quasi-solvable $M$-body model with up to the two-body interactions if we turn off all the coupling constants of the interactions except for the one- and two-body ones:

$$H_N = -\frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + \sum_{i=1}^{M} V_1(q_i) + \sum_{i<j}^{M} V_2(q_i, q_j).$$

The resultant model (14) should be, when we put $M = 2$, identical with one of the two-body models constructed from the $\mathfrak{sl}(3)$ generators. This comes from the fact that the gauged Hamiltonian (11) constructed from the $\mathfrak{sl}(M + 1)$ generators reduces to the one constructed from the $\mathfrak{sl}(3)$ generators if we put $M = 2$ and $h_i = 0$ for $i > 2$. 
Therefore, as far as up to two-body interactions are concerned, it is sufficient to solve the $M = 2$ case by virtue of the permutation symmetric construction. We have found that we can actually solve the canonical-form condition for $M = 2$ and that $\tilde{H}_N$ for $M \geq 2$ must have the following expression in terms of the variables $h_i$,

$$\tilde{H}_N(h) = - \sum_{i=1}^{M} P(h_i) \frac{\partial^2}{\partial h_i^2}$$

$$- \sum_{i=1}^{M} \left[ Q(h_i) - \frac{N - 2 + (M - 1)c}{2} P'(h_i) \right] \frac{\partial}{\partial h_i}$$

$$- 2c \sum_{i \neq j}^{M} \frac{P(h_i)}{h_i - h_j} \frac{\partial}{\partial h_i} - C(\sigma(h)),$$

where $C$ is given by,

$$C(\sigma(h)) = \frac{N - 1}{12} [N - 2 + 2(M - 1)c] \sum_{i=1}^{M} P''(h_i)$$

$$- \frac{N - 1}{2} \sum_{i=1}^{M} Q'(h_i) - \frac{N - 1}{2} c \sum_{i \neq j}^{M} P'(h_i) \frac{1}{h_i - h_j} + R.$$

The $P$ and $Q$ in Eqs. (15) and (16) are a fourth- and a second-degree polynomial, respectively:

$$P(h) = a_4 h^4 + a_3 h^3 + a_2 h^2 + a_1 h + a_0, \quad (17a)$$

$$Q(h) = b_2 h^2 + b_1 h + b_0. \quad (17b)$$

Thus, there are 10 parameters $a_n, b_n, c, R$, which characterize the quasi-solvable Hamiltonian (15). One can prove the quasi-solvability of the operator (15) by the convertibility of it into the form (11). The function $h(q)$, which determines the change of variables, is given by a solution of the differential equation,

$$h'(q)^2 = 2P(h(q)). \quad (18)$$

One may notice that the resultant Eqs. (15)–(18) have resemblance to those of the one-body quasi-solvable models constructed from $\mathfrak{sl}(2)$ generators [2, 3, 4], or equivalently, the type A $N$-fold supersymmetric models [13, 14, 16]. Indeed, we can easily see that the above results reduce to the one-body $\mathfrak{sl}(2)$ quasi-solvable and type A $N$-fold supersymmetric models if we set $M = 1$, where the double summation is understood as $\sum_{i \neq j} \equiv 0$. This is consistent with the fact that in the case of $M = 1$ the above procedure is essentially equivalent to that in the $\mathfrak{sl}(2)$ construction of type A $N$-fold supersymmetry [13, 14]. Under the above conditions (15)–(18) satisfied, the original Hamiltonian becomes the following Schrödinger type, 

$$H_N = - \frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + \frac{1}{2} \sum_{i=1}^{M} \left[ \left( \frac{\partial \mathcal{W}(q)}{\partial q_i} \right)^2 - \frac{\partial^2 \mathcal{W}(q)}{\partial q_i^2} \right] - C(\sigma(h)),$$

under the above conditions (15)–(18) satisfied, the original Hamiltonian becomes the following Schrödinger type,
and the superpotential $W(q)$ is given by,

$$W(q) = -\sum_{i=1}^{M} \int dh_i \frac{Q(h_i)}{2P(h_i)} + \mathcal{N} - 1 + (M - 1)c \sum_{i=1}^{M} \ln |h_i'| - c \sum_{i<j}^{M} \ln |h_i - h_j|.$$  

(20)

It is evident by the construction that the solvable wave functions $\psi(q)$ of the Hamiltonian (19) take the following form,

$$\psi(q) = \tilde{\psi}(q) e^{-W(q)}, \quad \tilde{\psi}(q) \in \tilde{\mathcal{V}}_{\mathcal{N}}.$$  

(21)

The Hamiltonian (19) with Eqs. (16) and (20) is the most general quasi-solvable many-body systems with two-body interactions which can be constructed from the $\mathfrak{sl}(M+1)$ generators (10).

Before investigating what kind of particular models emerges from the general Hamiltonian (19), we will refer to an interesting feature of the result. If the algebraic Hamiltonian (3) does not contain any raising operator $E_{i0}$, it preserves the vector space $\tilde{\mathcal{V}}_{\mathcal{N}}$ for arbitrary $\mathcal{N}$ and becomes not only a quasi-solvable but also a solvable model [21, 22]. In this case, it turns out that $C(\sigma) = C$, one of the constants involved in Eq. (3), and thus the original Hamiltonian (19) becomes supersymmetric [31, 32]. A system is always quasi-solvable if it is supersymmetric, since the ground state is always solvable. From the above result, we can conclude that a system is always supersymmetric if it is solvable and all its states have the form (21).

III. CLASSIFICATION OF THE MODELS

It was shown that the one-body $\mathfrak{sl}(2)$ quasi-solvable models can be classified using the shape invariance of the Hamiltonian under the action of $GL(2, \mathbb{R})$ of linear fractional transformations [3, 4]. We can see that the many-body Hamiltonian (15) also has the same property of shape invariance. The linear fractional transformation of $h_i$ is introduced by,

$$h_i \mapsto \hat{h}_i = \frac{\alpha h_i + \beta}{\gamma h_i + \delta} \quad (\Delta \equiv \alpha \delta - \beta \gamma \neq 0).$$  

(22)

Then, it turns out that the Hamiltonian (15) is shape invariant under the following transformation induced by Eq. (22),

$$\tilde{H}_{\mathcal{N}'}(h) \mapsto \tilde{H}_{\mathcal{N}'}(\hat{h}) = \prod_{i=1}^{M} (\gamma h_i + \delta)^{\mathcal{N}'-1} \tilde{H}_{\mathcal{N}'}(\hat{h}) \prod_{i=1}^{M} (\gamma h_i + \delta)^{-(\mathcal{N}'-1)},$$  

(23)

where the polynomials $P(h)$ and $Q(h)$ in the $\tilde{H}_{\mathcal{N}'}(h)$ are transformed according to,

$$P(h) \mapsto \hat{P}(\hat{h}) = \Delta^{-2}(\gamma h + \delta)^4 P(\hat{h}),$$  

(24a)

$$Q(h) \mapsto \hat{Q}(\hat{h}) = \Delta^{-1}(\gamma h + \delta)^2 Q(\hat{h}).$$  

(24b)

For a given $P(h)$, the function $h(q)$ is determined by Eq. (18) and a particular model is obtained by substituting this $h(q)$ for Eqs. (16), (19) and (20). Under the transformation
of $GL(2, \mathbb{R})$, every real quartic polynomial $P(h)$ is equivalent to one of the following eight forms:

1). $\frac{1}{2}$, 2). $2h$, 3). $2\nu h^2$, 4). $2\nu (h^2 - 1)$,
5). $2\nu (h^2 + 1)$, 6). $\frac{\nu}{2} (h^2 + 1)^2$,
7). $2h^3 - \frac{g_2}{2} h - \frac{g_3}{2}$, 8). $\frac{\nu}{2} (h^2 + 1) [(1 - k^2) h^2 + 1]$.

where $\nu$, $k$, $g_2$ and $g_3$ are all real numbers satisfying $\nu \neq 0$, $0 < k < 1$ and $g_3^2 - 27g_2^3 \neq 0$. Thus, the quasi-solvable models \[ \text{(19)} \] can be classified into the above eight cases.

**Case 1).** $h(q) = q$:

This leads to the rational $A$ type Inozemtsev model \[ \text{(33, 34, 35)} \]. Inozemtsev models are known as a family of deformed CS models which preserve the classical integrability. The main difference between quantum and classical case is that the quantum quasi-solvability holds only for quantized values of the parameter, say, for integer $N$, while the classical integrability holds for continuous values. This is one of the common features that the quantum quasi-solvable models share.

**Case 2).** $h(q) = q^2$:

This leads to the rational $BC$ type Inozemtsev model. The quasi-exactly solvable model reported in Ref. \[ \text{(29)} \] is just this case.

**Case 3).** $h(q) = e^{2\sqrt{\nu}q}$:

This leads to the hyperbolic ($\nu > 0$) and trigonometric ($\nu < 0$) $A$ type Inozemtsev model.

**Case 4).** $h(q) = \cosh 2\sqrt{\nu}q$:

This leads to the hyperbolic ($\nu > 0$) and trigonometric ($\nu < 0$) $BC$ type Inozemtsev model. The quasi-solvability of the special cases of the above four were recently shown in Ref. \[ \text{(36)} \] by an ansatz method.

**Case 5).** $h(q) = \sinh 2\sqrt{\nu}q$:

This leads to a hyperbolic model being neither the Inozemtsev nor the Olshanetsky–Perelomov type \[ \text{(37)} \]. The paper Ref. \[ \text{(30)} \] covers most of the above five models.

**Case 6).** $h(q) = \tan \sqrt{\nu}q$:

This leads to a trigonometric model being neither the Inozemtsev nor the Olshanetsky–Perelomov type.

**Case 7).** $h(q) = \wp(q; g_2, g_3)$:

This case includes the elliptic $BC$ type Inozemtsev model and the twisted CS models \[ \text{(38, 39, 40)} \]. The elliptic model in Ref. \[ \text{(41)} \] may be also included in this case.

**Case 8).** $h(q) = \text{sn} (\sqrt{\nu}q|k)/ \text{cn} (\sqrt{\nu}q|k)$:

This leads to an elliptic model being neither the Inozemtsev nor the Olshanetsky–Perelomov type.
The two-body potentials in all the cases have singularities at \( q_i = q_j \) \((i \neq j)\). Thus, each of the models is naturally defined on a Weyl chamber if the potential is non-periodic or on a Weyl alcove if the potential is periodic [37]. Cases 1–5 with \( \nu > 0 \) and Case 6 with \( \nu < 0 \) correspond to the former while the others to the latter. In the latter case, a system can be quasi-exactly solvable unless a pole of a one-body potential in the system exists and is in the Weyl alcove. On the other hand, quasi-exact solvability in the former case depends mainly on the asymptotic behavior of Eq. (21) at \( |q_i| \to \infty \). Since this behavior is in general not dominated by the two-body term in the r.h.s. of Eq. (20), most of the results on the normalizability of the one-body \( \mathfrak{sl}(2) \) quasi-solvable models in Ref. [3] may also hold for our models.

More details on the results presented here and further development will be reported in the near future [42].

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