The initial-boundary problem for the system of 1D equations of non-Newtonian hemodynamics

Gerasim V Krivovichev
Faculty of Applied Mathematics and Control Processes Saint Petersburg State University, 7/9 Universitetskaya nab., Saint Petersburg, 199034, Russian Federation
E-mail: g.krivovichev@spbu.ru

Abstract. The paper is devoted to the analytical solution of the problem for 1D hemodynamical equations with periodic boundary conditions. The method of the solution is based on the asymptotic expansions on the small parameter and Fourier method. The attention is focused only on the first-order terms in the expansion. The solution, obtained for the particular case of initial conditions, is used for the comparison of rheological models of blood. It is demonstrated that the strongest damping takes place for the Power Law non-Newtonian model.

1. Introduction

In modern medicine, the methods of mathematical and computer modeling are widely used in the cardiovascular investigations [1]. The main purpose of the model application is the prediction of the results of operations and the effects of the pathologies (stenoses, aneurysms, etc.) [2, 3]. The 1D models are obtained by the averaging of the Navier–Stokes system in cylindrical coordinates on the vessel cross-section [2]. These models are successfully applied for the simulation of flows in a complete cardiovascular system or systems of single organs [4]. Most of the works, devoted to 1D models, are based on inviscid or Newtonian models of blood. The 1D blood flow model, which includes the non-Newtonian nature of blood (based on generalized Oldroyd-B model), is considered only by Ghigo et al [5].

Analytical solutions play a very important role in 1D modeling of blood flow as a powerful tool for the comparison of models and for the testing of programs, which realize numerical methods. In most of the works, the analytical solutions are obtained for the case of the inviscid fluid model, infinite vessel and attention is focused on the Riemann problem [6, 7, 8]. Ilyin [9] obtain the analytical solution for the inviscid model in the case of a semi-infinite vessel. The analytical solutions in the case of the viscous Newtonian model are obtained by Mukhin et al [10] and Ashmetkov et al [11]. In [10], the Cauchy problem for the case of the infinite vessel is solved by the perturbation method. In [11], the initial-boundary problem for the linearized system is solved by the method of characteristics. Some analytical investigations for the case of nonlinear viscosity are realized in [12].

The presented paper is devoted to the analytical solution of the initial-boundary problem for 1D hemodynamical equations with periodic boundary conditions. The approach is based on the perturbation technique and Fourier method. The proposed approach can be applied to the inviscid and viscous models, including non-Newtonian. The obtained solutions can be applied
for the comparison of different rheological models of blood and for the testing of programs, which realize the numerical algorithms for solution of nonlinear problems in more realistic situations.

The paper has the following structure. In Section 2, the 1D model of blood flow is presented. In Section 3, the perturbation technique and Fourier method are described. Section 4 is devoted to the numerical experiment, where the inviscid and viscous models are compared. Some concluding remarks are made in Section 5.

2. The one-dimensional model of hemodynamics

The hyperbolic system, obtained as the model of blood flow in 1D case, is written as [2]:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\alpha}{\partial z} \left( \frac{Q^2}{A} \right) + \frac{A}{\rho} \frac{\partial p}{\partial z} = f(A, Q),$$

(1)

where $t$ is a time, $z$ is a spatial coordinate, $A = A(t, z)$ is the vessel cross-section area, $Q = Q(t, z)$ is a flow rate, $p = p(t, z)$ is the pressure, $\rho$ is a constant density, $\alpha$ is a Boussinesq coefficient, $f(A, Q)$ represents the viscous term (the case of inviscid fluid corresponds to $f \equiv 0$).

The system (1) should be closed by the equation-of-state $p = p(A)$. For the case of the arterial flow, the following dependence is used [7]:

$$p(A) = p_{\text{min}} + \frac{\beta}{A_{\text{min}}} \left( \sqrt{A} - \sqrt{A_{\text{min}}} \right), \quad \beta = \frac{4}{3} \sqrt{\pi} Eh,$$

(2)

where $p_{\text{min}}$ and $A_{\text{min}}$ are the diastolic pressure and cross-section area, $E$ is the elastic modulus, $h$ is a vessel wall thickness. In the presented paper, the case of the uniform vessel is considered, where these characteristics are the constants.

After the substitution of (2) into (1) the system is rewritten as:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\alpha}{\partial z} \left( \frac{Q^2}{A} \right) + \gamma \sqrt{A} \frac{\partial A}{\partial z} + f(A, Q) = 0,$$

(3)

where $\gamma = \beta / (2A_{\text{min}} \rho)$.

In dimensionless variables system (3) is rewritten in the following form [13]:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\alpha}{\partial z} \left( \frac{Q^2}{A} \right) + \chi \frac{\partial A}{\partial z} + f(A, Q) = 0,$$

(4)

where $\chi = \gamma A_m / (\rho U_m^2)$, $A_m$ is a typical cross-section area and $U_m$ is a typical velocity.

For the viscous Newtonian model, $f(A, Q)$ in (4) is presented as $f(A, Q) = -\varepsilon Q/A$, where $\varepsilon = kT_m/A_m$, where $T_m$ is a typical time and $k = -2\pi \mu s'(1)/\rho$, $\mu$ is a dynamic viscosity, $s = s(y)$ is a dimensionless velocity profile, $y$ is a dimensionless radius, $s'(1) = -4$ for the Newtonian model.

For the case of viscous non-Newtonian Power Law (PL) model, $f(A, Q)$ is written as:

$$f(A, Q) = -\varepsilon Q|Q|^{n-1} A^{-\frac{n-1}{n}},$$

where $\varepsilon = KU_m^{-1} T_m A_m^{-\frac{n+1}{n}}$, $K = -ks'(1)|s'(1)|^{n-1} \pi^{n+1} / \rho$, where $k$ and $n$ are the parameters of PL model, $s'(1) = -(3 + 1/n)$. It is proposed, that $Q \geq 0 \forall (t, z)$.

As it is estimated in [13], the parameter $\varepsilon$ for the arteries near the heart can be considered as $\varepsilon \sim 0.01$, so the small parameter exists in the models, and the perturbation method can be applied.

In the presented paper, the initial-boundary problem for the system (4) is considered with the following periodic boundary conditions

$$A(t, 0) = A(t, 1), \quad Q(t, 0) = Q(t, 1),$$

(5)

and initial conditions

$$A(0, z) = A^0(z), \quad Q(0, z) = Q^0(z).$$

(6)
3. Method of the solution

According to the perturbation method, let the initial functions \( A^0(z) \) and \( Q^0(z) \) are presented as:

\[
A^0(z) = A_0 + \varepsilon \varphi_1(z) + \varepsilon^2 \varphi_2(z) + \ldots, \quad Q^0(z) = Q_0 + \varepsilon \psi_1(z) + \varepsilon^2 \psi_2(z) + \ldots, \quad (7)
\]

where \( A_0, Q_0 \) are the constants.

So, the solution of (4)–(6) is obtained in the following form:

\[
A(t, z) = A_0 + \varepsilon A_1(t, z) + \varepsilon^2 A_2(t, z) + \ldots, \quad Q(t, z) = Q_0 + \varepsilon Q_1(t, z) + \varepsilon^2 Q_2(t, z) + \ldots. \quad (8)
\]

In the presented paper, the attention is focused on the first-order terms, so only the problem for \( A_1(t, z) \) and \( Q_1(t, z) \) is considered. The problem is obtained by the substitution of (7) and (8) into (4)–(6):

\[
\frac{\partial A_1}{\partial t} + \frac{\partial Q_1}{\partial z} = 0, \quad \frac{\partial Q_1}{\partial t} + \left( \chi \sqrt{A_0 - \alpha \frac{Q_1^2}{A_0^2}} \right) \frac{\partial A_1}{\partial z} + \frac{\alpha Q_0}{A_0} \frac{\partial Q_1}{\partial z} = f_0, \quad (9)
\]

\[
A_1(t, 0) = A_1(t, 1), \quad Q_1(t, 0) = Q_1(t, 1), \quad (10)
\]

\[
A_1(0, z) = \varphi_1(z), \quad Q_1(0, z) = \psi_1(z), \quad (11)
\]

where \( f_0 \equiv 0 \) for inviscid model, \( f_0 = -Q_0/A_0 \) for the Newtonian model and \( f_0 = -Q_0^n/A_0^{3n-1} \) for the PL model.

For the general view, let the second equation in (9) is rewritten as:

\[
\frac{\partial Q_1}{\partial t} + b_1 \frac{\partial A_1}{\partial z} + b_2 \frac{\partial Q_1}{\partial z} = f_0,
\]

where \( b_1, b_2 \) are the constants.

At first, let the homogeneous problem \( (f_0 \equiv 0) \) (9)–(11) is considered. Its solution can be obtained by the Fourier method, based on the separation of variables:

\[
A_1(t, z) = a(t) Z(z), \quad Q_1(t, z) = q(t) Z(z). \quad (12)
\]

After the substitution of (12) into (9), from the first equation, the following relation is obtained:

\[
\frac{\dot{a}(t)}{q(t)} = -\frac{Z'(z)}{Z(z)}.
\]

So the following eigenvalue problem can be stated:

\[
Z'(z) - \lambda Z(z) = 0, \quad Z(0) = Z(1). \quad (13)
\]

This problem has the following eigenvalues: \( \lambda_m = 2\pi m i, \quad i^2 = -1 \) and eigenfunctions \( Z_m(z) = e^{i2\pi mz}, \quad m \in \mathbb{Z} \).

The solution of (9)–(11) is presented by Fourier series:

\[
A_1(t, z) = \sum_{m=-\infty}^{+\infty} a_m(t) e^{i2\pi mz}, \quad Q_1(t, z) = \sum_{m=-\infty}^{+\infty} q_m(t) e^{i2\pi mz}.
\]

Functions \( a_m(t) \) at \( m \neq 0 \) are obtained as the solutions of the second-order equations:

\[
\ddot{a}_m(t) + \lambda_m b_2 \dot{a}_m(t) - \lambda_m^2 b_1 a_m(t) = 0,
\]
Figure 1. The plots of real part of $Q(t, 0.5)$ for different models of blood: 1 — inviscid fluid; 2 — Newtonian fluid; 3 — PL non-Newtonian fluid at $n = 0.9$

which have the following solutions:

$$a_m(t) = C_{1m}e^{i\pi m \kappa_1 t} + C_{2m}e^{i\pi m \kappa_2 t},$$

where $\kappa_{1,2} = -b_2 \pm \sqrt{b_2^2 + 4b_1}$, and $a_0(t) = C_0 = const$.

Functions $q_m(t)$ are presented as $q_m(t) = -\ddot{a}_m(t)/\lambda_m$ at $m \neq 0$ and $q_0(t) = D_0 = const$. All constants $C_{1m}$, $C_{2m}$, $C_0$, $D_0$ are obtained from the Fourier expansions of the initial conditions: $a_m(0) = \varphi_{1m}$, $q_m(0) = \psi_{1m}$, where $\varphi_{1m}$ and $\psi_{1m}$ are written as:

$$\varphi_{1m} = \int_0^1 \varphi_1(z)e^{-i2\pi mz}dz, \quad \psi_{1m} = \int_0^1 \psi_1(z)e^{-i2\pi mz}dz.$$

At the case of nonhomogeneous problem ($f_0 \neq 0$), the expressions for $a_m(t)$ and $q_m(t)$ are the same, and in the case of $m = 0$ $a_0(t) = C_0 = const$, $q_0(t) = f_0 t + D_0$. All constants are obtained by the same procedure.

4. Numerical example

In this section, the method, described above, is used for the comparison of models of blood. The following parameters are considered: $A_0 = \pi$, $Q_0 = 1$, $\chi = 17$, $\varepsilon = 0.01$, $n = 0.9$. Initial conditions are stated as: $\varphi_1(z) = 1$, $\psi_1(z) = \sin(2\pi z)$. The time interval $[0, \pi]$ is considered.
Figure 2. The plots of real part of $Q(\pi, z)$ for different models of blood: 1 — inviscid fluid; 2 — Newtonian fluid; 3 — PL non-Newtonian fluid at $n = 0.9$.

At figures 1–2 the plots of the real part of flow rate $Q(t, z)$ are presented at fixed $z$ and final time moment, respectively. As can be seen, the strongest damping of the solution takes place in the PL non-Newtonian model. The existence of the deviations of the solutions corresponds to two viscous models demonstrate the importance of the inclusion of non-Newtonian effects in 1D models.

5. Conclusion
The paper is devoted to the analytical solution of the problem for 1D hemodynamical equations with periodic boundary conditions. The method of the solution is based on the asymptotic expansions on the small parameter and Fourier method. The attention is focused only on the first-order terms in the expansion. The solution, obtained for the particular case of initial conditions, is used for the comparison of rheological models of blood. It is demonstrated that the strongest damping takes place for the PL non-Newtonian model.

Despite the simplification of the considered problem (periodic boundary conditions), the presented approach for obtaining analytical solutions can be used for the following reasons:

1) As a tool for the comparison of different rheological models of blood. By the obtained solutions different effects, such as an amplitude behavior, decaying of the solution, the effect of hematocrit and others, can be demonstrated.

2) As a tool for the testing of programs, which realize the numerical algorithms for the solution of nonlinear 1D problems for the system (1) in realistic situations.
References

[1] Quarteroni A, Manzoni A and Vergara C 2017 Acta Numer. 26 365
[2] Formaggia L, Lamponi D and Quarteroni A 2003 J. Eng. Math. 47 251
[3] Razavi A, Shirani E and Sadeghi M 2011 J. Biomech. 44 2021
[4] Alastruey J, Parker K, Peiro J, Byrd S and Sherwin S 2007 J. Biomech. 40 1794
[5] Ghigo A, Lagree P and Fullana J 2018 J. Non-Newt. Fluid Mech. 253 36
[6] Toro E and Siviglia A 2013 Commun. Comput. Phys. 13 361
[7] Toro E 2016 Appl. Math. Comput. 272 542
[8] Spiller C, Toro E, Vasquez-Cendon M and Contarino C 2017 Appl. Math. Comput. 303 178
[9] Ilyin O 2019 Wave Motion 84 56
[10] Mukhin S, Sosnin N and Favorskii A 2006 Diff. Eq. 42 1041
[11] Ashmetkov I, Mukhin S, Sosnin N, Favorskii A and Khruleenko A 2000 Diff. Eq. 7 1021
[12] Mozokhina A and Mukhin S 2018 Diff. Eq. 54 938
[13] Tkachenko P and Krivovich G 2019 J. Phys.: Conf. Ser. 1400 044031