Asymmetric Lévy flights in nonhomogeneous environments

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Abstract. We consider stochastic systems involving general—non-Gaussian and asymmetric—stable processes. The random quantities, either a stochastic force or a waiting time in a random walk process, explicitly depend on the position. A fractional diffusion equation corresponding to a master equation for a jumping process with a variable jumping rate is solved in a diffusion limit. It is demonstrated that for some model parameters the equation is satisfied in that limit by the stable process with the same asymptotics as the driving noise. The Langevin equation containing a multiplicative noise, depending on the position as a power-law, is solved; the existing moments are evaluated. The motion appears subdiffusive and the transport depends on the asymmetry parameter: it is fastest for the symmetric case. As a special case, the one-sided distribution is discussed.

Keywords: stochastic particle dynamics (theory), stochastic processes (theory), diffusion
1. Introduction

Long jumps and divergent fluctuations are observed in various areas: in physics, biology, finance and sociology [1]–[3], indicating the existence of the power-law tails of the distributions, \( \sim |x|^{-1-\alpha} \), where \( 0 < \alpha < 2 \) is a stability index (Lévy flights). In contrast to the Gaussian distribution, the general stable Lévy distribution can be asymmetric and, in the limiting cases, assume a stretched exponential tail. In particular, the distribution may be restricted to a half-axis. Specifically the asymmetric processes, where the asymmetry is measured by a ‘skewness’ parameter \( \beta \), emerge in many problems and are frequently discussed in the literature. A complicated picture of the anomalous diffusion in the reaction–diffusion systems is due to asymmetric Lévy flights and the right-moving fronts accelerate exponentially. They develop an algebraic tail, while the left-moving fronts have exponential decaying tails and move at a constant speed [4]. A fractional advection–dispersion equation with any degree of skewness in several dimensions was analysed in [5], whereas the problem of the diffusion in porous media, which display a fractal structure, was studied by a stochastic equation driven by the asymmetric Lévy process in [6]. The first passage times for the asymmetric Lévy flights were evaluated in [7]–[9] and properties of the stochastic resonance discussed in [10, 11]. A model of a granular material, which takes into account disordered packings of rigid, frictionless disks in two dimensions under gradually varying stress, predicts a dependence of a strain on the stress direction in a form of the asymmetric Lévy distribution [12]. Strongly asymmetric Lévy flights were observed in cracking of heterogeneous materials [13, 14]. It was demonstrated in the field of finance that prices of derivatives satisfy a fractional partial differential equation corresponding to the asymmetric Lévy processes [15]. In the framework of the jumping processes, a diffusion equation, fractional both in space and time and corresponding to a master equation for the random walk with the asymmetric Lévy distribution, was discussed in [16]; the appropriate algorithms were derived in order to numerically simulate the time-evolution.
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Dynamical descriptions of materials containing impurities and defects must take into account random quantities which actually do not evolve with time if the timescale of the impurities’ diffusion is much larger than that of the measured variable (a quenched disorder). This introduces a correlation between the successive trapping times; the trapping time at a given site is the same for each visit of this site [2]. As a result, the hopping rates are position-dependent: the jump probability function in a random walk description has a coupled form and the corresponding Langevin equation possesses a multiplicative noise. Usually, the Langevin equation with the additive noise is studied and, in such approaches, the medium nonhomogeneity is actually reduced to a homogeneous distribution of the noise activation times [17]. Only a few studies are devoted to multiplicative noise. The anomalous diffusion in a composite medium was studied in terms of a fractional equation with a variable coefficient in [18]. A master equation, describing a thermal activation of jumping particles within folded polymers, also contains a variable diffusion coefficient [19]. A stochastic Lotka–Volterra model has been applied to the case when the extreme events exhibit Lévy statistics [20] and a Verhulst equation to a population density description [21]. A recent analysis [22] of a tumour growth includes a coupling between the tumour and an immune cell which leads to a multiplicative noise. Traps make the transport slower: anomalous diffusion is a subdiffusion, i.e. the variance, if it exists, rises slower than linearly. The accelerated diffusion, in turn, emerges when variance is infinite, due to the long jumps.

Studies of the Langevin equation with multiplicative Lévy noise [23, 24] for the symmetric case demonstrate that physical conclusions qualitatively depend on a particular interpretation of the stochastic integral. The dynamics in the Stratonovich interpretation may be characterized by a finite variance, even in the absence of any potential, and then the solution exhibits fast-falling power-law tails and a subdiffusive behaviour. On the other hand, the Itô interpretation predicts the same asymptotics as the driving noise has. In this paper, we consider a general, asymmetric case and demonstrate, in particular, how the asymmetry parameter influences the anomalous transport. We begin with the continuous time random walk (CTRW) theory, well-known for the Lévy flights, but usually restricted to homogeneous distributions of the waiting time.

2. Random walk with a variable jumping rate

CTRW is defined in terms of two distributions: the waiting-time distribution \( w(t) \) and the jump-length distribution \( Q(x) \). Usually, one assumes that they are independent stochastic variables (the decoupled version of CTRW) and the resulting process appears non-Markovian, except the case of the Poissonian \( w(t) \). If \( w(t) \) has long tails and \( Q(x) \) is the Gaussian, the Fokker–Planck equation, which emerges from the master equation in the limit of small wave numbers, is fractional in time [3]. Then the trapping times hamper the transport and one observes subdiffusion. When, on the other hand, \( Q(x) \) obeys the general Lévy statistics with \( \alpha < 2 \), the variance is infinite.

We consider the Markovian case and assume, in contrast to the usual approach, that the jumping rate depends on the process value, \( \nu = \nu(x) \) [25]. The Markovian property...
implies a Poissonian form,

$$w(t) = w(t|x) = \nu(x)e^{-\nu(x)t},$$

and by introducing the variable $\nu(x)$ we take into account that the waiting time depends on the medium structure. The process is defined by an infinitesimal stationary transition probability,

$$p_{tr}(x, \Delta t|x', 0) = \left\{1 - \nu(x')\Delta t\right\}\delta(x - x') + \nu(x')\Delta tQ(x - x').$$

The particle remains at rest for a time sampled from $w(t)$ after which it instantaneously jumps to a new position and then the process is stepwise constant. The first term on the right-hand side of equation (2) is the probability that no jump occurred in the time interval $(0, \Delta t)$ and the term $\nu(x')\Delta t$ means the probability that one jump occurred. The master equation derived from equation (2) is the following:

$$\frac{\partial}{\partial t}p(x, t) = -\nu(x)p(x, t) + \int Q(x - x')\nu(x')p(x', t)dx'.$$

The distribution $Q(x)$ represents the Lévy flights. Trajectories corresponding to that kinetics form a self-similar clustering at all scales and exhibit long jumps between clusters; such an intermittent behaviour is frequently observed in physical phenomena and modelled by CTRW ([9] and references therein). $Q(x) = Q_{\alpha,\beta}(x)$ is assumed as a general stable distribution defined by the parameters $\alpha$ and $\beta$: $0 < \alpha \leq 2$ and $|\beta| \leq \alpha$ for $0 < \alpha < 1$ and $|\beta| \leq 2 - \alpha$ for $1 < \alpha < 2$. The case $\alpha = 1$ is special and will not be considered; we also neglect parameters related to translation and scaling of the distribution. The characteristic function has the form

$$\tilde{Q}_{\alpha,\beta}(k) = \exp\left[-|k|^\alpha \exp\left(i\frac{\pi\beta}{2}\text{sign}(k)\right)\right]$$

and the density follows from the inverse Fourier transform,

$$Q_{\alpha,\beta}(x) = \frac{1}{\pi}\Re \int_0^\infty \tilde{Q}_{\alpha,\beta}(k)e^{-ikx}dk.$$
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that leads, after applying equation (5) and for \( \nu(x) = \text{const} \), to a fractional differential equation [29]. The nonhomogeneity of the medium is reflected by the \( x \)-dependence of the jumping rate: the sojourn time of the particle in a trap depends on the position and then the diffusion coefficient in equation (7) is variable. We assume

\[
\nu(x) = K|x|^{-\theta},
\]

where \( K \) has been introduced for dimensional reasons; in the following we take \( K = 1 \). In particular, when \( \theta > 0 \) the average jumping rate is large near the origin whereas the average waiting time becomes large at large distances. The power-law form of \( \nu(x) \) is natural for problems exhibiting self-similarity, which often happens for disordered materials; it has been applied e.g. to study diffusion on fractals [30, 31] and turbulent two-particle diffusion [32]. To solve equation (7) we assume \( \alpha > 1 \) but a generalization to the case \( \alpha < 1 \) is straightforward.

There is no general method to exactly solve the fractional equation with a variable diffusion coefficient. However, since the equation itself was derived by applying condition (6), we may restrict our considerations to the limit \( |k| \to 0 \) without introducing any additional idealization. The solution is not unique since one can construct, in principle, a family of solutions the characteristic functions of which differ at orders higher than \( |k|^\alpha \). Can this family include the stable distributions? We will demonstrate that it is indeed the case but only in a limited range of the model parameters. The stable distribution can always be expressed in a form of the Fox \( H \)-function with well-determined coefficients [33]–[35]. Therefore, the variability of the diffusion coefficient may manifest itself in the solution only as a time-dependent scaling factor and the required solution of equation (7) with the initial condition \( p(x,0) = \delta(x) \) must have the form

\[
p(x,t) = \frac{\epsilon}{\sigma(t)^{\epsilon}} H_{2,2}^{1,1} \left[ \frac{x}{\sigma(t)^{\epsilon}} \ , \ \left(1 - \epsilon, \epsilon, 1 - \gamma, \gamma \right) \right],
\]

where \( \epsilon = 1/\alpha, \gamma = (\alpha - \beta)/2\alpha \) and the function \( \sigma(t) \) is to be determined. The derivation, presented in appendix A, shows that (9) satisfies equation (7) to the lowest order in \( |k| \) and yields a power-law time-dependence

\[
\sigma(t) = [A(\alpha + \theta)t]^{\alpha/(\alpha + \theta)},
\]

where

\[
A = \frac{2}{\pi \alpha^2} \Gamma(\theta/\alpha) \Gamma(1 - \theta) \sin \left( \frac{\pi \theta}{2} \right) \cos \left( \frac{\beta \theta}{2\alpha} \right)
\]

and \(-\alpha < \theta < 1\). The latter inequality specifies the conditions under which it is possible to express the solution of equation (7), valid in the diffusion limit, in the form of the stable process. The asymptotic form of the distribution, \( \sim |x|^{-1-\alpha} \), follows from the expansion of the \( H \)-function in equation (9); it is the same as that of the driving noise. The solution (9) satisfies the following scaling relations:

\[
t \to \lambda t, \quad x \to \lambda^\alpha x \quad \text{and} \quad p(x,t) \to \lambda^\alpha p(\lambda^\alpha x, \lambda t),
\]

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Figure 1. The fractional moments as a function of the asymmetry parameter calculated from equation (12) for \( \delta = 1, \alpha = 1.5 \) and the following values of \( \theta \): 
\(-1.2, -1, -0.5, 0, 0.5 \) (from bottom to top).

where \( \kappa = 1/(\alpha + \theta) \) and \( \lambda > 0 \) is an arbitrary scaling parameter. On the other hand, the scaling relations (11) can be directly inferred from the fractional equation (7).

Equation (7) can be solved also for \( \theta \geq 1 \) and for that purpose the \( H \)-function coefficients must be modified similarly to the symmetric case [36, 37]. However, then we would leave a domain of stable distributions.

The asymptotic shape of the solution indicates that all moments of the order \( \delta \geq \alpha \) diverge and the transport properties may be quantified by fractional moments. The existing fractional moments, corresponding to the solution (9), are given by the Mellin transform from the \( H \)-function,

\[
\langle |x|^{\delta} \rangle \sim t^{\delta/(\alpha + \theta)}.
\]

The proportionality coefficient, presented in figure 1 as a function of \( \beta \) for some values of \( \theta \), rises with \( \theta \) and the maximum always corresponds to the symmetric case. For \( \theta = 0 \), \( \langle |x|^{\delta} \rangle (\beta) \) has a cosine shape.

The divergence of the variance may be unphysical if one considers the motion of a massive particle. However, this property does not violate physical principles for such problems as the diffusion in energy space in spectroscopy, or for the diffusion on a polymer chain in chemical space [3]. To get rid of the difficulty of divergent moments one introduces the Lévy walk [3] which relates the jump length to the velocity. It is possible to generalize the above model by introducing a dependence of the velocity variance on time; then
one obtains a strong anomalous diffusion and scaling relations different from those for the ordinary Lévy walk [38]. One the other hand, one can argue that every system is finite and introduce a truncation of the distribution in the form of a fast-falling tail. The truncated distribution agrees with the Lévy distribution up to an arbitrarily large value of the argument and has finite variance. The problem of truncated Lévy flights for the multiplicative processes (in the symmetric case) was discussed in [37]. Variance is always finite for $\alpha = 2$ and then all kinds of diffusion emerge. In particular, for $\theta < 0$, we observe enhanced diffusion, $\langle |x|^\theta \rangle \sim t^{\delta/(2+\theta)}$, which represents a strong anomalous diffusion in the sense of [39].

3. Langevin equation

The stochastic dynamics driven by a multiplicative, algebraic random force and a linear deterministic force is governed by the Langevin equation,

$$dx = -\lambda x \, dt + K|x|^{-\theta/\alpha} \, d\eta(t), \tag{13}$$

where we assume that the increments $d\eta(t)$ are distributed according to (5); $\lambda \geq 0$ and the constant $K = 1$ cm$^{\theta/\alpha}$ will be dropped in the following. Since $\eta$ represents the white noise, equation (13) requires clarification of how a stochastic integral is to be interpreted [40]. In the Itô interpretation, which is frequently used due to its simplicity, the noise term is evaluated before the noise acts and applies when the noise consists of clearly separated pulses; this is, e.g., the case for CTRW. Moreover, it has been demonstrated both for the Gaussian [41] and for the general Lévy case [42] that this interpretation is suited to problems with a large inertia. On the other hand, the Stratonovich interpretation, which takes into account a middle point between the subsequent noise activations, applies if a system has finite correlations and the white noise is only an approximation. For the Gaussian case, the difference between the above interpretations resolves itself to a drift term in the corresponding Fokker–Planck equation.

The dynamics involving multiplicative noise and an arbitrary potential can be expressed in the Itô interpretation by a fractional equation with a variable diffusion coefficient [17] which in our case takes the form [29]\(^1\)

$$\frac{\partial p}{\partial t} = -\lambda \frac{\partial}{\partial x}(xp) - (-\Delta)^{\alpha/2}(|x|^{-\theta}p) + \tan(\pi\beta/2) \frac{\partial}{\partial x}(-\Delta)^{(\alpha-1)/2}(|x|^{-\theta}p), \tag{14}$$

where

$$(-\Delta)^{\alpha/2} f(x) = F^{-1}(|k|^\alpha \tilde{f}(k)). \tag{15}$$

Equation (14) involves, beside the usual fractional diffusion term—present in the Fokker–Planck equation for the symmetric case—a contribution to the convection due to the existence of the preferred direction [29]. The equation which determines the characteristic function is a generalization of equation (7),

$$\frac{\partial \tilde{p}(k,t)}{\partial t} = -\lambda k \frac{\partial}{\partial k} \tilde{p}(k,t) - |k|^\alpha \exp \left(\frac{i\pi\beta}{2} \text{sign}(k)\right) F[|x|^{-\theta}p(x,t)]. \tag{16}$$

\(^1\) Equation (14) is not unique because the definition of the fractional operator itself is not unique [29].

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Figure 2. The density distributions calculated by integration of equation (13) in the Itô interpretation for $\lambda = 1$, $\alpha = 1.5$ and $t = 5$. The solid lines mark the dependence $|x|^{-1-\alpha}$. For each curve, $10^7$ trajectories were evaluated.

We look for a solution in the diffusion limit of small $|k|$ and apply a procedure similar to that in section 2. The solution with the initial condition $p(x,0) = \delta(x)$ is given by equation (9) and $\sigma(t)$ satisfies the equation

$$\dot{\sigma}(t) = -\alpha \lambda \sigma(t) + \alpha A \sigma(t)^{-\theta/\alpha}$$

which has the solution

$$\sigma(t) = \left[ \frac{A}{\lambda} (1-e)^{-\lambda(\alpha+\theta)t} \right]^{1/c_{\theta}}$$

where $c_{\theta} = 1 + \theta/\alpha$. $p(x,t)$ converges with time to a stationary state and the tails $\sim |x|^{-1-\alpha}$ ($|\beta| < 2 - \alpha$) make the variance divergent for any $t$. Numerical values of $p(x,t)$ can be obtained by expanding the $H$-function near $x = 0$ and $|x| = \infty$, by means of equation (A.5).

The case $|\beta| = 2 - \alpha$ has a different asymptotics which is presented in appendix B. On the other hand, the density distributions can be calculated from a direct numerical integration of equation (13). Figure 2 presents examples of such distributions, close to the stationary states, where the driving noise was sampled according to a standard procedure [43]. The tails indicate a power-law shape except the case $\beta = 2 - \alpha$ when the left tail falls faster than exponentially.

We have already mentioned that an important property of the above solution is that all moments of the order $\geq \alpha$ diverge. The existence of the infinite variance was reported for some physical problems, e.g. for rain and cloud fields [44]; according to that study, the experimental radar rain reflectivities indicate the divergence of all moments higher than the value 1.06. However, the infinite variance is often unphysical and its presence in such problems as the advecting field for porous media and atmospheric turbulence has
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been questioned [17]. Usually, one observes power-law distributions with the tails falling faster than for stable Lévy distributions. This is the case for the financial market [45]–[47] and the minority game [48]; fast-falling power-law tails result from a multifractal analysis of extreme events [49] and emerge when one introduces a power-law truncation to the distribution [37, 50]. We shall demonstrate that equation (13) predicts such fat tails, without introducing any truncation, but in a different interpretation of the stochastic integral.

The physical importance of the Stratonovich interpretation, in which the random driving is evaluated at a middle point between its consecutive activations, stems from the fact that it corresponds to a white-noise limit of the coloured noises. Then the usual change of the variable leads to the Langevin equation with additive noise—for one-dimensional systems and for Gaussian noise [40]. If \(\alpha < 2\), one can formally define the white noise \(\eta\) as a limit of coloured noise: construct a coloured-noise process, change the variable and finally take the white-noise limit. This procedure can be easily accomplished for the generalized Ornstein–Uhlenbeck process,

\[
d\eta_c(t) = -\gamma_n \eta_c(t) dt + \gamma_n dL(t),
\]

where \(dL(t)\) has the stable Lévy distribution [42]; then \(d\eta(t)\) is given by a limit of the vanishing relaxation time, \(d\eta(t) = \lim_{\gamma_n \to \infty} d\eta_c(t)\). The numerical analysis for the symmetric noise demonstrates [23, 24] that results obtained by means of the variable transformation agree with those for the white noise in the Stratonovich interpretation. Then we solve the equation

\[
\dot{y} = -\lambda c_\theta y + \eta(t),
\]

obtained from equation (13) by the transformation

\[
y(x) = \frac{1}{c_\theta} |x|^{c_\theta \lambda} \text{sign}(x),
\]

assuming \(\alpha + \theta > 0\). Equation (20) is easy to solve [51] and applying the identity \(p(x, t) = p(y(x), t) dy/dx\) yields the final solution. Since the higher and lower domains of \(\alpha\) are qualitatively different, it is expedient to consider them separately.

3.1. The case \(\alpha > 1\)

The solution of equation (20) for this case can be expressed in the same form as equation (9) [33]. After the variable transformation, the solution of equation (13) for \(x > 0\) reads

\[
p(x, t) = \frac{c_\theta}{\alpha x} H_{2,2}^{1,1} \left[ \frac{x^{c_\theta}}{c_\theta \sigma_s(t)^{1/\alpha}} \right]_{(1,1), (1,1), (1,1)},
\]

where

\[
\sigma_s(t) = \frac{1 - e^{-\lambda(\alpha+\theta)t}}{\lambda(\alpha+\theta)},
\]

\[
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\]
and one should change $\beta \to -\beta$ to get the solution for $x < 0$. The numerical values of $p(x, t)$ for small $|x|$ follow from the $H$-function expansion, equation (A.5). The derivation yields the series

$$p(x, t) = \frac{c_0}{\pi} \sum_{n=1}^{\infty} c_0^{-n} \sigma_s(t)^{-n/\alpha} \Gamma(n/\alpha) \sin(\pi n\gamma) \frac{(-1)^{n-1}}{(n-1)!} x^{c_0 n-1}. \quad (24)$$

Similarly, after transforming the argument of the $H$-function $x \to x^{-1}$, we get the asymptotic expansion,

$$p(x, t) = \frac{c_0}{\pi} \sum_{n=1}^{\infty} c_0^{n\alpha} \sigma_s(t)^{n\alpha} \Gamma(n\alpha) \sin(\pi n\alpha) \frac{(-1)^{n-1}}{(n-1)!} x^{-1-(\alpha+\theta)n}. \quad (25)$$

Therefore, the asymptotic form of the distribution, $\sim |x|^{-1-\alpha-\theta}$, differs from the Itô version: the slope depends on $\theta$ and may be arbitrarily large. The above results are valid for $\beta \neq |\alpha - 2|$; otherwise the distribution falls faster than exponentially (see appendix B). Although all the integer moments higher than the first of this extremely asymmetric process are still infinite, the existence of the right-sided Laplace transform for $\beta = \alpha - 2$ makes it useful for applications e.g. in finance, where it is known as the FMLS model. A particular feature of the FMLS process is that it only exhibits downwards jumps, while upwards movements have continuous paths [15].

Figure 3 presents a comparison of the analytical distributions, equation (24), with those obtained from the numerical simulations for some values of $\theta$; a relatively large time, sufficient to reach the stationary state, was chosen. For a positive $\theta$ the density vanishes at $x = 0$ and the peak is split, in contrast to $\theta < 0$ when $p(0, t)$ is infinite.

The dependence of the distribution slope on $\theta$ makes this parameter responsible for the existence of the moments, in contrast to the Itô case, and that property implies...
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Figure 4. The variance as a function of the asymmetry parameter calculated from equation (26) for $\alpha = 1.1$ and the following values of $\theta$: 1.5, 2, 3 and 4 (from top to bottom).

important consequences for the diffusion ($\lambda = 0$). It may be accelerated, as it was for the Itô interpretation, but if $\theta$ is chosen sufficiently large, the moment of an arbitrary high order exists, namely the $n$th moment exists if $\theta > n - \alpha$. Let us evaluate the variance assuming $\alpha + \theta > 2$. Using the Mellin transform from the $H$-function yields

$$
\langle x^2 \rangle (t) = \frac{1}{\alpha \pi c_\theta} \frac{t^{2/(\alpha + \theta)}}{\Gamma(1 + 2/(\alpha + \theta))} \sin \left( \frac{\beta \pi}{\alpha + \theta} \right) \cos \left( \frac{\beta \pi}{\alpha + \theta} \right) t^{2/(\alpha + \theta)}. \quad (26)
$$

Therefore, the variance, if it exists, always rises with time slower than linearly and we observe subdiffusion. Equation (26) is illustrated with figure 4. The coefficient $\langle x^2 \rangle / t^{2/(\alpha + \theta)}$ has a cosine shape as a function of $\beta$ and its value strongly depends on $\theta$ near $\beta = 0$. On the other hand, the dependence on $\theta$ is relatively weak for the strongly asymmetric cases.

The linear growth with time of the variance (the normal diffusion) is expected on time scales larger than a microscopic timescale and is a consequence of the central limit theorem. If, on the other hand, correlations decay slowly, that theorem does not apply and anomalous transport emerges. Subdiffusion in CTRW results from particle trapping, which effect increases with time since the waiting-time distribution has long tails [3]; the medium in that theory is regarded as homogeneous. The anomalous transport predicted by the Langevin equation with multiplicative noise has a different origin: it results from the noise intensity which decreases with the distance. Not only multiplicative noise is able to increase the distribution slope and make the variance finite; this also happens when one introduces a nonlinear deterministic force into the Langevin equation [24, 52]. However, then a stationary state exists and subdiffusion does not occur. The finite moments have been found in the Verhulst model for population density in which the random force is
multiplicative and given by the one-sided Lévy distribution [53]. Moreover, they emerge due to the trapping inside a potential when the dynamics is driven by short overdamped Josephson junctions and distributions of the noise signals have long tails [54].

Anomalous diffusion is a generic property of complex systems and emerges in many fields [3]. In these systems nonhomogeneity effects are important and the central limit theorem does not apply. This is the case, in particular, for transport in media with quenched disorder [2], as well as in biological systems, for example in cytoplasm and cellular membranes, where macromolecules are densely packed and exhibit heterogeneous structures; in vitro experiments reveal the subdiffusion in these systems (for a recent review see [55]).

3.2. The case \( \alpha < 1 \)

Very long jumps, i.e. possessing an infinite first moment, are also observed in realistic physical systems. For example, the molecular dynamics calculations in the framework of a granular material model [12], in which rigid spheres are densely and randomly packed under gradually varying stress, yield Lévy-distributed large strain increments with a value of \( \alpha \) in the 0.4–0.6 range. A numerical handling of the dynamics driven by a noise with \( \alpha < 1 \) is difficult. Moreover, those processes can be non-ergodic: it has been proved [56] that a weak non-ergodic behaviour emerges in the CTRW when the waiting-time distribution is given by a one-sided Lévy distribution.

Now, the \( H \)-function representation of the stable distribution is different from the case \( \alpha > 1 \) [33] and it leads to the following solution of equation (13):

\[
p(x, t) = \frac{c_\theta}{\alpha x} H^{1.1}_{2,2} \left[ \frac{x^{c_\theta}}{c_\theta \sigma_s(t)^{1/\alpha}} \right]^{(2, 1), (2\gamma, \gamma)}_{(2/\alpha, 1/\alpha), (2\gamma, \gamma)}, \tag{27}
\]

where \( \gamma \neq 0, 1 \). The series expansions for small and large arguments are the same as for \( \alpha > 1 \); they are given by equations (24) and (25), respectively. Also an expansion for the intermediate values can be derived (see [57] for the symmetric case).

The cases \( \beta = \alpha \) and \( \beta = -\alpha \) represent one-sided distributions, restricted to \( x < 0 \) and \( x > 0 \), respectively. These maximally asymmetric Lévy flights are useful to describe multifractal processes with \( \alpha < 2 \) [58], applied e.g. as a model of atmospheric phenomena [44]. The one-sided processes were discussed from the point of view of the first passage time and first passage leapover problems in [7, 59]. For \( \beta = -\alpha \), the terms on the main diagonal in equation (27) are identical and can be eliminated; the reduction formula of the \( H \)-function yields

\[
p(x, t) = \frac{c_\theta}{\alpha x} H^{1.0}_{1,1} \left[ \frac{x^{c_\theta}}{c_\theta \sigma_s^{1/\alpha}} \right]^{(2, 1)}_{(2/\alpha, 1/\alpha)}, \tag{28}
\]

Similarly to the two-sided case, the \( n \)th moment exists if \( \alpha + \theta > n \) and now the mean value does not vanish. The variance \( \text{var}(t) = \langle (x - \langle x \rangle)^2 \rangle \) for \( \lambda = 0 \), given by the Mellin transform from equation (28), rises with time slower than linearly,

\[
\text{var}(t) = \frac{1}{\alpha} c_\theta^{2/\alpha} \left[ \frac{\Gamma(-2/\alpha)}{-2} - \frac{1}{\alpha} \frac{\Gamma^2(-1/(\alpha + \theta))}{\Gamma(-1/c_\theta)} \right] t^{2/(\alpha + \theta)}. \tag{29}
\]
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Figure 5. The variance for the one-sided case $\beta = -\alpha$ as a function of $\alpha$ calculated from equation (29) for the following values of $\theta$: 1.3, 1.5, 1.8, 2, 2.1, 2.5, 3, 4 and 5 (from top to bottom). The limiting case $\theta = 2$ is marked by the dashed line.

The expression (29) is illustrated in figure 5 for some values of $\theta$ as a function of $\alpha$. The variance rapidly decreases with $\theta$ and the dependence on $\alpha$ is weak for large $\theta$, except in the vicinity of $\alpha = 1$. If $\theta > 2$, $\text{var}(t) < \infty$ for any $\alpha$.

The $H$-function becomes an elementary function for $\alpha = 1/2$ and $\beta = -1/2$ (the Lévy–Smirnov distribution),

$$p(x, t) = \frac{\sigma_s(t)}{2\sqrt{\pi}} (1 + 2\theta)^{3/2} x^{-3/2-\theta} \exp \left( -\frac{1}{4} (1 + 2\theta) \sigma_s(t)^2 x^{1-2\theta} \right).$$

The distributions corresponding to the stationary states are presented in figure 6. They exhibit a maximum at $x_m = \left[ \lambda^2 (3/2 + \theta) \right]^{-1/(1+2\theta)}$ which rises with $\theta$ and shifts towards $x = 1$. Then $\lim_{\theta \to \infty} p(x, t) = \delta(x - 1)$ for any $t > 0$ and we observe an instantaneous jump $x = 0 \to x = 1$. The curves corresponding to small $\theta$, in particular negative, disclose a uniform pattern with long tails. The diffusive case, $\lambda = 0$, also is characterized by a strongly localized density for large $\theta$ and the peak moves with time to infinity, $x_m \sim t^{1/(1/2+\theta)}$.

4. Summary and conclusions

We discussed stochastic processes driven by asymmetric stable distributions and medium nonhomogeneity was taken into account by introducing multiplicative noise. The process defined in that way may no longer be stable. However, in the diffusion limit of small wave numbers, when the master equation for a jumping process becomes the fractional Fokker–Planck equation, one can approximate them by the stable processes. We have considered a coupled version of CTRW where the jumping rate depends on the position as a power-law function, $|x|^{-\beta}$, and demonstrated that such an approximation is valid but
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Figure 6. The stationary distributions calculated from equation (30) for $\alpha = 1/2$ and $\theta = 1$.

only in a limited range of the parameters, $-\alpha < \theta < 1$. The resulting density has the same form as the jump-size distribution with the asymptotic slope independent of $\beta$ and $\theta$; only its time-dependence is affected by the variable jumping rate. This property is in contrast to the Gaussian case, $\alpha = 2$, characterized by a stretched exponential asymptotics. All fractional moments of the order $\delta \geq \alpha$ are infinite for any $\beta$ and $\theta$; they are largest for the symmetric case.

The Fokker–Planck equation for the coupled CTRW is fractional and has a variable diffusion coefficient. In the Langevin formulation of CTRW, the stochastic force is multiplicative and the equation requires the Itô interpretation. On the other hand, the solutions of the Langevin equation obtained by a transformation of the process variable—a procedure which corresponds to the Stratonovich interpretation—have different properties: they can possess finite moments. If the variance is finite, i.e. if $\alpha + \theta > 2$, it rises with time slower than linearly: $\langle x^2 \rangle (t) = D t^{2(\alpha+\theta)}$. We observe subdiffusion as a result of diminishing of the noise intensity with the distance. The coefficient $D$ is largest for the symmetric case and then strongly depends on $\theta$. On the other hand, the very asymmetric cases exhibit a moderate growth of $D$ with $\theta$. The above conclusions can be generalized to other forms of the multiplicative noise in equation (13) if they have a sufficiently large slope.

The extremely asymmetric case for $\alpha < 1$ corresponds to a one-sided distribution which also can possess finite variance. If $\theta$ is sufficiently large, $D$ appears almost constant as a function of $\alpha$ in a wide range of this parameter but the variance rapidly falls to zero for $\alpha \to 1$. The one-sided case has been illustrated with the Lévy–Smirnov density. The role of the multiplicative noise is clearly visible for this simple process: it makes the distribution wide and uniform when $\theta$ is negative, whereas for large values of $\theta$ the distribution shrinks to the delta function.
It is not \textit{a priori} clear which of the two solutions of equations (13), (9) or (22), applies to a concrete physical system; the main difference between them consists of a different slope of the tails. Some experimental work would be helpful. The shape of the probability distributions can be determined experimentally and such studies, applied to heterogeneous systems, could reveal effects of the variable diffusion coefficient. From that point of view, the analysis of fractures of disordered materials and measuring the crack propagation is promising since in these systems complex processes proceed on a broad range of scales and exhibit self-similar properties. A recent study [14] demonstrates that the experimental distribution of the local velocities of the crack front is characterized by a power-law tail, $v^{2.7}$, and the global velocity distribution converges by upscaling to the asymmetric stable distribution for scales larger than the spatial correlation length. At smaller scales, in turn, only the tail agrees with the stable distribution but behaviour near $x = 0$ may depend on $\theta$ [36]. A characteristic feature of systems governed by equation (13) ($\lambda = 0$) is a specific time-dependence of the fractional moments, similar for both interpretations, which can both decrease and increase the transport speed, compared to the homogeneous case. This property appears robust in respect of the nonhomogeneity parameter $\theta$ and has been observed not only for the stable solutions (9) [36]. It has been argued in [14] that the experimentally estimated variance of the global crack front velocity assumes a finite value since the system is actually finite. It would be interesting to verify whether that quantity, measured for a small resolution to make the self-similar structure apparent, obeys equation (12).

Appendix A

We will show that function (9) satisfies equation (7) to the lowest order in $|k|$ and evaluate $\sigma(t)$. First, the Fourier transform from both $p(x, t)$ and $p_\theta(x, t) \equiv |x|^{-\theta} p(x, t)$ is needed. Since for any stable density $f_\beta(-x) = f_{-\beta}(x)$, we only consider $x \geq 0$. Then we have,

$$\tilde{f}_\beta(k) = \int_0^\infty [(f_\beta(x) + f_{-\beta}(x)) \cos(kx) + i(f_\beta(x) + f_{-\beta}(x)) \sin(kx)] dx \tag{A.1}$$

and $\int_0^\infty f(x) \sin(kx) dx = -\left(\partial/\partial k\right) \int_0^\infty x^{-1} f(x) \cos(kx) dx$. The cosine transform from the $H$-function is given by the general formula

$$\int_0^\infty H_{m,n}^{p,q} \left[ x \begin{pmatrix} (a_p, A_p) \\ (b_q, B_q) \end{pmatrix} \right] \cos(kx) dx = \frac{\pi}{k} H_{q+1,m+2}^{n+1,0} \left[ k \begin{pmatrix} 1 - b_q, B_q, 1/2 \\ 1 - a_p, A_p, 1/2 \end{pmatrix} \right]. \tag{A.2}$$

Moreover, the multiplication rule,

$$x^\sigma H_{m,n}^{p,q} \left[ x \begin{pmatrix} (ap, A_p) \\ (b_q, B_q) \end{pmatrix} \right] = H_{m,n}^{p,q} \left[ x \begin{pmatrix} (ap + \sigma A_p, A_p) \\ (b_q + \sigma B_q, B_q) \end{pmatrix} \right], \tag{A.3}$$

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where the existence of $\gamma = (\alpha \mp \beta)/2\alpha$. Then we take the Fourier transform from equation (A.4); the existence of $\tilde{p}_\theta(k, t)$ requires that the singularity at $x = 0$ must not be essential which, in turn, implies the condition $\theta < 1$.

Next, we expand all the functions in equation (A.1), for $p$ and $p_\theta$, in powers of $|k|$ by applying the general formula,

$$H_{p,q}^{m,n}[\begin{pmatrix} a_p & A_p \\ b_q & B_q \end{pmatrix}] = \sum_{h=1}^{\infty} \sum_{\nu=0}^{\infty} \prod_{j=1, j \neq h}^{m} \Gamma(b_j - B_j((b_h + \nu)/B_h)) \prod_{j=n+1}^{p} \Gamma(1 - a_j + A_j((b_h + \nu)/B_h)) \times (-1)^\nu \frac{h^\pm \sigma^{1-|k|}}{\nu! B_h},$$

(A.5)

which is valid if $|\beta| < 2 - \alpha$ and $\sum B_i > \sum A_i$; the latter condition is satisfied for the case $\alpha > 1$. Evaluating the first two terms in (A.1), corresponding to the lhs of equation (7), yields

$$\text{Re} \tilde{p}(k, t) = 1 + \pi(h^+_\alpha + h^-_\alpha)\sigma^{-1}|k|^\alpha + O(k^2),$$

(A.6)

where the coefficient

$$h^\pm_\alpha = -\frac{\alpha}{2\pi} \cos(\pi\alpha/2) \sin(\pi\alpha\gamma^\pm)/\sin(\pi\alpha)$$

(A.7)

corresponds to the term $h = 2$ and $\nu = 1$ in equation (A.5). A similar calculation for the imaginary part yields

$$\text{Im} \tilde{p}(k, t) = -\alpha \sigma \sin(\pi\beta/2)|k|^\alpha + O(k^2).$$

(A.8)

To evaluate the rhs of equation (7) up to the required order, it is sufficient to determine the term $k^0$ which is real. The simplest method makes use of the Mellin transform, namely

$$\tilde{p}_\theta(k = 0, t) = \int_{-\infty}^{\infty} p_\theta(x, t)dx = \sigma^{-\theta/\alpha}[\chi_+(1) + \chi_-(1)$$

$$= \frac{2\sigma^{-\theta/\alpha}}{\pi} \Gamma(1 - \theta)\Gamma(\theta/\alpha) \sin(\pi\theta/2) \cos\left(\frac{\beta\theta}{2\alpha}\right),$$

(A.9)

where $\chi_\pm(s)$ denotes the Mellin transform corresponding to $\pm \beta$. Since, asymptotically, $p_\theta \sim |x|^{1-\alpha-\theta}$, the convergence of the integral in equation (A.9) imposes the condition $\alpha + \theta > 0$. Introducing the above results to equation (7) produces two identical equations for the real and imaginary parts which determine the function $\sigma(t)$,

$$\dot{\sigma}(t) = \frac{2}{\pi\alpha} \Gamma(\theta/\alpha) \Gamma(1 - \theta) \sin\left(\frac{\pi\theta}{2}\right) \cos\left(\frac{\beta\theta}{2\alpha}\right) \sigma(t)^{-\theta/\alpha},$$

(A.10)

and its solution with the initial condition $\sigma(0) = 0$ is given by equation (10).
Appendix B

For $\beta = \alpha - 2$ the $H$-function in equation (9) can be reduced to a lower order and the density expressed as

$$p(x, t) = \frac{\epsilon}{\sigma(t)^{\epsilon}} H_{1,1}^{1,0} \left[ \frac{x}{\sigma(t)^{\epsilon}} \left( 1 - \epsilon, \epsilon \right) (0, 1) \right].$$

(B.1)

Since $n = 0$, the power-law asymptotics is no longer valid and instead an exponential behaviour emerges [60]. In the case of equation (B.1), the Fox function has the following form for the large arguments:

$$H(z) = c_1 z^\lambda e^{-c_2 z^{c_3}},$$

(B.2)

where $c_1 = \left[ 2\pi(\alpha - 1)\alpha^{1/(\alpha - 1)} \right]^{-1/2}$, $c_2 = (\alpha - 1)\alpha^{-c_3}$, $c_3 = \alpha/(\alpha - 1)$ and $\lambda = (2 - \alpha)/(2(\alpha - 1))$; the distribution falls faster than exponentially.

A similar reduction applies to the case $\alpha < 1$ and the result, similar to equation (B.2), represents an expansion for small arguments [33]. If $\alpha$ is a rational number, the distribution for the one-sided cases can be expressed by the generalized hypergeometric functions and then relatively easily evaluated [61].

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