Superlinear convergence of Anderson accelerated Newton’s method for solving stationary Navier-Stokes equations

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Abstract

This paper studies the performance of Newton’s iteration applied with Anderson acceleration for solving the incompressible steady Navier-Stokes equations. We manifest that this method converges superlinearly with a good initial guess, and moreover, a large Anderson depth decelerates the convergence speed comparing to a small Anderson depth. We observe that the numerical tests confirm these analytical convergence results, and in addition, Anderson acceleration sometimes enlarges the domain of convergence for Newton’s method.

1 Introduction

This work studies the performance of Newton’s method with an acceleration technique, known as Anderson acceleration introduced by [1], for solving the incompressible Navier-Stokes equations. It is inspired by the work of [11, 7, 12], where [11] shows how Anderson acceleration can locally improve the convergence rate of the linearly converging Picard iteration for solving the NSE, the numerical tests from [7] shows that Anderson acceleration slows the convergence speed of Newton’s method but enlarges the domain of convergence for NSE, and [12] justifies, both theoretically and numerically, superlinear convergence of this method with depth $m = 1$ for several benchmark nonlinear problems. Our work manifests the superlinear convergence of Anderson accelerated Newton’s method (AAN) for solving steady Navier-Stokes problem analytically and numerically.

Navier-Stokes equations (NSE) are governed by the following:

\begin{align*}
-\nu \Delta u + u \cdot \nabla u + \nabla p &= f, \\
\nabla \cdot u &= 0,
\end{align*}

on a domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), with appropriate boundary conditions. Here $u$ is velocity, $p$ is pressure, $f$ is external force such as buoyancy, gravity, etc. $\nu$ is the kinetic viscosity, where its reciprocal is known as Reynolds number $Re$. The Newton’s method of NSE takes the form: Given $u_0$, find $(u_k, p_k)$ satisfying

\begin{align*}
-\nu \Delta u_k + u_k \nabla \cdot u_{k-1} + u_{k-1} \nabla \cdot u_k - u_{k-1} \nabla \cdot u_{k-1} + \nabla p_k &= f, \\
\nabla \cdot u_k &= 0.
\end{align*}

It is known that this iterative method is a local method and converges quadratically if the initial guess is good enough. Our work herein demonstrates the Newton’s method applied with Anderson acceleration sometimes enlarges the domain of convergence for Newton’s method.
acceleration with general depth \( m \) for solving NSE converges superlinearly both theoretically and numerically, if the initial guess is good enough. Moreover, a large depth decelerates the convergence speed due to more high order terms appeared in the one-step bound, comparing to a small depth.

This paper is organized as follows. In section 2, we provide notations, mathematical preliminary, finite element scheme for steady NSE, and then present some properties of the solution operator to Newton’s method. Section 3 gives the algorithms and convergence results of Anderson accelerated Newton’s method with varying depth for solving steady NSE. Several numerical tests are provided in section 4 that confirm our analytical results.

2 Notation and Mathematical preliminaries

This section provides notation, mathematical preliminaries, and background to allow for a smooth analysis in later sections. First, we will give function spaces and notational details, followed by finite element discretization preliminaries. Then we provide the Newton’s iteration of steady Navier-Stokes equations and some basic properties. Throughout this paper, we consider homogeneous Dirichlet boundary condition for velocity

\[ u = 0 \text{ on } \partial \Omega. \]

Of course, all results can be extended to other common boundary conditions with some extra work and these are not discussed here.

2.1 Discretization of NSE

The domain \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) is assumed to be simply connected and to either be a convex polytope or have a smooth boundary. The \( L^2(\Omega) \) norm and inner product will be denoted by \( \| \cdot \| \) and \((\cdot, \cdot)\), respectively, and all other norms will be labeled with subscripts. The natural function spaces for velocity and pressure are given by

\[ X = H^1_0(\Omega)^d := \{ v \in L^2(\Omega)^d, \nabla v \in L^2(\Omega)^{d \times d}, v = 0 \text{ on } \partial \Omega \}, \]
\[ Q = L^2_0(\Omega) := \{ q \in L^2(\Omega), \int_{\Omega} q \, dx = 0 \}. \]

The Poincaré inequality is known to hold in \( X \) [10]: there exists \( C_P > 0 \) dependent only on the domain \( \Omega \) satisfying

\[ \|v\| \leq C_P \|\nabla v\|, \]

for any \( v \in X \).

**Definition 2.1.** Define a trilinear form: \( b : X \times X \times X \to \mathbb{R} \) such that for any \( u, v, w \in X \)

\[ b(u, v, w) := \frac{1}{2}((u \cdot \nabla v, w) - (u \cdot \nabla w, v)). \]

The operator \( b \) is skew-symmetric

\[ b(u, v, v) = 0, \] (2.1)
and satisfies inequality
\[
b(u, v, w) \leq M \| \nabla u \| \| \nabla v \| \| \nabla w \| ,
\]
for any \( u, v, w \in X \), with \( M \) depending only on \( |\Omega| \), see [10].

We will denote by \( \tau_h \) a regular, conforming triangulation of \( \Omega \) with maximum element diameter \( h \). The finite element spaces will be denoted as \( X_h \subset X, Q_h \subset Q \), and we require that \((X_h, Q_h)\) pair satisfies the usual discrete inf-sup condition [10]. For example one could select Taylor-Hood elements, Scott-Vogelius elements on an appropriate mesh [3, 15, 14], or the mini element [2], etc.

The discrete stationary NSE is given by: Finding \((u, p) \in (X_h, Q_h)\) such that for and \((v, q) \in (X_h, Q_h)\)
\[
\begin{align*}
b(u, u, v) + \nu(\nabla u, \nabla v) - (p, \nabla \cdot v) &= (f, v), \\
(\nabla \cdot u, q) &= 0.
\end{align*}
\]
(2.3)

(2.4)

\[
\text{Let } V_h \text{ be the discretely divergence-free subspace as } V_h := \{ v \in X_h, (\nabla \cdot v, q) = 0, \forall q \in Q_h \},
\]
then an equivalent formulation of (2.3)-(2.4) is obtained: Find \( u \in V_h \) such that for any \( v \in V_h \)
\[
b(u, u, v) + \nu(\nabla u, \nabla v) = (f, v).
\]
(2.5)

This paper focuses on studying the convergence behavior after applying Anderson acceleration to the Newton’s method. We assume that systems (2.3)-(2.4) and (2.5) are well-posed for simplicity. In other words, the small data condition
\[
\kappa := \nu^{-2} M \| f \|_{-1} < 1,
\]
(2.6)
is satisfied, see [10]. However, all results presented here can be extended to the case where the discretized steady NSE has distinct solutions (\( \kappa \geq 1 \)). With deflation techniques [5, 8], one may find out distinct solutions to steady NSE model using nonlinear solvers. For the rest of the paper, we assume (2.6) holds.

2.2 Properties of Newton’s solution operator for steady NSE

In this subsection, we define the Newton’s solution operator for steady NSE and present some properties of it.

Definition 2.2. Given \( u \in V_h \), define a mapping \( G : V_h \to V_h \) satisfying
\[
b(u, G(u), v) + b(G(u), u, v) - b(u, u, v) + \nu(\nabla G(u), \nabla v) = (f, v),
\]
for any \( v \in V_h \) and called \( G \) the Newton solution operator.

Next lemma shows that this operator \( G \) is well-defined on a ball.
Lemma 2.3. Equation (2.7) has a unique solution for any \( u \in B(0, R) := \{ v \in V_0 \mid \| \nabla v \| \leq R \} \) with \( R < \nu M^{-1} \). Moreover, the following inequalities hold

\[
\| \nabla G(u) \| \leq C_0 R^2 + (1 + C_0 R)\nu^{-1}\| f \|_{-1} := R_G,
\]

(2.8)

\[
\| \nabla (G(u) - u) \| \leq 2(1 + C_0 R)^{1/2} R + 2\nu^{-1}(1 + C_0 R)\| f \|_{-1} := R_{res},
\]

(2.9)

where \( C_0 := \frac{\nu^{-1} M}{1 - \nu^{-1} MR} \).

Proof. We begin the proof by finding an upper bound of \( \| \nabla G(u) \| \). Setting \( v = G(u) \) in (2.7) eliminates the first trilinear form and yields

\[
\nu \| \nabla G(u) \|^2 = -b(G(u), u, G(u)) + b(u, u, G(u)) + (f, G(u))
\]

\[
\leq M \| \nabla G(u) \|^2 \| \nabla u \| + M \| \nabla u \|^2 \| \nabla G(u) \| + \| f \|_{-1} \| \nabla G(u) \|,
\]

thanks to (2.1), (2.2) and Cauchy-Schwarz inequality. Dividing both sides by \( \| \nabla G(u) \| \), it reduces to

\[
\nu(1 - \nu^{-1} M \| \nabla u \|) \| \nabla G(u) \| \leq M \| \nabla u \|^2 + \| f \|_{-1}.
\]

Thus from the assumptions \( \| \nabla u \| \leq R < \nu M^{-1} \) and (2.6), we have (2.8). Since the system (2.7) is linear and finite dimensional, (2.8) is sufficient to imply solution uniqueness and therefore existence.

Then we show inequality (2.9). Setting \( v = G(u) - u \) in equation (2.7) gives

\[
b(G(u), u, G(u) - u) + \nu(\nabla G(u), \nabla (G(u) - u)) = (f, G(u) - u),
\]

thanks to (2.1). Applying the polarization identity, (2.2) and Cauchy-Schwarz inequality, we have

\[
\frac{\nu}{2}(1 - \nu^{-1} MR) (\| \nabla G(u) \|^2 + \| \nabla (G(u) - u) \|^2) \leq \frac{\nu}{2} \| \nabla u \|^2 + \| f \|_{-1} \| \nabla (G(u) - u) \|,
\]

Dropping the term with \( \| \nabla G(u) \|^2 \) and utilizing the Young’s inequality, we obtain

\[
\| \nabla (G(u) - u) \|^2 \leq 2(1 - \nu^{-1} MR)^{-1} \| \nabla u \|^2 + 4\nu^{-2}(1 - \nu^{-1} MR)^{-2} \| f \|_{-1}^2.
\]

From the definition of \( C_0 \) and identity \((1 - \nu^{-1} MR)^{-1} = 1 + C_0 R\), (2.9) is achieved. \( \square \)

Now we state the Newton’s iteration for steady NSE and a few properties of \( G \) are followed.

Algorithm 2.4 (Newton’s iteration for steady NSE). The Newton’s method for steady Navier-Stokes equations is as below:

Step 0 Give \( w_0 \in B(0, R) \).

Step \( k \) Compute \( w_k = G(w_{k-1}) \).

Obviously, Algorithm 2.4 fails when ever \( w_k \notin B(0, R) \) for some integer \( k \). In order to discuss its convergence order, we make an assumption:
Assumption 2.5. Assume the sequence \{w_k\} from Algorithm 2.4 satisfies
\[ w_k \in B(0, R) := \{v \in V_h \mid \|\nabla v\| \leq R\} \]
for all \( k \in \mathbb{N} \), where \( R < \nu M^{-1} \).

It is well-known that Algorithm 2.4 converges quadratically [9]. Here we present a different point of view to the quadratical convergence, expressing the error bound in terms of residuals \( G(u) - u \), the following lemma will be used multiple times in the next section.

Lemma 2.6. Assume \( u, w \in B(0, R) \) with \( R < \nu M^{-1} \), thus
\[ \|\nabla (G(w) - G(u))\| \leq 2C_0\|\nabla (w - u)\|\|\nabla (G(u) - u)\| + C_0\|\nabla (w - u)\|^2. \] (2.10)
where \( C_0 \) is defined in Lemma 2.3.

Proof. For any \( u, w \in B(0, R) \), we rewrite equation (2.7) as
\[ b(u, G(u), v) + b(G(u) - u, u, v) + \nu(\nabla G(u), \nabla v) = (f, v), \]
\[ b(w, G(w), v) + b(G(w) - w, w, v) + \nu(\nabla G(w), \nabla v) = (f, v), \]
for any \( v \in V_h \). Subtracting the above two equations gives
\[ b(w, G(w) - G(u), v) + b(G(w) - G(u), w, v) + b(w - u, G(u) - u, v) \]
\[ + b(G(u) - w, w - u, v) + \nu(\nabla (G(w) - G(u)), \nabla v) = 0. \] (2.11)
Setting \( v = G(w) - G(u) \) eliminates the first term and produces
\[ \nu(1 - \nu^{-1}M\|\nabla w\|)\|\nabla (G(w) - G(u))\| \leq M\|\nabla (w - u)\|\|\nabla (G(u) - w)\| + \|\nabla (G(u) - u)\|, \]
thanks to inequality (2.2). Then we have
\[ \|\nabla (G(w) - G(u))\| \leq C_0\|\nabla (w - u)\|\|\nabla (G(u) - u)\| + \|\nabla (G(u) - w)\|, \] (2.12)
due to assumptions \( \|\nabla w\| \leq R < \nu M^{-1} \) and (2.6). From triangle inequality, we have (2.10) and finish the proof.

Easily, one can end up with inequality
\[ \|\nabla (w_{k+1} - w_k)\| \leq C_0\|\nabla (w_k - w_{k-1})\|^2, \]
by setting \( u = w_{k-1}, \; w = w_k := G(w_{k-1}) \) in (2.12).

Lastly, we show that the solution operator \( G \) is Fréchet differentiable.

Definition 2.7. Given \( u, u + h \in B(0, R) \) with \( R < \nu M^{-1} \), define \( G'(u; \cdot) : V_h \rightarrow V_h \) such that
\[ b(u, G'(u; h), v) + b(G'(u; h), u, v) + b(h, G(u) - u, v) + b(G(u) - (u + h), h, v) \]
\[ + \nu(\nabla G'(u; h), \nabla v) = 0, \] (2.13)
for any \( v \in V_h \).
Lemma 2.8. \( G' \) is well-defined over \( B(0, R) \). Moreover, \( G' \) is the Fréchet derivative of \( G \) satisfying
\[
\| \nabla (G(u + h) - G(u) - G'(u; h)) \| \leq 2C_0 \| \nabla h \| \| \nabla (G(u + h) - G(u)) \|,
\]
and is bounded by
\[
\| \nabla G'(u; h) \| \leq 2C_0 \| \nabla G(u - u) \| \| \nabla h \| + C_0 \| \nabla h \|^2,
\]
for any \( u, u + h \in B(0, R) \).

Proof. This proof includes two parts. First, we show \( G' \) is well-defined and has an upper bound. Setting \( v = G'(u; h) \) in (2.13) eliminates the first term and produces
\[
\nu(1 - \nu^{-1}M \| \nabla u \|) \| \nabla G'(u; h) \| \leq 2M \| \nabla G(u - u) \| \| \nabla h \| + M \| \nabla h \|^2,
\]
thanks to (2.2), which reduces to (2.15). Since system (2.13) is linear and finite dimensional, (2.15) guarantee \( G' \) is well-defined. Second, we manifest \( G' \) is the Fréchet derivative of \( G \). Denoting \( \xi = G(u + h) - G(u) - G'(u; h) \) and subtracting (2.13) from (2.11) with \( w = u + h \), we have
\[
b(u, \xi, v) + b(h, G(u + h) - G(u), \xi) + b(\xi, u, v) + b(G(u + h) - G(u), h, \xi) + \nu(\nabla \xi, \nabla v) = 0.
\]
Setting \( v = \xi \) eliminates the first term and yields
\[
\nu(1 - \nu^{-1}M \| \nabla u \|) \| \nabla \xi \| \leq 2M \| \nabla G(u + h) - G(u) \| \| \nabla h \|,
\]
due to (2.2). Thus from \( \| \nabla u \| \leq R < \nu M^{-1} \) and (2.6), it leads to (2.14). Therefore \( G' \) is the Fréchet derivative of \( G \) and we finish the proof.

3 Anderson accelerated Newton’s iteration for steady NSE

In this section, we state algorithms of Anderson accelerated Newton’s iteration (AAN) for steady NSE and give an analysis of superlinear convergence order. We start with the simplest case where the Anderson depth is 1 in the coming subsection, and then move on to the Anderson depth \( m = 2 \) and general depth cases in the second and third subsections respectively.

3.1 Anderson accelerated Newton’s method with depth \( m = 1 \)

Algorithm 3.1 (Anderson accelerated Newton’s iteration with depth \( m = 1 \) (AAN m=1)). The algorithm of Anderson accelerated Newton’s method with depth \( m = 1 \) is stated as below:

Step 0 Guess \( u_0 \in B(0, R) \).

Step 1 Compute \( \tilde{u}_1 = G(u_0) \) and set the residual \( y_1 = \tilde{u}_1 - u_0 \), update \( u_1 = \tilde{u}_1 \).

Step k For \( k = 2, 3, \ldots \)

a) Compute \( \tilde{u}_k = G(u_{k-1}) \) and set the residual \( y_k = \tilde{u}_k - u_{k-1} \).

b) Find \( \alpha_k \in \mathbb{R} \) minimizing
\[
\| \nabla ((1 - \alpha_k)y_k + \alpha_k y_{k-1}) \|.
\]
c) Update \( u_k = (1 - \alpha_k)\tilde{u}_k + \alpha_k\tilde{u}_{k-1} \).

We will use the residual sequence \( \{y_k\} \) to discuss the convergence behavior of Algorithm 3.1. For smooth analysis, we use the following notation throughout this subsection

\[
e_k = u_k - u_{k-1}, \quad \tilde{e}_k = \tilde{u}_k - \tilde{u}_{k-1}, \quad y_k^\alpha = (1 - \alpha_k)y_k + \alpha_ky_{k-1}.
\]

Comparing Algorithm 3.1 with the usual Newton’s Algorithm 2.4, we add a minimization step at each iteration. It is clear that Algorithm 3.1 is back to Algorithm 2.4 at step \( k \) whenever \( \alpha_k = 0 \).

We make the following assumption in order to study the behavior of Algorithm 3.1.

**Assumption 3.2.** For step \( k \geq 2 \), assuming \( \alpha_k \neq 0 \) and \( u_j \in B(0,R) \) with \( R < \nu M^{-1} \) for all \( j \leq k \).

We now give an expression of \( \alpha_k \) in terms of residuals in next lemma.

**Lemma 3.3.** For any step \( k \) with \( \alpha_k \neq 0 \), let Anderson gain \( \theta_k := \|\nabla y_k^\alpha\|/\|\nabla y_k\| \), then \( \theta_k \in [0,1) \) and

\[
|\alpha_k| = \sqrt{1 - \theta_k^2}\|\nabla y_k\|/\|\nabla(y_k - y_{k-1})\|.
\]

**Proof.** It is clearly \( \theta_k \in (0,1] \) by the choice of \( \alpha_k \). Let \( f(\alpha) = \|\nabla(1 - \alpha)y_k + \alpha y_{k-1}\|^2 \), then \( \alpha_k \) is the stationary point of \( f(\alpha) \). That is, \( f'(\alpha_k) = 0 \), which gives

\[
\alpha_k\|\nabla(y_k - y_{k-1})\|^2 = (\nabla y_k, \nabla(y_k - y_{k-1})).
\]

By the definition of \( \theta_k \), we have

\[
\theta_k^2\|\nabla y_k\|^2 = \|\nabla y_k^\alpha\|^2 = \|\nabla y_k\|^2 - \alpha_k^2\|\nabla(y_k - y_{k-1})\|^2,
\]

which leads to (3.1). \( \square \)

Before we give the main convergence result for Algorithm 3.1, we present a few lemmas that play key roles in the analysis. First, we list a few identities that will be used repeatedly in this subsection

\[
\tilde{u}_k - u_k = \alpha_k\tilde{e}_k, \quad y_k - y_{k-1} = \tilde{e}_k - e_{k-1}, \quad y_k^\alpha = e_k + \alpha_k e_{k-1}.
\]

Next, we show that the difference between solutions from successive iterations can be bounded by the residuals.

**Lemma 3.4.** Let Assumption 3.2 holds, there exists a positive constant \( C_1 = C_1(\alpha_k, \nu, |\Omega|, f, R) \) such that

\[
|\alpha_k|\|\nabla e_{k-1}\| \leq C_1\|\nabla y_k\|,
\]

and

\[
\|\nabla e_k\| \leq C_1\|\nabla y_k\|.
\]

To be more specific, \( C_1 = \sqrt{3 + 2C_0R + 4C_0^2(R + 2|\alpha_k|R_G)^2 + 4C_0^2R_{\text{res}}^2} \).
Proof. We begin by the following equation
\[ b(u_j, \tilde{u}_{j+1}, v) + b(\tilde{u}_{j+1}, u_j, v) - b(u_j, u_j, v) + \nu(\nabla \tilde{u}_{j+1}, \nabla v) = (f, v), \tag{3.7} \]
for any nonnegative integer \( j \). Subtracting (3.7) with \( j = k - 2 \) from (3.7) with \( j = k - 1 \) yields
\[ \nu(\nabla \tilde{e}_k, \nabla v) = -b(\tilde{e}_k, \tilde{u}_k, v) - b(u_{k-2}, \tilde{e}_k, v) + b(y_k - y_{k-1}, y_k, v) - b(y_{k-1}, e_{k-1}, v) = -b(\tilde{e}_k, u_k, v) - b(u_{k-2} + \alpha_k \tilde{e}_k, \tilde{v}_k, v) + b(y_k - y_{k-1}, y_k, v) - b(y_{k-1}, e_{k-1}, v), \tag{3.8} \]
thanks to (3.3) and (3.2). Setting \( v = e_{k-1} \) eliminates the last term and gives
\[ \frac{\nu}{2} \left( \| \nabla \tilde{e}_k \|^2 + \| \nabla e_{k-1} \|^2 - \| \nabla (y_k - y_{k-1}) \|^2 \right) \leq M \| \nabla u_k \| \| \nabla \tilde{e}_k \| \| \nabla e_{k-1} \| \]
\[ + M \| \nabla (u_{k-2} + \alpha_k \tilde{e}_k) \| \| \nabla \tilde{e}_k \| \| \nabla (y_k - y_{k-1}) \| + M \| \nabla (y_k - y_{k-1}) \| \| \nabla y_k \| \| \nabla e_{k-1} \|, \]
thanks the polarization identity, (3.3), (2.2) and (2.1). From inequalities \( ab \leq \frac{a^2 + b^2}{2} \), triangle inequality \( \| \nabla (u_{k-2} + \alpha_k \tilde{e}_k) \| \leq R + 2|\alpha_k| R_G \), and the Young’s inequality, we obtain
\[ \frac{\nu}{4} (1 - \nu^{-1} MR) \left( \| \nabla \tilde{e}_k \|^2 + \| \nabla e_{k-1} \|^2 \right) \leq \left( \frac{\nu}{2} + (1 - \nu^{-1} MR)^{-1} M^2 (R + 2|\alpha_k| R_G)^2 + \nu^{-1} (1 - \nu^{-1} MR)^{-1} M^2 \| \nabla y_k \|^2 \right) \| \nabla (y_k - y_{k-1}) \|^2. \]
Dropping the term with \( \| \nabla \tilde{e}_k \| \) yields
\[ \| \nabla e_{k-1} \| \leq \tilde{C} \| \nabla (y_k - y_{k-1}) \| \leq C_1 \| \nabla (y_k - y_{k-1}) \|, \]
where \( \tilde{C} = \sqrt{2(1 + C_0 R) + 4C_0^2 (R + 2|\alpha_k| R_G)^2 + 4C_0^2 R_{res}^2} \). Applying (3.1) gives
\[ |\alpha_k| \| \nabla e_{k-1} \| \leq \tilde{C} \sqrt{1 - \theta_k^2} \| \nabla y_k \|, \]
and (3.5). Furthermore, we obtain
\[ \| \nabla e_k \| \leq \| \nabla y_k^0 \| + |\alpha_k| \| \nabla e_{k-1} \| \leq (\theta_k + \tilde{C} \sqrt{1 - \theta_k^2}) \| \nabla y_k \|, \]
thanks to (3.4). Therefore (3.6) holds as \( \max_{\theta_k \in [0,1]} \theta_k + \tilde{C} \sqrt{1 - \theta_k^2} = \sqrt{1 + \tilde{C}^2} = C_1. \)

The next lemma uses the Fréchet derivative properties of \( G \) as presented in Lemma 2.8.

Lemma 3.5. Let Assumption 3.2 holds, then
\[ \| \nabla (G'(u_{k-1}; e_k) + \alpha_k G'(u_{k-2}; e_{k-1})) \| \leq C_0 C_1 \left( 4 + C_1 + (2 + C_1)/|\alpha_k| \right) \| \nabla y_k \|^2 = O(\| \nabla y_k \|^2). \tag{3.9} \]
and
\[ \| \nabla (y_{k+1} - G'(u_{k-1}; e_k) - \alpha_k G'(u_{k-2}; e_{k-1})) \| \leq 2C_0^2 C_1^2 (2 + C_1 + 2/|\alpha_k| + C_1/|\alpha_k|^2) \| \nabla y_k \|^3 = O(\| \nabla y_k \|^3). \tag{3.10} \]
Proof. Utilizing (2.15) produces
\[
\|\nabla G'(u_{k-1}; e_k)\| \leq 2C_0\|\nabla y_k\|\|\nabla e_k\| + C_0\|\nabla e_k\|^2 \leq C_0C_1 (2 + C_1) \|\nabla y_k\|^2,
\]
and
\[
|\alpha_k\|\nabla G'(u_{k-2}; e_{k-1})\| \leq 2C_0\|\nabla y_{k-1}\|\|\alpha_k\|\|\nabla e_{k-1}\| + C_0|\alpha_k|\|\nabla e_{k-1}\|^2
\leq 2C_0C_1(\|\nabla y_{k-1}\| + \|\nabla (y_{k-1} - y_{k-2})\|)\|\nabla y_{k-1}\| + C_0C_1^2/|\alpha_k|\|\nabla y_{k-1}\|^2
\leq 2C_0C_1\|\nabla y_{k-1}\|^2 + C_0C_1(2 + C_1)/|\alpha_k|\|\nabla y_{k-1}\|^2,
\]
thanks to triangle inequality, (3.1) and Lemma 3.4. Combining the above two inequalities yields (3.9).

Next we prove the inequality (3.10). For notation simplification, we denote \(\psi_k := G(u_k) - G(u_{k-1}) - G'(u_{k-1}; e_k)\). Utilizing (3.2), we have identity \(y_{k+1} - e_{k+1} - \alpha_k e_k = \psi_k + \alpha_k \psi_{k-1}\).

From equation (2.10) and Lemma 3.4, we have
\[
\|\nabla \psi_{k+1}\| \leq 2C_0\|\nabla y_k\|\|\nabla e_k\| + C_0\|\nabla e_k\|^2 \leq C_0C_1 (2 + C_1)\|\nabla y_k\|^2,
\]
\[
\|\nabla \psi_k\| \leq 2C_0\|\nabla y_k\|\|\nabla e_{k-1}\| + C_0\|\nabla e_{k-1}\|^2 \leq C_0C_1/|\alpha_k|(2 + C_1/|\alpha_k|)\|\nabla y_k\|^2.
\]
Combining the above inequalities with (2.14) and Lemma 3.4, we obtain
\[
\|\nabla (\psi_k + \alpha_k \psi_{k-1})\| \leq 2C_0\|\nabla e_k\|\|\psi_{k+1}\| + 2|\alpha_k|C_0\|\nabla e_{k-1}\|\|\nabla \psi_k\|
\leq 2C_0^2C_1^2(2 + C_1 + 2/|\alpha_k| + C_1/|\alpha_k|^2)\|\nabla y_k\|^3.
\]

Now we are ready to give the superlinearly convergence result of Algorithm 3.1.

**Theorem 3.6.** (One-step residual bound for \(m = 1\)) Let Assumption 3.2 holds, then the residual sequence \(\{y_k\}\) from Algorithm 3.1 satisfies
\[
\|\nabla y_{k+1}\| \leq C_2\|\nabla y_k\|^{3/2}, \quad \forall k \geq 2,
\]
where \(C_2\) depends on \(\nu, M, R, \|f\|_{-1}, \alpha_k, \theta_k\).

Proof. We begin by constructing an equation of \(y_{k+1}\) using (3.7). Adding \((1 - \alpha_k)\) multiple of (3.7) with \(j = k - 1\) to \(\alpha_k\) multiple of (3.7) with \(j = k - 2\) gives
\[
b(u_{k-1}, u_k, v) - b(e_{k-1}, \alpha_k \tilde{u}_{k-1}, v) + b(y_k^0, u_{k-1}, v) - \alpha_k b(y_{k-1}, e_{k-1}, v) + \nu(\nabla u_k, \nabla v) = (f, v). \tag{3.12}
\]

Subtracting (3.12) from (3.7) with \(j = k\) produces
\[
b(u_k, y_{k+1}, v) + b(y_{k+1}, u_k, v) + b(e_k, u_k, v) + \alpha_k b(e_{k-1}, \tilde{u}_{k-1}, v) - b(y_k^0, u_{k-1}, v) + \alpha_k b(y_{k-1}, e_{k-1}, v) + \nu(\nabla y_{k+1}, \nabla v) = 0.
\]
Setting $v = \chi := G'(u_{k-1}; e_k) + \alpha_k G'(u_{k-2}; e_{k-1})$ gives

$$\frac{\nu}{2}(\| \nabla y_{k+1} \|^2 + \| \nabla \chi \|^2 - \| \nabla (y_{k+1} - \chi) \|^2) \leq MR\| \nabla \chi \| \| \nabla (y_{k+1} - \chi) \| + \frac{MR}{2}(\| \nabla y_{k+1} \|^2 + \| \nabla \chi \|^2) + MR\| \nabla e_k \| \| \nabla \chi \|$$

$$+ MR\| \alpha_k \| \| \nabla e_{k-1} \| \| \nabla \chi \| + \theta_e MR\| \nabla y_k \| \| \nabla \chi \| + M|\alpha_k||\nabla e_{k-1}||\nabla y_{k-1}||\nabla \chi|,$$

thanks to the polarization identity, (2.1) and (2.2). Dropping the term with $\| \nabla \chi \|^2$ and dividing both sides by $\frac{\nu}{2}(1 - \nu^{-1}MR)$, this reduces to

$$\| \nabla y_{k+1} \|^2 \leq (1 + C_0R)\| \nabla (y_{k+1} - \chi) \|^2 + 2C_0R\| \nabla \chi \| \| \nabla (y_{k+1} - \chi) \|$$

$$+ \theta_e C_0R\| \nabla y_k \| \| \nabla \chi \| + 2C_0C_1(R + R_G + R_{res})\| \nabla y_k \| \| \nabla \chi|,$$

thanks to Lemma 3.4, (2.9). Thus from Lemma 3.5, we have (3.11).

\section{Anderson accelerated Newton’s method with depth $m = 2$}

In this subsection, we study the Anderson accelerated Newton’s method with depth $m = 2$ for solving steady NSE. The algorithm and analysis of convergence are presented here.

\textbf{Algorithm 3.7} (Anderson accelerated Newton’s iteration with depth $m = 2$ (AAN m=2) ).

Algorithm of Anderson accelerated Newton’s method with depth $m = 2$ is stated as below:

\textbf{Step 0} Guess $u_0 \in B(0,R)$.

\textbf{Step 1} Compute $\tilde{u}_1 = G(u_0)$ and set residual $y_1 = \tilde{u}_1 - u_0$, update $u_1 = \tilde{u}_1$.

\textbf{Step 2} This step consists three parts:

a) Compute $\tilde{u}_2 = G(u_1)$ and set residual $y_2 = \tilde{u}_2 - u_1$.

b) Find $\alpha_2 \in \mathbb{R}$ minimizing $\| \nabla((1 - \alpha_2)y_2 + \alpha_2 y_1) \|$.

c) Update $u_2 = (1 - \alpha_2)\tilde{u}_2 + \alpha_2 \tilde{u}_1$.

\textbf{Step k} For $k = 3, 4, \ldots$

a) Compute $\tilde{u}_k = G(u_{k-1})$ and set residual $y_k = \tilde{u}_k - u_{k-1}$.

b) Find $\beta_1^k, \beta_2^k \in \mathbb{R}$ minimizing $\| \nabla((1 - \beta_1^k - \beta_2^k)y_k + \beta_1^k y_{k-1} + \beta_2^k y_{k-2}) \|$.

c) Update $u_k = (1 - \beta_1^k - \beta_2^k)\tilde{u}_k + \beta_1^k \tilde{u}_{k-1} + \beta_2^k \tilde{u}_{k-2}$.

For smooth analysis, we will use the following notations for the rest of subsection

$$e_k = u_k - u_{k-1}, \quad \tilde{e}_k = \tilde{u}_k - u_{k-1}, \quad \beta_1^k = (1 - \beta_1^k - \beta_2^k)y_k + \beta_1^k y_{k-1} + \beta_2^k y_{k-2}.$$

It is obvious that Algorithm 3.7 is back to either Algorithm 2.4 or Algorithm 3.1 at step $k$ whenever $\beta_2^k = 0$. So it is reasonable to make the following assumption.

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Assumption 3.8. For step $k$ ($k \geq 3$), assume $\beta_k^2 \neq 0$ and $u_j \in B(0, R)$ with $R < \nu M^{-1}$ for all $j \leq k$.

Now let’s give three lemmas that are analogue to the depth $m = 1$ case.

Lemma 3.9. Assume the Assumption 3.8 holds, let $\theta_k = \|\nabla y_k\|/\|\nabla y_k\|$, then $\theta_k \in [0, 1)$ and $\beta_k^1, \beta_k^2$ from Algorithm 3.7 satisfy the following inequalities

$$|\beta_k^2\|\nabla (y_{k-1} - y_{k-2})\| \leq (\sqrt{1 - \theta_k^2} + |\beta_k^1 + \beta_k^2|)\nabla y_k + |\beta_k^1 + \beta_k^2|\nabla y_{k-1}.$$  

(3.13)

Proof. It is obviously $\theta_k \in (0, 1]$ from the minimization step. Let

$$f(\xi, \eta) = \|\nabla (y_k - \xi(y_k - y_{k-1}) - \eta(y_{k-1} - y_{k-2}))\|^2,$$

then $(\beta_k^1 + \beta_k^2, \beta_k^2)$ is a stationary point of function $f$. Setting

$$\frac{\partial}{\partial \xi} f = 0, \quad \frac{\partial}{\partial \eta} f = 0,$$

then $\xi = \beta_k^1 + \beta_k^2, \eta = \beta_k^2$ satisfy

$$\xi = \frac{\|\nabla (y_{k-1} - y_{k-2})\|^2(\nabla y_k, \|\nabla y_{k-1} - y_{k-2}) - (\nabla (y_k - y_{k-1}), \nabla (y_{k-1} - y_{k-2}))\|\nabla y_k, \nabla (y_{k-1} - y_{k-2})\|}{\|\nabla (y_k - y_{k-1})\|^2\|\nabla (y_{k-1} - y_{k-2})\|^2 - (\nabla (y_k - y_{k-1}), \nabla (y_{k-1} - y_{k-2}))\|\nabla (y_k, \nabla (y_{k-1} - y_{k-2})\|^2},$$

$$\eta = \frac{\|\nabla (y_{k-1} - y_{k-2})\|^2(\nabla y_k, \nabla (y_{k-1} - y_{k-2}) - (\nabla (y_k - y_{k-1}), \nabla (y_{k-1} - y_{k-2}))\|\nabla y_k, \nabla (y_{k-1} - y_{k-2})\|}{\|\nabla (y_k - y_{k-1})\|^2\|\nabla (y_{k-1} - y_{k-2})\|^2 - (\nabla (y_k - y_{k-1}), \nabla (y_{k-1} - y_{k-2}))\|\nabla (y_k, \nabla (y_{k-1} - y_{k-2})\|^2}.$$

From the definition of $\theta_k$, we have

$$(1 - \theta_k^2)\|\nabla y_k\|^2 = \|\nabla (\xi(y_k - y_{k-1}) + \eta(y_{k-1} - y_{k-2}))\|^2,$$

which implies

$$|\beta_k^2\|\nabla (y_{k-1} - y_{k-2})\| \leq \sqrt{1 - \theta_k^2}\|\nabla y_k\| + |\beta_k^1 + \beta_k^2|\|\nabla (y_k - y_{k-1})\|,$$

thanks to triangle inequality $\|\nabla (y_k - y_{k-1})\|$ $\leq \|\nabla y_k\| + \|\nabla y_{k-1}\|$. \hfill \Box

A few more useful identities given here besides (3.3):

$$y_{k-1} - y_{k-2} = \hat{e}_{k-1} - e_{k-2},$$  

(3.14)

$$y_k = e_k + (\beta_k^1 + \beta_k^2)\hat{e}_{k-1} + \beta_k^2 e_{k-2}.$$  

(3.15)

Note (3.2) and (3.4) do not hold here as the optimization step changes. Now we bound errors $e_k$ by residuals $y_k$.

Lemma 3.10. Let Assumption 3.8 holds, then we have

$$|\beta_k^2\|\|\nabla e_{k-2}\| \leq C_3(\sqrt{1 - \theta_k^2} + |\beta_k^1 + \beta_k^2|)\|\nabla y_k\| + (C_3|\beta_k^1 + \beta_k^2| + 8C_0R|\beta_k^2|)\|\nabla y_{k-1}\|, \quad (3.16)$$

$$\|\nabla e_{k-1}\| \leq C_3(\|\nabla y_k\| + \|\nabla y_{k-1}\|), \quad (3.17)$$

$$\|\nabla e_k\| \leq (1 + C_3^2 + 2C_3|\beta_k^1 + \beta_k^2|)\|\nabla y_k\| + (2C_3|\beta_k^1 + \beta_k^2| + 8C_0R|\beta_k^2|)\|\nabla y_{k-1}\|, \quad (3.18)$$

where $C_3 := \sqrt{2 + 2C_0R + 8C_0^2d^2}$.  

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Proof. We first show inequality (3.17). Subtracting (3.7) with \( j = k - 2 \) from (3.7) with \( j = k - 1 \) yields
\[
b(\tilde{e}_k, u_{k-2}, v) + b(\tilde{u}_k, e_{k-1}, v) + b(u_{k-1}, y_k, v) - b(u_{k-2}, y_{k-1}, v) + \nu(\nabla \tilde{e}_k, \nabla v) = 0.
\]
Setting \( v = e_{k-1} \) eliminates the second term and produces
\[
\frac{\nu}{2} (1 - \nu^{-1} MR) (\| \nabla \tilde{e}_k \|^2 + \| \nabla e_{k-1} \|^2) \\
\leq \frac{\nu}{2} \| \nabla (y_k - y_{k-1}) \|^2 + MR \| \nabla y_k \| \| \nabla e_{k-1} \| + MR \| \nabla y_{k-1} \| \| \nabla e_{k-1} \|,
\]
thanks to polarization identity, \( ab \leq \frac{a^2 + b^2}{2} \) and (2.2). Applying the Young’s inequality and dropping the term with \( \| \nabla \tilde{e}_k \|^2 \) yields
\[
\| \nabla e_{k-1} \|^2 \leq 2(1 + C_0 R) \| \nabla (y_k - y_{k-1}) \|^2 + 8C_0^2 R^2 (\| \nabla y_k \|^2 + \| \nabla y_{k-1} \|^2) \\
\leq 2(1 + C_0 R) (\| \nabla y_k \| + \| \nabla y_{k-1} \|) + 8C_0^2 R^2 (\| \nabla y_k \| + \| \nabla y_{k-1} \|)^2 \\
\leq 2(1 + C_0 R + 4C_0^2 R^2) (\| \nabla y_k \| + \| \nabla y_{k-1} \|)^2.
\]
Taking square on both sides leads to (3.17).

Now we prove (3.16). Subtracting (3.7) with \( j = k - 3 \) from (3.7) with \( j = k - 2 \) yields
\[
b(\tilde{e}_k, u_{k-2}, v) + b(\tilde{u}_k, e_{k-2}, v) + b(u_{k-3}, y_{k-1} - y_{k-2}, v) + b(e_{k-2}, y_{k-1}, v) + \nu(\nabla \tilde{e}_{k-1}, \nabla v) = 0,
\]
using (3.14). Setting \( v = e_{k-2} \) eliminate the second term and gives
\[
\frac{\nu}{2} (1 - \nu^{-1} MR) (\| \nabla \tilde{e}_{k-1} \|^2 + \| \nabla e_{k-2} \|^2) \\
\leq \frac{\nu}{2} \| \nabla (y_{k-1} - y_{k-2}) \|^2 + MR \| \nabla (y_{k-1} - y_{k-2}) \| \| \nabla e_{k-2} \| + 2MR \| \nabla y_{k-1} \| \| \nabla e_{k-2} \|,
\]
thanks to the polarization identity, (2.2) and (2.1). Applying the Young’s inequality and dropping the term with \( \| \nabla \tilde{e}_{k-1} \|^2 \) yields
\[
\| \nabla e_{k-2} \| \leq \sqrt{C_3^2 \| \nabla (y_{k-1} - y_{k-2}) \|^2 + 32C_0^2 R^2 \| \nabla y_{k-1} \|^2} \\
\leq C_3 \| \nabla (y_{k-1} - y_{k-2}) \| + 8C_0 R \| \nabla y_{k-1} \|.
\]
Multiplying both sides by \( |\beta_k^2| \) and applying (3.13) produces
\[
|\beta_k^2| \| \nabla e_{k-2} \| \leq C_3 (\sqrt{1 - \theta_k^2} + |\beta_k^1 + \beta_k^2|) \| \nabla y_k \| + (C_3 |\beta_k^1 + \beta_k^2| + 8C_0 R|\beta_k^2|) \| \nabla y_{k-1} \|.
\]
 Lastly, we show (3.18). From (3.15), we have
\[
\| \nabla e_k \| \leq \theta_k \| \nabla y_k \| + |\beta_k^1 + \beta_k^2| \| \nabla e_{k-1} \| + |\beta_k^2| \| \nabla e_{k-2} \| \\
\leq (\theta_k + 2C_3 |\beta_k^1 + \beta_k^2| + C_3 \sqrt{1 - \theta_k^2}) \| \nabla y_k \| + (2C_3 |\beta_k^1 + \beta_k^2| + 8C_0 R|\beta_k^2|) \| \nabla y_{k-1} \| \\
\leq (\sqrt{1 + C_3^2 + 2C_3 |\beta_k^1 + \beta_k^2|} \| \nabla y_k \| + (2C_3 |\beta_k^1 + \beta_k^2| + 8C_0 R|\beta_k^2|) \| \nabla y_{k-1} \|,
\]
due to \( \max_{\theta \in [0,1]} \{ \theta + C_3 \sqrt{1 - \theta^2} \} = \sqrt{1 + C_3^2} \).
Lemma 3.11. Let Assumption 3.8 holds, then we have
\[ \| \nabla (G'(u_{k-1}; e_k) + (\beta_k^1 + \beta_k^2)G'(u_{k-2}; e_{k-1}) + \beta_k^2 G'(u_{k-3}; e_{k-2})) \| \leq O((\| \nabla y_k \| + \| \nabla y_{k-1} \|)^2), \] (3.19)
and
\[ \| \nabla (y_{k+1} - G'(u_{k-1}; e_k) - (\beta_k^1 + \beta_k^2)G'(u_{k-2}; e_{k-1}) - \beta_k^2 G'(u_{k-3}; e_{k-2})) \| \leq O((\| \nabla y_k \| + \| \nabla y_{k-1} \|)^2). \] (3.20)

Proof. From (2.15) and Lemma 3.10, we have
\[ \| \nabla G'(u_{k-1}; e_k) \| \leq 2C_0 \| \nabla y_k \| \| \nabla e_k \| + C_0 \| \nabla e_k \|^2 \]
\[ \leq O((\| \nabla y_k \| (\| \nabla y_k \| + \| \nabla y_{k-1} \|)) + O((\| \nabla y_k \| + \| \nabla y_{k-1} \|)^2), \]
\[ | \beta_k^1 + \beta_k^2 | \| \nabla G'(u_{k-2}; e_{k-1}) \| \leq 2C_0 \| \nabla y_{k-1} \| (| \beta_k^1 | + \beta_k^2 | \| \nabla e_{k-1} \| + C_0 | \beta_k^1 | + \beta_k^2 | \| \nabla e_{k-1} \|)^2 \]
\[ \leq O((\| \nabla y_{k-1} \| (\| \nabla y_{k-1} \| + \| \nabla y_{k-1} \|)) + O((\| \nabla y_{k-1} \| + \| \nabla y_{k-1} \|)^2), \]
and
\[ | \beta_k^2 | \| \nabla G'(u_{k-3}; e_{k-2}) \| \leq 2C_0 \| \nabla y_{k-1} \| (| \beta_k^2 | \| \nabla y_{k-1} \| + C_0 | \beta_k^2 | \| \nabla e_{k-2} \|)^2 \]
\[ \leq O((\| \nabla y_{k-1} \| (\| \nabla y_{k-1} \| + \| \nabla y_{k-1} \|)) + O((\| \nabla y_{k-1} \| + \| \nabla y_{k-1} \|)^2). \]

Combining the above three inequalities, we have
\[ \| \nabla (G'(u_{k-1}; e_k) + (\beta_k^1 + \beta_k^2)G'(u_{k-2}; e_{k-1}) + \beta_k^2 G'(u_{k-3}; e_{k-2})) \| \]
\[ \leq \| \nabla G'(u_{k-1}; e_k) \| + | \beta_k^1 + \beta_k^2 | \| \nabla G'(u_{k-2}; e_{k-1}) \| + | \beta_k^2 | \| \nabla G'(u_{k-3}; e_{k-2}) \| \]
\[ \leq O((\| \nabla y_k \| + \| \nabla y_{k-1} \|)^2). \]

For notation simplification, denote \( \psi_k = G(u_k) - G(u_{k-1}; e_k) \). Utilizing identity \( y_{k+1} = \tilde{e}_{k+1} + (\beta_k^1 + \beta_k^2) \tilde{e}_k + \beta_k^2 \tilde{e}_{k-1} \) yields
\[ y_{k+1} - G'(u_{k-1}; e_k) - (\beta_k^1 + \beta_k^2)G'(u_{k-2}; e_{k-1}) - \beta_k^2 G'(u_{k-3}; e_{k-2}) = \psi_k + (\beta_k^1 + \beta_k^2) \psi_{k-1} + \beta_k^2 \psi_{k-2}. \]

From equation (2.10) and Lemma 3.10, we have the following three inequalities
\[ \| \nabla \tilde{e}_{k-1} \| \leq 2C_0 \| \nabla e_k \| (\| \nabla y_k \| + C_0 \| \nabla e_k \|)^2 \leq O((\| \nabla y_k \| + \| \nabla y_{k-1} \|)^2), \]
\[ \| \nabla \tilde{e}_k \| \leq 2C_0 \| \nabla e_{k-1} \| (\| \nabla y_k \| + C_0 \| \nabla e_{k-1} \|)^2 \leq O((\| \nabla y_k \| + \| \nabla y_{k-1} \|)^2), \]
\[ \| \nabla \tilde{e}_{k-1} \| \leq 2C_0 \| \nabla e_{k-2} \| (\| \nabla y_{k-1} \| + C_0 \| \nabla e_{k-2} \|)^2 \leq O((\| \nabla y_{k-1} \| + \| \nabla y_{k-2} \|)^2). \]

Utilizing (2.14) gives
\[ \| \nabla (y_{k+1} - G'(u_{k-1}; e_k) - (\beta_k^1 + \beta_k^2)G'(u_{k-2}; e_{k-1}) - \beta_k^2 G'(u_{k-3}; e_{k-2})) \| \]
\[ \leq \| \nabla \psi_k \| + (| \beta_k^1 + \beta_k^2 | \| \nabla \psi_{k-1} \| + | \beta_k^2 | \| \nabla \psi_{k-2} \|) \]
\[ \leq 2C_0 \| \nabla e_k \| (\| \nabla \tilde{e}_{k+1} \| + | \beta_k^1 + \beta_k^2 | C_0 \| \nabla e_{k-1} \| \| \nabla \tilde{e}_k \| + 2C_0 | \beta_k^2 | \| \nabla e_{k-2} \| \| \nabla \tilde{e}_{k-1} \| \]
\[ \leq O((\| \nabla y_k \| + \| \nabla y_{k-1} \|)^2). \]
Now we manifest that Anderson accelerated Newton’s method with depth 2 also converges superlinearly.

**Theorem 3.12** (One-step residual bound for \( m = 2 \)). Let Assumption 3.8 holds, then the residual sequence \( \{y_k\} \) from Algorithm 3.7 satisfies

\[
\|\nabla y_{k+1}\| \leq O((\|\nabla y_k\| + \|\nabla y_{k-1}\|)^{3/2}),
\]

where the bound depends on parameters \( \nu, [\Omega, R, f, \beta_k^1, \beta_k^2, \theta_k] \).

**Proof.** Start with constructing \( u_k \). Adding \( 1 - \beta_k^1 - \beta_k^2 \) multiple of (3.7) with \( j = k - 1 \), \( \beta_k^1 \) multiple of (3.7) with \( j = k - 2 \) to \( \beta_k^2 \) multiple of (3.7) with \( j = k - 3 \) gives

\[
b(y_k, u_{k-1}, v) - (\beta_k^1 + \beta_k^2)b(y_k - y_{k-1}, u_{k-2}, v) - (\beta_k^1 + \beta_k^2)b(y_k, e_{k-1}, v)
- (\beta_k^1 + \beta_k^2)b(y_{k-1} - y_{k-2}, u_{k-3}, v) - (\beta_k^1 + \beta_k^2)b(y_{k-1}, e_{k-2}, v) + b(u_{k-1}, u_k, v) - \beta_k^1b(e_{k-1}, \tilde{u}_{k-1}, v)
- \beta_k^2b(e_{k-1} + e_{k-2}, \tilde{u}_{k-2}, v) + \nu(\nabla u_k, \nabla v) = (f, v).
\]

Subtracting it from (3.7) with \( j = k \) yields

\[
b(y_{k+1}, u_k, v) + b(e_k, u_k, v) + b(u_k, y_{k+1}, v) - b(y_k, u_{k-1}, v) + (\beta_k^1 + \beta_k^2)b(y_k - y_{k-1}, u_{k-2}, v)
+ (\beta_k^1 + \beta_k^2)b(y_k, e_{k-1}, v) + (\beta_k^1 + \beta_k^2)b(y_{k-1} - y_{k-2}, u_{k-3}, v) + (\beta_k^1 + \beta_k^2)b(y_{k-1}, e_{k-2}, v)
+ b(e_{k-1}, \beta_k^1\tilde{u}_{k-1} + \beta_k^2\tilde{u}_{k-2}, v) + \beta_k^2b(e_{k-1} + e_{k-2}, \tilde{u}_{k-2}, v) + \nu(\nabla y_{k+1}, \nabla v) = 0.
\]

Setting \( v = \chi := G'(u_{k-1}; e_k) + (\beta_k^1 + \beta_k^2)G'(u_{k-2}; e_{k-1}) + \beta_k^2G'(u_{k-3}; e_{k-2}) \) produces

\[
\frac{\nu}{2}(\|\nabla y_{k+1}\|^2 + \|\nabla \chi\|^2 - \|\nabla (y_{k+1} - \chi)\|^2)
\leq MR\|\nabla y_{k+1}\|\|\nabla \chi\| + MR\|\nabla e_k\|\|\nabla \chi\| + MR\|\nabla (y_{k+1} - \chi)\|\|\nabla \chi\| + MR\|\nabla y_k\|\|\nabla \chi\|
+ |\beta_k^1 + \beta_k^2|MR\|\nabla (y_k - y_{k-1})\|\|\nabla \chi\| + |\beta_k^1 + \beta_k^2|M\|\nabla y_k\|\|\nabla e_{k-1}\|\|\nabla \chi\|
+ |\beta_k^2|MR\|\nabla (y_{k-1} - y_{k-2})\|\|\nabla \chi\| + |\beta_k^2|M\|\nabla y_{k-1}\|\|\nabla e_{k-2}\|\|\nabla \chi\|
+ M\|\nabla (u_k - (1 - \beta_k^1 - \beta_k^2)\tilde{u}_k)\|\|\nabla e_{k-1}\|\|\nabla \chi\| + |\beta_k^2|M\|R_G\|\nabla e_{k-2}\|\|\nabla \chi\|,
\]

thanks to polarization identity, (2.1), (2.2). Applying \( \|\nabla (u_k - (1 - \beta_k^1 - \beta_k^2)\tilde{u}_k)\| \leq R + (1 + |\beta_k^1 + \beta_k^2|)R_G, \) Young’s inequality, and dropping \( \|\nabla \chi\|^2 \) term, we obtain

\[
\|\nabla y_{k+1}\|^2 \leq (1 + C_0K\|\nabla y_{k+1} - \chi\|^2 + 2C_0K\|\nabla y_{k+1} - \chi\|\|\nabla \chi\|
+ 4C_0K(1 + |\beta_k^1 + \beta_k^2|)(\|\nabla y_k\| + \|\nabla y_{k-1}\|)\|\nabla \chi\|
+ 2C_0K\|\nabla e_k\|\|\nabla \chi\| + 2C_0\|\nabla y_k\|\beta_k^2\|\nabla e_{k-1}\|\|\nabla \chi\|
+ 2C_0(\|R_G + R_kK(1 + |\beta_k^1 + \beta_k^2|)\|\nabla e_{k-2}\|\|\nabla \chi\|
+ 2C_0\|\nabla y_{k-1}\|\beta_k^2\|\nabla e_{k-2}\|\|\nabla \chi\| + 2C_0K|\beta_k^2|\|\nabla e_{k-2}\|\|\nabla \chi\|
\leq O((\|\nabla y_{k+1}\| + \|\nabla y_{k+1}\|)^3).
\]

thanks to Lemma 3.9, Lemma 3.10 and Lemma 3.11.
3.3 Anderson accelerated Newton’s method with depth $m$

This subsection states the algorithm and one-step convergence result of the general Anderson acceleration applied to Newton’s method for solving steady Navier-Stokes equations. The analysis would be similar to the previous two subsections but involving more complicate constants, and so here we omit it.

**Algorithm 3.13** (Anderson accelerated Newton’s iteration with depth $m$ for NSE). Algorithm of Anderson accelerated Newton’s method with depth $m$ is stated as below:

*Step 0* Guess $u_0 \in B(0, R)$.

*Step 1* Compute $\tilde{u}_1 = G(u_0)$ and set residual $y_1 = \tilde{u}_1 - u_0$, update $u_1 = \tilde{u}_1$.

*Step k* For $k = 2, 3, \ldots$, set $m_k = \min\{k - 1, m\}$

a) Compute $\tilde{u}_k = G(u_{k-1})$ and set $y_k = \tilde{u}_k - u_{k-1}$.

b) Find $\{\gamma^i_k\}_{i=1}^{m_k} \subset \mathbb{R}$ minimizing

$$
\left\| \nabla \left( \left( 1 - \sum_{i=1}^{m_k} \gamma^i_k \right) y_k + \sum_{i=1}^{m_k} \gamma^i_k y_{k-i} \right) \right\|.
$$

c) Update $u_k = \left( 1 - \sum_{i=1}^{m_k} \gamma^i_k \right) \tilde{u}_k + \sum_{i=1}^{m_k} \gamma^i_k \tilde{u}_{k-i}$.

Clearly, Algorithm 3.13 is back to AAN with small Anderson depth or Newton’s method if $\gamma^m_k = 0$ for any $k \geq m + 1$. So we will assume $\gamma^m_k \neq 0$.

**Theorem 3.14** (One-step residual bound). Assume for any step $k > m$ with $\gamma^m_k \neq 0$ and $u_j \in B(0, R)$ with $R < \nu M^{-1}$ for all $j \leq k$, then

$$
\| \nabla y_{k+1} \| \leq O \left( \left( \sum_{i=1}^{m} \| \nabla y_{k-i+1} \| \right)^{3/2} \right),
$$

where the bound depends on parameters $\nu, |\Omega|, R, f, \gamma^i_k, \theta_k$.

This theorem tells the general Anderson accelerated Newton’s method for solving NSE converges superlinearly if the initial guess is good enough, and large depth algorithm converges slower than small depth algorithm.

4 Numerical tests

In this section, we test two benchmark problems to verify the superlinearly convergence of Anderson accelerated Newton’s method for solving steady Navier-Stokes equation.
4.1 2D cavity problem

The 2D driven cavity uses a domain $\Omega = [0, 1]^2$, with no slip boundary conditions on the sides and bottom, and a ‘moving lid’ on the top which is implemented by enforcing the Dirichlet boundary condition $u(x, 1) = (1, 0)^T$, no forcing ($f = 0$). We discretize with $(P_2, P_1)$ Taylor-Hood elements on a 1/64 mesh that provides 44,365 total degrees of freedom, and for the initial guess we used the Picard solution after 3 iterations on the same mesh with the same finite element setting. Newton’s method and Anderson accelerated Newton’s methods with several depths are tested with tolerance $1e-13$.

Streamline plots at $Re = 2500, 5000$ obtained by Anderson accelerated Newton’s method with depth $m = 5$ are given in Figure 1, which are in well agreement with literature [6]. In fact, similar plots can be observed for depth $m = 1, 2, 10$ too, and therefore are omitted here. Convergence plot at various Reynolds numbers are shown in Figure 2. For problem with $Re = 2500$, Anderson acceleration slows down the convergence speed to Newton’s method, but is still faster than linearly convergent solvers, say Picard iteration, or Anderson accelerated Picard method [11]. For $Re = 5000$, we observe that Anderson acceleration enlarges the domain of convergence for Newton’s method, which implies Anderson acceleration Newton’s method is worth to use in practice, especially for problems with small domain of convergence. For both cases $Re = 2500, 5000$, we find that large depth Anderson accelerated Newton’s method converges slower than small depth algorithm, that matches our analytical results. In addition, Table 1 shows that Anderson accelerated Newton’s method $m = 1$ converges superlinearly with order close to 1.5, whereas large depth decelerates the convergence speed for Anderson accelerated Newton’s method, which matches our theoretical results from Theorem 3.6, Theorem 3.12 and Theorem 3.14.

![Streamline plots](image)

Figure 1: Shown above are streamline plots of the solutions from Anderson accelerated Newton solvers at Reynolds number $Re = 2500$ (left), 5000 (right).

| conv. order | Newton | AAN $m = 1$ | AAN $m = 2$ | AAN $m = 5$ | AAN $m = 10$ |
|-------------|--------|-------------|-------------|-------------|--------------|
| $Re = 2500$ | 2.0192 | 1.2662      | 1.2519      | 1.3072      | 1.3936       |
| $Re = 5000$ | Fail   | 1.2519      | 1.3936      | 1.0203      | 0.82909      |

Table 1: This table summarizes the median of convergence order for Newton’s method, Anderson accelerated Newton’s method (AAN) for different $Re = 2500, 5000$ in the 2D cavity problem.
In this subsection, we test Anderson accelerated Newton’s method on a 3D lid driven cavity problem with Reynolds number $Re = 400, 1000$. We use a domain $\Omega = [0, 1]^3$, with no slip boundary conditions on all walls, and a unite ‘moving lid’ $u = (1, 0, 0)^T$ on the top, no forcing ($f = 0$). We discretize with $(P_3, P_2^{\text{disc}})$ Scott-Vogelius elements on a barycenter refined uniform mesh that provides 206,874 total degrees of freedom, and use zero interior initial guess but satisfying the boundary conditions. Newton’s method and Anderson accelerated Newton’s methods with several depths are tested with tolerance $1 \times 10^{-6}$.

We solve the saddle point linear systems that arise at each iteration via method in [4]. Decompose the coefficient matrix via a LU block factorization

$$
\begin{pmatrix}
A_k & B \\
B^T & 0
\end{pmatrix}
\begin{pmatrix}
U_k \\
P_k
\end{pmatrix} =
\begin{pmatrix}
A_k & 0 \\
B^T & -B^T A_k^{-1} B
\end{pmatrix}
\begin{pmatrix}
I & A_k^{-1} B^T \\
0 & I
\end{pmatrix}
\begin{pmatrix}
U_k \\
P_k
\end{pmatrix} =
\begin{pmatrix}
F \\
G
\end{pmatrix}.
$$

This leads to two solves of a smaller size linear system with coefficient matrix $A_k$ and one solve of a linear system with coefficient matrix to be the Schur complement $B^T A_k^{-1} B$. Direct solver and BICGSTAB with tolerance $1 \times 10^{-10}$ and preconditioner pressure mass matrix were used to solve these two types linear systems respectively.

Plots of centerline $x-$velocity and centerplane slices obtained from Anderson accelerated Newton’s method with $m = 1$ are given in Figure 3, which matches well with [13]. In fact, similar plots can be observed from Anderson accelerated Newton’s method with other depth provided the method converges and are omitted. Convergence plots for 3D cavity problem with $Re = 400, 1000$ are given in Figure 4. We observe superlinear convergence for Anderson accelerated Newton’s method when $Re = 400$, which matches our analytical results well. However, when $Re = 1000$ we find that Anderson accelerated Newton’s method with depth $m = 2, 5, 10$ fail due to smaller domain of convergence while $m = 1$ converges superlinearly. A safeguard strategy would be using Picard iteration or Anderson accelerated Picard iteration first and then switch to Newton’s method with or without Anderson acceleration when residual is small enough.
5 Conclusions

In this paper, we have studied the performance of Anderson acceleration to Newton’s method for solving steady Navier-Stokes equations. We find that Anderson accelerated Newton’s method with a good initial guess converges superlinearly, which is slower than the usual Newton’s method. Moreover, Anderson acceleration with large depth decelerates the convergence speed comparing to the one with small depth. The numerical tests confirm our analytical results. In addition, we observe that Anderson acceleration sometimes enlarges the domain of convergence from the 2D cavity experiment with $Re = 5000$, but sometimes narrow the domain of convergence from the 3D cavity experiment with $Re = 4000$.
cavity experiment with $Re = 1000$, this phenomenon is unexplained and will be studied in the near future.

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