REVISITING THE MODULI SPACE OF 8 POINTS ON $\mathbb{P}^1$

KLAUS HULEK AND YOTA MAEDA

Abstract. The moduli space of 8 points on $\mathbb{P}^1$, a so-called ancestral Deligne-Mostow space, is, by work of Kondo, also a moduli space of K3 surfaces. We prove that the Deligne-Mostow isomorphism does not lift to a morphism between the Kirwan blow-up of the GIT quotient and the unique toroidal compactification of the corresponding ball quotient. Moreover, we show that these spaces are not $K$-equivalent, even though they are natural blow-ups at the unique cusps and have the same cohomology. This is analogous to the work of Casalaina-Martin-Grushevsky-Hulek-Laza on the moduli space of cubic surfaces. The moduli spaces of ordinary stable maps, that is the Fulton-MacPherson compactification of the configuration space of points on $\mathbb{P}^1$, play an important role in the proof. We further relate our computations to new developments in the minimal model program and recent work of Odaka. We briefly discuss other cases of moduli space of points on $\mathbb{P}^1$ where a similar behaviour can be observed, hinting at a more general, but not yet fully understood phenomenon.

1. Introduction

It was shown by Casalaina-Martin-Grushevsky-Hulek-Laza [CMGHL24] that the Kirwan blow-up and the toroidal compactification of the moduli space of (non-marked) smooth cubic surfaces are not isomorphic. In this paper, we prove analogous results for the moduli space of unordered 8 points on $\mathbb{P}^1$, denoted by $\mathcal{M}^{\mathrm{GIT}}$. The proof we give here is inspired by that of [CMGHL24], but requires further ideas. As we shall discuss in Section 6, the behaviour observed here is shared by other ball quotients as well, thus pointing towards a much more general, and yet not fully understood, phenomenon.

The case of 8 points on $\mathbb{P}^1$ is of special interest for more than one reason. One is that it has more than one modular interpretation. Besides being a moduli space of points, it is, by work of Kondo [Ko07a], also closely related to moduli of K3 surfaces and automorphic forms. A further reason is that it is a so-called ancestral Deligne-Mostow variety in the sense of the discussion by Gallardo-Kerr-Schaffler [GKS21]. This means that any Deligne-Mostow variety over the Gaussian integers with arithmetic monodromy group, and which has cusps, can be embedded into this ball quotient. The other ancestral case is that of 12 points on $\mathbb{P}^1$, which plays the same role for the Eisenstein integers. In this paper, we shall concentrate on the Gaussian case and only briefly discuss the Eisenstein case, which is the contents of the subsequent paper [HKM24].

1.1. Main results. The Deligne-Mostow theory [DM86] gives us an isomorphism between $\mathcal{M}^{\mathrm{GIT}}$ and the Baily-Borel compactification of an appropriate 5-dimensional ball quotient $\overline{\mathbb{B}}^5/\Gamma$. We are interested in the lifting of the Deligne-Mostow isomorphism to the unique toroidal compactification. There exist two natural blow-ups, playing important roles here: the Kirwan blow-up $f : \mathcal{M}^{\mathrm{K}} \to \mathcal{M}^{\mathrm{GIT}}$ and the toroidal compactification $\pi : \overline{\mathbb{B}}^5/\Gamma_{\mathrm{tor}} \to \overline{\mathbb{B}}^5/\Gamma_{\mathrm{BB}}$. Here, the Kirwan blow-up $\mathcal{M}^{\mathrm{K}}$ is the partial desingularisation of $\mathcal{M}^{\mathrm{GIT}}$ whose
centre is located in the polystable orbits (which is a unique point \(\{c_{4,4}\}\) in our case). The toroidal compactification \(\mathbb{B}_5^{\text{tor}}\) is a blow-up of \(\mathbb{B}_5^{\text{BB}}\) at the point \(\{\xi\}\), which is the unique cusp, i.e., the Baily-Borel boundary. The above Deligne-Mostow isomorphism sends \(c_{4,4}\) to \(\xi\), thus restricting to an isomorphism \(\mathcal{M}^K \setminus f^{-1}(c_{4,4}) \cong \mathbb{B}_5^{\text{tor}} \setminus \pi^{-1}(\xi)\). In this setting, our first main result asserts that the birational map \(g : \mathcal{M}^K \to \mathbb{B}_5^{\text{tor}}\) does not extend to a morphism.

\[
\begin{align*}
\mathcal{M}^K & \to \mathbb{B}_5^{\text{tor}} \\
\mathcal{M}^{\text{GIT}} & \to \mathbb{B}_5^{\text{BB}}.
\end{align*}
\]

**Theorem 1.1** (Theorem 3.15, Remark 4.12). *Neither the Deligne-Mostow isomorphism \(\phi : \mathcal{M}^{\text{GIT}} \to \mathbb{B}_5^{\text{BB}}\) nor its inverse \(\phi^{-1}\) lift to a morphism between the Kirwan blow-up \(\mathcal{M}^K\) and the unique toroidal compactification \(\mathbb{B}_5^{\text{tor}}\).*

This result still leaves the possibility open that the Kirwan blow-up and the toroidal compactification are isomorphic as abstract varieties. One obstruction to this could be that the varieties are topologically different. Indeed, the topology of these varieties is of independent interest (and indeed this was the starting point of [CMGHL23] and [CMGHL24] in the case of cubic threefolds and cubic surfaces). We compute the cohomology of these varieties, according to the Kirwan method [Ki84, Ki85, Ki89] and Casalaina-Martin-Grushevsky-Hulek-Laza [CMGHL23]. Wherever a space \(X\) has at most finite quotient singularities, we work with singular cohomology with rational coefficients and denote this by \(H^k(X)\). In the other cases, notably the GIT quotient and the Baily-Borel compactification of ball quotients, we work with intersection cohomology (of middle perversity) and denote this by \(IH^k(X)\). Note that for spaces with finite quotient singularities singular cohomology and intersection cohomology coincide. The cohomology groups of the varieties under consideration are given as follows.

**Theorem 1.2** (Theorem 5.1, 5.2, 5.6, 5.8). *All the odd degree cohomology of the following projective varieties vanishes. In even degrees, their Betti numbers are given by:*

| \(j\) | 0 | 2 | 4 | 6 | 8 | 10 |
| --- | --- | --- | --- | --- | --- | --- |
| \(\dim H^j(\mathcal{M}^K)\) | 1 | 2 | 3 | 3 | 2 | 1 |
| \(\dim IH^j(\mathbb{B}_5^{\text{BB}})\) | 1 | 1 | 2 | 2 | 1 | 1 |
| \(\dim H^j(\mathbb{B}_5^{\text{tor}})\) | 1 | 2 | 3 | 3 | 2 | 1 |
| \(\dim H^j(\mathcal{M}^{\text{ord}})\) | 1 | 43 | 99 | 99 | 43 | 1 |
| \(\dim IH^j(\mathbb{B}_5^{\text{BB}})\) | 1 | 8 | 29 | 29 | 8 | 1 |
| \(\dim IH^j(\mathbb{B}_5^{\text{tor}})\) | 1 | 43 | 99 | 99 | 43 | 1 |

*thus, all the Betti numbers of \(\mathcal{M}^K\) and \(\mathbb{B}_5^{\text{tor}}\) are the same.*

Here, \(\mathbb{B}_5^{\text{BB}_{\text{ord}}}\) denotes the Baily-Borel compactification of a 5-dimensional ball quotient, which is an \(S_8\)-cover of \(\mathbb{B}_5^{\text{BB}}\) and isomorphic to \(\mathcal{M}^{\text{ord}}\), the moduli space of ordered 8 points on \(\mathbb{P}^1\). Also, we denote by \(\mathcal{M}^{\text{ord}}\) the Kirwan blow-up of \(\mathcal{M}^{\text{ord}}\) and by \(\mathbb{B}_5^{\text{ord}_{\text{tor}}}\) the toroidal
blow-up of $\mathbb{B}^5/\Gamma_{\ord}^{BB}$. For more precise descriptions of these varieties, as well as the bounded symmetric domain and arithmetic subgroups, see Section 2.

Again, this result leaves the possibility that $\mathcal{M}^K$ and $\mathbb{B}^5/\Gamma_{\tor}$ are isomorphic as abstract varieties. We rule this out by showing that these spaces are not $K$-equivalent. Recall that two projective normal $\mathbb{Q}$-Gorenstein varieties $X$ and $Y$ are called $K$-equivalent if there is a common resolution of singularities $Z$ dominating $X$ and $Y$ birationally

\[
\begin{array}{ccc}
Z & \overset{f_X}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
Y & \overset{f_Y}{\longrightarrow} & \end{array}
\]

such that $f_X^*K_X \sim_{\mathbb{Q}} f_Y^*K_Y$. For $K$-equivalent varieties, the top intersection numbers are equal: $K^n_X = K^n_Y$, where $n$ is the dimension of $X$ and $Y$. We shall use this property to show that $\mathcal{M}^K$ and $\mathbb{B}^5/\Gamma_{\tor}$ are not $K$-equivalent. Thus, these varieties are in particular not isomorphic as abstract varieties, even though they are the blow-ups at the same points of $\mathcal{M}^{\GIT} \simeq \mathbb{B}^5/\Gamma_{\ord}^{BB}$ and have the same Betti numbers.

**Theorem 1.3** (Theorem 4.6). The Kirwan blow-up $\mathcal{M}^K$ and the toroidal compactification $\mathbb{B}^5/\Gamma_{\tor}$ are not $K$-equivalent and hence, in particular, not isomorphic as abstract varieties.

**Remark 1.4.** We can interpret Theorems 1.1 and 1.3 in the context of the minimal model program and semi-toric compactifications. This gives, in particular, an independent proof of one direction of Theorem 1.1 using the characterization of (semi-)toric compactifications. This proof does not require the calculation of the second Betti number of these spaces, but it uses the Luna slice calculations. We refer the reader to Subsection 4.3 for more details.

As we shall see later, the situation is in contrast to the case of the moduli space of ordered points, where we have an isomorphism $\mathcal{M}^K_{\ord} \simeq \mathbb{B}^5/\Gamma_{\ord}^{BB}$.

**Remark 1.5.** Kudla-Rapoport [KR12] studied the descent problem of Deligne-Mostow isomorphisms and ball quotients at the level of moduli stacks over the natural field of definitions. In particular, in the case of 12 points, they gave an interpretation as a DM-stack parameterizing abelian varieties, showing the Deligne-Mostow isomorphism comes from a morphism between DM-stacks and its image is a complement of Kudla-Rapoport cycles [KR12, Theorem 8.1]. It seems interesting to ask whether a similar result holds in our (and similar) situations.

1.2. **Outline of the proof of Theorem 1.1.** The strategy of the proof of Theorem 1.1 is as follows. As in [CMGHL24] the argument is divided into two steps. We first prove that the discriminant divisor and the boundary divisor intersect non-transversally in the Kirwan blow-up. This is done in terms of a local computation by using the Luna slice. Secondly, we show that the corresponding divisors intersect generically transversally in the toroidal compactification of the 5-dimensional ball quotient. Here is a major difference to [CMGHL24]. This is because we cannot use Naruki’s compactification. Instead, we work on a sequence of blow-ups of the Baily-Borel compactification of the 5-dimensional ball quotient. This was studied in detail in [GKS21, KM11] and can be described in terms of moduli spaces of weighted pointed stable curves [Ha03]. The discriminant divisor and boundary divisor exist as normal crossing divisors in these spaces, thus we can use this to prove the generic transversality of the divisors in the toroidal compactification.
1.3. **Organization of the paper.** In Section 2, we describe the relationship between GIT quotients and ball quotients. In Section 3, we prove Theorem 1.1 through local computations. In Section 4, we compute the top self-intersection number of canonical bundles, deduce Theorem 1.3 and discuss the relation to the minimal model program. In Section 5, we compute the cohomology by using the Kirwan method. In Section 6, we will briefly discuss other Deligne-Mostow varieties.

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2. **GIT and ball quotients**

Below, we consider the moduli spaces of ordered and unordered 8 points on $\mathbb{P}^1$. Throughout this paper, the phrase “8 points on $\mathbb{P}^1$” will always mean “unordered 8 points on $\mathbb{P}^1$” for simplicity. Let

$$\mathcal{M}_{\text{ord}}^\text{GIT} := (\mathbb{P}^1)^8//\text{SL}_2(\mathbb{C}), \quad \mathcal{M}^\text{GIT} := \mathbb{P}^8//\text{SL}_2(\mathbb{C}).$$

Here, the GIT quotients are taken with respect to the symmetric linearisation $\Theta(1, \cdots, 1)$ and $\Theta(1)$. We also note, see [KMT1], that

$$\mathcal{M}_{\text{ord}}^\text{GIT}/\mathbb{S}_8 \cong \mathcal{M}^\text{GIT}.$$

We denote by $\varphi_1: \mathcal{M}_{\text{ord}}^K \to \mathcal{M}_{\text{ord}}^\text{GIT}$ and $f: \mathcal{M}^K \to \mathcal{M}^\text{GIT}$ the Kirwan blow-ups [Ki85].

As in [Ko07a], we consider the free $\mathbb{Z}[\sqrt{-1}]$-module of rank 2 equipped with the Hermitian forms defined by the following matrices

$$\begin{pmatrix} 0 & 1 + \sqrt{-1} \\ 1 - \sqrt{-1} & 0 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 - \sqrt{-1} \\ 1 + \sqrt{-1} & -2 \end{pmatrix},$$

isomorphic to $U \oplus U(2)$ and $D_4(-1)$, respectively, where $U$ denotes the hyperbolic plane, $U(2)$ is the hyperbolic plane where the form has been multiplied by 2 and $D_4(-1)$ is the negative $D_4$-lattice. By abuse of notation, we will also denote the Hermitian lattices by these symbols.

Here, let $L := U \oplus U(2) \oplus D_4(-1)^{\oplus 2}$ be the Hermitian lattice of signature $(1, 5)$ over $\mathbb{Z}[\sqrt{-1}]$, defined by the above Hermitian forms. Let $U(L)$ be the unitary group scheme over $\mathbb{Z}$ preserving the lattice $L$, and $\Gamma := U(L)(\mathbb{Z})$. Now, there is the Hermitian symmetric domain $\mathbb{B}^5$ associated with the reductive group $U(L)(\mathbb{R}) \cong U(1, 5)$ defined by

$$\mathbb{B}^5 := \{v \in L \otimes \mathbb{Z}[\sqrt{-1}] \otimes \mathbb{C} | \langle v, v \rangle > 0 \}/\mathbb{C}^\times$$

which is isomorphic to the 5-dimensional complex ball. Let $L^\vee$ be the dual lattice of $L$, which contains $L$ as a finite $\mathbb{Z}[\sqrt{-1}]$-module, and $A_L := L^\vee/L$ be the discriminant group, isomorphic to $(\mathbb{Z}[\sqrt{-1}]/(1 + \sqrt{-1})\mathbb{Z}[\sqrt{-1}])^6$ in this situation. Now, let us introduce an important arithmetic subgroup $\Gamma_{\text{ord}} \subset \Gamma$, which is called the discriminant kernel:

$$\Gamma_{\text{ord}} := \{g \in \Gamma | g(v) \equiv v \mod L \ (\forall v \in A_L)\}$$. 
This data gives us the notion of the ball quotients $\mathbb{B}^5/\Gamma_{\text{ord}}$ and $\mathbb{B}^5/\Gamma$ which are quasi-projective varieties over $\mathbb{C}$. We denote by $\mathbb{B}^5/\Gamma_{\text{ord}}^\text{BB}$ and $\mathbb{B}^5/\Gamma^\text{BB}$ (resp. $\mathbb{B}^5/\Gamma_{\text{ord}}^\text{tor}$ and $\mathbb{B}^5/\Gamma_{\text{tor}}$) the Baily-Borel compactifications (resp. toroidal compactifications) of the corresponding ball quotients. Note that the toroidal compactifications of ball quotients are canonical as there is no choice of a fan involved. Further, let

$$H := \bigcup_{\langle \ell, \ell \rangle = -2} H(\ell)$$

be the discriminant divisor where

$$H(\ell) = \{ v \in \mathbb{B}^5 \mid \langle v, \ell \rangle = 0 \}$$

is the special divisor with respect to a root $\ell \in L$, see [Ko07a Subsection 3.4].

Next, we describe the stable, semi-stable and polystable loci on $M_{\text{GIT}}^\text{ord}$ and $M_{\text{GIT}}$. This goes back to very classical results of GIT, in fact Mumford’s seminal work, see [MFK94, Chapter 4, §2]. In our cases, this is spelled out as follows. In the ordered case, 8 points define a stable (resp. semi-stable) GIT-point if and only if no 4 points (resp. 5 points) coincide, see also [Ko07a Subsection 4.4] or [Do88, Example 2, p31]. Polystable points (that is, strictly semi-stable points whose orbit is closed) correspond to the points (4, 4), which means that we have two different points, each with multiplicity 4; for the notation, see [Ko07a Subsection 4.4]. In the unordered case, stable, semi-stable and polystable points are described in the same way as above, see also [Mu03, Subsection 7.2 (c)].

A crucial result of Kondô, [Ko07a, Theorem 4.6], says that there are $\mathfrak{S}_8$-equivariant isomorphisms

$$\phi_{\text{ord}} : M_{\text{ord}}^\text{GIT} \xrightarrow{\sim} \mathbb{B}^5/\Gamma_{\text{ord}}^\text{BB}$$

$$\phi : M^\text{GIT} \xrightarrow{\sim} \mathbb{B}^5/\Gamma^\text{BB},$$

where the second isomorphism goes back to [DM86].

These isomorphisms also allow us to describe the subloci of 8-tuples consisting of different points, the discriminant locus of stable, but not distinct, 8-tuples and the properly polystable loci. For this, let $(M^\text{GIT})^\circ \subset M_{\text{ord}}^\text{GIT}$ (resp. $(M^\text{GIT})^\circ \subset M^\text{GIT}$) be the moduli space of distinct ordered 8 points on $\mathbb{P}^1$ (resp. the moduli space of distinct 8 points on $\mathbb{P}^1$). By [Ko07a Theorem 3.3], the morphisms $\phi_{\text{ord}}$ and $\phi$ restrict to isomorphisms:

$$\phi_{\text{ord}}|_{(M_{\text{ord}}^\text{GIT})^\circ} : (M_{\text{ord}}^\text{GIT})^\circ \xrightarrow{\sim} (\mathbb{B}^5 \setminus H)/\Gamma_{\text{ord}}$$

$$\phi|_{(M^\text{GIT})^\circ} : (M^\text{GIT})^\circ \xrightarrow{\sim} (\mathbb{B}^5 \setminus H)/\Gamma.$$

Also the isomorphisms $\phi_{\text{ord}}$ and $\phi$ identify the discriminant locus of stable, but not distinct 8 points on $M_{\text{ord}}^\text{GIT}$ and $M^\text{GIT}$ with $H/\Gamma_{\text{ord}}$ and $H/\Gamma$ respectively. It turns out that the discriminant divisor $H/\Gamma_{\text{ord}}$ has 28 irreducible components, whereas $H/\Gamma$ is irreducible. See also [Ko07a Subsection 4.2], asserting that $A_L$ contains 64 vectors: 1 zero vector, 35 isotropic vectors and 28 non-isotropic vectors.

Finally, the properly polystable points are identified with the cusps of the Borel compactification, namely $(\mathbb{B}^5/\Gamma_{\text{ord}}^\text{BB}) \setminus (\mathbb{B}^5/\Gamma_{\text{ord}})$ and $(\mathbb{B}^5/\Gamma)^\text{BB} \setminus (\mathbb{B}^5/\Gamma)$ respectively. There are 35 cusps on $\mathbb{B}^5/\Gamma_{\text{ord}}^\text{BB}$ (also corresponding to the 35 isotropic vectors in $A_L$), but $\mathbb{B}^5/\Gamma^\text{BB}$ has a unique cusp. This directly follows from [Ko07a Subsection 4.2, Proposition 4.4], but we will see this in detail when we study the blow-up sequences.
The moduli spaces under consideration are also closely related to moduli spaces of stable curves. We do not repeat all details of the general theory here, but recall some notions as they are relevant for our purposes. Let \( \overline{M}_{0,8(\frac{1}{4}+\epsilon)} \) be the smooth projective variety which is the coarse moduli space representing the moduli problem of weighted pointed stable curves of type \((0, 8(\frac{1}{4}+\epsilon))\) with \(0 < \epsilon \ll 1\) in the sense of [Ha03, Theorem 2.1] or [KM11, Definition 2.1, Theorem 2.2], see also [GKS21, Lemma 2.3, Remark 2.4, Remark 2.11, Example 2.12]. This is also realized as the KSBA compactification [GKS21, Subsection 3.2]. \( \overline{M}_{0,8} \) is defined in the same way, but in this case, this is exactly the GIT quotient of \( \mathbb{P}^1 \) [FM94], by \( \text{SL}_2 \); see also [MM07, p55]. More generally, this is interpreted as the wonderful compactification [Li09, p536, Subsection 4.2] (or the Deligne-Mumford compactification [GKS21, Remark 2.9]).

We describe the relation of these spaces in Figure 1.

1. \( \psi_i \) is a morphism by the above discussion about stable conditions for \( i = 1, 2, 3 \).
2. \( \phi \) is an isomorphism [DM86].
3. \( \phi_{\text{ord}} \) is an \( \mathcal{S}_8 \)-equivalent isomorphism [Ko07a, Theorem 4.6].
4. \( \varphi_{\text{ord}} \) is an isomorphism [KM11, Theorem 1.1].
5. \( \Phi_{\frac{1}{4}+\epsilon} \) is an isomorphism [GKS21, Theorem 1.1].
6. \( p \) is a morphism [GKS21, Proposition 2.13].
7. The blow-up sequences \( \varphi_1, \varphi_2 \) are considered in [KM11, Theorem 4.1 (i), (iii)]. About the contraction of divisors of these morphisms, see [Ha03, Proposition 4.5] or [KM11, p1121]. We study these morphisms in detail in Subsection 3.2.
3. (Non-) Extendability of the Deligne-Mostow isomorphism

3.1. Non-transversality in the Kirwan blow-up. In this subsection, we show that the discriminant divisor and the boundary divisor do not intersect transversally in $\mathcal{M}^K$.

To prove this statement, we will need a detailed analysis of stabilizer groups. For an algebraic group $G$ we will denote the connected component of the identity by $G^0$. The following two lemmas are modeled on [CMGHL24, Lemma 2.3] and [CMGHL24, Lemma 3.1]. The boundary divisor $T$ is irreducible (and maps to the unique cusp in the Baily-Borel compactification); see also [Ko07a, Proposition 4.7]. We study $T_{\text{ord}}$ and $T$ in detail in Lemma 3.11.

**Lemma 3.1.** The following equalities hold:

$$R := \text{Stab}(c_{4,4}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \right\} \cup \left\{ \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \right\} \cong \mathbb{C}^\times \rtimes S_2$$

$$R^0 := \text{Stab}(c_{4,4})^0 \cong \mathbb{C}^\times.$$

Now, let us prepare for the local computations. The Luna slice theorem gives us a tool to study them as handled in the case of the moduli space of cubic threefolds [CMGHL23, Subsection 4.3.1] or cubic surfaces [CMGHL24, Lemma 3.4]; see also [Zh05, Subsection 7.1].

**Lemma 3.2.** A Luna slice for $c_{4,4}$, normal to the orbit $\text{SL}_2(\mathbb{C}) \cdot \{c_{4,4}\} \subset \mathbb{P}^8$, is isomorphic to $\mathbb{C}^6$, spanned by the 6 monomials

$$x_0^8, \quad x_1^8, \quad x_0^7x_1, \quad x_0x_1^7, \quad x_0^6x_1^2, \quad x_0^2x_1^6$$

in the tangent space $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8))$. Projectively,

$$\mathbb{P}^6 = \{ \alpha_0x_0^8 + \alpha_1x_1^8 + \beta_0x_0^7x_1 + \beta_1x_0x_1^7 + \gamma_0x_0^6x_1^2 + \gamma_1x_0^2x_1^6 + kx_0^4x_1^4 \}$$

$$\subset \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)) = \mathbb{P}^8.$$ 

**Proof.** This can be proven in the same way as [CMGHL23, Subsection 4.3.1]. We note that the (affine) tangent space of the orbit is given by the entries of the matrix

$$\begin{pmatrix} x_0^4x_1^4 & x_0^3x_1^5 \\ x_0^3x_1^3 & x_0^2x_1^4 \end{pmatrix}.$$ 

\[ \square \]

Let

$$\text{diag}(\lambda, \lambda^{-1}) := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \text{antidiag}(\lambda, -\lambda^{-1}) := \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}.$$
Then, the action of an element of $\text{Stab}(c_{4,4})$ is given by
\begin{equation}
\text{diag}(\lambda, \lambda^{-1}) \cdot (\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1) = (\lambda^8 \alpha_0, \lambda^{-8} \alpha_1, \lambda^6 \beta_0, \lambda^{-6} \beta_1, \lambda^4 \gamma_0, \lambda^{-4} \gamma_1)
\end{equation}
\begin{equation}
\text{antidiag}(\lambda, -\lambda^{-1}) \cdot (\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1) = (\lambda^{-8} \alpha_1, \lambda^8 \alpha_0, -\lambda^{-6} \beta_1, -\lambda^6 \beta_0, \lambda^{-4} \gamma_1, \lambda^4 \gamma_0).
\end{equation}
We write the coordinates of the Kirwan blow-up $Bl_0 \mathbb{C}^6 \subset \mathbb{C}^6 \times \mathbb{P}^5$ of the Luna slice as $(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1)$ and $[S_0 : S_1 : T_0 : T_1 : U_0 : U_1]$.

**Lemma 3.3.** The unstable locus of the action of the stabilizer $\text{SL}(c_{4,4})$ of $c_{4,4}$ in $\text{SL}_2(\mathbb{C})$ is the codimension three locus
\[
\{S_0 = T_0 = U_0 = 0 \} \cup \{S_1 = T_1 = U_1 = 0 \} \subset \mathbb{P}^5.
\]

**Proof.** From (3.1), the action of $R^c \cong \mathbb{C}^*$ is given by
\[
\text{diag}(\lambda, \lambda^{-1}) \cdot (S_0, S_1, T_0, T_1, U_0, U_1) = (\lambda^8 S_0, \lambda^{-8} S_1, \lambda^6 T_0, \lambda^{-6} T_1, \lambda^4 U_0, \lambda^{-4} U_1).
\]
Thus, the representation of $\mathbb{C}^*$ on $\mathbb{C}^6$ decomposes into 6 characters. By the same discussion as in the proof of [CMGHL24, Lemma 3.6], the points in the unstable locus are characterized by the property that the convex hull spanned by the weights appearing in the above representation does not contain the origin. This condition holds if and only if $\{S_0 = T_0 = U_0 = 0 \}$ or $\{S_1 = T_1 = U_1 = 0 \}$. \hfill $\square$

We denote by $\mathcal{D}_{\text{ord}}$ (resp. $\mathcal{D}$) the discriminant divisor, corresponding to the closure of $H/\Gamma_{\text{ord}}$ (resp. $H/\Gamma$), through the isomorphism $\phi_{\text{ord}} : \mathcal{M}^{\text{GIT}}_{\text{ord}} \to \mathbb{P}^5/\Gamma_{\text{ord}}^{\text{BB}}$ (resp. $\phi : \mathcal{M}^{\text{GIT}} \to \mathbb{P}^5/\Gamma^{\text{BB}}$). Let $\tilde{\mathcal{D}}$ be the strict transform of the discriminant divisor $\mathcal{D}$ in the blow-up $\mathcal{M}^K \to \mathcal{M}^{\text{GIT}}$. Besides, let $\Delta_{\text{ord}}$ (resp. $\Delta$) be the union of boundary divisors of $\mathcal{M}^K_{\text{ord}}$ (resp. $\mathcal{M}^K$).

**Theorem 3.4.** The strict transform $\tilde{\mathcal{D}}$ and the boundary divisor $\Delta$ do not meet generically transversally in $\mathcal{M}^K$.

**Proof.** We work on the local computation via the Luna slice described in Lemma 3.2. Before taking the GIT quotient, we have the blow-up
\[Bl_0 \mathbb{C}^6 \to \mathbb{C}^6,
\]
where the coordinates of the affine space (the Luna slice) $\mathbb{C}^6$ are $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1$ (this is the first step of the Kirwan blow-up). In this Luna slice, $\mathcal{D}$ locally near the origin, is given by
\[
disc(x^4 + \alpha_0 x^2 + \beta_0 x + \gamma_0) \cdot disc(y^4 + \alpha_1 y^2 + \beta_1 y + \gamma_1)
\]
\[
= (256\gamma_0^3 - 128\alpha_0^2 \gamma_0 + 144\alpha_0 \beta_0^2 \gamma_0 - 27\beta_0^4 + 16\alpha_0^4 \gamma_0 - 4\alpha_0^3 \beta_0^2)
\]
\[
\cdot (256\gamma_1^3 - 128\alpha_1^2 \gamma_1 + 144\alpha_1 \beta_1^2 \gamma_1 - 27\beta_1^4 + 16\alpha_1^4 \gamma_1 - 4\alpha_1^3 \beta_1^2)
\]
\[
= 0.
\]
The reason for this is that we consider the polystable point given by $x^4 y^4$ and that the versal deformation of the quadruple point $x^4 = 0$ is given by $x^4 + \alpha_0 x^2 + \beta_0 x + \gamma_0 = 0$. We write this as $V := V_1 \cup V_2$ with $\mathcal{S}_2$ permuting the two components. We consider the affine loci
\[\mathcal{P} := (S_0 \neq 0), \quad \mathcal{Q} := (T_0 \neq 0), \quad \mathcal{R} := (U_0 \neq 0).
\]

First, on $\mathcal{P}$, the inverse image of $V$ is
\begin{equation}
\alpha_0^6(256u_0^3 - 128\alpha_0 u_0^2 + 144\alpha_0 t_0^2 u_0 - 27\alpha_0 t_0^4 + 16\alpha_0^2 u_0 - 4\alpha_0^3 t_0^2)
\end{equation}
\[ \cdot (256u_1^4 - 128\alpha_0 s_1^2 u_1^2 + 144\alpha_0 s_1 t_1^2 u_1 - 27\alpha_0 t_1^4 + 16\alpha_0^2 s_1^4 u_1 - 4\alpha_0^2 s_1^3 t_1^2) \]
\[ = 0, \]

where
\[ s_1 := \frac{S_1}{S_0}, \quad t_i := \frac{T_i}{S_0}, \quad u_i := \frac{U_i}{S_0} \]

and the coordinates of \( P \) are \((\alpha_0, s_1, t_0, t_1, u_0, u_1)\). Hence, the strict transform of \( V \) is given by
\[ \cdot (256u_0^3 - 128\alpha_0 u_0^2 + 144\alpha_0 t_0^2 u_0 - 27\alpha_0 t_0^4 + 16\alpha_0^2 u_0 - 4\alpha_0^2 t_0^2) \]
\[ \cdot (256u_1^3 - 128\alpha_0 s_1^2 u_1^2 + 144\alpha_0 s_1 t_1^2 u_1 - 27\alpha_0 t_1^4 + 16\alpha_0^2 s_1^4 u_1 - 4\alpha_0^2 s_1^3 t_1^2) \]
\[ = 0, \]

since the exceptional divisor of the blow-up is \((\alpha_0 = 0)\). The Luna slice for the action \( \mathbb{T} \subset R \) is given by \((s_1 = 1)\) in \( P \) because for any point \((\alpha_0, s_1, t_0, t_1, u_0, u_1)\) in \( P \) with \( s_1 \neq 0 \), there exists a complex number \( \lambda \) such that \( \lambda^{-16} = s_1 \). Thus, the intersection of the strict transform of \( V \) with this Luna slice is given by
\[ \{256u_0^3 - \alpha_0(128u_0^2 + 144t_0^2 u_0 - 27t_0^4 + 16\alpha_0 u_0 - 4\alpha_0 t_0^2)\} \]
\[ \cdot \{256u_1^3 - \alpha_0(128u_1^2 + 144t_1^2 u_1 - 27t_1^4 + 16\alpha_0 u_1 - 4\alpha_0 t_1^2)\} \]
\[ = 0. \]

This shows that the first (resp. second) factor intersect the exceptional divisor \((\alpha_0 = 0)\) non-transversally along \((u_0 = 0)\) (resp. \((u_1 = 0)\)).

Next, on \( Q \), the inverse image of \( V \) is
\[ \beta_0^6(256u_0^3 - 128\beta_0 s_0^2 u_0^2 + 144\beta_0 s_0 u_0 - 27\beta_0 + 16\beta_0^2 s_0 u_0 - 4\beta_0^2 s_0) \]
\[ \cdot (256u_1^3 - 128\beta_0 s_1^2 u_1^2 + 144\beta_0 s_1 t_1^2 u_1 - 27\beta_0 t_1^4 + 16\beta_0^2 s_1^4 u_1 - 4\beta_0^2 s_1^3 t_1^2) \]
\[ = 0, \]

where
\[ s_1 := \frac{S_1}{T_0}, \quad t_1 := \frac{T_1}{T_0}, \quad u_i := \frac{U_i}{T_0} \]

and the coordinates of \( P \) is \((s_0, s_1, \beta_0, t_1, u_0, u_1)\). Hence, the strict transform of \( V \) is given by
\[ (256u_0^3 - 128\beta_0 s_0^2 u_0^2 + 144\beta_0 s_0 u_0 - 27\beta_0 + 16\beta_0^2 s_0 u_0 - 4\beta_0^2 s_0) \]
\[ \cdot (256u_1^3 - 128\beta_0 s_1^2 u_1^2 + 144\beta_0 s_1 t_1^2 u_1 - 27\beta_0 t_1^4 + 16\beta_0^2 s_1^4 u_1 - 4\beta_0^2 s_1^3 t_1^2) \]
\[ = 0, \]

since the exceptional divisor of the blow-up is \((\beta_0 = 0)\). The Luna slice for the action \( \mathbb{T} \subset R \) is given by \((t_1 = 1)\) in \( P \) because for any point \((s_0, s_1, \beta_0, t_1, u_0, u_1)\) in \( Q \) with \( t_1 \neq 0 \), there exists a complex number \( \lambda \) such that \( \lambda^{-12} = t_1 \). Thus, the intersection of the strict transform of \( V \) with this Luna slice is given by
\[ \{256u_0^3 - \beta_0(128s_0^2 u_0^2 + 144s_0 u_0 - 27 + 16\beta_0 s_0 u_0 - 4\beta_0 s_0)\} \]
\[ \cdot \{256u_1^3 - \beta_0(128s_1^2 u_1^2 + 144s_1 u_1 - 27 + 16\beta_0 s_1^4 u_1 - 4\beta_0 s_1^3 t_1^2)\} \]
\[ = 0. \]

This shows that the first (resp. second) factor intersect the exceptional divisor \((\beta_0 = 0)\) non-transversally along \((u_0 = 0)\) (resp. \((u_1 = 0)\)).
Finally, on \( \mathcal{R} \), the inverse image of \( V \) is
\[
\gamma_0^6(256 - 128\gamma_0 s_0^2 + 144\gamma_0 s_0 t_0^2 - 27\gamma_0 t_0^4 + 16\gamma_0^2 s_0^4 - 4\gamma_0^2 s_0^3 t_0^2) \\
\cdot (256u_1^3 - 128\gamma_0 s_1^2 + 144\gamma_0 s_1 t_1^2 - 27\gamma_0 t_1^4 + 16\gamma_0^2 s_1^4 - 4\gamma_0^2 s_1^3 t_1^2) \\
= 0,
\]
where
\[
s_i := \frac{S_i}{U_0}, \quad t_i := \frac{T_i}{U_0}, \quad u_1 := \frac{U_1}{U_0}
\]
and the coordinates of \( \mathcal{R} \) are \((s_0, s_1, t_0, t_1, \gamma_0, u_1)\). Hence the strict transform of \( V \) is given by
\[
(256 - 128\gamma_0 s_0^2 + 144\gamma_0 s_0 t_0^2 - 27\gamma_0 t_0^4 + 16\gamma_0^2 s_0^4 - 4\gamma_0^2 s_0^3 t_0^2) \\
\cdot (256u_1^3 - 128\gamma_0 s_1^2 + 144\gamma_0 s_1 t_1^2 - 27\gamma_0 t_1^4 + 16\gamma_0^2 s_1^4 - 4\gamma_0^2 s_1^3 t_1^2) \\
= 0,
\]
since the exceptional divisor of the blow-up is \((\gamma_0 = 0)\). The Luna slice for the action \( \mathbb{T} \subset R \) is given by \((g_1 = 1)\) in \( \mathcal{R} \) because for any point \((s_0, s_1, t_0, t_1, \gamma_0, u_1) \in \mathcal{R} \) with \(u_1 \neq 0\), there exists a complex number \(\lambda\) such that \(\lambda^{-8} = \gamma_1\). Thus, the intersection of the strict transform of \( V \) with this Luna slice is given by
\[
(256 - 128\gamma_0 s_0^2 + 144\gamma_0 s_0 t_0^2 - 27\gamma_0 t_0^4 + 16\gamma_0^2 s_0^4 - 4\gamma_0^2 s_0^3 t_0^2) \\
\cdot \{256u_1^3 - \gamma_0(128s_1^2 + 144s_1 t_1^2 - 27t_1^4 + 16s_1^2 t_1^2 - 4s_1^3 t_1^2)\}.
\]
This shows that the first factor has an empty intersection with the exceptional divisor \((\gamma_0 = 0)\), whereas the second factor intersects the exceptional divisor non-transversally along \((u_1 = 0)\).

Next, we consider the action of the finite quotient \( \mathfrak{G}_2 \cong R/R^\ast \). We only consider the intersection \((\alpha_0 = u_0 = 0)\) on affine locus \( \mathcal{P} \) (the other cases being the same). If \(\text{diag}(\lambda, \lambda^{-1})\) fixes a general point in \( \mathcal{P} \cap (\alpha_0 = u_0 = 0) \), by the condition on \(t_0\), we have \(\lambda^2 = 1\). This implies that \(\text{diag}(\lambda, \lambda^{-1})\) is trivial as an element of \(\text{PGL}_2(\mathbb{C})\).

Thus, finally, let us consider the case of the form \(\text{antidiag}(\lambda, -\lambda^{-1})\). This element swaps the coordinates \(u_0\) and \(u_1\). However, a general point \(p = (s_1, t_0, t_1, u_1) \in \mathcal{P} \cap (\alpha_0 = u_0 = 0)\) satisfies \(u_1 \neq 0\), and this implies \(\text{antidiag}(\lambda, -\lambda^{-1})\) cannot stabilize any point. More explicitly, we have
\[
\text{antidiag}(\lambda, -\lambda^{-1}) \cdot (s_1, t_0, t_1, u_1) = (\lambda^{-16}s_1^{-1}, \lambda^2t_1s_1^{-1}, -\lambda^{14}t_0s_1^{-1}, 0)
\]
by \((3.2)\). This cannot be equal to \(p = (s_1, t_0, t_1, u_1)\) with \(u_1 \neq 0\).

\[\square\]

**Remark 3.5.** The situation in the ordered case is different. Indeed, a similar calculation, again using a Luna slice argument, shows that the discriminant divisors and the boundary divisors meet transversally everywhere on \( \mathcal{M}_{\text{ord}}^K \).

**Remark 3.6.** In Theorem 5.2, we shall see that \( \mathcal{M}_{\text{ord}}^K \) and \( \overline{\mathbb{B}^5/\Gamma_{\text{ord}}^\ast} \) have the same cohomology. Note that this proof does not require a priori knowledge that the two spaces are isomorphic. Using the information of their Betti numbers, we can give a short independent proof that \( \mathcal{M}_{\text{ord}}^K \cong \overline{\mathbb{B}^5/\Gamma_{\text{ord}}^\ast} \) which is independent of [GKS21]. This argument follows a similar argument given by Casalaina-Martin for cubic surfaces. By the Borel extension theorem \([\text{Bo}72\text{ Theorem A}]\), the map \( \mathcal{M}_{\text{ord}} \to \mathbb{B}^5/\Gamma_{\text{ord}} \) extends to a morphism \( \mathcal{M}_{\text{ord}}^K \to \overline{\mathbb{B}^5/\Gamma_{\text{ord}}^\ast} \).
Since both spaces have the same Betti numbers, this must be an isomorphism or a small contraction. But the latter is impossible since $\mathbb{B}^5/\Gamma_{\text{ord}}$ is $\mathbb{Q}$-factorial (and in fact smooth).

In the rest of this subsection, we work on the stabilizers of points in the exceptional divisor $\Delta$ in $\mathcal{M}^K$. The following proposition plays a critical role in the proof of Theorem 4.6.

**Proposition 3.7.** For any point in $x \in \Delta$, the order of its stabilizer $S_x := \text{Stab}_R(x)$ is not divisible by 5.

**Proof.** Since the order of the finite part of $R$ is not divisible by 5, it is enough to concentrate on the connected component $R^\circ$, which is isomorphic to $\mathbb{C}^\times$. For simplicity, we will also use $S_x$ to denote the stabilizer of $x$ in $R^\circ$. By the $\mathfrak{S}_2$ symmetry, it suffices to show the claim for the affine open sets $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$.

First, let us consider the points $(\alpha_0, s_1, t_0, t_1, u_0, u_1) \in \mathcal{P}$. In this locus, the exceptional divisor corresponds to $(\alpha_0 = 0)$, and the action of $\text{diag}(\lambda, \lambda^{-1})$ is given by

$$\text{diag}(\lambda, \lambda^{-1}) \cdot (0, s_1, t_0, t_1, u_0, u_1) = (0, \lambda^{-16} s_1, \lambda^{-2} t_0, \lambda^{-14} t_1, \lambda^{-4} u_0, \lambda^{-12} u_1).$$

Since the Kirwan blow-up is completed after one step, it is enough to consider the stable points after blowing up the orbit $\text{SL}_2(\mathbb{C}) \cdot \{c_1, c_2\}$. It follows from Lemma 3.3 that both $\{t_0 \neq 0$ or $u_0 \neq 0\}$ and $\{s_1 \neq 0$ or $t_1 \neq 0$ or $u_1 \neq 0\}$. If $t_0 \neq 0$, then $S_x \cong \mathbb{Z}/2\mathbb{Z}$. If $u_0 \neq 0$, then

$$S_x \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & (s_1 \neq 0$ or $u_1 \neq 0) \\ \mathbb{Z}/2\mathbb{Z} & (t_1 \neq 0). \end{cases}$$

The other cases are similar, but we nevertheless state them for completeness, starting with the points $(s_0, s_1, 0, t_1, u_0, u_1) \in \mathcal{P} \cap (\beta_0 = 0)$. The action of $\text{diag}(\lambda, \lambda^{-1})$ is given by

$$\text{diag}(\lambda, \lambda^{-1}) \cdot (s_0, s_1, 0, t_1, u_0, u_1) = (\lambda^2 s_0, \lambda^{-14} s_1, 0, \lambda^{-12} t_1, \lambda^{-2} u_0, \lambda^{-10} u_1).$$

Again by Lemma 3.3, we can assume that $\{s_0 \neq 0$ or $u_0 \neq 0\}$ and $\{s_1 \neq 0$ or $t_1 \neq 0$ or $u_1 \neq 0\}$. In all cases, we obtain $S_x \cong \mathbb{Z}/2\mathbb{Z}$. Finally, let $(s_0, s_1, t_0, t_1, 0, u_1) \in \mathcal{P} \cap (\gamma_0 = 0)$. The action of $\text{diag}(\lambda, \lambda^{-1})$ is given by

$$\text{diag}(\lambda, \lambda^{-1}) \cdot (s_0, s_1, t_0, t_1, 0, u_1) = (\lambda^4 s_0, \lambda^{-4} s_1, \lambda^2 t_0, \lambda^{-10} t_1, 0, \lambda^{-8} u_1).$$

As above, we study the case holding both of $\{s_0 \neq 0$ or $t_0 \neq 0\}$ and $\{s_1 \neq 0$ or $t_1 \neq 0$ or $u_1 \neq 0\}$. If $t_0 \neq 0$, then $S_x \cong \mathbb{Z}/2\mathbb{Z}$. If $s_0 \neq 0$, then

$$S_x \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & (s_1 \neq 0$ or $u_1 \neq 0) \\ \mathbb{Z}/2\mathbb{Z} & (t_1 \neq 0). \end{cases}$$

This calculation completes the proof. □

### 3.2. Transversality in the toroidal compactification.

In this subsection, we prove that the discriminant divisors and the boundary divisors intersect transversally in $\mathbb{B}^5/\Gamma_{\text{ord}}$ and generically transversally in $\mathbb{B}/\Gamma_{\text{tor}}$. We will see that this also implies the transversality at a generic point in $\mathbb{B}^5/\Gamma_{\text{ord}}$. Throughout this subsection, let $N_8 := \{1, 2, \cdots, 8\}$ and $I \subset N_8$. As before, $(\mathcal{M}^\text{GIT})^0_\text{ord}$ denotes the set of 8-tuples where all points are different; see Section 2. Below, we shall recall the construction of the blow-up sequence $\mathcal{M}_{0,8} \to \mathcal{M}^K_{\text{ord}} \to \mathcal{M}^\text{GIT}_{\text{ord}}$. By the explicit description of the blow-ups or the interpretation as the configuration space, the
locus \((\mathcal{M}_{\text{ord}}^{\text{GIT}})^o\) does not meet the centres of each blow-up step. Thus, we consider \((\mathcal{M}_{\text{ord}}^{\text{GIT}})^o\) to be also an open subset of \(\overline{\mathcal{M}}_{0,8}\) and \(\mathcal{M}_{\text{ord}}^K\) via birational morphisms.

First, we work on \(\mathcal{M}_{\text{ord}}^\text{GIT}\). The boundary divisor \(\mathcal{M}_{\text{ord}}^\text{GIT}\setminus(\mathcal{M}_{\text{ord}}^{\text{GIT}})^o\) is

\[
D_2^{(0)} := \bigcup_{|I|=2} D_2^{(0)}(I) = \mathcal{D}_{\text{ord}} \subset \mathcal{M}_{\text{ord}}^\text{GIT}
\]

by \([\text{KM11}, \text{p}1134]\) \((m = 4, k = 0)\). Here, \(D_2^{(0)}(I)\) is defined by

\[
D_2^{(0)}(I) := \{(x_1, \cdots, x_8) \in (\mathbb{P}^1)^8 \mid x_i = x_j \text{ if } i, j \in I\}/\text{SL}_2(\mathbb{C}).
\]

The number of such \(I\) is 28. As in Section 2, the morphism \(\varphi_1 : \mathcal{M}_{\text{ord}}^K \to \mathcal{M}_{\text{ord}}^{\text{GIT}}\) is the Kirwan blow-up whose centre is the locus of polystable orbits, consisting of 35 orbits (which in turn correspond to the 35 cusps, see below). We interpret \(\varphi_1\) in terms of configuration spaces as follows. Let

\[
\Sigma_4^{(0)}(I, I^\perp) := \{(x_1, \cdots, x_8) \in (\mathbb{P}^1)^8 \mid x_i = x_j \text{ if and only if } \{i, j\} \subset I \text{ or } \{i, j\} \subset I^\perp\}/\text{SL}_2(\mathbb{C})
\]

for \(|I| = |I^\perp| = 4\) and \(I \sqcup I^\perp = \mathbb{N}_8\). We also denote by \(\Sigma_4^{(0)}\) their union running through such \(I\) and \(I^\perp\). Note that there are 35 pairs \((I, I^\perp)\) satisfying \(|I| = |I^\perp| = 4\) and \(I \sqcup I^\perp = \mathbb{N}_8\). In this terminology, the centre of \(\varphi_1\) is described by

\[
\Sigma_4^{(0)} = \{c_{\text{ord}, i}^{(0)}\}_{i=1}^{35}
\]

where \(\{c_{\text{ord}, i}^{(0)}\}_{i=1}^{35}\) are the polystable points of \(\mathcal{M}_{\text{ord}}^{\text{GIT}}\), corresponding to 35 Baily-Borel cusps.

Next, we consider \(\mathcal{M}_{\text{ord}}^K(\cong \overline{\mathcal{M}}_{0,8(\frac{1}{2} + \epsilon)})\). Let

\[
D_4^{(1)}(I) := \varphi_1^{-1} \left( \Sigma_4^{(0)}(I, I^\perp) \right)
\]

for \(|I| = 4\). Then, the exceptional divisor of \(\varphi_1\) is

\[
D_4^{(1)} := \bigcup_{|I|=4} D_4^{(1)}(I) = \varphi_1^{-1} \left( \Sigma_4^{(0)} \right) = \Delta_{\text{ord}}.
\]

Note that each irreducible component of \(\Delta_{\text{ord}}\) is isomorphic to \(\mathbb{P}^2 \times \mathbb{P}^2\) by \([\text{Ha03, Proposition 4.5}], [\text{MS21, Remark 6}]\) or \([\text{GKS21, Example 2.12}]\). Besides, let

\[
D_2^{(1)}(I) := \varphi_1^{-1} \left( D_2^{(0)}(I) \setminus \Sigma_4^{(0)} \right)
\]

be the strict transform of \(D_2^{(0)}(I)\) for \(|I| = 2\), and \(D_2^{(1)}\) be their union. Then \(D_2^{(1)}\) is the strict transform of \(\mathcal{D}_{\text{ord}},\) i.e.,

\[
D_2^{(1)} = \overline{\mathcal{D}_{\text{ord}}}
\]

and has 28 irreducible components. In this setting, the boundary divisor \(\mathcal{M}_{\text{ord}}^K \setminus (\mathcal{M}_{\text{ord}}^{\text{GIT}})^o\) is

\[
D_2^{(1)} \cup D_4^{(1)} = \overline{\mathcal{D}_{\text{ord}}} \cup \Delta_{\text{ord}}
\]

by \([\text{KM11, p}1134]\) \((m = 4, k = 1)\).

Next, we describe the centre of the blow-up \(\varphi_2 := \varphi_2 \circ \varphi_{\text{ord}} : \overline{\mathcal{M}}_{0,8} \to \mathcal{M}_{\text{ord}}^K\), which is a codimension 2 locus. Let

\[
\Sigma_3^{(0)}(I) := \{(x_1, \cdots, x_8) \in (\mathbb{P}^1)^8 \mid x_i = x_j \text{ if and only if } i, j \in I\}/\text{SL}_2(\mathbb{C})
\]

\[
\Sigma_3^{(1)}(I) := \varphi_1^{-1} \left( \Sigma_3^{(0)}(I) \setminus \Sigma_4^{(0)} \right)
\]
for \(|I| = 3\) and
\[
\Sigma_3^{(1)} := \bigcup_{|I|=3} \Sigma_3^{(1)}(I).
\]

Then, the centre of the blow up \(\varphi_2 : \overline{\mathcal{M}_{0,8}} \to \mathcal{M}_{ord}^K\) is \(\Sigma_3^{(1)}\).

Finally, we study \(\mathcal{M}_{0,8}\). For \(|I| = 3\), let
\[
D_3^{(2)}(I) := \varphi_2^{-1}\left(\Sigma_3^{(1)}(I)\right)
\]
be an irreducible component of the exceptional divisor of \(\varphi_2\). Then, the variety
\[
D_3^{(2)} := \bigcup_{|I|=3} D_3^{(2)}(I) = \varphi_2^{-1}\left(\Sigma_3^{(1)}\right)
\]
is exactly the exceptional divisor of the blow-up \(\varphi_2\). For \(|I| = 2, 4\), we denote by \(D_{|I|}^{(2)}(I)\) the strict transform of \(D_{|I|}^{(1)}\) and define
\[
D_2^{(2)} := \bigcup_{|I|=2} D_2^{(2)}(I), \quad D_4^{(2)} := \bigcup_{|I|=4} D_4^{(2)}(I).
\]

Now, the boundary divisor \(\overline{\mathcal{M}_{0,8}} \backslash (\mathcal{M}^{GIT}_{ord})^o\) is
\[
D_2^{(2)} \cup D_3^{(2)} \cup D_4^{(2)}
\]
by [KM11, p1134] \((m = 4, k = 2)\).

The boundaries which are contracted through the map \(\varphi_2\) can also be calculated as follows. By [Ha03, Theorem 4.1], there exists the reduction map
\[
\varphi'_2 : \overline{\mathcal{M}_{0,8}} \to \overline{\mathcal{M}_{0,8(\frac{1}{4}+\epsilon)}}.
\]
The map \(\varphi'_2\) is a divisorial contraction, more precisely:

**Lemma 3.8** (c.f. [Ha03 Proposition 4.5]). *The morphism \(\varphi'_2\) contracts the boundary divisors \(D_3^{(2)}\).*

**Proof.** By [KM11, p1121], the exceptional locus of \(\varphi'_2\) is the union of \(D_{|I|}^{(2)}(I)\) with \(I = \{i_1, \ldots, i_r\}\) for \(r > 2\) so that
\[
r \times \left(\frac{1}{4} + \epsilon\right) \leq 1.
\]
This implies \(r = 3\). \(\square\)

By construction, \(D_2^{(2)} \cup D_3^{(2)} \cup D_4^{(2)}\) is normal crossing (since \(\overline{\mathcal{M}_{0,8}}\) is a normal crossing compactification of \((\mathbb{B}^5 \backslash H)/\Gamma_{ord}\)). We denote \(\mathcal{H}_{ord} := \overline{H/\Gamma_{ord}}\) and \(\mathcal{H} := \overline{H/\Gamma}\), where the closures are taken in the respective Baily-Borel compactifications. We further denote by \(\mathcal{H}_{ord}\) the strict transform of \(\mathcal{H}_{ord}\) under \(\pi_{ord} : \overline{\mathbb{B}^5/\Gamma_{ord}} \to \overline{\mathbb{B}^5/\Gamma_{ord}}^{BB}\). Since the contraction divisor of \(\varphi_2\) is only \(D_3^{(2)}\), we now obtain the following:

**Theorem 3.9.** *The boundary \(\overline{\mathcal{H}_{ord}} \cup T_{ord}\) is a normal crossing divisor. In particular, \(\overline{\mathcal{H}_{ord}}\) and \(T_{ord}\) intersect transversally everywhere in \(\overline{\mathbb{B}^5/\Gamma_{ord}}^{tor}\).*
Again, by this formulation, we mean that $\mathcal{H}_{\text{ord}}$ and $T_{\text{ord}}$ intersect transversally everywhere along any component of their intersection. As a consequence, we obtain the following corollary, where $\mathcal{H}$ is the strict transform of $\mathcal{H}$ under $\pi : \mathbb{P}^5/\Gamma^\text{tor} \rightarrow \mathbb{P}^5/\Gamma^\text{BB}$.

**Corollary 3.10.** The divisor $\mathcal{H} \cap T$ is a normal crossing divisor, up to finite quotients.

Next, we discuss the generic transversality of the intersection of $\mathcal{H}$ and $T$ in $\mathbb{P}^5/\Gamma^\text{tor}$. Note that $\Gamma/\Gamma_{\text{ord}} \cong \mathfrak{S}_8$ acts on $\{T_{\text{ord},i}\}_{i=1}^{35}$ transitively and

$$1 \rightarrow \mathfrak{S}_4 \times \mathfrak{S}_4 \rightarrow \text{Stab}_{\mathfrak{S}_8}(T_{\text{ord},i}) \rightarrow \mathfrak{S}_2 \rightarrow 1.$$  

Next, we study the description of the boundary and group actions via the Hermitian form. The claim of the following lemma is already known in terms of a moduli description by [MS21, Remark 6] or [GKS21, Example 2.12], but we need the details in the proof of Theorem 3.14.

**Lemma 3.11.** The following holds.

1. $T_{\text{ord},i} \cong \mathbb{P}^2 \times \mathbb{P}^2$.
2. $T \cong (\mathbb{P}^2/\mathfrak{S}_4 \times \mathbb{P}^2/\mathfrak{S}_4)/\mathfrak{S}_2$.

**Proof.** We orientate ourselves along the strategy of the proof of [CMGHL23, Proposition 7.8]. First, we take an isotropic vector $h = (1, 0, 0, 0, 0, 0) \in L$ and denote by $F$ the corresponding cusp. As the unitary group acts transitively on the set of all cusps, this means no loss of generality. Also, taking $h' = (0, 1, 0, 0, 0, 0)$ as a further basis vector, we can replace our Hermitian form by

$$\begin{pmatrix}
B & 1 - \sqrt{-1} \\
1 + \sqrt{-1} & \end{pmatrix},$$

where

$$B := \begin{pmatrix}
-2 & 1 + \sqrt{-1} \\
1 - \sqrt{-1} & -2 \\
-2 & 1 + \sqrt{-1} \\
1 - \sqrt{-1} & -2
\end{pmatrix}.$$  

Then,

$$N(F) := \text{Stab}_T(F) = \left\{ g = \begin{pmatrix} u & v & w \\ X & y & s \end{pmatrix} \bigg| s\overline{u} = 1, \quad \overline{X}^t B X = B \right\}.$$  

The unipotent radical of $N(F)$ is

$$W(F) = \left\{ g = \begin{pmatrix} 1 & v & w \\ I_4 & y & s \end{pmatrix} \bigg| By + (1 - \sqrt{-1})\overline{v}s = 0 \right\}.$$  

and the centre of $W(F)$ is

$$Z(F) = \left\{ g = \begin{pmatrix} 1 & \sqrt{-1}(1 - \sqrt{-1})w \\ I_4 & 1 \end{pmatrix} \bigg| w \in \mathbb{Z} \right\}.$$
We take the partial quotient of $\mathbb{B}^5$ by the action of $Z(F)$:

$$
\mathbb{B}^5 \overset{\sim}{\longrightarrow} \mathbb{C}^\times \times \mathbb{C}^4
\begin{pmatrix}
z_0, z_1, z_2, z_3, z_4
\end{pmatrix} \mapsto (t = \exp \left(2\pi z_0/(1 - \sqrt{-1})\right), z_1, z_2, z_3, z_4).
$$

We shall here consider the quotient of $\mathbb{C}^4$ by $W(F)$. For an element $g \in W(F)$, its action on $z := (z_1, z_2, z_3, z_4)$ is given by

$$
g \cdot z^t = \frac{1}{s}(Xz + y).
$$

A straightforward computation shows that for given $y^t \in \mathbb{Z}[\sqrt{-1}]^4$, we can find suitable elements $w \in \mathbb{Z}[\sqrt{-1}]$ and $v \in \mathbb{Z}[\sqrt{-1}]^4$ such that $g = \begin{pmatrix} 1 & v & w \\ I_4 & y \\ 1 \end{pmatrix} \in W(F)$. This implies that

$$
\mathbb{C}^4/W(F) \cong (E_{\sqrt{-1}})^4,
$$

where $E_{\sqrt{-1}}$ is the CM-elliptic curve $\mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$. Now, we consider the effect of an element of the form

$$
g = \begin{pmatrix} u \\ I_4 \\ s \end{pmatrix} \in N(F).
$$

Here, from the above action, $s \in \mathbb{Z}[\sqrt{-1}]^\times$ acts on $(E_{\sqrt{-1}})^4$ diagonally by multiplication with powers of $\sqrt{-1}$. However, this element is already in $U(D_4^{\oplus 2})$, thus it follows that $T \cong (E_{\sqrt{-1}})^4/U(D_4^{\oplus 2})$. Here, we note that $X = U(D_4^{\oplus 2})$. By [Do08 Table 2], we have

$$
U(D_4^{\oplus 2}) \cong ((\mathbb{Z}/2\mathbb{Z})^2 \times \mathfrak{S}_2) \times \mathfrak{S}_4) \times \mathfrak{S}_2.
$$

See also [Sh53 Subsection 6.4]. Since, the action of this group, described in [Do08 Subsection 3.2, Table 2], gives

$$
(E_{\sqrt{-1}})^2/U(D_4) \cong (\mathbb{P}^1)^2/(\mathfrak{S}_2 \times \mathfrak{S}_4) \cong \mathbb{P}^2/\mathfrak{S}_4,
$$

where $\mathfrak{S}_4$ acts on $\mathbb{P}^2$ by the standard representation, we obtain

$$
(E_{\sqrt{-1}})^4/U(D_4^{\oplus 2}) \cong (\mathbb{P}^2/\mathfrak{S}_4)^2/\mathfrak{S}_2.
$$

For the ordered case, a straightforward computation shows that $\tilde{U}(D_4) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathfrak{S}_2$, thus this gives

$$
T_{ord,i} \cong \mathbb{P}^2 \times \mathbb{P}^2.
$$

**Remark 3.12.** This description allows us to describe the geometry of the toroidal boundary $T$ explicitly. By the above Lemma 3.11 we know that $T = (\mathbb{P}^2/\mathfrak{S}_4 \times \mathbb{P}^2/\mathfrak{S}_4)/\mathfrak{S}_2$ where $\mathfrak{S}_4$ acts on $\mathbb{P}^2$ by the standard 3-dimensional representation and $\mathfrak{S}_2$ exchanges the two factors. We claim that $\mathbb{P}^2/\mathfrak{S}_4 \cong \mathbb{P}(1, 2, 3)$ where $\mathbb{P}(1, 2, 3)$ denotes the weighted projective space with weights $(1, 2, 3)$. This follows since the invariants are freely generated by the restriction of the elementary symmetric polynomials of degree $2, 3, 4$ on $\mathbb{P}^3$ restricted to the hyperplane $\sum_{i=0}^3 x_i = 0$. Hence $\mathbb{P}^2/\mathfrak{S}_4 \cong \mathbb{P}(2, 3, 4) \cong \mathbb{P}(1, 2, 3)$. In conclusion we find that $T \cong S^2(\mathbb{P}(1, 2, 3))$. □
Before discussing the intersection of divisors on the toroidal compactifications, we recall the discriminant form, see [Ko07a, Subsection 2.3] (where the lattice is called \(N\) compared to our \(L\)):

\[ q_L : A_L \to \mathbb{F}_2. \]

Associated with \(q_L\), there is an associated bilinear form \(b_L(\ , \ )\) on \(A_L\). Note that \(q_L\) is isomorphic to the direct sum of 3 copies of the hyperbolic plane \(u\) over \(\mathbb{F}_2\) which follows from [Ko07a, Subsection 2.2] or explicit computation in terms of the concrete form of \(L\). Here, \(u\) is defined by the matrix

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

over \(\mathbb{F}_2\). We have to pay attention to the norm of a vector because our quadratic form exists over \(\mathbb{F}_2\). In other words, the norm is measured by \(q_L\), not \(b_L(\ , \ )\).

**Lemma 3.13.** For a given isotropic vector \(h\) in the finite quadratic space \(\mathbb{P}(A_L) \cong \mathbb{P}(\mathbb{F}_2^6)\), the orthogonal complement \(h^\perp \cong \mathbb{P}(\mathbb{F}_2^3)\) contains 19 isotropic vectors and 12 non-isotropic vectors. In addition, the stabilizer of \(\text{Stab}(h)\) in \(\mathcal{S}_8\) acts transitively on the set consisting of all 12 non-isotropic vectors.

**Proof.** By [FM11, Section 3] or [MT04, Proposition 3.2], we have an isomorphism \(\Gamma/\Gamma_{\text{ord}} \cong \mathcal{S}_8 \cong \text{O}(\mathbb{F}_2^3)\). This naturally induces the action of the symmetric group \(\mathcal{S}_8\) on the discriminant group. By [Ko07a, Proposition 4.7 (ii)], the action on the set of isotropic vectors is transitive. Hence it suffices to consider one isotropic vector, say \(h = (1, 0, 0, 0, 0, 0) \in u^{\mathbb{F}_2}\). Then, the non-isotropic vectors in \(h^\perp\) are given by \((0, 0, 1, 1, 0, 0), (0, 0, 1, 1, 1, 0), (0, 0, 1, 1, 0, 1), (0, 1, 1, 1, 0, 0), (0, 1, 1, 1, 1, 0), (0, 1, 1, 1, 0, 1)\) and the vectors which arise from these by interchanging the last two components of \(u^{\mathbb{F}_2}\). Similarly, one obtains a complete list of isotropic vectors in \(h^\perp\) (which contains \(h\) itself). The latter half of the statement is clear because for any two non-isotropic vectors \(v_1\) and \(v_2\), orthogonal to \(h\), we can define an element \(g \in \text{Stab}(h)\) permuting \(v_1\) and \(v_2\), and extend it by the identity to \(\langle v_1, v_2, h \rangle^\perp \subset \mathbb{F}_2^6\). Here, we used the fact that there is no relation such as \(h = v_1 + v_2\), i.e., that \(v_1, v_2\) and \(h\) are independent.

The goal of this subsection is the following theorem.

**Theorem 3.14.** The divisors \(\mathcal{H}\) and \(T\) meet generically transversally in \(\mathbb{P}^{\Gamma_{\text{tor}}}\).

**Proof.** First, we take an irreducible component \(T_{\text{ord},i}\) of \(T_{\text{ord}}\), namely the divisor over the cusp corresponding to the isotropic vector \(h = (1, 1, 0, 0, 0, 0)\). Then, we choose the component of \(\mathcal{H}_{\text{ord}} \cap T_{\text{ord},i}\) given by taking the divisor orthogonal to the vector \(\ell = (0, 0, 1, 1, 0, 0)\). We can perform both choices without loss of generality due to Lemma 3.13 which tells us that the group \(\mathcal{S}_8\) acts transitively on the components of \(\mathcal{H}_{\text{ord}} \cap T_{\text{ord},i}\).

Thus, it suffices to consider the component \(T\) of \(\mathcal{H}_{\text{ord}} \cap T_{\text{ord},i}\) chosen above. Now, \(T\) is the fixed locus of the reflection with respect to \(\ell\). In addition, through the isomorphism \(\Gamma/\Gamma_{\text{ord}} \cong \mathcal{S}_8 \cong \text{O}(\mathbb{F}_2^3)\), the choice of \(\ell\) implies that this reflection acts on \(\mathbb{P}^2 \times \mathbb{P}^2\) by

\[
\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2
\]

\[
([a_1 : b_1 : c_1], [a_2 : b_2 : c_2]) \mapsto ([b_1 : a_1 : c_1], [a_2 : b_2 : c_2]).
\]

Also, a straightforward computation shows that \(T\) is not fixed by any other reflection with respect to a non-isotropic vector set-theoretically. Hence, we consider a general point \(p =
and Theorem 3.14, the birational map \( g \) satisfies \( \text{Stab}_S(p_1) = \langle (1 \ 2) \rangle \), where \( (1 \ 2) \) denotes the transposition in \( S_4 \) of the first two components, and \( p_2 \) is general in the sense that \( p_1 \neq p_2 \) and \( \text{Stab}_S(p_2) = 1 \). Clearly, the set of these points is non-empty. Here, we have used the fact that \( S_4 \) acts on \( \mathbb{P}^2 \) by the standard representation; see the proof of Lemma 3.11 and [Do08, Subsection 3.2]. By construction, the stabilizer of \( p \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), generated by a non-trivial involution in the first factor of \( S_4 \times S_4 \).

Using the coordinates taken in the proof of Lemma 3.11, by Theorem 3.9, taking the quotient, we can choose the defining equation of the Betti numbers of the Kirwan blow-up \( \mathcal{M}^k \) and the toroidal compactification \( \overline{\mathbb{B}^5/\Gamma_{\text{tor}}} \) which we will perform in Section 5.

**Theorem 3.15.** Neither the Deligne-Mostow isomorphism \( \phi : \mathcal{M}^\text{GIT} \to \overline{\mathbb{B}^5/\Gamma} \) nor its inverse \( \phi^{-1} \) lift to a morphism between the Kirwan blow-up \( \mathcal{M}^k \) and the unique toroidal compactification \( \overline{\mathbb{B}^5/\Gamma_{\text{tor}}} \).

**Proof.** We shall prove this for \( \phi \), the argument for \( \phi^{-1} \) being the same. By Theorem 3.4 and Theorem 3.14, the birational map \( g : \mathcal{M}^k \dashrightarrow \overline{\mathbb{B}^5/\Gamma_{\text{tor}}} \) cannot be an isomorphism. By Theorems 5.6 and 5.8 the Betti numbers \( b_2(\mathcal{M}^k) = b_2(\overline{\mathbb{B}^5/\Gamma_{\text{tor}}}) = 2 \) agree. Hence \( g \) cannot contract a divisor and must thus be a small contraction. This, however, contradicts the fact that both \( \mathcal{M}^k \) and \( \overline{\mathbb{B}^5/\Gamma_{\text{tor}}} \) are \( \mathbb{Q} \)-factorial. (See also the proof of [CMGHL24, Theorem 1.1]).

Since the compactifications concerned are \( S_8 \)-equivariant, we obtain as a byproduct that \( \mathcal{M}^k_{\text{ord}}/S_8 \not\cong \mathcal{M}^k \).

### 3.3. Proof of Theorem 1.1

We shall now restate one of the main results in this paper. Its proof uses our computation of the Betti numbers of the Kirwan blow-up \( \mathcal{M}^k \) and the \( \mathbb{Q} \)-Picard groups, which we shall show in this section, we focus on the canonical bundles, and as a result, we shall show Theorem 1.3.

#### 4. Canonical bundles and relation to the minimal model program

On the way, we shall use a modular form constructed by Kondo, which will be essential for us. In this section, we focus on the canonical bundles, and as a result, we shall show Theorem 1.3.

#### 4.1. Computation involving blow-ups

We first recall some basic facts about the birational geometry of the relevant moduli spaces and, noticeably, the maps \( \varphi_1 \) and \( \varphi_2 \). These two morphisms \( \varphi_1 : \mathcal{M}^k_{\text{ord}} \to \mathcal{M}^\text{GIT} \) and \( \varphi_2 : \mathcal{M}_{0,8} \to \mathcal{M}^k_{\text{ord}} \) are the blow-ups given by the reduction of the weights [Ha03, Theorem 4.1]; see Figure 1 and Subsection 3.2. From [KMT11, Lemma 5.3], in their \( \mathbb{Q} \)-Picard groups, we obtain

\[
\varphi_1^*(D_2^{(0)}) = D_2^{(1)} + 6D_4^{(1)}
\]

\[
= \tilde{\Delta}_{\text{ord}} + 6\Delta_{\text{ord}}
\]
where we simply relabel the divisors in the second equality. Using the isomorphism \( \mathcal{M}_{\text{ord}}^K \cong \mathbb{B}^5/\Gamma_{\text{ord}} \), this, of course, implies

\[
\pi_{\text{ord}}^*(\mathcal{H}_{\text{ord}}) = \mathcal{H}_{\text{ord}} + 6T_{\text{ord}}.
\]

We also note that [Ko07a, Lemma 5.3] implies that

\[
\varphi_2^*(D_2^{(1)}) = D_2^{(2)} + 3D_3^{(2)}, \quad \varphi_2^*(D_4^{(1)}) = D_4^{(2)}.
\]

For the sake of completeness, we also remark that [Ko07a, Lemma 5.5] implies that

\[
\varphi_2^*(D_2^{(2)}) = D_2^{(1)} = \widehat{\mathcal{D}}_{\text{ord}}, \quad \varphi_2^*(D_3^{(2)}) = 0, \quad \varphi_2^*(D_4^{(2)}) = D_4^{(1)} = T_{\text{ord}},
\]

\[
\varphi_1^*(D_2^{(1)}) = D_2^{(0)} = \mathcal{D}_{\text{ord}}, \quad \varphi_1^*(D_4^{(1)}) = 0.
\]

The pushforward formulae are not used in this paper. All of these equalities hold in the relevant \( \mathbb{Q} \)-Picard groups.

Moreover, the canonical divisors are described as

\[
K_{M_{0,8}} = -\frac{2}{7}D_2^{(2)} + \frac{1}{7}D_3^{(2)} + \frac{2}{7}D_4^{(2)}
\]

\[
K_{\mathcal{M}_{\text{ord}}} = -\frac{2}{7}D_2^{(1)} + \frac{2}{7}D_4^{(1)}
\]

\[
K_{\mathcal{M}_{\text{GR}}} = -\frac{2}{7}D_2^{(0)}
\]

where the number 7 in the denominators comes from \( n-1 \) in [Ko07a, Proposition 5.4, Lemma 5.5]. Combining (4.1), (4.3) and (4.4) it follows that

\[
K_{\mathcal{M}_{\text{ord}}} = \varphi_1^*(K_{\mathcal{M}_{\text{GR}}}) + 2D_4^{(1)}
\]

\[
K_{\mathcal{M}_{0,8}} = \varphi_2^*(K_{\mathcal{M}_{\text{ord}}}) + D_3^{(2)}.
\]

In addition, there is a modular form \( F \) of weight 14 on \( \mathbb{B}^5 \) vanishing exactly on \( H \) with vanishing order 1. It follows that \( \text{div}(F) = H \) on \( \mathbb{B}^5 \). Since the quotient map \( \mathbb{B}^5 \to \mathbb{B}^5/\Gamma_{\text{ord}} \) ramifies along only \( H \) with index 2, the construction of \( \mathcal{L}_{\text{ord}} \) by Baily and Borel implies that the above relation decends to

\[
14\mathcal{L}_{\text{ord}} = \frac{1}{2}\mathcal{H}_{\text{ord}}
\]

in \( \text{Pic}(\mathbb{B}^5/\Gamma_{\text{ord}}^{\text{BB}}) \otimes \mathbb{Q} \). Note that the factor 1/2 comes from the ramification index along \( \mathcal{H}_{\text{ord}} \) arising from the action of the arithmetic subgroup \( \Gamma_{\text{ord}} \). Here \( \mathcal{L}_{\text{ord}} \) denotes the automorphic line bundle of weight 1. By (standard) abuse of notation, we use the same notation for this line bundle on both the Baily-Borel and toroidal compactifications. Thus,

\[
K_{\mathbb{B}^5/\Gamma_{\text{ord}}^{\text{BB}}} = -\frac{2}{7}\mathcal{D}_{\text{ord}}
\]

\[
= -8\mathcal{L}_{\text{ord}}.
\]

Now, we compute the canonical bundles of \( \mathbb{B}^5/\Gamma_{\text{ord}}^{\text{tor}} \cong \mathcal{M}_{\text{ord}}^K \) in two ways: the realization as a ball quotient and the blow-up sequence.
**Remark 4.1.** The finite map $\mathbb{B}^5 \to \mathbb{B}^5/\Gamma_{\text{ord}}$ (resp. $\mathbb{B}^5/\Gamma_{\text{ord}} \to \mathbb{B}^5/\Gamma$) branches along $H/\Gamma_{\text{ord}}$ (resp. $H/\Gamma$) with branch index 2 Here we give a sketch of the proof. First, for $r \in L$ let

$$\sigma_{\ell, \zeta}(r) := r + (1 - \zeta) \frac{\langle \ell, r \rangle}{2} \ell \in L \otimes \mathbb{Q}(\sqrt{-1})$$

where $\ell \in L$ is a $(-2)$-vector and $\zeta \in \{-1, \sqrt{-1}\}$. By [Be12, Corollary 3 (ii)] every quasi-reflection comes from such an automorphism. A straightforward calculation shows $\sigma_{\ell, -1} \in \Gamma_{\text{ord}}$ and $\sigma_{\ell, \sqrt{-1}} \in \Gamma \backslash \Gamma_{\text{ord}}$. This implies that the ramification by the finite group $\Gamma/\Gamma_{\text{ord}}$ is induced by an order 2 element $\sigma_{\ell, \sqrt{-1}}$, and hence the ramification index of the map $\mathbb{B}^5 \to \mathbb{B}^5/\Gamma_{\text{ord}}$ is 2. The claim for $\mathbb{B}^5/\Gamma_{\text{ord}} \to \mathbb{B}^5/\Gamma$ also follows from this consideration.

On the one hand, by Remark 4.1, a standard application of Hirzebruch’s proportionality principle [Mu77] gives

$$K_{\mathbb{B}^5/\Gamma_{\text{ord}}^{\text{tor}}} = 6 \mathcal{L}_{\text{ord}} - \frac{1}{2} \mathcal{H}_{\text{ord}} - \mathcal{T}_{\text{ord}}$$

$$= 6 \mathcal{L}_{\text{ord}} - \frac{1}{2} \{ \pi_{\text{ord}}^*(\mathcal{H}_{\text{ord}}) - 6 \mathcal{T}_{\text{ord}} \} - \mathcal{T}_{\text{ord}} \quad \text{(by (4.2))}$$

$$= -8 \mathcal{L}_{\text{ord}} + 2 \mathcal{T}_{\text{ord}} \quad \text{(by (4.5)).}$$

On the other hand,

$$K_{M_{\text{ord}}}^\mathcal{K} = -\frac{2}{t} \mathcal{D}_{\text{ord}} + \frac{2}{t} \Delta_{\text{ord}} \quad \text{(by (4.4))}$$

$$= -\frac{2}{t} (\varphi^*_1(\mathcal{D}_{\text{ord}}) - 6 \Delta_{\text{ord}}) + \frac{2}{t} \Delta_{\text{ord}} \quad \text{(by (4.1))}$$

$$= - \frac{2}{t} \varphi^*_1 \varphi^*_1 \mathcal{D}_{\text{ord}} + 2 \Delta_{\text{ord}}$$

$$= -8 \varphi^*_1 \varphi^*_1 (\mathcal{L}_{\text{ord}}) + 2 \Delta_{\text{ord}} \quad \text{(by (4.5))}$$

$$= \pi^* (-8 \mathcal{L}_{\text{ord}} + 2 \mathcal{T}_{\text{ord}}) \quad \text{(by Figure 1),}$$

for $\tau := \Phi_{\frac{1}{t}} \circ \phi_{\text{ord}}$. Thus, this calculation recovers the fact $K_{M_{\text{ord}}}^\mathcal{K} = \tau^* (K_{\mathbb{B}^5/\Gamma_{\text{ord}}^{\text{tor}}})$ under the isomorphism $\tau : M_{\text{ord}}^\mathcal{K} \simeq \mathbb{B}^5/\Gamma_{\text{ord}}^{\text{tor}}$.

**Remark 4.2.** The above modular form constructed by Kondo is a “special reflective modular form” in the sense of [MO23, Assumption 2.1]. Hence, both $M_{\text{ord}}^\mathcal{G}\mathcal{T}$ and $M^\mathcal{G}_{\text{GIT}}$ are Fano varieties from the above computation or [MO23, Theorem 2.4].

Now, we need the description of normal bundles along the toroidal boundary. For this we recall from Lemma 3.11 that $T_{\text{ord}, i} \simeq \mathbb{P}^2 \times \mathbb{P}^2$.

**Proposition 4.3.** The normal bundle of $T_{\text{ord}, i} \simeq \mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{B}^5/\Gamma_{\text{ord}}^{\text{tor}}$ is given by

$$N_{\mathcal{T}_{\text{ord}, i}/\mathbb{B}^5/\Gamma_{\text{ord}}^{\text{tor}}} = \mathcal{O}(-1, -1).$$

**Proof.** First, we obtain

$$(K_{\mathbb{B}^5/\Gamma_{\text{ord}}^{\text{tor}}} + T_{\text{ord}, i})|_{T_{\text{ord}, i}} = (-8 \mathcal{L}_{\text{ord}} + 2 \mathcal{T}_{\text{ord}} + T_{\text{ord}, i})|_{T_{\text{ord}, i}}.$$ 

The left-hand side gives

$$(K_{\mathbb{B}^5/\Gamma_{\text{ord}}^{\text{tor}}} + T_{\text{ord}, i})|_{T_{\text{ord}, i}} = K_{T_{\text{ord}, i}}$$
Now, we choose an isotropic vector $h$ corresponding to the following isotropic vector $A$ of isotropic vectors in the arithmetic group in question contains an element of the center in the full unitary group. That there are no irregular cusps in the sense of [Ma23]. These can only occur when the unipotent radicals of $\text{Stab}\Gamma$ group (see Section 2) acts trivially on the discriminant of notation both line bundles are denoted by the same symbol) and hence is trivial along the exceptional divisors for the blow-up $\pi: \overline{B^5/\Gamma_{\text{ord}}}^\text{BB} \to \overline{B^5/\Gamma_{\text{ord}}}^\text{tor}$. This completes the proof. □

Remark 4.4. This is an analogue of Naruki’s result [Na82 Proposition 12.1] on the moduli spaces of cubic surfaces. He constructed a cross ratio variety and analysed its singularity at the boundary. Later, Gallardo-Kerr-Schaffler [GKS21 Theorem 1.4] showed that the toroidal compactification and Naruki’s compactification are isomorphic and Casalaina-Martin-Grushevsky-Hulek-Laza [CMGHL24, Theorem 1.2] used this to compute the top self-intersection number of the canonical bundles. In the case of the moduli spaces of 8 points, there also exists the cross ratio variety constructed by [FML11, Theorem 2.4], [Ko07a, Theorem 7.2] or [MT04, Theorem 1.1]. However, these coincide with the Baily-Borel compactification $\overline{B^5/\Gamma_{\text{ord}}}^\text{BB}$ of the ball quotient unlike the case of cubic surfaces. This is why we used the results on the moduli spaces of stable curves in our case.

Now, we study the behaviour of the boundary divisors along the finite covering $\overline{B^5/\Gamma_{\text{ord}}}^\text{tor} \to \overline{B^5/\Gamma_{\text{ord}}}$. We recall that the toroidal compactifications are constructed by taking a “partial compactification in the direction of each cusp” [AMRT10, Section III. 5]. Here, this is done by choosing a polyhedral decomposition of a cone in the centre of the unipotent part of the stabilizer of a cusp (which is canonical in our case). Hence, this group, which is denoted by $U(F)$ in [AMRT10], describes the toroidal boundary.

Lemma 4.5. The map $\overline{B^5/\Gamma_{\text{ord}}}^\text{tor} \to \overline{B^5/\Gamma_{\text{ord}}}^\text{tor}$ does not branch along $T$.

Proof. We recall from Lemma 3.13 and its proof that the quotient $\Gamma/\Gamma_{\text{ord}} \cong \mathfrak{S}_8$ acts transitively on the set of toroidal boundary components $\{T_{\text{ord},i}\}_{i=1}^{35}$, since its action on the set of isotropic vectors in $A_L$ is transitive. Hence, it suffices to take one component $T_{\text{ord},i}$, corresponding to the following isotropic vector $h \in L$, and prove that the centre, denoted as $Z(F)$ in Lemma 3.11 of the unipotent radical of $\text{Stab}_F(h)$ and $\text{Stab}_{F_{\text{ord}}}(h)$ are equal. Now, we choose an isotropic vector $h := (1,0,0,0,0) \in U \oplus U(2) \oplus D_4(-1)^{\otimes 2}$. Then, the corresponding centre of the unipotent part of $\text{Stab}_F(h)$ is given by

$$\left\{ \begin{pmatrix} 1 & \sqrt{-1}(1 - \sqrt{-1})w \\ -w & 1 \end{pmatrix} \middle| w \in \mathbb{Z} \right\}.$$  

Then, one can check that each matrix of the above form acts trivially on the discriminant group (see Section 2). $A_L = L^\vee/L$, which is isomorphic to $(\mathcal{O}_F/(1 + \sqrt{-1})\mathcal{O}_F)^6$. In other words, the unipotent radicals of $\text{Stab}_F(h)$ and $\text{Stab}_{F_{\text{ord}}}(h)$ are equal. We finally remark that there are no irregular cusps in the sense of [Ma23]. These can only occur when the arithmetic group in question contains an element of the center in the full unitary group.
In this case, however, the discriminant group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (as $\mathbb{Z}$-module), and $-\text{id}$ and $-\sqrt{-1}\text{id}$ act trivially on the discriminant and are thus already contained in $\Gamma_{\text{ord}}$. Altogether, this proves the claim. □

On the one hand, in a similar way as [CMGHL24, Proposition 5.8], it follows that

\begin{equation}
K_{B^5/T^{\text{tor}}} = \pi^*K_{B^5/T^{\text{BB}}} + 7T
\end{equation}

by Lemma 4.5. On the other hand, we can calculate the canonical bundle of $\mathcal{M}^K$ by [CMGHL24, Lemma 6.4], where a general approach to calculating the canonical bundle of Kirwan blow-ups was developed:

\begin{equation}
K_{\mathcal{M}^K} = f^*K_{\mathcal{M}^{\text{GIT}}} + 5\Delta,
\end{equation}

where $\Delta$ is the exceptional divisor of the blow-up $f : \mathcal{M}^K \to \mathcal{M}^{\text{GIT}}$. Here, we apply the method [CMGHL24, Lemma 6.4] for our case $c = 6$ (Lemma 3.2) and $|G_X| = |G_F| = 2$ (Lemma 3.1) in their notation. Note that there is no divisorial locus having a strictly bigger stabilizer than $G_X$.

4.2. **Proof of Theorem 1.3.** We can now prove that these two compactifications are not $\mathbb{K}$-equivalent.

**Theorem 4.6.** The compactifications $\mathcal{M}^K$ and $B^5/T^{\text{tor}}$ are not $\mathbb{K}$-equivalent.

**Proof.** It suffices to show that $K_{\mathcal{M}^K}^5 \neq K_{B^5/T^{\text{tor}}}^5$. By (4.6) and (4.7), we need to show that

\((5\Delta)^5 \neq (7T)^5\).

Now, $T_{\text{ord},i}^5 = 6$ by Proposition 4.3. Hence, we have $T_{\text{ord}}^5 = 210$ and

\[T^5 = \frac{210}{8!} = \frac{1}{192}.
\]

Here, if $(5\Delta)^5$ and $(7T)^5$ are equal, then the denominator of $\Delta^5$ must be divided by 5 from the above calculation. On the other hand, [CMGHL24, Proposition 6.10] implies

$$\Delta^5 \in \frac{1}{e}\mathbb{Z},$$

where $e$ is the least common multiple of the orders of $S_x$ for any $x \in \Delta$. However, the quantity $e$ is not divisible by 5 by Proposition 3.7. This contradicts to the above. □

**Remark 4.7.** In principle, there is also a way to compute $\Delta^5$ explicitly. This can be approached as in [CMGHL24, Proposition 6.1]. Using the methods of [CMGHL24, Lemma 8.2] one can exhibit $\Delta$ as a finite quotient of a toric variety. After identifying the normal bundle, one can then compute its top intersection number by toric methods. As we do not need the precise number, we do not pursue this lengthy computation.

4.3. **Relation to the minimal model program.** We gave a proof of Theorems 1.1 and 1.3 by a specific computation in our situation. In this subsection, we would like to explain a more systematic approach to relate this to the minimal model program; see [KM98, Fu17] for the basic definitions. This also clarifies the relationship between our work and semi-toric compactifications of arithmetic quotients of type I and IV domains, in the sense of Looijenga; see [Lo86, Lo03a, Lo03b] and [AE23, Od22]. We explain the strategy below.

Let us recall the basic definition of the minimal model program before getting to the discussion. We use the notion of canonical model in the sense of [KM98, Definition 3.50],
which is referred to as log canonical model in [Fu17, Definition 4.8.1]. In addition, in this paper, minimal model is defined as not imposing dlt on [KM98, Definition 3.50], which is precisely the (log) minimal model in [Fu17, Definition 4.3.1]. We shall first apply the general theory of several compactifications to our case.

On the one hand, by the construction of the Baily-Borel compactifications, the automorphic line bundle $L = K_{\mathbb{P}^5/T} + \frac{3}{4} K$ is ample, thus the pair $(\mathbb{P}^5/T_{BB}, \frac{3}{4} K)$ is a canonical model and a good minimal model (of itself). Here, good means that $K_{\mathbb{P}^5/T_{BB}} + \frac{3}{4} K$ is semi-ample (it is in fact ample). The map

$$\pi : (\mathbb{P}^5/T_{tor}, \frac{3}{4} K + T) \to (\mathbb{P}^5/T_{BB}, \frac{3}{4} K)$$

is log crepant by [Mu77, Proposition 3.4]; see also [HKM24, Theorem 5.3]. This implies that $(\mathbb{P}^5/T_{tor}, \frac{3}{4} K + T)$ is a minimal model (of itself), and more strongly, a good minimal model with quasi-divisorially log terminal singularities, see [Od22, Theorem 3.1 (ii)]. Here we note that the fan, being 1-dimensional, is automatically regular. On the other hand, let us consider the Kirwan blow-up. Here, the problem is whether $(\mathbb{P}^5/T_{tor}, \frac{3}{4} K + T)$ is a minimal model or not. For this we need to compute the discrepancy $a(\Delta, M_{\text{GIT}}, \frac{3}{4} \mathcal{D})$ with respect to the Kirwan blow-up $f : (\mathbb{P}^5/T_{tor}, \frac{3}{4} K + T) \to (M_{\text{GIT}}, \frac{3}{4} \mathcal{D})$:

$$K_M = f^*(K_{M_{\text{GIT}} + \frac{3}{4} \mathcal{D}}) + a(\Delta, M_{\text{GIT}}, \frac{3}{4} \mathcal{D}) + \frac{3}{4} \mathcal{D}.$$

**Proposition 4.8.** The discrepancy $a(\Delta, M_{\text{GIT}}, \frac{3}{4} \mathcal{D})$ is $\frac{1}{2}$.

**Proof.** To compute the quantity $a(\Delta, M_{\text{GIT}}, \frac{3}{4} \mathcal{D})$ one can use [CMGHL24, Remark 6.7], which reduces the problem to the calculation of the discrepancy $a'(\Delta, (\mathbb{P}^8)^{\text{ss}}, \frac{3}{4} \mathcal{D})$ of $\Delta$ with respect to the map

$$f' : (\mathbb{P}^8, \frac{3}{4} \mathcal{D} + \Delta) \to (\mathbb{P}^8)^{\text{ss}}, \frac{3}{4} \mathcal{D}$$

(here we use the notation analogous to [CMGHL24, Subsection 6.2]). Note that since the codimension of the strictly semistable loci $(\mathbb{P}^8)^{\text{ss}} \setminus (\mathbb{P}^8)^{ss}$ and $M_{\text{GIT}} \setminus (M_{\text{GIT}})^o$ is larger than 2, we can apply the [CMGHL24, Remark 6.7] to our setting. In fact, combining this with the computation of (4.7), we have $a'(\Delta, (\mathbb{P}^8)^{\text{ss}}, \frac{3}{4} \mathcal{D}) = a(\Delta, M_{\text{GIT}}, \frac{3}{4} \mathcal{D})$.

We claim that $a'(\Delta, (\mathbb{P}^8)^{\text{ss}}, \frac{3}{4} \mathcal{D}) = \frac{1}{2}$. This follows from the computations in the proof of Theorem 3.4. For this purpose, it suffices to consider an affine locus $\mathcal{P} := (S_0 \neq 0)$. We claim that (3.3) and the subsequent calculations imply

$$f'^*(\mathcal{D}) = \mathcal{D} + 6\Delta.$$

For this, we must take the involution $\mathfrak{G}_2$, which interchanges the two local analytic branches of the discriminant $V = V_1 \cup V_2$, into account. Locally analytically, we are in the following situation. We have a commutative diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\pi}} & \tilde{Y} \\
\sigma_X \downarrow & & \sigma_Y \\
X & \xrightarrow{\pi} & Y
\end{array}$$

The varieties in this diagram are defined as follows. Here $X$ is the 6-dimensional Luna slice. Further, $\sigma_X : \tilde{X} \to X$ is the blow-up of $X$ in the origin. By $W = W_1 \cup W_2$, we denote the
intersection of the discriminant $V$ with the Luna slice. The strict transform of $W$ will be denoted by $\tilde{W}$. The horizontal maps in this diagram are the quotient maps given by the $S_2$-action and $\sigma_Y$ is induced from $\sigma_X$. We denote the image of $W$ in $Y$ by $Z$ and similarly for $\tilde{W}$. Finally, we denote the exceptional divisor in $\tilde{X}$ by $E_X$ and its image in $\tilde{Y}$ by $E_Y$. We claim that the quotient map $\tilde{\pi}$ is not ramified along $\tilde{W}$ and $E_X$. Indeed, the action of $S_2$ interchanges the two branches of $W$ and hence there is no branching along $\tilde{W}$. Moreover, the local Luna slice calculation also shows that $S_2$ acts non-trivially on the projectivized normal space of the origin and thus also on the exceptional divisor $E_X$.

Now, let \[ \sigma^*_X(W) = \tilde{W} + b_X E_X \]
and \[ \sigma^*_Y(Z) = \tilde{Z} + b_Y E_Y. \]
We claim that $b_X = b_Y$. Indeed, we have
\[ \tilde{\pi}^* \sigma^*_Y(Z) = \tilde{\pi}^* (\tilde{Z} + b_Y E_Y) = \tilde{W} + b_Y E_X \]
where we used that $\tilde{\pi}^*(E_Y) = E_X$, since $\tilde{\pi}$ is not ramified along $E_X$. Comparing this with
\[ \sigma^*_X \tilde{\pi}^*(Z) = \sigma^*_X(W) = \tilde{W} + b_X E_X \]
gives $b_X = b_Y$. We finally recall that the calculations in the proof of Theorem 3.4 imply that the stabilizer group of a generic point of the intersection of the strict transform of the discriminant and the exceptional divisor is of order 2 and generated by the involution considered above.

The claim about $a(\Delta, \mathcal{M}^\text{GIT}, \frac{3}{4} \mathcal{D})$ now follows by combining the above calculation with the equality
\[ K_{\tilde{\mathcal{D}}^\text{ss}} = f^*(K_{(\mathcal{D})^\text{ss}}) + 5\Delta \]
from (4.7). A straightforward calculation shows that
\[ K_{\tilde{\mathcal{D}}^\text{ss}} = f^* \left( K_{(\mathcal{D})^\text{ss}} + \frac{3}{4} \mathcal{D} \right) + \frac{1}{2} \Delta - \frac{3}{4} \mathcal{D} \]
and hence $a(\Delta, \mathcal{M}^\text{GIT}, \frac{3}{4} \mathcal{D}) = \frac{1}{2}$. \qed

One can extend the notion of $K$-equivalence to pairs. Let $(X, \Delta_X)$ and $(Y, \Delta_Y)$ be pairs of projective normal $\mathbb{Q}$-Gorenstein varieties $X$ and $Y$ and $\mathbb{Q}$-divisors $\Delta_X \in \text{Pic}(X)$ and $\Delta_Y \in \text{Pic}(Y)$ with a birational morphism $g : X -\to Y$ and $g_* \Delta_X = \Delta_Y$. We call these pairs $K$-equivalent as pairs if there is a common resolution of singularities $Z$ dominating $X$ and $Y$ birationally such that $f_X : Z \to X$ and $f_Y : Z \to Y$ satisfy $f_X^*(K_X + \Delta_X) \sim_Q f_Y^*(K_Y + \Delta_Y)$. The above calculation implies the following proposition, which can be seen as a variation of Theorem 1.3.

**Proposition 4.9.** $(\mathcal{M}^K, \frac{3}{4} \tilde{\mathcal{D}} + \Delta)$ and $(\overline{\mathbb{B}^5/T^\text{tor}}, \frac{3}{4} \tilde{\mathcal{H}} + T)$ are not $K$-equivalent as pairs.

**Proof.** We give a proof by contradiction. We first observe that formula (4.9), combined with Proposition 4.8, shows that
\[ K_{\mathcal{M}^K} + \frac{3}{4} \tilde{\mathcal{D}} + \Delta = f^*(K_{\mathcal{M}^\text{GIT}} + \frac{3}{4} \mathcal{D}) + \frac{3}{2} \Delta. \]
Now assume that \((\mathcal{M}^K, \frac{3}{4} \tilde{\mathcal{D}} + \Delta)\) and \((\mathbb{B}^5/\Gamma^{tor}, \frac{3}{4} \tilde{\mathcal{H}} + T)\) are \(K\)-equivalent as pairs. Then there exists a common resolution \(Z\) with birational morphisms \(f_1 : Z \to \mathcal{M}^K\) and \(f_2 : Z \to \mathbb{B}^5/\Gamma^{tor}\) such that
\[
(4.11) \quad f_1^*(K_{\mathcal{M}^K} + \frac{3}{4} \tilde{\mathcal{D}} + \Delta) = f_2^*(K_{\mathbb{B}^5/\Gamma^{tor}} + \frac{3}{4} \tilde{\mathcal{H}} + T).
\]
Altogether, we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{M}^K & \xrightarrow{g} & \mathbb{B}^5/\Gamma^{tor} \\
\downarrow f & & \downarrow \pi \\
\mathcal{M}^{GIT} & \xrightarrow{\phi} & \mathbb{B}^5/\Gamma^{BB}.
\end{array}
\]
Using this, we obtain
\[
(4.12) \quad f_1^*(f^*(K_{\mathcal{M}^{GIT}} + \frac{3}{4} \mathcal{D})) = f_2^*(\pi^*((\phi^{-1})^*(K_{\mathcal{M}^{GIT}} + \frac{3}{4} \mathcal{D}))) = f_2^*(K_{\mathbb{B}^5/\Gamma^{tor}} + \frac{3}{4} \tilde{\mathcal{H}} + T).
\]
Combining formulae (4.10), (4.11) and (4.12) implies that \(f_1^*(\Delta) = 0\) in \(\text{Pic}(Z) \otimes \mathbb{Q}\). However, since \(f_1\) is proper birational and \(\mathcal{M}^K\) is normal, the Stein factorization implies that \(f_1_* \mathcal{O}_Z = \mathcal{O}_{\mathcal{M}^K}\). Now, let \(L\) be a line bundle on \(\mathcal{M}^K\). We assume that \(f_1^* L = \mathcal{O}_Z\). Then, by the projection formula, we have \(f_1_* f_1^* L = L \otimes f_1_* \mathcal{O}_Z\). Our assumption and the above observation imply that \(L \cong \mathcal{O}_{\mathcal{M}^K}\), which results in showing that the pullback \(f_1^* : \text{Pic}(\mathcal{M}^K) \to \text{Pic}(Z)\) is injective.

**Remark 4.10.** For the ordered case, Gallardo-Kerr-Schaffler [GKS21, Theorem 2.1] showed that \(\mathcal{M}^{ord}_{\mathcal{M}^K} \cong \mathbb{B}^5/\Gamma^{ord}\) through the natural morphism lifting the Deligne-Mostow isomorphism. Here we remark that our computation determining the discrepancies can be used to give a different proof of this statement. The argument goes as follows. First, we prove that the pair \((\mathcal{M}^{ord}_{\mathcal{M}^K}, \frac{1}{2} \mathcal{D}_{\text{ord}} + \Delta_{\text{ord}})\) is a log minimal model of itself with log terminal singularities, as is \((\mathbb{B}^5/\Gamma_{\text{ord}}^{tor}, \frac{1}{2} \mathcal{H}_{\text{ord}} + T_{\text{ord}})\), which can be shown using the same arguments as in [HKM24, Proposition 5.1, Theorem 5.3]. The discrepancy of the birational morphism
\[
\varphi_1 : (\mathcal{M}^{K}_{\text{ord}}, \frac{1}{2} \mathcal{D}_{\text{ord}} + \Delta_{\text{ord}}) \to (\mathcal{M}^{GIT}_{\text{ord}}, \frac{1}{2} \mathcal{D}_{\text{ord}})
\]
is \(-1\) by (4.1) and (4.4). More precisely, these formulae show that
\[
K_{\mathcal{M}^K_{\text{ord}}} = \varphi_1^*(K_{\mathcal{M}^{GIT}_{\text{ord}}} + \frac{1}{2} \mathcal{D}_{\text{ord}}) - \Delta_{\text{ord}} - \frac{1}{2} \tilde{\mathcal{D}}_{\text{ord}},
\]
which implies that
\[
\varphi_{\text{ord}} \circ \varphi_1 : (\mathcal{M}^{K}_{\text{ord}}, \frac{1}{2} \mathcal{D}_{\text{ord}} + \Delta_{\text{ord}}) \to (\mathbb{B}^5/\Gamma_{\text{ord}}^{BB}, \frac{1}{2} \mathcal{H}_{\text{ord}})
\]
is log crepant. This is the same situation as the blow-up
\[
\pi_{\text{ord}} : (\mathbb{B}^5/\Gamma_{\text{ord}}^{tor}, \frac{1}{2} \mathcal{H}_{\text{ord}} + T_{\text{ord}}) \to (\mathbb{B}^5/\Gamma_{\text{ord}}^{BB}, \frac{1}{2} \mathcal{H}_{\text{ord}});
\]
see also (4.8). Hence, a similar proof to [HKM24, Proposition 5.1 (3)] implies that the pair $(\mathcal{M}_{\text{ord}}^K, \frac{1}{2}\mathcal{D}_{\text{ord}} + \Delta_{\text{ord}})$ has log terminal singularities. To prove that it is a log minimal model of itself, we check that it satisfies [Fu17, Definition 4.3.1]. [Fu17, Definition 4.3.1 (i)-(iii)] is trivially satisfied as $(X, \Delta) = (X', \Delta') = (\mathcal{M}_{\text{ord}}^K, \frac{1}{2}\mathcal{D}_{\text{ord}} + \Delta_{\text{ord}})$ in the notation there. Since $6\mathcal{L}_{\text{ord}} = K_{\mathcal{B}^5/T_{\text{ord}}^{\text{BB}}} + \frac{1}{2}\mathcal{H}_{\text{ord}}$ is ample and $\phi_{\text{ord}} \circ \varphi_1$ is log crepant, the $\mathbb{Q}$-line bundle

$$K_{\mathcal{M}_{\text{ord}}^K} + \frac{1}{2}\mathcal{D}_{\text{ord}} + \Delta_{\text{ord}}$$

is nef, which shows that the pair $(\mathcal{M}_{\text{ord}}^K, \frac{1}{2}\mathcal{D}_{\text{ord}} + \Delta_{\text{ord}})$ satisfies [Fu17, Definition 4.3.1 (iv)]. [Fu17, Definition 4.3.1 (v)] follows from the above computation, showing $\phi_{\text{ord}} \circ \varphi_1$ is log crepant; see also [HKM24, Theorem 5.3]. Summarizing the above, the pair $(\mathcal{M}_{\text{ord}}^K, \frac{1}{2}\mathcal{D}_{\text{ord}} + \Delta_{\text{ord}})$ is a log minimal model with log terminal singularities. We can now finish the claim concerning the isomorphism $\mathcal{M}_{\text{ord}}^K \cong \mathcal{B}^5/T_{\text{ord}}^{\text{tor}}$. According to [Od22, Theorem 3.1], it remains to prove that $\mathcal{D}_{\text{ord}} \cup \Delta_{\text{ord}}$ is a normal crossing divisor. This follows from a straightforward explicit Luna slice computation; see also Remark 3.5. We omit the details.

Finally, we shall refer to a characterization as a semi-toric compactification.

**Proposition 4.11.**

(1) The pair $(\mathcal{M}^K, \frac{2}{5}\mathcal{D} + \Delta)$ is not log canonical.

(2) $\mathcal{M}^K$ is not a semi-toric compactification.

**Proof.** Odaka gave a characterization of semi-toric compactifications in terms of singularities of pairs [Od22, Theorem 3.1 (iii)]. From this we can deduce that (1) implies (2). Hence it suffices to prove the first of the two statements. This can be shown in the same way as [HKM24, Corollary 5.10]. We omit the details, but a sketch of the proof is as follows. Essentially, it follows from the fact that the log canonical centre of the Baily-Borel compactification is the Baily-Borel (unique in this case) cusp [MO23, Lemma 2.9 (1)]. This implies that an exceptional divisor of a resolution of singularities of $\mathcal{M}^K$ is mapped to the unique Baily-Borel cusp $\xi$ in $\mathcal{B}^5/T_{\text{ord}}^{\text{BB}}$. Combined with the inequality in [KM98, Lemma 2.27], we can deduce that the discrepancy around the exceptional divisor is strictly less than $-1$. This concludes the proof.

**Remark 4.12.**

(1) We note that this gives another proof of one direction of Theorem 1.1. Indeed, this can be deduced from [AE23, Theorem 7.18] or [AEC24, Subsection 3F], since semi-toric compactifications are characterized by the property that they lie between $\mathcal{B}^5/T_{\text{ord}}^{\text{tor}}$ and $\mathcal{B}^5/T^{\text{BB}}$. Note that in this case, one does not need to know the equality of the second Betti numbers of $\mathcal{M}^K$ and $\mathcal{B}^5/T_{\text{ord}}^{\text{tor}}$ which we used in our proof.

(2) The phenomenon that the pair, consisting of the Kirwan blow-up with the exceptional divisor and the discriminant divisor with standard coefficients has worse than log canonical singularities, so that, in particular, it is not a semi-toric compactification of ball a quotient, can be observed in other situations as well. One example is the case of 12 points, which corresponds to the Eisenstein ancestral Deligne-Mostow variety. This was studied in [HKM24, Corollary 5.10].
5. Cohomology

In this section, we compute the cohomology of the varieties appearing in this paper.

5.1. The cohomology of $M_{\text{ord}}^K$, $B^5/T_{\text{ord}}^\text{BB}$, $B^5/T_{\text{ord}}^\text{tor}$ and $B^5/T^\text{BB}$. We first collect the results due to Kirwan-Lee-Weintraub [KLW87] and Kirwan [Ki89] who determined the Betti numbers of $M_{\text{ord}}^K$ and $B^5/T_{\text{ord}}^\text{BB}$, and $M_{\text{GIT}} \cong B^5/T^\text{BB}$ respectively. We summarize this in Theorem 5.1 ([KLW87, Table III, Theorem 8.6], [Ki89, Table, p.40]). All the odd degree cohomology of $M_{\text{ord}}^K$, $B^5/T_{\text{ord}}^\text{BB}$ and $B^5/T^\text{BB}$ vanishes. In even degrees, the Betti numbers are as follows:

| $j$      | $0$ | $2$ | $4$ | $6$ | $8$ | $10$ |
|----------|-----|-----|-----|-----|-----|------|
| $\dim H^j(M_{\text{ord}}^K)$ | 1   | 43  | 99  | 99  | 43  | 1    |
| $\dim IH^j(B^5/T_{\text{ord}}^\text{BB})$ | 1   | 8   | 29  | 29  | 8   | 1    |
| $\dim IH^j(M_{\text{GIT}})$ | 1   | 1   | 2   | 2   | 1   | 1    |
| $\dim IH^j(B^5/T^\text{BB})$ | 1   | 1   | 2   | 2   | 1   | 1    |

By an application of an easy version of the decomposition theorem, we can also compute the cohomology of $B^5/T_{\text{ord}}^\text{tor}$ (without using that this space is isomorphic to $M_{\text{ord}}^K$).

Theorem 5.2. All the odd degree cohomology of $B^5/T_{\text{ord}}^\text{tor}$ vanishes. In even degrees, the Betti numbers are as follows:

| $j$      | $0$ | $2$ | $4$ | $6$ | $8$ | $10$ |
|----------|-----|-----|-----|-----|-----|------|
| $\dim H^j(B^5/T_{\text{ord}}^\text{tor})$ | 1   | 43  | 99  | 99  | 43  | 1    |

Proof. We use the form of the decomposition theorem as given in [GH17, Lemma 9.1]. Here we have 35 cusps and the toroidal boundary at each cusp is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. The even Betti numbers of this space are given by $1, 2, 3, 2, 1$ and the result then follows from the Betti numbers of $B^5/T_{\text{ord}}^\text{BB}$ together with the fact that there are 35 cusps. 

5.2. The cohomology of $M^K$. Now, we compute the cohomology of $M^K$. This will be done using the Kirwan method [Ki84, Ki85, Ki89], studying the cohomology of the Kirwan blow-ups. We mainly follow [CMGHL23, Chapter 3, 4], in particular, the case of cubic threefolds with precisely $2A_5$-singularities. Let us consider $X = \mathbb{P}^8$, acted on by $G = \text{SL}_2(\mathbb{C})$ with the usual linearization and let $Z_{R}^{\text{ss}}$ be the fixed locus of the action of $R$ on $X^{\text{ss}}$, which is the semi-stable locus. Here we recall that $R$ is the stabilizer of the strictly semi-stable point $c_{4,4}$ as introduced in Lemma 3.1. We denote by $\tilde{X}^{\text{ss}} := Bl_{G \cdot Z_{R}^{\text{ss}}}(X)$ the blow-up whose centre is the unique polystable orbit $G \cdot Z_{R}^{\text{ss}}$. From [Ki89, Section 3 Eq. 3.2] or [CMGHL23, Subsection 4.12, (4.22)], the Poincaré series of $\tilde{X}^{\text{ss}}$ is given by

$$P^G_t(\tilde{X}^{\text{ss}}) = P^G_t(X^{\text{ss}}) + A_R(t),$$

where $A_R(t)$ is a correction term consisting of a “main term” and an “extra term” with respect to the unique stabilizer $R$; see [CMGHL23, Section 4.1.2] for precise definitions.

This method reduces the computation of $H^k(M^K)$ to the estimation of
(1) the semi-stable locus (Subsection 5.2.1),
(2) the main correction term (Subsection 5.2.2) and
(3) the extra correction term (Subsection 5.2.3).

5.2.1. Equivariant cohomology of the semi-stable locus. Here we proceed according to [CMGHL23, Chapter 3]. We can compute the cohomology of the semi-stable locus by using the stratification introduced by Kirwan. We omit details, but will still need to introduce some notation in order to describe the outline. Let \( \{S_\beta\}_{\beta \in B} \) be the stratification defined in [Ki84, Theorem 4.16] and \( d(\beta) \) be the codimension of \( S_\beta \) in \( X^{ss} \). Here, the index set \( B \) consists of the point which is closest to the origin of the convex hull spanned by some weights in the closure of a positive Weyl chamber in the Lie algebra of a maximal torus in \( \text{SO}(2) \); see [CMGHL23, Chapter 3] or [Ki84, Definition 3.13] for details.

Proposition 5.3.

\[
P_t^G(X^{ss}) \equiv 1 + t^2 + 2t^4 \mod t^6.
\]

**Proof.** We shall prove \( 2d(\beta) \geq 6 \) for any \( 0 \neq \beta \in B \). This implies

\[
P_t^G(X^{ss}) \equiv P_t(X)P_t(B \text{SL}_2(\mathbb{C})) \mod t^6
\]

\[
\equiv (1 - t^2)^{-1}(1 - t^4)^{-1} \mod t^6
\]

\[
\equiv 1 + t^2 + 2t^4 \mod t^6.
\]

Here we denote by \( BG \) the classifying space for any topological space \( G \); see [CMGHL23, Appendix A]. In the same way, as in the proof of [CMGHL23, Proposition 3.5] we obtain

\[d(\beta) \geq 7 - r(\beta),\]

where \( r(\beta) \) is the number of weights \( \alpha \) satisfying \( \beta \cdot \alpha \geq ||\beta||^2 \). Now, we have

\[B = \{(1, -1), (2, -2), (3, -3), (4, -4)\}.
\]

For each \( (a, -a) \in B \), it easily follows

\[r(\beta) = 5 - a,
\]

and this implies \( d(\beta) \geq 3 \).

\[\square\]

5.2.2. The main correction term. The following is based on [CMGHL23, Chapter 4].

Proposition 5.4. The main correction term in \( A_R(t) \) is given by

\[(1 - t^4)^{-1}(t^2 + t^4) \equiv t^2 + t^4 \mod t^6.
\]

**Proof.** In the same way as in [CMGHL23, Proposition B.1 (4)], the normalizer of \( R \) is computed to be

\[N := N(R) \cong \mathbb{T} \times \mathbb{Z}/2\mathbb{Z}.
\]

Hence, it follows that

\[H^*_N(Z_R^{ss}) \cong (H^*_T(Z_R^{ss}))_{\mathbb{Z}/2\mathbb{Z}}
\]

\[= (H^*(BR) \otimes H^*_R(Z_R^{ss}))_{\mathbb{Z}/2\mathbb{Z}}
\]

\[= (H^*(BR) \otimes H^*(\ast))_{\mathbb{Z}/2\mathbb{Z}}
\]

\[= \mathbb{Q}[e^4]
\]
where \( * \) denotes a set of 1 point and the degree of \( c \) is 1. The last equation follows from the discussion in the proof of [CMGHL23, Proposition 4.4]. Hence,
\[
P_t^N(Z_R^{ss}) = (1 - t^4)^{-1}.
\]
Combining this with [CMGHL23, (4.24)] completes the proof. \( \square \)

5.2.3. **The extra correction term.** Let \( \mathcal{N} \) be the normal bundle to the orbit \( G \cdot Z_R^{ss} \). Then, for a generic point \( x \in Z_R^{ss} \), we have a representation \( \rho \) of \( R \) on \( \mathcal{N}_x \). Let \( \mathcal{B}(\rho) \) be the set consisting of the closest point to 0 of the convex hull of a nonempty set of weights of the representation \( \rho \). For \( \beta' \in \mathcal{B}(\rho) \), let \( n(\beta') \) be the number of weights less than \( \beta' \).

**Proposition 5.5.** The extra correction term vanishes modulo \( t^6 \), i.e., does not contribute to \( A_R(t) \).

**Proof.** In our case we have \( Z_R^{ss} = \{c_{4,4}\} \). Thus, to describe \( \mathcal{N}_x \), we have to compute
\[
\left( T_{c_{4,4}}(\text{SL}_2(\mathbb{C}) \cdot \{c_{4,4}\}) \right)^{\perp}.
\]
This was calculated in Lemma 3.2. Moreover, \( \text{diag}(\lambda, \lambda^{-1}) \) acts on \( T_{c_{4,4}} \mathbb{C}^9 \cong \mathbb{C}^9 \) by the weights
\[
0, \pm 2, \pm 4, \pm 6, \pm 8.
\]
It follows that \( T_{c_{4,4}}(\text{SL}_2(\mathbb{C}) \cdot \{c_{4,4}\}) \) is generated by the weights \( \{0, \pm 2\} \), and hence we obtain
\[
\mathcal{B}(\rho) = \{\pm 4, \pm 6, \pm 8\}.
\]
This shows that
\[
d(|\beta'|) = n(|\beta'|)
\]
\[
= 1 + \frac{|\beta'|}{2}
\]
\[
\geq 3
\]
for \( \beta' \in \mathcal{B}(\rho) \). This in turn implies that

“extra correction term” \( \equiv 0 \text{ mod } t^6 \)

by [CMGHL23, (4.25)]. \( \square \)

5.2.4. **Computation of the cohomology of \( \mathcal{M}^K \).** From Propositions 5.3, 5.4 and 5.5, it follows that
\[
P_t(\mathcal{M}^K) = P_t^G(\tilde{X}^{ss})
\]
\[
\equiv (1 + t^2 + 2t^4) + (t^2 + t^4) \text{ mod } t^6
\]
\[
\equiv 1 + 2t^2 + 3t^4 \text{ mod } t^6.
\]
Therefore, we obtain the following:

**Theorem 5.6.** All the odd degree cohomology of \( \mathcal{M}^K \) vanishes. In even degrees, its Betti numbers are given as follows:

| \( j \) | 0 | 2 | 4 | 6 | 8 | 10 |
|-------|---|---|---|---|---|----|
| \( \dim H^j(\mathcal{M}^K) \) | 1 | 2 | 3 | 3 | 2 | 1 |
5.3. The cohomology of $\overline{B^5/\Gamma^{tor}}$. Now, we compute the cohomology of the toroidal compactification of the 5-dimensional ball quotient. Our main tool is the decomposition theorem in the easy form stated in theorem [GH17, Lemma 9.1], see also [CMGHL23, chapter 6]. This allows us to combine the cohomology of $\overline{B^5/\Gamma^{BB}}$ and the toroidal boundary. To do this, we first study the cohomology of the toroidal boundary.

**Proposition 5.7.** All the odd degree cohomology of the boundary $T$ vanishes. In even degrees, its Betti numbers are given as follows:

| $j$ | 0 | 2 | 4 | 6 | 8 |
|-----|---|---|---|---|---|
| $\dim H^j(T)$ | 1 | 1 | 2 | 1 | 1 |

**Proof.** This amounts to the computation of the invariant cohomology of the action of the stabilizer of a toroidal boundary component as in the proof of [CMGHL23, Proposition 7.13]. More precisely, we have to determine the cohomology ring

$$H^\bullet(\mathbb{P}^2 \times \mathbb{P}^2 / (\mathcal{S}_4 \times \mathcal{S}_4) \times \mathcal{S}_2) = H^\bullet((\mathbb{P}^2 / \mathcal{S}_4)^2, \mathbb{Q}) \mathcal{S}_2 = H^\bullet((\mathbb{P}(1, 2, 3)^2, \mathbb{Q}) \mathcal{S}_2.$$

Since $H^\bullet(\mathbb{P}^2 / \mathcal{S}_4) = H^\bullet((\mathbb{P}(1, 2, 3)) \cong \mathbb{Q}[x]/(x^3)$, this is equivalent to compute the $\mathcal{S}_2$-invariant parts of the tensor product $\mathbb{Q}[x]/(x^3) \otimes \mathbb{Q}[y]/(y^3).$ Hence the invariant cohomology is given by

$$P_t(T) = 1 + t^2 + 2t^4 + t^6 + t^8.$$ 

We can now summarize the above computations in the

**Theorem 5.8.** All the odd degree cohomology of $\overline{B^5/\Gamma^{tor}}$ vanishes. In even degrees, the Betti numbers are given by the following table:

| $j$ | 0 | 2 | 4 | 6 | 8 | 10 |
|-----|---|---|---|---|---|----|
| $\dim H^j(\overline{B^5/\Gamma^{tor}})$ | 1 | 2 | 3 | 2 | 1 | |

In particular, all the Betti numbers of $\mathcal{M}_K$ and $\overline{B^5/\Gamma^{tor}}$ are the same.

**Proof.** This follows now from an application of the decomposition theorem as stated in [GH17, Lemma 9.1], applied to the last line in Theorem 5.1 and Proposition 5.7.

6. Other cases of the Deligne-Mostow list

Here we very briefly discuss some further cases of the Deligne-Mostow list where a similar analysis can be made. More concretely, we consider $N$ points on $\mathbb{P}^1$ for $5 \leq N \leq 12$ with symmetric weights; see [DM86] or [Th98, Appendix]. Note that the notions of stable and semi-stable coincide for odd $N$. Remarkably, the behaviour which was observed for the moduli spaces of cubic surfaces and 8 points on $\mathbb{P}^1$, can also be found in other cases, thus pointing towards a much more general phenomenon.
6.1. 5 points. The moduli space of 5 points on \( \mathbb{P}^1 \) is associated with K3 surfaces with an automorphism of order 5 [Ko07b]. In this case, the Deligne-Mostow isomorphism gives

\[ \mathcal{M}^{\text{GIT}}_{\text{ord}} \cong \mathbb{B}^2/\Gamma_{\text{ord}}^{BB} \]

for the discriminant kernel group \( \Gamma_{\text{ord}} \) [Ko07b, Subsection 6.3, (6.5)]. Here, the weight in the sense of Deligne-Mostow is

\[ \left( \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5} \right). \]

This is the quintic del Pezzo surface [Ko02, Proposition 6.2 (2)]. Now, \( \mathbb{B}^2/\Gamma' \) is compact ([Ko07b Subsection 6.5] or [Th98, Appendix]). Hence, we have

\[ \mathcal{M}^K = \mathcal{M}^{\text{GIT}}_{\text{ord}} \cong \mathbb{B}^2/\Gamma^{\text{BB}}_{\text{ord}} = \mathbb{B}^2/\Gamma^{\text{tor}}. \]

for the full modular unitary group \( \Gamma \).

6.2. 7, 9, 10 or 11 points. The moduli space of 7 points on \( \mathbb{P}^1 \) was studied in [DvGK05]. In this paper, we apply the theory of the moduli spaces of stable curves to analyse the geometry of our ball quotients. In order to apply the work by Hassett, Kiem-Moon and others, the weights appearing in the Deligne-Mostow theory, that is the linearization of a line bundle, must be linearised as \( \mathcal{O}(1, \cdots, 1) \); see [KM11, Section 1]. Thus, in particular, the case of 7, 9, 10, and 11 points are out of scope in this paper.

6.3. 6 points and 12 points. These are Eisenstein cases, which will be treated in upcoming work.

6.3.1. 6 points. The moduli space of 6 points on \( \mathbb{P}^1 \) is closely related to the theory of the Igusa quartic and the Segre cubic [Ko13, Ko16, Ma01]. It is known that the Segre cubic is realised as the Baily-Borel compactification of a 3-dimensional ball quotient. We recall the setting of [Ko13]. Let \( \Lambda := \mathbb{Z}[\omega]\mathbb{Z}^4 \) be the Hermitian lattice over \( \mathbb{Z}[\omega] \) of signature \( (1,3) \) equipped with the Hermitian matrix \( \text{diag}(1, -1, -1, -1) \), where \( \omega \) is a primitive third root of unity. Let \( \Gamma := \text{U}(\Lambda)(\mathbb{Z}) \) and

\[ \Gamma_{\text{ord}} := \{ g \in \Gamma \mid g|_{\Lambda/\sqrt{-3}A} = \text{id} \}. \]

The ball quotient \( \mathbb{B}^3/\Gamma^{BB} \) (resp. \( \mathbb{B}^3/\Gamma_{\text{ord}}^{BB} \)) is isomorphic to the moduli space of unordered (resp. ordered) 6 points on \( \mathbb{P}^1 \). Here, \( \mathbb{B}^3 \) is the 3-dimensional complex ball. The approach developed in the current paper can be fully carried over to this case. In particular, the analogues of Theorems 1.1 and 1.3 hold unchanged.

6.3.2. 12 points. The moduli space of unordered 12 points on \( \mathbb{P}^1 \) is known to be the moduli space of (non-hyperelliptic) curves of genus 4 [Ko02]. In particular, this moduli space is the 9-dimensional ball quotient taken by the full unitary group for the Hermitian lattice with underlying integral lattice \( U\otimes E_8(-1)^{\otimes 3} \). There is, however, an important difference here to the cases discussed previously: the arithmetic subgroup defining the moduli space of ordered 12 points on \( \mathbb{P}^1 \) is not known; see [HKM24, Remark 2.2], although it is expected to be the discriminant kernel as in the case of 6 or 8 points.

In this case, there is the blow-up sequence

\[ \overline{\mathcal{M}}_{0,12} \to \overline{\mathcal{M}}_{0,12(1/3+\varepsilon)} \to \overline{\mathcal{M}}_{12(1/3+\varepsilon)} \to \overline{\mathcal{M}}_{0,12(1/3+\varepsilon)} \cong \mathcal{M}_{\text{ord}} \overset{\varphi}{\to} \mathcal{M}^{\text{GIT}}_{\text{ord}}. \]
We have since analysed the case of 12 points in some detail, see [HKM24], showing that an analogue of Theorem 1.1 also holds in the 12 points case. This further confirms preliminary computations due to Casalaina-Martin (private communication).

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K.H: Institut für Algebraische Geometrie, Leibniz University Hannover, Welfengarten 1, 30060 Hannover, Germany
Email address: hulek@math.uni-hannover.de

Y.M: Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan/Advanced Research Laboratory, Research Platform, Sony Group Corporation, 1-7-1 Konan, Minato-ku, Tokyo, 108-0075, Japan
Email address: y.maeda.math@gmail.com