Quantumness of channels

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The reliability of quantum channels for transmitting information is of profound importance from the perspective of quantum information. This naturally leads to the question as how well a quantum state is preserved when subjected to a quantum channel. We propose a measure of quantumness of channels based on non-commutativity of quantum states that is intuitive and easy to compute. We apply the proposed measure to some well known noise channels, both Markovian as well as non-Markovian and find that the results are in good agreement with those from a recently introduced $l_1$-norm coherence based measure.

I. INTRODUCTION

Quantifying the degree of quantumness of a channel has both theoretical and practical significance in quantum information science [1]. The quantum channels are completely positive and trace preserving maps which describe processes like information transfer in a given environment [2]. Since quantum information is transmitted in the form of quantum states, it is important to quantify the degree to which a quantum state gets affected while subjected to a quantum channel [3]. The classical states are usually identified as those whose correlations can be described in terms of classical probabilities. This approach has lead to the quantification of some well known nonclassical correlations such as entanglement, discord and related quantities [4]. Alternatively, a different way of quantifying the quantumness of a single system is by exploiting the non-commutative algebra of observables, such that the mutual commutation of all the accessible states of the system identify with a classical system. This approach has advantages in that it make no reference to the correlations and no complicated optimization procedures are needed [5].

Noise is usually known for its negative role in reducing the degree of coherence in a system. However, they can show enhancement in nonclassical correlations for some states [6–8]. In [9–11], it was shown that local environments can enhance the average fidelity of quantum teleportation for certain entangled states. Enhancement in quantum discord by local Markovian (i.e., memoryless) noise channels was reported in [12, 13]. Quantum channels provide a platform for studying the interplay between quantumness of states and the underlying dynamics in presence of an ambient environment [14]. This has lead to several interesting observations. For example, in [15] it was shown that the quantum channels need not be decohering, but could have cohering power as well. The cohering power, that is, the ability of quantum operations to produce coherence, was given an operational interpretation in [16]. It was further shown that the cohering power of any quantum operation is upper bounded by the corresponding unitary operation. The entangling capabilities of unitary operations acting on bipartite systems was reported in [17], with the maximum entanglement being created with product input states [18]. The deteriorating effect of the environment on a quantum state has been studied in the context of coherence-breaking channels and coherence sudden death [19]. An interesting class of channels known as semi-classical channels $\Lambda_{SC}$ map all the input states $\rho$ to $\Lambda_{SC}(\rho)$, such that the later are diagonal in the same basis. Such channels are realized by complete decoherence after which only diagonal elements of the density matrix are non-zero [20]. Another well studied class of quantum channels are those based on Lindbladian evolution which focus on the dynamics at time scales well separated from that of the reservoir correlations. However, in a number of practical applications, this assumption is not true and one has to take into account the non-Markovian aspects of the underlying dynamics [21–23].

Recently, a coherence based measure of quantumness of channel was proposed in [24], by defining the measure as the average quantum coherence of the state after the quantum channel acts on it, and minimized over all orthonormal basis sets of the state space. This measure was studied in the context of various (non) Markovian channels [25]. Further, this measure connects different coherence and entanglement measures, and is also the upper bound for another important coherence measure called robustness of coherence for all qubit states [26].

In this work, we propose a simple measure for quantumness of channels, based on commutation properties of the states evolving under the relevant channels. A necessary and sufficient condition for the creation of quantum correlations via local channels in finite dimensions is that they should not be commutativity preserving [27]. Commutative quantum channels preserve the commutation relation of any two compatible states, i.e., if $[\rho, \sigma] = 0$,
then \( [\mathcal{E}(\rho), \mathcal{E}(\sigma)] = 0 \). It is clear that the semiclas-sical channels, defined above, are commutativity preserving, implying that a departure from semiclassicality is necessary to create quantum correlations.

This paper is organized as follows: In Sec. (II) we introduce a measure of quantumness of channels. Section (III) is devoted to applying this measure to various well-known quantum channels. The experimental relevance of this measure is discussed in Sec. (IV), Results and their discussion are presented in Sec. (V). We conclude in Sec. (VI).

II. QUANTUMNESS OF CHANNELS

Given two arbitrary states \( \rho \) and \( \sigma \), one can quantify their mutual incompatibility by the Hilbert-Schmidt (HS) norm of their commutator \( \mathcal{M}(\rho, \sigma) = 2||C||_{HS} = 2 \text{Tr}[C^\dagger C] \). The measure is defined in terms of the HS norm of their commutator \( C = \rho \sigma - \sigma \rho \). The HS norm for an operator \( O \) is defined as \( ||O||_{HS}^2 = \text{Tr}[O^\dagger O] \). This measure was motivated in [3] with the aim of identifying nonclassicality with the incompatibility of states, where it was also shown that \( 0 \leq \mathcal{M}(\rho, \sigma) \leq 1 \).

Here we try to exploit this approach to probe the quantumness of a channel. Consider a channel described by a linear, completely positive and trace preserving map \( \Phi : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B) \) [28, 29]. The action of this map on an input state \( \rho \) leads to an output state \( \rho' \) and can be summarized as

\[
\rho' = \Phi[\rho].
\]

In the context of quantum channels, we start with two states \( \rho_a \) and \( \rho_b \) which are maximally noncommuting in the sense that \( \mathcal{M}(\rho_a, \rho_b) = 1 \). By subjecting one of them, say \( \rho_a \), to a quantum channel, the state evolves to \( \rho_b \). The quantumness of the channel can be attributed to the extent to which \( \rho_a \) and \( \rho_b \) are incompatible

\[
\mathcal{M}(\rho_a, \rho_b') = 2||C||_{HS} = 2 \text{Tr}[C^\dagger C],
\]

with \( C = \rho_a \rho_b - \rho_b' \rho_a \). It should be noted that \( \mathcal{M}(\rho_a, \rho_b') = 0 \) when the output state \( \rho_b' \) is maximally mixed. This suggests that the quantumness of the channel is identified by its ability to restrict the state from being maximally mixed. As an example, the states \( |a\rangle = \cos(x/2)|0\rangle + e^{-i\phi} \sin(x/2)|1\rangle \) and \( |b\rangle = \cos(y/2)|0\rangle + e^{-i\xi} \sin(y/2)|1\rangle \), with \( y = x+\pi/2 \) and \( \xi = \phi \), can be represented by the following density matrices

\[
\rho_a = \begin{pmatrix}
\cos^2(x) & \sin^2(x) \\
\sin^2(x) & \cos^2(x)
\end{pmatrix}, \quad 
\rho_b = \begin{pmatrix}
\sin^2(x) & -\sin(2x) \\
\sin(2x) & \cos^2(x)
\end{pmatrix}.
\]

These lead to the commutator

\[
C = \rho_a \rho_b - \rho_b' \rho_a = \begin{pmatrix}
0 & e^{-i\phi} \\
e^{-i\phi} & 0
\end{pmatrix},
\]

and therefore \( \mathcal{M}(\rho_a, \rho_b) = 2 \text{Tr}[C^\dagger C] = 1 \). Thus the states are maximally noncommuting and in this sense share maximum nonclassicality.

III. APPLICATION TO QUANTUM CHANNELS

We will now apply the above definition to some well-known quantum channels. We consider the dephasing channels like random telegraph noise (RTN) [30], non-Markovian dephasing (NMD) [31], phase damping (PD) [32] and generalized depolarizing channel (GDC) [33]. The generalized amplitude damping channel (GAD) [6, 34], which represents a dissipative channel is also studied. The Kraus operators for these channels are given in Table (I).

Example 1. Random Telegraph Noise (RTN): The dynamical map is represented by the Kraus operators \( K_0(t) = k_+ I \) and \( K_1(t) = k_- \sigma_z \), where \( k_{\pm} = \sqrt{1+\Lambda(t)/2} \), such that the action on a general qubit state

\[
\rho = \begin{pmatrix}
1-p & x \\
x^* & p
\end{pmatrix},
\]

is given by

\[
\rho' = \Phi^{RTN} \begin{pmatrix}
1-p & x \\
x^* & p
\end{pmatrix} = \begin{pmatrix}
1-p & x\Lambda(t) \\
x^*\Lambda(t) & p
\end{pmatrix}.
\]

Let us use the maximally nonclassical pair of states given in Eq. (3). The state \( \rho_b = \rho' \) is subjected to RTN evolution

\[
\rho_b' = \begin{pmatrix}
\cos^2 \left( \frac{x}{2} \right) & \frac{1}{2} e^{-i\phi} \cos(x) \Lambda(t) \\
\frac{1}{2} e^{i\phi} \cos(x) \Lambda(t) & \sin^2 \left( \frac{x}{2} + \pi \right)
\end{pmatrix}.
\]

The pertinent commutator in this case becomes

\[
C = \begin{pmatrix}
\frac{1}{2} & \cos^2(x) \Lambda(t) + \sin^2(x) \\
\cos^2(x) \Lambda(t) + \sin^2(x) & \frac{1}{2}
\end{pmatrix}.
\]

Therefore, the quantumness measure for the RTN channel turns out to be \( \mathcal{M}(\rho_a, \rho_b') = 2 \text{Tr}[C^\dagger C] = \cos^2(x) |\Lambda(t)|^2 + \sin^2(x) \), which upon maximizing over \( x \) leads to \( |\Lambda(t)|^2 \).

Example 2. Generalized depolarizing channel (GDC): The generalized depolarizing channel is represented by the following Kraus operators \( M_i = \sqrt{p_i} \sigma_i \) with \( i = 0, 1, 2, 3 \).

The action of this channel on state \( \rho_b \), Eq. (3), results in
thereby leading to quantumness measure
\[
\mathcal{M}(\rho_a, \rho_b') = (p_1 - p_2)^2 \cos^4(x) + \frac{1}{4}(p_1 - p_2)^2 \sin^2(x) \sin^2(2\phi) \\
+ \frac{1}{4} [ -2p_0 + p_1 + p_2 - (p_1 + p_2 - 2p_3) \cos(x)]^2 \\
+ (p_1 - p_2) \cos^2(x) \cos(2\phi)(2p_0 - p_1 - p_2) \\
+ (p_1 - p_2) \cos^2(x)(p_1 + p_2 - 2p_3) \cos(2x)) \cos(2\phi).
\]

Choosing \( x = \phi = 0 \), this gives \( \mathcal{M}(\rho_a, \rho_b') = (p_0 + p_1 - p_2 - p_3)^2 \). Note that if one starts with the states \( |0\rangle \) and \( (|0\rangle + |1\rangle)/\sqrt{2} \) which in Eq. (3) correspond to \( x = \phi = \xi = 0 \), then the final results need not be optimized for the cases considered here.

IV. EXPERIMENTAL RELEVANCE OF THE MEASURE

It is important to note that the quantity \( \mathcal{M}(\rho_a, \rho_b') \) can be given an experimental interpretation using an interferometric setup [35]. This useful technique can be easily incorporated to our purpose of quantifying quantumness of channels. One can write
\[
\mathcal{M}(\rho_a, \rho_b') = 4 \text{Tr}[(\rho_a)^2(\rho_b')^2] - (\rho_a \rho_b')^2.
\]

The two quantities \( \text{Tr}[(\rho_a)^2(\rho_b')^2] \) and \( \text{Tr}[(\rho_a \rho_b')^2] \) can be obtained from two separate measurements. The input state \( \rho = |0\rangle |0\rangle \otimes \rho_a \otimes \rho_b \otimes \rho_b' \), where \(|0\rangle \) is the control qubit, is subjected to the controlled unitary gate \( U \). This modifies the interference of the controlled qubit by the factor \( \text{Tr}[\rho U] = ve^{i\alpha} \); with \( v \) and \( \alpha \) being the visibility and phase shift of the interference fringes, respectively [36–39]. Two such schemes (corresponding to \( \text{Tr}[(\rho_a)^2(\rho_b')^2] \) and \( \text{Tr}[(\rho_a \rho_b')^2] \) ) lead to the quantumness \( \mathcal{M}(\rho_a, \rho_b') = 4(v_1 - v_2) \), where \( v_1 \) and \( v_2 \) correspond to the respective visibilities obtained by the action of relevant unitary gates. We motivate the present discussion by illustrating this notion on some of the channels discussed above.

(a) RTN: For RTN, the two visibilities (with \( x = \phi = 0 \)) correspond to
\[
\text{Tr}[(\rho_a \rho_b')^2] = \frac{1}{4}, \\
\text{Tr}[(\rho_a)^2(\rho_b')^2] = \frac{1}{4}(1 + |\Lambda(t)|^2).
\]

Making use of these in Eq. (11), we obtain \( \mathcal{M}(\rho_a, \rho_b') = |\Lambda(t)|^2 \), consistent with the definition in Eq. (2).

(b) GDC: For GDC, the two visibilities (with \( x = \phi = 0 \)) turn out to be
\[
\text{Tr}[(\rho_a \rho_b')^2] = \frac{1}{4}, \\
\text{Tr}[(\rho_a)^2(\rho_b')^2] = \frac{1}{4}[(p_0 + p_1)^2 + (p_2 + p_3)^2].
\]

These lead to the expression \( \mathcal{M}(\rho_a, \rho_b') = (p_0 + p_1 - p_2 - p_3)^2 \) in accord with the definition in Eq. (2).

What makes this approach particularly attractive is that here the quantumness of the channel can be experimentally determined.

V. RESULTS AND DISCUSSION

The quantumness of two arbitrary states \( \rho \) and \( \sigma \) can be identified with their incompatibility and quantified by \( \mathcal{M}(\rho, \sigma) \) as defined in Sec. (II). For a mixed initial diagonal state \( \rho_0 = \sum_i \lambda_i |i\rangle \langle i| \), which evolves to \( \rho_1 \) under some dynamics, the following inequality holds [40]
\[
\mathcal{M}(\rho_0, \rho_1) \leq F(\rho_0, \rho_1) \leq \frac{C_{t_1}(\rho_0, \rho_1)}{2}.
\]

Here, \( F(\rho_0, \rho_1) \) is the quantum Fisher information and \( C_{t_1}(\rho_0, \rho_1) \) is the \( t_1 \)-norm coherence; both well known measures of quantumness. The commutator based measure provides a lower bound and a reliable witness of quantumness.

In this work, we extend the approach of quantifying the quantumness of states, in terms of their incompatibility, to explore the quantumness of channels. This method involves starting with two states which are maximally non-commuting and subjecting one of them to a quantum channel. The incompatibility of the resulting output state with the input state (not subject to the channel) can be attributed to the degree of quantumness of the channel. We have computed the quantumness of various well known channels and compared them with the analogous estimation of quantumness from coherence based measure [24]. These are listed in Table (I). It is interesting to note that quantumness from the proposed measure is in good agreement with that with the coherence based measure [25]. This is consistent with our intuition as coherence is related to the off-diagonal elements of the density matrix as would be the cause for noncommutativity.
The quantum channels provide a way to describe the processes where pure states go over to the mixed ones. Therefore, it is natural to ask how well a quantum channel preserves the quantumness of the states which are subjected to it. Recently, a measure based on the $l_1$-norm coherence was introduced to quantify the quantumness of channels. In this work, we have addressed the problem by using an intuitive approach based on the incompatibility of the states. The quantumness of a system is identified with the mutual non-commutation of all its accessible states. We illustrated the approach developed here by considering various examples of quantum channels, both Markovian as well as non-Markovian, and found that our results are in good agreement with the coherence based measure. An added attraction of this method is that it between the states. It should be noted that in the case of GDC, the coherence based measure leads to quantumness $(p_0 - p_1)^2 + (p_2 - p_3)^2$, different from that obtained by the commutation based measure adapted here. This is consistent with Eq. (14).

To better understand this issue, we consider a recently introduced basis independent measure of coherence [41], which used maximally mixed state as the reference state. For a state $\rho \in \mathcal{H}_N$, where $\mathcal{H}_N$ is the Hilbert space of dimension $N$, the coherence is defined as

$$\mathcal{C}(\rho) = \sqrt{S\left(\frac{\rho + I}{N}\right) - S(\rho) + \log_2 N}. \quad (15)$$

Here, $I$ is the identity operator. The above definition of coherence is independent under arbitrary unitary transformations. For the output state of RTN and GDC channels, given in Eqs. (7) and (9), respectively, the basis independent measure of coherence gives

$$\mathcal{C}(\rho_\theta) = \begin{cases} \frac{\sqrt{N}}{2} & \text{RTN,} \\ \sqrt{2 - p_2 - p_3} & \text{GDC}. \end{cases} \quad (16)$$

A comparison with the corresponding expressions given by Eq. (10) and Table (I) brings out the similarity between the commutator based measures of quantumness and basis independent measure of coherence.

However, the attractive feature here is that the measure proposed does not require any complicated optimization prescription, and is also amenable to experimental determination. Further, from the cases of the RTN and NMD channels, it is evident that quantumness reflects the non-Markovian nature of the channel under consideration.

### VI. CONCLUSION

TABLE I. Various quantum channels, introduced at the beginning of Sec. (III), with their Kraus operators and the quantumness using commutation based measure $\mathcal{M}(\rho_\alpha, \rho_\theta')$. For the sake of comparision, the corresponding results based on the coherence based measure $Q_{C_\xi}$ [25] are also provided. Here, $\xi = \frac{1}{2}(\alpha - 1)^2(1 - \xi)^2$ and $\tau = \frac{5}{2}\sqrt{2\ln(\frac{2\pi}{6+4n+n^2})}$. For a state $\rho \in \mathcal{H}_N$, where $\mathcal{H}_N$ is the Hilbert space of dimension $N$, the coherence is defined as

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can be probed experimentally.

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