Error Estimates of Integral Deferred Correction Methods for Stiff Problems

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Abstract. In this paper, we present error estimates of integral deferred correction (InDC) methods constructed with stiffly accurate implicit Runge-Kutta methods when applied to stiff problems characterized by a small positive parameter $\varepsilon$. In our error estimates, we expand the global error in powers of $\varepsilon$ and show the coefficients are global errors of the InDC method applied to a sequence of differential algebraic systems. A study of these errors and of the remainder of the expansion yields sharp error bounds for the stiff problem. Numerical results for the van der Pol equation are presented. They confirm our theoretical prediction.

Keywords: Stiff Problems, Runge Kutta methods, Integral deferred correction methods, Differential algebraic systems.

1 Introduction

Deferred correction (DC) methods for solving initial value problems

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \in \mathbb{R}^N,$$ (1.1)

were investigated intensively [2] [13] [1]. An advantage of DC methods is that one can use a simple numerical method, for instance a first order method, to compute a solution with higher order accuracy. This is accomplished by using a lower order numerical method to solve a series of correction equations during each time step; in each iteration step, the order of the method increases. In [6], a new variation of deferred correction methods called spectral deferred correction (SDC) method was proposed. In SDC, the differential equation (1.1) is replaced with the corresponding Picard integral equation and a deferred correction procedure is applied to an integral formulation of the error equation in DC methods. It has been showed that SDC outperforms DC in many problems with promising numerical results [6]. In [4], the quadrature nodes in the proposed SDC methods are chosen to be Gauss-Lobatto, Gauss-Radau or Gauss-Legendre points for high order of accuracy. When the quadrature nodes are uniform, the SDC method is called the integral deferred correction (InDC) method. There are various SDC/InDC methods with different implementation strategies, e.g. in selecting time integrators in prediction and correction steps [14] [13] [11] [10] [9] [3] and in coupling with the Krylov subspace methods [10]. Under the InDC framework, it is shown in [3] that if an $r^{th}$ order integrator is used to solve the error equation, then the accuracy of the scheme increases by $r$ orders after each correction loop. This analysis has recently been extended in [8] for InDC methods constructed with implicit and semi-implicit integrators. In [4], the InDC method constructed with high order Runge-Kutta (RK) methods has been reformulated as a RK method, whose Butcher tableau has been explicitly constructed.

The main goal of this paper is to study the convergence behavior of the InDC method constructed using implicit RK methods of different orders for the prediction and correction steps, when applied to a special class of stiff problems containing a parameter $\varepsilon$ called singular perturbation problems (SPPs). A typical SPP has the form

$$y'(t) = f(y(t), z(t)), \quad \varepsilon z'(t) = g(y(t), z(t)),$$ (1.2)

where $y$ and $z$ are vectors and $\varepsilon > 0$ is the stiffness parameter. We call these vectors the differential component for $y$ and algebraic one for $z$. Classical books on this subject are [17] [15]. In system (1.2) we assume that $0 < \varepsilon \ll 1$ and $f$ and $g$ are sufficiently differentiable vector-valued functions. The functions $f$, $g$ and the initial values $y(0)$, $z(0)$ may depend smoothly on $\varepsilon$. For simplicity of notation, we suppress such dependence. We require that system (1.2) satisfies

$$\mu(g_z(y, z)) \leq -1,$$ (1.3)

in an $\varepsilon$-independent neighbourhood of the solution, where $\mu$ denotes the logarithmic norm with respect to some inner product. From a classical result in SPPs theory, the condition (1.3) guarantees the existence of

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an $\varepsilon$-expansion, whose coefficients are the sum of a smooth function of the independent variable $t$ and an exponentially decaying function of the stretched variable $\tau = t/\varepsilon$ (initial layer). The exponentially decaying function is not present if the initial values of system (1.2) (which depend on $\varepsilon$) are on the smooth solution, see Chap. VI.3 of [9] for more details. We thus suppose, in our analysis, that the initial values lie on the smooth solution, that $\varepsilon \ll H$, where $H$ is the time step size, and that the initial layer is over. In fact, arbitrary initial values introduce an initial layer in the solution. One possible way to overcome this difficulty is simply to ensure that the numerical method resolves the initial layer by taking small step size of $O(\varepsilon)$.

System (1.2) allows us to understand many phenomena observed for very stiff problems. Indeed, in [9] and in the original paper [8], the authors showed that most of the RK methods presented in the literature suffer from the phenomenon of the order reduction in the stiff regime. To this aim, we investigate the same phenomenon when it appears in the InDC framework. In the past, such order reduction has been numerically investigated without much theoretical justification [11, 3].

In this paper, we will study the global error of the InDC method when it is applied to SPPs of the form (1.2), in order to seek an understanding on the order reduction phenomenon appears for this method. First we consider InDC method constructed with the help of the backward Euler (BE) method, denoted as InDC-BE, and then with implicit RK (IRK) methods, denoted as InDC-IRK.

The main idea is to expand the error in powers of $\varepsilon$, whose coefficients are called error terms, and show convergence results for these error terms. Order reduction phenomenon exists for both differential and algebraic components in the InDC framework. Specifically, under suitable assumptions, the order of convergence for the first term in the $\varepsilon$-expansion of global error increases with high order if a high order RK method is applied in the correction steps of the InDC method; whereas the order of convergence for the second term in $\varepsilon$-expansion is determined by the stage order of the RK method for the prediction step. We focus our analysis on the InDC method using uniform quadrature nodes, but excluding the left-most endpoint. The uniform distribution of nodes is important to increase accuracy with the corresponding high order, when a high order RK method is applied in the correction steps for classical problems; we refer readers to [5] for details. The use of quadrature nodes excluding the left-most endpoint leads to an important stability condition for stiff problems, i.e. the method becomes L-stable if A-stable; we refer readers to [12] for details. We also remark that an important property for IRK method, called stiff accuracy (we will define it in the next section), is an important ingredient for our analysis both in the prediction and in the correction steps for the InDC method. We will show that, if this property is not satisfied, the corresponding InDC method becomes unstable and the numerical scheme diverges.

The paper is organized in the following way. In the rest of this introductory section, we briefly present existing classical local and global error estimates of IRK methods for SPPs introduced in [9] (for more details and explanations, consult the original paper [8]). In Section 2 we introduce the InDC-BE method for SPPs (1.2). In Section 3 main theoretical results are stated in the form of two Theorems; numerical evidence supporting these theoretical results are summarized and presented. In Section 4 we prove convergence results for the InDC-BE method. We organize the description of InDC-IRK methods, as well as the corresponding error estimates, into the appendix for better readability of the paper. Conclusions are given in Section 5. Throughout the paper, for classical concepts and convergence results related to RK methods applied to SPPs, we will cite the classical book on the subject [9] (with the Chapter numbering) from time to time.

1.1 IRK method applied to SPPs

In order to get more insight in the convergence estimates of InDC methods, it is useful to consider the convergence results about RK methods when applied to (1.2). We observe that when the parameter $\varepsilon$ in system (1.2) is small, the corresponding differential equation is stiff, and when $\varepsilon$ tends to zero, the differential equations become a differential algebraic system. The corresponding reduce system, i.e. $\varepsilon = 0$, is the differential algebraic equation (DAE)

$$
\begin{align*}
  y' &= f(y, z), \\
  0 &= g(y, z),
\end{align*}
$$

(1.4)

whose initial values are consistent if $0 = g(y_0, z_0)$. We assume that the Jacobian

$$
g_z(y, z) \quad \text{is invertible},
$$

(1.5)
in a neighbourhood of the solution of (1.4). This assumption guarantees the solvability of (1.4) and that the equation $g(y, z) = 0$ possesses a locally unique solution $z = G(y)$ (Implicit Function Theorem), which inserted
A differential-algebraic equation of index 1. For a definition of the index of differential algebraic problems, we refer to [7, 9].

Below we give a review of the main convergence results for IRK methods applied to SPPs [12] [8, 9]. Our discussions on these methods are based on notations introduced in Chap. VI.3 of [9]. Before we state the main results for IRK methods, we recall the definition of a RK method, the concept of classical order $p$ and stage order $q$.

We consider an IRK method applied to the SPP (1.2)

\[
\begin{pmatrix}
  y_{n+1} \\
  z_{n+1}
\end{pmatrix} = \begin{pmatrix}
  y_n \\
  z_n
\end{pmatrix} + h \sum_{i=1}^{s} b_i \begin{pmatrix}
  k_{ni} \\
  \ell_{ni}
\end{pmatrix},
\]

(1.7)

where

\[
\begin{pmatrix}
  k_{ni} \\
  \ell_{ni}
\end{pmatrix} = \begin{pmatrix}
  f(Y_{ni}, Z_{ni}) \\
  g(Y_{ni}, Z_{ni})
\end{pmatrix},
\]

(1.8)

and the internal stages are given by

\[
\begin{pmatrix}
  Y_{ni} \\
  Z_{ni}
\end{pmatrix} = \begin{pmatrix}
  y_n \\
  z_n
\end{pmatrix} + h \sum_{j=1}^{s} a_{ij} \begin{pmatrix}
  k_{nj} \\
  \ell_{nj}
\end{pmatrix}.
\]

(1.9)

Such method is characterized by the coefficient matrix $A = (a_{ij})$ and vectors $c = (c_1, ..., c_s)^T$, $b = (b_1, ..., b_s)^T$. They can be represented by a tableau in the usual Butcher notation,

\[
c = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} b \end{pmatrix}^T.
\]

(1.10)

The coefficients $c$ are given by the usual relation $c_i = \sum_{j=1}^{s} a_{ij}$.

In this paper, we let $p$ denote the classical order of the method, when it is applied to a non-stiff equation. We let $q$ denote the stage order, which is an important concept for stiff differential equations. It is the largest number $q$, such that the conditions

\[
C(q) : \sum_{j=1}^{s} a_{ij}c_j^{k-1} = c_i^k, \quad i = 1, \cdots, s, \quad \text{for} \quad k = 1, \ldots, q,
\]

(1.11)

holds, see Chap. IV.5 of [9]. This is equivalent to the fact that $q = \min(q_1, ..., q_s)$ where, for a problem $y'(t) = f(t, y(t))$, with $0 \leq t \leq T$ and a smooth function $f$, the internal stages are $O(h^{q_i+1})$-approximations to the exact solution at $c_i h$, i.e. $y(t_n + c_i h) - Y_{ni} = O(h^{q_i+1})$ where $Y_{ni} = y(t_n) + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_{nj})$, for $1 \leq i \leq s$. For example, for an $s$-stage diagonally IRK (DIRK) method, the stage order is 1. Of special importance in this paper are stiffly accurate methods.

**Definition 1.1.** An RK method is called stiffly accurate if $b^T = e_s^T A$ with $e_s^T = (0, ..., 0, 1)$, i.e., methods for which the numerical solution is identical to the last internal stage.

**Remark 1.2.** The importance of this condition appears when we treat SPPs and differential algebraic equations. In particular, this property is important for the $L$-stability of the method. To see this, let $R(\infty) = \lim_{z \to \infty} R(z)$, with $R(z) = 1 + z b^T (I - z A)^{-1} \mathbf{1}$ being the stability function of an implicit scheme, where $b^T = (b_1, ..., b_s)$ and $\mathbf{1} = (1, ..., 1)^T$. If the $A$ matrix is invertible, then $R(\infty) = 1 - \sum_{i,j=1}^{s} b_{ij} \omega_{ij}$, with $\omega_{ij}$ being elements of the inverse of $(a_{ij})$. If the IRK method is stiffly accurate, then $R(\infty) = 0$. This makes $A$-stable methods $L$-stable. From now on, we use InDC SA-IRK to denote an InDC method constructed with stiffly accurate IRK methods.

We now suppose that the matrix $A$ is invertible, then one obtains from (1.9)

\[
h\ell_{ni} = \sum_{j=1}^{s} \omega_{ij} (Z_{nj} - z_n),
\]

(1.12)
where \( \omega_{ij} \) are elements of the inverse of \( A \). Now inserting (1.12) into the numerical solution \( z_{n+1} \) and putting \( \varepsilon = 0 \) in (1.8), we obtain

\[
Y_{ni} = y_n + \sum_{j=1}^{s} a_{ij} f(Y_{nj}, z_{nj}),
\]

(1.13)

\[
g(Y_{ni}, Z_{ni}) = 0,
\]

(1.14)

\[
y_{n+1} = y_n + \sum_{i=1}^{s} b_i f(Y_{ni}, Z_{ni}),
\]

(1.15)

\[
z_{n+1} = R(\infty)z_n + \sum_{i,j=1}^{s} b_{ij} \omega_{ij} Z_{nj},
\]

(1.16)

We note that equations (1.13)-(1.16) represent the numerical method for solving the reduced system (1.4). From (1.14) we have \( Z_{ni} = G(Y_{ni}) \) (Implicit Function Theorem). If the method is stiffly accurate, we have \( y_{n+1} = Y_{ns} \) and \( z_{n+1} = Z_{ns} = G(Y_{ns}) = G(y_{n+1}) \). Then for the numerical solutions we have,

\[
g(y_{n+1}, z_{n+1}) = 0.
\]

(1.17)

In this case, the solution from (1.13)-(1.15) together with (1.17) for solving the system (1.4), is identical to the solution for solving (1.6) obtained by the same RK method. Therefore we have,

\[
y_n - y(t_n) = O(h^p).
\]

(1.18)

Furthermore, by \( z_{n+1} = G(y_{n+1}) \) and by the Lipschitz condition for \( G \), it follows

\[
z_n - z(t_n) = G(y_n) - G(y(t_n)) = O(h^p).
\]

(1.19)

We summarize these results by the following theorem.

**Theorem 1.3.** (Theorem 1.1 part (a) in Chap. VI. 1 of [9]) Suppose that the system (1.4) satisfies (1.5) in a neighborhood of the exact solution and assume that the initial values are consistent. Consider a stiffly accurate IRK method of order \( p \), which has an invertible matrix \( A \) in the Butcher notation (1.10). Then the numerical solutions of (1.13)-(1.15) and (1.17) have global errors

\[
y_n - y(t_n) = O(h^p), \quad z_n - z(t_n) = O(h^p),
\]

(1.20)

for \( t_n - t_0 = nh \leq \text{Const} \).

Now we review the main result obtained in Chap. VI.3 of [9] about the error analysis of IRK methods for SPPs. We perform an asymptotic expansion of smooth solutions of system (1.2) and similarly for the numerical solutions of an IRK method applied to (1.2). The errors of the \( y \) and \( z \)-component are formally considered as

\[
y_n - y(t_n) = \sum_{\nu \geq 0} \varepsilon^\nu (y_{n,\nu} - y_\nu(t_n)), \quad z_n - z(t_n) = \sum_{\nu \geq 0} \varepsilon^\nu (z_{n,\nu} - z_\nu(t_n)),
\]

(1.21)

where values \( y_\nu(t) \), \( z_\nu(t) \) are coefficients of the \( \varepsilon \)-expansion of the smooth solution for (1.2) and coefficients \( y_{n,0}, z_{n,0}, y_{n,1}, z_{n,1}, \ldots \), represent the numerical solution of the RK method applied to DAEs of arbitrary order. Furthermore, the first differences \( y_{n,0} - y_0(t_n) \) and \( z_{n,0} - z_0(t_n) \) in the expansion (1.21) are the global errors of the RK method applied to the reduced system (1.4), i.e. system of index 1, and the error estimates are summarized in Theorem 1.3. The other differences for \( \nu > 0 \) in (1.21) are related to the numerical solutions of the RK method when applied to the DAEs of higher index. Finally, we state the main result of global errors estimate (1.21) in Theorem 1.4 below for IRK methods when applied to SPPs. For details, see Chap. VI. 3 in [9].

**Theorem 1.4.** Consider the stiff problem (1.2), (1.3) with initial values \( y(0), z(0) \) admitting a smooth solution. Apply the IRK method (1.7)-(1.9) of classical order \( p \) and stage order \( q \), \( 1 \leq q < p \). Assume that the method is \( A \)-stable, that the stability function satisfies \( |R(\infty)| < 1 \) and that the eigenvalues of the coefficient matrix \( A \) have positive real parts. Then for a fixed constant \( c > 0 \), the global error satisfies, for \( \varepsilon \leq ch \)

\[
y_n - y(t_n) = O(h^p) + O(\varepsilon h^{q+1}), \quad z_n - z(t_n) = O(h^{q+1}).
\]

(1.22)
If, in addition, the method is stiffly accurate, we have

$$z_n - z(t_n) = O(h^p) + O(\varepsilon h^q).$$  \hfill (1.23)

The estimates hold uniformly for $h \leq h_0$ and $nh \leq \text{Const}.$

A complete analysis for the convergence of IRK methods for differential algebraic systems of higher index is given in [9]. In this paper, for our analysis, we will require the results given in Lemma 4.4, Theorem 4.5 and Theorem 4.6 in Chap. VII.4 of [9]. Below we summarize these optimal error estimates. From Lemma 4.4 in Chap. VII.4 of [9], we obtain the local error estimates

$$\delta y_h(t) := y_1 - y(t_n + h) = O(h^{q+1})$$  \hfill (1.24)

and

$$\delta z_h(t) := z_1 - z(t_n + h) = O(h^q).$$  \hfill (1.25)

From these estimates, general global convergence results for the $y$- and $z$-component follow from Theorem 4.5 and 4.6. A summary of convergence results about local and global error for some important RK methods are collected in Table 4.1, Chap. VII.4 of [9]. Here, as an example, we consider DIRK methods with $p \geq 2$. Such methods have stage order $q = 1$, then by (1.24) and (1.25) we have:

$$\delta y_h(t) = O(h^2), \quad \delta z_h(t) = O(h) \quad \text{or},$$

$$\delta y_h(t) = O(h^2), \quad \delta z_h(t) = O(h) \quad \text{if the method is stiffly accurate},$$  \hfill (1.26)

for the local error and by Theorem 4.5 and 4.6 in Chap. VII.4 of [9], and

$$y_n - y(t_n) = O(h^2) \quad z_n - z(t_n) = O(h),$$  \hfill (1.27)

for the global error. We note that BE method has $p = q = 1$, then the local error is $\delta y_h(t) = O(h^2)$ and the global error is $y_n - y(t_n) = O(h)$, i.e. the method, when applied to a system of index 2, maintains the classical order.

## 2 InDC Formulations Applied to SPPs

In this section, we consider InDC-IRK method for the solution of SPPs written in the form of (1.2). The use of uniform nodes is important for the increase of high order of accuracy, if high order RK methods are used in correction loops. This is related to the concept of “smoothness of the rescaled error vector”, when we apply high order RK methods in correction loops, for more details see [5]. The use of quadrature nodes excluding the left-most endpoint leads to an important stability condition for stiff problems, i.e. the method is L-stable if A-stable with $R(\infty) = 0$, see [12]. Then, in this paper, we consider the InDC methods with uniform nodes excluding the left-most endpoint.

### 2.1 InDC Framework

We consider InDC procedure [6] applied to a SSP,

$$y'(t) = f(y, z), \quad y(t_0) = y_0,$$

$$z'(t) = g(y, z), \quad z(t_0) = z_0.$$  \hfill (2.1)

The time interval $[0, T]$ is discretized into intervals $[t_n, t_{n+1}]$, $n = 0, 1, ..., N - 1$ such that

$$0 = t_0 < t_1 < t_2 < ... < t_n < ... < t_N = T,$$

with the step size $H$. Then, each interval $[t_n, t_{n+1}]$ is discretized again into $M$ uniform subintervals with quadrature nodes denoted by

$$t_n = \tau_0 < \tau_1 < \cdots < \tau_M = t_{n+1}.$$

Let $h = \frac{H}{M}$ be the size of a substep. In this paper, the interval $[t_n, t_{n+1}]$ will be referred as a time step while a subinterval $[\tau_m, \tau_{m+1}]$ will be referred as a substep. We remark that the size of time interval $[t_n, t_{n+1}]$ may vary
as the InDC method is a one-step, multi-stage method. We assume the InDC quadrature nodes are uniform, which is a crucial assumption for high order improvement in accuracy, when we apply general high order IRK methods in prediction and correction steps for a classical ODE system \((1.1)\), (see discussions in [5]). We also note that since \(h = \frac{H}{M}\), we will use \(O(h^p)\) and \(O(H^p)\) interchangeably throughout the paper.

Let’s assume we have obtained numerical solutions \(y_m^{(0)}\) and \(z_m^{(0)}\) approximating the exact solution at \(\tau_m\) by using a low order numerical method for \((2.1)\). Here superscript \((0)\) is used to denote the prediction step in the InDC method. We build continuous polynomial interpolants \(\hat{y}(t)\) and \(\hat{z}(t)\) interpolating these discrete values. Now we define the error functions

\[
epsilon(t) = y(t) - \hat{y}(t), \quad d(t) = z(t) - \hat{z}(t).
\]

Note that \(\epsilon(t)\) and \(d(t)\) are not polynomials in general. We specify the residual function with respect to \(y\) and \(z\) via the following set of differential equations

\[
\delta(t) = f(\hat{y}(t), \hat{z}(t)) - (\hat{y}(t)), \\
\rho(t) = g(\hat{y}(t), \hat{z}(t)) - (\hat{z}(t)).
\]

Thus, by subtracting \((2.4)\) from \((2.1)\), the error equations about the error functions \((2.3)\) become

\[
(e(t)) - \delta(t) = f(\epsilon(t) + \hat{y}(t), d(t) + \hat{z}(t)) - f(\hat{y}(t), \hat{z}(t)), \\
e(\rho(t)) = g(\epsilon(t) + \hat{y}(t), d(t) + \hat{z}(t)) - g(\hat{y}(t), \hat{z}(t)).
\]

Suppose that we have obtained approximate solutions \(\epsilon_m^{(0)}\) and \(d_m^{(0)}\) at \(\tau_m\) by using a low order numerical method for error equations\((2.5)\), the numerical solution can then be improved as

\[
\hat{y}_m^{(1)} = \hat{y}_m^{(0)} + \epsilon_m^{(0)}, \quad \hat{z}_m^{(1)} = \hat{z}_m^{(0)} + d_m^{(0)}, \quad \forall m = 0, \cdots M.
\]

Such correction procedures can be repeated in each local time step \([t_n, t_{n+1}]\). In summary, the strategy of InDC methods is to use a simple numerical method to compute numerical solutions \(\hat{y}(t)\) and \(\hat{z}(t)\) as prediction, and then to solve a series of correction equations in the integral form based on equations \((2.5)\), each correction improves the accuracy of numerical solutions from the previous iteration.

**Remark 2.1.** (About notations.) In our description of InDC, we let \(y_m, z_m, \epsilon_m^{(k)}, d_m^{(k)}\) denote the exact solutions and exact error functions (without hat); and let \(\hat{y}_m^{(k)}, \hat{z}_m^{(k)}, \epsilon_m^{(k)}, d_m^{(k)}\) denote the numerical approximations (with hat) to the exact solutions and error functions. We use subscript \(m\) to denote the location \(t = \tau_m\) and use superscript \((k)\) to denote the prediction \((k = 0)\) and correction loops \((k = 1, \cdots)\). We let \(\tilde{\cdot}\) denote the vector on InDC quadrature nodes, for example, \(\tilde{\hat{y}} = (y_1, \cdots, y_M)\).

### 2.2 InDC-BE method

In this subsection, we consider InDC-BE method for the solution of system \((2.1)\). We use uniformly distributed quadrature nodes \(\tau_1, \cdots, \tau_M\) from equation \((2.2)\) excluding the left-most endpoint.

1. (Prediction step) Use a BE discretization to compute

\[
\tilde{y}(0) = (\hat{y}_1^{(0)}, \cdots, \hat{y}_M^{(0)}).
\]

as the approximation of the exact solution \(\hat{y} = (y_1, \cdots, y_M)\) for \((2.1)\) at quadrature nodes \(\tau_1, \cdots, \tau_M\).

We make the same for the \(z\)-component. This gives

\[
\tilde{z}_m^{(0)} = \hat{z}_m^{(0)} + hf(\hat{y}_{m+1}, \hat{z}_{m+1}), \\
\epsilon_{m+1}^{(0)} = \epsilon_m^{(0)} + hg(\hat{y}_{m+1}, \hat{z}_{m+1}),
\]

for \(m = 0, 1, \cdots M - 1\).

2. (Correction loop). For \(k = 1, \cdots, K\) \((K\) is the number of the correction step), Let \(\hat{y}^{(k-1)}\) and \(\hat{z}^{(k-1)}\) denote the numerical solutions at the \((k - 1)^{th}\) sequence correction.
(a) Denote the error function at the \((k-1)\)th correction \(e^{(k-1)}(t) = y(t) - \hat{y}^{(k-1)}(t)\), where \(y(t)\) is the exact solution and \(\hat{y}^{(k-1)}(t)\) is a \((M-1)\)th order polynomial interpolating \(\hat{y}^{(k-1)}\) at quadrature nodes \(\tau_1, ..., \tau_M\). Similarly denote \(d^{(k-1)}(t) = z(t) - \hat{z}^{(k-1)}(t)\). Let \(\delta^{(k-1)}(t)\) and \(\rho^{(k-1)}(t)\) be defined by equation \(\text{(2.4)}\), but with the upper script \(0\) replaced with \((k-1)\). We compute the numerical error vector \(\vec{e}^{(k-1)} = (\hat{e}_1^{(k-1)}, ..., \hat{e}_M^{(k-1)})\) with \(\hat{e}_m^{(k-1)}\) approximating \(e^{(k-1)}(\tau_m)\) by applying a BE method to the integral form of \(\text{(2.5)}\),

\[
\hat{e}_m^{(k+1)} = \hat{e}_m^{(k)} + h\Delta f_{m+1}^{(k)} + \int_{\tau_m}^{\tau_{m+1}} \delta^{(k)}(s)ds,
\]

where

\[
\Delta f_{m+1}^{(k)} = f(y_m^{(k-1)} + e_m^{(k-1)} + \delta_m^{(k-1)}) - f(y_m^{(k-1)} + \hat{e}_m^{(k-1)}),
\]

\[
\Delta g_{m+1}^{(k)} = g(y_m^{(k-1)} + e_m^{(k-1)} + \delta_m^{(k-1)}) - g(y_m^{(k-1)} + \hat{e}_m^{(k-1)}).
\]

and

\[
\int_{\tau_m}^{\tau_{m+1}} \delta^{(k)}(s)ds = \int_{\tau_m}^{\tau_{m+1}} f(\hat{y}(s), \hat{z}(s), s)ds - \hat{y}_{m+1}^{(k-1)} + \hat{y}_{m}^{(k-1)},
\]

\[
\int_{\tau_m}^{\tau_{m+1}} \rho^{(k)}(s)ds = \int_{\tau_m}^{\tau_{m+1}} g(\hat{y}(s), \hat{z}(s), s)ds - \hat{z}_{m+1}^{(k-1)} + \hat{z}_{m}^{(k-1)}.
\]

The integral term \(\int_{\tau_m}^{\tau_{m+1}}\) in equations \(\text{(2.9)}\) are approximated by a numerical quadrature. Especially, let \(S\) be the integration matrix; its \((m, k)\) element is

\[
S_{m,k} = \frac{1}{h} \int_{\tau_m}^{\tau_{m+1}} \alpha_k(s)ds, \quad \text{for} \quad m = 0, \ldots, M-1, \quad k = 1, \ldots, M,
\]

where \(\alpha_k(s)\) is the Lagrangian basis function based on the node \(\tau_k\). Let

\[
S^m = \sum_{j=1}^{M} S^{m,j} f(y_j, z_j),
\]

then

\[
hS^m(\bar{f}) - \int_{\tau_m}^{\tau_{m+1}} f(y(s), z(s))ds = O(h^{M+1}),
\]

for any smooth function \(f\). In other words, the quadrature formula given by \(hS^m(\bar{f})\) approximates the exact integration with \((M+1)\) order of accuracy locally.

(b) Update the approximate solutions \(\bar{y}^{(k)} = \bar{y}^{(k-1)} + \bar{e}^{(k-1)}\) and \(\bar{z}^{(k)} = \bar{z}^{(k-1)} + \bar{a}^{(k-1)}\).

Remark 2.2. Using the notation introduced in equation \(\text{(2.10)}\), we get from equation \(\text{(2.7)}\) and \(\text{(2.9)}\),

\[
\hat{y}_{m+1}^{(k)} = \hat{y}_m^{(k)} + h\Delta f_{m+1}^{(k)} + hS^m(\hat{f}^{(k)}),
\]

\[
\hat{z}_{m+1}^{(k)} = \hat{z}_m^{(k)} + h\Delta g_{m+1}^{(k)} + hS^m(\hat{g}^{(k)}).
\]

Remark 2.3. Since we consider the nodes excluding the left most quadrature point \(t_0\), the order of approximation for integration/interpolation will be one order lower than the usual one considered in \(\text{[4] 5}\).

Remark 2.4. The InDC-BE described above, can be generalized to the InDC-IRK method, for solving SPPs \(\text{[2] 1}\). To avoid heavy notations from the InDC-IRK method and for a better presentation of the paper, we organize the description of InDC-IRK method and the corresponding error estimates in Appendix.

2.3 \(\varepsilon\)-asymptotic expansion

In this section, we introduce the \(\varepsilon\)-asymptotic expansion of the exact and numerical solution for system \(\text{(2.1)}\). These \(\varepsilon\)-expansion will be useful to study the behavior of the local error for the InDC method. Therefore, assuming that the initial values are on the smooth solution, the following \(\varepsilon\)-expansion of the form

\[
y(t) = \sum_{\nu=0}^{\infty} y_{\nu}(t)\varepsilon^\nu, \quad z(t) = \sum_{\nu=0}^{\infty} z_{\nu}(t)\varepsilon^\nu,
\]

\[\text{(2.12)}\]

\[\text{Page 7}\]
holds. Furthermore we note that a sequence of DAEs arise in the study of (2.1). In fact, the coefficients in the expansion (2.12) are the solutions of DAEs of different indices, for more details see Chap. VI.3 of [3].

In order to show this, inserting the $\varepsilon$-expansion of the exact solution (2.12) into (2.1) and collecting terms of equal powers of $\varepsilon$, it yields

\begin{equation}
\varepsilon^0 : \begin{cases} 
y_0 = f(y_0, z_0) \\
0 = g(y_0, z_0) 
\end{cases}, \quad (2.13)
\end{equation}

\begin{equation}
\varepsilon^1 : \begin{cases} 
y'_1 = f_y(y_0, z_0)y_1 + f_z(y_0, z_0)z_1 \equiv F_1 \\
z'_0 = g_y(y_0, z_0)y_1 + g_z(y_0, z_0)z_1 \equiv G_1 
\end{cases}, \quad (2.14)
\end{equation}

\begin{equation}
\varepsilon^n : \begin{cases} 
y'_n = f_y(y_0, z_0)y_n + f_z(y_0, z_0)z_n + \phi_n(y_0, z_0, \cdots, z_{n-1}, z_{n-1}) = F_n \\
z'_{n-1} = g_y(y_0, z_0)y_n + g_z(y_0, z_0)z_n + \psi_n(y_0, z_0, \cdots, z_{n-1}, z_{n-1}) = G_n 
\end{cases}, \quad (2.15)
\end{equation}

with initial values $y_n(0), z_n(0)$ known from (2.12). We observe that system (2.13) under the condition (1.5) is a DAE of index 1. According to [9], if we consider (2.13) and (2.14) together, we have a differential algebraic system of index 2. In general (2.13)-(2.15) is a differential algebraic system of index $\nu$.

Now we assume the $\varepsilon$-expansion of the numerical solution at the $k^{th}$ correction step of the InDC-BE method as,

\begin{equation}
\hat{y}_m^{(k)} = \sum_{\nu=0}^{\infty} \hat{y}_{m,\nu}^{(k)} \varepsilon^\nu, \quad \hat{z}_m^{(k)} = \sum_{\nu=0}^{\infty} \hat{z}_{m,\nu}^{(k)} \varepsilon^\nu. \quad (2.16)
\end{equation}

The case of $k = 0$ corresponds to the prediction step of InDC method. By plugging the $\varepsilon$-expansion of numerical solution (2.16) into the numerical scheme (2.6)-(2.9), and collecting terms of equal powers of $\varepsilon$, we have the following:

- for the prediction step $(k = 0)$

\begin{equation}
\varepsilon^0 : \begin{cases} 
\hat{y}_{m+1,0}^{(0)} = \hat{y}_{m,0}^{(0)} + h f(\hat{y}_{m,0}^{(0)}, \hat{z}_{m,0}^{(0)}), \\
0 = g(\hat{y}_{m+1,0}^{(0)}, \hat{z}_{m+1,0}^{(0)}), 
\end{cases}, \quad (2.17)
\end{equation}

\begin{equation}
\varepsilon^1 : \begin{cases} 
\hat{y}_{m+1,1}^{(0)} = \hat{y}_{m,1}^{(0)} + h \hat{F}_{m+1,1}, \\
\hat{z}_{m+1,0}^{(0)} = \hat{z}_{m,0}^{(0)} + h \hat{G}_{m+1,1}, 
\end{cases}, \quad (2.18)
\end{equation}

where

\begin{equation}
\begin{cases}
\hat{F}_{m+1,1} = f_y(\hat{y}_{m,0}^{(0)}, \hat{z}_{m,0}^{(0)}) \hat{y}_{m+1,1}^{(0)} + f_z(\hat{y}_{m,0}^{(0)}, \hat{z}_{m,0}^{(0)}) \hat{z}_{m+1,1}^{(0)}, \\
\hat{G}_{m+1,1} = g_y(\hat{y}_{m,0}^{(0)}, \hat{z}_{m,0}^{(0)}) \hat{y}_{m+1,1}^{(0)} + g_z(\hat{y}_{m,0}^{(0)}, \hat{z}_{m,0}^{(0)}) \hat{z}_{m+1,1}^{(0)},
\end{cases} \quad (2.19)
\end{equation}

- for the correction steps $(k \geq 1)$

\begin{equation}
\varepsilon^0 : \begin{cases} 
\hat{y}_{m+1,0}^{(k)} = \hat{y}_{m,0}^{(k)} + h \Delta f_{m+1,0}^{(k-1)} + h S_m f(\hat{y}_{m,0}^{(0)}, \hat{z}_{m,0}^{(0)}), \\
0 = h \Delta g_{m+1,0}^{(k-1)} + h S_m g(\hat{y}_{m,0}^{(0)}, \hat{z}_{m,0}^{(0)}), 
\end{cases}, \quad (2.20)
\end{equation}

\begin{equation}
\varepsilon^1 : \begin{cases} 
\hat{y}_{m+1,1}^{(k)} = \hat{y}_{m,1}^{(k)} + h \Delta \hat{F}_{m+1,1}^{(k-1)} + h S_m \hat{F}(\hat{y}_{m,0}^{(0)}, \hat{z}_{m,0}^{(0)}), \\
\hat{z}_{m+1,0}^{(k)} = \hat{z}_{m,0}^{(k)} + h \Delta \hat{G}_{m+1,1}^{(k-1)} + h S_m \hat{G}(\hat{y}_{m,0}^{(0)}, \hat{z}_{m,0}^{(0)}), 
\end{cases}, \quad (2.21)
\end{equation}

where in (2.20),

\begin{equation}
\begin{cases}
\Delta f_{m,0}^{(k-1)} = f(\hat{y}_{m,0}^{(k)}, \hat{z}_{m,0}^{(k)}) - f(\hat{y}_{m,0}^{(k)}, \hat{z}_{m,0}^{(k)}), \\
\Delta g_{m,0}^{(k-1)} = g(\hat{y}_{m,0}^{(k)}, \hat{z}_{m,0}^{(k)}) - g(\hat{y}_{m,0}^{(k)}, \hat{z}_{m,0}^{(k)}),
\end{cases} \quad (2.22)
\end{equation}

and in (2.21),

\begin{equation}
\Delta \hat{F}_{m+1,1}^{(k-1)} = \hat{F}_{m+1,1}^{(k-1)} - \hat{F}_{m+1,1}^{(k-1)} \\
= \left( f_y(\hat{y}_{m+1,0}^{(k)}, \hat{z}_{m+1,0}^{(k)}) \hat{y}_{m+1,1}^{(k)} + f_z(\hat{y}_{m+1,0}^{(k)}, \hat{z}_{m+1,0}^{(k)}) \hat{z}_{m+1,1}^{(k)} \right) \\
- \left( f_y(\hat{y}_{m,0}^{(k)}, \hat{z}_{m,0}^{(k)}) \hat{y}_{m+1,1}^{(k)} + f_z(\hat{y}_{m,0}^{(k)}, \hat{z}_{m,0}^{(k)}) \hat{z}_{m+1,1}^{(k)} \right), \quad (2.23)
\end{equation}
where

\[ F_{m+1,1}^{(k)} = f_y(y_{m+1,0}, \hat{z}_{m+1,0})y_{m+1,1}^{(k)} + f_z(y_{m+1,0}, \hat{z}_{m+1,0})\hat{z}_{m+1,1}^{(k)}. \] (2.24)

We note that both equations (2.17), (2.18) for the prediction step \((k = 0)\), and equations (2.20), (2.21) for the correction step \((k \geq 1)\), are consistent discretizations of equation (2.13)-(2.14). It is possible to generalize the \(\varepsilon\)-asymptotic expansion to \(\varepsilon^\nu (\nu \geq 2)\), but we skip this to avoid heavy notations.

Finally, we assume the \(\varepsilon\)-expansion of error function \(e^{(k)}(t), d^{(k)}(t)\) at the \(k\)th iteration to be

\[
\begin{pmatrix}
  e_{m}^{(k)} \\
  d_{m}^{(k)}
\end{pmatrix} = \begin{pmatrix}
  \sum_{\nu=0}^{\infty} e_{m,\nu}^{(k)}\varepsilon^\nu \\
  \sum_{\nu=0}^{\infty} d_{m,\nu}^{(k)}\varepsilon^\nu
\end{pmatrix} = \begin{pmatrix}
  \sum_{\nu=0}^{\infty} (y_{m,\nu} - y_{m,\nu}^{(k)})\varepsilon^\nu \\
  \sum_{\nu=0}^{\infty} (z_{m,\nu} - z_{m,\nu}^{(k)})\varepsilon^\nu
\end{pmatrix}.
\] (2.25)

Similarly, we consider the \(\varepsilon\)-expansion of numerical approximations of error functions \(e^{(k)}(t), \hat{d}^{(k)}(t)\) at the \(k\)th iteration

\[
\begin{pmatrix}
  \hat{e}_{m}^{(k)} \\
  \hat{d}_{m}^{(k)}
\end{pmatrix} = \begin{pmatrix}
  \sum_{\nu=0}^{\infty} \hat{e}_{m,\nu}^{(k)}\varepsilon^\nu \\
  \sum_{\nu=0}^{\infty} \hat{d}_{m,\nu}^{(k)}\varepsilon^\nu
\end{pmatrix} = \begin{pmatrix}
  \sum_{\nu=0}^{\infty} (y_{m,\nu}^{(k+1)} - y_{m,\nu})\varepsilon^\nu \\
  \sum_{\nu=0}^{\infty} (z_{m,\nu}^{(k+1)} - z_{m,\nu})\varepsilon^\nu
\end{pmatrix}.
\] (2.26)

Combining (2.25) and (2.26), we get with \(k, \nu \geq 0, m = 0, \ldots, M,\)

\[
e_{m,\nu}^{(k)} = \hat{e}_{m,\nu}^{(k)} + e_{m,\nu}^{(k+1)}, \quad d_{m,\nu}^{(k)} = \hat{d}_{m,\nu}^{(k)} + d_{m,\nu}^{(k+1)}.\] (2.27)

Remark 2.5. Similar \(\varepsilon\)-asymptotic expansions can be given for the numerical solutions of the InDC-IRK method. Again, to avoid heavy notations, we organize them in Appendix.

3 Main results and numerical evidence

In this section, we present the main theoretical results in the form of theorems, and provide numerical evidence supporting the main theorems. We will provide a rigorous mathematical proof in the next section.

3.1 Main results

The aim of this section is to present convergence results of the InDC-BE and InDC-IRK method when applied to (1.1).

Theorem 3.1. Consider the stiff system (1.2), (1.3) with initial values \(y(0), z(0)\) admitting a smooth solution. Consider the InDC-BE method constructed with \(M\) uniformly distributed quadrature nodes excluding the leftmost point and \(K\) correction steps. Then the global error after \(K\) correction satisfies,

\[
e_n^{(K)} = y_n^{(K)} - y(t_n) = \mathcal{O}(H^{\min(K+1,M)}) + \mathcal{O}(\varepsilon H),
\]

\[
d_n^{(K)} = z_n^{(K)} - z(t_n) = \mathcal{O}(H^{\min(K+1,M)}) + \mathcal{O}(\varepsilon H),\] (3.1)

for \(\varepsilon \leq cH\) and for any fixed constant \(c > 0\), where \(H = Mh\) is one InDC time step. The estimates hold uniformly for \(H \leq H_0\) and \(nH \leq \text{Const}.\)

Theorem 3.2. Consider the stiff system (1.2), (1.3) with initial values \(y(0), z(0)\) admitting a smooth solution. Consider the InDC method constructed with \(M\) uniformly distributed quadrature nodes excluding the leftmost point and a stiffly accurate IRK method of order \(p^{(0)}\), stage order \(q^{(0)}\) with \((q^{(0)} < p^{(0)})\) for the prediction step. Apply IRK methods of different classical orders \((p^{(1)}, p^{(2)}, \ldots, p^{(K)})\) in the correction loops, \(k = 1, \ldots, K\). Assume that each of these IRK methods in the correction steps is stiffly accurate.

Then the global error after \(K\) correction loops satisfies the following estimates

\[
e_n^{(K)} = y_n^{(K)} - y(t_n) = \mathcal{O}(H^{\min(S_p,M)}) + \mathcal{O}(\varepsilon H^{s_p^{(0)})}),
\]

\[
d_n^{(K)} = z_n^{(K)} - z(t_n) = \mathcal{O}(H^{\min(S_p,M)}) + \mathcal{O}(\varepsilon H^{s_p^{(0)})},\] (3.2)

for \(\varepsilon \leq cH\) and for any fixed constant \(c > 0\), where \(S_p = \sum_{k=0}^{K} p^{(k)}\), and \(H = Mh\) is one InDC time step. The estimates hold uniformly for \(H \leq H_0\) and \(nH \leq \text{Const}.\)
We note that (2.25), but replacing $m$ with $n$, can be adopted to represent the $\varepsilon$-expansion of the global error functions $e^{(K)}_n$ and $d^{(K)}_n$ at the $K$-th correction step, where $e^{(K)}_{n,\nu}$ and $d^{(K)}_{n,\nu}$ for $\nu = 0, 1, \cdots$, are the global errors of InDC SA-IRK method applied to the differential algebraic systems of different indices (2.13)-(2.15). In section 4, we only prove Theorem 3.1 for estimating $e^{(K)}_{n,\nu}$ and $d^{(K)}_{n,\nu}$ with $\nu = 0, 1$ for the InDC-BE method. To avoid heavy notations and technical details, we prove Theorem 3.2 for general InDC-IRK methods and estimate the remainder of the expansion in Appendix.

3.2 Numerical evidence

We present some numerical evidence of the estimates given in Theorem 3.1 and Theorem 3.2. Below, we consider the following InDC methods constructed with $M$ quadrature points.

- The InDC-BE method with $k$ correction steps (InDC-BE-M-k). The BE method has order $p = 1$ and stage order $q = 1$.

- The InDC method constructed with a second order stiffly accurate DIRK method in $k$ correction steps (InDC-DIRK2-SA-M-k). The second order DIRK method ($DIRK2-SA$) has the Butcher tableau

  $$
  \begin{array}{ccc}
  \gamma & 0 & 0 \\
  1 & 1-\gamma & \gamma \\
  1 & 1-\gamma & \gamma \\
  \end{array}
  \tag{3.3}
  $$

  where $\gamma = 1 - \sqrt{2}/2$. This method is stiffly accurate with order $p = 2$ and stage order $q = 1$.

- The InDC method constructed with a second order non stiffly accurate midpoint method in $k$ correction steps (InDC-DIRK2-NSA-M-k). The second order midpoint method ($DIRK2-NSA$) has the Butcher tableau

  $$
  \begin{array}{ccc}
  1/2 & 1/2 & 0 \\
  1 & 1 & 1 \\
  \end{array}
  \tag{3.4}
  $$

  This method is not stiffly accurate, with order $p = 2$ and stage order $q = 1$.

- The InDC method constructed with a third order stiffly accurate Radau IIA method in the prediction step and with the BE method in $k$ correction steps (InDC-Radau-BE-M-k). The third order Radau IIA method has the Butcher tableau

  $$
  \begin{array}{ccc}
  1/3 & 5/12 & -1/12 \\
  1 & 3/4 & 1/4 \\
  3/4 & 1/4 & 0 \\
  \end{array}
  \tag{3.5}
  $$

  This method is stiffly accurate with order $p = 3$ and stage order $q = 2$.

The indicated order of convergence by Theorem 3.1 and Theorem 3.2 for the $y$ and $z$ components in the SPPs are summarized in Table 3.1 Below we discuss about the Table 3.1

- For the InDC-BE-M-k method, the order of convergence will increase with $k$ for the first error term in equation (2.25) when $\varepsilon \ll H$ and $k \leq M - 1$, leading to a term of $H^{\min(k+1,M)}$ for the differential and algebraic component in equation (3.1). The BE method has stage order $q = 1$. The order of convergence for the second error term in equation (2.25) will be determined by the stage order of the prediction $q^{(0)} = 1$ when $k$ increases, leading to a term of $\varepsilon H$ in equation (3.1).

- For the InDC-DIRK2-SA-M-k, the order of convergence will increase with $k$ by 2 for the first error term in equation (2.25) when $\varepsilon \ll H$ and $k \leq M - 1$, leading to a term of $H^{\min(2(k+1),M)}$ for the differential and algebraic component in equation (3.1). DIRK2-SA method has stage order $q = 1$. The order of convergence for the second error term in equation (2.25) will be determined by the stage order of the prediction $q^{(0)} = 1$ when $k$ increases, leading to a term of $\varepsilon H$ in equation (3.1).

- An important ingredient, suggested by the analysis, is to require methods to be stiffly accurate, i.e. $a_{sj} = b_j$ for $j = 1, \cdots, s$. Such a choice provides a significant benefit for the convergence of the numerical solution, without which the numerical solutions will diverge. For example, if we consider of using the second order non stiffly accurate DIRK method in both the prediction and $k$ correction steps of an InDC framework with $M$ quadrature points (InDC-DIRK2-NSA-M-k), divergence results are expected. Note that in the analysis for InDC-IRK method in the appendix, a satisfactory theoretical explanation of this fact is given.

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For the InDC-Radau-BE-M-k, the order of convergence will increase with \( k \) by 1 for the first error term in equation (2.25) when \( \varepsilon \ll H \) and \( k \leq M - 1 \), leading to a term of \( H^{\min(3+k,M)} \) for the differential and algebraic component in equation (3.1). Radau IIA method has stage order \( q = 2 \). The order of convergence for the second error term in equation (2.25) will be determined by the stage order of the prediction \( q^{(0)} = 2 \) when \( k \) increases, leading to a term of \( \varepsilon H^2 \) in equation (3.1).

Table 3.1: Global error predicted by Theorem 3.1 and Theorem 3.2 with \( H \gg \varepsilon \). Note that ‘SA’/’NSA’ means stiffly accurate/not stiffly accurate.

| Method            | \( y \)-comp     | \( z \)-comp     |
|-------------------|-------------------|-------------------|
| InDC-BE-M-k       | \( H^{\min(k+1,M)} + \varepsilon H \) | \( H^{\min(k+1,M)} + \varepsilon H \) |
| InDC-DIRK2-SA-M-k | \( H^{\min(2(k+1),M)} + \varepsilon H \) | \( H^{\min(2(k+1),M)} + \varepsilon H \) |
| InDC-DIRK2-NSA-M-k| diverges         | diverges         |
| InDC-Radau-BE-M-k | \( H^{\min(3+k,M)} + \varepsilon H^2 \) | \( H^{\min(3+k,M)} + \varepsilon H^2 \) |

For numerical verification, we first consider a scalar example \[9\]

\[ \varepsilon z' = -z + \cos(t) \] (3.6)

with the analytical solution

\[ z(t) = \frac{\cos(t) + \varepsilon \sin(t)}{1 + \varepsilon^2} + C \exp(-t/\varepsilon), \]

where \( C = z(0) - 1 \) is determined by the initial condition. For a consistent initial condition, let \( C = 0 \). This is a good example to investigate the order of convergence for the \( \varepsilon^1 \) term in equation (1.21), as the error for \( \varepsilon^0 \) is 0. Indeed, for stiff parameter \( \varepsilon = 10^{-6} \) only a region of first order convergence is observed for the BE method, where the global and local error given for the \( z \)-component in Theorem 1.4 is \( O(\varepsilon H) \). Figure 3.1 gives the one step error (local error) and global error of BE method; expected \( O(\varepsilon H) \) is observed. We also test the InDC-DIRK2-NSA-3-1 method. Numerical results are presented in Figure 3.2. Divergence is observed when time step is large compared to \( \varepsilon \) if an InDC-correction is performed.

Figure 3.1: Scalar example. Local, i.e. one step error (left plot) and global error at \( T = 0.5 \) (right plot) of BE method. \( O(\varepsilon H) \) is observed in both plots with \( \varepsilon = 10^{-6} \).

Now we consider the van der Pol equation \[9\] with the well-prepared initial data up to \( O(\varepsilon^3) \)

\[ \begin{cases} y' = z \\ \varepsilon z' = (1 - y^2)z - y \end{cases} \quad \begin{cases} y(0) = 2 \\ z(0) = -\frac{2}{3} + \frac{10}{87} \varepsilon - \frac{292}{2187} \varepsilon^2 \end{cases} \] (3.7)

- Numerical results of the InDC-BE-3-2 method are presented in the upper row of Figure 3.3. The order of convergence for \( \varepsilon^0 \) term would increase with the correction loops. The \( \varepsilon^1 \) term of error behaves like \( O(\varepsilon H) \) for both \( y \) and \( z \) components.

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The numerical results of the InDC-DIRK2-SA-4-1 method are presented in the middle row of Figure 3.3. The order of convergence for \( \varepsilon^0 \) term would increase with second order with the correction loop. The \( \varepsilon^1 \) term of error behaves like \( O(\varepsilon H) \) for both \( y \) and \( z \) components.

The numerical results of the InDC-Radau-BE-6-2 method are presented in the bottom row of Figure 3.3. The order of convergence for \( \varepsilon^0 \) term would increase with first order correction loop and is observed to be \( O(H^5) \). The \( \varepsilon^1 \) term of error behaves like \( O(\varepsilon H^2) \) for both \( y \) and \( z \) components.

Numerical observations in Figure 3.3 are consistent with Theorem 3.1, 3.2 and Table 3.1. Especially, it is observed that the InDC SA-IRK methods exhibit order reduction both in differential and algebraic components. They produce an estimate for the \( y \) and \( z \) component in the form of equation (3.2).

For example, in Figure 3.3, we observe a behavior like \( e^{(k)}_{m,0} = O(H^3) + O(\varepsilon H) \). Furthermore, if the step size \( H > \varepsilon^{-k-q(0)} \), \( O(H^5) \) is dominant, otherwise the term \( O(\varepsilon H^{q(0)}) \) is observed. We observe that in the neighborhood of \( H \approx \varepsilon^{-k-q(0)} \), we have a cancellation of error terms between \( O(H^5) \) and \( \varepsilon O(H^{q(0)}) \), if error constants are of opposite signs, see for example the plots in middle and bottom rows of Figure 3.3.

4 Proofs of main results

In this section, we prove Theorem 3.1, which is a special case of Theorem 3.2. By proving Theorem 3.1 through several lemmas, we demonstrate the basic ingredients of the general proof for Theorem 3.2 presented in the Appendix. Our error estimates are based on the \( \varepsilon \)-expansion outlined in Section 2.3.

4.1 Error estimates for Theorem 3.1

We perform local error estimate for Theorem 3.1 by two Lemmas. We again note that since \( h = \frac{H}{M} \), we use \( O(h^p) \) and \( O(H^p) \) interchangeably below in our proof. We then prove the global error estimate based on the two Lemmas.

Lemma 4.1. (\( \varepsilon^0 \) error term). Suppose that the reduce system (1.4) with \( \varepsilon = 0 \) satisfies (1.5) and that the initial values are consistent. Consider the InDC-BE method constructed with \( M \) uniformly distributed quadrature nodes excluding the left-most point and \( k \) correction steps, i.e. (2.17) for the prediction step and (2.20) for the correction one, with \( k = 1, \ldots, K \). Then the numerical solutions have the following local error estimates at each interior node of InDC \( \tau_m \) with \( m = 0, \ldots, M \),

\[
e^{(k)}_{m,0} = y_{m,0} - \hat{y}^{(k)}_{m,0} = O(h^{\min(k+2, M+1)}), \quad d^{(k)}_{m,0} = z_{m,0} - \hat{z}^{(k)}_{m,0} = O(h^{\min(k+2, M+1)}),
\]

with \( g(\hat{y}^{(k)}_{m,0}, \hat{z}^{(k)}_{m,0}) = 0 \),

for \( k = 0, \ldots, K \).
Figure 3.3: Van der Pol equation. Global error ($T = 0.5$) of the InDC-BE-3-2 method (upper row); and of the InDC-DIRK2-SA-4-1 method (middle row); and of the InDC-Radau-BE-6-2 method (bottom row). $\varepsilon = 10^{-6}$.
Proof. For $k = 0$, equation (4.2) is a consequence of the consistency of the initial conditions for system (2.13). Then we start to prove the local error estimate (4.1) for the prediction step ($k = 0$). For the exact solution, by equation (2.13) and assumption (1.5), we have for the $y_0(t)$ component, equation (1.6) and $g(y_0(t), z_0(t)) = 0$. By equation (1.5), it follows that $z_0(t) = G(y_0(t))$.

For the numerical solution, we have (2.17). By $g(y_0^{(0)}, z_0^{(0)}) = 0$, with $m = 0, \ldots, M$, we get $z_0^{(0)} = G(y_0^{(0)})$ with $y_0^{(0)}$, being numerical solution of the ordinary differential equation (1.6). Then from classical error estimates for the BE method, we have for the local truncation error $|\hat{y}_m^{(0)} - y_m| \leq C_m h^2$ with $m = 0, \ldots, M$, for some constant $C_m$ independent of $H$. Therefore, $|y_0^{(0)} - y_m| = O(h^2)$ and by $\hat{z}_m^{(0)} = G(y_0^{(0)})$ and the Lipschitz condition of $G$, it follows that $|z_0^{(0)} - z_m| = O(h^2)$ with $m = 0, \ldots, M$.

Now we prove the local error estimate (4.1) and equation (4.2) for the correction step $k = 1$, assuming a fixed $M \geq 1$. By $g(y_0^{(0)}, z_0^{(0)}) = 0$ in the prediction step, from the second equation in (2.20), we obtain $g(y_0^{(1)}, z_0^{(1)}) = 0$, with $m = 0, \ldots, M$, i.e. equation (4.2) with $k = 1$. Then, from the condition (1.5) it follows $z_0^{(1)} = G(y_0^{(1)})$, and this gives from (2.20)

$$\hat{y}_m^{(1)} - y_m = h(\hat{f}(\hat{y}_m^{(1)})) + hS\hat{z}_m^{(1)},$$

where $\hat{f}(\hat{y}_m^{(1)}) = f(y_0^{(1)}, G(y_0^{(1)}))$, and $S\hat{z}_m^{(1)} = S\hat{z}_0^{(1)}$. The method (4.3) for updating $\hat{y}_m^{(1)}$ represents the first correction step of the InDC-BE method to solve the non-stiff ordinary differential equation (1.6). Therefore, from classical error estimates of InDC-BE method when applied to a non-stiff constant $C_m$ independent of $H$ with $h \leq h_0$. Therefore $|y_0^{(1)} - \hat{y}_m^{(1)}| = O(h^3)$ and by $\hat{z}_m^{(1)} = G(y_0^{(1)})$ and Lipschitz condition of $G$, we get $|z_0^{(1)} - \hat{z}_m^{(1)}| = O(h^3)$, $\forall m = 1, \ldots, M$ and $h \leq h_0$. The estimate for general $k > 1$ can be proved in a similar fashion and by mathematical induction with respect to $k$.

Lemma 4.2. ($\varepsilon$ error term). Assume condition (1.3) holds and initial values of the differential algebraic system (2.13)-(2.14) are consistent. Consider the InDC-BE method constructed with $M$ uniformly distributed quadrature nodes excluding the left-most point, and with (2.17)-(2.18) for the prediction step and (2.20)-(2.21) for the correction step with $k = 1, \ldots, K$ for solving the differential algebraic system (2.13)-(2.14). Then the local error estimates of the InDC-BE method

$$e_m^{(1)} = y_m - \hat{y}_m^{(1)} = O(h^2), \quad d_m^{(1)} = z_m - \hat{z}_m^{(1)} = O(h),$$

hold for $m = 1, \ldots, M$ at the interior nodes of InDC, and for $k = 0, \ldots, K$.

Proof. The proof for the case of $k = 0$ (prediction step) is a consequence of Lemma 4.4 in Chap. VII.4 in [9]. We then consider the first correction step with $k = 1$ and assume a fixed $M \geq 1$. We prove (4.3) by mathematical induction w.r.t. $m$. Especially, we know $e_m^{(1)} = d_m^{(1)} = 0$, with $m = 0$. We assume (4.4) is valid for $0, 1, \ldots, m$. We will prove that (4.4) is valid for $m + 1$. The integration of (2.14) over $[\tau_m, \tau_{m+1}]$ gives

$$\varepsilon^1 : \left\{ \begin{array}{c}
y_m = y_m + \int_{\tau_m}^{\tau_{m+1}} F_1(\tau)d\tau, \\
z_m = z_m + \int_{\tau_m}^{\tau_{m+1}} G_1(\tau)d\tau,
\end{array} \right.$$
Then from (4.7) and (4.8) for \( k = 1 \) it follows
\[
\begin{align*}
\Delta \hat{\phi}^{(0)}_{m+1,1} &= \left( f_p \hat{c}^{(0)}_{m+1,1} + f_z \hat{d}^{(0)}_{m+1,1} \right) + \mathcal{O}(h^2), \\
\Delta \hat{G}^{(0)}_{m+1,1} &= \left( g_p \hat{e}^{(0)}_{m+1,1} + g_z \hat{d}^{(0)}_{m+1,1} \right) + \mathcal{O}(h^2).
\end{align*}
\]
(4.9)

Now subtracting equation (2.21) from equation (4.5) this gives
\[
\begin{align*}
\varepsilon^1 : \left\{
\begin{aligned}
\varepsilon^{(1)}_{m+1,1} &= \varepsilon^{(1)}_{m,1} - h \Delta \hat{\phi}^{(0)}_{m+1,1} - h S^m(\bar{F}_1^{(0)}) + \int_{\tau_m}^{\tau_{m+1}} F_1(\tau)d\tau, \\
d^{(1)}_{m+1,0} &= d^{(1)}_{m,0} - h \Delta \hat{G}^{(0)}_{m+1,1} - h S^m(\bar{G}_1^{(0)}) + \int_{\tau_m}^{\tau_{m+1}} G_1(\tau)d\tau.
\end{aligned}
\right.
\]
(4.10)

On the right-hand side of the equations in (4.10) we add and subtract the following quantities: \( h S^m(\bar{F}_1) \) and \( h S^m(\bar{G}_1) \), these are the integrals of \((M-1)^{th}\) degree interpolating polynomials on \((\tau_m, F_1(\tau_m))_{m=1}^M\) and \((\tau_m, G_1(\tau_m))_{m=1}^M\) over the subinterval \([\tau_m, \tau_{m+1}]\), hence they are accurate to the order \( \mathcal{O}(h^{M+1}) \) locally, i.e. \( \int_{\tau_m}^{\tau_{m+1}} F_1(\tau)d\tau - h S^m(\bar{F}_1) = \mathcal{O}(h^{M+1}) \). By the local error estimates in Lemma 4.1 as well as equation (4.4) for \( k = 0 \), it follows that \( S^m(\bar{F}_1) - S^m(\bar{F}_1) \) and \( S^m(\bar{G}_1) - S^m(\bar{G}_1) \) are accurate to the order \( \mathcal{O}(h) \). Thus, from (4.10) we get
\[
\begin{align*}
\varepsilon^{(1)}_{m+1,1} &= \varepsilon^{(1)}_{m,1} - h \left( f_p \hat{c}^{(0)}_{m+1,1} + f_z \hat{d}^{(0)}_{m+1,1} \right) + \mathcal{O}(h^2), \\
d^{(1)}_{m+1,0} &= d^{(1)}_{m,0} - h \left( g_p \hat{e}^{(0)}_{m+1,1} + g_z \hat{d}^{(0)}_{m+1,1} \right) + \mathcal{O}(h^2).
\end{align*}
\]
(4.11)

Now from (2.27) and (4.11), we have
\[
\begin{align*}
\varepsilon^{(0)}_{m,1} &= \dot{y}^{(1)}_{m,1} - \dot{y}^{(0)}_{m,1} = \varepsilon^{(0)}_{m,1} - \varepsilon^{(1)}_{m,1} = -\varepsilon^{(1)}_{m,1} + \mathcal{O}(h^2), \\
\dot{z}^{(0)}_{m,1} &= \dot{z}^{(1)}_{m,1} - \dot{z}^{(0)}_{m,1} = \varepsilon^{(0)}_{m,1} - \varepsilon^{(1)}_{m,1} = -\varepsilon^{(1)}_{m,1} + \mathcal{O}(h),
\end{align*}
\]
(4.12)

and put it into equation (4.11) gives
\[
\begin{align*}
\varepsilon^{(1)}_{m+1,1} &= \varepsilon^{(1)}_{m,1} + h \left( f_p \dot{c}^{(1)}_{m+1,1} + f_z \dot{d}^{(1)}_{m+1,1} \right) + \mathcal{O}(h^2), \\
d^{(1)}_{m+1,0} &= d^{(1)}_{m,0} + h \left( g_p \dot{e}^{(1)}_{m+1,1} + g_z \dot{d}^{(1)}_{m+1,1} \right) + \mathcal{O}(h^2).
\end{align*}
\]
(4.13)

Now using the estimate (4.1) about \( d^{(1)}_{m,0} \), from the second equation in (4.13) we obtain
\[
d^{(1)}_{m+1,1} = -g_z^{-1} g_y \varepsilon^{(1)}_{m+1,1} + \mathcal{O}(h),
\]
(4.14)

with the invertibility of \( g_z \). Inserting this into the first equation in (4.13) gives
\[
\varepsilon^{(1)}_{m+1,1} = (1 - h (f_y - f_z g_z^{-1} g_y))^{-1} \varepsilon^{(1)}_{m,1} + \mathcal{O}(h^2).
\]
(4.15)

Finally \( e^{(1)}_{m+1,1} = \mathcal{O}(h^2) \) follows from (4.15), and \( d^{(1)}_{m+1,1} = \mathcal{O}(h) \) from (4.14). The estimate for general \( k > 1 \) can be proved in a similar fashion and by mathematical induction with respect to \( k \). \( \blacksquare \)

**Remark 4.3.** In [1], the InDC method constructed with explicit RK methods in the prediction and correction steps has been reformulated as a high-order explicit RK method whose Butcher tableau is explicitly constructed. Similarly, the InDC-BE can be viewed as an IRK method with the corresponding Butcher tableau. Below we present the Butcher tableau for the InDC-BE method with one loop of correction step. This takes the form
\[
\begin{array}{c|cc}
\hat{c} & T & Z \\
\hat{c} & P & T \\
\end{array}
\]
(4.16)

where \( \hat{c} = \frac{1}{M} [1, \cdots, M]^T \), \( Z \) is a \( M \times M \) matrix of zeros, \( T \) and \( P \) are \( M \times M \) matrices, with
\[
T = \frac{1}{M} \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1
\end{bmatrix}
\]

\[ 
\hat{c} \]

\[ 
\hat{c} \]

\[ 
\]
\[ P = \begin{bmatrix} (\tilde{S}_{11} - \frac{1}{M}) & \tilde{S}_{12} & \cdots & \tilde{S}_{1,M-1} & \tilde{S}_{1,M} \\ (\tilde{S}_{21} - \frac{1}{M}) & (\tilde{S}_{22} - \frac{1}{M}) & \cdots & \tilde{S}_{2,M-1} & \tilde{S}_{2,M} \\ \vdots & \vdots & & \vdots & \vdots \\ (\tilde{S}_{M,1} - \frac{1}{M}) & (\tilde{S}_{M,2} - \frac{1}{M}) & \cdots & (\tilde{S}_{M,M-1} - \frac{1}{M}) & (\tilde{S}_{M,M} - \frac{1}{M}) \end{bmatrix}, \]

where the term \( \tilde{S}_{ij} = \int_{t_0}^{t_1} \alpha_j(s) ds \) with \( \alpha_j(s) \) the Lagrangian basis functions for the node \( \tau_j \), and the vector

\[ \bar{v}_1 = \left( (\tilde{S}_{M,1} - \frac{1}{M}), (\tilde{S}_{M,2} - \frac{1}{M}), \ldots, (\tilde{S}_{M,M} - \frac{1}{M}) \right), \quad \bar{v}_2 = \frac{1}{M}(1,1,\ldots,1). \]

Now from remark 4.3, the following proposition follows.

**Proposition 4.4.** The InDC-BE method with \( K \) correction steps is an implicit stiffly accurate IRK method with an invertible matrix \( A \) in the Butcher tableau (1.10). Especially when \( K = 1 \),

\[ A = \begin{pmatrix} T & Z & \cdot & \cdot & \cdot \\ P & T & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}. \]

**Remark 4.5.** In the estimates in Lemma 4.2, we show that there is no improvement for \( \epsilon_{m,1}^{(k)} \) and \( \epsilon_{m,1}^{(k)} \) as \( k \) increases, see equation (4.4). This is consistent with our numerical evidences presented in the previous section. The reason is that both the local and global error for the \( z \)-component in the prediction and correction steps is of first order. This sets the bottleneck for the order increase in the second equation of (4.11).

We are now in the position to prove Theorem 3.1 by the local error estimates of the two lemmas above. **Proof of Theorem 3.1.** Our first step here is to estimate \( \epsilon_{n,0}^{(K)} \) and \( d_{n,0}^{(K)} \). For this, from Lemma 4.1 we have after one step from \( t_0 \) to \( t_1 \), the local error estimate

\[ y_0(t_1) - \hat{y}_{M,0}^{(K)} = O(H^{\min(K+2,M+1)}), \]

with \( m = M \) and \( \tau_M = t_1 \) in equation (4.1). In the estimate of the global error from local error, we obtain

\[ \epsilon_{n,0}^{(K)} = y_0(nH) - \hat{y}_{n,0}^{(K)} = O(H^{\min(K+1,M)}). \]

It thus follows from (4.2), and by the Lipschitz condition of \( G \), that

\[ d_{n,0}^{(K)} = z_0(nH) - \hat{z}_{n,0}^{(K)} = O(H^{\min(K+1,M)}). \]

Now our next aim is to estimate \( \epsilon_{n,1}^{(K)} \) and \( d_{n,1}^{(K)} \). From Lemma 4.2 we have for the local error estimate

\[ y_1(t_1) - \hat{y}_{M,1}^{(K)} = O(H^2). \]

By Lemma 4.2, the proof of the global error estimates for \( y \) and \( z \) is similar to that of Theorem 4.5 and 4.6 in Chap. VII.4 of [9]. Thus we obtain

\[ \epsilon_{n,1}^{(K)} = y_1(nH) - \hat{y}_{n,1}^{(K)} = O(H), \quad d_{n,1}^{(K)} = z_1(nH) - \hat{z}_{n,1}^{(K)} = O(H), \]

which proves the statement. \( \blacksquare \)

**Remark 4.6.** Similar error estimates can be given for the InDC SA-IRK method. We present and prove these error estimates in Appendix.

### 5 Conclusions

This paper studies the order of convergence of the InDC-BE and InDC-IRK methods when applied to SSPs, using uniform distribution of quadrature points excluding the leftmost point. We applied the technique of asymptotic expansion in powers of \( \varepsilon \) for the smooth exact solution and for the corresponding numerical solution presented in [8] and [9]. Two Theorems on global error estimate in the form of \( \varepsilon \)-expansion are presented and proved. Especially, we point out that the InDC methods improve the order of the \( \varepsilon \)-independent error, but there is no order improvement on the higher order terms \( \varepsilon^r \ (r \geq 1) \). Such asymptotic analysis enables us to understand the phenomenon of order reduction for InDC methods when applied to stiff problems. Numerical results on van der Pol equations confirm these convergence results.
6 Appendix.

In the appendix, we extend the error estimates of the InDC-BE method to InDC-IRK method when applied to SPPs. We first describe the InDC-IRK method applied to \([1,2]\), then perform an \(\varepsilon\)-expansion of the numerical solution of this method, and finally we prove Theorem 3.2.

6.1 InDC-IRK method

We consider the InDC-IRK method constructed with \(s\)-stage IRK methods, where \(A\) matrices in the Butcher tableau \([1,10]\) are invertible. For the internal stages in the IRK method, we introduce the integration matrix and interpolation matrix as following

\[
\begin{align*}
\int_{\tau_m}^{\tau_m + c_{mi} h} P_{c_{mi}}^{k} &= \alpha_k(\tau_m + c_{mi} h), \\
hS^{c_{mi}, k} &= \int_{\tau_m}^{\tau_m + c_{mi} h} \alpha_k(s) ds,
\end{align*}
\]

\(\forall m = 0, \cdots, M - 1, \; \forall k = 1, \cdots, M\) and \(\forall mi = 1, \cdots, s\), where \(mi\) is index used for the \(i\)th stage of the IRK method over the subinterval \([\tau_m, \tau_m + \Delta]\). Here \(\alpha_k(s)\) is the Lagrangian basis function based on the node \(\tau_k\). Let

\[
S^{c_{mi}}(f) = \sum_{j=1}^{M} S^{c_{mi}, j} f(y_j, z_j), \quad P^{c_{mi}}(\hat{f}) = \sum_{j=1}^{M} P^{c_{mi}, j} f(y_j, z_j),
\]

then we have

\[
\begin{align*}
\int_{\tau_m}^{\tau_m + c_{ih} h} f(y(s), z(s)) ds &= \mathcal{O}(h^{M+1}), \\
\mathcal{O}(h_M), & \quad \mathcal{O}(h_M),
\end{align*}
\]

for any smooth function \(f\). In other words, the quadrature formula given by \(hS^{c_{mi}}(\hat{f})\) approximates the exact integration with \((M + 1)\)th order accuracy locally, while the interpolation formula given by \(P^{c_{mi}}(\hat{f})\) approximates the exact solution at RK internal stages with \(M\)th order accuracy locally.

To compute the numerical error approximating the error function \(\varepsilon^{(k-1)}(\tau_m), d^{(k-1)}(\tau_m)\) with a general IRK method to \([2,5]\), we obtain

\[
\begin{pmatrix}
\varepsilon_m^{(k-1)} \\
\varepsilon_{m+1}^{(k-1)}
\end{pmatrix} = \begin{pmatrix}
\varepsilon_m^{(k-1)} + h \int_0^{\tau_m} \rho(\tau_m + \tau h) d\tau \\
\varepsilon_{m+1}^{(k-1)} + h \int_0^{\tau_m} \rho(\tau_m + \tau h) d\tau
\end{pmatrix} + h \sum_{i=1}^{s} b_i \begin{pmatrix}
\Delta K_{mi}^{(k-1)} \\
\Delta D_{mi}^{(k-1)}
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
\varepsilon_{mi}^{(k-1)} \\
\varepsilon_{mi+1}^{(k-1)}
\end{pmatrix} = \begin{pmatrix}
\varepsilon_{mi}^{(k-1)} + h \int_0^{\tau_m} \rho(\tau_m + \tau h) d\tau \\
\varepsilon_{mi+1}^{(k-1)} + h \int_0^{\tau_m} \rho(\tau_m + \tau h) d\tau
\end{pmatrix} + h \sum_{j=1}^{s} a_{ij} \begin{pmatrix}
\Delta K_{mi}^{(k-1)} \\
\Delta D_{mi}^{(k-1)}
\end{pmatrix},
\]

with

\[
\begin{pmatrix}
\Delta K_{mi}^{(k-1)} \\
\Delta D_{mi}^{(k-1)}
\end{pmatrix} \equiv \begin{pmatrix}
f(Y^{(k)}(Y_{mi}^{(k)}), Z_{mi}^{(k)}) - P^{c_{mi}}(\bar{\hat{f}}^{(k-1)}(\bar{\hat{f}}^{(k-1)})) \\
g(Y^{(k)}(Y_{mi}^{(k)}), Z_{mi}^{(k)}) - P^{c_{mi}}(\bar{\hat{g}}^{(k-1)})
\end{pmatrix}
\]

\[
\begin{pmatrix}
f(Y^{(k)}(Y_{mi}^{(k)}), Z_{mi}^{(k)}) - P^{c_{mi}}(\bar{\hat{f}}^{(k-1)}), P^{c_{mi}}(\bar{\hat{f}}^{(k-1)})) \\
g(Y^{(k)}(Y_{mi}^{(k)}), Z_{mi}^{(k)}) - g(P^{c_{mi}}(\bar{\hat{g}}^{(k-1)}), P^{c_{mi}}(\bar{\hat{g}}^{(k-1)}))
\end{pmatrix} + \mathcal{O}(h^M),
\]

where we put

\[
\begin{align*}
\hat{Y}_{mi}^{(k)} &= P^{c_{mi}}(\bar{\hat{f}}^{(k)}), \hat{Z}_{mi}^{(k)} = P^{c_{mi}}(\bar{\hat{g}}^{(k-1)}) + \bar{D}_{mi}^{(k-1)},
\end{align*}
\]

and equation \([6.7]\) is due to the high order interpolation accuracy of \(P^{c_{mi}}\), see equation \([6.3]\). We can rewrite the system \([6.4]\) and \([6.5]\) as

\[
\begin{pmatrix}
\varepsilon_{mi}^{(k)} \\
\varepsilon_{mi+1}^{(k)}
\end{pmatrix} = \begin{pmatrix}
\varepsilon_{mi}^{(k)} + h \int_0^{\tau_m} \rho(\tau_m + \tau h) d\tau \\
\varepsilon_{mi+1}^{(k)} + h \int_0^{\tau_m} \rho(\tau_m + \tau h) d\tau
\end{pmatrix} + h \sum_{i=1}^{s} b_i \begin{pmatrix}
\Delta K_{mi}^{(k-1)} \\
\Delta D_{mi}^{(k-1)}
\end{pmatrix}.
\]
implies $R \varepsilon (6.9)$ into powers of $\varepsilon$. We formally expand the quantities $\Delta$ so on. Similarly, we have $\varepsilon S_{m,s}(g(6.9), f(6.9) - S_{m,s}(g(6.9)))$.

Remark 6.1. Under the assumption $A$ invertible, from the second equation of (6.10), we obtain in vectorial form

$$h \Delta \tilde{\Lambda}(k-1) = A^{-1}(\varepsilon \tilde{\Lambda}(k) - \varepsilon \tilde{S}_{m}(1 - hS\tilde{\varepsilon}(g(6.9)))),$$

with $\Delta \tilde{\Lambda}(k-1) = (\Delta \tilde{\Lambda}_{1}(k-1), \ldots, \Delta \tilde{\Lambda}_{m}(k-1))^{T}$, $1 = (1, \ldots, 1)^{T}$ and $\tilde{c} = (c_{m1}, \ldots, c_{ms})$. Inserting this into the second equation of (6.9), we get

$$\varepsilon S_{m+1} = \varepsilon R(\varepsilon) \tilde{S}_{m} + \varepsilon hA^{-1} \tilde{\Lambda}(k) + h(S\tilde{c}(g(6.9)) - b^{T}A^{-1}S\tilde{\varepsilon}(g(6.9))).$$

Of special importance now are stiffly accurate RK methods, i.e., methods which satisfy $b^{T}A^{-1} = \varepsilon e$. This implies $R(\varepsilon) = 0$ and $b^{T}A^{-1}S\tilde{c}(g(6.9)) = e_{s}^{T}S\tilde{c}(g(6.9)) = S\tilde{c}(g(6.9))$. Hence by (6.12) we have: $\varepsilon S_{m+1} = \tilde{S}_{m}(g(6.9))$.

6.2 \varepsilon-asymptotic expansion of InDC-IRK methods

We formally expand the quantities $\Delta \tilde{\Lambda}_{m,i}(k-1)$, $\Delta \tilde{\Lambda}_{m,0}(k-1)$ from (6.6) and $\tilde{Y}_{m,i}$, $\tilde{Z}_{m,i}$, $\tilde{\Lambda}_{m,i}$ from (6.8) and (6.9) into powers of $\varepsilon$ with $\varepsilon$-independent coefficients

$$\varepsilon 0: \quad \Delta \tilde{\Lambda}_{m,0}(k-1) = f(\tilde{Y}_{m,0}(k), \tilde{\Lambda}_{m,0}(k-1)) - f(P_{c_{m1}}(\tilde{y}_{0}(k)), P_{c_{m1}}(\tilde{\varepsilon}(\tilde{y}_{0}(k)))) + O(h^{M}),$$

$$\varepsilon 1: \quad \Delta \tilde{\Lambda}_{m,1}(k-1) = \left( f_{y}(\tilde{Y}_{m,0}(k), \tilde{\Lambda}_{m,0}(k), \tilde{\Lambda}_{m,1}(k)) + f_{\varepsilon}(\tilde{Y}_{m,0}(k), \tilde{\Lambda}_{m,1}(k)) \right)$$

$$+ \left( f_{g}(P_{c_{m1}}(\tilde{y}_{0}(k)), P_{c_{m1}}(\tilde{\varepsilon}(\tilde{y}_{0}(k))))P_{c_{m1}}(\tilde{\varepsilon}(\tilde{y}_{0}(k))) \right)$$

$$+ f_{z}(P_{c_{m1}}(\tilde{y}_{0}(k)), P_{c_{m1}}(\tilde{\varepsilon}(\tilde{y}_{0}(k))))P_{c_{m1}}(\tilde{\varepsilon}(\tilde{y}_{0}(k))) + O(h^{M}).$$

ans so on. Similarly, we have

$$\Delta \tilde{\Lambda}_{m,1}(k-1) = g(\tilde{Y}_{m,0}(k), \tilde{\Lambda}_{m,0}(k)) - g(P_{c_{m1}}(\tilde{y}_{0}(k)), P_{c_{m1}}(\tilde{\varepsilon}(\tilde{y}_{0}(k)))) + O(h^{M}),$$

$$\Delta \tilde{\Lambda}_{m,0}(k-1) = \left( g_{y}(\tilde{Y}_{m,0}(k), \tilde{\Lambda}_{m,0}(k))g_{\varepsilon}(\tilde{Y}_{m,0}(k), \tilde{\Lambda}_{m,0}(k)) \right)$$

$$+ g_{z}(P_{c_{m1}}(\tilde{y}_{0}(k)), P_{c_{m1}}(\tilde{\varepsilon}(\tilde{y}_{0}(k))))P_{c_{m1}}(\tilde{\varepsilon}(\tilde{y}_{0}(k))) + O(h^{M}).$$

(6.17)
ans so on.

Because of the linearity of relations (6.9) and (6.10), we have to order \( \varepsilon' \) with \( \nu = -1 \) in vectorial form

\[
h \Delta \mathcal{L}_{m-1}^{(k-1)} + h S^T (\bar{g}) = 0, \quad h b^T \Delta \mathcal{L}_{m-1}^{(k-1)} + h S^m (\bar{g}) = 0,
\]

and for \( \nu \geq 0, \)

\[
\begin{align*}
\left( \begin{array}{c}
\dot{y}_{m+1, \nu} - h S^m_{\nu} (\bar{g}) \\
\dot{z}_{m+1, \nu} - h S^m_{\nu} (\bar{g}) \\
\end{array} \right) &= \left( \begin{array}{c}
\dot{y}_{m, \nu} \\
\dot{z}_{m, \nu} \\
\end{array} \right) + h \sum_{i=1}^{s} b_i \left( \begin{array}{c}
\Delta K_{mi, \nu}^{(k-1)} \\
\Delta L_{mi, \nu}^{(k-1)} \\
\end{array} \right), \\
\end{align*}
\]

where

\[
S^m_{\nu} (\bar{g}) = S^m (\bar{g}^{(k-1)}), \quad S^m_{\nu} (\bar{g}) = S^m (\bar{g}^{(k-1)}).
\]

Similarly for \( S^m_{\nu} (\bar{g}^{(k-1)}) \) and \( S^m_{\nu} (\bar{g}^{(k-1)}) \).

6.3 Proof of Theorem 3.2.

We now prove Theorem 3.2 below. We remark that the crucial assumption in Theorem 3.2 is that the IRK method is stiffly accurate. In the case that this property is not satisfied, the method becomes unstable and the numerical solutions diverge, see Figure 3.2. In order to justify this, from the invertibility of matrix \( A \) and by the first formula of (6.18) we get

\[
\Delta \hat{z}^{(k)}_{m-1} = -A^{-1} S^T (\bar{g}^{(k)}),
\]

substituting now into the second formula of (6.18) yields

\[
b^T A^{-1} S^c (\bar{g}^{(k-1)}) + S^m (\bar{g}^{(k-1)}) = 0.
\]

Proposition 6.2. Equation (6.23) is automatically satisfied, if the IRK methods in the prediction and correction steps of the InDC method are stiffly accurate.

Proof. An IRK method is stiffly accurate if

\[
b^T A^{-1} = e^T_x,
\]

with \( e_x = (0, \ldots, 0, 1)^T \). From (6.23) we get

\[
-e^T_x S^c (\bar{g}^{(k-1)}) + S^m (\bar{g}^{(k-1)}) = 0.
\]

Since the last row of the spectral integration matrix is \( S^{m,k} = \int_{\tau_m}^{\tau_{m+c_h}} \alpha_k (\tau) d\tau \) by (6.24) we get \( e_x = 1 \) and then \( \int_{\tau_m}^{\tau_{m+c_h}} \alpha_k (\tau) d\tau = \int_{\tau_m}^{\tau_{m+c_h}} \alpha_k (\tau) d\tau \). This yields that \( e^T_x S^c (\bar{g}^{(k-1)}) = S^m (\bar{g}^{(k-1)}) \), and then the equation (6.25) is satisfied. \( \blacksquare \)

We prove the error estimates in Theorem 3.2 by two Lemmas below.

Lemma 6.3. (\( \varepsilon^0 \) error term) Consider the same assumptions as in Theorem 3.2 and the limiting case, \( \varepsilon = 0 \). The numerical solutions of the InDC method after \( k \) correction loops have the following local error estimates at the interior nodes \( \tau_m, \ m = 0, \ldots, M, \)

\[
\varepsilon_{m,0}^{(k)} = O(h^{\min(s_k+1,M+1)}), \quad d_{m,0}^{(k)} = O(h^{\min(s_k+1,M+1)}).
\]

Proof. Since the IRK method for the prediction is stiffly accurate, by definition (1.1), we have \( b^T A^{-1} = e^T_x \). This implies, by (1.16) and (1.15), \( \hat{z}_{m+1,0} = \hat{z}_{m,0} \) and \( \hat{y}_{m+1,0} = \hat{y}_{m,0} \). By (1.14) we get \( \hat{z}_{m,0} = \hat{z}_{m,0} = \hat{z}(\dot{y}_{m,0}) \) for all \( m \) and, in particular, \( \hat{z}(\dot{y}_{m,0}) = \hat{z}(\dot{y}_{m,0}) \).

Now for the first correction step, i.e. \( k = 1 \), by the fact that the IRK method applied is stiffly accurate and that \( \bar{y}_0 = (g(\dot{y}_{1,0}, \hat{z}_{1,0}), \cdots, g(\dot{y}_{M,0}, \hat{z}_{M,0})) = \bar{0} \) from the prediction step, from (6.19) it follows

\[
\begin{align*}
\hat{y}_{m+1,0}^{(1)} &= \hat{y}_{m,0}^{(1)} + h S^m (\bar{f}^{(0)}) + h \sum_{i=1}^{s} b_i \Delta \hat{z}_{m,0}^{(0)}, \\
g(\hat{y}_{m+1,0}, \hat{z}_{m+1,0}) &= 0.
\end{align*}
\]
where, from (6.20), we have for the internal stages
\[
\hat{y}_{m,0}^{(1)} = \hat{y}_{m,0}^{(1)} + hS_{m}^{(0)}(\tilde{f}_{0}^{(0)}) + h\sum_{j=1}^{s}a_{ij}\Delta \hat{y}_{m,0}^{(0)},
\]
\[
g(\hat{y}_{m,0}^{(1)}, \hat{Z}_{m,0}) = 0. \tag{6.28}
\]

Now, from the invertibility of function \(g_{z}\), by (6.27) and (6.28) we get \(\hat{Z}_{m,0}^{(1)} = G(\hat{y}_{m,0})\) and \(\hat{z}_{m,1,0}^{(1)} = G(\hat{y}_{m,1,0})\). Thus the IRK method reads
\[
\hat{y}_{m,0}^{(1)} = \hat{y}_{m,0}^{(1)} + h\sum_{j=1}^{s}a_{ij}\Delta \hat{y}_{m,0}^{(0)} + hS_{m}^{(0)}(\tilde{f}_{0}^{(0)}),
\]
\[
\hat{z}_{m,0}^{(1)} = \hat{y}_{m,0}^{(1)} + h\sum_{i=1}^{s}b_{i}\Delta \hat{y}_{m,0}^{(0)} + hS_{m}^{(0)}(\tilde{f}_{0}^{(0)}), \tag{6.29}
\]
where \(\tilde{f}_{0}^{(0)} = (f(\hat{y}_{0,0}^{(0)}, G(\hat{y}_{0,0}^{(0)})), \ldots, f(\hat{z}_{M,0}^{(0)}, G(\hat{z}_{M,0}^{(0)})))\). The scheme (6.29) of updating \(\hat{y}_{m,0}^{(1)}\) can be interpreted as applying a correction step of the InDC method to the ordinary differential equation (1.6). Therefore applying similar local truncation error estimates as in [4, 5] for InDC frameworks using RK methods when applied to a classical ordinary differential equation, we obtain the local error estimate
\[
e_{m,0} = O(h^{\min(s_{2}+1, M+1)}), \tag{6.30}
\]
for \(m = 0, \cdots, M\), with \(s_{2} = p^{(0)} + p^{(1)}\). By \(\hat{z}_{m,0}^{(1)} = G(\hat{y}_{m,0})\), using the Lipschitz condition of \(G\), we get
\[
d_{m,0}^{(1)} = \hat{z}_{m,0}^{(1)} - \hat{z}_{m,0}^{(0)} = O(h^{\min(s_{2}+1, M+1)}). \tag{6.31}
\]
Similarly, at internal stages of the IRK method, by \(\hat{Z}_{m,0}^{(1)} = G(\hat{Y}_{m,0})\), we have the following local error estimates,
\[
E_{m,0}^{(1)} = y_{0}(\tau_{m} + c_{i}h) - \hat{Y}_{m,0}^{(1)} = O(h^{\min(s_{1}+q^{(1)}+1, M+1)}), \tag{6.32}
\]
and
\[
D_{m,0}^{(1)} = z_{0}(\tau_{m} + c_{i}h) - \hat{Z}_{m,0}^{(1)} = O(h^{\min(s_{1}+q^{(1)}+1, M+1)}), \tag{6.33}
\]
where \(q^{(1)}\) is the stage order for the IRK method applied to the first correction loop. We note that the proof of each \(k\) is similar.

**Remark 6.4.** The local truncation error estimate (6.30) from [5] is quite technically involved; it is related to estimating the smoothness of re-scaled error functions. The estimate (6.32) follows a similar fashion. We refer readers to the original paper [5] for details.

**Remark 6.5.** With the estimates in the above Lemma, i.e. equations (6.30)-(6.33), it follows from equation (6.15)
\[
\Delta \hat{y}_{m,1}^{(k-1)} = f(\hat{y}_{m,0}, z_{m,0})\hat{E}_{m,1}^{(k-1)} + f_{z}(\hat{y}_{m,0}, z_{m,0})\hat{D}_{m,1}^{(k-1)} + O(h^{k-1})\tag{6.34}
\]
where \(\hat{E}_{m,1}^{(k-1)}\) and \(\hat{D}_{m,1}^{(k-1)}\) are defined by the corresponding \(\varepsilon\)-expansion of equation (6.8), \(s_{k} = \sum_{r=0}^{k}p^{(r)}\), and \(\Delta \hat{y}_{m,1}^{(k-1)} = f(\hat{y}_{m,0}, z_{m,0})\hat{E}_{m,1}^{(k-1)} + f_{z}(\hat{y}_{m,0}, z_{m,0})\hat{D}_{m,1}^{(k-1)}\). Here we have used the abbreviations \(\hat{y}_{m,0}\) and \(z_{m,0}\), i.e. the exact solution \(y(t)\) and \(z(t)\) at the position \(t = \tau_{m} + c_{i}h\) respectively. We note that, from (6.15), we replaced \(\hat{Y}_{m,0}^{(k)}\) and \(P_{m}^{(k)}(\hat{z}_{m,0}^{(k-1)})\) by adding and subtracting \(y_{m,0}\) with an error of \(O(h^{k-1}+q^{(1)}+1)\) and \(O(h^{k-1}+1)\), the same for \(\hat{Z}_{m,0}^{(k)}\) and \(P_{m}^{(k)}(\hat{z}_{m,0}^{(k-1)})\). Similarly, we have from (6.17)
\[
\Delta \hat{E}_{m,1}^{(k-1)} = g(\hat{y}_{m,0}, z_{m,0})\hat{E}_{m,1}^{(k-1)} + g_{z}(\hat{y}_{m,0}, z_{m,0})\hat{D}_{m,1}^{(k-1)} + O(h^{k-1}), \tag{6.35}
\]
where \(\Delta \hat{E}_{m,1}^{(k-1)} = g(\hat{y}_{m,0}, z_{m,0})\hat{E}_{m,1}^{(k-1)} + g_{z}(\hat{y}_{m,0}, z_{m,0})\hat{D}_{m,1}^{(k-1)}\).
Lemma 6.6. ($\varepsilon^n$ error term) Consider the same assumptions as in Theorem 3.2 with $0 < \varepsilon < < 1$. Then the numerical solutions of the InDC method after $k$ correction loops have the following local error estimates at the interior nodes $\tau_m$ with $m = 0, \ldots, M$

$$
\varepsilon_{m,v}^{(k)} = y_{m,v} - \hat{y}_{m,v}^{(k)} = \mathcal{O}(h^{q^{(0)}+2-\nu}), \quad d_{m,v}^{(k)} = z_{m,v} - \hat{z}_{m,v}^{(k)} = \mathcal{O}(h^{q^{(0)}+1-\nu}),
$$

with $1 \leq \nu \leq q^{(0)} + 1$.

Proof. We first prove (6.36) in the case $\nu = 1$. In the prediction step ($k = 0$), under the assumption of stably accurate IRK method, by Theorem 1.4, we have that the error estimates for $\varepsilon^1$ in (1.21) at the interior nodes of the InDC method with $m = 0, \ldots, M$ satisfy

$$
ev^{(0)}_{m,1} = y_{m,1} - \hat{y}^{(0)}_{m,1} = \mathcal{O}(h^{q^{(0)}+2}), \quad d^{(0)}_{m,1} = z_{m,1} - \hat{z}^{(0)}_{m,1} = \mathcal{O}(h^{q^{(0)})}.
$$

We consider $\varepsilon$-expansions of $\hat{y}^{(1)}_{m,1}$, $\hat{z}^{(1)}_{m,1}$ and $\hat{E}^{(1)}_{m,1}$ and $\hat{D}^{(1)}_{m,1}$ as in (6.13). Inserting them onto equations (6.34), (6.35), from (6.19) and (6.20) for the power $\varepsilon^1$ with $k = 1$ and $\nu = 1$, we have

$$
\left(\begin{array}{c}
\hat{y}^{(1)}_{m+1,1} \hfill - hS^{(0)}_{m,1,\hat{F}_1} \\
\hat{z}^{(1)}_{m+1,0} \hfill - hS^{(0)}_{m,1,\hat{G}_1}
\end{array}\right) = \left(\begin{array}{c}
\hat{y}^{(1)}_{m,1} \\
\hat{z}^{(1)}_{m,0}
\end{array}\right) + h \sum_{i=1}^{s} b_i \left(\begin{array}{c}
\Delta K^{(0)}_{m,1} \\
\Delta L^{(0)}_{m,1}
\end{array}\right) + \mathcal{O}(h^{q^{(0)}+2}),
$$

and

$$
\left(\begin{array}{c}
\hat{y}^{(1)}_{m+1,0} \hfill - hS^{(0)}_{m,0,\hat{F}_1} \\
\hat{z}^{(1)}_{m+1,0} \hfill - hS^{(0)}_{m,0,\hat{G}_1}
\end{array}\right) = \left(\begin{array}{c}
\hat{y}^{(1)}_{m,0} \\
\hat{z}^{(1)}_{m,0}
\end{array}\right) + h \sum_{j=1}^{s} a_{ij} \left(\begin{array}{c}
\Delta K^{(0)}_{m,1} \\
\Delta L^{(0)}_{m,1}
\end{array}\right) + \mathcal{O}(h^{q^{(0)}+2}).
$$

Now from (4.5) we have for $\varepsilon^1$

$$
y_{m+1,1} = y_{m,1} + \int_{\tau_m}^{\tau_{m+1}} F_1(t)dt, \quad z_{m+1,0} = z_{m,0} + \int_{\tau_m}^{\tau_{m+1}} G_1(t)dt.
$$

We subtract (6.38) from (6.40) and so obtain

$$
\left(\begin{array}{c}
ev^{(1)}_{m+1,1} + hS^{(0)}_{m,1,\hat{F}_1} - \int_{\tau_m}^{\tau_{m+1}} F_1(t)dt \\
d^{(1)}_{m+1,0} + hS^{(0)}_{m,0,\hat{G}_1} - \int_{\tau_m}^{\tau_{m+1}} G_1(t)dt
\end{array}\right) = \left(\begin{array}{c}
ev^{(1)}_{m,1} \\
d^{(1)}_{m,0}
\end{array}\right) - h \sum_{i=1}^{s} b_i \left(\begin{array}{c}
\Delta K^{(0)}_{m,1} \\
\Delta L^{(0)}_{m,1}
\end{array}\right) + \mathcal{O}(h^{q^{(0)}+2}).
$$

From Theorem 1.4 and (6.37), we have the following estimates for the local errors

$$
ev^{(0)}_{m,0} = y_{m,0} - \hat{y}^{(0)}_{m,0} = \mathcal{O}(h^{q^{(0)}+1}), \quad d^{(0)}_{m,0} = z_{m,0} - \hat{z}^{(0)}_{m,0} = \mathcal{O}(h^{q^{(0)+1)}),
$$

$$
ev^{(0)}_{m,1} = y_{m,1} - \hat{y}^{(0)}_{m,1} = \mathcal{O}(h^{q^{(0)}+2}), \quad d^{(0)}_{m,1} = z_{m,1} - \hat{z}^{(0)}_{m,1} = \mathcal{O}(h^{q^{(0)})}.
$$

Similarly as done in the proof of Lemma 4.2 on the right hand-side of (6.41) we add and subtract the quantities $S^m(\hat{F}_1)$ and $S^m(\hat{G}_1)$, these are the integrals of $(M-1)^{th}$ degree interpolating polynomials on $(\tau_m, F_1(\tau_m))_{M=1}^{\tau_m}$ and $(\tau_m, G_1(\tau_m))_{M=1}^{\tau_m}$ over the subinterval $[\tau_m, \tau_m + 1]$. Hence, $\int_{\tau_m}^{\tau_{m+1}} F_1(t)dt - hS^m(\hat{F}_1) = \mathcal{O}(h^{M+1})$ and by (6.42), we have $S^m(\hat{F}_1) - S^{m(0)}_{\hat{F}_1} = \mathcal{O}(h^{q^{(0)})}$ and $S^m(\hat{G}_1) - S^{m(0)}_{\hat{G}_1} = \mathcal{O}(h^{q^{(0)})}$. Then we have from (6.41)

$$
ev^{(1)}_{m+1,1} = e^{(1)}_{m,1} - h \sum_{i=1}^{s} b_i \Delta K^{(0)}_{m,1} + \mathcal{O}(h^{q^{(0)+1)}),
$$

$$
d^{(1)}_{m+1,0} = d^{(1)}_{m,0} - h \sum_{i=1}^{s} b_i \Delta L^{(0)}_{m,1} + \mathcal{O}(h^{q^{(0)+1)}).
$$

Now we consider the $\varepsilon$-expansion of the error at internal stages $\tau_m + c_i h$, and as in equation (4.12) we get

$$
E^{(1)}_{m,1} = P^{m(1)} e^{(0)}_{m,1} - \hat{E}^{(0)}_{m,1}, \quad D^{(1)}_{m,1} = P^{m(1)} d^{(0)}_{m,1} - \hat{D}^{(0)}_{m,1}, \quad \forall k \geq 0, m.
$$

where $e^{(0)}_{m,1} = (e^{(0)}_{m,1}, \ldots, e^{(0)}_{m,1})$, $d^{(0)}_{m,0} = (d^{(0)}_{m,0}, \ldots, d^{(0)}_{m,0})$, $s$ is the number of internal stages in an IRK method. Especially, by (6.42), it follows from (6.44),

$$
\hat{E}^{(0)}_{m,1} = -E^{(1)}_{m,1} + \mathcal{O}(h^{q^{(0)}+2}), \quad \hat{D}^{(0)}_{m,1} = -D^{(1)}_{m,1} + \mathcal{O}(h^{q^{(0)})}.
$$

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Similarly as equations (6.43), from the definition of stage order for the prediction step, we have for the internal stages in vectorial form
\[
\bar{E}_1^{(1)} = e_{m,1}^{(1)}(1 - hA\Delta\bar{E}_0^{(0)}) + O(h^{q^{(0)+1}}), \\
\bar{D}_0^{(1)} = d_{m,0}^{(1)}(1 - hA\Delta\bar{E}_0^{(0)}) + O(h^{q^{(0)+1}}),
\] (6.46)
where \(\bar{E}_1 = (E_{m,1,1}, \ldots, E_{m,s,1})\), \(\bar{D}_0 = (D_{m,0,1}, \ldots, D_{m,s,0})\) and \(1 = (1, 1, \ldots, 1)^T\) is a vector of size \(s\). Now from the second equation in (6.46) and using (6.26) and (6.45), we get
\[
A(g_{y}\{y_{m,0}, z_{m,0}\})\hat{E}_{m,1}^{(0)} + g_z\{y_{m,0}, z_{m,0}\}\hat{E}_{m,1}^{(0)} = O(h^{q^{(0)})},
\] (6.47)
Thus, from the invertibility of matrix \(A\) we have
\[
\hat{D}_{m,1}^{(0)} = -g_z^{-1}g_y\{y_{m,0}, z_{m,0}\}\hat{E}_{m,1}^{(0)} + O(h^{q^{(0)})},
\] (6.48)
for \(m = 1, \ldots, ms\). Plug the above equation (6.48) into equation (6.34) and replace \(\hat{E}_{m,1}^{(0)}\) by \(E_{m,1}^{(1)}\) with \(O(h^{q^{(0)+2})\) error and \(\hat{D}_{m,1}^{(0)}\) by \(D_{m,1}^{(1)}\) with \(O(h^{q^{(0)})\) error, by (6.45) we obtain
\[
\Delta K_{m,1}^{(0)} = (f_y - f_zg_z^{-1}g_y)\{y_{m,0}, z_{m,0}\}E_{m,1}^{(1)} + O(h^{q^{(0)})}. 
\] (6.49)
Our next aim now is to prove the local error \(e_{m,1}^{(1)} = O(h^{q^{(0)+1})\) by mathematical induction w.r.t. \(m\). Especially, we would like to show that \(e_{m,1}^{(1)} = O(h^{q^{(0)+1}),\) if we assume the local error \(e_{l,1}^{(1)} = O(h^{q^{(0)+1)},\) \(\forall l \leq m\). To show this, we plug equation (6.49) into the first equation (6.46) and obtain \(E_{m,1}^{(1)} = O(h^{q^{(0)+1}),\) for \(m = 1, \ldots, ms\). From (6.49), \(\Delta K_{m,1}^{(0)} = O(h^{q^{(0)})\) and plug this estimate into the first equation of (6.43), we obtain the desired estimate of
\[
e_{m,1}^{(1)} = O(h^{q^{(0)+1}),
\] (6.50)
Thus, from (6.48) and (6.45), it follows
\[
D_{m,1}^{(1)} = O(h^{q^{(0)})}. 
\] (6.51)
Now in order to prove the estimate \(d_{m,1}^{(1)} = O(h^{q^{(0)})\), we start to considering equation (6.12). Since the IRK method is stiffly accurate, from Remark 6.1, we have \(z_{m,1}^{(1)} = \hat{z}_{m,1}^{(1)}.\) Hence from (6.51),
\[
z_{m+1}^{(1)} - z_{m+1,1}^{(1)} = d_{m+1,1}^{(1)} = D_{m+1,1}^{(1)} \frac{6.51}{O(h^{q^{(0)})}, \quad m = 0, \ldots, M-1.
\] (6.52)

The above proof can be generalized for the InDC method with different IRK methods applied to \(k\) correction steps. The local error estimates at the interior nodes of the InDC method \(\tau_m\) with \(m = 0, \ldots, M\) are
\[
e_{m,1}^{(k)} = O(h^{q^{(0)+1}), \quad d_{m,1}^{(k)} = O(h^{q^{(0)})},
\]
We have thus proved equation (6.36) with \(\nu = 1.\) The general estimates for \(\nu > 1\) in equation (6.36) can be obtained in a similar fashion to the case of \(\nu = 1,\) as in the Theorem 3.4 in Chap.VI of [9].

**Proposition 6.7.** The InDC method constructed with stiffly accurate IRK methods can be considered again a stiffly accurate IRK method with a corresponding Butcher Tableau that has the matrix \(A\) in Butcher tableau \((1.10)\) invertible.

**Proof of Theorem 3.3.** The proof is similar to that for Theorem 3.1 by using the results of Lemmas 6.3 and 6.6. Thus, we omit it for brevity.

**Remark 6.8.** We remark that we can not improve the estimate of the global error for the \(y\)-component as done in Theorem 3.4 in [9] for high-indices. Indeed the reason for such loss of accuracy is related to the evaluation of the integrals in equation (6.21). These integrals are obtained from the prediction step; and it is the algebraic variable \(z\) obtained in the prediction step that reduces the order of the differential variable \(y\). This can be seen in the evaluation from equation (6.41) to (6.43) due to (6.42). We note that a similar conclusion for the remainder can be drawn.
Now in order to estimate the remainder in the $\varepsilon$-expansion of the global error in equation (2.26), we denote the finite sum by
\[
\varepsilon_n^K = \hat{\varepsilon}_{n,0} + \varepsilon \hat{\varepsilon}_{n,1} + \cdots + \varepsilon^n \hat{\varepsilon}_{n,\nu}, \quad d_n^K = \hat{d}_{n,0} + \varepsilon \hat{d}_{n,1} + \cdots + \varepsilon^n \hat{d}_{n,\nu}.
\] (6.53)

We note that the remainder is
\[
\Delta e_n^{(K)} = e_n^{(K)} - e_n = O(\varepsilon^{\nu+1}/H), \quad \Delta d_n^{(K)} = d_n^{(K)} - d_n = O(\varepsilon^{\nu+1}/H).
\] (6.54)

The following result gives the estimate of the remainder in (2.26).

**Theorem 6.9.** Under the same hypothesis as in Theorem 3.2 for any fixed constant $c > 0$, the global error satisfies for $\varepsilon \leq cH$
\[
e_n^{(K)} = \sum_{\nu=0}^{q^{(0)}+1} e_{n,\nu}^{(K)} + O(\varepsilon^{\nu+1}/H), \quad d_n^{(K)} = \sum_{\nu=0}^{q^{(0)}+1} d_{n,\nu}^{(K)} + O(\varepsilon^{\nu+1}/H).
\] (6.55)

These estimates hold uniformly for $H \leq H_0$ and $nH \leq \text{Const}$. 

In order to prove this theorem we consider the truncated series (6.53). Using (2.26), then the statement (6.55) is equivalent to prove (6.54), and by (6.36) it suffices to prove the result for $\nu = q^{(0)} + 1$. The proof of this theorem is similar to that of Theorem 3.8, Chap. VI in [8], but unlike the proof of Theorem 3.8, because of Remark 6.8, we are not in the position to improve the estimate of $\Delta e_n^{(K)}$, then the estimate (6.55) holds.

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