Uniqueness properties of the Kerr metric

Marc Mars*
Departament de Física Fonamental, Universitat de Barcelona,
Av. Diagonal 647, 08028 Barcelona, Spain.
Fax. + 34 93 402 11 49, e-mail: marc@fismat.ffn.ub.es

November 2, 2021

Abstract

We obtain a geometrical condition on vacuum, stationary, asymptotically flat spacetimes which is necessary and sufficient for the spacetime to be locally isometric to Kerr. Namely, we prove a theorem stating that an asymptotically flat, stationary, vacuum spacetime such that the so-called Killing form is an eigenvector of the self-dual Weyl tensor must be locally isometric to Kerr. Asymptotic flatness is a fundamental hypothesis of the theorem, as we demonstrate by writing down the family of metrics obtained when this requirement is dropped. This result indicates why the Kerr metric plays such an important role in general relativity. It may also be of interest in order to extend the uniqueness theorems of black holes to the non-connected and to the non-analytic case.

PACS numbers: 0420, 0240

*Also at Laboratori de Física Matemàtica, Societat Catalana de Física, IEC, Barcelona, Spain.
1 Introduction

The Kerr metric is of fundamental importance in general relativity. Its relevance comes mainly from the uniqueness theorem for black holes, which states that, under rather general conditions, the Kerr spacetime is the only asymptotically flat, stationary, vacuum black hole. Therefore, the Kerr metric describes the exterior endstate of any sufficiently massive collapsing isolated system which obeys the cosmic censorship conjecture (assuming that some equilibrium state is reached). Despite its importance, a clear understanding of why the Kerr metric is so special remains incomplete. In comparison, the Schwarzschild metric has nice uniqueness properties like Birkhoff’s theorem or the conformal flatness of the hypersurfaces of constant static time (this second property characterizes Schwarzschild among static, asymptotically flat, vacuum spacetimes).

Obtaining a similar characterization for the Kerr metric has interest not only in order to understand better the Kerr metric but also in order to try and generalize the black hole uniqueness theorem. Indeed, the known proofs for this theorem (in the non-static case) require hypotheses which are too strong both from the mathematical and from the physical point of view. First of all, connectedness of the event horizon (i.e. the existence of a single black hole) is a fundamental hypothesis. Dropping this condition is necessary in order to exclude the existence of an equilibrium configuration containing several black holes. In the static case, the non-connected case was solved by G. L. Bunting and Masood-ul-Alam [1] by exploiting the conformal flatness of the static slices of the Schwarzschild spacetime. Another requirement in the existing proofs is the analyticity of the metric and of the event horizon (see Chruściel [2]), which is clearly a very strong and little justified assumption. Analyticity is used in order to apply the Hawking rigidity theorem [3] which proves the existence of a second Killing vector. Since this is an early step in the proof, relaxing the analyticity condition would probably force a completely new approach to the problem. Although the black hole uniqueness theorem is generally believed to be true when these two conditions are relaxed, few results in this direction are known at present (see, however, Weinstein [4], [5] for some progress in the non-connected case). We believe that obtaining a suitable characterization of the Kerr metric among stationary, asymptotically flat, vacuum metrics would be an important step forward in this problem.

Recently, using a characterization of the Kerr metric in terms of the so-called Simon tensor [6] in the manifold of trajectories, we have been able [7] to find the following geometric characterization of the Kerr metric. Let us consider the class of vacuum spacetimes admitting a Killing vector (with no restriction on its causal character). From the Killing vector $\xi$, we construct the so-called Killing form $F_{\alpha\beta} \equiv \nabla_{\alpha} \xi_{\beta} + \frac{i}{2} \eta_{\alpha\beta\lambda\mu} \nabla^{\lambda} \xi^{\mu}$ ($\eta_{\alpha\beta\lambda\mu}$ is the volume form) and the Ernst potential $\chi \equiv \lambda - i \omega$ where $\lambda = -\xi^{\alpha} \xi_{\alpha}$ and $\nabla_{\alpha} \omega = \eta_{\alpha\beta\mu} \xi^{\beta} \nabla^{\mu} \xi^{\nu}$ ($\omega$ and hence $\chi$ are defined only locally). $F_{\alpha\beta}$ is a self-dual 2-form, i.e. it satisfies $F^* = -iF$ where $*$ is the Hodge dual operator. On the other
hand, we consider the self-dual Weyl tensor from the right $C_{\alpha\beta\lambda\mu} \equiv C_{\alpha\beta\lambda\mu} + \frac{i}{2} \eta_{\lambda\mu\rho\sigma} C_{\alpha\beta}^{\rho\sigma}$ viewed as a symmetric tensor in the space of self-dual two forms (as is usually done in obtaining the Petrov type of a spacetime, see e.g. [8]). Thus, we have at hand two geometrical objects which are, in general, unrelated to each other. The theorem in [9] states, roughly speaking, that the Kerr metric is characterized by the appropriate “matching” of these two objects. More precisely,

**Theorem 1** Let $(\mathcal{M}, g)$ be a smooth, vacuum spacetime admitting a Killing vector $\vec{\xi}$. Let the Killing form be defined as above and let $\mathcal{M}$ satisfy

1. There exists a non-empty region $\mathcal{M}_f$ where $\mathcal{F}^2 \equiv \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} \neq 0$.

2. The Killing form and the Weyl tensor are related by

$$C^{\alpha\beta}_{\gamma\delta} = H \left( \mathcal{F}^{\alpha\beta} \mathcal{F}_{\gamma\delta} - \frac{1}{3} \mathcal{F}^{\alpha\beta}_{\gamma\delta} \mathcal{F}^2 \right)$$

(1)

where $\mathcal{F}^{\alpha\beta}_{\gamma\delta}$ is the identity on the space of self-dual 2-forms and $H$ is a scalar function.

Then, there exist two complex constants $l$ and $c$ such that $H = -6/(c - \chi)$ and $\mathcal{F}^2 = -l(c - \chi)^4$.

If, in addition, $c = 1$ and $l$ is real and positive, then $(\mathcal{M}, g)$ is locally isometric to the Kerr spacetime.

In reference [7], this theorem was stated in a different form. There, the hypotheses included asymptotic flatness of the spacetime and the vanishing of the so-called spacetime Simon tensor. However, as it was noticed at the end of Remark 1, asymptotic flatness was used only in order to prove Lemma 5 in that paper, which just fixes the value of two complex constants. On the other hand, the vanishing of the spacetime Simon tensor can be replaced by hypothesis 2 above by virtue of lemma 4 in [7]. It is easy to see that theorem 1 holds following essentially the same steps as in the proof of the theorem stated in [7]. We have preferred to state the theorem in this new form in order to emphasize that asymptotic flatness plays no role in the characterization of Kerr obtained in [7] (i.e. the characterization is purely local). If asymptotic flatness is imposed on $(\mathcal{M}, g)$ then condition 1 in the theorem can be dropped and the constants $l$ and $c$ appearing in condition 2 are fixed automatically to its values $c = 1$ and $l > 0$ by a suitable rescaling of the Killing vector.

Theorem 1 characterizes Kerr in a neat and geometrical way. However, it is “too local” for our purposes. Indeed, the condition of asymptotic flatness on the spacetime is of fundamental importance in the proof of the black hole uniqueness theorem (if this condition is dropped, then the uniqueness of the Kerr black hole is not true). On the other hand, the characterization of Kerr in theorem 1 does not use asymptotic flatness.
at all. This suggests very strongly that asymptotic flatness has not been exploited in its full extent in order to characterize the Kerr metric among stationary, vacuum and *asymptotically flat* spacetimes. It is plausible to believe that by employing asymptotic flatness in a more essential way, the geometric condition (1) involving the Weyl tensor and the Killing form can be relaxed significantly. In that case, we would certainly be in a better position for generalizing the black hole uniqueness theorems.

In this paper we present a generalization of Theorem 1 along these lines. The main theorem in this paper is described in the next subsection, while its proof is left for the following sections.

### 1.1 Main result and discussion

The first task to consider is trying to guess which condition should replace (1). To do that let us describe in some more detail the geometric content of (1). An arbitrary self-dual 2-form like $\mathcal{F}_{\alpha\beta}$ can be algebraically classified by analyzing its associated eigenvalue problem. At points where $\mathcal{F}_{\alpha\beta}$ is regular (i.e. $\mathcal{F}^2 \neq 0$), there exist two principal null directions defined as the real eigenvectors of $\mathcal{F}_{\alpha\beta}$, i.e.

$$\mathcal{F}_{\beta}^\alpha l^3 \propto l^\alpha. \quad (2)$$

At points where $\mathcal{F}_{\alpha\beta}$ is singular (i.e $\mathcal{F}^2 = 0$ with $\mathcal{F}_{\alpha\beta} \neq 0$) there exists one principal null direction. Similarly, the Petrov classification of the Weyl tensor is the algebraic classification of the endomorphism of the space of self-dual 2-forms defined by $\mathcal{C}(\mathcal{X})_{\alpha\beta} = \mathcal{C}_{\mu\nu} \mathcal{X}^\mu_{\rho\sigma}$. There exist four principal null directions of the Weyl tensor (which degenerate when the Weyl tensor is algebraically special). A repeated principal null direction of the Weyl tensor is a non-zero null vector field $\vec{l}$ satisfying $\mathcal{C}_{\alpha\beta\gamma\delta} l^\beta l^\delta \propto l_\alpha l_\gamma$. The Petrov classification of a Weyl tensor satisfying (1) is very simple. The Petrov type is $D$ whenever $H \mathcal{F}^2 \neq 0$, $N$ when $H \neq 0$, $\mathcal{F}^2 = 0$ and $0$ when $H = 0$. Moreover, at points where $\mathcal{F}_{\alpha\beta}$ is regular, each principal null direction of $\mathcal{F}_{\alpha\beta}$ is a double principal null direction of the Weyl tensor (this is an alternative way of writing down the characterization of Kerr given in Theorem 1). So, Theorem 1 involves a condition on the Weyl tensor that already determines its Petrov type. This is a strong local condition. We would like to generalize it so that the Petrov type remains undetermined. A possible way of doing this is by demanding

$$\mathcal{C}^{\alpha\beta\gamma\delta} \mathcal{F}_{\gamma\delta} \propto \mathcal{F}^{\alpha\beta}. \quad (3)$$

This condition is obviously satisfied by (1) but it is much weaker than (1). Indeed, while condition (1) restricts the full form of the Weyl tensor in the spacetime (and therefore imposes strong conditions on the Petrov type), condition (3) demands just that the Killing form is an eigenvector of the self-dual Weyl tensor. A priori, this condition does not restrict the Petrov type of the spacetime whatsoever. Moreover,
the essence of condition (1) (i.e. that the two geometrical objects $F_{\alpha\beta}$ and $C_{\alpha\beta\gamma\delta}$ “match” together) is retained. This indicates that (3) may be the generalization we are seeking. The main objective of this paper is proving the validity of this result. The precise theorem we prove is

**Theorem 2** Let $(\mathcal{M}, g)$ be a smooth, vacuum spacetime with the following properties

1. $(\mathcal{M}, g)$ admits a Killing field $\vec{\xi}$ such that its Killing form $F_{\alpha\beta}$ is an eigenvector of the self-dual Weyl tensor $C_{\alpha\beta\gamma\delta}$

\[ C_{\alpha\beta\gamma\delta} F^{\gamma\delta} \propto F_{\alpha\beta}. \]  

2. $(\mathcal{M}, g)$ contains a stationary, asymptotically flat four-end $\mathcal{M}_\infty$, $\vec{\xi}$ tends to a time translation at infinity in $\mathcal{M}_\infty$ and the Komar mass of $\vec{\xi}$ in $\mathcal{M}_\infty$ is non-zero.

Then $(\mathcal{M}, g)$ is locally isometric to a Kerr spacetime.

**Remark 1.** A stationary asymptotically flat four-end is an open submanifold $\mathcal{M}_\infty \subset \mathcal{M}$ diffeomorphic to $I \times \left( \mathbb{R}^3 \setminus B(R) \right)$, $(I \in \mathbb{R}$ is an open interval and $B(R)$ is a closed ball of radius $R)$, such that, in the local coordinates $\{t, x^i\}$ defined by the diffeomorphism, the metric satisfies

\[ |g_{\mu\nu} - \eta_{\mu\nu}| + |r \partial_t g_{\mu\nu}| \leq Cr^{-\alpha}, \quad \partial_t g_{\mu\nu} = 0 \]

where $C, \alpha$ are positive constants, $r = \sqrt{\sum (x^i)^2}$ and $\eta_{\mu\nu}$ is the Minkowski metric. Usually, the definition of asymptotically flat four-end requires $I = \mathbb{R}$ but we do not need this restriction. The Einstein field equations together with the existence of a timelike Killing vector force $\alpha \geq 1$ (Kennefick and Ó Murchadha [9]). Then, it is well-known (see e.g. [10]) that the metric in the asymptotic region can be written in the form

\[ g_{00} = -1 + \frac{2M}{r} + O(r^{-2}), \quad g_{0i} = -\epsilon_{ijk} \frac{4SJ^j x^k}{r^3} + O(r^{-3}), \quad g_{ij} = \delta_{ij} + O(r^{-1}), \]  

where $M$ is the Komar mass [11] of $\vec{\xi}$ in the asymptotically flat end $\mathcal{M}_\infty$ (and hence non-zero by assumption) and $\epsilon_{ijk}$ is the alternating Levi-Civita symbol.

**Remark 2.** The Kerr spacetime is understood to be the maximal analytic extension of the Kerr metric, as described by Boyer and Lindquist [12] and Carter [13]. An element of the Kerr family will be denoted by $(\mathcal{M}_{M,a}, g_{M,a})$, where $M$ denotes the Komar mass and $a$ the specific angular momentum. The particular case when $a = 0$ corresponds to the Kruskal extension of the Schwarzschild spacetime.

**Remark 3.** The conclusion of the theorem is that $(\mathcal{M}, g)$ is locally isometric to the Kerr spacetime. This concept is standard; it means that for any point $p \in \mathcal{M}$, there
exists an open neighbourhood $U_p$ of $p$ which is isometrically diffeomorphic to an open submanifold of $M_{M,a}$. Despite the fact that the characterization of Kerr in the theorem involves a local condition (I) and a global condition (asymptotic flatness), we should not expect in principle that the local isometry extends to an isometric embedding of $(M, g)$ into $(M_{M,a}, g_{M,a})$. The reason is that there may exist suitable identifications in the Kerr spacetime that define a spacetime which is still asymptotically flat in the sense we are using (and the local condition 1 would obviously be still satisfied). Let us emphasize, however, that Theorem 2 is "semi-local" because the existence of the local isometry is shown everywhere. Actually, showing that the result holds everywhere constitutes a substantial part of the proof.

While the methods in the proof of Theorem 2 are similar to those in Theorem 1, the complexity of the proof increases notably. The reason is, of course, that the local assumptions are now much weaker and instead asymptotic flatness must be exploited in a stronger way. In particular, the local condition (I) does not restrict the Petrov type and several cases must be analyzed. Moreover, the interplay between different regions where the Petrov type may change requires special care. We refer to Hall [14] (see also [15] for a generalization to other continuous endomorphisms) for general restrictions on the regions with different Petrov types.

Our main objective in this paper is obtaining a characterization of the Kerr metric among stationary, vacuum, asymptotically flat spacetimes which uses the asymptotic properties in a fundamental way. To show that Theorem 2 achieves this, we will prove the theorem trying to separate the use of the local conditions from the use of asymptotic flatness. This will be particularly so in Proposition 2 below where the local condition 1 is imposed but asymptotic flatness is not used at all. This proposition gives, essentially, the local form of the metric on the region where the Petrov type is II (or D or 0). This family of metrics is rather large and contains arbitrary functions in general. This shows that the local condition 1 is weak (it is fulfilled by a large family of metrics) and only when combined with asymptotic flatness gives a characterization of Kerr. We consider it difficult to relax further the local condition 1 and still maintain the conclusion of the theorem. So, in some sense, asymptotic flatness has been exploited "as much as possible" to characterize Kerr. An eventual proof of the black hole uniqueness theorem would require exploiting the existence of a regular black hole in order to try and prove that the local condition 1 holds. This problem is under current investigation.

The paper is organized as follows. In section 2, we introduce our notation and write down several identities which are valid on any vacuum spacetime admitting a Killing vector $\vec{\xi}$. These identities and definitions will be used thoroughly in this paper. We also write down some of the consequences of (I) and discuss the different possibilities for the Petrov type at points where $\mathcal{F}^2 \neq 0$.

The strategy of the proof consists in defining first an open, connected and asymptotically flat subset $\mathcal{M}_f \subset \mathcal{M}$ where $\mathcal{F}^2 \neq 0$ everywhere and restricting the analysis to $(\mathcal{M}_f, g_{\mathcal{M}_f})$. The bulk of the proof is showing that the conclusion of the theorem
holds on $\mathcal{M}_f$. Afterwards, we prove that $\mathcal{M}_f = \mathcal{M}$. In section 3, we show that the region $\mathcal{M}_f$ must be of Petrov type II, D or 0 (thus excluding Petrov types III, N and I on $\mathcal{M}_f$). Actually, Petrov type III is easily seen to be incompatible with (4) on $\mathcal{M}_f$, so only Petrov types N and I must be studied. Petrov type N is dealt with easily but excluding Petrov type I requires a rather involved argument that already uses several properties of asymptotic flatness. In section 4 we analyze in detail the Petrov type II (and its particularization to Petrov type D). First we find the local form of the metric when the condition of asymptotic flatness is dropped. As discussed above, the resulting family of metrics is rather large. The remaining part of section 4 is devoted to show that $\left(\mathcal{M}_f, g|_{\mathcal{M}_f}\right)$ is locally isometric to Kerr. Finally we prove that $\mathcal{M}_f = \mathcal{M}$.

Before starting with the details, let us make two final comments. First, the proof of the theorem uses, obviously, the Einstein vacuum equations extensively. Since our local condition 1 involves directly the Weyl tensor, it turns out that the Newman-Penrose (NP) formalism is well-adapted to the proof. However, using such a formalism requires some care because we are not assuming analyticity of the spacetime and therefore adapting the null tetrad so that some spin coefficients vanish requires an existence proof which may be difficult. So, we will avoid adapting the tetrad except when this existence issue is easily solved. Second, since the null tetrads in the NP formalism are defined only locally (i.e. in a suitable neighbourhood of each point) we write explicitly the domain of validity of the expressions involved (sometimes we indicate this domain in the text immediately before the formulas). This makes the exposition more precise but forces a slightly cumbersome notation. For the sake of clarity we have collected the various definitions that we need, even though they may not be used immediately. These definitions appear at the end of sections 3 and 4 and they should be looked at for reference.

### 2 Notation, definitions and basic equations.

A $C^n$ spacetime denotes a paracompact, Hausdorff and connected $C^{n+1}$ four-dimensional manifold endowed with a $C^n$ metric of signature $(-1,1,1,1)$. Smooth means $C^\infty$. In this paper, all spacetimes are assumed to be oriented with metric volume form $\eta^{\alpha\beta\gamma\delta}$. The conventions and notation for the Riemann and Ricci tensors follow [3]. Throughout this paper $(\mathcal{M}, g)$ will denote a spacetime satisfying the hypotheses of Theorem 2. The norm and twist of the Killing vector $\xi$ are defined as $\lambda = -\xi^\alpha \xi_\alpha$ and $\omega_\alpha = \eta_{\alpha\beta\gamma\delta} \xi_\beta \nabla^\gamma \xi_\delta$ respectively. As mentioned in the introduction, it is convenient to employ self-dual 2-forms, which are complex 2-forms $\mathcal{B}$ satisfying $\mathcal{B}^* = -i \mathcal{B}$, where $*$ is the Hodge dual operator. In particular, the 2-form $F_{\alpha\beta} = \nabla_\alpha \xi_\beta$ and the so-called Killing form $\mathcal{F}_{\alpha\beta} \equiv F_{\alpha\beta} + i F^*_{\alpha\beta}$ (which is self-dual by definition) will play a fundamental role. The Ernst one-form is defined as $\chi_\mu \equiv 2 \xi^\alpha \mathcal{F}_{\alpha\mu} = \nabla_\mu \lambda - i \omega_\mu$. 


From the Killing equations $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$ it follows $\nabla_\mu \nabla_\alpha \xi_\beta = \xi^\nu C_{\nu \mu \alpha \beta}$, where $C_{\alpha \beta \gamma \delta}$ is the Weyl tensor. Consequently, we have

$$\nabla_\mu F_{\alpha \beta} = \xi^\nu C_{\nu \mu \alpha \beta},$$

(6)

where $C_{\nu \mu \alpha \beta} \equiv C_{\nu \mu \alpha \beta} + \frac{i}{2} \eta_{\alpha \beta \rho \sigma} C_{\nu \rho \sigma}$ is the so-called self-dual Weyl tensor from the right. In vacuum, this tensor is a double symmetric, 2-form, trace-free and satisfies the first and second Bianchi identities $C_{\alpha [\beta \gamma \delta]} = 0$, $\nabla^\nu C_{\nu \mu \alpha \beta} = 0$. Using well-known identities involving self-dual 2-forms (see e.g. [7] for a brief summary) we easily find

$$\chi_\alpha \chi^\alpha = -\lambda F^2,$$

$$\nabla_\alpha \chi^\beta - \nabla_\beta \chi^\alpha = 0,$$

$$\nabla^\alpha \chi_\alpha = -F^2,$$

(7)

where $F^2 \equiv F_{\alpha \beta} F^{\alpha \beta}$. The second expression shows that the Ernst one-form is closed and hence locally exact. Similar, although longer calculations, allow us to establish the following two identities, which will be useful in the following

$$\nabla_\mu F^2 = 2 \xi^\nu C_{\nu \mu \alpha \beta} F^{\alpha \beta},$$

$$\nabla_\alpha \nabla^\alpha F^2 = -C_{\alpha \beta \lambda \mu} F^{\alpha \beta} F^{\lambda \mu} - \frac{\lambda}{2} C_{\alpha \beta \lambda \mu} C^{\alpha \beta \lambda \mu}. $$

(8)

(9)

Let us now define the subset $\tilde{M}_f = \{ q \in M; F^2|_q \neq 0 \}$. Since by assumption $\tilde{\xi}$ tends to a time translation at infinity, $\tilde{\xi}$ can be normalized without loss of generality so that $\lambda \to 1$ at the asymptotically flat end $M_\infty$. Then, the asymptotic form of the metric in $M_\infty$ (see Remark 1 after Theorem 2) implies $F^2 = -4M^2/r^4 + O(r^{-5})$ which implies that $\tilde{M}_f$ is not empty (the Komar mass $M$ is non-zero by hypothesis). Thus, we can assume (perhaps after restricting $M_\infty$ to a proper subset) that $M_\infty \subset M_f$ and hence the spacetime $(M_f, g_f)$ can be defined as the connected component of $\tilde{M}_f$ containing $M_\infty$, with the induced metric. The scalar product of two vectors $\vec{u}$ and $\vec{v}$ on $(M_f, g_f)$ will be denoted simply by $(\vec{u}, \vec{v})$.

By definition, the complex function $F^2$ does not vanish anywhere on $M_f$. Let us consider the eigenvalue problem $F_{\alpha \beta} V^\beta \propto V_\alpha$ with $V^\alpha$ real (so, we are actually considering two different eigenvalue problems, namely the real and the imaginary parts of the previous one). At points where $F_{\alpha \beta}$ is regular (i.e. $F^2 \neq 0$) this eigenvalue problem is known to have two simple eigenvalues $f$ and $-f$ satisfying $f^2 = -F^2/4$ (this being a direct consequence of $F_{\mu \beta} F^{\nu \beta} = (1/4)g_{\mu \nu} F^2$, which is a standard property of a self-dual 2-form). Moreover, simple roots of a polynomial depend smoothly on the polynomial coefficients (see e.g. [16]). The characteristic polynomial of the eigenvalue problem $F_{\alpha \beta} V^\beta \propto V_\alpha$ (resp. $F^{* \alpha \beta} V^\beta \propto V_\alpha$ ) is biquadratic and has exactly two real simple roots (whenever $F^2 \neq 0$) which correspond to the real (resp. imaginary) parts of $f$ and $-f$. Consequently $f$ and $-f$ depend smoothly on the coefficients of $F_{\alpha \beta}$ and therefore are smooth functions on $M_f$ (this fact is not obvious a priory because $M_f$ need not be simply connected and therefore the square root of $F^2$ might not be globally...
defined in $\mathcal{M}_f$). Let us also call $\vec{k}$ and $\vec{l}$ the two corresponding real eigenvectors, i.e. $\mathcal{F}_{\alpha\beta}k^\beta = f k_\alpha$ and $\mathcal{F}_{\alpha\beta}l^\beta = -f l_\alpha$. These vectors are null and can be normalized, without loss of generality, so that $(\vec{l}, \vec{k}) = -1$. Using the fact that $\mathcal{F}_{\alpha\beta}$ is self-dual we can write

$$\mathcal{F}_{\alpha\beta}|_{\mathcal{M}_f} = f \left(-k_\alpha l_\beta + k_\beta l_\alpha - i\eta_{\alpha\beta\gamma\delta}k^\gamma l^\delta\right)|_{\mathcal{M}_f} \implies (10)$$

$$\chi_\beta|_{\mathcal{M}_f} = 2f \left[-(\vec{\xi}, \vec{k}) l_\beta + (\vec{\xi}, \vec{l}) k_\beta - i\eta_{\alpha\beta\gamma\delta}\xi^\alpha k^\gamma l^\delta\right]|_{\mathcal{M}_f} \tag{11}$$

This expression shows, in particular, that $\vec{k}$ and $\vec{l}$ are smooth vector fields on $\mathcal{M}_f$. A simple consequence of the first equation, together with the fact that $\mathcal{L}_\vec{\xi}\mathcal{F}_{\alpha\beta} = 0$ (where $\mathcal{L}$ denotes Lie derivative) and $(\vec{k}, \vec{l}) = -1$, is

$$\left[\vec{\xi}, \vec{k}\right] = C_1 \vec{k}|_{\mathcal{M}_f}, \quad \left[\vec{\xi}, \vec{l}\right] = -C_1 \vec{l}|_{\mathcal{M}_f} \tag{12}$$

where $C_1$ is a smooth function on $\mathcal{M}_f$. Let us define the self-dual 2-form $W_{\alpha\beta}|_{\mathcal{M}_f} \equiv f^{-1}\mathcal{F}_{\alpha\beta}|_{\mathcal{M}_f}$ which obviously satisfies $W_{\alpha\beta}W^{\alpha\beta} = -4$. The space of self-dual 2-forms at any point $q \in \mathcal{M}_f$ is a complex three-dimensional vector space. We can therefore complete $W|_q$ with two self-dual 2-forms $U|_q$ and $V|_q$ such that $(U, V, W)|_q$ forms a basis. Furthermore, it is well-known that $U|_q$ and $V|_q$ can be chosen so that $U_{\alpha\beta}V^{\alpha\beta}|_q = 2$ and all the remaining contractions vanish. This choice of $U$ and $V$ can be made locally smooth (i.e. given any point $q \in \mathcal{M}_f$ there exists an open neighbourhood of $q$ where $U$ and $V$ are smooth) but this choice may not exist globally on $\mathcal{M}_f$. So, all the expressions involving $U$ and $V$ should be understood to hold in a neighbourhood of each point in $\mathcal{M}_f$. Since $\mathcal{C}_{\alpha\beta\gamma\delta}$ is a double, symmetric, self-dual 2-form, it can be expanded in this basis. The hypothesis 1 of the theorem implies the existence of a function $\Psi_2$ on $\mathcal{M}$ defined by $\mathcal{C}_{\alpha\beta\gamma\delta}\mathcal{F}^{\gamma\delta} = -8\Psi_2\mathcal{F}_{\alpha\beta}$ (i.e. the eigenvalue corresponding to the eigenvector $\mathcal{F}_{\alpha\beta}$). Since $8\Psi_2\mathcal{F}^2 = -\mathcal{C}_{\alpha\beta\gamma\delta}\mathcal{F}^{\alpha\beta}\mathcal{F}^{\gamma\delta}$ and $\mathcal{F}^2$ is non-zero on $\mathcal{M}_f$ it follows that $\Psi_2$ is smooth on $\mathcal{M}_f$. Using the symmetries of $\mathcal{C}_{\alpha\beta\gamma\delta}$ we obtain the decomposition

$$\mathcal{C}_{\alpha\beta\gamma\delta}|_{\mathcal{M}_f} = 2\Psi_0 U_{\alpha\beta}U_{\gamma\delta} + 2\Psi_4 V_{\alpha\beta}V_{\gamma\delta} + 2\Psi_2 \left(W_{\alpha\beta}W_{\gamma\delta} + U_{\alpha\beta}V_{\gamma\delta} + V_{\alpha\beta}U_{\gamma\delta}\right)|_{\mathcal{M}_f}, \tag{13}$$

where $\Psi_0$ and $\Psi_4$ are locally smooth complex functions (i.e. they are smooth on the same neighbourhood where $(U, V, W)$ is a smooth basis). The complex scalar $\mathcal{C}_{\alpha\beta\gamma\delta}\mathcal{C}^{\alpha\beta\gamma\delta}|_{\mathcal{M}_f} = 32(\Psi_0\Psi_4 + 3\Psi_2^2)|_{\mathcal{M}_f}$ is obviously smooth everywhere and therefore $\Psi_0\Psi_4$ is also a global smooth function on $\mathcal{M}_f$. Let us now write down the identities (8), (9) on $\mathcal{M}_f$. In terms of $f$, (8) becomes simply

$$\nabla_\alpha f = \frac{\Psi_2}{f} \chi_\alpha |_{\mathcal{M}_f}, \tag{14}$$
which implies $\chi[\alpha \nabla \beta] \Psi_2|_{\mathcal{M}_f} = 0$ because $\chi_\alpha$ is a closed one-form. Since $\chi_\alpha$ cannot vanish identically on any non-empty open subset of $\mathcal{M}_f$ (otherwise the third equation in (7) would imply $\mathcal{F}^2 = 0$ which cannot happen on $\mathcal{M}_f$) it follows $\nabla_\alpha \Psi_2|_{\mathcal{M}_f} = G_1\chi_\alpha|_{\mathcal{M}_f}$ for some smooth complex function $G_1$. Inserting this expression into (9) a short calculation gives us an explicit expression for $G_1$, namely $G_1|_{\mathcal{M}_f} = (2f^2)^{-1}(\Psi_0\Psi_4 + 3\Psi_2^2)|_{\mathcal{M}_f}$. It turns out to be convenient to define a complex one-form $P_\alpha = (2f^2)^{-1}\chi_\alpha$ which will be used thoroughly from now on. Notice that (14) implies that $P_\alpha$ is closed and hence locally exact. In terms of $P_\alpha$, the expressions discussed above take the form

$$\chi_\alpha = 2fP_\alpha|_{\mathcal{M}_f}, \quad \nabla_\alpha f = 2\Psi_2P_\alpha|_{\mathcal{M}_f}, \quad \nabla_\alpha \Psi_2 = \left(\Psi_0\Psi_4 + 3\Psi_2^2\right)P_\alpha|_{\mathcal{M}_f},$$

$$P_\beta = -\left(\tilde{\xi}, \tilde{k}\right)l_\beta + \left(\tilde{\xi}, \tilde{l}\right)k_\beta - i\eta_{\alpha\beta\gamma\delta}\xi^\alpha k^\gamma l^\delta|_{\mathcal{M}_f}. \quad (15)$$

As discussed in the introduction, we will need to work on different subsets of $\mathcal{M}_f$ to prove the theorem. The relevant definitions are

$$\mathcal{M}_1 = \{p \in \mathcal{M}_f; \Psi_2|_p = 0\}, \quad \mathcal{M}_2 = \{p \in \mathcal{M}_f; \Psi_2|_p \neq 0 \text{ and } \Psi_0\Psi_4|_p \neq 0\},$$

$$\mathcal{M}_3 = \{p \in \mathcal{M}_f; \Psi_2|_p 
eq 0 \text{ and } \Psi_0\Psi_4|_p = 0\}. \quad (16)$$

By definition, these three sets are a partition of $\mathcal{M}_f$. A simple analysis of the Petrov types compatible with the Weyl tensor (13) on $\mathcal{M}_f$ shows that Petrov type III is impossible. The points where the Petrov type is N are contained in $\mathcal{M}_1$, the points where the Petrov type is I are contained in $\mathcal{M}_1 \cup \mathcal{M}_2$ and the Petrov type on $\mathcal{M}_3$ is II (or D wherever $\Psi_0 = \Psi_4 = 0$).

Given any subset $U$ of $\mathcal{M}_f$ we will denote its interior in the topology of $\mathcal{M}_f$ by $\overset{\circ}{U}$ (occasionally we will also use $\text{int}(U)$). The topological boundary of $U$ is denoted by $\partial U$.

To conclude this section we introduce a concept that will be useful later in order to write down several lemmas in a more compact form.

**Definition 1** An open subset $U \subset \mathcal{M}_f$ will be called **spatially bounded with respect to** $\mathcal{M}_\infty$ if there exists an asymptotically flat, open, submanifold $\hat{\mathcal{M}}_\infty \subset \mathcal{M}_\infty$ such that $U \cap \hat{\mathcal{M}}_\infty = \emptyset$.

### 3 Excluding Petrov type I and N on $\mathcal{M}_f$.

Let us start by proving two simple lemmas

**Lemma 1** Let $(\mathcal{M}_f, g_f)$ be constructed as in section [3]. Then, the set of points $\{q \in \mathcal{M}_f; \nabla_\alpha \lambda|_q = 0\}$ has empty interior.
Proof: Assume that there exists a non-empty open set \( U \subset \mathcal{M}_f \) where \( \nabla_\alpha \lambda \) vanishes identically. From the third identity in (7) it follows that \( \mathcal{F}^2 \) is purely imaginary. Then, the first identity in (7) reads \( \omega_\alpha \omega^\alpha|_U = \lambda \mathcal{F}^2|_U \) which immediately implies \( \lambda|_U = 0 \) and \( \omega_\alpha \omega^\alpha|_U = 0 \). Since \( \xi^\alpha \omega_\alpha = 0 \) everywhere and using the fact that the set of zeros of a Killing vector has empty interior [17], it follows that there exists a smooth function \( S \) on \( U \) such that \( \omega_\alpha S \xi^\alpha|_U = 0 \). Using \( \bar{\xi}^{(S)}(S) = 0 \), which follows from \( \mathcal{L} \xi \omega_\alpha = 0 \), the third identity in (7) implies

\[
\nabla^\alpha \chi_\alpha = -i (\xi^\alpha \nabla_\alpha S + S \nabla_\alpha \xi^\alpha) = 0 = -\mathcal{F}^2,
\]

which is impossible on \( U \subset \mathcal{M}_f \).

\[\square\]

Lemma 2 Let \((\mathcal{M}, g)\) be a spacetime satisfying the hypotheses of Theorem 2 and let \( \mathcal{M}_f \) and \( \mathcal{M}_1^0 \) be defined as in section 2. Then, the set \( \mathcal{M}_1^0 \) is spatially bounded with respect to \( \mathcal{M}_\infty \).

Proof: \( \mathcal{M}_1^0 \) is open by definition. If \( \mathcal{M}_1^0 \) is empty the statement is trivial. If \( \mathcal{M}_1^0 \) is non-empty, then equation (14) shows that \( f \) and hence \( \mathcal{F}^2 \) are constant on each connected component of \( \mathcal{M}_1^0 \). As we discussed above, the asymptotic form of the metric in the asymptotically flat end \( \mathcal{M}_\infty \) implies \( \mathcal{F}^2 = -4M^2/r^4 + O(r^{-5}) \). Hence, there exists an asymptotically flat open submanifold \( \mathcal{M}_\infty \) where \( \nabla_\alpha \mathcal{F}^2 \) is nowhere zero. Thus \( \mathcal{M}_1^0 \cup \mathcal{M}_\infty = \emptyset \) and the lemma is proven.

\[\square\]

Take now an arbitrary point \( p \in \mathcal{M}_f \) and choose an open, connected neighbourhood \( U_p \) of \( p \) where the basis \((U, V, W)\) can be chosen smoothly. It is well-known [18] that there exists a smooth complex vector \( \tilde{m} \) on \( U_p \) such that \( \{\tilde{k}, \tilde{l}, \tilde{m}, \tilde{m}\} \) (the bar denotes complex conjugate) is a positively oriented null tetrad (i.e. \( \eta_{\alpha\beta\delta}k^\alpha l^\beta m^\gamma \overline{m}^\delta = -i \)) and such that \( U, V \) and \( W \) take the form

\[
U_{\alpha\beta}|_{U_p} = -l_\alpha m_\beta + l_\beta m_\alpha|_{U_p}, \quad V_{\alpha\beta}|_{U_p} = k_\alpha m_\beta - k_\beta m_\alpha|_{U_p}, \quad W_{\alpha\beta}|_{U_p} = -k_\alpha l_\beta + k_\beta l_\alpha + m_\alpha m_\beta - m_\beta m_\alpha|_{U_p}.
\]

(17)

We shall use the Newman-Penrose (NP) notation to denote the Ricci rotation coefficients associated with this null basis. Our conventions will follow [8]. The names \( \Psi_0 \), \( \Psi_2 \) and \( \Psi_4 \) above were chosen precisely so that they correspond to the spin coefficients of the Weyl tensor. In particular, [13] shows that \( \Psi_1 = \Psi_3 = 0 \). For completeness, let us write down the Newman-Penrose equations. They are just the standard definition of the curvature in terms of the connection when everything is expressed in the null
basis \( \{ \vec{k}, \vec{l}, \vec{m}, \vec{m} \} \) and therefore they only hold on \( U_p \)

\[
D\rho - \delta\kappa = \rho^2 + \sigma\sigma + \rho (\epsilon + \tau) - \pi\tau - \kappa (3\alpha + \beta - \pi) \tag{18}
\]

\[
D\sigma - \delta\kappa = \sigma (\rho + \pi) + (3\epsilon - \tau) \sigma - (\tau - \pi + \pi + 3\beta) \kappa + \Psi_0 \tag{19}
\]

\[
D\tau - \Delta\kappa = \rho (\tau + \pi) + \sigma (\tau + \pi) + \tau (\epsilon - \tau) - \kappa (3\gamma + \gamma) + \Psi_1 \tag{20}
\]

\[
D\alpha - \delta\epsilon = \alpha (\rho + \theta - 2\epsilon) + \sigma\beta - \beta\epsilon - \kappa\lambda - \pi\gamma + \pi (\epsilon + \rho) \tag{21}
\]

\[
D\beta - \delta\epsilon = \sigma (\alpha + \pi) + \beta (\tau - \tau - \kappa (\mu + \gamma) - \epsilon (\alpha - \pi) + \Psi_1 \tag{22}
\]

\[
D\gamma - \Delta\epsilon = \alpha (\tau + \pi) + \beta (\tau + \pi) - \gamma (\epsilon + \tau) - \epsilon (\gamma + \gamma) + \pi\tau - \nu\kappa + \Psi_2 \tag{23}
\]

\[
D\lambda - \delta\pi = \rho \lambda + \sigma\mu + \pi\pi + \pi (\alpha - \beta) - \nu\pi - \lambda (3\epsilon - \tau) \tag{24}
\]

\[
D\mu - \delta\pi = \beta\mu + \sigma\lambda + \pi\pi - \mu (\epsilon + \tau) - \pi (\tau - \beta) - \nu\kappa + \Psi_2 \tag{25}
\]

\[
D\nu - \Delta\pi = \mu (\pi + \tau) + \lambda (\pi + \tau) + \pi (\gamma - \tau) - \nu (3\epsilon + \tau) + \Psi_3 \tag{26}
\]

\[
\Delta\lambda - \delta\nu = -\lambda (\mu + \pi) - \lambda (3\gamma - \tau) + \nu (3\alpha + \beta + \pi - \tau) - \Psi_4 \tag{27}
\]

\[
\delta\rho - \delta\sigma = \rho (\tau + \beta) - \sigma (3\alpha - \beta) + \tau (\rho - \pi) + \kappa (\mu - \pi) - \Psi_1 \tag{28}
\]

\[
\delta\alpha - \beta\beta = \mu\rho - \lambda\sigma + \alpha\alpha + \beta\beta - 2\alpha\beta + \gamma (\rho - \pi) + \epsilon (\mu - \pi) - \Psi_2 \tag{29}
\]

\[
\delta\lambda - \delta\mu = \nu (\rho - \pi) + \pi (\mu - \pi) + \mu (\alpha + \beta) + \lambda (\tau - 3\beta) \tag{30}
\]

\[
\delta\nu - \Delta\mu = \mu^2 + \lambda\lambda + \mu (\gamma + \gamma) - \tau\pi + \nu (\tau - 3\beta - \alpha) \tag{31}
\]

\[
\delta\gamma - \Delta\beta = \gamma (\tau - \alpha - \beta) + \mu\tau - \sigma\nu - \epsilon\pi - \beta (\gamma - \gamma - \mu) + \alpha\lambda \tag{32}
\]

\[
\delta\tau - \Delta\sigma = \mu\sigma + \lambda\rho + \tau (\beta + \beta - \tau) - \sigma (3\gamma - \tau) - \kappa\pi \tag{33}
\]

\[
\Delta\rho - \delta\tau = -\rho\pi - \sigma\lambda + \tau (\beta - \alpha - \tau) + \rho (\gamma + \gamma) + \nu\kappa - \Psi_2 \tag{34}
\]

\[
\Delta\alpha - \delta\gamma = \nu (\rho + \epsilon) - \lambda (\tau + \beta) + \alpha (\tau - \pi) + \gamma (\beta - \tau) \tag{35}
\]

Taking into account the third equation in (13), the only non-trivial second Bianchi identities for the Weyl tensor can be written on \( U_p \) as

\[
\bar{\delta}\Psi_0 = \Psi_0 (4\alpha - \pi) + 3\kappa\Psi_2, \tag{36}
\]

\[
\Delta\Psi_0 = \Psi_0 (4\gamma - \mu) + 3\sigma\Psi_2, \tag{37}
\]

\[
-D\Psi_4 = \Psi_4 (4\rho - \pi) + 3\lambda\Psi_2, \tag{38}
\]

\[
-\delta\Psi_4 = \Psi_4 (4\beta - \tau) + 3\nu\Psi_2, \tag{39}
\]

while the commutators on \( U_p \) between the vectors \( \{ \vec{k}, \vec{l}, \vec{m}, \vec{m} \} \) read

\[
\Delta D - D\Delta = (\gamma + \pi) D + (\epsilon + \tau) \Delta - (\tau + \pi) \delta - (\tau + \pi) \delta, \tag{40}
\]

\[
\delta D - D\delta = (\pi + \beta - \pi) D + \kappa\Delta - \sigma\delta - (\pi + \nu - \tau) \delta, \tag{41}
\]

\[
\delta\Delta - \Delta\delta = -\pi D + (\tau - \tau - \beta) \Delta + \lambda\delta + (\mu - \gamma + \gamma) \delta, \tag{42}
\]

\[
\bar{\delta}\delta - \delta\bar{\delta} = (\pi - \mu) D + (\pi - \rho) \Delta - (\tau - \beta) \delta - (\tau - \alpha) \delta. \tag{43}
\]
Let us now exploit the identity (6) in order to obtain information on the NP spin coefficients. In order to do that, we need to know the covariant derivative of the self-dual 2-form $W_{\alpha\beta}$. The following identity follows directly from the definition of the NP spin coefficients in any spacetime

$$\nabla_\mu W_{\alpha\beta} \equiv 2 (\nu k_\mu + \pi l_\mu - \mu m_\mu - \lambda m_\mu) V_{\alpha\beta} + 2 (-\kappa k_\mu - \tau l_\mu + \rho m_\mu + \sigma m_\mu) U_{\alpha\beta}. $$

Inserting this identity into equation (6), it is straightforward to find the following expressions (which hold on $U_p$)

$$\nu = \frac{-\left(\xi, \tilde{m}\right) \Psi_4}{f}, \quad \pi = \frac{\left(\xi, \tilde{m}\right) \Psi_2}{f}, \quad \mu = \frac{\left(\xi, \tilde{l}\right) \Psi_2}{f}, \quad \lambda = \frac{-\left(\xi, \tilde{k}\right) \Psi_4}{f}, $$

$$\kappa = \frac{-\left(\xi, m\right) \Psi_0}{f}, \quad \tau = \frac{\left(\xi, m\right) \Psi_2}{f}, \quad \rho = \frac{\left(\xi, k\right) \Psi_2}{f}, \quad \sigma = \frac{-\left(\xi, l\right) \Psi_0}{f}. $$

These equations simplify the set of NP equation because the number of unknowns is reduced considerably. Furthermore the condition $(\xi, \xi) = -\lambda$ can be rewritten in the null basis as

$$\left(\xi, \tilde{k}\right) \left(\xi, \tilde{l}\right) - \left(\xi, \tilde{m}\right) \left(\xi, \tilde{m}\right) = \frac{\lambda}{2} \bigg|_{U_p}. $$

In Lemma 2 we showed that the interior of the set $M_1$ cannot extend to infinity. Our next aim is to show that the same happens for $M_2$.

**Lemma 3** Let $(\mathcal{M}, g)$ be a spacetime satisfying the hypotheses of Theorem 4 and let $M_f$ and $M_2$ be defined as in section 3. Then, $M_2$ is spatially bounded with respect to $M_\infty$.

**Proof:** $M_2$ is open because $\Psi_2$ and $\Psi_0\Psi_4$ are global smooth functions on $M_f$. If $M_2$ is empty the proposition is trivial, so let us assume that $M_2$ is non-empty and that intersects every asymptotically flat open submanifold $\hat{M}_\infty$. An immediate consequence of (13) is the existence of a complex function $G_2$ on $M_f$ such that $\nabla_\alpha (\Psi_0\Psi_4) = G_2P_\alpha$. Using this expression together with the Bianchi identities (38) and (39) and equation (43), the combination $\Psi_0 \times (23) + \Psi_4 \times (13)$ takes the simple form

$$\left(\frac{G_2}{4f} - \frac{2\Psi_0\Psi_4}{f^2}\right) \lambda \bigg|_{M_2} = -\Psi_0\Psi_4 \bigg|_{M_2}. $$

This equation implies, first of all, that neither $G_2f - 8\Psi_0\Psi_4\Psi_2$ nor $\lambda$ can vanish anywhere on $M_2$. Then, taking the gradient of (16) and using $\nabla_\alpha \lambda = 1/2(\chi_\alpha + \bar{\chi}_\alpha)$ (which follows from the definition of the Ernst one-form) and the fact that $\nabla_\alpha G_2 \propto P_\alpha$ (which follows from the definition of $G_2$) we obtain

$$(G_2f - 8\Psi_0\Psi_4\Psi_2) \chi_\alpha \bar{\chi}_\beta \big|_{M_2} = 0 \quad \implies \quad \omega_\beta \nabla_\alpha \lambda \big|_{M_2} = 0.$$

13
From Lemma 1 we have that $\nabla_\alpha \lambda$ cannot vanish on any open subset of $\mathcal{M}_2$ and hence there exists a real function $L$ such that $\omega_\alpha = L \nabla_\alpha \lambda$ on $\mathcal{M}_2$. The identity $\chi_\alpha \lambda^\alpha = -\lambda \mathcal{F}^2$ becomes

$$4f^2\lambda(1 - iL)^{-2}ig|_{\mathcal{M}_2} = \nabla_\alpha \lambda \nabla^\alpha \lambda \big|_{\mathcal{M}_2}. \quad (47)$$

Thus, $4f^2(1 - iL)^{-2}$ is real and, since it cannot vanish anywhere (because $f \neq 0$ on $\mathcal{M}_f$), its sign must remain constant on any connected component of $\mathcal{M}_2$. Those connected components where $4f^2(1 - iL)^{-2} < 0$ are obviously spatially bounded with respect to $\mathcal{M}_\infty$ because $\lambda$ is necessarily negative there (if $\lambda > 0$ then $\nabla_\alpha \lambda$ is spacelike – from $\xi^\alpha \nabla_\alpha \lambda = 0$ – and (17) implies $4f^2(1 - iL)^{-2} > 0$). Thus we only need to consider those connected components $\mathcal{M}_2^{(a)}$ where $4f^2(1 - iL)^{-2} > 0$ (i.e. where $2f(1 - iL)^{-1}$ is real). Equation (17) reads, on $\mathcal{M}_2$

$$\nabla_\beta \lambda \big|_{\mathcal{M}_2} = \frac{2f}{(1 - iL)} \left[ (\tilde{\xi}, \tilde{k}) l_\beta + (\tilde{\xi}, \tilde{l}) k_\beta - i\eta_{\alpha\beta\gamma\delta} \xi^\alpha \xi^\gamma \xi^\delta \xi^\beta \right] \bigg|_{\mathcal{M}_2}. \quad (48)$$

The imaginary part of this equation tells us that, at every point on $\mathcal{M}_2^{(a)}$, $\tilde{\xi}$ lies on the two-plane generated by $\tilde{k}$ and $\tilde{l}$ and therefore (14) $\nu = \pi = \kappa = \tau = 0$. On $\mathcal{M}_2^{(a)}$, the one-form $P_\alpha$ defined in (13) takes the form $P_\alpha = (1 - iL)(2f)^{-1} \nabla_\alpha \lambda$ and therefore is real. Consider now an arbitrary point $p \in \mathcal{M}_2^{(a)}$ and consider an open, simply connected neighbourhood of $U_p$ of $p$ in $\mathcal{M}_2^{(a)}$ such that the basis $\{U, V, W\}$ is well-defined. We have the freedom to rescale $U$ and $V$ according to $U' = QU$ $V' = Q^{-1}V$ where $Q$ is a non-zero, complex, smooth function defined on $U_p$. This transformation does not change any of the properties we have used of the basis $\{U, V, W\}$ and therefore all the equations above still hold for $\{U', V', W\}$ if all quantities are substituted by primes (those which are invariant under this transformation will be written without primes). Using (17) we find that the corresponding vectors $\tilde{k}'$, $\tilde{l}'$ and $\tilde{m}'$ are related to $\tilde{k}$, $\tilde{l}$ and $\tilde{m}$ by

$$\tilde{k}' = \frac{1}{|Q|} \tilde{k}, \quad \tilde{l}' = |Q| \tilde{l}, \quad \tilde{m}' = e^{-i\phi} \tilde{m}, \quad \text{where} \quad Q = |Q|e^{i\phi}. \quad (49)$$

Under such a transformation, $\Psi_0$ and $\Psi_4$ become $\Psi'_0 = Q^2 \Psi_4$ and $\Psi'_0 = Q^{-2} \Psi_0$. Since $\Psi_4$ and $\Psi_0$ are nowhere vanishing on $U_p$, there exists a smooth choice of $Q$ such that $\Psi'_0 = \Psi_4|_{U_p}$. This choice will be assumed from now on. Now, equation (46) allows us to obtain $G_2$ explicitly

$$G_2 \bigg|_{U_p} = \frac{4\Psi'^2_4 (2\Psi_2 \lambda - f^2)}{f \lambda} \bigg|_{U_p} \quad \Rightarrow \quad \nabla_\alpha \Psi'_4 \bigg|_{U_p} = \frac{2\Psi'_4 (2\Psi_2 \lambda - f^2)}{f \lambda} P_\alpha \bigg|_{U_p}. \quad (50)$$

14
The Bianchi equations (39)-(41) immediately imply (recall that, on \( \mathcal{M}_2^{(\alpha)} \), \( \vec{\zeta} \) is a non-lightlike linear combination of \( \vec{k}' \) and \( \vec{l} \) from which \( (\vec{\xi}, \vec{k}') \neq 0 \) and \( (\vec{\xi}, \vec{l}) \neq 0 \)

\[
\alpha'|_{U_p} = \beta'|_{U_p} = 0, \quad \gamma'|_{U_p} = \frac{f}{4(\vec{\xi}, \vec{k}')|_{U_p}}, \quad \epsilon'|_{U_p} = \frac{f}{4(\vec{\xi}, \vec{l})|_{U_p}}.
\]

Combining equations (18), (23) and (31) in order to eliminate \( \text{sign}(\lambda) \) and therefore \( \alpha \equiv A_2 \) is also real from (48). Furthermore, the NP equation (33) becomes simply \( \Delta \left( \vec{\xi}, \vec{l} \right) \) and \( D \left( \vec{\xi}, \vec{k}' \right) \) we obtain, after dropping some non-zero factors,

\[
\left( \Psi_4 f \left[ \left( \vec{\xi}, \vec{k}' \right)^4 + \left( \vec{\xi}, \vec{l} \right)^4 \right] - 2\bar{\mathcal{F}}_4 \left( \vec{\xi}, \vec{k}' \right)^2 \left( \vec{\xi}, \vec{l} \right)^2 \right)|_{U_p} = 0,
\]

which is equivalent to

\[
\left( \vec{\xi}, \vec{k}' \right)^2|_{U_p} = \left( \vec{\xi}, \vec{l} \right)^2|_{U_p}, \quad f\Psi_4|_{U_p} = \bar{\mathcal{F}}_4|_{U_p} \tag{48}
\]

and therefore \( \left( \vec{\xi}, \vec{k}' \right)|_{U_p} = \pm |\lambda/2|^{1/2}|_{U_p} \) and \( \left( \vec{\xi}, \vec{l} \right)|_{U_p} = \epsilon_1 \left( \vec{\xi}, \vec{k}' \right)|_{U_p} \), where \( \epsilon_1 = \text{sign}(\lambda) \). Let us now define four functions \( A_i \ (i = 1, 2, 3, 4) \) on \( U_p \) by the following expressions

\[
f \equiv \frac{\lambda}{2} (A_1 + i A_2), \quad \Psi_0 = \Psi_4' \equiv f A_3, \quad \Psi_2 = f \left( A_4 + i \frac{A_2}{2} \right). \tag{49}
\]

\( A_1 \) and \( A_2 \) are real by definition, \( A_3 \) is also real from (48). Furthermore, the NP equation (33) becomes simply \( A_4 - \overline{A_4} = 0 \) and therefore all \( A_i \) are real. The fact that \( 2f(1 - iL)^{-1} \) is also real implies that \( A_1|_{U_p} \) is nowhere zero and that \( L = -A_2 A_1^{-1}|_{U_p} \).

It is now a matter of simple, if somewhat long, calculation to check that the full system of Newman-Penrose equations (18)-(35) are satisfied on \( U_p \) if and only if

\[
\nabla_\alpha A_2 = 2A_2 A_4 P_\alpha, \quad \nabla_\alpha A_1 = \left( 2A_1 A_4 - A_1^2 - A_2^2 \right) P_\alpha, \tag{50}
\]

\[
\nabla_\alpha A_3 = A_3 \left( 2A_4 - A_1 \right) P_\alpha, \quad \nabla_\alpha A_4 = \left( A_3^2 + A_2^2 - \frac{1}{4} A_4^2 \right) P_\alpha, \tag{51}
\]

\[
\omega_\alpha = -A_2 \lambda P_\alpha, \quad \nabla_\alpha \lambda = \lambda A_1 P_\alpha, \tag{52}
\]

\[
4 \left( A_4^2 - A_3^2 \right) - 4A_4 A_1 - A_2^2 = 0. \tag{53}
\]

Let us now assume that \( \mathcal{M}_2 \) is not spatially bounded with respect to \( \mathcal{M}_\infty \). Then it must intersect any asymptotically flat end \( \mathcal{M}_\infty \) and we can choose \( U_p \) so that \( U_p \subset \mathcal{M}_\infty \). Thus, we can coordinate \( U_p \) with the asymptotically Minkowskian coordinates described in Remark 1 after Theorem 2. We know from asymptotic flatness that \( \mathcal{F}^2 = -4M^2/r^4 + O(r^{-5}) \) and therefore \( f = \tilde{\epsilon} M/r^2 + O(r^{-3}) \) where \( \tilde{\epsilon} = \pm 1 \). Using the
fact that $\lambda = 1 - 2M/r + O(r^{-2})$ we deduce from the definition of $A_1$ and $A_2$ in (49) that

$$A_1 = \frac{2M\tilde{\epsilon}}{r^2} + O(r^{-3}), \quad A_2 = O(r^{-3})$$

Regarding $A_4$, we can obtain its asymptotic behavior directly from the second equation in (50) after using the second equation in (52) and the fact that $A_1$ is non-zero on $U_p$. We find $A_4 = -\tilde{\epsilon}/r + O(r^{-2})$. Finally, the constraint (53) fixes the asymptotic behaviour of $A_3$ which is $A_3 = \tilde{\epsilon}/r + O(r^{-2})$ where $\tilde{\epsilon} = \pm 1$. A direct consequence of the first equation in (51) and the second one in (52) is

$$\lambda A_1 \nabla_\alpha A_3 - A_3 (2A_4 - A_1) \nabla_\alpha \lambda = 0.$$ 

Inserting the asymptotic expansions into this equation, we obtain $2M\tilde{\epsilon}\tilde{\epsilon}/r^4 + O(r^{-5}) = 0$. Choosing $U_p$ so that $r$ takes values which are large enough this identity becomes impossible. Thus $M_2$ must be spatially bounded with respect to $M_\infty$. \[\square\]

The main result in this section is Proposition 1 below. Before proving it, we must discuss some of the basic properties of the set $M_3$ (where the Petrov type is II or D). In the following section we will study $M_3$ in detail, but, for now, we only need to notice that inserting $\Psi_0 \Psi_4 = 0$ and $\Psi_2 \neq 0$ (which define $M_3$) into (15) we get $\nabla_\beta \Psi_2|_{M_3} = (3/2)\Psi_2 f^{-1}\nabla_\beta f|_{M_3}$. This implies immediately that $\Psi_2^2 / f^3$ is constant on any connected component of $M_3$. Since $\Psi_2$ is a smooth, globally defined complex function on $M_3$, it follows easily that $f^{1/2}$ defines a smooth, global function on $M_3$. Thus we can write

$$\Psi_2|_{M_3} = -\sqrt{\frac{2}{b}} \left(f^{1/2}\right)^3|_{M_3},$$

where $b$ is a nowhere zero, complex function on $M_3$ which is constant on each connected component of $M_3$.

**Proposition 1** Let $(M, g)$ be a spacetime satisfying the hypotheses of Theorem 2 and let $M_f$, $M_1$, $M_2$ and $M_3$ be defined as in section 3. Then $M_f = M_3$.

**Proof:** Trivial topological properties of $M_1$, $M_2$ and $M_3$ together with the fact that they constitute a partition of $M_f$ show that $\partial(M_1 \cup M_2) = \partial M_3$. Moreover, it is clear that $\partial(M_1 \cup M_2) \subset \partial M_1 \cup \partial M_2$ and therefore we have

$$\partial M_3 = \partial M_3 \cap \partial (M_1 \cup M_2) \subset \partial M_3 \cap (\partial M_1 \cup \partial M_2) = (\partial M_3 \cap \partial M_1) \cup (\partial M_3 \cap \partial M_2).$$

So, if we prove $(\partial M_3 \cap \partial M_1) = (\partial M_3 \cap \partial M_2) = \emptyset$ then we would have $\partial M_3 = \emptyset$ which implies easily $M_3 = M_f$ (after using Lemmas 2 and 3 above). So, our aim is to
show that both $\partial M_3 \cap \partial M_1$ and $\partial M_3 \cap \partial M_2$ are empty. Showing $\partial M_3 \cap \partial M_1 = \emptyset$ is trivial because at any point $q \in \partial M_3 \cap \partial M_1$ we would have $\Psi_2|_q = 0$ from the continuity of $\Psi_2$ and the definition of $M_2$ and this is incompatible with the limit of $\Psi_2$ on $q$ coming from $M_3$ which is nonzero due to expression (54) after using the constancy of $b$ on any connected component of $M_3$ and the fact that $f$ cannot become zero anywhere on $M_f$. So, we only need to show that $\partial M_3 \cap \partial M_2 = \emptyset$. Assume, on the contrary, that this set is non-empty and take an arbitrary point $q \in \partial M_3 \cap \partial M_2$ and a sufficiently small open, connected neighbourhood $U_q$ of $q$ such that the null tetrad $\{\vec{k}, \vec{l}, \vec{m}, \vec{m}'\}$ is well-defined on $U_q$. In particular, $\Psi_0$ and $\Psi_4$ are smooth functions on $U_q$. Let us now find an equation for the function $\Psi_0\Psi_4^2f^{-2}$. On $U_q \cap M_2$ we know that the tetrad $\{\vec{k}', \vec{l}', \vec{m}', \vec{m}'\}$ introduced in the proof of the previous proposition is well-defined. From (49) and (52) and the fact that $\Psi_1$ is well-defined on $\{0\}$, we have

$$\nabla_\alpha \left( \frac{\Psi_0\Psi_4^2}{f^2} \right) = \nabla_\alpha \left( A_3^2\lambda^2 \right) = 4A_3^2\lambda^2A_4P_\alpha = 4 \left( \frac{\Psi_0\Psi_4^2\lambda^2}{f^2} \right) \text{Re} \left( \frac{\Psi_2}{f} \right) P_\alpha \bigg|_{U_q \cap M_2}, \quad (55)$$

where $\text{Re}$ denotes real part. Take now a smooth curve $\gamma$ entirely contained in $U_q \cap M_2$ and with endpoint on $q$ and such that $\lambda$ is non-zero at least on one point $r$ of $\gamma$. Since $\lambda$ cannot vanish on any open set (from Lemma [1]) the existence of such a curve is trivial to establish. Let us parametrize the curve $\gamma$ such that $\gamma(0) = r$ and $\gamma(1) = q$. Integrating equation (53) along $\gamma$ we obtain

$$\frac{\Psi_0\Psi_4\lambda}{f^2}(s) = \frac{\Psi_0\Psi_4\lambda}{f^2}(0) \exp \left( \int_0^s \left( 4\text{Re} \left( \frac{\Psi_2}{f} \right) P_\alpha s^\alpha \right) \bigg| ds \right),$$

where $\vec{\gamma}$ is the tangent vector of $\gamma$. Taking into account that $\text{Re} (\Psi_2f^{-1})$ remains bounded along $\gamma$ we conclude that $\Psi_0\Psi_4\lambda^2f^{-2}$ cannot vanish on $q$ which is incompatible with the fact that $\Psi_0\Psi_4$ vanishes on $q$. Thus, we have proven that $\Omega = \emptyset$ and the proof is completed. 

\[\square\]

4 Analysis of the Petrov types II or D and proof of the theorem.

In this section we will concentrate on the subset $M_3$. From the previous section we know that $M_3 = M_f$ and therefore $M_3$ is open and connected. Thus the function $b$ appearing in (54) is constant throughout $M_3$. The decomposition of $b$ into real and imaginary parts will be written as $b = b_1 + ib_2$. Equation (54) gives the form of $\Psi_2$ on $M_3$. Inserting it into the second equation in (15) we obtain

$$\nabla_\alpha f = -2\sqrt{2b^{-1}(b^{1/2})^3}P_\alpha.$$ 

This equation implies first of all that $P_\alpha$ is exact on $M_3$ and that $P_\alpha = \nabla_\alpha P$ with $P$
given by

\[ P = \left. \frac{b}{\sqrt{2f^{1/2}}} \right|_{\mathcal{M}_3} \quad \Rightarrow \quad \Psi_2 = -\frac{b^2}{2P^3} \bigg|_{\mathcal{M}_3}. \]

(56)

Similarly, the first equation in (15) shows that the Ernst one-form \( \chi_\alpha \) is also exact on \( \mathcal{M}_3 \) and that \( \chi_\alpha = \nabla_\alpha \chi \) with

\[ \chi = -\frac{b^2}{P} + c \bigg|_{\mathcal{M}_3}, \]

(57)

where \( c \) is a complex constant. Since the imaginary part of the Ernst potential (i.e. the twist potential) is defined up to an arbitrary additive constant, we can assume without loss of generality that \( c \) is real. Let us also define two smooth real functions \( y \) and \( Z \) on \( \mathcal{M}_3 \) by

\[ P = y + iZ, \]

(58)

so that (13) becomes

\[ \nabla_\beta y \big|_{\mathcal{M}_3} = -\left( \frac{\nabla_\beta}{y} \right) \frac{\beta}{P} \bigg|_{\mathcal{M}_3}, \quad \nabla_\beta Z \big|_{\mathcal{M}_3} = -\eta_{\alpha\beta\gamma\delta} \xi^\alpha k^\gamma l^\delta \bigg|_{\mathcal{M}_3}. \]

(59)

In order to proceed, we still have to define the following subsets of \( \mathcal{M}_3 \).

\[ \mathcal{M}_3^k = \text{int} \{ p \in \mathcal{M}_3; \, (\xi, \kappa) \big|_p \neq 0 \text{ and } \Psi_0|_p = 0 \}, \]

\[ \mathcal{M}_3^\pi = \{ p \in \mathcal{M}_3; \, \eta_{\alpha\beta\mu\nu} \xi^\beta k^\mu l^\nu \big|_p \neq 0 \}, \]

\[ \mathcal{M}_3^l = \text{int} \{ p \in \mathcal{M}_3; \, (\xi, \lambda) \big|_p \neq 0 \text{ and } \Psi_4|_p = 0 \}, \]

\[ \mathcal{M}_3^D = \text{int} \{ p \in \mathcal{M}_3; \, \Psi_0|_p = \Psi_4|_p = 0 \}, \]

\[ \mathcal{M}_3^4 = \{ p \in \mathcal{M}_3; \, \Psi_4|_p \neq 0 \}, \quad \mathcal{M}_3^0 = \{ p \in \mathcal{M}_3; \, \Psi_0|_p \neq 0 \}. \]

(60)

For later convenience, let us now write down two equations that hold locally on \( \mathcal{M}_3^k \) (i.e. on a sufficiently small neighbourhood of any point of \( \mathcal{M}_3^k \)). The first one is obtained directly from (24) after using \( \Psi_0 = 0 \) and (13), and the second one follows directly from (24)

\[ \Psi_4 \left( \xi, \kappa \right)^2 \bigg|_{\mathcal{M}_3^k} = \frac{b^2}{2P^2} \left[ \left( \beta - \alpha \right) \left( \xi, \overrightarrow{m} \right) - \delta \left( \xi, \overrightarrow{m} \right) \right] \bigg|_{\mathcal{M}_3^k}, \]

(61)

\[ D \left[ \left( \xi, \overrightarrow{m} \right) \overrightarrow{P} \right] \bigg|_{\mathcal{M}_3^k} = \left( \xi, \overrightarrow{m} \right) \overrightarrow{P} \left( \epsilon - \tau \right) \bigg|_{\mathcal{M}_3^k}. \]

(62)

The following proposition gives the local form of the metric in a certain subset of \( \mathcal{M}_3 \). As mentioned in the introduction, asymptotic flatness is not used anywhere in the proof of this proposition.
Proposition 2 Let \((S, \gamma)\) be a two-dimensional Riemannian space of constant curvature equal to \(2c\) and denote by \(<, >\) the scalar product with respect to \(\gamma\). Denote also by \(\Delta_\gamma\) the Laplace-Beltrami operator on \((S, \gamma)\) and by \(\ast_\gamma\) the Hodge dual of \((S, \gamma)\). Finally, let \(V = \mathbb{R} \times \mathbb{R} \times S\) and \(\{u, y\}\) Cartesian coordinates of \(\mathbb{R} \times \mathbb{R}\).

Assume that \(\mathcal{M}_3^\pi \cap (\mathcal{M}_3^k \cup \mathcal{M}_3^l)\) defined above is non-empty and let \(X\) be a connected component of this set. Then, there exist three smooth real functions \(Z, A\) and \(B\) defined on \(S\) satisfying the linear partial differential equations

\[
\begin{align*}
\Delta_\gamma Z &= -2cZ + 2b_1b_2, \quad (63) \\
\langle dZ, dA \rangle - \langle \ast_\gamma dZ, dB \rangle + 2Z - A(2cZ - 2b_1b_2) &= 0, \quad (64)
\end{align*}
\]

\((b_1\) and \(b_2\) are the two constants defined above) such that \((X, g|_X)\) is locally isometric to the spacetime \((V, h)\) where

\[
h = -\left( c - \frac{(b_1^2 - b_2^2)y + 2b_1b_2Z}{y^2 + Z^2} \right) \left[ du - Bdz - A \ast_\gamma dz \right]^2 + 2(\gamma - \ast_\gamma dZ)(du - Bdz - A \ast_\gamma dz) + \left( y^2 + Z^2 \right) \gamma. \quad (65)
\]

Remark. In the line-element \((65)\) the objects \(Z, dZ, \ast_\gamma dZ\) and \(\gamma\) denote the pull-back with respect to the canonical projection \(\tilde{\pi} : \mathbb{R} \times \mathbb{R} \times S \rightarrow S\) of the same objects on \(S\). A precise notation should require giving different names to those pull-backs, but for the sake of notational simplicity we have preferred not to do so.

Proof: Let \(p \in X\) be an arbitrary point and consider, as usual, an open, connected neighbourhood \(U_p \subset X\) of \(p\) such that null tetrad \(\{\vec{k}, \vec{l}, \vec{m}, \vec{m}\}\) is well-defined on \(U_p\). Define \(\pi_0\) on \(U_p\) as \(\pi_0 \equiv (\vec{\xi}, \vec{m})\). From the definition of \(\mathcal{M}_3^\pi\) we have that \(\pi_0\) is nowhere vanishing on \(U_p\). In addition, from \(X \subset \mathcal{M}_3^k \cup \mathcal{M}_3^l\) it follows that, by choosing \(U_p\) small enough, we can assume that either \((\vec{\xi}, \vec{k}) \neq 0\) and \(\Psi_0 = 0\) everywhere on \(U_p\) or \((\vec{\xi}, \vec{l}) \neq 0\) and \(\Psi_4 = 0\) everywhere on \(U_p\). We will make the proof explicitly only for the case \((\vec{\xi}, \vec{k}) \neq 0\) and \(\Psi_0 = 0\) on \(U_p\). The proof in the other case is identical step by step after interchanging the roles of \(\vec{k} \leftrightarrow \vec{l}, \vec{m} \leftrightarrow \vec{m}\) and \(\Psi_0 \leftrightarrow \Psi_4\). So, let us assume from now on that \((\vec{\xi}, \vec{k}) \neq 0\) and \(\Psi_0 = 0\) everywhere on \(U_p\). From the definition of \(\pi_0\) it follows that we can choose the null tetrad globally and uniquely on \(U_p\) such that \(\pi_0\) is purely imaginary (i.e. \(\vec{\pi}_0 = -\pi_0\)) and that \((\vec{\xi}, \vec{k}) = 1\). Our next aim is to write down several consequences of the NP equations in this tetrad on \(U_p\). First, expression \((13)\) can be rewritten on \(U_p\) as

\[
\begin{align*}
DP &= 1, \\
\Delta P &= -\left( \vec{\xi}, \vec{l} \right), \\
\delta P &= \frac{\pi_0}{P}, \\
\bar{\delta} P &= \frac{\bar{\pi}_0}{P}.
\end{align*}
\quad (66)
\]
Equations (18) and (20) read, respectively \( \epsilon + \tau = 0 \) and \( \pi \alpha_0 + \tilde{k}(\pi_0) - \epsilon \pi_0 = 0 \) which, after using \( \pi_0 = -\pi_0 \), imply \( \epsilon = 0, \tilde{k}(\pi_0) = 0 \). Let us now define a complex function \( \alpha_0 \) by

\[
\alpha_0 \equiv -\left( \alpha P + \frac{\pi_0}{P} \right) \bigg|_{U_p}.
\]

Applying the commutator (11) to \( P \) we obtain \( \beta = (\alpha_0/P) \), which introduced in (21) gives simply \( \tilde{k}(\alpha_0) = 0 \). Equation (21) reads now \( \Psi_4 = b^2/2 (P^{-3} \pi_0 - 2P^{-4} \alpha_0 \pi_0) \big|_{U_p} \) and therefore provides an explicit expression for \( \Psi_4 \). Similarly, \( (\xi, \tilde{\xi}) \) is obtained from (13) as \( (\xi, \tilde{\xi}) = c/2 - b^2/(4P) - \pi_0^2/(P\pi_0) \big|_{U_p} \). Inserting these two expressions into (23) the following partial differential equation for \( \pi_0 \) is obtained

\[
\delta \pi_0 = \frac{2c (P - P) + b^2 - b^2 - 8\pi_0 \alpha_0}{4P} \bigg|_{U_p}.
\]  

(67)

The only NP spin coefficients that remain to be determined are \( \gamma \) and \( \alpha^0 \) (besides \( \pi^0 \)). Regarding \( \gamma \), we consider equation (20) together with the two equations obtained by applying the commutators (11) and (42) on \( P \). A straightforward combination of these equations gives

\[
\gamma = \frac{b^2}{4P^2} + \frac{\pi_0^2}{P^2} - \frac{\pi_0 (\alpha_0 + \alpha_0)}{P \pi_0} \bigg|_{U_p}, \quad \tilde{\xi}(\pi_0) \big|_{U_p} = 0.
\]

Regarding \( \alpha^0 \), equations (30) and (27) give the following partial differential equations

\[
\delta \alpha_0 = \frac{(b^2 - b^2 - 2cP + 2cP)(\pi_0 - \alpha_0) - \pi_0 (c + 8\alpha_0 \pi_0)}{4\pi_0 P^2} \bigg|_{U_p}, \quad \tilde{\xi}(\alpha_0) = 0 \bigg|_{U_p}.
\]  

(68)

It is straightforward, although lengthy, to check that the full set on NP equations on \( U_p \) are satisfied provided \( \tilde{k}(\pi_0) = 0, \tilde{k}(\alpha_0) = 0, \tilde{\xi}(\pi_0) = 0, (67) \) and (68) are fulfilled. So, we must try and solve these non-linear partial differential equations. In order to do that, let us define a complex vector field \( \vec{u} \equiv \vec{P} \vec{\pi} \). From (20) and the decomposition (58) into real and imaginary parts we have \( \vec{k}(y) = 1, \vec{\xi}(y) = 0, \vec{u}(y) = 0, \vec{k}(Z) = 0, \vec{\xi}(Z) = 0, \vec{u}(Z) = -i\pi_0 \). From \( (\vec{\xi}, \vec{k}) \big|_{U_p} = 1 \) it follows that \( \{\vec{k}, \vec{\xi}, \vec{u}, \vec{\pi}\} \) is a basis of vectors at every point in \( U_p \). The commutators between these vectors can be easily computed from (11)–(13) after using the expressions above. The result reads simply

\[
[\vec{k}, \vec{\xi}] = [\vec{k}, \vec{u}] = [\vec{\xi}, \vec{u}] = 0 \big|_{U_p}, \quad [\vec{u}, \vec{\pi}] = -2iZ \vec{\xi} + 2\alpha_0 \vec{u} - 2\alpha_0 \vec{\pi} \bigg|_{U_p}.
\]

(69)
Now, $dy|_{U_p}$ is nowhere zero because $\bar{k}(y) = 1$ and therefore $U_p$ can be foliated by the family of hypersurfaces $\{ \Sigma_{y_0} \}$ defined by $y = y_0 = \text{const}$. Obviously, the three vector fields $\xi, \bar{u}, \bar{v}$ are tangent to each hypersurface $\Sigma_{y_0}$. Since $\xi$ and $\bar{k}$ commute, these two vector fields are surface-forming. Moreover $\xi$ and $\bar{k}$ have non-zero scalar product on $U_p$ and hence the vector space they generate at each point $q \in U_p$ is two-dimensional.

We can consider the quotient space of $U_p$ with respect to the two-surfaces generated by $\bar{\xi}$ and $\bar{k}$. This quotient will be denoted by $S_p = U_p/(\text{span}\{\bar{\xi}, \bar{k}\})$. By choosing $U_p$ sufficiently small we can assume that $S_p$ is a differentiable manifold. The triple $(U_p, S_p, \pi_p)$, where $\pi_p : U_p \to S_p$ is the canonical projection, is a bundle over $S_p$. After restricting $U_p$ further if necessary, we can assume that this bundle is a trivial fiber bundle, i.e. $U_p$ is diffeomorphic to $I_p \times \hat{I}_p \times S_p$, where $I_p, \hat{I}_p$ are open intervals of the real line. Any vector field $\bar{v}$ on $U_p$ satisfying $[\bar{\xi}, \bar{v}] = [\bar{k}, \bar{v}] = 0$ defines a vector field $\pi_* \bar{v}$ on $S_p$ by projection. Similarly, scalar functions on $U_p$ which are constant along both $\bar{\xi}$ and $\bar{k}$ define scalar functions on $S_p$. In order to simplify the notation we will keep the same symbol when a function is projected onto $S_p$. The meaning of the expression should be clear from the context.

As $[\bar{u}, \bar{\xi}] = [\bar{u}, \bar{k}] = 0$, we can define $\pi_* \bar{u}$. Similarly, $\alpha_0, \pi_0$ and $Z$ can also be projected onto $S_p$. Let us take an arbitrary coordinate system $\{x^1, x^2\}$ on $S_p$ and define the complex vector field $\bar{s}$ on $S_p$ by $\bar{s} = \partial_{x_1} + i \partial_{x_2}$. Since $dZ$ is nowhere zero on $X$ (because $\bar{u}(Z) = -i \pi_0$) it follows that $\bar{s}(Z) \neq 0$ everywhere on $S_p$. It is convenient to define a real function $N$ on $S_p$ by $\pi_0 = (i/2) N \sqrt{\bar{s}(Z) \bar{\bar{s}}(Z)}$ where the square root is defined with positive sign. Using $\bar{u}(Z) = -i \pi_0$ we immediately have

$$\pi_* \bar{u} = \left. \frac{N \sqrt{\bar{s}(Z) \bar{\bar{s}}(Z)}}{2 \bar{s}(Z) \bar{\bar{s}}(Z)} \bar{s} \right|_{S_p}. \tag{70}$$

Projecting equation (71) onto $S_p$ we can obtain an expression for $\alpha_0$ in terms of $N, Z$ and their derivatives. It reads

$$\alpha_0 = \left. -\frac{1}{4} \frac{\bar{s}(Z)}{\bar{s}(Z)} \sqrt{\bar{s}(Z) \bar{\bar{s}}(Z)} - \frac{N}{2 \sqrt{\bar{s}(Z) \bar{\bar{s}}(Z)}} \left[ \bar{s}(\bar{s}(Z)) + \frac{\bar{s}(Z)}{\bar{s}(Z)} \bar{\bar{s}}(\bar{s}(Z)) + \frac{8 c Z}{N^2} + \frac{2 i (b^2 - \bar{b}^2)}{N^2} \right] \right|_{S_p}. \tag{71}$$

The projection of $[\bar{u}, \bar{u}]$ on $S_p$ is (see Eq. (69)) $[\pi_* \bar{u}, \pi_* \bar{u}] = 2 \alpha_0 (\pi_* \bar{u}) - 2 \bar{\alpha}_0 (\pi_* \bar{u})|_{S_p}$.

Inserting (70) and (71) into this commutator and using $[\bar{s}, \bar{s}] = 0$, we obtain after a rather long calculation, the remarkably simple equation

$$\bar{s}(\bar{s}(Z)) = -\frac{4}{N^2} \left[ c Z + \frac{i}{4} (b^2 - \bar{b}^2) \right], \tag{72}$$

21
which is a second order, linear partial differential equation for $Z$. We still need to find
an equation for $N$. This is accomplished by projecting the equation for $\delta a_0$ (see Eq.
(75)) onto $S_p$ and using the expressions for $a_0$ and $\pi_0$ above. Now the calculation is
quite long, but the result is surprisingly simple, namely

$$N\tilde{s}\left(\tilde{s}(N)\right) - \tilde{s}(N)\tilde{s}(N) = 2c.$$  \hspace{1cm} (73)

This is a non-linear partial differential equation which involves the function $N$ only.
Moreover, this equation can be explicitly solved as follows. Let us introduce the fol-
lowing metric on $S$

$$dl^2 = \gamma_{\hat{A}\hat{B}}dx^\hat{A}dx^\hat{B} = \frac{2}{N^2}[\hat{x}^2 + \hat{y}^2], \quad \hat{A}, \hat{B} = 1, 2,$$  \hspace{1cm} (74)

which has the Ricci scalar $R(\gamma) = N(N_{,x1^2} + N_{,x2^2}) - (N_{,x1}^2 - N_{,x2}^2)$. Recalling
that $\tilde{s}|_{S_p} = \partial x_1 + i\partial x_2$ we have that equation (73) becomes $R(\gamma) = 2c$. Hence, the
two-dimensional space $(S_p, \gamma)$ is a space of constant curvature equal to $2c$. The local
form of the metric for such space is well-known and therefore $N$ can be determined
explicitly. The still arbitrary coordinate system $\{x^1, x^2\}$ on $S_p$ can be chosen so that
$N$ takes the standard form

$$N = 1 + \frac{c}{2}[(x^1)^2 + (x^2)^2].$$  \hspace{1cm} (75)

In terms of the metric $\gamma$, it is straightforward to check that the linear PDE for $Z$ (72)
can be rewritten as equation (73) in the proposition.

The next task is to go from the quotient space $S_p$ back to $U_p$. To do that recall
that the fibre bundle $(U_p, S_p, \pi_p)$ is trivial and therefore admits a global cross section
$\tilde{\sigma}$ (i.e. $\tilde{\sigma} : S_p \rightarrow U_p$ such that $\pi_p \circ \tilde{\sigma} = Id|_{S_p}$). Let us construct a coordinate system
on $U_p$ as follows. Take an arbitrary point $q \in U_p$ and consider the hypersurface $\Sigma_{y_q}$
where $y_q \equiv y(q)$. Consider the curve tangent to the vector field $\tilde{k}$ passing through the
point $\tilde{\sigma} \circ \pi_p(q)$. Since $\tilde{k}$ is transversal to $\Sigma_{y_q}$ and the fiber bundle is trivial it follows
that this curve must intersect $\Sigma_{y_q}$ at a single point $r$. Now, $r$ and $q$ belong to the same
hypersurface $\Sigma_{y_q}$ and they project onto the same point on $S_p$. Since $\tilde{\xi}$ is tangent to
$\Sigma_{y_q}$, it follows that there exists an integral line of $\tilde{\xi}$ that connects $q$ with $r$. Choose
the parametrization $\hat{u}$ associated to the tangent vector $\tilde{\xi}$ on this curve (i.e. $\tilde{\xi} = \partial_{\hat{u}}$
on this curve) and such that $\hat{u} = 0$ on $r$. We define a function $u(q)$ as the value of
$\hat{u}$ at $q$. Finally, let $x^1(q)$ and $x^2(q)$ be the values of the coordinates $\{x^1, x^2\}$ of the
point $\pi_p(q)$ on $S_p$. It is easy to check that $\{u(q), y(q), x^1(q), x^2(q)\}$ thus constructed is
a well-defined coordinate system on $U_p$ and that, in this coordinate system, $\tilde{\xi}|_{U_p} = \partial_u$
and $\tilde{k}|_{U_p} = \partial_y$. Regarding the vector field $\tilde{u}$, we know that $\tilde{u}(y) = 0$ and that its
projection onto $S_p$ is given by (74). It follows that

$$\tilde{u}|_{U_p} = \frac{N\sqrt{\tilde{s}(Z)\tilde{\xi}(Z)}}{2\tilde{s}(Z)}(\partial x_1 + i\partial x_2) + \frac{N(B - iA)\sqrt{\tilde{s}(Z)\tilde{\xi}(Z)}}{2}\partial_u,$$  \hspace{1cm} (76)

22
where \( A, B \) are real functions on \( U \) which remain to be determined. From \( \vec{u}, \vec{\xi} = [\vec{u}, \vec{k}] = 0 \) it follows \( \vec{\xi}(A) = \vec{\xi}(B) = \vec{k}(A) = \vec{k}(B) = 0 \) and hence \( A, B \) define functions on \( S \). The last commutator in (69) on \( U \) can be rewritten after a straightforward calculation as

\[
\langle dZ, dA \rangle - \langle \star \gamma dZ, dB \rangle + 2Z - A(2cZ - 2b_1b_2) = 0,
\]

where \( <, > \) denotes scalar product with respect to the metric (74) and \( \star \gamma \) is the Hodge dual with respect to the same metric (recall that \( Z, b_1 \) and \( b_2 \) are defined by \( P = y + iZ \) and \( b = b_1 + ib_2 \)). This equation is exactly the linear partial differential equation for \( A \) and \( B \) appearing in the proposition. So, we have constructed a coordinate system on \( U \) and have obtained the explicit form of \( \vec{\xi}, \vec{k}, \vec{l}, \vec{\eta} \) in this coordinate system.

Moreover, the scalar products of these vectors can be determined directly from their definition in terms of the null tetrad \( \{\vec{k}, \vec{l}, \vec{\eta}, \vec{\eta}\} \). Thus, the spacetime metric on \( U \) can be explicitly written. After some algebra, it is not difficult to check that this metric can be written in the compact form given in (73). This proves the local isometry claimed in the proposition.

\( \square \).

The explicit expression for \( \Psi_4 \) for the metric (82) in the open set \( \mathcal{M}^3 \cup \mathcal{M}^k \) will be needed later. Using the expressions obtained in the proof of Proposition 2 it is straightforward to check that the following expression holds on \( \mathcal{M}^3 \cup \mathcal{M}^k \)

\[
\Psi_4 = i \frac{(b_1 + ib_2)^2}{2(y + iZ)^4} \left[ \frac{dG}{2G} dZ + i \gamma dZ \right] + cZ - b_1 b_2,
\]

where \( G \equiv \langle dZ, dZ \rangle \). The metric (82) contains several parameters and one function \( Z \) defined on the two-surface \( S \). The partial differential equation for \( Z \) is of elliptic type but \( S \) may be non-compact (for instance, \( S \) could be the hyperbolic plane). So, in general, this metric contains arbitrary functions (which can be thought, for instance, as the boundary conditions of the partial differential equation (83)). Thus, we are far from having proven the local isometry with the Kerr metric which is claimed in Theorem 2. The reason is, of course, that we have made no use of asymptotic flatness in Proposition 2. This fact shows, in particular, that asymptotic flatness is an essential ingrediet of Theorem 2. The next lemma exploits the asymptotic conditions in order to show that \( \mathcal{M}^D \) is either empty or that it covers the whole of \( \mathcal{M}^3 \).

**Lemma 4** Let \( \mathcal{M}^3 \) and \( \mathcal{M}^D \) be defined as in (80). Then either \( \mathcal{M}^D = \emptyset \) or \( \mathcal{M}^D = \mathcal{M}^3 \).

**Proof:** Since \( \mathcal{M}^D \) is open, the conclusion of the lemma is equivalent to \( \partial \mathcal{M}^D = \emptyset \) in the topology of \( \mathcal{M}^3 \). Thus, let us assume that \( \partial \mathcal{M}^D \) is non-empty and let us find a
contradiction. From the fact that $Ψ_0 = Ψ_4 = 0$ on $\mathcal{M}_3^D$ we find that the Weyl tensor takes the form

$$
C_{\alpha\beta\gamma\delta}|_{\mathcal{M}_3^D} = 3\Psi_2 \left( W_{\alpha\beta} W_{\gamma\delta} + \frac{4}{3} T^{\alpha\beta}_{\gamma\delta} \right)|_{\mathcal{M}_3^D},
$$

(78)

where we used (13) and the identity $4T^{\alpha\beta}_{\gamma\delta} = -W^{\alpha\beta} W_{\gamma\delta} + 2 \left( U^{\alpha\beta} V_{\gamma\delta} + V^{\alpha\beta} U_{\gamma\delta} \right)$ (recall that $T^{\alpha\beta}_{\gamma\delta}$ is the identity on the space of self-dual two-forms). Thus, the Weyl tensor on $\mathcal{M}_3^D$ satisfies the hypotheses of Theorem [1]. Furthermore, $\mathcal{M}_3(= \mathcal{M}_I)$ contains the asymptotically flat end $\mathcal{M}_\infty$. From the asymptotic conditions it follows that $F^2$ tends to zero at infinity. Using (56) we find that $1/P$ also tends to zero at infinity (recall that $F^2 = -4f^2$). Consequently, equation (57) implies that $\chi$ tends to $c$ at infinity. On the other hand, recall that the Killing vector $\xi$ was normalized so that its norm $\lambda \to 1$ when we approach infinity in $\mathcal{M}_\infty$. This implies $c = 1$. Moreover, a simple consequence of (56) and (57) is $F^2 = -b^{-4}(1 - \chi)^4$. The asymptotic behaviour of $F^2$ near infinity forces $b^4 = 4M^2 > 0$ where $M$ is the Komar mass of $\tilde{C}$ in $\mathcal{M}_\infty$ (which is non-zero by hypothesis 2 of theorem [2]). Thus, Theorem [1] can be applied to conclude that $\mathcal{M}_3^D$ is locally isometric to the Kerr spacetime.

On the other hand, the Kerr spacetime obviously satisfies all the hypotheses of Theorem [2], and therefore the objects $\vec{k}, \vec{l}, y, Z, \ldots$ that we constructed on $\mathcal{M}_3$ can also be defined on the Kerr spacetime $\mathcal{M}_{M,a}$. The subscript $Kerr$ will be used to denote them. It is a matter of simple calculation using the Kerr geometry to find that $y_{Kerr}$ and $Z_{Kerr}$ are $y_{Kerr} = r$ and $Z_{Kerr} = a \cos \theta$ where $a$ is the specific angular momentum and $r$ and $\theta$ are standard Boyer-Lindquist coordinates of the Kerr spacetime.

From our assumption that $\partial \mathcal{M}_3^D \neq \emptyset$ and the fact that $\mathcal{M}_3^D \cup \mathcal{M}_3^0 \cup \mathcal{M}_3^D$ is dense in $\mathcal{M}_3$, it follows that either $\mathcal{M}_3^i$ or $\mathcal{M}_3^D$ is non-empty. The case $\mathcal{M}_3^D \neq \emptyset$ can be treated similarly to $\mathcal{M}_3^i \neq \emptyset$ just by interchanging $\vec{k} \leftrightarrow \vec{l}, \vec{m} \leftrightarrow \vec{m}$ and $\Psi_0 \leftrightarrow \Psi_4$. So, we can assume without loss of generality that $\mathcal{M}_3^i$ is non-empty and therefore $\partial \mathcal{M}_3^D \cap \partial \mathcal{M}_3^i \neq \emptyset$. Take a point $q \in \partial \mathcal{M}_3^D \cap \partial \mathcal{M}_3^i$ and a sufficiently small neighbourhood $U_q$ of $q$ such that the null tetrad $\{ \vec{k}, \vec{l}, \vec{m}, \vec{m} \}$ is smooth on $U_q$ (hence $\Psi_4$ is also smooth on $U_q$). Define $\Xi_q \equiv U_q \cap \partial \mathcal{M}_3^D \cap \partial \mathcal{M}_3^i$. Combining the general Bianchi equation (38) with the expressions for the NP coefficients (14) we easily obtain

$$
D\Psi_4 = -4\Psi_4 \left( \epsilon + \left( \vec{\xi}, \vec{k} \right) / P \right)|_{U_q}.
$$

(79)

Since $\epsilon + \left( \vec{\xi}, \vec{k} \right) / P$ is smooth on $U_q$, we can conclude that an integral line of $\vec{k}$ is either contained in $\Xi_q$ or else does not intersect $\Xi_q$ (this follows from integrating equation (73) along any integral curve of $\vec{k}$, which shows that $\Psi_4$ cannot become zero along the curve). Moreover, we know that the Petrov type of any spacetime cannot change along an integral curve of a Killing vector (see e.g. [14]). This implies that such a curve is also either contained in $\Xi_q$ or else does not intersect $\Xi_q$. In addition, since
$\Xi_q$ is the boundary of an open subset of $U_q$ (namely $U_q \cap \mathcal{M}_3^D$) it follows that $\Xi_q$ cannot be contained in any two-dimensional surface. With this information at hand we wish to show that there exists an open subset $U \subset U_q$ with the following properties: $U \cap \mathcal{M}_3^D \neq \emptyset$, $U \cap \mathcal{M}_3^4 \neq \emptyset$ and such that that $(\vec{\xi}, \vec{k})$ is non-zero everywhere on $U \cap \mathcal{M}_3^D$ and on $U \cap \mathcal{M}_3^4$. Indeed, if there exists a point $s \in \Xi_q$ where $(\xi, k)|_s \neq 0$ the claim is trivial (a suitable neighbourhood of $s$ would do it). So, we only need to consider the case when $(\xi, k)|_{\Xi_q} = 0$. We know that the metric on $\mathcal{M}_3^D$ is locally isometric to the Kerr metric, so let us evaluate $W_1 \equiv (\vec{\xi}, \vec{k})(\xi, l)(y^2 + Z^2)$ in the region $\mathcal{M}_3^D \cap U_q$ where the metric is known. We obtain

$$W_1|_{\mathcal{M}_3^D \cap U_q} = 2 \left(\vec{\xi}, \vec{k}\right) \left(y^2 + Z^2\right)|_{\mathcal{M}_3^D \cap U_q} = y^2 - b^2 y + a^2|_{\mathcal{M}_3^D \cap U_q}. \quad (80)$$

By assumption $W_1$ vanishes at $\Xi_q$. This implies $b^4 \geq 4a^2$ and $y|_{\Xi_q} = y_{\pm} \equiv (1/2)(b^2 \pm \sqrt{b^4 - 4a^2})$. So, the set $\Xi_q$ must be contained on one of the event horizons of the Kerr spacetime. Using standard properties of the Kerr metric, it is immediate to see that the following expression holds

$$\nabla_\alpha W_1|_{y_{\pm}} = \pm \sqrt{b^4 - 4a^2} \nabla_\alpha y|_{y_{\pm}}. \quad (81)$$

Furthermore $\nabla_\alpha y$ vanishes in the Kerr spacetime at most on a two-surface (the bifurcate horizon). So we can always find $s \in \Xi_q$ so that $\nabla_\alpha y|_s \neq 0$ (here we use the fact that $\Xi_q$ cannot be contained in any two-surface). Take a neighbourhood of $U \subset U_q$ of $s$ where $\nabla_\alpha y|_U$ is nowhere zero. If $b^4 - 4a^2 \neq 0$ (i.e. the Kerr metric is non-extreme) then (81) immediately implies that $W_1$ changes sign across $\Xi_q \cap U \neq 0$. It follows that $(\vec{\xi}, \vec{k})$ must be non-zero on $\mathcal{M}_3^4 \cap U$ as claimed. On the other hand, if $b^4 = 4a^2$, the argument can be repeated using $W_2 \equiv \left(W_1\right)^{1/2}$ instead (in this case, $W_2|_{U_q \cap \mathcal{M}_3^D} = y^2 - b^2/2$) and the same conclusion follows. In any case, we have that $(\vec{\xi}, \vec{k})$ must be non-zero on $\mathcal{M}_3^4 \cap U$ as claimed.

At this point two cases must be distinguished depending on whether $a = 0$ or $a \neq 0$. The case $a = 0$ is very simple because then $Z$ vanishes on $\mathcal{M}_3^D \cap U$ (and therefore the metric on this open set is locally the Kruskal-Schwarzschild metric). Using the fact that the vector field $\vec{l}$ is transversal to $\Xi_q \cap U$ (this follows from the geometry of the event horizons of Schwarzschild) and the equation $\vec{l}(Z) = 0$ (which holds everywhere on $\mathcal{M}_f$ as a trivial consequence of (59)) we conclude that $Z = 0$ everywhere on $U$. So, the Killing vector $\vec{\xi}$ is static on $U$ (because $P$ real implies (57) $\chi$ real). It is well-known that Petrov type II is incompatible with a static Killing vector. This excludes the case $a = 0$. Regarding $a \neq 0$, we know from the Kerr geometry that $\vec{k}$ and $\vec{\xi}$ are linearly independent on the event horizon. Furthermore, from standard properties of the Kerr geometry we find that $\eta_{\alpha \beta \lambda \mu} \xi^\beta k^\lambda l^\mu$ vanishes in the Kerr spacetime only at
points where $\bar{\xi} = 0$ or at the axis of symmetry of the axial Killing, which is also a two-dimensional surface. Thus, there exists a point $p \in \Xi \cap U$ where $\bar{\xi}$ and $\bar{k}$ are linearly independent and that $\eta_{\alpha\beta\mu\nu} \bar{e}^\alpha \bar{e}^\beta \bar{e}^\mu |_{U_p} \neq 0$. Hence, there exists an open neighbourhood $U_p \subset U$ of $p$ where $\eta_{\alpha\beta\mu\nu} \bar{e}^\alpha \bar{e}^\beta \bar{e}^\mu |_{U_p} \neq 0$ and such that the quotient $S_p = U_p / \text{span} \{ \bar{\xi}, \bar{k} \}$ is a differentiable manifold. As before, we will denote by $\pi_p$ the standard projection of $U_p \to S_p$. As in Proposition 2 we endow $S_p$ with a metric $\gamma$ of constant curvature equal to 2 (recall that $c = 1$ from asymptotic flatness). On the open region $U_p \cap M^3$, Proposition 2 can be applied and therefore the local metric on $U_p \cap M^3_3$ is given by (29). Since $c = 1$ and $b_1 b_2 = 0$ (recall that $b^4 = 4M^2$) we have that $Z$ satisfies $\Delta_\gamma Z = -2Z$ on $\pi_p(U_p \cap M^3_3)$. On the other hand, the metric on $M^3_3 \cap U_p$ is locally Kerr. Therefore $Z|_{M^3_3} = a \cos \theta$ which clearly satisfies $\Delta_\gamma Z = -2Z$ on $\pi_p(M^3_3 \cap U_p)$. By construction we have that $\pi_p(M^3_3 \cap U_p)$ and $\pi_p(M^4_4 \cap U_p)$ are both non-empty and dense in $S_p$. Since $Z$ is smooth on $S_p$ it follows that $\Delta_\gamma Z = -2Z$ holds everywhere on $S_q$. Now, $\Delta_\gamma Z = -2Z$ is a second order elliptic partial differential equation. Hence, $Z$ is analytic on $S_p$. Since $Z = a \cos \theta$ on the open non-empty subset of $S_p$ we can conclude that $Z = a \cos \theta$ everywhere on $S_p$. This gives the contradiction we need because on the one hand we have $\Psi_4 \neq 0$ on $M^3_3 \cap U_p$ and on the other hand substituting $Z = a \cos \theta$ into (77) we find $\Psi_4|_{U_p} = 0$. Thus we can conclude that $\partial M^3_3 \cap \partial M^3_3 = \emptyset$ (and $\partial M^3_3 \cap \partial M^3_3 = \emptyset$ by a similar argument) and the proof of the lemma is completed. □

So, we already know that $M^3_3$ is either empty or covers $M^3_3$ and that the metric on $M^3_3$ is locally isometric to Kerr. In order to complete the proof of Theorem 2 we need to show first that $M^3_3 = \emptyset$ is impossible and second that $M_f = M$. Proving the first claim uses asymptotic flatness in an essential way.

**Proof of the Theorem**

We must first exclude the possibility that $M^3_3 = \emptyset$. So, we suppose $M^3_3 = \emptyset$ and obtain a contradiction. Notice first that $M^3_3 = \emptyset$ does not imply yet that one of $\Psi_0$ or $\Psi_4$ is everywhere zero. It is still possible that $\Psi_0$ and $\Psi_4$ are nonzero on some (necessarily disjoint) regions of $M^3_3$ (i.e. that $M^3_3 \neq \emptyset$ and $M^4_4 \neq \emptyset$). We know that $M_3 = M_f$ and therefore there exists an asymptotically flat end $M_\infty \subset M_3$. By restricting $M_\infty$ if necessary we can assume that $\bar{\xi}$ is timelike in $M_\infty$. It is well-known [19] that a vacuum spacetime admitting a Killing vector field is analytic in the region where the Killing vector is timelike. Consequently, in the strictly stationary region (i.e. where $\bar{\xi}$ is timelike) the vector fields $\bar{k}$ and $\bar{l}$ are analytic. This follows from the fact that they are the eigenvectors corresponding to simple eigenvalues of an analytic endomorphism and therefore no multiplicity changes are permitted). Thus, there exists (locally) an analytic null tetrad $\{ \bar{k}, \bar{l}, \bar{m}, \bar{\bar{m}} \}$. In the local neighbourhood where this tetrad is analytic, $\Psi_0$ and $\Psi_4$ are also analytic. Thus, we can conclude that in the asymptotically flat end $M_\infty$ (where $\xi$ is timelike), either $\Psi_0$ or $\Psi_4$ vanish.
everywhere (because at least one of them vanishes on an open non-empty set). As before, the two cases \( \Psi_0 = 0 \) or \( \Psi_4 = 0 \) are related to each other by a transformation \( \bar{k} \leftrightarrow \bar{l}, \bar{m} \leftrightarrow \bar{m}, \Psi_0 \leftrightarrow \Psi_4 \) and hence we can assume without loss of generality that \( \Psi_0 \) vanishes everywhere on \( \mathcal{M}_\infty \) (i.e. \( \mathcal{M}_\infty \subset \mathcal{M}_3^D \)). From the fact that \( [\xi, \bar{k}] \propto \bar{k} \) (see (12)) and that \( \{\xi, \bar{k}\} \) span a two-dimensional vector subspace at every point in \( \mathcal{M}_\infty \), it follows that \( \xi \) and \( \bar{k} \) are tangent to families of two-surfaces. Furthermore, \( \mathcal{M}_\infty \) can be chosen to have topology \( \mathbb{R} \times \mathbb{R} \times S^2 \) (\( S^2 \) is the two-sphere). Using the fact that \( \bar{k} \) is a principal null direction of the Weyl tensor (this follows from (13) after taking into account that \( \Psi_0 = 0 \)), asymptotic flatness implies (possibly after restricting \( \mathcal{M}_\infty \)) that \( \mathcal{M}_\infty / \text{span}\{\bar{k}, \bar{l}\} \cong S^2 \). Let us introduce on \( S^2 \) the round metric \( \gamma \) with constant curvature \( R = 2 \). So \( (S^2, \gamma) \) is a complete and simply connected Riemannian space. Let us also define the canonical projection onto this quotient by \( \bar{\pi} : \mathcal{M}_\infty \rightarrow S^2 \).

**Proposition 2** takes care of the local form of the metric at points where \( \eta_{\alpha\beta\lambda\mu} \xi^\beta k^\lambda \mu^\mu \neq 0 \) (i.e. at points on \( \mathcal{M}_3^D \)). So, we must analyze what happens on the set of points where \( \eta_{\alpha\beta\lambda\mu} \xi^\beta k^\lambda \mu^\mu = 0 \). To do that, we define the set \( T = \{ p \in \mathcal{M}_\infty : \eta_{\alpha\beta\lambda\mu} \xi^\beta k^\lambda \mu^\mu|_p = 0 \} \) and prove that \( T \) must have empty interior. Indeed, suppose there exists a point \( p \in \mathcal{M}_\infty \) and take a sufficiently small open neighbourhood \( U_p \subset \mathcal{T}_p \) of \( p \) where the null tetrad \( \{\bar{k}, \bar{l}, \bar{m}, \bar{m}\} \) is well-defined. We have (from the definition of \( T \)) that \( (\bar{\xi}, \bar{m})|_{U_p} = 0 \). Since \( U_p \) is open, equation (11) implies that \( \Psi_4|_{U_p} = 0 \). Hence we have \( \Psi_0|_T = \Psi_4|_T = 0 \) and therefore \( \mathcal{T} \subset \mathcal{M}_3^D \) which we are assuming is empty. Therefore \( T \) has empty interior. Thus, the set \( \mathcal{M}_\infty \cap \mathcal{M}_3^D \) is dense in \( \mathcal{M}_\infty \). Let us prove that its projection is also dense on \( S^2 \). Indeed, a trivial consequence of (12) is \( \bar{\ell}(\eta_{\alpha\beta\lambda\mu} \xi^\beta k^\lambda \mu^\mu)|_{\mathcal{M}_\infty} = 0 \). Similarly, taking into account that, locally, \( \eta_{\alpha\beta\lambda\mu} \xi^\beta k^\lambda \mu^\mu = i \left[ (\bar{\xi}, \bar{m}) m_\alpha - (\bar{\xi}, \bar{m}) \bar{m}_\alpha \right] \), equation (12) shows that the vanishing of \( \eta_{\alpha\beta\lambda\mu} \xi^\beta k^\lambda \mu^\mu \) at any point in \( \mathcal{M}_\infty \) implies its vanishing everywhere along the integral lines of \( \bar{k} \). Putting these two things together, we can conclude \( \bar{\pi}^{-1} \circ \bar{\pi}(T) = T \), i.e. the set \( T \) is the anti image of certain subset of \( S^2 \). Therefore \( \bar{\pi}(\mathcal{M}_\infty \cap \mathcal{M}_3^D) \) is dense in \( S^2 \). Recalling that \( \mathcal{M}_\infty \subset \mathcal{M}_3^D \), we observe that **Proposition 3** can be applied on \( \bar{\pi}(\mathcal{M}_\infty \cap \mathcal{M}_3^D) \). Taking into account that \( c = 1 \) and \( b_1 b_2 = 0 \), we find from **Proposition 3** that \( Z \) fulfills \( \Delta_\gamma Z = -2Z \) on a dense subset of \( S^2 \) and hence everywhere. Thus, \( Z \) satisfies the linear equation \( \Delta_\gamma Z = -2Z \) on the whole two-sphere. This equation is an eigenvalue equation (with eigenvalue equal to \(-2\)) for the Laplacian on the round sphere. Since \( Z \) is defined in terms of \( P \), which is a smooth function on \( \mathcal{M}_3 \), the solution of this eigenvalue equation must be regular and smooth everywhere on \( S^2 \). The eigenvalues of the Laplace-Beltrami operator on the sphere are well-known and the general regular solution of \( \Delta_\gamma Z = -2Z \) can always be written as \( Z = a \cos \theta \) where \( a \) is a real constant and \( \theta \) is a suitably chosen colatitude angle on the sphere. This expression for \( Z \) shows that \( \Psi_4 \) (given in (7)) must be zero everywhere on \( \mathcal{M}_\infty \subset \mathcal{M}_3 \). Since \( \Psi_0 \) vanishes also on \( \mathcal{M}_\infty \), this contradicts the fact that \( \mathcal{M}_3^D = \emptyset \). Thus, **Lemma 3** shows \( \mathcal{M}_3^D = \mathcal{M}_3 \). Since **Proposition 4** tells us that
$M_3 = M_f$ we conclude that $(M_f, g_f)$ is locally isometric to the Kerr spacetime.

The final part of the proof consists in showing that $M_f = M$. Assume that $M_f$ is a proper subset of $M$ and take a point $q$ on the topological boundary of $M_f$ as a subset of $M$. From the continuity of $F^2$ and the fact that $F^2 = b^{-4}(1-\chi)^4$ on $M_f$, we must have $\lambda|q = 1$, hence $\vec{\xi}$ is timelike in some neighbourhood of $q$. Consider a smooth curve $\gamma_p(s)$ defined on some interval $s_0 < s \leq 1$ such that $\gamma_p(s) \in M_f$, $\forall s \in (s_0, 1)$, $\gamma_p(1) = p$ and such that the tangent vector $\dot{\gamma}(s)$ is orthogonal to $\vec{\xi}$ and of unit length (the existence of such a curve is easy to establish). Define the real function $Y(s) = (y \circ \gamma_p)(s)$. Since $\chi \to 1$ at $p$ we have from [57] that $P$ diverges at $p$. Taking into account that $Z$ remains bounded, we find $Y(s) \to \infty$ when $s \to 1$. In the Kerr metric we have $\nabla_\alpha y \nabla^\alpha y = (y^2 - 2My + a^2)/(y^2 + a^2 \cos^2 \theta)$ and therefore $\nabla_\alpha y \nabla^\alpha y|_{\gamma_p(s)} \to 1$ when $s \to 1$. Hence we can assume that $\nabla_\alpha y|_{\gamma_p(s)}$ is spacelike for $s \in (s_0, 1)$. Then

$$\left(\frac{dY}{ds}(s) \right)^2 = \left(\nabla_\alpha y|_{\gamma_p(s)} \dot{\gamma}_p^\alpha(s) \right)^2 \leq \nabla_\alpha y \nabla^\alpha y|_{\gamma_p(s)},$$

where we have used the fact that $\{\nabla_\alpha y|_{\gamma_p(s)}, \dot{\gamma}_p(s)\}$ define a spacelike two-plane and we have applied the Schwarz inequality. Hence $\frac{dY}{ds}$ stays bounded, which contradicts $Y \to \infty$ when $s \to 1$. This completes the proof of the theorem. \hfill $\square$

Acknowledgements

I wish to thank Raúl Vera and José M.M. Senovilla for a careful reading of the manuscript and for valuable comments. This work has been partially supported by projects UPV172.310-G02/99 and 1998SGR 00015.

References

[1] G. L. Bunting and Masood-ul-Alam Nonexistence of multiple black holes in asymptotically euclidean static vacuum space–times, Gen. Rel. Grav. 19 (1987) 147-154.

[2] P.T.Chruściel, “No hair” theorems- folklore, conjectures, results, in Differential geometry and mathematical physics (Contemporary Mathematics vol 170) Eds. J.Beem and K.L.Dugal, (Providence, RI: American Mathematical Society) (1994) pp 23-49.

[3] S.W.Hawking and G.F.R.Ellis The large scale structure of space-time, (1973) (Cambridge University Press, Cambridge).
[4] G. Weinstein, *On rotating black holes in equilibrium in general relativity*, Commun. Pure Appl. Math. XLIII, (1990) 903-948.

[5] G. Weinstein, *N-black hole stationary and axially symmetric solutions of the Einstein/Maxwell equations*, Commun. Part. Diff. Eqs. 21 (1996) 1389-1430.

[6] W. Simon, *Characterizations of the Kerr metric*, Gen. Rel.Grav. 16 (1984) 465-476.

[7] M. Mars, *A spacetime characterization of the Kerr metric*, Class. Quantum. Grav. 16, (1999) 2507-2523.

[8] D. Kramer, H. Stephani, E. Herlt and M. A. H. MacCallum *Exact solutions of Einstein’s field equations*, (1980) (Cambridge University Press, Cambridge).

[9] D. Kennefick and N. Ó Murchadha, *Weakly decaying asymptotically flat static and stationary solutions to the Einstein equations*, Class. Quantum Grav. (1995) 12 149-158.

[10] R. Beig and W. Simon, *The stationary gravitational field near spatial infinity*, Gen. Rel. Grav. 12 (1980) 1003-1013.

[11] A. Komar, *Covariant conservation laws in general relativity*, Phys.Rev 113 (1958) 934-936.

[12] R. H. Boyer and R. W. Lindquist, *Maximal analytic extension of the Kerr metric*, J. Math. Phys. 8 (1967) 265-281.

[13] B. Carter, *Global structure of the Kerr family of gravitational fields*, Phys. Rev. 174 (1968) 1559-1571.

[14] G. S. Hall, *Decomposition of spacetimes admitting symmetries*, Class. Quantum. Grav. 13 (1996) 1479-1485.

[15] J. M. M. Senovilla and R. Vera, *Segre decomposition of spacetimes*, Class. Quantum Grav. 16 (1999) 1185-1196.

[16] G. S. Hall and A. D. Rendall, *Int. J. Theor, Phys. 28* 365.

[17] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, Vol. II* (1963) (Interscience publishers, John Wiley & Sons, New York).

[18] W. Israel, *Differential forms in general relativity*, Commun. of the Dublin Institute for Advanced Studies, Series A, 19 (1970) 1-100.
[19] H. Müller zum Hagen, *On the analyticity of stationary vacuum solutions of Einstein’s equations*, Proc. Camb. Phil. Soc. 68 (1970) 199-201.

[20] R. Garabedian *Partial differential equations*, (1964) (Chelsey Publishing company, New York).