Nonextensive perfect hydrodynamics - a model of dissipative relativistic hydrodynamics?

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Abstract: We demonstrate that nonextensive perfect relativistic hydrodynamics ($q$-hydrodynamics) can serve as a model of the usual relativistic dissipative hydrodynamics ($\varpi$-hydrodynamics) facilitating therefore considerably its applications. As illustration we show how using $q$-hydrodynamics one gets the $q$-dependent expressions for the dissipative entropy current and the corresponding ratios of the bulk and shear viscosities to entropy density, $\zeta/s$ and $\eta/s$.

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1. Introduction

Hydrodynamics is well a established effective approach to flow phenomena. Here we shall present its nonextensive relativistic version from the point of view of high energy collision physics [1–6]. The characteristic feature of such a processes is the production of a large number of secondaries (their multiplicities at present approach $\sim 10^3$). Already in 1953, when there were only $\sim 10$ particles produced and registered, mostly in cosmic rays experiments, it was found [1] that such processes can be very effectively described by using a thermodynamic approach in conjunction with hydrodynamical flow. In this so called Landau Hydrodynamic Model the secondaries secondaries were considered to be a product of decay of some kind of hadronic fluid produced in such collisions, which was expanding before hadronization [1]. This model has recently been updated to describe recent experimental data [2]. Since then there was a number of successful attempts to develop new solutions for both the Landau model [3]

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and for the so called Hwa-Bjorken version of the hydrodynamical model [4], for which a new class of solutions has been found [5]. Hydrodynamic models of different types were therefore frequently used in the phenomenological description of multiparticle production processes at high energies, especially for high energy nuclear collisions [6]. These are of special interest to us (being currently investigated at RHIC at Brookhaven with the newly commissioned LHC at CERN joining soon) because it is widely believed that in such collisions a new form of hadronic matter, the so called Quark-Gluon-Plasma (QCD), will be produced [7].

So far all hydrodynamic models in this field were based on the usual Boltzmann-Gibbs (BG) form of statistical mechanics. The only works discussing some general features of the nonextensive hydrodynamics, which we are aware of [8], use a nonrelativistic approach and are therefore not suitable for the applications we are interested in. On the other hand it is known that an approach based on non-extensive statistical mechanics (used mainly in the form proposed by Tsallis [9] with only one new parameter, the nonextensivity parameter \( q \)) describes different sets of data in a better way than the usual statistical models based on BG statistics, cf., [9] for general examples and [10–15] for applications to multiparticle production processes. Roughly speaking, all observed effects amount to a broadening of the respective spectra of the observed secondaries (both in transverse momentum space and in the rapidity space), they take the form of \( q \)-exponents instead of the naively expected usual exponents:

\[
\exp(-X/T) \Rightarrow \exp_q(-X/T) = [1 - (1 - q)X/T]^{1/(1-q)}.
\]

From these studies emerged a commonly accepted interpretation of the nonextensivity parameter \( q \) (in fact, \(|q-1|\)) as the measure of some intrinsic fluctuations characteristic for the hadronizing systems under consideration [10, 11]. For \( q > 1 \) and in the transverse momentum space it could be fluctuation of the temperature \( T \) corresponding to some specific heat parameter \( C \). In this case \( q - 1 = C \) and therefore it should be inversely proportional to the volume of the interaction region. This effect is indeed observed [15]. In rapidity space these are fluctuations of the so called partition temperature, \( T_{pt} = E/\langle n \rangle \) [14], which are precisely the same fluctuations that lead to the Negative Binomial form of the observed multiplicity distributions, \( P(n; \langle n \rangle; k) \), with its characteristic parameter \( k \) given by \( k = 1/(q - 1) \) [14, 16]. In the case of \( q < 1 \), the interpretation is not at present clear [11]. It seems that the first role of the parameter \( q \) is to restrict the allowed phase space [17]. Actually, the conjecture associating \( q \) with fluctuations has already been formalized as a new branch of statistical mechanics called superstatistics [18]. It should be noted that there are also arguments connecting nonextensivity with some special dynamical correlations existing in the system under consideration [19], but their connection with fluctuations is not yet fully clear (cf., [16])\(^{endnote1}\).

The physical picture emerging from the above experience is that, instead of a strict local thermal equilibrium customarily assumed in all applications of statistical models (including hydrodynamic), one rather encounters a kind of stationary state, which already includes some interactions. It can be introduced in different ways. For example, in [20] it was a random distortion of the energy and momentum conservation caused by the surrounding system. This results in the emergence of some nonextensive equilibrium. In [21, 22] the two-body...
energy composition is replaced by some generalized energy sum \( h(E_1, E_2) \), which is assumed to be associative but which is not necessarily simple addition and contains contributions stemming from pair interaction (in the simplest case). It turns out that, under quite general assumptions about the function \( h \), the division of the total energy among free particles can be done. Different forms of the function \( h \) lead then to different forms of the entropy formula, among which one encounters also the known Tsallis entropy. The origin of this kind of thinking can be traced back to the analysis of the \( q \)-Hagedorn model proposed some time ago in [23].

All phenomenological applications of hydrodynamic models to the recent multiparticle production processes show that, although perfect (nonviscous) hydrodynamics successfully describes most RHIC data [3, 24], there are indications that a hadronic fluid cannot be totally ideal. For example, the perfect fluid dynamical calculations with color glass condensate initial state could not reproduce the elliptic flow data [25] indicating a necessity to use some kind of viscous fluid description. A nonrelativistic viscous fluid is usually described by the first order Navier-Stokes equations. However, one needs their special relativistic generalization and this turns out to be acausal and unstable [26] (see also [27–29]). One therefore looks towards the extended, second order theories accepting all problems connected with their formulation and practical applications [30–36]. Physically, the difference is in that first order theories are based on the local equilibrium hypothesis, in which the independent variables are used, whereas in higher order theories the fluxes of the local equilibrium theory appear as independent variables. In particular, the entropy vector is quadratic in the fluxes, containing terms characterizing the deviation from local equilibrium. This situation plus our experience with the nonextensive formalism [10–12, 14–16] prompted us to investigate the simple nonextensive formulation of the perfect hydrodynamic model, a perfect \( q \)-hydrodynamics [37]. It turned out that this describes the experimental data fairly well. In addition, an apparently unexpected feature appeared, namely the possibility that (relatively simple and first order) perfect \( q \)-hydrodynamics can serve as a model of (second order and complicated in practical use) viscous \( d \)-hydrodynamics. This is the point we would like to discuss in more detail in this work.

In the next Section we shall, for completeness, present the main points of \( q \)-hydrodynamics [37]. In Section 3 we propose a nonextensive/dissipative conjecture (NexDC), which allows us to connect ideal \( q \)-hydrodynamics with a \( d \)-hydrodynamics. Consequences of NexDC are discussed in Section 4 (entropy production) and in Section 5 (transport coefficients). Section 6 contains the summary.

## 2. Basic elements of \( q \)-hydrodynamics

As in [37], we shall limit ourselves to a 1 + 1 dimensional baryon-free version of hydrodynamic flow. This is derived following Lavagno [38], in which a nonextensive version of the Boltzmann equation has been proposed and investigated. Because no external currents are assumed, this corresponds to a kind of \emph{perfect \( q \)-fluid}. There
are two important points in [38]: (i) the Boltzmann equation is formulated for \( f_q^1(x,p) \) distribution rather than for the usual \( f_q(x,p) \); (ii) the usual molecular chaos hypothesis is now assumed in nonextensive form:

\[
\begin{aligned}
& h_q[f_q,f_{q1}] = \exp_q \left[ \ln_q f_q + \ln_q f_{q1} \right] \quad \text{where} \quad \exp_q(X) = [1 + (1 - q)X]^{\frac{1}{1-q}}, \quad \ln_q(X) = \frac{X^{(1-q)} - 1}{1 - q}. \\
\end{aligned}
\]  

(1)

Here \( f_q(x,p) \) is the \( q \)-version of the phase space distribution function, whereas \( h_q[f_q,f_{q1}] \) is the \( q \)-version of the correlation function related to the presence of two particles in the same space-time position \( x \) but with different four-momenta \( p \) and \( p_1 \), respectively. By postulating Eq. (1) we are, in fact, assuming that, instead of a strict (local) equilibrium, a kind of stationary state is being formed, which already includes some interactions and which is summarily characterized by a parameter \( q \); very much in the spirit of [20–23] mentioned before. With this assumption we are already departing from the picture of the usual ideal fluid with its local thermal equilibrium, which is the prerogative of ideal hydrodynamics [26]. The consequences of this fact will be discussed below.

The most important ingredient for further discussion is now the corresponding nonextensive entropy (\( q \)-entropy) current [37, 38]:

\[
\begin{aligned}
\sigma_q^\mu(x) &= -k_B \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} \left\{ f_q^0 \ln_q f_q - f_q \right\}. \\
\end{aligned}
\]  

(2)

It turns out that \( \partial_\mu \sigma_q^\mu \geq 0 \) at any space-time point, i.e., the relativistic local H-theorem is valid in this case [37–39]. Assuming \( \partial_\mu \sigma_q^\mu \equiv 0 \) one finds that (\( k_B \) is Boltzmann constant):

\[
\begin{aligned}
f_q(x,p) &= \left[ 1 - (1 - q) \frac{p_\mu u^\mu_q(x)}{k_BT_q(x)} \right]^{1/(1-q)} \equiv \exp_q \left[ -\frac{p_\mu u^\mu_q(x)}{k_BT_q(x)} \right], \\
\end{aligned}
\]  

(3)

where \( T_q(x) \) is the \( q \)-temperature [37] and \( u^\mu_q(x) \) is the \( q \)-hydrodynamic flow four-vector. Actually, one should be aware of the fact that there is still an ongoing discussion on the meaning of temperature in nonextensive systems.

However, the small values of the parameter \( q - 1 \) deduced from data allow us to argue that, in the first approximation, \( T_q \) can be regarded as the hadronizing temperature in such a system (cf., [40] for a thorough discussion of the temperature of nonextensive systems). Finally, we get the \( q \)-version of the local energy-momentum conservation

\[
\begin{aligned}
\partial_\nu T^\nu_q(x) &= 0, \quad \text{with} \quad T^\nu_q(x) \equiv \frac{1}{(2\pi\hbar)^3} \int \frac{d^3p}{p^0} p^\nu f_q^0(x,p) \quad \text{and} \quad f_q(x,p) \equiv \exp_q \left[ -\frac{p_\mu u^\mu_q(x)}{k_BT_q(x)} \right]. \\
\end{aligned}
\]  

(4)

In what follows we shall use covariant derivative notation in which the vector \( u^\mu \) and tensor \( g^{\mu\nu} \) are defined as

\[
\begin{aligned}
u^\nu = \partial_\nu u^\nu + \Gamma^\nu_{\lambda\mu} u^\lambda \quad \text{and} \quad g^{\mu\nu} = \partial_\mu g^{\nu\nu} + \Gamma^\nu_{\sigma\mu} g^{\sigma\nu} + \Gamma^\nu_{\sigma\nu} g^{\mu\sigma}, \\
\end{aligned}
\]  

(5)

by means of the Christoffel symbols, \( \Gamma^\nu_{\lambda\mu} \equiv \frac{1}{2} g^{\nu\kappa}(\partial_\mu g_{\kappa\lambda} + \partial_\lambda g_{\mu\kappa} - \partial_\kappa g_{\lambda\mu}) \). In this notation Eq. (4) reads:

\[
\begin{aligned}
T^\mu_q(x) \equiv \left[ (\epsilon_q + P_q) u^\mu_q u^\nu_q - P_q g^{\mu\nu} \right], \mu &= \left[ \epsilon_q(T_q) u^\mu_q u^\nu_q - P_q(T_q) \Delta^\mu_q \right], \mu = 0, \\
\end{aligned}
\]  

(6)
where $\Delta_{\mu\nu}^q \equiv g_{\mu\nu}^q - u_{\mu q}^u u_{\nu q}^u$. Here it was assumed that the $q$-modified energy-momentum tensor $T_{\mu\nu}^q$ can be decomposed in the usual way in terms of the $q$-modified energy density and pressure, $\varepsilon_q \equiv u_{\mu q}^u T_{\mu\nu}^q u_{\nu q}^u$ and $P_q \equiv -\frac{1}{3} T_{\mu\nu}^q \Delta_{\mu\nu}^q$, by using the $q$-modified flow $u_{\mu q}^u$ (such that in the rest frame of the fluid $u_{\mu}^q = (1, 0, 0, 0)$ and for $q \rightarrow 1$ it becomes the usual hydrodynamic flow $u_{\mu}^q$). Notice that Eq. (6) is formally identical to the perfect hydrodynamic equation but with all the usual ingredients replaced by their $q$-counterparts (perfect means here that there is nothing on the r.h.s. of Eq. (6)). In this sense we can speak about the perfect $q$-fluid mentioned before.

Some remarks are in order before proceeding further.

- When applied to multiparticle production processes, each hydrodynamic model is supplemented with three ingredients, which must be considered together with Eq. (6) (or its equivalent): (i) - initial conditions (IC) setting the initial energy density which is going to evolve hydrodynamically; (ii) - equation of state (EoS) reflecting the internal dynamics of the fluid considered and (iii) freeze-out conditions describing transformation of the expanded fluid into observed hadrons. The problem is that all of them can, in principle, enter with their own intrinsic fluctuation pattern, i.e., with their own parameters $q$. In [37] where we provided preliminary comparison with experimental data, we have assumed, for simplicity, the same value of parameter $q$ throughout the whole collision. It is still to be checked how good this assumption is. However, in our present discussion this point is unimportant.

- Whereas in the usual perfect hydrodynamics (based on the BG statistics) entropy is conserved in the hydrodynamic evolution, both locally and globally, in the nonextensive approach it is only conserved locally. The total entropy of the whole expanding system is not conserved, because for any two volumes of the fluid, $V_{1,2}$, one finds that $S_q^{(V_1)} + S_q^{(V_2)} \neq S_q^{(V_1 \oplus V_2)}$ (where $S_q^{(V)}$ are the corresponding total entropies). This should be always remembered (albeit, strictly speaking, the hydrodynamic model requires only local, not global, entropy conservation). As a consequence of this fact, as we shall see below, contrary to the situation in usual perfect hydrodynamics, in the perfect $q$-hydrodynamics the entropy is produced (but not $q$-entropy).

- To guarantee that hydrodynamics makes sense, there should exists some spacial scale $L$ such that the volume $L^3$ contains enough particles. However, in the case when there are fluctuations and/or correlations characterized by some typical correlation length $l$ for which we expect that $l > L$, one has to use nonextensive entropy $S_q^{(L^3)}$ and its (locally defined) density, $s_q(x) = S_q^{(L^3)}/L^3$. When formulating the corresponding $q$-hydrodynamics one takes the limit $L \rightarrow 0$, in which case the explicit dependence on the scale $L$ vanishes, whereas the correlation length $l$ leaves its imprint as a parameter $q$. In this sense, perfect $q$-hydrodynamics can be considered as preserving causality and nonextensivity $q$ is then related with the correlation length $l$ (one can argue that, very roughly, $q \sim l/L_{\text{eff}} \geq 1$, where $L_{\text{eff}}$ is some effective spacial scale of the $q$-hydrodynamics). If the correlation length $l$ is compatible with the scale $L_{\text{eff}}$, i.e., $l \approx L_{\text{eff}}$, one recovers the condition of the usual local thermal equilibrium and in this case the $q$-hydrodynamics reduces to the usual
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The above considerations were limited to the case of \( q > 1 \) only, for which, as was said before, the clear correspondence with fluctuations was found. In what concerns the case of \( q < 1 \), it seems that following [17], where \( q < 1 \) was found as a main factor closing the allowed phase space, we can at the moment only propose that it could probably correspond to the case where, for some reason, the scale \( L \) cannot vanish but must stop at some value \( l < L \). In this case, analogously to what was said above, one could expect that, again, \( q \sim l/L_{\text{eff}} \), which this time would be smaller than unity. We shall not discuss this possibility further in this work.

3. Nonextensive/dissipative conjecture (NexDC)

As seen in Eq. (6), the structure of the perfect \( q \)-hydrodynamical flow is formally identical with the flow of ideal fluid described by the usual ideal hydrodynamics. We therefore regard the fluid described by Eq. (6) as a perfect \( q \)-fluid (in the sense already mentioned before). Our experience with applications of \( q \) statistics [12, 13] tells us that in cases of interest to us \(|q - 1| \ll 1\). It is therefore tempting to simply expand the corresponding quantities in powers of \(|q - 1|\) and to only keep the linear term [23]. The result one gets looks promising, namely

\[
T_q^{\mu\nu} \equiv T_{q=1}^{\mu\nu} + (q-1)\tau_q^{\mu\nu} \quad \text{where} \quad \tau_q^{\mu\nu} \equiv \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} p^{\mu} p^{\nu} \exp \left( -\frac{p \cdot u}{T} \right) \quad \text{and} \quad \tau_q^{\mu\nu} \equiv \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} p^{\mu} p^{\nu} \left( \frac{p \cdot u}{T} \right)^2 \exp \left( -\frac{p \cdot u}{T} \right),
\]

with \( T_{q=1} \) being the usual energy-momentum tensor describing an ideal fluid in the BG approach and \( \tau_q^{\mu\nu} \) representing a viscous correction caused by the nonextensivity. However, in order for Eq. (7) to be valid in the whole phase space, the \(|1-q| (\frac{p \cdot u}{T})^2 < 2 \) inequality must hold. This means that either such a procedure can be applied only to a limited domain of phase space, or that \( q = q(x, p) \), a possibility which is outside the scope of the present work. We must therefore proceed in a more general way. Let us formally decompose \( \varepsilon_q \), \( P_q \) and \( u_\mu^q \) in Eq. (6) into the, respectively, extensive and nonextensive parts:

\[
\varepsilon_q(T_q) \equiv \varepsilon(T_q) + \Delta \varepsilon_q(T_q), \quad P_q(T_q) \equiv P(T_q) + \Delta P_q(T_q), \quad u_\mu^q(x) \equiv u_\mu(x) + \delta u_\mu^q(x).
\]

Actually this can be only done approximately because our extensive \( \varepsilon \) and \( P \) still depend on \( T_q \), rather than on \( T \). We tacitly assume that \( T_q \) is not too far from \( T \) (so far they are independent parameters but later on we shall impose a condition on them, see Eq. (13) below). In Eq. (9) the four-velocity \( u_\mu(x) \) is formally a solution of the equation to which Eq. (6) is transformed by using Eq. (9):

\[
\left[ \bar{\varepsilon}(T_q) u^{\mu} u^{\nu} \tilde{P}(T_q) \Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu} \right]_{\rho} = 0.
\]

One can see that it has form of a dissipative hydrodynamic equation [30–36] (here \( \Delta^{\mu\nu} \equiv \delta^{\mu\nu} - u^{\mu} u^{\nu} \) and \( A^{(\mu} B^{\nu)} \equiv \frac{1}{2} (A^{\mu} B^{\nu} + A^{\nu} B^{\mu}) \) where \( \bar{\varepsilon} \) is the energy density, \( \tilde{P} \) is the pressure, \( W^{\mu} \) denotes the energy or heat
flow vector, whereas $\pi^{\mu\nu}$ is the shear (symmetric and traceless) pressure tensor. They are defined as (using the angular bracket notation: $a^{<\mu b^{\nu}>}$ def $= \frac{1}{2} (\Delta^{a}_{\mu} \Delta^{\nu}_{b} + \Delta^{a}_{\nu} \Delta^{\mu}_{b} - \frac{1}{3} \Delta^{a}_{\lambda} \Delta^{\lambda}_{\mu} \delta^{\nu}_{b} + \frac{1}{3} \Delta^{a}_{\lambda} \Delta^{\nu}_{\lambda} \delta^{\mu}_{b})$):

$$\tilde{\varepsilon} = \varepsilon_q + 3\Pi, \quad \tilde{P} = P_q + \Pi, \quad W^\mu = w_q [1 + \gamma] \Delta^{\mu}_{\lambda} \delta u^{\lambda}_q, \quad \pi^{\mu\nu} = \frac{W^\mu W^\nu}{w_q [1 + \gamma]} + \Pi \Delta^{\mu\nu} = w_q \delta u^{<\mu}_q \delta u^{\nu}_q, \quad (11)$$

and expressed in terms of $\Pi$ (being a kind of a $q$-dependent bulk pressure), $q$-enthalpy $w_q$ and a $q$-dependent variable $\gamma$,

$$\Pi \equiv \frac{1}{3} w_q [\gamma^2 + 2\gamma], \quad w_q \equiv \varepsilon_q + P_q, \quad \gamma \equiv u_\mu \delta u^\mu_q = -\frac{1}{2} \delta u_\mu u^\mu_q. \quad (12)$$

Notice that when $q \to 1$, the difference between the $q$ and ideal hydrodynamic flows vanishes, $\delta^\mu_q \to 0$, and with it also $\gamma \to 0$. It means that all dissipative fluxes of $d$-hydrodynamics (10) which are induced by the $q$-flow, like $W^\mu$, $\pi^{\mu\nu}$ and $\Pi$, vanish in this limit as well, and one recovers the equations of the usual perfect hydrodynamics. The variable $\gamma$ is easier to handle and to calculate than the differences in flows (see Eq. (18) below for its explicit form). Notice that, whereas the time evolution of $\Pi$ is controlled by $q$-hydrodynamics (via the respective time dependencies of $\varepsilon_q$, $P_q$ and $\gamma$) its form is determined by the assumed constraints which must assure that the local entropy production is never negative (as is always assumed in the standard 2nd order theory [31, 32]).

Now comes the crucial point. To finally link the usual $q$-hydrodynamics and its $d$ counterpart, one has to fix somehow the temperature $T_q$ and the flow velocity field $u_q$. This is done by assuming that there exists such a temperature $T$ and velocity difference $\delta u^\mu_q$ that the following two relations are satisfied:

$$P(T) = P_q(T_q), \quad \varepsilon(T) = \varepsilon_q(T_q) + 3\Pi. \quad (13)$$

We call them the nonextensive/dissipative relations, NexDC in short. Here $\varepsilon$ and $P$ are the energy density and pressure as defined in the usual BG statistics, i.e., for $q = 1$. Eq. (13) provides a definite relation between $T_q$ and $T$, therefore, in what follows, we shall mainly use $T$ for a description of dissipative effects (except of some expressions when it is easier to keep both $T$ and $T_q$, but always with the understanding that, because of Eq. (13) they are not independent). It is now straightforward to show [37] that in this case one can transform Eq. (10) into the equation of the usual $d$-hydrodynamics:

$$\left\{ \varepsilon(T) u^\mu u^\nu - [P(T) + \Pi] \Delta^{\mu\nu} + 2W^{(u^\mu u^\nu)} + \pi^{\mu\nu} \right\}_{\mu} = 0. \quad (14)$$

This completes a demonstration of equivalence of the perfect $q$-hydrodynamics represented by Eq. (6) and its $d$-hydrodynamical counterpart represented by Eq. (14). It means therefore that the perfect $q$-fluid is nothing but a viscous fluid which satisfies the $d$-hydrodynamic equation Eq. (14).
With the bulk pressure $\Pi$ given by Eq. (12) and using the NexDC relations (13) one can express the $q$-enthalpy $w_q$ by the usual enthalpy, $w \equiv T s = \varepsilon + P$, and the $q$-dependent variable $\gamma$:

$$w_q = \varepsilon_q(T_q) + P_q(T_q) = \frac{\varepsilon(T) + P(T)}{[1 + \gamma]^2} = \frac{w}{[1 + \gamma]^2}. \tag{15}$$

Notice that Eq. (13) leads to the following important relations between the heat flow vector $W^\mu$, the pressure tensor $\pi_{\mu\nu}$ and the bulk pressure $\Pi$:

$$W^\mu W_\mu = -3\Pi w, \quad \pi^\mu_{\nu} W_\nu = -2\Pi W^\mu, \quad \pi_{\mu\nu} \pi^{\mu\nu} = 6\Pi^2. \tag{16}$$

As mentioned before, we expect that in all cases of interest to us $|q-1| \ll 1$. This means then that we can regard the state characterized by $f_q(x, p)$ from Eq. (3) as some stationary state existing near equilibrium. This near equilibrium state is then defined by the correlation function $h_q$ in Eq. (1), for which the energy momentum tensor can be divided into two parts: the energy-momentum tensor of the usual ideal fluid, $T^\mu_{\nu;q=1} = T^\mu_{\nu;eq}$, and the $q$-dependent remaining $\delta T^\mu_{\nu}$ (the meaning of all components is the same as in Eqs. (11) and (12) and $\delta T^\mu_{\nu} \to 0$ when $q \to 1$):

$$T^\mu_{\nu;q} \equiv (\varepsilon_q + P_q)u_q^\mu u_q^\nu - P_q g^\mu_{\nu} \text{ def} = T^\mu_{\nu;eq} + \delta T^\mu_{\nu} \quad \text{where} \quad \delta T^\mu_{\nu} = -\Pi \Delta^\mu_{\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu}. \tag{17}$$

Using now Eq. (13) we get from Eq. (15) that

$$\gamma = \sqrt{1 + \delta \epsilon_q} - 1 \quad \text{where} \quad \delta \epsilon_q \equiv \frac{\varepsilon(T) - \varepsilon_q(T_q)}{\varepsilon_q(T_q) + P_q(T_q)}. \tag{18}$$

This relation allows us to connect the velocity field $u_q$ (which is solution of the $q$-hydrodynamics given by Eq. (6)) with the velocity field $u$ (which is solution of the dissipative hydrodynamics given by Eq. (14)). To make it more transparent, let us parameterize these velocity fields by using the respective fluid rapidities $\alpha_q$ and $\alpha$ (the metric used is $g^{\mu\nu} = (1, -1/\tau^2)$):

$$u_q^\mu(x) = \left[ \cosh (\alpha_q - \eta), \frac{1}{\tau} \sinh (\alpha_q - \eta) \right], \quad u^\mu(x) = \left[ \cosh (\alpha - \eta), \frac{1}{\tau} \sinh (\alpha - \eta) \right]. \tag{19}$$

Because $\gamma = u_\mu \delta u_q^\mu = \cosh(\alpha_q - \alpha) - 1$ the connection between $u$ and $u_q$ is given by

$$\cosh(\alpha_q - \alpha) = \sqrt{1 + \delta \epsilon_q} \quad \Rightarrow \quad \alpha = \alpha_q - \log \left( \epsilon_q + \sqrt{1 + \delta \epsilon_q} \right). \tag{20}$$

(we abandon the other solution, $\alpha = \alpha_q + \log \left( \epsilon_q + \sqrt{1 + \delta \epsilon_q} \right)$, because it leads to the reduction of entropy, not to its production, i.e., for it $[su^\mu]_\mu < 0$, for $q > 1$).
4. Entropy production in $q$-hydrodynamics

One of the important implications of the NexDC conjecture is that in the ideal $q$-hydrodynamics one observes entropy production. Taking the covariant derivative of Eq. (17) and multiplying it by $u_\nu$, one gets

$$u_\nu T_{\mu\nu} = \mu T_{[su\nu],\mu} + u_\nu \delta T_{\mu\nu} = 0 \implies [su\nu],\mu = -\frac{u_\nu}{T} \delta T_{\mu\nu}.$$ (21)

This means that, although in the ideal $q$-hydrodynamics the $q$-entropy is conserved, i.e., $[s_q u_q^\nu],\mu = 0$, the usual entropy is produced; the ideal $q$-fluid is therefore a kind of usual viscous fluid. This is illustrated in Fig. 1 where the expected entropy production, as given by Eq. (21), is shown. All curves presented in Fig. 1 were calculated using the $q$-hydrodynamical evolution described in [37] for $q = 1.08$ with the $q$-dependent initial conditions ($q$-IC). They were given assuming a $q$-gaussian shape (in rapidity) of the initial energy density out of which the hydrodynamic evolution started. Two types of ($q$-IC) were considered: in the first the maximal initial energy density was fixed as $\varepsilon^{(in)} = 22.3$ GeV/fm$^3$ (forcing the width of the $q$-gaussian to be equal to $\sigma = 1.28$), in the second the width of the $q$-gaussian was fixed as $\sigma = 1.25$ (forcing the initial energy density to be equal to $\varepsilon^{(in)} = 27.8$ GeV/fm$^3$).

Because both ($q$-IC) give reasonable fits to experimental data, they introduce only very small differences in the entropy production, cf., the left part of Fig. 1. These ($q$-IC) were accompanied by the $q$-dependent equation of state ($q$-EoS) for the relativistic pion gas. Finally, the usual Cooper-Frye prescription of the freeze-out (but with the corresponding distribution being the $q$-exponent rather than the usual one [37]) was used. All calculations were performed for $Au + Au$ collisions at $\sqrt{s_{NN}} = 200$ GeV (see [37] where it was shown that $q$-hydrodynamics with such $q$ and with the respective $q$-IC and $q$-EoS reproduces $dN/dy$ and $p_T$ distributions observed in RHIC experiment for $Au+Au$ collisions at $\sqrt{s_{NN}} = 200$ GeV energy). Notice that $su^\nu_{\mu} > 0$ for large $\eta$ region at any $\tau$ (but especially for the early stage of the hydrodynamical evolution). It supports therefore a dissipative character.
of the $q$-hydrodynamics mentioned before and leads us to the conclusion that the equilibrium state generated in heavy-ion collisions may, in fact, be the $q$-equilibrium state, i.e., some stationary state near the usual equilibrium state already containing some dissipative phenomena. Notice that the total multiplicity, which is usually treated as a measure of entropy, increases with $q$ as expected, namely $N = 11 \cdot (7 + 10q)$.

5. Transport coefficients of $q$-hydrodynamics.

The other implication of the NexDC conjecture is that there are some transport phenomena in an ideal $q$-fluid. We shall now identify the corresponding transport coefficients and compare them with the similar coefficients in the usual viscous $d$-hydrodynamics. Out of a number of different formulations of $d$-hydrodynamics [27–36] we shall choose the 2nd order theory of dissipative fluids presented in [31, 32]. It does not violate causality (at least not the global one over a distant scale given by the relaxation time) and it contains some dissipative fluxes like heat conductivity and bulk and shear viscosities. We shall now see, to what extent dissipative fluxes resulting from our $q$-hydrodynamics can be identified with the heat conductivity and with the bulk and shear viscosities introduced in [31, 32], and what are the resulting transport coefficients.

Let us start with the most general form of the off-equilibrium four-entropy current $\sigma^\mu$ which in our baryon-free scenario takes the following form [31, 32]:

$$\sigma^\mu = P(T)\beta^\mu + \beta_\nu T^\mu\nu + Q^\mu(\delta T^\mu\nu) \quad \text{with} \quad \beta^\mu \equiv \frac{u^\mu}{T}. \quad (22)$$

where $\delta T^\mu\nu$ is defined in Eq. (17). The function $Q^\mu = Q^\mu(\delta T^\mu\nu)$ characterizes the off-equilibrium state, which in our case is induced by the nonextensive effects and therefore depends on $q$, $Q \to 0$ when $q \to 1$. Using the NexDC conjecture, Eqs. (13), and Eq. (15) our $q$-entropy current is

$$\sigma_q^\mu(\equiv s_q^\mu) = su^\mu + \frac{W^\mu}{T} + Q^\mu_q \quad \text{where} \quad Q^\mu_q = Q^\mu_q(\equiv \chi \left\{ su^\mu + \frac{W^\mu}{T} \right\}). \quad (23)$$

Our near equilibrium state is thus characterized by the function $Q^\mu_q$ in which

$$\chi \equiv \frac{T}{T_q} \sqrt{1 - \frac{3\Pi}{u}} - \frac{1}{1 + \gamma} \frac{T}{T_q} - 1 \quad (24)$$

(temperatures $T$ and $T_q$ are not independent but connected, as was stated before, by the NexDC conjecture (13)). We stress that it is given by the dissipative part of our $q$-system, the one that leads to the increase of the usual entropy, cf. Fig. 1 (the $q$-entropy, as was discussed before, remains, however, strictly conserved, $\sigma_q^\mu_{q,\mu} = 0$).

The most general algebraic form of $Q^\mu$ for $d$-hydrodynamic that includes dissipative fluxes up to second order is [32]:

$$Q^\mu_{2nd} = \left[ -\beta_0 \Pi^2 + \beta_1 W_\nu W^\nu \right. - \left. \beta_2 \pi_{\nu\lambda} \pi^{\nu\lambda} \right] u^\mu - \frac{\alpha_0 \Pi W^\mu}{T} + \frac{\alpha_1 \pi^\mu W^\nu}{T}. \quad (25)$$
where $\beta_{i=0,1,2}$ are the corresponding thermodynamic coefficients for the, respectively, scalar, vector and tensor dissipative contributions to the entropy current, whereas $\alpha_{i=0,1}$ are the corresponding viscous/heat coupling coefficients. The corresponding expression calculated using the NexDC conjecture is:

$$Q^\mu_q = Q^\mu_{2nd} = \Gamma_{2nd} su^\mu + \Upsilon_{1st} \frac{W^\mu}{T}$$

where $\Gamma_{2nd} \equiv -\frac{3\beta_1}{2}\Pi - \frac{(\beta_0 + 6\beta_2)}{2w}\Pi^2$ and $\Upsilon_{1st} \equiv -(\alpha_0 + 2\alpha_1)\Pi$. (26)

This is given by the second order polynomial in the bulk pressure $\Pi$. Therefore, it is natural to expect (and this will be our assumption) that the most general entropy current in the NexDC approach is:

$$Q^\mu_{\text{full}} = \Gamma(\Pi) su^\mu + \Upsilon(\Pi) W^\mu$$

where $\Gamma \equiv \frac{1}{2}(\Gamma + \Upsilon)$ and $\Upsilon \equiv \frac{1}{2}(\Gamma - \Upsilon)$. (27)

and where $\Gamma, \Upsilon$ are (in general infinite) series in powers of the bulk pressure $\Pi$. In this sense $Q^\mu_{\text{full}}$ can be regarded as the full order dissipative current in $q$-hydrodynamics. In general one has entropy production/reduction, i.e., $\sigma^\mu_{\text{full}} \neq 0$, however, in the case when $\Gamma(\Pi) = \Upsilon(\Pi) = \chi$ one has $\sigma^\mu_{\text{full}} = 0$. Out of the two possible solutions for $(\Gamma, \Upsilon)$ only one is acceptable,

$$\Gamma = 2\frac{T}{T_q} \left(\sqrt{1 - 3\Pi/w} - 1\right) = -\frac{2\gamma}{1 + \gamma} \frac{T}{T_q}, \quad \Upsilon = 2\frac{T - T_q}{T_q},$$

(30)

because only for it $u_\mu Q^\mu_{\text{full}} \leq 0$ (i.e., the entropy is maximal in the equilibrium [32], this is because $(T - T_q)/T_q$ is always positive for $q \geq 1$ [37]). In this way we finally arrive at the following expression for the full order dissipative entropy current emerging from the NexDC approach:

$$\sigma^\mu_{\text{full}} \equiv su^\mu + \frac{W^\mu}{T} - \frac{2T}{T_q} \left[1 - \sqrt{1 - 3\Pi/w}\right] su^\mu + \frac{2(T - T_q)}{T_q} \frac{W^\mu}{T}.$$ (31)

Limiting ourselves to situations when $T/T_q \approx 1$ and neglecting terms higher than $O(3\Pi/w)^2$, one obtains that

$$Q^\mu_{\text{full}} \approx \left[-\frac{1}{2} \left(\frac{3\Pi}{w}\right)^2 \right] su^\mu.$$ (32)

Comparing now Eqs. (27) and (32) one gets that $\beta_1 = \frac{2}{w}$, $\beta_0 + 6\beta_2 = \frac{9}{w}$ and $\alpha_0 + 2\alpha_1 = 0$. Since in the Israel-Stewart theory [31] the relaxation time $\tau$ is proportional to the thermodynamic coefficients $\beta_{0,1,2}$, it is naturally to assume that in the case of NexDC it is proportional to the inverse of the enthalpy, $\tau \propto 1/w$ (notice that for the classical Boltzmann gas of massless particles $\beta_2 = 3/w$ [32, 36]).
Let us now see what kind of bulk and shear viscosities emerge from the NexDC approach. To this end let us write the full order entropy current Eq. (28) in the following form:

$$
\sigma_{\text{full};\mu} = [(1 + \chi)\Phi^\mu]_{;\mu} + [\xi\Psi^\mu]_{;\mu} \quad \text{where} \quad \Phi^\mu = su^\mu + \frac{W^\mu}{T} \quad \text{and} \quad \Psi^\mu = su^\mu - \frac{W^\mu}{T}.
$$

Because conservation of the $q$-entropy, $\sigma_{q,\mu}^\mu = 0$, is equivalent to $[(1 + \chi)\Phi^\mu]_{;\mu} = 0$, therefore, using Eq. (16) one gets that $\Psi^\mu = -\frac{W^\nu W^\mu}{\Pi T} u^\nu + \frac{W^\mu}{2\Pi T} \pi^{\mu\nu}$, therefore

$$
\sigma_{\text{full};\mu} = \frac{\Pi}{T}(wu^\mu X_\mu) - \frac{W^\mu}{T} \widetilde{Y}_\mu + \frac{\pi^{\mu\nu}}{T} Z_{\mu\nu},
$$

where:

$$
X_\mu = -\frac{\xi}{\Pi} \left[ \frac{\partial_\mu \Pi}{\Pi} + \frac{\partial_\mu T}{T} + \frac{\partial_\mu \xi}{\xi} \right], \quad \widetilde{Y}_\mu = \frac{\xi}{\Pi} \left[ \frac{2}{3} u^\nu W_{\mu;\nu} + \frac{1}{3} W_{\mu} u^\nu - \frac{1}{2} \pi^{\nu\mu} \right],
$$

$$
Z_{\mu\nu} = \frac{\xi}{\Pi} \left[ \frac{1}{2} W_{\nu;\mu} \right], \quad \widetilde{Z}_{\mu\nu} = Z_{\mu\nu} + \frac{\widetilde{Y}_\mu W_\nu}{2\Pi}.
$$
Eq. (16) allows to eliminate the term proportional to the heat flow, $\frac{W^\mu}{T}$. Finally one obtains

$$\sigma^{\mu}_{\text{full},\mu} = \frac{\Pi}{T}(wu^\mu X_\mu) + \frac{\pi^{\mu\nu}}{T} \tilde{Z}_{\mu\nu} \geq 0,$$  \hspace{1cm} (37)

where we have introduced the usual bulk and shear viscosities, $\zeta$ and $\eta$. Notice that, because of Eq. (16) one avoids the explicit contribution to the entropy production coming from the heat flow, $\frac{W^\mu}{T}$, which is present in Eq. (34) when one discusses a baryon free fluid, in which case the necessity to use the Landau frame would appear.

As one can see, Eq. (33) is covariant and therefore it does not depend on the frame used. One arrives at our main result: the sum rule connecting bulk and shear viscosity coefficients (expressed as their ratios over the entropy density $s$), see Fig. 2:

$$\frac{1}{\zeta/s} + 3\frac{\eta/s}{\Pi^2} = \frac{w\sigma^{\mu}_{\text{full},\mu}}{\Pi T}.$$

This is as much as one can get from the $q$-hydrodynamics alone. To disentangle this sum rule, one has to add some additional input. Suppose, therefore, that we are interested in an extremal case, when a total entropy is generated by action of the shear viscosity only. In this case one can rewrite the first part of Eq. (37) as

$$\sigma^{\mu}_{\text{full},\mu} = -\frac{\pi^{\mu\nu}}{6\Pi} \left( wu^\lambda X_\lambda \right) - \frac{\pi^{\mu\nu}}{T} \tilde{Z}_{\mu\nu}.$$

and arrive at

$$\eta/s = \frac{\gamma(\gamma + 2)}{(\gamma + 1)^2} \left( \frac{\pi^{\mu\nu}}{\Pi T} \tilde{Z}_{\mu\nu} - su^\lambda X_\lambda \right)^{-1}.$$

The predictions of Eq. (40) are shown in Fig. 3 for different values of the parameter $q$ and for different rapidities. They are also confronted with the known result on $\eta/s$, provided by the popular Ads/CFT conjecture [41], that $\eta/s \geq 1/4\pi$. Assuming the validity of this limitation we can use Eq. (38) only in the region where the r.h.s. of Eq. (40) is smaller than (or equal to) $1/4\pi$ (in which case we put $\eta/s = 1/4\pi$), otherwise (because of our assumption that the total entropy is generated by the shear viscosity only) we have to put $\zeta/s = 0$ and use Eq. (40) to evaluate $\eta/s$. The results for $\zeta/s$ and $\eta/s$ are shown in Fig. 3. Notice that, when the r.h.s of Eq. (40) approaches $1/4\pi$, $\zeta/s$ given by Eq. (38) approaches infinity. To avoid such a situation, $\eta/s$ should start to increase at higher temperatures, for example at $T \geq 75$ MeV

6. Summary

To summarize, we have discussed dissipative hydrodynamics from a novel point of view. This is provided by the nonextensive formulation of the usual perfect hydrodynamical model recently proposed by us [37]. Such a model can be solved exactly and contains terms which can be interpreted as due to some dissipative effects. They can be identified and are expressed by the nonextensivity parameter $q$ of the Tsallis formalism applied
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Figure 3. Upper panels: the ratio of the shear viscosity over the entropy density, $\eta/s$, as function of temperature $T$, calculated using Eq. (40) for different values of $q$ at the mid-rapidity region $y = 0$ (two upper-left panels) and for rapidity $y = 3$ (two upper-right panels). The same version of hydrodynamical model was used as in Fig. 1 with the first type of initial conditions. Notice that for $q \to 1$ this ratio vanishes, as expected, proving therefore correctness of our numerical calculations. The lower two panels summarize the rapidity dependence of both the shear and the bulk viscosity ratios, $\eta/s$ (left panel) and $\zeta/s$ (right panel). The $\eta/s$ is calculated using Eq. (40), i.e., assuming $\zeta/s = 0$, and presented only above the limit value $\eta/s = 1/4\pi$ found in the AdS/CFT approach [41]. The right panel shows the bulk viscosity, $\zeta/s$, calculated in this limit, i.e., assuming $\eta/s = 1/4\pi$ and using Eq. (38).

Here. This finding was used to propose a possible full order expression for the dissipative entropy current $\sigma^\mu_{\text{full}}$ resulting from the nonextensive approach. The corresponding bulk and shear transport coefficients resulting from $q$-hydrodynamics are connected by a kind of sum rule, Eq. (38). They were calculated for some specific simplified case, cf. Fig. 3. We close by noticing that there is still some uncertainty in the equation of state used in our numerical example (for example, in what concerns the role of the possible QGP phase transition, which, when included in EoS, could considerably affect $\eta/s$ presented in Fig. 3). We plan to address this subject elsewhere.

From our point of view it is interesting to observe that (at least some) remedies proposed to improve the formulation of $d$-hydrodynamic [26, 29] (like the use of some induced memory effects) introduce conditions which in statistical physics lead in a natural way to its nonextensive version described by the parameter $q$ [9]. It is then natural to expect that a nonextensive version of the hydrodynamical model with the nonextensivity parameter

...
Could help us to circumvent (at least to some extent) the problems mentioned. It is because equations of the perfect nonextensive hydrodynamics (or perfect $q$-hydrodynamics) can be formally solved in an analogous way as equations of the usual perfect hydrodynamics [37]. However, it turns out that, from the point of view of the usual (extensive) approach the new equations contain terms which can be formally identified with terms appearing in the usual dissipative hydrodynamics ($d$-hydrodynamics). Although this does not fully solve the problems of $d$-hydrodynamics, nevertheless it allows us to extend the usual perfect fluid approach (using only one new parameter $q$) well behind its usual limits, namely toward the regions reserved so far only for the dissipative approach.

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References

[1] L.D.Landau, Izv. Akad. Nauk SSSR 17, 51 (1953). S.Z.Belenkij and L.D.Landau, Usp. Fiz. Nauk 56, 309 (1955); Nuovo Cim. Suppl. 3, 15 (1956).
[2] C-Y. Wong, Landau Hydrodynamics Revisited, arXiv:0808.1294[hep-ph] and Lectures on Landau Hydrodynamics, arXiv:0809.0517.
[3] L.M.Satarow, I.N.Mishustin, A.V.Merdeev and H.Stöcker, Phys. Rev. C 75, 024903 (2007).
[4] R.C.Hwa, Phys.Rev. D 10, 2260 (1974); J.D.Bjorken, Phys. Rev. D 27, 140 (1983).
[5] T. Csörgő, M.J.Nagy and M.Csanád, Phys. Lett. B 663, 306 (2008); M.J.Nagy, T. Csörgő and M.Csanád, Phys.Rev. C 77, 024908 (2008) and references therein.
[6] For the most recent reviews see: P. Huovinen, P.V. Ruuskanen, Ann. Rev. Nucl. Part. Phys. 56, 163 (2006); T. Hirano, J.Phys. G 30, S845 (2004); T. Hirano, Y. Nara, J.Phys. G 31, S1 (2005); Y. Hama, T. Kodama and O. Socolowski Jr., Braz.J.Phys. 35, 24 (2005) and references therein.
[7] See for example: B. Müller, Nucl. Phys. A 774, 433 (2006) and references therein; cf. also M. Gyulassy, L. McLerran, Nucl. Phys. A 750, 30 (2005); I. Vitev, Int.J. Mod. Phys. A 20, 3777 (2005); R.D. Pisarski, Braz. J. Phys. 36, 122 (2006) and references therein.
[8] B.M. Boghosian, Braz.J.Phys. 29, 91 (1999); F.Q. Potiguar, U.M.S. Costa, Physica A 303, 457 (2002); I.Santamaria-Holek and R.F.Rodriguez, Physica A 366, 141 (2006).
[9] C.Tsallis, J. Stat. Phys. 52, 479 (1988), Braz. J. Phys. 29, 1 (1999), Physica A 340, 1 (2004) and Physica A 344, 718 (2004) and references therein. See also Nonextensive Statistical Mechanics and its Applications, S. Abe and Y. Okamoto (Eds.), Lecture Notes in Physics LPN560, Springer (2000); Nonexten-
Nonextensive perfect hydrodynamics - a model of dissipative relativistic hydrodynamics?

sive Entropy - Interdisciplinary Applications, Eds. M. Gell-Mann and C. Tsallis, Oxford University Press, New York 2004; Complexity, Metastability and Nonextensivity Eds. S. Abe, H. Herrmann, P. Quarati, A. Rapisarda and C. Tsallis, AIP Conference Proceedings, 965 (2007); for full updated bibliography see http://tsallis.cat.cbpf.br/TEMUCO.pdf.

[10] G. Wilk and Z. Włodarczyk, Phys. Rev. Lett. 84, 2770 (2000); T.S. Biró and A. Jakovác, Phys. Rev. Lett. 94, 132302 (2005).

[11] G. Wilk and Z. Włodarczyk, Chaos, Solitons and Fractals 13/3, 581 (2001).

[12] O.V. Utyuzh, G. Wilk and Z. Włodarczyk, J. Phys. G 26, L39 (2000); G. Wilk and Z. Włodarczyk, Nucl. Phys. B (Proc. Suppl.) 75A, 191 (1999); Physica A 305, 227 (2002); M. Rybczyński, Z. Włodarczyk and G. Wilk, Nucl. Phys. (Proc. Suppl.) B 97, 81 (2001) and 122, 325 (2003); T. Osada, O.V. Utyuzh, G. Wilk and Z. Włodarczyk, Europ. Phys. J. B 50, 7 (2006).

[13] I. Bediaga, E.M. Curado and J.M.de Miranda, Physica A 286, 156 (2000); T. Wibig and I. Kurp, Int. J. High Energy Phys. 0312, 039 (2003); W.M. Alberico, A. Lavagno and P. Quarati, Eur. Phys. J. C 12, 499 (2000); W. M. Alberico, P. Czerski, A. Lavagno, M. Nardi, and V. Somá, Physica A 387, 467 (2008).

[14] F.S. Navarra, O.V. Utyuzh, G. Wilk and Z. Włodarczyk, Phys. Rev. D 67, 114002 (2003); Physica A 344, 568 (2004); Nukleonika 49 (Supplement 2), S19 (2004); Physica A 340, 467 (2004).

[15] M. Biyajima, M. Kaneyama, T. Mizoguchi and G. Wilk, Eur. Phys. J. C 40, 243 (2005); M. Biyajima, T. Mizoguchi, N. Nakajima, N. Suzuki, and G. Wilk, Eur. Phys. J. C 48, 593 (2006).

[16] G. Wilk and Z. Włodarczyk, Physica A 376, 279 (2007).

[17] F.S. Navarra, O.V. Utyuzh, G. Wilk, and Z. Włodarczyk, Nuovo Cimento Soc. Ital. Fis., C 24, 725 (2001).

[18] C. Beck, E.G.D. Cohen, Physica A 322, 267 (2003); F. Sattin, Eur. Phys. J. B 49, 219 (2006).

[19] T. Kodama, H.-T. Elze, C.E. Augiar and T. Koide, Europhys. Lett. 70, 439 (2005); T. Kodama, J. Phys. G 31, S1051 (2005).

[20] T. Sherman and J. Rafelski, Lecture Notes in Physics 633, 377 (2004).

[21] T.S. Biró and G. Purcsel, Phys. Rev. Lett. 95, 162302 (2005); Phys. Lett. A 372, 1174 (2008) and Non-extensive equilibration in relativistic matter, arXiv: 0809.4768 [hep-ph]. See also: T.S. Biró, Abstract composition rule for relativistic kinetic energy in the thermodynamical limit, arXiv: 0809.4675 [nucl-th].

[22] T.S. Biró and G. Kaniadakis, Eur. Phys. J. B 50, 3 (2006) and references therein.

[23] C.Beck, Physica A 286, 164 (2000).

[24] T. Hirano and K. Tsuda, Phys. Rev. C 66, 054905 (2002); D. Teaney, Phys. Rev. C 68, 034913 (2003).

[25] T. Hirano and M. Gyulassy, Nucl. Phys. A 769, 71 (2006).

[26] P.Ván and T.S.Biro, Eur.Phys.J Special Topics 155, 201 (2008); T.S. Biro, E. Molnar and P. Ván, Phys. Rev. C 78, 014909 (2008) and references therein.

[27] See K. Tsumura and T. Kunihiro, arXiv:0709.3645 [nucl-th] and references therein.

[28] W. A. Hiscock and L. Lindblom, Ann. Phys. (N.Y.) 151, 466 (1983); Phys. Rev. D 31, 725 (1985); D 35,
[29] T. Koide, G.S. Denicol, Ph. Mota and T. Kodama, Phys. Rev. C 75, 034909 (2007); cf. also G. Denicol, T. Kodama, T. Koide and Ph. Mota, J. Phys. G 35, 115102 (2008) and references therein.

[30] C. Eckart, Phys. Rev. 58, 919 (1940).

[31] W. Israel, Ann. Phys. (N.Y.) 100, 310 (1976); J. M. Stewart, Proc. R. Soc. London, Ser. A 357, 59 (1977); W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

[32] A. Muronga, Phys. Rev. Lett. 88, 062302 (2002) [Erratum - ibid. 89, 15990 (2002)]; Phys. Rev. C 69, 034903 (2004); A. Muronga and D.H. Rischke, arXiv:nucl-th/0407114. See also A. Muronga, Eur. Phys. J Special Topics 155, 107 (2008) and references therein.

[33] U. Heinz, H. Song and A.K. Chaudhuri, Phys. Rev. C 73, 034904 (2006); H. Song and U. Heinz, Phys. Rev. C 77, 064901 (2008).

[34] R. Baier, P. Romatschke and U. A. Wiedmann, Phys. Rev. C 73, 064903 (2006); R. Baier and P. Romatschke, Eur. Phys. J. C 51, 677 (2007); P. Romatschke, Eur. Phys. J. C 52, 203 (2007); P. Romatschke and U. Romatschke, Phys. Rev. Lett. 99, 17230 (2007).

[35] A. K. Chaudhuri, Phys. Rev. C 74, 044904 (2006).

[36] A. Dumitru, E. Molnár and Y. Nara, Phys. Rev. C 76, 024910 (2007).

[37] T. Osada and G. Wilk, Phys. Rev. C 77, 044903 and Prog. Theor. Physics Suppl. 174, 168 (2008); cf. also: Dissipative or just Nonextensive hydrodynamics? - Nonextensive/Dissipative correspondence -, arXiv:0805.2253 [nucl-phys] (to be published in Indian Journal of Physics).

[38] A. Lavagno, Phys. Lett. A301, 13 (2002).

[39] J. A. S. Lima, R. Silva and A. R. Plastino, Phys. Rev. Lett. 86, 2938 (2001).

[40] S. Abe, Physica A 368, 430 (2006).

[41] P. Kovtun, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 94, 111601 (2005). For a recent review of this subject see D. T. Son ans A. O. Starinets, Annu. Rev. Nucl. Part. Sci. 57 (2007) 95 and references therein. For specific applications to heavy ion collisions see: Hong Liu, J. Phys. G 34, S361 (2007) or M.P. Heller, R.A. Janik and R. Peschanski, Hydrodynamic Flow of the Quark-Gluon Plasma and Gauge/Gravity Correspondence, talk at the 48th Cracow School of Theoretical Physics: Aspects of Duality, June 13-22, 2008, Zakopane, Poland, arXiv:0811.3113 [hep-th] (to be published in the proceedings) and references therein.

Notes

[42] One should keep in mind, however, that there are dualities in the non-extensive approach, i.e., that both $q$ and $1/q$ can be used as the nonextensivity parameter depending on the normalization of original or $q$-powered probabilities. Also, when considering the particle-hole symmetry in the $q$-Fermi distribution, $f(-E,T,\mu,q) = 1 - f(E,T,-\mu,2-q)$, in a plasma containing both particles and antiparticles both $q$ and
2 − q occur (µ denotes the chemical potential here). These dual possibilities warn us that a theory requiring that only q > 1 has physical meaning is still incomplete [9]. These points deserve further considerations which are, however, going outside the scope of this paper. On the other hand, in the present paper we are not considering plasmas containing both particles and antiparticles but only massive pions which are assumed to obey the q-Boltzmann distribution.

[43] Actually, this form of q-entropy current differs slightly from that in [38]. The reason is that only with such form we can, at the same time, both satisfy the H-theorem and reproduce thermodynamical relations resulting in our q-enthalpy, cf. Eq. (15), ((cf. also Eqs. (17a) and (17b) in [37]), which are crucial in all further derivations and which were not addressed in [38]).

[44] One must keep in mind therefore that, although to define the entropy current in q-statistics we require only that q > 0, the NexDC correspondence requires in addition that q > 1 to be consistent with d-hydrodynamics [32]. Only then, as witnessed by Fig. 1, constant (nonzero) initial Tsallis entropy results in increasing BG entropy (demonstrating itself in the total multiplicity increasing with (q − 1)). For q < 1 we could have decreasing BG entropy from some positive initial value (and, accordingly, total multiplicity decreasing with (q − 1)). This point is, however, not totally clear at present and we plan to address it elsewhere.

[45] It should be kept in mind that we have so far obtained only the relations between the Israel-Steward coefficients (β0,1,2 and α0,1) and not their individual values, therefore we cannot compare our results to those of the ideal Boltzmann gas case. This is due to the tensor form of relations (16) and to the fact that the original perfect q-hydrodynamics (Eq. (4)) does not contain any natural space-time scale. It is then natural that NexDC conjecture does not introduce per se any definite relaxation time or viscous-heat coupling length scale. Also, in examples shown here, only EoS for pionic gas was used as in [37]. Although it was shown there that such EoS depends only very weakly on the parameter q, it remains to be checked whether this is also true for a more realistic EoS with quarks and gluons and in the vicinity of the QGP → hadronic matter phase transition. We plan to address this point elsewhere.