An optimization problem for maximum vibration suppression in reconfigurable one dimensional metamaterials

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Abstract

Acoustic metamaterials have already been shown to be effective for vibration reduction and control. Local resonances in the metamaterial cause waves at frequencies within band gaps to become evanescent, thus preventing wave propagation through the material. Active and adaptable local resonances enables the band gaps to be shifted in frequency and increased in bandwidth. Since metamaterial local resonances are usually composite, methods to specify optimal component configurations are helpful for passive metamaterials and almost necessary for adaptable metamaterials, where the metamaterial must be reconfigured for optimal performance at various frequency ranges. To assess band gap locations and bandwidths for metamaterials, a wavenumber spectrum is commonly computed. Commonly, a parameter study of adaptable unit cell variables will be performed to assess optimal configurations of adaptable metamaterials. In this paper, the complex wavenumber is proposed as a direct optimization objective for reconfiguration of active adaptable acoustic metamaterials for maximum vibration suppression at a frequency range of choice. By directly maximizing the imaginary part of the wavenumber, associated with wave attenuation, the unit cell configuration maximum vibration suppression can be obtained for an operating frequency of choice. Additionally, since the optimization problem requires constraints for feasible solutions and the example active piezoelectric metamaterial system shown here is electrically unstable at some configurations, we also explore an experimental method for bounding the optimization problem. Numerical results of the optimization problem are presented.

1. Introduction

Previous literature has provided methods of modeling elastic wave propagation in spatially periodic elastodynamic systems. Of particular relevance are those of Mead [1–4], and Faulkner and Hong [5], and many others [6–11], where the \( k(\omega) \) method is used to calculate the complex band structure for the elastic system. Spatial periodicity (spatially repeating unit cells) and local resonances lead to band gaps wherein the system attenuates waves, and the complex band structure aids in determining the band gaps for the structure. Recently much attention has been given to acoustic metamaterials and phononic crystals as a means to acoustic cloaking, lensing, filtering, and vibration control, among other applications.

Several researchers have proposed using the complex wavenumber as a tool to design and optimize metamaterials [12–19]. In this work, we propose using the imaginary part of the wavenumber calculated from a unit cell substructure finite element model as a direct optimization objective through which transverse vibration in a one dimensional structure can be minimized using a built-in MATLAB optimizer. In contrast to parameter studies for specific reconfigurable metamaterial unit cells, we propose this method for direct optimization of any unit cell model with reconfigurable parameters.

The modeling methods utilized here have been discussed in the literature previously, specifically the wave finite element method for periodic elastodynamic systems. Manconi and others used a wave finite element scheme to analyze wave propagation structures which were uniform in the direction of propagation...
Numerical error in the wave finite element method was investigated by Waki et al. [24, 25]. Using substructuring and model reduction schemes, spatially periodic structures (that is, nonuniform in the direction of propagation) were also examined by several authors [26–32]. Computationally efficient calculation of a one-dimensional array of unit cells was formulated by Duhamel et al. [33, 34]. Waves in metamaterial structures with damping were investigated by several authors [35–42].

In this paper, we review the wave finite element method and relevant substructure model reduction techniques used to analyze a simple periodic metamaterial structure. The structure consists of a one-dimensional array of bimaterial unit cells containing piezoelectric actuators which are tunable by way of negative capacitance shunt circuits. The structure is excited by transverse elastic waves. The frequency response and complex wavenumber spectrum are calculated. Next, the variation in the attenuative part of the complex wavenumber (here, the imaginary part) as a function of unit cell electrical tuning parameters. Electrical stability of the shunted piezoelectric system is considered, and constraints on the optimization problem are imposed based on electrical limitations of the electrical shunt circuit. Having investigated both the behavior of the complex wavenumber spectrum and the shunted piezoelectric actuator, an optimization problem is proposed to specify the unit cell electrical parameters for maximum vibration suppression in the metamaterial system.

2. Finite element modeling of the unit cell

In this section, the finite element modeling and wave finite element modeling techniques are reviewed. First a finite element model of the unit cell is formulated. The dynamic stiffness matrix relating the generalized forces and generalized displacements is obtained. Next, the dynamic stiffness matrix is partitioned and condensed according to the boundary and internal degrees of freedom in order to apply the Floquet–Bloch boundary conditions to the boundary degrees of freedom. Additionally, the dynamic stiffness matrix is manipulated to relate the generalized forces and displacements on the left side of the unit cell to those on the right, i.e. a finite element transfer matrix is formulated. Finally, an eigenvalue problem for the complex wavenumber as a function of frequency, \( k(\omega) \) is shown along with the formulation for the frequency response of a periodic structure.

2.1. Dynamic stiffness matrix

The approach to modeling the entire elastic metamaterial structure is to substructure the metamaterial into a repeating unit cell. Here, it is demonstrated how the unit cell model can be used to obtain relevant information about the wave propagation characteristics of the complete multi-cell structure, specifically wave attenuation. A one dimensional finite element model of the metamaterial unit cell was developed in MATLAB. The wave finite element method was used to extract the dispersion relation and compute the forced response of the complete metamaterial structure.

Euler–Bernoulli beam elements were used to create a one dimensional model of the metamaterial unit cell. Using the finite element method the stiffness and mass matrices of the unit cell \((K\) and \(M\) respectively) can be obtained. The stiffness and mass matrices comprise the dynamic stiffness matrix, \( D \), given in equation (1)

\[
D(\omega) = K - \omega^2 M.  \tag{1}
\]

The dynamic stiffness matrix is a frequency dependent matrix that gives the natural frequencies, mode shapes, and response of the unit cell substructure. The wave finite element method will be used below to compute the properties of the whole metamaterial structure. The generalized nodal forces, \( F \), are related to the generalized nodal displacements, \( q \), through the finite element equation in equation (2).

\[
F = Dq \tag{2}
\]

2.2. Partitioning and condensation of the dynamic stiffness matrix

Using the dynamic stiffness matrix obtained from the finite element modeling scheme outlined above, the wave finite element method is invoked to perform an acoustic wave propagation analysis [32–34]. It should be noted that both the finite element model and the wave finite element model are here one dimensional, but could be extended to handle problems in higher dimensions.

The unit cell of the metamaterial has both boundary and internal degrees of freedom. Since the model discussed here is one dimensional, the boundary degrees of freedom are split into right and left, with \( n \) degrees of freedom on both right and left boundaries. The left, right, and internal degrees of freedom are denoted with subscripts L, R, and I respectively. The degrees of freedom and forces at the boundary of the unit cell become representative of the whole element when the finite element model is reduced. See figure 1
for a depiction of the generalized coupling coordinates and forces between the unit cells. The left and right generalized coordinates are represented by dots. The left and right generalized forces are represented by arrows.

The dynamic stiffness matrix can be partitioned according to the left, right, and internal degrees of freedom, as shown in equation (3). Additionally, since no forces are applied to the internal degrees of freedom of the cell, it can be assumed that the forces on the internal degrees of freedom, $F_I$, are zero. Equation (2) becomes:

$$\begin{bmatrix} F_L \\ 0 \\ F_R \end{bmatrix} = \begin{bmatrix} D_{LL} & D_{LI} & D_{LR} \\ D_{IL} & D_{II} & D_{IR} \\ D_{RL} & D_{RI} & D_{RR} \end{bmatrix} \begin{bmatrix} q_L \\ q_I \\ q_R \end{bmatrix}. \quad (3)$$

The internal degrees of freedom may be eliminated from the full unit cell dynamic stiffness matrix by using the second row of equation (3), which gives:

$$q_I = -D_{II}^{-1}(D_{IL}q_L + D_{IR}q_R). \quad (4)$$

Using the above, the $2n \times 2n$ reduced dynamic stiffness matrix is obtained

$$\begin{bmatrix} F_L \\ F_R \end{bmatrix} = \begin{bmatrix} D_{LL} & D_{LR} \\ D_{RL} & D_{RR} \end{bmatrix} \begin{bmatrix} q_L \\ q_R \end{bmatrix}. \quad (5)$$

The submatrices of equation (5) are computed as shown in equation (6).

$$\begin{align*}
D_{LL} &= D_{LL} - D_{LI}D_{II}^{-1}D_{IL} \\
D_{LR} &= D_{LR} - D_{LI}D_{II}^{-1}D_{IR} \\
D_{RL} &= D_{RL} - D_{RI}D_{II}^{-1}D_{IL} \\
D_{RR} &= D_{RR} - D_{RI}D_{II}^{-1}D_{IR}.
\end{align*} \quad (6)$$

An Euler–Bernoulli beam element is used in this work, which has four degrees of freedom (displacement and slope at each end of the element). Thus, independent of the number of elements used to construct the unit cell model in the finite element method, the unit cell will have four boundary degrees of freedom, and the reduced dynamic stiffness matrix for the unit cell is in this case $4 \times 4$.

### 2.3. An eigenproblem for the complex wavenumber

The following procedure is as performed by many other researchers [32–34]. The right and left degrees of freedom and forces of the unit cell must be related if the cell is arranged in a spatially periodic manner in the complete structure. Let a complex wavenumber, $kL$, be defined such that the Floquet–Bloch relations in equation (7) hold.

$$\begin{align*}
q_R &= e^{-ikL}q_L \\
F_R &= -e^{-ikL}F_L.
\end{align*} \quad (7)$$

Using the above relationship between the left and right boundaries, equation (5) can be rearranged as:

$$T \begin{bmatrix} q_L \\ F_L \end{bmatrix} = \lambda \begin{bmatrix} q_I \\ F_I \end{bmatrix}, \quad (8)$$

where:

$$\lambda = e^{-ikL}.$$
The relationship in equation (8) is an eigenproblem for eigenvalues \( \lambda \) from which the propagation constants and wavemodes may (theoretically) be obtained. The \( 2n \times 2n \) transfer matrix, \( T \), is defined in equation (10).

\[
T = \begin{bmatrix}
\mathcal{D}_{LL} & \mathcal{D}_{LR} & \mathcal{D}_{RL} & \mathcal{D}_{RR}
\end{bmatrix}
\]

2.4. Conditioning of the eigenproblem for the complex wavenumber

In fact, the eigenproblem of equation (8), while mathematically correct, is often ill-conditioned and prone to numerical error such that unit cell models with many elements or high frequency ranges are often very difficult to solve accurately. It is worthwhile to recast the problem for better numerical conditioning. This has been addressed in the literature, and several reformulations have been proposed. Those by Zhong [43], Waki et al [24], and Silva et al [31] all yield eigenproblems amenable to built-in eigenvalue solvers. Here, we use a method developed by Fan et al [32] which involves a similar linear problem proposed in the above sources along with a scaling factor to improve the condition of the linear eigenproblem. The scaling factor is:

\[
\sigma = \frac{\|\mathcal{D}_{LL}\|}{n^2},
\]

where the quantity \( \|\mathcal{D}_{LL}\| \) is the two-norm of the \( \mathcal{D}_{LL} \) submatrix. The scaling factor is used in a recast version of equation (8) to obtain the generalized eigenvalue problem given in equation (12).

\[
\begin{bmatrix}
0 & \sigma I \\
\mathcal{D}_{RL} & \mathcal{D}_{RR}
\end{bmatrix}
- \lambda
\begin{bmatrix}
\sigma I \\
\mathcal{D}_{LL}
\end{bmatrix}
\begin{bmatrix}
q_l \\
q_r
\end{bmatrix}
= 0
\]

With the eigenproblem of equation (12), the condition numbers of the matrices involved are smaller than that of \( T \). The eigenvalues of equation (12) are the eigenvalues of equation (8). The complex wavenumber is found from the eigenvalues as shown in equation (9).

2.5. Forced response of the whole periodic structure

It is possible to create a basis of wave mode shapes from the unit cell wave finite element model which can be used to construct a dynamic stiffness matrix of a complete periodic structure. This offers advantages over the traditional finite element method. For the classical finite element approach, the whole structure is meshed and its full dynamic stiffness matrix is inverted to find the forced response. For the wave finite element approach, using the reduced dynamic stiffness matrix of only one unit cell, the eigenvalues (propagation constants) and eigenvectors (wave mode shapes) of the transfer matrix are used to form a basis from which the frequency response of the periodic structure can be calculated. Several have published on the topic including [28, 29, 34, 44], and [31].

We desire a wave basis from the eigenproblem in equation (8), but have solved the recast problem in equation (12). The \( 2n \) eigenvalues obtained in the eigenproblem in equation (12) are the same as those obtained in equation (8) and occur in \( n \) inverse pairs of \( \lambda \) and \( 1/\lambda \). The right eigenvectors, \( \psi \), and left eigenvectors, \( \phi \), obtained from the recast equation (12) require some adjustment. The right eigenvectors of equation (12) have the form \([q^T, \lambda q^T]^T \). \( T \) is the non-conjugate transpose. The right eigenvectors of equation (8) associated with eigenvalue \( \lambda \) in terms of the recast eigenvectors are then:

\[
\phi = \begin{bmatrix}
q^{(\lambda)} \\
(\mathcal{D}_{LL} + \lambda \mathcal{D}_{LR}) q^{(\lambda)}
\end{bmatrix}
= \begin{bmatrix}
q^{(\lambda)} \\
\mathcal{H}^{(\lambda)}
\end{bmatrix}.
\]

The left eigenvectors of the original eigenproblem for eigenvalue \( \lambda \) are:

\[
\psi = \begin{bmatrix}
q^{(1/\lambda)^T} \\
(\mathcal{D}_{RR} + \lambda \mathcal{D}_{LR}) q^{(1/\lambda)^T}
\end{bmatrix}.
\]

Eigenvalues for which \( |\lambda| \leq 1 \) correspond to positive traveling waves. The eigenvalues associated with the negative traveling waves are those for which \( |\lambda| > 1 \), and are the inverses of the positive traveling wave eigenvalues. If \( |\lambda| = 1 \), then the positive going waves are those with \( \text{Re}\{\mathcal{H}^{(\lambda)}\} < 0 \), where \( \mathcal{H} \) is the conjugate transpose. Let the \( 2n \) eigenvalues be sorted in a diagonal matrix, \( \Lambda_T \):

\[
\Lambda_T = \text{diag} \left( \lambda_1, \lambda_2, \ldots, \lambda_n, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n} \right)
= \begin{bmatrix}
\Lambda & 0 \\
0 & \Lambda^{-1}
\end{bmatrix},
\]
where the set \{\lambda_1, \lambda_2, \ldots, \lambda_n\} have \(|\lambda| \leq 1\) and are ordered least to greatest. Using this ordering, the following matrices of eigenvector components are defined

\[
Q^+ = \begin{bmatrix} q(\lambda_1) & q(\lambda_2) & \cdots & q(\lambda_n) \end{bmatrix} \\
Q^- = \begin{bmatrix} q \left( \frac{1}{\lambda_1} \right) & q \left( \frac{1}{\lambda_2} \right) & \cdots & q \left( \frac{1}{\lambda_n} \right) \end{bmatrix}.
\]

Next, the propagation matrices, \(P_l\) and \(P_t\) can be computed

\[
P_l = Q^+ \Lambda (Q^-)^{-1} \\
P_t = Q^+ \Lambda (Q^+)^{-1}.
\]

Using the wave basis derived above, the following formulation for the dynamic stiffness matrix of a whole structure was presented exclusively by Duhamel et al [34] only the final result is presented here. For a structure of \(N\) consecutive unit cells, the dynamic stiffness matrix, \(D_T\), is:

\[
D_T = \begin{bmatrix} D_{LL} & 0 \\ 0 & D_{RL} \end{bmatrix} + \begin{bmatrix} D_{LR} & 0 \\ 0 & D_{RL} \end{bmatrix} \begin{bmatrix} P_l^{N-1} & P_t \\ P_t & P_t^{N-1} \end{bmatrix} \begin{bmatrix} I & P_l^{N} \\ 1 & P_t^{N} \end{bmatrix}^{-1}.
\]

This dynamic stiffness matrix is convenient is for several reasons. First, computing the dynamic stiffness matrix for a structure with more unit cells requires almost no increase in computation time. The effect of increasing the number of unit cells is confined to the power of a matrix, which is an easy operation. The dimension of \(D_T\) remains \(2n \times 2n\). For structures constructed of both periodic and non-periodic sections or multiple periodic sections with different unit cells, dynamic stiffness matrices of those sections can be assembled by the usual finite element assembly scheme. It should be noted that the above dynamic stiffness matrix only relates the generalized displacements at each end of the complete periodic structure. Therefore, if the response at some arbitrary location within the structure is desired, the structure should be sliced at the response probe location. Then the dynamic stiffness matrices for the sections on each side of the probe location can be assembled, and the resulting total dynamic stiffness matrix will contain degrees of freedom corresponding to the probe location. Boundary conditions can be applied to the dynamic stiffness matrix in the usual manner for the finite element method. The slicing and assembly method described above can also be used to apply forces at arbitrary locations within the structure.

3. Complex wavenumber

The above analysis considers wave propagation in systems which are spatially periodic. The spatially repeating unit cells may be coupled with multiple degrees of freedom (see figure 1), but waves in the structure travel in only one direction: to and fro along the length of the structure. With this in mind, we examine the wavenumber of a waveform propagating along one direction, \(y(x, t)\), in the form:

\[
y(x, t) = Ae^{-i(\omega t - kx)},
\]

where \(A\) is the complex amplitude, \(\omega\) is the angular frequency, \(k\) is the wavenumber, and \(t\) and \(x\) are time and space dimensions respectively. Here \(i = \sqrt{-1}\). If the wavenumber, \(k\), is complex, we have:

\[
k = \text{Re}\{k\} + i\text{Im}\{k\} = \alpha + i\beta
\]

\[
y(x, t) = Ae^{-\beta x}e^{-i(\omega t - \alpha x)}.
\]

Three possible cases for a complex wavenumber can be summarized in the following way [45, 46]. For a purely real wavenumber, the wave is a propagating wave with no spatial attenuation of amplitude as it propagates. A purely imaginary wavenumber indicates an evanescent wave which transfers no energy in the direction it points. A complex wavenumber implies the wave is propagating but with spatially decaying amplitude.

From here forward, we consider the complex wavenumber in a vibrating structure calculated, for example, by equation (8):

\[
kL = \text{Re}\{kL\} + i\text{Im}\{kL\}.
\]

The wavenumber has been nondimensionalized by the unit cell length, \(L\), sometimes referred to as the lattice constant. We have defined the eigenvalue problem in equation (8) such that the real part of \(kL\) is the
traditionally considered wavenumber, i.e.:

\[
\text{Re}\{kL\} = \frac{\omega}{c_{\text{ph}}} L
\]  

(22)

Since we are considering systems with wave propagation in only one direction, there exists only one nonzero wavenumber vector component. This analysis is valid in multiple dimensions as well if a vector wavenumber and wavenumber magnitude are defined. The imaginary part of the complex wavenumber, \(\text{Im}\{kL\}\), is the attenuation per unit length over the length, \(L\), of the unit cell. Both the real and imaginary part of the wavenumber are plotted in figure 2. The structure has free–free boundary conditions and is comprised of 10 unit cells with filling fraction 0.5. The real and imaginary parts of the wavenumber are compared to the frequency response of the finite system to understand the vibration reduction effects. The band gaps are the two visible frequency ranges, centered around 150 Hz and 600 Hz, wherein \(\text{Im}\{kL\} > 0\). Within the band gaps, the response is significantly lower. Additionally, corresponding to the 10 unit cells comprising the structure, 10 resonances are observed between each band gap.

4. Minimization of vibration through the complex wavenumber

As discussed above, the imaginary part of the complex wavenumber represents attenuation of the wave amplitude over the length of the unit cell. Periodic structures with many unit cells therefore exhibit greatly reduced vibration amplitude when the complex wavenumber has a nonzero imaginary part. In this section, we discuss implementation of an optimization problem to minimize vibration in a one dimensional periodic structure using the complex wavenumber.

Suppose a finite element model of a unit cell of a metamaterial has been constructed. Suppose that several pertinent configuration variables, \(x\), have been chosen which are known to affect the wave propagation characteristics of the unit cell. The unit cell configuration variables could theoretically be any variables which affect the structural acoustic properties of the unit cell: elastic modulus, density, damping, electrical parameters, physical dimensions, and the like.

The goal of the optimization is to specify the vibration frequency and optimize the unit cell configuration variables, \(x \in \mathbb{R}^n\), to minimize the amplitude of the wave as it traverses through the unit cell.
The optimization problem is the following

$$\max \text{ Im}\{kL(x)\}$$

subject to $x \in \Omega$.  \hspace{1cm} (23)

The feasible set, $\Omega$, may be practical limits on the unit cell configuration such as limits on weight, volume, or stability. Or perhaps $\Omega = \mathbb{R}^n$ if the configuration variables have no constraints. An optimization block diagram is shown in figure 3.

The above allows for systems with adaptable components to be optimized for maximum vibration suppression. This is especially useful for piezoelectric actuators with negative capacitance shunts. These systems are discussed in the following section.

5. An active adaptable metamaterial for vibration control

In this section, the principles of active piezoelectric actuator vibration control are outlined. A simple one dimensional unit cell is introduced to illustrate the effects the actuators on wave propagation in a single direction.

6. Active control using piezoelectric actuators

The metamaterial modeled here is a beam with piezoelectric actuator patches attached to negative capacitance shunt circuits. The piezoelectric patches are evenly spaced along the beam length, so the structure is periodic. The metamaterial is shown in figure 4.

The beam is aluminum, with thickness 1/8 in, width 1.5 in and length of 72 in. Its elastic modulus is 73 GPa. The piezoelectric actuators are modeled with thickness 0.25 mm, width 1.5 in and length 61 mm. The constant strain elastic modulus of the piezoelectric material is 6.2 GPa.

Attaching a shunt circuit to the vibrating piezoelectric actuator creates an electromechanical system. Each electrical element in the shunt has specific influence on the behavior of the mechanical system. The circuit elements thus present additional parameters by which the wave propagation characteristics of the metamaterial beam can be modified. For example, resistance elements have the effect of creating mechanical damping in the mechanical domain, and capacitive elements in the electrical circuit alter the stiffness in the mechanical domain. While the piezoelectric patch adds local stiffness to the beam even with no shunt circuit attached (open circuit configuration), a shunt circuit attached to the piezoelectric electrodes introduces electromechanical effects. If a shunt circuit (such as resistor–capacitor or resistor–capacitor–inductor) is attached to the piezoelectric electrodes, the piezoelectric actuator will...
exhibit a complex and frequency dependent elastic modulus, $E_p^{SU}$, dependent on both the frequency of the beam vibration, $\omega$, and the electrical admittance of the attached shunt circuit, $Y^{SU}$, according to the formula in equation (24) proposed by Hagood and von Flotow [47]

$$E_p^{SU}(\omega) = E_p^E i\omega C_p^T + Y^{SU} + \frac{Y^{SU}}{(1 - k_{31}^2) + Y^{SU}}.$$  \hspace{1cm} (24)

$E_p^E$ is the patch elastic modulus at constant strain, $C_p^T$ is inherent electrical capacitance of the patch, and $k_{31}$ is the 3–1 piezoelectric constant. For the metamaterial beam considered here, the shunt capacitance and resistance are in series, thus the shunt admittance is the following

$$Y^{SU} = \frac{R_S}{R_S^2 + \frac{1}{\omega C_S^R}}.$$  \hspace{1cm} (25)

Figure 5 shows a schematic of the shunt attached to the piezoelectric patch.

7. Optimization of an active adaptable piezoelectric unit cell

Here, we utilize the optimization method described above for optimal reconfiguration of a unit cell with a negative capacitance shunt. The active adaptable unit cell consists of a piezoelectric bending actuator bonded to a vibrating metal beam and shunted with a series resistance and capacitance circuit. The unit cell configuration variables are then:

$$x = \begin{bmatrix} R_S & C_S \end{bmatrix}^T,$$  \hspace{1cm} (26)

where $R_S$ and $C_S$ are the shunt circuit parameters. The two circuit parameters are usually different orders of magnitude. Thus, it is convenient to scale the parameters to aid in solution of the optimization problem.

Holding the resistance, $R_S$, and driving frequency, $\omega$, constant at $R_S = 10\, \Omega$ and $f = 25\, \text{Hz}$ and examining the behavior of the objective function versus capacitance ratio, $C_S/C_p^R$, produces the plot shown in figure 6. It is evident from the figure that $\text{Im}\{kL\}$ has a local maximum at approximately $C_S/C_p^R = -0.88$. However, for this particular frequency and shunt resistance, $\text{Im}\{kL\}$ is approximately flat far from the optimum. This near-zero gradient could be problematic for gradient-based optimizers. It is useful for this particular problem to condition the objective function to increase the gradient far from the optimum point. Figure 7 shows the original objective function, and figure 8 shows $1/\text{Im}\{kL\}$. Here, minimizing $1/\text{Im}\{kL\}$ is preferable to maximizing $\text{Im}\{kL\}$, as the gradient is steeper away from the optimum in the former case. Here, simply taking the inverse of the objective is sufficient to allow use of gradient optimization routines such as MATLAB’s ‘fmincon’. For other configuration variables, $x$, where the objective function, $\text{Im}\{kL(x)\}$, is conducive to gradient optimizers, no such measures may be necessary. In other cases, other modifications of the $\text{Im}\{kL(x)\}$ objective may be experimented with. Additionally, the option of non-gradient optimizers (such as MATLAB’s ‘fminsearch’) are available as well.

8. Experimentally obtained constraints for the optimization problem

In practice, not all values of $R_S$ and $C_S$ for this specific active shunt circuit are physically feasible because the negative capacitance circuit, which contains an operational amplifier, is electrically unstable for certain configurations. Stability of negative impedance converters is well explained in the literature [48].

The experimental stability of limit of the negative capacitance circuit for a prototypical shunted actuator was found as a function of series resistance. Figure 9 shows the upper bounds on normalized shunt capacitance, $C_S/C_p^R$, for varying shunt resistances, $R_S$. While a linear fit is shown in the figure, there is not
Figure 6. Optimization function behavior with mobility and piezoelectric behavior for comparison.

Figure 7. Im\{kL\}.

an appreciable dependence of the upper bound on the capacitance ratio on shunt resistance when the shunt resistance is low. For a piezoelectric capacitance at constant stress of $C_S^P = 64 \text{ nF}$, the mean value of the upper bounds was found to be the following

$$\text{mean} \left\{ \frac{C_{S,\text{sh}}}{C_P^S} \right\} = -1.01$$

$$C_{S,\text{sh}} = -64.6 \text{ nF}.$$  \hfill (27)

The data shows that the stability limit of the negative capacitance shunt is very near the theoretical optimal negative capacitance ratio. However, to maintain stability at all frequencies, a slightly larger negative capacitance ratio is required. It is not clear from the collected data that there exists a minimum resistance to comprise a stable negative capacitance circuit configuration.
9. Wave amplitude minimization problem for an active adaptable acoustic metamaterial with shunted piezoelectric actuators

With the above practical considerations of the shunt circuit stability limits in mind, a well-posed optimization problem can be written as follows to minimize the wave amplitude through the unit cell using a series shunted piezoelectric actuator.

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{\text{Im}\{kL\}} \\
\text{subject to} & \quad R_S > 0 \ \Omega \\
& \quad C_S/C_P^S < -1.01
\end{align*}
\] (28)

In practice, the lower bound on the shunt resistance, \(R_S\), should be set to a ‘small’ positive number, say 2Ω to assure boundedness of the analytical model of piezoelectric patch stiffness given in equation (24). The upper bound on the shunt capacitance is stated relative to the capacitance of the piezoelectric actuator at constant stress. To experimentally determine the limits of a representative shunt circuit in the present work, the piezoelectric capacitance was \(C_P^S = 64 \text{ nF}\), so the upper bound on the shunt capacitance is \(C_S < -64.6 \text{ nF}\).
10. Wavenumber optimization results for an active adaptable acoustic metamaterial

Figure 10 shows the mobility frequency response of the acoustic metamaterial system with the optimization objective, $\text{Im}\{kL\}$ plotted above it. Each driving frequency requires a slightly different unit cell configuration to optimally suppress vibration, and the figure shows the results of optimizing the unit cell configuration (in this case the shunt capacitance and resistance) at each frequency. The uncontrolled mobility is shown as a thin black line. The figure shows that the unconstrained and theoretical constraints allow the best control of the system response. Constraining the optimization with the practical stability constraints inhibits vibration control, but still gives the best control without the system becoming electrically unstable. At several points, the optimization objective ‘jumps’. This may be a result of the optimization objective being numerically sensitive to small changes in the shunt capacitance. Regardless of these discontinuities in the optimization results, in practice, the shunt parameters can still be specified for an approximation of the optimal setting. Both plots also clearly show that, for this particular system, high frequency vibration control is easier to achieve than low frequency vibration control. $\text{Im}\{kL\}$ is greater and the mobility response is smaller at high frequencies.

The optimization solutions are shown in figure 11. These plots show the shunt resistance and capacitance which realize the maximum $\text{Im}\{kL\}$, and thus the optimal vibration reduction. From these results, it can be seen that the required negative capacitance for optimal vibration suppression does not change much over the frequency range examined here. This is explained by the fact that the negative capacitance shunt performs best when the shunt capacitance is nearly the negative of the piezoelectric capacitance. The required shunt resistance, which does not as greatly effect the vibration control efficacy of the shunt, does change over frequency. Essentially, the resistance provides a mechanical damping effect in addition to the active negative capacitance vibration compensation effect. At low frequencies, where active vibration control is more difficult to attain, the optimization has specified a large resistance to compensate. Also evident in figure 11 are the discontinuities in the solution discussed above. These are most evident at very low frequencies (less than 20 Hz). The shunt resistance solution displays larger discontinuities than the shunt capacitance solution.

The above results (figures 10 and 11) show the results of optimizing the shunt resistance and capacitance at all frequencies. The result is a plot of optimal resistances and capacitances for every driving frequency in the range of the optimization. Given a system driving frequency, the optimal parameters $R_s$ and $C_s$ can be read from figure 11. Next, the system frequency response with the optimal resistance and capacitance settings from figure 11 is shown at a few chosen frequencies. It should be noted that in practice, only one optimal configuration can be physically realized at one time, and single frequency tunings are the topic of the discussion below.
Figure 11 suggests that for a driving frequency of 10 Hz, the optimal negative capacitance shunt configuration is $R_S = 2084 \ \Omega$ and $C_S = -64.2 \ \text{nF}$. That is, by setting the shunt parameters to the values above, the imaginary part of the wavenumber will be maximized, corresponding to optimal wave attenuation in the vicinity of 10 Hz. Figure 12 shows the $\text{Im}\{kL\}$ and mobility results of an optimization for a 10 Hz driving frequency. Note that the optimized shunt configuration does not guarantee that $\text{Im}\{kL\}$ will be greatest at 10 Hz relative to all other frequencies, but rather that $\text{Im}\{kL\}$ will be as large as it can be at 10 Hz relative to all other possible shunt configurations. $\text{Im}\{kL\}$ may still achieve higher values at other frequencies, as it does here. But from this optimization, there is no other shunt configuration that will yield a greater value of $\text{Im}\{kL\}$ at 10 Hz than the one described here. It should also be noted that the broadband nature of the negative capacitance shunt is visible in this plot. While we have achieved as high attenuation as possible at 10 Hz, other nearby frequencies (about a bandwidth of 100 Hz about the 10 Hz optimization
centering frequency) are also benefiting from increased attenuation. This is typical of negative capacitance shunts, and has been well described in previous literature [49–53].

Figure 13 shows the same visualization for an optimization at 2 kHz. The figure shows that for this structure, attenuation at higher frequencies is much easier to achieve: $\text{Im}\{kL\}$ is about an order of magnitude larger here. Notice that for both optimizations the Bragg gaps due to the impedance mismatch between the sections of beam with and without the piezoelectric actuators bonded are visible. For the 2 kHz optimization, they are significantly less visible: the attenuation due to the negative capacitance shunt is large enough to almost surpass the destructive interference due to the impedance mismatch. In either case, the impedance mismatch between the piezo-bonded and bare metal sections is so small that the Bragg gaps are very narrow. The negative capacitance shunt is the source of most of the wave attenuation.

These results show that a reconfigurable metamaterial (here, an acoustic metamaterial with a negative capacitance shunt circuit) can be effectively reconfigured for optimal vibration reduction at specific frequencies by using the attenuative part of the complex wavenumber, $\text{Im}\{kL\}$. Moreover, while the negative capacitance shunt exhibits broadband vibration reduction about the centering frequency of the tuning, the optimization is essential to achieve the best possible vibration control.

11. Conclusion

Using a complex wavenumber for optimizing metamaterial unit cells for vibration control allows a complete acoustic metamaterial system of unit cells to be optimally designed from the unit cell level. Additionally, the complex wavenumber is an objective function with physical meaning, and can be calculated for any one dimensional periodic structure. This work shows that the method can be used for an active adaptable piezoelectric metamaterial structure. Finite element modeling shows that the optimization method effectively specifies active electrical shunt configurations for minimization of vibration, and experimental stability considerations aid in constraining the optimization routine. Future work will consider applying the method to active adaptable plate structures with waves propagating in two dimensions.

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