Comments, corrections, and related references welcomed, as always!

\TeXed May 11, 2014

COMMUTING MATRICES, AND MODULES OVER ARTINIAN LOCAL RINGS

GEORGE M. BERGMAN

Abstract. Gerstenhaber [3] proves that any commuting pair of $n \times n$ matrices over a field $k$ generates a $k$-algebra $A$ of $k$-dimension $\leq n$. A well-known example shows that the corresponding statement for 4 matrices is false. The question for 3 matrices is open.

Gerstenhaber’s result can be looked at as a statement about the relationship between the length of a 2-generator finite-dimensional commutative $k$-algebra $A$, and the lengths of faithful $A$-modules. Wadsworth [17] generalizes this result to a larger class of commutative rings than those generated by two elements over a field. We recover his result, with a slightly improved argument.

We then explore some examples, raise further questions, and make a bit of progress toward answering some of these.

An appendix gives some lemmas on generation and subdirect decompositions of modules over not necessarily commutative Artinian rings, generalizing a special case noted in the paper.

When I drafted this note, I thought the main result, Theorem 3, was new; but having learned that it is not, I probably won’t submit this for publication unless I find further strong results to add. However, others may find interesting the observations, partial results, and questions noted below, and perhaps make some progress on them.

Noncommutative ring theorists might find the lemmas in the appendix, §3, of interest. (I would be interested to know whether they are known.)

1. Wadsworth’s generalization of Gerstenhaber’s result

To introduce Wadsworth’s strengthening of the result of Gerstenhaber quoted in the first sentence of the abstract, note that a $k$-algebra of $n \times n$ matrices generated by two commuting matrices can be viewed as an action of the polynomial ring $k[s, t]$ on the vector space $k^n$. A class of rings generalizing those of the form $k[s, t]$ are the rings $R[t]$ for $R$ a principal ideal domain. The analog of an action of $k[s, t]$ on a finite-dimensional $k$-vector space is then an $R[t]$-module of finite length. A key tool in our study of such actions will be

**Corollary 1** (to the Cayley-Hamilton Theorem; cf. [17] Lemma 1). Let $R$ be a commutative ring, $M$ an $R$-module which can be generated by $n < \infty$ elements, and $f$ an endomorphism of $M$. Then $f^n$ is an $R$-linear combination of $1_M$, $f$, \ldots, $f^{n-1}$.

**Proof.** Write $M$ as a homomorphic image of $R^n$. Since $R^n$ is projective, we can lift $f$ to an endomorphism $g$ thereof. Since $g$ satisfies its characteristic polynomial, $f$ satisfies the same polynomial. \hfill $\square$

This shows that the unital subalgebra of $\text{End}_R(M)$ generated by $f$ will be spanned over $R$ by the $f^i$ with $0 \leq i < n$, but doesn’t say anything about the size of the contribution of each $f^i$. In the next lemma, we obtain information of that sort by re-applying the above corollary to various $i$-generator submodules of $M$.

**Lemma 2.** Let $R$ be a commutative ring, $M$ a finitely generated $R$-module, and $f$ an endomorphism of $M$. For each $i \geq 0$, let $I_i$ be the ideal of $R$ generated by all elements $u$ such that $uM$ is contained in an $f$-invariant submodule of $M$ which is $i$-generated as an $R$-module. (So $I_0 \subseteq R$ is the annihilator of $M$, and we have $I_0 \subseteq I_1 \subseteq \ldots$, with the ideals becoming $R$ once we reach an $i$ such that $M$ is $i$-generated.)

Let $A$ be the unital $R$-algebra of endomorphisms of $M$ generated by $f$. For each $i \geq -1$, let $A_i$ be the $R$-submodule of $A$ spanned by $\{1, f, f^2, \ldots, f^i\}$. Then

---

2010 Mathematics Subject Classification. Primary: 13E10, 13E15, 15A27, Secondary: 13C13, 15A30, 16D60, 16P20.

Key words and phrases. Commuting matrices, lengths of faithful modules over commutative local Artinian rings.

This preprint is readable online at \url{http://math.berkeley.edu/~gbergman/papers/unpub}. 

arXiv:1309.0053v1 [math.AC] 31 Aug 2013.
(i) For all $i \geq 0$, the $R$-module $A_i/A_{i-1}$ is a homomorphic image of $R/I_i$. Hence
(ii) If $M$ has finite length, and can be generated by $d$ elements, then $A$ has length $\leq \sum_{i=0}^{d-1} \text{length}(R/I_i)$ as an $R$-module.

Proof. Since $A_i = Rf^i + A_{i-1}$, (i) will follow if we show that $I_i f^i \subseteq A_{i-1}$, which by definition of $I_i$ is equivalent to showing that, for each $u \in R$ such that $uM$ is contained in an $i$-generated $f$-invariant submodule $M'$ of $M$, we have $uf^i \in A_{i-1}$. Given such $u$ and $M'$, let $f'$ be the restriction of $f$ to $M'$. By Corollary 1, $f'$ is an R-linear combination of the lower powers of $f'$; say $\sum_{j<i} a_j f'^j$. Hence the restriction of $f$ to $uM \subseteq M'$ satisfies the same relation. This says that $uf^i = \sum_{j<i} a_j f'^ju$, equivalently, that $uf^i = \sum_{j<i} a_j f'^ju$, showing that $uf^i \in A_{i-1}$, as required.

We deduce (ii) by summing over the steps of the chain $\{0\} = A_{-1} \subseteq A_0 \subseteq \cdots \subseteq A_{d-1} = A$. □

We can now prove

**Theorem 3** (Wadsworth [17 Theorem 1], after Gerstenhaber [4 Theorem 2, p. 245]). Let $M$ be a module of finite length over a commutative principal ideal domain $R$, let $f$ be an endomorphism of $M$, and let $A$ be the unital $R$-algebra of $R$-module endomorphisms of $M$ generated by $f$. Then

\[ \text{length}_R(A) \leq \text{length}_R(M). \]

Proof. Because $R$ is a commutative principal ideal domain, we can write $M$ as $R/(q_0) \oplus \cdots \oplus R/(q_{d-1})$, where $q_0 \subseteq \cdots \subseteq q_d = 0$ (Theorem III.7.7, p. 151), with terms relabeled. Note that for $i = 0, \ldots, d-1$, the element $q_i$ annihilates the summands $R/(q_i), \ldots, R/(q_{d-1})$ of the above decomposition. Hence $q_i M$ is generated by $i$ elements, so in the notation of the preceding lemma, $q_i \in I_i$. (In fact, it is not hard to check that $I_i = (q_i)$.) In particular, $\text{length}_R(R/I_i) \leq \text{length}_R(R/(q_i))$.

By that lemma, the length of $A$ as an $R$-module is $\leq \sum \text{length}_R(R/I_i)$, which by the above inequality is $\leq \sum \text{length}_R(R/(q_i)) = \text{length}_R(M)$. □

We remark that in the above results, the length of a module is really a proxy for its image in the Grothendieck group of the category of finite-length modules. I did not so formulate the results for simplicity of wording, and to avoid excluding readers not familiar with that viewpoint. This has the drawback that when we want to pass from that lemma back to Gerstenhaber’s result on $k$-dimensions of matrix algebras, the lengths do not directly determine these dimensions, which the elements of the Grothendieck group would. (E.g., the $\mathbb{R}$-algebras $\mathbb{R}[s]/(s)$ and $\mathbb{R}[s]/(s^2 + 1)$ both have length 1 as modules, but their $\mathbb{R}$-dimensions are 1 and 2 respectively.)

However, we can get around this by applying Theorem 3 after extension of scalars to the algebraic closure of $k$. Indeed, if $A$ is a $k$-algebra of endomorphisms of a finite-dimensional $k$-vector-space $M$, then the $k$-dimensions of $A$ and $M$ are unaffected by extending scalars to the algebraic closure $k$ of $k$; and for $k$ algebraically closed, the length of $M$ as an $A$-module is just its $k$-dimension. Hence an application of Theorem 3 with $R = k[s]$ to these extended modules and algebras gives us Gerstenhaber’s result for the original modules and algebras.

2. Some notes on the literature

The hard part of Gerstenhaber’s proof of his result was a demonstration that the variety (in the sense of algebraic geometry) of all pairs of commuting $n \times n$ matrices is irreducible. Guralnick [5] notes that this fact had been proved earlier by Motzkin and Taussky [13], and gives a shorter proof of his own.

The first proof of Gerstenhaber’s result by non-algebraic-geometric methods is due to Barria and Hamas [1]. Wadsworth [17] abstracts that proof by replacing $k[s]$ by a general principal ideal domain $R$. His argument differs from ours in that he obtains Corollary 1 only for $R$ a principal ideal domain and $M$ a torsion $R$-module. This restriction arises from his calling on the fact that every such module $M$ embeds in a free module over some factor-ring of $R$ (in the notation of our proof of Theorem 3 the free module of rank $d$ over $R/(q_0)$, where we use, instead, the fact that any finitely generated $R$-module is a homomorphic image of a free $R$-module of finite rank, which is true for any commutative ring $R$. Of course, the restriction assumed in Wadsworth’s proof holds in the case to which we both apply the result; but the more general statement of Corollary 1 seemed worth recording.

I mentioned above that where I speak of the length of a module, a more informative statement would refer to its image in a Grothendieck group. Wadsworth uses an invariant of finite-length modules over PIDs that
is equivalent to that more precise information: in the notation of our proof of Theorem 3, the equivalence class, under associates, of the product $q_0 \ldots q_{d-1} \in R$. He also shows [17] Theorem 2 that as an $R$-module, $A$ can in fact be embedded in $M$. (However, we will note at the end of §4 that it cannot in general be so embedded as an $R[t]$-module.)

Gerstenhaber [4] proves a bit more about the algebra generated by two commuting $n \times n$ matrices than we have quoted: he also shows that it is contained in a commutative matrix algebra of dimension exactly $n$, at least after a possible extension of the base field. (He mentions that he does not know whether this is true without extension of the base field.) We shall not discuss that property further here.

Guralnick and others [6], [10] have continued the algebraic geometric investigation of these questions. Some investigations which, like this one, focus more on methods of linear algebra are [9], [8], [15]. For an extensive study of the subject, see O’Meara, Clark and Vinsonhaler [12, Chapter 5].

Returning to Theorem 3, the hypothesis that $R$ be a principal ideal domain can be weakened, with a little more work, to say that $R$ is a Dedekind domain, or even a Prüfer domain, since under these assumptions, every finite-length homomorphic image of $R$ is a direct product of uniserial rings, which is what is really needed to get the indicated description of finite-length modules (though I don’t know a reference stating this description in those cases). We shall discuss in §§5-6 wider generalizations that one may hope for, and will make some progress in those directions.

But for the next two sections, let us return to commuting matrices over a field, and examine what can happen in algebras generated by more than two such matrices.

### 3. Counterexamples with 4 Generators

The standard example showing that a 4-generator algebra of $n \times n$ matrices can have dimension $> n$ takes $n = 4$, and for $A$, the algebra of $4 \times 4$ matrices generated by $e_{13}$, $e_{14}$, $e_{23}$ and $e_{24}$. These commute, since their pairwise products are zero, and $A$ has for a basis these four elements and the identity matrix $1$, and so has dimension $5 > 4 = n$.

If the reader finds it disappointing that the extra dimension comes from the convention that algebras are unital, note that without that convention, one can obtain the same subalgebra from the four generators $1 + e_{13}$, $e_{14}$, $e_{23}$, $e_{24}$, using the fact that the not-necessarily-unital algebra generated by an upper triangular matrix (here $1 + e_{13}$) always contains both the diagonal part (here 1) and the above-diagonal part (here $e_{13}$) of that matrix.

One can modify this example to get commutative 4-generator matrix algebras in which the dimension of the algebra exceeds the size of the matrices by an arbitrarily large amount. Namely, for any $m$, let us form within the algebra of $4m \times 4m$ matrices a “union of $m$ copies” of each of the matrix units used in the above example. To do this, let $E_{13} = \sum_{j=0}^{m-1} e_{4j+1, 4j+3}$, $E_{14} = \sum_{j=0}^{m-1} e_{4j+1, 4j+4}$, $E_{23} = \sum_{j=0}^{m-1} e_{4j+2, 4j+3}$, $E_{24} = \sum_{j=0}^{m-1} e_{4j+2, 4j+4}$, and let us also choose a diagonal matrix $D$ having one value, $\alpha_0 \in k - \{0\}$, in the first four diagonal positions, a different value, $\alpha_i \in k - \{0\}$, in the next four, and so on; in other words, $D = \sum_{j=0}^{m-1} \alpha_j (\sum_{i=1}^{m} e_{4j+i, 4j+i})$. (Here we assume $|k| \geq m + 1$, so that the $\alpha_i$ can be taken distinct.) We then take as our four generators $D + E_{13}$, $E_{14}$, $E_{23}$ and $E_{24}$. From the fact about upper triangular matrices called on in the preceding paragraph, the algebra these generate contains $D$, $E_{13}$, $E_{14}$, $E_{23}$ and $E_{24}$. Using just $D$ and the $k$-algebra structure, one gets the $m$ diagonal idempotent elements $\sum_{i=1}^{m} e_{4j+i, 4j+i}$ $(j = 0, \ldots, m - 1)$, and with the help of these, one sees that our 4-generator algebra is the direct product of $m$ copies of the algebra of the preceding paragraphs. Thus, we have a 4-generated commutative algebra of dimension $5m$ within the ring of $4m \times 4m$ matrices.

Because of the strong way this construction used a diagonal matrix $D$, I briefly hoped that if we restricted attention to algebras $A$ generated by four commuting nilpotent matrices, the dimension of $A$ might never exceed the size of the matrices by more than 1. But the following family of examples contradicts that guess.

In describing it, I will again use the language of vector spaces and their endomorphisms. Let $m$ be any positive integer, and let $M$ be a $(5m^2 + 3m)/2$-dimensional $k$-vector-space, with basis consisting of elements which we name (proactively)

$$a^i \ b^j \ x, \text{ for } i, j \geq 0, \ i + j \leq 2m - 1, \text{ and } \ a^i \ b^j \ y, \text{ for } i, j \geq 0, \ i + j \leq m - 1.$$

(we have $2m(2m+1)/2 = m^2 + m$ basis elements $a^i \ b^j \ x$, and $m(m+1)/2$ basis elements $a^i \ b^j \ y$, totaling $(5m^2 + 3m)/2$ elements.)
We now define four linear maps, \( a, b, c \) and \( d \) on \( M \). Of these, \( a \) and \( b \) act in the obvious ways on the elements \( a^i b^j x \) with \( i + j < 2m - 1 \) and on the elements \( a^i b^j y \) with \( i + j < m - 1 \), namely, by increasing the formal exponent of \( a \), respectively, \( b \), by 1; while they annihilate the elements for which these formal exponents have their maximum allowed total value, \( i + j = 2m - 1 \), respectively \( i + j = m - 1 \). On the other hand, we let \( c \) annihilate \( x \), but take \( y \) to \( a^m x \), and hence (as it must if it is to commute with \( a \) and \( b \)) take \( a^i b^j y \) to \( a^{m+i} b^j x \), and we similarly let \( d \) annihilate \( x \), but take \( y \) to \( b^m x \), and hence \( a^i b^j y \) to \( a^i b^{m+j} x \).

It is easy to verify that these four linear maps commute, and are nilpotent. I claim that the unital algebra that they generate has for a basis the elements

\[
(3) \quad a^i b^j \text{ for } i, j \geq 0, \ i + j < 2m - 1, \quad \text{and} \quad a^i b^j c \quad \text{and} \quad a^i b^j d \text{ for } i, j \geq 0, \ i + j \leq m - 1
\]

(compare with (2)). Indeed, it is immediate that every monomial in \( a, b, c \) and \( d \) other than those listed in (3) is zero. Now suppose some \( k \)-linear combination of the monomials (3) were zero. By applying that linear combination to \( x \in M \), we see that the coefficients in \( k \) of the monomials \( a^i b^j \) (with no factor \( c \) or \( d \)) are all zero. Applying the same element to \( y \), and noting that the sets of basis elements \( a^m b^j x \) (\( i + j \leq m - 1 \)) and \( a^i b^{m+j} x \) (\( i + j \leq m - 1 \)) are disjoint, we conclude that the coefficients of the monomials \( a^i b^j c \) and \( a^i b^j d \) are also zero.

Counting the elements (3), we see that the dimension of our algebra is \((2m^2 + m) + (m^2 + m)/2 + (m^2 + m)/2 = 3m^2 + 2m\), and this exceeds that of \( M \) by \((m^2 + m)/2\), which is unbounded as \( m \) grows. The limit as \( m \to \infty \) of the ratio of the dimensions of \( A \) and \( M \) is \( \lim (3m^2 + 2m)/(5m^2 + 3m)/2 = 6/5 \).

Can we get a similar example with limiting ratio \( 5/4 \), the ratio occurring for our \( 4 \)-dimensional \( M \)? In fact we can. The description is formally like that of the above example, but with the basis (2) replaced by the slightly more complicated basis,

\[
(4) \quad a^i b^j x, \ \text{where } 0 \leq i, j < 2m \text{ and } \min(i, j) < m, \quad \text{and} \quad a^i b^j y, \ \text{where } 0 \leq i, j < m.
\]

We again define \( a \) and \( b \) to act by adding 1 to the relevant exponent symbol when this leads to another element of the above basis, while taking basis elements to zero when it does not; and we again let \( c \) and \( d \) annihilate \( x \), but carry \( y \) to \( a^m x \), respectively, \( b^m x \), and act on other basis elements as they must for our operators to commute. We find that \( \dim_k M = 4m^2 \) while \( \dim_k A = 5m^2 \), so indeed, \( \dim_k A/\dim_k M = 5/4 \).

Incidentally, the \( m = 1 \) case of both the construction using (2) and the one using (4) can be seen to be isomorphic to the \( 5 \)-dimensional algebra of \( 4 \times 4 \) matrices with which we began this section.

4. The recalcitrant 3-generator question

It is not known whether every 3-generator commutative \( k \)-algebra \( A \) of endomorphisms of a finite-dimensional \( k \)-vector-space \( V \) satisfies \( \dim_k A \leq \dim_k V \). Let me lead into the discussion of that question by starting with some observations applicable to any commutative algebra \( A \) of endomorphisms of a finite-dimensional vector space \( V \). We will again write \( M \) for \( V \) regarded as an \( A \)-module. Note that since \( A \) is an algebra of endomorphisms of \( M \), it acts faithfully on \( M \).

Since \( A \) is a finite-dimensional commutative \( k \)-algebra, it is a direct product of local algebras, and the idempotents arising from the decomposition of \( 1 \in A \) yield a corresponding decomposition of \( M \) as a direct product of modules over one or another of these factors. Thus the question of whether \( \dim_k A \) can exceed \( \dim_k M \) reduces, in general, to the corresponding question for local algebras; so we shall assume \( A \) local in what follows. Moreover, since passing to the algebraic closure of \( k \) does not affect the properties we are interested in, we can assume that \( k \) is algebraically closed, hence that the residue field of the local algebra \( A \) is \( k \) itself. (Having reduced our considerations to this case, we can now drop the hypothesis that \( k \) be algebraically closed, keeping only this condition on the residue field.) Thus, the \( k \)-algebra structure of \( A \) is determined by that of its maximal ideal, which, by finite-dimensionality, is nilpotent. We shall denote this ideal \( m \).

Next note that if \( M \) is cyclic as an \( A \)-module, then it will be isomorphic as an \( A \)-module to \( A \) itself. (This is a consequence of commutativity. If \( M = Ax \), and some nonzero element of \( A \) annihilated \( x \), then by commutativity, it would annihilate all of \( M \), contradicting the assumption that \( A \) acts faithfully.)

Note also that the dual space \( M^* = \operatorname{Hom}_k(M, k) \) acquires a natural structure of \( A \)-module (since the vector-space endomorphisms of \( M \) given by the elements of \( A \) induce endomorphisms of \( M^* \)), of the same \( k \)-dimension as \( M \). Using this duality, it follows from the preceding observations that \( \dim_k A = \dim_k M \) will
also hold if \( M^* \) is cyclic. The latter condition is equivalent to saying that the socle of \( M \) (the annihilator of \( \mathfrak{m} \) in \( M \)) is 1-dimensional; in this situation one calls \( M \) “cocyclic”. (Dually, the condition that \( M \) be cyclic is equivalent to saying that \( \mathfrak{m}M \) has codimension 1 in \( M \).)

Thus, if \( \dim_k A > \dim_k M \),

\[
\text{(5) } \dim_k A > \dim_k M,
\]

with \( M \) generated by two elements \( x_1 \) and \( x_2 \), so that it is a homomorphic image of \( A x_1 \oplus A x_2 \), then the construction of this homomorphic image must involve additional relations, i.e., the identification of a nonzero submodule of \( A x_1 \) with an isomorphic submodule of \( A x_2 \).

It turns out that a single relation, i.e., the identification of a cyclic submodule of \( A x_1 \) with an isomorphic cyclic submodule of \( A x_2 \), is still not enough to get \( \text{(5)} \). For let the common isomorphism class of the cyclic submodules that we identify be that of \( A/I_3 \). Then \( I_1 \) and \( I_2 \) are both contained in \( I_3 \), hence so is \( I_1 + I_2 \).

Now we have seen that the amount by which \( \dim_k A/(I_1 + I_2) \) exceeds \( \dim_k A \) is the codimension of \( I_1 + I_2 \) in \( A \); so setting to zero a submodule isomorphic to \( A/I_3 \), which has dimension at most that codimension, can at best give us equality.

Thus, we need to divide out by at least a 2-generator submodule to get \( \text{(5)} \). And indeed, the families of 4-generator examples we obtained in the latter half of the preceding section can be thought of as constructed by imposing two relations, \( cy = a^m x \) and \( dy = b^n x \), on a direct sum \( A x \oplus A y \cong A/I_1 \oplus A/I_2 \).

In the remainder of this section, I will display a few examples diagrammatically. In these examples, \( M \) will have a \( k \)-basis \( B \) such that each of our given generators of \( A \) carries each element of \( B \) either to another element of \( B \) or to 0. The actions of the various generators of \( A \) on basis elements will be shown as downward line segments of different slopes, with the matching of generator and slope shown to the right of the diagram (under the word “labeling”). Where no line segment of a given slope descends from a given element, this means that the corresponding generator of \( A \) annihilates that basis element. For instance, the 4-dimensional example with which we began the preceding section may be diagrammed

\[
\text{(6)}
\]

The fact that it can be obtained from a direct sum of two cyclic modules \( A x_1 \), \( A x_2 \) by two identifications is made clear in the representation below, where the labels on the lower vertices show the relations imposed.

\[
\text{(7)}
\]
the distinct nonzero monomials in \( A \),

Note the repetition of \( y \), becomes, as shown in the next diagram, a module whose submodules \( A x_1 \) and \( A x_2 \) are connected by only one relation, and which has \( \dim_k A = \dim_k M \) (both dimensions being 4), which, we have seen, is the best one can hope for from such a “one-relation” module.

If we drop one of the generators of the above \( A \), getting a 3-generator algebra, then the same vector space becomes, as shown in the next diagram, a module whose submodules \( A x_1 \) and \( A x_2 \) are connected by only one relation, and which has \( \dim_k A = \dim_k M \) (both dimensions being 4), which, we have seen, is the best one can hope for from such a “one-relation” module.

\[
\begin{align*}
\text{labeling:} & \quad a \quad b \quad c \\
\end{align*}
\]

\[
\begin{align*}
(8) & \quad c x_1 = a x_2 \\
(9) & \quad c^3 x_1 = a^2 x_2 \
\end{align*}
\]

We remark that we still get equality of dimensions if, in the above example, we replace one or more of the 1-step paths by multistep paths repeating the algebra generator in question; e.g.,

\[
\begin{align*}
\text{labeling:} & \quad a \quad b \quad c \\
\end{align*}
\]

which has \( \dim_k A = \dim_k M = 10 \).

I have attempted to find examples of 3-generator algebras \( A \) such that \( \dim_k A > \dim_k M \), by connecting two or more cyclic modules with the help of two or more relations, but without success; the best my fiddling with such examples has achieved is to get more examples of equality; for instance,

\[
\begin{align*}
\text{labeling:} & \quad a \quad b \quad c \\
\end{align*}
\]

Note the repetition of \( y_0 \) at the right end of the middle row; thus, \( y_0 \) is both \( a x_1 \) and \( c x_2 \). We find that the distinct nonzero monomials in \( A \) are

\[
1; \quad a, \quad b, \quad c; \quad a^2 = bc, \quad b^2 = ac, \quad c^2 = ab,
\]

which are linearly independent, so that \( A \), like \( M \), is 7-dimensional. (One must also verify commutativity of \( A \). This is fairly easy; there are three relations to be checked, \( ab = ba, \quad ac = ca \) and \( bc = cb \), and these need only be checked on \( x_1 \) and \( x_2 \), since on all other basis elements, both sides of each equation clearly give 0.)

One can modify this example by making the rightmost basis-element in the middle row be, not a repetition of \( y_0 = a x_1 \), but \( a^i y_0 = a^{i+1} x_1 \), for any \( i > 0 \). This adds exactly \( i \) basis elements to \( M \), namely \( a y_0, \ldots, a^i y_0 \), and likewise adds \( i \) monomials to the basis of \( A \), namely \( a^2, \ldots, a^{i+2} \) (with \( a^{i+2} \) rather than \( a^2 \) now coinciding with \( bc \)). In the new \( A \), the relation \( c^2 = ab \) no longer holds; rather, \( c^2 = 0 \), but \( ab \) remains nonzero.

We see that this modified example still satisfies \( \dim_k A = \dim_k M \). In fact, Kevin O’Meara has pointed out to me that by [12] Theorem 5.5.8, if a \( k \)-algebra \( A \) of endomorphisms of a \( k \)-vector-space \( M \) generated by three commuting elements \( a, \ b, \ c \) is to satisfy \( \dim_k A > \dim_k M \), then \( M \) must require at least 4 generators as a module over the subring \( k[a] \). (The wording of that theorem is that \( \dim_k A \leq \dim_k M \) holds for “3-regular” matrix algebras, i.e., those where \( M \) can be so generated by 3 elements.) The example of (10), and the variant just noted, are generated over \( k[a] \) by \( \{ x_1, x_2, y_2 \} \) confirming that we would need something more complicated to get a counterexample.

O’Meara suspects that the analog of the theorem just
quoted also holds for 4-regular algebras, but might fail in the 5-regular case. (Cf. [12] p.226, footnote 12.)

Incidentally, (9) is an example of a module that is not 3-generated over any of \( k[a], k[b], k[c] \), but which still satisfies \( \dim_k A = \dim_k M \).

We remark, for the benefit of the reader who wants to explore examples using diagrams like the above, that a consequence of our requirement of commutativity is that wherever the diagram shows distinct generators of \( A \) coming “into” and “out of” a vertex, e.g., \( \bullet \), these must be part of a parallelogram \( \square \). So, for instance, if we tried to improve on (10) by deleting the line segment from \( y_2 \) to \( z_2 \), so that \( b^2 \), though still nonzero because of its action on \( x_1 \), ceased to equal \( ac \), thus increasing \( \dim_k A \) by 1, the resulting algebra would not be commutative, because the configuration consisting of \( x_1, y_1 \) and \( z_2 \) and their connecting line segments would not be part of a parallelogram. (On the other hand, instances of \( \subseteq \) or of \( \subseteq \) do not need to belong to parallelograms, as illustrated by \( x_1, x_2, y_2 \) in (10).)

Our diagrammatic notation also allows us to illustrate the fact mentioned in (12) that a faithful module over a finite-length homomorphic image \( A \) of \( k[s, t] \) need not contain an isomorphic copy of \( A \) as a \( k[s, t] \)-module, though it will as a \( k[s] \)-module. Let \( A = k[s, t]/(s, t)^2 \), which has diagram \( \backslash / \). Let \( M \) be the dual module \( \text{Hom}_A(A, k) \). This is faithful, but has diagram \( \backslash \), which does not contain a copy of the diagram of \( A \). However, the diagrams for \( A \) and \( M \) as \( k[s] \)-modules are \( \backslash / \), and \( \backslash / \), which are isomorphic.

### 5. Some questions, and steps toward their answer

Theorem 3 has the unsatisfying feature that our \( R \) has absorbed one of the indeterminates of the original algebra \( k[s, t] \), but not the other. We may ask, without referring to indeterminates,

**Question 4.** For which commutative rings \( S \) does the statement,

\[
(12) \quad \text{length}_S(A) \leq \text{length}_S(M),
\]

hold?

Theorem 8 says roughly that the class of such rings includes the rings \( R[t] \) where \( R \) is a principal ideal domain. A plausible generalization would be that it contains all rings \( S \) such that every maximal ideal \( m \) of \( S \) satisfies \( \text{length}(m/m^2) \leq 2 \). If, in fact, Gerstenhaber’s result turns out to go over to 3-generators of commuting matrices, we can hope that (12) even holds for all \( S \) whose maximal ideals satisfy \( \text{length}(m/m^2) \leq 3 \).

(There is a slight difficulty with regarding (12) as a generalization of the property of Theorem 8. When \( S = R[t] \), Theorem 3 concerns \( S \)-modules of finite length over \( R \), while (12) concerns \( S \)-modules of finite length over \( S \), and these are not always the same. For instance, if \( R \) is a discrete valuation ring with maximal ideal \( (p) \), then the \( R[t] \)-module \( R[t]/(pt - 1) \) has length 1 as an \( R[t] \)-module, since the ring \( R[t]/(pt - 1) \) is a field, but has infinite length as an \( R \)-module. Since “length over \( R \)” has no meaning for a module over a ring \( S \) that is not assumed to be built from a subring \( R \), we shall take condition (12) as our focus from here on.)

Note that a commutative ring \( S \) satisfies (12) if and only if all of its finite-length homomorphic images \( A \) do; equivalently, if and only if all those images have the stated property for faithful \( A \)-modules \( M \). Now for a ring, being of finite length is equivalent to being Artinian, and every commutative Artinian ring is a finite direct product of local rings. This leads to the modified question,

**Question 5.** For which commutative Artinian local rings \( A \) does the statement

\[
(13) \quad \text{Every faithful \( A \)-module } M \text{ satisfies } \text{length}_A(M) \geq \text{length}_A(A)
\]

hold?

We can get further mileage on these questions by combining Theorem 8 with some theorems of I.S. Cohen [3]. (Note to the reader of that paper: a “local ring” there means what is now called a Noetherian local ring. Since the local rings we apply Cohen’s results to will be Artinian, this will be no problem to us. Incidentally, Cohen defines a generalized local ring to mean what we would call a (not necessarily Noetherian) local ring whose maximal ideal \( m \) is finitely generated and satisfies \( \bigcap m^n = \{0\} \), and he comments that he does not know whether every such ring is “local”, i.e., is also Noetherian. This has been answered in the negative [7].)
Recall that a local ring $A$ with maximal ideal $m$ is said to be equicharacteristic if the characteristics of $A$ and $A/m$ are the same. This is equivalent to saying that $A$ contains a field. (The implication from “contains a field” to “equicharacteristic” is clear. Conversely, note that since $A/m$ is a field, its characteristic is 0 or a prime number $p$. In the former case, every member of $\mathbb{Z} - \{0\}$ is invertible in $A/m$, and hence in $A$, so $A$ contains the field $\mathbb{Q}$; while in the latter, if $A$ is equicharacteristic, then, like $A/m$, it has characteristic $p$, and so contains the field $\mathbb{Z}/(p)$.)

Cohen shows in [3, Theorem 9, p. 72] that a complete Noetherian local ring which is equicharacteristic is a homomorphic image of the ring of formal power series in length($m/m^2$) indeterminates over a field, where $m$ is the maximal ideal of the ring. Using this, we can get

\begin{proposition}
Suppose $A$ is a commutative local Artinian ring with maximal ideal $m$, and that length($m/m^2$) $\leq 2$. Then if $A$ is equicharacteristic, it satisfies (13).

Hence, if $S$ is a commutative ring such that every maximal ideal $m \subseteq S$ satisfies length($m/m^2$) $\leq 2$, and $S$ contains a field, then $S$ satisfies (12).
\end{proposition}

\begin{proof}
We shall prove the first assertion. Clearly, the second will then follow by applying the first to local factor-rings of $S$.

Since the local ring $A$ is Artinian, it is complete, so by the result of Cohen’s cited, it is a homomorphic image of a formal power series ring in $\leq 2$ indeterminates over a field. But a finite-length homomorphic image of a formal power series ring is an image of the corresponding polynomial ring. Hence by the result of Gerstenhaber with which we started, $A$ satisfies (13).
\end{proof}

Cohen’s result for mixed characteristic is [3, Theorem 12, p. 84]. The case we shall use, that of the last sentence of that theorem, says that if $A$ is a complete Noetherian local ring whose residue field $A/m$ has characteristic $p$ (i.e., such that $p \in m$), but such that $p \not\in m^2$, then $A$ can be written as a homomorphic image of a formal power series ring in length($m/m^2$) $- 1$ indeterminates over a complete discrete valuation ring $V$ in which $p$ has valuation 1. (Intuitively, $p$ takes the place of one of the indeterminates in the result for the equicharacteristic case.) This gives us

\begin{proposition}
Again let $A$ be a commutative local Artinian ring with maximal ideal $m$, such that length($m/m^2$) $\leq 2$. If $p \in m$ but $p \not\in m^2$, then $A$ satisfies (14).

Hence, if $S$ is a commutative ring such that every maximal ideal $m \subseteq S$ satisfies length($m/m^2$) $\leq 2$, and such that no prime $p \in \mathbb{Z}$ belongs to the square of any maximal ideal of $S$, then $S$ satisfies (12).
\end{proposition}

\begin{proof}
We will prove the first assertion. The second will then follow by applying that assertion to local factor rings whose residue fields have prime characteristic, while applying the first assertion of Proposition 6 to local factor rings whose residue fields have characteristic zero.

In the situation of the first assertion, the result of Cohen cited, again combined with the observation that a finite-length homomorphic image of a formal power series ring is a homomorphic image of the corresponding polynomial ring, tells us that $A$ is a homomorphic image of a polynomial ring in at most one indeterminate over a discrete valuation ring $V$. Hence by Theorem (3) above, $A$ satisfies (13).
\end{proof}

If, in the mixed-characteristic case, we instead have $p \in m^2$, Cohen’s result only tells us that $A$ is a homomorphic image of a formal power series ring in length($m/m^2$) (rather than length($m/m^2$) $- 1$) indeterminates over a complete discrete valuation ring $V$; so in our case, $A$ is a homomorphic image of $V[[s, t]]$. In general, this is not enough to give us the conclusion we want, but there are cases where it is. Let $d$ be the integer such that $p \in m^d - m^{d+1}$, and suppose that

(14) $p$ has a $d$-th root $q$ in $A$.

Then this $d$-th root $q$ will lie in $m - m^2$, so via a change of variables, the indeterminate $s$ in $V[[s, t]]$ may be taken to be an element that maps to $q \in A$. Thus, $A$ is a homomorphic image of $V[[s, t]]/(s^d - p)$; so using, as before, the fact that $A$ has finite length, we see that $A$ is in fact a homomorphic image of $V[[s, t]]/(s^d - p)$. But $V[s]/(s^d - p)$ is a discrete valuation ring $V' \supseteq V$, so $A$ is a homomorphic image of $V'[t]$, and we can again conclude from Theorem (3) that it satisfies (13).

Can we generalize this further? It might seem harmless to weaken (14) to say that some associate of $p$ in $A$ has a $d$-th root $q \in A$. But then the problem arises of where the unit of $A$ that carries $p$ to $q^d$ lies. If it does not belong to the image of $V$, we can’t use it in constructing our extension $V'$. We might hope to incorporate the condition that that unit lie in the image of $V$ into a generalization of condition (14), but
a version of Proposition 4 based on such a condition would be awkward to formulate, since the V given by Cohen’s result is not part of the hypothesis of Proposition 7. One assumption that will clearly guarantee that we can argue as suggested is that the unit in question lie in the image of \( Z \) in \( A \). I will not try here to find the “best” result of this sort.

If we don’t assume any condition like (14), there are examples where \( A \) indeed cannot be generated by one element over a homomorphic image of a discrete valuation ring. For instance, let \( p \) be any prime, and within \( Z[p^{1/5}] \), let us take the subring \( Z[p^{2/5}, p^{3/5}] \) and divide out by the ideal \( (p^2) \), writing

\[
A = Z[p^{2/5}, p^{3/5}] / (p^2).
\]

We see that \( A \) is local and Artinian, with maximal ideal \( m \) generated by \( \{ p^{2/5}, p^{3/5}, p \} \); and since the last of these elements is the product of the first two, \( m \) is in fact 2-generated, and \( m/m^2 \) can be seen to have length 2. But I claim that \( A \) is not 1-generated over a homomorphic image \( B \) of a valuation ring \( V \). Roughly speaking, if it were, then that subring \( B \subseteq A \) would either have to have the property that all its elements are associates of powers of \( p^{2/5} \), or that they are associates of powers of \( p^{3/5} \); but \( p \in B \) cannot be either.

Nevertheless, I would be surprised if the ring (15) did not satisfy (13). Any way I can think of to construct a candidate counterexample could be duplicated over \( k[s^2, s^3]/(s^{10}) \) for \( k \) a field, though we know that no counterexample exists in that case by Gerstenhaber’s original result.

We can in fact show that for all but finitely many primes \( p \), the ring (15) does satisfy (13). For suppose we had counterexamples for an infinite set \( P \) of primes. Let us write \( A_p \) \((p \in P)\) for the corresponding rings (15), and choose for each \( p \in P \) an \( A_p \)-module \( M_p \) witnessing the failure of (13). Now let \( A \) be an ultraproduct of the \( A_p \) with respect to a nonprincipal ultrafilter on \( P \), and \( M \) the corresponding ultraproduct of the \( M_p \), an \( A \)-module. From the fact that the \( A_p \) all have the same length (namely 10), one can verify that \( A \) will also have that length, hence be Artinian, and from the fact that length \( m/m^2 = 2 \) for all \( A_p \), one finds that the same is true for \( A \). Moreover, the characteristic of \( A/m \) will be 0, because every prime integer is invertible in all but one of the \( A_p/m_p \); hence \( A \) is necessarily equicharacteristic. The ultraproduct \( M \) will be a faithful \( A \)-module, and since by assumption all the \( M_p \) have lengths less than the common length of the \( A_p \), the module \( M \) will also have length less than that common value. Hence \( M \) witnesses the failure of (13) for \( A \), contradicting Proposition 6 so there cannot be such an infinite set \( P \) of primes.

We see that the above method of reasoning in fact gives

**Proposition 8.** For every positive integer \( n \), there are at most finitely many primes \( p \) for which there exist commutative local Artinian rings \( A \) of length \( n \) and characteristic a power of \( p \) that satisfy length \( (m/m^2) \leq 2 \), but fail to satisfy (13). \( \square \)

Above, we have, brevity, been focusing on the more challenging aspects of our problem. One can also formally extend our results in more trivial ways. For instance, using the case of Cohen’s [3] Theorem 12, p. 84] that does not make the assumption \( p \not\in m^2 \) (quoted following Proposition 7, we see that any \( A \) having length \( (m/m^2) \leq 1 \) satisfies (13), with no need for a condition on the behavior of integer primes \( p \). Also, one can easily extend the final assertion of Proposition 6 to a commutative ring \( S \) which, rather than containing a field, contains a direct product of fields, or more generally, a von Neumann regular subring. Still more generally, using the first statements of both those propositions, we can extend the second statements thereof to rings \( S \) such that for every maximal ideal \( m \subseteq S \) and prime \( p \in Z \), either \( p \not\in m^2 \) or \( p \in \bigcap_m m^n \).

Let us now turn to rings \( S \) and \( A \) for which we can show that (12) or (13) does not hold. The first assertion of the next result generalizes our observations on the algebra described in (2) and (3). (This will be clearer from the proof than from the statement.)

**Proposition 9.** Suppose \( A \) is a commutative local Artinian ring, with maximal ideal \( m \). If \( A \) has ideals \( I_1 \) and \( I_2 \) with zero intersection, such that \( A/I_1 \) and \( A/I_2 \) have isomorphic submodules \( J_1/I_1 \cong J_2/I_2 \) satisfying

\[
(16) \quad \text{length}(A/I_1) + \text{length}(A/I_2) - \text{length}(J_1/I_1) < \text{length}(A),
\]

(equivalently, \( \text{length}(A) < \text{length}(J_1) + \text{length}(I_2) \)), then \( A \) does not satisfy (13).

In particular, this is the case if \( A \) is any commutative local Artinian ring satisfying \( m^2 = \{0\} \) and \( \text{length}(m) \geq 4 \).
Hence, no commutative ring $S$ having a maximal ideal $m$ with $\text{length}(m/m^2) \geq 4$ satisfies (12). (In this last statement, we do not require $\text{length}(m/m^2)$ to be finite.)

Proof. In the situation of the first paragraph, let $M$ be the module obtained from $A/I_1 \oplus A/I_2$ by identifying the isomorphic submodules $J_1/I_1 \cong A/I_1$ and $J_2/I_2 \cong A/I_2$. Each of $A/I_1$ and $A/I_2$ still embeds in $M$, so since the annihilators $I_1$ and $I_2$ of these modules have zero intersection, $M$ is a faithful $A$-module.

Proof. As stated in (10), hence that inequality shows the failure of (13). The parenthetical statement of equivalence on the line after (16) is seen by expanding the expressions of the form “$\text{length}(P/Q)$” in (16) as $\text{length}(P) - \text{length}(Q)$, and simplifying.

(In the example described in (2) and (3), we can take $I_1 = \text{Ann}_A(x) = (c, d)$, $I_2 = \text{Ann}_A(y) = (a^m, a^{-1}b, \ldots, b^n)$, $J_1 = \{ f \in A \mid fx \in Acy + Ady \} = I_1 + (a^m, b^n)$, and $J_2 = \{ f \in A \mid fy \in Acy + Ady \} = I_2 + (c, d)$.)

To get the assertion of the second paragraph, let $\text{length}(m) = d$, so that $m$ can be regarded as a $d$-dimensional vector space over $A/m$. Let $I_1$ and $I_2$ be any subspaces of $m$ of equal dimension $e \geq 2$, and having zero intersection (these exist because $d \geq 4$), and let $J_1 = J_2 = m$. By comparison of dimensions, $J_1/I_1 \cong J_2/I_2$ as $A/m$-modules, and hence as $A$-modules. Now $\text{length}(J_1) + \text{length}(I_2) = d + e > d + 1 = \text{length}(A)$, giving the inequality noted parenthetically as equivalent to (16).

For $S$ and $m$ as in the final statement, let $A_0$ be the local ring $S/m^2$, with square-zero maximal ideal $m_0 = m/m^2$. Since $A_0$ need not have finite length, let us divide out by an $A_0/m_0$-subspace of $m_0$ whose codimension is finite but $\geq 4$. The result is a homomorphic image $A$ of $S$ which has finite length and, by the second assertion of the lemma, fails to satisfy (13). Hence $S$ fails to satisfy (12).

If it should turn out that (13) holds for every $A$ with $\text{length}(m/m^2) \leq 3$, we would have a complete answer to Question 3 for comparing that fact with Proposition 9. We could conclude that the rings $S$ satisfying (12) are precisely those for which all maximal ideals $m$ satisfy $\text{length}(m/m^2) \leq 3$.

From the proof of Proposition 9 we can see that the existence of ideals satisfying (10) is necessary and sufficient for the existence of a $2$-generator $A$-module $M$ witnessing the failure of (13). For higher numbers of generators, it seems hard to formulate similar necessary and sufficient conditions; though one can give sufficient conditions, corresponding to necessary and sufficient conditions for the existence of such modules with particular sorts of structures (e.g., sums of three cyclic submodules, each pair of which is glued together along a pair of isomorphic submodules), and these might be useful in looking for examples.

We have seen that the algebras described in (3) are cases of Proposition 9. For a further example, suppose we adjoin to $\mathbb{Z}$ the 7-th root of a prime $p$, and then pass to the subring

\[(17) \quad S = \mathbb{Z}[p^{5/7}, p^{5/7}, p^{5/7}].\]

This has a maximal ideal $m$ generated by $\{ p^{5/7}, p^{5/7}, p^{5/7}, p \}$, and these generators are linearly independent modulo $m^2$, so by Proposition 9 $S$ does not satisfy (12).

The next result will give us a further class of rings $A$ that do satisfy (13). However, this class is not closed under homomorphic images, and can fail to satisfy (12). Thus, though the result will add to what we know regarding Question 5, it says little about Question 4, which inspired that question.

We recall that a commutative local Artinian ring $A$ is said to be Frobenius if it is cocyclic as an $A$-module, i.e., if its socle has length 1. (For an Artinian but not-necessarily-commutative, not-necessarily-local ring, the Frobenius condition is the statement that the socle is isomorphic as right and as left module to $A/J(A)$ (11. Theorem 16.14(4)).)

Lemma 10. Every Frobenius commutative local Artinian ring $A$ satisfies (13).

Proof. If $M$ is a faithful $A$-module, then $M$ has an element $x$ not annihilated by $\text{socle}(A)$. Since $\text{socle}(A)$ is simple, the annihilator of $x$ has trivial intersection with that socle, hence is zero. So $Ax$ is a faithful cyclic $A$-module, hence has length equal to the length of $A$, so length $M \geq \text{length} A$.

For an example of a ring as in the above lemma which has $\text{length}(m/m^2) \geq 4$, and therefore, though we have just seen that it satisfies (13), will not satisfy (12), let $k$ be a field, take any $n_1, \ldots, n_4 > 0$, and let $A = k[t_1, t_2, t_3, t_4]/(t_1^{n_1+1}, t_2^{n_2+1}, t_3^{n_3+1}, t_4^{n_4+1})$. Then the socle of $A$ is the 1-dimensional space spanned by the element $t_1 t_2 t_3 t_4$, so $A$ is Frobenius, but $m/m^2$ is 4-dimensional, with basis $t_1, t_2, t_3, t_4$.

Let us also note, in contrast with the above lemma, that a large socle does not prevent a ring from satisfying (13). For instance, for $k$ a field and $n$ any positive integer, the algebra $A = k[s, t]/(s^n, s^{n-1} t, \ldots, s t^{n-1}, t^n)$ has socle of length 2, with basis $\{ s^{n-1}, s^{n-2} t, \ldots, t^{n-2} \}$, but by Proposition 6 $A$ satisfies (13).
Question 11. Does every commutative local Artinian ring \( A \) whose socle has length \( \leq 2 \) (or even \( \leq 3 \)) satisfy \((13)\)?

In an appendix, \( [8] \) we shall obtain some results on modules over a not necessarily commutative ring \( A \), which, for \( A \) commutative Artinian, generalize Lemma \([10]\) to show that if \( A \) has socle of length \( n \) and does not satisfy \((13)\), then any minimal-length \( A \)-module \( M \) witnessing this failure must be generated by \( \leq n \) elements, and, dually, must have socle of length \( \leq n \).

We know from the module diagrammed in \([6]\) that an \( M \) generated by 2 elements and also having socle(\( M \)) of length 2 can witness the failure of \((13)\). But note that in that example, socle(\( A \)) has length 4, and the construction "economizes", using few vertices at the top and bottom to host a large number of edges representing elements of socle(\( A \)) in between. It seems likely that this is an instance of some general properties of modules witnessing the failure of \((13)\). If so, it may be possible to strengthen, for such modules, the bounds just mentioned, as suggested in

Question 12. Let \( A \) be a commutative local Artinian ring, with socle of length \( n \), and maximal ideal \( m \).

If \( A \) does not satisfy \((13)\), and \( M \) is an \( A \)-module witnessing this fact, must length(\( M/mM \)) and/or length(socle(\( M \))) be \( \leq n-1 \)?

In the above situation, and perhaps, more generally, if \( M \) is a faithful \( A \)-module satisfying length(\( M \)) \( \leq \) length(\( A \)), must length(\( M/mM \)) + length(socle(\( M \))) \( \leq n+1 \)?

The final part of the above question is suggested by the observation that faithful cyclic modules, faithful cocyclic modules, and all the modules described in \([3]\) and \([4]\) satisfy the stated inequality. A positive answer to either part of the question would immediately give a positive answer to the “length \( \leq 2 \)” case of Question \([11]\).

The final result of this section concerns rings \( A \) that are very small. We first note

Lemma 13. Any module \( M \) of length \( \leq 3 \) over a (not necessarily commutative or Artinian) ring \( A \) is either a direct sum of cyclic modules, or a direct sum of cocyclic modules.

Proof. If \( M \) is not itself cocyclic, this means length(socle(\( M \))) \( \geq 2 \), and dually, if \( M \) is not cyclic, then length(\( M/mM \)) \( \geq 2 \), which, subtracting from 3, gives length(\( mM \)) \( \leq 1 < \) length(socle(\( M \))). Hence socle(\( M \)) must have a simple submodule \( N \) not contained in \( mM \). If we take the inverse image \( L \subseteq M \) of a complement of the image of \( N \) in the semisimple module \( M/mM \), this will still not contain \( N \), and since \( N \) has length 1, we see that \( M = N \oplus L \). Now length(\( L \)) \( \leq 2 \), and it is easy to see that a module of that length is either simultaneously cyclic and cocyclic, or a direct sum of two simple submodules, in either case giving the desired decomposition of \( M \). \( \square \)

Corollary 14. Suppose \( A \) is a commutative local Artinian ring which has length \( \leq 4 \) (equivalently, whose maximal ideal \( m \) has length \( \leq 3 \)). Then \( A \) satisfies \((13)\).

Proof. Any \( M \) witnessing the failure of \((13)\) would have length < length(\( A \)), so by the above lemma, it would be a direct sum of cyclic or of cocyclic modules. We saw in \([4]\) that a direct sum of two cyclic or cocyclic modules cannot give a counterexample to \((13)\). One can generalize this result to any number of cyclic or cocyclic modules (using repeatedly the observation length(\( A/I \)) + length(\( A/J \)) > length(\( A/I \cap J \))), giving the desired result.

\( \square \)

So, for instance, this result applies to the ring of \([8]\) (essentially, the ring generated by matrix units \( e_{13}, e_{14}, e_{24} \)). Another example can be gotten by taking the ring \( \mathbb{Z}[[p^{2/5}, p^{3/5}] / (p^2) \), for which I noted above that I have not been able to prove \((13)\), and truncating it a bit further, to \( \mathbb{Z}[[p^{2/5}, p^{3/5}] / (p^{4/5}, p^{6/5}) \). This has maximal ideal of length 3, spanned modulo its square by \( \{ p^{2/5}, p^{3/5} \} \), and with square spanned by \( p = p^{2/5} p^{3/5} \). So this truncation does satisfy \((13)\). (We remark that, the explicit rational powers of \( p \) in our description of this ring are illusory; it could equally well be written \( \mathbb{Z} / (p^2) [s, t] / (s^2, t^2, st - p) \).

6. Other sorts of questions

Though the focus of this note has been the condition length \( A \leq \) length \( M \), one can ask, more generally, how big length \( A/length \ M \) can become in cases where it may exceed 1. To maximize the hope of positive results, I will pose the question here for algebras of endomorphisms of vector spaces.
Question 15. For each positive integer \( d \), let \( r_d \) be the supremum of the ratio \( \dim_k A/\dim_k V \), over all commutative \( d \)-generator algebras \( A \) of endomorphisms of nonzero finite-dimensional vector spaces \( V \) over arbitrary fields \( k \).

(Thus, the \( r_d \) form a nondecreasing sequence, whose terms are real numbers or \( +\infty \). We know that \( r_1 = r_2 = 1 \), and the examples of [1] and [3] respectively suggest that \( r_3 \) may be 1, and \( r_4 \) may be \( 5/4 \).)

Determine as much as possible about this sequence. In particular, are all its terms finite?

I don’t even see how to prove \( r_3 \) finite! (If we did not restrict ourselves to commuting endomorphisms, these suprema would become infinite for \( d = 2 \), since the full \( n \times n \) matrix algebra can be generated by two matrices, and the ratio of its dimension to that of the space on which it acts, \( n^2/n = n \), is unbounded.)

Something we can say is that as a function of \( d \), the \( r_d \) increase without bound:

Lemma 16. If \( d, e_0, e_1 \) are positive integers such that \( d \geq e_0 e_1 \), then \( r_d \geq (e_0 e_1 + 1)/(e_0 + e_1) \).

Hence, taking \( e_0 = 2m - 1 \), \( e_1 = 2m + 1 \), we see that \( r_{4m^2 - 1} \geq m \).

Proof. Let \( V \) be the direct sum of an \( e_0 \)-dimensional vector space \( V_0 \) and an \( e_1 \)-dimensional space \( V_1 \), and \( A \) be the algebra of endomorphisms of \( V \) spanned by the identity, and all endomorphisms that carry \( V_0 \) into \( V_1 \) and annihilate \( V_1 \). Since any two endomorphisms of the latter sort have product 0, \( A \) is commutative.

It is generated as an algebra by any basis of the \( e_0 e_1 \)-dimensional space of such endomorphisms, hence a fortiori it can be generated by \( d \geq e_0 e_1 \) elements. Since \( A \) has dimension \( e_0 e_1 + 1 \) and \( V \) has dimension \( e_0 + e_1 \), we get \( r_d \geq (e_0 e_1 + 1)/(e_0 + e_1) \), as claimed.

The final sentence clearly follows.

In the spirit of [2] §3, we might expect that the inequalities we have obtained for algebras of endomorphisms of vector spaces would entail analogous inequalities for monoids of endomaps of sets, with cardinalities replacing dimensions. But this is not the case. For instance, a 1-generator group of permutations of an \( n \)-element set can have order much larger than \( n \), if the generating permutation has many cycles of relatively prime lengths. (The reason why results on algebras don’t imply the corresponding results for monoids is that the matrices corresponding to a family of distinct endomaps of a finite set need not be linearly independent.)

I will end by repeating, in slightly generalized form, a question I asked in [2], which resembles the subject considered here (and differs from the subject considered there) in that it asks whether the size of a certain family of actions is bounded by the size of the object it acts on; but which is otherwise only loosely related to the topic of either paper.

Question 17 (after [2] Question 23]). Let \( R \) be a commutative algebra over a commutative ring \( k \), let \( V \) be a \( k \)-submodule of \( R \), and let \( n \) be a positive integer such that the \( k \)-submodule \( V^n \subseteq R \) of all sums of \( n \)-fold products of elements of \( V \) has finite length as a \( k \)-module. Then must

\[
\text{length}_k (V/\text{Ann}_V (V^n)) \leq \text{length}_k (V^n),
\]

where \( \text{Ann}_V (V^n) \) denotes \( \{ x \in V \mid x V^n = \{0\} \} \subseteq V \).

7. Acknowledgements

I am indebted to Arthur Ogus for asking whether two commuting \( n \times n \) matrices generate an algebra of dimension \( \leq n \) (when neither of us was aware that this was a known result), and for a suggestion he made in the ensuing discussion, which turned into the proof of Corollary [1] and to Kevin O’Meara for much helpful correspondence on these matters.

8. Appendix: Some submodules and factor modules

This section, except for the final corollary, can be read independently of the rest of this note. Rings are not here assumed commutative (but are still associative and unital), and “module” means left module. The Jacobson radical of a ring \( A \) is denoted \( J(A) \).

Statement (ii) in each of the next two results describes how certain modules can be decomposed into fairly “small” modules: in the first case, as a sum of submodules \( N \) such that \( N/J(A)N \) is simple; in the second, as a subdirect product of modules \( L \) with \( \text{socle}(L) \) simple.

Lemma 18. Let \( A \) be an Artinian ring, and \( M \) an \( A \)-module (not necessarily Artinian). Then

(i) If \( S \) is a simple submodule of \( M/J(A)M \), then \( M \) has a submodule \( N \) such that the inclusion \( N \subseteq M \) induces an isomorphism \( N/J(A)N \cong S \subseteq M/J(A)M \).
Hence

(ii) Given a decomposition of $M/J(A)M$ as a sum of simple modules $\sum_{i \in I} S_i$, one can write $M$ as the sum of a family of submodules $N_i$ (for $i \in I$), such that for each $i$, $N_i/J(A)N_i \cong S_i$, and $S_i$ is the image of $N_i$ in $M/J(A)M$.

Proof. In the situation of (i), let $x$ be any element of $M$ whose image in $M/J(A)M$ is a nonzero member of $S$. Thus, the image of $Ax$ in $M/J(A)M$ is $S$. Since $A$ is Artinian, $Ax$ has finite length, hence we can find a submodule $N \subseteq Ax$ minimal for having $S$ as its image in $M/J(A)M$. Now since $N/J(A)N$ is semisimple, it has a submodule $S'$ which maps isomorphically to $S$ in $M/J(A)M$. If $S'$ were a proper submodule of $N/J(A)N$, then its inverse image in $N$ would be a proper submodule of $N$ which still mapped surjectively to $S$, contradicting the minimality of $N$. Hence $N/J(A)N = S' \cong S$, completing the proof of (ii).

In the situation of (ii), choose for each $S_i$ a submodule $N_i \subseteq M$ as in (i). Then we see that $M = J(A)M + \sum_i N_i$, hence since $A$ is Artinian, $M = \sum_i N_i$ [Theorem 23.16(1) $\Rightarrow$ (2')], as required. □

The next result is of a dual sort, but the arguments can be carried out in a much more general context, so that the result we are aiming for (the final sentence) looks like an afterthought. Note that submodules called $S$ etc. are not here assumed simple.

Lemma 19. Let $A$ be a ring and $M$ an $A$-module. Then

(i) If $S$ is any submodule of $M$, then $M$ has a homomorphic image $M/N$ such that the composite map $S \rightarrow M \rightarrow M/N$ is an embedding, and the embedded image of $S$ is essential in $M/N$.

Hence

(ii) If $E$ is an essential submodule of $M$, and $f : E \rightarrow \prod_i S_i$ a subdirect decomposition of $E$, then there exists a subdirect decomposition $g : M \rightarrow \prod_i M_i$ of $M$, such that each $M_i$ is an overmodule of $S_i$ in which $S_i$ is essential, and $f$ is the restriction of $g$ to $E \subseteq M$.

In particular, every locally Artinian module can be written as a subdirect product of locally Artinian modules with simple socles.

Proof. In the situation of (i), let $N$ be maximal among submodules of $M$ having trivial intersection with $S$. The triviality of this intersection means that $S$ embeds in $M/N$, while the maximality condition makes the image of $S$ essential therein. (Indeed, if it were not essential, $M/N$ would have a nonzero submodule $T$ disjoint from the image of $S$, and the inverse image of $T$ in $M$ would contradict the maximality of $N$.)

In the situation of (ii), for each $j \in I$ let $K_j$ be the kernel of the composite $E \rightarrow \prod_j S_j \rightarrow S_j$. Applying statement (i) with $M/K_j$ in the role of $M$, and $E/K_j \cong S_j$ in the role of $S$, we get an image $M_j$ of $M/K_j$, and hence of $M$, in which $S_j$ is embedded and is essential. Now since $E$ is essential in $M$, every nonzero submodule $T \subseteq M$ has nonzero intersection with $E$, and that intersection has nonzero projection to $S_i$ for some $i$; so in particular, for that $i$, $T$ has nonzero image in $M_i$. Since this is true for every $T$, the map $M \rightarrow \prod_i M_i$ is one-to-one, and gives the desired subdirect decomposition.

The final assertion follows from the fact that the socle of a locally Artinian module is essential, and, being semisimple, can be written as a subdirect product (indeed, as a direct sum) of simple modules. □

We can now get the following result, showing that given a faithful module $M$ over an Artinian ring, we can carve out of $M$ a “small” faithful submodule, factor-module, or subfactor. Note that in the statement, though length has its usual meaning for modules, the length of socle($A$) as a bimodule may be less than its length as a left (or right) module.

Proposition 20. Let $A$ be an Artinian ring, let $n$ be the length of socle($A$) as a bimodule (equivalently, as a 2-sided ideal), and let $M$ be a faithful $A$-module. Then

(i) $M$ has a submodule $M'$ which is again faithful over $A$, and satisfies $\text{length}(M'/J(A)M') \leq n$. (In particular, $M'$ is generated by $n$ elements.)

(ii) $M$ has a homomorphic image $M''$ which is faithful over $A$, and satisfies $\text{length}(\text{socle}(M'')) \leq n$.

(iii) $M$ has a subfactor faithful over $A$ which has both these properties.

Proof. To get (i), write $M/J(A)M$ as a direct sum of simple $A$-modules $S_i$ (for $i \in I$), and take a generating family of submodules $N_i \subseteq M$ as in Lemma 18(ii). Since $M = \sum_i N_i$ is faithful, and socle($A$) has length $\leq n$ as a 2-sided ideal, the sum of some family of $\leq n$ of these submodules, say

\[ M' = N_{i_1} + \cdots + N_{i_m} \quad \text{where} \quad m \leq n, \]
must have the property that $M'$ is annihilated by no nonzero subideal of socle$(A)$. (Details: one chooses the $N_{ij}$ recursively. As long as $N_{ii} + \cdots + N_{ij}$ is annihilated by a nonzero subideal $I \subseteq$ socle$(A)$, one can choose an $N_{ij+1}$ which fails to be annihilated by some member of $I$. Thus, the annihilators in socle$(A)$ of successive sums $M' = N_{ii} + \cdots + N_{ij}$ ($j = 0, 1, \ldots$) form a strictly decreasing chain. By our assumption on the length of socle$(A)$, this chain must terminate after $\leq n$ steps with a sum $M'$ annihilated by no nonzero elements of socle$(A)$. But an ideal of an Artinian ring having zero intersection with the socle is zero, so $M'$ has zero annihilator, i.e., is faithful. Since each $N_i$ satisfies length$(N_i/J(A)N_i) = 1$, we have length$(M'/J(A)M') \leq m \leq n$.

Statement (ii) is proved in the analogous way from the final statement of Lemma 19 using images of $M$ in products of finite subfamilies of the $M_i$, in place of submodules of $M$ generated by finite subfamilies of the $N_i$.

Statement (iii) follows by combining (i) and (ii).

Returning to the ideas of the body of this note, we deduce from Proposition 20(iii) the following fact which was mentioned before Question 12.

**Corollary 21.** If a commutative local Artinian ring $A$ having socle of length $n$ fails to satisfy $L^3$, then any $M$ of minimal length among $A$-modules witnessing this failure is generated by $\leq n$ elements and has socle of length $\leq n$.

**Proof.** By the minimal-length assumption on $M$, no proper subfactor of $M$ can be faithful, hence the subfactor given by Proposition 20(iii) must be $M$ itself.

**References**

[1] José Barría and P. R. Halmos, *Vector bases for two commuting matrices*, Linear and Multilinear Algebra, 27 (1990) 147–157. MR 1064891 (91h:16053a).

[2] George M. Bergman, *Thoughts on Eggert's Conjecture*, 12 pp., to appear, readable at [http://math.berkeley.edu/~gbergman/papers/](http://math.berkeley.edu/~gbergman/papers/).

[3] I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc., 59 (1946) 54–106. MR 0016094 (7,590b).

[4] Murray Gerstenhaber, *On dominance and varieties of commuting matrices*, Annals of Mathematics, (2) 73 (1961) 324–348. [http://www.jstor.org/stable/1970336](http://www.jstor.org/stable/1970336). MR 0132079 (24#A1926).

[5] Robert M. Guralnick, *A note on commuting pairs of matrices*, Linear and Multilinear Algebra, 31 (1992) 71–75. MR 1199042 (94c:15021).

[6] Robert M. Guralnick and B. A. Sethuraman, *Commuting pairs and triples of matrices and related varieties*, Linear Algebra Appl., 310 (2000) 139–148. MR 1753173 (2001e:15015).

[7] William Heinzer and Moshe Roitman, *Generalized local rings and finite generation of powers of ideals*, pp. 287–312 in *Non-Noetherian commutative ring theory*, Math. Appl., 520, Kluwer, 2000. MR 1858167 (2002h:13036).

[8] J. Holbrook and K. C. O'Meara, *Some thoughts on Gerstenhaber’s Theorem*, in preparation (17 pp. as of June 2013).

[9] Thomas J. Laffey and Susan Lazarus, *Two-generated commutative matrix subalgebras*, Linear Algebra Appl., 147 (1991) 249–273. MR 1088666 (91m:15022).

[10] T. Y. Lam, *A first course in noncommutative rings*, Springer GTM, v.131, 1991. MR 92f:16001.

[11] T. Y. Lam, *Lectures on modules and rings*, Springer GTM, v.189, 1999. MR 1653294 (99i:16001).

[12] Kevin C. O'Meara, John Clark and Charles I. Vinsonhaler *Advanced topics in linear algebra. Weaving matrix problems through the Weyr form*, Oxford University Press, 2011. xxii + 400 pp. MR 2849857 (2012e:15003).

[13] Serge Lang, *Algebra*, Addison-Wesley, third edition, 1993, reprinted as Springer GTM, v.211, 2002. MR 1878556 (2003e:00003).

[14] T. S. Motzkin and Olga Taussky, *Pairs of matrices with property L, II*, Trans. Amer. Math. Soc., 80 (1955) 387–401. MR 0086781 (19,242c).

[15] Michael G. Neubauer and David J. Saltman, *Two-generated commutative subalgebras of $\mathbb{M}_n(F)$*, J. Algebra, 164 (1994) 545–562. MR 1271255 (95g:16039).

[16] B. A. Sethuraman and Klemen Sivic, *Jet schemes of the commuting matrix pairs scheme*, Proc. Amer. Math. Soc., 137 (2009) 3953–3967. MR 2538555 (2011b:14117).

[17] Adrian R. Wadsworth, *The algebra generated by two commuting matrices*, Linear and Multilinear Algebra, 27 (1990) 159–162. MR 1064892 (91h:16053b).

**University of California, Berkeley, CA 94720-3840, USA**

**E-mail address:** gbergman@math.berkeley.edu