Tilted Cone and Cylinder, Cone and Tilted Sphere

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July 16, 2021

Abstract

In this note, we will consider two classical volume problems related to elliptic integrals. The first problem has a neat formula by means of elliptic integrals. We remade it with details. In the second problem, we found a messy formula. On the other hand, it seems to be useful to find a good approximation for the volume.

Key Words. Cone, cylinder, sphere, elliptic, integral.

Mathematics Subject Classification. [2020] 51M25, 33E05.

1 Introduction

In this note, I discuss two classical volume problems. The first problem which, I saw in [2], dates back to 1932, has a neat solution formula by means of elliptic integrals. I reproduced the formula for the case \( k < 1 \) with some details for elliptic integrals. There is a key identity which also appeared in the second problem but is not available in [2]. I believe that Rhodes did this computations somewhere else. His purpose in this article was Landen transformations but I believe that they are sometimes complications.

The solution for the second problem I found is not very neat. I used Maclaurin’s series expansions of elliptic integrals of the first kind and the second kind and wrote the solution as an infinite series of trigonometric integrals. It seems to be useful to find a good approximation for the volume. I do not know whether this formula was known before. I have not seen.

I must also mention the beautiful book of Harris Hancock, [1], which helped me to understand the tricky identities about elliptic integrals.

WolframAlpha helped me a lot during my research. Its abilities are amazing.
2 Tilted cone and cylinder

Consider the cylinder $x^2 + y^2 = 1$ and the cone $z = \cot \alpha \sqrt{(x-k)^2 + y^2}$, $0 \leq k \leq 1$. We want to find the volume of the bounded region inside the cylinder, under the cone and above $z = 0$. Here $\alpha$ is the fixed angle of the cone, the angle between the cone and its axis, $0 \leq \alpha \leq \frac{\pi}{2}$.

Let the origin be $O = (0, 0, 0)$, the vertex of the cone be $T = (k, 0, 0)$ and let $P = (\cos \theta, \sin \theta, 0)$, $0 \leq \theta \leq 2\pi$, be a point one the unit circle of the $xy$-plane. Let the angle between $TP$ and positive side of the $x$-axis be $\phi$, $0 \leq \phi \leq 2\pi$. See [2] for some figures about this problem. If $TP = R$ then by law of cosines, $R = \sqrt{1 - k^2 \sin^2 \phi - k \cos \phi}$. The perpendicular from $P$ to $xy$-plane cuts the cone with height $R \cot \alpha$. Therefore, the parameterization of the region in tilted cylindrical coordinates is $0 \leq r \leq R$, $0 \leq \phi \leq 2\pi$ and $0 \leq z \leq r \cot \alpha$. And in tilted coordinates volume differential is $dV = r dr d\phi dz$. With two successive integrations, the volume integral can be reduced to $V = \frac{2 \cot \alpha}{3} \int_0^\pi R^3 d\phi$.

Now, by putting $R^3$ and observing that

$$
\int_0^\pi (-k^3 \cos^3 \phi - 3k \cos \phi + 3k^3 \cos \phi \sin^2 \phi) d\phi = 0
$$

we have

$$
V = \frac{4 \cot \alpha}{3} \int_0^\pi (3k^2 + 1 - 4k^2 \sin^2 \phi) \sqrt{1 - k^2 \sin^2 \phi} d\phi.
$$

By the definition of the elliptic integral of the second kind $E(k)$, $V$ becomes

$$
V = \frac{4(3k^2 + 1) \cot \alpha}{3} E(k) - \frac{16k^2 \cot \alpha}{3} \int_0^\pi \sin^2 \phi \sqrt{1 - k^2 \sin^2 \phi} d\phi.
$$

Next, $E_2(k) = \int_0^\pi \sin^2 \phi \sqrt{1 - k^2 \sin^2 \phi} d\phi$ must be computed in terms of elliptic integrals. Let $\Delta = \sqrt{1 - k^2 \sin^2 \phi}$, $S = \sin \phi$ and $C = \cos \phi$. The tricky identity is

$$
S^2 \Delta = \frac{2k^2 - 1}{3k^2} \Delta + \frac{1 - k^2}{3k^2} \frac{1}{\Delta} + \left[-\frac{1}{3}(1 - 2S^2)\Delta \pm \frac{k^2}{3} S^2 C^2 \frac{1}{\Delta}\right].
$$
Integrating from 0 to $\frac{\pi}{2}$, since

$$\int_0^{\pi/2} \left[ -\frac{1}{3} (1 - 2S^2) \Delta + \frac{k^2}{3} S^2 C^2 \frac{1}{\Delta} \right] d\phi = \left[ -\frac{1}{3} S C \right]_0^{\pi/2} = 0,$$

we find

$$E_2(k) = \frac{2k^2 - 1}{3k^2} E(k) + \frac{1 - k^2}{3k^2} K(k) \quad (\ast)$$

and

$$V = \frac{4}{9} \cot \alpha \left[ (k^2 + 7) E(k) + 4(k^2 - 1) K(k) \right]$$

where $K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{\frac{1}{2}} d\phi$. This formula is obtained in [2]. There, formula for $k > 1$ case is also obtained and then they are combined with a Landen transformation interpretation.

### 3 Cone and tilted sphere

Let us find the volume of the bounded region between the tilted sphere $(x + k)^2 + y^2 + z^2 = 1$, $0 \leq k \leq 1$ and the cone $z = \cot \alpha \sqrt{x^2 + y^2}$.

The set-up of the volume integral is easier than that of the first problem. So, we can skip figures. The sphere in spherical coordinates is $\rho^2 + 2k \rho \cos \theta \sin^2 \phi + k^2 - 1 = 0$ and the cone is $\phi = \alpha$. Thus, the volume of the region

$$0 \leq \rho \leq - k \cos \theta \sin \phi + \sqrt{1 - k^2 + k^2 \cos^2 \theta \sin^2 \phi}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \alpha$$

is found as

$$V = \int_0^{2\pi} \int_0^\alpha \left\{ \frac{(k^3 \cos \theta \sin^2 \phi - k \cos \theta \sin^2 \phi - 4 k^3 \cos^3 \theta \sin^4 \phi)}{3 k^2 \cos^2 \theta \sin^3 \phi + \frac{1 - k^2}{3} \sin \phi} \sqrt{1 - k^2 + k^2 \cos^2 \theta \sin^2 \phi} \right\} d\phi d\theta.$$

Since

$$\int_0^{2\pi} (k^3 \cos \theta \sin^2 \phi - k \cos \theta \sin^2 \phi - 4 k^3 \cos^3 \theta \sin^4 \phi) d\theta = 0$$

and due to symmetry, we find
\[
V = \int_0^\alpha \int_0^\infty \left( \frac{16}{3} k^2 \cos^2 \theta \sin^3 \phi + \frac{4}{3} (1 - k^2) \sin \phi \right) \sqrt{1 - k^2 + k^2 \cos^2 \theta \sin^2 \phi} d\theta d\phi.
\]

Let us define
\[
K = \frac{k \sin \phi}{\sqrt{1 - k^2 \cos^2 \phi}}
\]
and thus,
\[
V = \frac{\alpha}{2} \int_0^\infty \left( \frac{16}{3} k^2 \sin^3 \phi + \frac{4}{3} (1 - k^2) \sin \phi \right) \sqrt{1 - k^2 \cos^2 \phi} E(K) d\phi
\]
\[
- \frac{\alpha}{2} \int_0^\infty \frac{16}{3} k^2 \sin^3 \phi \sqrt{1 - k^2 \cos^2 \phi} \left( \int_0^{\pi/2} \sin^2 \theta \sqrt{1 - K^2 \sin^2 \theta} d\theta \right) d\phi.
\]

By using the star identity, (⋆) of the first problem, it can be written as
\[
V = \frac{4}{9} \int_0^\alpha (8k^2 \sin^3 \phi + 7(1 - k^2) \sin \phi) \sqrt{1 - k^2 \cos^2 \phi} E(K) d\phi - \frac{16}{9} \int_0^\alpha \sin \phi \sqrt{1 - k^2 \cos^2 \phi} K(K) d\phi.
\]

We can now insert infinite series of \(E(K)\) and \(K(K)\) and do term by term integration to obtain a formula which involves trigonometric integrals.

Let us recall that \(E(K) = \frac{\pi}{2} \sum_{n=0}^\infty \frac{c_n}{1 - 2n} K^{2n}\) and \(K(K) = \frac{\pi}{2} \sum_{n=0}^\infty c_n K^{2n}\)

where \(c_n = \left( \frac{(2n)!}{2^{2n} (n!)^2} \right)^2\). Putting these and \(K\) in the last equation we obtain
\[
V = \frac{2\pi}{9} \sum_{n=0}^\infty \frac{c_n k^{2n}}{1 - 2n} \int_0^\alpha \frac{8k^2 \sin^{2n+3} \phi + (3 - 7k^2 + 8n) \sin^{2n+1} \phi}{(1 - k^2 \cos^2 \phi)^{n+1}} d\phi.
\]

The zeroth term of the series gives the following approximation of the volume for small \(k\):
\[
\frac{2\pi}{9} \left( 1 + 2k^2 \right) \sqrt{1 - k^2} - \cos \alpha (1 + 4k^2 - 2k^2 \cos \alpha) \sqrt{1 - k^2 \cos^2 \alpha} + (2 - 3k^2) \frac{\arcsin k - \arcsin(k \cos \alpha)}{k}\).
\]

This gives the exact value \(\frac{2\pi}{3} (1 - \cos \alpha)\) in the limit \(k \to 0\).
4 Acknowledgement

I thank Paul Bracken who introduced me with AGM and elliptic integrals.

References

[1] Hancock Harris, Elliptic integrals, John Wiley and Sons Inc., New York 1917.

[2] C. E. Rhodes, A Geometric Interpretation of Landen’s Transformation The American Mathematical Monthly Vol. 39, No. 10 (Dec., 1932), pp. 594-596.