SPECTRA OF STRUCTURED DIFFUSIVE POPULATION EQUATIONS WITH GENERALIZED WENTZELL-ROBIN BOUNDARY CONDITIONS AND RELATED TOPICS

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Abstract. This paper provides two different extensions of a previous joint work "Time asymptotics of structured populations with diffusion and dynamic boundary conditions; Discrete Cont Dyn Syst, Series B, 23 (10) (2018)" devoted to asynchronous exponential asymptotics for bounded and weakly compact reproduction operators. The first extension considers bounded non weakly compact reproduction operators while the second extension deals with unbounded kernel reproduction operators and needs, as a preliminary step, a new generation result.

1. Introduction. A model of structured populations with generalized Wentzell-Robin boundary conditions

\[
\begin{cases}
  u_t(s,t) + (\gamma(s)u(s,t))_s = (d(s)u_s(s,t))_s - \mu(s)u(s,t) + \int_0^m \beta(s,y)u(y,t)dy \\
  \left[(d(s)u_s(s,t))_s\right]_{s=0} - b_0u_s(0,t) + c_0u(0,t) = 0, \\
  \left[(d(s)u_s(s,t))_s\right]_{s=m} + b_mu_s(m,t) + c_mu(m,t) = 0
\end{cases}
\]

with \( m < +\infty \) and

\[ b_0 - \gamma(0) > 0, \quad b_m + \gamma(m) > 0 \]

(and initial conditions) was considered first in [5] and then in [10]. We refer the reader to the introductions of these two papers and to the references therein for more motivation and information on such structured population models. From a mathematical point of view, the main issues are the well-posedness of this evolution system and the understanding of its time asymptotics. The goal of the present paper is to extend [10] to much more general reproduction operators.

The general assumptions in [10] are

\[ \gamma, d \in W^{1,\infty}(0, m) \text{ and } \mu \in L^\infty(0, m), \]

\[ \beta_0, \beta_m \in L^\infty(0, m), \]

\[ \mu, \gamma' \text{ and } s \mapsto \beta(s,.) \text{ are continuous at } s = 0 \text{ and } s = m, \]

the reproduction operator

\[ \beta : L^1(0, m) \ni u \to \int_0^m \beta(.,y)u(y)dy \in L^1(0, m) \text{ is bounded} \]

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and positive,
\[ b_0, b_m > 0, c_0, c_m \geq 0, \gamma, \mu \geq 0 \text{ and } d(s) \geq d_0 > 0\text{ a.e. } s \in [0, m]. \] (5)
The case \( m = +\infty \), i.e.
\[
\begin{cases}
u_t(s, t) + (\gamma(s)u(s, t))_s = (d(s)u_s(s, t))_s - \mu(s)u(s, t) + \int_0^{+\infty} \beta(s, y)u(y, t)dy,
\end{cases}
\]
\[
[(d(s)u_s(s, t))_s]_{s=0} - b_0u_s(0, t) + c_0u(0, t) = 0
\]
(and initial conditions) with
\[ b_0 - \gamma(0) > 0 \]
is also considered in [10] under the general assumptions
\[ \gamma, d \in W^{1,\infty}(0, \infty) \text{ and } \mu \in L^\infty(0, \infty), \] (6)
\[ \beta_0 \in L^\infty(0, \infty) \] (7)
\[ \mu, \gamma' \text{ and } s \mapsto \beta(s, .) \text{ are continuous at } s = 0, \] (8)
\[ \beta : L^1(0, \infty) \ni u \to \int_0^{+\infty} \beta(\cdot, y)u(y)dy \in L^1(0, \infty) \text{ is bounded}, \] (9)
positive and
\[ b_0 > 0, c_0 \geq 0, \gamma, \mu \geq 0 \text{ and } d(s) \geq d_0 > 0 \text{ a.e. } s \geq 0. \] (10)
In [5][10], the Cauchy problem above, for \( m < +\infty \), is written in the following matrix form
\[
\begin{cases}
u_t(t) = AU(t) = (A + K)U(t),
\end{cases}
\]
\[
\begin{cases}
u(0) = (u^0, u_0^0, u_0^m) \in \mathcal{X},
\end{cases}
\]
where
\[
A \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} = \begin{pmatrix}
(du')' - (\gamma u')' - \mu u \\
(b_0 - \gamma(0))u'(0) - \rho_0 u_0 \\
-(b_m + \gamma(m))u'(m) - \rho_m u_m \end{pmatrix}
\]
\[
K \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} = \begin{pmatrix}
\int_0^m \beta(\cdot, y)u(y)dy \\
\int_0^m \beta_0(\cdot, y)u(y)dy \\
\int_0^m \beta_m(\cdot, y)u(y)dy \end{pmatrix}
\]
(11)
in the space
\[ \mathcal{X} = (L^1(0, m) \times \mathbb{R}^2, \| \cdot \|_X) \]
edowed with the norm
\[ \|(u, u_0, u_m)\|_X = \|u\|_{L^1(0, m)} + c_1|u_0| + c_2|u_m| \]
where
\[ c_1 = \frac{d(0)}{b_0 - \gamma(0)}, \quad c_2 = \frac{d(m)}{b_m + \gamma(m)}. \]
The domain of \( A \) is
\[ D(A) = \{(u, u_0, u_m) \in W^{2,1}(0, m) \times \mathbb{R}^2 : u(0) = u_0, u(m) = u_m\} \]
and \( A \) is shown to be the generator of a positive \( C_0 \)-semigroup \( \left(T(t)\right)_{t \geq 0} \). Moreover, by using the Hopf maximum principle, \( \left(T(t)\right)_{t \geq 0} \) is shown to be irreducible, or equivalently the resolvent of its generator is positivity improving, see ([10] Theorem 2.4 and Proposition 2). Since the perturbing operator \( K \) is bounded then
\[ A := A + K \]
is a generator of a positive \( C_0 \)-semigroup \( \left(U(t)\right)_{t \geq 0} \). The fact that \( U(t) \geq T(t) \) implies trivially that \( \left(U(t)\right)_{t \geq 0} \) is also irreducible regardless of \( K \).
Note that the boundedness of $K$ amounts to
\[ \varphi \in L^1(0, m) \to \int_0^m \beta(s, y)\varphi(y) dy \in L^1(0, m) \text{ is bounded} \tag{12} \]
and
\[ \beta_0, \beta_m \in L^\infty(0, m). \tag{13} \]
Besides the generation theory, various mathematical results are given in [10]; in particular, the two semigroups have the same essential spectrum
\[ \sigma_{ess}(U(t)) = \sigma_{ess}(T(t)), \]
(and consequently the same essential type $\omega_{ess}(T) = \omega_{ess}(U)$) under the assumption that the reproduction operator
\[ \beta : \varphi \in L^1(0, m) \to \int_0^m \beta(s, y)\varphi(y) dy \in L^1(0, m) \text{ is weakly compact} \tag{14} \]
from which we can derive that $(U(t))_{t \geq 0}$ has a spectral gap
\[ \omega_{ess}(U) < \omega(U) \]
and exhibits an asynchronous exponential behaviour, see ([10] Theorem 3.2 and Theorem 2.9).

The object of the present paper is to give two different extensions of the general theory given in [10].

1. The first extension consists in replacing (14) by the much weaker conditions:
   (i) $\beta_0 + \beta_m$ is not identically zero and the reproduction operator $\beta$ is an arbitrary positive bounded operator
   or
   (ii) $\beta_0 = \beta_m = 0$ and the reproduction operator $\beta$ is a positive bounded operator which dominates a positive non trivial weakly compact operator $\hat{\beta}$.
   (Note that in both cases, $\beta$ need not be a kernel operator.)

2. Another extension consists in dealing with a class of unbounded kernel reproduction operators
\[ \beta : \varphi \to \int_0^m \beta(s, y)\varphi(y) dy \quad (0 < m \leq +\infty), \]
and unbounded functions $\beta_0, \beta_m$.

The strategy in the first extension consists in considering $\mathcal{A}$ as a (non trivial) weakly compact perturbation of $\tilde{\mathcal{A}} := \mathcal{A} + \tilde{K}$ where
\[ \tilde{K} \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} = \begin{pmatrix} \beta u \\ 0 \\ 0 \end{pmatrix} \]
in the case (i) or
\[ \tilde{K} \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} = \begin{pmatrix} (\beta - \hat{\beta}) u \\ 0 \\ 0 \end{pmatrix} \]
in the case (ii). It turns out that this slight change of point of view enlarges considerably the class of bounded positive reproduction operators $\beta$ for which the general theory in [10] is still valid, even if we don’t know the possible biological significance of non kernel reproduction operators.
Our second extension of [10] is much more involved. Firstly, we are faced with well posedness of the Cauchy problem. Our generation theory relies on Desh’s theorem [3][15] via suitable weak compactness arguments [9]. Secondly, the proof of a stability of essential spectra turns out to be much more technical than in [10].

Our paper is organized as follows. In Section 2, we provide some important functional analytic reminders; in particular, we recall a generation theorem for unbounded perturbations in $L^1$ spaces, see Lemma 2.3. In Section 3, we extend the asynchronous exponential behaviours (for $m < +\infty$ or $m = +\infty$) given in [10] when $\beta_0 + \beta_m$ is not identically zero and the reproduction operator $\beta$ is an arbitrary positive bounded operator or when $\beta_0 = \beta_m = 0$ and the reproduction operator $\beta$ is a positive bounded operator which dominates a non trivial positive weakly compact operator, see Theorem 3.1 and Theorem 3.2. The condition that $\beta$ dominates a non trivial positive weakly compact operator is satisfied e.g. if $\beta$ is a kernel operator whose kernel $\beta(.,.)$ is such that there exists some $m \in [0, m)$ such that $\beta(.,.)$ is not identically zero in a neighborhood of $(m, m)$ and there exists $\alpha \in [0, 1)$ such that

$$
\limsup_{(x,y) \to (m, m)} |s - y|^{\alpha} \beta(s, y) < +\infty,
$$

see Proposition 1. In Section 4, we deal with unbounded kernel reproduction operators $\beta$ such that

$$
\int_0^m \int_0^m \beta(s, y)dsdy < +\infty, \quad \int_0^m \beta_0(y)dy < +\infty
$$

and (when $m < +\infty$)

$$
\int_0^m \beta_m(y)dy < +\infty.
$$

We show that $K$ is $A$-weakly compact so that, by Lemma 2.3, $A := A + K$ is a generator of a positive $C_0$-semigroup $(U(t))_{t\geq 0}$, see Theorem 4.1. By using a suitable approximation procedure, we show that $U(t) - T(t)$ is weakly compact and consequently $(U(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ have the same essential spectrum and the same essential type, see Theorem 4.2. Finally, the proofs of the asynchronous exponential asymptotics for $m < +\infty$ or $m = +\infty$ (Theorem 4.3 and Theorem 4.4) follow the same strategy as for bounded perturbations. Notice that a further (more abstract) extension of [10] is still possible, see Remark 3.

This paper is dedicated to the celebration of the 60th birthday of Gisèle Ruiz Goldstein.

2. Some functional analytic reminders. Let $X$ be a Banach lattice. We start with:

**Lemma 2.1.** [15] Let $T : D(T) \subset X \to X$ be a resolvent positive linear operator. Let $B : D(T) \to X$ be linear and positive (i.e. $B : D(T) \cap X_+ \to X_+$). Then $A := T + B$ with $D(A) = D(T)$ is resolvent positive and

$$
(\lambda - A)^{-1} = (\lambda - T)^{-1} \sum_{j=0}^{+\infty} (B(\lambda - T)^{-1})^j, \quad \lambda > s(A),
$$

$$
s(A) = \inf \{ \lambda > s(T); \ r_r(B(\lambda - T)^{-1}) < 1 \}
$$

where $s(A)$ and $s(T)$ denote respectively the spectral bounds of $A$ and $T$. 

\[ \tag{15} \]
We recall that the spectral bound of a closed (unbounded) operator \( O \) is defined by
\[
s(O) := \begin{cases} 
\sup \{ \Re \lambda; \lambda \in \sigma(O) \} & \text{if } \sigma(O) \neq \emptyset \\
-\infty & \text{if } \sigma(O) = \emptyset.
\end{cases}
\]

It is well known that if \( O \in \mathcal{L}(X) \) is positive then
\[
\|O\|_{\mathcal{L}(X)} = \sup_{\{x \in X; \|x\| \leq 1\}} \|Ox\|.
\]

It follows easily that if \( O_i \in \mathcal{L}(X) \) (\( i = 1, 2 \)) are positive and if \( O_1 \leq O_2 \) (i.e. \( O_1 x \leq O_2 x \ \forall x \in X_+ \)) then \( r_{f}(O_1) \leq r_{f}(O_2) \). We complement this by:

**Lemma 2.2.** (\cite{7} Theorem 4.3) Let \( O_i \in \mathcal{L}(X) \) (\( i = 1, 2 \)) be positive such that \( O_1 \leq O_2 \) and \( O_1 \neq O_2 \). We assume that \( r_{f}(O_1) \) (resp. \( r_{f}(O_2) \)) is an isolated eigenvalue of \( O_1 \) (resp. of \( O_2 \)) and that \( O_2 \) is irreducible. Then
\[ r_{f}(O_1) < r_{f}(O_2). \]

We give now a generation result for unbounded perturbations.

**Lemma 2.3.** (\cite{9} Theorem 6) Let \( X \) be an \( L^1(\mu) \) space and \( T : D(T) \subset X \to X \) be the generator of a positive \( C_0 \)-semigroup. Let \( B : D(T) \to X \) be linear and positive (and therefore \( T \)-bounded). If \( B \) is weakly compact (with \( D(T) \) endowed with the graph norm) then \( A := T + B \) generates a positive \( C_0 \)-semigroup.

We recall also that the essential type \( \omega_{ess}(S) \) of a \( C_0 \)-semigroup \( (S(t))_{t \geq 0} \) satisfies
\[
r_{ess}(S(t)) = e^{\omega_{ess}(S)t} \quad (t \geq 0)
\]
where
\[
r_{ess}(S(t)) = \sup \{ \|\lambda\|; \lambda \in \sigma_{ess}(S(t)) \}
\]
see e.g. \cite{11}. Finally, we recall that, for a positive \( C_0 \)-semigroup \( (S(t))_{t \geq 0} \) on \( L^p \) spaces, the type coincides with the spectral bound of its generator, see e.g. \cite{17}.

3. **Bounded reproduction operators.** We start with a sufficient condition for a bounded kernel operator to dominate a weakly compact operator.

**Proposition 1.** Let \( \beta \) be a positive bounded integral operator on \( L^1(0,m) \) with kernel \( \beta(.,.) \). We assume that there exists some \( \overline{m} \in [0,m) \) such that \( \beta(.,.) \) is not identically zero in a neighborhood of \((\overline{m}, \overline{m})\) and there exists \( \alpha \in [0,1) \) such that
\[
\lim_{(s,y) \to (\overline{m}, \overline{m})} \sup_{s - y} |s - y|^\alpha \beta(s,y) < +\infty. \quad (16)
\]
Then \( \beta \) dominates a non trivial weakly compact operator.

**Proof.** We note that, in the statement above, \( \overline{m} = m \) is allowed if \( m < +\infty \). We observe that (16) implies the existence of a bounded interval \( I \subset [0,m) \) such that \( |s - y|^\alpha \beta(s,y) \) is bounded (and not identically zero) on \( I \times I \)
\[
\beta(s,y) \leq \frac{C}{|s - y|^\alpha}, \quad (s,y) \in I \times I.
\]
It is easy to see that the integral operator \( L^1(I) \to L^1(I) \) with kernel \( \frac{1}{|s - y|^\alpha} \) is compact. Define the kernel
\[
\hat{\beta}(s,y) := \begin{cases} 
\beta(s,y) & \text{if } (s,y) \in I \times I \\
0 & \text{otherwise}.
\end{cases}
\]
Then\[
\hat{\beta} : \varphi \in L^1(0, m) \to \int_0^m \hat{\beta}(s, y)\varphi(y)dy \in L^1(0, m)
\]
is a non trivial positive weakly compact operator dominated by $\beta$.

Remark 1. We suspect that the assumption that $\beta$ dominates a non trivial weakly compact operator has to do with the spectral condition $r_{ess}(\beta) < r_\sigma(\beta)$ but we have not yet a complete proof.

3.1. Asynchronous exponential behaviour. We already know that $(T(t))_{t \geq 0}$ and $(U(t))_{t \geq 0}$ are irreducible ([10] Theorem 2.4 and Proposition 2). We start with the case $m < +\infty$.

Theorem 3.1. Let $m < +\infty$ and let Assumptions (1)(2)(5) be satisfied. Let the reproduction operator $\beta$ be a positive bounded operator. We assume that either $\beta_0 + \beta_m$ is not identically zero, or $\beta_0 = \beta_m = 0$ and the reproduction operator $\beta$ dominates a positive non trivial weakly compact operator $\hat{\beta}$. Then\[
\omega_{ess}(U) < \omega(U) = s(A)
\]
and there exists $\varepsilon > 0$ and $C > 0$ such that\[
\|e^{-s(A)t}U(t) - P\| \leq Ce^{-\varepsilon t} \ (t \geq 0)
\]
where $P$ is the rank one spectral projection associated to the leading eigenvalue $s(A)$.

Proof. It suffices to show that $\omega_{ess}(U) < \omega(U)$ and to invoke a classical functional analytic result, e.g. [2] Theorem 9.11, p. 224. Note first that $(\lambda - A)^{-1}$ is compact (since the imbedding of $D(A)$ into $\mathcal{X}$ is compact) and positivity improving and therefore irreducible so\[
r_\sigma[(\lambda - A)^{-1}] > 0, \quad (\lambda > s(A))
\]
[12]. On the other hand\[
r_\sigma[(\lambda - A)^{-1}] = (\lambda - s(A))^{-1}
\]
(see [11]) whence $s(A) > -\infty$ and\[
s(A) \geq s(A) > -\infty.
\]

Consider first the case:

(i) $\beta_0$ or $\beta_m$ is not identically zero.

Let\[
\bar{A} := A + \tilde{K}
\]
where\[
\tilde{K} : \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} \to \begin{pmatrix} \beta u \\ 0 \\ 0 \end{pmatrix}.
\] (17)

Let $(\bar{T}(t))_{t \geq 0}$ be the semigroup generated by $\bar{A}$. We have the Duhamel equation\[
U(t) = \bar{T}(t) + \int_0^t U(t - s)\tilde{K}\bar{T}(s)ds
\]
where\[
\tilde{K} : \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} \to \begin{pmatrix} 0 \\ \int_0^m \beta_0(y)u(y)dy \\ \int_0^m \beta_m(y)u(y)dy \end{pmatrix}
\]
is compact. Hence the strong integral
\[
\int_0^t U(t - s)\tilde{K}\tilde{T}(s)ds
\]
is a compact operator (see e.g. [16]). This implies that \((U(t))_{t \geq 0}\) and \((\tilde{T}(t))_{t \geq 0}\) have the same essential spectrum [6] and consequently the same essential type
\[
\omega_{ess}(U) = \omega_{ess}(\tilde{T}).
\]
We note that the irreducible compact operators \((\lambda - A)^{-1}\) and \((\lambda - \tilde{A})^{-1}\) are such that
\[
(\lambda - \tilde{A})^{-1} \leq (\lambda - A)^{-1}
\]
and, since \(\tilde{K} \neq 0\),
\[
(\lambda - \tilde{A})^{-1} \neq (\lambda - A)^{-1}
\]
whence
\[
r_\sigma [(\lambda - A)^{-1}] < r_\sigma [(\lambda - A)^{-1}],
\]
by Lemma 2.2. Since \(r_\sigma [(\lambda - A)^{-1}] = (\lambda - s(A))^{-1}\) and \(r_\sigma [(\lambda - A)^{-1}] = (\lambda - s(A))^{-1}\) then
\[
s(\tilde{A}) < s(A).
\]
Finally
\[
\omega_{ess}(U) = \omega_{ess}(\tilde{T}) \leq \omega(\tilde{T}) = s(\tilde{A})
\]
implies
\[
\omega_{ess}(U) \leq s(\tilde{A}) < s(A) = \omega(U)
\]
and ends the proof in the case (i).

Consider now the case:
(ii) \(\beta_0 = \beta_m = 0\) and the reproduction operator \(\beta\) dominates a positive non trivial weakly compact operator \(\hat{\beta}\). Let now
\[
\tilde{A} := A + \tilde{K}
\]
where
\[
\tilde{K} : \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} \rightarrow \begin{pmatrix} \beta - \hat{\beta} & u \\ 0 & 0 \end{pmatrix}.
\]
(18)
Let \((\tilde{T}(t))_{t \geq 0}\) be the semigroup generated by \(\tilde{A}\). Let \(\tilde{\beta}(.,.)\) be the kernel of \(\hat{\beta}\), (note that a weakly compact operator is a kernel operator, see [4], p. 508). As previously, we have a Duhamel equation
\[
U(t) = \tilde{T}(t) + \int_0^t U(t - s)\tilde{K}\tilde{T}(s)ds
\]
where
\[
\tilde{K} : \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} \rightarrow \begin{pmatrix} \int_0^m \tilde{\beta}(.,y)u(y)dy \\ 0 \\ 0 \end{pmatrix}
\]
is weakly compact. Hence the strong integral
\[
\int_0^t U(t - s)\tilde{K}\tilde{T}(s)ds
\]
is a weakly compact operator (see [14] or [8]). This implies that \((U(t))_{t \geq 0}\) and \((\overline{T}(t))_{t \geq 0}\) have also the same essential spectrum [6] and consequently the same essential type
\[
\omega_{ess}(U) = \omega_{ess}(\overline{T}).
\]
Arguing as previously, one sees that \(s(\overline{A}) < s(A)\) (since \(\hat{K} \neq 0\)) and
\[
\omega_{ess}(U) = \omega_{ess}(\overline{T}) \leq \omega(\overline{T}) = s(\overline{A}) < s(A) = \omega(U)
\]
which ends the proof.

We consider now the case \(m = +\infty\).

**Theorem 3.2.** Let \(m = +\infty\) and let Assumptions (6)(7)(10) be satisfied. We assume that the reproduction operator \(\beta\) is a positive bounded operator. We assume that either (i) \(\beta_{0} + \beta_{m}\) is not identically zero or (ii) that \(\beta_{0} = \beta_{m} = 0\) and the reproduction operator \(\beta\) dominates a positive non trivial weakly compact operator \(\hat{\beta}\). Let \(\overline{A} := A + \overline{K}\) where \(\overline{K}\) is given by (17) in the case (i) or by (18) in the case (ii). If
\[
\lim_{\lambda \to s(A)} r_{\sigma} \left[ K(\lambda - \overline{A})^{-1} \right] > 1
\]
then \(\omega_{ess}(U) < \omega(U) = s(A)\) and there exists \(\varepsilon > 0\) and \(C > 0\) such that
\[
\left\| e^{-s(A)t} U(t) - P \right\| \leq C e^{-\varepsilon t} \quad (t \geq 0)
\]
where \(P\) is the rank one spectral projection associated to the leading eigenvalue \(s(A)\).

**Proof.** By combining (19) and the characterization of the perturbed spectral bound (15) we get
\[
s(\overline{A}) < s(A).
\]
As previously, \((U(t))_{t \geq 0}\) and \((\overline{T}(t))_{t \geq 0}\) have the same essential spectrum so
\[
\omega_{ess}(U) = \omega_{ess}(\overline{T}) \leq \omega(\overline{T}) = s(\overline{A}) < s(A) = \omega(U)
\]
and this ends the proof.

4. **Unbounded reproduction operators.** The first key question is the well posedness of the Cauchy problem. Let \(\beta(.,.)\), \(\beta_{0}(.)\) and \(\beta_{m}(.)\) be nonnegative and let
\[
K : \begin{pmatrix} u \\ u_{0} \\ u_{m} \end{pmatrix} \rightarrow \begin{pmatrix} \int_{0}^{m} \beta(.,y)u(y)dy \\ \int_{0}^{m} \beta_{0}(y)u(y)dy \\ \int_{0}^{m} \beta_{m}(y)u(y)dy \end{pmatrix}
\]
or
\[
K : \begin{pmatrix} u \\ u_{0} \end{pmatrix} \rightarrow \begin{pmatrix} \int_{0}^{m} \beta(.,y)u(y)dy \\ \int_{0}^{m} \beta_{0}(y)u(y)dy \end{pmatrix}
\]
(according as \(m < +\infty\) or \(m = +\infty\)) with suitable domains.
4.1. A generation theorem. The first key result is:

**Theorem 4.1.** Let \( m < +\infty \) and let Assumptions (1)(5) be satisfied or let \( m = +\infty \) and let Assumptions (6)(10) be satisfied. We assume that
\[
\int_0^m \int_0^m \beta(s, y) ds dy < +\infty.
\]

(i) If \( m < +\infty \) and if
\[
\int_0^m \beta_0(y) dy < +\infty, \quad \int_0^m \beta_m(y) dy < +\infty
\]
then \( K \) is \( A \)-bounded and \( A := A + K \) with \( D(A) = D(A) \) is the infinitesimal generator of a positive \( C_0 \)-semigroup \( (U(t))_{t \geq 0} \) on \( \mathcal{X} \).

(ii) If \( m = +\infty \) and if
\[
\int_0^m \beta_0(y) dy < +\infty
\]
then \( K \) is \( A \)-bounded and \( A := A + K \) with \( D(A) = D(A) \) is the infinitesimal generator of a positive \( C_0 \)-semigroup \( (U(t))_{t \geq 0} \) on \( \mathcal{X} \).

**Proof.** (i) We observe that
\[
\int_0^m \beta(x, y) u(y) dy = - \int_0^m \left( \int_0^y \beta(x, s) ds \right) u'(y) dy + \left( \int_0^m \beta(x, s) ds \right) u(m)
\]
so
\[
K \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} := \begin{pmatrix} - \int_0^m \left( \int_0^y \beta(x, s) ds \right) u'(y) dy + \hat{\beta}(x) u(m) \\ \int_0^m \beta_0(y) u(y) dy \\ \int_0^m \beta_m(y) u(y) dy \end{pmatrix}
\]
where
\[
\hat{\beta}(x) := \int_0^m \beta(x, s) ds.
\]
Note that \( W^{1,1}(0, m) \) imbeds continuously into \( C_0([0, m]) \) (see e.g. [1] Chap 8) so
\[
u \in W^{1,1}(0, m) \rightarrow u(m) \in \mathbb{R}
\]
\[
u \in W^{1,1}(0, m) \rightarrow \int_0^m \beta_0(y) u(y) dy \in \mathbb{R}
\]
and
\[
u \in W^{1,1}(0, m) \rightarrow \int_0^m \beta_m(y) u(y) dy \in \mathbb{R}
\]
are continuous linear functionals. Furthermore, the positive operator
\[
\varphi \in L^1(0, m) \rightarrow \int_0^m \left( \int_0^y \beta(x, s) ds \right) \varphi(y) dy \in L^1(0, m)
\]
is weakly compact since it is dominated by the rank-one operator
\[
\varphi \in L^1(0, m) \rightarrow \left( \int_0^m \varphi(y) dy \right) \hat{\beta}(x) \in L^1(0, m)
\]
so
\[
(u, u_0, u_m) \in W^{1,1}(0, m) \times \mathbb{R}^2 \rightarrow \begin{pmatrix} - \int_0^m \left( \int_0^y \beta(x, s) ds \right) u'(y) dy + \hat{\beta}(x) u(m) \\ \int_0^m \beta_0(y) u(y) dy \\ \int_0^m \beta_m(y) u(y) dy \end{pmatrix}
\]
is weakly compact. Thus
\[
K : D(A) \rightarrow \mathcal{X}
\]
is weakly compact, i.e. $K$ is $A$-weakly compact, and
\[ A := A + K : D(A) \to \mathcal{X} \]
is a generator of a positive semigroup $(U(t))_{t \geq 0}$ by virtue of Lemma 2.3.

(ii) We argue similarly when $m = +\infty$. \hfill \Box

**Remark 2.** We recall that $(U(t))_{t \geq 0}$ and $(T(t))_{t \geq 0}$ satisfy the Duhamel equation
\[ U(t)\varphi = T(t)\varphi + \int_0^t U(t-s)KT(s)\varphi ds, \quad \varphi \in D(A), \]
see [13]. In particular
\[ D(A) \ni \varphi \to \int_0^t U(t-s)KT(s)\varphi ds \in \mathcal{X} \]
extends uniquely as a bounded operator on $\mathcal{X}$. We will denote this extension symbolically by $\int_0^t U(t-s)KT(s)ds$.

4.2. **Stability of essential spectra.** Our second key result is:

**Theorem 4.2.** Let $m < +\infty$ and Assumptions (1)(5) be satisfied or let $m = +\infty$ and Assumptions (6)(10) be satisfied. Then
\[ U(t) - T(t) \]
is a weakly compact operator and therefore $U(t)_{t \geq 0}$ and $T(t)_{t \geq 0}$ have the same essential spectrum and consequently the same essential type.

**Proof.** We treat both cases $m < +\infty$ and $m = +\infty$ simultaneously. Let $\beta^k(x,.)$ be defined on $[0,m)^2$ with compact supports such that
\[ \beta^k(x,s) \leq \beta(x,s) \]
\[ \beta^k \to \beta \quad \text{in} \quad L^1((0,m)^2) \quad (k \to +\infty). \]
Let $\beta_0^k$ be defined on $[0,m)$ with compact supports such that
\[ \beta_0^k \leq \beta_0 \quad \text{and} \quad \beta_0^k \to \beta_0 \quad \text{in} \quad L^1(0,m) \quad (k \to +\infty). \]
If $m < +\infty$, we define also $\beta_m^k$ on $[0,m)$ with compact supports such that
\[ \beta_m^k \leq \beta_m \quad \text{and} \quad \beta_m^k \to \beta_m \quad \text{in} \quad L^1(0,m) \quad (k \to +\infty). \]

We introduce the bounded operator
\[ K^k \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} := \begin{pmatrix} \int_0^m \beta^k(y)u(y)dy \\ \int_0^m \beta_0^k(y)u(y)dy \\ \int_0^m \beta_m^k(y)u(y)dy \end{pmatrix} \]
where we drop the third component when $m = +\infty$. Since
\[ L^1(0,m) \ni u \to \int_0^m \beta^k(y)u(y)dy \in L^1(0,m) \]
is weakly compact and
\[ L^1(0,m) \ni u \to \begin{pmatrix} \int_0^m \beta_0^k(y)u(y)dy \\ \int_0^m \beta_m^k(y)u(y)dy \end{pmatrix} \in \mathbb{R}^2 \]
is compact then $K^k$ is a weakly compact operator and consequently so is the strong integral
\[ \int_0^t U(t-s)K^kT(s)ds \quad (0 < t < +\infty) \]
Note that \( T(s) \varphi \in D(A) \) so \( KT(s) \varphi \) is meaningful since \( K \) is \( A \)-bounded. There exists \( C \geq 1 \) and \( \lambda \in \mathbb{R} \) such that \( \| U(t) \| \leq Ce^{\lambda t} \). Then

\[
\left\| \int_0^t U(t-s)KT(s)\varphi ds - \int_0^t U(t-s)K^kT(s)\varphi ds \right\| \\
= \left\| \int_0^t U(t-s)(K-K^k) T(s)\varphi ds \right\| \\
\leq \int_0^t \|U(t-s)\| \| (K-K^k) T(s)\varphi \| ds \\
\leq C \int_0^t e^{\lambda(t-s)} \| (K-K^k) T(s)\varphi \| ds \\
= C e^{\lambda t} \int_0^t \| (K-K^k) e^{-\lambda s} T(s)\varphi \| ds.
\]

Since \( K - K^k \) is positive then, by the additivity of the \( L^1 \) norm on the positive cone,

\[
\int_0^t \| (K-K^k) e^{-\lambda s} T(s)\varphi \| ds = \int_0^t \( K-K^k \) e^{-\lambda s} T(s)\varphi ds \\
= \| (K-K^k) \int_0^t e^{-\lambda s} T(s)\varphi ds \|
\leq \| (K-K^k) \int_0^{+\infty} e^{-\lambda s} T(s)\varphi ds \|
= \| (K-K^k) (\lambda - A)^{-1}\varphi \|.
\]

Since \( D(A) \cap X_+ \) is dense in \( X_+ \) and \( (\lambda - A)^{-1} : X \rightarrow L^\infty(0, m) \times \mathbb{R}^2 \) is continuous then

\[
\left\| \left( \int_0^t U(t-s)KT(s)ds \right) \varphi - \left( \int_0^t U(t-s)K^kT(s)ds \right) \varphi \right\| \\
\leq C e^{\lambda t} \| (K-K^k) (\lambda - A)^{-1}\varphi \| \quad (\varphi \in X_+).
\]

We note that for any constant \( c > 0 \)

\[
\begin{align*}
\left( \int_0^m \beta_k(\cdot, y)u(y)dy \right) \\
\left( \int_0^m \beta_0(\cdot, y)u(y)dy \right) \\
\left( \int_0^m \beta_m(\cdot, y)u(y)dy \right)
\end{align*}
\rightarrow
\begin{align*}
\left( \int_0^m \beta(\cdot, y)u(y)dy \right) \\
\left( \int_0^m \beta_0(\cdot, y)u(y)dy \right) \\
\left( \int_0^m \beta_m(\cdot, y)u(y)dy \right)
\end{align*}
\quad (k \rightarrow +\infty)
\]

in \( X = L^1(0, m) \times \mathbb{R}^2 \) uniformly in \( \| u \| L^\infty(0, m) \leq c \). Since \( (\lambda - A)^{-1} : X \rightarrow L^\infty(0, m) \times \mathbb{R}^2 \) is continuous then

\[
\varphi \in X_+ \quad \| (K-K^k) (\lambda - A)^{-1}\varphi \| \rightarrow 0 \quad (k \rightarrow +\infty)
\]

and consequently

\[
\sup_{\varphi \in X_+, \| \varphi \| \leq 1} \left\| \left( \int_0^t U(t-s)KT(s)ds \right) \varphi - \left( \int_0^t U(t-s)K^kT(s)ds \right) \varphi \right\| \rightarrow 0 \quad (k \rightarrow +\infty).
\]
Since
\[ \int_0^t U(t-s)KT(s)ds - \int_0^t U(t-s)K^kT(s)ds \]
is a positive operator then
\[ \int_0^t U(t-s)K^kT(s)ds \to \int_0^t U(t-s)KT(s)ds \quad (k \to +\infty) \]
in operator norm. Since \( \int_0^t U(t-s)K^kT(s)ds \) (\( k \in \mathbb{N} \)) are weakly compact operators then so is \( \int_0^t U(t-s)KT(s)ds \). Finally, for any \( t > 0 \), \( U(t-T(t)) \) is a weakly compact operator and therefore \( (U(t))_{t \geq 0} \) and \( (T(t))_{t \geq 0} \) have the same essential spectrum, see [6].

4.3. Asynchronous exponential behaviour. We know that \( A \) is a generator of a positive semigroup \( (U(t))_{t \geq 0} \) (Theorem 4.1) since \( U(t) \geq T(t) \) and \( (U(t))_{t \geq 0} \) is also irreducible. Since \( (U(t))_{t \geq 0} \) and \( (T(t))_{t \geq 0} \) have the same essential spectrum (Theorem 4.2) then we have all the ingredients to deduce the following results whose proofs are similar to those of Theorem 3.1 and Theorem 3.2 and are omitted.

**Theorem 4.3.** Let \( m < +\infty \) and let Assumptions (1)(5) be satisfied. If \( K \neq 0 \) then \( \omega_{\text{ess}}(U) < \omega(U) = s(A) \) and there exists \( \epsilon > 0 \) and \( C > 0 \) such that
\[ \| e^{-(\lambda + \epsilon)t} U(t) - P \| \leq Ce^{-ct} \quad (t \geq 0) \]
where \( P \) is the rank one spectral projection associated to the leading eigenvalue \( s(A) \).

**Theorem 4.4.** Let \( m = +\infty \) and let Assumptions (6)(10) be satisfied. If
\[ \lim_{\lambda \to s(A)} r_\sigma [K(\lambda - A)^{-1}] > 1 \quad (20) \]
then \( \omega_{\text{ess}}(U) < \omega(U) = s(A) \) and there exists \( \epsilon > 0 \) and \( C > 0 \) such that
\[ \| e^{-(\lambda + \epsilon)t} U(t) - P \| \leq Ce^{-ct} \quad (t \geq 0) \]
where \( P \) is the rank one spectral projection associated to the leading eigenvalue \( s(A) \).

**Remark 3.** In the same spirit, we could combine the two extensions given in this paper into one (more general) abstract construction. For instance, for \( m < +\infty \), we may replace (11) by
\[ K \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} = \begin{pmatrix} \beta u_0 \\ (\beta u)(0) \\ (\beta u)(m) \end{pmatrix} \]
where
\[ \beta : W^{2,1}(0, m) \to L^1(0, m) \]
is a bounded operator such that
\[ W^{2,1}(0, m) \ni u \to ((\beta u)(0), (\beta u)(m)) \in \mathbb{R}^2 \quad (21) \]
is well defined and positive and
\[ \lim_{\lambda \to +\infty} r_\sigma [K(\lambda - A)^{-1}] < 1. \]
In this case \( A := A + K \) is a generator of a positive \( C_0 \)-semigroup \( (U(t))_{t \geq 0} \) on \( X \), see [3][15]. We can recover the asynchronous exponential behaviour of \( (U(t))_{t \geq 0} \) once (21) is not zero. If (21) is zero, we assume that \( \beta \) splits as \( \beta = \beta_1 + \beta_2 \) where \( \beta_1 \) is (say) a positive bounded operator \( L^1(0, m) \to L^1(0, m) \).
and $\beta_2 : W^{2,1}(0, m) \to L^1(0, m)$ is a non trivial positive weakly compact operator. In this case $\lim_{\lambda \to +\infty} \gamma_\sigma \left[ K(\lambda - A)^{-1} \right] = 0$ [9] and we can recover again the asynchronous exponential behaviour of $(U(t))_{t \geq 0}$. We do not try to elaborate on this point here.

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