GAUSS SUMS, STICKELBERGER’S THEOREM, AND THE GRAS CONJECTURE FOR RAY CLASS GROUPS

TIMOTHY ALL

Abstract. Let $k$ be a real abelian number field and $p$ an odd prime not dividing $[k : \mathbb{Q}]$. For a natural number $d$, let $E_d$ denote the group of units of $k$ congruent to 1 modulo $d$, $C_d$ the subgroup of $d$-circular units of $E_d$, and $\mathfrak{C}_d$ the ray class group of modulus $d$. Let $\varrho$ be an irreducible character of $G = \text{Gal}(k/\mathbb{Q})$ over $\mathbb{Q}_p$, and $e_{\varrho} \in \mathbb{Z}_p[G]$ the corresponding idempotent. We show that $|e_{\varrho} \text{Syl}_p(E_d/C_d)| = |e_{\varrho} \text{Syl}_p(\mathfrak{C}_d)|$. This is a ray class version of the Gras conjecture. In the case when $p | [k : \mathbb{Q}]$, similar but slightly less precise results are obtained. In particular, beginning with what could be considered a Gauss sum for real fields, we construct explicit Galois annihilators of $\text{Syl}_p(\mathfrak{C}_a)$ akin to the classical Stickelberger Theorem.

1. Introduction

Let $k$ denote a real abelian number field with Galois group $G$. Let $\mathfrak{o}_k = \mathfrak{o}$ denote the ring of integers of $k$. Fix $d \in \mathbb{N}$ and let $E_d$ denote the units of $\mathfrak{o}$ that are congruent to 1 modulo $d$. When $d = 1$ we typically omit subscripts. Let $\mathfrak{d}$ denote the square-free part of $d$, and for every $n \in \mathbb{N}$ we let $\zeta_n$ stand for a primitive $n$-th root of unity. For every $n > 1$ satisfying $n \nmid \mathfrak{d}$ let $k^n = \mathbb{Q}(\zeta_n) \cap k$ and

$$\delta_{n,d} := N_{k^n/k}^\mathbb{Q}(\zeta_n) \prod_{t \mid \mathfrak{d}} (1 - \zeta_n^t)^{\mu(t)d/t}$$

where $\mu(t)$ denotes the Möbius function. Let $D(d)$ denote the $G$-module generated by the $\delta_{n,d}$ for all $n \nmid \mathfrak{d}$ in $k^\times$. We let

$$C(d) = E \cap D(d)$$
$$C_d = E_d \cap D(d).$$

We call the modules $D(d)$, $C(d)$, and $C_d$ the $d$-cyclotomic numbers, units, and units congruent to 1 modulo $d$, respectively. These modules were originally introduced by Sinnott [S] (for $d = 1$) and by Schmidt [Sch0, Sch] (for $d > 1$).

For an ideal $\mathfrak{a} \subseteq \mathfrak{o}$, let $\mathfrak{C}(\mathfrak{a})$ denote the ray class group of $k$ of modulus $\mathfrak{a}$, and let $k(\mathfrak{a})$ denote the corresponding ray class field of $k$ so that

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\[ \mathrm{Gal}(k(a)/k) \simeq \mathcal{C}(a) \] via the Artin map. Let \( k_a = k(a) \cap \mathbb{Q}^{ab} \), the maximal sub-extension of \( k(a)/k \) abelian over \( \mathbb{Q} \), and let \( \mathcal{C}_a \leq \mathcal{C}(a) \) such that \( \mathcal{C}_a \simeq \mathrm{Gal}(k(a)/k_a) \). In the case when \( p \nmid [k: \mathbb{Q}] \), note that
\[ \text{Syl}_p(\mathcal{C}_a) = (1 - e_1) \text{Syl}_p(\mathcal{C}(a)), \]
where \( e_1 \in \mathbb{Z}[G] \) is the idempotent associate to the trivial character. The following theorem was proven by Sinnott [S] for \( d = 1 \), and by Schmidt [Sch] using similar methods.

**Theorem 1.1.** If \( p \nmid 2 \cdot |G| \), then
\[ |\text{Syl}_p(E_d/C_d)| = |\text{Syl}_p(\mathcal{C}_d)|. \]

One of the aims of this article is to prove a Galois-equivariant version of the theorem above. To be precise, let \( X \) be the \( \mathrm{Gal}(\mathbb{Q}_p(\zeta_{|G|})/\mathbb{Q}_p) \)-orbit of a non-trivial character \( \chi \) of \( G \) and define \( \varphi = \sum_{\psi \in X} \psi \). Let \( \epsilon_\varphi \) be the \( \mathbb{Z}_p \)-valued idempotent
\[ \epsilon_\varphi = \frac{1}{|G|} \sum_{\sigma \in G} \varphi(\sigma)\sigma^{-1} \in \mathbb{Z}_p[G]. \]

For a \( \mathbb{Z}_p[G] \)-module \( M \), we let \( M_\varphi \) denote the sub-module \( \epsilon_\varphi M \). One of our main results is the following

**Theorem 1.2.** If \( p \nmid 2 \cdot |G| \), then
\[ |\text{Syl}_p(E_d/C_d)_\varphi| = |\text{Syl}_p(\mathcal{C}_d)_\varphi|. \]

Theorem 1.2 is a ray class version of the Gras Conjecture (the statement of the claim when \( d = 1 \)). Greenberg [RG] observed that the Gras Conjecture followed from the Main Conjecture of Iwasawa Theory which was later on proven by Mazur-Wiles [MW]. In the case when \( p \) possibly divides the order of \( G \), we prove a result akin to Rubin’s [R] which itself was a generalization of a theorem of Thaine [Th]. Our method of proof follows along those same lines. In particular, we define a subgroup \( C(a) \) of \( E \) which we call the \( a \)-special numbers. These are akin to Rubin’s special numbers [R], and we show that \( C(d) \subseteq C(a) \) for an appropriate choice of \( d \). We then show

**Theorem 1.3.** Let \( \alpha : E \rightarrow \mathcal{O}[G] \) be any \( G \)-module map where \( \mathcal{O} \) is the valuation ring of any finite extension of \( \mathbb{Q}_p \), and let \( \delta_a \in C(a) \). Then \( \alpha(\delta_a) \) annihilates \( \mathcal{C}_a \otimes_{\mathbb{Z}} \mathcal{O} \).

As stated, our method of proof originates in the work of Thaine and Rubin. Thaine noticed that cyclotomic units could be used to generate elements \( \alpha \) that act like real analogues of Gauss sums much like roots of unity are used to generate classical Gauss sums. To generate \( \alpha \), Thaine relied on an invocation of Hilbert’s Theorem 90. A key feature here is that we give \( \alpha \) explicitly. This affords finer control over the ideal relations revealed by the factorization of \( \alpha \) thus paving the way towards annihilation results concerning ray classes. In particular we obtain the following ray class version of a conjecture of D. Solomon [Sol].
Theorem 1.4. Let \( O \) denote the valuation ring of a \( p \)-adic completion of \( k \), and let \( \pi \in O \) be a local parameter. Let \( e(p) \) denote the ramification index of \( p \) in \( k \) and \( p^j \) its \( p \)-part. For every \( \delta_a \in \mathcal{C}(a) \) we have that
\[
\pi^{je(p) - p^j} \sum_{\sigma \in G} \log_p (\delta_a^\sigma) \sigma^{-1} \in O[G]
\]
annihilates \( \mathfrak{C}_a \otimes \mathbb{Z} O \).

2. Preliminaries

In this section, we collect some results that will be useful in the sequel concerning the structure of relevant \( G \)-modules contained in \( k \). Until further notice, we consider \( p \) to be an odd prime not dividing \( n := [k : \mathbb{Q}] \). Let
\[
\mathcal{K} = \text{a local field containing the character values of } G
\]
\( \mathcal{O} = \text{the valuation integers of } \mathcal{K} \)
\( \mathbb{F} = \text{the residue field of } \mathcal{O} \)
\( \mathbb{F}_p = \text{the finite field with } p \text{-elements} \).

For any finite set \( X \), we use \( |X| \) to denote the number of elements in \( X \). We write \( \hat{G} \) for \( \text{Hom}_\mathbb{Z}(G, \mathbb{Z}[\zeta_n]) \). For every \( \chi \in \hat{G} \), we let \( e_\chi = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \in \mathcal{O}[G] \)
the idempotent associate to \( \chi \). We may naturally view \( e_\chi \in \mathbb{F}[G] \).

Throughout we use \( \otimes \) as an abbreviation for \( \otimes \mathbb{Z} \). For a \( \mathbb{Z}[G] \)-module \( M \) and commutative ring \( R \), we make \( M \otimes R \) into an \( R[G] \)-module in the obvious way. The following proposition generalizes \([A]\).

Proposition 2.1. Let \( M \subseteq k \) be a free \( \mathbb{Z} \)-module of finite rank such that \( \sigma(M) = M \) for all \( \sigma \in G \). For subfields \( F \subseteq k \), let \( M_F = M \cap F \). For every \( F \subseteq k \), suppose that
\[
M_F \otimes \mathbb{F}_p \hookrightarrow M \otimes \mathbb{F}_p.
\]
If there exists \( m \in M \) such that \( [M : (m)_{\mathbb{Z}[G]}] < \infty \), then \( M \otimes \mathbb{F}_p \) is a cyclic \( \mathbb{F}_p[G] \)-module.

Proof. Suppose \( k \) is cyclic with \( \sigma \) generating \( G \). Let \( m \in M \) such that the index \( [M : (m)_{\mathbb{Z}[G]}] \) is finite. Let \( r = \text{rank}_\mathbb{Z} M \) and let \( \rho : G \to \text{GL}(r, \mathbb{Z}) \)
be the representation induced by the action of \( G \) on a fixed \( \mathbb{Z} \)-basis, say \( \{m_1, m_2, \ldots, m_r\} \), for \( M \). Let \( m_\rho(\sigma) \) and \( h_\rho(\sigma) \) be the minimal and characteristic polynomials for \( \rho(\sigma) \), respectively. Since (ii) holds, it follows that \( r \leq n \) and
\[
h_\rho(\sigma)(x) = m_\rho(\sigma)(x) | x^n - 1.
\]
Now, let \( g(x) : G \to \text{GL}(r, \mathbb{F}_p) \) be the representation induced by the action of \( G \) on the \( \mathbb{F}_p \)-basis \( \{m_1 \mod pM, \ldots, m_r \mod pM\} \). Note that
\[
h_\rho(\sigma)(x) \equiv h_g(\sigma)(x) \mod p.
\]
Since $p \nmid n$, it follows that $h_{\rho(\sigma)}$ factors into a product of distinct irreducibles modulo $p$. So $M/pM \cong M \otimes \mathbb{F}_p$ is a cyclic $\mathbb{F}_p[G]$-module.

Now suppose $k$ is not necessarily cyclic. Let $\chi \in \hat{G}$, and let $F$ be the fixed field of $\ker \chi$. Then $\chi$ is a non-trivial character of $H = G/\ker \chi$, the Galois group of the cyclic extension $F/\mathbb{Q}$. Since (i) holds, we have that

$$M_F \otimes F \hookrightarrow M \otimes F.$$  

Let $e^*_\chi$ denote the idempotent associate to $\chi \in \text{Hom}_\mathbb{Z}(G/\ker \chi, \mathbb{Q}_p(\zeta_n))$, i.e.,

$$e^*_\chi = \frac{1}{|G/\ker \chi|} \sum_{\sigma \in G/\ker \chi} \chi(\sigma)\sigma^{-1}.$$

Note that

$$e_\chi = e^*_\chi : s(\ker \chi)$$

where $s(\ker \chi)$ denotes the sum of the elements of $\ker \chi$. Note that

$$e_\chi(m \otimes 1) = e^*_\chi(N_F^k(m) \otimes (\# \ker \chi)^{-1}).$$

Since $p \nmid n$, the norm map $N_F^k : M \otimes \mathbb{F} \to M_F \otimes \mathbb{F}$ is surjective. It follows that

$$e_\chi(M \otimes \mathbb{F}) = e^*_\chi(M_F \otimes \mathbb{F}).$$

We know that $M_F \otimes \mathbb{F}$ is cyclic since $M_F \otimes \mathbb{F}_p$ is cyclic. Let $g_F \in M_F \otimes \mathbb{F}$ such that $\langle g_F \rangle_{\mathbb{F}[G]} = M_F \otimes \mathbb{F}$. Then

$$e^*_\chi(M_F \otimes \mathbb{F}) = \{e^*_\chi g_F^\theta : \theta \in \mathbb{F}[G]\} = \langle e^*_\chi g_F \rangle_{\mathbb{F}}.$$

Hence

$$\dim_{\mathbb{F}} e_\chi(M \otimes \mathbb{F}) = \dim_{\mathbb{F}} e^*_\chi(M_F \otimes \mathbb{F}) = \begin{cases} 1 & \text{if } e^*_\chi g_F \neq 0 \\ 0 & \text{if } e^*_\chi g_F = 0. \end{cases}$$

For every $\chi \in \hat{G}$ such that $e_\chi(M \otimes \mathbb{F}) \neq 0$, let $g_\chi$ be a generator for the 1-dimensional space $e_\chi(M \otimes \mathbb{F})$. Then summing over all $\chi$ such that $e_\chi(M \otimes \mathbb{F})$ is non-trivial we obtain

$$g = \sum g_\chi,$$

a generator for $M \otimes \mathbb{F}$. Hence

$$M \otimes \mathbb{F} \cong \mathbb{F}[G]/\sum' e_\chi,$$

where the sum in the quotient is over all $\chi$ such that $e_\chi(M \otimes \mathbb{F})$ is trivial. Fix a character $\chi$, and let $X = \{\chi^g : g \in \text{Gal}(\mathbb{F}/\mathbb{F}_p)\}$. Let

$$e_X = \sum_{\chi \in X} e_\chi \in \mathbb{F}_p[G].$$

Note that

$$\dim_{\mathbb{F}_p} e_X(M \otimes \mathbb{F}_p) = \dim_{\mathbb{F}_p} e^*_X(M_F \otimes \mathbb{F}_p),$$
where $F$ is the field belonging to $\chi$ and $e^*_X$ is defined in analogy with $e^*_X$. Regarding $M_F \otimes F_p$ as an $F_p[x]$-module where $x$ acts like a generator for $\text{Gal}(F/\mathbb{Q})$, we have that

$$M_F \otimes F_p \simeq \bigoplus F_p[x]/q(x)$$

where the direct sum is taken over a finite set of distinct irreducibles each of which is a divisor of $x^{[F:\mathbb{Q}]} - 1$. Since $\text{Gal}(F/F_p)$ permutes the roots of each factor $q(x)$, it follows that either $e^*_X(M_F \otimes F_p)$ is trivial or that it corresponds with one of the irreducible subspaces $F_p[X]/q(x)$. Hence

$$\dim_{F_p} e^*_X(M \otimes F_p) = \begin{cases} 0 & \text{if } e^*_\chi(M \otimes F) = 0 \\ |X| & \text{if } e^*_\chi(M \otimes F) \neq 0. \end{cases}$$

It follows that $\sum' e^*_\chi \in F_p[G]$, and we obtain the following isomorphism of $F_p[G]$-modules:

$$\left( M \otimes F_p \right) \oplus \cdots \oplus \left( M \otimes F_p \right) \simeq \frac{F_p[G]}{\sum' e^*_\chi} \oplus \cdots \oplus \frac{F_p[G]}{\sum' e^*_\chi}.$$ 

Since $M \otimes F_p$ and $F_p[G]/\sum' e^*_\chi$ are finite $F_p[G]$-modules, they both decompose uniquely (up to isomorphism) into a direct sum of indecomposable modules. Combining this fact with the isomorphism above, it must be that

$$M \otimes F_p \simeq \frac{F_p[G]}{\sum' e^*_\chi}.$$ 

This completes the proof of the proposition. \hfill \Box

**Lemma 2.2.** Let $M$ be a $\mathbb{Z}[G]$-module. Then $M \otimes \mathbb{F}$ is a cyclic $\mathbb{F}[G]$-module if and only if $M \otimes \mathbb{O}$ is a cyclic $\mathbb{O}[G]$-module.

**Proof.** For $m \in M \otimes \mathbb{O}$, we have the exact sequence

$$\langle m \rangle_{\mathbb{O}[G]} \otimes \mathbb{O} \mathbb{F} \to (M \otimes \mathbb{O}) \otimes \mathbb{O} \mathbb{F} \to \left( M \otimes \mathbb{O} / \langle m \rangle \right) \otimes \mathbb{O} \mathbb{F} \to 0.$$ 

Since $\mathbb{F}$ is the residue field of $\mathbb{O}$, we obtain the chain of equivalences

$$M \otimes \mathbb{O} = \langle m \rangle_{\mathbb{O}[G]} \iff \left( M \otimes \mathbb{O} / \langle m \rangle_{\mathbb{O}[G]} \right) \otimes \mathbb{O} \mathbb{F} = 0$$

$$\iff \langle m \rangle_{\mathbb{O}[G]} \otimes \mathbb{O} \mathbb{F} \to (M \otimes \mathbb{O}) \otimes \mathbb{O} \mathbb{F}$$

$$\iff (M \otimes \mathbb{O}) \otimes \mathbb{O} \mathbb{F} \simeq M \otimes \mathbb{F} \text{ is cyclic.}$$

This completes the proof of the lemma. \hfill \Box

**Corollary 2.3.** The modules $E \otimes \mathbb{Z}_p$ and $\mathfrak{o} \otimes \mathbb{Z}_p$ are cyclic $\mathbb{Z}_p[G]$-modules. In fact

$$E \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p[G] / s(G) \quad \text{and} \quad \mathfrak{o} \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p[G].$$
Proof. Let $F$ be a subfield of $k$, then $E_F$ is the set of units of $F$ and $\mathfrak{o}_F$ is
the ring of integers of $F$. Since $k$ is real and Galois, it follows that
$$E_F \otimes \mathbb{F}_p \hookrightarrow E \otimes \mathbb{F}_p.$$  
Similarly, we have that
$$\mathfrak{o}_F \otimes \mathbb{F}_p = \mathfrak{o}_F / p \hookrightarrow \mathfrak{o} / p = \mathfrak{o} \otimes \mathbb{F}_p.$$  
The claim now follows from Proposition 2.1 and Lemma 2.2. □

Remark 2.4. The Hilbert-Speiser theorem states that $\mathfrak{o} \simeq \mathbb{Z}[G]$ whenever $k$
is tamely ramified at all finite primes. The latter half of Corollary 2.3 is a
sort of “up-to-multiples-of-$p^N$” version of this theorem.

For a prime $\ell$, we adopt the following notation to be used throughout.
$$\sigma_\ell = \text{a Frobenius automorphism in } G \text{ for } \ell,$$
$$I_\ell = \text{the inertia subgroup in } G \text{ of } \ell,$$
$$s(I_\ell) = \text{the sum in } \mathbb{Z}[G] \text{ of those automorphisms in } I_\ell,$$
$$e_\ell = \frac{s(I_\ell)}{|I_\ell|}.$$

Corollary 2.5. Suppose $\ell \neq p$ is a rational prime and $\mathfrak{L}$ the product of
primes of $\mathfrak{o}$ over $\ell$. Let $f = (\mathfrak{o}/1 : \mathbb{Z}(\ell))$ where $1$ is a prime above $\ell$, and let
$p^v$ be the largest power of $p$ dividing $\ell^f - 1$. Then
$$(\mathfrak{o}/\mathfrak{L})^\times \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p[G]/(p^v, \ell - \sigma_\ell, 1 - e_\ell).$$
In the case when $\ell = p$, we have
$$(\mathfrak{o}/p^e)^\times \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p[G]/(p^e, p^e - 1, e_p).$$

Proof. Suppose $\ell \neq p$. Then
$$(\mathfrak{o}/\mathfrak{L})^\times \otimes \mathbb{Z}_p \simeq (\mathfrak{o}/\mathfrak{L})^\times \otimes \mathbb{Z}_p \simeq \prod_{i=1}^r (\mathfrak{o}/l_i)^\times \otimes \mathbb{Z}_p,$$
where the $l_i$ are the primes of $\mathfrak{o}$ over $\ell$. Let $u \in \mathfrak{o}^\times$ such that $u \text{ mod } l_1$ is a
generator for the group $(\mathfrak{o}/l_1)^\times$ and $u \equiv 1 \text{ mod } l_i$ for all $i = 2, 3, \ldots, r$. We have
$$(\mathfrak{o}/\mathfrak{L})^\times \otimes \mathbb{Z}_p = \langle (u \text{ mod } \mathfrak{L}) \otimes 1 \rangle_{\mathbb{Z}_p[G]}.$$  
Consider the surjective map
$$\varphi : \mathbb{Z}_p[G] \to (\mathfrak{o}/\mathfrak{L})^\times \otimes \mathbb{Z}_p$$
$$\theta \mapsto ((u \text{ mod } \mathfrak{L}) \otimes 1)\theta.$$  
Clearly, $(p^v, \ell - \sigma_\ell, 1 - e_\ell) \subseteq \ker \varphi$. The claim will now follow from the fact
that the finite ring $\mathbb{Z}_p[G]$ modulo the ideal $(p^v, \ell - \sigma_\ell, 1 - e_\ell)$ has the correct
order. To compute this order, we extend scalars to $O$ and decompose into $\chi$-components. Note that
\[ e_\chi(1 - e_\ell) = \begin{cases} e_\chi, & I_\ell \not\subset \ker \chi \\ 0, & I_\ell \subset \ker \chi. \end{cases} \]
What’s more, $\min(p^v, \ell - \chi(\sigma_\ell)) = \ell - \chi(\sigma_\ell)$ ($p$-adically, of course) since $p^v$ is the $p$-part of $\prod (\ell - \chi(\sigma_\ell))$, the product taken over those characters giving distinct values for $\chi(\sigma_\ell)$. It follows that
\[ e_\chi(p^v, \ell - \sigma_\ell, 1 - e_\ell) = \begin{cases} O \cdot e_\chi, & I_\ell \not\subset \ker \chi \\ (\ell - \chi(\sigma_\ell)) \cdot e_\chi, & I_\ell \subset \ker \chi. \end{cases} \]
So
\[ \left| \mathbb{Z}_p[G]/(p^v, \ell - \sigma_\ell, 1 - e_\ell) \right| = \prod_{\chi \in \hat{G}} |\ell - \chi(\sigma_\ell)|_p, \]
where $\hat{G} = G/I_\ell$. Let $G_\ell = G/\ell I_\ell$ where $G_\ell$ is the decomposition group for $\ell$, and let $X = \{ \chi \in \hat{G} : G_\ell \subseteq \ker \chi \}$. Since the order of $\sigma_\ell \in G_\ell$ is $f$, it follows that
\[ \prod_{\chi \in \hat{G}} (\ell - \chi(\sigma_\ell)) = \prod_{\chi \in \hat{G}/X} (\ell - \chi(\sigma_\ell))^{X} \]
\[ = \prod_{a=0}^{f-1} (\ell - \zeta_\ell^a)^r \]
\[ = (\ell^f - 1)^r. \]
This proves the claim in the $\ell \neq p$ case.

Now, suppose $\ell = p$. For each prime $p_i$ of $\mathfrak{o}$ dividing $p$, let $U_{i,j} = \{ x \in \mathfrak{o} : x \equiv 1 \text{ mod } p_i^j \}$. Then we have
\[ (\mathfrak{o}/p^e)^\times \otimes \mathbb{Z}_p \simeq \prod_{i=1}^r \left( \mathfrak{o}/p_i^{e\cdot t_p} \right)^\times \otimes \mathbb{Z}_p \simeq \prod_{i=1}^r U_{i,1}/U_{i,e\cdot t_p}, \]
where $t_p$ is the ramification index of $p$. We have a $\mathbb{Z}_p[G_p]$-isomorphism
\[ U_{1,1}/U_{1,e\cdot t_p} \to p_1/p_1^{e\cdot t_p} \\
\begin{align*} x &\mapsto \log_p(x) \text{ mod } p_1^{e\cdot t_p}. \end{align*} \]
Since $p$ is tamely ramified in $k$, the above map makes sense. So we have
\[ (\mathfrak{o}/p^e)^\times \otimes \mathbb{Z}_p \simeq \prod_{i=1}^r p_i/p_i^{e\cdot t_p} \simeq \mathfrak{P}/p^e \]
as $\mathbb{Z}_p[G_p]$-modules where $\mathfrak{P}$ is the product of primes of $\mathfrak{o}$ over $p$. By Corollary 2.3, we have that $(\mathfrak{o}/p^e)^\times \otimes \mathbb{Z}_p$ is cyclic. Consider the module
\[ p_i/p_i^{t_p} \subset \mathfrak{o}/p_i^{t_p} \simeq \mathbb{Z}_p[G_p]/p. \]
Since
\[ \frac{o}{p_i} \simeq \mathbb{Z}_p[\mathcal{G}_p]/(p, 1 - e_p) , \]
it follows that
\[ p_i/p_t^{e_p} \simeq \mathbb{Z}_p[\mathcal{G}_p]/(p, e_p) \Rightarrow p_i/p_t^{e+e_p} \simeq \mathbb{Z}_p[\mathcal{G}_p]/(p^e, p^{e-1} . e_p) . \]
Whence
\[ (o/p^e)^{\times} \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p[\mathcal{G}]/(p^e, p^{e-1} . e_p) . \]
\[ \square \]

3. The Ray Class Gras Conjecture

To prove Theorem 1.1, Sinnott and then Schmidt embedded the units into \( \mathbb{R}[\mathcal{G}] \) via a logarithmic map \( l \) defined by
\[ l : k^\times \rightarrow \mathbb{R}[\mathcal{G}] \]
\[ x \mapsto -\frac{1}{2} \sum_{\sigma \in \mathcal{G}} \log |x^\sigma|^\sigma^{-1} . \]
The index \([E_d : C_d] = [l(E_d) : l(C_d)]\) is then dissected using various linear transformations. For example, we get
\[ [E_d : C_d] = (l(E_d) : \mathbb{R}[\mathcal{G}]_{tr=0})(\mathbb{R}[\mathcal{G}]_{tr=0} : l(C_d)) \]
where
\[ (l(E_d) : \mathbb{R}[\mathcal{G}]_{tr=0}) := |\det T| \]
and \( T \) is any linear transformation such that \( T(l(E_d)) = \mathbb{R}[\mathcal{G}]_{tr=0} \). The second index is similarly defined. For the proof of Theorem 1.2, our method will be highly analogous to those previously employed. The key differences will be that we replace \( \mathbb{R}[\mathcal{G}] \) with \( \mathcal{K}[\mathcal{G}] \) and the map \( l \) with the map \( \vartheta : k^\times \rightarrow \mathcal{K} \) defined by
\[ x \mapsto \sum_{\sigma \in \mathcal{G}} \log_p (x^\sigma)^\sigma^{-1} , \]
where \( \mathcal{K} \) is the extension of the topological closure of \( k \hookrightarrow \mathbb{Q}_p^{alg} \) containing all the character values of \( \mathcal{G} \) and \( \log_p \) is the Iwasawa logarithm.

Note that \( \log_p(k^\times) \subset \mathcal{O} \) since \( p \) is tamely ramified in \( \mathcal{K} \). For a character \( \chi \neq 1 \) with conductor \( f \), let \( L'_p(0, \chi) \) denote the special value
\[ L'_p(0, \chi) = \sum_{a=1}^{f} \log_p (1 - \zeta_p^a)^{\chi(a)} . \]
Let
\[ \omega'_p = \sum_{\chi \neq 1} L'_p(0, \chi)e_\chi \in \mathcal{K}[\mathcal{G}] . \]
We write $D_d = D(d) \cap \{ \alpha \in k^\times : \alpha \equiv 1 \text{ mod } d \}$, and we define the following $O[G]$-modules:

\[
\begin{align*}
T_0 &= \vartheta(E) \cdot O[G] & T_{d,0} &= \vartheta(E_d) \cdot O[G] \\
S(d) &= \vartheta(D(d)) \cdot O[G] & S_0(d) &= \vartheta(C(d)) \cdot O[G] \\
S_d &= \vartheta(D_d) \cdot O[G] & S_{d,0} &= \vartheta(C_d) \cdot O[G] \\
M(d) &= \{ \alpha_n, d : \delta_n, d \in D(d) \} \cdot O[G] & M_d &= \{ \theta \cdot \alpha_n, d : \delta_n, d \in D_d, \theta \in \mathbb{Z}[G] \} \cdot O[G].
\end{align*}
\]

We write $M_0(d)$ for $(1 - e_1)M(d)$, i.e., the trace zero subspace of $M(d)$ and $R_0$ for the trace zero subspace of $O[G]$. Let $m$ be the conductor of $k$, and let $d' = (d, m)$. The idea is to analyze the following sequence of maps

\[T_{d,0} \to T_0 \to R_0 \to S_{d',0} \to S_{d,0}.\]

This will be profitable since

\[E_d/C_d \otimes O \simeq E_d \otimes O/C_d \otimes O \simeq T_{d,0}/S_{d,0}.\]

First, we need the following $p$-adic formulation of $[S]$.

**Proposition 3.1** (Sinnott). Fix $\chi \neq 1$ and let $f$ be the conductor of $\chi$. Write $k^f$ for $\mathbb{Q}(\zeta_f) \cap k$, $G_f$ for $\text{Gal}(k/k^f)$. Then

\[ (3.1) \quad (1 - e_1) \cdot \vartheta \left( N_{k^f}^{\mathbb{Q}(\zeta_f)}(1 - \zeta_f) \right) = \omega_p \cdot s(G_f) \cdot \prod_{\ell \mid f} (1 - \sigma_\ell^{-1} \cdot e_\ell) \in O[G]. \]

**Proof.** Let $L$ and $R$ denote the respective left and right hand sides of Equation (3.1). For any $\psi \in \hat{G}$, let $L_\psi, R_\psi \in O$ such that

\[ e_\psi L = L_\psi e_\psi \quad \text{and} \quad e_\psi R = R_\psi e_\psi. \]

It suffices to show that $R_\psi = L_\psi$ for every $\psi \in \hat{G}$. For brevity, let $\delta_f = N_{k^f}^{\mathbb{Q}(\zeta_f)}(1 - \zeta_f)$. Since

\[ \vartheta(\delta_f) = s(G_f) \cdot \sum_{\sigma \in G/G_f} \log_p(\delta_f^\sigma)^{-1}, \]

it’s easy to see that $R_\psi = L_\psi = 0$ if $\psi$ is either the trivial character or is a non-trivial character on $G_f$.

Now, suppose $\psi \neq 1$ is trivial on $G_f$. Note that

\[ R_\psi = L'_p(0, \psi) \cdot |G_f| \cdot \prod_{\ell \mid f} (1 - \overline{\psi}(\ell)) \]

and

\[ e_\psi \cdot L = e_\psi \cdot \vartheta(\delta_f) = |G_f| \cdot \sum_{\sigma \in G/G_f} \log_p(\delta_f^\sigma) \overline{\psi}(\sigma) \cdot e_\psi, \]

so

\[ L_\psi = |G_f| \cdot \sum_{(a, f) = 1}^f \log_p(1 - \zeta_f^a) \overline{\psi}(a). \]
Write $f_\psi$ for the conductor of $\psi$. Since $G_f \leq \ker \psi$, it follows that $f_\psi \mid f$.

With $f_\psi t = f$, we get

$$
\sum_{a=1}^{f_\psi t} \log_p (1 - \zeta_f^a) \overline{\psi}(a) = \sum_{a=1}^{f_\psi t} \log_p (1 - \zeta_f^a) \overline{\psi}(a) \cdot \prod_{\ell \mid f} (1 - \overline{\psi}(\ell)) \\
= \sum_{a=1}^{f_\psi} \log_p (1 - \zeta_f^a) \overline{\psi}(a) \cdot \prod_{\ell \mid f} (1 - \overline{\psi}(\ell)) \\
= L'_p(0, \psi) \cdot \prod_{\ell \mid f} (1 - \overline{\psi}(\ell)).
$$

So $L_\psi = R_\psi$, and this completes the proof of the proposition. \qed

From Corollary 2.3, it follows that $E \otimes \mathcal{O} \simeq R_0$. Fix $\epsilon_0 \in E$ such that $\epsilon_0 \otimes 1$ generates $E \otimes \mathcal{O}$. Then $E \otimes \mathcal{O} = \langle \epsilon_0 \otimes 1 \rangle_{\mathcal{O}[G]}$. Let $\rho = \vartheta(\epsilon_0)$ so that $T_0 = \rho \cdot R_0$, and write $\rho = \sum_{\chi \neq 1} \rho_\chi e_\chi$. For any $\alpha \in \mathcal{K}[G]$, we define $\alpha^{-1} \in \mathcal{K}[G]$ to be the element such that

$$
\alpha \cdot \alpha^{-1} = \sum_{\epsilon_\chi \alpha \neq 0} e_\chi.
$$

**Corollary 3.2.** For every $\chi \neq 1$, let $f_\chi$ denote the conductor of $\chi$ and $\delta_\chi$ the cyclotomic number $N_{k_f(\chi)}(1 - \zeta_{f_\chi})$. Let

$$
\gamma = \sum_{\chi \neq 1} \vartheta(\delta_\chi) e_\chi.
$$

Then $S_0 = \gamma R_0$ and $S_0 = \gamma \rho^{-1} T_0$. In particular, we have the following isomorphism of $\mathcal{O}$-modules:

$$
e_\chi (E \otimes \mathcal{O} / C \otimes \mathcal{O}) \simeq \mathcal{O} / (r_\chi^{-1} L'_p(0, \chi)).
$$

**Proof.** Using notation from §1, we have $\vartheta(C)$ is the kernel in $\vartheta(D)$ of multiplication by $s(G)$. It follows that

$$(1 - e_1) S = S_0$$

Now, let $\pi \in \mathcal{O}$ be a generator of the prime ideal of $\mathcal{O}$. Since $\gamma = (1 - e_1) \gamma$, we apply Proposition 3.1 to obtain

$$e_\chi \cdot \rho^{-1} \gamma = (\rho_\chi^{-1} \cdot |G_{f_\chi}| \cdot L'_p(0, \chi)) e_\chi.
$$

Since $|G_{f_\chi}|$ is a $p$-adic unit, it follows that

$$(3.2) \quad \rho_\chi^{-1} L'_p(0, \chi) \in e_\chi (\rho^{-1} S_0) .$$

Since

$$E \otimes \mathcal{O} / C \otimes \mathcal{O} \simeq \rho^{-1} T_0 / \rho^{-1} S_0 = R_0 / \rho^{-1} S_0$$
Equation (3.2) and Theorem 1.1 give us that
\[ |\mathcal{C} \otimes \mathcal{O}| = |E \otimes \mathcal{O}/C \otimes \mathcal{O}| \leq \prod_{\chi \neq 1} \left| \mathcal{O}/ (\rho^{-1}_\chi L'_p(0, \chi)) \right|. \]

Note that
\[ \prod_{\chi \neq 1} \rho_\chi = \det(\alpha \mapsto \rho \alpha) = \text{Reg}_p' \]
where \( \text{Reg}_p' \) differs from \( \text{Reg}_p \), the Leopoldt regulator of \( k \), by a unit of \( \mathbb{Z}_p \).

Using the \( p \)-adic class number formula
\[ |\mathcal{C}| = \frac{1}{\text{Reg}_p} \prod_{\chi \neq 1} L'_p(0, \chi) \]
we get that
\[ |E \otimes \mathcal{O}/C \otimes \mathcal{O}| \leq \prod_{\chi \neq 1} \left| \mathcal{O}/ (\rho^{-1}_\chi L'_p(0, \chi)) \right| = |E \otimes \mathcal{O}/C \otimes \mathcal{O}|. \]

It follows that
\[ e_\chi \left( R_0/\rho^{-1}s_0 \right) = e_\chi \left( R_0/\rho^{-1}_0 \gamma R_0 \right). \]
So \( \rho^{-1}_0 \gamma R_0 = \rho^{-1}s_0 \) and the corollary follows. \( \square \)

For any proper divisor \( t \mid f \), let \( \alpha_f(t) \in \mathcal{O}[G] \) be defined by
\[ \alpha_f(t) := \left[ \mathbb{Q}(\zeta) : k^f \mathbb{Q}(\zeta_f/t) \right] \cdot s(G_f) \cdot \prod_{\ell \mid f/t} \left( 1 - \sigma_{\ell}^{-1} \cdot e_{\ell} \right). \]

Then Proposition 3.1 reads \( (1 - e_1)\vartheta(\delta_f) = \omega_p' \cdot \alpha_f(1) \). Let \( \alpha_{n,d} \in \mathcal{O}[G] \) be defined by
\[ \alpha_{n,d} := \frac{d}{\bar{d}} \cdot \kappa_{\bar{d}/(\bar{d},n)} \cdot \sum_{t \mid (\bar{d},n)} \mu(t) \cdot \frac{(d,n)}{t} \cdot \alpha_n(t) \]
where \( \bar{d} \) is the product of distinct prime divisors of \( d \) and
\[ \kappa_t = \prod_{\ell \mid t} (\ell - \sigma_{\ell} \cdot e_{\ell}). \]

Using the above notation we obtain another corollary of Proposition 3.1.

**Corollary 3.3.** If \( n \geq 1 \) and \( n \nmid \bar{d} \), then \( (1 - e_1)\vartheta(\delta_{n,d}) = \omega_p' \cdot \alpha_{n,d} \).

**Proof.** For \( n > 1 \) and \( n \nmid \bar{d} \), it’s straightforward to verify that
\[ \delta_{n,d} = \delta_{\frac{(d/\bar{d})}{(\bar{d},n)}} \cdot \kappa_{\bar{d}/(\bar{d},n)}. \]

The corollary now follows easily from Proposition 3.1. \( \square \)

**Lemma 3.4.** \( S_{d',0} = (1 - e_1)S_{d'} = \omega_p' \cdot M_0(d') \).
Proof. Suppose $\delta \in C_{d'}$. Then

$$s(G) \cdot \vartheta(\delta) = \vartheta(\delta^s(G)) = 0.$$ 

So $S_{d',0} \subseteq (1 - e_1)S_{d'}$. The reverse containment follows from the fact that $\delta_n$ (using notation from Proposition 3.1 is a non-unit only when $n$ is a prime power, in which case, we have $\delta_n^s = 1$ is a unit for any $\sigma \in G$.

Let $n > 1$ and $\theta \in \mathbb{Z}[G]$ be such that $\delta_{n,d'}^\theta \in D_{d'}$. By Corollary 3.3 we have

$$(1 - e_1)\vartheta(\delta_{n,d'}^\theta) = \omega'_p \cdot \theta_{\alpha_{n,d'}} \in \omega'_p M_{d'}.$$ 

If $M_0(d')$ and $M_{d'}$ are generated as $\mathbb{Z}[G]$-modules, then $M_{d'}$ is a $\mathbb{Z}[G]$-submodule of $M_0(d')$ of index co-prime to $p$ by [Sch]. In our case, this means that $M_0(d') = M_{d'}$. □

For a prime $\ell$, we let $\chi(\ell)$ denote the quantity

$$\frac{\chi(\sigma_\ell)}{|I_\ell|} \sum_{\tau \in I_\ell} \chi(\tau).$$

Lemma 3.5. Let $\mu \in \mathcal{O}[G]$ be defined by

$$\mu = \sum_{\chi \neq 1} \mu_\chi e_\chi \quad \text{where} \quad \mu_\chi := \prod_{\ell | d'} (\ell - \chi(\ell)).$$

Then $M_0(d') = \mu \cdot R_0$.

Proof. The idea is to compute the $\chi$-part of $\alpha_{n,d'}$, the generators for $M(d')$, for $\chi \neq 1$. Note that $M(d')$ is generated by those $\alpha_{n,d'}$ where $n = y \cdot m_r$ with

$$m_r = \prod_{\ell | r} p^{\ord_{\ell} m},$$

$\gcd(r, d') = 1$, $r \mid m_r$, and $y \mid m_d$ [Sch]. Of these, we have that

$$e_\chi \alpha_{ym_r,d'} = \begin{cases} \chi(\alpha_{ym_r,d'}) e_\chi, & f_\chi \mid ym_r, \ y = n \cdot (y, f_\chi), \ n = \pi, \ (n, f_\chi) = 1 \\ 0, & \text{otherwise} \end{cases}$$

where

$$\chi(\alpha_{ym_r,d'}) = \pm \chi(n)[k : k^{ym_r}] \cdot \prod_{\ell | r} (1 - \chi(\ell)) \cdot \prod_{\ell | d'/y} (\ell - \chi(\ell)) \cdot \prod_{\ell | y} (\chi(\ell) - \ell)$$

by [Sch]. The above quantity will have the smallest $p$-adic valuation precisely when we choose $y = (f_\chi, m_d)$ and $r = (f_\chi, y)$, in which case we get

$$\chi(\alpha_{ym_r,d'}) = \pm \chi(n)[k : k^{ym_r}] \cdot \prod_{\ell | d'} (\ell - \chi(\ell)),$$

and the claim follows. □
Again from Corollary 2.3, it follows that $E_d \otimes \mathcal{O} \simeq R_0$. Fix $\epsilon_d \in E_d$ such that $\epsilon_d \otimes 1$ generates $E_d \otimes \mathcal{O}$. Then $E_d \otimes \mathcal{O} = \langle \epsilon_d \otimes 1 \rangle_{\mathcal{O}[G]}$. Let $v \in \mathcal{O}[G]$ such that $(\epsilon_0 \otimes 1)^v = \epsilon_d \otimes 1$ so that $(E \otimes \mathcal{O})^v = E_d \otimes \mathcal{O}$. Per usual, we write $v = \sum_{\chi \neq 1} v\chi e_\chi$.

**Theorem 3.6.** Let $\Theta \in \mathcal{O}[G]$ be defined by

$$\Theta = v^{-1} \cdot \rho^{-1} \cdot \mu \cdot \omega'_p \cdot \frac{d}{d} \cdot \kappa_{d/d'}.$$

Then $S_{d,0} = \Theta \cdot T_{d,0}$.

**Proof.** The sequence $T_{d,0} \to T_0 \to R_0 \to S_{d',0} \to S_{d,0}$ is made explicit with the help of the preceding lemmas:

$$T_{d,0} \xrightarrow{v^{-1}} T_0 \xrightarrow{\rho^{-1}} R_0 \xrightarrow{\mu} M_0(d') \xrightarrow{\omega'_p} S_{d',0} \xrightarrow{\frac{d}{d} \cdot \kappa_{d/d'}} S_{d,0}.$$

The last arrow follows from [Sch].

We may now give the

**Proof of Theorem 1.2.** Let $\Theta =: \sum_{\chi \neq 1} \Theta \chi e_\chi$ be as in Theorem 3.6. We let $X$ be the $\text{Gal}(\mathbb{Q}_p(\zeta_{|G|})/\mathbb{Q}_p)$-orbit of $\chi \neq 1$. Let $v = \sum_{\psi \in X} \psi \psi$. Then

$$|\text{Syl}_p(E_d/C_d)_\psi| = \prod_{\chi \in X} \Theta \chi,$$

where $a =_{p} b$ means that $a$ and $b$ differ by a $p$-adic unit. On the other hand, by Lemma 3.5 we have

$$\Theta \chi = v^{-1}_\chi \cdot \rho^{-1}_\chi \cdot \prod_{\ell \mid d'} (\ell - \chi(\ell)) \cdot L'_p(0, \chi) \cdot \frac{d}{d} \cdot \prod_{\ell \mid d} (\ell - \chi(\ell))$$

$$= \rho^{-1}_\chi L'_p(0, \chi) \cdot \frac{d}{d} \prod_{\ell \mid d} (\ell - \chi(\ell)) \cdot v^{-1}.$$

By Corollary 3.2, we have

$$|\text{Syl}_p(E/C)_\psi| = \prod_{\chi \in X} \rho^{-1}_\chi L'_p(0, \chi).$$

Greenberg [RG] showed that the quantity on the left is equal to $|\text{Syl}_p(\mathcal{C})_\psi|$ assuming the Main Conjecture of Iwasawa Theory. That conjecture was proven by Mazur and Wiles in [MW]. By Corollary 2.5, we have

$$|\text{Syl}_p((\mathfrak{o}/d)^\times)_\psi| =_{p} (d/d')^{\left|X\right|} \prod_{\chi \in X} \prod_{\ell \mid d} (\ell - \chi(\ell)).$$

And finally,

$$|\text{Syl}_p(E/E_d)_\psi| =_{p} \prod_{\chi \in X} v\chi.$$
Putting everything together, we get

\[
|\text{Syl}_p(\mathfrak{C}_d)\varphi| = \frac{|\text{Syl}_p(E/C)\varphi| \cdot |\text{Syl}_p((\mathfrak{g}/d)^{\times})\varphi|}{|\text{Syl}_p(E_d)\varphi|}
= p \prod_{\chi \in X} \rho_\chi^{-1} L'_p(0, \chi) \cdot \prod_{\ell | d} (\ell - \chi(\ell)) \cdot \nu_\chi^{-1}
= p |\text{Syl}_p(E_d/C_d)\varphi|.
\]

This completes the proof of the theorem. \(\square\)

4. **Gauss Sums and Stickelberger’s Theorem for Ray Class Groups**

The results in this section are applicable to situations when \(p \mid [k : \mathbb{Q}]\). The main goal is to prove Theorem 1.3, a ray class version of a theorem of Rubin which itself generalized a theorem of Thaine.

The following lemma will act as an explicit version of Hilbert’s Theorem 90 for our purposes.

**Lemma 4.1.** Let \(\ell\) be a rational prime completely split in \(k\). For any \(\epsilon \in \mathfrak{o}_k^{\times}\) such that \(N_k(\zeta_\ell) = 1\), we have that the element

\[
\alpha := -\zeta_\ell - \zeta_\ell^2 \epsilon - \cdots - \zeta_\ell^{\tau - 2} \epsilon^{1 + \tau + \cdots + \tau^{\ell - 3}}
\]

is non-zero for some choice of \(\zeta_\ell\), moreover, \(\alpha^{-1} = \epsilon\) where \(\langle \tau \rangle = \text{Gal}(k(\zeta_\ell)/k)\).

**Proof.** Let \(\alpha(x) \in \mathbb{C}(x)\) be the rational function defined by

\[
x \mapsto -\frac{\zeta_\ell}{1 - x\zeta_\ell} - \frac{\zeta_\ell^2}{1 - x\zeta_\ell^2} \cdot \epsilon - \cdots - \frac{\zeta_\ell^{\tau - 2}}{1 - x\zeta_\ell^{\tau - 2}} \cdot \epsilon^{1 + \tau + \cdots + \tau^{\ell - 3}}.
\]

Since \(\alpha(x)\) has distinct poles, it follows that \(\alpha(x)\) is not identically zero. On the other hand, we may view \(\alpha(x) \in \mathbb{C}[[x]]\) and write

\[
\alpha(x) = \sum_{a=0}^{\infty} \left( -\zeta_\ell^{a+1} - \zeta_\ell^{(a+1)\tau} \epsilon - \cdots - \zeta_\ell^{(a+1)\tau^{\ell - 2}} \epsilon^{1 + \tau + \cdots + \tau^{\ell - 3}} \right) x^a.
\]

Note that the power series form of \(\alpha(x)\) has periodic coefficients of the form of the claim. Since \(\alpha(x)\) is not identically zero, we get that \(\alpha \neq 0\) for some choice of \(\zeta_\ell\). In fact, \(\alpha = 0\) for at least two choices of \(\zeta_\ell\) for if otherwise, then \(\alpha(x)\) has a pole at \(x = 1\), a contradiction. This proves the first claim.

Now, notice that

\[
\epsilon \alpha^\tau = -\zeta_\ell \epsilon - \zeta_\ell^2 \epsilon^{1 + \tau} - \cdots - \zeta_\ell^{\tau - 2} \epsilon^{1 + \tau + \cdots + \tau^{\ell - 2}}
= -\zeta_\ell \epsilon - \zeta_\ell^2 \epsilon^{1 + \tau} - \cdots - \zeta_\ell
= \alpha,
\]

since \(\tau^{\ell - 1} = \text{id}\) and \(1 + \tau + \cdots + \tau^{\ell - 2} = N_k^{k(\zeta_\ell)}\). This proves the lemma. \(\square\)
Fix an ideal \( a \subseteq \mathfrak{a} \). For odd primes \( \ell \) that are completely split in \( k \), let

\[
E(\ell, a) := \{ \varepsilon \in E_{k(\zeta_\ell)} : N_k^{k(\zeta_\ell)}(\varepsilon) = 1 \text{ and } \varepsilon \equiv 1 \text{ mod } a \}.
\]

The following theorem is the ray class version of a theorem of Rubin [R] which itself was a generalization of a theorem of Thaine [Th].

**Theorem 4.2.** Let \( n \in \mathbb{N} \) and \( \ell \) be an odd prime split completely in \( k \) such that \( \ell \equiv 1 \) mod \( n \). Fix a prime \( \lambda \) of \( k \) above \( \ell \), and let \( \mathcal{A} \subseteq (\mathbb{Z}/n\mathbb{Z})[G] \) be the annihilator of the cokernel of the natural map

\[
\phi : E(\ell, a) \rightarrow (\mathfrak{o}_{k(\zeta_\ell)}/L)^{\times} \otimes \mathbb{Z}/n\mathbb{Z},
\]

where \( L \) is the product of all primes of \( \mathfrak{o}_{k(\zeta_\ell)} \) above \( \ell \). Then \( \mathcal{A} \) annihilates the class of \( \lambda \) in \( \mathcal{E}(\mathfrak{a})/n\mathcal{E}(\mathfrak{a}) \).

**Proof.** Let \( \theta \in \mathcal{A} \), and let \( u \in \mathfrak{o}_{k(\zeta_\ell)} \) such that

\[
u \equiv s^{-1} \text{ mod } L \quad \text{ and } \quad u \equiv 1 \text{ mod } L^{\sigma} \quad \text{ for all } \sigma \neq \text{id},
\]

where \( L \) is the prime of \( \mathfrak{o}_{k(\zeta_\ell)} \) above \( \lambda \) and \((s) = \mathbb{Z}/\ell\mathbb{Z}^\times\). The element \( u \) has been chosen so that

\[
(\mathfrak{o}_{k(\zeta_\ell)}/L)^{\times} = \langle u \text{ mod } L \rangle_{\mathbb{Z}/(\ell-1)\mathbb{Z}[G]}.
\]

Now, \( u^\theta \equiv \eta^\theta \varepsilon \text{ mod } L \) for some \( \eta \in k(\zeta_\ell)^\times \) coprime to \( \ell \) and \( \varepsilon \in E(\ell, a) \). Let \( \tau \) be a generator for \( \text{Gal}(k(\zeta_\ell)/k) \) and

\[
\alpha := -\zeta_\ell - \zeta_\ell^{1+\tau} - \zeta_\ell^{1+\tau+\cdots+\tau^2} - \cdots - \zeta_\ell^{1+\tau+\cdots+\tau^3}.
\]

By Lemma 4.1, we may assume \( \alpha \neq 0 \). Since \( \varepsilon \in E(\ell, a) \) we have

\[
\alpha \equiv -\zeta_\ell - \zeta_\ell^{2+\cdots+\tau^2} \equiv 1 \text{ mod } a.
\]

Now, \((\alpha)\) is a non-zero ideal inert under the action imposed by \( \text{Gal}(k(\zeta_\ell)/k) \). It follows that

\[
(\alpha) = b \cdot \prod_{\sigma \in G} \mathfrak{L}^{a_\sigma - 1},
\]

where \( 0 \leq a_\sigma < \ell - 1 \) and \( b \) is an ideal of \( \mathfrak{o}_k \). Taking norms of both sides of the above we get

\[
\left(N_k^{k(\zeta_\ell)}(\alpha)\right) = b^{\ell - 1} \cdot \lambda^{\sum a_\sigma - 1}.
\]

Since \( \alpha \equiv 1 \text{ mod } a \), we have that \( N_k^{k(\zeta_\ell)}(\alpha) \equiv 1 \text{ mod } a \). By assumption we have \( n \mid (\ell - 1) \), so \( \sum a_\sigma - 1 \mod n\mathbb{Z}[G] \) annihilates the class of \( \lambda \) in \( \mathcal{E}(\mathfrak{a})/n\mathcal{E}(\mathfrak{a}) \).

It remains to relate the coefficients \( a_\sigma \) to \( \theta \). To that end, note that

\[
a_\sigma = \text{ord}_{\mathfrak{L}^{a_\sigma - 1}}(\alpha) = \text{ord}_{\mathfrak{L}^{a_\sigma - 1}}(1 - \zeta_\ell)^{a_\sigma}.
\]
Write \( \alpha = \beta (1 - \zeta_\ell)^{a_\sigma} \) where \( \beta \) is a \( L_{\sigma^{-1}} \)-unit. Without loss of generality, let’s suppose \( \tau : \zeta_\ell \to \zeta_\ell^s \). The primes above \( \ell \) are totally ramified in \( k(\zeta_\ell)/k \). So \( \tau \) acts trivially on \( L_{\sigma^{-1}} \)-units modulo \( L_{\sigma^{-1}} \). Hence

\[
\epsilon = \frac{\alpha}{\alpha^\tau} = \frac{\beta (1 - \zeta_\ell)^{a_\sigma}}{\beta^\tau (1 - \zeta_\ell^s)^{a_\sigma}} 
\equiv \left( \frac{1 - \zeta_\ell}{1 - \zeta_\ell^s} \right)^{a_\sigma} \mod L_{\sigma^{-1}} 
\equiv (s^{-1})^{a_\sigma} \mod L_{\sigma^{-1}}.
\]

This gives us that \( \epsilon \equiv u^{a_\sigma^{-1}} \mod L_{\sigma^{-1}} \), so

\[
\epsilon \equiv u^{\sum a_\sigma^{-1}} \equiv \eta^{-n} u^\theta \mod L.
\]

Hence \( \sum a_\sigma^{-1} \equiv \theta \mod n\mathbb{Z}[G] \). \( \square \)

**Remark 4.3.** If in Lemma 4.1 we take \( k = \mathbb{Q}(\zeta_m) \) and \( \ell \equiv 1 \mod m \) with \( \epsilon = \zeta_m \), then \( \alpha \) is the classical Gauss sum. In this case, we have that

\[
(\alpha^{\ell-1}) = \lambda \sum a_\sigma^{-1}
\]

where, similar to Theorem 4.2, we have \( \zeta_m \equiv u^{a_\sigma^{-1}} \mod \lambda_{\sigma^{-1}} \). The dependence of the coefficients \( a_\sigma \) on \( \ell \) is easy to tease out of this congruence, and we’re a hop, skip, and a jump away from the classical Stickelberger theorem. For the more general types of elements \( \alpha \) in Lemma 4.1 and Theorem 4.2, the dependence of the \( a_\sigma \) on \( \ell \) is more difficult to separate. Instead of reckoning with this obstacle, we step around it and show that any \( G \)-module map from \( E/E^{p^n} \) to \( \mathbb{Z}/p^n\mathbb{Z}[G] \) can be effectively filtered through \( \left( \mathfrak{o}_{k(\zeta_\ell)}/L \right)^{\mathbb{X}} \otimes \mathbb{Z}/p^n\mathbb{Z} \) for certain well chosen primes \( \ell \). This idea was first employed by Rubin in [R].

Theorem 4.2 inspires us to make the following definition.

**Definition 4.4.** For an ideal \( a \subseteq \mathfrak{o} \), let \( \mathcal{D}(a) \) denote the set of numbers \( \delta \in k^{\mathbb{X}} \) such that for all but finitely many primes \( \ell \) split completely in \( k \), we have that there is an \( \epsilon \in E_k(\ell, a) \) such that for all \( \sigma \in G \),

\[
\epsilon \equiv \delta \mod \mathcal{L}^\sigma
\]

where \( \mathcal{L} \subset \mathfrak{o}_{k(\zeta_\ell)} \) is a prime ideal such that \( \mathcal{L} \mid \ell \). We call \( \mathcal{D}(a) \) the \( a \)-special numbers of \( k \). Let

\[
\mathcal{C}(a) := \mathcal{D}(a) \cap E.
\]

We call \( \mathcal{C}(a) \) the \( a \)-special units of \( k \).

The 1-special numbers are, in fact, Rubin’s special numbers from [R]. It’s fair to ask if \( a \)-special units even exist. For an appropriate choice of \( d \), the following theorem will show that Schmidt’s \( d \)-cyclotomic units are contained in the \( a \)-special units. So \( \mathcal{C}(a) \) is a subgroup of finite index of \( E \).

**Theorem 4.5.** Let \( d(a) = d \in \mathbb{N} \) be the minimal integer such that \( a \mid d \). If \( \delta \in D(d) \), then \( \pm \delta \in \mathcal{D}(a) \), i.e., \( \pm D(d) \subseteq \mathcal{D}(a) \).
Proof. It suffices to show that $\pm \delta_{n,d} \in D(a)$ for all $n > 1$ and $n \nmid d = d(a)$ since these numbers generate $D(d)$. Let $\ell$ be a rational prime split completely in $k$ such that $(\ell, nd) = 1$. Define

$$\pm \epsilon_{n,d} = N^{k_n(\zeta_n)}_{k_n(\zeta)} \prod_{t \mid d} (\zeta^t - \zeta_n^t)^{\mu(t)d/t} \in k(\zeta).$$

Let $\lambda$ be a prime of $k$ above $\ell$ and $\mathcal{L}$ the prime of $k(\zeta)$ above $\lambda$. Since

$$(1 - \zeta_\ell)_{\mathcal{O}_{k(\zeta)}} = \prod_{\sigma \in G} \mathcal{L}^\sigma,$$

it follows that $\zeta_\ell \equiv 1 \mod \mathcal{L}^\sigma$ for all $\sigma \in G$, hence $\pm \epsilon_{n,d} \equiv \pm \delta_{n,d} \mod \mathcal{L}^\sigma$ for all $\sigma \in G$. Now, we note

$$N^{k_n(\zeta_n)}_{k_n(\zeta)}(\pm \epsilon_{n,d}) = N^{k_n(\zeta_n)}_{k_n(\zeta)} N^{k_n(\zeta_n)}_{k_n(\zeta)} \prod_{t \mid d} (\zeta^t - \zeta_n^t)^{\mu(t)d/t}$$

$$= N^{k_n(\zeta_n)}_{k_n(\zeta)} \prod_{t \mid d} \left( \frac{\zeta^t_{n,t} - 1}{\zeta_n^t - 1} \right)^{\mu(t)d/t}$$

$$= \sigma_{n,d}^{-1},$$

where $\sigma_\ell$ is the Frobenius automorphism for $\ell$ in $k$. Since $\ell$ splits completely in $k$, it follows that $\sigma_\ell = 1$ hence $N^{k_n(\zeta_n)}_{k_n(\zeta)}(\epsilon_{n,d}) = N^{k_n(\zeta_n)}_{k_n(\zeta)}(\epsilon_{n,d}) = 1$.

Now, let $q \mid d$ be a prime, let $q^j$ be the $q$-primary part of $d$, and let $d_q = d/q^j$. Then

$$\epsilon_{n,d} = N^{Q(\zeta_n)}_{k_n(\zeta)} \prod_{t \mid d_q} \left[ \frac{(\zeta^t_{n,t} - \zeta_n^t)q}{(\zeta^t_{q,t} - \zeta_n^t)} \right]^{q^j-1} \prod_{t \mid d_q} \mu(t)d_q/t.$$

For all $t \mid d/q$ we have that $\zeta^t_{\ell}$ and $\zeta^t_{q,t}$ are primitive $\ell$-th roots of unity since $(\ell, nd) = 1$, moreover, $\zeta^t_{n,t}$ and $\zeta^t_{q,n}$ are not equal to 1 since $n \nmid d$. It follows that $(\zeta^t_{\ell} - \zeta^t_{n,t})$ and $(\zeta^t_{q} - \zeta^t_{n,t})$ are units in $\mathbb{Z}[\zeta_n]$, hence $\epsilon_{n,d} \in \mathcal{O}_{k_n(\zeta)}. We also have

$$\frac{(\zeta^t_{\ell} - \zeta^t_{n,t})q}{\zeta^t_{q,t} - \zeta^t_{n,t}} \equiv 1 \mod q$$

from which it follows that

$$\left[ \frac{(\zeta^t_{\ell} - \zeta^t_{n,t})q}{\zeta^t_{q,t} - \zeta^t_{n,t}} \right]^{q^j-1} \equiv 1 \mod q^j,$$

whence $\epsilon_{n,d} \equiv 1 \mod d$. Since $q$ was an arbitrary divisor of $d$, it follows that $\epsilon_{n,d} \equiv 1 \mod d$, hence $\epsilon_{n,d} \equiv 1 \mod a$. Therefore, we have $\epsilon_{n,d} \in E(\ell, a)$, as desired. \qed
Now, let
\[ A_n(a) = \mathcal{C}(a)/p^n\mathcal{C}(a) \]
\[ A(a) = \text{Syl}_p(\mathcal{C}(a)) \]
\[ F_n(a) = \text{the ray class field over } k \text{ associate to } A_n(a). \]
\[ F(a) = \text{the ray class field over } k \text{ associate to } A(a). \]

Note that
\[ \text{Gal}(F_n(a)/k) \simeq A_n(a) \quad \text{and} \quad \text{Gal}(F(a)/k) \simeq A(a) \]
via the Artin map. Set \( A'_n(a) \leq A_n(a) \) such that \( A'_n(a) \simeq \text{Gal}(F_n(a)/(F_n(a) \cap k(\zeta_{p^n}))). \)

**Theorem 4.6.** Let \( \alpha : E/E^{p^n} \to \mathbb{Z}/p^n\mathbb{Z}[G] \) be a \( G \)-module map. Then
\[ \alpha(\mathcal{C}(a)E^{p^n}/E^{p^n}) \text{ annihilates } A'_n(a). \]

**Proof.** The argument here is essentially the same as in [R] but with straightforward adjustments, so we give a somewhat abbreviated version of the proof. Let \( \delta \in \mathcal{C}(a) \) and \( \epsilon \in A'_n(a) \). Let \( \mathcal{G} = \text{Gal}(k(\zeta_{p^n})/\mathbb{Q}) \) and
\[ \Gamma = \text{Gal}\left( k(\zeta_{p^n}, E^{1/p^n})/k(\zeta_{p^n}, (\ker \alpha)^{1/p^n}) \right). \]

Let \( \mathcal{G} \) act on \( \Gamma \) by conjugation, and let \( \gamma_1, \ldots, \gamma_j \) be a complete system of unique representatives of \( \Gamma/\mathcal{G} \). Since \( F_n(a) \) and \( k(\zeta_{p^n}, E^{1/p^n}) \) are linearly disjoint over \( F_n(a) \cap k(\zeta_{p^n}) \), we may choose \( \beta_i \in \text{Gal}(F_n(a)k(\zeta_{p^n}, E^{1/p^n})/k) \) such that
\[ \beta_i|_{F_n(a)} = \epsilon \quad \text{and} \quad \beta_i|_{k(\zeta_{p^n}, E^{1/p^n})} = \gamma_i. \]

By the Chebotarev Density Theorem, there exists infinitely many degree 1 non-conjugate \( j \)-tuples of primes \( \lambda_1, \ldots, \lambda_j \subseteq \mathfrak{o}_k \) such that \( (\lambda_i, a) = 1 \) and \( \beta_i \) is in the conjugacy class of Frobenius automorphisms for \( \lambda_i \) in \( \text{Gal}(F_n(a)k(\zeta_{p^n}, E^{1/p^n})/k) \). It follows that \( \lambda_i \in \mathfrak{c} \). We let \( \ell_i \) be the rational prime below \( \lambda_i \). Since \( \beta_i|_{k(\zeta_{p^n})} = \text{id} \), it follows that \( \ell_i \equiv 1 \mod p^n \).

Now, let \( \phi \) be the natural map from \( E/E^{p^n} \to (\mathfrak{o}/\mathfrak{L})^\times \otimes \mathbb{Z}/p^n\mathbb{Z} \) where \( \mathfrak{L} = \prod_{i=1}^{2} \ell_i \). Using the fact that \( \Gamma \) is saturated with Frobenius automorphisms for the \( \lambda_i \), we have that \( \epsilon \in \ker \alpha \) if and only if \( \epsilon \in \ker \phi \). This allows us to consider the well-defined map
\[ \alpha \circ \phi^{-1} : \text{im}(\phi) \to \mathbb{Z}/p^n\mathbb{Z}[G] \]
which we subsequently lift to a map \( f : (\mathfrak{o}/\mathfrak{L})^\times \otimes \mathbb{Z}/p^n\mathbb{Z} \to (\mathbb{Z}/p^n\mathbb{Z})[G] \)
obtaining the following commutative diagram:

\[
\begin{array}{ccc}
E/E^{p^n} & \xrightarrow{\alpha} & (\mathbb{Z}/p^n\mathbb{Z})[G] \\
\phi \downarrow & & \downarrow \ f \\
(\mathfrak{o}/\mathfrak{L})^\times \otimes \mathbb{Z}/p^n\mathbb{Z} & & \end{array}
\]
Now, without loss of generality, we may assume that for each $i$, there exists $\epsilon_i \in E(\ell_i, a)$ such that
\[
\epsilon_i \equiv \delta \mod \mathcal{L}_i^\sigma \quad \text{for all } \sigma \in G,
\]
where $\mathcal{L}_i \subset \mathfrak{o}_{k(\zeta_i)}$ is the prime above $\lambda_i$. Set
\[
L_i := \prod_{\sigma \in G} \mathcal{L}_i^\sigma.
\]
Since the primes of $\mathfrak{o}$ above $\ell_i$ are totally ramified in $k(\zeta_i)$, we have that
\[
(o/\mathcal{L})^\times \simeq \prod_{i=1}^j (o/\ell_i)^\times \simeq \prod_{i=1}^j \left( \mathfrak{o}_{k(\zeta_i)}/L_i \right)^\times.
\]
This association allows us to consider $\phi$ and $f$ as functions defined into and on
\[
\prod \left( \mathfrak{o}_{k(\zeta_i)}/L_i \right)^\times \otimes \mathbb{Z}/p^n \mathbb{Z},
\]
respectively. Let $u_i \in \mathfrak{o}_{k(\zeta_i)}$ such that
\[
u_i \equiv s_i^{-1} \mod \mathcal{L}_i \quad \text{and} \quad u_i \equiv 1 \mod \mathcal{L}_i^\sigma \quad \text{for all } \sigma \in G, \sigma \neq id,
\]
where $\langle s_i \rangle = \mathbb{Z}/\ell_i \mathbb{Z}$, as in Theorem 4.2. Note that
\[
\langle u_i \mod L_i \rangle (\mathbb{Z}/(\ell_i-1) \mathbb{Z})[G] \simeq \left( \mathfrak{o}_{k(\zeta_i)}/L_i \right)^\times.
\]
Let $\theta_i \in (\mathbb{Z}/p^n \mathbb{Z})[G]$ such that
\[
(\delta \mod L_i) \otimes 1 = (u_i \theta_i \mod L_i) \otimes 1 \in \left( \mathfrak{o}_{k(\zeta_i)}/L_i \right)^\times \otimes \mathbb{Z}/p^n \mathbb{Z}.
\]
Since $\delta \mod L_i \equiv \epsilon \mod L_i$, it follows that $\theta_i$ is an annihilator of the cokernel of the map
\[
E(\ell_i, a) \rightarrow \left( \mathfrak{o}_{k(\zeta_i)}/L_i \right)^\times \otimes \mathbb{Z}/p^n \mathbb{Z}.
\]
So $\theta_i$ annihilates the class of $\lambda_i$ in $\mathfrak{c}(a)/p^n \mathfrak{c}(a)$ by Theorem 4.2. But $\lambda_i \in \mathfrak{c}$, so $\theta_i$ annihilates $\mathfrak{c}$. Since
\[
\alpha(\delta \mod E^{p^n}) = \sum_{i=1}^j \theta_i \cdot f(\langle u_i \mod L_i \rangle \otimes 1),
\]
we get
\[
\mathfrak{c}^\alpha(\delta \mod E^{p^n}) = \prod_{i=1}^j \left( \mathfrak{c}^{\theta_i} \right)^{f(\langle u_i \mod L \rangle \otimes 1)} = 1.
\]
This completes the proof of the theorem. $\square$

We may now give a proof of Theorem 1.3
Proof of Theorem 1.3. Let \( \omega_1, \omega_2, \ldots, \omega_t \) be a \( \mathbb{Z}_p \)-basis for \( \mathcal{O} \), and let \( \omega_1^*, \ldots, \omega_t^* \) be a basis for the co-different of \( \mathcal{O} \) dual to \( \omega_1, \ldots, \omega_t \) with respect to the Galois trace. Write

\[
\alpha(\epsilon) = \sum_{\sigma \in G} a_\sigma(\epsilon) \sigma^{-1}.
\]

For every \( \sigma, \tau \in G \), we have \( a_\sigma(\epsilon^\tau) = a_{\sigma \tau} \). Let \( A = \alpha_{\text{id}} \) so that \( \alpha_{\sigma}(\epsilon) = A(\epsilon^\sigma) \). We decompose \( A(\epsilon) \) according to the \( \mathbb{Z}_p \)-basis \( \omega_1, \ldots, \omega_t \):

\[
A(\epsilon) = \sum_{i=1}^t A_i(\epsilon) \omega_i,
\]

with \( A_i(\epsilon) \in \mathbb{Z}_p \). For \( i = 1, \ldots, t \), define \( \alpha_i : E \to \mathbb{Z}_p[G] \) by

\[
\alpha_i(\epsilon) = \sum_{\sigma \in G} \text{tr} (A_i(\epsilon) \cdot \omega_i^*) \sigma^{-1}.
\]

Each \( \alpha_i \) is a \( G \)-module map, moreover, we have the decomposition of \( \alpha \):

\[
\alpha(\epsilon) = \sum_{i=1}^t \alpha_i(\epsilon) \omega_i.
\]

Since \( \text{Syl}_p(\mathfrak{C}_a) \leq A'_1(\mathfrak{a}) \), the corollary now follows from Theorem 4.6. \( \square \)

We now prove Theorem 1.4.

Proof of Theorem 1.4. Let \( \vartheta \) be the \( G \)-module map defined in the previous section. For every \( \alpha \in \text{Hom}_G(E, \mathcal{O}[G]) \), there exists \( \beta \in \mathcal{K}[G] \) such that \( \alpha = \beta \vartheta \) by [A]. So for every \( \beta \in \mathcal{K}[G] \) such that \( \beta \vartheta(E) \subseteq \mathcal{O}[G] \), \( \beta \vartheta(\mathcal{C}(\mathfrak{a})) \) annihilates \( \mathfrak{C}_a \otimes \mathcal{O} \). If the ramification index of \( p \) in \( k \) is \( e(p) = p^j b \) where \( (p, b) = 1 \), then we could take \( \beta = \pi^{je(p)-p^j} \) (see [A]). In particular, we get the explicit annihilation result: for every \( \delta_a \in \mathcal{C}(\mathfrak{a}) \) (perhaps a \( d \)-cyclotomic unit for appropriate \( d \)), we have

\[
\pi^{je(p)-p^j} \sum_{\sigma \in G} \log_p (\delta_a^\sigma) \sigma^{-1} \in \mathcal{O}[G]
\]

annihilates \( \mathfrak{C}_a \otimes \mathcal{O} \). \( \square \)

Remark 4.7. Using the proof of Theorem 1.3, we could construct explicit annihilators of \( \text{Syl}_p(\mathfrak{C}_a) \) from the element of Theorem 1.4.

5. Conclusion

We remark that the results in this article, particularly Theorem 1.3 and Theorem 1.4, confirm a suspicion by D. Solomon regarding a stronger annihilation result lying beyond his [Sol] (see [Sol]). In fact, Theorem 1.2 is possibly indicative of even stronger results. Let \( \mathfrak{a} \subseteq \mathfrak{o} \) be a \( G \)-stable ideal, and let \( \mathcal{C}_a = \{ \delta \in \mathcal{C}(\mathfrak{a}) : \delta \equiv 1 \mod \mathfrak{a} \} \). By Theorem 4.5, we know \( \mathcal{C}_a \subseteq \mathcal{C}_a \). Is this containment proper? When \( \mathfrak{a} = 1 \), this question was posed by Rubin [R] and is still an open problem. Considering Theorems 1.2 and 1.3, we make the following...
Conjecture. If $p \nmid 2 \cdot |G|$, then the order of $\text{Syl}_p(E_a/C_a)_\varrho$ equals the exponent of $\text{Syl}_p(C_a)_\varrho$.

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