A new treatment of mixed virtual and real IR-singularities

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based on work with:
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- Introduction: IR-singularities of massive $n$-point functions
- Mellin-Barnes representations for Feynman diagrams
- Mixed IR-singularities from loops and soft real emission
- Summary
Introduction: IR-singularities of massive $n$-point functions

- We collected some experience in using Mellin-Barnes (MB) representations for massive loop diagrams.
- They have proven very useful for the separation – and also evaluation – of the poles in $\epsilon = (4 - d)/2$ even for very complicated diagrams.
  Often quoted: V. Smirnov (and G. Heinrich) and B. Tausk, planar and non-planar massive double boxes.
- An interesting simpler application – with a potential of automatization – is demonstrated here:
  One-loop $n$-point functions with both virtual and real massless particles. They produce both $1/\epsilon$-poles from the virtual massless lines and the so-called end-point singularities from the phase space integrals with $\int dE/E \to \infty$ from $E = 0$.
- The MB-approach might be an ideal tools for the treatment of that at the amplitude level.
- The mathematica packages MB.m (Czakon, CPC 2005) and AMBRE.m (Gluza, Kajda, Riemann, arXiv:0704.2423, CPC) are well-suited for that.
- The result is not only numerical. We present here a representation in terms of inverse binomial sums and HPL’s.
Example since now: The 5-point function of Bhabha scattering

Radiative loop diagrams contribute to the NNLO corrections by interfering with radiative Born diagrams:

Figure 1: A pentagon topology and a Born topology
Five of the invariants are independent, e.g.:

\[ s = (p_1 + p_5)^2, \]
\[ t = (p_4 + p_5)^2, \]
\[ t' = (p_1 + p_2)^2, \]
\[ V_2 = 2p_2p_3 \sim E_\gamma, \]
\[ V_4 = 2p_4p_3 \sim E_\gamma \]

The invariants \( V_i = 2p_ip_3 \) appear also in the Born diagrams and produce the so-called endpoint singularities:

\[
\frac{1}{(p_2 + p_3)^2 - m^2} = \frac{1}{2p_2p_3 + [p_2^2 - m^2] + [p_3^2 - 0]} = \frac{1}{V_2} = \frac{1}{2E_\gamma E_2(1 - \beta_2 \cos \vartheta)} \sim \frac{1}{E_\gamma}
\]

The photon phase space integral is typically:

\[
\int \frac{d^3p_3}{2E_3} \frac{1}{V_2V_4} \sim \int_0^\omega \frac{dE}{E} = \ln(E)|^\omega_0 = \ln(\omega) - \ln(0) = \text{divergent}
\]

\[
\rightarrow \int_0^\omega \frac{dE}{E^{d-4}} = \frac{1}{d-4} E^{d-4}|_0^\omega = \frac{\omega^{2\epsilon} - 0}{2\epsilon} = \text{finite}
\]

We have to safely control the dependence on \( V_2, V_4 \) as part of the mixed infrared problem due to the common existence of virtual and real IR-sources.
Consider now only the scalar 5-point function.
the massless propagators are $d_5 = k^2$ and $d_2 = (k + p_1 + p_5)^2$.

The leading singularity is easily found algebraically:

$$\frac{1}{d_1 d_2 d_3 d_4 d_5} = -\frac{1}{s} \left[ \frac{2k(k + p_1 + p_5)}{d_1 d_2 d_3 d_4 d_5} - \frac{1}{d_1 d_2 d_3 d_4} - \frac{1}{d_1 d_3 d_4 d_5} \right]$$

The two IR-divergent 4-point functions trace to one IR-div. 3-point f. each, e.g.

$$\frac{1}{d_1 d_3 d_4 d_5} = -\frac{1}{V_2} \left[ \frac{2k(k + p_1 + p_4 + p_5)}{d_1 d_3 d_4 d_5} - \frac{1}{d_1 d_3 d_4} - \frac{1}{d_1 d_4 d_5} \right]$$

and the resulting IR-part is:

$$\int \frac{d^d k}{d_1 d_2 d_3 d_4 d_5} = \frac{1}{s V_2} \int \frac{d^d k}{d_1 d_4 d_5} + \frac{1}{s V_4} \int \frac{d^d k}{d_1 d_2 d_3} + \cdots$$

$$= \frac{1}{\epsilon} \left[ \frac{F(t')}{s V_2} + \frac{F(t)}{s V_4} \right] + \cdots$$

(6)

Evidently, one separates only a leading singularity, while we expect an expression like

$$\int \frac{d^d k}{d_1 d_2 d_3 d_4 d_5} = \frac{A_2}{s V_2 \epsilon} + \frac{A_4}{s V_4 \epsilon} + \frac{B_2}{s V_2} \ln(V_2) + \frac{B_4}{s V_4} \ln(V_4) + \frac{C_2}{s V_2} + \frac{C_4}{s V_4} + \cdots$$
Mellin-Barnes representation for the QED pentagon

The chords $q_i$ are defined from the propagators: $d_i = [(k - q_i)^2 - m_i^2]$

$$I_5[A(q)] = -e^{e\gamma_E} \int_0^1 \prod_{j=1}^5 dx_j \ \delta \left(1 - \sum_{i=1}^5 x_i \right) \frac{\Gamma(3 + \epsilon)}{F(x)^{3+\epsilon}} B(q),$$

with $B(1) = 1, B(q^\mu) = Q^\mu, B(q^\mu q^\nu) = Q^\mu Q^\nu - \frac{1}{2} g^{\mu\nu} F(x)/(2 + \epsilon)$, and $Q^\mu = \sum x_i q_i^\mu$.

The diagram depends on five variables and the $F$-form is:

$$F(x) = m_e^2(x_2 + x_4 + x_5)^2 + [-s]x_1x_3 + [-V_4]x_3x_5 + [-t]x_2x_4 + [-t']x_2x_5 + [-V_2]x_1x_4.$$ (7)

Henceforth, $m_e = 1$. Photon momentum is $p_3$.

The MB-representation,

$$\frac{1}{[A(x) + B x_i x_j]^R} = \frac{1}{2\pi i} \int_C dz [A(x)]^z [B x_i x_j]^{-R-z} \frac{\Gamma(R+z)\Gamma(-z)}{\Gamma(R)},$$

is used several times for replacing in $F(x)$ the sum over $x_i x_j$ by products of monomials in the $x_i x_j$, thus allowing the subsequent $x$-integrations in a simple manner.
Why the Mellin-Barnes integrals?

We want to apply a simple formula for integrating over the $x_i$:

$$\int_0^1 \prod_{j=1}^N dx_j \ x_j^{\alpha_j-1} \delta (1-x_1-\cdots-x_N) = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_N)}{\Gamma(\alpha_1 + \cdots + \alpha_N)}$$

with coefficients $\alpha_i$ dependent on $F$

For this, we have to apply several MB-integrals here:

$$F(x) = m_e^2 (x_2 + x_4 + x_5)^2 + [-s]x_1x_3 + [-V_4]x_3x_5 + [-t]x_2x_4 + [-t']x_2x_5 + [-V_2]x_1x_4.$$  \hspace{1cm} (8)

For each of the $+$-sign one MB-integral, so arrive at a 7-dimensional path integral.
\[
\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1} x_2^{b_2}]^\lambda} = \frac{1}{2\pi i} \frac{1}{\Gamma(\lambda)} \int_{-i\infty}^{i\infty} dz [A(s)x_1^{a_1}]^z [B(s)x_1^{b_1} x_2^{b_2}]^{\lambda+z} \Gamma(\lambda + z) \Gamma(-z)
\]

The integration path has to separate the chains of poles of $\Gamma(\lambda + z)$ and $\Gamma(-z)$:
\[
\text{Res}_F[z] \Gamma(A + z) \big|_{z=-n} = \frac{(-1)^{n-A}}{(n-A)!} F[-n], \quad n = -A, -A - 1, \ldots
\]
\[
\text{Res}_F[z] \Gamma(1 + z)^2 \big|_{z=-n} = \frac{1}{\Gamma[n]^2} (2F[-n] \text{PolyGamma}[n] + F'[-n])
\]
\[
\text{Res}_F[z] \Gamma[1 + z] \text{PolyGamma}[1 + z] \big|_{z=-n} = \frac{(-1)^n}{\Gamma[n]} F'[-n]
\]

with the definitions

\[
S_k[N] = \sum_{i=1}^{N} \frac{1}{i^k}
\]

and

\[
S_1[N] = \text{HarmonicNumber}[n-1] - \text{EulerGamma} = \text{PolyGamma}[n]
\]
Mellin, Robert Hjalmar, 1854-1933
Barnes, Ernest William, 1874-1953
**A little history**

- **N. Usyukina, 1975**: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;  
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral

- **E. Boos, A. Davydychev, 1990**: "A Method of evaluating massive Feynman integrals",  
  Theor. Math. Phys. 89 (1991);  
  N-point 1-loop functions represented by n-dimensional MB-integral

- **V. Smirnov, 1999**: "Analytical result for dimensionally regularized massless on-shell double box",  
  Phys. Lett. B460 (1999);  
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'

- **B. Tausk, 1999**: "Non-planar massless two-loop Feynman diagrams with four on-shell legs",  
  Phys. Lett. B469 (1999);  
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension $d$ for which the MB-integral is finite and well-defined

- **M. Czakon, 2005** (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);  
  Tausk’s approach realized in Mathematica program **MB.m**, published and available for use
We derive MB-representations with AMBRE, a publicly available Mathematica package

J. Gluza, K. Kajda, T. Riemann, arXiv:0704.2423 [hep-ph], to appear in CPC

**AMBRE** – **Automatic Mellin-Barnes Representations** for Feynman diagrams

For the Mathematica package AMBRE, many examples, and the program description, see:

http://prac.us.edu.pl/~gluza/ambre/
http://www-zeuthen.desy.de/theory/research/CAS.html

See also here:

http://www-zeuthen.desy.de/~riemann/Talks/capp07/

with additional material presented at the CAPP – School on Computer Algebra in Particle Physics, DESY, Zeuthen, March 2007
A  AMBRE functions list

The basic functions of AMBRE are:

- **Fullintegra**l[\{numerator\},\{propagators\},\{internal momenta\}] – is the basic function for input Feynman integrals

- **invariants** – is a list of invariants, e.g. invariants = \{p1*p1 → s\}

- **IntPart[iteration]** – prepares a subintegral for a given internal momentum by collecting the related numerator, propagators, integration momentum

- **Subloop[integral]** – determines for the selected subintegral the $U$ and $F$ polynomials and an MB-representation

- **ARint[result,i]** – displays the MB-representation number $i$ for Feynman integrals with numerators

- **Fauto[0]** – allows user specified modifications of the $F$ polynomial $fupc$

- **BarnesLemma[repr,1,Shifts→True]** – function tries to apply Barnes’ first lemma to a given MB-representation; when Shifts→True is set, AMBRE will try a simplifying shift of variables

  - **BarnesLemma[repr,2,Shifts→True]** – function tries to apply Barnes’ second lemma
**AMBRE - Automatic Mellin-Barnes REpresentation**

(arXiv:0704.2423)

To download 'right click' and 'save target as'.

- The package AMBRE.m
- Kinematics generator for 4-, 5- and 6-point functions with any external legs KinematicsGen.m
- Tarball with examples given below examples.tar.gz
  - example1.nb, example2.nb - Massive QED pentagon diagram.
  - example3.nb - Massive QED one-loop box diagram.
  - example4.nb - General one-loop vertex.
  - example5.nb - Six-point scalar functions;
    left: massless case,
    right: massive case.
  - example6.nb, example7.nb - right
    Massive two-loop planar QED box.
  - example8.nb - The loop-by-loop iterative procedure.
MB-representation for the scalar massive QED pentagon

In our example we get a seven-fold MB-representation, reduce to a four-fold representations after three times applying Barnes’ lemma in order to eliminate 2 spurious integrations from the mass term. and one from setting \( t' = t \) (Born kinematics assumed here).

\[
I_5 = \frac{-e^{\epsilon \gamma E}}{(2\pi i)^4} \prod_{i=1}^{4} \int_{-i\infty+u_i}^{+i\infty+u_i} dz_i (-s)^{z_2} (-t)^{z_4} (-V_2)^{z_3} (-V_4)^{-3-\epsilon-z_1-z_2-z_3-z_4} \prod_{j=1..12} \Gamma_j \frac{\prod_{\Gamma_{j=1..12}}}{\Gamma_0 \Gamma_{13} \Gamma_{14}},
\]

with a normalization \( \Gamma_0 = \Gamma[-1 - 2\epsilon] \), and the other \( \Gamma \)-functions are:

\[
\begin{align*}
\Gamma_1 &= \Gamma[-z_1], \quad \Gamma_2 = \Gamma[-z_2], \quad \Gamma_3 = \Gamma[-z_3], \quad \Gamma_4 = \Gamma[1 + z_3], \\
\Gamma_5 &= \Gamma[1 + z_2 + z_3], \quad \Gamma_6 = \Gamma[-z_4], \quad \Gamma_7 = \Gamma[1 + z_4], \quad \Gamma_8 = \Gamma[-1 - \epsilon - z_1 - z_2], \\
\Gamma_9 &= \Gamma[-2 - \epsilon - z_1 - z_2 - z_3 - z_4], \quad \Gamma_10 = \Gamma[-2 - \epsilon - z_1 - z_3 - z_4], \\
\Gamma_{11} &= \Gamma[-\epsilon + z_1 - z_2 + z_4], \quad \Gamma_{12} = \Gamma[3 + \epsilon + z_1 + z_2 + z_3 + z_4], \\
\end{align*}
\]

and

\[
\begin{align*}
\Gamma_{13} &= \Gamma[-1 - \epsilon - z_1 - z_2 - z_4], \quad \Gamma_{14} = \Gamma[-\epsilon - z_1 - z_2 + z_4].
\end{align*}
\]

This is a finite integral if all \( \Gamma \)-functions in the numerator have positive real parts of the arguments.
May be fulfilled with:

\[ \epsilon = -3/4 \]

The real shifts \( u_i \) of the integration strips \( r_i \) are:

\[
\begin{align*}
  u_1 &= -5/8 \\
  u_2 &= -7/8 \\
  u_3 &= -1/16 \\
  u_4 &= -5/8 \\
  u_5 &= -1/32
\end{align*}
\]
Analytical continuation in $\epsilon$ and deformation of integration contours

A well-defined MB-integral was found with the finite parameter $\epsilon$ and the strips parallel to the imaginary axis.

Now look at the real parts of arguments of $\Gamma$-functions (in the numerator only) and find out, which of them change sign (become negative) when $\epsilon \to 0$.

Rule:
Moving $\epsilon \to 0$ corresponds to a step-wise analytical continuation of the contour integral ($dimension = n$) and so we have to add or subtract the residues at these values of the integration variables.

The residues have the dimension of integration $n - 1, n - 2, \ldots$.

This procedure may be automatized "easily" and it is done in the publicly available Mathematica package MB.m (M. Czakon, hep-ph/0511200, CPC)
**Analytical continuation, $0 \neq \epsilon \ll 1$**

After the analytical continuation in $\epsilon$, the scalar pentagon function is represented by 11 MB-integrals. The IR-non-save parts are contained in only few and relatively simple of them:

\[ I_5^{IR} = I_5^{IR}(V_2) + I_5^{IR}(V_4), \]

\[ I_5^{IR}(V_2) = \frac{I_{-1}}{\epsilon} + I_0 \]

\[ \frac{I_{-1}}{\epsilon} = \frac{e^{\epsilon \gamma_E}}{2\pi i} \int_{-i\infty-5/8}^{+i\infty-5/8} dz_1 \frac{(-t)^{-1-z_1}}{2\epsilon s V_2} \Gamma[-z_1]^{3} \Gamma[1+z_1] \Gamma[-2z_1] \]

\[ I_0 = \frac{e^{\epsilon \gamma_E}}{2\pi i} \int_{-i\infty-5/8}^{+i\infty-5/8} \frac{dz_1}{2s V_2} [F_1[z_1] \Gamma[1+z_1] + F_2[z_1] \Gamma[1+z_1] \text{PolyGamma}[1+z_1] \]

\[ + \frac{e^{\epsilon \gamma_E}}{(2\pi i)^2} \int_{-i\infty-7/8}^{+i\infty-7/8} d z_2 \int d z_1 (-s)^{-z_2} (t)^{-z_1+z_2} (-V_2)^{-2-z_2} (-V_4)^{-1-z_2} \]

\[ \Gamma[-z_1] \Gamma[-1-z_2] \Gamma[-1-z_1-z_2] \Gamma[z_1-z_2] \Gamma[-z_2]^2 \Gamma[1+z_2] \Gamma[2+z_2] \Gamma[1-z_1+z_2] \]

\[ \Gamma[-2z_1] \Gamma[-1-2z_2] \]
Before taking sums of residua by closing contours to the left (anti-clockwise), look at powers of \((-V_2)\).

Its real part gives \((-V_2)^{-9/8}\), this would be not integrable for small \(V_2\).

Shift the contour \(z_2\) by a unit to the left.

This changes: \((-V_2)^{-9/8} \rightarrow (-V_2)^{-1/8}\) and after that, the 2-dim. integral is IR-safe.

One residue is crossed and has to be added to the resulting 2-dim. contour integral.

So take here instead of the original 2-dim. integral only the residue as the contribution of interest:

\[
I_0 = \frac{e^{\gamma E}}{2\pi i} \int_{-i\infty-5/8}^{+i\infty-5/8} \frac{dz_1}{2sV_2} \left[ (F_2 + F_4)\Gamma[1 + z_1] + (F_1 + F_5)[z_1]\Gamma[1 + z_1] \text{PolyGamma}[1 + z_1] \right]
\]

\[
F_1 = (-t)^{-1-z_1} \frac{\Gamma[-z_1]^3}{\Gamma[-2z_1]}
\]

\[
F_2 = F_1(\gamma E - 2 \ln[-s] - \ln[-t] + 2 \ln[-V_4])
\]

\[
F_4 = 2F_1(-\gamma E + \ln[-s] + \ln[-t] - \ln[-V_2] - \ln[-V_4])
\]

\[
F_5 = -2F_1
\]
IR-divergencies as inverse binomial sums

Now take the residues and get:

$$\frac{I_{-1}}{\epsilon} = \frac{1}{2sV_2\epsilon} \frac{1}{2\pi i} \int_{-i\infty+u}^{+i\infty+u} dr(-t)^{-1-r} \frac{\Gamma[-r]^3 \Gamma[1+r]}{\Gamma[-2r]}.$$ 

With Mathematica or using Kalmykov et al., Huber and Maitre:

$$\frac{I_{-1}}{\epsilon} = \frac{1}{2sV_2\epsilon} \sum_{n=0}^{\infty} \frac{(t)^n}{\binom{2n}{n}(2n+1)} = \frac{4 \arcsin(\sqrt{t/2})}{\sqrt{4-t\sqrt{t}}} = -\frac{2y\ln(y)}{1-y^2},$$

with

$$y \equiv y(t) = \frac{\sqrt{1-4/t} - 1}{\sqrt{1-4/t} + 1}.$$ 

and for the constant term in $\epsilon$:

$$I_0 = \frac{1}{2sV_2} \sum_{n=0}^{\infty} \frac{(t)^n}{\binom{2n}{n}(2n+1)} [-2\ln[-V_2] - 3S_1[n] + 2S_1[2n]]$$
Rewrite into Polylogs and/or Harmonic PolyLogs

The inverse binomial sums may be summed:

See Davydychiev, Kalmykov and quite recently also Huber, Maitre.

Here, the following question is of some interest:

→ Why these harmonic numbers?

Look at intermediate 11 MB-integrals, e.g.:

One of the 4 contributing MB-integrals – out of the 11 – is Int07:

$$
\text{Int07} = \text{Sum of residues} \\
= \frac{e^{\epsilon/\epsilon} \sqrt{\pi} (-s)^{-1-2\epsilon} (-V)^{2\epsilon}}{2^{2\epsilon} V_4} \\
\cdot \frac{\Gamma[3/2 + \epsilon] \Gamma[-2\epsilon] \Gamma[2\epsilon] \Gamma[1 + 2\epsilon]}{\Gamma[3/2 + \epsilon]} \\
\cdot \text{HypergeometricPFQ}[[1, 1 + 2\epsilon], [3/2 + \epsilon], t/4]
$$
Without taking the sum:

\[
\text{Int07} = \text{Sum of residues} = e^{\epsilon\gamma E (-s)^{-1} - 2\epsilon (-V_2)^{2\epsilon}} \frac{\Gamma[-2\epsilon]\Gamma[1 + 2\epsilon]}{V_4} \\
\sum_{n=1}^{\infty} t^{-1+n} \frac{\Gamma[\epsilon + n]\Gamma[2\epsilon + n]}{\Gamma[2\epsilon + 2n]}\]  

(10)

The well-known formula (Weinzierl 0402131 eq. 35 and maybe many others)

\[
\Gamma[n + 1 + \epsilon] = \Gamma[1 + \epsilon]\Gamma[1 + n] e^{-\sum_{k=1}^{\infty} \frac{(-\epsilon)^k}{k}\text{HarmonicNumber}[n,k]}
\]

shows why we meet the inverse harmonic sums with the harmonic numbers \(S_1[n]\) and \(S_1[2n]\).
Summary

• We present a general algorithm for the evaluation of mixed IR-divergencies from virtual and real emission in terms of inverse binomial sums.

• With AMBRE.m (May 2007) and MB.m (2005) and maybe in more complicated situations also with HypExp 2 on Expanding Hypergeometric Functions about Half-Integer Parameters, arXiv:0708.2443 [hep-ph] this may be automatized.

• The cases of more masses or more legs or more loops or of tensor integrals should not get much more complicated.

• For relatively simple applications like IR-divergent parts, an analytical treatment with MB-integrals may be quite useful.