ON THE TANGENT SPACE OF THE DEFORMATION
FUNCTOR OF CURVES WITH AUTOMORPHISMS

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Abstract. We provide a method to compute the dimension of the tangent space to the global infinitesimal deformation functor of a curve together with a subgroup of the group of automorphisms. The computational techniques we developed are applied to several examples including Fermat curves, $p$-cyclic covers of the affine line and to Lehr-Matignon curves.

The aim of this paper is the study of equivariant equicharacteristic infinitesimal deformations of a curve $X$ of genus $g$, admitting a group of automorphisms. This paper is the result of my attempt to understand the works of J.Berlin - A.Mézard [1] and of G. Cornelissen - F. Kato [3].

Let $X$ be a smooth algebraic curve, defined over an algebraically closed field of characteristic $p \geq 0$. The infinitesimal deformations of the curve $X$, without considering compatibility with the group action, correspond to directions on the vector space $H^1(X, T_X)$ which constitutes the tangent space to the deformation functor of the curve $X$ [8]. All elements in $H^1(X, T_X)$ give rise to unobstructed deformations, since $X$ is one-dimensional and the second cohomology vanishes.

In the study of deformations together with the action of a subgroup of the automorphism group, a new deformation functor can be defined. The tangent space of this functor is given by Grothendieck’s [7] equivariant cohomology group $H^1(X, G, T_X)$, [1, 3.1]. In this case the wild ramification points contribute to the dimension of the tangent space of the deformation functor and also posed several lifting obstructions, related to the theory of deformations of Galois representations.

The authors of [1], after proving a local-global principle, focused on infinitesimal deformations in the case $G$ is cyclic of order $p$ and considered liftings to characteristic zero, while the authors of [3] considered the case of deformations of ordinary curves without putting any other condition on the automorphism group. The ramification groups of automorphism groups acting on ordinary curves have a special ramification filtration, i.e., the $p$-part of every ramification group is an elementary Abelian group, and this makes the computation possible, since elementary Abelian group extensions are given explicitly in terms of Artin-Schreier extensions.

In this paper we consider an arbitrary curve $X$ with automorphism group $G$. By the theory of Galois groups of local fields, the ramification group at every wild ramified point, can break to a sequence of extensions of elementary Abelian groups $[20, IV]$. We will use this decomposition, together with the spectral sequence of Lyndon-Hochschild-Serre in order to reduce the computation, to a computation involving elementary Abelian groups.

We are working over an algebraically closed field of positive characteristic and for the sake of simplicity we assume that $p \geq 5$.

The dimension of the tangent space of the deformation functor, depends on the group structure of the extensions that appear in the series decomposition of the ramification groups at wild ramified points. We are able to give lower and upper bounds of the dimension of the tangent space of the deformation functor.

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In particular, if the decomposition group $G_P$ at a wild ramified point $P$ is the semidirect product of an elementary abelian group with a cyclic group, such that there is only a lower jump at the $i$-th position in the ramification filtration, then we are able to compute exactly the dimension of the local contribution $H^1(G_P, T_O)$ prop. 2.8 and example 4 on page 28.

We begin our exposition in section 1 by surveying some of the known deformation theory. Next we proceed to the most difficult task, namely the computation of the tangent space of the local deformation functor, by employing the low terms sequence stemming from the Lyndon-Hochschild-Serre spectral sequence.

The dimension of equivariant deformations that are locally trivial, i.e., the dimension of $H^1(X/G, \pi^G_0(T_X))$ is computed in section 3. The computational techniques we developed are applied to the case of Fermat curves, that are known to have large automorphism group, to the case of $p$-covers of $\mathbb{P}^1(k)$ and to the case of Lehr-Matignon curves. Moreover, we are able to recover the results of Cornelissen-Kato concerning deformations of ordinary curves. Finally, we try to compare our result with the results of R. Pries concerning the computation of unobstructed deformations of wild ramified actions on curves.

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1. SOME DEFORMATION THEORY

There is nothing original in this section, but for the sake of completeness, we present some of the tools we will need for our study. This part is essentially a review of [1], [3], [4].

Let $k$ be an algebraic closed field of characteristic $p \geq 0$. We consider the category $\mathcal{C}$ of local Artin $k$-algebras with residue field $k$.

Let $X$ be a non-singular projective curve defined over the field $k$, and let $G$ be a fixed subgroup of the automorphism group of $X$. We will denote by $(X, G)$ the couple of the curve $X$ together with the group $G$.

A deformation of the couple $(X, G)$ over the local Artin ring $A$ is a proper, smooth family of curves

$$\mathcal{X} \rightarrow \text{Spec}(A)$$

parametrized by the base scheme $\text{Spec}(A)$, together with a group homomorphism $G \rightarrow \text{Aut}_A(\mathcal{X})$ such that there is a $G$-equivariant isomorphism $\phi$ from the fibre over the closed point of $A$ to the original curve $X$:

$$\phi : \mathcal{X} \otimes_{\text{Spec}(A)} \text{Spec}(k) \rightarrow X.$$ 

Two deformations $\mathcal{X}_1, \mathcal{X}_2$ are considered to be equivalent if there is a $G$-equivariant isomorphism $\psi$, making the following diagram commutative:

$$\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{\psi} & \mathcal{X}_2 \\
\downarrow & & \downarrow \\
\text{Spec}A & & \text{Spec}A
\end{array}$$

The global deformation functor is defined:

$$D_{gl} : \mathcal{C} \rightarrow \text{Sets}, A \mapsto \left\{ \begin{array}{l}
\text{Equivalence classes} \\
\text{of deformations} \\
\text{of couples } (X, G) \text{ over } A
\end{array} \right\}$$

Let $D$ be a functor such that $D(k)$ is a single element. If $k[\epsilon]$ is the ring of dual numbers, then the Zariski tangent space $t_D$ of the functor is defined by $t_D :=$
be a reducible polynomial. If the functor $D$ satisfies the “Tangent Space Hypothesis”, i.e., when the mapping

$$ h : D(k[e] \times_k k[e]) \to D(k[e]) \times D(k[e]) $$

is an isomorphism, then the $D(k[e])$ admits the structure of a $k$-vector space \cite{[4, p.272]}. The tangent space hypothesis is contained in (H3)-Schlessinger’s hypothesis that hold for all the functors in this paper, since all the functors admit versal deformation rings \cite{[4, sect. 2]}. We will apply the classification of groups that can appear as Galois groups of \cite{[6, sect. 2]}. We will follow Green-Matignon \cite{[6]} on expressing the expansion of the Weierstrass polynomial of degree $j$ \cite{[7]}, VII 8. prop. 6] and $u(T)$ is a unit of $A[[T]]$. The reduction of the polynomial $f_j$ modulo $m_A$ gives the automorphism $\sigma$ on $G_j(P)$ but $\sigma$ when lifted on $X$ has in general more than one fixed points, since $f_j(T)$ might be a reducible polynomial. If $f_j(T)$, gives rise to only one horizontal branch divisor then we say that the corresponding deformation does not split the branch locus.

Let $\epsilon : A/(\mathfrak{m}^j) \to D(k[e])$ be a wild ramified point on the special fibre $X$, and let $\sigma \in G_j(P)$ where $G_j(P)$ denotes the $j$-ramification group at $P$. Assume that we can deform the special fibre to a deformation $\mathcal{X} \to A$, where $A$ is a complete local discrete valued ring that is a $k$-algebra. Denote by $m_A$ the maximal ideal of $A$ and assume that $A/m_A = k$. Moreover assume that $\sigma$ acts fibrewise on $\mathcal{X}$. We will follow Green-Matignon \cite{[6]}, on expressing the expansion

$$ \sigma(T) - T = f_j(T)u(T), $$

where $f_j(T) = \sum_{\nu=0}^j a_\nu T^\nu$ ($a_i \in m_A$ for $\nu = 0, \ldots, j-1$, $a_j = 1$) is a distinguished Weierstrass polynomial of degree $j$ \cite{[4]}, VII 8. prop. 6] and $u(T)$ is a unit of $A[[T]]$.

The category of $(G, \mathcal{O}_X)$-modules is the category of $\mathcal{O}_X$-modules with an additional $G$-module structure. We can define two left exact functors from the category of $(G, \mathcal{O}_X)$-modules, namely

$$ \pi_*^G \text{ and } \Gamma^G(X, \cdot), $$

where $\Gamma^G(X,F) = \Gamma(X,F)^G$. The derived functors $R^i\pi_*^G(X,\cdot)$ of the first functor are sheaves of modules on $X$, and the derived functors of the second are groups $H^i(X,G,F) = R^i\Gamma^G(X,F)$. J. Bertin and A. Mézard in \cite{[6, sect. 2]} proved the following

**Theorem 1.1.** Let $\mathcal{T}_X$ be the tangent sheaf on the curve $X$. The tangent space $t_{\mathcal{T}_X}$ to the global deformation functor, is given in terms of equivariant cohomology as $t_{\mathcal{T}_X} = H^1(X,G,\mathcal{T}_X)$. Moreover the following sequence is exact:

(1) \hspace{1em} 0 \to H^1(X/G, \pi_*^G(\mathcal{T}_X)) \to H^1(X,G,\mathcal{T}_X) \to H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \to 0.$$

For a local ring $k[[t]]$ we define the local tangent space $\mathcal{T}_O$, as the $k[[t]]$-module of $k$-derivations. The module $\mathcal{T}_O \cong k[[t]] \frac{d}{dt}$, where $\delta = \frac{d}{dt}$ is the derivation such that $\delta(t) = 1$. If $G$ is a subgroup of $\text{Aut}(k[[t]])$, then $G$ acts on $\mathcal{T}_O$ in terms of the adjoint representation. Moreover there is a bijection $D_p(k[e]) \cong H^1(G, \mathcal{T}_O)$ \cite{[6, sect. 2]}. In order to describe the tangent space of the local deformation space we will compute first the space of tangential liftings, i.e., the space $H^1(G, \mathcal{T}_O)$. This problem is solved when $G$ is a cyclic group of order $p$, by J. Bertin and A. Mézard \cite{[6, sect. 2]} and if the original curve is ordinary by G. Cornelissen and F. Kato in \cite{[6, sect. 2]}

We will apply the classification of groups that can appear as Galois groups of local fields in order to reduce the problem to elementary Abelian group case.
Moreover, if we reduce $\mathcal{X} \times_A \text{Spec} \frac{\mathcal{O}}{\mathfrak{m}_A}$ we obtain an infinitesimal extension that gives rise to a cohomology class in $H^1(G(P), T_G)$ by [3] prop. 2.3.

On the other hand cohomology classes in $H^1(X/G, \pi_*^G(T_X))$ induce trivial deformations on formal neighbourhoods of the branch point $P$ [4, 3.3.1] and do not split the branch points. In the special case of ordinary curves, the distinction of deformations that do or do not split the branch points does not occur since the polynomials $f_j$ are of degree 1.

1.2. Description of the ramification Group. The finite groups that appear as Galois groups of a local field $k((t))$, where $k$ is algebraically closed of characteristic $p$ are known [20].

Let $L/K$ be a Galois extension of a local field $K$ with Galois group $G$. We consider the ramification filtration of $G$.

$$G = G_0 \subset G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subset G_{n+1} = \{1\}.$$  

The quotient $G_0/G_1$ is a cyclic group of order prime to the characteristic, $G_1$ is $p$-group and for $i \geq 1$ the quotients $G_i/G_{i+1}$ are elementary Abelian $p$-groups. If a curve is ordinary then by [15] the ramification filtration is short, i.e., $G_2 = \{1\}$, and this gives that $G_1$ is an elementary Abelian group.

We are interested in the ramification filtrations of the decomposition groups acting on the completed local field at wild ramified points. We introduce the following notation: We consider the set of jumps of the ramification filtration

$$1 = t_f < t_{f-1} < \cdots < t_1 = n,$$

such that

$$G_1 = \ldots \supset G_{t_f} \supset G_{t_{f-1}} \supset \ldots \supset G_{t_{f-1}+1} \geq \cdots \geq G_{t_1} = G_n > \{1\},$$

i.e., $G_{t_i} > G_{t_i+1}$. For this sequence it is known that $t_i \equiv t_j$ mod $p$ [20] Prop. 10 p. 70.

1.3. Lyndon-Hochschild-Serre Spectral Sequences. In [10], Hochschild and Serre considered the following problem: Given the short exact sequence of groups

$$1 \to H \to G \to G/H \to 1,$$

and a $G$-module $A$, how are the cohomology groups

$$H^i(G, A), H^i(H, A) \text{ and } H^i(G/H, A^H)$$

related? They gave an answer to the above problem in terms of a spectral sequence. For small values of $i$ this spectral sequence gives us the low degree terms exact sequence:

$$0 \to H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(H, A)^{G/H} \xrightarrow{\text{tg}} H^2(G/H, A^H) \xrightarrow{\text{inf}} H^2(G, H),$$

where $\text{res, tg, inf}$ denote the restriction, transgression and inflation maps respectively.

Lemma 1.2. Let $H$ be a normal subgroup of $G$, and let $A$ be a $G$-module. The group $G/H$ acts on the cohomology group $H^1(H, A)$ in terms of the conjugation action given explicitly on the level of 1-cocycles as follows: Let $\bar{\sigma} = \sigma H \in G/H$. The cocycle

$$d : H \to A$$

$$x \mapsto d(x)$$

is sent by the conjugation action to the cocycle

$$d^\bar{\sigma} : H \to A$$

$$x \mapsto \sigma d(\sigma^{-1} x \sigma),$$

where $\sigma \in G$ is a representative of $\bar{\sigma}$. 


Proof. This explicit description of the conjugation action on the level of cocycles is given in [23, prop. 2-5-1, p.79] The action is well defined by [23, cor. 2-3-2]. □

Our strategy is to use equation (6) in order to reduce the problem of computation of \( H^1(G, T_O) \) to an easier computation involving only elementary abelian groups.

**Lemma 1.3.** Let \( A \) be a \( k \)-module, where \( k \) is a field of characteristic \( p \). For the cohomology groups we have \( H^1(G_0, A) = H^1(G_1, A)^{G_0/G_1} \).

**Proof.** Consider the short exact sequence

\[
0 \to G_1 \to G_0 \to G_0/G_1 \to 0.
\]

Equation (5) implies the sequence

\[
0 \to H^1(G_0/G_1, A^{G_1}) \to H^1(G_0, A) \to H^1(G_1, A)^{G_0/G_1} \to H^2(G_0/G_1, A^{G_1}).
\]

But the order of \( G_0/G_1 \) is not divisible by \( p \), and is an invertible element in the \( k \)-module \( A \). Thus the groups \( H^1(G_0/G_1, A^{G_1}) \) and \( H^2(G_0/G_1, A^{G_1}) \) vanish and the desired result follows [24, Cor. 6.59]. □

**Lemma 1.4.** If \( G = G_i, H = G_{i+1} \) are groups in the ramification filtration of the decomposition group at some wild ramified point, and \( i \geq 1 \) then the conjugation action of \( G \) on \( H \) is trivial.

**Proof.** Let \( L/K \) denote a wild ramified extension of local fields with Galois group \( G \), let \( \mathcal{O}_L \) denote the ring of integers of \( L \) and let \( m_L \) be the maximal ideal of \( \mathcal{O}_L \).

Moreover we will denote by \( L^* \) the group of units of the field \( L \). We can define [24, Prop. 7 p.67, Prop 9 p. 69] injections

\[
\theta_0 : \frac{G_0}{G_1} \to L^* \quad \text{and} \quad \theta_i : \frac{G_i}{G_{i+1}} \to \frac{m_L^i}{m_{i+1}},
\]

with the property

\[
\forall \sigma \in G_0 \text{ and } \forall \tau \in G_i/G_{i+1} : \theta_0(\sigma \tau \sigma^{-1}) = \theta_0(\sigma)^i \theta_i(\tau).
\]

If \( \sigma \in G_i \subset G_1 \) then \( \theta_0(\sigma) = 1 \) and since \( \theta_i \) is an injection, the above equation implies that \( \sigma \tau \sigma^{-1} = \tau \). Therefore, the conjugation action of an element \( \tau \in G_i/G_{i+1} \) on \( G_j \) is trivial, and the result follows.

\[\square\]

1.4. **Description of the transgression map.** In this section we will try to determine the kernel of the transgression map. The definition of the transgression map given in [23] is not suitable for computations. We will give an alternative description following [10].

Let \( A \) be a \( k \)-algebra that is acted on by \( G \) so that the \( G \) action is compatible with the operations on \( A \). Let \( \hat{A} \) be the set \( \text{Map}(G, A) \) of set-theoretic maps of the finite group \( G \) to the \( G \)-module \( A \). The set \( \hat{A} \) can be seen as a \( G \)-module by defining the action \( f^g(\tau) = gf(g^{-1}\tau) \) for all \( g, \tau \in G \). We observe that \( \hat{A} \) is projective. The submodule \( A \) can be seen as the subset of constant functions. Notice that the induced action of \( G \) on the submodule \( A \) seen as the submodule of constant functions of \( \hat{A} \) coincides with the initial action of \( G \) on \( A \). We consider the short exact sequence of \( G \)-modules:

\[
(6) \quad 0 \to A \to \hat{A} \to A_1 \to 0.
\]

Let \( H \triangleleft G \). By applying the functor of \( H \)-invariants to the short exact sequence (6) we obtain the long exact sequence

\[
(7) \quad 0 \to A^H \to \hat{A}^H \to A_1^H \xrightarrow{\psi} H^1(H, A) \to H^1(H, \hat{A}) = 0,
\]

where the last cohomology group is zero since \( \hat{A} \) is projective.
We split the above four term sequence, by defining $B = \ker \psi$ to two short exact sequences, namely:

$$0 \to A \to \bar{A} \to B \to 0,$$

(8) $$0 \to B \to A^H \xrightarrow{\psi} H^1(H, A) \to 0.$$

Now we apply the $G/H$-invariant functor to the above two short exact sequences in order to obtain:

$$H^i(G/H, B) = H^{i+1}(G/H, A^H),$$

and

(9) $$0 \to B^G/H \to A_1^G \xrightarrow{\delta} H^1(H, A)^G/H \xrightarrow{\delta} H^1(G/H, B) \xrightarrow{\phi} H^1(G/H, A_1^H) \cdots$$

It can be proved [16, Exer. 3 p.71] that the composition

$$H^1(H, A)^G/H \xrightarrow{\delta} H^1(G/H, B) \xrightarrow{\phi} H^1(G/H, A_1^H),$$

is the transgression map.

**Lemma 1.5.** Assume that $G$ is an abelian group. If the quotient $G/H$ is a cyclic group isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and the group $G$ can be written as a direct sum $G = G/H \times H$ then the transgression map is identically zero.

**Proof.** Notice that if $A^H = A$ then this lemma can be proved by the explicit form of the transgression map as a cup product [16, Exer. 2 p.71], [10].

The study of the kernel of the transgression is reduced to the study of the kernel of $\delta$ in (8). We will prove that the map $\phi$ in (8) is 1-1, and then the desired result will follow by exactness.

Let $\sigma$ be a generator of the cyclic group $G/H = \mathbb{Z}/p\mathbb{Z}$. We denote by $N_{G/H}$ the norm map $A \to A$, sending

$$A \ni a \mapsto \sum_{g \in G/H} \sigma^a.$$ 

By $I_{G/H}A$ we denote the submodule $(\sigma - 1) A$ and by $N_{G/H}A = \{ a \in A : N_{G/H}a = 0 \}$. Since $G/H$ is a cyclic group we know that [20, VIII 4], [24, Th. 6.2.2]:

$$H^1(G/H, B) = \frac{N_{G/H}B}{I_{G/H}B} \text{ and } H^1(G/H, A_1^H) = \frac{N_{G/H}A_1^H}{I_{G/H}A_1^H}.$$

Thus, the map $\phi$ is given by

$$\frac{N_{G/H}B}{I_{G/H}B} \to \frac{N_{G/H}A_1^H}{I_{G/H}A_1^H},$$

sending

$$b \bmod I_{G/H}B \mapsto b \bmod I_{G/H}A_1^H.$$ 

The map $\phi$ is well defined since $I_{G/H}B \subset I_{G/H}A_1^H$. The kernel of $\phi$ is computed:

$$\ker \phi = \frac{N_{G/H}B \cap I_{G/H}A_1^H}{I_{G/H}B}.$$ 

The short exact sequence in (8) is a short exact sequence of $k[G/H]$-modules. This sequence seen as a short exact sequence of $k$-vector spaces is split, i.e. there is a section $s : H^1(H, A) \to A_1^H$ so that $\psi \circ s = \text{Id}_{H^1(H, A)}$. This section map is only a $k$-linear map and not apriori compatible with the $G/H$-action.

Let us study the map $\psi$ more carefully. An element $x \in A_1^H$ is a class $a \bmod A$ where $a \in \bar{A}$, and since $x \in A_1^H$ we have that

$$a^h - a = ha - a = c[h] \in A.$$
It is a standard argument that \(c[h]\) is an 1-cocyle \(c[h] : H \to A\) and the class of this cocycle is defined to be \(\psi(x)\). Since the image of \(c[h]\) seen as a cocycle \(c[h] : H \to \tilde{A}\) is trivial, \(c[h]\) is a coboundary \(i.e.\) we can select \(\tilde{a}_c \in \tilde{A}\) so that

\[
(10) \quad c[h] = \tilde{a}_c^h - \tilde{a}_c.
\]

Obviously \(\tilde{a}_c \mod A\) is \(H\)-invariant and we define one section as

\[
s(c[h]) = \tilde{a}_c \mod A.
\]

We have assumed that the group \(G\) can be written as \(G = H \times G/H\) therefore we can write the functions \(\tilde{a}_c\) as functions of two arguments

\[
\tilde{a}_c : \left\{ \begin{array}{c} H \times G/H \to A \\ (h,g) \mapsto \tilde{a}_c(h,g) \end{array} \right.,
\]

Notice that (10) gives us that the for every \(h, h_1 \in H\) the quantity \(\tilde{a}_c(h_1, g_1)^h - \tilde{a}_c(h_1, g_1)\) does not depend on \(g_1 \in G/H\). Now for any element \(g \in G/H\) so that \(g = \sigma H\) the action of \(g\) on \(c[h]\) is given by lemma 1.2

\[
(11) \quad c[h]^g = \sigma c[\sigma h \sigma^{-1}] = \sigma (\tilde{a}_c^{\sigma^{-1}} \cdot \sigma - \tilde{a}_c).
\]

The function \(\tilde{a}_c^{\sigma^{-1}} \cdot \sigma - \tilde{a}_c\) is the function sending

\[
(12) \quad H \times G/H \ni (h_1, g_1) \mapsto \sigma^{-1} h \sigma \tilde{a}_c(h^{-1} h_1, \sigma^{-1} \sigma g_1) = \sigma^{-1} h \sigma \tilde{a}_c(h^{-1} h_1, g_1)
\]

By combining (11) and (12) we obtain

\[
c[h]^g = h \sigma \tilde{a}_c - \sigma \tilde{a}_c,
\]

since the function (notice the invariance on \(g\) in the second argument)

\[
H \times G/H \ni (h_1, g_1) \mapsto h \sigma \tilde{a}_c(h^{-1} h_1, g_1) - \sigma \tilde{a}_c(h_1, g_1) = \sigma^{-1} h \sigma \tilde{a}_c(h^{-1} h_1, g_1) - \sigma \tilde{a}_c(h_1, g_1)
\]

\[
= h \sigma \tilde{a}_c(h^{-1} h_1, g^{-1} g_1) - \sigma \tilde{a}_c(h_1, g^{-1} g_1) = (\tilde{a}_c^\sigma)^h - \tilde{a}_c^\sigma
\]

The above proves that

\[
s(c[h]^g) = s(c[h])^\sigma,
\]

\(i.e.\), the function \(s\) is compatible with the \(G/H\)-action. On the other hand, every element \(a \in A^H_1\) can be written as

\[
a = b_a + s(\psi(a)),
\]

where \(b_a := a - s\psi(a) \in B\), since \(\psi(b_a) = 0\). The arbitrary element in \(I_{G/H} A^H_1\) is therefore written as

\[
(13) \quad (\sigma - 1) a = (\sigma - 1) b_a + s \psi(a) - s(\psi(a)) = (\sigma - 1) b_a + s (\sigma \cdot \psi(a) - \psi(a)).
\]

If \((\sigma - 1) a \in N_{G/H} B \cap I_{G/H} A^H_1\) then since \(\text{Im}(s) \cap B = \{0\}\) we have that

\[
s(\sigma \cdot \psi(a) - \psi(a)) = 0 \iff (\sigma - 1) a = (\sigma - 1) b_a \in I_{G/H} B.
\]

Therefore, \(\phi\) is an injection and the desired result follows. \(\square\)
1.5. The $G$-module structure of $T_G$. Our aim is to compute the first order infinitesimal deformations, i.e., the tangent space $D_p(k[e])$ to the infinitesimal deformation functor $D_p$. \[14\] p.272. This space can be identified with $H^1(G, T_G)$. The conjugation action on $T_G$ is defined as follows:

$$ (f(t) \frac{dt}{dt})^\sigma = f(t)^\sigma \sigma \frac{d}{dt} \sigma^{-1} = f(t)^\sigma \left( \frac{d}{dt} \sigma^{-1}(t) \right) \frac{dt}{dt}. $$

where $\frac{d}{dt} \sigma^{-1}$ denotes the operator sending an element $f(t)$ to $\frac{d}{dt} f^{-1}(t)$, i.e., we first compute the action of $\sigma^{-1}$ on $f$ and then we take the derivative with respect to $t$. We will approach the cohomology group $H^1(G, T_G)$ using the filtration sequence given in \[2\] and the low degree terms of the Lyndon-Hochschild-Serre spectral sequence.

The study of the cohomology group $H^1(G, T_G)$ can be reduced to the study of the cohomology groups $H^1(V, T_G)$, where $V$ is an elementary Abelian group. These groups can be written as a sequence of Artin-Schreier extensions that have the advantage that the extension and the corresponding actions have a relatively simple explicit form:

**Lemma 1.6.** Let $L$ be a an elementary abelian $p$-extension of the local field $K := k((x))$, with Galois group $G = \oplus_{\nu=1}^n \mathbb{Z}_p$, such that the maximal ideal of $k[[x]]$ is ramified completely and the ramification filtration has no intermediate jumps i.e. is given by

$$ G = G_0 = \cdots = G_n > \{1\} = G_{n+1}. $$

Then the extension $L$ is given by $K(y_1, \ldots, y_n)$ where $1/y_i^p - 1/y_i = f_i(x)$, where $f_i \in k((x))$ with a pole at the maximal ideal of order $n$.

**Proof.** The desired result follows by the characterization of Abelian $p$-extensions in terms of Witt vectors, \[14\] 8.11. Notice that the exponent of the group $G$ is $p$ and we have to consider the image of $W_1(k((x))) = k((x))$, where $W_1(\cdot)$ denotes the Witt ring of order $\lambda$ as is defined in \[11\] 8.26.

**Lemma 1.7.** Every $\mathbb{Z}/p\mathbb{Z}$-extension $L = K(y)$ of the local field $K := k((x))$, with Galois group $G = \mathbb{Z}/p\mathbb{Z}$, such that the maximal ideal of $k[[x]]$ is ramified completely, is given in terms of an equation $f(1/y) = 1/x^n$, where $f(z) = z^p - z \in k[z]$. The Galois group of the above extension can be identified with the $\mathbb{F}_p$-vector space $V$ of the roots of the polynomial $f$, and the correspondence is given by

$$ \sigma_v : y \rightarrow \frac{y}{1 + vy} \text{ for } v \in V. $$

Moreover, we can select a uniformization parameter of the local field $L$ such that the automorphism $\sigma_v$ acts on $t$ as follows:

$$ \sigma_v(t) = \frac{t}{(1 + vt^n)^{1/n}}. $$

Finally, the ramification filtration is given by

$$ G = G_0 = \cdots = G_n > \{1\} = G_{n+1}, $$

and $n \neq 0 \mod p$.

**Proof.** By the characterization of Abelian extensions in terms of Witt vectors, \[11\] 8.11 we have that $f(1/y) = 1/x^n$, where $f(z) = z^p - z \in k[z]$ (look also \[22\] A.13). Moreover the Galois group can be identified with the one dimensional $\mathbb{F}_p$-vector space $V$ of roots of $f$, sending $\sigma_v : y \rightarrow \frac{y}{1 + vy}$.\]
The filtration of the ramification group $G$ is given by $G \cong G_0 = G_1 = \cdots G_n$, $G_i = \{1\}$ for $i \geq n+1$ \cite[prop. III.7.10 p.117]{[22]}. By computation
\[
\begin{align*}
    x^n &= \left((1/y)^p - 1/y\right)^{-1} = \frac{y^p}{1 - y^p},
\end{align*}
\]
hence $v_L(y) = n$, i.e., $y = \epsilon t^n$, where $\epsilon$ is a unit in $O_L$ and $t$ is the uniformization parameter in $O_L$.
Moreover, the polynomial $f$ can be selected so that $p \nmid n$ \cite[III. 7.8.]{[22]}. Since $k$ is an algebraically closed field, Hensel’s lemma implies that every unit in $O_L$ is an $n$-th power, therefore we might select the uniformization parameter $t$ such that $y = t^n$, and the desired result follows by (15).

**Lemma 1.8.** Let $H = \oplus_{i=1}^s \mathbb{Z}/p\mathbb{Z}$ be an elementary Abelian group with ramification filtration
\[
H = H_0 = \cdots = H_n > H_{n+1} = \{\text{Id}\} \text{ and } H_\kappa = \{\text{Id}\} \text{ for } \kappa \geq n+1.
\]
The upper ramification filtration in this case coincides with the lower ramification filtration.

**Proof.** Let $m$ be a natural number. We define the function $\phi : [0, \infty) \to \mathbb{Q}$ so that for $m \leq u < m + 1$
\[
\phi(u) = \frac{1}{|H_0|} \sum_{i=1}^m |H_i| + (u - m) \frac{|H_{m+1}|}{|H_0|},
\]
and since $H_{n+1} = \{\text{Id}\}$ we compute
\[
\phi(u) = \begin{cases}
    u & \text{if } m + 1 \leq n \\
    n + \frac{u - m - 1}{|H_0|} & \text{if } m + 1 > n
\end{cases}.
\]
The inverse function $\psi$ is computed by
\[
\psi(u) = \begin{cases}
    u & \text{if } u \leq n \\
    |H_0|u + (-n|H_0| + n + 1) & \text{if } u > n
\end{cases}.
\]
Therefore, by the definition of the upper ramification filtration we have $H^i = H_{\psi(i)} = H_i$ for $i \leq n$, while for $u > n$ we compute $\psi(u) = |H_0|u - n|H_0| + n \geq n$, thus $H^{u} = H_{\psi(u)} = \{\text{Id}\}$.

**Lemma 1.9.** Let $a \in \mathbb{Q}$. Then for every prime $p$ and every $\ell \in \mathbb{N}$ we have
\[
\left\lfloor \frac{a}{p^{\ell+1}} \right\rfloor = \left\lfloor \frac{a}{p^\ell} \right\rfloor.
\]

**Proof.** Let us write the $p$-adic expansion of $a$:
\[
a = \sum_{\nu=\lambda}^{\infty} a_\nu p^\nu + \sum_{\nu=0}^{\lambda} a_\nu p^\nu,
\]
where $\lambda \in \mathbb{Z}$, $\lambda < 0$. We compute
\[
\left\lfloor \frac{a}{p^\ell} \right\rfloor = \sum_{\nu=\ell}^{\infty} a_\nu p^\nu.
\]
and
\[
\left\lfloor \frac{a}{p^{\ell+1}} \right\rfloor = \sum_{\nu=\ell+1}^{\infty} a_\nu p^\nu = \left\lfloor \frac{a}{p^{\ell+1}} \right\rfloor.
\]
The arbitrary \( \sigma_v \in \text{Gal}(L/K) \) sends \( t^n \mapsto \frac{t^n}{1 + vt^n} \), so by computation
\[
\frac{d\sigma_v(t)}{dt} = \frac{1}{(1 + vt^n)^\frac{n+1}{n}}
\]

**Lemma 1.10.** We consider an Artin-Schreier extension \( L/k((x)) \) and we keep the notation from lemma 1.7. Let \( \sigma_v \in \text{Gal}(L/K) \). The corresponding action on the tangent space \( \mathcal{T}_\mathcal{O} \) is given by
\[
\left( f(t) \frac{d}{dt} \right)^{\sigma_v} = f(t)^{\sigma_v} \left( 1 + vt^n \right)^{\frac{n+1}{n}} \frac{d}{dt}.
\]

**Proof.** We have that \( \frac{d\sigma_v^{-1}(t)}{dt} = \frac{d\sigma_v^{-1}(t)}{dt} = \frac{1}{(1-vt^n)^\frac{n-1}{n}} \) and by computation
\[
\sigma_v \left( \frac{d\sigma_v^{-1}(t)}{dt} \right) = \left( 1 + vt^n \right)^{\frac{n-1}{n}}.
\]

Let \( \mathcal{O} = \mathcal{O}_L \), we will now compute the space of “local modular forms”
\[
\mathcal{T}_\mathcal{O}^{G_1} = \{ f(t) \in \mathcal{O} : f(t)^{\sigma_v} = f(t)(1 + vt^n)^{-\frac{n+1}{n}} \},
\]
for \( i \geq 1 \). First we do the computation for a cyclic \( p \)-group.

**Lemma 1.11.** Let \( L/k((x)) \) be an Artin-Schreier extension with Galois group \( H = \mathbb{Z}/p\mathbb{Z} \) and ramification filtration
\[
H_0 = H_1 = \ldots = H_n \supset \{1\}.
\]

Let \( t \) be the uniformizer of \( L \) and denote by \( \mathcal{T}_\mathcal{O} \) the set of elements of the form \( f(t) \frac{dt}{dt}, f(t) \in k[[t]] \) equipped with the conjugation action defined in \( \{14\} \). The space \( \mathcal{T}_\mathcal{O}^G \) is \( G \)-equivariantly isomorphic to the \( \mathcal{O}_K \)-module consisted of elements of the form
\[
f(x)x^{n+1-\left\lfloor \frac{n+1}{p} \right\rfloor} \frac{dx}{dx}, f(x) \in \mathcal{O}_K.
\]

**Proof.** Using the description of the action in lemma 1.10 we see that \( \mathcal{T}_\mathcal{O} \) is isomorphic to the space of Laurent polynomials of the form \( \{ f(t)/t^{n+1}, f(t) \in \mathcal{O} \} \), and the isomorphism is compatible with the \( G \)-action. Indeed, we observe first that \( t^{n+1} \frac{dt}{dt} \) is a \( G \)-invariant element in \( \mathcal{T}_\mathcal{O} \). Then, for every \( f(t) \frac{dt}{dt} \in \mathcal{T}_\mathcal{O} \), the map sending
\[
f(t) \frac{dt}{dt} = f(t) \frac{dt}{t^{n+1}} \frac{dt}{dt} \rightarrow f(t) \frac{dt}{t^{n+1}},
\]
is a \( G \)-equivariant isomorphism.

We have
\[
\{ f(t)/t^{n+1}, f(t) \in \mathcal{O} \}^G = \{ f(t)/t^{n+1}, f(t) \in \mathcal{O} \} \cap k((x)),
\]
so the \( G \)-invariant space consists of elements \( g(x) \) in \( K \) such that \( g \) seen as an element in \( L \) belongs to \( \mathcal{T}_\mathcal{O} \), i.e., \( v_L(g) \geq -(n+1) \). Consider the set of functions \( g(x) \in K \) such that \( v_L(g) = pv_K(g) \geq -(n+1) \), i.e., \( v_K(g) \geq -\frac{n+1}{p} \). Since \( v_K(g) \) is an integer the last inequality is equivalent to \( v_K(g) \geq \left\lfloor -(n+1)/p \right\rfloor = -(n+1)/p \).

On the other hand, a simple computation with the defining equation of the Galois extension \( L/K \) shows that
\[
t^{n+1} \frac{dt}{dt} = x^{n+1} \frac{dx}{dx},
\]
and the desired result follows.  

\[\square\]

Similarly one can prove the more general:
Lemma 1.12. We are using the notation of lemma 1.11. Let $A$ be the fractional ideal $k[[t]]^n \frac{d}{dt}$ where $a$ is a fixed integer. The $G$-module $A$ is $G$-equivariantly isomorphic to $t^{n-(n+1)}k[[t]]$. Moreover, the space $A^G$ is the space of elements of the form

$$f(x)x^{n+1-\left\lfloor \frac{n+1-a}{p} \right\rfloor} \frac{d}{dx}$$

Next we proceed to the more difficult case of elementary abelian $p$-groups.

Lemma 1.13. Let $G = \oplus_{i=1}^s \mathbb{Z}/p \mathbb{Z}$ be the Galois group of the fully ramified elementary abelian extension $L/k((x))$ and assume that the ramification filtration is of the form

$$G = G_0 = G_1 = \cdots = G_n \supset \{1\}.$$

Let $t$ denote the uniformizer of $L$. Denote by $T_G$ the set of elements of the form $f(t)^{\nu}$. Let $(f(t) \in k[[t]]$ equipped with the conjugation action defined in [14]. The space $T_G^G$ is $G$-equivariantly isomorphic to the $O_K$-module consisted of elements of the form

$$f(x)x^{n+1-\left\lfloor \frac{n+1}{p} \right\rfloor} \frac{d}{dx}, \quad f(x) \in O_K,$$

where $p^a = |G|$.

Proof. We will break the extension $L/k((x))$ to a sequence of extensions $L = L_0 > L_1 > \ldots L_s = k((x))$, such that $L_i/L_{i+1}$ is a cyclic $p$-extension. Denote by $\pi_i$ the uniformizer of $L_i$. According to lemma 1.12 the ramification extension $L_i/L_{i+1}$ is of conductor $n$, i.e. the conditions of [11] are satisfied. We will prove the result inductively. For the extension $L/L_1$ the statement is true by lemma 1.1. Assume that the lemma is true for $L/L_i$ so a $k[[\pi_i]]$ basis of $T_G^{O_{\pi_i} \mathbb{Z}/p \mathbb{Z}}$ is given by the element $\pi_i^{n+1-\left\lfloor \frac{n+1}{p} \right\rfloor} \frac{d}{d\pi_i}$. Then lemma 1.11 implies that a $k[[\pi_{i+1}]]$ basis for

$$T_G^{O_{\pi_{i+1}} \mathbb{Z}/p \mathbb{Z}} = T_G^{O_{\pi_i} \mathbb{Z}/p \mathbb{Z}} \mathbb{Z}/p \mathbb{Z},$$

is given by the element:

$$\pi_{i+1}^{n+1-\left\lfloor \frac{n+1}{p} \right\rfloor} \frac{d}{d\pi_{i+1}}.$$ 

The desired result follows by lemma 1.9. \hfill \Box

Similarly one can prove the more general:

Lemma 1.14. We are using the notation of lemma 1.13. Let $A$ be the fractional ideal $k[[t]]^n \frac{d}{dt}$ where $a$ is a fixed integer. The $G$-module $A$ is $G$-equivariantly isomorphic to $t^{n-(n+1)}k[[t]]$. Moreover, the space $A^G$ is the space of elements of the form

$$f(x)x^{n+1-\left\lfloor \frac{n+1-a}{p} \right\rfloor} \frac{d}{dx}.$$

By induction, the above computation can be extended to the following:

Proposition 1.15. Let $L = k((t))$ be a local field acted on by a Galois $p$-group $G$ with ramification subgroups

$$G_1 = \ldots = G_{t_f} > G_{t_f+1} > \ldots > G_{t_{j-1}} > G_{t_{j-1}+1} > \ldots \geq G_{t_1} = G_n > G_{t_0} = \{1\}.$$

We consider the tower of local fields

$$L^{G_0} = L^{G_1} \subseteq L^{G_2} \subseteq \ldots \subseteq L^{\{1\}} = L.$$
Let us denote by $\pi_t$ a local uniformizer for the field $L^G_i$, i.e. $L^G_i = k((\pi_t))$. The extension $L^{G_{t+1}}/L^{G_t}$ is Galois with Galois group the elementary abelian group $H(i) := G_i/G_{t+1}$. Moreover the ramification filtration of the group $H(i)$ is given by

$$H(i)_0 = H(i)_1 = \ldots = H(i)t_i > H(i)t_i+1 = \{\text{Id}\}$$

and the conductor of the extension is $t_i$. Let $O$ be the ring of integers of $L$. The invariant space $T_O^{G_{t_i}}$ is the $OG_{t_i}$-module generated by:

$$\pi_t^{\mu_i} \frac{d}{d\pi_i}$$

where $\mu_0 = 0$ and $\mu_i = t_i + 1 - \left\lfloor \frac{-\mu_{i-1} + \mu_{i} + 1}{G_{t_i}/G_{t_i-1}} \right\rfloor$

Proof. The first statements are clear from elementary Galois theory. What needs a proof is the formula for the dimensions $\mu_i$. For $i = 1$ we have that $G_{t_i} = G_n$ is an elementary abelian group and lemma 1.13 applies, under the assumption $G_{t_0} = \{\text{Id}\}$. Therefore,

$$T_O^{G_{t_1}} = \pi_1^{n+1-\left\lfloor \frac{\mu_0 + 1}{G_{t_1}} \right\rfloor} \frac{d}{d\pi_1}$$

Assume that the formula is correct for $i$, i.e.,

$$T_O^{G_{t_i}} = \pi_1^{\mu_i} \frac{d}{d\pi_i}$$

Then lemma 1.13 implies that

$$T_O^{G_{t_{i+1}}}/T_O^{G_{t_i}} = \pi_1^{\mu_{i+1}} \frac{d}{d\pi_{i+1}}$$

where $\mu_{i+1} = n_{i+1} + 1 - \left\lfloor \frac{n_{i+1} + 1 - \mu_i}{G_{t_{i+1}}/G_{t_i}} \right\rfloor$ and the inductive proof is complete. \hfill \Box

Let $k((t))/k([x])$ be a cyclic extension of local fields of order $p$, such that the maximal ideal $xk[[x]]$ is ramified completely in the above extension. For the ramification groups $G_i$ we have

$$Z/pZ = G = G_0 = \ldots = G_n > G_{n+1} = \{1\}.$$

Hence, the different exponent is computed $d = (n + 1)(p - 1)$. Let $E = t^a k[[t]]$ be a fractional ideal of $k((t))$. Let $N(E)$ denote the images of elements of $E$ under the norm map corresponding to the group $Z/pZ$. It is known that $N(E) = x^{[(d+a)/p]}k[[x]]$, and $E \cap k[[x]] = x^{[a/p]}k[[x]]$. The cohomology of cyclic groups is 2-periodic and J. Bertin and A. Mézard in \cite{1} Prop. 4.1.1, proved that

$$\dim_k H^1(G,E) = \dim_k H^2(G,E) = \frac{E \cap k[[x]]}{N(E)} = \left[\frac{d + a}{p}\right] - \left[\frac{a}{p}\right].$$

Remark: The reader might notice that in \cite{1} Prop. 4.1.1] instead of (18) the following formula is given:

$$\dim_k H^3(G,k[[x]]) \frac{d}{dx} = \left[\frac{2d}{p}\right] - \left[\frac{d}{p}\right].$$

But $k[[x]] \frac{d}{dx} \cong x^{-n-1}k[[x]]$, and $d = (n + 1)(p - 1)$, thus

$$\left[\frac{2d}{p}\right] - \left[\frac{d}{p}\right] = \left[\frac{d + (n+1)(p-n-1)}{p}\right] - \left[\frac{(n+1)(p-n-1)}{p}\right] = \left[\frac{d-n-1}{p}\right] - \left[\frac{n-1}{p}\right],$$

and the two formulas coincide.
Corollary 1.16. Let $G$ be an abelian group that can be written as a direct product $G = H_1 \times H_2$ of groups $H_1, H_2$, and suppose that $H_2 = \mathbb{Z}/p\mathbb{Z}$. The following sequence is exact:

$0 \to H^1(H_2, A^{H_1}) \to H^1(H_1 \times H_2, A) \to H^1(H_1, A)^{H_2} \to 0$

Proof. The group $H_2$ is cyclic of order $p$ so the transgression map is identically zero by lemma 1.5 and the desired result follows. □

Remark: It seems that the result of J. Bertin, A. Mézard, solves the problem of determining the dimension of the $k$-vector spaces $H^1(\mathbb{Z}/p\mathbb{Z}, A)$ for fractional ideals of $k[[x]]$. But in what follows we have to compute the $G/H$-invariants of the above cohomology groups, therefore an explicit description of these groups and of the $G/H$-action is needed.

2. Computing $H^1(\mathbb{Z}/p\mathbb{Z}, A)$.

We will need the following

Lemma 2.1. Let $a$ be a $p$-adic integer. The binomial coefficient $\binom{a}{i}$ is defined for $a$ as usual:

$$\binom{a}{i} = \frac{a(a-1) \cdot (a-i+1)}{i!}$$

and it is also a $p$-adic integer [5, Lemma 4.5.11]. Moreover, the binomial series is defined

(19) $$(1 + t)^a = \sum_{i=0}^{\infty} \binom{a}{i} t^i.$$ 

Let $i$ be an integer and let $\sum_{\mu=0}^{\infty} b_\mu p^\mu$ and $\sum_{\mu=0}^{\infty} a_\mu p^\mu$ be the $p$-adic expansions of $i$ and $a$ respectively. The $p$-adic integer $\binom{a}{i} \not\equiv 0 \mod p$ if and only if every coefficient $a_i \geq b_i$.

Proof. The only think that needs a proof is the criterion of the vanishing of the binomial coefficient mod $p$. If $a$ is a rational integer, then this is a known theorem due to Gauß [4, Prop. 15.21]. When $a$ is a $p$-adic integer we compare the coefficients mod $p$ of the expression

$$(1 + t)^a = (1 + t)^{\sum_{\mu=0}^{\infty} a_\mu p^\mu} = \prod_{\mu=1}^{\infty} (1 + t^{b_\mu})^{a_\mu}$$

and of the binomial expansion in (19) and the result follows. □

Lemma 2.2 (Nakayama map). Let $G = \mathbb{Z}/p\mathbb{Z}$ be a cyclic group of order $p$ and let $A = \pi^n k[[t]]$ be a fractional ideal of $k[[t]]$. Let $x$ be a local uniformizer of the field $k((t))/k[[t]]$. Let $\alpha \in H^2(G, A)$, and let $u[\sigma, \tau]$ be any cocycle representing the class $\alpha$. The map

(20) $$\phi : H^2(G, A) \to \frac{x^{[a/p]}k[[x]]}{x^{[\frac{(n+1)p-n-ax}{p-1}]k[[x]]}}$$

sending

$$\alpha \mapsto \sum_{\rho \in G} u[\rho, \tau] \quad \tau \in G,$$

is well defined and is an isomorphism.
Proof. Let $A$ be a $G$-module. Let us denote by $\hat{H}^0(G, A)$ the zero Tate-cohomology. We use Remark 4-5-7 and theorem 4-5-10 in the book of Weiss \[25\] in order to prove that the map $H^2(G, A) \ni \alpha \mapsto \sum_{\rho \in G} u[\rho, \tau] \in \hat{H}^0(G, A)$ is well defined and an isomorphism.

Let $\sigma$ be a generator of the cyclic group $\mathbb{Z}/p\mathbb{Z}$. We know that

$$\hat{H}^0(\mathbb{Z}/p\mathbb{Z}, t^\alpha k[[t]]) = \frac{\ker(\delta)}{N_{\mathbb{Z}/p\mathbb{Z}}(t^\alpha k[[t]])},$$

where $\delta = \sigma - 1$ and $N_{\mathbb{Z}/p\mathbb{Z}} = \sum_{i=0}^{p-1} \sigma^i$. We compute that

$$\ker(\delta) = t^\alpha k[[t]] \cap k((x)) = x^{[\alpha/p]} k[[x]]$$

and

$$N_{\mathbb{Z}/p\mathbb{Z}}(t^\alpha k[[t]]) = x^{\frac{a + (n+1)(p - 1)}{p}} k[[x]]$$

and this completes the proof. \(\Box\)

Let $A = t^\alpha k[[t]]$ be a fractional ideal of $k[[t]]$. We consider the fractional ideal $t^{\alpha + n + 1} k[[t]]$, and we form the short exact sequence:

$$0 \to t^{\alpha + n + 1} k[[t]] \to t^\alpha k[[t]] \to M \to 0,$$

where $M$ is an $n + 1$-dimensional $k$-vector space with basis $\{1, \frac{1}{1 - a}, \ldots, \frac{1}{1 - a - n}\}$.

Let $\sigma_v$ be the automorphism $\sigma_v(t) = t/(1 + vt^n)^{1/n}$, where $v \in \mathbb{F}_p$. The action of $\sigma_v$ on $1/t^\mu$ is given by

$$\sigma_v: \frac{1}{t^\mu} \mapsto \frac{(1 + vt^n)^{\mu/n}}{t^\mu} = \frac{1}{t^\mu} \left( \sum_{\nu = 0}^{\infty} \binom{\mu/n}{\nu} \nu^\nu t^{\nu n} \right).$$

The action of $\mathbb{Z}/p\mathbb{Z}$ on the basis elements of $M$ is given by

$$\sigma_v(1/t^\mu) = \begin{cases} 1/t^\mu & \text{if } -a < \mu \\ 1/t^{-a} + \frac{a}{n} t^{-a - n} & \text{if } \mu = -a \end{cases}$$

We consider the long exact sequence we obtain by applying the $G$-invariants functor on (21):

$$0 \to t^{\alpha + n + 1} k[[t]]^G \to t^\alpha k[[t]]^G \to M^G \to H^1(G, t^{\alpha + n + 1} k[[t]]) \to H^1(G, t^\alpha k[[t]]) \to H^1(G, M) \to H^2(G, t^{\alpha + n + 1} k[[t]]) \to \cdots$$

Lemma 2.3. Assume that the group $G = \mathbb{Z}/p\mathbb{Z}$ generated by $\sigma_v$. The map $\delta_1$ in (24) is onto.

Proof. By (23) we have:

$$\dim_k M^\mathbb{Z}/p\mathbb{Z} = \begin{cases} n + 1 & \text{if } p \mid a \\ n & \text{if } p \nmid a \end{cases}.$$  

On the other hand, if $x$ is a local uniformizer of the field $k((t))^\mathbb{Z}/p\mathbb{Z}$, then:

$$\left( \frac{1}{t^{-a - (n+1)}} \right)^\mathbb{Z}/p\mathbb{Z} = x^{\frac{a + (n+1)}{p}} k[[x]] = \frac{1}{x} x^{\frac{-a - (n+1)}{p}} k[[x]],$$

and similarly

$$\left( \frac{1}{t^{-a}} k[[t]] \right)^\mathbb{Z}/p\mathbb{Z} = \frac{1}{x} x^{\frac{-a}{p}} k[[x]].$$

The image of $\delta_1$ has dimension:

$$\dim_k M^\mathbb{Z}/p\mathbb{Z} - \left| \frac{-a}{p} \right| + \left| \frac{-a - (n + 1)}{p} \right|.$$
Moreover for the dimension of $H^1(\mathbb{Z}/p\mathbb{Z}, \frac{1}{t-a-(n+1)}k[[t]])$ we compute:

$$h := \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, \frac{1}{t-a-(n+1)}k[[t]]) =$$

$$= \left[ \frac{(n+1)(p-1) + a + (n+1)}{p} \right] - \left[ \frac{a + (n+1)}{p} \right] =$$

$$(n+1) - \left[ \frac{-a}{p} \right] + \left[ \frac{-a - (n+1)}{p} \right].$$

We distinguish the following two cases:

- If $p \mid a$ then $\left[ \frac{-a}{p} \right] = \left[ \frac{-a}{p} \right]$, $\dim_k M^G = n + 1$ and we observe that $\dim_k \text{Im}(\delta_1) = h$.
- If $p \nmid a$ then $\left[ \frac{-a}{p} \right] = \left[ \frac{-a}{p} \right] + 1$, and $\dim_k M^G = n$ and in this case it also holds $\dim_k \text{Im}(\delta_1) = h$.

\[\square\]

**Proposition 2.4.** The cohomology group $H^1(\mathbb{Z}/p\mathbb{Z}, M)$ is isomorphic to

$$H^1(\mathbb{Z}/p\mathbb{Z}, M) \cong \begin{cases} \bigoplus_{\nu=-a-n} \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z}, & \text{if } p \mid a \\ \bigoplus_{\nu=-a-n+1} \mathbb{Z}/p\mathbb{Z}, & \text{if } p \nmid a \end{cases}.$$

**Proof.** Assume that the arbitrary automorphism $\sigma_v \in \mathbb{Z}/p\mathbb{Z}$ is given by $\sigma_v(t) = t/(1 + vt^n)^{1/n}$ where $v \in \mathbb{F}_p$.

Let us write a cocycle $d$ as
d

$$d\sigma_v = \sum_{i=-a}^{n} \alpha_i(\sigma_v) \frac{1}{t^i}.$$

By computation,

$$d(\sigma_v)^{\sigma_w} = \sum_{i=-a}^{n} \alpha_i(\sigma_v) \frac{1}{t^i} + \alpha_a(\sigma_v) \frac{-a}{n} \frac{1}{t-a-n}.$$

Moreover, the cocycle condition $d(\sigma_v + \sigma_w) = d(\sigma_w) + d(\sigma_v)^{\sigma_w}$ for $d(\sigma_v) = \sum_{i=-a}^{n} \alpha_i(\sigma_v) \frac{1}{t^i}$ gives:

$$\sum_{i=-a}^{n} \alpha_i(\sigma_v + \sigma_w) \frac{1}{t^i} = \left( \sum_{i=-a}^{n} \alpha_i(\sigma_v) \frac{1}{t^i} \right)^{\sigma_w} + \sum_{i=-a}^{n} \alpha_i(\sigma_w) \frac{1}{t^i} =$$

$$\sum_{i=-a}^{n} \alpha_i(\sigma_v) \frac{1}{t^i} + \alpha_a(\sigma_v) \frac{-a}{n} \frac{1}{t-a-n} + \sum_{i=-a}^{n} \alpha_i(\sigma_w) \frac{1}{t^i}.$$

By comparing coefficients we obtain:

$$\alpha_i(\sigma_w + \sigma_v) = \alpha_i(\sigma_w) + \alpha_i(\sigma_v) \text{ for } i \neq -a-n,$$

and

$$\alpha_{-a-n}(\sigma_w + \sigma_v) = \alpha_{-a-n}(\sigma_w) + \alpha_{-a-n}(\sigma_v) + \alpha_a(\sigma_v) \frac{-a}{n}.$$

The last equation allows us to compute the value of $\alpha_{-a-n}$ on any power $\sigma_v^{\nu}$ of the generator $\sigma_v$ of $\mathbb{Z}/p\mathbb{Z}$. Indeed, we have:

$$\alpha_{-a-n}(\sigma_v^{\nu}) = \nu \alpha_{-a-n}(\sigma_v) + (\nu - 1) \alpha_a(\sigma_v) \frac{-a}{n}.$$

This proves that the function $\alpha_{-a-n}$ depends only on the selection of $\alpha_{-a-n}(\sigma_v) \in k$.  

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We will now compute the coboundaries. Let \( b = \sum_{i=-a}^{-a+n} b_i \frac{1}{t} \), \( b_i \in k \) be an element in \( M \). By computation,
\[
b^{\sigma_r} - b = b_{-a} - b \frac{1}{n} v \frac{1}{t^{a-n}}.
\]
We distinguish the following cases:
- If \( p \mid a \) then the \( \mathbb{Z}/p\mathbb{Z} \) action on \( M \) is trivial, so
  \[
  H^1(\mathbb{Z}/p\mathbb{Z}, M) = \text{Hom}(\mathbb{Z}/p\mathbb{Z}, M) = \bigoplus_{i=-a}^{-a} \text{Hom}(\mathbb{Z}/p\mathbb{Z}, k).
  \]
  The dimension of \( H^1(\mathbb{Z}/p\mathbb{Z}, M) \) in this case is \( n+1 \).
- If \( p \nmid a \), then the coboundary kills the contribution of the cocycle on the \( 1/t^{a-n} \) basis element and the cohomology group is
  \[
  H^1(\mathbb{Z}/p\mathbb{Z}, M) = \text{Hom}(\mathbb{Z}/p\mathbb{Z}, M) = \bigoplus_{i=-a-n+1}^{-a} \text{Hom}(\mathbb{Z}/p\mathbb{Z}, k).
  \]

\[\Box\]

**Lemma 2.5.** Assume that \( p \geq 3 \). Let \( e = 1 \) if \( p \nmid a \) and \( e = 0 \) if \( p \mid a \). If \( n \geq 2 \) then an element
\[
\sum_{i=-a}^{-a+n} a_i(\cdot) \frac{1}{t^i} \in H^1(\mathbb{Z}/p\mathbb{Z}, M)
\]
is in the kernel of \( \delta_2 \) if and only if \( a_i(\cdot) \frac{i}{p-1} = 0 \) for all \( i \). If \( n = 1 \) then an element
\[
\sum_{i=-a}^{-a+n} a_i(\cdot) \frac{1}{t^i} \in H^1(\mathbb{Z}/p\mathbb{Z}, M)
\]
is in the kernel of \( \delta_2 \) if and only if \( a_i(\cdot) \frac{i}{n} = 0 \) for all \( -a - n + e \leq i \leq -a \) and
\[
a_i(\cdot) \frac{i}{n} \neq 0 \text{ for all } -a - n + e \leq i \leq -a \text{ such that } 2(p-1)n-i < (n+1)p+p \left\lceil \frac{a}{p} \right\rceil.
\]

**Proof.** A derivation \( a_i(\sigma_v) \frac{1}{t^i}, -a-n+e \leq i \leq -a \) representing a cohomology class in \( H^1(\mathbb{Z}/p\mathbb{Z}, M) \) is mapped to
\[
(25) \quad \delta_2(a_i(\cdot) \frac{1}{t^i}) |_{\sigma_v} = a_i(\sigma_v) \frac{1}{t^i} \sigma_v - a_i(\sigma_v + \sigma_w) \frac{1}{t^i} = a_i(\sigma_v) \frac{1}{t^i} \left( \sum_{\nu=1}^{\infty} \frac{i}{\nu} w^{\nu} t^{\nu} \right)
\]
We consider now the map \( \phi \) defined in (20) in the proof of lemma 2.2. The map \( \delta_2 : H^1(G, M) \to H^2(G, k) \) is composed with \( \phi \) and the the image of \( \phi \circ \delta_2 \) in \( x^{\frac{a+n+1}{p}} k[[x]] \) is given by
\[
\phi \circ \delta_2(a_i(\cdot) \frac{1}{t^i}) = \sum_{w \in \mathbb{Z}/p\mathbb{Z}} a_i(\sigma_v) \left( \sum_{\nu=1}^{\infty} \frac{i}{\nu} w^{\nu} t^{\nu} \right).
\]
On the other hand recall that
\[
\sum_{w \in \mathbb{Z}/p\mathbb{Z}} w^{\nu} = \begin{cases} 0 & \text{if } p-1 \nmid \nu \\ -1 & \text{if } p-1 \mid \nu \end{cases}
\]
and every homomorphism \( a_i : (\mathbb{Z}/p\mathbb{Z}, \cdot) \to (k, +) \) is given by \( a_i(\sigma_w) = \lambda_i w \), where \( \lambda_i \in k \). Therefore,
\[
\phi \circ \delta_2(a_i(\cdot) \frac{1}{t^i}) = \sum_{\nu=1}^{\infty} \left( \frac{i}{\nu} \right) \left( \sum_{w \in \mathbb{Z}/p\mathbb{Z}} w^{\nu} \right) a_i(\sigma_v) t^{\nu-i} =
\]
Observe that $p - 1 | \nu$ is equivalent to $\nu = \mu p - \mu$, and since $\nu \geq 1$, we have $\mu \geq 1$. Thus, equation (24) becomes

$$
\sum_{\nu=1}^{\infty} \frac{i/n}{\nu} (-1) \lambda_i t^{(\mu p - \mu) n - i} = \left( \frac{i/n}{p-1} \right) (-1) \lambda_i t^{(p-1) n - i} + \left( \frac{i/n}{2p-2} \right) (-1) \lambda_i t^{(2p-2) n - i} + \text{higher order terms}
$$

Claim: If $n \geq 2$ and $p \geq 3$ then for all $a \leq -i \leq a + n$ and for $\mu \geq 2$

$$
\mu(p-1)n - i \geq p \left[ \frac{(n+1)p + a}{p} \right].
$$

If $n = 1$ and $p \geq 3$ then (27) holds for $a \leq -i \leq a + n$ and for $\mu \geq 3$. Moreover

$$
(p-1)n - i < p \left[ \frac{(n+1)p + a}{p} \right],
$$

for $a \leq -i \leq a + n$.

Indeed, the inequality

$$
\mu \geq \frac{n + 1}{n} \frac{p}{p-1},
$$

holds for $p \geq 3, n \geq 2$ and $\mu \geq 2$ or for $p \geq 3, n = 1, \mu \geq 3$. Therefore, (28) implies that

$$
(n+1)p + \left\lfloor \frac{a}{p} \right\rfloor p \leq \mu(p-1)n + a \leq \mu(p-1)n - i
$$

and the first assertion is proved. On the other hand

$$
\frac{a}{p} < 1 + \left\lfloor \frac{a}{p} \right\rfloor \Rightarrow \frac{a}{p} + n < n + 1 + \left\lfloor \frac{a}{p} \right\rfloor \Rightarrow
$$

$$
(p-1)n - i < a + pn < p(n+1) + p \left\lfloor \frac{a}{p} \right\rfloor,
$$

and the second assertion is proved.

Since for elements $g \in k[[x]] \subset k[[t]]$ we have $pv_x(g) = \psi_t(g)$ we observe that all elements in $k[[t]]$ that have valuation greater or equal to $(n+1)p + \left\lfloor \frac{a}{p} \right\rfloor$ are zero in the lift of the ideal $x^{(n+1)+\left\lfloor \frac{a}{p} \right\rfloor} k[[x]]$ on $k[[t]]$. Therefore the claim gives us that that for $p \geq 3, n \geq 2$,

$$
\phi \circ \delta_2(a_i(\cdot) \frac{1}{t^{i/n}}) = \left( \frac{i/n}{p-1} \right) (-1) \lambda_i t^{(p-1) n - i}
$$

so $\sum_{i=-a-n}^{-a-n} a_i(\cdot) \frac{1}{t^{i/n}}$ is in the kernel of $\delta_2$ if and only if

$$
\left( \frac{i/n}{p-1} \right) (-1) \lambda_i = 0 \text{ for all } i.
$$

The case $n = 1$ follows by a similar argument. □
Proposition 2.6. The cohomology group \( H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]]) \) is isomorphic to the \( k \)-vector space generated by

\[
\left\{ \frac{1}{i^a}, b \leq i \leq -a, \text{ such that } \left( \frac{i}{p-1} \right) = 0 \right\},
\]

where \( b = -a - n \) if \( p \mid a \) and \( b = -a - n + 1 \) if \( p \nmid a \).

**Proof.** If \( n \geq 2 \) and \( p \geq 3 \) then the result is immediate by the exact sequence (24), lemmata 2.3 and 2.5 and by the computation of \( H^1(\mathbb{Z}/p\mathbb{Z}, M) \) given in proposition 2.4.

Assume that \( n = 1 \), and let \( a = a_0 + a_1p + a_2p^2 + \cdots \) be the \( p \)-adic expansion of \( a \). Then the inequality

\[
(n + 1)p + p \left\lfloor \frac{a}{p} \right\rfloor \leq 2(p - 1)n + a
\]

holds if \( a_0 \neq 0, 1 \). Indeed, in this case we have that \( \frac{a}{p} \leq \left\lfloor \frac{a}{p} \right\rfloor < 1 \) and (29) holds. Therefore, for the case \( p \mid a \) and \( a = 1 + pb \), \( b \in \mathbb{Z} \) we have to check the binomial coefficients \( \binom{a}{2p-2} \) as well. We will prove that in these cases if \( \binom{a}{2p-2} = 0 \) then \( \binom{a}{2p-2} = 0 \) and the proof will be complete.

Assume, first that \( p \mid a \) and \( n = 1 \). Then, \( -a - 1 \leq i \leq -a, i.e. i = -a - 1 \) or \( i = -a \). We compute that \( \binom{-a}{p-1} = 0 \) since there is no constant term in the \( p \)-adic expansion of \(-a\). Moreover the \( p \)-adic expansion of \( 2p - 2 = p - 2 + p \), and since \( p \neq 2 \) we have that \( \binom{-a}{2p-2} = 0 \) as well. For \( i = -a - 1 \) we have that \( i = -p - 1 + pb \) for some \( b \in \mathbb{Z} \) therefore by comparing the \( p \)-adic expansions of \(-a - 1 + pb \) we obtain that \( \binom{-a}{p-1} \neq 0 \) and this value of \( i \) does not contribute to the cohomology.

Assume now that \( a = 1 + pb \), \( b \in \mathbb{Z} \). We have that \( i = -a \), and \(-a = p - 1 + pb \). Therefore by comparing the \( p \)-adic expansions of \(-a, p - 1 \) we obtain that \( \binom{-a}{p-1} \neq 0 \) and this value of \( i \) does not contribute to the cohomology.

\( \square \)

Proposition 2.7. Let \( A = t^a k[[t]] \) be a fractional ideal of the local field \( k((t)) \). Assume that \( H = \oplus_{\nu=1}^{\nu_1} \mathbb{Z}/p\mathbb{Z} \) is an elementary Abelian group with ramification filtration

\[
H = H_0 = \ldots = H_n > H_{n+1} = \{ \mathrm{Id} \}.
\]

Let \( \pi_i \) be the local uniformizer of the local field \( k((t)) \oplus_{\nu=1}^{\nu_1} \mathbb{Z}/p\mathbb{Z} \), and \( a_1 = \left[ \frac{a_0}{p} \right] \), \( a_1 = a \). The cohomology group \( H^1(H, A) \) is generated as a \( k \)-vector space by the following basis elements:

\[
\left\{ \nabla_{\lambda=1}^{s} \frac{1}{\pi_i^a}, \text{ } \lambda = 1, \ldots, s \text{ such that } \left( \frac{i_\lambda/a}{p-1} \right) = 0 \right\},
\]

where \( b_i = -a_i - n \) if \( p \mid a_i \) and \( b_i = -a_i - n + 1 \) if \( p \nmid a_i \). Moreover, let \( H(i) := H/\oplus_{\nu=1}^{\nu_1} \mathbb{Z}/p\mathbb{Z} \). The groups \( H^1(H, A) \) are trivial \( H(i) \)-modules with respect to the conjugation action.

**Proof.** For \( A = t^a k[[t]] \), we compute the invariants \( t^a k[[t]] \cap k((t)) \mathbb{Z}/p\mathbb{Z} = x^{\frac{1}{p}} k[[x]] \), where \( x \) is a local uniformizer for the ring of integers of \( k((t)) \mathbb{Z}/p\mathbb{Z} \).

The modules \( A^{\oplus_{\nu=1}^{\nu_1} \mathbb{Z}/p\mathbb{Z}} \) can be computed recursively:

\[
A^{\oplus_{\nu=1}^{\nu_1} \mathbb{Z}/p\mathbb{Z}} = \pi_i^{a_i} k[[\pi_i]],
\]

where \( \pi_i \) is a uniformizer for the local field \( k((t)) \oplus_{\nu=1}^{\nu_1} \mathbb{Z}/p\mathbb{Z} \) and \( a_i = \left[ \frac{a_0}{p} \right], a_1 = a \).
In order to compute the ramification filtration of quotient groups we have to employ the upper ramification filtration for the ramification group \[118\] IV 3, p. 73-74. But according to lemma \[118\] the upper ramification filtration coincides with the lower ramification filtration therefore the ramification filtration for the groups \(H(i)\) is

\[ H(i) = \ldots = H(i)_0 > \{\text{Id}\}. \]

For the group \(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}\) corollary \[116\] implies that

\[ H^1(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, t^n k[[t]]) = H^1\left(\frac{\mathbb{Z}/p\mathbb{Z}}{t^n k[[t]]}, \mathbb{Z}/p\mathbb{Z}\right) \oplus H^1(\mathbb{Z}/p\mathbb{Z}, t^n k[[t]]) \text{ if } t^n k[[t]] \text{ is of conductor } n. \]

The dimension of \(\text{the lower ramification filtration}\) therefore the ramification filtration for the groups \(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}\) is generated over \(k\) by

\[ a_1 \leq 0, a_2 \leq j \leq -\left\lfloor \frac{a_1}{p} \right\rfloor \text{ and } \left(\frac{j/a_2}{p}\right) = 0. \]

The desired result follows by induction.

**Proposition 2.8.** Let \(A = t^n k[[t]]\) be a fractional ideal of the local field \(k((t))\). Assume that \(H = \oplus_{i=1}^s \mathbb{Z}/p\mathbb{Z}\) is an elementary Abelian group with ramification filtration

\[ H = H_0 = \ldots = H_n > H_{n+1} = \{\text{Id}\}. \]

The dimension of \(H^1(H, A)\) can be computed as

\[ \dim_k H^1(H, A) = \sum_{i=1}^s \left( \left\lceil \frac{(n+1)(p-1) + a_i}{p} \right\rceil - \left\lfloor \frac{a_i}{p} \right\rfloor \right), \]

where \(a_i\) are defined recursively by \(a_1 = a\) and \(a_i = \left\lceil \frac{a_{i-1}}{p} \right\rceil\). In particular if \(A = k[[t]]\), then

\[ \dim_k H^1(H, k[[t]]) = s \left\lfloor \frac{(n+1)(p-1)}{p} \right\rfloor. \]

**Proof.** By induction on the number of direct summands, corollary \[116\] and proposition \[2.7\] we can prove the following formula:

\[ H^1(H, A) = \oplus_{i=1}^s H^1(\mathbb{Z}/p\mathbb{Z}, A^{\oplus_{i=1}^s \mathbb{Z}/p\mathbb{Z}}). \]

In order to compute the dimensions of the direct summands \(H^1(\mathbb{Z}/p\mathbb{Z}, A^{\oplus_{i=1}^s \mathbb{Z}/p\mathbb{Z}})\), for various \(i\) we have to compute the ramification filtration for the groups defined as \(H(i) = \frac{H}{A^{\oplus_{i=1}^s \mathbb{Z}/p\mathbb{Z}}}, \) since \(\oplus_{i=1}^s \mathbb{Z}/p\mathbb{Z} = H/H(i)\). But the upper ramification filtration coincides with the lower ramification filtration \[118\] Thus, the dimension of \(H^1(H, A)\) can be computed as

\[ \dim_k H^1(H, A) = \sum_{i=1}^s \left( \left\lceil \frac{(n+1)(p-1) + a_i}{p} \right\rceil - \left\lfloor \frac{a_i}{p} \right\rfloor \right). \]

In particular if \(A = k[[t]]\), then

\[ \dim_k H^1(H, k[[t]]) = s \left\lfloor \frac{(n+1)(p-1)}{p} \right\rfloor. \]
Proposition 2.9. Let $G$ be the Galois group of the extensions of local fields $L/K$, with ramification filtration $G_i$ and let $(t_\lambda)_{1 \leq \lambda \leq f}$ be the jump sequence in (3). For the dimension of $H^1(G_1, \mathcal{T}_0)$ the following bound holds:

$$H^1(G_1/G_{t_{f-1}}, \mathcal{T}_0^{G_{t_{f-1}}}) \leq \dim_k H^1(G_1, \mathcal{T}_0) \leq \sum_{i=1}^f \dim_k H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_0^{G_{t_{i-1}}}) \leq \sum_{i=1}^f \dim_k H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_0^{G_{t_{i-1}}}),$$

where $G_{n+1} = \{1\}$. The left bound is best possible in the sense that there are ramification filtrations such that the first inequality becomes equality.

Proof. Using the low-term sequence in (3) we obtain the following inclusion for $i \geq 1$:

$$H^1(G_{t_i}, \mathcal{T}_0) = H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_0^{G_{t_{i-1}}}) + \ker t\gamma \subseteq H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_0^{G_{t_{i-1}}}) \oplus H^1(G_{t_{i-1}}, \mathcal{T}_0^{G_{t_{i-1}}}/G_{t_{i-1}}).$$

We start our computation from the end of the ramification groups:

$$H^1(G_{t_2}, \mathcal{T}_0) \subseteq H^1(G_{t_2}/G_{t_1}, \mathcal{T}_0^{G_{t_1}}) \oplus H^1(G_{t_1}, \mathcal{T}_0^{G_{t_1}}).$$

Observe here that $\mathcal{T}_0$ is not $G_{t_1}$-invariant so there in no apriori well defined action of $G_{t_2}/G_{t_1}$ on $\mathcal{T}_0$. But since the group $G_{t_2}$ is of conductor $n$ using the explicit form of $H^1(G_{t_1}, \mathcal{T}_0)$ we see that that $H^1(G_{t_1}, \mathcal{T}_0)$ is a trivial $G_1$-module. Of course this is also clear from the general properties of the conjugation action [23, cor. 2-3-2].

We go on to the next step:

$$H^1(G_{t_3}, \mathcal{T}_0) \subseteq H^1(G_{t_3}/G_{t_2}, \mathcal{T}_0^{G_{t_2}}) \oplus H^1(G_{t_2}/G_{t_1}, \mathcal{T}_0^{G_{t_1}}).$$

The combination of (33) and (34) gives us

$$H^1(G_{t_3}, \mathcal{T}_0) \subseteq H^1(G_{t_3}/G_{t_2}, \mathcal{T}_0^{G_{t_2}}) \oplus H^1(G_{t_2}/G_{t_1}, \mathcal{T}_0^{G_{t_1}}) \oplus H^1(G_{t_1}, \mathcal{T}_0^{G_{t_1}}).$$

Using induction based on (38) we obtain:

$$H^1(G_{t_1}, \mathcal{T}_0) \subseteq \bigoplus_{i=1}^f H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_0^{G_{t_{i-1}}}) \subseteq \bigoplus_{i=1}^f H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_0^{G_{t_{i-1}}}),$$

and the desired result follows.

Notice that in the above proposition $G_{t_{f-1}}$ appears in the ramification filtration of $G_0$ thus the corollary to proposition IV.1.3 in [20] implies that the ramification filtration of $G_{t_i}/G_{t_{i-1}}$ is constant. Namely, if $Q = G_{t_i}/G_{t_{i-1}}$ the ramification filtration of $Q$ is given by:

$$Q_0 = Q_1 = \cdots = Q_i > \{1\}.$$
Proposition 2.10. Let $\log_p(\cdot)$ denote the logarithmic function with base $p$. Let $s(\lambda) = \log_p [G_{t_{\lambda}}/|G_{t_{\lambda-1}}]$ and let $\mu_i$ be as in proposition I.12. Then

$$\dim_k H^1 \left( \frac{G_{t_{\lambda}}}{G_{t_{\lambda-1}}}, \mathcal{T}_O \right) = \sum_{i=1}^{s(\lambda)} \left( \left\lfloor \frac{(t_{\lambda} + 1)(p - 1) + a_i}{p} \right\rfloor - \left\lfloor \frac{a_i}{p} \right\rfloor \right),$$

where $a_1 = -t_{\lambda} - 1 + \mu_{\lambda-1}$, and $a_i = \left\lfloor \frac{a_{i-1}}{p} \right\rfloor$.

Proof. The module $\mathcal{T}_O \frac{G_{t_{\lambda}}}{G_{t_{\lambda-1}}}$ is computed in proposition I.12 to be isomorphic to $\pi_{\lambda-1}^{\mu_{\lambda-1}} \frac{d}{dx_{\lambda-1}}$, which in turn is $\pi_{\lambda-1}^{\mu_{\lambda-1}}$-equivariantly isomorphic to $\pi_{\lambda-1}^{-t_{\lambda} - 1 + \mu_{\lambda-1}} k[[x_{\lambda-1}]]$. The desired result follows by using proposition 2.8. □

Remark 2.11. If $n = 1$, i.e. $G_2 = \{1\}$ then the left hand side and the right hand of (37) are equal and the bound becomes the formula in [3].

Proposition 2.12. We will follow the notation of 2.10. Suppose that for every $i$, $\frac{G_{t_i}}{G_{t_{i-1}}}$ is a cyclic $p$-group. Then the following equality holds:

$$\dim_k H^1(G_1, \mathcal{T}_O) = \sum_{i=1}^{f} \dim_k H^1 \left( \frac{G_{t_i}}{G_{t_{i-1}}}, \mathcal{T}_O ^{G_{t_i}/G_{t_{i-1}}} \right) \leq \sum_{i=1}^{f} \left( \left\lfloor \frac{(t_i + 1)(p - 1) - t_i - 1 + \mu_{t_{i-1}}}{p} \right\rfloor - \left\lfloor \frac{-t_i - 1 + \mu_{t_{i-1}}}{p} \right\rfloor \right).$$

Proof. The kernel of the transgression at each step is by lemma 3.5 the whole $H^1(\frac{G_{t_i}}{G_{t_{i-1}}}, \mathcal{T}_O ^{G_{t_i}/G_{t_{i-1}}})$. Therefore the right inner inequality in equation (37) is achieved. The other inequality is trivial by the computation done in proposition 2.11 but it is far from being best possible. □

3. Global Computations

We consider the Galois cover of curves $\pi : X \rightarrow Y = X/G$, and let $b_1, \ldots, b_r$ be the ramification points of the cover. We will denote by

$$e_0^{(\mu)} \geq e_1^{(\mu)} \geq e_2^{(\mu)} \geq e_{n_{\mu}}^{(\mu)} > 1$$

the orders of the higher ramification groups at the point $b_{\mu}$. The ramification divisor $D$ of the above cover is a divisor supported at the ramification points $b_1, \ldots, b_r$ and equals to

$$D = \sum_{\mu=1}^{r} \sum_{i=0}^{n_{\mu}} (e_i^{(\mu)} - 1)b_{\mu}.$$  

Let $\Omega^1_X, \Omega^1_Y$ be the sheaves of holomorphic differentials at $X$ and $Y$ respectively. The following formula holds [3, IV. 2.3]:

$$\Omega^1_X \cong \mathcal{O}_X(D) \otimes \pi^*(\Omega^1_Y)$$

and by taking duals

$$\mathcal{T}_X \cong \mathcal{O}_X(-D) \otimes \pi^*(\mathcal{T}_Y)$$

Thus $\pi_*(\mathcal{T}_X) \cong \mathcal{T}_Y \otimes \pi_*(\mathcal{O}_X(-D))$ and $\pi_*(\mathcal{T}_X) \cong \mathcal{T}_Y \otimes (\mathcal{O}_{\pi^{-1}(D)} \cap \mathcal{O}_X(-D))$. We compute (similarly with [3 prop. 1.6]),

$$\pi_*(\mathcal{T}_X) = \mathcal{T}_Y \otimes \mathcal{O}_Y \left( -\sum_{\mu=1}^{r} \left\lfloor \sum_{i=0}^{n_{\mu}} \frac{e_i^{(\mu)} - 1}{e_0^{(\mu)}} \right\rfloor b_{\mu} \right)$$
Therefore, the global contribution to \( H^1(G, \mathcal{T}_X) \) is given by

\[
H^1(Y, \pi_Y^G(\mathcal{T}_X)) \cong H^1(Y, T_Y \otimes \mathcal{O}_Y) \left( - \sum_{\mu=1}^r \left[ \sum_{i=0}^{n_\mu} \frac{(e_\mu(i) - 1)}{e_\mu(i)} \right] b_i \right) \\
\cong H^0(Y, \Omega_Y^2) \left( - \sum_{\mu=1}^r \left[ \sum_{i=0}^{n_\mu} \frac{(e_\mu(i) - 1)}{e_\mu(i)} \right] b_i \right)
\]

and by Riemann-Roch formula

(41) \( \dim_k H^1(Y, \pi_Y^G(\mathcal{T}_X)) = 3g_Y - 3 + \sum_{\mu=1}^r \left[ \sum_{i=0}^{n_\mu} \frac{(e_\mu(i) - 1)}{e_\mu(i)} \right] \).

On the other hand the local contribution can be bounded by proposition 2.4 and by combining the local and global contributions, we arrive at the desired bound for the dimension.

3.1. Examples. Let \( V = \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z} \) be an elementary abelian group acted on by the group \( \mathbb{Z}/n\mathbb{Z} \). Assume that \( G := V \rtimes \mathbb{Z}/n\mathbb{Z} \) acts on the local field \( k((t)) \) and assume that the ramification filtration is given by

\[ G_0 > G_1 = G_2 = \cdots G_j > G_{j+1} = \{1\}. \]

Let \( H := \mathbb{Z}/p\mathbb{Z} \) be the first summand of \( V \). Let \( \sigma \) be a generator of the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) and assume that \( \sigma(t) = \zeta t \), where \( \zeta \) is a primitive \( n \)-th root of one.

The inflation-restriction sequence implies the short exact sequence:

\[ 0 \rightarrow H^1(G/H, x^a k[[x]]) \frac{d}{dx} \rightarrow H^1(G, t^a k[[t]]) \frac{d}{dt} \rightarrow H^1(H, t^a k[[t]]) \frac{d}{dt} \rightarrow 0 \]

where \( x^a k[[x]] \frac{d}{dx} = \left( t^a k[[t]] \frac{d}{dt} \right)^H \). The group \( \mathbb{Z}/n\mathbb{Z} \) acts on \( t^a k[[t]] \) but there is no apriori well defined action of \( \mathbb{Z}/n\mathbb{Z} \) on \( x^a k[[x]] \frac{d}{dx} = \left( t^a k[[t]] \frac{d}{dt} \right)^H \), since the group \( H \) might not be normal in \( G \). An element \( d \in H^1(G/H, x^a k[[x]] \frac{d}{dx}) \) is send by the inflation map on the 1-cocycle \( \text{inf}(d) \) that is a map

\[ \text{inf}(d) : G \rightarrow t^a g(t) \frac{d}{dt} \in t^a k[[t]] \frac{d}{dt}, \]

and the action of \( \sigma \) can be considered on the image of the inflation map, sending \( \text{inf}(d)(g) \rightarrow \sigma(\text{inf}(d)(g)) \). We observe that \( \sigma(\text{inf}(d)(g)) \) is zero for any \( g \in H \), by the definition of the inflation map, therefore there is an element \( a \in t^a k[[t]] \frac{d}{dt} \) such that

\[ \sigma(\text{inf}(d)(g)) + a^g - a \in x^a k[[x]] \frac{d}{dx}, \]

therefore we can consider the element

\[ \sigma(\text{inf}(d)) + a^g - a = \text{inf}(d'). \]

This means that although there is no well defined action of \( \mathbb{Z}/n\mathbb{Z} \) on \( k[[x]] \) we can define \( \sigma(d) = d' \) modulo cocycles. In what follows we will try to compute the element \( d' \in H^1(G/H, x^a k[[x]] \frac{d}{dx}) \).

Assume that the Artin-Schreier extension \( k((t))/k((x)) \) is given by the equation

\[ 1/y^p - 1/y = 1/x^j. \]

Then, we have computed that if \( g \) is a generator if \( H \) then

\[ g(t) = \frac{t}{(1 + t^j)^{1/j}}, \quad \text{and} \quad x = \frac{t^p}{(1 - t^j(t-1))^{1/j}}. \]

The action of \( \sigma \) on \( x \), where \( x \) is seen as an element in \( k[[t]] \) is given by

\[ \sigma(x) = \frac{t^p}{(1 - t^{j(p-1)})^{1/j}} = \zeta^p x \frac{(1 - t^{j(p-1)})^{1/j}}{(1 - \zeta^{j(p-1)} t^{j(p-1)})^{1/j}} = \zeta^p x u, \]

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where \( u = \frac{(1-t^{(p-1)\frac{1}{2}})^{1/4}}{(1-\zeta^{(p-1)\frac{1}{2}}t^{(p-1)\frac{1}{2}})^{1/4}} \) is a unit of the form \( 1 + y \), where \( y \in t^{2j(p-1)}k[[t]] \).

The cohomology group \( H^1(V/H, x^n k[[x]]) \) is generated by the elements \( \{1/x^n, b \leq \mu \leq a, (\mu/n) = 0 \} \). Each element \( 1/x^n \) is written as \( 1/x^n x^{j+1} \frac{dt}{dt} \) and it is lifted to

\[
\frac{1}{x^n x^{j+1}} \frac{dt}{dt} \zeta^{-p\mu+j} 1/x^n u^{-\mu} x^{j+1} \frac{dt}{dt}.
\]

In the above formula we have used the fact that the adjoint action of \( \sigma \) on \( t^r \frac{dt}{dt} \) is given by \( t^r \frac{dt}{dt} \zeta^{(r-1)t^r} \frac{dt}{dt} \) [3, 3.7].

Obviously the unit \( u \) is not \( H \)-invariant but we can add to \( u \) a 1-coboundary so that it becomes the \( H \)-invariant element \( \inf(d') \). We observe that this coboundary is of the form \( a^g - a \), and obviously \( a^g - a \) has to be in \( t^{2(p-1)}k[[t]] \). This gives us that

\[
(1/x^n)^j = \zeta^j 1/x^n + o,
\]

where \( o \) is a sum of terms \( 1/x^\nu \) with \( -\alpha < \nu \) and therefore \( o \) is cohomologous to zero. Using induction one can prove the following

**Lemma 3.1.** Let \( 1/p^i \), \( \lambda = 1, \ldots, s \), \( b_i \leq i \lambda \leq -a_i \) so that \( (i_{\lambda}/n) = 0 \) and \( b_i = -a_i - j \) if \( p \mid a_i \), \( b_i = -a_i - j + 1 \) if \( p \nmid a_i \) be the basis elements of the cohomology group \( H^1(V/H) \). Then the action of the generator \( \sigma \in \mathbb{Z}/n\mathbb{Z} \) on \( T_0 \) induces the following action on the basis elements:

\[
\sigma(\frac{1}{x^n}) = \zeta^{-p^i\mu+j} \frac{1}{x^{n+1}}.
\]

1. The Fermat curve

\[
F : x^n_1 + x^n_2 + x^n_3 = 0
\]

defined over an algebraically closed field \( k \) of characteristic \( p \), such that \( n - 1 = p^i \) is a power of the characteristic is a very special curve. Concerning its automorphism group, the Fermat curve has maximal automorphism group with respect to the genus [21]. Also it leads to Hermitian function fields, that are optimal with respect to the number of \( \mathbb{F}_{p^2} \)-rational points and Weil’s bound.

It is known that the Fermat curve is totally supersingular, i.e., the Jacobian variety \( J(F) \) of \( F \) has \( p \)-rank zero, so this curve cannot be studied by the tools of [3]. The group of automorphism of \( F \) was computed in [3] to be the projective unitary group \( G = \text{PGU}(3, q^2) \), where \( q = p^i = n - 1 \). H. Stichtenoth [21] p. 535 proved that in the extension \( F/F^G \) there are two ramified points \( P, Q \) and one is wildly ramified and the other is tamely ramified. For the ramification group \( G(P) \) of the wild ramified point \( P \) we have that \( G(P) \) consists of the \( 3 \times 3 \) matrices of the form

\[
(42) \quad \begin{pmatrix}
1 & 0 & 0 \\
\alpha & \chi & 0 \\
\gamma & -\chi^{-q} & \chi^{1+q}
\end{pmatrix},
\]

where \( \chi, \alpha, \gamma \in \mathbb{F}_{q^2} \) and \( \gamma + \gamma^{q} = \chi^{1+q} - 1 - \alpha^{1+q} \). Moreover Leopoldt proves that the order of \( G(P) \) is \( q^3(q^2-1) \) and the ramification filtration is given by

\[
G_0(P) > G_1(P) > G_2(P) = \cdots = G_{1+q}(P) > \{1\},
\]

where

\[
G_1(P) = \ker(\chi : G_0(P) \to \mathbb{F}_{q^2}^*)
\]

and

\[
G_2(P) = \ker(\alpha : G_1(P) \to \mathbb{F}_{q^2}).
\]

In this section we will compute the dimension of tangent space of the global deformation functor. Namely, we will prove:
Proposition 3.2. Let $p$ be a prime number, $p > 3$ let $X$ be the Fermat curve
\[ x_0^{1+p} + x_1^{1+p} + x_2^{1+p} = 0. \]
Then $\dim_k H^1(X, G, T_X) = 0$.

Proof. By the assumption $q = p$ and by the computations of Leopoldt mentioned
above we have $G_2 = \cdots = G_{p+1} = \mathbb{Z}/p\mathbb{Z}$. The different of $G_{p+1}$ is computed
$(p+2)(p-1)$. Hence, according to (18)
\[ \dim_k H^1(G_{p+1}, T_G) = \frac{(p+2)(p-1) - (p+2)}{p} - \left\lfloor \frac{-p-2}{p} \right\rfloor = p \]
Proposition 2.6 implies that the set
\[ \left\{ \frac{1}{t^i}, 2 \leq i \leq p+2 \text{ where } \left( \frac{1}{p+1} \right) = 0 \right\} \]
is a $k$-basis of $H^1(G_{p+1}, T_G)$. Indeed, the group $G_{1+p}$ has conductor $1+p$ and
$T_G$ is $G_{1+p}$-equivariantly isomorphic to $t^{-p-2}k[[t]]$. Thus following the notation of
proposition 2.6 $-a = p+2$ and $b = 2$. The rational number $(1+p)^{-1}$ has the following $p$-adic expansion:
\[ \frac{1}{1+p} = 1 + (p-1)p + (p-1)p^3 + (p-1)p^5 + \ldots \]
and using lemma 2.1 we obtain that for $2 \leq i \leq p+2$ the only integer $i$ such that
\(\left( \frac{1}{p+1} \right) \neq 0\) is $i = p-1$. Thus, the elements
\[ \left\{ \frac{1}{t^i}, 2 \leq i \leq p+2, i \neq p-1 \right\} \]
form a $k$-basis of $H^1(G_{p+1}, T_G)$.
Leopoldt in [13] proves that the $G_0(P)$ acts faithfully on the $k$-vector space
$L((p+1)P)$ that is of dimension 3 with basis functions $1, v, w$ and the representation
matrix is given by (12). Moreover, the above functions have $t$-expansions of the
following form $v = \frac{u}{t^2}, w = \frac{u}{t^3}$, for a suitable choice of the local uniformizer $t$ at the point $P$. The functions $v, w$ generate the
function field coresponding to the Fermat curve and they satisfy the relation $w^n = w^{n+(w+1)^n}$, therefore one can compute that the unit $u$ can be written as
\[ u = 1 + t^{p+1}g, \ g \in k[[t]]. \]
Let $\sigma$ be an element given by a matrix as in equation (12). The action of $\sigma \in G_1 = G_1(P)$ on powers of $\frac{1}{t}$ is given by
\[ \frac{1}{t^i} = \frac{(1+\gamma t^{p+1} - a^qut)^{1+i}}{t^i} \]
and the action on the basis elements $\{1/t^i, 2 \leq i \leq p+2, i \neq p-1\}$ is given by
\[ \frac{1}{t^i} \mapsto \frac{1}{t^i} + \sum_{\nu=1}^{i-2} a^q w^\nu \left( \frac{1}{t^{\nu+1}} \right) \frac{1}{t^{i-\nu}}. \]
We observe that the matrix of this action is given by
\[ A_\sigma = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]
We observe that $\sigma(1/t^2) = 1/t^2$ and $\sigma(1/t^p) = 1/t^p$, and moreover that all elements below the diagonal of the matrix $A_\sigma$ are $\frac{1}{t^{p+1}}$ and are non-zero unless $i = p$. Therefore the eigenspace of the eigenvalue 1 is 2-dimensional, and we can give a basis:

$$H^1(G_{1+p}, \mathcal{T}_G)^{G_{1+p}} = k \left\{ \frac{1}{t^2}, \frac{1}{t^p} \right\}$$

In order to compute $H^1(G_1(P, \mathcal{T}_G))$ we consider the exact sequence

$$1 \to G_2 \to G_1 \xrightarrow{\alpha} G_1/G_2 \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to 1$$

and the corresponding low-degree-term Lyndon-Hochschild-Serre sequence. The group $G_2$ is of conductor $p + 1$ thus $\mathcal{T}_G = \mathcal{T}_G^{Z/p\mathbb{Z}}$ is given by proposition 1.13 $(p > 2)$:

$$\mathcal{T}_G = \frac{x^{p+2} - \frac{1}{t^2}[x[x]]}{d\chi} = \frac{x^{p+1}k[[x]]}{dx},$$

where $x$ is a local uniformizer for $\mathcal{O}^{G_2}$. By [14 Cor. p.64] the ramification filtration for $G_2/G_1$ is

$$\mathcal{T}_G = \mathcal{T}_G^{Z/p\mathbb{Z}}$$

and the dimension of the cohomology group is computed:

$$H^1(G_1/G_2, \mathcal{T}_G) = H^1(Z/p\mathbb{Z}, x^{p-1}k[[x]]) \oplus H^1(Z/p\mathbb{Z}, (x^{p-1}k[[x]])^{Z/p\mathbb{Z}})$$

We compute

$$\dim_k H^1(Z/p\mathbb{Z}, x^{p-1}k[[x]]) = \left\lfloor \frac{2(p-1) + p - 1}{p} \right\rfloor - \left\lfloor \frac{p-1}{p} \right\rfloor = 1.$$ 

On the other hand, if $\pi$ is a local uniformizer for $k((x))^{Z/p\mathbb{Z}}$ then

$$(x^{p-1}k[[x]])^{Z/p\mathbb{Z}} = \pi^{\left\lfloor \frac{p-1}{p} \right\rfloor} k[[\pi]] = \pi k[[\pi]].$$

The conductor is 2, and the dimension of the cohomology group is computed:

$$\dim_k H^1(Z/p\mathbb{Z}, (x^{p-1}k[[x]])^{Z/p\mathbb{Z}}) = \dim_k H^1(Z/p\mathbb{Z}, \pi k[[\pi]]) = \left\lfloor \frac{2(p-1) + 1}{p} \right\rfloor - \left\lfloor \frac{1}{p} \right\rfloor = 0.$$ 

Using the bound for the kernel of the transgression we see that

$$1 = \dim_k H^1(G_1/G_2, \mathcal{T}_G) \leq \dim_k H^1(G_1, \mathcal{T}_G) \leq \dim_k H^1(G_1, \mathcal{T}_G) + \dim_k H^1(G_2, \mathcal{T}_G) = 3.$$

(44)

In order to compute the action of $G_0$ on $G_1/G_2$ we observe that

$$\chi a^p = \chi a$$

(45)

$$\begin{pmatrix} 1 & 0 & 0 \\ \star & \chi & 0 \\ \star & -b^p & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \star & -b^p & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \star & -b^p & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \star & -b^p & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \chi & 1 & 0 \\ \bar{-}\chi b^p & 1 \end{pmatrix}$$

If $b$ is an element of $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p \subset \mathbb{F}_p^\times$ then $b^p = b$. By looking at the computation of (45) we see that the conjugation action of $G_0/G_1$ to $\mathbb{F}_p$ is given by multiplication $b \to \chi^1 + qb$. Observe that $(\chi^{1+p})^{p-1} = \chi^{p-1} = 1$, and $(\chi^{1+p})^{p-1} = \chi^{1+p} \in (\chi^{p-1})^{p-1}$. The action on the cocycles is given by sending the cocycle $d(\tau)$ to $d(\sigma \tau \sigma^{-1})^{p-1}$ therefore the basis cocycle $\frac{1}{t^2}$ of the one dimensional cohomology group $H^1(Z/p\mathbb{Z}, x^{p-1}k[[x]])$ goes to $\chi^p t^{p-1} \frac{1}{t^2}$ under the
conjugation action, as one sees by applying lemma \[\text{Lemma 3.1}.\] Lemma 1.3 implies that $H^1(G_1/G_2, \mathcal{T}_\mathcal{O})_{G_1/G_2} = 0$. Similarly the conjugation action of $G_0/G_1$ on an element of $G_2$ can be computed to be multiplication of $\tau$ by $\chi^{1+p} \in F_p$, and the same argument shows that $H^1(G_1+p, \mathcal{T}_\mathcal{O})_{G_1/G_1+p, G_0/G_1} = 0$.

Finally the global contribution is computed by formula (11)

$$\dim_k H^1(F^G, \pi_*$$(\mathcal{T}_F)) = -3 + \left\lfloor \frac{p^2+2}{2} \right\rfloor - \left\lfloor \frac{1}{[G(Q)]} \right\rfloor = -3 + 2 + 1 = 0$$

The fact that the tangent space of the deformation functor is zero dimensional is compatible with the fact that there is only one isomorphism class of curves $C$ such that $|Aut(C)| \geq 16gC + \lfloor \frac{1}{2} \rfloor$.

2. $p$-covers of $\mathbb{P}^1(k)$ We consider curves $C_f$ of the form

$$C_f : w^p - w = f(x),$$

where $f(x)$ is a polynomial of degree $m$. We will say that such a curve is in reduced form if the polynomial $f(x)$ is of the form

$$f(x) = \sum_{i=1, (i,p)=1}^{m-1} a_i x^i + x^m.$$ 

Two such curves $C_f, C_g$ in reduced form are isomorphic if and only if $f = g$. The group $G := Gal(C_f/\mathbb{P}^1(k)) \cong \mathbb{Z}/p\mathbb{Z}$ acts on $C_f$. We observe that the number of independent monomials $\neq x^m$ in the above sums is given by:

\[
m - \left\lfloor \frac{m}{p} \right\rfloor - 1,
\]

since $\#\{1 \leq i \leq m, p \mid i\} = \left\lfloor \frac{m}{p} \right\rfloor$.

We will compute the tangent space of the deformation functor of the curve $C_f$ together with the group $C_f$. Let $P$ be the point above $\infty \in \mathbb{P}^1(k)$. This is the only point that ramifies in the cover $C_f \rightarrow \mathbb{P}^1(k)$, and the group $G$ admits the following ramification filtration:

$$G_0 = G_1 = G_2 = \cdots = G_m > G_{m+1} = \{1\}.$$ 

The different is computed $(p-1)(m+1)$ and $\mathcal{T}_\mathcal{O} \cong t^{-m-1}k[[t]]$. Thus the space $H^1(G, \mathcal{T}_\mathcal{O})$ has dimension $d$

$$d = \left\lfloor \frac{(p-1)(m+1) - (m+1)}{p} \right\rfloor - \left\lfloor \frac{-(m+1)}{p} \right\rfloor = m + 1 - \left\lfloor \frac{2m + 2}{p} \right\rfloor + \left\lfloor \frac{m+1}{p} \right\rfloor = 2p + 1.$$ 

Let $a_0 + a_1p + a_2p^2 + \cdots$ be the $p$-adic expansion of $m + 1$. We observe that

$$\left\lfloor \frac{2m + 2}{p} \right\rfloor - \left\lfloor \frac{m+1}{p} \right\rfloor = \left\lfloor \frac{2a_0 + \sum_{i=1}^\infty 2a_ip^{i-1}}{p} \right\rfloor - \sum_{i=1}^\infty a_ip^{i-1},$$

therefore, if $p \mid m + 1$

$$\left\lfloor \frac{2m + 2}{p} \right\rfloor - \left\lfloor \frac{m+1}{p} \right\rfloor = \left\lfloor \frac{m+1}{p} \right\rfloor + \delta,$$

where

$$\delta = \begin{cases} 2 & \text{if } 2a_0 > p \\ 1 & \text{if } 2a_0 \leq p. \end{cases}$$
Thus,
\[
d = \begin{cases} 
  m + 1 - \left\lfloor \frac{m+1}{p} \right\rfloor & \text{if } p \mid m + 1 \\
  m - \left\lfloor \frac{m+1}{p} \right\rfloor - \delta & \text{otherwise}
\end{cases}
\]

Finally, we compute that
\[
dim_k H^1(Y, \pi_c^G(T_X)) = -3 + \left\lceil \frac{(m+1)(p-1)}{p} \right\rceil = m - 2 - \left\lfloor \frac{m+1}{p} \right\rfloor.
\]

3. Lehr-Matignon Curves. Let us consider the curve
\[
C : y^p - y = \sum_{i=0}^{m-1} t_i x^{1+p^i} + x^{1+p^m},
\]
defined over the algebraically closed field \(k\) of characteristic \(p > 2\). Let \(n = 1 + p^m\) denote the degree of right hand side of the above equation. The automorphism group of these curves were studied by Matignon-Lehr in [12] and these curves were considered also by G. van der Geer, van der Vlught in [23] in connection with coding theory. Notice that the extreme Fermat curves studied in example 1 can be written in this form by a suitable transformation [22, VI.4.3 p.203]. Let \(H = \text{Gal}(C/\mathbb{P}^1(k))\).

The automorphism group \(G\) of \(C\) can be expressed in the form
\[
1 \to H \to G \to V \to 1,
\]
where \(V\) is the vector space of roots of the additive polynomial
\[
(47) \quad \sum_{0 \leq i \leq m} (t_i^{p^m-i}Y^{p^{m-i}} + t_i^mY^{p^{m+i}})
\]
[12 prop. 4.15]. Moreover there is only one point \(P \in C\) that ramifies in the cover \(C \to C^G\), namely the point above \(\infty \in \mathbb{P}^1(k)\).

In order to simplify the calculations we assume that \(t_0 = \cdots = t_{m-1} = 0\) so the curve is given by
\[
(48) \quad y^p - y = x^{p^m+1}.
\]

The polynomial in [12] is given by \(Y^{p^m} + Y\) and the vector space \(V\) of the roots is \(2m\)-dimensional. Moreover, according to [12] any automorphism \(\sigma_v\) corresponding to \(v \in V\) is given by
\[
\sigma_v(x) = x + v, \quad \sigma(y) = y + \sum_{k=0}^{m-1} v^{p^m+k} x^k.
\]

Observe that \(w\) (resp. \(x\)) has a unique pole of order \(p^m + 1\) (resp. \(p\)) at the point above \(\infty\), so we can select the local uniformizer \(\pi\) so that
\[
y = \frac{1}{\pi^{p^m+1}}, \quad x = \frac{1}{\pi} u,
\]
where \(u\) is a unit in \(k[[\pi]]\). By replacing \(x, y\) in (48) we observe that the unit \(u\) is of the form \(u = 1 + \pi^{p^m}\).

A simple computation based on the basis \(\{1, x, \ldots, x^{p^m-1}, y\}\), of the vector space \(L((1+p^m)P)\) given in [12] prop. 3.3. shows that the ramification filtration of \(G\) is
\[
G = G_0 = G_1 > G_2 = \ldots = G_{p^m+1} > \{1\},
\]
where \(G_2 = H\) and \(G_1/G_2 = V\). Using proposition 2.6 we obtain the following basis for \(H^1(G_2, T_G)\):
\[
\left\{ \frac{1}{\pi_i^1}, 2 \leq i \leq p^m + 2, \text{such that } \frac{p^m+1}{p-1} = 0 \right\}.
\]

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We have to study the action of $G_1/G_2$ on $H^1(G_2, T_O)$. From the action of $\sigma_v$ on $y$ we obtain that the action on the basis elements of $H^1(G_{p^m+1}, T_O)$ is given by

$$\sigma_v(\frac{1}{\pi^i}) = \frac{1}{\pi^i} + \left(\frac{\sum_{\kappa=0}^{\lfloor m/2 \rfloor} u^p v^{p^{m+\kappa} - \pi^i(p^{m+1} - p^{k+1})}}{\pi^i}\right) = \frac{1}{\pi^i} + \frac{i}{p^m + 1} \frac{1}{\pi^i} + \ldots\,.$$

We observe that if $p \mid i$ then all binomial coefficients $\binom{\frac{n}{p^m}}{\frac{\kappa}{p^m}}$ that contribute a coefficient $1/\pi^\kappa$, $2 \leq \kappa \leq p^m + 2$ are zero. Therefore, the elements $1/\pi, 1/\pi^\kappa$ are invariant. Moreover, by writing down the action of $\sigma_v$ as a matrix we see that there are no other invariant elements, so the dimension is computed ($p > 2$):

$$\dim_k H^1(G_{p^m+1}, T_O)^{G_1/G_{p^m+1}} = 1 + \left\lfloor \frac{p^m + 2}{p} \right\rfloor = 1 + p^{m-1}.$$

We observe that this dimension coincides with the computation done on the Fermat curves $m = 1$.

We proceed by computing $H^1(V, T_O^H)$. The space $T_O^H$ is computed by proposition 2.3

$$x^{p^m + 2} \left(\frac{m+2}{p^m} \right) k[[x]] \frac{d}{dx} = x^{p^m + 2 - p^{m-1}} k[[x]] \frac{d}{dx}.$$

Thus,

$$\dim_k H^1(V, T_O^H) = \sum_{\nu=1}^{2m} \left\lfloor \frac{2(p-1) + \nu}{p} \right\rfloor - \left\lfloor \frac{\nu}{p} \right\rfloor,$$

where $a_1 = p^m - p^{m-1}$, and $a_i = \left\lfloor \frac{\nu}{p} \right\rfloor$. By computation $a_{\nu} = p^{m-\nu + 1} - p^{m-\nu}$ for $1 \leq \nu \leq m$, and $a_{\nu} = 1$ for $\nu > m$. On the other hand, an easy computation shows that

$$\left\lfloor \frac{2(p-1) + \nu}{p} \right\rfloor - \left\lfloor \frac{\nu}{p} \right\rfloor = \begin{cases} 1 & \text{if } 1 \leq \nu < m \\ 2 & \text{if } \nu = m \\ 0 & \text{if } m < \nu \end{cases},$$

thus the dimension of the tangent space is $m + 1$.

We have proved that the dimension of $H^1(G_1, T_O)$ is bounded by

$$m + 1 = \dim_k H^1(G_1/G_2, T_O^{G_2}) \leq H^1(G_1, T_O) \leq \dim_k H^1(G_1/G_2, T_O^{G_2}) + H^1(G_2, T_O)^{G_1/G_2} = 2 + m + p^{m-1}.$$

Unfortunately we cannot be more precise here: an exact computation involves the computation of the kernel of the transgression and such a computation requires new ideas and tools.

To this dimension we must add the contribution of

$$\dim_k H^1(Y, \pi_*^G(T_X)) = 3g_Y - 3 + \sum_{\kappa=1}^{r} \left(\sum_{i=0}^{\lfloor r/2 \rfloor} \frac{e_i^{(1)} - 1}{e_i^{(1)}} \right) = -3 + \left(\frac{2p^{m+1} - 1}{p^{m+1}} + m \frac{p^m - 1}{p^{m+1}}\right) - 1 + \left\lfloor \frac{m}{p} \right\rfloor + 2 + m + p^{m-1}.$$

The later contribution is $> 0$ if $m \gg p$.

4. **Elementary Abelian extensions of $\mathbb{P}^1(k)$**. Consider the curve $C$ so that $G_0 = (\mathbb{Z}/p^2)^* \times \mathbb{Z}/n\mathbb{Z}$ is the ramification group of wild ramified point, and moreover the ramification filtration is given by

$$G_0 > G_1 = \ldots = G_j > G_{j+1} = \{1\}.$$
An example of such a curve is provided by the curve defined by
\[ C : \sum_{i=0}^{s} a_i y^i = f(x), \]
where \( f \) is a polynomial of degree \( j \) and all monomial summands \( a_k x^k \) of \( f \) have exponent congruent to \( j \) modulo \( n \). Let \( V \) be the \( \mathbb{F}_p \)-vector space of the roots of the additive polynomial \( \sum_{i=0}^{s} a_i y^i \). Assume that the automorphism group of the curve defined by (50) is \( G := \mathbb{V} \ltimes \mathbb{Z}/n\mathbb{Z} \). Thus \( C \to \mathbb{P}^1(k) \) is Galois cover ramified only above \( \infty \), with ramification group \( G \) and ramification filtration is computed to be as in equation (54).

Let us now return to the general case. Let us denote by \( V \) the group \( \mathbb{Z}/p\mathbb{Z}^* \). The group \( V \) admits the structure of a \( \mathbb{F}_p \) vector space, where \( \mathbb{F}_p \) is the finite field with \( p \)-elements. The conjugation action of \( \mathbb{Z}/n\mathbb{Z} \) on \( V \) implies a representation \( \rho : \mathbb{Z}/n\mathbb{Z} \to GL(V) \).

Since \((n,p) \equiv 1 \), Mascke’s Theorem gives that \( V \) is the direct sum of simple \( \mathbb{Z}/n\mathbb{Z} \)-modules, i.e., \( V = \bigoplus_{i=1}^{r} V_i \). On the other hand, lemma 2.3 implies that the conjugation action is given by multiplication by \( \zeta^j \), where \( \zeta \) is an appropriate primitive \( n \)-th root of one and \( j \) is the conductor of the extension. If \( \zeta \in \mathbb{F}_p \) then all the \( V_i \) are one dimensional. In the more general case one has to consider representations \( \rho_i : \mathbb{Z}/n\mathbb{Z} \to GL(V_i) \), where \( \dim_{\mathbb{F}_p} V_i = d \).

Notice that the dimension \( d \) is the degree of the extension \( \mathbb{F}_q/\mathbb{F}_p \), where \( \mathbb{F}_q \) is the smallest field containing \( \zeta \). Let \( e^{(i)}_1, \ldots, e^{(i)}_d \) be an \( \mathbb{F}_p \)-basis of \( V_i \), and let us denote by \( (a^{(i)}_{\mu \nu}) \) the entries of the matrix corresponding to \( \rho_i(\sigma) \), where \( \sigma \) is a generator of \( \mathbb{Z}/n\mathbb{Z} \). The conjugation action on the arbitrary
\[ v = \sum_{\mu=1}^{r} \sum_{\mu=1}^{d} \lambda_{\mu}^{(i)} e^{(i)}_{\mu} \in V \]
is given by:
\[ \sigma v \sigma^{-1} = \sum_{\mu=1}^{r} \sum_{\mu=1}^{d} \lambda_{\mu}^{(i)} e^{(i)}_{\mu} \sigma^{-1} = \sum_{\mu=1}^{r} \sum_{\mu=1}^{d} \lambda_{\mu}^{(i)} \sum_{\nu=1}^{d} a_{\mu \nu}^{(i)} e^{(i)}_{\nu} . \]
In particular,
\[ \sigma e^{(i)}_{\mu} \sigma^{-1} = \sum_{\nu=1}^{d} a_{\mu \nu}^{(i)} e^{(i)}_{\nu} . \]

For the computation of \( H^1(G, \mathcal{T}_O) \), we notice first that the group \( H^1(V, \mathcal{T}_O) \) can be computed using proposition 2.3 and the isomorphism \( \mathcal{T}_O \cong \mathbb{T}^{-1} k[[t]] \).

Next we consider the conjugation action of \( \mathbb{Z}/n\mathbb{Z} \) on \( H^1(V, \mathcal{T}_O) \), in order to compute \( H^1(G, \mathcal{T}_O) = H^1(V, \mathcal{T}_O) \mathbb{Z}/n\mathbb{Z} \). By (42) we have
\[ H^1(V, \mathcal{T}_O) = \bigoplus_{i=1}^{r} H^1(\mathbb{Z}/p\mathbb{Z}, \mathcal{T}_O \mathbb{Z}/p\mathbb{Z}) \mathbb{Z}/p\mathbb{Z} , \]
i.e., the arbitrary cocycle \( d \) representing a cohomology class in \( H^1(V, \mathcal{T}_O) \) can be written as a sum of cocycles \( d_i \), representing cohomology classes in \( H^1(\mathbb{Z}/p\mathbb{Z}, \mathcal{T}_O \mathbb{Z}/p\mathbb{Z}) \).

Let us follow a similar to (51) notation for the decomposition of \( d \), and write \( d = \sum_{i=1}^{r} \sum_{\nu=1}^{d} b_{\nu}^{(i)} d^{(i)}_{\nu} \), where \( b_{\nu}^{(i)} (e_{\mu}^{(i)}) = 0 \) if \( i \neq j \) or \( \nu \neq \mu \). Therefore,
\[ d(\sigma e_{\mu}^{(i)} \sigma^{-1}) = d \left( \sum_{\nu=1}^{d} a_{\mu \nu}^{(i)} e_{\nu}^{(i)} \right) = \sum_{\nu=1}^{d} b_{\nu}^{(i)} a_{\mu \nu}^{(i)} d^{(i)}_{\nu} (e_{\nu}^{(i)}) . \]
We have now to compute the $\mathbb{Z}/n\mathbb{Z}$-action on $d_k^{(i)}$. By lemma 3.1 the element $\sigma$ acts on the basis elements $\frac{1}{\pi_i}$ of $H^1(V, T_\mathcal{O})$ as follows

\begin{equation}
\sigma\left(\frac{1}{\pi_i}\right) = \zeta^{-p^i \mu + j} \frac{1}{\pi_i}.
\end{equation}

By the above remarks we arrive at

\begin{equation}
\sigma(d)(e^{(i)}_\mu) = d(\sigma e^{(i)}_\mu)^{\sigma^{-1}} = \sum_{\nu=1}^d b^{(i)}_{\nu}\sigma^{(i)} \zeta^{-c(\nu, i)} d^{(i)}(e^{(i)}_\nu),
\end{equation}

where $c(\nu, i)$ is the appropriate exponent, defined in (56). Let us denote by $A^{(i)}$ the $d \times d$ matrix $(a^{(i)}_{\mu, \nu})$. By (57) $\sigma(d)(e^{(i)}_\mu) = d(e^{(i)}_\mu)$ if and only if $b := (b^{(i)}_1, \ldots, b^{(i)}_d)$ is a solution of the linear system

\begin{equation}
(A^{(i)} \cdot \text{diag}^{-1}(\zeta^{c(1,i)}, \zeta^{c(2,i)}, \ldots, \zeta^{c(d,i)}) - I_d)b = 0.
\end{equation}

This proves that the dimension of the solution space is equal to the dimension of the eigenspace of the eigenvalue 1 of the matrix: $A^{(i)} \cdot \text{diag}^{-1}(\zeta^{c(1,i)}, \zeta^{c(2,i)}, \ldots, \zeta^{c(d,i)})$.

Moreover using a basis of the form $1, \zeta, \zeta^2, \ldots, \zeta^{d-1}$ for the simple space $V(i)$, we obtain that

\begin{equation}
A^{(i)} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & -a_{d-1} \\
0 & 0 & \cdots & 1 & -a_d
\end{pmatrix}
\end{equation}

It can be proved by induction that the characteristic polynomial of $A^{(i)}$ is $x^d + \sum_{\nu=0}^{d-1} a_\nu x^{\nu}$, and under an appropriate basis change $A^{(i)}$ can be written in the form $\text{diag}(\zeta, \zeta^2, \ldots, \zeta^{d-1})$. Moreover, the characteristic polynomial of the matrix $A^{(i)} \cdot \text{diag}^{-1}(\zeta^{c(1,i)}, \zeta^{c(2,i)}, \ldots, \zeta^{c(d,i)})$ can be computed inductively to be

\begin{equation}
f_i(x) := x^d + \zeta^{c(1,i)} a_{d-1} x^{d-1} + \zeta^{c(2,i)} x^{d-2} + \cdots + \zeta^{c(d,i)} a_0.
\end{equation}

If $f_i(1) \neq 0$, then we set $\delta(i) = 0$. If $f_i(1) = 0$ we set $\delta(i)$ to be the multiplicity of the root 1. The total invariant space has dimension

\begin{equation}
dim_k H^1(G, T_\mathcal{O}) = \sum_i \delta(i).
\end{equation}

Comparison with the work of Cornelissen-Kato

We will apply the previous calculation to the case of ordinary curves $j = 1$ and we will obtain the formulas in 3. We will follow the notation of proposition 2.3. The number $a_1 = -j - 1 = -2$. Thus, $a_2 = -2/p = -[2/p] = 0$ (recall that we have assumed that $p \geq 5$). Furthermore $a_i = 0$ for $i \geq 2$. For the numbers $b_i$ we have $b_1 = -a_1 - j + 1 = 2$, and $b_1 \leq i_1 \leq -a_1$, so there is only one generator, namely $(j_1^{1/p})$. Moreover, for $i \geq 2$ we have $b_i = -a_j - j = -1$ and there are two possibilities for $-1 \leq i_\lambda \leq 0 = -a_i$, namely $-1, 0$. But only $(j^{1/p}) = 0$, and we finally have that

\begin{equation}
H^1(V, T_\mathcal{O}) \cong \frac{1}{\pi_1^i} \langle 1 \rangle_k \times \cdots \times \langle 1 \rangle_k,
\end{equation}

a space of dimension $\log_p |V|$.

Let $d$ be the dimension of each simple direct summand of $H^1(V, T_\mathcal{O})$ considered as a $\mathbb{Z}/n\mathbb{Z}$-module. Of course $d$ equals the degree of the extension $F_\ell(\zeta)/\mathbb{F}_p$, where $\zeta$ is a suitable primitive root of 1. For the matrix $\text{diag}(\zeta^{c(1,i)}, \ldots, \zeta^{c(d,i)})$ we have that

\begin{equation}
\text{diag}(\zeta^{c(1,i)}, \ldots, \zeta^{c(d,i)}) = \begin{cases}
\text{diag}(\zeta^2, \zeta^{1+i}, \ldots, \zeta^{d-1}) & \text{if } i = 1, \\
\zeta \cdot I_d & \text{if } i \geq 2.
\end{cases}
\end{equation}

The characteristic polynomial in the first case is computed to be:

\[ f_1(x) = x^d + \sum_{\nu=1}^{d-1} \zeta^{d-\nu} a_{\nu} x^\nu + a_0 \zeta^{1+d}. \]

By setting \( x = \zeta y \) the above equation is written as

\[ \zeta^d \left( y^d + \sum_{\nu=0}^{d-1} a_{\nu} y^\nu \right) + (\zeta^{d+1} - \zeta^d) a_0. \]

Therefore, for \( x = 1, y = \zeta \), so \( f_1(1) = (\zeta^{d+1} - \zeta^d) a_0 \neq 0 \), therefore \( \delta(1) = 0 \).

In the second case, we observe that

\[ f_i(x) = x^d + \sum_{\nu=0}^{d-1} \zeta^{d-\nu} a_{\nu} x^\nu. \]

If we set \( x = y/\zeta \), we obtain that 1 is a simple root of \( f_i \), so \( \delta(i) = 1 \) for \( i \geq 2 \).

Thus, only the \( s/d - 1 \) blocks \( i \geq 2 \) admit invariant elements and \( \dim_k H^1(G, T_\mathcal{O}) = s/d - 1 \).

Comparison with the work of R. Pries. Let us consider the curve:

\[ C : y^p - y = f(x), \]

where \( f(x) \) is a polynomial of degree \( j \), \( (j, p) = 1 \). This, gives rise to a ramified cover of \( \mathbb{P}^1(k) \) with \( \infty \) as the unique ramification point. Moreover if all the monomial summands of the polynomial \( f(x) \) have exponents congruent to \( j \) mod \( m \), then the curve \( C \) admits the group \( G := \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z} \) as a subgroup of the group of automorphisms. In [17] R. Pries constructed a configuration space of deformations of the above curve and computed the dimension of this space. In what follows we will compute the dimension of \( H^1(G, T_\mathcal{O}) \) and we will compare with the result of Pries.

According to proposition 2.6 the tangent space of the deformation space is generated as a \( k \)-vector space by the elements of the form \( \frac{1}{x} \) where \( b \leq i \leq j + 1 \) and

\[ b = \begin{cases} 1 & \text{if } p \mid -j - 1 \\ 2 & \text{if } p \nmid -j - 1. \end{cases} \]

By lemma 1.4 the \( \mathbb{Z}/m\mathbb{Z} \)-action on \( \mathbb{F}_p \) is given by multiplication by \( \zeta^j \) where \( \zeta \) is an appropriate primitive \( m \)-th root of one. This gives us that \( \zeta^{jp} = \zeta^j \), i.e. \( jp \equiv j \mod \text{mod} \). If \( d_i \) is the cocyle corresponding to the element \( \frac{1}{x^i} \) then

\[ d_i(\sigma \tau \sigma^{-1}) = \zeta^j d_i(\tau)^{\sigma^{-1}}. \]

But the element \( \frac{1}{x^i} \) coresponds to the element \( x^{j+1-i} \frac{d}{dx} \). The \( \zeta^{-1} \)-action is given by

\[ x^{j+1-i} \frac{d}{dx} \mapsto \zeta^{-j} x^{j+1-i} \frac{d}{dx}. \]

Therefore, the action of \( \sigma \) on the cocyle corresponding to \( \frac{1}{x^i} \) is given by

\[ \frac{1}{x^i} \mapsto \zeta^i \frac{1}{x^i}. \]

Thus, \( \dim_k H^1(G, T_\mathcal{O}) = \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, T_\mathcal{O})^{\mathbb{Z}/p\mathbb{Z}} \) is equal to

\[ \# \left\{ i : b \leq i \leq j + 1, \left( \frac{i}{p-1} \right) = 0, i \equiv 0 \mod \text{mod} \right\} \]
By (11) we have
\[
\dim_k H^1(Y, \pi_*^G(T_X)) = 3gy - 3 + \sum_{\kappa=1}^r \left[ \sum_{i=0}^{n_{\kappa}} \frac{\left( c_i^{(\kappa)} - 1 \right)}{c_i^{(\kappa)}} \right],
\]
and by computation
\[
\dim_k H^1(Y, \pi_*^G(T_X)) = -3 + \left( 1 + \frac{-1}{mp} + \frac{j(p - 1)}{mp} \right).
\]

On the other hand the configuration space constructed by R. Pries is of dimension
\[
r := \#\{ e \in E_0 : \forall \nu \in \mathbb{N}^+, p' e \notin E_0 \}
\]
where
\[
E_0 := \{ e : 1 \leq e \leq j, e \equiv j \mod m \}.
\]
The above dimensions look quite different but using map\textit{l}\textit{e} we computed the following table:

| \( p \) | \( j \) | \( m \) | \( r \) | \( \dim_k H^1(G, T_O) \) | \( \dim_k H^1(Y, \pi_*^G(T_X)) \) | \( \dim_k D(k[\epsilon]) \) |
|---|---|---|---|---|---|---|
| 13 | 19 | 6 | 3 | 3 | 1 | 4 |
| 13 | 35 | 6 | 5 | 4 | 9 | 13 |
| 13 | 51 | 6 | 8 | 8 | 6 | 14 |
| 13 | 36 | 3 | 12 | 11 | 10 | 21 |
| 7 | 81 | 3 | 24 | 23 | 22 | 45 |
| 7 | 90 | 3 | 26 | 26 | 24 | 50 |

We observe that the \( r + a = \dim_k H^1(G, T_O) \), where \( a = 1, 0 \) and also the dimension of \( H^1(Y, \pi_*^G(T_X)) \) is near the above two values.

Conclusions: in her paper R. Pries considered deformations, where the ramification point does not split to ramification points that are ramified with smaller jumps at their ramification filtrations. By the difference of the formulas and by the fact that all infinitesimal deformations in \( H^1(Y, \pi_*^G(T_X)) \) are unobstructed we obtain that the difference in the dimensions \( r \) and \( \dim_k D(k[\epsilon]) \) can be explained either as obstructed deformations or as deformations splitting the branch points, see 1.1.

Let \( P \) be a \( p \)-group. In [18] R. Pries studied the dimension of unobstructed deformations of the general case of \( P \rtimes \mathbb{Z}/m \mathbb{Z} \), acting on a curve, without splitting the branch points. A comparison of this dimension with the dimension computed by the tools of this paper might be an interesting result, although it is a quite complicated task.

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\footnote{The maple code used for this computation is available on my web page http://eloris.samos.aegean.gr/preprints for download}
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