Bounds on Non-Symmetric Divergence Measures in Terms of Symmetric Divergence Measures

Inder Jeet Taneja

Departamento de Matemática
Universidade Federal de Santa Catarina
88.040-900 Florianópolis, SC, Brazil.
e-mail: taneja@mtm.ufsc.br
http://www.mtm.ufsc.br/~taneja

Abstract

There are many information and divergence measures exist in the literature on information theory and statistics. The most famous among them are Kullback-Leibler relative information and Jeffreys J-divergence. Sibson Jensen-Shannon divergence has also found its applications in the literature. The author studied a new divergence measures based on arithmetic and geometric means. The measures like harmonic mean divergence and triangular discrimination are also known in the literature. Recently, Dragomir et al. also studies a new measure similar to J-divergence, we call here the relative J-divergence. Another measures arising due to Jensen-Shannon divergence is also studied by Lin. Here we call it relative Jensen-Shannon divergence. Relative arithmetic-geometric divergence (Taneja) is also studied here. All these measures can be written as particular cases of Csiszár f-divergence. By putting some conditions on the probability distribution, the aim here is to obtain bounds among the measures.

Key words: J-divergence; Jensen-Shannon divergence; Arithmetic-geometric divergence; Relative J-divergence; Relative Jensen-Shannon divergence; Harmonic mean divergence; Triangular divergence; Csiszár f—divergence; Information inequalities.

1 Introduction

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \ldots, p_n) \left| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right. \right\}, \quad n \geq 2,$$

be the set of all complete finite discrete probability distributions. There are many information and divergence measures exists in the literature on information theory and statistics. Some of them are symmetric with respect to probability distributions, while others are not. Here we have divided these measure in these two categories. Through out the paper it is under stood that the probability distributions $P, Q \in \Gamma_n$. 

1
1.1 Non-Symmetric Measures

Here we shall give some non-symmetric measures of information. The most famous among them are $\chi^2$--divergence and Kullback-Leibler relative information. We understand by non symmetric measures are those that are not symmetric with respect to probability distributions $P, Q \in \Gamma_n$. These measures as follows.

- **$\chi^2$–Divergence** (Pearson [16])
  \[ \chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 \] (1)

- **Relative Information** (Kullback and Leibler [13])
  \[ K(P||Q) = \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} \] (2)

- **Relative J-Divergence** (Dragomir et al. [10])
  \[ D(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln \left( \frac{p_i + q_i}{2q_i} \right) \] (3)

- **Relative Jensen-Shannon divergence** (Sibson [17])
  \[ F(P||Q) = \sum_{i=1}^{n} p_i \ln \left( \frac{2p_i}{p_i + q_i} \right) \] (4)

- **Relative arithmetic-geometric divergence** (Taneja [20])
  \[ G(P||Q) = \sum_{i=1}^{n} \left( \frac{p_i + q_i}{2} \right) \ln \left( \frac{p_i + q_i}{2p_i} \right) \] (5)

1.2 Symmetric Measures of Information

Here we shall give some symmetric measures of information. Some of them can be obtained from subsection 1.1. These measures as follows.

- **Hellinger Discrimination** (Hellinger [11])
  \[ h(P||Q) = 1 - B(P||Q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2 \] (6)
  where
  \[ B(P||Q) = \sum_{i=1}^{n} \sqrt{p_i q_i} \] (7)
is the well-known Bhattacharyya distance.

- **Triangular Discrimination** (Dacunha-Castelle)

\[
\Delta(P||Q) = 2 [1 - W(P||Q)] = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}.
\]  

(8)

where

\[
W(P||Q) = \sum_{i=1}^{n} \frac{2p_i q_i}{p_i + q_i}.
\]  

(9)

is the well-known harmonic mean divergence.

- **Symmetric Chi-square Divergence** (Dragomir et al.)

\[
\Psi(P||Q) = \chi^2(P||Q) + \chi^2(Q||P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2(p_i + q_i)}{p_i q_i}.
\]  

(10)

- **J-divergence** (Jeffreys; Kullback and Leibler)

\[
J(P||Q) = K(P||Q) + K(Q||P) = D(P||Q) + D(Q||P) = \sum_{i=1}^{n} (p_i - q_i) \ln\left(\frac{p_i}{q_i}\right).
\]  

(11)

- **Jensen-Shannon divergence** (Sibson; Burbea and Rao)

\[
I(P||Q) = \frac{1}{2} \left[ F(P||Q) + F(Q||P) \right] = \frac{1}{2} \left[ \sum_{i=1}^{n} p_i \ln \left( \frac{2p_i}{p_i + q_i} \right) + \sum_{i=1}^{n} q_i \ln \left( \frac{2q_i}{p_i + q_i} \right) \right].
\]  

(12)

\[
T(P||Q) = \frac{1}{2} \left[ G(P||Q) + G(Q||P) \right] = \sum_{i=1}^{n} \left( \frac{p_i + q_i}{2} \right) \ln \left( \frac{p_i + q_i}{2\sqrt{p_i q_i}} \right).
\]  

(13)

After simplification, we can write

\[
J(P||Q) = 4 \left[ I(P||Q) + T(P||Q) \right]
\]
and

\[ D(Q||P) = \frac{1}{2} [F(P||Q) + G(P||Q)]. \]

The measures \( I(P||Q), J(P||Q), T(P||Q), D(P||Q), F(P||Q) \) and \( G(P||Q) \) can be written in terms of \( K(P||Q) \) as follows:

\[
\begin{align*}
I(P||Q) &= \frac{1}{2} \left[ K \left( P\left|\left|\frac{P+Q}{2}\right) + K \left( Q\left|\left|\frac{P+Q}{2}\right) \right) \right. \right], \\
J(P||Q) &= K(P||Q) + K(Q||P), \\
T(P||Q) &= \frac{1}{2} \left[ K \left( \frac{P+Q}{2}\left|\left|P\right) + K \left( \frac{P+Q}{2}\left|\left|Q\right) \right) \right. \right], \\
D(P||Q) &= \frac{1}{2} \left[ K \left( Q\left|\left|\frac{P+Q}{2}\right) + K \left( \frac{P+Q}{2}\left|\left|Q\right) \right) \right. \right], \\
F(P||Q) &= K \left( \frac{P+Q}{2}\left|P\right) \right)
\end{align*}
\]

and

\[ G(P||Q) = K \left( \frac{P+Q}{2}\left|P\right) \right). \]

respectively.

The following parallelogram identity is also famous in the literature [5]:

\[ K(P||U) + K(Q||U) = K \left( P\left|\left|\frac{P+Q}{2}\right) + K \left( Q\left|\left|\frac{P+Q}{2}\right) \right) \right. \right] + 2 K \left( \frac{P+Q}{2}\left|\left|U\right) \right. \right), \]

for all \( P, Q, U \in \Gamma_n \).

Some studies on information and divergence measures can be seen in Taneja [18], [19], [20]. Also see on line book by Taneja [21].

From the symmetric measures we observe that the measure (7) is a part of measure (6) and the measure (9) is a part of measure (8). Thus we have the six measures (6), (8), (10)-(13) symmetric with respect to probability distributions.

The following inequalities are already known:

\[
\begin{align*}
\frac{1}{2} h(P||Q) &\leq \frac{1}{4} \Delta(P||Q) \leq h(P||Q), \\
\frac{1}{4} \Delta(P||Q) &\leq h(P||Q) \leq \frac{1}{16} \Psi(P||Q), \\
\Delta(P||Q) &\leq \frac{1}{2} J(P||Q) \leq \frac{1}{4} \Psi(P||Q)
\end{align*}
\]
and
\[
\frac{1}{4} \Delta(P||Q) \leq I(P||Q) \leq \frac{\log 2}{2} \Delta(P||Q).
\]

The inequalities (14) are due to LeCam [14] and Dacunha-Castelle [6]. The inequalities (15) are due to Taneja [23]. The inequalities (16) are due to Dragomir et al. [9]. Finally, the inequalities (17) are due to Topsøe [28].

Recently, Taneja [25, 26] proved the following inequalities:
\[
\frac{1}{4} \Delta(P||Q) \leq I(P||Q) \leq h(P||Q) \leq \frac{1}{8} J(P||Q) \leq T(P||Q) \leq \frac{1}{16} \Psi(P||Q).
\]

In this paper, our aim is to relate the non-symmetric divergence measures with the symmetric measures given by (8), (11)-(13). In order to obtain these relationship we shall use the idea of Csiszár f-divergence and in some cases making restrictions on the probability distributions.

2 \hspace{1em} f–Divergence and Information Measures

Given a convex function \( f : (0, \infty) \to \mathbb{R} \), the \( f \)–divergence measure introduced by Csiszár [3] is given by
\[
C_f(P||Q) = \sum_{i=1}^{n} q_i f \left( \frac{p_i}{q_i} \right),
\]
where \( P, Q \in \Gamma_n \).

The following theorem is well known in the literature (ref. Csiszár [3, 4]).

**Theorem 2.1.** Let the function \( f : (0, \infty) \to \mathbb{R} \) is differentiable convex and normalized, i.e., \( f(1) = 0 \), then the Csiszár \( f \)–divergence, \( C_f(P||Q) \), given by (17) is nonnegative and convex in the pair of probability distribution \((P, Q) \in \Gamma_n \times \Gamma_n \).}

The following theorem is due to Dragomir [7, 8]. It gives bounds on Csiszár \( f \)–divergence.

**Theorem 2.2.** (Dragomir [7, 8]). Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be differentiable convex and normalized i.e., \( f(1) = 0 \). Then
\[
0 \leq C_f(P||Q) \leq E_{C_f}(P||Q)
\]
where
\[
E_{C_f}(P||Q) = \sum_{i=1}^{n} (p_i - q_i) f' \left( \frac{p_i}{q_i} \right),
\]
for all \( P, Q \in \Gamma_n \).

Let \( P, Q \in \Gamma_n \) be such that there exists \( r, R \) with \( 0 < r \leq \frac{p_i}{q_i} \leq R < \infty \), \( \forall i \in \{1, 2, \ldots, n\} \), then
\[
0 \leq C_f(P||Q) \leq A_{C_f}(r, R),
\]
where
\[ A_{C_f}(r, R) = \frac{1}{4}(R - r) [f'(R) - f'(r)]. \] (23)

Further, if we suppose that \( 0 < r \leq 1 \leq R < \infty, r \neq R \), then
\[ 0 \leq C_f(P||Q) \leq B_{C_f}(r, R), \] (24)

where
\[ B_{C_f}(r, R) = \frac{(R - 1)f(r) + (1 - r)f(R)}{R - r}. \] (25)

Moreover, the following inequalities hold:
\[ E_{C_f}(P||Q) \leq A_{C_f}(r, R), \] (26)
\[ B_{C_f}(r, R) \leq A_{C_f}(r, R) \] (27)

and
\[ 0 \leq B_{C_f}(r, R) - C_f(P||Q) \leq A_{C_f}(r, R). \] (28)

The inequalities (26) and (28) can be seen in Dragomir [8], while the inequality (27) can be proved easily.

The following theorem is due to Taneja [25, 26]. It relates two \( f \)-divergence measures.

**Theorem 2.3.** Let \( f_1, f_2 : I \subset \mathbb{R}_+ \rightarrow \mathbb{R} \) be two differentiable convex functions which are normalized, i.e., \( f_1(1) = f_2(1) = 0 \) and suppose that:
(i) \( f_1 \) and \( f_2 \) are twice differentiable on \((r, R)\);
(ii) there exists the real constants \( m, M \) such that \( m < M \) and
\[ m \leq \frac{f''_1(x)}{f''_2(x)} \leq M, \quad f''_2(x) > 0, \quad \forall x \in (r, R) \]

then we have
\[ m C_{f_2}(P||Q) \leq C_{f_1}(P||Q) \leq M C_{f_2}(P||Q). \] (29)

### 3 Bound on Divergence Measures

Based on Theorems 2.1 and 2.2 we have the particular cases for the measures given in Section 1. These particular cases are given as examples, where the following the expression is frequently used:
\[ L_{-1}(a, b) = \begin{cases} \frac{\ln b - \ln a}{b - a}, & a \neq b \\ a & a = b \end{cases} \] (30)

for all \( a > 0, b > 0 \).
Example 3.1. (Relative J-Divergence). Let us consider

\[ f_D(x) = (x - 1) \ln \left( \frac{x + 1}{2} \right), \quad x \in (0, \infty) \]  

(31)

in (19), then we have \( C_J(P||Q) = D(P||Q) \).

Moreover,

\[ f'_D(x) = \frac{x - 1}{x + 1} + \ln \left( \frac{x + 1}{2} \right) \]  

(32)

and

\[ f''_D(x) = \frac{x + 3}{(x + 1)^2}. \]  

(33)

In view of (31), (32), Theorems 2.1 and 2.2, we have the following bounds on relative J-divergence:

\[ 0 \leq D(P||Q) \leq E_D(P||Q) \leq A_D(r, R) \]  

(34)

and

\[ 0 \leq D(P||Q) \leq B_D(r, R) \leq A_D(r, R), \]  

(35)

where

\[ E_D(P||Q) = D(P||Q) + \Delta(P||Q), \]

\[ A_D(r, R) = \frac{1}{4}(R - r)^2 \left[ \frac{2}{(R + 1)(r + 1)} + \text{L}_{-1}(r + 1, R + 1) \right] \]

and

\[ B_D(r, R) = (R - 1)(1 - r)\text{L}_{-1}(r + 1, R + 1). \]

Example 3.2. (Relative Jensen-Shannon divergence). Let us consider

\[ f_F(x) = x \ln \left( \frac{2x}{x + 1} \right) - \frac{x - 1}{2}, \quad x \in (0, \infty) \]  

(36)

in (19), then we have \( C_J(P||Q) = F(P||Q) \).

Moreover,

\[ f'_F(x) = \frac{1}{2} \frac{x - 1}{x + 1} + \ln \left( \frac{2x}{x + 1} \right) \]  

(37)

and

\[ f''_F(x) = \frac{1}{x(x + 1)^2}. \]  

(38)

In view of (36), (37), Theorems 2.1 and 2.2, we have the following bounds on relative Jensen-Shannon divergence:

\[ 0 \leq F(P||Q) \leq E_F(P||Q) \leq A_F(r, R) \]  

(39)
and
\[
0 \leq F(P\|Q) \leq B_F(r, R) \leq A_F(r, R),
\] (40)
where
\[
E_F(P\|Q) = D(Q\|P) - \frac{1}{2} \Delta(P\|Q),
\]
\[
A_F(r, R) = \frac{(R - r)^2}{4(R + 1)(r + 1)} \left[ L_{-1}^{-1}\left( \frac{r}{r + 1}, \frac{R}{R + 1} \right) - 1 \right]
\]
and
\[
B_F(r, R) = \frac{1}{R - r} \left[ R \ln\left( \frac{2R}{R + 1} \right) - r \ln\left( \frac{2r}{r + 1} \right) \right]
\]
\[
- \frac{rR}{(R + 1)(r + 1)} L_{-1}^{-1}\left( \frac{r}{r + 1}, \frac{R}{R + 1} \right).
\]

Example 3.3. (Relative arithmetic-geometric divergence). Let us consider
\[
f_G(x) = \frac{x + 1}{2} \ln\left( \frac{x + 1}{2x} \right) + \frac{x - 1}{2}, \quad x \in (0, \infty)
\] (41)
in (14), then we have \( C_f(P\|Q) = G(P\|Q) \).
Moreover,
\[
f_G'(x) = \frac{1}{2} \left[ \ln\left( \frac{x + 1}{2x} \right) - \frac{x - 1}{x} \right]
\] (42)
and
\[
f_G''(x) = \frac{1}{2x^2(x + 1)}.
\] (43)

In view of (41), (42), Theorems 2.1 and 2.2 we have the following bounds on relative
arithmetic-geometric divergence:
\[
0 \leq G(P\|Q) \leq E_G(P\|Q) \leq A_G(r, R)
\] (44)
and
\[
0 \leq G(P\|Q) \leq B_G(r, R) \leq A_G(r, R),
\] (45)
where
\[
E_G(P\|Q) = \frac{1}{2} \left[ \chi^2(Q\|P) - D(Q\|P) \right],
\]
\[
A_G(r, R) = \frac{(R - r)^2}{8rR} \left[ 1 - L_{-1}^{-1}\left( \frac{r + 1}{r}, \frac{R + 1}{R} \right) \right]
\]
and
\[
B_G(r, R) = \frac{1}{2} \ln\left( \frac{(r + 1)(R + 1)}{4rR} \right) - \frac{1 - Rr}{2rR} L_{-1}^{-1}\left( \frac{r + 1}{r}, \frac{R + 1}{R} \right).
\]
Example 3.4. (Triangular discrimination). Let us consider
\[ f_{\Delta}(x) = \frac{(x-1)^2}{x+1}, \quad x \in (0, \infty) \] (46)
in (19), then we have \( C_f(P||Q) = \Delta(P||Q) \).
Moreover,
\[ f'_{\Delta}(x) = \frac{(x-1)(x+3)}{(x+1)^2} \] (47)
and
\[ f''_{\Delta}(x) = \frac{8}{(x+1)^3}. \] (48)

In view of (46), (47), Theorems 2.1 and 2.2, we have the following bounds on triangular
discrimination:
\[ 0 \leq \Delta(P||Q) \leq E_{\Delta}(P||Q) \leq A_{\Delta}(r, R) \] (49)
and
\[ 0 \leq \Delta(P||Q) \leq B_{\Delta}(r, R) \leq A_{\Delta}(r, R), \] (50)
where
\[ E_{\Delta}(P||Q) = \sum_{i=1}^{n} \left( \frac{p_i - q_i}{p_i + q_i} \right)^2 (p_i + 3q_i), \]
\[ A_{\Delta}(r, R) = \frac{(R-r)^2(R+r+2)}{(R+1)^2(r+1)^2} \]
and
\[ B_{\Delta}(r, R) = \frac{2(R-1)(1-r)}{(R+1)(1+r)}. \]

Example 3.5. (J-divergence). Let us consider
\[ f_J(x) = (x-1) \ln x, \quad x \in (0, \infty), \] (51)
in (19), then we have \( C_f(P||Q) = J(P||Q) \).
Moreover,
\[ f'_J(x) = 1 - x^{-1} + \ln x, \] (52)
and
\[ f''_J(x) = \frac{x+1}{x^2}. \] (53)

In view of (51), (52), Theorems 2.1 and 2.2, we have the following bounds on J-
divergence:
\[ 0 \leq J(P||Q) \leq E_J(P||Q) \leq A_J(r, R) \] (54)
and
\[ 0 \leq J(P||Q) \leq B_J(r, R) \leq A_J(r, R), \tag{55} \]

where
\[
E_J(P||Q) = J(P||Q) + \chi^2(Q||P),
\]
\[
A_J(r, R) = \frac{1}{4}(R - r)^2 [r^{-1} + L_{-1}^{-1}(r, R)]
\]

and
\[
B_J(r, R) = (R - 1)(1 - r)L_{-1}^{-1}(r, R).
\]

Example 3.6. (Jensen-Shannon divergence). Let us consider
\[
f_I(x) = \frac{x}{2} \ln x + \frac{x + 1}{2} \ln \left( \frac{2}{x + 1} \right), \quad x \in (0, \infty), \tag{56}\]
in (59), then we have \(C_J(P||Q) = I(P||Q)\).
Moreover,
\[
f_I'(x) = \frac{1}{2} \ln \left( \frac{2x}{x + 1} \right), \tag{57}\]
and
\[
f_I''(x) = \frac{1}{2x(x + 1)}. \tag{58}\]

In view of (56), (57), Theorems 2.1 and 2.2, we have the following bounds on Jensen-Shannon divergence:
\[ 0 \leq I(P||Q) \leq E_I(P||Q) \leq A_I(r, R) \tag{59} \]
and
\[ 0 \leq I(P||Q) \leq B_I(r, R) \leq A_I(r, R), \tag{60} \]

where
\[
E_I(P||Q) = \frac{1}{2} D(Q||P),
\]
\[
A_I(r, R) = \frac{1}{8 (R + 1)(r + 1)} L_{-1}^{-1} \left( \frac{r}{r + 1}, \frac{R}{R + 1} \right)
\]
and
\[
B_I(r, R) = \frac{1}{2(R - r)} \left[ (R - 1) \left( r \ln r + (r + 1) \ln \left( \frac{2}{r + 1} \right) \right) \right. \tag{61}
\]
\[ - (1 - r) \left( R \ln R + (R + 1) \ln \left( \frac{2}{R + 1} \right) \right) \right]. \]
Example 3.7. (arithmetic-geometric divergence). Let us consider
\[ f_T(x) = \left( \frac{x + 1}{2} \right) \ln \left( \frac{x + 1}{2\sqrt{x}} \right), \quad x \in (0, \infty), \quad (62) \]
in (19), then we have \( C_T(P||Q) = T(P||Q) \).
Moreover,
\[ f'_T(x) = \frac{1}{4} \left[ 1 - x^{-1} + 2 \ln \left( \frac{x + 1}{2\sqrt{x}} \right) \right], \quad (63) \]
and
\[ f''_T(x) = \frac{1}{4} \left( \frac{1 + x^2}{x^2 + x^3} \right). \quad (64) \]

In view of (63), (64), Theorems 2.1 and 2.2 we have the following bounds on arithmetic-geometric divergence:
\[ 0 \leq T(P||Q) \leq E_T(P||Q) \leq A_T(r, R) \quad (65) \]
and
\[ 0 \leq T(P||Q) \leq B_T(r, R) \leq A_T(r, R), \quad (66) \]
where
\[ E_T(P||Q) = \frac{1}{4} \chi^2(Q||P) + \frac{1}{2} \sum_{i=1}^{n} (p_i - q_i) \ln \left( \frac{p_i + q_i}{2\sqrt{p_i q_i}} \right), \]
\[ A_T(r, R) = \frac{1}{16} (R - r)^2 \left[ (rR)^{-1} + L_{-1}^{-1} (r + 1, R + 1) - L_{-1}^{-1} (r, R) \right] \]
and
\[ B_T(r, R) = \frac{1}{2(R - r)} \left[ (R - 1)(r + 1) \ln \left( \frac{r + 1}{2\sqrt{r}} \right) \right. \]
\[ + (1 - r)(R + 1) \ln \left( \frac{R + 1}{2\sqrt{R}} \right) \] .

4 Relative J-Divergence and Inequalities

In this section we shall present bound on relative J-divergence in terms of symmetric measures (8), (11)-(13).

Proposition 4.1. (Relative J-divergence and triangular discrimination). We have the following bounds:
\[ \frac{(r + 1)(r + 3)}{8} \Delta(P||Q) \leq D(P||Q) \leq \frac{(R + 1)(R + 3)}{8} \Delta(P||Q), \quad (67) \]
Proof. Let us consider
\[ g_{D\Delta}(x) = \frac{f''_D(x)}{f''_\Delta(x)} = \frac{(x+1)(x+3)}{8}, \quad x \in (0, \infty), \] (68)
where \( f''_D(x) \) and \( f''_\Delta(x) \) are as given by (33) and (48) respectively.

From (68), we have
\[ g'_{D\Delta}(x) = \frac{2 + x}{4} > 0, \quad x \in (0, \infty). \] (69)

In view of (69), we conclude that
\[ m = \inf_{x \in [r, R]} g_{D\Delta}(x) = \frac{(r+1)(r+3)}{8} \] (70)
and
\[ M = \sup_{x \in [r, R]} g_{D\Delta}(x) = \frac{(R+1)(R+3)}{8}. \] (71)

Expressions (70) and (71) together with (29) give the required result.

Proposition 4.2. (Relative J-divergence and J-divergence). We have the following bounds:
\[ \frac{r^2(r+3)}{(r+1)^3} J(P||Q) \leq D(P||Q) \leq \frac{R^2(R+3)}{(R+1)^3} J(P||Q), \] (72)

Proof. Let us consider
\[ g_{DJ}(x) = \frac{f''_D(x)}{f''_J(x)} = \frac{x^2(x+3)}{(x+1)^3}, \quad x \in (0, \infty), \] (73)
where \( f''_D(x) \) and \( f''_J(x) \) are as given by (33) and (53) respectively.

From (73), we have
\[ g'_{DJ}(x) = \frac{6x}{(x+1)^4} > 0, \quad x \in (0, \infty). \] (74)

In view of (74), we conclude that
\[ m = \inf_{x \in [r, R]} g_{DJ}(x) = \frac{r^2(r+3)}{(r+1)^3} \] (75)
and
\[ M = \sup_{x \in [r, R]} g_{DJ}(x) = \frac{R^2(R+3)}{(R+1)^3}. \] (76)

Expressions (75) and (76) together with (29) give the required result.
Proposition 4.3. (Relative J-divergence and Jensen-Shannon divergence). We have the following bounds:

\[
\frac{2r(r+3)}{r+1} I(P\|Q) \leq D(P\|Q) \leq \frac{2R(R+3)}{R+1} I(P\|Q),
\]

(77)

Proof. Let us consider

\[
g_{DI}(x) = \frac{f''_D(x)}{f''_I(x)} = \frac{2r(x+3)}{x+1}, \quad x \in (0, \infty),
\]

(78)

where \( f''_D(x) \) and \( f''_I(x) \) are as given by (33) and (58) respectively.

From (78), we have

\[
g'_{DI}(x) = \frac{2(x^2 + 2x + 3)}{(x+1)^2} > 0, \quad x \in (0, \infty).
\]

(79)

In view of (79), we conclude that

\[
m = \inf_{x \in [r,R]} g_{DI}(x) = \frac{2r(r+3)}{r+1}
\]

(80)

and

\[
M = \sup_{x \in [r,R]} g_{DI}(x) = \frac{2R(R+3)}{R+1}.
\]

(81)

Expressions (80) and (81) together with (29) give the required result. \( \square \)

5 Relative Jensen-Shannon divergence and Inequalities

In this section we shall present bound on relative Jensen-Shannon divergence in terms of symmetric measures (8), (11)-(13).

Proposition 5.1. (Relative Jensen-Shannon divergence and triangular discrimination). We have the following bounds:

\[
\frac{R+1}{8R} \Delta(P\|Q) \leq F(P\|Q) \leq \frac{r+1}{8r} \Delta(P\|Q),
\]

(82)

Proof. Let us consider

\[
g_{FD}(x) = \frac{f''_F(x)}{f''_D(x)} = \frac{x+1}{8x}, \quad x \in (0, \infty),
\]

(83)

where \( f''_F(x) \) and \( f''_D(x) \) are as given by (38) and (48) respectively.
From (83), we have
\[ g'_{F\Delta}(x) = -\frac{1}{8x^2} < 0, \quad x \in (0, \infty). \tag{84} \]

In view of (84), we conclude that
\[ m = \inf_{x \in [r,R]} g_{F\Delta}(x) = \frac{R + 1}{8R} \tag{85} \]
and
\[ M = \sup_{x \in [r,R]} g_{F\Delta}(x) = \frac{r + 1}{8r} \tag{86} \]

Expressions (85) and (86) together with (29) give the required result.

**Proposition 5.2.** *(Relative Jensen-Shannon divergence and J-divergence)*. We have the following bounds:
\[ 0 \leq F(P||Q) \leq \frac{4}{27} J(P||Q), \tag{87} \]

**Proof.** Let us consider
\[ g_{F,J}(x) = \frac{f''_F(x)}{f''_J(x)} = \frac{x}{(x+1)^3}, \quad x \in (0, \infty), \tag{88} \]
where \( f''_F(x) \) and \( f''_J(x) \) are as given by (38) and (53) respectively.

From (88), we have
\[ g'_{F,J}(x) = -\frac{2x - 1}{(x+1)^4} \begin{cases} 
  \geq 0, & x \leq \frac{1}{2} \\
  \leq 0, & x \geq \frac{1}{2}.
\end{cases} \tag{89} \]

In view of (89), we conclude that the function \( g_{F,J}(x) \) is increasing in \((0, \frac{1}{2})\) and decreasing in \((\frac{1}{2}, \infty)\), and hence
\[ M = \sup_{x \in [r,R]} g_{F,J}(x) = g_{F,J}(\frac{1}{2}) = \frac{4}{27}. \tag{90} \]

Now (90) together with (29) give the required result.

**Proposition 5.3.** *(Adjoint of relative Jensen-Shannon divergence and Jensen-Shannon divergence)*. We have the following bounds:
\[ \frac{2}{R+1} I(P||Q) \leq F(P||Q) \leq \frac{2}{r+1} I(P||Q), \tag{91} \]
Proof. Let us consider
\[ g_{FI}(x) = \frac{f''_F(x)}{f''_I(x)} = \frac{2}{x+1}, \quad x \in (0, \infty), \]  
(92)
where \( f''_F(x) \) and \( f''_I(x) \) are as given by (38) and (58) respectively. From (92), we have
\[ g_{FI}^\prime(x) = -\frac{2}{(x+1)^2} < 0, \quad x \in (0, \infty). \]  
(93)
In view of (93), we conclude that
\[ m = \inf_{x \in [r,R]} g_{FI}(x) = \frac{2}{R+1}, \]  
(94)
and
\[ M = \sup_{x \in [r,R]} g_{FI}(x) = \frac{2}{r+1}. \]  
(95)
Expressions (94) and (95) together with (29) give the required result. \( \square \)

Remark 5.1. The inequalities (82) and (91) can also be as
\[ r \leq \zeta_t(P||Q) \leq R, \quad t = 1 \text{ and } 2, \]
where
\[ \zeta_1(P||Q) = \frac{\Delta(P||Q)}{8F(P||Q) - \Delta(P||Q)}, \]
and
\[ \zeta_3(P||Q) = \frac{2I(P||Q) - F(P||Q)}{F(P||Q)} \]
respectively.

6 Relative arithmetic-geometric divergence and Inequalities

In this section we shall present bound on relative arithmetic-geometric divergence in terms of symmetric measures (8), (11)-(13).

Proposition 6.1. (Relative arithmetic-geometric divergence and triangular discrimination). We have the following bounds:
\[ \frac{(R+1)^2}{16R^2} \Delta(P||Q) \leq G(P||Q) \leq \frac{(r+1)^2}{16r^2} \Delta(P||Q), \]  
(96)
Proof. Let us consider
\[ g_{G\Delta}(x) = \frac{f''_G(x)}{f''_\Delta(x)} = \frac{(x+1)^2}{16x^2}, \quad x \in (0, \infty), \tag{97} \]
where \( f''_G(x) \) and \( f''_\Delta(x) \) are as given by (38) and (48) respectively.

From (97), we have
\[ g'_{G\Delta}(x) = -\frac{x+1}{8x^3} < 0, \quad x \in (0, \infty). \tag{98} \]

In view of (98), we conclude that
\[ m = \inf_{x \in [r,R]} g_{G\Delta}(x) = \frac{(R+1)^2}{16R^2} \tag{99} \]
and
\[ M = \sup_{x \in [r,R]} g_{G\Delta}(x) = \frac{(r+1)^2}{16r^2}. \tag{100} \]

Expressions (99) and (100) together with (29) give the required result.

Proposition 6.2. (Relative arithmetic-geometric divergence and J-divergence). We have the following bounds:
\[ \frac{1}{2(R+1)^2} J(P||Q) \leq G(P||Q) \leq \frac{1}{2(r+1)^2} J(P||Q), \tag{101} \]

Proof. Let us consider
\[ g_{GJ}(x) = \frac{f''_G(x)}{f''_J(x)} = \frac{1}{2(x+1)^2}, \quad x \in (0, \infty), \tag{102} \]
where \( f''_G(x) \) and \( f''_J(x) \) are as given by (38) and (53) respectively.

From (102), we have
\[ g'_{GJ}(x) = -\frac{1}{(x+1)^3} < 0, \quad x \in (0, \infty). \tag{103} \]

In view of (98), we conclude that
\[ m = \inf_{x \in [r,R]} g_{GJ}(x) = \frac{1}{2(R+1)^2} \tag{104} \]
and
\[ M = \sup_{x \in [r,R]} g_{GJ}(x) = \frac{1}{2(r+1)^2}. \tag{105} \]

Expressions (104) and (105) together with (29) give the required result.
Proposition 6.3. (Relative arithmetic-geometric divergence and Jensen-Shannon divergence). We have the following bounds:

\[
\frac{1}{R} I(P\|Q) \leq G(P\|Q) \leq \frac{1}{r} I(P\|Q),
\]

(106)

Proof. Let us consider

\[
g_{GI}(x) = \frac{f''_G(x)}{f''_I(x)} = \frac{1}{x}, \quad x \in (0, \infty),
\]

(107)

where \( f''_G(x) \) and \( f''_I(x) \) are as given by (38) and (58) respectively.

From (107), we have

\[
g'_{GI}(x) = -\frac{1}{x^2} < 0, \quad x \in (0, \infty).
\]

(108)

In view of (108), we conclude that

\[
m = \inf_{x \in [r,R]} g_{GI}(x) = \frac{1}{R}
\]

(109) and

\[
M = \sup_{x \in [r,R]} g_{GI}(x) = \frac{1}{r}.
\]

(110)

Expressions (109) and (110) together with (29) give the required result.

Proposition 6.4. (Relative arithmetic-geometric divergence and arithmetic-geometric divergence). We have the following bounds:

\[
\frac{2}{1 + R^2} T(P\|Q) \leq G(P\|Q) \leq \frac{2}{1 + r^2} T(P\|Q),
\]

(111)

Proof. Let us consider

\[
g_{GT}(x) = \frac{f''_G(x)}{f''_T(x)} = \frac{2}{1 + x^2}, \quad x \in (0, \infty),
\]

(112)

where \( f''_G(x) \) and \( f''_T(x) \) are as given by (38) and (64) respectively.

From (112), we have

\[
g'_{GT}(x) = -\frac{4x}{(1 + x^2)^2} < 0, \quad x \in (0, \infty).
\]

(113)

In view of (113), we conclude that

\[
m = \inf_{x \in [r,R]} g_{GT}(x) = \frac{2}{1 + R^2}
\]

(114) and

\[
M = \sup_{x \in [r,R]} g_{GT}(x) = \frac{2}{1 + r^2}.
\]

(115)

Expressions (114) and (115) together with (29) give the required result.
Remark 6.1. The inequalities (96), (101), (106) and (111) can also be as

\[ r \leq \xi_t(P||Q) \leq R, \quad t = 1, 2, 3 \text{ and } 4, \]

where

\[
\xi_1(P||Q) = \frac{\sqrt{\Delta(P||Q)}}{4\sqrt{G(P||Q)} - \sqrt{\Delta(P||Q)}},
\]

\[
\xi_2(P||Q) = \frac{\sqrt{J(P||Q)} - \sqrt{2G(P||Q)}}{\sqrt{2G(P||Q)}},
\]

\[
\xi_2(P||Q) = \frac{I(P||Q)}{G(P||Q)},
\]

and

\[
\xi_4(P||Q) = \frac{\sqrt{2T(P||Q)} - G(P||Q)}{\sqrt{G(P||Q)}},
\]

respectively.

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