Resolvent Operator Transformations and Bound–State Solutions for Confluent Natanzon Potentials

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Abstract

An algebraic method of constructing the confluent Natanzon potentials endowed with position–dependent mass is presented. This is possible by identifying the scaling resolvent operator (Green’s function) to nonrelativistic position–dependent mass Schrödinger equation.

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1 Introduction

The (confluent) Natanzon potentials [1,2] are known nowadays as a general class of solvable potentials related to the (confluent) hypergeometric functions and cover all well–known potentials in quantum mechanics for which an exact solutions can be found [3]. The (Radial) harmonic oscillator potential [4], the Coulomb potential [4] and the Morse potential [5] may all be considered as particular cases of the confluent hypergeometric potentials, while the hypergeometric potentials consist of the Pöschl–Teller potential [6], Eckart potential [7], Rosen–Morse potential [8], etc.

However, different series of papers have been devoted in recent years to the study of the dynamical symmetries [9–17]. It was proved that the bound and scattering states of these potentials are related to the representation of the $\mathfrak{so}(2,1) \sim \mathfrak{su}(1,1) \sim \mathfrak{sl}(2,\mathbb{R})$ Lie algebras; these algebras play the role of dynamical algebras. They are used as a spectrum generating algebra [9–13] and as a potential algebra [14–17] in order to obtain the analytic solutions.

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We have succeeded in generating the Natanzon hypergeometric potentials by making use of conformal mappings within the framework of position–dependent mass (PDM) [18]. In this paper, we will explain the form in which the \( \mathfrak{so}(2,1) \) Lie algebra can be enlarged as wide as possible in order to generate the confluent Natanzon hypergeometric potentials endowed with PDM. This is accomplished by scaling the resolvent operator (Green’s function) [19], in such a way that can be related to non-relativistic PDM Schrödinger equation [20–24]. This explicit connection between the resolvent operator transformations to PDM quantum systems.

The organization of the present paper is as follows. We introduce in section 2 a specific differential realization of the \( \mathfrak{so}(2,1) \) Lie algebra. Then applying a scaling in the resolvent operator, through a scale function that preserves the wavefunctions, will map it into the wave equation endowed with position–dependent mass. This is possible if the scale function is identified to the mass distribution. This correspondence gives, in section 3, the (effective) potential expression for confluent Natanzon potentials in the framework of position–dependent mass. The last section is kept for our concluding remarks and an appendix was added in order to point out the correct use of the scalar product under this scaling.

## 2 Resolvent operator transformations and \( \mathfrak{so}(2,1) \) algebra

We are going to build a realization of the \( \mathfrak{so}(2,1) \) Lie algebra according to [12]. Let us start from a couple of operators \( a \) and \( b \) satisfying the commutation relation

\[
[a, b] = 1,
\]

and defining the following realization for the generators \( J_k \) \((k = 0, 1, 2)\)

\[
J_0 = \frac{1}{4} \left[ -a^2 + \left( 4c + \frac{3}{4} \right) b^{-2} + b^2 \right],
\]
\[
J_1 = \frac{1}{4} \left[ -a^2 + \left( 4c + \frac{3}{4} \right) b^{-2} - b^2 \right],
\]
\[
J_2 = -\frac{i}{4} (2ba + 1),
\]

where \( c \) are eigenvalues of the Casimir’s operator of the algebra. The relations (2) satisfy the commutation relations

\[
[J_0, J_1] = iJ_2, \quad [J_2, J_0] = iJ_1, \quad [J_1, J_2] = -iJ_0.
\]

It might be of interest to consider the operators \( a \) and \( b \) in terms of a variable \( x \) as \( a \equiv \frac{d}{dx} + K(x) \) and \( b \equiv x \). Performing the point canonical transformation \( x = \sqrt{\xi(u)} \), the operators \( a \) and \( b \) become

\[
a = \frac{2\sqrt{\xi(u)}}{\xi'(u)} \frac{d}{du} - \frac{1}{2} \left( \frac{2\sqrt{\xi(u)}}{\xi'(u)} \right)', \quad b = \sqrt{\xi(u)}.
\]
where here and below the prime stands for derivatives in variable \( u \). In terms of this realization, the generators (2) become [12]

\[
J_0 = -\frac{\xi'(u)}{\xi'^2(u)} d^2 - \frac{1}{2} \frac{\xi''(u) \xi(u)}{\xi'^3(u)} - \frac{3}{4} \frac{\xi''(u) \xi(u)}{\xi'^4(u)} + \frac{c}{\xi(u)} + \frac{\xi(u)}{4}, \tag{5.1}
\]

\[
J_1 = -\frac{\xi(u)}{\xi'^2(u)} d^2 - \frac{1}{2} \frac{\xi''(u) \xi(u)}{\xi'^3(u)} - \frac{3}{4} \frac{\xi''(u) \xi(u)}{\xi'^4(u)} + \frac{c}{\xi(u)} - \frac{\xi(u)}{4}, \tag{5.2}
\]

\[
J_2 = -i \frac{\xi(u)}{\xi'(u)} d - \frac{i}{2} \frac{\xi''(u) \xi(u)}{\xi'^2(u)}. \tag{5.3}
\]

The Green's function for a wide class of potentials is given following expression

\[
(H - E) G(u, \overline{u}) = \delta(u - \overline{u}), \tag{6}
\]

and due to the form of the generators in (5), it is easy to give the resolvent operator

\[
\Omega(E) = \frac{\xi(u)}{\xi'^2(u)} (H - E) = q_0 + \sum_{k=0}^{2} p_k J_k, \tag{7}
\]

where \( q_0 \) and \( p_k \) \((k = 0, 1, 2)\) are arbitrary parameters that will be fixed below.

The realization in (7) may be implemented by scaling the resolvent operator on the generators (2) in such a way to explicit connection between the resolvent operator (7), and then the spectrum generating algebra method, to PDM Schrödinger equation. Under such transformations, the operator \( \Omega(E) \) becomes

\[
\overline{\Omega}(E) = \left( q_0 + \sum_{k=0}^{2} p_k J_k \right) P(u, \overline{u}) = P(u, \overline{u}) \left( q_0 + \sum_{k=0}^{2} p_k T_k \right), \tag{8}
\]

where \( P(u, \overline{u}) = P_1(u) P_2(\overline{u}) \) is called two–point scale function, and \( T_k \) are generators which are determined through identification.

Under this change and by identifying both sides of (8) in terms of parameters \( q_0 \) and \( p_k \), the generators \( T_k \) read as

\[
T_0 = -\frac{\xi}{\xi'^2}(d^2 - 2 \frac{\xi}{\xi'^2} \frac{P'}{P} d - \frac{\xi}{\xi'^2} \frac{P''}{P}) - \frac{1}{2} \frac{\xi''}{\xi'^3} + \frac{3}{4} \frac{\xi''}{\xi'^4} + \frac{c}{\xi} + \frac{\xi}{4}, \tag{9.1}
\]

\[
T_1 = -\frac{\xi}{\xi'^2}(d^2 - 2 \frac{\xi}{\xi'^2} \frac{P'}{P} d - \frac{\xi}{\xi'^2} \frac{P''}{P}) - \frac{1}{2} \frac{\xi''}{\xi'^3} + \frac{3}{4} \frac{\xi''}{\xi'^4} + \frac{c}{\xi} - \frac{\xi}{4}, \tag{9.2}
\]

\[
T_2 = -i \frac{\xi}{\xi'} d - i \frac{\xi}{\xi'} \frac{P'}{P} - \frac{i}{2} \frac{\xi''}{\xi'^2}, \tag{9.3}
\]

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where \( P \) here is referred to the function \( P_1(u) \) due to differentiation with respect to the variable \( u \).

By introducing the eigenfunctions

\[
\psi(u) = 2\frac{\xi^2(u)}{\xi'^2(u)}\phi(u),
\]

the operator \( \overline{\Omega}(E) \) becomes after some straightforward algebras

\[
\overline{\Omega}(E) = P(u,\overline{u})\frac{\xi}{\xi'^2} \left[ -2p_+\frac{\xi^2}{\xi'^2}\frac{d^2}{du^2} - \frac{\xi}{\xi} \left( 4p_+ \left( 2 + \frac{P'}{P} \frac{\xi}{\xi'^2} - 2\frac{\xi''}{\xi'^2} \right) + 2ip_2\xi \right) \right] \frac{d}{du} - p_+ \left( \frac{8p'}{\xi} + 2 - 2p'' \xi - 8\xi'' \xi'\xi'' - 12\xi''\xi'^2\xi'' + \frac{21\xi^2\xi'^2}{\xi'^4} + \frac{3\xi^2\xi''}{\xi'^2} \right)
+ 2q_0\xi + \frac{p_+^2\xi^2 - 2ip_2\xi\left( \frac{P'}{P}\xi^2 + 2\xi - \frac{3\xi''\xi^2}{2\xi'^2} \right) \right],
\]

where \( p_\pm = p_0 \pm p_1 \).

Without any loss of generality, let us take the appropriate–parameter choices on \( p_\pm \) and \( p_2 \) such as

\[
p_+ \equiv p_0 + p_1 = \frac{\alpha}{2}, \quad p_- \equiv p_0 - p_1 = \frac{\beta}{2}, \quad p_2 \equiv 0,
\]

then the operator \( \overline{\Omega}(E) \) in (11) becomes

\[
\overline{\Omega}(E) \equiv P(u,\overline{u})\frac{\xi}{\xi'^2} \left( \hat{H} - E \right)
= P(u,\overline{u})\frac{\xi}{\xi'^2} \left[ -\alpha\frac{\xi^2}{\xi'^2}\frac{d^2}{du^2} - \alpha\frac{\xi}{\xi} \left( 4 + 2\frac{P'}{P} \frac{\xi}{\xi'^2} - 4\frac{\xi''}{\xi'^2} \right) \frac{d}{du} - \frac{2q_0\xi + \frac{p_+^2\xi^2 - 2ip_2\xi\left( \frac{P'}{P}\xi^2 + 2\xi - \frac{3\xi''\xi^2}{2\xi'^2} \right) \right]}{4\xi'^4 + \frac{3\alpha\xi^2\xi''}{\xi'^2}} \right].
\]

On the other hand, in the case of varying mass which will be denoted by \( M(u) = m_0m(u) \), the Hamiltonian introduced by von Roos [20] reads as

\[
H_{\text{VR}} = \frac{1}{4} \left( \eta \epsilon m \frac{\partial}{\partial u} \right)^n m^{n-1} + V_{\text{eff}}(u),
\]

where \( \eta, \epsilon, \) and \( \rho \) are three parameters which obey to the restriction \( \eta + \epsilon + \rho = -1 \) in order to grant the classical limit. Here \( p = -i\frac{\partial}{\partial u} \) is a momentum with \( \hbar = m_0 = 1 \), and \( m(u) \) is dimensionless–real valued mass.

Applying the eigenfunctions [18,23]

\[
\Psi(u) \equiv m(u)\psi(u)
= 2m(u)\frac{\xi^2(u)}{\xi'^2(u)}\phi(u)
\]

(15)
the resolvent operator of $H_{vR}$ reads [18]

$$H_{vR} - E = -\frac{\xi^2}{\xi^2 du^2} - \frac{\xi}{\xi} \left( 4 + \frac{m'}{m' \xi} - \frac{4 \xi''}{\xi^2} \right) \frac{d}{du} + \frac{2 \xi}{\xi^2} \left( 3 \xi'' + \frac{\xi''''}{\xi} - \frac{3 \xi'''}{\xi^2} \right) \right) + \frac{m'}{m' \xi^2} \left( 2 \frac{\xi''}{\xi^2} - \frac{\xi''^2}{\xi^2} \right) + \frac{\epsilon - 1}{2} m' - (1 + \eta) (\eta + \epsilon) \frac{m'}{m} \right) - 2$$

$$+ 2 m \frac{\xi^2}{\xi^2} (V_{eff} (x) - E).$$

(16)

Then comparing (13) to (16) in their differential terms, we obtain

$$P(u, \pi) = \sqrt{m(u) m(\pi)},$$

(17)

leading to defining the one–point scale function $P(u, u) \equiv P(u) = m(u)$ as $u = \pi$. By substituting (17) into the remainder term leads, after long calculations, to the following general expression for the potential

$$V [\xi (u)] - E = \frac{1}{2m(u)} \xi^2(\xi)(2q_0 \xi (u) + \frac{\beta}{4} \xi^2 (u) + c) - \frac{1}{4m(u)} \left\{ \xi (u), u \right\}_S,$$

(18)

and which corresponds to the specific–parameter choice $\alpha = 1$. Here, $\left\{ \xi (u), u \right\}_S$ is the Schwarz derivative of $\xi (u)$ with respect to $u$ defined by

$$\left\{ \xi (u), u \right\}_S = \frac{\xi'' (u)}{\xi (u)} \left( \frac{\xi''' (u)}{2 \xi (u)} - \frac{3 \xi'' (u)}{2 \xi (u)} \right),$$

(19)

and $V [\xi (u)] = V_{eff} [\xi (u)] - \mathcal{V}_m^{(n,c)} (u)$, with

$$\mathcal{V}_m^{(n,c)} (u) = \frac{(1 + 2 \eta)^2 + 4 \epsilon (1 + \eta) m^2 (u)}{8 \epsilon m^2 (u)} - \frac{\epsilon m'' (u)}{4 m^2 (u)},$$

(20)

which is attributed to the dependence of the mass $m$ on $u$.

### 3 Generating confluent Natanzon potentials

Since we have to get a constant term in left–hand side of (18), we must divide the right–hand side into two parts: the former represents a constant term and which is identified to the energy $E$, while the latter corresponds to the potential $V [\xi (u)]$. This condition amounts to setting up differential equation for $\xi (u)$ and satisfying

$$\frac{\xi^2(\xi) \lambda_2 \xi^2(u) + \lambda_1 \xi (u) + \lambda_0}{4 \xi^2 (u)} = 1,$$

(21)

where $\lambda_i (i = 0, 1, 2)$ are arbitrary parameters. Inserting (21) into (18), we get

$$V [\xi (u)] - E = \frac{\beta \xi^2 (u) + 8 q_0 \xi (u) + 4 \epsilon}{\lambda_2 \xi^2 (u) + \lambda_1 \xi (u) + \lambda_0} - \frac{1}{4m(u)} \left\{ \xi (u), u \right\}_S.$$

(22)
However in order to get a constant term in right-hand side of (22), it is useful to make some particularizations on the parameters $\beta$, $q_0$, and $c$. As a consequence, it will turn out to be useful to express the resolvent operator $(\tilde{H} - E)$ in (13) as the exponential function of the generator $T_2$, since $T_2$ is essentially a generator of scale transformations [25].

To this end, taking into account (12), the resolvent operator (8) reads

$$\left(\tilde{H} - E\right)\psi(u) = \frac{\xi'^2(u)}{\xi(u)} \left[2(\beta + 1)T_0 - 2(\beta - 1)T_1 + 8q_0\right]\psi(u) = 0. \quad (23)$$

It follows from the identity [26]

$$e^{i\theta T_2} T_0 e^{-i\theta T_2} \equiv \sum_{k=0}^{\infty} \frac{1}{n!} [i\theta T_2, T_0]_{(n)}$$

$$= T_0 \cosh \theta - T_1 \sinh \theta, \quad (24)$$

where

$$[i\theta T_2, T_0]_{(n)} = \left[i\theta T_2, [i\theta T_2, T_0]_{(n-1)}\right] = \left[i\theta T_2, [i\theta T_2, [i\theta T_2, T_0]_{(n-2)}]\right] = \cdots \quad (25)$$

that the right-hand side of (23) becomes

$$T_0\psi(u) \equiv e^{i\delta T_2} T_0 e^{-i\delta T_2} \psi(u) = -\frac{2q_0}{\sqrt{\beta}} \psi(u), \quad (26)$$

where $\theta$ and $\delta$ are expressed in terms of $\beta$ as

$$\theta = \frac{\beta - 1}{\beta + 1}, \quad \text{and} \quad \delta = \frac{1}{2} \ln \beta, \quad (27)$$

and are allowed to depend on $n$ and $c$.

Since $J_0$ is the only compact generator with eigenvalues $\langle J_0 \rangle = j_0$, then the allowed values of $j_0$ are related to $n$ and $c$ through [12]

$$j_0 = n + \frac{1}{2} \left(1 + \sqrt{1 + 4c}\right), \quad (28)$$

where $n = 0, 1, 2, \ldots$ Combining (28) to (26), we get the basic equation

$$q_0 + \frac{\sqrt{\beta}}{4} \left(2n + 1 + \sqrt{1 + 4c}\right) = 0. \quad (29)$$

It is then obvious that the simplest set of parameters associated with $\beta$, $q_0$ and $c$ and compatible with (29) is

$$\beta = a^2, \quad q_0 = -a \left(\frac{b}{2} + n\right), \quad c = b \left(\frac{b}{2} - 1\right), \quad (30)$$
where the equation (29) can be obtained using the equalities (30) by the straight substitution.

Equation (22) becomes

$$V[\xi(u)] - E = \frac{a^2 \xi^2(u) - 2a(2n + b) \xi(u) + b(b - 2)}{\lambda_2 \xi^2(u) + \lambda_1 \xi(u) + \lambda_0} - \frac{1}{4m(u)} \{\xi(u), u\}_S. \quad (31)$$

which in particular case requires that coefficients in numerator are linear with those of denominator with respect to $E$ following

$$a^2 = -\lambda_2 E + \sigma_\beta, \quad (32.1)$$

$$2a(2n + b) = \lambda_1 E - \sigma_{\varrho_0}, \quad (32.2)$$

$$b(b - 2) = -\lambda_0 E + \sigma_c, \quad (32.3)$$

leading to express the potential $V(u)$ into

$$V[\xi(u)] = \frac{\sigma_\beta \xi^2(u) + \sigma_{\varrho_0} \xi(u) + \sigma_c}{R[\xi(u)]} - \frac{1}{4m(u)} \{\xi(u), u\}_S, \quad (33)$$

with $R(\xi) = \lambda_2 \xi^2 + \lambda_1 \xi + \lambda_0$. For convenience, we introduce the auxiliary function from (21) as [12, 18]

$$\mathcal{G}[\xi(u)] \equiv \frac{\xi^2(u)}{2m(u)} = \frac{4\xi^2(u)}{R[\xi(u)]}, \quad (34)$$

therefore the Schwarz derivative–term can be expressed in terms of $\xi(u)$

$$\frac{1}{4m(u)} \{\xi(u), u\}_S = -\frac{1}{R} - \frac{\xi^2 \ddot{R} + \xi \dot{R}}{R^2} + \frac{5}{4} \frac{\xi^2 \dot{R}^2}{R^3} - U_m(u), \quad (35)$$

where the dot refers to the derivative with respect to $\xi$ and $U_m(u) = \frac{5}{32} \frac{m^2(u)}{m^4(u)} - \frac{1}{8} \frac{m''(u)}{m^2(u)}$.

Inserting (35) into (33), taking into account the first and the second derivative, we get

$$V(u) = \frac{\sigma_\beta \xi^2 + \sigma_{\varrho_0} \xi + \sigma_c + 1}{R(\xi)} + \frac{4\lambda_2 \xi^2 + \lambda_1 \xi}{R^2(\xi)} - \frac{5}{4} \xi^2 \frac{(4\lambda_2 \xi + \lambda_1)^2}{R^3(\xi)}, \quad (36)$$

where a effective–mass potential is now given

$$U_m^{(\eta, \epsilon)}(u) = V_m^{(\eta, \epsilon)}(u) + U_m(u)$$

$$= \frac{4(1 + 2\eta)^2 + 16\epsilon(1 + \eta) + 5m^2(u)}{32m^4(u)} - \frac{2\epsilon + 1m''(u)}{8m^2(u)}. \quad (37)$$

Now consider simultaneous addition and subtraction of $\lambda_1^2 - \lambda_0 \lambda_2$, this procedure allows us to get $R(\xi)$ in the numerator of the third term. We obtain after some straightforward algebras

$$V(u) = \frac{\sigma_\beta \xi^2 + \sigma_{\varrho_0} \xi + \sigma_c + 1}{R(\xi)} + \frac{\lambda_1 \xi - \lambda_2 \xi^2}{R^2(\xi)} - \frac{5}{4} \xi^2 \frac{\Delta}{R^3(\xi)}, \quad (38)$$

where $\Delta = \lambda_1^2 - 4\lambda_0 \lambda_2$. We recognize in (38) the general expression of the confluent Natanzon potentials. Note that the potential deduced is completely independent of ambiguity parameters $\eta$, $\epsilon$, and $\rho$. However, the potential differs substantially from that usually known in the literature [11, 12].
3.1 Energy eigenvalues $E_n$

The corresponding energy eigenvalues of the confluent Natanzon potentials can be determined from (32). In fact, proceeding to squaring (32.a) and (32.c) and by inserting it into (32.b), we see then that

$$2n + 1 = -\frac{\sigma_q + \lambda_1 E_n}{2\sqrt{\sigma_{12} - \lambda_2 E_n}} - \sqrt{1 + \sigma_{12} - \lambda_0 E_n},$$

(39)

is an equality of the fourth degree in $E_n$ and agree with that obtained by [11,12]. It is obvious that parameters $a$ and $b$ depend on quantum number $n$.

3.2 Wavefunctions $\overline{\psi}_n [\xi (u)]$

The two sets of operators $T_i$ and $T_i$ are obviously different but have the same effects on the wavefunctions. Using (24) and (26), it turns out that the parameter $\sqrt{\beta} = a_n$ plays the role of scaling parameter in such a way that $\xi (u)$ can be mapped into $a_n \xi (u)$, but does not affect the general form of wavefunctions.

The wavefunctions belong to the Hilbert space, not with respect to the usual scalar product of quantum mechanics but with respect to the new scalar product (see the appendix)

$$\langle \overline{\psi}_j \overline{\psi}_i \rangle_W = \int_{-\infty}^{+\infty} \overline{\psi}_i (u) \overline{\psi}_j (u) du$$

$$= \int_{-\infty}^{+\infty} \chi_i (u) \frac{1}{W (u)} \chi_j^* (u) du,$$

(40)

where $\overline{\psi}_i (u) = \frac{1}{\sqrt{W(u)}} \chi_i (u) (i = 1, 2, \ldots)$ and $W (u) \equiv \frac{1}{p(u)} = \frac{1}{m(u)}$ represents the weight factor.

Then, and following [9], the assumption of an unnormalized wavefunction $\overline{\psi}_n [\xi (u)]$ gets transformed as

$$\overline{\psi}_n [\xi (u)] \sim \sqrt{\frac{m(u)}{\xi(u)}} \xi^{b_n/2} (u) e^{-\xi(u)/2} \frac{1}{F_1 (-n; b_n; a_n \xi (u))}$$

$$= m^{1/4} (u) R^{1/4} (u) \xi^{(b_n - 1)/2} (u) e^{-\xi(u)/2} \frac{1}{F_1 (-n; b_n; a_n \xi (u))},$$

(41)

where the identity (34) is used. The wavefunction (41) has an extra term $\sqrt{m(u)}$ compared with that for the constant mass (see the appendix). This achieves the bound–state problem for the confluent Natanzon potentials.

4 Conclusion

We have investigated another method to generate the confluent Natanzon potentials and its bound–states. The method consists to enlarge the $so (2,1)$ Lie group generators in order to relate a resolvent operator transformations to the position–dependent
mass Schrödinger equation. We have shown that if a particular correspondence between the scale function and mass distribution exists, i.e., \( P(u, u) = \sqrt{m(u)m(u)} \), then there is a connection procedure between the resolvent operator transformations and position–dependent mass Schrödinger equation leading to generate the confluent Natanzon potentials and their bound–states. We have also pointed out the correct use of the \( W(u) \)–scalar product when dealing with position–dependent mass.

5 Appendix: The correct scalar product

As outlined in the introduction, this appendix was added in order to bring some mathematical details were not approached in subsection 3.2 relating to the correct use of the scalar product under the \( P(u) \)–scaling.

The time–dependent Schrödinger equation for the functions \( \psi(u, t) \) and \( \psi^*(u, t) \) and associated to the Hamiltonian (14) read, respectively, as [27]

\[
\begin{align*}
  i \frac{\partial \psi(u, t)}{\partial t} &= -\frac{1}{2m(u)} \frac{\partial}{\partial u} \psi(u, t) + V_{\text{eff}}(u) \psi(u, t), \\
  -i \frac{\partial \psi^*(u, t)}{\partial t} &= -\frac{1}{2m(u)} \frac{\partial}{\partial u} \psi^*(u, t) + V_{\text{eff}}(u) \psi^*(u, t),
\end{align*}
\]

where \( V_{\text{eff}}(u) \in \mathbb{R} \).

Let \( \psi_1(u, t) \) and \( \psi_2^*(u, t) \) be the solutions of (A.1) and (A.2) respectively. By multiplying (A.1) by \( W(u) \psi_2^*(u, t) \) and (A.2) by \( W(u) \psi_1(u, t) \) on the left–hand side, we end up on subtracting both equations to the equality

\[
i \frac{\partial}{\partial t} [\psi_1(u, t) W(u) \psi_2^*(u, t)] = -W(u) \left[ \psi_2^*(u, t) \frac{1}{2m(u)} \frac{\partial}{\partial u} \psi_1(u, t) - \psi_1(u, t) \frac{1}{2m(u)} \frac{\partial}{\partial u} \psi_2^*(u, t) \right],
\]

due to the fact that \( W(u) \) is independent of time \( t \). Then (A.3) can be recast into the differential form as

\[
i \frac{\partial}{\partial t} [\psi_1(u, t) W(u) \psi_2^*(u, t)] = -\frac{N}{2} W(u) \frac{\partial}{\partial u} \left\{ W(u) \left[ \psi_2^*(u, t) \frac{\partial \psi_1(u, t)}{\partial u} - \psi_1(u, t) \frac{\partial \psi_2^*(u, t)}{\partial u} \right] \right\},
\]

with a specific choice of the weight function \( W(u) \) chosen as

\[
W(u) \equiv \frac{1}{N} \frac{1}{P(u)} = \frac{1}{N} \frac{1}{m(u)}, \quad \text{for any } N \in \mathbb{N} - \{0\}. \tag{A.5}
\]

Let us introduce the wavefunctions \( \chi_i(u, t) (i = 1, 2, \ldots) \) related to the eigenfunctions \( \psi_i(u, t) \) by

\[
\chi_i(u, t) = \sqrt{W(u)} \psi_i(u, t),
\]

\[ \tag{A.6} \]
then (A.4) can be rewritten as

\[
    i \frac{\partial}{\partial t} \left( \chi_1(u, t) \frac{1}{W(u)} \chi_2^*(u, t) \right) = -\frac{N}{2} \frac{\partial}{\partial u} \left( \chi_2^*(u, t) \frac{\partial \chi_1(u, t)}{\partial u} - \chi_1(u, t) \frac{\partial \chi_2^*(u, t)}{\partial u} \right).
\]

By integrating (A.7) with respect to the variable \( u \), we get

\[
    \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \chi_1(u, t) \frac{1}{W(u)} \chi_2^*(u, t) \, du = 0,
\]

where, as is excepted for bound-state wavefunctions, \( \chi_1(u, t) \to 0 \) and \( \chi_2^*(u, t) \to 0 \) as \( u \to \pm \infty \). It becomes clear that these wavefunctions are not orthonormal as they stand; rather they are orthonormal with respect to the weight factor \( \frac{1}{W(u)} \) defined in (A.5). Dealing only with spatial wavefunctions, i.e. \( \chi_i(u) \), (A.8) leads to the new scalar product

\[
    \langle \psi_2 | \psi_1 \rangle_W \equiv \int_{-\infty}^{+\infty} \chi_1(u) \frac{1}{W(u)} \chi_2^*(u) \, du,
\]

which is more general than that given usually in quantum mechanics. Through (A.9), taken into account (A.5) and \( N = 1 \), the eigenfunctions \( \psi_i(u) \) are mapped into

\[
    \bar{\psi}_i(u) \equiv \frac{1}{\sqrt{W(u)}} \chi_i(u) = \sqrt{m(u)} \chi_i(u),
\]

which explain the manifestation of the term \( \sqrt{m(u)} \) in (41). Then due to the redefinition of the scalar product, it is obvious that \( \chi_i(u) \) are the physical wavefunctions, while \( \psi_i(u) \) are called the group wavefunctions.

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