PICARD GROUPS OF DIFFERENTIAL OPERATORS ON CURVES

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Abstract. These notes are a supplement to the first part of [CH2], concerning the Picard group of $D(X)$, where $X$ is an affine curve. The main new fact is that the exact sequence of [CH2] describing $\text{Pic } D$ is split.

1. Introduction

Let $X$ be a smooth irreducible complex affine curve. To make the statements that follow as simple as possible, we shall assume that $X$ has no nontrivial automorphisms, and also that the ring $\mathcal{O}(X)$ has no nonconstant units (that is the “case of general position”; we shall remove these assumptions in section 5 below). Let $D ≡ D(X)$ be the ring of differential operators on $\mathcal{O}(X)$, and let $\text{Pic } D$ be the group of (isomorphism classes of) autoequivalences of the category of left $D$-modules. In [CH2], Cannings and Holland proved the following.

Theorem 1.1. There is an exact sequence of groups

$$0 \to \Omega^1(X) \to \text{Pic } D \to \text{Pic } X \to 0.$$  

Here (as usual) $\Omega^1(X)$ is the additive group of regular differentials on $X$, and $\text{Pic } X$ is the group of (algebraic) line bundles.

Theorem 1.1 almost determines $\text{Pic } D$; however, there remains the question of the group extension in (1.1).

Proposition 1.2. The sequence (1.1) is split.

This sequence takes on a more familiar appearance if we rephrase some of the material of [CH2] in terms of line bundles with connection. We denote by $\text{Pic}^e X$ the group of isomorphism classes of line bundles with (flat algebraic) connection over $X$, so that we have an exact sequence of abelian groups

$$0 \to \Omega^1(X) \to \text{Pic}^e X \to \text{Pic } X \to 0;$$

here the map $p$ assigns to a line bundle with connection the underlying line bundle, and $\Omega^1(X)$ is considered as the space of connections on the trivial bundle. Now, if $\mathcal{L}$ is a line bundle on $X$, with space of global sections $\Gamma_\mathcal{L}$, a connection on $\mathcal{L}$ may be viewed as a structure of left $D$-module on $\Gamma_\mathcal{L}$, extending the given structure of $\mathcal{O}(X)$-module. The functor $\Gamma_\mathcal{L} \otimes \mathcal{O}(X)$ — then defines an element of $\text{Pic } D$, so we have a natural homomorphism

$$\chi : \text{Pic}^e X \to \text{Pic } D.$$ 

It is easy to see that this map $\chi$ is injective: Cannings and Holland proved (in effect) the following.

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Theorem 1.3. The map (1.3) is an isomorphism.

Using this isomorphism, the exact sequence (1.1) becomes the sequence (1.2), which is quite amenable to study. The proof of Theorem 1.3 consists in combining the main results of [CH1] and of [ML]; we give the outline in section 6 below.

Most of the discussion above still holds, mutatis mutandis, if $X$ is a complete (that is, projective) curve: in that case we have to replace the ring $D(X)$ by the sheaf $D_X$ of differential operators on $X$. I do not know whether Theorem 1.3 still holds in the complete case; however, we certainly still have the exact sequence (1.2), provided we replace $\text{Pic} X$ by the group $\text{Pic}^0 X$ of line bundles of degree zero (since only these admit a connection). Our proof that the sequence is split still holds in the complete case. That contrasts with the known fact (see [M]) that $\text{Pic} \neq X$ is the universal extension of the Jacobian $\text{Pic}^0 X$ by a vector group, thus in some sense as far as possible from being split. The explanation is that this last statement considers (1.2) as an extension of complex algebraic groups, and the distinguished splitting described below is a splitting only of real Lie groups. In the affine case, there is no natural algebraic structure on the groups in (1.2); indeed, $\text{Pic} X$ is typically a quotient of a torus by a countably infinite subgroup, so the only possibility seems to be to regard it just as a huge abstract group. However, if $X$ is obtained from a complete curve $\Sigma$ by removing just one point, then $\text{Pic}^0 X$ is canonically identified with $\text{Pic}^0 \Sigma$, so we are in an awkward intermediate situation.

The paper is organized as follows. In section 2 I review the main technical device used in the proof of Proposition 1.2, namely the description of $\text{Pic}^0 X$ in terms of differentials of the third kind on $X$ (see [CH1]). I give a self-contained account of this which is less sophisticated than the one in [M]; it uses arguments that will be familiar to readers of [CH2]. Section 3 gives two constructions of splittings of the sequence (1.2). The first is purely algebraic, but involves an arbitrary choice of basis for an infinite-dimensional vector space. The second construction gives the distinguished splitting mentioned above; however, it involves analytic considerations. From an algebraic point of view there seems to be no natural splitting of the sequence (1.1), which is no doubt why none was found in [CH2]. In section 4 I explain very briefly the claim above that for a complete curve our distinguished splitting is a splitting of real Lie groups; and section 5 gives the small changes needed to treat the case of a general affine curve (possibly with automorphisms and units). Finally, in section 6 we sketch the proof of the basic Theorem 1.3.
differentials with integer residues\(^1\) at each pole. There is an obvious map
\[
\text{res} : \Omega^3_\mathbb{Z}(X) \to \text{Div} X
\]
which assigns to a differential \(\omega\) the divisor \(\sum_{x \in X} (\text{res}_x \omega) x\): its kernel is \(\Omega^1(X)\).
We have also the map \(\text{dlog} : \mathbb{K}^* \to \Omega^3_\mathbb{Z}(X)\): we shall identify its image with \(P\), so that the map \((2.1)\) restricts to the identity on \(P\).

**Proposition 2.1.** The quotient \(\Omega^3_\mathbb{Z}(X)/P\) can be canonically identified with \(\text{Pic}^0 X\).

The identification is such that the diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1(X) & \longrightarrow & \Omega^3_\mathbb{Z}(X) & \longrightarrow & \text{Div} X & \longrightarrow & 0 \\
\text{Id} & & \downarrow & & \text{res} & & \downarrow & & \text{Id} \\
0 & \longrightarrow & \Omega^1(X) & \longrightarrow & \text{Pic}^0 X & \longrightarrow & \text{Pic} X & \longrightarrow & 0
\end{array}
\]
commutes. Thus we can use the top sequence in this diagram to study the bottom one.

The rest of this section is devoted to explaining Proposition 2.1; we shall concentrate on the case where \(X\) is affine. Let us first review the notion of a (necessarily flat) connection on a line bundle \(L\): roughly speaking it is a way of making \(D(X)\) act on (sections of) \(L\). To be precise, let \(D_L\) be the ring of differential operators on \(L\): it contains \(O(X)\) as the subalgebra of operators of degree 0, and we may define a connection on \(L\) to be an isomorphism \(\varphi : D(X) \to D_L\) such that the restriction of \(\varphi\) to \(O(X)\) is the identity. Let us look at the special case where \(L\) is the trivial bundle \(X \times \mathbb{C}\), so that \(D_L \equiv D(X)\). Then a connection is just an automorphism of \(D(X)\) which fixes \(O(X)\). Since \(D(X)\) is generated by \(O(X)\) and its derivations, such an automorphism \(\varphi\) is determined by its action on derivations \(\partial\); this action necessarily takes the form
\[
\varphi(\partial) = \partial + \langle \omega, \partial \rangle
\]
where \(\omega \in \Omega^1(X)\) and \(\langle \cdot, \cdot \rangle\) is the natural pairing between 1-forms and vector fields. In this way, connections on the trivial bundle are in 1-1 correspondence with regular 1-forms.

For a general line bundle \(L\), a connection is usually described in terms of locally defined 1-forms as above, using local trivializations of \(L\); however, in our algebraic situation, we can use the fact that \(L\) always has a rational trivialization to describe a connection by a single rational differential, much as above. More precisely, let us fix a divisor \(D = \sum n_x x\) in the class of \(L\). Corresponding to \(D\) we have the fractional ideal of \(O(X)\)
\[
I_D := \{ f \in \mathbb{K} : \nu_x(f) \geq -n_x \ \forall x \in X\}
\]
(as usual \(\nu_x(f)\) is the order to which \(f\) vanishes (or minus the order of pole) at \(x\)). Then \(I_D\) is isomorphic to the \(O(X)\)-module \(\Gamma_L\) of sections of \(L\); indeed, choosing a divisor in the class of \(L\) is equivalent to choosing a (fractional) ideal \(I \subset \mathbb{K}\) isomorphic to \(\Gamma_L\). We may now identify \(D_L\) with the algebra
\[
D(I_D) := \{ \theta \in D(\mathbb{K}) : \theta . I_D \subseteq I_D\}.
\]

\(^1\)Some authors include this in the definition of “third kind”; others call any rational differential “of the third kind”.
So a connection on $L$ is an isomorphism $\varphi : D(X) \to D(I_D)$. This extends uniquely to an automorphism $\varphi$ of $D(\mathbb{K})$ (restricting to the identity on $\mathbb{K}$), which must have the form \((2.3)\), with $\omega \in \Omega_{\text{rat}}(X)$ now a rational differential. If we change the choice of ideal $I$ by a factor $f \in \mathbb{K}^\times$, the corresponding differential changes by the gauge transformation $\omega \mapsto \omega + \text{dlog}f$. Thus so far we have seen that $\text{Pic}^0_X$ embeds into the space $\Omega_{\text{rat}}(X)/\text{dlog} \mathbb{K}^\times$. To complete the explanation of Proposition \ref{prop:2.1} we have only to see what is the image of this embedding; that is, which rational differentials give rise to automorphisms of $\mathbb{K}$ that map $D(X)$ onto $D(I_D)$.

**Proposition 2.2.** Let $\omega$ be a rational differential, $\varphi$ the corresponding automorphism of $D(\mathbb{K})$. Then $\varphi$ maps $D(X)$ onto $D(I_D)$ if and only if (i) $\omega \in \Omega_{\text{rat}}^2(X)$ and (ii) $\text{res} \omega = D$.

**Proof.** Note first that if $\omega_1$ and $\omega_2$ are two differentials such that the corresponding automorphisms $\varphi_1$ and $\varphi_2$ both map $D(X)$ onto $D(I_D)$, then $\varphi_2^{-1} \varphi_1$ restricts to an automorphism of $D(X)$: it follows that $\omega_1 - \omega_2$ is a regular differential on $X$. Thus it is enough to find just one automorphism which maps $D(X)$ onto $D(I_D)$, and such that the corresponding differential $\omega$ has the principal parts specified by the properties (i) and (ii) in Proposition \ref{prop:2.2}. For this, let $I_D$ be the fractional ideal inverse to $I_D$: it corresponds to the divisor $-D$. Since $I_D I_D^* = \mathcal{O}(X)$, we can choose $\alpha_i \in I_D$ and $\beta_i \in I_D^*$ such that $\sum \alpha_i \beta_i = 1$. We claim that the differential $\omega := \sum \alpha_i d\beta_i$ has the required properties. Indeed, the automorphism $\varphi$ corresponding to $\omega$ acts on derivations $\partial$ of $\mathbb{K}$ by

$$\varphi(\partial) = \partial + \sum \alpha_i \langle d\beta_i, \partial \rangle$$

$$= \partial + \sum \alpha_i (d\beta_i, \partial)$$

$$= \partial + \sum \alpha_i \partial(\beta_i)$$

$$= \sum \alpha_i \partial \beta_i ,$$

where the last step used that $\sum \alpha_i \beta_i = 1$. If now $\partial$ is a derivation of $\mathcal{O}(X)$, then

$$(\alpha_i \partial \beta_i).I_D \subseteq (\alpha_i \partial).\mathcal{O}(X) \subseteq \alpha_i \mathcal{O}(X) \subseteq I_D ;$$

that is, $\varphi(\partial) \in D(I_D)$, whence $\varphi$ maps $D(X)$ to $D(I_D)$. Similarly, $\varphi^{-1}$ maps $D(I_D)$ to $D(X)$, so $\varphi$ does indeed give an isomorphism between these two rings. It remains to check that $\omega$ has the properties (i) and (ii). Fix a point $x$ in the support of $D$, and let $z$ be a local parameter near $x$. Then near $x$ the $\alpha_i$ and $\beta_i$ have the form

$$\alpha_i = a_i z^{-n_x} + \ldots, \quad \beta_i = b_i z^{n_x} + \ldots ,$$

where $a_i$ and $b_i$ are constants and the $\ldots$ denote higher order terms. Thus $d\beta_i = n_x b_i z^{n_x-1}dz + \ldots$, so

$$\omega := \sum \alpha_i d\beta_i = (\sum \alpha_i b_i) n_x z^{-1}dz + \ldots .$$

But since $\sum \alpha_i \beta_i = 1$, we have $\sum a_i b_i = 1$; it follows that $\omega$ has a simple pole at $x$ with residue $n_x$. That finishes the proof of Proposition \ref{prop:2.2} \qed
3. Splitting

Let us return to the diagram (2.2). The bottom map $p$, which we want to show is split, is obtained from the top map $\text{res}$ by dividing out by the (common) subgroup $P$ of principal divisors. Thus it is enough to construct a splitting $s : \text{Div} X \to \Omega^2_3(X)$ of the top map which extends the identity map on $P$, for this will then descend to the quotient to give a splitting of the bottom map.

That is almost trivial, but not quite, since $P$ is not a direct factor in $\text{Div} X$ (the quotient $\text{Pic} X$ has elements of finite order, while $\text{Div} X$ is a free abelian group). However, we can consider the map of larger groups $\text{res} : \Omega^3(X) \to (\text{Div} X) \otimes \mathbb{Z}$. These are now vector spaces, so we can certainly choose a $\mathbb{C}$-linear splitting $s$ with $s(p \otimes 1) = p$ for $p \in P$. The very fact that $s$ is a splitting implies that it maps $\text{Div} X \otimes 1$ into $\Omega^2_3(X)$, so we are finished.

It is more satisfactory to describe a “natural” splitting of our sequence. Let us consider first the case of a complete curve $X$: in that case we interpret $\text{Div} X$ to be group of divisors of degree zero on $X$. To define $s$, for each such divisor $D$ we have to choose a differential with principal parts as prescribed by (i) and (ii) in Proposition 2.2. There are several ways to normalize a differential with prescribed principal parts: the one we need is to make all its periods pure imaginary. It is clear that the resulting map $s$ is additive; further, if $D$ is the divisor of a rational function $f$, then we have $s(D) = d\log(f)$; that is, $s$ extends the identity map on the group $P$ of principal divisors, so again we are finished.

If $X$ is obtained from a complete curve $\Sigma$ by removing a single point, the situation is equally good. Indeed, if $D \in \text{Div} X$, then $D$ has a unique extension to a divisor $\overline{D}$ of degree zero on $\Sigma$; if we take the above normalized differential $s(\overline{D})$ and restrict it to $X$, we again get a distinguished splitting of the sequence (1.2).

To extend this construction to the general case, when $X$ is obtained by removing several points from $\Sigma$, we need to choose some way of extending divisors on $X$ to divisors of degree zero on $\Sigma$. For example, we could single out one of the points “at infinity” and let the others have multiplicity 0 in the extension; or, more democratically, we could let the extended divisor have the same multiplicity at each of the points at infinity. I leave the choice to the reader.

4. The complete case

Let us return to the case where $X$ is complete, and explain the claim that in that case we have a splitting of (finite dimensional) real Lie groups. We take an analytic point of view. Recall that (as for any complex manifold) there is a canonical identification $\text{Pic}^0 X \simeq H^1(X, \mathbb{C}^\times)$. From this point of view (1.2) comes from the cohomology sequence of the exact sequence of analytic sheaves

$$0 \to \mathbb{C}^\times \to \mathcal{O}^\times \xrightarrow{d\log} \Omega^1 \to 0.$$ 

These sheaves could be interpreted algebraically, but then the map $d\log$ would not be surjective.
The map \( \pi : \Omega^2 \to \Pic X \) also has a very simple description from this point of view: if we identify

\[
\Pic^\flat X \simeq H^1(X, \mathbb{C}^\times) \simeq \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^\times),
\]
then \( \pi \) sends a differential \( \omega \) to the homomorphism

\[
\pi(\omega) : [c] \mapsto \exp \int_c \omega,
\]
where \( [c] \) is the homology class of a 1-cycle \( c \). Clearly, \( \omega \) is normalized (to have imaginary periods) if and only if \( \pi(\omega) \) takes values in the unit circle \( \mathbb{S}^1 \subset \mathbb{C} \).

Now, as a real Lie group we have the polar decomposition \( \mathbb{C} \times \simeq \mathbb{R} \times \mathbb{S}^1 \), in which a pair \( (\lambda, e^{i\theta}) \in \mathbb{R} \times \mathbb{S}^1 \) corresponds to the complex number \( e^{\lambda+i\theta} \). Our decomposition of \( \Pic^\flat X \) is just the product of \( 2g \) copies of this one.

5. The general affine case

We have assumed so far that \( \mathcal{O}(X) \) has no nontrivial units, and that \( X \) has no nontrivial automorphisms; however, neither of these assumptions is essential. In the general case, let \( U \) be the group of units in \( \mathcal{O}(X) \). If \( u \in U \), then \( \text{dlog} \ u \in \Omega^1(X) \); set

\[
\overline{\Omega} := \Omega^1(X)/\text{dlog} \ U.
\]

Recall that \( \Pic^\flat X \) is the group of isomorphism classes (that is, gauge equivalence classes) of lines bundles with connection. Each \( u \in U \) gives an isomorphism of connections on the trivial bundle, changing the corresponding differential \( \omega \) by the gauge transformation \( \omega \mapsto \omega + \text{dlog} \ u \); thus the space of isomorphism classes of connections on the trivial bundle is \( \overline{\Omega} \), so in the exact sequence (1.2) we have to replace \( \Omega^1(X) \) by \( \overline{\Omega} \). Similarly, in the diagram (2.2), we have to change \( \Omega^2(X) \) to \( \overline{\Omega} \), and also \( \Omega^2_\mathbb{Z}(X) \) has to be replaced by the quotient \( \Omega^2_\mathbb{Z}(X)/\text{dlog} \ U \). The main part of our discussion is then unaffected by the presence of \( U \).

The automorphism group \( \text{Aut} X \) equally easy to deal with, but more interesting, since it gives us extra elements of \( \Pic D \). This group acts compatibly on all the groups in the split exact sequence (1.2), so we get a split exact sequence

\[
0 \to \overline{\Omega} \to \Pic^\flat X \times \text{Aut} X \to \Pic X \times \text{Aut} X \to 0.
\]

Further, there is a natural inclusion

\[
\chi : \Pic^\flat X \times \text{Aut} X \to \Pic D
\]

generalizing (1.3): Cannings and Holland show\(^3\) that it is an isomorphism. Inserting this isomorphism into (5.1), we get the exact sequence of \( \text{[CH2, Theorem 1.15]}. \)

Remark. It follows from Theorem (1.3) that if the group \( \text{Aut} X \) is trivial, then \( \Pic D \) is abelian. On the other hand, a nontrivial automorphism of \( X \) cannot act trivially on \( \overline{\Omega} \) (because it does not act trivially on the subgroup \( \text{dO}(X) \)). So from the exact sequence (5.1) we get the following curious fact: \( \Pic D \) is abelian if and only if \( \text{Aut} X \) is trivial.

\(^3\)At this point we have to exclude the case where \( X \) is isomorphic to the affine line.
6. Outline of proof of Theorem 1.3

We first translate Theorem 1.3 into the language of bimodules used in [CH2]. Recall that any autoequivalence \( \mathcal{T} \) of the category of left \( \mathcal{D} \)-modules is given by tensoring with the invertible \( \mathcal{D} \)-bimodule \( T(\mathcal{D}) \) (that is the case for any algebra; see, for example [B], p. 60 et seq.). Given a line bundle \( \mathcal{L} \) with connection, choose an ideal \( I \subseteq \mathcal{O} \) isomorphic to \( \Gamma_\mathcal{L} \); then as in section 2 the connection can be regarded as an isomorphism \( \varphi: \mathcal{D} \rightarrow \mathcal{D}(I) \), and the corresponding bimodule is \( I \otimes_\mathcal{O} \mathcal{D} = I\mathcal{D} \) with the obvious right \( \mathcal{D} \)-module structure and the left \( \mathcal{D} \)-module structure defined via \( \varphi \). Note that the algebra \( \mathcal{D}(I) = I\mathcal{D}^{\ast} \) can be identified with the endomorphism ring of the right ideal \( I\mathcal{D} \subseteq \mathcal{D} \). Now (as for any Noetherian domain \( \mathcal{D} \)) if we are given an invertible \( \mathcal{D} \)-bimodule; we may consider it first just as a right \( \mathcal{D} \)-module; it can then be embedded as a right ideal \( M \) in \( \mathcal{D} \), and the structure of left \( \mathcal{D} \)-module is given by some isomorphism \( \varphi: \mathcal{D} \rightarrow \text{End}_\mathcal{D} M \). So Theorem 1.3 amounts to the claim that (in our case) we can always choose \( M \) to be of the form \( I\mathcal{D} \), and furthermore that the isomorphism \( \varphi \) then restricts to the identity map on \( \mathcal{O} \). It is in this form that the theorem is proved in [CH2].

In broad outline, the proof goes as follows. By [St], Lemma 4.2, we may assume that \( M \) is fat, that is, \( M \cap \mathcal{O} \neq 0 \). The main result of [CH1] is that the assignment \( M \mapsto V := M \cap \mathcal{O} \) defines a bijection between the fat right ideals in \( \mathcal{D} \) and certain subspaces \( V \subseteq \mathcal{O} \) (called “primary decomposable”); and furthermore that \( \text{End}_\mathcal{D} M \) then gets identified with the algebra \( \mathcal{D}(V) := \{ D \in \mathcal{D}(\mathbb{K}) : D.V \subseteq V \} \). Thus the map defining the left \( \mathcal{D} \)-module structure on \( M \) can be regarded as an isomorphism \( \varphi: \mathcal{D} \rightarrow \mathcal{D}(V) \). Concerning \( V \), we need only know that the subalgebra \( \mathcal{D}_0(V) := \mathcal{D}(V) \cap \mathbb{K} \) is contained in \( \mathcal{O} \), and that the inclusion \( \mathcal{D}_0(V) \subseteq \mathcal{O} \) is a birational isomorphism. Now we use the main result of [ML], which states that \( \mathcal{O} \) is the unique maximal abelian ad-nilpotent (mad) subalgebra of \( \mathcal{D} \). Since \( \mathcal{D}_0(V) \) is a mad subalgebra of \( \mathcal{D}(V) \), that implies that \( \varphi \) must map \( \mathcal{O} \) isomorphically onto \( \mathcal{D}_0(V) \), so that we have a birational isomorphism \( \mathcal{O} \rightarrow \mathcal{D}_0(V) \subseteq \mathcal{O} \). Because \( X \) is smooth, this must be a genuine isomorphism, so under our assumption that \( \text{Aut}_X \) is trivial, it must be the identity. Thus \( \mathcal{D}_0(V) = \mathcal{O} \), and \( V \) is an ideal \( I \) of \( \mathcal{O} \). Under the bijection mentioned above, the fat right ideal of \( \mathcal{D} \) corresponding to \( I \) is \( I\mathcal{D} \). That completes the proof.

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