Topological Symmetries of Twisted N=2 Chiral Supergravity in Ashtekar Formalism

P. L. Paul

3208 7th St. East
Saskatoon, Saskatchewan
S7H 1B3

Abstract In this paper a topological theory of gravity is studied on a four-manifold using the formalism of Capovilla et al. We show that it is fact equivalent to Anselmi and Fre’s topological gravity using the topological symmetries. Using this formalism gives us a new way to study topological gravity and the intersection theory of gravitational instantons if the (3+1) decomposition with respect to local coordinates is performed.
1 Introduction

In recent times, Topological Field theory \[1\] has become a very popular way to study or compute topological invariances of oriented differentiable manifolds. These topological invariances are computed by counting the intersection of submanifolds of moduli space of Instantons having zero modes. More recently Witten found that the same invariances can be obtained by counting the solutions to a non-linear equation. But now the gauge group is abelian \[18\]. The first such theory was constructed in the action-packed paper of Witten \[17\]. It was constructed because of the need for a quantum field theoretic explanation of topological properties of some manifolds. There have been many types of TFT that are constructed much of the work being in two dimensions. The topological Yang-Mills theory was constructed from twisted \( N = 2 \) Supersymmetric Yang-Mills theory having observables that are the Donaldson’s Polynomials.

It has been shown in \[2\] that twisted \( N = 2 \) Supergravity leads to a topological theory of gravity(TG). Topological gravity is just the theory given in \[8\] when the algebra is Poincaré. This paper aims at another approach in this direction. I will keep close contact with their paper.

Another formulation of TG has been constructed, as a reformulation of Witten’s original form \[3, 4, 5, 7\], namely Topological Conformal Gravity. This Theory uses the action which is a analogue of TYMT, where one replaces Yang-Mills curvature by Weyl conformal curvature tensor. In \[8\], the use of Universal bundles was used to derive the action and study the curvatures. This was done, for Yang-Mills theory, in \[8\] and also using Batalin-Vilkovisky algorithm in \[9\]. With the connection between Ashtekar formulation and Yang-Mills theory, we can hope to use the results in TYMT to extend it to TG in Ashtekar variables.

At around the same time that these TQFT were being studied, new variables were found by Ashtekar for General Relativity \[25, 24\]. It has been used to study quantum gravity and has lead to new results. These variables simplified the constraints of (3+1) complex GR and have been shown to also work with supergravity for \( N = 1 \) \[11\] and \( N = 2 \) \[12\]. Furthermore, since topological gravity is exactly solvable and Ashtekar variables simply the equations it seems to be an appropriate application.

The problem that I would be studying in this paper is the application of these variables
to the study of a topological theory of gravity in *four dimensions*. The paper is ordered in the following way. First we look at the analysis of a two-form in general relativity. In the next section we look at the extension to supergravity. Finally we look at the topological twist of \( N = 2 \) supergravity.

## 2 Chiral Form of General Relativity

In Ashtekar formulation of general relativity, one considers complex quantities, such as the action. Being complex does not really change the equations of motion. The indices \( a, b \ldots \) are tangent space indices (i.e., \( SO(4) \) internal indices). We will use the two-spinor notation of Penrose and Rindler [28].

\( M \) is an oriented four-manifold and vanishing second Steifel-Witney class to admit spin structure. The structure group of the theory is \( SO(4)_C \). By analogy with real groups, and using the local isomorphism \( SU(2)^C \approx SL(2, C) \) and \( SO(4) \approx SU(2)_L \times SU(2)_R \) we state that \( SU(2)^C_L \times SU(2)^C_R \approx SO(4)_C \) [23]. These are the left and right handed part of Lorentz group. The fields are now sections of a complexified vector bundle. Many of what can be said with real general relativity can be restated in complex language.

In four dimensions [30, 27], the \(*\)-operator maps 2-forms to 2-forms,

\[
* : \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M
\]

(2.1)

which depends on the metric and has signature \((++++)\) and \(*^2 = +I\). This causes the bundle, \( \Lambda^2 T^* M \), to split into self-dual and anti-self-dual part,

\[
\Lambda^2 T^* M = \Lambda^2_{+} T^* M \oplus \Lambda^2_{-} T^* M.
\]

(2.2)

Therefore any two-form can be written as sum of its self-dual and anti-self-dual part. Corresponding elements are defined by the application of the projection operator, \( P^\pm = \frac{1}{2}(1 \mp *) \), to any element of the bundle. For example, the projection operator acting on the 2-form made out of tetrads, \( e^a \wedge e^b \) and \((su(2)_L \times su(2)_R \rightarrow SO(4))\) give us

\[
P^+(e^a \wedge e^b) = \frac{1}{2}(e^a \wedge e^b - *e^a \wedge e^b)
\]

\[
= \frac{1}{2}(e^a \wedge e^b - e^{ab} e^c \wedge e^d).
\]

(2.3)

notice that this is a complex quantity therefore reality conditions are required.
We are on a manifold that has a metric of euclidean signature therefore the projections created from projection operator onto the self-dual and anti-self-dual parts has a complex form. The tetrad $e^{\alpha\beta},(\alpha,\beta...$ are the spinor indices)is the dynamic variable rather than the metric of traditional gravity. In four dimensional general relativity the action can be taken to be

$$S[e^{\alpha\alpha'}, \omega_{\alpha\beta}] = \int_M e^{\alpha\alpha'} \wedge e^{\beta} \wedge R_{\alpha\beta}$$

(2.4)

and $R_{\alpha\beta}$, the curvature two-form which has the definition,

$$R_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_{\gamma\alpha} \wedge \omega_{\gamma\beta}. \quad (2.5)$$

and $\omega_{\alpha\beta}$ is $SU(2)$ connection (the self-dual part of $SO(4)$). The definition of Ashtekar self-dual $SU(2)$ spin connection is

$$\omega^{\alpha\beta} = \frac{1}{2}(\omega^{ab} - \ast \omega^{ab}) = \frac{1}{2}(\omega^{ab} - \varepsilon^{ab}_{\cd\cd} \omega^{\cd\cd}). \quad (2.6)$$

This notion of self-duality is with respect to internal indices.

When self-duality is in both space-time and internal indices the Ashtekar connection satisfies the self-dual Yang-Mills equations that come from the variations of the action. Therefore general relativity can be considered equivalent to self-dual Yang-Mills theory [24].

From the connection we can see that the curvature becomes

$$R^{\alpha\beta} = \frac{1}{2}(R^{ab} - \ast R^{ab}). \quad (2.7)$$

which are self-dual part of the curvature.

The tetrad can be seen to have the following combination,

$$\Sigma^{\alpha\beta} := e^{\alpha\beta'} \wedge e^{\beta'} \quad (2.8)$$

which in space-time of euclidean signature is hermitian. It can be written as an expansion in Pauli matrices, $\sigma^{\alpha\beta}$,

$$\Sigma^{\alpha\beta} = \Sigma^k \sigma^{\alpha\beta}_k \quad (2.9)$$

where $\Sigma^k$ is the three components as defined below. Equations 2.8 can be used as a variable [20], but in order for it to have such status there must be a constraint. This
constraint, which is \( \Sigma^{(\alpha \beta} \wedge \Sigma^{\gamma \delta)} \), be put into the action with a Lagrange multiplier \( \Psi_{\alpha \beta \gamma \delta} \). This makes the definition (2.8) possible. Now the action can be written down as

\[
S[\Sigma^{\alpha \beta}, \omega_{\alpha \beta}, \Psi_{\alpha \beta \gamma \delta}] = \int \Sigma^{\alpha \beta} \wedge R_{\alpha \beta} - \frac{1}{2} \Psi_{\alpha \beta \gamma \delta} \Sigma^{\alpha \beta} \wedge \Sigma^{\gamma \delta},
\]

which is chiral, that is, no primes appear. The field equations derived from this action by variation with respect to the fields are

\[
\begin{align*}
\Sigma^{(\alpha \beta} \wedge \Sigma^{\gamma \delta)} &= 0, \\
D \Sigma^{\alpha \beta} &= 0, \\
R_{\alpha \beta} &= \Psi_{\alpha \beta \gamma \delta} \Sigma^{\gamma \delta}.
\end{align*}
\]

As shown in [22], \( \Psi_{\alpha \beta \gamma \delta} \) is the spinor part of the Weyl Curvature so the curvature \( R_{\alpha \beta} \) is pure Weyl. Therefore the Weyl curvature is self-dual. These equations can be seen to be gravitational instantons equations [20, 22]. The three \( \Sigma^{\alpha \beta} \) satisfy the relation of quaterionic algebra when put in the form \{ \( 2i\Sigma^{01}, (\Sigma^{00} + \Sigma^{11}), i(\Sigma^{00} - \Sigma^{11}) \) \}. They are the three Kähler forms of a Hyper-Kähler manifold [20], and integrability condition, (ie \( d\Sigma^{\alpha \beta} = 0 \) \). This makes the manifold hyper-Kähler(if the term with \( \Psi_{\alpha \beta \gamma \delta} \Sigma^{\gamma \delta} \) vanishes it is conformally (anti) self-dual) and \( \Sigma^{\alpha \beta} \) is a integrable complex structure on \( M \). As a result, the Riemann curvature tensor is self-dual(in the space-time indices) and the Ricci tensor if flat. Any metric that has its Riemann tensor satisfying

\[
R_{abcd} = \frac{1}{2} \varepsilon_{ablm} R_{lm}^{cd}
\]

is called half-flat. Using the Bianchi identity this implies

\[
R_{ab} = 0
\]

therefore gravitational instantons satisfy the vacuum Einstein equations. Furthermore, \( R_{ab} = \frac{1}{2} \varepsilon_{abcd} R^{cd} \) if \( P^+ \omega_{ab} = \omega_{ab} - \frac{1}{2} \varepsilon_{abcd} \omega^{cd} = 0 \) the spin connection is self-dual.

These gravitational instantons need not obey the self-dual Weyl tensor [24].

With a Cosmological constant, \( \Lambda \), we get a term in the curvature field equation,

\[
R_{\alpha \beta} = \Psi_{\alpha \beta \gamma \delta} \Sigma^{\gamma \delta} - \frac{1}{6} \Lambda \Sigma_{\alpha \beta}
\]
The study of TFT of gravitational instantons using Ashtekar variables was undertaken by Torre [10]. For \( R_{\alpha\beta} = -\frac{1}{6} \Lambda \Sigma_{\alpha\beta} \) and \( \Lambda > 0 \) it was shown that the moduli space of gravitational instantons is discrete, this is the same as having its dimension to be zero.

This form was used to study some aspects of Topological Gravity when the term with \( \Psi_{\alpha\beta\gamma\delta} \Sigma_{\gamma\delta} \) vanish making the manifold anti-self-dual and the cosmological constant \( \Lambda \neq 0 \).

\[
R_{\alpha\beta} = -\frac{1}{6} \Lambda \Sigma_{\alpha\beta} \tag{2.17}
\]

It can be seen to be a TYMT. This gives us

\[
S[\omega_{\alpha\beta}] = -\frac{6}{\Lambda} \int R^{\alpha\beta} \wedge R_{\alpha\beta} \tag{2.18}
\]

this is the "topological charge" and can be gauge-fixed to give Witten’s Action for TYMT.

And the fermionic symmetry is just those of [17] and Diffeomorphism i.e. the gauge group is the semi-direct product of local \( su(2) \) and diffeomorphism. Combining (2.11) and (2.17) gives us five quadratic conditions that the curvature satisfies and replacing the connection by \( \omega + C \) gives us a perturbation that gave the results in [10]. We will not go any further in this direction.

### 3 \( N = 2 \) Chiral Supergravity in Ashtekar variables

In this section we will recap some results on the Ashtekar formulation of \( N = 2 \) Supergravity. This theory unifies gravitation with electromagnetism, and it also contains interaction with spin-\( \frac{3}{2} \) particles. See [31, 19] for notation and useful formulas. We will follow closely the paper by Kunitomo and Sano [12]. Let \( \mathbf{M} \) be a compact oriented manifold.

In the complex formulation, the groups \( SU(2) \) are complexified. The \( N = 2 \) Supergravity theory has two gravitinos (the superpartner of the graviton) forming a \( SU(2) \) doublet and we denote it’s indices by \( i, j = 1, 2 \). The left(right) component of the gravitino will be denoted by \( \psi^i_\alpha (\psi^i_{\alpha'}) \). The indices \( \alpha, \beta ...(\alpha', \beta', ...) = 1, 2 \) are in the representation of \( SU(2)_{L(R)} \). These are complex extensions of their real counterparts. We need to put the gravitino \( \psi^i_{\alpha'} \) in the chiral form, which simply means redefining a field that has only unprimed indices. We define a 2-form field as

\[
\chi^\alpha_i = e^\alpha_{\alpha'} \wedge \psi^\alpha_i. \tag{3.1}
\]
This theory possesses a $U(1)$ symmetry giving us gauge field, $A = A_\mu dx^\mu$, hence the Maxwell equation is included.

The chiral action for $N = 2$ supergravity is

$$S = \int_M \Sigma^{\alpha\beta} \wedge R_{\alpha\beta} - \frac{1}{2} \Psi_{\alpha\beta\gamma\delta} \Sigma^{\alpha\beta} \wedge \Sigma^{\gamma\delta} + \chi_i^\alpha \wedge D\psi_i^i - \kappa_i^i \Sigma^{\alpha\beta} \wedge \chi_i^\gamma$$

$$+ \phi_{\alpha\beta} F' \wedge \Sigma^{\alpha\beta} - \frac{1}{2} \phi_{\alpha\beta} \phi_{\gamma\delta} \Sigma^{\alpha\beta} \wedge \Sigma^{\gamma\delta} + \frac{1}{2} \phi_{\alpha\beta} \chi_i^\alpha \wedge \chi_i^{i\beta}$$

$$+ \frac{1}{2} F' \wedge \psi_i^\alpha \wedge \psi_i^i + \frac{1}{8} \psi_i^\alpha \wedge \psi_i^i \wedge \psi_i^\beta \wedge \psi_i^j,$$

(3.2)

and the auxiliary field $\phi_{\alpha\beta}$ is added to give a chiral action. $F'$ is the field strength,

$$F' = dA - \frac{1}{2} \psi_i^i \wedge \psi_i^\alpha.$$  

(3.3)

This equation defines the supercovariant derivative.

Variations with respect to it's fields give us the equations of motion,

$$R_{\alpha\beta} = \Psi_{\alpha\beta\gamma\delta} \Sigma^{\gamma\delta} + \kappa_i^i \Sigma^{\alpha\beta} \wedge \chi_i^\gamma - \phi_{\alpha\beta} F' + \phi_{\alpha\beta} \phi_{\gamma\delta} \Sigma^{\gamma\delta}$$

$$D\Sigma^{\alpha\beta} = \chi_i^{\alpha \wedge \psi_i^{\beta\gamma}}$$

$$D\psi_i^i = \kappa_i^i \Sigma^{\beta\gamma} - \phi_{\alpha\beta} \chi_i^{i\beta}$$

$$D\chi_i^\alpha = -dA \wedge \psi_i^\alpha$$  

(3.4)

and the constraints are, when variation is with respect to lagrange multipliers $\Psi$, $\kappa$, and $\phi$,

$$\Sigma^{\alpha\beta} \wedge \Sigma^{\gamma\delta} = 0$$

$$\Sigma^{\alpha\beta} \wedge \chi_i^\gamma = 0$$

$$F' \wedge \Sigma^{\alpha\beta} = \phi_{\gamma\delta} \Sigma^{\alpha\beta} \wedge \Sigma^{\gamma\delta} - \frac{1}{2} \chi_i^\alpha \wedge \chi_i^{i\beta}$$  

(3.5)

The formulæwill be useful later. A comment on the last equation, $F'$ can be expanded in $\Sigma$'s and arrive at the condition that $\phi$ is self-dual part of $U(1)$ curvature $F$.

The supersymmetries of the theory are, after combining the left and right supersymmetries,

$$\delta \Sigma^{\alpha\beta} = -\chi_i^{\alpha \eta^\beta \gamma} + \psi_i^{\alpha \wedge \tau^\beta \gamma}$$

$$\delta \psi_i^i = D\eta_i^i - \phi_{\alpha\beta} \tau^i_j$$
\[
\delta \chi_i^\alpha = F_i^\alpha - \phi_{\gamma \delta} \Sigma_i^{\gamma \delta} \eta_i^\alpha + \mathcal{D} \tau_i^\alpha - \psi_i^\alpha \wedge \xi
\]
\[
\delta A = -\psi_i^\alpha \eta_i^\alpha + \xi
\]
\[
\delta \omega_{\alpha \beta} = \kappa_{\alpha \beta \gamma} \tau_i^\gamma - \phi_{\alpha \beta} \xi
\]
\[
\delta \Psi_{\alpha \beta \gamma \delta} = 0
\]
\[
\delta \kappa_i^{\alpha \beta \gamma} = \Psi_{\alpha \beta \gamma \delta} \eta_i^\delta
\]
\[
\delta \phi_{\alpha \beta} = \kappa_i^{\alpha \beta \gamma} \eta_i^\gamma.
\]

Where both the left and right supersymmetries, \(\eta_i^\beta, \eta_{i'}^\beta\), respectively, have been combined and \(C_{\alpha \beta}\) denote the antisymmetric tensor (indices also for \(i, j\) and \(\alpha', \beta'\)). \(\mathcal{D}\) is local Lorentz covariant derivative [29]. Following [12], we have the definitions

\[
\tau_i^\alpha \sim e_{\alpha'}^{\alpha} \eta_i^\alpha',\tag{3.7}
\]
\[
\chi_i^\alpha = e_{\alpha'}^{\alpha} \wedge \psi_i^\alpha',\tag{3.8}
\]
and
\[
\xi \sim \psi_i^\alpha \eta_i^{\alpha'},\tag{3.9}
\]

where \(\tau_i^\alpha\) is a fermionic spinor 1-form, and \(\xi\) is a bosonic 1-form and the proportionality constant can be taken to be \(\frac{1}{2}\) to make the results match. Both defined by conditions,

\[
\Sigma^{(\alpha \beta \wedge \tau_i^\gamma)} = 0\tag{3.10}
\]
and
\[
\Sigma^{\alpha \beta \wedge \xi} = \chi^{i(\alpha \wedge \tau_i^\beta)},\tag{3.11}
\]

respectively. The covariant derivative is \(\mathcal{D}\) and the cosmological constant is zero.

### 4 Topological twist of chiral \(N = 2\) Supergravity

In [8 9], the action for TYMT was derived by BRST gauge fixing of a topological action. It was then shown to be to related to universal bundles.

The procedure of topological twisting \(N = 2\) Super Yang-Mills theory [17 16 15] can be performed in a similar way for \(N = 2\) Supergravity [2], but of course there is a little generalization. The action is a gauge-fixed version of some topological action. The
gauge-fixing condition being the self-dual part of the spin connection vanishing. This twist changes the statistic of fields, from half integer to integer and vice versa.

First, I will explain some of their results using the notation of this paper. The Lorentz group is $SU(2)_L \times SU(2)_R$, where $SU(2)_L$ ($SU(2)_R$) is the Left-handed (Right-handed) part (these groups being local). Also the theory contains internal symmetry group $SU(2)_I$.

Topological twist is performed as follows: replace $SU(2)_R$ by the diagonal subgroup of $SU(2)_L \times SU(2)_I$. The twisted version of $SU(2)_R$ is denoted by $SU(2)'_R$. Therefore the Lorentz Group gets replaced by $SU(2)_L \times SU(2)'_R$, which is equivalent to internal indices the $i, j...$ being replaced by $\alpha', \beta'...$, the right-handed indices. This makes primed indices appear all over the place.

We can set as in [16]

$$\eta^{\alpha \beta'} = 0, \quad \eta^{\alpha' \beta'} = -C^{\alpha' \beta'} \rho$$

(4.1)

$\eta^{\alpha \beta'}$ is the Killing spinor used in [7] and from the theorem in [21] implies that the manifold is non-Kahler if the $\eta^{\alpha \beta'}$ is real but it is a complex quantity in our case. Furthermore, $\eta^{\alpha' \beta'}$ is covariantly constant.

With these formulas at our disposal we perform the twist of the supersymmetries and obtaining the topological symmetries. These topological symmetries be related to the equivariant cohomology on the space of gravitational instantons. We try to get formulas similar to [2] but, of course, we only get half of the results being complex version of theirs. They also defined a shift to get the right results.

First the equations above are redefined as,

$$\tau_{\alpha'}^{\beta'} \sim -\rho e^{\alpha}_{\alpha'} \delta_{\beta'}^{\alpha'}$$

$$\chi_{\alpha'}^{\alpha} = e^{\alpha}_{\alpha'} \land \psi^{\alpha'}_{\beta'}$$

$$\xi \sim -\rho \psi^{\alpha'}_{\beta'} \delta_{\alpha'}^{\beta'}$$

(4.2)

these are twisted version of the corresponding fields defined in the previous section. We can keep the $\xi$ as a constant (an anticommuting parameter) still carrying the proportionality constant. This is the place that fermionic symmetry and is singlet supercharge arise and it is the component $(0, 0)^0_0$ of the twisted supersymmetry generators. This is just translating TYMT [17] into supergravity, in fact, we just generalized the procedure.
supersymmetries now become a BRST-like symmetry, 

\[ \delta \Sigma^{\alpha\beta} = -\frac{1}{2} \rho (\psi^{\beta'}_\beta \wedge e^{\beta'}_\alpha + \psi^{\beta}_\alpha \wedge e^{\alpha'}_\beta) \]

\[ \delta \psi^{\beta'}_\alpha = \rho \phi_{\alpha\beta} e^{\beta'}_\alpha \]

\[ \delta \chi^{\alpha}_{\beta'} = D r^{\alpha}_{\beta'} + \rho \psi^{\alpha}_{\beta'} \wedge (\delta^{\beta'}_\alpha \psi^{\alpha'}_{\beta'}) \]

\[ \delta A = -\rho (\delta^{\beta'}_\alpha \psi^{\alpha'}_{\beta'}) \]

\[ \delta \omega_{\alpha\beta} = \kappa^{\beta'}_{\alpha\beta} \tau^{\delta}_{\beta'} + \rho \phi_{\alpha\beta} (\delta^{\alpha'}_\beta \psi^{\beta'}_{\alpha'}) \]

\[ \delta \Psi_{\alpha\beta\gamma\delta} = 0 \]

\[ \delta \kappa^{\beta'}_{\alpha\beta\gamma} = 0 \]

\[ \delta \phi_{\alpha\beta} = 0 \]

where we can see that \( \delta^{\alpha'}_{\beta'} \psi^{\beta'}_{\alpha'} \) is already used in \([2]\) and they called it \( \tilde{\psi} \). We therefore have another transformation

\[ \delta \tilde{\psi} = 0. \] (4.4)

These symmetries can now be put into the form of BRST in suitable redefinition of fields. Now \( \delta \) is the BRST-like charge \( Q \). The first transformation has the form \([Q, \text{field}] = \text{ghost}\) as in \([17]\).

Now put

\[ \psi^{\alpha\beta} = \frac{1}{2} (\psi^{\beta'}_{\beta} \wedge e^{\beta'}_\alpha + \psi^{\beta}_\alpha \wedge e^{\alpha'}_\beta) \] (4.5)

where \( \psi^{\alpha\beta} \) is a two-form. Also \( \phi_{\alpha\beta} \) is the spinor form of the self-dual part of the Maxwell's curvature, \( F_{\alpha\beta} \).

The term contains the covariant derivative is proportional to the anti-self-dual spin connection \([20]\) minus the gravitinos,

\[ D e^{\alpha}_{\beta'} = \omega_{\beta\gamma} \wedge e^{\alpha'} - \frac{1}{2} \psi_{\beta'} \wedge \psi^{\alpha i}. \] (4.6)

which came from the covariant derivative written in two-spinor form and \( \omega \) equations of motion \( D \Sigma^{\alpha\beta} = \chi^{(\alpha} \wedge \psi^{\beta)} \) expanded in the definition of components. This can be twisted and the gravitinos cancel each other which is the reason for the proportionality constant appearing in the definition of \( \xi \), i.e the term with the singlet ghost \( \tilde{\psi} \) and \( \psi^{\beta'}_{\beta} \) vanish.

The equations of motion

\[ D \psi^{\beta'}_\alpha - \kappa^{\beta'}_{\alpha\beta\gamma} \Sigma^{\beta'} + \phi_{\alpha\beta} \chi^{\beta'} = 0 \] (4.7)
can be used to make the identification that $\kappa_{\alpha\beta'_r}^{\beta'} e^{\tilde{\gamma}}_{\gamma'}$ is the self-dual part of $\chi^{ab}$ that appears in [2] (see equation (4.4)). We write this as $\chi^\alpha{}_{\alpha\beta}$.

The transformations become

$$
\delta \Sigma^{\alpha\beta} = \rho \psi^{\alpha\beta},
\delta \psi^\beta = -\rho \phi^{\alpha\beta} e^{\beta'} e^{\alpha'}
\delta \chi^{\alpha}_{\beta'} = \rho \omega^{\beta'} \wedge e^{\alpha'}
\delta A = -\rho \tilde{\psi}
\delta \omega_{\alpha\beta} = -\rho (\chi'_{\alpha\beta} + \phi_{\alpha\beta} \tilde{\psi})
$$

(4.8)

which can be seen to be just the unshifted BRST symmetry in [2] (see equations (4.3)) with some constants absorbed by the $\rho$’s. By defining $\delta \Phi = \rho Q \Phi$, with $\Phi$ arbitrary field in the multiplet, we have BRST-like operator $Q$. These equations correspond to ref.[2] if one takes the variation as the exterior derivative on some moduli space. For example, the first one imples

$$
\delta \Sigma^{\alpha\beta} = \delta (e^{\alpha\alpha'} \wedge e^{\beta\beta'} C^{\alpha\beta'}_{\alpha'})
= (\delta e^{\alpha\alpha'} C^{\alpha\beta'}_{\alpha'}) \wedge e^{\beta\beta'} + (\delta e^{\beta\beta'} C^{\alpha\beta'}_{\alpha'}) \wedge e^{\alpha\alpha'}
$$

(4.9)

and putting the topological ghost (equation 4.5) into the transformation, we get

$$
Q e^{\alpha}_{\beta'} = \psi^{\alpha}_{\beta'}
$$

(4.10)

It is interesting to see that the twisting procedure gave us the same transformations as Anselmi and Fré.

By comparing the transformations above to Witten’s in [17], $\omega^{\beta'}_{\beta'}$ is the gauge fixing condition and was realized by Anselmi and Fré. They used it in the construction of their action for topological gravity.

The action for pure gravity can be seen to be of the following form

$$
L = R^{\alpha\beta} \wedge \Sigma^{\alpha\beta}
= 2 \omega^{(\alpha} \wedge \Sigma^{\beta)} \wedge \omega_{\alpha\beta} - \omega^{\delta}_{\alpha} \wedge \Sigma^{\alpha\beta} \wedge \omega_{\delta\beta} + d(\Sigma^{\alpha\beta} \wedge \omega_{\alpha\beta})
$$

(4.11)

using the gravitational equations of motion for $\Sigma$. But it is just half the story. This is true but remember that we have taken account the other half if the complex part of
the projection onto the self-dual vector bundle is ignored. Giving us essentially the same theory as Anselmi and Fré. We may as well have rewritten the theory of Anselmi and Fré in terms of spinors.

The path integral can be defined formally [13] as

\[
Z = \int \mathcal{D}\omega \mathcal{D}\Psi \mathcal{D}\Sigma \exp\{-\int \Sigma_{\alpha\beta} \wedge R^{\alpha\beta} + \frac{1}{2} \int \Psi_{\alpha\beta\gamma} \Sigma^{\alpha\beta} \wedge \Sigma^{\delta\gamma} - S_{fg} - S_{FP}\} \delta(\text{tr}\Psi) \\
= \int \mathcal{D}\omega \mathcal{D}\Psi \mathcal{D}\Sigma \exp\{-\int 2\omega_\delta^{(\alpha} \wedge \Sigma^{\beta)\delta} \wedge \omega_{\alpha\beta} - \omega_\delta^{(\alpha} \wedge \Sigma^{\beta)\delta} \wedge \omega_{\alpha\beta} + \frac{1}{2} \int \Psi_{\alpha\beta\gamma} \Sigma^{\alpha\beta} \wedge \Sigma^{\delta\gamma} - S_{fg} - S_{FP}\} \delta(\text{tr}\Psi)
\]

(4.12)

where $S_{fg}$ and $S_{FP}$ are the gauge-fixing and Fadeev-Popov terms. The surface terms are ignored. It remains to be seen if such a quantity gives a topological invariant.

5 Discussion

We have been studying the symmetries of topological gravity from the point of view of self-dual two forms. Which makes the Ashtekar formalism readily available. That is achieved after the (3+1) decomposition of space-time coordinate.

We can show that topological gravity comes by simply twisting the supersymmetries of $N = 2$ supergravity. The self-dual abelian gauge field is coupled to the gravitational instantons making connection with [18] but things have change. The gauge group includes the diffeomorphism group. This may be the technique of Witten in [18] applied to the construction of Kronheimer and Nakajima [32]. We thus find the topological symmetries which need to be fixed in order that we find the partition function. A few remarks about future prospect. First, in the loop representation of the constraints, there is no notion of a metric. In topological field theory, observables are metric independent [11]. The metric is now replaced by the densitized triad and self-dual $SU(2)$ connection. Therefore metric independence must be formulated in another way which is independence of those variables. The area derivative of [14] seems to be very convenient, also the volume can be used but it seems that it will be more difficult. This is due to the fact that surfaces are better understood. Furthermore, in Donaldson’s theory we have Reimann surfaces immersed
in a four-manifold and studying Dirac operators on these surfaces. This will be left for
future work.

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References

[1] Birmingham D, Blau M, Rakowski M and Thompson G, 1991 Phys. Reps. 209 129.

[2] Anselmi D and Fré P, 1993 Nucl. Phys. B392 401.

[3] Perry M J and Teo E, 1993 Nucl. Phys. B401 206.

[4] Witten E, 1988 Phys. Lett. 206B 601.

[5] Labastida J M F and Pernici M, 1988 Phys. Lett. 213B 319.

[6] Wu S, 1993 J. Geo. Phys. 12 205.

Birmingham D and Rakowski M 1991 Phys. Lett. 272B 223

[7] Karlhede A and Roček M, 1988 Phys. Lett. 212B 51.

[8] Baulieu L and Singer I M, 1988 Nucl. Phys. B5 12.

[9] Labastida J M F and Pernici M, 1988 Phys. Lett. 212B 56.

[10] Torre C G, 1990 Phys. Lett. 252B 242; (1990) Phys. Rev. D41 3620.

1990 J. Math. Phys. 31 2983

[11] Jacobson T, 1988 Class. Quan. Grav. 5 923.

[12] Kunitomo H and Sano T, 1993 Intl. J. Mod. Phys. D1 559.

[13] Kshirsagar A K, 1993 Class. Quan. Grav. 10 1859.

[14] Brümann B and Pullin J, Nucl. Phys. B390 399

[15] Galperin A and Ogievetsky O, 1991 Commun. Math. Phys. 139 377.

[16] Witten E, 1994 J. Math. Phys. 35 5101.

[17] Witten E, 1988 Commun. Math. Phys. 117 353. 1991 Intl. J. Mod. Phys. A16 2775

[18] Witten E, 1994 Math.Res.Lett. 1 769

[19] Alverez M and Labastida J M F, “Topological Matter in Four Dimensions”, Preprint hep-th/9404113
[20] Kronheimer P B, J.Diff.Geo. 1989 29 665.
    Swann A, Math.Anm. 1991 289421

[21] Hijazi O, 1993 Proc.Sym.Pure.Math.54 Pt.2 325

[22] Capovilla R, Dell J, Jacobson T and Mason L, 1991 Class. Quan. Grav. 841.

[23] Gotzes S and Hirshfeld A C, 1992 Topological and Geometrical Methods in Field Theory (Jyväskylä, 1991) ed J Mickelsson and O Pekonen (Singapore: World Scientific) p 163

[24] Rovelli C, 1991 Class. Quan. Grav. 8, 1613

[25] Ashtekar A, 1986 Phys. Rev. Lett. 57 2244, 1987 Phys. Rev. D36 1587; 1991 Lectures on Non-Perturbative Canonical Gravity,( World Scientific: Singapore).

[26] Israel W 1968, 1979 Differential Forms In General Relativity, Commun. Dublin Inst. Adv. Stud. Series A No 26 (Dublin)

[27] Donaldson S K and Kronheimer P B, 1991 The Geometry of Four-Manifolds, (Oxford: Oxford University Press)

[28] Penrose R and Rindler W, 1984 Spinors and Space-Time, Vol I (Cambridge: Cambridge University Press)

[29] Eguchi T, Gilkey P B and Hanson A J, 1980 Phys. Repts. 66 214.

[30] Atiyah M F, Hitchin N and Singer I M, 1978 Proc.R.Soc.Lond.A. 363, 425

[31] Wess J and Bagger J, 1992 Supersymmetry and Supergravity 2nd Ed., (Princeton: Princeton University Press)

[32] Kronheimer P B and Nakajima H, 1990 Math.Anm. 288 263