Research Article

Qumri H. Hamko, Nehmat K. Ahmed, and Alias B. Khalaf*

On soft $p_c$-separation axioms

https://doi.org/10.1515/dema-2020-0003
received October 22, 2019; accepted March 9, 2020

Abstract: Many mathematicians defined and studied soft separation axioms and soft continuity in soft spaces by using ordinary points of a topological space $X$. Also, some of them studied the same concepts by using soft points. In this paper, we introduce the concepts of soft $p_c - T_i$ and soft $p_c - T_i^*$, $i = 0, 1, 2$ by using the concept of soft $p_c$-open sets in soft topological spaces. We explore several properties of such spaces. We also investigate the relationship among these spaces and provide a counter example when it is needed.

Keywords: soft $p_c$-open set, soft $p_c - T_i$ spaces $i = 0, 1, 2$, $p_c - T_i^*$, spaces $i = 0, 1, 2$

MSC 2010: 54A05, 54A10, 54C05

1 Introduction

After the introduction of soft set theory for the first time by Molodtsov [1] in 1999 as a new tool in mathematics to deal with several kinds of vagueness in complicated problems in sciences, the study of soft sets and their properties was applied to many branches of mathematics such as probability theory, algebra, operation research, and mathematical analysis. In [2,3], some types of soft Baire spaces and some other mathematical structures were studied and investigated. However, there are analogous theories that can be taken into account as mathematical tools for dealing with uncertainties but each theory has its own difficulties. In the last two decades, mathematicians turned their studies towards soft topological spaces and they reported in several papers different and many interesting topological concepts. Shabir and Naz [4] in 2011 introduced the concept of soft topological spaces which are defined over an initial universe with fixed set of parameters. They indicated that a soft topological space gives a parameterized family of topological spaces and introduced the concept of soft open sets, soft closed sets, soft interior point, soft closure and soft separation axioms. Shi and Pang [5] reported some important results on soft topological spaces. It is noticed that a soft topological space gives a parametrized family of topologies on the initial universe but the converse is not true, i.e., if some topologies are given for each parameter, we cannot construct a soft topological space from the given topologies. Consequently, we can say that the soft topological spaces are more generalized than the classical topological spaces. Georgiou et al. [6] in 2013 defined and studied some soft separation axioms, soft continuity in soft topological spaces using ordinary points of a topological space $X$.

Zorlutuna et al. in [7] and [8] defined and introduced soft neighbourhood and soft continuity in soft spaces using soft points. Hussain and Ahmad [9] continued investigating the properties of soft open (soft closed), soft neighbourhood and soft closure. They also defined and discussed the properties of soft interior, soft exterior and soft boundary.

* Corresponding author: Alias B. Khalaf, Department of Mathematics, College of Science, University of Duhok, Kurdistan-Region, Iraq, e-mail: aliasbkhkhalaf@uod.ac

Qumri H. Hamko: Department of Mathematics, College of Education, Salahaddin University, Kurdistan-Region, Iraq, e-mail: qumri.hamko@su.edu.krd

Nehmat K. Ahmed: Department of Mathematics, College of Education, Salahaddin University, Kurdistan-Region, Iraq, e-mail: nehmat.ahmed@su.edu.krd

Open Access. © 2020 Qumri H. Hamko et al., published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 Public License.
Husain and Ahmed [10] in 2015 introduced separation axioms by using distinct point in the universal set, while in 2018, Bayramov and Aras [11] defined some separation axioms by using distinct soft points. Also, El-shafei et al. in [12] introduced different types of soft separation axioms.

Hamko and Ahmed [13] introduced the concepts of soft $p_c$-open (soft $p_c$ closed) sets, soft $p_c$-neighbourhood and soft $p_c$-closure. They also defined and discussed the properties of soft $p_c$-interior, soft $p_c$-exterior and soft $p_c$-boundary. Also, they defined and studied soft continuity and almost soft continuity in soft spaces using soft points and soft $p_c$-open sets in a soft topological space. Recently, several types of soft separation axioms were studied by Al-shami et al. [14–16]. Also, Al-shami and El-shafei [17,18] introduced other types of soft separation axioms and obtained many characterization theorems, while in [19] some relations on soft Hausdorff spaces are given and corrected some other relations that were written before by other authors. The aim of this paper is to introduce and discuss a study of soft separation axioms, soft $P_c - T_i$, soft $P_c - T_{2i}^*$ $(i = 0, 1, 2)$, soft $p_c$-regular and soft $p_c$-normal spaces, which are defined over an initial universe with a fixed set of parameters by using soft points defined in [11]. Characterizations and properties of these spaces are discussed.

Throughout the present paper, $X$ is a nonempty initial universal set and $E$ is a set of parameters. A pair $(F, E)$ is called a soft set over $X$, where $F$ is a mapping $F: E \rightarrow P(X)$. The collection of soft sets $(F, E)$ over a universal set $X$ with the parameter set $E$ is denoted by $SP(X)_E$. Any logical operation $(\lambda)$ on soft sets in soft topological spaces is denoted by usual set of theoretical operations with symbol $(\tilde{\lambda}(\lambda))$.

## 2 Preliminaries

For the definitions and results on the soft set theory and soft topological spaces, we refer to [7–11] and [20,21]. However, we recall some definitions and results on soft topology, which are used in the following sections.

**Definition 2.1.** A soft set $(F, E)$ over $X$ is said to be an empty soft set denoted by $\tilde{\phi}$, if for all $e \in E$, $F(e) = \emptyset$, and $(F, E)$ over $X$ is said to be an absolute soft set denoted by $\tilde{X}$, if for all $e \in E$, $F(e) = X$.

**Definition 2.2.** The complement of a soft set $(F, E)$ is denoted by $(F, E)^c$ or $\tilde{X} \setminus (F, E)$ and is defined by $(F, E)^c = (F^c, E)$, where $F^c: E \rightarrow P(X)$ is a mapping given by $F^c(e) = X \setminus F(e)$, for all $e \in E$.

It is clear that $((F, E)^c)^c = (F, E)$, $\tilde{\phi}^c = \tilde{X}$ and $\tilde{X}^c = \tilde{\phi}$.

**Definition 2.3.** For two soft sets $(F, E)$ and $(G, B)$ over a common universe $X$, we say that $(F, E)$ is a soft subset of $(G, B)$, if

1. $E \subseteq B$ and
2. for all $e \in E$, $F(e) \subseteq G(e)$.

We write $(F, E) \subseteq (G, B)$.

**Definition 2.4.** The union of two soft sets of $(F, E)$ and $(G, B)$ over the common universe $X$ is the soft set $(H, C) = (F, E) \cup (G, B)$, where $C = E \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e): & \text{if } e \in E \setminus B, \\ G(e): & \text{if } e \in B \setminus E, \\ F(e) \cup G(e): & \text{if } e \in E \cap B. \end{cases}$$
The intersection $(H, C)$ of two soft sets $(F, E)$ and $(G, B)$ over a common universe $X$, denoted $(F, E) \cap (G, B)$, is defined as $C = A \cap B \neq \emptyset$, and $H(e) = F(e) \cap G(e)$ for all $e \in C$.

Let $x \in X$, then $(x, E)$ denotes the soft set over $X$ for which $x(e) = \{x\}$, for all $e \in E$. Let $(F, E)$ be a soft set over $X$ and $x \in X$. We say that $x \notin (F, E)$ means $x$ belongs to the soft set $(F, E)$ whenever $x \in F(e)$ for all $e \in E$.

The soft set $(F, E)$ is called a soft point, denoted by $(x_e, E)$ or $x_e$, if for the element $e \in E$, $F(e) = \{x\}$ and $F(e) = \emptyset$ for all $e \in E \setminus \{e\}$.

We say that $x_e \in (G, E)$ if $x \in G(e)$.

Two soft points $x_e$ and $y_e$ are distinct if either $x \neq y$ or $e \neq e'$.

From Definitions 2.6 and 2.7, it is clear that:
1. $(x, E)$ is the smallest soft set containing $x$.
2. If $x \notin (F, E)$, then $x_e \notin (F, E)$ and $x_e \notin (x, E)$ always.
3. $(F, E) = \bigcup\{(x_e, E): e \in E\}$.

Let $\mathcal{E}$ be a collection of soft sets over a universe $X$ with a fixed set $E$ of parameters. Then, $\mathcal{E} \subseteq \text{SP}(X)_E$ is called a soft topology if,
1. $\emptyset$ and $X$ belong to $\mathcal{E}$.
2. The union of any number of soft sets in $\mathcal{E}$ belongs to $\mathcal{E}$.
3. The intersection of any two soft sets in $\mathcal{E}$ belongs to $\mathcal{E}$.

The triplet $(X, \mathcal{E}, E)$ is called a soft topological space over $X$. The members of $\mathcal{E}$ are called soft open sets in $X$ and complements of them are called soft closed sets in $X$ and they are denoted by $\text{SO}(X)$ and $\text{SC}(X)$, respectively. Soft interior and soft closure are denoted by $\text{sint}$ and $\text{ scl}$, respectively.

Let $(X, \mathcal{E}, E)$ be a soft topological space and let $(G, E)$ be a soft set. Then,
1. The soft closure of $(G, E)$ is the soft set $\text{scl}(G, E) = \cap \{(K, B) \in \text{SC}(X): (G, E) \subseteq (K, B)\}$.
2. The soft interior of $(G, E)$ is the soft set $\text{sint}(G, E) = \cup \{(H, B) \in \text{SO}(X): (H, B) \subseteq (G, E)\}$.

Let $(X, \mathcal{E}, E)$ be a soft topological space, $(G, E)$ be a soft set over $X$ and $x_e \in \text{X}$. Then, $(G, E)$ is said to be a soft neighbourhood of $x_e$ if there exists a soft open set $(H, E)$ such that $x_e \in (H, E) \subseteq ((G, E)$.

Let $(Y, \mathcal{E}, E)$ be a soft subspace of a soft topological space $(X, \mathcal{E}, E)$ and $(F, E) \in \text{SP}(X)_E$. Then,
1. If $(F, E)$ is a soft open set in $\tilde{Y}$ and $\tilde{Y} \in \mathcal{E}$, then $(F, E) \in \mathcal{E}$.
2. $(F, E)$ is a soft open set in $\tilde{Y}$ if and only if $(F, E) = \tilde{Y} \cap (G, E)$ for some $(G, E) \in \mathcal{E}$.
3. $(F, E)$ is a soft closed set in $\tilde{Y}$ if and only if $(F, E) = \tilde{Y} \cap (H, E)$ for some soft closed $(H, E)$ in $\tilde{X}$.

A soft subset $(F, E)$ of a soft space $X$ is said to be soft pre-open if $(F, E) \subseteq \text{sint}[\text{scl}(F, E)]$. The complement of soft pre-open set is said to be soft pre-closed. The family of soft pre-open set and soft pre-closed set is denoted by $\text{spO}(X)$ and $\text{spC}(X)$, respectively.

Arbitrary union of soft pre-open sets is a soft pre-open set.
Lemma 2.15. [21] A subset $(F, E)$ of a soft topological space $(X, \tilde{r}, E)$ is a soft pre-open set if and only if there exists a soft open set $(G, E)$ such that $(F, E) \subseteq (G, E) \subseteq \scl(F, E)$.

Lemma 2.16. [21] Let $(F, E) \subseteq \tilde{Y} \subseteq \tilde{X}$, where $(X, \tilde{r}, E)$ is a soft topological space and $\tilde{Y}$ is a soft pre-open subspace of $\tilde{X}$. Then $(F, E) \in \scl(X)$, if and only if $(F, E) \in \scl(Y)$.

Theorem 2.17. [23] If $(U, E)$ is soft open and $(F, E)$ is soft pre-open in $(X, \tilde{r}, E)$, then $(U, E) \cap (F, E)$ is soft pre-open.

Lemma 2.18. [23] Let $(F, E) \subseteq \tilde{Y} \subseteq \tilde{X}$, where $(X, \tilde{r}, E)$ is a soft topological space and $\tilde{Y}$ is a subspace of $\tilde{X}$. If $(F, E) \in \scl(X)$, then $(F, E) \in \scl(Y)$.

Definition 2.19. [25] A soft topological space $(X, \tilde{r}, E)$ is said to be:
1. Soft $T_0$, if for each pair of distinct soft points $x, y \in X$, there exist soft open sets $(F, E)$ and $(G, E)$ such that either $x \in (F, E)$ and $y \notin (F, E)$ or $y \in (G, E)$ and $x \notin (G, E)$.
2. Soft $T_1$, if for each pair of distinct soft points $x, y \in X$, there exist two soft open sets $(F, E)$ and $(G, E)$ such that $x \in (F, E)$ but $y \notin (F, E)$ and $y \in (G, E)$ but $x \notin (G, E)$.
3. Soft $T_2$, if for each pair of distinct soft points $x, y \in X$, there exist two disjoint soft open sets $(F, E)$ and $(G, E)$ containing $x$ and $y$, respectively.

In [11], Bayramov and Aras redefined soft $T_i$-spaces as in the following definition.

Definition 2.20. [11] A soft topological space $(X, \tilde{r}, E)$ is said to be:
1. Soft $T_0$, if for each pair of distinct soft points $x_e, y_e \in \scl(X)_E$, there exist soft open sets $(F, E)$ and $(G, E)$ such that either $x_e \in (F, E)$ and $y_e \notin (F, E)$ or $y_e \in (G, E)$ and $x_e \notin (G, E)$.
2. Soft $T_1$, if for each pair of distinct soft points $x_e, y_e \in \scl(X)_E$, there exist two soft open sets $(F, E)$ and $(G, E)$ such that $x_e \in (F, E)$ but $y_e \notin (F, E)$ and $y_e \in (G, E)$ but $x_e \notin (G, E)$.
3. Soft $T_2$, if for each pair of distinct soft points $x_e, y_e \in \scl(X)_E$, there exist two disjoint soft open sets $(F, E)$ and $(G, E)$ containing $x_e$ and $y_e$, respectively.

Proposition 2.21. [11]
1. Every soft $T_2$-space $\Rightarrow$ soft $T_1$-space $\Rightarrow$ soft $T_0$-space.
2. A soft topological space $(X, \tilde{r}, E)$ is soft $T_1$ if and only if each soft point is soft closed.

In [4], a soft regular space is defined by using ordinary points as follows.

Definition 2.22. [4] If for every $x \in X$ and every soft closed set $(F, E)$ not containing $X$, there exist two soft open sets $(G, E)$ and $(H, E)$ such that $x \in (G, E), (F, E) \subseteq (H, E)$ and $(G, E) \cap (H, E) = \emptyset$ then $X$ is called soft regular.

In [9], a soft regular space is defined by using soft points as follows.

Definition 2.23. [9] If for every $x_e \in \tilde{X}$ and every soft closed set $(F, E)$ not containing $x_e$, there exist two soft open sets $(G, E)$ and $(H, E)$ such that $x_e \in (G, E), (F, E) \subseteq (H, E)$ and $(G, E) \cap (H, E) = \emptyset$ then $\tilde{X}$ is called soft regular.

Definition 2.24. [24] A soft pre-open set $(F, E)$ in a soft topological space $(X, \tilde{r}, E)$ is called soft $p_c$-open if for each $x_e \in (F, E)$, there exists a soft closed set $(K, E)$ such that $x_e \in (K, E) \subseteq (F, E)$. The soft complement of each soft $p_c$-open set is called the soft $p_c$-closed set.

The family of all soft $p_c$-open (resp., soft $p_c$-closed) sets in a soft topological space $(X, \tilde{r}, E)$ is denoted by $\tilde{sp}_cO(X, \tilde{r}, E)$ (resp., $\tilde{sp}_cC(X, \tilde{r}, E)$) or $\tilde{sp}_cO(X)$ (resp., $\tilde{sp}_cC(X)$).

Definition 2.25. [21] Let $(X, \tilde{r}, E)$ be a soft topological space and let $(G, E)$ be a soft set. Then,
1. The soft pre-closure of \((G, E)\) is the soft set
\[ \bar{sp}(G, E) = \bigcap \{(K, B) \in SPC(\tilde{X}) : (G, E) \subseteq (K, B)\}. \]

2. The soft pre-interior of \((G, E)\) is the soft set
\[ \bar{sp}(G, E) = \bigcup \{(H, B) \in SPO(\tilde{X}) : (G, E) \subseteq (H, B)\}. \]

**Definition 2.26.** [13] Let \((X, \tilde{r}, E)\) be a soft topological space and let \((G, E)\) be a soft set. Then,
1. A soft point \(x_e \in \tilde{X}\) is said to be a soft \(p_c\)-limit soft point of a soft set \((F, E)\) if for every soft \(p_c\)-open set \((G, E)\) containing \(x_e\), \((G, E) \cap [(F, E) \setminus \{x_e\}] \neq \emptyset\).

The set of all soft \(p_c\)-limit soft points of \((F, E)\) is called the soft \(p_c\)-derived set of \((F, E)\) and is denoted by \(sp_cD(F, E)\).

2. The soft \(p_c\)-closure of \((G, E)\) is the soft set
\[ \bar{sp}_c cl(G, E) = \bigcap \{(K, B) \in SPC(\tilde{X}) : (G, E) \subseteq (K, B)\}. \]

3. The soft \(p_c\)-interior of \((G, E)\) is the soft set
\[ \bar{sp}_c int(G, E) = \bigcup \{(H, B) \in SPO(\tilde{X}) : (G, E) \subseteq (H, B)\}. \]

**Lemma 2.27.** [24] If \((F, E) \subseteq \tilde{Y} \subseteq \tilde{X}\) and \(Y\) is soft clopen, then \((F, E) \subseteq \bar{sp}_c O(Y)\) if and only if \((F, E) \subseteq \bar{sp}_c O(X)\).

**Lemma 2.28.** [24] Let \((F, E), \tilde{Y} \subseteq \tilde{X}\) and \(\tilde{Y}\) be soft clopen. If \((F, E) \subseteq \bar{sp}_c O(X)\), then \((F, E) \cap \tilde{Y} \subseteq \bar{sp}_c O(Y)\).

**Lemma 2.29.** [13] Let \((F, E) \subseteq \tilde{Y} \subseteq \tilde{X}\). If \(\tilde{Y}\) is soft clopen, then \(\bar{sp}_c cl_{\tilde{Y}}(F, E) = \bar{sp}_c cl_{\tilde{X}}(F, E) \cap \tilde{Y}\).

**Definition 2.30.** [21] A soft topological space \((X, \tilde{r}, E)\) is said to be:
1. Soft \(P_0\), if for each pair of distinct soft points \(x, y \in X\), there exist soft pre-open sets \((F, E)\) and \((G, E)\) such that either \(x \in (F, E)\) and \(y \notin (F, E)\) or \(y \in (G, E)\) and \(x \notin (G, E)\).

2. Soft \(P_1\), if for each pair of distinct soft points \(x, y \in X\), there exist two soft pre-open sets \((F, E)\) and \((G, E)\) such that \(x \notin (F, E)\) but \(y \in (F, E)\) and \(y \notin (G, E)\) but \(x \in (G, E)\).

3. Soft \(P_2\), if for each pair of distinct soft points \(x, y \in X\), there exist two disjoint soft pre-open sets \((F, E)\) and \((G, E)\) containing \(x\) and \(y\), respectively.

### 3 Soft \(p_c - T_i\) spaces for \((i = 0, 1, 2)\)

In this section, we define \(\bar{sp}_c - T_i\) spaces for \((i = 0, 1, 2)\) by using \(\bar{sp}_c\)-open sets and separating the soft points of the soft topological space. Several relations between these soft spaces and other types of soft separation axioms are investigated.

**Definition 3.1.** A soft topological space \((X, \tilde{r}, E)\) is said to be
1. \(\bar{sp}_c - T_0\), if for each pair of distinct soft points \(x_e, y_e \in \text{SP}(X)_E\), there exist \(\bar{sp}_c\)-open sets \((F, E)\) and \((G, E)\) such that \(x_e \in (F, E)\) and \(y_e \notin (F, E)\) or \(y_e \in (G, E)\) and \(x_e \notin (G, E)\).

2. \(\bar{sp}_c - T_1\), if for each pair of distinct soft points \(x_e, y_e \in \text{SP}(X)_E\), there exist two \(\bar{sp}_c\)-open sets \((F, E)\) and \((G, E)\) such that \(x_e \notin (F, E)\) but \(y_e \in (F, E)\) and \(y_e \notin (G, E)\) but \(x_e \in (G, E)\).

3. \(\bar{sp}_c - T_2\), if for each pair of distinct soft points \(x_e, y_e \in \text{SP}(X)_E\), there exist two disjoint \(\bar{sp}_c\)-open sets \((F, E)\) and \((G, E)\) containing \(x_e\) and \(y_e\), respectively.

**Proposition 3.2.** A soft topological space \((X, \tilde{r}, E)\) is \(\bar{sp}_c - T_0\) if and only if the \(\bar{sp}_c\)-closure of any two soft points is distinct.
Proof. Let \((X, \bar{r}, E)\) be an \(\tilde{sp}_c - T_0\) space and \(x_e, y_e \in \text{SP}(X)_E\) with \(x_e \neq y_e\). Then, there exist an \(\tilde{sp}_c\)-open set \((F, E)\) containing one of the soft points, say \(x_e\), but not the other. Then, \(X \setminus (F, E)\) is an \(\tilde{sp}_c\)-closed set which does not contain \(x_e\) but contains \(y_e\). Since, \(\tilde{sp}_c\text{cl}(|y_e|)\) is the smallest \(\tilde{sp}_c\)-closed set containing \(y_e\), \(\tilde{sp}_c\text{cl}(|y_e|) \subseteq X \setminus (F, E)\) and therefore \(x_e \notin \tilde{sp}_c\text{cl}(|y_e|)\). Consequently, \(\tilde{sp}_c\text{cl}(|x_e|) \neq \tilde{sp}_c\text{cl}(|y_e|)\).

Conversely, suppose that \(x_e, y_e \in \text{SP}(X)_E\) such that \(x_e \neq y_e\) and \(\tilde{sp}_c\text{cl}(|x_e|) \neq \tilde{sp}_c\text{cl}(|y_e|)\). Let \(z_a\) be a soft point in \(\text{SP}(X)_E\) such that \(z_a \notin \tilde{sp}_c\text{cl}(|x_e|)\), but \(z_a \notin \tilde{sp}_c\text{cl}(|y_e|)\). We claim that \(x_e \notin \tilde{sp}_c\text{cl}(|y_e|)\). For, if \(x_e \in \tilde{sp}_c\text{cl}(|y_e|)\), then \(\tilde{sp}_c\text{cl}(|x_e|) \subseteq \tilde{sp}_c\text{cl}(|y_e|)\). This contradicts the fact that \(z_a \notin \tilde{sp}_c\text{cl}(|y_e|)\). Consequently, \(x_e\) belongs to the \(\tilde{sp}_c\)-open set \((G, E) = X \setminus \tilde{sp}_c\text{cl}(|y_e|)\). Then, \((G, E)\) being the complement of \(\tilde{sp}_c\)-closed set. Thus \((G, E)\) is an \(\tilde{sp}_c\)-open space which contains \(x_e\) but not \(y_e\). Hence, \((X, \bar{r}, E)\) is an \(\tilde{sp}_c - T_0\) space.

\[\square\]

Proposition 3.3. If \((X, \bar{r}, E)\) is \(\tilde{sp}_c - T_0\) space, then \(\tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|x|)) \cap \tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|y|)) = \tilde{\phi}\), for each pair of distinct soft points \(x_e, y_e \in \text{SP}(X)_E\).

Proof. Let \((X, \bar{r}, E)\) be an \(\tilde{sp}_c - T_0\) space and \(x_e, y_e \in \text{SP}(X)_E\) such that \(x_e \neq y_e\). Then, there exists an \(\tilde{sp}_c\)-open set \((F, E)\) containing \(x_e\) or \(y_e\), say \(x_e\) but not \(y_e\), which implies that \(x_e \notin (F, E)\) and \(y_e \notin (F, E)\), then \(y_e \notin X \setminus (F, E)\) and \(X \setminus (F, E)\) is an \(\tilde{sp}_c\)-closed. Now \(\tilde{sp}_c\text{int}(|y_e|) \subseteq \tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|y_e|)) \subseteq X \setminus (F, E)\), which implies that \((F, E) \cap \tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|y_e|)) = \tilde{\phi}\), then \((F, E) \subseteq X \setminus \tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|y_e|))\). Since \(x_e \notin (F, E) \subseteq X \setminus \tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|y_e|))\), then \(\tilde{sp}_c\text{cl}(|x_e|) \subseteq X \setminus \tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|y_e|))\), which implies that \(\tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|x_e|)) \subseteq \tilde{sp}_c\text{cl}(|x_e|) \subseteq X \setminus \tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|y_e|))\). Therefore, \(\tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|x_e|)) \cap \tilde{sp}_c\text{cl}(\tilde{sp}_c\text{int}(|y_e|)) = \tilde{\phi}\).

\[\square\]

Proposition 3.4. Every soft \(\tilde{sp}_c - T_i\) space is soft \(T_i\) for \(i = 0, 1\).

Proof. We shall prove the case when \((X, \bar{r}, E)\) is \(\tilde{sp}_c - T_0\), the other proof is similar.

Let \((X, \bar{r}, E)\) be an \(\tilde{sp}_c - T_0\) space and \(x_e, y_e \in \text{SP}(X)_E\), with \(x_e \neq y_e\), so there exists an \(\tilde{sp}_c\)-open set \((F, E)\) containing one of them say \(x_e\). Since \((F, E)\) is an \(\tilde{sp}_c\)-open set, there exists a soft closed set \((K, E)\) such that \(x_e \in (K, E) \subseteq (F, E)\), so \(X \setminus (K, E)\) is a soft open set containing \(y_e\) but not \(x_e\). Therefore, \((X, \bar{r}, E)\) is a soft \(T_0\)-space.

The next example shows that the converse of Proposition 3.4 is not true in general.

Example 3.5. Let \(X = \{x_1, x_2\}, E = \{e_1, e_2\}\) and let \(\bar{r} = \{\bar{x}, \bar{\phi}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E)\}\), where \((F_1, E) = \{(e_1, \{x_1\}), (e_2, \{\bar{x}\})\}, (F_2, E) = \{(e_1, X), (e_2, \{x_3\})\}, (F_3, E) = \{(e_1, \phi), (e_2, X)\}, (F_4, E) = \{(e_1, \{x_1\}), (e_2, \{x_3\})\}, (F_5, E) = \{(e_1, \phi), (e_2, \{x_3\})\}\) and \((F_6, E) = \{(e_1, x_2), (e_2, \{x_3\})\}\). Then, \(\tilde{sp}_c\text{O}(X) = \{\bar{x}, \bar{\phi}\}\). It is easy to show that this space is a soft \(T_0\)-space but it is not \(\tilde{sp}_c - T_0\).

Proposition 3.6. Every soft \(T_i\) space is \(\tilde{sp}_c - T_i\) for \(i = 1, 2\).

Proof. If a soft space is soft \(T_2\) or soft \(T_1\), then by Proposition 2.21(2) every soft point is soft closed and hence every soft open set is an \(\tilde{sp}_c\)-open set. Therefore, if \((X, \bar{r}, E)\) is soft \(T_2\) (resp., soft \(T_1\)), then it is an \(\tilde{sp}_c - T_2\) (resp., \(\tilde{sp}_c - T_1\)) space.

Corollary 3.7. A soft topological space \((X, \bar{r}, E)\) is soft \(T_1\) if and only if it is \(\tilde{sp}_c - T_1\).

Proof. Follows directly from Propositions 3.4 and 3.6.

\[\square\]

Proposition 3.8. A space \((X, \bar{r}, E)\) is \(\tilde{sp}_c - T_i\) if and only if every soft point of the space \((X, \bar{r}, E)\) is an \(\tilde{sp}_c\)-closed set.
Proof. Let \((X, \bar{r}, E)\) be an \(\mathcal{SP}_1 - T_1\) space, so by Proposition 3.4, \((X, \bar{r}, E)\) is soft \(T_1\) and by Proposition 2.21(2), every soft point is soft closed hence soft pre-closed. Since every soft point is closed, \(\mathcal{SP}\((X) = \mathcal{SP}\((X).\) Hence, every soft point is an \(\mathcal{SP}_1\)-closed set.

Conversely, let \(x_e\) be a soft point of \((X, \bar{r}, E)\) which is \(\mathcal{SP}_1\)-closed, then \(X \setminus \{x_e\}\) is an \(\mathcal{SP}_1\)-open. Then, for distinct soft points \(x_e \) and \(y_e\), \(X \setminus \{x_e\}\) and \(X \setminus \{y_e\}\) are \(\mathcal{SP}_1\)-open sets such that \(x_e \notin X \setminus \{y_e\}\) and \(y_e \notin X \setminus \{x_e\}\) but \(x_e \notin X \setminus \{y_e\}\) and \(y_e \notin X \setminus \{x_e\}\). Thus, \((X, \bar{r}, E)\) is an \(\mathcal{SP}_1 - T_1\) space. \(\square\)

Proposition 3.9. A soft topological space \((X, \bar{r}, E)\) is an \(\mathcal{SP}_1 - T_1\) space if and only if \(\mathcal{SP}(\{x_e\}) = \bar{x}\), for each \(x_e \in \bar{X}\).

Proof. Let \(\bar{X}\) be an \(\mathcal{SP}_1 - T_1\) space. To prove \(\mathcal{SP}(\{x_e\}) = \bar{x}\), for each \(x_e \in \bar{X}\). If \(\mathcal{SP}(\{x_e\}) \neq \bar{x}\), then there is a soft point say \(y_e \in \mathcal{SP}(\{x_e\})\) and \(x_e \neq y_e\). Since \(\bar{X}\) is \(\mathcal{SP}_1 - T_1\), then there exists an \(\mathcal{SP}_1\)-open set \((F, E)\) such that \(y_e \in (F, E)\) and \(x_e \notin (F, E)\), then \(F, E \cap \{x_e\} = \bar{x}\) and hence \(y_e \notin \mathcal{SP}(\{x_e\})\), which is a contradiction. Thus, \(\mathcal{SP}(\{x_e\}) = \bar{x}\), for each \(x_e \in \bar{X}\).

Conversely, let \(\mathcal{SP}(\{x_e\}) = \bar{x}\), for each \(x_e \in \bar{X}\). Since \(\mathcal{SP}(\{x_e\}) = \{x_e\} \cup \mathcal{SP}(\{x_e\})\) and \(\mathcal{SP}(\{x_e\}) = \bar{x}\), \(\mathcal{SP}(\{x_e\}) = \{x_e\}\), which implies that \(\{x_e\}\) is an \(\mathcal{SP}_1\)-closed set and hence by Proposition 3.8, \(\bar{X}\) is an \(\mathcal{SP}_1 - T_1\) space. \(\square\)

Proposition 3.10. For a soft space \((X, \bar{r}, E)\), the following statements are equivalent.
1. \(\bar{X}\) is \(\mathcal{SP}_1 - T_2\).
2. For each \(x_e \in \bar{X}\) and each \(x_e \neq y_e\), there exists an \(\mathcal{SP}_1\)-open set \((F, E)\) of \(x_e\) such that \(y_e \notin \mathcal{SP}_1\) Cl\((F, E)\).
3. For each \(x_e \in \bar{X}\), \(\cap \mathcal{SP}_1\) Cl\((F, E)\) : \(x_e \in (F, E) \in \mathcal{SP}_1\) O\((X) = \{x_e\}\).

Proof.
(1) \(\Rightarrow\) (2). Since \(\bar{X}\) is an \(\mathcal{SP}_1 - T_2\) space, there exists two disjoint \(\mathcal{SP}_1\)-open sets \((F, E)\) and \((G, E)\) such that \(x_e \in (F, E)\) and \(y_e \in (G, E)\). This implies that \((F, E) \subseteq X \setminus (G, E)\). Therefore, \(\mathcal{SP}_1\) Cl\((F, E) \subseteq X \setminus (G, E)\). So, \(y_e \notin \mathcal{SP}_1\) Cl\((F, E)\).

(2) \(\Rightarrow\) (3). Let for some \(x_e \neq y_e\), we have \(y_e \in \mathcal{SP}_1\) Cl\((F, E)\) for every \(\mathcal{SP}_1\)-open set \((F, E)\) containing \(x_e\), which contradicts (2).

(3) \(\Rightarrow\) (1). Let \(x_e, y_e \in \bar{X}\) with \(x_e \neq y_e\). Then, there exists an \(\mathcal{SP}_1\)-open set \((F, E)\) containing \(x_e\) such that \(y_e \notin \mathcal{SP}_1\) Cl\((F, E)\). Let \((G, E) = X \setminus \mathcal{SP}_1\) Cl\((F, E)\), then \(y_e \in (G, E)\) and \(x_e \in (F, E)\) and so \((F, E) \cap (G, E) = \bar{X}\). \(\square\)

Proposition 3.11. A soft space \(\bar{X}\) is \(\mathcal{SP}_1 - T_2\) if for each pair of distinct soft points \(x_e, y_e \in \bar{X}\) there exists an \(\mathcal{SP}_1\) clopen set \((F, E)\) containing one of them but not the other.

Proof. Let for each pair of distinct soft points \(x_e, y_e \in \bar{X}\), there exists an \(\mathcal{SP}_1\) clopen set \((F, E)\) containing \(x_e\), but not \(y_e\), which implies that \(X \setminus (F, E)\) is also an \(\mathcal{SP}_1\) clopen set and \(y_e \in X \setminus (F, E)\), since \((F, E) \cap X \setminus (F, E) = \bar{X}\), \(\bar{X}\) is an \(\mathcal{SP}_1 - T_2\) space. \(\square\)

Proposition 3.12. Every soft clopen subspace of an \(\mathcal{SP}_1 - T_1\) space is an \(\mathcal{SP}_1 - T_1\) space for \((i = 0, 1, 2)\).

Proof. We prove only the case for \(\mathcal{SP}_1 - T_0\) space and the other cases are similar. Let \((Y, \tilde{r}, E)\) be a soft clopen subspace of \(\mathcal{SP}_1 - T_0\) space \((X, \bar{r}, E)\) and \(x_e, y_e\) be two distinct soft points in \(\bar{X}\). Since \((X, \bar{r}, E)\) is an \(\mathcal{SP}_1 - T_0\) space and \(x_e \neq y_e\), then there exists an \(\mathcal{SP}_1\)-open set \((F, E)\) containing one of them say \(x_e\) but not \(y_e\). So, by Lemma 2.28, \((F, E) \cap Y\) is an \(\mathcal{SP}_1\)-open set in \((Y, \tilde{r}, E)\), which contains \(x_e\) but not \(y_e\). Hence, \((Y, \tilde{r}, E)\) is \(\mathcal{SP}_1 - T_0\). \(\square\)

Proposition 3.13. If for each \(x_e \in \bar{X}\), there exists a soft clopen set \((F, E)\) containing \(x_e\) such that \((F, E)\) is an \(\mathcal{SP}_1 - T_i\) subspace of \(\bar{X}\), then the soft space \(\bar{X}\) is also an \(\mathcal{SP}_1 - T_i\) space for \((i = 0, 1, 2)\).
Proof. We prove only the case for the $\tilde{sp}_c - T_0$ space and the proofs of other cases are similar. Let $x_c$ and $y_c$ be two distinct soft points in $\tilde{X}$, then by hypothesis there exist soft regular open sets $(F, E)$ and $(G, E)$ such that $x_c \in (F, E)$, $y_c \in (G, E)$ and $(F, E), (G, E)$ are $\tilde{sp}_c - T_0$ subspaces of $\tilde{X}$. Now, if $y_c \notin (F, E)$, then the proof is completed, but if $y_c \in (F, E)$ and since $(F, E)$ is $\tilde{sp}_c - T_0$ subspace of $\tilde{X}$, there exists an $\tilde{sp}_c$-open set $(H, E)$ in $(F, E)$ such that $y_c \in (H, E)$ and $x_c \notin (H, E)$ and since $(F, E)$ is soft regular open, by Lemma 2.27, $(H, E)$ is an $\tilde{sp}_c$-open set in $\tilde{X}$ containing $y_c$ but not $x_c$. Therefore, $\tilde{X}$ is an $\tilde{sp}_c - T_0$ space. □

4 Soft $p_c - T^*_i$ spaces for $(i = 0, 1, 2)$

In this section, we define $\tilde{sp}_c - T^*_i$ spaces for $(i = 0, 1, 2)$ by using $\tilde{sp}_c$-open sets and separating the usual points of the space. Several relations between these soft spaces and other types of soft separation axioms are investigated. Relations between $\tilde{sp}_c - T^*_i$ spaces and $\tilde{sp}_c - T_i$ spaces for $(i = 0, 1, 2)$ are discussed.

Definition 4.1. A soft topological space $(X, \tilde{r}, E)$ is said to be
1. $\tilde{sp}_cT^*_0$, if for each pair of distinct points $x, y \in X$, there exist $\tilde{sp}_c$-open sets $(F, E)$ and $(G, E)$ such that $x \in (F, E)$ and $y \notin (F, E)$ or $y \in (G, E)$ and $x \notin (G, E)$.
2. $\tilde{sp}_cT^*_1$, if for each pair of distinct points $x, y \in X$, there exist two $\tilde{sp}_c$-open sets $(F, E)$ and $(G, E)$ such that $x \in (F, E)$ but $y \notin (F, E)$ and $y \in (G, E)$ but $x \notin (G, E)$.
3. $\tilde{sp}_cT^*_2$, if for each pair of distinct soft points $x, y \in X$, there exist two disjoint $\tilde{sp}_c$-open sets $(F, E)$ and $(G, E)$ containing $x$ and $y$, respectively.

Proposition 4.2. Every soft $\tilde{sp}_c - T^*_i$ space is a soft $p_i$-space for $i = 0, 1, 2$.

Proposition 4.3. If a soft topological space $(X, \tilde{r}, E)$ is a $\tilde{sp}_c - T^*_i$ space, then it is soft $\tilde{sp}_c - T^*_i$.

Proof. Let $(X, \tilde{r}, E)$ be an $\tilde{sp}_c - T^*_i$ space and $x, y \in X$ with $x \neq y$, then $x_c \neq y_c$ for all $e \in E$. Since $(X, \tilde{r}, E)$ is an $\tilde{sp}_c - T^*_i$, by Proposition 3.8, $\{x_e \}, \{y_e \}$ are $\tilde{sp}_c$-closed sets for all $e \in E$. Hence, $\tilde{X}\setminus \{y_e \}$ and $\tilde{X}\setminus \{x_e \}$ are the required $\tilde{sp}_c$-open sets containing $x$ and $y$, respectively. Therefore, $(X, \tilde{r}, E)$ is soft $\tilde{sp}_c - T^*_i$. □

From Definition 3.1, Definition 4.1, Proposition 4.2, Corollary 3.7 and Proposition 4.3, we obtain the following diagram of implications.

\[
\begin{array}{c}
\text{soft-}P_2 \rightarrow \text{soft-}P_1 \rightarrow \text{soft-}P_0 \\
\uparrow \hspace{1cm} \uparrow \hspace{1cm} \uparrow \\
\tilde{sp}_c T^*_2 \rightarrow \tilde{sp}_c T^*_1 \rightarrow \tilde{sp}_c T^*_0 \\
\uparrow \\
\tilde{sp}_c T_2 \rightarrow \tilde{sp}_c T_1 \rightarrow \tilde{sp}_c T_0 \\
\uparrow \\
\tilde{ST}_2 \rightarrow \tilde{ST}_1 \rightarrow \tilde{ST}_0
\end{array}
\]

Any other implication except those resulting from transitivity is not true in general as it is shown in the following examples.

Example 4.4. Let $X = \{x, y\}, E = \{e_1, e_2\}$ and let $\tilde{r} = \{(\tilde{X}, \tilde{r}), (F_1, E), (F_2, E)\}$, where

\[(F_1, E) = \{(e_1, \{x\}), (e_2, \{x\})\}, (F_2, E) = \{(e_1, \{y\}), (e_2, \{y\})\}.
\]

Then, it can be checked that $\tilde{sp}_c O(X) = \tilde{r}$. Hence, the space is $\tilde{sp}_c - T^*_i$ but it is not $\tilde{sp}_c - T_i$ for $i = 0, 1, 2$. 


Example 4.5. Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ and let
\[
\tilde{\tau} = \{\tilde{X}, \tilde{\phi}, (F_1, E), (F_2, E), (F_3, E)\},
\]
where
\[
(F_1, E) = \{(e_1, \{x\}), (e_2, \{x, y\})\},
(F_2, E) = \{(e_1, \{y\}), (e_2, \{x, y\})\},
(F_3, E) = \{(e_1, \phi), (e_2, \{x, y\})\}.
\]

Then, it can be checked that $\tilde{\sp E (O(X) = \tilde{\tau})$ and $\tilde{\sp (X) = \sp (X)_{\tilde{E}}$. Since $x_e \neq y_e$ and there is no soft open set containing one of them but not the other, it is not $\tilde{\sp E - T_i^\tau$ for $i = 0, 1, 2$. This space is $\tilde{\sp E - T_1^\tau$ for $i = 0, 1, 2$, but it is not $\tilde{\sp E - T_2^\tau$.

Example 4.6. Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ and let
\[
\tilde{\tau} = \{\tilde{X}, \tilde{\phi}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E)\},
\]
where
\[
(F_1, E) = \{(e_1, \{x\}), (e_2, \phi)\},
(F_2, E) = \{(e_1, \{y\}), (e_2, \phi)\},
(F_3, E) = \{(e_1, \{x, y\}), (e_2, \phi)\},
(F_4, E) = \{(e_1, \phi), (e_2, \{x, y\})\}.
\]

Then, it can be checked that $\tilde{\sp E (O(X) = \tilde{\tau}) and $\tilde{\sp (X) = \sp (X)_{\tilde{E}}$. Since $x_e \neq y_e$ and there is no soft open set containing one of them but not the other, it is not $\tilde{\sp E - T_i^\tau$ for $i = 0, 1, 2$. This space is $\tilde{\sp E - T_1^\tau$ for $i = 0, 1, 2$, but it is not $\tilde{\sp E - T_2^\tau$.

Example 4.7. Let $X$ be any infinite set, $E$ consists of infinite parameters and let $P_e$ be a fixed soft point in $\tilde{X}$ and let $\tilde{\tau}$ be the family of all soft subsets $(F, E)$ such that either $P_e \notin (F, E)$ or if $P_e \in (F, E)$, then
\[
\bigcup_{e \in E} X \setminus F(e)
\]
is finite. This space is both $\tilde{\tau} - T_2$ and $\tilde{\sp E - T_2$, but it is not $\tilde{\sp E - T_2^\tau$ because if $x \neq y$ in $X$ and $x_e = P_e$, if the soft open set containing $x$ is $(F, E)$, which implies that $x_e = P_e \notin (F, E)$, hence
\[
\bigcup_{e \in E} X \setminus F(e)
\]
is finite but there is no soft open set containing $y$ which is disjoint from $(F, E)$.

From Examples 4.6 and 4.7, we conclude that $\tilde{\sp E - T_2$ and $\tilde{\sp E - T_2^\tau$ are incomparable spaces.

Example 4.8. Let $X = \{1, 2, 3, \ldots\}$, $E = \{0, 1, 2\}$ and $\tilde{\tau} = \{G_m, E\}: n \in \{1, 2, \ldots\} \cup \{\tilde{\phi}, \tilde{X}\}$, such that
\[
G_m: E \rightarrow P(X), \text{ where } G_m(e) = \{n, n + 1, \ldots\} \text{ for every } e \in E.
\]
The triplet $(X, \tilde{\tau}, E)$ is a soft topological space and $\tilde{\sp E (O(X) = \{\tilde{\phi}, \tilde{X}\}$.

Let $x$ and $y$ be two distinct points of $X$ and $e \in E$. We suppose that $y < x$. Then, $x \in \{x, x + 1, \ldots\} = G_0(e)$ and $y \notin \{x, x + 1, \ldots\}$. This means that $x \in (G_0(e), E)$ and $y \notin (G_0(e), E)$. Thus, the soft topological space $(X, \tilde{\tau}, E)$ is a soft $T_0$-space. Also, we observe that this soft topological space is not an $\tilde{\sp E - T_0^\tau$-space.

Example 4.9. Let $X$ be any infinite set, $E = \{e_1, e_2\}$ and $\tilde{\tau}$ a topology consists of $\tilde{\phi}$ and all soft sets $(F, E)$, where $(F, E)$ is defined as: $X \setminus F(e)$ is a finite subset of $X$ for each $e \in E$. Then, $(X, \tilde{\tau}, E)$ is a soft topological space over $X$. It can be easily shown that this space is $\tilde{\sp E - T_1$ and $\tilde{\sp E - T_1^\tau$ space which is not $\tilde{\sp E - T_2$ and not $\tilde{\sp E - T_2^\tau$.}
Proposition 4.10. Let \((X, \tau, E)\) be a soft topological space and \(x, y \in X\) such that \(x \neq y\). If there exist \(\tilde{sp}_c\)-open sets \((F, E)\) and \((G, E)\) such that \(x \notin (F, E)\) and \(y \notin (G, E)\) or \(x \notin (G, E)\) and \(y \notin (F, E)\) then \((X, \tau, E)\) is \(\tilde{sp}_c - T^*_0\).

Proof. Let \(x, y \in X\) such that \(x \neq y\) and \((F, E), (G, E)\) are \(\tilde{sp}_c\)-open sets such that \(x \notin (F, E)\) and \(y \notin (G, E)\) or \(x \notin (G, E)\) and \(y \notin (F, E)\). If \(y \notin (F, E)\), then \(y \notin X\setminus(F(e))\) for each \(e \in E\). This implies that \(y \notin F(e)\) for each \(e \in E\). Therefore, \(y \notin (F, E)\). Similarly, we can show that if \(x \notin (G, E)\), then \(x \notin (F, E)\). Hence, \((X, \tau, E)\) is \(\tilde{sp}_c - T^*_0\).

Remark 4.11. In Example 4.6, it can be easily seen that the converse of Proposition 4.10 is not true in general. Because \(x \notin (F_2, E)\) but \(y \notin \tilde{X}\setminus(F_2, E)\) and also \(y \notin (F_0, E)\) but \(x \notin \tilde{X}\setminus(F_0, E)\).

Proposition 4.12. A soft topological space \((X, \tau, E)\) is \(\tilde{sp}_c - T^*_0\) if and only if for each pair of distinct points \(x, y \in X\), \(\tilde{sp}_c cl (x, E) \neq \tilde{sp}_c cl (y, E)\).

Proof. Let \((X, \tau, E)\) be an \(\tilde{sp}_c - T^*_0\) space and \(x, y\) be any two distinct points of \(X\). There exists an \(\tilde{sp}_c\)-open set \((F, E)\) containing \(x\) but not \(y\). Then, \(\tilde{X}\setminus(F, E)\) is an \(\tilde{sp}_c\)-closed set which does not contain \(x\) but contains \(y\). Since \(\tilde{sp}_c cl (y, E)\) is the smallest \(\tilde{sp}_c\)-closed set containing \(y\), \(\tilde{sp}_c cl (y, E) \subseteq \tilde{X}\setminus(F, E)\) and therefore \(x \notin \tilde{sp}_c cl (y, E)\). Consequently, \(\tilde{sp}_c cl (x, E) \neq \tilde{sp}_c cl (y, E)\).

Conversely, suppose that \(x, y \in X\), \(x \neq y\) and \(\tilde{sp}_c cl (x, E) \neq \tilde{sp}_c cl (y, E)\). Let \(z\) be a point of \(X\) such that \(z \notin \tilde{sp}_c cl (x, E)\), but \(z \notin \tilde{sp}_c cl (y, E)\). We claim that \(x \notin \tilde{sp}_c cl (y, E)\). For, if \(x \notin \tilde{sp}_c cl (y, E)\), then \(\tilde{sp}_c cl (x, E) \subseteq \tilde{sp}_c cl (y, E)\). This contradicts the fact that \(z \notin \tilde{sp}_c cl (y, E)\). Consequently, \(x\) belongs to the \(\tilde{sp}_c\)-open set \(\tilde{X}\setminus\tilde{sp}_c cl (y, E)\) and \(y\) does not belong to it.

Proposition 4.13. For each pair of distinct points \(x, y \in X\). If a soft topological space \((X, \tau, E)\) is \(\tilde{sp}_c - T^*_0\), then \(\tilde{sp}_c cl (\tilde{sp}_c int (x, E)) \cap \tilde{sp}_c cl (\tilde{sp}_c int (y, E)) = \phi\).

Proof. Let \((X, \tau, E)\) be \(\tilde{sp}_c - T^*_0\) and \(x, y \in X\) such that \(x \neq y\). Then, there exists an \(\tilde{sp}_c\)-open set \((F, E)\) containing one of the points, say \(x\), and does not contain the other, which implies that \(x \notin (F, E)\) and \(y \notin (F, E)\), then \(y \notin \tilde{X}\setminus(F, E)\) and \(X\setminus(F, E)\) is \(\tilde{sp}_c\)-closed. Now we have,

\[\tilde{sp}_c \text{ int} (y, E) \subseteq \tilde{sp}_c \text{ cl} (\tilde{sp}_c \text{ int} (y, E)) \subseteq X\setminus(F, E),\]

which implies that \((F, E) \cap \tilde{sp}_c \text{ cl} (\tilde{sp}_c \text{ int} (y, E)) = \phi\), then \((F, E) \cap \tilde{sp}_c \text{ cl} (\tilde{sp}_c \text{ int} (y, E)) = \phi\). So, \(x \notin (F, E) \subseteq X\setminus\tilde{sp}_c \text{ cl} (\tilde{sp}_c \text{ int} (y, E))\), then \(\tilde{sp}_c \text{ cl} (x, E) \subseteq X\setminus\tilde{sp}_c \text{ cl} (\tilde{sp}_c \text{ int} (y, E))\), which implies that \(\tilde{sp}_c \text{ cl} (\tilde{sp}_c \text{ int} (x, E)) \subseteq X\setminus\tilde{sp}_c \text{ cl} (\tilde{sp}_c \text{ int} (y, E))\). Therefore,

\[\tilde{sp}_c \text{ cl} (\tilde{sp}_c \text{ int} (x, E)) \cap \tilde{sp}_c \text{ cl} (\tilde{sp}_c \text{ int} (y, E)) = \phi.\]

Proposition 4.14. If a soft topological space \((X, \tau, E)\) is \(\tilde{sp}_c - T^*_0\) (resp., \(\tilde{sp}_c - T^*_0\)), then it is a soft \(T_0\) (resp., soft \(T_0\)) space due to [10].

Proof. Let \((X, \tau, E)\) be an \(\tilde{sp}_c - T^*_0\) space and \(x, y\) be any two distinct points in \(X\), there exists an \(\tilde{sp}_c\)-open set \((F, E)\) containing one of them say \(x\). Since \((F, E)\) is an \(\tilde{sp}_c\)-open set, there exists a soft closed set \((K, E)\) such that \(x \notin (K, E) \subseteq (F, E)\), so \(X\setminus(K, E)\) is a soft open set containing \(y\) but not \(x\). Therefore, \((X, \tau, E)\) is a soft \(T_0\)-space. The other proof is similar.

The next example shows that the converse of Proposition 4.14 is not true in general.

Example 4.15. In Example 4.8, we can see that \((X, \tau, E)\) is a soft \(T_0\)-space but it is not a \(\tilde{sp}_c - T^*_0\) space.
Proposition 4.16. Let \((X, \tilde{r}, E)\) be a soft topological space and \(x, y \in X\) such that \(x \neq y\). If there exist \(\tilde{sp}_c\)-open sets \((F, E)\) and \((G, E)\) such that \(x \in (F, E), y \in \tilde{X}(F, E)\) and \(y \in (G, E), x \in \tilde{X}(G, E)\), then \((X, \tilde{r}, E)\) is \(\tilde{sp}_c - T^*_1\).

Proof. It is similar to the proof of Proposition 4.10.

Remark 4.17. In Example 4.6, it is easy to see that the converse of Proposition 4.16 is not true in general.

Proposition 4.18. If \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_1\) space and \(x \in X\), then for each \(\tilde{sp}_c\)-open set \((F, E)\) with \(x \in (F, E)\). The following statements are true:

1. \((x, E) \subseteq \cap \{(F, E): x \notin \tilde{sp}_c O(X)\}\).
2. For all \(y \neq x\), we have \(y \notin \cap \{(F, E): x \in (F, E) \notin \tilde{sp}_c O(X)\}\).

Proof.

(1) Since \(x \in \cap \{(F, E): (F, E) \in \tilde{sp}_c O(X)\}\), then by Remark 2.8, it is clear that \((x, E) \subseteq \cap \{(F, E): (F, E) \in \tilde{sp}_c O(X)\}\).

(2) Let \(x \neq y\) for \(x, y \in X\), then there exists an \(\tilde{sp}_c\)-open set \((G, E)\) such that \(x \in (G, E)\) and \(y \notin (G, E)\). This implies that \(y \not\in G(e)\) for some \(e \in E\), and we have \(y \notin \cap \{(F, E): e \in E\}\). Therefore, \(y \notin \cap \{(F, E): x \in (F, E) \notin \tilde{sp}_c O(X)\}\).

Proposition 4.19. Every singleton soft set of a soft topological space \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c\)-closed set, if and only if \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_1\) space.

Proof. Suppose that \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_1\) space and \(x \in X\), we have to show that \((x, E)\) is an \(\tilde{sp}_c\)-closed set or alternatively \(X \setminus (x, E)\) is an \(\tilde{sp}_c\)-open set, let \(y \notin \tilde{X}(x, E)\), then clearly \(x \neq y\), since the space is an \(\tilde{sp}_c - T^*_1\) space, there must exist \(\tilde{sp}_c\)-open sets \((F, E)\) such that \(y \notin (F, E)\) but \(x \notin (F, E)\). Thus, corresponding to each \(y \in \tilde{X}(x, E)\) there exists an \(\tilde{sp}_c\)-open \((F, E)\) such that \(y \in (F, E) \subseteq X \setminus (x, E)\), therefore \(X \setminus (x, E) = \cup (F, E) \subseteq X \setminus (x, E)\). Hence \(X \setminus (x, E) = \cup (F, E)\), since \((F, E)\) is an \(\tilde{sp}_c\)-open set, and the union of an arbitrary collection of \(\tilde{sp}_c\)-open sets is an \(\tilde{sp}_c\)-open set. Therefore, \((x, E)\) is an \(\tilde{sp}_c\)-closed set.

Conversely, suppose that \((x, E)\) is an \(\tilde{sp}_c\)-closed set for each \(x \in X\). Let \(x, y \in X\) such that \(x \neq y\). Now, for \(x \in X\), \(X \setminus (x, E)\) is an \(\tilde{sp}_c\)-open set such that \(y \in X \setminus (x, E)\) and \(y \notin X \setminus (x, E)\). Similarly, \(X \setminus (y, E)\) is an \(\tilde{sp}_c\)-open set containing \(x\) but not \(y\). Thus, \((x, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_1\) space.

Proposition 4.20. If every soft point of a soft topological space \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c\)-closed set, then \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_1\) space.

Proof. Let \(x_e\) be a soft point of \((X, \tilde{r}, E)\) which is an \(\tilde{sp}_c\)-closed. Let \(x, y \in X\) such that \(x \neq y\). Now, for \(x \in X\), \(X \setminus (x_e)\) is an \(\tilde{sp}_c\)-open set such that \(y \in X \setminus (x_e)\) and \(x \notin X \setminus (x_e)\). Similarly, \(X \setminus (y_e)\) is an \(\tilde{sp}_c\)-open set containing \(x\) but not \(y\). Thus, \((x, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_1\) space.

Proposition 4.21. A soft topological space \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_1\) space if and only if \(\tilde{sp}_c D(x, E) = \tilde{D}\), for each \(x \in X\).

Proof. Let \(\tilde{sp}_c D(x, E) = \tilde{D}\), for each \(x \in X\). Since, \(\tilde{sp}_c cl(x, E) = (x, E)\cup \tilde{sp}_c D(x, E)\) and \(\tilde{sp}_c D(x, E) = \tilde{D}\), \(\tilde{sp}_c cl(x, E) = (x, E)\), which implies that \((x, E)\) is \(\tilde{sp}_c\)-closed and hence by Proposition 4.19, \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_1\) space. Conversely, let \((X, \tilde{r}, E)\) be an \(\tilde{sp}_c - T^*_1\) space. If \(\tilde{sp}_c D(x, E) \neq \tilde{D}\), for some \(x \in X\), then there exists a point, say \(y \in (x, E)\), and \(x \neq y\), since \((X, \tilde{r}, E)\) is a \(\tilde{sp}_c - T^*_1\) space, then there exists an \(\tilde{sp}_c\)-open set \((F, E)\) such that \(y \in (F, E)\) and \(x \notin (F, E)\), then \((F, E) \cap (x, E) = \tilde{D}\) and hence \(y \notin \tilde{sp}_c D(x, E)\), which is contradiction. Thus, \(\tilde{sp}_c D(x, E) = \tilde{D}\), for each \(x \in X\).
Remark 4.22. A soft topological space \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_2\) space if and only if \(\tilde{sp}_c \text{cl}(x, E) = (x, E)\), for each \(x \in X\).

Proposition 4.23. Let \((X, \tilde{r}, E)\) be a soft topological space and \(x \in X\). If \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_2\) space, then \((x, E) = \cap \{(F, E) : x \notin \tilde{sp}_c \text{O}(X) \}\).

Proof. Assume that there exist \(y \in X\) with \(x \neq y\) and \(y \notin \cap F(e)\) for some \(e \in E\). Since \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_2\) space, then there exist \(\tilde{sp}_c\)-open sets \((G, E)\) and \((H, E)\) such that \(x \notin (G, E)\) and \(y \notin (H, E)\) with \((G, E) \cap (H, E) = \emptyset\) and so \((G, E) \cap (y, E) = \emptyset\) and \(G(a) \cap y(a) = \emptyset\). This contradicts the fact that \(y \in \cap F(a)\) for some \(e \in E\). Hence proved.

Proposition 4.24. The following statements are equivalent for a soft topological space \((X, \tilde{r}, E)\):
1. \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_2\) space.
2. For each \(x \in X\) and each \(y \neq x\), there exists an \(\tilde{sp}_c\)-open set \((F, E)\) containing \(x\) such that \(y \notin \tilde{sp}_c \text{cl}(F, E)\).
3. For each \(x \in X\), \(\cap \{\tilde{sp}_c \text{cl}(F, E) : x \notin (F, E) \notin \tilde{sp}_c \text{O}(X)\} = (x, E)\).

Proof.

(1) \(\Rightarrow\) (2). Since \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_2\) space, then there exist disjoint \(\tilde{sp}_c\)-open sets \((F, E)\) and \((G, E)\) containing \(x\) and \(y\), respectively. Thus, \((F, E) \subseteq X \setminus (G, E)\). Therefore, \(\tilde{sp}_c \text{cl}(F, E) \subseteq X \setminus (G, E)\). Hence, \(y \notin \tilde{sp}_c \text{cl}(F, E)\).

(2) \(\Rightarrow\) (3). If possible for some \(y \neq x\), we have \(y \notin \tilde{sp}_c \text{cl}(F, E)\) for every \(\tilde{sp}_c\)-open set \((F, E)\) containing \(x\), which contradicts (2).

(3) \(\Rightarrow\) (1). Let \(x, y \in X\) and \(x \neq y\). Then, there exists an \(\tilde{sp}_c\)-open set \((F, E)\) containing \(x\) such that \(y \notin \tilde{sp}_c \text{cl}(F, E)\). Let \((G, E) = X \setminus \tilde{sp}_c \text{cl}(F, E)\), then \(y \notin (G, E)\) and \(x \in (F, E)\) and also \((F, E) \cap (G, E) = \emptyset\). Therefore, \((X, \tilde{r}, E)\) is an \(\tilde{sp}_c - T^*_2\) space.

Proposition 4.25. A soft space \((X, \tilde{r}, E)\) is \(\tilde{sp}_c - T^*_1\) if for each pair of distinct points \(x, y \in X\), there exists an \(\tilde{sp}_c\)-clopen set \((F, E)\) containing one of them but not the other.

Proof. Let for each pair of distinct points \(x, y \in X\), there exists an \(\tilde{sp}_c\)-clopen set \((F, E)\) containing \(x\) but not \(y\), which implies that \(X \setminus (F, E)\) is also an \(\tilde{sp}_c\)-open set and \(y \notin X \setminus (F, E)\), since \((F, E) \cap X \setminus (F, E) = \emptyset\). \(X\) is an \(\tilde{sp}_c - T^*_2\) space.

5 Conclusion

In the last decade, the concept of soft topological spaces has been introduced. After that, several topological concepts are extended to the soft set theory, new definitions, new classes of soft sets and properties for soft continuous mappings between different classes of soft sets are introduced and studied we refer to [14–16,26,27]. Also, many types of soft separation axioms are investigated [17,18,28,29]. This paper continues the study of some strong types of soft separation axioms. In Sections 3 and 4, we present the notion of soft \(p_c - T^*_1\) and soft \(p_c - T^*_2\) spaces for \(i = 0, 1, 2\) and we get several characterizations and properties of these spaces. Also, we discuss the relationship among these spaces and other existing soft separation axioms.

References

[1] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999), 19–31.
[2] A. Ghareeb, Soft weak Baire spaces, J. Egyptian Math. Soc. 26 (2018), no. 3, 395–405, DOI: 10.21608/joems.2018.2729.1028.
[3] A. H. Zakari, A. Ghareeb, and S. Omran, On soft weak structures, Soft Comput. 21 (2017), 2553–2559, DOI: 10.1007/s00500-016-2136-8.

[4] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011), 1786–1799, DOI: 10.1016/j.cam.2011.02.006.

[5] F.-G. Shi and B. Pang, A note on soft topological spaces, Iran. J. Fuzzy Syst. 12 (2015), no. 5, 169–155.

[6] D. N. Georgiou, A. C. Megaritis, and V. I. Petropoulos, On soft topological spaces, Appl. Math. Inf. Sci. 7 (2013), no. 2, 1889–1901.

[7] I. Zorlutuna, M. Akdag, W. K. Min, and S. Atmaca, Remarks on soft topological spaces, Ann. Fuzzy Math. Inform. 3 (2012), no. 2, 171–185.

[8] I. Zorlutuna and H. Cakir, On continuity of soft mappings, Appl. Math. Inf. Sci. 9 (2015), no. 1, 403–409.

[9] S. Hussain and B. Ahmad, Some properties of soft topological spaces, Comput. Math. Appl. 62 (2011), 4058–4067.

[10] S. Hussain and B. Ahmad, Soft separation axioms in soft topological spaces, Hacet. J. Math. Stat. 44 (2015), no. 3, 559–568.

[11] S. Bayramov and C. G. Aras, A new approach to separability and compactness in soft topological spaces, TWMS J. Pure Appl. Math. 9 (2018), no. 1, 82–93.

[12] M. E. El-Shafei, M. Abo-Elhamayel, and T. M. Al-Shami, Partial soft separation axioms and soft compact spaces, Filomat 32 (2018), no. 13, 4755–4771.

[13] Q. H. Hamko and N. K. Ahmed, Characterizations of $\mathcal{S}_p\alpha$-open sets and $\mathcal{S}_p\alpha$-almost continuous mapping in soft topological spaces, Eurasian J. Sci. Eng. 4 (2018), no. 2, 192–209.

[14] T. M. Al-shami and L. D. R. Kocinac, The equivalence between the enriched and extended soft topologies, Appl. Comput. Math. 18 (2019), no. 2, 149–162.

[15] T. M. Al-shami, M. E. El-Shafei, and M. Abo-Elhamayel, On soft topological ordered spaces, J. King Saud Univ. Sci. 31 (2019), no. 4, 556–566, DOI: 10.1016/j.jksus.2018.06.005.

[16] T. M. Al-shami and M. E. El-Shafei, On supra soft topological ordered spaces, Arab J. Basic Appl. Sci. 26 (2019), no. 1, 433–445.

[17] T. M. Al-shami and M. E. El-Shafei, Two new types of separation axioms on supra soft separation spaces, Demonstr. Math. 52 (2019), no. 1, 147–165.

[18] T. M. Al-shami and M. E. El-Shafei, Partial belong relation on soft separation axioms and decision making problem: two birds with one stone, Soft Comput. 24 (2019), 5377–5387.

[19] T. M. Al-shami, Comments on “soft mappings spaces”, Sci. World J. 2019 (2019), 2, Article ID 6903809, DOI: 10.1155/2019/6903809.

[20] P. K. Maji, R. Biswas, and R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003), 555–562.

[21] M. Akdag and A. Ozkan, On soft preopen sets and soft pre-separation axioms, Gazi Univ. J. Sci. 27 (2014), no. 4, 1077–1083.

[22] G. Ilango and M. Ravindran, On soft pre-open sets in soft topological spaces, Int. J. Math. Research 5 (2013), no 4, 399–409.

[23] M. Ravindran and G. Ilango, A note on soft pre-pen sets, Int. J. Pure Appl. Math. 106 (2016), no. 5, 63–78.

[24] N. K. Ahmed and Q. H. Hamko, $\mathcal{S}_p\alpha$-open sets and $\mathcal{S}_p\alpha$-continuity in soft topological spaces, ZANCO J. Pure Appl. Sci. 30 (2017), no. 6, 72–84.

[25] O. Gocur and A. Kopuzlu, On soft separation axioms, Ann. Fuzzy Math. Inform. 9 (2015), no. 5, 817–822.

[26] T. M. Al-shami, Corrigendum to “On soft topological space via semi-open and semi-closed soft sets”, Kyungpook Math. J. 54 (2014), 221–236”, Kyungpook Math. J. 58 (2018), no. 3, 583–588.

[27] T. M. Al-shami, Investigation and corrigendum to some results related to g-soft equality and gf-soft equality relations, Filomat 33 (2019), no. 11, 3375–3383.

[28] M. E. El-Shafei, M. Abo-Elhamayel, and T. M. Al-shami, Two notes on “On soft Hausdorff spaces”, Ann. Fuzzy Math. Inf. 16 (2018), no. 3, 333–336.

[29] T. M. Al-shami, Corrigendum to “Separation axioms on soft topological spaces, Ann. Fuzzy Math. Inf. 11 (2016), no. 4, 511–525”, Ann. Fuzzy Math. Inf. 15 (2018), no. 3, 309–312.