GOOD LAMBDA INEQUALITIES FOR NON-DOUBLING MEASURES IN $\mathbb{R}^n$

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Abstract. We establish a good lambda inequality relating to the distribution function of Riesz potential and fractional maximal function on $(\mathbb{R}^n, d\mu)$ where $\mu$ is a positive Radon measure which doesn’t necessarily satisfy a doubling condition. This is extended to weights $w$ in $A_\infty(\mu)$ associated to the measure $\mu$. We also derive potential inequalities as an application.

1. INTRODUCTION

In this paper we will discuss the good-$\lambda$ inequality for Riesz potentials associated to a Radon measure not necessarily doubling but satisfies only a mild condition, which we call a growth condition in $\mathbb{R}^n$.

Definition 1.1. (See [17]) Let $(X, d)$ be a metric measure space equipped with a metric $d$ and a Radon Measure $\mu$. The Riesz potential operator of order $\alpha$ on $X$ is given by

\begin{equation}
I\alpha f(x) = \int_X \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}}, x \in X
\end{equation}

where $N$ is a fixed positive integer and $0 < \alpha < N$.

Definition 1.2. (See [10]) A Borel measure $\mu$ on a measure metric space $(X, d)$ is said to satisfy the growth condition if

\begin{equation}
\mu(B(x,r)) \leq Cr^N
\end{equation}

where the constant $C$ is independent of $x$ and $r$. This allows, in particular, non-doubling measures.

Definition 1.3. (See [10]) A measure $\mu$ on a measure metric space $(X, d)$ is said to satisfy the so-called “doubling condition” if there exists a constant $C = C(\mu) \geq 1$, such that, for every ball $B(x, r)$ of center $x$ and radius $r$

\begin{equation}
\mu(B(x, 2r)) \leq C \mu(B(x, r))
\end{equation}

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Definition 1.4. A pair of non-negative measurable functions \( f \) and \( g \) defined on \( \mathbb{R} \) are said to satisfy a good-\( \lambda \) inequality if there exists a constant \( K > 1, 0 < \epsilon_0 \leq 1 \) such that for every \( \lambda > 0 \)

\[
|\{x \in \mathbb{R} : f(x) > K\lambda, g(x) < \epsilon_0 \lambda\}| \leq b(\epsilon)\{x \in \mathbb{R} : f(x) > \lambda\} \tag{1.4}
\]

where \( b(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

In their ground-breaking paper [1], Burkholder and Gundy showed that the random variables \( X^* \) and \( S(X) \) satisfy certain inequalities relating to their distributions. These are now commonly called good-\( \lambda \) inequalities. In the harmonic function setting the first good-\( \lambda \) inequalities were also proved by Burkholder and Gundy [2]. They were subsequently improved and refined by, among others, Burkholder [6], Dahlberg [11], Fifferman, Gundy, Silverstein and Stein [12]. The variations of good-\( \lambda \) inequalities and its applications can be found in the work of Benjamin Muckenhoupt and Richard L. Wheeden (see [14]), S. D. Jaka (see [3]), D. L. Burkholder (see [4]), Rodrigo Bañuelos (see [7]), and Richard F. Bass (see [8]). A fair amount of deal about such inequalities is found in the book by Rodrigo Bañuelos and Charles N. Moore (see [9]). Let us recall the classical good lambda inequalities of Burkholder and Gundy [1] for continuous time martingales.

Theorem 1.1. Let \( X_t \) be a continuous time martingale with maximal function \( X^* \) and square function \( S(X) \). Then for all \( 0 < \epsilon < 1, \delta > 0 \) and \( \lambda > 0 \),

\[
P\{X^* > \delta \lambda, S(X) \leq \epsilon \lambda\} \leq \frac{\epsilon^2}{(\delta - 1)^2}P\{X^* > \lambda\}
\]

and

\[
P\{S(X) > \delta \lambda, X^* \leq \epsilon \lambda\} \leq \frac{\epsilon^2}{(\delta - 1)^2}P\{S(X) > \lambda\}
\]

As they are expressed here, these are actually a refinement, due to Burkholder (see [5]), of the inequalities of [1]. The usefulness of such inequalities is already amply demonstrated by the following lemma, which is but one of the many applications of these type of inequalities. For this lemma we consider a non-decreasing function \( \Phi \) defined on \([0, \infty]\) with \( \Phi(0) = 0 \), \( \Phi \) is not identically 0, and which satisfies the condition \( \Phi(2\lambda) \leq c\Phi(\lambda) \) for every \( \lambda > 0 \), where \( c \) is a fixed constant. This lemma is from [5].

Lemma 1.1. Suppose that \( f \) and \( g \) are nonnegative measurable functions on a measurable space \((\mathcal{Y}, \mathcal{A}, \mu)\), and \( \delta > 0, 0 < \epsilon < 1, \) and \( 0 < \gamma < 1 \) are real numbers such that

\[
\mu\{g > \delta \lambda, f \leq \epsilon \lambda\} \leq \gamma \mu\{g > \lambda\}
\]

for every \( \lambda > 0 \). Let \( \rho \) and \( \nu \) be real numbers which satisfy

\[
\Phi(\delta \lambda) \leq \rho \Phi(\lambda), \quad \Phi(\epsilon^{-1} \lambda) \leq \nu \Phi(\lambda)
\]

for every \( \lambda > 0 \). Finally, suppose \( \rho \gamma < 1 \) and \( \int_{\mathcal{Y}} \Phi(\min\{1, g\})d\mu < \infty \). Then

\[
\int_{\mathcal{Y}} \Phi(g)d\mu \leq \frac{\rho \nu}{1 - \rho \gamma} \int_{\mathcal{Y}} \Phi(f)d\mu.
\]
We need the following definition for our purpose:

**Definition 1.1.** Let $(\mathbb{X}, d)$ be a metric space and $\mu$ be any measure on $\mathbb{X}$. Let $f$ be a locally integrable function on $\mathbb{X}$. The maximal function of $f$, $M(f)$, is defined by

$$M(f)(x) = \sup_{r > 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y)| d\mu(y)$$

The operator $M$ is called Hardy-Littlewood maximal operator. The fractional maximal function of $f$ is defined for $0 < \alpha < N$ by

$$M_\alpha(x) = \sup_{r > 0} \frac{1}{\mu(B(x, r))^{N-\alpha/N}} \int_{B(x, r)} |f(y)| d\mu(y).$$

Muckenhoupt and Wheeden in [9] proved the following version of good $\lambda$-inequality in 1972.

**Theorem 1.2.** Let $\mu$ be a positive Radon measure in $\mathbb{R}^n$. Then there exists $a > 1$, $b > 0$ such that for every $\lambda > 0$ and for every $\epsilon, 0 < \epsilon \leq 1$

$$\mu(\{x : I_\alpha \mu(x) > a\lambda\}) \leq b\epsilon^{n/(n-\alpha)} \mu(\{x : I_\alpha \mu(x) > \lambda\}) + \mu(\{x : M_\alpha f(x) > \epsilon\lambda\}).$$

where

$$I_\alpha \mu(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x - y|^{n-\alpha}}$$

is the Riesz potential of the measure $\mu$.

As a special case, the above theorem is still true if $d\mu(y)$ is replaced by $f(y)dm(y)$ where $f$ is a measurable function and $m$ is the Lebesgue measure in $\mathbb{R}^n$. Note that Lebesgue measure is doubling. In general, the above inequality is true for any positive Radon measure which is doubling. The following theorem reveals this.

**Theorem 1.3.** Let $0 < \alpha < N$, $\frac{1}{q} = 1 - \frac{\alpha}{N}$, and $I_\alpha(x) = \int_{\mathbb{R}^n} \frac{f(y)dm(y)}{|x - y|^{n-\alpha}}$ where $\mu$ is a positive Radon measure satisfying the doubling condition (1.3) and $f$ is a measurable function on $\mathbb{R}^n$. Then there exist constants $a > 1$, $b > 0$ such that for every $\epsilon, 0 < \epsilon \leq 1$ and $\lambda > 0$ we have

$$\mu(\{x : I_\alpha f(x) > a\lambda\}) \leq b\epsilon^{N/N-\alpha} \mu(\{x : I_\alpha > \lambda\}) + \mu(\{x : M_\alpha f(x) > \epsilon\lambda\}).$$

Our main objective in this paper is to establish the above inequality for a Radon measure $\mu$ which is not necessarily doubling but only satisfies a mild condition that we defined as “growth condition” in (1.2).

2. USEFUL REMARKS AND RESULTS

In this section, we will mention some useful remarks and results in order to use them for our main result. We denote by $Q$ a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes and by $\ell(Q)$ the side length of $Q$. Xavier Tolsa in [16] makes the following remarks.
Remark 2.1. If \( \mu \) satisfies the growth condition (1.2) then there are lots of big doubling cubes. Precisely speaking, given any point \( x \in \text{supp}(\mu) \) and \( c > 0 \), there exists some \( (\alpha, \beta) \)-doubling cube \( Q \) (that is, \( \mu(\alpha Q) \leq \beta \mu(Q) \) where \( \alpha > 1 \) and \( \beta > \alpha^n \)) centered at \( x \) with \( \ell(Q) \geq c \).

Note that if \( \alpha, \beta \) are not specified then by a doubling cube we will mean a \((2, \beta)\)-doubling cube where \( \beta > 2^n \).

Remark 2.2. There are always small doubling cubes for a Radon measure \( \mu \). That is, for \( \mu \)-a.e. \( x \in \mathbb{R}^n \) there is a sequence of \( (\alpha, \beta) \)-doubling cubes \( \{Q_j\}_j^n \) centered at \( x \) such that \( \ell(Q_j) \to 0 \) as \( j \to \infty \).

José García-Cuerva and A. Eduardo Gatto proved the following theorem in 2003 in [10].

Theorem 2.1. Let \( (\mathbb{X}, d, \mu) \) be a metric measure space where the measure \( \mu \) is Borel which satisfies the growth condition (1.2). Then, for \( 1 \leq p < n \) and \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \), we have

\[
\mu \left( \{ x \in \mathbb{X} : |I_\alpha f(x)| > \lambda \} \right) \leq \left( \frac{C||f||_{L^p(\mu)}}{\lambda} \right)^q,
\]

that is, \( I_\alpha \) is a bounded operator from \( L^p(\mu) \) into the Lorentz space \( L^{q, \infty}(\mu) \).

The above theorem establishes the weak type estimate for the Riesz potentials in a metric space associated to Borel measure \( \mu \) for \( 1 \leq p < \infty \).

3. Main Results

In this section, we will prove the good-\( \lambda \) inequality for Riesz potentials in \( \mathbb{R}^n \) equipped with a measure not necessarily doubling but satisfies only the mild condition called the “growth condition” (1.2).

Theorem 3.1. Let \( 0 < \alpha < N \). The Riesz potential operator of order \( \alpha \) on \( \mathbb{R}^n \) is given by

\[
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{d(x,y)^{N-\alpha}} , x \in \mathbb{R}^n
\]

for measurable function \( f \) on \( \mathbb{R}^n \). Let \( \mu \) be a positive Radon measure on \( \mathbb{R}^n \) satisfying the growth condition (1.2). Then there exists constants \( k \geq 1 \) and \( C \) such that for every \( \lambda > 0 \) and \( \epsilon, 0 < \epsilon \leq 1 \),

\[
(3.1) \quad \mu \left( \{ x : I_\alpha f(x) > k\lambda, M_\alpha f(x) \leq \epsilon \lambda \} \right) \leq C \epsilon^{\frac{N}{\alpha}} \mu \left( \{ x : I_\alpha f(x) > \lambda \} \right).
\]

Proof. Let \( E_\lambda = \{ x \in \mathbb{R}^n : I_\alpha f(x) > \lambda \} \). Then \( E_\lambda \) is open. So it has the following Whitney decomposition (see appendix J in [13]).

There exists a countable family of dyadic cubes \( \{Q_j\} \) such that

(i) \( E_\lambda = \bigcup_j Q_j \) where \( Q_j \)’s have disjoint interiors.
(ii) \( diam(Q_j) \leq dist(Q_j, E_0^\lambda) \leq 4diam(Q_j) \), for every \( j \). That is,

\[
\sqrt{n} \ell(Q) \leq dist(Q_j, E_0^\lambda) \leq 4\sqrt{n} \ell(Q).
\]
(iii) If the boundaries of two cubes $Q_j$ and $Q_k$ touch, then
\[ \frac{1}{4} \leq \frac{\ell(Q_j)}{\ell(Q_k)} \leq 4. \]

(iv) For a given $Q_j$, there exists at most $12^n$ $Q_k$’s that touch it.

(v) For every $0 < \delta < \frac{1}{4}$, $(1 + \delta)Q \subseteq E_{\lambda}$, and $\sum_Q \chi_{(1+\delta)Q}(x) \leq 12^n \chi_{E_{\lambda}}(x)$ for every $x \in E_{\lambda}$.

This implies that
\[ \sum_Q \mu((1 + \delta)Q)(x) \leq 12^n \mu(E_{\lambda}). \]

Fix $x \in E_{\lambda}$ and a $Q$ in $\{Q_j\}_1^\infty$ with $x \in Q$. Let $k > 1$ and consider the set $\{x \in Q : l(x) > k\lambda \}$. Suppose $Q \cap \{M_{\alpha}f(x) \leq \epsilon \lambda\} \neq \emptyset$. Let $Q_x$ be the cube center at $x$ and $\ell(Q_x) = 2^m$. We may fix $\delta = \frac{1}{8}$. Then $Q_x \subseteq (1 + \delta)Q$ and $16Q_x \geq 2Q_x$. Consider $\frac{1}{2^j}Q_x, j = 1, 2, 3, \ldots$ Take the first which is doubling and denote it by $\hat{Q}_x := \frac{1}{2^j}Q_x$, where $m$ depends on $x$. Thus, $\hat{Q}_x \subseteq 2^{j+1}Q_x$.

(a) For every $x \in E_{\lambda} \cap \{M_{\alpha}f(x) \leq \epsilon \lambda\}$ there exists a doubling cube $\hat{Q}_x$ such that $\mu(2\hat{Q}_x) \leq \beta \mu(Q_x)$ where $\beta > 2^n$. We may assume $\beta = 2^{n+\epsilon}$. (b) $\mu(2\frac{1}{2^j}Q_x) \geq \beta \mu(\frac{1}{2^j}Q_x)$ for every $j < m$.

Therefore,
\[ \{x \in Q : l(x) > k\lambda, M_{\alpha}f(x) < \epsilon \lambda\} \subseteq \bigcup_{x \in Q} \hat{Q}_x. \]

The Besicovitch Covering Lemma implies that there exists $\hat{Q}_{x_j}, j = 1, 2, 3, \ldots$ such that
\[ \{x \in Q : l(x) > k\lambda, M_{\alpha}f(x) < \epsilon \lambda\} \subseteq \bigcup_j \hat{Q}_{x_j} \]
where
\[ \sum_j \chi_{\hat{Q}_{x_j}} \leq 4^n \chi_{(1+\delta)Q}. \]

That is,
\[ \mu \left( \bigcup_j \hat{Q}_{x_j} \right) \leq \mu \left( (1 + \delta)Q \right). \]

Let $f = f_1 + f_2$ where $f_1 = \chi_{2Q}f$ and $f_2 = f - f_1$. Then,
\[ \{x \in Q : l(x) > k\lambda, M_{\alpha}f(x) < \epsilon \lambda\} \leq \bigcup_j \{x \in \hat{Q}_{x_j} : l(x) > k\lambda, M_{\alpha}f(x) < \epsilon \lambda\}. \]
Fix a doubling cube $\tilde{Q}_{x_j}$. Write $f_1 = \chi_{2\tilde{Q}_{x_j}}f_1 + \chi_{(2\tilde{Q}_{x_j})^c} f_1 = f_{11} + f_{12}$. Then, by using the weak type inequality (see theorem (2.1), [10]), we obtain

$$\mu(\{x \in 2\tilde{Q}_{x_j} : |I_\alpha f_{11}| > k\lambda\}) \leq \left(\frac{1}{k\lambda} \|f_{11}\|_1\right)^{N/N-\alpha}$$

$$= \left(\frac{1}{k\lambda} \int_{2\tilde{Q}_{x_j}} |f_1(x)| d\mu(x)\right)^{N/N-\alpha}$$

$$= \left(\frac{1}{k\lambda} \mu(2\tilde{Q}_{x_j})^{\frac{N-\alpha}{N}} \frac{1}{\mu(2\tilde{Q}_{x_j})^{\frac{N-\alpha}{N}}} \int_{2\tilde{Q}_{x_j}} |f_1(x)| d\mu(x)\right)^{N/N-\alpha}$$

$$= \left(\frac{1}{k\lambda} \right)^{\frac{N-\alpha}{N}} \mu(2\tilde{Q}_{x_j})^{\frac{N-\alpha}{N}} M_\alpha f_1(x_j)^{N/N-\alpha}$$

$$\leq C \epsilon^{N/N-\alpha} \mu(\tilde{Q}_{x_j}).$$

Note that $\mu(Q) \leq c \ell(Q)^N$ for any cube $Q$. From this, it follows that

$$\left(\frac{c}{\mu(Q)}\right)^{\frac{N-\alpha}{N}} \geq \frac{1}{\ell(Q)^{N-\alpha}}.$$

Also,

$$\mu(Q_{x_j}) \geq \beta \mu\left(\frac{1}{2}Q_{x_j}\right) \geq \ldots \geq \beta^i \mu\left(\frac{1}{2^i}Q_{x_j}\right)$$

where $i < m$. This follows by the choice of our $Q_{x_j}$'s. So, if $x \in \tilde{Q}_{x_j}$
\[ I_\alpha f_{12}(x) = \int_{(2Q_x)^c} \frac{f_1(y)}{d(x, y)^{N-\alpha}} \, d\mu(y) \]

\[ \leq \sum_{k=0}^{m-1} \int_{\frac{Q_x}{2^k} \setminus \frac{2Q_x}{2^k+1}} \frac{|f_1(y)|}{d(x, y)^{N-\alpha}} \, d\mu(y) + \int_{2Q_x \setminus 2Q} \frac{|f_1(y)|}{d(x, y)^{N-\alpha}} \, d\mu(y) \]

\[ \leq \sum_{k=0}^{m-1} \frac{C}{(2Q_x)^{N-\alpha}} \int_{\frac{Q_x}{2^k} \setminus \frac{2Q_x}{2^k+1}} |f_1(y)| \, d\mu(y) + C \frac{\mu(16Q_x) (N-\alpha)}{\mu(16Q_x) (N-\alpha)} \int_{2Q_x \setminus 2Q} f_1(y) \, d\mu(y) \]

\[ \leq \sum_{k=0}^{m-1} C 2^{k(N-\alpha)} \frac{1}{\mu(Q_x)^{N-\alpha}} \int_{\frac{Q_x}{2^k} \setminus \frac{2Q_x}{2^k+1}} |f_1(y)| \, d\mu(y) + C M_\alpha f_1(x_j) \]

\[ \leq \sum_{k=0}^{m-1} C 2^{k(N-\alpha)} \frac{1}{\beta \mu(Q_x)^{N-\alpha}} \int_{\frac{Q_x}{2^k} \setminus \frac{2Q_x}{2^k+1}} |f_1(y)| \, d\mu(y) + C \lambda \]

\[ = C \sum_{k=0}^{m-1} \left( \frac{2^{N-\alpha}}{2^{N-\alpha} 2^k(1/2)} \right)^k M_\alpha f_1(x_j) + C \lambda \]

\[ = C \sum_{k=0}^{m-1} 2^{-c(\frac{N-\alpha}{2})} M_\alpha f_1(x_j) + C \lambda \]

\[ = C N_\alpha M_\alpha f_1(x_j) \leq C N_\alpha = C \lambda. \]

Note that the constant C in different occurrences above are not necessarily the same. Therefore

\[ I_\alpha f_{12}(x) \leq C \lambda \]

for every \( x \in \hat{Q}_{x_j} \). We now estimate \( I_\alpha f_2 \). Fix a point \( x \in Q \). Consider the ball \( B = B(x, 6diam(Q)) \supseteq 2Q \). Let \( x_0 \in B \cap E_\lambda^c \) such that

\[ diam(Q) \leq dist(Q, x_0) \leq 4diam(Q). \]

Then for any \( y \in (2Q)^c \)
\[ d(x_0, y) \leq d(x, x_0) + d(x, y) \]
\[ \leq C\text{diam}(Q) + d(x, y) \]
\[ \leq C\text{dist}(Q, E_\lambda) + d(x, y) \]
\[ \leq C d(x, y) + d(x, y). \]

Thus
\[ d(x_0, y) \leq C d(x, y) \]
for \( x_0 \in B \cap E_\lambda^c \) and \( x \in Q \). Therefore,
\[ I_\alpha f_2(x) = \int_{(2Q)^c} \frac{f(y) d\mu(y)}{d(x, y)^{N-\alpha}} \leq CI_\alpha f(x_0) < C\lambda. \]

Finally, summing over all Whitney cubes yields the inequality. Indeed,
\[
\begin{align*}
\mu \left( \{ x : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda \} \right) \\
= \sum_Q \mu \left( \{ x \in Q : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda \} \right) \\
\leq \sum_Q \mu \left( \{ x \in Q : I_\alpha f_1 > (k - C)\lambda, M_\alpha f < \epsilon\lambda \} \right) \\
= \sum_Q \sum_j \mu \left( \{ x \in \hat{Q}_x^i : I_\alpha f_1 > (k - C)\lambda, M_\alpha f < \epsilon\lambda \} \right) \\
< \sum_Q \sum_j \mu \left( \{ x \in \hat{Q}_x^i : I_\alpha f_11 > (k - 2C)\lambda, M_\alpha f < \epsilon\lambda \} \right) \\
\leq \sum_Q \sum_j C e^{N-\alpha} \mu(\hat{Q}_x^i) \\
\leq \sum_Q C e^{N-\alpha} 4^n \mu((1 + \delta)Q) \\
= 4^n 12^n C e^{N-\alpha} \mu(\{ I_\alpha f > \lambda \}).
\end{align*}
\]

Thus, finally we have
\[
\begin{align*}
\mu \left( \{ x : I_\alpha f > k\lambda \} \right) &\leq \mu \left( \{ I_\alpha f > k\lambda, M_\alpha f \leq \epsilon\lambda \} \right) + \mu \left( \{ M_\alpha f > \epsilon\lambda \} \right) \\
&\leq C e^{N-\alpha} \mu(\{ I_\alpha f > \lambda \}) + \mu(\{ M_\alpha f > \epsilon\lambda \})
\end{align*}
\]
where the constant \( C = C(n, N, \alpha) \). This completes the proof of the theorem. \( \square \)
4. Good-\(\lambda\) inequality for weights

Next, we extend this inequality for the weights \(w \in A_p(\mu)\) where \(\mu\) satisfies the growth condition (1.2). That is for any measurable set \(E\),

\[
w(E) = \int_E w(x) d\mu(x),
\]

where \(\mu\) is a positive Radon measure which satisfies the growth condition (1.2). Details about this type of weights can be obtained from the paper [15] by Joan Orobitg and Carlos Pérez. \(A_\infty(\mu)\) is defined as \(A_\infty(\mu) = \bigcup_{p>1} A_p(\mu)\) in a classical way.

**Theorem 4.1.** Let \(w \in A_\infty(\mu), \alpha > 0, \text{ and } 0 < \lambda < \infty\). Then there exists a positive constant \(a \geq 1\) for which, for every \(\eta > 0\), there is an \(\delta, 0 < \delta \leq 1\), such that the inequality

\[
w(\{x \in \mathbb{R}^n : I_\alpha f(x) > a\lambda\}) \leq \eta w(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}) + w(\{x \in \mathbb{R}^n : M_\alpha f(x) > \epsilon\lambda\})
\]

holds for every \(\lambda > 0\).

**Proof.** Let \(E_\lambda = \{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}, \lambda > 0\). Then \(E_\lambda\) is open because \(I_\alpha\) is lower semi-continuous. Then there exists a family of dyadic cubes \(\{Q_j\}\), called Whitney cubes, such that \(E_\lambda = \bigcup_j Q_j\) and

\[
diam(Q_j) \leq dist(Q_j, E_\lambda) \leq 4diam(Q_j).
\]

Because \(w \in A_\infty(\mu)\), it follows that, for every \(\eta > 0\), there is a \(\delta\) such that if \(Q\) is a cube and \(E\) is a measurable subset of \(Q\) then there is a constant \(C_0\) such that

\[
\frac{w(E)}{w(Q)} \leq C_0 \left( \frac{\mu(E)}{\mu(Q)} \right)^\delta.
\]

(see [15]). That is for every \(Q \in \{Q_j\}\),

\[
\frac{w(\{x \in Q : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\})}{w(Q)} \leq C_0 \left( \mu(\{x \in Q : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\}) \right)^\delta \leq C_0 \left( C_1 \epsilon^{N/N-\alpha} \right)^\delta.
\]

This implies that

\[
w(\{x \in Q : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\}) \leq C_\epsilon^{(N/N-\alpha)\delta} w(Q).
\]

It then follows that

\[
w(\{x \in Q : I_\alpha(x) > k\lambda\}) \leq C_\epsilon^{(N/N-\alpha)\delta} w(Q) + w(\{x \in Q : M_\alpha(x) > \epsilon\lambda\}).
\]

The theorem now follows from summing over all \(Q \in \{Q_j\}\). \(\Box\)
. We observe that the above good lambda inequality is true for the weights \( w \in A_\infty \) associated to a measure \( \mu \) which is doubling. This is because the good-\( \lambda \) inequality holds for doubling measure as well (see theorem (1.3)).

5. Applications

In this section we will provide some applications of our main results. Basically, we will derive the norm inequalities for fractional integrals and maximal functions. Note that the associated measure \( \mu \) that follows in this section satisfies either the growth condition (1.2) or the doubling condition (1.3). This is because our main results are true in either case. Here follows the two theorems as an application of our main results.

**Theorem 5.1.** Let \( 1 < p < \infty \) and \( 0 < \alpha < N \). Then there is a constant \( A \) such that for any measurable function \( f \) on \( \mathbb{R}^n \) and a Radon measure \( \mu \) satisfying either the growth condition (1.2) or the doubling condition (1.3), we have

\[
\| I_\alpha f \|_p \leq A \| M_\alpha f \|_p.
\]

**Proof.** We assume that \( \mu \) has compact support. The right hand inequality is a consequence of "good lambda inequality" (3.1). Multiplying the good lambda inequality (3.1) by \( \lambda^{p-1} \), we obtain for any positive \( R \),

\[
\int_0^R \mu(\{ x : I_\alpha f(x) > k\lambda \}) \lambda^{p-1} d\lambda \\
\leq b e^{N/N - \alpha} \int_0^R \mu(\{ x : I_\alpha f(x) > \lambda \}) \lambda^{p-1} d\lambda \\
+ \int_0^R \mu(\{ x : M_\alpha f(x) > \epsilon \lambda \}) \lambda^{p-1} d\lambda.
\]

After changing variables we obtain,

\[
k^{-p} \int_0^{kR} \mu(\{ I_\alpha f(x) > \lambda \}) \lambda^{p-1} d\lambda \\
\leq \frac{1}{2} k^{-p} \int_0^{kR} \mu(\{ I_\alpha f(x) > \lambda \}) \lambda^{p-1} d\mu + \epsilon^{-p} \int_0^{kR} \mu(\{ M_\alpha f(x) > \lambda \}) \lambda^{p-1} d\lambda.
\]

That is,

\[
k^{-p} \int_0^{kR} \mu(\{ I_\alpha f(x) > \lambda \}) \lambda^{p-1} d\lambda \leq 2 \epsilon^{-p} \int_0^{kR} \mu(\{ M_\alpha f(x) > \lambda \}) \lambda^{p-1} d\lambda.
\]

Letting \( R \to \infty \) and using the definition

\[
\int_X |f|^p d\mu = p \int_0^\infty t^{p-1} \mu(\{ x : f(x) > t \}) dt
\]

we obtain,
k^{-p} \int_X |I_\alpha f(x)|^p d\mu(x) \leq 2\epsilon^{-p} \int_X |M_\alpha f(x)|^p d\mu

This yields the right hand inequality. If \( \mu \) doesn’t have compact support, we let \( \mu_n \) be the restriction of \( \mu \) to the ball \( B(x_0, n) \) for \( n = 1, 2, 3, \ldots \) where \( x_0 \) is some point in \( X \). This yields \( \|I_\alpha f\|_{L^p(X, \mu_n)} \leq A \|M_\alpha f\|_{L^p(X, \mu_n)} \) for all \( n \), where \( A \) is independent of \( n \). Then the theorem follows by Monotone Convergence Theorem.

\[ \square \]

**Theorem 5.2.** Let \( \mu \) be a positive radon measure satisfying either the doubling condition (1.3) or the growth condition (1.2), \( w \in A_\infty(\mu) \), \( 0 < \alpha < N \), and let \( 0 < p < \infty \). Then there exists a positive constant \( C \), that only depends on \( N, p, \) and \( A_\infty(\mu) \) constants of \( w \), such that

\[
\int_{\mathbb{R}^N} |I_\alpha f|^p w d\mu \leq C \int_{\mathbb{R}^N} (M_\alpha)^p w d\mu
\]

for every measurable function \( f \).

**Proof.** Without loss of generality, we may assume that \( f \geq 0 \). We multiply the inequality (4.1) by \( \lambda^{p-1} \) and integrate from 0 to \( R (R > 0) \) with respect to \( \lambda \) to obtain

\[
\int_0^R w \left( \{ x \in \mathbb{R}^N : I_\alpha f(x) > a\lambda \} \right) \lambda^{p-1} d\lambda \leq \eta \int_0^R w \left( \{ x \in \mathbb{R}^N : I_\alpha f(x) > \lambda \} \right) \lambda^{p-1} d\lambda + \int_0^R w \left( \{ x \in \mathbb{R}^N : M_\alpha f(x) > \epsilon \lambda \} \right) \lambda^{p-1} d\lambda.
\]

Applying change of variable and \( a > 1 \) yields

\[
a^{-p} \int_0^{aR} w(\{ x \in \mathbb{R}^N : I_\alpha f(x) > a\lambda \}) \lambda^{p-1} d\lambda \leq \eta \int_0^{aR} w \left( \{ x \in \mathbb{R}^N : I_\alpha f(x) > \lambda \} \right) \lambda^{p-1} d\lambda + \epsilon^{-p} \int_0^{aR} w \left( \{ x \in \mathbb{R}^N : M_\alpha f(x) > \epsilon \lambda \} \right) \lambda^{p-1} d\lambda.
\]

Now we choose \( \eta \leq \frac{1}{2} a^{-p} \). This yields

\[
a^{-p} \int_0^{aR} w(\{ x \in \mathbb{R}^N : I_\alpha f(x) > a\lambda \}) \lambda^{p-1} d\lambda \leq 2\epsilon^{-p} \int_0^{aR} w \left( \{ x \in \mathbb{R}^N : M_\alpha f(x) > \epsilon \lambda \} \right) \lambda^{p-1} d\lambda.
\]

(5.1)
Now let
\[ \chi(x, \lambda) = \begin{cases} 1, & \text{if } I_\alpha f(x) > \lambda > 0; \\ 0, & \text{otherwise} \end{cases} \]
Then,
\[
\int_0^a w \left( \{ x \in \mathbb{R}^N : I_\alpha f(x) > \lambda \} \right) \lambda^{p-1} d\lambda \\
= \int_0^a \left( \int_{\{ x \in \mathbb{R}^N : I_\alpha f(x) > \lambda \}} w(x) d\mu(x) \right) \lambda^{p-1} d\lambda. \\
= \int_0^a \left( \int_{\mathbb{R}^N} (\chi(x, \lambda) w(x)) d\mu(x) \right) \lambda^{p-1} d\lambda. \\
= \int_{\mathbb{R}^N} w(x) \int_0^{\lambda} \lambda^{p-1} d\lambda d\mu(x) \\
= \int_{\mathbb{R}^N} \left( \min\{aR, I_\alpha f(x)\} \right)^p w(x) d\mu(x).
\]
Similarly,
\[
\int_0^a w \left( \{ x \in \mathbb{R}^N : M_\alpha f(x) > \lambda \} \right) \lambda^{p-1} d\lambda = \int_{\mathbb{R}^N} \left( \min\{\epsilon R, M_\alpha f(x)\} \right)^p w(x) d\mu(x).
\]
Therefore,
\[
\int_0^a w \left( \{ x \in \mathbb{R}^N : I_\alpha f(x) > \lambda \} \right) \lambda^{p-1} d\lambda \leq C \int_{\mathbb{R}^N} \left( \min\{aR, I_\alpha f(x)\} \right)^p w(x) d\mu(x).
\]
Using Fatou’s lemma,
\[
\int_{\mathbb{R}^N} (I_\alpha f(x))^p w(x) d\mu(x) = \int_{\mathbb{R}^N} \liminf_{R \to \infty} \left( \min\{aR, I_\alpha f(x)\} \right)^p w(x) d\mu(x) \\
\leq \liminf_{R \to \infty} \int_{\mathbb{R}^N} \left( \min\{aR, I_\alpha f(x)\} \right)^p w(x) d\mu(x) \\
\leq C \liminf_{R \to \infty} \int_{\mathbb{R}^N} \left( \min\{\epsilon R, M_\alpha f(x)\} \right)^p w(x) d\mu(x) \\
\leq \int_{\mathbb{R}^N} (M_\alpha f(x))^p w(x) d\mu(x).
\]

6. Future Motivation

Muckenhoupt and Wheeden in [14] have proved a weighted version of Sobolev imbedding theorem as an application of good-\lambda inequality associated to a Lebesgue measure. Following their footprints, it is expected to obtain the Sobolev imbedding
theorem in more general context associated to a Radon measure which satisfies either the growth condition (1.2) or the doubling condition (1.3). This will open a wide range of spectrum for the future research on this line.

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