Hawking Radiation in the Spacetime of White Holes

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A white hole (WH) is a time-reversed black hole (BH) solution in General relativity with a spacetime region to which cannot be entered from the outside. Recently they have been proposed as a possible solution to the information loss problem [Haggard and Rovelli, 2015]. In particular it has been argued that the quantization of the gravitational field may prevent a BH from collapsing entirely with an exponential decay law associated to the black-hole-to-white-hole (BHWH) tunneling scenario [Barcelo, Carballo, and Garay, 2017]. During this period of BHWH transition the Hawking radiation should take place. Taking this possibility into account, we utilize the Hamilton-Jacobi and Parkih-Wilczek methods to study the Hawking radiation viewed as a quantum tunneling effect to calculate the tunneling rate of vector particles tunneling inside (outside) the horizon of a WH (BH), respectively. We show that there is a Hawking radiation associated to a WH spacetime equal to the BH Hawking temperature when viewed from the outside region of the WH geometry. In the framework of Parkih-Wilczek method, surprisingly, we show that Hawking temperature is affected by the initial radial distance at which the gravitational collapse starts.

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I. INTRODUCTION

Black holes undoubtedly have attracted and continue to attract great interest among the physicists. Even thought they were predicted 100 years ago by Einstein’s theory of relativity, historically, black holes were met with wide-spread scepticism. Today, however, the situation has changed substantially due to the indirect experimental data supporting their existence. In the context of classical general relativity, black holes are thought as a region of spacetime that nothing can escape from inside it—not even light. Forty years ago, Stephen Hawking changed this view. He showed that black holes should radiate particles due to the quantum effects near the event horizon [1–3]. Hawking radiation is a well understood phenomenon which has been extensively studied in the past. One such an interesting method is the quantum tunneling method where Hawking temperature can be found by simply estimating the tunneling rate of particles tunneling outside the black hole [4–39].

Although the mathematics of Hawking radiation is shown to be correct by several methods, it was realized by Hawking himself that a serious conceptual problem arises due to the black hole evaporation known as the information loss problem. To solve this problem, several solutions have been proposed over the years. In a recent work by Haggard and Rovelli [42] a new possibility of releasing the information form the black hole has been proposed. This idea involves a BHWH tunneling scenario and certainly remains as an open possibility which has yet to be achieved by some quantum gravity theory. It is speculated that when gravitational collapse reaches the Planckian scale, a quantum bounce may take place leading to the BHWH tunneling [43]. It is speculated that a WH acts as a long-lived remnant, in this way WHs can be shed light on the black-hole information paradox [44–47]. White holes are of primary phenomenological interest in the analogue gravity, in this context Hawking radiation from dispersive theories in acoustic white holes is studied in Ref. [48], while acoustic white holes in flowing atomic Bose-Einstein condensates has been studied in Ref. [49]. From the astrophysical point of view, high energy radiation from white holes was studied in Ref. [50].

In a very interesting work [51], authors have put forward this idea by considering a path integral approach they found an exponential law associated to the BHWH tunneling probability. However, the framework used by the authors is highly idealized and not free from ambiguity, for example, they have assumed that the spacetime is static before and after the bounce. Yet there is a major problem from the physical point of view concerning the BHWH transition due to the second law of thermodynamics. Namely, as we know, the gravitational collapse can lead to a black hole with an enormous amount of entropy. Contrary to this, a possible WH formation should be followed by a decrease of entropy which makes their existence very unlikely in nature. During the evolution of BHWT quantum transition the Hawking effect has not been taken into account. Although this effect is negligible, nevertheless, it will be interesting to study this effect in the context of WH spacetime which is the main purpose of this paper.

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This paper is organised as follows. In Sec. II, we review the generalized Painlevé coordinates [54] and the BHWH tunneling considered in Ref. [51]. In Sec. III, we shall solve the Proca equation to find the radial motion of massive vector particles using Hamilton-Jacobi (HJ) method. In Sec. IV, we study Hawking radiation in the spacetime of a BH. In Sec. V, we shall focus on the Hawking radiation in the spacetime of a WH. In Sec. VI, we study the tunneling of scalar particles. In Sect. VII we solve this problem in the framework of Parkikh-Wilczek (PW) method. Finally, in Sect. VIII we present our conclusions.

II. BLACK-HOLE-TO-WHITE-HOLE

Let us start by briefly reviewing a bouncing spacetime geometry outside a collapsing gravitational process which can be given by a time-symmetric bounce in terms of the following metric [51–54]

$$ds^2 = -dt^2 + \left(\frac{dr - f(u)v(r)dt}{1 - 2M/r_i} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right),$$

provided \( r_i > 2M \). It is worth noting that the metric (1) represents a generalized Painlevé type metric which is different from the standard Painlevé metric. There are two particular cases, \( f(u) = \pm 1 \), corresponding to the WH (BH) spacetimes, respectively. The function \( v(r) \) is given by

$$v(r) = \left(\frac{2M}{r} - \frac{2M}{r_i}\right)^{1/2}, \quad r_s(t) < r < r_i. \quad (2)$$

In the above equation, \( r_i \) gives the initial radial distance at which the gravitational collapse starts. One should keep in mind that the value of \( r_i \) is far above the Schwarzschild radius. Furthermore \( r_s(t) \) represents the trajectory of the surface of the star \( r_s(t) \) at a given time \( t \). In the BH case, from a classical point of view (singularity theorems) it is well known that such a collapse ends with a singularity. Here, in contrast, the singularity \( r_s(t) = 0 \) is not achieved. One way to achieve this is to simply continue \( r_s(t) \) with it’s time-reversal solution with the spacetime geometry to which cannot be entered from the outside i.e., a WH geometry.

From a quantum mechanical point of view, the amplitude between two configurations, say \( h_- \) and \( h_+ \), with the corresponding hypersurfaces \( \Sigma_- \) and \( \Sigma_+ \), related to the BHWH tunneling is given by [51]

$$(h_+, \Sigma_+|h_-, \Sigma_-) = \frac{1}{\mathcal{N}} \int_{\Sigma_-|h_-} \mathcal{D}g \exp(-\mathcal{L}_{EH}[g]), \quad \Sigma_+ = \text{the spacetime of a BH}. \quad (3)$$

where \( \mathcal{L}_{EH}[g] \) is the Einstein-Hilbert action of a Euclidean geometry \( g \), while \( \mathcal{N} \) being a normalization constant. Then the probability amplitude for the BHWH transition can be given as [51]

$$\mathcal{P}_{BH\rightarrow WH}(M, \Delta_0) = \int_{0}^{\Delta_0} |\langle WH|BH,M,\Delta_0 \rangle|^2 d\Delta_0, \quad (4)$$

where \( k_l \in [1, 3] \) and \( \Delta_0 \in [0, \infty) \). In Ref. [51], the authors were able to estimate the BHWH transition probability given by an exponential decay law

$$\mathcal{P}_{BH\rightarrow WH}(M, \Delta_0) = 1 - \exp(-2(k_l + 1/3)M\Delta_0 \sqrt{1 - 2M/r_i}). \quad (5)$$

Then taking into account that \( k_l + 1/3 \) is of the order of unity, the mean lifetime was found to be \( \tau \leq 1/2M \).

III. TUNNELING OF VECTOR PARTICLES WITH HJ METHOD

In order to proceed to study the quantum tunneling let us first introduce a new radial function given by

$$\mathcal{F}(r) = 1 - \frac{2M}{r}, \quad (6)$$

which after we substitute into equation (2) gives

$$v(r) = \left(1 - \mathcal{F}(r) - \frac{2M}{r_i}\right)^{1/2}. \quad (7)$$

More specifically metric (1) has the following metric tensor components

$$g_{\mu\nu} = \begin{bmatrix}
\frac{r_i - 2M - f^2(u)v^2(r)r_i}{2M - r_i} & \frac{f(u)v(r)r_i}{2M - r_i} & 0 & 0 \\
\frac{f(u)v(r)r_i}{2M - r_i} & \frac{r_i}{r_i - 2M} & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta 
\end{bmatrix}, \quad (8)$$

with the determinant given by

$$g = \det(g_{\mu\nu}) = \frac{r_i r^4 \sin^2 \theta}{r_i - 2M}. \quad (9)$$

We shall study the tunneling of massive vector particles described by the vector field \( \Psi^\mu \) in a curved spacetime metric given by the Proca equation (PE) as follows [18]

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \Psi^{\mu\nu} - \frac{m^2}{h^2} \Psi^{\nu} \right) = 0, \quad (10)$$

with the following relation

$$\Psi_{\mu\nu} = \partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu. \quad (11)$$
One way to solve PE is to apply the WKB approximation method with the solution proposed as
\[
\Psi_\nu(x^\mu) = C_\nu(x^\mu) \exp \left(\frac{i}{\hbar} \left( S_0(x^\mu) + \hbar S_1(x^\mu) + \ldots \right) \right).
\]

(12)

We now take into consideration the spacetime symmetries which lead to the following ansatz for the action of the particle
\[
S_0(t, r, \theta, \varphi) = -E t + R(r, \theta) + j \varphi,
\]

(13)

with \( E \) being the energy, and \( j \) being the angular momentum of the particle. Inserting Eq. (12) into the Proca equation and keeping only the leading order of \( \hbar \) results with the following four differential equations:

\[
0 = \left[ \frac{E r^2 \sin^2 \theta (2 M - r_i) R'(r) - r_i K_1}{r_i r^2 \sin^2 \theta} \right] C_1 + \left[ \frac{f v (\partial_\theta R)' - E (\partial_\theta R)}{r^2} \right] C_2 + \left[ \frac{r v R' j - E j}{r^2 \sin^2 \theta} \right] C_3 + \left[ \frac{\sin^2 \theta (2 M - r_i) r^2 R^2 - r_i K_2}{r_i r^2 \sin^2 \theta} \right] C_4,
\]

(14)

\[
0 = \left[ -f v (\partial_\theta R)^2 - (2 M - r_i) (\partial_\theta R)^2 \sin^2 \theta - 2 K_3 + K_4 \right] C_1 + \left[ \frac{f v (2 M - r_i) (\partial_\theta R) R' - r_i f v E (\partial_\theta R)}{r_i r^2 \sin^2 \theta} \right] C_2 + \left[ \frac{(f v (2 M - r_i) j) R' - E f v j r}{r_j r_i r^2 \sin^2 \theta} \right] C_3 + \left[ -r_j \sin^2 \theta f v (\partial_\theta R)^2 + (2 M - r_i) r^2 E \sin^2 \theta + K_5 \right] C_4,
\]

(15)

\[
0 = \left[ \frac{(2 M - r_i) (\partial_\theta R) R' - r_i f v E (\partial_\theta R)}{r_j r^2 \sin^2 \theta} \right] C_1 + \left[ -r_j \sin^2 \theta f v (\partial_\theta R)^2 + (2 M - r_i) r^2 E \sin^2 \theta + K_5 \right] C_2 + \left[ \frac{(\partial_\theta R) j}{r^2 \sin^2 \theta} \right] C_3 + \left[ \frac{f v (\partial_\theta R) R' - E (\partial_\theta R)}{r^2} \right] C_4,
\]

(16)

\[
0 = \left[ \frac{(f v (2 M - r_i) j) R' - E f v j r}{r_j r_i \sin^2 \theta} \right] C_1 + \left[ \frac{(\partial_\theta R) j}{r^2 \sin^2 \theta} \right] C_2 + \left[ \frac{\sin^2 \theta (f v (2 M - r_i) R^2 + 2 M - r_i) R^2 - K_7}{r_j r^2 \sin^2 \theta} \right] C_3 + \left[ \frac{f v R' j - E j}{r^2 \sin^2 \theta} \right] C_4,
\]

(17)

where we have used the following relations

\[
K_1 = f v \sin^2 \theta \partial_\theta (2 M - r_i)^2 + m^2 r^2 f v \sin^2 \theta + j^2 v f,
\]

\[
K_2 = \sin^2 \theta (\partial_\theta R)^2 + m^2 r^2 \sin^2 \theta + j^2,
\]

\[
K_3 = \cos^2 \theta (f v^2 r^2 r_i + (2 M - r_i) (E^2 - m^2)) + K_5,
\]

\[
K_4 = (2 M - r_i) ((E^2 - m^2) r^2 - j^2) - \nu^2 r^2 r_i (m^2 r^2 + j^2),
\]

\[
K_5 = f v m^2 r^2 r_i \cos \theta - f v r_i (m^2 r^2 + j^2),
\]

\[
K_6 = 2 E f v r^2 \sin^2 \theta R' - [(E^2 - m^2) r^2 \sin^2 \theta - j^2],
\]

\[
K_7 = r_i (E^2 - m^2) r^2 - (\partial_\theta R)^2.
\]

Resulting with the following non-zero matrix elements

\[
M_{11} = \frac{E r^2 \sin^2 \theta (2 M - r_i) R'(r) - r_i K_1}{r_i r^2 \sin^2 \theta},
\]

\[
M_{21} = -\frac{(f v^2 r_i + 2 M - r_i) (\partial_\theta R)^2 \sin^2 \theta - r^2 K_3 + K_4}{r_i r^2 \sin^2 \theta},
\]

\[
M_{31} = M_{22} = \frac{(f v^2 r_i + 2 M - r_i) (\partial_\theta R) R' - r_i f v E (\partial_\theta R)}{r_j r^2 \sin^2 \theta},
\]

\[
M_{41} = M_{23} = \frac{(f v^2 j r_i + (2 M - r_i) j) R' - E f v j r}{r_j r_i r^2 \sin^2 \theta},
\]

\[
M_{12} = M_{34} = \frac{f v (\partial_\theta R)' - E (\partial_\theta R)}{r^2},
\]

\[
M_{32} = -\frac{r^2 \sin^2 \theta (f v^2 r_i + 2 M - r_i) R^2 + r_i K_6}{r_j r^4 \sin^2 \theta},
\]

\[
M_{42} = M_{33} = -\frac{(\partial_\theta R) j}{r^2 \sin^2 \theta},
\]

\[
M_{13} = M_{44} = \frac{f v R' j - E j}{r^2 \sin^2 \theta},
\]

\[
M_{14} = \frac{\sin^2 \theta (2 M - r_i) r^2 R^2 - r_i K_2}{r_j r^2 \sin^2 \theta},
\]

\[
M_{24} = -\frac{r_j \sin^2 \theta f v (\partial_\theta R)^2 + (2 M - r_i) r^2 E \sin^2 \theta + K_5}{r_j r^2 \sin^2 \theta},
\]

\[
M_{43} = -\frac{r_j (f v^2 r_i^2 + 2 M - r_i) R^2 + 2 E f v r_i R' - K_7}{r_j r^4 \sin^2 \theta}.
\]

The radial motion of the particles is easily found if we solve the determinant

\[
\det M(C_1, C_2, C_3, C_4)^T = 0,
\]

(18)

which yields

\[
\frac{m^2 (2 M - r_i) [H - 2 E R' v^2 r_i f \sin^2 \theta - r_i G]^3}{r^{10} r_i^3 \sin^8 \theta} = 0,
\]

(19)

with

\[
H = r^2 \sin^2 \theta (f v^2 r_i + 2 M - r_i) R^2,
\]

\[
G = \sin^2 \theta (\partial_\theta R)^2 - (E^2 - m^2) r^2 \sin^2 \theta + j^2.
\]
Finally, after solving for the radial trajectories results with
\[
R_{\pm}(r) = \int \frac{E f(u) v(r) r_i \pm \sqrt{E^2 r_i^2 \left(1 - \frac{2M}{r_i}\right) - \Delta N}}{r_i \Delta(r)} \, dr,
\]
with
\[
\Delta = v^2(r) f^2(u) - \left(1 - \frac{2M}{r_i}\right), \quad (21)
\]
\[
N = \left[ m^2 + \frac{(\partial_u \theta)^2 r_i^2}{r^2} + \frac{\beta^2 r_i^2}{r^2 \sin^2 \theta} \right]. \quad (22)
\]

**IV. QUANTUM TUNNELING IN A BH GEOMETRY**

Let us now consider the quantum tunneling in the spacetime of a BH \([f(u) = -1]\). The function \(\Delta\) can be written as
\[
\Delta = \left( -v(r) - \sqrt{1 - \frac{2M}{r_i}} \right) \left( -v(r) + \sqrt{1 - \frac{2M}{r_i}} \right).
\]

Near the horizon we may expand this function which yields
\[
\Delta(r_h) \simeq -2\kappa_{BH}(r - r_h) + ..., \quad (24)
\]
in which we have used the following identification for the surface gravity of the black hole
\[
\kappa_{BH} = \frac{F'(r_h)}{2} > 0. \quad (25)
\]

The radial solution (61) near the horizon reads
\[
R_{\pm}(r_h) = \int \frac{-Ev(r_h) r_i \pm \sqrt{E^2 r_i^2 \left(1 - \frac{2M}{r_i}\right) - \Delta(r_h) N}}{-2r_i \kappa_{BH}(r - r_h)} \, dr.
\]

At this point, we can use the following identity
\[
\lim_{\epsilon \to 0} \text{Im} \frac{1}{r - r_h + i\epsilon} = \pi \delta(r - r_h), \quad (27)
\]
which lead to a non-zero contribution only for the outgoing solution \(R_-(r)\)
\[
\text{Im} R_-(r) = \frac{\pi E}{\kappa_{BH}}, \quad \text{Im} R_+(r) = 0. \quad (28)
\]

where \(\mathcal{E} = E v(r_h)\), is the net energy of the particle with \(v(r_h) = \sqrt{1 - 2M/r_i}\). Of course, this result is to be expected as a consequence of the coordinate system used in our setup. There is no barrier experienced by the ingoing particle across the event horizon. But, clearly, this is not the case for the outgoing particle. In this way if we define the tunneling rate from the inside to the outside
\[
\Gamma_{BH} = \frac{\Gamma_{out}}{\Gamma_{in}} = \exp(-2\text{Im} R_-) = \exp\left(-\frac{2\pi \mathcal{E}}{\kappa_{BH}}\right), \quad (29)
\]
after we compare the tunneling rate with the Boltzmann equation \([\Gamma_B = \exp(-\mathcal{E}/T_H)]\), we easily find the Hawking temperature
\[
T_H = \frac{\kappa_{BH} \mathcal{F}'(r_h)}{2\pi} = \frac{\mathcal{F}'(r_h)}{4\pi}. \quad (30)
\]

This result was to be expected, except that this procedure is not free from ambiguity. It has been shown that the above method can lead to the factor–two problem [40, 41]. As was pointed out in Refs. [40] the factor–two problem can be solved when we consider the invariance under canonical transformations by considering a closed path we can write
\[
\oint p_r \, dr = \int_{r_i}^{r_f} p_r^{in} \, dr + \int_{r_f}^{r_i} p_r^{out} \, dr. \quad (31)
\]

Note that the path goes from just outside the horizon, say \(r = r_i\), to \(r = r_f\) which is located just inside of the horizon. Next, we shall first work out the spatial contribution to the tunneling rate (we temporarily introduce the Planck constant) [40]
\[
\Gamma = \exp \left(-\frac{1}{\hbar} \text{Im} \oint p_r \, dr \right)
\]
\[
= \exp \left[-\frac{1}{\hbar} \text{Im} \left( \int p_r^{in} \, dr + \int p_r^{out} \, dr \right) \right], \quad (32)
\]
with \(p_r = \partial_r R\). Next one can shift the pole into the upper half plane \(r_h \to r_h + i\epsilon\) and rewrite the last equation as follows
A non-zero contribution gives only the second term leading to

\[
\text{Im} \int p_r \, d\bar{r} = \lim_{\epsilon \to 0} \left\{ \text{Im} \left[ \int_{r_f}^{r_i} \left( -Ev(r)r_i + \sqrt{E^2r_i^2 + 2M} - \Delta(r) \left( m^2 + \frac{(\partial \varphi)^2 r_i^2}{r^2} + \frac{\beta^2 r_i^2}{r^2 \sin^2 \varphi} \right) \right] \right. \right. \\
+ \left. \left. \text{Im} \left[ \int_{r_f}^{r_i} \left( -Ev(r)r_i - \sqrt{E^2r_i^2 + 2M} - \Delta(r) \left( m^2 + \frac{(\partial \varphi)^2 r_i^2}{r^2} + \frac{\beta^2 r_i^2}{r^2 \sin^2 \varphi} \right) \right] \right] \right. \right. \\
\left. \left. \left. \left. \left. \left. \frac{dr}{-2r_i \kappa_{BH}(r - r_h + i\epsilon)} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
After we compare with the Boltzmann distribution law
\[
\Gamma_B = \exp(-\mathcal{E}/T_H),
\]
we end up with an interesting result
\[
T_H = -\frac{\kappa_{WH}}{2\pi} = \frac{\mathcal{J}''(r_h)}{4\pi}. \tag{46}
\]

Now we shall derive this result in terms of the invariance under canonical transformations. In this case the path goes from just inside the WH horizon, say \( r = r_i \), to \( r = r_f \), just outside the WH horizon
\[
\int p_r dr = \int_{r_i}^{r_f} p_r^{\text{out}} dr + \int_{r_f}^{r_i} p_r^{\text{in}} dr. \tag{47}
\]

In particular after we shift the pole \( r_h \to r_h - i\epsilon \), we can write
\[
\text{Im} \int p_r dr = \lim_{\epsilon \to 0} \left\{ \text{Im} \left[ \int_{r_i}^{r_f} \frac{E\nu(r_h) r_i + \sqrt{E^2 r_i^2 \left( 1 - \frac{2M}{r_i} \right)} - \Delta(r_h) \left[ m^2 + \frac{(\partial_{\theta} R)^2}{r^2} + \frac{j^2 \Phi^2}{r^2 \sin^2 \theta} \right]}{2 r_i \kappa_{WH} (r - r_h - i\epsilon)} dr \right] \right\} + \lim_{\epsilon \to 0} \left\{ \text{Im} \left[ \int_{r_f}^{r_i} \frac{E\nu(r_h) r_i - \sqrt{E^2 r_i^2 \left( 1 - \frac{2M}{r_i} \right)} - \Delta(r_h) \left[ m^2 + \frac{(\partial_{\theta} R)^2}{r^2} + \frac{j^2 \Phi^2}{r^2 \sin^2 \theta} \right]}{2 r_i \kappa_{WH} (r - r_h - i\epsilon)} dr \right] \right\}. \tag{48}
\]

To ensure that the positive result for the Hawking temperature we change the direction in which we shift the pole, that is equivalent to change the direction in which we deform the contour. More precisely we shall use the following equation
\[
\lim_{\epsilon \to 0} \frac{1}{r - r_h - i\epsilon} = -\pi \delta(r - r_h), \tag{49}
\]

Again, there is a non-zero contribution for the ingoing particle given by the first term. Put it differently, in the WH spacetime, only the ingoing particle experiences barrier across the horizon. The spatial contribution to the tunneling rate gives
\[
\text{Im} \int p_r dr = -\frac{\pi \mathcal{E}}{\kappa_{WH}}. \tag{50}
\]

It remains to be seen the temporal contribution. Let us introduce the following coordinates
\[
dt \to d\tilde{t} + \frac{v(r) dr}{v^2(r) f^2(u) \left( 1 - \frac{2M}{r_i} \right)}. \tag{51}
\]

Again, \( t \) corresponds to the Painlevé time, while \( \tilde{t} \) corresponds to the time measured by a far-away observer outside the WH geometry. The action of the particle gives
\[
S_{0}^{WH} = -E \tilde{t} - \int \frac{v(r_h) E}{2 \kappa_{WH} (r - r_h)} dr + R(r, \theta) + j\varphi. \tag{52}
\]

We find
\[
\text{Im}(E\Delta t^{\text{out}}) = \text{Im}(E\Delta t^{\text{in}}) = -\frac{\pi \mathcal{E}}{\kappa_{WH}}. \tag{53}
\]

Hence the total tunneling rate is found to be
\[
\Gamma_{WH} = \exp \left[ -\frac{1}{\hbar} (\text{Im}(E\Delta t^{\text{out}}) + \text{Im}(E\Delta t^{\text{in}})) + \text{Im} \int p_r dr \right] = \exp \left( \frac{2\pi \mathcal{E}}{\kappa_{WH}} \right). \tag{54}
\]

And the expected result reads
\[
T_H = -\frac{\kappa_{WH}}{2\pi} = \frac{\mathcal{J}''(r_h)}{4\pi}. \tag{55}
\]

This result suggest that an observer outside the spacetime of a WH should detected Hawking quanta, in other words a flux of radiation from inside to outside the WH. In addition, the equation of Hawking temperature is form invariant to the BH temperature.

\section*{VI. TUNNELING OF SCALAR PARTICLES}

In this section we shall explore in details the tunneling of spinless particles, namely massive scalar particles. The relativistic scalar field equation can be written as follows
\[
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( g^{\mu\nu} \partial_{\nu} \Phi \right) - \frac{m^2}{\hbar^2} \Phi = 0. \tag{56}
\]
The scalar wave solution can be chosen as follows

$$\Phi(t, r, \theta, \varphi) = C(t, r, \theta, \varphi) \exp \left( \frac{i}{\hbar} S_0(t, r, \theta, \varphi) + \ldots \right),$$

(57)

with the same action

$$S_0(t, r, \theta, \varphi) = -Et + R(r, \theta) + j\varphi.$$  

(59)

we find

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \right) \Phi(r, \theta, \varphi) = \left( \frac{\partial^2}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{\cos^2 \theta}{r^2} \right) \Phi(r, \theta, \varphi) = 0.$$  

(60)

Solving this equation we find

$$R_\pm(r) = \int^{\epsilon(r)} \frac{E f(u) v(r) r_i \pm \sqrt{E^2 r_i^2 \left( 1 - \frac{2M}{r_i} \right) - \Delta N}}{r_i \Delta (r)} dr,$$

(61)

with $\Delta$ and $N$ are given by Eqs. (20) and (21). Specializing the WH (BH) solution we need to set $f(u) = \pm 1$, which leads to the same conclusions as in the case of vector field.

VII. TUNNELING WITH PW METHOD

The basic idea behind this method is to apply the radial null geodesics which can be found by the metric (1). In our case we are left with the following result

$$r \approx f(u) v(r) \pm \sqrt{1 - \frac{2M}{r_i}},$$

(62)

in which the $+(-)$ gives the outgoing (ingoing) geodesics. The tunneling rate is related to the imaginary part of the action in the classically forbidden region. In our paper the black hole mass is held fixed and the total ADM mass allowed to vary. When a shell of energy $\omega$ tunnels from the black hole, $M$ should be replaced by $M - \omega$. The imaginary part of the action is written as

$$\text{Im} S = \text{Im} \int_{r_i}^{r_f} p_i dr = \text{Im} \int_{r_i}^{r_f} \int_0^{p_r} p_r' dr$$

(63)

Solving the KG equation in leading order terms we find the differential equation

$$\frac{1}{r^2 r_i \sin^2 \theta} \left[ \sin^2 \theta (f^2(u) v^2(r) r_i + 2M - r_i) \right] - 2R' E f(u) v(r) - \frac{\sin^2 \theta (\partial_\theta R)^2 - r^2 (E^2 - m^2) \sin^2 \theta + J^2}{r^2 \sin^2 \theta} = 0$$

(58)

with the following result

$$\text{Im} = \int_M^{r_f} \int_{r_i}^{r_f} \frac{dr}{dH}$$

(64)

$$= \int_0^\omega \int_{r_i}^{r_f} \frac{dr}{r} - v(r) + \sqrt{1 - \frac{2(M - \omega'^i}{r_i}},$$

(65)

Note that the integral becomes singular at the horizon. In particular we need to choose a positive sign which corresponds to the outgoing particles. Note that to ensure the positive result we deform the a semi-circle to give $-\pi i \text{Res} [f(x)]$, after solving this integral we end up with the following result

$$\text{Im} S \approx 4\pi \omega M \left( 1 - \frac{M}{r_i} \right),$$

(66)

with the tunneling rate

$$\Gamma_{BH} = \exp (-2\text{Im} S) = \exp \left[ -8\pi \omega M \left( 1 - \frac{M}{r_i} \right) \right].$$

(67)

Making use of the Boltzmann equation $[\Gamma_{BH} = \exp(-\omega/T_H)]$ we find

$$T_H = \frac{1}{8\pi M} \left( 1 + \frac{M}{r_i} + \ldots \right).$$

(68)

Considering the fact that $r_i \gg M$, we end up with the same equation

$$T_H = \frac{k_{BH}}{2\pi} = F'(r_h) \frac{4\pi}{4\pi}. \quad (69)$$
In the white hole case \( f(u) = 1 \), when we apply the same procedure we find

\[
\text{Im} S = \text{Im} \int_{r_i}^{r_f} \frac{dr}{f} dH
\]

\[
= \text{Im} \int_0^\omega \int_{r_i}^{r_f} \frac{dr}{v(r)} \sqrt{1 - \frac{2(M - \omega')}{r_i}} d(-\omega').
\]

Note that a minus sign which corresponds to the ingoing particles has been chosen. To ensure the positive result we deform the a semi-circle to give \( \pi i \text{Res} [f(x)] \) which gives

\[
\text{Im} S = \text{Im} \int_0^\omega 4\pi i (M - \omega') \sqrt{1 - \frac{2(M - \omega')}{r_i}} d\omega'
\]

\[
\simeq +4\pi \omega \left( M - \omega^2 \right) - \frac{4\pi}{r_i} \left( M^2 \omega - M\omega^2 + \omega^3 + \ldots \right)
\]

Considering only the terms linear in \( \omega \), this result can be approximated as

\[
\text{Im} S \simeq 4\pi \omega M \left( 1 - \frac{M}{r_i} \right),
\]

with the tunneling rate

\[
\Gamma_{WH} = \exp (-2\text{Im} S) = \exp \left[ -8\pi \omega M \left( 1 - \frac{M}{r_i} \right) \right].
\]

Finally approximating the solution when \( r_i >> M \), yields

\[
T_H = -\frac{\kappa_{WH}}{2\pi} = \frac{f'(r_b)}{4\pi}.
\]

**VIII. CONCLUSION**

In this letter we have considered the Hawking radiation associated to the BH/WH geometry. We have used a generalized Painlevé coordinates together with the WKB and HJ methods. We have shown that a Hawking radiation is associated to the WH spacetime. Although particles can tunnel from the outside to the inside, based on the infinite valued flow analogue model one can argue that they will be spit out again implying that only the outgoing mode is of primary interest. Based on this, we have shown that the Hawking temperature outside the WH is the same as the BH temperature. Furthermore we have verified our result using the PW method. It is worth noting that, besides the mass, HT is affected by the initial radial distance \( r_i \) at which the gravitational collapse starts. In general, HR can be considered as a negligible effect, but it remains to be seen if this effect could have any impact on the BHWH transition process. Finally we wish to point out that quantum gravity effects may effect this picture, in particular one may incorporate the GUP effects during the BH-to-WH transition, and see whether the infinite valued flow can really disappear. We plan in the near future to study the problem of Hawking radiation from acoustic WHs in the tunneling approach with GUP effects.

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[1] S.W. Hawking, Commun. Math. Phys. 43, 199 (1975).
[2] G.W. Gibbons and S.W. Hawking, Phys. Rev. D 15, 275 (1977).
[3] G.W. Gibbons and S.W. Hawking, Phys. Rev. D 15, 2738 (1977).
[4] P. Kraus, F. Wilczek, Phys. Rev. Lett. A 9, 3713 (1994).
[5] P. Kraus, F. Wilczek, Nucl. Phys. B 437, 231 (1995).
[6] M.K. Parikh, F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000).
[7] M.K. Parikh, Phys. Lett. B 546, 189 (2002).
[8] M.K. Parikh, Int. J. Mod. Phys. D 13, 2351 (2004).
[9] M. Angheben, M. Nadalini, L. Vanzo and S. Zerbini, J. High Energy Phys. 05, 014 (2005).
[10] K. Srivinasan and T. Padmanabhan, Phys. Rev. D 60, 024007 (1999).
[11] M. Angheben, M. Nadalini, L. Vanzo, and S. Zerbini, JHEP 0505, 037 (2005).
[12] M.K. Parikh and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000).
[13] P. Kraus and F. Wilczek, Nucl. Phys. B 433, 403 (1995).
[14] Banerjee R, Majhi B R. JHEP 0806:095, 2008; Bibhas Ranjan Majhi, Phys. Rev. D 81, 044005, 2009, Rabin Banerjee, Bibhas Ranjan Majhi, Elias C. Vagenas, Phys. Lett. B 686:279-282, 2010; Rabin Banerjee, Bibhas Ranjan Majhi Phys. Lett. B 674:218-222, 2009
[15] L. Vanzo, G. Acquaviva and R. Di Criscienzo, Class. Quantum Gravity 28, 18 (2011).
[16] R. Kerner and R. B. Mann, Phys. Rev. D 79, 104010 (2006).
[17] A. Yale and R. B. Mann, Phys. Lett. B 673, 168-172, (2009).
[18] Xiao-Mei Kuang, Joel Saavedra, Ali Ovgun, Eur. Phys. J. C (2017) 77:613 ; I. Sakalli and A. Ovgun, Astrophys. Space Sci. 361, no.10, 330 (2016); I. Sakalli and A. Ovgun, Eur. Phys. J. Plus 131, no.6, 184 (2016); I. Sakalli and A. Ovgun, Europhys.Lett. 110, no.1, 10008 (2015); I. Sakalli and A. Ovgun, Gen.Rel.Grav. 48, no.1, 1 (2016); I. Sakalli and A. Ovgun, J. Astrophys. Astron. 37, 21 (2016).
[19] S.I Kruglov, Mod. Phys. Lett. A 29, 1450203 (2014); S.I Kruglov, Int. J. Mod. Phys. A 29, 1450118 (2014).
[20] Canisius Bernard, Phys. Rev. D 94, 085007 (2016)
[21] Kimet Jusufi, Ali Ovgun, Gordana Apostolovska, Advances in High Energy Physics, vol. 2017, Article ID 8798657, 7 pages, 2017; Kimet Jusufi, Gordana Apostolovska, Astrophys Space Sci (2016) 361: 374
