Variable population manipulations of reallocation rules in economies with single-peaked preferences

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Abstract

In a one-commodity economy with single-peaked preferences and individual endowments, we study different ways in which reallocation rules can be strategically distorted by affecting the set of active agents. We introduce and characterize the family of iterative reallocation rules and show that each rule in this class is withdrawal-proof and endowments-merging-proof, at least one is endowments-splitting-proof and that no such rule is pre-delivery-proof.

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1 Introduction

This paper studies a variable population model of an economy consisting of one non-disposable commodity and a set of agents with individual endowments of that commodity. In this context, we investigate different ways in which reallocation rules, i.e., systematic ways of selecting reallocations for each possible configuration of agents’

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preferences and endowments, can be strategically distorted by affecting the set of active agents. We limit our analysis to the case where agents’ preferences are single-peaked: up to some critical level, called the peak, an increase in an agent’s consumption raises her welfare; beyond that level, the opposite holds. This model has been extensively studied (see, for example, Bonifacio, 2015; Klaus et al., 1997, 1998). We allow for variable population as in Moreno (1996, 2002).\(^1\) To illustrate this type of problem, consider the distribution of a task (e.g., teaching hours) among the members of a group with concave disutility of labor (which induces single-peaked preferences over the time they dedicate to work). From one period to the next one, external factors (e.g., research and administrative duties) might affect preferences and a reallocation of the time assigned to each agent in the first period (taken as a benchmark for the second period) could benefit everyone. Another application of this model is a pollution problem in which countries have different rates of pollution and could trade via money transfers their pollution quotas.

Our analysis will be conducted over reallocation rules which are own-peak-only (the sole information collected by the rule from an agent’s preference to determine her reallocation is her peak amount) and meet the endowments lower bound (no agent is made worse off than at her endowment). Two monotonicity properties are appealing in this model. First, a population monotonicity. Since variable population is allowed, it is natural to ask for a monotonicity condition requiring the arrival of new agents to affect all agents present before the change in the same direction. Adding the proviso that agents entering the economy do not change the sign of the excess demand of the economy, we get the property of one-sided population monotonicity (see Thomson, 1995b). Second, a resource monotonicity. If, in case of excess demand, the individual endowments decrease (or increase in case of excess supply), then no individual is better off after the change. We call this property one-sided endowments monotonicity (see Thomson, 1994b).

Our first result is a characterization of the family of reallocation rules that satisfy the four previously mentioned properties (Theorem 1). This family of reallocation rules, which we call “iterative”, resembles the family of weakly sequential reallocation rules (Bonifacio, 2015) in that each rule in the family can be described by a step-by-step procedure that, at each stage, guarantees levels of consumption to the agents which are adjusted throughout the process. Weakly sequential reallocation rules, in turn, follow closely the definition of the sequential rules presented in Barberà et al. (1997) for economies with single-peaked preferences and a social endowment to be allotted.\(^2\) Our characterization turns out to be different from Barberà et al. (1997)’s since their characterization of sequential rules marries efficiency (allocations chosen by the rule satisfy

\(^1\)One-commodity economies with single-peaked preferences and a social endowment are studied, for example, by Sprumont (1991) assuming a fixed population. The first studies of the extension of this model to allow for variable populations are Thomson (1994a, 1995b).

\(^2\)Ehlers (2002b) extends the rules of Barberà et al. (1997) to the domain of single-plateaued preferences.
the Pareto criterion) with strategy-proofness (no agent ever has incentives to misrepresent preferences) and replacement monotonicity (if a change in one agent’s preferences results in that agent receiving at least as much as before, then all other agents receive at most as much as before). Iterative reallocation rules are always efficient (Lemma 1) but need not be strategy-proof nor replacement monotonic.³

Most of the literature on asymmetric rationing with single-peaked preferences has been focused on a specific property of immunity to manipulation: strategy-proofness. Moulin (1999) characterizes the family of rationing rules along fixed paths by means of efficiency, strategy-proofness, a resource monotonicity, and consistency (see also Ehlers, 2002a).⁴ We deliberately depart from that stance. Instead, and following Thomson (2014), we examine the robustness of iterative reallocation rules to various types of manipulations by affecting the set of active agents. The manipulations we consider are the following:

(i) Instead of participating, an agent withdraws with her endowment. The rule is applied without her. She then trades with one of the agents that did participate the resources they control between the two of them in such a way that both end up better off. A rule immune to this type of manipulation is called withdrawal-proof.

(ii) Two agents can merge their endowments, and one of them withdraw. The rule is applied without this second agent. The agent who stays may be assigned an amount that can be divided between the two in such a way that both become at least as well off as they would have been without the manipulation, and at least one of them is better off. A rule immune to this type of manipulation is called endowments-merging-proof.

(iii) An agent may split her endowment with some outsider (an agent with no endowment). The rule is applied and the guest then transfers her assignment to the agent who invited her in. The first agent may prefer her final assignment to what she would have received without the manipulation. A rule immune to this type of manipulation is called endowments-splitting-proof.

(iv) An agent may make a pre-delivery to some other agent of the trade the latter would be assigned if she participated. The rule is applied without the second agent. At her final assignment, the first agent may be better off than she would have been without the manipulation. A rule immune to this type of manipulation is called pre-delivery-proof.

It turns out that all iterative rules satisfy both withdrawal-proofness and endowments-merging-proofness (Corollary 1 and Lemma 4). Endowments-splitting-proofness is satisfied

³See Appendix B for more details.
⁴See also Moulin (2017), and references therein, for a general treatment of strategy-proof allocation with single-peaked preferences.
by the proportional reallocation rule (Remark 2). Finally, no iterative reallocation rule satisfies pre-delivery-proofness (Corollary 2).

The rest of the paper is organized as follows. In Section 2 the model and some basic properties of reallocation rules are presented. In Section 3, iterative reallocation rules are defined and characterized. The different variable population manipulations are discussed in Section 4. Final comments are gathered in Section 5.

2 Preliminaries

2.1 Model

We consider the set of natural numbers \( \mathbb{N} \) as the set of potential agents. Denote by \( \mathcal{N} \) the collection of all finite subsets of \( \mathbb{N} \). Each \( i \in \mathbb{N} \) is characterized by an endowment \( \omega_i \in \mathbb{R}_+ \) of the good and a continuous preference relation \( R_i \) defined over \( \mathbb{R}_+ \). Call \( P_i \) and \( I_i \) to the strict preference and indifference relations associated with \( R_i \), respectively. We assume that agents’ preferences are single-peaked, i.e., each \( R_i \) has a unique maximum \( p(R_i) \in \mathbb{R}_+ \) such that, for each pair \( \{x_i, x'_i\} \subset \mathbb{R} \), we have \( x_i p_i x'_i \) as long as either \( x'_i < x_i \leq p(R_i) \) or \( p(R_i) \leq x_i < x'_i \) holds. Denote by \( \mathcal{R} \) the domain of single-peaked preferences defined on \( \mathbb{R}_+ \). Given \( N \in \mathcal{N} \), an economy consists of a profile of preferences \( R \in \mathcal{R}^N \) and an individual endowments vector \( \omega = (\omega_i)_{i \in N} \in \mathbb{R}_+^N \), and is denoted by \( e = (R, \omega) \). If \( S \subset N \) and \( R \in \mathcal{R}^N \), let \( R_S = (R_j)_{j \in S} \) denote the restriction of \( R \) to \( S \). We often write \( N \setminus S \) by \( -S \). Using similar notation for the vector of endowments, \( e' = (R'_s, R_{-S}, \omega'_s, \omega_{-S}) \) stands for the economy where the preference and endowment of agent \( i \in S \) are \( R'_i \) and \( \omega'_i \), and those of agent \( i \notin S \) are \( R_i \) and \( \omega_i \). Let \( \mathcal{E}^N \) be the domain of economies with agents in \( N \). Given \( e = (R, \omega) \in \mathcal{E}^N \), let \( z(e) = \sum_{j \in N}(p(R_j) - \omega_j) \). If \( z(e) \geq 0 \) we say that economy \( e \) has excess demand whereas if \( z(e) < 0 \) we say that economy \( e \) has excess supply. Let \( \mathcal{E} = \bigcup_{N \in \mathcal{N}} \mathcal{E}^N \) denote the set of all potential economies. For each \( N \in \mathcal{N} \) and each \( e \in \mathcal{E}^N \), let \( X(e) = \{x \in \mathbb{R}_+^N : \sum_{j \in N} x_j = \sum_{j \in N} \omega_j\} \) be the set of reallocations for economy \( e \), and let \( X = \bigcup_{e \in \mathcal{E}} X(e) \). A reallocation rule is a function \( \varphi : \mathcal{E} \rightarrow X \) such that \( \varphi(e) \in X(e) \) for each \( e \in \mathcal{E} \). For each \( N \in \mathcal{N} \), each \( i \in N \), and each \( e \in \mathcal{E}^N \), let \( \Delta \varphi_i(e) = \varphi_i(e) - \omega_i \) be agent \( i \)'s net trade at \( e \).

2.2 Basic properties

The next informational simplicity property states that if an agent unilaterally changes her preference for another one with the same peak, then her allotment remains unchanged.

**Own-peak-only:** For each \( e = (R, \omega) \in \mathcal{E}^N \), each \( i \in N \), and each \( R'_i \in \mathcal{R} \) such that \( p(R'_i) = p(R_i) \), if \( e' = (R'_i, R_{N \setminus \{i\}}, \omega) \) then \( \varphi_i(e') = \varphi_i(e) \).
This property is weaker than the “peak-only” property,\(^5\) that has been imposed in a number of axiomatic studies. Analyzing the uniform rule, Sprumont (1991) derives the own-peak-only property from other axioms (see also Ching, 1992, 1994).

The following property requires respecting ownership of the resource, and also can be seen as giving incentive to participate in the exchange process. It says that no agent can get a reallocation that she finds worse than her endowment.

**Endowments lower bound:** For each \(e = (R, \omega) \in \mathcal{E}^N\), and each \(i \in N, \varphi_i(e)R_i\omega_i\).\(^6\)

Next, we present our two monotonicity properties. The first one requires that as population enlarges, and the new resources and preferences considered are not as disruptive as to modify the status of the economy from excess demand to excess supply or vice versa, the welfare of each of the initially present agents should move in the same direction.

**One-sided population monotonicity:** For each \(e = (R, \omega) \in \mathcal{E}^N\), each \(N' \subset N\), and each \(e' = (R_{N'}, \omega_{N'}) \in \mathcal{E}^{N'}\), \(z(e)z(e') \geq 0\) implies either \(\varphi_i(e)R_i\varphi_i(e')\) for each \(i \in N'\) or \(\varphi_i(e')R_i\varphi_i(e)\) for each \(i \in N'\).

The second one requires all agents to benefit from a favorable change in the amount to allocate. Given two vectors \(x, y \in \mathbb{R}^N\), define \(x \succeq y\) if and only if \(x_i \geq y_i\) for each \(i \in N\).

**One-sided endowments monotonicity:** For each \(e = (R, \omega) \in \mathcal{E}^N\), and each \(\omega' \in \mathbb{R}^N\) such that \(\omega' \succeq \omega\), if \(e' = (R, \omega')\), then \(z(e') \geq 0\) implies \(\varphi_i(e')R_i\varphi_i(e)\) for each \(i \in N\), and \(z(e) \leq 0\) implies \(\varphi_i(e)R_i\varphi_i(e')\) for each \(i \in N\).

The usual Pareto optimality property states that, for each economy, the reallocation selected by the rule should be such that there is no other reallocation that all agents find at least as desirable and at least one agent prefers. In this model, it is equivalent to the following same-sidedness condition:

**Efficiency:** For each \(e = (R, \omega) \in \mathcal{E}^N\), \(z(e) \geq 0\) implies \(\varphi_i(e) \leq p(R_i)\) for each \(i \in N\), and \(z(e) \leq 0\) implies \(\varphi_i(e) \geq p(R_i)\) for each \(i \in N\).

**Lemma 1** Each own-peak-only and one-sided endowments monotonic reallocation rule that meets the endowments lower bound is efficient.

**Proof.** Let \(\varphi\) be an own-peak-only, one-sided endowments monotonic rule that meets the endowments lower bound, and assume \(\varphi\) is not efficient. Then, there is \(e = (R, \omega) \in \mathcal{E}^N\), that without loss of generality we assume \(z(e) \geq 0\), and \(i \in N\) such that \(\varphi_i(e) > p(R_i)\). This implies that \(\varphi_i(e) \leq \omega_i\).\(^7\)

To see that (1) holds, first assume \(p(R_i) \leq \omega_i < \varphi_i(e)\). By single-peakedness, \(\omega_iP_i\varphi_i(e)\), contradicting the endowments lower bound. Second, assume \(\omega_i < p(R_i)\). Let \(\bar{R}_i \in \mathcal{R}\).

\(^5\)See Appendix B for a discussion on this fact.

\(^6\)This property is commonly known in the literature as individual rationality.
be such that \( p(\bar{R}_i) = p(R_i) \), and \( \omega_i P_i \phi_i(e) \) and let \( \bar{e} = (\bar{R}_i, R_{-i}, \omega) \). By the own-peak-only property, \( \phi_i(\bar{e}) = \phi_i(e) \). Hence, \( \omega_i P_i \phi_i(\bar{e}) \), contradicting the endowments lower bound. Therefore, (1) holds. Next, let \( \omega'_i \in \mathbb{R}_+ \) be such that \( \omega'_i = p(R_i) \), and let \( e' = (R, \omega'_i, \omega_{-i}) \). By the endowments lower bound, \( \phi_i(e') = p(R_i) \) and therefore \( \phi_i(e') P_i \phi_i(e) \), contradicting one-sided endowments monotonicity since \( z(e') \geq 0 \). □

The following result will be useful in the rest of the paper.

**Lemma 2** Let \( \varphi \) be an efficient and own-peak-only reallocation rule that meets the endowments lower bound. Let \( e = (R, \omega) \in \mathcal{E}^N \) and \( i \in N \). If either \( z(e) \geq 0 \) and \( p(R_i) \leq \omega_i \), or \( z(e) \leq 0 \) and \( p(R_i) \geq \omega_i \), then \( \varphi_i(e) = p(R_i) \).

**Proof.** Let \( \varphi \) satisfy the properties in the lemma and let \( e \in \mathcal{E}^N \) and \( i \in N \). Assume \( z(e) \geq 0 \) and \( p(R_i) \leq \omega_i \). Since \( z(e) \geq 0 \), by efficiency, \( \varphi_i(e) \leq p(R_i) \). If \( p(R_i) = \omega_i \), \( \varphi_i(e) = p(R_i) \) by the endowments lower bound. Suppose, then, that \( p(R_i) < \omega_i \) and \( \phi_i(e) < p(R_i) \). Let \( R'_i \in \mathcal{R} \) and \( x_i \in \mathbb{R}_+ \) be such that \( p(R'_i) = p(R_i) \), \( \phi_i(e) < x_i < p(R_i) \) and \( x_i \omega_i \). Let \( e' = (R'_i, R_{N\setminus\{i\}}, \omega) \). By the own-peak-only property, \( \phi_i(e') = \phi_i(e) \). Then, \( \omega_i P_i \phi_i(e') \), contradicting the endowments lower bound. A similar reasoning establishes the same conclusion when \( z(e) \leq 0 \) and \( p(R_i) \geq \omega_i \). □

### 3 Iterative reallocation rules

In this section we present a well-behaved family of reallocation rules. They resemble the weakly sequential reallocation rules\(^7\) (Bonifacio, 2015) in that both families of rules are defined through an easy step-by-step procedure. The two families, although overlap, are different. This is discussed thoroughly in Appendix B.

#### 3.1 Definition

For each \( N \in \mathcal{N} \) and each \( e = (R, \omega) \in \mathcal{E}^N \), let \( Q(e) \equiv \{ q \in \mathbb{R}^N : \sum_{j \in N} q_j = 0 \text{ and } \omega + q \geq 0 \} \) be the possible net trades of endowments in economy \( e \) and let \( Q \equiv \bigcup_{e \in \mathcal{E}} Q(e) \). Next, define \( \mathcal{Q} \equiv \{(q, e) \in Q \times \mathcal{E} : q \in Q(e)\} \). Each element of \( \mathcal{Q} \) specifies a net trade in a particular economy. An iterative reallocator is a function that, for each \( N \in \mathcal{N} \) and each economy \( e = (R, \omega) \in \mathcal{E}^N \), starting from the individual endowments of the agents (i.e., from a net trade \( q^0 \) equal to zero for each agent \( i \in N \)), generates iteratively a sequence of net trades \( q^0, q^1, \ldots, q^{N-1} \). Its repeated application is constrained to follow monotonic features.

**Definition 1** An *iterative reallocator* is a function \( g : \mathcal{Q} \to \mathcal{Q} \) such that \( g(q, e) \in Q(e) \) and, for each \( N \in \mathcal{N} \), and each \( e = (R, \omega) \in \mathcal{E}^N \), if \( (q^t, e) = g(q^{t-1}, e) \equiv g^t(0, e) \), then:\(^8\)

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\(^7\)These rules, in turn, follow closely the definition of the sequential rules presented in Barberà et al. (1997) for economies with a social endowment to be allotted.

\(^8\)Here \( g^t \) denotes \( g \) compose with itself \( t \) times.
(i) for each \( i \in N \), and each \( t \geq 1 \),

\[
q_i^t = p(R_i) - \omega_i \quad \text{whenever} \quad \begin{cases} 
z(e) \geq 0 & \text{and} & p(R_i) \leq \omega_i + q_i^{t-1} \\
z(e) < 0 & \text{and} & p(R_i) \geq \omega_i + q_i^{t-1}.
\end{cases}
\]

(ii) for each \( i \in N \), and each \( t \geq 1 \),

\[
q_i^t \geq q_i^{t-1} \quad \text{whenever} \quad z(e) \geq 0 \quad \text{and} \quad p(R_i) > \omega_i + q_i^{t-1}
\]

\[
q_i^t \leq q_i^{t-1} \quad \text{whenever} \quad z(e) < 0 \quad \text{and} \quad p(R_i) < \omega_i + q_i^{t-1}.
\]

(iii) for each \( \tilde{e} = (R, \tilde{\omega}) \in \mathcal{E}^N \) such that \( \tilde{\omega} \geq \omega \) and \( (q_i^{\lfloor N \rfloor - 1}, \tilde{\omega}) = g^{\lfloor N \rfloor - 1}(0, \tilde{e}) \),

\[
\tilde{\omega} + q_i^{\lfloor N \rfloor - 1} \geq \omega + q_i^{\lfloor N \rfloor - 1} \quad \text{whenever} \quad z(e) \leq 0 \quad \text{or} \quad z(\tilde{e}) \geq 0.
\]

(iv) for each \( \tilde{N} \subset N \), each \( \tilde{e} = (R_{\tilde{N}}, \omega_{\tilde{N}}) \in \mathcal{E}^{\tilde{N}} \), and \( (q_i^{\lfloor \tilde{N} \rfloor - 1}, \tilde{e}) = g^{\lfloor \tilde{N} \rfloor - 1}(0, \tilde{e}) \),

\[
\begin{bmatrix}
q_i^{\lfloor \tilde{N} \rfloor - 1} - q_i^{\lfloor N \rfloor - 1} \\
q_j^{\lfloor \tilde{N} \rfloor - 1} - q_j^{\lfloor N \rfloor - 1}
\end{bmatrix} \geq 0 \quad \text{whenever} \quad \begin{cases} 
z(e) \geq 0 & \text{and} & z(\tilde{e}) \geq 0 \\
z(e) \leq 0 & \text{and} & z(\tilde{e}) \leq 0
\end{cases}
\]

for each \( \{i, j\} \subset \tilde{N} \).

Let us put in words the above definition for the case of excess demand (this is, when \( z(e) \geq 0 \)). The first two conditions relate to the behavior of the net trades of an economy throughout the iterations of \( g \). Condition (i) says that if at stage \( t - 1 \) agent \( i \)’s peak is not higher than her endowment plus her net trade, i.e. \( p(R_i) \leq \omega_i + q_i^{t-1} \), then agent \( i \)’s net trade is set at \( p(R_i) - \omega_i \) from stage \( t \) onward. Condition (ii) establishes that if at stage \( t - 1 \) agent \( i \)’s peak is higher than her endowment plus her net trade, i.e. \( p(R_i) > \omega_i + q_i^{t-1} \), then her net trade should not decrease from stage \( t - 1 \) to stage \( t \), i.e. \( q_i^t \geq q_i^{t-1} \). The last two conditions relate to the behavior of the iterations of \( g \) between two different economies. Condition (iii) states that, in another economy \( \tilde{e} \) with the same agents and preferences where no agent has lower endowment and the increase in the resources is not disruptive, i.e. \( z(\tilde{e}) \geq 0 \), the resources available to each agent in the last stage of the iterations cannot be smaller than the resources available to each agent in the last stage of the iterations in the original economy. Finally, Condition (iv) says that for any subeconomy with excess demand, if the net trade of one agent in the last stage of the iteration is not smaller (bigger) than the net trade that same agent gets in the original economy, then the net trade of each of the other agents in the subeconomy should not be smaller (bigger) than the net trade that agent gets in the original economy either.

**Remark 1** Note that, as there are \( |N| \) agents in the economy, at most \( |N| - 1 \) adjustments take place. Therefore, \( (q_i^{\lfloor N \rfloor + t - 1}, e) = g^{\lfloor N \rfloor + t - 1}(0, e) \) implies \( q_i^{\lfloor N \rfloor + t - 1} = q_i^{\lfloor N \rfloor - 1} \) for each \( t \geq 1 \).
Each iterative reallocator induces a reallocation rule in a straightforward way:

**Definition 2** A reallocation rule \( \varphi : \mathcal{E} \to X \) is **iterative** if there is an iterative reallocator \( g : \mathcal{Q} \to \mathcal{Q} \) such that, for each \( N \in \mathcal{N} \) and each \( e \in \mathcal{E}^N \), \((q^{[N]-1}, e) = g^{[N]-1}(0, e) \) implies \( \Delta \varphi(e) = q^{[N]-1} \).

A prominent member of the class of iterative reallocation rules is the uniform reallocation rule, first studied by Thomson (1995a) (see also Klaus et al., 1997, 1998), that adapts the celebrated uniform rule characterized by Sprumont (1991) to the model with individual endowments:

**Uniform reallocation rule, \( u \):** for each \( N \in \mathcal{N} \), each \( e \in \mathcal{E}^N \), and each \( i \in N \),

\[
 u_i(e) = \begin{cases} 
 \min\{p(R_i), \omega_i + \lambda(e)\} & \text{if } z(e) \geq 0 \\
 \max\{p(R_i), \omega_i - \lambda(e)\} & \text{if } z(e) < 0 
\end{cases}
\]

where \( \lambda(e) \geq 0 \) and solves \( \sum_{j \in N} u_j(e) = \sum_{j \in N} \omega_j \).

Within the class of iterative reallocation rules, the uniform reallocation rule is the only one that supports envy-free redistributions, meaning by this that for no \( N \in \mathcal{N} \), \( e \in \mathcal{E}^N \), and pair of agents \( \{i, j\} \subset N \) such that \( \omega_i - \Delta u_j(e) \in \mathbb{R}_+ \), we have \( (\omega_i - \Delta u_j(e))\, P_i \, u_i(e) \) (see Moreno, 2002, Theorem 1).

To see that the uniform reallocation rule is an iterative reallocation rule, given \( e = (R, \omega) \in \mathcal{E}^N \) and \( q^0 = (0, \ldots, 0) \), consider the iterative reallocator \( g : \mathcal{Q} \to \mathcal{Q} \) defined as follows. If \((q^t, e) = g(q^{t-1}, e)\) then, for each \( i \in N \) and each \( t = 1, \ldots, |N| - 1 \),

\[
 q_i^t = \begin{cases} 
 p(R_i) - \omega_j & \text{if } i \in N^t \\
 \lambda^t & \text{if } i \in N \setminus N^t 
\end{cases}
\]

where

\[
 \lambda^t = \lambda^{t-1} + \frac{\sum_{j \in N \setminus (t=0 \cup N^t)} [\omega_j + \lambda^{t-1} - p(R_j)]}{|N \setminus N^t|},
\]

\( \lambda^0 = 0, N^0 = \emptyset \), and

\[
 N^t = \begin{cases} 
 \{j \in N : p(R_j) \leq \omega_j + q_j^{t-1}\} & \text{if } z(e) \geq 0 \\
 \{j \in N : p(R_j) > \omega_j + q_j^{t-1}\} & \text{if } z(e) < 0 
\end{cases}
\]

It is easy to see that \((q^{[N]-1}, e) = g^{[N]-1}(0, e) \) implies \( q^{[N]-1} = \Delta u(e) \) for each \( N \in \mathcal{N} \) and each \( e \in \mathcal{E}^N \).

**Example 1** Consider \( e = (R, \omega)^{\{1,2,3,4\}} \) with \( p(R_1) = 0, p(R_2) = 2, p(R_3) = 3.5 \), and \( p(R_4) = 10 \); and \( \omega_1 = 9, \omega_2 = 1, \omega_3 = 0 \), and \( \omega_4 = 2 \). Then, as \( z(e) = 15.5 - 12 > 0 \),

\[
 N^1 = \{1\}, \ \lambda^1 = 0 + \frac{9 + 0 - 0}{3} = 3, \ \text{and thus } q^1 = (-9, 3, 3, 3),
\]

\[
 N^2 = \{1, 2\}, \ \lambda^2 = 3 + \frac{1 + 3 - 2}{2} = 4, \ \text{and thus } q^2 = (-9, 1, 4, 4),
\]
\[ N^3 = \{1, 2, 3\}, \quad \lambda^3 = 4 + \frac{0 + 4 - 3.5}{1} = 4.5, \quad \text{and thus } q^3 = (-9, 1, 3.5, 4.5). \]

Therefore, \( u_1(e) = 0, u_2(e) = 2, u_3(e) = 3.5, \) and \( u_4(e) = 6.5. \)

Another iterative reallocation rule, that will be analyzed in Section 4.3, is the proportional reallocation rule that we present next.

**Proportional reallocation rule, \( \varphi^p \):** for each \( N \in \mathcal{N} \), each \( e \in \mathcal{E}^N \), and each \( i \in N \),

\[
\varphi_i^p(e) = \begin{cases} 
\min \{ p(R_i), \lambda(e) \omega_i \} & \text{if } z(e) \geq 0 \\
\max \{ p(R_i), \lambda(e) \omega_i \} & \text{if } z(e) \leq 0 
\end{cases}
\]

where \( \lambda(e) \geq 1 \) and solves \( \sum_{j \in N} \varphi_j^p(e) = \sum_{j \in N} \omega_j \).

### 3.2 Characterization

The next result states that the class of iterative rules is characterized by the own-peak-only property, the endowments lower bound, one-sided endowments monotonicity and one-sided population monotonicity:

**Theorem 1** A reallocation rule satisfies the own-peak-only property, the endowments lower bound, one-sided endowments monotonicity, and one-sided population monotonicity if and only if it is an iterative reallocation rule.

**Proof.** (\( \implies \)) Let \( \varphi \) be an own-peak-only, one-sided endowments monotonic, and one-sided population monotonic reallocation rule that meets the endowments lower bound. By Lemma 1, \( \varphi \) is also efficient. The iterative reallocator \( g : Q \rightarrow Q \) is constructed as follows. Given \((q^{t-1}, e) \in Q\), define \( q^t \) such that \((q^t, e) = g(q^{t-1}, e)\) as

\[
q^t = \varphi(q^t) - \omega
\]

where economy \( e^t = (R, \omega^t) \in \mathcal{E}^N \) is such that, for each \( i \in N \),

\[
\omega_i^t = \begin{cases} 
p(R_i) & \text{if } z(e)[p(R_i) - \omega_i^{t-1} - q_i^{t-1}] \leq 0 \\
\omega_i^{t-1} + q_i^{t-1} & \text{otherwise,}
\end{cases}
\]

with \( \omega_i^0 = \omega_i \) and \( q_i^0 = 0 \).

Let us assume that \( e = (R, \omega) \in \mathcal{E}^N \) is such that \( z(e) \geq 0 \). The other case is similar. We need to see that \( \Delta \varphi(e) = q^{\lvert N \rvert - 1} \) where \( q^{\lvert N \rvert - 1} \) is such that \((q^{\lvert N \rvert - 1}, e) = g^{\lvert N \rvert - 1}(0, e)\). In order to do this, let \( t \in \{1, \ldots, \lvert N \rvert - 1\} \) be such that \((q^t, e) = g(q^{t-1}, e) = g^t(0, e)\) (notice that \( q^0 = 0 \)).

**Claim 1:** Let \( t \in \{1, \ldots, \lvert N \rvert - 1\} \). If \( p(R_i) \geq \omega_i^{t-1} + q_i^{t-1} \) for each \( i \in N \), then \( q^t = \Delta \varphi(e) \).

Consider first the case \( t = 1 \). As \( q^0 = 0 \) by the hypothesis \( p(R_i) \geq \omega_i \) for each \( i \in N \).

\(^9\)For this rule to be well-defined, we need to constrain the domain of economies to those in which all individual endowments are always strictly positive.
The endowments lower bound, efficiency, and feasibility imply \( \varphi(e) = \omega \), and therefore \( \Delta \varphi(e) = 0 \). Note that, in \( e^1 \) no agent’s peak is less than her endowment and \( z(e^1) \geq 0 \), by the same reasoning as before \( \varphi(e^1) = \omega \). Thus, \( q^1 = \omega - \omega = 0 \). Next, assume the claim is true for each \( t < T \). Then \( q^T_{i} = 0 \) for each \( i \in N \) and, again, since in \( e^T \) no agent’s peak is less than her endowment and \( z(e^T) \geq 0 \), we get \( q^T_{i} = 0 = \Delta \varphi(e) \). This proves the claim.

Claim 2: Let \( t \in \{1, \ldots, |N| - 1 \} \). If \( i \in N \) is such that \( p(R_i) \leq \omega^{t-1}_i + q^{t-1}_i \), then \( q^t_i = \Delta \varphi_i(e) \). Let \( i \in N \) be such that \( p(R_i) \leq \omega^{t-1}_i + q^{t-1}_i \). First, notice that when \( t = 1 \), \( p(R_i) \leq \omega^0_i + q^0_i \) implies, as \( q^0_i = 0 \) and \( \omega^0_i = \omega_i \), that \( p(R_i) \leq \omega_i \). Then, by Lemma 2, \( \varphi_i(e) = p(R_i) \), and, therefore,

\[
\Delta \varphi_i(e) = p(R_i) - \omega_i.
\]  

Next, let \( t \in \{1, \ldots, |N| \} \). Since \( \omega^t_i = p(R_i) \) and \( z(e^t) \geq 0 \), by Lemma 2 applied to economy \( e^t \), we have \( \varphi_i(e^t) = p(R_i) \) and then \( q^t_i = p(R_i) - \omega_i \). Hence, by (3), \( q^t_i = \Delta \varphi_i(e) \). This proves the claim.

Claims 1 and 2 show that if \( q^{|N|} = 1 \) is such that \( (q^{|N|} - 1,e) = g^{|N| - 1}(0,e) \), then \( q^{|N| - 1} = \Delta \varphi(e) \). It remains to be checked that function \( g \) satisfies conditions (i)-(iv) in Definition 1. Condition (i) is clear by Claims 1 and 2. Condition (ii) follows from the next claim.

Claim 3: for each \( t = 1, \ldots, |N| - 1 \), if \( i \in N \) is such that \( p(R_i) > \omega^{t-1}_i + q^{t-1}_i \), then \( q^t_i \geq q^{t-1}_i \). Let \( i \in N \) be such that \( p(R_i) > \omega^{t-1}_i + q^{t-1}_i \). Consider first the case \( t = 1 \). Since \( q^0_i = 0 \) and \( \omega^0_i = \omega_i \), by the hypothesis \( p(R_i) > \omega_i \). Then, \( \omega^1_i = \omega_i \). This implies, as \( z(e^1) \geq 0 \), that \( q_1(e^1) \geq \omega^1_i \) by the endowments lower bound. Hence, \( q^1_i = q_i(e^1) - \omega_i \geq 0 = q^0_i \) and thus \( q^1_i \geq q^0_i \). Next, assume the claim is true for each \( t < T \). Then \( q^T_{i} \geq q^{T-1}_{i} \geq \cdots \geq q^0_i = 0 \). Since \( p(R_i) > \omega^{T-1}_i + q^{T-1}_i \) implies \( \omega^T = \omega^{T-1}_i + q^{T-1}_i \) and \( z(e^T) \geq 0 \), by the endowments lower bound \( \varphi_i(e^T) \geq \omega^T_i = \omega_i + \sum_{k=1}^{T-1} q^T_k \). Then,

\[
q^T_i = \varphi_i(e^T) - \omega_i \geq \sum_{k=1}^{T-1} q^T_k \geq q^{T-1}_i.
\]

This proves the claim.

Condition (iii) follows from the definition of \( g \) and one-sided endowments monotonicity of \( \varphi \), whereas condition (iv) is a consequence of the definition of \( g \) and one-sided population monotonicity of \( \varphi \).

\( \iff \) Let \( \varphi \) be an iterative reallocation rule. Then there exists an iterative reallocator \( g \) such that, for each \( e = (R, \omega) \in \mathcal{E}^N \), if \( q^{|N|} - 1 \) is such that \( (q^{|N|} - 1,e) = g^{|N| - 1}(0,e) \), then \( \Delta \varphi(e) = q^{|N|} - 1 \). We will consider only the case \( z(e) \geq 0 \), since an analogous argument can be used in the case \( z(e) < 0 \). Next, we prove that \( \varphi \) is efficient,\(^{10} \) one-sided endowments monotonic, and one-sided population monotonic.

**Efficiency:** We need to show that \( \varphi_i(e) \leq p(R_i) \) for each \( i \in N \). Suppose \( \varphi_i(e) \neq p(R_i) \).

\(^{10}\)This is used to prove the two monotonicity properties.
Then $ω_i + q_i^{[N]} ≠ p(R_i)$. If $q_i^{[N]} > p(R_i) − ω_i$ by (i) in Definition 1 we have $q_i^{[N]} ≠ p(R_i) − ω_i$. Then, $q_i^{[N]} ≠ q_i^{[N]}$, contradicting Remark 1. Thus, $q_i^{[N]} ≤ p(R_i) − ω_i$ implying $Δϕ_i(e) = q_i^{[N]} − ω_i ≤ p(R_i) − ω_i$ and $ϕ_i(e) ≤ p(R_i)$.

One-sided endowments monotonicity: Let $\tilde{e} = (R, ω_e) ∈ E^N$ be such that $ω_e ≥ ω$ and $z(\tilde{e}) ≥ 0$, let $\tilde{q}^{[N]}$ be such that $(\tilde{q}^{[N]}(\tilde{e}) = g^{[N]}(\tilde{e})$ and consider $i ∈ N$ such that $p(R_i) > ω_i$. By condition (iii) in Definition 1, $ω_i + \tilde{q}_i^{[N]} ≥ ω_i + q_i^{[N]}$. Then, $ϕ_i(\tilde{e}) ≥ ϕ_i(e)$ and, as by efficiency $p(R_i) ≥ ϕ_i(e)$, we have $ϕ_i(\tilde{e})R_iϕ_i(e)$.

One-sided population monotonicity: Let $\tilde{N} ⊂ N$ and $\tilde{e} = (R_{\tilde{N}}, ω_{\tilde{N}}) ∈ E^{\tilde{N}}$ be such that $z(\tilde{e}) ≥ 0$. Take $\{i, j\} ⊆ \tilde{N}$. By condition (iv) in Definition 1, $[q_i^{[N]} − q_i^{[N]}][\tilde{q}_i^{[N]} − q_i^{[N]}] ≥ 0$. Assume, without loss of generality, that $q_i^{[N]} ≥ q_i^{[N]}$. Then, $\tilde{q}_i^{[N]} ≥ q_i^{[N]}$. This implies $ϕ_i(\tilde{e}) ≥ ϕ_i(e)$ and $ϕ_j(\tilde{e}) ≥ ϕ_j(e)$. As $z(\tilde{e}) ≥ 0$, by efficiency, $p(R_i) ≥ ϕ_i(e)$ and $p(R_j) ≥ ϕ_j(\tilde{e})$. Thus, $ϕ_i(\tilde{e})R_iϕ_i(e)$ and $ϕ_j(\tilde{e})R_iϕ_j(e)$.

To complete the proof, notice that $ϕ$ satisfies the own-peak-only property because $g$ does, and meets the endowments lower bound because, for each agent, the adjustment process at each step guarantees an amount at least as good as the individual endowment. □

The independence of the axioms involved in the characterization of Theorem 1 is analyzed in Appendix A.

4 Variable population manipulations

In this section, we analyze each of the four properties of immunity to manipulation presented in the introduction and its relations with the family of iterative reallocation rules.

4.1 Withdrawal-proofness

Consider an economy and suppose that an agent withdraws with her endowment and the reallocation rule is applied without her. It could be the case that the amount that some other agent received in the reallocation together with the endowment of the agent that withdrew could be re-divided between the two of them in such a way that both agents get (strictly) better off with respect to the assignments they would have obtained if the first agent had not withdrawn. We require immunity to this sort of behavior:

Withdrawal-proofness: For each $e = (R, ω) ∈ E^N$, each $\{i, j\} ⊂ N$ and each $(x_i, x_j) ∈ \mathbb{R}_+^2$ such that $x_i + x_j = ϕ_i(e') + ω_j$, where $e' = (R_{N\{j\}}, ω_{N\{j\}})$, it is not the case that $x_kp_kϕ_k(e)$ for each $k ∈ \{i, j\}$.

The next result shows that withdrawal-proofness is implied by some of the properties discussed in Subsection 2.2.
Lemma 3 Each efficient, own-peak-only, and one-sided population monotonic reallocation rule that meets the endowments lower bound is withdrawal-proof.

Proof. Let \( \varphi \) satisfy the hypothesis of the Theorem. By Lemma 1, \( \varphi \) is also efficient. Assume \( \varphi \) is not withdrawal-proof. Then, there are \( e = (R, \omega) \in \mathcal{E}^N, \{i, j\} \subset N, \) and \( (x_i, x_j) \in \mathbb{R}^2_+ \) such that, if \( e' = (R_{N\setminus\{j\}}, \omega_{N\setminus\{j\}}), \) then
\[
 x_i + x_j = \varphi_i(e') + \omega_j,
\]
and
\[
x_k \varphi_k(e) \text{ for each } k \in \{i, j\}.
\]
Assume \( z(e) \geq 0. \) The case \( z(e) \leq 0 \) can be handled similarly. By (5), \( z(e) > 0. \) By efficiency, \( \varphi_k(e) < p(R_k) \) for each \( k \in N. \) By (5), \( \varphi_k(e) < x_k \) for each \( k \in \{i, j\} \) and therefore, by (4),
\[
 \varphi_i(e) + \varphi_j(e) < x_i + x_j = \varphi_i(e') + \omega_j.
\]

Claim: there is \( k^* \in N \setminus \{i, j\} \) such that \( \varphi_{k^*}(e') < \varphi_{k^*}(e) \). Otherwise,
\[
 \sum_{k \in N\setminus\{i,j\}} \varphi_k(e') \geq \sum_{k \in N\setminus\{i,j\}} \varphi_k(e) \tag{7}
\]
and, since \( \sum_{k \in N\setminus\{j\}} \omega_k = \sum_{k \in N\setminus\{j\}} \varphi_k(e') \), by (6) and (7) we have
\[
 \sum_{k \in N} \omega_k = \sum_{k \in N\setminus\{j\}} \varphi_k(e') + \omega_j > \sum_{k \in N} \varphi_k(e) = \sum_{k \in N} \omega_k,
\]
which is absurd. This proves the Claim.

Now, by the Claim and efficiency, \( \varphi_{k^*}(e') < \varphi_{k^*}(e) \leq p(R_{k^*}) \). This implies
\[
 \varphi_{k^*}(e)P_{k^*} \varphi_{k^*}(e') \tag{8}
\]
and also, by efficiency, \( z(e') \geq 0. \) By (5), \( \varphi_i(e) \neq p(R_i) \) holds. Then, by Lemma 2, \( \omega_j \leq p(R_j); \) and by the endowments lower bound, \( \varphi_j(e) \geq \omega_j. \) It follows from this and (6) that \( 0 \leq \varphi_j(e) - \omega_j < \varphi_i(e') - \varphi_i(e). \) Therefore, \( \varphi_i(e') > \varphi_i(e). \) As \( z(e') \geq 0, \) by efficiency, \( \varphi_i(e') \leq p(R_i). \) Thus, \( \varphi_i(e) < \varphi_i(e') \leq p(R_i) \) and
\[
 \varphi_i(e')P_{k^*} \varphi_i(e) \tag{9}
\]
Note that, as \( z(e) \geq 0 \) and \( z(e') \geq 0, \) (8) and (9) contradict one-sided population monotonicity. We conclude that \( \varphi \) is withdrawal-proof. \( \Box \)

As a consequence of the previous result and Theorem 1, the whole class of iterative reallocation rules precludes this kind of manipulation.

Corollary 1 Each iterative reallocation rule is withdrawal-proof.
4.2 Endowments-merging-proofness

Another manipulation involving variable population is the following. Consider an economy and a pair of agents in that economy. One of those agents gives her endowment to the other and withdraws. The reallocation rule is applied without the first agent and with the second agent’s enlarged endowment. The allocation that the second agent obtains could be divided between the two agents in such a way that each agent is at least as well off as she would have been if the merging had not taken place, and at least one of them is better off. We require immunity to this sort of behavior:

**Endowments-merging-proofness:** For each \( e = (R, \omega) \in \mathcal{E}^N \), each \( \{i, j\} \subset N \) and each \( (x_i, x_j) \in \mathbb{R}^2_+ \) such that \( x_i + x_j = \varphi_i(e) \), where \( e' = (R_{N \setminus \{j\}}, \omega'_i, \omega_{N \setminus \{i, j\}}) \) and \( \omega'_i = \omega_i + \omega'_j \), it is not the case that \( x_k R_k \varphi_k(e) \) for each \( k \in \{i, j\} \), and \( x_k P_k \varphi_k(e) \) for at least one \( k \in \{i, j\} \).

Each rule in the class of iterative reallocation rules precludes such manipulations.

**Lemma 4** Each iterative reallocation rule is endowments-merging-proof.

**Proof.** Let \( \varphi \) be an iterative reallocation rule. By Theorem 1, \( \varphi \) is one-sided endowments monotonic. By Lemma 1, \( \varphi \) is also efficient. Assume \( \varphi \) is not endowments-merging-proof. Then, there are \( e = (R, \omega) \in \mathcal{E}^N \), \( \{i, j\} \subset N \), and \( (x_i, x_j) \in \mathbb{R}^2_+ \) such that, if \( e' = (R_{N \setminus \{j\}}, \omega'_i, \omega_{N \setminus \{i, j\}}) \), then

\[
x_i + x_j = \varphi_i(e'),
\]

\[
x_k R_k \varphi_k(e) \text{ for each } k \in \{i, j\},
\]

and

\[
x_k P_k \varphi_k(e) \text{ for at least one } k \in \{i, j\}.
\]

Assume \( z(e) \geq 0 \). By (12), \( z(e) > 0 \). By efficiency, \( \varphi_k(e) \leq p(R_k) \) for each \( k \in N \). By (11) and (12), \( x_k \geq \varphi_k(e) \) for each \( k \in \{i, j\} \) and \( x_k > \varphi_k(e) \) for at least one \( k \in \{i, j\} \). Therefore, by (10),

\[
\varphi_i(e') = x_i + x_k > \varphi_i(e) + \varphi_j(e).
\]

Claim 1: there is \( k^* \in N \setminus \{i, j\} \) such that \( \varphi_{k^*}(e') < \varphi_{k^*}(e) \). Otherwise,

\[
\sum_{k \in N \setminus \{i, j\}} \varphi_k(e') \geq \sum_{k \in N \setminus \{i, j\}} \varphi_k(e)
\]

and, since \( \sum_{k \in N \setminus \{j\}} \omega_k = \sum_{k \in N \setminus \{j\}} \varphi_k(e') \), by (13) and (14) we have

\[
\sum_{k \in N} \omega_k = \sum_{k \in N \setminus \{j\}} \varphi_k(e') > \sum_{k \in N} \varphi_k(e) = \sum_{k \in N} \omega_k,
\]

which is absurd.
Now, by Claim 1 and efficiency, \( \varphi_k(e') \leq \varphi_k(e) \leq p(R_k) \), which implies \( z(e') \geq 0 \). Let \( e'' = (R_{|N| \setminus \{j\}}, \omega_{|N| \setminus \{j\}}) \). As \( z(e') \geq 0 \), it follows that \( z(e'') \geq 0 \). By Corollary 1, \( \varphi \) is withdrawal-proof, which implies that

\[
\varphi_i(e) + \varphi_j(e) \geq \varphi_i(e'') + \omega_j.
\]  

(15)

Since \( (\omega', \omega_{|N| \setminus \{j\}}) \geq (\omega_{|N| \setminus \{j\}}) \), by one-sided endowments monotonicity, \( \varphi_k(e') R_k \varphi_k(e'') \) for each \( k \in N \setminus \{j\} \). Then efficiency implies

\[
\varphi_k(e') \geq \varphi_k(e'') \text{ for each } k \in N \setminus \{j\}.
\]  

(16)

Combining (13) and (15) we obtain

\[
\varphi_i(e') > \varphi_i(e'') + \omega_j.
\]  

(17)

Claim 2: there is \( k^{**} \in N \setminus \{i, j\} \) such that \( \varphi_{k^{**}}(e'') > \varphi_{k^{**}}(e') \). Otherwise,

\[
\sum_{k \in N \setminus \{i, j\}} \varphi_k(e'') \leq \sum_{k \in N \setminus \{i, j\}} \varphi_k(e')
\]  

(18)

and, since \( \sum_{k \in N \setminus \{j\}} \omega_k = \sum_{k \in N \setminus \{j\}} \varphi_k(e'') \), by (17) and (18) we have

\[
\sum_{k \in N} \omega_k = \omega_j + \sum_{k \in N \setminus \{j\}} \varphi_k(e'') < \sum_{k \in N \setminus \{j\}} \varphi_k(e') = \sum_{k \in N} \omega_k,
\]

which is absurd.

Therefore, by Claim 2, there is \( k^{**} \in N \setminus \{i, j\} \) such that \( \varphi_{k^{**}}(e'') > \varphi_{k^{**}}(e') \). This contradicts (16). We conclude that \( \varphi \) is endowments-merging-proof.

\[
\square
\]

### 4.3 Endowments-splitting-proofness

Consider an economy and assume that an agent in the economy transfers some of her endowment to another agent that was not initially present; the rule is applied, and the guest transfers her assignment to the agent who invited her in. The first agent could obtain an amount that she prefers to her initial assignment. We require immunity to this type of behavior:

**Endowments-splitting-proofness:** For each \( e = (R, \omega) \in \mathcal{E}^N \), each \( i \in N \), each \( j \notin N \), each \( R_j \in \mathcal{R} \), and each \( (\omega'_i, \omega'_j) \in \mathbb{R}_+^2 \) such that \( \omega'_i + \omega'_j = \omega_i \), we have \( \varphi_i(e) R_i [\varphi_i(e') + \varphi_j(e')] \), with \( e' = (R, R_j, \omega'_i, \omega_{N \setminus \{i\}}, \omega'_j) \).

Not all iterative reallocation rules satisfy this property. The following example shows that the uniform reallocation rule violates endowments-splitting-proofness.

**Example 2** Let \( e = (R, \omega) \in \mathcal{E}^{1,2,3} \) be such that \( p(R_1) = 4, p(R_2) = 0, p(R_3) = \omega_1 = \omega_2 = 2 \) and \( \omega_3 = 1 \). Then, \( u_1(e) = 3, u_2(e) = 0, \) and \( u_3(e) = 2 \). Next, let \( R_4 \in \mathcal{R} \) be such that \( p(R_4) = 4 \) and let \( \omega'_4 = 1 \). Consider the economy \( e' = (R, R_4, \omega'_i, \omega_{|N| \setminus \{i\}}, \omega'_j) \in \mathbb{R}_+^2 \).
\[ E^{1,2,3,4} \text{ with } \omega_1' = 1 \text{ (notice that } \omega_1 = \omega_4' + \omega_1'). \] It follows that \( u_1(e') = u_3(e') = u_4(e') = \frac{5}{3}, u_2(e') = 0, \text{ and } u_1(e') + u_4(e') = \frac{40}{9} P_1 3 = u_1(e). \) This implies that \( u \) is not \textit{endowments-splitting-proof}. \hfill \Box

\textbf{Priority reallocation rules}\textsuperscript{11} violate the property as well.

\textbf{Example 3} Consider \( \prec \) as the usual “less than” order in \( \mathbb{N} \). Let \( e = (R, \omega) \in E^{1,2,3,4} \) be such that \( p(R_1) = 0, \omega_1 = 4, p(R_3) = p(R_4) = 6, \text{ and } \omega_3 = \omega_4 = 2 \). Then, \( \varphi_1^\prec(e) = 0, \varphi_2^\prec(e) = 6, \text{ and } \varphi_4^\prec(e) = 2 \). Next, let \( R_2 \in \mathcal{R} \) be such that \( p(R_2) = 4 \) and let \( \omega_2 = 1 \).

Consider the economy \( e' = (R, R_2, \omega_{1,2}, \omega_2, \omega_4') \in E^{1,2,3,4} \) with \( \omega_4' = 1 \) (notice that \( \omega_4 = \omega_4' + \omega_2 = 2 \)). It follows that \( \varphi_1^\prec(e') = 0, \varphi_2^\prec(e') = 4, \varphi_3^\prec(e') = 3, \text{ and } \varphi_4^\prec(e') = 1 \). However, \( \varphi_2^\prec(e') + \varphi_4^\prec(e') = 4 + 1 = 5 P_4 2 = \varphi_4^\prec(e) \). This implies that \( \varphi^\prec \) is not \textit{endowments-splitting-proof}. \hfill \Box

However, the proportional reallocation rule is immune to endowments’ splitting:

\textbf{Remark 2} The proportional reallocation rule is \textit{endowments-splitting-proof}.

\textbf{Proof.} Suppose \( \varphi^p \) is not \textit{endowments-splitting-proof}. Then, there are \( e = (R, \omega) \in E^N, i \in N, j \notin N, R_j \in \mathcal{R}, \text{ and } (\omega_i', \omega_j') \in \mathbb{R}_+^2 \) with \( \omega_i' + \omega_j' = \omega_i \) such that, if \( e' = (R, R_j, \omega_i', \omega_j', \omega_{N \setminus \{i\}, \omega_j'}), \text{ then} \)

\[
[\varphi_i^p(e') + \varphi_j^p(e')] P_i \varphi_i^p(e).
\] (19)

Consider first the case \( z(e) \geq 0 \). By (19), \( \varphi_i^p(e') < p(R_i) \) and therefore

\[
\lambda(e) \omega_i = \varphi_i^p(e') < \varphi_i^p(e') + \varphi_j^p(e').
\] (20)

Since \( z(e') \geq z(e) \geq 0, \)

\[
\varphi_i^p(e') + \varphi_j^p(e') \leq \lambda(e') \omega_i + \lambda(e') \omega_j = \lambda(e') \omega_i.
\] (21)

By (20) and (21),

\[
\lambda(e) < \lambda(e').
\] (22)

It follows that there is \( k \in N \setminus \{i\} \) such that \( \varphi_k^p(e') < \varphi_k^p(e) \). Otherwise, there is a violation of feasibility by (20). Then, \( \lambda(e') \omega_k = \varphi_k^p(e') < \varphi_k^p(e) \leq \lambda(e) \omega_k \), which implies \( \lambda(e') < \lambda(e) \), contradicting (22). If \( z(e) \leq 0 \) and \( z(e') \leq 0 \), the proof is similar to the previous one. Assume then that \( z(e) \leq 0 \) and \( z(e') \geq 0 \). By (19), \( \varphi_i^p(e) > p(R_i) \geq \varphi_i^p(e') \). This implies the existence of \( k \in N \setminus \{i\} \) such that \( \varphi_k^p(e) < \varphi_k^p(e') \). But then,

\[
p(R_k) \leq \varphi_k^p(e) < \varphi_k^p(e') \leq p(R_k),
\]

which is a contradiction. Therefore, \( \varphi^p \) is \textit{endowments-splitting-proof}. \hfill \Box

\textsuperscript{11}Given a linear \( \prec \) order over the set of potential agents \( \mathbb{N} \), the \textit{priority reallocation rule} \( \varphi^\prec \) for economies with excess demand (supply) satiates all suppliers (demanders) and demanders (suppliers) according to order \( \prec \), respecting the endowments lower bound. For economies with excess supply, a symmetric procedure is performed. See Appendix A for a formal definition. It is easy to see that such reallocation rules are iterative in our sense.
4.4 Pre-delivery-proofness

Consider now the case in which one agent makes a “pre-delivery” to some other agent of the trade that this second agent would be assigned if she had participated with everyone else. After the rule is applied, the first agent may end up with an amount she prefers to his assignment if she had not carried out the pre-delivery. We require immunity to this sort of behavior.

**Pre-delivery-proofness:** For each \( e = (R, \omega) \in \mathcal{E}^N \) and each \( \{i,j\} \subset N \) such that \( \omega_i + \omega_j - \varphi_j(e) \geq 0 \), \( \varphi_i(e)R_i\varphi_i(e') \) where \( e' = (R_{N\setminus\{j\}}, \omega_i', \omega_{N\setminus\{i,j\}}) \) and \( \omega_i' = \omega_i + \omega_j - \varphi_j(e) \).

The endowments rule, that in each economy assigns to agents their own endowment, is trivially pre-delivery-proof. However, as long as we require the rule to be efficient and own-peak-only, and meet the endowments lower bound, we reach an impossibility.

**Theorem 2** No efficient and own-peak-only reallocation rule that meets the endowments lower bound is pre-delivery-proof.

**Proof.** Let reallocation rule \( \varphi \) be efficient, own-peak-only and meet the endowments lower bound. Let \( e = (R, \omega) \in \mathcal{E}^{\{1,2,3\}} \) be such that \( 0 < p(R_1) = \omega_2 = \omega_3 < \omega_1 < p(R_2) = p(R_3) \). Then \( z(e) > 0 \) and, as \( p(R_1) < \omega_1 \), by Lemma 2 we have \( \varphi_1(e) = p(R_1) \). By feasibility, there is \( i^* \in \{2,3\} \) such that \( \varphi_{i^*}(e) < \omega_1 \). Assume, without loss of generality, that \( i^* = 2 \). Let \( \omega_2' = \omega_2 + \omega_1 - \varphi_1(e) \). Then \( \omega_2' = \omega_1 \). Consider now the economy \( e' = (R_{\{2,3\}}, \omega_2', \omega_3) \). It follows that \( z(e') > 0 \). By efficiency, \( \varphi_2(e') \leq p(R_2) \), and since \( p(R_2) > \omega_2' \), by the endowments lower bound we have \( \varphi_2(e') \geq \omega_2' = \omega_1 \). By feasibility then, \( \varphi_2(e') = \omega_1 \). Therefore, \( \varphi_2(e') < \omega_1 = \varphi_2(e') < p(R_2) \), which implies \( \varphi_2(e') \neq \omega_1 \) and \( \varphi \) is not pre-delivery-proof. \( \square \)

Of course, the previous result extends to the whole class of iterative reallocation rules.

**Corollary 2** No iterative reallocation rule is pre-delivery-proof.

5 Final comments

We conclude with some remarks. One may ask whether the definition of withdrawal-proofness can be made with just one of the agents involved in the manipulation strictly improving. However, not even the uniform reallocation rule satisfies this variant, as the following example shows:

**Example 4** Consider \( e = (R, \omega)^{\{1,2,3,4\}} \) with \( p(R_1) = p(R_4) = 1 \), \( p(R_2) = 4 \), \( p(R_3) = 3 \), and \( \omega_1 = \omega_4 = 3 \), \( \omega_2 = \omega_3 = 1 \). Then, \( z(e) > 0 \) and \( u_1(e) = u_4(e) = 1 \), \( u_2(e) = u_3(e) = 3 \). When agent 4 withdraws, if \( e' = (R_{\{1,2,3\}}, \omega_{\{1,2,3\}}) \), then \( u_1(e') = 1 \), \( u_2(e') = 1 \).
In $\text{Thomson (2014)}$, the property of withdrawal-proofness is presented in this variant: one of the agents involved in the manipulation can be indifferent between the amount she gets from the rule and the amount she gets after the manipulation is performed. However, in all the impossibility examples presented there both agents get strictly better off, so our version does not hold in those examples (classical multi-commodity exchange model with homothetic and quasi-linear preferences) either.

The results obtained in our paper are in sharp contrast with the findings in models with several goods and classical preferences. The Walrasian reallocation rule is neither withdrawing-proof, nor endowments-merging-proof, nor endowments-splitting-proof. These negative results are obtained by $\text{Thomson (2014)}$ in two classical subdomains: (i) the domain of economies in which preferences are homothetic and strictly convex, and individual endowments are proportional, and (ii) the domain of economies in which preferences are quasi-linear and strictly convex. The Walrasian reallocation rule, however, is pre-delivery-proof on the classical domain (see $\text{Thomson, 2014}$). “Constrained dictatorial rules”, defined by maximizing the welfare of a particular agent subject to each of the others finding their assignment at least as desirable as their endowment, satisfy none of these various requirements either (see $\text{Thomson, 2022}$).

Several questions are left open for further research. For example, finding out which properties are required for a rule to be endowments-merging-proof or endowments-splitting-proof, and knowing whether iterative reallocation rules exhaust the rules that are withdrawal-proof or endowments-merging-proof. Coalitional versions of the manipulations studied in Section 4 seem worth studying as well.

\section*{A Independence of axioms in Theorem 1}

In order to study the independence of axioms in the characterization of Theorem 1, next we consider several reallocation rules. For each $N \in \mathcal{N}$ and each $e \in \mathcal{E}^N$, let $N^+(e) = \{i \in N : p(R_i) > \omega_i\}$ be the set of demanders of $e$. Agents in $N \setminus N^+$ are called suppliers. Let $S(e) \equiv \sum_{j \in N \setminus N^+} (\omega_j - p(R_j))$.

Given a linear $\preceq$ order over the set of potential agents $\mathbb{N}$, the priority reallocation rule $\varphi^\preceq$ for each economy with excess demand (supply) satiates all suppliers (demanders) and demanders (suppliers) according to order $\preceq$, respecting the endowments lower bound. So if $e = (R, \omega) \in \mathcal{E}^N$ is such that $z(e) \geq 0$,

$$\varphi^\preceq_i(e) = \begin{cases} p(R_i) & \text{if } i \in N \setminus N^+(e) \\ \min \left\{ p(R_j), \omega_i + S(e) - \sum_{j \in N^+ : j < i} \Delta \varphi^\preceq_j(e) \right\} & \text{if } i \in N^+(e) \end{cases}$$

In case $z(e) < 0$, the rule is defined similarly. Priority reallocation rules allow the definition of a reallocation rule which is not one-sided population monotonic.
**Reallocation rule \( \varphi \):** for each \( N \in \mathcal{N} \), each \( e \in \mathcal{E}^N \), and each \( i \in N \),

\[
\varphi_i^+(e) = \begin{cases} 
\varphi_i^<(e) & \text{if } |N| \text{ is odd} \\
\varphi_i^>(e) & \text{if } |N| \text{ is even}
\end{cases}
\]

where \( \geq \) is the dual of \( \leq \).

Next, let us recall the celebrated **uniform rule**, first characterized by Sprumont (1991).

**Uniform rule, \( \varphi^u \):** for each \( N \in \mathcal{N} \), each \( e \in \mathcal{E}^N \), and each \( i \in N \),

\[
\varphi_i^u(e) = \begin{cases} 
\min \{ p(R_i), \lambda(e) \} & \text{if } z(e) \geq 0 \\
\max \{ p(R_i), \lambda(e) \} & \text{if } z(e) < 0
\end{cases}
\]

where \( \lambda(e) \) and solves \( \sum_{j \in N} \varphi_j^u(e) = \sum_{j \in N} \omega_j \).

Since this rule does not take into account individual endowments, it trivially does not meet the **endowments lower bound**.

The following rule satiates as many agents as possible. For economies with excess demand, demanders are satiated according to their claims. First minimal demands are satiated uniformly. If there is some supply left, then the next smallest demands are satiated, and so on. This reallocation rule is not **one-sided endowments monotonic**.

**Maximally satiating reallocation rule \( \varphi^\max \):** For each \( N \in \mathcal{N} \) and each \( e \in \mathcal{E}^N \) such that \( z(e) \geq 0 \), partition \( N^+(e) \) into subsets \( N_1, N_2, \ldots, N_s \) such that (i) for each \( t \in \{1, \ldots, s\} \), \( p(R_i) - \omega_i = p(R_j) - \omega_j \) for each \( i, j \in N_t \), and (ii) \( p(R_i) - \omega_i < p(R_j) - \omega_j \) if \( i \in N_r, j \in N_s, \) and \( r < s \). Then,

\[
\varphi_i^\max(e) = p(R_i)
\]

if \( i \in N \setminus N^+ \), and

\[
\varphi_i^\max(e) = \min \left\{ \frac{1}{|N_t|} \left( S(e) - \sum_{j \in \bigcup_{r=1}^{t-1} N_r} \Delta \varphi_j(e) \right) \right\}
\]

if \( i \in N_t \) and \( t \in \{1, \ldots, s\} \). The formula when \( z(e) < 0 \) is obtained similarly.

| \( \varphi \) | Own-peak-only | Endow LB | OS endow mon | OS pop mon |
|-------------|----------------|---------|--------------|-------------|
| \( \varphi^\max \) | + | + | - | + |
| \( \varphi^u \) | + | - | + | + |
| ? | - | + | + | + |

Table 1: **Independence of axioms in the characterization of Theorem 1**.

Each one of the previously presented rules satisfies all properties of the characterization in Theorem 1 except one. This is shown in Table 1. For example, reallocation
Next, define a linear order

For each

Strategy-proofness: For each

regardless of the preferences declared by the other agents. Formally ,

is a strong property . Together with

Strategy-proofness

Peak-only: For each

if two profiles of preferences have the same peaks, then the reallocations recommended by the rule in both profiles are the same.

Peak-only: For each

\((R, \omega) \in \mathcal{E}^N\) and each \(R' \in \mathcal{R}\) such that \(p(R'_i) = p(R_i)\) for each \(i \in N\), if \(e' = (R', \omega)\) then \(\phi(e') = \phi(e)\).

Iterative reallocation rules are not in general peak-only, because they can be bossy: it could be that a change in one agent’s preferences (preserving the peak amount) only affects other agents’ reallocations (see Example 5). When a rule is own-peak-only and non-bossy\(^{12}\), then it is also peak-only. Sequential allocation rules in Barberà et al. (1997) satisfy replacement monotonicity, which in turn implies non-bossiness. For this reason, own-peak-onliness is equivalent to peak-onliness for those rules.

A reallocation rule is strategy-proof if, for each agent, truth-telling is always optimal, regardless of the preferences declared by the other agents. Formally,

Strategy-proofness: For each

\((R, \omega) \in \mathcal{E}^N\), each \(i \in N\), and each \(R'_i \in \mathcal{R}\), if \(e' = (R'_i, R_{-i}, \omega)\) then \(\phi_i(e) R_i \phi_i(e')\).

Strategy-proofness is a strong property. Together with efficiency, it implies the own-peak-only property for reallocation rules (this is proven, for example, in Lemma 3 of Klaus et al., 1998). Since we do not impose strategy-proofness in our paper, we need to explicitly invoke the own-peak-only property.

To see that iterative reallocation rules need not be peak-only nor strategy-proof, consider reallocation rule \(\phi^*\) that works as follows.\(^{13}\) For each \(N \in \mathcal{N}\) and each \((R, \omega) \in \mathcal{E}^N\) with \(z(e) \geq 0\), let \(i^*\) be the agent in \(N \setminus N^+(e)\) (the set of suppliers) with the lowest index.\(^{14}\) Next, define a linear order \(\preceq\) over \(N^+(e)\) (the set of demanders) saying that \(i \preceq j\) if and only if either \(p(R_i) - \omega_i < p(R_j) - \omega_j\) or \(p(R_i) - \omega_i = p(R_j) - \omega_j\) and \(i < j\). If \(0P_i^* \sum_{i \in N} \omega_i\), then we apply the priority reallocation rule according to \(\preceq\), i.e., \(\phi^* (e) = \phi^\preceq (e)\). If \(\sum_{i \in N} \omega_j R_i^* 0\), then we apply the uniform reallocation rule, i.e., \(\phi^* (e) = \phi^u (e)\). The definition of the rule for economies with excess supply is similar.

\(^{12}\)Non-bossiness: For each \(e = (R, \omega) \in \mathcal{E}^N\), each \(i \in N\), and each \(R'_i \in \mathcal{R}\), if \(e' = (R'_i, R_{-i}, \omega)\) and \(\phi_i(e') = \phi_i(e)\) then \(\phi(e') = \phi(e)\).

\(^{13}\)Here we use the terminology presented in Appendix A.

\(^{14}\)If \(N \setminus N^+(e) = \emptyset\), simply allocate to each agent their endowment.
is easy to see that reallocation rule $\varphi^*$ belongs to the class of iterative reallocation rules. The following example shows that $\varphi^*$ is neither peak-only nor strategy-proof.

**Example 5** Let $e = (R, \omega) \in \mathcal{E}^{\{1,2,3\}}$ be such that $\omega_1 = 9$, $\omega_2 = 1$, $\omega_3 = 4$, $p(R_1) = 1$ and $0$, $p(R_2) = 7$, and $p(R_3) = 9$. Since $p(R_3) - \omega_3 = 9 - 4 = 5 < 6 = 7 - 1 = p(R_2) - \omega_2$, we have $\varphi_1^*(e) = 1$, $\varphi_2^*(e) = 4$, and $\varphi_3^*(e) = 9$. Let $\tilde{R}_1 \in \mathcal{R}$ be such that $p(\tilde{R}_1) = p(R_1)$ and $14$, $\tilde{P}_1 = 0$, and consider economy $\tilde{e} = (\tilde{R}_1, R_{-1}, \omega)$. Then, $\varphi_1(\tilde{e}) = 1$, $\varphi_2(\tilde{e}) = 5$, and $\varphi_3(\tilde{e}) = 8$. Thus, although $\varphi^*$ is own-peak-only, it is certainly not peak-only.

To see that $\varphi^*$ does not satisfy strategy-proofness, let $R_2' \in \mathcal{R}$ be such that $p(R_2') = 5.5$ and consider economy $e' = (R_2', R_{-2}, \omega)$. Since $p(R_2') - \omega_2 = 4.5 < 5 = 9 - 4 = p(R_3) - \omega_3$, we have $\varphi_2^*(e') = 5.5$. Therefore, $\varphi_2^*(e') = 5.5 \neq \varphi_2^*(e)$. \hfill $\diamond$

Finally, it is worth mentioning that the weakly sequential rules in Bonifacio (2015) are immune to misrepresentation of preferences and endowments in a very general way: they are bribe-proof.\footnote{This concept was introduced by Schummer (2000) and applied to the model with a social endowment by Massó and Neme (2007).} This property does not allow any group of agents to compensate one of its subgroups to misrepresent their preferences or endowments in order that, after an appropriate redistribution of what the rule reallocates to the group (adjusted by the resource surplus or deficit they all engage in by misreporting), (i) each agent in the misrepresenting subgroup obtain a preferred amount, and (ii) the rest of the agents in the group are not made worse-off.

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