HURWITZ QUATERNION ORDER AND ARITHMETIC RIEHMANN SURFACES

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ABSTRACT. We clarify the explicit structure of the Hurwitz quaternion order, which is of fundamental importance in Riemann surface theory and systolic geometry.

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1. Congruence towers and the 4/3 bound

Hurwitz surfaces are an important and famous family of Riemann surfaces.

We clarify the explicit structure of the Hurwitz quaternion order, which is of fundamental importance in Riemann surface theory and
systolic geometry. A Hurwitz surface \( X \) by definition attains the upper bound of \( 84(g_X - 1) \) for the order \( |\text{Aut}(X)| \) of the holomorphic automorphism group of \( X \), where \( g_X \) is its genus. In [30], we proved a systolic bound

\[
\text{sys}_{\pi_1}(X) \geq \frac{4}{3} \log(g_X)
\]

(1.1)

for Hurwitz surfaces in a principal congruence tower (see below). Here the systole \( \text{sys}_{\pi_1} \) is the least length of a noncontractible loop in \( X \). The question of the existence of other congruence towers of Riemann surfaces satisfying the bound (1.1), remains open.

Marcel Berger’s monograph [7, pp. 325–353] contains a detailed exposition of the state of systolic affairs up to '03. More recent developments are covered in [26].

While (1.1) can be thought of as a differential-geometric application of quaternion algebras, such an application of Lie algebras may be found in [5].

We will give a detailed description of a specific quaternion algebra order, which constitutes the arithmetic backbone of Hurwitz surfaces. The existence of a quaternion algebra presentation for Hurwitz surfaces is due to G. Shimura [42, p. 83]. An explicit order was briefly described by N. Elkies in [16] and in [17], with a slight discrepancy between the two descriptions, see Remark 2.3 below. We have been unable to locate a more detailed account of this important order in the literature. The purpose of this note is to provide such an account.

A Hurwitz group is by definition a (finite) group occurring as the (holomorphic) automorphism group of a Hurwitz surface. Such a group is the quotient \( \Delta_{2,3,7}/\Gamma \) of a pair of Fuchsian groups. Here \( \Delta_{2,3,7} \) is the \((2,3,7)\) triangle group, while \( \Gamma \triangleleft \Delta_{2,3,7} \) is a normal subgroup.

Let \( \eta = 2 \cos \frac{2\pi}{7} \) and \( K = \mathbb{Q}[\eta] \), a cubic extension of \( \mathbb{Q} \). A class of Hurwitz groups arise from ideals in the Hurwitz quaternion order \( \mathcal{Q}_{\text{Hur}} \subset (\eta, \eta)_K \), see Definition 2.1 for details.

Recall that for a division algebra \( D \) over a number field \( k \), the discriminant \( \text{disc}(D) \) is the product of the finite ramification places of \( D \). Let \( \mathcal{Q} \) be an order of \( D \), and let \( O_K \) be the ring of algebraic integers

\[ ^1 \text{In the literature, the term “Hurwitz quaternion order” has been used both in the sense used in the present text, and in the sense of the unique maximal order of Hamilton’s rational quaternions.} \]
in $K$. By definition $Q^1$ is the group of elements of norm 1 in $Q$, and a principal congruence subgroup of $Q^1$ is a subgroup of the form

$$Q^1(I) = \{ x \in Q^1 : x \equiv 1 \pmod{IQ} \}$$

(1.2)

where $I \triangleleft O_K$. Any subgroup containing such a subgroup is called a congruence subgroup.

The Hurwitz order is described in Section 2. In Section 3 we verify its maximality. The precise relationship between the order and the $(2, 3, 7)$ group is given in Section 4. In Section 5 we note that the Hurwitz order is Azumaya, which implies that every ideal of the order is generated by a central element, and every automorphism is inner. It also follows that all quotient rings are matrix rings, and in Section 6 we present some explicit examples of these quotients and their associated congruence subgroups.

Recent developments in systolic geometry include [1, 3, 4, 5, 6, 8, 9, 10, 11, 14, 15, 18, 19, 22, 24, 25, 27, 28, 31, 34, 35, 37, 38, 39, 40, 45].

### 2. The Hurwitz Order $Q_{\text{Hur}}$

Let $H^2$ denote the hyperbolic plane. Let $\text{Aut}(H^2) = \text{PSL}_2(\mathbb{R})$ be its group of orientation-preserving isometries. Consider the lattice $\Delta_{2,3,7} \subset \text{Aut}(H^2)$, defined as the even part of the group of reflections in the sides of the $(2, 3, 7)$ hyperbolic triangle, i.e. geodesic triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{7}$. We follow the concrete realization of $\Delta_{2,3,7}$ in terms of the group of elements of norm one in an order of a quaternion algebra, given by N. Elkies in [16, p. 39] and in [17, Subsection 4.4].

Let $K$ denote the real subfield of $\mathbb{Q}[ho]$, where $\rho$ is a primitive $7^{th}$ root of unity. Thus $K = \mathbb{Q}[\eta]$, where the element $\eta = \rho + \rho^{-1}$ satisfies the relation

$$\eta^3 + \eta^2 - 2\eta - 1 = 0.$$  

(2.1)

Note the resulting identity

$$(2 - \eta)^3 = 7(\eta - 1)^2.$$  

(2.2)

There are three embeddings of $K$ into $\mathbb{R}$, defined by sending $\eta$ to any of the three real roots of (2.1), namely

$$2 \cos \left( \frac{2\pi}{7} \right), 2 \cos \left( \frac{4\pi}{7} \right), 2 \cos \left( \frac{6\pi}{7} \right).$$
We view the first embedding as the ‘natural’ one $K \hookrightarrow \mathbb{R}$, and denote the others by $\sigma_1, \sigma_2 : K \rightarrow \mathbb{R}$. Notice that $2 \cos(2\pi/7)$ is a positive root, while the other two are negative.

From the minimal polynomial we have $\text{Tr}_{K/F}(\eta) = -1$. Multiplying (2.1) by the ‘conjugate’ $\eta^3 - \eta^2 - 2\eta + 1$ gives

$$(\eta^2)^2 - 5(\eta^2)^2 + 6(\eta^2) - 1 = 0,$$

so similarly $\text{Tr}_{K/F}(\eta^2) = 5$. The recursion relation

$$\text{Tr}(\eta^{3+i}) = -\text{Tr}(\eta^{2+i}) + 2\text{Tr}(\eta^{1+i}) + \text{Tr}(\eta^i)$$

provides $\text{Tr}(\eta^3) = -4$ and $\text{Tr}(\eta^4) = 13$. Traces of multiples in the integral basis $1, \eta, \eta^2$ then give the discriminant

$$\text{disc}(K/\mathbb{Q}) = \begin{vmatrix} 3 & -1 & 5 \\ -1 & 5 & -4 \\ 5 & -4 & 13 \end{vmatrix} = 49.$$  

By Minkowski’s bound [23, Subsection 30.3.3], it follows that every ideal class contains an ideal of norm $< \frac{3}{2} \sqrt{49} < 2$, which proves that $O_K = \mathbb{Z}[\eta]$ is a principal ideal domain. The only ramified prime is $7O_K = (2-\eta)^3$, cf. (2.2) and using the fact that $\eta - 1 = (\eta^2 + 2\eta)^{-1}$ is invertible in $O_K$. Note that the minimal polynomial $f(t) = t^3 + t^2 - 2t - 1$ remains irreducible modulo 2, so that

$$O_K/(2) \cong \mathbb{F}_2[\bar{\eta}] = \mathbb{F}_8,$$  

the field with 8 elements.

**Definition 2.1.** We let $D$ be the quaternion $K$-algebra

$$(\eta, \eta)_K = K[i, j \mid i^2 = j^2 = \eta, ji = -ij].$$  

(2.4)

As mentioned above, the root $\eta > 0$ defines the natural imbedding of $K$ into $\mathbb{R}$, and so $D \otimes \mathbb{R} = M_2(\mathbb{R})$, and thus the imbedding is unramified. On the other hand, we have $\sigma_1(\eta) < 0$ and $\sigma_2(\eta) < 0$, so the algebras $D \otimes_{\sigma_1} \mathbb{R}$ and $D \otimes_{\sigma_2} \mathbb{R}$ are isomorphic to the standard Hamilton quaternion algebra over $\mathbb{R}$.

Moreover, $D$ is unramified over all the finite places of $K$ [30, Prop. 7.1].

**Remark 2.2.** By the Albert-Brauer-Hasse-Noether theorem [36, Theorem 32.11], $D$ is the only quaternion algebra over $K$ with this ramification data.
Let $\mathcal{O} \subseteq D$ be the order defined by
\[ \mathcal{O} = O_K[i, j]. \]
Clearly, the defining relations of $D$ serve as defining relations for $\mathcal{O}$ as well:
\[ \mathcal{O} \cong \mathbb{Z}[\eta][i, j] | i^2 = j^2 = \eta, ji = -ij]. \quad (2.5) \]
Fix the element $\tau = 1 + \eta + \eta^2$, and define an element $j' \in D$ by setting
\[ j' = \frac{1}{2}(1 + \eta i + \tau j). \]
Notice that $j'$ is an algebraic integer of $D$, since the reduced trace is 1, while the reduced norm is
\[ \frac{1}{4}(1 - \eta \cdot \eta^2 - \eta \cdot \tau^2 + \eta^2 \cdot 0) = -1 - 3\eta, \]
so that both are in $O_K$. In particular, we have the relation
\[ j'^2 = j' + (1 + 3\eta). \quad (2.6) \]
We define an order $\mathcal{Q}_{\text{Elk}} \subset D$ by setting
\[ \mathcal{Q}_{\text{Elk}} = \mathbb{Z}[\eta][i, j']. \quad (2.7) \]
Finally, we define a new order $\mathcal{Q}_{\text{Hur}} \subset D$ by setting
\[ \mathcal{Q}_{\text{Hur}} = \mathbb{Z}[\eta][i, j, j']. \quad (2.8) \]

Remark 2.3. There is a discrepancy between the descriptions of a maximal order of $D$ in [16, p. 39] and in [17, Subsection 4.4]. According to [16, p. 39], $\mathbb{Z}[\eta][i, j, j']$ is a maximal order. Meanwhile, in [17, Subsection 4.4], the maximal order is claimed to be the order $\mathcal{Q}_{\text{Elk}} = \mathbb{Z}[\eta][i, j']$, described as $\mathbb{Z}[\eta]$-linear combinations of the elements 1, $i$, $j'$, and $ij'$, on the last line of [17, p. 94]. The correct answer is the former, i.e. the order (2.8).

We correct this minor error in [17], as follows.

Lemma 2.4. The order $\mathcal{Q}_{\text{Hur}}$ strictly contains $\mathcal{Q}_{\text{Elk}} = \mathbb{Z}[\eta][i, j'].$

Proof. The identities (2.6) and
\[ j' i = \eta^2 + i - ij' \quad (2.9) \]
show that the module
\[ \mathcal{Q}'_{\text{Elk}} = \mathbb{Z}[\eta] + Z[\eta]i + Z[\eta]j' + Z[\eta]ij', \quad (2.10) \]
which is clearly contained in $Q_{Elk}$, is closed under multiplication, and thus equal to $Q_{Elk}$. Moreover, the set $\{1, i, j, ij\}$ is a basis of $D$ over $K$, and a computation shows that

$$j = \frac{-9 + 2\eta + 3\eta^2}{7} + \frac{3 - 3\eta - \eta^2}{7}i + \frac{18 - 4\eta - 6\eta^2}{7}j',$$

with non-integral coefficients. Therefore $j \notin Q_{Elk}$. □

3. Maximality of the order $Q_{Hur}$

**Theorem 3.1.** The order $Q_{Hur}$ is a maximal order of $D$.

**Proof.** By Lemma 2.4 it is enough to show that every order containing $Q_{Elk}$ is contained in $Q_{Hur}$. Let $M \supseteq Q_{Elk}$ be an order, namely a ring which is finite as a $O_K$-module, and let $x \in M$. Since $\{1, i, j, ij\}$ is a $K$-basis for the algebra $D$, we can write

$$x = \frac{1}{2}(a + bi + cj + dij)$$

for suitable $a, b, c, d \in K$. Recall that every element of an order satisfies a monic polynomial over $O_K$, so in particular it has integral trace. Since we have $x, ix, jx, ijx \in M$, with traces $a, \eta b, \eta c, -\eta^2 d$, respectively, while the element $\eta = (\eta^2 + \eta - 2)^{-1}$ is invertible in $O_K$, we conclude that, in fact, $a, b, c, d \in O_K$. Now,

$$\text{tr}(xj') = \frac{1}{4}\text{tr}((a + bi + cj + dij)(1 + \eta i + \tau j)) = \frac{1}{2}(a + \eta^2 b + \eta \tau c)$$

and

$$\text{tr}(xij') = \frac{1}{4}\text{tr}((a + bi + cj + dij)i(1 + \eta i + \tau j)) = \frac{1}{2}(\eta^2 a + \eta b - \eta^2 \tau d).$$

Since these are integers, and since $\eta \tau \equiv \eta + 1$ and $\eta^3 \tau \equiv 1$ modulo $2O_K$, we have that $a \equiv \eta^2 b + (\eta + 1)c$, and $d \equiv \eta^3 a + \eta^2 b \equiv \tau b + \eta c$. It then follows that

$$x - (\eta^2 + 2\eta + 1)cj' - ((\eta^2 + 3\eta + 1)c + b)i j' \in O_K[i, j],$$

so that $x \in Q_{Hur}$. □

**Remark 3.2.** Since $KQ_{Hur} = D$, the center of $Q_{Hur}$ is $\text{Cent}(Q_{Hur}) = Q_{Hur} \cap \text{Cent}(D) = Q_{Hur} \cap K = O_K$.

While $O$ admits the presentation (2.5), typical of symbol algebras, it should be remarked that $Q_{Hur}$ cannot have such a presentation.
Remark 3.3. There is no pair of anticommuting generators of $Q_{\text{Hur}}$ over $\mathbb{Z}[\eta]$.

Proof. One can compute that $Q_{\text{Hur}}/2Q_{\text{Hur}}$ is a $2 \times 2$ matrix algebra [30, Lemma 4.3], and in particular non-commutative; however anticommuting generators will commute modulo 2.

The prime 2 poses the only obstruction to the existence of an anticommuting pair of generators. Indeed, adjoining the fraction $\frac{1}{2}$, we clearly have

$$Q_{\text{Hur}}[\frac{1}{2}] = \mathcal{O}[\frac{1}{2}] = O_K[\frac{1}{2}][i, j \mid i^2 = j^2 = \eta, ji = -ij],$$

and this is an Azumaya algebra over $O_K[\frac{1}{2}]$, see Definition 5.1. A presentation of $Q_{\text{Hur}}$ is given in Theorem 4.2.

4. The $(2, 3, 7)$ group inside $Q_{\text{Hur}}$

The group of elements of norm 1 in the order $Q_{\text{Hur}}$, modulo the center $\{\pm 1\}$, is isomorphic to the $(2, 3, 7)$ group [17, p. 95]. Indeed, Elkies gives the elements

$$g_2 = \frac{1}{\eta}ij,$$

$$g_3 = \frac{1}{2}(1 + (\eta^2 - 2)j + (3 - \eta^2)ij),$$

$$g_7 = \frac{1}{2}(\tau - 2) + (2 - \eta^2)i + (\tau - 3)ij),$$

satisfying the relations $g_2^2 = g_3^3 = g_7^7 = -1$ and $g_2 = g_7g_3$, which therefore project to generators of $\Delta_{2,3,7} \subset \text{PSL}_2(\mathbb{R})$.

Theorem 4.1. The Hurwitz order is generated, as an order, by the elements $g_2$ and $g_3$, so that we can write $Q_{\text{Hur}} = O_K[g_2, g_3]$.

Proof. We have $g_2, g_3, g_7 \in Q_{\text{Hur}}$ by the invertibility of $\eta$ in $O_K$ and the equalities

$$g_3 = (3 + 6\eta - \eta^2) + (1 + 3\eta)i - (2 + \eta^2)jj' - 2(ijj' - (1 - \eta)ij),$$

$$g_7 = (\tau + 3\eta) + 2(1 + \eta)i - (\eta + \eta^2)jj' + (2 - \tau)(ijj' - (1 - \eta)ij).$$

Conversely, we have the relations

$$i = (1 + \eta)(g_3g_2 - g_2g_3),$$

$$j = (1 + \eta)(1 + (\eta^2 + \eta - 1)g_2 - 2g_3),$$

$$j' = (1 + \eta)i + (\eta^2 - 2)ij + j,$$
proving the lemma. □

**Theorem 4.2.** A basis for the order $\mathcal{Q}_{\text{Hur}}$ as a free module over $\mathbb{Z}[\eta]$ is given by the four elements $1$, $g_2$, $g_3$, and $g_2g_3$.

The defining relations $g_2^2 = -1$, $g_3^2 = g_3 - 1$ and $g_2g_3 + g_3g_2 = g_2 \cdot (\eta^2 + \eta - 1)$ provide a presentation of $\mathcal{Q}_{\text{Hur}}$ as an $O_K$-order.

**Proof.** The module spanned by $1$, $g_2$, $g_3$, $g_2g_3$ is closed under multiplication by the relations given in the statement (which are easily verified); thus $O_K[g_2, g_3] = \text{span}_{O_K} \{1, g_2, g_3, g_2g_3\}$. The relations suffice since the ring they define is a free module of rank 4, which clearly project onto $\mathcal{Q}_{\text{Hur}}$. □

**Remark 4.3.** An alternative basis for the order $\mathcal{Q}_{\text{Hur}}$ as a free module over $\mathbb{Z}[\eta]$ is given by the four elements $1$, $i$, $jj'$, and $\ell = ijj' - (1-\eta)ij$.

**Remark 4.4.** Since $\mathcal{O}$ is a free module of rank 4 over $O_K$, so is $\frac{1}{2}\mathcal{O}$, and $\frac{1}{2}\mathcal{O}/\mathcal{O}$ is a 4-dimensional vector space over $O_K/2O_K$, which is the field of order 8. Furthermore, one can check that $\mathcal{Q}_{\text{Hur}}/\mathcal{O}$ is a two-dimensional subspace, namely $[\frac{1}{2}\mathcal{O} : \mathcal{Q}_{\text{Hur}}] = [\mathcal{Q}_{\text{Hur}} : \mathcal{O}] = 2^6$ where we are calculating the indices of the orders as abelian groups.

## 5. AZUMAYA ALGEBRAS

We briefly describe a useful generalization of the class of central simple algebras over fields, to algebras over commutative rings.

**Definition 5.1** (e.g. [41, Chapter 2]). Let $R$ be a commutative ring. Let $A$ be an $R$-algebra which is a faithful finitely generated projective $R$-module. If the natural map $A \otimes_R A^{\text{op}} \rightarrow \text{End}_R(A)$ (action by left and right multiplication) is an isomorphism, then $A$ is an Azumaya algebra over $R$.

Suppose every non-zero prime ideal of $R$ is maximal (which is the case with every Dedekind domain, such as $O_K = \mathbb{Z}[\eta]$), and let $F$ denote the ring of fractions of $R$. It is known that if $A$ is an $R$-algebra, which is a finite module, such that

(1) for every maximal ideal $M \triangleleft R$, $A/MA$ is a central simple algebra, of fixed degree, over $R/MA$; and

(2) $A \otimes_R F$ is central simple, of the same degree, over $F$,

then $A$ is Azumaya over $R$ [41 Theorem 2.2.a]. The second condition clearly holds for $\mathcal{Q}_{\text{Hur}}$ over $O_K$ since $\mathcal{Q}_{\text{Hur}} \otimes_{O_K} K \cong D$. 


In [30, Lemma 4.3] we proved the following theorem.

**Theorem 5.2.** For every ideal $I \triangleleft O_K$, we have an isomorphism

$$Q_{Hur}/IQ_{Hur} \cong M_2(O_K/I).$$

This was proved in [30] for an arbitrary maximal order in a division algebra with no finite ramification places, by decomposing $I$ as a product of prime power ideals, applying the isomorphism $Q/p^t = Q_p/p^tQ_p$ [36, Section 5] for $Q = Q_{Hur}$ ($Q_p$ being the completion with respect to the $p$-adic valuation), and using the structure of maximal orders over a local ring [36, Section 17].

We therefore obtain the following corollary.

**Corollary 5.3.** The order $Q_{Hur}$ is an Azumaya algebra.

This fact has various ring-theoretic consequences. In particular, there is a one-to-one correspondence between two-sided ideals of $Q_{Hur}$ and ideals of its center, $O_K$ [41, Proposition 2.5.b]. Since $O_K$ is a principal ideal domain, it follows that every two-sided ideal of $Q_{Hur}$ is generated by a single central element.

Another property of Azumaya algebras is that every automorphism is inner (namely, induced by conjugation by an invertible element) [41, Theorem 2.10], in the spirit of the Skolem-Noether theorem, cf. [33, p. 107].

**6. Quotients of $Q_{Hur}$**

The examples discussed in this section have not appeared in an explicit form in the published literature.

In order to make Theorem 5.2 explicit, suppose $I \triangleleft O_K$ is an odd ideal (namely $I + 2O_K = O_K$). By the inclusion $2Q_{Hur} \subseteq \mathcal{O}$, we have that $\mathcal{O} + IQ_{Hur} = Q_{Hur}$ and $\mathcal{O} \cap IQ_{Hur} = I\mathcal{O}$. Therefore

$$\mathcal{O}/I\mathcal{O} = \mathcal{O}/(\mathcal{O} \cap IQ_{Hur}) \cong (IQ_{Hur} + \mathcal{O})/IQ_{Hur} = Q_{Hur}/IQ_{Hur},$$

and so $\mathcal{O}/I\mathcal{O} \cong M_2(L)$ for $L = O_K/I$. From the presentation of $\mathcal{O}$, see (2.5), it follows that

$$\mathcal{O}/I\mathcal{O} \cong L[i,j \mid i^2 = j^2 = \eta, ji = -ij],$$

which allows for an explicit isomorphism $\mathcal{O}/I\mathcal{O} \to M_2(L)$. 
Example 6.1 (First Hurwitz triplet). The quotient \( Q_{\text{Hur}}/13Q_{\text{Hur}} \) can be analyzed as follows. Since the minimal polynomial \( \lambda^3 + \lambda^2 - 2\lambda - 1 \) factors over \( \mathbb{F}_{13} \) as \( (\lambda - 7)(\lambda - 8)(\lambda - 10) \), we obtain the ideal decomposition

\[
13O_K = \langle 13, \eta - 7 \rangle \langle 13, \eta - 8 \rangle \langle 13, \eta - 10 \rangle,
\]

(6.1)

and the isomorphism \( O_K/\langle 13 \rangle \to \mathbb{F}_{13} \times \mathbb{F}_{13} \times \mathbb{F}_{13} \), defined by \( \eta \mapsto (7, 8, 10) \). In fact, one has

\[
13 = \eta(\eta + 2)(2\eta - 1)(3 - 2\eta)(\eta + 3),
\]

where \( \eta(\eta + 2) \) is invertible, and the other factors generate the ideals given above, in the respective order; therefore, (6.1) can be rewritten as

\[
13O_K = (2\eta - 1)O_K \cdot (3 - 2\eta)O_K \cdot (\eta + 3)O_K.
\]

An embedding \( O_K[i]/\langle 13 \rangle \hookrightarrow M_2(\mathbb{F}_{13}) \times M_2(\mathbb{F}_{13}) \times M_2(\mathbb{F}_{13}) \) is obtained by mapping the generator \( i \) via

\[
i \mapsto \begin{pmatrix}
0 & 1 \\
7 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
8 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
10 & 0
\end{pmatrix},
\]

satisfying the defining relation \( i^2 = \eta \). In order to extend this embedding to \( Q_{\text{Hur}}/13Q_{\text{Hur}} \), we need to find in each case a matrix which anti-commutes with \( i \), and whose square is \( \eta \). Namely, we seek a matrix

\[
\begin{pmatrix}
a & b \\
-\eta b & -a
\end{pmatrix},
\]

such that \( a^2 - \eta b^2 = \eta \) (\( \eta \) stands for 7, 8 or 10, respectively). Solving this equation in each case, the map

\[
Q_{\text{Hur}}/13Q_{\text{Hur}} \to M_2(\mathbb{F}_{13}) \times M_2(\mathbb{F}_{13}) \times M_2(\mathbb{F}_{13})
\]

may then be defined as follows:

\[
j \mapsto \begin{pmatrix}
1 & 1 \\
6 & 12
\end{pmatrix}, \begin{pmatrix}
4 & 1 \\
5 & 9
\end{pmatrix}, \begin{pmatrix}
6 & 0 \\
0 & 7
\end{pmatrix}.
\]

The map is obviously onto each of the components, and thus, by the Chinese remainder theorem, onto on the product.

The three prime ideals define a triplet of principal congruence subgroups, as in (1.2). One therefore obtains a triplet of distinct Hurwitz surfaces of genus 14. All three differ both in the value of the systole and in the number of systolic loops \([43]\).
**Example 6.2** (Klein quartic). Consider the ramified prime, \( p = 7 \). The minimal polynomial of \( \eta \) factors modulo 7 as \((t - 2)^3\), and so \( 7O_K = p^3 \) for \( p = \langle \eta - 2 \rangle \), see identity (2.2). The quotient

\[
L = O_K/p^3 \cong \mathbb{F}_7[t, \varepsilon^3 = 0]
\]

is in this case a local ring, with \( O_K/7O_K \cong L \) via \( \eta \mapsto 2 + \varepsilon \). The isomorphism \( \mathcal{Q}_{\text{Hur}}/7\mathcal{Q}_{\text{Hur}} \cong M_2(L) \) can be defined by \( i \mapsto \begin{pmatrix} 0 & 1 \\ 2 + \varepsilon & 0 \end{pmatrix} \) and \( j \mapsto (3 - \varepsilon + \varepsilon^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), taking advantage of the square root \((3 - \varepsilon + \varepsilon^2)^2 = 2 + \varepsilon \) in \( L \).

The Hurwitz surface defined by the principal congruence subgroup associated with the ideal \( \langle 2 - \eta \rangle \) is the famous Klein quartic, a Hurwitz surface of genus 3.

Having computed \( \mathcal{Q}_{\text{Hur}}/I\mathcal{Q}_{\text{Hur}} \) for \( I \) an odd ideal, it remains to consider the even prime, 2. Recall that the order \( \mathcal{Q}_{\text{Elk}} \) was defined in (2.7).

**Theorem 6.3.** Let \( I = 2^tO_K \) and \( L = O_K/I \). Then

\[
\mathcal{Q}_{\text{Hur}}/I\mathcal{Q}_{\text{Hur}} \cong \mathcal{Q}_{\text{Elk}}/I\mathcal{Q}_{\text{Elk}} \cong M_2(L).
\]

**Proof.** Since \( \tau j = 2j' - 1 - \eta i \), we have that \( \tau \mathcal{Q}_{\text{Hur}} \subseteq \mathcal{Q}_{\text{Elk}} \); it then follows that \( \mathcal{Q}_{\text{Elk}} + I\mathcal{Q}_{\text{Hur}} = \mathcal{Q}_{\text{Hur}} \) and \( \mathcal{Q}_{\text{Elk}} \cap I\mathcal{Q}_{\text{Hur}} = I\mathcal{Q}_{\text{Elk}} \). As before,

\[
\mathcal{Q}_{\text{Elk}}/I\mathcal{Q}_{\text{Elk}} \cong \mathcal{Q}_{\text{Hur}}/I\mathcal{Q}_{\text{Hur}},
\]

and so \( \mathcal{Q}_{\text{Elk}}/I\mathcal{Q}_{\text{Elk}} \cong M_2(L) \) for \( L = O_K/I \).

Let us make this isomorphism explicit. Let \( \eta \in L \) denote the image of \( \eta \) under the projection \( O_K \to L \). By the relations obtained in Lemma 2.4, we have the following presentation:

\[
\mathcal{Q}_{\text{Elk}}/I\mathcal{Q}_{\text{Elk}} \cong L[x, y \mid x^2 = \eta, y^2 = 1 + 3\eta + y, yx = \eta^2 + x - xy]
\]

Let \( b \in L \) be a solution to \( b = 2 + 2b^2 \) (such a solution exists by Hensel’s lemma, or one can iterate the defining equation). Taking \( x' = \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix} \) and \( y' = \begin{pmatrix} \eta^2 \\ \eta^2(1 - b) \\ 1 - \eta^2 \end{pmatrix} \) in \( M_2(L) \), one verifies that \( x' \) and \( y' \) satisfy the required relations, and so \( x \mapsto x' \) and \( y \mapsto y' \) define an isomorphism. \( \Box \)
Example 6.4 (Fricke-Macbeath curve). Let $I = 2O_K$. Then $L = O_K/I = \mathbb{F}_8$ by (2.3), and so $\mathcal{Q}_{\text{Hur}}/2\mathcal{Q}_{\text{Hur}} = M_2(\mathbb{F}_8)$, as was already mentioned in Remark 3.3. The associated quotient is
\[ \mathcal{Q}_{\text{Hur}}^1/I^1 = \text{SL}_2(\mathbb{F}_8) = \text{PSL}_2(\mathbb{F}_8), \]
which is the automorphism group of the corresponding Hurwitz surface of genus 7, called the Fricke-Macbeath curve [16, p. 37].

Remark 6.5. The quotient $\mathcal{O}/2\mathcal{O}$ is a local commutative ring, with residue field $\mathbb{F}_8$ and a radical $J$ whose dimension over $\mathbb{F}_8$ is 3, satisfying $\dim(J^2) = 1$ and $J^3 = 0$.

Proof. Since $ji = -ij$, the elements $i$ and $j$ commute modulo 2, so the quotient ring is commutative. Also, $\eta^8 \equiv \eta \mod{2}$. Taking the defining relations of $\mathcal{O}$ modulo 2 we obtain $\mathcal{O}/2\mathcal{O} = \mathbb{F}_8[i, j \mid (i-\eta^4)^2 = (j-\eta^4)^2 = 0]$, so take $J = \mathbb{F}_8 \cdot (i-\eta^4) + \mathbb{F}_8 \cdot (j-\eta^4) + \mathbb{F}_8 \cdot (i-\eta^4)(j-\eta^4)$. □

References

[1] Ambrosio, L.; Katz, M.: Flat currents modulo $p$ in metric spaces and filling radius inequalities, Comm. Math. Helv., to appear. See [arXiv:1004.1374]
[2] Ambrosio, L.; Wenger, S.: Rectifiability of flat chains in Banach spaces with coefficients in $\mathbb{Z}_p$. Mathematische Zeitschrift (online first). See [arXiv:0905.3372]
[3] Babenko, I.; Balacheff, F.: Distribution of the systolic volume of homology classes. See [arXiv:1009.2835]
[4] Balacheff, F.: A local optimal diastolic inequality on the two-sphere. J. Topol. Anal. 2 (2010), no. 1, 109–121. See [arXiv:0811.0330]
[5] Bangert, V; Katz, M.; Shnider, S.; Weinberger, S.: $E_7$, Wirtinger inequalities, Cayley 4-form, and homotopy. Duke Math. J., 146 (2009), no. 1, 35-70. See [arXiv:math.DG/0608006]
[6] Belolipetsky, M.; Thomson. S.: Systoles of Hyperbolic Manifolds. See [arXiv:1008.2646]
[7] Berger, M.: A panoramic view of Riemannian geometry. Springer-Verlag, Berlin, 2003.
[8] Berger, M.: What is... A Systole? Notices of the AMS 55 (2008), no. 3, 374-376.
[9] Brunnbauer, M.: Homological invariance for asymptotic invariants and systolic inequalities. Geometric and Functional Analysis (GAFA), 18 (2008), no. 4, 1087–1117. See [arXiv:math.GT/0702789]
[10] Brunnbauer, M.: Filling inequalities do not depend on topology. J. Reine Angew. Math. 624 (2008), 217–231. See [arXiv:0706.2790]
[11] Brunnbauer M.: On manifolds satisfying stable systolic inequalities. Math. Annalen 342 (2008), no. 4, 951–968. See [arXiv:0708.2589]
[12] Conder, M.: The genus of compact Riemann surfaces with maximal automorphism group. *Jour. of Alg.* **108** (1987), 204–247.

[13] Conder, M.: The Hurwitz groups: A brief survey. *Bull. Am. Math. Soc.* **23** (1990), no. 2, 359–370.

[14] Dranishnikov, A.; Katz, M.; Rudyak, Y.: Small values of the Lusternik-Schnirelmann category for manifolds. *Geometry and Topology* **12** (2008), 1711–1727. See [arXiv:0805.1527](http://arxiv.org/abs/0805.1527).

[15] Dranishnikov, A.; Rudyak, Y.: Stable systolic category of manifolds and the cup-length. *Journal Journal of Fixed Point Theory and Applications* **6** (2009), no. 1, 165–177.

[16] Elkies, N.: Shimura curve computations. Algorithmic number theory (Portland, OR, 1998), 1–47, *Lecture Notes in Comput. Sci.*, **1423**, Springer, Berlin, 1998. See [arXiv:math.NT/0005160](http://arxiv.org/abs/math.NT/0005160).

[17] Elkies, N.: The Klein quartic in number theory. The eightfold way, 51–101, *Math. Sci. Res. Inst. Publ.* **35**, Cambridge Univ. Press, Cambridge, 1999.

[18] Elmir, C.: The systolic constant of orientable Bieberbach 3-manifolds. See [arXiv:0912.3894](http://arxiv.org/abs/0912.3894).

[19] Elmir, C.; Lafontaine, J.: Sur la géométrie systolique des variétés de Bieberbach. *Geometriae Dedicata* **136** (2008), no. 1, 95–110. See [arXiv:0804.1419](http://arxiv.org/abs/0804.1419).

[20] Gromov, M.: Filling Riemannian manifolds. *J. Differential Geom.* **18** (1983), 1–147.

[21] Guth, L.: Volumes of balls in large Riemannian manifolds. *Annals of Mathematics*, to appear. See [arXiv:math/0610212](http://arxiv.org/abs/math/0610212).

[22] Guth, L.: Systolic inequalities and minimal hypersurfaces. *Geometric and Functional Analysis*, **19** (2010), no. 6, 1688–1692. See [arXiv:0903.5299](http://arxiv.org/abs/0903.5299).

[23] Hasse, H.: Number Theory. *A Series of Comprehensive Studies and Monographs*, **229**, Springer-Verlag, 1980. Translated from Hasse’s Zahlentheorie, 3rd edition, 1969.

[24] Katz, Karin Usadi; Katz, M.: Hyperellipticity and Klein bottle companionship in systolic geometry. See [arXiv:0811.1717](http://arxiv.org/abs/0811.1717).

[25] Katz, Karin Usadi; Katz, M.: Bi-Lipschitz approximation by finite-dimensional imbeddings. *Geometriae Dedicata* (online first DOI: 10.1007/s10711-010-9497-4). See [arXiv:0902.3126](http://arxiv.org/abs/0902.3126).

[26] Katz, M.: Systolic geometry and topology. With an appendix by Jake P. Solomon. *Mathematical Surveys and Monographs*, **137**, American Mathematical Society, Providence, RI, 2007.

[27] Katz, M.: Systolic inequalities and Massey products in simply-connected manifolds. *Israel J. Math.* **164** (2008), 381–395. [arXiv:math.DG/0604012](http://arxiv.org/abs/math.DG/0604012).

[28] Katz, M.; Rudyak, Y.: Bounding volume by systoles of 3–manifolds. *J. Lond. Math. Soc.* **78** (2008), no 2, 407–417. See [arXiv:math.DG/0504008](http://arxiv.org/abs/math.DG/0504008).

[29] Katz, M.; Rudyak, Y.; Sabourau, S.: Systoles of 2–complexes, Reeb graph, and Grushko decomposition. *International Math. Research Notices* **2006** (2006). Art. ID 54936, pp. 1–30. See [arXiv:math.DG/0602009](http://arxiv.org/abs/math.DG/0602009).
[30] Katz, M.; Schaps, M.; Vishne, U.: Logarithmic growth of systole of arithmetic Riemann surfaces along congruence subgroups. *J. Differential Geom.* **76** (2007), no. 3, 399–422. Available at arXiv:math.DG/0505007.

[31] Katz, M.; Shnider, S.: Cayley 4-form comass and triality isomorphisms. *Israel J. Math.* **178** (2010), 187–208. See arXiv:0801.0283.

[32] Lang, S.: Algebraic Number Theory, Graduate Texts in Mathematics **110**, Springer, 1970.

[33] Maclachlan, C.; Reid, A.: The Arithmetic of Hyperbolic 3-Manifolds. *Graduate Texts in Math.*, **219**. Springer, 2003.

[34] Nabutovsky, A.; Rotman, R.: The length of the second shortest geodesic. *Comment. Math. Helv.* **84** (2009), no. 4, 747–755.

[35] Parlier, H.: The homology systole of hyperbolic Riemann surfaces, preprint. See arXiv:1010.0358

[36] Reiner, I.: Maximal Orders. Academic Press, 1975.

[37] Rotman, R.: The length of a shortest geodesic loop at a point. *J. Differential Geom.* **78** (2008), no. 3, 497–519.

[38] Rudyak, Y.; Sabourau, S.: Systolic invariants of groups and 2–complexes via Grushko decomposition. *Ann. Inst. Fourier* **58** (2008), no. 3, 777–800. See arXiv:math.DG/0609379.

[39] Sabourau, S.: Asymptotic bounds for separating systoles on surfaces. *Comment. Math. Helv.* **83** (2008), no. 1, 35–54.

[40] Sabourau, S.: Local extremality of the Calabi-Croke sphere for the length of the shortest closed geodesic. *J. London Math. Soc.* (2010), online first doi: 10.1112/jlms/jdq045. See arXiv:0907.2223.

[41] Saltman, D.: Lectures on Division Algebras. CBMS Regional Conference Series in Mathematics **94**, 1999.

[42] Shimura, G.: Construction of class fields and zeta functions of algebraic curves. *Ann. of Math.* (2) **85** (1967), 58–159.

[43] Vogeler, R.: On the Geometry of Hurwitz Surfaces. Thesis, the Florida State University, College of Arts and Sciences, 2003.

[44] Vogeler, R.: Combinatorics of curves on Hurwitz surfaces. Dissertation, University of Helsinki, Helsinki, 2004. *Ann. Acad. Sci. Fenn. Math. Diss.* No. 137, (2004), 40 pp.

[45] Wenger, S.: A short proof of Gromov’s filling inequality. *Proc. Amer. Math. Soc.* **136** (2008), no. 8, 2937–2941.

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