ON $\mathbb{Q}$-FANO THREEFOLDS OF FANO INDEX 2

YURI PROKHOROV AND MILES REID

To Shigefumi Mori in friendship and admiration

1. Introduction

1.1. Recall that a projective threefold $X$ is a $\mathbb{Q}$-Fano threefold if it has only terminal singularities and its anticanonical divisor $-K_X$ is ample. Here we assume also that $X$ is $\mathbb{Q}$-factorial and has rank 1, that is, Pic $X \simeq \mathbb{Z}$ or equivalently, Cl $X \otimes \mathbb{Q} \simeq \mathbb{Q}$. In this situation we define the Fano–Weil and $\mathbb{Q}$-Fano index of $X$ as follows:

$$q_W(X) := \max \{ q \in \mathbb{Z} \mid -K_X \sim qA, \ A \text{ is a Weil divisor} \},$$

$$q_Q(X) := \max \{ q \in \mathbb{Z} \mid -K_X \sim_q qA, \ A \text{ is a Weil divisor} \},$$

where $\sim$ (resp. $\sim_q$) is linear equivalence (resp. $\mathbb{Q}$-linear equivalence). Clearly, $q_W(X)$ divides $q_Q(X)$, and $q_W(X) = q_Q(X)$ unless $K_X + qA$ is a nontrivial torsion element of Cl $X$. Another important invariant of a Fano variety $X$ is its genus $g(X) := \dim |-K_X| - 1$.

It is known that

$$q_Q(X) \in \{1, \ldots, 11, 13, 17, 19\}$$

(see [Suz04], [Pro10b, Lemma 3.3]). Moreover, we have the following results.

1.2. Theorem ([Pro10b]). Let $X$ be a $\mathbb{Q}$-Fano threefold with $q := q_Q(X) \geq 9$. Then Cl $X \simeq \mathbb{Z}$.

(i) If $q = 19$, then $X \simeq \mathbb{P}(3, 4, 5, 7)$.

(ii) If $q = 17$, then $X \simeq \mathbb{P}(2, 3, 5, 7)$.

(iii) If $q = 13$ and $g(X) > 4$, then $X \simeq \mathbb{P}(1, 3, 4, 5)$.

(iv) If $q = 11$ and $g(X) > 10$, then $X \simeq \mathbb{P}(1, 2, 3, 5)$.

(v) $q \neq 10$.

1.3. Theorem ([Pro10c]). Let $X$ be a $\mathbb{Q}$-Fano threefold and set $q := q_Q(X)$ for its $\mathbb{Q}$-Fano index.

Y.P. acknowledges partial support from RFBR grants No. 11-01-92613-KO_a, 11-01-00336-a, the grant of Leading Scientific Schools No. 4713.2010.1, and AG Laboratory SU-HSE, RF government grant ag. 11.G34.31.0023.

M.R. is partially funded by Korean Government WCU Grant R33-2008-000-10101-0.
(i) If \( q = 9 \) and \( g(X) > 4 \) then \( X \cong X_6 \subset \mathbb{P}(1,2,3,4,5) \).
(ii) If \( q = 8 \) and \( g(X) > 10 \) then \( X \cong X_6 \subset \mathbb{P}(1,2,3,5) \) or \( X_{10} \subset \mathbb{P}(1,2,3,5,7) \).
(iii) If \( q = 7 \) and \( g(X) > 17 \) then \( X \cong \mathbb{P}(1^2,2,3) \).
(iv) If \( q = 6 \) and \( g(X) > 15 \) then \( X \cong X_6 \subset \mathbb{P}(1^2,2,3,5) \).
(v) If \( q = 5 \) and \( g(X) > 18 \) then \( X \cong \mathbb{P}(1^3,2) \) or \( X_4 \subset \mathbb{P}(1^2,2^2,3) \).
(vi) If \( q = 4 \) and \( g(X) > 21 \) then \( X \cong \mathbb{P}^3 \) or \( X_4 \subset \mathbb{P}(1^3,2,3) \).
(vii) If \( q = 3 \) and \( g(X) > 20 \) then \( X \cong X_2 \subset \mathbb{P}^4 \) or \( X_3 \subset \mathbb{P}(1^4,2) \).

In this paper we study \( Q \)-Fano threefolds with \( q(Q(X)) = 2 \).

1.4. Theorem ([BS07b]). There are at most 1492 power series that are numerical candidates for the Hilbert series of a \( Q \)-Fano threefold with \( q = q(Q(X)) = q(W(X)) = 2 \).

Our main result is the following.

1.5. Theorem. Let \( X \) be a \( Q \)-Fano threefold of rank 1 such that \( q(Q(X)) = q(W(X)) = 2 \), and assume \( K_X \) is not Cartier. Let \( A \) be a Weil divisor on \( X \) such that \(-K_X = 2A\).

Then \( \dim |A| \leq 4 \). Moreover, if \( \dim |A| = 4 \), then \( X \) belongs to the single irreducible family described in Section 3.

1.5.1. Corollary. Let \( X \) be a \( Q \)-Fano threefold with \( q(Q(X)) = q(W(X)) = 2 \) and \( K_X \) not Cartier. Then \( g(X) \leq 16 \).

1.5.2. Remark. Gorenstein \( Q \)-Fano threefolds \( X \) with \( q(W(X)) = 2 \) are particular cases of so-called del Pezzo varieties [Fuj90]. The bound \( \dim |A| \leq 4 \) does not hold for them. More precisely, there are two further cases with \( \dim |A| = 5 \) and 6:

(i) the complete intersection of two quadrics \( X = X_{2,2} \subset \mathbb{P}^5 \)
(ii) \( X = X_5 \subset \mathbb{P}^6 \), a smooth [Pro10a, Cor. 5.3] section of the Grassmanian \( \text{Gr}(2,5) \subset \mathbb{P}^9 \) by a subspace of codimension 3.

1.6. Background. In the study of \( Q \)-Fanos, the two main methods are the biregular and birational approaches. The biregular methods work in terms of projective embedding by multiples of \( A \), or more precisely, by the study of Gorenstein rings \( R(X,A) \). This is effective when this model has small codimension, especially when \( R(X,A) \) is a hypersurface or codimension 2 complete intersection etc. In contrast, the birational methods are effective when the linear system \( |A| \) is large, since then the canonical threshold is low, giving scope for imposing noncanonical singularities on \( |A| \) and studying \( X \) birationally in terms of the resulting Sarkisov links, aiming for either a birational construction or nonexistence results. The main interest of this paper is that this is a point where the two methods meet.
A surface section $F \in |A|$ of a $\mathbb{Q}$-Fano threefold $X$ of index 2 is a del Pezzo surface. In a small number of cases where $F$ has the simplest quotient singularities such as $\frac{1}{3}(2, 2)$ or $\frac{1}{5}(2, 4)$, the paper [RS03] studied such surfaces by means of projections from nonsingular points; this study foreshadows the constructions of our main example in Section 3 and hints at other possible examples; it would be interesting to study other cases of $X$ with $\dim |A| \geq 2$.

2. Preliminaries

2.1. Notation. We work throughout over the complex numbers $\mathbb{C}$.

$\text{Cl} X$ denotes the Weil divisor class group;
$\mathbb{P}(a_1, \ldots, a_n)$ is weighted projective space;
$X_d \subset \mathbb{P}(a_1, \ldots, a_n)$ is a hypersurface of weight $d$.

2.2. Construction [Ale94]. Let $\mathcal{M}$ be a mobile linear system without fixed components and $c := \text{ct}(X, \mathcal{M})$ the canonical threshold of $(X, \mathcal{M})$. Thus the pair $(X, c\mathcal{M})$ is canonical but not terminal. Assume that $-(K_X + c\mathcal{M})$ is ample. Let $f: \tilde{X} \to X$ be a $(K + c\mathcal{M})$-crepant blowup in the Mori category so that $\tilde{X}$ has only terminal $\mathbb{Q}$-factorial singularities, $\rho(\tilde{X}/X) = 1$, and

$$K_{\tilde{X}} + c\tilde{\mathcal{M}} = f^*(K_X + c\mathcal{M}).$$

The exceptional locus $E \subset \tilde{X}$ is an irreducible divisor. As in [Ale94], run a $(K+c\mathcal{M})$-MMP on $\tilde{X}$. We get the following diagram (a Sarkisov link of type I or II)

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & X \\
\downarrow & & \downarrow \bar{f} \\
\tilde{X} & &
\end{array}$$

where $\tilde{X}$ and $\bar{X}$ have only $\mathbb{Q}$-factorial terminal singularities, $\rho(\tilde{X}) = \rho(\bar{X}) = 2$, $f$ is a Mori extremal divisorial contraction, $\tilde{X} \dashrightarrow \bar{X}$ a chain of log flips, and $\bar{f}$ a Mori extremal contraction, which is either a divisorial contraction to a $\mathbb{Q}$-Fano 3-fold $\hat{X}$ or a Mori fibre space over a curve or surface $\tilde{X}$. In either case, $\rho(\hat{X}) = 1$.

In what follows, for a divisor or linear system $D$ on $X$, we write $\tilde{D}$ and $\bar{D}$ respectively for the birational transform of $D$ on $\tilde{X}$ and $\bar{X}$.

Assume that $K_X + \lambda\mathcal{M} + \Xi \sim_\mathbb{Q} 0$ for some $\lambda > c$ and an effective $\mathbb{Q}$-divisor $\Xi$. We can write

$$K_{\tilde{X}} + \lambda\tilde{\mathcal{M}} + \tilde{\Xi} + aE \sim_\mathbb{Q} f^*(K_X + \lambda\mathcal{M} + \Xi) \sim_\mathbb{Q} 0,$$
where $a > 0$ is the log discrepancy of $f$. Note that if $K_X + \lambda M + \Xi \sim 0$ then it is a Cartier divisor and $\lambda M$ and $\Xi$ are integral Weil divisors, so that $a$ is an integer.

2.2.4. Assume that $\overline{f}$ is not birational. Then $\hat{X}$ is either a smooth rational curve or a del Pezzo surface with at worst Du Val singularities and $\rho(\hat{X}) = 1$ [MP08]. We also have $\overline{f}(E) = \hat{X}$, because no multiple $nE$ of the exceptional divisor $E$ of $f$ moves on $\hat{X}$. In this case we write $\overline{f}$ for a general fiber of $\overline{f}$. Let $\Theta$ be an ample Weil divisor on $\hat{X}$ whose class generates $\text{Cl}\hat{X}/\text{Tors}$. If $\hat{X}$ is a surface with $K^2_{\hat{X}} = 1$, we take $\Theta = -K_{\hat{X}}$.

2.2.5. For $\hat{X}$ a surface, one of the following holds:

(i) $-K_{\hat{X}} \cdot \Theta = 3$, $-K_{\hat{X}} \sim 3\Theta$, $\hat{X} \simeq \mathbb{P}^2$ and $\dim |\Theta| = 2$;
(ii) $-K_{\hat{X}} \cdot \Theta = 2$, $-K_{\hat{X}} \sim 4\Theta$, $\hat{X} \simeq \mathbb{F}(1,1,2)$ and $\dim |\Theta| = 1$;
(iii) $-K_{\hat{X}} \cdot \Theta = 1$, $-K_{\hat{X}} \sim d\Theta$, where $d := K^2_{\hat{X}} \leq 6$, and the minimal resolution of $\hat{X}$ is a blowup of $\mathbb{P}^2$ at $9 - d$ points in almost general position. In this case, $\dim |\Theta| \leq 1$. Moreover, by Kawamata–Viehweg vanishing and orbifold Riemann–Roch [Rei87], for an ample Weil divisor $B \sim t\Theta$ we have

$$\dim |B| \leq \frac{t(t+d)}{2d}.$$ 

2.2.6. Assume that the contraction $\overline{f}$ is birational. In this case, $\hat{X}$ is a $\mathbb{Q}$-Fano threefold and $\overline{f}$ contracts a unique exceptional divisor $\overline{E}$. Write $\overline{F} \subset \hat{X}$ and $F := f(\overline{F})$ for its birational transform. Set $\hat{q} := q(\hat{X})$. For a divisor $\overline{D}$ on $\overline{X}$, we put $\hat{D} := \overline{f}_* \overline{D}$. One sees that $\overline{E} \neq \overline{F}$ (for otherwise $X \dashrightarrow \hat{X}$ would be an isomorphism in codimension one).

2.3. Computer search for $\mathbb{Q}$-Fano threefolds. All $\mathbb{Q}$-Fano threefolds lie in a finite number of algebraic families [Kaw92]. In fact, Kawamata’s proof gives a method of listing all possible “candidate” $\mathbb{Q}$-Fano threefolds, although the volume of computations makes a computer search appropriate. This method was used in [Suz04], [BS07a, BS07b], [Pro07], [Pro10b], [Pro10c]. See [BZ] for the database of candidates for the numerical types of graded rings. We now outline the algorithm, starting with a useful remark.

2.3.1. Remark. The local Weil divisor class group of a threefold $\mathbb{Q}$-factorial terminal point $P \in X$ is cyclic $\text{Cl}(X,P) \simeq \mathbb{Z}_r$, generated by the canonical divisor $K_X$ [Kaw88 Lemma 5.1]. In particular, if $X$ is a $\mathbb{Q}$-Fano threefold $X$, its local Gorenstein index $r$ at every terminal point is coprime to the $\mathbb{Q}$-Fano index $q = qW(X)$. 

4
2.3.2. Let $X$ be a $Q$-Fano threefold. For simplicity we assume that $q := qQ(X) = qW(X) \geq 3$ (the only case we need). Let $A$ be a Weil divisor such that $-K_X \sim qA$ and $B(X) = \{(r_P, b_P)\}$ the basket of orbifold points of $X$ [Rei87].

**Step 1.** We have the equality

$$-K_X \cdot c_2(X) + \sum_{P \in B} \frac{r_P - 1}{r_P} = 24,$$

where $-K_X \cdot c_2(X) > 0$ [Kaw92]. Hence there is only a finite (but huge) number of possibilities for the basket $B(X)$ and $-K_X \cdot c_2(X)$. Let $r := \text{lcm}\{r_P\}$ be the Gorenstein index of $X$.

**Step 2.** (1.1.1) says that $q \in \{3, \ldots, 11, 13, 17, 19\}$. Remark 2.3.1 implies that gcd($q, r$) = 1, which eliminates some possibilities.

**Step 3.** In each case we compute $A^3$ by the formula

$$A^3 = \frac{12}{(q - 1)(q - 2)} \left(1 - \frac{A \cdot c_2}{12} + \sum_{P \in B} c_P(-A)\right),$$

(see [Suz04]), where $c_P$ is the correction term in the orbifold Riemann–Roch formula [Rei87]. The number $rA^3$ must be an integer [Suz04, Lemma 1.2].

**Step 4.** Next, the Bogomolov–Miyaoka inequality (see [Kaw92]) implies that

$$(4q^2 - 3q) A^3 \leq -4K_X \cdot c_2(X)$$

[Suz04 Prop. 2.2].

**Step 5.** Finally, by the Kawamata–Viehweg vanishing theorem we have $\chi(tA) = h^0(tA) = 0$ for $-q < t < 0$. We check this condition by using orbifold Riemann–Roch [Rei87].

See [B^+] for lists.

3. An Example

3.1. **Overview.** This section treats the exceptional family of $Q$-Fano threefolds $X$ mentioned in Theorem 1.5. More precisely, it gives two independent constructions of two families of index 2 Fano 3-folds $X$ and $Y$, each having a single orbifold point of type $\frac{1}{3}(1, 2, 2)$, and satisfying

$$-K_X^3 = \frac{8}{3}, \quad \dim|-\frac{1}{2}K_X| = 4, \quad \text{and} \quad -K_Y^3 = \frac{5}{3}, \quad \dim|-\frac{1}{2}K_Y| = 3.$$

For a general nonsingular point $P \in X$, there is a birational projection $\psi_P : X \dasharrow Y'$ that blows $P$ up to a plane in $\mathbb{P}^2 \subset Y'$. Here $Y'$ is a special member of the family of $Y$, obtained by imposing a
plane $\mathbb{P}^2$ on $Y'$. This projection $X \dasharrow Y'$ is analogous to the familiar “internal” projections between del Pezzo varieties $\psi_P: V_d \dasharrow V_{d-1}$.

Later in this section, we use this to give our second construction of $X$ by unprojection from $Y'$, written as a $5 \times 5$ Pfaffian variety specialised to contain $\mathbb{P}^2$ by a Jerry format [BKR10]. An interesting point is that there is also a Tom construction, but that it gives rise to a $\mathbb{Q}$-Fano threefold of Picard rank 2.

At the same time, $X$ and $Y$ have Sarkisov links $X \dasharrow Q \subset \mathbb{P}^4$ and $Y \dasharrow \mathbb{P}^3$ initiated by the Kawamata blowup [CPR00, 3.4.2] of their respective $\frac{1}{3}(1,2,2)$ orbifold points. The aforementioned birational maps all fit together into a commutative diagram

\[
\begin{array}{ccc}
P \in X & \xrightarrow{\psi_P} & Q \ni P \\
\downarrow & \searrow & \downarrow \\
Y' & \xrightarrow{\pi_P} & \mathbb{P}^3
\end{array}
\]

where $\pi_P: Q \dasharrow \mathbb{P}^3$ is the usual linear projection from $P$ of the smooth quadric $Q \subset \mathbb{P}^4$.

3.2. Construction of $Q \dasharrow X$ and $\mathbb{P}^3 \dasharrow Y$. Our first construction of $X$ and $Y$ works via the inverse map $Q \dasharrow X$ and $\mathbb{P}^3 \dasharrow Y$. Both of these blow up a curve $\Gamma \subset Q$ (resp. $\Gamma \subset \mathbb{P}^3$) where $\Gamma$ is a rational quintic curve having (in general) a triple point with distinct tangent directions. In (3.1.1), the blown up curve $\Gamma$ is the same up to isomorphism in the two cases. In either case, $\Gamma \subset S$ is contained in a quadric cone $S = T_{Q,P_0} \cap Q$ (the tangent plane at $P_0 \in Q$) resp. $S \subset \mathbb{P}^3$. We identify $S$ with $\mathbb{P}(1,1,2)_{(u_1,u_2,v)}$, and $\Gamma \subset \mathbb{P}(1,1,2)$ is the quintic curve given by $vu_3(u_1, u_2) + b_5(u_1, u_2) = 0$.

Since the two constructions are very similar, and our second construction gives $Y$ directly, we concentrate on the case $Q \dasharrow X$. More precisely, we prove the following.

3.3. Theorem. There exists a Sarkisov link

\[
\begin{array}{ccc}
\overline{Q} & \xrightarrow{\overline{f}} & X \\
\downarrow & \searrow & \downarrow \\
Q & \xrightarrow{f} & X
\end{array}
\]

where $Q \subset \mathbb{P}^4$ is the smooth quadric, and $\overline{f}$ and $f$ are extremal divisorial contractions in the Mori category with respective exceptional
divisors $\mathcal{F}, \mathcal{S} \subset \mathcal{Q}$. The endpoint $X$ is a $\mathcal{Q}$-Fano threefold with

$$\text{Cl} X \simeq \mathbb{Z}, \quad q\mathcal{Q}(X) = 2, \quad A^\mathcal{Q} = 10/3, \quad \dim |A| = 4, \quad g(X) = 14;$$

having as its only singularity a terminal cyclic quotient point of type $\frac{1}{3}(1,1,2)$ at $P_3 = f(\mathcal{S})$. The map $f$ is the Kawamata blowup of $P_3$, with exceptional divisor $\mathcal{S} \simeq \mathbb{P}(1,1,2)$. The contraction $\tilde{f}$ maps $\mathcal{S}$ isomorphically to the section $S = Q \cap T_{P_0,Q}$ of $Q$ by the tangent hypersection at a point $P_0$; it blows up a rational quintic curve $\Gamma \subset S$ as specified below.

3.4. Notation. Let $S = Q \cap T_{P_0,Q}$ be a singular hyperplane section of $Q$, a quadratic cone with vertex $P_0 \in S$. We identify $S$ with the weighted projective plane $\mathbb{P}(1,1,2)$ with homogeneous coordinates $u_1, u_2, v$. Let $\Gamma \subset S$ be an irreducible quintic curve given by the equation $va_3(u_1,u_2) + b_5(u_1,u_2) = 0$, where $a_3$ and $b_5$ are homogeneous polynomials of the indicated degrees with no common factor. One sees that $\Gamma$ is smooth outside $P$ and has a triple point at $P$ with (in general) three linearly independent tangent branches.

We first construct the birational extraction $\tilde{f}$. According to [KM92, Th. 4.9] such an extraction, if it exists, is unique up to isomorphism over $Q$.

3.5. Proposition. In the above notation there exists a divisorial extraction $\tilde{f}: \widetilde{Q} \rightarrow Q$ in the Mori category whose exceptional divisor $\mathcal{F}$ is contracted to $\Gamma$. The only singular point $P \in Q$ is a terminal cyclic quotient point of type $\frac{1}{3}(1,1,1)$ and the divisor $-K\widetilde{Q}$ is ample, that is, $\widetilde{Q}$ is a $\mathbb{Q}$-Fano 3-fold.

Proof. Let $\sigma: \tilde{Q} \rightarrow Q$ be the blowup of $P$. The proper transform $\tilde{S} \subset \tilde{Q}$ of $S$ is isomorphic to the Hirzebruch surface $\mathbb{F}_2$. The proper transform $\tilde{\Gamma} \subset \tilde{S}$ of $\Gamma$ is a smooth rational curve $\tilde{\Gamma} \simeq \Sigma + 5\Upsilon$, where $\Sigma$ and $\Upsilon$ are the negative section and fiber of $\mathbb{F}_2$. Denote $S^* := \sigma^* S$ and let $D$ be the $\sigma$-exceptional divisor. Then

$$\tilde{S} \sim S^* - 2D, \quad -K\tilde{Q} \sim 3S^* - 2D \sim \tilde{S} + 2S^*, \quad \tilde{S} \cap D = \Sigma.$$

Since $|S^* - D|$ is a free linear system, $-K\tilde{Q}$ is ample, that is, $\tilde{Q}$ is a $\mathbb{Q}$-Fano threefold. By Kodaira vanishing $H^1(\tilde{Q}, \mathcal{O}_{\tilde{Q}}(-K\tilde{Q} - \tilde{S})) = 0$. From the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{Q}}(-K\tilde{Q} - \tilde{S}) \rightarrow \mathcal{O}_{\tilde{Q}}(-K\tilde{Q}) \rightarrow \mathcal{O}_{\tilde{S}}(-K\tilde{Q}) \rightarrow 0$$

we get surjectivity of the restriction map

$$H^0(\tilde{Q}, \mathcal{O}_{\tilde{Q}}(-K\tilde{Q})) \rightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(-K\tilde{Q})).$$
Note that
\[-K_{\tilde{Q}}|_{\tilde{S}} \sim 3S^*|_{\tilde{S}} - 2D|_{\tilde{S}} \sim 3(\Sigma + 2\Upsilon) - 2\Sigma = \Sigma + 6\Upsilon.\]

This implies that the linear system \( |-K_{\tilde{Q}}|_{\tilde{S}} - \tilde{\Gamma} |\) is free on \( \tilde{S} \) and \( \tilde{\Gamma} \) is a scheme theoretic intersection of members of \( |-K_{\tilde{Q}}| \).

Now let \( \sigma': Q' \to \tilde{Q} \) be the blowup of \( \tilde{\Gamma} \) and let \( F' \subset Q' \) be the \( \sigma' \)-exceptional divisor. Let \( S' \subset Q' \) (resp. \( D' \subset Q' \)) be the proper transform of \( \tilde{S} \) (resp. \( D \)) and let \( S^{**} := \sigma^*(S^*) \). Clearly, \( S' \sim \tilde{S} \sim \mathbb{F}_2 \). Since \( \tilde{\Gamma} \) is a scheme-theoretic intersection of members of \( |-K_{\tilde{Q}}| \), the linear system \( |-K_{Q'}| = |\sigma^*(\tilde{\Gamma}) - F'| \) is base point free. In particular, \( -K_{Q'} \) is nef. For the normal bundle of \( \tilde{\Gamma} \) we have \( c_1(N_{\tilde{\Gamma}/\tilde{Q}}) = 2g(\tilde{\Gamma}) - 2 - K_{\tilde{Q}} \cdot \tilde{\Gamma} = 7 \). Hence,
\[
(3.5.1) \quad -K_{Q'}^3 = -K_{\tilde{Q}}^3 - 3(-K_{\tilde{Q}}) \cdot \tilde{\Gamma} + c_1(N_{\tilde{\Gamma}/\tilde{Q}}) = 26
\]
and so \(-K_{Q'}\) is big, that is, \( Q' \) is a weak Fano threefold. Since \( D \cdot \tilde{\Gamma} = (\Sigma \cdot \tilde{\Gamma})_{\tilde{S}} = 3 \), there is a 2-secant (or tangent) line \( l \subset D \simeq \mathbb{P}^2 \) to \( \tilde{\Gamma} \). For its proper transform \( l' \subset D' \) we have \(-K_{Q'} \cdot l' = -K_{\tilde{Q}} \cdot l - F' \cdot l' \leq 0 \). Since \(-K_{Q'}\) is nef, \(-K_{Q'} \cdot l' = 0 \) and \( l' \) generates an extremal ray \( R \) of the Mori cone \( \overline{NE}(Q'/Q) \). Clearly, \( \text{Supp}(R) \subset F' \cup D' \). Since
\[
(-K_{Q'})^2 \cdot D' = (\sigma^*(-K_{\tilde{Q}}) - F')^2 \cdot \sigma^*D = (-K_{\tilde{Q}})^2 \cdot D - D \cdot \tilde{\Gamma} = 1,
\]
we have \( \text{Supp}(R) \neq D' \). Similarly, \( \text{Supp}(R) \neq F' \). Therefore, \( R \) is a flopping extremal ray. Consider the corresponding flop \( \chi: Q' \dashrightarrow Q^+ \). We have \( \rho(Q^+/Q) = 2 \), \( Q^+ \) is nonsingular, and \(-K_{Q^+}\) is nef. Running the MMP over \( Q \) gives the diagram
\[
\begin{array}{ccc}
Q' & \xrightarrow{\chi} & Q^+ \\
\downarrow{\sigma'} & & \downarrow{\varphi} \\
\tilde{Q} & \xrightarrow{\sigma} & \overline{Q} \\
\end{array}
\]
Here \( \varphi \) is a divisorial contraction. Since \( Q \) is smooth, \( K_{\tilde{Q}} \) cannot be nef over \( Q \). Therefore, \( \overline{Q} \) is also a divisorial extremal contraction.

If \( \varphi \) contracts \( F^+ \), the proper transform of \( F^+ \), then \( \tilde{Q} \) and \( \overline{Q} \) are isomorphic in codimension one over \( Q \). Since \( \rho(\tilde{Q}/Q) = \rho(\overline{Q}/Q) = 1 \), they must be isomorphic over \( Q \). Then \( Q' \) and \( Q^+ \) also must be isomorphic over \( Q \). On the other hand, \( F' \) is ample with respect to the flopping extremal ray \( R \). Hence, its proper transform \( F^+ \) must be anti-ample with respect to the corresponding extremal ray, a contradiction.
Therefore, \( \varphi \) contracts \( D^+ \), the proper transform of \( D' \), and \( \mathcal{F} \) contracts \( F := \varphi(F^+) \). Moreover, \( \mathcal{F}(\mathcal{F}) = \Gamma \) and \( \mathcal{F} := \varphi(D^+) \) is a point. Since \( \Gamma \) is not a locally complete intersection, the divisor \( K_{\mathcal{Q}} \) is not Cartier [Cut88]. By [Mors2] there is only one possibility: \( D^+ \simeq \mathbb{P}^2 \), \( \mathcal{O}_{D^+}(D^+) \simeq \mathcal{O}_{\mathbb{P}^2}(-2) \), and \( \mathcal{F} \in \mathcal{Q} \) is a cyclic quotient singularity of index 2. Finally, \( -K_{Q^+} \) is nef. Since the \( \varphi \)-exceptional divisor \( D^+ \) is contracted to a point and meets flopped curves, the divisor \( -K_{\mathcal{Q}} \) is ample.

3.5.2. Corollary. In the above notation the following holds.

(i) \(-K_{\mathcal{Q}} \sim 3F \sim 2F + S\);

(ii) \(-K_{Q^+}^3 = 53/2\);

(iii) for the proper transform \( \mathcal{S} \subset \mathcal{Q} \) of \( S \), the restriction \( \mathcal{F}|_{\mathcal{S}} : \mathcal{S} \to S \) is an isomorphism and \( \text{Sing}(\mathcal{S}) = \text{Sing}(Q) = \{\mathcal{F}\}\);

(iv) \( \mathcal{O}_{\mathcal{S}}(-K_{\mathcal{Q}}) \simeq \mathcal{O}_{\mathbb{P}(1,1,2)}(1), \mathcal{O}_{\mathcal{S}}(S) \simeq \mathcal{O}_{\mathbb{P}(1,1,2)}(-3). \)

Proof. Since outside of \( P \) the map \( \mathcal{F} \) is just the blowup of \( \Gamma \), we have \( \square \)

By (3.5.1) we have

\[-K_{Q^+}^3 = -K_{Q^+}^3 = 26, \quad -K_{Q}^3 = -K_{Q^+}^3 + \frac{1}{2} = \frac{53}{2}.\]

This proves (ii).

Let \( l' \subset Q^+ \) be a flopping curve. We have

\[-K_{Q^+} \cdot D' \cdot S' = -K_{Q^+} \cdot D \cdot \Sigma + \sigma^* D \cdot F'^2 = 1.\]

The contradiction shows that \( S' \cap l' = \emptyset \), that is, \( \chi \) is an isomorphism near \( S' \) and so \( S^+ \simeq S' \simeq \mathbb{P}^2 \). Thus \( S^+ \) intersects \( D^+ \) along a smooth rational curve, the negative section of \( S^+ \simeq \mathbb{P}^2 \) and so \( S := \varphi(S^+) \simeq \mathbb{P}(1,1,2) \), \( \varphi(D^+) = \text{Sing}(S) = \text{Sing}(Q) \). This proves (iii).

By (3.5.3) we have \( \mathcal{O}_{S^+}(-K_{Q^+}) \simeq \mathcal{O}_{\mathbb{P}^2}(\mathcal{Y}) \) and so \( \mathcal{O}_{\mathcal{S}}(-K_{\mathcal{Q}}) \simeq \mathcal{O}_{\mathbb{P}(1,1,2)}(1) \).

By (i) and because \( \mathcal{O}_{\mathcal{S}}(\mathcal{F}) = \mathcal{O}_{\mathbb{P}(1,1,2)}(2) \) we have \( \mathcal{O}_{\mathcal{S}}(S) \simeq \mathcal{O}_{\mathbb{P}(1,1,2)}(-3) \). This proves (iv).

3.5.4. Remark. If the polynomial \( a_3(u_1, u_2) \) in (3.4) has distinct roots, we can make our construction more explicit. In this case the intersection \( \tilde{\Gamma} \cap D \) consists of three points \( P_1, P_2, P_3 \) in general position on \( D \simeq \mathbb{P}^2 \). Thus \( D' \) is a smooth del Pezzo surface of degree 6. For \( 1 \leq i < j \leq 3 \), let \( l_{i,j} \) be the line of \( D \simeq \mathbb{P}^2 \) through points \( P_i \) and \( P_j \), and set \( l'_{i,j} \subset D' \) for its proper transform. Let \( m_k := \sigma^{-1}(P_k) \cap D' \).
Clearly, $m_k$ is a $-1$-curve on $D'$. Thus the six $-1$-curves $m_1, l_{1,2}', m_2, l_{2,3}, m_3, l_{1,3}'$, form a hexagon on $D'$. Moreover, they generate the Mori cone $\overline{\text{NE}}(D')$. One sees that $-K_{Q'} \cdot l_{i,j}' = 0$ and $-K_{Q'} \cdot m_k = 1$.

This shows that the curves $l_{i,j}'$ generate a flopping extremal ray $R \subset \overline{\text{NE}}(Q'/Q)$. Since the normal bundle $N_{l_{i,j}'/Q'}$ has a subbundle $N_{l_{i,j}'/D'} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, we have $N_{l_{i,j}'/Q'} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ [Rei83, Remark 5.2]. Hence $\chi$ is the simplest Atiyah–Kulikov flop along each $l_{i,j}'$.

The restriction $\chi|_{D'}$ extends to a morphism $\chi|_{D'}: D' \to D^+$ which is a contraction of the $-1$-curves $l_{i,j}'$ [Rei83, Remark 5.13]. Hence $D^+ \cong \mathbb{P}^2$ and the composition map $\mathbb{P}^2 \to D^+$ is just the classical Cremona quadratic involution.

**Proof of Theorem 3.3.** By (iv) of Corollary 3.5.2 the curves in $S$ generate a $K$-negative extremal ray. Moreover, the divisor $S$ satisfies the contractibility criterion [Kaw96]. Hence there is a Mori contraction $f: Q \to X$ such that $f(S)$ is a point of type $\frac{1}{3}(1,1,2)$. By (i) $qQ(X) = 2$ and $\text{Cl}X \cong \mathbb{Z}$ (because the class of $S$ is a primitive element of $\text{Cl}(Q)$). Finally,

$$-K^3_X = -K^3_Q + \frac{1}{6} = \frac{80}{3}$$

and by the orbifold Riemann–Roch $\dim |A| = 4$. □

### 3.9. Second construction via unprojection.

The codimension 3 model $Y \subset \mathbb{P}(1,1,1,1,2,2,3)_{(x_1,x_2,x_3,x_4,y_1,y_2,z)}$ is given by the maximal Pfaffians of a general $5 \times 5$ skew matrix $M$ of degrees

$$
\begin{pmatrix}
1 & 1 & 2 & 2 \\
1 & 2 & 2 \\
2 & 2 \\
3
\end{pmatrix}
$$

(3.9.1)

see [CR02] and [BKR10] for conventions and background. As in [CR02], this variety is a regular pullback from a weighted Grassmann variety $\text{wGr}(2,5)$, and the general $Y$ is a quasismooth $Q$-Fano threefold with a single $\frac{1}{3}(1,2,2)$ point at $P_z$. Indeed, the three Pfaffians partners of $m_{45} = z$ are $z x_i = \cdots$ for $i = 1, 2, 3$, so the point $P_z \in Y$ is the orbifold point $\frac{1}{3}(1,2,2)_{(x_4,y_1,y_2)}$. Smoothness outside $P_z$ comes from Bertini’s theorem or explicit computation. One reads the Hilbert series

$$
\frac{1 - 2t^3 - 3t^4 + 3t^5 + 2t^6 - t^8}{\prod_{a \in [1,1,1,1,2,3]}(1 - t^a)}
$$

(3.9.2)

directly from the Pfaffian format.
The birational map $Y' \rightarrow X$ that we use to construct $X$ contracts an unprojection divisor, the plane $\mathbb{P}^2 \subset Y'$. We impose $\mathbb{P}^2$ on $Y'$ by specializing the entries of the matrix $M$, moving the general $Y$ to a special $Y'$ containing $D = \mathbb{P}^2_{(x_1,x_2,x_3)}$. The key point of [BKR10] is that there are several different formats that arrange for $Y'$ to contain $D$, and they lead to topologically different $X$. Our construction of $X$ is a routine but interesting exercise in these techniques.

The ideal of $D$ is the complete intersection ideal $I_D = (x_4, y_1, y_2, z)$. We construct $Y'$ using Jerry 45: we require the 7 entries in the 4th and 5th column of $M'$ to be in $I_D$. A simple case is

$$M' = \begin{pmatrix} x_3 & -x_2 & y_1 & n_2x_4 \\ x_1 & y_2 & y_1 & z \\ n_1x_4 & y_2 \\ z \end{pmatrix}$$

where $n_1$ and $n_2$ are linear forms (for example, $n_2 = x_4 - x_1$ and $n_1 = x_4 - \lambda x_3$ with $\lambda \neq 1$). The methods of [BKR10] 6.1 give that $Y'$ defined by the Pfaffians of $M'$ is smooth except for the $1/3$ orbifold point at $P_2$ and ordinary nodes at 5 points of $D$. This constructs $Y'$ containing $D = \mathbb{P}^2$, and hence its unprojections $X$. The map $Y' \rightarrow X$ blows up the ideal of $D$ in $Y'$ to make it a Cartier divisor (introducing flopping $(-1,-1)$-curves over the 5 nodes); this makes $D$ into a copy of $\mathbb{P}^2$ with normal bundle $O(-1)$, which contracts to a smooth point of $X$.

3.10. Diagram (3.1.1) in equations. The existence of $X$ is now established. However, it is interesting to expand on how the maps of (3.1.1) come out in coordinates. As in 2.2 above, we start from the general rational quintic curve $\Gamma \subset \mathbb{P}^3$ with a triple point. It is contained in the quadric cone $S : (x_1x_3 = x_2^2)$; we identify $S$ with $\mathbb{P}(1,1,2)_{(u_1,u_2,v)}$ by setting

$$x_1 = u_1^2, \quad x_2 = u_1u_2, \quad x_3 = u_2^2, \quad x_4 = v.$$  

and take $\Gamma : (v a_3(u_1,u_2) + b_5(u_1,u_2) = 0)$. Every term in $a_3$ is divisible by $u_1$ or $u_2$, and every term in $b_5$ is divisible by $u_1^3$ or $u_2^3$, so we write

$$a_3 = u_1m_2 - u_2m_1 \quad \text{and} \quad b_5 = u_1^3n_2 - u_2^3n_1$$

with $m_i, n_i$ linear in $x_1, x_2, x_3$. Then $\Gamma \subset \mathbb{P}^3$ has equations

$$\bigwedge^2 N = 0, \quad \text{where} \quad N = \begin{pmatrix} x_1 & x_2 & x_3n_1 + x_4m_1 \\ x_2 & x_3 & x_1n_2 + x_4m_2 \end{pmatrix}.$$
The rational map \( \mathbb{P}^3 \longrightarrow Y \) adjoins \( y_1, y_2, z \) subject to the equations \( \text{Pf} \, M = 0 \) with

\[
M = \begin{pmatrix}
x_3 & -x_2 & y_1 + \mu_{11}x_4^2 & x_4n_2 + \mu_{21}x_4^2 \\
x_1 & y_2 + \mu_{12}x_4^2 & y_1 + \mu_{22}x_4^2 \\
x_4n_1 + \mu_{13}x_4^2 & y_2 + \mu_{23}x_4^2 \\
\end{pmatrix},
\]

where \( m_1 = \mu_{11}x_1 + \mu_{12}x_2 + \mu_{13}x_3 \) and \( m_2 = \mu_{21}x_1 + \mu_{22}x_2 + \mu_{23}x_3 \).

Each entry of the last two columns of \( M \) is in \( I_D = (x_4, y_1, y_2, z) \), where \( D \) corresponds to the plane \( \mathbb{P}^2 \subset \mathbb{P}^3 \) given by \( x_4 = 0 \). Thus our description of \( Y' \) as the blowup of \( \mathbb{P}^3 \) along \( \Gamma \subset S \), together with our choice of coordinates, puts it directly in Jerry 45 format, containing the plane \( x_4 = 0 \), and with 5 nodes corresponding to the locus \( v = b_5 = 0 \) in \( S \).

In (3.10.2), the two Pfaffians without \( z \) give \( y_1, y_2 \) as the solutions of

\[
N \begin{pmatrix} y_1 \\ y_2 \\ x_4 \end{pmatrix} = 0
\]

(with denominator \( x_1x_3 - x_2^2 \)) and the three Pfaffians involving \( z \) are

\[
z \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \bigwedge^2 \begin{pmatrix} y_1 + \mu_{11}x_4^2 & x_4n_2 + \mu_{21}x_4^2 \\ y_2 + \mu_{12}x_4^2 & y_1 + \mu_{22}x_4^2 \\ x_4n_1 + \mu_{13}x_4^2 & y_2 + \mu_{23}x_4^2 \end{pmatrix}.
\]

Compare the “Double Jerry” format of [BKR10, 9.2].

The unprojection \( Y' \longrightarrow X \) is defined by adjoining \( x_5 \) to the homogeneous coordinate ring of \( Y' \). The equations involving \( x_5 \) are

\[
(3.10.4) \quad x_5 \begin{pmatrix} y_1 \\ y_2 \\ x_4 \end{pmatrix} = \bigwedge N,
\]

There is also an equation for \( x_5z \) that exists by the Kustin–Miller unprojection theorem, but we do not know any neat derivation of it. In the general setting, it is a “long equation”, with the right hand side having 144 terms [BKR10, 9.2]. In the “fairly typical” case (3.9.3), a little game of syzygies gives that \( x_1(x_5z + n_1x_3y_1 + n_2x_1y_2 + n_1n_2x_2x_4) \) is in the ideal generated by the Pfaffians of \( M' \) and (3.10.4), so that the \( x_5z \) equation is

\[
(3.10.5) \quad x_5z + n_1x_3y_1 + n_2x_1y_2 + n_1n_2x_2x_4 = 0.
\]
3.11. **The Tom1 case.** In cases such as these, our experience is that there is usually a different method of imposing the plane \( \mathbb{P}^2_{(x_1,x_2,x_3)} \) on \( Y \): namely Tom1 requires that all the entries except those in Row 1 are in the ideal \( I_D = (x_4,y_1,y_2,y_3) \). The nonsingularity calculation of [BKR10] shows that a general Tom1 matrix defines \( Y' \) that is quasi-smooth except for 4 nodes on \( P^2 \). As before, its unprojection is a \( \mathbb{Q} \)-Fano threefold \( X \) of index 2 with a single \( \frac{1}{3}(1,2,2) \) quotient singularity and \( \dim |A| = 4 \). However, one checks that it is a regular pullback of a weighted form of \( \mathbb{P}^2 \times \mathbb{P}^2 \), and has Picard rank 2. The point is that one can use row and column operations to put the matrix in the normal form

\[
M_0 = \begin{pmatrix}
x_1 & x_2 & q_1 & q_2 \\
x_4 & y_1 & 0 & y_2 \\
0 & y_2 & z
\end{pmatrix}
\]

where \( x_3 \) appears only in the quadratic terms \( q_1, q_2 \) (we omit the details). The zeros imply that several of the Pfaffians are monomial equations, so \( X \) has rank > 1.

The Pfaffians of \( M_0 \) are the \( 2 \times 2 \) minors of the \( 3 \times 3 \) array

\[
M_0 = \begin{pmatrix}
* & x_1 & q_2 \\
x_2 & x_4 & y_2 \\
q_1 & y_1 & z
\end{pmatrix},
\]

and the unprojection just puts the unprojection variable \( x_5 \) of degree 1 in place of the *. The result is the Segre embedding of the product of two copies of the weighted projective plane \( \mathbb{P}^2(\frac{1}{2},\frac{1}{2},\frac{3}{2}) \). Taking a regular pullback by setting \( q_1, q_2 \) to be forms of degree 2 in \( x_i, y_j \) gives \( X \).

From the point of view of diagram (3.1.1), the mechanism seems to be that the blowup of a general quintic curve in the quadric cone \( S \cong \mathbb{P}(1,1,2) \) (a curve of genus 2) initiates a Sarkisov link to a general codimension 3 Pfaffian \( Y \subset \mathbb{P}(1,1,1,2,2,3) \). The Jerry45 and specialization to \( Y' \) containing a plane \( \mathbb{P}^2 \) corresponds to \( \Gamma \) acquiring a triple point at the cone point of \( S \), as discussed above. The Tom1 specialization presumably corresponds to \( \Gamma \) breaking up as a line plus an elliptic quartic.

4. **On Fano threefolds of large Fano index**

4.1. Recall that a *polarized variety* is a pair \((X, S)\) consisting of a projective algebraic variety \( X \) and an ample Cartier divisor \( S \) on \( X \). The *\( \Delta \)-genus* of \((X, S)\) is defined as follows [Fuj90]:

\[
\Delta(X, S) = \dim X + S^{\dim X} - \dim H^0(X, \mathcal{O}_X(S)).
\]
It is known that $\Delta(X, S) \geq 0$ and polarized varieties of small $\Delta$-genera are classified [Fuj90]. The following easy consequence of Fujita’s classification is very useful for us.

4.2. Lemma. Let $X$ be a $\mathbb{Q}$-Fano threefold and $S$ an ample Weil divisor on $X$ such that $\dim |S| > 0$, $|S|$ has no fixed components, and $-K_X \sim_q \lambda S$ with $\lambda \geq 2$. Assume that the pair $(X, |S|)$ is terminal. Then one of the following holds:

(i) $X \cong \mathbb{P}^3$, $\lambda = 4$, $\dim |S| = 3$;
(ii) $X \cong \mathbb{P}^3$, $\lambda = 2$, $\dim |S| = 9$;
(iii) $X \cong X_2 \subset \mathbb{P}^4$ is a smooth quadric, $\lambda = 3$, $\dim |S| = 4$;
(iv) $X$ is a del Pezzo threefold of degree $1 \leq d \leq 5$, $\lambda = 2$, $\dim |S| = d + 1$;
(v) $X \cong \mathbb{P}(1^3, 2)$, $\lambda = 5/2$, $\dim |S| = 6$.

Proof. Replace $S$ with a general member of $|S|$. Since $(X, |S|)$ is terminal, the surface $S$ is smooth and contained in the smooth locus of $X$ [Ale94, 1.22]. By the adjunction formula we have $-K_S \sim (\lambda - 1)|S|$. Hence $S$ is a (smooth) del Pezzo surface and $(\lambda - 1)^2S^3 = K_S^2$. Since $H^i(X, \mathcal{O}_X) = 0$ and $H^i(S, \mathcal{O}_S(S)) = 0$ for $i > 0$, by Riemann–Roch we have

$$\dim H^0(X, \mathcal{O}_X(S)) = \dim H^0(S, \mathcal{O}_S(S)) + 1 = \frac{\lambda}{2}S^3 + 2.$$

Therefore,

$$\Delta(X, S) = 3 + S^3 - \frac{\lambda}{2}S^3 - 2 = 1 + \frac{(2 - \lambda)S^3}{2} = 1 + \frac{(2 - \lambda)K_S^2}{2(\lambda - 1)^2}.$$

If $S \cong \mathbb{P}^2$, then $\mathcal{O}_S(S) = \mathcal{O}_{\mathbb{P}^2}(l)$, where $3 = (\lambda - 1)l \geq l$. Then $\Delta(X, S) = 0$ and [Fuj90, Th. 5.10 and 5.15] gives cases (i) and (v).

If $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $\mathcal{O}_S(S) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, k)$, where $k(\lambda - 1) = 2$. So, $\lambda = 2$ or 3, $\Delta(X, S) = 0$, and [Fuj90, Th. 5.10 and 5.15] gives cases (ii) or (iii).

Finally, if $S \not\cong \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, then $K_S$ is a primitive element of Pic $S$. Hence $\lambda = 2$ and $\Delta(X, S) = 1$. Then we have case (iv) [Fuj90, Ch. 1, §9].

4.3. Lemma (cf. [Pro10b, Th. 1.4 (vii)]). Let $X$ be a Fano threefold with terminal singularities and with $q := qQ(X) \geq 5$. Let $A$ be a Weil divisor such that $-K_X \sim_q qA$. If $\dim |A| \geq 2$, then $X \cong \mathbb{P}(1^3, 2)$.

Proof. We first consider the case rank $Cl X = 1$ and $qQ(X) = qW(X)$ (in particular, $X$ is a $\mathbb{Q}$-Fano threefold and $-K_X \sim qA$). Running the computer search [2.3] we get $-K_X^3 \geq 125/2$. Then by [Pro07] we have $X \cong \mathbb{P}(1^3, 2)$.

Next consider the case $Cl X \cong \mathbb{Z}^r$ with $r > 1$ (in particular, $Cl X$ is torsion free). We get a contradiction in this case. Run the MMP. At
the end we get a $\mathbb{Q}$-Fano threefold with $q\mathbb{Q}(X) = q \geq 5$ and $\dim |A| \geq 2$, where $-K_X \sim qA$. By the above $X \simeq \mathbb{P}(1^3, 2)$ and $\dim |A| = \dim |A| = 2$. Let $P \in X$ be the point of type $1/2(1, 1, 1)$. Consider the final step $g: \tilde{X} \to X$ of the MMP, a divisorial contraction, and let $E \subset \tilde{X}$ be its exceptional divisor. There are the following possibilities:

(a) $g(\tilde{E}) = P$. Then $K_{\tilde{X}} \sim q^*K_X + \frac{1}{2}\tilde{E}$, $\tilde{E} \simeq \mathbb{P}^2$, and $\mathcal{O}_{\tilde{E}}(\tilde{E}) \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$ [Kaw96]. Hence, $\mathcal{O}_{\tilde{E}}(-K_{\tilde{X}}) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$. We get a contradiction because $-K_{\tilde{X}}$ is divisible by $q \geq 5$.

(b) $g(\tilde{E})$ is either a smooth point or a curve. In this case $g(\tilde{E}) \not\subset \text{Bs}|A| = \{P\}$. On the other hand, $g$ is a $K_{\tilde{X}}$-negative contraction, a contradiction.

Finally assume that $\text{Cl}X$ has a torsion element $\xi \in \text{Cl}X$ of order $n \geq 2$, defining a $\mu_n$-cover $\pi: X' \to X$ that is étale in codimension 2. By the above, $X' \simeq \mathbb{P}(1^3, 2)$. Since $\dim H^0(X', \pi^*A) = \dim H^0(X, A) = 3$, the action of $\mu_n$ on $H^0(X', \pi^*A) = H^0(\mathcal{O}_{\mathbb{P}(1^3, 2)}(1))$ is trivial. On the other hand, we can take independent sections $x_1, x_2, x_3 \in H^0(\mathcal{O}_{\mathbb{P}(1^3, 2)}(1))$ as orbit representatives at the $1/2(1, 1, 1)$-point $P' \in X'$. This contradicts that the point $(X', P')/\mu_n$ is terminal.

\[\square\]

Similar to Lemma 4.3 one can prove the following.

4.4. Lemma (cf. Pro10b Th. 1.4 (vi)). Let $X$ be a Fano threefold with terminal singularities and with $q := \mathbb{Q}(X) \geq 7$. Let $A$ be a Weil divisor such that $-K_X \sim qA$. If $\dim |A| \geq 1$, then $X \simeq \mathbb{P}(1^2, 2, 3)$.

4.5. Proposition. Let $X$ be a $\mathbb{Q}$-Fano threefold and let $q := \mathbb{Q}(X)$. Let $\mathcal{M}$ be a linear system on $X$ such that $\dim \mathcal{M} \geq 4$ and $-K_X \sim 2\mathcal{M} + \Xi$, where $\Xi$ is an effective Weil divisor, $\Xi \neq 0$. Then $\text{Cl}X \simeq \mathbb{Z}$, the class of $\Xi$ generates $\text{Cl}X$, $q = 2n + 1$ is odd, and $\mathcal{M} \sim n\Xi$. Moreover, one of the following holds:

(i) $q = 13$, $X \simeq \mathbb{P}(1, 3, 4, 5)$;
(ii) $q = 11$, $X \simeq \mathbb{P}(1, 2, 3, 5)$;
(iii) $q = 9$, $X \simeq X_6 \subset \mathbb{P}(1, 2, 3, 4, 5)$;
(iv) $q = 7$, $X \simeq \mathbb{P}(1^2, 2, 3)$;
(v) $q = 5$, $X \simeq X_4 \subset \mathbb{P}(1^2, 2^2, 3)$;
(vi) $q = 5$, $X \simeq \mathbb{P}(1^3, 2)$;
(vii) $q = 3$, $X \simeq X_2 \subset \mathbb{P}^4$.

Proof. By our assumption $q \geq 3$. If $q \geq 9$, then the assertion follows by Pro10b Prop. 3.6] and Theorem 4.3(i). So we assume that $3 \leq q \leq 8$.

Let $A$ be a Weil divisor such that $-K_X \sim qA$ and let $n$ be the integer such that $\mathcal{M} \sim qnA$. If $\text{Cl}X$ is torsion free, we can run the computer search for $q := \mathbb{Q}(X) = q \geq 4$ and $g(\tilde{X}) \geq 21$. Then by Theorem 4.3 we
get one of cases \((\text{iv})\) \((\text{vii})\). Thus from now on we assume that \(\text{Cl} \, X\) contains a nontrivial torsion element.

We may assume that \(\mathcal{M}\) has no fixed components. If the pair \((X, \mathcal{M})\) is terminal, then \(X\) is as in \((\text{vi})\) or \((\text{vii})\) by Lemma 4.2. Assume that the pair \((X, \mathcal{M})\) is not terminal. Apply construction 2.2 to \((X, \mathcal{M})\). We can write

\[
K_{\tilde{X}} + 2\tilde{M} + \tilde{\Xi} + a\tilde{E} \sim f^*(K_X + 2\mathcal{M} + \Xi) \sim 0,
\]

where \(a \in \mathbb{Z}_{>0}\). Hence,

\[
(4.5.1) \quad K_{\tilde{X}} + 2\tilde{M} + \tilde{\Xi} + a\tilde{E} \sim 0.
\]

First, we consider the case where \(f\) is not birational. Then we are in the situation of 2.2.4. In particular, \(\tilde{X}\) is either \(\mathbb{P}^1\) or a del Pezzo surface as in 2.2.5.

Assume that \(\mathcal{M}\) is \(f\)-horizontal. Restricting the relation \((4.5.1)\) to a general fiber \(F\) of \(f\) we get

\[
-K_{\tilde{X}} \sim_q 2\tilde{M}|_F + \tilde{\Xi}|_F + a\tilde{E}|_F,
\]

where the divisors \(\tilde{M}|_F\) and \(\tilde{E}|_F\) are ample. This is possible only if \(F \simeq \mathbb{P}^2\), \(\tilde{X} \simeq \mathbb{P}^1\), \(O_F(\tilde{M}) \simeq O_F(\tilde{E}) \simeq O_{\mathbb{P}^2}(1)\), and \(a = 1\). From the exact sequence

\[
0 \rightarrow O_{\mathbb{P}^2}(\tilde{M} - F) \rightarrow O_{\mathbb{P}^2}(\tilde{M}) \rightarrow O_{\mathbb{P}^2}(\tilde{M}) \rightarrow 0
\]

we get

\[
\dim H^0(O_{\mathbb{P}^2}(\tilde{M} - F)) \geq \dim H^0(O_{\mathbb{P}^2}(\tilde{M})) - \dim H^0(O_{\mathbb{P}^2}(\tilde{M})) \geq 2.
\]

Thus \(\tilde{M} \geq F + L\), where \(L \in |M - F|\) is a moveable divisor. Hence there is a decomposition \(-K_X \sim 2F + 2L + \Xi\). In particular, \(q \geq 5\) and \(F \sim_{q} L \sim_{q} A\). This implies that \(f\) has no multiple fibers. So, the group \(\text{Cl} \, \tilde{X}\) is torsion free. Since \(O_{\mathbb{P}^2}(\tilde{E}) \simeq O_{\mathbb{P}^2}(1)\), the class of \(\tilde{E}\) is not divisible in \(\text{Cl} \, \tilde{X}\). Hence \(\text{Cl} \, X\) is also torsion free, a contradiction.

Therefore, \(\tilde{M}\) is \(f\)-vertical. Then \(\tilde{M} = \tilde{f}^*B\), where \(B\) is a linear system of Weil divisors on \(\tilde{X}\) with \(\dim B \geq 4\). We use the notation of 2.2.4. Let \(G = \tilde{f}^*\Theta\). We can write \(B \sim_{q} t\Theta\) for some \(t \in \mathbb{Z}_{>0}\). Then

\[
-K_{\tilde{X}} \sim_{q} 2t\tilde{C} + \tilde{\Xi} + a\tilde{E},
\]

so \(8 \geq q \geq 2t + 1\) and \(t \leq 3\). If \(\tilde{X} \simeq \mathbb{P}^1\), we obviously have \(\dim B \leq 2\). Therefore, \(\tilde{X}\) is a surface. Now we use 2.2.5.

If \(t = 1\), then \(\dim B \leq 2\), a contradiction. Consider the case \(t = 2\). Then \(\dim B \geq 4\) only in the case \(\tilde{X} \simeq \mathbb{P}^2\). Then \(q \geq 5\), \(G \sim_{q} A\), and \(m = 2\). Since \(\dim |G| \geq 2\), by Lemma 1.3 we have \(X \simeq \mathbb{P}(1^5, 2)\). Consider the case \(t = 3\). Then \(q \geq 7\) and \(G \sim_{q} A\). Since \(\dim B \geq 4\), we have either \(\tilde{X} \simeq \mathbb{P}^2\), \(\tilde{X} \simeq \mathbb{P}(1, 1, 2)\), or \(K_{\tilde{X}}^2 = 1\). In either case
dim |G| ≥ 1 (recall that if $K_X^2 = 1$, we take $\Theta = -K_X$). By Lemma 4.4 we get $X \simeq \mathbb{P}(1^2, 2, 3)$.

Now assume that $\overline{f}$ is birational. We have

$$-K_X \sim 2\hat{M} + \hat{\Xi} + a\hat{E},$$

where $\dim \hat{M} \geq \dim M$. If $(\hat{X}, \hat{M})$ is not terminal, we can repeat the procedure 2.2 and continue. Thus we may assume that $(\hat{X}, \hat{M})$ is as in (i)–(vii). In particular, $\text{Cl} \hat{X}$ is torsion free and $\hat{\Xi} + a\hat{E} \sim \hat{\Theta}$, where $\hat{\Theta}$ is the ample generator of $\text{Cl} \hat{X}$. So, $\hat{\Xi} = 0$, $a = 1$, and $\hat{E} \sim \hat{\Theta}$. In particular, the class of $\hat{E}$ is a primitive element of $\text{Cl} \hat{X} \simeq \mathbb{Z} \oplus \mathbb{Z}$. In this case, $\text{Cl} X$ is also torsion free. □

5. Proof of Theorem 1.5

5.1. Let $X$ be a $\mathbb{Q}$-Fano threefold such that $-K_X \sim 2A$ for a primitive element $A \in \text{Cl} X$. Assume that $K_X$ is not Cartier and $\dim |A| \geq 4$. Apply Construction 2.2 with $M := |A|$ and $\Xi = 0$. By Lemma 4.2 the pair $(X, M)$ is not terminal. Hence in the notation of (2.2.3), the discrepancy $a > 0$. On the other hand, $a$ is an integer. Therefore, $a \geq 1$.

5.2. Lemma. The map $\overline{f}$ in (2.2.2) is birational.

Proof. Suppose that $\overline{f}$ is not birational. Let $\overline{F}$ be a general fiber of $\overline{f}$. If $\overline{M}$ is $\overline{f}$-vertical, then $\overline{M} = \overline{f}^*\hat{B}$, where $\hat{B}$ is a linear system on $\hat{X}$ whose class generates $\text{Cl} \hat{X}/\text{Tors}$. But then $\dim \overline{M} = \dim \hat{B} \leq 2$ by 2.2.5 contradicting our assumption.

Thus $\overline{M}$ is $\overline{f}$-horizontal. Then $-K_{\overline{F}} = 2\overline{M}|_{\overline{F}} + a\overline{E}|_{\overline{F}}$. This implies that $\overline{F} \simeq \mathbb{P}^2$, that is, $\overline{f}$ is a generically $\mathbb{P}^2$-bundle and $\mathcal{O}_{\overline{F}}(\overline{M}) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$. From the exact sequence

$$0 \rightarrow \mathcal{O}_{\overline{X}}(\overline{M} - \overline{F}) \rightarrow \mathcal{O}_{\overline{X}}(\overline{M}) \rightarrow \mathcal{O}_{\overline{F}}(\overline{M}) \rightarrow 0$$

we get

$$\dim H^0(\mathcal{O}_{\overline{X}}(\overline{M} - \overline{F})) \geq 2.$$

Therefore, $\overline{M} \ni \overline{F} + \overline{T}$, where $\overline{F}$ and $\overline{T}$ are moveable divisors. This contradicts $q \mathbb{Q}(X) = 2$. □

5.3. Thus $\overline{f}$ is birational. In this case, $\hat{X}$ is a $\mathbb{Q}$-Fano and

$$(5.3.1) \quad -K_{\hat{X}} \sim 2\hat{M} + a\hat{E} \quad \text{with} \quad a > 0, \ \dim \hat{M} \geq 4.$$

By Proposition 4.5 the class of $\hat{E}$ is the ample generator of $\text{Cl} \hat{X} \simeq \mathbb{Z}$, $\hat{q} = 2n + 1$, and $\hat{M} \subset |n\hat{E}|$. Moreover, $\hat{X}$ belongs to one of the possibilities listed in Proposition 4.5.
Assume first that \( q > 3 \). The case \( q = 3 \) will be considered in the next section. We make frequent use of the following easy observation.

**5.3.2. Remark.** Assume that in the notation of 5.3 there is a member \( \hat{M} \in \mathcal{M} \) such that \( \hat{M} = \hat{L}_1 + \hat{L}_2 \), where \( \hat{L}_1 \) and \( \hat{L}_2 \) are effective ample Weil divisors. Then either \( \text{Supp} \hat{L}_1 = \hat{E} \) or \( \text{Supp} \hat{L}_2 = \hat{E} \). Indeed, we can write

\[
\mathcal{M} \sim q \mathcal{T}_1 + \mathcal{T}_2 + \gamma \mathcal{F},
\]

where \( \mathcal{T}_i \) is the proper transform of \( \hat{L}_i \) and \( \gamma \geq 0 \). Therefore,

\[
\mathcal{M} \sim f_\ast \chi_\ast^{-1} \mathcal{M} \sim q f_\ast \chi_\ast^{-1} \mathcal{T}_1 + f_\ast \chi_\ast^{-1} \mathcal{T}_2 + \gamma \mathcal{F}.
\]

Since the class of \( A \) is a primitive element of \( \text{Cl} \ X \), we have either \( f_\ast \chi_\ast^{-1} \mathcal{T}_1 = 0 \) or \( f_\ast \chi_\ast^{-1} \mathcal{T}_2 = 0 \) (and \( \gamma = 0 \)).

**5.3.3. Corollary.** Assume that in the notation of 5.3 we have \( \dim |n \hat{E}| = 4 \). Then for any partition \( n = n_1 + n_2 \), \( n_i \in \mathbb{Z} \) either \( \dim |n_1 \hat{E}| \leq 0 \) or \( \dim |n_2 \hat{E}| \leq 0 \).

**Proof.** In this case \( \mathcal{M} = |n \hat{E}| \) is a complete linear system. Hence, one can take \( \hat{L}_i \in |n_i \hat{E}| \).

We consider the cases of Proposition 4.5 separately.

**5.3.4. Cases [i], [iii] and [v].** Then \( \dim |n \hat{E}| = 4 \) and \( n \) is even. Apply Corollary 5.3.3 with \( n_1 = n_2 = n/2 \). We get a contradiction because \( \dim |n_i \hat{E}| > 0 \).

**5.3.5. Case [ii], that is, \( \hat{X} \simeq \mathbb{P}(1,2,3,5) \).** Then \( n = 5 \) and \( \dim |n \hat{E}| = 5 \). Thus \( \mathcal{M} \subset |5 \hat{E}| \) is a subsystem of codimension \( \leq 1 \). Since \( \dim 2 \hat{E} = 1 \), we can take \( \hat{L}_1 \in |2 \hat{E}| \) so that \( \hat{L}_1 \neq 2 \hat{E} \). Since \( \dim |3 \hat{E}| = 2 \), there exists a one-dimensional family of divisors \( \hat{L}_2 \in |3 \hat{E}| \) such that \( \hat{L}_1 + \hat{L}_2 \in \mathcal{M} \). So we may assume that \( \hat{L}_2 \neq 3 \hat{E} \). By Remark 5.3.2 we get a contradiction.

**5.3.6. Case [iv], that is, \( \hat{X} \simeq \mathbb{P}(1^2,2,3) \).** Then \( n = 3 \) and \( \dim |n \hat{E}| = 6 \). Thus \( \mathcal{M} \subset |3 \hat{E}| \) is a subsystem of codimension \( \leq 2 \). Since \( \dim |2 \hat{E}| = 1 \), we can take \( \hat{L}_1 \in |2 \hat{E}| \) so that \( \hat{L}_1 \neq 2 \hat{E} \). Since \( \dim |2 \hat{E}| = 3 \), there exists a one-dimensional family of divisors \( \hat{L}_2 \in |2 \hat{E}| \) such that \( \hat{L}_1 + \hat{L}_2 \in \mathcal{M} \). So we may assume that \( \hat{L}_2 \neq 2 \hat{E} \). By Remark 5.3.2 we get a contradiction.

**5.3.7. Case [vi], that is, \( \hat{X} \simeq \mathbb{P}(1^3,2) \).** Then \( n = 2 \) and \( \dim |n \hat{E}| = 6 \). Thus \( \mathcal{M} \subset |2 \hat{E}| \) is a subsystem of codimension \( \leq 2 \).

Assume that \( \mathcal{O}(\hat{F}) \) is a curve. Then

\[
K_{\hat{X}} = \mathcal{T}^* K_{\hat{X}} + \mathcal{F}, \quad \mathcal{E} = \mathcal{T}^* \hat{E} - \gamma \mathcal{F}.
\]
Since any member of $|\hat{E}|$ is smooth in codimension one, $\gamma \leq 1$. Moreover, since $n\hat{E}$ is not moveable for any $n$, we have $\gamma > 0$. Hence, $\gamma = 1$. So,

$$K_{\hat{X}} + 5\hat{E} + 4\hat{F} = \hat{f}'(K_{\hat{X}} + 5\hat{E}) \sim 0.$$ 

This implies that $-K_X$ is divisible by 4, a contradiction.

Hence $\hat{f}(\hat{F}) \in \hat{X}$ is a point, say $\hat{P}$. If $\hat{P} \in \hat{X}$ is the points of index 2, then $\hat{f}$ is the blowup of the maximal ideal [Kaw90]. In this case $\hat{X}$ has exactly two the extremal contractions: $\hat{f}$ and the $\mathbb{P}^1$-bundle induced by the projection $\mathbb{P}(1^3, 2) \to \mathbb{P}^2$. On the other hand, since the second contraction must be birational, a contradiction. Hence $\hat{P} \in \hat{X}$ is a smooth point.

Let $\hat{L} := |\hat{E}|$. Take a general member $\hat{L}_1 \in \hat{L}$. Dimension count shows that there exists $\hat{L}_2 \in \hat{L}$ such that $\hat{L}_1 + \hat{L}_2 \in \hat{M}$. If $\hat{L}_2 \neq \hat{E}$, we get a contradiction by Remark 5.3.2. Thus $\hat{L}_2 = \hat{E}$ for any choice of $\hat{L}_1 \in \hat{L}$. Therefore, $\hat{E} + \hat{L} \subset \hat{M}$ and we can write $\hat{M} \sim_{\hat{q}} \hat{L} + \hat{E} + \gamma \hat{F}$, where $\gamma \geq 0$. Then

$$0 \sim K_{\hat{X}} + 2\hat{M} + \hat{E} \sim_{\hat{q}} K_{\hat{X}} + 2\hat{L} + 3\hat{E} + 2\gamma \hat{F}.$$ 

Note that the only base point of $\hat{L}$ is the point of index 2. Hence, $\hat{L} \sim_{\hat{q}} \hat{f}' \hat{L}$. Let $\hat{L}' \subset \hat{L}$ be the subsystem consisting of elements passing through $\hat{P}$. Then we can write

$$\hat{L} \sim_{\hat{q}} \hat{f}' \hat{L}' - \delta \hat{F} \sim_{\hat{q}} \hat{L} - \delta \hat{F}, \quad \delta > 0.$$ 

Therefore,

$$0 \sim_{\hat{q}} K_{\hat{X}} + 2\hat{L} + 3\hat{E} + 2\gamma \hat{F} \sim_{\hat{q}} K_{\hat{X}} + 2\hat{L}' + 3\hat{E} + 2(\delta + \gamma)\hat{F}.$$ 

This gives us $-K_X \sim_{\hat{q}} 2\hat{L}' + 2(\delta + \gamma)\hat{F}$ which contradicts $q\mathbb{Q}(X) = 2$.

### 6. Proof of Theorem 1.5 (continued)

#### 6.1. In this section we consider the case [vii] that is, we assume that $\hat{X} = Q \subset \mathbb{P}^4$ is a smooth quadric. Then $\hat{M} = |\mathcal{O}_Q(1)|$ is a complete linear system. In particular, $\hat{M}$ is base point free. Hence, $\hat{M} \sim_{\hat{q}} \hat{f}' \hat{M}$. We also have $\hat{E} \in |\mathcal{O}_Q(1)|$ and $\hat{f}(\hat{F}) \subset \hat{E}$.

#### 6.2. Lemma. $\Gamma := \hat{f}(\hat{F})$ is a curve.

*Proof.* Assume that $\hat{f}(\hat{F})$ is a point. Let $\hat{M}' \subset \hat{M}$ be the subsystem consisting of elements passing through $\hat{f}(\hat{F})$. Then we can write

$$\hat{M}' \sim_{\hat{q}} \hat{f}' \hat{M}' - \delta \hat{F} \sim_{\hat{q}} \hat{M} - \delta \hat{F}, \quad \delta > 0.$$
Therefore,

\[ 0 \sim_\mathbb{Q} \mathcal{I} \left( K_{\tilde{X}} + 2\mathcal{M}' + \tilde{E} \right) \sim_\mathbb{Q} \mathcal{I} \left( K_{\tilde{X}} + 2\mathcal{M} + \tilde{E} \right) \sim_\mathbb{Q} K_{\tilde{X}} + 2\mathcal{M} + \tilde{E} \sim_\mathbb{Q} K_{\tilde{X}} + 2\mathcal{M}' + \tilde{E} + 2\delta \mathcal{F}. \]

This gives us \(-K_X \sim_\mathbb{Q} 2\mathcal{M}' + 2\delta \mathcal{F}\) which contradicts \(\mathbb{Q}(X) = 2\). \(\square\)

6.3. Lemma. \(\mathcal{E} \simeq \mathbb{P}(1, 1, 2), \mathbb{F}_2, \text{ or } \mathbb{P}^1 \times \mathbb{P}^1.\)

Proof. Clearly, \(\tilde{\mathcal{E}} \simeq \mathbb{P}(1, 1, 2)\) or \(\mathbb{P}^1 \times \mathbb{P}^1.\) In particular, the pair \((\tilde{X}, \tilde{S})\) is plt. Since \(K_{\tilde{X}} \sim_\mathbb{Q} \mathcal{I} K_{\tilde{X}} + \mathcal{F}\) and \(\tilde{S}\) is smooth at the generic point of \(\Gamma\), we have

\[ (6.3.1) \quad K_{\tilde{X}} + \tilde{S} \sim_\mathbb{Q} \mathcal{I} \left( K_{\tilde{X}} + \tilde{S} \right). \]

Hence the pair \((\tilde{X}, \tilde{S})\) is plt and the divisor \(K_{\tilde{X}} + \tilde{S}\) is Cartier. By the adjunction, the surface \(\tilde{S}\) has at worst Du Val singularities. Moreover, \(K_{\tilde{S}} = \mathcal{I} \left( K_{\tilde{S}} \right)\), that is, the restriction \(\mathcal{I} \tilde{S}\) is either an isomorphism or the minimal resolution of \(\tilde{S}\). \(\square\)

6.4. Lemma. \(-K_{\overline{X}}\) is nef.

Proof. Recall that by our construction \(\overline{X}\) has exactly two extremal rays. Denote them by \(R_1\) and \(R_2\). One of them, say \(R_1\), is generated by nontrivial fibers of \(\mathcal{I}\). Let \(C\) be an extremal curve on \(X\) that generates \(R_2\). Assume that \(-K_{\overline{X}}\) is not nef. Then \(K_{\overline{X}} \cdot C > 0\) and \(C\) must be a flipped curve (because a divisorial contraction must be \(K\)-negative in our situation). Since \(-K_{\overline{X}} \sim_\mathbb{Q} \tilde{S} + 2\mathcal{I} \tilde{S}\), we have \(\overline{S} \cdot C < 0\). In particular, \(C \subset \overline{S}\). Since \(C\) is a flipped curve, it cannot be moveable on \(\overline{S}\), that is, \(\dim |C| = 0\). By Lemma 6.3, the only possibility is \(\mathcal{E} \simeq \mathbb{F}_2\) and \(C\) is the negative section of \(\mathbb{F}_2\). But in this case \(C\) is contracted by \(\mathcal{I}\) to a point, that is, the class of \(C\) lies in \(R_1\), a contradiction. \(\square\)

6.5. Lemma. \(K_{\overline{X}}\) is not Cartier at some point of \(\overline{S}\).

Proof. By (6.3.1) the divisor \(K_{\overline{X}}\) is Cartier outside of \(\overline{S}\). Assume that \(K_{\overline{X}}\) is Cartier near \(\overline{S}\). Since \(-K_{\overline{X}}\) is nef, the map \(\overline{X} \dashrightarrow \tilde{X}\) is either an isomorphism or a flop. In both cases \(\tilde{X}\) has the same type of singularities as \(X\), that is, \(K_{\tilde{X}}\) is Cartier. By the classification of extremal contractions of Gorenstein terminal treefolds [Cut88] the divisor \(2K_{\tilde{X}}\) is Cartier. This contradicts the following remark. \(\square\)

6.5.1. Corollary. The singularities of \(\Gamma\) are worse than locally complete intersection points.

Proof. Indeed otherwise by [KM92, Prop. 4.10.1] the map \(\overline{f}\) is the blowup of \(\Gamma\) and \(K_{\overline{X}}\) is Cartier. \(\square\)
6.5.2. Corollary. \( \hat{E} \simeq \mathbb{P}(1,1,2) \), the curve \( \Gamma \) is not a Cartier divisor on \( \hat{E} \), and \( \Gamma \) is singular at the vertex of \( \mathbb{P}(1,1,2) \).

6.6. Lemma. \( \deg \Gamma = 5 \).

Proof. Let \( \hat{C} \subset \hat{E} \) be a general hyperplane section. Since \( -K_X \) is nef,
\[
0 \leq -K_X \cdot \hat{C} = -K_X \cdot \hat{C} - (\Gamma \cdot \hat{C})_{\hat{E}} = 6 - \deg \Gamma.
\]
Since \( \Gamma \) is not a Cartier divisor on \( \hat{E} \), its degree should be odd. If \( \deg \Gamma \neq 5 \), then \( \Gamma \) is either a line or a twisted cubic. In particular, it is smooth, a contradiction. \( \square \)

6.7. Thus \( \deg \Gamma = 5 \) and \( \Gamma \) is singular. Then the curve \( \Gamma \) can be given, in some coordinate system \( x_1, x_1', x_2 \) in \( S \simeq \mathbb{P}(1,1,2) \), by the equation \( x_2\phi_3 + \phi_5 \), where \( \phi_k = \phi_k(x_1, x_1') \) is a homogeneous polynomial of degree \( k \). So, \( P \) is a triple point of \( \Gamma \) and \( \Gamma \) has no singular points other than \( P \). Thus \( \Gamma \) is as in Theorem 3.3. According to \[\text{KM92 \ Th. 4.9} \] the extraction \( \overline{f} : X \rightarrow Q = \hat{X} \) is unique up to isomorphism over \( Q \). Since \( \rho(\overline{X}/Q) = 2 \), the Sarkisov link \( Q \leftarrow \overline{X} \rightarrow \hat{X} \rightarrow X \) is uniquely determined. This completes the proof of Theorem 3.3.

References

[Al94] Valery Alexeev. General elephants of \( \mathbb{Q} \)-Fano 3-folds. Compositio Math., 91(1):91–116, 1994.

[B\+\] Gavin Brown et al. Graded ring database.

[BKR10] G. Brown, M. Kerber, and M. Reid. Fano 3-folds in codimension 4, Tom and Jerry, Part I. ArXiv e-print, 1009.4313, 2010. to appear in Comp. Math.

[BS07a] Gavin Brown and Kaori Suzuki. Computing certain Fano 3-folds. Japan J. Indust. Appl. Math., 24(3):241–250, 2007.

[BS07b] Gavin Brown and Kaori Suzuki. Fano 3-folds with divisible anticanonical class. Manuscripta Math., 123(1):37–51, 2007.

[CPR00] Alessio Corti, Aleksandr Pukhlikov, and Miles Reid. Fano 3-fold hypersurfaces. In Explicit birational geometry of 3-folds, volume 281 of London Math. Soc. Lecture Note Ser., pages 175–258. Cambridge Univ. Press, Cambridge, 2000.

[CR02] Alessio Corti and Miles Reid. Weighted Grassmannians. In Algebraic geometry, pages 141–163. de Gruyter, Berlin, 2002.

[Cut88] Steven Cutkosky. Elementary contractions of Gorenstein threefolds. Math. Ann., 280(3):521–525, 1988.

[Fuj90] Takao Fujita. Classification theories of polarized varieties, volume 155 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1990.

[Kaw88] Yujiro Kawamata. Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. Ann. of Math. (2), 127(1):93–163, 1988.
[Kaw92] Yujiro Kawamata. Boundedness of $\mathbb{Q}$-Fano threefolds. In *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 439–445, Providence, RI, 1992. Amer. Math. Soc.

[Kaw96] Yujiro Kawamata. Divisorial contractions to 3-dimensional terminal quotient singularities. In *Higher-dimensional complex varieties (Trento, 1994)*, pages 241–246. de Gruyter, Berlin, 1996.

[KM92] János Kollár and Shigefumi Mori. Classification of three-dimensional flips. *J. Amer. Math. Soc.*, 5(3):533–703, 1992.

[Mor82] S. Mori. Threefolds whose canonical bundles are not numerically effective. *Ann. Math.*, 115:133–176, 1982.

[MP08] Shigefumi Mori and Yuri Prokhorov. On $\mathbb{Q}$-conic bundles. *Publ. Res. Inst. Math. Sci.*, 44(2):315–369, 2008.

[Pro07] Yu. Prokhorov. The degree of $\mathbb{Q}$-Fano threefolds. *Russian Acad. Sci. Sb. Math.*, 198(11):1683–1702, 2007.

[Pro10a] Yu. Prokhorov. G-Fano threefolds, I. *ArXiv e-print*, 1012.4959, 2010. to appear in Advances in Geometry.

[Pro10b] Yuri Prokhorov. $\mathbb{Q}$-Fano threefolds of large Fano index. I. *Doc. Math.*, J. DMV, 15:843–872, 2010.

[Pro10c] Yuri Prokhorov. $\mathbb{Q}$-Fano threefolds of large Fano index, II. *ArXiv e-print*, 1010.3404, 2010.

[Rei83] Miles Reid. Minimal models of canonical 3-folds. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 131–180. North-Holland, Amsterdam, 1983.

[Rei87] Miles Reid. Young person’s guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[RS03] Miles Reid and Kaori Suzuki. Cascades of projections from log del Pezzo surfaces. In *Number theory and algebraic geometry*, volume 303 of *London Math. Soc. Lecture Note Ser.*, pages 227–249. Cambridge Univ. Press, Cambridge, 2003.

[Suz04] Kaori Suzuki. On Fano indices of $\mathbb{Q}$-Fano 3-folds. *Manuscripta Math.*, 114(2):229–246, 2004.

Department of Algebra, Faculty of Mathematics, Moscow State University, Moscow 117234, Russia
Laboratory of Algebraic Geometry, SU-HSE, 7 Vavilova Str., Moscow 117312, Russia

E-mail address: prokhor@gmail.com

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

E-mail address: Miles.Reid@warwick.ac.uk