Topological expansion of mixed correlations in the Hermitian 2-matrix model and $x$–$y$ symmetry of the $F_g$ algebraic invariants

B Eynard and N Orantin

Service de Physique Théorique de Saclay, F-91191 Gif-sur-Yvette Cedex, France
E-mail: eynard@spht.saclay.cea.fr and orantin@spht.saclay.cea.fr

Received 1 October 2007
Published 12 December 2007
Online at stacks.iop.org/JPhysA/41/015203

Abstract
We compute expectation values of mixed traces containing both matrices in a two matrix model, i.e. a generating function for counting bicolored discrete surfaces with non-uniform boundary conditions. As an application, we prove the $x$–$y$ symmetry of Eynard and Orantin (2007 Invariants of algebraic curves and topological expansion Preprint math-ph/0702045).

PACS numbers: 02.10.Ox, 02.30.Ik, 02.10.Yn, 11.15.Pg

1. Introduction

Formal matrix integrals can be regarded as an efficient toy model to explore the link between algebraic geometry and integrable systems [3, 31]. The theory of quantum gravity [11, 12, 27] is based on the idea that matrix models provide a generating function to measure ‘volumes’ of moduli spaces of Riemann surfaces, and random matrix models were introduced in the 1980s [6] as a discretized version of 2D quantum gravity, i.e. conformal field theory coupled to gravity.

The formal matrix integral is at the same time a tau-function of some integrable hierarchy [12], and it has a ‘t Hoof topological expansion [1, 12, 33]:

$$\ln \int_{\text{formal}} dM e^{-N \text{Tr} V(M)} = \sum_{g=0}^{\infty} N^{2-2g} F(g),$$

(1.1)

which is related to algebraic geometry (see [3, 5, 14, 29]).

In a recent work [7, 8, 13, 19, 21], we have developed a method to compute $F(g)$’s for various formal Hermitian matrix models (1-matrix model, 2-matrix model, matrix model with an external field, double scaling limits of a 2-matrix model) out of the data of an algebraic equation (called the classical spectral curve):

$$\mathcal{E}(x, y) = 0, \quad \mathcal{E} = \text{polynomial}.$$
The construction of [21] extends beyond matrix models, and $F(g)$’s can be computed for any algebraic equation of the type $E(x, y) = 0$.

However, the construction of [21] assumes an embedding of the curve into $\mathbb{C}^2$, i.e. the choice of two meromorphic functions $x$ and $y$ on the curve. It was claimed in [21] that $F(g)$ is invariant under the exchange $x \leftrightarrow y$, and the proof was announced to be published separately.

This is what we do in the present paper, together with additional results.

### Mixed correlations

In order to prove this claim, we first explore the case where $F(g)$’s come from a formal 2-matrix model (the symmetry $x \leftrightarrow y$ holds almost by definition in that case, see [8]). We write the loop equation relations (W-algebra) [18, 32], which we solve, and we are led to define new mixed correlation functions ($W_{k,l}$ and $H_{k,l}$), which did not appear in [21].

In the application of the 2-matrix model to quantum gravity and conformal field theory, those mixed correlation functions were known to play an important role in the understanding of boundary operators. But their explicit computation has been a challenge until recently. The main reason is that they do not reduce to eigenvalues of the matrices and could not be computed by standard methods. The first explicit computations were obtained in [4] and [17]. Here in this paper, we show how to compute the topological expansion of a family of mixed correlation functions of the 2-matrix model. In a coming work [23], we shall show how to compute all mixed correlations, and introduce a link with the group theory and Bethe ansatz (this is a generalization of [22]).

Then, for the general case (i.e. if $E$ was not obtained from a matrix model), we mimic these mixed correlation functions and this allows us to prove the $x \leftrightarrow y$ symmetry of $F(g)$.

### 2. Mixed traces of matrix models

Consider the formal 2-matrix integral:

$$ Z = \int dM_1 \, dM_2 \, e^{-N \text{tr}(V_1(M_1) + V_2(M_2) - M_1 M_2)}, $$

where we assume in this section that $V_1$ is a polynomial of degree $d_1 + 1$ and $V_2$ is a polynomial of degree $d_2 + 1$.

Our goal is to compute the following connected expectation values:

$$ W_{k,l}(x_1, \ldots, x_k | y_1, \ldots, y_l) = \left\langle \text{tr} \frac{1}{x_1 - M_1} \frac{1}{x_2 - M_1} \cdots \frac{1}{x_k - M_1} \frac{1}{y_1 - M_2} \frac{1}{y_2 - M_2} \cdots \frac{1}{y_l - M_2} \right\rangle_c $$

$$ = \sum_{g=0}^{\infty} N^{2 - 2g - k - l} W_{k,l}^{(g)}(x_1, \ldots, x_k | y_1, \ldots, y_l) $$

and

$$ H_{k,l}(x, y; x_1, \ldots, x_k | y_1, \ldots, y_l) = \left\langle \text{tr} \frac{1}{x - M_1} \frac{1}{y_1 - M_2} \frac{1}{y_2 - M_2} \cdots \frac{1}{x_k - M_1} \frac{1}{y_1 - M_2} \frac{1}{y_2 - M_2} \cdots \frac{1}{y_l - M_2} \right\rangle_c $$

$$ = \sum_{g=0}^{\infty} N^{2 - 2g - k - l} H_{k,l}^{(g)}(x, y; x_1, \ldots, x_k | y_1, \ldots, y_l). $$

---

1. A formal integral is defined as a formal power series in some expansion parameter $t$, as explained in [20] or [21]. Formal matrix integrals always have a $1/N^2$ expansion order by order in $t$, called the topological expansion.
The curve of equation: was introduced in [28] as a discrete version of the Ising model on a random surface.

Note that in $\overline{W}_{k,l}^{(g)}$, the first trace contains both matrices $M_1$ and $M_2$; we call it a mixed trace because it cannot be expressed in terms of eigenvalues of $M_1$ and $M_2$. In applications of matrix models to conformal field theories, such objects correspond to the insertion of a pair of boundary operators, and are thus very interesting. $\overline{W}_{0,0}^{(g)}$ was computed in many works [9, 18], and in the context of convergent integrals (instead of formal integrals), $\overline{W}_{0,0}^{(g)}$ was computed in [2, 4, 17].

$\overline{W}_{k,0}^{(g)}$'s were already computed in [8, 13, 19], and are given by the algebraic invariants defined in [21]; they are the non-mixed traces.

It is known (see for instance [8]) that all these functions are multivalued functions of their $x$ or $y$ variables, and they are in fact functions living on a Riemann surface called the spectral curve of equation:

$$\mathcal{E}(x, y) = 0.$$  

(2.4)

On this curve, we chose a canonical basis of cycles $A_i \cap B_j = \delta_{i,j}$, $i, j = 1, \ldots, G$, where $G$ denotes the genus of the curve $\mathcal{E}$. We will note by $p^i$ (respectively $\tilde{p}^i$) the different points of $\mathcal{E}$ whose projection in the complex plane by the meromorphic function $x$ (respectively $y$) are equal:

$$\forall i = 1 \cdots d_2, \quad x(p^i) = x(p^0), \quad \forall i = 1 \cdots d_1, \quad y(\tilde{p}^i) = x(\tilde{p}^0),$$

(2.5)

where the superscript $0$ refers to the $x$- and $y$-physical sheets.

It is thus more convenient to redefine $\overline{W}_{k,0}^{(g)}$ and $\overline{W}_{k,l}^{(g)}$ in terms of meromorphic forms on the curve:

$$W_{k,l}^{(g)}(p_1, \ldots, p_k|q_1, \ldots, q_l) = \overline{W}_{k,l}^{(g)}(x(p_1), \ldots, x(p_k)|y(q_1), \ldots, y(q_l)) \, dx(p_1) \cdots dx(p_k) \, dy(q_1) \cdots dy(q_l)$$

$$+ \delta_{k,0} \delta_{l,0} (y(p_1) - V_1(x(p_1))) \, dx(p_1) + \delta_{k,0} \delta_{l,1} (x(q_1))$$

$$- V_2'(y(q_1)) \, dy(q_1) + \delta_{k,0} \delta_{l,0} \delta_{l,0} \, dx(p_1) \, dx(p_2)$$

$$+ \delta_{k,0} \delta_{l,0} \delta_{l,2} \, dy(q_1) \, dy(q_2)$$

$$((y(q_1) - y(q_2))^2)$$

(2.6)

where $p_i$'s and $q_j$'s are now points on the curve $\mathcal{E}$, instead of points in the complex plane. We have also ‘renormalized the unstable functions’ with $2 - 2g - k - l \geq 0$.

With those notations, we have [8, 5]

$$W_{1,0}^{(0)} = W_{0,1}^{(0)} = 0,$$  

(2.7)

$$W_{2,0}^{(0)}(p, q) = - W_{1,1}^{(0)}(p, q) = W_{0,2}^{(0)}(p, q) = B(p, q),$$

(2.8)

where $B$ is the Bergmann kernel, i.e. the unique bilinear form on $\mathcal{E}$ with a double pole at $p = q$ and no other pole, with vanishing residue, and normalized on $A$-cycles:

$$B(p, q) \sim \frac{dz(q)}{p - q} \frac{dz(q)}{(z(p) - z(q))^2} + \text{finite}, \quad \forall i = 1 \cdots G, \quad \oint_A B = 0.$$  

(2.9)

2 All required definitions relative to algebraic geometry can be found in [21] or more generally in [24, 25]. We will use all along these notes the notations of [21]. The $A$ and $B$ cycles may be the modified cycles of [21].
We also define the differentials corresponding to the mixed correlation functions:

\[ H_{k,l}^{(q)}(p, q; p_1, \ldots, p_k, q_1, \ldots, q_l) \]

\[ = \frac{\delta_{k,0}}{H_{0,0}^{(q)}(p, q)} \sum_{g=0}^{\infty} N^{2g-k-l-1} H_{k,l}^{(q)}(p, q; p_1, \ldots, p_k | q_1, \ldots, q_l). \]  

(2.10)

and we normalize them by the leading order of the simplest mixed correlation function:

\[ h_{k,l}^{(q)}(p, q; p_1, \ldots, p_k | q_1, \ldots, q_l) = \frac{H_{k,l}^{(q)}(p, q; p_1, \ldots, p_k | q_1, \ldots, q_l)}{H_{0,0}^{(q)}(p, q)}. \]  

(2.11)

It is well known [9, 14, 18] (and it can be rederived from equations (2.18) and (2.21)) that

\[ H_{0,0}^{(q)}(p, q) = \frac{\delta_{k,0}}{H_{0,0}^{(q)}(p, q)}. \]  

(2.12)

We also need to introduce

\[ U_{k,l}(p, y; p_1, \ldots, p_k, q_1, \ldots, q_l) \]

\[ = \left\{ \frac{1}{\text{tr}} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \frac{dx(p_1)}{x(p_1) - M_1} \cdots \frac{dx(p_k)}{x(p_k) - M_1} \right. \]

\[ \left. \frac{\text{tr}}{y(q_1) - M_2} \cdots \frac{\text{tr}}{y(q_l) - M_2} \right\} e^{\delta_{k,0} \delta_{l,0} (V_2^1(y) - x(p))} \]

\[ = \sum_{g=0}^{\infty} N^{2g-k-l-1} U_{k,l}^{(q)}(p, y; p_1, \ldots, p_k | q_1, \ldots, q_l), \]  

(2.13)

which is a polynomial of y of degree at most \( d_2 - 1 \),

\[ \tilde{U}_{k,l}(x, q; p_1, \ldots, p_k, q_1, \ldots, q_l) \]

\[ = \left\{ \frac{1}{\text{tr}} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{dx(p_1)}{x(p_1) - M_1} \right. \]

\[ \left. \frac{\text{tr}}{y(q_1) - M_2} \cdots \frac{\text{tr}}{y(q_l) - M_2} \right\} e^{\delta_{k,0} \delta_{l,0} (V_1^1(x) - y(p))} \]

\[ = \sum_{g=0}^{\infty} N^{2g-k-l-1} \tilde{U}_{k,l}^{(q)}(x, q; p_1, \ldots, p_k | q_1, \ldots, q_l), \]  

(2.14)

which is a polynomial of x of degree at most \( d_1 - 1 \) and

\[ -E_{k,l}(x, y; p_1, \ldots, p_k, q_1, \ldots, q_l) \]

\[ = \left\{ \frac{1}{\text{tr}} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{dx(p_1)}{x(p_1) - M_1} \right. \]

\[ \left. \frac{\text{tr}}{y(q_1) - M_2} \cdots \frac{\text{tr}}{y(q_l) - M_2} \right\} e^{\delta_{k,0} \delta_{l,0} ((V_1^1(x) - y(p))(V_2^1(y) - x(p)) - 1)} \]

\[ = -\sum_{g=0}^{\infty} N^{2g-k-l-1} E_{k,l}^{(q)}(x, y; p_1, \ldots, p_k | q_1, \ldots, q_l), \]  

(2.15)

which is a polynomial of x of degree \( d_1 - 1 \) and of y of degree \( d_2 - 1 \).
We have
\[ E_{0,0}(x, y) = \mathcal{E}(x, y), \quad U_{0,0}(p, y) = \frac{\mathcal{E}(x(p), y)}{y - y(p)}, \quad \tilde{U}_{0,0}(x, q) = \frac{\mathcal{E}(x, y(q))}{x - x(q)}, \]
(2.16)
and
\[ P_{0,0}(x, y) = -\mathcal{E}(x, y). \]
(2.17)

2.1. Loop equations

In order to obtain a closed set of equations computing these mixed correlation functions, we consider four families of loop equations [16, 18, 32] corresponding to different infinitesimal changes of variables \(M_i \rightarrow M_i + \epsilon \delta M_i\) in the matrix integral.

\[ \delta M_2 = \frac{1}{x(p) - M_1} \frac{1}{y(q) - M_2} \prod_{i=1}^{k} \frac{1}{x(p_i) - M_1} \prod_{j=1}^{l} \frac{1}{y(q_j) - M_2} \]
gives:
\[ -U_{k,l}^{(g)}(p, y(q); pK | qL) = (x(p) - x(q)) H_{k,l}^{(g)}(p, q; pK | qL) \]
\[ + \sum_{h} \sum_{i,j} W_{i,j+1}^{(h)}(p_i, q_j) H_{k-l,i-j}^{(g-h)}(p, q; pK | qL,j) \]
\[ + \frac{H_{k,l+1}^{(g-1)}(p, q; pK | qL) \delta y(q)}{dx(p)} = \sum_{m} d_{qm} H_{k,l}^{(g)}(p, q; pK | qL/m), \]
(2.18)

\[ \delta M_1 = \frac{1}{x(p) - M_1} \frac{1}{y(q) - M_2} \prod_{i=1}^{k} \frac{1}{x(p_i) - M_1} \prod_{j=1}^{l} \frac{1}{y(q_j) - M_2} \]
gives:
\[ -\tilde{U}_{k,l}^{(g)}(x(p), q; pK | qL) = (y(q) - y(p)) H_{k,l}^{(g)}(p, q; pK | qL) \]
\[ + \sum_{h} \sum_{i,j} W_{i,j+1}^{(h)}(p_i, q_j) H_{k-l,i-j}^{(g-h)}(p, q; pK | qL,j) \]
\[ + \frac{H_{k+1,l}^{(g-1)}(p, q; pK | qL) \delta x(p)}{dy(q)} = \sum_{m} d_{pm} H_{k,l}^{(g)}(p, q; pK | qL/m), \]
(2.19)

\[ \delta M_2 = \frac{1}{x(p) - M_1} \frac{1}{y(q) - M_2} \prod_{i=1}^{k} \frac{1}{x(p_i) - M_1} \prod_{j=1}^{l} \frac{1}{y(q_j) - M_2} \]
gives:
\[ E_{k,l}^{(g)}(x(p), y(q); pK | qL) = (x(p) - x(q)) \tilde{U}_{k,l}^{(g)}(x(p), q; pK | qL) \]
\[ + \sum_{h} \sum_{i,j} W_{i,j+1}^{(h)}(p_i, q_j) \Gamma_{k-l,i-j}^{(g-h)}(x(p), q; pK | qL,j) \]
\[ + \frac{\tilde{U}_{k,l+1}^{(g-1)}(x(p), q; pK | qL) \delta y(q)}{dy(q)} = \sum_{m} d_{qm} \tilde{U}_{k,l}^{(g)}(x(p), qm; pK | qL/m), \]
(2.20)
and \( \delta M = \frac{1}{x(p) - M_1} \frac{V_2'(y(q)) - V_2'(M_2)}{y(q) - M_2} \prod_{i=1}^{k} \text{tr} \frac{1}{x(p_i) - M_1} \prod_{j=1}^{l} \frac{1}{y(q_j) - M_2} \) gives:

\[
E^{(g)}_{k,l}(x(p), y(q); p_k q_l) = (y(q) - y(p))U^{(g)}_{k,l}(p, y(q); p_k q_l)
+ \sum_h \sum_{I \subset K} \sum_{J \subset L} W^{(g)}_{i+1,j}(p, p_I | q_J) h^{(g-h)}_{k-i,l-j}(r, q; p_k q_l / I | J)
+ \sum_h \sum_{I \subset K} \sum_{J \subset L} W^{(g)}_{i+1,j}(p, p_I | q_J) h^{(g-h)}_{k-i,l-j}(r, q; p_k q_l / I | J)
\]

where \( \text{Res}_{r \to \tilde{q}_j, p_k} \) means that one takes the residues around all the points \( \tilde{q}_j \neq q \) such that \( y(\tilde{q}_j) = y(q) \).

\[
M_{k,l}^{(g)}(p_{k+1}, \ldots, p_k) = W^{(g)}_{k}(p_{k+1}, \ldots, p_k) | E,
\]

where \( W^{(g)}_{k}(p_{k+1}, \ldots, p_k) | E \) is the function defined in [21], the above system is triangular and computes univocally any \( h^{(g)}_{k,l} \) and \( W^{(g)}_{k,l} \) in at most \( k + l + \frac{g}{2} \) steps.

One easily proves by recursion on \( 2g + k + l \) that

\[
H^{(g)}_{k,l}(p, q; p_k q_l) \text{ has poles } \begin{cases} \text{in } p = a, q, q_l \text{ in } q = b, p, p_k \text{ in } q_j = a, q, q_l \text{ in } p_j = a, q, q_l \end{cases}
\]

and

\[
W^{(g)}_{k,l}(p_k q_l) \text{ has poles } \begin{cases} \text{in } p_j = a, q_l \text{ in } q_j = b, p_k \end{cases}
\]
Proof. Since \( \tilde{U}_{k,l}^{(q)}(x(p), q; p_k|q_L) \) is a polynomial in \( x(p) \) of degree at most \( d_1 = 2 \), it is
given by the Lagrange interpolation formula:
\[
\tilde{U}_{k,l}^{(q)}(x(p), q; p_k|q_L) = \tilde{U}_{0,0}^{(0)}(x(p), q) \sum_{j=1}^{d_1} \frac{\tilde{U}_{k,l}^{(q)}(x(q^j), q; p_k|q_L)}{(x(p) - x(q^j)) \tilde{U}_{0,0}^{(0)}(x(q^j), q)} \sum_{j=1}^{d_1} \frac{\tilde{U}_{k,l}^{(q)}(x(q^j), q; p_k|q_L)}{(x(p) - x(q^j)) \tilde{U}_{0,0}^{(0)}(x(q^j), q)}
\]
\[
= \tilde{U}_{0,0}^{(0)}(x(p), q) \sum_{j=1}^{d_1} \frac{\tilde{U}_{k,l}^{(q)}(x(q^j), q; p_k|q_L) d_x(r)}{x(r) - x(p_m)}.
\]
(2.27)

Then we replace \( \tilde{U}_{k,l}^{(q)}(x(q^j), q; p_k|q_L) \) by its value from the loop equation (2.19):
\[
\tilde{U}_{k,l}^{(q)}(x(p), q; p_k|q_L) = \sum_{j=1}^{d_1} \frac{\tilde{U}_{0,0}^{(0)}(x(p), q)}{(x(p) - x(r)) \tilde{U}_{0,0}^{(0)}(x(r), q)}
\]
\[
× \sum_{\substack{h \in J \cup I \setminus \{q_L, q\} \setminus \{p_L, p_K, l\} \cup \{x(r), q\} \setminus \{x(q^j), q\} \setminus \{x(p), q\} \setminus \{x(p_m), q\} \cup \{x(r), q\}\cup \{x(q^j), q\}\cup \{x(p), q\}\cup \{x(p_m), q\}}
\]
\[
+ H_{k+1,l}^{(q)}(r, q; p_k|q_L) = \sum_{j=1}^{d_1} \frac{H_{k+1,l}^{(q)}(p_m, q; p_k|l|m) d_x(r)}{x(r) - x(p_m)}.
\]
(2.28)

Note that the same residue computed at \( r \rightarrow p \) gives the terms in the rhs of the loop equation (2.19), and therefore
\[
(y(q) - y(p)) H_{k,l}^{(q)}(p, q; p_k|l|m) = \frac{\tilde{U}_{0,0}^{(0)}(x(p), q)}{(x(p) - x(r)) \tilde{U}_{0,0}^{(0)}(x(r), q)}
\]
\[
× \sum_{\substack{h \in J \cup I \setminus \{q_L, q\} \setminus \{p_L, p_K, l\} \cup \{x(r), q\} \setminus \{x(q^j), q\} \setminus \{x(p), q\} \setminus \{x(p_m), q\} \cup \{x(r), q\}\cup \{x(q^j), q\}\cup \{x(p), q\}\cup \{x(p_m), q\}}
\]
\[
+ H_{k+1,l}^{(q)}(r, q; p_k|q_L) = \sum_{j=1}^{d_1} \frac{H_{k+1,l}^{(q)}(p_m, q; p_k|l|m) d_x(r)}{x(r) - x(p_m)}.
\]
(2.29)

Moreover, the last term \( d_{p_m} H_{k+1,l}^{(q)}(p_m, q; p_k|l|m) d_x(r) \) can be computed explicitly:
\[
d_{p_m} \frac{d_{p_m} H_{k+1,l}^{(q)}(p_m, q; p_k|l|m) d_x(r)}{x(r) - x(p_m)} = \frac{\tilde{U}_{0,0}^{(0)}(x(p), q)}{(x(p) - x(r)) \tilde{U}_{0,0}^{(0)}(x(r), q)} x(r) - x(p_m)
\]
\[
= d_{p_m} \frac{\tilde{U}_{0,0}^{(0)}(x(p), q)}{(x(p) - x(r)) \tilde{U}_{0,0}^{(0)}(x(r), q)} x(r) - x(p_m)
\]
\[
+ \frac{H_{k+1,l}^{(q)}(p_m, q; p_k|l|m) d_x(r)}{x(r) - x(p_m)}.
\]
(2.30)

Under this form, one can see that the integrant is a rational function of \( x(r) \). Thus, the residue can be computed on the complex plane obtained by the projection \( x \) and we can move the integration contours on the complex plane instead of the curve \( \mathcal{E} \) itself. This term is then equal to
\[
d_{p_m} \frac{\tilde{U}_{0,0}^{(0)}(x(p), q)}{(x(p) - x(r)) \tilde{U}_{0,0}^{(0)}(x(r), q)} x(r) - x(p_m)
\]
\[
= -d_{p_m} \frac{\tilde{U}_{0,0}^{(0)}(x(p), q)}{(x(p) - x(r)) \tilde{U}_{0,0}^{(0)}(x(r), q)} x(r) - x(p_m)
\]
\[
+ \frac{H_{k+1,l}^{(q)}(p_m, q; p_k|l|m) d_x(r)}{x(r) - x(p_m)}.
\]
\begin{aligned}
\times H_{k-1,l}^{(q)}(pm, q; PK_{j|m}|QL) d_x \\
&\times (x - x(pm)) E(x(p), y(q))(x(pm) - x(q)) H_{k-1,l}^{(q)}(pm, q; PK_{j|m}|QL) \\
&= - d_{pm} E(x(p), y(q))(x(p) - x(q)) E(x(p), y(q)) H_{k-1,l}^{(q)}(pm, q; PK_{j|m}|QL) \\
&= - \text{Res}_{r \to pm} \frac{\tilde{U}_{0,0}^{(h)}(x(p), q)}{x(r) - x(p)} \tilde{U}_{0,0}^{(h)}(x(p), q) H_{k-1,l}^{(q)}(r, q; PK_{j|m}|QL) W_{2,0}^{(0)}(p, pm) \\
&= - \text{Res}_{r \to pm} \frac{\tilde{U}_{0,0}^{(h)}(x(p), q)}{x(r) - x(p)} \tilde{U}_{0,0}^{(h)}(x(p), q)
\times \left( \sum_{h, I, J} W_{h+1, j}^{(i)}(r, p I | q J) H_{k-1,l-j}^{(q-h)}(r, q; PK_{j|m}|QL) + H_{k-1,l}^{(q-1)}(r, q; r, PK_{j|m}|QL) \right),
\end{aligned}

(2.31)

where the last equality holds, thanks to the loop equation (2.19). Therefore,

\begin{align}
(y(q) - y(p)) H_{k,l}^{(q)}(p, q; PK_{j|m}|QL) &= \text{Res}_{r \to p, q, px} \frac{\tilde{U}_{0,0}^{(h)}(x(p), q)}{x(r) - x(p)} \tilde{U}_{0,0}^{(h)}(x(r), q) \\
&\times \left( \sum_{h, I, J} W_{h+1, j}^{(i)}(r, p I | q J) H_{k-1,l-j}^{(q-h)}(r, q; PK_{j|m}|QL) + H_{k-1,l}^{(q-1)}(r, q; r, PK_{j|m}|QL) \right). 
\end{align}

(2.32)

If we divide by \( \tilde{U}_{0,0}^{(h)}(x(p), q) \), we obtain

\begin{align}
-h_{k,l}^{(h)}(p, q; PK_{j|m}|QL) &= \text{Res}_{r \to p, q, px} \frac{1}{x(r) - x(p)} (y(r) - y(q)) \\
&\times \left( \sum_{h, I, J} W_{h+1, j}^{(i)}(r, p I | q J) h_{k-1,l-j}^{(q-h)}(r, q; PK_{j|m}|QL) + h_{k-1,l}^{(q-1)}(r, q; r, PK_{j|m}|QL) \right). 
\end{align}

(2.33)

The other half of the theorem is obtained from the fact that for large \( x \),

\begin{equation}
\text{tr} \frac{1}{x - M_1} \frac{1}{y - M_2} \to \frac{1}{x - M_1} \frac{1}{y - M_2}
\end{equation}

and thus,

\begin{equation}
H_{k,l}^{(q)}(p, q; PK_{j|m}|QL) \to \frac{1}{x(p)} \frac{W_{k,l+1}^{(q)}(PK_{j|m}, q)}{dy(q)}
\end{equation}

when \( p \to \infty \).

\( \square \)

2.3. Examples, first few terms

Let us solve the recursive definition and give explicit formulae for the simplest functions.

\( \infty_3 \) is the only point on the curve where the meromorphic function \( x \) has a simple pole (see [15] for further details).
Example $W^{(0)}_{1,1}$.

In particular, definitions equations (2.22) and (2.23) give

$$W^{(0)}_{1,1}(p_1|q) = \text{Res}_{r \to q, p_1} \frac{dy(q)B(r, p_1)}{(y(r) - y(q))}$$

$$= - \text{Res}_{r \to q} \frac{dy(q)B(r, p_1)}{(y(r) - y(q))}$$

$$= -B(q, p_1).$$  \hspace{1cm} (2.36)

Therefore, we recover

$$W^{(0)}_{2,0}(p_1, q) + W^{(0)}_{1,1}(p_1|q) = 0. \hspace{1cm} (2.37)$$

Example $H^{(0)}_{1,0}$.

$$h^{(0)}_{1,0}(p, q; p_1) = \text{Res}_{r \to q', p, p_1} \frac{B(r, p_1)}{(x(p) - x(r))(y(r) - y(q))}$$

$$= - \text{Res}_{r \to p', q} \frac{B(r, p_1)}{(x(p) - x(r))(y(r) - y(q))}. \hspace{1cm} (2.38)$$

Example $H^{(0)}_{0,1}$.

$$h^{(0)}_{0,1}(p, q; p_1) = \text{Res}_{r \to q', p} \frac{W^{(0)}_{1,1}(r|p_1)}{(x(p) - x(r))(y(r) - y(q))}$$

$$= - \text{Res}_{r \to q'} \frac{B(r, p_1)}{(x(p) - x(r))(y(r) - y(q))}$$

$$= \text{Res}_{r \to p', q, p_1} \frac{B(r, p_1)}{(x(p) - x(r))(y(r) - y(q))}. \hspace{1cm} (2.39)$$

Moreover, we have

$$h^{(0)}_{1,0}(p, q; p_1) + h^{(0)}_{0,1}(p, q; p_1) = \text{Res}_{r \to p_1} \frac{B(r, p_1)}{(x(p) - x(r))(y(r) - y(q))}$$

$$= d_{p_1} \left( \frac{1}{(x(p) - x(p_1))(y(p_1) - y(q))} \right). \hspace{1cm} (2.40)$$

Example $W^{(0)}_{2,1}$.

$$W^{(0)}_{2,1}(p_1, p_2|q) = \text{Res}_{r \to q', p_1, p_2} \frac{B(r, p_1)h^{(0)}_{1,0}(r, q; p_2) + B(r, p_2)h^{(0)}_{1,0}(r, q; p_1) + W^{(0)}_{3,0}(r, p_1, p_2)}{(y(r) - y(q))}$$

$$= - \text{Res}_{r \to q, a} \frac{B(r, p_1)h^{(0)}_{1,0}(r, q; p_2) + B(r, p_2)h^{(0)}_{1,0}(r, q; p_1) + W^{(0)}_{3,0}(r, p_1, p_2)}{(y(r) - y(q))}. \hspace{1cm} (2.41)$$

2.4. Conclusion of section 2

Therefore, through theorem 2.1, we have an effective explicit method to compute any $H^{(g)}_{k,l}$ and any $W^{(g)}_{k,l}$ for the 2-matrix model.

This is an interesting result in itself, since none of these quantities were computed before, and these quantities are of importance in applications of random matrices to combinatorics of maps with colored boundaries, i.e. boundary conformal field theory.
An important remark is that we have chosen to emphasize the role of the loop equation (2.19), rather than equation (2.18), i.e. we have used the Lagrange interpolation formula for a polynomial in \(x\), whereas we could have done the same thing with a polynomial in \(y\). In other words, we have chosen the \(x\)-representation rather than the \(y\)-representation, although both methods must give the same answer. In particular, given \(W_{k,0}\), theorem 2.1 allows us to compute \(W_{0,l}\). \(W_{k,0}\) can be computed with the method of \([8, 21]\) using the \(x\)-representation, while \(W_{0,l}\) can be computed with the method of \([8, 21]\) using the \(y\)-representation, i.e. under the exchange

\[
x \leftrightarrow y.
\]

Therefore, in the following section, we improve the result of theorem 2.1, in order to prove that the diagrammatic rules of \([8, 21]\) are indeed symmetric under the exchange of \(x\) and \(y\). In other words we prove theorem 7.1 of \([21]\), as announced in that article.

3. Proof of the symmetry \(x \leftrightarrow y\) of the algebraic invariants \(F^{(g)}(\mathcal{E})\)

Consider the two algebraic curves:

\[
\check{\mathcal{E}}(x, y) = \mathcal{E}(x, y) \quad \text{and} \quad \tilde{\mathcal{E}}(x, y) = \mathcal{E}(y, x).
\]

(3.1)

In \([21]\), for any curve \(\mathcal{E}\) an infinite sequence of invariants \(F^{(g)}\) was defined. Here we consider these invariants for the two curves \(\check{\mathcal{E}}\) and \(\tilde{\mathcal{E}}\) respectively.

In this section, we prove the following theorem (which was announced in \([21]\)).

**Theorem 3.1.** Symmetry under the exchange \(x \leftrightarrow y\):

\[
F^{(g)}(\check{\mathcal{E}}) = F^{(g)}(\tilde{\mathcal{E}})
\]

(3.2)

where the functional \(F^{(g)}(\mathcal{E})\) is defined for any curve \(\mathcal{E}(x, y)\) in \([21]\).

3.1. Preliminaries

For the curve \(\check{\mathcal{E}}(x, y) = 0\), we have defined in \([21]\) an infinite sequence of meromorphic forms:

\[
\check{W}^{(g)}(p, \ldots, p_{k}) = W^{(g)}(p, \ldots, p_{k})|_{\check{\mathcal{E}}}
\]

(3.3)

with poles only at the zeros \(a = \{a_{i}\}\) of \(dx\) and some free energies

\[
\check{F}^{(g)} = \frac{1}{2 - 2g} \text{Res}_{p \to a} \Phi(p) \check{W}^{(g)}(p),
\]

(3.4)

where \(\Phi\) is any anti-derivative of \(y\ dx\), \(d\Phi = y\ dx\) and \(\text{Res}_{p \to a}\) stands for \(\sum_i \text{Res}_{p \to a_i}\).

Likewise, for the curve \(\tilde{\mathcal{E}}(x, y) = 0\), we have defined an infinite sequence of meromorphic forms:

\[
\tilde{W}^{(g)}(q, \ldots, q_{k}) = W^{(g)}(q, \ldots, q_{k})|_{\tilde{\mathcal{E}}}
\]

(3.5)

with poles only at the zeros \(b = \{b_{i}\}\) of \(dy\) and some free energies

\[
\tilde{F}^{(g)} = \frac{1}{2 - 2g} \text{Res}_{q \to b} \Psi(q) \tilde{W}^{(g)}(q),
\]

(3.6)

where \(d\Psi = x\ dy\).

Our first step is to extend these forms into two families of multilinear meromorphic forms similar to those of section 2 (i.e. mimicking the mixed traces of matrix models):

\[
\check{W}_{k,l}^{(g)}(p, \ldots, p_{k}|q, \ldots, q_{l}) \quad \text{and} \quad \tilde{W}_{k,l}^{(g)}(p, \ldots, p_{k}|q, \ldots, q_{l})
\]

(3.7)
such that
\[ \tilde{W}_{k,0}^{(g)} = \tilde{W}_{k}^{(g)}, \quad \tilde{W}_{0,l}^{(g)} = \tilde{W}_{l}^{(g)}. \] (3.8)

Our second step is to prove that
\[ \tilde{W}_{k,l}^{(g)} = \tilde{W}_{k,l}^{(g)}. \] (3.9)

Our third step is to prove that
\[ \tilde{W}_{k+1,l}(p, q_L) + \tilde{W}_{k,l+1}(p, q_L) = d_p \left( \frac{A_{k,l}(p; pK|q_L)}{dx(p) dy(p)} \right), \] (3.10)

where \( A_{k,l}(p; pK|q_L) \) has poles of degree at most 2 at the poles of \( y \) \( dx \), so that in particular for \( k = l = 0 \) we have
\[ \tilde{W}_{1,0}^{(g)}(p) + \tilde{W}_{0,1}^{(g)}(p) = d_p \left( \frac{A_{0,0}^{(g)}(p)}{dx(p) dy(p)} \right), \] (3.11)

where \( A_{0,0}^{(g)} \) has poles of degree at most 2 at the poles of \( y \) \( dx \).

This last step is sufficient to prove that
\[ F^{(g)} = F^{(g)}. \] (3.12)

3.2. Definitions of mixed correlators \( \tilde{W}_{k,l}^{(g)} \) and \( \tilde{W}_{k,l}^{(g)} \)

We define the initial terms
\[ \tilde{E}_{0,0}^{(g)}(x, y) = \tilde{E}_{0,0}^{(g)}(x, y) = \mathcal{E}(x, y), \] (3.13)
\[ \tilde{F}_{0,0}^{(g)}(p, q) = \tilde{F}_{0,0}^{(g)}(p, q) = \frac{\mathcal{E}(x(p), y(q))}{(x(p) - x(q))(y(p) - y(q))}, \] (3.14)
\[ \tilde{W}_{0,0}^{(g)}(p) = \tilde{W}_{0,0}^{(g)}(p) = \tilde{W}_{0,0}^{(g)}(p) = 0, \] (3.15)
\[ \tilde{W}_{1,0}^{(g)}(p, q) = \tilde{W}_{0,1}^{(g)}(p, q) = -\tilde{W}_{1,1}^{(g)}(p, q) = B(p, q) \] (3.16)

and
\[ \tilde{W}_{2,0}^{(g)}(p, q) = \tilde{W}_{0,2}^{(g)}(p, q) = -\tilde{W}_{1,1}^{(g)}(p, q) = B(p, q). \] (3.17)

Let us recursively define the following quantities for any \( g, k, l \geq 0 \):
\[ J_{k,l}^{(g)}(p, q; pK|q_L) := \sum_{m_1, m_2 = 0}^{k} \sum_{n_1, n_2 = 0}^{l} \sum_{h, h' = 1}^{g} \tilde{W}_{m_1, n_1}^{(h)}(p, pM_1|q_{L_1}) \]
\[ \times \tilde{W}_{m_2, n_2}^{(h)}(p, q_{L_2}, q_L) H_{k-m_1-l_1-n_1-h_1}^{(g-h-h')} \]
\[ + \sum_{h=1}^{g-1} \left[ (x(p) - x(q)) \tilde{W}_{m_1+1, n_1}^{(h)}(p, q_L) dy(q) + (y(q) - y(p)) \tilde{W}_{m_2+1, n_2}^{(h)}(p, q_L) dx(p) \right] \]
\[ \times H_{k-m-l-n}^{(g-h-h')}(p, q; pK|q_{L_1}/N) \]
\[ \times \left[ (x(p) - x(q)) \tilde{W}_{m_1+1, n_1}^{(h)}(p, q_{L_2}, q_L) dy(q) \right. \]
\[ + (y(q) - y(p)) \tilde{W}_{m_2+1, n_2}^{(h)}(p, q_{L_2}, q_L) dx(p) \]
\[ (x(p) - x(q))H_{k+1,j}^{(g-1)}(p, q; p, PK|QL) \, dy(q) + (y(q) - y(p)) \]
\[ \times H_{k,j+1}^{(g-1)}(p, q; PK|QL, q) \, dx(p) \]
\[ + \sum_{m=0}^{k} \sum_{n=0}^{l} \sum_{h=0}^{g-1} \hat{W}_{m+1,n+1}^{(h)}(p, PM|QN)H_{k-m,l-n+1}^{(g-h-1)}(p, q; PK/M|QN, q) \]
\[ + \frac{1}{2}(\hat{W}_{m+1,n+1}^{(h)}(p, PM|QN, q) + \hat{W}_{m+1,n+1}^{(h)}(p, PM|QN, q)) \]
\[ \times H_{k-m,l-n}^{(g-h-1)}(p, q; PK/M|QN/N) \]
\[ + \hat{W}_{m+1,n+1}^{(h)}(PM|QN, q)H_{k-m+l-j-a}^{(g-h-1)}(p, q; PK/M|QN/N) \]
\[ + H_{k+1,j+1}^{(g-2)}(p, q; PK|QL, q) \]

and
\[ J_{k,l}^{(g)}(p, q; PK|QL) := J_{k,l}^{(g)}(p, q; PK|QL) \]
\[ = \sum_{a=1}^{k} d_{pa} \{ \frac{dx(p)}{x(p) - x(pa)} \left[ (x(pa) - x(q)) \, dy(q) \right] H_{k-1,j}^{(g)}(pa, q; PK-[a]|QL) \]
\[ + \sum_{h=1}^{g} \hat{W}_{0,1}^{(h)}(q)H_{k-1,j}^{(g-b)}(pa, q; PK-[a]|QL) \]
\[ + \sum_{h=1}^{k-l} \sum_{h=1}^{b} \left\{ \hat{W}_{h-1,j-1}^{(h)}(pa, q; PK-[a]|QL) \right\} \]
\[ = \sum_{a=1}^{k} d_{pa} \{ \frac{dy(q)}{y(q) - y(qa)} \left[ (y(qa) - y(p)) \, dx(p) \right] H_{k,l-1}^{(g-h)}(p, qa; PK|QL-[b]) \]
\[ + \sum_{h=1}^{g} \hat{W}_{0,1}^{(h)}(p)H_{k,l-1}^{(g-h)}(p, qa; PK|QL-[b]) \]
\[ + \sum_{h=1}^{k-l} \sum_{h=1}^{b} \left\{ \hat{W}_{h-1,j-1}^{(h)}(p, qa; PK[1]|QL-[b]) \right\} \]
\[ = \sum_{a=1}^{k} d_{pa} \{ \frac{dx(p)}{x(p) - x(pa)} \left[ \frac{dy(q)}{y(q) - y(qa)} \right] H_{k-1,l-1}^{(g-h-1)}(pa, qa; PK-[a]|QL-[b]) \}
\[ + \sum_{a=1}^{k} \sum_{b=1}^{l} d_{pa} d_{qa} \left\{ \frac{dx(p)}{x(p) - x(pa)} \left[ \frac{dy(q)}{y(q) - y(qa)} \right] H_{k-1,l-1}^{(g-h-1)}(pa, qa; PK-[a]|QL-[b]) \right\} \]
\[ = \sum_{a=1}^{k} d_{pa} \left( H_{k-1,l-1}^{(g-1)}(pa, q; PK-[a]|QL, q) \, dx(p) \right) \]
\[ \times (x(p) - x(pa)) \]
\[ - \sum_{a=1}^{k} d_{pa} \left( H_{k-1,l-1}^{(g-1)}(pa, qa; PK-[a]|QL, qa) \, dx(p) \right) \]
\[ \times (x(p) - x(pa)) \]
\[ = - \sum_{a=1}^{k} d_{pa} \left( H_{k-1,l-1}^{(g-1)}(pa, qa; PK-[a]|QL, qa) \, dx(p) \right) \]
\[ \times (x(p) - x(pa)) \]
\[ \times (x(p) - x(pa)) \]

Remark 3.1. These expressions are not as complicated as they look. They are inspired from section 2. In the matrix model case of section 2, these expressions contain nearly all the terms we would obtain from inserting loop equation (2.19) into loop equation (2.20) or, equivalently,
from inserting loop equation (2.18) into loop equation (2.21). However, here we are not in a matrix model, and we do not assume any of equations (2.19) to (2.21); in fact we are going to prove them.

Now we define

$$\hat{W}_{k,l}^{(q)}(p,k|q,l) := \text{Res}_{s \to a.q} \text{d}S_{s,o}(p) \times \left[ 1 \sum_{j=1}^{d_1} \frac{\mathcal{J}_{k,j}^{(q)}(s, \tilde{s}'; p|k|q,l)}{U_{0,0}(x(s), s)} + 1 \sum_{i=1}^{d_2} \frac{\mathcal{J}_{k,l}^{(q)}(s', s; p|k|q,l)}{U_{0,0}(y(s), s)} \right],$$

(3.20)

$$\hat{W}_{k,l}^{(q)}(p,k|q,l) := \text{Res}_{r \to b.p} \text{d}S_{s,o}(q) \times \left[ 1 \sum_{j=1}^{d_1} \frac{\mathcal{J}_{k,j}^{(q)}(s, \tilde{s}'; p|k|q,l)}{U_{0,0}(x(s), s)} + 1 \sum_{i=1}^{d_2} \frac{\mathcal{J}_{k,l}^{(q)}(s', s; p|k|q,l)}{U_{0,0}(y(s), s)} \right],$$

(3.21)

$$G_{k,j}^{(q)}(p,q; p_k|q_l) := \mathcal{J}_{k,j}^{(q)}(p,q; p_k|q_l) + H_{0,0}^{(q)}(p,q) \times \left[ (x(p) - x(q)) \hat{W}_{k,l}^{(q)}(p,p_k|q_l) \text{d}y(q) + (y(q) - y(p)) \hat{W}_{k,l}^{(q)}(p,p_k|q_l) \text{d}x(p) \right],$$

(3.22)

$$G_{k,j}^{(q)}(p,q; p_k|q_l) := \mathcal{J}_{k,j}^{(q)}(p,q; p_k|q_l) + H_{0,0}^{(q)}(p,q) \times \left[ (x(p) - x(q)) \hat{W}_{k,l}^{(q)}(p,p_k|q_l) \text{d}y(q) + (y(q) - y(p)) \hat{W}_{k,l}^{(q)}(p,p_k|q_l) \text{d}x(p) \right],$$

(3.23)

$$\hat{H}_{k,l}^{(q)}(p,q; p_k|q_l) := \text{Res}_{r \to a.p} \frac{\mathcal{J}_{k,j}^{(q)}(p,r; p_k|q_l)}{\mathcal{E}(x(p), y(q))} \times \left[ (y(q) - y(p))(y(q) - y(r))(x(p) - x(r)) H_{0,0}^{(q)}(r,p) \text{d}x(p) \right].$$

(3.24)

$$\hat{H}_{k,l}^{(q)}(p,q; p_k|q_l) := \text{Res}_{r \to a.p} \frac{\mathcal{J}_{k,j}^{(q)}(r,q; p_k|q_l)}{\mathcal{E}(x(p), y(q))} \times \left[ (x(p) - x(q))(x(p) - x(r))(y(q) - y(r)) H_{0,0}^{(q)}(r,q) \text{d}y(q) \right].$$

(3.25)

and

$$H_{k,l}^{(q)} = \frac{\hat{H}_{k,l}^{(q)} + \hat{H}_{k,l}^{(q)}}{2}$$

(3.26)

(we prove below that $\hat{H}_{k,l}^{(q)} = \hat{H}_{k,l}^{(q)} = H_{k,l}^{(q)}$) as well as

$$\hat{E}_{k,j}^{(q)}(p,q; p_k|q_l) := \text{Res}_{r \to a.p} \frac{\mathcal{J}_{k,j}^{(q)}(p,r; p_k|q_l)}{\mathcal{E}(x(p), y(q))} \times \left[ (y(q) - y(p))(y(q) - y(r))(x(p) - x(r)) H_{0,0}^{(q)}(r,p) \text{d}x(p) \right],$$

(3.27)

$$\hat{E}_{k,j}^{(q)}(p,q; p_k|q_l) := \text{Res}_{r \to a.p} \frac{\mathcal{J}_{k,j}^{(q)}(r,q; p_k|q_l)}{\mathcal{E}(x(p), y(q))} \times \left[ (x(p) - x(q))(x(p) - x(r))(y(q) - y(r)) H_{0,0}^{(q)}(r,q) \text{d}y(q) \right].$$

(3.28)
The quantity $U_{g,l}^{(g)}(p, q; \mathbf{p}_k | \mathbf{q}_l) := (y(q) - y(p)) H_{g,l}^{(g)}(p, q; \mathbf{p}_k | \mathbf{q}_l)$

\[
\begin{align*}
&+ \sum_{h} \sum_{i,l} \frac{\tilde{W}_{e,i,j}^{(h)}(p, \mathbf{p}_i | \mathbf{q}_j) \tilde{H}_{e-l,j}^{(h-1)}(p, q; \mathbf{p}_k | \mathbf{q}_l, \mathbf{q}_n)}{dx(p)} \\
&+ \frac{H_{k,l+1}^{(g-1)}(p, q; \mathbf{p}_k | \mathbf{q}_l, \mathbf{q}_n)}{dy(q)} = \sum_{m} d_{pm} \frac{H_{k,l}^{(g)}(p_m, q; \mathbf{p}_k | \mathbf{q}_l, \mathbf{q}_n)}{x(p) - x(p_m)} \tag{3.29}
\end{align*}
\]

and

\[
\begin{align*}
&-U_{g,l}^{(g)}(p, q; \mathbf{p}_k | \mathbf{q}_l) := (x(p) - x(q)) H_{g,l}^{(g-1)}(p, q; \mathbf{p}_k | \mathbf{q}_l) \\
&+ \sum_{h} \sum_{i,l} \frac{\tilde{W}_{e,i,j}^{(h)}(p, \mathbf{q}_i | q) \tilde{H}_{e-l,j}^{(h-1)}(p, q; \mathbf{p}_k | \mathbf{q}_l, \mathbf{q}_n)}{dy(q)} \\
&+ \frac{H_{k,l+1}^{(g-1)}(p, q; \mathbf{p}_k | \mathbf{q}_l, \mathbf{q}_n)}{dy(q^2)} = \sum_{m} d_{q_m} \frac{H_{k,l}^{(g)}(p, q_m; \mathbf{p}_k | \mathbf{q}_l, \mathbf{q}_n)}{y(q) - y(q_m)} \tag{3.30}
\end{align*}
\]

These definitions form a triangular system of definitions, and each term is well defined in a unique recursive way.

**Remark 3.2.** Definitions equations (3.29) and (3.30) coincide with loop equations (2.19) and (2.18) in the matrix model case, i.e. when $E$ is the classical spectral curve of the 2-matrix model.

**3.3. Theorems**

**Theorem 3.2.** For $2g + k + l \geq 3$, one has the following properties:

- $\tilde{W}_{g,l}^{(g)}(\mathbf{p}_k | \mathbf{q}_l)$ (respectively $\tilde{H}_{g,l}^{(g)}(\mathbf{p}_k | \mathbf{q}_l)$) has poles only when $p_i \to a, q_j$ and \( q_j \to b, \mathbf{p}_k; \)
- in any of the $k + l$ variables, the $A$-cycle integrals vanish: $\tilde{f}_A^{(g)} \tilde{W}_{g,l}^{(g)} = \tilde{f}_A^{(g)} \tilde{H}_{g,l}^{(g)} = 0$;
- $\tilde{H}_{g,l}^{(g)}(p, q; \mathbf{p}_k | \mathbf{q}_l) = \tilde{H}_{g,l}^{(g-1)}(p, q; \mathbf{p}_k | \mathbf{q}_l, \mathbf{q}_l)$ has poles only when $p \to q, a, \mathbf{q}_l$ and \( q \to q, a, \mathbf{q}_l$ and \( q \to a, \mathbf{p}_k,$ and
- $\tilde{E}_{g,l}^{(g)}(x(p), q; \mathbf{p}_k | \mathbf{q}_l) = \tilde{E}_{g,l}^{(g)}(x(p), y(q); \mathbf{p}_k | \mathbf{q}_l) := E^{(g)}(x(p), y(q); \mathbf{p}_k | \mathbf{q}_l)$ is a polynomial of degree $d_1 - 1$ in $x(p)$ and $d_2 - 1$ in $y(q)$;
- $U_{g,l}^{(g)}(p, y(q); \mathbf{p}_k | \mathbf{q}_l)$ (respectively $\tilde{U}_{g,l}^{(g)}(x(p), q; \mathbf{p}_k | \mathbf{q}_l)$) is a polynomial in $y(q)$ (respectively $x(p)$) of degree $d_2 - 1$ (respectively $d_1 - 1$).

**Proof.** Let us proceed by induction on $2g + k + l$. Suppose that the properties are satisfied for any $g', k', l'$ such that $2g' + k' + l' < 2g + k + l$. Let us prove that they are true for $g, k, l$. In order to make the proof more readable, we split it into pieces. Nevertheless, for every step, the global recursion hypothesis is needed. □

We need the following lemma.

**Lemma 3.1.** The quantity

$$ f_{g,l}^{(g)}(s; \mathbf{p}_k | \mathbf{q}_l) := \frac{\mathcal{J}_{g,l}^{(g)}(s, \mathbf{y}_l; \mathbf{p}_k | \mathbf{q}_l)}{U_{0,0}^{(0)}(s, y(s)) dy(s)} \tag{3.32} $$

is independent of $j \neq 0$; it is a meromorphic 1-form in the variable $s$, with poles at $s = a, \mathbf{q}_l$, and it vanishes to order at least $\deg(y dx) - 1$ near the poles of $y dx$. 

14
Similarly, the quantity
\[ \tilde{f}_{k,l}^{(g)}(s; p_k; q_l) := \frac{J_{\tilde{f}_{k,l}}^{(g)}(s^i; s; p_k; q_l)}{U_{0,0}^{(g)}(x(s), s) \, dx(s)} \]  
(3.33)
is independent of \( i \neq 0 \); it is a meromorphic 1-form in the variable \( s \), with poles at \( s = b, q_l \), and it vanishes to order at least \( \deg(x \, dy) - 1 \) near the poles of \( x \, dy \). Moreover, one has
\[ \oint_A \left( f_{k,l}^{(g)}(s; p_k; q_l) + \tilde{f}_{k,l}^{(g)}(s; p_k; q_l) \right) = 0. \]  
(3.34)
\[ \oint_B \left( f_{k,l}^{(g)}(s; p_k; q_l) + \tilde{f}_{k,l}^{(g)}(s; p_k; q_l) \right) = 0 \]  
(3.35)
and
\[ f_{k,l}^{(g)}(s; p_k; q_l) + \tilde{f}_{k,l}^{(g)}(s; p_k; q_l) = \text{Res}_{q \to a, b, p_k, q_l} dS_{q,o}(s) \left( f_{k,l}^{(g)}(q; p_k; q_l) + f_{k,l}^{(g)}(q; p_k; q_l) \right). \]  
(3.36)

**Proof of the lemma.** First of all, one can remark that the definition of \( J_{\tilde{f}_{k,l}}^{(g)} \) involves only quantities whose properties are known by the recursion hypothesis. One can note that it can be written under the following forms:
\[ J_{\tilde{f}_{k,l}}^{(g)}(p, q; p_k; q_l) := - \sum_{h=1}^{g-1} \sum_{m=0}^{k} \sum_{n=0}^{l} \tilde{W}_{m,n+1}^{(h)}(p_m; q, q_N) \tilde{U}_{k-m,l-n}^{x-h}(x(p), q; p_k/M; q_l/N) \, dx(p) \]
\[ - \sum_{m=0}^{k} \sum_{n=0}^{l} \tilde{W}_{m,n+1}^{(0)}(p_m; q, q_N) \tilde{U}_{k-m,l-n}^{x-h}(x(p), q; p_k/M; q_l/N) \, dx(p) \]
\[ - \sum_{m=0}^{k} \sum_{n=0}^{l} \tilde{W}_{m,n+1}^{(g)}(p_m; q, q_N) \tilde{U}_{k-m,l-n}^{x-h}(x(p), q; p_k/M; q_l/N) \, dx(p) \]
\[ - \tilde{U}_{k,l+1}^{(g-1)}(x(p), q; p_k; q_l) \, dx(p) + \sum_{h=1}^{g-1} \sum_{m=0}^{k} \sum_{n=0}^{l} d_{p_a} \]
\[ \times \left( \tilde{W}_{m,n+1}^{(h)}(p_m; q, q_N) H_{k-m-1,l-n}^{x-h}(p_a; q; p_k/M; q_l/N) \, dx(p) \right) x(p) - x(p_a) \]
\[ + \sum_{m=0}^{k} \sum_{n=0}^{l} \tilde{W}_{m,n+1}^{(0)}(p_m; q, q_N) H_{k-m-1,l-n}^{x}(p_a; q; p_k/M; q_l/N) \, dx(p) \]
\[ + \sum_{m=0}^{k} \sum_{n=0}^{l} \tilde{W}_{m,n+1}^{(g)}(p_m; q, q_N) H_{k-m-1,l-n}^{x}(p_a; q; p_k/M; q_l/N) \, dx(p) \]
\[ + d_{p_a} \left( \tilde{W}_{m,n+1}^{(g)}(p_m; q, q_N) H_{k-m-1,l-n}^{x}(p_a; q; p_k/M; q_l/N) \, dx(p) \right) x(p) - x(p_a) \]
\[ + d_{p_a} \left( H_{k-1,l+1}^{x}(p_a; q; p_k/M; q; q_l) \, dx(p) \right) + (x(p) - x(q)). \]
\[
\times \left[ \sum_{h=1}^{g-1} \sum_{m=0}^{k} \sum_{n=0}^{l} \hat{W}_{m+n+1}^{(h)}(p, \mathbf{p}_m; \mathbf{q}_n) \, dy(q) \, H_{k-m-j-n}^{(g-h)}(p, q; \mathbf{p}_k; \mathbf{q}_l) \\
+ \sum_{m=0}^{k} \sum_{n=0}^{l} \hat{W}_{m+n+1}^{(0)}(p, \mathbf{p}_m; \mathbf{q}_n) \, dy(q) \, H_{k-m-j-n}^{(0)}(p, q; \mathbf{p}_k; \mathbf{q}_l) \\
+ \sum_{m=0}^{k} \sum_{n=0}^{l} \hat{W}_{m+n+1}^{(0)}(p, \mathbf{p}_m; \mathbf{q}_n) \, dy(q) \, H_{k-m-j-n}^{(0)}(p, q; \mathbf{p}_k; \mathbf{q}_l) \\
+ \hat{J}_{k+1,j}^{(g-1)}(p, q; \mathbf{p}_k; \mathbf{q}_l) \, dy(q) \right]
\] (3.37)

And
\[
\mathcal{J}_{k,l}^{(g)}(p, q; \mathbf{p}_k; \mathbf{q}_l) = - \sum_{h=1}^{g-1} \sum_{m=0}^{k} \sum_{n=0}^{l} \hat{W}_{m+n+1}^{(h)}(p, \mathbf{p}_m; \mathbf{q}_n) \tilde{U}_{k-m-j-n}^{(h)}(x(p), q; \mathbf{p}_k; \mathbf{q}_l) \, dx(p) \\
- \sum_{m=0}^{k} \sum_{n=0}^{l} \tilde{U}_{m+n+1}^{(0)}(p, \mathbf{p}_m; \mathbf{q}_n) \tilde{U}_{k-m-j-n}^{(0)}(x(p), q; \mathbf{p}_k; \mathbf{q}_l) \, dx(p) \\
- \tilde{U}_{k+1,j}^{(g-1)}(x(p), q; \mathbf{p}_k; \mathbf{q}_l) \, dx(p) \\
+ d_{\beta} \left( \frac{\hat{U}_{k+1,j}^{(g-1)}(p, q; \mathbf{p}_k; \mathbf{q}_l)}{y(q) - y(q_\beta)} \, dx(p) \, dy(q) \right) \\
- d_{\alpha} \left( \frac{x(p_\alpha) - x(q)}{x(p) - x(p_\alpha)} \frac{H_{k-1,j}^{(g)}(p_\alpha, q; \mathbf{p}_k; \mathbf{q}_l)}{dx(p) \, dy(q)} \right) + (x(p) - x(q)) \\
\times \left[ \sum_{h=1}^{g-1} \sum_{m=0}^{k} \sum_{n=0}^{l} \hat{W}_{m+n+1}^{(h)}(p, \mathbf{p}_m; \mathbf{q}_n) \, dy(q) \, H_{k-m-j-n}^{(g-h)}(p, q; \mathbf{p}_k; \mathbf{q}_l) \\
+ \sum_{m=0}^{k} \sum_{n=0}^{l} \hat{W}_{m+n+1}^{(0)}(p, \mathbf{p}_m; \mathbf{q}_n) \, dy(q) \, H_{k-m-j-n}^{(0)}(p, q; \mathbf{p}_k; \mathbf{q}_l) \\
+ \sum_{m=0}^{k} \sum_{n=0}^{l} \hat{W}_{m+n+1}^{(g)}(p, \mathbf{p}_m; \mathbf{q}_n) \, dy(q) \, H_{k-m-j-n}^{(0)}(p, q; \mathbf{p}_k; \mathbf{q}_l) \\
+ \hat{J}_{k+1,j}^{(g-1)}(p, q; \mathbf{p}_k; \mathbf{q}_l) \, dy(q) \right].
\] (3.38)

Thanks to the properties implied by the recursion hypothesis \((U \text{ and } \tilde{U} \text{ are polynomials})\), one has
\[
\mathcal{J}_{k,l}^{(g)}(q', q; \mathbf{p}_k; \mathbf{q}_l) = - \sum_{h=1}^{g-1} \sum_{m=0}^{k} \sum_{n=0}^{l} \hat{W}_{m+n+1}^{(h)}(\mathbf{p}_m; q \cdot \mathbf{q}_n) \tilde{U}_{k-m-j-n}^{(h)}(x(q), q; \mathbf{p}_k; \mathbf{q}_l) \, dx(q) \\
- \sum_{m=0}^{k} \sum_{n=0}^{l} \hat{W}_{m+n+1}^{(0)}(\mathbf{p}_m; q \cdot \mathbf{q}_n) \tilde{U}_{k-m-j-n}^{(0)}(x(q), q; \mathbf{p}_k; \mathbf{q}_l) \, dx(q)
\]
\[ - \sum_{m=0}^{k} \sum_{n=0}^{l} \tilde{W}_{m+1}^{(g)}(p_{M}; q, q_{N}) \tilde{U}_{k-m-l-n}^{(g)}(x(q), q; p_{K}/M, q_{L}/N) \, dx(q) \]
\[ - \tilde{U}_{k+1}^{(g-1)}(x(q), q; p_{K}; q, q_{L}) \, dx(q) \]
\[ + d_{p_{a}} \left( \frac{\tilde{U}_{k+1}^{(g-1)}(x(q), q_{\beta}; p_{K}/[a], q_{L})}{y(q) - y(q_{\beta})} \right) \, dx(q) \, dy(q) \]
\[ + d_{q_{\beta}} \left( \frac{U_{k+1}^{(g)}(p_{a}, x(p); p_{K}/[a], q_{L})}{x(p) - x(p_{a})} \right) \, dx(p) \, dy(p) \]
\[ + d_{p_{a}} \left( H_{k+1}^{(g)}(p_{a}, q_{\beta}; p_{K}/[a], q_{L}/[\beta]) \right) \, dx(p) \, dy(p). \] (3.39)

for any non-vanishing \( i \). Thus this quantity does not depend on \( i \), and \( f \) is clearly a meromorphic 1-form, whose poles can be easily seen on this expression using the recursion hypothesis. The same considerations give the equivalent through the exchange of \( x \leftrightarrow y \):

\[ \mathcal{F}_{k,l}^{(g)}(p, \tilde{p}^{j}; p_{K}; q_{L}) = - \sum_{h=1}^{g-1} \sum_{m=0}^{k} \sum_{n=0}^{l} \tilde{W}_{m+1, n}^{(h)}(p, p_{M}; q_{N}) U_{k-m-l-n}^{g-h}(p, y(p); p_{K}/M, q_{L}/N) \, dy(p) \]
\[ - \sum_{m=0}^{k} \sum_{n=0}^{l} \tilde{W}_{m+1, n}^{(g)}(p, p_{M}; q_{N}) U_{k-m-l-n}^{(g-h)}(p, y(p); p_{K}/M, q_{L}/N) \, dy(p) \]
\[ - U_{k+1, l}^{(g-1)}(p, y(p); p, p_{K}; q_{L}) \, dy(p) \]
\[ + d_{q_{\beta}} \left( \frac{U_{k+1}^{(g)}(p_{a}, y(p); p_{K}/[a], q_{L})}{x(p) - x(p_{a})} \right) \, dx(p) \, dy(p) \]
\[ + d_{p_{a}} \left( H_{k+1}^{(g)}(p_{a}, q_{\beta}; p_{K}/[a], q_{L}/[\beta]) \right) \, dx(p) \, dy(p). \] (3.40)

This quantity does not depend on \( j \), and \( f \) is clearly a meromorphic 1-form, whose poles can be easily seen on this expression using the recursion hypothesis.

The fact that the \( A \) and \( B \) cycle integrals vanish comes from the symmetry \( x \leftrightarrow y \). Indeed under the symmetry \( x \leftrightarrow y \), \( \hat{f} \) is changed to \( f \) and \( \tilde{f} \) is changed to \( f \). At the same time the \( A \)-cycles are changed to \( -A \) because \( 2i \pi \epsilon = \oint_{A} y \, dx = -\oint_{A} x \, dy \), and the \( B \)-cycles are changed to \( -B \) in order to form a canonical basis. Therefore, the \( A \) and \( B \) cycle integrals of \( f + \tilde{f} \) vanish.

Equation (3.36) simply comes from the Cauchy residue formula and Riemann’s bilinear identity.

The fact that \( f \) vanishes to order at least \( \deg(y \, dx) - 1 \) near a pole \( \alpha \) of \( y \, dx \) follows from the definition of \( \mathcal{F} \):

\[ \mathcal{F}_{k,l}^{(g)}(p, \tilde{p}^{j}; p_{K}/q_{L}) \sim_{p \to \alpha} \frac{x(p) - x(\tilde{p}^{j})}{dx(p)} \left( \sum_{m=0}^{k} \sum_{n=0}^{l} \sum_{h=0}^{g} \tilde{W}_{m+1, n}^{(h)}(p, p_{M}; q_{N}) \right) \]
\[ \times H_{k-m-l-n}^{(g-h)}(p, \tilde{p}^{j}; p_{K}/M, q_{L}/N) + H_{k+1, l}^{(g-1)}(p, \tilde{p}^{j}; p, p_{K}/q_{L}) \]
\[ - \sum_{a=1}^{k} d_{p_{a}} \left( \frac{x(p_{a}) - x(\tilde{p}^{j})}{x(p) - x(p_{a})} \right) H_{k-l}^{(g)}(p_{a}, \tilde{p}^{j}; p_{K}/[a], q_{L}/[\beta]) \]
\[ - \sum_{\beta=1}^{l} d_{q_{\beta}} \left( \frac{y(q_{\beta}) - y(p)}{y(p) - y(q_{\beta})} \right) H_{k,l}^{(g)}(p_{a}, q_{\beta}; p_{K}/q_{L}/[\beta]), \] (3.41)
By a symmetric argument, the same holds for $\tilde{W}(\cdot)$

Note that the first term corresponds exactly to

which is at most finite if $p$ approaches a pole of $y \, dx$. Then it implies that $\tilde{f}_k^{(q)}(p; \text{pk}; \text{ql}) = \tilde{g}_k^{(q)}(p,p'; \text{pk}; \text{ql})$ vanishes at order at least $\text{deg}(y \, dx) - 1$. The same holds for $\tilde{f}$.

- $W_{k,l}^{(q)}$ has poles only when $p_i \to a_i, q_j \to b_i, \text{pk},$ and $\tilde{f}_{k,l}^{(q)} \equiv 0$.

From definition (3.20), it is clear that $W_{k+1}^{(q)}(p, p_1, \ldots, p_k|q_1, \ldots, q_l)$ is finite when $p$ is not close to a branch point of $q_j$'s and becomes infinite only if the integration contour is pinched. Thus in the variable $p$, the only poles of $\tilde{W}_{k+1}^{(q)}(p, p_1, \ldots, p_k|q_1, \ldots, q_l)$ are at $p = a_i, q_j$.

The poles of $\tilde{W}_{k+1}^{(q)}(p, p_1, \ldots, p_k|q_1, \ldots, q_l)$ in any other variable follow from the recursion hypothesis, and thus they are at $p_i = a_i, q_j$. The fact that $\tilde{f}_{k,l}^{(q)} = 0$ when one integrates over the first variable comes from the fact that this is a property of $ds$, and in the other variables it comes from the recursion hypothesis. By a symmetric argument, the same holds for $\tilde{W}_{k+1}^{(q)}(p, p_1, \ldots, p_k|q_1, \ldots, q_l, p)$, and we see that $\tilde{W}_{k,l}^{(q)}$ and $\tilde{W}_{k+1,l}^{(q)}$ have the same poles.

We have (from the Cauchy residue formula and Riemann bilinear identity):

$$\tilde{W}_{k+1,l}^{(q)}(p_1, \ldots, p_k|q_1, \ldots, q_l) = \tilde{f}_{k,l}^{(q)}(p_1; \text{pk}; \text{ql}) + \tilde{f}_{k,l}^{(q)}(p_2; \text{pk}; \text{ql}) + \cdots + \tilde{f}_{k,l}^{(q)}(p_k; \text{pk}; \text{ql}).$$

One has

$$\frac{\tilde{H}_{k,l}^{(q)}(p-q; \text{pk}; \text{ql})}{\mathcal{E}(x(p), y(q))} = \text{Res}_{r-q, r'} \frac{\tilde{g}_{k,l}^{(q)}(p-r; \text{pk}; \text{ql})}{\mathcal{E}(x(p), y(q))} \sum_{s-r} \frac{\tilde{g}_{k,l}^{(q)}(s-r; \text{pk}; \text{ql})}{\mathcal{E}(x(s), y(r))} \cdot \mathcal{E}(x(s), y(r))$$

where the last equality holds because the integrant has no pole when $s \to \tilde{q}_i$. Then

$$\frac{\tilde{H}_{k,l}^{(q)}(p-q; \text{pk}; \text{ql})}{\mathcal{E}(x(p), y(q))} = \text{Res}_{r-q, r'} \text{Res}_{s-r, s'} \frac{\tilde{g}_{k,l}^{(q)}(s-r; \text{pk}; \text{ql})}{\mathcal{E}(x(p), y(q))} \frac{\tilde{g}_{k,l}^{(q)}(s-r; \text{pk}; \text{ql})}{\mathcal{E}(x(s), y(r))}.$$
\[
\begin{align*}
\text{Res}_{r \to q', p' \to p} & = \text{Res}_{r \to q', p' \to p} + \text{Res}_{r \to q', q' \to q} + \text{Res}_{r \to q', q' \to q} + \text{Res}_{r \to q', q' \to q} + \text{Res}_{r \to q', q' \to q} \\
& = \text{Res}_{r \to q', p' \to p} + \sum_{j \neq 0} \text{Res}_{r \to q', j' \to j} + \sum_{j \neq 0} \text{Res}_{r \to q', j' \to j} \\
& = \text{Res}_{r \to q', p' \to p} + \sum_{j \neq 0} \text{Res}_{r \to q', j' \to j} + \sum_{j \neq 0} \text{Res}_{r \to q', j' \to j} \\
& + \sum_{j \neq 0} \text{Res}_{r \to q', p' \to p} + \sum_{j \neq 0} \text{Res}_{r \to q', p' \to p} \\
& = \text{Res}_{r \to q', p' \to p} + \sum_{j \neq 0} \text{Res}_{r \to q', p' \to p} + \sum_{j \neq 0} \text{Res}_{r \to q', p' \to p}.
\end{align*}
\]

(3.45)

The last term does not contribute because the integrant is regular when \( r' \to s \); thus

\[
\frac{\tilde{H}^{(g)}_{k,l}(p, q; \mathbf{p}_k; \mathbf{q}_L)}{\mathcal{E}(x(p), y(q))} = \frac{\hat{H}^{(g)}_{k,l}(p, q; \mathbf{p}_k; \mathbf{q}_L)}{\mathcal{E}(x(p), y(q))} + \sum_{j=1}^{d_1} \frac{g^{(g)}_{k,l}(p, q; \mathbf{p}_k; \mathbf{q}_L)}{\mathcal{E}(x(p), y(q))} + \sum_{j=1}^{d_1} \frac{\tilde{f}^{(g)}_{k,l}(p, q; \mathbf{p}_k; \mathbf{q}_L)}{\mathcal{E}(x(p), y(q))} \\
+ \sum_{j=1}^{d_1} \frac{f^{(g)}_{k,l}(p, q; \mathbf{p}_k; \mathbf{q}_L) - \tilde{W}^{(g)}_{k,l}(\mathbf{p}_k; \mathbf{q}_L, p')}{\mathcal{E}(x(p), y(q))} \\
+ \sum_{j=1}^{d_1} \frac{W^{(g)}_{k+l+1}(\mathbf{p}_k; \mathbf{q}_L, p')}{\mathcal{E}(x(p), y(q))}.
\]

(3.46)

Note from equation (3.42) that

\[
g^{(g)}_{k,l}(s; \mathbf{p}_k; \mathbf{q}_L) := \tilde{f}^{(g)}_{k,l}(s; \mathbf{p}_k; \mathbf{q}_L) - \tilde{W}^{(g)}_{k+l+1}(\mathbf{p}_k; \mathbf{q}_L, s) \\
= -f^{(g)}_{k,l}(s; \mathbf{p}_k; \mathbf{q}_L) + \tilde{W}^{(g)}_{k+l+1}(s, \mathbf{p}_k; \mathbf{q}_L)
\]

(3.47)

is a holomorphic 1-form in \( s \), i.e., it has no poles. We have

\[
\frac{\tilde{H}^{(g)}_{k,l}(p, q; \mathbf{p}_k; \mathbf{q}_L)}{\mathcal{E}(x(p), y(q))} = \frac{\hat{H}^{(g)}_{k,l}(p, q; \mathbf{p}_k; \mathbf{q}_L)}{\mathcal{E}(x(p), y(q))} - \frac{\tilde{f}^{(g)}_{k,l}(p, q; \mathbf{p}_k; \mathbf{q}_L)}{\mathcal{E}(x(p), y(q))} \\
- \frac{f^{(g)}_{k,l}(p, q; \mathbf{p}_k; \mathbf{q}_L) - \tilde{W}^{(g)}_{k+l+1}(\mathbf{p}_k; \mathbf{q}_L, p')}{\mathcal{E}(x(p), y(q))} \\
+ \frac{W^{(g)}_{k+l+1}(\mathbf{p}_k; \mathbf{q}_L, p')}{\mathcal{E}(x(p), y(q))}.
\]
\[ E(g) = \sum_{i=1}^{d_1} \text{Res} \frac{(y(p) - y(s)) s_k^i(s; p_k|q_l)}{(y(q) - y(p))(y(q) - y(s))(x(s) - x(p))} \]
\[ - \sum_{j=1}^{d_1} \text{Res} \frac{(x(q) - x(s)) s_k^i(s; p_k|q_l)}{(x(p) - x(q))(x(p) - x(s))(y(s) - y(q))} \]
\[ = \sum_{i=0}^{d_1} \text{Res} \left( \frac{(x(q) - x(s))}{(x(p) - x(q))(x(p) - x(s))(y(s) - y(q))} \right) s_k^i(s; p_k|q_l) \]
\[ = \sum_{i=0}^{d_1} \text{Res} \left( \frac{1}{(x(s) - x(p))(y(q) - y(s))} - \frac{1}{(x(s) - x(p))(y(q) - y(s))} \right) s_k^i(s; p_k|q_l) \]
\[ = 0. \tag{3.48} \]

Therefore, \( \hat{H}_{k,l}^{(g)}(p, q; p_k|q_l) = H_{k,l}^{(g)}(p, q; p_k|q_l) = H_{k,l}^{(g)}(p, q; p_k|q_l). \)

- \( E_{k,l}^{(g)}(p, q; p_k|q_l) = \hat{E}_{k,l}^{(g)}(p, q; p_k|q_l). \)

We have from equation (3.27)
\[ \hat{E}_{k,l}^{(g)}(p, q; p_k|q_l) = (x(p) - x(q))(y(p) - y(q)) \hat{H}_{k,l}^{(g)}(p, q, p_k|q_l) - \frac{G_{k,l}^{(g)}(p, q; p_k|q_l)}{dx(p)dy(q)}, \tag{3.49} \]
and from equation (3.28)
\[ \hat{E}_{k,l}^{(g)}(p, q, p_k|q_l) = (x(p) - x(q))(y(p) - y(q)) \hat{H}_{k,l}^{(g)}(p, q, p_k|q_l) - \frac{G_{k,l}^{(g)}(p, q; p_k|q_l)}{dx(p)dy(q)}, \tag{3.50} \]
so that \( \hat{E}_{k,l}^{(g)} = E_{k,l}^{(g)}. \)

Moreover, one can see from equation (3.27) that \( \hat{E}_{k,l}^{(g)}(p, q; p_k|q_l) \) is a polynomial of \( y(q) \) while \( \hat{E}_{k,l}^{(g)}(p, q; p_k|q_l) \) is a polynomial of \( x(p) \); therefore
\[ E_{k,l}^{(g)}(x(p), y(q); p_k|q_l) = \hat{E}_{k,l}^{(g)}(p, q; p_k|q_l) = \hat{E}_{k,l}^{(g)}(p, q; p_k|q_l) \tag{3.51} \]
is a polynomial in two variables.

- \( U_{k,l}^{(g)} \) and \( \tilde{U}_{k,l}^{(g)} \) are polynomials.

Equations (3.49), (3.50), (3.37) and (3.38) imply that
\[ E_{k,l}^{(g)}(x(p), y(q); p_k|q_l) = (x(p) - x(q)) \tilde{U}_{k,l}^{(g)}(x(p), q; p_k|q_l) \]
\[ + \sum_{h=0}^{d_1} \sum_{l=1}^{d_1} \frac{W_{l,j+1}(p_l; q_l, q) \tilde{U}_{k,k-l-h}^{(g-h)}(x(p), q; p_k|q_l)}{dy(q)} \]
\[ + \tilde{U}_{k,l+1}^{(g-l)}(x(p), q; p_k|q_l) - \sum_{m} d_{pm} H_{k,l-1}^{(g)}(p_m, q; p_k|q_l) \]
and
\[
E_{k,l}^{(g)}(x(p), y(q); \mathbf{p}_k | \mathbf{q}_l) = (y(q) - y(p))U_{k,l}^{(g)}(p, y(q); \mathbf{p}_k | \mathbf{q}_l) \\
+ \sum_h \sum_{l,j} \hat{W}_{l+1,j}^{(h)}(p, \mathbf{p}_l; \mathbf{q}_j) U_{k-l-j}^{(g-h)}(p, y(q); \mathbf{p}_{k-l-j} | \mathbf{q}_l) \\
+ \frac{U_{k+1,l}^{(g-1)}(p, y(q); p, \mathbf{p}_k | \mathbf{q}_l)}{dx(p)} - \sum_m d_{pm} \frac{U_{k-1,l}^{(g-1)}(p_m, y(q); \mathbf{p}_{k-1} | \mathbf{q}_l)}{x(p) - x(p_m)} \\
- \sum_m d_{pm} H_{k,l-1}^{(g)}(p_m, q; \mathbf{p}_k | \mathbf{q}_l/m).
\]  
(3.53)

from which (together with the recursion hypothesis) we deduce that \( U_{k,l}^{(g)} \) and \( \hat{U}_{k,l}^{(g)} \) are polynomials. This proves theorem 3.2.

**Theorem 3.3.** Symmetry of \( \hat{W}_{k,l}^{(g)} \).

For any \( k, l, g \), we have
\[
\hat{W}_{k+1,l+1}^{(g)}(p, \mathbf{p}_k | \mathbf{q}_l, q) = \hat{W}_{k+1,l+1}^{(g)}(p, \mathbf{p}_k | \mathbf{q}_l, q).
\]  
(3.54)

**Proof.** Let us prove it by recursion on \( 2g + k + l \). Assume that we have already proved it for any \( g', k', l' \) such that \( 2g' + k' + l' < 2g + k + l \).

Insert equation (3.30) into equation (3.53) in order to eliminate \( U \)'s, and then insert the result into equation (3.49). Most of the terms cancel (in fact the definitions of \( J_{k,l}^{(g)}, \tilde{J}_{k,l}^{(g)}, G_{k,l}^{(g)} \) were designed for that purpose), and using the recursion hypothesis, the only term left is
\[
\hat{W}_{k+1,l+1}^{(g-1)}(p, \mathbf{p}_k | \mathbf{q}_l, q) = \frac{1}{2} \left( \hat{W}_{k+1,l+1}^{(g-1)}(p, \mathbf{p}_k | \mathbf{q}_l, q) + \hat{W}_{k+1,l+1}^{(g-1)}(p, \mathbf{p}_k | \mathbf{q}_l, q) \right).
\]  
(3.55)

which proves the theorem.

**Corollary 3.1.** \( \hat{W}_{k,l}^{(g)}(\mathbf{p}_k | \mathbf{q}_l) \) is a symmetric function of its variables \( p_1, \ldots, p_k \) and a symmetric function of its variables \( q_1, \ldots, q_l \).

**Proof.** It is clear from the definitions that \( \hat{W}_{k,l}^{(g)}(\mathbf{p}_k | \mathbf{q}_l) \) is a symmetric function of its variables \( p_1, \ldots, p_k \) and that \( \hat{W}_{k,l}^{(g)}(\mathbf{p}_k | \mathbf{q}_l) \) is a symmetric function of its variables \( q_1, \ldots, q_l \).

Now, we prove the following theorem.

**Theorem 3.4.**
\[
\hat{W}_{k,0}^{(g)}(\mathbf{p}_k) = \hat{W}_{k}^{(g)}(\mathbf{p}_k)
\]  
(3.56)

and
\[
\hat{W}_{0,l}^{(g)}(\mathbf{q}_l) = \hat{W}_{l}^{(g)}(\mathbf{q}_l).
\]  
(3.57)

**Proof.** Write equation (3.53) for \( l = 0 \):
\[
E_{k,0}^{(g)}(x(p), y(q); \mathbf{p}_k) = (y(q) - y(p))U_{k,0}^{(g)}(p, y(q); \mathbf{p}_k) \\
+ \sum_h \sum_{l} \hat{W}_{l+1,0}^{(h)}(p, \mathbf{p}_l) U_{k-l,0}^{(g-h)}(p, y(q); \mathbf{p}_{k-l} | \mathbf{q}_l) \\
+ \frac{U_{k+1,0}^{(g-1)}(p, y(q); p, \mathbf{p}_k | \mathbf{q}_l)}{dx(p)} - \sum_m d_{pm} \frac{U_{k-1,0}^{(g-1)}(p_m, y(q); \mathbf{p}_{k-1} | \mathbf{q}_l)}{x(p) - x(p_m)}.
\]  
(3.58)
Using lemma B.2, we obtain
\[ \hat{W}^{(g)}_{k,0}(p|p_K) = \hat{W}^{(g)}_k(p_K). \] (3.59)

The other equality is obtained by writing equation (3.52) for \( k = 0 \) and exchanging the roles of \( x \) and \( y \) in lemma B.2. □

**Theorem 3.5.**
\[ \hat{W}^{(g)}_{k+1,l}(p, p_K|q_L) + \hat{W}^{(g)}_{k,l+1}(p_K|q_L, p) = d_p \frac{A^{(g)}_{k,l}(p; p_K|q_L)}{dx(p) dy(p)}, \] (3.60)

where \( A^{(g)}_{k,l}(p; p_K|q_L) \) has at most simple poles when \( p \to \alpha \).

**Proof.** From equation (3.42), it is easy to see that all contour integrals of \( \hat{W}^{(g)}_{k+1,l}(p, p_K|q_L) + \hat{W}^{(g)}_{k,l+1}(p_K|q_L, p) \) are vanishing, and thus it is the differential of some function.

The fact that \( A^{(g)}_{k,l}(p; p_K|q_L) \) has at most simple poles when \( p \to \alpha \) follows from lemma 3.1. □

**Theorem 3.6.**
\[ \text{Res}_{p \to \alpha} x(p)y(p) \hat{W}^{(g)}_{k+1,l}(p, p_K|q_L) = 0, \] (3.61)
\[ \text{Res}_{p \to \alpha} x(p)y(p) \hat{W}^{(g)}_{k,l+1}(p_K|q_L, p) = 0. \] (3.62)

**Proof.** By definition,
\[ \hat{W}^{(g)}_{k+1,l}(p, p_K|q_L) = \text{Res}_{s \to a, q_L} dS_{x,o}(p) f^{(g)}_{k+l}(s; p_K|q_L), \] (3.63)
and we have
\[ \text{Res}_{p \to \alpha} x(p)y(p) \hat{W}^{(g)}_{k+1,l}(p, p_K|q_L) = \text{Res}_{p \to \alpha} x(p)y(p) dS_{x,o}(p) f^{(g)}_{k+l}(s; p_K|q_L) \]
\[ = \text{Res}_{s \to a, q_L} \text{Res}_{p \to \alpha} x(p)y(p) dS_{x,o}(p) f^{(g)}_{k+l}(s; p_K|q_L) \]
\[ = - \text{Res}_{s \to a, q_L} (x(s)y(s) - x(o)y(o)) f^{(g)}_{k+l}(s; p_K|q_L) \] (3.64)

since \( f^{(g)}_{k+l} \) vanishes near the poles of \( y dx \) to order at least \( dy - 1 \), the expression above has no other poles than \( a, q_L \) and thus the total residue is zero. □

**Theorem 3.7.** For any \( k, l, g \) such that \( k + l + g \leq 1 \), one has
\[ \text{Res}_{p \to \alpha, q_L} \Phi(p) \hat{W}^{(g)}_{k+1,l}(p, p_K|q_L) = \text{Res}_{q \to b, p_K} \Psi(q) \hat{W}^{(g)}_{k,l+1}(p_K|q_L, q) \]
\[ = (2 - 2g - k - l) \hat{W}^{(g)}_{k,l}(p_K|q_L). \] (3.65)

**Proof.** We have
\[ \text{Res}_{p \to \alpha, q_L} \Phi(p) \hat{W}^{(g)}_{k+1,l}(p, p_K|q_L) - \text{Res}_{p \to b, p_K} \Psi(p) \hat{W}^{(g)}_{k,l+1}(p_K|q_L, p) \]
\[ = \text{Res}_{p \to \alpha, q_L} x(p)y(p) \hat{W}^{(g)}_{k+1,l}(p, p_K|q_L) - \text{Res}_{p \to a, q_L} \Psi(p) \hat{W}^{(g)}_{k+1,l}(p, p_K|q_L) \]
\[ - \text{Res}_{p \to b, p_K} \Psi(p) \hat{W}^{(g)}_{k,l+1}(p_K|q_L, p) \]
\[
\begin{align*}
\Psi(p)\left(\tilde{W}_{k,l}^{(g)}(p, p_K|q_L) + \tilde{W}_{k,l+1}^{(g)}(p_K|q_L, p)\right) \\
= \text{Res}_{p \to a, b, p_K, q_L} x(p) \frac{A_{k,l}^{(g)}(p; p_K|q_L)}{dx(p) dy(p)} \\
= \text{Res}_{p \to a} x(p) \frac{A_{k,l}^{(g)}(p; p_K|q_L)}{dx(p) dy(p)} \\
= 0.
\end{align*}
\] (3.66)

The fact that \(\text{Res}_{p \to a, q_L} \Phi(p) \tilde{W}_{k+1,l}^{(g)}(p, p_K|q_L) = (2 - 2g - k - l)\tilde{W}_{k,l}^{(g)}(p_K|q_L)\) can be proved by recursion on \(2g + k + l\) and using corollary 3.1.

This allows us to prove our main theorem.

**Theorem 3.8.** \(F^{(g)}\)'s are symmetric under the exchange \(x \leftrightarrow y\):
\[
\tilde{F}^{(g)} = \tilde{F}^{(g)}
\] (3.67)

**Proof.** Indeed, we have
\[
(2 - 2g) \tilde{F}^{(g)} = \text{Res}_a \Phi(p) \tilde{W}_{1,0}^{(g)}(p), \quad (2 - 2g) \tilde{F}^{(g)} = \text{Res}_b \Psi(p) \tilde{W}_{0,1}^{(g)}(p).
\] (3.68)

### 3.4. Additional properties

The following theorem relates \(H\) and \(W\).

**Theorem 3.9.** We have
\[
\begin{align*}
\tilde{W}_{k+1,l}^{(g)}(p, p_K|q_L) &= \text{Res}_{q \to a} \frac{H_{k,l}^{(g)}(p, q; p_K|q_L)}{H_{0,0}^{(0)}(p, q)} \, dy(q) \\
\tilde{W}_{k,l+1}^{(g)}(p_K|q_L, q) &= \text{Res}_{p \to a} \frac{H_{k,l}^{(g)}(p, q; p_K|q_L)}{H_{0,0}^{(0)}(p, q)} \, dx(p).
\end{align*}
\] (3.69)

**Proof.** Multiply equation (3.30) by \(dx(p) dy(q)/(y(q) - y(p))H_{0,0}^{(0)}(p, q)\) and take the residues at \(q \to \alpha\).

**Remark 3.3.** This theorem was expected from the matrix model property that
\[
\text{tr} \frac{1}{x - M_1} \frac{1}{y - M_2} \rightarrow \frac{1}{x} \text{ tr} \frac{1}{y - M_2}
\] (3.71)

when \(x \to \infty\).

### 4. Conclusion

In this paper, we have proved the \(x \leftrightarrow y\) symmetry which was announced in [21]. This symmetry has many applications; for instance in [21] it was used to recover the \((p, q) \leftrightarrow (q, p)\) duality of minimal models [30], or to give a very short proof that the Kontsevitch integral indeed depends only on odd times and satisfies the KdV hierarchy [26].

In addition, we have shown how to compute some family of mixed correlation functions of the 2-matrix model.

This could open the route to some matrix model approach to the understanding of the boundary conformal field theory in higher genus. In a forthcoming article, we shall introduce a similar algebraic geometry method to compute all possible mixed correlation functions [23].
This work also raises many questions and calls the following prospects.

- It would be interesting to see what $H_{k,l}$ and $W_{k,l}$ correspond to for other matrix models (e.g. Kontsevitch’s integral, chain of matrices), although we may guess that they also correspond to mixed traces expectation values in those cases.
- More interesting would be to understand what $H^{(g)}_{k,l}$ and $W^{(g)}_{k,l}$ compute in algebraic geometry. These should correspond to ‘volume’ or ‘intersection numbers of some moduli spaces’.

Acknowledgments

We would like to thank Michel Bergère and Aleix Prats Ferrer for fruitful discussions on this subject. This work is partly supported by the Enigma European network MRT-CT-2004-5652, by the ANR project Géométrie et intégrabilité en physique mathématique ANR-05-BLAN-0029-01, by the Enrage European network MRTN-CT-2004-005616, by the European Science foundation through the Misgam program, by the French and Japanese governments through PAI Sakura, by the Quebec government with the FQRNT.

Appendix A. Spectral curve

We recall that the curve $E(x, y)$, called the classical spectral curve, is given by a polynomial of the form

$$E(x, y) = \sum_{j=0}^{d+1} E_j(x) y^j.$$  \hfill (A.1)

We define the ‘quantum spectral curve’ as the formal power series:

$$E_N(x, y) = \sum_{g} N^{-2g} E^{(g)}(x, y),$$  \hfill (A.2)

where

$$E^{(g)}(x, y) = E_{d+1}(x) \sum_{r=1}^{d} \sum_{J_0 \subseteq K, |J_0| = k} \sum_{g_1, \ldots, g_r} \delta_{\sum_{J_0 \cup \cdots \cup J_r = K} g_1, \ldots, g_r} \prod_{l=1}^{r} W^{(g_l)}_{l}(p^{J_l}).$$ \hfill (A.3)

with

$$K = \{1, \ldots, d\}$$ \hfill (A.4)

and

$$W^{(g)}_{k}(p_K) := W^{(g)}_{k}(p_K) + \delta_{k,1} \delta_{g,0} (y - Y(p_1)).$$ \hfill (A.5)

where $W^{(g)}_{k}(p_K)$ is the meromorphic form defined in [21] for the curve $E(x, y)$.

**Lemma 2.** For any $g$, $E^{(g)}(x, y)$ is a polynomial in $x$ and $y$, whose degrees are at most those of $E$.

**Proof.** It is clear that $E_N(x, y)$ is a polynomial in $y$ and a rational function of $x$. Let us prove that $E^{(g)}(x, y)$ is indeed a polynomial in $x$ for $g \geq 1$. The coefficient of $y^k$ in $E^{(g)}(x, y)$ is

$$\frac{E^{(g)}_{k}(x)}{E^{(g)}_{d+1}(x)} = \sum_{J_0 \subseteq K, |J_0| = k} \prod_{j \in J_0} y(p^j) \sum_{r=1}^{d-k} \sum_{J_1 \cup \cdots \cup J_r = K \setminus J_0} \sum_{g_1, \ldots, g_r} \delta_{\sum_{J_0 \cup \cdots \cup J_r = K} g_1, \ldots, g_r} \prod_{l=1}^{r} W^{(g_l)}_{l}(p^{J_l}).$$ \hfill (A.6)
First, note that the product of $W$’s can have poles only at branch points and the product of $y$’s can have poles only at poles of $y$. The poles of $y$ which are not poles of $x$ are killed by the prefactor $E_{d+1}(x)$, as they are in the classical curve $E(x, y)$. Let us consider the poles at a branch point $a$. The only terms which might diverge at $p \to a$ are of either of the following forms.

- $\left( W_{i+j}^{(h)}(p, p') + W_{i+j}^{(h)}(\mathcal{T} p, p') \right) \times \text{reg}$ where reg means a term with no poles at $p \to a$. This term is regular because of theorem 4.4 in [21].

- Or $\left( W_{i+j}^{(h)}(p, p') + W_{i-j}^{(h)}(\mathcal{T} p, p') \right) \times \text{reg}$ again; this expression is regular when $p \to a$, because of theorems 4.4 and 4.5 in [21].

Thus, we have proved that $E_{d+1}(x)$ is a rational function of $x$ whose only poles are the poles of $x$, i.e. it is a polynomial in $x$.

Consider a pole $\infty$ of $x$, the behavior of $E_{d+1}(x, y)$ when $p \to \infty$ is at most that of $\sum_{J_0 \subset K} \prod_{j \in J_0} y(p_j)$. Note that $J_0$ cannot be equal to $K$ itself, because the product of the corresponding $W$’s vanishes (it contains no term), and $|J_0|$ cannot be equal to $|K| - 1$, because the prefactor vanishes due to theorem 4.4 in [21]. Thus, $|J_0| \leq |K| - 2$, which implies that $E_{d+1}(x(\rho), y(\rho)) d\rho$ has a pole of degree at most that of $E_{d}(x(\rho), y(\rho))$, i.e. $E_{d+1}(x, y)$ is contained in Newton’s polytope of $E(x, y)$. This means that

$$E_{d+1}(x(p), y(p)) \frac{d}{dx(p)}$$

is a holomorphic differential.

**Appendix B. Lemma: unicity of the solution of loop equations**

**Lemma B.2.** The system of equations

$$E_{d+1}^{(g)}(x(p), y(q); p_K) = (y(q) - y(p))U_{d+1}^{(g)}(p, y(q); p_K)$$

$$+ \sum_{h} \sum_{I} W_{i+j}^{(h)}(p, p_I)U_{d+1-j}^{(g-h)}(p, y(q); p_K) \frac{d}{dx(p)}$$

$$+ U_{d+1-j}^{(g-h)}(p, y(q); p_K) \frac{d}{dx(p)} - \sum_{m} d_{pm} \frac{U_{d+1-j}^{(g-h)}(p_m, y(q); p_K)}{x(p) - x(p_m)},$$

where

- if $2g + k > 2$, $W_{d+1}(p, p_K)$ has poles only at branch points in any of its variables and vanishing $A$-cycle integrals,
- $E_{d+1}^{(g)}(x(p), y(q); p_K)$ is a polynomial in $x(p)$ of degree at most $d_1 - 1$ and a polynomial in $y(q)$ of degree at most $d_2 - 1$,
- $U_{d+1}^{(g)}(p, y(q); p_K)$ is a polynomial in $y(q)$ of degree at most $d_2 - 1$,
- has a unique solution.

This solution is such that

$$W_{d+1}^{(g)}(p_K) = \tilde{W}_{d+1}^{(g)}(p_K).$$

**Proof of the lemma:**

**Unicity.**

We prove it by recursion on $2g + k$. Assume that it is already proved for any $g', k'$ such that $2g' + k' < 2g + k$. 25
At $p = q$, equation (B.1) gives
\[
W_{k+1}^{(g)}(p, \mathbf{p}_K) = \frac{E_k^{(g)}(x(p), y(p); \mathbf{p}_K) \, dx(p)}{U_0^{(0)}(p, y(p))} - \sum_h \sum_l \frac{W_{h+1}^{(g)}(p, \mathbf{p}_1) U_k^{(g-h)}(p, y(p); \mathbf{p}_{K\backslash l})}{U_0^{(0)}(p, y(p))}
\]
\[
- \frac{U_{k+1}^{(g-1)}(p, y(p); \mathbf{p}_K)}{U_0^{(0)}(p, y(p))} + \sum_m \delta_{p_m}(x(p) - x(p_m)) \, dx(p)
\]
\[
+ \sum_m \delta_{h}(x(p) - x(p_m)) \, dx(p)
\]
(\text{B.3})

Then write the Cauchy residue formula:
\[
W_{k+1}^{(g)}(p, \mathbf{p}_K) = - \text{Res}_{r \rightarrow p} dS_{r,o}(p) W_{k+1}^{(g)}(r, \mathbf{p}_K).
\]
(\text{B.4})

Since we know the poles of $W_{k+1}^{(g)}(p, \mathbf{p}_K)$ and its $\mathcal{A}$-cycle integrals, we may move the integration contour using Riemann’s bilinear identity and get
\[
W_{k+1}^{(g)}(p, \mathbf{p}_K) = \text{Res}_{r \rightarrow p} dS_{r,o}(p) W_{k+1}^{(g)}(r, \mathbf{p}_K).
\]
(\text{B.5})

Now, we replace $W_{k+1}^{(g)}(r, \mathbf{p}_K)$ by its value in equation (B.3). We see that the term
\[
\frac{E_k^{(g)}(x(r), y(r); \mathbf{p}_K) \, dx(r)}{U_0^{(0)}(r, y(r))}
\]
has no pole at the branch points and does not contribute to the residue, and similarly the last term of equation (B.3) does not contribute to the residue. We get
\[
W_{k+1}^{(g)}(p, \mathbf{p}_K) = - \text{Res}_{r \rightarrow p} dS_{r,o}(p) W_{k+1}^{(g)}(r, \mathbf{p}_K)
\]
\[
\times \left( U_{k+1}^{(g-1)}(r, y(r); \mathbf{p}_K) + \sum_h \sum_l W_{h+1}^{(g)}(r, \mathbf{p}_l) U_{k+1}^{(g-h)}(r, y(r); \mathbf{p}_{K\backslash l}) \right).
\]
(\text{B.6})

Since all the terms in the rhs are already known from the recursion hypothesis, this determines $W_{k+1}^{(g)}(p, \mathbf{p}_K)$ uniquely. Then, we write equation (B.1) for $p = \tilde{q}^j$ with $j = 1, \ldots, d_1$:
\[
E_k^{(g)}(x(\tilde{q}^j), y(q); \mathbf{p}_K) = \sum_h \sum_l \frac{W_{h+1}^{(g)}(\tilde{q}^j, \mathbf{p}_l) U_{k+1}^{(g-h)}(\tilde{q}^j, y(q); \mathbf{p}_{K\backslash l})}{dx(\tilde{q}^j)}
\]
\[
+ \frac{U_{k+1}^{(g-1)}(\tilde{q}^j, y(q); \mathbf{p}_K)}{dx(\tilde{q}^j)} - \sum_m \delta_{h}(x(\tilde{q}^j) - x(p_m)) \, dx(p_m)
\]
(\text{B.7})

since all terms in the rhs are uniquely determined, so is the lhs. Since $E_k^{(g)}(x(p), y(q); \mathbf{p}_K)$ is a polynomial in $x(p)$ of degree $d_1 - 1$ and we know its value in $d_1$ points, then $E_k^{(g)}(x(p), y(q); \mathbf{p}_K)$ is uniquely determined.

Then, using equation (B.1) once again, we uniquely determine $U_k^{(g)}(p, y(q); \mathbf{p}_K)$.

This proves the unicity for $g$ and $k$.

**Existence.**

Start from the meromorphic form $W_k^{(g)}(p_K)$ defined in [21] for the curve $\mathcal{E}(x, y)$, and define
\[
\tilde{W}_k^{(g)}(p_K) := W_k^{(g)}(p_K) / dx(p_K) + \delta_{k,1} \delta_{g,0}(y - y(p_1)).
\]
(\text{B.8})

Then, let $K_0 = \{0, 1, \ldots, d_2\} \cup K$ and $K_1 = \{1, \ldots, d_2\} \cup K$, and define
\[
\mathcal{E}_k^{(g)}(x(p_K), y; p_K) = \mathcal{E}_{d_1+1}(x) \sum_{j_1=0}^{d_2+1-k} \sum_{j_2=0}^{d_2} \sum_{j_3=0}^{d_2} \delta_{\sum_{l=1}^3 |j_l| - 1, g} \prod_{l=1}^{r-1} \tilde{W}_l^{(g)}(p^{j_l})
\]
(\text{B.9})
and:

\[ U_k^{(g)}(p^0, y; p_K) = \mathcal{E}_{d_2+1}(x) \sum_{r=1}^{d_1+k} \prod_{J \cup \cdots \cup J_r = K, g_1, \ldots, g_r} \delta_{\sum_{|J_l|=1-r}} x^r \prod_{l=1}^{N} W_{|J_l|}^{(g)}(p^0). \]  

(B.10)

It is clear that both \( \mathcal{E}_k^{(g)}(x, y; p_K) \) and \( U_k^{(g)}(p, y; p_K) \) are polynomials in \( y \) of degree at most \( d_2 - 1 \). Following the same line as in lemma A.1, it is easy to get that \( \mathcal{E}_k^{(g)}(x, y; p_K) \) is also a polynomial in \( x \) of degree at most \( d_1 - 1 \).

Therefore, the functions \( \mathcal{E}_k^{(g)}(x, y; p_K) \), \( U_k^{(g)}(p, y; p_K) \) and \( W_k^{(g)}(p) \) obey the requirements of lemma B.2, and equation (B.1) is clearly satisfied from the definitions of \( \mathcal{E}_k^{(g)}(x, y; p_K) \) and \( U_k^{(g)}(p, y; p_K) \). Thus, we have found an explicit solution of the system of lemma B.2, which proves the existence.

References

[1] Ambjørn J, Chekhov L, Kristjansen C F and Makeenko Yu 1993 Matrix model calculations beyond the spherical limit Nucl. Phys. B 404 127–72
[2] Ambjørn J, Chekhov L, Kristjansen C F and Makeenko Yu 1995 Matrix model calculations beyond the spherical limit Nucl. Phys. B 449 681 (Preprint hep-th/9302014) (Erratum)
[3] Bertola M 2006 Two-matrix model with semiclassical potentials and extended Toda tau-function Preprints CRM-2921, hep-th/0306184
[4] Bertola M and Eynard B 2003 Mixed correlation functions of the 2-matrix model Preprint math-ph/0504058
[5] Chekhov L and Eynard B 2006 Hermitian matrix model free energy: Feynman graph technique for all genera J. High Energy Phys. JHEP12(2006)053 (Preprint math-ph/0603003)
[6] Daul J M, Kazakov V and Kostov I 1993 Rational theories of 2D gravity from the two-matrix model Nucl. Phys. B 409 311–38 (Preprint hep-th/9303093)
[7] David F 1990 Loop equations and nonperturbative effects in two-dimensional quantum gravity Mod. Phys. Lett. A 5 1019
[8] David F 1985 Planar diagrams, two-dimensional lattice gravity and surface models Nucl. Phys. B 257 45–58
[9] Di Francesco P, Ginsparg P and Zinn-Justin J 1995 2D gravity and random matrices Phys. Rep. 235 1
[10] Eynard B and Orantin N 2004 Topological expansion for the 1-hermitian matrix model correlation functions J. High Energy Phys. JHEP12(2004)041 (Preprint math-ph/0403019)
[11] Eynard B, Orantin N 2007 Invariants of algebraic curves and topological expansion Preprint math-ph/0702045
[12] Eynard B and Orantin N 2005 Mixed correlation functions in the 2-matrix model, and the Bethe ansatz J. High Energy Phys. JHEP12(2005)034 (Preprint math-ph/0504058)
[13] Eynard B and Orantin N 2007 Invariants of algebraic curves and topological expansion Preprint math-ph/0702045
[14] Eynard B and Orantin N 2005 Mixed correlation functions in the 2-matrix model, and the Bethe ansatz J. High Energy Phys. JHEP12(2005)034 (Preprint math-ph/0504058)
[23] Eynard B and Orantin N Whole topological expansion of any correlation function in the two matrix model, in preparation.
[24] Farkas H M and Kra I 1992 Riemann Surfaces 2nd edn (Berlin: Springer)
[25] Fay J D 1973 Theta Functions on Riemann Surfaces (Berlin: Springer)
[26] Itzykson C and Zuber J B 1992 Combinatorics of the modular group: II. The Kontsevich integrals Int. J. Mod. Phys. A 7 5661–705 (Preprint hep-th/9201001)
[27] Kazakov V A 1985 Bitocal regularization of models of random surfaces Phys. Lett. B 150 282–4
[28] Kazakov V A 1986 Ising model on a dynamical planar random lattice: exact solution Phys. Lett. A 119 140–4
[29] Kazakov V A and Marshakov A 2003 Complex curve of the two matrix model and its tau-function J. Phys. A: Math. Gen. 36 3107–36 (Preprint hep-th/0211236)
[30] Kharchev S and Marshakov A 1993 On p − q duality and explicit solutions in c < 1 2d gravity models Preprint hep-th/9303100
[31] Krichever I 1992 The τ-function of the universal Whitham hierarchy, matrix models and topological field theories Commun. Pure Appl. Math. 47 437 (Preprint hep-th/9205110)
[32] Staudacher M 1993 Combinatorial solution of the 2-matrix model Phys. Lett. B 305 332–8
[33] ’t Hooft G 1974 Nucl. Phys. B 72 461
[34] Tutte W T 1962 A census of planar triangulations Can. J. Math. 14 21–38
[35] Tutte W T 1963 A census of planar maps Can. J. Math. 15 249–71