MF-PROPERTY FOR COUNTABLE DISCRETE GROUPS

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Abstract. In this article we study MF-property for countable discrete groups, i.e. groups which admit embedding into unitary group of $\prod M_n/\oplus M_n$. We prove that Baumslag group $\langle a, b | a^a b = a^2 \rangle$ has MF-property and check some permanent facts about MF-groups.

1. Introduction

By definition MF-groups are countable groups which admit embedding into $U(\prod_{n=1}^{\infty} M_n/\oplus_{n=1}^{\infty} M_n)$ where $M_n$ - algebra of $n$-by-$n$ complex-valued matrices, $U(A)$ - group of unitary elements of $C^*$-algebra $A$. MF-groups were first considered in [7], where it was proved that for amenable groups MF-property equivalent to quasidiagonality of $C^*_r(G)$. In [7] it was also proved that LEF-groups have MF-property. The main motivation for considering and studying amenable MF-groups is famous conjecture that for amenable group $G$ algebra $C^*_r(G)$ is quasidiagonal. Recently this conjecture was proved in [17], i.e. all countably amenable groups are MF. So it is very natural to examine what non-amenable groups have this property. The first reason is the connection with vector bundles. From [14] we know how to construct a vector bundle on $BG$ from homomorphism $G \rightarrow U(\prod M_n/\oplus M_n)$ (such homomorphisms are usually called asymptotic homomorphisms or MF-representation). This vector bundle has some good properties which make it similar to vector bundle which is constructed from finitedimensional representation of group $G$. But on the other hand this construction often produce whole $K^0(BG)$, which is in some sense finitedimensional way to Novikov conjecture.

Our second motivation - is connection between MF-property and hyperlinearity. It is easy to see that MF-property for $G$ is equivalent to possibility of embedding $G \hookrightarrow \prod U_n/\oplus U_n$, where $U_n$ is usual $n$-unitary group and $\oplus U_n = \{u \in U_n : \|u-1\| \rightarrow 0\}$. Recall that countable group $G$ is called hyperlinear if it admits embedding $G \hookrightarrow \prod U_n/\oplus_2 U_n$, where $\oplus_2 U_n = \{u \in U_n : \|u-1\|_2 \rightarrow 0\}$ and $\|a\|_2 = \sqrt{\tau(a^*a)}$ and $\tau$ - normalized trace on $M_n$ (it seems to be interesting to consider more general approximation in $GL_n$ instead of $U_n$. Apriori we get another classes of groups after replacing $U_n$ by $GL_n$ in definition of hyperlinear or MF-group. Concept of linear sofic groups is example of such generalizations, see [1] in this direction). Due to similarity of definitions we can check some facts about MF-groups just rewriting proof of similar facts about hyperlinear groups into MF-language, but not always we can do that. For example, it is known that amalgamated product of two hyperliner group over amenable group us hyperlinear; while we don’t even know is it true that amalgamated product of two MF-groups over finite group is MF-group. While we have $\|a\|_2 \leq \|a\|$ and so every MF-representation is hyperlinear representation, faithful MF-representation could be

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non-faithful hyperlinear representation and MF-property does not automatically imply hyperlinearity. Now nobody knows example of non hyperlinear group (existence of non MF-groups is also open question), but one of the main candidates for this role is Higman group \(\langle a,b,c,d|a^b = a^2, b^c = b^2, c^d = c^2, d^a = d^2 \rangle\). The famous property of this group is nonexistence of nontrivial finitedimensional representation and it is not clear, how we can construct nontrivial MF-representation or hyperlinear representation. There is homomorphisms \(\langle a,b,c,d|a^b = a^2, b^c = b^2, c^d = c^2, d^a = d^2 \rangle \rightarrow G = \langle x,y| x^{x^2} = x^2, [x,y]^4 = 1 \rangle\) defined via \(a \mapsto x, b \mapsto x^y, c \mapsto x^{y^2}, d \mapsto x^{y^4}\). So if we have nontrivial representation of group \(G\) - it is a great chance to construct a representation of Higman group, but group \(G\) is too complicated and for partial understanding its representations we consider less complicated Baumslag group \(\langle x,y| x^{x^2} = x^2 \rangle\).

2. PERMANENT FACTS

We will consider only countable groups, all maps between groups are assumed to be unital. We also use notation \(a^b = b^{-1}ab\).

**Definition 1.** Countable group \(G\) is called MF-group (or has MF-property) if there is injective homomorphism \(G \hookrightarrow \prod U_n/\oplus U_n\), where \(\oplus U_n = \{\{u_n\} \in \prod U_n : \|u_n - 1\| \rightarrow 0\}\).

**Proposition 2.** The following conditions are equivalent

1) \(G\) is MF-group.
2) There are maps \(\alpha_n : G \rightarrow U_n\) such that for every \(g,h \in G\) we have \(\|\alpha_n(gh) - \alpha_n(g)\alpha_n(h)\| \rightarrow 0\) and for every \(g \neq 1\) we have \(\|\alpha_n(g) - 1\| \rightarrow 0\).
3) For every finite set \(F \subset G\) there is \(\delta\) such that for every \(\varepsilon > 0\) there is \(n \in \mathbb{N}\) and map \(\alpha : G \rightarrow U_n\) such that for every \(g,h \in F\) we have \(\|\alpha(gh) - \alpha(g)\alpha(h)\| < \varepsilon\) and for every \(g \in F\) such that \(g \neq 1\) we have \(\|\alpha(g) - 1\| > \delta\).
4) For some subsequence \(\{n_k\}\) we have inclusion \(G \hookrightarrow \prod U_{n_k}/\oplus U_{n_k}\).
5) \(G \hookrightarrow U(\prod M_n/\oplus M_n)\).
6) \(G \hookrightarrow U(A)\) for some MF-algebra \(A\).

**Proof.** Easy exercise. \(\square\)

**Definition 3.** We will call homomorphism \(\alpha : G \rightarrow \prod U_n/\oplus U_n = U(\prod M_n/\oplus M_n)\) by asymptotic homomorphisms (or MF-representation). We will call maps \(\alpha_n\) (which appear from some lift \(G \rightarrow \prod U_n\) of \(\alpha\)) by almost representations.

There is very important homomorphism \(U_n \rightarrow U_{n^2}\), \(u \mapsto \text{Ad}(u)\), where \(\text{Ad}(u)\) is unitary matrix of conjugate by \(u\) in the space \(\mathbb{C}^{n^2} = M_n(\mathbb{C})\) with inner product \(\langle A,B \rangle = \sum_{i,j} \bar{A}_{i,j} B_{i,j}\).

**Proposition 4.** For every \(\delta > 0\) there is \(k_\delta \in \mathbb{N}\) such that for every \(u \in U_n\) with \(\text{diam}(\sigma(u)) > \delta\) there is \(k < k_\delta\) such that \(\|\gamma^k(u) - 1\| \geq \sqrt{2}\).

**Proof.** Put \(k_\delta = \lceil \log_2(\frac{\delta}{\pi}) \rceil + 1\). We may assume that \(u = \text{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_n})\). We have \(\gamma(u) = \text{diag}(e^{i(\alpha_i - \alpha_j)})\) in the basis consisting of matrix units in \(M_n(\mathbb{C}) = \mathbb{C}^{n^2}\). Let \(x, y\) be arguments of eigenvalues of \(u\) such that \(|e^{ix} - e^{iy}| = \text{diam}(\sigma(u))\). So, we have \(e^{i(x-y)}\) and \(e^{i(y-x)}\) among eigenvalues of \(\gamma(u)\). By induction we have \(e^{i2k(x-y)}\) and \(e^{i2k(y-x)}\) among eigenvalues of \(\gamma^k(u)\). Consider minimal \(k\) such that \(2^k(x-y) \in \left[\frac{\pi}{2}, \pi\right)\). In this case we have \(|e^{i2k(x-y)} - 1| \geq \sqrt{2}\) and so \(\|\gamma^{k+1}(u) - 1\| \geq \sqrt{2}\). It is easy to see that \(k < k_{\text{diam}(\sigma(u))}\). \(\square\)
Later for operator we will use standard notation $x =_\varepsilon y$ when $\|x - y\| < \varepsilon$.

**Proposition 5.** Let $\alpha' : G \to \prod U_n / \oplus U_n$ - some MF-representation such that $\alpha'(g) \neq 1$ for some $g \in G$. Then there is MF-representation $\beta$ such that $\|\beta(g) - 1\| \geq \sqrt{2}$.

**Proof.** For convenience we will consider representation $\alpha'$ as set $\{\alpha'_n\}$ of almost representation. Putting $\alpha_n = \alpha'_n \oplus 1$ we have for every $g \in G$ that $1 \in \sigma(\alpha_n(g))$. Since $\alpha(g) \neq 1$, there is $\delta > 0$ and $n_0$ such that for every $n > n_0$ we have $\|\alpha_n(g) - 1\| > \delta$ (if it is necessary we consider some subsequence). Since $1 \in \sigma(\alpha_n(g))$ we have $\text{diam}(\sigma(\alpha_n(g))) > \delta$. By Proposition 4 we can construct $k_\delta$ and numbers $k(n)$ with $k(n) < k_\delta$ such that for every $n > n_0$ we have $\|\beta_n(g) - 1\| \geq \sqrt{2}$ where $\beta_n = \gamma^{k(n)} \circ \alpha_n$. Moreover $\{\beta_n\}$ is asymptotic homomorphism. Indeed, for every $\varepsilon > 0$ and every finite set $K \subset G$ we can ensure that $\alpha_n(q)\alpha_n(h)\alpha_n^{-1}(qh) = _\varepsilon 1$ for every $q, h \in K$ and big enough. Since $\gamma^k$ -homomorphism then $\beta_n(q)\beta_n(h)\beta_n^{-1}(qh) = \gamma^{k(n)}(\alpha_n(q)\alpha_n(h)\alpha_n^{-1}(qh))$. From the formula $\gamma(\text{diag}(\{e^{i\alpha_n}\})) = \text{diag}(\{e^{i(\alpha_n - \alpha_n)}\})$ it is easy follows that $\|\gamma(u) - 1\| \leq 2\|u - 1\|$, so $\beta_n(q)\beta_n(h)\beta_n^{-1}(qh) = _{2\varepsilon} 1$. It means that $\beta$ - asymptotic representation. □

**Proposition 6.** Residually MF-group is MF-group.

**Proof.** Let $G$ - residually MF-group. It means that for every $g \neq 1$ there is MF-homomorphisms $\alpha^g$ (here $g$ is index) such that $\alpha^g(g) \neq 1$. By Proposition 5 we can find some another MF-homomorphism $\beta^g$ such that $\|\beta^g(g) - 1\| \geq \sqrt{2}$. Consider $\varepsilon > 0$ and finite set $K \subset G$ and let $\beta^K = \oplus_{g \in K} \beta^g$. As finite direct sum of MF-representation $\beta^K$ is also MF-representation, so for some big enough $N = N(K, \varepsilon)$ we have $\beta^K_N(g, h) = _\varepsilon \beta^K_N(g)\beta^K_N(h)$ for every $g, h \in K$ and $\|\beta^K_N(g) - 1\| \geq 1$ for every $g \in K$ such that $g \neq 1$.

Consider our group $G = \bigcup K_n$ as union os increasing sequence of finite sets. Then $\omega_n = \beta^K_{N(\frac{1}{n})}$ is faithful asymptotic representation. It is easy to see that $\|\omega(g) - 1\| \geq 1$ for every $g \neq 1$, and for every $\varepsilon$ and every finite $K \subset G$ we can find $n$ such that $\varepsilon > \frac{1}{n}$ and $K \subset K_n$. So $\|\omega_n(g)\omega_n(h) - \omega_n(gh)\| < \varepsilon$ for every $g, h \in K$. □

Proofs of Propositions 4-6 are almost the same as in the case of hyperlinear groups (see [8]).

Using $C^*$-theory we can present shorter proof of Proposition 6. Let $G$ -residually MF-groups, so there is MF-algebras $A_n$ such that $G \hookrightarrow \prod A_n$ (where image lies in unitary group). Consider $B_n = A_1 \oplus \ldots \oplus A_n$, homomorphism $\alpha : \prod A_n \hookrightarrow \prod B_n$ defined via $\alpha(\{a_n\}) = \{a_1 \oplus \ldots \oplus a_n\}$. Consider composition map $\beta : G \hookrightarrow \prod A_n \hookrightarrow B_n \to \prod B_n / \oplus B_n$. It is easy to see that $\beta$ - injective and $C^*(\beta(G))$ is separable subalgebra of $\prod B_n / \oplus B_n$. So $C^*(\beta(G))$ is MF-algebra by [5].

**Proposition 7.** Let $G$ - group. Then $G$ is MF-group iff for every $\varepsilon > 0$ and every finite $K \subset G$ there is map $\alpha : G \to U_n$ to some finitedimensional unitary group such that for every $g, h \in K$ inequality $\|\alpha(gh) - \alpha(g)\alpha(h)\| < \varepsilon$ holds and for every nontrivial $g \in G$ we have $\|\alpha(g) - 1\| \geq \sqrt{2}$.

**Proof.** $2 \Rightarrow 1$ is obvious. For $1 \Rightarrow 2$ we can consider $\alpha = \beta^K_{N(\frac{1}{n})}$ from the proof of Proposition 6. □

Proposition 7 means that for MF-groups there is injective MF-homomorphism $\alpha$ such that $\alpha(G)$ is discrete in the induced topology of $U(\prod M_n / \oplus M_n)$. 
Proposition 11. Let $G = \langle a_1, \ldots, a_n | r_1, \ldots, r_m = 1 \rangle$ be finite presented group. Then $G$ is MF-group iff for every $\varepsilon > 0$ and every finite set $K \subset G$ there is $N \in \mathbb{N}$ and matrices $A_1, \ldots, A_n \in U_N$ such that

1) For every nontrivial $k \in F$ there is some corresponding word $\omega_k \in \mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$ in free group (i.e. for natural quotient homomorphism $\pi : \mathbb{F}_n \to G$ we have $\pi(\omega_k) = k$).

For convenience we also assume $\omega_{a_j} = a_j$ such that $\|\omega_k(A_1, \ldots, A_n) - 1\| \geq 1$.

2) $r_j(A_1, \ldots, A_n) = \varepsilon^{-1}$

Proof. Let us prove $\Leftarrow$. Let $\varepsilon > 0$ and $F \subset G$. Consider matrices $A_1, \ldots, A_n \in U_N$ corresponding to $\varepsilon$ and finite set $K = F \cdot F = \{gh : g, h \in F\}$. Define our almost representation $\alpha : G \to U_N$ in such way: $\alpha(k) = \omega_k(A_1, \ldots, A_n)$ (for nontrivial $k \in K$) and arbitrary on $G \setminus K$. Let $C_{g,h}$ - minimal number of operations of type $r_j \leftrightarrow 1, a_j^{-1}a_j \leftrightarrow 1, a_ja_j^{-1} \leftrightarrow 1$ which is necessary to transform $\omega_{gh}$ into $\omega_g\omega_h$ (since $\pi(\omega_{gh}) = \pi(\omega_g\omega_h)$ this transformation is possible). Put $C = \max_{g,h \in F} C_{g,h}$. Then it is easy to see that $\|\alpha(gh) - \alpha(g)\alpha(h)\| < C\varepsilon$ for all $g, h \in F$. These almost representations $\alpha$ generate faithful asymptotic representation because $\|\omega_k(A_1, \ldots, A_n) - 1\| \geq 1$ for nontrivial $k$ by assumption.

Let us prove $\Rightarrow$. Let $G$ - MF-group and $\alpha_N : G \to U_N$ is faithful asymptotic representation. Using Proposition 7 we may assume that $\|\alpha_N(g) - 1\| \geq \sqrt{2}$ for all $N$ and $g \neq 1$. Put $\omega_k \in \mathbb{F}_n$ arbitrary with property $\pi(\omega_k) = k$. As $\|\alpha_N(gh) - \alpha_N(g)\alpha_N(h)\| \to 0$ we have $\|\alpha_N(\omega_{g}(a_1, \ldots, a_n)) - \omega_{g}(\alpha_N(a_1), \ldots, \alpha_N(a_n))\| \to 0$ as $N \to \infty$. Since $\limsup_N \|\alpha_N(\omega_{g}(a_1, \ldots, a_n)) - 1\| \geq \sqrt{2}$ we have $\|\omega_{g}(\alpha_N(a_1), \ldots, \alpha_N(a_n)) - 1\| \geq 1$ for every nontrivial $g \in F$ and $N$ big enough. Analogously using $\|\alpha_N(gh) - \alpha_N(g)\alpha_N(h)\| \to 0$ we deduce $\|r_j(\alpha_N(a_1), \ldots, \alpha_N(a_n)) - 1\| < \varepsilon$ for every $j$ and $N$ big enough. It means that there is some number $N_0$ such that matrices $A_i = \alpha_{N_0}(a_i)$ satisfy properties 1) and 2). \qed

As for amenable groups MF-property equivalent to quasidiagonality of $C^*(G)$ (see [7]) we have the following important version of main theorem from [17]:

Theorem 9. Amenable groups are MF-groups.

Using Proposition 6 we immediately deduce the following fact:

Corollary 10. Residually MF-groups are MF-groups.

This corollary covers very wide class of countable groups, for example Baumslag-Solitar groups $B(n, m) = \langle a, b | b^{-1}a^n b = a^m \rangle$ (from [12] we know that $B(n, m)^m = \mathbb{F}$. This imply that group $B(n, m)$ is residually solvable and so residually amenable because solvable groups are amenable (see [4, Example 2.6.5])). Another way for checking MF-property for Baumslag-Solitar groups $\langle a, b | b^{-1}a^n b = a^m \rangle$ is direct following the proof of the main theorem of [16] with using [13] for appearing amalgamated products. This works because all approximations in [16] are norm approximations.

The easiest example of non-residually solvable group is Baumslag group $\langle a, b | a^{ab} = a^2 \rangle$. This group is also MF-group and last paragraph is devoted to proof of this fact. We do not know is this group is residually amenable.

Proposition 11. Let $G_1, G_2, \ldots$ - MF-groups. Then $\bigoplus G_j$ also MF-group.

Proof. If $\alpha_j$ are $(\varepsilon, F_j)$-almost representations of group $G_j$ then $\alpha_1 \oplus \cdots \oplus \alpha_m$ is $(\varepsilon, F_1 \oplus \cdots \oplus F_m)$-almost representation of group $G_1 \oplus \cdots \oplus G_m$. It is easy to see that every finite subset of $\bigoplus G_j$ is consisted in some finite direct subproduct. \qed
Proposition 12. Let $G$ be $MF$-group, $F$ - finite group. Then $G \times F$ is $MF$-group.

Remark that we can prove only this weak permanent fact about cross product. We do not know answer also in the case of $K = \mathbb{Z}$, while in the case of hyperlinear groups it is true that if $G$ is hyperlinear and $F$ is amenable then $G \times F$ is also hyperlinear.

Proof. Let $\alpha_n : G \to U_n$ be faithful asymptotic representation and $\gamma_k(g) = k^{-1}gk$ for $k \in F$, $g \in G$. Put $\beta_n(g) = \oplus_k \alpha_n(k^{-1}gk) \in U_{n|K|}$ for $g \in G$ and let $\beta_n(k) \in U_{n|F|}$ be shift such that $\beta_n(k^{-1})(\oplus y_h)\beta_n(k) = \oplus_k y_{kh}$ for every $y_h \in U_n$, i.e. shift $\beta_n(k)$ move "$h$-block" to "$kh$-block". Now define $\beta_n : G \times F \to U_{n|F|}$ on whole $G \times F$ by the formula $\beta_n^N(g) = \beta_n(g)\beta_n(k)$ for $k \in F$, $g \in G$.

Since $\alpha_n$ is asymptotic representation and $\beta_n(k_1k_2) = \beta_n(k_1)\beta_n(k_2)$ for every $k_1, k_2 \in F$ we have $\beta_n^N(g_1k_1g_2k_2) = \beta_n(g_1\gamma_k^{-1}(g_2)k_1k_2) = \beta_n(g_1\gamma_k^{-1}(g_2))\beta_n(k_1)\beta_n(k_2) = o(1)$.

Let us deduce faithfulness of $\beta_n$. Consider arbitrary nontrivial $\omega \in G \times F$. It has the following form $\omega = gk$ for some $g \in G$, $k \in F$. If $k = 1$ then $||\beta_n(\omega) - 1|| \geq ||\alpha_n(g) - 1|| \geq \sqrt{2}$. Consider case $k \neq 1$. As $\beta_n(k)$ is shift we get that unitary matrix $\beta_n(\omega)$ has only zeros on diagonal. It follows that $||\beta_n(\omega) - 1|| \geq 1$, i.e. $\beta_n$ is faithful asymptotic representation and so $G \times F$ is $MF$-group.

Proposition 13. Let $G_j$ be $MF$-groups. Then $G = \lim \implies G_j$ is also $MF$-group.

We have only $C^*$-algebraic proof.

Proof. We have embeddings $\beta_j : G_j \hookrightarrow U(A_j)$ for some $MF$-algebras $A_j$. Moreover by Proposition 7 we may assume $||\beta_j(g) - 1|| \geq 1$ for every nontrivial $g \in G_j$. Let homomorphisms $\alpha_j^k : G_j \to G_{j+k}$ determine our direct limit. Define $\gamma_j : G_j \to \coprod A_i/ \oplus A_i$ by formula

$$\gamma_j(g) = (*) \cdots (*) \beta_j, \beta_{j+1}(\alpha_j^1(g)), \beta_{j+2}^2(\alpha_j^2(g)), \ldots$$

where values of $*$ are not important. Since $\gamma_j+k \circ \alpha_j^k = \gamma_j$ we have that homomorphisms $\gamma_j$ define homomorphism $\gamma : \lim \implies G_j \to \coprod A_i/ \oplus A_i$. Consider arbitrary nontrivial $g \in G$. For some $j$ we have $g \in ImG_j$ where $ImG_j$ - image of $G_j$ under natural map $G_j \to G$. It is easy to see that $||\gamma(g) - 1|| = \lim sup_k ||\beta_j+k(\alpha_j^k(g)) - 1|| \geq 1$ since $g$ is nontrivial. So $\gamma$ is injective. Let $A_G$ be $C^*$-algebra generating by $\gamma(G)$. It is separable and $A_G \hookrightarrow \coprod A_i/ \oplus A_i$ so it is $MF$-algebra (see [4]). So $G$ is $MF$-group.

Proposition 14. Let $G, H$ be $MF$-groups. Then $G \star H$ is also $MF$-group.

Proof. Consider injective homomorphisms $\alpha_G : g \hookrightarrow U(A_G)$ and $\alpha_H : H \hookrightarrow U(A_H)$ for some $MF$-algebras $A_G$ and $A_H$. We have $G \star H \xrightarrow{\alpha_G \alpha_H} A_G \star_0 A_H \xrightarrow{\gamma} A_G \star A_H$ where $A \star B$ - unital free product of $C^*$-algebras $A$ and $B$, $A \star_0 B$ - unital free algebraic product (i.e. product without completion) of $C^*$-algebras $A$ and $B$, homomorphism $\alpha_G \star \alpha_H$ is defined via formulas $\alpha_G \star \alpha_H(g) = \alpha_G(g)$ for $g \in G$ and $\alpha_G \star \alpha_H(h) = \alpha_H(h)$ for $h \in H$. Injectivity of $\alpha_G \star \alpha_H$ is obvious, injectivity of $\gamma$ follows from [2]. We know from [13] that unital free product of $MF$-algebras is $MF$-algebra, so $A_G \star A_H$ is $MF$-algebra and $G \star H$ is $MF$-group.
It is known that if $G, H$ is hyperlinear groups and $K$ is amenable then $G \star_K H$ is hyperlinear group. But we do not know it is true that $G \star_K H$ is MF-group when $G, H$ is MF-groups and $K$ is finite group.

**Proposition 15.** Let $\varphi_t : G \rightarrow U(B(\mathbb{H}))$ - pointwise continuous family of homomorphisms (i.e. for every $g \in G$ function $\varphi_t(g)$ is continuous) where $t \in [0, \infty)$ and $B(\mathbb{H})$ - algebra of bounded operators on some Hilbert space. Let $C \subset B(\mathbb{H})$ be some quasidiagonal algebra and $g \in G$ is hyperlinear group. But we do not know is it true that $G \star H$ is MF-group when $G, H$ is MF-groups and $K$ is finite group.

**Proof.** Let $Q$ be countable dense subset of $[0, \infty)$ and $A$ be separable $C^*$-algebra generated by $\{\varphi_q(g)\}_{q \in Q, g \in G}$. Since $\varphi_t$ is pointwise continuous then $\varphi_t(g) \in A$ for every $t \in [0, \infty)$. Consider $C^*$-algebra $\Omega = \{f \in C_b([0, \infty), A) : f(\infty) \in C\}$ of continuous bounded $A$-valued function which tend to some element of $C$ at infinity. Remark that $C \subset A$. Define homomorphism $\varphi : G \rightarrow U(\Omega)$ via formula $\varphi(g)(t) = \varphi_t(g)$. This homomorphism is injective since $\varphi_0$ is injective. It is easy to see that $\Omega$ is homotopic to $C$, the construction of homotopy equivalence is following: $\alpha : C \rightarrow \Omega$ via $\alpha(c)(t) = c$ and $\beta : \Omega \rightarrow C$ via $\beta(f) = f(\infty)$. Obviously $\alpha \circ \beta \simeq \text{id}_A$ and $\beta \circ \alpha \simeq \text{id}_C$. Since quasidiagonality is homotopy invariant (see [6], Theorem 7.3.6), algebra $\Omega$ is quasidiagonal, so algebra which is generated by $\varphi(G)$ is also quasidiagonal as subalgebra of quasidiagonal $\Omega$. Since separable quasidiagonal algebras are MF-algebras, we deduce that $G$ is MF-group. 

\[ \square \]

3. **Baumslag group**

In this section we prove that Baumslag group is MF-group. Its hyperlinearity follows from [10] (see also [4],[15]), where it is also proved that soficity is closed under extension by amenable groups (it is well known that sofic groups are hyperlinear).

We will use following notation:

- $x^y = y^{-1} xy = \text{Ad}_y x$
- $B = \langle a, b | a^b = a^2 \rangle$ - Baumslag group
- $H = \langle a, b | a^b = a^2 \rangle$
- $H_j = \langle a_{-j}, ..., a_j | a_{i+1}^i = a_i^2, i = -j, ..., j - 1 \rangle \cong \mathbb{Z} \star ... \star \mathbb{Z}$ where multiplication factors of amalgamated product are numbered from $-j$ to $j - 1$, $i$-th factor generated by $a_i$ and $a_{i+1}$ and the generator of common subgroup $\mathbb{Z}$ of $i$-th and $(i + 1)$-th factors is $a_{i+1}$.
- $H_\infty = \langle ...a_{-j}, ..., a_j, ... | a_i^{a_i+a} = a_i^2 \rangle = \lim H_j$
- $D_n = \text{diag}\{1, e^{2\pi i n}, ..., e^{2\pi i (n-1)}\}$
- $T_n$ - standard shift matrix in $M_n(\mathbb{C})$ i.e. $T_n = e_{1,2} + e_{2,3} + ... + e_{(n-1),n} + e_{n,1}$ where $e_{i,j}$ - standard matrix units.
- $U_n = U(M_n(\mathbb{C}))$
- $\max(K)$ for finite subset $K \subset H_\infty$ is minimal $j$ such that $K \subset H_j$. It easy to see that $\max(K)$ also equals to maximal absolute values of such $j$ for which letter $a_j$ nonreducible appears in words in $K$.
- $x \sim y$ if $x$ and $y$ are unitary conjugate.
- $x =_\varepsilon y$ if $\|x - y\| < \varepsilon$. 

\[ \]
We will write $x \sim_\varepsilon y$ for unitary $x, y \in U_n$ if there is unitary $y' \in U_n$ such that $\|y - y'\| < \varepsilon$ and $x \sim y'$. It means that after small perturbation $y$ become unitary conjugate to $x$.

**Proposition 16.** There is isomorphism $B \cong H_\infty \rtimes \mathbb{Z}$ where action of generator of $\mathbb{Z}$ on $H_\infty$ is defined via formula $a_i \mapsto a_{i+1}$.

**Proof.** The proof is easy exercise. □

**Lemma 17.** Consider automorphism $\varphi$ of group $G$, homomorphism $\alpha : G \rtimes \mathbb{Z} \rightarrow F$ where $F$ - some group (possibly non-countable). Let for every nonzero $k \in \mathbb{Z}$ automorphism $\varphi^k$ be non-inner. We have that if $\alpha|_G$ is faithful then $\alpha$ is also faithful.

**Proof.** Assume that $\alpha$ is not faithful. So we can find $g \in G$ and $z \in \mathbb{Z}$ for which $\alpha(gz) = 1$. If $z = 0$ than this contradict with faithfulness of $\alpha|_G$ (by 0 we denote neutral element of $\mathbb{Z}$). So $z \neq 0$ and $\alpha(z) = \alpha(g^{-1})$. It means that for every $h \in G$ we have $\alpha(\varphi^z(h)) = \alpha(\varphi \circ \ldots \circ \varphi(h)) = \alpha(z^{-1}hz) = \alpha(ghg^{-1})$. Since $\alpha|_G$ is faithful then $\varphi^z(h) = ghg^{-1}$, i.e. automorphism $\varphi^z$ is inner which is contradiction. □

**Proposition 18.** All powers of automorphism $\varphi : H_\infty \rightarrow H_\infty$, $\varphi(a_j) = a_{j+1}$ are non-inner.

**Proof.** Assume that for some nonzero $k \in \mathbb{Z}$ and $\omega \in H_\infty$ we have $\varphi^k(h) = \omega^{-1}h\omega$ for every $h \in H_\infty$. Put $N = \max(\{\omega\})$. So $a_{N+k} = \varphi^k(a_N) = \omega^{-1}a_N\omega \in H_N$ since $\omega \in H_N$. But $a_{N+k} \notin H_N$. Contradiction. □

We need the following theorem from [3]:

**Theorem 19.** Let $A, B$ be residually solvable groups, $D$ be solvable. Consider common subgroup $C \subset A, B$. If there is homomorphism $\beta : A \rtimes B \rightarrow D$ such that $\beta|_C$ is faithful. Then $A \rtimes_B C$ is residually solvable.

As easy corollary we can deduce the following proposition:

**Proposition 20.** For every $j$ group $H_j$ is residually solvable (so it is MF-group).

**Proof.** Let us use induction to prove this fact.

Base case: $H$ is solvable and so residually solvable.

Inductive step: Group $H_j$ has $2j + 1$ amalgamated multiplication factors. Put $L_N = \langle a_{-j}, \ldots, a_N | a_i^{a_{i+1}} = a_i^2 \rangle$. Trivially we have $L_{j+1} = H$ and $L_j = H_j$. To prove proposition it is enough to show that if $L_N$ is residually solvable then $L_{N+1} = L_N \rtimes H$ is also residually solvable. By Theorem 19 it is enough to construct homomorphism $\beta : L_{N+1} \rightarrow H$ which is injective on common subgroup $Z$ of $L_N$ and $H$. Define $\beta$ on generators in such way: $\beta(a_i) = 1$ for $i < N$, $\beta(a_N) = a$, $\beta(a_{N+1}) = b$. It is easy that this map extends to homomorphism, which satisfy necessary conditions. □

Remark, that neither $B$ nor $H_\infty$ can be residually solvable because residual solvability is closed under taking extensions by solvable groups (see [11]).

Let $f(N) = 2^{p_N} - 1$ where $p_N$ is $N$-th prime number.

**Proposition 21.** There exists matrix $T$ such that $T^{-1}D_{f(N)}T = D^2_{f(N)}$ and $T \sim 1 \oplus D_{p_N} \oplus \ldots \oplus D_{p_N}$. In other words spectrum of $T$ is set of all $p_N$-th roots of unity with the same multiplicity and additional 1 with multiplicity 1.
Proof. We have \( D_{f(N)} = \text{diag}(e^0, e^{1c}, e^{2c}, \ldots, e^{(f(N) - 1)c}) \), \( D_{f(N)}^2 = \text{diag}(e^0, e^{2c}, e^{4c}, \ldots, e^{2(f(N) - 1)c}) \), where \( c = \frac{2\pi i}{f(N)} \). Since \( f(N) \) is odd number then there exists bijection \( \sigma \) between \((0, 1, 2, \ldots, f(N) - 1)\) and \((0, 2, 4, \ldots, 2(f(N) - 1))\) modulo \( f(N) \). Consider matrix \( T \) of permutation of basis vectors \( e_j \) corresponding to permutation \( \sigma \). It is easy to see that every disjoint \( n \)-cycle of \( \sigma \) corresponds to set \( \sqrt{n}I \) in spectrum of \( T \) (because \( T|_L \) is usual shift where \( L = \text{span}\{T^k e_j\}_k \) for some \( j \in (0, 1, 2, \ldots, f(N) - 1) \) which belongs to our disjoint cycle). To show that spectrum of \( T \) has desired properties let us examine structure of disjoint cycles of \( \sigma \). We have \( 0 \mapsto 0 \) - this trivial orbit corresponds to 1 with multiplicity 1 in spectrum. Let us prove that every nonzero \( x \) has orbit of length \( p_N \). As permutation \( \sigma \) is defined via formula \( x \mapsto 2x \,(\text{mod } f(N)) \) and since \((2^p - 1)x = f(N)x = 0 \,(\text{mod } f(N))\) we have that length of orbit divides \( p_N \). But since \( p_N \) is prime number and orbit is nontrivial we have that length of every nontrivial orbit is \( p_N \).

\[ \square \]

Proposition 22. For every \( \varepsilon > 0 \), every finite set \( K \subset H_\infty \) and every \( j > \max(K) \) there exists natural number \( n \) and map \( \varphi : H_\infty \to U_n \) such that:

1) \( \|\varphi(kh) - \varphi(k)\varphi(h)\| < \varepsilon \) for every \( k, h \in K \).

2) \( \|\varphi(k) - 1\| \geq 1 \) for every nontrivial \( k \in K \).

3) \( \|\text{Ad}_{\varphi(a_{i+1})} \varphi(a_i) - \varphi(a_i)^2\| \leq \varepsilon \) where \( |i| \leq j \).

4) \( \varphi(a_i) \sim \varepsilon \varphi(a_i) \) for every \( i, l \) with \( i, l \leq j + 1 \).

Proof. As \( K \subset H_j \) and there is no occurrence of elements of \( H_\infty \setminus H_{j+1} \) in conditions 1)-4) then we can define \( \varphi \) on \( H_\infty \setminus H_{j+1} \) arbitrary. Since \( H_{j+1} \) is MF-group then we can construct \( \psi : H_{j+1} \to U_m \) which satisfy conditions 1)-3).

Idea is following: we construct asymptotic representation \( \pi_N \) of \( H_j \) such that spectrum of \( \pi_N(a_i) \) would be uniform. Then \( \varphi = \psi \oplus \pi_N \oplus \ldots \oplus \pi_N \) has desired properties, because \( \psi \)-summand secures faithfulness conditions 2) and a lot of \( \pi_N \)-summands make spectrum of \( \varphi(a_j) \) to be almost uniform, so guarantees condition 4).

Put \( \pi_N(a_{j-1}) = D_{f(N)} \). On other generators generators of \( H_{j+1} \) define \( \pi_N \) by induction. Let us have already construct \( \pi_N(a_i) \) such that \( \pi_N(a_i) \sim D_{f(N)} \). By Proposition 21 we can find matrix \( V \) such that \( V^{-1}\pi_N(a_i)V = \pi_N(a_i)^2 \) and spectrum of \( V \) consists of 1 with multiplicity 1 and \( p_N \)-th root of unity with multiplicity \( \frac{2p_N - 2}{p_N} \) (the reason of using \( f(N) \) instead of usual \( N \) is that spectrum of \( V \) is very simple. We do not know for odd \( N \) good characterization of spectrum of matrix \( R \) such that \( R^{-1}DNR = D_R^2 \)). It is easy to see that \( D_{f(N)} \sim \frac{2\pi i}{p_N} V \) (because we can eigenvalue \( e^{\frac{2\pi i}{p_N}} \) with multiplicity \( \frac{2p_N - 2}{p_N} \) uniformly spread on interval \((e^{\frac{2\pi i}{p_N}}, e^{\frac{2\pi i(k+1)}{p_N}}) \subset S^1 \). Then we should shift all eigenvalues to clear "space" for eigenvalue 1 with multiplicity 1). So we can make matrix \( S \) with properties \( S = \frac{2\pi i}{p_N} V \) and \( S \sim D_{f(N)} \sim \pi_N(a_i) \sim \ldots \sim \pi_N(a_{j-1}) \). Put \( \pi_N(a_{i+1}) = S \). It is clear that \( \text{Ad}_{\pi_N(a_{i+1})} \pi_N(a_i) = O(1) \pi_N(a_i)^2 \) so \( \pi_N \) is asymptotic representation and we can find \( N \) large enough that almost representation \( \pi_N \) satisfy conditions 1)-3).

Put \( \varphi = \psi \oplus \pi_N \oplus \ldots \oplus \pi_N \) where in direct sum there are \( m \) direct \( \pi_N \)-summands, \( N \) as in the previous paragraph, \( m \) is dimension of \( \psi \). Obviously \( \phi \) satisfy conditions 1)-3). It is easy to see that \( e^{\lambda} \oplus D_\lambda \sim \frac{1}{\lambda} D_{n+1} \) for every \( \lambda \). Consider arbitrary generators \( a_i \) and \( a_{i+1} \) of \( H_{j+1} \). Every normal matrix can be diagonalized so \( \psi(a_i) \sim e^{i\lambda_i} + \ldots + e^{i\lambda_m} \) and \( \phi(a_i) \sim e^{i\lambda_1} + \ldots + e^{i\lambda_m} \pi_N(a_i) + \ldots + \pi_N(a_i) \sim (e^{i\lambda_1} + \pi_N(a_i)) \oplus \ldots \oplus \pi_N(a_i) \).
\[ \cdots (e^{i\lambda_m} \oplus \pi_N(a_i)) \sim (e^{i\lambda_1} \oplus D_{f(N)}) \oplus \cdots (e^{i\lambda_m} \oplus D_{f(N)}) \sim \frac{1}{f(N)} \cdot D_{f(N)+1} \oplus \cdots \oplus D_{f(N)+1}. \]

Similarly we can deduce \( \phi(a_i) \sim \frac{1}{f(N)} \cdot D_{f(N)+1} \oplus \cdots \oplus D_{f(N)+1} \) and so \( \phi(a_i) \sim \epsilon \phi(a_i). \)

**Proposition 23.** Let \( u \in U_n \) and \( \epsilon > 0 \). Then there exist matrices \( u_{-k}, u_{-k+1}, \ldots, u_k \in U_n \) where \( k = \left[ \frac{1}{\epsilon} \right] \) with properties:

1) \( u_0 = 1 \).
2) \( u_i = 4\epsilon u_{i+1} \) for every \( i \).
3) \( u = u_{-k}u_n^{-1} \).

**Proof.** Put \( u_i = 1 \) for \( i \leq 0 \) Minimal length of path between 1 and \( u^{-1} \) in the unitary group is not greater than 4 (because all unitary matrices can be diagonalized and so geodesic distance between 1 and \( u^{-1} \) is not greater than geodesic diameter of unit circle which is equal to \( \pi \)). So we can find matrices \( u_0, u_1, \ldots, u_k \) with \( u_0 = 1, u_k = u^{-1} \) and \( u_i = 4\epsilon u_{i+1} \). \( \square \)

**Theorem 24.** Group \( B = \langle a, b | a^ab^b = a^2 \rangle \) is MF-group.

**Proof.** By Lemma 17 it is enough to construct asymptotic representation of \( B \) which is faithful on \( H_\infty \subset B \). Due to Proposition 8 it is enough to find for every \( \epsilon > 0 \) and every finite \( K \subset H_\infty \) matrices \( A \) and \( B \) with properties:

1) \( \| \omega(\{B^{-i}AB^i\}) - 1 \| \geq 1 \) for every nontrivial \( \omega \in K \)
2) \( \text{Ad}_{B^{-1}AB} A = \text{O}(\epsilon) A^2 \)

Put \( k_0 = \max(K), N = \left[ \frac{1}{\epsilon} \right] \). Let \( j \) be natural number such that \( 2j+1 = (2k_0+1)(2N+1) \). Consider almost representation \( \varphi : H_\infty \to U_n \) from the Proposition 22. Due to condition 4) of Proposition 22 there exists unitary \( u \in U_n \) such that \( \text{Ad}_u \varphi(a_{-j}) = \varphi(a_{j+1}) \).

Applying Proposition 23 to \( u \) we get matrices \( u_{-N}, u_{-N+1}, \ldots, u_N \). Let us construct matrices \( u_{-j}, u_{-j+1}, \ldots, u_j \) in the following way: \( u_{-j} = u_{-j+1} = \cdots = u_{j+j+2k_0} = u_{-N}, u_{j+j+2k_0+1} = \cdots = u_{j+j+4k_0+1} = u_{-N+1}, \ldots, u_{j-j-2k_0} = \cdots = u_j = u_N \). More precisely \( u_i = u \frac{1}{N_{k_{i+1}-1}} \). Let us put

\[
A = \text{Ad}_{u_{-j}} \varphi(a_{-j}) \oplus \cdots \oplus \text{Ad}_{u_j} \varphi(a_j)
\]

\[
B = \text{id} \otimes T_{2j+1}^a
\]

where \( A, B \in U(M_n \otimes M_{2j+1}) = U(M_n(2j+1)) \), \( B \) is shift matrix which permute blocks in block structure of matrix \( A \), i.e.

\[
B^{-1}AB = \text{Ad}_{u_{-j+1}} \varphi(a_{-j+1}) \oplus \cdots \oplus \text{Ad}_{u_j} \varphi(a_j) \oplus \text{Ad}_{u_{-j}} \varphi(a_{-j})
\]

Let \( P_i \) be projections corresponding to block structure of \( A \) i.e. for which \( P_i P_j = 0 \oplus \cdots \oplus 0 \oplus \text{Ad}_{u_i} \varphi(a_i) \oplus 0 \oplus \cdots \oplus 0 \). For notation convenience we may think that \( P_i CP_i \in M_n \) for every matrix \( C \in M_{n(2j+1)} \) and in these terms \( P_i P_j = \text{Ad}_{u_i} \varphi(a_i) \).

Let us check property \( \text{Ad}_{B^{-1}AB} A = \text{O}(\epsilon) A^2 \). Put \( R = \text{Ad}_{B^{-1}AB} A - A^2 \). If \( i \neq j \) then since \( u_i = 4\epsilon u_{i+1} \) and \( \text{Ad}_{\varphi(a_{i+1})} \varphi(a_i) = \varphi(a_i) \) we have inequality \( \| P_i RP_i \| = 1 \). If \( i = j \) then since \( \text{Ad}_{\varphi(a_{i+1})} \varphi(a_j) = \varphi(a_j) \) we have \( \| P_i RP_i \| = \| \text{Ad}_{u_i^{-1}} \varphi(a_{-j}) \| \leq 3\epsilon \).

Since \( R \) has the same block structure as \( A \) it is easy to see that \( R = \oplus P_i R P_i \) and \( \text{Ad}_{B^{-1}AB} A = \text{O}(\epsilon) A^2 \).

Let us check property \( \| \omega(\{B^{-i}AB^i\}) - 1 \| \geq 1 \) for all nontrivial \( \omega \in K \subset H_{k_0} \subset H_\infty \). Arbitrary \( \omega \) has the following form \( \omega(\{B^{-i}AB^i\}) = B^{-i_1}AB^{i_1}B^{-i_2}AB^{i_2} \cdots B^{-i_n}AB^{i_n} \) with
$|i_0| \leq k_0$. But since $B^{-i}AB^i$ has the same block structure as $A$, $P_0(B^{-i}AB^i)P_0 = \varphi(a_i)$ and $v_{-k_0} = \ldots = v_k = u_0 = 1$ we have that
\[
\|\omega(\{B^{-i}AB^i\}) - 1\| \geq \|P_0\omega(\{B^{-i}AB^i\})P_0 - P_0\| = \|\omega(\{\varphi(a_i)\}) - 1\| \geq 1
\]
This finishes the proof. □

Remark 25. We think that it would be interesting to examine MF-properties for groups of the form $\langle a, b | \omega(a, a^b) = 1 \rangle$ where group $\langle a, b | \omega(a, b) = 1 \rangle$ is not very difficult. If we trying to follow similarly way we see the main difficulties in construction asymptotic representation of $\langle a, b | \omega(a, b) = 1 \rangle$ for which spectrums of generators are almost uniform and checking MF-property for groups of the form $\langle a, b | \omega(a, b) = 1 \rangle \ast \ast \langle a, b | \omega(a, b) = 1 \rangle$.

In our case of Baumslag group the second problem was not consider able due to theorem of Azarov and Tieudjo and nilpotence of group $\langle a, b | b^{-1}ab = a^2 \rangle$.

Remark 26. We use in proof exotic matrix size $2^n - 1$ because of simplicity of spectrum of corresponding shift matrix. But we think it is very interesting to examine uniform properties for unitary matrix $T$ for which $T^{-1}D_2T = D_2$. For example we think that it is important to compute $\lim_{n \to \infty} \# \{z \in \sigma(T) | z \in (a, b) \}$ for segment of circle $(a, b) \subset S^1$.

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