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Conditions for Quantum Interference in Cognitive Sciences

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\textbf{Abstract}

We present a general classification of the conditions under which cognitive science, concerned e.g. with decision making, requires the use of quantum theoretical notions. The analysis is done in the frame of the mathematical approach based on the theory of quantum measurements. We stress that quantum effects in cognition can arise only when decisions are made under uncertainty. Conditions for the appearance of quantum interference in cognitive sciences and the conditions when interference cannot arise are formulated.
1 Introduction

The special issue, devoted to quantum-like effects in cognition, is certainly, of great interest giving hope that such formulations could explain a variety of empirical phenomena that do not find explanations in classical terms. The purpose of the special issue is well described in the introductory article by Wang, Busemeyer, Atmanspacher, and Photos (2013). All papers in this issue appeal, in one way or another, to the existence of quantum interference effects in cognitive phenomena. Thus, the present Comment aims at formulating in precise mathematical terms under what conditions the interference effects can arise and when they are prohibited. The clear understanding of these conditions will help avoiding incorrect interpretations and unnecessary misuse of the quantum formalism.

The idea that human psychological processes could be described by quantum theory was advanced by Bohr (1933, 1958), who assumed that human cognition, involving deliberations between several possible actions, could be modeled as a quantum process. Deliberations between several complementary actions is analogous, Bohr argued, to interference between several quantum states. Since then, possible interference effects in human decision making have been discussed in a number of papers that suggested different models for resolving the so-called paradoxes in classical decision making and suggesting explanations of different cognitive phenomena. The history of this trend and the related models have been recently summarized in the book by Busemeyer and Bruza (2012).

As mentioned above, in the papers of the issue, interference is considered as a major quantum effect required for the explanation of a variety of cognitive phenomena. Even the order effects arising in different polls are also suggested as being connected with the manifestation of quantum features in decision making (Wang and Busemeyer, 2013), similarly to these effects in quantum phenomena.

The typical way of incorporating quantum effects into cognitive phenomena, which is employed by the majority of authors, is as follows. One first considers some particular cognitive empirical observations, then constructs a model for their explanation, postulating the occurrence of interference, and finally fits the model parameters to the same empirical observations.

This approach however is clearly unsatisfactory as a genuine scientific endeavor since the models: (a) lack generality, being mostly ad hoc, and (b) cannot claim predictive power as the same observations that motivated the models are used to qualify them. A real explanation of cognitive processes must be based on the following three necessary requirements:

(i) There should exist a general theory developed for arbitrary situations. This theory has to be applicable to any cognitive phenomenon, instead of inventing particular models for each empirical case.

(ii) The theory has to clearly prescribe the conditions when quantum effects, such as interference, appear and when they cannot arise. This prevents that interferences are postulated ad hoc.

(iii) The theory has to be able to make quantitative predictions without fitting parameters. In other words, the process of validation must be developed, which consists in making novel predictions to empirical situations that have not been previously used to motivate the theory. Fitting a model to empirical observations can serve only as a first trivial step in the whole
chain of model validation (Sornette et al., 2007, 2008).
In the next Sec. 2, we give a brief account of such a general theory. In Sec. 3, to avoid confusion, we specify the terminology and notation employed in defining quantum events. Section 4 emphasizes the conditions when interference can arise and when it cannot. Particular models, which are suggested for interpreting cognitive phenomena, are to be tested against these conditions. Section 5 then considers the Gallup polls treated in the contribution by Wang and Busemeyer (2013) in the special issue, as an example illustrating our general classification.

2 Decision making as a measurement procedure

Formulating the theory of quantum measurements, von Neumann (1955) noticed that a measurement procedure can be interpreted as a logical proposition satisfying the rules of quantum logic (Birkhoff and von Neumann, 1936). In quantum measurements, the observable measurable quantities are defined as the averages of self-adjoint operators from the algebra of local observables. More precisely, if an observable is represented by an operator $\hat{A}$ from the algebra of local observables, defined on a complex Hilbert space $\mathcal{H}$, then the observable quantity that can be measured is given by $\text{Tr}\hat{\rho}\hat{A}$, where the trace is over $\mathcal{H}$ and $\hat{\rho}$ is a positive trace-one operator on $\mathcal{H}$, termed statistical operator, or system state, characterizing the considered system.

The rigorous definition of quantum probabilities can be done in line with the theory of quantum measurements. According to the Gleason (1957) theorem, a quantum probability measure has a unique extension to a positive linear functional defined for any bounded linear operator on a Hilbert space having a dimension larger than two, so that, if $A$ is a proposition, then the related quantum probability is uniquely given by $\text{Tr}\hat{\rho}\hat{A}$. Quantum probabilities, under different axiomatics, have been studied in a number of publications (Pitowsky, 1989). The achievable accuracy of quantum measurements has been analyzed by Dyakonov (2012). Psychological problems in the interpretation of quantum measurements have been discussed by Mermin (2012).

The most general and mathematically rigorous way of developing decision theory with quantum probabilities is not by constructing some particular models, but by following the general theory of quantum measurements. Such a quantum decision theory has been recently advanced (Yukalov and Sornette, 2008) and developed in a series of papers (Yukalov and Sornette, 2009a, 2009b, 2010a, 2010b, 2011). This theory uniquely defines the conditions, when the interference effects can arise and when they do not occur. To explain this, we need to give a brief sketch of the approach, omitting mathematical details that can be found in the cited papers.

Consider a given set of prospects $\pi_j$, forming a complete lattice

$$ \mathcal{L} = \{\pi_j : j = 1, 2, \ldots, L\} , $$

where the prospects are ordered by means of their probabilities, to be given below. Each of the prospects is represented in a Hilbert space $\mathcal{H}$ as a vector $|\pi_j\rangle$. The prospect operators on $\mathcal{H}$, defined as

$$ \hat{P}(\pi_j) \equiv |\pi_j\rangle\langle\pi_j| , \quad (1) $$

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play the role of the operators of observables. Note that the prospect operators are not required to be either orthogonal or to be projectors. A decision maker is characterized by a decision-maker state $\hat{\rho}$ on $\mathcal{H}$. The observable quantities, given by the averages

$$p(\pi_j) \equiv \text{Tr}\hat{\rho}\hat{P}(\pi_j),$$

define the prospect probabilities. Note that the state $\hat{\rho}$ can be either pure, if the decision maker is isolated, or mixed, when the decision maker interacts with a surrounding society (Yukalov and Sornette, 2012).

Accomplishing the trace operation in definition (2), and separating the diagonal and non-diagonal terms, results in the expression

$$p(\pi_j) = f(\pi_j) + q(\pi_j),$$

where the first term corresponds to the diagonal representation, while the second term gives the off-diagonal contribution. The latter term is caused by quantum interference, so that we refer to it as the interference factor or coherence factor.

According to the generalized correspondence principle, first advanced by Bohr (1913) when analyzing atomic spectra, quantum theory must reduce to classical theory in the limit where quantum effects become negligible. In the more general version, the quantum-classical correspondence principle is understood as the requirement that the results of quantum measurements would be reducible to those of classical measurements when the quantum effects, such as interference, vanish. This reduction is called decoherence (Wheeler and Zurek, 1983; Zurek, 2003).

By the quantum-classical correspondence principle, the quantum probability, which is a measurable quantity, has to reduce to the classical probability, when the interference (coherence) factor tends to zero:

$$p(\pi_j) \to f(\pi_j), \quad q(\pi_j) \to 0.$$

Therefore, the diagonal term $f(\pi_j)$ corresponds to the classical probability. The latter can be understood as a frequentist probability associated with the utility factor representing the objective utility of the prospect.

The interference term $q(\pi_j)$ is caused by the quantum nature of probability (2). In decision theory, this term represents the subjective inclinations of a decision maker caused by his/her emotions as well as possible behavioral biases, because of which it can also be called the attraction factor.

It is clear that subjective feelings can vary in a wide range, being rather different for different decision makers. Even for the same decision maker, under the same conditions, with the same information, but at different times, subjective feelings often change (Anderson et al., 1992; Anand et al., 2010). This, it seems, would make it impossible to quantify the attraction factor. However, it has been shown (Yukalov and Sornette, 2009a, 2009b, 2010b, 2011) that the attraction factor, arising in the definition of quantum probability (2), enjoys the following general properties. It varies in the interval

$$-1 \leq q(\pi_j) \leq 1.$$
The following *alternation condition* holds:

\[ \sum_{j=1}^{L} q(\pi_j) = 0. \]  

(6)

And the average absolute value of the *aggregate* attraction factor can be estimated by the *quarter law*:

\[ \frac{1}{L} \sum_{j=1}^{L} |q(\pi_j)| = \frac{1}{4}. \]  

(7)

provided not all terms are zero.

As has been shown in our papers (Yukalov and Sornette, 2009a, 2009b, 2011), the quarter law holds true in a variety of experimental observations with a very good accuracy. However it is important to emphasize that this quarter law is not merely an empirical fact, but it can be theoretically derived as a *non-informative prior* (Yukalov and Sornette, 2009b, 2011). Because of the importance of the quarter law for quantitative prediction, we briefly sketch its derivation. Recall that the attraction factor \( q \) is, generally, different for different decision makers, being in that sense a random quantity. Let \( \varphi(q) \) be a distribution of the attraction factor over the manifold of attraction factors corresponding to a large set of decision makers and/or of their mental states. According to the factor range (5), the distribution \( \varphi(q) \) is to be normalized over the domain \([-1, 1]\) and has to satisfy the alternation condition (6), so that

\[ \int_{-1}^{1} \varphi(q) \, dq = 1, \quad \int_{-1}^{1} \varphi(q)q \, dq = 0. \]

Then the values

\[ q_+ \equiv \int_{0}^{1} \varphi(q)q \, dq, \quad q_- \equiv \int_{-1}^{0} \varphi(q)q \, dq, \]

in view of the alternation condition, are related as \( q_+ + q_- = 0 \). The absence of any a priori information implies that the distribution \( \varphi(q) \) is uniform. Then the normalization condition yields \( \varphi(q) = 1/2 \). As a result, from the above definition, we have \( q_+ = 1/4 \) and \( q_- = -1/4 \). Thus, the non-informative prior for the absolute value of the aggregate attraction factor is \( 1/4 \).

These properties make it straightforward to give quantitative predictions, without any fitting parameters, for the behavior of decision makers. For instance, it is easy to give quantitative explanations for classical paradoxes in decision making, such as the disjunction effect or the conjunction fallacy (Yukalov and Sornette, 2009b, 2011).

### 3 On terminology and notation for quantum events

Defining quantum events or quantum measurements, one has to be accurate with terminology and notation that may differ in the works of different authors. To avoid misunderstanding, it is necessary to provide some precisions on our terminology and notations. We shall be very brief, not going into mathematical details that can be found in the cited papers.
In quantum theory, an event $A$ is represented by a self-adjoint operator $\hat{A}$. For defining the probability of the event, one usually interprets the latter operator as a projector. There are then two ways of introducing quantum probabilities.

One way is to consider all operators, representing events, being defined on the same Hilbert space $\mathcal{H}$, with the system state $\hat{\rho}$. Then the probability of each event $A$ is $p(A) = \text{Tr}\hat{\rho}\hat{A}$, with the trace over $\mathcal{H}$. For two disjoint events, say $A$ and $B$, their union $A \cup B$ is represented by the sum of the orthogonal operators $\hat{A} + \hat{B}$ acting on $\mathcal{H}$. The probability of this union $p(A \cup B)$ is the sum $p(A) + p(B)$.

One calls the events disjoint, or mutually exclusive, or orthogonal, such that $A \mathcal{H} B = 0$, when the related operators are mutually orthogonal projectors. In classical probability theory, the disjoint events are often termed as incompatible. The latter term, however, has a different meaning in quantum theory, where one calls incompatible, or incommensurable, such events whose related operators do not commute. And events are named compatible, or commensurable, when their operators commute. Therefore, to avoid confusion, we shall use the term disjoint for the orthogonal events, such that $A \cap B = 0$, while we shall call the events compatible when $[\hat{A}, \hat{B}] = 0$. Respectively, two events are incompatible, if the related operator commutator $[\hat{A}, \hat{B}]$ is not zero.

In the procedure of quantum measurements of two incompatible observables, say $A$ and $B$, the order of their measurement is important, so that measuring $A \cap B$, generally, is not the same as measuring $B \cap A$. In order to avoid ambiguity, it is necessary to emphasize that, according to the standard notation accepted in quantum theory, the order of operations is understood as being accomplished from right to left. That is, when one writes the operations as $\hat{A}\hat{B}$, one assumes that $B$ acts first, while $\hat{A}$ acts second. This order of actions is natural for quantum theory where actions are represented by operators acting on functions from a Hilbert space. Thus, the product of operators $\hat{A}\hat{B}$, acting on a function $\psi$, is uniquely defined as $\hat{A}\cdot\hat{B}\psi \equiv \hat{A}(\hat{B}\psi)$. We always use this standard definition.

There exists the well known problem in defining the joint probability of two incompatible events. In this respect, it has been shown (Niestegge, 2008) that the quantum joint probability $p(A \cap B)$ can be correctly defined either for compatible events, for which $\hat{A}$ and $\hat{B}$ commute, or for incompatible events by assigning the operators to the Jordan algebra, where the operator product $\hat{A}\cdot\hat{B}$ is given by the Jordan symmetric form $(\hat{A}\cdot\hat{B} + \hat{B}\cdot\hat{A})/2$. However, such a symmetric definition of the joint probability possesses the properties of the classical probability yielding no interference terms.

When an event $A$ occurs after an event $B$, then, instead of the joint probability, one considers the transition probability defined as the product of the Lüders projection-rule probability of successive measurements $p_L(A|B)$ and the probability $p(B)$. This transition probability was first introduced by Wigner (1932), so that one can denote it as $p_W(A|B) = p_L(A|B)p(B)$. It is this Wigner transition probability that is used by Wang and Busemeyer (2013). The Lüders probability is often interpreted as an extension to the quantum region of the classical conditional probability. However, when the events $A$ and $B$ are represented by one-dimensional orthogonal projectors, the Lüders form $p_L(A|B)$ is symmetric, such that $p_L(A|B) = p_L(B|A)$. Contrary to this, the classical conditional probabilities are not symmetric. Therefore, the Lüders form cannot be treated as an extension of the classical conditional probability. As a consequence, the Wigner transition probability cannot be accepted as an extension of the
joint probability.

There is another way of introducing quantum probabilities for several events, which is free of the problem of defining the joint probabilities. In this approach, each event \( A \) is represented by a proposition operator \( \hat{A} \) acting on a Hilbert space \( \mathcal{H}_A \). Respectively, another event \( B \) is represented by an operator \( \hat{B} \) acting on a Hilbert space \( \mathcal{H}_B \). The system state is defined by a statistical operator \( \hat{\rho}_{AB} \) acting on the tensor-product space \( \mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B \). The probability of a separate event \( p(A) \equiv \text{Tr}\hat{\rho}_{AB}\hat{A} \), with the trace over the tensor product \( \mathcal{H}_{AB} \), reduces to the previous definition above.

The sequence of two events \( A \) and \( B \) is called a composite event and is denoted by \( A \otimes B \) or simply as \( AB \), and is represented by the tensor-product operator \( \hat{A} \otimes \hat{B} \) acting on \( \mathcal{H}_{AB} \).

By this definition, the joint probability of these two events is \( p(AB) = \text{Tr}\hat{\rho}_{AB}\hat{A} \otimes \hat{B} \), with the trace over the tensor-product space \( \mathcal{H}_{AB} \). It is this definition of composite events that we have used from the very beginning in our approach to quantum decision theory (Yukalov and Sornette, 2008, 2009a, 2009b, 2010a, 2010b, 2011). Further mathematical information on the properties of composite events can be found in the literature (Holevo, 1973; Wilce, 1992; Niestegge, 2004; Harding, 2009).

In the majority of cases studied in experiments performed in cognitive sciences, one considers the situation where the prospect lattice is formed by just two prospects, \( \pi_1 = A_1B \) and \( \pi_2 = A_2B \), where \( B \equiv \{B_1, B_2\} \) is represented as a set of two possible actions (or events), \( B_1 \) and \( B_2 \). By the general theory, for this case, the prospect states are given by \( |\pi_j\rangle = |A_j\rangle \otimes |B\rangle \).

The prospect probabilities are

\[
p(A_jB) = p(A_jB_1) + p(A_jB_2) + q(A_jB),
\]

where \( j = 1, 2 \). The first two terms, whose sum is \( f(A_jB) = p(A_jB_1) + p(A_jB_2) \), are the classical joint probabilities that are uniquely defined through the conditional probabilities:

\[
p(A_i|B_j)p(B_j).
\]

Hence, the interference term (attraction factor) is

\[
q(A_jB) = p(A_jB) - p(A_jB_1) - p(A_jB_2).
\]

The alternation condition (6) now reads as \( q(A_1B) = -q(A_2B) \), and the quarter law (7) becomes \( |q(A_1B)| + |q(A_2B)| = 1/2 \).

Interference effects exist only when the interference factor (10) is not zero. Only then, there is the necessity of invoking quantum notions for cognitive phenomena. When this factor is zero, there is no interference effects and all cognitive phenomena can be described with a classical formalism.

### 4 Interference appears in decisions under uncertainty

As has been mentioned, postulating the existence of interference effects may lead to incorrect conclusions, since interference does not necessarily occur for any prospect. Let us now clarify this and formulate the conditions under which interference cannot arise and those when it
can. These conditions follow from the general theory delineated above. The mathematical details can be found in our earlier publications (Yukalov and Sornette, 2008, 2009a, 2009b, 2010b, 2011).

Prospects are composed of elements that, depending on applications, may be called events (in probability theory), propositions (in logic), measurements (in natural sciences), or decisions and actions (in decision theory). Keeping this in mind, we use the term “event”.

(i) Simple events.
A simple event $A_i$ is such that it cannot be represented as a union or conjunction of several events. There is no interference for a simple event:

$$q(A_i) = 0 .$$

(ii) Unions of mutually disjoint simple events.
Let $\bigcup_i A_i$ be a union of mutually disjoint simple events $A_i$ all represented by orthogonal projectors on the same space $\mathcal{H}$. There is no interference for this union:

$$q\left( \bigcup_i A_i \right) = 0 .$$

(iii) Factorized composite events.
An event is composite if it can be represented as a conjunction of several events as defined above. A composite event is factorized if it is a conjunction $\bigotimes_i A_i$ of simple events $A_i$, whose representing operators $\hat{A}_i$ are defined each on its own space $\mathcal{H}_i$. There is no interference for these factorized composite events:

$$q\left( \bigotimes_i A_i \right) = 0 .$$

(iv) Entangled composite events.
A composite event is entangled if it cannot be represented as a factorized event. An example of an entangled event is $A_iB$, where $B = \{B_1, B_2\}$, so that the prospect state is $|A_i\rangle \bigotimes |B\rangle$. Interference can occur only for entangled events. For illustration, let us recall the double-slit experiment in physics. A particle is emitted in the direction of a screen having two slits. From another side of the screen, there are detectors registering the arrival of the particle. Let the registration of the particle by an $i$-detector be denoted as $A_i$ and the passage of the particle through one of the slits be denoted as $B_1$ or $B_2$, respectively. When the passage of the particle through a slit $B_j$ is certain, then the event $A_iB_j$ is factorized and displays no interference, with the event probability given by $p(A_iB_j)$ and the vanishing factor $q(A_iB_j)$. However, when it is not known through which of the slits the particle passes, then the event $A_iB$, is entangled and demonstrates interference, that is, a nonzero factor $q(A_iB)$, giving rise to the remarkable quantum effect of interference fringes, or periodic modulations of the probability of finding the particle along the dimension of the screen.
5 Is there anything quantum in Gallup polls?

As an application of the above analysis, let us consider the Gallup polls treated in the contribution by Wang and Busemeyer (2013). In accordance with the considerations we have presented above, this contribution is the most interesting, since it attempts to give an explanation for the necessity of involving quantum notions in order to interpret empirical tests, such as Gallup polls, without introducing fitting parameters.

The Gallup polls, demonstrating order effects, have been described by Moore (2002). In the polls, conducted in September 6, 1997, the respondents were asked to answer the questions of the following type: "Do you generally think Clinton (Gore) is honest and trustworthy?" The questions were asked in two separate contexts: non-comparative and comparative. The non-comparative context for a question occurs when that question is asked first, without any mention of the other item. The comparative context for a question occurs when that question is asked after the other question in the pair, so that the second question may be influenced by the response to the first question. The Gallup polls, as described by Moore, have demonstrated noticeable order effects. Thus, when respondents were asked about Clinton first, 50% said he was honest and trustworthy. When the other group of the sample was asked about Gore first, 68% said he was honest and trustworthy. However, in the comparative context, when the question about Clinton was asked second, 57% said he was honest. And when the question about Gore was asked second, 60% said he was honest. This demonstrated the questions-order effect, or more precisely, the difference between the comparative and non-comparative contexts.

At this point, it is worth noting that these numbers, to our mind, should not be treated as having absolute value. They are actually to a large extent rather random. For instance, the responses to the same questions about Clinton and Gore, given at very close times, fluctuated around 50% for both of them, since, generally, people did not see much difference between these two democrats furthermore associated in the US Administration as president and vice-president respectively (Gallup poll, 1997). For example, in the Gallup poll of October 3, 1997, Clinton was considered as honest by 56% of respondents and Gore, by 47%, which shows a relation opposite to the previous September poll, when respondents classified Gore as being more honest than Clinton. And in the poll of October 30, 1997, Clinton was considered honest by 62% of respondents, while Gore, by 53%, again with a relation opposite to the September poll. This shows that these polls exhibit very strong fluctuations with time. While, at each fixed poll, the order effects were found to be statistically significant (Moore, 2002), this does not contradict the fact that a poll as a whole can be random, displaying different data at close times, when the external conditions are practically the same.

The typical structure of the polls is as follows. Respondents first answer a question $B$, choosing either $B_1$ or $B_2$, and after this, answer a question $A$, choosing either $A_1$ or $A_2$. Suppose a population of $N$ respondents is interrogated. Answering the question $B$, a part $N(B_1)$ answers $B_1$, with $N(B_1) + N(B_2) = N$. The corresponding fractions give the unconditional frequentist probabilities $p(B_j) = N(B_j)/N$. Then each part $N(B_j)$ of the population is questioned on $A$, which separates the population into the subpopulations $N(A_i|B_j)$ of those who answer $A_i$ under the condition that before they have answered $B_j$. Clearly, $N(A_1|B_j) + N(A_2|B_j) = N(B_j)$. The related fractions define the classical conditional prob-
abilities $p(A_i|B_j) = N(A_i|B_j)/N(B_j)$, satisfying the standard normalization for classical conditional probabilities $p(A_1|B_j) + p(A_2|B_j) = 1$. Classical joint probabilities can be defined as $p(A_iB_j) = p(A_i|B_j)p(B_j)$. In the same way, the other group of respondents is interrogated, first on $A$ and then on $B$, giving the classical probabilities $p(B_jA_i)$.

The classical joint probabilities are symmetric, so that $p(A_iB_j) = p(B_jA_i)$ should hold. But the Gallup polls have shown that this is not always the case. To explain the order effects, Wang and Busemeyer (2013) suggested to treat $p(A_iB_j)$ as the Wigner transition probability $p_W(A_i|B_j) = p_L(A_i|B_j)p(B_j)$. Their main result, that they call the ”q-test”, is the demonstration that the so interpreted probabilities satisfy the equality

$$p(A_1B_2) + p(A_2B_1) = p(B_1A_2) + p(B_2A_1),$$

(11)

which is in good agreement with the Gallup polls.

However, to satisfy this equality, there is no need to resort to quantum notions. This is because Eq. (11) is also valid for classical probabilities due to their symmetry. Therefore, the validity of this test, as such, does not prove yet the quantum origin of the phenomenon. What is specific for the Gallup polls is not this equation, but the asymmetry of the related frequentist probabilities. That is, the proof would be if it could be possible to calculate somehow quantum probabilities $p_{quantum}(A_iB_j)$ and to show that they coincide with the empirical frequencies $p(A_iB_j)$.

The events $A_i$ or $B_j$, from the mathematical point of view defined above, are not entangled. Consequently, the corresponding probabilities $p(A_i)$ and $p(B_j)$, according to our classification, involve no interference. Thus, we are not convinced of the need to invoke interference effects to explain Gallup polls, due to a lack of genuine uncertainty in the related decision making. While Wang and Busemeyer (2013) mention that the questions posed in the polls can contain some kind of uncertainty, the general theory that we have outlined above specifies rigorously the type of uncertainty that ensures the existence of interference: it is the mathematically defined uncertainty related to the occurrence of entangled composite events. In the absence of such events, there can be no interference terms. The analysis of the order effects by Wang and Busemeyer (2013) is certainly of interest. However, it remains unclear why the Gallup polls would require quantum interpretation.

In conclusion, the rigorous approach, based on the theory of quantum measurements, allows us to find the conditions when quantum effects, such as interference, can appear in cognitive sciences. Possible occurrence of quantum effects in cognitive empirical tests should be checked against these conditions. Generally, quantum effects can arise only when one makes decisions under uncertainty, in the presence of entangled composite events.
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