OPTIMAL CONVERGENCE SPEED OF BERGMAN METRICS ON SYMPLECTIC MANIFOLDS

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Abstract. It is known that a compact symplectic manifold endowed with a prequantum line bundle can be embedded in the projective space generated by the eigensections of low energy of the Bochner Laplacian acting on high \( p \)-tensor powers of the prequantum line bundle. We show that the Fubini-Study metrics induced by these embeddings converge at speed rate \( 1/p^2 \) to the symplectic form.

0. Introduction

A very useful tool in the study of canonical Kähler metrics is the use of Bergman metrics to approximate arbitrary Kähler metrics in a given integral cohomology class, see e.g., \([7, 11, 17]\).

Let \((X, \omega)\) be a compact Kähler manifold endowed with a Hermitian holomorphic line bundle \((L, h^L)\) such that \(\sqrt{\frac{-1}{2\pi}} R^L = \omega\). Since the bundle \(L\) is positive, Kodaira’s theorem shows that high powers \(L^p\) give rise to holomorphic embeddings \(\Phi_p : X \to \mathbb{P}(H^0(X, L^p)^*)\). The Bergman metric \(\omega_p\) at level \(p\) is defined as the rescaled induced Fubini-Study metric \(\frac{1}{p} \Phi_p^* \omega_{\text{FS}}\), where \(\omega_{\text{FS}}\) is the natural Fubini-Study metric on \(\mathbb{P}(H^0(X, L^p)^*)\). Tian \([17]\) showed that \(\omega_p\) converges to \(\omega\) in the \(C^2\) topology with speed rate \(p^{-1/2}\), as \(p \to \infty\), that is, there exists \(C > 0\) such that for any \(p \in \mathbb{N}^*\) we have

\[
\left|\frac{1}{p} \Phi_p^* (\omega_{\text{FS}}) - \omega\right|_{C^2} \leq C \cdot \frac{1}{p^{1/2}}.
\]

This was improved by Ruan \([15]\) to convergence in \(C^\infty\) with speed rate \(p^{-1}\) (see also \([13, \text{Theorem 5.1.4}]\)). Tian’s result was motivated by a problem of Yau \([18]\).

The proof of the convergence in \([15, 17]\) is based on the diagonal expansion of the Bergman kernel up to second order. A full asymptotic expansion in powers of \(p\) in the \(C^\infty\) topology was obtained by Catlin \([5]\) and Zelditch \([19]\) as an application of Boutet de Monvel and Sjöstrand’s work \([4]\), see also \([6, 14]\) for different approaches and generalizations. We refer to \([13]\) for a comprehensive study of several analytic and geometric aspects of Bergman kernel. One advantage of the expansion in the \(C^\infty\) topology is that it easily implies the convergence of the Bergman metrics \(\omega_p\) to \(\omega\) with speed rate \(p^{-2}\), see \([13, (5.1.23)]\). This

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convergence speed is optimal and is attained only when the scalar curvature of $\omega$ is constant. Note that the scalar curvature is up to a multiplicative constant the coefficient of the second term of the Bergman kernel expansion. The purpose of this paper is to extend this optimal result to the case of symplectic manifolds.

The Bergman kernel of a holomorphic line bundle $L$ on a complex manifold is the smooth kernel of the orthogonal projection from the space of square integrable sections on the space of holomorphic sections, or, equivalently, on the kernel of the Kodaira Laplacian $\Box^L = \overline{\partial}^L \overline{\partial}^{L*} + \overline{\partial}^{L*} \overline{\partial}^L$ on $L$. In order to find a suitable notion of “holomorphic section” of a prequantum line bundle on a compact symplectic manifold, Guillemin and Uribe [9] introduced a renormalized Bochner Laplacian $\Delta_{p,0}$ (cf. (0.5)) which reduces to $2\Box^L$ in the Kähler case.

We describe this construction in detail. Let $(X, \omega)$ be a compact symplectic manifold of real dimension $2n$. Let $(L, h^L)$ be a Hermitian line bundle on $X$, and let $\nabla^L$ be a Hermitian connection on $(L, h^L)$ with the curvature $R^L = (\nabla^L)^2$. We will assume throughout the paper that $(L, h^L, \nabla^L)$ is a prequantum line bundle of $(X, \omega)$, i.e.,

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega.$$  

(0.2)

We choose an almost complex structure $J$ such that $\omega$ is $J$-invariant and $\omega(\cdot, J\cdot) > 0$. The almost complex structure $J$ induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of $J$ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Let $g^{TX}(\cdot, \cdot) := w(\cdot, J\cdot)$ be the Riemannian metric on $TX$ induced by $\omega$ and $J$. The Riemannian volume form $dv_X$ of $(X, g^{TX})$ has the form $dv_X = \omega^n/n!$. The $L^2$-Hermitian product on the space $\mathcal{C}^\infty(X, L^p)$ of smooth sections of $L^p$ on $X$, with $L^p := L^{\otimes p}$, is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1, s_2 \rangle(x)dv_X(x).$$  

(0.3)

Let $\nabla^{TX}$ be the Levi-Civita connection on $(X, g^{TX})$ with curvature $R^{TX}$, and let $\nabla^{L^p}$ be the connection on $L^p$ induced by $\nabla^L$. Let $\{e_k\}$ be a local orthonormal frame of $(TX, g^{TX})$. The Bochner Laplacian acting on $\mathcal{C}^\infty(X, L^p)$ is given by

$$\Delta^{L^p} = -\sum_k \left[ (\nabla^{L^p}_{e_k})^2 - \nabla^{L^p}_{\nabla^{TX}_{e_k} e_k} \right].$$  

(0.4)

Given $\Phi \in \mathcal{C}^\infty(X, \mathbb{R})$, the renormalized Bochner Laplacian is defined by

$$\Delta_{p, \Phi} = \Delta^{L^p} - 2\pi np + \Phi.$$  

(0.5)

By [9], [12] Corollary 1.2, there exists $C_L > 0$ independent of $p$ such that

$$\text{Spec}(\Delta_{p, \Phi}) \subset [-C_L, C_L] \cup [4\pi p - C_L, +\infty),$$  

(0.6)

where $\text{Spec}(A)$ denotes the spectrum of the operator $A$. Since $\Delta_{p, \Phi}$ is an elliptic operator on a compact manifold, it has discrete spectrum and its eigensections are smooth. Let $\mathcal{H}_p$ be the direct sum of eigenspaces of $\Delta_{p, \Phi}$ corresponding to the eigenvalues lying in $[-C_L, C_L]$. In mathematical physics terms, the operator $\Delta_{p, \Phi}$ is a semiclassical Schrödinger operator and the space $\mathcal{H}_p$ is the space of its bound states as $p \to \infty$. The space $\mathcal{H}_p$ proves to be an.
appropriate replacement for the space of holomorphic sections $H^0(X, L^p)$ from the Kähler case. In particular, we have for $p$ large enough (cf. [13] (8.3.3)),

$$(0.7) \quad \dim \mathcal{H}_p = \int_X \text{Td}(T^{(1,0)}X)e^{p\omega},$$

where $\text{Td}(T^{(1,0)}X)$ is the Todd class of $T^{(1,0)}X$, which corresponds to the Riemann-Roch-Hirzebruch formula from complex geometry.

Let $\mathbb{P}(\mathcal{H}_p^*)$ be the projective space associated to the dual space of $\mathcal{H}_p$; we identify $\mathbb{P}(\mathcal{H}_p^*)$ with the Grassmannian of hyperplanes in $\mathcal{H}_p$. The base locus of $\mathcal{H}_p$ is the set $\text{Bl}(\mathcal{H}_p) = \{ x \in X : s(x) = 0 \text{ for all } s \in \mathcal{H}_p \}$. We define the Kodaira map

$$(0.8) \quad \Phi_p : X \setminus \text{Bl}(\mathcal{H}_p) \to \mathbb{P}(\mathcal{H}_p^*), \quad \Phi_p(x) = \{ s \in \mathcal{H}_p : s(x) = 0 \},$$

which sends $x \in X \setminus \text{Bl}(\mathcal{H})$ to the hyperplane of sections vanishing at $x$. Note that $\mathcal{H}_p$ is endowed with the induced $L^2$ Hermitian product (0.3) so there is a well-defined Fubini-Study metric $g_{FS}$ on $\mathbb{P}(\mathcal{H}_p^*)$ with the associated form $\omega_{FS}$.

The symplectic Kodaira embedding theorem [14, Theorem 3.6], [13, Theorem 8.3.12], states that for large $p$ the Kodaira maps $\Phi_p : X \to \mathbb{P}(\mathcal{H}_p^*)$ are embeddings and the Bergman metrics converge to the symplectic form with speed rate $p^{-1}$. We note that in this case the near-diagonal expansion of the Bergman kernel is essential for the proof, in contrast to the the Kähler case, where the diagonal expansion already implies the result. Let us also observe that the results in [14, Theorem 3.6], [13, Theorem 8.3.12] are valid in a more general context, namely when $g^{TX}$ is an arbitrary $J$-invariant Riemannian metric.

There exist in the literature another replacement of the notion of holomorphic section, e.g., [2, 16]. It is based on a construction by Boutet de Monvel and Guillemin [3] of a first-order pseudodifferential operator $D_b$ on the circle bundle of $L^*$. The associated Szegő kernels are well defined modulo smooth operators on the associated circle bundle, even though $D_b$ is neither canonically defined nor unique. Indeed, Boutet de Monvel–Guillemin define the Szegő kernels first, and construct the operator $D_b$ from the Szegő kernels. Also for these spaces the Bergman metrics converge to the symplectic form with speed rate $p^{-1}$.

The main result of this paper is as follows.

**Theorem 0.1.** Let $(X, \omega)$ be a compact symplectic manifold and $(L, h^L)$ be a Hermitian line bundle endowed with a Hermitian connection $\nabla^L$ such that $\frac{2\pi}{h} R^L = \omega$. Let $J$ be an almost complex structure on $TX$ such that $g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot)$ is a $J$-invariant Riemannian metric on $TX$. Then for any $\ell \in \mathbb{N}$, there exists $C_\ell > 0$ such that

$$(0.9) \quad \left| \frac{1}{p} \Phi^*_p(\omega_{FS}) - \omega \right|_{\ell} \leq C_\ell p^2,$$

where $\Phi_p$ is the Kodaira map (0.8) defined by the space $\mathcal{H}_p$ of bound states of the renormalized Bochner Laplacian $\Delta_{p, \Phi}$ associated with $g^{TX}, \nabla^L, \Phi$ in (0.5).

The proof is based on the near diagonal expansion of the Bergman kernel of $\mathcal{H}_p$ from [13] [14]. The sharp bound of $O(p^{-2})$ is due to some remarkable cancellations of the coefficients in this expansions, reminiscent of the local properties of the curvature of Kähler metrics.

The main motivation for approximating Kähler metrics by Fubini-Study metrics comes from the questions about the existence and uniqueness of Kähler metrics of constant scalar curvature, or more generally, Kähler-Einstein metrics, see [7] [17]. It is natural to study this
questions also in the symplectic framework, for example, it is interesting to generalize to the almost-Kähler case the lower bounds on the Calabi functional given by Donaldson [8]. This is done by Lejmi and Keller [10]. Theorem 0.1 plays a crucial role in their proof in the symplectic case.

The organization of this paper is as follows. In Section 1, we recall the formal calculus on \( \mathbb{C}^n \) for the model operator \( \mathcal{L} \), which is the main ingredient of our approach. In Section 2, we review the asymptotic expansion of the generalized Bergman kernel. In Section 3, we reduce the proof of Theorem 0.1 to Theorem 3.3. In Section 4, we prove Theorem 3.3 and thus finish the proof of Theorem 0.1.

We shall use the following notations. For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), \( z \in \mathbb{C}^n \), we set \( |\alpha| = \sum_{j=1}^n \alpha_j \), \( \alpha! = \prod_j (\alpha_j!) \) and \( z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n} \). Moreover, when an index variable appear twice in a single term, it means that we are summing over all its possible values.

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1. **Kernel calculus on \( \mathbb{C}^n \)**

In this section, we recall the formal calculus on \( \mathbb{C}^n \) for the model operator \( \mathcal{L} \) introduced in [14, §7.1], [13, §4.1.6] (with \( a_j = 2\pi \) therein). This calculus is the main gradient of our approach.

Let us consider the canonical coordinates \( (Z_1, \ldots, Z_{2n}) \) on the real vector space \( \mathbb{R}^{2n} \). On the complex vector space \( \mathbb{C}^n \) we consider the complex coordinates \( (z_1, \ldots, z_n) \). The two sets of coordinates are linked by the relation \( z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}, \ j = 1, \ldots, n \).

We consider the \( L^2 \)-norm

\[
\| \cdot \|_{L^2} = \left( \int_{\mathbb{R}^{2n}} |\cdot|^2 dZ \right)^{1/2}
\]

on \( \mathbb{R}^{2n} \), where \( dZ = dZ_1 \cdots dZ_{2n} \) is the Lebesgue measure. We define the differential operators:

\[
b_j = -2 \frac{\partial}{\partial z_j} + \pi \overline{z}_j, \quad b_j^+ = 2 \frac{\partial}{\partial \overline{z}_j} + \pi z_j, \quad b = (b_1, \ldots, b_n), \quad \mathcal{L} = \sum_{j=1}^n b_j b_j^+,
\]

which extend to closed densely defined operators on \( (L^2(\mathbb{R}^{2n}), \| \cdot \|_{L^2}) \). As such, \( b_j^+ \) is the adjoint of \( b_j \) and \( \mathcal{L} \) defines as a densely defined self-adjoint operator on \( (L^2(\mathbb{R}^{2n}), \| \cdot \|_{L^2}) \).

The following result was established in [14, Theorem 1.15] (cf. also [13, Theorem 4.1.20]).

**Theorem 1.1.** The spectrum of \( \mathcal{L} \) on \( L^2(\mathbb{R}^{2n}) \) is given by

\[
\text{Spec}(\mathcal{L}) = \{ 4\pi |\alpha| : \alpha \in \mathbb{N}^n \},
\]

and an orthogonal basis of the eigenspace of \( 4\pi |\alpha| \) is given by

\[
b^\alpha \left( e^{\beta \exp \left( -\pi \sum_j |z_j|^2 / 2 \right)} \right), \quad \text{with } \beta \in \mathbb{N}^n.
\]

In particular, an orthonormal basis of \( \text{Ker}(\mathcal{L}) \) is

\[
\left\{ \phi_\beta(z) = \left( \frac{|\beta|}{\beta!} \right)^{1/2} e^{-\pi \sum_j |z_j|^2 / 2} : \beta \in \mathbb{N}^n \right\}.
\]
Let \( \mathcal{P}(Z, Z') \) denote the kernel of the orthogonal projection \( \mathcal{P} : L^2(\mathbb{R}^{2n}) \to \text{Ker}(\mathcal{L}) \) with respect to \( dZ' \). Set \( \mathcal{P}^\perp = \text{Id} - \mathcal{P} \).

Obviously \( \mathcal{P}(Z, Z') = \sum_\beta \phi_\beta(z) \phi_\beta(z') \), so we infer from (1.5) that

\[
\mathcal{P}(Z, Z') = \exp \left( -\frac{\pi}{2} \sum_{j=1}^n \left( |z_j|^2 + |z_j'|^2 - 2z_jz_j' \right) \right). \tag{1.6}
\]

By (1.2) and (1.6), we obtain

\[
(b_+^j \mathcal{P})(Z, Z') = 0, \quad (b_j \mathcal{P})(Z, Z') = 2\pi(z_j - z_j') \mathcal{P}(Z, Z'). \tag{1.7}
\]

The following commutation relations are very useful in the computations. Namely, for any polynomial \( g(z, \bar{z}) \) in \( z \) and \( \bar{z} \), we have

\[
[b_j, b_k^+] = b_j b_k^+ - b_k^+ b_j = -4\pi\delta_{jk},
\]

\[
[b_j, b_k] = [b_j^+, b_k^+] = 0,
\]

\[
[g(z, \bar{z}), b_j] = 2\frac{\partial}{\partial z_j} g(z, \bar{z}),
\]

\[
[g(z, \bar{z}), b_j^+] = 2\frac{\partial}{\partial \bar{z}_j} g(z, \bar{z}). \tag{1.8}
\]

For a polynomial \( F \) in \( Z, Z' \), we denote by \( F \mathcal{P} \) the operator on \( L^2(\mathbb{R}^{2n}) \) defined by the kernel \( F(Z, Z') \mathcal{P}(Z, Z') \) and the volume form \( dZ \).

In the calculations involving the kernel \( \mathcal{P}(\cdot, \cdot) \), we prefer however to use the orthogonal decomposition of \( L^2(\mathbb{R}^{2n}) \) given in Theorem 1.1 and the fact that \( \mathcal{P} \) is an orthogonal projection, rather than integrating against the expression (1.6) of \( \mathcal{P}(\cdot, \cdot) \). This point of view helps simplify a lot the computations and understand better the operators. As an example, Theorem 1.1 implies that

\[
(\mathcal{P}^{\alpha \beta} z^\beta \mathcal{P})(Z, Z') = \begin{cases} (z^\beta \mathcal{P})(Z, Z'), & \text{if } |\alpha| = 0, \\ 0, & \text{if } |\alpha| > 0. \end{cases} \tag{1.9}
\]

We will also identify \( z \) to \( \sum_j z_j \frac{\partial}{\partial z_j} \) and \( \bar{z} \) to \( \sum_j \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \) when we consider \( z \) and \( \bar{z} \) as vector fields, and

\[
\mathcal{R} = \sum_j Z_j \frac{\partial}{\partial Z_j} = z + \bar{z} = Z. \tag{1.10}
\]

2. Asymptotic expansion of the generalized Bergman kernel

Let \( a^X \) be the injectivity radius of \((X, g^{TX})\). We denote by \( B^X(x, \varepsilon) \) and \( B^{T_xX}(0, \varepsilon) \) the open balls in \( X \) and \( T_xX \) with center \( x \) and radius \( \varepsilon \), respectively. Then the exponential map \( T_xX \ni Z \to \exp^X_z(Z) \in X \) is a diffeomorphism from \( B^{T_xX}(0, \varepsilon) \) onto \( B^X(x, \varepsilon) \) for \( \varepsilon \leq a^X \).

From now on, we identify \( B^{T_xX}(0, \varepsilon) \) with \( B^X(x, \varepsilon) \) via the exponential map for \( \varepsilon \leq a^X \).

We fix \( x_0 \in X \). For \( Z \in B^{T_{x_0}X} \) we identify \( (L_Z, h^Z_Z) \) to \( (L_{x_0}, h_{x_0}^L) \) by parallel transport with respect to the connection \( \nabla^L \) along the curve \( \gamma_Z : [0, 1] \ni u \to \exp^X_{x_0}(uZ) \).

In general, for functions in normal coordinates, we will add a subscript \( x_0 \) to indicate the base point \( x_0 \in X \). Similarly, \( P_{h^L}(x, y) \) induces in terms of the above trivialization (note that
End\((L_{x_0}^p) = \mathbb{C}\) a smooth function

\[
\{ (Z, Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon \} \ni (Z, Z') \mapsto P_{H, x_0}(Z, Z') \in \mathbb{C},
\]

which also depends smoothly on the parameter \(x_0\).

Let us choose an orthonormal basis \(\{w_j\}_{j=1}^n\) of \(T_{x_0}(1,0)X\). Then \(e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)\) and \(e_{2j} = \frac{1}{\sqrt{2}}(w_j - \overline{w}_j), j = 1, \ldots, n\), forms an orthonormal basis of \(T_{x_0}X\). We use coordinates on \(T_{x_0}X \cong \mathbb{R}^{2n}\) given by the identification

\[
(2.1) \quad \mathbb{R}^{2n} \ni (Z_1, \ldots, Z_{2n}) \mapsto \sum_{j=1}^{2n} Z_j e_j \in T_{x_0}X.
\]

In the sequel we also use complex coordinates \(z = (z_1, \ldots, z_n)\) on \(\mathbb{C}^n \cong \mathbb{R}^{2n}\).

Let \(dv_{TX}\) be the Riemannian volume form on \((T_{x_0}X, g_{T_{x_0}X})\). Let \(\kappa_{x_0} : T_{x_0}X \to \mathbb{R}, Z \mapsto \kappa_{x_0}(Z)\) be a smooth positive function defined by

\[
(2.2) \quad dv_X(Z) = \kappa_{x_0}(Z)dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1,
\]

where the subscript \(x_0\) of \(\kappa_{x_0}(Z)\) indicates the base point \(x_0 \in X\).

Rescaling \(\Delta p, \Phi\) and Taylor expansion. For \(s \in C^\infty(\mathbb{R}^{2n}, \mathbb{C})\), \(Z \in \mathbb{R}^{2n}\), \(|Z| \leq \varepsilon\), and for \(t = \frac{1}{\sqrt{s}}\), set

\[
(2.3) \quad (S_t s)(Z) := s(Z/t), \quad L_t := S^{-1}_t k^{1/2} t^2 P_{H, \Phi} k^{-1/2} S_t.
\]

For \(U \in T_{x_0}X\), we denote \(\nabla_U\) the ordinary differential in direction \(U\). Set

\[
(2.4) \quad \nabla_{0, \ast} = \nabla_\ast + \frac{1}{2} \Delta_{x_0}^{\ast}(Z, \ast), \quad L_0 = -\sum_{j=1}^{2n} (\nabla_{0, e_j})^2 = \sum_{j=1}^n b_j b_j^* = L.
\]

By [14] Theorem 1.4, there exist second order differential operators \(O_i\) such that we have an asymptotic expansion in \(t\) when \(t \to 0\),

\[
(2.5) \quad L_t = L_0 + \sum_{r=1}^m t^r O_r + O(t^{m+1}).
\]

Moreover,

\[
(2.6) \quad O_1(Z) = -\frac{2}{3} (\partial_j R^L_{x_0}(\mathcal{R}, e_i) Z_j \nabla_{0, e_i} - \frac{1}{3} (\partial_i R^L_{x_0}(\mathcal{R}, e_i),
\]

and

\[
(2.7) \quad O_2(Z) = \frac{1}{3} \left( R^T_{x_0}(\mathcal{R}, e_i) R, e_j \right) x_0 \nabla_{0, e_i} \nabla_{0, e_j}
\]

\[
+ \frac{2}{3} \left( R^T_{x_0}(\mathcal{R}, e_j) e_j, e_i \right) x_0 - \frac{1}{2} \sum_{|\alpha| = 2} (\partial^\alpha R^L_{x_0}(\mathcal{R}, e_i) Z_\alpha^{\ast} \mathcal{R}^i \mathcal{R}, e_i) \right] \nabla_{0, e_i}
\]

\[
- \frac{1}{4} \nabla_{e_i} \left( \sum_{|\alpha| = 2} (\partial^\alpha R^L_{x_0}(\mathcal{R}, e_i) Z_\alpha^{\ast} \mathcal{R}^i \mathcal{R}, e_i) \right) Z_i^{\ast} \mathcal{R}_{\alpha} - \frac{1}{9} \sum_{i} \left[ \sum_{j}(\partial_j R^L_{x_0}(\mathcal{R}, e_i) Z_j \right]^2
\]

\[
- \frac{1}{12} \left[ L_0, \left( R^T_{x_0}(\mathcal{R}, e_i) R, e_i \right) x_0 \right] + \Phi_{x_0}.
\]
From (2.1) and (2.3), as in [13, Remark 4.1.8], $\mathcal{L}_t$ is a formally self-adjoint elliptic operator with respect to $\| \cdot \|_{L^2}$ on $\mathbb{R}^{2n}$ and is a smooth family of operator with respect to the parameter $x_0 \in X$. Thus $\mathcal{L}_t$, $\mathcal{L}_0$ and $\mathcal{O}_r$ in (2.5) are formally self-adjoint with respect to $\| \cdot \|_{L^2}$.

By [13, Theorem 8.3.8], the following asymptotic expansion of the generalized Bergman kernel holds.

**Theorem 2.1.** There exists $J_r(Z, Z')$ polynomials in $Z, Z'$ with the same parity as $r$ and $\deg J_r(Z, Z') \leq 3r$, such that if we define

\[
J_r(Z, Z') = J_r(Z, Z')\mathcal{P}(Z, Z'), \quad J_0 = 1,
\]

then for any $k, \ell, m \in \mathbb{N}$, $q > 0$, there exists $C > 0$ such that if $p \geq 1$, $Z, Z' \in T_{x_0} X$ and $|Z|, |Z'| \leq \frac{q}{\sqrt{p}}$, we have

\[
\sup_{|\alpha| + |\alpha'| \leq m} \left| \frac{\partial |\alpha| + |\alpha'|}{\partial Z^\alpha \cdot Z^\alpha'} \left( \frac{1}{p^n} P_{\mathcal{H}_r}(Z, Z') - \sum_{r=0}^k J_r(\sqrt{p} Z, \sqrt{p} Z') \kappa^{-\frac{1}{2}}(Z) \kappa^{-\frac{1}{2}}(Z') p^{-\frac{r}{2}} \right) \right|_{\tilde{\mathcal{E}}(X)} \leq C p^{-\frac{k-1}{2}},
\]

where $\mathcal{C}^\ell(X)$ is $\mathcal{C}^\ell$-norm for the parameter $x_0 \in X$.

Moreover, by [13, (4.1.93), (8.3.45)], $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by (cf. [13, (8.3.65)], [14, (1.111)])

\[
\begin{align*}
\mathcal{F}_1 &= -\mathcal{P}^\perp \mathcal{L}^{-1} \mathcal{O}_1 \mathcal{P} - \mathcal{P} \mathcal{O}_1 \mathcal{L}^{-1} \mathcal{P}^\perp, \\
\mathcal{F}_2 &= \mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_1 \mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_1 \mathcal{P} - \mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_2 \mathcal{P} \\
&\quad + \mathcal{P} \mathcal{O}_1 \mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_1 \mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_2 \mathcal{P} - \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P}^\perp \\
&\quad + \mathcal{P} \mathcal{L}^{-1} \mathcal{O}_1 \mathcal{P} \mathcal{O}_1 \mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_1 \mathcal{P} - \mathcal{P} \mathcal{O}_1 \mathcal{L}^{-2} \mathcal{P}^\perp \mathcal{O}_1 \mathcal{P}.
\end{align*}
\]

From Theorem 2.1, we get in particular [13, Theorem 8.3.3]: there exist $b_r \in C^\infty(X, \mathbb{R})$ such that for any $k, \ell \in \mathbb{N}$, there exists $C_{k, \ell} > 0$ such that

\[
\left| \frac{1}{p^n} P_{\mathcal{H}_r}(x, x) - \sum_{r=0}^k b_r(x) p^{-r} \right|_{\mathcal{E}} \leq C_{k, \ell} p^{-k-1},
\]

and

\[
b_0(x_0) = \mathcal{F}_0(0, 0) = 1, \quad b_r(x_0) = \mathcal{F}_2r(0, 0), \quad \mathcal{F}_{2r+1}(0, 0) = 0.
\]

### 3. PROOF OF THEOREM 0.1

In this section we reduce Theorem 0.1 to Theorem 3.3. Let us fix $x_0 \in X$. As in section 2 we identify a small geodesic ball $B^X(x_0, \varepsilon)$ to $B^{T_{x_0} X}$ by means of the exponential map and we trivialize $\mathcal{L}$ by using a unit frame $e_L(Z)$ which is parallel with respect to $\nabla^L$ along the curve $[0, 1] \ni u \mapsto u Z$ for $Z \in B^{T_{x_0} X}(0, \varepsilon)$.

Set $d_p := \dim \mathcal{H}_p$ and for $v = (v_1, \ldots, v_{d_p}) \in \mathbb{C}^{d_p}$, set $\|v\|^2 = \sum_{j=1}^{d_p} |v_j|^2$. We can now express the Fubini-Study metric in the homogeneous coordinate $[v] = [v_1, \ldots, v_{d_p}] \in \mathbb{P}(\mathcal{H}_p)$ as

\[
\frac{-1}{2\pi} \partial \bar{\partial} \log (\|v\|^2) = \frac{-1}{2\pi} \left[ \frac{1}{\|v\|^2} \sum_{j=1}^{d_p} dv_j \wedge d\bar{v}_j - \frac{1}{\|v\|^4} \sum_{j,k=1}^{d_p} \overline{v}_j v_k dv_j \wedge d\overline{v}_k \right].
\]
Let \( \{s_j\} \) be an orthonormal basis of \( \mathcal{H}_p \), and let \( \{s^j\} \) be its dual basis. We write locally \( s_j = f_j e^{\frac{\omega}{L}_j} \), then by (3.3), as in [13 (5.1.17)], we have

\[
(3.2) \quad \Phi_p(x) = \left[ \sum_{j=1}^{\nu} f_j(x)s^j \right] \in \mathbb{P}(\mathcal{H}_p^*) .
\]

Set

\[
(3.3) \quad f_p(x, y) = \sum_{i=1}^{\nu} f_i(x) \tilde{f}_i(y) \quad \text{and} \quad |f_p(x)|^2 = f_p(x, x).
\]

Then

\[
(3.4) \quad P_{\mathcal{H}_p}(x, y) = f_p(x, y)e^{\frac{\omega}{L}_y} \otimes e^{\frac{\omega}{L}_x}(y)^*, \quad |f_p(x)|^2 = P_{\mathcal{H}_p}(x, x).
\]

By (3.1), (3.2) and (3.3), we get

\[
(3.5) \quad \Phi_p^e(\omega_{FS})(x) = \frac{\sqrt{-1}}{2\pi} \left[ \frac{1}{|f_p|^2} \sum_{j=1}^{\nu} df_j \wedge df_j - \frac{1}{|f_p|^2} \sum_{j,k=1}^{\nu} f_j f_k df_j \wedge df_k \right](x)
\]

\[
= \frac{\sqrt{-1}}{2\pi} \left[ |f_p(x)|^{-2} d_x d_y f_p(x, y) - |f_p(x_0)|^{-4} d_x f_p(x, y) \right]_{x=y=x_0},
\]

where \( |x=y=x_0 \) means the pull-back by the diagonal map \( j : X \to X \times X, x_0 \mapsto (x_0, x_0) \).

By (3.4), \( P_{\mathcal{H}_p}(x, y) \) is \( f_p(x, y) \) under our trivialization of \( L \). Since we work with normal coordinates, we get from (2.2) (cf. [13 (4.1.101)])

\[
(3.6) \quad \kappa(Z) = 1 + O(|Z|^2).
\]

By (2.9), (2.12), (3.5) and (3.6), we get

\[
\frac{1}{p} \Phi_p^e(\omega_{FS})(x) = \frac{\sqrt{-1}}{2\pi} \left\{ \left[ \frac{1}{\mathcal{F}_0} d_x d_y \mathcal{F}_0 - \frac{1}{\mathcal{F}_0^2} d_x \mathcal{F}_0 \wedge d_y \mathcal{F}_0 \right](0, 0) + p^{-1/2} \left[ \frac{1}{\mathcal{F}_0} d_x d_y \mathcal{F}_1 - \frac{1}{\mathcal{F}_0^2} (d_x \mathcal{F}_1 \wedge d_y \mathcal{F}_0 + d_x \mathcal{F}_0 \wedge d_y \mathcal{F}_1) \right](0, 0) + p^{-1} \left[ \frac{1}{\mathcal{F}_0} d_x d_y \mathcal{F}_2 - \frac{\mathcal{F}_2}{\mathcal{F}_0^2} d_x d_y \mathcal{F}_0 + \frac{2 \mathcal{F}_2}{\mathcal{F}_0} d_x \mathcal{F}_0 \wedge d_y \mathcal{F}_0 \right.(0, 0) - \frac{1}{\mathcal{F}_0^3} (d_x \mathcal{F}_0 \wedge d_y \mathcal{F}_2 + d_x \mathcal{F}_1 \wedge d_y \mathcal{F}_1 + d_x \mathcal{F}_2 \wedge d_y \mathcal{F}_0) \left. \right](0, 0) + p^{-3/2} \left[ \frac{1}{\mathcal{F}_0} d_x d_y \mathcal{F}_3 - \frac{2 \mathcal{F}_2}{\mathcal{F}_0} d_x d_y \mathcal{F}_1 - \frac{1}{\mathcal{F}_0^2} (d_x \mathcal{F}_0 \wedge d_y \mathcal{F}_3 + d_x \mathcal{F}_1 \wedge d_y \mathcal{F}_2 + d_x \mathcal{F}_2 \wedge d_y \mathcal{F}_1 + d_x \mathcal{F}_3 \wedge d_y \mathcal{F}_0) + \frac{2 \mathcal{F}_2}{\mathcal{F}_0^2} (d_x \mathcal{F}_0 \wedge d_y \mathcal{F}_1 + d_x \mathcal{F}_1 \wedge d_y \mathcal{F}_0) \right](0, 0) \right\} + O(p^{-2}).
\]

From (1.6) and (2.8), we obtain

\[
(3.8) \quad d_x \mathcal{F}_0(0, 0) = d_y \mathcal{F}_0(0, 0) = 0.
\]
As $J_r$ is a polynomial in $Z, Z'$ with the same parity as $r$, we know from (1.6) and (2.8) that for $\alpha, \alpha' \in \mathbb{N}^{2n}$, there exists a polynomial $J_{r,\alpha,\alpha'}$ in $Z, Z'$ with the same parity as $r - |\alpha| - |\alpha'|$ such that

$$
\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} F_r(Z, Z') = (J_{r,\alpha,\alpha'} \mathcal{P})(Z, Z').
$$

In particular, (3.9) yields

$$
d_x d_y F_1(0, 0) = d_x d_y F_3(0, 0) = 0, \quad d_y F_2(0, 0) = d_x F_2(0, 0) = 0.
$$

By (1.6) and (2.8), we get

$$
\frac{\sqrt{-1}}{2\pi} (d_x d_y F_0)(0, 0) = \frac{\sqrt{-1}}{2\pi} (d_x d_y \mathcal{P})(0, 0) = \frac{\sqrt{-1}}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j = \omega(x_0).
$$

Substituting (2.12), (3.8), (3.10) and (3.11) into (3.7) yields

$$
\frac{1}{p} \Phi^*_p(\omega_{FS})(x_0) = \omega(x_0) + \frac{\sqrt{-1}}{2\pi} (d_x d_y F_2 - d_x F_1 \wedge d_y F_1)(0, 0) \bigg|_{p=1} - b_1(x_0) \omega(x_0) + \mathcal{O}(p^{-2}).
$$

Recall that for a tensor $\psi$, $\nabla^X \psi$ is the covariant derivative of $\psi$ induced by the Levi-Civita connection $\nabla^TX$. We will denote by $\langle \cdot, \cdot \rangle$ the $\mathbb{C}$-bilinear form on $TX \otimes_{\mathbb{R}} \mathbb{C}$ induced by $g^TX$.

The following observation [13, (8.3.54)] is very useful.

**Lemma 3.1.** For $U \in T_{x_0}X$, $\nabla^X J$ is skew-adjoint and the tensor $\langle \langle \nabla^X J \rangle \cdot, \cdot \rangle$ is of type $(T^{s(1,0)}X)^{\otimes 3} \oplus (T^{s(0,1)}X)^{\otimes 3}$.

**Lemma 3.2.** We have

$$
(d_x F_1)(0, 0) = (d_y F_1)(0, 0) = 0.
$$

**Proof.** By (1.2) and (2.6), we have (cf. [13, (8.3.51)])

$$
\mathcal{O}_1 = -\frac{2}{3} \left[ \frac{\langle \nabla^X J \rangle R, \frac{\partial}{\partial z_i} b_i^+ - b_i \langle \nabla^X J \rangle R, \frac{\partial}{\partial \bar{z}_i} \rangle }{\langle \nabla^X J \rangle R, \frac{\partial}{\partial z_i} \rangle } \right], \quad J = -2\pi \sqrt{-1} J.
$$

From Theorem 1.1, (1.7), (3.14) and Lemma 3.1, we get (cf. [13, (8.3.67)])

$$
\left( \mathcal{L}^{-1} \mathcal{P} - \mathcal{O}_1 \mathcal{P} \right)(Z, Z')
$$

$$
= -\frac{\sqrt{-1}}{3} \left[ \left( \frac{b_i b_j}{4\pi} \langle \nabla^X \mathcal{P}_r, \frac{\partial}{\partial z_i} \rangle + b_i \langle \nabla^X \mathcal{P}_r, \frac{\partial}{\partial \bar{z}_i} \rangle \right) \mathcal{P} \right](Z, Z')
$$

$$
= -\frac{\sqrt{-1}}{3} \left[ \langle \nabla^X \mathcal{P}_r, \mathcal{P}_r \rangle + \langle \nabla^X \mathcal{P}_r, \mathcal{P}_r \rangle \right] \mathcal{P}(Z, Z').
$$

Note that if $K$ is an operator on $(\mathbb{R}^{2n}, \| \cdot \|_{L^2})$ with smooth kernel $K(Z, Z')$ with respect to $dZ'$, then the kernel $K^*(Z, Z')$ of the adjoint $K^*$ of $K$, with respect to $dZ'$, is given by

$$
K^*(Z, Z') = \overline{K(Z', Z)}.
$$
As $L, O_1$ are formally self-adjoint with respect to $\|\cdot\|_{L^2}$, thus $PO_1L^{-1}P^\perp$ is the adjoint of $L^{-1}P^\perp O_1P$. From Lemma 3.1, (3.15) and (3.16), we get
\begin{equation}
(PO_1L^{-1}P^\perp)(Z, Z') = \frac{\sqrt{-1}}{2\pi} \left[ \left\langle (\nabla^X J) z, z' \right\rangle + \left\langle (\nabla^X J) z^\prime, z \right\rangle \right] P(Z, Z').
\end{equation}
(3.17)
\begin{equation}
= \frac{\sqrt{-1}}{2\pi} \left\langle \left( \nabla^X \frac{\partial}{\partial z_k} \right) \frac{\partial}{\partial z_l} \right\rangle \left( z'_j z_k z'_l + z_j z_k z'_l \right) P(Z, Z').
\end{equation}
(3.18)
As the coefficient of $P(Z, Z')$ in (3.15) and (3.17) are polynomials of degree 3, from (2.10), (3.15) and (3.17), we get (3.13). The proof of Lemma 3.2 is completed.

\textbf{Theorem 3.3.} The following identity holds,
\begin{equation}
\sqrt{-1} \left( d_x d_y F_2 \right)(0, 0) = b_1(x_0) \omega(x_0).
\end{equation}
(4.1)

By Lemma 3.2, Theorem 3.3 and (3.12), we get Theorem 0.1. Section 4 is devoted to the proof of Theorem 3.3.

\section{Proof of Theorem 3.3}

This section is devoted to the proof of Theorem 3.3. We will compute the contribution of each term in (2.10) to $F_2$. Set
\begin{align}
I_1 &= L^{-1}P^\perp O_1L^{-1}P^\perp O_1P, \quad I_2 = -L^{-1}P^\perp O_2P,
I_3 &= PO_1L^{-1}P^\perp O_1L^{-1}P^\perp, \quad I_4 = -PO_2L^{-1}P^\perp,
I_5 &= P^\perp L^{-1}O_1PO_1L^{-1}P^\perp, \quad I_6 = -PO_1L^{-2}P^\perp O_1P.
\end{align}
(4.2)

For $j \in \{1, \ldots, 6\}$, let $I_j(Z, Z')$ be the smooth kernel of the operator $I_j$ with respect to $dZ'$. By (2.10),
\begin{equation}
(d_x d_y F_2)(0, 0) = \sum_{j=1}^6 (d_x d_y I_j)(0, 0).
\end{equation}
(4.3)

In the context of (3.16), if we denote $b_{jk} = \frac{\partial^2 K}{\partial Z_j \partial Z_k^*}(Z, Z')|_{Z=Z'=0}$, then
\begin{equation}
(d_x d_y K^*)(0, 0) = \sum_{j,k} dZ_j \wedge dZ_k \left( \frac{\partial^2 K^*}{\partial Z_j \partial Z_k^*}(Z, Z') \right)|_{Z=Z'=0}
= \sum_{j<k} (b_{kj} - b_{jk}) dZ_j \wedge dZ_k = -(d_x d_y K)(0, 0).
\end{equation}
(4.4)

As $O_r$ in (2.5) are formally self-adjoint with respect to $\|\cdot\|_{L^2}$, we get by (4.1) that $I_1$ and $I_2$ are adjoints of $I_3$ and $I_4$, respectively, as operators acting on $(\mathbb{R}^{2n}, \|\cdot\|_{L^2})$. Hence by (4.3),
\begin{equation}
(d_x d_y I_3)(0, 0) = -(d_x d_y I_1)(0, 0), \quad (d_x d_y I_4)(0, 0) = -(d_x d_y I_2)(0, 0).
\end{equation}
(4.5)
4.1. **Evaluation of** \((d_x d_y I_j)(0, 0)\) **for** \(j = 1, 3, 5, 6\). To simplify the notation, for polynomials \(Q_1, Q_2\) in \(Z, Z'\), we will denote

\[(4.5) \quad (Q_1 \mathcal{P})(Z, Z') \sim (Q_2 \mathcal{P})(Z, Z'), \]

if the constant coefficient and for all \(j\) the coefficient of \(Z_j'\) in \(Q_1 - Q_2\) as a polynomial in \(Z'\) are zero; we denote

\[(4.6) \quad (Q_1 \mathcal{P})(Z, Z') \approx (Q_2 \mathcal{P})(Z, Z'), \]

if (4.5) holds and the constant coefficient and for all \(j\) the coefficient of \(Z_j\) in \(Q_1 - Q_2\) as a polynomial in \(Z\) are zero.

Set

\[(4.7) \quad J_{jir} := \left( \langle \nabla^X J \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_r} \rangle \right), \quad J_{jir}' := \left( \langle \nabla^X J \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_r} \rangle \right). \]

From (1.7), (3.14), Lemma 3.1 and (3.15), we get

\[(4.8) \quad \left( O_1 \mathcal{L}^{-1} \mathcal{P} \perp O_1 \mathcal{P} \right)(Z, Z') = 4\pi \left\{ \left( \left( \langle \nabla^X J \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_r} \rangle \right) b_i^+ - b_i \left( \langle \nabla^X J \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_r} \rangle \right) \right) \right\}(Z, Z') \]

in the last equation of (4.8), we use \(b_i \left( \langle \nabla^X J \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_r} \rangle \right) \mathcal{P} = \left( \langle \nabla^X J \frac{\partial}{\partial z_i}, -2\pi \frac{\partial}{\partial z_r} \rangle \right) \mathcal{P}. \]

By Theorem 1.1 (1.7) and (1.8), we get \(b_i^+ b_j b_k = b_{ki} b_i^+ + 4\pi (\delta_{ij} b_k + \delta_{ik} b_j)\) and

\[(4.9) \quad \mathcal{L}^{-1} \mathcal{P} \perp \left( z_{s t} b_i^+ b_j b_k \mathcal{P} \right) = 4\pi \mathcal{L}^{-1} \mathcal{P} \perp \left( z_{s t} (\delta_{ij} b_k + \delta_{ik} b_j) \mathcal{P} \right) \]

By (1.7), (1.8), (4.1), (4.8) and (4.9), we get

\[(4.10) \quad I_1(Z, Z') \approx 9 \left( \langle \delta_{ij} b_k z_s z_t + \delta_{ik} b_j z_s z_t \rangle \mathcal{P} \right)(Z, Z') \]

Recall that \(J_{sti}\) is anti-symmetric on \(t\) and \(i\), thus the contribution of \(-2\delta_{ij} \delta_{kl} z_s - 2\delta_{ik} \delta_{jl} z_s\) in (4.10) is zero. Thus from (4.10), we get

\[(4.11) \quad I_1(Z, Z') \approx -\frac{2}{9} \langle J_{sri} J_{rj} \left( \delta_{ik} \delta_{js} + \delta_{ij} \delta_{ks} \right) z_r \mathcal{P} \rangle(Z, Z'). \]
By (1.6), Lemma 3.1 and (4.11), we get

\[(d_x d_y I_1)(0,0) = -\frac{2}{9} \mathcal{J}_{jir} \left( \mathcal{J}_{\eta \eta} + \mathcal{J}_{\eta \eta} \right) dz_r \wedge d\bar{z}_q. \tag{4.12} \]

From (4.4), (4.7) and (4.12), we get

\[(d_x d_y I_3)(0,0) = (d_x d_y I_4)(0,0). \tag{4.13} \]

By (1.6), (3.15), (3.17) and (4.1), we get

\[(4.14) \]

\[I_5(Z, Z') \approx \frac{\pi^2}{9} \left\{ \left( \left( \nabla^X J \right) \mathcal{P}' + \left( \nabla^X J \right) \mathcal{P}' \right) \circ \left( \left( \nabla^X J \right) \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_l} \right) \mathcal{P}(Z, Z') \right\} (Z, Z') \approx \frac{\pi^2}{9} \left\{ \left( \nabla^X J \right) \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_l} \right\} \mathcal{P}(Z, Z') \left( \mathcal{P} \circ \left( z_j z_k z_l \right) \right)(0,0), \]

where in the last equation we use \( \mathcal{P}(0,0) = 1 \), since we need to compute the constant coefficient of \( \mathcal{P} \) in \( \mathcal{P} \circ \left( z_j z_k z_l \right) \).

By (1.7) and (1.8), we get

\[(4.15) \]

\[\mathcal{P}(Z, Z') = \frac{1}{4\pi^2} (z_j z_k b_l b_l) (Z, Z'), \]

where in the last equation we use \( \mathcal{P}(0,0) = 1 \), since we need to compute the constant coefficient of \( \mathcal{P} \) in \( \mathcal{P} \circ \left( z_j z_k z_l \right) \).

From Theorem 1.1 and (4.15), we get

\[(4.16) \]

\[\left( \mathcal{P} \circ \left( z_j z_k z_l \right) \right)(0,0) = \frac{1}{4\pi^2} (4\delta_{ij} \delta_{kl} + 4\delta_{jk} \delta_{kl}), \mathcal{P}(0,0) \]

\[= \frac{1}{\pi^2} (\delta_{ij} \delta_{kl} + \delta_{jk} \delta_{kl}). \]

From (4.7), (4.14) and (4.16), we obtain

\[(4.17) \]

\[(d_x d_y I_3)(0,0) = -\frac{1}{9} \mathcal{J}_{jir} \left( \mathcal{J}_{\eta \eta} + \mathcal{J}_{\eta \eta} \right) dz_r \wedge d\bar{z}_q. \]

By (3.15), (3.17) and (4.1), we get

\[(4.18) \]

\[I_6(Z, Z') \approx -\frac{\pi^2}{9} \left\{ \left( \nabla^X J \right) z_j, \frac{\partial}{\partial z_l} \right\} \left( \nabla^X J \right) \mathcal{P}(Z, Z') \left( \mathcal{P} \circ \left( z_j z_k z_l \right) \right)(0,0). \]

Thus by (4.7), (4.16) and (4.18), we get

\[(4.19) \]

\[(d_x d_y I_6)(0,0) = \frac{1}{9} \mathcal{J}_{jir} \left( \mathcal{J}_{\eta \eta} + \mathcal{J}_{\eta \eta} \right) dz_r \wedge d\bar{z}_q. \]
4.2. Evaluation of \((d_x d_y I_2)(0, 0)\): part I. Recall that [14] Lemma 2.1,

\[
O_2 \mathcal{P} = \left\{ \frac{1}{2} b_i b_j \left\langle R_{x_0}^{TX}(\mathcal{R}, \frac{\partial}{\partial z_i}) \mathcal{R}, \frac{\partial}{\partial z_j} \right\rangle + \frac{1}{2} b_i \left[ \sum_{|\alpha|=2} \left( \partial^\alpha R^j \right)(\mathcal{R}, \frac{\partial}{\partial z_i}) \frac{Z^\alpha}{\alpha!} \right] \right.
\]
\[
+ \frac{4}{3} b_j \left[ \left\langle R_{x_0}^{TX} \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) \mathcal{R}, \frac{\partial}{\partial z_j} \right\rangle - \left\langle R_{x_0}^{TX} \left( \mathcal{R}, \frac{\partial}{\partial z_i} \right), \frac{\partial}{\partial z_j} \right\rangle \right]
\]
\[
- 2\pi \sqrt{-1} \left\langle \left( \nabla X \nabla X J \right)_{(R, R)} \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle + 4 \left\langle R_{x_0}^{TX} \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right), \frac{\partial}{\partial z_j} \right\rangle \right\}
\mathcal{P}
\]
\[
+ \left[ -\frac{1}{3} \mathcal{L} \left\langle R_{x_0}^{TX} \left( \mathcal{R}, \frac{\partial}{\partial z_j} \right), \frac{\partial}{\partial z_j} \right\rangle + \frac{4\pi^2}{9} |(\nabla R^X J)|^2 + \Phi_{x_0} \right\} \mathcal{P}.
\]

Set

\[
I_{21}(Z, Z') = \frac{1}{3} \left( \mathcal{L}^{-1} \mathcal{P} \left\langle R_{x_0}^{TX} \left( \mathcal{R}, \frac{\partial}{\partial z_i} \right), \frac{\partial}{\partial z_j} \right\rangle \right) (Z, Z'),
\]

\[
I_{22}(Z, Z') = \frac{1}{2} \left( \mathcal{L}^{-1} \mathcal{P} b_i \left[ \sum_{|\alpha|=2} \left( \partial^\alpha R^j \right) \frac{Z^\alpha}{\alpha!} \right] \right) (Z, Z'),
\]

\[
I_{23}(Z, Z') = \frac{4}{3} \left\{ \mathcal{L}^{-1} b_j \left\langle R_{x_0}^{TX} \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) \right\rangle \mathcal{R} - R_{x_0}^{TX} \left( \mathcal{R}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_j} \mathcal{P} \right\} (Z, Z'),
\]

\[
I_{24}(Z, Z') = -2\pi \sqrt{-1} \left( \mathcal{L}^{-1} \mathcal{P} \left\langle \left( \nabla X \nabla X J \right)_{(R, R)} \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle \right) (Z, Z'),
\]

\[
I_{25}(Z, Z') = -\frac{1}{3} \left( \mathcal{L}^{-1} \mathcal{P} \left\langle R_{x_0}^{TX} \left( \mathcal{R}, \frac{\partial}{\partial z_i} \right), \frac{\partial}{\partial z_j} \right\rangle \mathcal{P} \right) (Z, Z'),
\]

\[
I_{26}(Z, Z') = \frac{4\pi^2}{9} \left( \mathcal{L}^{-1} \mathcal{P} \left| (\nabla R^X J)^2 \right| \mathcal{P} \right) (Z, Z').
\]

By Theorem 1.1, \(\mathcal{P} \left\langle R_{x_0}^{TX} \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right), \frac{\partial}{\partial z_j} \right\rangle \mathcal{P} = \mathcal{P} \| x_0 \mathcal{P} = 0 \). Thus by (4.1), (4.20) and (4.21), we get

\[
(4.22)
- I_2(Z, Z') = \sum_{j=1}^{6} I_{2j}(Z, Z').
\]

We evaluate first the contribution of \(I_{2j}, j = 1, 3, 5, 6, \) in \((d_x d_y I_2)(0, 0)\). We recall the following well-known symmetry properties of the curvature \(R^{TX}\): for \(U, V, W, Y \in TX\), we have

\[
(4.23)
\left\langle R^{TX}(U, V)W, Y \right\rangle = \left\langle R^{TX}(W, Y)U, V \right\rangle,
\]

\[R^{TX}(U, V)W + R^{TX}(V, W)U + R^{TX}(W, U)V = 0.\]

Using (1.8) and (4.23), we have

\[
(4.24)
b_i b_j \left\langle R^{TX} \left( \mathcal{R}, \frac{\partial}{\partial z_i} \right) \mathcal{R}, \frac{\partial}{\partial z_j} \right\rangle = b_i b_j \left\langle R^{TX} \left( \mathcal{R}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_j} \right) z_i z_j \right\rangle
\]
\[
+ 2b_i b_j \left\langle R^{TX} \left( \mathcal{R}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_j} \right) z_i z_j \right\rangle + b_i b_j \left\langle R^{TX} \left( \mathcal{R}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_j} \right) z_i z_j \right\rangle.
\]
By (1.7) and (1.8), we get

\[ (4.25) \quad z_s z_t \mathcal{P}(Z, Z') = z_s \left( \frac{b_l}{2\pi} + \mu_l \right) \mathcal{P}(Z, Z') = \left( \frac{b_l}{2\pi} z_s + \frac{\delta_{st}}{\pi} + z_s \mu_l \right) \mathcal{P}(Z, Z'). \]

By Theorem 1.1, (4.21), (4.24) and (4.25), we get

\[ (4.26) \quad 3I_{21}(Z, Z') = \frac{1}{8\pi} \left< R^{TX} \left( \frac{\partial}{\partial z_s}, \frac{\partial}{\partial \mu_s} \right) \mathcal{P}(Z, Z') \right> \left( b_l b_j z_s \mathcal{P}(Z, Z') \right) \]

\[ = \left< R^{TX} \left( \frac{\partial}{\partial z_s}, \frac{\partial}{\partial \mu_s} \right) \mathcal{P}(Z, Z') \right> \left( \frac{b_l b_j b_s + b_j b_j}{4\pi} z_s \mathcal{P}(Z, Z') \right) \]

where

\[ (4.27) \quad I_{27}(Z, Z') = \left< R^{TX} \left( \frac{\partial}{\partial z_s}, \frac{\partial}{\partial \mu_s} \right) \mathcal{P}(Z, Z') \right> \left( \mathcal{L}^{-1} \mathcal{P} + b_l b_j z_s \mathcal{P}(Z, Z') \right). \]

Note that by Theorem 1.1, (1.7) and (1.8),

\[ (4.28) \quad 4\pi^2 \mathcal{L}^{-1} \mathcal{P} + b_l b_j z_s \mathcal{P} = \mathcal{L}^{-1} \mathcal{P} + b_l b_j b_s + 2\pi z_s \mathcal{P} + \frac{b_j b_j}{4\pi} z_s \mathcal{P} \]

Thus, from (1.7), (4.6) and (4.28), we get

\[ (4.29) \quad I_{27}(Z, Z') \approx 0. \]

From (1.7) and (1.8), we get

\[ (b_l b_j \mathcal{P})(Z, Z') = 4\pi^2 (\mu_l - \mu_s) (\mu_j - \mu_j) \mathcal{P}(Z, Z'), \]

\[ (b_l b_j z_s \mathcal{P})(Z, Z') = \left[ -4\pi \delta_{js} \mu_l (\mu_j - \mu_s) - 4\pi \delta_{js} \mu_j (\mu_j - \mu_s) \right] \mathcal{P}(Z, Z'), \]

\[ (4.30) \quad (b_l b_j z_s \mathcal{P})(Z, Z') = \left[ 4\delta_{ls} \delta_{st} - 4\pi \delta_{js} \mu_l (\mu_j - \mu_s) + 4\delta_{jt} \delta_{js} - 4\pi \delta_{jl} \mu_s (\mu_j - \mu_s) \right] \mathcal{P}(Z, Z'), \]

\[ + 4\pi \delta_{js} \mu_s (\mu_j - \mu_s) (\mu_j - \mu_j) \mathcal{P}(Z, Z'), \]

and

\[ (4.31) \quad (b_l b_l b_j z_s \mathcal{P})(Z, Z') = \left[ -2\delta_{ls} b_l b_j - 2\delta_{js} b_l b_l - 2\delta_{ls} b_l b_l + z_s b_l b_l \mathcal{P} \right](Z, Z'). \]

By (1.7), (4.30) and (4.31), we get

\[ (4.32) \quad (d_s d_y (b_l b_j \mathcal{P}))(0, 0) = 0, \quad (d_s d_y (b_l b_j b_l z_s \mathcal{P}))(0, 0) = 0. \]
Substituting (3.11), (4.23), (4.29)–(4.32) into (4.26), we obtain

\[
(d_x d_y I_{21})(0, 0) = -\frac{\sqrt{-1}}{3} \left\langle 2 R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_i} \right\rangle \omega(x_0) \\
\quad + \frac{1}{3} \left\langle 2 R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_j} \right\rangle \omega(x_0) \\
\quad + \frac{1}{3} \left\langle 2 R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_j} \right\rangle d z_r \wedge d z_q
\]

(4.33)

By (4.21),

\[
\frac{3}{4} I_{23} = \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_j} \right\rangle \mathcal{L}^{-1} b_j z_s P \\
\quad + \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_j} \right\rangle \mathcal{L}^{-1} b_j z_s P.
\]

By Theorem 1.1 (1.7) and (1.8),

\[
(\mathcal{L}^{-1} P^\perp b_j z_s P)(Z, Z') = \frac{1}{4\pi} (b_j z_s P)(Z, Z') \\
\quad = \frac{1}{4\pi} \left( -2\delta_{js} + 2\pi z_s (z_j - z_j) \right) \mathcal{P}(Z, Z').
\]

(4.35)

Note that by (1.7), \( z_s P = \left( \frac{b_j}{2\pi} + z_s \right) \mathcal{P}. \) Thus from Theorem 1.1 we get

\[
(\mathcal{L}^{-1} b_j z_s P)(Z, Z') = \left[ \left( \frac{b_j}{16\pi^2} + \frac{b_j}{4\pi} z_s \right) \mathcal{P} \right](Z, Z') \\
\quad = \left[ \frac{1}{4} (z_j - z_j) (z_j - z_j) + \frac{1}{2} z_s^2 (z_j - z_j) \right] \mathcal{P}(Z, Z').
\]

(4.36)

As in (4.32), we get

\[
(d_x d_y (\mathcal{L}^{-1} b_j z_s P))(0, 0) = \frac{1}{2} dz_j \wedge dz_s.
\]

(4.37)

From (3.11), (4.23), (4.34), (4.35) and (4.37), we get

\[
(d_x d_y I_{23})(0, 0) = \frac{4\sqrt{-1}}{3} \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_i} \right\rangle \omega(x_0) \\
\quad - \frac{2}{3} \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_j} \right\rangle \omega(x_0) \\
\quad - \frac{2}{3} \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_j} \right\rangle d z_r \wedge d z_q
\]

(4.38)

Clearly, by Theorem 1.1 and (4.21),

\[
-3 I_{25}(Z, Z') = \left( P^\perp \left\langle R^{TX} \left( R, \frac{\partial}{\partial z_k} \right) R, \frac{\partial}{\partial z_i} \right\rangle \mathcal{P} \right)(Z, Z') \\
\quad = \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial z_k} \right\rangle \mathcal{P}^\perp \circ (z_j z_k P)(Z, Z')
\]

(4.39)
From Theorem 1.1 (1.7) and (4.25), we get
\begin{equation}
(4.40) \quad \frac{1}{2\pi} \left( \mathcal{P}^\perp \circ \left( z_j \bar{z}_k \mathcal{P} \right) \right)(Z, Z') = \frac{1}{2\pi} \left( b_k z_j + \frac{1}{2\pi} (b_j \bar{z}_k + b_k \bar{z}_j) \right) \mathcal{P}(Z, Z'),
\end{equation}
and
\begin{equation}
(4.41) \quad \left( \mathcal{P}^\perp \circ \left( z_j \bar{z}_k \mathcal{P} \right) \right)(Z, Z') = \mathcal{P}^\perp \left[ \frac{1}{4\pi} (b_j \bar{z}_k + b_k \bar{z}_j) \right] \mathcal{P}(Z, Z')
\end{equation}
\begin{equation}
= \left( (z_j - \bar{z}_j)(\bar{z}_k - z_k) + \bar{z}_k (\bar{z}_j - z_j) + z_j (\bar{z}_k - z_k) \right) \mathcal{P}(Z, Z')
\end{equation}
\begin{equation}
= (z_j \bar{z}_k - z_k \bar{z}_j) \mathcal{P}(Z, Z').
\end{equation}
As in [13, (8.3.56), (8.3.63)], we have
\begin{equation}
(4.42) \quad \langle R^T \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) \rangle = \frac{1}{32} |\nabla X J|^2.
\end{equation}
By (3.11), (4.23), (4.39), (4.40), (4.41) and (4.42), we get
\begin{equation}
(4.43) \quad (dz dz I_{25})(0, 0) = -\frac{1}{3} \langle R^T \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) \rangle \mathcal{P}(Z, Z')
\end{equation}
\begin{equation}
= \left[ -\frac{1}{96} |\nabla X J|^2 + \frac{1}{3} \langle R^T \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) \rangle \mathcal{P}(Z, Z') \right] 2\sqrt{-1} \omega(x_0)
\end{equation}
\begin{equation}
+ \frac{1}{3} \langle R^T \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) \rangle \mathcal{P}(Z, Z')
\end{equation}
\begin{equation}
= \frac{1}{3} \langle R^T \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) \rangle \mathcal{P}(Z, Z')
\end{equation}
\begin{equation}
+ \frac{1}{3} \langle R^T \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) \rangle \mathcal{P}(Z, Z')
\end{equation}
\begin{equation}
\frac{1}{2\pi} \left( \mathcal{P}^\perp \circ \left( z_i z_j b_s \mathcal{P} \right) \right)(Z, Z') = \frac{1}{4\pi} \left( -2 \delta_{is} z_j - 2 \delta_{js} z_i + z_i z_j b_s \right) \mathcal{P}(Z, Z'),
\end{equation}
By Theorem 1.1 (1.7), (1.8) and (4.15), we get
\begin{equation}
(4.45) \quad \left( \mathcal{L}^{-1} \mathcal{P}^\perp z_i z_j b_s \mathcal{P} \right)(Z, Z') = \frac{1}{4\pi} \left( -2 \delta_{is} z_j - 2 \delta_{js} z_i + z_i z_j b_s \right) \mathcal{P}(Z, Z'),
\end{equation}
\begin{equation}
(4.46) \quad \left( \mathcal{L}^{-1} \mathcal{P}^\perp z_i z_j b_i b_s \mathcal{P} \right)(Z, Z') = \frac{1}{4\pi} \left( -3 \delta_{js} \delta_i + 3 \delta_{js} \delta_i + \frac{1}{2} \delta_{it} z_i z_j + \frac{1}{2} \delta_{jt} z_i z_j + \frac{1}{2} \delta_{is} z_j b_i + \frac{1}{2} \delta_{ts} z_j b_i + \frac{1}{4} z_i z_j b_i b_s \right) \mathcal{P}(Z, Z').
\end{equation}
By (4.47), we get
\[
\left(\mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{P}\right)(Z, Z') = \left(\mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{P}\right)(Z, Z') = 1
\]
\[
= \frac{1}{4\pi^2} \left\{ -\frac{3}{2\pi} \delta_{it} \delta_{js} - \frac{3}{2\pi} \delta_{jt} \delta_{is} + \frac{3}{2\pi} \delta_{it} \delta_{js} + \frac{3}{2\pi} \delta_{jt} \delta_{is} - \frac{1}{2\pi} \delta_{it} \delta_{js} + \frac{1}{2\pi} \delta_{jt} \delta_{is} - \frac{1}{2\pi} \delta_{it} \delta_{js} - \frac{1}{2\pi} \delta_{jt} \delta_{is} \right\} \mathcal{P}(Z, Z')
\]
\[+ \frac{1}{8\pi^2} \left\{ -2\delta_{is} \delta_{jt} - 2\delta_{is} \delta_{jt} - 2\pi \delta_{is} \delta_{jt} - 2\pi \delta_{is} \delta_{jt} \right\} \mathcal{P}(Z, Z').
\]

By (3.11), (4.44) and (4.47), we get
\[
9(d_{x_0}d_{y_0}I_26)(0, 0) = -\left\langle \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial z_j}, \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z_j}} \right\rangle \omega(x_0)
\]
\[+ \delta_{it} d_{z_j} \wedge d\bar{z}_s + \delta_{jt} d_{z_i} \wedge d\bar{z}_s + \delta_{is} d_{z_j} \wedge d\bar{z}_t + \delta_{js} d_{z_i} \wedge d\bar{z}_t + 2\delta_{it} d_{z_j} \wedge d\bar{z}_t + 2\delta_{jt} d_{z_i} \wedge d\bar{z}_s + 2\delta_{is} d_{z_j} \wedge d\bar{z}_t + 2\delta_{js} d_{z_i} \wedge d\bar{z}_s.
\]

By (4.49),
\[
(d_{x_0}d_{y_0}I_26)(0, 0) = \frac{2}{3} \sqrt{-1} \left\langle \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial z_j}, \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z}_j} \right\rangle \omega(x_0)
\]
\[+ \frac{1}{3} \left\langle \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial z_j}, \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z}_q} + \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z}_i} \right\rangle d_{z_r} \wedge d\bar{z}_q.
\]

Note that for $U, V, W \in TX, \langle JU, V \rangle = \omega(U, V)$, thus (cf. [13, (8.3.48)]),
\[
\langle (\nabla_X^X J)V, W \rangle + \langle (\nabla_Y^X J)W, U \rangle + \langle (\nabla_Z^X J)U, V \rangle = d\omega(U, V, W) = 0.
\]
From Lemma 3.1, (4.50) and $|\frac{\partial}{\partial \bar{z}_j}|^2 = \frac{1}{2}$, we have
\[
\left\langle \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial z_j}, \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z}_j} \right\rangle = 2 \left\langle \left(\nabla_{x_0}^X J \right) J \frac{\partial}{\partial z_j}, \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z}_j} \right\rangle
\]
\[+ \left\langle \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial z_j}, \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z}_q} + \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z}_i} \right\rangle \frac{1}{16} |\nabla_X J|^2.
\]
When we sum (4.51) over $r = q$, we get by (4.42) (cf. [13, (8.3.58)]),
\[
\left\langle \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial z_j}, \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z}_j} \right\rangle = \frac{1}{16} |\nabla_X J|^2.
\]
By (4.7) and (4.51), we get
\[
\left\langle \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial z_j}, \left(\nabla_{x_0}^X J \right) \frac{\partial}{\partial \bar{z}_q} \right\rangle = 2 \mathcal{J}_{ijr} (2 \mathcal{J}_{ijr} - \mathcal{J}_{ijr}).
\]
By Lemma 3.1 (4.50) and $|\frac{\partial}{\partial \bar{z}_j}|^2 = \frac{1}{2}$, we obtain
In particular, we have

\[ (\nabla^X J)_{(z)} u_1 \bigg|_{\overline{z}} = 0, \]

\[ (\nabla^X J)_{(z)} u_1 \bigg|_{\overline{z}} = \left( \nabla^X J \right)_{(z)} u_1 \bigg|_{\overline{z}} = 0, \]

\[ \left( \nabla^X J \right)_{(z)} u_1 \bigg|_{\overline{z}} = \left( \nabla^X J \right)_{(z)} u_1 \bigg|_{\overline{z}} = 0. \]
By (4.57) and (4.60), we get
\[
\left\langle \left( \nabla^X \nabla^X J \right)_{(z, \bar{z}), \bar{z}}, \frac{\partial}{\partial \bar{z}_i} \right\rangle = \left\langle \left[ R^{TX}(z, \bar{z}), J \right] \bar{z}, \frac{\partial}{\partial \bar{z}_i} \right\rangle
\]
(4.61)
\[= -2\sqrt{-1} \left\langle R^{TX}(z, \bar{z}) \bar{z}, \frac{\partial}{\partial \bar{z}_i} \right\rangle.
\]
By (4.57) and (4.59) we get (cf. [13] (8.3.62)),
\[
\left\langle \left( \nabla^X \nabla^X J \right)_{(u_1, u_2), \bar{z}}, \frac{\partial}{\partial \bar{z}_i} \right\rangle = \frac{1}{2\sqrt{-1}} \left\langle \left( \nabla^X_{u_1} J \right) u_2, \left( \nabla^X_{\bar{z}} J \right) \frac{\partial}{\partial \bar{z}_i} - \left( \nabla^X_{\bar{z}} J \right) \bar{z} \right\rangle.
\]
(4.62)
By (4.59), (4.60), (4.61) and (4.62), we get
\[
-\pi \sqrt{-1} \left\langle \left( \nabla^X \nabla^X J \right)_{(R, \bar{R}), R}, \frac{\partial}{\partial \bar{z}_i} \right\rangle = -\frac{\pi}{2} \left\langle \left( \nabla^X_{\bar{z}} J \right) z, \left( \nabla^X_{\bar{z}} J \right) \frac{\partial}{\partial \bar{z}_i} - \left( \nabla^X_{\bar{z}} J \right) \bar{z} \right\rangle
\]
\[= -2\pi \left\langle R^{TX}(z, \bar{z}) \bar{z}, \frac{\partial}{\partial \bar{z}_i} \right\rangle - \pi \sqrt{-1} \left\langle \left( \nabla^X \nabla^X J \right)_{(z, \bar{z}), \bar{z}}, \frac{\partial}{\partial \bar{z}_i} \right\rangle.
\]
By (4.21), (4.58) and (4.63), we get
\[
I_{22}(Z, Z') = -\frac{\pi}{4} \left\langle \left( \nabla^X_{\bar{z}} J \right) \frac{\partial}{\partial \bar{z}_k}, \left( \nabla^X_{\bar{z}} J \right) \frac{\partial}{\partial \bar{z}_s} - \left( \nabla^X_{\bar{z}} J \right) \frac{\partial}{\partial \bar{z}_s} \right\rangle
\]
\[+ \frac{\pi}{3} \left\langle R^{TX} \left( \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial \bar{z}_s}, \frac{\partial}{\partial \bar{z}_s} \right) \frac{\partial}{\partial \bar{z}_i} \right\rangle \left( \mathcal{L}^{-1} b_{i} z_{j} z_{k} \mathcal{P} \right)(Z, Z')
\]
(4.64)
\[= -\frac{4\pi}{3} \left\langle R^{TX} \left( \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial \bar{z}_s}, \frac{\partial}{\partial \bar{z}_s} \right) \frac{\partial}{\partial \bar{z}_i} \right\rangle \left( \mathcal{L}^{-1} b_{i} z_{j} z_{k} \mathcal{P} \right)(Z, Z')
\]
\[= -\frac{\pi}{2} \sqrt{-1} \left\langle \mathcal{L}^{-1} b_{i} \left( \nabla^X \nabla^X J \right)_{(z, \bar{z}), \bar{z}}, \frac{\partial}{\partial \bar{z}_i} \right\rangle \mathcal{P}(Z, Z').
\]
By (1.7) and (1.8), we get
\[
b_{i} z_{j} z_{k} \mathcal{P} = b_{i} z_{j} z_{k} \left( \frac{b_{s}}{2\pi} + \mathcal{P} \right)
\]
(4.65)
\[= \left( \frac{b_{i} b_{s}}{2\pi} z_{j} z_{k} + \frac{\delta_{js}}{\pi} b_{i} z_{k} + \frac{\delta_{ks}}{\pi} b_{i} z_{j} + b_{i} z_{j} z_{k} \mathcal{P} \right) \mathcal{P}.
\]
Thus, by Theorem 1.1 (1.7), (1.8) and (4.65), as in (4.47), we get
\[
\left( \mathcal{L}^{-1} b_{i} z_{j} z_{k} \mathcal{P} \right)(Z, Z')
\]
\[= \left( \frac{b_{i} b_{s}}{16\pi^2} z_{j} z_{k} + \frac{\delta_{js}}{4\pi^2} b_{i} z_{k} + \frac{\delta_{ks}}{4\pi^2} b_{i} z_{j} + \frac{b_{i}}{4\pi} z_{j} z_{k} \mathcal{P} \right) \mathcal{P}(Z, Z')
\]
(4.66)
\[= \frac{1}{4\pi^2} \left( -\delta_{js} \delta_{ik} + \pi \delta_{js} z_{k} (\mathcal{P} - \mathcal{P}) - \delta_{ks} \delta_{ij} + \pi \delta_{ks} z_{j} (\mathcal{P} - \mathcal{P}) \right)
\]
\[= \frac{1}{4\pi^2} \left( -2\delta_{ij} z_{k} - 2\delta_{ik} z_{j} + 2\pi z_{j} z_{k} (\mathcal{P} - \mathcal{P}) \right) \mathcal{P}(Z, Z').
\]
By (4.6) and (4.66), we get

\[
(4.67) \quad \left( \mathcal{L}^{-1} b_i z_j z_k \overline{z}_s P \right)(Z, Z') \approx -\left[ \frac{1}{4\pi^2} \left( \delta_{js} \delta_{ik} + \delta_{ks} \delta_{ij} \right) + \frac{1}{4\pi} \left( \delta_{js} \delta_{ik} \overline{z}_t + \delta_{ks} \delta_{ij} z_t + \delta_{ij} z_t \overline{z}_s + \delta_{ik} z_t \overline{z}_s \right) \right] P(Z, Z').
\]

Again by (1.7) and (1.8),

\[
(4.68) \quad b_i z_j \overline{z}_s \overline{z}_t P \ = b_i z_j \left( \frac{b_i}{2\pi} + \overline{z}_s \right) \left( \frac{b_i}{2\pi} + \overline{z}_t \right) P
\]

\[= \left[ \frac{b_i b_i}{4\pi^2} z_j + \frac{1}{2\pi^2} \left( \delta_{js} b_i b_t + \delta_{jt} b_i b_s \right) + \frac{b_i}{2\pi} \left( b_s z_j + 2\delta_{js} \right) \overline{z}_t + \frac{b_i}{2\pi} \left( b_t z_j + 2\delta_{jt} \right) \overline{z}_s + b_i z_j \overline{z}_s \overline{z}_t \right] P.
\]

By Theorem 1.1 and (4.68), we get

\[
(4.69) \quad \mathcal{L}^{-1} b_i z_j \overline{z}_s \overline{z}_t P = \left[ \frac{b_i b_i b_i}{48\pi^3} z_j + \frac{1}{16\pi^3} \left( \delta_{js} b_i b_t + \delta_{jt} b_i b_s \right) + \frac{b_i}{16\pi^2} \left( b_s z_j + 4\delta_{js} \right) \overline{z}_t + \frac{1}{16\pi^2} \left( b_t z_j + 4\delta_{jt} \right) \overline{z}_s + \frac{b_i}{4\pi} z_j \overline{z}_s \overline{z}_t \right] P.
\]

By (1.7), (1.8) and (4.6), we get

\[
(4.70) \quad \left( \frac{b_i}{16\pi^2} \left( b_s z_j + 4\delta_{js} \right) \overline{z}_t P \right)(Z, Z') \approx \frac{1}{4\pi} \left( \delta_{js} \overline{z}_s - \delta_{ij} \overline{z}_s \right) \overline{z}_t P(Z, Z').
\]

By (4.32), (4.69) and (4.70), we obtain

\[
(4.71) \quad \left( d_x d_y \left( \mathcal{L}^{-1} b_i z_j \overline{z}_s \overline{z}_t P \right) \right)(0, 0) = \frac{1}{4\pi} \left( \delta_{js} d \overline{z}_t - \delta_{ij} d \overline{z}_s \right) \wedge d \overline{z}_t + \frac{1}{4\pi} \left( \delta_{jj} d \overline{z}_t - \delta_{ij} d \overline{z}_s \right) \wedge d \overline{z}_s
\]

\[= \frac{1}{4\pi} \left( \delta_{js} d \overline{z}_t + d \overline{z}_t + \delta_{jj} d \overline{z}_s + d \overline{z}_s \right).
\]

Finally, by Theorem 1.1, (1.7) and (1.8), as in (4.29), we get

\[
(4.72) \quad \left( \mathcal{L}^{-1} b_i \overline{z}_j \overline{z}_s \overline{z}_t P \right)(Z, Z') \approx \left( \mathcal{L}^{-1} b_i \left( \frac{b_j}{2\pi} + \overline{z}_j \right) \left( \frac{b_j}{2\pi} + \overline{z}_s \right) \left( \frac{b_j}{2\pi} + \overline{z}_t \right) P \right)(Z, Z')
\]

\[\sim \left\{ \mathcal{L}^{-1} b_i \left[ \frac{b_j b_i b_t}{8\pi^3} + \frac{1}{4\pi^2} \left( b_j b_t \overline{z}_t + b_j b_i \overline{z}_s + b_j b_i \overline{z}_j \right) \right] P \right\}(Z, Z') \approx 0.
\]
From (3.11), (4.64), (4.67), (4.71) and (4.72), we obtain

\[
(4.73) \quad (d_x d_y I_{22})(0, 0) = \left\{ \frac{1}{8\pi} \left( \left\langle \left( \nabla^X_{\bar{z}_i} J \right) \frac{\partial}{\partial z_i}, \left( \nabla^X_{\bar{z}_j} J \right) \frac{\partial}{\partial \bar{z}_j} + \left( \nabla^X_{\bar{z}_j} J \right) \frac{\partial}{\partial z_i} \right) \right\} \left( -2\pi \sqrt{-1} \right) \omega(x_0)
\]

\[
+ \frac{1}{12\pi} \left\langle R^{TX} \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) \frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_j} \right\rangle \left( -2\pi \sqrt{-1} \right) \omega(x_0)
\]

\[
+ \frac{1}{8} \left\langle \left( \nabla^X_{\bar{z}_i} J \right) \frac{\partial}{\partial z_i}, \left( \nabla^X_{\bar{z}_j} J \right) \frac{\partial}{\partial \bar{z}_j} + \left( \nabla^X_{\bar{z}_j} J \right) \frac{\partial}{\partial z_i} \right\rangle \left( dz_r \wedge d\bar{z}_q \right)
\]

\[
+ \frac{1}{6} \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_i} \right) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right\rangle \left( dz_r \wedge d\bar{z}_q \right)
\]

\[
+ \frac{1}{3} \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_i} \right) \frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_j} \right\rangle \left( d\bar{z}_r \wedge d\bar{z}_q \right)
\]

Note that by (4.23),

\[
(4.74) \quad R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_i} \right) \frac{\partial}{\partial \bar{z}_i} = R^{TX} \left( \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial \bar{z}_i} + R^{TX} \left( \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial \bar{z}_i} \right) \frac{\partial}{\partial z_i}.
\]

By (4.23), (4.42), (4.52), (4.53), (4.54), (4.55), (4.73) and (4.74), we get

\[
(4.75) \quad (d_x d_y I_{22})(0, 0) = -\sqrt{-1} \left[ \frac{5}{96} \left( \nabla^X J \right)^2 + \frac{1}{3} \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_i} \right) \frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_j} \right\rangle \right] \omega(x_0)
\]

\[
+ \frac{1}{4} \left[ \mathcal{J}_{jir} \left( 5 \mathcal{J}_{j\bar{z}_r} - 4 \mathcal{J}_{i\bar{z}_r} \right) \right] \left( dz_r \wedge d\bar{z}_q \right)
\]

\[
+ \frac{1}{6} \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_r} \right) \frac{\partial}{\partial \bar{z}_r}, \frac{\partial}{\partial z_j} \right\rangle \left( dz_r \wedge d\bar{z}_q \right)
\]

\[
+ \frac{1}{3} \left\langle R^{TX} \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_r} \right) \frac{\partial}{\partial \bar{z}_r}, \frac{\partial}{\partial \bar{z}_j} \right\rangle \left( d\bar{z}_r \wedge d\bar{z}_q \right)
\]

By (4.57), (4.59) and (4.60), we get (cf. [14] (2.28)),

\[
\left\langle \left( \nabla^X \nabla^X J \right)_{(R, \bar{R})} \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right\rangle = 2 \left\langle \left( \nabla^X \nabla^X J \right)_{(z, \bar{\tau})} \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial \tau_i} \right\rangle
\]

\[
= -\sqrt{-1} \left\langle \left( \nabla^X J \right)_{(\bar{z}, \tau)} \frac{\partial}{\partial \tau_i}, \left( \nabla^X J \right) \frac{\partial}{\partial \bar{z}_j} \right\rangle.
\]

By (4.25) and (4.35), we get

\[
(4.77) \quad \left( \mathcal{L}^{-1} \mathcal{P} \right)_{z_{\bar{z}}} (Z, Z') \left( \mathcal{P} (Z, Z') \right) = -\frac{1}{4\pi^2} \left( \delta_{zt} - \pi z_t \left( \bar{z}_{\bar{t}} - z_{\bar{t}} \right) \right) \mathcal{P} (Z, Z').
\]
By (3.11), (4.42), (4.54), (4.76) and (4.77), we obtain
\[
(d_x d_y I_{24})(0,0) = \left(\nabla^X J \frac{\partial}{\partial z_j}, \nabla^X J \frac{\partial}{\partial \overline{z}_i}\right)(-\sqrt{-1}) \omega(x_0)
\]
(4.78)
\[
+ \frac{1}{2} \left(\nabla^X J \frac{\partial}{\partial z_j}, \nabla^X J \frac{\partial}{\partial \overline{z}_i}\right) dz_r \wedge d\overline{z}_q
\]
\[
= -\frac{\sqrt{-1}}{8} |\nabla^X J|^2 \omega(x_0) + 2 \mathcal{J}_{ijr} (\mathcal{J}_{\overline{r}q} - \mathcal{J}_{\overline{r}i}) dz_r \wedge d\overline{z}_q.
\]

Combining (4.22), (4.23), (4.33), (4.38), (4.42), (4.43), (4.56), (4.74), (4.75) and (4.78), we obtain
\[
(d_x d_y I_2)(0,0) = -\sqrt{-1} \left(\nabla^X J \frac{\partial}{\partial z_j}, \nabla^X J \frac{\partial}{\partial \overline{z}_i}\right) \omega(x_0)
\]
(4.79)
\[
- \frac{1}{2} \left(\nabla^X J \frac{\partial}{\partial z_j}, \nabla^X J \frac{\partial}{\partial \overline{z}_i}\right) dz_r \wedge d\overline{z}_q - \frac{1}{12} \mathcal{J}_{ijr} (\mathcal{J}_{\overline{r}q} + 4 \mathcal{J}_{\overline{r}i}) dz_r \wedge d\overline{z}_q.
\]

By Lemma 3.1, (4.50), (4.57), (4.51), (4.54), (4.55) and (4.62), we get (cf. [13, (8.3.63)])
\[
\left(\nabla^X J \frac{\partial}{\partial z_j}, \nabla^X J \frac{\partial}{\partial \overline{z}_i}\right) = \frac{-\sqrt{-1}}{2} \left(\nabla^X J \frac{\partial}{\partial z_j}, \nabla^X J \frac{\partial}{\partial \overline{z}_i}\right)
\]
(4.80)
\[
= \frac{1}{4} \left(\nabla^X J \frac{\partial}{\partial z_j} - \nabla^X J \frac{\partial}{\partial \overline{z}_i}, \nabla^X J \frac{\partial}{\partial \overline{z}_i} - \nabla^X J \frac{\partial}{\partial z_j}\right)
\]
\[
= \frac{1}{2} \mathcal{J}_{ijr} \mathcal{J}_{\overline{r}q}.
\]

Substituting (4.80) into (4.79) yields
\[
(d_x d_y I_2)(0,0) = -\sqrt{-1} \left(\nabla^X J \frac{\partial}{\partial z_j}, \nabla^X J \frac{\partial}{\partial \overline{z}_i}\right) \omega(x_0) + \frac{1}{3} \mathcal{J}_{ijr} (\mathcal{J}_{\overline{r}q} + 4 \mathcal{J}_{\overline{r}i}) dz_r \wedge d\overline{z}_q.
\]

4.4. Proof of Theorem 3.3

By (4.4) and (4.81), we get
\[
(d_x d_y I_4)(0,0) = (d_x d_y I_2)(0,0).
\]

Substituting (4.12), (4.13), (4.17), (4.19), (4.81) and (4.82) into (4.2), we finally obtain
\[
(d_x d_y \mathcal{F}_2)(0,0) = -\frac{6}{9} \mathcal{J}_{ijr} (\mathcal{J}_{\overline{r}q} + 4 \mathcal{J}_{\overline{r}i}) dz_r \wedge d\overline{z}_q + 2 (d_x d_y I_2)(0,0)
\]
(4.83)
\[
= -2\sqrt{-1} \left(\nabla^X J \frac{\partial}{\partial z_j}, \nabla^X J \frac{\partial}{\partial \overline{z}_i}\right) \omega(x_0).
\]

By [13] Theorem 8.3.4, Lemma 8.3.10,
\[
8 \left(\nabla^X J \frac{\partial}{\partial z_j}, \nabla^X J \frac{\partial}{\partial \overline{z}_i}\right) = \nu^X + \frac{1}{4} |\nabla^X J|^2 = 8\pi b_1(x_0).
\]

The identities (4.83) and (4.84) yield Theorem 3.3. This concludes the proof of Theorem 0.1.
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