GABRIEL-KRULL DIMENSION AND MINIMAL ATOMS IN GROTHENDIECK CATEGORIES

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ABSTRACT. Let \( \mathcal{A} \) be a Grothendieck category. In this paper, we classify localizing subcategories of a semi-noetherian category \( \mathcal{A} \) through open subsets of \( \text{ASpec} \mathcal{A} \). For a semi-noetherian locally coherent category \( \mathcal{A} \), we introduce a new topology on \( \text{ASpec} \mathcal{A} \) and we show that it is homeomorphic to the Ziegler spectrum \( \text{Zg} \mathcal{A} \). Moreover, for a locally coherent category \( \mathcal{A} \), we provide a new characterization of localizing subcategories of finite type of \( \mathcal{A} \). We define a dimension for objects using the preorder \( \leq \) on \( \text{ASpec} \mathcal{A} \), which serves as a lower bound of the Gabriel-Krull dimension. Finally, we investigate the minimal atoms of a noetherian object \( M \) in \( \mathcal{A} \) and establish sufficient conditions for the finiteness of the number of minimal atoms of \( M \).

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1. Introduction

The Gabriel spectrum of a Grothendieck category \( \mathcal{A} \) equipped with a topology is the set of isomorphism classes of indecomposable injective objects which can be viewed as a generalization of the spectrum of a commutative ring. This topology plays a key role in identifying localizing subcategories of a Grothendieck category (see [G,Kr]).

For a locally coherent Grothendieck category, there is an alternative topology on the set of isomorphism classes of indecomposable injective objects. Ziegler [Z] associated to a ring \( A \), a topological space whose points are the isomorphism classes of pure-injective indecomposable left \( A \)-modules. This space is homeomorphic to the \( \text{Zg}(\mathcal{C}) \) whose points are the isomorphism classes of the indecomposable injective objects of \( \mathcal{C} = \text{mod}\mathcal{A}, \mathcal{A}b \) and the collection \( O(C) = \{ E \in \text{Zg}(\mathcal{C}) | \text{Hom}(C,E) \neq 0 \} \) forms a basis for \( \text{Zg}(\mathcal{C}) \) in which \( C \) ranges over coherent objects in \( \mathcal{C} \). Herzog [H] extended the Ziegler spectrum to the locally coherent Grothendieck categories.

For an abelian category \( \mathcal{A} \), which does not have necessarily enough injective objects, Kanda [K1, K2], introduced the atom spectrum \( \text{ASpec} \mathcal{A} \) of \( \mathcal{A} \). This construction is inspired by monoform modules and their equivalence relation over non-commutative rings, as explored by Storerr [St]. When \( \mathcal{A} \) is a Grothendieck category, \( \text{ASpec} \mathcal{A} \) is a set. Kanda [K2] constructed a topology on \( \text{ASpec} \mathcal{A} \) in which the open subsets of \( \text{ASpec} \mathcal{A} \) correspond to the specialization closed subsets of \( \text{Spec} \mathcal{A} \) when \( \mathcal{A} \) is a commutative ring.

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Unfortunately, Grothendieck categories do not generally possess enough atoms, which limits our ability to find out further insights about $\mathcal{A}$. However, in the case where $\mathcal{A}$ is a locally noetherian Grothendieck category, Kanda [K4] proved that $\mathcal{A}$ does have enough atoms and $\text{ASpec}\mathcal{A}$ is homeomorphic to $Zg\mathcal{A}$. In this paper, we show that a semi-noetherian category $\mathcal{A}$ possesses enough atoms and establish a one-to-one correspondence between localizing subcategories of $\mathcal{A}$ and open subsets of $\text{ASpec}\mathcal{A}$. For a locally coherent Grothendieck category $\mathcal{A}$, we provide a classification of localizing subcategories of finite type of $\mathcal{A}$. We study the Gabriel-Krull dimension of objects and we introduce a new dimension for objects based on the preorder $\preceq$ on $\text{ASpec}\mathcal{A}$. Furthermore, we study the minimal atoms of noetherian objects in a Grothendieck category. Throughout this paper, except for Section 2, we assume that $\mathcal{A}$ is a Grothendieck category.

Section 2 is devoted to study Alexandroff and Kolmogorov spaces. As Alexandroff spaces are uniquely determined by their specialization preorders $[\mathcal{A}]$, we study category of preorder sets. In Theorem 2.2, we show that there exist functors $T: \mathcal{T} \to \mathcal{P}$ and $S: \mathcal{P} \to \mathcal{T}$ between the category of topological spaces $\mathcal{T}$ and the category of preorder sets $\mathcal{P}$ such that $S$ is the left adjoint of $T$. As a conclusion, a topological space $Y$ is Alexandroff if and only if the canonical morphism $\psi_Y: STY \to Y$ is homeomorphism. Furthermore, if $P$ is a partially ordered set, then $SP$ is an Alexandroff Kolmogorov space.

In Section 3, we study the preorder $\preceq$ on $\text{ASpec}\mathcal{A}$. We show if $\mathcal{X}$ is a localizing subcategory of $\mathcal{A}$ and $\text{ASpec}\mathcal{A}$ is Alexandroff, then $\text{ASpec}\mathcal{A}/\mathcal{X}$ is Alexandroff. An atom $\alpha \in \text{ASpec}\mathcal{A}$ is maximal, if there exists a simple object $S$ in $\mathcal{A}$ such that $\alpha = S$. If $\mathcal{A}$ is a locally finitely generated Grothendieck category with the Alexandroff space $\text{ASpec}\mathcal{A}$, then we show that an atom in $\text{ASpec}\mathcal{A}$ is maximal if and only if it is maximal under $\preceq$; see Lemma 3.1.

In Section 4, we study semi-noetherian categories. A Gabriel-Krull filtration $\{A_\alpha\}_\sigma$ of $\mathcal{A}$ is defined by a transfinite induction on ordinals $\sigma$. The category $\mathcal{A}$ is said to be semi-noetherian if $\mathcal{A} = \cup \sigma A_\sigma$. The Gabriel-Krull dimension of an object $M$ in $\mathcal{A}$, denoted by $\text{GK-dim} M$, is the least ordinal $\sigma$ such that $M \in A_\sigma$. In Proposition 4.5, we show that every noetherian object $M$ in $\mathcal{A}$ has Gabriel-Krull dimension. For an ordinal $\sigma$, an object $M$ in $\mathcal{A}$ is $\sigma$-critical if $\text{GK-dim} M = \sigma$ while $\text{GK-dim} M/N < \sigma$ for every nonzero subobject $N$ of $M$. We prove the following theorem.

**Theorem 1.1.** Let $\sigma$ be an ordinal. Then $A_\sigma$ is generated by all $\delta$-critical objects in $\mathcal{A}$ with $\delta \leq \sigma$.

We show that semi-noetherian categories have enough atoms; see Corollary 4.11. The following theorem is one of the main results of this section.

**Theorem 1.2.** Let $\mathcal{A}$ be a semi-noetherian category. Then the map $\mathcal{X} \mapsto \text{ASupp}\mathcal{X}$ provides a one-to-one correspondence between localizing subcategories of $\mathcal{A}$ and open subsets of $\text{ASpec}\mathcal{A}$. The inverse map is given by $U \mapsto \text{ASupp}^{-1}U$.

The Grothendieck category $\mathcal{A}$ is said to be locally coherent provided $\mathcal{A}$ has a generating set of finitely presented objects and the full subcategory $\text{fp-}\mathcal{A}$ of finitely presented objects in $\mathcal{A}$ is abelian. Assume that $\text{Sp}\mathcal{A}$ is the set of isomorphism classes of indecomposable injective objects in $\mathcal{A}$. Krause [Kr] has constructed a topology on $\text{Sp}\mathcal{A}$ in which for a subset $\mathcal{U}$ of $\text{Sp}\mathcal{A}$, the closure of $\mathcal{U}$ is defined as $\overline{\mathcal{U}} = (\mathcal{U} \cup \text{fp-}\mathcal{A})^\wedge$. The subsets $\mathcal{U}$ of $\text{Sp}\mathcal{A}$ satisfying $\overline{\mathcal{U}} = \mathcal{U}$ form the closed subsets of $\text{Sp}\mathcal{A}$. In Section 5, we show that $Zg(\mathcal{A})$ and $\text{Sp}\mathcal{A}$ have the same topologies; see Proposition 5.1. We define a new topology on $\text{ASpec}\mathcal{A}$ in which $\{\text{ASupp}\ M \mid M \in \text{fp-}\mathcal{A}\}$ forms a basis of open subsets for $\text{ASpec}\mathcal{A}$. We use the symbol $\text{ZASpec}\mathcal{A}$ instead of $\text{ASpec}\mathcal{A}$ with this topology. We show that $\text{ZASpec}\mathcal{A}$ is a topological subspace of $Zg\mathcal{A}$ and for semi-noetherian categories we have the following theorem.

**Theorem 1.3.** Let $\mathcal{A}$ be a semi-noetherian locally coherent category. Then $\text{ZASpec}\mathcal{A}$ is homeomorphic to $Zg\mathcal{A}$.

Moreover, there is a one-to-one correspondence between open subsets of $\text{ZASpec}\mathcal{A}$ and Serre subcategories of $\text{fp-}\mathcal{A}$; see Proposition 5.10. For an object $M$ in $\mathcal{A}$, we define $\text{ZSupp}\ M$, the Ziegler support of $M$ that is $\text{ZSupp}\ M = \{I \in Zg\mathcal{A} \mid \text{Hom}(M, I) \neq 0\}$. Also, for a subcategory $\mathcal{X}$ of $\mathcal{A}$, we define $\text{ZSupp}\mathcal{X} = \bigcup_{M \in \mathcal{X}} \text{ZSupp}\ M$ and for every subset $\mathcal{U}$ of $Zg\mathcal{A}$, we define $\text{ZSupp}^{-1}\mathcal{U} = \{M \in \text{ZSupp}\mathcal{X} \mid \text{ZSupp}\ M \subseteq \mathcal{U}\}$. We use the symbol $\text{ZSSpec}\mathcal{A}$ instead of $\text{ZASpec}\mathcal{A}$ with this topology. We show that $\text{ZSSpec}\mathcal{A}$ is a topological subspace of $Zg\mathcal{A}$ and for semi-noetherian categories we have the following theorem.
A | ZSupp M ⊆ U \}. These new concepts allow us to classify localizing subcategories of finite type of A as follows.

**Theorem 1.4.** The map \( U \mapsto ZSupp^{-1}U \) provides a one-to-one correspondence between open subsets of \( ZgA \) and localizing subcategories of finite type of A. The inverse map is given by \( \mathcal{X} \mapsto ZSupp \mathcal{X} \).

As a conclusion of the above theorem, a localizing subcategory \( \mathcal{X} \) of A is of finite type if and only if \( ZSupp \mathcal{X} \) is an open subset of \( ZgA \). Furthermore, if A is semi-noetherian, then \( \mathcal{X} \) is of finite type if and only if \( ASupp \mathcal{X} \) is an open subset of \( ZASpec A \). As \( fpA \) is an abelian category, the atom spectrum of \( fpA \) can be investigated independently. For every object \( M \) in \( A \), we use the symbol \( fASupp M \) for atom support of \( M \) in \( ASpec fpA \) instead of \( ASupp M \) and \( fAAss M \) instead of \( AAss M \). We show that if \( A \) is semi-noetherian, then \( ASpec fpA \) is a topological subspace of \( ZASpec A \). Moreover, if \( M \) is a finitely presented object in \( A \), we have \( AAss M \subseteq fAAss M \) and if \( A \) is semi-noetherian, then the equality holds. We show that a monoform object in \( fpA \) is uniform in A.

The concept of Krull dimension, traditionally defined for modules over a commutative ring through chains of prime ideals, has been extensively studied and utilized. Gabriel and Rentschler [GRe] defined this notion for certain modules over noncommutative rings, showing that it coincides with the classical definition for modules over commutative rings (for example see [GR, GW, MR]).

In Section 6, for every object \( M \) in \( A \), we define the dimension of \( M \), denoted by \( dim M \), in a manner analogous to the Krull dimension of modules over a commutative ring, using the preorder \( \leq \) on \( ASpec A \). We show that \( dim M \) can be served as a lower bound for \( GK-dim M \). To be more precise, we have the following theorem.

**Theorem 1.5.** Let \( M \) be an object in \( A \) with Gabriel-Krull dimension. Then \( dim M \leq GK-dim M \). Moreover, If \( A \) is locally finitely generated with Alexandroff space \( ASpec A \) and \( GK-dim M \) is finite, then \( dim M = GK-dim M \).

It should be noted that these two dimensions may not coincide if \( ASpec A \) is not Alexandroff even if \( A \) is locally noetherian; see Example 6.10. It is a natural question to ask whether Gabriel-Krull dimension of an object is finite if its dimension is finite. As a Grothendieck category does not have enough atoms, the question may have a negative answer. However, for a locally finitely generated category \( A \) with Alexandroff space \( ASpec A \), a slightly weaker result exists. More precisely, if \( M \) is an object in \( A \) and \( n \) is a non-negative integer such that \( dim M = n \), then \( ASupp M \subseteq ASupp \Lambda \). In particular, if \( M \) has Gabriel-Krull dimension, then \( GK-dim M = n \).

In Section 7, we investigate the minimal atoms of an object in \( A \). We show that if \( A \) is a semi-noetherian category and \( M \) is an object in \( A \), then for every \( \alpha \in ASupp M \), there exists an atom \( \beta \) in \( AMin M \) such that \( \beta \leq \alpha \). The main aim of this section is to study finiteness of the number of minimal atoms of a noetherian object. We prove the following theorem.

**Theorem 1.6.** Let \( M \) be a noetherian object in \( A \). If \( A(\alpha) \) is an open subset of \( ASpec A \) for every \( \alpha \in AMin M \), then \( AMin M \) is a finite set.

If \( ASpec A \) is Alexandroff, then the assumption in the above theorem are satisfied. We remark that the above theorem may not hold if \( ASpec A \) is not Alexandroff even if \( A \) is a locally noetherian category; see Example 7.20. We also concern to study the compressible modules \([Sm] \) in a fully right bounded ring \( A \) which have a key role in the finiteness of the number of minimal atoms. We show that for a fully right bounded ring \( A \), the atom spectrum \( ASpec Mod-A \) is Alexandroff and \( AMin M \) is a finite set for every noetherian right \( A \)-module \( M \). We give an example which shows this result is not true if \( A \) is not fully right bounded. We prove that if \( M \) is a noetherian object in \( A \), then the set of minimal atom of \( M \) is finite provided that \( A \) has a noetherian projective generator \( U \) such that \( End(U) \) is a fully right bounded ring; see Corollary 7.10.
2. The category of preorder sets

In this section, we recall form [A] some well-known results about the preorder sets and that they are in close relation with the topological spaces.

A set $X$ is said to be a preorder set if whenever it is equipped with a preorder relation $\leq$ (i.e. transfinite and reflexive relation $\leq$). Let $X$ be a topological space and $x \in X$. We denote by $U_x$, the intersection of all open subsets of $X$ containing $x$. We define a preorder relation $\leq$ on $X$ as follows: for every $x,y \in X$ we have $x \leq y$ if for every open subset $U$ of $X$, the condition $x \in U$ implies that $y \in U$; in other words if $U_y \subseteq U_x$. It is easy to see that if a map $f : X \rightarrow Y$ of topological spaces is continuous, then it is preorder-preserving (i.e. for every $x_1, x_2 \in X$ the condition $x_1 \leq x_2$ implies that $f(x_1) \leq f(x_2)$). Let $\mathcal{T}$ be the category of topological spaces and let $\mathcal{P}$ be the category of preorder sets in which the morphisms are preorder-preserving maps. Then in view of the above argument, there exists a functor $T : \mathcal{T} \rightarrow \mathcal{P}$ given by $T(X) = X$.

**Definition 2.1.** A topological space $X$ is said to be Alexandroff if the intersection of any family of open subsets of $X$ is open.

Every preorder set $X$ can be equipped with a topology as follows: for any $x \in X$, let $A(x) = \{ y \in X | x \leq y \}$. The system $\{ A(x) | x \in X \}$ forms a basis for a topology on $X$ that makes $X$ into an Alexandroff space. Given a preorder-preserving map $g : P_1 \rightarrow P_2$ of preorder sets, for every $x \in P_1$, we have $A(x) \subseteq g^{-1}(A(g(x)))$. Hence, it is straightforward to show that $g$ is a continuous map of topological spaces, when $P_1$ and $P_2$ are considered as topological spaces as mentioned. Then we have a functor $S : \mathcal{P} \rightarrow \mathcal{T}$ such that for any preorder set $P$, the topological space $SP = P$ is defined as mentioned above. We now have the following theorem.

**Theorem 2.2.** There exist adjoint functors $T : \mathcal{T} \rightarrow \mathcal{P}$ and $S : \mathcal{P} \rightarrow \mathcal{T}$ between the category of topological spaces $\mathcal{T}$ and the category of preorder sets $\mathcal{P}$ such that $S$ is a left adjoint of $T$.

**Proof.** We define the functors $T$ and $S$ as above. Suppose that $X \in \mathcal{P}$, $Y \in \mathcal{T}$, and $f : SX \rightarrow Y$ is a continuous function of topological spaces. We assert that $f : X \rightarrow TY$ is a preorder-preserving map. For any $a,b \in X$ with $a \leq b$, assume that $U$ is open subset $Y$ such that $f(a) \in U$. The condition $a \leq b$ implies $f(b) \in U$ so that $f(a) \leq f(b)$. Now assume that $g : X \rightarrow TY$ is a preorder-preserving map of preorder sets. For every open subset $U$ of $Y$ and any $a \in g^{-1}(U)$, it is straightforward that $A(a) \subseteq g^{-1}(U)$ so that $g^{-1}(U)$ is an open subset of $SX$; consequently $g : SX \rightarrow Y$ is continuous.

For any $X \in \mathcal{P}$ and $Y \in \mathcal{T}$, assume that $\Theta_{X,Y} : \text{Hom}_\mathcal{P}(X,TY) \rightarrow \text{Hom}_\mathcal{T}(SX,Y)$ is the bijective function in Theorem [2.2]. Then $\eta_X = \Theta_{X,SX}^{-1}(1_{SX}) = X \rightarrow TSX$ is a preorder-preserving function of preorder sets which is natural in $X$. It is clear that $\eta_X$ is isomorphism for any preorder set $X$. On the other hand, $\psi_Y = \Theta_{TY,Y}(1_{TY}) = STY \rightarrow Y$ is a continuous function of topological spaces which is natural in $Y$. We have the following corollary.

**Corollary 2.3.** Let $Y$ be a topological space. Then $Y$ is Alexandroff if and only if $\psi_Y$ is homeomorphism (i.e. an isomorphism of topological spaces).

**Proof.** Straightforward.

A topological space $X$ is said to be Kolmogorov (or $T_0$-space) if for any distinct points $x, y$ of $X$, there exists an open subset of $X$ containing exactly one of them; in other words $U_x \neq U_y$. We have the following corollary.

**Corollary 2.4.** The following conditions hold.

(i) If $X$ is a Kolmogorov space, then $TX$ is a partially ordered set.

(ii) If $P$ is a partially ordered set, then $SP$ is an Alexandroff Kolmogorov space.
Remark 2.5. Let $X$ be a topological space. An equivalence relation on $X$ is defined as follows: for every $x, y \in X$, we say $x \sim y$ if and only if $U_x = U_y$; equivalently, if $x \leq y$ and $y \leq x$. We denote by $\sim$, the quotient topological space $X/ \sim$ together with the canonical continuous function $\nu : X \to X/ \sim$. For every $x \in X$, it is straightforward that $\nu^{-1}(\nu(U_x)) = U_x$. If $X$ is Alexandroff, $U_x$ is an open subset of $X$ and so $\nu(U_x)$ is an open subset of $X/ \sim$. This fact forces $\nu(U_x) = U_{\nu(x)}$ so that $X/ \sim$ is a Kolmogorov space.

3. Atom spectrum and Alexandroff topological spaces

In this section we recall from [K1, K2] some definitions on atom spectrum of an abelian category $\mathcal{A}$. We also give some basic results in this area.

Definition 3.1. (1) An abelian category $\mathcal{A}$ with a generator is said to be a Grothendieck category if it has arbitrary direct sums and direct limits of short exact sequences are exact, this means that if a direct system of short exact sequences in $\mathcal{A}$ is given, then the induced sequence of direct limits is a short exact sequence.

(2) An object $M$ in $\mathcal{A}$ is finitely generated if whenever there are subobjects $M_i \subseteq M$ for $i \in I$ satisfying $M = \sum M_i$, then there is a finite subset $J \subseteq I$ such that $M = \sum M_i$. A category $\mathcal{A}$ is said to be locally finitely generated if it has a small generating set of finitely generated objects.

(3) A category $\mathcal{A}$ is said to be locally noetherian if it has a small generating set of noetherian objects.

Definition 3.2. (i) A nonzero object $M$ in $\mathcal{A}$ is monoform if for any nonzero subobject $N$ of $M$, there exists no common nonzero subobject of $M$ and $M/N$ which means that there does not exist a nonzero subobject of $M$ which is isomorphic to a subobject of $M/N$. We denote by $\text{ASpec}_0 \mathcal{A}$, the set of all monoform objects in $\mathcal{A}$.

Two monoform objects $H$ and $H'$ in $\mathcal{A}$ are said to be atom-equivalent if they have a common nonzero subobject. The atom equivalence establishes an equivalence relation on monoform objects. For every monoform object $H$ in $\mathcal{A}$, we denoted by $\overline{\text{H}}$, the equivalence class of $H$, that is

$\overline{\text{H}} = \{G \in \text{ASpec}_0 \mathcal{A} | H \text{ and } G \text{ have a common nonzero subobject}\}$.

Definition 3.3. The atom spectrum $\text{ASpec} \mathcal{A}$ of $\mathcal{A}$ is the quotient class of $\text{ASpec}_0 \mathcal{A}$ consisting of all equivalence classes induced by this equivalence relation; in other words

$\text{ASpec} \mathcal{A} = \{\overline{\text{H}} | H \in \text{ASpec}_0 \mathcal{A}\}$.

Any equivalence class is called an atom of $\text{ASpec} \mathcal{A}$.

In the rest of this section $\mathcal{A}$ is a Grothendieck category and so $\text{ASpec} \mathcal{A}$ is a set in this case. The main propose of this section is to determine when the topological space $\text{ASpec} \mathcal{A}$ is Alexandroff. It follows from [K2, Proposition 3.3] that for any commutative ring $\mathcal{A}$, the topological space $\text{ASpec}(\text{Mod} \mathcal{A})$ is Alexandroff.

From the previous section, we have the following corollary.

Corollary 3.4. $\text{ST}(\text{ASpec} \mathcal{A})$ is an Alexandroff Kolmogrov space.

Proof. According to [K2, Proposition 3.5], the atom spectrum $\text{ASpec} \mathcal{A}$ is a Kolmogrov space. On the other hand, it follows from Corollary [2.4] that $\text{T}(\text{ASpec} \mathcal{A})$ is a partially ordered set. Using again Corollary [2.4], $\text{ST}(\text{ASpec} \mathcal{A})$ is an Alexandroff Kolmogrov space. \qed

The notion support and associated prime of modules over a commutative ring can be generalized to objects in a Grothendieck category $\mathcal{A}$.

Definition 3.5. Let $M$ be an object in $\mathcal{A}$.

(1) The atom support of $M$, denoted by $\text{ASupp} M$, is defined as

$\text{ASupp} M = \{\alpha \in \text{ASpec} \mathcal{A} | \text{there exists } H \in \alpha \text{ which is a subquotient of } M\}$. 

(2) The \textit{associated atoms} of $M$, denoted by $\text{AAss} M$, is defined as
\[
\text{AAss} M = \{\alpha \in \text{ASupp} M | \text{there exists } H \leq \alpha \text{ which is a subobject of } M\}.
\]

In view of [Sto, p.631], for a commutative ring $A$, there is a bijection between $\text{ASpec}(\text{Mod} A)$ and $\text{Spec} A$. A subset $\Phi$ of $\text{Spec} A$ is said to be \textit{closed under specialization} if for any prime ideals $p$ and $q$ of $A$ with $p \subseteq q$, the condition $p \in \Phi$ implies that $q \in \Phi$. A corresponding subset in $\text{ASpec} A$ can be defined as follows.

\textbf{Definition 3.6.} A subset $\Phi$ of $\text{ASpec} A$ is said to be \textit{open} if for any $\alpha \in \Phi$, there exists a monoform $H$ with $\alpha = \overline{p}$ and $\text{ASupp} H \subset \Phi$. For every nonzero object $M$ in $A$, it is clear that $\text{ASupp} M$ is an open subset of $\text{ASpec} A$. For any subcategory $X$ of $A$, we set $\text{ASupp}X = \bigcup_{M \in X} \text{ASupp} M$ which is an open subset of $\text{ASpec} A$. Also, for every subset $U$ of $\text{ASpec} A$, we set $\text{ASupp}^{-1}U = \{M \in A| \text{ASupp} M \subseteq U\}$.

We recall from [K2] that $\text{ASpec} A$ can be regarded as a preordered set together with a specialization order $\leq$ as follows: for any atoms $\alpha$ and $\beta$ in $\text{ASpec} A$, we have $\alpha \leq \beta$ if and only if for any open subset $\Phi$ of $\text{ASpec} A$ satisfying $\alpha \in \Phi$, we have $\beta \in \Phi$.

\textbf{Definition 3.7.} An atom $\alpha$ in $\text{ASpec} A$ is said to be \textit{maximal} if there exists a simple object $H$ in $A$ such that $\alpha = \overline{p}$. The class of all maximal atoms in $\text{ASpec} A$ is denoted by $\text{m-ASpec} A$. If $\alpha$ is a maximal atom, then $\alpha$ is maximal in $\text{ASpec} A$ under the order $\leq$; see [Sa, Remark 4.7].

We describe the atom spectrum of the quotient category $A/X$ of a Grothendieck category $A$ induced by a localizing subcategory $X$ of $A$. We first recall some basic definitions.

\textbf{Definition 3.8.} A full subcategory $X$ of an abelian category $A$ is said to be \textit{Serre} if for any exact sequence $0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$ in $A$, the object $N$ belongs to $X$ if and only if $M$ and $K$ belong to $X$.

\textbf{Definition 3.9.} For a Serre subcategory $X$ of $A$, we define the \textit{quotient category} $A/X$ in which the objects are those in $A$ and for objects $M$ and $N$ in $A$, we have
\[
\text{Hom}_{A/X}(M, N) = \lim_{\longrightarrow} \text{Hom}_A(M', N/N')
\]
where $S_{M,N}$ is a directed set defined by
\[
S_{M,N} = \{(M', N') | M' \subset M, N' \subset N \text{ with } M/M', N' \in X\}.
\]
We observe that $A/X$ is a Grothendieck category together with a canonical exact functor $F : A \rightarrow A/X$. We refer the reader to [G] or [Po, Chap 4] for more details about the properties of the quotient categories.

A Serre subcategory $X$ of $A$ is said to be \textit{localizing} if the canonical functor $F : A \rightarrow A/X$ has a right adjoint functor $G : A/X \rightarrow A$.

The functors $F$ and $G$ induce functorial morphisms $u : 1_A \rightarrow GF$ and $v : FG \rightarrow 1_{A/X}$ such that $Gv \circ uG = 1_G$ and $vF \circ Fu = 1_F$. An object $M$ in $A$ is said to be \textit{closed} if $u_M$ is an isomorphism. It follows from [Po, chap 4, Corollary 4.4] that $G(M)$ is closed for any $M \in A/X$.

For every $\alpha \in \text{ASpec} A$, the topological closure of $\alpha$, denoted by $\overline{\{\alpha\}}$ consists of all $\beta \in \text{ASpec} A$ such that $\beta \leq \alpha$. According to [K1, Theorem 5.7], for each atom $\alpha$, there exists a localizing subcategory $X(\alpha) = \text{ASupp}^{-1}(\text{ASpec} A \setminus \{\alpha\})$ induced by $\alpha$.

For any object $M$ in $A$, we denote $F_\alpha(M)$ by $M_\alpha$ where $F_\alpha : A \rightarrow A/X(\alpha)$ is the canonical functor. Furthermore, assume that $t_\alpha : A \rightarrow X(\alpha)$ is the right adjoint functor of the inclusion functor $i : X(\alpha) \rightarrow A$. We notice that $t_\alpha$ is known as the radical of $A$ corresponding to $X(\alpha)$ and $t_\alpha(M)$ is the largest subobject of $M$ contained in $X(\alpha)$. Notice that $t_\alpha(M)$ satisfies $t_\alpha(M)$ is closed for any $M \in A/X$.

Let $X$ be a localizing subcategory of $A$ and $\alpha \in \text{ASpec} A \setminus \text{ASupp} X$. Then for any monoform $H$ in $A$ with $\overline{\alpha} = \alpha$, we have $H \notin X$ and so it follows from [K2, Lemma 5.14] that $F(H)$ is a
Lemma 3.10. The function \( F \) is the inverse of \( G \).

Proof. See [K2, Theorem 5.17].

In the rest of this section \( \mathcal{X} \) is a localizing subcategory of \( \mathcal{A} \) with the canonical functor \( F: \mathcal{A} \to \mathcal{A}/\mathcal{X} \). We also assume that \( G: \mathcal{A}/\mathcal{X} \to \mathcal{A} \) is the right adjoint functor of \( F \).

Lemma 3.11. If \( \alpha_1, \alpha_2 \in \text{ASpec} \mathcal{A} \setminus \text{ASupp} \mathcal{X} \) such that \( \alpha_1 \leq \alpha_2 \), then \( F(\alpha_1) \leq F(\alpha_2) \)

Proof. If \( \alpha_1 \leq \alpha_2 \), by the same notation as the previous section, we have \( U_{\alpha_2} \subseteq U_{\alpha_1} \). Since by [K2, Theorem 5.17], the map \( F \) is homeomorphism, we have \( U_{F(\alpha_2)} \subseteq U_{F(\alpha_1)} \) which implies that \( F(\alpha_1) \leq F(\alpha_2) \).

A similar proof yields the following lemma.

Lemma 3.12. Let \( \alpha_1, \alpha_2 \in \text{ASpec} \mathcal{A}/\mathcal{X} \) such that \( \alpha_1 \leq \alpha_2 \). Then \( G(\alpha_1) \leq G(\alpha_2) \).

For any \( \alpha \in \text{ASpec} \mathcal{A} \), we define \( \Lambda(\alpha) = \{ \beta \in \text{ASpec} \mathcal{A} \mid \alpha \leq \beta \} \). According to [SaS, Proposition 2.3], we have \( \Lambda(\alpha) \equiv \bigcap_{\alpha=1} \text{ASupp} H \). If \( \mathcal{A} \) is locally noetherian, since any object contains a nonzero noetherian subobject, we have \( \Lambda(\alpha) = \bigcap_{\alpha=1} \text{ASupp} H \), where noeth\( \mathcal{A} \) is the class of noetherian objects in \( \mathcal{A} \). The openness of \( \Lambda(\alpha) \) in \( \text{ASpec} \mathcal{A} \) has a crucial role in the finiteness of the number of minimal atoms of a noetherian object in \( \mathcal{A} \). We show that openness of \( \Lambda(\alpha) \) is preserved when passing quotient categories.

Lemma 3.13. For every atom \( \alpha \in \text{ASpec} \mathcal{A} \), we have \( F(\Lambda(\alpha)) = \Lambda(F(\alpha)) \) where the function \( F \) is as in Lemma 3.11. Moreover, if \( \Lambda(\alpha) \) is an open subset of \( \text{ASpec} \mathcal{A} \), then \( F(\Lambda(\alpha)) \) is open in \( \text{ASpec} \mathcal{A}/\mathcal{X} \).

Proof. If \( \alpha \in \text{ASupp} \mathcal{X} \), there is nothing to prove and so we may assume that \( \alpha \notin \text{ASupp} \mathcal{X} \). The first assertion is straightforward by using Lemma 3.10 and Lemmas 3.11 and 3.12. Given \( F(\beta) \in \Lambda(F(\alpha)) \), according to Lemma 3.11 and Lemma 3.12, we have \( \alpha < \beta \) so that \( \beta \in \Lambda(\alpha) \). Then by the assumption, there exists a monoform object \( H \) in \( \mathcal{A} \) such that \( \beta = 1 \) and \( \text{ASupp} H \subseteq \Lambda(\alpha) \). It follows from Lemma 3.10 and the first assertion that \( \text{ASupp} F(H) = F(\text{ASupp} H \setminus \text{ASupp} \mathcal{X}) \subseteq \Lambda(F(\alpha)) \). Therefore the result follows as \( F(\beta) = F(H) \).

The following result holds for a general topology as well.

Lemma 3.14. \( \text{ASpec} \mathcal{A} \) is an Alexandroff topological space if and only if \( \Lambda(\alpha) \) is open for all \( \alpha \in \text{ASpec} \mathcal{A} \).

Proof. "Only if" holds according to [SaS, Proposition 2.3]. Assume that \( \{ U_i \}_{i \in I} \) is a family of open subsets of \( \text{ASpec} \mathcal{A} \) and \( \alpha \in \bigcap_{i} U_i \) is an arbitrary atom. As \( \alpha \in U_i \) for each \( i \), there is a monoform \( H_i \) such that \( \alpha = \Gamma_i \) and \( \text{ASupp} H_i \subseteq U_i \) for each \( i \). Since \( \Lambda(\alpha) \) is open, using again [SaS, Proposition 2.3], we have \( \alpha \in \Lambda(\alpha) = \bigcap_{\alpha=1} \text{ASupp} H_i \subseteq \bigcap_{i} U_i \).
As ASpec $\mathcal{A}/\mathcal{X}$ is a closed subset of ASpec $\mathcal{A}$, the topological space ASpec $\mathcal{A}/\mathcal{X}$ with the induced topology is Alexandroff as well.

**Lemma 3.15.** If ASpec $\mathcal{A}$ is Alexandroff, then so is ASpec $\mathcal{A}/\mathcal{X}$.

**Proof.** It is straightforward by using Lemma 3.14 and Lemma 3.13.

In a locally finitely generated category with Alexandroff topological space ASpec $\mathcal{A}$, the maximal atoms are precisely those that are maximal under $\leq$.

**Lemma 3.16.** Let $\mathcal{A}$ be locally finitely generated and let $\alpha$ be an atom in ASpec $\mathcal{A}$ such that $\Lambda(\alpha)$ is an open subset of ASpec $\mathcal{A}$. Then $\alpha$ is maximal if and only if it is maximal under $\leq$. In particular, there exists a maximal atom $\beta$ in ASpec $\mathcal{A}$ such that $\alpha \leq \beta$.

**Proof.** If $\alpha$ is a maximal atom, in view of [Sa, Remark 4.7], it is maximal under $\leq$. Now, assume that $\alpha \in$ ASpec $\mathcal{A}$ is maximal under $\leq$. According to [SaS, Proposition 2.3] and the assumption, $\Lambda(\alpha) = \bigcap_{H \in \alpha} \text{ASupp } H = \{\beta \in$ ASpec $\mathcal{A} | \alpha \leq \beta\} = \{\alpha\}$ is open and so there exists a finitely generated monoform object $H$ such that $\text{ASupp } H = \{\alpha\}$ and $\alpha = \overline{M}$. If $H$ is not simple, it has a maximal subobject $N$ which is a contradiction as $H/N$ has a common nonzero subobject.

Then $\alpha$ is maximal. To prove the second assertion, there exists a finitely generated monoform object $M$ such that $\alpha = \overline{M}$ and $\text{ASupp } M = \{\beta \in$ ASpec $\mathcal{A} | \alpha \leq \beta\}$. Since $M$ is finitely generated, it has a maximal subobject $N$. Thus $S = M/N$ is a simple object and $\beta = \overline{S} \in$ ASupp $M$ is a maximal atom.

4. CRITICAL OBJECTS AND SEMI-NOETHERIAN CATEGORIES

Let us begin this section with a definition.

**Definition 4.1.** For a Grothendieck category $\mathcal{A}$, we define the Gabriel-Krull filtration of $\mathcal{A}$ as follows. For any ordinal (i.e., ordinal number) $\sigma$ we denote by $\mathcal{A}_\sigma$, the localizing subcategory of $\mathcal{A}$ which is defined in the following manner:

- $\mathcal{A}_{-1}$ is the zero subcategory.
- $\mathcal{A}_0$ is the smallest localizing subcategory containing all simple objects.

Let us assume that $\sigma = \rho + 1$ and denote by $F_{\rho} : \mathcal{A} \to \mathcal{A}/\mathcal{A}_\rho$ the canonical functor and by $G_{\rho} : \mathcal{A}/\mathcal{A}_\rho \to \mathcal{A}$ the right adjoint functor of $F_{\rho}$. Then an object $X$ in $\mathcal{A}$ will belong to $\mathcal{A}_\sigma$ if and only if $F_{\rho}(X) \in \text{Ob}(\mathcal{A}/\mathcal{A}_\rho)$. The left exact radical functor (torsion functor) corresponding to $\mathcal{A}_\rho$ is denoted by $t_{\rho}$. If $\sigma$ is a limit ordinal, then $\mathcal{A}_\sigma$ is the localizing subcategory generated by all localizing subcategories $\mathcal{A}_\rho$ with $\rho < \sigma$. It is clear that if $\sigma \leq \sigma'$, then $\mathcal{A}_\sigma \subseteq \mathcal{A}_{\sigma'}$. Moreover, since the class of all localizing subcategories of $\mathcal{A}$ is a set, there exists an ordinal $\tau$ such that $\mathcal{A}_\sigma = \mathcal{A}_\tau$ for all $\sigma \leq \tau$. Let us put $\mathcal{A}_\tau = \bigcup_{\sigma} \mathcal{A}_\sigma$. Then $\mathcal{A}$ is said to be semi-noetherian if $\mathcal{A} = \mathcal{A}_\tau$. We also say that the localizing subcategories $\{\mathcal{A}_\sigma\}_\sigma$ define the Gabriel-Krull filtration of $\mathcal{A}$. We say that an object $M$ in $\mathcal{A}$ has the Gabriel-Krull dimension defined or $M$ is semi-noetherian if $M \in \text{Ob}(\mathcal{A}_\tau)$. The smallest ordinal $\sigma$ so that $M \in \text{Ob}(\mathcal{A}_\sigma)$ is denoted by $\text{GK-dim } M$. Because the class of ordinals is well-ordered, throughout this paper, $\omega$ is denoted the smallest limit ordinal. We observe that $\text{GK-dim } 0 = -1$ and $\text{GK-dim } M \leq 0$ if and only if $\text{ASupp } M \subseteq m$-ASpec $\mathcal{A}$.

We note that any locally noetherian Grothendieck category is semi-noetherian; see [Po, Chap. 5, Theorem 8.5]. To be more precise, if $\mathcal{A} \neq \mathcal{A}_{\infty}$, then $\mathcal{A}/\mathcal{A}_{\infty}$ is also locally noetherian and so it has a nonzero noetherian object $X$. Then $X$ has a maximal subobject $Y$ so that $S = X/Y$ is simple. Therefore, $\sigma = \overline{S} \in \text{ASupp}(\mathcal{A}/\mathcal{A}_\rho)$ which is a contradiction by the choice of $\tau$.

For every atom $\alpha \in$ ASpec $\mathcal{A}$, the Gabriel-Krull dimension of $\alpha$, is the least ordinal $\sigma$ such that $\alpha \in$ ASupp $\mathcal{A}_\sigma$. If such an ordinal exists, we denote it by $\text{GK-dim } \alpha$ and by definition, it is a non-limit ordinal. We observe that if $\mathcal{A}$ is semi-noetherian, then every $\alpha \in$ ASpec $\mathcal{A}$ has Gabriel-Krull dimension.

**Lemma 4.2.** Let $\alpha, \beta$ be two atoms in ASupp $\mathcal{A}$ such that $\text{GK-dim } \alpha = \text{GK-dim } \beta$. Then $\alpha \notin \beta$. 

Proof. Assume that $\text{GK-dim}\, \alpha = \text{GK-dim}\, \beta = \sigma$ where $\sigma$ is a non-limit ordinal by definition and assume that $\alpha < \beta$. Using Lemma 3.11, we have $F_{\sigma-1}(\alpha) < F_{\sigma-1}(\beta)$. Since $\alpha, \beta \in \text{ASupp}\, \mathcal{A}_\sigma$, we have $F_{\sigma-1}(\alpha), F_{\sigma-1}(\beta) \in \text{ASupp}\, F_{\sigma-1} (\mathcal{A}_\sigma) = \text{ASupp}(\mathcal{A}/\mathcal{A}_\sigma)$ so that $F_{\sigma-1}(\alpha)$ and $F_{\sigma-1}(\beta)$ are maximal. Thus $F_{\sigma-1}(\alpha) = F_{\sigma-1}(\beta)$; and consequently $\alpha = G_{\sigma-1} F_{\sigma-1}(\alpha) = G_{\sigma-1} F_{\sigma-1}(\beta) = \beta$ by Lemma 3.10 which is a contradiction.

Corollary 4.3. If $\alpha, \beta$ are two atoms in $\text{ASpec}\, \mathcal{A}$ such that $\alpha < \beta$, Then $\text{GK-dim}\, \beta < \text{GK-dim}\, \alpha$.

Proof. Assume that $\text{GK-dim}\, \alpha = \sigma$ for some ordinal $\sigma$. Since $\text{ASupp}\, \mathcal{A}_\sigma$ is an open subset of $\text{ASpec}\, \mathcal{A}$, by definition $\beta \in \text{ASupp}\, \mathcal{A}_\sigma$. Therefore $\text{GK-dim}\, \beta \leq \sigma$. Now Lemma 3.12 implies that $\text{GK-dim}\, \beta < \sigma$.

For an object in $\mathcal{A}$ of finite Gabriel-Krull dimension, we have the following proposition.

Proposition 4.4. If $M$ is of finite Gabriel-Krull dimension, then any ascending chain of atoms in $\text{ASupp}\, M$ stabilizes.

Proof. Assume that $\text{GK-dim}\, M = n$ and that $\alpha_1 < \alpha_2 < \ldots$ is an ascending chain of atoms in $\text{ASupp}\, M$. Then $\text{GK-dim}\, \alpha_1 \leq n$ and so Corollary 1.3 implies that the length of this chain is at most $n$.

The following result shows that a noetherian object in $\mathcal{A}$ always possesses Gabriel-Krull dimension.

Proposition 4.5. Every noetherian object $M$ in $\mathcal{A}$ has Gabriel-Krull dimension. In particular, $\text{GK-dim}\, M$ is a non-limit ordinal.

Proof. Assume that $M$ does not have the Gabriel-Krull dimension. Since $M$ is noetherian, there exists a subobject $N$ of $M$ such that $M/N$ does not have the Gabriel-Krull dimension but all proper quotients of $M/N$ has the Gabriel-Krull dimension. Replacing $M/N$ by $M$ we may assume that $N = 0$ and let

$$\sigma = \sup\{\text{GK-dim}\, M/N \mid N \text{ is a nonzero submodule of } M\}.$$ 

We assert that $\text{GK-dim}\, M \leq \sigma + 1$ and thus we obtain a contradiction. Since $M$ is noetherian, using [Po, Chap 5, Lemma 8.3], the object $F_\sigma(M)$ is noetherian and hence it suffices to show that $F_\sigma(M)$ has finite length. Given a descending chain of objects $N_1 \supseteq N_2 \supseteq \ldots$ of $F_\sigma(M)$, it follows from [Po, Chap 4, Corollary 3.10] that there exists a descending chain $M_1 \supseteq M_2 \supseteq \ldots$ of subobjects of $M$ such that $F_\sigma(M_i) = N_i$ and $F_\sigma(M_i/M_{i+1}) = N_i/N_{i+1}$ for all $i \geq 1$. If for some $n$, we have $M_n = 0$, there is nothing to prove. If $M_i$ are nonzero for all $i$, we have $\text{GK-dim}\, (M_i/M_{i+1}) \leq \text{GK-dim}\, (M/M_{i+1}) \leq \sigma$. Hence for each $i$, we have $F_\sigma(M_i/M_{i+1}) = N_i/N_{i+1} = 0$ as $M_i/M_{i+1} \in \mathcal{A}_\sigma$. To prove the second assertion, if $\text{GK-dim}\, M = \sigma$ is a limit ordinal, since $M$ is noetherian and $\sigma = \sum_{\delta < \sigma} t_\delta(M)$, there exists $\rho < \sigma$ such that $M = t_\rho(M)$ which is a contradiction.

Definition 4.6. Given an ordinal $\sigma \geq 0$, we recall from [MR or GW] that an object $M$ in $\mathcal{A}$ is said to be $\sigma$-critical provided $\text{GK-dim}\, M = \sigma$ while $\text{GK-dim}\, M/N < \sigma$ for all nonzero subobjects $N$ of $M$. It is clear that any nonzero subobject of a $\sigma$-critical object is $\sigma$-critical. An object $M$ is said to be critical if it is $\sigma$-critical for some ordinal $\sigma$. We note that any critical object is monoform.

Lemma 4.7. Let $M$ be a $\sigma$-critical object in $\mathcal{A}$. Then $\sigma$ is a non-limit ordinal.

Proof. Assume that $\sigma$ is a limit ordinal. Then there exists some $\rho < \sigma$ such that $t_\rho(M) \neq 0$ and so $\text{GK-dim}\, t_\rho(M) \leq \rho$. But $t_\rho(M)$ is $\sigma$-critical which is a contradiction.

The following lemma is crucial in this section.

Lemma 4.8. Let $\sigma$ be a non-limit ordinal and let $M$ be an object in $\mathcal{A}$. If $F_{\sigma-1}(M)$ is simple, then $M/t_{\sigma-1}(M)$ is $\sigma$-critical.
Proof. Observe that \( F_{\sigma^{-1}}(M) \cong F_{\sigma^{-1}}(M/t_{\sigma^{-1}}(M)) \) and so we may assume that \( t_{\sigma^{-1}}(M) = 0 \). Let \( N \) be a nonzero subobject of \( M \). Then \( F_{\sigma^{-1}}(N) \) is nonzero because \( t_{\sigma^{-1}}(M) = 0 \). Since \( F_{\sigma^{-1}}(M) \) is simple, we have \( F_{\sigma^{-1}}(M/N) = 0 \) and hence \( \text{GK-dim } M/N < \sigma \). On the other hand, by the definition and the fact that \( F_{\sigma^{-1}}(M) \) is simple, we have \( \text{GK-dim } M = \sigma \). \( \square \)

For every ordinal \( \sigma \), the localizing subcategory \( \mathcal{A}_\sigma \) of \( \mathcal{A} \) is generated by critical objects.

**Theorem 4.9.** Let \( \sigma \) be an ordinal. Then \( \mathcal{A}_\sigma \) is generated by all \( \delta \)-critical objects in \( \mathcal{A} \) with \( \delta \leq \sigma \).

**Proof.** If \( \sigma \) is a limit ordinal, then \( \mathcal{A}_\sigma \) is generated by \( \bigcup_{\rho < \sigma} \mathcal{A}_\rho \) and so we may assume that \( \sigma \) is a non-limit ordinal. Let \( \mathcal{C} \) be the subclass of all \( \delta \)-critical objects in \( \mathcal{A} \) with \( \delta \leq \sigma \). We have to prove \( \mathcal{A}_\sigma = \langle \mathcal{C} \rangle_{\text{loc}} \), where \( \langle \mathcal{C} \rangle_{\text{loc}} \) is the localizing subcategory of \( \mathcal{A} \) generated by \( \mathcal{C} \). We prove the claim by transfinite induction on \( \sigma \). The case \( \sigma = 0 \) is clear and so we assume that \( \sigma > 0 \). Let \( \mathcal{D} \) be the subclass of \( \sigma \)-critical objects in \( \mathcal{A} \). Then we have the following equalities
\[
F_{\sigma^{-1}}(\mathcal{A}_\sigma) = \langle F_{\sigma^{-1}}(C) | F_{\sigma^{-1}}(C) \text{ is simple for } C \in \mathcal{A} \rangle_{\text{loc}} = F_{\sigma^{-1}}(\langle \mathcal{A}_{\sigma^{-1}} \cup \mathcal{D} \rangle_{\text{loc}}) = F_{\sigma^{-1}}(\langle \mathcal{C} \rangle_{\text{loc}})
\]
where the first equality holds by the definition and the second equality holds by [K1, Proposition 4.18] and Lemma 4.13 and the last equality holds using the induction hypothesis. It now follows from [K1, Proposition 4.14] that \( \mathcal{A}_\sigma = \langle \mathcal{C} \rangle_{\text{loc}} \). \( \square \)

**Proposition 4.10.** If \( M \) is a nonzero object in \( \mathcal{A} \) with Gabriel-Krull dimension, then \( M \) has a critical subobject (and so a monoform subobject).

**Proof.** Since ordinals satisfy the descending chain condition, we can choose a nonzero subobject \( N \) of \( M \) of minimal Gabriel-Krull dimension \( \sigma \). Clearly \( \sigma \) is non-limit ordinal and \( t_{\sigma^{-1}}(N) = 0 \). Since \( F_{\sigma^{-1}}(N) \in (\mathcal{A}/\mathcal{A}_{\sigma^{-1}})_{0} \), it follows from [St, Chap VI, Proposition 2.5] that \( F_{\sigma^{-1}}(N) \) contains a simple subobject \( S \). Then \( N \) contains a subobject \( H \) such that \( F_{\sigma^{-1}}(H) = S \) by [Po, Chap4, Corollary 3.10]. Now, Lemma 4.13 implies that \( H \) is a \( \sigma \)-critical.

The above proposition yields the following conclusion.

**Corollary 4.11.** Let \( \mathcal{A} \) be a semi-noetherian category. Then any nonzero object \( M \) in \( \mathcal{A} \) has a critical subobject.

In a semi-noetherian category, any atom has a representative by a critical object. More generally we have the following result.

**Corollary 4.12.** Let \( \sigma \) be an ordinal and \( \alpha \) be an atom in \( \text{ASpec } \mathcal{A} \) such that \( \text{GK-dim } \alpha = \sigma \). Then \( \alpha \) is represented by a \( \sigma \)-critical object in \( \mathcal{A} \).

**Proof.** Since \( \alpha \in \text{ASupp } \mathcal{A}_\sigma \), there exists \( X \in \mathcal{A}_\sigma \) such that \( \alpha \in \text{ASupp } X \). Then there exists a monoform object \( M \in \mathcal{A} \) such that \( \alpha = M \) and \( M \) is a subquotient of \( X \). This implies that \( M \) has Gabriel-Krull dimension and \( \text{GK-dim } M = \sigma \). Now, the assumption and Proposition 4.10 imply that \( M \) contains a \( \sigma \)-critical subobject \( H \).

The following lemma is crucial to prove the main theorem of this section.

**Lemma 4.13.** Let \( \mathcal{X} \) be a localizing subcategory of \( \mathcal{A} \) and let \( M \) be an object in \( \mathcal{A} \) with Gabriel-Krull dimension. If \( \text{ASupp } M \subset \text{ASupp } \mathcal{X} \), then \( M \in \mathcal{X} \).

**Proof.** Assume that \( M \) is not in \( \mathcal{X} \) and \( t(M) \) is the largest subobject of \( M \) belonging to \( \mathcal{X} \). By the assumption, \( M/t(M) \) has Gabriel-Krull dimension and by Proposition 4.10 it contains a monoform subobject \( N/t(M) \). Then \( N/t(M) \in \text{ASupp } \mathcal{X} \) and so there exists an object \( X \in \mathcal{X} \) such that \( N/t(M) \in \text{ASupp } X \). Thus \( N/t(M) \) contains a nonzero subobject isomorphic to a subquotient of \( X \). But this implies that \( t(N/t(M)) \) is nonzero which is a contradiction. \( \square \)

We are ready to present the main result of this section.
Theorem 4.14. Let $\mathcal{A}$ be a semi-noetherian category. Then the map $X \mapsto \text{ASupp}_X$ provides a one-to-one correspondence between localizing subcategories of $\mathcal{A}$ and open subsets of $\text{ASpec}\, \mathcal{A}$. The inverse map is given by $U \mapsto \text{ASupp}^{-1}_U$.

Proof. Using Lemma 4.13, the proof is straightforward. □

5. The spectrum of a locally coherent Grothendieck category

Throughout this section, we remind that $\mathcal{A}$ is a Grothendieck category with a generating set.

A finitely generated object $Y$ in $\mathcal{A}$ is finitely presented if every epimorphism $f : X \to Y$ in $\mathcal{A}$ with $X$ finitely generated has a finitely generated kernel $\text{Ker} \, f$. A finitely presented object $Z$ in $\mathcal{A}$ is coherent if every its finitely generated subobject is finitely presented. We denote by $\text{fg-} \mathcal{A}, \text{fp-} \mathcal{A}$ and $\text{coh-} \mathcal{A}$, the full subcategories of $\mathcal{A}$ consisting of finitely generated, finitely presented and coherent objects, respectively.

We recall that a Grothendieck category $\mathcal{A}$ is locally coherent if every object in $\mathcal{A}$ is a direct limit of coherent objects; or equivalently finitely generated subobjects of finitely presented objects are finitely presented. According to [Ro, 2] and [H] a Grothendieck category $\mathcal{A}$ is locally coherent if and only if $\text{fp-} \mathcal{A} = \text{coh-} \mathcal{A}$ is an abelian category.

Throughout this section $\mathcal{A}$ is a locally coherent Grothendieck category. For this case, topological space $Zg(\mathcal{A})$, the Ziegler spectrum of $\mathcal{A}$, has been studied by Herzog [H]. The set $Zg(\mathcal{A})$ consists of indecomposable injective objects in $\mathcal{A}$. For every finitely presented object $M$ in $\mathcal{A}$, we associate a subset $\mathcal{O}(M) = \{I \in Zg(\mathcal{A}) \mid \text{Hom}(M, I) \neq 0\}$ which the collection of these subsets satisfies the axioms for a basis of open subsets of $Zg(\mathcal{A})$. On the other hand, the collection of the isomorphism classes of indecomposable injective objects in $\mathcal{A}$, denoted as $\text{Sp}_\mathcal{A}$, forms a set. This is because every indecomposable injective object is the injective envelope of some quotient of an element of the generating set of $\mathcal{A}$. For a locally coherent category $\mathcal{A}$, Krause [Kr] has constructed a topology on $\text{Sp}_\mathcal{A}$ in which for a subset $U$ of $\text{Sp}_\mathcal{A}$, the closure of $U$ is defined as $\overline{U} = (\cap \{ U \cap \text{fp-} \mathcal{A} \})^{\perp}$. The subsets $U$ of $\text{Sp}_\mathcal{A}$ satisfying $U = \overline{U}$ form the closed subsets of $\text{Sp}_\mathcal{A}$. We observe that $Zg \, \mathcal{A}$ and $\text{Sp}_\mathcal{A}$ have the same objects with relatively different topologies. The following proposition shows that the topologies of $Zg \, \mathcal{A}$ and $\text{Sp}_\mathcal{A}$ are identical.

Proposition 5.1. Let $\mathcal{A}$ be a locally coherent Grothendieck category. Then $Zg(\mathcal{A})$ and $\text{Sp}_\mathcal{A}$ have the same topologies.

Proof. We show that $Zg \, \mathcal{A}$ and $\text{Sp}_\mathcal{A}$ have the same open subsets. Given an open subset $O$ of $Zg(\mathcal{A})$, it suffices to show that $\langle \cap O^{\perp} \cap \text{fp-} \mathcal{A} \rangle^{\perp} = O^{\perp}$ and so $O^{\perp}$ will be a closed subset of $\text{Sp}_\mathcal{A}$, where $O^{\perp} = \text{Sp}_\mathcal{A} \setminus O$. If $I \in O^{\perp}$, then it is clear that $\text{Hom}(\cap O^{\perp} \cap \text{fp-} \mathcal{A}, I) = 0$ and so $I \in \langle \cap O^{\perp} \cap \text{fp-} \mathcal{A} \rangle^{\perp}$.

Conversely, if $I \in \langle \cap O^{\perp} \cap \text{fp-} \mathcal{A} \rangle^{\perp} \setminus O^{\perp}$, there exists $M \in \text{fp-} \mathcal{A}$ such that $I \in \mathcal{O}(M) \subseteq O$; and hence $M \not\in \langle \cap O^{\perp} \cap \text{fp-} \mathcal{A} \rangle^{\perp}$. Then $\text{Hom}(M, O^{\perp}) \neq 0$ so that there exists $J \in O^{\perp}$ such that $\text{Hom}(M, J) \neq 0$. But this implies that $J \in \mathcal{O}(M) \subseteq O$ which is a contradiction. Now suppose that $O$ is an open subset of $\text{Sp}_\mathcal{A}$ and so $O^{\perp} = \langle \cap O^{\perp} \cap \text{fp-} \mathcal{A} \rangle^{\perp}$. We now show that $O$ is an open subset of $Zg \, \mathcal{A}$. Given $I \in O$, we have $\text{Hom}(\cap O^{\perp} \cap \text{fp-} \mathcal{A}, I) \neq 0$ and so there exists $M \in \langle \cap O^{\perp} \cap \text{fp-} \mathcal{A} \rangle^{\perp}$ such that $\text{Hom}(M, I) \neq 0$. Thus $I \in \mathcal{O}(M)$ and $\text{Hom}(M, O^{\perp}) = 0$. For every $J \in \mathcal{O}(M)$, we have $\text{Hom}(M, J) \neq 0$ which implies that $J \in O$. Therefore, $\mathcal{O}(M) \subset O$; and consequently $O$ is an open subset of $Zg \, \mathcal{A}$. □

For every $I \in Zg \, \mathcal{A}$, the localizing subcategory associated to $I$ is

$$\mathcal{X}(I) = \langle \cap I = \{ M \in \mathcal{A} \mid \text{Hom}(M, I) = 0 \}. $$

For any $I, J \in Zg \, \mathcal{A}$, we define a specialization preorder as follows:

$$I \leq J \text{ if and only if } \langle \cap J \cap \text{fp-} \mathcal{A} \subseteq \langle \cap I \cap \text{fp-} \mathcal{A}. $$
For every indecomposable injective object \( I \in Zg\mathcal{A} \), we denote by \( \alpha(I) \), the intersection of all open subsets of \( Zg\mathcal{A} \) containing \( I \).

In view of Section 3, the Ziegler spectrum of a locally coherent Grothendieck category admits a canonical preorder relation as follows: for \( I \) and \( J \in Zg(\mathcal{A}) \) we have \( I \leq J \) if \( \alpha(J) \subseteq \alpha(I) \). The following lemma shows that these two preorder relations are the same.

**Lemma 5.2.** Let \( I, J \in Zg\mathcal{A} \). Then \( \perp I \cap \text{fp-} \mathcal{A} \subseteq \perp J \cap \text{fp-} \mathcal{A} \) if and only if \( \alpha(J) \subseteq \alpha(I) \).

**Proof.** Assume that \( I \leq J \) and \( \mathcal{O} \) is an open subset of \( Zg\mathcal{A} \) containing \( I \). It suffices to consider that \( \mathcal{O} = \mathcal{O}(M) \) for a finitely presented object \( M \) in \( \mathcal{A} \). If \( J \notin \mathcal{O}(M) \), we have \( M \in \perp I \cap \text{fp-} \mathcal{A} \subseteq \perp J \cap \text{fp-} \mathcal{A} \) which is a contradiction. The converse is straightforward.

**Lemma 5.3.** For every \( I \in Zg\mathcal{A} \), we have \( \alpha(I) = \{ J \in Zg\mathcal{A} | I \leq J \} \).

**Proof.** Straightforward.

The following lemma shows that the closure defined by Krause coincides with the closure defined by \( \perp \) on \( Zg\mathcal{A} \).

**Lemma 5.4.** Let \( I \) be an indecomposable injective module. Then \( \{ J | J \in \text{Sp}\mathcal{A} \} = \{ J | J \leq I \} \).

**Proof.** We should prove that \( \perp I \cap \text{fp-} \mathcal{A} \) is the closure of \( I \) in \( \text{Sp}\mathcal{A} \). Given \( J \in \perp I \cap \text{fp-} \mathcal{A} \), we have \( \text{Hom}(\perp I \cap \text{fp-} \mathcal{A}) = 0 \) and so \( \perp I \cap \text{fp-} \mathcal{A} \subseteq \perp J \) which forces that \( J \leq I \). Conversely if \( J \leq I \), by definition, we have \( \perp I \cap \text{fp-} \mathcal{A} \subseteq \perp J \cap \text{fp-} \mathcal{A} \) and so \( J \in \perp I \cap \text{fp-} \mathcal{A} \).

For \( \alpha \in \text{ASpec} \mathcal{A} \) and monoform objects \( H_1 \) and \( H_2 \) in \( \mathcal{A} \) satisfying \( \alpha = \overline{H_1} = \overline{H_2} \), we have \( E(H_1) = E(H_2) \). The isomorphism class of all such \( E(H) \) such that \( \alpha = \overline{H} \) is denoted by \( \text{E}(\alpha) \). We observe that \( \text{E}(\alpha) \) is an indecomposable injective object in \( \mathcal{A} \). Because if \( \text{E}(\alpha) = E(H) \) for some monoform object \( H \) in \( \mathcal{A} \) and \( E(\alpha) = E_1 \oplus E_2 \), then \( E_1 \cap H \) and \( E_2 \cap H \) are nonzero monoform subobjects of \( H \). Thus \( E_1 \cap H \cap E_2 \cap H \) is nonzero which is a contradiction. We now show that for any object \( M \) in \( \mathcal{A} \), \( \text{ASupp} \mathcal{M} \) can be determined in terms of indecomposable injective objects.

**Lemma 5.5.** If \( M \) is a nonzero object in \( \mathcal{A} \), then \( \text{ASupp} \mathcal{M} = \{ \alpha \in \text{ASpec} \mathcal{A} | \text{Hom}(M, \text{E}(\alpha)) \neq 0 \} \). In particular, \( X(\alpha) = \perp \text{E}(\alpha) \).

**Proof.** Given \( \alpha \in \text{ASupp} \mathcal{M} \), there exist subobjects \( K < L \subseteq M \) such that \( H = L/K \) is a monoform object with \( \alpha = \overline{H} \). Since \( \text{Hom}(H, \text{E}(\alpha)) \neq 0 \), we have \( \text{Hom}(L, \text{E}(\alpha)) \neq 0 \) and consequently \( \text{Hom}(M, \text{E}(\alpha)) \neq 0 \). The converse and the second assertion is clear.

The following lemma due to Krause [Kr, Lemma 1.1] is crucial in our investigation.

**Lemma 5.6.** An object \( X \in \mathcal{A} \) is finitely generated if and only if for any epimorphism \( \varphi : Y \rightarrow X \) in \( \mathcal{A} \), there is a finitely generated subobject \( U \) of \( Y \) such that \( \varphi(U) = X \).

For every subcategory \( \mathcal{S} \) of \( \mathcal{A} \), we denote by \( \overline{\mathcal{S}} \), the full subcategory of \( \mathcal{A} \) consisting of direct limits \( \lim X_i \), with \( X_i \in \mathcal{S} \) for each \( i \). For \( \mathcal{S} \subseteq \text{fp-} \mathcal{A} \), we denote by \( \sqrt{\mathcal{S}} \), the smallest Serre subcategory of \( \text{fp-} \mathcal{A} \) containing \( \mathcal{S} \). For a locally coherent category \( \mathcal{A} \), the following lemma establishes another topology on \( \text{ASpec} \mathcal{A} \).

**Lemma 5.7.** The set \( \{ \text{ASupp} \mathcal{M} | M \in \text{fp-} \mathcal{A} \} \) forms a basis of open subsets for \( \text{ASpec} \mathcal{A} \).

**Proof.** Since \( \mathcal{A} \) is locally coherent, it is clear that for every \( \alpha \in \text{ASpec} \mathcal{A} \), there exists a finitely presented object \( M \) in \( \mathcal{A} \) such that \( \alpha \in \text{ASupp} \mathcal{M} \). If \( M_1 \) and \( M_2 \) are finitely presented objects in \( \mathcal{A} \) and \( \alpha \in \text{ASupp} M_1 \cap \text{ASupp} M_2 \), then by Lemma 5.2 there exists a nonzero morphism \( f_i : M_i \rightarrow \text{E}(\alpha) \) for \( i = 1, 2 \). Since \( \text{E}(\alpha) \) is uniform, \( \text{Im} f_1 \cap \text{Im} f_2 \) is a nonzero subobject of \( \text{E}(\alpha) \) and so it contains a nonzero finitely generated subobject \( X \) as \( \mathcal{A} \) is locally coherent. Using the pull-back diagram and Lemma 5.6, there exists a finitely presented subobject \( L_i \) of \( M_i \) such that
Lemma 5.8. There exists a continuous injective map \( f : ZASpec A \to Zg A \), given by \( \alpha \mapsto E(\alpha) \) which is a morphism of preordered sets. In particular, ZASpec \( A \) is homeomorphic to a topological subspace of Zg \( A \).

Proof. The injectivity of \( f \) is clear. For every \( M \in \text{fp-} A \), it follows from Lemma 5.5 that \( f^{-1}(\text{O}(M)) = \text{ASupp } M \). Then \( f^{-1}(\text{O}) \) is an open subset of ZASpec \( A \) for any open subset \( \text{O} \) of Zg \( A \). It is straightforward to prove that \( f \) is a morphism of preordered sets.

Suppose that every finitely presented object in \( A \) has Gabriel-Krull dimension. Since \( A \), the subcategory of all objects in \( A \) having Gabriel-Krull dimension is localizing, \( A \) is semi-noetherian. In this case, the following theorem shows that ZASpec \( A \) is homeomorphic to Zg \( A \).

Theorem 5.9. Let \( A \) be a semi-noetherian category. Then the map \( f : ZASpec A \to Zg A \), given by \( \alpha \mapsto E(\alpha) \) is a homeomorphism. Moreover, this map is an isomorphism of ordered sets.

Proof. Let \( E \) be an indecomposable injective object in \( A \). Using Proposition 4.10, the object \( E \) contains a monoform subobject \( H \) and so \( E = E(\alpha) \), where \( \alpha = H \). This implies that \( f \) is surjective. Therefore, it follows from Lemma 5.8 that \( f \) is homeomorphism. The second assertion is clear.

For a subset \( U \) of ZASpec \( A \), we set \( \text{ASupp}^{-1} U = \{ M \in \text{fp-} A | \text{ASupp } M \subset U \} \).

Proposition 5.10. Let \( A \) be a semi-noetherian category. The map \( U \mapsto \text{ASupp}^{-1} U \) provides a one-to-one correspondence between open subsets of ZASpec \( A \) and Serre subcategories of \( \text{fp-} A \). The inverse map is \( X \mapsto \text{ASupp } X \).

Proof. Assume that \( U \) is an open subset of ZASpec \( A \) and \( X \) is a Serre subcategory of \( \text{fp-} A \). It is clear that \( \text{ASupp}^{-1} U \) is a Serre subcategory of \( \text{fp-} A \) and \( \text{ASupp } X \) is an open subset of ZASpec \( A \). In order to prove \( \text{ASupp}^{-1}(\text{ASupp } X) = X \), it suffices to show that \( \text{ASupp}^{-1}(\text{ASupp } X) \subset X \).

Given \( M \in \text{ASupp}^{-1}(\text{ASupp } X) \), we have \( \text{ASupp } M \subset \text{ASupp } X \). Thus \( \text{ASupp } M \subset \text{ASupp } X \) and so it follows from Lemma 4.10 that \( M \in X \). Hence \( M = \lim M_i \) is the direct limit of objects \( M_i \) of \( X \). Since \( M \) is finitely presented, it is a direct summand of some \( M_i \) so that \( M \in X \). The fact that \( \text{ASupp}(\text{ASupp}^{-1} U) = U \) is straightforward.

Definition 5.11. (1) For every indecomposable injective object \( I \) in \( A \), the localizing subcategory \( X(I) \) admits a canonical exact functor \( (\cdot)_I : A \to A / X(I) \). The image of every object \( M \) under this functor is said to be the localization of \( M \) at \( I \) and we denote it by \( M_I \).

(2) The Ziegler support of an object \( M \) in \( A \) is denoted by \( ZSupp(M) \), that is
\[
ZSupp M = \{ I \in Zg A | M_I \neq 0 \}.
\]
The definition forces that \( \text{ZSupp} M = \{ I \in \text{Zg} \mathcal{A} \mid \text{Hom}(M, I) \neq 0 \} \). Then, for every finitely presented object \( M \) in \( \mathcal{A} \), we have \( \text{ZSupp} M = \text{O}(M) \). For a subcategory \( \mathcal{X} \) of \( \mathcal{A} \), we define \( \text{ZSupp} \mathcal{X} = \bigcup_{M \in \mathcal{X}} \text{ZSupp} M \). For every subset \( \mathcal{U} \) of \( \text{Zg} \mathcal{A} \), we define \( \text{ZSupp}^{-1} \mathcal{U} = \{ M \in \mathcal{A} \mid \text{ZSupp} M \subset \mathcal{U} \} \).

It is clear that \( \text{ZSupp}^{-1} \mathcal{U} \) is a localizing subcategory of \( \mathcal{A} \).

(3) A localizing subcategory \( \mathcal{X} \) of \( \mathcal{A} \) is said to be of finite type provide that the corresponding right adjoint functor of the inclusion \( \mathcal{X} \to \mathcal{A} \) commutes with direct limits. If \( \mathcal{A} \) is a locally noetherian Grothendieck category, then \( \text{fg-} \mathcal{A} = \text{noeth-} \mathcal{A} = \text{fp-} \mathcal{A} = \text{coh-} \mathcal{A} \) so that \( \mathcal{A} \) is locally coherent. In this case, any localizing subcategory \( \mathcal{X} \) of \( \mathcal{A} \) is of finite type. We recall that a topological space \( \mathcal{V} \) is quasi-compact provide that for every family \( \{ \mathcal{U}_i \}_{i \in I} \) of open subsets \( \mathcal{V} = \bigcup_i \mathcal{U}_i \) implies that \( \mathcal{V} = \bigcup_i \mathcal{U}_i \) for some finite subset \( I' \) of \( I \).

In terms of this new definition, we establish a one-to-one correspondence between open subsets of \( \text{Zg} \mathcal{A} \) and localizing subcategories of finite type of \( \mathcal{A} \).

**Theorem 5.12.** The map \( \mathcal{U} \mapsto \text{ZSupp}^{-1} \mathcal{U} \) provides a one-to-one correspondence between open subsets of \( \text{Zg} \mathcal{A} \) and localizing subcategories of finite type of \( \mathcal{A} \). The inverse map is \( \mathcal{X} \mapsto \text{ZSupp} \mathcal{X} \).

**Proof.** Given an open subset \( \mathcal{U} \) of \( \text{Zg} \mathcal{A} \), it is clear by the definition that \( \text{ZSupp}^{-1} \mathcal{U} = \mathcal{U} \). Then [Kr, Corollary 4.3] and Proposition [5.1] imply that \( \text{ZSupp}^{-1} \mathcal{U} \) is a localizing subcategory of finite type of \( \mathcal{A} \). Given a localizing subcategory \( \mathcal{X} \) of finite type of \( \mathcal{A} \), by [Kr, Lemma 2.3], we have \( \mathcal{X} = \mathcal{X} \cap \text{fp-} \mathcal{A} \). Then \( \text{ZSupp} \mathcal{X} = \bigcup_{i \in \mathcal{X}} \text{ZSupp} M \) that is an open subset of \( \text{Zg} \mathcal{A} \). It is clear that \( \text{ZSupp}(\text{ZSupp}^{-1} \mathcal{U}) \subset \mathcal{U} \). On the other hand, for every \( i \in \mathcal{U} \), there exists a finitely presented object \( M \) such that \( i \in \text{O}(M) = \text{ZSupp} M \subset \mathcal{U} \). This implies that \( M \in \text{ZSupp}^{-1} \mathcal{U} \) and so the previous argument forces that \( \text{ZSupp} \mathcal{M} \subset \text{ZSupp}(\text{ZSupp}^{-1} \mathcal{U}) \). Therefore \( i \in \text{ZSupp}(\text{ZSupp}^{-1} \mathcal{U}) \); and hence \( \mathcal{U} = \text{ZSupp}(\text{ZSupp}^{-1} \mathcal{U}) \). To prove \( \mathcal{X} = \text{ZSupp}^{-1}(\text{ZSupp} \mathcal{X}) \), clearly \( \mathcal{X} \subset \text{ZSupp}^{-1}(\text{ZSupp} \mathcal{X}) \). For the other side, by the previous argument, \( \text{ZSupp}^{-1}(\text{ZSupp} \mathcal{X}) \) is a localizing subcategory of finite type of \( \mathcal{A} \). Thus, for every \( M \in \text{ZSupp}^{-1}(\text{ZSupp} \mathcal{X}) \), we have \( M = \lim M_i \), where each \( M_i \) belongs to \( \text{ZSupp}^{-1}(\text{ZSupp} \mathcal{X}) \cap \text{fp-} \mathcal{A} \) by [Kr, Lemma 2.3]. Then \( \text{ZSupp} M_i \subset \text{ZSupp} \mathcal{X} \) for each \( i \). Fixing \( i \), since \( \text{ZSupp} M_i \) is a quasi-compact open subset of \( \text{Zg} \mathcal{A} \) by [Kr, Corollary 4.6], there exists \( N \in \mathcal{X} \cap \text{fp-} \mathcal{A} \) such that \( \text{ZSupp} M_i \subset \text{ZSupp} N \) and hence it follows from [H, Corollary 3.12] that \( M_i \in \sqrt{N} \), where \( \sqrt{N} \) is the smallest Serre subcategory of \( \text{fp-} \mathcal{A} \) containing \( N \). Clearly \( \sqrt{N} \subset \mathcal{X} \) and hence \( M_i \in \mathcal{X} \). Finally, this forces that \( M \in \mathcal{X} \) as \( M = \lim M_i \).

The above theorem gives a characterization for localizing subcategories of finite type of \( \mathcal{A} \).

**Corollary 5.13.** Let \( \mathcal{X} \) be a localizing subcategory of \( \mathcal{A} \). Then \( \mathcal{X} \) is of finite type if and only if \( \text{ZSupp} \mathcal{X} \) is an open subset of \( \text{Zg} \mathcal{A} \).

**Proof.** *Only if* is clear. Conversely, if \( \text{ZSupp} \mathcal{X} \) is an open subset of \( \text{Zg} \mathcal{A} \), then a similar proof of Theorem 5.12 shows that \( \mathcal{X} = \text{ZSupp}^{-1}(\text{ZSupp} \mathcal{X}) \). Consequently, \( \mathcal{X} \) is of finite type of \( \mathcal{A} \) by Theorem 5.12.

For every localizing subcategory \( \mathcal{X} \) of finite type of \( \mathcal{A} \), it is clear that \( \text{ASupp} \mathcal{X} \) is an open subset of \( \text{ZASpec} \mathcal{A} \). The above theorem also yields an immediate result for semi-noetherian categories.

**Corollary 5.14.** Let \( \mathcal{A} \) be a semi-noetherian category. The map \( \mathcal{U} \mapsto \text{ASupp}^{-1} \mathcal{U} \) provides a one-to-one correspondence between open subsets of \( \text{ZASpec} \mathcal{A} \) and localizing subcategories of finite type of \( \mathcal{A} \). The inverse map is \( \mathcal{X} \mapsto \text{ASupp} \mathcal{X} \).

**Proof.** The proof is straightforward using Theorem 5.12 and Theorem 5.13.

The above corollary provides a characterization for localizing subcategories of finite type of \( \mathcal{A} \) in terms of atoms.

**Corollary 5.15.** Let \( \mathcal{A} \) be a semi-noetherian category and let \( \mathcal{X} \) be a localizing subcategory of \( \mathcal{A} \). Then \( \mathcal{X} \) is of finite type if and only if \( \text{ASupp} \mathcal{X} \) is an open subset of \( \text{ZASpec} \mathcal{A} \).
Proof. By Lemma 4.13 we have \( \text{ASupp}^{-1}(\text{ASupp} X) = X \). Hence, if \( \text{ASupp} X \) is an open subset of \( \text{ZASpec} A \), then \( X \) is of finite type by Corollary 5.14. The converse is clear.

For any localizing subcategory \( X \) in \( A \), we denote by \( \langle X \rangle \) the largest localizing subcategory of finite type of \( A \) contained in \( X \). In view of [Kr, Theorem 2.8], it is clear that \( \langle X \rangle = S \), where \( S = X \cap \text{fp-} A \).

The following proposition shows that the preorder relation \( \leq \) is defined on \( \text{Zg} A \) can be redefined in terms of the localizing subcategories of finite type of \( A \) associated with indecomposable injective objects.

**Proposition 5.16.** Suppose that \( I, J \in \text{Zg} A \). Then \( I \leq J \) if and only if \( \langle J \rangle \cap I \leq \langle I \rangle \). In particular, \( \overline{I} = \langle I \rangle \).

**Proof.** Straightforward.

As for a locally coherent Grothendieck category \( A \), the subcategory \( \text{fp-} A \) is abelian, the atom spectrum of \( \text{fp-} A \) can be investigated independently. To avoid any mistakes, for every object \( M \) in \( A \), we use the symbol \( \text{fAAss} M \) for atom support of \( M \) in \( \text{ASpec} \text{fp-} A \) instead of \( \text{ASupp} M \). Similarly we use the symbol \( \text{fAss} M \) instead of \( \text{Ass} M \). If \( A \) is semi-noetherian, \( \text{ASpec} \text{fp-} A \) is a topological subspace of \( \text{ZASpec} A \).

**Proposition 5.17.** Let \( A \) be a semi-noetherian category. Then \( \text{ASpec} \text{fp-} A \) is a topological subspace of \( \text{ZASpec} A \).

**Proof.** Given \( \alpha \in \text{ASpec} \text{fp-} A \), there exists a monoform object \( H \) of the abelian category \( \text{fp-} A \) such that \( \alpha = \overline{H} \). Since \( A \) is a semi-noetherian locally coherent category, by Proposition 4.10, the object \( H \) contains a finitely generated monoform subobject \( H_1 \) in \( A \). Thus \( H_1 \) is finitely presented and \( \alpha = \overline{H_1} \in \text{ZASpec} A \). Therefore \( \text{ASpec} \text{fp-} A \) is a subset of \( \text{ZASpec} A \). Now, assume that \( M \) is a finitely presented object in \( A \) and we prove that \( \text{fAAss} M = \text{ASupp} M \cap \text{ASpec} \text{fp-} A \). The above argument indicates that \( \text{fAAss} M \subseteq \text{ASupp} M \cap \text{ASpec} \text{fp-} A \). To prove the converse, assume that \( \alpha \in \text{ASupp} M \cap \text{ASpec} \text{fp-} A \). Then there exists a finitely presented monoform object \( H \) in \( A \) such that \( \alpha = \overline{H} \) and \( H \) is a subquotient of \( M \). Using Lemma 5.6 we can choose such \( H \) such that \( H = L/K \), where \( L \) is a finitely presented subobject of \( M \). Since \( \text{fp-} A \) is abelian, we deduce that \( K \) is finitely presented; and hence \( \alpha \in \text{fAAss} M \).

We further have the following result.

**Lemma 5.18.** Let \( M \) be an object of \( \text{fp-} A \). Then \( \text{fAss} M \subseteq \text{fAAss} M \). In particular, if \( A \) is semi-noetherian, then the equality holds.

**Proof.** If \( \alpha \in \text{fAAss} M \), then \( M \) contains a monoform subobject \( H \) in \( A \) such that \( \alpha = \overline{H} \). Since \( A \) is locally coherent, we may assume that \( H \) is finitely generated. Hence \( H \) is finitely presented because \( M \) is finitely presented. Moreover, it is clear that \( H \) is a monoform object of \( \text{fp-} A \) and consequently \( \alpha \in \text{fAss} M \). To prove the second claim, if \( \alpha \in \text{fAss} M \), then \( M \) contains a monoform subobject of \( \text{fp-} A \) such that \( \alpha = \overline{H} \). Since \( A \) is semi-noetherian, \( H \) contains a finitely presented monoform object \( H_1 \) in \( A \). Clearly, \( H_1 \) is a monoform object of \( \text{fp-} A \) and \( \alpha = \overline{H_1} \in \text{fAss} M \).

**Corollary 5.19.** Let \( A \) be a semi-noetherian category. Then every monoform object of \( \text{fp-} A \) is a uniform object in \( A \).

**Proof.** If \( H \) is a monoform object of \( \text{fp-} A \), then \( \text{Ass} H = \{ \alpha \} \), where \( \alpha = \overline{H} \). Then for any nonzero subobjects \( K, L \) of \( H \), we have \( \alpha \in \text{Ass} K \cap L \) so that \( K \cap L \) is a nonzero subobject of \( H \). □
6. A LOWER BOUND FOR THE GABRIEL-KRULL DIMENSION OF OBJECTS

We begin this section with the following lemma which is crucial in our investigation in .

**Lemma 6.1.** For any ordinal \( \sigma \), there is \( F_0(\mathcal{A}_\sigma) \cong \mathcal{A}_\sigma/\mathcal{A}_0 \). Moreover we have

\[
F_0(\mathcal{A}_\sigma) = \begin{cases} 
(\mathcal{A}/\mathcal{A}_0)_{\sigma-1} & \text{if } \sigma < \omega \\
(\mathcal{A}/\mathcal{A}_0)_\sigma & \text{if } \sigma \geq \omega.
\end{cases}
\]

**Proof.** The equivalence follows from [K1, Proposition 4.17] and so it suffices to prove the equalities. We proceed by induction on \( \sigma \). If \( \sigma < \omega \), then the cases \( \sigma = 0, 1 \) are clear by the definition. Assume that \( \sigma > 1 \) and so by the induction hypothesis, there are the equivalence and equality of categories

\[
\mathcal{A}/\mathcal{A}_{\sigma-1} \cong \mathcal{A}/\mathcal{A}_0/\mathcal{A}_0(\mathcal{A}/\mathcal{A}_0)_{\sigma-2}.
\]

If \( F'_{\sigma-2} : \mathcal{A}/\mathcal{A}_0 \to \mathcal{A}/\mathcal{A}_0/(\mathcal{A}/\mathcal{A}_0)_{\sigma-2} \) is the canonical functor, it suffices to show that \( F'_{\sigma-2}(F_0(\mathcal{A}_\sigma)) \) is the smallest subcategory of \( \mathcal{A}/\mathcal{A}_0/(\mathcal{A}/\mathcal{A}_0)_{\sigma-2} \) generated by simple objects. Suppose that \( \theta : \mathcal{A}/\mathcal{A}_0/(\mathcal{A}/\mathcal{A}_0)_{\sigma-2} \to \mathcal{A}/\mathcal{A}_{\sigma-1} \) is the equivalence functor. Thus \( F_{\sigma-1} = \theta \circ F'_{\sigma-2} \circ F_0 \); and hence there are the following equalities and equivalences of categories

\[
(\mathcal{A}/\mathcal{A}_{\sigma-1})_0 = F_{\sigma-1}(\mathcal{A}_\sigma) = \theta(F'_{\sigma-2}(F_0(\mathcal{A}_\sigma))) \cong F'_{\sigma-2}(F_0(\mathcal{A}_\sigma))
\]

which proves the assertion. We now prove the case \( \sigma \geq \omega \). If \( \sigma \) is limit ordinal, then \( \mathcal{A}_\sigma = (\cup_{\rho < \sigma} \mathcal{A}_\rho)_{\text{loc}} \) is the localizing subcategory of \( \mathcal{A} \) generated by \( \cup_{\rho < \sigma} \mathcal{A}_\rho \) and so \( F_0 \) is exact and preserves arbitrary direct sums, the induction hypothesis yields \( F_0(\mathcal{A}_\sigma) = (\cup_{\rho < \sigma} (\mathcal{A}/\mathcal{A}_0)_\rho)_{\text{loc}} = (\mathcal{A}/\mathcal{A}_0)_\sigma \). If \( \sigma \) is a non-limit ordinal, then \( F_{\sigma-1} \) can be factored as \( \mathcal{A} \xrightarrow{F_0} \mathcal{A}/\mathcal{A}_0 \xrightarrow{F'_{\sigma-1}} \mathcal{A}/\mathcal{A}_0/(\mathcal{A}/\mathcal{A}_0)_{\sigma-1} \cong \mathcal{A}/\mathcal{A}_{\sigma-1} \). Thus \( F_{\sigma-1}(\mathcal{A}_\sigma) = F'_\sigma(F_0(\mathcal{A}_\sigma)) \) so that \( F_0(\mathcal{A}_\sigma) = (\mathcal{A}/\mathcal{A}_0)_\sigma \).

It is straightforward by Lemma 6.1 that if \( \text{GK-dim } \alpha \) exists, then \( \text{GK-dim } \alpha \geq \text{GK-dim } F_0(\alpha) + 1 \). The proposition gives the following corollary.

**Corollary 6.2.** Let \( \sigma \) be an ordinal and let \( M \) be an object in \( \mathcal{A} \) such that \( \text{GK-dim } M = \sigma \). Then

\[
\text{GK-dim } M = \begin{cases} 
\text{GK-dim } F_0(M) + 1 & \text{if } \sigma < \omega \\
\text{GK-dim } F_0(M) & \text{if } \sigma \geq \omega
\end{cases}
\]

**Proof.** Assume that \( \text{GK-dim } M = \sigma \) for some ordinal \( \sigma \). If \( \sigma < \omega \), then we have \( M \in \mathcal{A}_\sigma \) and so Lemma 6.1 implies that \( F_0(M) \in (\mathcal{A}/\mathcal{A}_0)_{\sigma-1} \) so that \( \text{GK-dim } F_0(M) \leq \sigma - 1 \). If \( \text{GK-dim } F_0(M) = \rho < \sigma - 1 \), then \( F_0(M) \in (\mathcal{A}/\mathcal{A}_0)_\rho = F_0(\mathcal{A}_\rho+1) \) by Lemma 6.1. Thus, there exists an object \( N \in \mathcal{A}_{\rho+1} \) such that \( F_0(M) = F_0(N) \). For every \( \alpha \in \text{ASupp } M \), if \( \alpha \) is a maximal atom, then \( \alpha \in \text{ASupp } \mathcal{A}_\rho \subseteq \text{ASupp } \mathcal{A}_{\rho+1} \). If \( \alpha \) is not maximal, then \( F_0(\alpha) \in \text{ASupp } F_0(M) = \text{ASupp } F_0(N) = F_0(\text{ASupp } N \setminus \text{ASupp } \mathcal{A}_\rho) \) so that \( \alpha \in \text{ASupp } N \) by Lemma 3.10. Hence \( \text{ASupp } M \subseteq \text{ASupp } \mathcal{A}_{\rho+1} \) and so \( M \in \mathcal{A}_{\rho+1} \) by Lemma 4.13 Therefore \( \text{GK-dim } M \leq \rho + 1 < \sigma \) which is a contradiction. If \( \sigma \geq \omega \), it follows from Lemma 6.1 that \( F_0(M) \in (\mathcal{A}/\mathcal{A}_0)_\sigma \) and so \( \text{GK-dim } F_0(M) \leq \sigma \). If \( \text{GK-dim } F_0(M) = \rho < \sigma \), it follows from the first case that \( \rho \geq \omega \). Thus according to Lemma 6.1, we have \( F_0(M) \in (\mathcal{A}/\mathcal{A}_0)_\rho = F_0(\mathcal{A}_\rho) \) and so \( \text{GK-dim } M \leq \rho < \sigma \) which is a contradiction. \( \square \)

The following result shows that the functor \( F_0 \) preserves critical objects.

**Proposition 6.3.** Let \( M \) be a \( \sigma \)-critical object in \( \mathcal{A} \). Then we have the following conditions.

(i) If \( \sigma < \omega \) then \( F_0(M) \) is \( \sigma - 1 \)-critical.

(ii) If \( \sigma \geq \omega \) then \( F_0(M) \) is \( \sigma \)-critical.

**Proof.** (i) By Corollary 6.2 we have \( \text{GK-dim } F_0(M) = \sigma - 1 \). Given a nonzero subobject \( X \) of \( F_0(M) \), it follows from [Po, Chap 4. Corollary 3.10] that there exists a nonzero subobject \( N \) of \( M \) such that \( F_0(N) = X \). Since \( M \) is \( \sigma \)-critical, \( \text{GK-dim } M/N \leq \sigma - 1 \) and hence Corollary 6.2 implies that \( \text{GK-dim } F_0(M)/X = \text{GK-dim } F_0(M/N) \leq \sigma - 2 \). (ii) The proof is similar to (i) using Corollary 6.2. \( \square \)
Definition 6.4. For every $\alpha \in \text{ASpec}\,A$, we define $\dim \alpha$ by transfinite induction. We say that $\dim \alpha = 0$ if $\alpha$ is maximal under $\leq$. For an ordinal $\sigma > 0$, we say that $\dim \alpha \leq \sigma$ if for every $\beta \in \text{ASpec}\,A$ with $\alpha < \beta$, we have $\dim \beta < \sigma$. The least such an ordinal $\sigma$ is said to be the dimension of $\alpha$ and we say that $\dim \alpha = \sigma$. We set $\dim 0 = -1$. If $\dim \alpha = n$ is finite, then there exists a chain of atoms $\alpha < \alpha_1 < \cdots < \alpha_n$ in $\text{ASpec}\,A$ and this chain has the largest length among those starting with $\alpha$. For every object $M$ in $\mathcal{A}$, the dimension of $M$, denoted by $\dim M$, is the supremum of all $\dim \alpha$ such that $\alpha \in \text{ASupp}\,M$. For every subobject $N$ of $M$, it is clear that $\dim M = \max\{\dim N, \dim M/N\}$.

Lemma 6.5. Let $\alpha$ be an atom in $\text{ASpec}\,A$ such that $\Lambda(\alpha)$ is an open subset of $\text{ASpec}\,A$. Then there exists a monoform object $M$ in $\mathcal{A}$ such that $\alpha = \overline{M}$ and $\dim M = \dim \alpha$.

Proof. Since $\Lambda(\alpha)$ is an open subset of $\text{ASpec}\,A$, there exists a monoform object $M$ in $\mathcal{A}$ such that $\alpha = \overline{M}$ and $\text{Supp}\,M = \Lambda(\alpha)$. Therefore $\dim M = \dim \alpha$. \hfill \Box

Lemma 6.6. Let $\alpha \in \text{ASpec}\,A$. Then we have the following inequalities:

$$\dim F_0(\alpha) \geq \begin{cases} 
\dim \alpha - 1 & \text{if } \dim \alpha < \omega \\
\dim \alpha & \text{if } \dim \alpha \geq \omega.
\end{cases}$$

Moreover, if $\mathcal{A}$ is locally finitely generated such that $\text{ASpec}\,A$ is Alexandroff, then the inequalities are equalities.

Proof. We proceed by transfinite induction on $\dim \alpha = \sigma$. We first assume that $\sigma < \omega$. The case $\sigma = 0$ is clear. If $\sigma > 0$, there exists an atom $\beta \in \text{ASpec}\,A$ such that $\alpha < \beta$ and $\dim \beta = \sigma - 1$. The induction hypothesis implies that $\dim F_0(\beta) \geq \sigma - 2$ so that $\dim F_0(\alpha) \geq \sigma - 1$. To prove the second claim in this case, assume that $\mathcal{A}$ is locally finitely generated with Alexandroff space $\text{ASpec}\,A$. If $\sigma = 0$, by Lemma 6.10, the atom $\alpha$ is maximal and so there exists a simple object $S$ in $\mathcal{A}$ such that $\alpha = S$. Then $F_0(S) = 0$ and so $\dim F_0(\alpha) = -1$ by the definition. If $\sigma > 0$ and $\dim F_0(\alpha) > \sigma - 1$, there exists $\beta \in \text{ASpec}\,A$ such that $F_0(\alpha) < F_0(\beta)$ and $\dim F_0(\beta) = \sigma - 1$. But Lemma 6.12 and Lemma 6.10 imply that $\alpha < \beta$ and the induction hypothesis implies that $\dim \beta = \sigma$ which is a contradiction. We now assume that $\sigma \geq \omega$. If $\sigma = \omega$, then for any non-negative integer $n$ there exists $\beta \in \text{ASpec}\,A$ such that $\alpha < \beta$ and $\dim \beta = n + 1$ and so the first case implies that $\dim F_0(\beta) \geq n$ so that $\dim F_0(\alpha) \geq \omega$. Now, assume that $\sigma > \omega$. If $\sigma$ is a non-limit ordinal, then there exists $\beta \in \text{ASpec}\,A$ such that $\alpha < \beta$ and $\dim \beta = \sigma - 1$. Thus the induction hypothesis implies that $\dim F_0(\beta) \geq \sigma - 1$ and consequently $\dim F_0(\alpha) \geq \sigma$. If $\sigma$ is a limit ordinal, then for every ordinal $\rho < \sigma$ there exists $\beta \in \text{ASpec}\,A$ such that $\alpha < \beta$ and $\dim \beta \geq \rho + 1$. The induction hypothesis implies that $\dim F_0(\beta) \geq \rho$ so that $\dim F_0(\alpha) \geq \sigma$. To prove the second claim in this case, assume that $\text{ASpec}\,A$ is Alexandroff and $\sigma = \omega$. Then for every $\beta \in \text{ASpec}\,A \setminus \text{ASpec}\,A_0$ with $\alpha < \beta$, we have $\dim \beta < \omega$. Then using the first case, $\dim F_0(\beta) < \omega$ and hence $\dim F_0(\alpha) = \omega$. If $\sigma > \omega$ and $\dim F_0(\beta) > \sigma$, then there exists $\beta \in \text{ASpec}\,A \setminus \text{ASpec}\,A_0$ with $\alpha < \beta$ and $\dim F_0(\beta) \geq \sigma$. But the induction hypothesis implies that $\dim \beta = \dim F_0(\beta) \geq \sigma$ which is a contradiction. \hfill \Box

Corollary 6.7. Let $M$ be an object in $\mathcal{A}$. Then we have the following inequalities:

$$\dim F_0(M) \geq \begin{cases} 
\dim M - 1 & \text{if } \dim M < \omega \\
\dim M & \text{if } \dim M \geq \omega.
\end{cases}$$

Moreover, if $\mathcal{A}$ is locally generated such that $\text{ASpec}\,A$ is Alexandroff, then the inequalities are equalities.

Proof. Straightforward using Lemma 6.6. \hfill \Box
The following theorem shows that the dimension of an object serves as a lower bound for its Gabriel-Krull dimension. Specifically, if $\text{ASpec } A$ is Alexandroff and Gabriel-Krull dimension of an object in $A$ is finite, then it is equal to its dimension.

**Theorem 6.8.** Let $M$ be an object in $A$ with the Gabriel-Krull dimension. Then $\dim M \leq \text{GK-dim } M$. Moreover, if $A$ is locally finitely generated with Alexandroff space $\text{ASpec } A$ and $\text{GK-dim } M$ is finite, then $\dim M = \text{GK-dim } M$.

**Proof.** Assume that $\text{GK-dim } M = \sigma$ for some ordinal $\sigma$. We proceed by transfinite induction on $\sigma$. If $\sigma = 0$, then $M \in A_0$ and so using [Sa, Remark 4.7], every atom in $\text{ASupp } M$ is maximal. Therefore, every atom in $\text{ASupp } M$ is maximal under $\leq$ so that $\dim M = 0$. Suppose inductively that $\sigma > 0$ and $\alpha$ is an arbitrary atom in $\text{ASupp } M$. We prove that $\dim \alpha \leq \sigma$; and consequently $\dim M \leq \sigma$. For every $\beta \in \text{ASpec } A$ with $\alpha < \beta$ and $\text{GK-dim } \beta = \rho$, according to Corollary 6.13 we have $\rho < \text{GK-dim } \alpha \leq \sigma$. We observe that $\beta$ is represented by a $\rho$-critical object $G$ in $A$ by Corollary 6.12. Now, the induction hypothesis implies that $\dim \beta \leq \dim G \leq \text{GK-dim } G = \rho < \sigma$. To prove the equality, assume that $\text{ASpec } A$ is Alexandroff and $\sigma$ is a finite number. We proceed again by induction on $\text{GK-dim } M = \sigma$. If $\sigma = 0$, then as previously mentioned, we have $\dim M = 0$ and so the equality holds in this case. If $\sigma > 0$, it follows from Corollary 6.14 that $\text{GK-dim } F_0(M) = \sigma - 1$. The induction hypothesis, Corollary 6.7 and Corollary 6.2 imply that $\text{GK-dim } M = \text{GK-dim } F_0(M) + 1 = \dim F_0(M) + 1 = \dim M$. □

**Example 6.9.** We remark that the equality in the above theorem may not hold if $\text{ASpec } A$ is not Alexandroff even if $A$ is locally noetherian. To be more precise, if we consider the locally noetherian Grothendieck category $A = \text{GrMod } k[x]$ of graded $k[x]$-modules, where $k$ is a field and $x$ is an indeterminate with $\deg x = 1$. According to [K2, Example 3.4], $\dim k[x] = 0$ while $\text{GK-dim } k[x] = 1$.

For an atom $\alpha$, the following lemma determines a relation between $\dim \alpha$ and $\text{GK-dim } \alpha$.

**Corollary 6.10.** Let $\alpha$ be an atom in $\text{ASpec } A$ with the Gabriel-Krull dimension. Then $\dim \alpha \leq \text{GK-dim } \alpha$. In particular, if $A$ is a locally finitely generated with the Alexandroff space $\text{ASpec } A$ and $\text{GK-dim } \alpha$ is finite, then $\dim \alpha = \text{GK-dim } \alpha$.

**Proof.** According to Corollary 6.12 there exists a monoform object $M$ in $A$ such that $\alpha = M$ and $\text{GK-dim } \alpha = \text{GK-dim } M$. Clearly $\dim \alpha \leq \dim M$ and so the result follows by using Theorem 6.8. If $\text{ASpec } A$ is Alexandroff, by Lemma 6.5 we can choose such $M$ such that $\dim M = \dim \alpha$ and so it follows from Theorem 6.8 that $\text{GK-dim } \alpha = \text{GK-dim } M$. □

It is a natural question to ask whether Gabriel-Krull dimension of an object is finite if its dimension is finite. As a Grothendieck category does not have enough atoms, the question may have a negative answer. However, for a locally finitely generated Grothendieck category $A$ with the Alexandroff space $\text{ASpec } A$, we have the following slightly weaker result.

**Proposition 6.11.** Let $A$ be locally finitely generated such that $\text{ASpec } A$ is Alexandroff, $M$ be an object in $A$ and let $n$ be a non-negative integer such that $\dim M = n$. Then $\text{ASupp } M \subset \text{ASupp } A_n$. In particular, if $M$ has Gabriel-Krull dimension, then $\text{GK-dim } M = n$.

**Proof.** Assume that $\alpha$ is an arbitrary atom in $\text{ASupp } M$ and we prove by induction on $n$ that $\alpha \in \text{ASupp } A_n$. If $n = 0$, then $\alpha$ is maximal under $\leq$ and so $\alpha$ is maximal by Lemma 6.10. Therefore $\alpha \in \text{ASupp } A_0$. We now assume that $n > 0$. By Lemma 6.5, there exists a monoform object $H$ in $A$ such that $\alpha = \overline{H}$ and $\dim \alpha = \dim H$. If $\dim \alpha < n$, the induction hypothesis implies that $\text{ASupp } H \subset \text{ASupp } A_n$ so that $\alpha \in \text{ASupp } A_n$. If $\dim \alpha = n$, then $F_0(\alpha) = F_0(H)$ and by Lemmas 6.13 and 6.14 we have $\dim F_0(\alpha) = \dim F_0(H) = n - 1$. The induction hypothesis and Lemma 6.11 imply that $F_0(\alpha) \in \text{ASupp } (A/A_n)_{n-1} = \text{ASupp } F_0(A_n)$. Hence $\alpha \in \text{ASupp } A_n$; and consequently $\text{ASupp } M \subset \text{ASupp } A_n$. For the second assertion, according to Lemma 6.14 we have $M \in A_n$. Thus the result follows by Theorem 6.8. □
For a locally finitely generated category \( \mathcal{A} \) such that \( \text{ASpec}\mathcal{A} \) is Alexandroff, the Gabriel-Krull dimension of an atom is finite if its dimension is finite.

**Corollary 6.12.** Let \( \mathcal{A} \) be locally finitely generated such that \( \text{ASpec}\mathcal{A} \) is Alexandroff and let \( \alpha \) be an atom in \( \text{ASpec}\mathcal{A} \) such that \( \dim\alpha \) is finite. Then \( \dim\alpha = \text{GK-dim}\alpha \).

**Proof.** Assume that \( \dim\alpha = n \) for some non-negative integer \( n \). According to Lemma 6.5, there exists a monoform object \( M \) in \( \mathcal{A} \) such that \( \alpha = M \) and \( \dim M = n \). It follows from Proposition 7.11 that \( \text{ASupp} M \subset \mathcal{A}_\alpha \) so that \( \text{GK-dim}\alpha \leq n \). Now, Corollary 6.10 implies that \( \text{GK-dim}\alpha = n \).

### 7. Minimal atoms of objects

In this section we study the minimal atoms of objects in the Grothendieck category \( \mathcal{A} \). Given an object \( M \) in \( \mathcal{A} \), an atom \( \alpha \in \text{ASupp} M \) is said to be minimal if it is minimal in \( \text{ASupp} M \) under \( \leq \). We denote by \( \text{AMin} M \), the set of all minimal atoms of \( M \).

In the following proposition due to Kanda [K2, Proposition 3.6], his proof works without requiring the condition that \( \mathcal{A} \) is locally noetherian.

**Proposition 7.1.** If \( M \) is a noetherian object in \( \mathcal{A} \), Then \( \text{ASupp} M \) is a quasi-compact subset of \( \text{ASpec}\mathcal{A} \).

Also [K2, Proposition 4.7] holds for every noetherian object in a Grothendieck category.

**Proposition 7.2.** Let \( M \) be a noetherian object in \( \mathcal{A} \) and let \( \alpha \) be an atom in \( \text{ASupp} M \). Then there exists \( \beta \in \text{AMin} M \) such that \( \beta \leq \alpha \).

When an object in \( \mathcal{A} \) has the Gabriel-Krull dimension, the subset of its minimal atoms can be identified as follows.

**Lemma 7.3.** Let \( \sigma \) be a non-limit ordinal and let \( M \) be an object in \( \mathcal{A} \) with \( \text{GK-dim} M = \sigma \). Then every \( \alpha \in \text{ASupp} M \) with \( \text{GK-dim} \alpha = \sigma \) belongs to \( \text{AMin} M \). Additionally, if \( M \) is noetherian, there are only a finite number of such \( \alpha \).

**Proof.** If \( \alpha \notin \text{AMin} M \), then there exists some \( \beta \in \text{ASupp} M \) such that \( \beta < \alpha \) and it follows from Lemma 4.2 that \( \beta \in \text{ASupp} \mathcal{A}_{\sigma - 1} \). But this forces \( \alpha \in \text{ASupp} \mathcal{A}_{\sigma - 1} \) which is a contradiction. To prove the first claim, if \( M \) is noetherian, then \( F_\sigma(M) \) has finite length and so \( \text{ASupp} F_{\sigma - 1}(M) \) is a finite set. On the other hand, \( F_{\sigma - 1}(\{ \alpha \in \text{ASupp} M \mid \text{GK-dim} \alpha = \sigma \}) \subset \text{ASupp} F_{\sigma - 1}(M) \); and hence \( \{ \alpha \in \text{ASupp} M \mid \text{GK-dim} \alpha = \sigma \} \) is a finite set.

Proposition 7.2 can be extended for every object of a semi-noetherian category \( \mathcal{A} \).

**Proposition 7.4.** Let \( \mathcal{A} \) be a semi-noetherian category and let \( M \) be an object in \( \mathcal{A} \). Then for every \( \alpha \in \text{ASupp} M \), there exists an atom \( \beta \) in \( \text{AMin} M \) such that \( \beta \leq \alpha \).

**Proof.** Assume that \( \alpha \in \text{ASupp} M \) and assume that \( \mathcal{A} \rightarrow \mathcal{A}/\mathcal{X}(\alpha) \) is the canonical functor. We notice that \( \text{ASupp} F(M) = \text{F(ASupp} M \cap [\alpha]) \). It follows from [Po, Chap 5, Corollary 5.3] that \( \mathcal{A}/\mathcal{X}(\alpha) \) is semi-noetherian and so \( F(M) \) has Gabriel-Krull dimension. Assume that \( \text{GK-dim} F(M) = \sigma \). Then using Lemma 5.13, there exists \( F(\beta) \in \text{ASupp} F(M) \) such that \( \text{GK-dim} F(\beta) = \sigma \). Hence Lemma 7.3 implies that \( F(\beta) \in \text{AMin} F(M) \). Now, Lemma 3.10 and Lemma 6.11 indicate \( \beta \in \text{AMin} M \).

We now present the first main theorem of this section which provides a sufficient condition for finiteness of the number of minimal atoms of a noetherian object.

**Theorem 7.5.** Let \( M \) be a noetherian object in \( \mathcal{A} \). If \( \Lambda(\alpha) \) is an open subset of \( \text{ASpec}\mathcal{A} \) for any \( \alpha \in \text{AMin} M \), then \( \text{AMin} M \) is a finite set.
Proof. Let $\alpha \in \text{AMin}\, M$ and set $W(\alpha) = \{\beta \in \text{ASpec}\, A | \alpha < \beta\}$. It is straightforward to show that $W(\alpha) = A(\alpha) \setminus \{\alpha\}$; and hence $W(\alpha)$ is an open subset of $\text{ASpec}\, A$. Consider $\Phi = \cup_{\alpha \in \text{AMin}\, M} W(\alpha)$, the localizing subcategory $X = \text{ASupp}^{-1}(\Phi)$ and the canonical functor $F : A \to A/X$. It follows from [K2, Lemma 5.16] that $\text{ASupp}\, F(M) = F(\text{AMin}\, M)$. We notice that for any $\alpha \in \text{AMin}\, M$, we have $A(\alpha) \cap (\text{ASpec}\, A \setminus \Phi) = \{\alpha\}$; and hence using Lemma 5.13, $F(A(\alpha)) = \{F(\alpha)\}$ is an open subset of $\text{ASpec}\, A/X$ so that $F(\alpha)$ is a maximal atom of $\text{ASpec}\, A/X$ by using [Sa, Proposition 3.2]. On the other hand, according to [Po, Chap 5, Lemma 8.3], the object $F(M)$ is noetherian. Thus the previous argument implies that $F(M)$ has finite length so that $F(\text{AMin}\, M)$ is a finite set. Since $\text{AMin}\, M \subseteq \text{ASpec}\, A \setminus \text{ASupp}\, X$, the set $\text{AMin}\, M$ is finite using Lemma 5.14. 

Let $M$ be an object in $A$. We define a subset $A(M)$ of $\text{ASpec}\, A$ as follows

$$A(M) = \{\alpha \in \text{ASpec}\, A | t_\alpha(M) = 0\}.$$ 

It is straightforward that if $M$ is a nonzero object in $A$, then $A(M) \subseteq \text{ASupp}\, M$.

**Lemma 7.6.** If $M$ is an object in $A$ and $N$ is a subobject of $M$, then $A(M) \subseteq A(N)$. In particular, if $N$ is a nonzero essential subobject of $M$, then $A(N) = A(M)$.

**Proof.** The first assertion is straightforward by the definition. To prove the second, if $\alpha \in A(N)$, we have $0 = t_\alpha(N) = t_\alpha(M) \cap N$ which implies that $t_\alpha(M) = 0$.

We now have the following lemma.

**Lemma 7.7.** Let $H$ be a monoform object in $A$ with $\alpha = \overline{\alpha}$. Then $A(H) = A(\alpha)$.

**Proof.** We observe that $t_\alpha(H) = 0$ and so $\alpha \in A(H)$. For any $\beta \in A(\alpha)$, since $\alpha \leq \beta$, we have $X(\beta) \subseteq X(\alpha)$ so that $t_\beta \leq t_\alpha$. Therefore $t_\beta(H) = 0$ so that $\beta \in A(H)$. Conversely assume that $\beta \in A(H)$. For any monoform object $H'$ with $\overline{\beta} = \alpha$, there exists a nonzero subobject $H_1$ of $H'$ isomorphic to a subobject of $H$. Since $t_\beta(H) = 0$, we have $t_\beta(H_1) = 0$ and since $H_1$ is an essential subobject of $H'$, we have $t_\beta(H') = 0$ so that $\beta \in \text{ASupp}\, H'$. It now follows from [K2, Proposition 4.2] that $\alpha \leq \beta$.

**Proposition 7.8.** Let $M$ be an object in $A$. Then $A(M) = \bigcap_{\alpha \in \text{AAss}\, M} A(\alpha)$. In particular, if $A(M)$ contains an atom $\alpha \in \text{AMin}\, M$, then $\text{AAss}\, M = \{\alpha\}$ and $A(M) = A(\alpha)$.

**Proof.** For any $\alpha \in \text{AAss}\, M$, there exists a monoform subobject $H$ of $M$ such that $\overline{\alpha} = \alpha$. Then using Lemma 7.6 and Lemma 7.7, we have $A(M) \subseteq A(\alpha)$. Conversely assume that $\beta \in \text{ASpec}\, A$ such that $\alpha \leq \beta$ for all $\alpha \in \text{AAss}\, M$. If $t_\beta(M) \neq 0$, there exists $\alpha \in \text{AAss}\, t_\beta(M)$ and hence $\alpha \leq \beta$. Since $\text{ASupp}\, t_\beta(M)$ is open, we deduce that $\beta \in \text{ASupp}\, t_\beta(M)$ which is a contradiction. The second claim is straightforward by the first part.

The proposition provides an immediate corollary about minimal atoms of objects in $A$.

**Corollary 7.9.** Let $M$ be an object in $A$ and $\alpha \in \text{AMin}\,(M)$. Then $\text{AAss}\, M/t_\alpha(M) = \{\alpha\}$.

**Proof.** Since $\alpha \in \text{AMin}\, M$, we deduce that $\alpha \in \text{AMin}\, M/t_\alpha(M)$. Clearly $\alpha \in A(M/t_\alpha(M))$ and so Proposition 7.8 implies that $\text{AAss}\, M/t_\alpha(M) = \{\alpha\}$.

The above proposition gives also the following corollary.

**Corollary 7.10.** Let $M$ be an object in $A$. Then $A(M) = \text{ASupp}\, M$ if and only if $\text{AAss}\,(M) = \text{AMin}\, M$ has only one element.

In the rest of this section we assume that $A$ is a right noetherian ring. At first we recall the classical Krull dimension of right $A$-modules [GW].
Definition 7.11. In order to define Krull dimension for right $A$-modules, we define by transfinite induction, classes $\mathcal{K}_\sigma$ of modules, for all ordinals $\sigma$. Let $\mathcal{K}_{-1}$ be the class containing precisely of the zero module. Consider an ordinal $\sigma \geq 0$ and suppose that $\mathcal{K}_\beta$ has been defined for all ordinals $\beta < \alpha$. We define $\mathcal{K}_\alpha$, the class of those modules $M$ such that, for every (countable) descending chain $M_0 \supseteq M_1 \supseteq \ldots$ of submodules of $M$, we have $M_i/M_{i+1} \in \bigcup_{\beta < \alpha} \mathcal{K}_\beta$ for all but finitely many indices $i$. The smallest such $\alpha$ such that $M \in \mathcal{K}_\alpha$ is the Krull dimension of $M$, denoted by $\text{K-dim} M$ and we say that $\text{K-dim} M$ exists.

The following lemma shows that the Gabriel-Krull dimension of modules serves as a lower bound for the classical Krull dimension as defined above.

Proposition 7.12. Let $M$ be a right $A$-module with $\text{K-dim} M = \sigma$. Then

$$\text{GK-dim} M \leq \begin{cases} \sigma & \text{if } \sigma < \omega \\ \sigma + 1 & \text{if } \sigma \geq \omega. \end{cases}$$

In particular, if $M$ is noetherian, the inequalities are equalities.

Proof. We proceed by induction on $\sigma$. We first consider $\sigma < \omega$. If $\sigma = 0$, then $M$ is artinian and so $\text{GK-dim} M = 0$. If $\sigma > 0$ and $\text{GK-dim} M \notin \mathcal{A}_\sigma$, we have $M \notin \mathcal{A}_{\sigma}$ and so $F_{\sigma-1}(M)$ is not artinian. Then there exists an unstable descending chain $M_0' \supseteq M_1' \supseteq \ldots$ of submodules of $F_{\sigma-1}(M)$. According to [Po, Chap 4, Corollary 3.10], there exists a descending chain $M_0 \supseteq M_1 \supseteq \ldots$ of submodules of $M$ such that $F_{\sigma-1}(M_i) = M_i'$ for each $i$ and since $F_{\sigma-1}(M_i/M_{i+1}) \neq 0$ for infinitely many indexes $i$, the induction hypothesis implies that $M_i/M_{i+1} \notin \mathcal{K}_{\sigma-1}$ for infinitely many $i$ which is a contradiction. To prove the second assertion, assume that $M$ is noetherian and so by Proposition 7.13, there exists a non-limit ordinal $\delta$ such that $\text{GK-dim} M = \delta$. We proceed by induction on $\delta$ that $\text{K-dim} M \leq \text{GK-dim} M$. If $\delta = 0$, then $M$ has finite length and so K-dim $M = 0$. If $\delta > 0$, since $F_{\delta-1}(M)$ has finite length, for any descending chain $M_0 \supseteq M_1 \supseteq \ldots$ of submodules of $M_0$, there exists some non-negative integer $n$ such that $F_{\delta-1}(M_i/M_{i-1}) = 0$ for all $i \geq n$ and so the induction hypothesis implies that $\text{K-dim}(M_i/M_{i-1}) \leq \delta - 1$ so that $\text{K-dim} M \leq \delta$. We now assume that $\sigma \geq \omega$. Then for any descending chain $M_0 \supseteq M_1 \supseteq \ldots$ of submodules of $M$, there exists some non-negative integer $n$ such that $\text{K-dim}(M_i/M_{i-1}) < \sigma$ for all $i \geq n$. Hence $F_{\sigma}(M_i/M_{i-1}) = 0$ for all $i \geq n$ by induction hypothesis. This implies that $F_{\sigma}(M)$ is artinian and so $\text{GK-dim} M \leq \sigma + 1$ as $M \in \mathcal{A}_{\sigma+1}$. If $M$ is noetherian and $\text{GK-dim} M = \delta$, we prove by transfinite induction on $\delta$ that $\text{K-dim} M + 1 \leq \delta$. If $\delta = \omega + 1$, then $M$ has finite length and so for any descending chain $M_0 \supseteq M_1 \supseteq \ldots$ of submodules of $M$, there exists some non-negative integer $n$ such that $F_{\omega}(M_i/M_{i-1}) = 0$ for all $i \geq n$ so that $\text{GK-dim}(M_i/M_{i-1}) \leq \omega$ for all $i \geq n$. Since the Gabriel-Krull dimension of noetherian modules are non-limit ordinals, using the first case we deduce that $\text{K-dim}(M_i/M_{i-1}) = \text{GK-dim}(M_i/M_{i-1}) < \omega$ for all $i \geq n$; and hence $\text{K-dim} M \leq \omega$. Similar to the induction step, $F_{\delta-1}(M)$ has finite length and so for any descending chain $M_0 \supseteq M_1 \supseteq \ldots$ of submodules of $M$, there exists some non-negative integer $n$ such that $F_{\delta-1}(M_i/M_{i-1}) = 0$ for all $i \geq n$ so that $\text{GK-dim}(M_i/M_{i-1}) \leq \delta - 1$ for all $i \geq n$. Now, the induction hypothesis implies that $\text{K-dim}(M_i/M_{i-1}) = \text{GK-dim}(M_i/M_{i-1}) - 1 < \delta - 1$ for all $i \geq n$; and hence $\text{K-dim} M \leq \delta - 1$. \[\square\]

We recall that a right noetherian ring $A$ is said to be fully right bounded if for every prime ideal $p$, the ring $A/p$ has the property that every essential right ideal contains a nonzero two sided ideal.

We show that if $A$ is a fully right bounded ring, then $\text{ASpec Mod-}A$ is Alexandroff where Mod-$A$ denotes the category of right $A$-modules. At first, we recall the compressible objects which have a key role in our studies.

Definition 7.13. We recall from [Sm] that a nonzero object $M$ in $\mathcal{A}$ is said to be compressible if each nonzero subobject $L$ of $M$ has some subobject isomorphic to $M$.

In the fully right bounded rings, irreducible prime ideals are closely related to the compressible modules.
Proposition 7.14. Let $A$ be a fully right bounded ring and let $p$ be a prime ideal of $A$. Then the following conditions are equivalent.  
(1) $p$ is an irreducible right ideal. 
(2) $A/p$ is compressible. 
(3) $A/p$ is monoform.

Proof. (1)⇒(2). If $p$ is an irreducible right ideal, then every nonzero submodule of $A/p$ is essential. Given a nonzero submodule $K$ of $A/p$, since $\text{Ass}(K) = \{p\}$, there exists a nonzero element $x \in K$ such that $\text{Ann}(xA) = p$. Observe that $p \subseteq \text{Ann}(x)$. If $p \neq \text{Ann}(x)$, since $A$ is fully right bounded, there exists a two-sided ideal $b$ such that $p \subseteq b \subset \text{Ann}(x)$. But this implies that $b \subseteq \text{Ann}(xA) = p$ which is impossible. Thus $p = \text{Ann}(x)$; and hence $xA \cong A/p$. (2)⇒(3). Assume that $A/p$ is compressible. Then using [K3, Proposition 2.12], the module $A/p$ is monoform. (3)⇒(1). Since $A/p$ is monoform, any nonzero submodule is essential. Thus $p$ is an irreducible right ideal. \qed

For any ring $A$, the atom spectrum $\text{ASpec Mod-}A$ is denoted by $\text{ASpec } A$. Now, we have the following proposition.

Lemma 7.15. Let $A$ be a fully right bounded ring. Then for any $\alpha \in \text{ASpec } A$, there exists a compressible monoform right $A$-module $H$ such that $\overline{H} = \alpha$.

Proof. Assume that $\alpha$ is an atom in $\text{ASpec } A$ and $M$ is a monoform right $A$-module such that $\alpha = \overline{M}$. Since $A$ is right noetherian, it follows from [GW, Lemma 15.3] that Krull dimension of $A$ exists and so by virtue of [Sm, Proposition 26.5.10], the module $M$ contains a compressible monoform submodule $H$ such that $\alpha = \overline{H}$. \qed

Proposition 7.16. If $A$ is a fully right bounded ring and $M$ is a right $A$-module, then $A(M)$ is an open subset of $\text{ASpec } A$. In particular, $\text{ASpec } A$ is an Alexandroff topological space.

Proof. Let $\alpha \in \text{A}(M)$. Then according to Lemma 7.15, there exists a compressible module $H$ such that $\alpha = \overline{H}$. Therefore $\bigcap_{H' \subseteq H} \text{Ass } H' = \text{Ass } H = \Lambda(\alpha)$ by [SaS, Proposition 2.3]. For any $\beta \in \text{Ass } H$, we have $\alpha \leq \beta$ and hence $X_\beta \subseteq X_\alpha$, which implies that $t_\beta \leq t_\alpha$. Thus $\beta \in \text{A}(M)$. The second claim follows by the first part and Lemma 7.14 and Lemma 7.7. \qed

As applications of Theorem 7.3 we have the following corollaries.

Corollary 7.17. Let $A$ be a fully right bounded ring and $M$ be a noetherian right $A$-module. Then $\text{AMin } M$ is a finite subset of $\text{ASpec } A$.

Proof. The result follows from Proposition 7.16 and Theorem 7.5. \qed

The following example due to Goodearl [Go] shows that if $A$ is not a fully right bounded ring, then Corollary 7.17 may not hold even for a cyclic module. An analogous example has been given by Musson [M].

Example 7.18. Let $k$ be an algebraically close field of characteristic zero and let $B = k[[t]]$ be the formal power series ring over $k$ in an indeterminate $t$. Define a $k$-linear derivation $\delta$ on $S$ according to the rule $\delta(\sum_{n=0}^\infty a_n t^n) = \sum_{n=0}^\infty na_n t^n$. Now, assume that $A = B[\theta]$ is the formal linear differential operator ring (the Ore extension) over $(B, \delta)$. Thus additively, $A$ is the abelian group of all polynomials over $B$ in an indeterminate $\theta$, with a multiplication given by $\theta b = b\theta + \delta(b)$ for all $b \in B$. Since $B$ is noetherian, using [R, Théorème 2, p.65], the ring $A$ is right and left noetherian and there is a $B$-isomorphism $B = A/\theta A$. In view of [Go], the nonzero right $A$-submodules of $B$ form a strictly descending chain $B > tB > t^2B > \ldots$ and $B$ is a critical right $A$-module of Krull dimension one and so all factors $t^n B/t^{n+1}B$ have Krull dimension zero. Also none of these submodules can embed in any strictly smaller submodule; and hence none of these submodule is compressible. It therefore follows from [GR, Theorem 8.6, Corollary 8.7] that $A$ is not a fully right bounded ring. Since $k$ is algebraically close field, the maximal two-sided ideals are precisely
we have $A_{\text{Min}}(M)$ is locally noetherian.

Example 7.20. (ii) $A$ is a noetherian Grothendieck category. Let $A_{\text{Spec}}$ be the category of right bounded rings. According to [St, Chap X, p.223, Example 2], the full faithful functor $T(-) = \text{Hom}_A(U, -) : A \to \text{Mod-A}$ establishes an equivalence between $A$ and $\text{Mod-A}$, the category of right $A$-modules. According to [Po, Chap 5, Lemma 8.3], $A$ is a right noetherian ring and $T(M)$ is a noetherian right $A$-module. It follows from Corollary 7.17 that $A_{\text{Min}}(T(M))$ is a finite set, say $A_{\text{Min}}(T(M)) = \{\alpha_1, \ldots, \alpha_n\}$. If $a : \text{Mod-A} \to A$ is the left adjoint functor of $T$, then according to Lemma 8.11 and Lemma 8.12, we have $A_{\text{Min}}(M) = \{a(\alpha_i)| 1 \leq i \leq n\}$. □

The following example shows that the above result may not hold in a more general case even if $A$ is locally noetherian.

Example 7.20. (a) is a noetherian generator. To be more precise, let $A = \text{GrMod}(k[x])$ be the category of graded $k[x]$ modules, where $k$ is a field and $x$ is a indeterminate with $\deg x = 1$. We notice that $A$ is a locally noetherian Grothendieck category. For each $i \in \mathbb{Z}$, the object $S_i = x^i k[x]/x^{i+1}k[x]$ is 0-critical; and hence $S_i$ is a minimal atom of $A$ for each $i \in \mathbb{Z}$. Furthermore, the set of minimal atom of a noetherian object is not finite in general even if $A$ is locally noetherian. If we consider the noetherian $k[x]$-module $M = k[x]$, then it is easy to see that $A_{\text{Min}}(M) = \text{ASupp}(M) = \{S_j| j \leq 0\} \cup \{M\}$.

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