Sample distribution theory using Coarea Formula

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ABSTRACT
Let $(\Omega, \Sigma, p)$ be a probability measure space and let $X : \Omega \rightarrow \mathbb{R}^k$ be a (vector valued) random variable. We suppose that the probability $p_X$ induced by $X$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^k$ and set $f_X$ as its density function. Let $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a $C^1$-map and let us consider the new random variable $Y = \phi(X)$: $X \rightarrow \mathbb{R}^n$. Setting $m := \max \{ \text{rank } (J\phi(x)) : x \in \mathbb{R}^k \}$, we prove that the probability $p_Y$ induced by $Y$ has a density function $f_Y$ with respect to the Hausdorff measure $H^m$ on $\phi(\mathbb{R}^k)$ which satisfies

$$f_Y(y) = \int_{\phi^{-1}(y)} f_X(x) \frac{1}{J_m \phi(x)} \, dH^{k-m}(x), \quad \text{for } H^m - \text{a.e. } y \in \phi(\mathbb{R}^k).$$

Here $J_m \phi$ is the $m$-dimensional Jacobian of $\phi$. When $J\phi$ has maximum rank we allow the map $\phi$ to be only locally Lipschitz. We also consider the case of $X$ having probability concentrated on some $m$-dimensional sub-manifold $E \subseteq \mathbb{R}^k$ and provide, besides, several examples including algebra of random variables, order statistics, degenerate normal distributions, Chi-squared and “Student’s t” distributions.

1. Introduction

The theory of Sample Distribution is an important branch of Statistics and Probability which study the general problem of determining the distribution of functions of random vectors. It provides a formal framework for modeling, simulating and making statistical inference. To be more precise, let us fix a probability measure space $(\Omega, \Sigma, p)$ and let $X$ be a (vector valued) random variable, i.e., a $\Sigma$-measurable function from $\Omega$ to $\mathbb{R}^k$, where $k \in \mathbb{N}$; $X$ is usually referred to as the data. Let $Y$ be any measurable function of the data $X$, i.e., $Y$ is a random variable which satisfies $Y = \phi(X)$ for some measurable function $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ with $n \in \mathbb{N}$; $Y$ is usually called a statistic. The problem which we address consists in finding the probability distribution of $Y$ knowing the distribution of $X$.

Depending on the nature of the data, there are in general different approaches for finding the distribution of the statistic $Y$, including the distribution function technique, the moment-generating function technique and the change of variable technique (see,
e.g., Hogg and Craig 1978, Section 4.1, 122). In the last case, let us suppose, for example, that the probability measure induced by \( X \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^k \) and let \( f_X \) be its density function. Then if \( k = n \) and \( \phi : \mathbb{R}^k \to \mathbb{R}^k \) is a \( C^1 \)-diffeomorphism, then the change of variable \( y = \phi(x) \) yields, for every Borel subset \( A \subseteq \mathbb{R}^k \),

\[
p(\phi(X) \in A) = p(X \in \phi^{-1}(A)) = \int_{\phi^{-1}(A)} f_X(x) \, dx = \int_A f_X(\phi^{-1}(y)) \frac{1}{|\det J_\phi(\phi^{-1}(y))|} \, dy.
\] (1)

The last equation implies that the probability measure induced by \( Y \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^k \) and its density function \( f_Y \) is determined by the equation

\[
f_Y(y) = f_X(\phi^{-1}(y)) \frac{1}{|\det J_\phi(\phi^{-1}(y))|}, \quad y \in \mathbb{R}^k.
\] (2)

This change of variables formula is widely used, for example, in machine learning and is essential for some recent results in density estimation and generative modeling like normalizing flows (Rezende and Mohamed 2015), NICE (Dinh, Krueger, and Bengio 2015), or Real NVP (Dinh, Sohl-Dickstein, and Bengio 2017). However all uses of this formula in the machine learning literature that we are aware of are constrained by the bijectivity and the differentiability of the map \( \phi \).

In this expository paper we extend formula (2) to the more general case of statistics \( Y = \phi(X) \) defined by locally Lipschitz functions \( \phi : \mathbb{R}^k \to \mathbb{R}^n \). The approach presented is mainly based upon the Coarea Formula proved by Federer in Federer (1959) which provides, in our setting, an elegant tool to derive the distribution of \( Y \). This also allows to get rid of the differentiability and invertibility assumptions on \( \phi \) and to treat the case \( k \neq n \) : this is quite useful in many problems of statistical inference and of machine learning (see, e.g., Cvitkovic and Koliander 2019).

When \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) is a \( C^1 \)-map and \( m := \max\{\text{rank } (J_\phi(x)) : x \in \mathbb{R}^k\} \), our result states that the probability \( p_Y \) induced by \( Y = \phi(X) \) has a density function \( f_Y \) with respect to the Hausdorff measure \( \mathcal{H}^m \) on \( \phi(\mathbb{R}^k) \) which satisfies

\[
f_Y(y) = \int_{\phi^{-1}(y)} f_X(x) \frac{1}{|J_m \phi(x)|} \, d\mathcal{H}^{k-m}(x), \quad \text{for } \mathcal{H}^m - \text{a.e. } y \in \phi(\mathbb{R}^k),
\] (3)

where \( J_m \phi(x) \) is the \( m \)-dimensional Jacobian of \( \phi \) (see Definition 3.1). When the Jacobian matrix \( J_\phi \) has, at any point, maximum rank we allow the map \( \phi \) to be only locally Lipschitz. We also consider the case of \( X \) having probability concentrated on some \( m \)-dimensional sub-manifold \( E \subseteq \mathbb{R}^k \). This case has many applications in directional and axial statistics, morphometrics, medical diagnostics, machine vision, image analysis and molecular biology (see, e.g., Bhattacharya and Bhattacharya 2008 and references there in).

We do not make any attempt to reach any novelty but the evidence suggests that this result is not as universally known as it should be. Besides this we give also several examples.
Let us briefly describe the content of the sections. In Section 2 we introduce the main definitions and notation that we use throughout the paper. Section 3 collects the main results we need about the theory of Hausdorff measures and the Area and Coarea Formulas. In Section 4 we develop the method proving Formula (3): when \( J \phi \) has maximum rank we also allow the map \( \phi \) to be locally Lipschitz. In Section 5 we gives a further generalization considering the case of random variables \( X \) having probability density functions \( f_X \) with respect to the Hausdorff measure \( \mathcal{H}^m \) concentrated on some \( m \)-dimensional sub-manifold \( E \subseteq \mathbb{R}^k \). Finally, in Section 6, we provide several examples which show how to apply the latter results in order to characterize the distribution of algebra of random variables and how to compute, in an easy way, the probability densities of some classic distributions including order statistics, degenerate normal distributions, Chi-squared and “Student’s t.” distributions.

Notation. We write \( \langle \lambda, \mu \rangle = \sum_i \lambda_i \mu_i \) to denote the inner product of \( \mathbb{R}^k \). When \( f : \mathbb{R}^k \to \mathbb{R}^n \) is a Lipschitz map we write \( Jf \) to denote its Jacobian matrix \( \left( \frac{\partial f_i}{\partial x_j} \right)_{i,j} \), which is defined a.e. on \( \mathbb{R}^k \). When \( A = (a_{ij}) \in \mathbb{R}^{n \times k} \) is a real matrix we write \( Ax \) to denote the linear operator

\[
\phi : \mathbb{R}^k \to \mathbb{R}^n, \quad x = (x_1, ..., x_k) \mapsto x \cdot A^t = (y_1, ..., y_n), \quad y_i = \sum_{j=1}^{k} a_{ij} x_j.
\]

With this notation the Jacobian matrix \( J(Ax) \) of \( Ax \) satisfies \( J(Ax) = A \). \( I_k \) is the identity matrix of \( \mathbb{R}^{k \times k} \). If \( (\Omega_1, \Sigma_1) \) and \( (\Omega_2, \Sigma_2) \) are measurable spaces, a function \( f : \Omega_1 \to \Omega_2 \) is said to be \( (\Sigma_1, \Sigma_2) \)-measurable if \( f^{-1}(B) \in \Sigma_1 \) for all \( B \in \Sigma_2 \). Unless otherwise specified when \( \Omega_2 = \mathbb{R}^k \) we always choose \( \Sigma_2 \) as the \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^k) \) of all the Borel subsets of \( \mathbb{R}^k \) and in this case we simply say that \( f \) is \( \Sigma_1 \)-measurable. We finally write \( \mathcal{L}^k \) and \( \mathcal{H}^s \) to denote respectively the Lebesgue measure and the \( s \)-dimensional Hausdorff measure on \( \mathbb{R}^k \): under this notation we have in particular that \( \mathcal{L}^k = \mathcal{H}^k \) and that \( \mathcal{H}^0 \) is the counting measure on \( \mathbb{R}^k \).

2. Preliminaries

In this section we fix the main notation and collect the main results we use concerning the Probability theory. For a good survey on the topic we refer the reader, for example, to Halmos (1950, Chapter IV) and Schervish (1995, Appendix A and B).

Let \( \mu, \nu \) two (positive) measures defined on a measurable space \( (\Omega, \Sigma) \). \( \nu \) is said to be absolutely continuous with respect to \( \mu \) and we write \( \nu \ll \mu \) if and only if \( \nu(B) = 0 \) for every \( B \in \Sigma \) such that \( \mu(B) = 0 \). \( \nu \) is said to have a density function \( f \) with respect to \( \mu \) if and only if there exists a measurable positive function \( f : \Omega \to \mathbb{R}^+ \) such that

\[
\nu(B) = \int_B f \, d\mu, \quad \text{for all } A \in \Sigma,
\]

(note that \( f \) is uniquely defined up to zero measure sets). When \( \mu \) is \( \sigma \)-finite, thanks to the Radon-Nikodym Theorem, the latter two definitions coincide and \( \frac{d\nu}{d\mu} := f \) is called
the Radon-Nikodym derivative of $\nu$ with respect to $\mu$ (see, e.g., Ambrosio, Fusco, and Pallara 2000, Theorem 1.28).

Let now $(\Omega, \Sigma, p)$ be a probability measure space, i.e., a measure space with $p(\Omega) = 1$ and let $k \in \mathbb{N}$. A $\Sigma$-measurable function $X$ from $\Omega$ to $\mathbb{R}^k$ is called a (vector) random variable; in statistical inference problems, $X$ is sometimes referred to as the given data. We write $p_X$ to denote the distribution of $X$, i.e., the measure induced by $X$ on $(\mathbb{R}^k, B(\mathbb{R}^k))$ defined by

$$p_X(A) = p(X \in A) := p(X^{-1}(A)), \quad \text{for all Borel set } A \subseteq \mathbb{R}^k.$$ 

With a little abuse of terminology, $X$ is said to be an absolutely continuous random variable if and only if $p_X \ll \mathcal{L}^k$, i.e., $p_X$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^k$. In this case the non-negative Radon-Nikodym derivative $f_X := \frac{d\nu_k}{d\mathcal{L}^k}$ is called the density function of $X$ and it is defined through the relation

$$p_X(A) = \int_A f_X(x) \, dx, \quad \text{for all Borel set } A \subseteq \mathbb{R}^k.$$ 

$X$ is said to be a discrete random variable if and only if there exists a countable subset $I = (a_i)_{i \in \mathbb{N}}$ of $\mathbb{R}^k$ such that $p_X \ll \mathcal{H}^0|_I$, i.e., $p_X$ is absolutely continuous with respect to the counting measure $\mathcal{H}^0$ on $I$. In this case the density $f_X := \frac{d\nu_k}{d\mathcal{H}^0|_I}$ is also called the probability mass function and it is defined through the relation

$$p_X(A) = \sum_{a_i \in A} f_X(a_i), \quad \text{for all subset } A \subseteq \mathbb{R}^k.$$ 

In particular $p_X(a) = f_X(a)$, for all $a \in A$.

Let $X : \Omega \to \mathbb{R}^k$ be a fixed random variable and let $n \in \mathbb{N}$; a random variable $Y : \Omega \to \mathbb{R}^n$ is called a statistic (of the data $X$) if it is a measurable function of $X$, i.e., $Y$ satisfies $Y = \phi^X$ for some $(B(\mathbb{R}^k), B(\mathbb{R}^n))$ measurable function $\phi : \mathbb{R}^k \to \mathbb{R}^n$.

Finally, let $X_1, \ldots, X_n$ be $n$ random variables where $X_i : \Omega \to \mathbb{R}^k$, for $i = 1, \ldots, n$. $X_1, \ldots, X_n$ are said to be independent if for every Borel subset $A_1, \ldots, A_n$ of $\mathbb{R}^k$ and for every $J \subseteq \{1, \ldots, n\}$ one has

$$p\left(\bigcap_{i \in J} X_i^{-1}(A_i)\right) = \prod_{i \in J} p_X(A_i).$$ 

In this case, if every $X_i$ is absolutely continuous with density function $f_i$, then $X$ is absolutely continuous and its density function satisfies $f_X(x_1, \ldots, x_n) = \prod_{i=1, \ldots, n} f_i(x_i)$ for every $x_i \in \mathbb{R}^k$.

If, moreover, $X_1, \ldots, X_n$ are identically distributed, i.e., $p_{X_i} = p_{X_j} := q$ for every $i, j$, then $X = (X_1, \ldots, X_n)$ is called a random sample from the distribution $q$; in this case, if every $X_i$ is absolutely continuous with density $f_i = f$, then the density function of $X$ satisfies $f_X(x_1, \ldots, x_n) = \prod_{i=1, \ldots, n} f(x_i)$. 

3. Area and Coarea Formulas

In this section we provide a brief introduction to the theory of Hausdorff measures and we collect the main results about the Area and Coarea Formulas proved by Federer in Federer (1959). For the related proofs, we refer the reader, for example, to (Ambrosio, Fusco, and Pallara 2000; Federer 1969; Giaquinta and Modica 2009) (and references therein).

We begin with the definition of the $s$-dimensional Hausdorff measure. Let $E \subseteq \mathbb{R}^n$, $\varepsilon > 0$ and let $(B(x_i, r_i))_{i \in \mathbb{N}}$ be a coverings of $E$ by a countable collections of balls $B(x_i, r_i)$ whose radii satisfy $r_i \leq \varepsilon$. For each $s \geq 0$, let

$$
\sigma_s(\varepsilon) = \frac{\pi^{s/2}}{\Gamma(1 + s/2)} \inf \sum_{i \in \mathbb{N}} r_i^s,
$$

where the infimum is taken over all such coverings. The monotonicity of $\sigma_s$, with respect to $s$, implies that there exists the limit (finite or infinite)

$$
\mathcal{H}^s(E) := \lim_{\varepsilon \to 0^+} \sigma_s(\varepsilon).
$$

This limit is called the $s$-dimensional Hausdorff measure of $E$. The Hausdorff measure $\mathcal{H}^s$ satisfies Caratheodory’s criterion therefore, the $\sigma$-algebra of all the $\mathcal{H}^s$-measurable sets contains all the Borel subsets of $\mathbb{R}^n$ (see, e.g., Ambrosio, Fusco, and Pallara 2000, Proposition 2.49).

This class of measures is very useful in geometric measure theory since it allows to define $s$-dimensional areas in a purely intrinsic way without any reference to parametrizations and which apply to general subsets $E \subseteq \mathbb{R}^n$ which can have a complicated geometric description but which can be nevertheless equipped with a notion of $s$-dimension in the sense that $0 < \mathcal{H}^s(E) < \infty$. In some particular cases of great interests, $\mathcal{H}^s$ gives back the classic well known notions of surfaces areas (i.e., induced measures on sub-manifolds), Lebesgue measures and of counting measures:

(i) If $0 < s \leq n$ is a positive integer and $B$ is a borel subset of an $s$-dimensional smooth sub-manifold embedded in $\mathbb{R}^n$ (in what follows we briefly write sub-manifold of $\mathbb{R}^n$), then by Theorem 3.3, $\mathcal{H}^s(B)$ coincides with the $s$-dimensional volume of $B$ given, using local charts, by the classic formula (4). In particular, $\mathcal{H}^n(E) = \mathcal{L}^n(E)$ for Lebesgue measurable subsets $E \subseteq \mathbb{R}^n$ (see, e.g., Federer 1969, Sections 3.1.19, 3.1.20 and Section 3.2).

(ii) When $s = 0$, $\mathcal{H}^0$ coincides with the counting measure on $\mathbb{R}^n$ which associates to any $E \subseteq \mathbb{R}^n$, its number of elements $|E|$ ($+\infty$ in case the subset is infinite): for a proof we refer e.g., to Maggi (2012, Proposition 3.3). In particular for every function $f : \mathbb{R}^n \to \mathbb{R}$ one has

$$
\int_E f(x) \, d\mathcal{H}^0(x) = \sum_{x \in E} f(x).
$$

Let now $\phi : \mathbb{R}^k \to \mathbb{R}^n$ be a locally Lipschitz map; we remark that Rademacher’s Theorem assures that $\phi$ is a.e. differentiable (see, e.g., Ambrosio, Fusco, and Pallara 2000, Proposition 2.12 and 2.14).

**Definition 3.1** ($k$-dimensional Jacobian). Let $k, m, n \in \mathbb{N}$ and let $\phi : \mathbb{R}^k \to \mathbb{R}^n$ be a locally Lipschitz map. The $m$-dimensional Jacobian of $\phi$ is defined by
\[ I_m \phi(x) := \sup \left\{ \frac{\mathcal{H}^m(J\phi(x)(P))}{\mathcal{H}^m(P)} \colon P \text{ is a } m \text{- dimensional parallelepiped of } \mathbb{R}^k \right\}, \]

where \( J\phi(x)(P) \) is the image of \( P \) under the Jacobian matrix \( J\phi(x) \) of \( \phi \) which exists for a.e. \( x \in \mathbb{R}^k \). When \( \operatorname{rank} (J\phi(x)) \leq m \) then
\[
I_m \phi(x) = \sqrt{\sum_B (\det B)^2},
\]
where the sum in the last equality runs along all \( m \times m \) minors \( B \) of \( J\phi(x) \) (see, e.g., Krantz and Parks 2008, 124; Morgan 2016, Section 3.6).

Note that \( I_m \phi(x) = 0 \) if and only if \( \operatorname{rank}(J\phi(x)) < m \) and that the Cauchy-Binet formula gives in particular
\[
J_n \phi = \sqrt{\det(J\phi \cdot J\phi^T)}, \quad J_k \phi = \sqrt{\det(J\phi^T \cdot J\phi)}.
\]

In the conventional situation \( k = n = m \), the above definition gives back the classic Jacobian \( J_k \phi = |\det \phi| \).

**Remark 3.2.** Setting \( m = \operatorname{rank}(J\phi(x)) \), when \( m < N \), \( m < K \), computations are simplified by viewing the differential \( J\phi(x) \) as a bijective map \( T : (\text{Ker } T)^\perp \rightarrow \text{rg } T \) from the orthogonal complement of its kernel onto its image: in this case \( I_m \phi(x) = |\det (T)| \) (see, e.g., Krantz and Parks 2008, 124; Morgan 2016, Section 3.6).

The next Theorem handles with the case \( k \leq n \) and it can be seen as a generalization of the change of variables formula (1) when the invertibility and \( C^1 \)-regularity assumptions on \( \phi \) are dropped. For its proof we refer the reader to Federer (1959, Theorem 3.2.11) and Giaquinta and Modica (2009, Theorem 2.80).

**Theorem 3.3 (Area formula).** Let \( \phi : \mathbb{R}^k \rightarrow \mathbb{R}^n \) be a locally Lipschitz map with \( k \leq n \).

(i) For any \( L^k \)-measurable set \( E \subseteq \mathbb{R}^k \) the multiplicity function \( y \mapsto \mathcal{H}^n(E \cap \phi^{-1}(y)) \) is \( \mathcal{H}^k \)-measurable in \( \mathbb{R}^n \) and
\[
\int_E I_k \phi(x) \, dx = \int_{\mathbb{R}^n} \mathcal{H}^n(E \cap \phi^{-1}(y)) \, d\mathcal{H}^k(y).
\]

(ii) If \( u \) is a positive measurable function, or \( u J_k \phi \in L^1(\mathbb{R}^k) \), then
\[
\int_{\mathbb{R}^k} u(x) I_k \phi(x) \, dx = \int_{\mathbb{R}^n} \int_{\phi^{-1}(y)} u(x) \, d\mathcal{H}^n(x) \, d\mathcal{H}^k(y) = \int_{\mathbb{R}^n} \sum_{x \in \phi^{-1}(y)} u(x) \, d\mathcal{H}^k(y).
\]

When \( \phi : E \rightarrow \mathbb{R}^n \) is injective, the last formula allows the computation of the area of the Lipschitz parametrized \( k \)-dimensional manifold \( \phi(E) \) of \( \mathbb{R}^n \):
\[
\mathcal{H}^k(\phi(E)) = \int_E J_k \phi \, dx = \int_E \sqrt{\det(J\phi^T \cdot J\phi)} \, dx,
\]
(4)
\[
\int_{\phi(E)} g(y) \, d\mathcal{H}^k(y) = \int_E g(\phi(x)) I_k \phi \, dx = \int_E g(\phi(x)) \sqrt{\det(J\phi^T \cdot J\phi)} \, dx,
\]
(5)
where \( g \) is any positive measurable function, or \( g \in L^1(\phi(E), \mathcal{H}^k) \). Note that in the
particular case of a Cartesian parametrization \( \phi(x) = (x, \psi(x)) \) one has \( J_k \phi = \sqrt{1 + \sum_B \det(B)^2} \) where this times the sum runs along all square minors \( B \) of the Jacobian matrix \( J\psi \) of \( \psi \) (see Ambrosio, Fusco, and Pallara 2000, 88).

The next theorem treats, conversely, the case \( k > n \) and it can be seen as a generalization of the Fubini’s theorem about the reduction of integrals. For its proof we refer the reader to Federer (1959, Theorem 3.2.3) and Giaquinta and Modica (2009, Theorem 2.86).

**Theorem 3.4 (Coarea formula).** Let \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) be a locally Lipschitz map with \( k > n \).

(i) For any \( \mathcal{L}^k \)-measurable set \( E \subseteq \mathbb{R}^k \) the function \( y \mapsto \mathcal{H}^{k-n}(E \cap \phi^{-1}(y)) \) is \( \mathcal{L}^n \)-measurable in \( \mathbb{R}^n \) and

\[
\int_E J_n \phi(x) \, dx = \int_{\mathbb{R}^n} \mathcal{H}^{k-n}(E \cap \phi^{-1}(y)) \, dy.
\]

(ii) If \( u \) is a positive measurable function, or \( u J_n \phi \in L^1(\mathbb{R}^k) \), then

\[
\int_{\mathbb{R}^k} u(x) J_n \phi(x) \, dx = \int_{\mathbb{R}^n} \int_{\phi^{-1}(y)} u(x) \, d\mathcal{H}^{k-n}(x) \, dy.
\]

When \( \phi \) is an orthogonal projection (e.g., \( \phi(x_1, \ldots, x_k) = (x_{i_1}, \ldots, x_{i_n}) \) where \( \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, k\} \)), then \( J_n \phi = 1 \), the level sets of \( \phi \) are \( (n-k) \)-planes and the last formula corresponds to Fubini’s theorem.

Applying Theorem 3.4 in the particular case \( n = 1 \), then one has \( J_1 \phi(x) = |\nabla \phi(x)| \) and the formula in (ii) becomes

\[
\int_{\mathbb{R}^k} u(x) \, |\nabla \phi(x)| \, dx = \int_{\mathbb{R}^n} \int_{\phi^{-1}(y)} u(x) \, d\mathcal{H}^{k-1}(x) \, dy. \tag{6}
\]

In the special case \( \phi(x) = |x|, J_n \phi(x) = 1 \) for every \( x \neq 0 \) and, since the map sending \( x \mapsto rx \) changes \( \mathcal{H}^{k-1} \) by the factor \( r^{k-1} \) (see, e.g., Ambrosio, Fusco, and Pallara 2000, Proposition 2.49), one has

\[
\int_{\mathbb{R}^k} u(x) \, dx = \int_0^\infty \int_{|x|=r} u(x) \, d\mathcal{H}^{k-1}(x) \, dr = \int_0^\infty r^{k-1} \int_{|x|=1} u(x) \, d\mathcal{H}^{k-1}(x) \, dr.
\]

We end the section by stating a generalization of the Coarea Formula to the case where the Lebesgue measure on the right hand side of the equations in Theorem 3.4 is replaced by the Hausdorff measure \( \mathcal{H}^m \), where \( m := \max\{\text{rank } (J\phi(x)) : x \in \mathbb{R}^k\} \). For simplicity we suppose \( f \) to be \( C^1 \)-differentiable.

**Theorem 3.5.** Let \( k, n \in \mathbb{N} \) and let \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) be a \( C^1 \)-map and let \( m := \max\{\text{rank } (J\phi(x)) : x \in \mathbb{R}^k\} \). The following properties hold.

(i) For every \( \mathcal{L}^k \)-measurable set \( E \subseteq \mathbb{R}^k \) the function \( y \mapsto \mathcal{H}^{k-m}(E \cap \phi^{-1}(y)) \) is \( \mathcal{H}^m \)-measurable in \( \mathbb{R}^n \) and one has
\[
\int_E J_m \phi(x) \, dx = \int_{\mathbb{R}^n} \mathcal{H}^{k-m}(E \cap \phi^{-1}(y)) \, d\mathcal{H}^m(y).
\]

(ii) If \( u \) is a positive measurable function, or \( uJ_m \phi \in L^1(E) \), then
\[
\int_E u(x) J_m \phi(x) \, dx = \int_{\mathbb{R}^n} \int_{\phi^{-1}(y) \cap E} u(x) \, d\mathcal{H}^{k-m}(x) \, d\mathcal{H}^m(y).
\]

**Proof.** The proof is a consequence of (Hajłasz, Korobkov, and Kristensen 2017, Theorem 5.1, Theorem 5.2).

The next Remark clarifies some positivity properties about \( k \)-dimensional Jacobians. It can be seen as a generalization of Sard’s Theorem (see also Ambrosio, Fusco, and Pallara 2000, Lemma 2.73, Lemma 2.96, Remark 2.97; Hajłasz, Korobkov, and Kristensen 2017, Theorem 1.1).

**Remark 3.6.** Let \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) be a locally Lipschitz map and let us suppose, preliminarily, that \( J\phi(x) \) has maximum rank for a.e. \( x \in \mathbb{R}^k \).

(i) if \( k \leq n \), then using (i) of Theorem 3.3 with \( E := \{ x \in \mathbb{R}^k : J_k \phi(x) = 0 \} \) we get
\[
\int_{\mathbb{R}^n} \mathcal{H}^0(E \cap \phi^{-1}(y)) \, d\mathcal{H}^k(y) = 0.
\]

This implies \( \mathcal{H}^0(E \cap \phi^{-1}(y)) = 0 \) (i.e., \( \phi(E) \cap \{ y \} = \emptyset \)) for \( \mathcal{H}^k \)-a.e. \( y \in \mathbb{R}^n \). This yields \( \mathcal{H}^k(\phi(E)) = 0 \) and it implies, in particular, that \( J_k \phi > 0 \) on \( \phi^{-1}(y) \) for \( \mathcal{H}^k \)-a.e. \( y \in \mathbb{R}^n \).

(ii) if \( k \geq n \), then using (i) of Theorem 3.4 with \( E := \{ x \in \mathbb{R}^k : J_n \phi(x) = 0 \} \) we get
\[
\int_{\mathbb{R}^n} \mathcal{H}^{k-n}(E \cap \phi^{-1}(y)) \, dy = 0.
\]

This yields \( \mathcal{H}^{k-n}(E \cap \phi^{-1}(y)) = 0 \) for a.e. \( y \in \mathbb{R}^n \) and it implies, in particular, that \( J_n \phi > 0 \mathcal{H}^{k-n} \)-a.e. on \( \phi^{-1}(y) \) for a.e. \( y \in \mathbb{R}^n \).

Let us suppose, now, \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) to be \( C^1 \) and let \( m := \max \{ \text{rank} \ (J \phi(x)) : x \in \mathbb{R}^k \} \). Then setting \( E := \{ x \in \mathbb{R}^k : J_m \phi(x) = 0 \} \) and using Theorem 3.5 we get
\[
\int_{\mathbb{R}^n} \mathcal{H}^{k-m}(E \cap \phi^{-1}(y)) \, d\mathcal{H}^m(y) = 0.
\]

This implies that \( J_m \phi > 0 \mathcal{H}^{k-m} \)-a.e. on \( \phi^{-1}(y) \) for \( \mathcal{H}^m \)-a.e. \( y \in \mathbb{R}^n \).

4. Sample distribution theory

Let \((\Omega, \Sigma, p)\) be a probability measure space, let \( k \in \mathbb{N} \) and let \( X : \Omega \to \mathbb{R}^k \) be an absolutely continuous random variable. Let \( Y := \phi \circ X \) be a statistic, where \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) is a
measurable map and $k \in \mathbb{N}$. In this section we prove that when \( \phi \) is locally Lipschitz then the probability measure \( p_Y \) induced by \( Y \) has a density function, with respect to some Hausdorff measure \( H^m \) on \( \phi(\mathbb{R}^k) \subseteq \mathbb{R}^n \), which can be computed explicitly in terms of an integral involving the density function \( f_X \) of \( X \). We recall preliminarily that the Radon-Nikodym derivative of a measure is uniquely defined up to zero measure sets: since, by definition, \( p_Y \) is concentrated on \( \phi(\mathbb{R}^k) \), in what follows we can always set \( f_Y(y) = 0 \) for \( y \notin \phi(\mathbb{R}^k) \).

We start with the case \( k \leq n \).

**Theorem 4.1.** Let \((\Omega, \Sigma, p)\) be a probability measure space, let \( k, n \in \mathbb{N} \) with \( k \leq n \) and let \( X : \Omega \to \mathbb{R}^k \) be an absolutely continuous random variable with probability density function \( f_X \). If \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) is a locally Lipschitz map such that rank \((J\phi) = k\) a.e., then the probability measure induced by the statistic \( Y := \phi \circ X \) is absolutely continuous with respect to the Hausdorff measure \( H^k \) on \( \mathbb{R}^n \), i.e., \( p_Y \ll H^k \). Its Radon-Nykodym derivative \( \frac{dp_Y}{dH^k} \) is defined through the relation

\[
p_Y(A) = p(Y^{-1}(A)) = \int_A \frac{dp_Y}{dH^k}(y) \, dH^k(y), \quad \text{for all Borel subset } A \subseteq \mathbb{R}^n.
\]

It satisfies

\[
\frac{dp_Y}{dH^k}(y) = \sum_{i \in \mathbb{N}} f_X(\phi_i^{-1}(y)) \frac{1}{J_k \phi(x)} \, dH^0(x) = \sum_{\phi(x) = y} f_X(x) \frac{1}{J_k \phi(x)}, \quad \text{for } H^k - \text{a.e. } y \in \phi(\mathbb{R}^k)
\]

and it is 0 otherwise.

Moreover let us suppose that there exists a countable disjoint covering \( \cup_{i \in \mathbb{N}} E_i \) of a.e point of \( \mathbb{R}^k \) (i.e., the set of points of \( \mathbb{R}^k \) which are not covered has \( \mathcal{L}^k \)-measure zero) such that on each measurable subset \( E_0 \) the restriction map \( \phi_i := \phi|_{E_i} \) is a.e injective. Then

\[
\frac{dp_Y}{dH^k}(y) = \sum_{i \in \mathbb{N}} f_X(\phi_i^{-1}(y)) \frac{1}{J_k \phi(\phi^{-1}(y))}, \quad \text{for } H^k - \text{a.e. } y \in \phi(\mathbb{R}^k).
\]

**Proof.** Let \( A \subseteq \mathbb{R}^n \) be a Borel set. Recalling Definition 3.1 and Remark 3.6, \( J_k \phi > 0 \) on \( \phi^{-1}(y) \) for \( H^k \)-a.e. \( y \in \mathbb{R}^n \). Then using the Area Formula of Theorem 3.3 one has

\[
p_Y(A) = p(Y^{-1}(A)) = p(\{\phi^{-1}(A)\}) = \int_{\phi^{-1}(A)} f_X(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} \int_{\phi^{-1}(y) \cap \phi^{-1}(A)} f_X(x) \frac{1}{J_k \phi(x)} \, dH^0(x) \, dH^k(y)
\]

\[
= \int_A \int_{\phi^{-1}(y)} f_X(x) \frac{1}{J_k \phi(x)} \, dH^0(x) \, dH^k(y) = \int_A \sum_{x \in \phi^{-1}(y)} f_X(x) \frac{1}{J_k \phi(x)} \, dH^k(y).
\]

This proved the first required claim. The second assertion follows after observing that, under the given hypothesis, \( \phi^{-1}(y) = \cup_i \{\phi_i^{-1}(y)\} \) for every \( y \in \phi(\mathbb{R}^k) \). \( \square \)
Remark 4.2. We note that if $k < n$, then $H^k$ is not $\sigma$-finite on $\mathbb{R}^n$, so we cannot directly use the Radon-Nikodym theorem in order to deduce from $p_Y \ll \mathcal{H}^k$ the existence of $\frac{dp_Y}{d\mathcal{H}^k}(y)$. Nevertheless in this case $p_Y$ is concentrated on $\phi(\mathbb{R}^k)$ which has $\sigma$-finite $\mathcal{H}^k$-measure and therefore the Radon-Nikodym theorem applies using $p_Y \ll \mathcal{H}^k_{\phi(\mathbb{R}^k)}$.

Indeed if $\mathbb{R}^k \subseteq \bigcup_{i \in \mathbb{N}} E_i$, where each $E_i$ is a Borel subset of $\mathbb{R}^k$ such that $\mathcal{L}^k(E_i) < \infty$, then $\phi(\mathbb{R}^k) \subseteq \bigcup_{i \in \mathbb{N}} \phi(E_i)$ and from (Ambrosio, Fusco, and Pallara 2000, Proposition 2.49) one has $\mathcal{H}^k(\phi(E_i)) < \text{Lip}(\phi)^k \mathcal{L}^k(E_i) < \infty$.

When $k = n$, then recalling that $H^k = \mathcal{L}^k$, the previous theorem implies that $p_Y \ll \mathcal{L}^k$.

Corollary 4.3. Let $(\Omega, \Sigma, p)$ be a probability measure space, let $k \in \mathbb{N}$ and let $X : \Omega \to \mathbb{R}^k$ be an absolutely continuous random variable with probability density function $f_X$. If $\phi : \mathbb{R}^k \to \mathbb{R}^k$ is a locally Lipschitz map such that $\text{rank } (J\phi) = k$ a.e., then the statistic $Y := \phi^*X$ is an absolutely continuous random variable and its probability density function $f_Y$ satisfies

$$f_Y(y) = \sum_{\phi(x) = y} f_X(x) \frac{1}{J_k\phi(x)}, \quad \text{for a.e. } y \in \phi(\mathbb{R}^k)$$

and it is 0 otherwise. Moreover let us suppose that there exists a countable disjoint covering $\bigcup_{i \in \mathbb{N}} E_i$ of a.e point of $\mathbb{R}^k$ such that on each measurable subset $E_i$, $\phi_i := \phi|_{E_i}$ is a.e injective. Then

$$f_Y(y) = \sum_{i \in \mathbb{N}} f_X(\phi_i^{-1}(y)) \frac{1}{J_k\phi(\phi_i^{-1}(y))}, \quad \text{for a.e. } y \in \phi(\mathbb{R}^k).$$

Let us now consider the case $k > n$.

Theorem 4.4. Let $(\Omega, \Sigma, p)$ be a probability measure space, let $k, n \in \mathbb{N}$ with $k \geq n$ and let $X : \Omega \to \mathbb{R}^k$ be an absolutely continuous random variable with probability density function $f_X$. If $\phi : \mathbb{R}^k \to \mathbb{R}^n$ is a locally Lipschitz map such that $\text{rank } (J\phi) = n$ a.e., then the statistic $Y := \phi^*X$ is an absolutely continuous random variable (i.e., $p_Y \ll \mathcal{L}^n$) and its probability density function $f_Y$ satisfies

$$f_Y(y) = \int_{\phi^{-1}(y)} f_X(x) \frac{1}{J_n\phi(x)} \, d\mathcal{H}^{k-n}(x), \quad \text{for a.e. } y \in \phi(\mathbb{R}^k)$$

and it is 0 otherwise.

Proof. The case $k = n$ is the result of the previous Corollary. Let us suppose $k > n$. Recalling Definition 3.1 and Remark 3.6, $J_n\phi > 0H^{k-n}$-a.e on $\phi^{-1}(y)$ for a.e. $y \in \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ be a Borel set. Then using the Coarea Formula of Theorem 3.4 one has
\[ p_Y(A) = p(Y^{-1}(A)) = p(X^{-1}(\phi^{-1}(A))) \]
\[ = \int_{\phi^{-1}(A)} f_X(x) \, dx = \int_{\mathbb{R}} \int_{\phi^{-1}(y) \cap \phi^{-1}(A)} f_X(x) \frac{1}{f_n \phi(x)} \, d\mathcal{H}^{k-n}(x) \, dy \]
\[ = \int_{A \cap \phi^{-1}(y)} f_X(x) \frac{1}{f_n \phi(x)} \, d\mathcal{H}^{k-n}(x) \, dy. \]

This proved the required claim. \(\Box\)

For the reader’s convenience we enlighten in the following corollary the particular case \(n=1\) which is very useful in the applications and which follows from formula (6).

**Corollary 4.5.** Let \((\Omega, \Sigma, p)\) be a probability measure space, let \(k \in \mathbb{N}\) and let \(X : \Omega \to \mathbb{R}^k\) be an absolutely continuous random variable with probability density function \(f_X\). If \(\phi : \mathbb{R}^k \to \mathbb{R}^n\) is a locally Lipschitz map such that \(|\nabla \phi| > 0\) a.e., then the statistic \(Y := \phi^\circ X\) is an absolutely continuous random variable and has probability density function

\[ f_Y(y) = \int_{\phi^{-1}(y)} f_X(x) \frac{1}{|\nabla \phi(x)|} \, d\mathcal{H}^{k-1}(x), \quad \text{for a.e. } y \in \phi(\mathbb{R}^k). \]

**Remark 4.6.** The assumptions \(p(\Omega) = 1\) was never used in the proof of the Theorems 4.1 and 4.4. Indeed analogous results hold with \(p_X\) replaced by an absolutely continuous measure on \(\mathbb{R}^k\). More precisely let \(\mu\) be a measure defined on \((\mathbb{R}^k, \mathcal{B})\) such that \(\mu \ll \mathcal{L}^k\) and let \(\frac{d\mu}{d\mathcal{L}^k}\) its Radon-Nykodym derivative. Let \(\phi : \mathbb{R}^k \to \mathbb{R}^n\) be a locally Lipschitz map whose Jacobian matrix \(J\phi\) has a.e. maximum rank.

(i) \(\text{If } k \leq n \text{ then } \mu \phi^{-1} \ll \mathcal{H}^k \text{ and for } \mathcal{H}^k\text{-a.e. } y \in \phi(\mathbb{R}^k) \text{ one has} \)

\[ \frac{d\mu \phi^{-1}}{d\mathcal{H}^k}(y) = \int_{\phi^{-1}(y)} \frac{d\mu}{d\mathcal{L}^k}(x) \frac{1}{f_n \phi(x)} \, d\mathcal{H}^0(x) = \sum_{\phi(x) = y} \frac{d\mu}{d\mathcal{L}^k}(x) \frac{1}{f_n \phi(x)}. \]

(ii) \(\text{If } k > n \text{ then } \mu \phi^{-1} \ll \mathcal{L}^n \text{ and for a.e. } y \in \mathbb{R}^n \text{ one has} \)

\[ \frac{d\mu \phi^{-1}}{d\mathcal{L}^n}(y) = \int_{\phi^{-1}(y)} \frac{d\mu}{d\mathcal{L}^k}(x) \frac{1}{f_n \phi(x)} \, d\mathcal{H}^{k-n}(x). \]

We end the section by applying Theorem 3.5 in order to extend Theorem 4.4 to the case of a \(C^1\)-map \(\phi\) whose Jacobian could possibly have not maximum rank. In this case, setting \(m := \max\{\text{rank } (J\phi(x)) : x \in \mathbb{R}^k\}\), the induced probability \(p_Y\) has a density function \(f_Y\) with respect to the Hausdorff measure \(\mathcal{H}^m\) on \(\phi(\mathbb{R}^k) \subseteq \mathbb{R}^n\).

**Theorem 4.7.** Let \((\Omega, \Sigma, p)\) be a probability measure space, let \(k,n \in \mathbb{N}\) and let \(X : \Omega \to \mathbb{R}^k\) be an absolutely continuous random variable with probability density function \(f_X\). Let \(\phi : \mathbb{R}^k \to \mathbb{R}^n\) be a \(C^1\)-map and let \(m := \max\{\text{rank } (J\phi(x)) : x \in \mathbb{R}^k\}\). Then the induced probability measure \(p_Y\) of the statistic \(Y := \phi^\circ X\) has a density function \(f_Y\) with respect to the Hausdorff measure \(\mathcal{H}^m\) which satisfies
\[ f_Y(y) = \int_{\phi^{-1}(y)} f_X(x) \frac{1}{J_m \phi(x)} \, d\mathcal{H}^{k-m}(x), \quad \text{for } \mathcal{H}^m \text{-a.e. } y \in \phi(\mathbb{R}^k) \]

and it is 0 otherwise.

**Proof.** Recalling Definition 3.1 and Remark 3.6, \( J_m \phi > 0 \mathcal{H}^{k-m}\text{-a.e.} \) on \( \phi^{-1}(y) \) for \( \mathcal{H}^m\text{-a.e.} \) \( y \in \mathbb{R}^n \). Let \( A \subseteq \mathbb{R}^n \) be a Borel set. Then using Theorem 3.5 one has

\[
p_Y(A) = p(Y^{-1}(A)) = p(X^{-1}(\phi^{-1}(A))) = \int_{\phi^{-1}(A)} f_X(x) \, dx = \int_{\mathbb{R}^n} \int_{\phi^{-1}(y) \cap \phi^{-1}(A)} f_X(x) \frac{1}{J_m \phi(x)} \, d\mathcal{H}^{k-m}(x) \, d\mathcal{H}^m(y) = \int_A \phi^{-1}(y) \frac{1}{J_m \phi(x)} \, d\mathcal{H}^{k-m}(x) \, d\mathcal{H}^m(y).
\]

This proves the required claim.

### 5. Further generalizations

In this section we briefly expose, for the interested reader, a further generalization of Theorem 4.4 which covers the case of probabilities \( p_X \) concentrated on some \( m \)-dimensional subset \( E \subseteq \mathbb{R}^k \). This includes, for example, the cases of a random variable \( X \) which is uniformly distributed over a generic subset \( E \) and of random variables taking values on \( m \)-dimensional manifolds. Statistical analysis on manifolds has many applications in directional and axial statistics, morphometrics, medical diagnostics, machine vision and image analysis (see, e.g., Bhattacharya and Bhattacharya 2008 and references therein). Amongst the many important applications, those arising, for example, from the analysis of data on torus play also a fundamental role in molecular biology in the study of the Protein Folding Problem.

Although the following Theorem is valid for countably \( m \)-rectifiable sets (see Krantz and Parks 2008, Definition 5.4.1, Lemma 5.4.2) and in the context of abstract Riemannian manifolds (see Federer 1959, Theorem 3.1), we suppose for simplicity \( E \) to be an \( m \)-dimensional sub-manifold of \( \mathbb{R}^k \).

We state first the Area and Coarea formula relative to sub-manifolds of \( \mathbb{R}^k \). For simplicity we also suppose \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) to be a \( C^1 \)-map although the next results are valid for locally Lipschitz maps. If \( J\phi^E \) is the tangential Jacobian matrix of \( \phi \) with respect to \( E \) (see Maggi 2012, Formula 11.1), the \( k \)-tangential Jacobian \( J_k^E \phi \) is defined as in Definition 3.1. For a rigorous introduction to tangential Jacobians as well as for all the other details we refer the reader to Federer (1969, Chapter 3), Krantz and Parks (2008, Section 5.3), and Maggi (2012, Section 11.1).

**Remark 5.1.** If \( \phi \in C^1(\mathbb{R}^k; \mathbb{R}^n) \), \( E \) is an \( m \)-dimensional sub-manifold of \( \mathbb{R}^k \), and \( x \in E \), then \( J\phi^E(x) \) is the restriction of \( J\phi(x) \) (viewed as a linear map) over the tangent space \( T_xE \) of \( E \) at \( x \). In particular when \( n = 1 J\phi^E(x) \) is simply the orthogonal projection \( \nabla \phi(x) \perp \) of \( \nabla \phi(x) \) over the tangent space \( T_xE \) and in this case \( J_k^E \phi(x) = ||\nabla \phi(x) \perp|| \) (see Maggi 2012, Remark 11.2; Krantz and Parks 2008, Remark 5.3.4).
**Theorem 5.2.** Let \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) be a \( C^1 \)-map and let \( E \subseteq \mathbb{R}^k \) an \( m \)-dimensional submanifold. The following properties hold.

(i) If \( m \leq n \) and \( u \) is a positive measurable function, or \( u \in L^1(E, \mathcal{H}^m) \), one has
\[
\int_E u \, J_m^E \phi \, d\mathcal{H}^m(x) = \int_{\mathbb{R}^n} \int_{E \cap \phi^{-1}(y)} u \, d\mathcal{H}^0(x) \, d\mathcal{H}^m(y).
\]

(ii) If \( m \geq n \) and \( u \) is a positive measurable function, or \( u \in L^1(E, \mathcal{H}^m) \), one has
\[
\int_E u \, J_m^E \phi \, d\mathcal{H}^m(x) = \int_{\mathbb{R}^n} \int_{E \cap \phi^{-1}(y)} u \, d\mathcal{H}^{m-n}(x) \, dy.
\]

**Proof.** See Ambrosio, Fusco, and Pallara (2000, Theorem 2.91, Theorem 2.93) and Krantz and Parks (2008, Theorem 5.4.7, Theorem 5.4.8). We also recall that, in case (ii), the Hausdorff measure \( \mathcal{H}^n \) on \( \mathbb{R}^n \) coincides with the Lebesgue measure.

The same methods of proof used in Section 4, yield finally the next result. Note that every \( m \)-dimensional submanifold \( E \subseteq \mathbb{R}^k \) has \( \mathcal{H}^m - \sigma \)-finite measure.

**Theorem 5.3.** Let \( (\Omega, \Sigma, \mathbb{P}) \) be a probability measure space, let \( k, n \in \mathbb{N} \) and let \( E \subseteq \mathbb{R}^k \) an \( m \)-dimensional submanifold. Let \( X : \Omega \to E \) be a random variable having a probability density function \( f_X \) with respect to the Hausdorff measure \( \mathcal{H}^m_{|E} \) on \( E \). Let \( \phi : \mathbb{R}^k \to \mathbb{R}^n \) be a \( C^1 \)-map whose tangential Jacobian \( J^E \phi(x) \) has maximum rank at any point \( x \in E \). The following properties hold.

(i) If \( m \leq n \) then the probability measure \( \mathbb{P}_Y \) induced by the statistic \( Y := \phi^* X \) is absolutely continuous with respect to the Hausdorff measure \( \mathcal{H}^m_{|\phi(E)} \) on \( \phi(E) \) and its density function \( f_Y \) satisfies
\[
f_Y(y) = \int_{\phi^{-1}(y)} f_X(x) \frac{1}{J_m^E \phi(x)} \, d\mathcal{H}^0(x) = \sum_{\phi(x)=y} f_X(x) \frac{1}{J_m^E \phi(x)},
\]
for \( \mathcal{H}^m - \text{a.e.} \ y \in \phi(E) \).

(ii) If \( m \geq n \) then the probability measure \( \mathbb{P}_Y \) induced by the statistic \( Y := \phi^* X \) is absolutely continuous with respect to the Lebesgue measure \( \mathcal{L}^n \) on \( \phi(E) \) (i.e., \( \mathbb{P}_Y \ll \mathcal{L}^n_{|\phi(E)} \)) and its probability density function \( f_Y \) satisfies
\[
f_Y(y) = \int_{\phi^{-1}(y)} f_X(x) \frac{1}{J_m^E \phi(x)} \, d\mathcal{H}^{m-n}(x), \text{ for a.e. } y \in \phi(E).
\]

6. Some applications

In this section we apply the results of the previous sections in order to compute the density functions of some distributions in some cases of relevant interest.
6.1. First examples

In these first examples we provide a density formula for random variables which are algebraic manipulations of absolutely continuous random variables. The first example, in particular, is used in Proposition 6.12 in order to find the density of the chi-squared distribution.

Example 6.1 (Square function). Let \((\Omega, \Sigma, \mu)\) be a probability measure space and let \(X : \Omega \to \mathbb{R}\) be an absolutely continuous random variable with probability density function \(f_X\). We employ Corollary 4.3 with \(\phi : \mathbb{R} \to [0, \infty[, \quad \phi(t) = t^2, \quad J_1(t) = 2|t|\).

Then the statistic \(Y := X^2\) is an absolutely continuous random variable and its probability density function \(f_Y\) satisfies

\[
f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}, \quad \text{for any } y > 0.
\]

More generally let \(k \in \mathbb{N}\) and let \(X = (X_1, \ldots, X_k) : \Omega \to \mathbb{R}^k\) be an absolutely continuous (vector valued) random variable with probability density function \(f_X\). Then employing Theorem 4.4 with \(\phi : \mathbb{R}^k \to [0, \infty[, \quad \phi(x) = \|x\|^2 = x_1^2 + \cdots + x_k^2, \quad J_1(x) = 2\|x\|\), we get that the statistic \(Y := \|X\|^2 = X_1^2 + \cdots + X_k^2\) is an absolutely continuous random variable whose probability density function satisfies

\[
f_Y(y) = \int_{\|x\|^2 = \frac{y}{2}\|x\|} f_X(x) \, d\mathcal{H}^{k-1}(x), \quad \text{for any } y > 0.
\]

Example 6.2 (Affine transformations). Let \((\Omega, \Sigma, \mu)\) be a probability measure space, let \(k \in \mathbb{N}\) and let \(X : \Omega \to \mathbb{R}^k\) be an absolutely continuous random variable with probability density function \(f_X\). Let us consider the affine transformation \(\phi : \mathbb{R}^k \to \mathbb{R}^n, \quad \phi(x) = Ax + y_0\), where \(A \in \mathbb{R}^{n \times k}\), \(\text{rank}(A) = m\) and \(y_0 \in \mathbb{R}^n\). Recalling Definition 3.1, the \(m\)-dimensional Jacobian of \(\phi\) is given by

\[
J_m \phi(x) = \sqrt{\sum_B (\det B)^2} =: A_m
\]

where the sum runs along all \(m \times m\) minors \(B\) of \(A\). Then, using Theorem 4.7, the induced probability measure \(\mu_Y\) of the statistic \(Y := AX + y_0\) has a density function \(f_Y\) with respect to the Hausdorff measure \(\mathcal{H}^m\) on the \(m\)-dimensional hyper-surface \(\phi(\mathbb{R}^k) = \{y = Ax + y_0 : x \in \mathbb{R}^k\}\) which satisfies

\[
f_Y(y) = \frac{1}{A_m} \int_{Ax + y_0 = y} f_X(x) \, d\mathcal{H}^{k-m}(x)
= \frac{1}{A_m} \int_{\text{Ker}(A) + x_{y}} f_X(x) \, d\mathcal{H}^{k-m}(x), \quad \text{for } y \in \phi(\mathbb{R}^k).
\]
Here for $y \in \phi(R^k), x_t \in \mathbb{R}^k$ is any fixed solution of the equation $y = Ax_t + y_0$.

When $m = n$, then $A_n = \sqrt{\det(AA^T)}$ and the map $\phi$ is surjective, i.e., $\phi(R^k) = \mathbb{R}^n$. In this case Theorem 4.4 implies that $p_y \ll L^n$, i.e., $Y$ is an absolutely continuous random variable. If moreover $k = n$ and $A \in \mathbb{R}^{k \times k}$ is not-singular then $A_k = |\det A|$ and in this case we have

$$f_Y(y) = \frac{1}{|\det A|} \int_{Ax + y_0 = y} f_X(x) \, d\mathcal{H}^0(x) = \frac{f_X(A^{-1}(y-y_0))}{|\det A|}, \quad \text{for } y \in \mathbb{R}^k.$$

**Example 6.3 (Sum of variables and Sample mean).** Let $(\Omega, \Sigma, \rho)$ be a probability measure space, let $k \in \mathbb{N}$ and let $X = (X_1, \ldots, X_k) : \Omega \to \mathbb{R}^k$ be an absolutely continuous random variable with probability density function $f_X$. We employ Corollary 4.5 with

$$\phi : \mathbb{R}^k \to \mathbb{R}, \quad \phi(t) = \sum_{i=1}^k t_i, \quad J_1(t) = |\nabla \phi| = \sqrt{k}.$$

Then the statistic $Y := \sum_{i=1}^k X_i$ is an absolutely continuous random variable and its probability density function $f_Y$ satisfies

$$f_Y(y) = \int_{\sum_{i=1}^k x_i = y} \frac{f_X(x)}{\sqrt{k}} \, d\mathcal{H}^{k-1}(x), \quad \text{for a.e. } y \in \mathbb{R}.$$

Let us set $x^{k-1} := (x_1, \ldots, x_{k-1})$ and let $\psi(x^{k-1}) = (x^{k-1}, y - \sum_{i=1}^{k-1} x_i)$ be a parameterization of the hyperplane $\sum_{i=1}^k x_i = y$. Using the area formula (5), the last integral becomes

$$f_Y(y) = \int_{\mathbb{R}^{k-1}} f_X(x^{k-1}, y - \sum_{i=1}^{k-1} x_i) \, dx^{k-1}, \quad \text{for a.e. } y \in \mathbb{R}.$$

In the particular case $k = 2$ and if $X_1, X_2$ are independent, the last formula gives the well-known convolution form for the distribution of the random variable $X_1 + X_2$:

$$f_{X_1 + X_2}(y) = \int_{\mathbb{R}} f_{X_1}(t) f_{X_2}(y - t) \, dt, \quad \text{for a.e. } y \in \mathbb{R},$$

where $f_{X_1}, f_{X_2}$ are respectively the density function of the distribution generated by $X_1, X_2$.

Moreover if $X_1, \ldots, X_k$ are identically distributed and independent with common probability density function $f : \Omega \to \mathbb{R}$, then (using also Example 6.2 with $\phi(x) = \frac{1}{k} x$), the density function of the sample mean $Z := \frac{1}{k} \sum_{i=1}^k X_i$ is

$$f_Z(y) = k \, f_Y(ky) = k \int_{\sum x_i = ky} \prod_{i=1}^k f(x_i) \, d\mathcal{H}^{k-1}(x), \quad \text{for a.e. } y \in \mathbb{R}.$$
Example 6.4 (Product and ratio of random variables). Let \((\Omega, \Sigma, \rho)\) be a probability measure space and let \(X : \Omega \rightarrow \mathbb{R}^2\), \(X = (X_1, X_2)\) be an absolutely continuous random variable with probability density function \(f_X\).

(i) Let us employ Corollary 4.5 with

\[
\phi : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \phi(x_1, x_2) = x_1 x_2, \quad J_1 \phi(x_1, x_2) = |\nabla \phi(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}.
\]

Then the statistic \(X_1X_2\) is an absolutely continuous random variable whose probability density function satisfies

\[
f_{X_1X_2}(y) = \int_{x_1x_2 = y} \frac{f_X(x_1, x_2)}{\sqrt{x_1^2 + x_2^2}} \, dH^1(x_1, x_2) = \int_{\mathbb{R}\setminus\{0\}} f_X\left(t, \frac{y}{t}\right) \frac{1}{|t|} \, dt,
\]

for a.e. \(y \in \mathbb{R}\),

where we parametrized the hyperbole \(x_1x_2 = y\) by \(\psi(t) = (t, \frac{y}{t})\) and we used Formula (5) to evaluate the last integral.

(ii) Let us suppose \(X_2 \neq 0\) a.e. and let us employ Corollary 4.5 with

\[
\phi : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \phi(x_1, x_2) = \frac{x_1}{x_2}, \quad J_1 \phi(x_1, x_2) = |\nabla \phi(x_1, x_2)| = \frac{1}{x_2^2} \sqrt{x_1^2 + x_2^2}.
\]

Then the statistic \(\frac{X_1}{X_2}\) is an absolutely continuous random variable whose probability density function satisfies

\[
f_{\frac{X_1}{X_2}}(y) = \int_{\frac{x_1}{x_2} = y} f_X(x_1, x_2) \frac{x_2^2}{\sqrt{x_1^2 + x_2^2}} \, dH^1(x_1, x_2) = \int_{\mathbb{R}} f_X(ty, t)|t| \, dt,
\]

for a.e. \(y \in \mathbb{R}\),

where we parametrized the line \(x_1 = yx_2\) by \(\psi(t) = (ty, t)\) and we used (5) to evaluate the last integral.

(iii) Let \(X : \Omega \rightarrow \mathbb{R}\) be an absolutely continuous random variable such that \(X \neq 0\) a.e. and let \(f_X\) its probability density function. We employ Corollary 4.3 with

\[
\phi : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}, \quad \phi(t) = \frac{1}{t}, \quad J_1(t) = |\phi'(t)| = \frac{1}{t^2}.
\]

Then the statistic \(\frac{1}{X}\) is an absolutely continuous random variable whose probability density function satisfies

\[
f_{\frac{1}{X}}(y) = f_X\left(\frac{1}{y}\right) \frac{1}{y^2}, \quad \text{for any} \quad y \neq 0.
\]

6.2. Examples on manifolds

In the following examples we show how to apply Theorem 4.7 and Theorem 5.3 in the simple case of the unitary sphere \(S^{k-1}\) of \(\mathbb{R}^k\).
Example 6.5 (Projection onto the sphere). Let $(\Omega, \Sigma, p)$ be a probability measure space, let $k \geq 2$ and let $X : \Omega \to \mathbb{R}^k$ be an absolutely continuous random variable with probability density function $f_X$ such that $X \neq 0$ a.e. Let $S^{k-1} = \{ x \in \mathbb{R}^k : ||x|| = 1 \}$ be the unitary sphere and let us consider the projection

$$
\phi : \mathbb{R}^k \setminus \{0\} \to S^{k-1}, \quad \phi(x) = \frac{x}{||x||}.
$$

By standard computation one has $J\phi(x) = \frac{1}{||x||} \left( I_k - \frac{x x^T}{||x||^2} \right)$ where $I_k$ is the identity matrix and $x \otimes x$ is the matrix $(x_i x_j)_{i,j=1,\ldots,k}$. The matrix $A = \left( I_k - \frac{x x^T}{||x||^2} \right)$ has eigenvalues 0 with eigenvector $x$ and 1 with eigenspace the orthogonal complement of $x$. In particular $\text{Ker}(J\phi(x)) = \text{Lin} \{x\}$, $\text{rg}(J\phi(x)) = x^\perp$ and $\dim \text{Ker}(J\phi(x))^\perp = \dim (\text{rg}(J\phi(x))) = k - 1$. This implies that, as a map from the orthogonal complement of its kernel onto its image, $J\phi(x)$ is the multiplication operator $y \mapsto \frac{1}{||x||} y$. Recalling Remark 3.2, the $k - 1$-dimensional Jacobian of $\phi$ is then given by

$$
J_{k-1} \phi(x) = \frac{1}{||x||^{k-1}}.
$$

Then, using Theorem 4.7, the induced probability measure $p_Y$ of the statistic $Y := \frac{X}{||X||}$ has a density function $f_Y$ with respect to the Hausdorff measure $\mathcal{H}^{k-1}$ on $S^{k-1}$ which satisfies

$$
f_Y(y) = \int_{R_y} f_X(x) ||x||^{k-1} \, d\mathcal{H}^{1}(x), \quad \text{for } y \in S^{k-1},
$$

where $R_y$ is the half-line passing through 0 and $y$, parametrized as $\{ x \in \mathbb{R}^k : \exists t > 0, x = ty \}$. Using (5) to compute the integral, the last equality simplifies to

$$
f_Y(y) = \int_0^\infty f_X(ty) \, t^{k-1} \, dt, \quad \text{for } y \in S^{k-1}.
$$

Example 6.6 (Semicircle). Let $S^1_+ = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, y > 0 \}$ be the unitary semicircle of $\mathbb{R}^2$ which is parametrized by the map $x \in (-1,1) \mapsto (x, \sqrt{1 - x^2}) \in S^1_+$. Let $(\Omega, \Sigma, p)$ be a probability measure space and let $X : \Omega \to S^1_+$ be a random variable having a probability density function $f_X$ with respect to the Hausdorff measure $\mathcal{H}^1_{|S^1_+|}$ on $S^1_+$.

Let us consider the map

$$
\phi : \mathbb{R}^2 \to \mathbb{R}^1, \quad \phi(x,y) = 2y.
$$

By Remark 5.1, if $\omega = (x, \sqrt{1 - x^2}) \in S^1_+$, then $f^{S^1_+} \phi(\omega)$ is the orthogonal projection of $\nabla \phi(\omega) = (0, 2)$ over the tangent space $T_\omega S^1_+ = \omega^\perp$ and therefore the 1-dimensional tangential Jacobian of $\phi$ is
\[
 f_1^g \cdot \phi (\omega) = \left| \nabla \phi (\omega) : \left( \sqrt{1-x^2}, -x \right) \right| = 2|x|.
\]

Then, using Theorem 5.3, the probability measure \( p_Y \) induced by the statistic \( Y := \phi^0 X \) is absolutely continuous with respect to the Lebesgue measure \( \mathcal{L}^1 \) on \([0,2)\) has its density function \( f_Y \) given by

\[
f_Y(y) = \int f_X(\omega) \frac{1}{f_1^g \cdot \phi (\omega)} \, d\mathcal{H}^0 (\omega), \quad \text{for } \ y \in [0,2).
\]

Noticing that if \( \omega = (x, \sqrt{1-x^2}) \in S_1 \), then \( \phi (\omega) = y \) if and only if \( x = \pm \frac{\sqrt{4-y^2}}{2} \), then the last equality simplifies to

\[
f_Y(y) = \frac{1}{\sqrt{4-y^2}} \left[ f_X \left( \frac{\sqrt{4-y^2}}{2}, \frac{y}{2} \right) + f_X \left( -\frac{\sqrt{4-y^2}}{2}, \frac{y}{2} \right) \right], \quad \text{for } \ y \in [0,2).
\]

### 6.3. Order statistics

Let \( S_k \) be the set of all the permutations of the set \( \{1,\ldots,k\} \). Let \( X = (X_1,\ldots,X_k) : \Omega \rightarrow \mathbb{R}^k \) be a random variable and let us consider the map

\[
\phi : \mathbb{R}^k \rightarrow \mathbb{R}^k, \quad x = (x_1,\ldots,x_k) \mapsto (x_{(1)},\ldots,x_{(k)}),
\]

which associates to any vector \( x \) its increasing rearrangement \( (x_{(1)},\ldots,x_{(k)}) \), i.e., \( x_{(1)} \leq \ldots \leq x_{(k)} \). The random variable \( \phi^0 X := (X_{(1)},\ldots,X_{(k)}) \) is the random vector of the so-called Order Statistics of \( X \). In what follows, as an easy application of the results of the previous sections, we deduce their well known density functions. We start with the following Lemma which shows, in particular, that \( \phi \) has unitary Jacobian.

**Lemma 6.7.** Let \( n \in \mathbb{N} \) such that \( n \leq k \), let \( I = \{i_1, i_2,\ldots,i_n\} \subseteq \{1,\ldots,k\} \) a subset of indexes, where \( |I| = n \) and \( i_1 < i_2 < \ldots < i_n \). Let

\[
\phi_I : \mathbb{R}^k \rightarrow \mathbb{R}^n, \quad x = (x_1,\ldots,x_k) \mapsto (x_{(i_1)},\ldots,x_{(i_n)})
\]

where \( (x_{(i_1)},\ldots,x_{(i_n)}) \) is the vector obtained by extracting from the increasing rearrangement \( (x_{(1)},\ldots,x_{(k)}) \) the component corresponding to the indexes of \( I \) (note that when \( I = \{1,\ldots,k\} \) then \( \phi_I = \phi \)). Then the n-Jacobian \( J_n \phi_I \) of \( \phi_I \) satisfies \( J_n \phi_I = 1 \).

**Proof.** Let us suppose, without any loss of generality, \( I = \{1,2,\ldots,n\} \). For every fixed permutation \( \sigma \in S_k \), let us consider the Borel subset \( A_{\sigma} \subseteq \mathbb{R}^k \) defined by

\[
A_{\sigma} := \{x = (x_1,\ldots,x_k) \in \mathbb{R}^k : x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(k)}\}.
\]

The complementary set of \( \bigcup_{\sigma \in S_k} A_{\sigma} \) is the set

\[
F = \{x = (x_1,\ldots,x_k) \in \mathbb{R}^k : \exists \ i \neq j \text{} s.t. x_i = x_j\}
\]

which satisfies \( \mathcal{L}^k (F) = 0 \); therefore the collection of all \( A_{\sigma} \) is a finite disjoint covering of a.e point of \( \mathbb{R}^k \).
Let us fix $\sigma \in S_k$; on $A_{\sigma}$, $\phi_1$ is injective and it coincides with the permutation of indexes operator $T_{\sigma}$ defined by $\sigma$:

$$
\phi_1(x) := T_{\sigma}(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \forall x \in A_{\sigma}.
$$

The Jacobian matrix of $T_{\sigma}$ at any point $x \in A_{\sigma}$ is then a permutation of the matrix \( \begin{pmatrix} I_n & 0 \end{pmatrix} \). Recalling Definition 3.1, this implies in particular $J_n \phi = 1$ on $A_{\sigma}$ which, by the arbitrariness of $\sigma$, proves the required claim.

Let now $X_1, \ldots, X_k$ be a sequence of absolutely continuous, independent and identically distributed random variables. Let $f$ be the common density function of each $X_i : \Omega \rightarrow \mathbb{R}$ and let $F(y) := \int_{-\infty}^{y} f(t) \, dt$ be the associated distribution function. In the following Proposition we compute the density of the Order statistics of the vector $X = (X_1, \ldots, X_k) : \Omega \rightarrow \mathbb{R}^k$. We remark that $X$ is an absolutely continuous random variable having density function given by $f_X(y) = \prod_{i=1}^{k} f(y_i)$.

**Proposition 6.8.**

(i) The density function of the distribution of the vector $Y = (X_{(1)}, \ldots, X_{(k)})$ of the all order statistics satisfies

$$
f_Y(y) = k! \prod_{i=1}^{k} f(y_i), \quad \forall y \in \mathbb{R}^k \text{ s.t. } y_1 < y_2 < \ldots < y_k
$$

and it is 0 otherwise.

(ii) Let $i \in \mathbb{N}$ such that $i \leq k$. The density function of the distribution of the order statistic $X_{(i)}$ satisfies

$$
f_{X_{(i)}}(y) = i \binom{k}{i} f(y)^{i-1} (1 - F(y))^{k-i}, \quad \forall y \in \mathbb{R}.
$$

(iii) Let $i, j \in \mathbb{N}$ such that $i < j \leq k$. The density function of the distribution of the vector of the two order statistics $(X_{(i)}, X_{(j)})$ satisfies

$$
f_{(X_{(i)}, X_{(j)})}(y_1, y_2) = \frac{k!}{(i-1)!(j-i-1)!(k-j)!} f(y_1)^{i-1} f(y_2)^{j-i-1} (1 - F(y_1))^{k-j} (1 - F(y_2))^{j-i-1}, \quad \forall y \in \mathbb{R}^2 \text{ s.t. } y_1 < y_2
$$

and it is 0 otherwise.

**Proof.** Using Corollary 4.3 and the previous Lemma, we get that the random vector $Y = (X_{(1)}, \ldots, X_{(k)})$ of the Order Statistics of $X$ is an absolutely continuous random variable and its probability density function $f_Y$ satisfies

$$
f_Y(y) = \sum_{\sigma \in S_k} f_X(T_{\sigma}^{-1}(y)) = \sum_{\sigma \in S_k} f_X(T_{\sigma^{-1}}(y)) = \sum_{\sigma \in S_k} \prod_{i=1}^{k} f(y_{\sigma^{-1}(i)})
$$

$$
= \sum_{\sigma \in S_k} \prod_{j=1}^{k} f(y_j) = k! \prod_{j=1}^{k} f(y_j), \quad \forall y \in \mathbb{R}^k \text{ s.t. } y_1 < y_2 < \cdots < y_k.$$
and it is 0 otherwise. This proves (i).

Let \( i \in \mathbb{N} \) such that \( i \leq k \). Claim (ii) can be proved directly by applying, as in the previous step, Theorem 4.4 and the previous Lemma or alternatively by integrating the joint density \( f_Y \). Indeed if we write \( \tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \in \mathbb{R}^{k-1} \) to denote the variable \( x \) without the \( x_i \) component and if we set \( F = \{ \tilde{x}_i \in \mathbb{R}^{k-1} : x_1 < x_2 < \cdots < x_{i-1} < y < x_{i+1} < \cdots < x_k \} \) then we obtain

\[
f_{X_{(i)}}(y) = \int_{\{z \in \mathbb{R}^k : x_i = y\}} f_Y(z) \, d\tilde{x}_i = k! f_y(y) \int_F \prod_{j \neq i} f(x_j) \, d\tilde{x}_i
\]

\[
= k! f_y(y) \int_{\{x_1 < \cdots < x_{i-1} < y\}} f(x_j) \, dx_1 \cdots dx_{i-1} \int_{\{y < x_{i+1} < \cdots < x_k\}} \prod_{j=1}^k f(x_j) \, dx_{i+1} \cdots dx_k.
\]

Since the integrand of the first integral of the right hand side of the last equation is invariant under any permutations of its variables then

\[
\int_{\{x_1 < \cdots < x_{i-1} < y\}} \prod_{j=1}^{i-1} f(x_j) \, dx_1 \cdots dx_{i-1} = \frac{1}{(i-1)!} \int_{\{x_j < y, \forall j \leq i-1\}} \prod_{j=1}^{i-1} f(x_j) \, dx_1 \cdots dx_{i-1}
\]

\[
= \frac{1}{(i-1)!} \prod_{j=1}^{i-1} \int_{-\infty}^{y} f(x_j) \, dx_j = \frac{1}{(i-1)!} F(y)^{i-1}.
\]

Analogously one has

\[
\int_{\{y < x_{i+1} < \cdots < x_k\}} \prod_{j=i+1}^k f(x_j) \, dx_{i+1} \cdots dx_k = \frac{1}{(k-i)!} \left(1 - F(y)\right)^{k-i}.
\]

This gives

\[
f_{X_{(i)}}(y) = k! \frac{1}{(i-1)!(k-i)!} f(y) F(y)^{i-1} \left(1 - F(y)\right)^{k-i} = i \binom{k}{i} f(y) (1 - F(y))^{k-i}
\]

which is the required claim.

The proof of (iii) follows similarly. \( \square \)

**Remark 6.9.** (i) The joint density function of three or more order statistics could be derived using similar arguments.

(ii) The same methods applies also when the random variables \( X_1, \ldots, X_k \) are independent but not identically distributed. For example, let \( f_i(y) \) be the density function of \( X_i : \Omega \rightarrow \mathbb{R} \). Then in this case \( f_X(y) = \prod_{i=1}^k f_i(y_i) \) for every \( y \in \mathbb{R}^k \). If \( Y = (X_{(1)}, \ldots, X_{(k)}) \) then one obtains as before

\[
f_Y(y) = \sum_{\sigma \in S_k} \prod_{i=1}^k f_i(y_{\sigma(i)}) \quad \forall y \in \mathbb{R}^k \text{ s.t. } y_1 < y_2 < \cdots < y_k.
\]
6.4. Normal distributions

Let \( a \in \mathbb{R} \), \( \sigma > 0 \) and let \( X : \Omega \to \mathbb{R} \) be a random variable. \( X \) is said to have a Normal (or Gaussian) distribution \( p_X \), and we write \( p_X \sim \mathcal{N}(a, \sigma^2) \), if \( p_X \) has density

\[
 f_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{|t-a|^2}{2\sigma^2} \right), \quad t \in \mathbb{R}.
\]

When \( \sigma = 0 \) we also write \( p_X \sim \mathcal{N}(a,0) \) with the understanding that \( p_X \) is the dirac measure \( \delta_a \) at the point \( a \). The parameters \( a \) and \( \sigma^2 \) are called the mean and the variance of \( X \), respectively.

Let \( k \in \mathbb{N} \), \( a \in \mathbb{R}^k \) and let \( \Sigma \in \mathbb{R}^{k,k} \) be a symmetric, and positive semi-definite matrix. A random variable \( X = (X_1, \ldots, X_k) : \Omega \to \mathbb{R}^k \) is said to have a (multivariate) normal distribution \( \mathcal{N}(a, \Sigma) \) if

\[
 \langle \lambda, X \rangle \sim \mathcal{N}(\langle \lambda, a \rangle, \langle \Sigma \lambda, \lambda \rangle), \quad \forall \lambda \in \mathbb{R}^k
\]

(we write \( \langle \lambda, \mu \rangle = \sum_i \lambda_i \mu_i \) to denote the inner product of \( \mathbb{R}^k \)). Here \( a := E(X) \) is the mean vector and \( \Sigma = (\sigma_{ij})_{i,j} := \text{Cov}(X) \) is the covariance matrix of \( X \), i.e., \( \sigma_{i,j} = \text{Cov}(X_i, X_j) \). The following very well-known properties about Gaussian vectors are direct consequences of their definition (see, e.g., Bogachev 1998, Chapter 1).

**Proposition 6.10.** Let \( X = (X_1, \ldots, X_k) : \Omega \to \mathbb{R}^k \) be a random variable such that \( X \sim \mathcal{N}(a, \Sigma) \).

(i) The mean vector \( a \) and the Covariance matrix \( \Sigma \) uniquely characterized the Gaussian measure \( p_X \).

(ii) \( X_i, X_j \) are independent if and only if \( \sigma_{ij} = \text{Cov}(X_i, X_j) = 0 \).

(iii) For every matrix \( A \in \mathbb{R}^{m,k} \) one has \( AX \sim \mathcal{N}((AA)_a, A\Sigma A^t) \).

(iv) When \( \Sigma \) is positive definite we say that \( p_X \) is not-degenerate: in this case \( X \) is absolutely continuous and has density function

\[
 f_X(x) = \frac{1}{(2\pi)^{k/2} \det(\Sigma)^{1/2}} \exp \left( -\frac{\|x-a\|^2}{2} \right).
\]

In the following Proposition we show that, when the Covariance matrix \( \Sigma \) is degenerate, \( p_X \) has a density function with respect to the Hausdorff measure \( \mathcal{H}^m \) on some hyperplane of \( \mathbb{R}^k \). In what follows we say that that a matrix \( P \in \mathbb{R}^{k,m} \), with \( m \leq k \), is orthogonal if it has orthonormal columns; in this case \( |Qy| = |y| \) for every \( y \in \mathbb{R}^m \).

**Proposition 6.11.** Let \( k \in \mathbb{N} \), \( a \in \mathbb{R}^k \) and let \( \Sigma \in \mathbb{R}^{k,k} \) be a positive semi-definite matrix with \( m = \text{rank}(\Sigma) \geq 1 \). Let \( X = (X_1, \ldots, X_k) : \Omega \to \mathbb{R}^k \) be a random variable. Then \( X \sim \mathcal{N}(a, \Sigma) \) if and only if there exists an orthogonal matrix \( P \in \mathbb{R}^{k,m} \) and \( m \) independent random variables \( Y_1, \ldots, Y_m \) which satisfies \( Y_i \sim \mathcal{N}(0,1) \) for every \( i = 1, \ldots, m \) and such that

\[
 X = \Sigma^{1/2} P Y + a, \quad Y = (Y_1, \ldots, Y_m).
\]
Moreover the probability measure \( p_X \) has a density function \( f_X \) with respect to the Hausdorff measure \( \mathcal{H}^m \) on the hyperplane

\[
\Sigma^\perp P(\mathbb{R}^m) + a = \{ x \in \mathbb{R}^k \mid x = \Sigma^\perp P y + a, \ y \in \mathbb{R}^m \}
\]

which satisfies

\[
f_X(x) = \frac{1}{(2\pi)^{\frac{k}{2}} \Sigma_m^\frac{1}{2}} \exp \left( -\frac{|y|^2}{2} \right), \quad x = \Sigma^\perp P y + a,
\]

where \( \Sigma_m^\frac{1}{2} = \prod_i \sqrt{\lambda_i} \) and the product runs over all positive eigenvalues of \( \Sigma \) (counted with their multiplicities).

**Proof.** Let us prove the first claim and let us suppose, preliminarily, that the Covariance matrix \( \Sigma \) is a diagonal matrix and, without any loss of generality, let us assume that its entries in the main diagonal are

\[
(\sigma_{11}, \ldots, \sigma_{mm}, 0, \ldots, 0),
\]

where \( \sigma_{ii} > 0 \) for \( i \leq m \). Then from Proposition 6.10, \( X_1, \ldots, X_m \) are independent and \( X_i \sim \mathcal{N}(a_i, \sigma_{ii}) \); moreover \( X_i = a_i \) a.e. for \( i > m \). The required claim then immediately follows setting \( Y = (Y_1, \ldots, Y_m) \), with \( Y_i = \frac{X_i - a_i}{\sqrt{\sigma_{ii}}} \) and \( P = \left( \begin{array}{c} I_m \\ 0 \end{array} \right) \), where \( I_m \) is the identity matrix of \( \mathbb{R}^{m, m} \).

In the general case let us diagonalize the Covariance matrix \( \Sigma \): Let \( Q \in \mathbb{R}^{k,k} \) be an orthogonal matrix such that \( Q \Sigma Q^t = D \), where \( D \) is the diagonal matrix whose entries in the main diagonal are \( (\lambda_1, \ldots, \lambda_m, 0, \ldots, 0) \), where the \( \lambda_i > 0 \) are the positive eigenvalues of \( \Sigma \). From Proposition 6.10 the vector \( Z = QX \) satisfies \( Z \sim \mathcal{N}(Qa, D) \); from the previous step there exists \( Y = (Y_1, \ldots, Y_m) \sim \mathcal{N}(0, I_m) \) such that

\[
Z = D^\frac{1}{2} \left( \begin{array}{c} I_m \\ 0 \end{array} \right) Y + Qa.
\]

Then since \( Q \Sigma^\perp Q^t = D^\frac{1}{2} \) we get

\[
X = Q^t Z = Q^t D^\frac{1}{2} \left( \begin{array}{c} I_m \\ 0 \end{array} \right) Y + a = \Sigma^\perp Q^t \left( \begin{array}{c} I_m \\ 0 \end{array} \right) Y + a
\]

and the claim follows with \( P = Q^t \left( \begin{array}{c} I_m \\ 0 \end{array} \right) \).

Finally, to prove the second claim, let us apply the first step and Example 6.2 with \( A = \Sigma^\perp P \) and \( y_0 = a \). Then we get that \( X \) has a density function \( f_X \) with respect to the Hausdorff measure \( \mathcal{H}^m \) on the hyperplane

\[
\Sigma^\perp P(\mathbb{R}^m) + a = \{ x \in \mathbb{R}^k \mid x = \Sigma^\perp P y + a, \ y \in \mathbb{R}^m \}
\]

which satisfies

\[
f_X(x) = \frac{1}{(\Sigma^\perp P)^\frac{1}{m}} \int_{\Sigma^\perp P y + a = x} f_Y(y) \ d\mathcal{H}^0(y), \quad \text{for } x \in \Sigma^\perp P(\mathbb{R}^m) + a.
\]
Since $P$ has orthogonal columns then from Definition 3.1 we have $(\Sigma^\frac{1}{2}P)_m = \prod \sqrt{\lambda_i} = \Sigma^\frac{1}{2}$, where the product runs over all positive eigenvalues of $\Sigma$. Moreover since $\Sigma^\frac{1}{2}P$ has maximum rank, the equation $x = \Sigma^\frac{1}{2}Py + a$ has a unique solution. Then

$$f_X(x) = \frac{1}{(2\pi)^{\frac{m}{2}}\Sigma^\frac{1}{2}} \exp\left(-\frac{|y|^2}{2}\right), \quad x = \Sigma^\frac{1}{2}Py + a.$$  

6.5. Chi-squared and Student’s distributions

Let $X : \Omega \to \mathbb{R}^k$ be a Gaussian random vector whose covariance matrix is the identity matrix $I_k$. If $X \sim \mathcal{N}(0, I_k)$ then the probability measure $p_{\chi^2(k)}$ induced by $|X|^2$ is called Chi-squared distribution with $k$-degrees of freedom and we write $|X|^2 \sim \chi^2(k)$.

If $X$ is not-centered, i.e., $X \sim \mathcal{N}(\mu, I_k)$ for some $\mu \in \mathbb{R}^k \setminus \{0\}$, then the measure $p_{\chi^2(k, \mu)}$ induced by $|X|^2$ is called Non-central Chi-squared distribution with $k$-degrees of freedom and non-centrality parameter $\lambda = |\mu|^2 > 0$ and we write $|X|^2 \sim \chi^2(k, \lambda)$.

In the next Proposition, we derive the density function of $|X|^2$. In what follows we consider the gamma function $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t} \, dt$, $r > 0$ (see, e.g., Abramowitz and Stegun 1964, 255) and the modified Bessel function of the first kind $I_v$ defined for $y > 0$ as

$$I_v(y) = (y/2)^v \sum_{j=0}^{\infty} \frac{(y^2/4)^j}{j! \Gamma(\nu+j+1)} = \frac{(y/2)^v}{\pi^\frac{1}{2} \Gamma(\nu+\frac{1}{2})} \int_0^{\infty} e^{-y \cos \theta} (\sin \theta)^{2\nu} \, d\theta,$$

(see, e.g., Abramowitz and Stegun 1964, Section 9.6 and Formula 9.6.20, 376).

**Proposition 6.12 (Chi-squared Distribution).** Let $X : \Omega \to \mathbb{R}^k$ be a Gaussian random vector. If $X \sim \mathcal{N}(0, I_k)$ then the Chi-squared distribution $p_{\chi^2(k)}$ induced by $|X|^2$ has density function

$$f_{\chi^2(k)}(y) = \frac{1}{2^\frac{k}{2}\Gamma\left(\frac{k}{2}\right)} y^{\frac{k}{2}-1} \exp\left(-\frac{y}{2}\right), \quad \text{for any } y > 0.$$

If $X \sim \mathcal{N}(\mu, I_k)$ for some $\mu \in \mathbb{R}^k \setminus \{0\}$ then, setting $\lambda = |\mu|^2 > 0$, the Non-central Chi-squared distribution $p_{\chi^2(k, \mu)}$ induced by $|X|^2$ has density function

$$f_{\chi^2(k, \mu)}(y) = \frac{1}{2} \exp\left(-\frac{y + \lambda}{2}\right) \left(\frac{y}{\lambda}\right)^{\frac{k}{2}+\frac{1}{2}} I_{\frac{k}{2}-1}\left(\sqrt{\lambda y}\right), \quad \text{for any } y > 0.$$

**Proof.** Let $X \sim \mathcal{N}(0, I_k)$; using Example 6.1 we have for any $y > 0$

$$f_{\chi^2(k)}(y) = \frac{1}{2\sqrt{y}} \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{|X|=\sqrt{y}} \exp\left(-\frac{|X|^2}{2}\right) \, d\mathcal{H}^{k-1}(x)$$

$$= \frac{1}{2\sqrt{y}} \frac{1}{(2\pi)^{\frac{m}{2}}} \exp\left(-\frac{y}{2}\right) \mathcal{H}^{k-1}(\mathbb{S}^{k-1}) y^{\frac{k}{2}-1} = \frac{1}{2^\frac{k}{2}\Gamma\left(\frac{k}{2}\right)} y^{\frac{k}{2}-1} \exp\left(-\frac{y}{2}\right)$$
which is the first claim. If \( X \sim \mathcal{N}(\mu, I_k) \) for some \( \mu \in \mathbb{R}^k \setminus \{0\} \), then using Example 6.1 again and the elementary equality \(|x - \mu|^2 = |x|^2 + |\mu|^2 - 2\langle x, \mu \rangle\) we have for any \( y > 0 \)

\[
f_{Z^2(k)}(y) = \frac{1}{2\sqrt{y}} \int_{|x|=\sqrt{y}} \frac{1}{(2\pi)^{\frac{k}{2}}} \exp \left( -\frac{|x - \mu|^2}{2} \right) \, d\mathcal{H}^{k-1}(x)
= \frac{1}{2\sqrt{y}(2\pi)^{\frac{k}{2}}} \exp \left( -\frac{y + \lambda}{2} \right) \int_{|x|=\sqrt{y}} \exp \left( \langle x, \mu \rangle \right) \, d\mathcal{H}^{k-1}(x)
= \frac{y^{\frac{k}{2}-1}}{2(2\pi)^{\frac{k}{2}}} \exp \left( -\frac{y + \lambda}{2} \right) \int_{|z|=1} \exp \left( \sqrt{y}z, \mu \right) \, d\mathcal{H}^{k-1}(z).
\]

Since \( \mathcal{H}^{k-1}_{|z|=1} \) is rotationally invariant, up to an orthogonal transformation of \( \mathbb{R}^k \) which maps \( \frac{\mu}{|\mu|} \) to \( e_1 = (1, 0, \ldots, 0) \), we can suppose \( \frac{\mu}{|\mu|} = e_1 \). Using \( k \)-dimensional spherical coordinates to evaluate the last integral then we have

\[
\int_{|z|=1} \exp \left( \sqrt{y}z, \mu \right) \, d\mathcal{H}^{k-1}(z) = \int_{|z|=1} \exp \left( \sqrt{y}z_1 \right) \, d\mathcal{H}^{k-1}(z)
= \mathcal{H}^{k-2}(S^{k-2}) \int_0^\pi \exp \left( \sqrt{y} \cos \theta \right) \left( \sin \theta \right)^{k-2} \, d\theta
= (2\pi)^{\frac{k}{2}} (k) \frac{k-1}{2} \left( \sqrt{\frac{y}{k}} \right).
\]

Combining the latter equalities gives the required last claim.

Finally let \( X, Y : \Omega \to \mathbb{R} \) be two independent random variables such that \( X \sim \mathcal{N}(0, 1) \) and \( Y \sim \chi^2(k) \). The probability measure \( p_T \) induced by the random variable \( T = \frac{X}{\sqrt{Y/k}} \) is called a (Student’s) \textit{t-distribution with} \( k \)-degrees of freedom. In the next Proposition, we use Corollary 4.5 and example 6.4 in order to derive the density function of \( p_T \).

**Proposition 6.13 (Student’s \textit{t}-Distribution).** Let \( X, Y : \Omega \to \mathbb{R} \) two independent random variables such that \( X \sim \mathcal{N}(0, 1) \) and \( Y \sim \chi^2(k) \). The \textit{t-distribution} \( p_T \) induced by \( T = \frac{X}{\sqrt{Y/k}} \) has density function

\[
f_T(y) = \frac{\Gamma \left( \frac{k+1}{2} \right)}{\sqrt{k\pi} \Gamma \left( \frac{k}{2} \right)} \left( 1 + \frac{y^2}{k} \right)^{-\frac{k+1}{2}}, \quad \text{for any} \quad t \in \mathbb{R}.
\]

**Proof.** Using Corollary 4.5 with \( \phi : \mathbb{R}^+ \to \mathbb{R}^+, \phi(t) = \sqrt{t/k} \), we have

\[
f_{\sqrt{Y/k}}(y) = \frac{2k^2}{2^k \Gamma \left( \frac{k}{2} \right)} y^{k-1} \exp \left( -\frac{ky^2}{2} \right), \quad \forall y > 0.
\]
Then applying example 6.4 we get for $y \in \mathbb{R}$,

$$f_T(y) = \int_0^\infty f_X(t) \sqrt{\frac{j}{k}}(t) t \, dt = \frac{2k^\frac{3}{2}}{2^\frac{3}{2} \Gamma\left(\frac{k}{2}\right) \sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{t^2(y^2 + k)}{2}\right) t^k \, dt,$$

which with the substitution $s = t^2 \frac{y^2 + k}{2}$ becomes

$$f_T(y) = \frac{1}{\Gamma\left(\frac{k}{2}\right) \sqrt{k\pi}} \left(1 + \frac{y^2}{k}\right)^{-\frac{k+1}{2}} \int_0^\infty e^{-s} s^{-\frac{k+1}{2}} \, ds = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right) \sqrt{k\pi}} \left(1 + \frac{y^2}{k}\right)^{-\frac{k+1}{2}}.$$

\[\square\]

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