On discrete generalized nabla fractional sums and differences

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Abstract

This article investigates a class of discrete nabla fractional operators by using the discrete nabla convolution theorem. Inspired by this, we define the discrete generalized nabla fractional sum and differences of Riemann-Liouville and Caputo types. In the process, we give a relationship between the generalized discrete delta fractional operators introduced by Ferreira [12] and the proposed discrete generalized nabla fractional operators via the dual identities. Also, we present some test examples to justify the relationship. Moreover, we prove the fundamental theorem of calculus for the defined discrete generalized nabla fractional operators. Inspired by the above operators, we define discrete generalized nabla Atangana-Baleanu-like (or Caputo-Fabrizio-like) fractional sum and differences at the end of the article.

1 Introduction

The operators of fractional calculus (FC) and discrete fractional calculus (DFC) are used in various applications to address problems of mathematical analysis. As usual in the use of fractional operators, their applications to real-world problems involve singular or nonsingular kernels to determine new approaches arising from continuous or discrete fractional calculus models. The use of singular or nonsingular kernels implies that important model properties are satisfied on the discrete level. To introduce new models of DFC with new singular or nonsingular kernels, constrictions of discrete fractional analogues of the continuous fractional operators has been an area of active research in recent years (see Refs. [1, 2]).

Different fractional operators to DFC theory and their applications have been introduced and examined by several authors in recent years, such as Riemann-Liouville (see Ref. [3]), Caputo (see Ref. [4, 5]), Weighted (see Ref. [6]), Caputo-Fabrizio (CF) (see Ref. [7, 8]), Atangana-Baleanu (AB) (see Refs. [9, 10]), generalized AB (see Ref. [11]) fractional operators.

It is important to introduce an discrete operator which collect all of the above discrete fractional operators. Recently, Ferreira [12] defined a class of discrete delta fractional operators by means of the following set of pair-of-functions:

\[ e_{x_0}^\Delta := \{ p, q : N_{x_0} \to \mathbb{R} \text{ such that } (p \ast q)_{x_0}^\Delta(x) = 1 \quad \forall \ x \in N_{x_0+1} \}, \]

where \((p \ast q)_{x_0}^\Delta\) is the discrete delta convolution of two functions \(p\) and \(q\) and defined by

\[ (p \ast q)_{x_0}^\Delta(x) := \sum_{r=x_0}^{x-1} p(x - \sigma(r) + x_0)q(r) \quad (\forall \ x \in N_{x_0+1}, \ x_0 \in \mathbb{R}), \]

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where $N_{x_0} := \{x_0, x_0 + 1, x_0 + 2, \ldots\}$.

Furthermore, he defined the following generalized discrete delta fractional sum and differences:

**Definition 1.1** (see [12]). Let $p, q : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $y : N_{x_0} \rightarrow \mathbb{R}$ be three functions. Then, the generalized discrete delta fractional sum is defined by

$$
\left( \mathcal{S}_{\{p; x_0\}}^\alpha \right) y(x) := \sum_{r=x_0}^{x-1} p(x-\sigma(r)+x_0)y(r) \quad (\forall x \in N_{x_0+1}).
$$

(1.3)

Also, the generalized discrete delta fractional difference of Riemann-Liouville type is defined by

$$
\left( \mathcal{D}_{\{q; x_0\}}^\alpha \right) y(x) := \Delta_x \sum_{r=x_0}^{x-1} q(x-\sigma(r)+x_0)y(r) \quad (\forall x \in N_{x_0+1}),
$$

(1.4)

and the Caputo type is defined by

$$
\left( \mathcal{C}_{\{q; x_0\}}^\alpha \right) y(x) := \sum_{r=x_0}^{x-1} q(x-\sigma(r)+x_0)(\Delta_r y(r)) \quad (\forall x \in N_{x_0+1}).
$$

(1.5)

It is important to note that Definitions (1.4) and (1.5) can generalize discrete nabla Riemann-Liouville and Caputo fractional differences, defined in Ref. [3–5], for specific choices of $p$ and $q$ as discussed in his article. Finally, he proved the fundamental theorem of calculus for the above operators.

Motivated by Ferreira [12] results, the goal of this article is to introduce a new class of discrete nabla Riemann-Liouville and Caputo fractional operators that generalizes the existing discrete delta Riemann-Liouville and Caputo fractional operators. Also, we establish dual identities to make a connection between the discrete convolutions. The fundamental theorem of calculus for our new operators are also considered. Furthermore, we extend our results to define discrete generalized nabla ABR-like (or CFR-like) fractional sum and differences.

## 2 Preliminary concepts

This section gives some notation and definitions of discrete fractional calculus which are used in our paper.

**Definition 2.1** (see [3,5,6]). The forward difference operator $(\Delta y)(x) = y(\sigma(x)) - y(x)$ with $\sigma(x) := x + 1$ and the backward difference operator $(\nabla y)(x) = y(x) - y(\rho(x))$ with $\rho(x) := x - 1$ are two basic operators of a discrete function $y$ defined on $N_{x_0}$.

**Definition 2.2** (see [3,5,6]). Let $\alpha > 0$ be the order and $y : N_{x_0} \rightarrow \mathbb{R}$ be a function with a starting point $x_0$ in $\mathbb{R}$. Then, the delta Riemann-Liouville fractional sum is given by

$$
(x_0^\Delta \alpha y)(x) = \frac{1}{\Gamma(\alpha)} \sum_{r=x_0}^{x-\alpha} (x-\sigma(r))^{(\alpha-1)}y(r) \quad [\forall x \in N_{x_0+\alpha}],
$$

(2.1)

and the nabla Riemann-Liouville fractional sum is given by

$$
(x_0^\nabla \alpha y)(x) = \frac{1}{\Gamma(\alpha)} \sum_{s=x_0+1}^{x} (x-\rho(s))^{\alpha-1}y(s) \quad [\forall x \in N_{x_0+1}],
$$

(2.2)

where $x^{(\alpha)}$ and $x^{[\alpha]}$ are the falling and rising factorial functions, respectively, defined by

$$
x^{(\alpha)} = \frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)} \quad \text{for } x, \alpha \in \mathbb{R},
$$

(2.3)

and

$$
x^{[\alpha]} = \frac{\Gamma(x+\alpha)}{\Gamma(x)} \quad \text{for } \alpha \in \mathbb{R} \text{ and } x \in \mathbb{R} \setminus \{-\alpha, -1, 0\},
$$

(2.4)
where we use the standard conventions that these tend to zero at the poles and
\[ \sum_{r=m}^{n} a_r := 0, \]
whenever \( m > n \).

**Definition 2.3** (see [6]). Let \( 0 \leq \alpha < 1 \) be the order. Then, the delta Riemann-Liouville fractional difference is given by
\[
\left( RL_{x_0}^\alpha y \right)(x) = \left( \Delta_{x_0}^\alpha \Delta^{-(1-\alpha)} y \right)(x) = \frac{1}{\Gamma(1-\alpha)} \Delta_{x_0} \left( \sum_{r=x_0}^{x+\alpha-1} (x - \sigma(r))^{(-\alpha)} y(r) \right) \quad [\forall x \in N_{x_0+1-\alpha}],
\]
and the nabla Riemann-Liouville fractional difference is given by
\[
\left( RL_{x_0}^\alpha y \right)(x) = \left( \nabla_{x_0}^\alpha \nabla^{-(1-\alpha)} y \right)(x) = \frac{1}{\Gamma(1-\alpha)} \nabla_{x_0} \left( \sum_{s=x_0+1}^{x} (x - \rho(s))^{-\alpha} y(s) \right) \quad [\forall x \in N_{x_0+1}].
\]

On the other hand, the delta Caputo fractional difference is given by
\[
\left( C_{x_0}^\alpha y \right)(x) = \left( x_0 \Delta^{-(1-\alpha)} \Delta \right)(x) = \frac{1}{\Gamma(1-\alpha)} \sum_{r=x_0}^{x+\alpha-1} (x - \sigma(r))^{-\alpha} (\Delta y)(r) \quad [\forall x \in N_{x_0+1-\alpha}],
\]
and the nabla Caputo fractional difference is given by
\[
\left( C_{x_0}^\alpha y \right)(x) = \left( x_0 \nabla^{-(1-\alpha)} \nabla \right)(x) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=x_0+1}^{x} (x - \rho(s))^{-\alpha} (\nabla y)(s) \quad [\forall x \in N_{x_0+1}].
\]

**Lemma 2.1** (see [13]). Let \( \alpha > 0, \mu > -1 \) and \( \mu + \alpha \notin \mathbb{N}_- \), then
\[
a_{\mu}^{\alpha} \Delta^{-\alpha} \left[ \frac{(x-x_0)_{[\mu]}^{(\alpha)}}{\Gamma(\mu+1)} \right] = \frac{(x-x_0)^{(\alpha+\mu)}}{\Gamma(\mu+1+\alpha)} \quad [\forall x \in N_{x_0+\mu+\alpha}], \quad (2.5)
\]
and
\[
x_{\mu}^{\alpha} \nabla^{-\alpha} \left[ \frac{(x-x_0)_{[\mu]}^{(\alpha)}}{\Gamma(\mu+1)} \right] = \frac{(x-x_0)^{(\alpha+\mu)}}{\Gamma(\mu+1+\alpha)} \quad [\forall x \in N_{x_0+1}], \quad (2.6)
\]

**Lemma 2.2** (see [6, Dual identities]). For any function \( y \) defined on \( N_{x_0} \), the following dual identities hold:

1. \( \left( x_{0+1} \Delta^{-\alpha} y \right)(x + \alpha) = \left( x_{0} \nabla^{-\alpha} y \right)(x) \quad [\forall x \in N_{x_0}], \)
2. \( \left( x_{0+1} RL_{x_0}^\alpha y \right)(x - \alpha) = \left( RL_{x_0}^\alpha y \right)(x) \quad [\forall x \in N_{x_0+1}]. \)

for each \( x_0 \in \mathbb{R}, \alpha > 0 \) and \( x \in N_{x_0+1} \).

Since another part of our article is dedicated to discrete generalized nabla AB and CF fractional operators, we recall the discrete nabla ABC and ABR fractional differences with discrete ML and exponential function kernels, and their corresponding discrete nabla fractional sums.
Definition 2.4 (see [9]). Let $\lambda = -\frac{\alpha}{x_0}$, $\alpha \in [0, 0.5)$ and $y$ be defined on $N_{x_0}$ with $x_0 \in \mathbb{R}$. Then, for any $x \in N_{x_0+1}$, the discrete nabla ABC and ABR fractional differences are defined by

$$(^{AB_{x_0}}C_{x_0}^{\alpha}y)(x) = \frac{B(\alpha)}{1-\alpha} \sum_{s=x_0+1}^{x} (\nabla_{s}y)(s) E_{\alpha}(\lambda, x - \rho(s)), \quad (3.2)$$

and

$$(^{AB_{x_0}}D_{x_0}^{\alpha}y)(x) = \frac{B(\alpha)}{1-\alpha} \nabla_{x} \sum_{s=x_0+1}^{x} y(s) E_{\alpha}(\lambda, x - \rho(s)), \quad (3.3)$$

respectively. Also, their corresponding discrete nabla ABC fractional sum is given by

$$(^{AB_{x_0}}C_{x_0}^{\alpha}y)(x) = \frac{1-\alpha}{B(\alpha)} y(x) + \frac{\alpha}{B(\alpha)} (x_{0}^{\alpha}y)(x), \quad (3.4)$$

where $B(\alpha)$ is a normalizing positive constant.

Definition 2.5 (see [8, 14]). Let $\alpha \in (0, 1)$ and $y$ be defined on $N_{x_0}$ with $x_0 \in \mathbb{R}$. Then, for any $x \in N_{x_0+1}$, the discrete nabla CFC and CFR fractional differences are defined by

$$(^{C_{x_0}}C_{x_0}^{\alpha}y)(x) = \frac{B(\alpha)}{1-\alpha} \sum_{s=x_0+1}^{x} (\nabla_{s}y)(s)(1-\alpha)^{\frac{s-x_{0}}{\alpha}}, \quad (3.5)$$

and

$$(^{C_{x_0}}D_{x_0}^{\alpha}y)(x) = \frac{B(\alpha)}{1-\alpha} \nabla_{x} \sum_{s=x_0+1}^{x} y(s)(1-\alpha)^{\frac{s-x_{0}}{\alpha}}, \quad (3.6)$$

respectively. On the other hand, its corresponding discrete nabla CF fractional sum is given by

$$(^{C_{x_0}}C_{x_0}^{\alpha}y)(x) = \frac{1-\alpha}{B(\alpha)} y(x) + \frac{\alpha}{B(\alpha)} (x_{0}^{\alpha}y)(x). \quad (3.7)$$

3 Discrete nabla convolution and fractional operator class

We consider the discrete nabla convolution of two functions $p$ and $q$:

$$(p \ast q)_{x_0}^{\nabla}(x) := \sum_{s=x_0+1}^{x} p(x - \rho(s) + x_{0})q(s) \quad (\forall x \in N_{x_0+1}, \ x_0 \in \mathbb{R}). \quad (3.8)$$

Particularly, for $p(x) = \frac{(x-x_{0})^{\alpha-1}}{\Gamma(\alpha)}$, we get

$$(p \ast q)_{x_0}^{\nabla}(x) = \frac{1}{\Gamma(\alpha)} \sum_{s=x_0+1}^{x} (x - x_{0})^{\alpha-1}q(s) \quad (3.9)$$

Moreover, from Lemma 2.2(1) and (3.2), we can deduce

$$(p \ast q)_{x_0}^{\nabla}(x) = (x_0 + \Delta_{x_0}^{\alpha}q)(x + \alpha). \quad (3.10)$$

Now, we introduce the following class of discrete nabla fractional operators:

$$\mathcal{E}_{x_0}^{\nabla} := \{ p, q : N_{x_0} \rightarrow \mathbb{R} \text{ such that } (p \ast q)_{x_0}^{\nabla}(x) = 1 \quad \forall x \in N_{x_0+1} \}. \quad (3.11)$$

The following example clarifies the above definitions.
Example 3.1. Let \( x_0 \in \mathbb{R} \) and \( \alpha \in \mathbb{R}^+ \setminus N_1 \). Consider the functions

\[
p(x) = \frac{(x - x_0)^{\alpha - 1}}{\Gamma(\alpha)} \quad \text{and} \quad q(x) = \frac{(x - x_0)^{-\alpha}}{\Gamma(1 - \alpha)} \quad (\forall x \in N_{x_0 + 1}).
\]

By using these in the discrete nabla convolution of (3.1), it follows that

\[
(p \ast q)^{\nabla}_{x_0}(x) = \sum_{s=x_0+1}^{x} \frac{(x - s)^{\alpha - 1}}{\Gamma(\alpha)} \frac{(s - x_0)^{-\alpha}}{\Gamma(1 - \alpha)}
\]

\[
= \frac{1}{\Gamma(\alpha)} \sum_{s=x_0+1}^{x} (x - s)^{\alpha - 1} (s - x_0)^{-\alpha} \frac{1}{\Gamma(1 - \alpha)}
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} \left[ x_0^{-\alpha} (x - x_0)^{-\alpha} \right]
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} \left[ \frac{\Gamma(1 - \alpha)}{\Gamma(1 - \alpha + \alpha)} (x - x_0)^{-\alpha} \right] = 1,
\]

where for the part in the square brackets we used the fractional sum power rule (2.6). Thus, we see that the given functions are in the class of discrete nabla fractional operators (3.4).

Remark 3.1. If we consider

\[
p(x) = \frac{(x - x_0 + \alpha - 2)^{(\alpha - 1)}}{\Gamma(\alpha)} \quad \text{and} \quad q(x) = \frac{(x - x_0 - \alpha - 1)^{(-\alpha)}}{\Gamma(1 - \alpha)} \quad (\forall x \in N_{x_0 + 1}, \ x_0 \in \mathbb{R}, \ \alpha \in \mathbb{R}^+ \setminus N_1),
\]

then, the identity \( x^{\alpha} = (x + \alpha - 1)^{(\alpha)} \) and the same steps used in Example 3.1 enable us to prove that \( (\tilde{p}, \tilde{q}) \in \mathbb{C}_{x_0}^\alpha \) as well.

Now, we give a relationship between the discrete convolutions (1.2) and (3.1).

Theorem 3.1 (Dual identities). Let the functions \( p \) and \( q \) be defined on \( N_{x_0} \) and \( x_0 \in \mathbb{R} \). Then, we have the following relation between discrete convolutions:

\[
(p \ast q)^{\nabla}_{x_0+1}(x + 1) = (p \ast q)^{\Delta}_{x_0}(x) \quad (\forall x \in N_{x_0+1}).
\]

Proof. From the discrete delat convolution (1.2), we have

\[
(p \ast q)^{\Delta}_{x_0+1}(x + 1) = \sum_{r=x_0+1}^{x} p(x + 1 - \sigma(r) + x_0 + 1)q(r)
\]

\[
= \sum_{r=x_0+1}^{x} p(x - \rho(r) + x_0)q(r)
\]

\[
= (p \ast q)^{\nabla}_{x_0}(x),
\]

as required. \( \square \)

Corollary 3.1. Let \( p(x) = \frac{(x - x_0)^{\alpha - 1}}{\Gamma(\alpha)} \) and \( q(x) \) be a function defined on \( N_{x_0} \). Then, we have

\[
(p \ast q)^{\Delta}_{x_0+1}(x + 1) = (x_{x_0+1})^{\Delta^{-\alpha}}q(x + \alpha).
\]

Moreover, if \( q(x) = \frac{(x - x_0)^{-\alpha}}{\Gamma(1 - \alpha)} \), we find that \( (p \ast q)^{\Delta}_{x_0+1}(x + 1) = 1 \).

Proof. The first part can be proved by Theorem 3.1 and (3.3), and the second part follows directly from Theorem 3.1 and Example 3.1. \( \square \)
4 Discrete generalized nabla fractional operators

Having introduced the new class of discrete nabla fractional operators using the discrete nabla convolution of two functions, we are now in a position to define discrete generalized nabla fractional sum and differences using the discrete nabla convolution of two functions, following in the footsteps of Ferreira [12] which defined discrete delta generalized fractional sum and differences using the discrete delta convolution of two functions.

4.1 Discrete Riemann-Liouville- and Caputo-type fractional operators

Definition 4.1. Let \( p, q : D \subseteq \mathbb{R} \to \mathbb{R} \) and \( y : x_0 \to \mathbb{R} \) be three functions. Then, we define the discrete generalized nabla fractional sum as follows:

\[
(S_{\{p \}; x_0}^\nabla y)(x) := \sum_{s=x_0+1}^{x_0} p(x - \rho(s) + x_0)y(s) \\
= (p \ast y)_{x_0}^\nabla (x) \quad [\forall x \in \mathbb{N}_{x_0+1}].
\] (4.1)

Also, we define the discrete generalized nabla fractional difference in the sense of Riemann-Liouville as follows:

\[
(RL_D_{\{q \}; x_0}^\nabla y)(x) := \nabla_x \sum_{s=x_0+1}^{x} q(x - \rho(s) + x_0)y(s) \\
= \nabla_x [(q \ast y)_{x_0}^\nabla (x)] \quad [\forall x \in \mathbb{N}_{x_0+1}],
\] (4.2)

and in the sense of Caputo as follows:

\[
(C_D_{\{q \}; x_0}^\nabla y)(x) := \sum_{s=x_0+1}^{x} q(x - \rho(s) + x_0)(\nabla_s y)(s) \\
= (q \ast (\nabla y))_{x_0}^\nabla (x) \quad [\forall x \in \mathbb{N}_{x_0+1}].
\] (4.3)

Example 4.1. For \( p(x) = \frac{(x-x_0)^{\gamma-1}}{\Gamma(\gamma)} \) for each \( x \in \mathbb{N}_{x_0+1} \), we have

\[
(S_{\{p \}; x_0}^\nabla y)(x) = \sum_{s=x_0+1}^{x_0} p(x - \rho(s) + x_0)y(s) \\
= \frac{1}{\Gamma(\gamma)} \sum_{s=x_0+1}^{x} (x - \rho(s))^{\alpha-1}y(s) = (x_0 \nabla^{-\alpha} y)(x).
\]

Meanwhile, for \( q(x) = \frac{(x-x_0)^{\alpha-1}}{\Gamma(1-\alpha)} \) for each \( x \in \mathbb{N}_{x_0+1} \), we have

\[
(RL_{D_{\{q \}; x_0}^\nabla y)(x) = \nabla_x \sum_{s=x_0+1}^{x} q(x - \rho(s) + x_0)y(s) \\
= \nabla_x \frac{1}{\Gamma(\alpha)} \sum_{s=x_0+1}^{x} (x - \rho(s))^{-\alpha}y(s) = (RL_{x_0}^\nabla y)(x).
\]

Analogously, for the above \( q(x) \), we can find that

\[
(C_{D_{\{q \}; x_0}^\nabla y)(x) = (C_{x_0}^\nabla y)(x).
\]

Now, we give relationships between discrete generalized nabla and delta fractional sums and differences.
Theorem 4.1. Let the functions \( p \) and \( q \) be defined on \( N_{x_0} \) and \( x_0 \in \mathbb{R} \). Then, we have the following relations:

\[
\left( S_{\{p; x_0+1\}}^\Delta \right) (x + 1) = \left( S_{\{q; x_0\}}^\nabla \right) (x) \quad (\forall \, x \in N_{x_0 + 1}),
\]

and

\[
\left( RL D_{\{q; x_0+1\}}^\Delta \right) (x) = \left( RL D_{\{q; x_0\}}^\nabla \right) (x) \quad (\forall \, x \in N_{x_0 + 1}).
\]

Proof. The proof of (4.4) follows from the definition of the generalized discrete fractional sums (1.3) and (4.1). In proving (4.5), we should proceed by using the definition of generalized discrete fractional differences (1.4) and (4.2):

\[
\left( RL D_{\{q; x_0+1\}}^\Delta \right) (x) = \Delta_x \sum_{s=x_0+1}^{x-1} q(x - \sigma(s) + x_0 + 1)y(s)
\]

\[
= \nabla_x \sum_{s=x_0+1}^{x-1} q(x + 1 - s + x_0)y(s)
\]

\[
= \nabla_x \sum_{s=x_0+1}^{x} q(x - \rho(s) + x_0)y(s) = \left( RL D_{\{q; x_0\}}^\nabla \right) (x),
\]

where we have used the identity \((\Delta y)(x) = (\nabla y)(x + 1)\). \(\square\)

Remark 4.1. From Theorem 4.1, Example 4.1 and Lemma 2.2, it is clear that

\[
\left( S_{\{p; x_0+1\}}^\Delta \right) (x + 1) = \left( x_0 \nabla^{-\alpha} y \right) (x) \quad \text{for} \quad p(x) = \left( \frac{x - x_0}{} \right),
\]

and

\[
\left( RL D_{\{q; x_0+1\}}^\Delta \right) (x) = \left( RL D_{\{q; x_0\}}^\nabla \right) (x) \quad \text{for} \quad q(x) = \left( \frac{x - x_0}{\Gamma(1 - \alpha)} \right),
\]

for each \( x \in N_{x_0 + 1} \).

4.2 Discrete generalized fractional operators with nonsingular-like kernels

Having introduced the discrete generalized nabl fractional sum and differences using the discrete nabl convolution of two functions, we are now in a position to define discrete generalized nabl fractional sums and differences with exponential-like (or Mittag-Liffler-like) kernels.

At first, we should introduce two further classes of discrete nabl fractional operators with the Mittag-Liffler-like kernel as follows:

\[
AB_{\{p; x_0\}} := \left\{ p, q : N_{x_0} \to \mathbb{R} \quad \text{s.t.} \right. (p * q)_{\{x_0\}}(x) = \frac{E_{\alpha, \alpha + 1}(\lambda, x - x_0)}{} \quad \forall \, x \in N_{x_0 + 1}, \, |\lambda| < 1\},
\]

and with the exponential-like kernel as follows:

\[
CF_{\{x_0\}} := \left\{ p, q : N_{x_0} \to \mathbb{R} \quad \text{s.t.} \right. (p * q)_{\{x_0\}}(x) = \frac{1 - \alpha - (1 - \alpha)^{x - x_0 + 1}}{\alpha} \quad \forall \, x \in N_{x_0 + 1}, \, \alpha \in (0, 1)\}.
\]

The following examples are to confirm the above definitions.

Example 4.2. Consider

\[
p(x) = \left( \frac{x - x_0}{\Gamma(\alpha)} \right) \quad \text{and} \quad q(x) = E_{\alpha}(\lambda, x - x_0),
\]
Thus, we conclude that the pair \((p, q)\) where we have used the fact that \(E_{\alpha,\beta}^{\lambda,R}\) and the discrete generalized nabla \(\nabla_{\alpha,\beta}^{R}\) where we have used the geometric series sum. Hence, we conclude that the pair \((p, q)\):

\[
x_0 \nabla_{\alpha,\beta}^{-\alpha} E_{\alpha,\beta}(\lambda, x - x_0) = E_{\alpha,\beta}^{\lambda,R}(\lambda, x - x_0),
\]

where we have used the fact that \(E_{\alpha,\beta}(\lambda, \cdot) = E_{\alpha,\beta}^{\lambda,R}(\lambda, \cdot)\) and the identity (see [11]):

\[
x_0 \nabla_{\alpha,\beta}^{-\alpha} E_{\alpha,\beta}(\lambda, x - x_0) = E_{\alpha,\beta}^{\lambda,R}(\lambda, x - x_0).
\]

Thus, we conclude that the pair \((p, q)\) is in the class of \((4.6)\).

**Example 4.3.** Let us consider

\[
p(x) = 1 \quad \text{and} \quad q(x) = (1 - \alpha)^{x-x_0},
\]

for \(x_0 \in \mathbb{R}\) and \(\alpha \in (0, 1)\). Then, by using the discrete nabla convolution of \((3.1)\), we can deduce that

\[
(p \ast q)_{x_0}^{\nabla}(x) = \sum_{s=x_0+1}^{x} (1 - \alpha)^{x-p(s)} = (1 - \alpha)^{x-x_0+1} \sum_{s=0}^{(x-x_0)-1} \left( \frac{1}{1-\alpha} \right)^s
\]

\[
= (1 - \alpha)^{x-x_0+1} \cdot \frac{1 - \left( \frac{1}{1-\alpha} \right)^{x-x_0}}{1 - \frac{1}{1-\alpha}}
\]

\[
= \frac{1 - \alpha (1 - \alpha)^{x-x_0+1}}{\alpha},
\]

where we have used the geometric series sum. Hence, we conclude that the pair \((p, q)\) is in the class of \((4.7)\).

**Definition 4.2.** Let \(p, q : D \subseteq \mathbb{R} \to \mathbb{R}, y : N_{x_0} \to \mathbb{R}\) be three functions and \(0 < \alpha < 1\). Then, we define the discrete generalized nabla \(A_{\beta}\)-like (or \(CF\)-like) fractional sum as follows:

\[
C^{\nabla}_{(p,x_0)y}(x) := \frac{1-\alpha}{B(\alpha)} y(x) + \alpha \frac{B(\alpha)}{B(\alpha)} \sum_{s=x_0+1}^{x} p(x - \rho(s) + x_0) y(s)
\]

\[
= \frac{1-\alpha}{B(\alpha)} y(x) + \alpha \frac{B(\alpha)}{B(\alpha)} (C^{\nabla}_{(p,x_0)y})(x) \quad [\forall x \in N_{x_0}]. \quad (4.8)
\]

Also, we define the discrete generalized nabla \(A_{\beta}\)-like (or \(CF\)-like) fractional difference as follows:

\[
\left( C^{\nabla}_{(q,x_0)y} \right)(x) := \frac{B(\alpha)}{1-\alpha} \nabla_x \left( \sum_{s=x_0+1}^{x} q(x - \rho(s) + x_0) y(s) \right)
\]

\[
= \frac{B(\alpha)}{1-\alpha} \left( C^{\nabla}_{(q,x_0)y} \right)(x) \quad [\forall x \in N_{x_0+1}], \quad (4.9)
\]

and the discrete generalized nabla \(A_{\beta}\)-like (or \(CF\)-like) fractional difference as follows:

\[
\left( C^{\nabla}_{(q,x_0)y} \right)(x) := \frac{B(\alpha)}{1-\alpha} \sum_{s=x_0+1}^{x} q(x - \rho(s) + x_0) (\nabla_s y(s))
\]

\[
= \frac{B(\alpha)}{1-\alpha} \left( C^{\nabla}_{(q,x_0)y} \right)(x) \quad [\forall x \in N_{x_0+1}], \quad (4.10)
\]

where \(B(\alpha)\) is as before.
Lemma 5.1 confirm the validate of our obtained results. The Leibniz rule helps us to obtain the discrete version of the fundamental theorem of calculus, which will

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Example 4.4. For \( p(x) = \frac{x-x_0^{\frac{n}{1-n}}}{{\Gamma}(\alpha)} \) and \( \alpha \in (0, \frac{1}{2}) \), we have

\[
\left( \overset{G}{S}^{\nabla}_{\{p:(x)\}y} \right)(x) = \frac{1-\alpha}{B(\alpha)} y(x) + \frac{\alpha}{B(\alpha)} \sum_{s=x_0+1}^{x} p(x - \rho(s) + x_0) y(s) \\
= \frac{1-\alpha}{B(\alpha)} y(x) + \frac{\alpha}{B(\alpha)} \frac{1}{\Gamma(\alpha)} \sum_{s=x_0+1}^{x} (x - \rho(s))^{\frac{n}{1-n}} y(s) \\
= \frac{1-\alpha}{B(\alpha)} y(x) + \frac{\alpha}{B(\alpha)} (x_0 \nabla^{-\alpha} y)(x) \quad \forall \ x \in N_{x_0+1}.
\]

Also, for \( q(x) = \mathbb{E}(\lambda, x - x_0) \), \( \lambda = -\frac{\alpha}{1-\alpha} \) and \( \alpha \in [0, \frac{1}{2}) \), we have

\[
\left( \overset{G}{D}^{\nabla}_{\{q:(x)\}y} \right)(x) = \frac{B(\alpha)}{1-\alpha} \nabla_x \left( \sum_{s=x_0+1}^{x} q(x - \rho(s) + x_0) y(s) \right) \\
= \frac{B(\alpha)}{1-\alpha} \nabla_x \sum_{s=x_0+1}^{x} \mathbb{E}(\lambda, x - \rho(s)) y(s) = (\overset{A}{B}^{\nabla}_{R} x_0 \nabla^{-\alpha} y)(x) \quad \forall \ x \in N_{x_0+1}.
\]

Similarly, for the above \( q(x) \) with \( \lambda = -\frac{\alpha}{1-\alpha} \) and \( \alpha \in [0, \frac{1}{2}) \), we see that

\[
\left( \overset{G}{D}^{\nabla}_{\{q:(x)\}y} \right)(x) = (\overset{A}{B}^{\nabla}_{R} x_0 \nabla^{-\alpha} y)(x) \quad \forall \ x \in N_{x_0+1}.
\]

Example 4.5. For \( p(x) = 1 \) and \( \alpha \in (0, 1) \), we have

\[
\left( \overset{G}{S}^{\nabla}_{\{p:(x)\}y} \right)(x) = \frac{1-\alpha}{B(\alpha)} y(x) + \frac{\alpha}{B(\alpha)} \sum_{s=x_0+1}^{x} y(s) \\
= (\overset{C}{F}^{\nabla}_{\alpha} x_0 \nabla^{-\alpha} y)(x) \quad \forall \ x \in N_{x_0+1}.
\]

Moreover, for \( q(x) = (1 - \alpha)^{x-x_0} \) and \( \alpha \in (0, 1) \), we have

\[
\left( \overset{G}{D}^{\nabla}_{\{q:(x)\}y} \right)(x) = \frac{B(\alpha)}{1-\alpha} \nabla_x \left( \sum_{s=x_0+1}^{x} q(x - \rho(s) + x_0) y(s) \right) \\
= \frac{B(\alpha)}{1-\alpha} \nabla_x \sum_{s=x_0+1}^{x} (1 - \alpha)^{x - \rho(s)} y(s) = (\overset{C}{F}^{\nabla}_{R} x_0 \nabla^{-\alpha} y)(x) \quad \forall \ x \in N_{x_0+1}.
\]

Analogously, for the same \( q(x) \) and \( \alpha \), we have that

\[
\left( \overset{G}{D}^{\nabla}_{\{q:(x)\}y} \right)(x) = (\overset{C}{F}^{\nabla}_{\alpha} x_0 \nabla^{-\alpha} y)(x) \quad \forall \ x \in N_{x_0+1}.
\]

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The Leibniz rule helps us to obtain the discrete version of the fundamental theorem of calculus, which will confirm the validate of our obtained results.

Lemma 5.1 (Leibniz rule). For the function \( y : N_{x_0} \times N_{x_0} \rightarrow \mathbb{R} \), one can have

\[
\nabla_x \sum_{s=x_0+1}^{x} y(x, s) = y(x-1, x) + \sum_{s=x_0+1}^{x} (\nabla_y)(x, s) \quad \text{for all} \ x \in N_{x_0+1}. \quad (5.1)
\]

Proof. This can be deduced directly from the definition of backward difference operator, which is \((\nabla y)(x) = y(x) - y(x-1)\). \( \square \)
Theorem 5.1 (Fundamental Theorem of Calculus). Let a pair $(p, q)$ be in the class $\mathcal{C}_x^\Sigma$ with $x_0 \in \mathbb{R}$. Then, we have

$$
\left( RL \mathcal{D}^\Sigma_{\{q\}; x_0} S^\Sigma_{\{p\}; x_0} y \right)(x) = \left( C \mathcal{D}^\Sigma_{\{q\}; x_0} S^\Sigma_{\{p\}; x_0} y \right)(x) = y(x) \quad \text{for all } x \in \mathbb{N}_{x_0+1}.
$$

Further, we have

$$
\left( S^\Sigma_{\{p\}; x_0} RL \mathcal{D}^\Sigma_{\{q\}; x_0} y \right)(x) = y(x) \quad \text{for all } x \in \mathbb{N}_{x_0},
$$

and

$$
\left( S^\Sigma_{\{p\}; x_0} C \mathcal{D}^\Sigma_{\{q\}; x_0} y \right)(x) = y(x) - y(x_0) \quad \text{for all } x \in \mathbb{N}_{x_0}.
$$

Proof. Considering $(p, q) \in \mathcal{C}_x^\Sigma$ by assumption, we have that $(p * q)^\Sigma_x(x) = 1$ for each $x \in \mathbb{N}_{x_0+1}$. Then, we have from Definitions (3.1), (4.1) and (4.2):

$$
\left( RL \mathcal{D}^\Sigma_{\{q\}; x_0} S^\Sigma_{\{p\}; x_0} y \right)(x) = \nabla_x \left( q * \left( S^\Sigma_{\{p\}; x_0} y \right)(x) \right)_{x_0} = \nabla_x \left( q * (p * y)^\Sigma_{x_0}(x) \right)_{x_0}
$$

$$
= \nabla_x \left( \sum_{s=x_0+1}^x y(s) = y(x) \right)_{x_0}.
$$

To prove the other part of (5.2), we use Lemma 5.1:

$$
\left( RL \mathcal{D}^\Sigma_{\{q\}; x_0} y \right)(x) = \nabla_x \sum_{s=x_0+1}^x y(x - \rho(s) + x_0)q(s) = q(x_0)y(x) + \sum_{s=x_0+1}^x (\nabla_x y)(x - \rho(s) + x_0)q(s)
$$

$$
= q(x_0)y(x) + \left( C \mathcal{D}^\Sigma_{\{q\}; x_0} y \right)(x).
$$

This together with (5.5) imply that

$$
\left( C \mathcal{D}^\Sigma_{\{q\}; x_0} S^\Sigma_{\{p\}; x_0} y \right)(x) = \left( RL \mathcal{D}^\Sigma_{\{q\}; x_0} S^\Sigma_{\{p\}; x_0} y \right)(x) - q(x_0) \left( S^\Sigma_{\{p\}; x_0} y \right)(x_0) = y(x).
$$

For the proof of (5.3), we should proceed by using Definitions (3.1), (4.1), (4.2), the assumption, and the commutativity property of the convolution operator in (5.6):

$$
\left( RL \mathcal{D}^\Sigma_{\{q\}; x_0} y \right)(x) = \nabla_x \sum_{s=x_0+1}^x y(x - \rho(s) + x_0)q(s) = \nabla_x \sum_{s=x_0+1}^x q(x - \rho(s) + x_0)y(s)
$$

$$
= y(x_0)q(x) + \sum_{s=x_0+1}^x (\nabla_x q)(x - \rho(s) + x_0)y(s)
$$

$$
= y(x_0)q(x) + \sum_{s=x_0+1}^x (\nabla_x y)(x - \rho(s) + x_0)q(s)
$$

$$
= y(x_0)q(x) + \left( C \mathcal{D}^\Sigma_{\{q\}; x_0} y \right)(x).
$$

$$
= y(x_0)q(x) + (q * (\nabla y))^\Sigma_{x_0}(x),
$$
to get

\[
\left( S_{(p);x_0}^{\nabla} \right) \left( R L \left( q; x_0 \right) \right) = \left( p \ast R L \mid \left( q; x_0 \right) \right)
\]

\[
\left( p \ast \left( y(x_0) q(x) + (q \ast (\nabla y)) \right) \right) \left( x \right)
\]

\[
y(x_0) (p \ast q) x_0 (x) + \left( (p \ast q) x_0 (x) \ast (\nabla y) \right)_{\left( x_0 \right)} (x)
\]

\[
y(x_0) + \left( 1 \ast (\nabla y) \right)_{\left( x_0 \right)} (x)
\]

\[
y(x_0) + \sum_{s=x_0+1}^{x} (\nabla y)(s)
\]

\[
y(x_0) + y(x) = y(x).
\]

For the last proof, we will proceed by using the assumption, and Definitions (3.1), (4.1), (4.2), (5.6) and (5.2):

\[
\left( S_{(p);x_0}^{\nabla} \right) \left( C \left( q; x_0 \right) \right) (x) = \left( S_{(p);x_0}^{\nabla} \right) \left( \left( R L \mid \left( q; x_0 \right) \right) - y(x_0) q(x) \right) (x)
\]

\[
y(x) - y(x_0) \left( S_{(p);x_0}^{\nabla} q \right) (x)
\]

\[
y(x) - y(x_0) (p \ast q) x_0 (x)
\]

\[
y(x) - y(x_0),
\]

which trivially rearranges to the required identity (5.4). Thus, the proof of this theorem is done.

6 Conclusions and further work

In this work, we have investigated a class of discrete nabla fractional operators for the discrete nabla convolution of two functions. Inspired by the new class, discrete generalized nabla fractional sum and differences have been defined. We have proved relationships between the proposed discrete generalized nabla fractional operators with the generalized discrete delta fractional operators defined by Ferreira [12]. We illustrated the relations by some examples using particular functions. In addition, we have proved the fundamental theorem of calculus to our new defined discrete generalized nabla fractional operators. Finally, we have extend our work by defining discrete nabla AB fractional sum and differences via the new class.

In the future, we are planning to extend the results of this article by considering other types of discrete fractional calculus. As we know that the work here is set within the Riemann-Liouville and Caputo models, but it may be possible to modify and extend it, applying the same arguments in some general class of discrete fractional operators such as those operators investigated in [6, 11], so that we will obtain further results which would be useful in different types of discrete modelling problems.

Availability of data and materials

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Competing interests

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