A Relaxed Approach to Estimating Large Portfolios

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Abstract

This paper considers three aspects of the estimation of large portfolios, namely variance, gross exposure, and risk, when the number of assets, \( p \), is larger than sample size, \( n \). Using a novel technique called nodewise regression to estimate the inverse of a large covariance matrix of excess asset returns, we show that ratio of the estimated variance of a large portfolio to its population counterpart tends to one in probability. Building upon this result we show that the estimation error of the optimal portfolio variance decreases when the number of assets increases under the assumption that the inverse covariance matrix is sparse. We generalize this result to non-sparse inverse covariance matrices, showing that estimation error of the optimal portfolio variance converges in probability to zero, albeit at a slower rate than in the sparse case. We then establish that the gross exposure of a large portfolio is consistently estimated whether the true exposure is assumed to be growing or constant. Finally, our approach provides consistent estimates of the risk of large portfolios. Our main results are established under the assumption that excess asset returns are sub-Gaussian, and generalized under the weaker assumption of returns with bounded moments. Simulations verify and illustrate the theoretical results. Nodewise regression based estimator is compared to factor models and shrinkage based methods in an out-of-sample exercise, where we do well in terms of portfolio variance. Our method is complementary to existing factor based and shrinkage based models.

Keywords: high-dimensionality, penalized regression, precision matrix, portfolio optimization

JEL Classification: C13, C55, G11, G17

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1 Introduction

Three of the major problems related to large portfolio analysis are the estimation of variance, gross exposure, and the risk. One of the key issues pertaining to all three problems is the estimation of the inverse of the covariance matrix of excess asset returns. Inverting the sample covariance matrix becomes impossible when the number of assets, \( p \), is larger than the sample size, \( n \), since the sample covariance matrix of excess asset returns is singular.

This paper analyzes these three issues when \( p > n \), using a new technique in statistics called nodewise regression to estimate the inverse covariance matrix. We show that ratio of the variance of the estimate of the large portfolio variance to its population counterpart tends to one in probability. Using that result, estimation error stemming from the difference of the large portfolio variance and its estimator also goes to zero in probability even in the case of non-sparsity. This is due to variance decreasing by the number of assets. Also, gross exposure of a large portfolio is estimated consistently even in the case of growing exposure. Our last result is consistent estimation of large portfolio risk.

We impose a sparsity assumption on the inverse of covariance matrix. But there is no assumption on sparsity on the covariance matrix of asset returns. This still allows for all correlated assets such as Toeplitz type of covariance structure, hence it is a mild restriction. A weak sparsity assumption on the inverse, which allows many small coefficients on the inverse, is possible but will lengthen the proofs considerably, hence not attempted in this paper. Also, in simulations we have all non-sparse inverse of covariance, and non-sparse covariance DGPs that mimic S&P 500 returns, and we compare our model with factor model based and shrinkage based approaches. Out-of-sample empirical analysis show the low variance of our portfolio.

To understand the recent literature: there are two major problems in setting up a large portfolio. The first problem is the non-invertibility of large sample covariance matrix, and the second one is how to incorporate a new method that provides a solution to first problem into the large portfolios. In statistical literature, in order to handle singularity of the sample covariance matrix, the vast majority of work impose sparsity conditions, that is, many off-diagonal components are either zero or close to zero; see e.g. Smoothly Clipped Absolute Deviation (SCAD) method in Fan and Li (2001), Adaptive Least Absolute Shrinkage and Selection Operator (Lasso) method in Zou (2006) and thresholding method in Bickel and Levina (2008), generalized thresholding in Rothman et al. (2009) and adaptive thresholding in Cai and Liu (2011).

Furthermore, Meinshausen and Bühlmann (2006) developed nodewise regression technique to handle inverses of large matrices. Recently, van de Geer et al. (2014) constructed asymptotically honest confidence intervals for low-dimensional components of a large-dimensional parameter vector.

\footnote{In particular, \( \ell_1 \) penalized likelihood estimator has been considered by several authors, see e.g. Rothman et al. (2008), Friedman et al. (2008), Ravikumar et al. (2011), Cai et al. (2011), d’Aspremont et al. (2008), Javanmard and Montanari (2013), O.Banerjee et al. (2008) and Fan et al. (2009). There are also banding methods including re-parameterization of the covariance matrix and inverse through the Cholesky decomposition, see e.g. Wu and Fourahmadi (2000), Wong et al. (2003), Huang et al. (2006), Yuan and Lin (2007), Levina et al. (2008), Rothman et al. (2010).}
using nodewise regression. Moreover, Caner and Kock (2014) examined the same issue for high-dimensional models in the presence of conditional heteroscedasticity in the error terms in which they derived conservative lasso estimator. They also constructed an approximate inverse matrix with nodewise regression based on conservative lasso rather than plain lasso.

Related to the issue of inversion of sample covariance matrix, is the study of large dimensional portfolios. Empirical evidence that is related to poor performance of the mean-variance portfolios due to insufficient sample size can be found in Michaud (1989), Green and Hollifield (1992), Brodie et al. (2009), Jagannathan and Ma (2003) and DeMiguel et al. (2009a). 2

There are several approaches for estimating large covariance matrix of a portfolio. The most convenient way is to use higher frequency of data, i.e. using daily returns instead of monthly returns, see e.g. Jagannathan and Ma (2003), Liu (2009).

Other methods particularly focused on the sample covariance matrix of asset returns. For example, an effective shrinkage approach has been proposed by Ledoit and Wolf (2003, 2004), where they have used a convex combination of sample covariance matrix of excess asset returns with (Sharpe, 1963)’s Capital Asset Pricing Model (CAPM) based covariance matrix and identity matrix, respectively. Recently, Ledoit and Wolf (2012) proposed a nonlinear shrinkage approach of the sample eigenvalues for the case where $p < n$. Ledoit and Wolf (2015) extended this nonlinear shrinkage approach to the case $p > n$. Ledoit and Wolf (2004) and Ledoit and Wolf (2015) are only concerned about covariance estimations and no portfolio theories are studied. 3

Another approach is to impose factor structure to reduce high-dimensionality of the covariance matrices, see e.g. Fan et al. (2011), Fan et al. (2013). Fan et al. (2008) proposed the factor-based covariance matrix estimator depending on the observable factors and impose zero cross-correlations in the residual covariance matrix. Conditional sparsity is assumed in these papers. They also derived the rates of convergence of portfolio risk in large portfolios. Fan et al. (2015) proposed principal orthogonal complement thresholding (POET) estimator when the factors are unobservable. We see that Fan et al. (2008) and Fan et al. (2015) are very valuable in understanding the variance and exposure of large portfolios. Fan et al. (2016), Ait-Sahalia and Xiu (2016) considered estimation of precision matrix via several norms when $p > n$ in a factor model structure. There is no estimation of portfolio variance or gross exposure in these last two papers when $p > n$. In case of $p > n$, Li et al. (2017), and Ao et al. (2017) provide important and valuable findings in optimal global variance ratio via factor models, and portfolio selection via lasso in heteroskedastic asset returns. Our paper uses nodewise regression, and hence is different and complementary to theirs.

Instead of dealing with large covariance matrices, some approaches directly address the portfolio weight vector, $w$ which is the outcome in the portfolio optimization. Jagannathan and Ma (2003) showed that imposing a non-negativity (no short-sale) constraint on the portfolio weight vector turn

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2In a high-dimensional case, estimation accuracy can be affected adversely by distorted eigenstructure of covariance matrix as expressed by Daniels and Kass (2001). As noted by Markowitz (1990), this implies it will not be feasible to estimate an efficient portfolio when $p > n$.

3 From similar point of view, Kourtis et al. (2012) introduced a linear combination of shrinking the inverse of sample covariance matrix of excess asset returns with a target matrix in the case $p < n$. 

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out to be shrinkage-like effect on the covariance matrix estimation. On the contrary, DeMiguel et al. (2009b) indicated that even imposing short-sale constraints did not reduce the estimation error and hence, they suggested a naive portfolio diversification strategy based on equal proportion of wealth to each asset. Directly analyzing weights has the problem of possibly feeding large estimation errors in the inverse of the covariance matrix to the weights. To get better weights, we need to estimate the inverse of covariance matrix of excess asset returns. Our method can do that when \( p > n \).

The rest of paper is organized as follows. Section 2 introduces the nodewise regression with \( \ell_1 \) penalty, the approximate inverse of the empirical Gram matrix and its asymptotic properties. In Section 3 we show convergence rates in estimating the global minimum portfolio and mean-variance optimal portfolio by usage of nodewise regression. Section 4 relaxes the assumptions on data. In Section 5 we give a brief overview of the existing literature of the covariance matrix estimators and present simulation results. An empirical study is considered in Section 6. Finally, Section 7 concludes. All the proofs are given in the Appendix A and B. Throughout the paper \( \| \nu \|_\infty, \| \nu \|_1, \| \nu \|_2 \) denote the sup, \( l_1 \), and Euclidean norm of a generic vector \( \nu \), respectively.

### 2 The Lasso for Nodewise Regression

In this section, we employ lasso nodewise regression to construct the approximate inverse \( \hat{\Theta} \) of \( \hat{\Sigma} \) developed by Meinshausen and Bühlmann (2006) and van de Geer et al. (2014) along the lines of plain lasso of Tibshirani (1994). We employ the nodewise regression method since it allows for \( p > n \), and builds an approximate inverse for a large covariance matrix by deleting irrelevant terms.

#### 2.1 Method

For each \( j = 1, \ldots, p \), lasso nodewise regression is defined as:

\[
\hat{\gamma}_j := \arg\min_{\gamma \in \mathbb{R}^{p-1}} \langle \rho_j - r_{-j}\gamma, \rho_j - r_{-j}\gamma \rangle / n + 2\lambda_j \| \gamma \|_1, \quad (2.1)
\]

with \( n \times p \) design matrix of excess asset returns \( \mathbf{r} = [r_1, \ldots, r_p] \). \( r_j \) is \( (n \times 1) \) vector and denotes the \( j \)th column in \( \mathbf{r} \) and \( r_{-j} \) denotes all columns of \( \mathbf{r} \) except for the \( j \)th one. \( \hat{\gamma}_j \) is \( (p - 1) \) regression coefficient estimate, and specifically \( \hat{\gamma}_j = \{\hat{\gamma}_{jk}; k = 1, \ldots, p, k \neq j\} \) which will be used in estimating the relaxed inverse of the covariance matrix. \( \lambda_j \) is a positive tuning parameter which determines the size of penalty on the parameters. As in van de Geer et al. (2014), \( \gamma_j := \arg\min_{\gamma \in \mathbb{R}^{p-1}} \mathbb{E}\|r_j - r_{-j}\gamma\|_2^2 \). We denote by \( S_j := \{j; \gamma_j \neq 0\} \) which will be the main building

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Brodie et al. (2009) reformulated the Markowitz’s risk minimization problem as a regression problem by adding \( \ell_1 \) penalty to the constrained objective function to obtain sparse portfolios. DeMiguel et al. (2009a) proposed optimal portfolios with \( \ell_1 \) and \( \ell_2 \) constraints and allowing their strategy to nest different benchmarks such as Jagannathan and Ma (2003), Ledoit and Wolf (2003, 2004), DeMiguel et al. (2009b). Li (2015) formulated \( \ell_1 \) norm and a combination of \( \ell_1 \) and \( \ell_2 \) norm constraints known as elastic-net (Zou and Hastie, 2005) on \( w \). Fan et al. (2012) introduced \( \ell_1 \) regularization to identify optimal large portfolio selection by imposing gross-exposure constraints.
block of the $j$th row of inverse covariance matrix of excess asset returns in the portfolio and its cardinality is given by $s_j := |S_j|$. The idea is to use a reasonable approximation of an inverse of \( \hat{\Sigma} = n^{-1} \sum_{t=1}^{n} (r_t - \bar{r})(r_t - \bar{r})' \) with $\bar{r} = n^{-1} \sum_{t=1}^{n} r_t$. Let $\hat{\Theta}$ be such an inverse. $\hat{\Theta}$ is given by nodewise regression with $\ell_1$ penalty on the design $r$. We run the lasso $p$ times for each regression of $r_j$ on $r_{-j}$, where the latter is the design submatrix without the $j$th column.

To derive $\hat{\Theta}$, first define

$$
\hat{C} := \begin{pmatrix}
1 & -\hat{\gamma}_{1,2} & \cdots & -\hat{\gamma}_{1,p} \\
-\hat{\gamma}_{2,1} & 1 & \cdots & -\hat{\gamma}_{2,p} \\
\vdots & \vdots & \ddots & \vdots \\
-\hat{\gamma}_{p,1} & -\hat{\gamma}_{p,2} & \cdots & 1
\end{pmatrix}
$$

and write $\hat{T}^2 = \text{diag}(\hat{\tau}_{1}^2, \ldots, \hat{\tau}_{p}^2)$ which is $p \times p$ diagonal matrix for all $j = 1, \ldots, p$. Note that elements of $\hat{T}^2$ are:

$$
\hat{\tau}_{j}^2 := \frac{\| r_j - r_{-j} \hat{\gamma}_j \|^2_{2}}{n} + \lambda_j \| \hat{\gamma}_j \|_1.
$$

(2.2)

Then define the relaxed inverse $\hat{\Theta}$ as:

$$
\hat{\Theta} := \hat{T}^{-2} \hat{C}.
$$

(2.3)

$\hat{\Theta}$ is considered a relaxed (approximate) inverse of $\hat{\Sigma}$. While $\hat{\Sigma}$ is self-adjoint, $\hat{\Theta}$ is not. Define the $j$th row of $\hat{\Theta}$ as a $1 \times p$ vector and analogously for $\hat{C}_j$. Thus, $\hat{\Theta}_j = \frac{\hat{C}_j}{\hat{\tau}_{j}^2}$.

The specific algorithm to compute $\hat{\Theta}$ is given below, and we only use $p$ regressions.

1. Apply lasso estimator for the regression of $r_j$ on $r_{-j}$ as in (2.1) and get $\hat{\gamma}_j$ vector with ($p - 1$) dimension given $\lambda_j$.

2. $\lambda_j$ is chosen by modified BIC in the nodewise regressions via (5.1).

3. Repeat steps 1-2 $j = 1, \cdots, p$ times.

4. Take the transpose of $\hat{\gamma}_j$ and let it be a $(p - 1)$ dimensional matrix consisting of each column of $\hat{\gamma}_j$ and multiply all elements with minus one.

5. Compute $\hat{C}$ matrix.

6. Estimate $\hat{T}^2$ matrix with $\hat{\gamma}_j$ values by (2.2) and optimally chosen $\lambda_j$. $\hat{T}^2 = \text{diag}(\hat{\tau}_{1}^2, \ldots, \hat{\tau}_{p}^2)$ which is $(p \times p)$ diagonal matrix for all $j = 1, \ldots, p$.

7. Use (2.3) to get $\hat{\Theta}$.

Next, we show why $\hat{\Theta}$ is an approximate inverse for $\hat{\Sigma}$. The Karush-Kuhn-Tucker conditions for the nodewise lasso (2.1) imply that,

$$
\hat{\tau}_{j}^2 = \frac{(r_j - r_{-j} \hat{\gamma}_j)' r_j}{n}
$$

(2.4)
Divide each side by $\hat{\tau}_j^2$ and using the definition $\hat{\Theta}_j$ shows that,

$$\frac{(r_j - r_{-j}\hat{\gamma}_j)'r_j}{\hat{\tau}_j^2 n} = \frac{\hat{\gamma}' r_j}{\hat{\tau}_j^2 n} = 1$$  \hspace{1cm} (2.5)

which shows that $j$th diagonal term of $\hat{\Theta}\hat{\Sigma}$ exactly equals to one. Karush-Kuhn-Tucker conditions for the nodewise lasso (2.1) for the off-diagonal terms of $\hat{\Theta}\hat{\Sigma}$ imply that,

$$\|r_{-j}'\hat{\Theta}'_j\|_\infty \leq \frac{\lambda_j}{\hat{\tau}_j^2}$$ \hspace{1cm} (2.6)

Hence, we combine (2.5) and (2.6) results for diagonal and off-diagonal terms in $\hat{\Theta}\hat{\Sigma}$ to have,

$$\|\hat{\Sigma}\hat{\Theta}'_j - e_j\|_\infty \leq \frac{\lambda_j}{\hat{\tau}_j^2}$$ \hspace{1cm} (2.7)

where, $e_j$ is the $j$th unit column vector ($p \times 1$).

### 2.2 Asymptotics

In this section we provide assumptions and we will show that nodewise regression estimate of the inverse of variance covariance matrix is consistent. These results are all uniform over $j = 1, \ldots, p$.

To derive our theorems, we need the following assumptions.

**Assumption 1.** The $n \times p$ matrix of excess asset returns $r$ has iid sub-Gaussian rows.

**Assumption 2.** The following condition holds

$$\max_j s_j \sqrt{\log p/n} = o(1).$$

**Assumption 3.** The smallest eigenvalue of $\Sigma$, $\Lambda_{\min}$ is strictly positive and $1/\Lambda_{\min} = O(1)$. Moreover, $\max_j \Sigma_{j,j} = O(1)$. Also the minimal eigenvalue of $\Theta = \Sigma^{-1}$ is strictly positive.

The assumptions above are Assumptions (B1)-(B3) of van de Geer et al. (2014). Assumption 1 can be relaxed to non i.i.d. non sub-Gaussian cases as shown in Caner and Kock (2014) and at subsection 4 we generalize our results to excess asset returns with bounded moments. Assumption 2 allows us to have $p > n$. Note that when $p > n$, $\max_j s_j = o(\sqrt{n/\log p})$. In the simple case of $p = an$, where $a > 1$ is constant, then $\max_j s_j = o(\sqrt{n/\log(an)})$ which is growing with $n$, but less than maximum number of possible nonzeros $p - 1$ in a row. If the problem is such that $p < n$, then it is possible that $\max_j s_j = p - 1$ (no sparsity in the inverse of the covariance matrix), so in that case, Assumption 2 can be written as $p\sqrt{\log p/n} = o(1)$. Note that our Assumption 2 has extra $\sqrt{\max_j s_j}$ factor compared to van de Geer et al. (2014). This is due to portfolio optimization problem.

Assumption 2 is a mild sparsity condition on the inverse of the covariance matrix of excess asset
returns. This does not imply sparsity of covariance matrix of excess asset returns. For example, if the excess asset returns have an autoregressive structure of order one, the covariance matrix will be non-sparse but its inverse will be sparse. Another prominent example is if covariance matrix of asset returns: $\Sigma$, has a block diagonal or a Toeplitz structure, $\Sigma_{i,j} = \rho_{|i-j|}$, $-1 < \rho_1 < 1$, $\rho_1$ being the correlation among assets. Then again the inverse in these cases will be sparse.

Fan et al. (2008) assumed sparsity of residual covariance matrix of excess asset returns. A specific case of Fan et al. (2015) on the other hand, had the structure-formula as our Assumption 2 but sparsity is imposed on the residual covariance matrix rather than on its inverse. Assumption 3 allows population covariance matrix to be nonsingular. Strictly positive minimal eigenvalue means this is not a local to zero sequence.

The following Lemma shows one of the main results of our paper, and can be deduced from proof of Theorem 2.4 of van de Geer et al. (2014). We provide a proof in our Appendix to be complete. It shows that nodewise regression estimate of the inverse of the covariance matrix can be uniformly (over $j$) consistently estimated.

**Lemma 2.1.** Under Assumptions 1-3 with $\lambda_j = O(\sqrt{\log p/n})$ uniformly in $j \in 1, \ldots, p$,

$$\|\hat{\Theta} \hat{\Sigma} - I_p\|_\infty = O_p(\sqrt{\log p/n}) = o_p(1).$$

Note that $\lambda_j = O(\sqrt{\log p/n})$ is a standard assumption in van de Geer et al. (2014), and can be found also in Bühlmann and van de Geer (2011). This rate is derived from concentration inequalities. This Lemma shows that we can estimate inverse matrix even when $p > n$.

It is clear that from the previous lemma and our assumptions that we allow $p = an$, where $0 < a < \infty$. Specifically, $a$ can be much larger than one, allowing assets to dominate sample size in dimensions. Also, our assumptions allow $p = \exp(n^b)$, where $0 < b < 1$. Note that in practice, relaxed inverse in finite samples may not be positive definite, but our asymptotic proofs are not affected by that as can be seen from Lemma 2.1 and $\Sigma^{-1}$ being positive definite, and the proof of Lemma A.1. We can ensure positive definiteness of the constructed $\hat{\Theta}$ by eigenvalue cleaning as in Callot et al. (2016); Hautsch et al. (2012).

### 3 Nodewise Regression in Large Asset Based Portfolios

Nodewise regression is a technique that helps us in getting an approximate estimate for the inverse of covariance matrix of excess asset returns when the number of assets ($p$) is larger than time period ($n$). One clear advantage of this technique that will be shown below is that, we can form weights, variance, and risk of large portfolios when $p > n$. Second advantage is that sparsity of covariance matrix is not assumed, even conditionally. However, as can be seen sparsity of inverse matrix is assumed, which can happen when excess returns are correlated with an autoregressive with lag one structure. This is a mild restriction. Then another issue is stability. For example from the formula below for global minimum variance portfolio, nodewise regression based estimate
of inverse of covariance matrix will be more stable when \( p \) is near \( n \), with \( p < n \) and may provide a smaller variance than a very standard approach that uses inverse of sample covariance matrix in that case. Compared with factor model based approaches, we do not form a structure on returns but assume sparsity of inverse of the covariance matrix. But with both factor models and shrinkage based methods it will be difficult to compare theoretically since assumptions and problems may differ, so a second best response is simulation based evidence which we provide.

3.1 Optimal Portfolio Allocation and Risk Assessment

Consider a given set of \( p \) risky assets with their excess returns at time \( t \) by \( r_{it}, i = 1,\ldots,p \). We denote \((p \times 1)\) vector of excess returns as \( r_t = (r_{1t},\ldots,r_{pt})'\). We assume that excess returns are stationary and \( \mathbb{E}[r_t] = \mu \), where \( \mu = (\mu_1,\ldots,\mu_p)' \). Full rank covariance matrix of excess returns is expressed as \( \Sigma \in \mathbb{R}^{pxp} \), where \( \Sigma = \mathbb{E}[(r_t - \mu)(r_t - \mu)'] \). A portfolio allocation is defined as composition of weights, \( w \in \mathbb{R}^p \). Specifically, \( w = (w_1,\ldots,w_p)' \) represents the relative amount of invested wealth in each asset.

3.1.1 Global Minimum Variance Portfolio

First, we start with unconstrained optimization. The aim is to minimize the variance without an expected return constraint. Denote the global minimum variance portfolio as \( w_u \)

\[
 w_u = \arg\min_w (w'\Sigma w), \quad \text{such that} \quad w'1_p = 1.
\]

Note that p.370 of Fan et al. (2015) showed that weights of the global minimum variance portfolio is:

\[
 w_u = \frac{\Sigma^{-1}1_p}{1_p'\Sigma^{-1}1_p}. \tag{3.1}
\]

We also give the following estimate for the weights in global minimum variance portfolio

\[
 \hat{w}_u = \frac{\hat{\Theta}1_p}{1_p'\hat{\Theta}1_p}. \tag{3.2}
\]

Note that global minimum variance without constraint on the expected return is

\[
 w_u'\Sigma w_u = (1_p'\Sigma^{-1}1_p)^{-1}, \tag{3.3}
\]

as shown in equation (11) of Fan et al. (2008). So denote the global minimum variance portfolio by

\[
 \Phi_G = (1_p'\Sigma^{-1}1_p)^{-1},
\]

and its estimate by

\[
 \hat{\Phi}_G = (1_p'\hat{\Theta}1_p)^{-1}. \tag{3.4}
\]
In other words, we estimate global minimum variance directly. This is also the approach used by Theorem 5 in Fan et al. (2008). A less straightforward estimate of the global minimum variance could have been using (3.2) directly in estimating left side of (3.3). This last way uses the estimates of weights and unnecessarily complicates the estimator.

Next result shows estimation of global minimum variance using nodewise regression method for inverse of covariance matrix. This is one of the main results of this paper.

Theorem 3.1. Under Assumptions 1-3, uniformly in $j \lambda_j = O(\sqrt{\log p}/n), \frac{\hat{\Phi}_G}{\Phi_G} - 1 = O_p(\max_j s_j \sqrt{\log p}/n) = o_p(1)$.

Remarks:

1. We allow $p > n$ in global minimum variance portfolio.

2. As far as we know, the ratio of estimate of large portfolio variance to its population counterpart is new without any factor model structure, and standardizes the variance estimate.

3. In the factor model case, very recently, Li et al. (2017), provides this ratio converges to one in probability. Their paper makes important and valuable contributions in optimality of their ratio result. With high dimensional approaches like theirs and us we are learning more about the portfolio variance with very large number of assets.

We provide the following Corollary that shows direct difference between two portfolio variances, which shows that error converges to zero in probability at a faster than $p$, which is due to rate of global minimum variance. So using Theorem 3.1 with (A.7) in Appendix we have the following Corollary. In Remark 2 below, we show that even in case of non-sparsity of population inverse covariance matrix, consistent estimation of the variance is possible, which is an important finding.

Corollary 3.1. Under Assumptions 1-3, uniformly in $j \lambda_j = O(\sqrt{\log p}/n), |\hat{\Phi}_G - \Phi_G| = \frac{O_p(\max_j s_j \sqrt{\log p}/n)}{O(p)} = o_p(-\frac{1}{p}).$

Remarks:

1. Note that Theorem 5 of Fan et al. (2008) derived the result in Corollary 3.1 above using factor model based covariance inverse with no sparsity assumption on inverse of the covariance matrix. Their estimation error converges to zero in probability when $n$ is much larger than $p$ only. Since the assumptions of that paper is different from us, a direct comparison of these two papers in theory is not suitable.
2. A key issue is what happens when Assumption 2 is violated. Assume now \( \max_j s_j = p - 1 \), then the estimation error still converges in probability to zero, as long as \( \log p/n \to 0 \), which allows \( p > n \)

\[
|\hat{\Phi}_G - \Phi_G| = \frac{O_p(\max_j s_j \sqrt{\log p/n})}{O(p)} = O_p(\sqrt{\log p/n}) = o_p(1).
\]

3.1.2 Markowitz Mean-Variance Framework

Markowitz (1952) has defined the portfolio selection problem to find the optimal portfolio that has the least variance i.e. portfolio variance for a given expected return \( \rho_1 \). At time \( t \), an investor determines the portfolio weights to minimize the mean-variance objective function:

\[
w = \arg\min_w (w'\Sigma w), \quad \text{subject to} \quad w'1_p = 1 \quad \text{and} \quad w'\mu = \rho_1,
\]

where \( 1_p = (1, \ldots, 1)' \). For a given portfolio \( w \), \( w'\mu \) and \( w'\Sigma w \) equal to the expected rate of return and variance, respectively. Full investment constraint (\( w'1_p = 1 \)) requires that weights should sum up to 1. The target return constraint (\( w'\mu = \rho_1 \)) indicates a certain level of desired expected portfolio return. Throughout the paper, we assume short-selling is allowed and hence the value of weights could be negative in the portfolio.

The well-known solution from the Lagrangian and the first order conditions from constrained quadratic optimization is (as equation (9) in Fan et al. (2008)):

\[
w^* = \frac{D - \rho_1 B}{AD - B^2} \Sigma^{-1} 1_p + \frac{\rho_1 A - B}{AD - B^2} \Sigma^{-1} \mu,
\]

(3.5)

where \( A = 1_p' \Sigma^{-1} 1_p \), \( B = 1_p' \Sigma^{-1} \mu \) and \( D = \mu' \Sigma^{-1} \mu \). Since \( \Sigma \) is positive-definite, \( A > 0 \) and \( D > 0 \). By virtue of Cauchy-Schwarz inequality, the system has a solution if \( AD - B^2 > 0 \).

Now we derive our second result. This uses optimal weights \( w^* \) to get the variance of the optimal portfolio. We form the following estimate of the optimal weight \( w^* \) using (3.5):

\[
\hat{w} = \frac{\hat{D} - \rho_1 \hat{B}}{AD - B^2} \hat{\Theta} 1_p + \frac{\rho_1 \hat{A} - \hat{B}}{AD - B^2} \hat{\Theta} \hat{\mu},
\]

where \( \hat{A} = 1_p' \hat{\Theta} 1_p \), \( \hat{B} = 1_p' \hat{\Theta} \hat{\mu} \), \( \hat{D} = \hat{\mu}' \hat{\Theta} \hat{\mu} \), and \( \hat{\mu} = n^{-1} \sum_{t=1}^n r_t \).

Now define \( \Psi_{OPV} = \frac{A\rho_1^2 - 2B\rho_1 + D}{AD - B^2} \), and its estimate \( \hat{\Psi}_{OPV} = \frac{\hat{A}\rho_1^2 - 2\hat{B}\rho_1 + \hat{D}}{AD - B^2} \). These will be used immediately below. Note that the variance of the optimal portfolio from the constrained optimization above is

\[
w^*\Sigma w^* = \frac{A\rho_1^2 - 2B\rho_1 + D}{AD - B^2} = \Psi_{OPV}.
\]

(3.6)

The estimate for the above optimal portfolio variance is

\[
\frac{\hat{A}\rho_1^2 - 2\hat{B}\rho_1 + \hat{D}}{AD - B^2} = \hat{\Psi}_{OPV}.
\]
Note that we have not used an estimate of the optimal portfolio variance based on estimating $w^*$ first from (3.5) and the equation immediately below. This could have created a much complicated estimator. Instead, like Fan et al. (2008), we use optimal portfolio variance expression and estimate the closed form solution in (3.6).

The following Theorem provides the rate for the error in estimating the optimal portfolio variance.

**Theorem 3.2.** Under Assumptions 1-3, and assuming uniformly in $j$, $\lambda_j = O(\sqrt{\log p/n})$ with, $p^{-2}(AD - B^2) \geq C_1 > 0$, where $C_1$ is a positive constant, and $p^{-1}(A\rho_1^2 - 2B\rho_1 + D) \geq C_1$, $\rho_1$ being bounded, we get

$$\left| \frac{\hat{\Psi}_{OPV}}{\Psi_{OPV}} - 1 \right| = O_p(\max_j s_j \sqrt{\log p/n}) = o_p(1).$$

**Remarks:**

1. Our result shows that we allow for $p > n$ in estimating optimal portfolio variance. The ratio of estimated optimal variance to population counterpart is new in large portfolio analysis, and we show that this ratio converges to one in probability.

2. We also restrict $p^{-1}(A\rho_1^2 - 2B\rho_1 + D) \geq C_1 > 0$, and $p^{-2}(AD - B^2) \geq C_1 > 0$. This is not a major restriction since by Lemma A.4 all terms $A, B, D$ grow at rate $p$, so this excludes a pathological case that with a certain return and implies that optimal portfolio variance $\Psi_{OPV}$ in (3.6) is positive.

**Corollary 3.2.** Under Assumptions 1-3, and assuming uniformly in $j$, $\lambda_j = O(\sqrt{\log p/n})$ with, $p^{-2}(AD - B^2) \geq C_1 > 0$, where $C_1$ is a positive constant, and $p^{-1}(A\rho_1^2 - 2B\rho_1 + D) \geq C_1$, $\rho_1$ being bounded, we get

$$\left| \hat{\Psi}_{OPV} - \Psi_{OPV} \right| = O_p(\max_j s_j \sqrt{\log p/n}) = o_p(1).$$

**Remarks:**

1. Corollary 3.2 shows that estimation error converges in probability to zero at a rate faster than $1/p$, which shows that adding assets to portfolio helps. However this is mainly due to optimal variance declining at rate $p$. To our knowledge these are new results in the literature.

2. When Assumption 2 is violated, assuming instead $\max_j s_j = p - 1$, the estimation error still converges to zero in probability as long as $\log p/n \to 0$ even when $p > n$:

$$\left| \hat{\Psi}_{OPV} - \Psi_{OPV} \right| = O_p(\max_j s_j \sqrt{\log p/n}) = O_p(\sqrt{\log p/n}) = o_p(1).$$
3. In a factor model setting, Theorem 6 of Fan et al. (2008) shows the variance estimation error tends to zero in probability when $n > p$.

### 3.2 Estimating Gross Exposure

In this subsection we estimate the gross exposure of portfolios based on the nodewise estimator of the inverse covariance matrix. We begin with Theorem 3.3 by showing that we can consistently estimate the weights of the global minimum variance portfolio in a high dimensional setting with $p > n$.

**Theorem 3.3.** Under Assumptions 1,3 and the sparsity assumption $(\max_j s_j)^{3/2} \sqrt{\log p}/n = o(1)$, with $\lambda_j = O(\sqrt{\log p}/n)$ uniformly in $j$, we have

$$\|\hat{w}_u - w_u\|_1 = O_p((\max_j s_j)^{3/2} \sqrt{\log p}/n) = o_p(1).$$

**Remarks:**

1. Note that we have a very mild sparsity assumption. Instead of Assumption 2, our sparsity Assumption here needs an extra $\sqrt{s_j}$ due to approximation error for $\hat{\Theta}$ in $\ell_1$ norm in our proof. Note that even though $p > n$, $\max_j s_j$ has to be constrained so that this result in Theorem 3.3 holds true. Next we consider what if this constraint is relaxed.

2. One important case is non-sparse inverse covariance matrix. In other words, what if for all $j = 1, \cdots, p$, $s_j = p - 1$? This is the case with all nonzero cells in inverse of covariance matrix. Then of course, $p$ has to be less than $n^{1/3}$, meaning $p(\log p)^{1/3} = o(n^{1/3})$ so that we satisfy the assumption: $(\max_j s_j)^{3/2} \sqrt{\log p}/n = (p - 1)^{3/2} \sqrt{\log p}/n = o(1)$.

3. We consider whether we have growing or finite gross exposure, in other words we need to analyze $\|w_u\|_1$. By (A.60) and a simple eigenvalue inequality

$$\|w_u\|_1 = O(\max_j \sqrt{s_j}),$$

which may grow with $n$. In this sense, by Theorem 3.3 consistent estimation of portfolio weights is possible even in the case of growing exposure.

But if we further assume $\max_j s_j = O(1)$, meaning if the number of nonzero elements in each row of inverse variance matrix is finite then we have $\|w_u\|_1 = O(1)$. So finite gross exposure is possible with one extra assumption. With finite gross exposure Theorem 3.3 result is

$$\|\hat{w}_u - w_u\|_1 = O_p(\sqrt{\log p}/n) = o_p(1).$$

This means that with finite gross exposure we get a better approximation compared with rate in Theorem 3.3.
4. In Remark 3 above we find rate of approximation for weights in the case of constant exposure. Fan et al. (2015), by using factor models, estimates $\Sigma$ by a sparse structure in rows and then estimates $\Sigma^{-1}$. Basically, applying $\hat{\Sigma}^{-1}$ to weight estimation, p.372 of Fan et al. (2015) has derived the same rate as Remark 3 above, as long as maximum number of non vanishing elements in each row in $\Sigma$ is finite (i.e. case of parameter $q = 0$ in Assumption 4.2 of Fan et al. (2015)).

We now turn to the estimator of the gross exposure of the portfolio, $\|w^*\|_1$. The estimator is $\hat{w}$, where we use $\ell_1$ norm as in Fan et al. (2015) among others. The next Theorem established consistency of the portfolio gross exposure estimator with $p > n$, it is one of our main results and a novel contribution to the literature.

Theorem 3.4. Under Assumptions, 1, 3 with $(\max_j s_j)^{3/2} \sqrt{\log p/n} = o(1)$ and assuming uniformly in $j$, $\lambda_j = O(\sqrt{\log p/n})$ with $p^{-2}(AD - B^2) \geq C_1 > 0$, where $C_1$ is a positive constant, and $\rho_1$ being bounded as well, we have

$$\|\hat{w} - w^*\|_1 = O_p((\max_j s_j)^{3/2} \sqrt{\log p/n}) = o_p(1).$$

Remarks:

1. To see whether we allow for growing gross exposure or not we need to know the rate of $\|w^*\|_1$. Note that

$$w^* = \frac{D - \rho_1 B}{AD - B^2} \Theta_1 p + \frac{\rho_1 A - B}{AD - B^2} \Theta \mu.$$

See that

$$\|w^*\|_1 \leq \frac{|D - \rho_1 B| \|\Theta_1 p\|_1}{p^2 C_1} + \frac{|\rho_1 A - B| \|\Theta \mu\|_1}{p^2 C_1} = O(\max_j s_j).$$

With (A.60)-(A.63) we have $\|\Theta_1 p\|_1 = O(p \sqrt{\max_j s_j})$, $\|\Theta \mu\|_1 = O(p \sqrt{\max_j s_j})$, and for the other terms in the numerator we use Lemma A.4 $A = O(p)$, $D = O(p)$, $|B| = O(p)$, with $\rho_1$ being bounded and also we use $AD - B^2 \geq p^2 C_1$ for the denominator. This shows that we have growing gross exposure.

2. With the Assumptions 1 and 3, with imposing now $\max_{1 \leq j \leq p} s_j = O(1)$, $\sqrt{\log p/n} = o(1)$, $p^{-2}(AD - B^2) \geq C_1 > 0$, $\rho_1$ being bounded and $\max_j \lambda_j = O(\sqrt{\log p/n})$ leading to finite gross exposure, as can be seen from the proof of Theorem 3.4, and

$$\|\hat{w} - w^*\|_1 = O_p(\sqrt{\log p/n}) = o_p(1).$$

So the rate of approximating the optimal portfolio improves greatly. This result shows even for very large portfolios we can estimate the weights successfully controlling the error.

3. Here, we show that with all possible nonzero coefficients in rows of $\Sigma^{-1}$, and growing exposure,
(for all $j = 1, \cdots, p$: $s_j = p - 1$), the rate in Theorem 3.4 is:

$$\|\hat{w} - w^*\|_1 = O_p(p^{3/2} \sqrt{\log p / n}),$$

which means we need $p^{3/2} \sqrt{\log p / n} = o(1)$. This result shows that $p \log p^{1/3} = o(n^{1/3})$, so $p$ should be much less than $n$ in this case.

### 3.3 Estimating Risk Error of a Large Portfolio

In this subsection we are interested in estimating the risk error of a large portfolio. This error is defined as $|\hat{w}'(\hat{\Sigma} - \Sigma)\hat{w}|$ and $|\hat{w}'(\hat{\Sigma} - \Sigma)\hat{w}|$ for global minimum variance portfolio, and Markowitz portfolios respectively, where $p > n$. This is defined in section 3.2 of Fan et al. (2015) by using factor model based approach with constant exposure. Here, we want to consider what happens when we have growing exposure, and with relaxed inverse of covariance matrix without a factor structure for returns. However, we still make a sparsity assumption on inverse of covariance matrix, whereas Fan et al. (2015) does not, but has sparsity on residual error matrix.

The following theorem provides risk error for a large portfolio with growing exposure in case of global minimum variance portfolio. We are still able to show that this error will converge in probability to zero, but is affected by maximum of number of nonzero elements across rows in the inverse of covariance matrix.

**Theorem 3.5.** Under Assumptions 1,3 and the sparsity assumption $(\max_j s_j)^{3/2} \sqrt{\log p / n} = o(1)$, with $\lambda_j = O(\sqrt{\log p / n})$ uniformly in $j$, we have

$$|\hat{w}'(\hat{\Sigma} - \Sigma)\hat{w}| = O_p(\max_j s_j \sqrt{\log p / n}) = o_p(1).$$

**Remarks:**

1. This is a new result, and shows the tradeoff between number of assets, sample size, and $\max_j s_j$. We allow for $p > n$, as well as growing exposure, $\|w_u\|_1 = O(\max_j \sqrt{s_j})$.

2. When we have constant exposure $\|w_u\|_1 = O(1)$, then we have

$$|\hat{w}'(\hat{\Sigma} - \Sigma)\hat{w}| = O_p(\sqrt{\log p / n}) = o_p(1),$$

which is the same result as in Fan et al. (2015).

Next we conduct the same analysis for the Markowitz based portfolio, in case of growing exposure. We are still able to show, even in the case of extreme positions in the portfolio, the risk estimation error converges to zero in probability.
Theorem 3.6. Under Assumptions 1, 3 with \( (\max_j s_j)^{3/2} \sqrt{\log p/n} = o(1) \) and assuming uniformly in \( j \), \( \lambda_j = O(\sqrt{\log p/n}) \) with \( p^{-2} (AD - B^2) \geq C_1 > 0 \), where \( C_1 \) is a positive constant, and \( \rho_1 \) being bounded as well, we have

\[
|\hat{w}'(\hat{\Sigma} - \Sigma)\hat{w}| = O_p(\max_j s_j \sqrt{\log p/n}) = o_p(1).
\]

Remark. This is a new result, and again shows that we cannot have large \( \max_j s_j \). In the case of constant exposure we get the result also in Remark 2 of global minimum variance portfolio case above.

4 Asset Returns with Bounded Moments

In this section we relax assumption of the sub-Gaussianity of the asset returns. We build our results on the work of Caner and Kock (2014). Assumptions 1-2 are modified and Assumption 3 is not changed. The new assumptions involve bounded moments and a more restrictive sparsity assumption, so there is a trade-off.

Assumption 1*. The excess asset return vector \( r_i \) is iid, but \( \max_{1 \leq j \leq p} E|r_{ij}|^l \leq C \), with \( l > 4 \).

Assumption 2*. The sparsity condition is

\[
(\max_j s_j) \frac{p^{2/l}}{n^{1/2}} = o(1).
\]

Caner and Kock (2014) shows that, uniformly in \( j \), \( \lambda_j = O(p^{2/l}/n^{1/2}) \) through their Lemma 2. In sub-Gaussian setting, the tuning parameter was of order \( (\log p)^{1/2}/n^{1/2} \), this shows the price of relaxing the sub-Gaussianity of the returns: we allow fewer assets in our portfolio. In the Appendix B we show which parts of the proofs in Theorems 3.1-3.6 are affected. Also see that our tuning parameter choice affects Assumption 2*, where compared to Assumption 2 we replace \( (\log p)^{1/2} \) with \( p^{2/l} \).

Theorem 4.1. Under Assumptions 1*,2*,3, assuming uniformly in \( j \), \( \lambda_j = O(p^{2/l}/\sqrt{n}) \)

\[
\left| \frac{\hat{\Phi}_G}{\Phi_G} - 1 \right| = o_p(1).
\]
Theorem 4.2. Under Assumptions 1*, 2*, 3, and assuming uniformly in $j$, $\lambda_j = O\left(\frac{p^2/l}{\sqrt{n}}\right)$ with $p^{-2}(AD - B^2) \geq C_1 > 0$, where $C_1$ is a positive constant, $p^{-1}(A\rho_1^2 - 2B\rho_1 + D) \geq C_1 > 0$, $\rho_1$ is bounded, we get

$$\left| \frac{\hat{\Psi}_{OPV}}{\Psi_{OPV}} - 1 \right| = o_p(1).$$

Note that counterparts of Corollaries 3.1-3.2 can be shown with ease as long as we have $p^{2/l}n^{1/2} = o(1)$. So again a nonsparse result, without Assumption 2* is possible.

Now we provide counterparts to Theorems 3.3-3.4, in terms of estimating gross exposure of the portfolios.

Theorem 4.3. Under Assumptions 1*, 3 and the sparsity assumption $\left(\max_j s_j\right)^{3/2} \frac{p^{2/l}}{\sqrt{n}} = o(1)$, assuming uniformly in $j$, $\lambda_j = O\left(\frac{p^{2/l}}{\sqrt{n}}\right)$ we have

$$\|\hat{w}_u - w_u\|_1 = O_p\left(\left(\max_j s_j\right)^{3/2} \frac{p^{2/l}}{\sqrt{n}}\right) = o_p(1).$$

Note that we see here it is possible to have $p > n$ in Theorem 3.3 as long as $l$ is large, or $\max_j s_j = O(1)$. Theorem 3.4 allows also $p > n$.

Theorem 4.4. Under Assumptions, 1*, 3, with $\left(\max_j s_j\right)^{3/2} \frac{p^{2/l}}{\sqrt{n}} = o(1)$, and assuming uniformly in $j$, $\lambda_j = O\left(\frac{p^{2/l}}{\sqrt{n}}\right)$ with $p^{-2}(AD - B^2) \geq C_1 > 0$ where $C_1$ is a positive constant and $\rho_1$ being bounded as well, we have

$$\|\hat{w} - w^*\|_1 = O_p\left(\left(\max_j s_j\right)^{3/2} \frac{p^{2/l}}{\sqrt{n}}\right) = o_p(1).$$

We provide the counterparts of Theorems 3.5-3.6 of risk estimation errors, in the case of data with bounded moments.

Theorem 4.5. Under Assumptions 1*, 3 and the sparsity assumption $\left(\max_j s_j\right)^{3/2} \frac{p^{2/l}}{\sqrt{n}} = o(1)$, assuming uniformly in $j$, $\lambda_j = O\left(\frac{p^{2/l}}{\sqrt{n}}\right)$ we have

$$|\hat{w}_u'(\hat{\Sigma} - \Sigma)\hat{w}_u| = O_p\left(\left(\max_j s_j\right) \frac{p^{2/l}}{\sqrt{n}}\right) = o_p(1).$$

Theorem 4.6. Under Assumptions, 1*, 3, with $\left(\max_j s_j\right)^{3/2} \frac{p^{2/l}}{\sqrt{n}} = o(1)$, and assuming uniformly in $j$, $\lambda_j = O\left(\frac{p^{2/l}}{\sqrt{n}}\right)$ with $p^{-2}(AD - B^2) \geq C_1 > 0$ where $C_1$ is a positive constant and $\rho_1$ being
bounded as well, we have

$$|\hat{w}'(\hat{\Sigma} - \Sigma)\hat{w}| = O_p((\max_j s_j \frac{p^{2/l}}{\sqrt{n}}) = o_p(1).$$

The main difference between Theorems 3.1-3.6 and Theorems 4.1-4.6 is that we replace $\sqrt{\log p}$ everywhere in approximations with $\frac{p^2}{l}$, $l > 4$, as consequence of the rate of the tuning parameter in non-sub Gaussian returns being: $\max_j \lambda_j = O(p^{2/l}/n^{1/2})$

5 Simulations

This section begins by discussing the implementation of the nodewise estimator, as well as alternative estimators. We then present the setup of this simulation study and follow by discussing the results.

5.1 Implementation of the nodewise estimator

The nodewise regression approach is implemented using the coordinate descent algorithm implemented in the \textit{glmnet} package (Friedman et al., 2010). The penalty parameter $\lambda_j$, $j = 1, \cdots, p$ is chosen by minimizing the Bayesian information criterion of Wang et al. (2009):

$$BIC(\lambda_j) = \log(\hat{\sigma}^2_{\lambda_j}) + \left| \hat{S}_j(\lambda_j) \right| \frac{\log(n)}{n} \log(\log(p)), \quad (5.1)$$

where $\hat{\sigma}^2_{\lambda_j} = ||r_j - \hat{r}_j \hat{\gamma}_j||_2^2/n$ is the residual variance for asset $j$ and $\left| \hat{S}_j(\lambda_j) \right|$ represents the estimated number of nonzero cells in $(p-1)$ vector of $\hat{\gamma}_j$ in a given nodewise regression. Alternative information criterion were investigated, including those proposed by Caner et al. (2017), Zou et al. (2007) yielding only minor differences in the results. The entries of the covariance matrix are estimated equation by equation and a different penalty parameter is selected for each equation.

5.2 Alternative Covariance Matrix Estimation methods

5.2.1 Ledoit-Wolf Shrinkage

Ledoit and Wolf (2004) proposed an estimator for high-dimensional covariance matrices that is invertible and well-conditioned. Their estimator is a linear combination of the sample covariance matrix and an identity matrix, and they showed that their estimator is asymptotically optimal with respect to the quadratic loss function. The properties of this covariance matrix estimator, and of its inverse, when used to construct portfolios have not been investigated in the finance literature.
5.2.2 Multi-factor Estimator

The Arbitrage Pricing Theory is derived by Ross (1976, 1977) and the multi-factor models are proposed by Chamberlain and Rothschild (1983). These studies have motivated the use of factor models for the estimation of excess return covariance matrices. The model takes the form

\[ r_F = Bf + \varepsilon, \]  

(5.2)

where \( r_F \) is a matrix of dimensions \( p \times n \) of excess return of the assets over the risk-free interest rate and \( f \) a \( K \times n \) matrix of factors. \( B \) is a \( p \times K \) matrix of unknown factor loadings and \( \varepsilon \) is the matrix of idiosyncratic error terms uncorrelated with \( f \). This model yields an estimator for the covariance matrix of \( r_F \):

\[ \Sigma_{FAC} = B \text{cov}(f) B' + \Sigma_{n,0}, \]  

(5.3)

where \( \Sigma_{n,0} \) is the covariance matrix of errors \( \varepsilon \). When the factors are observed, as in Fan et al. (2008), the matrix of loadings \( B \) can be estimated by least squares. In the case where the factors are unobserved, Fan et al. (2013, 2015) proposed the POET, to estimate \( \Sigma_{FAC} \). In our simulations we compare the nodewise regression approach to the POET estimator as in Fan et al. (2015), which we describe below.

Let \( \hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_p \) to be ordered eigenvalues of the sample covariance matrix \( \hat{\Sigma} \) and \( \hat{\xi}_{j,1}^{\hat{\cdot}} \) to be its corresponding eigenvectors. The estimated covariance matrix \( \hat{\Sigma}_{POET} \) is defined as:

\[ \hat{\Sigma}_{POET} = \sum_{j=1}^{K} \hat{\lambda}_j \hat{\xi}_j \hat{\xi}'_j + \hat{\Omega}, \]  

(5.4)

\[ \hat{\Omega}_{ij} = \begin{cases} \frac{\sum_{k=K+1}^{p} \hat{\lambda}_k \hat{\xi}_k^2 i^i}{s_{ij} \left( \sum_{k=K+1}^{p} \hat{\lambda}_k \hat{\xi}_k \hat{\xi}_j \right)}, & i = j, \\ \left(\sum_{k=K+1}^{p} \hat{\lambda}_k \hat{\xi}_k \hat{\xi}_j \right), & i \neq j, \end{cases} \]  

(5.5)

where \( \hat{\Omega} = (\hat{\Omega}_{ij})_{p \times p} \) and \( s_{ij}(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) is the soft-thresholding function \( s_{ij} = (z - \tau_{ij})_+ \) (Antoniadis and Fan, 2001; Rothman et al., 2009; Cai and Liu, 2011), and \( (\cdot)_+ \) represents the larger of zero and a scalar with \( z \) denoting the \( \{ij\} \) th cell in the standard residual sample covariance matrix of \( \varepsilon \). The residual is defined as least squares estimator residual in the factor model. The entry dependent thresholding parameter is

\[ \tau_{ij} = \zeta \sqrt{\hat{\Omega}_{ii} \hat{\Omega}_{jj}} \left( \frac{\sqrt{\log p}}{n} + \frac{1}{\sqrt{p}} \right), \]  

(5.6)

where \( \zeta > 0 \) is a user-specified positive constant to maintain the finite sample positive definiteness of \( \hat{\Omega} \). In our implementation we initialize at \( \zeta = 0 \), and increase \( \zeta \) by increments of 0.1 until \( \hat{\Omega} \) is invertible and well conditioned.
The number of factors \( \hat{K} \) is chosen using the information criterion of Bai and Ng (2002)

\[
\hat{K} = \text{argmin}_{0 \leq k \leq M} \frac{1}{p} \text{tr} \left( \sum_{j=k+1}^{p} \hat{\lambda}_j \hat{\xi}_j \hat{\xi}_j' \right) + \frac{k(p + n)}{pn} \log \left( \frac{pm}{p + n} \right),
\]

where \( M \) is a user-defined upper bound which we set to \( M = 7 \).

5.3 Simulation Setup

In every simulation we use \( n = 252 \) observations, corresponding to one year of daily returns, and generate \( p \) assets, with \( p = 50, 100, ..., 300, 400, ..., 1000 \). For each value of \( p \) we perform 1000 replications. Computations are carried using \( R 3.3.3 \) (R Core Team, 2015), and are fully reproducible.

5.3.1 Data Generating Processes

We consider four data generating processes (DGP), one based on a three factor model, two based on randomly generated covariance matrices, and one based on sparse random Cholesky factors. All of them have non-sparse inverse covariance matrix when we checked DGPs. So all of these designs violate our sparsity assumption. In this sense we also see how the non-sparsity of inverse matrix affects our results.

The factor based DGP, noted Factor, is a replication of the data generating process of Fan et al. (2008). The process is based on 3 randomly generated factors and loadings with a Gaussian distribution and known moments. The same factors are used for every value of \( p \) considered, new loadings and standard deviations of the innovations are generated for each value of \( p \). The innovations are \( \text{iid} \) Gaussian with Gamma distributed standard errors.

The two random covariance matrix DGPs use the estimated moments (noted \( \mu \) for the mean and \( \sigma^2 \) for the variance) of the excess returns from 466 stocks of the S&P 500 observed from March 17, 2011 to March 17, 2016 (1257 observations), using the 13 weeks (3 month) treasury bill to compute the risk free returns. The two first random covariance matrix DGPs use non-sparse covariance matrices with returns generated using either a Gaussian distribution or a Student \( t \) distribution with 9 degrees of freedom.

The algorithm we use to generate means and covariance matrices for the returns is as follows:

1. Generate \( p \) means for the excess returns, \( \mu_{RCV} \), from \( N(\mu, \sigma^2) \).
2. Generate \( p \) standard deviations \( \sigma_1, ..., \sigma_p \) for the errors by sampling from a Gamma distribution \( G(\alpha, \beta) \) where \( \alpha \) and \( \beta \) are estimated from the S&P data.
3. Generate off-diagonal elements for the covariance matrix by drawing from a Gaussian distribution while ensuring symmetry.
4. Ensure positive definiteness of the constructed covariance matrix by eigenvalue cleaning as in Callot et al. (2016); Hautsch et al. (2012). To implement this procedure, perform a spectral decomposition of the form \( \Sigma = V'AV \) where \( V \) is the matrix of eigenvectors and \( A \) the (diagonal) matrix of eigenvalues, replace the negative eigenvalues as well as those below \( 10^{-6} \) by the smallest eigenvalue larger than \( 10^{-6} \) yielding \( \tilde{A} \), and construct \( \tilde{\Sigma} = V'\tilde{A}V \).

5. Generate the matrix of excess returns by drawing from \( N(\mu_{RCV}, \tilde{\Sigma}) \) or by drawing from a multivariate \( t \) distribution with 9 degrees of freedom and covariance matrix \( \tilde{\Sigma} \) to which the mean returns \( \mu_{RCV} \) is then added.

The final DGP uses sparse Cholesky factors to generate the covariance matrix, returns are then generated using a multivariate Gaussian distribution with mean \( \mu_{RCV} \) as above. The Cholesky factors are generated by filling the off-diagonal entries of the matrix with either zeros (with probability 80%) or random numbers uniformly distributed between -0.1 and 0.1. The diagonal is filled with random numbers uniformly distributed between 0 and 1.

5.3.2 Portfolio Weights Estimation

We consider the global minimum variance portfolio as well as the mean-variance portfolios introduced by Markowitz (1952). For the Markowitz portfolios we use a daily return target of 0.0378%, corresponding to a 10% return when cumulated over 252 days. Closed form solutions for the portfolio allocation vectors and its variance are given in section 3.1.2 for the Markowitz portfolio, and in section 3.1.1 for the global minimum variance portfolio.

5.3.3 Reported Measures

Our results are reported in five figures with the number of assets \( p \) on the horizontal axis and the median over 1000 iterations of one of the five following statistics on the vertical axis.

- Figure 1: is the absolute ratio of estimated to true variances minus one defined in Theorems 3.1 and 3.2.

- Figure 2: the estimation error of estimated portfolio variance \( \hat{\Phi}_G \) for the global minimum variance portfolio and \( \hat{\Psi}_{OPV} \) for the Markowitz portfolio in Corollary 3.1, Corollary 3.2.

- Figure 3: the absolute risk error as defined in Theorems 3.5 and 3.6.

- Figure 4: the absolute difference between portfolio weights based on the estimated and true covariance matrices defined in Theorems 3.3 and 3.4.

- Figure 5: the estimated portfolio exposure \( \|\hat{w}\|_1 \).
5.4 Simulation Results

Figure 1 shows that the ratio of the estimated to the true portfolio variances (minus one) is stable when the weights are based on the Nodewise estimator under the factor DGP and the sparse Cholesky factor. This variance ratio (minus one) is increasing under the two random covariance DGPs for all estimators but the increase is faster for the POET estimator. The Ledoit-Wolf estimator performs poorly under the Cholesky factor DGP. We note that our nodewise regression based portfolio variance ratio (minus one) approaches zero as expected in Cholesky factor DGP.

In case of portfolio variance estimation errors, our nodewise regression based variance error converges to zero fast unlike other methods in Figure 2. POET estimator based variance error does not perform well under Gaussian data, and $t$ distributed data. Ledoit-Wolf method does not do well under Cholesky factor based DGP.

Figure 3 clearly shows the superior performance of our nodewise regression based estimator in terms of large portfolio risk. We see that POET estimator’s risk error converges to zero slower than the other methods except Cholesky factor based DGP.

Figure 4 shows that the portfolio weights estimation error by computed using the POET estimator is lower than weights computed using alternative estimators under the Factor DGP. Under the other three DGPs the POET error is always higher than Nodewise errors, the Ledoit-Wolf based weights are the most accurate under the non-sparse DGPs though the difference with the Nodewise estimator decreases when the number of assets increases. The Ledoit-Wolf estimator performs poorly under the Sparse Cholesky factor DGP.

Figure 5 shows that the Nodewise estimator yields portfolios with an exposure close to 1 under every DGP and for any number of assets. The exposure of portfolios based on the Ledoit-Wolf estimator is as high as 5 under the factor DGP and is only marginally higher than that of the Nodewise based portfolios under the other DGPs. The exposure of portfolios based on the POET estimator is consistently higher under the factor DGP as well as under the other DGPs though it decreases rapidly when the number of assets increases.

Overall, we see good performance of our estimator, it is mostly robust to several DGP’s even though all DGP’s violate the sparsity Assumption 2. POET, on the other hand, as expected performs well with factor model based DGP, but not so well with others. Ledoit-Wolf based estimator does well in Gaussian and $t$ based data, but not in factor model and sparse Cholesky factor based DGP.

6 Empirics

6.1 Performance Measures

In this section we perform an out-of-sample forecasting exercise. We focus on four metrics commonly used in finance. These metrics are the Sharpe ratio (SR from now on), portfolio turnover, and the average returns and variances of portfolios. We compare our approach against the POET estimator
Figure 1: Portfolio Variance Ratio Minus One, Median Absolute Value
Figure 2: Portfolio Variance Estimation Error, Median Absolute Value
Figure 3: Portfolio Risk Error, Median Absolute Value
Figure 4: Portfolio Weights Estimation Error, Median Absolute Value
Figure 5: Portfolio Exposure, Median Absolute Value
and Ledoit and Wolf (2004) based estimator. We consider portfolio formation with and without transaction costs.

Note that variance of large portfolio is analyzed in this paper, hence it will be interesting to see how our method performs against POET and Ledoit and Wolf (2004) based estimator. The SR, portfolio turnover, average returns are not analyzed in our paper theoretically, they need an entirely different mathematical setup, and technique. We will analyze SR in a future project.

We use a rolling horizon method for out-of-sample forecasting. The samples of length \( n \) are split into a training part, in-sample indexed \((1 : n_I)\) and a testing, or out-of-sample, part indexed \((n_I + 1 : n)\). The rolling window method works as follows: the portfolio weights are calculated in-sample for the period in between \((1 : n_I)\) and denoted as \( \hat{w}_{n_I} \), then this is multiplied by the return in \( n_I + 1 \) period to have the forecast portfolio return for \( n_I + 1 \) period: \( \hat{w}_{n_I}^t r_{n_I+1} \). Then we roll the window by one period and form the portfolio weight for the period: \((2 : n_I + 1)\) which we denote \( \hat{w}_{n_I+1} \). This is again multiplied by returns at \( n_I + 2 \) period to get the forecast for \( n_I + 2 \) period: \( \hat{w}_{n_I+1}^t r_{n_I+2} \). In this way, we go until (including) \( n - 1 \) period and get \( \hat{w}_{n-1}^t r_n \). So, in the case of no transaction costs, out-of-sample average portfolio returns across all out of sample observations is:

\[
\hat{\mu}_{os} = \frac{1}{n - n_I} \sum_{t=n_I}^{n-1} \hat{w}_t^t r_{t+1},
\]

and variance for out of sample is:

\[
\hat{\sigma}_{os}^2 = \frac{1}{(n - n_I) - 1} \sum_{t=n_I}^{n-1} (\hat{w}_t^t r_{t+1} - \hat{\mu}_{os})^2.
\]

We analyze these two measures above, average return and the variance in our Tables. Next measure is SR:

\[
SR = \hat{\mu}_{os}/\hat{\sigma}_{os}.
\]

For transaction cost based Sharpe ratio, let \( c \) be the proportional transaction cost. This is chosen to be 50 basis points in DeMiguel et al. (2009b). Excess portfolio return at time \( t \) with transaction cost is (see Ban et al. (2016)),

\[
Return_t = \hat{w}_t^t r_{t+1} - c(1 + \hat{w}_t^t r_{t+1}) \sum_{j=1}^p |\hat{w}_{t+1,j} + \hat{w}_{t,j}^+|,
\]

where \( \hat{w}_{t,j}^+ = \hat{w}_{t,j}(1 + R_{t+1,j})/(1 + R_{t+1,p}) \) and \( R_{t+1,j} \) is the excess return added to risk free rate for \( j \)th asset, and \( R_{t+1,p} \) is the portfolio excess return plus risk free rate. For this definition, see Li (2015).

The SR with transaction costs is:

\[
SR_c = \hat{\mu}_{os,c}/\hat{\sigma}_{os,c}.
\]
where
\[ \hat{\mu}_{os,c} = \frac{1}{n - n_I} \sum_{t=n_I}^{n-1} Return_t, \]
and
\[ \hat{\sigma}^2_{os,c} = \frac{1}{(n - n_I) - 1} \sum_{t=n_I}^{n-1} (Return_t - \hat{\mu}_{os,c})^2. \]

The next measure that we analyze is portfolio turnover (PT):
\[ PT = \frac{1}{n - n_I} \sum_{t=n_I}^{n-1} \sum_{j=1}^{p} |\hat{w}_{t+1,j} - \hat{w}^+_{t,j}|. \]

6.2 Data

We use daily and monthly empirical datasets of S&P500 index with major assets. We use two different periods each for daily and monthly data. We have also tried other possibilities and time periods, but our tables are representative and similar to other period results.

1. First Monthly Data-Table 1: Full Sample: January 2000 to March 2017 with \( n = 207 \) and \( p = 384 \).
   a) In-Sample period 1: January 2000-March 2011 (\( n_I = 135 \)), Out-Of-Sample 1: April 2011-March 2017 (\( n - n_I = 72 \)).
   b) In-Sample Period 2: January 2000-March 2014 (\( n_I = 171 \)), Out-Of-Sample 2: April 2014-March 2017 (\( n - n_I = 36 \)).

2. Second Monthly Data-Table 2: Full Sample: January 2000 to December 2013 with \( n = 168 \) and \( p = 384 \).
   a) In-Sample period 1: January 2000-March 2007 (\( n_I = 96 \)), Out-Of-Sample 1: January 2008-December 2013 (\( n - n_I = 72 \)).
   b) In-Sample Period 2: January 2000-March 2007 (\( n_I = 96 \)), Out-Of-Sample 2: January 2008-December 2010 (\( n - n_I = 36 \)).

3. First Daily Data-Table 3: Full Sample: February 4 2013 to March 31 2017 with \( n = 1047 \) and \( p = 458 \).
   a) In-Sample period 1: February 4 2013-March 31 2016 (\( n_I = 795 \)), Out-Of-Sample 1: April 1 2016-March 31 2017 (\( n - n_I = 252 \)).
   b) In-Sample Period 2: February 4 2013-September 30 2016 (\( n_I = 921 \)), Out-Of-Sample 2: October 3 2016-March 31 2017 (\( n - n_I = 126 \)).

4. Second Daily Data-Table 4: Full Sample: February 4 2013 to August 8 2015 with \( n = 640 \) and \( p = 458 \).
a) In-Sample period 1: February 4 2013-August 18 2014 \((n_I = 388)\), Out-Of-Sample 1: August 19 2014-August 18 2015 \((n - n_I = 252)\).

b) In-Sample Period 2: February 4 2013-February 18 2015 \((n_I = 514)\), Out-Of-Sample 2: February 19 2015-August 18 2015 \((n - n_I = 36)\).

The idea of first period of monthly data is: we cover all available months that we can obtain starting 2000. With the second monthly sample, we evaluate portfolios in the context of a recession and of its aftermath. For daily data, we use available data for a large number of assets. The idea of the first daily data is to forecast last year or six months of daily data based on in-sample period. In the second daily data, the span of data is shorter than the first daily data period, and includes a period where out-of-sample daily data has negative returns.

The portfolios are rebalanced on a daily and monthly basis. At each rebalancing point, the expected return vector and the covariance matrices are re-estimated. For instance, for a six-year \((n - n_I = 72)\) rolling window forecast horizon, we estimate expected returns and covariance matrices and formulate global minimum and Markowitz portfolios 72 times. Portfolios are held for one month and rebalanced at the beginning of the next month. As a return constraint, we use monthly target of 0.7974% and daily target of 0.0378% which are equivalent of 10% return for a year when compounded.

6.3 Results

We report the global minimum and Markowitz portfolio empirical results with and without transaction costs (TC in tables) based on the POET, Nodewise and Ledoit-Wolf estimators.

Table 1 reports monthly portfolio performances for the full evaluation period of January 2000 - March 2017. According to the results, Nodewise based portfolios without transaction costs provide the highest Sharpe Ratio (SR) and the lowest portfolio variance in six-year \((n - n_I = 72)\) out-of-sample period. In Table 1, Ledoit-Wolf estimator has the highest variance levels for all cases. Also, Nodewise based portfolios are generally associated with higher portfolio turnover in Table 1. With transaction costs at Table 1, SR of POET is the best in the second part of Table that covers three year out-of-sample period \((n - n_I = 36)\), but the difference between Nodewise and POET is not that large.

Table 2 represents the out-of-sample results of January 2008 - December 2013 and January 2008 - December 2010 forecast horizon periods. Nodewise estimator based portfolios yield better out-of-sample portfolio variances in portfolios with transaction costs than the other estimators generally. Without transaction costs, POET performs lower out-of-sample variance results. However, the difference between Nodewise and POET estimators at variances is not large. Ledoit-Wolf based portfolios have the highest variance levels for both global minimum and Markowitz portfolios but better SR.

Table 3 indicates the first daily portfolio performances. Performance results of Nodewise estimator based portfolios are more striking in terms of SR for portfolios with transaction cost for
|                     | Global Minimum Portfolio | Markowitz Portfolio |
|---------------------|--------------------------|---------------------|
|                     | Return | Variance | Sharpe | Turnover | Return | Variance | Sharpe | Turnover |
| **In-Sample: Jan 2000-Mar 2011, Out-Of-Sample: Apr 2011-Mar 2017, n_I = 135, n - n_I = 72** |                       |                     |
| **without TC**      | POET   | 0.008701 | 0.0008940 | 0.2910   | 0.09777 | 0.008008 | 0.0008958 | 0.2676 | 0.11961 |
|                     | Nodewise | 0.008894 | 0.0008830 | 0.2903   | 0.13829 | 0.008116 | 0.0008825 | 0.2732 | 0.15773 |
|                     | Ledoit-Wolf | 0.009683 | 0.0014824 | 0.2515   | 0.04718 | 0.008725 | 0.0014975 | 0.2255 | 0.04808 |
| **with TC**         | POET   | 0.008376 | 0.0008849 | 0.2816   | -       | 0.007543 | 0.0008883 | 0.2531 | -       |
|                     | Nodewise | 0.008362 | 0.0008916 | 0.2800   | -       | 0.007457 | 0.0008921 | 0.2497 | -       |
|                     | Ledoit-Wolf | 0.009683 | 0.0014824 | 0.2515   | -       | 0.008662 | 0.0015152 | 0.2225 | -       |
| **In-Sample: Jan 2000-Mar 2014, Out-Of-Sample: Apr 2014-Mar 2017, n_I = 171, n - n_I = 36** |                       |                     |
| **without TC**      | POET   | 0.006152 | 0.0006870 | 0.2347   | 0.05755 | 0.005714 | 0.0007088 | 0.2146 | 0.0746  |
|                     | Nodewise | 0.006221 | 0.0007406 | 0.2286   | 0.15019 | 0.005771 | 0.0007605 | 0.2093 | 0.1624  |
|                     | Ledoit-Wolf | 0.005703 | 0.0009777 | 0.1824   | 0.04663 | 0.005699 | 0.0009774 | 0.1823 | 0.0467  |
| **with TC**         | POET   | 0.006180 | 0.0007022 | 0.2332   | -       | 0.005626 | 0.0007271 | 0.2086 | -       |
|                     | Nodewise | 0.005751 | 0.0007574 | 0.2090   | -       | 0.005197 | 0.0007799 | 0.1861 | -       |
|                     | Ledoit-Wolf | 0.005745 | 0.0010032 | 0.1814   | -       | 0.005741 | 0.0010029 | 0.1813 | -       |

Table 1: Monthly Portfolio Performance: First Monthly Results

|                     | Global Minimum Portfolio | Markowitz Portfolio |
|---------------------|--------------------------|---------------------|
|                     | Return | Variance | Sharpe | Turnover | Return | Variance | Sharpe | Turnover |
| **In-Sample: Jan 2000-Dec 2007, Out-Of-Sample: Jan 2008-Dec 2013, n_I = 96, n - n_I = 72** |                       |                     |
| **without TC**      | POET   | 6.971e-03 | 0.002257 | 0.1467184 | 0.48679 | 0.007094 | 0.002088 | 0.15523 | 0.49969 |
|                     | NodeWise | 7.842e-03 | 0.002287 | 0.1639747 | 0.15139 | 0.007971 | 0.002252 | 0.16797 | 0.19371 |
|                     | Ledoit-Wolf | 4.537e-02 | 0.056525 | 0.1908248 | 0.08746 | 0.044104 | 0.052465 | 0.19255 | 0.08715 |
| **with TC**         | POET   | 4.498e-03 | 0.002391 | 0.0919897 | -       | 0.004579 | 0.002222 | 0.09715 | -       |
|                     | NodeWise | 7.038e-03 | 0.002315 | 0.1462547 | -       | 0.006980 | 0.002281 | 0.14615 | -       |
|                     | Ledoit-Wolf | 4.509e-02 | 0.056269 | 0.1900708 | -       | 0.043816 | 0.052232 | 0.19172 | -       |
| **In-Sample: Jan 2000-Dec 2007, Out-Of-Sample: Jan 2008-Dec 2010, n_I = 96, n - n_I = 36** |                       |                     |
| **without TC**      | POET   | 0.0015308 | 0.003489 | 0.025915 | 0.83965 | 0.0002891 | 0.002871 | 0.005396 | 0.84461 |
|                     | NodeWise | 0.0024552 | 0.003592 | 0.040963 | 0.16816 | 0.0006144 | 0.003023 | 0.011174 | 0.23004 |
|                     | Ledoit-Wolf | 0.0246339 | 0.014686 | 0.203272 | 0.09226 | 0.0243036 | 0.014522 | 0.20179 | 0.09199 |
| **with TC**         | POET   | -0.0042556 | 0.003689 | -0.070065 | -       | -0.0054897 | 0.003055 | -0.099320 | -       |
|                     | NodeWise | 0.0001145 | 0.003609 | 0.001907 | -       | -0.0020261 | 0.003028 | -0.036821 | -       |
|                     | Ledoit-Wolf | 0.0228039 | 0.014859 | 0.187075 | -       | 0.0224671 | 0.014692 | 0.185353 | -       |

Table 2: Monthly Portfolio Performance: Second Monthly Results
Table 3: Daily Portfolio Performance: First Daily Results

|                      | Global Minimum Portfolio |                      | Markowitz Portfolio |
|----------------------|--------------------------|----------------------|---------------------|
|                      | Return | Variance | Sharpe | Turnover | Return | Variance | Sharpe | Turnover |
| **In-Sample: Feb 4 2013-Mar 31 2016, Out-Of-Sample: Apr 1 2016-Mar 31 2017, \( n_I = 795, n - n_I = 252 \)** |                  |          |        |          |                  |          |        |          |
| without TC           |        |          |        |          |                  |          |        |          |
| POET                 | -1.129e-03 | 3.651e-04 | -0.059075 | 2.23978 | -3.109e-04 | 7.379e-05 | -0.036193 | 0.97720 |
| Nodewise             | 3.442e-04 | 3.997e-05 | 0.054450 | 0.05740 | 3.397e-04 | 3.865e-05 | 0.054645 | 0.06796 |
| Ledoit-Wolf          | 5.961e-04 | 3.367e-05 | 0.102729 | 0.32489 | 5.601e-04 | 3.620e-05 | 0.093094 | 0.33374 |
| with TC              |        |          |        |          |                  |          |        |          |
| POET                 | -1.100e-02 | 6.475e-03 | -0.136666 | -      | -5.113e-03 | 5.709e-04 | -0.214004 | -      |
| Nodewise             | 6.191e-05 | 4.008e-05 | 0.009778 | -      | 5.586e-06 | 3.887e-05 | 0.000896 | -      |
| Ledoit-Wolf          | -1.002e-03 | 3.349e-05 | -0.173219 | -      | -1.078e-03 | 3.767e-05 | -0.175671 | -      |

In Table 4 all estimators have negative SR with transaction costs. But POET has the best performance among three in terms of SR with transaction costs. Ledoit-Wolf estimator has the best variance in this second daily data.

In summary, Nodewise estimator shows good out-of-sample variance properties compared to other methods. In terms of SR, the results are mixed. POET estimator, when there are transaction costs, does well in daily data, followed by Nodewise. Ledoit-Wolf estimator has large variance generally but good SR with monthly data in case of transaction costs. Nodewise estimator performs well with larger out-of-sample horizon forecasts in terms of SR.

7 Conclusion

In this paper, we analyze three aspects of a large portfolio. Namely, we consider variance, gross exposure, and the risk. These cases are shown when the number of assets is larger than the time...
Global Minimum Portfolio Markowitz Portfolio

|                   | Return | Variance | Sharpe | Turnover | Return | Variance | Sharpe | Turnover |
|-------------------|--------|----------|--------|----------|--------|----------|--------|----------|
| In-Sample: Feb 4 2013-Aug 18 2014, Out-Of-Sample: Aug 19 2014-Aug 18 2015, n_I = 388, n_n = 252 |
| POET              | 2.160e-04 | 5.432e-05 | 0.029310 | 0.04707 | 1.721e-04 | 5.314e-05 | 0.0236040 | 0.07448 |
| NodeWise          | 2.059e-04 | 5.393e-05 | 0.028041 | 0.11554 | 1.950e-04 | 5.299e-05 | 0.0267847 | 0.12763 |
| Ledoit-Wolf       | 6.439e-04 | 3.636e-05 | 0.106788 | 0.45766 | 6.531e-04 | 3.700e-05 | 0.1073679 | 0.46179 |
| with TC           | -1.273e-05 | 5.479e-05 | -0.001720 | - | -1.927e-04 | 5.358e-05 | -0.0263184 | - |
| NodeWise          | -3.613e-04 | 5.419e-05 | -0.049077 | - | -4.323e-04 | 5.321e-05 | -0.0592633 | - |
| Ledoit-Wolf       | -1.650e-03 | 3.522e-05 | -0.278010 | - | -1.661e-03 | 3.710e-05 | -0.2727097 | - |

|                   | Return | Variance | Sharpe | Turnover | Return | Variance | Sharpe | Turnover |
|-------------------|--------|----------|--------|----------|--------|----------|--------|----------|
| In-Sample: Feb 4 2013-Feb 18 2015, Out-Of-Sample: Feb 19 2015-Aug 18 2015, n_I = 514, n_n = 126 |
| POET              | -0.0001218 | 4.388e-05 | -0.01839 | 0.01142 | -0.0001801 | 4.232e-05 | -0.02768 | 0.03569 |
| NodeWise          | -0.0001987 | 4.346e-05 | -0.03013 | 0.09489 | -0.0002007 | 4.228e-05 | -0.03087 | 0.10744 |
| Ledoit-Wolf       | 0.0002105 | 3.701e-05 | 0.03461 | 0.43991 | 0.0002350 | 3.844e-05 | 0.03790 | 0.44033 |
| with TC           | -0.0001671 | 4.416e-05 | -0.02515 | - | -0.000357 | 4.277e-05 | -0.05285 | - |
| NodeWise          | -0.0006546 | 4.371e-05 | -0.09900 | - | -0.0007194 | 4.269e-05 | -0.11011 | - |
| Ledoit-Wolf       | -0.0019979 | 3.765e-05 | -0.32563 | - | -0.0019757 | 4.038e-05 | -0.31089 | - |

|                   | Return | Variance | Sharpe | Turnover | Return | Variance | Sharpe | Turnover |
|-------------------|--------|----------|--------|----------|--------|----------|--------|----------|
| In-Sample: Feb 4 2013-Aug 18 2014, Out-Of-Sample: Aug 19 2014-Aug 18 2015, n_I = 388, n_n = 252 |
| POET              | 2.160e-04 | 5.432e-05 | 0.029310 | 0.04707 | 1.721e-04 | 5.314e-05 | 0.0236040 | 0.07448 |
| NodeWise          | 2.059e-04 | 5.393e-05 | 0.028041 | 0.11554 | 1.950e-04 | 5.299e-05 | 0.0267847 | 0.12763 |
| Ledoit-Wolf       | 6.439e-04 | 3.636e-05 | 0.106788 | 0.45766 | 6.531e-04 | 3.700e-05 | 0.1073679 | 0.46179 |
| with TC           | -1.273e-05 | 5.479e-05 | -0.001720 | - | -1.927e-04 | 5.358e-05 | -0.0263184 | - |
| NodeWise          | -3.613e-04 | 5.419e-05 | -0.049077 | - | -4.323e-04 | 5.321e-05 | -0.0592633 | - |
| Ledoit-Wolf       | -1.650e-03 | 3.522e-05 | -0.278010 | - | -1.661e-03 | 3.710e-05 | -0.2727097 | - |

Table 4: Daily Portfolio Performance: Second Daily Results

We show that increasing number of assets in a portfolio decreases estimation error for the portfolio variance, unlike the previous literature, under a sparsity assumption on the population inverse covariance matrix. Even without this sparsity assumption, still consistent estimation of variance is possible which is an important finding. Furthermore, we show consistent estimation of the gross exposure of the portfolio. Risk of the large portfolio is also estimated. We generalize the results by relaxing sub-Gaussianity of the returns assumption. We compare our estimator to the factor model based and shrinkage estimators in simulations and an application.

Appendix A.

In this section we provide proofs. Here, we repeat the proof of Theorem 3.2.4 in van de Geer et al. (2014) with more clarifying steps for the reader.

Proof of Lemma 2.1. First by (2.7)

\[ \|\hat{\Theta} - I_p\|_\infty \leq \max_j \frac{\lambda_j}{\tau_j^2}. \]  

Then uniformly in j, \( \lambda_j = O(\sqrt{\log p/n}) \). Also see that by Theorem 3.2.4 of van de Geer et al. (2014), by Assumption \( \max_j s_j \sqrt{\log p/n} = o(1) \),

\[ |\hat{\tau}_j^2 - \tau_j^2| = o_p(1). \]

Note we define \( \Theta = \Sigma^{-1} \). Then, since \( \tau_j^2 = (\Theta_{jj}^{-1}) \) in van de Geer et al. (2014), via Assumption 3 \( \|\Theta_j\|_2 \leq \Lambda_{min}^{-1} = O(1) \) uniformly in j. This last result implies \( \min_j \tau_j^2 > 0 \). Then by Assumption
3, \( \tau_j^2 \leq \Sigma_{j,j} = O(1) \) uniformly in \( j \). We combine these last two components to have

\[
\| \hat{\Theta} \hat{\Sigma} - I_p \|_\infty = O_p(\sqrt{\log p/n}) = o_p(1).
\]

Q.E.D.

Before going through the main steps, the following norm inequality is used in all of the proofs. Take a \( p \times p \) generic matrix: \( M \), and a generic \( p \times 1 \) vector \( x \). Note that \( M_j' \) represents \( 1 \times p \), \( j \)th row vector in \( M \), and \( M_j \) is \( p \times 1 \) vector (i.e. transpose of \( M_j' \), or column version of \( M_j' \))

\[
\| Mx \|_1 = |M_1'x| + |M_2'x| + \cdots + |M_p'x| \\
\leq \| M_1 \|_1 \| x \|_\infty + \| M_2 \|_1 \| x \|_\infty + \cdots + \| M_p \|_1 \| x \|_\infty \\
= \sum_{j=1}^{p} \| M_j \|_1 \| x \|_\infty \\
\leq p \max_j \| M_j \|_1 \| x \|_\infty,
\]

(A.2)

where we use Holders inequality to get each inequality.

The following Lemma A.1 is useful for the proof of Theorem 3.1. Before that, we need a result from Theorem 3.2.4 of van de Geer et al. (2014). This is an \( l_1 \) bound on the estimation error between \( \hat{\Theta}_j \) and \( \Theta_j \). Under Assumption 1-3, we have

\[
\| \hat{\Theta}_j - \Theta_j \|_1 = O_p(s_j \sqrt{\log p/n}).
\]

We define \( \hat{A} = 1_p' \hat{\Theta} 1_p \), also note that \( A = 1_p' \Theta 1_p \), where the population quantity \( \Theta = \Sigma^{-1} \).

**Lemma A.1.** Under Assumptions 1-3, uniformly in \( j \in \{1, \cdots, p\} \),

\[
\frac{1}{p} |\hat{A} - A| = o_p(1).
\]

**Proof of Lemma A.1.** First, see that

\[
\hat{A} - A = 1_p' \hat{\Theta} 1_p - 1_p' \Theta 1_p = 1_p' (\hat{\Theta} - \Theta) 1_p
\]

(A.4)

Now consider the the right side of (A.4)

\[
|1_p' (\hat{\Theta} - \Theta) 1_p| \leq \| (\hat{\Theta} - \Theta) 1_p \|_1 \| 1_p \|_\infty \\
\leq p \| \hat{\Theta}_j - \Theta_j \|_1 \\
= O_p(p \max_j s_j \sqrt{\log p/n}) = o_p(p),
\]

(A.5)

where Holders inequality is used in the first inequality, and (A.2) is used for the second inequality and the last equality is obtained by imposing Assumption 2. Q.E.D.
**Proof of Theorem 3.1.** We consider

\[
\left| \frac{\hat{A}^{-1}}{A^{-1}} - 1 \right| = \frac{|\hat{A}^{-1} - A^{-1}|}{|\hat{A}^{-1}|}.
\]  

(A.6)

First, use Assumption 3 to have, (where \(C_0 = \text{Eigmin}(\Sigma^{-1}) > 0\), \(C_0\) is a positive constant, and it represents the minimal eigenvalue of \(\Theta = \Sigma^{-1}\))

\[
A = 1_p' \Sigma^{-1} 1_p \geq pC_0 > 0,
\]

which shows by Assumption 3

\[
\frac{A}{p} \geq C_0 > 0.
\]  

(A.7)

By Lemma A.1 and its proof we have \(p^{-1}||\hat{A} - A|| = O_p(\max_j s_j \sqrt{\log p/n}) = o_p(1)\), where we use Assumption 2 in last equality. Then use this last equation for the numerator in (A.6) via Assumption 2

\[
\left| \frac{\hat{A}^{-1}}{A^{-1}} - 1 \right| \leq \frac{O_p(\max_j s_j \sqrt{\log p/n})}{o_p(1) + C_0} = \frac{o_p(1)}{(o_p(1) + C_0)} = o_p(1),
\]  

(A.8)

where the denominator is bounded away from zero by \(A/p\) being bounded away from zero as shown in (A.7). Also use \(\hat{A}/p = A/p + o_p(1)\) by Lemma A.1 to get the denominator’s rate and the result.

Q.E.D.

The result below is needed for the subsequent lemmata. First, let \(r_t\) be the asset return for all \(p\) assets at time \(t\), and \(r_t\) is \(p \times 1\) vector, and \(\mu\) is the \(p \times 1\) population return vector. This result can be obtained by using Lemma 2.14.16 of Bühlmann and van de Geer (2011) since our returns are deemed to be sub-Gaussian.

\[
\|n^{-1} \sum_{t=1}^{n} r_t - \mu\|_{\infty} = O_p(\sqrt{\log p/n}) = o_p(1).
\]  

(A.9)

Note that we use sample mean to predict population mean. Before the next Lemma, we define \(\hat{B} = 1_p' \hat{\Theta} \hat{\mu}\), and \(B = 1_p' \Theta \mu\).

**Lemma A.2.** Under Assumptions 1-3, uniformly in \(j \in \{1, \ldots, p\}\)

\[
\frac{1}{p} |\hat{B} - B| = o_p(1).
\]

Proof of Lemma A.2. We can decompose \(\hat{B}\) by simple addition and subtraction into

\[
\hat{B} - B = 1_p' (\hat{\Theta} - \Theta)(\hat{\mu} - \mu) + 1_p' (\hat{\Theta} - \Theta) \mu + 1_p' \Theta (\hat{\mu} - \mu)
\]  

(A.10)  

(A.11)  

(A.12)
Now we analyze each of the terms above. Defining \( \hat{\mu} = n^{-1} \sum_{t=1}^{n} r_t \),

\[
|1_p' (\hat{\Theta} - \Theta)(\hat{\mu} - \mu)| \leq \| (\hat{\Theta} - \Theta)1_p \|_1 \| \hat{\mu} - \mu \|_\infty \\
\leq p \max_{1 \leq j \leq p} \| \hat{\Theta}_j - \Theta_j \|_1 \| \hat{\mu} - \mu \|_\infty \\
= pO(\max_j s_j \sqrt{\log p/n}) O_p(\sqrt{\log p/n}),
\] (A.13)

where we use Holder’s inequality in the first inequality, and the norm inequality in (A.2) with \( M = \hat{\Theta} - \Theta, x = 1_p \) in the second inequality above, and the rate is by (A.3) and (A.9).

So we consider (A.11) above. Note that by Assumption 1, \( \| \hat{\mu} \|_\infty < C \), where \( C \) is a positive constant.

\[
|1_p' (\hat{\Theta} - \Theta)\mu| \leq \| (\hat{\Theta} - \Theta)1_p \|_1 \| \mu \|_\infty \\
\leq C p \max_{1 \leq j \leq p} \| \hat{\Theta}_j - \Theta_j \|_1 \\
= C p O_p(\max_j s_j \sqrt{\log p/n}),
\] (A.14)

where we use Holder’s inequality in the first inequality, and the norm inequality in (A.2) with \( M = \hat{\Theta} - \Theta, x = 1_p \) in the second inequality above, and the rate is by (A.3).

Before the next proof, we need an additional result. First, by the proof of Theorem 3.2.4 in van de Geer et al. (2014), or proof of Lemma 5.3 of van de Geer et al. (2014), we have \( \| \gamma_j \|_1 = O(\sqrt{s_j}) \). Note that \( \Theta_j = C_j/\tau_j^2 \). Also remember taking \( j = 1, C_1 = (1, -\gamma_1) \) (without losing any generality here). By our Assumption 2, \( \min_j \tau_j^2 > 0 \). So

\[
\max_j \| \Theta_j \|_1 = O(\max_j \sqrt{s_j}).
\] (A.15)

Now consider (A.12).

\[
|1_p' \Theta (\hat{\mu} - \mu)| \leq \| \Theta 1_p \|_1 \| \hat{\mu} - \mu \|_\infty \\
\leq p \max_{1 \leq j \leq p} \| \Theta_j \|_1 \| \hat{\mu} - \mu \|_\infty \\
= p O_p(\max_j \sqrt{s_j}) O_p(\sqrt{\log p/n}),
\] (A.16)

where we use Holder’s inequality in the first inequality, and the norm inequality in (A.2) with \( M = \Theta, x = 1_p \) in the second inequality above, and the rate is from (A.15) and (A.9). Combine (A.13)(A.14)(A.16) in (A.10)-(A.12), and note that the largest rate is coming from (A.14). So use Assumption 2, \( \max_j s_j \sqrt{\log p/n} = o(1) \) to have

\[
p^{-1} |\hat{B} - B| = O_p(\max_j s_j \sqrt{\log p/n}) = o_p(1).
\] (A.17)

\[Q.E.D.\]

Next, we show the uniform consistency of another term in the estimated optimal weights. Note that \( D = \mu' \Theta \mu \), and its estimator is \( \hat{D} = \hat{\mu}' \hat{\Theta} \hat{\mu} \).

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Lemma A.3. Under Assumptions 1-3, uniformly in $j \in \{1, \cdots, p\}$

$$p^{-1}|\hat{D} - D| = o_p(1).$$

Proof of Lemma A.3. By simple addition and subtraction

$$\hat{D} - D = (\hat{\mu} - \mu)'(\hat{\Theta} - \Theta)(\hat{\mu} - \mu) \quad (A.18)$$

$$+ (\hat{\mu} - \mu)'\Theta(\hat{\mu} - \mu) \quad (A.19)$$

$$+ 2(\hat{\mu} - \mu)'\Theta \mu \quad (A.20)$$

$$+ 2\mu'(\hat{\Theta} - \Theta)(\hat{\mu} - \mu) \quad (A.21)$$

$$+ \mu'(\hat{\Theta} - \Theta)\mu. \quad (A.22)$$

We start with (A.18).

$$|(\hat{\mu} - \mu)'(\hat{\Theta} - \Theta)(\hat{\mu} - \mu)| \leq \|\hat{\Theta} - \Theta\|_1 \|\hat{\mu} - \mu\|_\infty \quad (A.23)$$

$$\leq p\|\hat{\mu} - \mu\|_\infty \max_j \|\Theta_j - \Theta_j\|_1 \quad (A.23)$$

$$= pO_p(\log p/n)O_p(\max_j \sqrt{s_j} \sqrt{\log p/n}) \quad (A.23)$$

$$= O_p(p \max_j \sqrt{s_j} (\log p/n)^{1/2}), \quad (A.23)$$

where Holder’s inequality is used for the first inequality above, and the inequality (A.2), with $M = \hat{\Theta} - \Theta$ and $x = \hat{\mu} - \mu$ for the second inequality above, for the rates we use (A.3), (A.9).

We continue with (A.19).

$$|(\hat{\mu} - \mu)'(\Theta)(\hat{\mu} - \mu)| \leq \|\Theta\|_1 \|\hat{\mu} - \mu\|_\infty \quad (A.24)$$

$$\leq p\|\hat{\mu} - \mu\|_\infty \max_j \|\Theta_j\|_1 \quad (A.24)$$

$$= pO_p(\log p/n)O_p(\max_j \sqrt{s_j}) \quad (A.24)$$

$$= O_p(p \max_j \sqrt{s_j} (\log p/n)^{3/2}), \quad (A.24)$$

where Holder’s inequality is used for the first inequality above, and the inequality (A.2), with $M = \Theta$ and $x = \hat{\mu} - \mu$ for the second inequality above, for the rates we use (A.9), (A.15).

Then we consider (A.20), with using $\|\mu\|_\infty \leq C$,

$$|(\hat{\mu} - \mu)'(\Theta)(\mu)| \leq \|\Theta\|_1 \|\hat{\mu} - \mu\|_\infty \quad (A.25)$$

$$\leq C p\|\hat{\mu} - \mu\|_\infty \max_j \|\Theta_j\|_1 \quad (A.25)$$

$$= pO_p(\sqrt{\log p/n})O_p(\max_j \sqrt{s_j}) \quad (A.25)$$

$$= O_p(p \max_j \sqrt{s_j} (\log p/n)^{1/2}), \quad (A.25)$$

where Holder’s inequality is used for the first inequality above, and the inequality (A.2), with
\( M = \Theta \) and \( x = \hat{\mu} - \mu \) for the second inequality above, for the rates we use (A.9), (A.15).

Then we consider (A.21).

\[
|\mu'(\hat{\Theta} - \Theta)(\hat{\mu} - \mu)| \leq \|\mu'(\hat{\Theta} - \Theta)(\hat{\mu} - \mu)\|_1 \leq p\|\mu\|_{\infty} \max_j \|\hat{\Theta}_j - \Theta_j\|_1 \|\hat{\mu} - \mu\|_{\infty} \\
\leq Cp\max_j \|\hat{\Theta}_j - \Theta_j\|_1 \|\hat{\mu} - \mu\|_{\infty} \\
= pO_p(\max_j s_j \sqrt{\log p/n})O_p(\sqrt{\log p/n}) \\
= O_p(p \max_j s_j \sqrt{\log p/n}), \quad (A.26)
\]

where Holder’s inequality is used for the first inequality above, and the inequality (A.2), with \( M = \hat{\Theta} - \Theta \) and \( x = \mu \) for the second inequality above, and for the third inequality above we use \( \|\mu\|_{\infty} \leq C \), and for the rates we use (A.3), (A.9).

Then we consider (A.22),

\[
|\mu'(\hat{\Theta} - \Theta)(\mu)| \leq \|\mu'(\hat{\Theta} - \Theta)(\mu)\|_1 \|\mu\|_{\infty} \\
\leq p\|\mu\|_{\infty}^2 \max_j \|\hat{\Theta}_j - \Theta_j\|_1 \\
\leq Cp\max_j \|\hat{\Theta}_j - \Theta_j\|_1 \\
= pO_p(\max_j s_j \sqrt{\log p/n}) \\
= O_p(p \max_j s_j \sqrt{\log p/n}) ^{1/2}, \quad (A.27)
\]

where Holder’s inequality is used for the first inequality above, and the inequality (A.2), with \( M = \hat{\Theta} - \Theta \) and \( x = \mu \) for the second inequality above, and for the third inequality above we use \( \|\mu\|_{\infty} \leq C \), and for the rate we use (A.3). Note that the last rate above in (A.27) derives our result, since it is the largest rate by Assumption 2.

Combine (A.23)-(A.27) in (A.18)-(A.22) and the rate in (A.27) to have

\[
p^{-1}|\hat{D} - D| = O_p(\max_j s_j \sqrt{\log p/n}) = o_p(1). \quad (A.28)
\]

Q.E.D.

The following lemma establishes orders for the terms in the optimal weight, A, B, D. This is useful to understand the implications of the assumptions in Theorems 1-2. Note that both A, D are positive by Assumption 3, and bounded away from zero.

**Lemma A.4.** Under Assumptions 1, 3

\[
A = O(p). \\
|B| = O(p). \\
D = O(p).
\]
Proof of Lemma A.4. We do the proof for term $D = \mu' \Theta \mu$. The proof for $A = 1' \Theta 1$, is the same.

$$D = \mu' \Theta \mu \leq \text{Eigmax}(\Theta) \|\mu\|_2^2 = O(p),$$

where we use the fact that each $\mu_j$ is a constant by Assumption 1, and the maximal eigenvalue of $\Theta = \Sigma^{-1}$ is finite by Assumption 3. For term B, the proof can be obtained by using Cauchy-Schwartz inequality first and the using the same analysis for terms A and D. Q.E.D.

Next we need the following technical lemma, that provides the limit and the rate for the denominator in optimal portfolio.

Lemma A.5. Under Assumptions 1-3, uniformly over $j$, in $\lambda_j = O(\sqrt{\log p / n})$

$$p^{-2}|(\hat{A} \hat{D} - \hat{B}^2) - (AD - B^2)| = o_p(1).$$

Proof of Lemma A.5. Note that by simple adding and subtracting

$$\hat{A} \hat{D} - \hat{B}^2 = [(\hat{A} - A) + A][(\hat{D} - D) + D] - [(\hat{B} - B) + B]^2.$$

Then using this last expression and simplifying, $A, D$ being both positive

$$p^{-2}|(\hat{A} \hat{D} - \hat{B}^2) - (AD - B^2)| \leq p^{-2}\{|\hat{A} - A| |\hat{D} - D| + |\hat{A} - A|D + AD - D| + (\hat{B} - B)^2 + 2B||\hat{B} - B||\} = O_p(\max_j s_j \sqrt{\log p / n}) = o_p(1),$$

where we use (A.5)(A.17)(A.28), Lemma A.4, and Assumption 2: $\max_j s_j \sqrt{\log p / n} = o(1)$. Q.E.D.

Proof of Theorem 3.2. Now we define notation to help us in the proof here. First set

$$\hat{x} = \hat{A} \rho_1^2 - 2\hat{B} \rho_1 + \hat{D}. \quad (A.30)$$

$$x = A \rho_1^2 - 2B \rho_1 + D. \quad (A.31)$$

$$\hat{y} = \hat{A} \hat{D} - \hat{B}^2. \quad (A.32)$$

$$y = AD - B^2. \quad (A.33)$$

Then we can write the estimate of the optimal portfolio variance as

$$\hat{\Psi}_{OPV} = \frac{\hat{x}}{\hat{y}},$$

and the optimal portfolio variance is

$$\Psi_{OPV} = \frac{x}{y}.$$

To start the main part of the proof we need a rate for a limit fraction: $y/x$. Note that the
fraction is positive by Assumptions $AD - B^2 \geq p^2 C_1 > 0$, $A\rho_1^2 - 2B\rho_1 + D \geq pC_1 > 0$.

\[
\frac{y}{x} = \frac{AD - B^2}{A\rho_1^2 - 2B\rho_1 + D} \leq \frac{AD}{A\rho_1^2 - 2B\rho_1 + D} = \frac{O(p^2)}{O(p)} = O(p), \tag{A.34}
\]

where we use $B^2 > 0$ and the assumption $A\rho_1^2 - 2B\rho_1 + D \geq C_1p > 0$ and Lemma A.4.

So we can setup the problem as, by adding and subtracting $xy$ from the numerator, and $y/x > 0$ by assumption, and use (A.34) for the second equality below

\[
\frac{\hat{\Psi}_{OPV}^2 - \Psi_{OPV}^2}{\hat{\Psi}_{OPV}} = \frac{\hat{x} - x}{\hat{y} - y} \frac{y}{x} = pO(1) \left| \frac{\hat{x} - x}{\hat{y} - y} \right| = p \left| \frac{\hat{x}y - xy + xy - x\hat{y}}{\hat{yy}} \right| O(1) = \frac{p^{-3}(\hat{x} - x)y + x(y - \hat{y})}{p^{-4}\{\hat{yy}\}} O(1) = \frac{\hat{x} - x}{p - 2} \frac{\hat{y} - y}{p^2} + \frac{x}{p^2} \frac{y - \hat{y}}{p} O(1). \tag{A.35}
\]

We consider each term in the numerator in (A.35). Via Lemma A.1-A.3, and $\rho_1$ being bounded, and (A.5)(A.17)(A.28)

\[
p^{-1}|\hat{x} - x| = p^{-1}|\hat{A}\rho_1^2 - 2\hat{B}\rho_1 + \hat{D} - (A\rho_1^2 - 2B\rho_1 + D)| \leq p^{-1}\{|\hat{A} - A|\rho_1^2 + 2|\hat{B} - B|\rho_1 + |\hat{D} - D|\} = O_p(\max_j s_j \sqrt{\log p/n}) = o_p(1), \tag{A.36}
\]

where we use Assumption 2 in the rate above. Now analyze the following term in the numerator

\[
p^{-2}y = p^{-2}AD - B^2 \leq p^{-2}AD = O(1), \tag{A.37}
\]

where we use $B^2 > 0$ in the inequality, and Lemma A.4 for the rate result, which is the final equality above in (A.37). Next, consider the following in the numerator

\[
p^{-1}x = p^{-1}A\rho_1^2 - 2B\rho_1 + D \leq p^{-1}(A\rho_1^2 + 2|B|\rho_1 + D) = O(1), \tag{A.38}
\]

where we use $A, D$ being positive, and Lemma A.4, with $\rho_1$ being bounded. Then Lemma A.5 and (A.29) provides

\[
p^{-2}|\hat{y} - y| = p^{-2}|\hat{A}\hat{D} - \hat{B}^2 - (AD - B^2)| = O_p(\max_j s_j \sqrt{\log p/n}) = o_p(1). \tag{A.39}
\]
So the numerator in (A.35) is,

$$\frac{|\hat{\beta} - \hat{\gamma}| y}{p^2} + \frac{\hat{\beta} - \hat{\gamma}}{p^2} = O_p(\max_j s_j \sqrt{\log p/n}) = o_p(1), \quad (A.40)$$

where we use (A.36)-(A.39) and $x > 0, y > 0$ by Assumption.

We consider the denominator in (A.35)

$$\left|\frac{\hat{\beta} y}{p^2} \right| = \left|\left[(\hat{\beta} - y) + y\right] \frac{y}{p^2}\right| = \left(\frac{y}{p^2}\right) y \geq \left(\frac{y}{p^2}\right) + C_1 > 0, \quad (A.41)$$

where we add and subtract $y$ in the first equality, and Lemma A.5 in the second equality, and $p^{-2}y = p^{-2}(AD - B^2) \geq C_1 > 0$, and $C_1$ is a positive constant by assumption. Next, combine (A.40)(A.41) in (A.35) with Assumption 2 to have

$$\left|\frac{\hat{\Psi}_{OPV} - \Psi_{OPV}}{\Psi_{OPV}}\right| \leq O_p\left(\max_j s_j \sqrt{\log p/n}\right) = \frac{o_p(1)}{C_1^2 + o_p(1)} = o_p(1).$$

Q.E.D.

Proof of Theorem 3.3. We start with the definition of the global minimum variance portfolio weight vector estimate

$$\hat{w}_u = \frac{\hat{\Theta}_1 p}{1'\hat{\Theta}_1 p},$$

where we can write

$$\hat{w}_u - w_u = \frac{\hat{\Theta}_1 p}{1'\hat{\Theta}_1 p} - \frac{\Theta_1 p}{1'\Theta_1 p}.$$

Using the definition of $\hat{A} = 1'\hat{\Theta}_1 p$, and $A = 1'\Theta_1 p$, via adding and subtracting $A\Theta_1 p$ from the numerator below

$$\hat{w}_u - w_u = \frac{p^{-2}[(A\hat{\Theta}_1 p) - (A\hat{\Theta}_1 p)]}{p^{-2}(A\hat{A})} = \frac{p^{-2}[(A\Theta_1 p) - (A\Theta_1 p) + (A\Theta_1 p) - (A\Theta_1 p)]}{p^{-2}(A\hat{A})}.$$

Using the above result

$$\|\hat{w}_u - w_u\|_1 \leq \frac{A \|\hat{\Theta} - \Theta\|_1 p + |A - \hat{A}| \|\Theta_1 p\|_1 p}{|A| A' p} \cdot \quad (A.42)$$

Then in (A.48) consider the numerator. By (A.59)(A.60)

$$\|\hat{\Theta} - \Theta\|_1 p = O_p\left(p \max_j s_j \sqrt{\log p/n}\right). \quad (A.43)$$
∥Θ_1p∥_1 = O(p\sqrt{\max_j s_j}). \tag{A.44}

and via Lemma A.4, \( A = O(p) \), also by (A.5)

\[ p^{-1}|\hat{A} - A| = O_p(\max_j s_j \sqrt{\log p/n}). \tag{A.45} \]

By (A.43)-(A.45)

\[
\frac{A}{p} \frac{\| (\hat{\Theta} - \Theta)_p \|_1}{p} + \frac{|A - \hat{A}|}{p} \frac{\| \Theta_1p \|_1}{p} = O(1)O_p(\max_j s_j \sqrt{\log p/n})
+ O_p(\max_j s_j \sqrt{\log p/n})O(\sqrt{\max_j s_j})
= O_p((\max_j s_j)^{3/2} \sqrt{\log p/n}) = o_p(1), \tag{A.46} \]

where we use sparsity assumption \((\max_j s_j)^{3/2} \sqrt{\log p/n} = o(1)\) in the last step. Then for the denominator in (A.42) from (A.6)-(A.8) we have, for \( C_0 > 0 \), is a positive constant,

\[
\frac{|\hat{A}|}{p} \frac{A}{p} \geq (o_p(1) + C_0)C_0. \tag{A.47} \]

Now combine (A.46)(A.47) in (A.42) to have the desired result. Q.E.D.

**Proof of Theorem 3.4.** Denote \( w^* = \Delta_1 \Theta_1p + \Delta_2 \Theta\mu \), where

\[
\Delta_1 = \frac{D - \rho_1 B}{AD - B^2},
\]

\[
\Delta_2 = \frac{\rho_1 A - B}{AD - B^2}.
\]

Next, denote \( \hat{w} = \hat{\Delta}_1 \hat{\Theta}1p + \hat{\Delta}_2 \hat{\Theta}\hat{\mu} \), where \( \hat{\Delta}_1, \hat{\Delta}_2 \) represent estimators for \( \Delta_1, \Delta_2 \) respectively. We get \( \hat{\Delta}_1, \hat{\Delta}_2 \) by replacing A, B, D, in the formula for \( \Delta_1, \Delta_2 \) with their estimators shown in above Theorems. Next, by adding and subtracting

\[
\hat{w} - w^* = \hat{\Delta}_1 \hat{\Theta}_1p + \hat{\Delta}_2 \hat{\Theta}\hat{\mu} - \Delta_1 \Theta_1p - \Delta_2 \Theta\mu
= [(\hat{\Delta}_1 - \Delta_1) + \Delta_1][(\hat{\Theta} - \Theta) + \Theta] \Theta_1p
+ [(\hat{\Delta}_2 - \Delta_2) + \Delta_2][(\hat{\Theta} - \Theta) + \Theta][(\hat{\mu} - \mu) + \mu]
- \Delta_1 \Theta_1p - \Delta_2 \Theta\mu
= (\hat{\Delta}_1 - \Delta_1)(\hat{\Theta} - \Theta) \Theta_1p
+ \Delta_1(\hat{\Theta} - \Theta) \Theta_1p
+ (\hat{\Delta}_2 - \Delta_2)(\hat{\Theta} - \Theta)(\hat{\mu} - \mu)
+ (\hat{\Delta}_2 - \Delta_2) \Theta(\hat{\mu} - \mu) + (\hat{\Delta}_2 - \Delta_2)(\hat{\Theta} - \Theta)\mu
+ (\hat{\Delta}_2 - \Delta_2) \Theta\mu + \Delta_2(\hat{\Theta} - \Theta)(\hat{\mu} - \mu)
+ \Delta_2 \Theta(\hat{\mu} - \mu) + \Delta_2(\hat{\Theta} - \Theta)\mu. \tag{A.48} \]
Using (A.48), and since $\hat{\Delta}_1, \Delta_1, \Delta_2, \Delta_2$ are all scalars,

\[
\|\hat{w} - w^\ast\|_1 \leq |(\hat{\Delta}_1 - \Delta_1)\|((\hat{\Theta} - \Theta)1_p)\|_1 + |(\hat{\Delta}_1 - \Delta_1)\|1_p\|_1 + |(\hat{\Delta}_2 - \Delta_2)\|((\hat{\Theta} - \Theta)(\mu - \mu))_1 + |(\hat{\Delta}_2 - \Delta_2)\|_1 + \|\hat{\Delta}_2\|((\hat{\Theta} - \Theta)(\mu - \mu))_1 + \|\Delta_2\|((\hat{\Theta} - \Theta)(\mu - \mu))_1.
\]

We consider each term above. But rather than analyzing them one by one, we analyze common elements and then determine the order of each term on the right side of (A.49). Using the definitions of $\hat{y}, y$ in (A.32)(A.33) respectively, and adding and subtracting $y(D - \rho_1 B)$ respectively from the numerator, with $\rho_1$ being bounded, $y > 0$ by assumption

\[
p|\hat{\Delta}_1 - \Delta_1| = p \left| \frac{y(\hat{\Delta} - \rho_1 \hat{B}) - \hat{y}(D - \rho_1 B)}{y^y} \right|
\]  
\[
= p \left| \frac{y(\hat{\Delta} - \rho_1 \hat{B}) - y(D - \rho_1 B) + y(D - \rho_1 B) - \hat{y}(D - \rho_1 B)}{y^y} \right|
\]  
\[
\leq \left| \frac{y \hat{\Delta}D - D - \rho_1 \hat{B}}{p^2} \hat{B} + \frac{y \rho_1 B}{p^2} \hat{B} - \hat{B} + \frac{y \hat{y} D - \rho_1 B}{p^2} \right|.
\]

Now we analyze each term in the numerator. By Lemma A.4 with $y > 0$ by assumption

\[
\frac{y}{p^2} = AD - \frac{B^2}{p^2} \leq \frac{AD}{p^2} = O(1).
\]

Next, by (A.17)(A.28)

\[
\frac{y}{p^2} |\hat{\Delta} - D| + \frac{y}{p^2} \rho_1 |B - \hat{B}| = O_p(\max_j s_j \sqrt{\log p/n}).
\]

Then

\[
\frac{|y - \hat{y}| |D - \rho_1 B|}{p^2} \leq \left( \frac{|\hat{\Delta} - B^2 - (AD - B^2)|}{p^2} \right) \left( \frac{D + |\rho_1 B|}{p} \right) = O_p(\max_j s_j \sqrt{\log p/n}),
\]

where we use $y, \hat{y}$ definitions in the inequality, and to get the rate Lemma A.4 with (A.29) is used. Combine now (A.52)(A.53) in the numerator in (A.50) to have

\[
\frac{y}{p^2} |\hat{\Delta} - D| + \frac{y}{p^2} \rho_1 |B - \hat{B}| + \frac{|y - \hat{y}| |D - \rho_1 B|}{p^2} = O_p(\max_j s_j \sqrt{\log p/n}).
\]

Then combine (A.41)(A.54) to have

\[
p|\hat{\Delta}_1 - \Delta_1| = O_p(\max_j s_j \sqrt{\log p/n}).
\]

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which simply implies
\[ |\hat{\Delta}_1 - \Delta_1| = O_p\left(\frac{\max_j s_j}{p}\sqrt{\log p/n}\right). \] (A.55)

Exactly following the same way we derive
\[ |\hat{\Delta}_2 - \Delta_2| = O_p\left(\frac{\max_j s_j}{p}\sqrt{\log p/n}\right). \] (A.56)

Consider, by using \( AD - B^2 \geq p^2 C_1 > 0 \) by assumption
\[ |\Delta_1| = \left| \frac{D - \rho_1 B}{AD - B^2} \right| \leq \left| \frac{D + \rho_1 B}{p^2 C_1} \right| = O(1/p), \] (A.57)
where we use Lemma A.4 to have \( D = O(p), |B| = O(p) \), and \( \rho_1 \) being bounded. In the same way we obtain
\[ |\Delta_2| = O(1/p). \] (A.58)

Next consider,
\[ \| (\hat{\Theta} - \Theta)(\hat{\mu} - \mu) \|_1 \leq p \max_j \| \hat{\Theta}_j - \Theta_j \|_1 \]
\[ = O_p(p \max_j s_j \sqrt{\log p/n}), \] (A.59)
where we use (A.2) for the inequality, and (A.3) for the rate result in (A.59). Now we analyze
\[ \| \Theta_1 \|_1 \leq p \max_j \| \Theta_j \|_1 = O(p \sqrt{\max_j s_j}), \] (A.60)
where the inequality is obtained by (A.2), and the rate is derived by (A.15).

Next, we consider the following term:
\[ \| (\hat{\Theta} - \Theta)(\hat{\mu} - \mu) \|_1 \leq p \| (\hat{\mu} - \mu) \|_\infty \max_j \| \hat{\Theta}_j - \Theta_j \|_1 \]
\[ = O_p(p \max_j s_j \log p/n), \] (A.61)
where we use (A.2) for the first inequality, and the rate is derived from (A.3)(A.9).

Then consider given \( \| \mu \|_\infty \leq C \)
\[ \| (\hat{\Theta} - \Theta)\mu \|_1 \leq C p \max_j \| \hat{\Theta}_j - \Theta_j \|_1 \]
\[ = O_p(p \max_j s_j \sqrt{\log p/n}), \] (A.62)
where we use (A.2) for the first inequality, and the rate is derived from (A.3).

Note that
\[ \| \Theta \mu \|_1 = O_p(p \sqrt{\max_j s_j}), \] (A.63)
where we use the same analysis in (A.60).
Next,
\[
\|\Theta(\hat{\mu} - \mu)\|_1 \leq p\|\hat{\mu} - \mu\|\infty \max_j \|\Theta_j\|_1 \\
= pO_p(\sqrt{\log p/n})O(\max_j \sqrt{s_j}) \\
= O_p(p\sqrt{\max_j s_j \sqrt{\log p/n}}), \tag{A.64}
\]
where we use (A.2) for the first inequality and (A.9)(A.15) for rates.

Use (A.55)-(A.64) in (A.49) to have
\[
\|\hat{w} - w^*\|_1 = O_p((\max_j s_j)^{3/2}\sqrt{\log p/n}) + O_p((\max_j s_j)^{3/2}(\log p/n)^{1/2}) \\
+ O_p((\max_j s_j)^{3/2}(\log p/n)^{1/2}) + O_p((\max_j s_j)^2(\log p/n)) \\
+ O_p((\max_j s_j)^2(\log p/n)^{1/2}) + O_p((\max_j s_j)(\log p/n)) \\
+ O_p((\max_j s_j)(\log p/n)^{1/2}) + O_p((\max_j s_j)(\log p/n)^{1/2}) \\
= O_p((\max_j s_j)^{3/2}(\log p/n)^{1/2}) = o_p(1), \tag{A.65}
\]
where we use the fact that \((\max_j s_j)^{3/2}(\log p/n)^{1/2}\) is the slowest rate of convergence on the right hand side terms. Then by \((\max_j s_j)^{3/2}(\log p/n)^{1/2} = o(1)\) we have the last result.\textbf{Q.E.D}

\textbf{Proof of Theorem 3.5}. We consider
\[
|\hat{w}'(\hat{\Sigma} - \Sigma)\hat{w}| \leq \|\hat{w}_u\|_1^2 \|\hat{\Sigma} - \Sigma\|_\infty. \tag{A.66}
\]

In (A.66) we analyze each right side term. First,
\[
\|\hat{w}_u\|_1 \leq \|\hat{w}_u - w_u\|_1 + \|w_u\|_1. \tag{A.67}
\]

Then from the definition of global minimum variance portfolio
\[
\|w_u\|_1 = \frac{\|\Theta_1\|_1}{1_p'\Theta_1}. \tag{A.68}
\]

Apply (A.7)(A.60) in (A.68) to have
\[
\|w_u\|_1 \leq \frac{O(p \max_j \sqrt{s_j})}{pC_0} = O(\max_j \sqrt{s_j}). \tag{A.69}
\]

Then use (A.69) and Theorem 3.3 in (A.67) to have
\[
\|\hat{w}_u\|_1 = O_p((\max_j s_j)^{3/2}\sqrt{\log p/n}) + O(\max_j \sqrt{s_j}) = o_p(1) + O(\max_j \sqrt{s_j}), \tag{A.70}
\]
where we use Assumption that \((\max_j s_j)^{3/2}\sqrt{\log p/n} = o(1)\) in the second equality.

Then use p.1195 of van de Geer et al. (2014) to have \(\|\hat{\Sigma} - \Sigma\|_\infty = O_p(\sqrt{\log p/n})\) and (A.70) in
(A.66) to have
\[ |\hat{w}_u' (\hat{\Sigma} - \Sigma) \hat{w}_u| \leq O_p(\max_j s_j) O_p(\sqrt{\log p/n}) = o_p(1), \]  
(A.71)
where we use Assumption that \((\max_j s_j)^{3/2} \sqrt{\log p/n} = o(1)\) in the second equality. Q.E.D.

**Proof of Theorem 3.6.** We consider
\[ |\hat{w}' (\hat{\Sigma} - \Sigma) \hat{w}| \leq \|\hat{w}\|^2 \|\hat{\Sigma} - \Sigma\|_\infty. \]  
(A.72)
In (A.72) we analyze each right side term. First,
\[ \|\hat{w}\|_1 \leq \|\hat{w} - w^*\|_1 + \|w^*\|_1. \]  
(A.73)
Then from the definition of Markowitz portfolio
\[ \|w^*\|_1 \leq \frac{|D - \rho_1 B|}{AD - B^2} \|\Theta_1 p\|_1 + \frac{\rho_1 A - B}{AD - B^2} \|\Theta\|_1 \]  
\[ \leq \frac{|D| + |\rho_1| |B|}{AD - B^2} \|\Theta_1 p\|_1 + \frac{|\rho_1||A| + |B|}{AD - B^2} \|\Theta\|_1. \]  
(A.74)
On the right side of (A.74) above, we use the analysis in (A.60), (A.63) for \|\Theta_1 p\|_1, \|\Theta\|_1, and \rho_1 \] is bounded, and by Lemma A.4 with assumption \(AD - B^2 \geq C_1 p^2 > 0\), to have
\[ \|w^*\|_1 \leq O(p) \frac{O(p)}{C_1 p^2} O(p \max_j \sqrt{s}_j) + \frac{O(p)}{C_1 p^2} O(p \max_j \sqrt{s}_j) = O(\max_j \sqrt{s}_j). \]  
(A.75)
The rest of proof follows exactly as in the proof of Theorem 3.5, given the result in Theorem 3.4 to be used in (A.73). Q.E.D.

**Appendix B.**
In this part of the Appendix, we analyze what happens when we relax sub-Gaussian returns assumption. Given Assumptions 1*, 2*, 3 going over proof of Lemma 2.1, with \(\max_j \lambda_j = O(p^2/l/n^{1/2})\), we get
\[ \|\hat{\Theta} \hat{\Sigma} - I_p\|_\infty = O_p(p^{2/l}/n^{1/2}) = o_p(1). \]  
Next, one of the most important inputs to proofs of Theorems 3.1-3.6 are (A.3). Lemma 2 of Caner and Kock (2014) show that
\[ \max_j \|\hat{\Theta}_j - \Theta_j\|_1 \leq O_p((\max_j s_j)p^{2/l}/n^{1/2}). \]
Note that the result above is a subcase of Lemma 2 in Caner and Kock (2014) with no restrictions, \(h = 1\) in their notation. After that we need to adjust (A.9) to reflect the new return structure. Following Lemma A.4 of Caner and Kock (2014) we have
\[ \|\hat{\mu} - \mu\|_\infty = O_p(p^{2/l}/n^{1/2}) = o_p(1). \]
The rest of the proofs for Theorems 3.1-3.6 follow from these equations in Appendix A. So the main difference with Theorems 3.1-3.6 will be that in approximation rates, we replace \(\sqrt{\log p}\) with \(p^{2/l}\).
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