SOLVABLE BASE CHANGE
AND RANKIN-SELBERG CONVOLUTIONS

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Abstract

In this paper we define a Rankin-Selberg L-function attached to automorphic cuspidal representations of $GL_m(\mathbb{A}_E) \times GL_m(\mathbb{A}_F)$ over solvable algebraic number fields $E$ and $F$ which are invariant under the Galois action, using a result proved by C.S. Rajan, and prove a prime number theorem for this L-function.

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1. Introduction

A prime number theorem for Rankin-Selberg L-functions has already been studied by several authors. In the classical case of [12] the proof requires much of what is known about the classical Rankin-Selberg L-function: meromorphic continuation, functional equation and the zero free region due to Moreno [13]. Unlike the prime number theorem for the zeta function, where non-vanishing on the line $\text{Re}(s) = 1$ is equivalent to the asymptotic formula

$$\sum_{n \leq x} \Lambda(n) \sim x$$

this non-vanishing result does not suffice to give the same estimate for any Rankin-Selberg L-function, and this is the reason for the self-contragredient assumption made in [12]. If one uses the automorphic induction functor to define a "Rankin-Selberg product" as in [2] then the main obstacle is to obtain the multiplicity of the poles of such a product. We now recall the basic notation of [2], so let $\pi$ be an automorphic cuspidal representation of $GL_n(\mathbb{A}_E)$ where $E$ is a cyclic algebraic number field of prime degree $\ell$. Suppose also that $\pi^\sigma \cong \pi$ for $\sigma$ a generator of the Galois group $Gal(E/\mathbb{Q})$, then we have the factorization

$$L(s, \pi) = L(s, \pi_Q)L(s, \pi_Q \otimes \eta_{E/Q})...L(s, \pi_Q \otimes \eta_{E/Q}^{\ell-1})$$

where $\{\pi_Q \otimes \eta_{E/Q}^i\}_{i=0,\ldots,\ell-1}$ are automorphic cuspidal representations on $GL_n(\mathbb{A}_Q)$ and $\eta_{E/Q}$ is an idele class character obtained using the reciprocity isomorphism

$$\mathbb{A}_Q^\times/\mathbb{Q}^\times N_{E/Q}(\mathbb{A}_E^\times) \cong Gal(E/\mathbb{Q})$$

Similarly, let $\pi'$ be an automorphic cuspidal representation of $GL_m(\mathbb{A}_F)$ where $F$ is a cyclic algebraic number field of prime degree $q$ and $\pi' \cong \pi'^\tau$ where $\tau$ is a generator of $Gal(F/\mathbb{Q})$. Then as before we have

$$L(s, \pi') = L(s, \pi'_Q)L(s, \pi'_Q \otimes \psi_{F/Q})...L(s, \pi'_Q \otimes \psi_{F/Q}^{q-1})$$
for an idele class character $\psi_{F/Q}$. Then we define the Rankin-Selberg L-function over different fields by

$$L(s, \pi \times_{BC} \tilde{\pi}') = \prod_{i=0}^{\ell-1} \prod_{j=0}^{q-1} L(s, \pi_Q \otimes \eta_{E/Q}^i \times \pi_Q' \otimes \psi_{F/Q}^j)$$

$$= L(s, \mathbb{T}_{i=0}^{\ell-1}(\pi_Q \otimes \eta_{E/Q}^i) \times \mathbb{T}_{j=0}^{q-1}(\pi_Q' \otimes \psi_{F/Q}^j))$$

$$= L(s, AI_{E/Q}(\pi) \times AI_{F/Q}(\pi'))$$

where $AI_{K/Q}$ denotes the automorphic induction functor for any number field $K/Q$. The Euler product for $L(s, \pi \times_{BC} \tilde{\pi}')$ converges absolutely for $Re(s) > 1$, so that we can write

$$\frac{L'}{L}(s, \pi \times_{BC} \tilde{\pi}') = - \sum_{n \geq 1} \frac{\Lambda(n)a_{\pi \times_{BC} \tilde{\pi}'}(n)}{n^s}$$

see the next section for a precise definition of $a_{\pi \times_{BC} \tilde{\pi}'}(n)$. For future reference we let

$$T = \{(\sigma_Q, \sigma_Q') | \sigma_Q \in BC_{E/Q}^{-1}(\pi), \sigma_Q' \in BC_{F/Q}^{-1}(\pi'), \sigma_Q \cong \sigma_Q' \otimes | \det \tau \exists \tau \in \mathbb{R}\}$$

where $BC_{K/Q}$ denotes the base change functor for any number field $K$. Our first result is a prime number theorem for $L(s, \pi \times \tilde{\pi}')$ in the cyclic case when $\ell = q$.

**Theorem 1.1.** Let $\pi$ and $\pi'$ be unitary automorphic cuspidal representations of $GL_n(\mathbb{A}_E)$ and $GL_n(\mathbb{A}_F)$, respectively, with $E/Q$ and $F/Q$ of prime degree $\ell$ such that $E \neq F$. Suppose that $\pi$ and $\pi'$ are invariant under the action of $Gal(E/Q)$ and $Gal(F/Q)$, respectively, and with notation as above suppose at least one of $\pi_Q$ or $\pi_Q'$ is self-contragredient. Then

$$\sum_{n \leq x} \Lambda(n)a_{\pi \times_{BC} \tilde{\pi}'}(n) =$$

$$= \begin{cases} x^{1+\rho(\pi, \pi')} + O\{x \exp(-c\sqrt{\log x})\} & \text{if } T \neq \phi \text{ and } BC_{EF/E}(\pi) \text{ is cuspidal} \\
\frac{x^{1+\rho(\pi, \pi')}}{1+\rho(\pi, \pi')} + O\{x \exp(-\sqrt{\log x})\} & \text{if } T \neq \phi \text{ and } BC_{EF/E}(\pi) \text{ is not cuspidal} \\
O\{x \exp(-\sqrt{\log x})\} & \text{if } T = \phi \end{cases}$$

**Remark:** By Lemma 4.1 of [2] if $T$ is nonempty there is a unique $\tau(\pi, \pi')$ so that $\pi_Q \otimes n_{E/Q}^{i_0} \cong \pi_Q' \otimes \psi_{F/Q}^{j_0} \otimes | \det \tau(\pi, \pi')$ for some $0 \leq i_0, j_0 \leq \ell - 1$. This follows from multiplicity one for characters and the fact that $n_{E/Q}$ and $\psi_{F/Q}$ have finite order.

More generally let $E/Q$ and $F/Q$ be any solvable Galois extensions of degrees $\ell$ and $\ell'$, and let $\pi$ and $\pi'$ denote unitary automorphic cuspidal representations of $GL_n(\mathbb{A}_E)$ and $GL_n(\mathbb{A}_F)$, respectively. Moreover suppose that both $\pi$ and $\pi'$ admit base change from $Q$; in other words assume that the sets $BC_{E/Q}^{-1}(\pi)$ and $BC_{F/Q}^{-1}(\pi')$ are nonempty. Then by Theorem 2 of [15] we can write

$$BC_{E/Q}^{-1}(\pi) = \{\pi_Q \otimes \chi_i \}_{i \in I}$$
for some idele class characters \( \chi_i \) trivial on \( \mathbb{Q}^\times N_{E_{ab}/Q}(\mathbb{A}_{E_{ab}}) \), where \( E_{ab} \) denotes the fixed field of the commutator subgroup \([Gal(E/Q), Gal(E/Q)]\). Similarly we can write

\[
BC_{E/Q}^{-1}(\pi') = \{\pi'_Q \otimes \psi'_j\}_{j \in J}
\]

for some idele class characters of \( \mathbb{A}_Q^\times \) trivial on \( \mathbb{Q}^\times N_{E_{ab}/Q}(\mathbb{A}_{E_{ab}}) \). Consider the towers of extensions coming from the cyclic composition factors of \( Gal(E/Q) \) and \( Gal(F/Q) \) using the Galois correspondence.

\[
E = E_0 \supset E_1 \supset \ldots \supset E_k \supset E_{k+1} = \mathbb{Q} \quad (1.1)
\]

\[
F = F_0 \supset F_1 \supset \ldots \supset F_t \supset F_{t+1} = \mathbb{Q} \quad (1.2)
\]

with \( [E_i, E_{i+1}] = \ell_{i+1} \) of prime degree for \( 0 \leq i \leq k \) and \( [F_j : F_{j+1}] = q_{j+1} \) of prime degree for \( 0 \leq j \leq t \). We will actually need stronger assumptions on the Galois invariance of the representations in the fibers \( BC_{E_i/E_1}^{-1}(\pi) \) for all \( i \). More precisely suppose that \( \pi \) is invariant under the action of \( Gal(E_1/E_1) \) and that the representations in the fiber \( BC_{E_i/E_1}^{-1}(\pi) \) are invariant under the action of \( Gal(E_i/E_{i+1}) \) for any \( 2 \leq i \leq k \), then we define the Rankin-Selberg L-function over the fields \( E \) and \( F \) by

\[
L(s, \pi \times_{BC} \pi') = \prod_{i \in I} \prod_{j \in J} L(s, \pi_Q \otimes \chi_i \times \pi'_Q \otimes \psi'_j) = L(s, AI_{E/Q}(\pi) \times AI_{F/Q}(\pi))
\]

To simplify notation first consider the two step case

\[
E = E_0 \supset E_1 \supset E_2 = \mathbb{Q} \quad (1.3)
\]

Then by assumption the \( \ell_1 \) distinct representations

\[
BC_{E_i/E_1}^{-1}(\pi) = \{\pi_{E_i} \otimes \eta_{E_i/E_1}^\sigma\}_{\sigma = 0}^{\ell_1 - 1}
\]

are all invariant under the action of \( Gal(E_1/E_2) \) and from the proof of Theorem 2 in \[15\] this forces \( \eta_{E_i/E_1}^\sigma \cong \eta_{E/E_1}^\sigma \) for all \( \sigma \in Gal(E_1/E_2) \) so that \( \eta_{E_i/E_1} = \eta_{E_i/E_1} \circ N_{E_i/Q} \) for some idele character on \( \mathbb{A}_Q^\times \) trivial on \( \mathbb{Q}^\times N_{E_{ab}/Q}(\mathbb{A}_{E_{ab}}) \). Thus we can write

\[
BC_{E/Q}^{-1}(\pi) = \{\pi_Q \otimes \eta_{E/Q}^\ell \circ \eta_{E_1/Q}^\ell \}_{0 \leq \ell \leq \ell_1 - 1}
\]

for some cuspidal automorphic \( \pi_Q \) on \( GL_n(\mathbb{A}_Q) \). Note the above representations are distinct, which can be seen using the fact that \( BC_{E_i/E_1}(\pi_Q \otimes \eta_{E/Q}^\ell) = BC_{E_i/E_1}(\pi_Q) \otimes \eta_{E_i/E_1}^\ell \). In other words if we have \( \pi_Q \otimes \eta_{E_1/Q}^{i_1} \otimes \eta_{E_1/Q}^{j_1} \cong \pi_Q \otimes \eta_{E_1/Q}^{i_2} \otimes \eta_{E_1/Q}^{j_2} \) then the preceding remark implies that \( i_1 = i_2 \mod(\ell_1) \) so that \( j_1 = j_2 \mod(\ell_2) \). Thus inductively we get that in the general case we have the \( \ell \) distinct representations which lift to \( \pi \) from \( Q \)

\[
BC_{E/Q}^{-1}(\pi) = \{\pi_Q \otimes (\otimes_{a=0}^k \eta_{E_{ab}/Q}^a)\}_{k=0}^{\ell_1 - 1}
\]

If we make similar Galois invariance assumptions for \( \pi' \) then we can write

\[
BC_{F/Q}^{-1}(\pi') = \{\pi'_Q \otimes (\otimes_{b=0}^t \psi_{F_{ab}/Q}^b)\}_{t=0}^{q_{t+1} - 1}
\]
and these are also distinct. By a similar calculation as in Theorem 1.2 of [2] we put a
group structure on the above representations and show that the set of twisted equivalent
pairs divides the order of this group. From this we obtain a prime number theorem for
\( L(s, \pi \times_{BC} \tilde{\pi}') \)

**Theorem 1.2.** Let \( E \) and \( F \) be solvable Galois extensions of degrees \( \ell \) and \( \ell' \) with
\( (\ell, \ell') = 1 \). Let \( \pi \) and \( \pi' \) be unitary automorphic cuspidal representations on \( GL_n(\mathbb{A}_E) \) and
\( GL_m(\mathbb{A}_F) \) respectively. Suppose that the representations in the fibers \( BC^{-1}_{E_i/E}(\pi) \),
\( BC^{-1}_{F_j/F}(\pi') \) are invariant under the action of \( Gal(E_i/E_{i+1}) \) and \( Gal(F_j/F_{j+1}) \), respec-
tively for \( 1 \leq i \leq k \), \( 1 \leq j \leq t \). Also suppose that for some \( \pi_Q \in BC^{-1}_{E/Q}(\pi) \) that \( \pi_Q \) is
self-contragredient, then

\[
\sum_{n \leq x} a_{\pi \times_{BC} \tilde{\pi}'}(n)\Lambda(n) = \left\{ \begin{array}{ll}
\frac{x^{1+i\tau(\pi,\pi')}}{1+i\tau(\pi,\pi')} + O(x \exp(-c\sqrt{\log x})} & \text{if } T \text{ is nonempty} \\
O\left(x \exp(-c\sqrt{\log x})\right) & \text{if } T \text{ is empty} 
\end{array} \right.
\]

The method used in proving Theorem 1.1 is to calculate the multiplicity of a pole of
\( L(s, \pi \times_{BC} \tilde{\pi}') \) and apply the main theorem in [12]. We rely heavily on the de-
scription of the fibers of the base change map as proved in [15], and using Theorem 2 from [15]
combined with Lemma 4.1 from [2] we get that there is at most one distinct pole of
\( L(s, \pi \times_{BC} \tilde{\pi}') \) with multiplicity equal to one.

**2. Notation**

We will use the L-functions as in [3]. For \( \pi \) an automorphic cuspidal representation
on \( GL_n(\mathbb{A}_E) \) with \( E/\mathbb{Q} \) Galois recall that one can define \( L(s, \pi) \) as a product of local factors

\[
L(s, \pi) = \prod_p \prod_{\nu|p} \prod_{i=1}^{n} (1 - \alpha_\pi(i, \nu)p^{-fs})^{-1}
\]

where \( \{\alpha_\pi(i, \nu)\}_{i=1}^{n} \) is a collection of complex numbers for any place \( \nu \) of \( E \) and \( f_p \) denotes
the modular degree of any place \( \nu \) lying over \( p \). The above product converges absolutely
for \( Re(s) > 1 \) so we can write

\[
\frac{L'(s, \pi)}{L(s, \pi)} = -\sum_{n \geq 1} \frac{\Lambda(n)a_\pi(n)}{n^s}
\]

where \( \Lambda(n) \) denotes the Von-Mangoldt function, and for \( n = p^{f_p}e \) a prime power

\[
a_\pi(n) = f_p \sum_{\nu|p} \sum_{i=1}^{n} \alpha_\pi(i, \nu)^k
\]

The Galois group \( Gal(E/\mathbb{Q}) \) acts on \( \pi \) by \( \pi^\sigma(g) = \pi(g^{\sigma^{-1}}) \) for \( g \in GL_n(\mathbb{A}_E) \). In the
special case when \( E \) is a cyclic algebraic number field of prime degree \( \ell \), if we let \( < \sigma > =
Gal(E/\mathbb{Q}) \) by the results in [1] either \( \pi \cong \pi^\sigma \) in which case \( \pi \) is the base change lift of
exactly $\ell$ non-equivalent cuspidal representations $\{\pi_Q \otimes \eta_{E/Q}^j\}_{j=0}^{\ell-1}$ where $\pi_Q \otimes \eta_{E/Q}^j$ is on $GL_n(\mathbb{A}_Q)$, or we have $\pi \not\cong \pi^\sigma$ so that

$$L(s, \pi) = L(s, \pi_Q)$$

with $\pi_Q$ an automorphic cuspidal representation of $GL_n(\mathbb{A}_Q)$. We will also need some results regarding Rankin-Selberg $L$-functions. We will use the Rankin-Selberg $L$-functions $L(s, \pi \times \overline{\pi}')$ as developed by Jacquet, Piatetski-Shapiro, and Shalika [9], Shahidi [18], and Moeglin and Waldspurger [14] where $\pi$ and $\pi'$ are unitary automorphic cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_m(\mathbb{A}_E)$, respectively. Recall $L(s, \pi \times \overline{\pi}')$ is defined as the product of local factors

$$\prod_p L_p(s, \pi \times \overline{\pi}') = \prod_p \prod_{\nu | p} \prod_{i=1}^m \prod_{j=1}^{m'} \left(1 - \alpha_{i, \nu}(j, \nu)p^{-f_p s}\right)^{-1}$$

where $f_p$ denotes the modular degree of any place $\nu | p$. We will need the following properties of the $L$-functions $L(s, \pi \times \overline{\pi}')$.

RS1. The Euler product defining $L(s, \pi \times \overline{\pi}')$ converges absolutely for $\sigma > 1$ (Jacquet and Shalika [9]).

RS2. $L(s, \pi \times \overline{\pi}')$ admits meromorphic continuation to all of $\mathbb{C}$, and if we denote $\alpha(g) = |\det(g)|$ and $\pi' \not\cong \pi \otimes \alpha^t$ for any $t \in \mathbb{R}$, then $L(s, \pi \times \overline{\pi}')$ is holomorphic. When $m = m'$ and $\pi' \cong \pi \otimes \alpha^{i\tau_0}$ for some $\tau_0 \in \mathbb{R}$, the only poles of $L(s, \pi \times \overline{\pi}')$ are simple poles at $s = i\tau_0$ and $1 + i\tau_0$ (Jacquet and Shalika [9], Moeglin and Waldspurger [14]).

RS3. $L(s, \pi \times \overline{\pi}')$ is non-zero in $\sigma \geq 1$ (Shahidi [18]). Furthermore, if at least one of $\pi$ or $\pi'$ is self-contragredient, it is zero-free in the region

$$\sigma > 1 - \frac{c}{\log(Q_{\pi Q_{\overline{\pi}'}}(|t| + 2))}, \quad |t| \geq 1$$

(2.6)

where $c$ is an explicit constant depending only on $m$ and $n$ (see Sarnak [17], Moreno [13] or Gelbert, Lapid and Sarnak [8]).

3. PROOF OF THEOREM 1.1

Proof. First note that since $E$ and $F$ are of prime degree, if $E \neq F$ then we have an isomorphism of Galois groups $Gal(EF/Q) \cong Gal(E/Q) \times Gal(F/Q)$ given by restriction. We also have that $EF/Q$ is a solvable extension of degree $\ell^2$, so that the base change map $BC_{EF/Q}$ exists, and we denote $\pi_{EF} = BC_{EF/E}(\pi)$. Suppose first that $\pi_{EF}$ is cuspidal, then from [15] we get $BC_{EF/Q}^{-1}(\pi_{EF}) = \{\pi_Q \otimes \chi_i\}_{i \in I}$ for some idele class characters of the quotient

$$\chi_i : \mathbb{A}_Q^\times/(\mathbb{Q}^\times N_{EF/Q}(\mathbb{A}_EF^\times)) \longrightarrow \mathbb{C}^\times$$

Note that there are $\ell^2$ distinct representations which lift to $\pi_{EF}$ if $\pi_{EF}$ is cuspidal. To see this take

$$BC_{EF/E}^{-1}(\pi_{EF}) = \{\pi \otimes \eta_{EF/E}^j\}_{j=0}^{\ell^2}$$
and these representations are distinct. Now let \( C_K = \mathbb{A}_K^\times / K^\times \) for any number field \( K \), and consider the character \( \psi_{F/Q} \circ N_{E/Q} \) which is a character on \( C_E \) trivial on \( N_{EF/E}(C_E) \). Thus we have that for some \( 0 \leq i \leq \ell - 1 \), \( \eta^i_{EF/E} = \psi_{F/Q} \circ N_{E/Q} \) and if the character on the right hand side is trivial we get that \( N_{E/Q}(C_E) \subseteq \ker(\psi_{F/Q}) = N_{F/Q}(C_F) \) so by class field theory \( F \subseteq E \) which is a contradiction, so that \( \psi_{F/Q} \circ N_{E/Q} \) is nontrivial and so has order \( \ell \). Hence \( \eta_{EF/E} = \psi_{EF/E} \circ N_{E/Q} \) for some class field theory character \( \eta_{EF/Q} \), in other words \( \eta_{EF/E} \) lies in the image of the base change map so that we can take

\[
BC^{-1}_{E/Q}(\pi \otimes \eta^i_{EF/E}) = \{ \pi^j \otimes \eta^i_{E/Q} \}_{i=0}^{\ell-1}
\]

and again these are distinct for each fixed \( j \). Consider the collection \( \{ \pi^j \otimes \eta^i_{E/Q} \}_{0 \leq i,j \leq \ell-1} \), which all lift to \( \pi_{EF} \) by the transitivity of the base change map; moreover they are distinct since given \( \pi^{i_1} \otimes \eta^i_{E/Q} \cong \pi^{i_2} \otimes \eta^j_{E/Q} \) then applying \( BC_{E/Q} \) gives

\[
\pi \otimes \eta^{i_1}_{EF/E} \cong \pi \otimes \eta^{j}_{EF/E}
\]

which implies \( j_1 = j_2 \mod(\ell) \) so that \( i_1 = i_2 \mod(\ell) \) and this proves the claim. Finally using the isomorphism of Galois groups we get that any character on \( \mathbb{A}_Q^\times / (Q^\times N_{E/Q}(\mathbb{A}_E^\times)) \) may be written as \( \eta^i_{E/Q} \otimes \psi^j_{F/Q} \) for some \( 0 \leq i,j \leq \ell - 1 \). From this and the preceding remarks it follows that the representations

\[
\{ \pi_Q \otimes \eta^{i}_{E/Q} \otimes \psi^{j}_{F/Q} \}_{0 \leq i,j \leq \ell-1}
\]

are distinct. Now suppose the set \( T \) of twisted equivalent pairs is nonempty, so that for some \( 0 \leq i_0, j_0 \leq \ell - 1 \) and \( \tau_0 \in \mathbb{R} \) we have

\[
\pi_Q \otimes \eta^{i_0}_{E/Q} \cong \pi_Q' \otimes \psi^{j_0}_{F/Q} \otimes | \det |^{\imath_{\tau_0}}
\]

If we have another twisted equivalent pair, say

\[
\pi_Q \otimes \eta^{i_1}_{E/Q} \cong \pi_Q' \otimes \psi^{j_1}_{F/Q} \otimes | \det |^{\imath_{\tau_1}}
\]

Then by Lemma 1 of [2] we may suppose that \( \tau_0 = \tau_1 \), thus we get

\[
\pi_Q \otimes \eta^{i_0}_{E/Q} \otimes \psi^{j_0}_{F/Q} \cong \pi_Q' \otimes \psi^{j_0}_{F/Q} \otimes \psi^{j_0}_{F/Q} \otimes | \det |^{\imath_{\tau_0}}
\]

so that \( i_1 = i_0 \mod(\ell) \) and \( j_1 = j_0 \mod(\ell) \) as desired. Finally suppose that \( \pi_{EF} \) is not cuspidal, then from [1] we get that \( \ell | n \) and

\[
\pi \otimes \eta_{EF/E} \cong \pi
\]

for some nontrivial character of \( \mathbb{A}_E^\times / (E^\times N_{EF/E}(\mathbb{A}_E^\times)) \). As before we get that \( \eta_{EF/E} = \psi^k_{F/Q} \circ N_{E/Q} \) for some \( 1 \leq k \leq \ell - 1 \), hence \( \pi_Q \cong \pi_Q \otimes \eta^s_{E/Q} \otimes \psi^r_{F/Q} \) for some \( 0 \leq s \leq \ell - 1 \) again from [1]. As before suppose that

\[
\pi_Q \otimes \eta^{i_0}_{E/Q} \cong \pi_Q' \otimes \psi^{j_0}_{F/Q} \otimes | \det |^{\imath_{\tau_0}}
\]

(3.7)
Then by a simple calculation we get another twisted equivalent pair
\[ \pi_Q \otimes \eta^{i_0}_{E/Q} \cong \pi'_Q \otimes \psi_{F/Q} \otimes | \det |^{i_0} \]
and this pair is distinct from (3.7), so by Lemma 4.2 from [2] we get \(|T| = \ell\). The rest of

the proof follows as in the proof of Theorem 1.2 of [2]. □

4. Proof of Theorem 1.2

For completeness we first state a Lemma which is the analogue of Lemma 4.1 of [2], we omit the proof as it is almost identical to that given in [2]

Lemma 4.1. Suppose that \( \pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i_{a,0}}_{E_a/Q}) \cong \pi'_Q \otimes (\otimes_{a=0}^{k} \psi_{F_a/Q}) \otimes | \det |^{i_0} \) for some \( 0 \leq i_{a,0} \leq \ell_{a} \) and \( 0 \leq j_{b,0} \leq q_{b} \) then

\[ \pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i_{a,0}}_{E_a/Q}) \cong \pi'_Q \otimes (\otimes_{a=0}^{k} \psi_{F_a/Q}) | \det |^{i_0} \]

implies that \( j_{b} = j_{b,0} \) for \( b = 0, \ldots, t \) and \( \tau = \tau_0 \). Moreover, if for some \( i_a \) and \( j_b \) with \( a = 0, \ldots, k \), \( b = 0, \ldots, t \)

\[ \pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i_{a,0}}_{E_a/Q}) \cong \pi'_Q \otimes (\otimes_{a=0}^{k} \psi_{F_a/Q}) \otimes | \det |^{i_0} \]

then \( \tau = \tau_0 \)

By Lemma 4.1 if the set \( T \) is nonempty the exponent \( \tau \) is uniquely determined and we denote this by \( \tau_{(\pi, \pi')} \). We will use the notation as in the introduction. Since the representations

\[ BC_{E/Q}^{-1}(\pi) = \{ \pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i_{a,0}}_{E_a/Q}) \}_{i_{a,0}=0}^{\ell_{a}+1} \]

are distinct, we get a well-defined group operation by setting

\[ (\pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i_{a,0}}_{E_a/Q})) \ast (\pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i'_{a,0}}_{E_a/Q})) \]

\[ = \pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i_{a,0}+i'_{a,0}}_{E_a/Q}) \]

and we denote this group of order \( \ell \) by \((\mathcal{G}, \ast)\). Now suppose the set \( T \) is nonempty, then

\[ \sigma_Q \cong \sigma'_Q \otimes | \det |^{i_{\tau_{(\pi, \pi')}}} \]

for any \((\sigma_Q, \sigma'_Q) \in T\), and by relabeling if necessary we may assume that \((\pi_Q, \pi'_Q \otimes (\otimes_{b=0}^{k} \psi_{F_b/Q})) \in T\) for some \( \pi'_Q \otimes (\otimes_{b=0}^{k} \psi_{F_b/Q}) \in BC_{F/Q}^{-1}(\pi') \). Moreover given two twisted equivalent pairs

\[ \pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i_{a,0}}_{E_a/Q}) \cong \pi'_Q \otimes (\otimes_{b=0}^{k} \psi_{F_b/Q}) \otimes | \det |^{i_{\tau_{(\pi, \pi')}}} \]

\[ \pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i'_{a,0}}_{E_a/Q}) \cong \pi'_Q \otimes (\otimes_{b=0}^{k} \psi_{F_b/Q}) \otimes | \det |^{i_{\tau_{(\pi, \pi')}}} \]

we obtain

\[ \pi_Q \otimes (\otimes_{a=0}^{k} \eta^{i_{a,0}-i'_{a,0}}_{E_a/Q}) \cong \pi'_Q \otimes (\otimes_{b=0}^{k} \psi_{F_b/Q}) \otimes (\otimes_{a=0}^{k} \eta^{i'_{a,0}}_{E_a/Q}) \otimes | \det |^{i_{\tau_{(\pi, \pi')}}} \]

\[ \cong \pi_Q \otimes (\otimes_{b=0}^{k} \psi_{F_b/Q}) \cong \pi'_Q \otimes (\otimes_{b=0}^{k} \psi_{F_b/Q}) \otimes | \det |^{i_{\tau_{(\pi, \pi')}}} \]

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so it follows that the subset \( H \subset G \) defined by
\[
H = \{ \sigma_Q \in G | (\sigma_Q, \sigma_Q') \in T, \exists \sigma_Q' \in BC_{F/Q}(\pi') \}
\]
forms a subgroup of \( G \) so by LaGrange's theorem we get \(|H|\) divides \( \ell \). Moreover Lemma 4.1 gives that for \((\sigma_Q, \sigma_Q') \in T\) then \(\sigma_Q\) is twisted equivalent to at most one \(\sigma_Q' \in BC_{F/Q}(\pi')\), hence \(|T| = |H|\). Thus applying the same argument above to the collection
\[
\{ \pi_Q \otimes (\otimes_{k=0}^{\ell} \psi_{F_k/Q}) \}_{\ell+1}^{-1}
\]
we get that the cardinality of \( T \) also divides \( \ell' \), so that \(|T| = 1\) since \((\ell, \ell') = 1\). Finally, assuming \(\pi_Q\) to be self-contragredient and using the equality
\[
L(s, \pi_Q \times \chi \times \pi_Q' \otimes \xi) = L(s, \pi_Q \times \pi_Q' \otimes \chi^{-1} \xi)
\]
valid for any finite order idele class characters we can apply the zero-free region to obtain the desired error term, and the theorem follows. \( \square \)

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