EPIC SUBSTRUCTURES AND PRIMITIVE POSITIVE FUNCTIONS

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Abstract. For $A \leq B$ first order structures in a class $\mathcal{K}$, say that $A$ is an epic substructure of $B$ in $\mathcal{K}$ if for every $C \in \mathcal{K}$ and all homomorphisms $g, g' : B \to C$, if $g$ and $g'$ agree on $A$, then $g = g'$. We prove that $A$ is an epic substructure of $B$ in a class $\mathcal{K}$ closed under ultraproducts if and only if $A$ generates $B$ via operations definable in $\mathcal{K}$ with primitive positive formulas. Applying this result we show that a quasivariety of algebras $\mathcal{Q}$ with an $n$-ary near-unanimity term has surjective epimorphisms if and only if $SP_n^{a,P}(Q_{RSI})$ has surjective epimorphisms. It follows that if $\mathcal{F}$ is a finite set of finite algebras with a common near-unanimity term, then it is decidable whether the (quasi)variety generated by $\mathcal{F}$ has surjective epimorphisms.

1. Introduction

Let $\mathcal{K}$ be a class of first order structures in the same signature, and let $A, B \in \mathcal{K}$. We say that $A$ is an epic substructure of $B$ in $\mathcal{K}$ provided that $A$ is a substructure of $B$, and for every $C \in \mathcal{K}$ and all homomorphisms $g, g' : B \to C$ such that $g|_A = g'|_A$, we have $g = g'$. That is, if $g$ and $g'$ agree on $A$, then they must agree on all of $B$. At first glance the definition may suggest that $A$ generates $B$, but on closer inspection this does not make sense. As $A$ is a substructure of $B$, generating with $A$ will yield exactly $A$. However, as the main result of this article shows, the intuition that $A$ acts as a set of generators of $B$ is not far off. In fact, if $\mathcal{K}$ is closed under ultraproducts, we prove that $A$ actually “generates” $B$, only that the generation is not through the fundamental operations but rather through primitive positive definable functions. Let’s take a look at an example. Write $\mathcal{D}$ for the class of bounded distributive lattices. There are several ways to show that both of the three-element chains contained in the bounded distributive lattice $B := 2 \times 2$ are epic substructures of $B$ in $\mathcal{D}$. One way to do this is via definable functions. Note that the formula

$$\varphi(x, y) := x \land y = 0 \& x \lor y = 1$$

defines the complement (partial) operation in every member of $\mathcal{D}$. Let $A$ be the sublattice of $B$ with universe $\{(0, 0), (0, 1), (1, 1)\}$, and suppose there are $C \in \mathcal{D}$ and $g, g' : B \to C$ such that $g|_A = g'|_A$. Clearly $B \models \varphi((0, 1), (1, 0))$, and since $\varphi$ is open and positive, it follows that $C \models \varphi(g(0, 1), g(1, 0))$ and $C \models \varphi(g'(0, 1), g'(1, 0))$. Now $\varphi(x, y)$ defines a function in $C$, and $g(0, 1) = g'(0, 1)$, so $g(1, 0) = g'(1, 0)$. Theorem II below says that every epic substructure in a class closed under ultraproducts is of this nature (although the formulas defining the generating operations may be primitive positive).

Key words and phrases. Epimorphism, epic substructure, Beth definability, definable function.
The notion of epic substructure is closely connected to that of epimorphism. Recall that a homomorphism $h : A \to B$ is a $K$-epimorphism if for every $C \in K$ and homomorphisms $g, g' : B \to C$, if $gh = g'h$ then $g = g'$. That is, $h$ is right-cancellable in compositions with $K$-morphisms. Of course every surjective homomorphism is an epimorphism, but the converse is not true. Revisiting the example above, the inclusion of the three-element chain $A$ into $2 \times 2$ is a $D$-epimorphism. This also illustrates the connection between epic substructures and epimorphisms. It is easily checked that $A$ is an epic substructure of $B$ in $K$ if and only if the inclusion $\iota : A \to B$ is a $K$-epimorphism. A class $K$ is said to have the surjective epimorphisms if every $K$-epimorphism is surjective. Although this property is of an algebraic (or categorical) nature it has an interesting connection with logic. When $K$ is the algebraic counterpart of an algebraizable logic $\vdash$ then: $K$ has surjective epimorphisms if and only if $\vdash$ has the (infinite) Beth property ([2, Thm. 3.17]). For a thorough account on the Beth property in algebraic logic see [2]. We don’t go into further details on this topic as the focus of the present article is on the algebraic and model theoretic side.

The paper is organized as follows. In the next section we establish our notation and the preliminary results used throughout. Section 3 contains our characterization of epic substructures (Theorem 5), the main result of this article. We also take a look here at the case where $K$ is a finite set of finite structures. In Section 4 we show that checking for the presence of proper epic subalgebras (or, equivalently, surjective epimorphisms) in certain quasivarieties can be reduced to checking in a subclass of the quasivariety. An interesting application of these results is that if $F$ is a finite set of finite algebras with a common near-unanimity term, then it is decidable whether the quasivariety generated by $F$ has surjective epimorphisms (see Corollary 12).

2. Preliminaries and Notation

Let $L$ be a first order language and $K$ a class of $L$-structures. We write $\mathbb{I}, \mathbb{S}, \mathbb{H}, \mathbb{P}$ and $\mathbb{P}_u$ to denote the class operators for isomorphisms, substructures, homomorphic images, products and ultraproducts, respectively. We write $\mathbb{V}(K)$ for the variety generated by $K$, that is $\mathbb{HSP}(K)$; and with $\mathbb{Q}(K)$ we denote the quasivariety generated by $K$, i.e., $\mathbb{HSP}_u(K)$.

**Definition 1.** Let $A, B \in K$.
- $A$ is an epic substructure of $B$ in $K$ if $A \leq B$, and for every $C \in K$ and all homomorphisms $g, g' : B \to C$, we have $g|_A = g'|_A$, we have $g = g'$.
- A homomorphism $h : A \to B$ is a $K$-epimorphism if for every $C \in K$ and homomorphisms $g, g' : B \to C$, if $gh = g'h$ then $g = g'$.

We say that $A$ is a proper epic substructure of $B$ in $K$ (and write $A <_e B$ in $K$), if $A \leq_e B$ in $K$ and $A \neq B$.

The next lemma explains the connection between epic substructures and epimorphisms.

**Lemma 2.** If $h : A \to B$ with $A, B, h(A) \in K$, then t.f.a.e.:
1. $h$ is a $K$-epimorphism.
2. The inclusion map $\iota : h(A) \to B$ is a $K$-epimorphism.
3. $h(A) \leq_e B$ in $K$. 

Proof. Immediate from the definitions. □

Here are some straightforward facts used in the sequel.

**Lemma 3.** Let $A, B \in \mathcal{K}$.

1. $A \leq_e B$ in $\mathcal{K}$ iff $A \leq_e B$ in $\mathbb{ISP}(\mathcal{K})$
2. Let $A \leq_e B$ in $\mathcal{K}$ and suppose $h : B \to C$ is such that $h(A), h(B) \in \mathcal{K}$.
   Then $h(A) \leq_e h(B)$ in $\mathcal{K}$.
3. Let $\mathcal{Q}$ be a quasivariety. T.f.a.e.:
   (a) $\mathcal{Q}$ has surjective epimorphisms.
   (b) For all $A, B \in \mathcal{Q}$ we have that $A \leq_e B$ in $\mathcal{Q}$ implies $A = B$.

3. **Main Theorem**

Recall that a primitive positive (p.p. for brevity) formula is one of the form $\exists \bar{y} \alpha(\bar{x}, \bar{y})$ with $\alpha(\bar{x}, \bar{y})$ a finite conjunction of atomic formulas. We shall need the following fact.

**Lemma 4.** ([6, Thm. 6.5.7]) Let $A, B$ be $\mathcal{L}$-structures. T.f.a.e.:

1. Every primitive positive $\mathcal{L}$-sentence that holds in $A$ holds in $B$.
2. There is a homomorphism from $A$ into an ultrapower of $B$.

Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. We say that the $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ defines a function in $\mathcal{K}$ if

$$\mathcal{K} \models \forall \bar{x}, \bar{y}, \bar{z}. \varphi(\bar{x}, \bar{y}) \land \varphi(\bar{x}, \bar{z}) \rightarrow \bigwedge_{j=1}^{m} y_j = z_j.$$ 

In that case, for each $A \in \mathcal{K}$ we write $[\varphi]^A$ to denote the $n$-ary partial function defined by $\varphi$ in $A$.

If $X$ is a set disjoint with $\mathcal{L}$, we write $\mathcal{L}_X$ to denote the language obtained by adding the elements in $X$ as new constant symbols to $\mathcal{L}$. If $\mathcal{B}$ is an $\mathcal{L}$-structure and $A$ is a subset of $B$, let $\mathcal{B}_A$ be the expansion of $\mathcal{B}$ to $\mathcal{L}_A$ where each new constant names itself. If $\mathcal{L} \subseteq \mathcal{L}^+$ and $A$ is an $\mathcal{L}^+$-model, let $A|_{\mathcal{L}}$ denote the reduct of $A$ to $\mathcal{L}$.

Next we present the main result of this article.

**Theorem 5.** Let $\mathcal{K}$ be a class closed under ultraproducts and $A \leq B$ structures. T.f.a.e.:

1. $A$ is an epic subalgebra of $B$ in $\mathcal{K}$.
2. For every $b \in B$ there are a primitive positive formula $\varphi(\bar{x}, y)$ and $\bar{a}$ from $A$ such that:
   (a) $\varphi(\bar{x}, y)$ defines a function in $\mathcal{K}$
   (b) $[\varphi]^{\mathcal{B}}(\bar{a}) = b$.

Proof. (1)⇒(2). We can assume that $\mathcal{K}$ is axiomatizable (replacing $\mathcal{K}$ by $\mathbb{ISP}(\mathcal{K})$ if necessary). Suppose $A \leq_e B$ in $\mathcal{K}$ and let $b \in B$. Define

$$\Sigma(x) := \{ \varphi(x) \mid \varphi(x) \text{ is a p.p. formula of } \mathcal{L}_A \text{ and } \mathcal{B}_A \models \varphi(b) \}.$$ 

Let $c, d$ be two new constant symbols and take

$$\mathcal{K}^* := \{ M \mid M \text{ is a } \mathcal{L}_A \cup \{ c, d \}-\text{model and } M|_{\mathcal{L}} \in \mathcal{K} \}.$$
Let $C$ be a model of $K^*$ such that $C \models \Sigma(c) \cup \Sigma(d)$. By Lemma 4, there are elementary extensions $E, E'$ of $C$, and homomorphisms

\[
\begin{align*}
h : B_A &\rightarrow E|_{L_A} \\
h' : B_A &\rightarrow E'|_{L_A}
\end{align*}
\]

such that $h(b) = c^C$ and $h'(b) = d^C$. The elementary amalgamation theorem [6, Thm. 6.4.1] provides us with an algebra $D$ and elementary embeddings $g : E \rightarrow D$, $g' : E' \rightarrow D$ such that $g$ and $g'$ agree on $C$. Next, observe that $gh : B \rightarrow D|_L$ and $g'h' : B \rightarrow D|_L$ are homomorphisms that agree on $A$, and since $D|_L \in K$ we must have $gh = g'h'$.

In particular $gh(b) = g'h'(b)$, which is $g(c^C) = g'(d^C)$. So, as $g$ is 1-1, and $g$ and $g'$ are the same on $C$ we have $c^C = d^C$.

Thus we have shown $K^* \models \bigwedge (\Sigma(c) \cup \Sigma(d)) \rightarrow c = d$.

By compactness (and using that the conjunction of p.p. formulas is equivalent to a p.p. formula), there is single p.p. $L$-formula $\varphi(\bar{x}, \bar{y})$ such that

\[
K^* \models \varphi(\bar{a}, c) \land \varphi(\bar{a}, d) \rightarrow c = d,
\]

and hence

\[
K \models \forall \bar{x}, y, z \varphi(\bar{x}, z) \land \varphi(\bar{x}, z) \rightarrow y = z.
\]

This completes the proof of (1)⇒(2).

(2)⇒(1). Suppose (2) holds for $A, B$ and $K$. Let $C \in K$ and $h, h' : B \rightarrow C$ homomorphisms agreeing on $A$. Fix $b \in B$. There are a p.p. formula $\varphi(\bar{x}, y)$ and $\bar{a}$ elements from $A$ such that

\[
B \models \varphi(\bar{a}, b) \\
K \models \forall \bar{x}, y, z \varphi(\bar{x}, y) \land \varphi(\bar{x}, z) \rightarrow y = z.
\]

Hence

\[
C \models \varphi(h\bar{a}, hb) \land \varphi(h'\bar{a}, h'b),
\]

and as $h\bar{a} = h'\bar{a}$ we have $hb = h'b$. □

It is worth noting that (2)⇒(1) in Theorem 5 always holds, i.e., it does not require for $K$ to be closed under ultraproducts. On the other hand, as the upcoming example shows, the implication (1)⇒(2) may fail if $K$ is not closed under ultraproducts.

**Example 6.** Let $L = \{s, 0\}$ where $s$ is a binary function symbol and $0$ a constant. Let $B$ be the $L$-structure with universe $\omega \cup \{\omega\}$ such that $0^B = 0$ and

\[
s^B(a, b) = \begin{cases} 0 & \text{if } b = a + 1, \\ 1 & \text{otherwise}. \end{cases}
\]

Take $A$ the subalgebra of $B$ with universe $\omega$. It is easy to see that the identity is the only endomorphism of $B$. Thus, in particular, we have that $A \leq_e B$ in $\{B\}$.

We prove next that there is no p.p. formula with parameters from $A$ defining $\omega$ in
B. Take $\mathcal{L}^+ := \mathcal{L}_B \cup \{\omega'\}$, where $\omega'$ is a new constant, and let $\Gamma$ be the $\mathcal{L}^+$-theory obtained by adding to the elementary diagram of $B$ the following sentences:

$$\{s(n, \omega') = 1 \mid n \in \omega\} \cup \{s(\omega', n) = 1 \mid n \in \omega\} \cup \{\omega \neq \omega'\}.$$ 

It is a routine task to show that $\Gamma$ is consistent. Fix a model $C$ of $\Gamma$ and define $h, h' : B \to C$ by $h(n) = h'(n) = n^C$ for all $n \in \omega$, $h(\omega) = \omega^C$ and $h'(\omega) = \omega^C$. Again, it is easy to see that $h$ and $h'$ are homomorphisms from $B$ to $C|_{\mathcal{L}}$. Since they agree on $A$ and $h(\omega) \neq h'(\omega)$, we conclude that there is no p.p. formula with parameters from $A$ defining $\omega$ in $B$.

3.1. The finite case. When $\mathcal{K}$ is (up to isomorphisms) a finite set of finite structures, we can sharpen Theorem 5. In this case it is possible to avoid the existential quantifiers in the definable functions at the cost of adding parameters from $B$.

**Theorem 7.** Let $\mathcal{K}$ be (up to isomorphisms) a finite set of finite structures, and let $A \leq B$ be finite. T.f.o.e.: 

(1) $A$ is an epic substructure of $B$ in $\mathcal{K}$.
(2) For every $b_1 \in B$ there are a finite conjunction of atomic formulas $\alpha(\bar{x}, \bar{y})$, $a_1, \ldots, a_n \in A$ and $b_2, \ldots, b_m \in B$, with $m \geq 1$, such that
   (a) $\alpha(\bar{x}, \bar{y})$ defines a function in $\mathcal{K}$
   (b) $[\alpha]^B(\bar{a}) = b$.
(3) For every $b \in B$ there are a primitive positive formula $\varphi(\bar{x}, \bar{y})$ and $\bar{a}$ from $A$ such that:
   (a) $\varphi(\bar{x}, \bar{y})$ defines a function in $\mathcal{K}$
   (b) $[\varphi]^B(\bar{a}) = b$.

**Proof.** (1)$\Rightarrow$(2). If $b_1 \in A$ the formula $x_1 = y_1$ does the job. Suppose $b_1 \notin A$, and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ be enumerations of $A$ and $B \setminus A$ respectively. Let

$$\Delta(\bar{x}, \bar{y}) := \{\delta(\bar{x}, \bar{y}) \mid \delta(\bar{x}, \bar{y}) \text{ is an atomic formula and } B \models \delta(\bar{a}, \bar{b})\}.$$ 

Since $\mathcal{K}$ is a finite set of finite structures, there are finitely many formulas in $\Delta(\bar{x}, \bar{y})$ up to logical equivalence in $\mathcal{K}$. Thus, there is a finite conjunction of atomic formulas $\alpha(\bar{x}, \bar{y})$ such that

$$\mathcal{K} = \alpha(\bar{x}, \bar{y}) \iff \bigwedge \Delta(\bar{x}, \bar{y}).$$

Take $C \in \mathcal{K}$ and suppose $C \models \alpha(\bar{c}, \bar{d}) \land \alpha(\bar{c}, \bar{e})$. Then the maps $h, h' : B \to C$, given by $h : \bar{a}, \bar{b} \mapsto \bar{c}, \bar{d}$ and $h' : \bar{a}, \bar{b} \mapsto \bar{c}, \bar{e}$, are homomorphisms. Since $h$ and $h'$ agree on $A$, it follows that $h = h'$. Hence $\bar{d} = \bar{e}$, and we have shown that $\alpha(\bar{x}, \bar{y})$ defines a function in $\mathcal{K}$.

(2)$\Rightarrow$(3). The p.p. formulas in (3) can be obtained by adding existential quantifiers to the formulas given by (2).

(3)$\Rightarrow$(1). This is the same as (2)$\Rightarrow$(1) in Theorem 5.

Again, it is worth noting that implications (2)$\Rightarrow$(3)$\Rightarrow$(1) hold for any $A$, $B$ and $\mathcal{K}$.

The example below shows that, in the general case, the existential quantifiers in (2) of Theorem 5 are necessary.

**Example 8.** Let $B$ be the Browerian algebra whose lattice reduct is depicted in Figure 3.1, and let $A$ be the subalgebra of $B$ with universe $\{a_0, a_1, \ldots\} \cup \{\top\}$. It is proved in [1] Thm. 6.1] that $A \leq_{ep} B$ in $\mathbb{V}(B)$. We show that (2) in Theorem 7 does not hold for $A$, $B$ and $\mathbb{V}(B)$. Towards a contradiction fix $d_1 \in B \setminus A$, and suppose
there are a conjunction of equations \( \alpha(x_1, \ldots, x_n, y_1, \ldots, y_m), c_1, \ldots, c_n \in A \) and \( d_2, \ldots, d_m \in B \) such that

- \( \alpha(\bar{x}, \bar{y}) \) defines a function in \( V(B) \)
- \( B \models \alpha(\bar{c}, \bar{d}) \).

Let \( C \) and \( D \) be the subalgebras of \( B \) generated by \( \bar{c} \) and \( \bar{c}, \bar{d} \) respectively. Note that \( D \) is finite and \( C < D \). Also note that \( \alpha(\bar{x}, \bar{y}) \) defines a function in \( V(D) \), and \( D \models \alpha(\bar{c}, \bar{d}) \), because \( \alpha \) is quantifier-free. So we have \( C <_e D \) in \( V(D) \); but this is not possible, as Corollary 5.5 in [1] implies that there are no proper epic subalgebras in finitely generated varieties of Browerian algebras.

4. Checking for epic subalgebras in a subclass

In the current section all languages considered are algebraic, i.e., without relation symbols. Given a quasivariety \( Q \) it can be a daunting task to determine whether \( Q \) has surjective epimorphisms, or equivalently, no proper epic subalgebras. In this section we prove two results that, under certain assumptions on \( Q \), provide a (hopefully) more manageable class \( S \subseteq Q \) such that \( Q \) has no proper epic subalgebras iff \( S \) has no proper epic subalgebras.

Our first result provides such a class \( S \) for quasivarieties with a near-unanimity term. The second one for arithmetical varieties whose class of finitely subdirectly irreducible members is universal.

4.1. Quasivarieties with a near-unanimity term. An \( n \)-ary term \( t(x_1, \ldots, x_n) \) is a near-unanimity term for the class \( K \) if \( n \geq 3 \) and \( K \) satisfies the identities

\[
t(x, \ldots, x, y) = t(x, \ldots, x, y, x) = \cdots = t(y, x, \ldots, x) = x.
\]

When \( n = 3 \) the term \( t \) is called a majority term for \( K \). In every structure with a lattice reduct the term \( (x \lor y) \land (x \lor z) \land (y \lor z) \) is a majority term. This example is specially relevant since many classes of structures arising from logic algebrizations have lattice reducts.
For the sake of the exposition the results are presented for quasivarieties with a majority term. They are easily generalized to quasivarieties with an arbitrary near-unanimity term.

For functions \( f : A \to A' \) and \( g : B \to B' \) let \( f \times g : A \times B \to A' \times B' \) be defined by \( f \times g(a,b) := (f(a),g(b)) \).

**Theorem 9.** Let \( K \) be a class of structures with a majority term and suppose \( \varphi(\bar{x},y) \) defines a function in \( K \). T.f.a.e.:

(1) There is a term \( t(\bar{x}) \) such that \( K \models \forall \bar{x},y \varphi(\bar{x},y) \to y = t(\bar{x}) \).

(2) For all \( A, B \in \mathbb{P}_u(K) \), all \( S \subseteq A \times B \) and all \( s_1, \ldots, s_n \in S \) such that \( \varphi^A \times \varphi^B(\bar{s}) \) is defined, we have that \( \varphi^A \times \varphi^B(\bar{s}) \in S \).

An algebra \( A \) in the quasivariety \( Q \) is relatively subdirectly irreducible provided its diagonal congruence is completely meet irreducible in the lattice of \( Q \)-congruences of \( A \). We write \( Q_{RSI} \) to denote the class of relatively subdirectly irreducible members of \( Q \). For a class \( K \) let \( K \times K := \{ A \times B \mid A, B \in K \} \).

**Theorem 10.** Let \( Q \) be a quasivariety with a majority term and let \( S = \mathbb{P}_u(Q_{RSI}) \). T.f.a.e.:

(1) \( Q \) has surjective epimorphisms.

(2) For all \( A, B \in Q \) we have that \( A \leq_e B \) in \( Q \) implies \( A = B \).

(3) For all \( A, B \in S(S \times S) \) we have that \( A \leq_e B \) in \( S \times S \) implies \( A = B \).

(4) \( S(S \times S) \) has surjective epimorphisms.

**Proof.** The equivalences (1)\( \Leftrightarrow \) (2) and (3)\( \Leftrightarrow \) (4) are immediate, and (2) clearly implies (3). We prove (3)\( \Rightarrow \) (2). Suppose \( A \leq_e B \) in \( Q \) and let \( b \in B \). We shall see that \( b \in A \). By Theorem 9 there is a p.p. \( L \)-formula \( \varphi(\bar{x},y) \) defining a function in \( Q \), and such that \( \varphi^B(\bar{a}) = b \) for some \( \bar{a} \in A^n \). Let

\[
\Sigma := \{ \varepsilon \mid \varepsilon \text{ is a p.p. formula of } \mathcal{L}_A \text{ and } B_A \models \varepsilon \},
\]

and define

\[
K := \{ C \in \text{Mod}(\Sigma) \mid C|_{L} \in S \}.
\]

Let \( \psi(y) := \varphi(\bar{a},y) \), and note that \( \psi(y) \) defines a nullary function in \( K \). Note as well that \( \exists \psi(y) \in \Sigma \), and hence \( \psi^K \) is defined for every \( K \in K \). We aim to apply Theorem 9 to \( K \) and \( \psi(y) \). To this end fix \( C, D \in \mathbb{P}_u(K) = K \) and let \( S \subseteq C \times D \). Note that as \( \Sigma \) is a set of p.p. formulas we have \( C \times D \vdash \Sigma \), and thus by Lemma 4 there is an ultrapower \( E \) of \( C \times D \) and a homomorphism \( h : B_A \to E \). We have that \( E \in \mathbb{P}_u(K \times K) \subseteq \mathbb{P}_u(K) \times \mathbb{P}_u(K) = K \times K \), and so

\[
E|_{L} \in K|_{L} \times K|_{L} \subseteq S \times S.
\]

Next observe that since \( h(A) \leq_e h(B) \) in \( Q \), and \( h(A), h(B) \leq E|_{L} \), by (3) it follows that \( h(A) = h(B) \). Also, as \( S \) is an \( \mathcal{L}_A \)-subalgebra of \( E \), we have that

\[
h(B_A) = h(A_A) \leq S.
\]

The fact that \( B \models \psi(b) \) implies \( E \models \psi(hb) \), and so \( [\psi]^E = hb \in S \). We know that \( \{ C, D, C \times D \} \models \exists y \psi(y) \); furthermore, since \( \psi \) is p.p., we have \( [\psi]^C \times [\psi]^D = [\psi]^{C \times D} \). Putting all this together

\[
[\psi]^C 	imes [\psi]^D = [\psi]^{C \times D} = [\psi]^E \in S.
\]

Thus, Theorem 9 produces an \( \mathcal{L}_A \)-term \( t \) such that

\[
(4.1) \quad K \models \forall y \psi(y) \to y = t.
\]
In particular, for all $C \in Q_{RSI}$ and all $c_1, \ldots, c_n \in C$ such that $[\varphi]^C(c)$ is defined, we have

$$[\varphi]^C(c) = t^C(c).$$

Next let $\{B_i \mid i \in I\} \subseteq Q_{RSI}$ such that $B \leq \prod_{i} B_i$ is a subdirect product. For every $i \in I$ let $B^A_i$ be the expansion of $B_i$ to $\mathcal{L}_A$ given by $aB^A_i = \pi_i(a)$, where $\pi_i : B \to B_i$ is the projection map. It is clear that

$$(4.2) \quad B_A \leq \prod_i B^A_i.$$ 

Now, each $B^A_i$ is a homomorphic image of $B_A$, so $B^A_i \models \Sigma$ and thus $B^A_i \in K$ for all $i \in I$. Since $\forall y \psi(y) \to y = t$ is (equivalent to) a quasi-identity, from (4.1) and (4.2) we have

$$B_A \models \forall y \psi(y) \to y = t.$$ 

Hence $b = t^{B_A} \in A$, and the proof is finished. \(\square\)

Observe that Theorem 10 holds for any $S \subseteq Q$ closed under ultraproducts and containing $Q_{RSI}$.

**Corollary 11.** Let $Q$ be a finitely generated quasivariety with a majority term. T.f.a.e.:

1. $Q$ has surjective epimorphisms.
2. $\mathcal{S}(Q_{RSI} \times Q_{RSI})$ has surjective epimorphisms.

**Proof.** For any class $K$ we have $Q(K)_{RSI} \subseteq \mathcal{ISP}_u(K)$. Thus if $Q$ is finitely generated, then $Q_{RSI}$ is (up to isomorphic copies) a finite set of finite algebras, and the corollary follows at once from Theorem 10. \(\square\)

Recall that an algebra $A$ is **finitely subdirectly irreducible** if its diagonal congruence is meet irreducible in the congruence lattice of $A$. It is **subdirectly irreducible** if the diagonal is completely meet irreducible. For a variety $V$ we write $(V_{FSI})_V$ to denote its class of (finitely) subdirectly irreducible members.

An interesting consequence of Corollary 11 is the following.

**Corollary 12.** Let $F$ be a finite set of finite algebras with a common majority term. It is decidable whether the (quasi)variety generated by $F$ has surjective epimorphisms.

**Proof.** Let $\mathcal{V}$ be the variety generated by $F$. By Jónsson’s lemma 17, $V_{SI} \subseteq \mathcal{HISP}_u(F) = \mathcal{H}(F)$ is a finite set of finite structures, and by Corollary 11 it suffices to decide whether $\mathcal{S}(V_{SI} \times V_{SI})$ has surjective epimorphisms, and this is clearly a decidable problem. If $Q$ is the quasivariety generated by $F$, then $Q_{RSI} \subseteq \mathcal{ISP}_u(F) = \mathcal{I}(F)$, and the same reasoning applies. \(\square\)

4.2. **Arithmetical varieties whose FSI members form a universal class.** A variety $\mathcal{V}$ is **arithmetical** if for every $A \in \mathcal{V}$ the congruence lattice of $A$ is distributive and the join of any two congruences is their composition. For example, the variety of boolean algebras is arithmetical.

**Lemma 13.** Let $\mathcal{V}$ be an arithmetical variety such that $V_{FSI}$ is a universal class, and let $\varphi(x, y)$ be a p.p. formula defining a function in $\mathcal{V}$. Suppose that for all $A \in V_{FSI}$, all $S \leq A$ and all $s_1, \ldots, s_n \in S$ such that $A \models \exists y \varphi(s, y)$, we have $S \models \exists y \varphi(s, y)$. Then there is a term $t(x)$ such that $\mathcal{V} \models \forall x, y \varphi(x, y) \to y = t(x)$. 
Proof. Add new constants $c_1, \ldots, c_n$ to the language of $V$ and let $K := \{ (A, \vec{a}) \mid A \models \forall y \varphi(\vec{c}, y) \text{ and } A \in \mathcal{V}_{FSI} \}$. Note that $\psi(y) := \varphi(\vec{c}, y)$ defines a nullary function in $K$, and this function is defined for every member of $K$. Also note that by our assumptions $K$ is a universal class. Using Jónsson's lemma \[7\] it is not hard to show that $\forall(K)_{FSI} = K$. Since $K|_L$ is contained in an arithmetical variety it has a Pixley Term \[3\] Thm. 12.5, which also serves as a Pixley Term for $K$, and thus $\forall(K)$ is arithmetical. Next we show that $\psi(y)$ is equivalent to a positive open formula in $K$. By \[4\] Thm. 3.1 it suffices to show that

- For all $A, B \in K$, all $S \leq A$, all $h : S \to B$ and every $a \in A$ we have that $A \models \psi(a)$ implies $B \models \psi(ha)$.

So suppose $A \models \psi(a)$. From our hypothesis and the fact that $\psi(y)$ defines a function we have $S \models \psi(a)$, and as $\psi(y)$ is p.p. we obtain $B \models \psi(ha)$. Hence there is a positive open formula $\beta(y)$ equivalent to $\psi(y)$ in $K$. Now, \[5\] Thm. 2.3 implies that there is an $\mathcal{L} \cup \{ c_1, \ldots, c_n \}$-term $t'$ such that $\forall(K) \models \alpha(t')$. Let $(x_1, \ldots, x_n)$ be an $\mathcal{L}$-term such that $t' = (\vec{c})$. So, if $\Gamma$ is a set of axioms for $\mathcal{V}_{FSI}$, we have

$$\Gamma \cup \{ \exists y \varphi(\vec{c}, y) \} \models \varphi(\vec{c}, t(\vec{c})),$$

and this implies

$$\Gamma \models \exists y \varphi(\vec{c}, y) \to \varphi(\vec{c}, t(\vec{c})),$$

or equivalently

$$\mathcal{V}_{FSI} \models \forall y(\varphi(\vec{c}, y) \to \varphi(\vec{c}, t(\vec{c}))).$$

This and the fact that that $\varphi(\vec{x}, y)$ defines a function in $V$ yields

$$\mathcal{V}_{FSI} \models \forall \vec{x}, y \varphi(\vec{x}, y) \to y = t(\vec{x}).$$

To conclude, note that $\forall \vec{x}, y \varphi(\vec{x}, y) \to y = t(\vec{x})$ is logically equivalent to a quasi-identity, and since it holds in $\mathcal{V}_{FSI}$ it must hold in $\mathcal{V}$. \[\square\]

Theorem 14. Let $V$ be an arithmetical variety such that $\mathcal{V}_{FSI}$ is a universal class $T.f.a.e.$:

1. $\mathcal{V}$ has surjective epimorphisms.
2. For all $A, B \in \mathcal{V}$ we have that $A \leq_e B$ in $\mathcal{V}$ implies $A = B$.
3. For all $A, B \in \mathcal{V}_{FSI}$ we have that $A \leq_e B$ in $\mathcal{V}_{FSI}$ implies $A = B$.
4. $\mathcal{V}_{FSI}$ has surjective epimorphisms.

Proof. We prove (3)⇒(2) which is the only nontrivial implication. Suppose $A \leq_e B$ in $\mathcal{V}$ and let $b \in B$. We shall see that $b \in A$. By Theorem\[5\] there is a p.p. $\mathcal{L}$-formula $\varphi(\vec{x}, y)$ defining a function in $\mathcal{V}$, and such that $[\varphi]^B(\vec{a}) = b$ for some $\vec{a} \in A^n$. Let

$$\Sigma := \{ \varepsilon \mid \varepsilon \text{ is a p.p. sentence of } \mathcal{L}_A \text{ and } B_A \models \varepsilon \},$$

and define

$$K := \{ C \in \text{Mod}(\Sigma) \mid C|_L \in \mathcal{V}_{FSI} \}.$$

Claim. $K$ is a universal class.

Since $K$ is axiomatizable we only need to check that $K$ is closed under substructures. Let $C \leq D \in K$; clearly $C|_L \in \mathcal{V}_{FSI}$, so it remains to see that $C \models \Sigma$. As $D \models \Sigma$, Lemma\[4\] yields a homomorphism $h : B_A \to E$ with $E$ an ultrapower of $D$. Note that $E \in K$. Since $h(A) \leq_e h(B)$ in $\mathcal{V}$ and $h(A), h(B) \in \mathcal{V}_{FSI}$, it follows...
that \( h(A) = h(B) \), because there are no proper epic subalgebras in \( V_{FSI} \). Now \( C \) is an \( E_A \)-subalgebra of \( D \), so \( h(B) = h(A) \subseteq C \). Finally, since \( h(B) \models \Sigma \) and every sentence in \( \Sigma \) is existential, we obtain \( C \models \Sigma \). This finishes the proof of the claim.

**Claim.** \( V(K) \) is arithmetical and \( V(K)_{FSI} = K \).

To show that \( V(K) \) is arithmetical we can proceed as in the proof of Lemma 13. We prove \( V(K)_{FSI} = K \). Note that for \( C \in K \) we have that \( C \) and \( C|_E \) have the same congruences; hence every algebra in \( K \) is FSI. For the other inclusion, Jonsson’s lemma \([7]\) produces \( V(K)_{FSI} \subseteq \text{HSP}_u(K) \), and by the first claim \( \text{HSP}_u(K) = \text{H}(K) \). So, as \( \text{H}(K) \models \Sigma \), we have that \( V(K)_{FSI} \models \Sigma \) and thus \( V(K)_{FSI} \subseteq K \).

Next we want to apply Lemma 13 to \( V(K) \) and \( \varphi(\bar{a}, y) \), so we need to check that the hypothesis hold. Take \( C \in K \) and \( S \leq C \). Since \( K \) is universal we have \( S \in K \), and thus \( S \models \exists y \varphi(\bar{a}, y) \). Let \( t \) be a term such that \( V(K) \models \forall y \varphi(\bar{a}, y) \to y = t \). Then \( b = t^B \in A \), and we are done. \( \square \)

Every discriminator variety (see \([3, \text{Def. 9.3}]\) for the definition) satisfies the hypothesis in Theorem 14. Furthermore, in such a variety every FSI member is simple (i.e., has exactly two congruences). Writing \( V_S \) for the class of simple members in \( V \) we have the following immediate consequence of Theorem 14.

**Corollary 15.** For a discriminator variety \( V \) the following are equivalent.

1. \( V \) has surjective epimorphisms.
2. For all \( A, B \in V \) we have that \( A \leq_e B \) in \( V \) implies \( A = B \).
3. For all \( A, B \in V_S \) we have that \( A \leq_e B \) in \( V_S \) implies \( A = B \).
4. \( V_S \) has surjective epimorphisms.

It is not uncommon for a variety arising as the algebrization of a logic to be a discriminator variety; thus the above corollary could prove helpful in establishing the Beth definability property for such a logic.

Another special case relevant to algebraic logic to which Theorem 14 applies is given by the class of Heyting algebras and its subvarieties (none of these are discriminator varieties with the exception of the class of boolean algebras). Heyting algebras constitute the algebraic counterpart to intuitionistic logic, and have proven to be a fertile ground to investigate definability and interpolation properties of intuitionistic logic and its axiomatic extensions by algebraic means (see \([1]\) and its references).

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