On the zeros of $L$-functions

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Abstract

We generalize our recent construction of the zeros of the Riemann $\zeta$-function to two infinite classes of $L$-functions, Dirichlet $L$-functions and those based on level one modular forms. More specifically, we show that there are an infinite number of zeros on the critical line which are in one-to-one correspondence with the zeros of the cosine function, and thus enumerated by an integer $n$. We obtain an exact equation on the critical line that determines the $n$-th zero of these $L$-functions. We show that the counting formula on the critical line derived from such an equation agrees with the known counting formula on the entire critical strip. We provide numerical evidence supporting our statements, by computing numerical solutions of this equation, yielding $L$-zeros to high accuracy. We study in detail the $L$-function for the modular form based on the Ramanujan $\tau$-function, which is closely related to the bosonic string partition function. The same analysis for a more general class of $L$-functions is also considered.

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1. INTRODUCTION

Dirichlet $L$-series are functions of a complex variable $z$ defined by the series

$$L(z) = \sum_{n=1}^{\infty} \frac{a(n)}{n^z}, \quad \Re(z) > 1$$

(1)

where $a(n)$ is an arbitrary arithmetic function. In this paper we will consider two infinite classes of important $L$-functions, the Dirichlet $L$-functions where $a(n) = \chi(n)$ is a Dirichlet character, and $L$-functions associated with modular forms. The former have applications primarily in multiplicative number theory, whereas the latter in additive number theory.

The Dirichlet $L$-functions are generalizations of the Riemann $\zeta$-function, the latter being the simplest example \[1\]. They can be analytically continued to the entire complex plane. The Generalized Riemann Hypothesis (GRH) is the conjecture that all non-trivial zeros of these functions are on the critical line, i.e. have real part equal to $\frac{1}{2}$. Proving the validity of the GRH would have many implications in number theory. Much less is known about the zeros of Dirichlet $L$-functions in comparison with the $\zeta$-function, however let us mention a few works. Selberg \[2\] obtained the analog of Riemann-von Mangoldt counting formula $N(T, \chi)$ for the number of zeros up to height $T$ within the entire critical strip $0 \leq \Re(z) \leq 1$. Based on this result, Fujii \[3\] gave an estimate for the number of zeros on the critical strip with the ordinate between $[T + H, T]$. The distribution of low lying zeros of $L$-functions near and at the critical line was examined in \[4\], assuming the GRH. The statistics of the zeros, i.e. the analog of the Montgomery-Odlyzko conjecture, were studied in \[5, 6\]. It is also known that more than half of the non-trivial zeros are on the critical line \[7\]. For a more detailed introduction to $L$-functions see \[8\].

Besides the Dirichlet $L$-functions, there are more general constructions of $L$-functions based on arithmetic and geometric objects, like varieties over number fields and modular forms \[9, 10\]. Some results for general $L$-functions are still conjectural. For instance, it is not even clear if some $L$-functions can be analytically continued into a meromorphic function. We will only consider the additional $L$-functions based on modular forms here. Thus the $L$-functions considered in this paper have similar properties, namely, they possess an Euler product, can be analytic continued into the (upper half) complex plane, except for possible poles at $z = 0$ and $z = 1$, and satisfy a functional equation.

In our previous work \[12\], a new approach to the characterization of zeros of the $\zeta$-function
was developed, building on the earlier work [11]. Enumerating the zeros on the critical line as \( \rho_n = \frac{1}{2} + iy_n \), with \( n = 1, 2, \ldots \), an exact transcendental equation for the imaginary parts of the zeros \( y_n \) was derived which depends only on \( n \). An asymptotic version of these equations was first proposed in [11]. From these equations for the zeros on the critical line, one can derive the Riemann-van Mangoldt and the exact Backlund counting formulae for the zeros on the entire strip, thus this result strongly indicates that all non-trivial zeros of the \( \zeta \)-function are on the critical line. These transcendental equations can easily be solved numerically to very high precision, even for high zeros such as the billion’th, with simple implementations such as are available on Mathematica. Also, various approximate versions of the equation were presented.

In this work, we extend the results obtained in [12] to the two infinite classes of \( L \)-functions mentioned above. In particular, we derive an exact equation for the \( n \)-th zero which only depends on \( n \) and some very elementary properties of the mathematical object that the \( L \)-function is based on, for instance, the Gauss sum of the Dirichlet character or the weight of its modular form. As for the \( \zeta \)-function, the derivation presents an argument that all the zeros are on the critical line. We will not present our derivation in as much detail as we did for the \( \zeta \)-function case [12], since our analysis follows precisely the same steps.

Our results are presented as follows. In section II A we review some of the main properties of Dirichlet characters and their \( L \)-functions, all of which are well-known. In section II B we derive the exact equation for the \( n \)-th zero of Dirichlet \( L \)-functions (24). One interesting new feature, in comparison with the \( \zeta \)-function, is that if the characters are complex numbers, then the zeros on the negative \( y \)-axis are not symmetrical to the ones lying on the positive \( y \)-axis. In section II C, we consider an approximation to the exact equation based on the leading order of the generalized Riemann-Siegel \( \vartheta_{k,a} \) function (21) for large \( y \). From this, one can derive counting formulas \( N_0^\pm(T, \chi) \) for both the positive and negative imaginary \( y \)-axes. An approximate formula for the zeros can be found explicitly in terms of the Lambert \( W \) function. In section II G we find numerical solutions to our equation (24) to high accuracy, considering two different examples of Dirichlet \( L \)-functions based on characters with modulus 7. In section III we repeat the same steps but for \( L \)-functions based on modular forms, and we consider a specific example based on the Ramanujan \( \tau \)-function, connected with the Dedekind \( \eta \)-function and the bosonic string partition function. In appendix A we derive the main formulas for a more general, but not specific, class of \( L \)-functions.
II. DIRICHLET L-FUNCTIONS

A. Dirichlet characters and $L$-functions

In this section we recall some of the main properties of Dirichlet characters and $L$-functions based on it [1].

Dirichlet $L$-series are defined as

$$ L(z, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z}, \quad \Re(z) > 1 \quad (2) $$

where the arithmetic function $\chi(n)$ is a Dirichlet character. They can all be analytically continued to the entire complex plane, and are then referred to as Dirichlet $L$-functions.

There are an infinite number of distinct Dirichlet characters which are primarily characterized by their modulus $k$, which determines their periodicity. They can be defined axiomatically, which leads to specific properties, some of which we now describe. Consider a Dirichlet character $\chi \mod k$, and let the symbol $(n,k)$ denote the greatest common divisor of the two integers $n$ and $k$. Then $\chi$ has the following properties:

1. $\chi(n + k) = \chi(n)$.

2. $\chi(1) = 1$ and $\chi(0) = 0$.

3. $\chi(nm) = \chi(n)\chi(m)$.

4. $\chi(n) = 0$ if $(n, k) > 1$ and $\chi(n) \neq 0$ if $(n, k) = 1$.

5. If $(n, k) = 1$ then $\chi(n)^{\varphi(k)} = 1$, where $\varphi(k)$ is the Euler totient arithmetic function. This implies that $\chi(n)$ are roots of unity.

6. If $\chi$ is a Dirichlet character so is the complex conjugate $\chi^*$.

For a given modulus $k$ there are $\varphi(k)$ distinct Dirichlet characters, which essentially follows from property [5] above. They can thus be labeled as $\chi_{k,j}$ where $j = 1, 2, \ldots, \varphi(k)$ denotes an arbitrary ordering. If $k = 1$ we have the trivial character where $\chi(n) = 1$ for every $n$, and (2) reduces to the Riemann $\zeta$-function. The principal character, usually denoted $\chi_1$, is defined as $\chi_1(n) = 1$ if $(n, k) = 1$ and zero otherwise. In the above notation the principal character is always $\chi_{k,1}$. 
Characters can be classified as *primitive* or *non-primitive*. Consider the Gauss sum

\[ G(\chi) = \sum_{m=1}^{k} \chi(m)e^{2\pi im/k}. \]  

If the character \( \chi \) mod \( k \) is primitive, then \( |G(\chi)|^2 = k \). This is no longer valid for a non-primitive character. Consider a non-primitive character \( \overline{\chi} \) mod \( \overline{k} \). Then it can be expressed in terms of a primitive character of smaller modulus as \( \overline{\chi}(n) = \overline{\chi_1}(n)\chi(n) \), where \( \overline{\chi_1} \) is the principal character mod \( \overline{k} \) and \( \chi \) is a primitive character mod \( k < \overline{k} \), where \( k \) is a divisor of \( \overline{k} \). More precisely, \( k \) must be the *conductor* of \( \overline{\chi} \) (see [1] for further details). In this case the two \( L \)-functions are related as

\[ L(z, \overline{\chi}) = L(z, \chi)\Pi_p \left( 1 - \chi(p)/p^z \right). \]

Thus \( L(z, \overline{\chi}) \) has the same zeros as \( L(z, \chi) \). The principal character is only primitive when \( k = 1 \), which yields the \( \zeta \)-function. The simplest example of non-primitive characters are all the principal ones for \( k \geq 2 \), whose zeros are the same as the \( \zeta \)-function. Let us consider another example with \( \overline{k} = 6 \), where \( \varphi(6) = 2 \), namely \( \overline{\chi}_{6,2} \), whose components are \([19]\)

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| \( \overline{\chi}_{6,2}(n) \) | 1 | 0 | 0 | 0 | -1 | 0 |

In this case, the only divisors are 2 and 3. Since \( \chi_1 \) mod 2 is non-primitive, it is excluded. We are left with \( k = 3 \) which is the conductor of \( \overline{\chi}_{6,2} \). Then we have two options; \( \chi_{3,1} \) which is the non-primitive principal character mod 3, thus excluded, and \( \chi_{3,2} \) which is primitive. Its components are

| \( n \) | 1 | 2 | 3 |
|---|---|---|---|
| \( \chi_{3,2}(n) \) | 1 | -1 | 0 |

Note that \( |G(\chi_{6,2})|^2 = 3 \neq 6 \) and \( |G(\chi_{3,2})|^2 = 3 \). In fact one can check that \( \overline{\chi}_{6,2}(n) = \overline{\chi}_{6,1}(n)\chi_{3,2}(n) \), where \( \overline{\chi}_{6,1} \) is the principal character mod \( \overline{k} = 6 \). Thus the zeros of \( L(z, \overline{\chi}_{6,2}) \) are the same as those of \( L(z, \chi_{3,2}) \). Therefore, it suffices to consider primitive characters, and we will henceforth do so.

We will need the functional equation satisfied by \( L(z, \chi) \). Let \( \chi \) be a *primitive* character. Define its *order* \( a \) such that

\[ a \equiv \begin{cases} 
1 & \text{if } \chi(-1) = -1 \quad \text{(odd order)} \\
0 & \text{if } \chi(-1) = 1 \quad \text{(even order)}
\end{cases} \]  

(6)
Let us define the meromorphic function

\[ \Lambda(z, \chi) \equiv \left( \frac{k}{\pi} \right)^{\frac{z+a}{2}} \Gamma \left( \frac{z + a}{2} \right) L(z, \chi). \]  

(7)

Then \( \Lambda \) satisfies the well known functional equation \( \Pi \)

\[ \Lambda(z, \chi) = \frac{i^{-a} G(\chi)}{\sqrt{k}} \Lambda(1 - z, \chi^*). \]  

(8)

The above equation is only valid for primitive characters.

B. The exact transcendental equation for the \( n \)-th zero on the critical line

In this section we derive an exact equation satisfied by zeros enumerated by an integer \( n \). The analysis that leads to this equation is the same as for the \( \zeta \)-function in \( \Pi \Pi \), \( \Pi \Pi \), consequently we do not provide as detailed a derivation, since such details can be surmised from \( \Pi \Pi \).

For a primitive character, since \( |G(\chi)| = \sqrt{k} \), the factor on the right hand side of (8) is a phase. It is thus possible to obtain a more symmetric form of (8) through a new function defined as

\[ \xi(z, \chi) \equiv \frac{i^{a/2} k^{1/4}}{\sqrt{G(\chi)}} \Lambda(z, \chi). \]  

(9)

It then satisfies

\[ \xi(z, \chi) = \xi^*(1 - z, \chi) \equiv (\xi(1 - z^*, \chi^*))^*. \]  

(10)

Above, the function \( \xi^* \) of \( z \) is defined as the complex conjugation of all coefficients that define \( \xi \), namely \( \chi \) and the \( i^{a/2} \) factor, evaluated at a non-conjugated \( z \).

Note that \((\Lambda(z, \chi))^* = \Lambda(z^*, \chi^*)\). Using the well known result \( G(\chi^*) = \chi(-1) (G(\chi))^* \) we conclude that

\[ (\xi(z, \chi))^* = \xi(z^*, \chi^*). \]  

(11)

This implies that if the character is real, then if \( \rho \) is a zero of \( \xi \) so is \( \rho^* \), and one needs only consider \( \rho \) with positive imaginary part. On the other hand if \( \chi \neq \chi^* \), then the zeros with negative imaginary part are different than \( \rho^* \). For the trivial character where \( k = 1 \) and \( a = 0 \), implying \( \chi(n) = 1 \) for any \( n \), then \( L(z, \chi) \) reduces to the Riemann \( \zeta \)-function and \( \Pi \Pi \) yields the well known functional equation \( \xi(z) = \xi(1 - z) \) with \( \xi(z) = \pi^{-z/2} \Gamma(z/2) \zeta(z) \).
Let \( z = x + iy \). Then the function (9) can be written as

\[
\xi(z, \chi) = A(x, y, \chi) \exp \{i\theta(x, y, \chi)\},
\]

where

\[
A(x, y, \chi) = \left( \frac{k}{\pi} \right)^{x+\frac{a}{2}} \left| \Gamma \left( \frac{x + a + iy}{2} \right) \right| |L(x + iy, \chi)|,
\]

\[
\theta(x, y, \chi) = \arg \Gamma \left( \frac{x + a + iy}{2} \right) - \frac{y}{2} \log \left( \frac{\pi}{k} \right) - \frac{1}{2} \arg G(\chi) + \arg L(x + iy, \chi) + \frac{\pi a}{4}.
\]

From (11) we also conclude that \( A(x, y, \chi) = A(x, -y, \chi^*) \) and \( \theta(x, y, \chi) = -\theta(x, -y, \chi^*) \).

Denoting

\[
\xi^*(1 - z, \chi) = A'(x, y, \chi) \exp \{-i\theta'(x, y, \chi)\}
\]

we therefore have

\[
A'(x, y, \chi) = A(1 - x, y, \chi), \quad \theta'(x, y, \chi) = \theta(1 - x, y, \chi).
\]

Taking the modulus of (10) we also have that \( A(x, y, \chi) = A'(x, y, \chi) \) for any \( z \).

On the critical strip, the functions \( L(z, \chi) \) and \( \xi(z, \chi) \) have the same zeros. Thus the zeros can be defined by \( \xi(\rho, \chi) = 0 \). Thus on a zero we clearly have

\[
\lim_{\delta \to 0^+} \{ \xi(\rho + \delta, \chi) + \xi^*(1 - \rho - \delta, \chi) \} = 0.
\]

Let us denote

\[
B(x, y, \chi) = e^{i\theta(x, y, \chi)} + e^{-i\theta'(x, y, \chi)}.
\]

Since \( A = A' \) everywhere, from (17) we conclude that on a zero we have

\[
\lim_{\delta \to 0^+} A(x + \delta, y, \chi) B(x + \delta, y, \chi) = 0.
\]

This equation is satisfied if \( A = 0, B = 0 \) or both, where the \( \delta \to 0^+ \) limit is implicit here and in the following. The function \( A \) has the same zeros as \( L(z, \chi) \) since \( A(x, y, \chi) \propto |L(z, \chi)| \).

There is more structure in \( B \), so let us consider its zeros. This equation has the general solution \( \theta + \theta' = (2n + 1)\pi \), which is a family of curves \( y(x) \) and thus cannot correspond to the zeros of a complex analytic function which must be isolated points. Therefore, let us choose the particular solution within this general class given by

\[
\theta = \theta', \quad \lim_{\delta \to 0^+} \cos \theta = 0.
\]
A more lengthy discussion of why the above particular solution corresponds to the Riemann zeros was given in [12].

Let us define the function

\[
\vartheta_{k,a}(y) \equiv \arg \Gamma \left( \frac{1}{4} + \frac{a}{2} + i \frac{y}{2} \right) - \frac{y^2}{2} \log \left( \frac{\pi}{k} \right) - \Im \left[ \log \Gamma \left( \frac{1}{4} + \frac{a}{2} + i \frac{y^2}{2} \right) \right] - \frac{y^2}{2} \log \left( \frac{\pi}{k} \right).
\]

When \( k = 1 \) and \( a = 0 \), the function (21) is just the usual Riemann-Siegel \( \vartheta \) function.

Since the function \( \log \Gamma \) has a complicated branch cut, one can use the following series representation in (21) [13]

\[
\log \Gamma(z) = -\gamma z - \log z - \sum_{n=1}^{\infty} \left\{ \log \left( 1 + \frac{z}{n} \right) - \frac{z}{n} \right\},
\]

(22)

where \( \gamma \) is the Euler-Mascheroni constant. Nevertheless, most numerical packages already have the \( \log \Gamma \) function implemented.

On the critical line the first equation in (20) is already satisfied. From the second equation we have \( \lim_{\delta \to 0^+} \theta = (n + \frac{1}{2}) \pi \), therefore

\[
\vartheta_{k,a}(y_n) + \lim_{\delta \to 0^+} \arg L \left( \frac{1}{2} + \delta + iy_n, \chi \right) - \frac{1}{2} \arg G(\chi) + \frac{\pi a}{4} = \left( n + \frac{1}{2} \right) \pi.
\]

(23)

Analyzing the left hand side of (23) we can see that it has a minimum, thus we shift \( n \to n - (n_0 + 1) \) for a specific \( n_0 \), to label the zeros according to the convention that the first positive zero is labelled by \( n = 1 \). Thus the upper half of the critical line will have the zeros labelled by \( n = 1, 2, \ldots \) corresponding to positive \( y_n \), while the lower half will have the negative values \( y_n \) labelled by \( n = 0, -1, \ldots \). The integer \( n_0 \) depends on \( k, a \) and \( \chi \), and should be chosen according to each specific case. In the cases we analyze below \( n_0 = 0 \), whereas for the trivial character \( n_0 = 1 \). In practice, the value of \( n_0 \) can always be determined by plotting (23) with \( n = 1 \), passing all terms to its left hand side. Then it is trivial to adjust the integer \( n_0 \) such that the graph passes through the point \( (y_1, 0) \) for the first jump, corresponding to the first positive solution. Henceforth we will omit the integer \( n_0 \) in the equations, since all cases analyzed in this paper have \( n_0 = 0 \). Nevertheless, the reader should bear in mind that for other cases, it may be necessary to replace \( n \to n - n_0 \) in the following equations.
In summary, these zeros have the form \( \rho_n = \frac{1}{2} + iy_n \), where for a given \( n \in \mathbb{Z} \), the imaginary part \( y_n \) is the solution of the equation

\[
\vartheta_{k,a}(y_n) + \lim_{\delta \to 0^+} \arg L \left( \frac{1}{2} + \delta + iy_n, \chi \right) - \frac{1}{2} \arg G(\chi) = \left( n - \frac{1}{2} - \frac{a}{4} \right) \pi. 
\] (24)

It is important to note that the above limit is defined with a positive \( \delta \). This limit is well defined, is generally not equal to zero, and consistent with other definitions of \( \arg L \). It also controls its wild oscillation when solving the equation numerically.

C. An asymptotic equation for the \( n \)-th zero

From Stirling’s formula we have the following asymptotic form for \( y \to \pm \infty \):

\[
\vartheta_{k,a}(y) = \text{sgn}(y) \left\{ \frac{|y|}{2} \log \left( \frac{k|y|}{2\pi e} \right) + \frac{2a - 1}{8} \pi + O(1/y) \right\}. 
\] (25)

The first order approximation of (24), i.e. neglecting terms of \( O(1/y) \), is therefore given by

\[
\sigma_n \frac{|y_n|}{2\pi} \log \left( \frac{k|y_n|}{2\pi e} \right) + \frac{1}{\pi} \lim_{\delta \to 0^+} \arg L \left( \frac{1}{2} + \delta + i\sigma_n|y_n|, \chi \right) - \frac{1}{2\pi} \arg G(\chi) = n + \sigma_n - \frac{4 - 2a(1 + \sigma_n)}{8}, 
\] (26)

where \( \sigma_n = 1 \) if \( n > 0 \) and \( \sigma_n = -1 \) if \( n \leq 0 \). For \( n > 0 \) we have \( y_n = |y_n| \) and for \( n \leq 0 \) \( y_n = -|y_n| \).

D. An explicit approximate solution in terms of the Lambert function

Using the definition of the Lambert \( W \) function, \( W(z)e^{W(z)} = z \), if we neglect the much smaller \( \arg L \) term in (26) we can find an exact solution. Let

\[
A_n(\chi) = \sigma_n \left( n + \frac{1}{2\pi} \arg G(\chi) \right) + \frac{1 - 4\sigma_n - 2a(\sigma_n + 1)}{8}. 
\] (27)

Considering the transformation \( |y_n| = 2\pi A_n x_n^{-1} \), equation (26) can be written as \( x_n e^{x_n} = kA_n e^{-1} \). Thus the approximate solution that takes into account only the smooth part of (24) is explicitly given by

\[
\tilde{y}_n = \frac{2\pi \sigma_n A_n(\chi)}{W[k e^{-1} A_n(\chi)]}, 
\] (28)

where \( W \) is the principal branch of the Lambert \( W \) function over real values. The \( W \) function is implemented in most numerical packages, thus (28) can easily estimate arbitrarily high
zeros on the line. In \((28)\) \(n = 1, 2, \ldots\) correspond to positive \(y_n\) solutions, while \(n = 0, -1, \ldots\) correspond to negative \(y_n\) solutions.

E. Counting formulas

Let us define \(N_0^+ (T, \chi)\) as the number of zeros on the critical line with \(0 < \Im(\rho) < T\) and \(N_0^- (T, \chi)\) as the number of zeros with \(-T < \Im(\rho) < 0\). As explained in section II A, \(N_0^+ (T, \chi) \neq N_0^- (T, \chi)\) if the characters are complex numbers, since the zeros are not symmetrically distributed between the upper and lower half of the critical line.

The counting formula \(N_0^+ (T, \chi)\) is obtained from \((24)\) by replacing \(y_n \rightarrow T\) and \(n \rightarrow N_0^+ 1/2\), therefore

\[
N_0^+(T, \chi) = \frac{1}{\pi} \psi_{k,a}(T) + \frac{1}{\pi} \arg L \left( \frac{1}{2} + iT, \chi \right) - \frac{1}{2\pi} \arg G(\chi) + \frac{a}{4}. \tag{29}
\]

Comparing with the counting formulas, which are staircase functions, the left hand side of \((24)\) is a monotonically increasing function. Assuming that such a function is continuous with the \(\delta\) limit, then equation \((24)\) should have a unique solution for every \(n\). This justifies the passage from \((24)\) to \((29)\). Analogously, the counting formula on the lower half line is given by

\[
N_0^- (T, \chi) = \frac{1}{\pi} \psi_{k,a}(T) - \frac{1}{\pi} \arg L \left( \frac{1}{2} - iT, \chi \right) + \frac{1}{2\pi} \arg G(\chi) - \frac{a}{4}. \tag{30}
\]

Note that in \((29)\) and \((30)\) \(T\) is positive. Both cases are plotted in FIG. 1 for the character \(\chi_{7,2}\) shown in \((36)\). One can notice that they are precisely staircase functions, jumping by one at each zero. Note also that the functions are not symmetric about the origin, since for a complex \(\chi\) the zeros on upper and lower half lines are not simply complex conjugates.

From \((25)\) we also have the first order approximation for \(T \rightarrow \infty\),

\[
N_0^+(T, \chi) = \frac{T}{2\pi} \log \left( \frac{k T}{2\pi e} \right) + \frac{1}{\pi} \arg L \left( \frac{1}{2} + iT, \chi \right) - \frac{1}{2\pi} \arg G(\chi) - \frac{1}{8} + \frac{a}{2}. \tag{31}
\]

Analogously, for the lower half line we have

\[
N_0^- (T, \chi) = \frac{T}{2\pi} \log \left( \frac{k T}{2\pi e} \right) - \frac{1}{\pi} \arg L \left( \frac{1}{2} - iT, \chi \right) + \frac{1}{2\pi} \arg G(\chi) - \frac{1}{8}. \tag{32}
\]

As in \((24)\), again we are omitting \(n_0\) since in the cases below \(n_0 = 0\), but for other cases one may need to include \(\pm n_0\) on the right hand side of \(N_0^\pm\), respectively.
It is known that the number of zeros on the whole critical strip up to height $T$, i.e. $0 < x < 1$ and $0 < y < T$, is given by \( \text{(14)} \)

$$N^+(T, \chi) = \frac{1}{\pi} \vartheta_{k,a}(T) + \frac{1}{\pi} \arg L \left( \frac{1}{2} + iT, \chi \right) - \frac{1}{\pi} \arg L \left( \frac{1}{2}, \chi \right).$$  \hspace{1cm} (33)

From Stirling’s approximation and noticing that $2a - 1 = -\chi(-1)$, for $T \to \infty$ we obtain the asymptotic approximation \( \text{(2)} \) \( \text{(14)} \)

$$N^+(T, \chi) = \frac{T}{2\pi} \log \left( \frac{kT}{2\pi e} \right) + \frac{1}{\pi} \arg L \left( \frac{1}{2} + iT, \chi \right) - \frac{1}{\pi} \arg L \left( \frac{1}{2}, \chi \right) - \frac{\chi(-1)}{8} + O(1/T).$$  \hspace{1cm} (34)

Both formulas (33) and (34) are exactly the same as (29) and (31), respectively. This can be seen as follows. From (10) we conclude that $\xi$ is real on the critical line. Thus \( \arg \xi \left( \frac{1}{2} \right) = 0 = -\frac{1}{2} \arg G(\chi) + \arg L \left( \frac{1}{2}, \chi \right) + \frac{\pi a}{4} \). Then, replacing $\arg G$ in (24) we obtain

$$\vartheta_{k,a}(y_n) + \lim_{\delta \to 0^+} \arg L \left( \frac{1}{2} + \delta + iy_n, \chi \right) - \arg L \left( \frac{1}{2}, \chi \right) = (n - \frac{1}{2}) \pi. \hspace{1cm} (35)$$

Replacing $y_n \to T$ and $n \to N^+_0 + 1/2$ in (35) we have precisely the expression (33), and also (34) for $T \to \infty$. Therefore, we conclude that $N^+_0(T, \chi) = N^+(T, \chi)$ exactly. From (11) we see that negative zeros for character $\chi$ correspond to positive zeros for character $\chi^*$. Then for $-T < \Im(\rho) < 0$ the counting on the strip also coincides with the counting on the line, since $N^-_0(T, \chi) = N^+_0(T, \chi^*)$ and $N^-(T, \chi) = N^+(T, \chi^*)$. Therefore, the number of zeros on the whole critical strip is the same as the number of zeros on the critical line obtained as solutions of (24).


F. The generalized Riemann hypothesis

Our previous conclusions were based on the particular solution (20) of \( \lim_{\delta \to 0^+} B = 0 \). A 
zero of \( B = 0 \) is also a zero of \( L(z, \chi) = 0 \). The number of \( L \)-zeros on the entire critical strip 
is given by (33). As discussed above, if the equation (24) has a unique solution for every \( n \), 
then \( N_0^+(T, \chi) = N^+(T, \chi) \), implying that all zeros are on the critical line. It means that 
(20) captures all non-trivial zeros of \( L \). Since (24), or equivalently (35), arises from \( B = 0 \), 
there are no additional zeros from \( A = 0 \) nor from the most general solution to \( B = 0 \) since 
the counting formula \( N(T) \) is already saturated.

Another important fact follows from the one-to-one correspondence between the zeros of 
\( L(z, \chi) \) with the zeros of \( \cos \theta = 0 \), which are all simple. Thus, under the validity of (24) and 
(29), the non-trivial zeros of \( L(z, \chi) \) are simple. Note that (24) gives a different equation 
for each \( n \), so there are no repeated solutions.

G. Numerical solutions

We can solve the equation (24) starting with the approximation given by (28). We 
will illustrate this for some specific examples, using the root finder function provided by 
Mathematica [20].

The numerical procedure is carried out as follows:

1. We solve (24) looking for the solution in a region centered around the number \( \tilde{y}_n \) 
   provided by (28), with a not so small \( \delta \), for instance \( \delta \sim 10^{-5} \).

2. We solve (24) again but now centered around the solution obtained in step 1 above, 
   and we decrease \( \delta \), for instance \( \delta \sim 10^{-8} \).

3. We repeat the procedure in step 2 above, decreasing \( \delta \) again.

4. Through successive iterations, and decreasing \( \delta \) each time, it is possible to obtain 
solutions as accurately as desirable. In carrying this out, it is important to not allow 
\( \delta \) to be exactly zero.

We will illustrate our formulas with the primitive characters \( \chi_{7,2} \) and \( \chi_{7,3} \), since they 
possess the full generality of \( a = 0 \) and \( a = 1 \) and complex components. There are actually 
\( \varphi(7) = 6 \) distinct characters mod 7.
The solutions are accurate to 50 decimal places and verified to \( |L(\tfrac{1}{2} + iy_n, \chi_{7, 2})| \sim 10^{-50} \).

**Example \( \chi_{7, 2} \).** Consider \( k = 7 \) and \( j = 2 \), i.e. we are computing the Dirichlet character \( \chi_{7, 2}(n) \). For this case \( a = 1 \). Then we have the following components:

\[
\begin{array}{c|ccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
  \chi_{7, 2}(n) & 1 & e^{2\pi i/3} & e^{\pi i/3} & e^{-2\pi i/3} & e^{-\pi i/3} & -1 & 0 \\
\end{array}
\]  

(36)

The first few zeros, positive and negative, obtained by solving (24) are shown in TABLE I. The solutions shown are easily obtained with 50 decimal places of accuracy, and agree with the ones in [15], which were computed up to 20 decimal places.

**Example \( \chi_{7, 3} \).** Consider \( k = 7 \) and \( j = 3 \), such that \( a = 0 \). In this case the components of \( \chi_{7, 3}(n) \) are the following:

\[
\begin{array}{c|ccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
  \chi_{7, 3}(n) & 1 & e^{-2\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} & e^{-2\pi i/3} & 1 & 0 \\
\end{array}
\]  

(37)

The first few solutions of (26) are shown in TABLE II and are accurate up to 50 decimal places, and agree with the ones obtained in [15]. As stated previously, the solutions to
The equation (24) can be calculated to any desired level of accuracy. For instance, continuing with the character \( \chi_{7,3} \), we can easily compute the following number for \( n = 1000 \), accurate to 100 decimal places, i.e. 104 digits:

\[
y_{1000} = 1037.56371706920654296560046127698168717112749601359549 \\
01734503731679747841764715443496546207885576444206
\]

We also have been able to solve the equation for high zeros to high accuracy, up to the millionth zero, some of which are listed in Table III and were previously unknown.
TABLE III: Higher zeros for the Dirichlet character \( (37) \). These solutions to \( (24) \) are accurate to 50 decimal places.

III. \( L\)-FUNCTIONS BASED ON MODULAR FORMS

A. General level one forms

The modular group can be represented by the set of \( 2 \times 2 \) integer matrices,

\[
SL_2(\mathbb{Z}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \det A = 1 \right\},
\]

provided each matrix \( A \) is identified with \(-A\), i.e. \( \pm A \) are regarded as the same transformation. Thus for \( \tau \) in the upper half complex plane, it transforms as \( \tau \mapsto A\tau = \frac{a\tau + b}{c\tau + d} \) under the action of the modular group. A modular form \( f \) of weight \( k \) is a function that is analytic in the upper half complex plane which satisfies the functional relation \[16\]

\[
f \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau).
\]

If the above equation is satisfied for all of \( SL_2(\mathbb{Z}) \), then \( f \) is referred to as being of level one. It is possible to define higher level modular forms which satisfy the above equation for a subgroup of \( SL_2(\mathbb{Z}) \). Since our results are easily generalized to the higher level case, henceforth we will only consider level 1 forms.

For the \( SL_2(\mathbb{Z}) \) element \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), the above implies the periodicity \( f(\tau) = f(\tau + 1) \), thus it has a Fourier series

\[
f(\tau) = \sum_{n=0}^{\infty} a_f(n) q^n, \quad q \equiv e^{2\pi i \tau}.
\]

If \( a_f(0) = 0 \) then \( f \) is called a cusp form.

From the Fourier coefficients, one can define the Dirichlet series

\[
L_f(z) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^z}.
\]
The functional equation for \( L_f (z) \) relates it to \( L_f (k - z) \), so that the critical line is \( \Re(z) = \frac{k}{2} \), where \( k \geq 4 \) is an even integer. One can always shift the critical line to \( \frac{1}{2} \) by replacing \( a_f(n) \) by \( a_f(n)/n^{(k-1)/2} \), however we will not do this here. Let us define

\[
\Lambda_f(z) \equiv (2\pi)^{-z} \Gamma(z) L_f(z). \tag{42}
\]

Then the functional equation is given by \([16]\)

\[
\Lambda_f(z) = (-1)^{k/2} \Lambda_f(k - z). \tag{43}
\]

There are only two cases to consider since \( \frac{k}{2} \) can be an even or an odd integer. As in \((9)\) we can absorb the extra minus sign factor for the odd case. Thus we define \( \xi_f(z) \equiv \Lambda_f(z) \) for \( k \) even, and we have \( \xi_f(z) = \xi_f(k - z) \), and \( \xi_f(z) \equiv e^{-i\pi/2} \Lambda_f(z) \) for \( k \) odd, implying \( \xi_f(z) = e^{-i\pi/2} \Lambda_f(k - z) \). Representing \( \xi_f(z) = |\xi_f(z)| e^{i\vartheta(x,y)} \) where \( z = x + iy \), we follow exactly the same steps as in the last section. From the particular solution \((20)\) we conclude that there are infinite zeros on the critical line \( \Re(\rho) = \frac{k}{2} \) determined by \( \lim_{\delta \to 0^+} \theta \left( \frac{k}{2} + \delta, y, \chi \right) = \left( n - \frac{k}{2} \right) \pi \). Therefore, these zeros have the form \( \rho_n = \frac{k}{2} + iy_n \), where \( y_n \) is the solution of the equation

\[
\vartheta_k(y_n) + \lim_{\delta \to 0^+} \arg L_f \left( \frac{k}{2} + \delta + iy_n \right) = \left( n - \frac{1 + (-1)^{k/2}}{4} \right) \pi, \tag{44}
\]

where we have defined

\[
\vartheta_k(y) \equiv \arg \left( \frac{k}{2} + iy \right) - y \log 2\pi. \tag{45}
\]

Repeating the argument of the last section, if equation \((44)\) has a unique solution for every \( n \), then this implies that the number of zeros with imaginary part less that \( T \) is given by

\[
N_0(T) = \frac{1}{\pi} \vartheta_k(T) + \frac{1}{\pi} \arg L_f \left( \frac{k}{2} + iT \right) - \frac{1 - (-1)^{k/2}}{4}. \tag{46}
\]

In the limit of large \( y_n \), neglecting terms of \( O(1/y) \), the equation \((44)\) becomes

\[
y_n \log \left( \frac{y_n}{2\pi e} \right) + \lim_{\delta \to 0^+} \arg L_f \left( \frac{k}{2} + \delta + iy_n \right) = \left( n - \frac{k + (-1)^{k/2}}{4} \right) \pi. \tag{47}
\]

If one ignores the small \( \arg L_f \) term, then the approximate solution is given by

\[
\tilde{y}_n = \frac{A_n \pi}{W[(2e)^{-1}A_n]}, \quad A_n = n - \frac{k + (-1)^{k/2}}{4}. \tag{48}
\]
B. An example with weight $k = 12$

The simplest example is based on the Dedekind $\eta$-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

(49)

Up to a simple factor, $\eta$ is the inverse of the chiral partition function of the free boson conformal field theory [17], where $\tau$ is the modular parameter of the torus. The modular discriminant

$$\Delta(\tau) = \eta(\tau)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

(50)

is a weight $k = 12$ modular form. It is closely related to the inverse of the partition function of the bosonic string in 26 dimensions, where 24 is the number of light-cone degrees of freedom [18]. The Fourier coefficients $\tau(n)$ correspond to the Ramanujan $\tau$-function, and the first few are

| $n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-----|----|----|----|----|----|----|----|----|----|
| $\tau(n)$ | 1  | -24 | 252 | -1472 | 4830 | -6048 | -16744 | 84480 | -113643 |

(51)

We then define the Dirichlet series

$$L_\Delta(z) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^z}.$$ 

(52)

From (44) the zeros are $\rho_n = 6 + iy_n$, where the $y_n$ satisfy the exact equation

$$\vartheta_{12}(y) + \lim_{\delta \to 0^+} \arg L_\Delta(6 + \delta + iy_n) = \left(n - \frac{1}{2}\right) \pi.$$ 

(53)

The counting function (46) and its asymptotic approximation are

$$N_0(T) = \frac{1}{\pi} \vartheta_{12}(T) + \frac{1}{\pi} \arg L_\Delta(6 + iT)$$

$$\approx \frac{T}{\pi} \log \left( \frac{T}{2\pi e} \right) + \frac{1}{\pi} \arg L_\Delta(6 + iT) + \frac{11}{4} + O(1/T).$$

(54)

(55)

A plot of (54) is shown in FIG. 2 and we can see that it is a perfect staircase function.

The approximate solution (48) in terms of the Lambert function is given by

$$\tilde{y}_n = \frac{(n - \frac{13}{4}) \pi}{W \left[ (2e)^{-1} (n - \frac{13}{4}) \right]} \quad (n = 2, 3, \ldots).$$

(56)

Note that the above equation is valid for $n > 1$, since $W(x)$ is not defined for $x < -1/e$.

We follow exactly the same procedure discussed in the beginning of section II G to solve equation (53) numerically, starting with the approximation provided by (56). Some of these solutions are shown in TABLE IV and are accurate to 50 decimal places [21].
FIG. 2: Exact counting formula (54) based on Ramanujan $\tau$-function.

| $n$ | $\tilde{y}_n$ | $y_n$ |
|-----|---------------|-------|
| 1   | 9.22237939992110252224376719274347813552877062243201 |       |
| 2   | 12.46         | 13.90754986139213440644668132877021949175755235351449 |
| 3   | 16.27         | 17.442776978234473313551525137127627187088652427527 |
| 4   | 19.30         | 19.656513419549610001272817563213028016155091200324 |
| 5   | 21.94         | 22.33610363720986727568267445923624619245504695246527 |
| 6   | 24.35         | 25.27463654811236535674532419313346311859592673122941 |
| 7   | 26.60         | 26.8043911583504030325757492358456474715206800497933 |
| 8   | 28.72         | 28.83168262241868754450219619129848972569093668609124 |
| 9   | 30.74         | 31.178209498360259064492188890774055854655119866267 |
| 10  | 32.68         | 32.77487538223120744183045567331198999909916163721260 |
| 100 | 143.03        | 143.0835552634784550737397797694664120256210342087127 |
| 200 | 235.55        | 235.7471014399921366770380713073362103592120614210694 |
| 300 | 318.61        | 318.36169446742310747533323741641236307865855919162340 |

TABLE IV: Non-trivial zeros of the modular $L$-function based on the Ramanujan $\tau$-function, obtained from (53) starting with the approximation (56). These solutions are accurate to 50 decimal places.

IV. CONCLUSION

We have generalized the approach proposed in [11, 12] for the Riemann $\zeta$-function, to all primitive Dirichlet $L$-functions. We showed that there are infinite zeros on the critical line in one-to-one correspondence with the zeros of $\cos \theta = 0$, equation (20). In this way, the zeros are enumerated and their imaginary parts satisfy the equation (24).

Under the weak assumption that (24) is well defined for every $y_n$, noting that it is a
monotonically increasing function, then it has a unique solution for every $n$. Thus one can obtain the counting formula (29) on the critical line. This agrees with the counting formula for the number of zeros on the entire critical strip. Thus the zeros from (24) already saturate the counting formula on the strip, indicating that all zeros must be on the critical line.

We have computed some numerical solutions of (24) to high accuracy, shown in section II G. Thus this gives additional strong numerical support for the validity of (24).

Furthermore, we have also employed the same analysis for $L$-functions based on level one modular forms in section III, and considered a specific example based on the Ramanujan $\tau$-function. In appendix A we unify this analysis for a more general class of $L$-functions.

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Appendix A: General $L$-functions

With the aim of unifying the previous results, we extend the main equations to a more general, but not specific, class of $L$-functions. We must of course assume that the analysis in section II B, and also discussed more thoroughly in [12], is valid, and this must be checked case by case. Thus one must bear in mind that since we are not specific about the $L$-function, but just assume some elementary properties, there is no guarantee that the latter analysis is valid for every $L$-function with the properties below. We are thus simply going to assume that the analysis in section II B is valid and present the resulting equations that would follow.

We are going to consider $L$-functions with the properties outlined in [9, Chap. 5]. The general $L$-function has a Dirichlet series

\[ L(z, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^z}, \]

(A1)

where $\lambda_f(1) = 1$ and $\lambda_f(n)$ is a complex number. This series is convergent for $\Re(z) > 1$, it has
an Euler product of degree \( d \geq 1 \), where \( d \) is an integer, and admits an analytic continuation in the whole complex plane, except for poles at \( z = 0 \) and \( z = 1 \). The arithmetic object \( f \) has no specific meaning here. For instance, \( f \) can be a modular or cusp form, or it can be associated to Dirichlet characters \( \lambda_f(n) = \chi(n) \). The object \( f \) defines the particular class of \( L \)-functions. Let

\[
\gamma(z, f) = \pi^{-dz/2} \prod_{j=1}^{d} \Gamma \left( \frac{z + a_j}{2} \right),
\]

where the complex numbers \( a_j \), which come in conjugate pairs, are the so called local parameters at infinity. Let us define

\[
\Lambda(z, f) \equiv k z/2 \gamma(z, f) L(z, f),
\]

where \( k = k(f) \geq 1 \) is an integer, the conductor of \( L(z, f) \). Then it satisfies the functional equation \([9]\)

\[
\Lambda(z, f) = \alpha(f) \Lambda^*(1 - z, f) \equiv \alpha(f) (\Lambda(1 - z^*, f))^*.
\]

Here \( \alpha \) is a complex phase, i.e. \( |\alpha| = 1 \). The symbol \( f^* \) denotes the dual of \( f \), associated to the Dirichlet series with \( \lambda_{f^*}(n) = (\lambda_f(n))^* \). We also have the relations \( \gamma(z, f^*) = \gamma(z, f) \) and \( k(f^*) = k(f) \).

We can write the functional equation in a more symmetric form by introducing

\[
\xi(z, f) \equiv e^{-i\beta/2} \Lambda(z, f),
\]

where \( \alpha(f) = e^{i\beta(f)} \). Then we have the functional equation

\[
\xi(z, f) = \xi^*(1 - z, f) = (\xi(1 - z^*, f))^*.
\]

Writing the polar form \( \xi(z, f) = A(x, y, f)e^{i\theta(x, y, f)} \) we have

\[
A(x, y, f) = k^{x/2} \pi^{-dx/2} |L(x + iy, f)| \prod_{j=1}^{d} \left| \Gamma \left( \frac{x + a_j + iy}{2} \right) \right|,
\]

\[
\theta(x, y, f) = \sum_{j=1}^{d} \arg \Gamma \left( \frac{x + a_j + iy}{2} \right) + \arg L(x + iy, f) + \frac{y}{2} \log \left( \frac{k}{\pi^d} \right) - \frac{\beta}{2}.
\]

Denoting \( \xi^*(1 - z, f) = A'(x, y, f)e^{-i\theta'(x, y, f)} \), we then have \( A'(x, y, f) = A(1 - x, y, f) \) and \( \theta'(x, y, f) = \theta(1 - x, y, f) \). Let us define the generalized Riemann-Siegel \( \vartheta_{k, a_j} \) function

\[
\vartheta_{k, a_j}(y) \equiv \arg \Gamma \left( \frac{1}{4} + \frac{a_j}{2} + iy/2 \right) - \frac{y}{2} \log \left( \frac{\pi}{k^{1/4}} \right).
\]
Following the same previous analysis, by imposing the identity
\[ \lim_{\delta \to 0^+} (\xi(\rho + \delta, f) + \xi^*(1 - \rho - \delta, f)) = 0 \]  
where \( \rho = x + iy \) is a non-trivial \( L \)-zero, we take the particular solution \( \theta = \theta' \) and \( \lim_{\delta \to 0^+} \cos \theta = 0 \). Thus \( \theta = \theta' \) is satisfied by \( \Re(\rho) = \frac{1}{2} \) and then \( \lim_{\delta \to 0^+} \cos \theta \left( \frac{1}{2}, y, f \right) = 0 \) yields the equation for the \( n \)-th zero. Introducing a shift \( n \to n - (n_0 + 1) \), where \( n_0 \) should be determined by each specific case according to the convention that the first positive zero is labelled by \( n = 1 \) (we will omit \( n_0 \) in the following), we then conclude that these zeros have the form \( \rho_n = \frac{1}{2} + iy_n \) for \( n \in \mathbb{Z} \), and \( y_n \) is the solution of the equation
\[ \sum_{j=1}^{d} \vartheta_{k,a_j}(y_n) + \lim_{\delta \to 0^+} \arg L \left( \frac{1}{2} + \delta + iy_n, f \right) - \frac{\beta}{2} = \left( n - \frac{1}{2} \right) \pi. \]  
It is also possible to replace \( \beta \) by noting that \( \xi \left( \frac{1}{2} + iy, f \right) \) is real, thus \( \arg \xi \left( \frac{1}{2} \right) = 0 \) then
\[ \frac{\beta(f)}{2} = \sum_{j=1}^{d} \arg \Gamma \left( \frac{1}{4} + \frac{a_j}{2} \right) + \arg L \left( \frac{1}{2}, f \right). \]  
For real \( a_j \) the first term vanishes. If \( f^* = f \) then \( L(z, f) \) is said to be self-dual. If besides this \( \alpha(f) = -1 \), then \( L \left( \frac{1}{2}, f \right) = 0 \).

The counting formula on the critical line, for \( 0 < \Im(z) < T \), can be obtained from (A11) by replacing \( y_n \to T \) and \( n \to N_0^+ (T, f) + \frac{1}{2} \), thus
\[ N_0^+ (T, f) = \frac{1}{\pi} \sum_{j=1}^{d} \left\{ \vartheta_{k,a_j}(T) - \arg \Gamma \left( \frac{1}{4} + \frac{a_j}{2} \right) \right\} + \frac{1}{\pi} \arg L \left( \frac{1}{2} + iT, f \right) - \frac{1}{\pi} \arg L \left( \frac{1}{2}, f \right). \]  
The same counting on the whole strip \( N^+(T, f) \) can be obtained through the standard Cauchy’s argument principle. Thus \( N_0^+(T, f) = N^+(T, f) \), justifying that the particular solution captures all non-trivial zeros on the critical strip, therefore they must be all on the critical line. The counting on the negative half line \( -T < \Im(z) < 0 \) can be obtained from \( N_0^-(T, f) = N_0^+(T, f^*) \). Expanding (A9) from Stirling’s formula we have
\[ \vartheta_{k,a_j}(y) = \frac{y}{2} \log \left( \frac{k^{1/d} y}{2\pi e} \right) + \frac{2a_j - 1}{8} \pi + O(1/y). \]  
Then from (A13) we have
\[ N_0^+ (T, f) = \frac{T}{2\pi} \log \left( \frac{k T^d}{(2\pi e)^d} \right) + \sum_{j=1}^{d} \left\{ \frac{2a_j - 1}{8} - \frac{1}{\pi} \arg \Gamma \left( \frac{1}{4} + \frac{a_j}{2} \right) \right\} + \frac{1}{\pi} \arg L \left( \frac{1}{2} + iT, f \right) - \frac{1}{\pi} \arg L \left( \frac{1}{2}, f \right) + O(1/y). \]
Using (A14) in (A11) and neglecting the small \( \arg L \left( \frac{1}{2}, f \right) \) term, it is possible to obtain an approximate solution in closed form, which reads
\[
\tilde{y}_n = \frac{2\pi A_n}{dW \left[ k^{1/d}(d e)^{-1}A_n \right]},
\]
where \( W(x) \) is the principal value of the Lambert function over real values, and
\[
A_n = n - \frac{1}{2} + \frac{1}{\pi} \arg L \left( \frac{1}{2}, f \right) + \sum_{j=1}^{d} \left\{ \frac{1}{\pi} \arg \Gamma \left( \frac{1}{4} + \frac{a_j}{2} \right) - \frac{2a_j - 1}{8} \right\}.
\]

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[17] P. Francesco, P. Mathieu, D. Senechal, *Conformal Field Theory*, Springer 1999

[18] M. B. Green, J. H. Schwarz, E. Witten, *Superstring Theory: Volume 1*, Cambridge University Press 1988

[19] Our enumeration convention for the $j$-index of $\chi_{k,j}$ is taken from Mathematica.

[20] The Mathematica notebook we used to carry out these computations has only about a dozen lines of code and is available on the arXiv in math.NT as an ancillary file to this submission.

[21] See also the Mathematica notebooks attached to this submission on arXiv.