From PET to SPLIT.

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Abstract The polynomial ergodic theorem (PET) which appeared in [1] attracted substantial attention in ergodic theory studies the limits of expressions having the form $\frac{1}{N}\sum_{n=1}^{N} T^{q_1(n)} f_1 \cdots T^{q_\ell(n)} f_\ell$ where $T$ is a weakly mixing measure preserving transformation, $f_i$'s are bounded measurable functions and $q_i$'s are polynomials taking on integer values on the integers. Motivated partially by this result we obtain a central limit theorem for expressions of the form

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} (X_1(q_1(n))X_2(q_2(n)) \cdots X_\ell(q_\ell(n)) - a_1a_2\cdots a_\ell)$$

(sum-product limit theorem—SPLIT) where $X_i$'s are fast $\alpha$-mixing bounded stationary processes, $a_i = EX_i(0)$ and $q_i$'s are positive functions taking on integer values on integers with some growth conditions which are satisfied, for instance, when $q_i$'s are polynomials of growing degrees. This result can be applied to the case when $X_i(n) = T^nf_i$ where $T$ is a mixing subshift of finite type, a hyperbolic diffeomorphism or an expanding transformation taken with a Gibbs invariant measure, as well, as to the case when $X_i(n) = f_i(\xi_n)$ where $\xi_n$ is a Markov chain satisfying the Doeblin condition considered as a stationary process with respect to its invariant measure.

Keywords central limit theorem, polynomial ergodic theorem, $\alpha$-mixing.

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1 Introduction

The polynomial ergodic theorem (PET) appeared in [1] sais that in the $L^2$-sense

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{q_1(n)} f_1 \cdots T^{q_\ell(n)} f_\ell = \prod_{i=1}^{\ell} \int f_i d\mu$$

where $T$ is a measure preserving weakly mixing transformation, $f_i$'s are bounded measurable functions and $q_i$'s are polynomials taking on integer values on the integers and satisfying $q_{i+1}(n) - q_i(n) \to \infty$ as $n \to \infty$, $i = 1, \ldots, \ell - 1$. This and related results (see, for instance, [9], [8] and references there) where motivated originally by the study of multiple recurrence for dynamical systems. Namely, if $f_i = 1_{A_i}$, $i = 1, \ldots, \ell$ are indicators of some measurable

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sets $A_i$ of positive measure $\mu$ then PET implies that for $\mu$-almost all (a.a.) $x$ the event $\cap_{i=1}^{\ell} \{ T^n x \in A_i \}$ occurs with the frequency $\prod_{i=1}^{\ell} \mu(A_i)$, in particular, infinitely often.

The probability theory name for the ergodic theorem is the law of large numbers and after verifying it the next natural question to ask is whether a central limit theorem type result holds also true in this framework though, as usual, under somewhat stronger assumptions. In this paper we will obtain convergence in distribution to the normal law as $N \to \infty$ of expressions having the form

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} (X_1(q_1(n))X_2(q_2(n)) \cdots X_\ell(q_\ell(n)) - \prod_{i=1}^{\ell} a_i)$$

(sum–product limit theorem: SPLIT) where $a_i = EX_i(0)$, $X_i$’s are exponentially fast $\alpha$-mixing bounded stationary processes and $q_i$’s are positive increasing for large $n$ functions taking on integer values on the integers with some growth conditions which are satisfied, for instance, when $q_i$’s are polynomials of increasing degrees. We observe that unlike PETs our SPLITs do not require $q_i$’s to be polynomials, and so we obtain also some new sum–product ergodic theorems paying the price of much stronger mixing assumptions than in PETs. As in other cases with central limit theorem our SPLIT describes, in particular, fluctuations of the number of multiple currencies mentioned above from its average frequency. In fact, we will derive a functional central limit theorem type extension of the above result.

Our results are applicable, for instance, to the case when $X_i(n) = f_i(\xi_n)$ for bounded measurable $f_i$’s and a Markov chain $\xi_n$ in a space $M$ satisfying the Doeblin condition (see [11]) taken with its invariant measure $\mu$ which yields, in particular, that for any measurable sets $A_i \subset M$ with $\mu(A_i) > 0$, $i = 1, \ldots, \ell$ if $N(n)$ is the number of events $\cap_{i=1}^{\ell} \{ \xi_{q_i(k)} \in A_i \}$ for $k$ running between 1 and $n$ then $n^{-1/2}(N(n) - \prod_{i=1}^{\ell} \mu(A_i))$ is asymptotically normal. Our SPLITs seem to be new even when $X_i(n)$, $n = 0, 1, 2, \ldots$ are independent identically distributed (i.i.d.) random variables though in this case the proof is much easier and the result holds true in more general circumstances (see Section [6]). Another important class of processes satisfying our conditions comes from dynamical systems where $X_i(n) = f_i(T^n x)$ with $T$ being a topologically mixing subshift of finite type or a $C^2$ expanding endomorphism or an Axiom A (in particular, Anosov) (see [3]) diffeomorphisms considered in a neighborhood of an attractor taken with a Gibbs invariant measure. Some other dynamical systems which fit our setup will be mentioned in the next section. For a particular case of $T x = \theta x \pmod{1}$, $\theta > 1$, $x \in [0, 1]$, polynomial $q_i$’s and fast approximable by trigonometric polynomials $f_i$’s a corresponding central limit theorem appears in [2] whose specific setup allows application of the Fourier analysis machinery.

Our methods are completely different from the ones in the ergodic theory papers cited above and we rely on splitting the products into weakly dependent factors (so SPLIT is not only an abbreviation here) so that our main tool which is the inequality estimating the difference between expectation of a product and a product of expectations via the $\alpha$-mixing coefficient could be applied. Observe that the martingale approximation methods which are popular in modern proofs of the central limit theorem do not seem to work (at least, directly) in our setup in view of strong dependencies between past and future terms of sums here.
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2 Preliminaries and main results

Our setup consists of $\ell$ bounded stationary processes $X_1, X_2, \ldots, X_\ell$, $|X_j(n)| \leq D < \infty$, $j = 1, \ldots, \ell; n = 0, 1, \ldots$ on a probability space $(\Omega, \mathcal{F}, P)$ and of a family of $\sigma$-algebras $\mathcal{F}_{kl} \subset \mathcal{F}$, $-\infty \leq k \leq l \leq \infty$ such that $\mathcal{F}_{kl} \subset \mathcal{F}_{k' l'}$ if $k' \leq k$ and $l' \geq l$. Given such family of $\sigma$-algebras the $\alpha$-mixing coefficient is defined by

$$\alpha(n) = \sup_{k \geq 0} \sup_{A \in \mathcal{F}_{-\infty k}, B \in \mathcal{F}_{k+n, \infty}} |P(A \cap B) - P(A)P(B)|, \ n \geq 0.$$ 

Set also

$$\beta_j(n) = \sup_{m \geq 0} E|X_j(m) - E(X_j(m)|\mathcal{F}_{m-n, m+n})|.$$ 

We assume that for some $\kappa > 0$,

$$\alpha(n) + \max_{1 \leq j \leq \ell} \beta_j(n) \leq \kappa^{-1}e^{-\kappa n}. \quad (2.1)$$

In what follows we can always consider $X_j$ and $\mathcal{F}_{kl}$ with $m, k, l \geq 0$ only and just set formally in the above definitions $\mathcal{F}_{kl} = \mathcal{F}_{kl}$ for $k < 0$ and $l \geq 0$.

Next, let $q_1(n), q_2(n), \ldots, q_\ell(n)$ be nonnegative functions taking on integer values on the integers and such that $q_i(n)$ is linear, i.e.,

$$q_i(n) = rn + p \quad \text{for integer} \quad r > 0, \ p \geq 0, \quad (2.2)$$

and there exists $\gamma \in (0, 1)$ so that for all $n \geq n_0 > 1$,

$$q_j(n + 1) \geq q_j(n) + n^\gamma, \ j = 2, \ldots, \ell \quad (2.3)$$

and

$$q_{j+1}([n^{1-\gamma}]) \geq q_j(n)^{n^\gamma}, \ j = 1, \ldots, \ell - 1. \quad (2.4)$$

Observe that (2.3) and (2.4) are satisfied when $q_i$’s are polynomials of positive degrees growing with $i$.

**Theorem 2.1** Set $a_j = EX_j(0)$ and assume that the above conditions $(2.2)$–$(2.3)$ on the processes $X_j$ and the functions $q_j$, $j = 1, \ldots, \ell$ hold true. Then, as $N \to \infty$,

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N} \left( \prod_{j=1}^{\ell} X_j(q_j(n)) - \prod_{j=1}^{\ell} a_j \right). \quad (2.5)$$
converges in distribution to a normal random variable with zero mean and the variance
\[ \sigma^2 = \sigma_1^2 = EX_1^2(0)\left( \prod_{j=2}^\ell EX_j^2(0) - \prod_{j=2}^\ell a_j^2 \right) + \sigma_0^2 \prod_{j=2}^\ell a_j^2 \] (2.6)
where
\[ \sigma_1^2 = \lim_{N \to \infty} \frac{1}{N} E \left( \sum_{n=1}^N (X_1(q_1(n)) - a_1)^2 \right) = EX_1^2(0) - a_1^2 + 2 \sum_{n=1}^\ell E \left( (X_n(0) - a_1)(X_1(0) - a_1) \right), \] (2.7)
\[ \sigma^2 = \sigma_1^2 \text{ if } \ell = 1 \text{ and the last series in (2.7) converges. Furthermore, } \sigma = 0 \text{ if and only if either } X_j(0) = 0 \text{ almost surely (a.s.) for some } j \geq 1 \text{ or } X_j(0) = a_j \text{ a.s. for all } j \geq 2 \text{ and } \sigma_1 = 0. \text{ Finally, } \sigma_1 = 0 \text{ if and only if for all } m = 0, 1, 2, \ldots, \]
\[ X_1(\rho_m + p) - a_1 = U^{m+1}X - U^mX \text{ a.s.} \] (2.8)
where \( U \) is the unitary operator associated with the stationary process \( \{X_1(\rho_m + p), m = 0, 1, 2, \ldots\} \) and \( X \) belongs to the Hilbert space of random variables with finite second moments which are measurable with respect to the \( \sigma \)-algebra generated by \( \{X_1(\rho_m + p), m = 0, 1, 2, \ldots\} \) (see, for instance, [11], Ch. 16).

Observe that since \( EX_j^2(0) \geq a_j^2 \) by the Cauchy–Schwarz inequality the last assertion of Theorem 2.1 concerning \( \sigma = 0 \) follows from (2.6) and (2.7) while the equivalence of \( \sigma_1 = 0 \) and the representation (2.8) is rather well known since it concerns the standard central limit theorem for
\[ \frac{1}{\sqrt{N}} \sum_{n=0}^N (X_1(\rho_n + p) - a_1). \]
Still, for readers’ convenience we recall the argument that (2.8) follows from \( \sigma_1 = 0 \) in Corollary 3.7 while the opposite implication is clear.

Note also that the case when \( q_1(n) \) grows faster than linearly in \( n \) also fits our setup since we can take \( X_1 \equiv 1 \) which would mean that, in fact, we start with \( X_2 \) and \( q_2 \). In this case
\[ \sigma^2 = \prod_{j=1}^\ell EX_j^2(0) - \prod_{j=1}^\ell a_j^2 \] (2.9)
and \( \sigma^2 > 0 \) unless all \( X_j \)'s are constants with probability one.

In Section 5 we will extend Theorem 2.1 to a more general result where two linear functions \( q_j \) are allowed. Namely, set \( q_0(n) = n \) and \( q_j, j = 1, 2, \ldots, \ell \) as above where \( q_1 \) is given by (2.2) with \( r \geq 2 \). We add another stationary process \( X_0 \) with \( X_0(\rho) \leq D \) for all \( \rho \) and set \( a_0 = EX_0(0) \). Then we have the following assertion.

**Theorem 2.2** As \( N \to \infty \) the sequence of random variables
\[ \frac{1}{\sqrt{N}} \sum_{n=0}^N \left( \prod_{j=0}^\ell X_j(q_j(n)) - \prod_{j=0}^\ell a_j \right), \] (2.10)
converges in distribution to a normal random variable with zero mean and the variance
\[ \sigma^2 = EX_0^2(0)EX_1^2(0)\left(\prod_{j=2}^{\ell} EX_j^2(0) - \prod_{j=2}^{\ell} a_j^2\right) + \sigma_{01}^2 \prod_{j=2}^{\ell} a_j^2 \] (2.11)

where
\[ \sigma_{01}^2 = \lim_{N \to \infty} \frac{1}{N} E \left( \sum_{n=1}^{N} (X_0(n)X_1(q_1(n)) - a_0a_1) \right)^2 \] (2.12)

\[ = (EX_0^2(0) - a_0^2)EX_1^2(0) + a_0^2(\sum_{j=2}^{\ell} a_j^2) + 2\sum_{n=1}^{\infty} E((X_0(n) - a_0)(X_0(0) - a_0)) \]

\[ + 2a_0^2 \sum_{n=1}^{N} E((X_1(m) - a_1)(X_1(0) - a_1)) + \mathcal{E} \]

and
\[ \mathcal{E} = 2a_0a_1 E\left( \sum_{n=0}^{m} (X_0(n) - a_0)(X_1(0) - a_1) + \sum_{n=1}^{m} (X_0(0) - a_0)(X_1(n) - a_1) \right). \]

If we take \( X_0 \equiv 1 \) then Theorem 2.2 reduces to Theorem 2.2 where we need only \( r \geq 1 \). Furthermore, we can take instead \( X_j \equiv 1 \) for all \( j \geq 2 \) which yields a nontrivial particular case of Theorem 2.2 saying that
\[ \frac{1}{\sqrt{N}} \sum_{n=0}^{N} (X_0(n)X_1(rn + p) - a_0a_1), \quad r \geq 2, \quad p, r \in \mathbb{N} \]

is asymptotically normal.

For the readers’ sake we will present first a complete proof of Theorem 2.1 and then in Section 5 we explain additional elements of the proof needed for Theorem 2.2 since a direct exposition from the beginning of the latter more general case would make the reading more difficult. Our main tool is splitting the products of \( X_j(q_j(n)) \) into \( \tilde{a}_j \), where \( \tilde{a}_j = 0 \) or \( \tilde{a}_j = a_j \), in the way which enables us to replace the expectation of a product by a product of expectations with a sufficiently small error which will yield, first, Gaussian type moment estimates for the expression in (2.5). Then we break the whole sum into a sum of blocks plus terms which can be disregarded but play the role of gaps between blocks. This will enable us to replace the characteristic function of a sum of these blocks by a product of their characteristic functions making only a small error. This is a standard method of proving central limit theorem type results when such blocks can be made sufficiently weakly dependent but in our case the terms of sums depend on the far away future so our blocks are strongly dependent and still, somewhat surprisingly, using the Taylor expansion of characteristic functions and splitting products as described above we can rely on this method in our case, as well. We observe that in the case of Theorem 2.2 we will need, in fact, certain sequences of blocks so that the numbers \( q_1(n), q_1(q_1(n)), q_1(q_1(q_1(n))) \), etc. stay within the same sequence.

Our \( \alpha \)-mixing condition is formulated in the form which allow functions depending on the whole path of a stochastic process and the exponentially fast decay (2.1) holds true for many important models. Let, for instance, \( \xi_n \) be a Markov chain on a space \( M \) satisfying the Doeblin condition (see, for instance, 11, p.p. 367–368) and \( f_j, j = 1, \ldots, \ell \) be a bounded measurable functions on the space of sequences \( x = (x_i, i = 0, 1, 2, \ldots), x_i \in M \) such that \( |f_j(x) - f_j(y)| \leq C e^{-cn} \) provided
x = (x_i), y = (y_i) and x_i = y_i for all i = 0, 1, ..., n where c, C > 0 do not depend on n
and j. Set X_j(n) = f_j(ξ_n, ξ_{n+1}, ξ_{n+2}, ...), and let σ-algebras \( \mathcal{F}_k \), k < l be generated
by ξ_k, ξ_{k+1}, ..., ξ_l then the condition \( (2.1) \) will be satisfied considering \( \{ \xi_n, n \geq 0 \} \) with
its invariant measure as a stationary process.

Important classes of processes satisfying our conditions come from dynamical systems. Let T be a
C^2 Axiom A diffeomorphism (in particular, Anosov) in a
neighborhood of an attractor or let T be an expanding C^2 endomorphism of a Riemmanian
manifold M (see [3]). f_j’s are Hölder continuous functions and X_j(n) = f_j(T^n x).
Here the probability space is (M, \( \mathcal{B} \), \( \mu \)) where \( \mu \) is a Gibbs invariant measure corre-
sponding to some Hölder continuous function. Let ζ be a finite Markov partition for
T then we can take \( \mathcal{F}_k \) to be the finite σ-algebra generated by the partition \( \Gamma \). Among other dynamical systems with exponentially fast
- mixing we can mention also the Gauss map

A functional central limit theorem extension of Theorem 2.1 can be derived by
essentially the same method. Namely, for each \( u \in [0, 1] \) set

\[
W_N(u) = N^{-1/2} \sum_{n=0}^{[uN]} \left( \prod_{j=0}^{\ell} X_j(q_j(n)) - \prod_{j=0}^{\ell} a_j \right).
\]

The process \( W_N \) is a cádlág, i.e. its paths belong to the space \( D[0, 1] \) of right continu-
ous functions on \( [0, 1] \) which have left limits and, as usual, we consider \( D[0, 1] \) with
the Skorokhod topology (see [2]). Denote by \( W \) the standard one dimensional Brown-
ian motion and let \( P_{W_N} \) and \( P_{\sigma W} \) be the distributions of \( W_N \) and of \( \sigma W(u), u \in [0, 1] \)
on \( D[0, 1] \), respectively, i.e.

\[
P_{W_N} = P\{W_N \in \Gamma\} \quad \text{and} \quad P_{\sigma W}(\Gamma) = P\{\sigma W \in \Gamma\}
\]

for any Borel subset \( \Gamma \) of \( D[0, 1] \).

**Theorem 2.3** Under the conditions of Theorem 2.1,

\[
P_{W_N} \Rightarrow P_{\sigma W} \quad \text{as} \quad N \to \infty
\]

where \( \Rightarrow \) denotes the weak convergence of measures.
We will derive in Section 4 Theorem 3.3, first, for the setup of Theorem 2.1, i.e. when \( X_0 \equiv 1 \) and \( \sigma = \sigma \), and the additional arguments of Section 5 will yield the result in the full generality of the setup of Theorem 2.2. The proof proceeds in the traditional way which consists of two ingredients. First, we show by the block technique of Section 4 (and by the corresponding modification of Section 5) that finite dimensional distributions of \( W_Y \) weakly converge to corresponding finite dimensional distributions of \( \hat{\sigma}W \) which identifies the limit in (2.15) uniquely (if it exists). Secondly, relying on Lemma 3.8 (and its generalisation in Section 5) we obtain tightness of the family \( \{ P_{W_Y}, N = 1, 2, ... \} \) which yields the convergence.

3 Gaussian type moment estimates

We start with the well known \( \alpha \)-mixing inequality (see, for instance, [4] or [5]) saying that for any nonnegative integers \( k, n \) and random variables \( Y \) and \( Z \) which are \( \mathcal{F}_{\infty, k} \)- and \( \mathcal{F}_{k+n, \infty} \)-measurable, respectively,

\[
|E(YZ) - E(Y)E(Z)| \leq 4\alpha(n)\|Y\|_{\infty}\|Z\|_{\infty}
\]  

(3.1)

where \( \| \cdot \|_{\infty} \) is the \( L^\infty \)-norm. This inequality yields the following "splitting" lemma which will be our main working tool throughout this paper.

**Lemma 3.1** Let \( Y(j), j = 0, 1, ... \) be bounded random variables and set

\[
\beta(n) = \sup_{j \geq 0} E|Y(j) - E(Y(j)|\mathcal{F}_{j-n, j+n})|.
\]  

(3.2)

Then for any \( 0 \leq n_1 \leq ... \leq n_i < n_{i+1} \leq n_{i+2} \leq ... \leq n_m \),

\[
|E [E_{l=1}^m Y(n_l) - E \prod_{i=1}^l Y(n_i) E \prod_{i=l+1}^m Y(n_i)] |
\]  

(3.3)

\[
\leq 2(m\beta(k) + 2\alpha(k)) \prod_{i=1}^m \max(1, \|Y(n_i)\|_{\infty})
\]

where \( k = \lfloor (n_{i+1} - n_i)/3 \rfloor \) and \( \lfloor \cdot \rfloor \) denotes the integral part.

**Proof** Clearly,

\[
|E \prod_{i=1}^m Y(n_i) - E \prod_{i=1}^l Y(n_i) E \prod_{i=l+1}^m Y(n_i) |
\]  

(3.4)

\[
\leq I_1 \prod_{i=l+1}^m \|Y(n_i)\|_{\infty} + I_2 \prod_{i=1}^l \|Y(n_i)\|_{\infty} + I_3
\]

where

\[
I_1 = E|\prod_{i=1}^l Y(n_i) - E(\prod_{i=1}^l Y(n_i)|\mathcal{F}_{\infty,n_{l+2}})|, 
\]  

(3.5)

\[
I_2 = E|\prod_{i=l+1}^m Y(n_i) - E(\prod_{i=l+1}^m Y(n_i)|\mathcal{F}_{n_{i+1},k,\infty})| 
\]  

(3.6)

and by (3.1),

\[
I_3 = |E(\prod_{i=1}^l Y(n_i)|\mathcal{F}_{\infty,n_{l+2}})E(\prod_{i=l+1}^m Y(n_i)|\mathcal{F}_{n_{i+1},k,\infty}) - E \prod_{i=1}^l Y(n_i) E \prod_{i=l+1}^m Y(n_i)| \leq 4\alpha(k) \prod_{i=1}^m \|Y(n_i)\|_{\infty}.
\]  

(3.7)
Observe that
\[
\left| \prod_{j=1}^{l} Y(n_i) - \prod_{i=1}^{l} E(Y(n_i) | \mathcal{F}_{-\infty, n_j+1}) \right| \leq \sum_{j=1}^{l} \left| \prod_{j=1}^{l} Y(n_i) (Y(n_j) - E(Y(n_j) | \mathcal{F}_{-\infty, n_j+1})) \prod_{j=1}^{l} E(Y(n_i) | \mathcal{F}_{-\infty, n_j+1}) \right|
\]
which together with (3.2) and (3.5) yields that
\[
I_1 \leq 2l \beta(k) \prod_{i=1}^{l} \max(1, \|Y(n_i)\|_{\infty}). \tag{3.8}
\]

Similarly,
\[
I_2 \leq 2(m - l) \beta(k) \prod_{i=1}^{m} \max(1, \|Y(n_i)\|_{\infty}), \tag{3.9}
\]
and so (3.3) follows from (3.4)–(3.7), (3.8) and (3.9).

Next, set
\[
R(n) = \prod_{j=1}^{\ell} X_j(q_j(n)) - \prod_{j=1}^{\ell} a_j
\]
\[
= \sum_{j=1}^{\ell} a_1 \cdots a_{j-1}(X_j(q_j(n)) - a_j)X_{j+1}(q_{j+1}(n)) \cdots X_{\ell}(q_{\ell}(n)). \tag{3.10}
\]

Here and in what follows if \( \ell = 1 \) and a formula includes products of undefined factors such as \( X_{\ell+1}, X_{\ell-1}, a_{\ell+1}, a_{\ell-1} \) with \( 1 \leq j \leq \ell \) then such products should be replaced by 1. Observe that by (2.3) and (2.4) for any \( j = 1, \ldots, \ell - 1 \) and \( n \geq n_0, \)
\[
q_{j+1}(n) - q_{j}(n) \geq n - \lfloor n^{1-\gamma} \rfloor, \tag{3.11}
\]
and so by (3.3) for such \( n, \)
\[
|ER(n)| \leq 4l \beta(\ell) (\lfloor n^{1-\gamma} \rfloor) + 2(\lfloor n^{1-\gamma} \rfloor/3)). \tag{3.12}
\]

The following result provides a Gaussian type estimate for the second moment of sums of \( R(n) \)'s.

**Lemma 3.2** There exists \( C > 0 \) such that for all \( n \in \mathbb{N}, \)
\[
E \left( \sum_{k=0}^{n} R(k) \right)^2 \leq Cn. \tag{3.13}
\]

**Proof** By (3.10) for any \( k_1, k_2 \leq n, \)
\[
|ER(k_1)R(k_2)| \leq \sum_{j_1, j_2=1}^{\ell} D^{j_1 + j_2 - 2} |EQ_{j_1,j_2}(k_1,k_2)| \tag{3.14}
\]
where, recall, \( D \) is an upper bound on all \( |X_j(k)| \)'s and
\[
Q_{j_1,j_2}(k_1,k_2) = \sum_{i=1}^{2} [(X_{j_1}(q_{j_1}(k_i)) - a_{j_1})X_{j_1+1}(q_{j_1+1}(k_i)) \cdots X_{\ell}(q_{\ell}(k_i))].
\]
Suppose that \( q_{j_1}(k_1) < q_{j_2}(k_2) \) and \( k_1, k_2 > n_0 \) where \( n_0 \) is the same as in (2.3) and (2.4). Then by (2.4),
\[
q_{j_1}(k_1) < q_{j_2}(k_2) < ... < q_{j_2}(k_2).
\]
Hence, we can apply (3.3) with \( k_i \) in place of \( n_i \), \( Y(n_1) = X_j(q_{j_1}(k_1)) - a_{j_1} \), \( n_1 = q_{j_1}(k_1) \), \( l = 1 \) and \( Y(n_i), i > 1 \) being other factors in the product for \( Q_{j_1j_2}(k_1, k_2) \) deriving that
\[
|E Q_{j_1j_2}(k_1, k_2)| \leq 16D2^l (\ell \beta(q_{j_1j_2}(k_1, k_2)) + \alpha(q_{j_1j_2}(k_1, k_2)))
\] (3.15)
where
\[
v_{j_1j_2}(k_1, k_2) = \min\left(\{(q_{j_1+1}(k_1) - q_{j_1}(k_1))/3, [(q_{j_2}(k_2) - q_{j_1}(k_1))/3]\}\right).
\]
This together with (2.1), (2.3) and (3.11) yields that there exists a constant \( C_1 > 0 \) such that
\[
\sum_{1 \leq j_1, j_2 \leq \ell, n_0 \leq k_1, k_2 \leq n \text{ and } q_{j_1}(k_1) < q_{j_2}(k_2)} |E Q_{j_1j_2}(k_1, k_2)| \leq C_1 (n + 1).
\] (3.16)
Now, if
\[
q_{j_1}(k_1) = q_{j_2}(k_2)
\] (3.17)
for some \( k_1, k_2 \geq n_0 \) then by (2.3),
\[
q_{j_1}(k_1) < q_{j_2}(k_2 + m) \quad \text{for all } m \geq 1,
\]
and so the number of pairs \( (j_2, k_2) \) such that \( 1 \leq j_2 \leq \ell, n_0 \leq k_2 \leq n \) and (3.17) is satisfied does not exceed \( \ell \). Hence, we obtain from here and (3.16) that
\[
\sum_{1 \leq j_1, j_2 \leq \ell, n_0 \leq k_1, k_2 \leq n} |E Q_{j_1j_2}(k_1, k_2)| \leq C_2 (1 + 2\ell^2 n_0 + 2\ell^2)(n + 1)
\] (3.18)
for some \( C_2 > 0 \) and (3.13) follows.

**Remark 3.3** The estimates (3.11) and (3.15) enable us to obtain (3.13) under a weaker than (2.1) condition, namely, a polynomial decay of \( \alpha(n) \) and \( \beta(n) \) so that either
\[
\sum_{n=1}^{\infty} \alpha(n^{\gamma}) + \beta(n^{\gamma}) \quad \text{or} \quad \sum_{n=1}^{\infty} \alpha(n^{1-\gamma}) + \beta(n^{1-\gamma})
\]
converges would already suffice. If we were interested only in (3.13) we could also weaken the boundedness condition on the stationary processes \( X_j, j = 1, ..., \ell \) assuming only existence of their sufficiently high moments and using in place of (3.1) the inequality (see [4] or [5]),
\[
|E(YZ) - EYZ| \leq 10||Y||_p ||Z||_q (\alpha(n))^{1 - \frac{1}{p} - \frac{1}{q}}
\] (3.19)
which holds true provided \( Y \) and \( Z \) are \( \mathcal{F}_{\infty,k} \)- and \( \mathcal{F}_{k+\gamma,n} \)-measurable random variables, respectively, such that \( E|Y|^p < \infty, E|Z|^q < \infty \) and \( \frac{1}{p} + \frac{1}{q} < 1 \). Furthermore, (3.13) does not require the full strength of the assumption (2.4) as we use only (3.11) so that in place of (2.4) we can assume here, for instance, that \( q_{j+1}(n) - q_j(n) \geq \delta n^\delta \) for some \( \delta > 0 \) and all \( n \geq n_0 \).
Remark 3.4 Lemma 3.4 yields that in the $L^2$-sense,
\[
\frac{1}{n} \sum_{k=0}^{n} X_j(q_j(k)) \longrightarrow \prod_{j=1}^{\ell} a_j \quad \text{as} \quad n \rightarrow \infty
\] (3.20)
which seems to be new when $q_j$’s are not polynomials.

The following result justifies the formula (2.6) for the variance in our SPLIT.

Lemma 3.5 Suppose that $N \geq n > m \geq [N^{1-\gamma}] \geq n_0$. Then
\[
|E\left(\sum_{k=m+1}^{n} R(k)\right)^2 - (n-m)\sigma^2| \leq \hat{C}
\] (3.21)
for some constant $\hat{C} > 0$ independent of $n, m$ and $N$, where $\sigma$ is given by (2.6).

Proof By (3.10)
\[
E(R(k_1)R(k_2)) = \sum_{j_1=1}^{\ell} a_{j_1}^{2}\cdots a_{j_2-1}^{2} EQ_{jj}(k_1, k_2) \tag{3.22}
\]
+ $\sum_{j_2 > j_1} a_{j_1}\cdots a_{j_2-1}(EQ_{jj}(k_1, k_2) + EQ_{j_1 j_2}(k_1, k_2))$
where $EQ_{j_1 j_2}(k_1, k_2)$ is the same as in (3.14). First, we estimate $EQ_{jj}(k_1, k_2)$ for $j \geq 2$ and $k_1 \neq k_2$, say, when $k_2 > k_1$. Assuming that $k_1 \geq m$ it follows from (2.3) and (3.11) that

\[
q_j(k_2) \geq q_j(k_1) + m^{\gamma} \text{ and } q_{j+1}(k_1) \geq q_j(k_1) + m - [m^{1-\gamma}],
\] (3.23)

and so we can apply (3.15) in order to obtain
\[
|EQ_{jj}(k_1, k_2)| \leq 16D^{2\ell}(\ell \beta(\rho_1(N)) + \alpha(\rho_1(N)))
\] (3.24)
where
\[
\rho_1(N) = \min\left([N^{1-\gamma}]^{\gamma}/3, \left[[N^{1-\gamma}] - [N^{1-\gamma}]^{\gamma}\right]/3\right)
\]
since $m \geq [N^{1-\gamma}]$.

Next, if $j_2 > j_1$ and $k, l \geq [N^{1-\gamma}]$ then
\[
q_{j_2}(l) \geq q_{j_1}(k)N^{\gamma} \geq q_{j_1}(k) + N^{\gamma} - 1,
\] (3.25)
and so by (3.15) we conclude that
\[
|EQ_{j_1 j_2}(k, l)| \leq 16D^{2\ell}(\ell \beta(\rho_2(N)) + \alpha(\rho_2(N)))
\] (3.26)
where $\rho_2(N) = ([N^{\gamma} - 1])/3$.

It remains to deal with the terms $Q_{jj}(k, k)$ and $Q_{11}(k_1, k_2)$. Taking into account (3.25) we apply (3.3) with $Y(n_1) = (X_j(q_j(k)) - a_j)^2$, $n_1 = q_j(k)$, $l = 1$ and $Y(n_{j+i}) = X_j^{2}(q_{j+i}(k))$, $n_{j+i} = q_{j+i}(k), i = 1, ..., \ell - j$. It follows that
\[
|EQ_{jj}(k, k) - E(X_j(q_j(k)) - a_j)^2E\prod_{i=1}^{j-1} X_{j+i}^{2}(q_{j+i}(k))| \leq 16D^{2\ell}(\ell \beta(\rho_2(N)) + \alpha(\rho_2(N))).
\] (3.27)
Applying the same argument $\ell - j - 1$ times to the expectation of the product in (3.27) and taking into account stationarity of the processes $X_j$, we obtain that

$$|E Q_{ij}(k, k) - E (X_j(0) - a_j)^2 \Pi_{j+1}^{(i-1)} E X_j^2(0) - a_j^2| \leq 16(\ell + 1)D^{2j}(\ell \beta (p_2(N)) + \alpha (p_2(N))).$$

and since $E(X_j(0) - a_j)^2 = EX_j^2(0) - a_j^2$ it follows that

$$|\sum_{j=0}^{\ell} a_j^2 \cdot a_j^{j-1} E Q_{ij}(k, k) - \Pi_{j=1}^{(i-1)} EX_j^2(0) + \Pi_{j=1}^{(i-1)} a_j^2| \leq 16(\ell + 1)D^{2j}(\ell \beta (p_2(N)) + \alpha (p_2(N))).$$

Finally, in view of (2.2) and (2.3) for $k_2 > k_1 \geq \lfloor N^{1-\gamma} \rfloor$ we obtain relying on (3.3)

$$|E Q_{11}(k_1, k_2) - E (X_1(r(k_2 - k_1)) - a_1)(X_1(0) - a_1)| \Pi_{j=2}^{(2)} a_j^2| \leq 32D^{2j}(\ell \beta (p_2(N)) + \alpha (p_2(N))).$$

Again, by (3.3) we have also

$$|E (X_1(r(k_2 - k_1)) - a_1)(X_1(0) - a_1)| \leq 16D^{2j}(\beta (r(k_2 - k_1)/3) + \alpha (r(k_2 - k_1)/3)).$$

This together with (2.1) yields that for some constant $C_3 > 0$ independent of $n, m$ and $N,

$$|(n-m)\sum_{0}^{\infty} E (X_1(ri) - a_1)(X_1(0) - a_1)| - \sum_{k_1=m+1}^{n} \sum_{k_1=0}^{n-k_1} E (X_1(ri) - a_1)(X_1(0) - a_1) \leq C_3.$$ 

Collecting (3.24), (3.26) and (3.28)–(3.30) we arrive at (3.21) taking into account (2.1) which completes the proof of the lemma.

**Corollary 3.6**

$$\lim_{N \to \infty} \frac{1}{N} E \left( \sum_{n=0}^{N} R(n) \right)^2 = \sigma^2$$

and if $\sigma = 0$ then $N \to \infty$ the expression (2.5) converges to zero in distribution.

**Proof** By (3.13) for any $M < N,$

$$|E \left( \sum_{n=0}^{N} R(n) \right)^2 - E \left( \sum_{n=M}^{N} R(n) \right)^2| = \left| E \left( \sum_{n=0}^{M-1} R(n) \right) \left( \sum_{n=0}^{N} R(n) \right) + \sum_{n=M}^{N} R(n) \right| \leq \sqrt{2} \left( E \left( \sum_{n=0}^{N} R(n) \right)^2 \right)^{1/2} \left( E \left( \sum_{n=0}^{N} R(n) \right)^2 \right)^{1/2} \leq 2\sqrt{2} \sqrt{MN}$$

and (3.31) follows from (3.21) and (3.22) taking $M = \lfloor N^{1-\gamma} \rfloor + 1.$ If $\sigma = 0$ then (3.31) together with the Chebyshev inequality yields that as $N \to \infty$ the expression (2.5) converges to zero in probability, and so in distribution, and in this case the main assertion of Theorem 2.1 follows.
Corollary 3.7 If \( \sigma_1 = 0 \) then

\[
\sup_n E \left( \sum_{j=0}^n (X_j (rj + p)) - a_1 \right)^2 < \infty, \tag{3.33}
\]

and the representation \( 2.3 \) holds true.

Proof The inequality \( 3.33 \) follows from \( 2.7, 3.29 \) and \( 3.30 \), and so by Theorem 18.2.2 from \( 11 \) the representation \( 2.8 \) takes place.

The following result gives the 4th moment Gaussian type estimate needed to bound the error in the Taylor expansions of the characteristic functions.

Lemma 3.8 There exists \( \tilde{C} > 0 \) such that whenever \( N \geq n > m \geq [N^{1-\gamma}] \geq n_0 \) then

\[
E \left( \sum_{k=m+1}^n R(k) \right)^4 \leq \tilde{C} (n - m)^2. \tag{3.34}
\]

Proof We have

\[
E \left( \sum_{k=m+1}^n R(k) \right)^4 \leq \sum_{k_1,k_2,k_3,k_4=m+1}^n A_{k_1,k_2,k_3,k_4} \tag{3.35}
\]

where by \( 3.10 \) for any \( k_1,k_2,k_3,k_4 \),

\[
A_{k_1,k_2,k_3,k_4} = |E \left( R(k_1)R(k_2)R(k_3)R(k_4) \right)| \leq \sum_{j_1,j_2,j_3,j_4=1}^{n} D_{j_1+j_2+j_3+j_4-4} |Q_{j_1,j_2,j_3,j_4}(k_1,k_2,k_3,k_4)| \tag{3.36}
\]

with

\[
Q_{j_1,j_2,j_3,j_4}(k_1,k_2,k_3,k_4) = E \prod_{i=1}^{4} \left( (X_i(q_{j_i}(k_i)) - a_{j_i})X_{i+1}(q_{j_i+1}(k_i)) \cdot \cdot \cdot X_{j_i}(q_{j_i}(k_i)) \right). \tag{3.27}
\]

In estimating the terms in the right hand side of \( 3.36 \) we assume without loss of generality that \( j_1 \leq j_2 \leq j_3 \leq j_4 \). If \( j_1 < j_2 \) then taking into account that \( k_1,k_2,k_3,k_4 > m \geq [N^{1-\gamma}] \) we conclude relying on \( 3.3 \) and using \( 3.25 \) similarly to \( 3.26 \) that in this case

\[
|Q_{j_1,j_2,j_3,j_4}(k_1,k_2,k_3,k_4)| \leq 64D^{44(2\tilde{\beta}(\rho_2(N)) + \alpha(\rho_2(N)))} \tag{3.37}
\]

with the same \( \rho_2(N) \) as in \( 3.26 \).

Next, consider the case \( j_1 = j_2 < j_3 = j_4 \). Then

\[
Q_{j_1,j_2,j_3,j_4}(k_1,k_2,k_3,k_4) = \prod_{i=1}^{4} \left( X_i(q_{j_i}(k_i)) - a_{j_i} \right) Z_1 Z_2 \prod_{j=3}^{j_4} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i}) Z_3 \tag{3.38}
\]

where \( Z_1 \) is the product of terms \( X_i(q_{j_i}(k_i)) \) with \( i = 1,2 \) and \( j_1 < j < j_3 \), \( Z_2 \) is the product of terms \( X_i(q_{j_i}(k_i)) \) with \( i = 1,2 \) and \( Z_3 \) is the product of terms \( X_j(q_j(k_i)) \)
with \( j_3 < j \leq \ell \) and \( i = 1, 2, 3, 4 \). Then employing 3 times (3.35) and using again (3.25) we obtain in this case that

\[
|Q_{j_1, j_2, j_3, j_4}(k_1, k_2, k_3, k_4) - E \left( \prod_{i=1}^{4} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i}) \right) EZ_1 \times E(Z_2) \prod_{i=3}^{4} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i})|EZ_3| \\
\leq 192D^{4\ell} (2\ell \beta (\rho_2(N)) + \alpha (\rho_2(N)))
\]  

(3.39)

By (2.2) and (2.3) we see that for any \( E \),

\[
|q_{j_i}(k) - q_{j_i}(l)| \geq |k - l|,
\]

(3.40)

and so we derive from (3.3) that

\[
|E \left( \prod_{i=1}^{2} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i}) \right) - 16D^2 \left( \beta(|k_1 - k_2|/3) + \alpha(|k_1 - k_2|/3) \right)|.
\]

(3.41)

Applying the same argument twice we obtain also that

\[
|E \left( Z_2 \prod_{i=3}^{4} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i}) \right) - E(Z_2) \prod_{i=3}^{4} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i})| \leq 16D^{2\ell} (\ell \beta (\rho_3(k_1, k_2, k_3, k_4)) + \alpha (\rho_3(k_1, k_2, k_3, k_4)))
\]

(3.42)

where

\[
\rho_3(k_1, k_2, k_3, k_4) = \frac{1}{3} \min_{i_1=1, i_2=2} |k_{i_1} - k_{i_2}|
\]

and

\[
|E \left( \prod_{i=3}^{4} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i}) \right) - 16D^2 \left( \beta(|k_3 - k_4|/3) + \alpha(|k_3 - k_4|/3) \right)|.
\]

(3.43)

Next, if \( j_1 = j_2 < j_3 < j_4 \) then we represent \( Q_{j_1, j_2, j_3, j_4}(k_1, k_2, k_3, k_4) \) again in the form (3.33) but now applying 3 times (3.3) we obtain

\[
|Q_{j_1, j_2, j_3, j_4}(k_1, k_2, k_3, k_4) - E \left( \prod_{i=1}^{4} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i}) \right) EZ_1 \times E(Z_2) \prod_{i=3}^{4} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i})|EZ_3| \\
\leq 192D^{4\ell} (2\ell \beta (\rho_2(N)) + \alpha (\rho_2(N)))
\]  

(3.44)

Similarly to (3.42) and (3.43) it follows that

\[
|E \left( Z_2(X_{j_i}(q_{j_i}(k_i)) - a_{j_i}) \right)| \\
\leq 8D^{2\ell-1} (\ell \beta (\rho_4(k_1, k_2, k_3)) + \alpha (\rho_4(k_1, k_2, k_3)))
\]

(3.45)

where

\[
\rho_4(k_1, k_2, k_3) = \frac{1}{3} \min(|k_1 - k_3|, |k_2 - k_3|).
\]

Now, if \( j_1 = j_2 = j_3 < j_4 \) then

\[
Q_{j_1, j_2, j_3, j_4}(k_1, k_2, k_3, k_4) = \prod_{i=1}^{3} (X_{j_i}(q_{j_i}(k_i)) - a_{j_i})Z_4
\]

(3.46)
where $Z_4$ is the product of $X_j(q_j(k_i)) - a_{j_i}$ and the terms of the form $X_j(q_j(k_i))$ with $\ell \geq j > j_1$ and $i = 1, 2, 3, 4$. In this case by (3.3) and (3.23),

$$|Q_{j_1,j_2,j_3,j_4}(k_1,k_2,k_3,k_4) - E \prod_{i=1}^4 (X_j(q_j(k_i)) - a_{j_i})EZ_4| \leq 192D^{e\ell}(2\ell\beta(p_2(N)) + \alpha(p_2(N))).$$

Applying (3.3) and (3.40) we obtain that

$$|E \prod_{i=1}^4 (X_j(q_j(k_i)) - a_{j_i})| \leq 16D^3(3\beta(p_2(k_1,k_2,k_3)) + 2\alpha(p_2(k_1,k_2,k_3)))$$

where

$$\rho_5(k_1,k_2,k_3)) = \frac{1}{6}(\max(k_1,k_2,k_3) - \min(k_1,k_2,k_3)).$$

Finally, in the case $j_1 = j_2 = j_3 = j_4$ we can write

$$Q_{j_1,j_2,j_3,j_4}(k_1,k_2,k_3,k_4) = \prod_{i=1}^4 (X_j(q_j(k_i)) - a_{j_i})Z_5$$

where $Z_5$ is the product of the terms $X_j(q_j(k_i))$ with $\ell \geq j > j_1$ and $i = 1, 2, 3, 4$. Then by (3.3) and (3.23) we have that

$$|Q_{j_1,j_2,j_3,j_4}(k_1,k_2,k_3,k_4) - E \prod_{i=1}^4 (X_j(q_j(k_i)) - a_{j_i})EZ_4| \leq 192D^{e\ell}(2\ell\beta(p_2(N)) + \alpha(p_2(N))).$$

Suppose that $k_{i_1} \leq k_{i_2} \leq k_{i_3} \leq k_{i_4}$ where $i_1,i_2,i_3,i_4$ are different integers between 1 and 4. Then by (3.3) and (3.23),

$$|E \prod_{i=1}^4 (X_j(q_j(k_{i})) - a_{j_i})| \leq 64D^4(2\beta(p_6(k_1,k_2,k_3,k_4)) + \alpha(p_6(k_1,k_2,k_3,k_4)))$$

where

$$\rho_6(k_1,k_2,k_3,k_4) = \frac{1}{3} \max(|k_{i_2} - k_{i_1}|,|k_{i_4} - k_{i_3}|).$$

Collecting (3.35)–(3.39) and (3.41)–(3.51) and taking into account (2.1) we arrive at (3.34) completing the proof of the lemma. \hfill \Box

Remark 3.9 It is clear from the above arguments that the proofs of Lemmas 3.5 and 3.8 still go through if in place of (3.1) and boundedness of $X_j$’s we assume that $\alpha(n)$ and $\beta(n)$ decay with sufficiently fast polynomial speed and some high enough moments of $X_j$’s are finite so that we could apply (3.19) sufficiently many times. This would not suffice in the next section where we have to apply (3.1) in the form of (3.3) the number of times growing in $N$, and so (3.19) with any fixed $p$ and $q$ will not work.

Remark 3.10 Lemma 3.8 yields the convergence (3.20) with probability one. Indeed, (3.34) together with Chebyshev’s inequality gives that

$$P\left(\frac{1}{n} \sum_{k=0}^n R(k) \geq \frac{1}{n^q}\right) \leq \tilde{C}n^{-3/2}$$

which in view of the Borel–Cantelly lemma implies the above assertion.
4 Blocks and characteristic functions

Choose a small positive \( \varepsilon \) and a large \( L \geq 4 \) so that \( L \varepsilon < \gamma / 4 \). Set \( \tau(N) = [N^{1-\varepsilon}] \), \( \theta(N) = [N^{1-L\varepsilon}] \), \( m(N) = \lfloor N / (\theta(N) + \tau(N)) \rfloor \) and introduce the sets of integers

\[
\Gamma_k(N) = \{ n : \theta(N) + (k-1)(\theta(N) + \tau(N)) \leq n \leq k(\theta(N) + \tau(N)) \}
\]

and

\[
\tilde{\Gamma}_k(N) = \{ n : (k-1)(\theta(N) + \tau(N)) + 1 \leq n \leq \theta(N) + (k-1)(\theta(N) + \tau(N)) \}.
\]

Assuming that \( N \geq \exp(2/\varepsilon) \) which ensures that \( m(N) \geq 1 \) set for \( k = 1, 2, \ldots, m(N) \),

\[
Y_k = \sum_{n \in \Gamma_k(N)} R(n) \quad \text{and} \quad Z_k = \sum_{n \in \tilde{\Gamma}_k(N)} R(n)
\]

where \( R(n) \) is the same as in (3.10). Till the end of this section our goal will be to show that the characteristic function \( \Phi_N(t) = E \exp \left( \frac{it}{\sqrt{N}} \sum_{n=0}^{N} R(n) \right) \) converges to \( \exp(-\sigma^2t^2/2) \) which will yield Theorem 2.1. In doing so we employ the blocks (partial sums) introduced above and the estimates of Section 3 so that we will deal mainly with the larger blocks \( Y_k \) showing that the smaller blocks \( Z_k \) can be disregarded and they will be treated as gaps between \( Y_k \)'s.

First, setting

\[
\Psi_N(t) = E \exp \left( \frac{it}{\sqrt{N}} \sum_{1 \leq n \leq m(N)} Y_n \right)
\]

and relying on the inequality

\[
|e^{ix+y} - e^y| = |e^{ix} - 1| \leq |x|
\]

we obtain from (3.13) and (3.34) that

\[
|\Phi_N(t) - \Psi_N(t)| \leq \frac{|t|}{\sqrt{N}} E \left| \sum_{n=0}^{\theta(N)} R(n) \right| + \left| \sum_{2 \leq n \leq m(N)} Z_n \right| + \left| \sum_{n=m(N)}^{\theta(N)+\tau(N)+1} R(n) \right|
\]

\[
\leq \frac{|t|}{\sqrt{N}} \left( \sqrt{C} \sqrt{\theta(N) + 1} + \tilde{C}^{1/4} m(N) \sqrt{\theta(N) + \tilde{C}^{1/4} \sqrt{\theta(N) + \tau(N)}} \right)
\]

for some constant \( \tilde{C} > 0 \) independent of \( N \).

The main part of this section is the following result showing that up to a small error the characteristic function of the sum of blocks \( Y_k \) is close to the product of characteristic functions of \( Y_k \)'s themselves. When blocks are weakly dependent this step follows immediately from (3.1) but our blocks are strongly dependent, and so the proof requires some work. Set

\[
\psi_N^{(k)}(t) = E \exp \left( \frac{it}{\sqrt{N}} Y_k \right), \quad k \leq m(N).
\]
**Lemma 4.1** For any \( t \) and each small \( \varepsilon > 0 \) there exists \( K_\varepsilon(t) > 0 \) such that for all \( N \geq \exp(2/\varepsilon) \),

\[
|\Psi_N(t) - \prod_{1 \leq k \leq m(N)} \psi_N^{(k)}(t)| \leq K_\varepsilon(t)N^{-\frac{2}{\varepsilon}\sqrt{N}}. \tag{4.2}
\]

**Proof** Set \( \hat{Y}_k = Y_k + \tau(N)\prod_{j=1}^k a_j \),

\[
\hat{\Psi}_N(t) = E \exp \left( \frac{it}{\sqrt{N}} \sum_{1 \leq k \leq m(N)} \hat{Y}_k \right) \text{ and } \hat{\psi}_N^{(k)}(t) = E \exp \left( \frac{it}{\sqrt{N}} \hat{Y}_k \right).
\]

Then, clearly,

\[
|\Psi_N(t) - \prod_{1 \leq k \leq m(N)} \psi_N^{(k)}(t)| = |\hat{\Psi}_N(t) - \prod_{1 \leq k \leq m(N)} \hat{\psi}_N^{(k)}(t)|. \tag{4.3}
\]

By the reminder formula for the Taylor expansion

\[
|e^z - \sum_{k=0}^n \frac{(iz)^k}{k!}| \leq \frac{|z|^{n+1}}{(n+1)!}. \tag{4.4}
\]

With the same \( \varepsilon > 0 \) as above set

\[
n(N) = n_\varepsilon(N) = \lceil N^{\frac{1}{2}+\varepsilon} \rceil \tag{4.5}
\]

and denote

\[
i_N^{(k)}(t) = \sum_{l=0}^{n(N)} \frac{(it)^l}{N^{l/2}l!} \hat{Y}_k^l. \tag{4.6}
\]

Then by (4.3),

\[
|\exp \left( \frac{it}{\sqrt{N}} \hat{Y}_k \right) - i_N^{(k)}(t)| \leq \frac{|t|D\sqrt{N}n(N)+1}{n(N)+1!} \leq C_4^{n(N)}|t|^{n(N)}N^{-\varepsilon n(N)} \tag{4.7}
\]

for some constant \( C_4 > 0 \) independent of \( N \geq 4 \). Then

\[
|\hat{\Psi}_N(t) - \prod_{1 \leq k \leq m(N)} \hat{\psi}_N^{(k)}(t)| \leq J(t,N) + \delta(t,N) \tag{4.8}
\]

where

\[
J(t,N) = |E \prod_{1 \leq k \leq m(N)} i_N^{(k)}(t) - \prod_{1 \leq k \leq m(N)} Ei_N^{(k)}(t)|
\]

and

\[
\delta(t,N) = 2m(N)C_4^{n(N)}|t|^{n(N)}N^{-\varepsilon n(N)} \times (1 + C_4^{n(N)}|t|^{n(N)}N^{-\varepsilon n(N)})m(N) \leq C(\varepsilon,t)N^{-\frac{2}{\varepsilon}\sqrt{N}}
\]

for some \( C(\varepsilon,t) > 0 \) independent of \( N \).
It remains to estimate $J(t, N)$ which is the main point of the proof. We have

$$J(t, N) = \sum_{0 \leq l_1, \ldots, l_m(N) \leq n(N)} |tN^{-1/2} \Sigma 1 \leq k \leq m(N) l_k!^{-1} G_{l_1, \ldots, l_m(N)}(t, N) \tag{4.9}$$

where

$$G_{l_1, \ldots, l_m(N)}(t, N) = |E \prod_{k=1}^{m(N)} \hat{Y}^k_l - \prod_{k=1}^{m(N)} E \hat{Y}^k_l|.$$ 

Next, we represent the $l_k$-th power of the sum $\hat{Y}_k$ in the form

$$\hat{Y}^l_k = \sum_{\sigma^{(k)}} \beta^{(k)}_{\sigma^{(k)}} \prod_{n \in \Gamma_k(N)} \ell \prod_{j=1}^{l} X_{j}^{\alpha_{j}}(q_j(n)) \tag{4.10}$$

where $\beta^{(k)}_{\sigma^{(k)}}$ are $l_k$-nomial coefficients and $\sigma^{(k)} = (\sigma_n^{(k)}, n \in \Gamma_k(N))$ satisfies

$$\sigma_n^{(k)} \geq 0 \text{ and } \sum_{n \in \Gamma_k(N)} \sigma_n^{(k)} = l_k \leq n(N). \tag{4.11}$$

Then

$$G_{l_1, \ldots, l_m(N)}(t, N) \leq \sum_{\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(m(N))}} \prod_{k=1}^{m(N)} \beta^{(k)}_{\sigma^{(k)}} H_{l_1, \ldots, l_m(N)}(t, N) \tag{4.12}$$

where

$$H_{l_1, \ldots, l_m(N)}(t, N) = |E \prod_{k=1}^{m(N)} \prod_{n \in \Gamma_k(N)} \prod_{j=1}^{l} X_{j}^{\alpha_{j}}(q_j(n))$$

$$- \prod_{k=1}^{m(N)} E \prod_{n \in \Gamma_k(N)} \prod_{j=1}^{l} X_{j}^{\alpha_{j}}(q_j(n))|.$$ 

Next, we change the order of products in the two expectations above so that the product $\prod_{j=1}^{l}$ appear immediately after the expectation and apply the ”splitting” Lemma [5.1] $\ell$ times to the latter product for both expectations. Since $n \geq [N^{1-\ell}]$ in the above expressions then relying $\ell$ times on [5.3] and the second part of [4.2] we obtain taking into account (4.11) that

$$|E \prod_{k=1}^{m(N)} \prod_{n \in \Gamma_k(N)} \prod_{j=1}^{l} X_{j}^{\alpha_{j}}(q_j(n)) $$

$$- \prod_{k=1}^{m(N)} E \prod_{n \in \Gamma_k(N)} \prod_{j=1}^{l} X_{j}^{\alpha_{j}}(q_j(n))| \leq 2\ell D^{m(N)m(N)}(\ell m(N) \beta(\rho_6(N)) + 2\alpha(\rho_6(N))) \tag{4.13}$$

where

$$\rho_6(N) = \left\lfloor \frac{1}{3} \left( [N^{1-\ell}] - [N^{1-\gamma}] [1-\ell] \right) \right\rfloor.$$ 

Similarly,

$$|E \prod_{n \in \Gamma_k(N)} \prod_{j=1}^{l} X_{j}^{\alpha_{j}}(q_j(n)) - \prod_{j=1}^{l} E \prod_{n \in \Gamma_k(N)} X_{j}^{\alpha_{j}}(q_j(n))| \leq 2\ell D^{m(N)}(\ell m(N) \beta(\rho_6(N)) + 2\alpha(\rho_6(N))) \tag{4.14}$$
Next, for each fixed \( j \) we apply (3.3) \( m(N) \) times to the product \( \prod_{k=1}^{m(N)} \) appearing after the expectation and in view of (3.40) and the size of the gaps \( Z_k \) between the blocks \( Y_k \) it follows that

\[
|E \prod_{k=1}^{m(N)} \prod_{\ell \in F_k(N)} X_{j}^{(k)}(q_j(n)) - \prod_{k=1}^{m(N)} E \prod_{\ell \in F_k(N)} X_{j}^{(k)}(q_j(n))| \leq 2m(N)D^{m(N)n(N)}(m(N)n(N)\beta([N^{1-\epsilon}]/3) + 2\alpha([N^{1-\epsilon}]/3)).
\]

(4.15)

Collecting (4.3), (4.5)–(4.15) and taking into account that for each \( k \),

\[
\sum_{\sigma(k) \in \sigma(k)} \beta_{\sigma(k)} \leq N^{(1-\epsilon)l_k}
\]

and

\[
\sum_{1 \leq l, \ldots, \sum_{m(N) \leq n(N)}} |N^{\frac{1}{2}-\epsilon} l_k| \leq \exp(N^{\frac{1}{2}-\epsilon}l)\}
\]

we arrive at (4.2),

Now we can complete the proof of Theorem 2.1. Using the inequalities

\[
|e^{ix} - 1 - ix + \frac{x^2}{2}| \leq |x|^3 \quad \text{and} \quad |e^{-x} - 1 + x| \leq x^2
\]

which hold true for any real \( x \) we derive from (3.12), (3.21) and (3.34) together with the Hölder inequality that

\[
|\psi_{\delta}(t) - \exp\left(-\sigma_{0}^{2} t/N\right)| \leq 4\ell D^{\frac{1}{2}-\epsilon} |t| (\epsilon \beta(\rho_{0}(N)) + 2\alpha(\rho_{0}(N))) + C^{3/4} |t|^{3} N^{-3/2} + \sigma_{1}^{2} \tau(\tau(N))^2
\]

(4.16)

where \( \rho_{0} \) is the same as in (4.13). Taking into account that

\[
|\prod_{1 \leq k \leq l} \gamma_{k} - \prod_{1 \leq k \leq l} h_{k}| \leq \sum_{1 \leq k \leq l} |\gamma_{k} - h_{k}|
\]

(4.17)

whenever \( 0 \leq |\gamma_{k}|, |h_{k}| \leq 1, k = 1, \ldots, l \) we obtain from (4.16) that

\[
|\prod_{1 \leq k \leq m(N)} \psi_{\delta}^{(k)}(t) - \exp\left(-\sigma_{0}^{2} t/N\right)| \leq 4\ell D^{\frac{1}{2}-\epsilon} m(N)|t|^3 (\epsilon \beta(\rho_{0}(N)) + 2\alpha(\rho_{0}(N))) + C^{3/4} |t|^3 N^{-3/2} m(N) + \sigma_{1}^{2} \tau(\tau(N))^2 m(N)
\]

(4.18)

and since \( m(N) \) is of order \( N^{\epsilon} \) while \( \tau(\tau(N)) \) is of order \( N^{1-\epsilon} \) we obtain that the right hand side of (4.18) is bounded by \( \text{const}(t^4 + 1)N^{-\epsilon/2} \). This together with (4.1) and (4.2) gives

\[
|\Phi_{\delta}(t) - \exp\left(-\frac{1}{2} \sigma^{2} t^2\right)| \leq K(t)N^{-\epsilon/2}
\]

(4.19)

for some \( K(t) > 0 \) independent of \( N \) and the assertion of Theorem 2.1 follows. \( \Box \)
Next, we explain the proof of Theorem 2.3. In order to show that finite dimensional distributions of $W_N$ converge to corresponding finite dimensional distributions of $\sigma W$ we fix $0 = u_0 < u_1 < u_2 < \cdots < u_k \leq 1$ and some real $t_1, t_2, \ldots, t_k$ proving that

$$
\Phi_N^{u_1, \ldots, u_k}(t_1, \ldots, t_k) = E \exp \left( i \sum_{j=1}^k t_j W_N(u_j) \right) \rightarrow \phi_{\sigma W}^{u_1, \ldots, u_k}(t_1, \ldots, t_k)
$$

(4.20)

as $N \rightarrow \infty$. First, we have

$$
\Phi_N^{u_1, \ldots, u_k}(t_1, \ldots, t_k) = E \exp \left( i \sum_{j=1}^k \left( \sum_{l=j}^k t_l \right) (W_N(u_j) - W_N(u_{j-1})) \right).
$$

(4.21)

Then similarly to (4.11) we show that

$$
|\Phi_N^{u_1, \ldots, u_k}(t_1, \ldots, t_k) - \Psi_N^{u_1, \ldots, u_k}(t_1, \ldots, t_k)| \rightarrow 0 \text{ as } N \rightarrow \infty.
$$

(4.22)

Next, similarly to Lemma 4.1 we obtain that

$$
|\Psi_N^{u_1, \ldots, u_k}(t_1, \ldots, t_k) - \prod_{j=1}^k \psi_N^{(j)}(t_1, \ldots, t_k)| \rightarrow 0 \text{ as } N \rightarrow \infty.
$$

(4.23)

where

$$
\psi_N^{(j)}(t_1, \ldots, t_k) = E \exp \left( i N^{-1/2} \sum_{j=1}^k \left( \sum_{l=j}^k t_l \right) \sum_{m \in A_j(N)} Y_m \right).
$$

(4.24)

Now in the same way as in (4.16) we see that

$$
\psi_N^{(j)}(t_1, \ldots, t_k) \rightarrow \exp \left( -\frac{1}{2} \sigma^2(u_j - u_{j-1})^2 \sum_{l=j}^k t_l^2 \right) \text{ as } N \rightarrow \infty.
$$

(4.25)

which together with (4.22), (4.23) and (4.17) yields (4.20).

Next, let $0 \leq u_1 \leq u \leq u_2 \leq 1$ then by Lemma 3.8

$$
E \left( (W_N(u) - W_N(u_1))^2 (W_N(u_2) - W_N(u))^2 \right)
$$

\leq (E(W_N(u) - W_N(u_1))^4)^{1/2} (E(W_N(u_2) - W_N(u))^4)^{1/2}

\leq \tilde{C} N^{-2} (|uN| - [u_1 N]) (|u_2 N| - |u N|) \leq \tilde{C} \left( \frac{|u_2 N| - |u_1 N|}{N} \right)^2.
$$
Now, either \( u_2 - u_1 \geq 1/N \) and then the right hand side of (4.25) is bounded by 
\[ 4 \tilde{C}(u_2 - u_1)^2 \] 
or \( u_2 - u_1 < 1/N \) and then the left hand side of (4.25) is zero. Hence, 
the left hand side of (4.25) is always bounded by \( 4 \tilde{C}(u_2 - u_1)^2 \) and by Ch. 15 of 
[2] the family \( \{P_{W_N}, N \geq 1 \} \) of distributions of \( W_N \)'s is tight. This together with the 
convergence of finite dimensional distributions of \( W_N \)'s established above completes 
the proof of Theorem 2.3 (cf. Ch. 15 in [2]). \( \square \)

5 Extension to the two linear terms case

In this section we enhance arguments of Sections 3 and 4 in order to derive Theorem 2.2. Set
\[
R(n) = \prod_{j=0}^{\ell} X_j(q_j(n)) - \prod_{j=0}^{\ell} a_j \\
= \sum_{j=0}^{\ell} a_1 \cdots a_{j-1}(X_j(q_j(n)) - a_j)X_{j+1}(q_{j+1}(n)) \cdots X_{\ell}(q_{\ell}(n)).
\]

Lemma 5.1 There exists \( C > 0 \) such that for all \( n \in \mathbb{N} \),
\[
E\left( \sum_{k=0}^{n} R(k)^2 \right) \leq Cn. \tag{5.1}
\]

Proof Relying on (3.14) where the summation starts with \( j_1, j_2 = 0 \) and estimating 
\( EQ_{j_1,j_2}(k_1, k_2) \) essentially by the same argument as in Lemma 3.2 we arrive at (5.1). \( \square \)

Next, we obtain the 4th moment Gaussian estimate.

Lemma 5.2 There exists \( \tilde{C} > 0 \) such that for all \( n \) and \( m \) satisfying \( N \geq n > m \geq [N^{1-\gamma}] \geq n_0 \),
\[
E\left( \sum_{k=m+1}^{n} R(k)^4 \right) \leq \tilde{C}(n-m)^2. \tag{5.2}
\]

Proof Similarly to (3.35) and (3.36),
\[
E\left( \sum_{k=m+1}^{n} R(k)^4 \right) \leq \sum_{k_1,k_2,k_3,k_4=m+1}^{n} \sum_{j_1,j_2,j_3,j_4=0}^{\ell} D^{j_1+j_2+j_3+j_4} E|Q_{j_1,j_2,j_3,j_4}(k_1,k_2,k_3,k_4)|. \tag{5.3}
\]

In estimating \( |Q_{j_1,j_2,j_3,j_4}(k_1,k_2,k_3,k_4)| \) here we can assume without loss of generality 
that \( 0 \leq j_1 \leq j_2 \leq j_3 \leq j_4 \leq \ell \). If \( j_1 < j_2 \) then as in (3.25) we still have here that for 
large \( N \) and \( k, l \geq [N^{1-\gamma}] \),
\[
q_{j_2}(l) \geq q_{j_1}(k) + N_\gamma - 1, \tag{5.4}
\]
and so similarly to (5.25) taking into account that \( k_1, k_2, k_3, k_4 > m \geq [N^{1-\gamma}] \) we obtain 
the estimate (3.37) in this case too. Other estimates of Lemma 3.8 hold true here, as 
well, since in addition to (3.3) and (5.4) we needed there only (3.40) which is satisfied 
in the circumstances of Theorem 2.2 as well. \( \square \)
Next, we derive a version of Lemma 3.5 which holds true under the conditions of Theorem 2.2.

**Lemma 5.3** There exists $\hat{C} > 0$ such that if $N \geq n > m \geq [N^{1-\epsilon}] \geq n_0$ and $n - m \leq \frac{1}{2}[N^{1-\epsilon}]$ for $\epsilon \in (0, \gamma)$ then

$$|E \left( \sum_{k=m+1}^n R(k) \right)^2 - (n-m)(\hat{\sigma}^2 - \Xi)\prod_{j=2}^\ell a_j^2| \leq \hat{C}$$ (5.5)

where $\Xi$ is the same as in (2.12).

**Proof** We start with (3.22) only now the summation in $j$ should begin there from 0. For $j \geq 2$ and $k_2 \geq k_1 \geq [N^{1-\epsilon}]$ we still have the estimate (3.24) while for $j \geq j_1, j_2 \geq 2$ and $k, l \geq [N^{1-\epsilon}]$ the estimate (3.26) holds true though in both cases $\ell$ should be replaced by $\ell + 1$. Since (3.27) remains true also for $j = 0$ we obtain (3.28) with the summation in $j$ starting with 0 and $\ell$ (in the right hand side) replaced by $\ell + 1$. Next, (3.29) and (3.30) remain valid too. Similarly to (3.29) we obtain that for $k_2 > k_1 \geq [N^{1-\epsilon}]$,

$$|EQ_{00}(k_1, k_2) = E(\mathcal{X}_0(k_2 - k_1) - a_0)(\mathcal{X}_0(0) - a_0)E(\mathcal{X}_1(\mathcal{r}(k_2 - k_1))|\mathcal{X}_1(0))\prod_{j=2}^\ell a_j^2| \leq 32D^{2(\ell+1)}((\ell+1)(\beta(\rho_2(N)) + \alpha(\rho_2(N))))$$

Observe that if $\frac{1}{2}[N^{1-\epsilon}] \geq k_1, k_2 \geq [N^{1-\epsilon}]$ then $|q_1(k_i) - k_j| \geq \frac{1}{2}[N^{1-\epsilon}]$, $i, j = 1, 2$ and relying on (3.15) and (3.23) we obtain that

$$|EQ_{01}(k_1, k_2)| \leq 16D^{2(\ell+1)}(\ell+1)(\beta(\min(\rho_1(N), [N^{1-\epsilon}] / 6))) + \alpha(\min(\rho_1(N), [N^{1-\epsilon}] / 6)))$$ (5.7)

where $\rho_1$ is the same as in (3.24). The same estimate holds true for $|EQ_{10}(k_1, k_2)|$ which together with (5.6), (5.7) and other estimates mentioned above yield (5.5) similarly to Lemma 3.5. □

Next, we enhance arguments of Section 4 to make them work in the situation of Theorem 2.2. Choose a small $\epsilon > 0$ and a large $L \geq 4$ so that $L\epsilon < \gamma/4$. Set $\kappa(N) = [N^{1-\frac{\epsilon}{2}}], \tau(N) = [N^{1-\epsilon}], \theta(N) = [N^{1-L\epsilon}]$ and $\mu(N) = \lfloor \frac{(L-1)\kappa(N) + p}{\tau(N) + \theta(N)} \rfloor$ recalling that $q_1(n) = rn + p$ and assuming that $N \geq \exp(2/\epsilon)$ which ensures that $\mu(N) \geq 1$.

Using the notation $q_1^{(s)}(n) = q_1(q_1^{(s-1)}(n)), q_1^{(1)} = q_1$ for iterates of $q_1$ define

$$L_{kl}(N) = q_1^{(l)}(\kappa(N) + (k-1)(\tau(N) + \theta(N)))$$

and

$$L_{kl}(N) = q_1^{(l)}(\kappa(N) + \tau(N) + (k-1)(\tau(N) + \theta(N))).$$

Introduce the sets of integers

$$\Gamma_k(N) = \{n : L_{kl}(N) \leq n < L_{kl}(N)\} \text{ and } \tilde{\Gamma}_k(N) = \{n : L_{kl}(N) \leq n < L_{kl+1}(N)\}$$
where \( k = 1, 2, \ldots, \mu(N) \) and \( l = 1, 2, \ldots, v_k(N) \) with \( v_k(N) = \max\{I : L_{k+1,l}(N) \leq N\} \). The block sequences \( \{\Gamma_k(N)\}_{k=1}^{\mu(N)} \) and \( \{\Gamma_k(N)\}_{k=1}^{\mu(N)} \) will play the same role as the blocks \( \Gamma_k(N) \) and \( \Gamma_k(N) \) in Section 4.

Set
\[
Y_{kl} = \sum_{n \in \Gamma_{kl}(N)} R(n), \quad Z_{kl} = \sum_{n \in \Gamma_{kl}(N)} R(n), \quad Y_k = \sum_{1 \leq l \leq v_k(N)} Y_{kl} \\
\Phi_N(t) = E \exp \left( \frac{it}{\sqrt{N}} \sum_{n=0}^{N} R(n) \right) \quad \text{and} \quad \Psi_N(t) = E \exp \left( \frac{it}{\sqrt{N}} \sum_{1 \leq k \leq \mu(N)} Y_k \right).
\]

Similarly to (4.1) we obtain from (5.1) and (5.2) that for some constant \( \tilde{C} > 0 \) and all \( t \) and \( N \),
\[
|\Phi_N(t) - \Psi_N(t)| \leq \tilde{C}|t| \left( N^{-\frac{1}{2}} + N^{1-\frac{L}{2}} \ln N + N^{-\frac{(L-1)}{2\sqrt{L}}} \ln N \right). \tag{5.8}
\]

Set
\[
\Psi_N^{(k)}(t) = E \exp \left( \frac{it}{\sqrt{N}} Y_k \right).
\]

**Lemma 5.4** For any \( t \) and each small \( \varepsilon > 0 \) there exists \( K_{\varepsilon}(t) > 0 \) such that for all \( N \geq \exp(2/\varepsilon) \),
\[
|\Psi_N(t) - \prod_{1 \leq k \leq \mu(N)} \Psi_N^{(k)}(t)| \leq K_{\varepsilon}(t) N^{-\varepsilon \sqrt{N}}. \tag{5.9}
\]

**Proof** The argument goes on, essentially, in the same way as in Lemma 4.1 Namely, we set
\[
\hat{Y}_{kl} = Y_{kl} + \tau(N) \sum_{j=0}^{f} a_j, \quad \hat{Y}_k = \sum_{1 \leq l \leq v_k(N)} \hat{Y}_{kl}
\]
and proceed as in (4.3)–(4.9). Next, we write
\[
\left( \sum_{1 \leq m \leq v_k(N)} \hat{Y}_{km} \right)^J_k = \sum_{\sigma^{(k)}} B_{\sigma^{(k)}}^{(k)} I_{\sigma^{(k)}}^{(k)} J_{\sigma^{(k)}}^{(k)} \tag{5.10}
\]
where
\[
I_{\sigma^{(k)}}^{(k)} = \prod_{1 \leq m \leq v_k(N)} \prod_{n \in \Gamma_{km}(N)} \chi_{\sigma^{(k)}}^{(m)}(n) \chi_{1}^{(m)}(q_1(n)), \quad J_{\sigma^{(k)}}^{(k)} = \prod_{1 \leq m \leq v_k(N)} \prod_{n \in \Gamma_{km}(N)} \prod_{j=2}^{f} \chi_{\sigma^{(k)}}^{(m)}(q_j(n)),
\]
\( \beta_{\sigma^{(k)}}^{(k)} \) are \( L^{n} \)-nominal coefficients and \( \sigma^{(k)} = (\sigma_n^{(k)}, n \in \Gamma_{km}(N), m = 1, \ldots, v_k(N) ) \) satisfies
\[
\sigma_n^{(k)} \geq 0 \quad \text{and} \quad \sum_{1 \leq m \leq v_k(N)} \sum_{n \in \Gamma_{km}(N)} \sigma_n^{(k)} = l_k \leq n(N) \tag{5.11}
\]
with \( n(N) \) defined by (4.5). Then we obtain the estimate (4.12) with \( \mu(N) \) in place of \( m(N) \) and
\[
H_{l_1, \ldots, l_{\mu(N)}}(t, N) = E \prod_{k=1}^{\mu(N)} I_{\sigma^{(k)}}^{(k)} J_{\sigma^{(k)}}^{(k)} - \prod_{k=1}^{\mu(N)} E I_{\sigma^{(k)}}^{(k)} J_{\sigma^{(k)}}^{(k)} \tag{5.12}
\]
Since all \( n \in \Gamma_{k, m}(N) \) satisfy \( n \geq N^{\frac{1}{k}} \), we obtain from (3.3) and (3.23) similarly to (4.13) that

\[
\left| E \prod_{k=1}^{\mu}(I_{(k)}^{(k)} \sigma_{(k)}) - E(\prod_{k=1}^{\mu}(I_{(k)}^{(k)} \sigma_{(k)})E(\prod_{k=1}^{\mu}(I_{(k)}^{(k)} \sigma_{(k)}) \right| \\
\leq 2D(\ell + 1)n(\mu)\beta(\rho_\gamma(N)) + 2\alpha(\rho_\gamma(N)),
\]

(5.13)

where

\[
\rho_\gamma(N) = \frac{1}{3}([N^{\frac{1}{k}}] - [N^{\frac{\gamma}{k}}]),
\]

and

\[
\left| E I_{(k)}^{(k)} \sigma_{(k)} - E(\prod_{k=1}^{\mu}(I_{(k)}^{(k)} \sigma_{(k)}) \right| \\
\leq 2D(\ell + 1)n(\mu)\beta(\rho_\gamma(N)) + 2\alpha(\rho_\gamma(N)),
\]

(5.14)

Observe that if \( n_1 \in \Gamma_{k, m_1}(N) \) and \( n_2 \in \Gamma_{k, m_2}(N) \) with either \( k_1 \neq k_2 \) or \( m_1 \neq m_2 \) then \( |n_1 - n_2| > \theta(N) = [N^{\frac{1}{k}}] \). Thus using (3.25), (3.41) and applying (3.3) no more than 2 \( \sum_{1 \leq \mu \leq N} v_k(N) \) times we obtain that

\[
\left| E \prod_{k=1}^{\mu}(I_{(k)}^{(k)} \sigma_{(k)}) - E(\prod_{k=1}^{\mu}(I_{(k)}^{(k)} \sigma_{(k)}) \right| \leq 4(\sum_{1 \leq \mu \leq N} v_k(N))D(\ell + 1)n(\mu)\beta(\rho_\gamma(N))
\]

\[
\times (\ell + 1)n(\mu)\beta(\rho_\gamma(N)) + 2\alpha(\rho_\gamma(N))
\]

(5.15)

where

\[
\rho_\gamma(N) = \frac{1}{3}\min(\theta(N), \rho_\gamma(N)).
\]

By our construction if \( n \in \Gamma_{k, m}(N) \) then \( q_1(n) \in \Gamma_{k, m+1}(N) \), and so we can represent \( I_{(k)}^{(k)} \sigma_{(k)} \) in the form

\[
I_{(k)}^{(k)} = \prod_{1 \leq n \leq v_k(N)} \prod_{n \in \Gamma_{m+k}(N)} X_0^{(k)}(n)X_1^{(k)}(n)
\]

which together with (5.3) and the above argument that \( |n_1 - n_2| > \theta(N) \) for \( n_j \in \Gamma_{k, m_j}, j = 1, 2 \) from different blocks yields that

\[
\left| E \prod_{k=1}^{\mu}(I_{(k)}^{(k)} \sigma_{(k)}) - E(\prod_{k=1}^{\mu}(I_{(k)}^{(k)} \sigma_{(k)}) \right| \leq 8(\sum_{1 \leq \mu \leq N} v_k(N))D(\ell + 1)n(\mu)\beta(\rho_\gamma(N)) + 2\alpha(\rho_\gamma(N))
\]

(5.16)

Similarly,

\[
\left| E I_{(k)}^{(k)} - \prod_{1 \leq n \leq v_k(N)} E(\prod_{n \in \Gamma_{m+k}(N)} X_0^{(k)}(n)X_1^{(k)}(n) \right| \leq 8v_k(N)D(\ell + 1)n(\mu)\beta(\rho_\gamma(N)) + 2\alpha(\rho_\gamma(N))
\]

(5.17)

Collecting (5.12)-(5.17) we obtain that

\[
H_{t_1, \ldots, t_n}(t, N) \leq 28(\sum_{1 \leq \mu \leq N} v_k(N))D(\ell + 1)n(\mu)\mu(N)
\]

\[
\times ((\ell + 1)n(\mu)\beta(\rho_\gamma(N)) + 2\alpha(\rho_\gamma(N))).
\]

(5.18)
Observe that for each $k$, 

$$\sum_{\sigma(k)} B_{\sigma(k)}^{(k)} \leq (\tau(N) \sum_{l=0}^{v_k(N)} \mu_l)^k \leq (N^{1-\varepsilon} \nu_k(N)^n)^k \leq N^{1-(1-\frac{1}{r})\varepsilon}N^{\frac{1}{r}+\varepsilon}$$  \hspace{1cm} (5.19)$$
since, clearly,

$$v_k(N) \leq \frac{\varepsilon \ln N}{L \ln r}. \hspace{1cm} (5.20)$$

In addition, we see by (5.20) that

$$\sum_{1 \leq l_1, \ldots, l_{\mu(N)} \leq n(N)} \sum_{k=1}^{\mu(N)} \prod_{k=1}^{N^{\frac{1}{r}+\varepsilon}} \left| \frac{\nu_k(N)^{l_k}}{l_k!} \right| \leq \exp(N^{\frac{1}{r}+\varepsilon(1-\frac{1}{r})}\varepsilon\mu(N)). \hspace{1cm} (5.21)$$

Employing (4.3)–(4.9) and (4.12) with $\mu(N)$ in place of $m(N)$ and $H_{l_1, \ldots, l_{\mu(N)}}$ given by (5.12) together with (5.18)–(5.21) we arrive at (5.9). \hfill \Box

Next, in order to complete the proof of Theorem 2.2 in the same way as at the end of Section 4, proceeding via (4.16)–(4.19) we observe that by (5.3) if $M_{km}(N)$ denotes the number of integers in $I_{km}(N)$ then

$$E\left(\sum_{m=1}^{v_k(N)} \sum_{n \in I_{km}(N)} R(n)\right)^4 \leq \left(\nu_k(N)^n\right)^3 E\left(\sum_{n \in I_{km}(N)} R(n)\right)^4 \hspace{1cm} (5.22)$$

$$\leq \hat{C}\left(\nu_k(N)^n\right)^3 \sum_{m=1}^{v_k(N)} (M_{km}(N))^2 \leq \hat{C}\left(\nu_k(N)^n\right)^3 \left(\sum_{m=1}^{v_k(N)} M_{km}(N)\right)^2$$

which in view of (5.20) is still sufficient for the estimate of the form (4.16).

Finally, we show that

$$A_k(N) = \left| E\left(\sum_{m=1}^{v_k(N)} \sum_{n \in I_{km}(N)} R(n)\right)^2 - \hat{\alpha}^2 M_k(N) \right| \leq C \nu_k(N) \hspace{1cm} (5.23)$$

where $M_k(N) = \sum_{1 \leq m \leq v_k(N)} M_{km}(N)$ and $C > 0$ does not depend on $N$ and $k$. Indeed, by (5.5),

$$\left| \sum_{m=1}^{v_k(N)} E\left(\sum_{n \in I_{km}(N)} R(n)\right)^2 - M_k(N)(\hat{\alpha}^2 - \varepsilon \prod_{j=2}^{l} a_j^2) \right| \leq \hat{C} \nu_k(N). \hspace{1cm} (5.24)$$

Observe that if $m_2 - m_1 \geq 2$ then for any $n_1 \in I_{km_1}$ and $n_2 \in I_{km_2}$ we have that $n_2 - q_1(n_1) \leq \lfloor N^{1-L} \rfloor$. Thus using (3.15), (3.23) and (3.14) (the latter with the summation starting with $j_1, j_2 = 0$) we obtain for such $n_1$ and $n_2$ that

$$|R(n_1)R(n_2)| \leq 16D^{4(\ell+1)}(\ell+1)^2((\ell+1)\beta(\rho_0(N)) + \alpha(\rho_0(N))) \hspace{1cm} (5.25)$$

where

$$\rho_0(N) = \frac{1}{3} \min([N^{1-L}], N^{1-\frac{e}{2}} - [N^{1-\frac{e}{2}}(1-\gamma)])].$$

It remains to estimate

$$B_{km} = E\left(\sum_{n \in I_{km}(N)} R(n)\right)\left(\sum_{\hat{n} \in I_{km+1}(N)} R(\hat{n})\right) = \sum_{n \in I_{km}(N), \hat{n} \in I_{km+1}(N)} ER(n)R(\hat{n}). \hspace{1cm} (5.26)$$
Using (3.15), (3.22) and (3.23) we obtain that
\[
|B_{km} - a_0 \sum_{n \in \Gamma_{km}(N), \tilde{n} \in \Gamma_{km+1}(N)} E Q_{01}(\tilde{n}, n)| \leq 48 \tilde{D}^{(\ell+1)}(\ell + 1)^2 \bigl( (\ell + 1) \beta (\rho_0(N)) + \alpha (\rho_0(N)) \bigr).
\]
(5.27)

In the same way as in (5.6) it follows that for \( n \in \Gamma_{km}(N) \) and \( \tilde{n} \in \Gamma_{km+1}(N) \),
\[
|E Q_{01}(\tilde{n}, n) - a_1 E(X_0(\tilde{n}) - a_0)(X_1(q_1(n) - n) - a_1) \prod_{j=2}^\ell a_j^2 | \leq 32 \tilde{D}^{(\ell+1)}(\ell + 1)^2 \bigl( (\ell + 1) \beta (\rho_2(N)) + \alpha (\rho_2(N)) \bigr).
\]
(5.28)

Furthermore,
\[
\sum_{n \in \Gamma_{km}(N), \tilde{n} \in \Gamma_{km+1}(N)} E(X_0(\tilde{n}) - a_0)(X_1(q_1(n) - n) - a_1)
\]
\[
= \sum_{n \in \Gamma_{km}(N)} \bigl( \sum_{\tilde{n} \geq q_1(n)} E(X_0(\tilde{n} - q_1(n)) - a_0)(X_1(0) - a_1) + \sum_{\tilde{n} < q_1(n)} E(X_0(0) - a_0)(X_1(q_1(n) - \tilde{n}) - a_1) - V_1 - V_2 \bigr)
\]
\[
= \sum_{n \in \Gamma_{km}(N)} \sum_{\tilde{n} \geq \ell q_1(n)} E(c(\tilde{n} - q_1(n))) \left( \sum_{n \in \Gamma_{km+1}(N)} \sum_{\tilde{n} \geq \ell q_1(n)} \exp(c(q_1(n) - \tilde{n})) \right) \leq \tilde{C}
\]
(5.29)

where by (2.1) and (3.3),
\[
|V_1| + |V_2| \leq \sum_{n \in \Gamma_{km}(N)} \sum_{\tilde{n} \geq \ell q_1(n)} \exp(c(q_1(n) - \tilde{n})) \leq \tilde{C}
\]
(5.30)

for some \( c, \tilde{C} > 0 \) independent of \( N, k \) and \( m \). Collecting (5.24)–(5.30) we obtain (5.23). In view of (5.20) this enables us to complete the proof of Theorem 2.2 in the same way as at the end of Section 3. \( \square \)

6 Concluding remarks

The condition (2.4) was crucial for our proof of Theorem 2.1 since its, essentially, equivalent form (3.25) arranges \( q_j(n) \), \( j = 1, \ldots, \ell \) for big \( n \) into \( \ell \) sets separated by large gaps which was necessary in our splitting arguments. This property is lost when more than one of \( q_j \)'s are linear but, still, the block sequences construction of Section 5 enabled us to carry out the proof of Theorem 2.2 for two linear terms. Lemmas 5.1–5.3 still can be carried out when more than two \( q_j \)'s are linear but it is not clear how to make an appropriate block sequences construction in this case, for instance, when \( q_1(n) = n, q_2(n) = 2n, q_3(n) = 3n \) and \( \ell = 3 \). Probably, in a special algebraic situation, for instance, when \( X_j(n) = X(n) = f(T^nx) \) with \( T \) being a hyperbolic automorphism or an expanding (algebraic) endomorphism of a torus, the Fourier analysis technique in the spirit of [17] may still lead to a SPLIT in the form of Theorems 2.1–2.2. Nevertheless, for more general stationary processes \( X(n) \) it is not clear whether a Theorems 2.1–2.2 type result holds true for expressions of the form
\[
N^{-1/2} \sum_{0 \leq n \leq N} \bigl( X(n)X(2n)X(3n) - (EX(0))^3 \bigr).
\]
(6.1)
On the other hand, if \( X(0), X(1), X(2), \ldots \) are i.i.d. random variables such results can be easily proved. Namely, let \( q_1 = 1 < q_2 < \ldots < q_t \) be some prime numbers and set \( EX^2(0) = b^2 \) assuming for simplicity that \( EX(0) = 0 \). Then as \( N \to \infty \),

\[
W_N = N^{-1/2} \sum_{0 \leq n \leq N} X(q_1 n)X(q_2 n) \cdots X(q_t n) \tag{6.2}
\]

converges in distribution to the centered normal random variable with the variance \( \sigma^2 = b^2 \). Indeed, let \( 1 \leq k_1 < k_2 < \ldots < k_{m_N} \leq N \) be all integers which are not divisible by any of \( q_j \)'s, \( j \geq 2 \). Then we can define disjoint sets \( A_k, l = 1, \ldots, m_N \) so that \( A_k \subset \{1, \ldots, N\} \) and any \( n \in A_k \) is obtained from \( k_l \) by multiplication by some of \( q_j \)'s. It is clear that the number \( r_l(N) \) of elements of each \( A_k \) does not exceed \( \log_2 N \).

Set

\[
S_N(l) = \sum_{n \in A_k} X(q_1 n)X(q_2 n) \cdots X(q_t n). \tag{6.3}
\]

Then \( S_N(l), l = 1, 2, \ldots, m_N \) are independent random variables with zero mean and the variance \( r_l(N)b^2 \). Applying the standard central limit theorem for triangular arrays (see, for instance, [14]) to

\[
W_N = N^{-1/2} \sum_{0 \leq l \leq m_N} S_N(l) \tag{6.4}
\]

and taking into account that \( \sum_{0 \leq l \leq m_N} r_l(N) = N \) we obtain the required result. If \( EX(0) \neq 0 \) then this method still works using the representation (3.10) for computation of variances.

Observe that, in principle, we could ask whether under appropriate conditions our results could be extended to continuous time processes trying to obtain central limit theorems for integrals

\[
\int_0^T X_1(q_1(t))X_2(q_2(t)) \cdots X_l(q_l(t))dt
\]

in place of sums. Nevertheless, the answer to this question is not clear yet and the approach of this paper does not seem to work in this case.

Another result which can be derived for i.i.d. bounded random variables \( X(0), X(1), X(2), \ldots \) is a corresponding sum-product large deviations (SPLAD) theorem. Namely, we are interested in the asymptotic behavior of

\[
Q_N(U) = \frac{1}{N} \log P\{ \frac{1}{N} S_N \in U \} \tag{6.5}
\]

as \( N \to \infty \) where \( S_N = \sum_{n=0}^N X(q_1(n)) \cdots X(q_t(n)) \) and \( U \subset \mathbb{R} \). Here, \( q_1(n), \ldots, q_t(n) \) are nonnegative strictly increasing functions taking on integer values on the integers and such that for some \( \gamma \in (0, 1) \) and \( n_0 \in \mathbb{Z} \) we have

\[
q_{j+1}(\lceil n^{\gamma} \rceil) > q_j(n) \quad \text{for all } n \geq n_0. \tag{6.6}
\]

Let \( M_N(t) = E \exp(iS_N) \) be the moment generating function of \( S_N \). It is well known (see, for instance, Theorem 2.3.6 in [6]) that if the limit

\[
\eta(t) = \lim_{N \to \infty} \frac{1}{N} \log M_N(t) \tag{6.7}
\]
exists and it is differentiable in $t$ then
\[ \limsup_{N \to \infty} Q_N(F) \leq - \inf_{x \in F} \Lambda(x) \] (6.8)
for any closed set $F \subset \mathbb{R}$ and
\[ \liminf_{N \to \infty} Q_N(G) \geq - \inf_{x \in G} \Lambda(x) \] (6.9)
for any open set $G \subset \mathbb{R}$ where
\[ \Lambda(x) = \sup_t (tx - \eta(t)) \]
is the Legendre transform of $\eta$.

Set $\tilde{S}_N = \sum_{N \geq n \geq |N|} X(q_1(n)) \cdots X(q_l(n))$ and $\bar{M}_N(t) = E \exp(t\tilde{S}_N)$. Then
\[ \bar{M}_N(t) \exp(-D\gamma N) \leq M_N(t) \leq \bar{M}_N(t) \exp(D\gamma N), \] (6.10)
where we assume that $|X(0)| \leq D$ a.s., and so
\[ \lim_{N \to \infty} \frac{1}{N} M_N(t) = \lim_{N \to \infty} \frac{1}{N} \log \bar{M}_N(t) \] (6.11)
whenever one of these limits exists. Set $m_n(t) = E \exp(t \prod_{j=1}^l X(q_j(n)))$. By (6.6) and the strict monotonicity of the functions $q_j(n)$ it follows that the terms $\exp(t \prod_{j=1}^l X(q_j(n)))$ are independent for different $n$, and so
\[ \bar{M}_N(t) = \prod_{n=|N|}^N m_n(t). \] (6.12)

Next, by (6.6) the factors in the product appearing in the definition of $m_n(t)$ with $n \geq |n|_0$ are independent, and so for such $n$,
\[ m_n(t) = E \sum_{k=0}^\infty t^k \prod_{j=1}^l X^k(q_j(n)) \] (6.13)
\[ = \sum_{k=0}^\infty \frac{t^k}{k!} \prod_{j=1}^l E X^k(0) = \sum_{k=0}^\infty \frac{t^k}{k!} \prod_{j=1}^l E X^k(0) = m_{[n]_0}(t). \]

Thus, we obtain
\[ \eta(t) = \lim_{N \to \infty} \frac{1}{N} \log \bar{M}_N(t) = \log m_{[n]_0}(t), \] (6.14)
which is, clearly, differentiable in $t$ since $X(k)$’s are bounded, and so (6.8) and (6.9) follow. SPLAD in other situations will be treated in another paper.

For i.i.d. $X(j)$, $j = 0, 1, 2, \ldots$ it is easy to prove the existence of a differentiable limit $\eta(t)$ in (6.7) also for moment generating functions $M_N(t)$ of the sums
\[ S_N = \sum_{0 \leq n \leq N} X(q_1(n)) X(q_2(n)) \cdots X(q_l(n)), \]
where $q_i, i = 1, \ldots, \ell$ are primes as in (6.2), by using the sets $A_{k_i}$ and partial sums $S_N(l)$ appearing in (6.3).

In conclusion, remark that using the thermodynamic formalism and decay of correlations results for random transformations from [12] and [13] we can obtain the corresponding (quenched or fiberwise) SPLIT for random subshifts of finite type, random expanding transformations and for Markov chains with random transitions.
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