On a novel evaluation of the hadronic contribution to the muon’s $g - 2$ from QCD

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Abstract

We evaluate the hadronic contribution to the $g - 2$ of the muon by deriving the low-energy limit of quantum chromodynamics (QCD) and computing in this way the hadronic vacuum polarization. The low-energy limit is a non-local Nambu–Jona-Lasinio (NJL) model that has all the parameters fixed from QCD, and the only experimental input used is the confinement scale that is known from measurements of hadronic physics. Our estimations provide a novel analytical alternative to the current lattice computations and we find that our result is close to the similar computation performed from experimental data. We also comment on how this analytical approach technique, in general, may provide prospective estimates for hadronic computations from dark sectors and its implication in BSM model-building in future.

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I. INTRODUCTION

Since the original computation for the electron from first principles [1], originating from Dirac equation, the lepton anomalous magnetic moments continue to be very important observables for precision tests of the Standard Model (SM) [2]. Recent data seems to indicate a tension with the theoretical prediction for the anomalous magnetic moment of the muon with the recent experimental value for the anomalous magnetic moment of the muon is [3, 4]

\[ g_\mu /2 = 1 + a_\mu = 1.0011659208(6). \]  \hspace{1cm} (1)

The Particle Data Group (PDG) gives an updated value for the muon anomaly in the form [5]

\[ a_\mu^{\text{exp}} = 116592091(54)(33) \times 10^{-11}. \]  \hspace{1cm} (2)

This precision clearly is a challenge for the theoretical side to increase the precision of the prediction [4].

The theoretical results for the muon anomalous magnetic moment in the SM are traditionally represented as a sum of three parts,

\[ a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{had}} \]  \hspace{1cm} (3)

with \( a_\mu^{\text{QED}} \), \( a_\mu^{\text{EW}} \) being the leptonic and electroweak parts, respectively, and \( a_\mu^{\text{had}} \) is the contribution involving the electromagnetic currents of quarks.

The leptonic part is computed in perturbation theory and reads [5]

\[ a_\mu^{\text{QED}} = 116584718.95(0.08) \times 10^{-11}. \]  \hspace{1cm} (4)

The electroweak part is known to two loops and reads [5]

\[ a_\mu^{\text{EW}} = 153.6(1.0) \times 10^{-11}. \]  \hspace{1cm} (5)

The hadronic part \( a_\mu^{\text{had}} \) in the SM is related to quark contributions to the electromagnetic currents.

The current total SM prediction reads [5]

\[ a_\mu^{\text{SM}} = 116591823(1)(34)(26) \times 10^{-11}. \]  \hspace{1cm} (6)

The difference

\[ \Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 268(63)(43) \times 10^{-11} \]  \hspace{1cm} (7)
could be due to new unknown physics beyond the SM, but it is not statistically significant
off yet [6], the main idea behind this being that contributions from unknown virtual particles
not part of the SM might enter the calculations.

In general, theoretical estimates are very precise for what one should expect from quantum
electrodynamics (QED), but fall short in the case of the hadronic contributions, due to the
known difficulties to treat quantum chromodynamics (QCD) at low energies. The general
accepted technique is to use experimental results from \( e^+e^- \) scattering into hadrons measured
in collider experiments [6]. Two key ingredients of this contribution are the hadronic vacuum
polarization (HVP) and the high-order hadronic light-by-light scattering (HLbL). Of these
two contributions, the former is the most critical one, due to the current inability to compute
this contribution starting right from the Lagrangian of QCD. Some recent evaluation of the
HVP from experimental data is given in Ref. [7–9] for the \( \pi\pi \) part which is the most relevant
contribution. The value of the HVP contribution determines in a critical way whether there
is room for Beyond-Standard Model (BSM) physics or not in the context of observed values
for muon \( g - 2 \).

The only independent technique for calculations in QCD is by using large computer
facilities to solve the equations, a technique known as lattice QCD. Still, the Muon \( g - 2 \)
Theory Initiative [6] decided to not use this technique, as there are large differences between
the results of different collaborations, disclosing the technique not to be yet trustworthy. For
instance, the Budapest–Marseille–Wuppertal (BMW) Collaboration has put forward their
latest results [10], showing that the HVP correction they obtain moves the ballpark of the
muon \( g - 2 \) value back into the SM field. In turn, this would imply that the technique using
experimental values from the colliders probably underestimates this contribution.

Working on QCD, one generally makes the use of effective models. However, it is often
unknown if such models could be straightforwardly obtained from the original Lagrangian.
Still, successful results have been obtained from some of these models. In the very early
days of the study of the muon \( g - 2 \) problem, attempts were made to derive the HVP
contribution from such effective models like for instance the Nambu–Jona-Lasinio (NJL)
model [11], detailed by de Rafael in Ref. [12] and more recently [13]. However, due to
the large set of undetermined parameters entering in such effective theories, this kind of
approach in this early, primitive stage was abandoned in favor of the use of experimental
data and lattice QCD calculations.
Inspired by such an approach, in this article we will show how an effective field theory can be derived from QCD, starting directly from the Lagrangian level. The model is a non-local Nambu–Jona-Lasinio model, having all the parameters properly fixed. A first attempt in this direction was given in Ref. [14] in order to determine the proper low-energy limit of the theory\(^1\). In this work, we fix an error in this publication and show how the effective NJL model comes out naturally from QCD. Based on these first principles, we will evaluate the HVP contribution to the muon \((g - 2)\).

II. BASIC EQUATIONS FOR NJL MODEL

Our starting point is the well-known QCD Lagrangian

\[
L_{QCD} = \sum_i \bar{q}_i (i \gamma^\mu D_\mu + m) q_i - \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{2} \xi (\partial_\mu A^a_\mu)(\partial_\nu A^a_\nu)
\]

(8)

with covariant derivative \(D_\mu = \partial_\mu + igT^a A^a_\mu\) and the field strength tensor components \(F^a_{\mu\nu}\) defined by \(igT^a F^a_{\mu\nu} = [D_\mu, D_\nu]\). The sum over \(i\) is understood to run over the quark flavors and colors. Throughout this paper we work with the Minkowskian metric \(g^{\mu\nu} = \text{diag}(1, -1, -1, -1)\). Calculating the Euler–Lagrange equations, one obtains

\[
0 = \frac{\partial L_{QCD}}{\partial A^a_\mu} - \partial_{\mu} \frac{\partial L_{QCD}}{\partial (\partial_\mu A^a_\nu)} = \\
= \partial_\mu (\partial^\nu A^a_\nu - \partial^\nu A^a_\mu) + \frac{1}{\xi} \partial^\nu (\partial_\mu A^a_\nu) + gf_{abc}(A^b_\mu A^c_\nu) \\
+ gf_{abc}(\partial^\mu A^b_\nu - \partial^\nu A^b_\mu)A^c_\mu + g^2 f_{abc} f_{cde} A^b_\mu A^c_\nu A^e_\mu - g \sum_i \bar{q}_i \gamma^\nu T_a q_i,
\]

(9)

These classical equations of motion are the starting point for a tower of Dyson–Schwinger equations. In order to study these equations, we use the method proposed by Bender, Milton and Savage [15], details of which can found in Refs. [14, 16–19]. For the purpose of this publication we sketch the main steps here, skipping contributions from BRST ghosts for simplicity.

Enlarging the Lagrangian of the classical action by adding corresponding source terms \(A^a_\mu J^{\mu}_{a}\), \(\bar{q}_i \eta_i\) and \(\eta_i q_i\), one obtains the exponential of the generating functional. Functional

\(^1\) Unfortunately, this publication contained a mistake that made the conclusions unreliable.
derivatives of this generating functional lead to the Dyson–Schwinger analogue of the Euler–Lagrange equations, expressed in terms of Green functions for the fields. The set of equations in Landau gauge \( \xi = 0 \) we start with is given by

\[
\partial^2 A^a_\mu(x) + g f_{abc} (\partial^\mu A^{bc}_{\mu
u}(x, x) + \partial^\nu A^{b\nu}_\mu(x) A^c_\mu(x) - \partial^\nu A^{2bc}_{\mu\nu}(x, x) - \partial^\mu A^{1b}_\nu(x) A^1_{\mu\nu}(x)) \\
+ g f_{abc} \partial^\mu A^{2bc}_{\mu\nu}(x, x) + g f_{abc} \partial^\nu (A^{b\mu}_\nu(x) A^1_{\mu\nu}(x)) + g^2 f_{abc} f_{cde} \left( g^{\mu\rho} A^{3de}_{\mu\nu\rho}(x, x, x) \\
+ A^{2\mu\rho}(x, x) A^1_{\mu\rho}(x) + A^{2b\nu}(x, x) A^{1b\nu}(x) + A^{2de}(x, x) A^{1d\mu}(x) A^{1\mu\nu}(x) \right) = \\
= g \sum_i \gamma_\nu T^a_{\mu i} q_i^1(x) + g \sum_i \bar{q}_i^1(x) \gamma_\nu T^a_{\nu i} q_i^1(x), \\
(i\partial - m_q) q_i^1(x) + g^\gamma A^1_{\mu i}(x) T_a q_i^1(x) = 0,
\]

(10)

where the one-, two- and three-point Green functions are given by \( A^1_{\mu a}(x) = \langle A^a_\mu(x) \rangle \), \( A^{2ab}_\mu(x, y) = \langle A^a_\mu(x) A^b_\nu(y) \rangle \), \( A^{3abc}_{\mu\rho\nu}(x, y, z) = \langle A^a_\mu(x) A^b_\nu(y) A^c_\rho(z) \rangle \), \( q_i^1(x) = \langle q_i(x) \rangle \) and \( q_{ij}^2(x, y) = \langle q_i(x) q_j(y) \rangle \). The expected solutions can be written in the form

\[
A^a_\mu(x) = \eta^a_\mu \phi(x), \quad A^{2ab}_\mu(x, y) = \left( g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \delta^{ab} \Delta(x - y),
\]

(11)

where \( \eta^a_\mu \) are the coefficients of the polarization vector with \( \eta^a_\mu \eta^\mu_b = \delta_{ab} \), \( \phi(x) \) is a scalar field and \( \Delta(x - y) \) is the propagator of the scalar field. The three-point function can be set to zero. For the one-point functions we obtain

\[
\eta^a_\mu \partial^2 \phi(x) + 2 N_c g^2 \Delta(0) \eta^a_\mu \phi(x) + N_c g^2 \eta^a_\mu \phi^3(x) = g \sum_i \gamma_\nu T_a q_i^2(x, x) + g \sum_i \bar{q}_i^1(x) \gamma_\nu T^a_{\nu i} q_i^1(x), \\
(i\gamma^\mu \partial_\mu - m_q) q_i^1(x) + g^\gamma \eta^a_\mu T_a \phi(x) q_i^1(x) = 0.
\]

(12)

Using \( \eta^a_\mu \eta^\mu_a = N_c^2 - 1 \) and \( \sum_i q_{ii}^2(x, x) = N_c N_f S(0) \), the first differential equation (12) takes the form

\[
\partial^2 \phi(x) + 2 N_c g^2 \Delta(0) \phi(x) + N_c g^2 \phi^3(x) = \frac{g}{N_c^2 - 1} \left[ N_c N_f \gamma^\nu \eta^a_\mu T_a S(0) + \sum_i \bar{q}_i^1(x) \gamma^\nu \eta^a_\mu T_a q_i^1(x) \right].
\]

(13)

In the ’t Hooft limit \( N_c \to \infty \), \( \lambda := N_c g^2 \gg 1 \) finite but large, this set of equations yields a Nambu–Jona-Lasinio model in a straightforward way. Indeed, for this case, we can perform a perturbation series expansion \( \phi(x) = \phi_0(x) + \phi_1(x) + O(g^2) \) in \( g \), obtaining at leading order

\[
\partial^2 \phi_0(x) + 2 \lambda \Delta(0) \phi_0(x) + \lambda \phi_0^3(x) = 0,
\]

(14)
while the next-to-leading order yields
\[
\partial^2 \phi_1(x) + 2\lambda \Delta(0) \phi_1(x) + 3\lambda \phi_0^2(x) \phi_1(x) = \\
= \frac{g}{N_c^2 - 1} \left[ N_c N_f \gamma^\nu \eta^\alpha \gamma_{\alpha} T_a S(0) + \sum_i \bar{q}_i^1(x) \gamma^\nu \eta^\alpha T_a q_i^1(x) \right].
\] (15)

A. Zeroth order solution and Green function

Note that \( \Delta(0) \) is a constant. Therefore, \( m^2 = 2\lambda \Delta(0) \) can be considered as the mass square of the scalar field. The leading order differential equation \( \partial^2 \phi_0(x) + m^2 \phi_0(x) + \lambda \phi_0^3(x) = 0 \) is nonlinear, but a solution in terms of Jacobi’s elliptic functions exists,
\[
\phi_0(x) = \sqrt{\frac{2(p^2 - m^2)}{\lambda}} \text{sn} (p \cdot x + \theta | \kappa)
\] (16)

with
\[
p^2 = \frac{1}{2} \left( \sqrt{m^4 + 2\lambda \mu^4 + m^2} \right) \quad \text{and} \quad \kappa = \frac{m^2 - p^2}{p^2},
\] (17)

where \( \mu \) and \( \theta \) are integration constants. \( \text{sn}(z|\kappa) \) is Jacobi’s elliptic function of the first kind.

Given this solution, the second differential equation can be solved by noting that
\[
(\partial^2 + m^2 + 3\lambda \phi_0^2(x)) \Delta(x - y) = i\delta^4(x - y)
\] (18)
is solved by a Green function written in momentum space as
\[
\tilde{\Delta}(p) = \tilde{Z}(p^2, m^2) \frac{2\pi^3}{K^3(\kappa)} \sum_{n=0}^{\infty} (-1)^n \frac{e^{-(n+1/2)\varphi(\kappa)}}{1 - e^{-(2n+1)\varphi(\kappa)}} \frac{(2n + 1)^2}{p^2 - m_n^2 + i\epsilon}
\] (19)

with
\[
\varphi(\kappa) = \frac{K^*(\kappa)}{K(\kappa)}, \quad K^*(z) = K(1 - z)
\] (20)

and
\[
\tilde{Z}(p^2, m^2) = \frac{2p^8 \sqrt{p^2(p^6 + 2p^4m^2 - 3p^2m^4 + m^6)}}{\sqrt{2p^2 - m^2(2p^{12}(2p^2 - m^2) - 5p^2m^8(2p^2 - m^2)(p^2 - m^2) - m^{14})}}.
\] (21)

The mass spectrum is given by
\[
m_n = \frac{(2n + 1)\pi}{2K(\kappa)} \sqrt{2p^2} =: (2n + 1)m_G(\kappa).
\] (22)

At this point the circle for the mass of the scalar field is closed. Inserting back the Fourier transform of the propagator \( \tilde{\Delta}(p) \) into \( m^2 = 2\lambda \Delta(0) \) results in
\[
m^2 = 2\lambda \int \frac{d^4p}{(2\pi)^4} \tilde{Z}(p^2, m^2) \frac{2\pi^3}{K^3(\kappa)} \sum_{n=0}^{\infty} (-1)^n \frac{e^{-(n+1/2)\varphi(\kappa)}}{1 - e^{-(2n+1)\varphi(\kappa)}} \frac{(2n + 1)^2}{p^2 - (2n + 1)^2m_G^2(\kappa) + i\epsilon}.
\] (23)
This self-consistency equation provides the proper spectrum of a Yang-Mills theory with no fermions \[19\], in very close agreement with lattice data.

**B. First order solution**

The convolution of the propagator \(\Delta\) with the right hand side of Eq. (15) leads to

\[
\phi_1(x) = \frac{g}{N_c^2 - 1} \int d^4 y \Delta(x - y) \left[ N_c N_f \gamma^\nu \eta^a \gamma^\eta T_a S(0) + \sum_i \bar{q}_i^1(y) \gamma^\nu \eta^a \gamma^\eta T_a q_i^1(y) \right].
\]

The first term renormalizes the fermion mass and can taken to be zero by choosing the renormalization condition \(S(0) = 0\). The second term yields a Nambu–Jona-Lasinio (NJL) interaction in the equation of motion of the quark.

Inserting \(\phi(x)\) into the Dirac equation (12), in the 't Hooft limit the term \(\phi_0\) is negligible small compared to the NJL interaction term \(\phi_1\). This can be realized by noting that \(\phi_0 \sim \lambda^{1/4}\) while \(\phi_1 \sim \lambda\). In the strong coupling limit \(\lambda \gg 1\), for the quark one-point function we have to retain only the NJL term. Therefore, we obtain

\[
(i\gamma^\mu \partial_\mu - m_q)q_i^1(x) + \frac{g^2}{N_c^2 - 1} \sum_\eta \int d^4 y \Delta(x - y) \gamma^\eta \eta^a \gamma^\eta T_a q_i^1(x) \sum_j \bar{q}_j^1(y) \gamma^\eta T_a q_i^1(y) = 0.
\]

The equation turns out to be the Euler–Lagrange equation with respect to the one-point function of the quark, obtained for the Nambu–Jona-Lasinio model with a non-local Lagrangian \([20, 21]\)]

\[
\mathcal{L}_{\text{NJL}}'' = \sum_\iota \bar{q}_i^1(x)(i\gamma^\mu \partial_\mu - m_q)q_i^1(x)
\]

\[
+ \frac{g^2}{N_c^2 - 1} \sum_\eta \sum_\iota \bar{q}_i^1(x) \gamma^\eta \eta^a \gamma^\eta T_a q_i^1(x) \int d^4 y \Delta(x - y) \sum_j \bar{q}_j^1(y) \gamma^\eta T_a q_i^1(y).
\]

Note that \(\sum_\eta \eta^a \eta^b = \delta_{ab} g_{\mu\nu}\), where \(\eta\) symbolizes the polarizations. In addition, one traces out the color degrees of freedom with \(\text{tr}(T_a T_a) = N_c C_F\), \(C_F = (N_c^2 - 1)/(2 N_c)\), and \(\sum_\eta \bar{q}_i^1(x) \gamma^\eta q_i^1(x) = N_c \sum_\iota \bar{\psi}_i(x) \gamma^\iota \psi_i(x)\), where \(\psi_i(x)\) are spinors in Dirac and flavor space, only. This leads us to the NJL lagrangian

\[
\mathcal{L}_{\text{NJL}}' = \sum_\iota \bar{\psi}_i(x)(i\gamma^\mu \partial_\mu - m_q)\psi_i(x)
\]

\[
+ \frac{N_c g^2}{2} \sum_\iota \bar{\psi}_i(x) \gamma^\iota \psi_i(x) \int d^4 y \Delta(x - y) \sum_j \bar{\psi}_j(y) \gamma_\mu \psi_j(y).
\]
The Fierz rearrangement of the quark fields yields

\[\mathcal{L}'_{\text{NJL}} = \sum_i \bar{\psi}_i(x)(i\gamma^\mu\partial_\mu - m_q)\psi_i(x) + \frac{N_cg^2}{2} \int d^4y \Delta(x-y) \sum_{i,j} \bar{\psi}_i(x)\psi_j(y)\bar{\psi}_j(y)\psi_i(x) + \frac{Ncg^2}{2} \int d^4y \Delta(x-y) \sum_{i,j} \bar{\psi}_i(x)i\gamma_5\psi_j(y)\bar{\psi}_j(y)i\gamma_5\psi_i(x) \]

\[+ \frac{Ncg^2}{4} \int d^4y \Delta(x-y) \sum_{i,j} \bar{\psi}_i(x)\gamma^\mu\psi_j(y)\bar{\psi}_j(y)\gamma_5\psi_i(x) - \frac{Ncg^2}{4} \int d^4y \Delta(x-y) \sum_{i,j} \bar{\psi}_i(x)\gamma^\mu\gamma_5\psi_j(y)\bar{\psi}_j(y)\gamma_5\gamma_5\psi_i(x). \tag{28}\]

C. Bosonization

Let \(\Gamma_\alpha\) be a set of Dirac and flavor matrices containing not only the Dirac structures \(1, i\gamma_5, \gamma_\mu\) and \(\gamma_\mu\gamma_5\) from the Fierz rearrangement but also the flavor matrices \(\mathbb{1}\) and \(\frac{1}{2}\lambda_\alpha\) relating quarks of equal and different flavor \(i\) and \(j\) in adjoint representation. \(\Gamma_\alpha\) obeys the conjugation rule \(\gamma^0\Gamma_\alpha^\dagger\gamma^0 = \Gamma_\alpha\), where \(\alpha\) denotes the components of the adjoint flavor representation. Accordingly, the spinor \(\psi(x)\) spans over all these spaces. The most prominent degrees of freedom are the scalar–isoscalar and pseudoscalar–isovector degrees which can formally be combined as four vector. As the coefficients of these two contributions are the same, one can reinterpret the sum over these \(1 + 3 = 4\) degrees of freedom as a sum over four-vector components. The next step is to apply the bosonization procedure exemplified in Ref. [22] by adding scalar–isoscalar and pseudoscalar–isovector mesonic fields as auxiliary fields \(M_\alpha(w) = (\sigma(w); \vec{\pi}(w))\) at an intermediate space-time location \(w = (x+y)/2\), coupled to the nonlocal fermionic currents. The result of the Fierz rearrangement can be expressed as NJL action

\[S_{\text{NJL}} = -\frac{Ncg^2}{2G^2} \int d^4z \Delta(z) \int d^4w M_\alpha^*(w)M_\alpha(w) \]

\[+ \int d^4x \left[ \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m_q)\psi(x) + \frac{Ncg^2}{2} \int d^4y \Delta(x-y)\bar{\psi}(x)\Gamma_\alpha\psi(y)\bar{\psi}(y)\Gamma_\alpha\psi(x) \right] \]

\((G = 2 \int d^4z \Delta(z))\). By performing a nonlocal functional shift

\[M_\alpha\left(\frac{x+y}{2}\right) \rightarrow M_\alpha\left(\frac{x+y}{2}\right) + G\bar{\psi}(x)\Gamma_\alpha\psi(y), \tag{30}\]
the nonlocal quartic fermionic interaction can be removed. Instead, the fermion field starts to interact nonlocally with the mesonic fields,

\begin{equation}
S_{NJL} = -\frac{N_c g^2}{2G} \int d^4z \Delta(z) \int d^4w M_\alpha^*(w) M_\alpha(w) + \int d^4x \bar{\psi}(x)(i \gamma^\mu \partial_\mu - m_q)\psi(x)
\end{equation}

\begin{equation}
- \frac{N_c g^2}{2G} \int d^4x \int d^4y \bar{\psi}(x)\Delta(x-y) \left(M_\alpha \left(\frac{x+y}{2}\right) + M_\alpha^* \left(\frac{x+y}{2}\right)\right) \Gamma^\alpha \psi(y).
\end{equation}

After Fourier transform, in momentum space one obtains

\begin{equation}
S_{NJL} = -\frac{N_c g^2}{4G} \int \frac{d^4q}{(2\pi)^4} \tilde{M}_\alpha^*(q) \tilde{M}_\alpha(q) + \int \frac{d^4p}{(2\pi)^4} \tilde{\psi}(p)(\vec{p} - m_q)\tilde{\psi}(p)
\end{equation}

\begin{equation}
- \frac{N_c g^2}{2G} \int d^4p^' \int \frac{d^4p}{(2\pi)^4} \tilde{\psi}(p)\tilde{\Delta} \left(\tilde{M}_\alpha(p-p') + \tilde{M}_\alpha^*(p-p')\right) \Gamma^\alpha \tilde{\psi}(p'),
\end{equation}

where the symbols with tilde are used for the Fourier transformed quantities. The final step in the bosonization is to integrate out the fermionic fields, in the general case leading to \[22\]

\begin{equation}
S_{bos} = -\frac{N_c g^2}{4G} \int \frac{d^4q}{(2\pi)^4} \tilde{M}_\alpha^*(q) \tilde{M}_\alpha(q)
\end{equation}

\begin{equation}
- \ln \det \left(\frac{2\pi}4 \delta^4(p-p')(\vec{p} - m_q) - \frac{N_c g^2}{2G} \tilde{\Delta} \left(\frac{p+p'}{2}\right) \left(\tilde{M}_\alpha(p-p') + \tilde{M}_\alpha^*(p-p')\right) \Gamma^\alpha\right),
\end{equation}

where \(\det\) denotes the direct product of a functional and an analytical determinant, the former in the Fock space transition between space-time points \(x\) and \(y\), the latter in the Dirac and flavor indices.

\section*{D. Mean field approximation}

Expanding the bosonic fields \(\sigma(x) = \bar{\sigma} + \delta\sigma(x)\) and \(\bar{\pi}(x) = \delta\bar{\pi}(x)\) about the vacuum expectation value \(\bar{\sigma} = \langle \sigma \rangle\), the zeroth order expansion coefficient is the mean field approximation, leading to the simplified NJL action

\begin{equation}
S_{NJL} = -\frac{N_c g^2 \bar{\sigma}^2}{4G} V^{(4)} + \int d^4x \bar{\psi}(x)(i \gamma^\mu \partial_\mu - m_q)\psi(x)
\end{equation}

\begin{equation}
- \frac{N_c g^2 \bar{\sigma}}{G} \int d^4x \int d^4y \bar{\psi}(x)\Delta(x-y)\psi(y).
\end{equation}

After Fourier transform, in momentum space one has

\begin{equation}
S_{NJL} = -\frac{N_c g^2 \bar{\sigma}^2}{4G} V^{(4)} + \int \frac{d^4p}{(2\pi)^4} \tilde{\psi}(p)(\vec{p} - m_q)\tilde{\psi}(p) - \frac{N_c g^2 \bar{\sigma}}{G} \int \frac{d^4p}{(2\pi)^4} \tilde{\psi}(p)\tilde{\Delta}(p)\tilde{\psi}(p) =
\end{equation}

\begin{equation}
= -\frac{N_c g^2 \bar{\sigma}^2}{4G} V^{(4)} + \int \frac{d^4p}{(2\pi)^4} \tilde{\psi}(p)(\vec{p} - M_q(p))\tilde{\psi}(p)
\end{equation}
with the unit space-time volume $V^{(4)}$, where $(G = 2\tilde{\Delta}(0))$

$$M_q(p) = m_q + \frac{N_c g^2}{G} \tilde{\Delta}(p) \tilde{\sigma} = m_q + \frac{N_c g^2 \tilde{\Delta}(p)}{2\Delta(0)} \tilde{\sigma}$$  \hspace{1cm} (36)

is the dynamical mass of the quark. The bosonization leads to

$$S_{\text{bos}}^{V^{(4)}} = -\frac{N_c g^2 \tilde{\sigma}^2}{4G} - \int \frac{d^4 p}{(2\pi)^4} \ln \det(\not{p} - M_q(p)).$$ \hspace{1cm} (37)

On the other hand, one has $\ln \det(\not{p} - M_q(p)) = \text{tr} \ln(\not{p} - M_q(p)) = \frac{1}{4} 4N_f \ln(p^2 - M^2_q(p))$. The quantity $\tilde{\sigma}$ can be determined by variation of the action $S_{\text{bos}}$ with respect to this quantity. Taking into account the dependence of $M_q(p)$ on $\tilde{\sigma}$, one obtains

$$0 = -\frac{N_c g^2 \tilde{\sigma}}{2G} + 2N_f \int \frac{d^4 p}{(2\pi)^4} \frac{2M_q(p)}{p^2 - M^2_q(p)} \frac{N_c g^2}{G} \tilde{\Delta}(p) \Rightarrow \tilde{\sigma} = 8N_f \int \frac{d^4 p}{(2\pi)^4} \frac{\tilde{\Delta}(p) M_q(p)}{p^2 - M^2_q(p)}.$$ \hspace{1cm} (38)

Finally, this result can be re-inserted to Eq. (36) to obtain the dynamical mass equation

$$M_q(p) = m_q + 4N_f N_c g^2 \frac{\tilde{\Delta}(p)}{\Delta(0)} \int \frac{d^4 p'}{(2\pi)^4} \frac{\tilde{\Delta}(p') M_q(p')}{p^2 - M^2_q(p')}.$$ \hspace{1cm} (39)

A similar gap equation for the $g-2$ problem was shown in Ref. [13]. In this article, however, we derived the gap equation directly from the QCD Lagrangian.

**III. SOLVING THE GAP EQUATION**

At this point we can insert $\tilde{\Delta}(p)$ from Eq. (19) into Eq. (39) in order to obtain the gap equation for the dynamical quark mass – or to be more precise the couple of gap equations, if taking into account Eq. (23) as well. However, in order to make the calculation feasible, we recognize that the dependence on the mass $m$ of the scalar field is subdominant, and this mass can be neglected compared to the mass of the quark. For $m = 0$ one has $\kappa = -1$, $\varphi(\kappa = -1) = (1 - i)\pi$ and

$$\tilde{\Delta}(p) = \sum_{n=0}^{\infty} \frac{iB_n}{p^2 - m^2_n + i\epsilon}, \quad B_n = \frac{(2n+1)^2 \pi^3}{4K(-1)^3} \frac{e^{-(n+1/2)\pi}}{1 + e^{-(2n+1)\pi}}.$$ \hspace{1cm} (40)

$m_n = (2n + 1)\sqrt{2p^2/2K(-1)} = (2n + 1)m_0$ is the glue ball spectrum, with the ground state given by $m_0 = m_G(-1) = \sqrt{2p^2/2K(-1)}$ and $K(z)$ is the complete elliptic integral.
of the first kind. As a further simplification we calculate the dynamical quark mass at zero momentum, \( p = 0 \). In this case we obtain

\[
M_q = m_q + 4N_f N_c g^2 \sum_{n=0}^\infty \int_0^\Lambda \frac{d^4 p}{(2\pi)^4} \frac{B_n}{p^2 + m_n^2} \frac{M_q}{p^2 + M_q^2},
\]

where we have performed a Wick rotation to the Euclidean domain. As this integral is UV singular, we integrate the momentum up to a cut \( \Lambda \) to obtain

\[
\int_0^\Lambda \frac{d^4 p}{(2\pi)^4} \frac{B_n}{p^2 + m_n^2} \frac{M_q}{p^2 + M_q^2} = \frac{\pi^2}{(2\pi)^4} \int_0^{\Lambda^2} \frac{B_n M_q p^2 dp^2}{(p^2 + m_n^2)(p^2 + M_q^2)} =
\]

\[
= \frac{1}{(4\pi)^2 (m_n^2 - M_q^2)} \left[ m_n^2 \ln \left( 1 + \frac{\Lambda^2}{m_n^2} \right) - M_q^2 \ln \left( 1 + \frac{\Lambda^2}{M_q^2} \right) \right] =
\]

\[
= \frac{1}{(4\pi)^2 ((2n+1)x^2 - y^2)} \left[ (2n+1)^2 x^2 \ln \left( 1 + \frac{1}{(2n+1)^2 x^2} \right) - y^2 \ln \left( 1 + \frac{1}{y^2} \right) \right],
\]

where we have used the dimensionless quantities \( x = m_0/\Lambda \) and \( y = M_q/\Lambda \), assuming that \( M_q \ll \Lambda \). Reinserting into Eq. (41) leads to the gap equation

\[
y = \frac{m_q}{\Lambda} + \kappa \alpha_s \sum_{n=0}^\infty \frac{B_n y}{2(n+1)^2 x^2 - y^2} \left[ (2n+1)^2 x^2 \ln \left( 1 + \frac{1}{(2n+1)^2 x^2} \right) - y^2 \ln \left( 1 + \frac{1}{y^2} \right) \right],
\]

where \( \kappa = N_f N_c / \pi \) and \( \alpha_s = g^2 / 4\pi \). We note that the cut-off completely disappeared except for the ratio \( m_q/\Lambda \) that, for the light quarks, is negligible small.

For the QCD cut-off \( \Lambda = 1 \) GeV, the average mass of the \( u \) and \( d \) quarks is taken to be \( m_q = 0.003415(48) \) GeV \[5\]. The ground state of the glue ball spectrum is given by the \( f_0(500) \) resonance, measured as \( m_0 = 0.512(15) \) GeV \[23\]. Using \( N_c = 3 \), \( N_f = 6 \) and \( \alpha_s(3.1 \text{ GeV}) = 0.256506 \) we obtain \( M_q = 0.427(29) \) GeV.

\section*{IV. Hadronic Vacuum Polarization}

Inspired by the approach in Ref. \[12\], next we will evaluate the contribution to the hadronic vacuum polarization, assuming that a NJL approximation holds \[14, 17\]. Looking at the Fierz decomposition as shown in Eqn. (28), one obtains

\[
G_S = G_P = \pi \alpha G, \quad G_V = G_A = -\frac{1}{2} \pi \alpha G,
\]

where

\[
G = 2 \Delta(0) = - \sum_{n=0}^\infty \frac{B_n}{(2n+1)^2 m_0^2},
\]
which agrees well with the analysis in the preceding section, provided we evaluate the gap equation as in Eq. (43). Using Ref. [12], we evaluate

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^2 m_\mu^2 \frac{4\pi^2}{3} P_1.$$  

(46)

The coefficient $P_1$ determines the contribution called “had 1a”. It is defined by

$$P_1 = -\frac{\partial \Pi_R^{(H)}(Q^2)}{\partial Q^2} \bigg|_{Q^2=0}$$

(47)

where $\Pi_R^{(H)}(Q^2) = \frac{2}{3} \left( \Pi_V^{(1)}(Q^2) - \Pi_V^{(1)}(0) \right)$,

$$\Pi_V^{(1)}(Q^2) = \frac{\bar{\Pi}_V^{(1)}(Q^2)}{1 + Q^2(8\pi^2 G_V/N_c \Lambda_\chi)\bar{\Pi}_V^{(1)}(Q^2)},$$

(48)

and

$$\bar{\Pi}_V^{(1)}(Q^2) = \frac{N_c}{2\pi^2} \int_0^1 dy y(1-y) \Gamma \left(0, \frac{M_q^2 + Q^2 y(1-y)}{\Lambda_\chi^2}\right), \quad \Gamma(n, \varepsilon) = \int_{\varepsilon}^{\infty} dz z^{n-1} e^{-z}.$$  

(49)

$\Gamma(n, \varepsilon)$ is the incomplete gamma function, but $\Gamma(1, \varepsilon)$ is an analytic expression,

$$\Gamma(1, \varepsilon) = \int_{\varepsilon}^{\infty} e^{-z} dz = \left[ -e^{-z} \right]_{z=\varepsilon}^{\infty} = e^{-\varepsilon}.$$  

(50)

Using these formulas, we obtain

$$P_1 = -\frac{\partial \Pi_R^{(H)}(Q^2)}{\partial Q^2} \bigg|_{Q^2=0} = -\frac{2}{3} \frac{\partial \Pi_V^{(1)}(Q^2)}{\partial Q^2} \bigg|_{Q^2=0} =$$

$$= -\frac{2}{3} \left[ \frac{\partial \Pi_V^{(1)}(Q^2)}{\partial Q^2} \right]_{Q^2=0} =$$

$$= -\frac{2}{3} \left[ \frac{\partial \Pi_V^{(1)}(Q^2)}{\partial Q^2} - \frac{8\pi^2 G_V N_c \Lambda_\chi^2}{M_q^2 + Q^2 y(1-y)} \right]_{Q^2=0} =$$

$$= -\frac{2}{3} \left[ \frac{\partial \Pi_V^{(1)}(Q^2)}{\partial Q^2} - \frac{8\pi^2 G_V N_c \Lambda_\chi^2}{M_q^2 + Q^2 y(1-y)} \right]_{Q^2=0} =$$

$$= -\frac{2}{3} \left[ \frac{\partial \Pi_V^{(1)}(Q^2)}{\partial Q^2} - \frac{8\pi^2 G_V N_c \Lambda_\chi^2}{M_q^2 + Q^2 y(1-y)} \right]_{Q^2=0} =$$

$$= -\frac{2}{3} \left[ -\frac{N_c}{2\pi^2} \int_0^1 dy y(1-y) \Gamma \left(0, \frac{M_q^2 + Q^2 y(1-y)}{\Lambda_\chi^2}\right) e^{-\left(\frac{M_q^2 + Q^2 y(1-y)}{\Lambda_\chi^2}\right)} \right]_{Q^2=0} =$$

$$= -\frac{2}{3} \left[ -\frac{N_c}{60\pi^2 M_q^2} \Gamma \left(1, \frac{M_q^2}{\Lambda_\chi^2}\right) - \frac{N_c G_V}{18\pi^2 \Lambda_\chi^2} \Gamma \left(0, \frac{M_q^2}{\Lambda_\chi^2}\right) \right] =$$

$$= \frac{N_c}{3\pi^2} \left[ \Gamma \left(1, \frac{M_q^2}{\Lambda_\chi^2}\right) + \frac{10G_V M_q^2}{3\Lambda_\chi^2} \Gamma \left(0, \frac{M_q^2}{\Lambda_\chi^2}\right) \right].$$  

(51)
With the values given above, for the $u$ and $d$ quarks we obtain

$$a_{\mu}^{u,d}(\text{had 1a}) = 452(67) \cdot 10^{-10}. \quad (52)$$

This result is in close agreement with the evaluation given in eq. (3.3) in Ref. [7] and eq.(18) in Ref. [8] and Eq.(6) in Ref. [9].

In order to have a clearer understanding of the meaning of this result, we present also the strange quark contribution. This will yield

$$a_{\mu}^{s}(\text{had 1a}) = 232(34) \cdot 10^{-10}. \quad (53)$$

The overall is

$$a_{\mu}^{HVP} = 684(75) \cdot 10^{-10}. \quad (54)$$

The error bar is not yet competitive to decide if BSM physics is needed but nevertheless in closed agreement with the experimental value as obtained in [6–8] from experiments in hadron physics.

Finally, we want to analyze the contribution to the error due to the choice of the ’t Hooft limit: $Ng^2$ constant and $N \to \infty$. There have been several studies on lattice to estimate the error of such an approximation ([24, 25] and references therein). The main conclusion is that the next-to-leading order correction to any observable goes like

$$A = A(\infty) + \frac{c_1}{N^2} + \ldots, \quad (55)$$

being $c_1 = O(1)$, a numerical factor. This same pattern is seen in the spectrum of a Yang-Mills theory without quarks where, for the ground state, one sees [26]

$$\frac{m_{0^++}}{\sqrt{\sigma}} = 3.28(8) + \frac{2.1(1.1)}{N^2}, \quad (56)$$

where $\sigma$ is a mass scale proper to strong interactions and obtained by experiment. So, this can be estimated of the same magnitude as the error we obtained from QCD data at worst.

V. CONCLUSIONS AND OUTLOOK

To summarise, using technique devised by Bender, Milton and Savage, in Ref. [15] the Dyson-Schwinger equations for quantum chromodynamics in differential form was revisited. Following Ref. [15], in this article we discussed the hadronic contributions to the muon
anomalous magnetic moment following NJL model as the low energy effective theory description of QCD, as shown in Eq. (26). We provided a full derivation of the HVP contribution to the anomalous magnetic moment $a = (g - 2)/2$ of the muon from first principles, starting from the QCD partition function and the effective mass for the quarks as shown in Eq. (39). Our result as obtained in Eq. (52) is in close agreement with the Muon $g - 2$ Theory Initiative [6], as obtained from experimental data in Ref. [7–9]. In doing so, we have shown a possible new analytical approach as an alternative to lattice calculations. Our approach provides a theoretical framework for the application of QCD to several other applications and the opportunity to investigate future studies model-building for BSM physics in the dark sector just by using analytical methods. The next step will be to include other quark flavors which is beyond the scope of the current manuscript. Moreover, following the same approach and using NJL model as the low energy EFT for QCD, we also can perform a complete proof of confinement in QCD in our future studies.

We hope to improve our computations in the near future to reduce the error bar significantly.

VI. ACKNOWLEDGEMENTS

The research was supported in part by the European Regional Development Fund under Grant No. TK133.

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