GROUND STATES FOR THREE-LAYER HYBRID SUPERCONDUCTOR

by

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Abstract

A superconducting hybrid structure composed of three layers is considered. The 2D layers interact mutually by higher grade inter-layer couplings. We determine the possible superconducting modes. Those solutions enable to discuss the conditions for the onset and enhancement of 3D superconductivity in such a structure.

1 Introduction

We are concerned with a system of potentially superconducting 2D layers interacting with one another by possibly distant interlayer couplings. The general aim of our considerations is the study of the nanoelectronic properties of such structures, including the dependence of their electronic properties on the structural and mechanical parameters.

We shall make use of the higher grade hybrid model (HM) of superconductors formulated in [11] and developed in [12, 14, 13, 15, 16]. In the present paper we confine ourselves to three-layer structures, with the aim to determine the complete set of solutions. According to [11], in the framework of HM the layered superconductor is

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considered as a one-dimensional chain with "atoms" (being here identical atomic planes described by 2D Ginsburg-Landau theory with parameters $\alpha_0$ and $\beta$) and with Josephson’s bonds (called J-links) between them. The grade $K$, expressed by an arbitrary (but specified for any particular case) integer, defines the admitted range of Josephson’s interaction in terms of interplanar gaps.

We denote by $\psi_n$ the order parameter associated to the layer indexed by the number $n$. Its complex conjugate (c.c.) is denoted by $\bar{\psi}_n$. In the considered variant of HM (identical atomic planes), besides the GL parameters characterising isolated planes there are two classes of long-range coupling constants: Josephson parameters $\gamma_q$ and proximity effect parameters $\zeta_q$. In the present paper we intend to examine the influence of these long-range coupling, under assumption that there is no magnetic field and no currents.

Following [11] we shall now briefly present the equations for plane-uniform states of our hybrid model in the absence of magnetic field. The order parameter is then independent of the in-plane variables and the net supercurrents vanish. We shall confine our attention to the grade $K = 2$. For $K = 2$ the condition of vanishing Josephson current is equivalent to

$$\gamma_1(\bar{\psi}_n\psi_{n+1} - \text{c.c.}) + \gamma_2(\bar{\psi}_n\psi_{n+2} + \bar{\psi}_{n-1}\psi_{n+1} - \text{c.c.}) = 0,$$

(1)

and the field equations take the form

$$\tilde{\alpha}_n\psi_n + \beta|\psi_n|^2\psi_n - \frac{1}{2}[\gamma_1(\psi_{n+1} + \psi_{n-1}) + \gamma_2(\psi_{n+2} + \psi_{n-2})] = 0,$$

(2)

where instead of $\alpha_0$ we have introduced

$$\tilde{\alpha}_n = \alpha_0 + \frac{1}{2} \sum_q \zeta_q,$$

(3)

with the summation running over the planes which are J-linked to the plane number $n$. The plane index $n$ belongs to a given set $P$.

2 Exact solutions for N=3

Let us consider the simplest non-trivial instance of higher grade system composed of a finite number of layers: the case $K = 2$ (hence interactions
of nearest and next nearest neighbours only, characterized by parameters 
\(\zeta_1, \gamma_1, \zeta_2, \gamma_2\), the number of layers equal 3. We shall 
index the planes with the integers from the set \(P = \{-1, 0, 1\}\). The 
interlayer gaps will be then numbered by \(-\frac{1}{2}\) and \(\frac{1}{2}\). The 
energy functional takes the form

\[
E = \tilde{\alpha}_1|\psi_1|^2 + \tilde{\alpha}_-|\psi_-|^2 + \tilde{\alpha}_0|\psi_0|^2 - \frac{\alpha_0}{2}\bar{(\psi_- + \psi_1)}\psi_0 + \psi_0(\psi_- + \psi_1)] \\
-\frac{\alpha_1}{2}(\bar{\psi_-}\psi_1 + \bar{\psi_1}\psi_-) + \frac{\delta}{2}(|\psi_1|^4 + |\psi_0|^4 + |\psi_-|^4)
\]

and the system of equations for the order parameters \(\psi_-\), \(\psi_0\) and \(\psi_1\) reads

\[
(\tilde{\alpha}_1 + \beta|\psi_1|^2)\psi_1 - \frac{1}{2}[\gamma_1\psi_0 + \gamma_2\psi_-] = 0, \quad (5) \\
(\tilde{\alpha}_0 + \beta|\psi_0|^2)\psi_0 - \frac{1}{2}\gamma_1(\psi_1 + \psi_-) = 0, \quad (6) \\
(\tilde{\alpha}_- + \beta|\psi_-|^2)\psi_- - \frac{1}{2}\gamma_1\psi_0 + \gamma_2\psi_1 = 0. \quad (7)
\]

Due to the finiteness of the system of layers under consideration, the conditions (1) for vanishing Josephson’s currents \(J_{-\frac{1}{2}}\) and \(J_{\frac{1}{2}}\) are automatically satisfied. For the present system they take the form

\[
\gamma_1\bar{\psi_-}\psi_0 + \gamma_2\bar{\psi_-}\psi_1 - c.c. = 0, \quad (8) \\
\gamma_1\bar{\psi_0}\psi_1 + \gamma_2\bar{\psi_-}\psi_1 - c.c. = 0. \quad (9)
\]

Note that, according to (3), \(\tilde{\alpha}_1\) and \(\tilde{\alpha}_-\) are equal to one another; hence we shall use the symbol \(\tilde{\alpha}_1\) for both the cases. We have

\[
\tilde{\alpha}_1 = \alpha_0 + \frac{1}{2}(\zeta_1 + \zeta_2) = \tilde{\alpha}_0 - \frac{1}{2}\delta, \quad (10)
\]

where

\[
\delta = \zeta_1 - \zeta_2. \quad (11)
\]

For the central layer we have

\[
\tilde{\alpha}_0 = \alpha_0 + \zeta_1. \quad (12)
\]

The trivial solution \(\psi_j = 0, \ j \in P\) describes the normal state. We are interested in discussing the stability of the normal solution, and in finding stable nontrivial solutions to the system (5-7) describing the superconducting states.
3 Properties

The eqns. (5-7) have a number of specific properties which facilitate complete solution of the system. To compactify the phrasing, we shall use the following terminology.

Definition 1. A solution to system (5-7) is called \( T \)-invariant iff it is gauge-equivalent to a real solution \( \psi_j \):

\[
\bar{\psi}_j = \psi_j.
\]  

(13)

Definition 2. A solution to system (5-7) is called \( TP \)-invariant iff it can be gauge-transformed to the form

\[
\bar{\psi}_{-j} = \psi_j.
\]  

(14)

It is also convenient to introduce the symbols for real and imaginary parts of the order parameters:

\[
\psi_j = a_j + ib_j, \quad j \in P.
\]  

(15)

Property 1. Any solution to eqns. (5-7) is gauge-equivalent to a solution which satisfies the following conditions

\[
\psi_0 = a_0
\]  

(16)

and

\[
b_1 + b_{-1} = 0.
\]  

(17)

Moreover, any such solution fulfills

\[
b_1[\gamma_1 a_0 + \gamma_2 (a_1 + a_{-1})] = 0.
\]  

(18)

Property 2. Any real solution satisfies

\[
(a_1 - a_{-1})[\alpha_1^* + \beta(a_1^2 + a_1 a_{-1} + a_{-1}^2)] = 0.
\]  

(19)

where

\[
\alpha_1^* = \tilde{\alpha}_0 + \frac{1}{2}(\gamma_2 - \delta).
\]  

(20)

Property 3. For any solution \( \psi_0 = 0 \) iff \( \psi_1 = \psi_{-1} \).

Property 4. Any solution which is not \( T \)-invariant, is \( TP \)-invariant, and vice versa.
4 Classification of solutions

Due to the Property 1 we can confine further consideration to the nontrivial solutions satisfying (16) and (17). From the remaining properties it follows that this set of solutions to the system (8-7) is partitioned into the following four disjoint classes:

Class (A): \( b_1 = 0, \ a_1 = -a_{-1} \neq 0, \ a_0 = 0, \)
Class (B): \( b_1 \neq 0, \ a_1 = a_{-1}, \ a_0 \neq 0, \)
Class (C1): \( b_1 = 0, \ a_1 = a_{-1}, \ a_0 \neq 0, \)
Class (C2): \( b_1 = 0, \ |a_1| \neq |a_{-1}|, \ a_0 \neq 0. \)

The classes (A) and (C2) are T-invariant, the class (B) is TP-invariant, and the class (C1) is T- and TP-invariant. The classes (B) and (C2) are 3D, the class (A) is 1D and the class (C1) is 2D. In the respective classes, the field equations (8-7) reduce to simplified forms. Only equations of the class (B) contain imaginary parts.

In the class (A) the situation is particularly simple. We have

\[ a_1^2 = -\frac{\alpha_1^*}{\beta}, \]

so that (taking into account the positiveness of \( \beta \)) the class is non-empty iff

\[ 2\tilde{\alpha}_0 + \gamma_2 - \delta < 0. \]

In the remaining classes the situation is more complicated.

Class (B): The equations take the form

\[ a_0^2 = -\frac{1}{\beta}(\tilde{\alpha}_0 + \frac{\gamma_1^2}{2\gamma_2}), \]
\[ a_1 = a_{-1} = -\frac{\gamma_1}{2\gamma_2}a_0, \]
\[ b_1^2 = |\psi_1|^2 - a_1^2 = b_{-1}^2, \]

where

\[ |\psi_1|^2 = |\psi_{-1}|^2 = -\frac{\alpha_1^*}{\beta}. \]
and \( \alpha_1^* \) is given by the eqn. (20). The necessary and sufficient condition for the existence of solutions in this class is the conjunction of (22) and the following two inequalities

\[
\tilde{\alpha}_0 + \frac{\gamma_1^2}{2\gamma_2} < 0, 
\]

\[
(\gamma_1^2 - 4\gamma_2^2)\tilde{\alpha}_0 - 2\gamma_2^2(\gamma_2 - \delta) + \frac{\gamma_1^4}{2\gamma_2} > 0. 
\]

Class (C1): The system of equations for \( a_0 \) and \( a_1 \) can be written as

\[
a_0 = \frac{2}{\gamma_1}(\alpha_1^* - \gamma_2 + \beta a_1^2)a_1, 
\]

\[
a_1 = \frac{1}{\gamma_1}(\tilde{\alpha}_0 + \beta a_0^2)a_0, 
\]

where \( \alpha_1^* \) is given by the eqn. (20). Let us multiply by sides the equations (29) and (30). The result is

\[
(\alpha_1^* - \gamma_2 + \beta a_1^2)(\tilde{\alpha}_0 + \beta a_0^2) = \frac{1}{2}\gamma_1^2. 
\]

Now we make the ansatz

\[
\tilde{\alpha}_0 + \beta a_0^2 = x, 
\]

and

\[
2(\alpha_1^* - \gamma_2 + \beta a_1^2) = \gamma_1^2 \frac{1}{x}. 
\]

We are looking for non-zero solutions of the equations (29) and (30), hence we can divide the equations by sides. We obtain

\[
\beta a_0^2 = x - \tilde{\alpha}_0 > 0 
\]

and

\[
\beta a_1^2 = \gamma_1^2 \frac{1}{x} - \alpha_1^* - \gamma_2 > 0, 
\]

where both the inequalities formulate the necessary conditions for the existence of solutions in the class (C1). Further

\[
\gamma_1^2 \left( \frac{a_1}{a_0} \right)^2 = x^2. 
\]
and, taking into account (20),
\[
\frac{x^2}{\gamma^2} = \frac{\gamma_1^2 - \tilde{\alpha} + \frac{1}{2} (\delta + \gamma_2)}{x - \tilde{\alpha}}.
\] (37)

Geometrically, the set of solutions from class (C1) are determined by the points of intersection of two curves:
\[
y = \frac{2}{\gamma_1^2} x^3,
\] (38)
and
\[
y = \frac{\gamma_1^2 - 2(\tilde{\alpha} - \delta - \gamma_2)x}{x - \tilde{\alpha}}.
\] (39)

Let us note, that \(x = \tilde{\alpha}\) corresponds to the zero-solution, describing the transition to the normal state. We have then the equation for \(\tilde{\alpha}\)
\[
2(\tilde{\alpha} - \delta - \gamma_2)\tilde{\alpha} - \gamma_1^2 = 0,
\] (40)
which always has two real roots. We shall denote them by
\[
\alpha_{01} = \frac{1}{4} (\delta + \gamma_2 - \sqrt{(\delta + \gamma_2)^2 + 8 \gamma_1^2}),
\] (41)
and
\[
\alpha_{02} = \frac{1}{4} (\delta + \gamma_2 + \sqrt{(\delta + \gamma_2)^2 + 8 \gamma_1^2}),
\] (42)

For a given material, the number of solutions in the class (C1) depends on the temperature parameter \(\tilde{\alpha}\) in the following manner.
1) When \(\tilde{\alpha} < \alpha_{01}\), there are two solutions
2) When \(\alpha_{01} < \tilde{\alpha} < \frac{1}{2} (\delta + \gamma_2)\) or \(\frac{1}{2} (\delta + \gamma_2) < \tilde{\alpha} < \alpha_{02}\), there exists one solution
3) For \(\alpha_{02} < \tilde{\alpha}\) there are no solutions.

Class (C2): The solutions of this class fulfill the condition
\[
a_1^2 + a_1 a_{-1} + a_{-1}^2 = -\frac{\alpha^*_1}{\beta}.
\] (43)
By combining it with real parts of eqns. (5)-(7) we obtain

$$a_1 = \frac{1}{2}(a_+ + a_-), \quad a_{-1} = \frac{1}{2}(a_+ - a_-),$$  \hspace{1cm} (44)$$

where the variables $a_+$, $a_-$ satisfy the equations

$$a_+^2 = -4\frac{a_1^*}{\beta} - 3a_+^2,$$  \hspace{1cm} (45)$$

$$a_0 = -\frac{2}{\gamma_1}(\alpha_1^* + \frac{\gamma_2}{2} + \beta a_+^2)a_+,$$  \hspace{1cm} (46)$$

$$a_+ = \frac{2}{\gamma_1}(\tilde{\alpha}_0 + \beta a_0^2)a_0.$$  \hspace{1cm} (47)$$

Because of (43), the inequality (22) is the necessary condition for the existence of solutions in the class (C2). To simplify the discussion of the sufficient conditions let us first introduce some additional symbols:

$$\alpha_1 = \frac{2\alpha_1^* + \gamma_2}{\gamma_1}, \quad \lambda = \alpha_1^4, \quad \kappa = \frac{1}{4}\frac{\tilde{\alpha}_0}{\gamma_1}\alpha_1$$  \hspace{1cm} (48)$$

and once more change variables. We shall express the quotient and the product of $a_+$ and $a_0$ by $\xi$ and $\eta$ defined as follows.

$$\xi = \frac{a_+}{a_0}\alpha_1, \quad \eta = \frac{2\beta}{\gamma_1}\frac{1}{\alpha_1^2}a_+a_0.$$  \hspace{1cm} (49)$$

Then the equations (46) and (47) will be transformed into

$$\xi^2 - 4\kappa\xi - \lambda\eta = 0$$  \hspace{1cm} (50)$$

and

$$\eta\xi^2 + \xi + 1 = 0.$$  \hspace{1cm} (51)$$

Eliminating $\eta$ one obtains the following equation for $\xi$

$$\xi^4 - 4k\xi^3 + l(\xi + 1) = 0,$$  \hspace{1cm} (52)$$

which can be transformed into the form

$$w^4 = (w - p)^2 - q.$$  \hspace{1cm} (53)$$
by introducing the variable:

\[ w = \sqrt{6\kappa}(\xi - \kappa) \]  

(54)

and the coefficients:

\[ p = \frac{\sqrt{6}\lambda - 8\kappa^3}{36 - 2\kappa^3}, \quad q = p^2 + \frac{-3\kappa^4 + \kappa\lambda + \lambda}{36\kappa^4}. \]  

(55)

The request of tangency of curves

\[ \eta = \frac{1}{\lambda}(\xi^2 - 4\kappa\xi) \]  

(56)

and

\[ \eta = -\frac{\xi + 1}{\xi^2} \]  

(57)

implies the condition

\[ 2\xi^4 - 4\kappa\xi^3 - \lambda(\xi + 2) = 0. \]  

(58)

Combining (58) with (52) one can express \( \kappa \) and \( \lambda \) by \( \xi \):

\[ \lambda = \frac{\xi^4}{2\xi + 3} \]  

(59)

and

\[ \kappa = \frac{\xi(3\xi + 4)}{2\xi + 3}. \]  

(60)

Now we can solve (60) with respect to \( \xi \):

\[ \xi_{1,2} = \frac{1}{3}(\kappa - 2 \mp \sqrt{(k + 2)^2 + 2}) \]  

(61)

and then, apropriately substituting \( \xi_1 \) or \( \xi_2 \), calculate \( \lambda_{1,2} \) as well as \( \eta_{1,2} \). The curves (56) and (57) are tangent to each other at points \((\xi_1, \eta_1)\) and \((\xi_2, \eta_2)\). For \( \lambda > \lambda_1 \) there are no points of intersection of curves (56) and (57), hence no superconducting solutions to equations (46) and (47). For \( \lambda = \lambda_1 \) there are two solutions. Finally, for \( \lambda > \lambda_1 \) there are four solutions.
5 Onset of superconductivity

Let us first discuss the stability of the normal state. Due to Properties the second variation of energy can be represented in the form

\[
\delta^2 \mathcal{E} = (\tilde{\alpha}_1 + 2\beta|\psi_1|^2)[(\delta a_1)^2 + (\delta b_1)^2] + \\
(\tilde{\alpha}_1 + 2\beta|\psi_{-1}|^2)[(\delta a_{-1})^2 + (\delta b_1)^2] + \\
(\tilde{\alpha}_0 + 3\beta a_0^2)(\delta a_0)^2 + \beta \text{Re}[\bar{\psi}_1^2(\delta\psi_1)^2 + \bar{\psi}_{-1}^2(\delta\psi_{-1})^2] + \\
-\gamma_1(\delta a_1 + \delta a_{-1})\delta a_0 - \gamma_2(\delta a_1 \delta a_{-1} + (\delta b_1)^2).
\]

Introducing variables \(a_+\) and \(a_-\) and their variations, we obtain for the normal state (all fields equal zero)

\[
\delta^2 \mathcal{E} = \frac{1}{2}(\tilde{\alpha}_1 - \frac{2\gamma_2}{\delta})(\delta a_+)^2 + \frac{1}{2}(\tilde{\alpha}_1 + \frac{2\gamma_2}{\delta})(\delta a_-)^2 + \\
2(\tilde{\alpha}_1 + \gamma_2)(\delta b_1)^2 + \tilde{\alpha}_0(\delta a_0)^2 - \gamma_1 \delta a_+ \delta a_0.
\]

The necessary and sufficient condition for stability of the normal state is the positive definiteness of the second variation of energy, which implies the conjunction of the following inequalities

\[
\tilde{\alpha}_0 > 0, \quad (64)
\]

\[
\tilde{\alpha}_0 > \frac{1}{2}(\delta - |\gamma_2|), \quad (65)
\]

\[
\tilde{\alpha}_0 > \alpha_{02}, \quad (66)
\]

where \(\alpha_{02}\) is defined by the eqn. (42).

Let us introduce the material parameters plane with coordinates \(\gamma_2\) and \(\delta\). For every point of the plane \((\gamma_2, \delta)\) there exists a stable normal state, depending on the temperature. The highest temperature in which (for a given material) the normal state becomes unstable determines the onset of superconductivity. Instability of the normal state implies stability of a superconducting state.
6 Stability of the superconducting states

We shall say that a solution from a given class has the property of internal stability if it is stable with respect to variations preserving the class.

For the classes with real solutions we have
\[
\delta^2 E = (\tilde{\alpha}_1 + 3\beta a_1^2)(\delta a_1)^2 + (\tilde{\alpha}_1 + 3\beta a_{-1}^2)(\delta a_{-1})^2 +
(\tilde{\alpha}_0 + 3\beta a_0^2)(\delta a_0)^2 + [\tilde{\alpha}_1 + \beta(a_1^2 + a_{-1}^2) - \gamma_2]\delta(b_1)^2 +
-\gamma_1(\delta a_1 + \delta a_{-1})\delta a_0 - \gamma_2\delta a_1\delta a_{-1}.
\] (67)

hence for the class (A), taking into account the solution (21)
\[
\delta^2 E = -(\tilde{\alpha}_1 + \gamma_2)(\delta a^+)^2 - (\tilde{\alpha}_1 + \frac{\gamma_2}{2})(\delta a^-)^2 + \tilde{\alpha}_0(\delta a_0)^2 - \gamma_1\delta a_+\delta a_0.
\] (68)

Note that the coefficient by \((\delta b_1)^2\) equals zero. The conditions for stability of mode (A) read
\[
\tilde{\alpha}_0 < \frac{\delta}{2} - \gamma_2, \quad (69)
\]
\[
\tilde{\alpha}_0 < \frac{1}{2}(\delta + \gamma_2), \quad (70)
\]
\[
\tilde{\alpha}_0(\tilde{\alpha}_0 - \frac{\delta}{2} - \gamma_2) + \frac{1}{4}\gamma_1^2 < 0, \quad (71)
\]
and the condition (64). To fulfill (69) and (71) the following restriction of the material parameters is necessary
\[
\frac{\delta}{2} - \gamma_2 > |\gamma_1|. \quad (72)
\]

For such materials, the real roots of the polynomial from (71) have the form
\[
\alpha_{1,2} = \frac{1}{2}\left(\frac{\delta}{2} - \gamma_2 \pm \sqrt{\left(\frac{\delta}{2} - \gamma_2\right)^2 - \gamma_1^2}\right). \quad (73)
\]

Within the class (C1) the second variation of energy has the form
\[
\delta^2 E = (2\tilde{\alpha}_1 + \gamma_2 + 3\beta a_1^2)(\delta a_1)^2 + (2\tilde{\alpha}_1 - \gamma_2 + 6\beta a_{-1}^2)(\delta a_{-1})^2 +
(2\tilde{\alpha}_1 - \gamma_2 + 2\beta a_0^2)(\delta b_1)^2 + (\tilde{\alpha}_0 + 3\beta a_0^2)(\delta a_0)^2 - 2\gamma_1\delta a_+\delta a_0.
\] (74)
and the stability conditions read

\[ 2\tilde{\alpha}_0 - \delta + \gamma_2 + 3\beta a_+^2 > 0, \quad (75) \]
\[ 2\tilde{\alpha}_0 - \delta - \gamma_2 + 6\beta a_+^2 > 0, \quad (76) \]
\[ \tilde{\alpha}_0 + 3\beta a_0^2 > 0, \quad (77) \]
\[ (2\tilde{\alpha}_0 - \delta - \gamma_2 + 6\beta a_+^2)(\tilde{\alpha}_0 + 3\beta a_0^2) - \gamma_1^2 > 0, \quad (78) \]

The conditions of positive definiteness of the second variations for the class (C2)

\[ \delta^2 E = (2\tilde{\alpha}_1 + \gamma_2 + 3\beta(a_+^2 + a_-^2))(\delta a_-)^2 + (2\tilde{\alpha}_1 - \gamma_2 + 3\beta(2a_+^2 + a_-^2))(\delta a_+)^2 + \]
\[ (2\tilde{\alpha}_1 - \gamma_2 + 2\beta(a_+^2 + a_-^2))(\delta b_1)^2 + (\tilde{\alpha}_0 + 3\beta a_0^2)(\delta a_0)^2 - \]
\[ 2\gamma_1 \delta a_+ \delta a_0 + 6\beta a_+ a_- \delta a_+ \delta a_-, \quad (79) \]

and for the class (B)

\[ \delta^2 E = (2\tilde{\alpha}_1 + \gamma_2 + \beta(3a_+^2 + 2b_1^2))(\delta a_-)^2 + (2\tilde{\alpha}_1 - \gamma_2 + 2\beta(3a_+^2 + b_1^2))(\delta a_+)^2 + \]
\[ (2\tilde{\alpha}_1 - \gamma_2 + 2\beta(a_+^2 + 3b_1^2))(\delta b_1)^2 + (\tilde{\alpha}_0 + 3\beta a_0^2)(\delta a_0)^2 - \]
\[ 2\gamma_1 \delta a_+ \delta a_0 + 2\beta a_+ b_1 \delta a_+ \delta b_1, \quad (80) \]

are necessary and sufficient to ensure the stability in the corresponding classes. Those modes, however, do not appear at the onset of the superconductivity in the system under consideration.

## 7 Conclusion

1. The isolated atomic planes are all described by the model 2D GL with the same parameters \( \alpha_0 \) and \( \beta \). The parameter \( \alpha_0 \) depends on the temperature. If \( \alpha_0 > 0 \), then the 2D state of such a plane is N (normal), in the opposite case it is the state S (superconducting). We interpret the parameter \( \alpha_0 \) as a measure of empiric temperature, and introduce the notation \( \tau = \tau_0 + \alpha_0 \). The
result for an isolated plane: state N for $\tau > \tau_0$, state S for $\tau < \tau_0$.

2. Each material is characterised by two empiric temperatures $\tau_A$ and $\tau_C$ defined by the formulae

$$\tau_A = \tau_0 - \zeta_1 + \frac{1}{2}(\delta - \gamma_2),$$

(81)

and

$$\tau_C = \tau_0 - \zeta_1 + \alpha_{02},$$

(82)

where $\alpha_{02}$ is given by the eqn. (42). As long as $\tau > \max(\tau_A, \tau_C)$, the normal state is stable. Below this limit a stable superconducting mode appears. If $\tau_A > \tau_C$, the mode is A; in the opposite case - it is the mode C. The other superconducting modes do not appear in the onset.

3. On the material plane ($\gamma_2, \delta$) the limiting curve between the regions of onset A and onset C is placed at the half-plane $\gamma_2 < 0$, and given by the formula

$$\delta = \gamma_2 - \gamma_2^2.$$

(83)

4. The solutions belonging to the class (A) exist in the whole plane ($\gamma_2, \delta$). The unique condition for their existence is $\tilde{\alpha}_0 < \frac{1}{2}(\delta - \gamma_2)$, hence $\tau < \tau_A$. This condition ensures also the internal stability of the solutions.

5. Under the above circumstances the necessary and sufficient condition for the stability of solutions from the class (A) is their C-stability expressed by the eqn. (71). However, C-stability of the solutions from the class (A) is possible only for the materials fulfilling the inequality (72).

6. Then the necessary condition of such stability is $\tilde{\alpha}_0 > \alpha_1$, (with $\alpha_1$ given by (73)), hence $\tau > \tau_{CA}$. Violation of the condition implies the loss of stability on behalf of the mode (C2).

7. If material is of the type (A), the solutions from the class (A) are stable in the interval $\alpha_1 < \tilde{\alpha}_0 < \alpha_{02}$, hence in the interval $\tau_{CA} < \tau > \tau_A$.

8. If the material is of the type (C), then the solutions from the class (A) are stable in the interval $\alpha_1 < \tilde{\alpha}_0 < \alpha_2$, hence $\tau_{CA} < \tau < \tau_{AC}$. At the both
ends the stability is lost on behalf of the mode (C2).

9. Now we shall discuss the enhancement of superconductivity in the system under consideration. We consider the enhancement from the following two points of view:
   a) The enhancement due to long distance coupling with respect to the short distance ones,
   b) The enhancement of 3D superconductivity with respect to the 2D.

10. Consider first increments of the onset temperatures \( \tau_A \) and \( \tau_C \) due to long distance couplings. Let \( \tau_{A0} \) and \( \tau_{C0} \) denote the onset temperatures \( \tau_A \) and \( \tau_C \) for the first grade material, i.e. for \( \zeta_2 = 0 \) and \( \gamma_2 = 0 \). Then, from the equations (81)–(82) it follows that

   \[
   \tau_A - \tau_{A0} = -\frac{1}{2}(\zeta_2 + \gamma_2),
   \]

   \[
   \tau_C - \tau_{C0} = -\frac{1}{4}[\sqrt{(\zeta_1 - (\zeta_2 - \gamma_2))^2 + 8\gamma_2^2} - \sqrt{\zeta_1^2 + 8\gamma_1^2 + (\zeta_2 - \gamma_2)}].
   \]

11. In consequence of the above formulae, the mode A superconductivity is enhanced provided that

   \[\zeta_2 + \gamma_2 < 0.\]

   Similarly, due to the fact, that the right hand side of the eqn. (85) is a monotically increasing function of the difference \( \zeta_2 - \gamma_2 \), the mode C superconductivity is enhanced provided that

   \[\zeta_2 - \gamma_2 > 0.\]

   In consequence, negative values of \( \gamma_2 \) are in favour of both the modes A and C. At the same time, negative values of \( \zeta_2 \) enhance the mode A and tend to suppress the mode C of superconductivity.

12. For some materials, characterised by appropriate values of the parameters \( \zeta_1, \zeta_2 \) and \( \gamma_2 \), the formulae (81) and (82) can give the onset temperatures \( \tau_A \) and/or \( \tau_C \) greater than the 2D critical temperature \( \tau_0 \). In such a situation the out-of-plane superconductivity appears in spite of the fact that all the layers remain in overcritical in-plane states.
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