HIGHER VARIATIONS FOR FREE LÉVY PROCESSES

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ABSTRACT. For a general free Lévy process, we prove the existence of its higher variation processes as limits in distribution, and identify the limits in terms of the Lévy-Itô representation of the original process. For a general free compound Poisson process, this convergence holds in probability. This implies joint convergence in distribution to a k-tuple of higher variation processes, and so the existence of k-fold stochastic integrals as limits in probability. If the existence of moments of all orders is assumed, the result holds for free additive (not necessarily stationary) processes and more general approximants.

In the appendix we note relevant properties of symmetric polynomials in non-commuting variables.

1. INTRODUCTION

A free (additive) Lévy process (in law; we will typically omit this qualifier) is a family of self-adjoint random variables \( \{X(t) : t \geq 0\} \) affiliated to a non-commutative probability space \((\mathcal{A}, \tau)\) which starts at zero, has free, stationary increments, and is stochastically continuous:

(a) \( X(0) = 0 \),
(b) For all \( n \in \mathbb{N} \) and \( t_0 < t_1 < \ldots < t_n \),
\[ \{X(t_0), X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})\} \]
are free,
(c) The distribution of the increment \( X(t + h) - X(t) \) depends only on \( h \) (and will be denoted \( \mu_h \)),
(d) For all \( \varepsilon > 0 \), \( \lim_{h \to 0} \mu_h(|x| > \varepsilon) = 0 \).

The distributions of increments of a free Lévy process form a semigroup with respect to the additive free convolution \( \boxplus \), and so are \( \boxplus \)-infinitely divisible. This implies that the Voiculescu transform of the distribution \( \mu_t \) of \( X(t) \) has the form

\[
\varphi_{\mu_t}(z) = t\eta + tz + t \int_{\mathbb{R}} \left[ \frac{z^2}{z-x} - z - x \mathbf{1}_{[-1,1]}(x) \right] d\rho(x),
\]

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where \( \eta \in \mathbb{R} \), \( a \in \mathbb{R}_+ \), and \( \rho \) is a Lévy measure. Barndorff-Nielsen and Thorbjørnsen proved that a free Lévy process has a free Lévy-Itô decomposition.

**Theorem 1** (Theorems 6.4, 6.5 in \cite{BNT05}). Let \( \{X(t) : t \geq 0\} \) be a free Lévy process, with the generating triple \((\eta, a, \rho)\) as above. Then, \( X(t) \) is equal in distribution to a sum of three freely independent parts. In general,

\[
(2) \quad X(t) \overset{d}{=} \eta t 1_{\mathcal{A}^0} + \sqrt{a} S(t) + \lim_{\epsilon \searrow 0} \left( \int_{(0,t] \times \{|x| > \epsilon\}} x dM(t, x) - \int_{(0,t] \times \{\epsilon < |x| \leq 1\}} x (\text{Leb} \otimes \rho)(dt, dx) 1_{\mathcal{A}^0} \right).
\]

In particular, when \( \int_{[-1,1]} |x| \rho(dx) \) is finite and \( \tilde{\eta} := \eta - \int_{-1}^{1} x \rho(dx) \), then

\[
(3) \quad X(t) \overset{d}{=} \tilde{\eta} t 1_{\mathcal{A}^0} + \sqrt{a} S(t) + \int_{(0,t] \times \mathbb{R}} x dM(t, x).
\]

Here, \( S(t) \) is the free Brownian motion (in some \( W^* \)-probability space \((\mathcal{A}^0, \tau^0)\)) and \( M \) is a free Poisson random measure on the measure space \((\mathbb{R}_+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}), \text{Leb} \otimes \rho)\) with values in \((\mathcal{A}^0, \tau^0)\). The limit is taken in probability.

In the representation in the theorem above, define the \( k \)'th variation of the process by \( X^{(1)}(t) = X(t) \) and for \( k \geq 2 \),

\[
(4) \quad X^{(k)}(t) = at \delta_{k,2} 1_{\mathcal{A}} + \int_{(0,t] \times \mathbb{R}} x^k dM(t, x).
\]

We will show that these objects are well defined, and again form a free Lévy process. Later in the article we will define the corresponding object when \( x^k \) is replaced by a more general function \( p(x) \).

Our first main result concerns convergence in distribution to a higher variation process.

**Theorem 2.** For each \( N \in \mathbb{N} \), let \( \{X_{i,N} : i \in \mathbb{N}\} \) be free, identically distributed, self-adjoint random variables affiliated to \((\mathcal{A}, \tau)\). Suppose that for \( t \geq 0 \),

\[
\lim_{N \to \infty} \sum_{i=1}^{[Nt]} X_{i,N} \overset{d}{=} X(t).
\]

Then for each \( k \),

\[
\lim_{N \to \infty} \sum_{i=1}^{[Nt]} X_{i,N}^k \overset{d}{=} X^{(k)}(t),
\]

the limits being taken in distribution.

We next discuss joint convergence in distribution. In the non-commutative case, there is at this point no universally accepted definition of this notion. Recall the following.
**Definition 1.** A family of self-adjoint operators \((a_{1,N}, \ldots, a_{k,N})\) affiliated to a non-commutative probability space \((\mathcal{A}, \tau)\) converges to \((a_1, \ldots, a_k)\) jointly in moments if for any non-commutative self-adjoint polynomial \(P(x_1, \ldots, x_k)\),
\[
\tau[P(a_{1,N}, \ldots, a_{k,N})] \rightarrow \tau[P(a_1, \ldots, a_k)]
\]
The family converges jointly in distribution if for any \(P\) as above,
\[
P(a_{1,N}, \ldots, a_{k,N}) \rightarrow P(a_1, \ldots, a_k)
\]
in distribution (see \[MS13\] for a related notion).

Recall that convergence in distribution and convergence in moments coincide for bounded operators, but in general neither implies the other.

The next result applies to free additive processes whose increments are not necessarily stationary.

**Theorem 3.** For each \(N \in \mathbb{N}\), let \(\{X_{i,N} : i \in \mathbb{N}\}\) be free self-adjoint random variables affiliated to \((\mathcal{A}, \tau)\) all of whose moments are finite. Suppose that for \(t \geq 0\),
\[
\sum_{i=1}^{[Nt]} X_{i,N}
\]
converges in moments to \(X(t)\) as \(N \to \infty\). Suppose in addition that

\[
\sum_{i=1}^{N} \tau[X_{i,N}^k]^2 \to 0
\]
as \(N \to \infty\), for all \(k\). Then there exist free additive processes \(\{X^{(j)}(t)\}\) such that we have joint convergence in moments
\[
\left(\sum_{i=1}^{[Nt]} X_{i,N}, \sum_{i=1}^{[Nt]} X_{i,N}^2, \ldots, \sum_{i=1}^{[Nt]} X_{i,N}^k\right) \rightarrow \left(X(t), X^{(2)}(t), \ldots, X^{(k)}(t)\right)
\]
as \(N \to \infty\).

**Remark 1.** For triangular arrays of centered random variables with finite variance, the standard condition for convergence is \(\max_{1 \leq i \leq N} \tau[X_{i,N}^2] \to 0\) and \(\sum_{i=1}^{N} \tau[X_{i,N}^2] \leq c < \infty\), see for example Section 22 in \[Loe77\]. The assumption \([5]\) is clearly significantly stronger. On the other hand, it is significantly weaker that assuming that all \(X_{i,N}\) are identically distributed. In the latter case, the result follows from the limit theorem 13.1 in \[NS06\], itself based on a result of Speicher \[Spe90\].
The second case where we can prove joint convergence is when individual convergence holds in probability.

**Theorem 4.** Let $\rho$ be a finite probability measure, and

$$X(t) = \int_{(0,t] \times \mathbb{R}} x \, dM(t, x)$$

the corresponding free compound Poisson process. Then for $X_{i,N} = X\left(\tfrac{i}{N}\right) - X\left(\tfrac{i-1}{N}\right)$, we have

$$\lim_{N \to \infty} \sum_{i=1}^{[Nt]} X_{i,N}^k = X^{(k)}(t),$$

the limit being taken in probability.

We expect similar convergence for general free Lévy processes. At this point we have the following partial result.

**Theorem 5.** Let $\{X(t) : t \geq 0\}$ be a free Lévy process whose increments have symmetric distributions, and $X_{i,N} = X\left(\tfrac{i}{N}\right) - X\left(\tfrac{i-1}{N}\right)$. Then

$$\lim_{N \to \infty} \sum_{i=1}^{[Nt]} X_{i,N}^2 = X^{(2)}(t),$$

the limit being taken in probability.

**Corollary 6.** For a free compound Poisson process $\{X(t) : t \geq 0\}$ and increments $X_{i,N}$ as above, we have joint convergence in distribution

$$\left(\sum_{i=1}^{[Nt]} X_{i,N}, \sum_{i=1}^{[Nt]} X_{i,N}^2, \ldots, \sum_{i=1}^{[Nt]} X_{i,N}^k\right) \rightarrow (X(t), X^{(2)}(t), \ldots, X^{(k)}(t))$$

as $N \rightarrow \infty$.

**Corollary 7.** Let $\{X_{i,N} : 1 \leq i \leq N, N \in \mathbb{N}\}$ be as in either Theorem 3 or in Corollary 6. Then for $t \geq 0$,

$$\lim_{N \to \infty} \sum_{1 \leq i(1), i(2), \ldots, i(k) \leq [Nt]} X_{i(1),N}X_{i(2),N} \ldots X_{i(k),N} = \sum_{j=1}^{k} (-1)^{k-j} \sum_{m_1, \ldots, m_j \geq 1 \atop m_1 + \ldots + m_j = k} X^{(m_1)}(t) \ldots X^{(m_j)}(t).$$

Here under the assumptions of Theorem 3 the limit is in moments, while under the assumptions of Corollary 6 the limit is in probability, and so also in distribution.
It was shown in Proposition 1 of [Ans00] that for free Lévy processes with bounded, centered increments, the limits (in norm) of the left-hand side of (6) and of

\[
\sum_{1 \leq i(1),i(2),\ldots,i(k) \leq N t \atop \{(i(1),i(2),\ldots,i(k))\}=k} X_{i(1),N} X_{i(2),N} \ldots X_{i(k),N}.
\]

coincide. These limits should be interpreted as the free stochastic integral

\[
\int_{[0,t]^k} dX(s_1) \ldots dX(s_k).
\]

See the end of the introduction, and the appendix, for the explanation of why the expression (6) is more appropriate in the free case.

**Prior results.** The initial motivation for our analysis was the article [AT86] by Avram and Taqqu. We briefly compare some of their results with ours; the reader should consult their article for more details. Let \( \{X(t)\} \) be a Lévy process, and define its higher variations pathwise using jumps. Note that such a definition is unavailable in the non-commutative case. Representation (4), which we use instead, is closely related to Theorem 36 in section I of [Pro90] (where it is stated only for the jumps of a Lévy process), and is (as Protter points out) obvious in the classical case. Let \( \{X_{i,N} : 1 \leq i \leq N, N \in \mathbb{N}\} \) be a triangular array with i.i.d. rows, such that

\[
\sum_{i=1}^{N} X_{i,N} \rightarrow X(t)
\]

in distribution as \( N \rightarrow \infty \). Then a multivariate limit theorem implies that

\[
\sum_{i=1}^{[N t]} (X_{i,N}, X_{i,N}^2, \ldots, X_{i,N}^k) \rightarrow (X(t), X^{(2)}(t), \ldots, X^{(k)}(t))
\]

jointly in distribution. At this point, in the non-commutative case such a theorem is only available for convergence in moments. On the other hand, we actually prove Theorem 2 not just for powers but for polynomials, that is, linear combinations of powers. For commuting variables, convergence in distribution of linear combinations is equivalent to joint convergence in distribution (an easy exercise left to the reader). So the appropriate commutative analog of Theorem 2 also implies the joint convergence in (8).

Next, recall that the elementary symmetric polynomial

\[
e_k(x_1, \ldots, x_N) = \sum_{1 \leq i(1)<i(2)<\ldots<i(k) \leq N} x_{i(1)} x_{i(2)} \ldots x_{i(k)}
\]
is a polynomial $P_k(p_1, \ldots, p_k)$ in the power sum symmetric polynomials

$$p_j(x_1, \ldots, x_N) = \sum_{i=1}^{N} x_i^j$$

(the polynomial $P_k$ can be written down explicitly). Consequently,

$$\sum_{1 \leq i(1) < i(2) < \cdots < i(k) \leq [Nt]} X_{i(1),N}X_{i(2),N} \cdots X_{i(k),N}$$

$$= P_k \left( \sum_{i=1}^{[Nt]} X_{i,N}, \sum_{i=1}^{[Nt]} X_{i,N}^2, \ldots, \sum_{i=1}^{[Nt]} X_{i,N}^k \right)$$

converges in distribution as $N \to \infty$. Its limit is naturally identified with the multiple integral

$$\int_{0 \leq s_1 < s_2 < \cdots < s_k \leq t} dX(s_1) \, dX(s_2) \cdots dX(s_k).$$

Note that as explained in the appendix, if the variables $\{x_i\}$ do not commute, $e_k$ is not a polynomial in the $p_j$’s. Its natural replacement in the non-commutative setting is

$$\tilde{e}_k(x_1, \ldots, x_N) = \sum_{1 \leq i(1), i(2), \ldots, i(k) \leq N \atop i(1) \neq i(2), i(2) \neq i(3), \ldots, i(k-1) \neq i(k)} x_{i(1)}x_{i(2)} \cdots x_{i(k)}$$

used in equation (6).

Motivated by [RW97], the first author studied related objects in [Ans00], but only for the case of free Lévy processes with compactly supported distributions. We are not aware of other sources where these specific topics are studied in the free probability setting. See however the study of homogeneous sums in [DN14, Sim15].

The article is organized as follows. After the introduction and background in Section 2, Section 3 treats, for general free Lévy processes, convergence in distribution to the higher variation processes, and their generalization from powers to more general continuous functions. The key result is Theorem 18. Section 4 treats joint convergence in moments for more general additive processes. Section 5 contains results about convergence in probability, as well as an alternative definition of joint convergence in distribution for non-commuting variables. Finally, in the appendix we explain which symmetric polynomials in non-commuting variables can be expressed in terms of the basic power sum symmetric polynomials.
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2. Background and the Free Poisson Random Measure

2.1. Unbounded Operators and Affiliated Operators. A $W^*$-probability space is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a von Neumann algebra acting on a Hilbert space and $\tau$ is a faithful normal tracial state on $\mathcal{A}$. Throughout most of the paper, we will work with possibly unbounded operators affiliated to $\mathcal{A}$. A self-adjoint operator $a$ is affiliated to $\mathcal{A}$ if all of its spectral projections are in $\mathcal{A}$. Equivalently, for any bounded Borel function, $f(a) \in \mathcal{A}$. We denote the collection of all self-adjoint operators affiliated to $\mathcal{A}$ by $\tilde{\mathcal{A}}_{sa}$. A general closed, densely defined operator $a$ is affiliated to $\mathcal{A}$ if in its polar decomposition $a = u |a|$, we have $u \in \mathcal{A}$ and $|a| \in \tilde{\mathcal{A}}_{sa}$. The collection of all such operators is denoted by $\tilde{\mathcal{A}}$. Murray and von Neumann [MVN36] proved that $\tilde{\mathcal{A}}_{sa}$ is an algebra, that is, if $a, b \in \tilde{\mathcal{A}}_{sa}$, then $a + b$ and $ab$ are densely defined and closable, and their closures are in $\tilde{\mathcal{A}}$.

For $a \in \tilde{\mathcal{A}}_{sa}$, its distribution is the unique probability measure $\mu_a$ on $\mathbb{R}$ such that for any bounded Borel function,

\[ \tau[f(a)] = \int_{\mathbb{R}} f(x) \, d\mu_a(x). \]

**Definition 2.** ([BNT02]) Let $(\mathcal{A}, \tau)$ be a $W^*$-probability space and $(a_n)_{n \in \mathbb{N}}$ be a sequence of operators affiliated with $\mathcal{A}$. We say that $a_n \to a$ in probability if $|a_n - a| \to 0$ in distribution as $n \to \infty$.

Here, $|a| := \sqrt{a^*a}$, which is self-adjoint. When $a_n$ and $a$ are self-adjoint operators affiliated with $\mathcal{A}$, $a_n \to a$ in probability if and only if $a_n - a$ converges to zero in distribution, i.e. the distribution of $a_n - a$ as a probability measure on $\mathbb{R}$ converges weakly to probability measure $\delta_0$.

We list the following proposition for completeness. See for example Proposition 2.18 in [BNT02].

**Proposition 8.** The following are equivalent.

(a) $a_n \to a$ in probability.

(b) $\forall \varepsilon > 0$, the traces of the spectral projections $\tau[1_{(\varepsilon, \infty)}(|a_n - a|)] \to 0$. 
Denote
\[ \mathcal{N}(\varepsilon, \delta) = \left\{ b \in \tilde{A} : \exists \ projection \ p \in A \ s.t. \ \tau[1 - p] < \delta, bp \in A, \ ||bp|| < \varepsilon \right\}. \]

Then \( \forall \varepsilon, \delta > 0, \) for sufficiently large \( n, \) \( a_n - a \in \mathcal{N}(\varepsilon, \delta). \)

This mode of convergence is also called convergence in measure.

We also recall the following part of Theorem 1 from [Nel74].

**Lemma 9.** In the notation of the preceding proposition,
\[ \mathcal{N}(\varepsilon_1, \delta_1) + \mathcal{N}(\varepsilon_2, \delta_2) \subseteq \mathcal{N}(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2) \]
and
\[ \mathcal{N}(\varepsilon_1, \delta_1) \mathcal{N}(\varepsilon_2, \delta_2) \subseteq \mathcal{N}(\varepsilon_1 \varepsilon_2, \delta_1 + \delta_2). \]
In particular, if \( a_n \to a \) and \( b_n \to b \) in probability, then \( a_n + b_n \to a + b \) and 
\( a_n b_n \to ab \) in probability.

### 2.2. Freely infinitely divisible distributions and limit theorems

As already mentioned in the introduction, a probability measure \( \mu \) on \( \mathbb{R} \) is \( \boxplus \)-infinitely divisible if and only if its Voiculescu transform has a representation

\[ \varphi_\mu(z) = \eta + \frac{a}{z} + \int_{\mathbb{R}} \left[ \frac{z^2}{z - x} - z - x \mathbf{1}_{[-1,1]}(x) \right] d\rho(x), \]

where \( \eta \in \mathbb{R}, \) \( a \in \mathbb{R}^+, \) and \( \rho \) is a Lévy measure, that is,

\[ \rho(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} \min(1, x^2) \ d\rho(x) < \infty. \]

\( \varphi_\mu \) also has an alternative representation

\[ \varphi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1 + xz}{z - x} d\sigma(x). \]

For future reference, we record the relation between the generating triple \((a, \eta, \rho)\) and the generating pair \((\gamma, \sigma)\) for the same measure \( \mu:\)

\[ \begin{cases} 
\sigma(dx) = a\delta_0(dx) + \frac{x^2}{1 + x^2} \rho(dx) \\
\gamma = \eta - \int_{\mathbb{R}} x \left[ \mathbf{1}_{[-1,1]}(x) - \frac{1}{1 + x^2} \right] d\rho(x)
\end{cases} \]

and, conversely,

\[ \begin{cases} 
a = \sigma(\{0\}) \\
\eta = \gamma + \int_{\mathbb{R}\setminus\{0\}} \frac{1 + x^2}{x} \left[ \mathbf{1}_{[-1,1]}(x) - \frac{1}{1 + x^2} \right] d\sigma(x) \\
\rho(dx) = \frac{1 + x^2}{x^2} \mathbf{1}_{\mathbb{R}\setminus\{0\}}(x) \sigma(dx).
\end{cases} \]
The following fundamental limit theorem was proved by Bercovici and Pata in [BP99].

**Theorem 10.** For a sequence of probability measures \( \{\mu_n\} \) and a strictly increasing sequence of positive integers \((k_n)\), the following assertions are equivalent:

(a) the sequence of \( k_n \)-fold free convolutions \( \mu_n^{\boxplus k_n} \) converges weakly to a probability measure \( \mu \);

(b) there exist a finite positive Borel measure \( \sigma \) on \( \mathbb{R} \) and a real number \( \gamma \) such that

\[
\lim_{n \to \infty} k_n \int_{\mathbb{R}} \frac{x^2}{1+x^2} d\mu_n(x) \to d\sigma(x)
\]

and

\[
\lim_{n \to \infty} k_n \int_{\mathbb{R}} \frac{x}{1+x^2} d\mu_n(x) = \gamma.
\]

The pair of parameters \((\gamma, \sigma)\) comes from the Voiculescu transform \([11]\) of \( \mu \). This also implies the \( \boxplus \)-infinite divisibility of \( \mu \).

### 2.3. Free Poisson Random Measures.

**Definition 3** (Free Poisson Random Measures). Let \((\Theta, E, \nu)\) be a measure space and put \( E_0 = \{ E \in E : \nu(E) < \infty \} \). Let further \((\mathcal{A}, \tau)\) be a \( W^* \)-probability space and let \( \mathcal{A}_+ \) denote the cone of positive operators in \( \mathcal{A} \). A free Poisson random measure on \((\Theta, E, \nu)\) with values in \((\mathcal{A}, \tau)\) is a mapping \( M : E_0 \to \mathcal{A}_+ \) with the following properties:

(a) the distribution of \( M(E) \) is a free Poisson distribution \( \text{Poisson}^{\boxplus}(\nu(E)) \);

(b) for mutually disjoint sets \( A_1, \ldots, A_n \) in \( E_0 \), the random variables

\[
M(A_1), M(A_2), \ldots, M(A_n)
\]

are freely independent and \( M(\bigcup_{j=1}^n A_j) = \sum_{j=1}^n M(A_j) \).

Here, the free Poisson distribution \( \text{Poisson}^{\boxplus}(\lambda) \) is obtained by the limit in distribution of

\[
\left( (1 - \frac{\lambda}{N}) \delta_0 + \frac{\lambda}{N} \delta_1 \right)^{\boxplus N},
\]

as \( N \to \infty \) (see Lecture 12 in [NS06]). The existence of free Poisson random measures is proved by Barndorff-Nielsen and Thorbjørnsen in [BNT05]. For an alternative approach, see Remark [3] below.

We next discuss integration with respect to a free Poisson random measure.
**Definition 4.** Let $s$ be a real-valued simple function in $L^1(\Theta, \mathcal{E}, \nu)$ of the form $s = \sum_{j=1}^{r} a_j 1_{E_j}$, where $a_j \in \mathbb{R} \setminus \{0\}$ and $E_j$ are disjoint sets from $\mathcal{E}_0$. Then, we define the integral of $s$ with respect to $M$ as

$$\int_{\Theta} s\,dM = \sum_{j=1}^{r} a_j M(E_j) \in \mathcal{A}.$$ 

Because $M(E_j)$ are positive in $\mathcal{A}$, the element $\int_{\Theta} s\,dM$ is self-adjoint in $\mathcal{A}$, for any real-valued simple function in $L^1(\Theta, \mathcal{E}, \nu)$. Next, we can extend this integration to general functions in $L^1(\Theta, \mathcal{E}, \nu)$.

**Lemma 11.** [BNT05, Proposition 4.3] Let $f$ be a real-valued function in the space $L^1(\Theta, \mathcal{E}, \nu)$. Choose a sequence of real-valued simple functions $(s_n)$ in $L^1(\Theta, \mathcal{E}, \nu)$ which satisfies the assumptions of the Dominated Convergence Theorem, such that $s_n(\theta) \to f(\theta)$, for all $\theta \in \Theta$. Then, $\int_{\Theta} s_n\,dM$ converges in probability to a self-adjoint (possibly unbounded) operator affiliated with $\mathcal{A}$. This operator is independent of the choice of approximating sequence $(s_n)$. We denote this operator by $\int_{\Theta} f\,dM$.

The proof of the following lemma follows by the same techniques as Proposition 4.3 and Corollary 4.5 in [BNT05].

**Lemma 12.** Let $f$ be a real-valued function in $L^1(\Theta, \mathcal{E}, \nu)$. Choose a sequence of real-valued functions $(f_n)$ in $L^1(\Theta, \mathcal{E}, \nu)$ which satisfies the assumptions of the Dominated Convergence Theorem, such that $f_n(\theta) \to f(\theta)$, for all $\theta \in \Theta$. Then, $\int_{\Theta} f_n\,dM$ converges in probability to $\int_{\Theta} f\,dM$.

In fact, we only use a special measure space with a concrete intensity measure in our situation. Let $D = \mathbb{R}_+ \times \mathbb{R}$ and $\mathcal{B}(D)$ be the set of all Borel subsets of $D$. In our case,

$$(\Theta, \mathcal{E}, \nu) = (D, \mathcal{B}(D), \text{Leb} \otimes \rho),$$

where $\rho$ is a Lévy measure. The free Poisson random measure $M$ that we will use is defined on $(D, \mathcal{B}(D), \text{Leb} \otimes \rho)$ with values in a $W^*$-probability space $(\mathcal{A}, \tau)$. Besides, the integration with respect to this free Poisson measure $M$ we will use is also a special case.

**Lemma 13.** Let $\rho$ be a Lévy measure on the real line, and let $M$ be a free Poisson random measure on $(D, \mathcal{B}(D), \text{Leb} \otimes \rho)$ with values in the $W^*$-probability space $(\mathcal{A}, \tau)$. Suppose that $p(x)$ is any continuous function on $\mathbb{R}$. 
(a) For any $\epsilon > 0$ and $0 \leq s < t < \infty$, the integral
\[
\int_{(s,t] \times \{\epsilon < |x| \leq n\}} p(x) M(dt, dx)
\]
converges in probability, as $n \to \infty$, to some self-adjoint operator affiliated with $A$, which is denoted by
\[
\int_{(s,t] \times \{\epsilon < |x| < \infty\}} p(x) M(dt, dx).
\]

(b) If $\int_{[-1,1]} |p(x)|\rho(dx) < \infty$, then for any $\epsilon > 0$ and $0 \leq s < t < \infty$, the integral
\[
\int_{(s,t] \times \{|x| \leq n\}} p(x) M(dt, dx)
\]
converges in probability to some self-adjoint operator affiliated with $A$, as $n \to \infty$. We denote it by
\[
\int_{(s,t] \times \mathbb{R}} p(x) M(dt, dx).
\]

The statement of Lemma 13 is quite similar with Lemma 6.3 of [BNT05]. In the paper [BNT05], the authors only proved the situation when $p(x) = x$ but their methods in Lemma 6.1 and Lemma 6.2 of [BNT05] still work well for Lemma 13. According to Lemma 6.3 of [BNT05], there are only two things for us to check. Since $\rho$ is a Lévy measure, we have that
\[
\int_{(s,t] \times \{\epsilon < |x| \leq n\}} |p(x)| \text{Leb} \otimes \rho(du, dx) = (t - s) \int_{\{\epsilon < |x| \leq n\}} |p(x)| \rho(dx) < \infty.
\]

If $\int_{[-1,1]} |p(x)|\rho(dx) < \infty$, we have that
\[
\int_{(s,t] \times \{|x| \leq n\}} |p(x)| \text{Leb} \otimes \rho(du, dx)
\]
\[
= (t - s) \left[ \int_{\{|x| \leq 1\}} |p(x)| \rho(dx) + \int_{\{1 < |x| \leq n\}} |p(x)| \rho(dx) \right] < \infty.
\]

Thus, integrals $\int_{(s,t] \times \{\epsilon < |x| \leq n\}} p(x) M(dt, dx)$ and $\int_{(s,t] \times \{|x| \leq n\}} p(x) M(dt, dx)$ are well-defined by Proposition 4.3 of [BNT05]. Then, we can copy the proof of Lemma 6.3 of [BNT05] and replace the function $f(x) = x$ by arbitrary continuous function $p(x)$ directly to prove Lemma 13. The idea for proving Lemma 6.3 is employing the Bercovici-Pata bijection to transform the statement into classical sense and then using Lebesgue’s dominated convergence theorem.
3. THE HIGHER VARIATIONS OF FREE LÉVY PROCESSES

**Proposition 14.** If there exist a finite Borel measure $\sigma$ and a constant $\gamma$ such that

$$N \frac{x^2}{x^2 + 1} d\mu_N(x) \xrightarrow{w} d\sigma(x)$$

and

$$\lim_{N \to \infty} N \int_{\mathbb{R}} \frac{x}{1 + x^2} d\mu_N(x) = \gamma,$$

then there exists a family $\{\mu_t\}_{t \geq 0}$ of probability measures on $\mathbb{R}$ such that

$$\mu_N \ast [Nt] \xrightarrow{w} \mu_t,$$

for any $t \in [0, \infty)$. Each $\mu_t$ is $\boxplus$-infinitely divisible and its Voiculescu transform is $\varphi_{\mu_t}(z) = t\gamma + t \int_{\mathbb{R}} \frac{1 + xz}{x - z} d\sigma(x) = t \varphi_{\mu}(z)$, where $\mu := \mu_1$ is the distribution of $X(1)$.

Moreover, there exists a free Lévy process $\{X(t)\}_{t \geq 0}$ such that the distribution of each $X(t)$ is $\mu_t$, for all $t \geq 0$.

**Proof.** By Theorem [10] we know that if there exist a finite Borel measure $\sigma$ and a constant $\gamma$ such that (16) and (17) hold, then $\mu_N \xrightarrow{w} \mu_1$. For any $t \in [0, \infty)$, we have that

$$[Nt] \frac{x^2}{x^2 + 1} d\mu_N(x) \xrightarrow{w} t d\sigma(x) =: d\sigma_t(x)$$

and

$$\lim_{N \to \infty} [Nt] \int_{\mathbb{R}} \frac{x}{1 + x^2} d\mu_N(x) = t \lim_{N \to \infty} N \int_{\mathbb{R}} \frac{x}{1 + x^2} d\mu_N(x) = t \gamma =: \gamma_t.$$

Therefore, for any $t \in [0, \infty)$, there exists a probability measure $\mu_t$ such that $\mu_N \xrightarrow{w} \mu_t$. According to Theorem [10], for any $t \in [0, \infty)$, $\mu_t$ is $\boxplus$-infinitely divisible since the Voiculescu transform of $\mu_t$ is

$$\varphi_{\mu_t}(z) = \gamma_t + \int_{\mathbb{R}} \frac{1 + xz}{x - z} d\sigma_t(x) = t \varphi_{\mu}(z),$$

where $\mu := \mu_1$. Therefore, $\varphi_{\mu_t} = \varphi_{\mu_t - s} + \varphi_{\mu_s}$, when $t > s \geq 0$. In other words, $\mu_t = \mu_{t - s} \boxplus \mu_s$. Meanwhile, $\varphi_{\mu_t} \to 0$ when $t \to 0$, which means $\mu_t \xrightarrow{w} \delta_0$, as $t \to 0$. Then, by Remark 6.7 in [BNT05], we can conclude that there exists a free Lévy process $\{X(t)\}_{t \geq 0}$, which is a family of self-adjoint operators affiliated with some $W^*$-probability space $(\mathcal{A}^0, \tau^0)$, such that the distribution of each $X(t)$ is $\mu_t$, for all $t \geq 0$. \hfill \Box

**Lemma 15.** Let $(\mathcal{A}, \tau)$ be a $W^*$-probability space. Let $a \in \tilde{\mathcal{A}}_{sa}$ with distribution $\mu$, and $p(x)$ be a continuous real-valued function. Then the distribution $\mu^{(p)}$ of
operator \( p(a) \) (obtained via continuous functional calculus) can be obtained by the following formula:
\[
\int_{\mathbb{R}} f(p(x))d\mu(x) = \int_{\mathbb{R}} f(x)d\mu^{(\rho)}(x),
\]
for any bounded Borel function \( f : \mathbb{R} \to \mathbb{R} \).

**Proof.** By definition of the distribution, for any bounded Borel function \( f : \mathbb{R} \to \mathbb{R} \),
\[
\tau(f(a)) = \int_{\mathbb{R}} f(x)d\mu(x).
\]
Then, \( f \circ p(x) \) is still a bounded Borel function. Thus,
\[
\int_{\mathbb{R}} f(p(x))d\mu(x) = \tau(f(p(a))) = \int_{\mathbb{R}} f(x)d\mu^{(\rho)}(x).
\]
\[\square\]

Generally, Lemma 15 shows how to change variables between different probability measures.

Note the difference between the notation \( \mu^{(\rho)} \) in the preceding lemma and \( \rho^p \) in the following one.

**Lemma 16.** Let \( p(x) \) be any real-valued continuous function such that \( p(0) = 0 \) and \( p'(0) \) exists. Suppose that \( M \) is a free Poisson random measure determined by a Lévy measure \( \rho \) on the Borel measure space \((D, \mathcal{B}(D), \text{Leb} \otimes \rho)\) with values in some \( W^* \)-probability space \( A \). If \( \rho^{p} \) is another measure defined by
\[
(18) \quad \int_{\mathbb{R}} f(x)d\rho^{p}(x) = \int_{\mathbb{R}} f(p(x))1_{\mathbb{R}\setminus\{0\}}(p(x))d\rho(x),
\]
for any bounded Borel function \( f(x) \) on \( \mathbb{R} \), then \( \rho^{p} \) is a Lévy measure. The free Poisson random measure \( M^{(\rho)} \) defined by \( \rho^p \) on \((D, \mathcal{B}(D), \text{Leb} \otimes \rho^{p})\) has the following relation with \( M \):
\[
(19) \quad \int_{(0,t] \times \{\epsilon < |x|\}} xdM^{(\rho)}(t, x) \overset{d}{=} \int_{(0,t] \times \{|p(x)|\}} p(x)dM(t, x),
\]
for any \( t, \epsilon > 0 \), and
\[
(20) \quad \int_{(0,t] \times \mathbb{R}} xdM^{(\rho)}(t, x) \overset{d}{=} \int_{(0,t] \times \mathbb{R}} p(x)dM(t, x), \quad \forall t > 0,
\]
provided that \( \int_{[-1,1]} |x|d\rho^{p}(x) < \infty \).

**Proof.** Since \( p(0) = 0 \), there exists an \( \epsilon > 0 \) such that \( |p(x)| \leq 1 \) when \( |x| \leq \epsilon \).
Since \( p'(0) \) exists, the function
\[
h(x) := \begin{cases} 
\frac{p(x)}{x}, & x \neq 0 \\
p'(0), & x = 0,
\end{cases}
\]
is continuous on $\mathbb{R}$. First, we show that $\rho^p$ is a Lévy measure. If $f(x) = 1_{\{0\}}(x)$, then $\rho^p(\{0\}) = \int_{\mathbb{R}} 1_{\{0\}}(x)d\rho^p(x)$ is zero by the definition (18). Next, if $f(x) = \min\{1, x^2\}$, then we can get the following conclusion:

$$\int_{\mathbb{R}} \min\{1, x^2\}d\rho^p(x)$$

$$= \int_{\mathbb{R}} 1_{[-1,1]}(x)x^2d\rho^p(x) + \int_{\mathbb{R}} 1_{\mathbb{R}\setminus[-1,1]}(x)d\rho^p(x)$$

$$= \int_{\mathbb{R}} 1_{[-1,1]\setminus\{0\}}(p(x))(p(x))^2dp(x) + \int_{\mathbb{R}} 1_{\mathbb{R}\setminus[-1,1]}(p(x))d\rho(x)$$

$$\leq \int_{\{x\in\mathbb{R}:|p(x)|<1\}} p(x)^2dp(x) + \int_{-\varepsilon}^{\varepsilon} h(x)^2x^2d\rho(x) + \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} 1d\rho(x)$$

$$\leq \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} 1d\rho(x) + \max_{-\varepsilon< x< \varepsilon} |h(x)|^2 \int_{-1}^{1} x^2d\rho(x) + \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} 1d\rho(x) < \infty.$$  

Therefore, $\rho^p$ is a Lévy measure.

Second, we show that the relation (20) holds. If $\int_{[-1,1]} |x|d\rho^p(x)$ is finite, then so is $\int_{-1}^{1} |p(x)|d\rho(x)$, since

$$\int_{-1}^{1} |p(x)|d\rho(x) = \int_{\mathbb{R}} 1_{[-1,1]\setminus\{0\}}(p(x)) \cdot |p(x)|d\rho(x) = \int_{-1}^{1} |p(x)|d\rho(x) < \infty.$$  

Thus, the right-hand side and left-hand side of (20) make sense by Lemma [13]. According to Lemma [13] we only need to show that

$$\int_{\{x\in\mathbb{R}:|p(x)|<n\}} x\cdot dM^{(p)}(t, x) = \int_{\{x\in\mathbb{R}:|p(x)|<n\}} p(x)\cdot dM(t, x),$$  

for all $t \geq 0$ and $n \in \mathbb{N}$. For any $N \in \mathbb{N}$, consider mutually disjoint intervals

$$E_m^N = \left[-n + \frac{2nm(m-1)}{N}, -n + \frac{2nm}{N}\right),$$  

where $1 \leq m \leq N$ and $m \in \mathbb{N}$. Then, the simple functions

$$s_N(x) = \sum_{m=1}^{N} \left(-n + \frac{2nm(m-1)}{N}\right)1_{E_m^N}(x)$$  

converge to $f(x) = x$, for any $x \in [-n, n)$ as $N \to \infty$. Thus,

$$\int_{\{x:|p(x)|<n\}} s_N(x)dM^{(p)}(t, x) \to \int_{\{x:|p(x)|<n\}} x\cdot dM^{(p)}(t, x)$$  

in probability. Let

$$J_m^N = \{x : p(x) \in E_m^N\}, (1 \leq m \leq N, m \in \mathbb{N}).$$  

Then, $\bigcup_{m=1}^{N} J_m^N = \{x : -n \leq |p(x)| < n\}$ and $\{J_m^N\}$ are mutually disjoint. The simple functions

$$g_N(x) = \sum_{m=1}^{N} \left(-n + \frac{2nm(m-1)}{N}\right)1_{J_m^N}(x) \to p(x)$$  

for any $x \in [-n, n)$. Thus, we can conclude that $\rho^p$ is a Lévy measure.
for any \( x \in \{ x : -n \leq |p(x)| < n \} \), as \( N \to \infty \). Therefore, when \( N \to \infty \),
\[
\int_{(0,t) \times \{ x : -n \leq |p(x)| < n \}} g_N(x) dM(t, x) \to \int_{(0,t) \times \{ x : -n \leq |p(x)| < n \}} p(x) dM(t, x)
\]
in probability. We conclude that it suffices to show the equality in distribution
\[
\int_{(0,t) \times \{ x : -n \leq |p(x)| < n \}} g_N(x) dM(t, x) \overset{d}{=} \int_{(0,t) \times \{ x : -n \leq |p(x)| < n \}} s_N(x) dM^{(p)}(t, x).
\]
Let \( F^N_m = (0, t] \times E^N_m \) and \( K^N_m = (0, t] \times J^N_m \). By Definition \( 4 \) we know that
\[
\int_{(0,t) \times \{ x : -n \leq x < n \}} s_N(x) dM^{(p)}(t, x) = \sum_{m=1}^{N} \left( -n + \frac{2n(m - 1)}{N} \right) M^{(p)}(F^N_m),
\]
and
\[
\int_{(0,t) \times \{ x : -n \leq p(x) < n \}} g_N(x) dM(t, x) = \sum_{m=1}^{N} \left( -n + \frac{2n(m - 1)}{N} \right) M(K^N_m).
\]
By Definition \( 3 \), the distribution of \( M(K^N_m) \) is \( \text{Poiss}^m(t \rho(J^N_m)) \) and the distribution of \( M^{(p)}(F^N_m) \) is \( \text{Poiss}^m(t \rho^p(E^N_m)) \). According to (18), we conclude that, when \( m \neq \frac{N}{2} + 1, 0 \notin E^N_m \), so
\[
\rho^p(E^N_m) = \int_R 1_{E^N_m}(x) d\rho^p(x) \\
= \int_R 1_{E^N_m \setminus \{0\}}(p(x)) d\rho(x) \\
= \int_R 1_{\{x : p(x) \in E^N_m\}}(x) d\rho(x) = \rho(J^N_m).
\]
So, \( \text{Poiss}^m(t \rho(J^N_m)) = \text{Poiss}^m(t \rho^p(E^N_m)), m \neq \frac{N}{2} + 1 \). Notice that the coefficients in front of \( M(K^N_m) \) and \( M^{(p)}(F^N_m) \) are zero. Then, we get the final result (20). In general, for any \( t, \epsilon > 0 \) and \( n \in \mathbb{N} \), we can apply the same method and show that
\[
\int_{(0,t) \times \{ \epsilon < |x| < n \}} x dM^{(p)}(t, x) \overset{d}{=} \int_{(0,t) \times \{ \epsilon < |p(x)| < n \}} p(x) dM(t, x),
\]
to prove equation (19). \( \square \)

**Lemma 17.** Let \( p(x) \) be any real-valued continuous function such that \( p(0) = p'(0) = 0 \) and \( p''(0) = 2c \) exists. Then whether or not \( \int_{-1}^{1} |x| d\rho(x) \) is finite, \( \int_{-1}^{1} |x| d\rho^p(x) \) is finite.

**Proof.** Denote
\[
q(x) := \begin{cases} \frac{p(x)}{x}, x \neq 0 \\ c, x = 0. \end{cases}
\]
Then $q(x)$ is a continuous function. So we can check that
\[
\int_{-1}^{1} |x|d\rho^p(x)
\]
\[
= \int_{\mathbb{R}} 1_{[-1,1]}(x) |x|d\rho^p(x)
\]
\[
= \int_{\mathbb{R}} 1_{[-1,1]\setminus\{0\}}(p(x))|p(x)|d\rho(x)
\]
\[
= \int_{-1}^{1} 1_{[-1,1]\setminus\{0\}}(p(x))|p(x)|d\rho(x) + \int_{\mathbb{R}\setminus[-1,1]} 1_{[-1,1]\setminus\{0\}}(p(x))|p(x)|d\rho(x)
\]
\[
\leq \|q\|_{C([-1,1])} \int_{-1}^{1} x^2d\rho(x) + \int_{\mathbb{R}\setminus[-1,1]} 1_{\{0<|p(x)|\leq 1\}}(x)|p(x)|d\rho(x)
\]
\[
\leq C \int_{\mathbb{R}} \min\{1, x^2\}d\rho(x) < \infty. \quad \Box
\]

Theorem 2 follows from the following more general result by taking $p(x) = x^k$.

**Theorem 18.** For each $N \in \mathbb{N}$, let $\{X_{i,N} : i \in \mathbb{N}\}$ be free, identically distributed, self-adjoint random variables affiliated to $(\mathbb{A}, \tau)$. Suppose that for $t \geq 0$,
\[
\sum_{r=1}^{[Nt]} X_{N,r} \xrightarrow{d} X(t),
\]
where $X(t)$ is a free Lévy process. Let $p(x)$ be any real-valued continuous function such that $p(0) = 0$, $p'(0) = b$, and $p''(0) = 2c$. Then, there exists a Lévy process $X^p(t)$ such that
\[
\sum_{r=1}^{[Nt]} p(X_{N,r}) \xrightarrow{d} X^p(t).
\]

In addition, if $X(t)$ has the Lévy-Itô decomposition (2) with the generating triple $(\eta, a, \rho)$, then $X^p(t)$ has a representation in the form:
\[
X^p(t) \xrightarrow{d} bX(t) + act1_{A^0} + \int_{(0,t] \times \mathbb{R}} (p(x) - bx)dM(t, x),
\]
where $M$ is a free Poisson random measure coming from the Lévy-Itô decomposition of $X(t)$. This is the case whether or not $\int_{-1}^{1} |x|d\rho(x)$ is finite. In particular, if $p'(0) = 0$,
\[
X^p(t) \xrightarrow{d} act1_{A^0} + \int_{(0,t] \times \mathbb{R}} p(x)dM(t, x).
\]

**Proof.** Let $X(1)$ be generated by the pair $(\gamma, \sigma)$. Let $\mu_N$ and $\mu^p_N$ be the distributions of $X_{N,r}$ and $p(X_{N,r})$ respectively. Recall that by Lemma 15,
\[
\int_{\mathbb{R}} f(x)d\mu^p_N(x) = \int_{\mathbb{R}} f(p(x))d\mu_N(x),
\]
for any real-valued and bounded Borel function \( f(x) \). Let
\[
q(x) := \begin{cases} \frac{p(x) - bx}{x^2}, & x \neq 0 \\ c, & x = 0. \end{cases}
\]
Then \( q(x) \) is a continuous function. Therefore,
\[
\lim_{N \to \infty} [Nt] \int \frac{x}{1 + x^2} d\mu_N^p(x) = t \lim_{N \to \infty} N \int \frac{p(x)}{1 + p(x)^2} d\mu_N(x)
= t \lim_{N \to \infty} N \left[ \int \frac{bx}{1 + x^2} d\mu_N(x) + \int \left( \frac{p(x)}{1 + p(x)^2} - \frac{bx}{1 + x^2} \right) d\mu_N(x) \right]
= t b \gamma + t \lim_{N \to \infty} N \int g_p(x) \frac{x^2}{1 + x^2} d\mu_N(x),
\]
where \( g_p(x) = \frac{p(x) + q(x) - b(b + xq(x))p(x)}{1 + p(x)^2} \in Cb(\mathbb{R}) \) and \( g_p(0) = c \). So \( \gamma^p \) is defined by
\[
\gamma^p := \lim_{N \to \infty} N \int \frac{x}{1 + x^2} d\mu_N^p(x) = b \gamma + \int g_p(x) d\sigma(x),
\]
where \( \gamma \) and \( \sigma \) are defined by (14) and (15). Define
\[
h(x) := \begin{cases} \frac{p(x)}{x}, & x \neq 0 \\ b, & x = 0. \end{cases}
\]
Then, \( h(x) \) is a continuous function and \( xh(x) = p(x) \). For any \( f(x) \in Cb(\mathbb{R}) \),
\[
[Nt] \int f(x) \frac{x^2}{x^2 + 1} d\mu_N^p(x)
= [Nt] \int f(p(x)) \frac{p(x)^2}{p(x)^2 + 1} d\mu_N(x)
= [Nt] \int f(p(x)) \frac{p(x)^2}{p(x)^2 + 1} \frac{x^2}{x^2 + 1} \frac{x^2 + 1}{x^2} d\mu_N(x)
\rightarrow_{N \to \infty} t \int f(p(x)) \frac{p(x)^2 + h(x)^2}{p(x)^2 + 1} d\sigma(x).
\]
Let \( h_p(x) := \frac{p(x)^2 + h(x)^2}{p(x)^2 + 1} \), which is a positive bounded Borel function on \( \mathbb{R} \). We denote by \( d\tilde{\sigma}(x) \) the measure \( h_p(x) d\sigma(x) \). The measure \( d\sigma^p(x) \) is defined by
\[
\int f(x) d\sigma^p(x) = \int f(p(x)) d\tilde{\sigma}(x),
\]
for any bounded Borel function \( f(x) \). Then,
\[
N \frac{x^2}{x^2 + 1} d\mu_N^p(x) \xrightarrow{w} d\sigma^p(x),
\]
as \( N \to \infty \). Since \( \sigma \) is a finite positive Borel measure on \( \mathbb{R} \), we know that \( \sigma^p \) is also a finite positive Borel measure. Thus, the conclusion (21) follows immediately from Theorem 10. By Proposition 14 we know that \( \{X^p(t)\}_{t \geq 0} \) can be a free Lévy process affiliated with some \( W^* \)-probability space. Denote the free generating triplet of \( X^p(1) \) by \( (a^p, \eta^p, \rho^p) \).
Next, based on Theorem 1, Lévy process \( X^p(t) \) has a decomposition in the form of (2) with free generating triplet \((a^p, \eta^p, \rho^p)\). Hence, to prove the representation (18) of \( X^p(t) \), it is necessary to compute the free generating triplet \((a^p, \eta^p, \rho^p)\) in terms of free generating pair \((\gamma, \sigma)\) or free generating triplet \((a, \eta, \rho)\) of \( X(t) \). Firstly, \( a^p = \sigma^p(\{0\}) = \int_\|X\|=0 1_{\{0\}}(x) d\sigma^p(x) = \sigma(\{0\}) h(0)^2 = ab^2 \). Secondly, for Lévy measure \( \rho^p \) and any bounded Borel function \( f(x) \), we have that

\[
\int_{\mathbb{R}} f(x) d\rho^p(x) = \int_{\mathbb{R}} f(x) \frac{1 + x^2}{x^2} 1_{\mathbb{R}\backslash\{0\}}(x) d\sigma^p(x)
\]

\[
= \int_{\mathbb{R}} f(p(x)) \frac{1 + (p(x))^2}{p(x)^2} 1_{\mathbb{R}\backslash\{0\}}(p(x)) \frac{p(x)^2 + h(x)^2}{1 + p(x)^2} d\sigma(x)
\]

\[
= \int_{\mathbb{R}} f(p(x)) 1_{\mathbb{R}\backslash\{0\}}(p(x)) \frac{1 + x^2}{x^2} d\sigma(x)
\]

\[
= \int_{\mathbb{R}} f(p(x)) 1_{\mathbb{R}\backslash\{0\}}(p(x)) d\rho(x).
\]

Therefore, \( \rho^p \) is precisely the measure from Lemma 16 and in particular a Lévy measure. Thirdly, by the relation \( \eta^p = \gamma^p + \int_{\mathbb{R}\backslash\{0\}} \frac{1 + x^2}{x} (1_{[-1,1]}(x) - \frac{1}{1 + x^2}) d\sigma^p(x) \), and the corresponding relation between \( \eta \) and \( \gamma \), using also \( a = \sigma(\{0\}) \) and relation (12), we can deduce that

\[
\eta^p = b\gamma + \int_{\mathbb{R}} 1_{\{p(x) \neq 0\}}(x) g_p(x) d\sigma(x) + \int_{\mathbb{R}} 1_{\{p(x) = 0\}}(x) g_p(x) d\sigma(x)
\]

\[
+ \int_{\mathbb{R}} 1_{\{p(x) \neq 0\}}(x) h^2 + \frac{p^2}{p} \left( 1_{\{1 \leq p(x) \leq 1\}}(x) - \frac{1}{1 + (p(x))^2} \right) d\sigma(x)
\]

\[
= b\gamma - \int_{\mathbb{R}\backslash\{0\}} 1_{\{p(x) = 0\}}(x) b x d\sigma(x) + \int_{\mathbb{R}} 1_{\{x = 0\}}(x) c d\sigma(x)
\]

\[
+ \lim_{\epsilon \to 0} \left[ \int_{\{x \leq |p(x)| \}} \left( h^2 + \frac{p^2}{p} 1_{\{-1 \leq p \leq 1\}}(x) \right) d\sigma - \int_{\{|p(x)| \geq \epsilon\}} \frac{b}{x} d\sigma(x) \right]
\]

\[
= b\gamma + ac + \lim_{\epsilon \to 0} \left[ \int_{\{x \leq |p(x)| \}} \left( h^2 + \frac{p^2}{p} 1_{\{-1 \leq p \leq 1\}}(x) \right) d\sigma(x) - \int_{\{|x| \geq \epsilon\}} \frac{b}{x} d\sigma(x) \right]
\]

\[
= b\eta + ac + \left( \int_{\mathbb{R}} 1_{\{0 < |p(x)| \leq 1\}}(x) p(x) - 1_{\{0 \leq x \leq 1\}}(x) bx \right) d\rho(x).
\]

Note that for some \( \varepsilon > 0, |p(x)| \leq 1 \) for \( |x| \leq \varepsilon \). So

\[
\int_{\mathbb{R}} |1_{\{0 < |p(x)| \leq 1\}}(x) p(x) - 1_{\{0 \leq x \leq 1\}}(x) bx| d\rho(x)
\]

\[
= \int_{-\varepsilon}^{\varepsilon} |q(x)| x^2 d\rho(x)
\]

\[
+ \int_{\mathbb{R}} |1_{\{0 < |p(x)| \leq 1\}}| |x| \geq \varepsilon \} p(x) - 1_{\{\varepsilon \leq x \leq 1\}} bx| d\rho(x)
\]

\[
\leq \sup_{-\varepsilon \leq x \leq \varepsilon} |q(x)| \int_{-\varepsilon}^{\varepsilon} x^2 d\rho(x) + 2 \int_{\{|x| \geq \varepsilon\}} d\rho(x) < \infty
\]
since $\rho$ is a Lévy measure, and so the expression above makes sense.

Combine three results we got above and recall the general free Lévy-Itô decomposition of $X^p(t)$ with the free generating triplet $(a^p, \eta^p, \rho^p)$. Let $M^{(p)}$ be the free Poisson random measure on $(D, \mathcal{B}(D), \text{Leb} \otimes \rho^p)$. Then, we can simplify the last part of Lévy-Itô decomposition of $M(t)$ with respect to the free Poisson random measure $M^{(p)}$:

$$
\lim_{\epsilon \to 0} \left[ \int_{(0,t] \times \{|x|>\epsilon\}} x dM^{(p)}(t, x) - \int_{(0,t] \times \{|x|<1\}} x \text{Leb} \otimes \rho^p(dt, dx) 1_{A^0} \right]
$$

$$= \lim_{\epsilon \to 0} \left[ \int_{(0,t] \times \{|p(x)|>\epsilon\}} p(x) dM(t, x) - t \int_{\mathbb{R}} x 1_{\{\epsilon<|x|<1\}}(x) d\rho^p(x) 1_{A^0} \right]
$$

$$= \lim_{\epsilon \to 0} \left[ \int_{(0,t] \times \{|p(x)|>\epsilon\}} p(x) dM(t, x) - t \int_{\mathbb{R}} p(x) 1_{\{\epsilon<|x|<1\}}(p(x)) d\rho(x) 1_{A^0} \right].
$$

Here, we employ Lemma 16, integration by substitution with respect to free Poisson random measures and relation (13). Thus finally,

$$X^p(t) = \mathcal{L} \left( b\eta + ac + \lim_{\epsilon \to 0} \left( \int_{\mathbb{R}} 1_{\{\epsilon<|p(x)|<1\}}(x) p(x) - 1_{\{\epsilon<x<1\}}(bx) \right) d\rho(x) \right) 1_{A^0}$$

$$+ \sqrt{ab} S(t)$$

$$+ \lim_{\epsilon \to 0} \int_{(0,t] \times \{|x|>\epsilon\}} x dM(t, x) - t \int_{\{\epsilon<|x|<1\}} x d\rho(x) 1_{A^0})
$$

$$+ act + \lim_{\epsilon \to 0} \int_{(0,t] \times \{|p(x)|>\epsilon\}} p(x) dM(t, x) - b \int_{(0,t] \times \{|x|>\epsilon\}} x dM(t, x)
$$

$$= bX(t) + act + \lim_{\epsilon \to 0} \int_{(0,t] \times \{|p(x)|>\epsilon\}} (p(x) - bx) dM(t, x)
$$

$$+ \lim_{\epsilon \to 0} \int_{(0,t] \times \mathbb{R}} p(x)(1_{\{|p(x)|>\epsilon\}} - 1_{\{|x|>\epsilon\}}) dM(t, x).
$$

Here we used the fact that the distribution of $S(t)$ is symmetric. Since

$$\int_{(0,t] \times \mathbb{R}} (p(x) - bx) dM(t, x) = \int_{(0,t] \times \mathbb{R}} x dM(p(x)-bx)(t, x)$$

exists by Lemmas 17 and 13 and the functions $(p(x) - bx)1_{|x|\leq \varepsilon}$ have a uniform integrable bound and converge to zero pointwise as $\varepsilon \to 0$, by Lemma 12 we
have
\[
\lim_{\epsilon \to 0} \int_{(0,t] \times \{ |x| > \epsilon \}} (p(x) - bx) dM(t,x) = \int_{(0,t] \times \mathbb{R}} (p(x) - bx) dM(t,x).
\]

Finally, the functions
\[
p(x)(1_{\{ |p(x)| > \epsilon \}} - 1_{\{ |x| > \epsilon \}}) = -p(x)(1_{\{ |p(x)| \leq \epsilon \}} - 1_{\{ |x| \leq \epsilon \}})
\]
also have a uniform integrable bound and converge to zero pointwise as \(\epsilon \to 0\). Therefore by Lemma 12,
\[
\lim_{\epsilon \to 0} \int_{(0,t] \times \mathbb{R}} p(x)(1_{\{ |p(x)| > \epsilon \}} - 1_{\{ |x| > \epsilon \}}) dM(t,x) = 0.
\]

Remark 2. It is natural to consider, more generally, free additive (not necessarily) stationary processes approximated by free, non-identically distributed triangular arrays which are infinitesimal, that is, their distributions \(\mu_{i,N}\) satisfy
\[
\lim_{N \to \infty} \max_{1 \leq i \leq k_N} \mu_{i,N}(\{|x| \geq \epsilon\}) = 0
\]
for every \(\epsilon > 0\). The following very simple example shows how without additional assumptions, the results immediately break down. Let
\[
X_{i,N} = \frac{1}{N} + (-1)^i \frac{1}{N^\alpha}, i = 1, \ldots, 2N.
\]
Then clearly the array \(\{X_{i,N}\}\) is infinitesimal, and \(\lim_{N \to \infty} \sum_{i=1}^{[2Nt]} X_{i,N} = 2t\). But
\[
\sum_{i=1}^{[2Nt]} X_{i,N}^2 \sim \frac{2t}{N^{2\alpha-1}}
\]
diverges for \(\alpha < \frac{1}{2}\). So while the quadratic variation of a non-random process is zero, these sums do not converge to it. Compare with the remarks on page 494 of [AT86].

4. Convergence in moments

For a non-crossing partition \(\pi \in NC(n)\), denote
\[
\tau_\pi [a_1, \ldots, a_n] = \prod_{V \in \pi} \tau \left[ \prod_{i \in V} a_i \right].
\]
Recall that the free cumulant functional is defined by
\[
R[a_1, \ldots, a_n] = \sum_{\pi \in NC(n)} \text{M"ob} (\pi) \tau_\pi [a_1, \ldots, a_n],
\]
where $\text{Möb}$ is the Möbius function on the lattice of non-crossing partitions. The key property of the free cumulant functional is that if $a_1, \ldots, a_k$ are free, then

$$R[a_{u(1)}, \ldots, a_{u(n)}] = 0$$

unless $u(1) = \ldots = u(n)$.

**Proof of Theorem 3** Note first that by freeness and the free moment-cumulant formula,

$$R \left( \sum_{i=1}^{[N]} X_{i,N}^{u(1)}, \ldots, \sum_{i=1}^{[N]} X_{i,N}^{u(k)} \right) - \tau \left[ \sum_{i=1}^{[N]} X_{i,N}^{u(1)+\ldots+u(k)} \right]$$

$$= \sum_{i=1}^{[N]} \left( R \left( X_{i,N}^{u(1)}, \ldots, X_{i,N}^{u(k)} \right) - \tau \left[ X_{i,N}^{u(1)+\ldots+u(k)} \right] \right)$$

$$= \sum_{i=1}^{[N]} \sum_{\pi \in \text{NC}(k)} \text{Möb}(\pi) \tau \left[ X_{i,N}^{u(1)}, \ldots, X_{i,N}^{u(k)} \right].$$

The absolute value of this expression is bounded by

$$\sum_{\pi \in \text{NC}(k) \atop \pi \neq 1_k} |\text{Möb}(\pi)| \left| \sum_{i=1}^{[N]} \prod_{V \in \pi} \tau \left[ X_{i,N}^{\sum_{j \in V} u(j)} \right] \right|$$

$$\leq \sum_{\pi \in \text{NC}(k) \atop \pi \neq 1_k} |\text{Möb}(\pi)| \left( \sum_{i=1}^{[N]} \tau \left[ X_{i,N}^{\sum_{j \in V_1} u(j)} \right]^2 \right)^{1/2} \left( \sum_{i=1}^{[N]} \tau \left[ X_{i,N}^{\sum_{j \in V_2} u(j)} \right]^2 \right)^{1/2}$$

$$\times \prod_{V \in \pi \setminus \{V_1, V_2\}} \max_{1 \leq i \leq [N]} \left| X_{i,N}^{\sum_{j \in V} u(j)} \right|,$$

which goes to zero as $N \to \infty$, by assumption. So to prove joint convergence in moments, it suffices to show that the limit

$$\lim_{N \to \infty} \tau \left[ \sum_{i=1}^{[N]} X_{i,N}^k \right]$$

exists for each $k$. Indeed, applying the derivation above to the case $u(1) = \ldots = u(k) = 1$,

$$\tau \left[ \sum_{i=1}^{[N]} X_{i,N}^k \right] - R_k \left( \sum_{i=1}^{[N]} X_{i,N} \right) \to 0$$
as $N \to \infty$. Finally, by assumption
\[ R_k \left( \sum_{i=1}^{[Nt]} X_{i,N} \right) \to R_k(X(t)). \]
The statement about processes follows as in Proposition 14. □

5. CONVERGENCE IN PROBABILITY

We first quote a result from [BV93].

Lemma 19 (Lemma 4.4). Let $(\mathcal{A}, \tau)$ be a $W^*$-probability space, $T_1, T_2, T'_1, T'_2 \in \tilde{\mathcal{A}}$, and $p_1, p_2 \in \mathcal{A}$ orthogonal projections. Suppose $T'_j = T_j p_j$, for $j = 1, 2$.

Then there exist projections $p, q \in \mathcal{A}$ such that

(a) $(T_1 T_2) p = (T'_1 T'_2) p$,
(b) $(T_1 + T_2) q = (T'_1 + T'_2) q$, and
(c) $\tau[p], \tau[q] \geq \tau[p_1] + \tau[p_2] - 1$.

Remark 3. Let $\rho$ be a probability measure on $\mathbb{R}$. In the tracial non-commutative probability space $\mathcal{C} = L^\infty((0, 1] \times \mathbb{R}, \text{Leb} \otimes \rho)$, consider the projections $P(B) = \chi_B$ for every Borel set $B$. Let $s$ be a semicircular element free from $\mathcal{C}$. Then according to [NS96], the family of operators $M : B \to s P(B) s$ satisfies all the properties of a free Poisson random measure in Definition 3. Next, let
\[ e(t) = \int_{\mathbb{R}} x P((0, t] \times dx), \]
meaning that the spectral projections of $e_t$ are $\{P((0, t] \times (-\infty, x))\}$. Then
\[ \{e(t) : t \in (0, 1]\} \]
is a process with orthogonal increments, and $\{se(t) s : t \in (0, 1]\}$ is a free compound Poisson process. Note that
\[ se(t) s = \int_{\mathbb{R}} x s P((0, t] \times dx) s = \int_{(0, t] \times \mathbb{R}} x dM(t, x). \]

Proposition 20. Let $Z_1, \ldots, Z_k$ be bounded and centered, free from a stationary process $\{e(t)\}$ with orthogonal increments. Then
\[ \sum_{i=1}^{N} e_{i,N}^{m_0} Z_1 e_{i,N}^{m_1} Z_2 \ldots e_{i,N}^{m_{k-1}} Z_k e_{i,N}^{m_k} \to 0 \]
in probability as $N \to \infty$. Here we denote as usual $e_{i,N} = e(i/N) - e((i-1)/N)$. 
Proof. Without loss of generality, assume that \(\{e(t)\}\) has the form in Remark 3. For arbitrary \(\varepsilon > 0\), choose \(T\) so that \(\rho((-T, T)^c) < \varepsilon\). Denote

\[ q_{i,N} = \chi((0,1] \times \mathbb{R}) \setminus \left(\left[\frac{i-1}{N}, \frac{i}{N}\right] \times (-T, T)^c\right), \]

so that \(\tau(q_{i,N}) > 1 - \varepsilon/N\). Denote \(e'_i,N = e_i,N q_{i,N}\). Then \(\{e'_{i,N} : 1 \leq i \leq N\}\) are still orthogonal, and

\[ \left\| \sum_{i=1}^{N} e'_{i,N} \right\| = \left\| \int_{-T}^{T} x \ P((0, 1] \times dx) \right\| \leq T. \]

According to Lemma 19, there is a projection \(p_{i,N}\) with

\[ \tau(p_{i,N}) > 1 - \sum_{j=0}^{k} m_j \varepsilon / N \]

such that

\[ e_{i,N}^{m_0} Z_1 e_{i,N}^{m_1} Z_2 \ldots e_{i,N}^{m_{k-1}} Z_k e_{i,N}^{m_k} p_{i,N} \]

\[ = (e'_{i,N})^{m_0} Z_1 (e'_{i,N})^{m_1} Z_2 \ldots (e'_{i,N})^{m_{k-1}} Z_k (e'_{i,N})^{m_k} p_{i,N}. \]

Therefore for \(p_N = \bigwedge_{i=1}^{N} p_{i,N}\), \(\tau(p_N) > 1 - \sum_{j=0}^{k} m_j \varepsilon\) and

\[ \sum_{i=1}^{N} e_{i,N}^{m_0} Z_1 e_{i,N}^{m_1} Z_2 \ldots e_{i,N}^{m_{k-1}} Z_k e_{i,N}^{m_k} p_N \]

\[ = \sum_{i=1}^{N} (e'_{i,N})^{m_0} Z_1 (e'_{i,N})^{m_1} Z_2 \ldots (e'_{i,N})^{m_{k-1}} Z_k (e'_{i,N})^{m_k} p_N. \]

On the other hand, according to Theorem 3 from [Ans00],

\[ \left\| \sum_{i=1}^{N} (e'_{i,N})^{m_0} Z_1 (e'_{i,N})^{m_1} Z_2 \ldots (e'_{i,N})^{m_{k-1}} Z_k (e'_{i,N})^{m_k} \right\| \]

\[ \leq 4^{2k} (\max \|Z_j\|)^k T \sum_{j=0}^{k} m_j N^{-k/2}. \]

The result follows. \(\square\)

Proof of Theorem 4 By using the addition part of Lemma 9, we may assume that \(t \in (0, 1]\). Note first that by Lemma 12

\[ \int_{\left[0, \frac{1}{N^2}\right] \times \mathbb{R}} x^k dM(t, x) \to \int_{(0,1] \times \mathbb{R}} x^k dM(t, x) \]

in probability as \(N \to \infty\). Next, write \(X(t) = se(t)s\) as before. By the same reasoning as in Remark 3

\[ \int_{(0,1] \times \mathbb{R}} x^k dM(t, x) = se(t)^k s. \]
Therefore
\[
\sum_{i=1}^{[Nt]} X_{i,N}^k - \int_{0}^{[Nt]} x^k \, dM(t, x) = \sum_{i=1}^{[Nt]} (s e_{i,N} s)^k - \sum_{i=1}^{[Nt]} s e_{i,N}s
\]
\[
= \sum_{j=1}^{k-1} \sum_{m_0, m_1, \ldots, m_j \geq 1 \atop m_0 + m_1 + \ldots + m_j = k} s \left( \sum_{i=1}^{[Nt]} e_{i,N}^m \left( s^2 - 1 \right) \ldots e_{i,N}^{m_{j-1}} \left( s^2 - 1 \right) e_{i,N}^{m_j} \right). \]

Now note that \( \tau(s^2 - 1) = 0 \) and apply Proposition 20. \( \square \)

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**Proof of Corollary 6.** Combining Theorem 4 with Lemma 9, polynomials in the variables \( \{ \sum_{i=1}^{[Nt]} X_{i,N}^j \} \) converge to the corresponding polynomials in \( \{ X(j)(t) \} \) in probability. Finally, by Proposition 2.19 in [BNT02] (see also Proposition 2.1 in [LP97]), convergence in probability implies convergence in distribution. \( \square \)

**Proof of Corollary 7.** According to Corollary 25
\[
\sum_{1 \leq i_1, i_2, \ldots, i_k \leq [Nt] \atop i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k} X_{i_1,N} X_{i_2,N} \ldots X_{i_k,N}
\]
\[
= \sum_{j=1}^{k} (-1)^{k-j} \sum_{m_1, \ldots, m_j \geq 1 \atop m_1 + \ldots + m_j = k} \left( \sum_{i(1)=1}^{[Nt]} X_{i(1),N}^{m_1} \right) \ldots \left( \sum_{i(j)=1}^{[Nt]} X_{i(j),N}^{m_j} \right). \]

Now apply either Theorem 3 or Corollary 6. \( \square \)

See the second author’s thesis for a direct proof.

**Remark 4.** In the case of a process which is not necessarily centered, normalizing it so that \( \tau[X(t)] = t \), a more natural definition of an \( n \)-fold stochastic integral \( \psi_n \), according to Theorem 4 of [Ans00], is
\[
\psi_n = X \psi_{n-1} + \sum_{j=2}^{n} (-1)^{j-1} \sum_{k=0}^{n-j} \binom{k+j-2}{j-2} X^{(j)} \psi_{n-j-k}.
\]

The recursion
\[
P_n = \left( \sum_{i=1}^{N} x_i \right) P_{n-1} + \sum_{j=2}^{n} (-1)^{j-1} \sum_{k=0}^{n-j} \binom{k+j-2}{j-2} \left( \sum_{i=1}^{N} x_i^j \right) P_{n-j-k}.
\]

for polynomials \( P_n(x_1, \ldots, x_N, t) \) can be solved explicitly, but we find the resulting formula complicated and not particularly illuminating, and omit it from the article.
We can similarly upgrade various results proven in [Ans00] for bounded free Lévy processes and uniform limits to general free compound Poisson processes and limits in probability. This applies to Theorem 1 (stochastic measures corresponding to crossing partitions are zero), Proposition 1 (for a centered process, stochastic measures corresponding to partitions with inner singletons are zero) and its corollary on the equality of expressions (6) and (7).

**Remark 5.** Let \( \mu, \nu \) be probability measures on \( \mathbb{R} \), such that \( \mu = \mu_a, \nu = \mu_b \) for free \( a, b \in (\bar{A}_{sa}, \tau) \). The additive free convolution \( \mu \boxplus \nu \) is the distribution of \( a + b \). If \( \mu \) is supported on \( \mathbb{R}^+ \) (so that \( a \) is positive), the multiplicative free convolution \( \mu \boxdot \nu \) is the distribution of \( a^{1/2}ba^{1/2} \), which we identify (since \( \tau \) is a trace) with the distribution of \( ab \).

According to Proposition 3.5 in [BN08], we have the relation

\[
(\mu^{\boxplus t}) \boxdot (\nu^{\boxplus t}) = (\mu \boxdot \nu)^{\boxplus t} \circ D_{1/t},
\]

where \( D_{1/t} \) is the dilation operator corresponding to multiplying the operator by \( t \). Note that in the proposition, the relation is stated for \( t \geq 1 \), but the same argument shows that it holds whenever all the convolution powers on the left-hand side are defined and at least one of them is supported on \( \mathbb{R}^+ \).

**Proposition 21.** Let

\[
\left\{X_{i,N}^{(1)} : 1 \leq i \leq N, N \in \mathbb{N}\right\} \cup \left\{X_{i,N}^{(2)} : 1 \leq i \leq N, N \in \mathbb{N}\right\} \subset \left((\bar{A}, \tau)_{sa}\right)
\]

be two triangular arrays with free, identically distributed rows, free from each other, the first of which consists of positive operators. Denote

\[
\sum_{i=1}^{N} X_{i,N}^{(j)} = X_N^{(j)}, \quad j = 1, 2
\]

and suppose that

\[
\lim_{N \to \infty} X_N^{(j)} = X^{(j)}, \quad j = 1, 2
\]

in distribution, for some \( \{X^{(1)}, X^{(2)}\} \). Then as \( N \to \infty \),

\[
\sum_{i=1}^{N} \left(X_{i,N}^{(1)}\right)^{1/2} X_{i,N}^{(2)} \left(X_{i,N}^{(1)}\right)^{1/2} \to 0
\]

in distribution, and so also in probability.

**Proof.** Using the identity from the preceding remark,

\[
\mu_{X_{i,N}^{(1)}}^{1/2} X_{i,N}^{(2)} \left(X_{i,n}^{(1)}\right)^{1/2} = (\mu_{X_{i,N}^{(1)}}^{\boxplus (1/N)}) \boxdot (\mu_{X_{i,N}^{(2)}}^{\boxplus (1/N)}) = (\mu_{X_{i,N}^{(1)}} \boxdot \mu_{X_{i,N}^{(2)}})^{\boxplus (1/N)} \circ D_N
\]

and so

\[
\mu_{\sum_{i=1}^{N} \left(X_{i,N}^{(1)}\right)^{1/2} X_{i,N}^{(2)} \left(X_{i,n}^{(1)}\right)^{1/2}}(z) = (\mu_{X_{i,N}^{(1)}} \boxdot \mu_{X_{i,N}^{(2)}}) \circ D_N.
\]
As \( N \to \infty, \mu_{X(1)}^{(N)} \boxplus \mu_{X(2)}^{(N)} \to \mu_{X(1)} \boxplus \mu_{X(2)} \) weakly, and so the distribution above converges to \( \delta_0 \) weakly. \( \square \)

**Remark 6.** Denote \( C_\mu(z) = z \varphi_\mu(1/z) \) the free cumulant transform. A measure \( \sigma \) is free regular if

\[
C_\sigma(z) = \eta' z + \int_{\mathbb{R}} \left( \frac{1}{1-zx} - 1 \right) \nu(dx)
\]

for some \( \eta' \geq 0 \) and \( \nu((-\infty, 0]) = 0 \). By Proposition 6.2 in [AHS13], if \( \mu \) is \( \boxplus \)-infinitely divisible and symmetric, then

\[
\mu^2 = m \boxplus \sigma.
\]

Here \( \mu^2 = \mu(x^2) \) in our earlier notation, \( m \) is the standard free Poisson distribution, and \( \sigma \) is a free regular measure. Moreover by Theorem 11 from [PAS12], this is equivalent to

\[
C_\mu(z) = C_\sigma(z^2).
\]

Next, let \( \mu, \nu \) be probability measures on \( \mathbb{R} \), such that \( \mu = \mu_a, \nu = \mu_b \) for free \( a, b \in (\tilde{A}_{sa}, \tau) \). Denote by \( \mu \boxplus \nu \) the distribution of the anti-commutator \( ab + ba \). If \( \mu, \nu \) are both symmetric, it coincides with the distribution of the commutator \( i(ab - ba) \), and satisfies

\[
((\mu \boxplus \nu)^{\boxplus 1/2})^2 = \mu^2 \boxplus \nu^2.
\]

See [NS98], Lectures 15 and 19 in [NS06], and Corollary 6.5 in [AHS13].

We also note that if in Remark 5 \( \mu \) is free regular, then by Theorem 4.2 in [AHS13], \( \mu^\boxplus t \) is the distribution of a positive operator for all \( t > 0 \). So if in addition \( \nu \) is \( \boxplus \)-infinitely divisible, the identity (23) holds for all such \( t \).

**Proposition 22.** Let

\[
\left\{ X_{i,N}^{(1)} : 1 \leq i \leq N, N \in \mathbb{N} \right\} \cup \left\{ X_{i,N}^{(2)} : 1 \leq i \leq N, N \in \mathbb{N} \right\} \subset (\tilde{A}, \tau)_{sa}
\]

be two triangular arrays with free, identically distributed rows, free from each other, all of whose distributions are symmetric. Denote

\[
\sum_{i=1}^{N} X_{i,N}^{(j)} = X_N^{(j)}, \quad j = 1, 2
\]

and suppose that the distribution of each \( X_N^{(j)} \) is \( \boxplus \)-infinitely divisible and

\[
\lim_{N \to \infty} X_N^{(j)} = X^{(j)}, \quad j = 1, 2
\]

in distribution, for some \( \{ X^{(1)}, X^{(2)} \} \). Then as \( N \to \infty \),

\[
\sum_{i=1}^{N} \left( X_{i,N}^{(1)} X_{i,N}^{(2)} + X_{i,N}^{(2)} X_{i,N}^{(1)} \right) \to 0
\]
in distribution, and so also in probability, and

\[(24)\quad \sum_{i=1}^{N} X_{i,N}^{(1)} X_{i,N}^{(2)} \to 0\]

in probability.

**Proof.** Denote by \(\mu_{j,N}\) the distribution of \(X_{N}^{(j)}\). Using the preceding remark, we may write

\[\mu_{j,N}^{2} = m \boxplus \sigma_{j,N},\]

where \(\sigma_{j,N}\) is a free regular measure, such that

\[C_{\mu_{j,N}}(z) = C_{\sigma_{j,N}}(z^{2}).\]

Note that

\[C_{\mu_{j,N}} \boxplus (1/N) = 1 = C_{\sigma_{j,N}}(z^{2}).\]

Thus

\[\left(\mu_{j,N}^{(1/N)}\right)^{2} = m \boxplus \sigma_{j,N}^{(1/N)}.\]

Next,

\[\left(\left(\mu_{1,N}^{(1/N)} \boxminus \mu_{2,N}^{(1/N)}\right)^{(1/2)}\right)^{2} = \left(\mu_{1,N}^{(1/N)}\right)^{2} \boxtimes \left(\mu_{2,N}^{(1/N)}\right)^{2}\]

\[= m \boxtimes \sigma_{1,N}^{(1/N)} \boxtimes m \boxtimes \sigma_{2,N}^{(1/N)}\]

Therefore

\[C_{\left(\mu_{1,N}^{(1/N)} \boxminus \mu_{2,N}^{(1/N)}\right)^{(N/2)}}(z) = NC_{\sigma_{1,N}^{(1/N)} \boxtimes m \boxtimes \sigma_{2,N}^{(1/N)}}(z^{2}).\]

Applying the relation (23) twice and distributing the dilation, we get

\[\left(\sigma_{1,N}^{(1/N)} \boxtimes m \boxtimes \sigma_{2,N}^{(1/N)}\right)^{N} = \left(\sigma_{1,N} \boxtimes m \boxtimes \sigma_{2,N}\right) \circ D_{N^{2}}\]

\[= \left(m^{(N)} \circ D_{N}\right) \boxtimes \left(\left(\sigma_{1,N} \boxtimes \sigma_{2,N}\right) \circ D_{N}\right).\]

Using the (noncommutative) law of large numbers, or by a direct calculation, \(m^{(N)} \circ D_{N} \to \delta_{1}\), so these measures converge to \(\delta_{0}\) weakly. Therefore their free cumulant transforms converge to zero pointwise, which implies that

\[\left(\mu_{1,N}^{(1/N)} \boxminus \mu_{2,N}^{(1/N)}\right)^{(N/2)} \to \delta_{0}.\]

Since the same convergence in probability holds for the commutators

\[\sum_{j=1}^{N} i \left(X_{j,N}^{(1)} X_{j,N}^{(2)} - X_{j,N}^{(2)} X_{j,N}^{(1)}\right),\]

it holds for their linear combination (24). \(\square\)
Proof of Theorem 5. Let

\[ X(t) = \eta t \mathbf{1}_{\mathcal{A}^0} + \sqrt{a} S(t) \]

\[ + \lim_{\epsilon \searrow 0} \left( \int_{(0,t] \times \{ |x| > \epsilon \}} x dM(t, x) - \int_{(0,t] \times \{ |x| < \epsilon \}} x((\text{Leb} \otimes \rho))(dt, dx) \mathbf{1}_{\mathcal{A}^0} \right). \]

Fix \( \alpha \in (0, 1) \). Denote

\[ X'(t) = \left( \eta - \int_{\{ |x| \leq 1 \}} x \rho(dx) \right) t \mathbf{1}_{\mathcal{A}^0} + \sqrt{a} S(t) \]

\[ + \lim_{\epsilon \searrow 0} \left( \int_{(0,t] \times \{ |x| < \alpha \}} x dM(t, x) - \int_{(0,t] \times \{ |x| < \epsilon \}} x((\text{Leb} \otimes \rho))(dt, dx) \mathbf{1}_{\mathcal{A}^0} \right). \]

and

\[ X''(t) = \int_{|x| \geq \alpha} x dM(t, x). \]

Note that \( \{X''(t)\} \) is an (unbounded) free compound Poisson process, \( X(t) = X'(t) + X''(t) \), \( \{X'(t)\} \) and \( \{X''(t)\} \) are free from each other, and all of their distributions are \( \mathbb{E} \)-infinitely divisible and symmetric. Then

\[ \sum_{i=1}^{[Nt]} X_{i,N}^2 = \sum_{i=1}^{[Nt]} \left( X'_{i,N} \right)^2 + \sum_{i=1}^{[Nt]} \left( X''_{i,N} \right)^2 + \sum_{i=1}^{[Nt]} \left( X'_{i,N} X''_{i,N} + X''_{i,N} X'_{i,N} \right). \]

By Theorem 4, the second term converges to \( (X''(t))^2 \) in probability. By Proposition 22 the third term converges to zero in probability. By Theorem 2 for fixed \( \alpha \), the first term converges in distribution to

\[ (X'(t))^2 = at \mathbf{1}_{\mathcal{A}} + \int_{(0,t] \times (-\alpha, \alpha)} x^2 dM(t, x). \]

Finally, as \( \alpha \to 0 \), \( (X'(t))^2 \to at \mathbf{1}_{\mathcal{A}} \) in probability. Thus, given \( \epsilon, \delta > 0 \), we may choose \( \alpha \) small so that \( (X'(t))^2 - at \mathbf{1}_{\mathcal{A}} \in \mathcal{N}(\epsilon, \delta) \). Then for sufficiently large \( N \), \( \sum_{i=1}^{[Nt]} \left( X'_{i,N} \right)^2 - at \mathbf{1}_{\mathcal{A}} \in \mathcal{N}(\epsilon, \delta) \) and

\[ \sum_{i=1}^{[Nt]} X_{i,N}^2 - (X''(t))^2 - at \mathbf{1}_{\mathcal{A}} \in \mathcal{N}(\epsilon, \delta). \]

It remains to note that also

\[ X(t)^2 - (X''(t))^2 - at \mathbf{1}_{\mathcal{A}} = (X'(t))^2 - at \mathbf{1}_{\mathcal{A}} \in \mathcal{N}(\epsilon, \delta). \]

We finish this section with another possible definition of joint convergence in distribution. As already noted, for commuting variables, convergence in distribution of linear combinations is equivalent to joint convergence in distribution. As pointed out by Édouard Maurel-Segala and Maxime Fevrier, this is not the case for non-commuting variables. However the following matricial version is its natural replacement. By the well-known linearization trick [HT05] (see also...
Chapter 10 of [MS17], it implied the definition in the introduction; we do not
know if they are in general equivalent. We show that convergence in probability
implies joint convergence in this possibly stronger sense as well.

**Definition 5.** Let

\[
\{x_{i,N} : 1 \leq i \leq k, N \in \mathbb{N}\} \cup \{x_i : 1 \leq i \leq k\} \subset (\tilde{A}_{sa}, \tau).
\]

We say that \((x_{1,N}, \ldots, x_{k,N}) \rightarrow (x_1, \ldots, x_k)\) jointly in distribution if for any \(d\) and any Hermitian matrices \(A_1, \ldots, A_k \in M_d(\mathbb{C})\), and any \(B \in M_d(\mathbb{C})\) with \(\Re B > \varepsilon I\) for some \(\varepsilon > 0\), the Cauchy transforms

\[
(I \otimes \tau) \left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_{i,N} \right)^{-1} \rightarrow (I \otimes \tau) \left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_i \right)^{-1}
\]

in norm in \(M_N(\mathbb{C})\).

**Proposition 23.** If for each \(i\), \(x_{i,N} \rightarrow x_i\) in probability, then \((x_{1,N}, \ldots, x_{k,N}) \rightarrow (x_1, \ldots, x_k)\) in the sense of Definition 5.

**Proof.** The argument in Proposition 2.19 in [BNT02] largely goes through; we
outline it for the reader’s convenience. Note first that for \(X \in M_d(\tilde{A}_{sa})\),

\[
\| (B \otimes 1 - X)^{-1} \| \leq \| (\Re B)^{-1} \|,
\]

and in particular this operator is bounded. By the resolvent identity,

\[
\left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_{i,N} \right)^{-1} - \left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_i \right)^{-1}
\]

\[
= \left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_i \right)^{-1} \left( \sum_{i=1}^{k} A_i \otimes x_i - \sum_{i=1}^{k} A_i \otimes x_{i,N} \right) \left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_i \right)^{-1}.
\]

By assumption and a short argument, for any \(\varepsilon, \delta > 0\) there is an \(n\) such that for
\(N \geq n\), there is a projection \(p_N\) with \(\tau[p_N] > 1 - \delta\) and

\[
\left\| \left( \sum_{i=1}^{k} A_i \otimes x_i - \sum_{i=1}^{k} A_i \otimes x_{i,N} \right) (I \otimes p_N) \right\| < \varepsilon \sum_{i=1}^{k} \| A_i \|.
\]

Thus for some projection \(q_N\) with the same property,

\[
\left\| \left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_{i,N} \right)^{-1} - \left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_i \right)^{-1} \right\| (I \otimes q_N)
\]

\[
\leq \varepsilon \| (\Re B)^{-1} \|^2 \sum_{i=1}^{k} \| A_i \|.
\]

In particular, the same estimate holds on each matrix entry on the left-hand side.
Applying the rest of the argument from Proposition 2.19 in [BNT02] entry-wise,
it follows that
\[(I \otimes \tau) \left[ \left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_{i,N} \right)^{-1} - \left( B \otimes 1 - \sum_{i=1}^{k} A_i \otimes x_i \right)^{-1} \right] \to 0. \]

\[\square\]

\section{Appendix A. Symmetric polynomials in non-commuting variables}

Symmetric functions in non-commuting variables (not to be confused with non-commutative symmetric functions) have been considered in [RS06, BRRZ08] and subsequent work. We need the following observation, whose explicit statement we could not find in the literature.

\textbf{Proposition 24.} Let \( p_k = \sum_{i=1}^{N} x_i^k \) be the basic power sum symmetric polynomials. In the algebra of non-commutative polynomials \( \mathbb{C}\langle x_1, \ldots, x_N \rangle \), the subalgebra generated by \( \{ p_k : k \geq 1 \} \) is the linear span of polynomials
\[
P_u(\mathbf{x}) = \sum_{i(1), i(2), \ldots} x_{u(1)}^{i(1)} x_{u(2)}^{i(2)} \ldots
\]
for all choices of \( u \) with coordinates \( u(i) \geq 1 \). Note that these polynomials are obviously linearly independent. In particular, the elementary symmetric functions
\[
e_k = \sum_{i(1) \neq i(2) \neq \ldots \neq i(r)} x_{i(1)} x_{i(2)} \ldots x_{i(r)}
\]
are not in this subalgebra for \( k > 1 \).

\textbf{Proof.} Clearly the algebra generated by all \( p_k \) is the span of all
\[
Q_u(\mathbf{x}) = p_{u(1)}(\mathbf{x}) p_{u(2)}(\mathbf{x}) \cdots = \sum_{i(1), i(2), \ldots} x_{u(1)}^{i(1)} x_{u(2)}^{i(2)} \ldots,
\]
where the \( i(j) \) are not necessarily distinct. Denote by \( \text{Int}(n) \) the interval partitions of \([n]\). Then we may re-index these polynomials as
\[
Q_\pi(\mathbf{x}) = \sum_{i(1), i(2), \ldots, i(r)=1} x_{i(j)}^{\mathbf{V}_j} = \prod_{j=1}^{r} p_{\mathbf{V}_j}(\mathbf{x})
\]
for \( \pi = \{ V_1, \ldots, V_r \} \in \text{Int}(n) \) for some \( n \). For \( u \in [N]^r \), denote \( \ker(u) \in \mathcal{P}(n) \) the partition such that \( u(i) = u(j) \) if and only if \( i, j \) lie in the same block of \( \ker(u) \). Note that for \( V \in \ker(u) \), the notation \( u(V) \) is unambiguous. Also, for \( \pi \in \mathcal{P}(n) \), let \( I(\pi) \) be the largest interval partition such that \( I(\pi) \leq \pi \). Note that
$I(\pi) = \tau$ if $\pi \geq \tau$ and if $V, V'$ are neighboring blocks of $\tau$, they lie in different blocks of $\pi$. Finally, for $\pi = \{V_1, \ldots, V_r\} \in \text{Int}(n)$, denote

$$P_\pi(x) = \sum_{\pi \in \text{Int}(n)} \prod_{V \in \pi} x_{i(V)}^{\vert V \vert},$$

Then for $\sigma \in \text{Int}(n)$,

$$Q_\sigma(x) = \sum_{\pi \in \text{P}(n)} \sum_{\tau \geq \sigma} \prod_{V \in \pi} x_{i(V)}^{\vert V \vert} = \sum_{\tau \in \text{Int}(n)} \sum_{\pi \in \text{P}(n)} \sum_{I(\pi) = \tau} \prod_{V \in \pi} x_{i(V)}^{\vert V \vert} = \sum_{\tau \in \text{Int}(n)} \sum_{\tau \geq \sigma} P_\tau(x).$$

Then by Möbius inversion on the lattice $\text{Int}(n)$, the spans of $\{Q_\pi\}$ and of $\{P_\pi\}$ are the same. □

**Corollary 25.** In the notation of the preceding proof,

$$P_\sigma = \sum_{\pi \in \text{Int}(n)} (-1)^{\vert \sigma \vert - \vert \pi \vert} \prod_{V \in \pi} p_{\vert V \vert}(x).$$

In particular,

$$\sum_{i(1), i(2), \ldots, i(n) = 1} \prod_{j=1}^{n} x_{i(j)} = \sum_{\pi \in \text{Int}(n)} (-1)^{n - \vert \pi \vert} \prod_{V \in \pi} p_{\vert V \vert}(x).$$

**Proof.** The first statement follows by Möbius inversion, since the Möbius function on the lattice $\text{Int}(n)$ is $\text{Möb}(\sigma, \pi) = (-1)^{\vert \sigma \vert - \vert \pi \vert}$. The second statement follows from the fact that the left-hand side is $P_{\hat{0}_n}(x)$. □

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