A Novel Statistical Independence Test for Dynamic Causal Discovery with Rare Events

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Abstract
Causal phenomena associated with rare events occur across a wide range of engineering problems, such as risk-sensitive safety analysis, accident analysis and prevention, and extreme value theory. However, current methods for causal discovery are often unable to uncover causal links between random variables in a dynamic setting that manifest only when the variables first experience low-probability realizations. To address this issue, we introduce a novel statistical independence test on data collected from time-invariant dynamical systems in which rare but consequential events occur. In particular, we exploit the time-invariance of the underlying data to superimpose the occurrences of rare events, allowing these occurrences to be better represented. We then perform a novel conditional independence test on the superimposed data. We provide non-asymptotic sample complexity bounds for the consistency of our method, and validate its performance across various simulated scenarios, with applications to traffic accidents.

Keywords: Causal Discovery, Time-Series Data, Rare Events, Statistical Independence Tests, Sample Complexity Bounds.

1. Introduction
The occurrence of rare yet consequential events during the evolution of a dynamical system is ubiquitous in many fields of engineering and science. Examples include natural disasters, vehicular accidents, and stock market crashes. When studying such phenomena, understanding the causal links between the disruptive event and the underlying system dynamics is crucial for controlling the system. In particular, if certain values of the system state are found to increase the probability that the disruptive event occurs, control strategies should be implemented to steer the state away from such values. This can be accomplished, for instance, by incorporating a description of this causal relationship into the cost function that generates these control inputs in an optimization-based control theory framework. In general, it is important to consider the following question:

Main Question (Q) Given a rare event associated with the evolution of a dynamical system, does the onset of the event become more likely when the system state assumes certain values?

Below, we present a running example invoked throughout subsequent sections as context.
**Running Example**  Consider a traffic engineer who wishes to reduce accident occurrences by identifying their causes. For example, if poor paving causes disruptive accidents on the link to occur more frequently, then the road should be re-paved to insulate the road from poor weather. In particular, consider the scenario in which the amount of traffic in a network of roads has a causal effect on the occurrence of accidents. For example, on some busy streets, high traffic flow induces congestion and thus renders chain collisions more likely. In this case, since steady-state flows in a traffic network can be controlled via tolling, regulators can adjust the toll on each network link to redistribute flow and reduce the number of accidents that transpire Maheshwari et al. (2022a,b). Conversely, on other roads, low traffic flow may incentivize drivers to exceed the speed limit and create more opportunities for accidents to occur; thus, traffic engineers can choose to enforce speed limits more stringently at times of low traffic flow.

Although many well-established methods in the causal discovery literature can efficiently learn causal relationships from data, most only apply to data generated from probability distributions associated with static, acyclic Bayesian networks Glymour et al. (2019); Pearl (2009); Spirtes et al. (2000). Moreover, most of the algorithms developed for uncovering causal relationships from the time series data rely on stringent assumptions, such as linear and additive Gaussian noise models, or aggregate data along time indices, and treat data points at different times as independent Glymour et al. (2019); Gnecco et al. (2021); Granger (1969); Pérez-Ariza et al. (2012). Often, however, rare events occur sparsely at any fixed time, and cannot be easily modeled using linear dynamics.

To address these shortcomings, we present a novel approach for aggregating and analyzing time series data in which consequential events of interest occur sparsely. Our method rests on the observation that, whereas the occurrence of a rare event at each fixed time may occur extremely infrequently, the probability of the event occurring at some time along the entire horizon is often much higher. Thus, we aggregate the time series data along the times of the event’s first occurrence. This renders the dataset more informative, by better representing the rare events of interest. Next, we present an algorithm that uses the curated data to analyze the causal relationships governing the occurrence of the rare event. The question of whether the system state affects the probability that the rare event occurs is formally posed as a binary hypothesis test, with the null hypothesis $H_0$ corresponding to the negative answer, and the alternative hypothesis $H_1$ corresponding to the positive one. We mathematically prove that our proposed method is consistent against all alternatives Lehmann (1951). In other words, if $H_0$ were true, then as the number of data trajectories $N$ in the dataset approaches infinity, our approach would reject $H_1$ with probability 1. We validate the performance of our algorithm on simulated and real traffic datasets. Finally, we summarize our work and present directions for future research.

2. Related Work

We situate the problem of causal discovery for rare events at the intersection of literature on causal discovery for time series, conditional independence testing, and rare event analysis.

**Causal Discovery for Static and Time Series Data**  Causal discovery aims to identify causal relationships between a collection of random variables from a dataset of their realizations. Common methods include constraint-based methods, which use statistical independence tests, score-based methods, which pose causal discovery as an optimization problem, and hybrid methods Glymour et al. (2019); Pearl (2009); Peters et al. (2017). However, most of these works focus on inferring
causal links between non-temporal data. For time series data, Granger causality uses vector autoregression to study whether one time series data can be used to predict another Granger (1969). Other common methods treat random variables with different time indices as independent, aggregate different data trajectories by matching time index Pérez-Ariza et al. (2012); Entner and Hoyer (2010); Malinsky and Spirtes (2018), or aggregate data from distinct causal graphical models with structural similarities Löwe et al. (2022). However, these methods do not tackle the problem of inferring causal relationships between rare events and dynamical systems, across sample trajectories on which the rare event may occur at different times, or not at all.

**Extreme Value Theory and Analysis of Rare Events** Extreme value theory aims to characterize dependences between random variables that exist only when certain low-probability events occur, such as rare meteorological events, or financial crises Engelke and Volgushev (2022); Asadi et al. (2018). Most closely related to our work are Gnecco et al. (2021), which studies causal links between heavy-tailed random variables, and Jana et al. (2021), which explores causal relationships between characteristics of London cycle superhighways, such as density, length, type, and collision rate, and abnormal congestion. However, Gnecco et al. (2021) relies on restrictive assumptions, such as the linearity of the dynamical model governing the variables’ evolution, while the discussion in Jana et al. (2021) on accidents occurrences is restricted to empirical studies. In contrast, our proposed algorithm applies a non-parametric conditional independence test that is capable of inferring relationships between a general dynamical system, and the onset of a rare event.

**Traffic Network Analysis** Traffic network analysis aims to mathematically describe and control the flow of traffic in urban networks of roads, bridges, and highways Baillon and Cominetti (2008); Krichene et al. (2014); Ahipaşaoğlu et al. (2019). Recent literature has proposed the design of tolling mechanisms that drive a traffic network to a socially optimal steady state flow Maheshwari et al. (2022b); Como and Maggistro (2022). However, these methods do not model or predict the occurrence of sudden yet consequential events that introduce abrupt shocks in the traffic flow, such as extreme weather events, earthquakes, car accidents, and other causes of sharp spikes in congestion. To address this issue, our paper uses the occurrence of rare but consequential vehicular accidents in traffic networks as a running example, to illustrate the applicability of our method on analyzing causal relationships between dynamical systems and associated rare events.

3. Preliminaries

Consider a stochastic, discrete-time dynamical system with state variable \(X_t \in \mathbb{R}^n\), event variable \(A_t \in \{0, 1\}\) with \(\mathbb{P}(A_t = 1) \in [p_1, p_2]\) for some \(p_1, p_2 \in (0, 1)\) with \(p_1 < p_2\) for all \(t\), and dynamics \(X_{t+1} = f(X_t, A_t, W_t)\) for each \(t \geq 0\), where \(W_t \in \mathbb{R}^w\) denotes i.i.d. noise, and \(f: \mathbb{R}^n \times \{0, 1\} \times \mathbb{R}^w \to \mathbb{R}^n\) denotes the dynamics of the system state. With a slight abuse of notation, we use \(A_{1:t} = 0\) to denote the event that \(A_1 = \cdots = A_t = 0\). Moreover, we assume that the first occurrence of the rare event is governed by a time-invariant probability distribution, i.e.,:

\[
\mathbb{P}(A_{t+1} = 1|X_t \preceq x, A_{1:t} = 0) = \mathbb{P}(A_{t'+1} = 1|X_{t'} \preceq x, A_{1:t'} = 0), \quad \forall t, t' \geq 0,
\]

where, for each \(x, y \in \mathbb{R}^n\), the notation \(x \preceq y\) represents \(x_i \leq y_i\) for each \(i \in [n] := \{1, \ldots, n\}\).

We restate \(Q\), first defined in the introduction, as the following hypothesis testing problem.
**Definition 1**  The binary hypothesis test, with null hypothesis \( H_0 \) defined below, is a mathematically rigorous characterization of \( Q \):

\[
H_0 : \quad P(A_{t+1} = 1|X_t \leq x, A_{1:t} = 0) = P(A_{t+1} = 1|A_{1:t} = 0), \quad \forall x \in \mathbb{R}.
\]

(Note that, if \( H_0 \) were true, then \( P(A_{t+1} = 1|A_{1:t} = 0) \) would inherit the time-invariance property of \( P(A_{t+1} = 1|X_t \leq x, A_{1:t} = 0) \).)

In words, the null hypothesis \( H_0 \) is true if and only if the first occurrence of the rare event transpires independently of the system state at that time. For convenience, define the left and right hand sides of \( H_0 \) by:

\[
a_1(x) := P(A_{t+1} = 1|X_t \leq x, A_{1:t} = 0), \quad a_2 := P(A_{t+1} = 1|A_{1:t} = 0).
\]

**Running Example**  Consider a parallel link traffic network of \( R \) links that straddle a single source and a single destination. Let \( X_{t,i} \in \mathbb{R} \) denote the traffic flow on every link \( i \in [R] := \{1, \cdots, R\} \) at time \( t \). (In general, one can define \( X_{t,i} \in \mathbb{R}^d \) to encapsulate other observed quantities relevant to link \( i \) at time \( t \), including vehicle speed and pavement quality). The event variable \( A_t = 1 \) corresponds to the occurrence of an accident in the network at time \( t \).

In this context, Definition 1 corresponds to checking whether the first occurrence of an accident on the \( R \)-link network at time \( t \) is affected by the flow level at time \( t - 1 \). This question may be of interest to traffic authorities, since if the occurrences of costly accidents becomes more likely at certain levels of traffic flow \( X_t \), then the flow should be monitored to decrease the chance that such accidents occur. Flow management can be applied by dynamically tolling the links, as in Maheshwari et al. (2022a). As accidents are relatively rare in most traffic datasets, it can be difficult to construct accurate estimates of accident probabilities and flows before accidents at any given time \( t \). Instead, we propose a novel method of data aggregation that allows the use of information on accident occurrences across all times.

Since \( X_t \) is a continuous random variable, a direct comparison of (1) and (2) would necessitate computing (1) for uncountably many values of \( x \in \mathbb{R}^n \). Instead, we use the laws of conditional and total probability to reformulate the problem. In the spirit of Bayes’ rule we compare the state distribution immediately before the rare event occurred, instead of the rare event probabilities under different states. Formally, we observe that the state distribution immediately before the first accident can be decomposed as the following infinite sum; for each \( x \in \mathbb{R}^n \):

\[
P(X_{T-1} \leq x) = \sum_{t=1}^{\infty} P(X_{t-1} \leq x, T = t)
\]

\[
= \sum_{t=1}^{\infty} P(X_{t-1} \leq x, A_t = 1, A_{1:t-1} = 0)
\]

\[
= \sum_{t=1}^{\infty} P(X_{t-1} \leq x, A_{1:t-1} = 0) \cdot P(A_t = 1|X_{t-1} \leq x, A_{1:t-1} = 0).
\]

Intuitively, if \( H_0 \) were true, then the condition \( X_{t-1} \leq x \) in the term \( P(X_{t-1} \leq x, A_{1:t-1} = 0) \) can be dropped. This observation is rigorously formulated as Proposition 2, stated below.
Proposition 2  The null hypothesis \( H_0 \) in Definition 1 holds if and only if, for each \( x \in \mathbb{R}^n \):

\[
P(X_{T-1} \preceq x) = \sum_{t=1}^{\infty} P(X_{t-1} \preceq x|A_{1:t-1} = 0) \cdot P(T = t).
\]  

(3)

Proof  Please see Appendix A.

For convenience, we define, for each \( t \in \mathbb{N} \) and \( x \in \mathbb{R} \):

\[
b_1(x) := P(X_{T-1} \preceq x),
\]

\[
\beta_t(x) := P(X_{t-1} \preceq x, A_{1:t-1} = 0),
\]

\[
\gamma_t := P(A_t = 1|A_{1:t-1} = 0),
\]

\[
b_2(x) := \sum_{t=1}^{\infty} \beta_t(x) \cdot \gamma_t = \sum_{t=1}^{\infty} P(X_{t-1} \preceq x, A_{1:t-1} = 0) \cdot P(A_t = 1|A_{1:t-1} = 0)
\]

\[
= \sum_{t=1}^{\infty} P(X_{t-1} \preceq x|A_{1:t-1} = 0) \cdot P(T = t).
\]

Running Example  In the traffic network example, \( b_1(x) \) corresponds to the probability that the network flows before the first accident, \( X_{T-1} \), is component-wise less than or equal to \( x \). Meanwhile, \( b_2(x) \) described the weighted average of traffic flows at each time \( t \), conditioned on the first accident occurring after \( t \), with the first accident time distribution of \( T \) as weights. Section 4 describes sample-efficient methods for constructing empirical estimates of \( b_1(x) \) and \( b_2(x) \) from a dataset of independent traffic flows.

4. Methods

4.1. Main Algorithm

We present Algorithm 1, which solves the hypothesis testing problem in Definition 1 from a dataset of \( N \) independent trajectories, by constructing and comparing finite-sample estimates \( \hat{b}_N^1(x) \) and \( \hat{b}_N^2(x) \) for the expressions \( b_1(x) \) and \( b_2(x) \), respectively, and verifying whether or not (3) holds (in accordance with Proposition 2). Observe that \( \hat{b}_N^1(x) \) and \( \hat{b}_N^2(x) \) are empirical cumulative distribution functions (CDFs) constructed from the dataset of \( N \) independent trajectories, and can thus be completely specified by \( N \) state values.

Note  The common baseline method for resolving the problem in Definition 1 is to fix \( t \geq 1 \), and compare the CDF values \( P(X_{t-1} \preceq x|T = t) \) and \( P(X_{t-1} \preceq x) \), for each \( x \in \mathbb{R}^n \). This is effectively a “static variant” of Algorithm 1 that only utilizes dynamical state values immediately before accidents that occur at time \( t \).

4.2. Theoretical Guarantees:

Theorem 3 below illustrates that, if \( H_0 \) in Definition 1 holds, and the distributions (4) and (5) are identical, then as the number of sample trajectories \( N \) approaches infinity, the empirical distributions of (4) and (5) converge at an exponential rate. The theorem follows by carefully applying concentration bounds for light-tailed random variables, and invoking the Glivenko-Cantelli Theorem, which describes for the convergence of empirical CDFs.
Algorithm 1: Hypothesis Testing with Reorganized Dataset

Data: Dataset of system state and rare event variables: \( \{(X_t^i, A_t^i) : t \geq 0, i \in [N]\} \)

Result: Distribution gap: \( \sup_{x \in \mathbb{R}} |\hat{b}_1^N(x) - \hat{b}_2^N(x)| \)

\[ T^N \leftarrow \text{Value of } T \text{ for data trajectory } i, \forall i \in [N] \]

\[ \hat{b}_1^N(x) \leftarrow \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{X_{t_{i}} \leq x\}. \]

\[ \hat{\beta}_t^N(x) \leftarrow \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{X_{t_i} \leq x, A_{1,t-1} = 0\}. \]

\[ \hat{\gamma}_t^N \leftarrow \begin{cases} \frac{\sum_{i=1}^{N} \mathbf{1}\{A_{1,t-1}=0\}}{\sum_{i=1}^{N} \mathbf{1}\{A_{1,t-1}=0\}}, \quad \text{if } \sum_{i=1}^{N} \mathbf{1}\{A_{1,t-1}=0\} \neq 0, \\ 0, \quad \text{else.} \end{cases} \]

\[ \hat{b}_2^N(x) \leftarrow \sum_{t=1}^{\infty} \hat{\beta}_t^N(x) \cdot \hat{\gamma}_t^N. \]

Return \( \sup_{x \in \mathbb{R}} |\hat{b}_1^N(x) - \hat{b}_2^N(x)| \).

Theorem 3 (Exponential Convergence to Consistency Against all Alternatives) Suppose the null hypothesis holds, i.e., \( b_1(x) = b_2(x) \).

1. If \( n = 1 \), i.e., \( X_t \in \mathbb{R} \) for each \( t \geq 0 \), then for each \( \epsilon > 0 \), there exists continuous, positive functions \( C_1(\epsilon), C_2(\epsilon) > 0 \) such that:

\[
\mathbb{P} \left( \sup_{x \in \mathbb{R}} \left\{ |\hat{b}_1^N(x) - \hat{b}_2^N(x)| \right\} > \epsilon \right) \leq C_1(\epsilon) \cdot e^{-N \cdot C_2(\epsilon)}.
\]

2. If \( n > 1 \), then there exists continuous, positive functions \( C_3(\epsilon), C_4(\epsilon) > 0 \) such that:

\[
\mathbb{P} \left( \sup_{x \in \mathbb{R}^n} \left\{ |\hat{b}_1^N(x) - \hat{b}_2^N(x)| \right\} > \epsilon \right) \leq \left[ C_3(\epsilon) (N + 1)n + C_4(\epsilon) \right] \cdot e^{-N \cdot C_5(\epsilon)}.
\]

Moreover, for sufficiently large \( N \), the factor \( N + 1 \) can be replaced by the constant 2.

Proof Please see Appendix B.

5. Results

In this section, we illustrate the numerical performance of our proposed method, and its superiority over naive aggregation methods of concatenating data points along a single, fixed time \( t \).

5.1. Simulated Data

In our first set of experiments, we construct synthetic data for single-link and multi-link traffic networks. For the single-link network, we use the dynamics:

\[
x[t + 1] = (1 - \mu(A[t])) \cdot x[t] + \mu(A[t]) \cdot u[t] + w[t], \quad \forall t \in [T],
\]

\[
A[t + 1] \sim \mathcal{P}(x[t])
\]

where \( x[t] \in \mathbb{R} \) denotes the traffic flow at time \( t \), \( A[t] \in \{0, 1\} \) is the Boolean random variable that indicates whether or not an accident has occurred at time \( t \), \( u[t] \in \mathbb{R} \) denotes the total input traffic.
flow, $w[t] \in \mathbb{R}$ is a zero-mean noise term, and $T_h$ is the finite time horizon. In our experiments, we set $T_h = 500$, $\mu(0) = 0.3$, $\mu(1) = 0.2$, $u(t) = 100$ for each $t \in [T]$, and draw $w(t)$ i.i.d. from the continuous uniform distribution on $(-10, 10)$. We create datasets corresponding to the null and alternative hypothesis. For the null hypothesis dataset, we fix $P(x[t]) = \text{Bernoulli}(0.01)$, regardless of the value of $x[t]$. This simulates a scenario where the likelihood of an accident occurring has no dependence on traffic flow. For the alternative hypothesis dataset, we set $P(x[t]) = \text{Bernoulli}(0.01)$ when $x[t] < 109$ and $P(x[t]) = \text{Bernoulli}(0.10)$ when $x[t] \geq 109$. This represents a scenario where higher traffic loads increase the likelihood that an accident occurs.

To contrast the performance of our algorithm with the baseline, we compute the following quantities from datasets of independent trajectories corresponding to $H_0$ and $H_1$, in accordance with Proposition 2 and Theorem 3:

- **For our method**—We compute the empirical estimates $\hat{b}_1^N(x)$ and $\hat{b}_2^N(x)$ of the functions $b_1(x)$ and $b_2(x)$ as functions of $x$ (Figure 1), and the maximum CDF gap $\sup_{x \in \mathbb{R}} |\hat{b}_1^N(x) - \hat{b}_2^N(x)|$ as functions of $N$ (Figure 2).

- **For the baseline method**—We compute the empirical estimates of the CDFs of $X_{t-1}|T = t$ and $X_{t-1}$, with $t$ fixed at 1, as functions of $x$ (Figure 1), and the corresponding maximum gap as functions of $N$ (Figure 2). Note that for $N < 500$, it is difficult to obtain the CDF of $X_{t-1}|T = t$ for any fixed $t$, due to the low value of each $P(T = t)$.

Figures 1 and 2 demonstrate that, compared to the baseline method, our approach is able to distinguish between the null and alternative hypotheses from a far smaller dataset. This illustrates the superiority of our method, compared to the baseline, in identifying distinguishing the dependence between the occurrence of a rare event and the state values immediately preceding the event.

For further empirical results on synthetic datasets corresponding to multi-link networks, please see Appendix C.
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5.2. PEMS Dataset

We now demonstrate the efficacy of our algorithm for data aggregation to conduct the hypothesis test in Proposition 2 on real traffic flow and incident data collected from the Caltrans Performance Measurement System (PeMS). We consider traffic flow and speed data collected from January to June 2022 from 6am to 6pm daily, at 5-minute intervals, on a selection of bridges in the Bay Area. That is, we consider single link networks connecting a source and destination with the continuous variables $X_t \in \mathbb{R}_+$ corresponding to average flows or average speeds on the link. Correspondingly, we use incident data collected on these bridges by PeMS in the same time interval.

6. Conclusion

We present a novel method for identifying causal links between the state evolution of a dynamical system and the onset of an associated rare event. Crucially, we leverage the time-invariance of frequently encountered dynamical models to construct a reorganized data set, in which the rare event is better represented. We then formulate a novel statistical independence test for inferring causal dependencies between the states of the dynamical system and the rare event. Empirical results on time-series data indicate that our method outperforms a baseline approach that performs independence tests only a single time slice of the original dataset, in which rare events occur sparsely.

Future work will explore the use of our method to more effectively control the evolution of a dynamical system associated with a rare but disastrous event, and thus improve the utility associated with the system. By establishing causal links between the dynamical state and the rare disaster, control strategies can be redesigned to steer the dynamical state away from regions of the state space where the event occurs more frequently. Important engineering applications include incentive design and flow control methods in the network traffic systems literature, such as dynamic tolling and rerouting. We will also develop more flexible formulations of our method to study more complicated models of rare events, such as continuous and multivariate random variables.

For the appendix, please see the ArXiV version of our work. The authors will ensure that the above link stays active.

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Appendix A. Appendix A: Preliminaries

Here, we present the proof of Proposition 2.

**Proof (Proof of Proposition 2)** Observe that:

\[
P(X_{T-1} \leq x) \tag{4}
\]

\[
= \sum_{t=1}^{\infty} P(X_{t-1} \leq x \mid T = t) \cdot P(T = t)
\]

\[
= \sum_{t=1}^{\infty} P(X_{t-1} \leq x, T = t)
\]

\[
= \sum_{t=1}^{\infty} P(X_{t-1} \leq x, A_{1:t-1} = 0, A_t = 1)
\]

\[
= \sum_{t=1}^{\infty} P(A_t = 1 \mid X_t \leq x, A_{1:t-1}) \cdot P(X_{t-1} \leq x, A_{1:t-1} = 0)
\]

\[
= a_1(x) \cdot \sum_{t=1}^{\infty} P(X_{t-1} \leq x, A_{1:t-1} = 0).
\]

and:

\[
\sum_{t=1}^{\infty} P(X_{t-1} \leq x, A_{1:t-1} = 0) \cdot P(A_t = 1 \mid A_{1:t-1} = 0) \tag{5}
\]

\[
= a_2 \cdot \sum_{t=1}^{\infty} P(X_{t-1} \leq x, A_{1:t-1} = 0).
\]

Thus, the null hypothesis \( H_0 \) in Definition 1 holds if and only if (4) and (5) are equal, as claimed. ■

Appendix B. Appendix B: Methods

**Proposition 4 (Corollary to the Dvoretzky-Kiefer-Wolfowitz Inequality)** Let \( X \in \mathbb{R}^n \) be a random variable defined on the probability space \( (\Omega, \Sigma, \mathbb{P}) \), and let \( E \in \Sigma \). Fix \( \epsilon > 0 \) and \( N \in \mathbb{N} \), and let \( X_1, \ldots, X_N \) be i.i.d. copies of \( X \). Then, for each \( N \geq \mathbb{N} \) and each \( \epsilon > 0 \):

\[
\mathbb{P} \left( \sup_{x \in \mathbb{R}^n} \left| \mathbb{P}(X \leq x, E) - \frac{1}{N} \sum_{i=1}^{N} 1\{X_i \leq x, E\} \right| \right) \leq 2e^{-2N\epsilon^2}.
\]

**Proof** Let \( G_1, \ldots, G_N \) be drawn i.i.d. from the continuous uniform \((0, 1)\) distribution. Then:

\[
\mathbb{P} \left( \sup_{x \in \mathbb{R}^n} \left| \mathbb{P}(X \leq x, E) - \frac{1}{N} \sum_{i=1}^{N} 1\{X_i \leq x, E\} \right| > \epsilon \right)
\]

\[
= \mathbb{P} \left( \sup_{x \in \mathbb{R}^n} \left| \mathbb{P}(X \leq x, E) - G_n(\mathbb{P}(X \leq x, E)) \right| > \epsilon \right)
\]

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\begin{align*}
\leq & \mathbb{P}\left( \sup_{t \in [0,1]} \left| t - G_n(t) \right| > \epsilon \right) \\
\leq & 2e^{-2N\epsilon^2},
\end{align*}

where the final inequality follows by applying the Dvoretzky-Kiefer-Wolfowitz Inequality to the continuous uniform (0, 1) distribution.

**Proof** Fix \( \epsilon > 0 \), and take:

\[
T_c(\epsilon) := \left\lceil \frac{1}{\ln(1 - p_1)} \ln \left( \frac{e p_1^2}{16 \rho_2} \right) \right\rceil.
\]

First, to show that \( \hat{b}_1^N(x) \to b_1(x) \) at an exponential rate in \( N \), we invoke the Dvoretzky-Kiefer-Wolfowitz inequality:

\[
\mathbb{P}\left( \sup_{x \in \mathbb{R}} \left| \hat{b}_1^N(x) - b_1(x) \right| > \frac{1}{2} \epsilon \right) \leq 2 \cdot e^{-\frac{1}{2} N \epsilon^2}
\]

Next, to show that \( \hat{b}_2^N(x) \to b_2(x) \) at an exponential rate in \( N \), we have:

\[
\begin{align*}
\sup_{x \in \mathbb{R}} \left| \left\{ \hat{b}_2^N(x) - b_2(x) \right\} \right| &= \sup_{x \in \mathbb{R}} \left| \sum_{t=1}^{T_c} \left[ \beta_t^N(x) \hat{\gamma}_t^N - \beta_t(x) \gamma_t \right] \right| \\
&= \sum_{t=1}^{T_c} \left[ \sup_{x \in \mathbb{R}} \left| \beta_t^N(x) - \beta_t(x) \right| \right] \hat{\gamma}_t^N + \left( \sum_{t=1}^{T_c} \sup_{x \in \mathbb{R}} \left| \hat{\beta}_t^N(x) \right| + \sup_{x \in \mathbb{R}} \left| \gamma_t \right| \right) \beta_t(x) \\
&\leq \sum_{t=1}^{T_c} \sup_{x \in \mathbb{R}} \left| \beta_t^N(x) - \beta_t(x) \right| + \sum_{t=1}^{T_c} \sup_{x \in \mathbb{R}} \left| \hat{\gamma}_t^N - \gamma_t \right| \cdot \mathbb{P}(A_{1:t-1} = 0) \\
&+ \frac{1}{N} \sum_{n=1}^{N} \sum_{t=T_c+1}^{\infty} 1\{T_n \geq t\} + \sum_{t=T_c+1}^{\infty} \mathbb{P}(T \geq t).
\end{align*}
\]

Below, we upper bound each of the four terms in the final expression above.

- First, by the Dvoretzky-Kiefer-Wolfowitz inequality, we have, for each \( t \in [T_c] := \{1, \cdots, T_c\} \):

\[
\mathbb{P}\left( \sum_{t=1}^{T_c} \sup_{x \in \mathbb{R}} \left| \beta_t^N(x) - \beta_t(x) \right| \geq \frac{1}{8} \epsilon \right) \leq \sum_{t=1}^{T_c} \mathbb{P}\left( \sup_{x \in \mathbb{R}} \left| \beta_t^N(x) - \beta_t(x) \right| \geq \frac{1}{8T_c} \epsilon \right) \\
\leq 2T_c \exp \left( -\frac{\epsilon^2}{32T_c^2} \cdot N \right).
\]
• Let $N_t \in [N]$ denote the number of trajectories with $A_{1:t-1} = 0$. We first show that, with high probability, $N_t \geq N \cdot \mathbb{P}(A_{1:t-1} = 0)^2$. We then show that, under this condition on $N_t$ taking a sufficiently large value, $\hat{\gamma}_t^N(x) \to \gamma_t(x)$ exponentially in $N$.

First, by the Hoeffding bound for general bounded random variables (Vershynin Vershynin (2018), Theorem 2.2.6), we have:

$$
\mathbb{P} \left( \frac{1}{N} N_t \leq \mathbb{P}(A_{1:t-1} = 0)^2 \right)
\leq \mathbb{P} \left( \left| \frac{1}{N} N_t - \mathbb{P}(A_{1:t-1} = 0) \right| > \mathbb{P}(A_{1:t-1} = 0) - \mathbb{P}(A_{1:t-1} = 0)^2 \right)
\leq \exp \left( -2 \left[ \mathbb{P}(A_{1:t-1} = 0) - \mathbb{P}(A_{1:t-1} = 0)^2 \right]^2 \cdot N \right)
$$

Then, if $N_t \geq N \cdot \mathbb{P}(A_{1:t-1} = 0)$:

$$
\mathbb{P} \left( |\hat{\gamma}_t^N(x) - \gamma_t(x)| > \frac{\epsilon}{8T_c \cdot \mathbb{P}(A_{1:t-1} = 0)} \right)
\leq \exp \left( -2 \cdot \mathbb{P}(A_{1:t-1} = 0)^2 \cdot N \cdot \frac{\epsilon^2}{64T_c^2 \cdot \mathbb{P}(A_{1:t-1} = 0)^2} \right)
\leq \exp \left( -\frac{\epsilon^2}{32T_c^2} \cdot N \right).
$$

• To bound the third term, $\frac{1}{N} \sum_{n=1}^N \sum_{\tau=T_c+1}^{\infty} 1 \{ \hat{T}_n \geq t \}$, define:

$$
B_{T_c} := \sum_{\tau=T_c+1}^{\infty} \sum_{t=T_c+1}^{\infty} 1 \{ T = \tau \}
= \sum_{\tau=T_c+1}^{\infty} \sum_{t=T_c+1}^{\infty} 1 \{ T = \tau \}
= \sum_{\tau=T_c+1}^{\infty} \sum_{t=T_c+1}^{\infty} 1 \{ T = \tau \}
= \sum_{\tau=T_c+1}^{\infty} (\tau - T_c) 1 \{ T = \tau \}
= \sum_{\tau=1}^{\infty} \tau 1 \{ T = \tau + T_c \}.
$$

Thus, we have:

$$
\mathbb{E}[B_{T_c}] = \sum_{\tau=1}^{\infty} \tau \cdot \mathbb{P}(T = \tau + T_c) \leq \sum_{\tau=1}^{\infty} \tau \cdot (1 - p_1)^{\tau+T_c-1} p_2
$$

$$
\leq (1 - p_1)^{T_c} \cdot \frac{p_2}{p_1} \sum_{\tau=1}^{\infty} \tau (1 - p_1)^{\tau-1} p_1
$$

$$
= \frac{(1 - p_1)^{T_c} \cdot p_2}{p_1^2},
$$

and:

$$
\mathbb{P}(B_{T_c} \leq t) \leq (1 - p_1)^{t+k-1} p_2 \leq 2 \exp \left( -\ln \left( \frac{1}{1 - p_1} \right) t \right).
$$
Thus, $B_{T_c} - \mathbb{E}[B_{T_c}]$ is sub-exponential, with: (see Vershynin (2018), Proposition 2.7.1)

$$\|B_{T_c} - \mathbb{E}[B_{T_c}]\|_{\psi_1} = 6 \sqrt{\frac{e}{\ln 2}} \left( 1 + \frac{6}{\ln 2} \right) \cdot \frac{-1}{\ln(1 - p_1)}.$$  

Applying the Bernstein inequality for zero-mean sub-exponential variables (Vershynin, Vershynin (2018)), we obtain:

$$\mathbb{P} \left( \frac{1}{N} \sum_{n=1}^{N} B_{T_c}^N > \frac{1}{8} \epsilon \right) = \mathbb{P} \left( \frac{1}{N} \sum_{n=1}^{N} B_{T_c}^N - \mathbb{E}[B_{T_c}] > \frac{1}{8} \epsilon - \mathbb{E}[B_{T_c}] \right) \leq \mathbb{P} \left( \frac{1}{N} \sum_{n=1}^{N} B_{T_c}^N - \mathbb{E}[B_{T_c}] > \frac{1}{16} \epsilon \right)$$

$$= \exp \left( - \min \left\{ - \frac{1}{96} \sqrt{\frac{2 \ln 2}{e}} \left( \frac{\ln 2}{6 + \ln 2} \right) \ln(1 - p_1) C_1 \cdot \epsilon, \frac{\ln 2}{9216 e} \left( \frac{\ln 2}{6 + \ln 2} \right)^2 \left[ \ln(1 - p_1) \right]^2 C_2 \cdot \epsilon^2 \right\} \cdot N \right),$$

where:

$$C_1 = \sqrt{\frac{2 \pi}{48 e^2 + 1/e}} \approx 0.00489,$$

$$C_2 = \frac{1}{4 e^{1 + 1/(2 e)}} \approx 0.0765.$$  

• Finally, note that by definition of $\epsilon > 0$:

$$\sum_{t=T_c+1}^{\infty} \mathbb{P}(T \geq t) \leq \sum_{t=T_c+1}^{\infty} (1 - p_1)^{t-1} = \frac{1}{p_1} (1 - p_1)^{T_c} \leq \frac{P_2}{P_1} (1 - p_1)^{T_c} = \frac{1}{8} \epsilon.$$  

For the multivariate version (i.e., $n > 1$), the same proof follows, albeit with the multivariate version of the Dvoretzky-Kiefer-Wolfowitz-Massart inequality. For more details, see Naaman, 2021 Naaman (2021).

**Appendix C. Appendix C: Experiment Results**

For the multi-link traffic network, we use the dynamics: (Maheshwari et al. (2022a))

$$x_i[t + 1] = (1 - \mu) \cdot x_i[t] + \mu \cdot \frac{e^{-\beta \cdot x_i[t]}}{\sum_{j=1}^{R} e^{-\beta \cdot x_j[t]}} \cdot u[t] + w[t], \quad \forall t \in [T], i \in [R],$$
$A[t] \sim \mathcal{P}(x[t])$, 

where $x_i[t]$ denotes the traffic flow on each link $i \in [R]$, $u[t] \in \mathbb{R}$ and $w[t] \in \mathbb{R}$, and $T_h$, are the input, zero-mean noise terms, and time horizon, as before. Here, we set $T = 250$, $\mu(0) = 0.3$, $\mu(1) = 0.2$, $u(t) = 100R$ for each $t \in [T]$, and we again draw $w[t]$ i.i.d. from the continuous uniform distribution on $(-10, 10)$. As with the single-link case, we created two datasets for the null and alternative hypotheses. For the null hypothesis, we fix $\mathcal{P}(x[t])$ to be Bernoulli(0.02); for the alternative hypothesis, we set $\mathcal{P}(x[t])$ to be Bernoulli(0.02) when $x[t] < 105$, and Bernoulli(0.30) when $x[t] \geq 105$. Again, this setting encodes the situation where higher traffic loads cause higher accident probabilities.