Multi-Step Model-Agnostic Meta-Learning: Convergence and Improved Algorithms

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Abstract

As a popular meta-learning approach, the model-agnostic meta-learning (MAML) algorithm has been widely used due to its simplicity and effectiveness. However, the convergence of the general multi-step MAML still remains unexplored. In this paper, we develop a new theoretical framework, under which we characterize the convergence rate and the computational complexity of multi-step MAML. Our results indicate that $N$-step MAML attains the convergence with linearly increasing complexity with $N$ under a properly chosen inner stepsize. We then take a further step to develop a more efficient Hessian-free MAML. We first show that the existing zeroth-order Hessian estimator contains a constant-level estimation error so that the MAML algorithm can perform unstably. To address this issue, we propose a novel Hessian estimator via a gradient-based Gaussian smoothing method, and show that it achieves a much smaller estimation bias and variance, and the resulting algorithm achieves the same performance guarantee as the original MAML under mild conditions. Our experiments validate our theory and demonstrate the effectiveness of the proposed Hessian estimator.

1 Introduction

Meta-learning or learning to learn (Thrun and Pratt, 2012; Naik and Mammone, 1992; Bengio et al., 1991) is a powerful tool for quickly learning new tasks by using the prior experience from related tasks. Recent works have empowered this idea with neural networks, and their proposed meta-learning algorithms have been shown to enable fast learning over unseen tasks using only a few samples by efficiently extracting the knowledge from a range of observed tasks (Santoro et al., 2016; Vinyals et al., 2016; Finn et al., 2017a). Current meta-learning algorithms can be generally categorized into metric-learning based (Koch et al., 2015; Snell et al., 2017), model-based (Vinyals et al., 2016; Munkhdalai and Yu, 2017), and optimization-based (Finn et al., 2017a; Nichol and Schulman, 2018; Rajeswaran et al., 2019) approaches. Among them, optimization-based meta-learning is a simple and effective approach used in a wide range of domains including classification/regression (Rajeswaran et al., 2019), reinforcement learning (Finn et al., 2017a), robotics (Al-Shedivat et al., 2017), federated learning (Chen et al., 2018), and imitation learning (Finn et al., 2017b).

Model-agnostic meta-learning (MAML) (Finn et al., 2017a) is a popular optimization-based method, which is simple and compatible generally with models trained with gradient descents. MAML consists of

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two nested stages, where the inner stage runs a few steps of (stochastic) gradient descent for each individual task, and the outer stage updates the meta parameter over all the sampled tasks. The goal of MAML is to find a good meta initialization $w^*$ based on the observed tasks such that for a new task, starting from this $w^*$, a few (stochastic) gradient steps suffice to find a good model parameter. Although having been widely used (Antoniou et al., 2018; Grant et al., 2018; Zintgraf et al., 2018; Nichol et al., 2018), the theoretical convergence of MAML is not well explored except a few attempts recently. In particular, Finn et al. (2019) extended MAML to the online setting, and analyzed the regret for the strongly convex objective function. More recently, Fallah et al. (2019) provided an analysis for one-step MAML for general nonconvex functions, where each inner stage takes only a single stochastic gradient descent (SGD) step. However, the convergence of the more general and frequently-used multi-step MAML has not been explored yet. Several new challenges arise here in theoretical analysis due to the multi-step inner stage updates in MAML. First, the meta gradient of multi-step MAML has a nested and recursive structure, which requires the analysis of the performance of an optimization path over a nested structure. In addition, multi-step update also yields a complicated bias error in the Hessian estimation as well as the statistical correlation between the Hessian and gradient estimators, both of which cause further difficulty in the analysis of the meta gradient.

- The first contribution of this paper lies in the development of a new theoretical framework for analyzing multi-step MAML with techniques for handling the above challenges. For the resampling case where each iteration needs sampling of fresh data (e.g., in reinforcement learning), our analysis enables to decouple the Hessian approximation error from the gradient approximation error based on a novel bound on the distance between two different inner optimization paths, which facilitates the analysis of the overall convergence of MAML. For the finite-sum case where the objective function is based on pre-assigned samples (e.g., supervised learning), we develop novel techniques to handle the difference between two losses over the training and test sets in the analysis.

Our analysis provides a guideline for choosing the inner-stage stepsize at the order of $O(1/N)$ and shows that $N$-step MAML is guaranteed to converge with the gradient and Hessian computation complexities growing only linearly with $N$. In addition, for problems where Hessians are small, e.g., most classification/regression meta-learning problems (Finn et al., 2017a), we show that the inner stepsize $\alpha$ can be set larger while still maintaining the convergence, which explains the empirical findings for MAML training in Finn et al. (2017a); Antoniou et al. (2018); Rajeswaran et al. (2019).

Although achieving promising performance in many areas, MAML has a high computation and memory cost due to the computation of Hessians, and such a cost increases dramatically if the inner stage takes multiple gradient steps (Rajeswaran et al., 2019). To address such an issue, Hessian-free MAML algorithms have gained a lot of attention recently (Nichol and Schulman, 2018; Finn et al., 2017a; Song et al., 2020). Among them, Song et al. (2020) recently proposed an evolution strategies based MAML (ES-MAML) approach by using the zeroth-order (i.e., the function value) information for Hessian approximation. Though ES-MAML exhibits promising empirical performance, Song et al. (2020) did not provide the convergence guarantee for this algorithm, and their experiments demonstrated that the zeroth-order Hessian approximation sometimes leads to an inferior and unstable training in some RL problems.

- The second contribution of this paper includes two parts. We first provide a theoretical analysis of the zeroth-order Hessian approximation (Song et al., 2020) and show that such estimation contains a constant-level error so that ES-MAML cannot converge exactly to a stationary point but with a
possibly large error. This explains the inferior and unstable training by ES-MAML observed in Song et al. (2020).

We then propose a first-order Hessian estimator by Gaussian smoothing, which we show to achieve a much smaller bias and variance. More importantly, we show the resulting algorithm (which we call as GGS-MAML) achieves the same performance guarantee as the original MAML under mild conditions. Our analysis develops novel techniques for charactering the variance of Gaussian smoothing based Hessian estimators by random matrix theory, which can be of independent interest. Our experiments validate the effectiveness of GGS-MAML.

1.1 Related Work

Optimization-based meta-learning. Optimization-based meta-learning approaches have been widely used due to its simplicity and efficiency (Li et al., 2017; Ravi and Larochelle, 2016; Finn et al., 2017a). As a pioneer along this line, MAML (Finn et al., 2017a) aims to find an initialization such that gradient descent from it achieves fast adaptation. Many follow-up studies (Grant et al., 2018; Finn et al., 2019; Jerfel et al., 2018; Finn and Levine, 2017; Finn et al., 2018; Mi et al., 2019; Liu et al., 2019; Rothfuss et al., 2018; Foerster et al., 2018; Fallah et al., 2019; Collins et al., 2020; Fallah et al., 2020) have extended MAML from different perspectives. For example, Finn et al. (2019) provided a follow-the-meta-leader extension of MAML for online learning. Alternatively to meta-initialization algorithms such as MAML, meta-regularization approaches aim to learn a good bias for a regularized empirical risk minimization problem for intra-task learning (Alquier et al., 2016; Denevi et al., 2018b,a, 2019; Rajeswaran et al., 2019; Balcan et al., 2019; Zhou et al., 2019). For example, Rajeswaran et al. (2019) proposed efficient iMAML using a conjugate gradient (CG) based solver. Balcan et al. (2019) formalized a connection between meta-initialization and meta-regularization from an online learning perspective. Zhou et al. (2019) proposed an efficient meta-learning approach based on a minibatch proximal update.

Theory for MAML-type algorithms. There have been only a few studies on the statistical and convergence performance of MAML-type algorithms. Finn and Levine (2017) showed that MAML is a universal learning algorithm approximator under certain conditions. Finn et al. (2019) analyzed online MAML for a strongly convex objective function under a bounded-gradient assumption. Rajeswaran et al. (2019) proposed a meta-regularization variant of MAML named iMAML, and analyzed its convergence by assuming that the regularized empirical risk minimization problem in the inner optimization stage is strongly convex. Fallah et al. (2020) proposed a variant of RL-MAML named stochastic gradient meta-reinforcement Learning (SG-MRL), and analyzed the convergence and complexity performance of one-step SG-MRL. Fallah et al. (2019) developed a convergence analysis for one-step MAML for a general nonconvex objective in the resampling case. Our study here provides a new convergence analysis for multi-step MAML in the nonconvex setting for both the resampling and finite-sum cases.

Hessian-free MAML. Various Hessian-free MAML algorithms have been proposed, which include but not limited to FOMAML (Finn et al., 2017a), Reptile (Nichol and Schulman, 2018), ES-MAML (Song et al., 2020), and HF-MAML (Fallah et al., 2019). In particular, FOMAML (Finn et al., 2017a) omits all second-order derivatives in its meta-gradient computation, ES-MAML (Song et al., 2020) approximates Hessian matrices in RL using a zeroth-order smoothing method, and HF-MAML (Fallah et al., 2019) estimates the meta gradient in one-step MAML using Hessian-vector product approximation. In this paper, we propose a new
Hessian-free MAML algorithm based on a first-order Hessian estimator and study its analytical and empirical performance.

2 Problem Setup

In this paper, we study the convergence of the multi-step MAML algorithm. We consider two types of objective functions that are commonly used in practice: (a) **resampling case** (Finn et al., 2017a; Fallah et al., 2019), where loss functions take the form in expectation and new data are sampled as the algorithm runs; and (b) **finite-sum case** (Antoniou et al., 2018), where loss functions take the finite-sum form with given samples. The resampling case occurs often in reinforcement learning where data are continuously sampled as the algorithm iterates, whereas the finite-sum case typically occurs in classification problems where the datasets are already sampled in advance. Examples are given in Appendix A for these two types of problems.

2.1 Resampling Case: Problem Setup and Multi-Step MAML

Suppose a set $\mathcal{T} = \{T_i, i \in \mathcal{I}\}$ of tasks are available for learning and tasks are sampled based on a probability distribution $p(\mathcal{T})$ over the task set. Assume that each task $T_i$ is associated with a loss $l_i(w) : \mathbb{R}^d \rightarrow \mathbb{R}$ parameterized by $w$.

The goal of multi-step MAML is to find a good initial parameter $w^*$ such that after observing a new task, a few gradient descend steps starting from such a point $w^*$ can efficiently approach the optimizer (or a stationary point) of the corresponding loss function. Towards this end, multi-step MAML consists of two nested stages, where the inner stage consists of multiple steps of (stochastic) gradient descent for each individual tasks, and the outer stage updates the meta parameter over all the sampled tasks. More specifically, at each inner stage, each $T_i$ initializes at the meta parameter, i.e., $\overline{w}_i^0 := w$, and runs $N$ gradient descent steps as

$$\overline{w}_{i,j+1}^j = \overline{w}_{i,j}^j - \alpha \nabla l_i(\overline{w}_{i,j}^j), \quad j = 0, ..., N - 1.$$  \hspace{1cm} (1)

Thus, the loss of task $T_i$ after the $N$-step inner stage iteration is given by $l_i(\overline{w}_{i,N}^N)$, where $\overline{w}_{i,N}^N$ depends on the meta parameter $w$ through the iteration updates in (1), and can hence be written as $\overline{w}_{i,N}^i(w)$. We further define $L_i(w) := l_i(\overline{w}_{i,N}^N(w))$, and hence the overall meta objective is given by

$$\min_{w \in \mathbb{R}^d} L(w) := \mathbb{E}_{i \sim p(\mathcal{T})} [L_i(w)] := \mathbb{E}_i[l_i(\overline{w}_{i,N}^N(w))].$$  \hspace{1cm} (2)

Then the outer stage of meta update is a gradient decent step to optimize the above objective function. Using the chain rule, we provide a simplified form (see Appendix B for its derivations) of gradient $\nabla L_i(w)$ by

$$\nabla L_i(w) = \left[ \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\overline{w}_{j}^j)) \right] \nabla l_i(\overline{w}_{N}^N),$$  \hspace{1cm} (3)

where $\overline{w}_0^0 = w$ for all task $i$.

Hence, the full gradient descent step of the outer stage for (2) can be written as

$$w_{k+1} = w_k - \beta_k \mathbb{E}_{i \sim p(\mathcal{T})} \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\overline{w}_{k,j}^j)) \nabla l_i(\overline{w}_{k,N}^N),$$  \hspace{1cm} (4)
Algorithm 1 Multi-step MAML in the resampling case

1: **Input:** Initial parameter $w_0$, inner stepsize $\alpha > 0$
2: while not done do
3:   Sample $B_k \subset I$ of i.i.d. tasks by distribution $p(T)$
4:   for all tasks $T_i$ in $B_k$ do
5:     for $j = 0, 1, \ldots, N - 1$ do
6:       Sample a training set $S^i_{k,j}$
7:       Update $w^i_{k,j+1} = w^i_{k,j} - \alpha \nabla l_i(w^i_{k,j}; S^i_{k,j})$
8:     end for
9:   end for
10: end while

where the index $k$ is added to $\tilde{w}^i_j$ in (3) to denote that these parameters are at the $k^{th}$ iteration of the meta parameter $w$.

The inner- and outer-stage updates of MAML given in (1) and (4) involve the gradient $\nabla l_i(\cdot)$ and the Hessian $\nabla^2 l_i(\cdot)$ of the loss function $l_i(\cdot)$, which takes the form of the expectation over the distribution of data samples as given by

$$l_i(\cdot) = \mathbb{E}_{\tau} l_i(\cdot; \tau), \quad (5)$$

where $\tau$ represents the data sample. In practice, these two quantities based on the population loss function are estimated by samples. In specific, each task $T_i$ samples a batch $\Omega$ of data under the current parameter, and uses $\nabla l_i(\cdot; \Omega) := \frac{1}{|\Omega|} \sum_{\tau \in \Omega} \nabla l_i(\cdot; \tau)$ and $\nabla^2 l_i(\cdot; \Omega) := \frac{1}{|\Omega|^2} \sum_{\tau \in \Omega} \nabla^2 l_i(\cdot; \tau)$ as unbiased estimates of the gradient $\nabla l_i(\cdot)$ and the Hessian $\nabla^2 l_i(\cdot)$, respectively.

For practical multi-step MAML as shown in Algorithm 1, at the $k^{th}$ outer stage, we sample a set $B_k$ of tasks. Then, at the inner stage, each task $T_i \in B_k$ samples a training set $S^i_{k,j}$ for each iteration $j$ in the inner stage, uses $\nabla l_i(w^i_{k,j}; S^i_{k,j})$ as an estimate of $\nabla l_i(\tilde{w}^i_{k,j})$ in (1), and runs a SGD update as

$$w^i_{k,j+1} = w^i_{k,j} - \alpha \nabla l_i(w^i_{k,j}; S^i_{k,j}), \quad j = 0, \ldots, N - 1, \quad (6)$$

where the initialization parameter $w^i_{k,0} = w_k$ for all $i \in B_k$.

At the $k^{th}$ outer stage, we draw a batch $T^i_k$ and $D^i_{k,j}$ of data samples independent from each other and both independent from $S^i_{k,j}$ (for $j = 0, \ldots, N - 1$) and use $\nabla l_i(w^i_{k,j}; T^i_k)$ and $\nabla^2 l_i(w^i_{k,j}; D^i_{k,j})$ to estimate $\nabla l_i(\tilde{w}^i_{k,N})$ and $\nabla^2 l_i(\tilde{w}^i_{k,j})$ in (4), respectively. Then, the meta parameter $w_{k+1}$ at the outer stage is updated by the following SGD step

$$w_{k+1} = w_k - \frac{\beta_k}{|B_k|} \sum_{i \in B_k} \tilde{G}_i(w_k), \quad (7)$$

where $\tilde{G}_i(w_k)$ of task $T_i$ is given by

$$\tilde{G}_i(w_k) = \prod_{j=0}^{N-1} \left( I - \alpha \nabla^2 l_i(w^i_{k,j}; D^i_{k,j}) \right) \nabla l_i(w^i_{k,N}; T^i_k). \quad (8)$$

For simplicity, we assume the sizes of the sample sets $S^i_{k,j}$, $D^i_{k,j}$ and $T^i_k$ are $S$, $D$ and $T$ in this paper.
2.2 Finite-Sum Case: Problem Setup and Multi-Step MAML

In this case, each task $T_i$ is pre-assigned with a support/training sample set $S_i$ and a query/test sample set $T_i$. Differently from the resampling case, these sample sets are fixed and no additional fresh data are sampled as the algorithm runs. The goal here is to learn an initial parameter $w$ such that for each task $i$, after $N$ gradient descent steps on data from $S_i$ starting from this $w$, we can find a parameter $w_N$ that performs well on the test data set $T_i$. Thus, each task $T_i$ is associated with two fixed loss functions $l_S(w) := \frac{1}{|S_i|} \sum_{t \in S_i} l_t(w; \tau)$ and $l_T(w) := \frac{1}{|T_i|} \sum_{t \in T_i} l_t(w; \tau)$ with a finite-sum structure, where $l_t(w; \tau)$ is the loss on a single sample point $\tau$ and a parameter $w$. Then, the meta objective function takes the form given by

$$\min_{w \in \mathbb{R}^d} \mathcal{L}(w) := \mathbb{E}_{i \sim p(T)}[\mathcal{L}_i(w)] = \mathbb{E}_{i \sim p(T)}[l_T(\tilde{w}_i^N)],$$

where $\tilde{w}_i^N$ is obtained by

$$\tilde{w}_{j+1}^i = \tilde{w}_j^i - \alpha \nabla l_S(\tilde{w}_j^i), \quad j = 0, 1, ..., N - 1 \text{ with } \tilde{w}_0^i := w.$$

Similarly to the resampling case, we define the expected loss functions $l_S(w) = \mathbb{E}_{i \sim p(T)}[l_S(w)]$ and $l_T(w) = \mathbb{E}_{i \sim p(T)}[l_T(w)]$. Similarly to (4), the meta gradient step of the outer stage for (9) can be written as

$$w_{k+1} = w_k - \beta_k \mathbb{E}_{i \sim p(T)} \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_S(\tilde{w}_{k,j}^i)) \nabla l_T(\tilde{w}_{k,N}^i),$$

where the index $k$ is added to $\tilde{w}_j^i$ in (10) to denote that these parameters are at the $k$th iteration of the meta parameter $w$.

As shown in Algorithm 2, MAML in the finite-sum case has a nested structure similar to that in the resampling case except that it does not sample fresh data at each iteration. In the inner stage, MAML performs a sequence of full gradient descent steps (instead of stochastic gradient steps as in the resampling case) for each task $i \in B_k$ given by

$$w_{k,j+1} = w_{k,j} - \alpha \nabla l_S(w_{k,j}), \quad j = 0, ..., N - 1$$

where $w_{k,0} = w_k$ for all $i \in B_k$. As a result, the parameter $w_{k,j}$ (which denotes the parameter due to the full gradient update) in the update step (12) is equal to $\tilde{w}_{k,j}$ in (11) for all $j = 0, ..., N$.

At the outer-stage iteration, the meta optimization of MAML performs stochastic gradient descent by

$$w_{k+1} = w_k - \beta_k \frac{1}{|B_k|} \sum_{i \in B_k} \hat{G}_i(w_k), \quad \hat{G}_i(w_k) := \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_S(w_{k,j}^i)) \nabla l_T(\tilde{w}_{k,N}^i).$$

Note that $\hat{G}_i(w_k)$ here is an estimate of the true gradient $\nabla \mathcal{L}(w_k) = \mathbb{E}_{i \sim p(T)} \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_S(\tilde{w}_{k,j}^i)) \nabla l_T(\tilde{w}_{k,N}^i)$ of the objective function (9), where $\tilde{w}_{k,0}^i = w_k$ for all tasks $i \in \mathcal{I}$ and $\tilde{w}_{k,j}^i$ are given by the gradient descent steps in (10).

Compared with the resampling case, the biggest difference for analyzing Algorithm 2 in the finite-sum case is that the losses $l_S(\cdot)$ and $l_T(\cdot)$ used in the inner and outer stages respectively are different from each other, whereas in the resampling case, they both are equal to $l_t(\cdot)$ which takes the expectation over the corresponding samples. Thus, the convergence analysis for the finite-sum case requires to develop different techniques. For simplicity, we assume that the sizes of all $B_k$ are $B$. 

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Algorithm 2 Multi-step MAML in the finite-sum case

1: Input: Initial parameter $w_0$, inner stepsize $\alpha > 0$
2: while not done do
3: Sample $B_k \subset \mathcal{I}$ of i.i.d. tasks by distribution $p(T)$
4: for all tasks $\mathcal{T}_i$ in $B_k$ do
5: for $j = 0, 1, ..., N - 1$ do
6: Update \[ w_{k,j+1} = w_{k,j} - \alpha \nabla l_{T_i}(w_{k,j}^i) \]
7: end for
8: end for
9: Update $w_{k+1} = w_k - \beta_k \frac{1}{|B_k|} \sum_{i \in B_k} \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{T_i}(w_{k,j}^i)) \nabla l_{T_i}(w_{k,N}^i)$
10: end while

3 Convergence of Multi-Step MAML in Resampling Case

3.1 Basic Assumptions

We first make the following standard assumptions (Fallah et al., 2019; Rajeswaran et al., 2019).

Assumption 1. The loss $l_i(\cdot)$ of task $\mathcal{T}_i$ given by (5) satisfies

1. The loss $l_i(\cdot)$ is bounded below, i.e., \( \inf_{w \in \mathbb{R}^d} l_i(w) > -\infty. \)
2. The gradient $\nabla l_i(\cdot)$ is $L_i$-Lipschitz for any $i \in \mathcal{I}$, i.e., for any $w, u \in \mathbb{R}^d$,
\[ \| \nabla l_i(w) - \nabla l_i(u) \| \leq L_i \| w - u \|. \]
3. The Hessian $\nabla^2 l_i(\cdot)$ is $\rho_i$-Lipschitz for any $i \in \mathcal{I}$, i.e., for any $w, u \in \mathbb{R}^d$,
\[ \| \nabla^2 l_i(w) - \nabla^2 l_i(u) \| \leq \rho_i \| w - u \|. \]

By the definition of the objective function $\mathcal{L}(\cdot)$ in (2), item 1 of Assumption 1 implies that $\mathcal{L}(\cdot)$ is bounded below. In addition, item 2 implies $\| \nabla^2 l_i(w) \| \leq L_i$ for any $w \in \mathbb{R}^d$. For notational convenience, we take $L = \max_i L_i$ and $\rho = \max_i \rho_i$.

The following assumptions impose the bounded-variance conditions on $\nabla l_i(w)$, $\nabla l_i(w; \tau)$ and $\nabla^2 l_i(w; \tau)$.

Assumption 2. The stochastic gradient $\nabla l_i(\cdot)$ (with $i$ uniformly randomly chosen from set $\mathcal{I}$) has bounded variance, i.e., there exists a constant $\sigma > 0$ such that, for any $w \in \mathbb{R}^d$,
\[ \mathbb{E}_{i \sim p(T)} \| \nabla l_i(w) - \nabla l_i(w) \|^2 \leq \sigma^2, \]
where the expected loss function $l(w) := \mathbb{E}_{i \sim p(T)}[l_i(w)]$.

Assumption 3. For any $w \in \mathbb{R}^d$ and $i \in \mathcal{I}$, there exist constants $\sigma_g, \sigma_H > 0$ such that
\[ \mathbb{E}_\tau \| \nabla l_i(w; \tau) - \nabla l_i(w) \|^2 \leq \sigma_g^2, \]
\[ \mathbb{E}_\tau \| \nabla^2 l_i(w; \tau) - \nabla^2 l_i(w) \|^2 \leq \sigma_H^2. \]

Note that the above assumptions are made only on individual loss functions $l_i(\cdot)$ rather than on the total loss $\mathcal{L}(\cdot)$, because some conditions do not hold for $\mathcal{L}(\cdot)$, as shown later.
3.2 Challenges of Analyzing Multi-Step MAML

Several new challenges arise when we analyze the convergence of multi-step MAML (with $N \geq 2$) compared to the one-step case (with $N = 1$).

First, each iteration of the meta parameter affects the overall objective function via a nested structure of $N$-step SGD optimization paths over all tasks. Hence, our analysis of the convergence of such a meta parameter needs to characterize the nested structure and the recursive updates.

Second, the meta gradient estimator $\hat{G}_i(w_k)$ given in (8) involves $\nabla^2 l_i(w^i; D^i_{k,j})$ for $j = 1, ..., N - 1$, which are all biased estimators of $\nabla^2 l_i(w^i; D^i_{k,j})$ in terms of the randomness over $D^i_{k,j}$. This is because $w^i_{k,j}$ is a stochastic estimator of $w^i_{k,j}$ obtained via random training sets $S^i_{k,i}$, $t = 0, ..., j - 1$ along an $N$-step SGD optimization path in the inner stage. In fact, such a bias error occurs only for multi-step MAML with $N \geq 2$ (which equals zero for $N = 1$), and requires additional efforts to handle.

Third, both the Hessian term $\nabla^2 l_i(w^i; D^i_{k,j})$ for $j = 2, ..., N - 1$ and the gradient term $\nabla l_i(w^i_{k,N}; T^i_k)$ in the meta gradient estimator $\hat{G}_i(w_k)$ given in (8) depend on the sample sets $S^i_{k,i}$ used for inner stage iteration to obtain $w^i_{k,N}$, and hence they are statistically correlated even conditioned on $w_k$. Such complication also occurs only for multi-step MAML with $N \geq 2$ and requires new treatment (the two terms are independent for $N = 1$).

In Section 3.3, we develop a theoretical framework to handle the above challenges and establish the convergence for $N$-step MAML.

3.3 Properties of Meta Gradient

Differently from the conventional gradient whose corresponding loss is evaluated directly at the current parameter $w$, the meta gradient has a more complicated nested structure with respect to $w$, because its loss is evaluated at the final output of the inner optimization stage, which is $N$-step SGD updates. As a result, analyzing the meta gradient is very different and more challenging compared to analyzing the conventional gradient. In this subsection, we establish some important properties of the meta gradient which are useful for characterizing the convergence of multi-step MAML.

Recall that $\nabla \mathcal{L}(w) = \mathbb{E}_{\tilde{p}(\mathcal{T})}[\nabla \mathcal{L}_i(w)]$ with $\nabla \mathcal{L}_i(w)$ given by (3). The following proposition characterizes the Lipschitz property of the gradient $\nabla \mathcal{L}(\cdot)$.

**Proposition 1.** Suppose that Assumptions 1, 2 and 3 hold. Then, for any $w, u \in \mathbb{R}^d$, we have

$$\|\nabla \mathcal{L}(w) - \nabla \mathcal{L}(u)\| \leq ((1 + \alpha L)^{2N} L + C_L \mathbb{E}_i\|\nabla l_i(w)\|)\|w - u\|,$$

where $C_L$ is a positive constant given by

$$C_L = ((1 + \alpha L)^{N-1} \alpha \rho + \frac{\rho}{L}(1 + \alpha L)^N((1 + \alpha L)^{N-1} - 1))(1 + \alpha L)^N. \quad (14)$$

The proof of Proposition 1 handles the first challenge described in Section 3.2. More specifically, we bound the differences between $\tilde{w}^i_j$ and $\tilde{w}^i_j$ along two separate paths ($\tilde{w}^i_j$, $j = 0, ..., N$) and ($\tilde{w}^i_j$, $j = 0, ..., N$), and then connect these differences to the distance $\|w - u\|$. Proposition 1 shows that the objective $\mathcal{L}(\cdot)$ has a gradient-Lipschitz parameter $L_w = (1 + \alpha L)^{2N} L + C_L \mathbb{E}_{\tilde{p}(\mathcal{T})}\|\nabla l_i(w)\|$, which can be unbounded due to the fact that $\nabla l_i(w)$ may be unbounded. Similarly to Fallah et al. (2019), we use

$$\hat{L}_{w_k} = (1 + \alpha L)^{2N} L + \frac{C_L}{|B_k|^2} \sum_{i \in B_k} \|\nabla l_i(w_k; D^i_{L_k})\| \quad (15)$$
to estimate $L_{w_k}$ at the meta parameter $w_k$, where we independently sample the data sets $B_i^k, D_i^k$. As will be shown in Theorem 1, we set the meta stepsize $\beta_k$ to be inversely proportional to $L_{w_k}$ to handle the possibly unboundedness. In the experiments, we find that the gradients $\nabla l_i(w_k), k \geq 0$ are well bounded during the optimization process, and hence a constant outer-stage stepsize is sufficient in practice.

We next characterize several estimation properties of the meta gradient estimator $\hat{G}_i(w_k)$ in (8). Here, we address the second and third challenges described in Section 3.2. We first quantify how far $w_{k,j}^i$ is away from $\tilde{w}_{k,j}^i$, and then provide upper bounds on the first- and second-moment distances between $w_{k,j}^i$ and $\tilde{w}_{k,j}^i$ for all $j = 0,...,N$ as below.

**Proposition 2 (Point-wise distance between two paths).** Suppose that Assumptions 1, 2 and 3 hold. Then, for any $i \in B_k$ and $j = 0,...,N$, we have

- **First-moment:** $\mathbb{E}(\|w_{k,j}^i - \tilde{w}_{k,j}^i\| | w_k) \leq ((1 + \alpha L)^j - 1) \frac{\sigma_g}{L\sqrt{S}}$.
- **Second-moment:** $\mathbb{E}(\|w_{k,j}^i - \tilde{w}_{k,j}^i\|^2 | w_k) \leq ((1 + \alpha L + 2\alpha^2 L^2)^j - 1) \frac{\sigma_g^2}{(1 + \alpha L) L S}$.

Proposition 2 shows that we can effectively upper-bound the point-wise distance between two paths by choosing $\alpha$ and $S$ properly. Based on Proposition 2, we provide an upper bound on the first-moment estimation error of $\hat{G}_i(w_k)$.

**Proposition 3 (First-moment error).** Suppose Assumptions 1, 2 and 3 hold. Define constants

$$C_{err_1} = (1 + \alpha L)^N((1 + \alpha L)^N - 1)\sigma_g, \quad C_{err_2} = \frac{(1 + \alpha L)^{2N-1}((1 + \alpha L)^N-1)^2 \rho g}{2 - (1 + \alpha L)^{2N}}.$$ (16)

Let $e_k := \mathbb{E}[\hat{G}_i(w_k)] - \nabla L(w_k)$ denote the estimation error. If the inner stepsize $\alpha < (2\pi - 1)/L$, then conditioning on $w_k$, we have

$$\|e_k\| \leq \frac{C_{err_1}}{\sqrt{S}} + \frac{C_{err_2}}{\sqrt{S}} (\|\nabla L(w_k)\| + \sigma).$$ (17)

In contrast to the one-step case, i.e., Lemma 4.12 in Fallah et al. (2019), the estimation error for the multi-step case shown in Proposition 3 involves an additional term $O(\|\nabla L(w_k)\|)$, which cannot be avoided due to the Hessian approximation error caused by the randomness over the samples sets $S_{k,j}^i$. Somewhat interestingly, our later analysis shows that this term does not affect the final convergence rate if we choose the size $S$ properly. The following proposition provides an upper-bound on the second moment of the meta gradient estimator $\hat{G}_i(w_k)$.

**Proposition 4 (Second-moment error).** Suppose that Assumptions 1, 2 and 3 hold. Define constants

$$C_{squ_1} = 3\left(\frac{\alpha^2 g^2}{D} + (1 + \alpha L)^2\right)^N\sigma_g^2, \quad C_{squ_2} = \frac{2C_{squ_1}(1 + \alpha L)^{2N}}{(2 - (1 + \alpha L)^{2N})^2\sigma_g^2}, \quad C_{squ_3} = (1 + 2\alpha L + 2\alpha^2 L^2)^N - 1)\alpha L(1 + \alpha L)^{-1}.$$ (18)

If the inner stepsize $\alpha < (2\pi - 1)/L$, then conditioning on $w_k$, we have

$$\mathbb{E}\|\hat{G}_i(w_k)\|^2 \leq \frac{C_{squ_1}}{T} + \frac{C_{squ_2}}{S} + C_{squ_3} (\|\nabla L(w_k)\|^2 + \sigma^2).$$ (19)

By choosing set sizes $D, T, S$ and the inner stepsize $\alpha$ properly, the factor $C_{squ_3}$ in the second-moment error bound in (19) can be made at a constant level and the first two error terms $\frac{C_{squ_1}}{T}$ and $\frac{C_{squ_2}}{S}$ can be made sufficiently small so that the variance of the meta gradient estimator can be well controlled in the convergence analysis, as shown later.
3.4 Main Convergence Result

By using the properties of the meta gradient established in Section 3.3, we provide the convergence rate for multi-step MAML of Algorithm 1 in the following theorem.

**Theorem 1.** Suppose that Assumptions 1, 2 and 3 hold. Set the meta stepsize \( \beta_k = \frac{1}{C_L w_k} \) with \( \hat{L}_{wk} \) given by (15), where \( |B'_1| > \frac{4C^2 \sigma^2}{3(1+\alpha L)^{2N}} \) and \( |D'_1| > \frac{64\sigma^2 C^2}{(1+\alpha L)^{2N}} \) for all \( i \in B'_1 \). Define \( \chi = \frac{(2-(1+\alpha L)^{2N})(1+\alpha L)^{2N} L}{C_L^2} \) and \( \sigma \) and

\[
\xi = \frac{6}{C_L^2} \left( \frac{1}{5} + \frac{2}{C_L} \right) \left( C_{err_1} + C_{err_2} \sigma^2 \right), \\
\theta = \frac{2}{C_L^2} \left( \frac{2(2-(1+\alpha L)^{2N})}{5} \right) \left( \frac{3}{5} + \frac{6}{C_L} \right) C_{err_2} - \frac{C_{squ_1}}{C_L^2} - \frac{2}{C_L}, \\
\phi = \frac{2}{C_L^2} \left( \frac{C_{squ_1}}{T} + \frac{C_{squ_2}}{S} + \frac{C_{squ_3}}{T} \right),
\]

where \( C_{err_1}, C_{err_2} \) are given in (16) and \( C_{squ_1}, C_{squ_2}, C_{squ_3} \) are given in (18). Choose the inner stepsize \( \alpha < \frac{(2^{N/2} - 1)}{L} \), and choose \( C_L, S \) and \( B \) such that \( \theta > 0 \). Then, Algorithm 1 finds a solution \( w_\chi \) such that

\[
E\|\nabla \mathcal{L}(w_\chi)\| \leq \frac{\Delta}{\theta K} + \frac{\xi}{\theta S} + \frac{\phi}{\theta B} + \sqrt{\frac{N}{2}} \sqrt{\frac{\Delta}{\theta}} \frac{1}{K} + \frac{\xi}{\theta S} + \frac{\phi}{\theta B},
\]

where \( \Delta = \mathcal{L}(w_0) - \mathcal{L}^* \) with \( \mathcal{L}^* = \inf_{w \in \mathbb{R}^d} \mathcal{L}(w) \).

The proof of Theorem 1 (see Appendix C for details) consists of four main steps: step 1 of bounding an iterative meta update by the meta-gradient smoothness established by Proposition 1; step 2 of characterizing first-moment estimation error of the meta-gradient estimator \( \hat{G}_i(w_k) \) by Proposition 3; step 3 of characterizing second-moment estimation error of the meta-gradient estimator \( \hat{G}_i(w_k) \) by Proposition 4; and step 4 of combining steps 1-3, and telescoping to yield the convergence.

In Theorem 1, the convergence rate given by (21) mainly contains three parts: the first term \( \frac{\Delta}{\theta K} \) indicates that the meta parameter converges sublinearly with the number \( K \) of meta iterations, the second term \( \frac{\xi}{\theta S} \) captures the estimation error of \( \nabla l_i(w^i_{k,j}; S^i_{k,j}) \) for approximating the full gradient \( \nabla l_i(w^i_{k,j}) \) which can be made sufficiently small by choosing a large sample size \( S \), and the third term \( \frac{\phi}{\theta B} \) captures the estimation error and variance of the stochastic meta gradient, which can be made small by choosing large \( B, T \) and \( D \) (note that \( \phi \) is proportional to both \( \frac{1}{T} \) and \( \frac{1}{B} \)).

Our analysis reveals several insights for the convergence of multi-step MAML as follows. (a) To guarantee convergence, we require \( \alpha L < 2^{N/2} - 1 \) (e.g., \( \alpha = \Theta(\frac{1}{\sqrt{T}}) \)). Hence, if the number \( N \) of inner gradient steps is large and \( L \) is not small (e.g., for some RL problems), we need to choose a small inner stepsize \( \alpha \) so that the last output of the inner stage has a strong dependence on the initialization (i.e., meta parameter), as also shown and explained in Rajeswaran et al. (2019). (b) For problems with small Hessians such as many classification/regression problems (Finn et al., 2017a), \( L \) (which is an upper bound on the spectral norm of Hessian matrices) is small, and hence we can choose a larger \( \alpha \). This explains the empirical findings in Finn et al. (2017a); Antoniou et al. (2018).

We next specify the selection of parameters to simplify the convergence result in Theorem 1 and derive the complexity of Algorithm 1 for finding an \( \varepsilon \)-accurate stationary point.
Corollary 1 (Stepsize \( \alpha = \Theta(\frac{1}{NL}) \)). Under the setting of Theorem 1, choose \( \alpha = \frac{1}{8NL}, C_{\beta} = 100 \) and let batch sizes \( S \geq \frac{15\beta^{2}\sigma_{g}^{2}}{L^{2}} \) and \( D \geq \sigma_{h}^{2}L^{2} \). Then we have

\[
\mathbb{E}[\|\nabla L(w_{\zeta})\|] \leq O\left(\frac{1}{K} + \frac{\sigma_{g}^{2}(\sigma_{g}^{2} + 1)}{S} + \frac{\sigma_{g}^{2}}{B} + \frac{\sigma_{g}^{2}}{TB} + \sqrt{\sigma + 1}\sqrt{\frac{1}{K} + \frac{\sigma_{g}^{2}(\sigma_{g}^{2} + 1)}{S} + \frac{\sigma_{g}^{2}}{B} + \frac{\sigma_{g}^{2}}{TB}}\right).
\]

To achieve \( \mathbb{E}[\|\nabla L(w_{\zeta})\|] < \epsilon \), Algorithm 1 requires at most \( O\left(\frac{1}{\epsilon^{2}}\right) \) iterations, and \( O\left(\frac{N}{\epsilon^{2}} + \frac{1}{\epsilon^{2}}\right) \) gradient computations and \( O\left(\frac{N}{\epsilon^{2}}\right) \) Hessian computations per meta iteration.

Differently from the conventional SGD that requires a gradient complexity of \( O\left(\frac{1}{\epsilon}\right) \), MAML requires a higher gradient complexity by a factor of \( O\left(\frac{1}{\epsilon^{2}}\right) \), which is unavoidable because MAML requires \( O\left(\frac{1}{\epsilon}\right) \) tasks to achieve an \( \epsilon \)-accurate meta point, whereas SGD runs only over one task.

Corollary 1 shows that given a properly chosen inner stepsize, e.g., \( \alpha = \Theta(\frac{1}{NL}) \), MAML is guaranteed to converge with both the gradient and the Hessian computation complexities growing only linearly with \( N \). These results explain some empirical findings for MAML training in Rajeswaran et al. (2019). The above results can also be obtained by using a larger stepsize such as \( \alpha = \Theta(c\frac{1}{\epsilon} - 1)/L > \Theta(\frac{1}{NL}) \) with a certain constant \( c > 1 \).

4 Convergence of Multi-Step MAML in Finite-Sum Case

4.1 Basic Assumptions

In this section, we provide convergence analysis for Algorithm 2. We first state several standard assumptions for the analysis.

Assumption 4. For each task \( T_i \), the loss functions \( l_{S_i}(\cdot) \) and \( l_{T_i}(\cdot) \) in (13) satisfy

1. \( l_{S_i}(\cdot) \) and \( l_{T_i}(\cdot) \) are bounded below, i.e., \( \inf_{w \in \mathbb{R}^{d}} l_{S_i}(w) > -\infty \) and \( \inf_{w \in \mathbb{R}^{d}} l_{T_i}(w) > -\infty \).

2. \( l_{S_i}(\cdot) \) and \( l_{T_i}(\cdot) \) both have \( L \)-Lipschitz continuous gradients, i.e., for any \( w, u \in \mathbb{R}^{d} \)

\[
\|\nabla l_{S_i}(w) - \nabla l_{S_i}(u)\| \leq L\|w - u\| \quad \text{and} \quad \|\nabla l_{T_i}(w) - \nabla l_{T_i}(u)\| \leq L\|w - u\|.
\]

3. \( l_{S_i}(\cdot) \) and \( l_{T_i}(\cdot) \) are twice differentiable, and have \( \rho \)-Lipschitz continuous Hessians, i.e., for any \( w, u \in \mathbb{R}^{d} \)

\[
\|\nabla^{2} l_{S_i}(w) - \nabla^{2} l_{S_i}(u)\| \leq \rho\|w - u\| \quad \text{and} \quad \|\nabla^{2} l_{T_i}(w) - \nabla^{2} l_{T_i}(u)\| \leq \rho\|w - u\|.
\]

Note that \( \nabla l_{S_i}(\cdot) \) and \( \nabla l_{T_i}(\cdot) \) are stochastic approximations of \( \nabla l_{S_i}(\cdot) = \mathbb{E}_{i \sim p(T)} [\nabla l_{S_i}(\cdot)] \) and \( \nabla l_{T_i}(\cdot) = \mathbb{E}_{i \sim T} [\nabla l_{T_i}(\cdot)] \). The following assumption provides the conditions on the estimation properties of the gradients \( \nabla l_{S_i}(\cdot) \) and \( \nabla l_{T_i}(\cdot) \).

Assumption 5. For all \( w \in \mathbb{R}^{d} \), gradients \( \nabla l_{S_i}(w) \) and \( \nabla l_{T_i}(w) \) satisfy

1. \( \nabla l_{T_i}(\cdot) \) has a bounded variance, i.e., there exists a constant \( \sigma > 0 \) such that

\[
\mathbb{E}_{i \sim p(T)} \|\nabla l_{T_i}(w) - \nabla l_{T_i}(w)\|^{2} \leq \sigma^{2}.
\]

2. For each \( i \in T \), there exists a constant \( b_{i} > 0 \) such that

\[
\|\nabla l_{S_i}(w) - \nabla l_{T_i}(w)\| \leq b_{i}.
\]
Instead of imposing a bounded variance condition on the stochastic gradient $\nabla l_{S_i}(w)$, we alternatively assume the difference $\|\nabla l_{S_i}(w) - \nabla l_{T_i}(w)\|$ to be upper-bounded by a constant term. We note that the second condition also implies $\|\nabla l_{S_i}(w)\| \leq \|\nabla l_{T_i}(w)\| + b_i$, which is much weaker than the bounded gradient assumption made in papers such as Finn et al. (2019). It is worthwhile mentioning that the second condition can be relaxed to $\|\nabla l_{S_i}(w)\| \leq c_i \|\nabla l_{T_i}(w)\| + b_i$ for a constant $c_i > 0$. Without the loss of generality, we consider $c_i = 1$ for simplicity.

4.2 Properties of Meta Gradient

In this subsection, we develop several important properties of the meta gradient for the finite-sum case. The following proposition characterizes a Lipschitz property of the gradient of the objective function

$$\nabla \mathcal{L}(w) = E_{i \sim p(T)} \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}^i_j)) \nabla l_{T_i}(\tilde{w}^i_N),$$

where the weights $\tilde{w}^i_j, i \in \mathcal{I}, j = 0, \ldots, N$ are given by the gradient descent steps in (10).

Proposition 5. Suppose that Assumptions 4 and 5 hold. Then, for any $w, u \in \mathbb{R}^d$, we have

$$\|\nabla \mathcal{L}(w) - \nabla \mathcal{L}(u)\| \leq ((1 + \alpha L)^2 N L + C_b + C_L E_{i \sim p(T)} \| \nabla l_{T_i}(w) \|) \| w - u \|,$$

where $b = E_{i \sim p(T)} [b_i]$ and $C_b, C_L > 0$ are constants given by

$$C_b = (1 + \alpha L)^{N-1} \alpha \rho + \frac{\rho}{L} (1 + \alpha L)^N (1 + \alpha L)^{N-1} - 1),$$

$$C_L = (1 + \alpha L)^{N-1} \alpha \rho + \frac{\rho}{L} (1 + \alpha L)^N (1 + \alpha L)^{N-1} - 1) (1 + \alpha L)^N.$$

(22)

Proposition 5 shows that the gradient $\nabla \mathcal{L}(w)$ has a Lipschitz parameter $L_{w} := (1 + \alpha L)^2 N L + C_b + C_L E_{i \sim p(T)} \| \nabla l_{T_i}(w) \|$. Similarly to (15) for the resampling case, we use the following construction

$$\hat{L}_{w_k} = (1 + \alpha L)^2 N L + C_b + \frac{C_L}{|B_k^l|} \sum_{i \in B_k^l} \| \nabla l_{T_i}(w_k) \|,$$

(23)

at the $k^{th}$ outer-stage iteration to approximate $L_{w_k}$, where $B_k^l \subset \mathcal{I}$ is chosen independently from $B_k$. It can be verified that the gradient estimator $\hat{G}_i(w_k)$ given in (13) is an unbiased estimate of $\nabla \mathcal{L}(w_k)$. Thus, we next only need to provide an upper bound on the second moment of $\hat{G}_i(w_k)$.

Proposition 6. Suppose that Assumptions 4 and 5 are hold, and define two constants

$$A_{\text{squ}_1} = \frac{4(1 + \alpha L)^4}{(2 - (1 + \alpha L)^2)^2},$$

$$A_{\text{squ}_2} = \frac{4(1 + \alpha L)^4(1 + \alpha L)^2 - 1)^2}{(2 - (1 + \alpha L)^2)^2} (\sigma + b)^2 + 2(1 + \alpha)^4 (\sigma^2 + \bar{b}),$$

(24)

where $\bar{b} = E_{i \sim p(T)} [b_i^2]$. Then, if the inner stepsize $\alpha < (2 \frac{1}{N} - 1)/L$, then conditioning on $w_k$, we have

$$\mathbb{E}\|\hat{G}_i(w_k)\|^2 \leq A_{\text{squ}_1} \| \nabla \mathcal{L}(w_k) \|^2 + A_{\text{squ}_2}.$$

Based on the above properties, we next characterize the convergence behavior of multi-step MAML in the finite-sum setting.
4.3 Main Convergence Results

In this subsection, we provide the convergence and complexity analysis for Algorithm 2 based on the properties established in the previous subsection.

**Theorem 2.** Let Assumptions 4 and 5 hold, and apply Algorithm 2 to solve the objective function (9). Choose the meta stepsize \( \beta_k = \frac{1}{C_\beta L_{w_k}} \) with \( \tilde{L}_{w_k} \) given by (23), where \( C_\beta > 0 \) is a constant and the batch size \( |B'_k| \) satisfies \( |B'_k| \geq \frac{2C_\beta^2 \sigma^2}{(C_\beta b + (1 + \alpha L)^2 \epsilon N)} \). Define constants

\[
\theta = \frac{2}{C_\beta} - \frac{1}{C_\beta^2} \left( \frac{A_{\text{squ}}}{B} + 1 \right), \quad \phi = \frac{A_{\text{squ}}}{C_\beta L},
\]

\[
\xi = \frac{2}{C_\beta} - \frac{1}{C_\beta^2} (1 + \alpha L)^{2N} + (2 - (1 + \alpha L)^{2N} \frac{C_\beta b}{C_\beta L} + (1 + \alpha L)^N ((1 + \alpha L)^{2N} - 1) \epsilon b,
\]

where \( C_\beta, C_\beta^2, A_{\text{squ}}, \) and \( A_{\text{squ}}^2 \) are given by (22) and (24). Choose the inner stepsize \( \alpha < \frac{1}{2N - 1} \), and choose \( C_\beta \) and \( B \) such that \( \theta > 0 \). Then, Algorithm 2 attains a solution \( w_\zeta \) such that

\[
\mathbb{E} \| \nabla L(w_\zeta) \| \leq \frac{\Delta}{2\theta} + \frac{\phi}{2\theta B} + \sqrt{\xi \left( \frac{\Delta}{2\theta} + \frac{\phi}{2\theta B} \right) + \left( \frac{\Delta}{2\theta} + \frac{\phi}{2\theta B} \right)^2},
\]

where \( \Delta = L(w_0) - L^* \) with \( L^* = \inf_{w \in \mathbb{R}^d} L(w) \).

The parameters \( \theta, \phi \) and \( \xi \) in Theorem 2 take complicate forms. The following corollary specifies the parameters \( C_\beta, \alpha \) in Theorem 2, and provides a simplified convergence and complexity result for Algorithm 2.

**Corollary 2.** Under the same setting of Theorem 2, choose the inner stepsize \( \alpha = \frac{1}{8N} \) and \( C_\beta = 80 \). Then we have

\[
\mathbb{E} \| \nabla L(w_\zeta) \| \leq O \left( \frac{1}{K} + \frac{\sigma^2}{B} + \sqrt{\frac{1}{K} + \frac{\sigma^2}{B}} \right).
\]

In addition, suppose the batch size \( B \) further satisfies \( B \geq C_B \sigma^2 \epsilon^{-2} \), where \( C_B > 0 \) is a sufficiently large constant. Then, to achieve an \( \epsilon \)-approximate stationary point, i.e., \( \mathbb{E} \| \nabla L(w_\zeta) \| < \epsilon \), Algorithm 2 requires at most \( K = O(\epsilon^{-2}) \) iterations, and a total number \( O((T + NS)^{-1}) \) of gradient computations and a number \( O(NS \epsilon^{-2}) \) of Hessian computations per iteration, where \( T \) and \( S \) correspond to the sample sizes of the pre-assigned sets \( T_i, i \in I \) and \( S_i, i \in I \).

5 Efficient Hessian-Free MAML Algorithms

The presence of Hessian in MAML causes significant computational complexity and storage cost, especially in the multi-step case. More recently, a Hessian-free MAML algorithm based on a zeroth-order Hessian estimator has been proposed in Song et al. (2020). In this section, we first theoretically show the performance limitation of the zeroth-order Hessian estimator, and then propose a new Hessian-free MAML algorithm that has provable performance guarantee.
5.1 Limitations of Zeroth-Order Hessian Estimation

The zeroth-order Hessian estimator\(^1\) for the resampling case has the following form at the point \(w_{k,j}^i\):

\[
H_{k,j}^i = \frac{1}{U} \sum_{u \in U_{k,j}^i} \frac{\Delta \delta u + \Delta \delta u}{2\delta^2} uu^T, \tag{27}
\]

where \(\Delta \delta u = l_i(w_{k,j}^i + \delta u; \tilde{D}_{k,j}^i) - l_i(w_{k,j}^i; D_{k,j}^i)\) and \(\Delta \delta u = l_i(w_{k,j}^i - \delta u; \tilde{D}_{k,j}^i) - l_i(w_{k,j}^i; D_{k,j}^i)\) with \(\tilde{D}_{k,j}^i, D_{k,j}^i\) sampled under parameters \(w_{k,j}^i + \delta u, w_{k,j}^i - \delta u, w_{k,j}^i\), \(u \in U_{k,j}^i\) are i.i.d. standard Gaussian vectors, \(U\) is the batch size of \(U_{k,j}^i\), and \(\delta\) is a positive smoothing parameter.

We next provide an analysis on the estimation error of \(H_{k,j}^i\), which characterizes its performance limitation.

**Proposition 7.** Suppose Assumption 1 holds. Let \(l_{i,x}(x) := \mathbb{E}_{w \sim N(0, I_d)} l_i(x + \delta u)\) be the smoothed approximation of the function \(l_i(x)\). Then, conditioning on \(w_{k,j}^i\), we have

\[
\mathbb{E}(H_{k,j}^i) = \nabla^2 l_{i,x}(w_{k,j}^i) + \frac{1}{2} \text{Tr}(\nabla^2 l_i(w_{k,j}^i) I_d) + \frac{c_H}{\delta^2} I_d,
\]

where \(c_H\) is bounded by \(|c_H| < \frac{\delta^3}{3} \rho(d + 3)^{3/2}\).

Though the first term in the above equality can be sufficiently close to the true Hessian for a small \(\delta\), and the error term \(\frac{c_H}{\delta^2} I_d\) can also be small for small enough \(\delta\), there exists a constant-level error term \(\frac{1}{2} \text{Tr}(\nabla^2 l_i(w_{k,j}^i) I_d)\). Due to such a constant bias term, the corresponding algorithm can converge only to a neighborhood around a stationary point with an error \(O(d\sigma((1 + \alpha)N - 1))\), which can be substantial.

The following proposition shows that the constant error term in the zeroth-order Hessian approximation, as characterized by Proposition 7, leads to a large estimation bias for the following stochastic meta gradient

\[
\hat{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha H_{k,j}^i) \nabla l_i(w_{k,N}^i, T_k^i).
\]

**Proposition 8.** Suppose that Assumptions 1, 2, and 3 hold. Define

\[
C_e = \alpha (1 + \alpha L)^{N-1} C_r + \frac{((1 + \alpha L)^{N-1} - 1)(1 + \alpha L + \alpha C_r)^{N-1}}{L} \rho^2 \sigma_g \frac{\rho}{L\sqrt{S}} + C_r,
\]

where the constant \(C_r = \rho(d + 1)^{1/2} + \frac{4}{3} \rho(d + 3)^{3/2} + \frac{1}{2} \sup_{w \in \mathbb{R}^d} \text{Tr}(\nabla^2 l_i(w))\).

If the inner stepsize \(\alpha < \frac{2\sqrt{N} - 1}{L}\), conditioning on \(w_k\), we have

\[
\|\mathbb{E} \hat{G}_i(w_k) - \nabla L(w_k)\| \leq (1 + \alpha L + \alpha C_r)^N ((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{\sqrt{S}} + \frac{(1 + \alpha L)^N C_e \|

\]

To deal with the second term in the above bound, using an approach similar to (63) in the proof of Theorem 1, it requires that \((1 + \alpha L)^N C_e = \Theta(d((1 + \alpha L)^N - 1)) \leq O(1)\), which implies that \(\alpha < \frac{(1 + d)^{1/2}}{L}\) in original MAML (as shown in Corollary 1). In the meanwhile, the above error contains an additional term \(\frac{c_H}{\delta^2} I_d\) which is much more restricted than \(\alpha < \frac{1}{NL}\) and is hence much more restricted than \(\alpha < \frac{1}{NL}\) in original MAML (as shown in Corollary 1). In the meanwhile, the above error contains an additional term \(\frac{c_H}{\delta^2} I_d\) which is much more restricted than \(\alpha < \frac{1}{NL}\). Consequently, the corresponding algorithm can only guarantee to converge to a neighborhood around a stationary point with a possibly large error \(O((1 + \alpha L)^N - 1))\). Such an issue still exists in the finite-sum case.

\(^1\) The zeroth-order estimator used by Song et al. (2020) has the form of \(H_{k,j}^i = \frac{1}{U} \sum_{u \in U_{k,j}^i} l_i(w_{k,j}^i + \delta u; \tilde{D}_{k,j}^i) uu^T - l_i(w_{k,j}^i; D_{k,j}^i)I\), which achieves a mean almost the same as that of (27) (with a small gap of \(O(\delta)\)), but has a higher variance. To this end, this paper focuses on the zeroth-order Hessian estimator in (27) with lower variance, but the analysis is also applied to the one in Song et al. (2020).
Algorithm 3 GGS-MAML in the resampling case

1: **Input:** Initial parameter \( w_0 \), inner stepsize \( \alpha > 0 \)

2: **while** not done **do**

3: Sample \( B_k \subset I \) of i.i.d. tasks according to \( p(T) \)

4: **for** all tasks \( T_i \) in \( B_k \) **do**

5: Sample a support set \( S_{i,k,j} \)

6: **Update** \( w_{i,k,j}^{+1} = w_{i,k,j}^1 - \alpha \nabla l_i(w_{i,k,j}^1; S_{i,k,j}) \)

7: **end for**

8: **end for**

9: Sample \( T^i_k, \hat{D}_{i,k,j}^i, D_{i,k,j}^i \) and Gaussian vectors \( u \in U_{i,k,j}^i \)

10: **Update** \( w_{k+1} \) according to (29)

11: Update \( k \leftarrow k + 1 \)

12: **end while**

5.2 GGS-MAML in Resampling Case and Convergence

Our analysis in Section 5.1 shows that zeroth-order MAML does not provide desired performance due to a possibly large estimation error of the Hessian. Here, we develop a more accurate Hessian estimator via a gradient-based Gaussian smoothing (GGS) method, and propose an easy-to-implement MAML variant (GGS-MAML). We show in the next two subsections that GGS-MAML admits a much bigger stepsize \( \alpha \) and achieves better convergence guarantee than the zeroth-order MAML.

As shown in Algorithm 3, GGS-MAML has a similar structure as the original MAML, but constructs Hessian approximation using Gaussian random vectors \( u \in U_{i,k,j}^i \) by

\[
H_{i,k,j}^i = \frac{1}{U} \sum_{u \in U_{i,k,j}^i} \Delta_{\delta u} u^T,
\]

where \( U \) is the batch size of \( U_{i,k,j}^i \) and \( \Delta_{\delta u} = \nabla l_i(w_{i,k,j} + \delta u; \hat{D}_{i,k,j}^i) - \nabla l_i(w_{i,k,j}; D_{i,k,j}^i) \). We then update \( w_{k+1} \) by

\[
w_{k+1} = w_k - \frac{\beta_k}{|B_k|} \sum_{i \in B_k} \hat{G}_i(w_k),
\]

where \( \hat{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha H_{i,k,j}^i) \nabla l_i(w_{i,k,N}^i; T_k^i) \).

The following proposition bounds the estimation error of \( H_{i,k,j}^i \) for approximating the true Hessian \( \nabla^2 l_i(w_{i,k,j}) \).

**Proposition 9.** Suppose that Assumption 1 hold. Then, conditioning on \( w_{i,k,j} \), we have

\[
\| \mathbb{E}(H_{i,k,j}^i) - \nabla^2 l_i(w_{i,k,j}) \| \leq \rho \delta (d + 3)^{3/2}.
\]

Proposition 9 shows that our Hessian estimator eliminates the constant error term in the zeroth-order Hessian estimation, and its estimation error can be made sufficiently small by choosing a small \( \delta \). We next provide an upper bound on the estimation variance of \( H_{i,k,j}^i \) by random matrix theory.
Proposition 10. Suppose Assumption 1 holds. Then, conditioning on \( w^i_{k,j} \), we have

\[
\text{Var}(H^i_{k,j}) = \mathbb{E}\|H^i_{k,j} - \mathbb{E}(H^i_{k,j})\|^2 \leq \frac{6d\sigma^2_g}{\delta^2 D} + 18p^2\delta^2(d + 6)^3 + 9L^2 \max\{\lambda, \lambda^2\},
\]

where the parameter \( \lambda \) is given by

\[
\lambda = C \max\{\log d, 2\} \sqrt{m} U^{-1},
\]

where \( m := \mathbb{E}(\max_{u \in U^i_{k,j}} \|u\|^4) \) and \( C \) is a constant independent of \( d \) and \( U \).

For the finite-sum case, we can remove the first error term in (30) as we detail in Section 5.3.

Our analysis of the variance bound involves novel techniques using random matrix theory and a generalized Rudelson’s inequality, which may be of independent interest.

Based on Propositions 9 and 10, we next provide the convergence and complexity analysis for GGS-MAML.

Theorem 3. Suppose that Assumptions 1, 2, and 3 hold. Set \( \alpha = \min\left\{ \frac{1}{8NL}, \frac{1}{8\sqrt{NL}\sqrt{\max(\lambda, \lambda^2)}}, \frac{\delta\sqrt{D}}{8\sigma_\gamma \sqrt{\lambda^4/4N}} \right\}, C \beta = 100 \) and choose \( \delta, S, U, B \) such that \( \delta \leq \frac{L}{18p(d+3)^2/4}, S > \frac{8\lambda^2\sigma^2_g}{L^2}, \) and \( B \geq 3 \), where \( C \) and \( m \) are given in Proposition 10. We then have

\[
\mathbb{E}\|\nabla L(w_\zeta)\| \leq O \left( \frac{1}{K} + \frac{\sigma^2_g(\sigma^2 + 1)}{S} + \delta^2 d^3 \sigma^2 + \frac{\sigma^2_g}{BT} + \frac{\sigma^2_g}{B} \right)
\]

(31)

To achieve an \( \epsilon \)-stationary point, i.e., \( \mathbb{E}\|\nabla L(w_\zeta)\| < \epsilon \), Algorithm 3 requires at most \( O\left(\frac{1}{\epsilon^2}\right) \) gradient computations per meta iteration.

Theorem 3 captures a trade-off between the inner stepsize \( \alpha \) and the number \( U \) of gradients for Hessian approximation in (28). To see this, consider the case with \( \lambda > 1 \) for illustration. The choice of \( \alpha \) in the above theorem requires \( \frac{\alpha L}{\delta^2} \leq O\left(\frac{1}{\sqrt{N}\log d/\sqrt{m}}\right) \), where \( m := \mathbb{E}(\max_{u \in U^i_{k,j}} \|u\|^4) \ll U(d + 4)^2 \). Such a requirement indicates that the increase of \( U \) yields more accurate Hessian approximation and hence admits a larger inner stepsize \( \alpha \), but causes higher gradient computation cost. For example, if we choose \( U = \Theta(\log d/\sqrt{m}/N) \) and \( D = \Theta\left(\frac{\sigma^2_g}{N^2}\right) \), then the corresponding inner stepsize \( \alpha = \theta\left(\frac{1}{N}\right) \) is the same as that in Corollary 1 used for original MAML, but the gradient computation complexity \( O\left(\frac{N}{\epsilon} + \frac{NU}{\epsilon} + \frac{1}{\epsilon^2}\right) \) becomes larger. On the other hand, if we choose a smaller \( \alpha \) or the norm of Hessian is small (i.e., \( L \) is small), we can choose a smaller \( U \) to save computation cost. This is because the Hessian-related components in the meta gradient estimator are in terms of \( \alpha H^i_{k,j} \), and thus a smaller \( \alpha L \) yields a smaller variance of meta-gradient estimation.

5.3 GGS-MAML in Finite-Sum Case and Convergence

In Section 5.2, we propose an efficient Hessian-free GGS-MAML algorithm for the resampling case. Here, we adapt this algorithm for the finish-sum case. In particular, GGS-MAML (Algorithm 4) for the finite-sum case adopts the following Hessian approximation of the loss function \( l_{S,\cdot} \) at the point \( w^i_{k,j} \) based on a gradient-based Gaussian smoothing method:

\[
H^i_{k,j} = \frac{1}{U} \sum_{u \in U^i_{k,j}} \nabla l_{S, w^i_{k,j} + \delta u} - \nabla l_{S, w^i_{k,j}} u^T.
\]

(32)
Algorithm 4 GGS-MAML in the finite-sum case

1: **Input:** Initial parameter $w_0$, inner stepsize $\alpha > 0$
2: while not done do
3:     Sample a batch $B_k \subset \mathcal{I}$ of i.i.d. tasks according to the distribution $p(\mathcal{T})$
4:     for all tasks $\mathcal{T}_i$ with $i \in B_k$ do
5:         for $j = 0, 1, ..., N - 1$ do
6:             Update $w_{k,j+1}^i = w_{k,j}^i - \alpha \nabla l_{\mathcal{T}_i}(w_{k,j}^i)$
7:         end for
8:     end for
9:     Sample Gaussian random vectors $u, u_i \in U_{k,j}$ for all $i \in B_k$ and $j = 0, ..., N$
10:    Compute Hessian estimator $H_{k,j}^i = \frac{1}{U} \sum_{u_i \in U_{k,j}} \nabla l_{\mathcal{T}_i}(w_{k,j}^i) - \nabla l_{\mathcal{T}_i}(w_{k,j}^i) + \delta u_i u_i^T$
11:    Update $w_{k+1} = w_k - \frac{\beta_k}{|B_k|} \prod_{i \in B_k} \prod_{j=0}^{N-1} (I - \alpha H_{k,j}^i) \nabla l_{\mathcal{T}_i}(w_{k,N}^i)$
12:    end while

Note that differently from the construction (28) in the resampling case, the above Hessian estimation does not involve the randomness over data sampling due to the fixed data samples. Then, similarly to the resampling case, we update the meta parameter $w_k$ by the following mini-batch stochastic gradient descent

$$w_{k+1} = w_k - \frac{\beta_k}{|B_k|} \prod_{i \in B_k} \prod_{j=0}^{N-1} (I - \alpha H_{k,j}^i) \nabla l_{\mathcal{T}_i}(w_{k,N}^i).$$

We next provide a convergence analysis of the proposed GGS-MAML in the finite-sum case, where the objective function is given by (9) and the Hessian estimator $H_{k,j}^i$ is given by (32). Our analysis for GGS-MAML adopts the same assumptions as in Section 4 for the original MAML.

**Proposition 11.** Suppose that Assumptions 4 and 5 hold. Then, conditioning on $w_k$, we have

- **First-moment bound:**
  $$\| \mathbb{E}(H_{k,j}^i) - \nabla^2 l_{\mathcal{T}_i}(w_{k,j}^i) \| \leq \rho \delta (d + 3)^{3/2}.$$

- **Second-moment bound:**
  $$\mathbb{E}\|H_{k,j}^i - \mathbb{E}(H_{k,j}^i)\|^2 \leq 6 \rho^2 \delta^2 (d + 6)^3 + 3L^2 \max\{\lambda, \lambda^2\}.$$

where the parameter $\lambda$ is given by

$$\lambda = C \max\{\log d, 2\} \sqrt{m U} U^{-1},$$

where $m := \mathbb{E}(\max_{u_i \in U_{k,j}} \| u_i \|^4)$ and $C$ is a constant independent of $d$ and $U$.

Based on the above proposition, we next provide two properties of the meta gradient estimator $\hat{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha H_{k,j}^i) \nabla l_{\mathcal{T}_i}(w_{k,N}^i)$. Compared with original MAML in the finite-sum case, $\hat{G}_i(w_k)$ is not an unbiased estimator anymore due to the Hessian approximations. Thus, the analysis of GGS-MAML is more challenging and requires a new treatment. The following proposition upper-bounds the first moment of the estimation error of $\hat{G}_i(w_k)$ for approximating $\nabla \mathcal{L}(w_k)$.
Proposition 12 (Biased meta gradient estimator). Suppose that Assumptions 4 and 5 hold. Define the constants
\[
C_\delta = (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1} \alpha \rho \delta (d + 3)^{3/2} + \left( (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1} - 1 \right) \frac{(1 + \alpha L)^{N-1} \rho \delta (d + 3)^{3/2}}{L + \rho \delta (d + 3)^{3/2}},
\]
\[
C_{e_1} = \frac{C_\delta(1 + \alpha L)^N}{2 - (1 + \alpha L)^{2N}}, \quad C_{e_2} = \frac{C_\delta(1 + \alpha L)^N(\sigma + b)}{2 - (1 + \alpha L)^{2N}},
\]
where \(b := \mathbb{E}_{i \sim p(T)}[b_i]\) with \(b_i\) given in Assumption 5. Then, conditioning on \(w_k\), the estimation error satisfies
\[
\|\mathbb{E} \tilde{G}_i(w_k) - \nabla \mathcal{L}(w_k)\| \leq C_{e_1} \|\nabla \mathcal{L}(w_k)\| + C_{e_2}.
\]

Note that the constant \(C_\delta\) in the estimation error bound (35) is proportional to the smoothing parameter \(\delta\). Thus, by choosing \(\delta\) carefully, we can still guarantee the convergence of GGS-MAML in the finite-sum setting, as shown later. We next provide an upper bound on the second moment of \(\tilde{G}_i(w_k)\).

Proposition 13 (Second-moment error). Suppose that Assumptions 4 and 5 hold. Define constants
\[
C_\lambda = 7\alpha^2 \rho^2 \delta^2 (d + 6)^3 + 3\alpha^2 L^2 \max \{\lambda, \lambda^2\} + (1 + \alpha L)^2 + 2(1 + \alpha L)\alpha \rho \delta (d + 3)^{3/2},
\]
\[
C_{s_1} = \frac{4(1 + \alpha L)^{2N} C_\lambda}{(2 - (1 + \alpha L)^{2N})^2},
\]
\[
C_{s_2} = \frac{4(1 + \alpha L)^{2N} C_\lambda ((1 + \alpha L)^{2N} - 1)^2}{(2 - (1 + \alpha L)^{2N})^2} (\sigma + b)^2 + 2C_\lambda (1 + \alpha L)^{2N} (\sigma^2 + \bar{b}),
\]
where \(\bar{b} := \mathbb{E}_i b_i^2\); \(b = \mathbb{E}_i b_i\) and \(\lambda\) is a small term given by (34). Then, conditioning on \(w_k\), we have
\[
\mathbb{E} \|\tilde{G}_i(w_k)\|^2 \leq C_{s_1} \|\nabla \mathcal{L}(w_k)\|^2 + C_{s_2}.
\]

Based on the above properties, we now develop a general convergence result for GGS-MAML in the finite-sum case.

Theorem 4. Suppose that Assumptions 4 and 5 hold. Choose the meta stepsize \(\beta_k = \frac{1}{C_{i,k} w_k}\) with \(\tilde{L}_{w_k}\) given by (23), where \(C_{i,k} > 0\) is a constant and the batch size \(|B_k'|\) satisfies \(|B_k'| \geq \frac{2C_\lambda^2 \sigma^2}{(C_\lambda b + (1 + \alpha L)^{2N})^2}\). Define constants
\[
\theta = \frac{2 - (1 + \alpha L)^{2N}}{C_L} \frac{1}{C_{i,k}} \left( 2 - C_{s_1}^2 - \frac{1}{C_{i,k}} \left( \frac{C_{s_1}}{B} + 2 + 4C_{e_1}^2 \right) \right), \quad \phi = \frac{1}{C_{i,k}} \left( \frac{C_{s_2}}{B} + 4C_{e_2}^2 \right),
\]
\[
\xi = \frac{2 - (1 + \alpha L)^{2N}}{C_L} \left( (1 + \alpha L)^{2N} L + \frac{2 - (1 + \alpha L)^{2N}}{C_L} C_\beta b + (1 + \alpha L)^N (1 + \alpha L)^2 N - 1) b, \right.
\]
where \(C_{s_1}, C_L\) are given by (22), \(C_{s_1}, C_{e_1}, C_{s_2}, C_{s_1}, C_{s_2}\) are given in Propositions 12 and 13, respectively. Choose the inner stepsize \(\alpha < (2N + 1)/L\), and choose \(\delta, C_{i,k}, B\) and \(U\) such that \(\theta > 0\). Then, Algorithm 4 attains a solution \(w_\xi\) such that
\[
\mathbb{E} \|\nabla \mathcal{L}(w_\xi)\| \leq \frac{\Delta}{\theta k} + \phi + \sqrt{\xi \left( \frac{\Delta}{\theta k} + \phi \right)},
\]
where \(\Delta = \mathcal{L}(w_0) - \mathcal{L}^*\) with \(\mathcal{L}^* = \inf_{w \in \mathbb{R}_d} \mathcal{L}(w)\).
By choosing a small $\delta$, GGS-MAML achieves a convergence rate similar to that of the original MAML, as demonstrated in the following corollary.

**Corollary 3.** Under the same setting of Theorem 4, choose parameters $\alpha,U,\delta$ such that $\alpha < \frac{1}{8NL}$, $\alpha^2 L^2 \max\{\lambda,\lambda^2\} < \frac{1}{64}$, $\rho(d+3)^{3/2} < \frac{L}{30}$ and $B > 3$. Then, we have

$$
\mathbb{E}\|\nabla L(w_{\zeta})\| \leq O\left(\frac{1}{K} + \left(\sigma^2 + 1\right)\left(\frac{1}{B} + \delta^2d^3\right) + \sqrt{\frac{1}{K} + \left(\sigma^2 + 1\right)\left(\frac{1}{B} + \delta^2d^3\right)}\right).$$  \tag{39}
$$

Furthermore, suppose that the batch size $B$ further satisfies $B \geq C_B(\sigma^2 + 1)\epsilon^{-2}$ and the smoothing parameter $\delta$ satisfies $\delta < \epsilon/(C_d\sigma^2\sqrt{\sigma^2 + 1})$, where $C_B$ and $C_d$ are sufficiently large constants. Then, to achieve $\mathbb{E}\|\nabla L(w_{\zeta})\| < \epsilon$, Algorithm 4 requires at most $K = \mathcal{O}(\epsilon^{-2})$ iterations, and a total number $\mathcal{O}(\epsilon^{-2}(T + NUS))$ of gradient computations per meta iteration, where $T$ and $S$ correspond to the sample sizes of the pre-assigned sets $T_i, i \in I$ and $S_i, i \in I$.

6 Experiments

In this section, we validate our theory and the effectiveness of the proposed GGS-MAML algorithm. We present our experiments on three meta-learning problems, i.e., rank-one matrix factorization, regression, and reinforcement learning. All experiments are run using PyTorch (Paszke et al., 2019). The code for matrix factorization and regression is available at https://github.com/JunjieYang97/GGS-MAML-DL and the code for reinforcement learning is available at https://github.com/JunjieYang97/GGS-MAML-RL.

We compare the performance among the following four algorithms: GGS-MAML proposed in this paper (Hessian-free and based on first-order Gaussian smoothing Hessian estimator), ZO-MAML\(^2\) (Hessian free and based on the same zeroth-order Hessian estimator as ES-MAML (Song et al., 2020)), FO-MAML (Finn and Levine, 2017) (Hessian free by dropping all Hessians in the meta-gradient computation) and the original MAML (Finn and Levine, 2017) (which uses the exact Hessian).

6.1 Rank-One Matrix Factorization

We study a rank-one matrix factorization problem. For each task $i$, consider a loss function $l_i(w) := \frac{1}{2\epsilon}\|ww^T - g_i g_i^T\|_F^2$, where $g_i$ is a random vector sampled from the Gaussian distribution $\mathcal{N}(0,\sigma^2)$. At each meta iteration, 10 different tasks are sampled for meta-training. For all algorithms, we run $N = 3$ inner-stage gradient updates with a fixed stepsize $\alpha = 0.01$. At the outer stage (meta iteration), we use SGD as the meta-optimizer with the learning rate choosing from $\{0.01,0.003\}$. For GGS-MAML and ZO-MAML, we use $U = 10$ Gaussian vectors for Hessian approximation and choose the best smoothing parameter $\delta$ from $\{10^{-2}, 10^{-4}\}$ for each algorithm.

In Figure 1, we plot how the loss function changes as the algorithms iterate. Since we sample new tasks at each iteration (which maximizes the expectation form of the objective function in the resampling case), the meta loss value also represents the test loss of the meta parameter. Figure 1 illustrates that the proposed GGS-MAML outperforms the other two Hessian-free algorithms ZO-MAML and FO-MAML, and achieves a comparable accuracy to the original MAML. This validates the effectiveness of the proposed Hessian estimator

\(^2\)ZO-MAML is a slightly modified/better version of ES-MAML proposed in Song et al. (2020), which uses the same zeroth-order Hessian estimator as ES-MAML, but uses the exact gradient rather than a zero-order gradient estimator used in ES-MAML.
used in GGS-MAML. The figure also indicates that ZO-MAML converges much slower than other algorithms due to a large bias of the zeroth-order Hessian estimator, as we theoretically show in Proposition 7.

![Graph](image1)

Figure 1: Comparison of various MAML-type algorithms on matrix factorization.

### 6.2 Sine Wave Regression

Following Finn and Levine (2017), we further consider a sine wave regression problem here. Each task \( i \) uses a loss function \( l_i(w; \mathcal{X}) := \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} (f(w; x) - a_i \sin(x + b_i))^2 \), where the amplitude \( a_i \) and the phase \( b_i \) are uniformly drawn from the intervals \([0, 1.5, 0]\) and \([0, \pi]\) respectively, \( x \in \mathcal{X} \) represents data samples drawn from the interval \([-5.0, 5.0]\), and \( f(w; x) \) is a neural network parameterized by \( w \) with a sample \( x \). During the meta-training, 25 different tasks are generated with each \((a_i, b_i)\) independently sampled. At the inner stage, we run \( N = 3 \) SGD steps with a fixed stepsize \( \alpha = 10^{-4} \) with 10 data samples. At the outer stage, we sample 10 data points for each task for testing, and adopt Adam (Kingma and Ba, 2014) as the meta-optimizer with a learning rate of \( 10^{-4} \). For GGS-MAML and ZO-MAML, we set \( U = 10 \) and choose the best \( \delta \) from \( \{10^{-2}, 10^{-5}, 10^{-7}\} \). We use the meta loss (i.e., test loss) for performance evaluation. We run two experiments respectively over two neural networks (NNs): two hidden-layer NN of size 40 with ReLU nonlinearities and one hidden-layer NN of size 200 with ReLU nonlinearities.

![Graph](image2)

Figure 2: Comparisons of various MAML-type algorithms on regression.

Figure 2 illustrates that the proposed GGS-MAML achieves comparable and sometimes better performance than FO-MAML and MAML. Moreover, GGS-MAML converges much faster and more stable than ZO-MAML due to the better estimation accuracy of our proposed Hessian estimator.
6.3 Comparison of Hessian Estimators for MAML in Reinforcement Learning

In this section, we compare two Hessian estimators for MAML: our gradient-based Gaussian smoothing (GGS) and zeroth-order (ZO) (Song et al., 2020) Hessian estimators. We evaluate the performance of the corresponding GGS-MAML and ZO-MAML on a 2D Navigation meta-RL problem (Finn and Levine, 2017), which consists of many similar tasks as below. In each task, a point agent, which is represented by a point in a 2D square, aims to move toward a target position randomly chosen on this square. The reward is measured by the distance to the target location and the algorithm stops once the agent locates within 0.01 of the target point or reaches the horizon of $H = 100$. At each meta iteration, 20 different tasks are sampled for meta-training. At the inner stage, we run one-step policy gradient descent with 20 trajectories with the stepsize chosen from $\{0.03, 0.1\}$. At the outer stage, we adopt TRPO (Schulman et al., 2015) as the meta-optimizer. The experiment is performed over a two hidden-layer neural network of size 100 with ReLU nonlinearities. We set $U = 10$ for both ZO-MAML and GGS-MAML. For the leftmost plot in Figure 3, we choose $\delta = 0.3$ optimized for ZO-MAML and $\delta = 0.01$ optimized for GGS-MAML. For the right two plots in Figure 3, we choose the same $\delta = 0.1$ for both ZO-MAML and GGS-MAML to make a fair comparison of the estimation accuracy achieved by the two estimators.

As shown in the left two plots in Figure 3, our GGS-MAML converges much faster and achieves a higher reward than ZO-MAML. The rightmost plot of Figure 3 compares the Frobenius norms of the Hessian approximated by the zeroth-order Hessian estimator and our GGS Hessian estimator under the same smoothing parameter $\delta = 0.1$. It can be seen that the Frobenius norm of Hessian of our GGS Hessian estimator is much lower and more stable than that of the zeroth-order Hessian estimator. Since the true Hessian norms are very close to zero during the entire meta training iteration (Finn and Levine, 2017), the rightmost plot implies that the zeroth-order Hessian estimator suffers from a large estimation bias and variance, which explains the inferior performance of ZO-MAML to our proposed GGS-MAML. This is also consistent with our theoretical results.

7 Conclusion and Future Work

In this paper, we provide a new theoretical framework for analyzing the convergence of multi-step MAML algorithm. Our analysis reveals that a properly chosen inner stepsize is crucial for guaranteeing MAML to converge with the complexity increasing only linearly with $N$ (the number of the inner-stage gradient updates). Moreover, for problems with small Hessians, the inner stepsize can be set larger while maintaining the
convergence. We also propose a Hessian-free MAML algorithm, and validate its performance by experiments. We expect that our analysis framework can be applied to understand the convergence of MAML in other scenarios such as various RL problems, and the proposed Hessian-free algorithm is useful for large-Hessian problems.

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Appendices

A Examples for Two Types of Objective Functions

A.1 RL Example for Resampling Case

RL problems are often captured by objective functions in the expectation form. Consider a RL meta learning problem, where each task corresponds to a Markov decision process (MDP) with horizon $H$. Each RL task $T_i$ corresponds to an initial state distribution $\rho_i$, a policy $\pi_w$ parameterized by $w$ that denotes a distribution over the action set given each state, and a transition distribution kernel $q_i(x_{t+1} | x_t, a_t)$ at time steps $t = 0, \ldots, H - 1$. Then, the loss $l_i(w)$ is defined as negative total reward, i.e.,

\[
(\text{RL example}) : \quad l_i(w) := -E_{\tau \sim p_i(\cdot | w)}[R(\tau)],
\]

where $\tau = (s_0, a_0, s_1, a_1, \ldots, s_{H-1}, a_{H-1})$ is a trajectory following the distribution $p_i(\cdot | w)$, and the reward $R(\tau) := \sum_{t=0}^{H-1} \gamma^t R(s_t, a_t)$ with $R(\cdot)$ given as a reward function. The estimated gradient here is $\nabla l_i(w; \Omega) := \frac{1}{|\Omega|} \sum_{\tau \in \Omega} g_i(w; \tau)$, where $g_i(w; \tau)$ is an unbiased policy gradient estimator s.t. $E_{\tau \sim p_i(\cdot | w)} g_i(w; \tau) = \nabla l_i(w)$, e.g., REINFORCE (Williams, 1992) or G(P)OMDP (Baxter and Bartlett, 2001). In addition, the estimated Hessian is $\nabla^2 l_i(w; \Omega) := \frac{1}{|\Omega|} \sum_{\tau \in \Omega} H_i(w; \tau)$, where $H_i(w; \tau)$ is an unbiased policy Hessian estimator, e.g., DiCE (Foerster et al., 2018) or LVC (Rothfuss et al., 2018).

A.2 Classification Example for Finite-Sum Case

The risk minimization problem in classification often has a finite-sum objective function. For example, the mean-squared error (MSE) loss takes the form of

\[
(\text{Classification example}) : \quad l_{S_i}(w) := \frac{1}{|S_i|} \sum_{(x_j, y_j) \in S_i} \|y_j - \phi(w; x_j)\|^2 \quad (\text{similarly for } l_{T_i}(w)),
\]

where $x_j, y_j$ are a feature-label pair and $\phi(w; \cdot)$ can be a deep neural network parameterized by $w$.

B Derivation of Simplified Form of Gradient $\nabla L_i(w)$ in (3)

First note that $L_i(w_k) = l_i(\overline{w}_{k,N}^i)$ and $\overline{w}_{k,N}^i$ is obtained by the following gradient descent updates

\[
\overline{w}_{k,j+1}^i = \overline{w}_{k,j}^i - \alpha \nabla l_i(\overline{w}_{k,j}^i), \quad j = 0, 1, \ldots, N - 1 \text{ with } \overline{w}_{k,0}^i := w_k.
\]

Then, by the chain rule, we have

\[
\nabla L_i(w_k) = \nabla_{w_k} l_i(\overline{w}_{k,N}^i) = \prod_{j=0}^{N-1} \nabla_{\overline{w}_{k,j}^i} \left( \overline{w}_{k,j+1}^i \right) \nabla l_i(\overline{w}_{k,N}^i),
\]

which, in conjunction with (40), implies that

\[
\nabla L_i(w_k) = \prod_{j=0}^{N-1} \nabla_{\overline{w}_{k,j}^i} \left( \overline{w}_{k,j}^i - \alpha \nabla l_i(\overline{w}_{k,j}^i) \right) \nabla l_i(\overline{w}_{k,N}^i) = \prod_{j=0}^{N-1} \left( I - \alpha \nabla^2 l_i(\overline{w}_{k,j}^i) \right) \nabla l_i(\overline{w}_{k,N}^i),
\]

which finishes the proof. A similar proof can be found in Equation (21) in Nichol et al. (2018).
C Proofs for Section 3: Convergence of Multi-Step MAML in Resampling Case

In this section, our ultimate goal is to prove Theorem 1, which characterizes the convergence for multi-step MAML in the resampling case. To this end, we first derive some useful lemmas in Section C.1, and then use these lemmas to prove the propositions given in Section 3.3 on the properties of the meta gradient, and finally we use these lemmas and the propositions to establish Theorem 1 and Corollary 1.

C.1 Auxiliary Lemmas

The first lemma provides a bound on the difference between $\|\tilde{w}_j^i - \tilde{u}_j^i\|$ for $j = 0, ..., N, i \in I$, where $\tilde{w}_j^i, j = 0, ..., N, i \in I$ are given through the gradient descent updates in (1) and $\tilde{u}_j^i, j = 0, ..., N$ are defined in the same way.

**Lemma 1.** For any $i \in I$, $j = 0, ..., N$ and $w, u \in \mathbb{R}^d$, we have

$$\|\tilde{w}_j^i - \tilde{u}_j^i\| \leq (1 + \alpha L)^j \|w - u\|.$$  

**Proof.** Based on the updates that $\tilde{w}_m^i = \tilde{w}_{m-1}^i - \alpha \nabla l_i(\tilde{w}_{m-1}^i)$ and $\tilde{u}_m^i = \tilde{u}_{m-1}^i - \alpha \nabla l_i(\tilde{u}_{m-1}^i)$, we obtain, for any $i \in I$,

$$\|\tilde{w}_m^i - \tilde{u}_m^i\| = \|\tilde{w}_{m-1}^i - \alpha \nabla l_i(\tilde{w}_{m-1}^i) - \tilde{u}_{m-1}^i + \alpha \nabla l_i(\tilde{u}_{m-1}^i)\|$$

$$(i) \leq \|\tilde{w}_{m-1}^i - \tilde{u}_{m-1}^i\| + \alpha L \|\tilde{w}_{m-1}^i - \tilde{u}_{m-1}^i\|$$

$$(i) \leq (1 + \alpha L) \|\tilde{w}_{m-1}^i - \tilde{u}_{m-1}^i\|,$$

where (i) follows from the triangle inequality. Telescoping the above inequality over $m$ from 1 to $j$, we obtain

$$\|\tilde{w}_j^i - \tilde{u}_j^i\| \leq (1 + \alpha L)^j \|\tilde{w}_0^i - \tilde{u}_0^i\|,$$

which, in conjunction with the fact that $\tilde{w}_0^i = w$ and $\tilde{u}_0^i = u$, finishes the proof.

The following lemma provides an upper bound on $\|\nabla l_i(\tilde{w}_j^i)\|$ for all $i \in I$ and $j = 0, ..., N$, where $\tilde{w}_j^i$ is defined in the same way as in Lemma 1.

**Lemma 2.** For any $i \in I$, $j = 0, ..., N$ and $w \in \mathbb{R}^d$, we have

$$\|\nabla l_i(\tilde{w}_j^i)\| \leq (1 + \alpha L)^j \|\nabla l_i(w)\|.$$  

**Proof.** For $m \geq 1$, we have

$$\|\nabla l_i(\tilde{w}_m^i)\| = \|\nabla l_i(\tilde{w}_{m-1}^i) - \nabla l_i(\tilde{w}_{m-1}^i) + \nabla l_i(\tilde{w}_{m-1}^i)\|$$

$$\leq \|\nabla l_i(\tilde{w}_{m-1}^i) - \nabla l_i(\tilde{w}_{m-1}^i)\| + \|\nabla l_i(\tilde{w}_{m-1}^i)\|$$

$$\leq L \|\tilde{w}_m^i - \tilde{w}_{m-1}^i\| + \|\nabla l_i(\tilde{w}_{m-1}^i)\| \leq (1 + \alpha L) \|\nabla l_i(\tilde{w}_{m-1}^i)\|,$$

where the last inequality follows from the update $\tilde{w}_m^i = \tilde{w}_{m-1}^i - \alpha \nabla l_i(\tilde{w}_{m-1}^i)$. Then, telescoping the above inequality over $m$ from 1 to $j$ yields

$$\|\nabla l_i(\tilde{w}_j^i)\| \leq (1 + \alpha L)^j \|\nabla l_i(\tilde{w}_0^i)\|,$$

which, combined with the fact that $\tilde{w}_0^i = w$, finishes the proof.

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The following lemma gives an upper bound on the quantity $\|I - \prod_{j=0}^{m}(I - \alpha V_j)\|$ for all matrices $V_j \in \mathbb{R}^{d \times d}$, $j = 0, \ldots, m$ that satisfy $\|V_j\| \leq L$.

**Lemma 3.** For all matrices $V_j \in \mathbb{R}^{d \times d}$, $j = 0, \ldots, m$ that satisfy $\|V_j\| \leq L$, we have

$$\left\|I - \prod_{j=0}^{m}(I - \alpha V_j)\right\| \leq (1 + \alpha L)^{m+1} - 1.$$

**Proof.** First note that the product $\prod_{j=0}^{m}(I - \alpha V_j)$ can be expanded as

$$\prod_{j=0}^{m}(I - \alpha V_j) = I - \sum_{j=0}^{m} \alpha V_j + \sum_{0 \leq p < q \leq m} \alpha^2 V_p V_q + \cdots + (-1)^{m+1} \alpha^m \prod_{j=0}^{m} V_j.$$

Then, by using $\|V_j\| \leq L$ for $j = 0, \ldots, m$, we have

$$\left\|I - \prod_{j=0}^{m}(I - \alpha V_j)\right\| \leq \left\|\sum_{j=0}^{m} \alpha V_j\right\| + \left\|\sum_{0 \leq p < q \leq m} \alpha^2 V_p V_q\right\| + \cdots + \left\|\alpha^m \prod_{j=0}^{m} V_j\right\|$$

$$\leq C_{m+1}^n \alpha + C_{m+1}^n (\alpha L)^2 + \cdots + C_{m+1}^n (\alpha L)^m$$

$$= (1 + \alpha L)^{m+1} - 1,$$

where the notation $C_k^n$ denotes the number of $k$-element subsets of a set of size $n$. Then, the proof is complete. \qed

Recall the gradient $\nabla L_i(w) = \prod_{j=0}^{n-1} (I - \alpha \nabla^2 l_i(\tilde{w}_j)) \nabla l_i(\tilde{w}_N)$, where $\tilde{w}_j^i, i \in \mathcal{I}, j = 0, \ldots, N$ are given by the gradient descent steps in (1) and $\tilde{w}_0^i = w$ for all tasks $i \in \mathcal{I}$. Next, we provide an upper bound on the difference $\|\nabla l_i(w) - \nabla L_i(w)\|$.

**Lemma 4.** For any $i \in \mathcal{I}$ and $w \in \mathbb{R}^d$, we have

$$\|\nabla l_i(w) - \nabla L_i(w)\| \leq C_i\|\nabla l_i(w)\|,$$

where $C_i$ is a positive constant given by

$$C_i = (1 + \alpha L)^{2N} - 1 > 0.$$  \hspace{1cm} (41)

**Proof.** First note that $\tilde{w}_N^i$ can be rewritten as $\tilde{w}_N^i = w - \alpha \sum_{j=0}^{N-1} \nabla l_i(\tilde{w}_j^i)$. Then, based on the mean value theorem (MVT) for vector-valued functions (McLeod, 1965), we have, there exist constants $r_t, t = 1, \ldots, d$ satisfying $\sum_{t=1}^{d} r_t = 1$ and vectors $w_t^i \in \mathbb{R}^d, t = 1, \ldots, d$ such that

$$\nabla l_i(\tilde{w}_N^i) = \nabla l_i\left(w - \alpha \sum_{j=0}^{N-1} \nabla l_i(\tilde{w}_j^i)\right) = \nabla l_i(w) + \left(\sum_{t=1}^{d} r_t \nabla^2 l_i(w_t^i)\right) - \alpha \sum_{j=0}^{N-1} \nabla l_i(\tilde{w}_j^i)$$

$$= \left(I - \alpha \sum_{t=1}^{d} r_t \nabla^2 l_i(w_t^i)\right) \nabla l_i(w) - \alpha \sum_{t=1}^{d} r_t \nabla^2 l_i(w_t^i) \sum_{j=1}^{N-1} \nabla l_i(\tilde{w}_j^i).$$  \hspace{1cm} (42)
For simplicity, we define \( K(N) := \prod_{j=0}^{N-1}(1 - \alpha \nabla^2 l_i(\tilde{w}_j^i)) \). Then, using (42), we write \( \|\nabla l_i(w) - \nabla L_i(w)\| \) as

\[
\|\nabla l_i(w) - \nabla L_i(w)\| = \|\nabla l_i(w) - K(N)\nabla l_i(\tilde{w}_N^i)\|
\]

\[
= \|\nabla l_i(w) - K(N) \left(I - \alpha \sum_{t=1}^{d} r_t \nabla^2 l_i(w'_t)\right)\nabla l_i(w) + \alpha K(N) \sum_{t=1}^{d} r_t \nabla^2 l_i(w'_t) \sum_{j=1}^{N-1} \nabla l_i(\tilde{w}_j^i)\|
\]

\[
\leq \left\|\left(I - K(N) \left(I - \alpha \sum_{t=1}^{d} r_t \nabla^2 l_i(w'_t)\right)\right)\nabla l_i(w)\right\| + \left\|\alpha K(N) \sum_{t=1}^{d} r_t \nabla^2 l_i(w'_t) \sum_{j=1}^{N-1} \nabla l_i(\tilde{w}_j^i)\right\|
\]

\[
\leq (i) \left\|\left(I - K(N) \left(I - \alpha \sum_{t=1}^{d} r_t \nabla^2 l_i(w'_t)\right)\right)\nabla l_i(w)\right\| + \alpha L(1 + \alpha L)^N \sum_{j=1}^{N-1} \left\|\nabla l_i(\tilde{w}_j^i)\right\|
\]

\[
\leq \left\|\left(I - K(N) \left(I - \alpha \sum_{t=1}^{d} r_t \nabla^2 l_i(w'_t)\right)\right)\nabla l_i(w)\right\| + \alpha L(1 + \alpha L)^N \sum_{j=1}^{N-1} \left\|\nabla l_i(\tilde{w}_j^i)\right\|
\]

\[
\leq (ii) \left\|\left(I - K(N) \left(I - \alpha \sum_{t=1}^{d} r_t \nabla^2 l_i(w'_t)\right)\right)\nabla l_i(w)\right\| + \alpha L(1 + \alpha L)^N \sum_{j=1}^{N-1} \left\|\nabla l_i(\tilde{w}_j^i)\right\|
\]

\[
\leq (iii) \left\|\nabla l_i(w)\right\| + (1 + \alpha L)^{N+1}((1 + \alpha L)^{N-1} - 1)\left\|\nabla l_i(w)\right\|
\]

\[
= ((1 + \alpha L)^{2N} - 1)\left\|\nabla l_i(w)\right\|
\]

where (i) follows from the fact that \( \|\nabla^2 l_i(u)\| \leq L \) for any \( u \in \mathbb{R}^d \) and \( \sum_{t=1}^{d} r_t = 1 \), and the inequality that \( \|\sum_{j=1}^{n} a_j\| \leq \sum_{j=1}^{n} |a_j| \), (ii) follows from Lemma 2, and (iii) follows from Lemma 3.

Recall that the expected value of the gradient of the loss \( \nabla l(w) := \mathbb{E}_{i \sim p(T)} \nabla l_i(w) \) and the objective function \( \nabla L(w) := \nabla L_i(w) \). Based on the above lemmas, we next provide an upper bound on \( \|\nabla l(w)\| \) using \( \|\nabla L(w)\| \).

**Lemma 5.** For any \( w \in \mathbb{R}^d \), we have

\[
\|\nabla l(w)\| \leq \frac{1}{1 - C_l} \|\nabla L(w)\| + \frac{C_l}{1 - C_l} \sigma,
\]

where the constant \( C_l \) is given by

\[
C_l = (1 + \alpha L)^{2N} - 1.
\]

**Proof.** Based on the definition of \( \nabla l(w) \), we have

\[
\|\nabla l(w)\| = \|\mathbb{E}_{i \sim p(T)}(\nabla l_i(w) - \nabla L_i(w) + \nabla L_i(w))\|
\]

\[
\leq \|\mathbb{E}_{i \sim p(T)}\nabla L_i(w)\| + \|\mathbb{E}_{i \sim p(T)}(\nabla l_i(w) - \nabla L_i(w))\|
\]

\[
\leq \|\nabla L(w)\| + \mathbb{E}_{i \sim p(T)}\|\nabla l_i(w) - \nabla L_i(w)\|
\]

\[
\leq (i) \|\nabla L(w)\| + C_l \mathbb{E}_{i \sim p(T)}\|\nabla l_i(w)\|
\]

\[
\leq (ii) \|\nabla L(w)\| + C_l(\|\nabla l(w)\| + \sigma),
\]

where (i) follows from Lemma 4, and (ii) follows from Assumption 2. Then, rearranging the above inequality completes the proof. \( \square \)
Recall from (15) that we choose the meta stepsize $\beta_k = \frac{1}{C_\beta L_{w_k}}$, where $C_\beta$ is a positive constant and $\hat{L}_{w_k} = (1 + \alpha L)^2 N L + C_\epsilon \frac{1}{|B_k|} \sum_{i \in B_k} \|\nabla l_i(w_k; \hat{D}_{L_k})\|$. Using an approach similar to Lemma 4.11 in Fallah et al. (2019), we establish the following lemma to provide the first- and second-moment bounds for $\beta_k$.

**Lemma 6.** Suppose that Assumptions 1, 2 and 3 hold. Set the meta stepsize $\beta_k = \frac{1}{C_\beta L_{w_k}}$ with $\hat{L}_{w_k}$ given by (15), where $|B'_k| > \frac{4C^2\sigma^2}{3(1+\alpha L)^2 L^2}$ and $|D'_{L_k}| > \frac{64\sigma^2 C^2}{(1+\alpha L)^2 L^2}$ for all $i \in B'_k$. Then, conditioning on $w_k$, we have

$$
E\beta_k \geq \frac{4}{C_\beta 5L_{w_k}}, \quad E\beta_k^2 \leq \frac{4}{C_\beta^2 L_{w_k}^2},
$$

where $L_{w_k} = (1 + \alpha L)^2 N L + C_\epsilon \mathbb{E}_{i \sim p(T)} \|\nabla l_i(w_k)\|$ with $C_\epsilon$ given in (14).

**Proof.** Let $\tilde{L}_{w_k} = 4L + \frac{4C_\epsilon}{(1+\alpha L)^2 N} \frac{1}{|B_k|} \sum_{i \in B_k} \|\nabla l_i(w_k; D_{L_k}^i)\|$. Note that $|B'_k| > \frac{4C^2\sigma^2}{3(1+\alpha L)^2 L^2}$ and $|D_{L_k}^i| > \frac{64\sigma^2 C^2}{(1+\alpha L)^2 L^2}$, $i \in B'_k$. Then, using an approach similar to (61) in Fallah et al. (2019) and conditioning on $w_k$, we have

$$
E\left(\frac{1}{L_{w_k}^2}\right) \leq \frac{\sigma^2_\beta/4L^2 + \mu^2_\beta/\mu_\beta^2}{\sigma^2_\beta + \mu^2_\beta}, \quad \text{(43)}
$$

where $\sigma^2_\beta$ and $\mu_\beta$ are the variance and mean of variable $\frac{4C_\epsilon}{(1+\alpha L)^2 N} \frac{1}{|B_k|} \sum_{i \in B_k} \|\nabla l_i(w_k; D_{L_k}^i)\|$. Using an approach similar to (62) in Fallah et al. (2019), conditioning on $w_k$ and using $|D_{L_k}^i| > \frac{64\sigma^2 C^2}{(1+\alpha L)^2 L^2}$, we have

$$
\frac{C_\epsilon}{(1 + \alpha L)^2 N} E_i \|\nabla l_i(w_k)\| - L \leq \mu_\beta \leq \frac{C_\epsilon}{(1 + \alpha L)^2 N} E_i \|\nabla l_i(w_k)\| + L, \quad \text{(44)}
$$

which implies that $\mu_\beta + 5L \geq \frac{4}{(1+\alpha L)^2 N} L_{w_k}$, and thus using (43) yields

$$
\frac{16}{(1 + \alpha L)^4 N} L_{w_k}^2 E\left(\frac{1}{L_{w_k}^2}\right) \leq \frac{\mu^2_\beta (25/16 + \sigma^2_\beta/(8L^2)) + 25\sigma^2_\beta/8}{\sigma^2_\beta + \mu^2_\beta}. \quad \text{(45)}
$$

Furthermore, conditioning on $w_k$, $\sigma_\beta$ is bounded by

$$
\sigma^2_\beta = \frac{16C^2_\epsilon}{(1 + \alpha L)^4 N |B'_k|} \operatorname{Var}(\|\nabla l_i(w_k; D_{L_k}^i)\|) \\
\leq \frac{16C^2_\epsilon}{(1 + \alpha L)^4 N |B'_k|} \left(\sigma^2 + \frac{\sigma^2_\beta}{|D_{L_k}^i|}\right) \\
\overset{(i)}{\leq} \frac{16C^2_\epsilon \sigma^2}{(1 + \alpha L)^4 N |B'_k|} + \frac{L^2}{4|B'_k|} \overset{(ii)}{\leq} 12L^2 + \frac{1}{4} L^2 < \frac{25}{2} L^2, \quad \text{(46)}
$$

where (i) follows from $|D_{L_k}^i| > \frac{64\sigma^2 C^2}{(1+\alpha L)^2 L^2}$, $i \in B'_k$ and (ii) follows from $|B'_k| > \frac{4C^2\sigma^2}{3(1+\alpha L)^2 L^2}$ and $|B'_k| \geq 1$. Then, plugging (46) in (45) yields

$$
\frac{16}{(1 + \alpha L)^4 N} L_{w_k}^2 E\left(\frac{1}{L_{w_k}^2}\right) \leq \frac{25}{8}.
$$

Then, noting that $\beta_k = \frac{1}{C_\beta L_{w_k}} = \frac{4}{C_\beta (1+\alpha L)^2 N L_{w_k}}$, using the above inequality and conditioning on $w_k$, we have

$$
E\beta_k^2 \leq \frac{16}{C_\beta^2 (1 + \alpha L)^4 N} E\left(\frac{1}{L_{w_k}^2}\right) \leq \frac{25}{8} \frac{1}{L_{w_k}^2} < \frac{4}{C_\beta^2 L_{w_k}^2}. \quad \text{(47)}
$$
In addition, by Jensen’s inequality and conditioning on \( w_k \), we have

\[
\mathbb{E} \beta_k = \frac{4}{C_\beta (1 + \alpha L)^{2N}} \mathbb{E} \left( \frac{1}{L_{w_k}} \right) \\
\geq \frac{4}{C_\beta (1 + \alpha L)^{2N}} \frac{1}{\mathbb{E} L_{w_k}} \\
= \frac{C_\beta (1 + \alpha L)^{2N}}{4L + \mu_\beta} \\
\geq \frac{4}{C_\beta 4L (1 + \alpha L)^{2N} + L_{w_k}} \\
\geq \frac{4}{C_\beta 5L_{w_k}},
\]

(48)

where (i) follows from (44) and (ii) follows from the fact \( L_{w_k} > (1 + \alpha L)^{2N} L \).

\[\square\]

### C.2 Proofs for Propositions in Section 3.3

In this subsection, we provide proofs for the propositions on the properties of the meta gradient given in Section 3.3 based on the lemmas established in the previous subsection. To simplify notations, we let \( \bar{S}_j^i \) and \( \bar{D}_j^i \) denote the randomness over \( S_{k,m}^i, m = 0, ..., j - 1 \) and \( D_{k,m}^i, m = 0, ..., j - 1 \), and let \( \bar{S}_j \) and \( \bar{D}_j \) denote all randomness over \( S_j^i, i \in \mathcal{I} \) and \( D_j^i, i \in \mathcal{I}, \) respectively.

**Proof of Proposition 1.** First recall that \( \nabla L_i(w) = \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \nabla l_i(\bar{w}_N^i) \). Then, we have

\[
\| \nabla L_i(w) - \nabla L_i(u) \| = \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \nabla l_i(\bar{w}_N^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \nabla l_i(\bar{u}_N^i) \right\|
\]

\[
\leq \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \nabla l_i(\bar{w}_N^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \nabla l_i(\bar{u}_N^i) \right\|
\]

\[
+ \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \nabla l_i(\bar{w}_N^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \nabla l_i(\bar{u}_N^i) \right\|
\]

\[
\leq \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \right\| \| \nabla l_i(\bar{w}_N^i) \|
\]

\[
+ (1 + \alpha L)^N \| \nabla l_i(\bar{w}_N^i) - \nabla l_i(\bar{u}_N^i) \|
\]

\[
\leq \left( i \right) \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \right\| (1 + \alpha L)^N \| \nabla l_i(w) \|
\]

\[
+ (1 + \alpha L)^N \| \bar{w}_N^i - \bar{u}_N^i \|
\]

\[
\leq \left( i \right) \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_j^i)) \right\| (1 + \alpha L)^N \| \nabla l_i(w) \|
\]

\[
+ (1 + \alpha L)^{2N} L \| w - u \|,
\]

(49)
where (i) follows from Lemma 2, and (ii) follows from Lemma 1. We next upper-bound the term $V(N)$ in the above inequality. Specifically, define a more general quantity $V(m) = \| \prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(\tilde{w}_j^i)) - \prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(\tilde{u}_j^i)) \|$. Then, we have

$$V(m) \leq \left\| \prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(\tilde{w}_j^i)) - \prod_{j=0}^{m-2} (I - \alpha \nabla^2 l_i(\tilde{w}_j^i))(I - \alpha \nabla^2 l_i(\tilde{u}_m^i)) \right\|$$

$$+ \left\| \prod_{j=0}^{m-2} (I - \alpha \nabla^2 l_i(\tilde{w}_j^i))(I - \alpha \nabla^2 l_i(\tilde{w}_m^i)) - \prod_{j=0}^{m-2} (I - \alpha \nabla^2 l_i(\tilde{u}_j^i)) \right\|$$

$$\leq \left\| \prod_{j=0}^{m-2} (I - \alpha \nabla^2 l_i(\tilde{w}_j^i)) \right\| \left\| \alpha \nabla^2 l_i(\tilde{w}_{m-1}^i) - \alpha \nabla^2 l_i(\tilde{u}_{m-1}^i) \right\|$$

$$+ \left\| \prod_{j=0}^{m-2} (I - \alpha \nabla^2 l_i(\tilde{u}_j^i)) - \prod_{j=0}^{m-2} (I - \alpha \nabla^2 l_i(\tilde{u}_j^i)) \right\|$$

$$\leq (1 + \alpha L)^{m-1} \alpha \| \tilde{w}_{m-1}^i - \tilde{u}_{m-1}^i \| \leq \alpha \| w - u \| + (1 + \alpha L)V(m - 1)$$

Telescoping the above inequality over $m$ from 1 to $N$ and noting $V(1) = \| \alpha \nabla^2 l_i(w) - \alpha \nabla^2 l_i(u) \| \leq \alpha \| w - u \|$, we obtain

$$V(N) \leq (1 + \alpha L)^{N-1} V(1) + \sum_{m=0}^{N-2} \alpha \rho(1 + \alpha L)^{2(N-2)-m} \| w - u \| (1 + \alpha L)^{m}$$

$$\leq (1 + \alpha L)^{N-1} \alpha \| w - u \| + \sum_{m=0}^{N-2} \alpha \rho(1 + \alpha L)^{2N-2-m} \| w - u \|$$

$$= (1 + \alpha L)^{N-1} \alpha \| w - u \| + \alpha \rho(1 + \alpha L)^{N} \sum_{m=0}^{N-2} (1 + \alpha L)^{m} \| w - u \|$$

$$\leq \left( (1 + \alpha L)^{N-1} \alpha \rho + \frac{\rho}{L} (1 + \alpha L)^{N} ((1 + \alpha L)^{N-1} - 1) \right) \| w - u \|. \ (50)$$

Recall that $C_L = \left( (1 + \alpha L)^{N-1} \alpha \rho + \frac{\rho}{L} (1 + \alpha L)^{N} ((1 + \alpha L)^{N-1} - 1) \right) (1 + \alpha L)^{N}$. Combining (49) and (50) yields

$$\| \nabla L_i(w) - \nabla L_i(u) \| \leq \left( C_L \| \nabla l_i(w) \| + (1 + \alpha L)^{2N L} \right) \| w - u \|.$$ 

Based on the above inequality, we have

$$\| \nabla L(w) - \nabla L(u) \| = \| \mathbb{E}_{i \sim p(T)} (\nabla L_i(w) - \nabla L_i(u)) \|$$

$$\leq \mathbb{E}_{i \sim p(T)} \| (\nabla L_i(w) - \nabla L_i(u)) \|$$

$$\leq \left( C_L \mathbb{E}_{i \sim p(T)} \| \nabla l_i(w) \| + (1 + \alpha L)^{2N L} \right) \| w - u \|,$$

which finishes the proof. \qed
Proof of Proposition 2. We first prove the first-moment bound. Conditioning on $w_k$, we have
\[
E_{S^i_m} \|w^i_{k,m} - \bar{w}^i_{k,m}\| \overset{(i)}{=} E_{S^i_m} \|w^i_{k,m-1} - \alpha \nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) \rangle - (\bar{w}^i_{k,m-1} - \alpha \nabla l_i(\bar{w}^i_{k,m-1}))\| \\
\leq E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\| + \alpha E_{S^i_m} \|\nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1})\| \\
+ \alpha E_{S^i_m} \|\nabla l_i(w^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1})\| \\
\leq \alpha L \|\nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1})\| \\
+ (1 + \alpha L) E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\| \\
\leq (1 + \alpha L) E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\| + \alpha \sigma_g \frac{S}{\sqrt{S}},
\]
where (i) follows from (1) and (6), and the last inequality follows from Assumption 3. Rearranging the above inequality, we have
\[
E_{S^i_m} \|w^i_{k,m} - \bar{w}^i_{k,m}\| \leq \frac{\sigma_g}{L \sqrt{S}} \leq (1 + \alpha L) \left( E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\| + \frac{\sigma_g}{L \sqrt{S}} \right).
\]
Telescoping the above inequality over $m$ from 1 to $j$ yields
\[
E_{S^j} \|w^j_{k,j} - \bar{w}^j_{k,j}\| \leq (1 + \alpha L) \left( E_{S^i_0} \|w^i_{k,0} - \bar{w}^i_{k,0}\| + \frac{\sigma_g}{L \sqrt{S}} \right),
\]
which, in conjunction with the fact that $w^i_{k,0} = \bar{w}^i_{k,0} = w_k$, implies that
\[
E_{S^j} \|w^j_{k,j} - \bar{w}^j_{k,j}\| \leq ((1 + \alpha L) - 1) \frac{\sigma_g}{L \sqrt{S}},
\]
which finishes the proof of the first-moment bound.

We next begin to prove the second-moment bound. Similarly, conditioning on $w_k$, we have
\[
E_{S^i_m} \|w^i_{k,m} - \bar{w}^i_{k,m}\|^2 \\
= E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1} - \alpha (\nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1}))\|^2 \\
= E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\|^2 + \alpha^2 E_{S^i_m} \|\nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1})\|^2 \\
- 2\alpha E_{S^i_m} \langle w^i_{k,m-1} - \bar{w}^i_{k,m-1}, \nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1}) \rangle \\
\leq E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\|^2 + \alpha^2 E_{S^i_m} \|\nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1})\|^2 \\
- 2\alpha E_{S^i_m} \langle w^i_{k,m-1} - \bar{w}^i_{k,m-1}, \nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1}) \rangle \\
\overset{(i)}{\leq E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\|^2 - 2\alpha E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}, \nabla l_i(w^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1})\|^2 \\
+ \alpha^2 E_{S^i_m} (2 \|\nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) - \nabla l_i(w^i_{k,m-1})\|^2 + 2\|\nabla l_i(w^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1})\|^2) \\
\overset{(ii)}{\leq E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\|^2 + 2\alpha E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}, \nabla l_i(w^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1})\|^2 \\
+ \alpha^2 E_{S^i_m} (2 \|\nabla l_i(w^i_{k,m-1}; S^i_{k,m-1}) - \nabla l_i(w^i_{k,m-1})\|^2 + 2\|\nabla l_i(w^i_{k,m-1}) - \nabla l_i(\bar{w}^i_{k,m-1})\|^2) \\
\leq E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\|^2 + 2\alpha L E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\|^2 \\
+ 2\alpha^2 E_{S^i_m} \left( \frac{\sigma_g^2}{S} + L^2 \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\|^2 \right) \\
\leq (1 + 2\alpha L + 2\alpha^2 L^2) E_{S^i_m} \|w^i_{k,m-1} - \bar{w}^i_{k,m-1}\|^2 + \frac{2\alpha^2 \sigma_g^2}{S},
\]
where (i) follows from \( E_{S^1_{k,m-1}} \nabla l_i(w_{k,m-1}^i ; S^1_{k,m-1}) = \nabla l_i(w_{k,m-1}^i) \) and (ii) follows from the inequality that \(-\langle a, b \rangle \leq \|a\| \|b\|\) for any vectors \(a, b\). Noting that \(w_{k,0}^i = \tilde{w}_{k,0}^i = w_k\) and telescoping the above inequality over \(m\) from 1 to \(j\), we obtain

\[
E_{S^j} \|w_{k,j}^i - \tilde{w}_{k,j}^i\|^2 \leq ((1 + 2\alpha L + 2\alpha^2 L^2)^j - 1) \frac{\alpha \sigma_g^2}{L(1 + \alpha L)^S}.
\]

Then, taking the expectation over \(w_k\) in the above inequality yields

\[
E\|w_{k,j}^i - \tilde{w}_{k,j}^i\|^2 \leq ((1 + 2\alpha L + 2\alpha^2 L^2)^j - 1) \frac{\alpha \sigma_g^2}{L(1 + \alpha L)^S},
\]

which finishes the proof.

**Proof of Proposition 3.** Recall the definition that \( \hat{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i ; D_{k,j}^i)) \nabla l_i(w_{k,j}^i ; T_k^i) \).

Then, conditioning on \(w_k\), we have

\[
\mathbb{E} \hat{G}_i(w_k) = \mathbb{E}_{S^N, \iota \sim p(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i ; D_{k,j}^i)) \mathbb{E}_{T_k} \nabla l_i(w_{k,N}^i ; T_k^i) \right) | S^N, \iota
\]

\[
= \mathbb{E}_{S^N, \iota \sim p(T)} \left( \prod_{j=0}^{N-1} \mathbb{E}_{D_{k,j}} \left( I - \alpha \nabla^2 l_i(w_{k,j}^i ; D_{k,j}^i) \right) | S^N, \iota \right) \nabla l_i(w_{k,N}^i , T_k^i)
\]

\[
= \mathbb{E}_{S^N, \iota \sim p(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i)) \nabla l_i(w_{k,N}^i) \right) ,
\]

which, in conjunction with \( \nabla \mathcal{L}(w_k) = \mathbb{E}_{\iota \sim p(T)} \nabla \mathcal{L}_i(w_k) = \mathbb{E}_{\iota \sim p(T)} \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\tilde{w}_{k,j}^i)) \nabla l_i(\tilde{w}_{k,N}^i) \) and the definition that \( e_k = \mathbb{E} \hat{G}_i(w_k) - \nabla \mathcal{L}(w_k) \), implies

\[
\|e_k\| \leq \mathbb{E}_{S^N, \iota \sim p(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i)) \nabla l_i(w_{k,N}^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\tilde{w}_{k,j}^i)) \nabla l_i(\tilde{w}_{k,N}^i) \right)
\]

\[
\leq \mathbb{E}_{S^N, \iota \sim p(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i)) \nabla l_i(w_{k,N}^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\tilde{w}_{k,j}^i)) \nabla l_i(\tilde{w}_{k,N}^i) \right)
\]

\[
+ \mathbb{E}_{S^N, \iota \sim p(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\tilde{w}_{k,j}^i)) \nabla l_i(\tilde{w}_{k,N}^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\tilde{w}_{k,j}^i)) \nabla l_i(\tilde{w}_{k,N}^i) \right)
\]

\[
\leq \mathbb{E}_{S^N, \iota \sim p(T)} (1 + \alpha L)^N \left( \nabla l_i(w_{k,N}^i) - \nabla l_i(\tilde{w}_{k,N}^i) \right)
\]

\[
+ \mathbb{E}_{S^N, \iota \sim p(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i)) \nabla l_i(w_{k,N}^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\tilde{w}_{k,j}^i)) \nabla l_i(\tilde{w}_{k,N}^i) \right)
\]

\[
\leq (1 + \alpha L)^N \mathbb{E}_{\iota \sim p(T)} \|w_{k,N}^i - \tilde{w}_{k,N}^i\|
\]

\[
+ (1 + \alpha L)^N \mathbb{E}_{S^N, \iota \sim p(T)} \|l_i(w_k)\| \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i)) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\tilde{w}_{k,j}^i)) \right)
\]

\[
\leq (1 + \alpha L)^N \mathbb{E}_{\iota \sim p(T)} \left( \|l_i(w_k)\| \mathbb{E}_{\tilde{S}^N} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i)) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\tilde{w}_{k,j}^i)) \right) \right)
\]

\[
+ (1 + \alpha L)^N ((1 + \alpha L)^N - 1) \frac{\sigma_g}{\sqrt{S}}.
\]

\[34\]
where (i) follows from the Jensen’s inequality, (ii) follows from Lemma 2 that $\|\nabla l_i(\hat{w}_{k,N}^i)\| \leq (1 + \alpha L)^N\|\nabla l_i(w_k)\|$, and the last inequality follows from item 1 in Proposition 2 with $j = N$.

Our next step is to upper-bound the term $R(N)$ in (52). To simplify notations, we define a general quantity $R(m) = \mathbb{E}_{S_m|i}\|\prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i)) - \prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(\hat{w}_{k,j}^i))\|$ for $m = 1, ..., N$, where we use $\mathbb{E}_{S_m|i}(\cdot)$ to denote $\mathbb{E}_{S_m}(\cdot|i)$. Then, we have

$$R(m) \leq \mathbb{E}_{S_m|i}\|\prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i)) - \prod_{j=0}^{m-2} (I - \alpha \nabla^2 l_i(\hat{w}_{k,j}^i))\|
+ \mathbb{E}_{S_m|i}\|\prod_{j=0}^{m-2} (I - \alpha \nabla^2 l_i(\hat{w}_{k,j}^i)) - \prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(\hat{w}_{k,j}^i))\|$$

$$\leq (1 + \alpha L)^{m-1} \alpha \mathbb{E}_{S_m|i}\|w_{k,m-1}^i - \hat{w}_{k,m-1}^i\| + (1 + \alpha L) R(m-1)
\leq \alpha \alpha (1 + \alpha L)^{m-1}((1 + \alpha L)^{m-1} - 1) \frac{\sigma_g}{L\sqrt{S}} + (1 + \alpha L) R(m-1)$$

(53)

where (i) follows from Proposition 2. Noting that $m \leq N$, we have

$$\alpha \alpha (1 + \alpha L)^{m-1}((1 + \alpha L)^{m-1} - 1) \frac{\sigma_g}{L\sqrt{S}} \leq \alpha \alpha (1 + \alpha L)^{N-1}((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{L\sqrt{S}},$$

which, in conjunction with (53), implies that

$$R(m) \leq \alpha \alpha (1 + \alpha L)^{N-1}((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{L\sqrt{S}} + (1 + \alpha L) R(m-1).$$

Rearranging the above inequality, we obtain

$$R(m) + \frac{\rho}{L} (1 + \alpha L)^{N-1}((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{L\sqrt{S}}$$

$$\leq (1 + \alpha L) \left( R(m-1) + \frac{\rho}{L} (1 + \alpha L)^{N-1}((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{L\sqrt{S}} \right).$$

Telescoping the above inequality over $m$ from 2 to $N$ yields

$$R(N) + \frac{\rho}{L} (1 + \alpha L)^{N-1}((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{L\sqrt{S}}$$

$$\leq (1 + \alpha L)^{N-1} \left( R(1) + \frac{\rho}{L} (1 + \alpha L)^{N-1}((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{L\sqrt{S}} \right),$$

which, combined with the fact $R(1) = \mathbb{E}_{S_1|i}\|\alpha \nabla^2 f_i(w_k) - \alpha \nabla^2 f_i(w_k)\| = 0$, yields

$$R(N) \leq ((1 + \alpha L)^{N-1} - 1) \frac{\rho}{L} (1 + \alpha L)^{N-1} \frac{\sigma_g}{L\sqrt{S}}.$$  

(54)
Thus, conditioning on \( w_k \) and combining (54) and (52), we have
\[
\|\epsilon_k\| \leq (1 + \alpha L)^{N-1} - 1 \frac{2\alpha}{L} (1 + \alpha L)^{2N-1} \frac{\sigma_g}{L \sqrt{S}} E_{\iota \sim \mathcal{P}(\mathcal{T})} (\| \nabla l_i(w_k) \|) \\
+ (1 + \alpha L)^{N} ((1 + \alpha L)^{N} - 1) \frac{\sigma_g}{\sqrt{S}}
\]
\[
\leq (1 + \alpha L)^{N-1} - 1 \frac{2\alpha}{L} (1 + \alpha L)^{2N-1} \frac{\sigma_g}{L \sqrt{S}} (\| \nabla l_i(w_k) \| + \sigma) \\
+ (1 + \alpha L)^{N} ((1 + \alpha L)^{N} - 1) \frac{\sigma_g}{\sqrt{S}}
\]
\[
\leq (1 + \alpha L)^{N-1} - 1 \frac{2\alpha}{L} (1 + \alpha L)^{2N-1} \frac{\sigma_g}{L \sqrt{S}} \left( \frac{1}{1 - C_l(\alpha, N)} \| \nabla L(w_k) \| + \frac{C_l(\alpha, N)}{1 - C_l(\alpha, N)} \| \nabla \| + \sigma \right) \\
+ (1 + \alpha L)^{N} ((1 + \alpha L)^{N} - 1) \frac{\sigma_g}{\sqrt{S}},
\]
where the last inequality follows from Lemma 5 and \( C_l(\alpha, N) \) is given by (41). Rearranging the above inequality and using the definitions of \( C_{err_1} \) and \( C_{err_2} \) in Proposition 3, we finish the proof.

**Proof of Proposition 4.** Recall \( \tilde{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i; D_{k,j}^i)) \nabla l_i(w_{k,N}^i; T_k^i) \). Then, conditioning on \( w_k \), we have
\[
\mathbb{E}\|\tilde{G}_i(w_k)\|^2 \leq \mathbb{E}_{S_N,i} \left( \mathbb{E}_{\bar{D}_N,T_k} \left( \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k,j}^i; D_{k,j}^i)) \right\| \| \nabla l_i(w_{k,N}^i; T_k^i) \|^2 \right) \right).
\]
\[
\leq \mathbb{E}_{S_N,i} \left( \prod_{j=0}^{N-1} \mathbb{E}_{\bar{D}_N} \left( \left\| I - \alpha \nabla^2 l_i(w_{k,j}^i; D_{k,j}^i) \right\| \| \tilde{S}_N, i \| \right) \mathbb{E}_{T_k^i} \left( \| \nabla l_i(w_{k,N}^i; T_k^i) \|^2 \right) \right).
\]
We upper-bound (P) and (Q) in the above inequality, respectively. Note that \( w_{k,j}^i, j = 0, \ldots, N - 1 \) are deterministic when conditioning on \( S_N, i \), and \( w_k \). Thus, conditioning on \( S_N, i \), and \( w_k \), we have
\[
\mathbb{E}_{\bar{D}_N} \left\| I - \alpha \nabla^2 l_i(w_{k,j}^i; D_{k,j}^i) \right\|^2 = \text{Var} \left( I - \alpha \nabla^2 l_i(w_{k,j}^i; D_{k,j}^i) \right) + \| I - \alpha \nabla^2 l_i(w_{k,j}^i) \|^2 \leq \frac{\alpha^2 \sigma_H^2}{D} + (1 + \alpha L)^2.
\]
We next bound (Q). Conditioning on \( \tilde{S}_N, i \) and \( w_k \), we have
\[
\mathbb{E}_{T_k^i} \| \nabla l_i(w_{k,N}^i; T_k^i) \|^2 \leq 3 \mathbb{E}_{T_k^i} \| \nabla l_i(w_{k,N}^i; T_k^i) - \nabla l_i(w_{k,N}^i) \|^2 + 3 \mathbb{E}_{T_k^i} \| \nabla l_i(w_{k,N}^i) - \nabla l_i(\tilde{w}_{k,N}^i) \|^2 \\
+ 3 \mathbb{E}_{T_k^i} \| \nabla l_i(\tilde{w}_{k,N}^i) \|^2 \leq \frac{3\sigma_H^2}{L} + 3L^2 \| w_{k,N}^i - \tilde{w}_{k,N}^i \|^2 + 3(1 + \alpha L)^2 N \| \nabla l_i(w_k) \|^2,
\]
where (i) follows from the inequality that \( \| \sum_{i=1}^n a \|^2 \leq n \sum_{i=1}^n \| a \|^2 \), and the second inequality follows from Lemma 2. Thus, conditioning on \( w_k \) and combining (56), (57) and (58), we have
\[
\mathbb{E}\|\tilde{G}_i(w_k)\|^2 \leq \left( \frac{\alpha^2 \sigma_H^2}{D} + (1 + \alpha L)^2 \right) \mathbb{E}_{T_k^i} \| \nabla l_i(w_{k,N}^i; T_k^i) \|^2 + 3 \mathbb{E}_{T_k^i} \| \nabla l_i(w_{k,N}^i) - \nabla l_i(\tilde{w}_{k,N}^i) \|^2 + 3(1 + \alpha L)^2 N \| \nabla l_i(w_k) \|^2
\]
which, in conjunction with Proposition 2 that
\[
\mathbb{E}\|w_{k,N}^i - \tilde{w}_{k,N}^i\|^2 = \mathbb{E}_i(\|w_{k,N}^i - \tilde{w}_{k,N}^i\|^2 | i) \leq ((1 + 2\alpha L + 2\alpha^2 L^2)^N - 1) \frac{\alpha \sigma_H^2}{L(1 + \alpha L)S},
\]
36
implies that
\[
\mathbb{E}[\|\hat{G}_i(w_k)\|^2] \leq \frac{C_{\text{squ}}}{T} + \frac{C_{\text{squ}}}{S} + 3(1 + \alpha L)^{2N} \left( \frac{\alpha^2 \sigma_H^2}{D} + (1 + \alpha L)^2 \right)^N \left(\|\nabla l(w_k)\|^2 + \sigma^2\right).
\]

Based on Lemma 5 and conditioning on \(w_k\), we have
\[
\|\nabla l(w_k)\|^2 \leq \frac{2}{(1 - C_l)^2} \|\nabla L(w_k)\| + \frac{2C_l^2}{(1 - C_l)^2} \sigma^2,
\]
which, in conjunction with the inequality that \(\frac{2x^2}{(1-x)^2} + 1 \leq \frac{2}{(1-x)^2}\) and (59), yields
\[
\mathbb{E}[\|\hat{G}_i(w_k)\|^2] \leq \frac{C_{\text{squ}}}{T} + \frac{C_{\text{squ}}}{S} + 3(1 + \alpha L)^{2N} \left( \frac{\alpha^2 \sigma_H^2}{D} + (1 + \alpha L)^2 \right)^N \frac{2}{(1 - C_l)^2} \left(\|\nabla L(w_k)\|^2 + \sigma^2\right).
\]
Combining the above inequality with \(C_l = (1 + \alpha L)^{2N} - 1\) yields the proof.

C.3 Proofs for Section 3.4: Convergence of Multi-Step MAML in Resampling Case

In this subsection, we provide proofs for the results in Section 3.4.

**Proof of Theorem 1.** The proof of Theorem 1 consists of four main steps: step 1 of bounding an iterative meta update by the meta-gradient smoothness established by Proposition 1; step 2 of characterizing first-moment estimation error of the meta-gradient estimator \(\hat{G}_i(w_k)\) by Proposition 3; step 3 of characterizing second-moment estimation error of the meta-gradient estimator \(\hat{G}_i(w_k)\) by Proposition 4; and step 4 of combining steps 1-3, and telescoping to yield the convergence.

To simplify notations, we let
\[
L_{w_k} = (1 + \alpha L)^{2N} L + C_L \mathbb{E}_{i \sim p(B)} \|\nabla l_i(w_k)\|,
\]
where \(C_L\) is given in (14).

Based on the smoothness of the gradient \(\nabla L(w)\) given by Proposition 1, we have
\[
\mathcal{L}(w_{k+1}) \leq \mathcal{L}(w_k) + \langle \nabla \mathcal{L}(w), w_{k+1} - w_k \rangle + \frac{L_{w_k}}{2} \|w_{k+1} - w_k\|^2
\]
\[
= \mathcal{L}(w_k) - \beta_k \langle \nabla \mathcal{L}(w_k), \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \rangle + \frac{L_{w_k} \beta_k^2}{2} \left\| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \right\|^2.
\]
Note that the randomness from \(\beta_k\) depends on \(B'_k\) and \(D_{L_k}^{i}, i \in B'_k\), and thus is independent of \(S_{k,j}^i, D_{k,j}^i\) and \(T_k^i\) for \(i \in B_k, j = 0, ..., N\). Thus, taking expectation over the above inequality, conditioning on \(w_k\), and recalling \(e_k := \mathbb{E}[\hat{G}_i(w_k) - \nabla \mathcal{L}(w_k)]\), we have
\[
\mathbb{E}(\mathcal{L}(w_{k+1}|w_k) \leq \mathcal{L}(w_k) - \left[\mathbb{E}(\beta_k)\right] \langle \nabla \mathcal{L}(w_k), \nabla \mathcal{L}(w_k) + e_k \rangle
\]
\[
+ \frac{L_{w_k}}{2} \left[\mathbb{E}(\beta_k^2)\right] \mathbb{E}\left(\left\| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \right\|^2 \right| w_k\right). \quad (61)
\]
Then, applying Lemma 6 in (61) yields
\[
\mathbb{E}(\mathcal{L}(w_{k+1}|w_k) \leq \mathcal{L}(w_k) - \frac{4}{5C_\beta L_{w_k}} \|\nabla \mathcal{L}(w_k)\|^2 - \frac{4}{5C_\beta L_{w_k}} \langle \nabla \mathcal{L}(w_k), e_k \rangle + \frac{2}{C_\beta} \mathbb{E}\left(\left\| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \right\|_2 w_k \right)
\]
\[
\leq \mathcal{L}(w_k) - \frac{4}{5C_\beta L_{w_k}} \|\nabla \mathcal{L}(w_k)\|^2 - \frac{4}{5C_\beta L_{w_k}} \langle \nabla \mathcal{L}(w_k), e_k \rangle + \frac{2}{C_\beta} \mathbb{E}\left(\left\| \hat{G}_i(w_k) \right\|^2 + \| \mathbb{E}\hat{G}_i(w_k) \|^2 \right)
\]
\[
\leq \mathcal{L}(w_k) - \frac{4}{5C_\beta L_{w_k}} \|\nabla \mathcal{L}(w_k)\|^2 + \frac{2}{5C_\beta} \frac{1}{L_{w_k}} \|\nabla \mathcal{L}(w_k)\|^2 + \frac{2}{5C_\beta} \frac{1}{L_{w_k}} \|e_k\|^2
\]
\[
+ \frac{2}{C_\beta} \frac{1}{L_{w_k}} \left( \frac{1}{B} \mathbb{E}\left( \left\| \hat{G}_i(w_k) \right\|^2 + \| \mathbb{E}\hat{G}_i(w_k) \|^2 \right) \right),
\]
(62)
where the last inequality follows from the fact that $-\langle a, b \rangle \leq \frac{1}{2}(\|a\|^2 + \|b\|^2)$. Then, applying Propositions 3 and 4 to the above inequality yields
\[
\mathbb{E}(\mathcal{L}(w_{k+1}|w_k) \leq \mathcal{L}(w_k) - \frac{2}{5C_\beta} \frac{1}{L_{w_k}} \|\nabla \mathcal{L}(w_k)\|^2 + \frac{2}{5C_\beta} \frac{1}{L_{w_k}} \left( \frac{3C_{err_2}}{S} \|\nabla \mathcal{L}(w_k)\|^2 + \frac{3C_{err_1}}{S} + \frac{3C_{err_2}^2}{S} \sigma^2 \right)
\]
\[
+ \frac{2}{C_\beta} \frac{1}{L_{w_k}} \left( \frac{1}{B} \left\| \hat{G}_i(w_k) \right\|^2 + \frac{1}{2} \| \mathbb{E}\hat{G}_i(w_k) \|^2 \right)
\]
\[
\leq \mathcal{L}(w_k) - \frac{2}{5C_\beta} \frac{1}{L_{w_k}} \|\nabla \mathcal{L}(w_k)\|^2 + \left( \frac{6}{5C_\beta L_{w_k}} + \frac{12}{C_\beta^2 L_{w_k}} \right) \left( \frac{C_{err_2}^2}{S} \|\nabla \mathcal{L}(w_k)\|^2 + \frac{C_{err_1}}{S} + \frac{C_{err_2}^2}{S} \right)
\]
\[
+ \frac{2}{C_\beta} \frac{1}{L_{w_k}} \left( \frac{1}{B} \left\| \hat{G}_i(w_k) \right\|^2 + \frac{4}{C_\beta} \frac{1}{L_{w_k}} \|\nabla \mathcal{L}(w_k)\|^2 \right)
\]
\[
\leq \mathcal{L}(w_k) - \frac{2}{5C_\beta} \frac{1}{L_{w_k}} \|\nabla \mathcal{L}(w_k)\|^2 + \left( \frac{6}{5C_\beta L_{w_k}} + \frac{12}{C_\beta^2 L_{w_k}} \right) \left( \frac{C_{err_2}^2}{S} \|\nabla \mathcal{L}(w_k)\|^2 + \frac{C_{err_1}}{S} + \frac{C_{err_2}^2}{S} \right)
\]
\[
+ \frac{2}{C_\beta} \frac{1}{L_{w_k}} \left( \frac{1}{B} \left\| \hat{G}_i(w_k) \right\|^2 + \frac{4}{C_\beta} \frac{1}{L_{w_k}} \|\nabla \mathcal{L}(w_k)\|^2 \right)
\]
\[
\leq \mathcal{L}(w_k) - \frac{2}{C_\beta L_{w_k}} \left( \frac{1}{5} - \frac{3}{5} \frac{C_{err_2}}{C_\beta} - \frac{C_{err_1}}{C_\beta} - \frac{2}{C_\beta} \right) \|\nabla \mathcal{L}(w_k)\|^2
\]
\[
+ \frac{6}{C_\beta L_{w_k}} \left( \frac{1}{5} + \frac{2}{C_\beta} \right) \left( \frac{C_{err_1}}{S} + \frac{C_{err_2}^2}{S} \right) \frac{1}{S} + \frac{2}{C_\beta L_{w_k}} \left( \frac{C_{err_2}}{T} + \frac{C_{err_1}}{S} + \frac{C_{err_2}^2}{S} \right) \frac{1}{B}.
\]
(63)
Recall that $L_{w_k} = (1 + \alpha L)^{2N} L + C_L\mathbb{E}_{i \sim \mathcal{P}(T)} \|\nabla l_i(w_k)\|$. Then, we have $L_{w_k} \geq L$ and
\[
L_{w_k} \leq (1 + \alpha L)^{2N} L + C_L \|\nabla l_i(w_k)\| + \sigma
\]
\[
\leq (1 + \alpha L)^{2N} L + C_L \left( \frac{1}{1 - C_l} \|\nabla \mathcal{L}(w_k)\| + \frac{C_l}{1 - C_l} \sigma + \sigma \right)
\]
\[
\leq (1 + \alpha L)^{2N} L + \frac{C_L \sigma}{1 - C_l} + \frac{C_L}{1 - C_l} \|\nabla \mathcal{L}(w_k)\|,
\]
(64)
where (i) follows from Assumption 2, and (ii) follows from Lemma 5. Combining (63) and (64) yields
\[
\mathbb{E}(\mathcal{L}(w_{k+1}|w_k) \leq \mathcal{L}(w_k) + \frac{6}{C_\beta L} \left( \frac{1}{5} + \frac{2}{C_\beta} \right) \left( \frac{C_{err_1}}{S} + \frac{C_{err_2}^2}{S} \right) \frac{1}{S} + \frac{2}{C_\beta L} \left( \frac{C_{err_2}}{T} + \frac{C_{err_1}}{S} + \frac{C_{err_2}^2}{S} \right) \frac{1}{B}
\]
\[
- \frac{2}{C_\beta} \frac{1}{(1 + \alpha L)^{2N} L + \frac{C_L \sigma}{1 - C_l} + \frac{C_L}{1 - C_l} \|\nabla \mathcal{L}(w_k)\|} \left( \frac{1}{5} - \frac{3}{5} \frac{C_{err_2}}{C_\beta} - \frac{C_{err_1}}{C_\beta} - \frac{2}{C_\beta} \right) \|\nabla \mathcal{L}(w_k)\|^2.
\]
(65)
To simplify notations, we define

\[
\xi = \frac{6}{C_\beta L} \left( \frac{1}{5} + \frac{2}{C_\beta} \right) (C_{err_1}^2 + C_{err_2}^2 \sigma^2), \quad \phi = \frac{2}{C_\beta^2 L} \left( \frac{C_{squ_1}}{T} + \frac{C_{squ_2}}{S} + C_{squ_3} \sigma^2 \right)
\]

\[
\theta = \frac{2(2 - (1 + \alpha L)^2N)}{C_\beta C_L} \left( \frac{1}{5} - \left( \frac{3}{5} + \frac{6}{C_\beta} \right) \frac{C_{err_2}}{S} - \frac{C_{squ_3}}{C_\beta B} - \frac{2}{C_\beta} \right)
\]

\[
\chi = \frac{(2 - (1 + \alpha L)^2N)(1 + \alpha L)^2N}{C_L} + \sigma.
\]

Based on the notations in (66), we rewrite (65) as

\[
\mathbb{E}(L(w_{k+1})|w_k) \leq L(w_k) + \frac{\xi}{S} + \frac{\phi}{B} - \theta \frac{\|\nabla L(w_k)\|^2}{\chi + \|\nabla L(w_k)\|}.
\]

Unconditioning on \( w_k \) in the above inequality, we have

\[
\mathbb{E}L(w_{k+1}) \leq \mathbb{E}L(w_k) + \frac{\xi}{S} + \frac{\phi}{B} - \mathbb{E} \left( \frac{\theta \|\nabla L(w_k)\|^2}{\chi + \|\nabla L(w_k)\|} \right).
\]

Telescoping the above inequality over \( k \) from 0 to \( K - 1 \) and using \( L(w_k) \geq L^* = \inf_{w \in \mathbb{R}^d} L(w) > -\infty \), we have

\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \left( \frac{\theta \|\nabla L(w_k)\|^2}{\chi + \|\nabla L(w_k)\|} \right) \leq \frac{\Delta}{K} + \frac{\xi}{S} + \frac{\phi}{B},
\]

where \( \Delta = L(w_0) - L^* \). Choosing \( \zeta \) from \( \{0, ..., K - 1\} \) uniformly at random, we obtain from (67) that

\[
\mathbb{E} \left( \frac{\theta \|\nabla L(w_{\zeta})\|^2}{\chi + \|\nabla L(w_{\zeta})\|} \right) \leq \frac{\Delta}{K} + \frac{\xi}{S} + \frac{\phi}{B}.
\]

Consider a function \( f(x) = \frac{x^2}{c+x} \), \( x > 0 \), where \( c > 0 \) is a constant. Simple computation shows that \( f''(x) = \frac{2^2}{(x+c)^3} > 0 \). Thus, using Jensen’s inequality in (68), we have

\[
\frac{\theta \mathbb{E}\|\nabla L(w_{\zeta})\|^2}{\chi + \mathbb{E}\|\nabla L(w_{\zeta})\|} \leq \frac{\Delta}{K} + \frac{\xi}{S} + \frac{\phi}{B}.
\]

Rearranging the above inequality yields

\[
\mathbb{E}\|\nabla L(w_{\zeta})\|^2 \leq \frac{\Delta}{2\theta K} + \frac{\xi}{2\theta S} + \frac{\phi}{2\theta B} + \sqrt{\chi \left( \frac{\Delta}{2\theta K} + \frac{\xi}{2\theta S} + \frac{\phi}{2\theta B} \right) + \left( \frac{\Delta}{2\theta K} + \frac{\xi}{2\theta S} + \frac{\phi}{2\theta B} \right)^2} \leq \Delta \frac{1}{\theta K} + \frac{\xi}{\theta S} + \frac{\phi}{\theta B} + \sqrt{\frac{\chi}{2} \left( \frac{\Delta}{\theta K} + \frac{\xi}{\theta S} + \frac{\phi}{\theta B} \right)^2},
\]

which finishes the proof.

**Proof of Corollary 1.** Since \( \alpha = \frac{1}{8N^2} \), we have

\[
(1 + \alpha L)^N = \left(1 + \frac{1}{8N^2}\right)^N = e^{N \log(1 + 1/N)} \leq e^{1/8} < \frac{5}{4},
\]

\[
(1 + \alpha L)^{2N} < e^{1/4} < \frac{3}{2}.
\]
which, in conjunction with (16), implies that

$$C_{\text{err}_1} < \frac{5\sigma_g}{16}, \quad C_{\text{err}_2} < \frac{3\rho\sigma_g}{4L^2}. \quad (71)$$

Furthermore, noting that $D \geq \sigma^2 g / L^2$, we have

$$C_{\text{squ}_1} \leq 3(1 + 2\alpha L + 2\alpha^2 L^2) N \sigma_g^2 < 3.6^{9/32} \sigma_g^2 < 4\sigma_g^2, \quad C_{\text{squ}_2} < \frac{1.3\sigma^2}{8} < \frac{\sigma_g^2}{5}, \quad C_{\text{squ}_3} \leq 11. \quad (72)$$

Based on (14), we have

$$C_C < \frac{75 \rho}{128 L} < \frac{3 \rho}{5 L}$$

and $C_C (\text{i}) \geq \frac{\rho}{L} ((N - 1) \alpha L) > \frac{1}{16} \frac{\rho}{L} \quad (73)$

where (i) follows from the inequality that $(1 + a)^n > 1 + an$. Then, using (71), (72) and (73), we obtain from (20) that

$$\xi = \frac{7}{500L} \left( \frac{1}{10} + \frac{9\rho\sigma^2}{16L^2} \right) \sigma_g^2, \quad \phi = \frac{1}{5000L} \left( \frac{3\sigma^2}{T} + \frac{\sigma^2}{5S} + 11\sigma^2 \right) < \frac{1}{1000L} \left( \sigma_g^2 + 3\sigma^2 \right)$$

$$\theta = \frac{L}{60\rho} \left( \frac{1}{5} - \frac{4\rho^2\sigma^2}{15L^4} \right) - \frac{11}{100B} - \frac{1}{50^2} \geq \frac{L}{60\rho} \left( \frac{20 - 2 - 11}{100} - \frac{9}{20} \right) = \frac{L}{60\rho} \left( \frac{20}{25} = \frac{L}{1500\rho} \right)$$

$$\chi \leq \frac{24L^2}{\rho} + \sigma. \quad (74)$$

Then, treating $\Delta, \rho, L$ as constants and using (21), we obtain

$$E[\nabla L(\omega_C)] \leq O\left( \frac{1}{K} + \frac{\sigma_g^2 (\sigma_g^2 + 1)}{S} + \frac{\sigma^2}{B} + \frac{\sigma^2}{\sigma g} + \sqrt{\frac{1}{K}} + \frac{\sigma_g^2 (\sigma_g^2 + 1)}{S} + \frac{\sigma^2}{B} + \frac{\sigma^2}{\sigma g} \right).$$

Then, choosing batch sizes $S \geq C_S \sigma_g^2 (\sigma^2 + 1) \max(\sigma, 1) \epsilon^{-2}, B \geq C_B (\sigma_g^2 + \sigma^2) \max(\sigma, 1) \epsilon^{-2}$ and $TB > C_T \sigma^2 \max(\sigma, 1) \epsilon^{-2}$, we have

$$E[\nabla L(\omega_C)] \leq O\left( \frac{1}{K} \left( \frac{1}{\sqrt{C_S}} + \frac{1}{\sqrt{C_B}} + \frac{1}{\sqrt{C_T}} \right) \right) + \sqrt{\frac{1}{K}} \left( \frac{1}{\sqrt{C_S}} + \frac{1}{\sqrt{C_B}} + \frac{1}{\sqrt{C_T}} \right)$$

After at most $K = C_K \max(\sigma, 1) \epsilon^{-2}$ iterations, the above inequality implies, for constants $C_S, C_B, C_T$ and $C_K$ large enough,

$$E[\nabla L(\omega_C)] \leq \epsilon.$$ 

Recall that we need $|B'_k| > \frac{4C^2 g^2}{3(1 + \alpha L)^4 N L^2}$ and $|D'_k| > \frac{64\sigma^2 C^2 L^2}{(1 + \alpha L)^4 N L^2}$ for building stepsize $\beta_k$ at each iteration $k$. Based on the selected parameters, we have

$$\frac{4C^2 g^2}{3(1 + \alpha L)^4 N L^2} \leq \frac{4\sigma^2}{3L^2} \frac{3 \rho}{2 L^2} \leq \Theta(\sigma^2), \quad \frac{64\sigma^2 C^2 L^2}{(1 + \alpha L)^4 N L^2} < \Theta(\sigma_g^2),$$

which implies that $|B'_k| = \Theta(\sigma^2)$ and $|D'_k| = \Theta(\sigma_g^2)$. In addition, recall that the batch size $D = \Theta(\sigma^2 g / L^2)$. Thus, the total number of gradient computations at each meta iteration $k$ is given by

$$B(NS + T) + |B'_k||D'_k| \leq O(N \epsilon^{-4} + \epsilon^{-2}).$$

Furthermore, the total number of Hessian computations at each meta iteration is given by

$$BND \leq O(N \epsilon^{-2}).$$

This completes the proof. □

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D Proofs for Section 4: Convergence of Multi-Step MAML in Finite-Sum Case

In this section, we provide the proof of Theorem 2, which characterizes the convergence for MAML algorithm in the finite-sum case. To this end, we first provide some useful lemmas, then establish a few properties on the meta gradient, and finally use these results for proving Theorem 2.

D.1 Auxiliary Lemmas

The following lemma provides an upper bound on $\|l_{S_i}(\tilde{w}_j)\|$ for all $i \in \mathcal{I}$ and $j = 0, \ldots, N$, where $\tilde{w}_j$ is defined by (10) with $\tilde{w}_0 = w$.

**Lemma 7.** For any $i \in \mathcal{I}$, $j = 0, \ldots, N$ and $w \in \mathbb{R}^d$, we have

$$\|\nabla l_{S_i}(\tilde{w}_j)\| \leq (1 + \alpha L)j \|\nabla l_{S_i}(w)\|.$$  

**Proof.** The proof is similar to that of Lemma 2, and thus omitted. □

We next provide a bound on $\|\nabla l_{T_i}(w) - \nabla l_i(w)\|$, where $\nabla l_i(w) = \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(w_j))\nabla l_{T_i}(w_N)$.  

**Lemma 8.** For any $i \in \mathcal{I}$ and $w \in \mathbb{R}^d$, we have

$$\|\nabla l_{T_i}(w) - \nabla l_i(w)\| \leq ((1 + \alpha L)^N - 1)\|\nabla l_{T_i}(w)\| + (1 + \alpha L)^N ((1 + \alpha L)^N - 1)\|\nabla l_{S_i}(w)\|.$$  

**Proof.** Using the mean value theorem (MVT), we have, there exist constants $r_t, t = 1, \ldots, d$ satisfying $\sum_{t=1}^{d} r_t = 1$ and vectors $w'_t \in \mathbb{R}^d$, $t = 1, \ldots, d$ such that

$$\nabla l_{T_i}(\tilde{w}_N) = \nabla l_{T_i}(w - \alpha \sum_{j=0}^{N-1} \nabla l_{S_i}(\tilde{w}_j)) = \nabla l_{T_i}(w) + \sum_{t=1}^{d} r_t \nabla^2 l_{T_i}(w'_t) (-(1 + \alpha L)^N \sum_{j=0}^{N-1} \nabla l_{S_i}(\tilde{w}_j)).$$

Based on the above equality, we have

$$\|\nabla l_{T_i}(w) - \nabla l_i(w)\| = \left\| \nabla l_{T_i}(w) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j))\nabla l_{T_i}(w_N) \right\|$$

$$= \left\| \nabla l_{T_i}(w) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j))\nabla l_{T_i}(w) + \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j))\alpha \sum_{t=1}^{d} r_t \nabla^2 l_{T_i}(w'_t) \sum_{j=0}^{N-1} \nabla l_{S_i}(\tilde{w}_j) \right\|$$

$$= \left\| (1 - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j)))\nabla l_{T_i}(w) \right\| + \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j))\alpha \sum_{t=1}^{d} r_t \nabla^2 l_{T_i}(w'_t) \sum_{j=0}^{N-1} \nabla l_{S_i}(\tilde{w}_j) \right\|$$

$$\leq ((1 + \alpha L)^N - 1)\|\nabla l_{T_i}(w)\| + \alpha L(1 + \alpha L) \sum_{j=0}^{N-1} \|\nabla l_{S_i}(\tilde{w}_j)\|$$

$$\leq ((1 + \alpha L)^N - 1)\|\nabla l_{T_i}(w)\| + \alpha L(1 + \alpha L)^N \sum_{j=0}^{N-1} (1 + \alpha L)^j \|\nabla l_{S_i}(w)\|$$

$$= ((1 + \alpha L)^N - 1)\|\nabla l_{T_i}(w)\| + (1 + \alpha L)^N ((1 + \alpha L)^N - 1)\|\nabla l_{S_i}(w)\|,$$
where (i) follows from Lemma 3 and \(\|\sum_{t=1}^{d} r_t \nabla^2 l_{T_i}(w'_t)\| \leq \sum_{t=1}^{d} r_t \|\nabla^2 l_{T_i}(w'_t)\| \leq L\), and (ii) follows from Lemma 7. Then, the proof is complete. \(\square\)

Recall that \(\nabla l_{T_i}(w) = E_{i \sim p(T)} \nabla l_{T_i}(w)\), \(\nabla L(w) = E_{i \sim p(T)} \nabla L_i(w)\) and \(b = E_{i \sim p(T)}[b_i]\). The following lemma provides an upper bound on \(\|\nabla l_{T_i}(w)\|\).

**Lemma 9.** For any \(i \in I\) and \(w \in \mathbb{R}^d\), we have

\[
\|\nabla l_{T_i}(w)\| \leq \frac{1}{C_1} \|\nabla L(w)\| + \frac{C_2}{C_1},
\]

where constants \(C_1, C_2 > 0\) are given by

\[
C_1 = 2 - (1 + \alpha L)^{2N}, \\
C_2 = ((1 + \alpha L)^2N - 1)\sigma + (1 + \alpha L)^N((1 + \alpha L)^N - 1)b.
\]

**Proof.** First note that

\[
\|\nabla l_{T_i}(w)\| = \|E_i(\nabla l_{T_i}(w) - \nabla L_i(w)) + \nabla L(w)\|
\]

\[
\leq \|\nabla L(w)\| + E_i \|\nabla l_{T_i}(w) - \nabla L_i(w)\|
\]

\[
\overset{(i)}{\leq} \|\nabla L(w)\| + E_i \left((1 + \alpha L)^N - 1\right)\|\nabla l_{T_i}(w)\| + (1 + \alpha L)^N((1 + \alpha L)^N - 1)\|\nabla L_i(w)\|
\]

\[
\overset{(ii)}{\leq} \|\nabla L(w)\| + \left((1 + \alpha L)^N - 1\right)\|\nabla l_{T_i}(w)\| + (1 + \alpha L)^N((1 + \alpha L)^N - 1)\|\nabla L_i(w)\|
\]

\[
+ (1 + \alpha L)^N((1 + \alpha L)^N - 1)\|\nabla l_{T_i}(w)\|
\]

\[
\leq \|\nabla L(w)\| + \left((1 + \alpha L)^2N - 1\right)\|\nabla l_{T_i}(w)\| + (1 + \alpha L)^2N - 1\sigma + (1 + \alpha L)^N((1 + \alpha L)^N - 1)b
\]

where (i) follows from Lemma 8, (ii) follows from Assumption 5. Based on the definitions of \(C_1\) and \(C_2\) in (76), the proof is complete. \(\square\)

The following lemma provides the first- and second-moment bounds on \(1/L_{w_k}\), where \(\hat{L}_{w_k}\) is given by

\[
\hat{L}_{w_k} = (1 + \alpha L)^2N L + C_b b + \frac{C_L}{|B'_k|} \sum_{i \in B'_{k}} \|\nabla l_{T_i}(w_k)\|.
\]

**Lemma 10.** If the batch size \(|B'_k| \geq \frac{2C^2 \sigma^2}{N + 2(1 + \alpha L)^2NL}I\), then conditioning on \(w_k\), we have

\[
E\left(\frac{1}{L_{w_k}}\right) \geq \frac{1}{L_{w_k}}, \quad E\left(\frac{1}{L_{w_k}^2}\right) \leq \frac{2}{L_{w_k}^2}
\]

where \(L_{w_k}\) is given by

\[
L_{w_k} = (1 + \alpha L)^2N L + C_b b + C_L E_{i \sim p(T)}\|\nabla l_{T_i}(w_k)\|
\]

**Proof.** Conditioning on \(w_k\) and using an approach similar to (43), we have

\[
E\left(\frac{1}{L_{w_k}^2}\right) \leq \frac{\sigma^2_b/(C_b b + (1 + \alpha L)^2NL)^2 + \mu^2_b/(\mu_b + C_b b + (1 + \alpha L)^2NL)^2}{\sigma^2_b + \mu^2_b},
\]

(77)
where $\mu_\beta$ and $\sigma_\beta^2$ are the mean and variance of variable $\frac{C_b}{|B_k|} \sum_{i \in B_k} \|\nabla l_{T_i}(w_k)\|$. Noting that $\mu_\beta = C_b \mathbb{E}_{i \sim p(T)} \|\nabla l_{T_i}(w_k)\|$, we have $L_{w_k} = (1 + aL)^2 N + C_b \mu_\beta$, and thus

$$L_{w_k}^2 \mathbb{E}\left( \frac{1}{L_{w_k}^2} \right) \leq \frac{\sigma_\beta^2 (1 + \alpha L)^2 N + C_b \mu_\beta + \mu_\beta^2}{\sigma_\beta^2 + \mu_\beta^2} \leq \frac{2 \sigma_\beta^2 + \mu_\beta^2 + \frac{2 \sigma_\beta^2 \mu_\beta}{(C_b + (1 + \alpha L)^2 N)^2}}{\sigma_\beta^2 + \mu_\beta^2},$$

(78)

where the second inequality follows from $(a + b)^2 \leq 2a^2 + 2b^2$. Also note that, conditioning on $w_k$,

$$\sigma_\beta^2 = \frac{C_b^2}{|B_k|^2} \text{Var} \|\nabla l_{T_i}(w_k)\| \leq \frac{C_b^2}{|B_k|^2} \sigma^2,$$

which, in conjunction with $|B_k| \geq \frac{2 C_b^2 \sigma^2}{(C_b + (1 + \alpha L)^2 N)^2}$, yields

$$\frac{2 \sigma_\beta^2}{(C_b + (1 + \alpha L)^2 N)^2} \leq 1.$$ \hfill (79)

Combining (79) and (78) yields

$$\mathbb{E}\left( \frac{1}{L_{w_k}^2} \right) \leq \frac{2}{L_{w_k}^2}.$$ \hfill (80)

In addition, conditioning on $w_k$, we have

$$\mathbb{E}\left( \frac{1}{L_{w_k}} \right) \overset{(i)}{=} \frac{1}{\mathbb{E} L_{w_k}} = \frac{1}{L_{w_k}},$$

(80)

where $(i)$ follows from Jensen’s inequality.

\[ \square \]

### D.2 Proofs for Propositions in Section 4.2

In this subsection, we provide proofs for the propositions on the properties of the meta gradient in Section 4.2.

**Proof of Proposition 5.** Recall that $\nabla L_i(w) = \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i)) \nabla l_{T_i}(\tilde{w}_N^i)$ and $\nabla L_i(u)$ is defined in the same way. Then, we have

$$\|\nabla L_i(w) - \nabla L_i(u)\| = \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i)) \nabla l_{T_i}(\tilde{w}_N^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i)) \nabla l_{T_i}(\tilde{u}_N^i) \right\|$$

$$\leq \left\| \prod_{j=0}^{N-1} \left( (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i)) \nabla l_{T_i}(\tilde{w}_N^i) - (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i)) \nabla l_{T_i}(\tilde{u}_N^i) \right) \right\| + \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i)) \nabla l_{T_i}(\tilde{w}_N^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i)) \nabla l_{T_i}(\tilde{u}_N^i) \right\|$$

$$\leq \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i)) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(\tilde{u}_j^i)) \right\| \cdot \|\nabla l_{T_i}(\tilde{w}_N^i)\| + (1 + \alpha L)^N \|\nabla l_{T_i}(\tilde{w}_N^i) - \nabla l_{T_i}(\tilde{u}_N^i)\|.$$ \hfill (81)
We next upper-bound \((A)\) in the above inequality. Specifically, we have
\[
(A) \leq \left\| \prod_{j=0}^{N-1} \left( I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i) \right) \right\| \prod_{j=0}^{N-2} \left( I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i) \right) \left( I - \alpha \nabla^2 l_{S_i}(\tilde{u}_{N-1}^i) \right) \\
+ \left\| \prod_{j=0}^{N-2} \left( I - \alpha \nabla^2 l_{S_i}(\tilde{w}_j^i) \right) \left( I - \alpha \nabla^2 l_{S_i}(\tilde{u}_{N-1}^i) \right) \right\| 
\]
which, using an approach similar to (50), yields
\[
(A) \leq \left( (1 + \alpha L)^{N-1} \alpha \rho + \frac{\rho}{L} (1 + \alpha L)^N \left( (1 + \alpha L)^{N-1} - 1 \right) \right) \|w - u\|. \tag{82}
\]
Combining (81) and (82) yields
\[
\|\nabla \mathcal{L}_i(w) - \nabla \mathcal{L}_i(u)\| \leq \left( (1 + \alpha L)^{N-1} \alpha \rho + \frac{\rho}{L} (1 + \alpha L)^N \left( (1 + \alpha L)^{N-1} - 1 \right) \right) \|w - u\| \|\nabla T_i(\tilde{w}_N^i)\| \\
+ (1 + \alpha L)^N \|\tilde{w}_N^i - \tilde{u}_N^i\|. \tag{83}
\]
To upper-bound \(\|\nabla T_i(\tilde{w}_N^i)\|\) in the above inequality, we have
\[
\|\nabla T_i(\tilde{w}_N^i)\| = \left\| \nabla T_i(w - \sum_{j=0}^{N-1} \alpha \nabla l_{S_i}(\tilde{w}_j^i)) \right\|
\leq \|\nabla T_i(w)\| + \alpha \left\| \nabla^2 T_i(\tilde{w}_j^i) \right\| \sum_{j=0}^{N-1} \|\nabla l_{S_i}(\tilde{w}_j^i)\|
\leq \|\nabla T_i(w)\| + \alpha L \sum_{j=0}^{N-1} \|\nabla l_{S_i}(\tilde{w}_j^i)\|
\leq \|\nabla T_i(w)\| + \alpha L \sum_{j=0}^{N-1} (1 + \alpha L)^j \|\nabla l_{S_i}(w)\|
\leq \left( (1 + \alpha L)^N \|\nabla T_i(w)\| + (1 + \alpha L)^N - 1 \] \right) b_i \tag{84}
\]
where (i) follows from Lemma 7, and (ii) follows from Assumption 5. In addition, using an approach similar to Lemma 1, we have
\[
\|\tilde{w}_N^i - \tilde{u}_N^i\| \leq \|w - u\|. \tag{85}
\]
Combining (83), (84) and (85) yields
\[
\|\nabla \mathcal{L}_i(w) - \nabla \mathcal{L}_i(u)\| \leq \left( (1 + \alpha L)^{N-1} \alpha \rho + \frac{\rho}{L} (1 + \alpha L)^N \left( (1 + \alpha L)^{N-1} - 1 \right) \right) \|\nabla T_i(w)\| \|w - u\|
+ \left( (1 + \alpha L)^{N-1} \alpha \rho + \frac{\rho}{L} (1 + \alpha L)^N \left( (1 + \alpha L)^{N-1} - 1 \right) \right) (1 + \alpha L)^N \|\tilde{w}_N^i - \tilde{u}_N^i\| \\
+ (1 + \alpha L)^{2N} \|\tilde{w}_N^i - \tilde{u}_N^i\|, 
\]
which, in conjunction with \(C_b\) and \(C_L\) given in (22), yields
\[
\|\nabla \mathcal{L}_i(w) - \nabla \mathcal{L}_i(u)\| \leq \left( (1 + \alpha L)^{2N} L + C_b b_i + C_L \|\nabla T_i(w)\| \right) \|w - u\|. 
\]
Based on the above inequality and Jensen’s inequality, we have
\[
\|\nabla L(w) - \nabla L(u)\| \leq \mathbb{E}_{t \sim \mathcal{P}(T)} \|\nabla L_t(w) - \nabla L_t(u)\| \\
\leq \left((1 + \alpha L)^{2N} L + C_0 b + C_L \mathbb{E}_{t \sim \mathcal{P}(T)} \|\nabla l_t(w)\|\right) \|w - u\|,
\]
which finishes the proof.

**Proof of Proposition 6.** Recall that \( \hat{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(w_{k,j}^i)) \nabla l_{T_i}(w_{k,N}^i) \). Then, conditioning on \( w_k \), we have
\[
\mathbb{E}[\|\hat{G}_i(w_k)\|^2] = \mathbb{E}\left[ \left\| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i}(w_{k,j}^i)) \nabla l_{T_i}(w_{k,N}^i) \right\|^2 \right] \\
\leq (1 + \alpha L)^{2N} \mathbb{E}[\|\nabla l_{T_i}(w_{k,N}^i)\|^2],
\]
which, using an approach similar to (84), yields
\[
\mathbb{E}[\|\hat{G}_i(w_k)\|^2] \leq (1 + \alpha L)^{2N} 2(1 + \alpha L)^{2N} \mathbb{E}[\|\nabla l_{T_i}(w_{k,N}^i)\|^2] + 2(1 + \alpha L)^{2N} (1 + \alpha L)^{N - 1})^2 \mathbb{E}[b_i^2] \\
= 2(1 + \alpha L)^{4N} \mathbb{E}[\|\nabla l_{T_i}(w_{k,N}^i)\|^2] + 2(1 + \alpha L)^{2N} (1 + \alpha L)^{N - 1})^2 \mathbb{E}[b_i^2] \\
\leq 2(1 + \alpha L)^{4N} (\|\nabla l_{T_i}(w_{k,N}^i)\|^2 + \sigma^2) + 2(1 + \alpha L)^{2N} (1 + \alpha L)^{N - 1})^2 \mathbb{E}[b_i^2] \\
\leq 4(1 + \alpha L)^{4N} \left( \frac{2}{C_1^2} \|\nabla l_{T_i}(w_{k,N}^i)\|^2 + \frac{2C_2^2}{C_1^2} \sigma^2 \right) + 2(1 + \alpha L)^{2N} (1 + \alpha L)^{N - 1})^2 \mathbb{E}[b_i^2],
\]
where (i) follows from Lemma 9, and constants \( C_1 \) and \( C_2 \) are given by (76). Noting that \( C_2 = ((1 + \alpha L)^{2N} - 1)\sigma + (1 + \alpha L)^{N} (1 + \alpha L)^{N - 1})b < (1 + \alpha L)^{2N - 1}(\sigma + b) \) and using the definitions of \( A_{sq,1}, A_{sq,2} \) in (24), we finish the proof.

**D.3 Proofs for Section 4.3: Convergence of Multi-Step MAML in Finite-Sum Case**

In this subsection, we provide detailed proofs for convergence results in Section 4.3.

**Proof of Theorem 2.** Based on the smoothness of \( \nabla L(\cdot) \) established in Proposition 5, we have
\[
L(w_{k+1}) \leq L(w_k) + \langle \nabla L(w_k), w_{k+1} - w_k \rangle + \frac{L_{w_k}}{2} \|w_{k+1} - w_k\|^2 \\
\leq L(w_k) - \beta_k \langle \nabla L(w_k), \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \rangle + \frac{L_{w_k} \beta_k^2}{2} \left\| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \right\|^2
\]
Taking the conditional expectation given \( w_k \) over the above inequality and noting that the randomness over \( \beta_k \) is independent of the randomness over \( \hat{G}_i(w_k) \), we have
\[
\mathbb{E}(L(w_{k+1})|w_k) \leq L(w_k) - \mathbb{E}(\beta_k|w_k)\langle \nabla L(w_k), \langle \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \rangle \rangle + \frac{L_{w_k}}{2} \mathbb{E}(\beta_k^2|w_k)\mathbb{E}(\left\| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \right\|^2|w_k) \\
\leq L(w_k) - \frac{1}{C_\beta} \mathbb{E}(\frac{1}{L_{w_k}}|w_k)\langle \nabla L(w_k), \langle \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \rangle \rangle + \frac{L_{w_k}}{2C_\beta} \mathbb{E}(\left\| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \right\|^2|w_k).
\]
Note that, conditioning on \( w_k \),
\[
\mathbb{E} \left\| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \right\|^2 \leq \frac{1}{B} \mathbb{E} \left\| \hat{G}_i(w_k) \right\|^2 + \mathbb{E} \left\| \hat{G}_i(w_k) \right\|^2
\]
\[
\leq \frac{1}{B} \mathbb{E} \left\| \hat{G}_i(w_k) \right\|^2 + \| \nabla L(w_k) \|^2
\]
\[
\leq \frac{1}{B} \left( A_{\text{squ}} + \| \nabla L(w_k) \|^2 + A_{\text{squ}_2} \right) + \| \nabla L(w_k) \|^2
\]
where the last inequality follows from Proposition 6. Then, combining (88), (87) and applying Lemma 10, we have
\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) - \frac{1}{L_{w_k} C_\beta} \| \nabla L(w_k) \|^2 + \frac{1}{L_{w_k} C_\beta} \left( \frac{A_{\text{squ}}}{B} \| \nabla L(w_k) \|^2 + \frac{A_{\text{squ}_2}}{B} + \| \nabla L(w_k) \|^2 \right)
\]
\[
\leq \mathcal{L}(w_k) - \left( \frac{1}{L_{w_k} C_\beta} - \frac{1}{L_{w_k} C_\beta} \left( \frac{A_{\text{squ}}}{B} + 1 \right) \right) \| \nabla L(w_k) \|^2 + \frac{1}{L_{w_k} C_\beta} \frac{A_{\text{squ}_2}}{B}.
\]
Recalling that \( L_{w_k} = (1 + \alpha L)^2 L + C_b + C_L \mathbb{E}_{\iota \sim \mathcal{P}(\mathcal{F})} \| \nabla l_{\mathcal{T}_i}(w_k) \| \) and conditioning on \( w_k \), we have \( L_{w_k} \geq L \) and
\[
L_{w_k} \leq (1 + \alpha L)^2 L + C_b + C_L \left( (1 + \alpha L)^2 L + C_b + C_L \left( \frac{1}{C_1} \| \nabla L(w_k) \| + \frac{C_2}{C_1} \sigma \right) \right) + C_L \| \nabla L(w_k) \|,
\]
where \((i)\) follows from Lemma 9. Combining (90) and (89) yields
\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) - \frac{C_1}{C_L} \left( \frac{1}{C_\beta} - \frac{1}{C_\beta} \left( \frac{A_{\text{squ}}}{B} + 1 \right) \right) \| \nabla L(w_k) \|^2 + \frac{1}{C_L} \left( \frac{A_{\text{squ}}}{B} + 1 \right) \| \nabla L(w_k) \|^2 \]
\[
= \mathcal{L}(w_k) - \frac{C_1}{C_L} \left( \frac{1}{C_\beta} - \frac{1}{C_\beta} \left( \frac{A_{\text{squ}}}{B} + 1 \right) \right) \| \nabla L(w_k) \|^2 + \frac{1}{C_L} \frac{A_{\text{squ}}}{B} \]
\[
= \mathcal{L}(w_k) - \frac{C_1}{C_L} \left( \frac{1}{C_\beta} - \frac{1}{C_\beta} \left( \frac{A_{\text{squ}}}{B} + 1 \right) \right) \| \nabla L(w_k) \|^2 + \frac{1}{C_L} \frac{A_{\text{squ}}}{B} + \frac{1}{C_L} \frac{A_{\text{squ}_2}}{B},
\]
where the last equality follows from the definitions of \( C_1, C_2 \) in (76). To simplify notations, let
\[
\theta = \frac{C_1}{C_L} \left( \frac{1}{C_\beta} - \frac{1}{C_\beta} \left( \frac{A_{\text{squ}}}{B} + 1 \right) \right), \quad \phi = \frac{A_{\text{squ}}}{C_L \beta}
\]
\[
\xi = \frac{C_1}{C_L} \left( (1 + \alpha L)^2 L + \frac{b C_b C_1}{C_L} + (1 + \alpha L)^N ((1 + \alpha L)^2 L - 1)b + \| \nabla L(w_k) \| \right) + \frac{1}{C_L} \frac{A_{\text{squ}_2}}{B},
\]
which, in conjunction with (91) and taking the expectation over \( w_k \), yields
\[
\mathbb{E} \frac{\theta \| \nabla L(w_k) \|^2}{\xi + \| \nabla L(w_k) \|^2} \leq \mathbb{E}(\mathcal{L}(w_k) - \mathcal{L}(w_{k+1})) + \frac{\phi}{B}.
\]
Telescoping the above bound over \( k \) from 0 to \( K - 1 \) and choosing \( \xi \) from \{0, ..., K - 1\} uniformly at random, we have
\[
\mathbb{E} \frac{\theta \| \nabla L(w_k) \|^2}{\xi + \| \nabla L(w_k) \|^2} \leq \frac{\Delta}{K} + \frac{\phi}{B}.
\]
Using an approach similar to (69), we obtain from (92) that
\[
\frac{\mathbb{E} \| \nabla L(w_\xi) \|^2}{\xi + \mathbb{E} \| \nabla L(w_\xi) \|} \leq \frac{\Delta}{\theta K} + \frac{\phi}{\theta B},
\]
which further implies that
\[
\mathbb{E} \| \nabla L(w_\xi) \| \leq \frac{\Delta}{2\theta K} + \frac{\phi}{2\theta B} + \sqrt{\xi \left( \frac{\Delta}{\theta K} + \frac{\phi}{\theta B} \right) + \left( \frac{\Delta}{2\theta K} + \frac{\phi}{2\theta B} \right)^2},
\tag{93}
\]
which finishes the proof.

\[\square\]

**Proof of Corollary 2.** Since \( \alpha = \frac{1}{8NL} \), we have \( (1 + \alpha L)^{4N} < e^{0.5} < 2 \), and thus
\[
A_{\text{squa}} < \frac{8}{1^4} = 32, \quad A_{\text{squa}} < 8(\sigma + b)^2 + 4(\sigma^2 + \bar{b}),
\]
\[
C_L < \left( \frac{5\rho}{32NL} + \frac{\rho}{L} \right) \frac{5}{4} < \frac{5\rho}{8L}, \quad C_L > \frac{\rho}{L} (1 + \alpha L)^{N-1} - 1 > \frac{\rho}{16L},
\]
\[
C_b < \frac{15\rho}{32L} < \frac{\rho}{8L},
\tag{94}
\]
which, in conjunction with (25), yields
\[
\theta \geq \frac{1}{80} \frac{4L}{5\rho} \left( 1 - \frac{33}{80} \right) \geq \frac{L}{200\rho}, \quad \phi \leq \frac{2(\sigma + b)^2 + (\sigma^2 + \bar{b})}{1600L},
\]
\[
\xi \leq \frac{24L^2}{\rho} + \frac{37b}{16}
\tag{95}
\]
Combining (95) and (26) yields
\[
\mathbb{E} \| \nabla L(w_\xi) \| \leq \frac{\Delta}{2\theta K} + \frac{\phi}{2\theta B} + \sqrt{\xi \left( \frac{\Delta}{\theta K} + \frac{\phi}{\theta B} \right) + \left( \frac{\Delta}{2\theta K} + \frac{\phi}{2\theta B} \right)^2} \leq O \left( \frac{1}{K} + \frac{\sigma^2}{B} + \sqrt{\frac{1}{K} + \frac{\sigma^2}{B}} \right).
\]
Then, based on the parameter selection that \( B \geq C_B \sigma^2 \epsilon^{-2} \) and after at most \( K = C_k \epsilon^{-2} \) iterations, we have
\[
\mathbb{E} \| \nabla L(w_\xi) \| \leq O \left( \left( \frac{1}{C_B} + \frac{1}{C_k} \right)^2 + \frac{1}{\epsilon} \sqrt{\left( \frac{1}{C_B} + \frac{1}{C_k} \right)^2} \right).
\]
Then, for \( C_B, C_k \) large enough, we obtain from the above inequality that
\[
\mathbb{E} \| \nabla L(w_\xi) \| \leq \epsilon.
\]
Thus, the total number of gradient computations is given by
\[
B(T + NS) = O(\epsilon^{-2}(T + NS)).
\]
Furthermore, the total number of Hessian computations is given by
\[
BNS = O(NS \epsilon^{-2})
\]at each iteration. Then, the proof is complete. \[\square\]
E Proofs in Section 5.1: Limitations of Zeroth-Order Hessian Estimation

In this section, we provide proofs for the propositions on the limitations of the zeroth-order Hessian estimator in Section 5.1.

Proof of Proposition 7. Recall the definition of $H_{k,j}^i$ that

$$H_{k,j}^i = \frac{1}{U} \sum_{u \in U_{k,j}^i} \frac{l_i(w_{k,j}^i + \delta u; \hat{D}_{k,j}^i) - l_i(w_{k,j}^i - \delta u; \overline{D}_{k,j}^i) - 2l_i(w_{k,j}^i; D_{k,j}^i)}{2\delta^2} uu^T. \quad (96)$$

Using the fact that $u$ is independent of $\hat{D}_{k,j}^i, \overline{D}_{k,j}^i$ and $D_{k,j}^i$, and conditioning on $w_{k,j}^i$, we have

$$\mathbb{E}(H_{k,j}^i) = \mathbb{E}\left( \mathbb{E}(H_{k,j}^i | U_{k,j}^i) \right) = \mathbb{E}\left[ \frac{1}{U} \sum_{u \in U_{k,j}^i} \frac{l_i(w_{k,j}^i + \delta u) - l_i(w_{k,j}^i - \delta u) - 2l_i(w_{k,j}^i)}{2\delta^2} uu^T \right]$$

$$= \mathbb{E}_u \left[ \frac{l_i(w_{k,j}^i + \delta u) - l_i(w_{k,j}^i - \delta u) - 2l_i(w_{k,j}^i)}{2\delta^2} \right]$$

$$= \nabla^2 l_i(w_{k,j}^i) - \frac{l_i(w_{k,j}^i) - l_i(w_{k,j}^i)}{\delta^2} I_d, \quad (97)$$

where the last inequality follows from equation (D.2) in Ye et al. (2018). Then, based on Theorem 3 in Nesterov and Spokoiny (2017), we have

$$l_i,\delta(w_{k,j}^i) - l_i(w_{k,j}^i) = \frac{\delta^2}{2} \mathbb{E}_u(\nabla^2 l_i(w_{k,j}^i)u, u) + \epsilon_H$$

$$= \frac{\delta^2}{2} \text{Tr}(\nabla^2 l_i(w_{k,j}^i)) + \epsilon_H$$

where the residual term $|\epsilon_H| < \frac{\delta^2}{4} \rho(d + 3)^{3/2}$. Combining the above inequality with (97) yields

$$\mathbb{E}(H_{k,j}^i) = \nabla^2 l_i(w_{k,j}^i) + \frac{1}{2} \text{Tr}(\nabla^2 l_i(w_{k,j}^i)) I_d + \frac{\epsilon_H}{\delta^2} I_d.$$ \hspace{1cm} \square

Proof of Proposition 8. For notational convenience, we let $\hat{S}_j$ and $\overline{D}_j$ denote the randomness over $\{S_{k,m}^j, m = 0, ..., j - 1, i \in T\}$ and $\{\hat{D}_{k,m}^j, \overline{D}_{k,m}^j, m = 0, ..., j - 1, i \in T\}$, respectively. Let $r_{k,j} = \mathbb{E}(H_{k,j}^i | w_{k,j}^i) - \nabla^2 l_i(w_{k,j}^i)$. Then, using Lemma 4 in Ye et al. (2018) and Proposition 7, we have

$$\|r_{k,j}\| \leq \rho\delta(d + 1)^{1/2} + \frac{\delta}{3} \rho(d + 3)^{3/2} + \frac{1}{2} \text{Tr}(\nabla^2 l_i(w_{k,j}^i)). \quad (98)$$

Define a constant $C_\tau = \rho\delta(d + 1)^{1/2} + \frac{\delta}{3} \rho(d + 3)^{3/2} + \frac{1}{2} \sup_{w \in \mathbb{R}^d} \{\text{Tr}(\nabla^2 l_i(w))\}$. Recall that $\hat{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha H_{k,j}^i) \nabla l_i(w_{k,N}; T_k)$ and note that conditioning on $\hat{S}_N$ and $w_k$, parameters $w_{k,j}, j = 0, ..., N$ are
deterministic. Then, using an approach similar to (51) and conditioning on \(w_k\), we have

\[
\mathbb{E} \tilde{G}_i(w_k) = \mathbb{E}_{S_N, i \sim p(T)} \left( \mathbb{E}_{D_N, \{u, u \in U_{k, j}^i\}} \left( \prod_{j=0}^{N-1} (I - \alpha H_{k, j}^i) \mathbb{E}_{T_k^i} \nabla l_i(w_{k, N}^i; T_k^i | S_N, i) \right) \right) \\
= \mathbb{E}_{S_N, i \sim p(T)} \left( \prod_{j=0}^{N-1} \mathbb{E}_{D_{k, j}^i, u} (I - \alpha H_{k, j}^i | S_N, i) \nabla l_i(w_{k, N}^i) \right) \\
= \mathbb{E}_{S_N, i \sim p(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k, j}^i) - \alpha r_{k, j}^i) \nabla l_i(w_{k, N}^i) \right),
\]

which, in conjunction with \(\nabla \mathcal{L}(w_k) = \mathbb{E}_{i \sim p(T)} \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_{k, j}^i)) \nabla l_i(\bar{w}_{k, N}^i)\) and the definition that \(e_k = \mathbb{E} \tilde{G}_i(w_k) - \nabla \mathcal{L}(w_k)\), implies

\[
\|e_k\| \leq \mathbb{E}_{S_N, i \sim p(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k, j}^i) - \alpha r_{k, j}^i) \nabla l_i(w_{k, N}^i) \right) \leq \mathbb{E}_{S_N, i \sim p(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_{k, j}^i)) \nabla l_i(\bar{w}_{k, N}^i) \right)
\]

\[
\leq \mathbb{E}_{S_N, i \sim p(T)} \left( 1 + \alpha L + \alpha C_r \right)^N \left\| \nabla l_i(w_{k, N}^i) - \nabla l_i(\bar{w}_{k, N}^i) \right\|
\]

\[
\leq (1 + \alpha L + \alpha C_r)^N \mathbb{E}_{S_N, i \sim p(T)} \left\| \nabla l_i(w_{k, N}^i) \right\|
\]

\[
\leq (1 + \alpha L + \alpha C_r)^N \mathbb{E}_{i \sim p(T)} \left\| \nabla l_i(w_k) \right\|
\]

\[
\leq (1 + \alpha L + \alpha C_r)^N \mathbb{E}_{S_N, i \sim p(T)} \left( \left\| \nabla l_i(w_k) \right\| \mathbb{E}_{S_N} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w_{k, j}^i) - \alpha r_{k, j}^i) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_{k, j}^i)) \right) \right)
\]

\[
+ \left( 1 + \alpha L + \alpha C_r \right)^N \left( 1 + \alpha L \right)^N \mathbb{E}_{S_N} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_{k, j}^i)) \right) \right)
\]

\[
\leq (1 + \alpha L)^N \mathbb{E}_{S_N} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}_{k, j}^i)) \right)
\]

\[
+ \left( 1 + \alpha L + \alpha C_r \right)^N \left( 1 + \alpha L \right)^N \frac{\sigma_g}{\sqrt{S}}.
\]

where (i) follows from Jensen’s inequality, (ii) follows from (98), (iii) follows from Lemma 2 that \(\left\| \nabla l_i(\bar{w}_{k, N}^i) \right\| \leq (1 + \alpha L)^N \left\| \nabla l_i(w_k) \right\|\), and the last inequality follows from item 1 in Proposition 2 with \(j = N\).

We next upper-bound the term \(R(N)\) in (99). To simplify notations, we define a general quantity \(R(m) = \mathbb{E}_{S_m} \left( \prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(\bar{w}_{k, j}^i) - \alpha r_{k, j}^i) - \prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(\bar{w}_{k, j}^i)) \right)\) for \(m = 1, ..., N\), where we use
$\mathbb{E}_{S_m|i}(\cdot)$ to denote $\mathbb{E}_{S_m}(\cdot|i)$. Then, conditioning on $w_k$ yields

$$R(m) \leq \mathbb{E}_{S_m|i} \left[ \prod_{j=0}^{m-1} \left( I - \alpha \nabla^2 l_i(w_{k,j}) - \alpha r_{k,j} \right) - \prod_{j=0}^{m-2} \left( I - \alpha \nabla^2 l_i(w_{k,j}) - \alpha r_{k,j} \right) (I - \alpha \nabla^2 l_i(\bar{w}_{k,m-1})) \right]$$

$$+ \mathbb{E}_{S_m|i} \left[ \prod_{j=0}^{m-2} \left( I - \alpha \nabla^2 l_i(w_{k,j}) - \alpha r_{k,j} \right) (I - \alpha \nabla^2 l_i(\bar{w}_{k,m-1})) - \prod_{j=0}^{m-1} \left( I - \alpha \nabla^2 l_i(\bar{w}_{k,j}) \right) \right]$$

$$\leq (1 + \alpha L + \alpha C_r)m^{-1} \left( \alpha \rho \mathbb{E}_{S_m|i} \|w_{k,m-1} - \bar{w}_{k,m-1}\| + \alpha C_r \right)$$

$$+ (1 + \alpha L)R(m-1)$$

$$(i) \leq (1 + \alpha L + \alpha C_r)m^{-1} \left( \alpha \rho ((1 + \alpha L)^{m-1} - 1) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + \alpha C_r \right) + (1 + \alpha L)R(m-1)$$

(100)

where (i) follows from Proposition 2. Noting that $m \leq N$, we have

$$(1 + \alpha L + \alpha C_r)^{m-1} \left( \alpha \rho ((1 + \alpha L)^{m-1} - 1) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + \alpha C_r \right)$$

$$\leq (1 + \alpha L + \alpha C_r)^{N-1} \left( \alpha \rho ((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + \alpha C_r \right),$$

which, in conjunction with (100), implies that

$$R(m) \leq (1 + \alpha L + \alpha C_r)^{N-1} \left( \alpha \rho ((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + \alpha C_r \right) + (1 + \alpha L)R(m-1).$$

Rearranging the above inequality, we obtain

$$R(m) + \frac{(1 + \alpha L + \alpha C_r)^{N-1}}{L} \left( \rho ((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + \alpha C_r \right)$$

$$\leq (1 + \alpha L) \left( R(m-1) + \frac{(1 + \alpha L + \alpha C_r)^{N-1}}{L} \left( \rho ((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + \alpha C_r \right) \right).$$

Telescoping the above inequality over $m$ from 2 to $N$ yields

$$R(N) + \frac{(1 + \alpha L + \alpha C_r)^{N-1}}{L} \left( \rho ((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + \alpha C_r \right)$$

$$\leq (1 + \alpha L)^{N-1} \left( R(1) + \frac{(1 + \alpha L + \alpha C_r)^{N-1}}{L} \left( \rho ((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + \alpha C_r \right) \right),$$

which, in conjunction with $R(1) \leq \alpha C_r$, yields

$$R(N) \leq \alpha (1 + \alpha L) \left( R(1) + (1 + \alpha L)^{N-1} - 1 \right) \frac{(1 + \alpha L + \alpha C_r)^{N-1}}{L} \left( \rho ((1 + \alpha L)^{N-1} - 1) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + \alpha C_r \right).$$

Combining the above inequality and (99) and using an approach similar to (55) yields

$$\|e_k\| \leq (1 + \alpha L + \alpha C_r)^N \left( (1 + \alpha L)^N - 1 \right) \frac{\sigma_g}{\sqrt{\mathcal{S}}} + (1 + \alpha L)^N C_e E_{i \sim p(T)} \|\nabla l_i(w_k)\|$$

$$\leq (1 + \alpha L + \alpha C_r)^N \left( (1 + \alpha L)^N - 1 \right) \frac{\sigma_g}{2 \sqrt{\mathcal{S}}} + \frac{(1 + \alpha L)^N C_e}{2 - (1 + \alpha L)^{2N}} \left( \|\nabla L(w_k)\| + \sigma \right),$$

(101)

which finishes the proof.
F Proofs for Section 5.2: Convergence of GGS-MAML in Resampling Case

F.1 Proof of Propositions in Section 5.2: Properties of Proposed Hessian Estimator

In this section, we provide proofs for the propositions on the properties of the proposed Hessian estimator in Section 5.2.

Proof of Proposition 9. Based on the definition of $H_{k,j}^i$ given by (28) and conditioning on $w_{k,j}^i$, we have

$$\mathbb{E}(H_{k,j}^i) = \mathbb{E} \left( \frac{1}{U} \sum_{u \in U_{k,j}^i} \nabla l_i(w_{k,j}^i + \delta u) - \nabla l_i(w_{k,j}^i) \right) u^T$$

$$= \mathbb{E} \left( \frac{\nabla l_i(w_{k,j}^i + \delta u) - \nabla l_i(w_{k,j}^i)}{\delta} \right) u^T$$

$$= \mathbb{E} \left( \frac{\sum_{t=1}^d r_t \nabla^2 l_i(w_{k,j}^i + a_t \delta u) uu^T}{\delta} \right), \quad r_t, a_t \in (0, 1), \sum_{t=1}^d r_t = 1$$

$$= \mathbb{E} \left( \left( \sum_{t=1}^d r_t \nabla^2 l_i(w_{k,j}^i + a_t \delta u) uu^T - \nabla^2 l_i(w_{k,j}^i) uu^T \right) + \nabla^2 l_i(w_{k,j}^i) \right)$$

(102)

where (i) follows from the fact that $\tilde{D}_{k,j}^i$ and $D_{k,j}^i$ are independent of Gaussian vectors $\{u \in U_{k,j}^i\}$, (ii) follows from mean value theorem (MVT), and (iii) follows from the fact that $\mathbb{E}[uu^T] = I$ for Gaussian random vector $u \sim N(0, I_d)$. Then, rearranging (102) yields, conditioning on $w_{k,j}^i$,

$$\|\mathbb{E}H_{k,j}^i - \nabla^2 l_i(w_{k,j}^i)\|$$

$$\leq \mathbb{E} \left\| \left( \sum_{t=1}^d r_t \nabla^2 l_i(w_{k,j}^i + a_t \delta u) uu^T - \nabla^2 l_i(w_{k,j}^i) uu^T \right) \right\|$$

$$\leq \mathbb{E} \left\| \left( \sum_{t=1}^d r_t \nabla^2 l_i(w_{k,j}^i + a_t \delta u) uu^T - \sum_{t=1}^d r_t \nabla^2 l_i(w_{k,j}^i) uu^T \right) \right\|$$

$$\leq \mathbb{E} \sum_{t=1}^d r_t a_t \rho \delta \|u\|^3$$

$$\leq \sum_{t=1}^d r_t \rho \delta \mathbb{E}[\|u\|^3]$$

$$\leq \rho \delta (d + 3)^{3/2}, \quad (103)$$

where the first inequality follows from Jensen’s inequality and the last inequality follows from Lemma 1 in Nesterov and Spokoiny (2017). Then, the proof is complete.

Proof of Proposition 10. We first establish the following lemma on a general systematization property of a random variable.
Lemma 11. Let $X_i, i = 1, ..., U$ be i.i.d. random variables, and let $\epsilon_i, i = 1, ..., U$ be i.i.d. Rademacher random variable taking values from $\{-1, 1\}$ with equal probability. Then, we have

$$E \left\| \sum_{i=1}^{U} (X_i - E(X_i)) \right\|^2 \leq 4E \left\| \sum_{i=1}^{U} \epsilon_i X_i \right\|^2.$$ 

Proof. Let $X'_i, i = 1, ..., U$ be independent copies of random variables $X_i, i = 1, ..., U$. Then, we have

$$E \left\| \sum_{i=1}^{U} (X_i - E(X_i)) \right\|^2 = E \left\| \sum_{i=1}^{U} (X_i - E(X'_i)) \right\|^2 \leq \left( i \right) E \left\| \sum_{i=1}^{U} (X_i - X'_i) \right\|^2 = \left( i \right) E \left\| \sum_{i=1}^{U} \epsilon_i (X_i - X'_i) \right\|^2 \leq 2E \left\| \sum_{i=1}^{U} \epsilon_i X_i \right\|^2 + 2E \left\| \sum_{i=1}^{U} \epsilon_i X'_i \right\|^2 = 4E \left\| \sum_{i=1}^{U} \epsilon_i X_i \right\|^2,$$

where (i) follows from Jensen’s inequality, (ii) follows from the symmetry between $X_i$ and $X'_i$, and the last equality follows from the fact that $X'_i$ has the same distribution as $X_i$. Then, the proof is complete.  

Based on Lemma 11, we next prove Proposition 10. For notational convenience, let $H_u = \frac{1}{\delta} (\nabla l_i(w_{k,j}^i + \delta u; \tilde{D}_{k,j}^i)) - \nabla l_i(w_{k,j}^i; D_{k,j}^i))u^T$. Based on Proposition 9, we now upper-bound $E \left\| H_{k,j}^i - E(H_{k,j}^i) \right\|^2$. In specific, conditioning on $w_{k,j}^i$, we have

$$E \left\| H_{k,j}^i - E(H_{k,j}^i) \right\|^2 \leq E \left\| \frac{1}{U} \sum_{u \in U_{k,j}^i} H_u - (\nabla^2 l_i(w_{k,j}^i) + E_H) \right\|^2, \text{ with } \| E_H \| \leq \rho \delta (d + 3)^{3/2}$$

$$\leq 3E \left\| \frac{1}{U} \sum_{u \in U_{k,j}^i} \frac{1}{\delta} (\nabla l_i(w_{k,j}^i + \delta u; \tilde{D}_{k,j}^i) - \nabla l_i(w_{k,j}^i + \delta u))u^T \right\|^2 + 3E \left\| \frac{1}{U} \sum_{u \in U_{k,j}^i} \frac{1}{\delta} (\nabla l_i(w_{k,j}^i; D_{k,j}^i) - \nabla l_i(w_{k,j}^i))u^T \right\|^2 + 3E \left\| \frac{1}{U} \sum_{u \in U_{k,j}^i} \frac{1}{\delta} (\nabla l_i(w_{k,j}^i + \delta u) - \nabla l_i(w_{k,j}^i))u^T - (\nabla^2 l_i(w_{k,j}^i) + E_H) \right\|^2$$

$$\leq \frac{3}{\delta^2} E \left\| (\nabla l_i(w_{k,j}^i + \delta u; \tilde{D}_{k,j}^i) - \nabla l_i(w_{k,j}^i + \delta u))u^T \right\|^2 + \frac{3}{\delta^2} E \left\| (\nabla l_i(w_{k,j}^i; D_{k,j}^i) - \nabla l_i(w_{k,j}^i))u^T \right\|^2 + \frac{3}{\delta^2} E \left\| \frac{1}{U} \sum_{u \in U_{k,j}^i} \frac{1}{\delta} (\nabla l_i(w_{k,j}^i + \delta u) - \nabla l_i(w_{k,j}^i))u^T - (\nabla^2 l_i(w_{k,j}^i) + E_H) \right\|^2$$

$$\leq \frac{6d\sigma_g^2}{\delta^2} + 3E \left\| \frac{1}{U} \sum_{u \in U_{k,j}^i} \frac{1}{\delta} (\nabla l_i(w_{k,j}^i + \delta u) - \nabla l_i(w_{k,j}^i))u^T - (\nabla^2 l_i(w_{k,j}^i) + E_H) \right\|^2,$$

(104)
where (i) follows from Proposition 9 and the last inequality follows from the bounded variance assumption, i.e., Assumption 3. Our next step is to upper-bound the second term in (104). Specifically, we have

$$3\mathbb{E}\left\| \frac{1}{U} \sum_{u \in U_{k,j}} \frac{1}{\delta} \left( \nabla l_i(w_{k,j}^i + \delta u) - \nabla l_i(w_{k,j}^i) \right) u^T - \left( \nabla^2 l_i(w_{k,j}^i) + E_H \right) \right\|^2$$

$$= 3\mathbb{E}\left\| \frac{1}{U} \sum_{u \in U_{k,j}} \left( \sum_{t=1}^{d} r_t \nabla^2 l_i(w_{k,j}^i + a_t \delta u) - \nabla^2 l_i(w_{k,j}^i) \right) uu^T \right\|^2$$

$$+ \nabla^2 l_i(w_{k,j}^i) \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - \nabla^2 l_i(w_{k,j}^i) \right\|^2 + 9\| E_H \|^2$$

$$(i) \leq 9\mathbb{E}\left\| \sum_{t=1}^{d} r_t \nabla^2 l_i(w_{k,j}^i + a_t \delta u) - \nabla^2 l_i(w_{k,j}^i) \right\|^2$$

$$+ 9\mathbb{E}\left\| \nabla^2 l_i(w_{k,j}^i) \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - \nabla^2 l_i(w_{k,j}^i) \right\|^2 + 9\| E_H \|^2$$

$$(i) \leq 9\rho^2 \delta^2 \mathbb{E}\| u \|^6 + 9L^2 \mathbb{E}\left\| \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - I \right\|^2 + 9\rho^2 \delta^2 (d + 3)^3$$

$$\leq 9\rho^2 \delta^2 (d + 3)^3 + 9\rho^2 \delta^2 (d + 3)^3 + 9L^2 \mathbb{E}\left\| \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - I \right\|^2,$$

(105)

where (i) follows from the mean value theorem, (ii) follows from the inequality that $\| \sum_{i=1}^{n} b_i \|^2 \leq n \sum_{i=1}^{n} \| b_i \|^2$ for any vector $b_i$, (iii) follows by combining Assumption 1, the inequality that $\| \sum_{i=1}^{n} b_i \| \leq \sum_{i=1}^{n} \| b_i \|$ and $r_t, a_t \in (0, 1)$, $\sum_{i=1}^{d} r_t = 1$, and the last inequality follows from Lemma 1 in Nesterov and Spokoiny (2017). Then, our next step is to upper-bound the expectation term in (105). Following from Lemma 11, we have

$$\mathbb{E}\left\| \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - I \right\|^2 \leq 4\mathbb{E}\left\| \frac{1}{U} \sum_{i=1}^{U} \epsilon_i u_i u_i^T \right\|^2,$$

(106)

where $u_i, i = 1, ..., U$ are elements in $U_{k,j}$. Based on the Rudelson’s inequality (Equation (3.4) in Rudelson (1999) with $p = 2$), and conditioning on $u_i, i = 1, ..., U$, we have

$$\mathbb{E}_\epsilon \left\| \sum_{i=1}^{U} \epsilon_i u_i u_i^T \right\|^2 \leq C_\epsilon \max\{\log d, 2\} \left( \max_{u \in U_{k,j}} \| u \|^2 \right) \sum_{u \in U_{k,j}} uu^T,$$

where $C_\epsilon$ is a positive constant independent of $d$ and $U$. Taking the expectation over the above inequality
Let $\epsilon_l$. Theorem 3.

Suppose that Assumptions 1, 2, and 3 hold. Define a constant

$$C = \frac{C_0}{\max \{\log d, 2\}} \left( \frac{\theta}{U} \right)^2$$

where $m := \mathbb{E} \left( \max_{u \in U_{k,j}} \|u\|^4 \right)$. Letting $C = 8C_0$, combining (107) with (106), and considering the cases $\mathbb{E}\left\| \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - I \right\|^2 \geq 1$ and $\mathbb{E}\left\| \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - I \right\|^2 < 1$ separately, we have

$$\mathbb{E}\left\| \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - I \right\|^2 \leq \max\{\lambda, \lambda^2\}, \quad \lambda = \frac{C \max \{\log d, 2\}}{U}.$$  

Combining the above inequality with (104) and (105) yields

$$\mathbb{E}\|H^i_{k,j} - \mathbb{E}(H^i_{k,j})\|^2 \leq \frac{6\rho^2 \sigma^2}{\delta^2 D} + 9\rho^2 \delta^2 (d + 3)^3 + 9\rho^2 \delta^2 (d + 3)^3 + 9L^2 \max\{\lambda, \lambda^2\},$$  

where $\lambda = \frac{C \max \{\log d, 2\}}{U}$. Then, using the fact $d + 3 < d + 6$ in the above inequality completes the proof. \(\square\)

### F.2 Properties of Meta Gradient of GGS-MAML

Based on the properties of the proposed Hessian estimator we establish in the previous subsection, we now establish some propositions on the properties of the meta gradient used in GGS-MAML for proving Theorem 3.

**Proposition 14.** Suppose that Assumptions 1, 2, and 3 hold. Define a constant

$$C_{e_1} = \left( 1 + \alpha L + \alpha \delta (d + 3)^{3/2} \right)^N \left( 1 + \alpha L \right)^N - 1 \frac{\sigma^2}{\sqrt{S}}$$

$$C_{e_2} = \frac{\left( 1 + \alpha L \right)^N \left( 1 + \alpha L \right)^N - 1 \left( 1 + \alpha L + \alpha \delta (d + 3)^{3/2} \right)^N \left( \rho \left( 1 + \alpha L \right)^N - 1 \right) \sigma^2}{L \sqrt{S}} + \rho \delta (d + 3)^{3/2} \right) + \alpha \left( 1 + \alpha L \right)^{N-1} \rho \delta (d + 3)^{3/2}.$$  

Let $e_k := \mathbb{E}(G_i(w_k) - \nabla \mathcal{L}(w_k))$. Then, if inner stepsize $\alpha < (2^{1/3} - 1)/L$, then conditioning on $w_k$, we have

$$\|e_k\| \leq C_{e_1} + C_{e_2} \|\nabla \mathcal{L}(w_k)\| + C_{e_3} \sigma.$$  

Note that the third term in the above estimation error has a order of $\mathcal{O}\left(\left( 1 + \alpha L \right)^N \left( \frac{1}{\sqrt{S}} + \delta d^{3/2} \right) \right)$, which can be controlled to be small by a sample size $S$ and a smoothing parameter $\delta$. In contrast, the meta gradient estimator using the zeroth-order Hessian approximating involves a constant-level bias $\Theta\left( \rho \sigma \left( 1 + \alpha L \right)^N - 1 \right)$. In the meanwhile, we emphasize here that by choosing $\delta$ properly, we enable to achieve a similar estimation accuracy as the original MAML without extra constraints on the inner stepsize $\alpha$. To see this, if we choose $\delta = \Theta\left( \left( 1 + \alpha \right)^N - 1 \right) d^{-3/2} S^{-1/2}$, then it can be verified that the above estimator error has the same order as that in (17) for original MAML.
Proof. Let \( \bar{S}_j \) and \( \bar{D}_j \) denote randomness over \( \{ S^i_{k,m}, m = 0, \ldots, j - 1, i \in \mathcal{I} \} \) and \( \{ \bar{D}^i_{k,m}, \bar{D}^i_{k,m}, m = 0, \ldots, j - 1, i \in \mathcal{I} \} \), respectively. Let \( r^i_{k,j} = E(H^i_{k,j} w^i_{k,j}) - \nabla^2 l_i(w^i_{k,j}) \). Then, based on Proposition 9, we have

\[
\| r^i_{k,j} \| \leq \rho d (d + 3)^{3/2}.
\] (111)

Note that conditioning on \( \bar{S}_N \) and \( w_k \), parameters \( \bar{w}^i_{k,j}, j = 0, \ldots, N \) are deterministic. Then, using an approach similar to (51) and conditioning on \( w_k \), we have

\[
\mathbb{E} \tilde{G}_i(w_k) = \mathbb{E}_{\bar{S}_N, i \sim \rho(T)} \left( \mathbb{E}_{D_N, \{ u^i_{k,j,q} \in U^i_{k,j} \}} \left( \prod_{j=0}^{N-1} (I - \alpha H^i_{k,j}) \mathbb{E}_{I^i_{k,j}} \nabla l_i(w^i_{k,N}; T^i_{k,j}) | \bar{S}_N, i) \right) \right)
\]

\[
= \mathbb{E}_{\bar{S}_N, i \sim \rho(T)} \left( \prod_{j=0}^{N-1} \mathbb{E}_{D_N, \bar{w}^i_{k,j,q}} (I - \alpha \bar{H}^i_{k,j} | \bar{S}_N, i) \nabla l_i(w^i_{k,N}) \right)
\]

which, using \( \nabla L(w_k) = \mathbb{E}_{i \sim \rho(T)} \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}^i_{k,j})) \nabla l_i(\bar{w}^i_{k,N}) \) and \( e_k \equiv \mathbb{E} \tilde{G}_i(w_k) - \nabla L(w_k) \), yields

\[
\| e_k \| \leq \mathbb{E}_{\bar{S}_N, i \sim \rho(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w^i_{k,j}) - \alpha r^i_{k,j}) \nabla l_i(w^i_{k,N}) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w^i_{k,j}) - \alpha r^i_{k,j}) \nabla l_i(\bar{w}^i_{k,N}) \right)
\]

\[
+ \mathbb{E}_{\bar{S}_N, i \sim \rho(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}^i_{k,j}) - \alpha r^i_{k,j}) \nabla l_i(\bar{w}^i_{k,N}) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}^i_{k,j})) \nabla l_i(\bar{w}^i_{k,N}) \right)
\]

\[
(1) \leq \mathbb{E}_{\bar{S}_N, i \sim \rho(T)} \left( 1 + \alpha L + \alpha \rho d (d + 3)^{3/2} \right)^N \| \nabla l_i(w^i_{k,N}) - \nabla l_i(\bar{w}^i_{k,N}) \|
\]

\[
+ \mathbb{E}_{\bar{S}_N, i \sim \rho(T)} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}^i_{k,j}) - \alpha r^i_{k,j}) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}^i_{k,j})) \right) \| \nabla l_i(\bar{w}^i_{k,N}) \|
\]

\[
(2) \leq (1 + \alpha L + \alpha \rho d (d + 3)^{3/2} \right)^N L \mathbb{E}_{\bar{S}_N, i \sim \rho(T)} \| w^i_{k,N} - \bar{w}^i_{k,N} \|
\]

\[
+ (1 + \alpha L)^N \mathbb{E}_{\bar{S}_N, i \sim \rho(T)} \| \nabla l_i(w_k) \| \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w^i_{k,j}) - \alpha r^i_{k,j}) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}^i_{k,j})) \| \nabla l_i(\bar{w}^i_{k,N}) \|
\]

\[
\leq (1 + \alpha L)^N \mathbb{E}_{i \sim \rho(T)} \left( \| \nabla l_i(w_k) \| \mathbb{E}_{\bar{S}_N} \left( \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(w^i_{k,j}) - \alpha r^i_{k,j}) - \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_i(\bar{w}^i_{k,j})) \right) \right) \right) \right)
\]

\[
+ (1 + \alpha L + \alpha \rho d (d + 3)^{3/2} \right)^N ((1 + \alpha L)^N - 1) \frac{\sigma_q}{\sqrt{S}}.
\] (112)

where (i) follows from (111), (iii) follows from Lemma 2 that \( \| \nabla l_i(\bar{w}^i_{k,N}) \| \leq (1 + \alpha L)^N \| \nabla l_i(w_k) \| \), and the last inequality follows from item 1 in Proposition 2.

To upper-bound the difference \( (P) \) in (112), we define \( \tilde{R}(m) = \mathbb{E}_{\bar{S}_m, i} \prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(w^i_{k,j}) - \alpha r^i_{k,j}) - \prod_{j=0}^{m-1} (I - \alpha \nabla^2 l_i(\bar{w}^i_{k,j})) \) for \( m = 1, \ldots, N \), where \( \mathbb{E}_{\bar{S}_m} \) denotes \( \mathbb{E}_{\bar{S}_m} \) (|i). Then, conditioning on \( w_k \), we
have

\[
R(m) \leq \mathbb{E}_{S_m|\cdot} \left( \prod_{j=0}^{m-1} \left( I - \alpha \nabla^2 l_i(w_{k,j}^i) - \alpha r_{k,j}^i \right) - \prod_{j=0}^{m-2} \left( I - \alpha \nabla^2 l_i(w_{k,j}^i) - \alpha r_{k,j}^i \right) \right) \left( I - \alpha \nabla^2 l_i(\tilde{w}_{k,m-1}^i) \right)
\]

\[
+ \mathbb{E}_{S_m|\cdot} \left( \prod_{j=0}^{m-2} \left( I - \alpha \nabla^2 l_i(w_{k,j}^i) - \alpha r_{k,j}^i \right) \right) \left( I - \alpha \nabla^2 l_i(\tilde{w}_{k,m-1}^i) \right)
\]

\[
\leq (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{m-1} \left( \alpha \rho \mathbb{E}_{S_m|\cdot} \|w_{k,m-1}^i - \tilde{w}_{k,m-1}^i\| + \alpha \rho \delta (d + 3)^{3/2} \right) + (1 + \alpha L) R(m-1)
\]

\[
(1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{m-1} \left( \frac{\alpha \rho ((1 + \alpha L)^{m-1} - 1) \sigma_g}{L \sqrt{S}} + \alpha \rho \delta (d + 3)^{3/2} \right) + (1 + \alpha L) R(m-1),
\]

where (i) follows from Proposition 2. Noting that \( m \leq N \), we have

\[
R(m) \leq (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1} \left( \frac{\rho ((1 + \alpha L)^{N-1} - 1) \sigma_g}{L \sqrt{S}} + \rho \delta (d + 3)^{3/2} \right) + (1 + \alpha L) R(m-1).
\]

Rearranging the above inequality, we obtain

\[
R(m) + \frac{(1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1}}{L} \left( \frac{\rho ((1 + \alpha L)^{N-1} - 1) \sigma_g}{L \sqrt{S}} + \rho \delta (d + 3)^{3/2} \right)
\]

\[
\leq (1 + \alpha L) \left( R(m-1) + \frac{(1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1}}{L} \left( \frac{\rho ((1 + \alpha L)^{N-1} - 1) \sigma_g}{L \sqrt{S}} + \rho \delta (d + 3)^{3/2} \right) \right).
\]

Telescoping the above inequality over \( m \) from 2 to \( N \) yields

\[
R(N) + \frac{(1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1}}{L} \left( \frac{\rho ((1 + \alpha L)^{N-1} - 1) \sigma_g}{L \sqrt{S}} + \rho \delta (d + 3)^{3/2} \right)
\]

\[
\leq (1 + \alpha L)^N \left( R(1) + \frac{(1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1}}{L} \left( \frac{\rho ((1 + \alpha L)^{N-1} - 1) \sigma_g}{L \sqrt{S}} + \rho \delta (d + 3)^{3/2} \right) \right),
\]

which, in conjunction with \( R(1) \leq \alpha \rho \delta (d + 3)^{3/2} \), yields

\[
R(N) \leq ((1 + \alpha L)^{N-1} - 1) \frac{(1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1}}{L} \left( \frac{\rho ((1 + \alpha L)^{N-1} - 1) \sigma_g}{L \sqrt{S}} + \rho \delta (d + 3)^{3/2} \right)
\]

\[
+ \alpha(1 + \alpha L)^{N-1} \rho \delta (d + 3)^{3/2}.
\]

Let \( C_e \) be the constant at the right side of the above inequality. Combining the above inequality and (112) and using an approach similar to (55) yields

\[
\|e_k\| \leq (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^N \left( (1 + \alpha L)^N - 1 \right) \frac{\sigma_g}{\sqrt{S}} + (1 + \alpha L)^N C_e \mathbb{E}_{\omega \sim p(\mathcal{T})} \|\nabla l_i(w_k)\|
\]

\[
\leq (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^N \left( (1 + \alpha L)^N - 1 \right) \frac{\sigma_g}{\sqrt{S}} + \frac{(1 + \alpha L)^N C_e}{2 - (1 + \alpha L)^{2N}} \left( \|\nabla L(w_k)\| + \sigma \right),
\]

which, in conjunction of definitions of \( C_{e_1} \) and \( C_{e_2} \), finishes the proof.
We next upper-bound the second moment of $\hat{G}_i(w_k)$ based on the variance bound we develop in Proposition 10.

**Proposition 15.** Suppose that Assumptions 1, 2, and 3 hold. Define constants

$$C_{s_1} = 3\left(\frac{6d\sigma_2^2\alpha^2}{\delta^2D} + 19\alpha^2\rho^2\delta^2(d + 6)^3 + 9\alpha^2L^2\max\{\lambda, \lambda^2\} + (1 + \alpha L)^2 + 2(1 + \alpha L)\alpha\rho\delta(d + 3)^{3/2}\right)^N\sigma_g^2,$$

$$C_{s_2} = \frac{\alpha L((1 + 2\alpha L + 2\alpha^2L^2)^N - 1)C_{s_1}}{(1 + \alpha L)} \quad \text{and} \quad C_{s_3} = \frac{2(1 + \alpha L)^2NC_{s_1}}{(2 - (1 + \alpha L)^{2N})^2\sigma_g^2}. \quad (115)$$

Then, if the inner stepsize $\alpha < (2\pi^2 - 1)/L$, then conditioning on $w_k$, we have

$$\mathbb{E}||\hat{G}_i(w_k)||^2 \leq \frac{C_{s_1}}{T} + \frac{C_{s_2}}{S} + C_{s_3}(\|\nabla L(w_k)\|^2 + \sigma^2).$$

Note that with a proper choice of the smoothing parameter $\delta$, the above proposition enables to achieve a similar bound as in Proposition 4 for original MAML algorithm.

**Proof.** Recall $\hat{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha H_{k,j})\nabla l_i(w_{k,N}^j; T_k^i)$. Then, conditioning on $w_k$, we have

$$\mathbb{E}||\hat{G}_i(w_k)||^2 \leq \mathbb{E}_{S_N, i} \left(\mathbb{E}_{D_{N,t_i}, T_k^i, \{u_{k,j,q}^i\}_{j\in U_k^i}} \left(\prod_{j=0}^{N-1} (I - \alpha H_{k,j})\right)^2 \|\nabla l_i(w_{k,N}^j; T_k^i)\|^2 | S_N, i\right) \right)$$

\[\begin{aligned}
&\leq \mathbb{E}_{S_N, i} \left(\prod_{j=0}^{N-1} \mathbb{E}_{D_{k,j}^i, u_{k,j,q}^i} \left(\|I - \alpha H_{k,j}\|^2 | S_N, i\right) \right) \mathbb{E}_{T_k^i} \left(\|\nabla l_i(w_{k,N}^j; T_k^i)\|^2 | S_N, i\right) \right) \quad (P)
\end{aligned}

We upper-bound $(P)$ and $(Q)$ in the above inequality, respectively. Note that $w_{k,j}^i, j = 0, ..., N - 1$ are deterministic when conditioning on $S_N, i, w_k$. Thus, conditioning on $S_N, i, w_k$, we have

$$\mathbb{E}_{D_{N,t_i}, u_{k,j,q}^i} \left\|I - \alpha H_{k,j}\right\|^2 = \text{Var} \left(I - \alpha H_{k,j}\right) + \left\|I - \alpha \mathbb{E}_{D_{N,t_i}, u_{k,j,q}^i} (H_{k,j}^i)\right\|^2$$

\[= \alpha^2 \text{Var} \left(H_{k,j}^i\right) + \|I - \alpha \nabla^2 l_i(w_{k,j}^i)\|^2 + 2\|I - \alpha \nabla^2 l_i(w_{k,j}^i)\|\|\alpha \nabla^2 l_i(w_{k,j}^i) - \alpha \mathbb{E}_{D_{N,t_i}, u_{k,j,q}^i} (H_{k,j}^i)\|$$

\[\leq \alpha^2 \text{Var} \left(H_{k,j}^i\right) + \|I - \alpha \nabla^2 l_i(w_{k,j}^i)\|^2 + 2(1 + \alpha L)\alpha\rho\delta(d + 3)^{3/2} + \alpha^2 \rho^2\delta^2(d + 3)^3 \quad (i)
\]

\[\leq \alpha^2 \left(\frac{6d\sigma_2^2}{\delta^2D} + 18\rho^2\delta^2(d + 6)^3 + 9L^2 \max\{\lambda, \lambda^2\}\right) + (1 + \alpha L)^2 + 2(1 + \alpha L)\alpha\rho\delta(d + 3)^{3/2} + \alpha^2 \rho^2\delta^2(d + 3)^3 \quad (ii)
\]

\[\leq \frac{6d\sigma_2^2}{\delta^2D} + 19\alpha^2 \rho^2\delta^2(d + 6)^3 + 9\alpha^2L^2 \max\{\lambda, \lambda^2\} + (1 + \alpha L)^2 + 2(1 + \alpha L)\alpha\rho\delta(d + 3)^{3/2} \quad (117)
\]

where (i) follows from Proposition 9 and (ii) follows from Proposition 10.

To simplify notations, let $C_{s_1} = 3\left(\frac{6d\sigma_2^2\alpha^2}{\delta^2D} + 19\alpha^2\rho^2\delta^2(d + 6)^3 + 9\alpha^2L^2 \max\{\lambda, \lambda^2\} + (1 + \alpha L)^2 + 2(1 + \alpha L)\alpha\rho\delta(d + 3)^{3/2}\right)^N\sigma_g^2$. We next upper-bound $(Q)$. Conditioning on $S_N, i, w_k$ and using an approach similar to (58), we have

$$\mathbb{E}_{T_k^i} \|\nabla l_i(w_{k,N}^i; T_k^i)\|^2 \leq \frac{3\sigma_g^2}{T} + 3L^2\|w_{k,N}^i - \tilde{w}_{k,N}^i\|^2 + 3(1 + \alpha L)^2N\|\nabla l_i(w_k)\|^2. \quad (118)$$

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Thus, conditioning on $w_k$ and combining (116), (117) and (118), we have
\[
E\|\tilde{G}_i(w_k)\|^2 \leq \frac{C_{s_1}}{3\sigma^2} \left( 3\sigma_g^2 \frac{1}{T} + 3L^2 \mathbb{E}\|w_{k,N} - \tilde{w}_{k,N}\|^2 + 3(1 + \alpha L)^2N \mathbb{E}\|\nabla l_i(w_k)\|^2 \right)
\]
which, in conjunction with Proposition 2 that
\[
E\|w_{k,N} - \tilde{w}_{k,N}\|^2 = E_i \mathbb{E}(\|w_{k,N} - \tilde{w}_{k,N}\|^2 | i) \leq (1 + 2\alpha L + 2\alpha^2 L^2)^N - 1)^{\frac{\alpha \sigma_g^2}{L(1 + \alpha L)^S}}.
\]
implies that
\[
E\|\tilde{G}_i(w_k)\|^2 \leq \frac{C_{s_1}}{T} + \frac{\alpha L C_{s_1}((1 + 2\alpha L + 2\alpha^2 L^2)^N - 1)}{(1 + \alpha L)} \frac{1}{S} + \frac{(1 + \alpha L)^{2N}}{\sigma^2} C_{s_1}(\|\nabla l(w_k)\|^2 + \sigma^2).
\]

Based on Lemma 5 and conditioning on $w_k$, we have
\[
\|\nabla l(w_k)\|^2 \leq \frac{2}{(1 - C_1)^2} \|\nabla L(w_k)\|^2 + \frac{2C_l^2}{(1 - C_1)^2} \sigma^2,
\]
which, in conjunction with the inequality that $\frac{2\alpha^2}{(1 - x)^2} + 1 \leq \frac{2}{(1 - x)^2}$, (119) and $C_l = (1 + \alpha L)^{2N} - 1$, yields
\[
E\|\tilde{G}_i(w_k)\|^2 \leq \frac{C_{s_1}}{T} + \frac{\alpha L C_{s_1}((1 + 2\alpha L + 2\alpha^2 L^2)^N - 1)}{(1 + \alpha L)} \frac{1}{S} + \frac{2(1 + \alpha L)^{2N} C_{s_1}}{(2 - (1 + \alpha L)^{2N})^2 \sigma^2}(\|\nabla L(w_k)\|^2 + \sigma^2).
\]

Then, the proof is complete.

\[ \square \]

F.3 Proof of Theorem 3 in Section 5.2: Convergence of GGS-MAML in Resampling Case

In this subsection, we provide the proof for Theorem 3 based on the propositions established in the previous subsection.

**Proof of Theorem 3.** Set the meta stepsize $\beta_k = \frac{1}{C_\beta L_{w_k}}$ with $\hat{L}_{w_k}$ given by (15), where we set $|B_k'| > \frac{4C_2^2 \sigma^2}{\pi (1 + \alpha L)^{1/2} L}$ and $|D_{L_{w_k}}| > \frac{64 \sigma^2 C_2^2}{(1 + \alpha L)^{1/2} L}$ for all $i \in B_k'$. Define parameters
\[
\xi = \frac{6}{C_\beta L} \left( \frac{1}{5} + \frac{2}{C_\beta} \right) \left( C_{e_1} + C_{e_2} \sigma^2 \right), \quad \phi = \frac{2}{C_\beta L} \left( C_{e_1} \frac{C_{s_1}}{T} + C_{s_2} + C_{s_3} \sigma^2 \right)
\]
\[
\theta = \frac{2(2 - (1 + \alpha L)^{2N})}{C_\beta C_L} \left( \frac{1}{5} - \frac{6}{C_\beta} \right) C_{e_2} - \frac{C_{s_3}}{C_\beta B} - \frac{2}{C_\beta} \right)
\]
\[
\chi = \frac{(2 - (1 + \alpha L)^{2N})(1 + \alpha L)^{2N} L}{C_L} + \sigma.
\]

where $C_{e_1}, C_{e_2}$ are given in (110) and $C_{s_1}, C_{s_2}, C_{s_3}$ are given in (115).
Let \( L_{w_k} = (1 + \alpha L)^2 N + C_L \mathbb{E}_{i \sim p(T)} \| \nabla l_i(w_k) \| \), where \( C_L \) is given in (14). Then, by Proposition 1, we have

\[
\mathcal{L}(w_{k+1}) \leq \mathcal{L}(w_k) + \langle \nabla \mathcal{L}(w), w_{k+1} - w_k \rangle + \frac{L_{w_k}}{2} \| w_{k+1} - w_k \|^2
\]

\[
= F(w_k) - \beta_k \langle \nabla \mathcal{L}(w_k), \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \rangle + \frac{L_{w_k} \beta_k^2}{2} \left\| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \right\|^2.
\]

Similarly to (61), conditioning on \( w_k \), and recalling \( e_k := \mathbb{E} \hat{G}_i(w_k) - \nabla \mathcal{L}(w_k) \), we have

\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) - \left[ \mathbb{E}(\beta_k) \right] \langle \nabla \mathcal{L}(w_k), \nabla \mathcal{L}(w_k) + e_k \rangle + \frac{L_{w_k}}{2} \left[ \mathbb{E}(\beta_k^2) \right] \mathbb{E} \left( \left\| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \right\|^2 \right) w_k. \quad (122)
\]

Using an approach similar to (47) and (48) and conditioning on \( w_k \), we have

\[
\mathbb{E} \beta_k^2 < \frac{4}{C_{\beta}^2} \frac{1}{L_{w_k}} \quad \text{and} \quad \mathbb{E} \beta_k \geq \frac{4}{C_{\beta}^2} \frac{1}{5 L_{w_k}}.
\]

Combining (123) with (122), and using an approach similar to (62) yields

\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) - \frac{2}{5C_{\beta}} \frac{1}{L_{w_k}} \| \nabla \mathcal{L}(w_k) \|^2 + \frac{2}{5C_{\beta}} \frac{1}{L_{w_k}} \| \nabla \mathcal{L}(w_k) \|^2 + \frac{2}{5C_{\beta}} \frac{1}{L_{w_k}} \| e_k \|^2
\]

\[
+ \frac{2}{C_{\beta}^2} \frac{1}{L_{w_k}} \mathbb{E} \left\| \hat{G}_i(w_k) \right\|^2 + \| \mathbb{E} \hat{G}_i(w_k) \|^2 \right),
\]

which, in conjunction with Propositions 14 and 15, yields

\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) - \frac{2}{5C_{\beta}} \frac{1}{L_{w_k}} \| \nabla \mathcal{L}(w_k) \|^2 + \frac{2}{5C_{\beta}} \frac{1}{L_{w_k}} \left( 3C_{e_1} + 3C_{e_2} \| \nabla \mathcal{L}(w_k) \|^2 + 3C_{e_2} \sigma^2 \right)
\]

\[
+ \frac{2}{C_{\beta}^2} \frac{1}{L_{w_k}} \mathbb{E} \left\| \hat{G}_i(w_k) \right\|^2 + \frac{2}{C_{\beta}^2} \frac{1}{L_{w_k}} \left( 2\| \nabla \mathcal{L}(w_k) \|^2 + 2\| e_k \|^2 \right) \leq \mathcal{L}(w_k) - \frac{2}{5C_{\beta}} \frac{1}{L_{w_k}} \| \nabla \mathcal{L}(w_k) \|^2 + \left( \frac{6}{5C_{\beta} L_{w_k}} + \frac{12}{C_{\beta}^2 L_{w_k}} \right) \left( C_{e_1} + C_{e_2} \| \nabla \mathcal{L}(w_k) \|^2 + C_{e_2} \sigma^2 \right)
\]

\[
+ \frac{2}{C_{\beta}^2} \frac{1}{L_{w_k}} \frac{1}{B} \left( \frac{C_{e_1}}{T} + \frac{C_{s_1}}{S} + C_{s_2} \left( \| \nabla \mathcal{L}(w_k) \|^2 + \sigma^2 \right) \right) + \frac{4}{C_{\beta}^2} \frac{1}{L_{w_k}} \| \nabla \mathcal{L}(w_k) \|^2
\]

\[
\leq \mathcal{L}(w_k) - \frac{2}{C_{\beta} L_{w_k}} \left( \frac{1}{5} - \left( \frac{3}{5} + \frac{6}{C_{\beta}} \right) \frac{C_{e_2}}{C_{\beta}} - \frac{C_{s_1}}{C_{\beta} B} - \frac{2}{C_{\beta}^2} \right) \| \nabla \mathcal{L}(w_k) \|^2
\]

\[
+ \frac{6}{C_{\beta} L_{w_k}} \left( \frac{1}{5} + \frac{2}{C_{\beta}} \right) \left( C_{e_1} + C_{e_2} \sigma^2 \right) + \frac{2}{C_{\beta}^2 L_{w_k}} \left( \frac{C_{e_1}}{T} + \frac{C_{s_1}}{S} + C_{s_2} \sigma^2 \right) \frac{1}{B},
\]

which, in conjunction with \( L_{w_k} = (1 + \alpha L)^2 N + C_L \mathbb{E}_{i \sim p(T)} \| \nabla l_i(w_k) \| \) and (64), yields

\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) + \frac{6}{C_{\beta} L_{w_k}} \left( \frac{1}{5} + \frac{2}{C_{\beta}} \right) \left( C_{e_1} + C_{e_2} \sigma^2 \right) + \frac{2}{C_{\beta}^2 L_{w_k}} \left( \frac{C_{e_1}}{T} + \frac{C_{s_1}}{S} + C_{s_2} \sigma^2 \right) \frac{1}{B}
\]

\[
- \frac{2}{C_{\beta} (1 + \alpha L)^2 N + \frac{C_{\beta} \sigma}{2C_{e_1}} + \frac{C_{e_2}}{2C_{e_1}}} \| \nabla \mathcal{L}(w_k) \|^2 \left( \frac{1}{5} - \left( \frac{3}{5} + \frac{6}{C_{\beta}} \right) \frac{C_{e_2}}{C_{\beta}} - \frac{C_{s_1}}{C_{\beta} B} - \frac{2}{C_{\beta}^2} \right) \| \nabla \mathcal{L}(w_k) \|^2, \quad (124)
\]

where \( C_{\beta} \) is given by (41) and \( C_L \) is given by (14). Then, combining (121) with (124) yields

\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) + \xi + \frac{\phi}{B} - \theta \frac{\| \nabla \mathcal{L}(w_k) \|^2}{\mathcal{L}(w_k) + \xi + \| \nabla \mathcal{L}(w_k) \|^2}.
\]
Unconditioning on $w_k$ in the above inequality, and telescoping the above inequality over $k$ from 0 to $K - 1$, we have

$$\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \left( \frac{\theta \|\nabla \mathcal{L}(w_k)\|^2}{\chi + \|\nabla \mathcal{L}(w_k)\|} \right) \leq \frac{\Delta}{K} + \xi + \frac{\phi}{B},$$

where $\Delta = \mathcal{L}(w_0) - \mathcal{L}^*$ with $\mathcal{L}^* = \inf_{w \in \mathbb{R}^d} \mathcal{L}(w) > -\infty$. Then, using an approach similar to (70), we have

$$\mathbb{E}\|\nabla \mathcal{L}(w_c)\| \leq \frac{\Delta}{20K} + \frac{\xi}{20} + \frac{\phi}{20B} + \sqrt{\chi \left( \frac{\Delta}{20K} + \frac{\xi}{20} + \frac{\phi}{20B} \right)^2 + \left( \frac{\Delta}{20K} + \frac{\xi}{20} + \frac{\phi}{20B} \right)^2},$$

(126)

Noting that $\alpha \leq \min \left( \frac{1}{8NL}, \frac{\delta \sqrt{D}}{8\sigma \sqrt{64N}} \right)$, $\delta \leq \frac{L}{8\rho(d+3)^{3/2}}$, and $\alpha^2L^2 \max\{\lambda, \lambda^2\} < \frac{1}{64N}$ and using the inequality that $(1 + c)^n = e^{n \log(1 + c)} < e^{nc}$, we have

$$C_{e_1} \leq \left( \frac{1}{4N} \right)^N (\sqrt{2} - 1) \frac{\sigma_g}{\sqrt{N}} \leq e^{1/4} \frac{1}{2} \frac{\sigma_g}{\sqrt{N}} < \frac{3\sigma_g}{4\sqrt{N}},$$

$$C_{e_2} \leq \left( \frac{\sqrt{2}(\sqrt{2} - 1)}{\sqrt{2}} \right)^e \frac{1}{2} \left( \frac{\rho\sigma_g}{2L} + \rho\theta(d+3)^{3/2} \right) + \frac{5}{32} \rho\delta(d+3)^{3/2} \leq \frac{1}{4} + \frac{5}{576} < \frac{1}{2},$$

$$C_{s_1} \leq 3 \left( \frac{1}{64N} + \frac{19}{64 \times 12^2 N^2} + \frac{9}{64N} + \left( 1 + \frac{1}{8N} \right) \frac{1}{72N} + \left( 1 + \frac{1}{8N} \right)^N \right)^{3} \sigma_g^2 < \frac{9}{2} \sigma_g^2,$$

$$C_{s_2} \leq \frac{1}{8N} \frac{3}{2} C_s, \quad C_{s_3} \leq 6 \frac{9\sigma_g^2}{2\sigma_g^2} < 27.$$  

(127)

In addition, based on (73), we have $\frac{1}{10} \frac{\xi}{L} < C \xi < \frac{3}{5} \frac{\xi}{L}$. Then, using the above inequalities, $B \geq 3$ and (121), we have

$$\xi \leq \mathcal{O} \left( \frac{\sigma_g^2}{S} + \frac{\sigma_g^2}{S^2} + \sigma^2 \right), \quad \phi \leq \mathcal{O} \left( \frac{\sigma_g^2}{T} + \frac{\sigma_g^2}{NS} + \sigma^2 \right),$$

$$\theta \geq \frac{1}{100} \frac{5L}{3} \left( \frac{1}{5} \frac{33}{50} \frac{12}{50} \frac{27}{300} \frac{1}{50} \right) = \frac{L}{60\rho} \frac{3}{100} = \frac{L}{2000\rho},$$

$$\chi \leq \mathcal{O}(1 + \sigma),$$

which, in conjunction with (126), yields

$$\mathbb{E}\|\nabla \mathcal{L}(w_c)\| \leq \mathcal{O} \left( \frac{1}{K} + \frac{\sigma_g^2(s^2 + 1)}{S} + \sigma^2 \frac{\delta d \sigma^2}{BT} + \frac{\sigma^2}{B} + \sqrt{1 + \sigma} \sqrt{\frac{1}{K} + \frac{\sigma_g^2(s^2 + 1)}{S} + \sigma^2 \frac{\delta d \sigma^2}{BT} + \frac{\sigma^2}{B}} \right).$$

Suppose batch sizes $S, T, B$ further satisfy $S \geq C_S \sigma_g^2(\sigma + 1)\epsilon^{-2}$, $B \geq C_B \sigma^2 \max(\sigma, 1)\epsilon^{-2}$, and $TB > C_T \sigma_g^2 \max(\sigma, 1)\epsilon^{-2}$ and smoothing parameter $\delta$ satisfies $\delta < \epsilon/(C_g d^{3/2} \sigma \sqrt{\max(\sigma, 1)})$, where $C_S, C_T, C_B, C_\delta > 0$ are sufficiently large constants. Then, using an approach similar to Corollary 1 yields the proof. □

G  Proofs for Section 5.3: Convergence of GGS-MAML in Finite-Sum Case

In this section, we first prove the propositions on the properties of the proposed Hessian estimator and the meta gradient, and then use these propositions for proving the convergence results of GGS-MAML.
Proof of Proposition 11. Using an approach similar to (102) and conditioning on $w_k$, we have

$$
E(H_{k,j}^i) = \frac{1}{U} \sum_{u \in U_{k,j}} \nabla l_{S_i}(w_{k,j}^i + \delta u) - \nabla l_{S_i}(w_{k,j}^i) u^T
$$

$$
= \frac{1}{U} \sum_{u \in U_{k,j}} \nabla l_{S_i}(w_{k,j}^i + \delta u) - \nabla l_{S_i}(w_{k,j}^i) u^T
$$

$$
= \frac{1}{U} \sum_{u \in U_{k,j}} \left( \sum_{t=1}^d r_t \nabla^2 l_{S_i}(w_{k,j}^i + a_t \delta u) - \nabla^2 l_{S_i}(w_{k,j}^i) \right) u u^T + \nabla^2 l_{S_i}(w_{k,j}^i) u u^T \right) a_t, r_t \in (0, 1), \sum_{t=1}^d r_t = 1
$$

$$
= \frac{1}{U} \sum_{u \in U_{k,j}} \left( \sum_{t=1}^d r_t \nabla^2 l_{S_i}(w_{k,j}^i + a_t \delta u) - \nabla^2 l_{S_i}(w_{k,j}^i) \right) u u^T + \nabla^2 l_{S_i}(w_{k,j}^i),
$$

which, using an approach similar to (103), yields the proof of item 1.

We next prove item 2. Let $H_u = \frac{1}{U} \left( \nabla l_{S_i}(w_{k,j}^i + \delta u) - \nabla l_{S_i}(w_{k,j}^i) \right) u^T$. Then, conditioning on $w_{k,j}^i$, we have

$$
E\|H_{k,j}^i - E(H_{k,j}^i)\|^2 = E\left[ \frac{1}{U} \sum_{u \in U_{k,j}} H_u - E(H_u) \right]^2
$$

where (i) follows from item 1, (ii) follows from the mean value theorem (MVT), (iii) follows from the inequality that $\sum_{i=1}^n a_i^2 \leq n \sum_{i=1}^n a_i^2$, (iv) follows from an approach similar to (iii) in (104), and the last inequality

$$
= 6 \rho^2 \delta^2 (d+3)^3 + 3 \rho^2 \delta^2 (d+3)^3 + 3 L^2 E\left[ \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - I \right]^2
$$

$$
\leq 6 \rho^2 \delta^2 (d+3)^3 + 3 \rho^2 \delta^2 (d+3)^3 + 3 L^2 E\left[ \frac{1}{U} \sum_{u \in U_{k,j}} uu^T - I \right]^2
$$

where (i) follows from item 1, (ii) follows from the mean value theorem (MVT), (iii) follows from the inequality that $\sum_{i=1}^n a_i^2 \leq n \sum_{i=1}^n a_i^2$, (iv) follows from an approach similar to (iii) in (104), and the last inequality
follows from Lemma 1 in Nesterov and Spokoiny (2017). Finally, combining the above inequality with (108) yields the proof. □

**Proof of Proposition 12.** Recall the meta gradient estimator

$$\hat{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha H_{k,j}) \nabla l_{T_i} (w_{k,N}^i)$$

and the true gradient

$$\nabla L(w_k) = \mathbb{E}_{i \sim p(T)} \prod_{j=0}^{N-1} (I - \alpha \nabla^2 l_{S_i} (\tilde{w}_{k,j}^i)) \nabla l_{T_i} (\tilde{w}_{k,N}^i).$$

Then, recalling $w_{k,j}^i = \tilde{w}_{k,j}^i$ for $j = 0, ..., N$ and conditioning on $w_k$, we have

$$\mathbb{E}\hat{G}_i(w_k) = \mathbb{E}_{i \sim p(T)} \left( \mathbb{E} \left( \prod_{j=0}^{N-1} (I - \alpha H_{k,j}) \nabla l_{T_i} (w_{k,N}^i) | i \right) \right)$$

$$= \mathbb{E}_{i \sim p(T)} \left( \prod_{j=0}^{N-1} \mathbb{E}_{w_{k,j}^i, \eta \in U_{k,j}^i} \left( I - \alpha H_{k,j}^i | i \right) \nabla l_{T_i} (w_{k,N}^i) \right)$$

$$(i) = \mathbb{E}_{i \sim p(T)} \left( \prod_{j=0}^{N-1} \left( I - \alpha (\nabla^2 l_{S_i} (w_{k,j}^i) + e_{k,j}^i) \right) \nabla l_{T_i} (w_{k,N}^i) \right), \quad \|e_{k,j}^i\| \leq \rho \delta (d + 3)^{3/2}, \quad (128)$$

where (i) follows from item 1 in Proposition 11. Then, based on (128) and the definition of $\nabla L(w_k)$, and conditioning on $w_k$, we have

$$\|\mathbb{E}\hat{G}_i(w_k) - \nabla L(w_k)\|$$

$$\leq \mathbb{E}_{i \sim p(T)} \left\| \prod_{j=0}^{N-1} \left( I - \alpha (\nabla^2 l_{S_i} (w_{k,j}^i) + e_{k,j}^i) \right) \nabla l_{T_i} (w_{k,N}^i) - \prod_{j=0}^{N-1} \left( I - \alpha \nabla^2 l_{S_i} (w_{k,j}^i) \right) \nabla l_{T_i} (w_{k,N}^i) \right\|$$

$$\leq \mathbb{E}_{i \sim p(T)} \left\| \prod_{j=0}^{N-1} \left( I - \alpha (\nabla^2 l_{S_i} (w_{k,j}^i) + e_{k,j}^i) \right) - \prod_{j=0}^{N-1} \left( I - \alpha \nabla^2 l_{S_i} (w_{k,j}^i) \right) \right\| \nabla l_{T_i} (w_{k,N}^i) \right\|$$

$$\leq \mathbb{E}_{i \sim p(T)} \left\| \prod_{j=0}^{N-1} \left( I - \alpha (\nabla^2 l_{S_i} (w_{k,j}^i) + e_{k,j}^i) \right) - \prod_{j=0}^{N-1} \left( I - \alpha \nabla^2 l_{S_i} (w_{k,j}^i) \right) \right\| \nabla l_{T_i} (w_{k,N}^i) \right\|, \quad (129)$$

where the first inequality follows from Jensen’s inequality. Using an approach similar to (84) and conditioning on $w_k$, we have

$$\|\nabla l_{T_i} (w_{k,N}^i)\| \leq (1 + \alpha L)^N \|\nabla l_{T_i} (w_k)\| + ((1 + \alpha L)^N - 1) b_i, \quad (130)$$

where $b_i$ is given in Assumption 5. We next upper-bound the term $R(N)$ in (129). To simplify notations, we define a more general quantity $R(t) = \left\| \prod_{j=0}^{t-1} \left( I - \alpha (\nabla^2 l_{S_i} (w_{k,j}^i) + e_{k,j}^i) \right) - \prod_{j=0}^{t-1} \left( I - \alpha \nabla^2 l_{S_i} (w_{k,j}^i) \right) \right\|$. Then,
conditioning on \( w_k \), and using the triangle inequality, we have

\[
R(t) \leq \left\| \prod_{j=0}^{t-1} (I - \alpha \nabla^2 l_{S_j} (w_{k,j}^i)) - \prod_{j=0}^{t-2} (I - \alpha (\nabla^2 l_{S_j} (w_{k,j}^i)) (I - \alpha (\nabla^2 l_{S_j} (w_{k,t-1}^i) + e_{k,t-1}^i))) \right\|
\]

\[
+ \left\| \prod_{j=0}^{t-2} (I - \alpha (\nabla^2 l_{S_j} (w_{k,j}^i)) (I - \alpha (\nabla^2 l_{S_j} (w_{k,t-1}^i) + e_{k,t-1}^i)) - \prod_{j=0}^{t-1} (I - \alpha (\nabla^2 l_{S_j} (w_{k,j}^i) + e_{k,j}^i))) \right\|
\]

\[(i) \leq (1 + \alpha L)^{t-1} \alpha \rho \delta (d + 3)^{3/2} + (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2}) R(t - 1)
\]

\[
\leq (1 + \alpha L)^{N-1} \alpha \rho \delta (d + 3)^{3/2} + (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2}) R(t - 1)
\]

where (i) follows from \( \| e_{k,j}^i \| \leq \rho \delta (d + 3)^{3/2} \). Rearranging the above inequality yields

\[
R(t) + \frac{(1 + \alpha L)^{N-1} \rho \delta (d + 3)^{3/2}}{L + \rho \delta (d + 3)^{3/2}} \leq (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2}) \left( R(t - 1) + \frac{(1 + \alpha L)^{N-1} \rho \delta (d + 3)^{3/2}}{L + \rho \delta (d + 3)^{3/2}} \right)
\]

Telescoping the above inequality over \( t \) from 2 to \( N \) and conditioning on \( w_k \), we have

\[
R(N) + \frac{(1 + \alpha L)^{N-1} \rho \delta (d + 3)^{3/2}}{L + \rho \delta (d + 3)^{3/2}} \leq (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1} \left( R(1) + \frac{(1 + \alpha L)^{N-1} \rho \delta (d + 3)^{3/2}}{L + \rho \delta (d + 3)^{3/2}} \right),
\]

which, in conjunction with \( R(1) = \| \alpha e_{k,0}^i \| \leq \alpha \rho \delta (d + 3)^{3/2} \), yields

\[
R(N) \leq (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1} \alpha \rho \delta (d + 3)^{3/2} +
\]

\[
+ \left( (1 + \alpha L + \alpha \rho \delta (d + 3)^{3/2})^{N-1} - 1 \right) \frac{(1 + \alpha L)^{N-1} \rho \delta (d + 3)^{3/2}}{L + \rho \delta (d + 3)^{3/2}}.
\]

Let \( C_\delta \) be the constant at the right side of the above inequality. Then, combining the above inequality, (129) and (130), and conditioning on \( w_k \), we have

\[
\| \mathbb{E}_{i \sim p(T)} \nabla l_{T_i}(w_k) \| \leq C_\delta (1 + \alpha L)^N \mathbb{E}_{i \sim p(T)} \| \nabla l_{T_i}(w_k) \| + C_\delta (1 + \alpha L)^N - 1) b
\]

\[
\leq C_\delta (1 + \alpha L)^N \left( \| \nabla l_{T_i}(w_k) \| + \sigma \right) + C_\delta (1 + \alpha L)^N - 1) b
\]

\[
= C_\delta (1 + \alpha L)^N \| \nabla l_{T_i}(w_k) \| + C_\delta (1 + \alpha L)^N \sigma + C_\delta (1 + \alpha L)^N - 1) b
\]

\[(i) \leq C_\delta (1 + \alpha L)^N \left( \frac{1}{C_1} \| \nabla l(w_k) \| + \frac{C_2}{C_1} \right) + C_\delta (1 + \alpha L)^N \sigma + C_\delta (1 + \alpha L)^N - 1) b
\]

\[
\leq C_\delta (1 + \alpha L)^N \left( \frac{1}{2} \| \nabla l(w_k) \| + \frac{1}{2 - (1 + \alpha L)^{2N}} C_\delta (1 + \alpha L)^N (\sigma + b)
\]

where \( b := \mathbb{E}_{i \sim p(T)} b_i \), (i) follows from Lemma 9, and the last inequality follows from

\[
C_2 = ((1 + \alpha L)^{2N} - 1) \sigma + (1 + \alpha L)^N ((1 + \alpha L)^N - 1) b < ((1 + \alpha L)^{2N} - 1) (\sigma + b).
\]

Then, the proof is complete.

\[\square\]

**Proof of Proposition 13.** Recalling

\[
\hat{G}_i(w_k) = \prod_{j=0}^{N-1} (I - \alpha H_{b,j}) \nabla l_{T_i}(w_{k,N})
\]

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and conditioning on \( w_k \) and \( i \), we have

\[
\mathbb{E}\| \hat{G}_i(w_k) \|^2 = \mathbb{E}\left\| \prod_{j=0}^{N-1} (I - \alpha H^i_{k,j}) \nabla l_{T_i}(w^i_{k,N}) \right\|^2 \\
\leq \mathbb{E}\left( \prod_{j=0}^{N-1} \left\| (I - \alpha H^i_{k,j}) \right\|^2 \| \nabla l_{T_i}(w^i_{k,N}) \|^2 \right)^{(i)} \\
\leq \prod_{j=0}^{N-1} \left( \mathbb{E}H^i_{k,j} \right)^2 \left( 2(1 + \alpha L)^2N \| \nabla l_{T_i}(w_k) \|^2 + 2((1 + \alpha L)^N - 1)\tilde{b}_i^2 \right) \tag{131}
\]

where (i) follows from (84) and the inequality \( \|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2 \). Then, similarly to (117), using Proposition 11 and conditioning on \( w_k \) and \( i \), we have

\[
\mathbb{E}H^i_{k,j} \left( I - \alpha H^i_{k,j} \right)^2 \leq 7\alpha^2 \rho^2 \delta^2 (d + 6)^3 + 3\alpha^2 L^2 \max\{ \lambda, \lambda^2 \} + (1 + \alpha L)^2 + 2(1 + \alpha L)\alpha \rho \delta (d + 3)^{3/2}. \tag{132}
\]

Let \( C_\lambda \) be the constant at the right side of the above inequality. Then, combining (131) and (132), conditioning on \( w_k \), and taking the expectation over \( i \), we have

\[
\mathbb{E}\| \hat{G}_i(w_k) \|^2 \leq 2(1 + \alpha L)^2N C_\lambda \mathbb{E}\| \nabla l_{T_i}(w_k) \|^2 + 2C_\lambda ((1 + \alpha L)^N - 1)\mathbb{E}\tilde{b}_i^2 \\
\leq 2(1 + \alpha L)^2N C_\lambda \| \nabla l_{T_k}(w_k) \|^2 + 2C_\lambda ((1 + \alpha L)^N - 1)\tilde{b} \\
\leq 2(1 + \alpha L)^2N C_\lambda \left( \frac{2}{C_1^2} \| \nabla L(w_k) \|^2 + \frac{2C_2^2}{C_1^2} + \sigma^2 \right) + 2C_\lambda ((1 + \alpha L)^N - 1)\tilde{b} \\
\leq \frac{4(1 + \alpha L)^2N C_\lambda}{(2 - (1 + \alpha L)^2N)^2} \| \nabla L(w_k) \|^2 + \frac{4(1 + \alpha L)^2N C_\lambda ((1 + \alpha L)^N - 1)\tilde{b}}{(2 - (1 + \alpha L)^2N)^2} (\sigma + b)^2 \\
\leq 2C_\lambda (1 + \alpha L)^2N (\sigma^2 + \tilde{b}), \tag{133}
\]

where \( \tilde{b} := \mathbb{E}_i \tilde{b}_i^2 \), \( b = \mathbb{E}_i \tilde{b}_i \), (i) follows from Lemma 9, and the definitions of \( C_1 \) and \( C_2 \) are given in (76), and the last inequality follows from the fact that

\[
C_2 = ((1 + \alpha L)^2N - 1)\sigma + (1 + \alpha L)^N ((1 + \alpha L)^N - 1)b < ((1 + \alpha L)^2N - 1)(\sigma + b).
\]

Then, the proof is complete. \( \square \)

Based on the propositions established above, we now prove the convergence for GGS-MAML in the finite-sum case.

**Proof of Theorem 4.** Based on the smoothness of \( \nabla L(\cdot) \) established in Proposition 5, we have

\[
L(w_{k+1}) \leq L(w_k) + \langle \nabla L(w_k), w_{k+1} - w_k \rangle + \frac{L_{\text{wk}}}{2} \| w_{k+1} - w_k \|^2 \\
\leq L(w_k) - \beta_k \langle \nabla L(w_k), \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \rangle + \frac{L_{\text{wk}} \beta_k^2}{2} \| \frac{1}{B} \sum_{i \in B_k} \hat{G}_i(w_k) \|^2. 
\]

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Taking the conditional expectation given \( w_k \) over the above inequality and conditioning on \( w_k \), we have

\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) - \mathbb{E}(\beta_k|w_k)\|\nabla \mathcal{L}(w_k)\|^2 + \mathbb{E}(\beta_k|w_k)\left(\nabla \mathcal{L}(w_k), \nabla \mathcal{L}(w_k) - \mathbb{E}\tilde{G}_i(w_k)\right) \\
+ \frac{L_{wk}}{2}\mathbb{E}(\beta_k^2|w_k)\mathbb{E}\left(\| \frac{1}{B} \sum_{i \in B_k} \tilde{G}_i(w_k) \| w_k \right) \\
\leq \mathcal{L}(w_k) - \frac{1}{2}\mathbb{E}(\beta_k|w_k)\|\nabla \mathcal{L}(w_k)\|^2 + \frac{1}{2}\mathbb{E}(\beta_k|w_k)\|\nabla \mathcal{L}(w_k) - \mathbb{E}\tilde{G}_i(w_k)\|^2 \\
+ \frac{L_{wk}}{2}\mathbb{E}(\beta_k^2|w_k)\mathbb{E}\left(\| \frac{1}{B} \sum_{i \in B_k} \tilde{G}_i(w_k) \| w_k \right) \\
\overset{(i)}{\leq} \mathcal{L}(w_k) - \frac{1}{2C_\beta^2}\mathbb{E}\left(\frac{1}{L_{wk}}|w_k\right)\|\nabla \mathcal{L}(w_k)\|^2 + \frac{1}{2C_\beta^2}\mathbb{E}\left(\frac{1}{L_{wk}}|w_k\right)(2C_{e_1}^2\|\nabla \mathcal{L}(w_k)\|^2 + 2C_{e_2}^2) \\
+ \frac{L_{wk}}{2C_\beta^2}\mathbb{E}\left(\frac{1}{L_{wk}}|w_k\right)\mathbb{E}\left(\left\| \frac{1}{B} \sum_{i \in B_k} \tilde{G}_i(w_k) \right\| w_k \right),
\]

(134)

where (i) follows from Proposition 12.

Then, using an approach similar to (88) and conditioning on \( w_k \), we have

\[
\mathbb{E}\left(\frac{1}{L_{wk}}|w_k\right) \geq \frac{1}{L_{wk}}, \quad \mathbb{E}\left(\frac{1}{L_{wk}^2}|w_k\right) \leq \frac{2}{L_{wk}^2},
\]

where the last inequality follows from Proposition 12. Then, combining (134), (135), using Lemma 10 that

\[
\mathbb{E}\left(\frac{1}{L_{wk}}|w_k\right) \geq \frac{1}{L_{wk}}, \quad \mathbb{E}\left(\frac{1}{L_{wk}^2}|w_k\right) \leq \frac{2}{L_{wk}^2},
\]

and conditioning on \( w_k \), we have

\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) - \frac{1}{2C_\beta L_{wk}}(1 - 2C_{e_1}^2)\|\nabla \mathcal{L}(w_k)\|^2 + \frac{1}{2C_\beta L_{wk}}\mathbb{E}\left(\frac{1}{L_{wk}}|w_k\right)2C_{e_2}^2 \\
+ \frac{L_{wk}}{2C_\beta^2 L_{wk}^2}\left(\frac{C_{s_1}}{B} + 2 + 4C_{e_1}^2\right)\|\nabla \mathcal{L}(w_k)\|^2 + \frac{C_{s_2}}{B} + 4C_{e_2}^2,
\]

which, in conjunction with (23) that \( \hat{L}_{wk}, L_{wk} > L \), yields

\[
\mathbb{E}(\mathcal{L}(w_{k+1})|w_k) \leq \mathcal{L}(w_k) - \frac{1}{2C_\beta L_{wk}}\left(1 - 2C_{e_1}^2 - \frac{2}{C_\beta}\left(\frac{C_{s_1}}{B} + 2 + 4C_{e_1}^2\right)\right)\|\nabla \mathcal{L}(w_k)\|^2 + \frac{C_{s_2}}{C_\beta L} \\
+ \frac{1}{C_\beta L}\left(\frac{C_{s_2}}{B} + 4C_{e_2}^2\right).
\]

(136)

Recall that

\[
L_{wk} := (1 + \alpha L)^N L + C_b + C_L\mathbb{E}_{i \sim \pi(T)} \|\nabla l_{T_i}(w_k)\|,
\]

where \( C_b \) and \( C_L \) are given by (22).

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Then, using an approach similar to (90) and (91) yields

$$
\mathbb{E}(L(w_{k+1})|w_k) \leq L(w_k) - \frac{C_1}{C_\epsilon} \frac{1}{\epsilon_\theta} \left( 1 - 2C_\epsilon^2 - \frac{2}{\epsilon_\theta} \left( \frac{C_\epsilon}{B} + 2 + 4C_\epsilon^2 \right) \right) \|\nabla L(w_k)\|^2
+ \frac{1}{C_\beta^2} \left( \frac{C_\epsilon^2}{B} + 4C_\epsilon^2 \right). 
$$

(137)

To simplify notations, define

$$
\theta = \frac{C_1}{C_\epsilon} \frac{1}{C_\beta} \left( \frac{1}{2} - C_\epsilon^2 - \frac{1}{C_\beta} \left( \frac{C_\epsilon}{B} + 2 + 4C_\epsilon^2 \right) \right),
\phi = \frac{1}{C_\beta^2} \left( \frac{C_\epsilon^2}{B} + 4C_\epsilon^2 \right),
\xi = \frac{C_1}{C_\epsilon} \left( 1 + \alpha L \right) \|

(1 + \alpha L)^{2N} L + b C_1 C_\beta \frac{1}{C_\epsilon} \left( 1 + \alpha L \right) \|

(1 + \alpha L)^{2N} (1 + \alpha L)^{2N} - 1) \frac{1}{B} + \|\nabla L(w_k)\|

which, in conjunction with (137), yields

$$
\mathbb{E}(L(w_{k+1})|w_k) \leq L(w_k) - \frac{\theta}{\xi} \|\nabla L(w_k)\|^2 + \phi.
$$

Taking the expectation over $w_k$ in the above inequality yields

$$
\mathbb{E}(L(w_{k+1})) \leq \mathbb{E}L(w_k) - \mathbb{E} \frac{\theta}{\xi} \|\nabla L(w_k)\|^2 + \phi.
$$

(138)

Then, using an approach similar to (93), we have

$$
\mathbb{E}\|\nabla L(w_\zeta)\| \leq \frac{\Delta}{\theta K} + \frac{\phi}{\theta K} + \sqrt{\xi \left( \frac{\Delta}{\theta K} + \frac{\phi}{\theta} \right) + \left( \frac{\Delta}{\theta K} + \frac{\phi}{\theta} \right)^2}
$$

$$
\leq \frac{\Delta}{\theta K} + \frac{\phi}{\theta} + \sqrt{\xi \left( \frac{\Delta}{\theta K} + \frac{\phi}{\theta} \right)}
$$

(139)

where the last inequality follows from $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$. Combining the above inequality with

$$
C_1 = 2 - (1 + \alpha L)^{2N}
$$

finishes the proof.

**Proof of Corollary 3.** Since $\alpha < \frac{1}{8NL}$, $\rho \delta (d + 3)^{3/2} < \frac{L}{20}$, $\alpha^2 L^2 \max\{\lambda, \lambda^2\} < \frac{1}{64}$ and $B > 3$, we obtain

$$
C_\delta < \frac{\rho}{L} \delta (d + 3)^{3/2},
C_\epsilon < \frac{5\rho}{2L} \delta (d + 3)^{3/2} < \frac{1}{8},
C_\epsilon < \frac{5\rho}{2L} (\sigma + b) \delta (d + 3)^{3/2}
C_\lambda < \frac{7}{64} + \frac{3}{64} + \frac{5}{4} + \frac{18}{64} < 2,
C_s < 48, C_{s_2} < 12(\sigma + b)^2 + 6(\sigma^2 + \tilde{b}),
$$

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which, in conjunction with (37) and \( \frac{\rho}{16L} < C_L < \frac{5\rho}{8L} \), yields

\[
\theta \geq \frac{L}{800\rho},
\]

\[
\phi \leq \frac{6(\sigma + b)^2 + 3(\sigma^2 + \tilde{b})}{3200L} \frac{1}{B} + \frac{1}{1600L} \frac{25\rho^2}{4L^2} (\sigma + b)^2 (d+3)^3 \leq \mathcal{O}\left((\sigma^2 + \tilde{b})\left(\frac{1}{B} + \delta^2 d^3\right)\right),
\]

\[
\xi \leq \mathcal{O}(1),
\]

which, in conjunction with (38), yields

\[
E\|\nabla \mathcal{L}(w_\zeta)\| \leq \mathcal{O}\left(\frac{1}{K} + (\sigma^2 + 1)\left(\frac{1}{B} + \delta^2 d^3\right) + \sqrt{\frac{1}{K} + (\sigma^2 + 1)\left(\frac{1}{B} + \delta^2 d^3\right)}\right). \tag{140}
\]

The remaining proof is similar to that of Corollary 2, and is thus omitted. \( \square \)