NULL-GEODESICS IN COMPLEX CONFORMAL MANIFOLDS AND THE LEBRUN CORRESPONDENCE

FLORIN ALEXANDRU BELGUN

ABSTRACT. In the complex-Riemannian framework we show that a conformal manifold containing a compact, simply-connected, null-geodesic is conformally flat. In dimension 3 we use the LeBrun correspondence, that views a conformal 3-manifold as the conformal infinity of a selfdual four-manifolds. We also find a relation between the conformal invariants of the conformal infinity and its ambient.

1. Introduction

On a complex manifold, the existence of a complex-Riemannian metric implies, in general, strong topological assumptions, especially if the manifold is compact (e.g. the — square of the — canonical bundle has to be trivial). However, any analytic (pseudo-) Riemannian (or conformal) manifold can be complexified, and a natural question is to see to what extent the global properties of the real manifold (e.g. existence of closed (null-) geodesics) hold for the complexified spaces. This complexification procedure naturally occurs in twistor theory (see below), which has been intensively studied for Riemannian space-times (see, e.g., [1], [3], [10], [15]); the complex-Riemannian setting, in which historically the twistor theory was first introduced [12], can provide a link to the Lorentzian geometry.

In complex conformal geometry (which implies weaker assumptions on the topology of the manifold), the conformal structure is determined by the set of null-geodesics, which can be organized as a complex manifold under some topological conditions [8], [11]. A natural question is which complex conformal manifolds admit compact null-geodesics; for example, if a self-dual manifold admits a globally-defined twistor space, then application of a twistorial interpretation of the Weyl tensor [2], implies that it is conformally flat, and the compact null-geodesic is simply-connected.

Our main result (section 4, Theorem 1) states that, if a conformal complex n-manifold admits a rational curve as a null-geodesic, then it is conformally flat (see also [7] for the case of a complex projective manifold). The proof uses the properties of Jacobi fields along the considered compact, simply-connected, null-geodesic : namely, we compute the normal bundle of a compact, simply-connected, null-geodesic, and we show that the small deformations of the latter as a compact curve, or as a null-geodesic, coincide (section 4, Proposition 3). In addition to that, we use, for the (more difficult) case of dimension 3, a criterion for conformal flatness from [3], and...
we apply it to a \textit{locally defined}, by the \textit{LeBrun correspondence} (see below), self-dual ambient.

The other topic of this paper uses implicitly another application of twistor theory: It has been shown by LeBrun \cite{LeBrun1}, \cite{LeBrun2}, that any conformal 3-manifold can be locally realized as the \textit{conformal infinity} of a self-dual Einstein (with non-zero scalar curvature) 4-manifold. We have, thus, a local correspondence assigning to a conformal structure in dimension 3 a self-dual Einstein metric in dimension 4, which we call the \textit{LeBrun correspondence}.

As conformal structures of both manifolds are encoded in the complex, resp. \textit{CR}, structure of their twistor spaces, they are implicitly related, for example if the 3-manifold $M$ is conformally flat, its ambient $N$ equally is. It is, however, difficult to obtain an explicit relation between the conformal invariants of $M$ and those of $N$ by twistorial methods, as there is no simple expression of the \textit{Cotton-York tensor} of $M^3$ in twistorial terms, and the twistorial interpretation of the \textit{Weyl tensor} of $N^4$ is highly non-linear \cite{LeBrun2}.

In this paper we find a relation between these two conformal invariants of the manifolds involved in the LeBrun correspondence, or, more generally, of an umbilic submanifold $M^3$ and of its self-dual ambient $N^4$. It appears that the Weyl tensor of $N^4$ identically vanishes along $M^3$, and thus the Cotton-York tensor of $N^4$, restricted to $M^3$, is conformally invariant and can be identified with the Cotton-York tensor of $M^3$; in this case, it is also equal to the normal derivative of the Weyl tensor of $N^4$ (section 3, Theorem \ref{thm:main}). This gives conditions for an open self-dual 4-manifold to admit a conformal infinity.

The paper is organized as follows: in section 2 we recall a few basic facts about complex- Riemannian and -conformal geometry, in section 3 we relate the conformal invariants of a 3-dimensional conformal infinity to those of its self-dual ambient (arising from the LeBrun correspondence), and in section 4 we state our results about conformal complex manifolds containing compact null-geodesics.

Throughout the paper we use the following conventions: in complex-Riemannian (or -conformal) geometry we use the same terminology as in the real framework (metric, Levi-Civita connection, curvature), and the holomorphic bundles are denoted like the corresponding bundles in real geometry (for example, the holomorphic tangent bundle of $M$ is denoted simply by $TM$, rather than the more precise $T^{1,0}M$); manifolds with holomorphic conformal structures are denoted by bold-face letters (except in section 3, where the results hold also in the real framework).

\section{2. Holomorphic conformal geometry}

\textbf{Definition 1.} Let $M$ be a complex manifold, let $n$ be its complex dimension. A complex-Riemannian metric $g$ on it is a \textit{holomorphic section} of $S^2T^*M$ which is non-degenerate at any point. A holomorphic conformal structure on $M$ is a holomorphic line subbundle $C$ in $S^2T^*M$ such that any non-vanishing local section of $C$ is a local complex-Riemannian metric.
During the rest of this section, and of the whole fourth section of this paper, we shall simply denote these structures as *metric* and *conformal structure* (therefore omitting any reference to the complex framework).

A metric on $TM$ induces metrics (i.e. non-degenerate symmetric bilinear forms) on all tensor bundles, in particular the square of the canonical bundle $\kappa := \Lambda^n T^* M$ is trivialized. If we can choose an *orientation* (defined as follows), then the canonical bundle itself can be trivialized by a *volume form* of norm 1. There are exactly 2 such forms in each fiber of $\Lambda^n T^* M$, and an *orientation* is the choice of one of them, depending continuously (thus holomorphically) on the base point.

**Remark.** The notion of *orientation* is generally related to the reduction (when possible) of the structure group of the frame bundle from $G$ to the connected component of the identity $G_0$; in absence of any structure, the group $G$ is simply the connected $GL(n, \mathbb{C})$, so the notion of orientation has no meaning in “raw” complex geometry. But a Riemannian metric on $M$ is equivalent to the reduction of its frame bundle to a $O(n, \mathbb{C})$-bundle, where $O(n, \mathbb{C}) := \{ A \in GL(n, \mathbb{C}) | A^t A = 1 \}$; a further choice of an orientation reduces the structure group of the frame bundle to the connected component of this group, containing the identity: $SO(n, \mathbb{C}) := O(n, \mathbb{C}) \cap SL(n, \mathbb{C})$.

Unlike in the real framework, these reductions, always possible on (small) contractible open sets, are submitted to some topological constraints if we want to define them globally on $M$.

Weaker constraints are implied by the existence of a conformal structure: the square of the canonical bundle needs to admit a $n$th order root $L^{-2} \simeq C$, where $L := \kappa^{-1/n}$ is the weight 1–density bundle of the manifold $M$. We can study the conformal structure $C$ using the formalism of density bundles and of Weyl derivatives [4]. From now on, we shall not make use of the weight 1–density bundle, but only of $C = L^{-2}$, which is enough to define the conformal structure.

**Remark.** A conformal structure is equivalent to the reduction of the structure group of the frame bundle to $CO(n, \mathbb{C}) := O(n, \mathbb{C}) \times \mathbb{C}^*/\{\pm 1\}$ (the quotient is due to the fact that $-1 \in O(n, \mathbb{C})$). This group is disconnected if $n$ is even, and the connected component of 1 is $CO_0(n, \mathbb{C}) = SO(n, \mathbb{C}) \times \mathbb{C}^*/\{\pm 1\}$, but it is connected if $n$ is odd (and in this case the right hand side of the previous equality coincides with $CO(n, \mathbb{C})$). Therefore, although a complex-Riemannian 3-manifold admits, locally, 2 possible orientations, they are conformally equivalent, fact that makes impossible a canonical way to associate an orientation to a metric in the conformal class.

**Remark.** In complex-, as in real-Riemannian geometry, the orientation determines (and is determined by) a family of compatible oriented orthonormal basis in $TM$; if dim $M$ is even, by multiplying all the vectors of such a basis by a non-zero complex number we obtain an oriented orthonormal basis compatible with another metric in the conformal class (if dim $M$ is odd, multiplication by $-1$ yields a basis compatible with the same metric, but with the opposite orientation).

\footnote{the dual of $L$ is a square root of $C$; the choice of such a square root is implied, if $n = 2m + 1$, by the conformal structure $C$ as $\kappa = C^m \otimes L^{-1}$; if $n$ is even, neither $C$ nor an orientation — see below — imply the choice of $L$.}
The notion of orientation, in four-dimensional conformal geometry, is important for the definition of anti-, resp. self-duality (see below). More generally, the Hodge $*$ operator (defined as in real Riemannian geometry) is conformally invariant, and gives an explicit expression for the splitting of the bundle of 2-forms and of the curvature tensor.\[2\], see also next section.

Maybe the simplest way to view a conformal structure is as an equivalence class $c$ of local metrics, two such local representants $g$ and $h$ satisfying, on the open set where both are defined, to $g = fh$, with $f$ a non-vanishing holomorphic function. Unlike in the real framework, global representants may not exist in general. From now on we shall consider a conformal structure on $M$ as being given by the conformal class $c$ rather than by the line bundle $C$.

Geometrically, a conformal structure is given by its isotropy cone $C \subset TM$ of vectors of norm 0. Because of the non-degeneracy of any local metric in $c$, the projective isotropy cone $\mathbb{P}(C)$ is a non-degenerate hyperquadric in $\mathbb{P}(TM)$. In dimension 3, $\mathbb{P}(C)$ is a conic (curve) in $\mathbb{CP}^2$, and in dimension 4 it is a ruled surface in $\mathbb{CP}^3$; therefore, in this latter case, there are 2 families — each of which can be characterized with respect to a given orientation — of isotropic planes in $TM$ called $\alpha$-, resp. $\beta$-planes.

For any local metric we have a Levi-Civita connection, whose curvature has the same components as in the real Riemannian geometry (see next section). In particular, the Weyl tensor is independent of the metric in the conformal class.

The geodesics for which the tangent direction at a point (and thus, at any point) is isotropic are called null-geodesics, and they are locally independent (up to a reparametrization) of the metric in the conformal class. The same is true for higher-dimensional totally geodesic and isotropic submanifolds — if they exist —, called null-submanifolds. In dimension 4 they are $\alpha$-, resp. $\beta$-surfaces (tangent to $\alpha$-, resp. $\beta$-planes, see above), and they exist if and only if the (oriented) conformal structure is anti-, resp. self-dual, i.e. the component $W^+$, resp. $W^-$, of the Weyl tensor $W$ of $(M, c)$ vanishes identically \[2\],\[3\],\[4\].

If the manifold is self-dual, one considers locally the twistor space $Z$ of $(M^4, c)$ as the set of $\beta$-surfaces. It is a 3-dimensional complex manifold containing rational curves whose normal bundle is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (called twistor lines) (where $\mathcal{O}(1)$ is the dual of the tautological bundle $\mathcal{O}(-1)$ of $\mathbb{CP}^1$). If, in addition, we can choose an Einstein metric $g$ in the conformal class $c$, we get an extra structure on $Z$, namely a distribution of 2-planes, which is totally integrable (and yields a foliation) if the scalar curvature of $g$ vanishes, otherwise it is a contact structure \[5\],\[6\],\[8\]. Conversely, from a manifold $Z$ containing twistor lines as above (called a twistor space), plus — possibly — the additional 2-planes distribution, one can recover — at least locally —, via the reverse Penrose construction \[1\], the self-dual manifold $(M^4, c)$.

\[2\] see \[8\], \[11\], \[2\] and section 4 for an explanation of the difficulties of a global definition of the twistor space.
In all generality, one can always consider locally (on a geodesically convex open set, for example, see section 4) the space of null-geodesics of a conformal manifold \((M^n, c)\), and the key point in the LeBrun correspondence (defined below) is that the space of null-geodesics of a 3-dimensional conformal manifold \((M^3, c)\) (also called the twistor space of \(M\)) is a twistor space endowed with a contact structure, therefore we get (locally again) a self-dual manifold \((N^4, c)\), in which \((M^3, c)\) is umbilic, and it is the conformal infinity of an Einstein metric \(g\) on \(N\) with non-zero scalar curvature [8], [9].

**Definition 2.** Let \((M, c)\) be a conformal 3-manifold, that we shall suppose civilized (e.g. geodesically connected for some metric in the conformal class). The LeBrun correspondence associates to \(M\) the (germ-unique) self-dual Einstein 4-manifold \(N\) such that the twistor spaces of \(M\) and \(N\) coincide.

**Proposition 1.** [8], [9] In the LeBrun correspondence, \((M^3, c)\) is an umbilic hypersurface of \((N^4, c)\) (and has the induced conformal structure) and the Einstein metric of \(N^4\) has a second order pole at \(M^3\) (conformal infinity). Conversely, in such a geometric setting, the twistor spaces of the manifolds \(M\) and \(N\) coincide.

**Remark.** There is no a priori definition of a conformal infinity of an open (real- or complex-) Riemannian manifold \(X\), even if the metric is complete (in the real framework). Here we consider uniquely the case when this infinity is an (umbilic) submanifold (or boundary) of a conformal extension of \(X\), \(\bar{X} \supset X\); the conformal structure extends smoothly beyond the infinity. In other cases, in which the notion of conformal infinity can still be defined, the conformal structure is singular at infinity, which, in these cases, is no longer conformal, but admits instead a \(CR\) [6], [3] or a quaternionic contact structure [3].

3. **Conformal infinity of a self-dual manifold**

The object of this section is to find a relationship between the conformal invariants of a conformal infinity and of its self-dual ambient arising from the LeBrun correspondence. The results are local, and they hold in the complex as well as in the real Riemannian or in the signature (2,2) pseudo-Riemannian framework. We begin by recalling a few facts about the conformally invariant tensors in Riemannian geometry.

For a \(n\)-dimensional Riemannian manifold \((M^n, g)\), the curvature has the following expression:

\[
R^M(X, Y) = (h \wedge I)(X, Y) + W(X, Y), \quad \text{where}
\]

\[
(h \wedge I)(X, Y) := h(X, \cdot) \wedge Y - h(Y, \cdot) \wedge X, \quad \forall X, Y \in TM,
\]

is the suspension by the identity \(I\) of the normalized Ricci tensor

\[
h = \frac{1}{2n(n-1)} \text{Scal}^g \cdot g + \frac{1}{n-2} \text{Ric}_0;
\]

\(\text{Scal}\) and \(\text{Ric}_0\) are the scalar curvature, resp. the trace-free Ricci tensor, and, together with the Weyl tensor \(W\), they are the irreducible components of the curvature under the orthogonal group if \(n \geq 5\). If \(n = 3\), \(W\) vanishes identically, and if \(n = 4\) it further decomposes in two irreducible components.
$W^+$, resp. $W^-$, called the self-dual, resp. anti-self-dual (or positive, resp. negative) Weyl tensor.

The Weyl tensor, viewed as a section in $\text{Hom}(\Lambda^2 TM \otimes TM, TM)$ (as a $(3,1)$–tensor), is conformally invariant, and, if $n \geq 4$, it completely determines, locally, the conformal structure of $(M,[g])$ (for a self-dual manifold, $W^- \equiv 0$, thus the Weyl tensor actually coincides with $W^+$). In dimension $n = 3$, this function is fulfilled by the Cotton-York tensor, which can be defined in all dimensions by

\begin{align}
C(X,Y)(Z) &:= (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \ \forall X,Y,Z \in TM,
\end{align}

and it can be shown that, for another metric $g' := e^{2\varphi}g$ in the same conformal class, the corresponding Cotton-York tensor $C'$ is related to $C$ by the formula

\begin{align}
C'(X,Y)(Z) &= C(X,Y)(Z) - d\varphi(W(X,Y)Z).
\end{align}

In particular $C$ is conformally invariant along the zero set of $W$, thus everywhere if $\dim M = 3$.

**Remark.** The Cotton-York tensor $C$ of $M$ is a 2-form with values in $T^* M$, and it satisfies a first Bianchi identity, as $h$ is a symmetric tensor, and also a contracted (second) Bianchi identity, coming from the second Bianchi identity in Riemannian geometry, \cite{4}:

\begin{align}
\sum C(X,Y)(Z) &= 0 \text{ circular sum}; \ \ \ (6) \\
\sum C(X,e_i)(e_i) &= 0 \text{ trace over an orthonormal basis}. \ (7)
\end{align}

This means that $C$ is an irreducible tensor if $n = 3$ or $n > 4$, and, if $n = 4$, $C$ has two irreducible components, the self-dual, resp. the anti-self-dual Cotton-York tensor

\begin{align}
C^+ &\in \Lambda^+ M \otimes \Lambda^1 M, \ \text{resp.} \ C^- \in \Lambda^- M \otimes \Lambda^1 M.
\end{align}

They both satisfy (6) and (7) (note that these two relations are equivalent in their case).

The Cotton-York tensor is related to the Weyl tensor of $M$ by the formula \cite{4}:

\begin{align}
\delta W = C, \ (8)
\end{align}

where $\delta : \Lambda^2 M \otimes \Lambda^2 M \to \Lambda^2 M \otimes \Lambda^1 M$ is induced by the codifferential on the second factor, and by the Levi-Civita connection $\nabla$. Then, again if $n = 4$, $C^+$ has to be the component of $\delta W$ in $\Lambda^+ M \otimes \Lambda^1 M$, and we know that the restriction of $W^-$ to $\Lambda^+ M \otimes \Lambda^2 M$ is identically zero. This means that

\begin{align}
\delta W^+ &= C^+, \text{ and also} \ (9) \\
\delta W^- &= C^- \ (10)
\end{align}

We have thus:

**Lemma 1.** On a self-dual manifold, $C^-$ vanishes identically.

We consider now the situation in the LeBrun correspondence: Let $(M,c)$ be a 3-dimensional conformal manifold, and we suppose, without any local loss of generality, that it is the conformal infinity of the self-dual manifold $(N,c)$ (no use will be made of the Einstein metric on it); $M \subset N$ is, thus,
an umbilic hypersurface, such that the restriction of the conformal structure \(c\) of \(N\) to \(M\) is non-degenerate (equivalently, \(TM\) is nowhere tangent to an isotropic cone) and coincides with the conformal structure, still denoted by \(c\), on \(M\).

If we introduce the Hodge operator \(\star^M : \Lambda^2 M \to \Lambda^1 M\), then the curvature tensor \(R^M\) is equivalent to the symmetric 2-tensor \(\star^M \circ R^M \circ \star^M\). A straightforward application of the above formula yields

\[
\star R^M := \star^M \circ R^M \circ \star^M = -h + (\text{tr}h)I.
\]

(11)

For the 4-dimensional manifold \(N\), the components of the Riemannian curvature can also be expressed as eigenspaces of \(\star\)-type operators. Namely, considering \(R := R^N\) as a symmetric endomorphism of \(\Lambda^2 N = \Lambda^+ N \oplus \Lambda^- N\), \(W^+\) is the trace-free component of \(R\) in \(\text{End}(\Lambda^+ N)\), and \(W^-\) is the trace-free component of \(R\) in \(\text{End}(\Lambda^- N)\) \(\llbracket 4 \rrbracket\). (The scalar curvature is four times the trace of \(R|_{\Lambda^+}\) or of \(R|_{\Lambda^-}\), and the trace-free Ricci tensor is identified to the component of \(R\) sending \(\Lambda^+\) into \(\Lambda^-\) \(\llbracket 4 \rrbracket\).)

We can canonically identify \(\Lambda^+ N\) and \(\Lambda^- N\), restricted to \(M \subset N\), to \(\Lambda^2 M\), by:

\[
\begin{align*}
\Lambda^2 M \ni \alpha & \mapsto \alpha + \star^N \alpha \in \Lambda^+ N \\
\Lambda^2 M \ni \alpha & \mapsto \alpha - \star^N \alpha \in \Lambda^- N.
\end{align*}
\]

(12)

Our first result is:

**Theorem 1.** Let \(M\) be an umbilic hypersurface of a self-dual manifold \(N\). Then:

(i) The Weyl tensor of \(N\) vanishes along \(M\):

\[
W^+|_M \equiv 0;
\]

(ii) The Cotton-York tensor of \(M\) is related to the self-dual Weyl tensor of \(N\) by the formula:

\[
g(\nabla_\nu W^+(A), B)_x = -C(A)(\star^M B)_x, \quad \forall x \in M
\]

where \(A, B \in \Lambda^2 T_x M\), \(\nu \perp T_x M\) is unitary for the metric \(g\), and the Hodge operator \(\star^M\) is induced by \(g\) and the orientation on \(M\) admitting \(\nu\) as an exterior normal vector.

(iii) The restriction to \(M\) of the (self-dual) Cotton-York tensor of \(N\) is equal to the Cotton-York tensor of \(M\):

\[
C^+(X, Y)(Z) = C^M(X, Y)(Z), \quad \forall X, Y, Z \in TM.
\]

**Proof.** The claimed identities are conformally invariant: for (i) it is obvious, and the conformal invariance of (iii) follows from (i) and (\(\llbracket 3\rrbracket\)); to see that for (ii), let \(X, Y, Z, \nu\) be a \(g\)-orthonormal oriented basis of \(N\), such that \(X, Y, Z\) is a \(g\)-orthonormal basis on \(M\) giving the orientation as above. Then \(\star^M(Z \wedge X) = Y\), and, if we take \(A := X \wedge Y, B := Z \wedge X\), the identity (ii) becomes

\[
\langle \nabla_\nu W^+(X, Y)Z, X \rangle = -C(X, Y)(Y),
\]

where angle brackets denote the scalar product induced by \(g\).

The tensors \(W^+, C\), in the above form, are independent of the chosen metric \(g\) \(\llbracket 3\rrbracket\), which depends on the normal vector \(\nu\), supposed to be \(g\)-unitary. If \(\nu' := \lambda \nu\), for \(\lambda \in \mathbb{C}^*\), then the corresponding metric \(g' = \lambda^{-2} g\),
and also $\phi^M = \lambda^{-1} \phi^M$, thus the identity (13) for $\nu', g'$ is equivalent to the one for $\nu, g$.

**Remark.** As $W^+$ is the trace-free component of the Riemannian curvature contained in $\text{End}(\Lambda^+ N)$, and is symmetric, it is enough to evaluate it on pairs $A, B \in \Lambda^2 M \simeq \Lambda^+ N$ which are unitary and orthogonal for the metric $g$, therefore the check of the equation (13) will prove the Theorem.

As $W^\pm$ are $\phi^N$-eigenvectors in $\text{End}_0(\Lambda^2 N)$ (the space of trace-free endomorphisms of $\Lambda^2 N$), they are determined by the following formulas, where $X, Y, Z, \nu$ is any oriented orthonormal basis of $\mathbb{T} M$:

\begin{align}
(W^+(X, Y)Z, X) &= \frac{1}{4}((R(X, Y)Z, X) + (R(Z, \nu)Y, \nu) + (R(X, Y)Y, \nu) + (R(Z, \nu)Z, X)) \\
(W^-(X, Y)Z, X) &= \frac{1}{4}((R(X, Y)Z, X) + (R(Z, \nu)Y, \nu) - (R(X, Y)Y, \nu) - (R(Z, \nu)Z, X)),
\end{align}

where $X, Y, Z, \nu$ is supposed to be a local extension, around a region of $M$, of the $g$-orthonormal frame used in (13). As $N$ is self-dual, $W^-$ is identically zero, thus, in the points $x \in M$, we have

\begin{equation}
(W^+(X, Y)Z, X)_x = \frac{1}{2}((R(X, Y)Y, \nu) + (R(Z, X)Z, \nu))_x.
\end{equation}

It is a standard fact that, if $M$ is umbilic, there is a local metric $g$ in the conformal class $c$ of $N$, such that, for $g, M$ is totally geodesic. Without any loss of generality, because of the conformal invariance of the claimed identities (see above), we fix such a metric. Then we have

\begin{equation}
R(X', Y')Z' = R^M(X', Y')Z', \ \forall X', Y', Z' \in \mathbb{T} M,
\end{equation}

which, together with (16), implies that $W^+|_M \equiv 0$, and thus proves the point (i) in the Theorem.

On the other hand, (17), together with (16) and (13), yield

\begin{equation}
(R(X, Y)Z, X)_x + (R(Z, \nu)Y, \nu)_x = 0, \ \forall x \in M.
\end{equation}

Let us compute now the normal derivative of $W^+$ in a point $x \in M$; we suppose that $X, Y, Z, \nu$ are locally extended by an orthonormal frame, and that they are parallel at $x$ (we omit, for simplicity of notation, the point $x$ in the following lines):

$$
\langle \nabla_\nu W^+(X, Y)Z, X \rangle = \frac{1}{2}(\langle \nabla_\nu R(X, Y)Y, \nu \rangle + \langle \nabla_\nu R(Z, X)Z, \nu \rangle),
$$

from (16). This is then equal to:

$$
\langle \nabla_\nu W^+(X, Y)Z, X \rangle = -\frac{1}{2}(\langle \nabla_X R(Y, \nu)Y, \nu \rangle + \langle \nabla_Y R(\nu, X)Y, \nu \rangle + \langle \nabla_Z R(X, \nu)Z, \nu \rangle + \langle \nabla_X R(\nu, Z)Z, \nu \rangle),
$$

from the second Bianchi identity. Then we have

$$
\langle \nabla_\nu W^+(X, Y)Z, X \rangle = \frac{1}{2}(\langle \nabla_X R(Z, X)Z, X \rangle + \langle \nabla_Y R(Z, Y)Z, X \rangle + \langle \nabla_Z R(Y, Z)X, Y \rangle + \langle \nabla_X R(Y, X)X, Y \rangle)
$$
from analogs of (18). Then

$$\langle \nabla_\nu W^+ (X,Y) Z, X \rangle = \frac{1}{2} (\langle \nabla_X R^M (Z, X) Z, X \rangle + \langle \nabla_Y R^M (Z, Y) Z, X \rangle + \langle \nabla_Z R^M (Y, X) X, Y \rangle)$$

from (17)

$$\langle \nabla_\nu W^+ (X,Y) Z, X \rangle = \frac{1}{2} (\nabla_X h (Z, Z) + \nabla_X h (X, X) + \nabla_Y h (Y, Y) - \nabla_Z h (Z, X) - \nabla_X h (X, X) - \nabla_X h (Y, Y)),$$

from (1). Finally, from (7), we get

$$\langle \nabla_\nu W^+ (X,Y) Z, X \rangle = \frac{1}{2} (C(X,Z)(Z) - C(X,Y)(Y)) = -C(X,Y)(Y)$$

This proves equation (13) and the point (ii) in the Theorem.

For the point (iii), we use (13) and (9); from the codifferential of $W^+$, only the derivative along the normal vector, $\nu$, can be non-zero, as $W^+$ vanishes along $M$. This proves the Theorem. \qed

**Remark.** The point (i) gives a condition for a self-dual manifold to admit an umbilic hypersurface: $W^+$ has to vanish along it, generically at order 0 (following (ii)), thus such a hypersurface, if it exists, is locally defined as the zero set of $W^+$.

Considering the umbilic hypersurfaces which arise from the LeBrun correspondence, the point (i) gives a condition for an open self-dual manifold to admit such a conformal infinity, namely it has to be asymptotically conformally flat.

4. **Null-geodesics of complex conformal manifolds**

4.1. **Properties of the twistor space of a 3-dimensional conformal manifold.** Consider $(M,c)$ a complex 3-dimensional conformal manifold. In some topological conditions ($M$ has to be civilized); as a geodesically convex set is always of this type, any point has a basis of civilized neighbourhoods), the *twistor space* of $M$ is defined as the space $Z$ of null-geodesics of $M$, and it is a complex 3-manifold, containing *twistor lines* (i.e. rational curves with normal bundle isomorphic to $O(1) \oplus O(1)$), and endowed with a distribution of 2-planes $F_\gamma \subset T_\gamma Z$ which is a contact structure.

We denote by $\bar{\gamma}$ the point of $Z$ corresponding, in the following way, to the null-geodesic $\gamma$: some twistor lines tangent to $F_\gamma$ (actually a non-empty open set in the space of curves of $Z$, tangent to $F$) correspond to the points of the null-geodesic $\bar{\gamma}$.

The first question we raise is whether there exist twistor lines tangent to any direction of a given 2-plane $F_\gamma$; the answer is:

**Theorem 2.** Let $Z$ be a twistor space of a conformal civilized 3-manifold $M$; let $F_\gamma \subset T_\gamma Z$ be its contact structure. Suppose there is a point $\bar{\gamma} \in Z$ such that there are twistor lines tangent to any direction in $F_\gamma$. Then $Z$ is projectively flat, and $M$ is conformally flat.

---

3Deformations of these twistor lines are, in general, not tangent to the distribution $F$; they correspond to points in the self-dual ambient $N$ (and outside of $M$) arising from the LeBrun correspondence.
This follows directly from \cite{2}, Theorem 3, which has a similar statement
referring to the twistor space of a self-dual manifold; we simply apply it
to the ambient $N$ from the LeBrun correspondence; its conformal flatness
implies the flatness of $M$ (Theorem 1).

\textbf{Remark.} The key point in the above cited Theorem is a
twistorial interpre-
tation of the Weyl tensor of a self-dual manifold $N$ \cite{2}, Theorem 2, together
with the remark that, for a given 2-plane $F$ in $T_{\bar{\beta}}Z$, the union of all twistor
lines tangent to it (supposing there exists one pointing in any direction of
$F$) is a complex surface which is \textit{smooth} at $\bar{\beta} \in Z$. The above cited Theorem
and the following one (Theorem 3' from \cite{2}) show that this situation implies
the vanishing of the Weyl tensor of $N$, $W^+$, in certain directions:

\textbf{Theorem 3.} \cite{4} Let $Z$ be the twistor space of the (civilized) self-dual mani-
fold $N$, and let $\beta \in Z$ be a point in $Z$, corresponding to the $\beta$-surface $\beta \subset N$;
let $F^\gamma \subset T_{\bar{\beta}}Z$ be a 2-plane, corresponding to the null-geodesic $\gamma \subset \beta$ \cite{10}.
Suppose that, for each direction $\sigma \in \mathbb{P}(F^\gamma) \subset \mathbb{P}(T_{\bar{\beta}}Z)$, there is a smooth
(non-necessarily compact) curve $Z_\sigma$ tangent to $\sigma$, such that:
(i) if $\sigma$ is tangent to a twistor line $Z_x$ at $\bar{\beta}$, then $Z_\sigma = Z_x$;
(ii) $Z_\sigma$ varies smoothly with $\sigma \in \mathbb{P}(F^\gamma)$.
Then
$$\bar{Z}^\gamma_{\bar{\beta}} := \bigcup_{\sigma \in \mathbb{P}(F^\gamma)} Z_\sigma$$
is a smooth surface around $\beta$, and $W^+(F^\gamma_x) = 0$, $\forall x \in \gamma$, where $F^\gamma_x \subset T_x M$
is the $\alpha$-plane containing $\dot{\gamma}$.

In other words, if the \textit{integral} $\alpha$-cone corresponding to the 2-plane $F \subset
T_{\bar{\gamma}}Z$ — defined as the union of all twistor lines tangent $F$ \cite{2} — can be
completed to a surface, smooth around its “vertex” $\bar{\gamma}$, then $W^+$ vanishes
along the null-geodesic $\gamma$ (whose points correspond to the twistor lines that
constitute the $\alpha$-cone \cite{3}).

4.2. Compact null-geodesics and conformal flatness. Our main result
is:

\textbf{Theorem 4.} Let $M$ be a conformal $n$-manifold containing an immersed
rational curve as null-geodesic. Then $M$ is conformally flat.

This fact has been proven by Ye \cite{17} for complex projective manifolds
— we are grateful to B. Klingler for having brought this paper into our
attention. In the above Theorem, we do not make any assumption on the
topology of the manifold, but only of one null-geodesic contained in it.

\textit{Proof.} The proof is different in the cases $n > 3$ and $n = 3$; one of the reasons
is that conformal flatness reduces, in higher dimensions, to the vanishing of
the Weyl tensor, while in dimension 3 it is a higher-order condition more
difficult to handle. The first step (common to all cases) is to prove that a small deformation (seen just as a compact submanifold of $M$, \cite{1}) of such a
compact null-geodesic $\gamma$ is still a compact null-geodesic, and to characterize
the global sections of the normal bundle of $\gamma$ as locally determined by \textit{Jacobi}
fields.
Lemma 2. Let \( \gamma \) be a (immersed) null-geodesic in \((M, c)\). Let \( J \) be a vector field along \( \gamma \). Then the condition \( \dot{J} \perp \dot{\gamma} \) (where \( \dot{J} := \nabla_{\dot{\gamma}} J \)) is independent of the metric \( g \) with respect to which we take the derivative \( \nabla^g \).

Proof. The relation between two Levi-Civita connections (or, more generally, Weyl structures) of metrics in the same conformal class, is given by [4]:

\[
\nabla' X Y - \nabla X Y = \theta(X)Y + \theta(Y)X - \theta^g g(X, Y),
\]

where \( \theta \) is a 1-form, and the rising of indices in \( \theta^g \) is made using the same (arbitrary) metric \( g \in c \) as in the scalar product \( g(X, Y) \). The Lemma immediately follows.

Denote by \( N(\gamma) \) the normal bundle of \( \gamma \) in \( M \), and by \( N^\perp(\gamma) \) its subbundle represented by vectors orthogonal to \( \dot{\gamma} \). Fix a metric \( g \) in the conformal class \( c \). Let \( J \) be a Jacobi field along \( \gamma \), satisfying to the Jacobi equation

\[\ddot{J} = R^g(\dot{\gamma}, J)\dot{\gamma} .\]

It represents an infinitesimal deformation of \( \gamma \) through null-geodesics if and only if \( \dot{J} \perp \dot{\gamma} \). \( J \) induces a section in the normal bundle \( N(\gamma) \), or in \( N^\perp(\gamma) \) if \( J \perp \dot{\gamma} \) in a point, hence everywhere. We want to prove that this section is independent of the connection \( \nabla^g \):

Proposition 2. The Jacobi equations on a null-geodesic \( \gamma \subset M \) induce a second order linear differential operator \( P \) on \( N(\gamma) \), which, restricted to the sections \( J \) such that \( \dot{J} \perp \dot{\gamma} \), depends only on the conformal structure \( c \) of \( M \). In particular, \( P \) restricted to \( N^\perp(\gamma) \) is conformally invariant.

Proof. For a Levi-Civita connection \( \nabla \) of a local metric on \( M \), we locally define the following differential operator on the sections of \( TM|_\gamma \):

\[
P : \Gamma(TM|_\gamma \otimes S^2(T\gamma)) \rightarrow \Gamma(TM|_\gamma),
\]

by \( P(Y; X, X) := \nabla_X \nabla_X Y - \nabla_{\nabla_X X} Y - R(X, Y)X \). Because \( \gamma \) is a null-geodesic, \( P \) induces a (local) differential operator on \( N(\gamma) \), and we need to relate \( P \) to the corresponding operator \( P' \) induced by another connection \( \nabla' \). First we write

\[
P(Y, X, X) = \nabla_X [X, Y] + \nabla_{[X, Y]} X - [\nabla_X X, Y],
\]

then we recall that another Levi-Civita connection \( \nabla' \) is related to \( \nabla \) by the formula [4], such that we get

\[
P'(Y; X, X) - P(Y; X, X) = 2[Y(\theta(X)) - \theta([X, Y])]X - g(\nabla_X Y, X)\theta^g,
\]

which is identically zero modulo \( T\gamma \), provided that \( \nabla_X Y \perp X \) (the latter condition being independent of the Levi-Civita connection, according to the previous Lemma).

Using the fact that \( \mathbb{CP}^1 \) is the union of two contractible sets \( U_1 \cup U_2 \) (on each of which the Jacobi equation, with any initial condition — the same for \( U_1 \) and for \( U_2 \) — in \( x_0 \in U_1 \cap U_2 \), has a unique solution — and these solutions necessarily coincide on the connected intersection \( U_1 \cap U_2 \)), we immediately get:
Proposition 3. Let $\gamma$ be an immersed null-geodesic, diffeomorphic to a projective line $\mathbb{CP}^1$. Then any local Jacobi field $J$ with $\dot{J} \perp \dot{\gamma}$ induces a global normal field $\nu^J$ on $\gamma$.

This has important consequences about the normal bundle of $\gamma$ in $M$, as Jacobi fields provide it with global sections; in particular, $N(\gamma)/N^\perp(\gamma)$ is a line bundle admitting nowhere-vanishing sections, hence it is trivial; on the other hand, $N^\perp(\gamma)$ is a $(n-2)$–rank bundle over $\mathbb{CP}^1$, admitting sections with any prescribed 1-jet (induced, again, by some appropriate Jacobi fields), hence

\[
N^\perp(\gamma) \simeq \bigoplus_{k=1}^{n-2} O(a_k); \quad N(\gamma) \simeq \bigoplus_{k=1}^{n-2} O(a_k) \oplus O(0); \quad a_k \in \mathbb{N}^*.
\]

For $a_k \in \mathbb{Z}$, this is the general form of a vector bundle over $\mathbb{CP}^1$, according to a theorem of Grothendieck; the condition $a_k \geq 1$ arises from the existence of sections of $N^\perp(\gamma)$ with prescribed 1-jet. We are going to show later that all $a_k$ are equal to 1. First we prove:

Proposition 4. Null-geodesics close to a compact, simply-connected one are also compact and simply-connected, and they are generically embedded.

Proof. We consider the projectivized bundle $\mathbb{P}(\mathbb{C})$ of the isotropic cone $\mathbb{C} \subset TM$. It is a standard fact [3] that any null-geodesic $\gamma \subset M$ has a canonical *horizontal* lift $\tilde{\gamma} \subset \mathbb{P}(\mathbb{C})$ (depending only on the conformal structure), such that $\pi_s(T_s\tilde{\gamma}) = T_s\gamma$, where $\pi : \mathbb{P}(\mathbb{C}) \rightarrow M$ is the projection, and $s \in \pi^{-1}(x)$.

Note that $\tilde{\gamma}$ is always embedded, even if $\gamma$ may have self-intersections (it is always immersed).

The lifts of the null-geodesics of $M$ consist in a foliation of $\mathbb{P}(\mathbb{C})$, which has a compact, simply-connected leaf, namely the lift $\tilde{\gamma}$ of our compact, simply-connected null-geodesic $\gamma$. By Reeb’s stability Theorem [13], then there is a saturated tubular neighbourhood of $\tilde{\gamma}$, diffeomorphic to $\tilde{\gamma} \times D$ (where $D \subset \mathbb{C}^{n-2}$ is a polydisc), such that the leaves close to $\tilde{\gamma}$ are identified, via the above diffeomorphism, to the (compact and simply-connected) curves $\tilde{\gamma} \times \{z\}, \; z \in D$.

So all null-geodesics close to $\gamma$ are compact and simply connected. If $\gamma$ has self-intersections at the points $x_1, \ldots, x_k$, we blow-up $M$ at those points, and the lift of $\gamma$ is now embedded. So must be then the lifts of the null-geodesics close to $\gamma$, as they are now deformations of the lifted (hence, embedded) curve. But, generically, such curves avoid the finite set of points $x_1, \ldots, x_k$; the corresponding null-geodesics must have been embedded from the beginning.

From now on, according to the previous Proposition, we may suppose that $\gamma$ is a compact, simply-connected, embedded null-geodesic.

We compute the normal bundle of $\gamma$ in $M$, using the relation (20) and the projection $\pi : \mathbb{P}(\mathbb{C}) \rightarrow M$, as follows: We have the following exact sequence of bundles:

\[
0 \rightarrow N^\pi(\tilde{\gamma}) \rightarrow N(\tilde{\gamma}) \rightarrow \pi^*N(\gamma) \rightarrow 0,
\]

where $N^\pi(\tilde{\gamma})$ is the normal subbundle of $\tilde{\gamma}$ represented by vectors tangent to the fibers of $\pi$ and $N(\tilde{\gamma})$ is the normal bundle of $\tilde{\gamma}$ in $\mathbb{P}(\mathbb{C})$. In a point
$T_x\gamma \in \tilde{\gamma} \subset \mathbb{P}(\mathcal{C})$, the fiber of $\pi$ is equal to $\mathbb{P}(\mathcal{C})_x$, so the tangent space to it is isomorphic to $\text{Hom}(T_x\gamma, N_x^\perp(\gamma))$, for the projective variety $\mathbb{P}(\mathcal{C})_x \subset \mathbb{P}(T_x\mathcal{M})$.

Thus

$$N^\pi(\tilde{\gamma}) \simeq \text{Hom}(T\gamma, N^\perp(\gamma)) \simeq \mathcal{O}(-2) \otimes N^\perp(\gamma),$$

as $T\gamma \simeq T\mathbb{C}P^1 \simeq \mathcal{O}(-2)$.

The central bundle in the exact sequence (21) is trivial, because $\tilde{\gamma}$ is a leaf of a foliation. (20) and (21) imply that the Chern numbers $a_1, \ldots, a_{n-2}$ are subject to the following constraint:

$$\sum_{k=1}^{n-2} (2a_k - 2) = 0,$$

thus, as $a_k \geq 1$, we have $a_k = 1, \forall k = 1, n-2$. We have then:

**Proposition 5.** The normal bundle of a compact, simply-connected, null-geodesic $\gamma$ in $\mathcal{M}$ is isomorphic to

$$N(\gamma) \simeq (\mathbb{C}^{n-2} \otimes \mathcal{O}(1)) \oplus \mathcal{O}(0),$$

and all its global sections are induced by Jacobi fields $J$ such that $\dot{J} \perp \dot{\gamma}$.

Moreover, the deformations of $\gamma$ as a compact curve coincide with the null-geodesics close to $\gamma$.

The last statement follows from the expression of the normal bundle, and a Theorem of Kodaira [7]: the normal bundle satisfies $H^1(N(\gamma)) = 0$, thus the space of deformations of $\gamma$ as a compact curve has dimension equal to $\dim H^0(N(\gamma)) = \dim \Gamma(N(\gamma)) = 2n - 3$, which is precisely the dimension of the space of null-geodesics, defined locally, over a geodesically convex open set, as the space of the leaves of the horizontal foliation of $\mathbb{P}(\mathcal{C})$ [11]. We conclude using the fact that all null-geodesics close to $\gamma$ are deformations of this one (as a compact, and simply-connected, curve).

We return to the proof of Theorem 4. Consider first the case when the dimension of $\mathcal{M}$, $n > 3$. We are going to show that the Weyl tensor of $\mathcal{M}$, $W$, is identically zero (a special sub-case is $n = 4$, when $W = W^+ + W^-$).

For simplicity, suppose first that $n > 4$, and consider the fiber of $N^\perp(\gamma)$ at an arbitrary point $x \in \gamma$: it has a non-degenerate conformal structure, induced from $\mathcal{M}$, and the isotropy cone spans the whole fiber $N^\perp(\gamma)_x$ (this still holds for $n = 4$, but not for $n = 3$). Let $L \subset N^\perp(\gamma)$ be an isotropic line. We have:

**Lemma 3.** Let $\gamma^U$ be an open set of the null-geodesic $\gamma$, on which local metrics $g, g' \in c$ are well defined. If a (locally defined, over $\gamma^U$) line subbundle $L \subset N^\perp(\gamma)$ is parallel (or stable) for $\nabla^g$, then it is parallel for $\nabla^{g'}$ as well.

The proof is a straightforward application of (19).

Let $(L_1)_x, \ldots, (L_{n-2})_x$ be linearly independent isotropic lines in $N^\perp(\gamma)_x$. According to the previous Lemma, and to the fact that $\gamma$ is simply-connected, their parallel transport over $\gamma$ does not depend on any Levi-Civita connection of a metric in the conformal class. We get, thus, a *global splitting*

$$N^\perp(\gamma) = L_1 \oplus \cdots \oplus L_{n-2},$$

where the line bundles $L_i, i = 1, n-2$ are all isotropic and parallel.
All these bundles are isomorphic to $O(b_i)$, $b_i \in \mathbb{Z}$. As their sections are also sections of $N^\perp(γ) \cong \mathbb{C}^{n-2} \otimes O(1)$, they cannot vanish at more that 1 point, for each $L_i$, thus $b_i \leq 1$, $i = 1, n - 2$. On the other hand, the sum of all $b_i$'s has to be $n - 2$, thus $b_i = 1$, $\forall i = 1, n - 2$.

Let $φ_i$ be a section of $L_i$; from Proposition 3, it is locally represented by a Jacobi field $J_i$, for the metric $g \in c$. From the Jacobi equation, by taking the scalar product with $J$, we get:

$$g(\dot{R}(γ, J)γ, J) = 0,$$

and it is easy to see that, because of the fact that all scalar products involving $γ$ and $J$ are 0, the term $h \wedge I$ of the curvature satisfies the above relation identically. This equation holds, at $x$, for any isotropic vector $J_x \perp γ_x$, but we may consider also other compact, simply-connected, null-geodesics $γ'$, containing $x$, and close to $γ$ (namely, small deformations of the compact curve $γ$).

For any 2-plane $F \subset T_x M$, we denote by $R^F$ the sectional curvature of $F$:

$$R^F : S^2(\Lambda^2 F) \to \mathbb{C}, \quad R^F(X \wedge Y, X \wedge Y) := \langle R(X \wedge Y), Y \wedge X \rangle, \quad \forall X \wedge Y \in \Lambda^2 F,$$

and we have seen that, if $F$ is totally isotropic, $R^F$ depends only on $W$ (and on the metric $g$ only via the scalar product $\langle \cdot, \cdot \rangle$).

**Lemma 4.** If $\dim M > 4$, the Weyl tensor at $x \in M$, $W_x$, is determined by the sectional curvatures $\{R^F, F \in U(F_0)\}$, for $U(F_0)$ a small neighbourhood of the totally isotropic arbitrary 2-plane $F_0 \subset T_x M$ in the Grassmanian of totally isotropic 2-planes at $x$.

**Remark.** A similar statement holds in dimension 4, but in that case, the Grassmanian of totally isotropic 2-planes has 2 connected components; as a consequence, $W^+$ is determined by the sectional curvatures of $α$-planes, and $W^-$ by the sectional curvatures of the $β$-planes [3]: Proposition 2.

**Proof.** This reduces to the claim that $W_x = 0$ if and only if

$$R^F = 0, \quad \forall F \in U(F_0),$$

which is a problem of linear algebra. If we consider the space $K$ of all curvature tensors $R'$ satisfying $(R')^F = 0, \forall F \in U(F_0)$, then this is a vector space, which is invariant to the action of $so(n, \mathbb{C})$ (which is the Lie algebra infinitesimal action corresponding to the action of $SO(n, \mathbb{C})$ — note that the Grassmanian of totally isotropic planes is preserved by this action). But there are only 3 $so(n, \mathbb{C})$-irreducible components of the space of curvature tensors, and we have seen that for the Ricci-like tensors $h \wedge I$, the totally isotropic planes always have zero sectional curvature. Then either any Weyl tensor satisfying (24) is zero at $x$, or $K$ contains the whole space of curvature tensors. The latter possibility can easily be excluded by an example of a curvature tensor $K$ satisfying:

$$K(X_0, Y_0)X_0 = A_0, \quad \text{where } \langle Y_0, A_0 \rangle = 1,$$

and

$$\langle X_0, X_0 \rangle = \langle X_0, Y_0 \rangle = \langle Y_0, Y_0 \rangle = \langle X_0, A_0 \rangle = 0.$$
From (23) and the previous Lemma we conclude that $W_x = 0$ for any $x$ contained in a compact, simply-connected, null-geodesic; but we know from Proposition 3 that the set of such points contains a neighbourhood of $\gamma$, thus, by analyticity of $W$, it vanishes identically.

The proof is similar in dimension 4 (note that, in the self-dual case, we can retrieve the result by applying Theorem 3; this is how we shall proceed for the case of dimension 3, using the LeBrun correspondence); the difference with the higher-dimensional case is that the splitting (22) is canonical, $L_1$ corresponding, say, to the $\alpha$-plane $F^\alpha$ containing $\dot{\gamma}$, and $L_2$ to the $\beta$-plane $F^\beta$ containing $\dot{\gamma}$. It is important now that each of $L_1$, $L_2$ is isomorphic to $\mathcal{O}(1)$, because the vanishing of $R_{F^\alpha}$ implies $W^+ \equiv 0$, and the vanishing of $R_{F^\beta}$ implies $W^- \equiv 0$ [2], Proposition 2. Thus the manifold $(M^4, c)$ is conformally flat.

Consider now the particular case where $n = 3$. We are going to use the LeBrun correspondence, then Theorem 3, to prove that $M$ is then conformally flat. Note that we cannot use directly Theorem 1 and the above proven result for self-dual manifolds, as the ambient self-dual manifold $N$ can only be defined for a civilized (e.g. geodesically convex) 3-manifold.

We cover $\gamma$ with geodesically convex open sets $U_i$, $i = 1, n$, such that:

$$\forall i \neq j \text{ such that } U_i \cap U_j \cap \gamma \neq \emptyset, \quad \exists U_{ij} \supset (U_i \cup U_j),$$

where $U_{ij}$ is still geodesically convex (with respect to some particular Levi-Civita connection). This is possible by choosing $U_i$, $i = 1, n$, small enough [16]. Then we choose a relatively compact tubular neighbourhood $N(r_0)$ of $\gamma$, such that its closure is covered by the $U_i$'s. We may choose this tubular neighbourhood small enough to be contained in the projection $U$, from $\mathbb{P}(C)$, of a saturated neighbourhood (see Proposition 3) of the lift $\tilde{\gamma}$.

We consider then the twistor spaces $Z_i$, the spaces of null-geodesics of $U_i$. The compact, simply-connected, null-geodesics close to $\gamma$ identify (diffeomorphically) the neighbourhoods of $\tilde{\gamma} \in Z_i$ with the space $Z$ of the deformations (contained in $U$) of $\gamma$ as a compact curve. We can see then (a small open set of) $Z$ as an open set common to all the $Z_i$'s:

Following LeBrun, we define the self-dual manifolds $N_i$ as the spaces of projective lines in $Z_i$, with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Then $U_i$ is an umbilic hypersurface in $N_i$.

The local twistor spaces $Z_i$ admit contact structures, which coincide on $Z$, and contain projective lines $Z^i_x$ corresponding to points $x \in \gamma \cap U_i$. If we denote by $Z_{ij}$ the twistor space of $U_{ij}$, then $Z_{ij}$ is identified to an open set in $Z_i$ and, at the same time, to an open set in $Z_j$, in particular the twistor
lines \( Z^i_x \subset Z^i \) and \( Z^j_x \subset Z^j \) (corresponding to the same point \( x \in U_{ij} \)) are identified. Thus \( Z^i_x \cap Z \) and \( Z^j_x \cap Z \) coincide and we denote by \( Z_x \) this (non-compact) curve in \( Z \), and by \( F \) the canonical contact structure of \( Z \) (restricted from the ones of \( Z_i \)).

**Remark.** We already have obtained that the integral \( \alpha \)-cone (i.e. the union of twistor lines passing through \( \gamma \) and tangent to \( F_\gamma \), see the comment after Theorem 3) corresponding to \( F_\gamma \) is a part of a smooth surface: the union of the lines \( Z_x \), \( x \in \gamma \). Thus, from Theorem 3, the Weyl tensor \( W^+_i \) of the self-dual manifold \( N_i \) vanishes on the \( \alpha \)-planes generated by \( T_\gamma \). But this is nothing new: we know, from Theorem 1, that \( W^+_i \) vanishes on \( U_i \).

We intend to apply Theorem 3 to prove that \( W^+_i \) vanishes on points close to \( U_i \), but generically in \( N_i \setminus U_i \). We do that by showing that the integral \( \alpha \)-cones corresponding to planes \( F' \subset T_\gamma Z \) close to \( F \) are parts of smooth surfaces, then we conclude using Theorem 3.

First we choose Hermitian metrics \( h_i \) on \( Z_i \), such that they coincide (with \( h \)) on \( Z \). We have a diffeomorphism between \( \gamma \) and \( \mathbb{P}(F_\gamma) \), so we choose relatively compact open sets in \( \mathbb{P}(F_\gamma) \), covering \( \mathbb{P}(F_\gamma) \), with the following properties: As the metrics \( h_i \) induce metrics on \( N_i \), we first choose a small enough distance \( r_1 > 0 \) such that

1. \( \forall i \), there is a sub-covering \( V_i \subseteq U_i \) of \( \gamma \) such that the “tubular neighbourhoods” \( Q_i := \{ y \in N_i \mid d(y, V_i) \leq r_1 \pi_i(y) \in V_i \cap \gamma \} \) are compact (\( d(y, V_i) \) is the distance between \( y \) and \( V_i \), and \( \pi_i \) is the “orthogonal projection” — for the Hermitian metric — from \( N_i \) to \( \gamma \cap U_i \); it is well defined because of the condition below);

2. \( r_1 \) is less than the bijectivity radius of the (Hermitian) exponentials for the points of \( V_i \) in \( N_i \), and for the points of \( V_i \cup V_j \) in \( N_{ij} \) (if \( U_i \cap U_j \cap \gamma \neq \emptyset \)).

We have then

**Lemma 5.** For any \( y_i \in Q_i \subset N_i \), \( y_j \in Q_j \subset N_j \) such that the curves \( Z_{y_i} := Z^i_{y_i} \cap Z \), \( Z_{y_j} := Z^j_{y_j} \cap Z \) are tangent to the same direction in \( \bar{\gamma} \in Z \), the respective curves \( Z_{y_i}, Z_{y_j} \) coincide.

**Proof.** We first note that the projection \( \pi_i \) from \( N_i \) to \( \gamma \cap U_i \) is equivalent to the \( h \)-orthogonal projection of the direction of \( T_\gamma Z_{y_i} \) to a direction in \( F_\gamma \), so \( \pi_i(y) = \pi_j(y) : y \in \gamma \); thus \( y \) belongs to both \( U_i \) and \( U_j \), and we use again the twistor space \( Z_{ij} \) to conclude that \( Z_{y_i} \) and \( Z_{y_j} \) are “restrictions” to \( Z \) of the same projective line (as they both have the same tangent space at \( \bar{\gamma} \)) \( Z_{y_{ij}}^{ij} \), for a point \( y_{ij} \in N_{ij} \). \( \square \)

Now we have a tubular neighbourhood \( S \subset \mathbb{P}(T_\gamma Z) \) of \( \mathbb{P}(F_\gamma) \), of radius \( r_1/2 \), such that, for any 2-plane \( F' \subset S \), the conditions in Theorem 3 are satisfied (considering any of the local twistor spaces \( Z_i \)).

We recall that, via the LeBrun correspondence, a point \( \bar{\gamma}_0 \) in the twistor space of \( M_0 \), \( Z_0 \), is identified to the point \( \bar{\beta}_0 \) in the twistor space of \( N_0 \), still denoted by \( Z_0 \). They correspond to the null-geodesic \( \gamma_0 \subset M_0 \), resp. to the \( \beta \)-surface \( \beta_0 \subset N_0 \), such that \( \gamma_0 \subset \beta_0 \). The planes \( F' \) above are included in \( T_\gamma Z = T_{\bar{\beta}^i} Z_i \), and they correspond to null-geodesics in \( N_i \) contained in \( \beta^i \).
By Theorem 3, we conclude that the Weyl tensor $W^+_i$ of $N_i$ vanishes along all null-geodesics of $N_i$, close (in $Q_i$) to $\gamma$ and included in the $\beta$-surface $\beta^i$, determined by $\gamma$. This means that $W^+$ vanishes everywhere on $\beta^i$. By deforming $\gamma$, we obtain that $W^+_i$ vanishes on a neighbourhood of $U_i$ in $N_i$, hence $N_i$ is conformally flat.

It follows from Theorem 4 that $U_i$, hence $M$, is conformally flat. 

References

[1] M.F. Atiyah, N.J. Hitchin, I.M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London, Series A 362 (1978), 425-461.
[2] F.A. Belgun, *On the Weyl tensor of a self-dual complex 4-manifold*, Preprint 98-17, École Polytechnique, Palaiseau (1998).
[3] O. Biquard, *Métriques d’Einstein asymptotiquement symétriques*, Preprint 98-14, École Polytechnique, Palaiseau (1998).
[4] P. Gauduchon, *Connexion canonique et structures de Weyl en géométrie conforme*, Preprint (1990).
[5] N.J. Hitchin, *Complex manifolds and Einstein’s equations*, in Twistor Geometry and Non-linear Systems, Lecture Notes in Mathematics 970 (1982) Springer-Verlag, 73-99.
[6] N.J. Hitchin, *Twistor spaces, Einstein metrics and isomonodromic deformations*, J. Diff. Geom. 42 (1995), 30-112.
[7] K. Kodaira, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. Math., II Ser., 75 (1962), 146-162.
[8] C.R. LeBrun, *H-Space with a cosmological constant*, Proc. Roy. Soc. London, Series A 380 (1982), 171-185.
[9] C.R. LeBrun, *Twistor CR manifolds and three-dimensional conformal geometry*, Trans. Am. Math. Soc., 284 (1984), 601-616.
[10] C.R. LeBrun, *Twistors, Ambitwistors, and Conformal Gravity*, in Twisters in mathematics and physics, London Math. Soc. Lecture Note Ser., 156 (1990), Cambridge Univ. Press, Cambridge, 71-86.
[11] C.R. LeBrun, *Spaces of Complex Null Geodesics in Complex-Riemannian Geometry*, Trans. Am. Math. Soc. 278 (1983), 209-231.
[12] R. Penrose, *The Structure of Space-Time*, in Battelle Rencontres (ed. C. DeWitt & J. Wheeler), New York: Benjamin (1968), 121-235.
[13] G. Reeb, *Stabilité des feuilles compactes à groupe de Poincaré fini*, C. R. Acad. Sci. Paris 228 (1949), 47-48.
[14] I.M. Singer, J.A. Thorpe, *The curvature of 4-dimensional Einstein spaces*, in Global Analysis, in honor of Kodaira, Princeton Math. Series 29 (1969), Princeton Univ. Press, 355-365.
[15] R.S. Ward, *Self-Dual Space-Times with Cosmological Constant*, Comm. Math. Phys. 78 (1980), 1-17.
[16] J.H.C. Whitehead, *Convex regions in the geometry of paths*, Quart. J. Math. 3 (1932), 33-42.
[17] Y.-G. Ye, *Extremal rays and null geodesics on a complex manifold*, Internat. J. Math. 5 (1994), 141-168.

Mathematisches Institut
Humboldt Universität zu Berlin
Unter den Linden 6, 10099 Berlin
Germany

E-mail: belgun@mathematik.hu-berlin.de