Information-theoretic measures for a position-dependent mass system in an infinite potential well

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Abstract. In this work we calculate the Cramér–Rao, the Fisher–Shannon and the López-Ruiz–Mancini–Calbert (LMC) complexity measures for eigenstates of a deformed Schrödinger equation, being this intrinsically linked with position-dependent mass (PDM) systems. The formalism presented is illustrated with a particle confined in an infinite potential well. Abrupt variation of the complexity near to the asymptotic value of the PDM-function $m(x)$ and erasure of its asymmetry along with negative values of the entropy density in the position space, are reported as a consequence of the interplay between the deformation and the complexity.

Keywords: PDM systems, Cramér-Rao complexity, Fisher-Shannon complexity, LMC complexity

1. Introduction

The complexity of a system is understood as a measure of internal order/disorder of its components and of the interrelations between them. From the standard description of considering the states of a system characterised by probability density distributions, several notions of complexity can be defined in terms of statistical measures as the information entropy, within the context of the information theory [1, 2, 3, 4]. Intuitively, a fully ordered system (for instance a crystal) does not present complexity or its complexity is zero. The same simple description can be assigned to a totally disordered system like an ideal gas, although their physical characteristics differ a lot from a crystal. In these basic examples we see that they share in common a minimal level of complexity. Several measures of complexity for finite systems in terms of the variance, the Shannon entropy [1], the Fisher information [2] and other statistical measures have been proposed.
Information-theoretic measures for a PDM system in an infinite potential well

In this context, two of the most popular are the Cramér-Rao [14, 15] and the Fisher-Shannon [16, 17] complexities, which are monotones with respect to a convolution with any Gaussian probability distribution, i.e., the complexity after the convolution decreases with respect to the original one. In addition, the LMC (López-Ruiz, Mancini and Calbert [18, 19]) complexity is monotonic regarding to a certain family of stochastic operations. Subsequently, quantum information measures and their properties against LOCC operations have motivated the following axiomatic definition of complexity [20]: a preexistent family of probability density distributions of minimal complexity that are preserved by a certain class of operations, and the monotonicity of the complexity respect to these operations.

Furthermore, along several decades an increasing application of these three measures of complexity (the Crâmer-Rao, the Fisher-Shannon, and the LMC) have been reported in order to characterise a wide class of phenomena and systems: Hartree-Fock wave functions [16], multiparticle systems [17], chemical reactions [21], Morse and Pöschl–Teller potentials [22], rigid rotator [23], hydrogenic-like systems [24, 25, 26], blackbody radiation problem [27], Dirac-delta-like quantum potentials [28], particle in an infinite well [29], etc. Among these systems, those which have a position-dependent mass (i.e., the so-calles position-dependent mass systems, or briefly PDM systems) have presented a particular interest due to applicability in multiple areas: astrophysics [30], semiconductors [31], quantum dots [32], quantum liquids [33], inversion potential for NH$_3$ [34], many body theory [35], relativistic quantum mechanics [36], superintegrable systems [37], supersymmetry [38], nuclear physics [39], nonlinear optics [40], etc. In addition, studies of information-theoretic measures for PDM systems have been made in references [41, 42, 43, 44, 45, 46].

One of the main achievements of the PDM systems is that they allow to model the dynamics of particles in an non-homogeneous media. Lately, a deformed Schrödinger equation associated with a position-dependent mass has been proposed within the context of generalised translation operators [47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57], where a real parameter $\gamma$ controls the deformation and whose mathematical background is provided by an algebraic structure [58, 59] inspired in nonextensive statistics [60, 61].

Motivated by previous works [47, 48, 51, 52, 53, 57] and following the definition of complexity given in [20], in this work we calculate the Cramér-Rao, the Fisher-Shannon and the LMC complexities for eigenstates of the deformed Schrödinger equation for a particle in an infinite potential well [47, 49, 51] and then, we analyse the interplay between the underlaying deformed structure and the behavior of the complexities. In particular, we extent the results presented in [29] for a particle with PDM.

The paper is organised as follows. In Section 2 we review the notion of complexity for probability density distributions as well as the entropic uncertainty relation. Section 3 is devoted to calculate the Cramér-Rao, the Fisher-Shannon and the LMC complexities for eigenstates of the deformed Schrödinger of a particle confined in an infinite potential well. Finally, in Section 4 we outline some conclusions and future directions are discussed.
2. Preliminaries

In this Section we give the necessary concepts and definitions for the development of the forthcoming sections.

2.1. Complexity measures for probability density distributions

We introduce some basic elements of the information-theoretic description of unidimensional probability density distributions. We consider continuous probability density distributions \( \rho(x) \) defined over a subset \( X \) of the real numbers \( \mathbb{R} \), mathematically defined by \( \rho : \mathbb{R} \rightarrow [0, +\infty) \) with \( \int_X \rho(x)dx = 1 \). The spread of the variable \( x \) over an interval \( X \subseteq \mathbb{R} \) is measured by its standard deviation or Heisenberg length given by the square mean deviation value of \( x \)

\[
L_H = \Delta x = \sqrt{V[\rho]} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}
\]  

where \( V[\rho] \) is the variance of \( x \). As usual, \( \langle f(x) \rangle \) stands for the mean value of \( f(x) \) with respect of the probability density distribution \( \rho(x) \), i.e., \( \langle f(x) \rangle = \int_X f(x)\rho(x)dx \). Given a probability density distribution \( \rho \), the Shannon entropy (or information entropy) \( S[\rho] \) and the Fisher information \( F[\rho] \) are found between the most relevant information theoretic measures. The former, introduced by C. Shannon in his foundational work on communication theory [1], is the expected value of the information entropy of the distribution \( \rho(x) \). The latter, whose role in estimation theory was pointed out by R. Fisher [2], measures the information of the variable \( x \) regarding an unknown parameter associated to the distribution \( \rho(x) \). These are given by

\[
S[\rho] = -\int_X \rho(x) \ln \rho(x)dx = -\langle \ln \rho(x) \rangle
\]

and

\[
F[\rho] = \int_X \frac{1}{\rho(x)} \left[ \frac{d\rho(x)}{dx} \right]^2 dx = \left\langle \left[ \frac{d\ln \rho(x)}{dx} \right]^2 \right\rangle
\]

respectively. Other information measure of interest is the Onicescu entropy or disequilibrium [62]:

\[
D[\rho] = \int_X [\rho(x)]^2 dx = \langle \rho(x) \rangle,
\]

which corresponds to the entropic moment of second order. The Shannon information and the disequilibrium are global quantifiers that measure the extent to which the density is concentrated, while the Fisher information is local (due to the derivative of \( \rho \) contained in its definition) and it characterises the oscillatory nature of \( \rho \). These measures complement each other and they characterise different aspects of the distribution \( \rho \). For comparing, the Shannon and Fisher lengths (denoted by \( L_S \) and \( L_F \)) are defined [28, 27]

\[
L_S = e^{S[\rho]} , \quad L_F = \frac{1}{\sqrt{F[\rho]}}.
\]
Information-theoretic measures for a PDM system in an infinite potential well

deserve to be mentioned. Heisenberg length measures the separation between the regions, where the probability density is concentrated, regarding the mean value $\langle x \rangle$. Shannon length quantifies the concentration of the density distribution $\rho$. Fisher length tends to zero for discontinuous distributions and it presents a high sensibility to fluctuations of $\rho$. Moreover, we have the following relations:

$$L_F \leq L_H \quad , \quad (2\pi e)^{\frac{1}{2}} L_F \leq L_S \leq (2\pi e)^{\frac{1}{2}} L_H$$

(6)

where the equality is satisfied for the Gaussian distributions. It is worth to be mentioned that other notions of the information length have been extended, for instance in the context of the Lagrangian formulation [63]. The complexity measures associated to the information measures $S[\rho], F[\rho], D[\rho]$ with their respective Heisenberg, Shannon and Fisher lengths are constructed by means of the product between the information measure and its corresponding length, thus capturing a joint balance of the features of $\rho$ described by the information measures and the lengths. In this way, the Cramér-Rao, Fisher-Shannon and LMC complexity measures are defined as [20]:

$$C_{CR}[\rho] = F[\rho](\Delta x)^2 = F[\rho] \times L_F^2, \quad (7a)$$

$$C_{FS}[\rho] = \frac{1}{2\pi e} F[\rho] \times e^{2S[\rho]} = \frac{1}{2\pi e} F[\rho] \times L_S^2, \quad (7b)$$

$$C_{LMC}[\rho] = D[\rho] \times e^{S[\rho]} = D[\rho] \times L_S. \quad (7c)$$

It should be noted that these complexity measures are (i) dimensionless, (ii) lower bounded by the unity, (iii) minimum for the Dirac delta (maximum order) and the uniform distribution (maximum disorder), and (iv) invariant under translation and scaling transformations.

2.2. Entropic uncertainty relations

Entropic uncertainty relations are important expressions that allow to characterise several dynamical features of a quantum system concerning two non-commuting observables. Generally, for two non-commuting observables $\hat{A}$ and $\hat{B}$ an entropic uncertainty relation presents the form [64, 24]

$$S_A + S_B \geq h_{\hat{A}\hat{B}}$$

(8)

where $S$ is a functional defined over the set of observables of the quantum system (typically the entropy), and $h_{\hat{A}\hat{B}}$ is a positive constant that contains information about the non-commutativity between the observables $\hat{A}, \hat{B}$ and it represents the lower bound of the uncertainty relation. As an example, let us consider $\hat{A} = \hat{x}$ and $\hat{B} = \hat{k}$ the position and wave-vector operators, $\psi(x)$ the wave-function in the position space, $\tilde{\psi}(k)$ the wave-function in the $k$-space, and $S$ a modified Shannon entropy for the distributions $\rho(x) = |\psi(x)|^2$ and $\tilde{\rho}(k) = |\tilde{\psi}(k)|^2$ [26], i.e.,

$$S_x[\rho] = -\int_{-\infty}^{\infty} \rho(x) \ln[\sigma \rho(x)]dx = -\langle \ln[\sigma \rho(x)] \rangle = S[\rho] - \ln \sigma \quad (9a)$$
Information-theoretic measures for a PDM system in an infinite potential well

and

\[ S_k[\hat{\rho}] = -\int_{-\infty}^{\infty} \hat{\rho}(k) \ln \frac{\hat{\rho}(k)}{\sigma} \, dk = -\langle \ln[\hat{\rho}(k)/\sigma] \rangle = S[\hat{\rho}] + \ln \sigma, \tag{9b} \]

where \( \sigma \) is introduced in order to have a dimensionless argument of the logarithm. Then, using the equations (9a) and (9b) the entropic uncertainty relation (8) adopts the BBM-inequality form [64]

\[ S_x[\rho] + S_k[\hat{\rho}] = S[\rho] + S[\hat{\rho}] \geq 1 + \ln \pi. \tag{10} \]

3. Complexity measures for a particle with a position-dependent effective mass in an infinite potential well

In this Section we calculate the Cramér-Rao, the Fisher-Shannon and the LMC complexities (given by (7a), (7b) and (7c)) as well as the entropic uncertainty relations for the eigenstates of a unidimensional particle with a PDM and confined in an infinite potential well within the displacement operator approach [47, 48, 49, 50, 51, 52, 53].

3.1. Schrödinger equation for position-dependent effective mass and deformed space

Costa Filho et al have introduced a quantum system with a PDM by using a generalised translation operator which produces nonadditive spatial displacements expressed by [47]

\[ \hat{\tau}_x(\varepsilon)|x\rangle = |x + \varepsilon + \gamma x\varepsilon\rangle, \tag{11} \]

with \( \varepsilon \) an infinitesimal displacement and \( \gamma \) a parameter with dimensions of inverse length. The operator \( \hat{\tau}_x(\varepsilon) \) sends a well-localised state around \( x \) into another well-localised state around \( x + \varepsilon + \gamma x\varepsilon \), without changing the other physical properties. In this sense, the parameter \( \gamma \) can be understood as a measure of coupling between the particle displacement \( \varepsilon \) and the original position \( x \). The generator of translations (11) is the Hermitian momentum operator [49, 50, 51]

\[ \hat{\Pi} = \frac{1}{2}\left[ m(\hat{x})\hat{\hat{p}} + \hat{\hat{p}}(1 + \gamma \hat{x}) \right] = (\hat{1} + \gamma \hat{x})^{1/2}\hat{\hat{p}}(1 + \gamma \hat{x})^{1/2}. \tag{12} \]

The canonically conjugate space operator for the deformed linear momentum \( \hat{\Pi} \) is

\[ \hat{\eta} = \frac{\ln(1 + \gamma \hat{x})}{\gamma}. \tag{13} \]

Thus, \((\hat{\eta}, \hat{\Pi}) \rightarrow (\hat{x}, \hat{p})\) results a point canonical transformation (PCT) which maps a particle with constant mass \( m_0 \) into other with position-dependent mass. In fact, the Hamiltonian operator \( \hat{K}(\hat{\eta}, \hat{\Pi}) = \frac{1}{2m_0}\hat{\Pi}^2 + \hat{U}(\hat{\eta}) \) is mapped into \( \hat{H}(\hat{x}, \hat{p}) = \hat{T} + \hat{V}(\hat{x}) \) with \( \hat{V}(\hat{x}) = \hat{U}(\hat{\eta}(\hat{x})) \) the potential energy operator,

\[ \hat{T} = \frac{1}{2}[m(\hat{x})]^{-1/4}\hat{\hat{p}}[m(\hat{x})]^{-1/2}\hat{\hat{p}}[m(\hat{x})]^{-1/4}, \tag{14} \]
the kinetic energy operator, and
\[ m(x) = \frac{m_0}{(1 + \gamma x)^2} \] (15)
the effective mass. This function has a singularity in \( x_d = -1/\gamma \) due to the definition of the deformed translation operator (11) which satisfies the property \( \hat{T}_{\gamma}(\varepsilon)|x_d\rangle = |x_d\rangle \), i.e., the particle has an inertia so high in \( x_d \) that it can not be displaced whatever the value of \( \varepsilon \). Since we are interested in probability density distributions given by stationary wave functions then we only consider the time-independent Schrödinger equation. Consequently, from (14) and (15) the time-independent Schrödinger equation \( \hat{H}|\alpha\rangle = E|\alpha\rangle \) in terms of the wave function 
\[ \psi(x) = \langle x|\alpha\rangle \]
results
\[ -\frac{\hbar^2}{2m_0}(1 + \gamma x)^2 \frac{d^2\psi(x)}{dx^2} - \frac{\hbar^2\gamma(1 + \gamma x)}{m_0} \frac{d\psi(x)}{dx} - \frac{\hbar^2\gamma^2}{8m_0}\psi(x) + V(x)\psi(x) = E\psi(x). \] (16)
Alternatively, the equation (16) can be expressed by means of a field \( \varphi(x) \) related to \( \psi(x) \) by [52, 53]
\[ \psi(x) = \sqrt{m(x)/m_0}\varphi(x) = \frac{\varphi(x)}{\sqrt{1 + \gamma x}}. \] (17)
for \( x > -1/\gamma \) and \( \psi(x) = 0 \) otherwise, in such a way that the region \((-\infty, -1/\gamma)\) is forbidden for the particle with the mass function given by (15). Therefore \( \rho(x) = |\psi(x)|^2 = |\varphi(x)|^2/(1 + \gamma x) \) remains nonnegative \( \forall x \), in accordance with the concept of probability density. Substituting (17) in (16), one obtains
\[ -\frac{\hbar^2}{2m_0}\hat{D}_\gamma^2\varphi(x) + V(x)\psi(x) = E\varphi(x), \] (18)
where \( \hat{D}_\gamma = (1 + \gamma x)\frac{d}{dx} \) is a deformed derivative operator [51, 52, 59, 53]. By means of the change of variable \( x \rightarrow \eta = \gamma^{-1}\ln(1 + \gamma x) \), we recover the time-independent Schrödinger equation for a particle with constant mass \( m_0 \) in the representation of the deformed space \( \{|\eta\rangle\} \) [47, 48, 52]:
\[ -\frac{\hbar^2}{2m_0} \frac{d^2\phi(\eta)}{d\eta^2} + U(\eta)\phi(\eta) = E\phi(\eta), \] (19)
described in terms of the wave function \( \phi(\eta) = \varphi(x(\eta)) \).

The eigenfunctions of the momentum operator (12) in the \( x \)-representation, with \( \hat{\Pi}|k\rangle = \hbar k|k\rangle \), are [51]
\[ \psi_k(x) = \langle x|k\rangle = \frac{\psi_0}{\sqrt{1 + \gamma x}}\exp\left[ \frac{ik}{\gamma}\ln(1 + \gamma x) \right], \] (20)
which are non-normalizable and constitute a deformation of the standard free particle plane waves, being this latter recovered for \( \gamma \rightarrow 0 \). A wave packet for position-dependent mass system can be defined by the deformed Fourier transform:
\[ \hat{\psi}(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{1 + \gamma x}}\int_{-\infty}^{+\infty} \tilde{\psi}(k)e^{ik\gamma^{-1}\ln(1+\gamma x)}dk, \] (21)
with \( \tilde{\psi}(k) \) a distribution function of the wave vectors \( k \). From relation (17) and the coordinate transformation \( x \rightarrow \eta \), the deformed inverse Fourier transform of (21) results

\[
\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(\eta) e^{-ik\eta} d\eta \\
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(x) e^{-ik\gamma^{-1} \ln(1+\gamma x)} dx \\
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x) e^{-ik\gamma^{-1} \ln(1+\gamma x)} dx.
\]

(22)

### 3.2. Particle in an infinite potential well

Consider a unidimensional particle with a PDM \( m(x) \) given by (15) in an infinite one-dimensional square potential well of width \( 2a \):

\[
V(x) = \begin{cases} 
0 & -a < x < a, \\
\infty, & \text{otherwise}.
\end{cases}
\]

(23)

Outside the square potential well the solution \( \psi(x) \) is null outside because this region is forbidden for the particle. Inside, the solution \( \psi(x) \) of the Schrödinger equation (16) is

\[
\psi(x) = \frac{1}{\sqrt{1+\gamma x}} \left[ C_+ e^{ik\gamma^{-1} \ln(1+\gamma x)} + C_- e^{-ik\gamma^{-1} \ln(1+\gamma x)} \right]
\]

(24)

where \( k = \sqrt{2m_0 E/\hbar} \) and \( C_\pm \) are constants. From the boundary conditions \( \psi(a) = \psi(-a) = 0 \), we obtain

\[
\begin{pmatrix}
C_+ e^{ik\gamma^{-1} \ln(1+\gamma a)} \\
C_- e^{-ik\gamma^{-1} \ln(1+\gamma a)}
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

(25)

This equation has nontrivial solutions only if the determinant of the \( 2 \times 2 \) matrix vanishes, which leads us to

\[
\sin \left[ \frac{k}{\gamma} \ln \left( \frac{1+\gamma a}{1+\gamma a} \right) \right] = 0,
\]

that is \( k_{\gamma,n} = n\pi/L_\gamma \) with \( n \) an integer

\[
L_\gamma = \frac{1}{\gamma} \ln \left( \frac{1+\gamma a}{1-\gamma a} \right) = 2a \left[ \text{atanh}(\gamma a) \right]^{-1}.
\]

(26)

\( L_\gamma \) is the length of box at the coordinate basis \( \{|\eta\} \) which can also be obtained by the transformation \( x \rightarrow \eta \). Therefore, the eigenfunctions for this problem are

\[
\psi_n(x) = \frac{A_\gamma}{\sqrt{1+\gamma x}} \sin \left[ \frac{k_{\gamma,n}}{\gamma} \ln \left( \frac{1+\gamma x}{1+\gamma a} \right) \right]
\]

(27)

at the interval \( |x| < a \) and \( \psi_n(x) = 0 \) otherwise with \( A_\gamma = \sqrt{2/L_\gamma} \) the normalisation constant. The energy levels are given by

\[
E_n = \frac{\hbar^2 \pi^2 n^2}{2m_0 L_\gamma^2} = \varepsilon_0 n^2 \left[ \frac{\gamma a}{\text{atanh}(\gamma a)} \right]^2.
\]

(28)
where \( \lim_{\gamma \to 0} \) corresponds to the standard case. The first moments of the position and the linear momentum for the classical distribution (30) are

\[
\rho_n(x) = |\psi_n(x)|^2 = \frac{2\gamma}{\ln \left(\frac{1+\gamma a}{1-\gamma a}\right)} \frac{1}{1+\gamma x} \sin^2 \left[ n\pi \ln \left(\frac{1+\gamma x}{1+\gamma a}\right) \right]
\]

for \( |x| \leq a \), and \( \rho_n(x) = 0 \) otherwise. As \( n \to \infty \) the classical probability density \( \rho_{\text{classic}}(x) \) coincides respectively with (32

\[
\rho_{\text{classic}}(x) = \frac{\gamma}{\ln \left(\frac{1+\gamma a}{1-\gamma a}\right)} (1+\gamma x),
\]

so the uniform probability density \( \rho_{\text{classic}}(x) = 1/(2a) \) is recovered for \( \gamma \to 0 \), as expected classically.

The expected values \( \langle \hat{x} \rangle \), \( \langle \hat{x}^2 \rangle \), \( \langle \hat{p} \rangle \) and \( \langle \hat{p}^2 \rangle \) for the particle in the one-dimensional symmetric infinite square well are

\[
\langle \hat{x} \rangle = \frac{1}{\gamma} \frac{\gamma a}{\text{atanh}(\gamma a)} - 1 - \frac{a \text{atanh}(\gamma a)}{\gamma a},
\]

\[
\langle \hat{x}^2 \rangle = -\frac{1}{\gamma^2} \frac{\gamma a}{\text{atanh}(\gamma a)} - 1 - \frac{4a \text{atanh}(\gamma a)}{\gamma [4 \text{atanh}^2(\gamma a) + (n\pi)^2]},
\]

\[
\langle \hat{p} \rangle = 0,
\]

\[
\langle \hat{p}^2 \rangle = \frac{h^2 k_n^2 \gamma a}{(1-\gamma a^2)^2 \text{atanh}(\gamma a)} \left[ 1 + \frac{\text{atanh}^2(\gamma a)}{4 \text{atanh}^2(\gamma a) + (n\pi)^2} \right].
\]

It can be shown that \( \lim_{\gamma \to 0} \langle \hat{x} \rangle = 0 \), \( \lim_{\gamma \to 0} \langle \hat{x}^2 \rangle = \frac{a^2}{3} \left( 1 - \frac{6}{n^2 \pi^2} \right) \) and \( \lim_{\gamma \to 0} \langle \hat{p}^2 \rangle = \left( \frac{n\pi}{2a} \right)^2 \) which correspond to the standard case. The first moments of the position and the linear momentum for the classical distribution (30) are

\[
\overline{x} = \frac{1}{\gamma} \frac{\gamma a}{\text{atanh}(\gamma a)} - 1,
\]

\[
\overline{x^2} = -\frac{1}{\gamma^2} \frac{\gamma a}{\text{atanh}(\gamma a)} - 1,
\]

\[
\overline{p} = 0,
\]

\[
\overline{p^2} = \frac{2m_0 E \gamma a}{(1-\gamma a^2)^2 \text{atanh}(\gamma a)},
\]

where \( \lim_{\gamma \to 0} \overline{x} = 0 \), \( \lim_{\gamma \to 0} \overline{x^2} = a^2/3 \) and \( \lim_{\gamma \to 0} \overline{p^2} = 2m_0 E \). It is straightforward to verify that (31a) to (31d) coincide respectively with (32a) to (32d) as \( n \to \infty \), thus showing the classical limit in terms of probability densities.

Substituting the eigenfunctions (27) in (22), we obtain the eigenfunctions in \( k \)-space

\[
\tilde{\psi}_n(k) = \sqrt{\frac{\pi n^2 L_n}{4}} \sin \left( \frac{kL_n}{2} - \frac{n\pi}{2} \right) e^{-\frac{1}{2} k(1-\gamma a) + \pi(n+1)},
\]
and consequently, the corresponding probability densities are
\[
\tilde{\rho}_n(k) = |\tilde{\psi}_n(k)|^2 = \frac{\pi n^2 L_\gamma}{4} \sin^2 \left( \frac{kL_\gamma}{2} - \frac{n\pi}{2} \right),
\]
(34)

From the probability densities (34), we have
\[
\langle \hat{\Pi} \rangle = \hbar \langle \hat{k} \rangle = 0,
\]
(35a)
\[
\langle \hat{\Pi}^2 \rangle = \hbar^2 \langle \hat{k}^2 \rangle = \left( \frac{n\pi \hbar}{L_\gamma} \right)^2.
\]
(35b)

Figure 1 illustrates the energy eigenstates for the three states of the lowest energies with different values of $\gamma a$, as well as their probability densities in the $x$ and $k$ spaces. For $n = 10$, we can see from the figure 2 that the average value of the quantum probability density (29) approaches to the classical probability density (30), in agreement with the correspondence principle. We also see that when $\gamma a$ increases the mass variation (in relation with the position) grows, which implies that the moment distribution turns out more localised around $k = 0$.

3.3. Complexities analysis and uncertainty relations

Inserting the eigenfunctions (29) and (34) in (9a), (9b), and considering $\sigma = a$, we obtain the quantum entropy densities in the position and the wave-vector spaces $\rho_S(x) \equiv -\rho_n(x) \ln[\rho_n(x)]$ and $\rho_S(k) \equiv -\tilde{\rho}_n(k) \ln[\tilde{\rho}_n(k)/a]$. Figure 3 shows the quantum entropy densities, in the position and the wave-vector representations, for the three first eigenstates and some values of the deformation parameter $\gamma a$. The entropy in position space is
\[
S_x[\rho_n] = \ln \left( \frac{2L_\gamma}{a} \sqrt{1 - \gamma^2 a^2} \right) - 1
\]
(36)

from which the usual case $S_x[\rho_n] = \ln(4) - 1$ is recovered for $\gamma \to 0$. For the case of particles with constant mass, the independence of the entropy $S_x[\rho_n]$ with $n$ is due to the periodicity of the probability density $\rho_n(x)$ (see Ref. [29]). Note that although the probability density is non-periodic the entropy in position (36) does not depend on $n$. From the classical probability distribution (30) we obtain $S_x[\rho_{\text{classical}}] = \ln(L_\gamma \sqrt{1 - \gamma^2 a^2}/a)$, and then it satisfies the relation between the entropies for the distributions in the classical and quantum formalisms, i.e., $S_x[\rho_n] = S_x[\rho_{\text{classical}}] + \ln(2) - 1$ [65].

For the entropy in the $k$-space, using the probability density (34) we have
\[
S_k[\tilde{\rho}_n] = -\ln \left( \frac{2L_\gamma}{a} \right) + f(n),
\]
(37)

which $f(n)$ is a trigonometric functional given by
\[
f(n) = \ln \left( \frac{8}{\pi} \right) - \pi \int_{-\infty}^{+\infty} \frac{n^2 \sin^2 u}{(u^2 + \pi nu)^2} \ln \left[ \frac{n^2 \sin^2 u}{(u^2 + \pi nu)^2} \right] du.
\]
(38)
Information-theoretic measures for a PDM system in an infinite potential well

Figure 1. (Color online) The first three energy eigenfunctions $\psi_n(x)$ (first line) along with their probability densities $\rho_n(x)$ (second line) and $\tilde{\rho}_n(k)$ (third line) for a particle with a PDM given by (15) in a symmetric infinite square well, and for different values of $\gamma a$ (for comparing, the standard case $\gamma a = 0$ is also represented). (a), (d) and (g): $n = 1$ (ground state). (b), (e) and (h): $n = 2$ (first excited state). (c), (f) and (i): $n = 3$ (second excited state).

Figure 2. (Color online) Probability densities (a) $\rho_n(x)$ and (b) $\tilde{\rho}_n(k)$ for a PDM (15) in a symmetric infinite square well for $\gamma a = 0.8$ and $n = 10$. The classical distribution $\rho_{\text{classical}}(x)$ (dot-dashed curve) given by (30) is shown for comparison in (a). The upper bound (dotted curve) corresponds to $2\rho_{\text{classical}}(x)$. In both figures (a) and (b), the usual case ($\gamma a = 0$) is shown for comparison.
The usual case $\mathcal{S}_k[\tilde{\rho}_n] = -\ln(4) + f(n)$ is recover as $\gamma \to 0$. An analytic expression for $f(n)$ has not yet been found. However, from a numerical analysis it is obtained that $f(n)$ increases with $n$. Specifically, we have that $f(1) \approx 3.21204$, $f(2) \approx 3.60700$ and $f(3) \approx 3.75314$. It is known that $\lim_{n \to \infty} f(n) = \ln(8\pi) + 2(1 - c)$, where $c$ is the Euler-Mascheroni constant.

Hence, the Shannon entropy sum for a particle with a PDM results

$$\mathcal{S}_x[\rho_n] + \mathcal{S}_k[\tilde{\rho}_n] = f(n) - 1 + \ln(\sqrt{1 - \gamma^2 a^2}).$$

(39)

For the usual case the sum $(\mathcal{S}_x[\rho_n] + \mathcal{S}_k[\tilde{\rho}_n])|_{\gamma=0} = f(n) - 1$ satisfies the entropic uncertainty relationship (10) once $f(n) - 1$ varies from 2.21204 to 3.06974. Thus, for a particle with a PDM within the range $|\gamma a| < 1$ we have that the sum (39) obeys also the entropic uncertainty relation (10).

From the expressions of $\mathcal{S}_x[\rho_n]$ and $\mathcal{S}_k[\tilde{\rho}_n]$, we obtain the following position and wave-vector Shannon length

$$L_S[\rho_n] = \exp(S[\rho_n]) = a \exp(\mathcal{S}_x[\rho_n]) = \frac{2L_\gamma}{e} \sqrt{1 - \gamma^2 a^2},$$

(40a)

$$L_S[\tilde{\rho}_n] = \exp(S[\tilde{\rho}_n]) = \frac{1}{a} \exp(\mathcal{S}_k[\tilde{\rho}_n]) = \frac{\exp(f(n))}{2L_\gamma}.$$  

(40b)

![Figure 3](image)

**Figure 3.** (Color online) Entropy densities in position (upper line) and wave-vector (bottom line) spaces for a particle with a PDM in an infinite potential quantum well for different parameters $\gamma a$ (the usual case, $\gamma a = 0$, is shown for comparison). [(a) and (d)] $n = 1$ (ground state), [(b) and (e)] $n = 2$ (first excited state), [(c) and (f)] $n = 3$ (second excited state). As a result of the deformation $\gamma a$ and due to the functional form of the PDM, the entropy densities present regions with negative values, which become more pronounced as the deformation and the quantum number increase.

Recalling the standard Fisher information in the position space $\{|\hat{x}\rangle\}$

$$F[\rho_n] = \int_{-\infty}^{+\infty} \frac{1}{\rho_n(x)} \left[ \frac{d\rho_n(x)}{dx} \right]^2 dx = 4 \int_{-\infty}^{+\infty} \left[ \frac{d\psi_n(x)}{dx} \right]^2 dx = \frac{4\langle \hat{p}^2 \rangle}{\hbar^2},$$

(41)
and using the moment (31d), we have
\[ F[\rho_n] = \frac{4k_{Thomas}^2 \gamma a}{(1 - \gamma^2 a^2)^2} \frac{\theta(\gamma)}{\mathrm{atanh}(\gamma)} \left[ 1 + \frac{\mathrm{atanh}^2(\gamma a)}{4 \mathrm{atanh}^2(\gamma a) + (n \pi)^2} \right]. \] (42)

The Fisher information for \(k\)-space is obtained in a similar way. Writing the wave function (33) in the form \( \psi_n(k) = \zeta_n(k)e^{-\gamma \alpha_n(k)} \), with \( \zeta_n(k) = \sqrt{\rho_n(k)} \) and \( \alpha_n(k) = 1/2[k\gamma^{-1} \ln(1 - \gamma^2 a^2) + \pi(n + 1)] \), one obtains
\[ F[\tilde{\rho}_n] = \int_{-\infty}^{+\infty} \frac{1}{\tilde{\rho}_n(k)} \left[ \frac{d\rho_n(k)}{dk} \right]^2 dk = 4 \int_{-\infty}^{+\infty} \frac{d\zeta_n(k)}{dk} \frac{d\zeta_n(k)}{dk} dk = 4\langle \hat{q}^2 \rangle, \] (43)

The probability distribution in the deformed space is \( \bar{\rho}_n(\eta) = |\phi_n(\eta)|^2 = A^2 \sin^2(k_{\gamma,n}(\eta)) \) for \( \gamma^{-1} \ln(1 - \gamma a) < \eta < \gamma^{-1} \ln(1 + \gamma a) \) and \( \bar{\rho}_n(\eta) = 0 \) otherwise. Therefore, it is straightforward verify that
\[ F[\bar{\rho}_n] = 4L_{\gamma} \left[ \frac{1}{3} - \frac{\ln(1 + \gamma a)}{\gamma L_{\gamma}} + \frac{\ln^2(1 + \gamma a)}{\gamma^2 L_{\gamma}^2} - \frac{1}{2(n \pi)^2} \right]. \] (44)

For \( \gamma = 0 \), equations (42) and (44) recover respectively the usual cases \( F[\rho_n] = \frac{\pi^2 n^2}{a^2} \) and \( F[\tilde{\rho}_n] = \frac{4a^2}{3} (1 - \frac{6}{\pi^2 n^2}) \), according to [29]. Also, we have
\[ L_F[\rho_n] = \frac{a}{n \pi} (1 - \gamma^2 a^2) \left( \frac{L_{\gamma}}{2a} \right)^{\frac{1}{2}} \left[ 1 + \frac{\mathrm{atanh}^2(\gamma a)}{4 \mathrm{atanh}^2(\gamma a) + (n \pi)^2} \right]^{-\frac{1}{2}} \] (45a)
and
\[ L_F[\bar{\rho}_n] = \frac{1}{2L_{\gamma}} \left[ \frac{1}{3} - \frac{\ln(1 + \gamma a)}{\gamma L_{\gamma}} + \frac{\ln^2(1 + \gamma a)}{\gamma^2 L_{\gamma}^2} - \frac{1}{2(n \pi)^2} \right]^{-\frac{1}{2}}. \] (45b)

In agree with (4), the disequilibrium of the particle with PDM for spaces \( x \) and \( k \) are
\[ D[\rho_n] = \frac{3}{4a} \frac{\gamma^2 a^2}{(1 - \gamma^2 a^2)^2} \frac{\mathrm{atanh}^2(\gamma a)}{4\pi^4 n^4} \] (46)
and
\[ D[\bar{\rho}_n] = L_{\gamma} \frac{6}{\pi} \left( 1 + \frac{15}{2\pi^2 n^2} \right). \] (47)

By using (31a), (31b) and (42), we get Cramér-Rao complexity, (7a), in position space is
\[ C_{\mathrm{CR}}[\rho_n] = \left( \frac{n \pi \gamma a}{1 - \gamma^2 a^2} \right)^2 \frac{1}{\mathrm{atanh}^2(\gamma a)} \left[ 1 + \frac{\mathrm{atanh}^2(\gamma a)}{4 \mathrm{atanh}^2(\gamma a) + (n \pi)^2} \right] \times \left\{ \frac{\mathrm{atanh}^2(\gamma a)(n \pi)^2}{4 \mathrm{atanh}^2(\gamma a) + (n \pi)^2} - \frac{(\gamma a)(n \pi)^4}{[\mathrm{atanh}^2(\gamma a) + (n \pi)^2]^2} \right\}, \] (48)
and from (35a), (35b) and (44) for wave vector space
\[ C_{\mathrm{CR}}[\bar{\rho}_n] = (2\pi n)^2 \left[ \frac{1}{3} - \frac{\ln(1 + \gamma a)}{2 \mathrm{atanh}(\gamma a)} + \frac{\ln^2(1 + \gamma a)}{4 \mathrm{atanh}^2(\gamma a)} - \frac{1}{2(n \pi)^2} \right]. \] (49)
Both complexities (48) and (49) recover the usual cases $C_{\text{CR}}[\rho_n] = C_{\text{CR}}[\tilde{\rho}_n] = 4 \left( \frac{\pi^2 n^2}{3} - 2 \right) > 1$ for $\gamma a = 0$. On the other hand, the Cramér-Rao complexities have different expressions in $x$ and $k$ spaces for $\gamma a \neq 0$. For Rydberg states (i.e., $n \gg 1$) the asymptotic behavior of (48) and (49) are respectively given by

$$C_{\text{CR}}[\rho_n] \approx \left( \frac{n\pi \gamma a}{1 - \gamma^2 a^2} \right)^2 \frac{1}{\text{atanh}^4(\gamma a)} \left[ 1 - \frac{\gamma a}{\text{atanh}(\gamma a)} \right], \quad (50)$$

$$C_{\text{CR}}[\tilde{\rho}_n] \approx (2\pi n)^2 \left[ \frac{1}{3} - \frac{\ln(1 + \gamma a)}{2\text{atanh}(\gamma a)} + \frac{\ln^2(1 + \gamma a)}{4\text{atanh}^2(\gamma a)} \right]. \quad (51)$$

From (40a), (40b), (45a) and (45b) the Fisher-Shannon complexities, (7b), for $x$ and $k$ spaces are respectively expressed by

$$C_{\text{FS}}[\rho_n] = \frac{8\pi n^2}{e^3} \left( 1 - \gamma^2 a^2 \right) \text{atanh}(\gamma a) \left[ 1 + \frac{\text{atanh}^2(\gamma a)}{4\text{atanh}^2(\gamma a) + (n\pi)^2} \right], \quad (52)$$

and

$$C_{\text{FS}}[\tilde{\rho}_n] = \frac{e^{2f(n) - 1}}{2\pi} \left[ \frac{1}{3} - \frac{\ln(1 + \gamma a)}{2\text{atanh}(\gamma a)} + \frac{\ln^2(1 + \gamma a)}{4\text{atanh}^2(\gamma a)} - \frac{1}{2(n\pi)^2} \right]. \quad (53)$$

For $\gamma a = 0$, one recovers $C_{\text{FS}}[\rho_n] = \frac{8\pi n^2}{e^3}$ and $C_{\text{FS}}[\tilde{\rho}_n] = \frac{e^{2f(n) - 1}}{24\pi} \left( 1 - \frac{6}{\pi^2 n^2} \right)$. The Rydberg states for (52) and (53) become

$$C_{\text{FS}}[\rho_n] \approx \frac{8\pi n^2}{e^3} \left( 1 - \gamma^2 a^2 \right) \text{atanh}(\gamma a), \quad (54a)$$

$$C_{\text{FS}}[\tilde{\rho}_n] \approx 32\pi e^{3 - 4c} \left[ \frac{1}{3} - \frac{\ln(1 + \gamma a)}{2\text{atanh}(\gamma a)} + \frac{\ln^2(1 + \gamma a)}{4\text{atanh}^2(\gamma a)} \right]. \quad (54b)$$

The LMC complexities (7c) are

$$C_{\text{LMC}}[\rho_n] = \frac{3}{e \text{atanh}(\gamma a) \text{atanh}^2(\gamma a) + \pi^2 n^2 \text{atanh}^2(\gamma a) + 4\pi^2 n^2}, \quad (55)$$

and

$$C_{\text{LMC}}[\tilde{\rho}_n] = \frac{e^{f(n)}}{12\pi} \left( 1 + \frac{15}{2\pi^2 n^2} \right), \quad (56)$$

with $C_{\text{LMC}}[\rho_n] = 3/e$ for $\gamma a = 0$. Interestingly, $C_{\text{LMC}}[\tilde{\rho}_n]$ depends only on $n$ as in the standard case. Again, considering the Rydberg states, we have

$$C_{\text{LMC}}[\rho_n] \approx \frac{3}{e} \frac{\gamma a}{\sqrt{1 - \gamma^2 a^2 \text{atanh}(\gamma a)}}, \quad (57)$$

$$C_{\text{LMC}}[\tilde{\rho}_n] \approx \frac{2}{3} e^{2(1-c)}. \quad (58)$$

Given a fixed value of the deformation $\gamma a$, from the expressions (48)-(56) we have that $C_{\text{CR}}[\rho_n], C_{\text{CR}}[\tilde{\rho}_n], C_{\text{FS}}[\rho_n] \propto n^2$, for $n \gg 1$, which expresses the quadratic dependence with $n$ for these complexities in the classical limit. By contrast, when $n \gg 1$ we have $C_{\text{FS}}[\tilde{\rho}_n], C_{\text{LMC}}[\rho_n]$ and $C_{\text{LMC}}[\tilde{\rho}_n]$ are independent of $n$. The figure 4 shows the
Information-theoretic measures for a PDM system in an infinite potential well

complexities (7a), (7b) and (7c) for distributions (29) and (34) of the first three excited states. For the probability density in position space, the Cramér-Rao and Fisher-Shannon complexities present a similar behavior, while the LMC complexity is very different from those by exhibiting more sensibility to the deformation. Although the entropy densities in position are asymmetric with respect the axis $x = 0$ (figure 3), all the complexities turn out symmetric around $\gamma a = 0$ (absence of deformation). We also observe that the curves have a similar shape between the Cramér-Rao and Fisher-Shannon complexities with respect to the deformation range of values analysed $-1 < \gamma a < 1$ (plots (a), (b), (c) and (d) of the figure 4), while the LMC complexity is the more affected by the deformation (plots (e) and (f) of the figure 4). The abrupt variation of the Cramér-Rao and Fisher-Shannon complexities observed from a certain value of $\gamma a$ (which depends on $n$) indicates the sensibility of the Fisher information. The oscillations of the entropy densities are reflected in the curves of $C_{CR}$ and $C_{FS}$ (in position and wave vector spaces) that are located from bottom to up as the quantum number increases. This does not occurs with the LMC complexity since the curves of $C_{LMC}[\rho_n]$ corresponding to $n = 2$ and $n = 3$ are almost superposed as well as $C_{LMC}[\tilde{\rho}_n]$ to $n = 1$ and $n = 2$. Plots (e) and (f) express the uncertainty principle between the position and the wave vector by means of their LMC complexities. We see that while the LMC's in position space of the second and the third excited states are almost superposed the corresponding LMC's curves in wave vector space are straight lines maximally separated. For comparison, the plots (g) and (h) illustrate the complexities of the ground state $n = 1$. It can be observed that in both $x$ and $k$ spaces the order $C_{CR} > C_{FS} > C_{LMC} > 1$ is satisfied.

4. Conclusions

We have studied three main complexity measures on continuous probability density distributions used in the literature, the Cramér-Rao, the Fisher-Shannon and the LMC ones, in the context of the deformed Schrödinger equation for PDM systems proposed originally by Costa Filho et al. in reference [47]. To analyse their behaviors we have considered some eigenstates of a particle with a PDM and confined in an infinite potential well.

The effect of the PDM is characterised by the arising of an asymmetry in the entropy density in position space around the value $x = 0$ (that depends on the functional form of the PDM) and by regions with negative values, that become more pronounced as the deformation increases for $\gamma a > 0$. In wave vector space the entropy density is symmetric around $k = 0$, but becomes more compressed when the deformation parameter increases.

The complexities are symmetric with respect to the deformation parameter in both position and wave vector spaces. The Cramér-Rao and Fisher-Shannon complexities allow to distinguish the eigenstates in the presence of the deformation, in such a way that they are well separated (see figure 4). In addition, the allowed range of the deformation is bounded $|\gamma a| \leq 1$, where the critical values $|\gamma a| = 1$ correspond to divergences of the
Figure 4. Cramér-Rao [(a) and (b)], Fisher-Shannon [(c) and (d)] and LMC complexities [(e) and (f)] for position and wave vector spaces densities probability \( \rho_n(x) \) and \( \tilde{\rho}_n(k) \) of the first three eigenstates of a particle with a PDM in a symmetric infinite square potential well of width \( 2a \), in function of the dimensionless deformation parameter \( \gamma a \). The three complexities for the ground state are illustrated for the \( x \) and \( k \) spaces in (g) and (h), respectively. The ordering \( C_{LMC} < C_{FS} < C_{CR} \) is satisfied for the probability distributions in both spaces.

Regarding the classical limit, the quadratic dependence with the quantum number is recovered for the Cramér-Rao and the Fisher-Shannon complexities. From our study we conclude that in the context of PDM systems the complexities in
Information-theoretic measures for a PDM system in an infinite potential well

16 position and wave vector space can provide the same information, as we see for the Cramér-Rao (plots (a) and (b) of figure 4) and the Fisher-Shannon ones (plots (c) and (d) of figure 4), by separating states almost in the same manner. Moreover, if the complexity curves are superposed (i.e., they can not distinguish between different states as we can see from the plot (e) of figure 4 for the LMC) the corresponding ones in the wave vector space must be separated due to the uncertainty principle. Hence, given a certain complexity, in practice it is necessary to calculate it in the position and the wave vector representations in such a way that, if one of them can not distinguish states the other will do it according to the uncertainty principle. This is a feature that has not classical analogue and then, it can not be used within an exclusively classical context.

Finally, we consider that the present work gives interesting features for studying the relationship between the PDM systems and the information theoretical measures: abrupt variation of the complexity near to the asymptotic value of the PDM, erasure of the asymmetry of the complexity and an entropic density with negative values. We hope this can be explored with more examples in future researches.

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