LCK METRICS ON ELLIPTIC PRINCIPAL BUNDLES

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Abstract. For elliptic principal bundles $\pi : X \to B$ over Kähler manifolds it was shown by Blanchard that $X$ has a Kähler metric if and only both Chern classes (with real coefficients) of $\pi$ vanish. For some elliptic principal bundles, when the span of these Chern classes is 1-dimensional, it was shown by Vaisman that $X$ carry locally conformally Kähler (LCK, for short) metrics. We show that in the case when the Chern classes are linearly independent, $X$ carries no LCK metric.

1. Introduction.

1.1. Motivation. Elliptic principal bundles, or, more generally, principal torus bundles (we will recall immediately the definitions) were always an excellent "reservoir" for interesting examples in complex geometry. A short list would certainly include:
- the very first non-Kähler manifolds (H. Hopf, 1948) - some Hopf surfaces
- which are such bundles over $\mathbb{P}^1$;
- the first examples of non-Kähler simply connected manifolds (Calabi-Eckmann manifolds) are elliptic bundles over a product of projective spaces $\mathbb{P}^r \times \mathbb{P}^s$,
- the first examples of manifolds with non-closed global holomorphic forms (Iwasawa manifolds) are again elliptic principal bundles over abelian surfaces,
- examples of manifolds for which the Fröhlicher spectral sequence degenerates arbitrarily high (Röllenske, [5]).

On the other hand, these kind of manifolds are abundant; for instance, in complex dimension 2 such compact surfaces exists in each class in Kodaira classification (except for class $VII_{>0}$).

A natural question one may ask about this type of manifolds is about the kind of hermitian metrics they can carry. A classical result of Blanchard (cf. [II]) states that if $\pi : X \to B$ is an elliptic principal bundle over a compact Kähler manifold, then $X$ carries a Kähler metric if and only if both Chern classes (with real coefficients) of the bundle vanish. On the other hand, on a rather large class of elliptic principal bundles which are non-Kähler, one can show the existence of a locally conformally Kähler metric.

1.2. Chern classes of elliptic principal bundles. In this section, we quickly recall some basic facts about elliptic principal bundles: more details...
can be found for instance in [2] or in [4]. Let $B$ be a differentiable manifold and $E$ a 2-dimensional real torus. The set of isomorphism classes of principal bundles over $B$ with fiber $E$ is classified by the cohomology group $H^1(B, \mathcal{C}_B(E))$ where $\mathcal{C}_B(E)$ is a (ad-hoc) notation for the sheaf of germs of differentiable, $E$-valued functions defined on (open subsets of) $B$. As $E \cong \mathbb{S}^1 \times \mathbb{S}^1$, letting $\mathbb{Z}_B$ (respectively $\mathcal{C}_B(\mathbb{C})$) for the sheaf of germs of differentiable integer-valued (respectively complex-valued) functions, from the exact sequence

$$0 \to \mathbb{Z}_B^{\oplus 2} \to \mathcal{C}_B(\mathbb{C}) \to \mathcal{C}_B(E) \to 0$$

and the fact that $\mathcal{C}_B(\mathbb{C})$ is a fine sheaf, we see

$$H^1(B, \mathcal{C}_B(E)) \cong H^2(B, \mathbb{Z}_B)$$

The image of a given $E$-principal bundle $X \to B$ under the above isomorphism will be denoted (keeping the notations in [2]) by $(c'_1(X), c''_1(X))$; its components will be called the Chern classes of $X$.

An alternative way of defining the Chern classes is as follows. As translations of $E$ act trivially in the cohomology of $E$, we see that for any principal bundle $\pi : X \to B$ with fiber $E$ one has $\mathcal{R}^1\pi_*(\mathbb{Z}_X) \cong \mathbb{Z}_B \otimes H^1(E, \mathbb{Z})$ for all $i$. Fix $\alpha, \beta$ generators of $H^2(E, \mathbb{Z})$ and consider the spectral sequence

$$E_{pq}^2 = H^q(B, \mathcal{R}^p\pi_*(\mathbb{Z}_X)) \Rightarrow H^{p+q}(B, \mathbb{Z}_B).$$

Then the images of $\alpha, \beta$ under the differential

$$d : H^0(B, \mathcal{R}^1\pi_*(\mathbb{Z}_X)) \to H^2(B, \pi_*(\mathbb{Z}_X))$$

are the above Chern classes (modulo a possible twist by an automorphism of $H^1(E, \mathbb{Z})$, that is, modulo action by an element in $SL_2(\mathbb{Z})$).

In the next lemma we gather some basic facts needed further; most likely, its content is well-known, but as we were not able to find a precise reference, we include a proof below. Notice that Chern classes here are viewed in $H^2(B, \mathbb{R})$ under the natural map $H^2(B, \mathbb{Z}) \to H^2(B, \mathbb{R})$.

**Lemma 1.** Let $\pi : X \to B$ be a principal bundle with fiber a 2-dimensional torus $E$.

a) If at least one of the Chern classes of it is nontrivial in $H^2(B, \mathbb{R})$ then the homology class (with $\mathbb{R}$-coefficients) of a fiber vanishes.

b) If the two Chern classes are linearly independent in $H^2(B, \mathbb{R})$ then the natural map $\pi^* : H^1(B, \mathbb{R}) \to H^1(X, \mathbb{R})$ is an isomorphism.

**Proof.** a) Let $\mathbb{R}_X$ (resp. $\mathbb{R}_B$) denote the sheaf of locally constant functions on $X$ (resp. $B$). Consider the spectral sequence

$$E_{pq}^2 = H^q(B, \mathcal{R}^p\pi_*(\mathbb{R}_X)) \Rightarrow H^{p+q}(B, \mathbb{R}_B).$$

This induces an exact sequence

$$0 \to \mathbb{R}_B^{\oplus 2} \to \mathcal{R}_B(\mathbb{C}) \to \mathcal{R}_B(E) \to 0$$

and the fact that $\mathcal{R}_B(\mathbb{C})$ is a fine sheaf, we see

$$H^1(B, \mathcal{R}_B(E)) \cong H^2(B, \mathbb{R}_B)$$

The image of a given $E$-principal bundle $X \to B$ under the above isomorphism will be denoted (keeping the notations in [2]) by $(c'_1(X), c''_1(X))$; its components will be called the Chern classes of $X$.
\[ H^2(X, \mathbb{R}) \overset{i^*}{\to} H^0(B, \mathcal{R}^2\pi_*(\mathbb{R}_X)) \overset{d}{\to} H^2(B, \mathcal{R}^1\pi_*(\mathbb{R}_X)) \]

which, under canonical identifications, becomes

\[ H^2(X, \mathbb{R}) \overset{i^*}{\to} H^2(E, \mathbb{R}) \overset{d}{\to} H^2(B, \mathbb{R}) \otimes H^1(E, \mathbb{R}). \]

We must show that the natural map \( H^2(E, \mathbb{R}) \to H^2(X, \mathbb{R}) \) induced by the inclusion of a fiber is zero. This map is the dual of the map \( i^* \) form (1).

But \( H^2(E, \mathbb{R}) \) is generated by \( \alpha \wedge \beta \) and as differentials in the above spectral sequence are multiplicative, we see

\[ d(\alpha \wedge \beta) = c'_1(X) \otimes \beta + c''_1(X) \otimes \alpha \]

Now if at least one of \( c'_1(X), c''_1(X) \) is non-vanishing, we see the map \( d \) is also nonvanishing; since \( H^2(E, \mathbb{R}) \) is one dimensional, we get \( d \) is actually injective, so \( i^* \) is the null map.

b). The proof is similar, only at this time we look at another exact sequence induced by the spectral sequence above, namely (again under canonical identifications) at

\[ 0 \to H^1(B, \mathbb{R}) \overset{\pi^*}{\to} H^1(X, \mathbb{R}) \overset{\delta}{\to} H^1(E, \mathbb{R}) \overset{c}{\to} H^2(B, \mathbb{R}) \]

The map \( c \) acts by \( c(\alpha) = c'_1, c(\beta) = c''_1 \) so our hypothesis implies \( c \) is injective, hence \( \delta \) is the null map thus \( \pi^* \) is surjective.

□

If \( X, B \) are complex manifolds and \( E \) is an elliptic curve (viewed as a complex 1-dimensional Lie group), a holomorphic principal bundle map \( \pi : X \to B \) with fiber \( E \) will be called elliptic principal bundle for short. From a) of the above Lemma and since no proper compact complex submanifold of a Kähler manifold can be homologous to zero, we have:

**Corollary 1.** If \( E \) is an elliptic curve and \( \pi : X \to B \) is an elliptic principal bundle such that at least one of the Chern classes is non-vanishing (with real coefficients) then \( X \) carries no Kähler metric.

In particular, we rediscover one implication in Blanchard’s theorem - but with no hypothesis on the base \( B \).
following. Let \( \Omega \) be the Kähler form of \( g \); then \( g \) is LCK iff there exists a closed 1–form \( \theta \) such that
\[
d\Omega = \theta \wedge \Omega.
\]
Moreover, \( g \) is globally conformal to a Kähler metric iff \( \theta \) is exact.

Historically, the first example of an LCK metric was given by Vaisman in 1976 in [3]; we include it here very briefly. Let \( n \geq 1 \) and let \( W \overset{\text{def}}{=} \mathbb{C}^n \setminus \{0\} \).

Let \( Z \) act on \( W \) by
\[
l \circ (z_1, \ldots, z_n) = (2^l z_1, \ldots, 2^l z_n), l \in \mathbb{Z}.
\]
Then the action is fixed-point-free and properly discontinuous, and the quotient \( X \overset{\text{def}}{=} W/Z \) is compact. One immediately checks that the hermitian metric
\[
g = \frac{1}{\sum |z_i|^2} (dz_1 d\overline{z}_1 + \cdots + dz_n d\overline{z}_n)
\]
on \( W \) is \( Z \)--invariant and defines an LCK metric on \( X \).

On the other hand, \( X \) has an obvious holomorphic projection \( \pi : X \rightarrow \mathbb{P}^{n-1}(\mathbb{C}) \); one immediately checks that \( \pi \) is actually an elliptic principal bundle, whose Chern classes are \((\alpha, 0)\) where \( \alpha \) is the Poincaré dual of a hyperplane section in \( \mathbb{P}^{n-1}(\mathbb{C}) \).

As a consequence, we get the following

**Corollary 2.** Let \( B \) be a complex projective manifold. Then there exists an elliptic principal bundle \( X \rightarrow B \) such that \( X \) has no Kähler metric but carries an LCK metric. For this bundle one of the Chern classes vanishes.

Indeed, just embed \( B \) into some projective space and take the restriction of the map \( \pi \) above to it.

### 2. The main result.

We begin by recalling, for the sake of simplicity, the following well-known statement:

**Lemma 2.** Let \((X, \Omega)\) be an LCK manifold, \( \theta \) the associated Lee form of \( \Omega \) and let \( \omega \) be a 1–form on \( X \) which is cohomologous to \( \theta \). Then there exists a metric \( \Omega' \) on \( X \) which is conformally equivalent to \( \Omega \) (in particular \( \Omega' \) is LCK too) such that the Lee form of \( \Omega' \) is \( \omega \).

We are now ready to prove the main result of this note.

**Theorem 1.** Let \( X, B \) be compact complex manifolds, \( X \overset{\pi}{\rightarrow} B \) an elliptic principal bundle with fiber \( E \). If the Chern classes of the bundle \( X \overset{\pi}{\rightarrow} B \) are linearly independent in \( H^2(B, \mathbb{R}) \) then \( X \) carries no locally conformally Kähler structure.

**Proof.** Assume \( X \) carries an LCK metric \( \Omega \) with Lee form \( \theta \). As the Chern classes are independent, we get that \( \pi^*: H^1(B, \mathbb{R}) \rightarrow H^1(X, \mathbb{R}) \) is an isomorphism. Hence there exists a 1–form \( \eta \) on \( B \) such that \( \theta \) and \( \pi^*(\eta) \) are cohomologous. By Lemma 2 we may assume that \( \theta = \pi^*(\eta) \). This yields
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\[ d\Omega = \pi^*(\eta) \wedge \Omega. \]

Now for each \( b \in B \) let \( \lambda(b) \overset{\text{def}}{=} \int_{E_b} \Omega = \text{vol}_\Omega(E_b); \) clearly \( \lambda \) is a differentiable map \( \lambda : B \to \mathbb{R}_{>0}. \)

Hence, replacing \( \Omega \) with \( \frac{1}{\lambda} \Omega \) we can assume that relation (2) holds good and also that the volume of the fibers with respect to \( \Omega \) is constant.

Let now \( [\gamma] \in \pi_1(B) \) be arbitrary and let \( \gamma \in [\gamma] \) be a smooth representative. Let \( M \overset{\text{def}}{=} \pi^{-1}(\gamma); \) we see \( M \) is a smooth real 3-submanifold of \( X. \)

We have
\[ 0 = \int_M d\Omega = \int_M \pi^*(\eta) \wedge \Omega. \]

But from the independence of \( \int_{E_b} \Omega \) on \( b \in B \) and from Fubini's theorem, we get furthermore
\[ 0 = \left( \int_{[\gamma]} \eta \right) \left( \int_{E_b} \Omega \right). \]

We obtain \( \int_{[\gamma]} \eta = 0 \) and hence \( \eta \) is exact, as \( [\gamma] \) was arbitrary. We derive that \( \Omega \) is globally conformally Kähler, so \( X \) would admit a Kähler metric; but this is impossible by Corollary [1].

\[ \square \]

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