The **Random Edge** Rule
on Three-Dimensional Linear Programs\(^1\)
(extended abstract)

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**Abstract**

The worst-case expected length of the path taken by the simplex algorithm
with the **Random Edge** pivot rule on a 3-dimensional linear program with \(n\)
constraints is shown to be bounded by

\[
1.3445 \cdot n \leq f(n) \leq 1.4943 \cdot n
\]

for large enough \(n\).

**1 Introduction**

The **Random Edge** pivot rule is undoubtedly the most natural, and simplest (randomized) pivot rule for the simplex algorithm: “At each iteration, proceed from the current vertex of the polyhedron \(P\) of feasible solutions to an improving neighbor, chosen uniformly at random in the one-dimensional skeleton (i.e., the graph) of \(P\).”

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Despite its simplicity, this algorithm until now has resisted almost all attempts to analyze its worst-case behavior, with a few exceptions for special cases, among them the linear assignment problem (Tovey [6]), certain linear programs on cubes, including the Klee-Minty cubes (Kelly [4], Gärtner, Henk & Ziegler [2]), and \(d\)-dimensional linear programs (i.e., \(\text{dim}(P) = d\)) with at most \(d + 2\) constraints (Gärtner et al. [3]). All known results leave open the possibility that the \textsc{Random Edge} pivot rule yields a strongly polynomial time algorithm – it might be even quadratic. In particular, it is not fooled by the deformed products (defined by Amenta and Ziegler [1]), which yield the well-known exponential examples for all the classical deterministic pivot rules.

Here, we only treat the case of 3-dimensional linear programs, which, of course, is solved in linear time by every (even deterministic) finite variant of the simplex algorithm. Nevertheless, due to the remarks made above, it seems interesting to analyze the \textsc{Random Edge} pivot rule for this case – and here too, it seems that accurate analysis of \textsc{Random Edge} is quite hard.

With the usual reductions (see, e.g., Ziegler [7, Lect. 3]), we may assume that our linear program is \(\min\{x_3 : x \in P\}\), where \(P\) is a 3-dimensional simple polytope (its graph is 3-regular) with exactly \(n\) facets, and hence \(2n - 4\) vertices and \(3n - 6\) edges, and no two vertices have the same objective function value \(x_3\). Thus we have an ordering of the vertices \(v_{2n-5}, v_{2n-6}, \ldots, v_1, v_0\) by decreasing objective function (i.e., by height). Here \(v_0 = v_{\min}\) is the unique minimal (lowest) vertex of the linear program, while \(v_{2n-5} = v_{\max}\) is the unique maximal (highest) vertex.

The expected length of the path (i.e., the number of pivot steps) taken by the simplex algorithm on the linear program, starting at vertex \(v\) of \(P\) and using the \textsc{Random Edge} rule, is then given by \(E(v_0) = 0\) and

\[
E(v) = 1 + \frac{1}{d_v} \sum_{j=1}^{d_v} E(w_j) \quad (v \neq v_0),
\]

where \(w_1, \ldots, w_{d_v}\) are the lower neighbors of \(v\). It is easy to see that (in addition to the unique maximal vertex and the unique minimal vertex) there are \(n - 3\) vertices \(v\) with \(d_v = 1\) (1-\textit{vertices}) and \(n - 3\) vertices \(v\) with \(d_v = 2\) (2-\textit{vertices}); this is the 3-dimensional case of the Dehn-Sommerville equations [7, Thm. 8.21].

Define \(f(n)\) to be the maximum expected number of pivot steps taken by the \textsc{Random Edge} algorithm on any 3-dimensional linear program with \(n\) constraints. While it is quite straightforward to construct a sequence of examples with \(E(v) \geq \frac{4}{3} \cdot n - \text{const}\), our results in Section 2 (Theorem 2.2) show

\[
f(n) \geq \frac{1721}{1280} \cdot n - \frac{4722}{1280}
\]

for infinitely many \(n\) in arithmetic progression (\(\frac{1721}{1280} = \frac{4}{3} + \frac{43}{3840}\)). In Section 3 we prove

\[
f(n) \leq \frac{130}{87} \cdot n - \frac{115}{29}
\]
(Theorem 3.2). Both results taken together, this yields that
\[ 1.3445 \cdot n \leq f(n) \leq 1.4943 \cdot n \]
holds for all large enough \(n\). In particular, asymptotically \(f(n)\) lies between \((\frac{4}{3} + \varepsilon) \cdot n\) and \((\frac{3}{2} - \varepsilon) \cdot n\) for some \(\varepsilon > 0\). Determining “the right coefficient” seems, however, to be very hard.

2 Lower Bounds

The expected number of pivot steps required by the simplex algorithm using RANDOM EDGE only depends on the graph of the polytope, directed via the objective function. Therefore, we will describe our examples yielding lower bounds on \(f(n)\) by the corresponding directed graphs. The following result provides a nice certificate for a directed graph to come from a 3-dimensional linear program.

**Theorem 2.1 (Mihalisin and Klee [5]).** A directed graph \(D\) (without loops and parallel arcs) is induced by a 3-dimensional linear program if and only if
- it is planar and 3-connected (as an undirected graph),
- it is acyclic with a unique source and a unique sink,
- it has a unique local sink in every face cycle (these are the non-separating induced cycles), and
- it admits three directed paths from its source to its sink that have disjoint sets of interior nodes.

2.1 Duals of Cyclic Polytopes

**Example 1.** Our first sequence of examples are wedges, i.e., they are combinatorially equivalent to duals of cyclic polytopes. Figure 1 depicts the orientations of the edges. Here, as well as in the sequel, our convention is that the ordering of the vertices from left to right in the figure defines the (decreasing) ordering of the vertices according to the objective function. It is easy to see that the conditions of the Mihalisin–Klee theorem are satisfied.

![Figure 1: The example on the dual cyclic polytope.](image-url)
For the expected number of pivot steps $E(v_i)$, we then have the starting values $E(v_0) = 0$ and $E(v_1) = 1$, and the recurrences

$$E(v_{2j}) = E(v_{2j-1}) + 1, \quad E(v_{2j+1}) = \frac{1}{2}(E(v_{2j}) + E(v_{2j-2})) + 1$$

for $0 < j \leq n - 4$. Thus, using induction, we obtain

$$E(v_{2j}) + 2E(v_{2j+1}) = 4j + 2$$

for $0 \leq j \leq n - 4$. In particular, for $j = n - 4$ this yields

$$\max\left\{E(v_{2n-8}), E(v_{2n-7})\right\} \geq \frac{4n}{3} - \frac{14}{3}.$$ 

2.2 Improved Lower Bounds

The next examples are based on the construction of a “backbone polytope”: This will be a simple 3-polytope $P_k$ with $k + 2$ facets and $2k$ vertices, of which $k$ vertices $v_{k-1}, \ldots, v_0$ form a decreasing chain, such that $v_0$ is the minimal vertex, and $v_{i-1}$ is the only lower neighbor of $v_i$, for $i > 0$.

Constructing the backbone. We start with the simplex, obtained for $k = 2$ with vertices $w_0, w_1, v_1, v_0$, as shown on the left. We then inductively cut off the vertex $v_{k-1}$ by a plane, replacing it by a small triangle, as shown on the right.

Example 2. Our second sequence of examples is obtained from the backbone polytopes $P_k$ by performing three specific vertex cuts at each vertex $v_i$, for $i = 0, \ldots, k - 1$. Before cutting (in $P_k$), each vertex $v_i$ ($i > 0$) has indegree 2, while $v_0$ has indegree 3. The two vertex cuts are supposed to create at each vertex $v_i$ the following configuration (where again all edges are directed “to the right”):

This creates a simple polytope $P'_k$ with $n = k + 2 + 3k = 4k + 2$ facets.
Our starting vertex for Random Edge on this example will be \( v_{k-1,6} \); the expected number of steps taken by Random Edge is the sum of the probabilities \( p_e \) that the edge \( e \) is traversed. We think of these probabilities as a “flow” from \( v_{k-1,6} \) to \( v_{0,0} \). Our figure indicates the flow values on the edges, for a flow of total value 8; equivalently, these are the transversal probabilities in units of \( \frac{1}{8} \).

We get the same values for each of the triple-vertex-cut-off configurations, except for the last one, which has no edge leaving the global sink \( v_{0,0} \). Thus the expected number of Random Edge steps, starting from \( v_{k-1,6} \), is

\[
E(v_{k-1,6}) = k \cdot \frac{43}{8} - 1 = \frac{43}{32}n - \frac{59}{16} ,
\]

with \( k = \frac{1}{4}(n - 2) \). Asymptotically, this yields a better lower bound, due to \( \frac{43}{32} > \frac{4}{3} \).

The graph of \( P'_k \) looks like this:

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**Example 3.** The last examples produced \( k = \frac{1}{4}(n - 2) \) vertices which are not used. We will further improve the lower bound by using more facets in each local configuration and thus reducing the number of unused vertices (though our final example will still have linearly many unused vertices).

We use the same backbone polytope as before, but we replace each vertex \( v_i \), for \( i = 0, \ldots, k - 1 \), with the following graph. (We do not give an explicit polytopal construction for this example, but it can be constructed by cutting off vertices and edges of the backbone polytope. Alternatively, such a construction is provided by the Mihalisin-Klee Theorem.) To analyze the random path length through this graph we send from \( v_{i,18} \) 128 units of flow through the network.
The total flow through all edges is 1721. For each configuration 9 facets are required. Together with the facets from the backbone construction, this yields $n = 10k + 2$. Hence we get

$$E(v_{k-1,18}) = k \cdot \frac{1721}{128} - 1 = \frac{1721}{1280}n - \frac{4722}{1280},$$

where $\frac{1721}{1280} = \frac{43}{32} + \frac{1}{1280} = \frac{4}{3} + \frac{1}{96} + \frac{1}{1280} > 1.3445$.

In contrast to the preceding examples, this example does not contain a directed Hamiltonian path.

Summarizing, we have proved the following bound.

**Theorem 2.2.** For $n = 10k + 2 \geq 12$,

$$f(n) \geq \frac{1721}{1280}n - \frac{4722}{1280}$$

holds.

**Starting from the source.** By splitting the maximal vertex, one can also construct examples where the expected number of steps starting at the maximal vertex is at least $(\frac{1721}{1280} - \varepsilon)n$. This observation is due to Günter Rote.

### 3 Upper Bounds

Consider any linear program on a simple 3-polytope with the notations as described in the introduction. For a vertex $v$, let $N_1(v)$ (resp., $N_2(v)$) denote the number of 1-vertices (resp., 2-vertices) that are not higher than $v$ (including $v$ itself). Put $N(v) = N_1(v) + N_2(v)$. This is the number of vertices lower than $v$. The core of our upper bound on $f(n)$ is the following result.

**Theorem 3.1.** For each vertex $v$, other than the maximal vertex $v_{2n-5}$, we have

$$E(v) \leq \frac{46}{87}N_1(v) + \frac{42}{87}N(v).$$

Theorem 3.1 implies

$$E(v) \leq \frac{130(n-3)}{87} + \frac{15}{29} = \frac{130}{87} \cdot n - \frac{115}{29}$$
for all \( v \). Here, the “\( \frac{15}{29} \)” comes from the fact that Theorem 3.1 is proved only for \( v \neq v_{2n-5} \); therefore, we bound \( E(v_{2n-5}) \) by \( 1 + \frac{1}{3} \sum_{i=1}^{3} E(w_i) \), where \( w_1, w_2, w_3 \) are the neighbors of \( v_{2n-5} \), and we exploit \( N(w_1) + N(w_2) + N(w_3) \leq 6n - 21 \).

**Theorem 3.2.** For every 3-dimensional linear program with \( n \) constraints, the expected number of pivot steps taken by Random Edge is not more than

\[
\frac{130}{87} \cdot n - \frac{115}{29} .
\]

In the remaining part of this section, we briefly sketch the proof of Theorem 3.1. It proceeds by deriving the generic inequality

\[
E(v) \leq \alpha N_1(v) + \beta N(v) ,
\]

where, for most of the proof, \( \alpha \) and \( \beta \) are treated as indeterminates. Each step of the proof yields a linear inequality on \( \alpha \) and \( \beta \) that needs to be satisfied in order to imply (1). The proof is then completed once it is shown that \((\alpha, \beta) = \left(\frac{46}{87}, \frac{42}{87}\right)\) satisfies all inequalities; in fact, it is optimal with respect to the objective function \( \alpha + 2\beta \) (see below for more details).

Inequality (1) is proved by induction on \( N(v) \). The base case \( N(v) = 0 \) is obvious, since \( v \) is the optimum in this case, and \( E(v) = 0 \). Suppose the theorem holds for all vertices lower than some vertex \( v \).

We express \( E(v) \) in terms of the expected costs \( E(w) \) of certain vertices \( w \) that are reachable from \( v \) via a few downward edges. The general form of such a recursive expression will be

\[
E(v) = c + \sum_{i=1}^{k} \lambda_i E(w_i) ,
\]

where \( \lambda_i > 0 \) for each \( i = 1, \ldots, k \), and \( \sum_i \lambda_i = 1 \).

Since we assume by induction that \( E(w_i) \leq \alpha N_1(w_i) + \beta N(w_i) \), for each \( i \), it suffices to show that

\[
\sum_{i=1}^{k} \alpha \lambda_i (N_1(v) - N_1(w_i)) + \sum_{i=1}^{k} \beta \lambda_i (N(v) - N(w_i)) \geq c .
\]

Write

\[
\Delta_1(w_i) = N_1(v) - N_1(w_i) ,
\]

\[
\Delta(w_i) = N(v) - N(w_i) ,
\]

for \( i = 1, \ldots, k \). (Of course, these terms are defined with respect to the currently considered vertex \( v \).) Note that \( \Delta(w_i) \) is the distance between \( v \) and \( w_i \), that is, one plus the number of vertices between \( v \) and \( w_i \).

We thus need to show that

\[
\sum_{i=1}^{k} \alpha \lambda_i \Delta_1(w_i) + \sum_{i=1}^{k} \beta \lambda_i \Delta(w_i) \geq c .
\]
This requires a quite extensive case analysis, of which we present only the beginning in this extended abstract in order to indicate the kinds of arguments used. The complete case analysis is given in the appendix.

**Case 1:** $v$ is a 1-vertex. 
Let $w_1$ denote the target of the unique downward edge emanating from $v$ as in the following figure, where (here and in all subsequent figures) each edge is labelled by the probability of reaching it from $v$.

![Diagram of Case 1](image)

In this case, $E(v) = 1 + E(w_1)$ holds. In the setup presented above, we have $c = 1$, $\Delta_1(w_1) \geq 1$, and $\Delta(w_1) \geq 1$, thus (2) is implied by

$$\alpha + \beta \geq 1.$$ (3)

**Case 2:** $v$ is a 2-vertex.
Let $w_1$ and $w_2$ denote the targets of the two downward edges emanating from $v$, where $w_2$ is lower than $w_1$.

![Diagram of Case 2](image)

We have

$$E(v) = 1 + \frac{1}{2} E(w_1) + \frac{1}{2} E(w_2),$$

hence we need to require that

$$\frac{\alpha}{2} \Delta_1(w_1) + \frac{\alpha}{2} \Delta_1(w_2) + \frac{\beta}{2} \Delta(w_1) + \frac{\beta}{2} \Delta(w_2) \geq 1.$$ (4)

Note that $\Delta(w_1) \geq 1$.

**Case 2.a:** $\Delta(w_2) \geq 4$.
Ignoring the effect of the $\Delta_1(w_j)$’s, it suffices to require that

$$\frac{\beta}{2} \Delta(w_1) + \frac{\beta}{2} \Delta(w_2) \geq 1,$$

which will follow if

$$\beta \geq \frac{2}{5}.$$ (4)
Case 2.b.i: $\Delta(w_2) = 3$ and one of the two vertices above $w_2$ and below $v$ is a 1-vertex. In this case $\Delta_1(w_2) \geq 1$ and $\Delta(w_1) + \Delta(w_2) \geq 4$, so (2) is implied by
$$\frac{1}{2} \alpha + 2\beta \geq 1.$$ (5)

We skip the remaining cases 2.b.ii and 2.c ($\Delta(w_2) = 2$) in this extended abstract; the latter one splits into quite a large number of subcases, which become slightly more involved. One ends up with roughly 24 linear inequalities in addition to (3), (4), (5).

Assuming we have $\alpha$ and $\beta$ that satisfy all these inequalities, each of the induction steps is justified, and the inequality (1) follows. Since we always have $N_1(v), N_2(v) \leq n - 3$, we obtain
$$E(v) \leq \alpha N_1(v) + \beta(N_1(v) + N_2(v)) = (\alpha + \beta)N_1(v) + \beta N_2(v) \leq (\alpha + 2\beta)(n - 3).$$

Hence we choose $(\alpha, \beta)$ to minimize $\alpha + 2\beta$, subject to all the derived inequalities. This is indeed the choice appearing in the statement of Theorem 3.1.

4 Discussion

The improved lower bounds of Section 2.2 arose from complete enumerations for small $n$. In particular, the lower bounds provided by examples 2 and 3 are tight for $n = 10, 12$, respectively. Thus, we have $f(10) = 39/4 = 9.75$ and $f(12) = 1593/128 \approx 12.45$.

We are convinced that the bound in Theorem 3.2 is not tight. In fact, precisely two of the inequalities of the proof of Theorem 3.1 are tight for $(\alpha, \beta) = (46/87, 42/87)$. The two corresponding subcases thus constitute the bottleneck for the current upper bound. In order to improve the bound, one should expand these two subcases further, aiming at replacing those two inequalities by weaker ones (at the cost of a longer proof). As a matter of fact, in an earlier (unpublished) version of this manuscript we had obtained an upper bound of $1.5 \cdot n$, using a somewhat more compact enumeration scheme. The current scheme is a refinement, based on further expansion of the preceding one.

At this point, we have no real sense of what the exact bound should be. The refinement of the approach in this section, as just outlined, is not likely to yield substantial improvements in the upper bound, so a radically different approach is probably called for. Such an improvement might be based on the observation that certain local structures involve 1-vertices with one of its upward neighbors lying above $v$. In fact, if the portion below $v$ contains $k$ such vertices, there must exist at least $k$ vertices higher than $v$, so the upper bound $\alpha N_1(v) + \beta N(v)$ is much smaller than $(\alpha + 2\beta)(n - 3)$. As a matter of fact, the lower bounds derived in Section 2 do take this constraint into consideration.

Another observation is that the proof of Theorem 3.1 (as detailed in the appendix) uses (twice) the 3-connectivity of the edge graph of $P$, but it does not use its planarity at all, although it does occasionally run into nonplanar configurations. It
is conceivable that further refinement stages might reach nonplanar configurations, whose exclusion would allow us to further improve the bound.

What if we also drop the 3-connectivity assumption? Then we need to consider additional cases, which cause our upper bound to increase. The best upper bound we have at the moment for this relaxed situation is \(13n/8 = 1.625 \cdot n\), but we are convinced that it too can be further improved.

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Appendix

This appendix contains the complete case analysis of the proof of Theorem 3.1 in the extended abstract (including the cases presented there).

**Case 1:** $v$ is a 1-vertex.
Let $w_1$ denote the target of the unique downward edge emanating from $v$ as in the following figure, where each edge is labelled by the probability of reaching it from $v$.

In this case, $E(v) = 1 + E(w_1)$ holds. In the setup presented above, we have $c = 1$, $\Delta_1(w_1) \geq 1$, and $\Delta(w_1) \geq 1$, thus (2) is implied by

$$\alpha + \beta \geq 1.$$  \hfill (3)

**Case 2:** $v$ is a 2-vertex.
Let $w_1$ and $w_2$ denote the targets of the two downward edges emanating from $v$, where $w_2$ is lower than $w_1$.

We have

$$E(v) = 1 + \frac{1}{2} E(w_1) + \frac{1}{2} E(w_2),$$

hence we need to require that

$$\frac{\alpha}{2} \Delta_1(w_1) + \frac{\alpha}{2} \Delta_1(w_2) + \frac{\beta}{2} \Delta(w_1) + \frac{\beta}{2} \Delta(w_2) \geq 1.$$  \hfill (4)

Note that $\Delta(w_1) \geq 1$.

**Case 2.a:** $\Delta(w_2) \geq 4$.
Ignoring the effect of the $\Delta_1(w_j)$’s, it suffices to require that

$$\frac{\beta}{2} \Delta(w_1) + \frac{\beta}{2} \Delta(w_2) \geq 1,$$

which will follow if

$$\beta \geq \frac{2}{5}. \hfill (4)$$
Case 2.b.i: \( \Delta(w_2) = 3 \) and one of the two vertices above \( w_2 \) and below \( v \) is a 1-vertex.
In this case \( \Delta_1(w_2) \geq 1 \) and \( \Delta(w_1) + \Delta(w_2) \geq 4 \), so (2) is implied by
\[
\frac{1}{2} \alpha + 2 \beta \geq 1 .
\] (5)

Case 2.b.ii: \( \Delta(w_2) = 3 \) and the two vertices between \( v \) and \( w_2 \) are 2-vertices. Denote the second intermediate vertex as \( v' \). We may assume that \( v' \) is reachable from \( v \), otherwise we can ignore it and reduce the situation to Case 2.c treated below (be choosing another ordering of the vertices producing the same oriented graph). Three subcases can arise.

First, assume that none of the three edges that emanate from \( w_1 \) and \( v' \) further down reaches \( w_2 \). Denote by \( x, y \) the two downward neighbors of \( v' \) and by \( z \) the downward neighbor of \( w_1 \) other than \( v' \). The vertices \( x, y, z \) need not be distinct but none of them coincides with \( w_2 \).

We have here \( c = 7/4 \).

To make the analysis simpler to follow visually, we present it in a table. Each row denotes one of the target vertices \( w_2, x, y, z \), ‘multiplied’ by the probability of reaching it from \( v \). The left (resp., right) column denotes a lower bound on the corresponding quantities \( \Delta_1(\cdot) \) (resp., \( \Delta(\cdot) \)). To obtain an inequality that implies (2), one has to multiply each entry in the left (resp., right) column by the row probability times \( \alpha \) (resp., times \( \beta \)), and require that the sum of all these terms be \( \geq c \).

| \( \frac{1}{2} w_2 \) | \( \frac{1}{8} x \) | \( \frac{1}{8} y \) | \( \frac{1}{4} z \) | \( \alpha \Delta_1 \) | \( \beta \Delta \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| 0              | 0              | 0              | 0              | 3              | 4              |
| 0              | 0              | 0              | 0              | 4              | 5              |
| 0              | 0              | 0              | 0              | 4              | 4              |

Note the following: (a) We do not assume that the rows represent distinct vertices (in fact, \( x = z \) is implicit in the table); this does not cause any problem in applying the rule for deriving an inequality from the table. (b) We have to squeeze the vertices so as to make the resulting inequality as sharp (and difficult to satisfy) as possible; thus we made one of \( x, y \) the farthest vertex, because making \( z \) the farthest vertex would have made the inequality easier to satisfy.
We thus obtain
\[
\left(\frac{3}{2} + \frac{4}{8} + \frac{5}{8} + \frac{4}{4}\right) \beta \geq \frac{7}{4},
\]
or
\[
\beta \geq \frac{14}{29}.
\]  
(6)

Next, assume that \(w_2\) is connected to \(v'\). In this case \(w_2\) is a 1-vertex, and we extend the configuration to include its unique downward neighbor \(w_3\).

Let \(x\) denote the other downward neighbor of \(v'\) and let \(y\) denote the other downward neighbor of \(w_1\). In the following table, the ‘worst’ case is to make \(w_3\) and \(y\) coincide, and make \(x\) the farthest vertex.

|            | \(\alpha \Delta_1\) | \(\beta \Delta\) |
|------------|---------------------|------------------|
| \(5/8w_3\) | 1                   | 4                |
| \(1/8x\)   | 1                   | 5                |
| \(1/4y\)   | 1                   | 4                |

We then obtain
\[
\alpha + \left(\frac{20}{8} + \frac{5}{8} + \frac{4}{4}\right) \beta \geq \frac{19}{8},
\]
or
\[
\alpha + \frac{33}{8} \beta \geq \frac{19}{8}.
\]  
(7)

Finally, assume that \(w_2\) is connected to \(w_1\). Here too \(w_2\) is a 1-vertex, and we extend the configuration to include its unique downward neighbor \(w_3\).

Denoting by \(x, y\) the two downward neighbors of \(v'\), our table and resulting inequality become

|            | \(\alpha \Delta_1\) | \(\beta \Delta\) |
|------------|---------------------|------------------|
| \(3/4w_3\) | 1                   | 4                |
| \(1/8x\)   | 1                   | 5                |

\[
\alpha + \frac{33}{8} \beta \geq \frac{5}{2},
\]  
(8)
which, by the way, is stronger than (7).

**Case 2.c:** $\Delta(w_2) = 2$. Hence, the only remaining case is that $w_1$ and $w_2$ are the two vertices immediately following $v$.

**Case 2.c.i:** $w_1$ is a 1-vertex (whose other upward neighbor lies above $v$). Its unique downward edge ends at some vertex which is either $w_2$ or lies below $w_2$.

Assume first that this vertex coincides with $w_2$, which makes $w_2$ a 1-vertex, whose unique downward neighbor is denoted as $v'$. The local structure, table, and inequality are

\[
\begin{array}{c|c|c|c|c}
 & \alpha \Delta & \beta \Delta \\
\hline
v' & 2 & 3 \\
\end{array}
\]

\[
2\alpha + 3\beta \geq \frac{5}{2}. \quad (9)
\]

Suppose next that the downward neighbor $w_3$ of $w_1$ lies below $w_2$. We get

\[
\begin{array}{c|c|c|c|c}
 & \alpha \Delta & \beta \Delta \\
\hline
1/2w_2 & 1 & 2 \\
1/2w_3 & 1 & 3 \\
\end{array}
\]

\[
\alpha + \frac{5}{2}\beta \geq \frac{3}{2}. \quad (10)
\]

**Case 2.c.ii:** $w_1$ is a 2-vertex, both of whose downward neighbors lie strictly below $w_2$. Denote these neighbors as $w_3, w_4$, with $w_3$ lying above $w_4$.

We may assume $\Delta(w_3) = 3$ (i.e., there is no vertex between $w_2$ and $w_3$), since $\Delta(w_3) \geq 4$ requires $\beta \geq \frac{6}{13}$ as the sharpest inequality, which is already implied by (6).

**Case 2.c.ii.1:** $w_2$ is a 1-vertex. Then the table and inequality become

\[
\begin{array}{c|c|c|c|c}
 & \alpha \Delta & \beta \Delta \\
\hline
1/2w_2 & 0 & 2 \\
1/4w_3 & 1 & 3 \\
1/4w_4 & 1 & 4 \\
\end{array}
\]

\[
\frac{1}{2}\alpha + \frac{11}{4}\beta \geq \frac{3}{2}. \quad (11)
\]
Case 2.c.ii.2: $w_2$ is a 2-vertex but $w_3$ is a 1-vertex. Then $w_3$ (which satisfies $\Delta(w_3) = 3$) is connected either to $w_2$ or to a vertex above $v$. In the former case, let $x$ denote the other downward neighbor of $w_2$, and let $y$ denote the unique downward neighbor of $w_3$. The local structure looks like this (with $x, y, w_4$ not necessarily distinct):

![Diagram](image)

The table depends on whether $x$ precedes or succeeds $w_3$. In the former case the (worst) table and inequality are

|     | $\alpha \Delta_1$ | $\beta \Delta$ |
|-----|-------------------|----------------|
| $1/4x$ | 0                  | 3              |
| $1/2y$ | 1                  | 5              |
| $1/4w_4$ | 1                  | 5              |

(12)

In the latter case the (worst) table and inequality are

|     | $\alpha \Delta_1$ | $\beta \Delta$ |
|-----|-------------------|----------------|
| $1/4x$ | 1                  | 4              |
| $1/2y$ | 1                  | 4              |
| $1/4w_4$ | 1                  | 5              |

(13)

The next case is where the other upward neighbor of $w_3$ lies above $v$. Let $x, y$ denote the two downward neighbors of $w_2$, and let $z$ denote the unique downward neighbor of $w_3$. The local structure is:

![Diagram](image)

The table depends on how many of $x, y$ precede $w_3$. If both precede $w_3$, the table and inequality become

|     | $\alpha \Delta_1$ | $\beta \Delta$ |
|-----|-------------------|----------------|
| $1/4x$ | 0                  | 3              |
| $1/4y$ | 0                  | 4              |
| $1/4z$ | 1                  | 6              |
| $1/4w_4$ | 1                  | 6              |

(14)
If only one of \(x, y\) precedes \(w_3\), say \(x\), the table and inequality become

\[
\begin{array}{c|cc}
\alpha \Delta_1 & \beta \Delta \\
\hline
{1/4}x & 0 & 3 \\
{1/4}y & 1 & 5 \\
{1/4}z & 1 & 5 \\
{1/4}w_4 & 1 & 6 \\
\end{array}
\]

\[
\frac{3}{4} \alpha + \frac{19}{4} \beta \geq \frac{9}{4}, \tag{15}
\]

which is weaker than (14).

Finally, if none of \(x, y\) precedes \(w_3\), the table and inequality become

\[
\begin{array}{c|cc}
\alpha \Delta_1 & \beta \Delta \\
\hline
{1/4}x & 1 & 4 \\
{1/4}y & 1 & 4 \\
{1/4}z & 1 & 5 \\
{1/4}w_4 & 1 & 5 \\
\end{array}
\]

\[
\alpha + \frac{9}{2} \beta \geq \frac{9}{4}. \tag{16}
\]

**Case 2.c.ii.3:** Both \(w_2\) and \(w_3\) are 2-vertices. We have to consider the following type of configuration (where \(x, y, z, t, w_4\) need not all be distinct, but \(x \neq y\) and \(z \neq t\), and we may assume \(x \neq t\), \(y \neq z\); also, because \(\Delta(w_3) = 3\), both \(x\) and \(y\) are lower than \(w_3\):

Intuitively, a worst table is obtained by ‘squeezing’ \(x, y, z, t, w_4\) as much to the left as possible, placing two of them at distance 4 from \(v\), two at distance 5, and one at distance 6. However, squeezing them this way will make some pairs of them coincide and form 1-vertices, which will affect the resulting tables and inequalities.

Suppose first that among the three ‘heavier’ targets \(x, y, w_4\), at most one lies at distance 4 from \(v\). The worst table and the associated inequality are (recall that \(x \neq y\)):

\[
\begin{array}{c|cc}
\alpha \Delta_1 & \beta \Delta \\
\hline
{1/4}x & 0 & 4 \\
{1/4}y & 0 & 5 \\
{1/8}z & 0 & 4 \\
{1/8}t & 0 & 6 \\
{1/4}w_4 & 0 & 5 \\
\end{array}
\]

\[
\frac{19}{4} \beta \geq \frac{9}{4}, \tag{17}
\]

Suppose then that among \(\{w_4, x, y\}\), two are at distance 4 from \(v\), say \(w_4\) and \(y\). Then \(w_4 = y\) is a 1-vertex, and we denote by \(w\) its unique downward neighbor. The local structure is:
Two equally worst tables, and the resulting common inequality are

\[
\begin{array}{|c|c|c|}
\hline
\text{Column} & \alpha \Delta_1 & \beta \Delta \\
\hline
\frac{1}{4}x & 1 & 5 \\
\frac{1}{8}z & 1 & 6 \\
\frac{1}{8}t & 1 & 7 \\
\frac{1}{2}w & 1 & 5 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{Column} & \alpha \Delta_1 & \beta \Delta \\
\hline
\frac{1}{4}x & 1 & 6 \\
\frac{1}{8}z & 1 & 6 \\
\frac{1}{8}t & 1 & 5 \\
\frac{1}{2}w & 1 & 5 \\
\hline
\end{array}
\]

\[
\alpha + \frac{43}{8} \beta \geq \frac{11}{4} . \quad (18)
\]

**Case 2.c.iii:** $w_1$ is a 2-vertex that reaches $w_2$. Then $w_2$ is a 1-vertex, and we denote by $x$ its unique downward neighbor.

\[
\begin{array}{|c|c|c|}
\hline
\text{Column} & \alpha \Delta_1 & \beta \Delta \\
\hline
\frac{3}{4}x & 1 & 4 \\
\frac{1}{4}w_4 & 1 & 3 \\
\hline
\end{array}
\]

\[
\alpha + \frac{15}{4} \beta \geq \frac{9}{4} . \quad (19)
\]

A crucial observation is that $x$ cannot be equal to $w_4$. Indeed, if they were equal, then $w_4$ would be a 1-vertex.

In this case, cutting the edge graph $G$ of $P$ at the downward edge emanating from $x$ and at the edge entering $v$ would have disconnected $G$, contradicting the fact that $G$ is 3-connected.

We first dispose of the case where $x$ lies lower than $w_4$. The table and inequality are

\[
\alpha + \frac{15}{4} \beta \geq \frac{9}{4} . \quad (19)
\]

In what follows we thus assume that $x$ lies above $w_4$. 

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Case 2.c.iii.1:  $x$ is a 1-vertex that precedes $w_4$. Suppose first that $w_4$ is the unique downward neighbor of $x$. Then $w_4$ is a 1-vertex, and we denote its unique downward neighbor by $z$. The local structure, table and inequality are:

\[ c = 4 \]

\[
\begin{array}{|c|c|c|}
\hline
z & 3 & 5 \\
\hline
\end{array}
\]

\[ 3\alpha + 5\beta \geq 4. \quad (20) \]

Suppose next that the unique downward neighbor $y$ of $x$ is not $w_4$. The local structure, table and inequality look like this ($y$ is drawn above $w_4$ because this yields a sharper inequality):

\[ c = 3 \]

\[
\begin{array}{|c|c|c|}
\hline
3/4y & 2 & 4 \\
1/4w_4 & 2 & 5 \\
\hline
\end{array}
\]

\[ 2\alpha + \frac{17}{4}\beta \geq 3. \quad (21) \]

Case 2.c.iii.2:  $x$ is a 2-vertex that precedes $w_4$. This subcase splits into several subcases, where we assume, respectively, that $\Delta(w_4) \geq 6$, $\Delta(w_4) = 4$, and $\Delta(w_4) = 5$.

Case 2.c.iii.2(a).  Suppose first that $\Delta(w_4) \geq 6$. The configuration looks like this:

\[ c = 9/4 \]

The table and inequality are

\[
\begin{array}{|c|c|c|}
\hline
3/4x & 1 & 3 \\
1/4w_4 & 1 & 6 \\
\hline
\end{array}
\]

\[ \alpha + \frac{15}{4}\beta \geq \frac{9}{4}. \quad (22) \]

Case 2.c.iii.2(b).  Suppose next that $\Delta(w_4) = 4$, and that one of the downward neighbors of $x$ is $w_4$. Let $z$ denote the other downward neighbor. $w_4$ is a 1-vertex, and we denote by $w$ its unique downward neighbor.
The 3-connectivity of the edge graph of $P$ implies, as above, that $w \neq z$. Since we assume that $\Delta(w_4) = 4$, $z$ also lies below $w_4$, and the table and inequality are

\[
\begin{array}{c|cc}
\alpha \Delta_1 & \beta \Delta \\
5/8w & 2 & 5 \\
3/8z & 2 & 6 \\
\end{array}
\quad\quad 2\alpha + \frac{43}{8}\beta \geq \frac{29}{8}.
\] (23)

Suppose next that $\Delta(w_4) = 4$ and $w_4$ is not a downward neighbor of $x$. Denote those two neighbors as $w$ and $z$, both of which lie lower than $w_4$, by assumption, and are clearly distinct. The configuration, table and inequality look like this:

\[
\begin{array}{c|cc}
\alpha \Delta_1 & \beta \Delta \\
1/4w_4 & 1 & 4 \\
3/8w & 1 & 5 \\
3/8z & 1 & 6 \\
\end{array}
\quad\quad \alpha + \frac{41}{8}\beta \geq 3.
\] (24)

**Case 2.c.iii.2(c).** It remains to consider the case $\Delta(w_4) = 5$. Let $z$ denote the unique vertex lying between $x$ and $w_4$. We may assume that $z$ is connected to $x$, for otherwise $z$ is not reachable from $v$, and we might as well reduce this case to the case $\Delta(w_4) = 4$ just treated.

Consider first the subcase where the other downward neighbor of $x$ is $w_4$ itself. Then $w_4$ is a 1-vertex, and we denote by $w$ its unique downward neighbor. This subcase splits further into two subcases: First, assume that $z$ is a 1-vertex, and let $y$ denote its unique downward neighbor. Clearly, $y$ must lie below $w_4$ (it may coincide with or precede $w$). The configuration looks like this:

\[
\begin{array}{c|cc}
\alpha \Delta_1 & \beta \Delta \\
3/8y & 3 & 6 \\
5/8w & 3 & 6 \\
\end{array}
\quad\quad 3\alpha + 6\beta \geq 4.
\] (25)
In the other subcase, $z$ is a 2-vertex; we denote its two downward neighbors as $y$ and $t$. The vertices $w, y, t$ all lie below $w_4$ and may appear there in any order. The configuration looks like this:

The table and inequality are

|       | $\alpha \Delta_1$ | $\beta \Delta$ |
|-------|---------------------|-----------------|
| $3/16y$ | 2                   | 6               |
| $3/16t$ | 2                   | 7               |
| $5/8w$  | 2                   | 6               |

\[ 2\alpha + \frac{99}{16} \beta \geq 4. \] (26)

Consider next the subcase where $w_4$ is not a downward neighbor of $x$. Denote the other downward neighbor of $x$ as $y$, which lies strictly below $w_4$. This subcase splits into three subcases. First, assume that $z$ is a 1-vertex, and denote its unique downward neighbor as $w$. The configuration looks like this:

The table and inequality are

|       | $\alpha \Delta_1$ | $\beta \Delta$ |
|-------|---------------------|-----------------|
| $1/4w_4$ | 2                   | 5               |
| $3/8y$  | 2                   | 6               |
| $3/8w$  | 2                   | 5               |

\[ 2\alpha + \frac{43}{8} \beta \geq \frac{27}{8}. \] (27)

Second, assume that $z$ is a 2-vertex, so that none of its two downward neighbors is $w_4$. Denote these neighbors as $w$ and $t$. All three vertices $y, t, w$ lie strictly below $w_4$. The configuration looks like this:
The table and inequality are

|       | $\alpha \Delta_1$ | $\beta \Delta$ |
|-------|------------------|----------------|
| $1/4w_4$ | 1                | 5              |
| $3/8y$    | 1                | 6              |
| $3/16w$   | 1                | 6              |
| $3/16t$   | 1                | 7              |

\[
\alpha + \frac{95}{16} \beta \geq \frac{27}{8}, \tag{28}
\]

Finally, assume that $z$ is a 2-vertex, so that one of its two downward neighbors is $w_4$. Denote the other neighbor as $w$. In this case $w_4$ is a 1-vertex, and we denote its unique downward neighbor as $t$. All three vertices $y, t, w$ lie strictly below $w_4$. The configuration looks like this:

The table and inequality are

|       | $\alpha \Delta_1$ | $\beta \Delta$ |
|-------|------------------|----------------|
| $3/8y$    | 2                | 6              |
| $3/16w$   | 2                | 7              |
| $7/16t$   | 2                | 6              |

\[
2\alpha + \frac{99}{16} \beta \geq \frac{61}{16}, \tag{29}
\]

which, by the way, is weaker than (26).