Quantum Theory as an Emergent Phenomenon: Foundations and Phenomenology

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Abstract. I review the proposal made in my 2004 book [1], that quantum theory is an emergent theory arising from a deeper level of dynamics. The dynamics at this deeper level is taken to be an extension of classical dynamics to non-commuting matrix variables, with cyclic permutation inside a trace used as the basic calculational tool. With plausible assumptions, quantum theory is shown to emerge as the statistical thermodynamics of this underlying theory, with the canonical commutation-anticommutation relations derived from a generalized equipartition theorem. Brownian motion corrections to this thermodynamics are argued to lead to state vector reduction and to the probabilistic interpretation of quantum theory, making contact with phenomenological proposals [2, 3] for stochastic modifications to Schrödinger dynamics.

1. Introduction
Quantum mechanics is our most successful physical theory, but its foundational issues have been under debate for over eighty years. There are two possibilities:

- **Quantum theory is exact**, with (perhaps) a need for reinterpretation at the foundational level. Popular interpretations are “Many worlds”, “Bohmian mechanics”, “Histories”, and “Quantum theory as information”. Experimentally, they are indistinguishable, and I will have nothing further to say about them.

- **Quantum theory is not exact**, but is a very accurate asymptotic approximation to a deeper level theory – a pre-quantum dynamics. This is the idea developed in my 2004 book [1], and forms the basis for this talk.

Motivations for considering the possibility that quantum theory is only approximate include:

- **Historical.** “Exact” theories have usually turned out to be approximations accurate only in a certain domain, such as Newtonian mechanics and thermodynamics.

- **The riddle of canonical quantization.** The standard approach to creating a quantum theory is to write down a classical theory and then to “quantize” it by replacing the Poisson bracket by \(-i/\hbar\) times the corresponding commutator. But this raises some questions. (i) If quantum theory is more fundamental, why can’t it be obtained directly from an operator theory? (ii) What is the origin of the Planck constant \(\hbar\)? In my opinion, canonical quantization looks like an algorithm to invert the classical limit of quantum theory, and not a fundamental principle.
• Measurement problem. Quantum mechanics is a linear theory, that is, the state vector at time \(t\) is related to the state vector at time 0 by the linear relation \(|\psi(t)\rangle = U(t)|\psi(0)\rangle\). On the other hand, “measurements” involve a nonlinear stochastic “state vector reduction” induced by interaction with a “classical” apparatus. This raises profound questions: Where do the probabilities come from? Is there a single dynamical framework that incorporates both the linear Schrödinger evolution part of quantum theory, and the nonlinear, stochastic state vector reduction part?

The proposal for a pre-quantum theory developed in [1] and described in this talk is illustrated schematically in Fig. 1.

2. Overview of Trace Dynamics

“Trace Dynamics” is a noncommutative generalization of classical Lagrangian and Hamiltonian dynamics. To illustrate how it works for bosonic variables, let \(\{q_r\}\) be a set of noncommuting coordinates, and \(\{\dot{q}_r\}\) (with the dot denoting \(\partial/\partial t\)) their time derivatives. Let \(L = L[\{q_r\}, \{\dot{q}_r\}]\) be a polynomial in its arguments; since the arguments do not commute, the derivative \(\delta L/\delta q_r\) is not defined. Now let us use cyclic invariance of the trace: define the trace Lagrangian \(L\) by \(L = \text{Tr}L\). Let us vary to form \(\delta L\), and reorder cyclically so that all \(\delta q_r\) and \(\delta \dot{q}_r\) stand on the
right. This gives the definition
\[ \delta L = \text{Tr} \sum_r \left( \frac{\delta L}{\delta q_r} \delta q_r + \frac{\delta L}{\delta \dot{q}_r} \delta \dot{q}_r \right), \tag{1} \]
which defines the derivatives of the trace Lagrangian \( \delta L/\delta q_r \) and \( \delta L/\delta \dot{q}_r \). One can now show that requiring a stationary trace action
\[ 0 = \delta S = \delta \int_{-\infty}^{\infty} dt L \tag{2} \]
directly gives the operator Euler-Lagrange equations
\[ 0 = \frac{\delta L}{\delta q_r} - \frac{d}{dt} \frac{\delta L}{\delta \dot{q}_r}. \tag{3} \]

2.1. Conserved quantities
Since the commutation relations of the \( q_r \)s are arbitrary, we have obtained a dynamics that is more general than quantum mechanics. To recover quantum mechanics, we consider the equilibrium statistical mechanics of this dynamics. This is obtained by noting that there are three generic conserved quantities in trace dynamics.

- **Trace Hamiltonian:** Let us define the canonical momentum conjugate to \( q_r \) by \( p_r = \frac{\delta L}{\delta \dot{q}_r} \), and introduce the trace Hamiltonian \( H \) defined by
  \[ H = \text{Tr} \sum_r p_r q_r - L \tag{4} \]
  Then \( \delta H/\delta q_r = -p_r \) and \( \delta H/\delta p_r = \dot{q}_r \), and a simple calculation gives
  \[ \frac{dH}{dt} = \text{Tr} \sum_r \left( \frac{\delta H}{\delta q_r} \dot{q}_r + \frac{\delta H}{\delta p_r} \dot{p}_r \right) = \text{Tr} \sum_r (-\dot{p}_r \dot{q}_r + \dot{q}_r \dot{p}_r) = 0, \tag{5} \]
  where in the first line we have used the definition of the trace derivative, and in the second line we have used cyclic invariance under the trace to cancel the two terms.

- **Operator \( \tilde{C} \):** My graduate student Andrew Millard first noted, in the context of Weyl-ordered Hamiltonians, that the operator \( \tilde{C} = \sum_r [q_r, p_r] \) is conserved, and this turns out to generalize. Suppose that \( H \) is *global unitary invariant*, that is, it involves no noncommutative constants in its definition. Then \( \tilde{C} \) as defined above is conserved, and if one introduces Grassmann variables to represent fermionic degrees of freedom, the operator
  \[ \tilde{C} = \sum_{r,B} [q_r, p_r] - \sum_{r,F} \{q_r, p_r\}, \tag{6} \]
  where the subscripts \( B, F \) indicating respectively sums over the bosonic and fermionic degrees of freedom, is the conserved Noether charge operator corresponding to the global unitary invariance of \( H \), that is \( d\tilde{C}/dt = 0 \). With standard adjointness assignments for the dynamical variables (see [1]) the operator \( \tilde{C} \) is anti-self-adjoint.
Phase space measure: Define the operator phase space measure
\[ d\mu \equiv \prod_{r,m,n,A} d\langle m|q_r|n\rangle^A d\langle m|p_r|n\rangle^A, \tag{7} \]
where \( A \) indexes the real valued components. Then one can show that \( d\mu \) is invariant under the canonical transformation with generator \( G \),
\[ \delta p_r = -\delta G / \delta q_r, \quad \delta q_r = \delta G / \delta p_r, \tag{8} \]
of which the trace Hamiltonian dynamics is a special case. This means that trace dynamics has a generalized Liouville theorem: operator phase space volumes are conserved under Hamiltonian flow. This permits the use of statistical mechanical concepts, since if one postulates a uniform a priori phase space distribution, this uniformity is preserved in time under the Hamiltonian dynamics.

2.2. Canonical ensemble

We now introduce the canonical ensemble for trace dynamics, which is the phase space distribution that maximizes the entropy subject to the constraints that \( H \) and \( \tilde{C} \) have specified average values, and is given by
\[ \rho = Z^{-1} \exp(-\tau H - \text{Tr} \tilde{\lambda} \tilde{C}) . \tag{9} \]
The normalization condition \( \int d\mu \rho = 1 \) fixes the partition function \( Z \) to be
\[ Z = \int d\mu \exp(-\tau H - \text{Tr} \tilde{\lambda} \tilde{C}) . \tag{10} \]
The ensemble parameters are the real number \( \tau \), which has the dimensions of inverse mass, and the anti-self-adjoint matrix \( \tilde{\lambda} \). Letting \( \langle ... \rangle_{AV} \) denote the average over the canonical ensemble, we can write the average of \( \tilde{C} \) in the canonical form
\[ \langle \tilde{C} \rangle_{AV} = i_{\text{eff}} D, \]
\[ i_{\text{eff}}^2 = -1, \quad i_{\text{eff}}^\dagger = -i_{\text{eff}}, \quad [i_{\text{eff}}, D] = 0 . \tag{11} \]
The simplest case, which we shall assume, is that no direction in the underlying operator space is preferred to any other, so that \( D \) is a multiple of the unit matrix, which with some foresight we write as
\[ D = \hbar 1 . \tag{12} \]
Since \( \langle \tilde{C} \rangle_{AV} \) is a function of \( \tau \) and \( \tilde{\lambda} \), with this assumption we have \( \tilde{\lambda} = i_{\text{eff}} \lambda \), with lambda a real number.

2.3. Unitary fixing

Because \( \tilde{\lambda} = i_{\text{eff}} \lambda \), the canonical ensemble is invariant under unitary transformations \( U_{\text{eff}} \) for which \( U_{\text{eff}}^\dagger i_{\text{eff}} U_{\text{eff}} = i_{\text{eff}} \), or equivalently, \( [U_{\text{eff}}, i_{\text{eff}}] = 0 \). This follows immediately from cyclic invariance of the trace,
\[ \text{Tr} \tilde{\lambda} U_{\text{eff}}^\dagger \tilde{C} U_{\text{eff}} = \text{Tr} U_{\text{eff}} \tilde{\lambda} U_{\text{eff}}^\dagger \tilde{C} = \text{Tr} \tilde{\lambda} \tilde{C} . \tag{13} \]
Thus the canonical ensemble only partially breaks the global unitary invariance, and this means that we will average too much if we use $\int d\mu \rho$ for forming ensemble averages. So we introduce a restricted measure

$$d\mu' = d\mu|_{\text{OVERALL U eff FROZEN}},$$

(14)

and will use this measure for forming thermodynamic averages.

With the preceding remarks in mind, it is useful to introduce the effective complex part of any operator $x$, which can be any function of the canonical variables. We define $x_{\text{eff}}$ to be the part of $x$ that commutes with $i_{\text{eff}}$; this is given by

$$x_{\text{eff}} = \frac{1}{2}(x - i_{\text{eff}} x i_{\text{eff}}),$$

(15)

as verified by noting that

$$i_{\text{eff}} x_{\text{eff}} = \frac{1}{2} (i_{\text{eff}} x + x i_{\text{eff}}).$$

(16)

### 2.4. Physical observables

Physical observables in trace dynamics are trace quantities, which are independent of the unitary fixing since

$$\text{Tr} O = \text{Tr} U_{\text{eff}}^\dagger O U_{\text{eff}}.$$

(17)

This suffices, since transition probabilities can be written as traces,

$$|\langle \alpha | O | \beta \rangle|^2 = \langle \alpha | O | \beta \rangle \langle \beta | O^\dagger | \alpha \rangle = \text{Tr} P_\alpha O P_\beta O^\dagger,$$

(18)

with $P_\alpha$ the projection operator

$$P_\alpha = |\alpha \rangle \langle \alpha | = \frac{1}{2\pi i} \oint_{\lambda_\alpha} \frac{dz}{z - O_\alpha},$$

(19)

where the contour integral over $z$ circles $\lambda_\alpha$, the eigenvalue of the operator $O_\alpha$ on the state $|\alpha \rangle$.

### 3. How Quantum Field Theory Emerges

We now set up a correspondence

$$\frac{\int d\mu' \rho x_{1\text{eff}} \ldots x_{n\text{eff}}}{\int d\mu' \rho} \leftrightarrow x_{1\text{eff}} \ldots x_{n\text{eff}},$$

(20)

where on the right the $x$s are operators in a quantum field theory with the role of $i$ played by $i_{\text{eff}}$. One can derive *equipartition theorems* or *Ward identities* for the averages on the left of Eq. (20). When $\tau$ is very small (so that it is the inverse of a very large mass – the Planck mass?), and when $\tilde{C}$ is replaced by its ensemble average $(\tilde{C})_{\text{AV}} = i_{\text{eff}} \hbar$, these Ward identities have the structure of quantum mechanics, so that inside ensemble averages we have

$$[q_{\text{eff}}, p_{\text{eff}}] = i\hbar \delta_{rs} + O(\tau),$$

$$[q_{\text{eff}}, q_{\text{eff}}] = 0 + O(\tau),$$

$$[p_{\text{eff}}, p_{\text{eff}}] = 0 + O(\tau),$$

$$\dot{x}_{\text{eff}} = \frac{i_{\text{eff}}}{\hbar} [H_{\text{eff}}, x_{\text{eff}}] + O(\tau).$$

(21)

Thus, although the initial dynamical degrees of freedom have general commutation relations, the emergent theory has the usual local commutation relations of quantum theory when $r$ is a spatial box label.
4. Sketch of General Ward Identity Derivation

We now give a brief sketch of how the Ward identities are derived. For general operator $O$, let

$$\frac{\int d\mu' \rho O}{\int d\mu' \rho} \equiv \langle O \rangle_{AV}$$

be the average defined with respect to the unitary fixed phase space measure. Taking $O = \{\tilde{C}, i_{\text{eff}}\}W$, we proceed as follows:

- Use translation invariance of the measure $d\mu'$ to write $\int d\mu' \delta[\rho O] = 0$, where $\delta$ is a variation with respect to a variable not frozen by the unitary fixing.
- Neglect the $\tau$ term coming from $\delta \rho$: this is an assumption of decoupling of the low energy from the high energy regime.
- Replace $\tilde{C} \rightarrow \langle \tilde{C} \rangle_{AV}$ in integrands.
- Make various choices for $W$ as follows: (i) Taking $W$ proportional to a canonical $q$ or $p$ gives the canonical algebra inside $\langle \rangle_{AV}$. (ii) Taking $W$ proportional to the operator Hamiltonian $H$ gives the Heisenberg equation of motion inside $\langle \rangle_{AV}$. (iii) Taking $W$ proportional to a self adjoint generator $G$ gives a unitary canonical transformation inside $\langle \rangle_{AV}$.
- One can include sources in $\rho$ so that $\langle \rangle_{AV}$ has source terms that can be varied.
- Some remarks on this derivation are in order:
  - The $O(\tilde{\lambda})$ term in the Ward identities vanishes because $\{i_{\text{eff}}, [\tilde{\lambda}, x]\} \propto [\tilde{\lambda}, x_{\text{eff}}] = 0$.
  - There are conditions for the neglect of the $O(\tau)$ terms in the Ward identities. The first [1] is that it suffices for $\dot{x}_{\text{eff}}$ and $\tilde{C}_{\text{eff}}$ to have disjoint support on operator phase space.
  - A second condition for neglect of the $O(\tau)$ terms is that boson–fermion balance is required. Suppose the theory has only bosons, then the effective canonical algebra with $O(\tau)$ terms neglected reads (omitting subs “eff” on $q,p$; this is justified for a second type of Ward identity derived in [1])

$$\langle [q_s, p_r] \rangle_{AV} = i_{\text{eff}} \hbar \delta_{rs} ,$$

(23)

Setting $r = s$ and summing gives then

$$\langle \sum_r [q_r, p_r] \rangle_{AV} = \langle \tilde{C}_{AV} \rangle = i_{\text{eff}} \hbar = i_{\text{eff}} \hbar \sum_r 1 = i_{\text{eff}} \hbar N_B ,$$

(24)

with $N_B$ the number of bosonic degrees of freedom, which is a contradiction. When both bosons and fermions are present, this is changed to

$$\langle \sum_{r,B} [q_r, p_r] - \sum_{r,F} [q_r, p_r] \rangle_{AV} = \langle \tilde{C}_{AV} \rangle = i_{\text{eff}} \hbar$$

(25)

the left hand side is now $i_{\text{eff}} \hbar (N_B - N_F) + \sum O(\tau)$, and so there is no contradiction in neglecting the $O(\tau)$ terms if $N_B = N_F$.

5. Why there are Brownian Motion Corrections

Let us write

$$\tilde{C} = \langle \tilde{C} \rangle_{AV} + \Delta \tilde{C} = i_{\text{eff}} \hbar + \Delta \tilde{C} ,$$

(26)

with $\Delta \tilde{C}$ a rapidly fluctuating quantity in operator phase space. Then the combination appearing in the Ward identities becomes

$$\frac{1}{2} \{\tilde{C}, i_{\text{eff}}\} = -\hbar + \frac{1}{2} \{\Delta \tilde{C}, i_{\text{eff}}\} = -\hbar - \hbar (K + N) ,$$

(27)
with $K$ a fluctuating c-number, and with $N$ a fluctuating matrix, with an operator analog. An appropriate Ansatz for the $N$ term then corresponds to the mass-proportional form of the “continuous spontaneous localization” (CSL [2]) stochastic Schrödinger equation. Thus we have plausibility arguments, but not a derivation, of the CSL model.

As a simple illustration of a stochastic Schrödinger equation, consider a pointer with center of mass $q$. (This is the leading small displacement approximation to the Ghirardi-Rimini-Weber and CSL models.) The stochastic Schrödinger equation is

$$d|\psi\rangle = -\frac{i}{\hbar}H|\psi\rangle - \eta \frac{(q - \langle q \rangle)^2}{2}|\psi\rangle dt + \sqrt{\eta}(q - \langle q \rangle)|\psi\rangle dW_t ,$$  \hspace{1cm} (28)

with $\langle q \rangle = \langle \psi|q|\psi\rangle$ the expectation value of $q$ in the state $|\psi\rangle$, and with $dW_t$ a Brownian noise. We note that:

- Because $|\psi\rangle$ appears in $\langle q \rangle$, this is a nonlinear stochastic differential equation.
- This equation can be proved to give state vector reduction on position eigenstates with Born rule probabilities.

6. Summary: Trace Dynamics as Pre-Quantum Mechanics

- Thermodynamics, via equipartition theorems (Ward identities) implies the unitary evolution of quantum mechanics in the form of the Heisenberg equation of motion, which can be transformed into the Schrödinger equation.
- Brownian motion corrections, via CSL phenomenology gives the probability interpretation and the Born rule of the reduction postulate of quantum theory.

Appendix A. Proof of Conservation of $\hat{C}$

Let us assume global unitary invariance of the trace Hamiltonian, so that

$$H[\{U^+q_rU\}, \{U^+p_rU\}] = H[\{q_r\}, \{p_r\}] ,$$  \hspace{1cm} (A.1)

with $U = e^\Lambda$, $U^+ = e^{-\Lambda^\dagger}$. Expanding to first order in $\Lambda$, this gives

$$H[\{q_r - [\Lambda, q_r]\}, \{p_r - [\Lambda, p_r]\}] = H[\{q_r\}, \{p_r\}] ,$$  \hspace{1cm} (A.2)

so using the definition of the trace derivative we get

$$\text{Tr} \sum_r \left( -\frac{\delta H}{\delta q_r} [\Lambda, q_r] - \frac{\delta H}{\delta p_r} [\Lambda, p_r] \right) = 0 .$$  \hspace{1cm} (A.3)

Since $\Lambda$ is arbitrary, this implies the operator relation

$$0 = \sum_r \left( \frac{\delta H}{\delta q_r}, q_r \right) + \left( \frac{\delta H}{\delta p_r}, p_r \right)$$

$$= \sum_r ([\hat{p}_r, q_r] + [\hat{q}_r, p_r]) = \sum_r ([\hat{q}_r, \hat{p}_r] + [\hat{q}_r, p_r]) = \hat{C} ,$$  \hspace{1cm} (A.4)

with $\hat{C} = \sum_r [q_r, p_r]$. When fermions (Grassmann odd $q_r, p_r$) are included, the same derivation gives conservation of $\hat{C} = \sum_{r,B} [q_r, p_r] - \sum_{r,F} [q_r, p_r]$. 

7
References

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