Open Strings in the SL(2,R) WZWN Model with Solution for a Rigidly Rotating String

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Abstract

Boundary conditions and gluing conditions for open strings and D-branes in the $SL(2,R)$ WZWN model, corresponding to $AdS_3$, are discussed. Some boundary conditions and gluing conditions previously considered in the literature are shown to be incompatible with the variation principle.

We then consider open string boundary conditions corresponding to a certain field-dependent gluing condition. This allows us to consider open strings with constant energy and angular momentum. Classically, these open strings naturally generalize the open strings in flat Minkowski space. For rigidly rotating open strings, we show that the torsion leads to a bending and an unfolding. We also derive the $SL(2,R)$ Regge relation, which generalizes the linear Minkowski Regge relation. For ”high” mass, it takes the form $L \approx \pm M/H$, where $H$ is the scale of the $SL(2,R)$ group manifold.
1 Introduction

Historically [1, 2, 3], one of the reasons for string theory to enter the field of theoretical high energy physics, was its ability to reproduce the Regge behaviour seen in the hadron spectrum: As is now well known, using the Nambu-Goto action [4, 5] in flat Minkowski space and imposing standard Neumann boundary conditions one obtains, for a rigidly rotating open string, the relation $L = M^2 \alpha'$, where $M$ is the mass, $L$ is the angular momentum and $\alpha'$ is the reciprocal string tension (in suitable units). Rigidly rotating open strings have been considered also in various black hole and cosmological backgrounds, and their physical properties have been analyzed in some detail [6, 7, 8].

In superstring theory it was for many years the closed strings that attracted most of the attention (for a review, see for instance [9]). However, open strings dramatically re-entered the scene due to the work by Polchinski on D-branes [10] (for a review, see for instance [11, 12]). Open strings and D-branes on group manifolds have attracted a lot of interest (see for instance [13, 14, 15, 16]), and recently, especially in connection with 2+1-dimensional Anti de Sitter space [17, 18, 19, 20, 21] (and references given therein).

$AdS_3$ and the corresponding $SL(2, R)$ WZWN model [22] plays an important role in string theory, since it represents a non-trivial (with curved space and curved time) exact string background (some of the original works include [23-32]). Moreover, Anti de Sitter space appears in connection with the Maldacena conjecture [33] relating supergravity and superstring theory with a conformal field theory on the boundary. In such constructions, $AdS_3$ often appears on the 10-dimensional supergravity/superstring side in a product with some compact spaces, for instance as $AdS_3 \times S^3 \times T^4$.

The classical WZWN action [22] is somewhat ambiguous for an open string. Basically the problem is that the variation of the action does not uniquely specify the open string surface terms (boundary terms). Adding different open string surface terms, which corresponds to a coupling of the string endpoints to different background vector fields, thus defines different open string theories. The open string surface terms, in turn, must be canceled by imposing appropriate open string boundary conditions; Neumann, Dirichlet or combinations or generalizations thereof.

Usually D-branes, on which open strings can end, in WZWN models are described in terms of gluing conditions. A key-problem concerns the relation between boundary conditions and gluing conditions [13, 14, 15, 16, 17].
In the present paper we show that the previously considered Neumann boundary condition \( \partial_\sigma g = 0 \) \cite{34, 35} and Neumann type gluing condition \( J = \bar{J} \) for open strings in \( AdS_3 \) are incompatible with the variation principle. More precisely, it is impossible to add an open string surface term to the variation of the action, which vanishes for such boundary/gluing conditions. The Dirichlet type gluing condition \( J = -\bar{J} \), on the other hand, is shown to be compatible with the variation principle, provided that the string endpoints are fixed to move on certain D-branes. These D-branes, however, are unphysical \cite{18}. Eventually, by using another set of gluing conditions at the string endpoints \cite{17, 18}, it is possible to describe open strings attached to physically well-defined D-branes, and at the same time being compatible with the variation principle. The latter case is of particular importance since these gluing conditions preserve the spectral flow \cite{31, 36}, and the open strings can be consistently quantized \cite{19, 20} by generalizing the procedure of \cite{36}.

Finally, we consider an alternative open string surface term as well as its corresponding boundary conditions. They correspond to a certain field-dependent gluing condition. This allows us to consider open strings with constant energy and angular momentum. These open strings naturally generalize the open strings in flat Minkowski space. For rigidly rotating open strings we show that the torsion leads to a bending and an unfolding of the strings. We also derive the \( SL(2, R) \) Regge relation, which generalizes the linear Minkowski Regge relation.

The paper is organized as follows. In Section 2, we set our notation and conventions, and we give a general discussion of open string boundary conditions and gluing conditions in the \( SL(2, R) \) WZWN model. In Section 3, we consider rigidly rotating open strings corresponding to a field-dependent gluing condition. The dynamics is solved explicitly and discussed in detail and compared with the Minkowski case. In Section 4, we give our conclusions.
Our starting point is the action for the WZWN model \([22]\):

\[
S_{\text{Closed}} = S_1 + S_{\text{Closed}}^2
\]

\[
S_1 = -\frac{k}{8\pi} \int_M d\tau d\sigma \eta^{\alpha\beta} \text{Tr} \left[ g^{-1} \partial_\alpha g^{-1} \partial_\beta g \right]
\]

\[
S_{\text{Closed}}^2 = \frac{k}{12\pi} \int_B \text{Tr} \left[ g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right]
\]

(2.1)

Here \(M\) is the string world-sheet and \(B\) is a manifold which has \(M\) as its boundary. As the superscript indicates, the WZWN term \(S_{\text{Closed}}^2\) is only defined for closed strings since a boundary cannot have a boundary. Hence, \(M\) cannot be an open string world-sheet. But if we take the variation of \(S_{\text{Closed}}^2\) with respect to the group element \(g\),

\[
\delta S_{\text{Closed}}^2 = -\frac{k}{4\pi} \int_B d\text{Tr} \left[ d(g^{-1} dg) g^{-1} \delta g \right]
\]

(2.2)

and use Stokes theorem \(\int_B d\omega = \int_M \omega\) for a two-form \(\omega\), with the convention \(\int_M F \, d\tau \wedge d\sigma = \int_M F d\tau d\sigma\) for a function \(F\), we end up with

\[
\delta S_{\text{Closed}}^2 = -\frac{k}{4\pi} \int_M d\tau d\sigma \epsilon^{\alpha\beta} \text{Tr} \left[ \partial_\alpha (g^{-1} \partial_\beta g) g^{-1} \delta g \right]
\]

(2.3)

This formula does not refer to the manifold \(B\), so one approach for making a model for open strings is to use this as a starting point. The problem is that we derived it for closed strings and therefore do not know anything about a possible surface term.

The \(S_1\) term makes perfectly sense for open strings, and taking the variation of it we get

\[
\delta S_1 = \frac{k}{4\pi} \int_M d\tau d\sigma \eta^{\alpha\beta} \text{Tr} \left[ \partial_\alpha (g^{-1} \partial_\beta g) g^{-1} \delta g \right]
\]

\[= -\frac{k}{4\pi} \int d\tau \text{Tr} \left[ g^{-1} \partial_\sigma g g^{-1} \delta g \right] |_{\sigma=\pi}^{\sigma=0}
\]

(2.4)

\(^1\)Our conventions are \(\eta^{00} = -1, \eta^{11} = 1\) and \(\epsilon^{01} = 1\).
Adding this to eq. (2.3), introducing world-sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$ and the unknown surface term $\delta S^\text{Surface}_2$, we have

$$
\delta S^\text{Open} = -\frac{k}{2\pi} \int_M d\sigma^- d\sigma^+ \text{Tr} \left[ \partial_- (g^{-1} \partial_+ g) g^{-1} \delta g \right] - \frac{k}{4\pi} \int d\tau \text{Tr} \left[ g^{-1} \partial_\sigma g g^{-1} \delta g \right] \bigg|_{\sigma=0} + \delta S^\text{Surface}_2
$$

(2.5)

The surface terms do not contribute to the equations of motion, so they are the same for both open and closed strings, namely

$$
\partial_- (g^{-1} \partial_+ g) = 0
$$

(2.6)

which is equivalent to

$$
\partial_+ (\partial_- g^{-1}) = 0
$$

(2.7)

We can therefore define the two quantities

$$
J = \frac{ik}{2} \partial_- g^{-1} , \quad \bar{J} = \frac{ik}{2} g^{-1} \partial_+ g
$$

(2.8)

depending only on $\sigma^-$ and $\sigma^+$, respectively. From now on concentrating on the $SL(2,R)$ case, the currents $J$ and $\bar{J}$ take values in the Lie algebra of $SL(2,R)$. They can be decomposed as

$$
J = \eta_{ab} J^a b , \quad \bar{J} = \eta_{ab} \bar{J}^a b
$$

(2.9)

where $J^a$, $\bar{J}^a$ are real valued fields, $\eta_{ab} = \text{diag}(1,1,-1)$ and the $t^a$ are the three generators of the $SL(2,R)$ Lie algebra

$$
t^1 = -\frac{i}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \quad t^2 = -\frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad t^3 = -\frac{i}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

(2.10)

The currents $J$ and $\bar{J}$ are conserved, as follows from the equations of motion, but for an open string they can generally not be obtained as Noether currents corresponding to some global symmetries. Notice also that for an open string, $J$ and $\bar{J}$ are not independent, since they will in general be related by the boundary conditions. Moreover, for an open string the corresponding ”charges”

$$
Q^a_\pm \propto \int_0^\pi d\sigma (J^a \pm \bar{J}^a)
$$

(2.11)
will in general not be constants of motion. Different open string surface terms will therefore in general define different open string theories with different constants of motion.

The equations of motion are as usual supplemented by the constraints

\[ \text{Tr}[g^{-1} \partial_\pm g \ g^{-1} \partial_\pm g] = 0 \] (2.12)

which correspond to vanishing world-sheet energy-momentum tensor.

### 2.1 The Surface Term

All we need to do now is to specify the surface term \( \delta S_2^{\text{Surface}} \). In Refs.\[34, 35] it was set equal to zero, but if we want \( \delta S_2^{\text{Open}} \) to be the variation of some action \( S_2^{\text{Open}} \), we cannot choose \( \delta S_2^{\text{Surface}} \) freely, and especially setting it to zero will not work. To see this for the case of \( SL(2, R) \), we parametrise the group elements as

\[ g = e^{2uit_1} e^{2\rho i t_2} e^{2vi t_3} \] (2.13)

where \( u, \rho \) and \( v \) are the new fields on the string world-sheet, and the \( t^a \) are the generators of the \( SL(2, R) \) Lie algebra introduced before. If \( \delta S_2^{\text{Open}} \) is the variation of some action \( S_2^{\text{Open}} \), then we should be able to take the variation of \( \delta S_2^{\text{Open}} \) and get something that is symmetric in the variations of the fields. That is, if we take the variation twice, with respect to for instance \( \rho \) and \( u \), we should get the same result independently of the order in which we take the variations. So we have the condition

\[ \delta_{\delta X^\mu} \delta_{X^\nu} S_2^{\text{Open}} = \delta_{\delta X^\nu} \delta_{X^\mu} S_2^{\text{Open}} \] (2.14)

where \( X^\mu \) and \( X^\nu \) are any of the fields \( u, \rho \) or \( v \). Concentrating on the WZWN term, where the problems might arise, we can write the variation as

\[ \delta S_2^{\text{Open}} = \delta S_2^{\text{Bulk}} + \delta S_2^{\text{Surface}} \] (2.15)

where \( \delta S_2^{\text{Bulk}} \) has the same expression as \( \delta S_2^{\text{Closed}} \) in eq.(2.3). If we set \( \delta S_2^{\text{Surface}} \) equal to zero (as in Refs.\[34, 35] ), we only have \( \delta S_2^{\text{Bulk}} \) left. Inserting the \( SL(2, R) \) parametrisation in eq.(2.3), we get

\[ \delta S_2^{\text{Bulk}} = \frac{k}{\pi} \int_M d\tau d\sigma \ \sinh 2\rho \ \epsilon^{\alpha\beta}[\partial_\alpha u \partial_\beta \rho \ \delta u + \partial_\alpha \rho \partial_\beta u \ \delta v + \partial_\alpha u \partial_\beta v \ \delta \rho] \] (2.16)
It follows that
\[
\begin{align*}
\delta_\delta_\rho S^\text{Bulk}_2 &= \frac{k}{\pi} \int_M d\tau d\sigma \epsilon^{\alpha\beta} \sinh 2\rho \partial_\alpha u \partial_\beta v \delta \rho \\
\delta_\delta_\mu S^\text{Bulk}_2 &= \frac{k}{\pi} \int_M d\tau d\sigma \epsilon^{\alpha\beta} \sinh 2\rho \partial_\alpha v \partial_\beta \rho \delta u
\end{align*}
\] (2.17)

After another variation we get
\[
\begin{align*}
\delta_\delta_\mu \delta_\delta_\rho S^\text{Bulk}_2 &= -\frac{k}{\pi} \int_M d\tau d\sigma \epsilon^{\alpha\beta} \sinh 2\rho \partial_\alpha v \partial_\beta \delta u \delta \rho \\
\delta_\rho \delta_\delta_\mu S^\text{Bulk}_2 &= -\frac{k}{\pi} \int_M d\tau d\sigma \epsilon^{\alpha\beta} \sinh 2\rho \partial_\alpha v \partial_\beta \delta \rho \\
&\quad + \frac{k}{\pi} \int d\tau \sinh 2\rho \partial_\tau \delta \rho \delta u \big|_{\sigma=\pi} \big|_{\sigma=0}
\end{align*}
\] (2.19)

We see that the condition (2.14) is not fulfilled, and hence there is no \( S^\text{Open}_2 \) with vanishing \( \delta S^\text{Surface}_2 \). If we want the open string theory to be well defined in terms of an action, we therefore cannot proceed as in Refs.\cite{34, 35} and forget about a surface term from the WZWN term.

### 2.2 An Open String Action

A way to avoid the above problems is to introduce a parametrisation in eq.\( (2.1) \), use Stokes theorem to write it as an integral over \( M \), and then use that expression to define an action for open strings. Inserting the \( SL(2, R) \) parametrisation in eq.\( (2.1) \), we find
\[
S^\text{Closed}_2 = \frac{k}{\pi} \int_B \sinh 2\rho \ d\rho \wedge du \wedge dv \\
= \frac{k}{\pi} \int_B d \left( \sinh^2 \rho \ du \wedge dv \right) \\
= \frac{k}{\pi} \int_M d\tau d\sigma \epsilon^{\alpha\beta} \sinh^2 \rho \partial_\alpha u \partial_\beta v
\] (2.21)

The last expression can then be used to define the open string action
\[
S^\text{Open}_2 = \frac{k}{\pi} \int_M d\tau d\sigma \epsilon^{\alpha\beta} \sinh^2 \rho \partial_\alpha u \partial_\beta v
\] (2.22)
It is important to stress that there is some ambiguity in the above procedure. The action for the closed string is the same if we add a total $\sigma$ derivative to the integrand of the action. This is not true for the open string action. Said in another way, there are in some sense many open string actions corresponding to a single closed string action. But the special choice (2.22) has some nice features.

If we introduce a new parametrisation (\(H\) is a scale of the \(SL(2, R)\) group-manifold)

\[
\sinh \rho = Hr, \quad u = \frac{1}{2}(Ht + \phi), \quad v = \frac{1}{2}(Ht - \phi)
\] (2.23)

in terms of which

\[
g = \begin{pmatrix}
\sqrt{1 + H^2 r^2} \cos Ht + H r \cos \phi & \sqrt{1 + H^2 r^2} \sin Ht - H r \sin \phi \\
-\sqrt{1 + H^2 r^2} \sin Ht - H r \sin \phi & \sqrt{1 + H^2 r^2} \cos Ht - H r \cos \phi
\end{pmatrix}
\] (2.24)

we get the following total action (well defined for both closed and open strings)

\[
S = -\frac{H^2 k}{4\pi} \int_M d\tau d\sigma \left\{ \eta^{\alpha\beta} \left[ -(1 + H^2 r^2) \partial_\alpha t \partial_\beta t + \frac{\partial_\alpha r \partial_\beta r}{1 + H^2 r^2} + r^2 \partial_\alpha \phi \partial_\beta \phi \right] + 2\epsilon^{\alpha\beta} H r^2 \partial_\alpha t \partial_\beta \phi \right\}
\] (2.25)

If we let \(H \to 0\), scaling \(k\) such that \(1/\alpha' = H^2 k\) is kept constant, (2.25) becomes the usual Minkowski action in polar coordinates. Also a nice feature is that (2.25) is invariant under global \(t\) and \(\phi\) translations, so we can define constant energy and angular momentum as the corresponding Nöther charges.

For \(SL(2, R)\) we get the same group element \(g\) if we translate \(t\) by \(2\pi\). We will unwrap the time coordinate so that \(t\) is not identified with \(t + 2\pi\). As usual, this corresponds to going to the universal cover of \(SL(2, R)\).
2.3 Boundary Conditions

The variation of (2.25) is (a prime/dot denotes a \( \sigma/\tau \) derivative)

\[
\delta S = - \frac{H^2 k}{2\pi} \int_M d\tau d\sigma \left\{ \left[ -\frac{r'' - \dot{r}}{1 + H^2 r^2} - H^2 r (t'l' - \ddot{t}t) + \frac{H^2 r (r'r' - \dot{r}t)}{(1 + H^2 r^2)^2} \right] + r (\phi'\phi' - \dot{\phi}\dot{\phi}) + 2H r (\phi'\dot{t} - t'\dot{\phi}) \right\} \delta r \\
+ \left[ (1 + H^2 r^2) (t'' - \ddot{t}) + 2H^2 r (t'r' - \dot{t}\dot{r}) + 2H r (r'\dot{\phi} - \phi'\dot{r}) \right] \delta t \\
+ \left[ -r^2 (\phi'' - \ddot{\phi}) - 2r (r'\phi' - \dot{\phi}\dot{r}) + 2H r (t'i - r'i) \right] \delta \phi \\
- \frac{H^2 k}{2\pi} \int d\tau \left\{ \left[ -(1 + H^2 r^2) t' - H r^2 \dot{\phi} \right] \delta t + \left[ \frac{r'}{1 + H^2 r^2} \right] \delta r \\
+ \left[ r^2 \phi' + H r^2 \dot{t} \right] \delta \phi \right\} \bigg|_{\sigma=\pi}^{\sigma=0} \tag{2.26}
\]

so, besides the equations of motion, we get the boundary conditions

\[
t' = -\frac{H r^2}{(1 + H^2 r^2)} \dot{\phi}; \quad \sigma = 0, \pi \\
r' = 0; \quad \sigma = 0, \pi \\
\phi' = -H \dot{t}; \quad \sigma = 0, \pi \tag{2.27}
\]

As remarked earlier, we can add to the action a total sigma derivative

\[
S_3 = - \int_M d\tau d\sigma \partial_\sigma A = - \int d\tau A \bigg|_{\sigma=\pi}^{\sigma=0} \tag{2.28}
\]

without affecting the equations of motion. If we want the boundary conditions to be linear in derivatives, then \( A \) must have the form \( A = A_\mu \partial_\tau X^\mu \) (there cannot be any \( \sigma \) derivatives if the Lagrange density does not contain any derivatives of higher order than one). If we introduce \( F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu} \) we can write \( S_3 \) in two equivalent ways

\[
S_3 = \int_M d\tau d\sigma F_{\mu\nu} \partial_\tau X^\mu \partial_\sigma X^\nu = - \int d\tau A_\mu \partial_\tau X^\mu \bigg|_{\sigma=\pi}^{\sigma=0} \tag{2.29}
\]

and we see that this term can be interpreted in two ways. Either as an addition of a total derivative to the antisymmetric tensor background, or as a coupling of the string endpoints to a vector field background.
For the variation of the action we get the extra term

$$\delta S_3 = \int d\tau F_{\mu\nu} \partial_\tau X^\mu \delta X^\nu |_{\sigma=0}^{\sigma=\pi}$$

(2.30)

From this expression and eq. (2.26), we see that we now have the more general set of boundary conditions at \(\sigma = 0, \pi\)

\[
\begin{align*}
(1 + H^2 r^2) t' & = -H r^2 \dot{\phi} - \tilde{F}_{\mu\nu} \partial_\tau X^\mu \\
\frac{r'}{1 + H^2 r^2} & = \tilde{F}_{\mu\nu} \partial_\tau X^\mu \\
r^2 \phi' & = -H r^2 \dot{t} + \tilde{F}_{\mu\nu} \partial_\tau X^\mu
\end{align*}
\]

(2.31)

where \(\tilde{F}_{\mu\nu} \equiv \frac{2\pi}{H r} F_{\mu\nu}\).

Now a question one can ask is the following: Given some boundary conditions, is it possible to find \(A_\mu\) such that the boundary conditions can be derived from the action (2.25) with \(S_3\) added to it? If this is possible, we should be able to write the boundary conditions with all the \(\sigma\) derivatives on the left hand side of the equality sign, in the same form as above, and read off the tensor \(\tilde{F}_{\mu\nu}\) from the right hand side. \(\tilde{F}_{\mu\nu}\) should then be antisymmetric and satisfy the condition (the square of the exterior derivative being zero)

$$\tilde{F}_{\mu\nu,\rho} + \tilde{F}_{\rho\mu,\nu} + \tilde{F}_{\nu\rho,\mu} = 0$$

(2.32)

As a simple example, one can ask if it is possible to choose \(A_\mu\) such that there are no \(\tau\) derivatives in the boundary conditions (corresponding to only taking the boundary conditions from the \(S_1\) part of the action). Such boundary conditions were considered in Refs. [34, 35]. They correspond to the standard Neumann boundary conditions, which are usually also imposed for open strings in flat Minkowski space (see for instance [9]). However, for the \(\tau\) derivatives to disappear in eq. (2.31), we must have

$$\tilde{F}_{\phi t} = -\tilde{F}_{t \phi} = -H r^2, \quad \tilde{F}_{t r} = \tilde{F}_{r t} = \tilde{F}_{r \phi} = \tilde{F}_{\phi r} = 0$$

(2.33)

This is antisymmetric, but

$$\tilde{F}_{\phi t, r} + \tilde{F}_{r \phi, t} + \tilde{F}_{t r, \phi} = -2 H r$$

(2.34)

do not vanish, and so we cannot find a \(A_\mu\) giving a boundary condition without \(\tau\) derivatives. Thus it is not possible to obtain the standard Neumann boundary conditions, from an action principle, in the case of the WZWN model corresponding to \(SL(2, R)\). This is of course consistent with the conclusion obtained in Subsection 2.1.
### 2.4 The Neumann Type Gluing Condition

A lot of work (see for instance Refs.\[14, 15, 16, 17\]) on D-branes in the WZWN models use the gluing conditions (evaluated at $\sigma = 0, \pi$. We will not write this in the rest of this section)

\[ J = \pm \bar{J} \quad (2.35) \]

where $J$ and $\bar{J}$ are defined in eq.(2.8). One of them is considered as a generalization of the Neumann boundary conditions $\partial_\sigma X^\mu = 0$ in flat space, and the other as a generalization of the corresponding Dirichlet boundary conditions $\partial_\tau X^\mu = 0$. To see which is which, consider the case where the group-manifold is Abelian. If the group elements $g$ are parametrised by $g = e^{iX}$, with $X$ in the Lie algebra, then

\[ J = + \bar{J} \iff \partial_- X = \partial_+ X \iff \partial_\sigma X = 0 \]
\[ J = - \bar{J} \iff \partial_- X = - \partial_+ X \iff \partial_\tau X = 0 \quad (2.36) \]

and we see that $J = \bar{J}$ corresponds to the Neumann conditions.

As another example of the procedure developed in the last subsection, we can try to see if the Neumann type gluing condition can be derived from an open string $SL(2, R)$ WZWN action. In terms of the $(t, r, \phi)$ parametrisation, we have for the decomposition (2.3)

\[
\begin{align*}
J^1 + iJ^2 &= k \left[ Hr\sqrt{1 + H^2r^2} \left( H\partial_- t - \partial_\phi \right) - i \frac{H\partial_- r}{\sqrt{1 + H^2r^2}} \right] e^{-i(Ht+\phi)} \\
J^3 &= k \left[ (1 + H^2r^2)H\partial_- t - H^2r^2\partial_\phi \right] \\
\bar{J}^1 + i\bar{J}^2 &= k \left[ Hr\sqrt{1 + H^2r^2} \left( -H\partial_+ t - \partial_\phi \right) - i \frac{H\partial_+ r}{\sqrt{1 + H^2r^2}} \right] e^{i(Ht-\phi)} \\
\bar{J}^3 &= k \left[ (1 + H^2r^2)H\partial_+ t + H^2r^2\partial_\phi \right] 
\end{align*}
\]

Thus, the Neumann type gluing conditions $J^3 = \bar{J}^3$, $J^1 + iJ^2 = \bar{J}^1 + i\bar{J}^2$ are equivalent to

\[
\begin{align*}
(1 + H^2r^2)t' &= -Hr^2 \dot{\phi} \\
\frac{r'}{1 + H^2r^2} &= -\frac{r\dot{\phi}}{1 + H^2r^2} \tan Ht \\
r^2 \phi' &= -Hr^2 i + \frac{r\ddot{r}}{1 + H^2r^2} \tan Ht 
\end{align*}
\]
Reading off the components of $\tilde{F}_{\mu\nu}$ we get

$$\tilde{F}_{\phi r} = -\tilde{F}_{r\phi} = -\frac{r}{1 + H^2 r^2} \tan H t , \quad \tilde{F}_{tr} = \tilde{F}_{rt} = \tilde{F}_{t\phi} = \tilde{F}_{\phi t} = 0$$  \quad (2.39)

Again, this is antisymmetric but

$$\tilde{F}_{\phi t, r} + \tilde{F}_{r, \phi t} + \tilde{F}_{tr, \phi} = \frac{H r}{1 + H^2 r^2} (1 + \tan^2 H t)$$  \quad (2.40)

so we cannot, in the framework considered in this article, find an action giving the Neumann type gluing conditions.

### 2.5 The Dirichlet Type Gluing Condition

We can also take a closer look at the Dirichlet type gluing conditions

$$J = -\tilde{J}$$  \quad (2.41)

In the $(t, r, \phi)$ parametrisation they are equivalent to the equations

$$(1 + H^2 r^2) t' - \frac{r'}{H r} \tan H t = -H r^2 \dot{\phi} - \frac{\dot{\phi}}{H}$$

$$r^2 \phi' = -H r^2 t - \frac{t}{H}$$  \quad (2.42)

$$0 = \frac{H^2 r^2}{\sqrt{1 + H^2 r^2}} \cos H t - \sqrt{1 + H^2 r^2} \sin H t H t$$

and we see that we obviously cannot choose $\tilde{F}_{\mu\nu}$, so that these equations become equivalent to eqs.$^\text{(2.31)}$. This is all right, since we decided to consider them as a generalization of the Dirichlet boundary conditions, and therefore they are not supposed to be derivable from an action, with the endpoints of the string being allowed to move freely. Instead, the endpoints should be allowed only to move on some submanifold of space-time, that is a D-brane.

By integrating the last equation, we see that we have a candidate in the D-string consisting of points satisfying

$$\sqrt{1 + H^2 r^2} \cos H t = C$$  \quad (2.43)

where $C$ is the integration constant. The rest of the equations in eqs.$^\text{(2.42)}$ are then supposed to be derivable from the variation of the action, eqs.$^\text{(2.26)}$
with the restriction from eq. (2.43) that there should be the following
relation between the variations of the endpoints

$$
\frac{H^2 r}{\sqrt{1 + H^2 r^2}} \cos H t \delta r = \sqrt{1 + H^2 r^2} \sin H t \delta t \quad (2.44)
$$

This is the case if we choose

$$
\tilde{F}_{\phi t} = - \tilde{F}_{t \phi} = \frac{1}{H}, \quad \tilde{F}_{\phi r} = \tilde{F}_{r \phi} = \tilde{F}_{r t} = \tilde{F}_{t r} = 0 \quad (2.45)
$$

and we see that we finally have a total exterior derivative, so the open string action is well defined. However, it was argued in Ref. [18] that this D-brane is not physical. The dynamics of a D-brane is governed by the Dirac-Born-
Infeld action

$$
S_{\text{DBI}} = - T_D \int d^{p+1} \xi \sqrt{- \det \left( \hat{G}_{ab} + \hat{B}_{ab} + 2 \pi \alpha' \hat{F}_{ab} \right)} \quad (2.46)
$$

Here $T_D$ is the tension of the D-brane, $\xi^0, \ldots, \xi^p$ is a parametrisation of the D-brane world-volume, and a hat denotes the pullback to the D-brane world-volume. For instance

$$
\hat{G}_{ab} = G_{\mu \nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \quad (2.47)
$$

The D-brane (2.43) solves the equations of motion derived from the Dirac-Born-Infeld action, but it is unphysical since the action is imaginary

$$
- \det \left( \hat{G}_{ab} + \hat{B}_{ab} + 2 \pi \alpha' \hat{F}_{ab} \right) = - \frac{C^2}{H^4 \cos^4 \xi^0} < 0 \quad (2.48)
$$

when $C \neq 0$ (we have parametrised the world-volume by $\xi^0 = H t, \xi^1 = \phi$).

As a last example, we note that the gluing conditions [17, 18]

$$
J^3 = \tilde{J}^3, \quad J^1 + i J^2 = e^{2i \alpha} (\tilde{J}^1 - i \tilde{J}^2) \quad (2.49)
$$

where $\alpha$ is an arbitrary constant, can be derived in the same way from eqs. (2.26, 2.30), if the string endpoints are restricted to D-branes of the form

$$
H r \cos(\phi + \alpha) = C \quad (2.50)
$$

where $C$ is again a constant. This time we have to choose

$$
\tilde{F}_{\phi t} = \tilde{F}_{t \phi} = \tilde{F}_{\phi r} = \tilde{F}_{r \phi} = \tilde{F}_{r t} = \tilde{F}_{t r} = 0 \quad (2.51)
$$
which is clearly a total derivative. The D-brane (2.50) solves the Dirac-Born-Infeld equations of motion, and is physical since the action is real (we have chosen \( \xi^0 = Ht, \xi^1 = \phi + \alpha \))

\[
- \det \left( G_{ab} + \dot{B}_{ab} + 2\pi \alpha' F_{ab} \right) = \frac{C^2}{H^4 \cos^4 \xi_1} \geq 0
\]

(2.52)
as it should be. This D-brane was also studied more extensively in Ref. [18].

Both sets of gluing conditions (2.41) and (2.49) are compatible with the spectral flow considered in [31, 36], which takes a solution \( \bar{g} \) of the equations of motion (2.6) and generates the new solution

\[
g = e^{w_R \sigma^- i^3} \bar{g} e^{w_L \sigma^+ i^3}
\]

(2.53)
or in terms of the currents (2.9)

\[
J^1 + iJ^2 = \left( \bar{J}^1 + i\bar{J}^2 \right) e^{-iw_R \sigma^-}
\]

(2.54)

\[
J^3 = \bar{J}^3 + \frac{k}{2} w_R
\]

(2.55)

\[
J^1 + iJ^2 = \left( \bar{J}^1 + i\bar{J}^2 \right) e^{iw_L \sigma^+}
\]

(2.56)

\[
\bar{J}^3 = J^3 + \frac{k}{2} w_L
\]

(2.57)

with the restriction for (2.41) that \( w_R = -w_L = w \) and for (2.49) that \( w_R = w_L = w \), \( w \) taking integer values. This observation was used in [19, 20] to generalize the quantization procedure of [36] to the case of open strings ending on D-branes of the form (2.50).

### 3 Rigidly Rotating Strings

In this section we shall consider rigidly rotating open strings in the \( SL(2, R) \) WZWN model. We choose to work with the open string action (2.25), which is invariant under global \( t \) and \( \phi \) translations. This way, both the energy and angular momentum are ensured to be constants of motion. This should be contrasted with the cases considered in Section 2.

The open string boundary conditions are given by eq.(2.27). Using eq.(2.37), it follows that they correspond to the following field-dependent gluing conditions

\[
J^3 = \bar{J}^3, \quad J^1 + iJ^2 = e^{-2iHt} \left( \bar{J}^1 + i\bar{J}^2 \right)
\]

(3.1)
The rigidly rotating open string ansatz is
\[ t = \tilde{t}(\sigma) + c_1 \tau \]
\[ \phi = \tilde{\phi}(\sigma) + c_2 \tau \]
\[ r = r(\sigma) \] (3.2)
where \( c_1 \) and \( c_2 \) are constants. We take \( c_1 > 0 \) to ensure forward propagation in time \( (\dot{t} > 0) \), while \( c_2 \) is arbitrary. The above ansatz describes the most general, with constant velocity, rotating rigid string (up to world-sheet coordinate transformations) \[7\]. The equations of motion and constraints, in this case, reduce to
\[ \frac{d\tilde{t}}{d\sigma} = \frac{k_1 - Hc_2r^2}{1 + H^2r^2} \] (3.3)
\[ \frac{d\tilde{\phi}}{d\sigma} = \frac{k_2 - Hc_1r^2}{r^2} \] (3.4)
\[ \left( \frac{dr}{d\sigma} \right)^2 + V(r) = 0 \] (3.5)
where the potential \( V(r) \) is given by
\[ V(r) = - \frac{(H^2k_2^2 - k_1^2)(c_1^2 + 2Hc_1k_2)}{r^2k_2^2} \cdot \left( r^2 - \frac{k_2^2}{c_1^2 + 2Hc_1k_2} \right) \cdot \left( r^2 + \frac{k_2^2}{H^2k_2^2 - k_1^2} \right) \] (3.6)
and the integration constants \((k_1, k_2)\) are constrained by
\[ c_1k_1 = c_2k_2 \] (3.7)
The boundary conditions (2.27) demand
\[ k_1 = k_2 = 0, \quad V(r)_{\sigma = 0, \pi} = 0 \] (3.8)
The latter condition gives rise to the relation \( c_2^2 = n^2 + H^2c_1^2 \), where \( n \) is an arbitrary integer. Then the solution is given by
\[ r = \frac{c_1}{n} \cos(n\sigma) \] (3.9)
\[ t = c_1\tau \mp \frac{\sqrt{n^2 + H^2c_1^2}}{H} \sigma \pm \frac{1}{H} \cot^{-1} \left( \frac{\sqrt{n^2 + H^2c_1^2}}{n} \cot(n\sigma) \right) \] (3.10)
\[ \phi = \pm \sqrt{n^2 + H^2c_1^2} \tau - Hc_1\sigma \] (3.11)
The upper sign corresponds to positive values of \( c_2 \), the lower to negative. Notice that the integer \( n \) plays the role of a winding number or, more precisely, it gives the number of "foldings" of the open string. Namely, for \( H \to 0 \), we get

\[
r = \frac{c_1}{n} \cos(n\sigma), \quad \phi = \pm \frac{|n|}{c_1} t; \quad H \to 0
\]

which is the standard \( n \)-folded rigidly rotating straight string in Minkowski space. For further comparison with the Minkowski case, it is also convenient to express \( r \) and \( \phi \) in terms of \( t \) and \( \sigma \), and then to consider the string at fixed coordinate time \( t \). In particular, eq.(3.11) leads to

\[
\phi(t, \sigma) = \pm \sqrt{\frac{n^2 + H^2 c_1^2}{c_1}} t + \frac{n^2}{H c_1} \sigma \\
- \sqrt{\frac{n^2 + H^2 c_1^2}{H c_1}} \cot^{-1} \left( \sqrt{\frac{n^2 + H^2 c_1^2}{n}} \cot(n\sigma) \right)
\]

and it follows that the torsion leads to a bending and an unfolding of the string; see Figure 1. This should be contrasted with the case of Minkowski space and with the case of "pure" anti de Sitter space without torsion [6, 8].

Invariance of the action (2.25) under constant \( t \) and \( \phi \) translations gives rise to conserved Nöther currents

\[
P^\alpha_t = \frac{\partial \mathcal{L}}{\partial t^\alpha}, \quad P^\alpha_\phi = \frac{\partial \mathcal{L}}{\partial \phi^\alpha}
\]

The corresponding constant charges are given by

\[
Q_t = \int_0^\pi d\sigma P^\tau_t = -\frac{H^2 k}{2\pi} \int_0^\pi d\sigma \left[ (1 + H^2 r^2) i + H r^2 \phi' \right] \\
Q_\phi = \int_0^\pi d\sigma P^\tau_\phi = \frac{H^2 k}{2\pi} \int_0^\pi d\sigma \left[ r^2 \phi' + H r^2 t' \right]
\]

The first charge is identified with minus the mass-energy \( M \) and the second one with angular momentum \( L \). Using the solution (3.9)-(3.11), we get

\[
M = \frac{c_1}{2 \alpha'}, \quad L = \pm \frac{1}{2 \alpha' H^2} \left( \sqrt{n^2 + H^2 c_1^2} - n \right)
\]
where we used again the relation $1/\alpha' = H^2 k$. It follows that we get the relation

$$L = \pm \frac{M^2 \alpha'}{|n|} \left[ \frac{\sqrt{n^4 + 4H^2 \alpha'^2 M^2 n^2 - n^2}}{2H^2 \alpha'^2 M^2} \right] \quad (3.18)$$

For $H \to 0 \ (k \to \infty)$, the square bracket goes to 1 and we recover, for $n = 1$, the famous Minkowski-space Regge behavior (see for instance Ref.\[9\]). Thus, the relation (3.18) can be considered as the $SL(2, R)$ generalization of the Minkowski Regge behavior. In the present case, the linear relationship between $L$ and $M^2$ is recovered only for "small" mass, while for "high" mass we get instead

$$L \approx \pm M/H \quad (3.19)$$

### 4 Conclusion

It was recently shown that open strings in $AdS_3$ can be quantized \[19, 20\], by generalizing the procedure for closed strings \[36\]. Quantization of open strings involves D-branes, on which the open strings can end, corresponding to certain gluing conditions \[17, 18\]. Previously in the literature, various other boundary conditions and gluing conditions have been considered.

We have shown that some of the previously considered boundary conditions and gluing conditions for open strings in $AdS_3$ are in fact incompatible with the variation principle. Other boundary conditions and gluing conditions are compatible with the variation principle, and get interpretations in terms of D-branes.

We then considered a certain field-dependent gluing condition, compatible with the variation principle. The corresponding open strings seem to give the most natural generalization, at least classically, of the open strings in flat Minkowski space. The open strings were analyzed in detail for an ansatz corresponding to rigid rotation. We showed, in particular, that the torsion leads to a bending and an unfolding of the strings. Finally, we derived the $SL(2, R)$ Regge relation, i.e. the relation between mass $M$ and angular momentum $L$ for rigidly rotating strings in $SL(2, R) \cong AdS_3$. It turned out to be quite different from the Minkowski Regge relation; the linear relationship between $L$ and $M^2$ is recovered only for "small" mass, while for "high" mass we found instead $L \approx \pm M/H$, where $H$ is the scale of the $SL(2, R)$ group manifold.
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CAPTIONS FOR FIGURES

Figure 1: The rigidly rotating string (3.9)-(3.11) for $n = 2$ and $H_{c1} = 1$. The string is shown at fixed coordinate time $t = 0$. The axes represent $Hr \cos \phi$ and $Hr \sin \phi$, respectively. Notice that the string is bended and unfolded.
Open Strings in the SL(2,R) WZWN Model
with Solution for a Rigidly Rotating String

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Figure 1