Energy transfer and position measurement in quantum mechanics

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The Dirac delta function can be defined by the limitation of the rectangular function covering a unit area with decrease of the width of the rectangle to zero, and in quantum mechanics the eigenvectors of the position operator take the form of the delta function. When discussing the position measurement in quantum mechanics, one is prompted by the mathematical convention that uses the rectangular wave function of sufficiently narrow width to approximate the delta function in order to making the state of the position physical. We argue that such an approximation is improper in physics, because during the position measurement the energy transfer to the particle might be infinitely large. The continuous and square-integrable functions of both sharp peak and sufficiently narrow width can then be better approximations of the delta function to represent the physical states of position. When the slit experiment is taken as an apparatus of position measurement, no matter what potential is used to model the slit, only the ground state of the slit-dependent wave function matters.

I. INTRODUCTION

The measurement postulate of quantum mechanics can be divided into two parts. [1] The first part is: the only possible result of the measurement of a physical quantity $A$ is one of the eigenvalues of the corresponding observable $\hat{A}$. The second is (case of a continuous non-degenerate spectrum): when the physical quantity $A$ is measured on a system in the normalized state $|\psi\rangle$, the probability $dP(\alpha)$ of obtaining a result included between $\alpha$ and $\alpha + d\alpha$ is equal to,

$$dP(\alpha) = |\langle u_\alpha | \psi \rangle|^2 d\alpha,$$

where $|u_\alpha\rangle$ is the eigenvector corresponding to the eigenvalue $\alpha$ of the observable $\hat{A}$ associated with $A$, determined by,

$$\hat{A} |u_\alpha\rangle = \alpha |u_\alpha\rangle.$$

For the case of a discrete non-degenerate spectrum, we need to replace Eq. (1) by $P(a_n) = |\langle u_n | \psi \rangle|^2$ where $n$ denotes the discrete index and the $n$-th eigen-value and eigen-state $\{a_n, |u_n\rangle\}$ are determined by $\hat{A} |u_n\rangle = a_n |u_n\rangle$. In our study, for simplicity, all operators $\hat{A}$ have non-degenerate spectrum only.

We apply the measurement postulate to the position measurement of a particle in a quantum state $|\psi\rangle$, and for simplicity we assume that this particle moves in a one-dimensional potential. The eigenvector of position operator $\hat{x}$ takes a delta function $\delta(x - x')$ with eigenvalue $x'$, obeying following equation,

$$\hat{x} \delta(x - x') = x' \delta(x - x').$$

When the position $x$ is measured in any quantum state $|\psi\rangle$, no experiment can yield a result with complete accuracy. Let us assume that the opposite is true, i.e., assume that after the measurement, the particle will collapse to an eigen-state $\delta(x - x')$ with eigenvalue $\hat{x}'$. We can expand this delta-function in terms of eigen-states of the Hamiltonian $\hat{H}$ of the particle,

$$\delta(x - x') = \sum_n \varphi^*_n(x') \varphi_n(x),$$

where $\varphi_n(x)$ is the $n$-th eigen-state of the Hamiltonian $\hat{H}$ with energy eigenvalue $E_n$, i.e., we have,

$$\hat{H} \varphi_n(x) = E_n \varphi_n(x).$$

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Result (1) shows that the probability of the particle in an arbitrary energy eigen-state $\varphi_n(x)$ is $|\varphi_n(x)|^2$. Not only the total probability diverges as $\sum_n |\varphi_n(x)|^2 = \delta(x' - x) = \delta(0) \to \infty$, but also does the expectation value of the energy as $\sum_n |\varphi_n(x)|^2 E_n = E_0 \sum_n |\varphi_n(x)|^2 (E_n/E_0) > E_0 \sum_n |\varphi_n(x)|^2 \to \infty$, where the fact that the Hamiltonian operator $H$ is in general self-adjoint and bounded from below, i.e., $0 \leq E_0 < E_1 < E_2 < ...$, is used and the smallest value of the eigenvalues is assumed not less than zero. These two results are absurd, all due to the non-normalizability of the delta function. A question arises immediately: If the normalizable rectangular wave function instead of the delta function. A question arises immediately: If the normalizable rectangular wave function is sufficient? If not, which one is proper? The present paper investigates these problems and those related.

In section II, we will demonstrate that the conventional normalizable rectangular wave function is unsatisfactory. After all, if one likes to use the rectangular wave functions for some purposes, he must bear in mind that these functions have fatal flaws.

### II. ENERGY DIVERGENCE WITH RECTANGULAR WAVE FUNCTIONS

In a realistic experiment of position measurement, the position of the particle can only be found to be within a sufficiently narrow interval that nevertheless contains uncountably infinite many position eigenvalues because of the continuous spectrum of the position, and possibly these eigenvalues may be distributed around a point that is more probable. The most common means to get over the divergence problem is to use some approximated representations of the delta function. The simplest and the conventional choice is the rectangular function. 

$$\chi_\varepsilon(x - x_0) = \begin{ cases} 1/\sqrt{\varepsilon} & x \in [x_0 - \varepsilon/2, x_0 + \varepsilon/2], \\ 0 & \text{otherwise} \end{ cases}, \varepsilon > 0. \quad (6)$$

This function is normalized to unity, i.e., $\int |\chi_\varepsilon(x - x_0)|^2 dx = 1$, and also we have,

$$\lim_{\varepsilon \to 0^+} \int_{x_0 - \varepsilon/2}^{x_0 + \varepsilon/2} |\chi_\varepsilon(x - x_0)|^2 f(x)dx = \lim_{\varepsilon \to 0^+} \int_{x_0 - \varepsilon/2}^{x_0 + \varepsilon/2} \frac{1}{\varepsilon} f(x_0) + f'(x_0)(x - x_0) + O((x - x_0)^2) dx = f(x_0), \quad (7)$$

where $f(x)$ is a continuous function defined in the same interval in which the wave function under study is defined. This choice of the approximation of the delta function is still improper. The problem lies in that the rectangular wave function (6) belongs to the differentiability class $C^0$ rather than the class higher than $C^1$. We demonstrate it in the following.

Let us measure the position of the particle in a one-dimensional infinitely deep well of width $a$ defined in an interval $x \in (0, a)$ and the position happens to be approximately in the middle $x_0 = a/2$. The stationary state energies the states for the particle in the well are, respectively,

$$E_n = \frac{(n\pi\hbar)^2}{2ma^2}, \varphi_n(x) = \sqrt{\frac{2}{a}} \sin \left(\frac{n\pi x}{a}\right), n = 1, 2, 3, ... \quad (8)$$

where $m$ is the mass of the particle. Once the delta function is approximated by the rectangular wave function (6), each value $E_n$ can be found with following probability,

$$P(E_n) = \left| \int \varphi_n^*(x) \chi_\varepsilon(x - a/2)dx \right|^2 = \frac{8a}{\varepsilon} \begin{ cases} \frac{1}{(n\pi)^2} \sin^2(\frac{2n\pi a}{2a}) & \text{odd } n \\ 0 & \text{even } n \end{ cases}. \quad (9)$$

It is normalized to unity. However, the energy expectation diverges as, no matter what value of $\varepsilon$ is given,

$$\sum_n P(E_n)E_n = \sum_{n=1}^{\infty} P(E_{2n-1})E_{2n-1} = \frac{4\hbar^2}{ma\varepsilon} \sum_{n=1}^{\infty} \sin^2 \left(\frac{(2n - 1)\pi \varepsilon}{2a}\right) \to \infty. \quad (10)$$

It is totally unacceptable for it implies that at the moment of the measurement, the particle gains an infinite amount of energy. It is impossible in physics.

It is easily to verify that utilization of the one-dimensional simple harmonic oscillator, instead of the one-dimensional infinitely deep well, can give the same divergence (11) as well. In fact, at the discontinuous points of the function (6), action of momentum operator leads to $\delta$-functions in the following way,

$$p_x \chi_\varepsilon(x - x_0) = -\frac{i\hbar}{\sqrt{\varepsilon}} \left( \delta(x - x_0 + \frac{\varepsilon}{2}) - \delta(x - x_0 - \frac{\varepsilon}{2}) \right). \quad (11)$$

Thus it is the discontinuity in (6) that results in the divergence.

After all, if one likes to use the rectangular wave functions for some purposes, he must bear in mind that these functions have fatal flaws.
III. POSITION MEASUREMENT AND UNCERTAINTY PRINCIPLE

In order to construct proper position function in realistic experiment, we must resort to continuous and square-integrable function of differentiability class not less than $C^1$ that has both sharp peak and sufficiently narrow width to approximate the delta function. The form of the continuous functions does not matter but the sharpness matters for it characterizes resolution power of the measuring apparatus. The simplest possible choose is,

$$
\chi_\varepsilon(x-x_0) = \begin{cases} 
\sqrt{\frac{2}{\varepsilon}} \sin \left( \frac{\pi(x-x_0+\varepsilon/2)}{\varepsilon} \right), & x \in [x_0-\varepsilon/2, x_0+\varepsilon/2], \varepsilon > 0. \\
0 & \text{otherwise}
\end{cases} \quad (12)
$$

Its probability density behaves like a delta function, for we have not only $\int |\chi_\varepsilon(x-x_0)|^2 \, dx = 1$ but also,

$$
\lim_{\varepsilon \to 0^+} \frac{\int_{x_0-\varepsilon/2}^{x_0+\varepsilon/2} \frac{2}{\varepsilon} \sin^2 \left( \frac{\pi(x-x_0+\varepsilon/2)}{\varepsilon} \right) f(x) \, dx}{|\chi_\varepsilon(x-x_0)|^2} = f(x_0) \int |\chi_\varepsilon(x-x_0)|^2 \, dx = f(x_0). \quad (13)
$$

Now the energy expectation values over state $\chi_\varepsilon(x-x_0)$ (12) is finite no matter what small values of $\varepsilon$ are given, and the values are evidently,

$$
\frac{(\pi\hbar)^2}{2m\varepsilon^2} \sim \frac{\hbar^2}{m\varepsilon^2} \sim \frac{\delta p^2}{m}, \quad (14)
$$

where $\delta p \sim \hbar/\varepsilon$, as indicated by the uncertainty principle $\delta x \delta p \sim \hbar$ with $\delta x \sim \varepsilon$.

One can instead take the Gaussian function approximation of the delta function, rather than (12), and the similar results will be found. Explicitly, we take the following Gaussian function approximation,

$$
\chi_\varepsilon(x-x_0) = \left( \frac{1}{\sqrt{\pi} \varepsilon} \right)^{1/2} \exp \left[ -\frac{(x-x_0)^2}{2\varepsilon^2} \right], \varepsilon > 0. \quad (15)
$$

The energy expectation value is also finite provided that $\varepsilon$ does not vanish, and the value is given by,

$$
\frac{\hbar^2}{4m\varepsilon^2} \sim \frac{\hbar^2}{m\varepsilon^2} \sim \frac{\delta p^2}{m}. \quad (16)
$$

Above results (14) and (15) are physically significant in following aspects: 1. If the particle before the position measurement has energy much less than $\hbar^2/(m\varepsilon^2)$, we must adjust $\varepsilon$ to be a large value such that the momentum uncertainty $\delta p \sim \hbar/\varepsilon$ becomes smaller. 2. The measurement apparatus acts as an energy conversion that at the moment of measurement converts part of the energy from the particle before measurement to the particle whose position is under measurement, and the amount of the energy is quite definite as shown in (14) or (16). These two points will be further discussed in the following section. 3. The position measurement must be of a very complicated process, which is far beyond full understanding. [4, 9] It seems to us that annihilations and creations happen during the measurement.

IV. COMMENTS ON THE SINGLE SLIT EXPERIMENT AS A POSSIBLE APPARATUS OF POSITION MEASUREMENT

Now we try to take the single slit experiment to measure the $x$-component position of a particle via detect its $x$-component momentum on the remote screen (the Fraunhofer diffraction) after passing through a single narrow slit of finite width. For this propose, the diaphragm is kept in the plane $z = 0$, a slit of small width $\delta x = a$ is made with center at $x = 0$, and let the slit be straight and very long along the $y$-axis. It is then that only the component along the $x$-axis of the incident free particle is affected by diffraction. A monochromatic wave travelling along $z$-axis with energy $E_0 = p_z^2/2m$ and incident on the first screen which contains a slit. The emerging wave then arrives at the screen to form a diffraction pattern. The bright bands correspond to interference maxima, and the dark bands interference minima. Since the zeroth maximum has dominant part of the probability, the first dark band offers the best characterization of uncertainty of the momentum $\delta p_x$ that is also the momentum gained along $x$-axis for the particle passing through the slit,

$$
\delta p_x \approx \frac{\hbar}{a}. \quad (17)
$$
This result is compatible with results [4] and [10]. To note that within the single slit experiment, for a given openness $a$ of the slit, once the incident waves have energies $E_0 = \frac{p_x^2}{2m}$ lower than $\frac{h^2}{2ma^2}$, there would be no quantum state that contains nonvanishing $x$-dependent part. In other words, the interference maxima are no longer appreciable. On the other hand, once the incident waves have energies $E_0 = \frac{p_x^2}{2m}$ lower than $\frac{h^2}{2ma^2}$, the single slit experiment is not suitable apparatus of position measurement.

It should be stressed that when above single slit experiment is taken as an apparatus of position measurement, no matter what potential is used to model the slit, only the ground state of the $x$-dependent wave function matters. This is because, with increase of the quantum numbers in the $x$-dependent stationary states of the wave function, the uncertainties $\delta x \delta p_x$ will increase, and the relation [10] will be broken. In other words, the measurement apparatus inevitably causes some disturbances in both momentum and energy and it is reasonable to require that these disturbances keep smallest possible.

V. CONCLUSIONS

The simplest and the conventional choice of taking the rectangular function as an approximation of delta function, following the mathematical convention, is improper as it is used to represent a quantum state. Calculations of this approximation in position measurement show that the amount of the energy gained at the moment of measurement might be infinitely large. Therefore, we have to choose the continuous and square-integrable functions of both sharp peak and sufficiently narrow width as better approximations of the delta function representing the physical states of position. The detailed structure of forms of the continuous functions does not matter but the sharpness matters for it characterizes resolution power of the measuring apparatus. Similar discussions are applicable to the approximation of delta function representing the physical states of momentum, which can be easily done by the readers as an exercise.

Since the single slit experiment is frequently taken as an apparatus of position measurement, our results indicate that no matter what potential is used to model the slit, only the ground state of the slit-dependent wave function matters.

Acknowledgments

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[1] C. Cohen-Tannoudji, B. Diu and F. Laloe, Quantum Mechanics (vol. one), (Hermann, Paris, 1977). p.220
[2] S. Weinberg, Lectures on Quantum Mechanics, 2nd ed., (Cambridge University Press, Cambridge, 2015). p.63
[3] T. V. Marcella Eur. J. Phys. 23(2002)615
[4] T. Rothman and S. Boughin Eur. J. Phys. 32(2011)107
[5] G. Dillon, Eur. Phys. J. Plus 127(2012)66
[6] B. Fabbro, arXiv:1710.09758v3, 2017
[7] M. V. John, and K. Mathew, arXiv:1807.03800v2, 2018
[8] J. C. Ye, Y. Li, Q. Chen, S. G. Chen, Q. H. Liu, arXiv:1902.00015, 2019
[9] H. M. Wiseman, G. J. Milburn, Quantum Measurement and Control, (Cambridge University Press, Cambridge, 2010)