Local monomialization conjecture of a singular foliation of Darboux type

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Abstract

After the nice result introduced by Belotto in [1] concerning the local monomialization of a singular foliation given by $n$ first integrals, this work is a continuation in the same spirit. In this paper, we introduce an important conjecture about local monomialization of a singular foliation of Darboux type (see section 1). This conjecture can be used to study pseudo-abelian integrals [2,4].

1 Introduction

Let $M$ be an analytic manifold of dimension $n + 2$. Given a families of first integrals of Darboux type $H_\epsilon$

$$H_\epsilon(x, y) = H(x, y, \epsilon_1, \ldots, \epsilon_n) = \prod_{i=1}^{k} P^a_i(x, y, \epsilon_1, \ldots, \epsilon_n), \quad a_i > 0. \quad (1)$$

Let $F$ be the foliation of codimension one in $M$ with coordinates $(x, y, \epsilon_1, \ldots, \epsilon_n)$ which is given by the analytic one form $\omega$

$$\omega = \frac{H_x}{\phi} dx + \frac{H_y}{\phi} dy + \sum_{i=1}^{n} \frac{H_{\epsilon_i}}{\phi} d\epsilon_i = 0, \quad (2)$$

where $H_x = \frac{\partial H}{\partial x}, H_y = \frac{\partial H}{\partial y}, H_{\epsilon_i} = \frac{\partial H}{\partial \epsilon_i}$ and $\phi = \prod_{i=1}^{k} P^{a_i - 1}_i(x, y, \epsilon_1, \ldots, \epsilon_n)$ (integrating factor).

Let $F_i, i = 1, \ldots, n$ be foliations of codimension one in $M$ with coordinates $(x, y, \epsilon_1, \ldots, \epsilon_n)$ which are given by the one forms $\omega_i$

$$\omega_i = d\epsilon_i = 0, \quad i = 1, \ldots, n. \quad (3)$$

Let $\mathcal{F} = (F, F_1, \ldots, F_n)$ be the result foliation of dimension one in $M$ where its leaves are given by the transversal intersection of leaves of $F, F_1, \ldots, F_n$. Otherwise speaking, the singular foliation $\mathcal{F}$ is given by

$$\Omega = \omega \wedge \omega_1 \wedge \ldots \wedge \omega_n = \prod_{i=1}^{k} Q^a_i(x, y, \epsilon_1, \ldots, \epsilon_n), \quad (4)$$

where $Q_1 = \frac{H_x}{\phi}, Q_2 = \frac{H_y}{\phi}$ are polynomials.

We shall say that $\Omega$ is a foliation of Darboux type with first integrals $(H, \epsilon_1, \ldots, \epsilon_n)$.

Example. Let $H_\epsilon(x, y) = H(x, y, \epsilon) = (x - \epsilon)^{a_1}(x - y)^{a_2}(x + y)^{a_3}$ be the first integral of Darboux type. The foliation $F$ of codimension one in three dimensional space $M$ with coordinates $(x, y, \epsilon)$ is given by the one form

$$\omega = (a_1(x - y)(x + y) + a_2(x - \epsilon)(x + y) + a_3(x - \epsilon)(x - y))dx$$

$$- (a_2(x - \epsilon)(x + y) - a_3(x - \epsilon)(x - y))dy - a_1(x - y)(x + y)d\epsilon = 0$$

and the foliation $F_1$ of codimension one in $M$ is given by the one form

$$\omega_1 = d\epsilon = 0.$$
The result foliation \( \mathcal{F} = (F, F_1) \) is given by the two-form
\[
\Omega = \omega \wedge \omega_1 = (a_1(x-y)(x+y) + a_2(x-\epsilon)(x+y) + a_3(x-\epsilon)(x-y))dx \wedge dy - (a_2(x-\epsilon)(x+y) - a_3(x-\epsilon)(x-y))dy \wedge dx = 0
\]

Observe that the foliation \( \mathcal{F} = (F, F_1) \) has a complicated singularity at the origin \((0, 0, 0) \subset D_0 = \{ \epsilon = 0 \} \).

**Conjecture.** There exist sequences of local blowings-up such that the total transform of the foliation \( \tilde{F} : \omega \wedge \omega_1 \wedge \ldots \wedge \omega_n = 0 \) has locally \( n + 1 \) monomial first integrals \((z^{\gamma_1}, z^{\gamma_2}, \ldots, z^{\gamma_{n+2}})\) and the exponents matrix
\[
m(a_1, \ldots, a_k) = \begin{pmatrix}
\gamma_0^1 & \ldots & \gamma_0^{n+2} \\
\gamma_1^1 & \ldots & \gamma_1^{n+2} \\
\vdots & \ldots & \vdots \\
\gamma_n^1 & \ldots & \gamma_n^{n+2}
\end{pmatrix}
\]

has a maximal rank.

## 2 Blowing-up of the foliation \( \mathcal{F} \)

In this section, we introduce the fundamental idea to prove the conjecture which is based in first step on Hironaka’s reduction of singularities [3]. Let \( D_0 = \{ \epsilon_1 = \epsilon_2 = \ldots = \epsilon_n = 0 \} \) be a initial exceptional divisor.

**Theorem 1.** There exist a morphism \( \Phi \) such that the pull-back foliation \( \Phi^* \mathcal{F} = \tilde{F} \) is given locally in neighborhood \( U_1 \) of the divisor \( \Phi^*(D_0) \) with coordinates \((z_1, \ldots, z_{n+2})\) by the following system
\[
\begin{align*}
\tilde{H} &= z^{\gamma_0}, \Delta_0, \\
\tilde{\epsilon}_1 &= z^{\gamma_1}, \Delta_1, \\
&\vdots \\
\tilde{\epsilon}_n &= z^{\gamma_n}, \Delta_n,
\end{align*}
\]

where \( \Delta_i, i = 0, \ldots, n \) are a units.

**Proof.**

1. In first step, we monomialize the principal ideal \( I_1 = < P_1 > \), Hironaka theorem’s guarantee the existence of a sequence of blow-ups \( \Phi_1 = \Phi^1_{n_1} \circ \Phi^1_{n_1-1} \circ \ldots \circ \Phi^1_1 \) with initial center \( C_0 \subset D_0 \) (which is possibly a submanifold of \( M \)) such that
\[
(\Phi^*_1 P_1)^{a_1} = \delta_1 \prod_{i=1}^{n+2} z_i^{a_i \beta_i^{1}}, \quad \delta_1(0) \neq 0.
\]

2. In the second step, we consider the principal ideal \( I_2 = < \Phi^*_1 P_2 > \) and by Hironaka theorem’s there exist a sequence of blow-ups \( \Phi_2 = \Phi^2_{n_2} \circ \Phi^2_{n_2-1} \circ \ldots \circ \Phi^2_1 \) such that
\[
(\Phi^*_2 \circ \Phi^*_1 P_2)^{a_2} = \delta_2 \prod_{i=1}^{n+2} z_i^{a_i \beta_i^{2}}, \quad \delta_2(0) \neq 0.
\]

In the \( k \)-th step there exist a sequence of blow-ups \( \Phi_k = \Phi^k_{n_k} \circ \Phi^k_{n_k-1} \circ \ldots \circ \Phi^k_1 \) such that the principal ideal \( I_k = < \Phi^*_k \circ \Phi^*_k \circ \ldots \circ \Phi^*_1 P_k > \) has a normal crossings i.e.
\[
(\Phi^*_k \circ \Phi^*_k \circ \ldots \circ \Phi^*_1 P_k)^{a_k} = \delta_k \prod_{i=1}^{n+2} z_i^{a_i \beta_i^{k}}, \quad \delta_k(0) \neq 0.
\]

Finally, after desingularisation of each polynomial \( P_i \) of the first integral \( H = \prod_{i=1}^{k} P_i^{a_i} \), the equations \( z_1 = 0, \ldots, z_{n+2} = 0 \) are corresponding the irreducibles components of the exceptional divisor. For this
raison after desingularisation of \( \Phi_{i-1}^* \circ \Phi_{i-2}^* \circ \ldots \circ \Phi_1^* P_i \), the polynomial \( \Phi_i^* \circ \Phi_{i-1}^* \circ \Phi_{i-2}^* \circ \ldots \circ \Phi_1^* P_{i-1} \) has a normal crossings. So locally we have

\[
\begin{cases}
\tilde{H} = z^{\gamma_0} \Delta_0, \\
\tilde{\epsilon}_1 = z^{\gamma_1} \Delta_1, \\
\vdots \\
\tilde{\epsilon}_n = z^{\gamma_n} \Delta_n,
\end{cases}
\]

where \( z = (z_1, \ldots, z_{n+2}) \), \( \gamma_0 = \sum_{i=1}^{n+2} a_i \beta_i \), \( \beta_i = (\beta_{i1}, \ldots, \beta_{in+2}) \), \( \gamma_i = (\gamma_{i1}, \ldots, \gamma_{in+2}) \).

To complete the proof it is necessary to eliminate the units \( \Delta_0, \Delta_1, \ldots, \Delta_n \) in the system (6). Now we define the resonant locus of the foliation \( (z^{\gamma_0} \Delta_0, z^{\gamma_1} \Delta_1, \ldots, z^{\gamma_n} \Delta_n) \)

\[
\mathcal{R} := \{ a = (a_1, \ldots, a_k) : \gamma_0 \wedge \sum_{j=1}^n \gamma_j = 0 \}.
\]

To prove the conjecture, we distinguish two cases

- **generic case** \( a \notin \mathcal{R} \).
- **nongeneric case** \( a \in \mathcal{R} \).

## 3 Some examples in dimension three

To more understand the problem, we see some examples in dimension three.

**Example 1:** Let \( F \) be the local foliation which is obtained by after \( k \) blow-ups. The foliation \( F \) is given by the following system

\[
\begin{align*}
H_a &= x^{a_1} y^{a_2} (1 + z) \\
f &= xy
\end{align*}
\]

(7)

In this example we have \( \gamma_0 : a_1 \beta_1 + a_2 \beta_2 \) where \( \beta_1 = (1, 0, 0), \beta_2 = (0, 1, 0) \), \( \gamma_1 = (1, 1, 0) \) and \( \mathcal{R} = \{ a = (a_1, a_2) : \gamma_0 \wedge \gamma_1 = 0 \} \). Our goal is to kill the unit \( 1 + z \) in the first integral \( H_a \) without modifying the second monomial \( f \) in the sense to preserve its monomiality structure. For this reason, we distinguish two different cases:

(a) The generic case \( a_1 \neq -a_2 \Leftrightarrow a \notin \mathcal{R} \): We take the change of variables \( \tilde{x} = x(1 + z)^{-1} \), \( \tilde{y} = y(1 + z)^{-1} \) and \( \tilde{z} = z \). Then, we obtain the following system

\[
\begin{align*}
H_a &= \tilde{x}^{a_1} \tilde{y}^{a_2} \\
f &= \tilde{x} \tilde{y}
\end{align*}
\]

(8)

**Question:** How to calculate the generator vector field of the monomial foliation (7)?

Let us assume that the foliation \( F \) is generated locally by the vector field \( X(\tilde{x}, \tilde{y}, z) = \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z} \) which satisfies

\[
X(H) = X(\tilde{x}^{a_1} \tilde{y}^{a_2}) = 0, \quad X(f) = X(\tilde{x} \tilde{y}) = 0
\]

To determine the vector \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) we use the two following relations

\[
< \alpha, \gamma_0 > = 0 \quad \text{(i.e.,) } \quad X(H) = X(\tilde{x}^{a_1} \tilde{y}^{a_2}) = 0, \quad < \alpha, \gamma_1 > = 0 \quad \text{(i.e.,) } \quad X(f) = X(\tilde{x} \tilde{y}) = 0.
\]

where \( <, > \) is scalar product in \( \mathbb{C}^3 \). Finally, the vector \( (\alpha_1, \alpha_2, \alpha_3) \in \{ e_3 \} \) and then

\[
F = \{ z \frac{\partial}{\partial z} \}
\]
Now, we express the vector field $X$ in the original coordinates $(x, y, z)$. If we write $X(x, y, z) = Ax \frac{\partial}{\partial x} + By \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$, to determine $A, B$ we use the fact that

$$X(xy) = 0 \iff A = -B$$

and so $X(x, y, z) = A(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}) + z \frac{\partial}{\partial z}$ on the other hand we have

$$Ax = X(x) = X(\hat{x}(1 + z) \frac{a_1 + a_2}{a_1}) = \hat{x}(1 + z) \frac{a_1 + a_2}{a_1} \frac{z}{a_1 + a_2}$$

Finally, we obtain

$$X(x, y, z) = \frac{1}{a_1 + a_2} \frac{z}{1 + z} (x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}) + z \frac{\partial}{\partial z} \Rightarrow Y(x, y, z) = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + (a_1 + a_2)(1 + z) \frac{\partial}{\partial z}$$

**Remark 1.** In dimension three, if we consider the foliation $\mathcal{F}$ which is given locally by

$$\begin{cases}
    f_1 = x^a y^b z^c \\
    f_2 = x^a y^b z^c
\end{cases}$$

where $\text{rank}\begin{pmatrix} a & b & c \\ \hat{a} & \hat{b} & \hat{c} \end{pmatrix} = 2$. The generator vector field $X$ of the form

$$X(x, y, z) = \hat{a} x \frac{\partial}{\partial x} + \hat{b} y \frac{\partial}{\partial y} + \hat{c} z \frac{\partial}{\partial z}$$

where

$$<(\hat{a}, \hat{b}, \hat{c}), (a, b, c)> = 0, \text{ and } <(\hat{a}, \hat{b}, \hat{c}), (\tilde{a}, \tilde{b}, \tilde{c})> = 0.$$ 

In our example we observe that in the neighborhood of the leaf $\{z = 0\}$ the vector field

$$Y \simeq x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + (a_1 + a_2)z \frac{\partial}{\partial z}$$

is linearizable and consequently $Y$ is transversal to the leaf $\{z = 0\}$.

(b) The problem suppose where $a_1 = -a_2$ i.e $a \in \mathcal{R} = \{a = (a_1, a_2) : \gamma_0 \land \gamma_1 = 0\}$. In this case near the leaf $\{z = 0\}$, we have

$$Y \simeq x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$ 

It is clear that the condition of transversality of $Y$ and the leaf $\{z = 0\}$ is not satisfied.

**Example 2:** let $\mathcal{F}$ be the local foliation which is given (after a sequence of blow-ups) by

$$\begin{cases}
    H = x a_1 y a_2 z a_3 (1 + g(x, y, z)) \\
    f = xyz.
\end{cases}$$

The foliation $\mathcal{F}$ is given also

$$\begin{cases}
    \frac{H}{f} = y^{a_2 - a_1} z^{a_3 - a_1} (1 + g(x, y, z)) \\
    f = xyz.
\end{cases}$$

If $a = (a_1, a_2, a_3) \notin \mathcal{R} = \{a : (a_1 \beta_1 + a_2 \beta_2 + a_3 \beta_3) \land (1, 1, 1) = 0\}$ (resonant locus), we can take the following variables change $x = \tilde{x}, y = \tilde{y}(1 + g(x, y, z))^{\frac{1}{a_3 - a_3}}$ and $z = \tilde{z}(1 + g(x, y, z))^{\frac{1}{a_3 - a_3}}$. and in this case the local foliation $\mathcal{F}$ is generated by the vector field

$$X(\tilde{x}, \tilde{y}, \tilde{z}) = (a_2 - a_3) \tilde{x} \frac{\partial}{\partial \tilde{x}} + (a_3 - a_1) \tilde{y} \frac{\partial}{\partial \tilde{y}} + (a_1 - a_2) \tilde{z} \frac{\partial}{\partial \tilde{z}}.$$
let us express the vector field $X$ in the original coordinates $(x, y, z)$, so we have

$$X(x) = X(\tilde{x}) = (a_2 - a_3)x \frac{\partial}{\partial x}$$

$$X(\tilde{y}(1 + g(x, y, z)^{\frac{1}{a_2-a_3}})) = (a_3 - a_1)y + \frac{1}{a_2-a_3} \frac{X(g(x, y, z))}{1 + g(x, y, z)}$$

and

$$X(\tilde{z}(1 + g(x, y, z)^{\frac{1}{a_2-a_3}})) = (a_1 - a_2)z + \frac{1}{a_3-a_2} \frac{X(g(x, y, z))}{1 + g(x, y, z)}$$

Finally the vector field $X$ of the form

$$X(x, y, z) = (a_2 - a_3)x \frac{\partial}{\partial x} + (a_3 - a_1)y \frac{\partial}{\partial y} + (a_1 - a_2)z \frac{\partial}{\partial z} + \frac{1}{a_2-a_3} \frac{X(g(x, y, z))}{1 + g(x, y, z)} (y \frac{\partial}{\partial x} - z \frac{\partial}{\partial z}).$$

Proposition 1. For $a \notin \mathcal{R}$, there exist a local diffeomorphism $\phi : z = (z_1, \ldots, z_{n+2}) \mapsto \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_{n+2})$ such that the foliation $\mathcal{F}$ is given locally by $(\tilde{z}_1^\gamma_0, \ldots, \tilde{z}_{n+2}^\gamma_{n+2})$

Proof. We just make a convenable change of variables. 

Open question. To complete the proof of Conjecture we must solve the nongeneric case $a = (a_1, \ldots, a_k) \in \mathcal{R}$ because in this case the rank of exponent matrix

$$m(a_1, \ldots, a_k) = \begin{pmatrix}
\gamma_0^1 & \cdots & \gamma_0^{n+2} \\
\gamma_1^1 & \cdots & \gamma_1^{n+2} \\
\vdots & \vdots & \vdots \\
\gamma_n^1 & \cdots & \gamma_n^{n+2}
\end{pmatrix}$$

is not maximal.

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