Spectral estimates for matrix-valued periodic Dirac operators

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March 30, 2022

Abstract

We consider the first order periodic systems perturbed by a $2N \times 2N$ matrix-valued periodic potential on the real line. The spectrum of this operator is absolutely continuous and consists of intervals separated by gaps. We define the Lyapunov function, which is analytic on an associated N-sheeted Riemann surface. On each sheet the Lyapunov function has the standard properties of the Lyapunov function for the scalar case. The Lyapunov function has branch points, which we call resonances. We prove the existence of real or complex resonances. We determine the asymptotics of the periodic, anti-periodic spectrum and of the resonances at high energy (in terms of the Fourier coefficients of the potential). We show that there exist two types of gaps: i) stable gaps, i.e., the endpoints are periodic and anti-periodic eigenvalues, ii) unstable (resonance) gaps, i.e., the endpoints are resonances (real branch points). Moreover, we determine various new trace formulae for potentials and the Lyapunov exponent.

Keywords: periodic systems, spectrum, high energy asymptotics.

1 Introduction and main results

Consider the self-adjoint operator $\mathcal{K}$ acting on the space $L^2(\mathbb{R})^{2N}$ and given by

$$\mathcal{K}y = -iJ_1y' + V_t y, \quad J_1 = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}, \quad N \geq 1,$$

here and below we use the notation $(\cdot)' = \partial/\partial t$ and $I_N$ is the identity $N \times N$ matrix; $v$ is the complex 1-periodic $N \times N$ matrix and $V = V_t$ belongs to the Hilbert space $\mathcal{H}$ given by

$$\mathcal{H} = \left\{ V = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}, \quad v = v^T = \{ v_{jk}(t) \}_{j,k=1}^N, \quad t \in \mathbb{R}/\mathbb{Z}, \quad \| V \|^2 = \int_0^1 \text{Tr} V_t^2 dt < \infty \right\}.$$

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Note that Re\(v\) and Im\(v\) are self-adjoint. **Without loss of generality we assume**

\[
\mathcal{V} = \int_0^1 V(t)^2 dt = \mathcal{Y}_0 \oplus \mathcal{Y}_0, \quad \mathcal{Y}_0 = \text{diag}\{\nu_1, \ldots, \nu_N\}, \quad 0 \leq \nu_1 \leq \nu_2 \leq \ldots \leq \nu_N, \tag{1.1}
\]

here \(\nu_1, \ldots, \nu_N\) are the eigenvalues of \(\mathcal{V}\), see Sect. 2 for the proof.

It is well known (see [DS] p.1486-1494, [Ge]) that the spectrum \(\sigma(\mathcal{K})\) of \(\mathcal{K}\) is absolutely continuous and consists of non-degenerated intervals \(\sigma_n, n \in \mathbb{Z}\). These intervals are separated by the gaps \(g_n = (z_n^-, z_n^+)\) with the length \(|g_n| > 0, N_g^- < n < N_g^+, \) where \(-\infty \leq N_g^- < N_g^+ \leq \infty\) and \(N_g = N_g^+ - N_g^- - 1\) is a total number of the gaps.

Introduce the fundamental \(2N \times 2N\)-matrix solutions \(\psi(t, z)\) of the equation

\[-iJ_1 \psi' + V_1 \psi = z\psi, \quad z \in \mathbb{C}, \quad \psi(0, z) = I_{2N}, \tag{1.2}\]

and the monodromy \(2N \times 2N\)-matrix \(\psi(1, z)\). The matrix valued function \(\psi(1, \cdot)\) is entire. An eigenvalue of \(\psi(1, z)\) is called a multiplier of \(\mathcal{K}\): to each of them corresponds a solution \(f\) of \(-iJ_1 f' + V_1 f = zf\) with \(f(t + 1) = \tau(z)f(t), t \in [0, 1)\). They are roots of the algebraic equation \(D(\tau, z) = 0, \tau, z \in \mathbb{C}\), where \(D(\tau, z) = \det(\psi(1, z) - \tau I_{2N})\), \(\tau, z \in \mathbb{C}\). Zeros of the function \(\det(\mathcal{M}(z) - I_N)\) (or \(\det(\mathcal{M}(z) - I_N)\)) are periodic (or anti-periodic) eigenvalues.

There exist many papers about the first order periodic systems \(N \geq 2\): Gel’fand and Lidskii [GL], Gohbert and Krein [GK], Krein [Kr], Potapov [Po], [YS] and we mention new papers of Gesztesy and coauthors [CG], [CHGL], [GKM]. The basic results for direct spectral theory for the matrix case were obtained by Lyapunov and Poincaré (see [GL],[Kr],[YS]).

**Theorem (Lyapunov, Poincaré).** For each \((V, z) \in \mathcal{H} \times \mathbb{C}\) the matrix-function \(\psi(1, z)\) satisfies:

\[
\psi^{-1}(1, \cdot) = -J_1 \psi^\top(1, \cdot)J, \tag{1.3}
\]

\[
D(\tau, \cdot) = \tau^{2N} D(\tau^{-1}, \cdot), \quad \text{any } \tau \neq 0, \quad \text{where } \quad D(\tau, \cdot) = \det(\psi(1, \cdot) - \tau I_{2N}). \tag{1.4}
\]

\[
\sigma(\mathcal{K}) = \{z \in \mathbb{C} : |\tau(z)| = 1 \quad \text{for some multiplier } \tau(z) \quad \text{of} \quad \psi(1, z)\}. \tag{1.5}
\]

If for some \(z \in \mathbb{C}\) (or \(z \in \mathbb{R}\)) \(\tau(z)\) is a multiplier of multiplicity \(d \geq 1\), then \(\tau^{-1}(z)\) (or \(\tau'(z)\)) is a multiplier of multiplicity \(d\). Moreover, each \(\psi(1, z), z \in \mathbb{C}\), has exactly \(2N\) multipliers \(\tau_j^\pm(z), j = 1, \ldots, N\). If \(\tau(z)\) is a simple multiplier and \(|\tau(z)| = 1\), then \(\tau'(z) \neq 0\).

The eigenvalues of \(\psi(1, z)\) are the zeros of the equation \(D(\tau, z) = 0\). This is an algebraic equation in \(\tau\) of degree \(2N\) with coefficients, which are entire in \(z \in \mathbb{C}\). It is well known (see e.g. [Fol]) that the zeros \(\tau_j(z), j = 1, \ldots, 2N\) of \(D(\tau, z) = 0\) are (some branches of) analytic functions of \(z\) with only algebraic singularities: the zeros \(\tau_j(z), j = 1, \ldots, 2N\) constitute one or several branches of one or several analytic functions that have only algebraic singularities in \(\mathbb{C}\). Thus the number of eigenvalues of \(\psi(1, z)\) is a constant \(N_e\) with the exception of some special values of \(z\) (see below the definition of a resonance). In general, there is an infinite number of such points on the plane. If the functions \(\tau_j(z), j = 1, \ldots, 2N\) are all distinct, then \(N_e = 2N\). If some of them are identical, then we get \(N_e < 2N\) and \(\psi(1, z)\) is permanently degenerate.

Introduce the matrix-valued function \(\mathcal{L}(z) = \frac{1}{2}(\psi(1, z) + \psi^{-1}(1, z)), z \in \mathbb{C}\) and the function \(\Phi(z, \nu) = \det(\mathcal{L}(z) - \nu I_{2N}), z, \nu \in \mathbb{C}\). Each zero of \(\Phi(\nu, z)\) has multiplicity \(\geq 2\).
and define the Lyapunov function by \( \Delta_j(z) = \frac{1}{2}(\tau_j(z) + \tau_j^{-1}(z)) \), \( j = 1, \ldots, N \). The Riemann surface for the multipliers \( \tau_j(z), j \in \mathbb{N}_N = \{1, \ldots, N\} \) has \( 2N \) sheets, see (1.4). If \( N = 1 \), then it has 2 sheets and the Lyapunov function is entire. Similarly, in the case \( N \geq 2 \) it is more convenient for us to construct the Riemann surface for the Lyapunov function, which has \( N \) sheets. We need the following results from [K4].

**Theorem 1.1.** Let \( V \in \mathcal{H} \). Then there exists an analytic function \( \widetilde{\Delta}_s, s = 1, \ldots, N_0 \leq N \) on the \( N_s \)-sheeted Riemann surface \( \mathcal{R}_s, N_s \geq 1 \) having the following properties:

i) There exist disjoint subsets \( \omega_s, s = 1, \ldots, N_0 \cup \omega_s = \mathbb{N}_N \) such that all branches of \( \widetilde{\Delta}_s, s = 1, 2, \ldots, N_0 \) has the form \( \Delta_j(z) = \frac{1}{2}(\tau_j(z) + \tau_j^{-1}(z)) \), \( j \in \omega_s \). Moreover, for any \( z, \tau \in \mathbb{C} \) the following relations hold true:

\[
\det(\mathcal{L}(z) - \nu I_{2N}) = \prod_{1}^{N_0} \Phi_s^2(\nu, z), \quad \Phi_s(\nu, z) = \prod_{j \in \omega_s} (\nu - \Delta_j(z)), \quad z, \nu \in \mathbb{C}, \quad (1.6)
\]

\[
\Delta_j(z) = \cos z + o(e^{\text{Im} z}) \quad \text{as} \quad |z| \to \infty, \quad (1.7)
\]

where the functions \( \Phi_s(\nu, z) \) are entire with respect to \( \nu, z \in \mathbb{C} \). Moreover, if \( \Delta_i = \Delta_j \) for some \( i \in \omega_s, j \in \omega_s \), then \( \Phi_k = \Phi_s \) and \( \Delta_k = \Delta_s \).

ii) (The monotonicity property). Let some \( \Delta_j, j = 1, \ldots, N \), be real analytic on some interval \( Y = (\alpha, \beta) \subseteq \mathbb{R} \) and \( -1 < \Delta_j(z) < 1 \) for any \( z \in Y \). Then \( \Delta'_j(z) \neq 0 \) for each \( z \in Y \).

iii) The functions \( \rho, \rho_s \) given by (1.8) are entire,

\[
\rho = \prod_{1}^{N_0} \rho_s, \quad \rho_s(\cdot) = \prod_{i \leq j, j \in \omega_s} (\Delta_i(\cdot) - \Delta_j(\cdot))^2. \quad (1.8)
\]

iv) The following identity holds true

\[
\sigma(\mathcal{K}) = \mathbb{R} \setminus \bigcup_{j=1}^{N} g_n, \quad g_n = (z_n^-, z_n^+), \quad N_g^- < n < N_g^+ \quad (1.9)
\]

where each gap \( g_n = (z_n^-, z_n^+) \) is a bounded interval and \( z_n^\pm \) are either periodic (anti-periodic) eigenvalues or real branch points of \( \Delta_j \) (for some \( j = 1, \ldots, N \)) which are zero of \( \rho \) (below we call such point a resonance).

**Remark.** 1) In the case of \( 2 \times 2 \) system the monodromy matrix has exactly 2 eigenvalues \( \tau, \tau^{-1} \). The Lyapunov function \( \frac{1}{2}(\tau + \tau^{-1}) \) is an entire function of the spectral parameter. It defines the band-gap structure of the spectrum. By Theorem 1.1, the Lyapunov function for \( 2N \times 2N \)-matrix operator \( \mathcal{K} \) also defines the band-gap structure of the spectrum, but it is the \( N \)-sheeted analytic function.

2) We have the following asymptotics (see Sect. 3)

\[
\Delta_j(z) = \cos z + \frac{\sin z}{2z} \nu_j + O\left(\frac{e^{\text{Im} z}}{z^2}\right), \quad \text{if} \quad V' \in \mathcal{H} \quad (1.10)
\]

as \( |z| \to \infty, \ j \in \mathbb{N}_N \). Then firstly, \( \rho \) is not a polynomial since \( \rho \) is bounded on \( \mathbb{R} \). Secondly, if \( \nu_j' \neq \nu_j, j' \neq j \), then (1.10) implies \( \Delta_{j'} \neq \Delta_j \).
3) In the case \( N = 2 \) we determine \( \Delta_1, \Delta_2, \rho \) in terms of the traces of the monodromy matrix. Using (1.4), we have 
\[
D(\tau, \cdot) = (\tau^2 - 2\Delta_1 \tau + 1) (\tau^2 - 2\Delta_2 \tau + 1), \quad \Delta_1 = \frac{T_1}{2} + \frac{\sqrt{\rho}}{2}, \quad \Delta_2 = \frac{T_1}{2} - \frac{\sqrt{\rho}}{2},
\]  
see [BBK], where \( \rho = \frac{T_{n+1}}{2} - \frac{T_n^2}{4} \) and \( T_m = \text{Tr} \psi(m, z), \ m = 1, 2. \)

**Definition.** A zero \( z_0 \in \mathbb{C} \) of \( \rho \) given by (1.8) is a resonance of \( K \).

The main goal of this paper is to describe the spectrum of \( K \) and to determine the asymptotics of gaps and resonances, periodic and anti-periodic eigenvalues at high energy. We show that all resonances are real at high energy. Moreover, we prove the existence of complex resonances for some specific periodic potential. We have to underline that in the case of large \( N \) the resonances create the gaps in the spectrum of periodic operators, see Theorem 1.2 and remark after Theorem 1.3. If \( N = 1 \), then 2-periodic eigenvalues create the gaps in the spectrum. In the present paper we use some techniques from [K4] and [BBK], [BK], [CK].

The periodic eigenvalues \((n \text{ is even})\) satisfy
\[
\ldots \leq z_N^{2+} \leq z_0^{2+} \leq z_2^{2+} \leq \ldots \leq z_N^{0+} \leq z_1^{2-} \leq \ldots \leq z_N^{2-} \leq z_N^{3+} \leq \ldots \leq z_N^{1-} \leq \ldots,
\]  
the anti-periodic eigenvalues \((n \text{ is odd})\) satisfy
\[
\ldots \leq z_N^{-1+} \leq z_1^{-1+} \leq z_1^{-1-} \leq \ldots \leq z_N^{-1-} \leq z_1^{1+} \leq \ldots \leq z_N^{3+} \leq z_1^{3-} \leq \ldots \]  
and they have asymptotics
\[
z_j^{n,\pm} = \pi n + o(1) \quad \text{as} \quad n \to \pm\infty, \quad j \in \mathbb{N}_N = \{1, 2, \ldots, N\}. \tag{1.14}
\]

If \( V = 0 \), then these eigenvalues have the form \( z_j^{n,\pm} = \pi n, (n, j) \in \mathbb{Z} \times \mathbb{N}_N \).

Let \( \tilde{V}_n = \int_0^1 V_i e^{i2\pi ntJ_1} dt \). We formulate our first main result

**Theorem 1.2.** Let \( V, V' \in \mathcal{H} \) and let \( z_j^{n,\pm} \in \mathbb{N}_N \) be eigenvalues of the matrix \( \mathcal{V} - iJ_1 \tilde{V}_n \).
Then the periodic and anti-periodic eigenvalues have the following asymptotics:
\[
z_j^{n,\pm} = \pi n + \frac{\nu_j^{n,\pm}}{2\pi n} + O(n^{-2}), \quad j \in \mathbb{N}_N \quad \text{as} \quad n \to \pm\infty. \tag{1.15}
\]

Assume that \( \nu_j \neq \nu_{j'} \) for all \( j \neq j' \in \omega_s \) for some \( s = 1, \ldots, N_0 \). Then the function \( \rho_s \) has the zeros \( z_s^{n,\pm}, \alpha = (j, j'), j < j', j, j' \in \omega_s, n \in \mathbb{Z}, \) which are real at large \(|n|\) and satisfy
\[
z_s^{n,\pm} = \pi n + \frac{\nu_j^{n} + \nu_{j'}^{n}}{4\pi n} + O\left(\frac{\tilde{V}_n}{n} + \frac{1}{n^2}\right), \quad \alpha = (j, j') \quad \text{as} \quad n \to \pm\infty. \tag{1.16}
\]
Let in addition \( \nu_1 < \ldots < \nu_N \). Then for each \( s = 1, \ldots, N_0 \) and for large \( n \to \pm \infty \) there exists a system of real intervals (gaps) \( g^n_\alpha = (z^n_\alpha^-, z^n_\alpha^+) \) such that

\[
z^n_{j,j} = z^n_{j',j'}, \quad \alpha = (j, j'), \quad j, j' \in \omega_s, \quad z^n_{j,j_1} \leq z^n_{j,j_2} \leq z^n_{j,j_3} \leq \ldots < z^n_{j,j_{N_s}} \leq z^n_{j,j_{N_s}},
\]

\[( -1)^n \Delta_j(z) > 1, z \in g^n_{j,j}, \quad \text{and} \quad \Delta_j(z) = \Delta_j'(z), z \in g^n_{j,j'}, \quad \text{if} \quad j \neq j' \quad (1.17)\]

\( i \) Each branch \( \Delta_j \) is real and is analytic on the set \((\pi n - \frac{\pi}{2}, \pi n + \frac{\pi}{2}) \setminus \cup_{p \neq j} g^n_{p,j} \) and is not real on \( \cup_{p \neq j} g^n_{p,j} \).

\( ii \) If \( z^n_\alpha^- \neq z^n_\alpha^+ \) for some \( \alpha = (j, j'), j \neq j' \), then \( z^n_\alpha \) is the simple branch point (resonance) for the functions \( \Delta_j, \Delta_j' \). If \( z^n_\alpha^- = z^n_\alpha^+ \), then \( \Delta_j, \Delta_j' \) are analytic at \( z^n_\alpha \).

\( iii \) The following asymptotics hold true:

\[
z^n_\alpha = \pi n + \frac{v_j + v_j' \pm |v'_{n,\alpha}|}{2\pi n} + O\left(|v'_{n,\alpha}| + \frac{1}{n}\right), \quad v'_{n,\alpha} = \int_0^1 v'(t)e^{-i\pi n t} dt, \quad \alpha = (j, j'). \quad (1.18)
\]

**Remark.**

1. \( N_g = N_g^+ - N_g^- - 1 \) is the total number of gaps in the spectrum of \( \mathcal{K} \).

2. If \( \nu_1 < \ldots < \nu_N \), then there exist infinite number of resonances \( z^n_\alpha^\pm, \alpha = (j, j'), j \neq j' \) which form the gaps in the spectrum of \( \mathcal{K} \), see below Theorem 1.3. Roughly speaking, resonances "form" the gaps, the number of periodic and anti-periodic eigenvalues is less than the number of resonances. Thus there exists big difference between \( N = 1 \) and large \( N \). In the first case the endpoints of the gaps are 2-periodic eigenvalues. In the second case, roughly speaking, the endpoints of the gaps are resonances.

The second main result we describe finite band potentials.

**Theorem 1.3.** Let \( V, V' \in \mathcal{H} \) and let \( \nu_1 < \ldots < \nu_N \).

\( i \) If the identity \( \nu_1 + \nu_N = \nu_2 + \nu_{N-1} = \ldots = \nu_N + \nu_1 \) is not fulfilled, then \( |N_g^+| < \infty \).

\( ii \) If \( \nu_1 + \nu_N = \ldots = \nu_N + \nu_1 \) holds true and there exists a sequence of indices \( n_k \to \pm \infty \) such that \( |v'_{n,\alpha}|^2 + |n_k|^{-1} = o(|v'_{n,\alpha}|) \) as \( k \to \pm \infty \), for each \( \alpha = (j, N + 1 - j), j \in \mathbb{N}_N \), then \( N_g^\pm = \pm \infty \).

**Remark.**

1. Consider \( v = \text{diag}\{v_1, v_2, \ldots, v_N\} \), i.e., the case when "variables are separated".

The transformation \( \mathcal{U}_0 : y = (y_1, \ldots, y_{2N}) \to (y_1, y_{N+1}, y_2, y_{N+2}, \ldots, y_N, y_{2N}) \) gives

\[
\mathcal{U}_0 \mathcal{K} \mathcal{U}_0^* = \otimes^N \mathcal{K}_j, \quad \mathcal{K}_j = -i \mathbf{j}_1 \frac{d}{dt} + V_j, \quad \mathbf{j}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V_j = \begin{pmatrix} 0 & v_j \\ v^*_j & 0 \end{pmatrix}.
\]

The operator \( \mathcal{K}_j \) for the case \( N = 1 \) is well studied [YS], [K1-3], [Mi]. We have \( \nu_j = \int_0^1 |v_j(t)|^2 dt \geq 0 \). If \( \nu_j < \nu_p \) for some \( j < p \), then the number of gaps is \( N_g < \infty \).

2. Note that the condition \( |v'_{n,\alpha}|^2 + |n|^{-1} = o(|v'_{n,\alpha}|) \), \( \alpha = (j, N + 1 - j), j \in \mathbb{N}_N \) as \( n \to \pm \infty \), holds true for "generic" potentials \( V, V' \in \mathcal{H} \). This yields the existence of the real resonance gaps \( (z^n_\alpha^-, z^n_\alpha^+) \) at high energy. The coefficients \( v'_{n,\alpha}, \alpha = (j, N + 1 - j), j \in \mathbb{N}_N \) (the second diagonal of the matrix \( v \)) "create" the gaps.

**Example of complex resonances.** Consider the operator \( \mathcal{K}_{\nu,\tau} = -i J_1 \frac{d}{dt} + V_{\nu,\tau}, \nu = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \tau \in \mathbb{R} \) acting in \( L^2(\mathbb{R})^4 \), where the real periodic potential \( V_{\nu,\tau} \) is given by

\[
V_{\nu,\tau} = \begin{pmatrix} 0 & v_{\nu,\tau} \\ v_{\nu,\tau}^* & 0 \end{pmatrix}, \quad v_{\nu,\tau} = -\left( \begin{array}{ccc} a \tau b_{\nu} \\ \frac{a}{2\pi} \end{array} \right), \quad \frac{a}{2\pi} \in \mathbb{R}_+ \setminus \mathbb{N}, \quad b_{\nu} \in C(\mathbb{T}), \quad (1.19)
\]
\[
\int_0^1 |b_\nu(t)|dt = 1, \quad \int_0^m b_\nu(t)f(t)dt \rightarrow \int_0^m \delta_{\text{per}}(t)f(t)dt, \quad \delta_{\text{per}}(t) \equiv \sum_{-\infty}^\infty \delta(t-n-\frac{1}{2}), \quad (1.20)
\]
as \nu \rightarrow 0, for any \( f \in C(0,m), m \in \mathbb{N} \).

If \( \tau = 0 \), then the operator \( K_{\tau,0} = -iJ_1 \frac{d}{dt} + V_{\tau,0} \) has the constant potential \( V_{\tau,0} \). In this case there are no gaps in the spectrum of \( K_{\tau,0} \) and all resonances are given by \( r_n^0 = \pi n + \frac{\pi^2}{4n}, n \in \mathbb{Z} \setminus \{0\} \) with multiplicity 2. We show that there exist the non-degenerated resonance gaps for small \( \tau, \nu \). In this example some resonances are real and some are complex.

**Proposition 1.4.** Let a potential \( V_{\tau,\nu} \) satisfy (1.19), (1.20). Then for each large integer \( n_0 \geq 1 + a \) there exist sufficiently small \( \nu, \varepsilon > 0 \) such that the following statements hold true:

i) Each function \( \rho(z, V_{\tau,\nu}), \tau \in (-\varepsilon, \varepsilon) \) in the disk \( z \in \mathbb{D}_{n_0} = \{ |z| < \pi n_0 + 1 \} \) has exactly \( 4n_0 \) simple zeros \( r_{n,\nu}^\pm(\tau), 1 \leq \pm n \leq n_0 \), where \( r_{n,\nu}^\pm(\tau) \) is analytic function in the disk \( \tau \in \{ |\tau| < \varepsilon \} \) and \( r_{n,\nu}^0(0) = r_n^0 \) and the following estimates hold

\[
r_{n,\nu}^\pm(\tau) = r_n^0 \pm \tau(\sqrt{R_n} + o(1)) \quad \text{as} \quad \tau \rightarrow 0, \quad \text{where} \quad \begin{cases} R_n < 0 & \text{if} \quad |n| < \frac{a}{2\pi}, \\ R_n > 0 & \text{if} \quad |n| > \frac{a}{2\pi}, \end{cases} \quad (1.21)
\]

for some constant \( R_n \) and if \( \tau \in (-\varepsilon, \varepsilon), |n| > \frac{a}{2\pi} \), then \( (r_{-n,\nu}^-(\tau), r_{n,\nu}^+(\tau)) \subset \mathbb{R} \) is a gap.

ii) Each function \( D(1, z, V_{\tau,\nu})D(-1, z, V_{\tau,\nu}) \) in the disk \( \mathbb{D}_{n_0} \) has exactly \( 4n_0 + 4 \) zeros \( z_{n,m}^\pm(\tau, \nu) \), \( -n_0 \leq n \leq n_0, m = 1, 2 \) and the asymptotics \( z_{n,m}^\pm(\tau, \nu) = z_{n,m}^0 + o(1) \) holds as \( \tau \rightarrow 0 \).

**Remark.** 1) If \( 0 < a < 2\pi \), then \( \rho(\cdot, V_{\tau,\nu}) \) has only real roots \( r_{n,\nu}^\pm(\tau) \) in each large disk \( \mathbb{D}_{n_0} \) for sufficiently small \( \tau, \nu \). If \( 1 < \frac{a}{2\pi} \), then \( \rho(z, V_{\tau,\nu}) \) has at least two non-real roots \( r_{n,\nu}^\pm(\tau) \) for small \( \tau, \nu \). 2) We show that operator \( K_{\tau,\nu} \) has new gaps (so-called resonance gaps). The endpoints of the resonance gap are the branch points of the Lyapunov function, and, in general, they are not the periodic (or anti-periodic) eigenvalues. These endpoints are not stable. If they are real (see (1.21)), then we have a gap. If they are complex \( (0 < n < \frac{a}{2\pi}) \), then we have not a gap, we have only the branch points of the Lyapunov function in the complex plane. 3) We have a similar complicated distribution of other resonances, which are poles of S-matrix for scattering for Schrödinger operator with compactly supported potentials on the real line see [K5], [Z].

**We consider the conformal mapping associated with the operator \( K \).** Introduce the simple conformal mapping \( \eta : \mathbb{C} \setminus [-1, 1] \rightarrow \{ \zeta \in \mathbb{C} : |\zeta| > 1 \} \) by

\[
\eta(z) = z + \sqrt{z^2 - 1}, \quad z \in \mathbb{C} \setminus [-1, 1], \quad \text{and} \quad \eta(z) = 2z + o(1) \quad \text{as} \quad |z| \rightarrow \infty. \quad (1.22)
\]

Note that \( \eta(z) = \overline{\eta(z)}, z \in \mathbb{C} \setminus [-1, 1] \) since \( \eta(z) > 1 \) for any \( z > 1 \). Due to the properties of the Lyapunov functions we have \( |\eta(\Delta_s(\zeta))| > 1, \zeta \in \mathbb{R}^-_s = \{ \zeta \in \mathbb{R}_s : \text{Im} \zeta > 0 \} \). Thus we can introduce the quasimomentum \( k_j, j = 1, 2, ..., N \) (we fix some branch of arccos and \( \Delta_j(z) \)) and the function \( q_j \) by

\[
k_j(z) = \text{arccos} \Delta_j(z) = i \log \eta(\Delta_j(z)), \quad q_j(z) = \text{Im} k_j(z) = \log |\eta(\Delta_j(z))|, \quad (1.23)
\]

\( z \in \mathbb{R}^+_0 = \mathbb{C}_+ \setminus \beta_+, \beta_+ = \bigcup_{\beta \in \mathbb{B}_\Delta} [\beta, \beta + i\infty) \) where \( \mathbb{B}_\Delta \) is the set of all branch points of the function \( \Delta \). The branch points of \( k_m \) belong to \( \mathbb{B}_\Delta \). Define the **averaged quasimomentum**
For the function $k(z) = p(z) + iq(z)$, $z \in \mathbb{C}$, we introduce formally integrals

$$
Q_n = \frac{1}{\pi} \int t^n q(t) dt, \quad I_n^S = \frac{1}{\pi} \int t^n q(t) dp(t), \quad I_n^D = \frac{1}{\pi} \int \int_{C_+} |k'(n)]^2 dxdy,
$$

$n = 0, 1, 2$, here and below $k(0) = k - z, k(1) = zk(0), z = x + iy \in \mathbb{C}$. Let $C_{us}$ denote the class of all real upper semi-continuous functions $h : \mathbb{R} \to \mathbb{R}$. With any $h \in C_{us}$ we associate the "upper" domain $\mathbb{K}(h) = \{k = p + iq \in \mathbb{C} : q > h(p), p \in \mathbb{R}\}$. We formulate our last result.

**Theorem 1.5.**  i) Let $V \in \mathcal{H}$. Then the averaged quasimomentum $k = \frac{1}{N} \sum_{j=1}^{N} k_j$ is analytic in $\mathbb{C}_+$ and $k : \mathbb{C}_+ \to k(\mathbb{C}_+) = \mathbb{K}(h)$ is a conformal mapping for some $h \in C_{us}$. Furthermore, for some branches $k_j, j = 1, \ldots, N$ the following asymptotics, identities and estimates hold true:

$$
k(z) - z = -\frac{Q_0 + o(1)}{z} \quad \text{as} \quad |z| \to \infty, \quad \text{if} \quad y > r|x|, \quad \text{for any} \quad r > 0, \quad (1.26)
$$

$$
Q_0 = I_0^D + I_0^S = \frac{||V||^2}{4N}, \quad (1.27)
$$

$$
q|_{\sigma(N)} = 0, \quad 0 < q^2|_{\sigma(I) \cup g} \leq 2Q_0, \quad \text{where} \quad \sigma(N) = \{z \in \mathbb{R} : \Delta_1(z), \ldots, \Delta_N(z) \in [-1, 1]\},
$$

$$
\sigma(I) = \{z \in \mathbb{R} : \Delta_m(z) \in [-1, 1], \quad \Delta_p(z) \notin [-1, 1] \quad \text{some} \quad m, p = 1, \ldots, N\}, \quad (1.28)
$$

ii) Let additionally $V' \in \mathcal{H}$. Then the following asymptotics, identities hold true:

$$
k(z) - z = -\frac{Q_0}{z} - \frac{Q_1}{z^2} - \frac{Q_2 + o(1)}{z^3} \quad \text{as} \quad |z| \to \infty, \quad \text{if} \quad y > r|x|, \quad \text{for any} \quad r > 0, \quad (1.29)
$$

$$
Q_1 = \frac{\text{Tr}}{8N} \int_0^1 -iJ_VV'dt, \quad Q_2 = I_1^D + I_2^S - \frac{Q_0^2}{2} = \frac{\text{Tr}}{16N} \int_0^1 (V^2 + V_i^4) dt. \quad (1.30)
$$

A priori estimates for various parameters of the Dirac operator (the norm of a periodic potential, effective masses, gap lengths, height of slits, action variables for NLS and so on) were obtained in [KK1-2], [KK1-3], [Mi] only for the case $N = 1$. In order to get the required estimates the authors of [KK1-2],[K1-3], [Mi],... used the global quasi-momentum as the conformal mapping, which was introduced into the spectral theory of the Hill operator by Marchenko-Ostrovski [MO].

The mapping $k : \mathbb{C}_+ \to \mathbb{K}(h)$ is illustrated in Figure 1 for the case $N = 2$. The integral $I_0^S \geq 0$ is the area between the boundary of $\mathbb{K}(h)$ and the real line. In Figure 1 the upper picture is a domain $\mathbb{K}(h)$ and the points $A = k(A), B = k(B), \ldots$. The spectral interval
(A, B) (with multiplicity 2) of the z-domain is mapped on the curve (\(\tilde{A}, \tilde{B}\)) of the k-domain, the interval (a gap) (B, C) of the z-domain is mapped on the vertical slits, which lies on the line \(\text{Re } k = 0\). The spectral interval (C, D) (with multiplicity 2) of the z-domain is mapped on the curve (\(\tilde{C}, \tilde{D}\)) of the k-domain. The spectral interval (D, E) (with multiplicity 4) of the z-domain is mapped on the interval (\(\tilde{D}, \tilde{C}\)) of the k-domain. The case of the interval (E, J) is similar. The resonance gap (K, L) of the z-domain is mapped on the vertical slits, which lies on the line \(\text{Re } k = 2\pi\). In fact we have the graph of the function \(h(p), p \in \mathbb{R}\), which coincides with the boundary of \(\mathbb{K}(h)\).

We describe the plan of our paper. In Sect. 2 we obtain the basic properties of fundamental solution \(\psi(t, z)\). In Sect. 3 we determine the asymptotics of the fundamental solution \(\psi(t, z)\) for the case \(V, V' \in \mathcal{H}\). In Sect. 4 we determine the asymptotics of the Lyapunov function and multipliers at high energy and prove Theorem 1.2, 1.5. In Sect. 5 we prove Proposition 1.4. In Sect. 5 we determine the asymptotics of \(\det \mathcal{L}(z)\) as \(\text{Im } z \to \infty\).

2 The fundamental solutions

In this section we study \(\psi\). We begin with some notational convention. A vector \(h = \{h_n\}_{n=1}^N \in \mathbb{C}^N\) has the Euclidean norm \(|h|^2 = \sum_{n=1}^N |h_n|^2\), while a \(N \times N\) matrix \(A\) has the operator norm given by \(|A| = \sup_{|h|=1} |Ah|\). Note that \(|A|^2 \leq \text{Tr } A^*A\).

Recall the identity: if \(A\) is \(N \times N\) matrix, then

\[
\det(A - \nu I_N) = (-1)^N \sum_{j=0}^{N} a_j \nu^{N-j}, \quad a_0 = 1, \quad a_1 = -A_1, \quad \phi_2 = -\frac{A_2 + A_1 \phi_1}{2}, \quad (2.1)
\]

and...

\[
\phi_j = -\frac{1}{j} \sum_{k=1}^{j} A_k a_{j-k}, \ldots, \quad a_N = \det A, \quad A_m(z) = \text{Tr } A^m.
\]
Below we need the identity
\[ J_1 V = -V J_1, \quad e^{z J_1} V = V e^{-z J_1}, \quad (2.2) \]
for any \( z \in \mathbb{C} \). The solution of the equation (2.2) satisfies the integral equation
\[ \psi(t, z) = \psi_0(t, z) - i \int_0^t e^{iz J_1(t-s)} J_1 V_s \psi(s, z) ds, \quad \psi_0(t, z) = e^{iz t J_1}, \quad t \geq 0, \quad z \in \mathbb{C}. \quad (2.3) \]
It is clear that Eq. (2.3) has a solution as a power series in \( V \) given by
\[ \psi(t, z) = \sum_{n \geq 0} \psi_n(t, z), \quad \psi_n(t, z) = -i \int_0^t e^{iz J_1(t-s)} J_1 V_s \psi_{n-1}(s, z) ds, \quad n \geq 1. \quad (2.4) \]
Using (2.2), (2.4) we have
\[ \psi_1(t, z) = -i \int_0^t e^{iz J_1(t-s)} J_1 V_s e^{iz s J_1} ds = -i \int_0^t e^{iz J_1(t-2s)} J_1 V_s ds, \quad (2.5) \]
\[ \psi_2 = -i \int_0^t e^{iz J_1(t-t_1)} J_1 V_{t_1} \psi_1(t_1, z) dt_1 = \int_0^t dt_1 \int_0^{t_1} e^{iz J_1(t-2t_1+2t_2)V_1 V_2} dt_2. \quad (2.6) \]
Proceeding by induction, we obtain
\[ \psi_{2n}(t, z) = \int_0^t dt_1 \cdots \int_0^{t_{2n-1}} e^{iz J_1(t-2t_1+2t_2 \cdots 2t_{2n})} V_{t_1} \cdots V_{t_{2n}} dt_{2n}, \quad (2.7) \]
\[ \psi_{2n+1}(t, z) = -i \int_0^t dt_1 \cdots \int_0^{t_{2n}} e^{iz J_1(t-t_1+2t_2 \cdots 2t_{2n+1})} J_1 V_{t_1} \cdots V_{t_{2n+1}} dt_{2n+1}. \quad (2.8) \]
We need the following results from [K4].

**Lemma 2.1.** Let \( V \in \mathcal{H} \). For each \( z \in \mathbb{C} \) there exists a unique solution \( \psi \) of Eq. (2.3) given by (2.4) and series (2.4) converge uniformly on bounded subsets of \( \mathbb{R} \times \mathbb{C} \times \mathcal{H} \). For each \( t \geq 0 \) the function \( \psi(t, z) \) is entire on \( \mathbb{C} \). Moreover, for any \( n \geq 0 \) and \( (t, z) \in [0, \infty) \times \mathbb{C} \) the following estimates and asymptotics hold true:
\[ |\psi_n(t, z)| \leq \frac{e^{(1-\text{Im}|z|/t)}(\int_0^t |V_s| ds)^n}{n!}, \quad (2.9) \]
\[ \left| \psi(t, z) - \sum_{j=0}^{n-1} \psi_j(t, z) \right| \leq \frac{e^{(1-\text{Im}|z|/t)}|V||\psi_{n-1}(t, z)|}{n!} e^{(1-\text{Im}|z|/t) \int_0^t |V_s| ds}, \quad (2.10) \]
\[ \psi(t, z) - e^{iz t J_1} = o(e^{1-\text{Im}|z|}) \text{ as } |z| \to \infty, \quad (2.11) \]
uniformly on bounded \( t \in \mathbb{R} \). If the sequence \( V^\nu \to V \) weakly in \( \mathcal{H} \), as \( \nu \to \infty \), then \( \psi(t, z, V^\nu) \to \psi(t, z, V) \) uniformly on bounded subsets of \( \mathbb{R} \times \mathbb{C} \).
Below we need the simple properties of matrices \(a, b, c \in \mathcal{A} = \{ A = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} : \)

\(a_1, a_2 \) is \(N \times N\) matrix given by

\[
abc, J_1a, e^{zJ_1}a \in \mathcal{A}, \quad \text{all } z \in \mathbb{C},
\]

\[
ab = (-a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)J_1,
\]

\[
\text{Tr} a = 0, \quad \text{Tr} Ja^n = 0, \quad n \geq 0.
\]

For any matrices \(A, B\) the following identities hold

\[
\text{Tr} AB = \text{Tr} BA, \quad \overline{\text{Tr} A} = \text{Tr} A^*.
\]

Using (2.6)-(2.8) we define the function

\[
T_{j,1}(z, V) = \text{Tr} \int_0^j V_s \int_0^{s_1} V_{s_2} e^{izJ_1(j-2s_1+2s_2)} ds,
\]

\[
T_{j,n}(z, V) = \text{Tr} \psi_{2n}(j, z) = \text{Tr} \int_0^j \ldots \int_0^{t_{2n-1}} e^{izJ_1(j-2s_1+2s_2\ldots+2s_{2n})} V_{s_1} \ldots V_{s_{2n}} ds, \quad n \geq 2,
\]

where \(s = (s_1, \ldots, s_{2n}) \in \mathbb{R}^{2n}.

**Lemma 2.2.** Let \(V \in \mathcal{H}\). The functions \(T_j(\cdot, V), j = 1, 2, \ldots, N\) are entire on \(\mathbb{C}\) and \(T(z, V) \in \mathbb{R}\) for all \(z \in \mathbb{R}\). Moreover, the function \(T(z, V)\) satisfies

\[
T_j(z, V) = T_j(z, -V) = 2N \cos jz + \sum_{n \geq 1} T_{j,n}(z, V),
\]

\[
|T_j(z, V)| \leq 2Ne^{j(|\text{Im}z|+||V||)},
\]

\[
T_j(z, V) = 2N \cos jz + o(e^{j|\text{Im}z|}) \quad \text{as} \quad |z| \to \infty,
\]

and \(\text{Tr} \psi_{2n+1}(t, z) = 0\). Series (2.18) converge uniformly on bounded subsets of \(\mathbb{C} \times \mathcal{H}\). If a sequence \(V'\) converges weakly to \(V\) in \(\mathcal{H}\) as \(\nu \to \infty\), then \(T_j(z, V') \to T_j(z, V)\) uniformly on bounded subsets of \(\mathbb{C}\).

**Proof.** By Lemma 2.2, series (2.18) converge uniformly and absolutely on bounded subsets of \(\mathbb{C} \times \mathcal{H}\). Each term in (2.18) is an entire function of \(z\), then \(T_j\) is an entire function of \(z, V\). Moreover, if the sequence \(V'\) converges weakly to \(V\) in \(\mathcal{H}\), as \(\nu \to \infty\), then \(T_j(z, V') \to T_j(z, V)\) uniformly on bounded subsets of \(\mathbb{C}\).

We have \(T_j = \text{Tr} \psi(j, z) = \text{Tr} \sum_{n \geq 0} \psi_n(j, z)\) and \(\text{Tr} \psi_0(j, z) = 2N \cos jz\) for \(j \geq 1\). The estimate (2.9) yields (2.10) and (2.11) gives (2.20).

The first relation in (2.12) yields \(V_{t_1} \ldots V_{t_{2n+1}} \in \mathcal{A}\) for any \(t_1, \ldots, t_{2n+1} \in \mathbb{R}\). Then relations in (2.12) give \(\text{Tr} e^{izJ_1(t-2t_1+2t_2-\ldots-2t_{2n+1})} J_1 V_{t_1} \ldots V_{t_{2n+1}} = 0\), for any \(t, t_1, \ldots, t_{2n+1} \in \mathbb{R}\), which together with (2.11) implies \(\text{Tr} \psi_{2n+1}(t, z) = 0\).

Below we will show that \(T(z, V) \in \mathbb{R}\) for all \(z \in \mathbb{R}\) (see (2.28)).
We will show (1.1). Consider the self-adjoint operator $Jy' + \Omega(t)y$, $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$, acting on the Hilbert space $\oplus_1^{2N}L^2(\mathbb{R})$, where the real matrix $\Omega$ is given by

$$\Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \\ \Omega_2 & -\Omega_1 \end{pmatrix} : \Omega_2 = \Omega^*_1, \quad \Omega_1 = \Omega^*_1, \quad \|\Omega\|^2 = \int_0^1 \text{Tr} \Omega^2(t) dt < \infty. \quad (2.21)$$

Let $M(t, z), t \in \mathbb{R}$ be the fundamental solution of the equation $JM' + \Omega M = zM, M(0, z) = I_{2N}$. Note that $M(t, z)$ is real for $z \in \mathbb{R}$.

Define the unitary matrix $U = \frac{1}{\sqrt{2}}(J + iJ) = U^*, U^2 = I_{2N}$. Using the identities

$$J_2 = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}, \quad UJU = -iJ_1, \quad UJ_1U = iJ, \quad UJ_2U = -J_2, \quad \Omega = J_1\Omega_1 + J_2\Omega_2, \quad (2.22)$$

we deduce that $M_c = U M U$ satisfy the equation $-iJ_1M_c' + \Omega_c M_c = zM_c, M_c(0, z) = I_{2N}$, where $\Omega_c$ is given by

$$\Omega_c = \Omega_c^* = U\Omega U = \begin{pmatrix} 0 & \omega \\ \omega^* & 0 \end{pmatrix} \in \mathcal{H}, \quad \omega = -\Omega_2 + i\Omega_1 = \omega^T. \quad (2.23)$$

Thus we obtain

$$\int_0^1 \Omega_c^2(t) dt = \mathcal{V}_1 \oplus \mathcal{V}_2, \quad \mathcal{V}_1 = \int_0^1 \omega(t)\omega^*(t) dt, \quad \mathcal{V}_2 = \int_0^1 \omega^*(t)\omega(t) dt, \quad (2.24)$$

and

$$\mathcal{V}_1 = E\mathcal{V}_0 E^* = \mathcal{V}_2 \geq 0, \quad \mathcal{V}_0 = \text{diag}\{\nu_1, \ldots, \nu_N\}, \quad (2.25)$$

for some unitary matrix $E$ and the diagonal matrix $\mathcal{V}_0$. Define the unitary matrix $\mathcal{E} = E \oplus \overline{E}$. The function $\psi(t, z) = \mathcal{E}^* M_c(t, z) \mathcal{E}$ satisfies $-iJ_1\psi' + V\psi = z\psi, \quad \psi(0, z) = I_{2N}$, where

$$\mathcal{E} = E \oplus \overline{E}, \quad V = \mathcal{E}^*\Omega_c \mathcal{E} = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}, \quad v = E^*\omega \overline{E}, \quad (2.26)$$

$$\int_0^1 V_c^2 dt = \mathcal{V}_0 \oplus \mathcal{V}_0, \quad (2.27)$$

$$\text{Tr} \psi(t, z) = \text{Tr} M(t, z), \quad (2.28)$$

which gives $V \in \mathcal{H}$ and $\text{Tr} \psi(1, z) \in \mathbb{R}$ for all $z \in \mathbb{R}$.

It is well known that for real $\Omega$ we have $M(t, z)JM(t, z)^T = J$ (see [GL], [YS]). Then $M_c = U M U$ and (2.22) give

$$-iJ_1 = M_c(1, z)(-iJ_1)U\mathcal{U}^\top(M_c(1, z)^\top)\mathcal{U}^\top U, \quad (-iJ_1)U\mathcal{U}^\top = -J,$$

which yields

$$M_c(1, z)JM_c(1, z)^T = J. \quad (2.29)$$

The similar arguments and $\psi = \mathcal{E}^* M_c \mathcal{E}$ imply

$$\psi \mathcal{E}^* J \mathcal{E}^{*\top} \psi^\top \psi^* = \mathcal{E}^* J, \quad \mathcal{E}^* J \mathcal{E}^{*\top} = J,$$

which yields $\psi J \psi^\top = J$ and (1.3) is proved for $V \in \mathcal{H}$. 

3 Estimates of $\psi$ for the case $V' \in \mathcal{H}$

Sect. 2 does not give the needed estimates of the fundamental solution $\psi$ at high energy. In order to determine the asymptotics of $\psi$ we will do some modification. Define the integral operator $K$ and the matrix-valued function $a_t(z)$ by

$$(K f)(t) = \int_0^t e_{t-s} W_s f(s) ds, \quad W = -i J_1 V^2 - V', \quad e_t = e_t(z) = e^{itz J_1}, \quad a_t(z) = I - \frac{V_t}{2z}. \quad (3.1)$$

where $e_t = e_t(z) = e^{itz J_1}$. Introduce

$${\mathcal{V}}^1(z) = \int_0^1 V'_t e^{itz J_1} dt, \quad {\mathcal{V}}^3(z) = \int_0^1 V^3_t e^{itz J_1} dt, \quad {\mathcal{V}} = \int_0^1 V^2 dt. \quad (3.2)$$

**Lemma 3.1.** For each $(z, V') \in (\mathbb{C} \setminus \{0\}) \times \mathcal{H}$ the solution $\psi = a^{-1} \Psi a_0$, where $\Psi$ satisfies

$$\Psi = \psi_0 + \varepsilon K a^{-1} \psi, \quad \Psi = a \psi a^{-1}, \quad a_0 = I - \varepsilon V_0, \quad \varepsilon = \frac{1}{2z}. \quad (3.3)$$

$$\Psi = \psi_0 + \sum_{n \geq 1} \Psi_n, \quad \Psi_n = \varepsilon^n (Ka^{-1})^n \psi_0, \quad (3.4)$$

where series (3.3) converge uniformly on bounded subsets of $\mathbb{R} \times (\mathbb{C} \setminus \{0\}) \times \mathcal{H}$. Moreover, if $\sup_{t \in \mathbb{R}} |V_t| \leq |z|$, then for any $j - 1, m \in \mathbb{N}$ the following estimates are fulfilled:

$$|\Psi_n(t, z)| \leq \frac{e^{\left| \Im z \right| t}}{n! |z|^n} \left( \int_0^t |W_s| ds \right)^n, \quad (3.5)$$

$$|\Psi(m, z) - \psi_0(m, z) - \sum_{n \geq 1} \Psi_n(m, z)| \leq \frac{\varepsilon^j}{j!} e^{n(\left| \Im z \right| + \kappa)}, \quad \kappa \equiv \frac{\|V\|^2 + \|V'\|}{|z|}, \quad (3.6)$$

$$\Psi_1(1, z) = -\frac{e^{iz J_1}}{2z} \left( i J_1 {\mathcal{V}} + {\mathcal{V}}^3(z) + \frac{1}{2z} \left( i J_1 {\mathcal{V}}^3(z) + \int_0^1 V'_t V_t dt \right) \right) + O(e^{\left| \Im z \right| z^3}), \quad (3.7)$$

$$\Psi_2(1, z) = -\frac{e^{iz J_1}}{4z^2} \int_0^1 dt \int_0^t \left( V^2 V_s - i J_1 V^2 V_s e_{2s} - i V'_t e_{2t} J_1 V^2 - V'_t e_{2t} V_s e_{2s} \right) ds + O(e^{\left| \Im z \right| z^3}) \quad (3.8)$$

as $|z| \to \infty$ and where $e_t = e^{itz J_1}$.

**Proof.** Using (2.2), (2.3), $\varepsilon = \frac{1}{2z}$ and integrating by parts we get

$$\psi(t, z) - e^{iz J_1} = -i \int_0^t J_1 e^{iz J_1(t-s)} V_s \psi(s, z) ds = -i \int_0^t e^{iz J_1(t-2s)} J_1 \left( V_s e^{-iz J_1} \psi(s, z) \right) ds$$

$$= \varepsilon e^{iz J_1(t-2s)} \left( V_s e^{-iz J_1} \psi(s, z) \right) \bigg|_0^t - \varepsilon \int_0^t e^{iz J_1(t-2s)} \left( V_s e^{-iz J_1} \psi(s, z) \right)' ds$$

$$= \varepsilon \left( V_t \psi(t, z) - e^{iz J_1(2s)} \right) - \varepsilon \int_0^t e^{iz J_1(t-2s)} \left( V_s e^{-iz J_1 + iez J_1} J_1 V_s \psi(s, z) \right) ds.$$
Thus we obtain

\[ a(t, z)\psi(t, z) = e^{izt}a(0, z) + \varepsilon \int_0^t e^{iz(t-s)}W_s\psi(s, z)\,ds \]

which yields (3.3). We will show (3.4)-(3.6). Using \(|1/a(t, z)| \leq 2\) for \(t \in [0, T]\) and \(|\psi_0(t, z)| \leq e^{1|z|t}\), we have

\[ |\Psi_n(t, z)| \leq 2|\varepsilon| \int_0^t e^{1|z|(t-t_1)}|W_{t_1}||\Psi_{n-1}(t_1, z)|\,dt_1 \]

\[ \leq (2|\varepsilon|) \int_0^t dt_1 \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{n-1}} e^{1|z|t_1}|W_{t_1}| \ldots |W_{t_n}|\,dt_1 \leq \frac{(2|\varepsilon|)^n}{n!} e^{1|z|t} (\int_0^t |W_{t_1}|\,dt_1)^n, \]

which gives (3.5). Estimates (3.5) and \(\int_0^m |W(t)|\,dt \leq m(\|V\| + \|V\|^2)\) imply (3.4) and (3.6). We get

\[ \Psi_1(1, z) = \varepsilon Ka^{-1}\psi_0 = \varepsilon^2 K\psi_0 + \varepsilon^2 KV\psi_0 + O(\varepsilon^3 e^{1|z|}). \] (3.9)

Recall \(e_t = e^{iztJ_1}\). (2.10) implies

\[ K\psi_0 = -\int_0^1 e_{-t}(iJ_1V_t^2 + V_t')e_t\,dt = -ie_1J_1\psi - e_1\hat{V}'(z), \]

\[ KV\psi_0 = -\int_0^1 e_{-t}(iJ_1V_t^2 + V_t')e_t\,dt = -ie_1J_1\hat{V}'(z) - e_1\int_0^1 V_t'V_t\,dt, \]

which yields (3.7). Consider the second term \(\Psi_2 = \varepsilon^2 (Ka^{-1})^2\psi_0 = \varepsilon^2 \Psi_20 + O(\varepsilon^3 e^{1|z|}),\) where

\[ \Psi_2 = \int_0^1 dt \int_0^t e_{-t}(iJ_1V_t^2 + V_t')e_t \int_0^1 (iJ_1V_s^2 + V_s')e_s\,ds \]

\[ = \int_0^1 dt \int_0^t e_{-t} \left[ -J_1V_t^2 e_{-s}J_1V_s^2 + iJ_1V_tV_s e_{-s}J_1V_s' - iV_t'e_{-s}J_1V_s^2 + V_t'\hat{V}'(z) \right] e_s \]

\[ = -e_1 \int_0^1 dt \int_0^t \left[ V_t^2V_s^2 - iV_{t}V_{t'}V_s - iV_{t'}V_{t}V'_{s} + V_{t'}V_{t}^2 \right] e_s \]

which gives (3.8). □

In order to determine the asymptotics of the Lyapunov function we need the following modification. Substituting \(\psi^{-1} = -J\psi^\top J\) into \(\Psi(1, z)^{-1} = a_0\psi(1, z)^{-1}a_0^{-1} \) we get

\[ \Psi(1, z)^{-1} = -a_0Ja_0\Psi(1, z)^\top a_0^{-1}Ja_0^{-1} = -cJ\Psi(1, z)^\top \hat{J}c^{-1}, \]

where \(c = I - \varepsilon^2 V^2_0, \varepsilon = \frac{1}{2z}, \)

which yields

\[ L = \frac{1}{2} \left( \psi(1, z) + \psi(1, z)^{-1} \right) = \frac{1}{2} \left( \psi(1, z) - cJ\Psi^\top (1, z)Jc^{-1} \right). \] (3.10)
We determine the asymptotics of $\Psi(1,z)$. Using (3.6), we have
\[
\Psi = \psi_0 + \varepsilon K a^{-1} \psi_0 + \varepsilon^2 K a^{-1} K a^{-1} \psi_0 + \varepsilon^2 K a^{-1} K a^{-1} \psi_0 + O(\varepsilon^4 e^{\text{Im} z})
\]
\[
= \psi_0 + \varepsilon K(I + \varepsilon V + \varepsilon^2 V^2)\psi_0 + \varepsilon^2 K(I + \varepsilon V)K(I + \varepsilon V)\psi_0 + \varepsilon^2 K^3 \psi_0 + O(\varepsilon^4 e^{\text{Im} z})
\]
where
\[
\varkappa_1 = K \psi_0, \quad \varkappa_2 = (KV + K^2) \psi_0, \quad \varkappa_3 = (KV^2 + K^2 V + KV K + K^3) \psi_0.
\] (3.11)

We get
\[
2L = \psi_0 + \varepsilon \varkappa_1 + \varepsilon^2 \varkappa_2 + \varepsilon^3 \varkappa_3 - (I - \varepsilon^2 V^2)J(\psi_0 + \varepsilon \varkappa_1 + \varepsilon^2 \varkappa_2 + \varepsilon^3 \varkappa_3)^\top J(I + \varepsilon^2 V^2) + O(\varepsilon^4 e^{\text{Im} z})
\]
\[
= (\psi_0 - J\psi_0^\top J) + \varepsilon(\varkappa_1 - J\varkappa_1^\top J) + \varepsilon^2(\varkappa_2 - J\varkappa_2^\top J + V^2 J\psi_0^\top J - J\psi_0^\top JV^2)
\]
\[
+ \varepsilon^3(\varkappa_3 - J\varkappa_3^\top J + V^2 J\varkappa_1^\top J - J\varkappa_1^\top JV^2) + O(\varepsilon^4 e^{\text{Im} z}).
\]
Thus we have
\[
L = \cos z + \varepsilon L_1 + \varepsilon^2 L_2 + \varepsilon^3 L_3 + O(\varepsilon^4 e^{\text{Im} z}),
\] (3.12)
where
\[
L_1 = \frac{\varkappa_1 - J\varkappa_1^\top J}{2}, \quad L_2 = \frac{\varkappa_2 - J\varkappa_2^\top J}{2}, \quad L_3 = \frac{1}{2}(\varkappa_3 - J\varkappa_3^\top J + V^2 J\varkappa_1^\top J - J\varkappa_1^\top JV^2). \quad (3.13)
\]

Below we need the identities
\[
J^* V^\top J = -V, \quad J^* W^\top J = -W, \quad J^* J_1 J = -J_1.
\] (3.14)

We determine the asymptotics of $L$. Let $u_t = \int_0^t V_s^2 ds$, $f_t = \int_0^t V_s^2 e_{x_2} ds$.

**Lemma 3.2.** If $V, V' \in H$, then asymptotics (1.10), the following identities and asymptotics are fulfilled:

\[
L_1 = \mathcal{V} \sin z, \quad z \in \mathbb{C}, \quad \text{where} \quad \mathcal{V} = \int_0^1 V_t^2 dt,
\] (3.15)

\[
L_2 = i \frac{e_1 J_1}{2} (\mathcal{V} \bar{V}' + \bar{V}' \mathcal{V}) - i \sin z \int_0^1 J_1 V_t' V_t dt + L_{21} + L_{22},
\] (3.16)

\[
L_{21} = -\frac{1}{2} \int_0^1 (e_1 u'u + e_{-1} uu')dt, \quad L_{22} = \frac{1}{2} \int_0^1 (e_1 f' f + e_{-1} f f')dt,
\]

\[
L_{22} = o(e^{\text{Im} z}),
\] (3.17)

\[
L_2(\pi n) = \frac{(-1)^n}{2} \left( - \mathcal{V}^2 + i J_1 (\mathcal{V} \bar{V}'_n + \bar{V}'_n \mathcal{V}) + (\bar{V}'_n)^2 \right), \quad \bar{V}'_n = \bar{V}'(\pi n),
\] (3.18)

\[
\Delta_j(z) = \cos(z - \frac{\nu_j}{2z}) + \frac{O(|V_n| + |n|^{-1})}{n^2}, \quad \text{as} \quad z = j \pi n + O\left(\frac{1}{n}\right), \quad j = 1, \ldots, N.
\] (3.19)
\textbf{Proof.} Recall $e_t = e^{iztJ}$. (3.11), (4.1) give

\[ J^* \mathbf{\xi}_1^\top J = J^* (K \psi_0)^\top J = J^* \int_0^1 (e_{-t} W_t e_t)^\top J dt = \int_0^1 e_{-t} J^* W_t^\top J e_{t-1} dt = -\int_0^1 e_{-t} W_t e_{t-1} dt. \]

Then

\[ 2L_1 = \int_0^1 (e_{-t} W_t e_t - e_{-t} W_t e_{t-1}) dt = -i J_1 \int_0^1 (e_{-t} V_t^2 e_t - e_{-t} V_t^2 e_{t-1}) dt = -i J_1 \int_0^1 (e_{-t-1})V_t^2 dt \]

which yields (3.15). We determine $L_2 = \frac{1}{2}(\mathbf{\xi}_2 - J \mathbf{\xi}_2^\top J)$. Using (3.11), (4.1) we get

\[ \mathbf{\xi}_2 = (K V + K^2) \psi_0 = \int_0^1 e_{-t} W_t (V_t e_t + \int_0^t (e_{t-s} W_s e_s ds) dt) \]

\[ J^* \mathbf{\xi}_2^\top J = J^* \int_0^1 (e_t V_t^\top + \int_0^t (e_s W_s^\top e_{s-t} ds) W_t^\top e_{t-1} dt) = \int_0^1 (e_{-t} V_t + \int_0^t (e_{s} W_{s-1} e_{s-t} ds) W_t e_{t-1} dt. \]

This yields

\[ L_2 = F + S, \quad S = \frac{1}{2} \int_0^1 dt \int_0^t \left( e_{-t} W_t e_{t-s} W_s e_s + e_{-s} W_s e_{s-t} W_t e_{t-1} \right) ds \]

and using $W = -i J_1 V^2 - V'$ we obtain

\[ F = \int_0^1 \left( e_{-t} W_t V_t e_t + e_{-t} V_t W_t e_{t-1} \right) \frac{dt}{2} = -\int_0^1 \left( e_{-t} (i J_1 V_t^2 + V') V_t e_t + e_{-t} V_t (i J_1 V_t^2 + V') e_{t-1} \right) \frac{dt}{2} \]

\[ = -\frac{1}{2} \int_0^1 \left( e_t V_t^2 V_t' + e_{-1} V_t V_t' \right) dt = \frac{e_{-1} - e_1}{2} \int_0^1 V_t^2 V_t' dt = -i J_1 \sin z \int_0^1 V_t^2 V_t' dt \]

since $\int_0^1 V_t^2 V_t' dt = -\int_0^1 V_t V_t' dt$. Consider the second term,

\[ 2S = \int_0^1 dt \int_0^t \left( e_{-t} (i J_1 V_t^2 + V') e_{t-s} (i J_1 V_s^2 + V_s') e_s + e_{-s} (i J_1 V_s^2 + V_s') e_{s-t} (i J_1 V_t^2 + V_t') e_{t-1} \right) ds \]

\[ = \int_0^1 dt \int_0^t \left[ -J_1 V_t^2 e_{t-s} V_s + i J_1 V_t V_s (e_{t-s} V_s' + i V_{t-1} V_s^2 + V_{t-1} e_{t-s} V_s') e_s + 
\quad + e_{-s} \left[ -J_1 V_s^2 e_{s-t} V_t + i J_1 V_s V_t (e_{s-t} V_t' + i V_{s-1} V_s^2 + V_{s-1} e_{s-t} V_t') e_{t-1} \right] \right] ds \]

\[ = \int_0^1 dt \int_0^t \left[ \left[ -e_t V_t V_s^2 + ie_{t-2} e_{t-1} V_s V_s' + ie_{t-2} V_s^2 V_s' + e_{t-2} V_s V_t V_s' \right] \right] ds \]

\[ + \left[ -e_{-t} V_s V_t^2 + ie_{-t} V_t V_s V_s' + ie_{-t} V_s V_t V_s' + e_{-t} V_s V_{t-1} V_s' \right] ds \]

\[ = -\int_0^1 dt \left( e_t u' u + e_{-1} uu' \right) dt + \frac{ie_1 J_1}{2} (V V' + V' V') + \int_0^1 dt \left( e_1 f' f + e_{-1} f f' \right) dt \]
where \( \hat{V}'(z) = \int_0^1 V'(t)e_{2t}dt \) and \( u = \int_0^t V'_s e_{2s}ds, \ f = \int_0^t V'_s e_{-2s}ds \), and here we used
\[
\int_0^1 \int_t^t \left( e_1 V^2 + e_2 V^2 \right) dt ds = \int_0^1 \int_0^t \left( e_1 u' u + e_2 u u' \right) dt,
\]
\[
\int_0^1 \left( u f + f u' + f u' \right) dt = u_1 f_1 + f_1 u = \left( \hat{V} \right) ' \left( \hat{V} \right) ' + \left( \hat{V} \right) ' \left( \hat{V} \right) ',
\]
\[
\int_0^1 \int_t^t \left( e_1 - 2t + 2s V'_s + e_2 t - 2s V'_s \right) ds = \int_0^1 \int_0^t \left( e_1 f f + e_1 f f' \right) dt.
\]

We have \( e_{\pm 1} = (-1)^n I_{2N} \) at \( z = \pi n \) and then
\[
L_{21}(\pi n) = -\frac{(-1)^n}{2} \int_0^1 \left( u' u + u u' \right) dt = -\frac{(-1)^n \nu^2}{2}, \tag{3.20}
\]
\[
L_{22}(\pi n) = \frac{(-1)^n}{2} \int_0^1 \left( f' f + f f' \right) dt = \frac{(-1)^n}{2} f^2(\pi n) = \frac{(-1)^n}{2} (\hat{V})^2(\pi n), \tag{3.21}
\]
which yields (3.18). Using (3.12) (3.15) (3.18) we obtain
\[
L(z) = \cos(z - \nu \nu) + \nu^2 \left( L_2(\pi n) + \frac{(-1)^n \nu^2}{2} \right) + O(\nu^3) \text{ as } z = \pi n + O(1/n). \tag{3.22}
\]

Recall the simple fact: Let \( A, B \) be matrices and and \( \sigma(B) \) be spectra of \( B \). If \( A \) be normal, then \( \text{dist}\{\sigma(A), \sigma(A + B)\} \leq |B| \) (see [Ka,p.291]).

The normal operator \( \cos(z - \nu \nu) \) has the eigenvalues \( \cos(z - \nu \nu_j), j = 1, ..., N \) with the multiplicity 2. Using the result from [Ka] and asymptotics (3.12) and identity (3.18) we deduce that the eigenvalues \( \Delta_j(z) \) of matrix \( L(z) \) satisfy the asymptotics (3.19). The proof of (1.10) is similar. ■

4 Proof of the main theorems

We need the following results from [K4].

**Lemma 4.1.** Let \( V, V' \in \mathcal{H} \). Then the following asymptotics hold true:
\[
\Phi(z, \nu) = (\cos z - \nu)^{2N} + o(e^{N|\text{Im} z|}) \quad \text{as} \quad |z| \to \infty, \tag{4.1}
\]
where \( |\nu| \leq A_0 \) for some constant \( A_0 > 0 \). Moreover, there exists an integer \( n_0 \) such that:

i) the function \( \Phi(z, 1) \) has exactly \( N(2n_0 + 1) \) roots, counted with multiplicity, in the disc \( \{ |z| < \pi(2n_0 + 1) \} \) and for each \( |n| > n_0 \), exactly 2N roots, counted with multiplicity, in the domain \( \{ |z - 2\pi n| < \frac{\pi}{2} \} \). There are no other roots.

ii) the function \( \Phi(z, -1) \) has exactly \( 2Nn_0 \) roots, counted with multiplicity, in the disc \( \{ |z| < 2\pi n_0 \} \) and for each \( |n| > n_0 \), exactly 2N roots, counted with multiplicity, in the domain \( \{ |z - \pi(2n + 1)| < \frac{\pi}{2} \} \). There are no other roots.
iii) Let in addition \( \nu_i \neq \nu_j \) for all \( i \neq j \) in \( \omega_s \) for some \( s = 1, \ldots, N_0 \). Then the function \( \rho_s \) has exactly \( 2N_s(N_s - 1)n_0 \) roots, counted with multiplicity, in the disc \( \{|z| < \pi(n_0 + \frac{1}{2})\} \). Moreover, let

\[
\rho_s(z) = c_s \left(\frac{\sin z + o(e^{\Im z})}{2z}\right)^{N_s(N_s - 1)}, \quad c_s = \prod_{j,k \in \omega_s} (\nu_j - \nu_k)^2, \quad |z| \to \infty. \quad (4.2)
\]

\( \Delta \) satisfies

\[
\Delta(z) = \cos(z - \frac{\nu_j}{2}) + O(1/z^2), \quad m = 1, \ldots, N \quad \text{as} \quad z = \pi n + O(1). \]

ii) All zeros of \( \rho_s \) are given by \( z_{\alpha}^{n,\pm} = \pi n + \nu_j + o(1) \), \( \alpha = (j, k), j \neq k, k \in \omega_s \) and \( n \in \mathbb{Z} \setminus \{0\} \). Furthermore, they satisfy

\[
z_{\alpha}^{n,\pm} = \pi n + \nu_j + o(1), \quad \alpha = (j, k), \quad n \to \pm \infty. \quad (4.3)
\]

**Proof of Theorem 1.2.**

i) We determine asymptotics \( \Psi(1) \) for \( z_{j}^{n,\pm} \) as \( n \to \pm \infty, j = 1, \ldots, N \). Lemma 3.2 yields \( |z_{j}^{n,\pm} - \pi n| < \frac{\pi}{2} \), as \( n \to \infty, j = 1, 2, \ldots, 2N \). Lemma 3.1 gives

\[
\Delta_m(z) = \cos(z - \frac{\nu_j}{2}) + O(1/z^2), \quad m = 1, \ldots, N \quad \text{as} \quad z = \pi n + O(1). \]

For each \( m = 1, \ldots, N \) there exists \( j \) such that \( \Delta_j(z_{m}^{n,\pm}) = (-1)^n \). Thus we have \( z_{m}^{n,\pm} = \pi n + O(1/n) \). Define the local parameter \( \mu \) by \( z = \pi n + \varepsilon \mu, \varepsilon = \frac{1}{2\pi n} \). In order to improve these asymptotics of \( z_{n}^{m,\pm} \) we need asymptotics of the \( \Psi(1) \) as \( z = \pi n + O(1/n), n \to \pm \infty \) given by \( \Psi \)

\[
\Psi(1, z) = e^{izJ_1 \left(1 - i\varepsilon J_1 \Gamma_n + O(\varepsilon^2)\right)}, \quad \Gamma_n = \mathcal{Y} - iJ_1 \hat{V}_{n}, \quad \hat{V}_{n} = \int_{0}^{1} V'(s)e^{i2\pi n J_1} ds
\]

where \( \mathcal{Y} = \int_{0}^{1} V^2(t) dt = \mathcal{Y}_0 \oplus \mathcal{Y}_0, \quad \mathcal{Y}_0 = \text{diag}\{\nu_1, \ldots, \nu_N\}, \quad 0 \leq \nu_1 \leq \nu_2 \leq \ldots \leq \nu_N \). Thus we get

\[
A = \frac{(-1)^n \Psi(1, z) - I}{-i\varepsilon} = \frac{e^{izJ_1 \left(1 - i\varepsilon J_1 \Gamma_n + O(\varepsilon^2)\right)} - I}{-i\varepsilon} = J_1 \left(\Gamma_n - \mu + O(\varepsilon)\right). \quad (4.4)
\]

Hence we study the zeros of the equation

\[
det \left( \mathcal{Y} - iJ_1 \hat{V}_{n} + O(\varepsilon) - \mu \right) = 0, \quad \hat{V}_{n} = \begin{pmatrix} 0 & \hat{v}_{n}^* \\hat{v}_{n} & 0 \end{pmatrix}, \quad \hat{v}_{n} = \int_{0}^{1} v'(t)e^{-i2\pi nt} dt. \quad (4.5)
\]

where \( \mu \in \mathbb{C} \). We will use the standard arguments from the perturbation theory (see [Ka,p.291]). Let \( A, B \) be bounded operators, \( A \) be a normal operator and \( \sigma(A), \sigma(B) \) be spectra of \( A, B \). Then \( \text{dist}(\sigma(A), \sigma(B)) \leq \|A - B\| \).

Let \( \zeta_{m}^{n,\pm}, (m, n) \in \{1, 2, \ldots, N\} \times \mathbb{Z} \) be the eigenvalues of the self-adjoint operator \( \mathcal{Y} - iJ_1 \hat{V}_{n} \). Using the arguments from the perturbation theory (see [Ka,p.291]), we obtain that Eq. \( \mathcal{Y} \) has zeros \( \omega_{m}^{n,\pm}, (m, n) \in \{1, 2, \ldots, N\} \times \mathbb{Z} \) such that \( \omega_{m}^{n,\pm} = \zeta_{m}^{n,\pm} + O(n^{-1}) \) as \( n \to \pm \infty \), which yields \( \mathcal{Y} \).

Consider the case \( \nu_1 < \ldots < \nu_N \). We shall determine asymptotics \( \mathcal{Y} \) for the case \( \alpha = (m, m), m = 1, \ldots, N \). Let \( z_{\alpha}^{n,\pm} = \pi n + \varepsilon \mu \) and \( \mu - \nu_m = \xi \to 0 \). Using the simple transformation (unitary), i.e., changing the lines and columns, we obtain

\[
det \left( \mathcal{Y} - iJ_1 \hat{V}_{n} + O(\varepsilon) - \mu \right) = det \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = det A_4 det K,
\]

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\[ A_1 = \begin{pmatrix} -\xi & b \\ b & -\xi \end{pmatrix} + O(\varepsilon), \quad b = -i\tilde{w}_{\alpha}, \alpha = (m, m), \quad A_2, A_3 = O(\delta_n), \quad \delta_n = |\tilde{V}_n^t| + |\varepsilon|, \]

\[ A_4 = \mathcal{V}_{0m} \oplus \mathcal{V}_{0m} + O(\delta_n), \quad \mathcal{V}_{0m} = \text{diag}\{\nu_j - \xi, j \neq m\}, \]

\[ K = A_1 - A_2 A_4^{-1} A_3 + O(\varepsilon) = A_1 + O(\phi), \quad \phi = |\varepsilon| + |\tilde{V}_n^t|^2, \]

which yields

\[ 0 = \det K = \xi^2 - |b|^2 - \xi b_1 + ab_2 + bb_3 + O(\phi^2), \quad b_1, b_2, b_3 = O(\phi), \]

where \( b_1, b_2, b_3 \) are analytic functions of \( \xi \). Rewriting the last equation in the form \((\xi + \alpha)^2 = (c + \beta)^2 + O(\phi^2), \alpha, \beta = O(\phi)\) and using the estimate \(\sqrt{x^2 + y^2} - x \leq y \) for \(x, y \geq 0\) we get \(\xi = \pm c + O(\phi)\), which yields \(\text{(1.18)}\) for \(i = j\).

Consider the resonances. We shall determine asymptotics \(\text{(1.18)}\) for the case \(\nu_j \neq \nu_{j'}\) for all \(j \neq j' \in \omega_\varepsilon\) for some \(s = 1, \ldots, N_0\). By Lemma \(\text{3.2}\) the zeros of \(\rho_\varepsilon\) have the form \(z_{\alpha}^{\pm}, \alpha = (j, j'), j, j' \in \omega_\varepsilon, j < j', n \in \mathbb{Z}\) and satisfy \(|z_{\alpha}^{\pm} - \pi n| < \pi/2\).

Asymptotics \(\text{(1.10)}\) yields \(\Delta_j(z) - \Delta_{j'}(z) = (\nu_j - \nu_{j'}) \sin \frac{\pm}{2} + O(z^{-2} e^{\pm n z}), \quad |z| \to \infty\). Then \(|z_{\alpha}^{\pm} - \pi n| < \pi/2\) yields \(z_{\alpha}^{\pm} = \pi n + O(1/n)\) as \(n \to \infty\).

We have the identity \(\Delta_j(z) - \Delta_{j'}(z) = 0\) at \(z = z_{\alpha}^{\pm}\). Then using \(\text{(3.19)}\) we have

\[
\cos\left(z_{\alpha}^{\pm} - \frac{\nu_j}{2\pi n}\right) - \cos\left(z_{\alpha}^{\pm} - \frac{\nu_{j'}}{2\pi n}\right) = 2(-1)^n \sin \frac{\nu_j}{4\pi n} \sin \left(z_{\alpha}^{\pm} - \pi n - \frac{\nu_j + \nu_{j'}}{4\pi n}\right) = O(\delta_n) /
\]

which yields \(\text{(1.16)}\), i.e.,

\[
z_{\alpha}^{\pm} = \pi n + \varepsilon (a_+ + z_{\alpha}^{\pm}), \quad a_+ = \frac{\nu_j + \nu_{j'}}{2}, \quad \varepsilon = \frac{1}{2\pi n}, \quad z_{\alpha}^{\pm} = O(\delta_n). \quad (4.6)
\]

We shall show that for large \(n\) in the neighborhood of each \(\pi n + \varepsilon a_+\) the function \((\Delta_j(z) - \Delta_{j'}(z))^2\) has two real zeros resonances (counted with multiplicity). Introduce the functions

\[
f_m(\mu) = 2(2\pi n)^2 (1 - (-1)^n \Delta_m (\pi n + \varepsilon \mu)) = (\mu - \nu_m)^2 + O(\delta_n). \quad (4.7)
\]

For the case \(\mu \to a_+\) we get

\[
f_m(\mu) = (a_+ - \nu_m)^2 + o(1), \quad m = 1, \ldots, N, \quad \text{and} \quad f_m(\mu) = a_+^2 + o(1), \quad m = j, j'. \quad (4.8)
\]

Hence the function \(f_j - f_{j'}\) (maybe) has the zeros, but the functions \(f_j - f_m, m \neq j, j'\) have not zeros in the neighborhood of the point \(a_+\).

Note that these functions are real outside the small neighborhood of \(a_+\), otherwise for any complex branches there exists a complex conjugate branch, but the asymptotics \(\text{(1.7)}\) show that such branches are absent.

We have two cases: (1) let \(f_s(\mu), s = j, j'\) be real in some small neighborhood of \(a_+\). Then the function \(f_j - f_{j'}\) has at least one real zero since by Theorem \(\text{1.1}\) the functions \(f_j, f_{j'}\) are strongly monotone. Thus \((f_j - f_{j'})^2\) has at least 2 real zeros.

(2) Let \(f_s(\mu), s = j, j'\) be complex in some small neighborhood of \(a_+\). Then they have at least two real branch points. Thus \((f_j - f_{j'})^2\) has at least 2 real zeros.
Then the Lyapunov-Poincaré Theorem yields 

\[ A = \frac{(-1)^n \Psi(1, z) - I}{-i\varepsilon} = J_1(\gamma - \mu - i\tilde{V}_n^\prime + O(\varepsilon), \quad \gamma = \gamma_0 \oplus \gamma_0. \]

The operator \( A - a_\perp \) has the eigenvalue \( \xi_0 = \frac{(-1)^n \tau_{n,s} - 1}{i\varepsilon} - a_\perp \) of multiplicity two, since \( \tau_{n,s} = (-1)^n e^{i\varepsilon(a_\perp + o(1))} \). The operator \( J_1(\gamma - a_\perp) - a_\perp = (\gamma_0 - \nu_\perp) \oplus (\nu_\perp - \gamma_0) \) has two eigenvalue \( (= 0) \) and other eigenvalues are not zeros. Using the simple transformation (unitary), i.e., changing the lines and columns, \( \mu = a_\perp + r \in \mathbb{R}, r = r_{n,s} \to 0 \), we obtain

\[
\det(A - a_\perp - \xi) = \det \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \det A_4 \det K,
\]

\[
A_1 = \begin{pmatrix} -r - \xi & ib \\ ib & r - \xi \end{pmatrix} + O(\varepsilon), \quad K = A_1 - A_2 A_4^{-1} A_3 = \begin{pmatrix} -r - \xi + a_1 & ib - \xi + a_4 \\ ib + a_3 & r + a_2 \end{pmatrix}
\]

\[
A_4 = \gamma_{(0)} + \gamma_{(0)}', + O(\delta_n), \quad b = -\tilde{v}_{n,a}'; \quad \alpha = (j, j'), \quad A_2, A_3 = O(\delta_n),
\]

the function \( a_1, a_2, a_3, a_4 = O(\phi), \phi = |\varepsilon| + |\tilde{V}_n^\prime|^2 \) and they analytic with respect to \( \xi \) in some small disk. The function \( \det K \) has the form

\[ \det K = \xi^2 - r^2 + |b|^2 + a_1(r - \xi) + a_2(-r - \xi) - iva_3 - ia_4\bar{v} - a_4a_3 \quad (4.9) \]

Then \( 0 = \det K = (\xi - \xi_0)^2(1 + O(\xi - \xi_0)) \) for \( \xi \to 0 \) where \( \xi_0 = \frac{(-1)^n \tau_{n,s} - 1}{i\varepsilon} - a_\perp \) is the zero of \( F \) of multiplicity two. Then \( \varphi_0 = O(\phi) \) and we have \( (r - \gamma)^2 = (|b| - \beta)^2 + O(\phi^2) \) where \( \gamma, \beta = O(\phi) \). Then using the estimate \( \sqrt{x^2 + y^2} \leq y \) for \( x, y \geq 0 \) we get \( r = \pm |b| + O(\phi) \).

**Proof of Theorem 1.3** We consider \( N^+ \), the proof for \( N^- \) is similar.

(i) Assume that \( N^{+}_g = \infty \). Then, due to the Lyapunov-Poincaré Theorem and Theorem 1.2 there exists a real sequence \( z_k \to \infty \) as \( k \to \infty \), such that \( z_k \in \gamma_{j(m),m} \) for each \( m = 1, \ldots, N \). Hence, \( \cap_{m=1}^N \gamma_{j(m),m} \neq \emptyset \). Using asymptotics (1.18) and \( k \to \infty \), we obtain \( \nu_1 + \nu_{j(1)} = \ldots = \nu_N + \nu_{j(N)} \). Moreover, the estimates \( \nu_1 < \ldots < \nu_N \) yield \( \nu_{j(1)} > \ldots > \nu_{j(N)} \), i.e. \( j(1) = N, j(2) = N - 1, \ldots \). Then, \( \nu_1 + \nu_N = \nu_2 + \nu_{N-1} = \ldots \), which gives a contradiction.

(ii) Let \( 2a = \nu_1 + \nu_N = \nu_2 + \nu_{N-1} = \ldots \). Due to (1.18), \( \pi n_k + a \in \cap_{m=1}^N \gamma_{j(N+1-m),m} \) as \( k \to \infty \). Then the Lyapunov-Poincaré Theorem yields \( \pi n_k + a \notin \sigma(K) \), \( k \to \infty \), i.e. \( N_g = \infty \).

**Proof of Theorem 1.5** i) We need the following result from [K4]: Let \( V \in \mathcal{H} \). Define the quasimomentum \( k_{j+N} = k_j = \arccos \Delta_j(z) = i \log \eta(\Delta_j(z)), j = 1, \ldots, N \), see (1.23), (1.22). Then the averaged quasimomentum \( k = \frac{1}{N} \sum_1^N k_j = \frac{1}{N} \sum_1^N \sum_1^N k_j \) is analytic in \( \mathbb{C}_+ \) and \( k : \mathbb{C}_+ \to k(\mathbb{C}_+) = \mathbb{R}(\mathbb{h}) \) is a conformal mapping for some \( \mathbb{h} \in C_{ua} \). Furthermore, and there exist branches \( k_j, j \in \{1, N\} \) such that (1.26)-(1.28) hold true.

ii) Let \( V, V' \in \mathcal{H} \). We need the following results from [K4]: let for some constants \( C_0, C_1, C_2 \) the following asymptotics hold

\[ \det(M(z) + M^{-1}(z)) = \exp -i2N \left( z - \frac{C_0}{z} - \frac{C_1}{z^2} - \frac{C_2 + o(1)}{z^3} \right) \], as \( z = iy, \ y \to \infty \). (4.10)
Then
\[ k(z) = z - \frac{Q_0}{z} - \frac{Q_1}{z^2} - \frac{Q_2 + o(1)}{z^3}, \quad \text{as } y > r_0|x|, \quad y \to \infty, \quad \text{for any } r_0 > 0, \]  
(4.11)
where \( C_j = Q_j, \ j = 0, 1, 2, \) \( Q_2 = I^P_2 + I^S_2 - \frac{Q_2}{2}. \) Using these results and asymptotics from Lemma 6.1 we obtain (1.29)–(1.30). \( \blacksquare \)

5 Example of complex resonances

Let below \( N = 2. \) Consider the operator \( K_{\nu, \tau} = -iJ_4 \frac{d}{dt} + V_{\nu, \tau}, \nu = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \tau \in \mathbb{R} \) acting in \( L^2(\mathbb{R})^4, \) where the real periodic potential \( V_{\nu, \tau} \) is given by

\[ V_{\nu, \tau} = \begin{pmatrix} 0 & v_{\nu, \tau} \\ v_{\nu, \tau} & 0 \end{pmatrix}, \quad v_{\nu, \tau} = -\begin{pmatrix} a & \tau b_{\nu}(t) \\ \tau b_{\nu}(t) & 0 \end{pmatrix}, \quad \frac{a}{2\pi} \in \mathbb{R} \setminus \mathbb{N}, \quad b_{\nu} \in C(\mathbb{T}). \]  
(5.1)

We need another representation of \( K_{\nu, \tau}. \) Recall that \( \mathcal{U} = \frac{1}{\sqrt{2}}(J_1 + iJ) = \mathcal{U}^* \) and identities (2.22) give \( K_{\nu, \tau} = \mathcal{U} K_{\nu, \tau} \mathcal{U}^* = J_4 \frac{d}{dt} - V_{\nu, \tau}. \) Using the unitary transformation \( y = (y_1, y_2, y_3, y_4)^T \to \mathcal{U}y = (y_1, y_3, y_2, y_4)^T \) in \( L^2(\mathbb{R})^4 \) we define the new operator \( P_{\nu, \tau} = \mathcal{U} K_{\nu, \tau} \mathcal{U}^* = J_4 \frac{d}{dt} + W_{\nu, \tau}, \) where

\[ W_{\nu, \tau} = -\mathcal{U} V_{\nu, \tau} \mathcal{U}^* = \begin{pmatrix} a j_2 & \tau b \ j_2 \\ \tau b \ j_2 & 0 \end{pmatrix}, \quad j_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]  
(5.2)

We rewrite \( W_{\nu, \tau} \) in the form \( W_{\tau, \nu} = W^0 + \tau b_{\nu}J_2, \) where \( W^0 = a \begin{pmatrix} j_2 & 0 \\ 0 & 0 \end{pmatrix}. \) If \( \tau = 0, \) then we have the unperturbed operator \( P^0 = j_4 \frac{d}{dt} + W^0 \) with a constant potential \( W^0. \)

1. The 2 \times 2 Dirac operator. We consider the simple example of the 2 \times 2 Dirac operator \( P^1_4 = j_4 \frac{d}{dt} + a j_2, \ a > 0 \) acting in \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}). \) The spectrum of \( P^0 \) is purely absolutely continuous and consists of two intervals \((\infty, -a), (a, \infty)\) separated by the gap \((-a, a). \) In this case only one gap is open and other gaps are closed. The solution of the system \( jy' + aj_2y = zy \) has the form \( y = e^{\pm ikt}y_0, \) for some constant vector \( y_0 \in \mathbb{C}^2 \) and \( k \) satisfies

\[ \det(ikj + aj_2 - zI_2) = -k^2 - a^2 + z^2, \quad k = k(z) = \sqrt{z^2 - a^2}, \]

where the quasimomentum \( k : \mathbb{C} \setminus (-a, a) \to \mathbb{C} \setminus [ia, -ia] \) is the conformal mapping with asymptotics \( k(z) = z + o(1) \) as \( |z| \to \infty. \) Note that

\[ A = j(z - aj_2), \quad A^2 = (j(z - aj_2))^2 = -z^2 + a^2 - za j_2 - za j_2 = a^2 - z^2 = -k^2. \]  
(5.3)

Then the fundamental solution of the equation \( j\psi'_1 + aj_2 \psi_1 = z\psi_1, \psi_1(0, z) = I_2 \) is given by

\[ \psi_1(t, z) = e^{-tA} = \sum_{n \geq 0} \frac{(-tA)^n}{n!} = \sum_{n \geq 0} \left( \frac{(tA)^{2n}}{(2n)!} - \frac{(tA)^{2n+1}}{(2n+1)!} \right). \]
Thus the Lyapunov function has the form

\[
\Delta^0(z) = \frac{\text{Tr} e^{-A}}{2} = \cos k(z), \quad k = k(z) = \sqrt{z^2 - a^2}, \quad z \in \mathbb{C} \setminus [-a, a].
\]  

(5.5)

2. The \(4 \times 4\) unperturbed operator. Consider the \(4 \times 4\) operator \(P^0 = j \frac{d}{dt} + W^0\) in \(L^2(\mathbb{R})^2 \oplus L^2(\mathbb{R})^2\), where \(W^0 = \left( \begin{array}{cc} a & j_2 \\ 0 & 0 \end{array} \right)\) is the \(4 \times 4\) matrix. We rewrite this operator in the form \(P^0 = P_1^0 \oplus P_2^0\), where the Dirac operators \(P_1^0 = j \frac{d}{dt} + a j_2\) and \(P_2^0 = j \frac{d}{dt}\) act in \(L^2(\mathbb{R}) \oplus L^2(\mathbb{R})\). The corresponding equation \(j \frac{d}{dt} \psi^0 + W^0 \psi^0 = z \psi^0\) has the fundamental solution \(\psi^0(t)\) given by

\[
\psi^0(t) = \psi_1^0(t) \oplus \psi_2^0(t), \quad \psi_1^0 = e^{-At}, \quad \psi_2^0 = e^{-jzt}, \quad A = j(z - a j_2), \quad t \geq 0.
\]  

(5.6)

Thus using (5.4)-(5.6), (1.11) we obtain that \(D^0(\tau, \cdot) = \det(\psi^0(1) - \tau I_4)\) satisfies

\[
D^0(\tau, \cdot) = (\tau^2 - 2 \Delta_1^0 \tau + 1)(\tau^2 - 2 \Delta_2^0 \tau + 1), \quad \Delta_1^0 = (T^0_1 - (-1)^m \sqrt{\rho^0})/2,
\]

\[
\Delta_2^0 = (T^0_2 - (-1)^m \sqrt{\rho^0})/2,
\]

(5.7)

for any \(z \in \mathbb{C}, m = 1, 2\). We will determine the zeros of \(\rho^0(z)\). We have 0 = \(\cos k - \cos z = 2 \sin \frac{k_2}{2} \sin \frac{b_1}{2}\), \(k = k(z)\). Then we obtain \(k \pm z = 2 \pi n, n \in \mathbb{Z} \setminus \{0\}\), which gives zeros \(r^0_n\) (each zero has multiplicity 2) of \(\rho^0\) by

\[
r_n^0 = \pi n \pm \frac{a^2}{4 \pi n}, \quad k(r_n^0) = \begin{cases} \pi n - \frac{a^2}{4 \pi n} & \text{if } |n| > \frac{a}{2 \pi} \\ -\pi n + \frac{a^2}{4 \pi n} & \text{if } |n| < \frac{a}{2 \pi} \end{cases}, \quad n \in \mathbb{Z} \setminus \{0\}.
\]  

(5.8)

We determine the periodic spectrum for the equation \(j y' + W^0 y = z y\). Using (5.6), (5.7) we have \(\det(\psi^0(1, z) \mp I_4) = 4(\cos k(z) \mp 1)(\cos z \mp 1)\), which yields the periodic and anti-periodic spectrum multiplicity 2

\[
z_{n,1}^0 = \pi n, n \in \mathbb{Z}, \quad \text{and} \quad z_{n,2}^0 = \pi n \sqrt{1 + \frac{a^2}{\pi^2 n^2}}, \quad n \in \mathbb{Z} \setminus \{0\}.
\]  

(5.9)

and \(z_{0,2}^0 = \pm a\) has multiplicity one. Note that the zeros \(z_{n,p}^0 \neq r_m^0\) for all \(n, m \in \mathbb{Z}, p = 1, 2\). In this case there are no gaps in the spectrum.

3. The perturbed case. Consider the \(4 \times 4\) operator \(P_{\tau, \nu} = j \frac{d}{dt} + W_{\tau, \nu}\) in \(L^2(\mathbb{R})^2 \oplus L^2(\mathbb{R})^2\), where \(4 \times 4\) potential \(W_{\tau, \nu} = W^0 + \tau b_1 j_2 j_2\) satisfies (5.1), (1.20). We show that there exist the non-degenerated resonance gaps for some \(W_{\tau, \nu}\). In this example some resonances are real and some are complex. The fundamental solution \(\psi^{\tau, \nu}(t, z)\) of the Eq. \(j y' + W_{\tau, \nu} y = z y\) satisfies the integral equation

\[
\psi^{\tau, \nu}(t, z) = \psi^0(t, z) + \tau \int_0^t \psi^0(t - s, z) j b_1(s) j_2 j_2 \psi^{\tau, \nu}(s, z) ds.
\]  

(5.10)
Then \( \psi^{\tau,\nu}(t, z) \) has asymptotics

\[
\psi^{\tau,\nu}(m, z) = \psi^0(m, z) + \tau \psi^1(m, z, \nu) + \tau^2 \psi^2(m, z, \nu) + O(\tau^3 e^{m|\text{Im} z|}) \quad \text{as} \quad \tau \to 0,
\]

uniformly in \( \nu = 1, \frac{1}{2}, \ldots, m = 1, 2, z \in \mathbb{C} \), where

\[
\psi^1(m, z, \nu) = \int_0^m \psi^0(m - t, z) b_\nu(t) j_1 J_2 \psi_0(t, z) dt,
\]

\[
\psi^2(m, z, \nu) = \int_0^m \psi^0(m - t, z) b_\nu(t) j_1 J_2 dt \int_0^t \psi^0(t - s, z) b_\nu(s) j_1 J_2 \psi_0(s, z) ds.
\]

The identity \( \text{Tr} \psi^0(t, z) j_1 J_2 = 0 \), \( (t, z) \in \mathbb{R} \times \mathbb{C} \) yields \( \text{Tr} \psi^1(t, z, \nu) = 0 \). Thus we obtain

\[
T_m^{\tau,\nu}(z) \equiv \text{Tr} \psi^{\tau,\nu}(m, z) = T_m^0(z) + \tau^2 T_m^2(z, \nu) + O(\tau^3 e^{m|\text{Im} z|}), \quad m = 1, 2,
\]

\[
T_m^2(z, \nu) = \int_0^m b_\nu(t) dt \int_0^t b_\nu(s) F_m(t, s, z) ds, \quad F_m(t, s, z) = \text{Tr} j_1 J_2 \psi^0(y, z) j_1 J_2 \psi_0(\zeta, z),
\]

uniformly in \( \nu = 1, \frac{1}{2}, \ldots, |\tau| < 1, z \in \mathbb{C} \), where \( y = t - s, \zeta = m - y \).

**Lemma 5.1.** Let \( V_{\tau, \nu} \) satisfy \((1.19), \ (1.20)\). Then the following asymptotics hold true

\[
T_{12}(z, \nu) = \frac{T_1^0(z)}{2} + o(e^{\text{Im} z}), \quad T_{22}(z, \nu) = T_2^0(z) + 4\phi(z) + o(e^{2|\text{Im} z|}),
\]

\[
T_1^{\tau,\nu}(z) = T_1^0(z)(1 + \frac{\tau^2}{2}) + o(\tau e^{\text{Im} z}),
\]

\[
T_2^{\tau,\nu}(z) = (1 + \tau^2)T_2^0(z) + \tau^2 4\phi(z) + o(\tau^2 e^{2|\text{Im} z|}),
\]

\[
\rho^{\tau,\nu}(z) = \frac{T_2^{\tau,\nu}(z)}{2} - \frac{T_1^{\tau,\nu}(z)^2}{4} = (1 + \tau^2)\rho^0(z) + \tau^2 2(\phi(z) - 1) + o(\tau^2 e^{2|\text{Im} z|}),
\]

\[
\det(\psi^{\tau,\nu}(1, z) \neq I_4) = (1 + \tau^2)^2 D^0(\pm 1, z) + \tau^2 \left( T_1^0 - 1 - \frac{\phi(z) - 1}{2} + o(\tau e^{2|\text{Im} z|}) \right),
\]

as \( \nu \to 0, \) uniformly on \( |\tau| < 1, z \in \mathbb{C} \), and where \( \phi(z) = \cos k(z) \cos z + \frac{\mp}{k(z)} \sin z \sin k(z) \).

**Proof.** Using \( b_\nu \to \delta_{\text{per}} = \sum \delta(t - \frac{1}{2} - n) \) and \( F_j(\cdot, \cdot) \in C(\mathbb{R}^2), j = 1, 2 \), we obtain

\[
\int_0^1 dt \int_0^t b_\nu(t) b_\nu(s) F_1(t, s) ds = \frac{1}{2} F_1\left(\frac{1}{2}, \frac{1}{2}\right) + o(e^{\text{Im} z}),
\]

\[
\int_0^2 dt \int_0^t b_\nu(t) b_\nu(s) F_2(t, s) ds = \frac{1}{2} F_2\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{2} F_2\left(\frac{3}{2}, \frac{3}{2}\right) + F_2\left(\frac{3}{2}, \frac{1}{2}\right) + o(e^{2|\text{Im} z|})
\]

as \( \nu \to 0 \). Thus, if \( m = 1 \), then \((5.21)\) gives

\[
T_{12}(z, \nu) = \frac{1}{2} \text{Tr} j_1 J_2 \psi^0(0, z) j_1 J_2 \psi_0(1, z) + o(e^{\text{Im} z}) = \frac{T_1^0(z)}{2} + o(e^{\text{Im} z}),
\]
If \( m = 2 \), then \( F_2(\frac{1}{2}, \frac{1}{2}) = F_2(\frac{3}{2}, \frac{3}{2}) = \text{Tr} \psi_0(2, z) \) and the identity \( J_2 \psi^0 = \begin{pmatrix} 0 & \psi^0_2 \\ \psi^0_1 & 0 \end{pmatrix} \) yields
\[
F_2(\frac{3}{2}, \frac{1}{2}) = \text{Tr} j_1 \begin{pmatrix} 0 & \psi^0_2(y) \\ \psi^0_1(y) & 0 \end{pmatrix} j_1 \begin{pmatrix} 0 & \psi^0_2(\zeta) \\ \psi^0_1(\zeta) & 0 \end{pmatrix} = 2 \text{Tr} j_1 \psi^0_2(1, z) j_1 \psi^0_1(1, z) = 2 \text{Tr} j_1 \psi^0(1, z) j_1 \psi^0(1, z)
\]
\[
= 2 \text{Tr} j_1 (\cos z - j \sin z) j_1 (\cos k - \frac{A}{k} \sin k) = 2 \text{Tr} (\cos z + j \sin z) (\cos k - \frac{A}{k} \sin k) = 4 \cos z \cos k - 2 \text{Tr} j_1 \frac{A}{2k} \sin z \sin k = 4(\cos z \cos k + \frac{z}{k} \sin z \sin k) = 4\phi.
\]

Then (5.22) gives
\[
T_{22}(z, \nu) = \text{Tr} \psi_0(2, z) + 4\phi(z) = T^0_2(z) + 4\phi(z) + o(e^{2|\Im z|}) \quad \text{as} \quad \nu \to 0.
\]
Substituting (5.16) into (5.14) we obtain (5.17), (5.18). The asymptotics (5.17), (5.18) imply
\[
\rho^{\tau, \nu}(z) = \frac{T^0_2(z) + 4}{2} - \frac{T^{\tau, \nu}_1(z)^2}{4} = (1 + \tau^2)\rho^0(z) + \tau^2 2(\phi(z) - 1) + o(\tau^2 e^{2|\Im z|}),
\]
and \( D^{\tau, \nu}(\pm 1, \cdot) = \det(\psi^{\tau, \nu}(1, \cdot) \mp I_4) = (T^{\tau, \nu}_1 - 1)^2 - \rho^{\tau, \nu} \) satisfies
\[
D^{\tau, \nu}(1, z) = \left((1 + \frac{\tau^2}{2})T^0_1(z) - 1\right)^2 - (1 + \tau^2)\rho^0(z) - \frac{\tau^2}{2} (\phi(z) - 1) + o(\tau^2 e^{2|\Im z|})
\]
\[
= D^0(1, z) + \tau^2 \left(T^0_1(z)(T^0_1(z) - 1) - \rho^0(z) - \frac{(\phi(z) - 1)}{2} + o(1)\right)
\]
as \( \nu \to 0 \), which gives (5.20). The proof for \( D^{\tau, \nu}(-1, \cdot) \) is similar. ■

**Proof of Proposition 1.4.** We have the simple asymptotics
\[
k(z) = z - \frac{a^2}{2z} + O(z^{-3}), \quad \cos k(z) = \cos z + \frac{a^2}{2z} \sin z + O(z^{-2}e^{2|\Im z|}), \quad (5.23)
\]
\[
\rho^0(z) = (\cos k(z) - \cos z)^2 = \left(\frac{a^2 \sin z}{2z}\right)^2 + O(z^{-3}e^{2|\Im z|}) \quad (5.24)
\]
as \( |z| \to \infty \). We take \( \varkappa > 0 \) such that the disks \( B_n = \{|z - r^0_n| < \varkappa\}, n \in \mathbb{Z} \) are not overlapping. Define a constants \( \eta_n = \min_{|z - r^0_n| = \varkappa} |\rho^0(z)| > 0 \). Thus using (5.20) we get
\[
|\rho^{\tau, \nu}(z) - \rho^0(z)| = \tau^2 O(1) = \frac{\tau^2}{\eta_n} \rho^0(z) O(1), \quad |z - r^0_n| = \varkappa \quad \text{as} \quad \tau \to 0, \quad (5.25)
\]
for each \( n \). We also obtain for \( |z| = \pi n_0 + 1 \) the following estimates
\[
|\rho^{\tau, \nu}(z) - \rho^0(z)| = \tau^2 e^{2|\Im z|} O(1) = \tau^2 |4 \sin z|^2 O(1) = \tau^2 |z|^2 |\rho^0(z)| O(1), \quad n_0 \to \infty. \quad (5.26)
\]
Thus we take large \( n_0 \) and sufficiently small \( \tau \) such that \( |\rho(z) - \rho^0(z)| \leq \frac{1}{2} |\rho^0(z)| \) on all contours \( |z| = \pi(n_0 + 1) \) and \( |z - r^0_n| = \varkappa, |n| \leq n_0 \). Then by the Rouché theorem, \( \rho \) has as
many roots, counted with multiplicity, as \( \rho^0 \) in the disks \( \{|z| < \pi n_0 + 1\}, \{|z - r^0_n| < \varepsilon\} \). Since \( \rho^0 \) has exactly one double root at \( r^0_n, n \neq 0 \), and since \( n_0 \) can be chosen arbitrarily large, we deduce that in each disk \( \{|z - r^0_n| < \varepsilon\} \), \( 1 \leq |n| \leq n_0 \) there exist two zeros \( r^\pm_n(\tau) \) of \( \rho^{r,\nu} \) for sufficiently small \( \tau, \nu \).

Consider the zeros \( r^\pm_n(\tau) \) of \( \rho^{r,\nu} \) in the disk \( \{|z - r^0_n| < \varepsilon\} \) for fixed \( n, 1 \leq n \leq n_0 \). The proof for the case \( n < 0 \) is similar. Recall \( \phi(z) = \cos k(z) \cos z + \frac{z^2}{2(\pi z)} \sin z \cos k(z) \). Consider \( \phi(z) - 1 \) at the point \( r^0_n = \pi n + x_n, n \geq 1 \), where \( x_n = \frac{a^2}{4\pi n} \). Using (5.3) we obtain

\[
\phi(r^0_n) - 1 = \cos^2 x_n - 1 + \frac{r^0_n}{k(r^0_n)} \sin^2 x_n = \left(\frac{r^0_n}{k(r^0_n)} - 1\right) \sin^2 x_n > 0, \quad \text{if} \quad 1 \leq n < \frac{a}{2\pi}, \tag{5.27}
\]

\[
\phi(r^0_n) - 1 = \cos^2 x_n - 1 - \frac{r^0_n}{k(r^0_n)} \sin^2 x_n = -\left(1 + \frac{r^0_n}{k(r^0_n)}\right) \sin^2 x_n < 0, \quad \text{if} \quad n > \frac{a}{2\pi}. \tag{5.28}
\]

We rewrite the function \( \rho^{r,\nu} \) in the disk \( \{|z - r^0_n| < \varepsilon\} \) in the form

\[
R(z, \tau) \equiv \frac{\rho^{r,\nu}(z)}{(1 + \tau^2)^2} = (z - r^0_n)^2 f(z) + r^2 \phi_1(z, \tau), \quad f(z) = \frac{\rho^0(z)}{(z - r^0_n)^2}, \quad z \in B_n, \tag{5.29}
\]

for sufficiently small fixed \( \nu, \tau \). The functions \( R(z, \tau), f(z), \phi_1(z, \tau) \) are analytic in \((z, \tau) \in B_n \times \{|\tau| < \varepsilon\} \) for some small \( \varepsilon > 0 \) and satisfy

\[
f(r^0_n) > 0, \quad \phi_1(r^0_n, 0) = 2(\phi(r^0_n) - 1) + o(1) \quad \text{as} \quad \nu \to 0. \tag{5.30}
\]

Applying the Implicit Function Theorem to \( R(z, \tau) = 0 \) and using (5.27)-(5.30) we obtain a unique solution \( r^{\pm}_{n,\nu}(\tau) \) of the equation \( \Phi(r^{\pm}_{n,\nu}(\tau), \tau) = 0, \tau \in (-\tau_0, \tau_0), r^{\pm}_{n,\nu}(0) = r^0_n \), for some \( \tau_0 > 0 \) and here \( r^{\pm}_{n,\nu}(\tau) \) is an analytic function in \( \{|t| < \tau_0\} \) and satisfies (1.21) with \( R_n = -2(\phi(r^0_n) - 1)/f(r^0_n) \). The proof of the statement ii) is similar. \( \blacksquare \)

## 6 Appendix

**Lemma 6.1.** Let \( V, V' \in \mathcal{H} \) and let \( r > 0 \). Then for \( y \geq r|x|, \ y \to \infty \) following asymptotics hold:

\[
\text{Tr} L_1(z) = \|V\|^2 \sin z, \tag{6.1}
\]

\[
\text{Tr} L_2(z) = \left(iH_1 - \frac{\text{Tr} Y^2}{2} + i \frac{\|V'\|^2 + o(1)}{2z}\right) \cos z, \tag{6.2}
\]

\[
\text{Tr} YL_2(z) = \cos z \text{Tr} \left(G_1 + G - \frac{Y^3}{2}\right) + o(e^{|\text{Im} z|}), \tag{6.3}
\]

\[
\text{Tr} L_3(z) = i \cos z \text{Tr} \left(\int_0^1 V_1^4 dt + G_1 + G - \frac{Y^3}{6}\right) + o(e^{|\text{Im} z|}), \tag{6.4}
\]

\[
G_1 = Y \int_0^1 J_1 V_1' V_1 dt, \quad G = Y \int_0^1 J_1 u_t' u_t dt, \quad u_t = \int_0^t V_s^2 ds,
\]

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\[
\det L(z) = 2^{-2N} \exp -i \left( 2Nz - \frac{H_0}{2z} - \frac{H_1}{(2z)^2} - \frac{H_2 + o(1)}{(2z)^3} \right),
\]  

where \( H_0 = \text{Tr} \int_0^1 V_t^2 dt, \quad H_1 = \text{Tr} \int_0^1 -iJ_1 V_t' V_t dt, \quad H_2 = \text{Tr} \int_0^1 \left(V_t^2 + V_t'^2\right) dt. \)

**Proof.** Identity (3.15) implies (6.1). We will show (6.2). Asymptotics (3.12) yields

\[
L_2 = -i \sin z \int_0^1 J_1 V_t' V_t dt + \frac{i e_i J_1}{2} (\mathcal{V}' + \hat{\mathcal{V}}), \quad L_{21} = L_{22} = \frac{1}{2} \int_0^1 \left( e_i f f' + e_{-1} f f' \right) dt, \quad f_i = \int_0^1 V_t' e_{2s} ds
\]

as \(|z| \to \infty, y \geq r|x|\). We determine \(\text{Tr} L_2\). Using (2.12) we have \(\text{Tr} e_i J_1 (\mathcal{V}' + \hat{\mathcal{V}}') = 0\) and then

\[
\text{Tr} L_2 = -i \sin z \mathcal{H}_1 + \text{Tr}(L_{21} + L_{22}).
\]  

Using (2.15) we get

\[
\text{Tr} L_{21} = -\frac{\text{Tr}}{2} \int_0^1 (e_i u'u + e_{-1} uu') dt = -\frac{\text{cos} z}{2} \text{Tr} \int_0^1 (u'u + uu') dt = -\frac{\text{cos} z}{2} \text{Tr} \mathcal{V}'.
\]

Due to (2.15), (2.2) we obtain

\[
\text{Tr} L_{22} = \frac{\text{Tr}}{2} \int_0^1 (e_i f f' + e_{-1} f f') dt = \text{Tr} \int_0^1 \int_0^t e^{i z J_1 (1 - 2t + 2s)} V_t' V_s' dt ds
\]

and using

\[
\int_0^1 \int_s^t e^{iz(t-s)} f_t r_t dt ds = \int_0^1 \frac{i f_i r_i dt}{2z} + o(1), \quad \int_0^1 \int_s^t e^{-iz(t-s)} f_t r_t dt ds = \frac{o(e^{|1mz|})}{z},
\]

for \(f, h \in L^2(0, 1)\) and \(\text{Tr} J_1 (V')^2 = 0\), we have \(\text{Tr} L_{22} = \frac{i \text{cos} z}{2z} \left(\|V'\|^2 + o(1)\right)\), which give (6.2). We will determine \(\text{Tr} \mathcal{V} L_2\). Using (6.6) we have

\[
\text{Tr} \mathcal{V} L_2 = \text{Tr} \mathcal{V}( -i \sin z \int_0^1 \mathcal{V}_t' V_t dt + L_{21} ) + o(e^{|1mz|}),
\]

since \(\text{Tr} \mathcal{V} e_i J_1 (\mathcal{V}' + \hat{\mathcal{V}}') = 0\) and \(L_{22}(z) = o(e^{|1mz|})\). Due to \(e^{iz J_1} = \cos z + i J_1 \sin z\) and \(\int_0^1 (u'u + uu') dt = \mathcal{V}'^2, \text{Tr} \mathcal{V}'^3 = 0\) we get

\[
\text{Tr} \mathcal{V} L_{21} = -\frac{\text{cos} z}{2} \text{Tr} \mathcal{V}'^3 - i \frac{\text{sin} z}{2} \text{Tr} \mathcal{V} J_1 \int_0^1 (u'u - uu') dt = -\frac{\text{cos} z}{2} \text{Tr} \mathcal{V}'^3 - i \text{sin} z G
\]

and together with (6.11) we obtain (6.3), since \(\text{sin} z = i \text{cos} + O(e^{-|1mz|})\).
We will determine $\text{Tr } L_3$. Using (3.13) we rewrite $\text{Tr } L_3$ in the form
\[
\text{Tr } L_3 = \text{Tr } \tau_3 = \text{Tr}(KV^2 + K^2V + KV K + K^3)\psi_0 = \sum_1^4 A_k,
\]
\[
A_1 = \text{Tr } KV^2\psi_0, \ A_2 = \text{Tr } K^2V\psi_0, \ A_3 = \text{Tr } KV K\psi_0, \ A_4 = \text{Tr } K^3\psi_0, \quad (6.12)
\]
and recall $(Kf)(t) = \int_0^t e^{-t-s}W_s f(s)ds$, $W = -iJ_1V^2 - V'$. Using (2.12), (2.14), (2.15) and $e_1 = \cos z + iJ_1 \sin z$ we have
\[
A_1 = \text{Tr } \int_0^1 e_1 W_t V_t^2 dt = \text{Tr } \int_0^1 e_1 (-iJ_1 V_t^2 - V_t')V_t^2 dt = \text{Tr } \int_0^1 -iJ_1 e_1 V_t^4 dt = \text{Tr } \int_0^1 -iJ_1 (\cos z + iJ_1 \sin z)V_t^4 dt = \sin z \text{Tr } \int_0^1 V_t^4 dt.
\]
The similar arguments give
\[
A_2 = \text{Tr } \int_0^1 \int_0^t e_{1-t-s} W_t e_{t-s} W_s V_s dtds = \text{Tr } \int_0^1 \int_0^t e_{1-t-s}(iJ_1 V_t^2 + V_t') e_{t-s}(iJ_1 V_s^2 + V_s') V_s dtds
\]
\[
= \text{Tr } \int_0^1 \int_0^t iJ_1 \left( e_1 V_t^2 V_s V_s - e_{1-2t+2s} V_t' V_s^2 \right) dtds \quad (6.13)
\]
and
\[
A_3 = \text{Tr } \int_0^1 \int_0^t e_{1-t-s} W_t V_t e_{t-s} W_s dtds = \text{Tr } \int_0^1 \int_0^t e_{1-t-s}(iJ_1 V_t^2 + V_t') V_t e_{t-s}(iJ_1 V_s^2 + V_s') dtds
\]
\[
= \text{Tr } \int_0^1 \int_0^t iJ_1 \left( e_1 V_t' V_t V_s^2 + e_{1-2t+2s} V_t^3 V_s' \right) dtds. \quad (6.14)
\]
Summing (6.13), (6.14) we get $A_2 + A_3 = F_0 + F_1$, where
\[
F_0 = \text{Tr } \int_0^1 \int_0^t iJ_1 e_1 \left( V_t^2 V_s' V_s + V_t' V_t V_s' \right) dtds = \text{Tr } iJ_1 e_1 \int_0^1 V_t' V_t dt, \quad (6.15)
\]
since $iJ_1 e_1 = iJ_1 \cos z - \sin z$ and $\text{Tr } \int_0^1 V_t' V_t dt = 0$. We will show
\[
F_1 = \text{Tr } \int_0^1 \int_0^1 iJ_1 e_{1-2t+2s} \left( -V_t' V_s^2 + V_t^3 V_s' \right) dtds = o(e^{\Im z}). \quad (6.16)
\]
We use the standard arguments. If $V, V', V'' \in \mathcal{H}$, then integration by parts gives (6.16). If $V, V' \in \mathcal{H}$, then there exists $P, P', P'' \in \mathcal{H}$ such that $\|V - P\| + \|V' - P'\| = h$ for some small $h \geq 0$. Then $F_1 = o(e^{\Im z})(1 + O(h))$, which yields (6.16), since $h$ is arbitrary small.

The similar arguments give
\[
A_4 = \text{Tr } \int_0^1 \int_0^t s \int_0^s e_1-t-p W_t e_{t-s} W_s e_{s-p} W_p dtdsdp = F_2 + F_3,
\]
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where
\[ F_2 = \text{Tr} \int_0^1 \int_0^l \int_0^p (iJ_1)e_1 V_s^2 V_p^2 dt ds dp, \]
\[ F_3 = \text{Tr} \int_0^1 \int_0^l \int_0^p (iJ_1) \left( e_1 V_s^2 V_p^2 \epsilon + e_{1-2t+2p} V_s^2 V_p^2 \right) dt ds dp. \]

Using \( u_s = \int_s^1 V_s^2 dt \) and \( \int_s^1 V_s^2 dt = \mathcal{V} - u_s \) we obtain
\[ F_2 = \text{Tr} (iJ_1 e_1) \oint_0^1 u_s' u_s' ds = \text{Tr} (iJ_1 e_1) \oint_0^1 (\mathcal{V} - u_s) u_s' ds = i \cos z \text{Tr} G - \frac{\sin z}{6} \text{Tr} \mathcal{V}^3, \]

(6.17)

since
\[ \text{Tr} (iJ_1 e_1) \oint_0^1 u_s' u_s' ds = \text{Tr} \frac{iJ_1 e_1}{3} \oint_0^1 (u_s' u_s' + u_s' u_s + u_s' u_s') ds = \text{Tr} \frac{iJ_1 e_1}{3} \mathcal{V}^3 = -\frac{\sin z}{3} \text{Tr} \mathcal{V}^3, \]
and
\[ \text{Tr} (iJ_1 e_1) \oint_0^1 u_s' u_s' ds = i \cos z \text{Tr} J_1 \mathcal{V} \oint_0^1 u_s' u_s ds - \sin z \text{Tr} \mathcal{V} \oint_0^1 u_s' u_s ds \]
\[ = i \cos z \text{Tr} G - \frac{\sin z}{2} \text{Tr} \mathcal{V} \oint_0^1 (u_s' u_s + u_s' u_s') ds = i \cos z \text{Tr} G - \frac{\sin z}{2} \text{Tr} \mathcal{V}^3, \]

where we used: \( \text{Tr} J_1 \mathcal{V}^3 = 0 \) and \( \text{Tr} ABC = \text{Tr} ACB \) for real self-adjoint matrix and real representations \((2.23), (2.26)\) of \( V \). Using standard arguments (see the proof of (6.16)) we have \( F_3 = o(|\text{Im} z|) \). Summing \( A_1, \ldots, A_4 \) we have (6.4).

We will determine (6.5). Asymptotics (3.12) yields
\[ \frac{L}{\cos z} = I_{2N} + S, \quad S = i \varepsilon \mathcal{V} + \frac{\varepsilon^2 L_2}{\cos z} + \frac{\varepsilon^3 L_3}{\cos z} + O(\varepsilon^4), \quad S = O(\varepsilon) \]

(6.18)
as \( \text{Im} z \to \infty \). In order to use the identity
\[ \det(I + S) = e^{\Phi}, \quad \Phi = \text{Tr} S - \frac{S^2}{2} + \frac{S^3}{3} + O(\varepsilon^4), \quad |S| = O(\varepsilon), \]

we need the traces of \( S^m, m = 1, 2, 3 \). Due to (6.18), we get \( \text{Tr} \frac{S^3}{3} = -i\varepsilon^3 \text{Tr} \frac{\mathcal{V}^3}{3} + O(\varepsilon^4) \).

Using (6.18), (6.1), (6.4) we get
\[ -\frac{S^2}{2} = \text{Tr} \left( \frac{\varepsilon^2 \mathcal{V}^2}{2} - i\varepsilon^3 \frac{\mathcal{L}_2}{\cos z} + O(\varepsilon^3) \right) = \text{Tr} \left( \frac{\varepsilon^2 \mathcal{V}^2}{2} - i\varepsilon^3 \left( G_1 + G - \frac{\mathcal{V}^3}{2} + o(1) \right) \right), \]

(6.19)
\[ \text{Tr} S = i\varepsilon \mathcal{H}_0 + \varepsilon^2 \text{Tr} \left( i\mathcal{H}_1 - \frac{\varepsilon^2 \mathcal{V}^2}{2} \right) + i\varepsilon^3 \left( \mathcal{H}_2 + G_1 + G - \frac{\mathcal{V}^3}{6} + o(1) \right), \]

(6.20)
and summing (??)- (6.20) we get \( \Phi = i\varepsilon \mathcal{H}_0 + i\varepsilon^2 \mathcal{H}_1 + i\varepsilon^3 \mathcal{H}_2 + o(\varepsilon^3) \), which yields (6.5).

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