INTRODUCTION TO COMPLETE SEGAL SPACES

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Abstract. The goal is to review the notion of a complete Segal space and how certain categorical
notions behave in this context. In particular, we study functoriality in complete Segal spaces via
fibrations. Then we use it to define limits and adjunctions in a complete Segal space. This is
mostly expository and the focus is on examples and intuition.

Contents

0. Introduction 1
1. Category Theory & Homotopy Theory: Two Paths towards Simplicial Spaces 3
2. Segal Spaces 16
3. Complete Segal Spaces 34
4. Functoriality in Higher Categories: Fibrations 39
5. Colimits and Adjunctions 50
6. Model Structures of Complete Segal Spaces 56
References 60

Introduction

0.1 Motivation. The theory of higher categories has become very important in modern mathematics.
From topological quantum field theory to derived algebraic geometry to symplectic geometry,
higher categories have proven to be a good way study objects in their proper contexts. The
theory of higher categories or (∞, 1)-categories, as it is sometimes called, however, can be very
intractable at times. That is why there are now several models which allow us to understand what
a higher category should be. Among these models is the theory of quasi-categories, introduced by
Bordman and Vogt ([BV73]) and much studied by Joyal and Lurie ([Jo08], [Jo09] or [Lu09]). There
are also other very prominent models such as simplicial categories, relative categories and Segal
categories. For a general survey on different models of (∞, 1)-categories see [Be10].
One of those models, complete Segal spaces, were introduced by Charles Rezk in his seminal paper "A model for the homotopy theory of homotopy theory" ([Re01]). Later they were shown to be a model for $(\infty, 1)$-categories (see [JT07] for a direct proof and [To05] for an axiomatic argument). Despite their importance, most basic categorical constructions for complete Segal spaces have never been written out in detail.

The goal of this note is to write an introduction to higher categorical concepts from the perspective of complete Segal spaces. It focuses on examples and giving an understanding of the ideas rather than technical proofs.

0.2 Outline. In the first section we start with the two concepts that motivated higher category theory: category theory and homotopy theory. We show how these seemingly different ideas can be generalized to one general concept.

In the second section we define Segal spaces which are our first approach to higher categories and show how they already have many categorical properties. In particular, we can study object, morphisms and composition in a Segal space. We end this section by discussing why that is not enough and why we need further conditions.

In the third section we define complete Segal spaces and show that is a model of a higher category.

In the fourth section we study functoriality in the realm of higher categories. In particular, we show how we can use fibrations to study functors valued in spaces and functors valued in higher categories.

In the fifth section we discuss colimits and adjunctions of complete Segal spaces.

In the last section we show that complete Segal spaces have their own model structure and review some important features of that model structure.

0.3 Background. The main background we assume is a general familiarity with category theory. In particular, topics such as the definition of categories and functors, colimits and adjunctions are required. Such material can be found in the first chapters of [ML98] or [Ri17].

In addition, we assume some familiarity with homotopy theory. In particular, concepts such as topological spaces and homotopy equivalences of spaces.

Finally, it would be very helpful to have some background in the theory of simplicial sets and simplicial homotopy theory. This in particular includes the definition of Kan complexes and homotopy equivalences.

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Category Theory & Homotopy Theory: Two Paths towards Simplicial Spaces

In this section we take a quick look at categories and topological spaces to see how both of them can be thought of as special cases of simplicial sets. This is an informal review of these subjects and serves as a motivation for our definition of a higher category, rather than a thorough introductory text. The section culminates in an introduction to simplicial spaces, which combines category theory and homotopy theory.

1.1 Review of Category Theory. The philosophy of categories is not to just focus on objects but also consider how they are related to each other. This leads to following definition of a category.

Definition 1.1. A category $\mathcal{C}$ is a set of objects $\mathcal{O}$ and a set of morphisms $\mathcal{M}$ along with following functions:

1. An identity map $id : \mathcal{O} \to \mathcal{M}$.
2. A source-target map $(s, t) : \mathcal{M} \to \mathcal{O} \times \mathcal{O}$.
3. A composition map $m : \mathcal{M} \times \mathcal{O} \mathcal{M} \to \mathcal{M}$.

These functions have to make the following diagrams commute:

(1) Source-Target Preservation:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\pi_1} & \mathcal{M} \times \mathcal{M} & \xrightarrow{\pi_2} & \mathcal{M} \\
\downarrow{s} & & \downarrow{m} & & \downarrow{t} \\
\mathcal{O} & \xleftarrow{s} & \mathcal{M} & \xrightarrow{t} & \mathcal{O}
\end{array}
\]

(2) Identity Relations:

\[
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{id} & \mathcal{M} \\
\downarrow{(id, id)} & & \downarrow{(s, t)} \\
\mathcal{O} \times \mathcal{O}
\end{array}
\]

(3) Identity Composition:

\[
\begin{array}{ccc}
\mathcal{O} \times \mathcal{M} & \xrightarrow{id \times id} & \mathcal{M} \times \mathcal{M} & \xrightarrow{id \times \pi_1} & \mathcal{M} \times \mathcal{O} \\
\downarrow{\pi_2} & & \downarrow{m} & & \downarrow{\pi_1} \\
\mathcal{M}
\end{array}
\]

(4) Associativity:
There are many examples of categories in the world of mathematics.

**Example 1.2.** Let \( \text{Set} \) be the category which has as objects all sets and as morphisms all functions of sets. Then the function \( \text{id} \) assigns to each set the identity function and the source target maps \((s, t)\) assigns to each function it’s source and target. Finally \( m \) is just the usual composition of functions.

**Example 1.3.** We can repeat the same example as above by replacing sets with a set that has additional structure. So, we can define the category \( \text{Top} \) of topological spaces and continuous maps, or groups and homomorphisms.

**Remark 1.4.** Very often we care about the morphisms between two specific objects. Concretely, for two objects \( c, d \in \mathcal{C} = (\mathcal{O}, \mathcal{M}) \) we want to define the set of maps with source \( c \) and target \( d \) and denote it as \( \text{Hom}_\mathcal{C}(c, d) \), which we define as the following pullback

\[
\text{Hom}_\mathcal{C}(c, d) = \star^c \times^s \mathcal{M}^t \times^d \star
\]

Using the philosophy of categories on categories themselves means we should consider studying maps between categories.

**Definition 1.5.** A functor \( F : \mathcal{C} \to \mathcal{D} \) is a tuple of two maps. One map for objects \( F_\mathcal{O} : \mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{D} \) and one map for morphisms \( F_\mathcal{M} : \mathcal{M}_\mathcal{C} \to \mathcal{M}_\mathcal{D} \), such that they satisfy following conditions:

1. **Respecting Identity:** \( \text{id}_\mathcal{D} F_\mathcal{O} = F_\mathcal{M} \text{id}_\mathcal{C} \).
2. **Respecting Source/Target:** \( s_D F_\mathcal{M} = F_\mathcal{O} s_\mathcal{C} \) and \( t_D F_\mathcal{M} = F_\mathcal{O} t_\mathcal{C} \).
3. **Respecting Composition:** \( F_\mathcal{M} m_\mathcal{C} = m_\mathcal{D}(F_\mathcal{M} \times F_\mathcal{M}) \).

**Example 1.6.** The definition above allows us to define the category \( \text{Cat} \) which has objects categories and morphisms functors.

Repeating the philosophy of categories for functors leads us to the definition of a **natural transformation**.

**Definition 1.7.** Let \( F, G : \mathcal{C} \to \mathcal{D} \) be two functors. A natural transformation \( \alpha : F \Rightarrow G \) is a collection of maps

\[
\alpha_c : F(c) \to G(c)
\]

for every object \( c \in \mathcal{C} \) such that for every map \( f : c \to d \) the diagram...
Using natural transformations we can even build more categories.

**Theorem 1.8.** Let $C$ and $D$ be two categories. The collection of functors from $C$ to $D$, denoted by $\text{Fun}(C, D)$ is a category with objects functors and morphisms natural transformations.

**Notation 1.9.** For two functors $F, G : C \to D$, we denote the hom set in this category as $\text{Nat}(F, G)$.

This finally leads to the famous Yoneda lemma, which is one of the most powerful results in category theory.

**Definition 1.10.** Let $c \in C$ be an object. There is a functor $\mathcal{Y}_c : C \to \text{Set}$ that send each object $d$ to the set $\text{Hom}_C(c, d)$. Functoriality follows from composition.

**Lemma 1.11.** Let $F : C \to \text{Set}$ be a functor. For each object $c \in C$, there is a bijection of sets

\[ \text{Nat}(\mathcal{Y}_c, F) \cong F(c) \]

induced by the map that sends each natural transformation $\alpha$ to the value at the identity $\alpha_c(id_c)$.

The definitions given up to here are quite cumbersome and necessitate the reader to keep track of a lot of different information. It would be helpful if we could package that same information and present it in a more elegant manner. The way we can achieve this goal is by using simplicial sets.

### 1.2 Simplicial Sets: A Second Look at Categories.

Simplicial sets are a very powerful tool that can help us study categories.

**Definition 1.12.** Let $\Delta$ be the category with objects all non-empty finite linearly ordered sets

- $[0] = \{0\}$,
- $[1] = \{0 \leq 1\}$,
- $[2] = \{0 \leq 1 \leq 2\}$, ...

and morphisms order-preserving maps of linearly ordered sets.

**Notation 1.13.** There are some specific morphisms in the category $\Delta$ that we will need later on.

- For each $n \geq 0$ and $0 \leq i \leq n + 1$ there is a unique injective map
  \[ d_i : [n] \to [n + 1] \]
  such that $i \in [n + 1]$ is not in the image. More explicitly $d_i(k) = k$ if $k < i$ and $d_i(k) = k + 1$ if $k \geq i$.

- For each $n \geq 1$ and $0 \leq i \leq n$ there is a unique surjective map
  \[ s_i : [n] \to [n - 1] \]
  defined as follows. $s_i(k) = k$ if $k \leq i$ and $s_i(k) = k - 1$ if $k > i$. Notice in particular that $s_i(i) = s_i(i + 1) = i$ and that $s_i$ is injective for all other values.
We have following amazing fact regarding these two classes of maps.

**Remark 1.14.** Every morphisms in $\Delta$ can be written as a finite composition of these two classes of maps stated above. The maps satisfy certain relations that can be found in [GJ09, Page 4].

**Notation 1.15.** Because of this remark we can depict the category $\Delta$ as the following

\[
\begin{array}{cccc}
0 & \xrightarrow{d_0} & 1 & \xrightarrow{d_0} \xrightarrow{d_0} \xrightarrow{d_0} \cdots \\
& \xleftarrow{d_1} & & \xleftarrow{d_2} \\
\end{array}
\]

Having studied $\Delta$ we can finally define a simplicial set.

**Definition 1.16.** A simplicial set is a functor $X : \Delta^{op} \to \text{Set}$.  

**Remark 1.17.** Recall that $\Delta^{op}$ is the opposite category of $\Delta$. It has the same objects but every morphism has reverse source and targets.

**Remark 1.18.** Concretely a simplicial set is a choice of sets $X_0, X_1, X_2, \ldots$ which have the appropriate functions between them. Using the diagram above, we can depict a simplicial set as:

\[
\begin{array}{cccc}
X_0 & \xrightarrow{d_0} & X_1 & \xrightarrow{d_0} \xrightarrow{d_0} \xrightarrow{d_0} \cdots \\
& \xleftarrow{d_1} & & \xleftarrow{d_2} \\
\end{array}
\]

notice that all arrows are reversed because this functor is mapping out of the opposite category of $\Delta$.

**Definition 1.19.** A simplicial set is a functor and so the collection of simplicial sets is itself a category with morphisms being natural transformations. We will denote this category by $s\text{Set}$.

A simplicial set is an amazing object of study. In the coming two sections we will see how, depending on which aspects we focus on, a simplicial set can have a very interesting and diverse behavior. For now we focus on the categorical aspects of simplicial sets.

First we show how we can build a simplicial set out of a category.

**Construction 1.20.** Let $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a category. Then we define $NC$ as the following simplicial set. First we define it level-wise as

\[
NC_0 = \mathcal{O}
\]

\[
NC_n = \mathcal{M} \times \cdots \times \mathcal{M}
\]

where there are $n$ factors of $\mathcal{M}$ and $n \geq 1$. So, the 0 level is the set of objects and at level $n$ we have the set of $n$ composable morphisms.

Now we construct the maps between them. It suffices to specify the maps $s_i$ and $d_i$. If $n = 0$, then $s_0 : NC_0 \to NC_1$ is defined as $s_0 = id_\mathcal{C}$. Moreover, $d_0, d_1 : NC_1 \to NC_0$ are defined as $d_0 = s, d_1 = t$.

Let $n \geq 1$ and let $(f_1, f_2, \ldots, f_n) \in NC_n$ be an element. For $0 \leq i \leq n + 1$, we define $d_i : NC_n \to NC_{n-1}$ for the following 3 cases:

1. $d_i((f_1, f_2, \ldots, f_n)) = (f_2, f_3, \ldots, f_n)$
INTRODUCTION TO COMPLETE SEGAL SPACES

(1 to n) \( d_i((f_1, f_2, ..., f_{i-1}, f_i, ..., f_n)) = (f_1, f_2, ..., f_{i-1}f_i, ..., f_n) \)

(n+1) \( d_i((f_1, f_2, ..., f_n)) = (f_1, f_2, ..., f_{n-1}) \)

Similarly, for \( 0 \leq i \leq n \) we define \( s_i : C_n \rightarrow C_{n+1} \) for the following two cases:

(0 to n) \( s_i((f_1, f_2, ..., f_i, ..., f_n)) = (f_1, f_2, ..., id_{s_i(f_i)}, f_i, ..., f_n) \)

(n+1) \( s_i((f_1, f_2, ..., f_i, ..., f_n)) = (f_1, f_2, ..., f_i, ..., f_n, id_{s_i(f_n)}) \)

It is an exercise in diagram chasing to show that \( NC \) satisfies the relations of a simplicial set with the \( d_i \) and \( s_i \) defined above.

**Remark 1.21.** Notice in order to define \( NC \) it did not suffice to have a two sets with 3 maps between them. We needed the existence of the composition map to be able to make the definition work.

This construction merits a new definition.

**Definition 1.22.** Let \( C \) be a category. The **nerve** of \( C \) is the simplicial set \( NC \) described above.

The nerve construction fits well into our philosophy of category theory.

**Theorem 1.23.** The nerve construction is functorial. Thus we get a functor

\[ N : Cat \rightarrow sSet \]

**Proof.** We already constructed the map on objects. For a functor \( F : C \rightarrow D \), the simplicial map \( NF : NC \rightarrow ND \) can be defined level-wise as

- \( NF_0 = F_C \)
- \( NF_n = F_M \times \cdots \times F_M \).

From here on it is a diagram chasing exercise to see that \( NF_n \) make all the necessary squares commute.

Note that it clearly follows that if \( IC : C \rightarrow C \) is the identity functor, then \( NI_C \) is the identity map. Moreover, \( N(F \circ G) = NF \circ NG \). \[ \square \]

**Example 1.24.** We have already introduced the linearly ordered set \([n]\) before (Definition 1.12). We can think of \([n]\) as a category, where the objects are the elements and a morphism are ordered 2-tuples \((i, j)\), where \( i \leq j \). The source of such map \((i, j)\) is \( i \) and the target is \( j \). The identity map of an element \( i \) is the tuple \((i, i)\). Finally, we can compose two morphisms \((i, j)\) and \((j, k)\) to the morphism \((i, k)\). This gives us a category, which we will still denote by \([n]\). Notice in this case for each chosen objects \( i, j \) there either is a unique morphism from \( i \) to \( j \) (if \( i \leq j \)) or there is no morphism at all.

There is a more direct way to think about the set of morphisms. The ordered set \([1]\) has two ordered elements \( 0 \leq 1 \). Given that a morphism is a choice of two ordered elements, we can think of a morphism as an order preserving map \([1] \rightarrow [n]\). But that is exactly a morphism in the category \( \Delta \). Thus the set of morphisms also corresponds to \( Hom_\Delta([1], [n]) \). Let us compute \( N([n]) \). By definition \( N([n])_0 = [n] \). Moreover, \( N([n])_1 = Hom_\Delta([1], [n]) \). Next notice that \( N([n])_m = N([n])_1 \times N([n])_0 \).
... × \mathcal{N}([n])_1, \text{ which corresponds to a choice of } m \text{ ordered numbers } (i_1, i_2, ..., i_m). \text{ Using the same argument as the last paragraph, we see that } \mathcal{N}([n])_m = \text{Hom}_\Delta([m], [n]). \text{ Thus, } \mathcal{N}([n]) \text{ is really just the representable functor }

\mathcal{N}([n]) = \text{Hom}_\Delta(-, [n]) : \Delta^{op} \to \text{Set}

This simplicial set is really special and thus deserves its own name.

**Definition 1.25.** For each \( n \) there is a representable functor, which maps \([i]\) to \( \text{Hom}_\Delta([i], [n]). \) We will denote this simplicial set by \( \Delta[n] \). By the Yoneda lemma, for any simplicial set \( X \) we have following isomorphism of sets:

\[ \text{Hom}_{\text{Set}}(\Delta[l], X) \cong X_n. \]

By now we have shown that we can take a category and build a simplicial set out of it. But can we build every simplicial set this way? If not then which ones do we get?

**Definition 1.26.** A simplicial set \( X \) satisfies the Segal condition if the map

\[ X_n \xrightarrow{\sim} X_1 \times_{X_0} ... \times_{X_0} X_1 \]

is a bijection for \( n \geq 2 \).

The nerve \( \mathcal{N} \) satisfies the Segal condition by its very definition. Thus not every simplicial set is equivalent to the nerve of a category. But what condition other than the Segal condition do we need?

**Theorem 1.27.** Let \( X \) be a simplicial set that satisfies the Segal condition. Then there exists a category \( \mathcal{C} \) such that \( X \) is equivalent to \( \mathcal{N} \).

**Proof.** We define the category \( \mathcal{C} \) as follows. It has objects \( \mathcal{O}_\mathcal{C} = X_0 \) and morphisms \( \mathcal{M}_\mathcal{C} = X_1 \). Then the source, target and identity maps are defined as \( s_\mathcal{C} = d_1 : X_1 \to X_0, t_\mathcal{C} = d_0 : X_1 \to X_0, \) \( i_\mathcal{C} = s_0 : X_0 \to X_1 \) and the product map is defined as \( m_\mathcal{C} = d_1 : X_2 \to X_1 \). Here we are using the fact that \( X_2 \cong X_1 \times_{X_0} X_1 \). Thus we can think of \( m \) as a map \( \mathcal{M}_\mathcal{C} \times \mathcal{O}_\mathcal{C} \mathcal{M}_\mathcal{C} \to \mathcal{M}_\mathcal{C} \), which is exactly what we wanted. The simplicial relations show that \( \mathcal{C} \) satisfies the conditions stated in Definition 1.1.

Finally, we have the following bijection.

\[ (\mathcal{N} \mathcal{C})_n = \mathcal{M}_\mathcal{C} \times_{\mathcal{O}_\mathcal{C} \mathcal{M}_\mathcal{C}} \mathcal{M}_\mathcal{C} = X_1 \times_{X_0} ... \times_{X_0} X_1 \cong X_n \]

This shows that \( \mathcal{N} \mathcal{C} \) is equivalent to \( X \) and finished the proof.

The upshot is that a simplicial set that satisfies the Segal condition has the same data as a category and so instead of keeping track of all the necessary data and maps between them it packages everything very nicely and it gives us much better control. This doesn’t just hold for the categories themselves, but also carries over to functors.
**Theorem 1.28.** Let $C$ and $D$ be two categories. Then the functor $N$ induces a bijection of hom sets

$$N : \text{Hom}_{\text{Cat}}(C, D) \to \text{Hom}_{\text{Set}}(NC, ND)$$

**Proof.** We prove the result by showing the map above has an inverse. Let $f : NC \to ND$ be a simplicial map. Then we define $P(f)$ as the map that is defined on objects as $f_0$ and defined on morphisms as $f_1$. The simplicial identities then show that it satisfies the conditions of a functors. Finally, for any functor $F : C \to D$, the composition $PN(F) = F$ by definition. On the other hand for any simplicial map $f : NC \to ND$, $NP(f) = f$ as they agree at level 0 and 1 and that characterizes the map completely. \qed

Up until now we have shown how we can use the data of a simplicial set to study categories and recover category theory. The next goal is to show we can use the same ideas to study homotopy theory.

### 1.3 Homotopy Theory of Topological Spaces

Homotopy theory can now be found in many forms, but one of the most famous examples of homotopy theories is the homotopy theory of spaces. Again, similar to the case of categories, there are various ways to study spaces. Let us first review the more familiar one: topological spaces. Recall the classical definition of homotopies of topological spaces.

**Definition 1.29.** Two maps of topological spaces $f, g : X \to Y$ are called homotopic if there exists a map $H : X \times [0, 1] \to Y$ such that $H|_{X \times \{0\}} = f$ and $H|_{X \times \{1\}} = g$.

**Definition 1.30.** A map $f : X \to Y$ is called a homotopy equivalence if there exists a map $g : Y \to X$ such that both $fg$ and $gf$ are homotopic to the identity map.

A key question in the homotopy theory of spaces is to determine whether a map is an equivalence or not. However topological spaces can be quite pathological and so we often look for suitable "replacements" i.e. equivalent spaces which have a simpler structure. One good example is a CW-complex.

**Theorem 1.31.** For each topological space $X$ there exists a CW-complex $\tilde{X}$ and map $\tilde{X} \to X$ that is a homotopy equivalence.

Thus from a homotopical perspective it often suffices to study CW-complexes rather than all spaces. However, a CW-complex is built out of simplices. Thus what we really care about is how many simplices we have and how they are attached to each other. This suggests that we can study spaces from the perspective of simplicial sets.

### 1.4 Simplicial Sets: A Second Look at Spaces

Here we show how we can use simplicial sets to study the homotopy theory of topological spaces. We have already defined simplicial sets in the previous section. So, first we show how to construct a simplicial set out of any topological space.

**Definition 1.32.** Let $S(l)$ be the standard $l + 1$-simplex. Concretely $S(l)$ is the convex hull of the $l + 1$ points $(1, 0, ..., 0), (0, 1, ..., 0), ..., (0, 0, ..., 1)$ in $\mathbb{R}^{l+1}$. In particular, $S(0)$ is a point, $S(1)$ is an interval and $S(2)$ is a triangle.
Remark 1.33. One important fact about those simplices is that the boundary is built out of lower dimensional simplices. For example, the boundary of a line is the union of two points or the boundary of a triangle is the union of three lines. This means we have two maps $d_0, d_1 : S(0) \to S(1)$ that map to the two boundary points or we have three maps $d_0, d_1, d_2 : S(1) \to S(2)$.

On the other side, we can always collapse one boundary component to lower the dimension of our simplex. Thus there are two ways to collapse our triangle $S(2)$ to a line $S(1)$, which gives us two maps $s_0, s_1 : S(2) \to S(1)$. It turns out these maps do satisfy the covariant version of the simplicial identities, which are also called the cosimplicial identities. This means we can thus define a functor $S : \Delta \to \text{Top}$

This functor can be depicted in the following diagram.

\[
\begin{array}{cccc}
S(0) & \xrightarrow{d_0} & S(1) & \xrightarrow{d_0} & S(2) & \xrightarrow{d_0} & \cdots \\
& & \xrightarrow{d_1} & & \xrightarrow{d_2} & & \\
\end{array}
\]

Definition 1.34. Let $X$ be a topological space. We define the simplicial set $S(X)$ as follows. Level-wise we define $S(X)$ as

\[
S(X)_n = \text{Hom}_{\text{Top}}(S(n), X).
\]

The functoriality of $I$ as described in the remark above shows that this indeed gives us a simplicial set.

Thus we can build a simplicial set out of every topological space. Each level indicates how many $n+1$-simplices can be mapped into our space. However, we cannot build every kind of simplicial set this way. Rather the simplicial set we constructed is called a Kan complex. In order to be able to give a definition we need to gain a better understanding of simplicial sets first.

Definition 1.35. We say $K$ is a subsimplicial set of $S$, if for any $l$ we have $K_l \subset S_l$.

Example 1.36. There are two important classes of sub simplicial sets of $\Delta[l]$ (Definition 1.25):

1. The first one is denoted by $\partial \Delta[l]$ and defined as follows: $\partial \Delta[l]_i$ is the subset of all non-surjective maps in $\text{Hom}_\Delta([i], [l])$. In particular, this implies that for $i < n$, we have $\partial \Delta[l]_i = \Delta[l]_i$ and for $i = l$ we have $\partial \Delta[l]_l = \Delta[l]_l - \{id|_{[l]}\}$. Intuitively it looks like the boundary of our convex space i.e. $\Delta[l]$ with the center $n$-dimensional cell removed.

2. The second is denoted by $\Lambda[l]_i$ ($0 \leq i \leq l$) and consists of non-surjective maps that satisfy the following condition: $\Lambda[n]_i$ is the subset of all maps in $\text{Hom}_\Delta([j], [l])$, that satisfy following condition. If $i$ is not in the image of the map then at least one other elements also has to be not in the image. Concretely, this means it is also a subspace of $\partial \Delta[l]$ and it excludes the face which is formed by all vertices except for $i$. Intuitively, this one looks like a boundary where one of the faces (the one opposing the vertex $i$) has been removed as well. Given the resulting shape it is very often called a ”horn”.

Having gone through these definitions we can finally define a Kan complex.

Definition 1.37. A simplicial set $K$ is called a Kan complex if for any $l \geq 0$ and $0 \leq i \leq l$, the map

\[
\text{Hom}_S(\Delta[l], K) \to \text{Hom}_S(\Delta[l]_i, K)
\]

is surjective.
Remark 1.38. Basically the definition is saying that following diagram lifts:

\[
\begin{array}{ccc}
\Lambda[l] & \longrightarrow & K \\
\downarrow & & \\
\Delta[l] & \longleftarrow & \\
\end{array}
\]

Example 1.39. For every topological space \(X\), the simplicial set \(SX\) is a Kan complex. We will not prove this fact here. It relies on the idea that a topological space has no sense of direction. Thus every path can be inverted. Concretely, for any map \(\gamma : I(1) \to X\), there is a map \(\gamma^{-1} : I(1) \to X\) that is defined as \(\gamma^{-1}(t) = \gamma(1-t)\). Thus every element \(\gamma \in S(X)_1\) has a reverse path. A similar concept applies to higher dimensional maps.

It is that idea that allows us to lift any map of the form above. For a rigorous argument see [GJ09, Chapter 1].

Example 1.40. Contrary to the example above \(\Delta[l]\) is not a Kan complex (if \(l > 0\)). For example the map \(\Lambda[2]_0 \to \Delta[l]\) that sends 0 to 0, 1 to 2 and 2 to 1 cannot be lifted.

The definition above is a special case of a **Kan fibration**.

Definition 1.41. A map of simplicial sets \(f : S \to T\) between Kan complexes is called a **Kan fibration** if any commutative square of the form

\[
\begin{array}{ccc}
\Lambda[l] & \longrightarrow & S \\
\downarrow & & \\
\Delta[l] & \longleftarrow & \\
\end{array}
\]

lifts, where \(n \geq 0\) and \(0 \leq i \leq n\).

Remark 1.42. This generalizes Kan complexes as \(K\) is a Kan complex if and only if the map \(K \to \Delta[0]\) is a Kan fibration. As a result, if \(K \to L\) is a Kan fibration and \(L\) is Kan fibrant, then \(K\) is also Kan fibrant

Kan complexes share many characteristics with topological spaces. In particular, we can talk about equivalences and homotopies.

Definition 1.43. Two maps \(f, g : L \to K\) between Kan complexes are called **homotopic** if there exists a map \(H : L \times \Delta[1] \to K\) such that \(H|_0 = f\) and \(H|_1 = g\).

Remark 1.44. This definition can be made for any simplicial set, but it is only an equivalence relation for the case of Kan complex.

Example 1.45. One particular instance of this definition is when \(L = \Delta[0]\). In this case we have two points \(x, y : \Delta[0] \to K\). We say \(x\) and \(y\) are homotopic or **equivalent** if there is a map \(\gamma : \Delta[1] \to K\) such that \(\gamma(0) = x\) and \(\gamma(1) = y\).
**Definition 1.46.** A map $f : L \to K$ between Kan complexes is called an *equivalence* if there are maps $g, h : K \to L$ such that $fg : K \to K$ is homotopic to $id_K$ and $hf : L \to L$ is homotopic to $id_L$.

Most importantly, in order to study equivalences of spaces it suffices to study equivalences of the analogous Kan complexes.

**Lemma 1.47.** A map of topological spaces $f : X \to Y$ is a homotopy equivalence if and only if the map of Kan complexes $Sf : SX \to SY$ is a homotopy equivalence.

Seeing how that result holds requires us to use much more machinery. One very efficient way is to use the language of *model categories*. A model structure can capture the homotopical data in the context of a category. Using model categories we can show that topological spaces and simplicial sets (if we focus on Kan complexes) have equivalent model structures. For a better understanding of model structures see Section 6.

**Remark 1.48.** Kan fibrations are important in the homotopy theory of simplicial sets. That is because base change along Kan fibrations is equivalence preserving. By that we mean that in the following pullback diagram

$$
\begin{array}{ccc}
K \times_{M} L & \xrightarrow{\simeq} & K \\
\downarrow & & \downarrow \, g \\
L & \xrightarrow{\simeq} & M
\end{array}
$$

if $f$ is an equivalence and $g$ is a Kan fibration then $g^*f$ is also an equivalence. Moreover, the pullback of a Kan fibration is also a Kan fibration. Thus we say such a pullback diagram is *homotopy invariant*.

**Remark 1.49.** The homotopy invariance of base change by a Kan fibration implies in particular that we can define a *homotopy pullback*. We say a diagram of Kan complexes

$$
\begin{array}{ccc}
A & \xrightarrow{r} & B \\
\downarrow & & \downarrow \, g \\
C & \xrightarrow{f} & D
\end{array}
$$

is a homotopy pullback if the induced map $A \to B \times_D C$ is a homotopy equivalence. In other words, we demand a pullback “up to homotopy” rather than a strict pullback. The fact that $g$ is a Kan fibration implies that this definition is well-defined.

Before we move on we will focus on one particular, yet very important instance of a homotopy equivalence.

**Definition 1.50.** A Kan complex $K$ is *contractible* if the map $K \to \Delta[0]$ is a homotopy equivalence.
Remark 1.51. The notion of a contractible Kan complex is central in homotopy theory. It is the homotopical analogue of uniqueness as it implies that every two points in \( K \) are equivalent. Moreover, any two paths are themselves equivalent in the suitable sense and this pattern continues.

A contractible Kan complex is again a special kind of Kan fibration.

**Definition 1.52.** We say a map \( K \to L \) is a **trivial Kan fibration** if it is a Kan fibration and a weak equivalence.

**Lemma 1.53.** A map \( K \to L \) is a trivial Kan fibration if and only if it is a Kan fibration and for every map \( \Delta[0] \to L \), the fiber \( \Delta[0] \times_L K \) is contractible.

**Remark 1.54.** Thus a trivial Kan fibration not only has lifts, but the space of lifts is contractible, meaning there is really only one choice of lift up to homotopy.

Having a homotopical notion of an isomorphism, namely an equivalence, we can also define the homotopical version of an injection, namely a \((-1\)-truncated map).

**Definition 1.55.** A Kan fibration \( K \to L \) is \((-1\)-truncated if for every map \( \Delta[0] \to L \), the fiber \( \Delta[0] \times_L K \) is either contractible or empty.

Before we move on there is one last property of Kan complexes that we need, namely that they are Cartesian closed.

**Remark 1.56.** The category of simplicial sets is Cartesian closed. For every two simplicial sets \( X, Y \) there is a mapping simplicial set, \( \text{Map}(X,Y) \) defined level-wise as

\[
\text{Map}(X,Y)_n = \text{Hom}(X \times \Delta[n], Y).
\]

**Proposition 1.57.** If \( K \) is a Kan complex, then for every simplicial set \( X \), the simplicial set \( \text{Map}(X,K) \) is also a Kan complex.

**Notation 1.58.** As we have established a well functioning homotopy theory with Kan complexes, we will henceforth exclusively use the word space to be a Kan complex.

1.5 Two Paths Coming Together. Until now we showed that we can think of categories as a simplicial set that satisfies the Segal condition and a topological space as a Kan complex. Thus simplicial sets have two different aspects to them.

We can either think of simplicial sets that have a notion of direction and allow us to do category theory. When we think of simplicial sets this way we denote them by \( s\text{Set} \) and pictorially we can depict them as:

\[
\begin{array}{cccc}
\ldots & C_0 & = & C_1 \\
\downarrow && \downarrow && \\
\downarrow && \downarrow && \\
\ldots & \swarrow & \swarrow & \ldots
\end{array}
\]

On the other side, we can think of simplicial sets that have homotopical properties. In this case we call them spaces and denote that very same category as \( \mathcal{S} \). This time we depict it as:

\[
\begin{array}{cccc}
\ldots & C_0 & \longrightarrow & C_1 \\
\downarrow && \downarrow && \\
\downarrow && \downarrow && \\
\ldots & \Rightarrow & \Rightarrow & \ldots
\end{array}
\]
A higher category should generalize categories and spaces at the same time. Thus our goal is to embed both versions of simplicial sets (categorical and homotopical) into a larger setting. We need to start with a category which can house two versions of simplicial sets in itself independent of each other so that we can give each the properties we desire and make sure one part has a categorical behavior and one part has a homotopical behavior. This point of view leads us to the study of simplicial spaces.

1.6 Simplicial Spaces. In this section we define and study objects that have enough room to fit two versions of simplicial sets inside of it. We will call this object a simplicial space, although they are also known as bisimplicial sets. The next subsection will justify why we have decided to use the term simplicial space.

**Definition 1.59.** We define the category of simplicial spaces as $\text{Fun}(\Delta^{op}, S)$ and denote it by $sS$.

**Remark 1.60.** We have the adjunction

$$\text{Fun}(\Delta^{op} \times \Delta^{op}, \text{Set}) \cong \text{Fun}(\Delta^{op}, \text{Fun}(\Delta^{op}, \text{Set})) = \text{Fun}(\Delta^{op}, S).$$

Thus on a categorical level a simplicial space is a bisimplicial set. Therefore, we can depict it at the same time as a bisimplicial set or as a simplicial space. We an depict those two as follows:
Notice that $X_{0\bullet}, X_{1\bullet}, \ldots$ are themselves simplicial sets.

Remark 1.61. There are two ways to embed simplicial sets into simplicial spaces.

1. There is a functor $i_F : \Delta \times \Delta \to \Delta$

that send $([n], [m])$ to $[n]$. This induces a functor $i_F^* : \text{sSet} \to \text{sSet}$

that takes a simplicial set $S$ to the simplicial space $i_F^*(S)$ defined as follows.

$$i_F^*(S)_{kl} = S_k$$

We call this embedding the \textit{vertical embedding}.

2. There is a functor $i_\Delta : \Delta \times \Delta \to \Delta$

that send $([n], [m])$ to $[m]$. This induces a functor $i_\Delta^* : \text{sSet} \to \text{sSet}$

that takes a simplicial set $S$ to the simplicial space $i_\Delta^*(S)$ defined as follows.

$$i_\Delta^*(S)_{kl} = S_l$$

We call this embedding the \textit{horizontal embedding}.

Given that there are two embeddings there are two ways to embed generators.

Definition 1.62. We define $F(n) = i_F^*(\Delta[n])$ and $\Delta[l] = i_\Delta^*(\Delta[l])$. Similarly, we define $\partial F(n) = i_F^*(\partial \Delta[n])$ and $L(n)_i = i_F^*(\Lambda[n]_i)$. 
The category of simplicial spaces has many pleasant features that we will need later on.

**Definition 1.63.** The category of simplicial spaces is Cartesian closed. For any two objects $X$ and $Y$ we define the simplicial space $Y^X$ as

$$(Y^X)_{n|l} = \text{Hom}_{sS}(F(n) \times \Delta[l], X, Y)$$

**Remark 1.64.** In particular, the previous statement implies that $sS$ is enriched over simplicial sets, as for every $X$ and $Y$, we have a mapping space $\text{Map}_{sS}(X,Y) = (Y^X)_0$.

**Remark 1.65.** Using the enrichment, by the Yoneda lemma, for any simplicial space $X$ we have following isomorphism of simplicial sets:

$$\text{Map}_{sS}(F(n), X) \cong X_n.$$ 

**Segal Spaces**

The goal of this section is to introduce Segal spaces and show how the conditions we impose on it actually allows us to do interesting higher categorical constructions. In the previous section we described that in order to study category theory and homotopy theory at the same time we need to expand our playing field and use simplicial spaces. However, clearly we cannot just use any simplicial space, but rather need to impose the right set of conditions to be able to develop a proper theory. We will achieve this goal in three steps:

1. Reedy fibrant simplicial spaces
2. Segal spaces
3. Complete Segal spaces

In this section we focus on the first two conditions. The next section will discuss the last condition.

2.1 **Reedy Fibrant Simplicial Spaces.** First we have to make sure that the vertical axis actually behave like a space as described in Subsection 1.4. This is achieved by the Reedy fibrancy condition.

**Definition 2.1.** A simplicial space $X \in sS$ is **Reedy fibrant** if for every $n \geq 0$, the induced map of spaces

$$\text{Map}_{sS}(F(n), X) \to \text{Map}_{sS}(\partial F(n), X)$$

is a Kan fibration.

**Remark 2.2.** By the Yoneda lemma $\text{Map}_{sS}(F(n), X) \cong X_n$. Moreover,

$$\text{Map}_{sS}(\partial F(0), X) = \text{Map}_{sS}(\emptyset, X) = \Delta[0]$$

so an inductive argument shows that $X_n$ is a Kan complex. Notice, the opposite does not necessarily hold. In other words, a level-wise Kan fibrant simplicial space is not necessarily Reedy fibrant.

**Remark 2.3.** At the beginning of Subsection 1.6 we stated that we chose to use the word simplicial space rather than bisimplicial set. The reason is exactly because we will focus on the Reedy fibrancy condition which guarantees to us that we have level-wise spaces.

**Remark 2.4.** Concretely, $X$ is Reedy fibrant if for every $n \geq 0, l \geq 0, 0 \leq i \leq n$ the following diagram lifts.
As is the case for Kan fibrations there is an analogous Reedy fibration. A map $Y \rightarrow X$ is a Reedy fibration if the following square lifts.

\[
\begin{array}{ccc}
\partial F(n) \times \Delta[l] \times \coprod_{\partial F(n) \times \Lambda[l],} F(n) \times \Lambda[l] & \rightarrow & X \\
\downarrow & & \downarrow \\
F(n) \times \Delta[l] & \rightarrow & X
\end{array}
\]

Remark 2.5. Note in particular $X$ is Reedy fibrant if and only if $X \rightarrow F(0)$ is a Reedy fibration. So, if $Y \rightarrow X$ is a Reedy fibration and $X$ is Reedy fibrant then $Y$ is also Reedy fibrant.

Remark 2.6. [Re01, 2.5] If $X$ is Reedy fibrant and $Y$ is any simplicial space then $X^Y$ is also Reedy fibrant.

As Reedy fibrant simplicial spaces are just level-wise spaces they have their own homotopy theory.

**Definition 2.7.** A map of $X \rightarrow Y$ of Reedy fibrant simplicial spaces is an equivalence if and only if for any $n \geq 0$, the map of spaces $X_n \rightarrow Y_n$ is an equivalence of spaces.

**Remark 2.8.** One important reason we use the Reedy fibrancy condition is outlined in Remark 1.48. We want all definitions to be homotopy invariant and as many of those definitions have pullback conditions involved, Reedy fibrancy is an effective way to guarantee that all definitions have that condition.

**Remark 2.9.** There is a more conceptual reason for the Reedy fibrancy condition. It allows us to use the tools from the theory of model categories to study higher categories (Theorem 6.8).

### 2.2 Defining Segal Spaces.

In the last subsection we made sure that the vertical axis of the simplicial space actually has the behavior of spaces, by adding the Reedy fibrancy condition. In the next step we will add the necessary condition to the horizontal axis to make sure it has the proper categorical behavior, which will lead to the definition of a Segal space. Segal spaces were originally defined in [Re01, Section 4]

Recall in Subsection 1.2 we showed how a category is really a simplicial set that satisfies the Segal condition. We want to repeat that argument for simplicial spaces.

**Construction 2.10.** Let $\alpha_i \in F(n) = Hom_\Delta([1],[n])$ be defined by $\alpha_i(k) = k + i$, where $0 \leq i \leq n - 1$. Concretely it is the map $[1] \rightarrow [n]$, which takes 0 to $i$ and 1 to $i + 1$. Let $G(n)$ be the
simplicial subspace of $F(n)$ generated by $A = \{ \alpha_i : 0 \leq i \leq n - 1 \}$. This means that

$$G(n) = F(1) \coprod_{F(0)} ... \coprod_{F(0)} F(1)$$

where there are $n$ factors of $F(1)$. Let $\varphi^n : G(n) \to F(n)$ be the natural inclusion map. Thus we think of $G(n)$ as a subobject of $F(n)$. This is commonly known as the "spine" of $F(n)$.

**Remark 2.11.** It easily follows that for a simplicial space $X$ we have

$$\text{Map}_{sS}(G(n), X) \cong X_1 \times X_0 \times X_1 \times X_0 \times ... \times X_0 X_1$$

where there are $n$ factors of $X_1$ and $d_0, d_1 : X_1 \to X_0$ are the simplicial maps. This gives us the following canonical Kan fibration:

$$\varphi_n = (\varphi^n)^* : X_n \cong \text{Map}_{sS}(F(n), X) \to \text{Map}_{sS}(G(n), X) \cong X_1 \times X_0 \times ... \times X_1.$$

Having properly defined our map we can now make following definition.

**Definition 2.12.** A Segal space $T_n \in sS$ is a Reedy fibrant simplicial space such that the canonical map

$$\varphi_n : T_n \xrightarrow{\cong} T_1 \times ... \times T_1$$

is a Kan equivalence for every $n \geq 2$.

Note that Reedy fibrancy implies that the maps $\varphi_n$ are actually fibrations and so, for Segal spaces, these maps are trivial fibrations. Also, Reedy fibrancy tells us that $d_1$ and $d_0$ are fibrations of spaces and so the pullbacks are already homotopy pullbacks.

**Intuition 2.13.** What is the idea of a Segal space? If $T$ is a Segal space then it is a simplicial space, thus has spaces $T_0, T_1, T_2, ...$. The Segal condition tells us that we should think of $T_0$ as the "space of objects", $T_1$ the "space of morphisms", $T_2$ the "space of compositions". Let us see how we can manifest those ideas in a more concretely:

**n=2:** The first condition states that $T_2 \xrightarrow{\cong} T_1 \times T_0 T_1$. Note that $T_2$ is the space of 2-cells. Concretely, we depict a 2-cell $\sigma$ as follows:

```
\begin{tikzpicture}
    \node (x) at (0,0) {$x$};
    \node (y) at (1,1) {$y$};
    \node (z) at (2,0) {$z$};
    \node (s) at (1,0) {$\sigma$};

    \draw[->] (x) -- (s);
    \draw[->] (s) -- (y);
    \draw[->] (s) -- (z);
    \draw[->] (x) -- (z);
    \draw[->] (y) -- (z);

    \end{tikzpicture}
```

Similarly, we think of $T_1 \times T_0 T_1$ as the space of *two composable arrows*, which we depict as:

```
\begin{tikzpicture}
    \node (x) at (0,0) {$x$};
    \node (y) at (1,1) {$y$};
    \node (z) at (2,0) {$z$};
    \node (s) at (1,0) {$g$};
    \node (t) at (1,1) {$f$};

    \draw[->] (x) -- (s);
    \draw[->] (s) -- (y);
    \draw[->] (s) -- (z);
    \draw[->] (x) -- (z);
    \draw[->] (y) -- (z);

    \end{tikzpicture}
```
The Segal condition states that every such diagram can be filled out to a complete 2-cell:

\[
\begin{array}{c}
\text{x} \\
\sigma \\
\text{y}
\end{array}
\begin{array}{ccc}
\text{f} & \sigma & \text{g} \\
\Downarrow & & \Downarrow \\
\text{h} & & \text{z}
\end{array}
\]

From this point of view, we think of \( h \) as the composition of \( f \) and \( g \), and we think of \( \sigma \) as a witness of that composition. Thus we often depict \( h \) as \( g \circ f \).

Right here we can already notice the difference between the classical setting and the higher categorical setting. In the classical setting every two composable maps have a unique composition, whereas in this case neither \( h \) nor \( \sigma \) are unique. Rather we only know such lift exists. In the next part we will explain how we are still justified in using a specific name ("\( g \circ f \)") despite its non-uniqueness.

**n=3:** The second condition is that \( T_3 \longrightarrow T_1 \times T_0 \times T_1 \times T_0 \times T_1 \).

\( T_3 \) is the space of 3-cells. We depict a 3-cell as a tetrahedron.

\[
\begin{array}{c}
\text{w} \\
\text{h} \\
\text{y}
\end{array}
\begin{array}{ccc}
\text{f} & \gamma & \text{g} \\
\Downarrow & & \Downarrow \\
\text{x} & & \text{z}
\end{array}
\]

where all 2-cells and the middle 3-cell are filled out. On the other hand \( T_1 \times T_0 \times T_0 \times T_1 \) can be depicted as three composable arrows.
The Segal condition then implies that this diagram can be completed to a diagram of the form:

In the diagram above the 2-cell $\sigma$ witnesses the composition of $f$ and $g$ and the 2-cell $\gamma$ witnesses the composition of $g$ and $h$. The middle 3-cell (and the two other 2-cells) witness an equivalence between the composition $h \circ (g \circ f)$ and $(h \circ g) \circ f$. So, the existence of this 3-cell witnesses the associativity of the composition operation.

The Segal condition for 3-cells has even more interesting implications. In the previous part we discussed that composition is not unique and that any lift is a possible composition. Using the Segal condition we can show that every two choices of equivalences are equivalent.

Let $f : x \to y$ $g : y \to z$, then the Segal condition implies we can fill in following diagram:
The two cells $\sigma$ and $\gamma$ represent two possible compositions, which we denoted by $c_1$ and $c_2$. The Segal condition (which fills in a 3-cell) tells us that those two compositions are equivalent.

Up to here the goal has been to give the reader an intuition on how a Segal space take familiar concepts from category theory and generalize them to a homotopical setting. The goal of the next subsection is to actually give precise definitions and make those intuitive arguments rigorous.

### 2.3 Category Theory of Segal Spaces

The goal of this subsection is to make our first steps towards studying the category theory of Segal spaces. In particular, we define objects, morphisms, compositions and homotopy equivalences. A lot of the concepts here are guided by the ideas introduced in Intuition 2.13. The work here was originally developed in [Re01, Section 5].

**Definition 2.14.** Let $T$ be a Segal space. We define the objects of $T$ as the set of objects of $T_0$. Thus, $\text{Obj}(T) = T_{00}$.

**Intuition 2.15.** In Intuition 2.13 we discussed how $T_0$ should be the “space of objects”. Consistent with that philosophy the points in $T_0$ are exactly the objects.

**Notation 2.16.** Following standard conventions we often use the notation $x \in T$ to denote an object $x$ in the Segal space $T$.

**Definition 2.17.** For two objects $x, y \in T$, we define the mapping space by the following pullback:

\[
\begin{array}{ccc}
\text{map}_T(x, y) & \to & T_1 \\
\downarrow & & \downarrow_{(d_0, d_1)} \\
\Delta[0] & \to & T_0 \times T_0 \\
(x, y) & &
\end{array}
\]

or in other words the fiber of $(d_0, d_1)$ over the point $(x, y)$.

**Intuition 2.18.** Again this definition is consistent with Intuition 2.13. $T_1$ should be thought of as the “space of morphism”. From that point of view the boundary map $d_0$ gives us the source of the morphism and $d_1$ is the target of the morphism. The space of morphisms that start at a certain object $x$ and end with the object $y$ is exactly the pullback constructed above.
Remark 2.19. We can compare the definition of a mapping space to the definition of a Hom set in a usual category as introduced in Remark 1.4. The overall definitions are very similar, giving one more evidence that this is a good way to define the space of maps.

Again by Reedy fibrancy the map \((d_0, d_1)\) is a fibration and so our pullback diagram is actually homotopy invariant.

Notation 2.20. In order to simplify our notation instead of writing \(f \in \text{map}_T(x, y)\) we will use the more familiar \(f : x \to y\).

Remark 2.21. The fact that \((d_0, d_1)\) is a Kan fibration also implies that \(\text{map}_T(x, y)\) is a Kan complex, which justifies using the word mapping space.

Our main goal now is to make the idea of composition, which we discussed in the last subsection, precise. In order to do so we need to define the space of compositions.

Definition 2.22. Let \(x_0, x_1, ..., x_n \in \text{Ob}(T)\) be objects in \(T\). We define the space of composition \(\text{map}_T(x_0, ..., x_n)\) as the pullback:

\[
\begin{array}{ccc}
\text{map}_T(x_0, ..., x_n) & \xrightarrow{\gamma} & T_n \\
\downarrow & & \downarrow \\
\ast & \xrightarrow{(x_0, x_1, ..., x_{n+1})} & (T_0)^{n+1}
\end{array}
\]

or, in other words, the fiber of the map \(T_n \to T_0^{n+1}\) over the point \((x_0, x_1, ..., x_n)\).

Intuition 2.23. We can think of a point in \(\text{map}_T(x_0, ..., x_n)\) as an \(n+1\)-simplex which has vertices \(x_0, x_1, ..., x_n\). In particular, a point in \(\text{map}_T(x_0, x_1, x_2)\) is the following triangle:

\[
\begin{array}{c}
x_1 \\
\sigma \downarrow \\
\downarrow \\
\sigma \downarrow \\
x_0 \xrightarrow{\sigma} x_2
\end{array}
\]

Remark 2.24. We have the commuting triangle

\[
\begin{array}{ccc}
T_n & \xrightarrow{\sim} & T_1 \times ... \times T_1 \\
\downarrow & & \downarrow \\
T_0 \times ... \times T_0
\end{array}
\]

where the top map is a trivial Kan fibration. Pulling back this equivalence along the point \((x_0, ..., x_n) : \Delta[0] \to (T_0)^{n+1}\) we get the trivial fibration.
\[ \varphi_n|_{\text{map}_T(x_0, \ldots, x_n)} : \text{map}_T(x_0, \ldots, x_n) \xrightarrow{\simeq} \text{map}_T(x_0, x_1) \times \cdots \times \text{map}_T(x_{n-1}, x_n). \]

**Intuition 2.25.** Intuitively, this map takes an \( n + 1 \)-simplex with vertices \( x_0, x_1, \ldots, x_n \) and restricts it to the spine. For example the triangle above we will be restricted to the diagram:

\[
\begin{array}{ccc}
  x_1 & \to & x_2 \\
  \downarrow & & \downarrow \\
  x_0 & \to & x_2
\end{array}
\]

**Construction 2.26.** Using this map we can give a rigorous definition of the composition map. Let us fix three objects \( x, y \) and \( z \) in \( T \). This gives us following diagram:

\[
\begin{array}{ccc}
  \text{map}(x, y, z) & \xrightarrow{d_1} & \text{map}(x, z) \\
  \downarrow (\alpha_0, \alpha_1) & \simeq & \downarrow \\
  \text{map}(x, y) \times \text{map}(y, z)
\end{array}
\]

Here \( (\alpha_0, \alpha_1) : T_2 \to T_1 \times_{T_0} T_1 \) and \( d_1 : W_2 \to W_1 \) are both restrictions of the actual maps. Now picking two morphisms \( f \in \text{map}_T(x, y), g \in \text{map}_T(y, z) \) is the same as picking a map \( \Delta[0] \to \text{map}_T(x, y) \times \text{map}_T(y, z) \). That allows us to expand our diagram to following pullback diagram:

\[
\begin{array}{ccc}
  \text{Comp}(f, g) & \xleftarrow{(f, g)} & \text{map}(x, y, z) & \xrightarrow{d_1} & \text{map}(x, z) \\
  \downarrow \simeq & & \downarrow (\alpha_0, \alpha_1) & \simeq & \downarrow \\
  \Delta[0] & \xrightarrow{(f, g)} & \text{map}(x, y) \times \text{map}(y, z)
\end{array}
\]

We can now take any point \( \nu \in \text{Comp}(f, g) \), then \( d_1 \nu \in \text{map}_T(x, z) \) is a composition morphism for \( (f, g) \). At first this definition might seem arbitrary as we are choosing a point. However, the map \( \text{Comp}(f, g) \to \Delta[0] \) is an equivalence, which means \( \text{Comp}(f, g) \) is contractible. Thus any two points \( \nu_1 \) and \( \nu_2 \) are equivalent. Thus we are justified in naming any such choice \( d_1 \nu = g \circ f \) as our composition map, with the understanding that there is a contractible space of such choices. This is the precise argument for why compositions are unique, which was already outlined in Intuition 2.13.

**Intuition 2.27.** The way to think about it is that \( f \) and \( g \) are the morphisms we are trying to compose \( d_1 \nu = g \circ f \) is the composition and \( \nu \) is the witness for that composition. This can be captured in the following diagram:
This exactly ties back to our discussion in Intuition 2.13. It is by using the properties of Kan fibrations and contractibility that can make those ideas into precise definitions.

In this subsection we focused on the categorical aspects of a Segal space in the sense that it has objects, morphisms and identity maps. Moreover, those satisfy a homotopical analogue of composition, identity rule and associativity. In the next subsection we will focus on homotopical aspects. However, there is one homotopical definition that we need right now.

**Definition 2.28.** Let \( x, y \in T \) be two objects. We say two morphisms \( f, g \in \text{map}_T(x, y) \) are homotopic and denote it by \( f \sim g \), if the two maps \( f, g : \Delta[0] \to \text{map}(x, y) \) are homotopic maps in the space \( \text{map}_T(x, y) \), as discussed in Example 1.45.

**Definition 2.29.** For every object there is a notion of an identity map. It is the image of \( x \) under the degeneracy map \( s_0 : T_0 \to T_1 \). Following standard notation, we denote the identity map of \( x \) as \( \text{id}_x \).

Having a definition of a composition and identity map in a Segal space, we can show they satisfy the right homotopical properties.

**Proposition 2.30.** Let \( f \in \text{map}_T(x, y) \), \( g \in \text{map}_T(y, z) \) and \( h \in \text{map}_T(z, w) \). Then, \( h \circ (g \circ f) \sim (h \circ g) \circ f \) and \( f \circ \text{id}_x \sim \text{id}_y \circ f \sim f \) i.e. composition is associative and has units up to homotopy.

**Proof.** We have following commutative diagram
If we take \((f, g, h) \in \text{map}_T(x, y) \times \text{map}_T(y, z) \times \text{map}_T(z, w)\) we can lift it to a \(\sigma \in \text{map}_T(x, y, z, w)\). Going the left hand map gives us \((h \circ g) \circ f\), but the right hand map gives us \(h \circ (g \circ f)\). This proves associativity.

For the identity relation, let \(f \in \text{map}_T(x, y)\), this gives us a 2-cell \(s_0(f) \in \text{map}_T(x, x, y)\), which satisfies \(\varphi_2(s_0(f)) = (\text{id}_x, f)\). Moreover, \(d_1(s_0(f)) = f\). This proves one side of the identity relation. The other side follows similarly. \(\square\)

**Remark 2.31.** For a different (but similar) way to prove the same proposition see [Re01, Proposition 5.4].

This proposition allows us construct an ordinary category out of every Segal space, confirming the connection between Segal spaces and categories.

**Construction 2.32.** Let \(T\) be a Segal space. We will define the category, called the *homotopy category* and denoted by \(\text{Ho}T\), as follows. We let the objects of \(\text{Ho}T\) to be the objects of \(T\). For two objects \(x, y \in \text{Ho}T\) we define the mapping space as

\[
\text{Hom}_{\text{Ho}T}(x, y) = \pi_0(\text{map}_T(x, y))
\]

in other words \(\text{Hom}_{\text{Ho}T}(x, y)\) is the set of path components of the space \(\text{map}_T(x, y)\).

For three objects \(x, y, z \in \text{Ho}T\), the composition map \(\text{map}_T(x, y) \times \text{map}_T(y, z) \to \text{map}_T(x, z)\) gives us a composition map

\[
\text{Hom}_{\text{Ho}T}(x, y) \times \text{Hom}_{\text{Ho}T}(y, z) \to \text{Hom}_{\text{Ho}T}(x, z).
\]

This construction gives us following theorem.

**Theorem 2.33.** Let \(T\) be a Segal space. Then \(\text{Ho}T\) is a category.

**Proof.** We already specified the objects, morphisms and composition. The previous proposition shows that this composition has identity maps and is associative. \(\square\)

### 2.4 Homotopy Equivalences

A Segal space should be an amalgamation of category theory and homotopy theory. Up to here we mostly focused on basic categorical phenomena. In this subsection we point to some homotopical aspects of a Segal space. In particular, we will discuss homotopic maps and homotopy equivalences in a Segal space. Most of the work in this subsection follows [Re01, Section 5].

In the last subsection we already discussed the definition of *homotopic morphisms*. That was made possible by the fact that we have mapping spaces (rather than sets) combined with the homotopy theory of spaces. This naturally leads to the next definition.

**Definition 2.34.** Let \(T\) be a Segal space. A morphism \(f \in \text{map}_T(x, y)\) is a *homotopy equivalence* if there exist maps \(g, h \in \text{map}_T(y, x)\) such that \(g \circ f \sim \text{id}_x\) and \(f \circ h \sim \text{id}_y\).

**Intuition 2.35.** Although it might appear as the definition only involves the existence of three maps, but in reality the nature of a Segal space demands that there are several other important pieces of information. In particular, each composition \(g \circ f\) and \(f \circ h\) has a 2-cell that witnesses the composition. Moreover, there are homotopies between the compositions and identities. The information can be captured in a diagram of the following form.
Remark 2.36. Note that Proposition 2.30 implies that
\[ g \sim g \circ id_y \sim g \circ f \circ h \sim id_x \circ h \sim h \]
and so the inverse is unique (as always only up to homotopy).

There is another way to define homotopy equivalences.

Construction 2.37. Let \( Z(3) \) be the simplicial space defined by the colimit of the following diagram.

Thus the space \( \text{Map}(Z(3), T) \) is the limit of the following diagram.

which we can also express as.

\[ \text{Map}(Z(3), T) \cong T_1^{d_1} \times_{T_0}^{d_0} T_1^{d_0} \times_{T_0}^{d_0} T_1 \]

This construction comes with the following map.

\[ (d_1 d_3, d_0 d_3, d_1 d_0) : T_3 \to T_1^{d_1} \times_{T_0}^{d_1} T_1^{d_0} \times_{T_0}^{d_0} T_1 \]

It follows from the simplicial identities that this map is well-defined, by which we mean that
\[ d_1 d_1 d_3 = d_1 d_0 d_3 \quad \text{and} \quad d_0 d_0 d_3 = d_0 d_1 d_0. \]

Also, for any \( f \in T_1 \) we have \( (s_0 d_0 f, f, s_0 d_1 f) \in T_1^{d_1} \times_{T_0}^{d_1} T_1^{d_0} \times_{T_0}^{d_0} T_1 \)

Lemma 2.38. A map \( f \) is an equivalence if and only if the element
\[ (s_0 d_0 f, f, s_0 d_1 f) \in T_1^{d_1} \times_{T_0}^{d_1} T_1^{d_0} \times_{T_0}^{d_0} T_1 \]

lifts to an element \( H \in T_3 \).
Proof. Let \( f : x \to y \) be a homotopy equivalence and \( g \) be its inverse. Then \((g,f,g) \in T_1 \times_{T_0} T_1 \times_{T_0} T_1 \simeq T_3 \) and \((d_1d_3, d_0d_4, d_1d_0)(g,f,g) = (id_x, f, id_y)\), which implies that \((g,f,g)\) is our lift. On the other side, assume that \((id_x, f, id_y)\) lifts to \(H \in T_3\). Let us denote \(d_2d_1H = g\) and \(d_0d_0H = h\). Now, \(d_3H\) gives a homotopy from the map \(gf\) to \(id_y\) and \(d_0H\) gives a homotopy from the map \(hf\) to \(id_x\). This means that \(f\) is a homotopy equivalence and so we are done. \(\square\)

Intuition 2.39. The description here can seem quite confusing and so a more detailed breakdown can be quite helpful. The element in \((id_x, f, id_y) \in T_1 \times_{T_0} T_1 \times_{T_0} T_1\) can be represented by the diagram:

\[
\begin{tikzpicture}
  \node (x) at (0,0) {x};
  \node (y) at (0,-3) {y};
  \node (x') at (-3,0) {x};
  \node (y') at (-3,-3) {y};
  \draw[->] (x) to node [left] {$id_x$} (x');
  \draw[->] (x) to node [above] {$id_y$} (y);
  \draw[->] (y) to node [below] {$f$} (x);
\end{tikzpicture}
\]

A lift to an element in \(T_3\) would imply the existence of a diagram of the following form.

\[
\begin{tikzpicture}
  \node (x) at (0,0) {x};
  \node (y) at (0,-3) {y};
  \node (x') at (-3,0) {x};
  \node (y') at (-3,-3) {y};
  \draw[->] (x) to node [right] {$id_x$} (x');
  \draw[->] (x) to node [above] {$id_y$} (y);
  \draw[->] (y) to node [below] {$f$} (x);
  \draw[->] (y) to node [left] {$\tau_1$} (x');
  \draw[->] (y) to node [right] {$\tau_2$} (x');
\end{tikzpicture}
\]

The data in this diagram is exactly that of two morphisms \(g, h : y \to x\) and two 2-cells \(\tau_1, \tau_2\) that give us the right and left inverses.

Remark 2.40. On a first look Definition 2.34 and Lemma 2.38 might seem different, but we can use the Segal condition to show that they are actually the same. Note that we have two maps

\[
\beta_1 : F(2) \to F(3)
\]
\[
\beta_2 : F(2) \to F(3)
\]
defined as \(\beta_1(0) = 0, \beta_1(1) = 1, \beta_1(2) = 2\) and \(\beta_2(0) = 1, \beta_2(1) = 2, \beta_2(2) = 3\). Notice that \(d_2\beta_1 = d_0\beta_2 : F(1) \to F(3)\), namely the map that sends 0 to 1 and 1 to 2, so we call it \(\beta_3\). Thus
this gives us following map:

\[ \beta_1 \coprod \beta_2 : F(2)^d_2 \coprod_{F(1)} F(2) \to F(3) \]

This gives us following commuting triangle.

\[ \beta_1 \coprod \beta_2 \]
\[ T_3 \]
\[ \searrow \phi_3 \simeq \nearrow (\phi_2, \phi_2) \]
\[ T_1 \times T_0 \]
\[ T_2 \times T_1 \]

By the Segal condition the down left map is an equivalence. Using the Segal condition twice also implies that the right map is an equivalence. By the Segal condition the two diagonal maps are equivalences which means the top horizontal map is an equivalence as well.

But we also have the equivalence \( F(2) \coprod_{F(1)} F(2) \cong F(1) \times F(1) \), which gives us equivalences

\[ T_3 \simeq T_2 \times T_1 \]
\[ T_2 \simeq \text{Map}(F(1) \times F(1), T) \]

This means that in a Segal space there is an equivalence between \( T_3 \) and maps of squares into \( T \). Therefore, we can impose the right conditions on either shape to define a notion of equivalence.

This second method gives us an easy way of showing that the notion of a homotopy equivalence is homotopy invariant.

**Lemma 2.41.** Let \( \gamma : \Delta[1] \to T_1 \) be a path from \( \gamma(0) = f \in T_1 \) to \( \gamma(1) = f' \in T_1 \) and \( f' \) is a homotopy equivalence. Then \( f \) is also an homotopy equivalence.

**Proof.** We have the following diagram.

\[ \Delta[0] \]
\[ \Delta[1] \]
\[ \Delta[3] \]
\[ \Delta[4] \]
\[ \Delta[5] \]

This diagram lifts because the right-hand map is a fibration. Thus \( \tilde{\gamma}(1) \) is the lift of \( (s_0d_0g, g, s_0d_1g) \) which we were looking for. \( \square \)
Definition 2.42. We define $i : T_{\text{hoequiv}} \hookrightarrow T_1$ as the subspace generated by the set of homotopy equivalences and call it the space of homotopy equivalences.

Remark 2.43. Lemma 2.41 implies that this map is $(-1)$-truncated (Definition 1.55).

There are other, yet equivalent ways to understand the space $T_{\text{hoequiv}}$ and its $(-1)$-truncated map from $T_{\text{hoequiv}} \to T_1$.

Theorem 2.44. In the following pullback diagram

\[
\begin{array}{ccc}
T_{\text{hoeqchoice}} & \xrightarrow{\gamma} & T_3 \\
\downarrow{i} & & \downarrow{j} \\
T_{\text{hoequiv}} & \xrightarrow{(s_0d_0, \text{id}_{T_1}, s_0d_1)} & T_1 \times_{T_0} T_1 \times_{T_0} T_1 \\
\end{array}
\]

the map $j$ factors through $i$ and the resulting map $U$ is a homotopy equivalence.

Proof. In order to prove our result we have to show that for every homotopy equivalence $f : x \to y \in T_{\text{hoequiv}}$ the fiber of $U$ over $f$, which we call $HEquiv(f)$, is a contractible space. This means we are looking at following pullback squares.

\[
\begin{array}{ccc}
HEquiv(f) & \xrightarrow{\gamma} & T_{\text{hoeqchoice}} \\
\downarrow{\Delta[0]} & & \downarrow{j} \\
\Delta[0] & \xrightarrow{f} & T_{\text{hoequiv}} \times_{T_1} T_1 \times_{T_0} T_1 \\
\end{array}
\]

$HEquiv(f)$ is really the subspace of $T_3$ generated by all point $H$ such that $(d_1d_3, d_0d_3, d_1d_0)(H) = (\text{id}_x, f, \text{id}_y)$. We already know that $f$ is an equivalence which means there exists maps $g, h$ such that $f \circ g$ and $h \circ f$ are equivalent to identity maps, which implies that $HEquiv(f)$ is non-empty.

Finally, we also have following homotopy pullback square:

\[
\begin{array}{ccc}
HEquiv(f) & \xrightarrow{\gamma} & T_3 \\
\downarrow{\Delta[0]} & & \downarrow{\varphi_3} \\
\Delta[0] & \xrightarrow{(h,f,g)} & T_1 \times_{T_0} T_1 \times_{T_0} T_1 \\
\end{array}
\]
This follows from the fact that $\varphi_3$ is a Kan fibration and so the pullback is homotopy invariant combined with the fact that any choice of inverses for $f$ are themselves equivalent maps (Remark 2.36). But the fiber of each trivial Kan fibration is itself contractible and hence we are done. □

**Intuition 2.45.** The idea of the proof is that $T_{\text{hoequiv}}$ is the space consisting of all maps that are equivalences in the sense that some inverses exist, whereas $T_{\text{hoeqchoice}}$ is the space of all maps with specifically chosen inverses. The map $U : T_{\text{hoeqchoice}} \to T_{\text{hoequiv}}$ forgets the specific chosen inverse and only remembers the map that is an equivalence. The proof above basically says that up to homotopy there is only one way to find inverses for an equivalence. This is in line with the philosophy we layed out in Remark 1.51.

**Remark 2.46.** The concept of $T_{\text{hoequiv}}$ as the subspace of $T_1$ was introduced in [Re01, Subsection 5.7]. We used that definition to define and study $T_{\text{hoeqchoice}}$.

Viewing the space of equivalences as a pullback gives us a more systematic way to study it. We can even simplify the pullback diagram to make computations of $T_{\text{hoeqchoice}}$ simpler.

**Lemma 2.47.** The following is a pullback square:

$$
\begin{array}{ccc}
T_1^{(s_0d_0, id_{T_1}, s_0d_1)} & \rightarrow & T_1^{d_1 \times d_1, T_1^{d_0 \times d_0}} \times T_1^{d_0} \\
(d_0, d_1) & \downarrow & \downarrow \pi_1, \pi_3 \\
T_0 \times T_0 & \rightarrow & T_1 \times T_1
\end{array}
$$

Thus the following is a pullback square:

$$
\begin{array}{ccc}
T_{\text{hoeqchoice}} & \rightarrow & T_3 \\
& \downarrow & \downarrow (d_1d_3,d_1d_0) \\
T_0 \times T_0 & \rightarrow & T_1 \times T_1
\end{array}
$$

There are two last definitions related to homotopy equivalences that play important roles.

**Definition 2.48.** We say a Segal space $T$ is a *Segal space groupoid* if every map is a homotopy equivalence.

We can also have a local definition of homotopy equivalences.

**Definition 2.49.** For every two objects $x$ and $y$ we can define the *space of homotopy equivalences between $x$ and $y$, $hoequiv_T(x, y)$*, as the pullback
as the fiber of the map \((d_0, d_1) : T_{\text{hoequiv}} \to T_0 \times T_0\) over the point \((x, y)\) and the lemma implies that the natural inclusion map \(\text{hoequiv}_T(x, y) \hookrightarrow \text{map}_T(x, y)\) is \((-1)\)-truncated.

### 2.5 Examples of Segal Spaces.

Now that we spent some time developing the category theory of Segal spaces, it is a good idea to see some examples and realize how each of the previous definitions manifest in those particular examples.

**Example 2.50.** Let \(\mathcal{C}\) be a category. Then \(i^\ast_F(N\mathcal{C})\) is a discrete simplicial space. The discreteness implies that it is Reedy fibrant. Moreover, it satisfies the Segal condition by definition. Thus \(i^\ast_F(\mathcal{C})\) is a Segal space.

Fortunately, the definitions we are used to from category theory perfectly match up with the ones for a Segal space. In particular, as object in the Segal space \(i^\ast_F(N\mathcal{C})\) is just an object in the category \(\mathcal{C}\). Same is true for morphisms.

However, as \(i^\ast_F(N\mathcal{C})_1\) is just a set, the mapping space is actually just a set as well. This in particular implies that composition is well-defined not just up to homotopy. In fact for any collection of objects \(x_0, \ldots, x_n \in i^\ast_F(N\mathcal{C})\). The space \(\text{map}(x_0, ..., x_n)\) is bijective to \(\text{map}(x_0, x_1) \times ... \times \text{map}(x_{n-1}, x_n)\). Thus the pullback

\[
\begin{array}{ccc}
\text{Comp}(f, g) & \longrightarrow & \text{map}(x_0, x_1, x_2) \\
\downarrow & & \downarrow \\
\Delta[0] & \longrightarrow & \text{map}(x_0, x_1) \times \text{map}(x_1, x_2)
\end{array}
\]

is not just contractible, but actually just a point.

In addition to all of these, as \(\text{map}(x_0, x_1)\) is just a set, two maps are homotopic if and only if they are equal to each other. This implies that a map is a homotopy equivalence if and only if it is an isomorphism.

In particular, the homotopy category of the Segal space, \(\text{HoNC}\) is exactly \(\mathcal{C}\), as the two categories have the same objects and same morphisms.

**Remark 2.51.** As expected in the case of an ordinary category, the corresponding Segal space has all the category theory we desire, but has no valuable homotopical information.

**Example 2.52.** Let \(K\) be a space. Our first guess might be to take \(i^\ast_\Delta(K)\). While it does satisfy the Segal condition, but it is not Reedy fibrant! Fortunately, there is an equivalent simplicial space that is Reedy fibrant. Namely, let \(K^\Delta[\bullet]\) be the simplicial space defined as

\[
\begin{array}{ccc}
hoequiv_T(x, y) & \longrightarrow & T_{\text{hoequiv}} \\
\downarrow & & \downarrow \\
\Delta[0] & \longrightarrow & (d_0, d_1)
\end{array}
\]
where the boundary maps are induced by the maps between the simplices. This simplicial space is actually Reedy fibrant. Moreover it satisfies the Segal condition as in the diagram

\[
\begin{array}{c}
K_{n} \
\downarrow \cong \\
K^{\Delta[n]} \\
\downarrow \\
\cong \\
\downarrow \\
K_{0} \times K_{0} \times \cdots \times K_{0}
\end{array}
\]

the vertical maps are equivalences, which means the horizontal map is also an equivalence.

How does the category theory of a Segal space look like in this case? An object in this Segal space is a point in \( K \). A morphism is now a point in the path space \( K^{\Delta[1]} \) and so is just a path in the space. Composition of morphisms corresponds to concatenation of paths in the space. Notice here we really the contractibility condition. In other words, when we concatenate two paths then we get another path that is determined only up to homotopy.

Two paths are homotopic in the mapping space if they are homotopic in the usual sense for spaces. As every path in a space is reversible, we see that every morphism is an equivalence. Thus a space is an example of a Segal space groupoid (Definition 2.48).

Notice that the homotopy category of this Segal space is the category which has objects the points in \( K \) and has morphisms homotopy classes of paths in \( K \). This category is commonly called the fundamental groupoid of \( K \) and is denoted by \( \Pi(K) \).

**Example 2.53.** Let us see one non-example. The simplicial space \( G(n) \) is not a Segal space, although it is Reedy fibrant. For the case \( n = 2 \) we can see this directly as \( G(2) \) is the following simplicial space:

\[
\begin{array}{c}
\{0, 1, 2\} \\
\downarrow \\
\{00, 01, 11, 12, 22\} \\
\downarrow \\
\{000, 001, 011, 112, 122, 222\} \\
\downarrow \\
\cdots
\end{array}
\]

where the numbers indicate how the simplicial maps act. Thus \( d_{i} \) drops the \( i \)th digit. So, we have

\[
G(2)_{1} \times_{G(2)_{0}} G(2)_{1} = \{(00, 00), (00, 01), (01, 11), (01, 12), (11, 11), (11, 12), (12, 22), (22, 22)\}
\]

So, clearly \( G(2) \) is not equivalent to \( G(2)_{1} \times_{G(2)_{0}} G(2)_{1} \) as \( G(2) \) has 7 elements and the other has 8 elements.

Concretely, \( G(2)_{1} \times_{G(2)_{0}} G(2)_{1} \) has the element \((01, 12)\) which wants to be composed to a \( 012 \) in \( G(2)_{2} \), which is the element in \( F(2)_{2} \) that is missing in \( G(2)_{2} \).

### 2.6 Why are Segal Spaces not Enough?

Until now we defined Segal spaces and showed how we can use them to define all kinds of categorical concepts. In the last subsection we will see where a Segal space falls short of what we expect.
The problem with a Segal space can be summarized thusly: A Segal space has a category theory and has a homotopy theory, however, they are not compatible with each other which causes major problems. We will lay out the case in several examples that focus on a central theme. Before we can do so we have to discuss one important construction.

**Construction 2.54.** Let $I(1)$ be the category which has two objects and one invertible arrow. We want to carefully understand the Segal space $i^*_p(NI(1))$, which we will denote as $E(1)$. Clearly it is a discrete simplicial space. We can describe it explicitly as

$$E(1)_n = \{x, y\}^{[n-1]}.$$ 

More concretely an element in the set $E(1)_n$ is a map from the set $\{0, ..., n - 1\}$ to the set $\{x, y\}$. Thus $E(1)_n$ has exactly $2^n$ elements. At the lower levels we can give a more explicit description.

$E(1)_0$ has two elements, $x$ and $y$, which correspond to the two objects in $I(1)$. $E(1)_1$ has four elements which can be depicted as $\{xx, xy, yx, yy\}$, where $xx$ and $yy$ correspond to the identity map, and $xy$ is a morphism from $x$ to $y$ that has inverse $yx$. This trend continues in the higher levels.

**Example 2.55.** The category $I(1)$ is equivalent to the category $[0]$, which has only one object. However, the corresponding Segal spaces $i^*_p(I(1)) = E(1)$ and $i^*_p(\Delta[0]) = F(0)$ are clearly not equivalent Segal spaces, as $E(1)$ is not level-wise contractible.

**Intuition 2.56.** What essentially happened here is that the category theory has an underlying homotopy theory of groupoids ($I(1)$ is a groupoid), which is completely ignored and thus missed by the Segal space.

**Example 2.57.** Let us go back to $E(1)$ once more. It is a discrete Segal space, with two objects $x, y$. Moreover, it has two morphisms, $xy, yx$ which are inverses of each other. Thus the two objects are equivalent to each other in the sense that there is a homotopy equivalence between them. However, they are NOT equivalent in the space $E(1)_0$, as there is no path between them.

**Intuition 2.58.** Here we see a clear mismatch between homotopy theory and category theory. Categorically the two points are equivalent, but homotopically they are not.

**Example 2.59.** One familiar fact from category theory is the following. A functor $F : C \to D$ is an equivalence if and only if:

1. $F$ is fully faithful, meaning that for any two objects $x, y$

$$\text{Hom}_C(x, y) \to \text{Hom}_D(Fx, Fy)$$

is a bijection.

2. $F$ is essentially surjective, meaning that for any object $d \in D$, there is an object $c \in C$ such that $Fc$ is equivalent to $d$.

However, this condition does not behave well for Segal spaces. As a clear example, any map $F(0) \to E(1)$ satisfies both conditions stated above, however we do not get an equivalence of Segal spaces.

**Intuition 2.60.** As in the previous example, the problem is that $x$ and $y$ are equivalent in the Segal space $E(1)$, but as points in the space $E(1)_0$ they are not homotopic.
Example 2.61. In Definition 2.48 we defined a Segal groupoid as a Segal space in which every morphism is an equivalence. In Example 2.52 we discussed how every spaces gives us a Segal groupoid. However, the opposite is not true, as indicated by the existence of $E(1)$, which is a Segal groupoid, but not equivalent to a space.

Intuition 2.62. This example is contrary to our understanding of higher category theory. Intuitively, a higher category has homotopical data and categorical data. However, in a groupoid every morphism is invertible, which means it does not contain any non-trivial categorical data. Therefore, our notion of groupoid should really correspond to just a space. The idea we just explained is commonly called the homotopy hypothesis and is one guiding idea in the realm of higher category theory.

Seeing those examples we realize that we need to impose one additional condition to make sure the homotopy and category theory of groupoid work well with each other.

**Complete Segal Spaces**

The goal of this section is to define and study complete Segal spaces. The notion of a complete Segal space relies on the notion of completeness that was defined in [Re01, Section 6].

3.1 Defining Complete Segal Spaces. In order to define a complete Segal space we first need to review some concepts related to homotopy equivalences.

Construction 3.1. Recall that for any Segal space $T$ we get the space $T_{hoequiv}$, which is the subspace of $T_1$ consisting of all homotopy equivalences. There is a natural map $s_0 : T_0 \to T_1$, which takes each object to the identity map. However an identity map is a homotopy equivalence. Thus the map will factor through $T_{hoequiv}$, which gives us following diagram

$$
T_0 \xrightarrow{s_0} T_{hoequiv} \xrightarrow{i} T_1.
$$

The map $s_0$ is always an injection, however, it does not have to be surjective.

Example 3.2. In the Segal space $E(1)$, we have two objects and so $E(1)_0 = \{x, y\}$, but four homotopy equivalences $\{xx, xy, yx, yy\}$ (Construction 2.54). Thus the map from objects to equivalences is clearly not surjective.

In order to fix this problem we give the next definition.

Definition 3.3. A complete Segal space (CSS) is a Segal space $W$ for which the map

$$s_0 : W_0 \to W_{hoequiv}$$

described above is an equivalence.

There are several other equivalent ways to define a complete Segal space.

Lemma 3.4. Let $W$ be a Segal space. The following are equivalent.

1) $W$ is a complete Segal space.
(2) The following is a homotopy pullback square.

\[
\begin{array}{ccc}
W_0 & \longrightarrow & W_3 \\
\downarrow & & \downarrow \\
W_1 & \longrightarrow & W_1 \times W_1
\end{array}
\]

(3) The map of spaces

\[\text{Map}(E(1), W) \to \text{Map}(F(0), W)\]

is a weak equivalence.

(4) For any two objects \(x, y\) the natural map

\[\Delta[0] \times W_0^{\Delta[1]} \times W_0 \Delta[0] \to \text{hoequiv}_W(x, y)\]

is a equivalence of spaces.

**Proof.** The proof is mostly comparing various definitions. 

(1 \iff 2) The actual pullback is the space \(W_{\text{hoequiv}}\choice\), which is equivalent to \(W_{\text{hoequiv}}\). Thus the diagram is a homotopy pullback if and only if \(W_0\) is equivalent to \(W_{\text{hoequiv}}\), which is exactly the completeness condition.

(1 \iff 3) The space \(W_{\text{hoequiv}}\) is equivalent to \(\text{Map}(E(1), W)\), thus \(W\) is complete if and only if

\[W_{\text{hoequiv}} \simeq \text{Map}(E(1), W) \to \text{Map}(F(0), W) = W_0.\]

(1 \iff 4) The map \(W_0 \to W_{\text{hoequiv}}\) is an equivalence if and only if for each two points \(x, y\) the map

\[\Delta[0] \times W_0^{\Delta[1]} \times W_0 \Delta[0] \to (W_{\text{hoequiv}})^{\Delta[1]} \times W_{\text{hoequiv}} \Delta[0] = \text{hoequiv}_W(x, y).\]

\[\square\]

**Intuition 3.5.** The completeness condition exactly addresses the problems we raised in Subsection 2.6. By adding the condition that every equivalence in the Segal space in \(W\) can be represented by a path in \(W_1\), we are making sure that homotopic points in \(W_0\) correspond to equivalent points in the Segal space \(W_1\).

Thus a complete Segal space is now a bisimplicial set where

(1) The vertical axis has a homotopical behavior (Reedy fibrancy condition)
(2) The horizontal axis has a categorical behavior (Segal condition)
(3) The two interact well with each other (Completeness condition)

This is exactly the definition we had been working towards from the start.

**Notation 3.6.** Henceforth we will use the short form \(\text{CSS}\) to describe a complete Segal space.
CSS satisfy several helpful conditions, some of which are analogues to the conditions a Segal space satisfied and some of which correct the problems we brought up in Subsection 2.6.

**Remark 3.7.** Let \( W \) be a CSS and \( X \) any simplicial space. Then \( W^X \) is also a CSS.

**Theorem 3.8.** Let \( f : W \rightarrow V \) be a map of CSS. The following are equivalent:

1. \( f \) is a level-wise equivalence, meaning \( f_n : W_n \rightarrow V_n \) is an equivalence of spaces.
2. \( f \) is fully faithful and essentially surjective.
   
   (I) **Fully Faithful:** For any two objects \( x, y \in W \) the induced map of spaces
   
   \[
   \text{map}_W(x, y) \rightarrow \text{map}_V(fx, fy)
   \]
   
   is an equivalence of spaces.

   (II) **Essentially Surjective:** For any object \( y \in V \) there is an object \( x \in W \) such that \( fx \) is equivalent to \( y \) in \( V \).

In Definition 2.48 we defined a Segal space groupoid. Analogously we can define a complete Segal space groupoid as a CSS in which every morphism is an equivalence. The next proposition confirms the homotopy hypothesis we discussed in Intuition 2.62.

**Proposition 3.9.** A CSS \( W \) is a CSS groupoid if and only if \( W \) is homotopically constant.

**Proof.** If \( W \) is homotopically constant then \( s_0 : W_0 \rightarrow W_1 \) is an equivalence of spaces. This means that every map is an equivalence. On the other side, if \( W \) is a CSS groupoid then \( s_0 : W_0 \rightarrow W_1 \) is an equivalence. This means that the maps \( d_0, d_1 : W_1 \rightarrow W_0 \) are also equivalences. This implies that \( W_1 \times_{W_0} \cdots \times_{W_0} W_1 \simeq W_1 \). By the Segal condition this implies that \( W_n \simeq W_n \) and so \( W \) is homotopically constant. \( \square \)

Our next goal is to discuss how we can build a CSS out of a category. Until now we used the horizontal embedding of the nerve, \( i^*_p N \), however, while it does give us a Segal space, it might not be complete. As we already discussed before \( E(1) \) is not a CSS as the two objects are equivalent in the Segal space, but not connected by a path in \( E(1)_0 \). Thus we have to completely change our approach.

The problem is that the embedding functor \( i^*_p \) only considers the categorical aspect of the underlying category, but completely ignores the homotopy theory. Thus there is no way to get a CSS. The way to adjust things is to consider the category and homotopy theory at the same time. In order to achieve that we need a completely new construction.

Before we can do so we need several important definitions.

**Definition 3.10.** A relative category \( (\mathcal{C}, W) \) is a category \( \mathcal{C} \) along with a subcategory \( W \) that satisfies following conditions:

1. Every object in \( \mathcal{C} \) is an object in \( W \).
2. Every isomorphism in \( \mathcal{C} \) is a morphism in \( W \).

**Intuition 3.11.** Intuitively, a relative category is a category that has some homotopical information in the sense that morphisms that are in \( W \) play the role of “weak equivalences”. Notice, there is
no notion of homotopy and these maps are not invertible up to some homotopy condition. Rather, this is an intuition on how to think about relative categories.

**Definition 3.12.** Let \( C \) be a category. We define \( C^{\text{core}} \) as the category which has the same objects, but only has invertible morphisms between any two objects. By definition it is the maximal subcategory of \( C \) that is a groupoid, or, in other words, the maximal subgroupoid of \( C \).

**Example 3.13.** Let \( C \) be any category, then \((C, C^{\text{core}})\) is a relative category.

**Definition 3.14.** Let \((C, W)\) be a relative category and \( D \) any category. Then we define the category \( \text{we}(C^D) \) as the category which has as objects functors \( F: D \to C \) and as morphisms natural transformations \( \alpha \) such that for every object \( d \), the morphisms \( \alpha_d \in C \) is actually a morphism in \( W \).

**Construction 3.15.** We are now in a position to construct a new and improved version of the nerve construction that takes the homotopy theory of a category into account. Right now we will do our construction for a relative category.

Let \((C, W)\) be a relative category. We define a simplicial space \( N(C, W) \) as follows.

\[
N(C, W)_n = N(\text{we}(C^{[n]}))
\]

The necessary simplicial maps of the simplicial space follow from the maps between the various \([n]\). We call \( N(C, W) \) the *classification diagram* of the relative category \((C, W)\).

**Intuition 3.16.** This construction is a good illustration of what a CSS is and so it is worth dwelling over. For a relative category \((C, W)\) we use following notation:

1. \( \bullet \): For objects
2. \( \rightarrow \): For morphisms
3. \( \longrightarrow \): For morphisms that are in the subcategory \( W \).

With this notation the classification diagram has the form of the following simplicial space:
Thus the vertical direction focuses on the subcategory $W$, which we think of as the homotopical direction, whereas the horizontal direction focuses on the whole category which is clearly the categorical direction.

In particular, we have following important special case.

**Example 3.17.** Let $\mathcal{C}$ be a category. We define the *classifying diagram* of $\mathcal{C}$,

$$N\mathcal{C} = N(\mathcal{C}, \mathcal{C}_{\text{core}}).$$

**Remark 3.18.** There is a more concrete way to define the classifying diagram of a category $\mathcal{C}$. Let $I(n)$ be the category with $n + 1$ objects and exactly one isomorphism between every two objects. We can define the $(n,m)$-simplices directly as follows:

$$N(\mathcal{C})(n, l) = \text{Hom}_{\text{Cat}}([n] \times I(l), \mathcal{C}).$$

**Intuition 3.19.** It is very helpful to consider the diagram above for this case. In particular, at the zero level $N(\mathcal{C})_0$ is just the core $\mathcal{C}_{\text{core}}$ as it is just the subcategory of all isomorphisms. Thus two objects are equivalent in the category if and only if there is a path in between them at the zero level.

**Remark 3.20.** The notion of a classification diagram was introduced in [Re01, Section 3] as an improvement to the nerve construction.

Having improved our definition of a nerve, we have following result.

**Theorem 3.21.** [Re01, Equation 3.6, Lemma 3.9, Proposition 6.1] Let $\mathcal{C}$ be a category, then $N(\mathcal{C})$ is a CSS.
**Proof.** The Reedy fibrancy condition is a technical condition and the proof can be found in [Re01, Lemma 3.6]. The Segal condition follows from the fact that the simplicial set $N\mathcal{C}$ satisfies the Segal condition. Thus $N(\mathcal{C})$ satisfies the Segal condition level-wise.

For the last part notice that a morphism in $N(\mathcal{C})$ is an equivalence if it is an isomorphism in $\mathcal{C}$. Thus $N(\mathcal{C})_{\text{hoequiv}}$ inside $N(\mathcal{C})_1 = N((\mathcal{C}^{[1]})^{\text{core}})$ is equivalent to the subcategory $N(\mathcal{C}^{(1)})^{\text{core}}$. Moreover, we know that $\mathcal{C}^{\text{core}}$ is categorically equivalent to $((\mathcal{C}^{[1]})^{\text{core}})$. Thus we have following diagram of equivalences

$$N(\mathcal{C})_0 = N(\mathcal{C}^{\text{core}}) \cong N((\mathcal{C}^{[1]})^{\text{core}}) \cong (N\mathcal{C})_{\text{hoequiv}}$$

Hence, $N(\mathcal{C})$ satisfies the completeness condition. \hfill \Box

**Functoriality in Higher Categories: Fibrations**

When studying categories, it does not suffice to just study them in isolation, rather we also want to understand how they relate to each other. That is why we introduce functors. We want to do the same in the realm of higher category theory. The goal of this section is to study functoriality of higher categories via the theory of fibrations. We start by motivating and reviewing the theory of fibrations and then move on to see some interesting examples.

**4.1 Why Functors Fail in Higher Categories.** In category theory functors are used effectively to understand the relation between categories. However, as is often the case, the approach we are familiar with does not work for higher categories and needs to be refined. This is best witnessed by the following example.

**Example 4.1.** Let $W$ be a CSS and $x$ an object in $W$. From our experience with classical category theory, we expect a functor

$$W \to \text{Spaces}$$

$$y \mapsto \text{map}_W(x,y)$$

Clearly, we can define the map above on an object level, but we also need a way to deal with the functoriality. In particular, for any map $f : y_1 \to y_2$ we need a corresponding map

$$\text{map}_W(x,y_1) \to \text{map}_W(x,y_2).$$

But all we have is the following diagram

$$\text{map}_W(x,y_1,y_2) \xrightarrow{} \text{map}_W(x,y_2)$$

$$\Downarrow \simeq$$

$$\text{map}_W(x,y_1) \xrightarrow{(id,f)} \text{map}_W(x,y_1) \times \text{map}(y_1,y_2)$$

which means we have no direct map from $\text{map}_W(x,y_1)$ to $\text{map}_W(x,y_2)$, but rather a zig-zag. Thus we cannot just define a functor from $W$ to the CSS of $\text{Spaces}$.

**Intuition 4.2.** It’s clear where the problem lies: composition. In a classical category composition is unique and leaves us with no choice, which allows us to define a functor. In a CSS composition is only defined up to contractible choice and that choice prevents us from actually getting a functor.
Remark 4.3. Note we never actually defined the CSS of spaces. However, the problem we described above arises regardless of how we define this CSS.

Remark 4.4. Because of this issue we also cannot just generalize the Yoneda lemma to higher categories, as we need a proper notion of representable functors first.

When we can relax our composition condition in the world of higher categories, not why not relax the functoriality condition as well? This leads us to the study of fibrations.

4.2 Fibrations in Categories. Our goal is to relax the functoriality condition in order to get a functioning notion of a functor suited for higher categories. Unfortunately, that is quite difficult for functors. Fortunately, however, there is a different way to think about functors, which can readily be generalized to a more general setting, namely the theory of fibrations. Thus in this subsection we review the theory of fibrations for ordinary categories.

Definition 4.5. A functor $p : D \to C$ is cofibered in sets if for any map $f : x \to y$ in $C$ and any object $x' \in D$ such that $p(x') = x$, there exists a unique lift $f' : x' \to y$ in $D$, such that $p(f') = f$.

Remark 4.6. If we let $f = \text{id}_x \in C$ and choose a target $x' \in D$ such that $p(x') = x$, then clearly $p(\text{id}_x) = \text{id}_x$. Thus by uniqueness, the fiber over each $x$ has to be a set. This in particular implies that any functor $p : D \to \Delta[0]$ is cofibered in sets if and only if $D$ is a set.

Intuition 4.7. How does this definition model functoriality? We can see this by taking $C = [1]$ and analyzing a functor $p : D \to [1]$ that is cofibered over sets. First some notation. Let $S_0$ be the subcategory of $D$ that maps to $0 \in [1]$ and $S_1$ be the subcategory that maps to $1 \in \Delta[1]$. By the previous remark both of those are sets.

Now, the fibration conditions says that for every choice of point $x \in S_0$, there is a unique map in the category $D$ that starts at $x$ and ends with an object $y \in S_1$. Thus we have a unique way to assign a value in $S_1$ for every point in $S_0$. That is the definition of a function of sets $f : S_0 \to S_1$.

We can depict this situation as follows.

```
0  01  1

x • • • • • f • • • • • y

• • • • • •
```

Having a function between sets, we can now define the functor

$$[1] \to \text{Set}$$

that maps 0 to the set $S_0$, 1 to the set $S_1$ and morphism 01 to the map $f : S_0 \to S_1$.

Example 4.8. The key example for a functor that is fibered in sets is the notion of an undercategory. For each object $c \in C$, we have the over category $C/c$, which has objects all maps $c \to d$ (maps with domain $c$) and morphisms commuting triangles of the form:
The natural projection map $p : C_c/ \to C$, which takes $c \to d$, is fibered in sets. Indeed, if we take any map $f : d_1 \to d_2$ in $C$ and chosen lift $c \to d_1$ there is a unique arrow in $C_{c/}$ namely the triangle depicted above that lifts $f$ to the category $C_{c/}$.

There is a much more rigorous way to see how to get a category cofibered sets out of a functor. This method is commonly called the \textit{Grothendieck construction}.

\textbf{Definition 4.9.} Let $F : C \to \text{Set}$ be a functor. We define the category $\int_C F$, called the Grothendieck construction, in the following way:

- **Objects**: An object is a tuple $(c, x)$ such that $c \in C$ is an object and $x \in F(c)$.
- **Morphisms**: For two objects $(c, x)$ and $(d, y)$, we define the maps as

$$\text{Hom}_{\int_C F}((c, x), (y, d)) = \{f \in \text{Hom}_C(c, d) : F(f)(x) = y\}.$$ 

Note that $F$ is a functor, thus we get a map of sets $F(f) : F(c) \to F(d)$.

The category comes with a natural functor $p : \int_C F \to C$, which maps each tuple $(c, x)$ to $c$.

Let us see one important example

\textbf{Example 4.10.} Let $\mathcal{Y}_x : C \to \text{Set}$ be the functor represented by $c$. Applying the Grothendieck construction we get

$$\int_C \mathcal{Y}_x = C_{x/}.$$ 

Thus, we can in some sense think of $C_{x/}$ as the \textit{representable functor cofibered in sets}.

The example suggests following result.

\textbf{Theorem 4.11.} For any functor $F : C \to \text{Set}$, the Grothendieck construction $p : \int_C F \to C$ is a functor cofibered in sets.

\textit{Proof.} Let $f : c \to d$ be a map in $C$ and $(c, x)$ a chosen lift of $c$ in $\int_C F$. Then there is a unique lift of $f$, namely $f : (c, x) \to (d, F(f)(x))$. \hfill $\square$

This theorem shows that we can transfer the whole concept of set valued functors into the language of functors fibered in sets and we even have a notion of a representable functor. This perspective on functoriality even comes with its own Yoneda lemma:
Lemma 4.12. (Yoneda Lemma for cofibered Categories). For any functor cofibered in sets \( p : \mathcal{D} \to \mathcal{C} \), we have following equivalence:

\[
(F_c)^* : \text{Fun}_C(\mathcal{C}/c, \mathcal{D}) \xrightarrow{\sim} \text{Fun}_C([0], \mathcal{D})
\]

where the map comes from precomposing by the functor \( F_c : [0] \to \mathcal{C}/c \), which maps the point to the identity map \( \text{id}_c : c \to c \).

The notion of functoriality introduced in this section relies on the existence of unique lifts. But we know exactly how to adjust the notion of uniqueness to get a functioning definition for higher categories, namely by replacing uniqueness with contractibility. The goal of the next subsection is to take that idea to show we this gives us a well-defined notion of a functor valued in spaces.

4.3 Left Fibrations. The idea of a left fibration is a way to generalize functors fibered in sets we discussed above and models functors valued in spaces. They key is to have unique lifts up to homotopy instead of demanding unique lifts. In other words there should be a space of lifts which is contractible. The material in this section is a summary of [Ra17a], which is a rigorous treatment of the theory of left fibrations.

Definition 4.13. A Reedy fibration \( p : L \to W \) between CSS is a left fibration if the following is a homotopy pullback square

\[
\begin{array}{ccc}
L_1 & \xrightarrow{p_1} & L_0 \\
\downarrow{s} & & \downarrow{s} \\
W_1 & \xrightarrow{p_0} & W_0
\end{array}
\]

where the map \( s : W_1 \to W_0 \) is the source map that takes each arrow to its source.

Remark 4.14. This is equivalent to saying that the map

\[
L_1 \xrightarrow{\sim} L_0 \times_{W_0} W_1
\]

is a trivial Kan fibration.

Intuition 4.15. How is this the proper generalization of functors cofibered in sets described in the previous subsection? We have following pullback diagram of spaces:

\[
\begin{array}{ccc}
\text{Lift}(x', f) & \xrightarrow{\sim} & L_1 \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
\Delta[0] & \xrightarrow{(x', f)} & L_0 \times_{W_0} W_1
\end{array}
\]

The map \( (x', f) : \Delta[0] \to L_0 \times_{W_0} W_1 \) picks a map \( f : x \to y \) in \( W \) (which is exactly what a point in \( W_1 \) is, an object \( x' \in L \) (a point in \( L_0 \)) such that \( p(x') = x \).
The space \( \text{Lift}(x', f) \) consists of all morphisms in \( f' : x' \to y' \) in \( L \) (points in \( L_1 \)) such that \( p(f') = f \). The cofibered of sets condition from the previous subsection exactly stated that such a map \( f' \) is unique. In the higher categorical situation \( \text{Lift}(x', f) \) is contractible as it is a pullback of an equivalence. Thus the definition of a left fibration exactly replaces the unique lifting condition with contractible lifting condition.

Let us see some important examples.

**Example 4.16.** Let \( W = F(0) \) (i.e. the point). Let \( L \to F(0) \) be a left fibration over \( F(0) \). Then the left fibration condition implies that

\[
L_1 \xrightarrow{\cong} L_0 \times_{(F(0))_0} F(0)_n = L_0.
\]

So, \( L \) is a just a homotopically constant CSS or in other words only has the data of a space. Thus a left fibration over the point is really equivalent to a map from the point into spaces, which is just a space.

**Example 4.17.** Let us see a more interesting example. Let \( L \to F(1) \) be a left fibration over \( F(1) \). We think of \( F(1) \) as the discrete simplicial space for which \( F(1)_n = \text{Hom}([n],[1]) \), thus for the first levels we have

\[
\{0,1\} \xleftarrow{\cong} \{00,01,11\} \xrightarrow{\cong} \{000,001,011,111\} \xrightarrow{\cong} \cdots.
\]

Given that \( F(1) \) is a discrete simplicial space (each level is a set), the map \( L \to F(1) \) can be expressed as a disjoint union of the fibers over each point in the underlying set. For the purposes of this example, we express the fiber over the point \( i \) as \( L/i \). So, the fiber over the point 001 is expressed as \( L/001 \). With this notation, \( L \) can be expressed as following simplicial space:

\[
L/0 \coprod L/1 \xleftarrow{\cong} L/00 \coprod L/01 \coprod L/11 \xrightarrow{\cong} L/000 \coprod L/001 \coprod L/011 \coprod L/111 \xrightarrow{\cong} \cdots.
\]

The left fibration condition implies that the following is a homotopy pullback square.

\[
\begin{array}{c}
L/00 \coprod L/01 \coprod L/11 \xrightarrow{\cong} L/0 \coprod L/1 \\
\downarrow \ \\
\{00,01,11\} \xrightarrow{\cong} \{0,1\}
\end{array}
\]

As the spaces are themselves disjoint unions of smaller spaces, the equivalence breaks down into following three equivalences:

\[
L/00 \xrightarrow{\cong} L/0 \times_{\{0\}} \{\} \cong L/0
\]

\[
L/01 \xrightarrow{\cong} L/0 \times_{\{0\}} \{\} \cong L/0
\]

\[
L/11 \xrightarrow{\cong} L/1 \times_{\{1\}} \{\} \cong L/1
\]

Thus we can disregard \( L/00 \) and \( L/11 \) as they are equivalent to \( L/0 \) and \( L/1 \) respectively. What remains is the following zig-zag
where the left map is an equivalence by the left fibrancy condition. So we have one non-equivalent map in this diagram

\[
L_{/01} \xrightarrow{t} L_{/1}.
\]

This justifies why we think of a left fibration over \( F(1) \) as a functor from the category \( \Delta[1] \) into spaces, which is exactly what we had hoped for.

*Intuition 4.18.* This example is very instructive in how a left fibration really is the appropriate notion of a functor in a higher categorical setting. Recall in Example 4.1 we described how the composition map really gives us the following zig-zag:

\[
\text{map}(x, y_1, y_2) \xrightarrow{s} \text{map}(x, y_1) \xrightarrow{W} \text{map}(x, y_2)
\]

A left fibration allows us to define functoriality while taking this zig-zag structure into account.

*Example 4.19.* The same example can be expanded to show that every right fibration over \( F(n) \) is just the data of a functor from \([n]\) into spaces.

In Example 4.10 we discussed how an under-category gives us the appropriate notion of a representable functor cofibered in sets. In the next example we will generalize this to the setting of higher categories.

*Definition 4.20.* Let \( x \in W \) be an object. We define the *under-CSS* \( W_{x/} \) as

\[
W_{x/} = W^F(0) \times^W W^F(1).
\]

*Intuition 4.21.* Let us see why we are justified in calling \( W_{x/} \) the under CSS. This is easiest by looking at \((W_{x/})_0\). We have following equivalences

\[
(W_{x/})_0 = \Delta[0] \times^W W_1
\]

Thus a point in \( W_{x/} \) corresponds to a point in \( W_1 \), which is just a morphism, such that the source of that morphism is \( x \). That is exactly what we expected.

*Example 4.22.* For an object \( x \) have following pullback square.
This gives us a map $p_x : W_{x/} \rightarrow W$.

We have the following fact about this map.

**Proposition 4.23.** [Ra17a, Example 3.11] The map $p_x : W_{x/} \rightarrow W$ is a left fibration.

For the proof we need a technical understanding of the equivalences of Segal spaces. For more details see [Ra17a, Example 3.11, Theorem 7.1]

Note that there is a map $x : F(0) \rightarrow W_{x/}$, which maps the point to the identity map $id_x$. We will call this left fibration the *representable left fibration* represented by $x$. The following theorem justifies our naming convention:

**Lemma 4.24.** [Ra17a, Theorem 4.2] (Yoneda Lemma for Left Fibrations) Let $L \rightarrow W$ be a left fibration. Then the map induced by $F(0) \rightarrow W_{x/}$ over $W$

$$\text{Map}_W(W_{x/}, L) \xrightarrow{\simeq} \text{Map}_W(F(0), L) = \Delta[0] \times L_0$$

is a trivial fibration of Kan complexes.

The proof of this theorem needs a serious treatment of the theory of left fibrations, which is far beyond the scope of this note. For a detailed account of the theory of left fibrations see [Ra17a].

Before we move on we should point out that we here focused on covariant functors and there is a way to define fibrations that model contravariant functors.

**Definition 4.25.** A Reedy fibration $p : R \rightarrow W$ between CSS is a left fibration if the following is a homotopy pullback square

$$\begin{array}{ccc}
R_1 & \xrightarrow{p_1} & R_0 \\
\downarrow s & & \downarrow s \\
W_1 & \xrightarrow{p_0} & W_0
\end{array}$$

where the map $s : W_1 \rightarrow W_0$ is the source map that takes each arrow to its source.

Using this definition everything we have done until here can now be adjusted to the contravariant setting and give us the same results.
4.4 CoCartesian Fibrations. Up until now we gave an accurate description for functors valued in spaces. However, we also have functors that are valued in higher categories. Using the definition of a left fibration as a guide, we need a certain lifting condition. This lifting condition must have two differences compared to left fibrations. First, it must relax the conditions on each fiber as they could be a CSS rather than just a space. Second, the fact that each fiber is a CSS implies that the lifting condition cannot be restricted to points, but also has to take arrows into account. More on this concept can be found in [Ra17b].

The right way to generalize the lifting condition to arrows is via the language of coCartesian morphism.

**Definition 4.26.** Let $W$ be a CSS, $f : x \rightarrow y$ a morphism in $W$ and $z$ an object in $W$. We define the space $\text{map}_W(f, z)$ as the following pullback:

$$
\begin{array}{ccc}
\text{map}_W(f, z) & \xrightarrow{\tau} & \text{map}_W(x, y, z) \\
\downarrow & & \downarrow \\
\text{map}_W(y, z) & \xrightarrow{\rho} & \text{map}_W(x, y) \times \text{map}_W(y, z)
\end{array}
$$

**Intuition 4.27.** A point in $\text{map}_W(f, z)$ is a triangle of the form:

$$
\begin{array}{ccc}
y & \xrightarrow{f} & y \\
\sigma & \downarrow & \downarrow \\
x & \xrightarrow{\sigma} & z
\end{array}
$$

The reason we care about this space is because it fits into the following zig zag of spaces where the down left map is an equivalence.

$$
\begin{array}{ccc}
\text{map}_W(f, z) & \xrightarrow{f^*} & \text{map}_W(x, z) \\
\downarrow & & \downarrow \\
\text{map}_W(y, z) & \xrightarrow{\rho} & \text{map}_W(x, y)
\end{array}
$$

The map labeled $f^*$ plays the role of “pre-composing with $f$”, however, unlike regular category theory there is no direct map from $\text{map}_W(y, z) \rightarrow \text{map}_W(x, z)$ and so we use $\text{map}_W(f, z)$ as the appropriate replacement for $\text{map}_W(y, z)$.

**Definition 4.28.** Let $p : V \rightarrow W$ be a map of CSS. We say the morphism $f : x \rightarrow y$ in $V$ is a $p$-coCartesian morphism if for each object $z \in V$, the following diagram is a homotopy pullback square.
**Intuition 4.29.** We can depict the situation as follows:

\[
\begin{array}{ccc}
  \text{map}_V(f, z) & \xrightarrow{f^*} & \text{map}_V(x, z) \\
  \downarrow{p} & & \downarrow{p} \\
  \text{map}_W(pf, pz) & \xrightarrow{pf^*} & \text{map}_W(px, pz)
\end{array}
\]

The coCartesian condition stipulates that \(\sigma\) can be lifted to \(\hat{\sigma}\). This in particular implies that \(g\) lifts to an arrow \(\hat{g}\). Notice how this condition tells us something about the existence of certain 2-cells that lift the diagram. Thus we can really think of it as a higher dimensional version of the lifting condition for left fibrations.

**Example 4.30.** Let \(p : V \to W\) be a map of CSS. For any object \(x \in V\), the map \(id_x\) is \(p\)-coCartesian.

**Example 4.31.** Let \(p : V \to W\) be map of CSS. Let \(f \in V\) be a coCartesian morphism such that \(p(f) = id_y\) for an object \(y \in W\). Then \(f\) is an equivalence in \(V\).

With the notion of a \(p\)-coCartesian morphism we finally define a fibration that models functors valued in CSS.

**Definition 4.32.** A Reedy fibration of CSS \(p : C \to W\) is a coCartesian fibration if for every morphism \(f : x \to y\) in \(W\) and chosen lift \(x' \in C\) there exists a \(p\)-coCartesian lift \(f' : x' \to y'\) of \(f\).

**Intuition 4.33.** The best way to gain some intuition on this example is to review the graph we used in Intuition 4.7. Let \(C \to F(1)\) be a coCartesian fibration. Intuitively it corresponds to a functor from the category \([1]\) into CSS, thus it corresponds to a map of CSS. Following Intuition 4.7 we think of the fiber over the object 0 as the domain CSS and the fiber over 1 as the target CSS. The lifting condition should help us get a map from the fiber over 0 to the fiber over 1. We already know...
how to get a map on objects, so let us understand how the condition of a coCartesian fibration gives us a map on morphisms. Let $g : x_1 \rightarrow x_2$ be an arrow in the fiber over 0.

As $p$ is a coCartesian fibration, this gives us two $p$-coCartesian lifts, namely $f_1 : x_1 \rightarrow y_1$ and $f_2 : x_2 \rightarrow y_2$. The fact that $f_1$ is $p$-coCartesian implies that this diagram lifts to a map $\hat{g} : y_1 \rightarrow y_2$. So the lifting condition gives us the necessary map on morphisms that we need. Thus we think of the morphism $\hat{g}$ as the “target” of the morphism $g$ under the “functor” $C$.

There is a second way of defining coCartesian fibrations that aligns more closely to our work with left fibrations. For that one we need some definitions and lemmas. First, we can generalize the definition of a $p$-coCartesian morphism.

**Definition 4.34.** Let $p : V \rightarrow W$ be a map of CSS. An $n$-simplex $\sigma \in V_n$ is $p$-coCartesian if for every map $s : V_n \rightarrow V_1$, $s(\sigma)$ is $p$-coCartesian.

**Example 4.35.** In light of Example 4.30, every point in $V_0$ is $p$-coCartesian.

**Definition 4.36.** Let $p : V \rightarrow W$ be a Reedy fibration between CSS. We define $LFib_W(V_1)$ as the subsimplicial space generated by all $p$-coCartesian arrows in $V$. Based on the previous example $LFib_W(V_0) = V_0$.

Here is a very crucial lemma that exemplifies the importance of this construction.

**Lemma 4.37.** The map $LFib_W(V_1) \rightarrow LFib_W(V_0) \times_{W_0} W_1$ is $(-1)$-truncated.

**Intuition 4.38.** The lemma is telling us that for a chosen point $\Delta[0] \rightarrow LFib_W(V_0) \times_{W_0} W_1$, the space $coCartLift(f)$ defined by the following pullback diagram is either empty or contractible.

\[
\begin{array}{ccc}
\Delta[0] & \xrightarrow{(x', f)} & LFib_W(V_0) \times_{W_0} W_1 \\
\downarrow & & \downarrow \cong \\
coCartLift(f) & \rightarrow & LFib_W(V_1)
\end{array}
\]

However, this space is just the space of $p$-coCartesian lifts of the map $f : x \rightarrow y$ with given lift $x'$. So, this lemma shows that either no such lift exists or there is a contractible space of choices of such lifts. In other words, if a $p$-coCartesian lift exists it is unique up to homotopy.
This gives us following way of identifying coCartesian fibrations.

**Proposition 4.39.** A Reedy fibration of CSS $p : C \to W$ is a coCartesian fibration if and only if the map of simplicial spaces $LFib_W(V) \to W$ is a left fibration.

**Proof.** By definition we have to show that the condition of being a coCartesian fibration is equivalent to the following being a homotopy pullback square:

$$
\begin{array}{ccc}
LFib_W(V)_1 & \longrightarrow & LFib_W(V)_0 \\
\downarrow & & \downarrow \\
W_1 & \longrightarrow & W_0
\end{array}
$$

But this is just equivalent to the following map being an equivalence.

$$
LFib_W(V)_1 \to LFib_W(V)_0 \times_{W_0} W_1
$$

We already know it is always $(-1)$-truncated, thus it is an equivalence if and only if it is a surjection. But being a surjection is exactly the condition that every arrow with a given lift for the source has a $p$-coCartesian lift. That gives us the desired result. \qed

**Intuition 4.40.** If $C \to W$ is a coCartesian fibration, then it models a functor valued in CSS. However, every CSS $W$ has an underlying maximal subgroupoid, namely $W_0$. The left fibration $LFib_W(C)$ exactly models the functor valued in spaces that maps each point to the maximal subgroupoid of its image. The lemma above says that analyzing the functoriality of that underlying fibration already suffices. Why is that?

In order to build a fibration that models a certain functor two conditions are necessary. First, we must make sure that each point has the right fiber i.e. the fiber has the value we desire such as a space or a CSS. Second, we must make sure that we have the right lifting conditions to get a good notion of functoriality. For a coCartesian fibration between CSS the first condition is already given without any extra condition. The only thing that we need to add is a lifting condition to get functoriality. However, this can already be achieved at the level of spaces, if the lifts are coCartesian morphisms, which is because such morphisms will give us the necessary functoriality property.

Let us complement this intuition by looking at some examples.

**Example 4.41.** Let $p : C \to F(0)$ be a coCartesian fibration. There is only one map in $F(0)$, namely the identity map, and for any chosen source $x \in C$ the identity map lifts to the identity map on that source $id_x : x \to x$, which is always $p$-coCartesian. Thus the map $p$ does not imposes any condition $C$ and $C$ is just a given CSS, which is exactly what we expected.

**Intuition 4.42.** Notice in this situation $LFib_{F(0)}(C) \simeq C_0$. This corroborates our point that $LFib_W(C)$ gives us the functor valued in the underlying maximal subgroupoids.

**Example 4.43.** Let $p : C \to F(1)$ be a coCartesian fibration. The fibers $C_0$ and $C_1$ are two CSS. Let

$$
C_{01} = C^{F(1)}_{F(1)} \times_{F(1)} F(0).
$$
The two maps $0, 1 : F(0) \to F(1)$ give us a diagram of CSS.

$$
\begin{array}{c}
\text{C/01} \\
\downarrow s \\
\text{C/0} \\
\text{C/1} \\
\uparrow t
\end{array}
$$

We had a similar diagram when we worked with left fibrations. In that case the source map was an equivalence. However, here this is not the case as $C/01$ has functoriality information that cannot be recovered from $C/0$. However, there is a way to salvage the equivalence. We define

$$L\text{Fib}_{01}(C) = (L\text{Fib}_{F(1)}(C))^{F(1)}_{F(1)^{F(1)}} \times F(0)$$

This gives us a map $L\text{Fib}_{01}C \to C/01$. Now, the composition map

$$L\text{Fib}_{01}(C) \xrightarrow{\simeq} C/0$$

is an equivalence of CSS, by the left fibration property.

As in the case of left fibrations, coCartesian fibrations also have a contravariant analogue.

**Definition 4.44.** Let $p : V \to W$ be a map of CSS. We say the morphism $f : x \to y$ in $V$ is a $p$-Cartesian morphism if for each object $z \in V$, the following diagram is a homotopy pullback square.

$$
\begin{array}{ccc}
\text{map}_V(z, f) & \xrightarrow{f_*} & \text{map}_V(z, y) \\
\downarrow p & & \downarrow p \\
\text{map}_W(pz, pf) & \xrightarrow{pf_*} & \text{map}_W(pz, py)
\end{array}
$$

**Definition 4.45.** A Reedy fibration of CSS $p : C \to W$ is a Cartesian fibration if for every morphism $f : x \to y$ in $W$ and chosen lift $y' \in C$ there exists a $p$-Cartesian lift $f' : x' \to y'$ of $f$.

**Colimits and Adjunctions**

In this section we discuss colimits and adjunctions in CSS using the work we have done before.

### 5.1 Colimits in Complete Segal Spaces

The goal of this subsection is to study colimits. We will proceed in two steps. First we study initial objects, which are the simplest example of a colimit. Then we generalize from an initial object to an arbitrary colimit using cocones.

For this subsection let $W$ be a fixed CSS.

**Definition 5.1.** Let $i \in W$. We say $i$ is initial in $W$ if the projection map $p_i : W_{i/} \to W$ is an equivalence of CSS.
**Intuition 5.2.** There is a way to make this definition look more familiar. Let \( y \in W \) be an object in \( W \). Then we have the following pullback diagram:

\[
\begin{array}{c}
\text{map}_W(i, y) \\
\downarrow \tau \\
F(0) \\
\downarrow y \\
W
\end{array} \quad \text{or} \quad \begin{array}{c}
W_{i/f} \\
\downarrow p_i \\
W
\end{array}
\]

The fact that \( p_i \) is an equivalence implies that \( \text{map}(i, y) \) is contractible. This means that there is a unique map from \( i \) to \( y \), up to homotopy. This is correct generalization of an initial object in a classical category.

In the classical case we know that if an initial object exists then it is unique. We need to show the same thing holds for CSS.

**Definition 5.3.** Let \( W \) be a CSS. Let \( W_{\text{init}} \), called the space of final objects, be the subspace of \( W_0 \) generated by all initial objects.

**Lemma 5.4.** The space \( W_{\text{init}} \) is \((-1)\)-truncated.

**Proof.** \( W_{\text{init}} \) is \((-1)\)-truncated if and only if the map \( \Delta : W_{\text{init}} \to W_{\text{init}} \times W_{\text{init}} \) is an equivalence. This is true if for every map \( (x, y) : \ast \to W_{\text{init}} \times W_{\text{init}} \) the following square is a homotopy pullback.

\[
\begin{array}{c}
\ast \\
\downarrow c \\
W_{\text{init}} \\
\downarrow \Delta \\
W_{\text{init}} \times W_{\text{init}} \\
\downarrow (x, y) \\
\ast
\end{array}
\]

Now the problem is \( \Delta : W_{\text{init}} \to W_{\text{init}} \times W_{\text{init}} \) is not a Kan fibration and thus we need to replace this map with an equivalent map that is a Kan fibration. Fortunately that is not too hard. Concretely, we know that \( W_{\text{init}} \to (W_{\text{init}})^{\Delta[1]} \) is a Kan equivalence and \( (s, t) : (W_{\text{init}})^{\Delta[1]} \to W_{\text{init}} \times W_{\text{init}} \) is a Kan fibration. Thus we need following to be a homotopy pullback.

\[
\begin{array}{c}
\ast \\
\downarrow c \\
(W_{\text{init}})^{\Delta[1]} \\
\downarrow (s, t) \\
W_{\text{init}} \times W_{\text{init}} \\
\downarrow (x, y) \\
\ast
\end{array}
\]

The actual pullback is the space of paths inside the space \( W_{\text{fin}} \) which start at \( x \) and end at \( y \). By the completeness condition this space is equivalent to \( \text{hoequiv}/W(x, y) \leftrightarrow \text{map}_W(x, y) \simeq \ast \) (Lemma 3.4). As we know that \( \text{hoequiv}_W(x, y) \leftrightarrow \ast \) is \((-1)\)-truncated all we have to do is
to show that $\text{hoequiv}_{/W}(x, y) \neq \emptyset$ and we can conclude that $\text{hoequiv}_{/W}(x, y) \simeq \ast$. Indeed, let $f \in \text{map}_{/W}(x, y) = \ast$ and $g \in \text{map}_{/W}(y, x) = \ast$. Then, $g \circ f \simeq \text{id}_x \in \text{map}_{/W}(x, x) = \ast$ and $f \circ g \simeq \text{id}_y \in \text{map}_{/W}(y, y) = \ast$ and so $x$ and $y$ are equivalent and so $\text{hoequiv}_{/W}(x, y) \neq \emptyset$ and we are done. \qed

This means that if $W_{\text{init}} \neq \emptyset$ then it is contractible, which implies that if an initial object exists then it is unique up to homotopy. Using initial objects we can define colimits. But first we have to define the complete Segal space of cocones.

**Definition 5.5.** Let $f : I \to W$ be a map of simplicial spaces. We define $W_{f/}$ as the following CSS

$$W_{f/} = F(0) \times_{W^I} W^{I \times F(1)} \times_{W^I} W.$$  

**Intuition 5.6.** Let us look at $(W_{f/})_0$. We have

$$(W_{f/})_0 = F(0) \times_{(W^I)_0} (W^{I \times F(1)})_0 \times_{(W^I)_0} W_0 = F(0) \times_{\text{Map}(I \times F(1), W)} \text{Map}(I, W) \times_{\text{Map}(I, W)} W_0.$$  

Thus a point in $W_{f/}$ is a map $\hat{f} : I \times F(1) \to W$ such that $\hat{f}|_{I \times \{0\}} = f$ and $\hat{f}|_{I \times \{1\}} = x$, where $x$ is an object in $W$. That is very similar to the definition of a cocone that we have seen in category theory.

**Example 5.7.** As an example, if $I = F(0)$ and $x : F(0) \to W$, then the CSS of cocones, $W_{x/}$ is exactly the under-CSS, $W_{x/}$ as defined in Definition 4.20.

**Definition 5.8.** Let $f : I \to W$ be map of CSS. The diagram $f$ has a colimit if $(W_{f/})_{\text{init}} \neq \emptyset$. We will denote any choice of point in this space by $\text{colim}_I f$ and call it the colimit of $f$ or colimit cocone of $f$.

**Definition 5.9.** A CSS is cocomplete if for any map $f : I \to W$ the diagram $f$ has a colimit.

**Remark 5.10.** Let us give a more detailed description of the colimit of a diagram $f : I \to W$. It is an initial object in $W_{f/}$ and as such it is a map $\bar{f} : I \times F(1) \to W$ such that $\bar{f}|_{I \times \{0\}} = f$ and $\bar{f}|_{I \times \{1\}} = v$, where $v$ is an object in $W$. We very often abuse notation and call this point in $W$ the colimit of $f$ and also denote it by $\text{colim}_I f$.

Here are some examples of important colimits.

**Example 5.11.** Let $I = \emptyset$ be the empty CSS. Then there is a unique map $e : \emptyset \to W$. In this case $W_{e/} = W$ and so the colimit of the diagram $e$ is just the initial object.

**Example 5.12.** Let $S$ be a set, thought of as a discrete simplicial space. Concretely, $S$ is a disjoint union of $F(0)$. Let $f : S \to W$ be any map. Notice we have following equivalence

$$W^S \cong \prod_{s \in S} W.$$  

So

$$W_{f/} = F(0) \times_{W^S} W^{S \times F(1)} \times_{W^S} W =$$
Thus a point in $W_{f/}$ is a choice of object $x \in W$ and morphisms $i_s : f(s) \to x$. This means the data of a colimit of a set diagram is the same as in ordinary categories.

Example 5.13. In a similar fashion, let $\mathcal{C}$ be the following category:

$$
\begin{array}{c}
1 & \leftarrow & t & \rightarrow & 0 & \rightarrow & r & \rightarrow & 2
\end{array}
$$

and let $I = NC$. Concretely, $I = F(1)^s \coprod_{F(0)} m_1 F(1)$. A map $f : I \to W$ is of the form:

$$
x_1 \leftarrow f \quad x_0 \rightarrow g \rightarrow x_2 .
$$

where $f : x_0 \to x_1$ and $g : x_0 \to x_2$ are two arrows in $W$. An object in the CSS of cocones, $W_{f/}$, is a diagram of the form:

Thus a colimit a diagram of the form above that is an initial. In such diagram, the object $v$ is often expressed as $x_1 \coprod x_0 x_2$.

Considering the fact the bottom horizontal maps are identity maps, we can reduce the diagram to the more familiar form:

On classic fact about colimits is that a map out of a colimit is determined by a map out of the diagram that formed the colimit. The same result holds for complete Segal spaces.

Theorem 5.14. [Ra17a, Theorem 5.13] Let $f : I \to W$ be a map of CSS which has colimit cocone $\tilde{f} : F(0) \to X_{\eta/}$ with vertex point $v$. Let $y$ be any object in $W$. This gives us a constant map $\Delta y : I \to F(0) \to W$. There is a Kan equivalence of spaces

$$
\text{map}_W (v, y) \xrightarrow{\simeq} \text{map}_{W_{f/}} (\tilde{f}, \Delta y).
$$
5.2 Adjunctions. In this subsection we discuss adjunctions of complete Segal spaces. Recall that for two ordinary categories $\mathcal{C}$ and $\mathcal{D}$ an adjunction are two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ along with a choice of bijections
\[ \text{Hom}_D(Fc,d) \cong \text{Hom}_C(c,Gd) \]
for two objects $c$ in $\mathcal{C}$ and $d$ in $\mathcal{D}$.

When we want to generalize this to the higher categorical setting we again run into coherence issues when trying to specify the equivalences. Fortunately, there is a way to manage this difficulty, namely by using (co)Cartesian fibrations.

Definition 5.15. A map $p : A \to F(1)$ is an adjunction if it is a coCartesian and Cartesian fibration.

Intuition 5.16. How does this definition relate to a notion of an adjunction that we are familiar with? First notice that $F(1)$ has two objects, namely 0 and 1, that have fibers
\[ W = A \times_{F(1)} F(0) \]
\[ V = A \times_{F(1)} F(0) \]
which are both CSS.

Next as $A \to F(1)$ is a coCartesian fibration it classifies a map $f : W \to V$ as described in Example 4.43. Similarly, as $A \to F(1)$ is a Cartesian fibration it classifies a map $g : V \to W$. The key is now to use the existence of (co)Cartesian lifts to get the structure of an adjunction.

If $x$ is an object in $W$ then there exists an object $fx$ in $V$ along with a map $f : x \to fx$ in $A$ that is coCartesian. The coCartesian property implies that for any other object $y$ in $V$ a morphism $x \to y$ in $A$ will factor through $fx$. We can depict the situation as follows:

\[ \begin{array}{ccc}
    x & \xrightarrow{f} & fx \\
    \downarrow & & \downarrow \\
    0 & \rightarrow & 1
\end{array} \]

This gives us an equivalence
\[ \text{map}_V(fx,y) \simeq \text{map}_A(x,y). \]

On the other side for an object $y$ in $V$ there exists a Cartesian lift $gy \to y$. Using a similar argument to the one above, for each object $x$ in $W$ we have an equivalence
\[ \text{map}_W(x,gy) \simeq \text{map}_A(x,y). \]

Combining these two equivalences we get that
\[ \text{map}(fx,y) \simeq \text{map}_V(x,gy). \]
This is exactly the familiar format of an adjunction that we would have expected. It should be noted that as always the equivalence is not direct but rather through a zig-zag of equivalences.

Having a definition of an adjunction we can recover some computational methods for how to determine adjunctions. Recall that in classical category theory a map of categories \( F : C \to D \) is a left adjoint if and only if for each object \( d \) the functor

\[
\text{Hom}_D(F(-), d) : C^{op} \to \text{Set}
\]

is representable. The representing object will then be \( G(d) \) and we can use the universality of a representing objects to define a functor \( G : D \to C \) that is the right adjoint.

We want a similar result for higher categories. However, in this case representable functors are modeled by representable right fibrations or, in other words, over-categories. Using that we can adjust the result from above thusly.

**Theorem 5.17.** [Ra17b, Theorem 7.57] Let \( f : W \to V \) be a functor of CSS. Then \( f \) is a left adjoint if and only if for each object \( y \) in \( V \) the CSS \( W_y \) defined by the pullback

\[
\begin{array}{ccc}
W/y & \longrightarrow & V/y \\
\downarrow & & \downarrow \pi_y \\
W & \underset{f}{\longrightarrow} & V
\end{array}
\]

has a final object, which implies that \( W_y \to W \) is a representable right fibration.

This gives us a very helpful computational method to determine whether a functor is a left adjoint. We can use this computational method to redefine limits and colimits.

**Theorem 5.18.** Let \( I \) and \( W \) be CSS and let \( \Delta_I : W \to W^I \) be the natural inclusion induced by the map \( I \to F(0) \). Then \( \Delta_I \) has a left adjoint if and only if each map \( f : I \to W \) has a colimit and has a right adjoint if each map \( f : I \to W \) has a limit.

**Proof.** \( \Delta_I : W \to W^I \) has a right adjoint if and only if for each map \( h : I \to W \) the pullback \( W \times_{W^I} (W^I)_h \) has a final object. However, this is just the category of cones over \( f \), namely, \( W/f \), which by definition means \( h \) has a limit. The case for colimits follows similarly. \( \square \)

Let us trace through the steps of the proof in one concrete example.

**Example 5.19.** Let \( I = \emptyset \) and \( W \) be a CSS. Then \( W^I = F(0) \) and \( \Delta_I : W \to F(0) \) is just the map to the point. In this case a map \( i : F(0) \to W \) is a left adjoint if the fiber formed by the pullback diagram is a representable right fibration over \( F(0) \).
However, $F(0)$ has only one object and so the only representable right fibration over $F(0)$ is itself. Thus $map(i, x)$ needs to be contractible for $i : F(0) \to W$ to be a left adjoint. But this is exactly the condition of being an initial object.

**Model Structures of Complete Segal Spaces**

One very efficient way of studying higher categories is via the language of model categories. A model category is an ordinary category with several distinguished classes of maps that allows us to recover some of the important homotopical data. In particular, complete Segal spaces have a model structure that allows us to study them very efficiently. The goal of this section is to define and study the model structure for complete Segal spaces.

Most results about this model structure can be found in [Re01]. We will not prove any of them, but rather provide proper references. The goal of this section is only to give the reader an overview of the complete Segal space model structure.

### 6.1 Review of Model Structures

We are not going to develop the whole theory of model structures, but rather focus on several important properties that will come up in the coming subsections. For a good introduction to the theory of model structures see [Ho98] or [DS95].

**Definition 6.1.** A model structure on a category is a complete and cocomplete category $\mathcal{M}$ along with three classes of maps:

1. Fibrations $\mathcal{F}$.
2. Cofibrations $\mathcal{C}$.
3. Weak Equivalences $\mathcal{W}$.

that satisfies following conditions.

- **(2-out-of-3)** If two out of the three maps $f$, $g$ and $g \circ f$ are weak equivalences then so is the third.
- **(Retracts)** If $f$ is a retract of $g$ and $g$ is a weak equivalence, cofibration, or fibration, then so is $f$.
- **(Lifting)** Let $i : A \to B$ be a cofibration and $p : Y \to X$ be a fibration. The diagram in $\mathcal{M}$

\[
\begin{array}{ccc}
A & \longrightarrow & Y \\
\downarrow & & \downarrow \\
B & \longrightarrow & X \\
\end{array}
\]

is such that $i$ is a cofibration and $p$ is a fibration.
lifts if either $i$ or $p$ are weak equivalences.

- (Factorization) Any morphism $f$ can be factored into $f = pi$, where $p$ is a fibration and $i$ is a cofibration and weak equivalence and $f = qj$, where $q$ is a fibration and weak equivalence and $j$ is a cofibration.

One primary example of a model structure is the Kan model structure:

**Theorem 6.2.** [GJ09, Theorem I.11.3] There is a model structure on simplicial sets, called the Kan model structure and defined as follows:

- **C** A map is a cofibration if it is an inclusion.
- **F** A map is a fibration if it is a Kan fibration.
- **W** A map $X \to Y$ is an equivalence if and only if the for every Kan complex $K$ the induced map of Kan complexes
  \[ \text{Map}(Y, K) \to \text{Map}(X, K) \]
  is a weak equivalence.

Model structures can simplify many computations. For that reason there are many important properties that model structures can satisfy. For the rest of this subsection we will review some of the more important properties.

**Definition 6.3.** We say a model structure is left proper if for every weak equivalence $A \to X$ and for every cofibration $A \to B$, the pushout map $X \to X \coprod_A B$ is a weak equivalence.

**Definition 6.4.** We say a model structure is right proper if for every weak equivalence $A \to X$ and for every cofibration $Y \to X$, the pullback map $Y \times_X A \to Y$ is a weak equivalence.

**Definition 6.5.** A model structure is proper if it is left proper and right proper at the same time.

Sometimes we have a model structure on a category that is Cartesian closed. In this case we might wonder how well they interact with each other.

**Definition 6.6.** Let $\mathcal{M}$ be a model structure on a category that is Cartesian closed. We say the model structure is compatible with Cartesian closure if for every cofibration $i : A \to B$, $j : C \to D$ and fibration $p : Y \to X$, the map

\[ i : A \times D \coprod_{A \times B} B \times C \to B \times D \]

is a cofibration and

\[ Y^B \to Y^A \times X^B \]

is a fibration either of which is a weak equivalence if any of the maps involved is a weak equivalence.

**Remark 6.7.** Notice that the Kan model structure satisfies all of the conditions stated above. It is proper and compatible with Cartesian closure.
6.2 Reedy Model Structure. We defined a complete Segal space as a simplicial space that satisfied three conditions.

(1) Reedy fibrancy
(2) Segal condition
(3) Completeness condition

Accordingly we will define three model structures, which correspond to these three conditions. Thus we first start with the Reedy model structure.

Theorem 6.8. [ReXX] [DKS93, Subsection 2.4-2.6] There is a model structure on the category $s\mathbb{S}$ called the Reedy model structure, defined as follows.

- A map $f : X \to Y$ is a cofibration if it is an inclusion.
- A map $f : X \to Y$ is a weak equivalence if it is a level-wise equivalence.
- A map $f : X \to Y$ is a fibration if the map

$$\text{Map}(F(n), X) \to \text{Map}(\partial F(n), X) \times_{\text{Map}(\partial F(n), Y)} \text{Map}(F(n), Y)$$

is a Kan fibration.

The Reedy model structure has many amazing features. In particular it satisfies following properties:

(1) Cofibrantly generated.
(2) Proper.
(3) Compatible with Cartesian closure.
(4) Simplicial.

One important property of the Reedy model structure on simplicial spaces is that we can use techniques of Bousfield localizations on it.

Theorem 6.9. [Re01, Proposition 9.1] Let $f : A \to B$ be an inclusion. There is a unique, simplicial, cofibrantly generated model structure on $s\mathbb{S}$ called the $f$-local model structure, characterized as follows.

(1) Cofibrations are inclusions.
(2) A simplicial space $W$ is fibrant if it is Reedy fibrant and the map

$$f^* : \text{Map}(B, W) \to \text{Map}(A, W)$$

is a Kan equivalence.
(3) A map $g : X \to Y$ is a weak equivalence if for every fibrant object $W$ the induced map

$$g^* : \text{Map}(Y, W) \to \text{Map}(X, W)$$

is a Kan equivalence.
(4) A map between fibrant objects $W \to V$ is a weak equivalence (fibration) if and only if it is a Reedy equivalence (Reedy fibration).

Using this property of Bousfield localizations we can define model structures for Segal spaces and complete Segal spaces.
6.3 Segal Space Model Structure. The Segal space model structure is a Bousfield localization of the Reedy model structure.

**Theorem 6.10.** [Re01, Theorem 7.1] There is a unique, simplicial model structure on $sS$ called the Segal space model structure, characterized as follows.

1. Cofibrations are inclusions.
2. A simplicial space $T$ is fibrant if it is Reedy fibrant and the map
   \[ \varphi_n^* : \text{Map}(F(n), T) \to \text{Map}(G(n), T) \]
   is a Kan equivalence for $n \geq 2$.
3. A map $g : Y \to X$ is a weak equivalence if for every Segal space $T$ the induced map
   \[ g^* : \text{Map}(Y, T) \to \text{Map}(X, T) \]
   is a Kan equivalence.
4. A map between Segal Spaces $T \to U$ is a weak equivalence (fibration) if and only if it is a Reedy equivalence (Reedy fibration).

The Segal space model structure also satisfies some crucial properties. In particular it satisfies following properties:

1. Cofibrantly generated.
2. Left proper.
3. Compatible with Cartesian closure.

Notice we did not say the Segal space model structure is right proper. Here is a counter-example:

**Example 6.11.** Let $c : F(1) \to F(2)$ be the unique map that sends 0 to 0 and 1 to 2. We have pullback square

\[
\begin{array}{ccc}
F(0) \amalg F(0) & \longrightarrow & F(1) \\
\downarrow & & \downarrow c \\
G(2) & \longrightarrow & F(2)
\end{array}
\]

The map $c$ is a Segal fibration as it is a Reedy fibration between Segal spaces. Moreover, $\varphi_2$ is a Segal equivalence. However, the pullback is clearly not a Segal equivalence, as $F(0) \amalg F(0)$ is not equivalent to $F(1)$.

6.4 Complete Segal Space Model Structure. The Segal space model structure is a Bousfield localization of the Reedy model structure.

**Theorem 6.12.** [Re01, Theorem 7.2] There is a unique, simplicial model structure on $sS$ called the complete Segal space model structure, characterized as follows.

1. Cofibrations are inclusions.
(2) A simplicial space \( W \) is fibrant if it is a Segal space and the map 
\[
0^* : \text{Map}(E(1), W) \to \text{Map}(F(0), W)
\]
is a Kan equivalence.

(3) A map \( g : X \to Y \) is a weak equivalence if for every complete Segal space \( W \) the induced map 
\[
g^* : \text{Map}(Y, W) \to \text{Map}(X, W)
\]
is a Kan equivalence.

(4) A map between complete Segal Spaces \( W \to V \) is a weak equivalence (fibration) if and only if it is a Reedy equivalence (Reedy fibration).

The complete Segal space model structure also satisfies some crucial properties. In particular it satisfies following properties:

1. Cofibrantly generated.
2. Left proper.
3. Compatible with Cartesian closure.

Using the completeness condition we can characterize CSS equivalences in a more simple manner.

**Theorem 6.13.** [Re01, Theorem 7.7] A map \( f : T \to U \) of Segal space is a CSS equivalence if and only if it is fully faithful and essentially surjective.

The CSS model structure is also not right proper, as the example above still holds in this case, however, there are some restricted cases where right properness condition holds

**Theorem 6.14.** [Ra17b, Theorem 7.26] Let \( C \to W \) be a coCartesian fibration and \( V \to W \) be a CSS (Segal) equivalence. Then the pullback map \( V \times_W C \to C \) is a CSS (Segal) equivalence.

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