Introduction

A wormhole is a hypothetical tunnel through space. Such a concept has long been a favorite theme in *Star Trek* and in science fiction. An extensive list of popular science books\cite{1} are devoted to topics related to wormholes. We might gauge the popularity of wormholes among students by the recurring references to them in *The Simpsons*: in one Halloween special\cite{2}, Homer was sucked into modern-day California while saying "there's so much I don't know about astrophysics, I wish I'd read that book by [Dr. Hawking]." The detailed calculations about a wormhole are complex and still under debate among physicists, but with the techniques taught in a standard calculus course\cite{3}, a student is capable of producing the images of wormholes based on sound physical principles. Because many issues concerning wormholes are not fully resolved yet, this paper is not meant to be rigorous. We merely attempt to motivate students to apply mathematical skills that they have learned in calculus, and to induce their interest in non-Euclidean geometry and general relativity through this fascinating subject.

1 Einstein’s Theory of Gravity

Newton’s law of gravity states that the gravitational force between two bodies is

\[ F = GMm/r^2, \]

where \( G \) is the gravitational constant, \( m \) and \( M \) are the masses of the two bodies, and \( r \) is the distance between them. Einstein had an entirely different view of gravity. Suppose we were to measure the length of the equator of the earth very carefully and obtain the circumference \( C \), we would predict that the radius of the earth is \( C/2\pi \) based on Euclidean geometry. However, if we indeed dig a hole to the center of the earth and measure the radius, we would have a value greater (about 1.5 millimeters) than our prediction. If we do the same for the sun, the discrepancy would be about one-half a kilometer\cite{4}.

From this thought experiment, we conclude that the space is curved. Therefore, the differential of arc length based on Euclidean geometry,

\[ ds^2 = dx^2 + dy^2, \]  

(1)
is no longer valid. Gauss has developed a generalization of the Cartesian coordinates by drawing a system of arbitrary curves on the surface, designated as $u = 1, u = 2, u = 3, \ldots, v = 1, v = 2, v = 3$, and so forth (see Figure 1).

Figure 1: Gaussian coordinates.

Using the Gaussian coordinates, the differential of arc length is

$$d\sigma^2 = g_{11} \, du^2 + 2g_{12} \, du \, dv + g_{22} \, dv^2,$$  \hspace{1cm} (2)

where the coefficients $g_{\mu\nu}$, called metric coefficients, are constants or functions of $u$ and $v$. This equation can be construed as a generalized “Pythagorean theorem.” As an example, with the polar coordinates $(r, \phi)$, we have

$$d\sigma^2 = dr^2 + r^2 \, d\phi^2,$$  \hspace{1cm} (3)

where the metric coefficients are $g_{rr} = 1$, $g_{\phi\phi} = r^2$ and $g_{r\phi} = 0$ (because the system is orthogonal). Riemann extended the Gaussian geometry to greater number of dimensions, and Riemannian geometry later became the mathematical framework for Einstein’s general relativity. In brief, Einstein conceived of gravity as a change in the geometry of spacetime due to mass-energy, and his greatest achievement was to derive a field equation which relates the metric coefficients $g_{\mu\nu}$ to mass-energy [5].

2 Schwarzschilnd Wormhole and Embedding Diagram

Shortly after Einstein published his general theory of relativity, Schwarzschild found a solution [6] for the exterior of a spherically symmetric gravitational source $M$. (A solution to Einstein’s field equation can be written as a differential of arc length which describes the spacetime warped by gravity.) In the equatorial plane at a fixed time, the solution is simplified to

$$d\sigma^2 = \left(1 - \frac{2M}{r}\right)^{-1} \, dr^2 + r^2 \, d\phi^2.$$  \hspace{1cm} (4)
We have adopted a convention in general relativity of measuring mass in units of length by multiplying the ordinary mass in kilogram by a factor of $G/c^2$, where $G$ is the gravitational constant and $c$ the speed of light. For example, the mass of the earth is 0.444 cm, and that of the sun 1.48 km. A reader may notice the presence of a singularity at $r = 2M$ in equation (4). If we compress the sun into a sphere of a radius less than $2 \times 1.48$ km, we will have a black hole. The physical significance of this singularity is that nothing, not even light, can escape the gravitational pull of the black hole once it crosses the boundary $r = 2M$ (the event horizon).

Figure 2: An embedding diagram of the equatorial plane of a massive spherical object. The measured radius is $l$, while the length of equator divided by $2\pi$ gives $r$.

Equation (4) is a two-dimensional curved surface, and it can be visualized by means of an embedding diagram. As mentioned earlier, the measured radius in a curved space is greater than the ratio $C/2\pi$ (the supposed radius in a flat space). To visualize this phenomenon, we imagine a fictitious depth $z$, see Figure 2, to accommodate the actual radius. We emphasize that this artificial $z$ dimension is purely for visual purpose and has nothing to do with real space.

To be more quantitative, the measured distance in a curved space is the integral $\int dr$. We want to find an embedding formula $z$, such that the geometry of a curved two-dimensional space is the same as a flat three-dimensional space. A flat (Euclidean) three-dimensional space means that the ordinary Pythagorean theorem is valid, or

$$d\sigma^2 = dx^2 + dy^2 + dz^2.$$  

It is more convenient to rewrite this differential of arc length using cylindrical coordinates:

$$d\sigma^2 = dz^2 + dr^2 + r^2 d\phi^2.$$  

Comparing equations (4) and (5), we can solve for $dz$:

$$dz = \pm \sqrt{\left(1 - \frac{2M}{r}\right)^{-1} - 1} \, dr = \sqrt{\frac{2M}{r - 2M}} \, dr,$$  

and integrate to obtain $z$:

$$z = \pm \int \sqrt{\frac{2M}{r - 2M}} \, dr = \pm 2\sqrt{2M} \sqrt{r - 2M}.$$  

Because of spherical symmetry, there is no explicit $\phi$ term. To restore the $\phi$ dimension, we rotation the curve $z$ around the $z$-axis, which gives the embedding surface. In parametric form, the embedding surface is written as $(r \cos \phi, r \sin \phi, \sqrt{8M(r - 2M)})$; computer programs are readily available for graphing this surface.

Einstein and Rosen proposed a topology that connects two universes at $r = 2M$, and they called such a connection a “bridge.” By joining the positive and negative solution of $z$ at $r = 2M$, we have the embedding diagram for the Einstein-Rosen bridge, as shown in Figure 3. Based of its shape, J. A. Wheeler coined the term “wormhole” for this type of geometry.

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![Figure 3: The Einstein-Rosen bridge, or the Schwarzschild wormhole, embedded in a three-dimensional Euclidean space.](image)

### 3 Morris-Thorne Wormhole

In 1985, Carl Sagan wrote a novel *Contact*, which was later adopted to a film of the same title released in 1997. Before Sagan published the book, he sought advice about gravitational physics from a Caltech physicist, Kip Thorne. In the original manuscript, Sagan had his heroine, Eleanor Arroway (played by Jodie Foster in the film), plunge into a black hole near the earth, travel through the space, and emerge an hour later near the star Vega, 26 light-years away. Thorne pointed out a well-established result that it is impossible to travel through the Schwarzschild wormhole because its throat pinches off too quickly; in other words, Figure 3 exists only for a brief moment which is too short to allow communicating with or traveling to the other part of the universe.

Sagan’s request, however, piqued Thorne’s curiosity about wormholes. Thorne devised a wormhole solution, which is simplified in the equatorial plane at a fixed time as

$$d\sigma^2 = \frac{1}{1 - b_0^2/r^2} dr^2 + r^2 d\phi^2,$$

where $b_0$ is the radius of the throat.

It is easy to write a mathematical solution for a geometry that we desire, but Einstein’s field equation relates geometry to mass-energy. If we attempt
to construct a wormhole which has a geometry as equation (8) and remains open and stable so that it allows two-way travel, we will need negative-energy material, called exotic matter by Thorne (and incorporated into Sagan’s novel). There is a debate about the possibility of such exotic matter, but it is certain that the energy required is far beyond the producing capacity of a present and foreseeable future civilization. The hope of space travel in short term is impractical at least, if not entirely impossible.

Nevertheless, we employ the same procedure as the preceding section to derive the embedding formula to visualize this geometry. From equations (8) and (9), we solve for $dz$:

$$dz = \pm \sqrt{\frac{1}{1 - b_0^2/r^2} - 1} \, dr = \pm \sqrt{\frac{b_0^2}{r^2 - b_0^2}} \, dr. \tag{9}$$

With a substitution $r = b_0 \sec \theta$, we integrate $dz$ to obtain

$$z = \pm b_0 \ln \left[ \frac{r}{b_0} + \sqrt{\left(\frac{r}{b_0}\right)^2 - 1} \right] \tag{10}$$

Alternatively, with a substitution $r = b_0 \cosh \theta$, we obtain

$$z = \pm b_0 \cosh^{-1} \left( \frac{r}{b_0} \right), \tag{11}$$

which is a catenary curve. The surface of revolution of this curve is shown in Figure 4.

![Figure 4: The Morris-Thorne wormhole embedded in a three-dimensional Euclidean space.](image)

### 4 Summary

The images that appear in science literature depicting how a massive body warps space are based on the concept of embedding a curved two-dimensional surface in a three-dimensional flat (Euclidean) space. Quantitatively, the Schwarzschild wormhole can be visualized as a paraboloid of revolution

$$r = 2M + \frac{z^2}{8M}. \tag{12}$$
and the Morris-Thorne wormhole as a catenoid of revolution

\[ r = b_0 \cosh \left( \frac{z}{b_0} \right). \]  

(13)

Acknowledgments

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References

[1] See, for example, Stephen Hawking, *A Brief History of Time: From the Big Bang to Black Holes*, Bantam, New York, 1988; Michio Kaku, *Hyperspace*, Oxford University Press, New York, 1994; Kip S. Thorne, *Black Holes and Time Warps*, Norton, New York, 1994; Alan H. Guth, *The Inflationary Universe*, Persens, Reading, MA, 1997; Brian Greene, *The Elegant Universe*, Norton, New York, 1999; Stephen Hawking, *The Universe in a Nutshell*, Bantam, New York, 2001; Brian Greene, *The Fabric of the Cosmos: Space, Time, and the Texture of Reality*, Knopf, New York, 2004.

[2] See The Simpsons Archive at http://www.snpp.com, Episode 3F04, “Treehouse of Horror VI: Homer³.”

[3] James Stewart, *Calculus*, 5th ed., Brooks/Cole, Belmont, CA, 2003; Deborah Hughes-Hallett, et al., *Calculus*, 3rd ed., Wiley, New York, 2002.

[4] With an assumption that the mass density is constant, the radius excess is \( GM/3c^2 \). This approach is taken from R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley, Reading, MA, 1965, Volume 2, Chapter 42.

[5] Charles W. Misner, Kip S. Thorne and John Archibald Wheeler, *Gravitation*, Freeman, San Francisco, 1973. The Einstein equation is

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}. \]

On the left-hand side, \( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor, which is reduced from the metric coefficients \( g_{\mu\nu} \) based on Riemannian geometry; on the right-hand side, \( T_{\mu\nu} \) is the energy-momentum tensor, which measures the mass-energy content.

[6] Misner et al., ibid, p. 607. The four-dimensional Schwarzschild spacetime is

\[ ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

[7] Misner et al., ibid, pp. 613–615.
[8] The Maple (http://www.maplesoft.com) command for the surface of revolution for the Schwarzschild embedding formula is `plot3d([r*cos(phi), r*sin(phi), sqrt(8*(r-2))], r=2..10, phi=0..2*Pi);` the Mathematica (http://www.wri.com) command is `ParametricPlot3D[{r Cos[phi], r Sin[phi], Sqrt[8 (r-2)]}, {r,2,10}, {phi,0,2 Pi}].`

[9] A. Einstein and N. Rosen, “The particle problem in the general theory of relativity,” *Physical Review*, 48, 73–77 (1935).

[10] Carl Sagan, *Contact*, Simon & Schuster, New York, 1985; see pp. 347, 348, and 406.

[11] Thorne in Reference 1, Chapter 14; Misner et al, ibid, pp. 836–840.

[12] Michael S. Morris and Kip S. Thorne, “Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity,” *American Journal of Physics*, 56, 395–412 (1988); Michael S. Morris, Kip S. Thorne, and Ulvi Yurtsever, “Wormholes, time machines, and the weak energy condition,” *Physical Review Letters*, 61, 1446–1449 (1988). The four-dimensional wormhole spacetime is

\[
ds^2 = -e^{2\Phi}dt^2 + \frac{1}{1 - \frac{b_0^2}{r^2}}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).
\]