HYDRODYNAMIC BEHAVIOR OF ONE DIMENSIONAL
SUBDIFFUSIVE EXCLUSION PROCESSES WITH RANDOM
CONDUCTANCES

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Abstract. Consider a system of particles performing nearest neighbor ran-
monic walks on the lattice $\mathbb{Z}$ under hard-core interaction. The rate for a jump
over a given bond is direction-independent and the inverse of the jump rates
are i.i.d. random variables belonging to the domain of attraction of an
$\alpha$–stable law, $0 < \alpha < 1$. This exclusion process models conduction in strongly
disordered one-dimensional media. We prove that, when vary ing over the dis-
order and for a suitable slowly varying function $L$, under the super-diffusive
time scaling $N^{1+1/\alpha}L(N)$, the density profile evolves as the solution of the
random equation

$$
\partial_t \rho = L W \rho,
$$

where $L W$ is the generalized second-order dif-
ferential operator

$$
d \frac{d}{du} d \frac{d}{dW}
$$
in which $W$ is a double sided $\alpha$–stable subordinator.

This result follows from a quenched hydrodynamic limit in the case that the
i.i.d. jump rates are replaced by a suitable array

$$
\{\xi_{N,x} : x \in \mathbb{Z}\}
$$
having same
distribution and fulfilling an a.s. invariance principle. We also prove a law of
large numbers for a tagged particle.

Key words: interacting particle system, hydrodynamic limit, $\alpha$–stable subor-
dinator, random environment, subdiffusion, quasi-diffusion.

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1. Introduction

Fix a sequence of positive numbers $\xi = \{\xi_x : x \in \mathbb{Z}\}$ and consider the random
walk $\{X_t : t \geq 0\}$ on $\mathbb{Z}$ which jumps from $x$ to $x + 1$ and from $x + 1$ to $x$ at rate
$\xi_x$. Assume that

$$
\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{x=1}^{\ell} \xi_x^{-1} = \gamma, \quad \lim_{\ell \to \infty} \frac{1}{\ell} \sum_{x=-\ell}^{-1} \xi_x^{-1} = \gamma
$$

(1.1)

for some $0 < \gamma < \infty$. It is well known that

$$
X_{tN}/\sqrt{N} \quad \text{converges in distribution to} \quad \gamma^{-1} B_t,
$$

(1.2)
as $N \uparrow \infty$, where $B_t$ is a standard Brownian motion (cf. [11], [7] for references).

In the particular case where $\{\xi_x : x \in \mathbb{Z}\}$ is an ergodic sequence of positive random variables (e.g. i.i.d. positive random variables) with $E[\xi_0^{-1}] < \infty$, a quenched
(i.e. a.s. with respect the environment $\xi$) invariance principle follows from the pre-
vious result setting $\gamma = E[\xi_0^{-1}]$. Notice that the noise survives in the limit only
through the expected value of $\xi_0^{-1}$. Moreover, in the non trivial case where $\xi$ is
not constant, by Jensen’s inequality, the diffusion coefficient $E[\xi_0^{-1}]^{-1}$ is strictly
smaller than the expected value of the conductance $\xi_x$.

If the positive i.i.d. random variables $\{\xi_x : x \in \mathbb{Z}\}$ are such that $E[\xi_0^{-1}] = \infty$,
the invariance principle (1.2) suggests that the random walk remains freezed in
the diffusive scale. As discussed in [1], a natural assumption is to suppose that
the distribution of $\xi_0^{-1}$ belongs to the domain of attraction of an $\alpha$-stable law,
$0 < \alpha < 1$. In this case, the partial sums of the sequence $\{\xi^{-1}_x\}$ converge in law,
when properly rescaled, to a double sided $\alpha$–stable subordinator $W$: there exists a
slowly varying function $L(\cdot)$ such that, for each $u \in \mathbb{R}$,
$$
\lim_{N \to \infty} \frac{S([Nu])}{N^{1/\alpha}L(N)} = W(u) \text{ in law},
$$
(1.3)
where $[a] = \max\{k \in \mathbb{Z} : k \leq a\}$ is the integer part of $a$, and where the function
$S : \mathbb{Z} \to \mathbb{R}$ is defined as
$$
S(j) = \begin{cases}
\sum_{x=0}^{j-1} \xi^{-1}_x, & j > 0, \\
0, & j = 0, \\
-\sum_{x=j}^{-1} \xi^{-1}_x, & j < 0.
\end{cases}
$$
(1.4)
This case is treated in [11], where the authors prove that varying over the envi-
ronment $\xi$ and for a suitably slowly varying function $L'(\cdot)$, the process $X(N^{1+1/\alpha}
L'(N)/N$ converges weakly in the Skorohod space $D([0, \infty), \mathbb{R})$ to a continuous
self-similar process.
In order to prove a quenched limiting behavior we need the limit to be almost sure as well. To transform the convergence in law into an almost sure convergence we can, for instance, replace the sequence $\{\xi_x : x \in \mathbb{Z}\}$ by an array
$\{\xi_{N,x} : x \in \mathbb{Z}\}$, $N \geq 1$, which has the same distribution as $\{\xi_x : x \in \mathbb{Z}\}$ for
each $N \geq 1$. The most natural array can be constructed as follows. Define the
nonnegative function $G$ on $[0, \infty)$ by $P(W(1) > G(x)) = P(\xi_0^{-1} > x)$. Since $W(1)$
has a continuous strictly increasing distribution, $G$ is well defined, nondecreasing
and right continuous. Call $G^{-1}$ the nondecreasing right continuous generalized
inverse of $G$ and set
$$
\xi^{-1}_{N,x} = G^{-1}\left(N^{1/\alpha}\{W(x + 1/N) - W(x/N)\}\right).
$$
Then (cf. [9] Section 3)) the array $\{\xi_{N,x} : x \in \mathbb{Z}\}$, defined as function of the
subordinator $W$, has the same distribution of $\{\xi_x : x \in \mathbb{Z}\}$ and
$$
\lim_{N \to \infty} \frac{S_N([Nu])}{N^{1/\alpha}} = W(u), \quad \forall u \in \mathbb{R}, \quad \text{for a.a. } W,
$$
where $S_N$ is defined as the function $S$ in [13], with $\xi^{-1}_x$ replaced by $\xi^{-1}_{N,x}$.
Fix $N \geq 1$ and consider the random walk $X_N(t)$ which jumps from $x$ to $x + 1$
and from $x + 1$ to $x$ at rate $\xi_{N,x}$. Kawazu and Kesten proved in [11] that for a.a.
realizations of the environment and for a suitable slowly varying function $L'(\cdot)$,
$X_N(N^{1+1/\alpha}L'(N)t)/N$ converges weakly to a process $\{Y_t : t \geq 0\}$. The first main
result of this article states that $\{Y_t : t \geq 0\}$ is a Markov process with continuous
paths which is not strongly Markov. In particular, $\{Y_t : t \geq 0\}$ is not a Feller
process. Furthermore, we show in Section 8 that the generator $\mathcal{L}_W$ of $\{Y_t : t \geq 0\}$
is the generalized second-order differential operator
$$
\mathcal{L}_W = \frac{d}{du} \frac{d}{dW}.
$$
We point out that in contrast with the i.i.d. case with $E[\xi_0^{-1}] < \infty$, the noise $W$
entirely survives in the limit. In fact, even the generator depends on the realization
$W$ of the subordinator.
The second main object of this article is the hydrodynamic behavior of a one-dimensional simple exclusion process with conductances \( \xi = \{ \xi_x : x \in \mathbb{Z} \} \). This is the Markov process on \( \{0, 1\}^\mathbb{Z} \) which can be informally described as follows. Denote by \( \eta \) the configurations of \( \{0, 1\}^\mathbb{Z} \) so that \( \eta(x) = 0 \) if site \( x \) is vacant and \( \eta(x) = 1 \) otherwise. We start from a configuration with at most one particle per site. At rate \( \xi_x \) the occupation variables \( \eta(x) \), \( \eta(x + 1) \) are exchanged. The generator \( L \) of this Markov process acts on local functions \( f \) as
\[
Lf(\eta) = \sum_{x \in \mathbb{Z}} \xi_x[f(\sigma^{x,x+1}\eta) - f(\eta)],
\]
where \( \sigma^{x,x+1}\eta \) is the configuration obtained from \( \eta \) by exchanging the variables \( \eta(x) \), \( \eta(x + 1) \):
\[
(\sigma^{x,x+1}\eta)(y) = \begin{cases} 
\eta(x + 1) & \text{if } y = x, \\
\eta(x) & \text{if } y = x + 1, \\
\eta(y) & \text{otherwise}. 
\end{cases}
\]

The hydrodynamic behavior of this exclusion process has been derived in [7], [10] under the law of large numbers [11] for the inverse of the conductances, and previously in [12] under more restrictive assumptions: Assume [13] for some \( 0 < \gamma < \infty \). Fix a continuous initial profile \( \rho_0 : \mathbb{R} \to [0,1] \) and consider a sequence of probability measures \( \mu^N \) on \( \{0, 1\}^\mathbb{Z} \) such that
\[
\lim_{N \to \infty} \mu^N \left\{ \left| \frac{1}{N} \sum_{x \in \mathbb{Z}} H(x/N)\eta(x) - \int H(u)\rho_0(u)du \right| > \delta \right\} = 0
\]
for every \( \delta > 0 \) and every continuous functions \( H \) with compact support. As proven in [7], [10], if \( \mathbb{P}^\mu_N \) stands for the probability measure on the path space \( D(\mathbb{R}_+, \{0, 1\}^\mathbb{Z}) \) induced by the initial state \( \mu^N \) and the Markov process speeded up by \( N^2 \), then for any \( t \geq 0 \),
\[
\lim_{N \to \infty} \mathbb{E}^\mu_N \left\{ \left| \frac{1}{N} \sum_{x \in \mathbb{Z}} H(x/N)\eta_t(x) - \int H(u)\rho(t, u)du \right| > \delta \right\} = 0
\]
for every \( \delta > 0 \) and every continuous functions \( H \) with compact support. Here \( \rho \) is the solution of the heat equation
\[
\partial_t \rho = \gamma^{-1} \partial_x^2 \rho
\]
with initial condition \( \rho_0 \), \( t \) stands for the time variable and \( u \) for the macroscopic space variable.

In view of the discussion of the first part of this introduction, assume that the environment consists of a sequence of i.i.d. random variables \( \{\xi_x : x \in \mathbb{Z}\} \) and that the distribution of \( \xi_0^{-1} \) belongs to the domain of attraction of an \( \alpha \)-stable law, \( 0 < \alpha < 1 \). Recall the definition of the array \( \{\xi_{N,x} : x \in \mathbb{Z}\} \). By extending the methods developed in [7], [10], we can prove a quenched hydrodynamic limit for the exclusion process with random conductance given by the array \( \{\xi_{N,x} : x \in \mathbb{Z}\} \). Theorem 2.5 below states that, for almost all trajectories \( W \), the density profile of the exclusion process with random conductances \( \{\xi_{N,x} : x \in \mathbb{Z}\} \) evolves on the time scale \( N^{1+1/\alpha} \) as the solution of
\[
\partial_t \rho = \mathcal{L}_W \rho ,
\]
where $\mathcal{L}_W$ is the generalized second order differential operator defined above. From this quenched result we deduce in Theorem 2.2 an annealed result for the original exclusion process with random conductances given by \{\xi_x : x \in \mathbb{Z}\}.

The asymptotic evolution of a tagged particle is also examined. Under some assumptions on the solution of the hydrodynamic equation (1.7), we show that the asymptotic behavior $u(t)$ of a tagged particle initially at the origin is described by the differential equation

$$
\frac{d}{dt} u(t) = \begin{cases} 
-\frac{1}{\rho_t(u(t))} \frac{d\rho_t}{dW}(u(t)) & \text{if } \frac{d\rho_t}{dW}(u(t)) < 0 \\
-\frac{1}{\rho_t(u(t)-)} \frac{d\rho_t}{dW}(u(t)) & \text{if } \frac{d\rho_t}{dW}(u(t)) > 0 \\
0 & \text{otherwise}.
\end{cases}
$$

In this formula $\rho_t$ is the solution of (1.7), the differential $d/dW$ is defined by (2.2), and $(df/dt)(t_0) = \lim_{\epsilon \to 0} \frac{f(t_0 + \epsilon) - f(t_0)}{\epsilon}$.

2. Notation and results

We state in this section the main results of the article. In what follows, for simplicity of notation, we assume that \{\xi_x^{-1} : x \in \mathbb{Z}\} is a sequence of i.i.d. non-negative $\alpha$–stable random variables, $0 < \alpha < 1$, defined on some probability state space $(E, \mathcal{E}, \mathbb{Q})$, i.e., we assume that

$$
E_\mathcal{E}\{\exp\{-\lambda \xi_x^{-1}\}\} = \exp\{-c_0 \lambda^\alpha\}, \quad \lambda > 0,
$$

for some positive constant $c_0$. The reader can check that all the results and proofs presented below can be easily extended to the general case where $\xi_x^{-1}$ belongs to the domain of attraction of an $\alpha$–stable law.

Let us fix some basic notation: given an interval $I \subset \mathbb{R}$ and a metric space $\mathcal{Y}$, we write $D(I, \mathcal{Y})$ for the space of c càdlàg functions $f : I \to \mathcal{Y}$, endowed with the Skorohod metric $d_S$ [3], [6], and we denote by $D(f)$ the set of discontinuity points of $f$. If $\mathcal{Y} = \mathbb{R}$, the generalized inverse of $f$ is defined as

$$
 f^{-1}(u) = \sup\{v \in I : f(v) \leq u\}.
$$

Moreover, we denote by $C(\mathcal{Y})$, $C_b(\mathcal{Y})$, $C_c(\mathcal{Y})$, $C_0(\mathcal{Y})$, respectively, the space of continuous real functions on $\mathcal{Y}$, the space of bounded continuous real functions on $\mathcal{Y}$, the space of continuous real functions on $\mathcal{Y}$ with compact support and the space of bounded continuous real functions on $\mathcal{Y}$ vanishing at infinity, i.e. such that for any $\varepsilon > 0$ the function has modulus smaller than $\varepsilon$ outside a suitable bounded subset $U \subset \mathcal{Y}$.

In what follows, we will introduce several processes defined in terms of the Brownian motion or the $\alpha$–stable subordinator. To this aim, let $B$ be the Brownian motion with $\mathbb{E}[B(t)^2] = 2t$, defined on some probability space $(\mathcal{X}, \mathcal{F}, \mathbb{P})$, and let $L(t,y)$ be the local time of $B$. Then, $\mathbb{P}$–almost surely,

$$
\int_a^b L(t,y)dy = \int_0^t \mathbf{1}\{a \leq B(s) \leq b\}ds
$$

for all $t \geq 0$, $a \leq b$. In this formula, $\mathbf{1}\{A\}$ stands for the indicator function of the set $A$. 
Let $W$ be a double sided $\alpha$–stable subordinator defined on some probability space $(\Omega, \mathcal{F}, P)$ \cite[Section III.2]{2}. Namely, $W(0) = 0$, $W$ has non-negative independent increments such that for all $s < t$

\[
E\left[\exp \left\{ -\lambda (W(t) - W(s)) \right\} \right] = \exp \left\{ -c_0 \lambda^\alpha (t - s) \right\}
\]

for all $\lambda > 0$ and the same positive constant $c_0$ as in (\ref{2.1}). The sample paths of $W$ are càdlàg, strictly increasing and of pure jump type, in the sense that

\[
W(u) = \sum_{0 < v \leq u} \{W(v) - W(v-\}) .
\]

The jumps at location $(u, W(u) - W(u-))$ have a Poisson distribution with intensity $c_0^{1/\alpha} w^{-\alpha} du dw$ on $\mathbb{R} \times \mathbb{R}_+$. Given a realization of the subordinator $W$, denote by $\nu$ the Radon measure $dW^{-1}$, so that

\[
\int f(u)\nu(du) = \int f(W(u)) du
\]

for all $f \in C_c(\mathbb{R})$. The support of $\nu$ is given by

\[
\text{supp}(\nu) = W(\mathbb{R}) = \{W(x), W(x-): x \in \mathbb{R}\}.
\]

Finally, given a Radon measure $\mu$ on $\mathbb{R}$ and a Borel function $f$, we set

\[
\int_{[u, v]} f(x)\mu(dx) = \begin{cases} 
\int_{(u, v]} f(x)\mu(dx) & \text{if } u \leq v, \\
-\int_{[v, u]} f(x)\mu(dx) & \text{if } u > v.
\end{cases}
\]

2.1. Random walk with random conductances. Given $N \geq 1$, $x \in \mathbb{Z}$ and a realization $\{\xi_x : x \in \mathbb{Z}\}$ of the environment, consider the random walk $X_N^\xi(t|x)$ on $\mathbb{Z}$ having starting point $x$ and generator $L_{\xi,N}$ given by

\[
(L_{\xi,N}f)(x/N) = N^{1+1/\alpha} \xi_x \left\{ f(x+1) - f(x) \right\} + N^{1+1/\alpha} \xi_{x-1} \left\{ f(x-1) - f(x) \right\}.
\]

To describe the asymptotic behavior of this random walk, fix a realization of the subordinator $W$ and set

\[
\psi(t|u) = \int_{\mathbb{R}} L(t, u - v)\nu(dv), \quad \psi^{-1}(t|u) = \sup \{s \geq 0 : \psi(s|u) \leq t\}.
\]

It is known \cite[V.2.11]{4} that

\[
Z(t|u) = u + B\left(\psi^{-1}(t|u)\right), \quad t \geq 0, u \in W(\mathbb{R}),
\]

is a strong Markov process on $W(\mathbb{R})$. Let $Y(t|u)$, with $t \geq 0$ and $u \in \mathbb{R}$, be the process defined by

\[
Y(t|u) = Y_W(t|u) = W^{-1}\left(Z(t|W(u))\right).
\]

For $u$ in $\mathbb{R}$, set $[u] = \min\{k \in \mathbb{Z} : k \geq u\}$. Kawazu and Kesten proved in \cite{11} that the law of $X_N^\xi(t|[uN])/N$ averaged over the environment $\xi$ converges in distribution to the law of $Y(t|u)$ averaged over $W$.

We examine in Section 3 the process $\{Y_t : t \geq 0\}$. We first show that for a.a. realizations of the subordinator $W$, $\{Y_t : t \geq 0\}$ is Markov process with continuous paths which is not strongly Markov and therefore not Feller.

The definition of the generator $\mathcal{L}_W = \frac{d}{du} \frac{d}{dW}$ of the process $\{Y_t : t \geq 0\}$ requires some notation. Fix a realization of the subordinator $W$. Denote by $C_{W,0}(\mathbb{R})$ (resp. $C_{W,1}(\mathbb{R})$) the set of bounded (resp. bounded which vanish at $\pm \infty$) càdlàg functions
$f : \mathbb{R} \to \mathbb{R}$ such that $D(f) \subset D(W)$. $C_{W,0}(\mathbb{R})$ is provided with the usual sup norm $\| \cdot \|_\infty$. Let $\mathcal{D}_W$ be the set of functions $f$ in $C_{W,0}(\mathbb{R})$ such that

$$f(x) = a + bW(x) + \int_0^x dW(y) \int_0^y g(z)dz \quad \forall x \in \mathbb{R},$$

for some function $g$ in $C_{W,0}(\mathbb{R})$ and some $a, b$ in $\mathbb{R}$. One can check that this function $g$ is unique.

Define the linear operator $\mathcal{L}_W : \mathcal{D}_W \to C_{W,0}(\mathbb{R})$ by setting $\mathcal{L}_W f = g$. Formally,

$$\mathcal{L}_W = \frac{d}{dx} \frac{d}{dW}.$$ Alternatively, one can introduce the generalized derivative $\frac{df}{dW}$ as follows

$$\frac{df}{dW}(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon) - f(x)}{W(x + \epsilon) - W(x)},$$

if the above limit exists and is finite. Due to Lemma 0.9 in [5], given a right continuous function $f$ and a continuous function $h$, the following identities are equivalent

$$\frac{df}{dW}(x) = h(x), \quad \forall x \in \mathbb{R},$$

$$f(b) - f(a) = \int_{[a,b]} h(y)dW(y), \quad \forall a < b.$$ Hence, a function $f \in C_{W,0}(\mathbb{R})$ belongs to $\mathcal{D}_W$ if and only if $\frac{df}{dW}(x)$ is well defined and derivable, and $\frac{df}{dW}(x) \in C_{W,0}(\mathbb{R})$. In this case

$$\mathcal{L}_W f = \frac{d}{dx} \frac{d}{dW}f = \frac{d}{dx} \left( \frac{df}{dW} \right).$$

We point out that a function $f$ in $\mathcal{D}_W \cap C(\mathbb{R})$ must be constant, otherwise due to $\left(\frac{df}{dW}\right)(x)$ would be 0 on $\mathcal{D}(W)$, which is a dense set of $\mathbb{R}$. Since $\frac{df}{dW}(x)$ is derivable it must be 0 everywhere. The equivalence between (2.3) and (2.4) allows to conclude.

Denote by $\{P_t : t \geq 0\}$ the semigroup of the Markov process $Y_t$ so that for a bounded Borel function $H : \mathbb{R} \to \mathbb{R}$,

$$(P_t H)(u) = E[H(Y(t\wedge u))], \quad u \in \mathbb{R}.$$ In Section 3 we prove

**Theorem 2.1.** The space $C_{W,0}(\mathbb{R})$ is $P_t$–invariant and $\{P_t : t \geq 0\}$ is a strongly continuous contraction semigroup on $C_{W,0}(\mathbb{R})$ with infinitesimal generator $\mathcal{L}_W = \frac{d}{dx} \frac{d}{dW}$ defined on the domain $\mathcal{D}_W$.

In particular, given $\rho_0$ in $C_0(\mathbb{R})$, the function $\rho_W(t, u) = P_t\rho_0(u), t \geq 0, u \in \mathbb{R}$, is continuous in $t$ and càdlàg in $u$, $D(\rho_W(t, \cdot)) \subset D(W)$ and $\rho_W(0, u) = \rho_0(u)$. Moreover, we show in [8] that $\rho_W(t, u)$ belongs to $\mathcal{D}_W$, $\rho_W(t, u)$ is strictly positive and

$$\frac{\partial}{\partial t}\rho_W(t, u) = \frac{d}{du} \frac{d}{dW}\rho_W(t, u)$$

for all $t > 0$. 


2.2. Annealed hydrodynamic limit. Let $\mathcal{X} = \{0, 1\}^\mathbb{Z}$ and denote by the Greek letter $\eta$ the configurations of $\mathcal{X}$ so that $\eta(x) = 0$ if site $x$ is vacant for the configuration $\eta$ and $\eta(x) = 1$ otherwise.

For each fixed realization $\{\xi_x : x \in \mathbb{Z}\}$, consider the exclusion process on $\mathcal{X}$ with random conductances $\{\xi_x : x \in \mathbb{Z}\}$. This is the Markov process on $\{0, 1\}^\mathbb{Z}$ with generator $L$ given by \eqref{eq:generator}. Given $T > 0$ and a probability measure $\mu$ on $\mathcal{X}$, let $\mathbb{P}^{\xi,N}_\mu$ be the law on the path space $D([0, T], \mathcal{X})$ of the exclusion process $\{\eta_t : t \geq 0\}$ with initial distribution $\mu$ and generator $L$ speeded up by $N^{1+1/\alpha}$. Expectation with respect to $\mathbb{P}^{\xi,N}_\mu$ is denoted by $\mathbb{E}^{\xi,N}_\mu$.

Denote by $\mathcal{M} = \mathcal{M}(\mathbb{R})$ the space of Radon measures on $\mathbb{R}$ endowed with the vague topology, i.e., $m_n \to m$ if and only if $m_n(f) \to m(f)$ for any $f \in C_c(\mathbb{R})$. Let $\pi^N_t \in \mathcal{M}$ be the empirical measure at time $t$. This is the measure on $\mathbb{R}$ obtained by rescaling space by $N$ and by assigning mass $N^{-1}$ to each particle at time $t$:

$$\pi^N_t = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta_t(x) \delta_x/N,$$

where $\delta_x$ is the Dirac measure concentrated on $u$. For $H$ in $C_c(\mathbb{R})$, $\langle \pi^N_t, H \rangle$ stands for the integral of $H$ with respect to $\pi^N_t$:

$$\langle \pi^N_t, H \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}} H(x/N) \eta_t(x).$$

A sequence of probability measures $\{\mu_N : N \geq 1\}$ on $\mathcal{X}$ is said to be associated to a profile $\rho_0 : \mathbb{R} \to [0, 1]$ if $\pi^N_t$ converges to $\rho_0(u) du$, as $N \uparrow \infty$:

$$\lim_{N \to \infty} \mu_N \left\{ \left| \langle \pi^N_t, H \rangle - \int_{\mathbb{R}} H(u) \rho_0(u) du \right| > \delta \right\} = 0$$

for all $H \in C_c(\mathbb{R})$ and for all $\delta > 0$.

The following theorem describes the hydrodynamic behavior of the exclusion process with random conductances $\xi_x$:

**Theorem 2.2.** Let $\rho_0 : \mathbb{R} \to [0, 1]$ be a uniformly continuous function and let $\{\mu_N : N \geq 1\}$ be a family of probability measures on $\mathcal{X}$ associated to $\rho_0$. Then, for all $T > 0$, all $\delta > 0$ and all $H \in C_c(\mathbb{R})$,

$$\lim_{N \to \infty} \int \Omega(d\xi) \mathbb{E}^{\xi,N}_{\mu_N} \left[ \sup_{0 \leq t \leq T} \left| \langle \pi^N_t, H \rangle - \int_{\mathbb{R}} H(u) (P^\xi_t \rho_0)([u]_N) du \right| > \delta \right] = 0,$$

where $P^\xi_t$ is the Markov semigroup associated to the random walk $X^\xi_N(t \cdot)/N$ and $[u]_N = [uN]/N$.

**Corollary 2.3.** Given a uniformly continuous function $\rho_0 : \mathbb{R} \to [0, 1]$, for each $N \geq 1$ on a common probability space $(\Theta_N, F_N, Q_N)$ one can define a double sided $\alpha$–stable subordinator $W$ and an exclusion process $\eta_t$ with law $\int \Omega(d\xi) \mathbb{P}^{\xi,N}_{\mu_N}$ such that, for all $T > 0$, all $\delta > 0$ and all $H \in C_c(\mathbb{R})$,

$$\lim_{N \to \infty} Q_N \left( \sup_{0 \leq t \leq T} \left| \langle \pi^N_t, H \rangle - \int_{\mathbb{R}} H(u) \rho_W (t, u) du \right| > \delta \right) = 0,$$

where $\rho_W (t, u) = P_t \rho_0(u)$.

The proof of Theorem 2.2 is presented in Section 5 and Corollary 2.3 is a straightforward consequence of Theorem 2.2.
The asymptotic behavior of a tagged particle can be recovered from the hydrodynamic limit of the process. Fix an initial density profile \( \rho_0 : \mathbb{R} \rightarrow [0,1] \) in \( C_c(\mathbb{R}) \) and let \( \{ \mu_N : N \geq 1 \} \) be a family of probability measures on \( \mathcal{X} \) associated to \( \rho_0 \) and with bounded support in the sense that there exists \( A > 0 \) such that \( \mu_N \{ \eta(x) = 1 \} = 0 \) for all \( |x/N| > A, N \geq 1 \). Assume that the origin is occupied at time 0. Tag the particle at the origin and let the process evolve according to the generator (1.5) speeded up by \( N^{1+1/\alpha} \). Denote by \( x_t^N \) the position of the tagged particle at time \( t \). Since particles cannot jump over other particles and since, according to the previous theorem, the density profile at time \( t \) is approximated by \( P_t^{\xi,N} \rho_0, x_t^N/N \) must be close to \( u_t^{\xi,N} \), the unique solution of
\[
\int_{-\infty}^{u_t^{\xi,N}} (P_t^{\xi,N} \rho_0)([u]_N) \, du = \int_{-\infty}^{0} \rho_0(u) \, du.
\]
Note that \( u_t^{\xi,N} \) is uniquely determined by this equation because \( P_t^{\xi,N} \rho_0 \) is strictly positive and, due to Lemma 4.4, is Lebesgue integrable. In Section 5 we prove

**Theorem 2.4.** Fix an initial density profile \( \rho_0 : \mathbb{R} \rightarrow [0,1] \) in \( C_c(\mathbb{R}) \) and let \( \{ \mu_N : N \geq 1 \} \) be a family of probability measures on \( \mathcal{X} \) associated to \( \rho_0 \), with bounded support and conditioned to have a particle at the origin. Then, for all \( t > 0 \) and all \( \delta > 0 \),
\[
\lim_{N \to \infty} \int \Omega(d\xi) \mathbb{P}_{\mu_N}^{\xi,N} \left[ \left| x_t^N/N - u_t^{\xi,N} \right| > \delta \right] = 0.
\]

### 2.3. Quenched hydrodynamic limit.

For \( N \geq 1 \), \( x \) in \( \mathbb{Z} \), set
\[
c_x = c_x(W, N) = \frac{1}{\gamma_x}, \quad \gamma_x = \gamma_x(W, N) = N^{1/\alpha} \left\{ W \left( \frac{x + 1}{N} \right) - W \left( \frac{x}{N} \right) \right\}.
\]

Note that \( c_x \) equals the constant \( \xi_{x,N} \) defined in the introduction. Trivially, \( \{ \gamma_x : x \in \mathbb{Z} \} \) is a family of i.i.d. \( \alpha \)-stable random variables such that
\[
E \left[ \exp \left\{ -\lambda \gamma_x \right\} \right] = \exp \{-c_0 \lambda^\alpha\}
\]
for \( \lambda > 0 \). In particular, for each \( N \geq 1 \), \( \{ \gamma_x(W, N) : x \in \mathbb{Z} \} \) has the same distribution as \( \{ \xi_{x,1}^{-1} : x \in \mathbb{Z} \} \).

Consider the exclusion process on \( \mathbb{Z} \) in which we exchange the occupation variables \( \eta(x), \eta(x+1) \) at rate \( c_x \). This is the Markov process on \( \mathcal{X} \) whose generator \( \mathcal{L}_N \) acts on local functions \( f : \mathcal{X} \rightarrow \mathbb{R} \) as
\[
(\mathcal{L}_N f)(\eta) = \sum_{x \in \mathbb{Z}} c_x \left\{ f(\sigma^{x,x+1}\eta) - f(\eta) \right\},
\]
where \( \sigma^{x,x+1}\eta \) is the configuration defined by (1.6).

Given \( T > 0 \) and a probability measure \( \mu \) on \( \mathcal{X} \), let \( \mathbb{P}_\mu^{W,N} \) be the law on the path space \( D([0,T], \mathcal{X}) \) of the exclusion process \( \{ \eta_t : t \geq 0 \} \) with initial distribution \( \mu \) and generator \( \mathcal{L}_N \) speeded up by \( N^{1+1/\alpha} \).

**Theorem 2.5.** Let \( \rho_0 : \mathbb{R} \rightarrow [0,1] \) be a continuous function in \( C_0(\mathbb{R}) \) and let \( \{ \mu_N : N \geq 1 \} \) be a family of probability measures on \( \mathcal{X} \) associated to \( \rho_0 \). Then, for almost all \( W \) and all \( T > 0 \) the empirical measure \( \pi_t^N \) converges in probability to the measure \( \rho_W(t,u) \, du \), where \( \rho_W(t,u) = (P_t \rho_0)(u) \):
\[
\lim_{N \to \infty} \mathbb{P}_{\mu_N}^{W,N} \left[ \sup_{0 \leq t \leq T} \left| \langle \pi_t^N, H \rangle - \int_{\mathbb{R}} H(u) \rho_W(t,u) \, du \right| > \delta \right] = 0.
\]
for all \( H \in C_c(\mathbb{R}) \) and for all \( \delta > 0 \).

As observed above in the annealed case, the asymptotic behavior of a tagged particle can be recovered from the hydrodynamic limit.

**Theorem 2.6.** Let \( \rho_0 : \mathbb{R} \to [0,1] \) be a continuous function with compact support. Let \( \{\mu_N : N \geq 0\} \) be a family of probability measures on \( X \) associated to \( \rho_0 \), with bounded support and conditioned to have a particle at the origin. Then, for almost all \( W \) and for all \( t > 0 \), \( x_1^N / N \) converges in \( \mathbb{P}^{W,N}_{\mu_N} \)-probability, as \( N \uparrow \infty \), to \( u_W(t) \) given by

\[
\int_{-\infty}^{u_W(t)} \rho_W(t,u) \, du = \int_{-\infty}^{0} \rho_0(u) \, du .
\]

By Proposition 3.3 and Corollary 3.4 (iv) with \( H = \rho_0 \) and \( \rho = 1 \), \( \rho_W(t,\cdot) \) is strictly positive and integrable so that \( u_W(t) \) is uniquely determined. We prove Theorem 2.6 in Subsection 2.5 and that \( u_W(t) \) is continuous. Moreover, we derive in Lemma 5.5, under some extra assumptions on \( \rho_W(t,\cdot) \), the following differential equation for \( u_W \):

\[
\frac{d}{dt}u_W(t) = \begin{cases} 
\frac{1}{\rho_t(u_W(t))} \frac{d\rho_t}{dW}(u_W(t)) & \text{if } \frac{d\rho_t}{dW}(u_W(t)) < 0 \\
1 - \frac{1}{\rho_t(u_W(t)-)} \frac{d\rho_t}{dW}(u_W(t)) & \text{if } \frac{d\rho_t}{dW}(u_W(t)) > 0 \\
0 & \text{otherwise} .
\end{cases}
\]

In this formula \( \rho_t(\cdot) = \rho_W(t,\cdot) \), the differential \( d/dW \) is defined by (2.2), and \( (df/dt+)(t_0) = \lim_{\epsilon \downarrow 0} \epsilon^{-1}[f(t_0 + \epsilon) - f(t_0)] \).

### 3. Quasi-diffusions

In this section we study the Markov processes \( Y \) and \( Z \), defined in Section 2.4, in terms of the Brownian motion \( B \) and the subordinator \( W \). All the results presented in this section hold for almost all realizations of the subordinator \( W \), although not always explicitly stated.

Fix a realization of the subordinator \( W \). Recall that \( \nu = dW^{-1} \) and that the support of \( \nu \) coincides with \( \overline{W(\mathbb{R})} = \{W(x), W(x-) : x \in \mathbb{R}\} \).

#### 3.1. The Markov process \( Z(t|u) \)

We briefly recall some results from [13], [14] applied to the Markov process \( Z(t|u) = u + B(\psi^{-1}(t|u)) \) with state space \( \overline{W(\mathbb{R})} \) (see in particular Theorems 1.2.1 and 3.3.1 of [13] and Theorem 3.2 of [14]). Denote by \( \{Q_t : t \geq 0\} \) the Markov semigroup associated to \( Z(t|u) \) acting on the space \( C_0(\overline{W(\mathbb{R})}) \):

\[
Q_t f(u) = \mathbb{E}[f(Z(t|u))]
\]

for all \( f \) in \( C_0(\overline{W(\mathbb{R})}) \). By endowing \( C_0(\overline{W(\mathbb{R})}) \) with the uniform norm \( \| \cdot \|_\infty \), \( \{Q_t : t \geq 0\} \) is a strongly continuous semigroup of contraction operators: for all \( f \in C_0(\overline{W(\mathbb{R})}) \), \( Q_t f \) belongs to \( C_0(\overline{W(\mathbb{R})}) \), \( \|Q_t f\|_\infty \leq \|f\|_\infty \) and

\[
\lim_{s \to 0} \|Q_{t+s} f - Q_t f\|_\infty = 0 ,
\]

for all \( t > 0 \). The same statement holds for \( t = 0 \) if \( s \) takes only positive values.
Let \( D_W \) be the set of functions \( f \) in \( C_0(W(\mathbb{R})) \) for which there exists a function \( h \) in \( C_0(W(\mathbb{R})) \) and \( a, b \) in \( \mathbb{R} \) such that
\[
f(u) = a + bu + \int_0^u dv \int_0^v h(w) \nu(dw)
\]
for all \( u \) in \( W(\mathbb{R}) \). One can check that \( h \) is univocally determined. We denote \( h \) by
\[
h = \frac{d}{dW} f = Lf.
\]
Then, \( L_W : D_W \to C_0(W(\mathbb{R})) \) is the generator of the Markov semigroup \( \{Q_t : t \geq 0\} \) on \( C_0(W(\mathbb{R})) \).

We prove in \([8]\) that the process \( Z \) admits a strictly positive symmetric transition density function \( q_t(x,y) \) w.r.t. \( \nu \).

**Theorem 3.1.** There exists a strictly positive Borel function \( q \),
\[
q : (0,\infty) \times W(\mathbb{R}) \times W(\mathbb{R}) \to (0,\infty),
\]
symmetric in \( x, y \), such that
\[
\mathbb{E}[f(Z(t|x))] = \int f(y)q_t(x,y)\nu(dy)
\]
for all \( t > 0 \), \( x \) in \( W(\mathbb{R}) \) and \( f \) in \( C_0(W(\mathbb{R})) \). Moreover,
\[
\int q_t(x,y)\nu(dy) = 1
\]
for every \( t > 0 \), \( x \) in \( W(\mathbb{R}) \) and \( q_t(\cdot,y) \in D_W \).

\[
\partial_t q_t(\cdot,y) = L_W q_t(\cdot,y)
\]
for every \( t > 0 \) and \( y \) in \( W(\mathbb{R}) \).

### 3.2. Markovian properties of the process \( Y(t|u) \)

Denote by \( \{x_j : j \geq 1\} \) the jump points of \( W \), which form a countable dense set in \( \mathbb{R} \). Since \( W^{-1}(W(x_j-)) = W^{-1}(W(x_j)) \), the function \( W^{-1} : W(\mathbb{R}) \to \mathbb{R} \) is not one to one and the process \( Y(t|u) = W^{-1}(Z(t|W(u))) \) with space state \( \mathbb{R} \) could be a non Markov process. The following proposition clarifies the Markovian properties of the process \( Y \).

**Proposition 3.2.** The stochastic process \( Y \) has continuous paths. It is Markov but not strongly Markov. In particular, it is not a Feller Markov process.

**Proof.** The continuity of paths can be proved by the same arguments used in Lemma 4 of \([3] \). In what follows we denote respectively by \( \mathbb{P}_y^Y, \mathbb{P}_z^Z \) the law of \( Y(\cdot|y) \) and \( Z(\cdot|z) \), where \( y \in \mathbb{R}, z \in W(\mathbb{R}) \), and by \( \mathbb{E}_y^{Y}, \mathbb{E}_z^{Z} \) the related expectations. Moreover, we write \( \omega \) for a generic path in \( D([0,\infty),\mathbb{R}) \) and \( \theta_t \omega \) for the time–translated path \( \theta_t \omega(s) = \omega(t+s) \).

First we prove that \( Y \) is a Markov process. To this aim, we fix \( y \in \mathbb{R}, t > 0 \) and let \( \mathcal{A}, \mathcal{B} \subset D([0,\infty),\mathbb{R}) \) be of the form
\[
\mathcal{A} = \{ \omega : \omega(t_i) \in [a_i, b_i] \forall 1 \leq i \leq n \},
\]
\[
\mathcal{B} = \{ \omega : \omega(s_i) \in [c_i, d_i] \forall 1 \leq i \leq k \},
\]
where \( 0 \leq t_1 < t_2 < \cdots < t_n < t \) and \( 0 \leq s_1 < s_2 < \cdots < s_k \).
Due to the definition of \(Y\),
\[ P^Y_y (\omega \in A, \theta_t \omega \in B) = P^Z_{W(y)} (W^{-1} \circ \omega \in A, \theta_t (W^{-1} \circ \omega) \in B). \]
Since \(Z\) is a Markov process, the previous expression is equal to
\[ E^Z_{W(y)} \left[ 1 \{ W^{-1} \circ \omega \in A \} P^Z_{Z(t)W(y)} (W^{-1} \circ \omega \in B) \right]. \]
We claim that \(Z(t|W(y)) = W(Y(t|y))\) with probability 1. In fact, we know that \(Z(t|W(y))\) has value \(W(Y(t|y))\) or \(W(Y(t|y))\). If \(W(Y(t|y)) = W(Y(t|y))\) the conclusion is trivial. Otherwise, it must be \(W(Y(t|y)) = W(x_j -)\) for some \(j\).
Since a.s. \(\nu\) has no atoms, the countable set \(\{W(x_j -) : j \geq 1\}\) has zero \(\nu\)-measure and due to Theorem 5.1,
\[ P \left[ Z(t|W(y)) \in \{W(x_j -) : j \geq 1\} \right] = 0. \]
This allows to conclude that \(Z(t|W(y)) = W(Y(t|y))\) with probability 1. Therefore,
\[ P^Z_{Z(t)W(y)} (W^{-1} \circ \omega \in B) = P^Z_{W(Y(t|y))} (W^{-1} \circ \omega \in B) \]
\(P^Z_{W(y)}\)-a.s. By definition of \(Y\), putting all previous identities together, we get that
\[ P^Y_y [\omega \in A, \theta_t \omega \in B] = E^Z_{W(y)} \left[ 1 \{ W^{-1} \circ \omega \in A \} P^Z_{W(Y(t|y))} [W^{-1} \circ \omega \in B] \right] \]
\[ = E^Y_y [1 \{ \omega \in A \} P^Y_{Y(t|y)} [\omega \in B]]. \]
This proves that \(Y\) is a Markov process.

We now show that \(Y(t|y)\) is not strongly Markovian with respect to the filtration \(\mathcal{F}^Y_t = \sigma(Y_s : s \leq t)\). Fix \(y\) in \(\mathbb{R}\) such that \(0 < W(y-) < W(y)\) and consider the sets
\[ A = \left\{ \exists \delta > 0 : Y_s \leq y \text{ for } 0 \leq s \leq \delta \right\}, \]
\[ B = \left\{ \exists \delta > 0 : Z_s \leq W(y) \text{ for } 0 \leq s \leq \delta \right\}. \]
Let \(\tau\) be the first time the process \(Y_t\) reaches \(y\): \(\tau = \inf\{t \geq 0 : Y_t = y\}\) and let
\(\sigma\) be the first time the process \(Z_t\) reaches \(W(y)\) or \(W(y-)\): \(\sigma = \inf\{t \geq 0 : Z_t = W(y)\text{ or } Z_t = W(y-)\}\). Since \(Y_t\) has continuous paths, \(\tau\) is a stopping time for the filtration \(\{\mathcal{F}^Y_t : t \geq 0\}\) and \(\sigma\) is a stopping time for the filtration \(\{\mathcal{F}^Z_t : t \geq 0\}\), where \(\mathcal{F}^Z_t = \sigma(Z_s : s \leq t)\).

Assume, by contradiction, that \(Y\) is strongly Markov. By the strong Markov property,
\[ P^Y_0 [\theta_\tau Y \in A] = P^Y_0 [P^Y_{Y_\tau} [A]]. \]
Since \(Y_\tau = y\), the previous expectation is equal to \(P^Y_0 [A]\). By definition of the process \(Y\), this probability corresponds to \(P^Z_{W(y)} [B]\). This last probability is equal to 0 in view of the construction of \(Z\) through the Brownian motion.

On the other hand, by construction of the Markov process \(Y(t|0)\), by definition of the random times \(\tau, \sigma\) and since \(\sigma\) is a stopping time,
\[ P^Y_0 [\theta_\tau Y \in A] = P^Y_0 [\theta_\tau Z \in B] = E^Y_0 [P^Y_{Z_\tau} [B]]. \]
Since \(0 < W(y-)\), \(Z_\tau = W(y-)\). In particular, \(P^Y_{Z_\tau}\) is equal to \(P^Z_{W(y-)} [B]\) and this probability is equal to 1 by construction of the process \(Z\) through the Brownian motion.
Finally, $Y$ cannot be a Feller Markov process since otherwise it would be strong Markov process.

3.3. The generator of the process $Y(t|u)$. We obtain in this subsection the generator of the Markov process $Y$. To keep notation simple, we denote $Z(tW(u))$ by $Z_t$ and $Y(t|u)$ by $Y_t$. Moreover, we write $\{x_j : j \geq 1\}$ for the jump points of $W$, which form a countable dense set.

Denote by $d_W$ the distance in $\mathbb{R}$ defined by $d_W(x, y) = |W(x) - W(y)|$ and by $\mathbb{R}_W$ the completion of $\mathbb{R}$ with respect to this distance. It is easy to check that $\mathbb{R}_W$ is obtained by dividing in two each jump point of $W$: $\mathbb{R}_W = \mathbb{R} \cup \{x_j : j \geq 1\}$ and

$$d_W(x_j^-, y) = |W(x_j^-) - W(y)|, \quad d_W(x_j^-, x_k^-) = |W(x_j^-) - W(x_k^-)|$$

for every $y$ in $\mathbb{R}$, $j, k \geq 1$.

Recall that $W(\mathbb{R}) = \{W(x) : x \in \mathbb{R}\} \cup \{W(x_j^-) : j \geq 1\}$. Let

$$W_e : \mathbb{R}_W \rightarrow \overline{W(\mathbb{R})}$$

be given by $W_e(x) = W(x)$ for $x$ in $\mathbb{R}$, $W_e(x_j^-) = W(x_j^-)$ for $j \geq 1$. $W_e$ is an isometry from $(\mathbb{R}_W, d_W)$ to $(\overline{W(\mathbb{R})}, d)$, where $d$ is the usual Euclidean distance in $\mathbb{R}$. Its inverse, $W_e^{-1} : \overline{W(\mathbb{R})} \rightarrow \mathbb{R}_W$, is given by $W_e^{-1}(W(x)) = x$, $W_e^{-1}(W(x_j^-)) = x_j^-$. Since $W_e$ is an isometry, all the results concerning the process $Z_t$ with state space $\overline{W(\mathbb{R})}$ can be trivially restated in terms of the pullback process $X_t = W_e^{-1}(Z_t)$ with state space $\mathbb{R}_W$. In particular, $X_t$ is a strong Markov process and, denoting by $Q_t$ its Markov semigroup, $\{Q_t : t \geq 0\}$ is a strongly continuous semigroup of contraction operators acting on the space $C_0(\mathbb{R}_W)$ with norm $\|\cdot\|_{\infty}$. Let us describe its generator $L_W$.

We have seen that the domain of the generator $L_W$ of the Markov process $Z_t$ is $D_W$. Hence the domain of the generator $L_W$ is given by the set $D_W$ where $D_W = \{f \circ W_e : f \in D_W\}$. Let $f, h \in C(\overline{W(\mathbb{R})})$ be as in (3.3) for suitable constants $a, b$. By a change of variables, it simple to check that the functions $F = f \circ W_e$ and $H = h \circ W_e$, belonging to $C(\mathbb{R}_W)$, satisfy the identity

$$F(x) = a + bW(x) + \int_0^x dW(y) \int_0^y H(z)dz, \quad x \in \mathbb{R}, \quad (3.3)$$

and therefore, by continuity,

$$F(x_j^-) = F(x_j^-), \quad \forall j \geq 1.$$

Viceversa, if $F, H \in C(\mathbb{R}_W)$ fulfill (3.3), then $f = F \circ W_e^{-1} \in C(\overline{W(\mathbb{R})})$ and $h = H \circ W_e^{-1} \in C(\overline{W(\mathbb{R})})$ satisfy (3.2). In particular, we get that

$$D_W = \{F \in C_0(\mathbb{R}_W) : \exists H \in C_0(\mathbb{R}_W), \exists a, b \in \mathbb{R} \text{ satisfying (3.3)} \}.$$  

Moreover, due to the above observations, we obtain that $L_W F = H$ for all $F \in D_W$ and $H \in C_0(\mathbb{R}_W)$ as in (3.3).

We now turn to the process $Y_t$. Recall the definition of the spaces $C_{W,0}(\mathbb{R})$ and $\mathcal{D}_W$ introduced in Section 2.4. $C_{W,0}(\mathbb{R})$ is the set of functions $F : \mathbb{R} \rightarrow \mathbb{R}$ which are càdlàg, whose discontinuities form a subset of $\{x_j : j \geq 1\}$ and which vanish at $\pm \infty$, while $\mathcal{D}_W$ denotes the set of functions $F$ in $C_{W,0}(\mathbb{R})$ such that

$$F(x) = a + bW(x) + \int_0^x dW(y) \int_0^y G(z)dz, \quad \forall x \in \mathbb{R},$$
for some function $G$ in $C_{W,0}(\mathbb{R})$ and some $a$, $b$ in $\mathbb{R}$. $G$ is univocally determined and the linear operator $\mathcal{L}_W : \mathcal{D}_W \to C_{W,0}(\mathbb{R})$ is defined by setting $\mathcal{L}_W F = G$. Formally, $\mathcal{L}_W = \frac{d}{dx} \frac{d}{dW}$.

**Proof of Theorem 2.1** Since $W^{-1}(W(x_j^-)) = W^{-1}(W(x_j))$ we can write $Y_t = \Phi(X_t)$ where $\Phi : \mathcal{D}_W \to \mathbb{R}$ is defined as $\Phi(x) = x$ for all $x \in \mathbb{R}$ and $\Phi(x_j^-) = x_j$ for all $j \geq 1$.

Given a function $H : \mathbb{R} \to \mathbb{R}$, define the function $\mathcal{E}H : \mathcal{D}_W \to \mathbb{R}$ as $\mathcal{E}H = H \circ \Phi$. Viceversa, given a function $h : \mathcal{D}_W \to \mathbb{R}$, define $\mathcal{P}h : \mathbb{R} \to \mathbb{R}$ as $\mathcal{P}h(x) = h(x)$ for all $x \in \mathbb{R}$. One can easily check that $\mathcal{E}$ maps $C_{W,0}(\mathbb{R})$, $\mathcal{D}_W$ bijectively onto $C_0(\mathbb{R})$, $\mathcal{D}_W$ with inverse function given by $\mathcal{P}$. Moreover,

$$\mathcal{L}_W H = \mathcal{P}\mathcal{L}_W \mathcal{E}H, \quad \forall H \in \mathcal{D}_W.$$  \hspace{1cm} (3.4)

Since $Y_t = \Phi(X_t)$, for all $H \in C_{W,0}(\mathbb{R})$ we can write

$$P_t H(u) = \mathbb{E}[H(Y(t|u))]=\mathbb{E}[H \circ \Phi(X(t|u))] = Q_t(\mathcal{E}H)(u), \quad \forall u \in \mathbb{R},$$

thus implying that

$$P_t H = \mathcal{P}Q_t \mathcal{E}H, \quad \forall H \in C_{W,0}(\mathbb{R}).$$  \hspace{1cm} (3.5)

Due to the above identity and since $C_0(\mathbb{R})$ is $\mathcal{Q}_t$-invariant, the space $C_{W,0}(\mathbb{R})$ is $P_t$-invariant. Since $\mathcal{E}$ is an isomorphism between the normed spaces $C_{W,0}(\mathbb{R})$ and $C_0(\mathbb{R})$ (endowed of the uniform norm $\| \cdot \|_\infty$) and since the identities (3.3) and (3.5) hold, the fact that $\{Q_t : t \geq 0\}$ is a strongly continuous contraction semigroup on $C_0(\mathbb{R})$ having generator $\mathcal{L}_W$ with domain $\mathcal{D}_W$ implies that $\{P_t : t \geq 0\}$ is a strongly continuous contraction semigroup on $C_{W,0}(\mathbb{R})$ with generator $\mathcal{L}_W$ having domain $\mathcal{D}_W$.

We conclude this subsection with a result which follows easily from Theorem 3.1 by a change of variables. It will be particularly useful in the study of the limiting behavior of the tagged particle in the exclusion process with random conductances.

**Proposition 3.3.** The Borel function $p$ defined on $(0, \infty) \times \mathbb{R} \times \mathbb{R}$ as

$$p_t(x,y) = q_t(W(x),W(y))$$

is the transition density function of the Markov process $Y$ w.r.t. the Lebesgue measure. $p_t(\cdot, \cdot)$ is a strictly positive symmetric function and

$$\int_{\mathbb{R}} p_t(x,y) dy = 1, \quad (P_t f)(x) = \int_{\mathbb{R}} p_t(x,y) f(y) dy,$$

for all $t > 0$, $x$ in $\mathbb{R}$ and functions $f$ in $C_{W,0}(\mathbb{R})$.

Next result is a consequence of Theorem 2.1 and this proposition.

**Corollary 3.4.** Fix a function $H$ in $C_0(\mathbb{R})$.

(i) For every $t \geq 0$, $P_t H$ is a càdlàg function vanishing at infinity.

(ii) If $H$ has compact support, $P_t H$ belong to $L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} du P_t H(u) = \int_{\mathbb{R}} du H(u).$$

(iii) As $t \downarrow 0$, $P_t H$ converges to $H$ pointwisely. If $H$ has compact support this limit takes also place in $L^1(\mathbb{R})$. In this case, $P_{t+s} H$ converges to $P_t H$ in $L^1(\mathbb{R})$ as $s \to 0$. 

(iv) For any function $H$ in $C_c(\mathbb{R})$ and any function $\rho$ in $C_b(\mathbb{R})$,\[
\int du \,(P_tH)(u) \,\rho(u) = \int du \,H(u) \,(P_t\rho)(u)
\]

Proof. Statement (i) follows from Theorem 2.1 and statement (ii) from Proposition 3.3. The first claim of (iii) follows from Theorem 2.1. To prove the second claim of (iii) assume, without loss of generality, that $H$ is positive and has compact support. By part (ii) of this corollary, $\int P_t H(u) du$ is equal to $\int H(u) du$. In particular, by Scheffé Theorem, $P_t H$ converges to $H$ in $L^1(\mathbb{R})$. It follows from Proposition 3.3 that the semigroup $P_t$ is a contraction in $L^1(\mathbb{R})$. The third claim of (iii) follows from this observation and the convergence of $P_t H$ to $H$ in $L^1(\mathbb{R})$. Claim (iv) follows from the symmetry of the transition density function $p_t(x, y)$ and Fubini’s theorem. □

4. Random walk with random conductances

Fix a realization of the subordinator $W$ and $N \geq 1$, and recall the definition of the random variables $\{c_x : x \in \mathbb{Z}\}$ given in (2.7). We examine in this section the limiting behavior of the continuous–time random walk on $\mathbb{Z}$ which jumps from $x$ to $x+1$ at rate $c_x$ and from $x$ to $x-1$ are rate $c_{x-1}$. We first recall some results due to Stone [17].

Given a Radon measure $\mu$ on $\mathbb{R}$ with support, denoted by $\text{supp}(\mu)$, unbounded from below and from above, for each $x \in \text{supp}(\mu)$ and $t \geq 0$ set

\[
\psi(t|x, \mu) = \int_\mathbb{R} L(t, y-x) \mu(dy), \quad \psi^{-1}(t|x, \mu) = \sup \{s \geq 0 : \psi(s|x, \mu) \leq t\}.
\]

(4.1)

Then $\psi(\cdot | x, \mu)$ is a continuous function and $\psi^{-1}(\cdot | x, \mu)$ is a nondecreasing càdlàg function. Set

\[
Z(t|x, \mu) = B(\psi^{-1}(t|x, \mu)) + x.
\]

$Z = \{Z(t|x, \mu) : t \geq 0\}$, defined on probability space $(\mathcal{X}, \mathcal{F}, \mathbb{P})$ as the Brownian motion $B$, is a strong Markov process with state space $\text{supp}(\mu)$ and paths in the Skohorod space $D([0, \infty), \mathbb{R})$ endowed of the Skohorod metric $d_S$ [4, V.2.11].

By Theorem 1 and Corollary 1 in [17], we have

**Proposition 4.1.** Let $\{\mu_n\}_{n \geq 0}$, $\mu$ be Radon measures on $\mathbb{R}$ with support unbounded from below and from above. Suppose that:
- $\mu_n \rightharpoonup \mu$ vaguely,
- if $y_n \in \text{supp}(\mu_n)$ is a converging sequence as $n \uparrow \infty$, then $\lim_{n \uparrow \infty} y_n \in \text{supp}(\mu)$.

Let $x_n \in \text{supp}(\mu)$ be a converging sequence with $\lim_{n \uparrow \infty} x_n = x$. Then,

\[
\lim_{n \uparrow \infty} d_S(Z(\cdot|x_n, \mu_n), Z(\cdot|x, \mu)) = 0 \quad \mathbb{P} \text{ a.s.}
\]

Let us recall another consequence of the results in [17] (see also Section 2 in [11]):

**Proposition 4.2.** Let $\{x_k\}_{k \in \mathbb{Z}}$ satisfy $x_k < x_{k+1}$, $\lim_{k \to \pm \infty} x_k = \pm \infty$. Fix positive constants $\{w_k\}_{k \in \mathbb{Z}}$ and set $\mu = \sum_{k \in \mathbb{Z}} w_k \delta_{x_k}$. Then $Z(\cdot|x, \mu)$ is the continuous–time random walk on $\{x_k\}_{k \in \mathbb{Z}}$ starting in $x_j$ such that after reaching site $x_k$ it remains in $x_k$ for an exponential time with mean

\[
\frac{w_k}{x_{k+1} - x_{k-1}} \frac{(x_{k+1} - x_k)(x_k - x_{k-1})}{x_{k+1} - x_{k-1}}
\]
and then it jumps to \( x_{k-1}, x_{k+1} \) respectively with probability
\[
\frac{x_{k+1} - x_k}{x_{k+1} - x_{k-1}} \quad \text{and} \quad \frac{x_k - x_{k-1}}{x_{k+1} - x_{k-1}}.
\]

Given \( N \geq 1, x \in \mathbb{Z} \) consider the random walk \( X_N(t|x) \) on \( \mathbb{Z} \) having starting point \( x \) and generator
\[
L_N f(x) = c_x N^{1+1/\alpha} \{ f(x+1) - f(x) \} + c_{x-1} N^{1+1/\alpha} \{ f(x-1) - f(x) \}.
\]

Denote the transition probabilities of \( X_N \) by \( p^N \):
\[
p^N_t(x, y) = P[X_N(t|x) = y] \quad (4.2)
\]
for \( x, y \in \mathbb{Z} \). By symmetry, \( p^N_t(x, y) = p^N_t(y, x) \).

For \( N \geq 1 \), let \( \nu_N \) be the discrete measure defined by
\[
\nu_N = \frac{1}{N} \sum_{x \in \mathbb{Z}} \delta_{W(x/N)}.
\]
As \( N \uparrow \infty \), \( \nu_N \) converges vaguely to \( \nu \). By definition of \( \nu_N \) and by Proposition 4.2 the random walk \( X_N/N \) can be expressed as a space–time change of the Brownian motion:
\[
N^{-1}X_N(\cdot | x) \sim W^{-1}(Z(\cdot | W(x/N), \nu_N)) ,
\]
where \( \sim \) means that the two processes have the same law.

Recall that \([u]_N = [uN]/N\) and define
\[
Y_N(t|u) = W^{-1}\left(Z(t | W([u]_N), \nu_N)\right).
\]
It follows from the two previous identities that
\[
N^{-1}X_N(\cdot | [u]_N) \sim Y_N(\cdot | u).
\]

**Lemma 4.3.** Let \( Y(t|u) = W^{-1}(Z(t | W(u), \nu)) \). For all \( u \in \mathbb{R} \),
\[
\lim_{N \to \infty} d_S(Y_N(\cdot | u), Y(\cdot | u)) = 0 \quad \text{P-a.s.}
\]
Moreover, for all \( u \in \mathbb{R} \) and for all \( T > 0 \),
\[
\lim_{N \to \infty} \sup_{0 \leq t \leq T} |Y_N(t|u) - Y(t|u)| = 0 \quad \text{P-a.s.}
\]

**Proof.** If \( y_N \in \supp(\nu_N) \) and \( y_N \to y \in \mathbb{R} \) then \( y \in \supp(\nu) \) (see the proof of Lemma 2 in [11]). Since \( \lim_{N \to \infty} W([u]_N) = W(u) \) for all \( u \in \mathbb{R} \), and since \( \nu_N \) converges vaguely to \( \nu \), by Proposition 4.1,
\[
\lim_{N \to \infty} d_S\left(Z(\cdot | W([u]_N), \nu_N), Z(\cdot | W(u), \nu)\right) = 0, \quad \text{P-a.s.}
\]
The first claim of the lemma follows by the same arguments used in the proof of Proposition 1 in [11]. On the other hand, since by Proposition 3.2 \( Y(\cdot | u) \) has continuous paths \( \mathbb{P} \)-a.s., the second statement of the lemma follows from the first one. \( \square \)

Recall that \( P_t \) stands for the semigroup of the process \( \{ Y(t) : t \geq 0 \} \) and let \( \{ P^N_t : t \geq 0 \} \) be the semigroup of the process \( \{ Y_N(t) : t \geq 0 \} \). Hence, given a bounded Borel function \( H \),
\[
P^N_t H(u) = \mathbb{E}[H(Y_N(t|u))].
\]
It follows from Lemma 4.3 and the dominated convergence theorem that $P_t^N H$ converges pointwisely to $P_t H$ for every bounded continuous function $H$ and every $t \geq 0$.

Since $W$ is strictly increasing, $W^{-1}$ is a continuous function. In particular, since $\lim_{x \to \pm \infty} W(x) = \pm \infty$, $H \circ W^{-1}$ belongs to $C_c(\mathbb{R})$, $C_0(\mathbb{R})$ as soon as $H$ belongs. In the next three lemmata, we prove properties of the operators $P_t^N$ and $P_t$ and some convergence results of $P_t^N$ to $P_t$.

**Lemma 4.4.** Fix a continuous function $H : \mathbb{R} \to \mathbb{R}$ with compact support. For every $t \geq 0$, $P_t^N H$ belongs to $L^1(\mathbb{R})$ and

$$\int_\mathbb{R} du \ P_t^N H(u) = \frac{1}{N} \sum_{x \in \mathbb{Z}} H(x/N) .$$

*Proof.* Assume without loss of generality that $H \geq 0$. Since the transition probability $p_t(x,y)$ is symmetric,

$$\int_\mathbb{R} du \ P_t^N H(u) = N^{-1} \sum_{x,y \in \mathbb{Z}} p_t^N(x,y) H(y/N)$$

$$= N^{-1} \sum_{y \in \mathbb{Z}} H(y/N) \sum_{x \in \mathbb{Z}} p_t^N(y,x) = N^{-1} \sum_{y \in \mathbb{Z}} H(y/N) .$$

This proves the identity and that $P_t^N H$ belongs to $L^1(\mathbb{R})$. □

**Lemma 4.5.** Fix a function $H$ in $C_c(\mathbb{R})$ and $t \geq 0$.

(i) $P_t^N H$ converges in $L^1(\mathbb{R})$ to $P_t H$.

(ii) If $H$ has bounded variation in $\mathbb{R}$ then, for every $t \geq 0$, $P_t H$ has also bounded variation.

(iii) $P_t^N H$ also converges to $P_t H$ with respect to the counting measure:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}} |(P_t^N H)(x/N) - (P_t H)(x/N)| = 0 .$$

(iv) For any $\varepsilon > 0$ there exists $\Psi \in C_c(\mathbb{R})$ such that

$$\int_\mathbb{R} du | P_t H(u) - \Psi(u) | \leq \varepsilon , \quad \frac{1}{N} \sum_{x \in \mathbb{Z}} | P_t H(x/N) - \Psi(x/N) | \leq \varepsilon$$

for $N$ large enough.

*Proof.* Without loss of generality, fix a positive function $H$ in $C_c(\mathbb{R})$. Since $P_t^N H$ converges pointwisely to $P_t H$, in view of Corollary 3.4 (ii) and Lemma 4.4, $P_t^N H$ converges in $L^1(\mathbb{R})$ to $P_t H$ by Scheffé Theorem.

It is easy to couple two copies of the random walk $Y_N$ in such a way that

$$Y_N (t|u) \leq Y_N (t|v) , \quad \forall t \geq 0 , \quad \forall u \leq v .$$

To prove (ii), assume that $H$ is a continuous function of bounded variation in $\mathbb{R}$. Then there exist bounded, continuous, increasing functions, $H_+$ and $H_-$, such that $H = H_+ - H_-$. By the coupling, $P_t^N H_\pm$ are bounded increasing functions. Taking the pointwise limit as $N \uparrow \infty$ of the identity $P_t^N H = P_t^N H_+ - P_t^N H_-$ we get that $P_t H = P_t H_+ - P_t H_-$ where $P_t H_\pm$ are bounded increasing functions. Therefore, $P_t H$ has bounded variation.

For $N \geq 1$, and a right continuous function $f : \mathbb{R} \to \mathbb{R}$, let $T_N f : \mathbb{R} \to \mathbb{R}$ be given by $(T_N f)(u) = f([uN]/N)$. We claim that $T_N P_t H$ converges in $L^1(\mathbb{R})$
to $P_tH$. By Corollary 3.2 (i), $P_tH$ is right continuous. In particular, $T_N P_t H$ converges pointwisely to $P_t H$. For $x$ in $\mathbb{Z}$, denote by $V_x$ the total variation of $P_t H$ on $[x, x+1]$. Let $V : \mathbb{R} \to \mathbb{R}_+$ be given by $V(u) = V_{\lfloor u \rfloor}$. $V$ belongs to $L^1(\mathbb{R})$ because $P_t H$ has bounded variation due to (ii). Moreover, $T_N P_t H \leq P_t H + V$ which implies that $T_N P_t H$ belongs to $L^1(\mathbb{R})$. By the dominated convergence theorem, $T_N P_t H$ converges to $P_t H$ in $L^1(\mathbb{R})$ because $T_N P_t H$ converges pointwisely to $P_t H$.

The sum appearing in (iii) can be rewritten as

$$\int_{\mathbb{R}} du \left| (P_t^N H)(u) - (T_N P_t H)(u) \right|.$$ 

Since $P_t^N H$ and $T_N P_t H$ converge to $P_t H$ in $L^1(\mathbb{R})$, statement (iii) follows.

Fix $\varepsilon > 0$. By Corollary 3.4 (ii), $P_t H$ belongs to $L^1(\mathbb{R})$. In particular, there exists a continuous function with compact support $\Psi$ which approximates the first part of this lemma it vanishes as $N \to \infty$.

By Lemma 4.4 the sum inside the integral is uniformly bounded in $P_t H$.

Moreover, $\lambda > 0$, denote by $\{R_N^\lambda : \lambda > 0\}$ the resolvent associated to the semigroup $\{P_t^N : t \geq 0\}$:

$$R_N^\lambda H = \int_0^\infty dt e^{-\lambda t} P_t^N H$$

for $H$ in $C_c(\mathbb{R})$.

**Lemma 4.6.** Fix a function $g$ in $C_c(\mathbb{R})$. Then,

$$\lim_{N \to \infty} \limsup_{t \to 0} \frac{1}{N} \sum_{x \in \mathbb{Z}} \left| P_t^N g(x/N) - g(x/N) \right| = 0,$$

$$\lim_{\lambda \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}} \left| \lambda R_N^\lambda g(x/N) - g(x/N) \right| = 0.$$ 

Moreover, for every $\lambda > 0$,

$$\frac{1}{N} \sum_{x \in \mathbb{Z}} \left| \lambda R_N^\lambda g(x/N) \right| \leq \frac{1}{N} \sum_{x \in \mathbb{Z}} \left| g(x/N) \right|.$$ 

**Proof.** The first expression is bounded above by

$$\| P_t^N g - P_t g \|_1 + \| P_t g - g \|_1 + \frac{1}{N} \sum_{x \in \mathbb{Z}} \left| N \int_{x-1/N}^{x/N} g(u) du - g(x/N) \right|,$$

where $\| \cdot \|_1$ stands for the $L_1(\mathbb{R})$ norm. By Lemma 4.3 (i), the first expression vanishes as $N \uparrow \infty$. By Corollary 3.4 (iii), the second expression vanishes as $t \downarrow 0$. Since the third expression vanishes as $N \uparrow \infty$, the first claim of the lemma is proved.

By definition of the resolvent, the second expression is less than or equal to

$$\int_0^\infty dt \lambda e^{-\lambda t} \frac{1}{N} \sum_{x \in \mathbb{Z}} \left| P_t^N g(x/N) - g(x/N) \right|.$$ 

By Lemma 4.4 the sum inside the integral is uniformly bounded in $t$ and $N$. By the first part of this lemma it vanishes as $N \uparrow \infty$, $t \downarrow 0$. This proves the second claim.

The third claim follows from the definition of the resolvent and Lemma 4.3. \(\quad\Box\)
5. HYDRODYNAMIC BEHAVIOR

We prove in this section the main Theorems of the article. We first examine the convergence of the empirical measure $\pi^N$.

Recall that $M$ stands for the space of Radon measures endowed with the vague topology. Fix a realization of the subordinator $W$ and $T > 0$. For each probability measure $\mu$ on $\{0, 1\}^Z$, denote by $Q_{W, N}^\mu$ the measure on the path space $D([0, T], M)$ induced by the measure $\mu$ and the process $\pi^N_t$, introduced in (2.6), evolving according to the generator (2.8) speeded up by $N^{1+1/\alpha}$. Fix a continuous profile $\rho_0: \mathbb{R} \rightarrow [0, 1]$ and consider a sequence $\{\mu_N: N \geq 1\}$ of measures on $\{0, 1\}^Z$ associated to $\rho_0$. Let $Q_W$ be the probability measure on $D([0, T], M)$ concentrated on the deterministic path $\pi(t, du) = \rho_W(t, u)du$, where $\rho_W(t, u) = P_t \rho_0$.

**Proposition 5.1.** The sequence of probability measures $Q_{W, N}^\mu$ converges, as $N \uparrow \infty$, to $Q_W$.

The proof of this result is divided in two parts. In Section 5.1, we show that the sequence $\{Q_{W, N}^\mu: N \geq 1\}$ is tight and in Section 5.2 we characterize the limit points of this sequence.

5.1. Tightness. Recall that we denote by $M$ the space of positive Radon measures endowed with the vague topology. In particular, a sequence $\mu_n$ in $M$ converges to $\mu$ if

$$\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu$$

for all continuous functions $f$ with compact support.

This topology can be defined through a metric. It is indeed not difficult to find a countable family of functions $\{f_j: j \geq 1\}$ in $C_c(\mathbb{R})$ (even in $C^\infty_c(\mathbb{R})$) such that

(i) For each $\epsilon > 0$, integer $k \geq 1$ and continuous function $f$ with support contained in $[-k, k]$, there exists $j$ such that $\|f_j - f\|_\infty \leq \epsilon$ and such that the support of $f_j$ is contained in $[-k - 1, k + 1]$.

(ii) For each integer $k \geq 1$, there exists $j$ such that $f_j$ is a non-negative function with support contained in $[-k - 1, k + 1]$ and greater or equal to 1 in $[-k, k]$.

It is easy to check that $\mu_n$ converges to $\mu$ vaguely if and only if $\int f_j d\mu_n$ converges to $\int f_j d\mu$ for all $j$. In particular, $\mu_n$ converges to $\mu$ if and only if $d(\mu_n, \mu)$ vanishes, where $d$ is the metric defined by

$$d(\mu, \mu') = \sum_{j \geq 1} \frac{1}{2j} \left\{ 1 \wedge \left| \int f_j d\mu - \int f_j d\mu' \right| \right\}.$$  

The space $M$ endowed with this metric is a complete separable metric space.

The closure of a subset $M$ of $M$ is compact if and only if

$$\sup_{\mu \in M} \mu(K) < \infty$$

for all compact sets $K$ of $\mathbb{R}$. In particular, if $g_j$ is a non-negative smooth function with support contained in $[-j - 1, j + 1]$ and greater or equal to 1 in $[-j, j]$, a set $M$ such that

$$\sup_{\mu \in M} \sum_{j \geq 1} \frac{1}{2j} \int g_j d\mu < \infty$$

is compact. We refer to Section A.10 of [16] for the proofs of all the previous statements concerning the vague topology in $M$. 
We prove in this subsection that the sequence of probability measures \( \{ \mathbb{Q}^{W,N}_t : N \geq 1 \} \) is tight in \( D([0,T], \mathcal{M}) \). The method in \cite{10} consists in proving relative compactness of an auxiliary process and then showing that both processes are close.

For \( \lambda > 0 \), let \( \{ X^\lambda,N_t : t \geq 0 \} \) be the \( \mathcal{M} \)-valued Markov process such that
\[
X^\lambda,N_t(H) = \langle \pi^N_t, R^N_\lambda H \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}} (R^N_\lambda H)(x/N)\eta_t(x)
\]
for every \( H \in C_c(\mathbb{R}) \), where \( \{ R^N_\lambda : \lambda > 0 \} \) is the resolvent associated to the semi-group \( \{ P^N_t : t \geq 0 \} \). Here and below, we do not distinguish between a continuous function \( H : \mathbb{R} \to \mathbb{R} \) and its restriction to \( \mathbb{Z}_N = \{ x/N : x \in \mathbb{Z} \} \).

**Lemma 5.2.** For each \( T > 0 \) and \( \lambda > 0 \), the sequence of processes \( \{ X^\lambda,N : N \geq 1 \} \) is tight in \( D([0,T], \mathcal{M}) \).

**Proof.** By Proposition IV.1.7 in \cite{12}, it is enough to show that \( \{ X^\lambda,N_t(g) : 0 \leq t \leq T \} \) is tight in \( D([0,T], \mathbb{R}) \) for all functions \( g \) in \( C^1_c(\mathbb{R}) \). Note that the underlying space in \cite{12} is compact, while we are working here on \( \mathbb{R} \). However, in both cases the topology is given by a metric of type \( (5.1) \) and in both cases the compacts are characterized by integral of functions. It is easy to adapt the proof of Proposition IV.1.7 to the present case.

Fix a function \( g \) in \( C^1_c(\mathbb{R}) \) and let \( g^N_\lambda = R^N_\lambda g \). Note that \( g^N_\lambda \) belongs to \( l^2(\mathbb{Z}_N) \), the space of square summable functions \( f : \mathbb{Z}_N \to \mathbb{R} \), because so does \( g \). Since \( g^N_\lambda \) is the solution of
\[
\lambda g^N_\lambda - \mathbb{L}_N g^N_\lambda = g \quad (5.2)
\]
There is here a slight abuse of notation. We are using the same symbol \( \mathbb{L}_N \) for the generator of \( X^\lambda,N_t \) and the generator of \( N^{-1}X^\lambda,N_t \). The context makes clear to which operator we are referring to. Multiplying \( (5.2) \) by \( N^{-1}g^N_\lambda(\cdot/N) \) and summing over \( x \), we obtain that
\[
\frac{\lambda}{N} \sum_{x \in \mathbb{Z}} g^N_\lambda(x/N)^2 + N^{1/\alpha} \Theta \sum_{x \in \mathbb{Z}} c_x(\nabla N g^N_\lambda(x/N)^2) = \frac{1}{N} \sum_{x \in \mathbb{Z}} g^N_\lambda(x/N)g(x/N),
\]
where \( \nabla N \) stands for the discrete gradient: \( \langle \nabla N h(x/N) = N \{ h(x + 1/N) - h(x/N) \} \). By Schwarz inequality, we obtain that
\[
\frac{\lambda}{2N} \sum_{x \in \mathbb{Z}} g^N_\lambda(x/N)^2 + \frac{N^{1/\alpha}}{N^2} \sum_{x \in \mathbb{Z}} c_x(\nabla N g^N_\lambda(x/N)^2) \leq \frac{1}{2\lambda N} \sum_{x \in \mathbb{Z}} g(x/N)^2.
\]
Since \( g \) is continuous with compact support, \( N^{-1} \sum_{x \in \mathbb{Z}} g(x/N)^2 \) is bounded uniformly over \( N \). Hence,
\[
\sup_N \frac{N^{1/\alpha}}{N^2} \sum_{x \in \mathbb{Z}} c_x(\nabla N g^N_\lambda(x/N)^2) \leq \frac{C(g)}{\lambda} \quad (5.3)
\]
In this formula and below, \( C(g) \) stands for some finite constant depending only on \( g \).

An elementary computation shows that \( \mathcal{L}_N(\pi^N_t, H) = \langle \pi^N_t, \mathbb{L}_N H \rangle \), where \( \mathcal{L}_N \) is the operator defined in \cite{23}. In particular, the process \( M^N_{t,\lambda} \) defined by
\[
M^N_{t,\lambda} = X^\lambda,N_t(g) - X^\lambda,N_0(g) - \int_0^t ds \{ \lambda(\pi^N_s, g^N_\lambda) - \langle \pi^N_s, g^N_\lambda \rangle \} \quad (5.4)
\]
is a martingale with quadratic variation
\[
\langle M^{N,\lambda} \rangle_t = \int_0^t ds \frac{N^{1/\alpha}}{N^3} \sum_{x \in \mathbb{Z}} c_x (\nabla_N g^N(x/N))^2 \{\eta_s(x+1) - \eta_s(x)\}^2.
\]
Due to the previous estimate (5.3), the quadratic variation \(\langle M^{N,\lambda} \rangle_t\) satisfies
\[
\langle M^{N,\lambda} \rangle_t \leq C(g) T/\lambda N, \quad \forall 0 \leq t \leq T.
\]
Moreover, by Lemma 4.6,
\[
\sup_{N \geq 1} \frac{\lambda}{N} \sum_{x \in \mathbb{Z}} |g^N(x/N)| \leq C(g).
\]
Hence, given \(N \geq 1\) and constants \(0 \leq a < b\),
\[
\left| \int_a^b ds \left\{ \lambda \langle \pi^N_s, g^N \rangle - \langle \pi^N_s, g \rangle \right\} \right| \leq C(g) (b - a).
\]
Due to the decomposition (5.4), to the estimates (5.5) and (5.7), and to Doob inequality, it is simple to prove (cf. [12] Chap. IV, p. 55) that, given \(\varepsilon > 0\),
\[
\lim_{N \to \infty} \sup_{\tau, \theta} \mathbb{P}_{\mu^N} \left( sup_{0 \leq t \leq T} |X^{\lambda,N}_t(g) - X^{\lambda,N}_{\tau+\theta}(g)| > \varepsilon \right) = 0,
\]
where the supremum \(\sup_{\tau, \theta}\) is over all stopping times \(\tau\) bounded by \(T\), and over \(\theta\) with \(0 \leq \theta \leq \gamma\).
In addition, due to (5.6),
\[
\sup_{\tau \geq 0} \sup_{N \geq 1} |X^{\lambda,N}_t(g)| \leq \frac{C(g)}{\lambda}.
\]
As discussed in [12], Chap. IV, p. 51, (5.8) and (5.9) allow to apply Prohorov’s theorem, thus concluding the proof of the tightness of \(X^{\lambda,N}_t(g)\).

**Corollary 5.3.** The sequence of measures \(\{Q^{W,N}_{\mu^N} : N \geq 1\}\) is tight.

**Proof.** It is enough to show that for every function \(g\) in \(C^1_c(\mathbb{R})\) and every \(\varepsilon > 0\), there exists \(\lambda > 0\) such that
\[
\lim_{N \to \infty} \mathbb{P}^{W,N}_{\mu^N} \left( sup_{\tau \geq 0} \sup_{0 \leq t \leq T} |X^{\lambda,N}_t(\lambda g) - \langle \pi^N_t, g \rangle| > \varepsilon \right) = 0
\]
because in this case the tightness of \(\pi^N_t\) follows from the tightness of \(X^{\lambda,N}_t\). Since there is at most one particle per site the expression inside the absolute value is less than or equal to
\[
\frac{1}{N} \sum_{x \in \mathbb{Z}} |\lambda g^N(x/N) - g(x/N)|.
\]
By Lemma 4.6 this expression vanishes as \(N \uparrow \infty, \lambda \uparrow \infty\). \(\square\)
5.2. Proof of Theorem 2.5. We start this subsection with a generalization of [15], [2].

Lemma 5.4. Fix a function $H$ in $C_c(\mathbb{R})$ and a sequence of probability measures \( \{\mu_N : N \geq 1\} \) in \( \{0,1\}^2 \). For each \( t \geq 0 \),
\[
\lim_{N \to \infty} E_{\mu_N}^{W,N} \left[ \left\{ \frac{1}{N} \sum_{x \in \mathbb{Z}} H(x/N) \eta_t(x) - \frac{1}{N} \sum_{x \in \mathbb{Z}} (P_t^N H)(x/N) \eta_0(x) \right\}^2 \right] = 0 ,
\]
where \( E_{\mu_N}^{W,N} \) denotes the expectation w.r.t. \( \mu_N \).

Proof. We assume that the number of particles is finite \( \mu_N \)-a.s. One can extend the proof to the general case by the same arguments used in [15]. For each \( x \) in \( \mathbb{Z} \), denote by \( J_{t,x+1} \) the net current of particles through the bond \( \{x, x + 1\} \) in the time interval \( [0,t] \). This is the total number of particles which jumped from \( x \) to \( x+1 \) minus the total number of particles which jumped from \( x+1 \) to \( x \) in the time interval \( [0,t] \). Denote by \( M_{t,x+1}^x \) the martingale associated to \( J_{t,x+1}^x \): \( M_{t,x+1}^x = J_{t,x+1}^x - N^{1+1/\alpha} c_x \int_0^t \{ \eta_t(x) - \eta_t(x + 1) \} \, ds \) and let \( M_t^x = M_{t-1}^x - M_{t,x+1}^x \).

With this notation,
\[
\eta_t(x) = \sum_{y \in \mathbb{Z}} p_t^N(x,y) \eta_0(y) + \sum_{y \in \mathbb{Z}} \int_0^t p_{t-s}^N(x,y) \, dM_s^y ,
\]
where \( p_t^N(\cdot,\cdot) \) is the transition probability defined in (4.2). In particular, since \( p_t^N \) is symmetric, the expression inside the square in the statement of the proposition can be rewritten as
\[
\Gamma_t^N = \frac{1}{N} \sum_{y \in \mathbb{Z}} \int_0^t (P_{t-s}^N H)(y/N) \, dM_s^y .
\]

To prove the proposition it is therefore enough to show that \( E_{\mu_N}^{W,N}[\Gamma_t^N]^2 \) vanishes as \( N \uparrow \infty \). Since the martingales \( M_{t,z+1}^z \) are orthogonals, \( E_{\mu_N}^{W,N}[\Gamma_t^N]^2 \) is equal to
\[
\frac{N^{1/\alpha}}{N} \sum_{y \in \mathbb{Z}} \int_0^t ds \left\{ (P_{t-s}^N H)((y + 1)/N) - (P_{t-s}^N H)(y/N) \right\}^2 c_y E_{\mu_N}^{W,N}[a_{y,y+1}(\eta_s)] ,
\]
where \( a_{z,z+1}(\eta) = \{ \eta(z) - \eta(z+1) \}^2 \). Since this function is bounded by 1, the previous expression is less than or equal to
\[
\frac{N^{1/\alpha}}{N} \sum_{y \in \mathbb{Z}} \int_0^t ds c_y \left\{ (P_s^N H)((y + 1)/N) - (P_s^N H)(y/N) \right\}^2 .
\]

Let \( H_t(x/N) = (P_t^N H)(x/N) \) and observe that \( \partial_t H_t = \mathbb{L}_N H_t \). Hence the previous expression can be rewritten as
\[
- \frac{1}{N^2} \int_0^t (H_s,\mathbb{L}_N H_s)_N ds = - \frac{1}{2N^2} \int_0^t \partial_s (H_s, H_s)_N ds = \frac{1}{2N^2} \sum_{y \in \mathbb{Z}} H^2(y/N) \, ds - \frac{1}{2N^2} \sum_{y \in \mathbb{Z}} \left\{ (P_t^N H)(y/N) \right\}^2 ,
\]
where \( (\cdot,\cdot)_N \) denotes the scalar product on \( \ell^2(\mathbb{Z}_N) \) w.r.t. the counting measure. This proves the lemma. \( \square \)
We are now in a position to show that the sequence of probability measures $\mathbb{Q}_{\mu_N}^{W,N}$ converges, as $N \to \infty$, to $\mathbb{Q}_W$.

**Proof of Proposition 5.1** By Corollary 5.3, the sequence $\{\mathbb{Q}_{\mu_N}^{W,N} : N \geq 1\}$ is tight. To prove the lemma we only need to characterize the limit points of this sequence.

Fix a function $H$ in $C_c(\mathbb{R})$. We claim that

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N}^{W,N} \left[ \left| \langle \pi_t^{N}, H \rangle - \int_{\mathbb{R}} H(u) \rho_W(t,u) \, du \right| > \delta \right] = 0 \quad (5.10)$$

for all $0 \leq t \leq T$, $\delta > 0$.

By Lemma 5.4 we only need to prove that

$$\lim_{N \to \infty} \mu_N \left( \frac{1}{N} \sum_{x \in \mathbb{Z}} P_t^N H(x/N) \eta(x) - \int H(u) \rho_W(t,u) \, du \right) = 0.$$

Since there is at most one particle per site, by Lemma 4.6 (iii), we may replace $P_t^N H$ by $P_t H$. By assumption on $\mu_N$ and by Lemma 4.5 (iv), we may also replace $N^{-1} \sum_{x \in \mathbb{Z}} (P_t H)(x/N) \eta(x)$ by $\int (P_t H)(u) \rho_W(u) \, du$. Since $\rho_0$ is continuous and bounded, by Corollary 3.4 (iv), this expression is equal to

$$\int H(u) (P_t \rho_0)(u) \, du = \int H(u) \rho_W(t,u) \, du.$$

This concludes the proof of (5.10).

By (5.10), the finite dimensional distributions (f.d.d.) of $\mathbb{Q}_{\mu_N}^{W,N}$ converge to the f.d.d. of $\mathbb{Q}_W$. Since the f.d.d. characterize the measure, the proposition is proved.

**Proof of Theorem 2.5** Fix a function $H$ in $C_c(\mathbb{R})$. On the one hand, by Theorem 2.1 and Corollary 3.4 (iv), $\int_{\mathbb{R}} H(u) \rho_W(t,u) \, du$ is a bounded continuous function of time. On the other hand, for any continuous function $A : [0,T] \to \mathbb{R}$, the map in $D([0,T], \mathcal{M})$

$$\pi \to \sup_{0 \leq t \leq T} \left| \langle \pi_t, H \rangle - A(t) \right|$$

is bounded and continuous for the Skorohod topology. In particular, Theorem 2.5 follows from Proposition 5.1.

**5.3. Proof of Theorem 2.2** We first claim that

$$\lim_{N \to \infty} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} H(u) \left( P_t \rho_0(u) - P_t^N \rho_0(u) \right) \, du = 0 \quad (5.11)$$

for every function $H$ in $C_c(\mathbb{R})$. Fix $\varepsilon > 0$ and recall that we denote by $(\mathbb{X}, \mathcal{F}, \mathbb{P})$ the probability space in which the processes $Y(t|u)$ and $Y_N(t|u)$ are defined. Expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. If $K$ stands for a compact subset of $\mathbb{R}$ which contains the support of $H$, the previous supremum is bounded above by

$$C_0(H) \sup_{0 \leq t \leq T} \int_K \mathbb{E} \left[ \left| \rho_0(Y(t|u)) - \rho_0(Y_N(t|u)) \right| \right] \, du.$$

Since $\rho_0$ is uniformly continuous, there exists $\delta > 0$ for which the previous is less than or equal to

$$C_0(H) \varepsilon + C_0(H, \rho_0) \int_K \mathbb{P} \left[ \sup_{0 \leq t \leq T} \left| Y(t|u) - Y_N(t|u) \right| > \delta \right] \, du.$$
By Lemma 4.3 and the dominated convergence theorem, the second expression vanishes as \( N \uparrow \infty \) for every \( \delta > 0 \). This proves Claim (5.11).

It follows from (5.11) and Theorem 2.5 that

\[
\lim_{N \to \infty} \mathbb{P}_{\mu}^{W, N} \left[ \sup_{0 \leq t \leq T} \left| \langle \pi_t^N, H \rangle - \int_{\mathbb{R}} H(u) P_t^N \rho_0(u) \, du \right| > \delta \right] = 0.
\]

Since, for each \( N \geq 1 \), \( \{ \gamma_x(W, N) : x \in \mathbb{Z} \} \) has the same distribution as \( \{ \xi_x^{-1} : x \in \mathbb{Z} \} \),

\[
\mathbb{P}_{\mu}^{W, N} \left[ \sup_{0 \leq t \leq T} \left| \langle \pi_t^N, H \rangle - \int_{\mathbb{R}} H(u) P_t^N \rho_0(u) \, du \right| > \delta \right] = \mathbb{P}_{\mu}^{\xi, N} \left[ \sup_{0 \leq t \leq T} \left| \langle \pi_t^N, H \rangle - \int_{\mathbb{R}} H(u) P_t^N \xi_\rho_0(u) \, du \right| > \delta \right]
\]

in distribution. In particular, Theorem 2.6 follows from Theorem 2.5.

\[ \square \]

5.4. **The tagged particle.** We examine in this subsection the asymptotic behavior of the tagged particle. As mentioned in Section 2, the law of large numbers for the tagged particle in the case of compactly supported initial density profiles is a direct consequence of the hydrodynamic limit and the fact the relative order among particles is preserved by the dynamics.

We first prove that the position of the tagged particle, \( u_W(t) \), is uniquely determined.

Recall the notation introduced in Section 2 and assume that the initial density profile \( \rho_0 \) belongs to \( C_c(\mathbb{R}) \). It follows from Proposition 3.3 that

\[ \rho_W(t, u) = (P_t \rho_0)(u) = \int p_t(u, v) \rho_0(v) \, dv. \]

Since \( p_t(u, v) \) is strictly positive, \( \rho_W(t, \cdot) \) is strictly positive as soon as \( \rho_0 \) is not identically equal to 0. On the other hand, since \( P_t \) is a contraction in \( L^1(\mathbb{R}) \), \( \rho_W(t, \cdot), t \geq 0 \), belongs to \( L^1(\mathbb{R}) \). In particular, for each \( s \geq 0 \), there exists a unique \( u_W(s) \) in \( \mathbb{R} \) such that

\[ \int_{-\infty}^{u_W(s)} \rho_W(s, v) \, dv = \int_{-\infty}^{0} \rho_0(v) \, dv. \tag{5.12} \]

The function \( u_W(t) \) is continuous in time. Indeed, on the one hand, it follows from (5.12) with \( s = t, t_n \), that

\[ \left| \int_{u_W(t)}^{u_W(t_n)} \rho_W(t, v) \, dv \right| \leq \int_{\mathbb{R}} \left| \rho_W(t, v) - \rho_W(t_n, v) \right| \, dv. \]

On the other hand, by Corollary 3.3 (iii), \( \rho_W(t_n, \cdot) \) converges in \( L^1(\mathbb{R}) \) to \( \rho_W(t, \cdot) \) if \( t_n \to t \). Since \( \rho_W(t, \cdot) \) is strictly positive, \( u_W(t_n) \) must converge to \( u_W(t) \).

**Proof of Theorem 2.6.** Fix a density profile \( \rho_0 \) in \( C_c(\mathbb{R}) \) and let \( \rho_W(t, u) = P_t \rho_0 \). Observe that \( \rho_W(t, \cdot) \) belongs to \( L^1(\mathbb{R}) \) in virtue of Proposition 3.3. Consider a sequence \( \{ \mu_N : N \geq 1 \} \) of measures associated to \( \rho_0 \), conditioned to have a particle at the origin and such that \( \mu_N \{ \eta(x) = 1 \} = 0 \) for \( |x/N| \) large enough. For \( a \) in \( \mathbb{R} \), denote by \( H_a \) the indicator function of the interval \([a, \infty)\): \( H_a(u) = 1 \{ [a, \infty) \}(u) \).

We first claim that Theorem 2.5 can be extended to such test functions:

\[ \lim_{N \to \infty} \mathbb{P}_{\mu}^{W, N} \left[ \sup_{0 \leq t \leq T} \left| \langle \pi_t^N, H_a \rangle - \int_{a}^{\infty} \rho_W(t, u) \, du \right| > \delta \right] = 0 \]
for all $\delta > 0$, $a$ in $\mathbb{R}$. The same statement holds for $H_a = 1 - H_a$ in place of $H_a$.

Indeed, consider a sequence of compactly supported continuous functions $G_k$ (resp. $\tilde{G}_k$) increasing to $H_a$ (resp. $H_a$). On the one hand, $\langle \pi_{t, N}^\infty, H_a \rangle \geq \langle \pi_{t, N}^\infty, G_k \rangle$. On the other hand, $\langle \pi_{t, N}^\infty, H_a \rangle \leq N^{-1} \sum_x \eta_t(x) - \langle \pi_{t, N}^\infty, \tilde{G}_k \rangle$. Since the total number of particles is conserved, this expression is equal to $N^{-1} \sum_x \eta_0(x) - \langle \pi_{t, N}^\infty, \tilde{G}_k \rangle$. To conclude the proof of the claim it remains to let $N \uparrow \infty$ and then $k \uparrow \infty$ and to recall that $\int \rho_0(u) du = \int \rho_W(t, u) du$.

We are now in a position to prove Theorem 2.4. Fix $\delta > 0$ and assume that $x_t^N/N \geq u_W(t) + \delta$. Since the total numbers to the right of the tagged particle doesn’t change in time, $\sum_{x \geq 0} \eta_0(x) = \sum_{x \geq x_t^N} \eta(x) \leq \sum_{x/N \geq u_W(t) + \delta} \eta(x)$. Dividing by $N$ and letting $N \uparrow \infty$, by the previous observation, we get that

$$\int_0^\infty \rho_0(u) du \leq \int_{u_W(t) + \delta}^\infty \rho_W(t, u) du.$$ 

This contradicts the definition of $u_W(t)$ because $\rho_W(t, \cdot)$ is strictly positive in view of Proposition 5.3. Similarly, one can prove that the event $x_t^N/N \leq u_W(t) - \delta$ has negligible probability as $N \uparrow \infty$. \qed

**Proof of Theorem 2.4.** Let $u_{t, N}^W$, the unique solution of

$$\int_{-\infty}^{u_{t, N}^W} (P_t^N \rho_0)([u]_N) du = \int_{-\infty}^0 \rho_0(u) du.$$ 

Note that $u_{t, N}^\xi$ is uniquely determined by this equation because $P_t^N \rho_0$ is strictly positive and, due to Lemma 4.1, is Lebesgue integrable.

As in the proof of Theorem 2.2, since $\gamma_t(W, N) : x \in \mathbb{Z}$ has the same distribution as $\lbrace \xi_{x-1} : x \in \mathbb{Z} \rbrace$, the random variables

$$P_{t, N}^\xi \left[ x_t^N/N - u_{t, N}^\xi(t) > \delta \right] \quad \text{and} \quad P_{t, N}^W \left[ x_t^N/N - u_{t, N}^W(t) > \delta \right]$$

depending on $\xi$ and $W$, respectively, have the same law.

We claim that for each $t > 0$ and realization $W$, $u_{t, N}^W(t)$ converges to $u_W(t)$ as $N \uparrow \infty$. Indeed, since $\int_{-\infty}^{u_W(t)} \rho_W(t, v) dv = \int_{-\infty}^{u_{t, N}^W(t)} (P_t^N \rho_0)(v) dv$,

$$\left| \int_{u_W(t)}^{u_{t, N}^W(t)} \rho_W(t, v) dv \right| \leq \int_{\mathbb{R}} dv \left[ (P_t^N \rho_0)(v) - (P_t \rho_0)(v) \right].$$

By Lemma 1.3 (i), the right hand side vanishes as $N \uparrow \infty$. Since, by Proposition 5.3, $P_t \rho_0$ is strictly positive, the claim is proved.

In particular, by Theorem 2.5 for all $t > 0$, $\delta > 0$ and a.a. $W$,

$$\lim_{N \to \infty} P_{t, N}^W \left[ x_t^N/N - u_{t, N}^W(t) > \delta \right] = 0.$$ 

Theorem 2.4 follows from the dominated convergence theorem, from the first observation of the proof and from this last observation. \qed

We conclude this section deriving a differential equation for the asymptotic position of the tagged particle. Recall the definition of the derivative $d/dW$ introduced in (2.2) and let $\rho_0(\cdot) = \rho_W(\cdot, \cdot) = P_t \rho_0$.

**Lemma 5.5.** Fix a profile $\rho_0 : \mathbb{R} \to [0, 1]$. Assume that $\rho_0$ belongs to $C_c(\mathbb{R})$ and that

$$\lim_{h \to 0} \int_{\mathbb{R}} \frac{P_h \rho - \rho}{h} = \frac{\partial W \rho_0}{\partial t} = 0.$$
for all \( t > 0 \). Then, \( u_W(t) \) is differentiable both from the right and from the left for \( t > 0 \) and

\[
\frac{d}{dt} u_W(t) = \begin{cases} 
- \frac{1}{\rho_t(u_W(t))} \frac{d\rho_t}{dW}(u_W(t)) & \text{if } \frac{d\rho_t}{dW}(u_W(t)) < 0 \\
- \frac{1}{\rho_t(u_W(t)-)} \frac{d\rho_t}{dW}(u_W(t)) & \text{if } \frac{d\rho_t}{dW}(u_W(t)) > 0 \\
0 & \text{otherwise}. 
\end{cases}
\]

Notice that while \( \rho_t = P_t \rho_0 \), which belongs to the domain of the generator \( \mathfrak{L}_W \) in view of [8], may have discontinuities at \( \{ a_j : j \geq 1 \} \), its derivative \( \frac{d\rho_t}{dW} \) is continuous. On the other hand, the definition of the infinitesimal operator \( \mathfrak{L}_W \) and the fact that \( \rho_t \) belongs to the domain of the generator imply the uniform convergence of \( h^{-1}(P_h \rho_t - \rho_t) \) to \( \mathfrak{L}_W \rho_t \) on the whole real line. We are requiring here the convergence to take place in \( L^1(\mathbb{R}) \). In particular, \( \mathfrak{L}_W \rho_t \) belongs to \( L^1(\mathbb{R}) \).

**Proof.** Fix \( t > 0 \) and take \( h \in \mathbb{R} \) small. Denote \( \mathfrak{L}_W \rho_t \) by \( \lambda_t \). Since

\[
\int_{(-\infty,u_W(t))} \rho_t(v) \, dv = \int_{(-\infty,u_W(t+h))} \rho_{t+h}(v) \, dv,
\]

\[
-\frac{1}{h} \int_{u_W(t)}^{u_W(t+h)} \rho_t(v) \, dv = \int_{-\infty}^{u_W(t+h)} \frac{\rho_{t+h}(v) - \rho_t(v)}{h} \, dv.
\]

By the main assumption of the lemma, we may replace on the right hand side the ratio \( \{\rho_{t+h}(v) - \rho_t(v)\}/h \) by \( \lambda_t \) paying a price of order \( o(h) \). Since \( u_W(t+h) \) converges to \( u_W(t) \) as \( h \to 0 \), we get that

\[
\int_{-\infty}^{u_W(t)} \lambda_t(v) \, dv = -\lim_{h \to 0} \frac{1}{h} \int_{u_W(t)}^{u_W(t+h)} \rho_t(v) \, dv.
\]

We assume that \( h > 0 \) and compute the right derivative of \( u_W(t) \). The left derivative is left to the reader. Assume that \( \int_{(-\infty,u_W(t))} \lambda_t(v) \, dv > 0 \) so that \( u_W(t+h) < u_W(t) \) for \( h \) sufficiently small. Since \( \rho_t \) is a càdlàg function and \( u_W(t+h) < u_W(t) \) for \( h \) sufficiently small, it follows from the previous identity that

\[
\lim_{h \downarrow 0} \frac{u_W(t+h) - u_W(t)}{h} = -\frac{1}{\rho_t(u_W(t)-)} \int_{-\infty}^{u_W(t)} \lambda_t(v) \, dv.
\]

Similar identities can be obtained if \( \int_{(-\infty,u_W(t))} \lambda_t(v) \, dv \) vanishes or is less than 0 and for \( h \uparrow 0 \). Thus, \( u_W(t) \) is differentiable both from the right and from the left. To prove the lemma it remains to show that

\[
\frac{d\rho_t}{dW}(u_W(t)) = \int_{-\infty}^{u_W(t)} \lambda_t(v) \, dv.
\]

Since \( \lambda_t = \mathfrak{L}_W \rho_t \), we have that

\[
\rho_t(u) = a_t + b_t W(u) + \int_0^u W(v) \int_0^v \lambda_t(w) \, dw
\]

for some finite constants \( a_t, b_t \). In particular, for any \( u < v \),

\[
\frac{d\rho_t}{dW}(v) = \frac{d\rho_t}{dW}(u) + \int_u^v \lambda_t(w) \, dw.
\]
Since $\lambda_t$ belongs to $L^1(\mathbb{R})$, letting $u \downarrow -\infty$, we find that $d\rho_t/dW(u)$ converges to some constant $c_t$ and that

$$\frac{d\rho_t}{dW}(u) = c_t + \int_{-\infty}^{u} \lambda_t(w) \, dw.$$ 

Take $u > 0$. Since $W(0) = 0$, integrating this identity with respect to $dW$ in the interval $(0, u]$ we get that

$$\rho_t(u) - \rho_t(0) = c_t W(u) + \int_{(0,u]} W(dv) \int_{-\infty}^{v} \lambda_t(w) \, dw.$$ 

Dividing by $W(u)$, since $\rho_t$ is uniformly bounded and since $\lim_{u \to \infty} W(u) = \infty$

$$c_t = \lim_{u \to \infty} \frac{-1}{W(u)} \int_{(0,u]} W(dv) \int_{-\infty}^{v} \lambda_t(w) \, dw.$$ 

Since the $L^1(\mathbb{R})$ norm of $\rho_t$ is constant in time, $\int_{-\infty}^{\infty} \lambda_t(w) \, dw = 0$ for all $t > 0$. In particular, the previous expression vanishes. This concludes the proof of the lemma. □

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