Noncommutative geometry of phase space

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Abstract

A version of noncommutative geometry is proposed which is based on phase-space rather than position space. The momenta encode the information contained in the algebra of forms by a map which is the noncommutative extension of the duality between the tangent bundle and the cotangent bundle.
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1 Introduction

It has been long conjectured [1] that quantum fluctuations of the gravitational field might soften the singularities which appear in almost all solutions to the gravitational field equations without source. We here assume that this role is to be played by the effect of noncommutativity. We assume that is that noncommutativity is an effective if but partial description of quantum fluctuations; it would play in this respect the same role that thermodynamics plays with respect to statistical physics. Noncommutative geometry to a large extent follows a path previously taken by quantum mechanics with one essential difference; there is no experimental motivation for it. We consider the subject of interest in so far that it can be considered as an extension of the theory of gravity which enables us to better understand the role of the gravitational field as a regulator for classical as well as quantum singularities.

Over the past few years a noncommutative generalization of the Cartan frame formalism has been developed [3, 4] and applied [5, 6, 7, 8] with varying degrees of success to problems in gravitational physics, notably to the possibility of blowing up [9] the Big Bang. One distinguishing feature of this formalism is that the field equations are not derived from an action principle but rather from constraints imposed on the frame arising from Jacobi identities. These identities, we conjecture, fix the Ricci tensor. In particular then the Einstein tensor is determined and the value it takes can be interpreted as an effective source due to the existence of the noncommutative structure. In the quasi-classical approximation this can be more precisely interpreted as the energy of the Poisson structure. One of the motivations of this modification of the field equations is the further conjecture that curvature as is usually defined in differential geometry cannot be used without modification in the noncommutative generalization. It is hoped that once the correct expression is found then the Ricci contraction will yield an Einstein tensor which includes the Poisson energy.

The notation is drawn from previous publications [3]. A tilde is used to designate commutative geometry or a classical limit of a noncommutative one. We use the convention of distinguishing the frame components of a vector from the natural components by a choice in index: the Greek or Latin indices from the middle of the alphabet are coordinate and those from the beginning are frame indices. For example if $J^{\mu\nu}$ are the natural components of an antisymmetric tensor then the frame components will be written as $J^{\alpha\beta}$. If $J^{\mu\nu}$ is a commutator this is quite consistent provided we remain at the quasi-classical approximation. We shall however apply the convention also when describing the nonperturbative solutions in Section 2 in which case it can be ambiguous.
1.1 Algebraic approach

Consider a smooth manifold $M$ with a moving frame $\tilde{\theta}^\alpha$. Let $\mathcal{A}$ be a noncommutative deformation of the algebra $C(M)$ of smooth functions on $M$ defined by a symplectic structure $J$ and let $\theta^\alpha$ be a noncommutative deformation of the moving frame. As we shall see, the connection on $\mathcal{A}$ can be defined to satisfy both a left and right Leibniz rule, a condition which is intimately connected with the existence of a reality condition; the metric can be defined to be a bilinear map, a condition connected with locality. The classical limit of the geometry is thus naturally equipped with a linear connection and a metric as well as with a Poisson structure. Under the assumptions which we shall impose the Poisson structure is non-degenerate. We shall be more precise about these extensions below. We would like to be able to show that in the weak-field, quasi-classical approximation they imply that the metric defined by the frame cannot be arbitrary and that the Ricci tensor is fixed by Jacobi identities.

Let $\mathcal{A}$ be a noncommutative $\ast$-algebra generated by four hermitian elements $x^\mu$ which satisfy the commutation relations:

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu}(x^\sigma). \quad (1.1)$$

As a measure of noncommutativity, and to recall the many parallels with quantum mechanics, we use the symbol $k$, which designates the square of a real number whose value could lie somewhere between the Planck length and the proton radius $m_P^{-1}$. The value becomes important when we consider perturbations. The $J^{\mu\nu}$ are of course restricted by Jacobi identities; we shall see below that there are two other natural requirements which also restrict them.

Let $L$ be a macroscopic length scale. In the Schwarzschild geometry defined by a source of mass $\mu$ the gravitational field is weak if the parameter $G_N\mu L^{-1}$ is small. In de Sitter geometry with cosmological constant $\Lambda$ the corresponding parameter is $\Lambda L^2$. In the microscopic domain we have two length scales determined by respectively the square $G_N\hbar$ of the Planck length and by $k$. These two scales are not necessarily related, although both are of course much smaller than $L^2$. It would be reasonable to identify $k$ with $G_N\hbar$ thus presumably identifying quantum gravity with noncommutativity; a weaker assumption, that noncommutativity gives just an effective description of quantum gravity, would correspond to a vague inequality $G_N\hbar \simeq k$.

One can also compare a classical gravitational field with noncommutativity. As noted, the gravitational field is weak if the dimensionless parameter $\epsilon_{GF} = (G_N\mu)^2 L^{-2}$ is small; on the other hand, the space-time is almost commutative if the dimensionless parameter $\epsilon_{NC} = kL^{-2}$ is small. If noncommutativity is not related to gravity then it makes sense to speak of ordinary gravity as the limit $k \to 0$ with $G_N\mu$ non vanishing. However one could
assume that noncommutativity and gravity are directly related: in that case, both should vanish with $k$. We shall consider the first situation with

$$k \ll (G_N \mu)^2 \ll L^2, \quad \epsilon = \epsilon_{NC}/\epsilon_{GF} \ll 1,$$

and expand in the parameter $\epsilon$ which is a measure of the relative dimension of a typical ‘space-time cell’ compared with a typical ‘quantity of gravitational energy’.

### 1.2 Differential calculi

Assume that there is over $\mathcal{A}$ a differential calculus which is such [4] that the module of 1-forms is free as a left or right module and possesses a preferred basis $\theta^\alpha$ which commutes

$$[x^\mu, \theta^\alpha] = 0$$

with the algebra. Such a basis we call a frame. The space which one obtains in the commutative limit is therefore parallelizable, with a global moving frame $\tilde{\theta}^\alpha$ which is the commutative limit of $\theta^\alpha$. We can write the differential

$$dx^\mu = e^\mu_\alpha \theta^\alpha, \quad e^\mu_\alpha = e_\alpha x^\mu.$$ (1.4)

The algebra is defined by product (1.1) that is by the matrix of elements $J^{\mu\nu}$; the metric is defined we shall see below as in the commutative case, by the matrix of elements $e^\mu_\alpha$. Consistency requirements impose relations between these two matrices which in simple situations allow us to find a one-to-one correspondence between the structure of the algebra and the metric.

The input of which we shall make the most use is the Leibniz rule

$$ike_\alpha J^{\mu\nu} = [e^\mu_\alpha, x^\nu] - [e^\nu_\alpha, x^\mu].$$ (1.5)

One can see here a differential equation for $J^{\mu\nu}$ in terms of $e^\mu_\alpha$. The momenta $p_\alpha$ introduced as in quantum mechanics stand in duality to the position operators $x^\mu$ by the relation

$$[p_\alpha, x^\mu] = \hbar e_\alpha x^\mu = \hbar e^\mu_\alpha.$$ (1.6)

The right-hand side of this identity defines the gravitational field. The left-hand side must obey Jacobi identities. These identities yield relations between quantum mechanics in the given space-time and the noncommutative structure of the algebra.

The three aspects of reality then, the curvature of space-time, quantum mechanics and the noncommutative structure are intimately connected. We shall explore here an even more exotic possibility that the field equations of general relativity are encoded in the structure
of the algebra so that the relation between general relativity and quantum mechanics can be understood by the relation which each of these theories has with noncommutative geometry.

In spite of the rather lengthy formalism the basic idea is simple. We start with a classical geometry described by a moving frame \( \theta^\alpha \) and we quantize it by replacing the moving frame by a frame \( \tilde{\theta}^\alpha \), as we shall describe in some detail below. The easiest cases would include those frames which could be quantized without ordering problems. Let \( \tilde{e}_\alpha \) be the vector fields dual to the frame \( \tilde{\theta}^\alpha \); we quantize them as in (1.6) by imposing the rule

\[
\tilde{e}_\alpha \mapsto e_\alpha = \hbar^{-1} \text{ad} p_\alpha.
\]

Finally, we must construct a noncommutative algebra consistent with the assumed differential calculus; this defines the image of the map

\[
\theta^\alpha \longrightarrow \tilde{\theta}^\alpha \longrightarrow J^{\mu\nu}.
\]

More details of this map will be given in the Appendix. The algebra we identify with ‘position space’.

To construct phase space we must add the momenta \( p_\alpha \). In ordinary geometry there is but one way to do so; the derivations \( e_\alpha \) are outer and the associated momenta do not belong to the algebra generated by the position variables. That is, we add the extra elements which are necessary in order that the derivations be inner, as one does in ordinary quantum mechanics. In noncommutative geometry there are more possibilities; in particular, \( p_\alpha \) can belong to the initial algebra \( A \). Also the new element is that the consistency relations in the algebra such as Jacobi identities

\[
 i\hbar [p_\alpha, J^{\mu\nu}] = [x^{[\mu}, [p_\alpha, x^{\nu]}],
\]

in principle restrict \( \theta^\alpha \) and \( J^{\mu\nu} \).

If the space is flat, \( e_\alpha^\mu = \delta_\alpha^\mu \) and the frame is the canonical flat frame then the right-hand side of (1.9) vanishes and it is possible to consistently choose the expression \( J^{\mu\nu} \) to be equal to a constant. On the other hand, if the space is curved the right-hand side cannot vanish except of course in the limit \( \hbar \to 0 \). The map (1.8) is not single valued since any constant \( J \) has flat space as inverse image.

We must insure also that the differential is well defined. A necessary condition is that

\[
d[x^\mu, \theta^\alpha] = 0,
\]

from which it follows that the momenta \( p_\alpha \) must satisfy the consistency condition

\[
2p_\gamma p_\beta P^{\gamma\delta}_{\alpha\beta} - p_\gamma F^{\gamma\alpha\beta} - K_{\alpha\beta} = 0.
\]

The \( P^{\gamma\delta}_{\alpha\beta} \) define the exterior product in the algebra of forms,

\[
\theta^\gamma \theta^\delta = P^{\gamma\delta}_{\alpha\beta} \theta^\alpha \otimes \theta^\beta.
\]
We write $P^{\alpha \beta \gamma \delta}$ in the form

$$P^{\alpha \beta \gamma \delta} = \frac{1}{2} \delta^{[\alpha \beta}_{\gamma \delta] + i \epsilon Q^{\alpha \beta \gamma \delta} \tag{1.12}$$

of a standard antisymmetrizer plus a perturbation. If further [10] we decompose $Q^{\alpha \beta \gamma \delta}$ as the sum of two terms

$$Q^{\alpha \beta \gamma \delta} = Q^{\alpha \beta \gamma \delta}_- + Q^{\alpha \beta \gamma \delta}_+ \tag{1.13}$$

symmetric (antisymmetric) and antisymmetric (symmetric) with respect to the upper (lower) indices then the condition that $P^{\alpha \beta \gamma \delta}$ be a projector is satisfied to first order in $\bar{k}$ because of the property that

$$Q^{\alpha \beta \gamma \delta} = P^{\alpha \beta \gamma \delta} \zeta \eta Q^{\zeta \eta \gamma \delta}_- + P^{\alpha \beta \gamma \delta} \zeta \eta Q^{\zeta \eta \gamma \delta}_+. \tag{1.14}$$

The compatibility condition

$$(P^{\alpha \beta \gamma \delta}_-)^* P^{\beta \alpha \gamma \delta}_+ = P^{\beta \alpha \gamma \delta}_+ \tag{1.15}$$

with the product is satisfied provided $Q^{\alpha \beta \gamma \delta}$ is real.

From (1.3) it follows that

$$d[x^\mu, \theta^\alpha] = [dx^\mu, \theta^\alpha] + [x^\mu, d\theta^\alpha] = e^\mu_\beta [\theta^\beta, \theta^\alpha] - \frac{1}{2} [x^\mu, C^{\alpha \beta \gamma \delta}] \theta^\gamma \theta^\delta = 0. \tag{1.16}$$

We have here introduced the Ricci rotation coefficients $C^{\alpha \beta \gamma \delta}$. We find then that multiplication of 1-forms must satisfy

$$[\theta^\alpha, \theta^\beta] = \frac{1}{2} \theta^\beta_\mu [x^\mu, C^{\alpha \gamma \delta}] \theta^\gamma \theta^\delta. \tag{1.17}$$

Consistency requires then that

$$\theta^\beta_\mu [x^\mu, C^{\alpha \gamma \delta}] = 0; \tag{1.18}$$

because of the condition (1.3) consistency also requires that

$$\theta^\beta_\mu [x^\mu, C^{\beta \gamma \delta}] = Q^{\alpha \beta \gamma \delta}_-. \tag{1.19}$$

We have in general four consistency relations which must be satisfied in order to obtain a noncommutative extension of a commutative manifold. They are the Leibniz rule (1.5), the Jacobi identity and the conditions (1.18) and (1.19) on the differential. The first two constraints are not completely independent of the differential calculus since one involves the momentum operators. The condition (1.19) follows in general from the expression [11]

$$C^{\alpha \beta \gamma \delta} = F^{\alpha \beta \gamma \delta} - 4i \epsilon p_\delta Q^{\alpha \beta \gamma \delta}_- \tag{1.20}$$

for the rotation coefficients. It follows also from general considerations that the rotation coefficients must satisfy the gauge condition

$$e^\alpha_\gamma C^{\alpha \beta \gamma} = 0. \tag{1.21}$$

We shall refer to all these conditions as the Jacobi constraints.
2 Phase space

The classical phase space associated to Minkowski space-time has eight dimensions. The associated algebra $A$ can be considered as the algebra with four position generators $\tilde{x}^\mu$ and four momentum generators $\tilde{p}_\alpha$ subject to the sole relation that they commute. The quantized phase space is the same but with the Heisenberg commutation relations

$$[p_\alpha, x^\mu] = i\hbar \delta_\alpha^\mu.$$  \hspace{1cm} (2.1)

Over this algebra there is a natural moving frame $\theta^\alpha = \delta^\alpha_\mu dx^\mu$ the dual derivations of which are given by

$$e_\beta = -\frac{1}{\hbar} \text{ad} p_\beta.$$ \hspace{1cm} (2.2)

We shall set $\hbar = 1$ so that the momenta are normalized as we wish them to be, with $dx^\mu(e_\beta) = \delta^\mu_\beta$. The moving frame $\theta^\alpha$ is clearly a frame in the sense we have defined it. As in the commutative case, since the frame is exact the curvature vanishes: the associated geometry is flat. We have thus a map

$$\text{Minkowski} \mapsto \text{Heisenberg}$$ \hspace{1cm} (2.3)

the extension of which to a geometry with generic curvature we wish to construct. There exists the commutative limit as special case; the map takes each 4-geometry into a subalgebra of the algebra of sections of the form bundle defined by the moving frame. We shall extend this map to one which includes the corrections of first order in the noncommutativity parameter $\bar{k}$. We shall see that the existence of the extension imposes integrability conditions on the classical limit of the map. These conditions we conjecture (guess) replace the field equations. An analogous situation exists in classical gravity when one attempts to find solutions as perturbation expansions in the gravitational coupling constant: the existence for example of the first-order corrections implies that the sources obey the conservation laws of the flat space-time.

2.1 Momenta and representations

To make transparent the dual roles played by the position and the momenta we introduce the index $i = (\lambda, \alpha)$ and write a point in phase space as $y^i = (x^\lambda, p_\alpha)$. We lower the index with the metric components $g_{ij} = (g_{\mu\nu}, g^{\alpha\beta})$. The classical Heisenberg commutation relations can be written then as

$$[y^i, y^j] = J^{ij},$$ \hspace{1cm} (2.4)

with

$$J^{ij} = \begin{pmatrix} 0 & \delta^\mu_\beta \\
-\delta^\nu_\alpha & 0 \end{pmatrix}. \hspace{1cm} (2.5)$$
The matrix $J$ contains all the information of the system. The even elements $J^+$ describe the algebra and the noncommutative differential calculus, the odd elements $J^-$ depend directly on the frame which in turn determines the limiting commutative (de Rham) differential calculus. The extension of the map (2.3) is equivalent then to a map

$$J^+ \mapsto J^-.$$ (2.6)

The diagonal elements $J^+$ consist of the six position commutators

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu}$$ (2.7)

as well as of the ‘dual’ momentum commutators

$$[p_\alpha, p_\beta] = K_{\alpha\beta} + F^\gamma_{\alpha\beta} p_\gamma - 2i\epsilon_{Q}^{\gamma\delta} p_\gamma p_\delta.$$ (2.8)

which measure the curvature. There seems to be no obvious property which would characterize the map (2.6). As solution to a differential equation it is non-local. Even in the semi-classical limit it has no obvious characterization.

The unusual new feature of the noncommutative extension is the possibility that the momenta which are functions of coordinates $p_\alpha(x^\sigma)$ and satisfy all the constraints exist. In the commutative limit this would correspond to a section of the frame bundle; locally at least there are many. We shall refer to the $p_\alpha(x^\sigma)$ as a ‘section’ of the noncommutative frame bundle although we have not defined the latter. If represented by operators on some Hilbert space, the representation will be irreducible; the commutant will reduce to the identity. For example if the matrix $J$ is constant and invertable then the elements $\pi_\alpha$ defined by

$$\pi_\alpha = p_\alpha - p_{0\alpha}, \quad p_{0\alpha} = -(i\hbar J)^{-1}_{\alpha\mu} x^\mu$$ (2.9)

commute with all the position generators. They can then be set to zero; the momenta and the position operators generate then the same algebra. We shall assume this case to be generic. We assume that is there be a solution $p_{0\alpha}(x^\mu)$ to the equations

$$[p_{0\alpha}, x^\mu] = e^\mu_\alpha.$$ (2.10)

But if we have independent momenta $p_\alpha$ then also

$$[p_\alpha, x^\mu] = e^\mu_\alpha.$$ (2.11)

We have then two solutions and the differences $\pi_\alpha = p_\alpha - p_{0\alpha}$ commute with the generators $x^\mu$.

It would be natural to set $\pi_\alpha = 0$ to obtain an irreducible representation of the position algebra; this would lead however to a manifestly singular commutative limit. We resolve this by requiring only that the ‘phase algebra’ $T$ be irreducible. The projection of the tangent bundle onto the manifold is in the algebraic transcription an injection of $\mathcal{A}$ into $\mathcal{T}$. 
The \( p_\alpha \) then are decomposed as a sum of a section \( p_{0\alpha} \) of this bundle and a remainder \( \pi_\alpha \). The subalgebra \( \mathcal{A}' \) of \( \mathcal{T} \) generated by the \( \pi \) commutes with \( \mathcal{A} \). The condition we impose is

\[
\mathcal{T} = \mathcal{A} \otimes \mathcal{A}'.
\]  
(2.12)

This is certainly true in the commutative limit. The conditions we have placed on the manifold imply that the tangent bundle is trivial; the condition could be considered as the statement that this remains so in the non-commutative extension.

We can write the relation (2.9) as the definition of a ‘covariant momentum’

\[
\pi_\alpha = p_\alpha + Z_\alpha(x^\mu), \quad Z_\alpha = -p_{0\alpha}
\]  
(2.13)

This is somewhat analogous to gauge transformations; a covariant derivative is constructed to commute with them.

### 2.2 The correspondence

We shall here consider phase space with eight generators, the position generators \( x^\mu \) and the momenta \( p_\alpha \) defining the exterior derivations. The latter we suppose admits to first order a bracket of the form

\[
[p_\alpha, p_\beta] = C^\gamma_{\alpha\beta} p_\gamma + K_{\alpha\beta}.
\]  
(2.14)

This is a central extension of the classical relation satisfied by the derivations. In general the relation is given by (1.10). The center is non-trivial and is generated by the elements \( \pi_\alpha \).

The rotation coefficients are directly related to the commutators of the momentum generators. We have seen that the former are given by the expression (1.20) and the latter by (1.10), which we write in the form

\[
[p_\alpha, p_\beta] = \frac{1}{ik} L_{\alpha\beta}
\]  
(2.15)

with

\[
L_{\alpha\beta} = K_{\alpha\beta} + ikF^\gamma_{\alpha\beta} p_\gamma - 2(ik)^2 \mu^2 Q^\delta_{\alpha\beta} p_\gamma p_\delta.
\]  
(2.16)

There is therefore a direct connection between the rotation coefficients and the commutators \( J^{\mu\nu} \). This relation can be derived directly without explicitly referring to the momenta.

It is easy to see that the Jacobi identity

\[
[x^\nu, [p_\alpha, p_\beta]] + [p_\alpha, [p_\beta, x^\nu]] + [p_\beta, [x^\nu, p_\alpha]] = 0
\]  
(2.18)
is in fact an identity. Consider the Jacobi identity

\[ [p_\alpha, [x^\mu, x^\nu]] + [x^\mu, [x^\nu, p_\alpha]] + [x^\nu, [p_\alpha, x^\mu]] = 0. \]  \hspace{1cm} (2.19)

It can be written as

\[ ik[p_\alpha, J^{\mu\nu}] - [x^{[\mu}, e^{\nu]}] = 0, \]  \hspace{1cm} (2.20)

from which one derives in the semi-classical approximation a differential equation

\[ e_\alpha J^{\mu\nu} - J^{\mu\rho} \partial_\rho e^{\nu}_{\alpha} = 0. \]  \hspace{1cm} (2.21)

This condition can be expressed uniquely in terms of frame components using the sequence of identities

\[
\begin{align*}
\theta^\beta_\mu \theta^\gamma_\nu (e_\alpha J^{\mu\nu} - J^{\mu\rho} \partial_\rho e^{\nu}_{\alpha}) &= e_\alpha J^{\beta\gamma} - J^{\mu\nu} e_\alpha (\theta^\beta_\mu \theta^\gamma_\nu) - J^{[\beta\delta} [p_\delta, [x^\nu, p_\alpha]] \theta^\gamma_{\nu} \\
&= e_\alpha J^{\beta\gamma} - J^{\mu\nu} e_\alpha (\theta^\beta_\mu \theta^\gamma_\nu) - J^{[\beta\delta} e_\alpha e^{\gamma}_{\delta} \theta^\gamma_{\nu} + J^{[\beta\delta} [x^\nu, [p_\delta, p_\alpha]] \theta^\gamma_{\nu} \\
&= e_\alpha J^{\beta\gamma} + C^{[\beta}_{\alpha\delta} J^{\delta\gamma}] = 0. 
\end{align*}
\]  \hspace{1cm} (2.22)

which follow immediately from the Leibniz rule. We find then the condition

\[ e_\alpha J^{\beta\gamma} + C^{[\beta}_{\alpha\delta} J^{\delta\gamma}] = 0. \]  \hspace{1cm} (2.23)

The extension of the map (2.3) which one could propose is therefore a map

\[ e^\mu_\alpha \mapsto J^{\mu\nu} \]  \hspace{1cm} (2.24)

which one can consider as a map

\[ C^\gamma_{\alpha\beta} \mapsto J^{\alpha\beta} \]  \hspace{1cm} (2.25)

obtained by solving (2.23).

### 2.3 Metrics and connections

We have defined a notion of antisymmetry by the array \( P^{\alpha\delta}_{\gamma\beta} \). To define symmetry we introduce a flip \( \sigma \) which exchanges in a twisted way the two factors of a tensor product. In terms of the frame it can be written

\[ \sigma(\theta^\alpha \otimes \theta^\delta) = S^{\alpha\delta}_{\gamma\beta} \theta^\gamma \otimes \theta^\beta. \]  \hspace{1cm} (2.26)

If we require that the map be bilinear then the coefficients must be constant. The relation between \( P^{\alpha\delta}_{\gamma\beta} \) and \( S^{\alpha\delta}_{\gamma\beta} \) is the condition

\[ \pi \circ (1 + \sigma) = 0, \]  \hspace{1cm} (2.27)
the antisymmetric part of a symmetrized tensor should vanish. The conditions satisfied by the flip are quite simple in the first approximation we are here considering. If we write

$$S^{\alpha\beta}_{\gamma\delta} = \delta^{\beta}_{\gamma}\delta^{\alpha}_{\delta} + i\epsilon T^{\alpha\beta}_{\gamma\delta},$$

(2.28)

then we find that

$$Q^{\alpha\beta}_{\gamma\delta} + T^{[\alpha\beta}_{\gamma\delta} = 0.$$

(2.29)

Some further relations are given in the Appendix.

With the frame one can construct a metric just as one does in the commutative case. It is the bilinear map of the tensor product of the module of 1-forms by itself into the algebra. It associates therefore a function to each pair of vector fields. We consider this metric in the classical approximation. It is defined by the frame and a set of coefficients, necessarily in the center, by the expression

$$g^{\alpha\beta} = g(\theta^{\alpha} \otimes \theta^{\beta}).$$

(2.30)

We choose the frame to be orthonormal in the commutative limit; we can write therefore

$$g^{\alpha\beta} = \eta^{\alpha\beta} - i\epsilon h^{\alpha\beta}.$$

(2.31)

In the linear approximation, the condition of the reality of the metric becomes

$$h^{\alpha\beta} + \overline{h}^{\alpha\beta} = T^{\beta\alpha}_{\gamma\delta}\eta^{\gamma\delta}.$$

(2.32)

The covariant derivative is given by

$$D\xi = \sigma(\xi \otimes \theta) - \theta \otimes \xi.$$

(2.33)

In particular

$$D\theta^{\alpha} = -\omega^{\alpha}_{\gamma} \otimes \theta^{\gamma} = -(S^{\alpha\beta}_{\gamma\delta} - \delta^{\beta}_{\gamma}\delta^{\alpha}_{\delta})p_{\beta}\theta^{\gamma} \otimes \theta^{\delta} = -i\epsilon T^{\alpha\beta}_{\gamma\delta}p_{\beta}\theta^{\gamma} \otimes \theta^{\delta},$$

(2.34)

so the connection-form coefficients are linear in the momenta

$$\omega^{\alpha}_{\beta\gamma} = \omega^{\alpha}_{\beta\gamma}\theta^{\beta} = i\epsilon p_{\delta}T^{\alpha\beta}_{\beta\gamma}\theta^{\beta}.$$

(2.35)

On the left-hand side of the last equation is a quantity $\omega^{\alpha}_{\gamma}$ which measures the variation of the metric; on the right-hand side is the array $T^{\alpha\beta}_{\gamma\delta}$ which is directly related to the anticommutation rules for the 1-forms, and more importantly the momenta $p_{\delta}$ which define the frame. As $k \to 0$ the right-hand side remains finite and

$$\omega^{\alpha}_{\gamma} \to \tilde{\omega}^{\alpha}_{\gamma}.$$

(2.36)

The connection is torsion-free if the components satisfy the constraint

$$\omega^{\alpha}_{\gamma\delta}p^{\gamma\delta}_{\beta\gamma} = \frac{1}{2}C^{\alpha}_{\beta\gamma}.$$

(2.37)
The connection is metric if
\[ \omega^\alpha_{\beta\gamma} g^{\gamma\delta} + \omega^\delta_{\gamma\eta} S^\alpha_{\beta\delta} g^{\eta\gamma} = 0, \]
(2.38)
or linearized,
\[ T^{(\alpha\gamma\beta)} = 0. \]
(2.39)
The equation can be solved, so in the linear approximation every metric has associated to it a unique torsion-free metric connection.

## 3 The perturbation expansion

We must now examine the conditions under which the noncommutative frame can be considered as having the classical frame as limit. By classical we refer here to ordinary quantum mechanics. We have three commutators, position space, momentum space and the cross terms, given respectively by
\[ [x^\mu, x^{\nu}] = i \kbar J^{\mu\nu} \]
(3.1)
\[ [p_\alpha, p_\beta] = \frac{1}{i \kbar} L_{\alpha\beta} \]
(3.2)
\[ [p_\alpha, x^\mu] = e^\mu_\alpha \]
(3.3)
with \( L_{\alpha\beta} \) given by Equation (2.16).

### 3.1 General quasi-classical limit

Let us discuss in more detail the limit of the frame geometry. Recall the definitions
\[ f \theta^\alpha = \theta^\alpha f, \quad \theta^\alpha = \theta^\alpha_\mu (x^\sigma) dx^\mu, \]
(3.4)
the second being the inverse of (1.4). As functions of coordinates are given through the Taylor expansion, the commutator \([x^\lambda, f(x^\sigma)]\) can be expressed in terms of the basic commutators \( J^{\lambda\mu} \). Neglecting the operator ordering that is in linear order in \( \kbar \) we obtain
\[ [x^\lambda, f(x^\sigma)] = i \kbar J^{\lambda\mu} \partial_\mu f = i \kbar J^{\lambda\alpha} e_\alpha f. \]
(3.5)
This is the quasi-classical approximation. We have denoted
\[ J^{\lambda\mu} = J^{\alpha\beta} e^\lambda_\alpha e^\mu_\beta. \]
(3.6)
In particular,
\[ [x^\lambda, dx^\mu] = -e^\mu_\alpha [x^\lambda, \theta^\alpha] dx^\nu = -i k J^{\lambda \beta} e^\mu_\alpha e_\beta \theta^\alpha dx^\nu = i k J^{\lambda \beta} e_\beta e^\mu_\alpha \theta^\alpha. \]  

(3.7)

Using the quasi-classical approximation we can obtain an equation for \( J \). From
\[ dJ^{\lambda \mu} = dJ^{\alpha \beta} e^\lambda_\alpha e^\mu_\beta + J^{[\lambda \beta} e_\alpha e^{\mu]}_\beta \theta^\alpha \]
and (3.5) we have
\[ dJ^{\alpha \beta} + J^{\gamma [\alpha C^\beta]} \gamma_\delta \theta^\delta = 0. \]

(3.8)

This equation has the integrability condition
\[ d(J^{\gamma [\alpha C^\beta]} \gamma_\delta \theta^\delta) = 0, \]
that is
\[ d(J^{\gamma [\alpha C^\beta]} \gamma_\delta \theta^\delta) + J^{[\gamma [\alpha C^\beta]} \gamma_\delta d\theta^\delta = 0. \]

(3.9)

Using the known expression for \( dJ^{\alpha \beta} \) and the Bianchi identities
\[ \epsilon^{\alpha \beta \gamma \delta} (e_\delta C^\zeta \gamma_\beta + C^\zeta \delta_\eta C^{\eta \gamma_\beta}) = 0 \]
one shows that the first term is identically zero. The condition reduces to
\[ J^{\gamma [\alpha C^\beta]} \gamma_\delta d\theta^\delta = 0. \]

(3.10)

4 Conclusion

We have argued in favor of considering noncommutative geometry as a deformation of phase space rather than position space. From this point of view the algebra and the calculus are on the same footing and in fact one could avoid to a certain extent at least using the later since the 1-forms have been encoded in the momenta. Derivations can be almost identified with momenta but only if one neglects an additive constant in the latter. We have shown that in noncommutative geometry there is a preferred origin to the momenta, somewhat analogous to the preferred origin in the module of 1-forms. This is quite consistent with previous results to the effect that the commutators determine not only the structure of the algebra but also the metric of the associated geometry.
5 Appendices

5.1 Gauge dependence

We have found a map between the symplectic and the metric structures on a manifold. The definition is valid only in the semi-classical approximation and relies essentially on the existence of a frame. An interesting problem is the study of the variation of the map. For example one might inquire into the type of variations of the frame which leave the symplectic structure invariant and inversely. Both of these variations could be considered as 'gauge' transformations. We have succeeded in solving only partially this problem.

Consider two choices of commutator \( J^{\alpha\beta} \) and \( J'^{\alpha\beta} \) with

\[
J^{\alpha\beta} = J^{\alpha\beta} + \delta J^{\alpha\beta} \quad (5.1)
\]

Then the corresponding variation of the rotation coefficients \( \delta C^{\gamma}_{\alpha\beta} \) is given by a solution to the constraints

\[
\delta e_{\gamma} J^{\alpha\beta} + e_{\gamma} \delta J^{\alpha\beta} - \delta C^{\gamma\delta}_{\alpha\beta} J^{\beta\delta} - C^{\gamma\delta}_{\alpha\beta} \delta J^{\beta\delta} = 0, \quad (5.2)
\]

\[
\epsilon_{\alpha\beta\gamma\delta} \delta J^{\alpha\gamma} e_{\epsilon} J^{\beta\gamma} + \epsilon_{\alpha\beta\gamma\delta} J^{\alpha\gamma} \delta e_{\epsilon} J^{\beta\gamma} + \epsilon_{\alpha\beta\gamma\delta} J^{\alpha\gamma} e_{\epsilon} \delta J^{\beta\gamma} = 0, \quad (5.3)
\]

\[
[\delta e_{\alpha}, e_{\beta}] + [e_{\alpha}, \delta e_{\beta}] = \delta C^{\gamma}_{\alpha\beta} e_{\gamma} + C^{\gamma}_{\alpha\beta} \delta e_{\gamma}. \quad (5.4)
\]

This can be simplified if we use the momenta. We conclude from (5.7) that a variation of the momenta must satisfy the constraint

\[
[\delta p_{\alpha}, p_{\beta}] + [p_{\alpha}, \delta p_{\beta}] = \delta C^{\gamma}_{\alpha\beta} p_{\gamma} + C^{\gamma}_{\alpha\beta} \delta p_{\gamma}. \quad (5.5)
\]

with

\[
\delta C^{\gamma}_{\alpha\beta} = \delta F^{\gamma}_{\alpha\beta} - 4i\epsilon \delta Q^{\gamma\delta}_{\alpha\beta} p_{\delta}. \quad (5.6)
\]

In particular the rotation coefficients vary only with a change in the coefficients.

5.2 Rotation coefficients

One can show quite generally that the momenta \( p_{\alpha} \) have a bracket of the form

\[
[p_{\alpha}, p_{\beta}] = K_{\alpha\beta} + F^{\gamma}_{\alpha\beta} p_{\gamma} - 2i\epsilon Q^{\gamma\delta}_{\alpha\beta} p_{\gamma} p_{\delta}. \quad (5.7)
\]

For the formalism to work we must be able to impose the gauge condition

\[
E_{\alpha\beta} = 0, \quad E_{\alpha\beta} = e_{\gamma} C^{\gamma}_{\alpha\beta}. \quad (5.8)
\]
There is yet another stronger condition to be considered below.

Consider the equations

\[ [e_\alpha, e_\beta] = C^{\gamma}_{\alpha\beta} e_\gamma \]  \hspace{1cm} (5.9)

To obtain them as the commutative limits of a set of noncommutative extensions we introduce the commutators

\[ [p_\alpha, p_\beta] = X_{\alpha\beta}. \]  \hspace{1cm} (5.10)

We can consider the left-hand side of (5.9) as a limit of the left-hand side of (5.10) in the sense that

\[ [[p_\alpha, p_\beta], f] = [[e_\alpha, e_\beta], f]. \]  \hspace{1cm} (5.11)

For the right-hand sides to satisfy such a relation we require coefficients \( K_{\alpha\beta}, F^\gamma_{\alpha\beta}, Q^\delta_{\alpha\beta} \) such that the matrix of quadratic polynomials

\[ X_{\alpha\beta} = K_{\alpha\beta} + F^\gamma_{\alpha\beta} p_\gamma - 2i\epsilon Q^\delta_{\alpha\beta} p_\delta \]  \hspace{1cm} (5.12)

have the property that

\[ \lim_{k \to 0} [X_{\alpha\beta}, f] = C^\gamma_{\alpha\beta} \lim_{k \to 0} [p_\gamma, f]. \]  \hspace{1cm} (5.13)

and so we must find coefficients \( Q^\delta_{\alpha\beta} \) as well as momenta \( p_\alpha \) such that

\[ C^\gamma_{\alpha\beta} \lim_{k \to 0} [p_\gamma, f] = F^\gamma_{\alpha\beta} \lim_{k \to 0} [p_\gamma, f] - 4 \lim_{k \to 0} i\epsilon Q^\delta_{\alpha\beta} p_\delta [p_\gamma, f]. \]  \hspace{1cm} (5.14)

Since \( f \) is arbitrary we must have therefore to lowest order in \( k \)

\[ C^\gamma_{\alpha\beta} = F^\gamma_{\alpha\beta} - 4i\epsilon Q^\delta_{\alpha\beta} p_\delta. \]  \hspace{1cm} (5.15)

This is formally and to first order the same as

\[ C^\gamma_{\alpha\beta} = \frac{\partial}{\partial p_\gamma} X_{\alpha\beta}. \]  \hspace{1cm} (5.16)

If we take another derivative we obtain

\[ \frac{\partial}{\partial p_\delta} C^\gamma_{\alpha\beta} = \frac{\partial^2}{\partial p_\delta \partial p_\gamma} X_{\alpha\beta} = -4i\epsilon Q^\delta_{\alpha\beta}. \]  \hspace{1cm} (5.17)

From this follows the condition

\[ \frac{\partial}{\partial p_\delta} C^\gamma_{\alpha\beta} - \frac{\partial}{\partial p_\gamma} C^\delta_{\alpha\beta} = 0. \]  \hspace{1cm} (5.18)

We see then also that

\[ e_\delta C^\gamma_{\alpha\beta} = X^\zeta_{\delta} \frac{\partial}{\partial p_\delta} C^\gamma_{\alpha\beta} = -4i\epsilon X^\zeta_{\delta} Q^\zeta_{\alpha\beta} \]  \hspace{1cm} (5.19)

and from this the integrability conditions

\[ e_\gamma C^\gamma_{\alpha\beta} = 0 \]  \hspace{1cm} (5.20)

If the classical rotation coefficients do not satisfy this condition for some choice of \( p \)-variables they cannot be 'quantized'. We see here that the \( p \)-variables are somewhat analogous to the special variables of the classical Darboux theorem. In this case the transformation to the special coordinates is the Fourier transform \( x^\mu \mapsto p_\alpha \).
5.3 The cocycle

We return now to the cocycle condition (5.31). Either \( F = dA \) for some 1-form \( A \) or there is no such 1-form. We know of no \( F \) which is not of the form \( F = dA \) but there is a case with \( F = dA_D \) where \( A_D \) has no regular commutative limit. We know that the Dirac operator \( \theta = -p_\alpha \theta^\alpha \) diverges in the commutative limit and that the limit of

\[
A_D = -ik\theta
\]

is finite but not everywhere well defined. We notice then that the square of \( \theta \) can be written as

\[
\theta^2 = \frac{1}{2} [p_\alpha, p_\beta]\theta^\alpha \theta^\beta.
\]

From this it follows that

\[
d\theta + \theta^2 = -\frac{1}{2} [p_\alpha, p_\beta] \theta^\alpha \theta^\beta + \frac{1}{2} p_\gamma C^\gamma_{\alpha\beta} \theta^\alpha \theta^\beta
\]

\[
= -\frac{1}{2} [p_\alpha, p_\beta] \theta^\alpha \theta^\beta + \frac{1}{2} p_\gamma F^\gamma_{\alpha\beta} \theta^\alpha \theta^\beta - 2i\epsilon p_\gamma p_\delta Q^{\gamma\delta}_{\alpha\beta}
\]

\[
= -\frac{1}{2} ([p_\alpha, p_\beta] - p_\gamma F^\gamma_{\alpha\beta} + 4i\epsilon p_\gamma p_\delta Q^{\gamma\delta}_{\alpha\beta}) \theta^\alpha \theta^\beta
\]

\[
= -K.
\]

where we have set \( K = \frac{1}{2} K_{\alpha\beta} \theta^\alpha \theta^\beta \). We can conclude then that

\[
dA_D + A^2_D = iK
\]

Equations (2.23) can be written also in terms of the dual quantities

\[
J^*_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} J^{\gamma\delta}
\]

as

\[
e_\alpha J^*_{\beta\gamma} + C^\delta_{\alpha[\beta} J^*_{\gamma]\delta} + C^\delta_{\alpha\delta} J^*_{\beta\gamma} = 0.
\]

\[
C^\alpha_{[\alpha_1\gamma\beta\delta]} J^*_{\beta\gamma} = 0.
\]

It will be convenient to introduce the suggestive notation

\[
F_{\alpha\beta} = (J^{-1})_{\alpha\beta}.
\]

We could also have written

\[
F_{\alpha\beta} = |J|^{-1} J^*_{\alpha\beta}, \quad |J|^2 = \frac{1}{4} J^*_{\alpha\beta} J^{\alpha\beta}.
\]

We can now rewrite the equations in terms of the inverse.
From Equation (2.23) one can derive the identity
\[ e_\alpha F_{\beta \gamma} + F_{\alpha \delta} C^\delta_{\beta \gamma} = 0 \]  
(5.30)
for the derivative of the inverse if it exists. This can also be written as a ‘cocycle condition’
\[ dF = 0 \]  
(5.31)
if we introduce the 2-form
\[ F = \frac{1}{2} F_{\alpha \beta} \theta^\alpha \theta^\beta. \]  
(5.32)
One can solve (5.30) for the rotation coefficients. One obtains
\[ C^\alpha_{\beta \gamma} = J^{\alpha \eta} e_\eta F_{\beta \gamma}. \]  
(5.33)
It follows that in the quasi-classical approximation, the linear curvature is a polynomial in the commutator \( J \) and its inverse and their derivatives.

If we consider \( F \) as a Maxwell field strength then there is a source given by
\[ e^\alpha F_{\alpha \beta} = F^{\alpha \gamma} C_{\alpha \beta \gamma}. \]  
(5.34)
It follows also from the condition (1.21) that the commutator must necessarily satisfy the constraint
\[ e_\alpha (J^{\alpha \eta} e_\eta F_{\beta \gamma}) = 0. \]  
(5.35)
This can also be written as
\[ (e_\alpha J^{\alpha \zeta} + J^{\alpha \eta} C^{\zeta}_{\alpha \eta}) e_\zeta F_{\beta \gamma} = 0. \]  
(5.36)
If we equate the Expression (5.33) for the rotation coefficients with that in terms of the components of the frame we find after a few simple applications of the Leibniz rule that
\[ (dF)_{\alpha \beta \gamma} = e^\mu_{[\beta} e^\gamma_{\alpha]} F_{\alpha \mu}. \]  
(5.37)
The cocycle condition (5.31) is equivalent to the condition
\[ e^\mu_{[\beta} e^\gamma_{\alpha]} F_{\alpha \mu} = 0. \]  
(5.38)
An interesting particular solution is given by constants:
\[ F_{\alpha \mu} = F_{0 \alpha \mu}. \]  
(5.39)
It follows then that
\[ J^{\mu \nu} = J_0^{\mu \alpha} e^{\nu}_{\alpha}, \quad J_0^{(\mu \alpha} e^{\nu)}_{\alpha} = 0. \]  
(5.40)
One verifies that
\[ C^{\alpha \beta}_{\alpha \gamma} = J^{\alpha \eta} e_\eta F_{\alpha \gamma} = e_\eta J^{\alpha \eta} F_{\alpha \gamma} \]  
(5.41)
and so the left-hand side vanishes if and only if
\[ e_\beta J^{\alpha \beta} = 0. \]  
(5.42)
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