THE TYPICAL CELL OF A VORONOI TESSELLATION ON THE SPHERE

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Abstract. The typical cell of a Voronoi tessellation generated by \( n + 1 \) uniformly distributed random points on the \( d \)-dimensional unit sphere \( S^d \) is studied. Its \( f \)-vector is identified in distribution with the \( f \)-vector of a beta’ polytope generated by \( n \) random points in \( \mathbb{R}^d \). Explicit formulae for the expected \( f \)-vector are provided for any \( d \) and the low-dimensional cases \( d \in \{2, 3, 4\} \) are studied separately. This implies an explicit formula for the total number of \( k \)-dimensional faces in the spherical Voronoi tessellation as well.

1. Introduction and statement of results

1.1. Introduction. Let \( E \) be a metric space and \( \{x_i : i \in I\} \) a finite (or, more generally, locally finite) collection of points in \( E \), where \( I \) is some index set. The Voronoi cell of a point \( x_i \) is the set of all points in \( E \) whose distance to \( x_i \) is not greater than the distance to any other point \( x_j \) with \( i \neq j \). The Voronoi tessellation or Voronoi diagram associated with the set \( \{x_i : i \in I\} \) is then just the collection of all such Voronoi cells. The study of Voronoi tessellations has attracted a lot of attention in computational as well as in stochastic geometry. To a great extent this is because of their various applications ranging from the modelling of biological tissues or polycrystalline microstructures in metallic alloys to classification problems in machine learning. We refer the reader to the monographs \([11, 28, 29, 30]\) for details and many more references.

In this note we consider Voronoi tessellations of the unit sphere that are generated by a (finite) collection of uniformly distributed, independent random points. Unlike their Euclidean counterparts, for which there exists an extensive literature (see \([29, 30, 33, 34]\) and the references cited therein), the mathematical properties of spherical Voronoi tessellations are only poorly understood. Just a few results for Voronoi tessellation on the 2-dimensional unit sphere are available in the classical reference \([27]\). On the other hand, Voronoi tessellations induced by points on a general manifold become increasingly important in computational geometry, see \([10, 11]\). Our goal is to partially fill the resulting gap by considering the combinatorial structure of what is called the typical cell of a Voronoi tessellation on the \( d \)-dimensional unit sphere for general \( d \geq 2 \). More precisely, we shall study the \( f \)-vector of the typical spherical Voronoi cell. We do this by establishing and exploiting a new connection of such typical Voronoi cells with the classes of random beta and beta’ polytopes. These have recently been under intensive investigation \([5, 6, 9, 14, 20, 21, 22, 23, 24, 25]\). In fact, as it will turn out, the \( f \)-vector of the typical spherical Voronoi cell can be identified in distribution with the \( f \)-vector of (the dual of) a particular random beta’ polytope. Also the explicit expected values can be determined from this distributional identity and the known results for beta’ polytopes. We establish in addition a link between the expected \( f \)-vector of typical spherical Voronoi cells and that of a special beta polytope. Of special interest are the low-dimensional cases \( d \in \{2, 3, 4\} \) which will be examined separately.

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We would like to point out that our paper continues a recent line of research in stochastic geometry which focuses on the study of non-Euclidean geometric random structures. As examples we mention the studies of random convex hulls in spherical convex bodies or on half-spheres [3, 11, 23, 22], the results on random tessellations by great hyperspheres [1, 16, 17, 27], the central and non-central limit theorems for Poisson hyperplanes in hyperbolic spaces [15], the papers [12, 18] on splitting tessellations on the sphere, the asymptotic investigation of Voronoi tessellations on general Riemannian manifolds [8] and the general limit theory for stabilizing functionals of point processes in manifolds [31].

1.2. The typical Voronoi cell. We are now going to introduce our framework. Let \( S^d \) be the \( d \)-dimensional unit sphere, which we think of being embedded in \( \mathbb{R}^{d+1} \) in such a way that it is centred at the origin of \( \mathbb{R}^{d+1} \). A typical point in \( \mathbb{R}^{d+1} \) is denoted by \( x = (x_0, x_1, \ldots, x_d) \). The dimension of the sphere, \( d \in \mathbb{N} \), is fixed once and for all. The normalized spherical Lebesgue measure on \( S^d \) is denoted by \( \sigma_d \). Let \( X_1, \ldots, X_n \) be \( n \in \mathbb{N} \) independent random points sampled on \( S^d \) according to \( \sigma_d \) and defined over some underlying probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \). The binomial process \( \xi_n := \{X_1, \ldots, X_n\} \) is the point process on \( S^d \) with atoms at \( X_1, \ldots, X_n \). We can now construct the spherical Voronoi tessellation based on \( \xi_n \) as follows. If \( \rho(\cdot, \cdot) \) denotes the geodesic distance on \( S^d \), we let \( C_{i,n} \) be the Voronoi cell of a point \( X_i \in \xi_n \), that is,

\[
C_{i,n} := \{z \in S^d : \rho(X_i, z) \leq \rho(X_j, z) \text{ for all } j \in \{1, \ldots, n\}\}, \quad i \in \{1, \ldots, n\}.
\]

As in the Euclidean case (see [33, Chapter 10]), one shows that the sets \( C_{1,n}, \ldots, C_{n,n} \) are in fact spherical polytopes covering \( S^d \) and having disjoint interiors. Here, we recall that a spherical polytope is defined as an intersection of \( S^d \) and a polyhedral convex cone and that the latter is defined as an intersection of finitely many half-spaces whose bounding hyperplanes contain the origin. The
collection \(\{C_{1,n}, \ldots, C_{n,n}\}\) of all Voronoi cells of points of \(\xi_n\) is what we call the spherical Voronoi tessellation \(\mathfrak{m}_{n,d}\), see Figure [1.1] for two sample realizations.

In this note we are interested in the typical cell of such a spherical Voronoi tessellation. Roughly speaking, the typical cell arises by picking one of the cells \(C_{i,n}\) uniformly at random and rotating it such that its “center” \(X_i\) becomes the north pole \(e := (1,0,\ldots,0)\) of \(S^d\). To make this precise, let \(U = U_n\) be a random variable with uniform distribution on the set \(\{1, \ldots, n\}\) and assume that \(U\) is independent of the binomial process \(\xi_n\). Also, for every point \(v \in S^d\) we fix some orthogonal transformation \(O_v : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}\) such that \(O_v e = e\) and assume that the matrix elements of \(O_v\) are Borel functions of \(v\). Then, the typical cell of the Voronoi tessellation \(\mathfrak{m}_{n,d}\) is a random spherical polytope \(\mathcal{V}_{n,d}\) defined by

\[
\mathcal{V}_{n,d} := O_{X_U} C_{U,n}. 
\]

Since \(X_1, \ldots, X_n\) are exchangeable, the tuple \((\xi_n, X_U)\) has the same joint law as \((\xi_n, X_1)\) and we arrive at the following distributional equality:

\[
\mathcal{V}_{n,d} \overset{d}{=} O_{X_1} C_{1,n}. 
\]

In the following, it will be more convenient to consider binomial process with \(n + 1\) rather than with \(n\) points. The next proposition states that the typical Voronoi cell \(\mathcal{V}_{n+1,d}\) of the binomial process \(\xi_{n+1}\) has the same distribution as the Voronoi cell of the north pole \(e\) in the point process \(\xi_n \cup \{e\}\). Note that it also proves that the distribution of the typical cell does not depend on the choice of the family of orthogonal transformations \((O_v)_{v \in S^d}\).

**Proposition 1.1.** We have the distributional equality

\[
\mathcal{V}_{n+1,d} \overset{d}{=} \{ z \in S^d : \rho(e, z) \leq \rho(X_j, z) \text{ for all } j \in \{1, \ldots, n\} \}. 
\]

**Proof.** Conditioning on \(X_1 = v\) and integrating over all \(v \in S^d\), we can write the distribution of \(\mathcal{V}_{n+1,d}\) as follows:

\[
P[\mathcal{V}_{n+1,d} \in B] = \int_{S^d} P[O_{X_1} C_{1,n+1} \in B | X_1 = v] \sigma_d(dv),
\]

for every Borel set \(B\) in the space of compact subsets of \(S^d\) endowed with the usual Hausdorff distance. Recalling the definition of \(C_{1,n+1}\), we can write

\[
P[O_{X_1} C_{1,n+1} \in B | X_1 = v] = P \left[ O_v \left\{ z \in S^d : \rho(v, z) \leq \min_{j=2,\ldots,n+1} \rho(X_j, z) \right\} \in B \right]
\]

\[
= P \left[ \left\{ y \in S^d : \rho(v, O_v^{-1} y) \leq \min_{j=2,\ldots,n+1} \rho(X_j, O_v^{-1} y) \right\} \in B \right]
\]

\[
= P \left[ \left\{ y \in S^d : \rho(e, y) \leq \min_{j=2,\ldots,n+1} \rho(O_v X_j, y) \right\} \in B \right]
\]

\[
= P \left[ \left\{ y \in S^d : \rho(e, y) \leq \min_{j=1,\ldots,n} \rho(X_j, y) \right\} \in B \right],
\]

where we defined \(y := O_v z\) and used that \((O_v X_2, \ldots, O_v X_{n+1})\) has the same law as \((X_1, \ldots, X_n)\). Since the right-hand side does not depend on \(v \in S^d\), we arrive at

\[
P[\mathcal{V}_{n+1,d} \in B] = P \left[ \left\{ y \in S^d : \rho(e, y) \leq \min_{j=1,\ldots,n} \rho(X_j, y) \right\} \in B \right],
\]

which completes the proof. \(\square\)
For stationary tessellations in the Euclidean space $\mathbb{R}^d$, where the number of cells is almost surely infinite, one usually defines the typical cell using the concept of Palm distribution, which is a common device in stochastic geometry [33]. The Palm approach can be applied on the sphere, too. The Palm distribution $P_{\xi_{n+1}}^e$ of the binomial process $\xi_{n+1}$ with respect to a fixed point on the sphere (which we choose to be the north pole $e$) will be defined in Section 3 below. In Lemma 3.1 we shall show that it is explicitly given as

$$P_{\xi_{n+1}}^e(\cdot) = P_{\xi_n}(\xi_n \cup \{e\} \in \cdot),$$

where $P_{\xi_{n+1}}$ denotes the distribution of the binomial process $\xi_{n+1}$. Thus, the definition of the typical cell given above coincides with the definition based on the Palm approach.

1.3. Total number of faces. Our goal is to describe the $f$-vector of the typical Voronoi cell $V_{n,d}$. More precisely, consider a spherical polytope $P \subset S^d$ represented as an intersection of $S^d$ and a polyhedral convex cone $C$. The $k$-dimensional faces of $P$ are defined as intersections of $(k+1)$-dimensional faces of $C$ with $S^d$, where $k \in \{0,1,\ldots,d\}$. We denote by $F_k(P)$ the set of $k$-dimensional faces of $P$ and by $f_k(P) := |F_k(P)|$ their number. Here, $|A|$ stands for the number of elements of a set $A$. The $d$-dimensional vector $(f_0(P), f_1(P), \ldots, f_{d-1}(P))$ is called the $f$-vector of $P$.

Before stating the results on the expected $f$-vector of the typical Voronoi cell, let us point out its connection to another natural quantity. The total number of $k$-dimensional faces of the tessellation $m_{n,d}$ is denoted by

$$f_k(m_{n,d}) := \left| \bigcup_{i=1}^n F_k(C_{i,n}) \right|, \quad k \in \{0,1,\ldots,d\}.$$ 

Note that even if some face $F$ belongs to more than one cell $C_{i,n}$, it is counted only once in the above definition.

**Proposition 1.2.** For all $n \geq d + 1$ and $k \in \{0,\ldots,d\}$, we have

$$\mathbb{E}f_k(m_{n,d}) = \frac{n}{d + k - 1} \mathbb{E}f_k(V_{n,d}).$$

**Proof.** We use a double-counting argument. Let $N := \sum_{i=1}^n f_k(C_{i,n})$ be the number of pairs $(C_{i,n}, F)$, where $C_{i,n}$ is a cell of the tessellation $m_{n,d}$, and $F \subset C_{i,n}$ a $k$-dimensional face of $C_{i,n}$. On the one hand, the above definition [1.1] of the typical cell implies that

$$\mathbb{E}f_k(V_{n,d}) = \mathbb{E}f_k(O_{X_U} C_{U,n}) = \mathbb{E}f_k(C_{U,n}) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}f_k(C_{i,n}) = \frac{1}{n} \sum_{i=1}^n f_k(C_{i,n}) = \frac{EN}{n}.$$ 

On the other hand, the spherical Voronoi tessellation is normal, that is, every $k$-dimensional face belongs to $(d - k + 1)$ cells of dimension $d$, with probability one (cf. Theorem 10.2.3 in [33] for a similar statement in the Euclidean case). It follows that almost surely

$$N = (d - k + 1)f_k(m_{n,d}).$$

By taking the expectations and comparing both identities, we arrive at the claim. $\Box$
1.4. **Reduction to beta' polytopes.** As anticipated above, our goal will be to identify the expected $f$-vector of the typical Voronoi cell $\mathcal{V}_{n+1,d}$ generated by $n + 1$ uniformly distributed random points on the $d$-dimensional unit sphere. We do this first in terms of the $f$-vector of random beta' polytopes, a notion we are going to explain next. For $\beta > d/2$ we define the probability density $\tilde{f}_{d,\beta}$ on $\mathbb{R}^d$ by

$$
\tilde{f}_{d,\beta}(x) := \tilde{c}_{d,\beta} (1 + \|x\|^2)^{-\beta}, \quad \tilde{c}_{d,\beta} = \frac{\Gamma(\beta)}{\pi^{d/2}\Gamma(\beta - d/2)},
$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. We let $\tilde{P}_{n,d}^\beta := \text{conv}(\tilde{X}_1, \ldots, \tilde{X}_n)$ be the convex hull of $n \in \mathbb{N}$ independent random points $\tilde{X}_1, \ldots, \tilde{X}_n$ distributed in $\mathbb{R}^d$ according to the density $\tilde{f}_{d,\beta}$. This random polytope is known as a so-called beta' polytope. In our notation we follow [21, 24, 25], where these polytopes were studied. As in the spherical case we denote by $(f_0(P), f_1(P), \ldots, f_{d-1}(P))$ the $f$-vector of a polytope $P \subset \mathbb{R}^d$, where $f_k(P)$, $k \in \{0, 1, \ldots, d\}$, is the number of $k$-dimensional faces of $P$. Our main result relates the $f$-vector of $\mathcal{V}_{n+1,d}$ to that of $\tilde{P}_{n,d}^\beta$ with $\beta = d$ and can be formulated as follows, the proof is postponed to Section 2.

**Theorem 1.3.** For each $n \geq d + 1$ we have that

$$
(f_k(\mathcal{V}_{n+1,d}))_{k=0}^{d-1} = (f_{d-k-1}(\tilde{P}_{n,d}^\beta))_{k=0}^{d-1},
$$

where $\equiv$ denotes equality in distribution of random vectors.

1.5. **Reduction to beta polytopes.** Recall that $X_1, \ldots, X_n$ are independent and uniformly distributed random points on $\mathbb{S}^d$. Denote their convex hull by $P_{n,d+1}^{-1} := \text{conv}(X_1, \ldots, X_n)$. This random polytope is a particular case of a beta polytope with parameter $\beta = -1$ studied in [21, 24, 25]. We follow the notation used there. Our next theorem expresses the expected $f$-vector of $\mathcal{V}_{n,d}$ in terms of that of $P_{n,d+1}^{-1}$.

**Theorem 1.4.** For each $n \geq d + 2$ and $k \in \{0, 1, \ldots, d\}$ we have that

$$
\mathbb{E}f_k(\mathcal{V}_{n,d}) = \frac{d - k + 1}{n} \mathbb{E}f_{d-k}(P_{n,d+1}^{-1}).
$$

**Proof.** There is a duality between the faces of the Voronoi tessellation $m_{n,d}$ and the faces of the convex hull of $X_1, \ldots, X_n$, which was stated already in the work of Edelsbrunner and Nikitenko [13, pp. 3226–3227]. It says that for arbitrary $\ell \in \{0, \ldots, d\}$ and $1 \leq i_0 < \ldots < i_\ell \leq n$, the convex hull of $X_{i_0}, \ldots, X_{i_\ell}$ is a face of the convex hull of $X_1, \ldots, X_n$ if and only if the cells $C_{i_0}, \ldots, C_{i_\ell}$ have a non-empty intersection. This intersection is then a common face of these cells of dimension $d - \ell$, with probability 1. To explain this connection properly we follow [33, pp. 472–473]. Let $E = E(X_{i_0}, \ldots, X_{i_\ell})$ be the $\ell$-dimensional affine subspace through the points $X_{i_0}, \ldots, X_{i_\ell}$, put $F = E^\perp$ and note that $F$ is a linear subspace of dimension $d + 1 - \ell$. The intersection of $F$ with $\mathbb{S}^d$ is a $(d - \ell)$-dimensional subsphere of $\mathbb{S}^d$. If we denote by $\text{Cap}(x, r) := \{y \in \mathbb{S}^d : \rho(x, y) < r\}$ the open spherical cap centred at $x \in \mathbb{S}^d$ with radius $r > 0$ and put $S(X_{i_0}, \ldots, X_{i_\ell}) := \{y \in F \cap \mathbb{S}^d : \text{Cap}(y, \rho(y, X_{i_0})) \cap \{X_1, \ldots, X_n\} = \emptyset\}$, we have that $S(X_{i_0}, \ldots, X_{i_\ell}) \neq \emptyset$ if and only if $F \cap \mathbb{S}^d$ contains a $(d - \ell)$-dimensional common face of $C_{i_0}, \ldots, C_{i_\ell}$. On the other hand, $S(X_{i_0}, \ldots, X_{i_\ell}) \neq \emptyset$ also means by definition that the points $X_{i_0}, \ldots, X_{i_\ell}$ form a face of dimension $\ell$ of the convex hull of $X_1, \ldots, X_n$. Thus, taking $\ell := d - k$ we conclude that

$$
f_{d-k}(P_{n,d+1}^{-1}) = f_k(m_{n,d}) \text{ a.s.}
$$

(1.4)
On the other hand, by Proposition 1.2 we have
\[ \mathbb{E}f_k(m_{n,d}) = \frac{n}{d-k+1} \mathbb{E}f_k(V_{n,d}). \]
Putting these results together, we arrive at the claim. 

1.6. Explicit formula for the expected $f$-vector. The expected $f$-vectors of beta and beta' polytopes have been explicitly determined in the series of works \cite{19, 21, 20, 22, 25}. The main results we shall rely on are stated in Theorems 7.1 and 7.3 of \cite{21}. Combining these formulae with Theorem 1.3 or Theorem 1.4 we arrive at the following explicit expression for the $f$-vector of the typical Voronoi cell $V_{n+1,d}$.

**Theorem 1.5.** For all $d \geq 2$, $n \geq d + 1$ and $\ell \in \{1, \ldots, d\}$ we have

\[ \mathbb{E}f_{d-\ell}(V_{n+1,d}) = \frac{1}{\pi} \left( \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} \right)^{n-\ell} \sum_{\substack{m \in \{\ell, \ldots, d\} \\ m \equiv \ell \pmod{2}}} \frac{I_d(n, m)(md - 1)J_d(m, \ell)}{n - \ell \sum_{\substack{m \in \{\ell, \ldots, d\} \\ m \equiv \ell \pmod{2}}} (m - 1) \tilde{a}(m, \ell - 1)} \quad (1.5) \]

where

\[ \begin{align*}
\tilde{I}_d(n, m) &:= \binom{n}{m} \int_{-\pi/2}^{+\pi/2} (\cos x)^{dm - 1}(\tilde{F}_d(x))^{n-m} \, dx, \\
\tilde{J}_d(m, \ell) &:= \binom{m}{\ell} \int_{-\infty}^{+\infty} (\cosh y)^{-(d+1)(\tilde{F}_d(iy))^{n-\ell}} \, dy, \\
I_{d-1}(n, m) &:= \binom{n}{m} \int_{-\pi/2}^{+\pi/2} (\cos x)^{(d-1)(m+1)}(F_{d-1}(x))^{n-m} \, dx, \\
J_{d-1}(m, \ell) &:= \binom{m}{\ell} \int_{-\infty}^{+\infty} (\cosh y)^{-(d+1)(m+1)-2}(F_{d-1}(iy))^{n-\ell} \, dy, \\
\tilde{F}_d(z) &= F_{d-1}(z) := \int_{-\pi/2}^{z} (\cos y)^{d-1} \, dy, \quad z \in \mathbb{R}. 
\end{align*} \]

**Proof.** We can give two proofs based on reduction of the spherical Voronoi tessellation to beta' and beta polytopes. These proofs yield (1.5) and (1.6), respectively. Let us start with the approach based on beta' polytopes. By Theorem 1.3 we have

\[ \mathbb{E}f_{d-\ell}(V_{n+1,d}) = \mathbb{E}f_{\ell-1}(\tilde{P}_{n,d}). \]

By \cite{21} Theorem 7.3] applied with $\alpha = \beta = d$, we obtain

\[ \mathbb{E}f_{\ell-1}(\tilde{P}_{n,d}) = \frac{2 \cdot n!}{\ell!} \left( \frac{\Gamma\left(\frac{d+1}{2}\right)}{d \sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} \right)^{n-\ell} \sum_{\substack{m \in \{\ell, \ldots, d\} \\ m \equiv \ell \pmod{2}}} \frac{\tilde{b}(n, m)}{m - \frac{1}{d}} \tilde{a}\left[m - \frac{2}{d}, \ell - \frac{2}{d}\right]. \]
Proposition 1.7. 

According to our numerical computations, the individual summands in (1.5) and (1.6) are, in general, not equal. Finding a direct proof of this equality seems non-trivial. Let us also mention that, 

\[ a \left[ m - \frac{2}{d}, \ell - \frac{2}{d} \right] = \binom{d-1}{m-\ell+1} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (\cosh y)^{d-1} (F_{d-1}(iy))^{m-\ell} \, dy, \]

and where \( \tilde{F}_d \) is as above. After straightforward transformations, we arrive at (1.5). On the other hand, we can give an alternative proof based on Theorem 1.4 which states that 

\[ \mathbb{E}f_{d-\ell}(\mathcal{Y}_{n+1,d}) = \frac{\ell + 1}{n + 1} \mathbb{E}f_{\ell}(P_{n+1,d+1}). \]

The expected \( f \)-vector of the beta polytope is given explicitly in [21, Theorem 7.1]. Applying this theorem with \( \beta = -1 \) and \( \alpha = d - 1 \), we obtain 

\[ \mathbb{E}f_{\ell}(P_{n+1,d+1}) = \frac{2 \cdot (n + 1)!}{(\ell + 1)!} \left( \frac{\Gamma \left( \frac{d-1}{2} \right)}{2^{\frac{d}{2}} \Gamma \left( \frac{d}{2} \right)} \right)^{n-\ell} \times \sum_{\substack{m \in \{\ell, \ldots, d\} \atop m \equiv \ell \pmod{2}}} b\{n + 1, m + 1\} \left( m + 1 + \frac{1}{d-1} \right), \]

where 

\[ b\{n + 1, m + 1\} = \binom{d-1}{m-\ell+1} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (\cosh y)^{d-1} (F_{d-1}(iy))^{m-\ell} \, dy. \]

After some transformations, we arrive at (1.6). \( \square \)

Remark 1.6. In particular, we obtained an indirect proof that the right-hand sides of (1.5) and (1.6) are equal. Finding a direct proof of this equality seems non-trivial. Let us also mention that, according to our numerical computations, the individual summands in (1.5) and (1.6) are, in general, not equal.

Proposition 1.7. Let \( d \geq 2 \), \( n \geq d + 2 \) and \( k \in \{0, \ldots, d - 1\} \). If \( d \) is even, then \( \mathbb{E}f_k(\mathcal{Y}_{n,d}) \) is a rational number. If \( d \) is odd, then \( \mathbb{E}f_k(\mathcal{Y}_{n,d}) \) is a linear combination of the numbers \( \pi^{-2r} \), where \( r = 0, 1, \ldots, \left\lfloor \frac{n-d+k-1}{2} \right\rfloor \) with rational coefficients.

Proof. This follows from Theorem 1.4 together with [21, Theorem 7.2]. The same result could be deduced by combining Theorem 1.3 with [21, Theorem 7.4]. \( \square \)

Remark 1.8. Along with the Voronoi tessellation it is natural to consider the so-called spherical hyperplane tessellation which is defined as follows. As before, let \( X_1, \ldots, X_n \) be \( n \) independent, uniformly distributed random points on \( S^d \), where \( n \geq d + 1 \). Let \( X_i^\perp = \{ z \in \mathbb{R}^{d+1} : \langle z, X_i \rangle = 0 \} \) be the hyperplane orthogonal to \( X_i \). Here, \( \langle \cdot, \cdot \rangle \) stands for the standard scalar product in \( \mathbb{R}^{d+1} \). The hyperplanes \( X_1^\perp, \ldots, X_n^\perp \) dissect the sphere \( S^d \) into spherical polytopes which constitute the spherical hyperplane tessellation. The spherical Crofton cell \( Z_{n,d} \) is defined as the a.s. unique cell of this tessellation that contains the north pole \( e \). We have \( Z_{n,d} = S^d \cap (G_1 \cap \ldots \cap G_n) \), where \( G_i \) is the half-space bounded by \( X_i^\perp \) and containing the north pole \( e \). The expected \( f \)-vector of the
spherical Crofton cell $Z_{n,d}$ can be computed as follows. We observe that the dual of the convex cone $G_1 \cap \ldots \cap G_n$ is the positive hull $D_n := \text{pos}(X^+_1, \ldots, X^+_n)$ of the points $X^+_i := -X_i \cdot \text{sgn}(X_i, e)$. The points $X^+_1, \ldots, X^+_n$ are independent and uniformly distributed on the lower half-sphere $S^n_- := \{ z \in S^d : \langle z, e \rangle \leq 0 \}$. The corresponding $f$-vectors satisfy
\[
\mathbb{E}f_k(Z_{n,d}) = \mathbb{E}f_{k+1}(G_1 \cap \ldots \cap G_n) = \mathbb{E}f_{d-k}(D_n) = \mathbb{E}f_{d-k-1}(S^d \cap D_n)
\]
for all $k \in \{0, \ldots, d-1\}$. The expected face numbers of the random spherical polytopes $S^d \cap D_n$ that appear on the right-hand side have been explicitly computed in [22]. These polytopes are also closely related to the beta' polytopes $\tilde{P}_{n,d}^\beta$, but this time with $\beta = \frac{d+1}{2}$, see [23] [22].

1.7. **Low-dimensional cases.** Let us consider the low-dimensional cases separately. For example, in dimension $d = 2$, if we take $\ell = 1$ in Theorem 1.5 we arrive at the following result of Miles [27].

**Corollary 1.9.** For $d = 2$ and $n \geq 3$ we have
\[
\mathbb{E}f_0(\mathcal{V}_{n+1,2}) = \mathbb{E}f_1(\mathcal{V}_{n+1,2}) = 6 \cdot \frac{n-1}{n+1} = 6 \left(1 - \frac{2}{n+1}\right), \tag{1.7}
\]

**Proof.** According to Theorem 1.5 we have that $\mathbb{E}f_1(\mathcal{V}_{n+1,2}) = \frac{1}{n-1} \tilde{I}_2(n,2) \cdot 3 \cdot \tilde{J}_2(2,1)$. Moreover, $\tilde{F}_2(z) = 1 + \sin z$, which implies that $\tilde{J}_2(2,1) = \pi$. In addition,
\[
\begin{align*}
\binom{n}{2}^{-1} \tilde{I}_2(n,2) &= \int_{-1/2}^{\pi/2} (\cos x)^3(1 + \sin x)^{n-2} \, dx = \sum_{k=0}^{n-2} \binom{n-2}{k} \int_{-1/2}^{\pi/2} (\cos x)^3(\sin x)^k \, dx \\
&= \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{2(1 + (-1)^k)}{(k+1)(k+3)} = \frac{2^{n+1}}{n(n+1)},
\end{align*}
\]
which implies $\mathbb{E}f_1(\mathcal{V}_{n+1,2}) = \frac{3}{2^{n+1}} \binom{n}{2} \frac{2^{n+1}}{n(n+1)} = 6 \cdot \frac{n-1}{n+1}. \quad \Box$

As observed already by Miles [27], it is not surprising that $\mathbb{E}f_0(\mathcal{V}_{n+1,2}) \rightarrow 6$, as $n \rightarrow \infty$, which is the expected number of vertices of the typical cell of a Poisson-Voronoi tessellation in the plane, see [33] Theorem 10.2.5.

**Remark 1.10.** We note that (1.7) can alternatively be obtained by purely combinatorial means. Indeed, by the Euler relation and the fact that the Voronoi tessellation on the sphere is a.s. simple (which, for $d = 2$, means that each vertex of the tessellation belongs to exactly 3 edges), we have
\[
f_2(m_{n+1,d}) - f_1(m_{n+1,d}) + f_0(m_{n+1,d}) = 2 \quad \text{and} \quad 2f_1(m_{n+1,d}) = 3f_0(m_{n+1,d}) \quad \text{a.s.}
\]
Also, $f_2(m_{n+1,d}) = n + 1$ since each cell corresponds to its centre. Altogether, it follows that
\[
f_0(m_{n+1,d}) = 2(n-1) \quad \text{and} \quad f_1(m_{n+1,d}) = 3(n-1) \quad \text{a.s.}
\]
Taking the expectations and recalling Proposition 1.2 yields (1.7).

On the other hand, in dimensions $d > 2$ the $f$-vector of $m_{n+1,d}$ is not deterministic. For $d = 3$ and $d = 4$, we present exact formulae for the expected $f$-vector of the typical spherical Voronoi cell and refer to Table I for some exact and numerical values for small values of $n$. 

## References

[22] 

[23] 

[27]
Corollary 1.11. For $d = 3$ and all $n \geq 4$ we have
\[
E_0(\mathcal{V}_{n+1,3}) = \frac{256 \pi}{35} \left( \frac{1}{2\pi} \right)^{n-3} \binom{n}{3} \int_{-\pi/2}^{+\pi/2} (\cos x)^8 (2x + \sin(2x) + \pi)^{n-3} \, dx,
\]
\[
E_1(\mathcal{V}_{n+1,3}) = \frac{3}{2} E_0(\mathcal{V}_{n+1,3}),
\]
\[
E_2(\mathcal{V}_{n+1,3}) = \frac{1}{2} E_0(\mathcal{V}_{n+1,3}) + 2.
\]

Proof. The first formula follows from Theorem [1.5] with $d = 3$ and $\ell = 3$:
\[
E_0(\mathcal{V}_{n+1,3}) = \frac{1}{\pi} \left( \frac{2}{\pi} \right)^{n-3} I_3(n, 3) \cdot 8 \cdot J_3(3, 3).
\]

It remains to note that $\tilde{F}_3(z) = \frac{1}{2}(2z + \sin(2z) + \pi)$, which implies that $\tilde{J}_3(3, 3) = \frac{32}{35}$ and
\[
\tilde{I}_3(n, 3) = \left( \frac{1}{4} \right)^{n-3} \binom{n}{3} \int_{-\pi/2}^{+\pi/2} (\cos x)^8 (2x + \sin(2x) + \pi)^{n-3} \, dx.
\]

Since $\mathcal{V}_{n+1,3}$ is a simple polytope with probability one, we have that almost surely $2f_1(\mathcal{V}_{n+1,3}) = 3f_0(\mathcal{V}_{n+1,3})$. Finally, the formula for $E_0(\mathcal{V}_{n+1,3})$ follows from Euler’s relation, which says that almost surely $f_0(\mathcal{V}_{n+1,3}) - f_1(\mathcal{V}_{n+1,3}) + f_2(\mathcal{V}_{n+1,3}) = 2$. \hfill \qed

Corollary 1.12. For $d = 4$ and all $n \geq 5$ we have
\[
E_0(\mathcal{V}_{n+1,4}) = \frac{6435 \pi}{2048} \left( \frac{3}{48} \right)^{n-4} \binom{n}{4} \int_{-\pi/2}^{+\pi/2} (\cos x)^{15}(8 + 9\sin x + \sin(3x))^{n-4} \, dx,
\]
\[
E_1(\mathcal{V}_{n+1,4}) = 2E_0(\mathcal{V}_{n+1,4}),
\]
\[
E_2(\mathcal{V}_{n+1,4}) = 6 \frac{n-1}{n+1} + \frac{6}{5} E_0(\mathcal{V}_{n+1,4}),
\]
\[
E_3(\mathcal{V}_{n+1,4}) = 6 \frac{n-1}{n+1} + \frac{1}{5} E_0(\mathcal{V}_{n+1,4}).
\]
Proof. The identity for \( \mathbb{E} f_0(\mathcal{Y}_{n+1,4}) \) follows from Theorem 1.5. In fact, taking \( d = 4 \) and \( \ell = 4 \) we obtain
\[
\mathbb{E} f_0(\mathcal{Y}_{n+1,4}) = \frac{1}{\pi} \left( \frac{3}{4} \right)^{n-4} I_4(n, 4) \cdot 15 \cdot \mathbb{J}_4(4, 4).
\]
Moreover, \( \tilde{F}_4(z) = \frac{1}{12} (8 + 9 \sin z + \sin(3z)) \), which in turn implies that \( \mathbb{J}_4(4, 4) = \frac{429\pi}{2048} \) and
\[
\tilde{I}_4(n, 4) = \left( \frac{1}{12} \right)^{n-4} \left( \frac{n}{4} \right) \int_{-\pi/2}^{+\pi/2} (\cos x)^{n} (8 + 9 \sin x + \sin(3x))^{n-4} \, dx.
\]
To derive the other identities, we use the 3 linearly independent Dehn-Sommerville equations for simplicial 5-dimensional polytopes [17, Corollary 17.8]. Applied to \( P_{n+1,5}^{-1} \) they say that almost surely
\[
2 = f_0(P_{n+1,5}^{-1}) - f_1(P_{n+1,5}^{-1}) + f_2(P_{n+1,5}^{-1}) - f_3(P_{n+1,5}^{-1}) + f_4(P_{n+1,5}^{-1}),
\]
\[
2f_1(P_{n+1,5}^{-1}) = 3f_2(P_{n+1,5}^{-1}) - 6f_3(P_{n+1,5}^{-1}) + 10f_4(P_{n+1,5}^{-1}),
\]
\[
5f_4(P_{n+1,5}^{-1}) = 2f_3(P_{n+1,5}^{-1}).
\]
Using (1.4) these identities translate into the almost sure relations
\[
2 = f_4(m_{n+1,4}) - f_3(m_{n+1,4}) + f_2(m_{n+1,4}) - f_1(m_{n+1,4}) + f_0(m_{n+1,4}),
\]
\[
2f_3(m_{n+1,4}) = 3f_2(m_{n+1,4}) - 6f_1(m_{n+1,4}) + 10f_0(m_{n+1,4}),
\]
\[
5f_0(m_{n+1,4}) = 2f_1(m_{n+1,4})
\]
for the random Voronoi tessellation \( m_{n+1,4} \) on \( S^4 \). In addition, we have that almost surely \( f_4(m_{n+1,4}) = n + 1 \), since each cell of \( m_{n+1,4} \) corresponds to its centre. This implies that \( f_1(m_{n+1,4}), f_2(m_{n+1,4}) \) and \( f_3(m_{n+1,4}) \) can be expressed in terms of \( f_0(m_{n+1,4}) \) only. In fact, we have that almost surely
\[
f_1(m_{n+1,4}) = \frac{5}{2} f_0(m_{n+1,4}),
\]
\[
f_2(m_{n+1,4}) = 2(n - 1) + \frac{1}{2} f_0(m_{n+1,4}),
\]
\[
f_3(m_{n+1,4}) = 3(n - 1) + \frac{1}{2} f_0(m_{n+1,4}).
\]
We finally apply Proposition 1.2 to conclude that \( 5 \mathbb{E} f_0(m_{n+1,4}) = (n + 1) \mathbb{E} f_0(\mathcal{Y}_{n+1,4}) \) and
\[
\mathbb{E} f_1(\mathcal{Y}_{n+1,4}) = \frac{4}{n + 1} \mathbb{E} f_1(m_{n+1,4}) = \frac{4}{n + 1} \cdot \frac{5}{2} \cdot \mathbb{E} f_0(m_{n+1,4}) = 2 \mathbb{E} f_0(\mathcal{Y}_{n+1,4}).
\]
The identities for \( \mathbb{E} f_2(\mathcal{Y}_{n+1,4}) \) and \( \mathbb{E} f_3(\mathcal{Y}_{n+1,4}) \) follow similarly:
\[
\mathbb{E} f_2(\mathcal{Y}_{n+1,4}) = \frac{3}{n + 1} (2(n - 1) + 2 \mathbb{E} f_0(m_{n+1,4})) = 6 \frac{n - 1}{n + 1} + \frac{6}{5} \mathbb{E} f_0(\mathcal{Y}_{n+1,4}),
\]
\[
\mathbb{E} f_3(\mathcal{Y}_{n+1,4}) = \frac{2}{n + 1} \left( 3(n - 1) + \frac{1}{2} \mathbb{E} f_0(m_{n+1,4}) \right) = 6 \frac{n - 1}{n + 1} + \frac{1}{5} \mathbb{E} f_0(\mathcal{Y}_{n+1,4}).
\]
This completes the argument. \( \square \)

Remark 1.13. It is interesting to note that if we would apply the Dehn-Sommerville equations directly to the typical Voronoi cell \( \mathcal{Y}_{n+1,4} \), this would not yield enough relations to express all \( \mathbb{E} f_i(\mathcal{Y}_{n+1,4}) \) through \( \mathbb{E} f_0(\mathcal{Y}_{n+1,4}) \).
2. Proof of Theorem 1.3

2.1. Preliminaries. Let us first introduce some notation. Recall that \( \xi_n = \{X_1, \ldots, X_n\} \) is a binomial process on \( S^d \) induced by \( n \in \mathbb{N} \) independent random points \( X_1, \ldots, X_n \) with the uniform distribution \( \sigma_d \). For each \( i \in \{1, \ldots, n\} \) we let \( h_i \in [-1, 1] \) be the projection of \( X_i \) onto the 0-th coordinate of \( \mathbb{R}^{d+1} \) which is shown as the vertical direction in Figure 2.1. Recalling that \( e = (1, 0, \ldots, 0) \) is the vector pointing to the north pole of \( S^d \), and denoting the angle between \( e \) and \( X_i \) by \( \theta_i \), we have that

\[
h_i = \langle X_i, e \rangle = \cos \theta_i,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product in \( \mathbb{R}^d \). We can then decompose \( X_i \) as follows:

\[
X_i = e \cos \theta_i + U_i \sin \theta_i, \quad i \in \{1, \ldots, n\},
\]

where \( U_i \) is a suitable unit vector in the \( d \)-dimensional hyperplane \( e^\perp = \{x_0 = 0\} \) which we identify with \( \mathbb{R}^d \).

Since the distribution of \( X_i \) is rotationally invariant (in particular, it is invariant with respect to all rotations preserving \( e \)), we have that \( U_1, \ldots, U_n \) are uniformly distributed over the unit sphere in \( \mathbb{R}^d \) and that \( h_1, \ldots, h_n, U_1, \ldots, U_n \) are independent. In the next lemma we determine the density of \( h_i \). It is well known and can be found in [23, Lemma 7.6], where it is deduced from the slice integration formula for spheres, see Corollary A.5 in [2]. Alternatively, it can be deduced from Lemma [24, Lemma 4.4].

**Lemma 2.1.** For each \( i \in \{1, \ldots, n\} \) the random variable \( h_i \) has density

\[
f(h) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} \left(1 - h^2\right)^{\frac{d}{2}-1}, \quad h \in [-1, 1],
\]

with respect to the Lebesgue measure on \([-1, 1]\).
2.2. **Proof of Theorem 1.3.** The starting point of our proof is the representation of the typical Voronoi cell on $S^d$ given in Proposition 1.1:

$$V_{n+1,d} = \bigcap_{i=1}^{n} \{ z \in S^d : \rho(e,z) \leq \rho(X_i,z) \}.$$ 

Recalling that the geodesic distance on $S^d$ is given by $\rho(x,y) = \arccos \langle x,y \rangle$, $x,y \in S^d$, and using that the function $u \mapsto \arccos u$ is decreasing on $[-1,1]$, we can write the above representation as

$$V_{n+1,d} = \bigcap_{i=1}^{n} (L^+_i \cap S^d),$$ \hspace{1cm} (2.2)

where $L^+_1, \ldots, L^+_n \subset \mathbb{R}^{d+1}$ are half-spaces defined by

$$L^+_i := \{ z \in \mathbb{R}^{d+1} : \langle e,z \rangle \geq \langle X_i,z \rangle \} = \{ z \in \mathbb{R}^{d+1} : \langle X_i - e, z \rangle \leq 0 \}, \hspace{0.5cm} i \in \{1, \ldots, n\}.$$ 

The bounding hyperplane of $L^+_i$ is denoted by

$$L_i := \{ z \in \mathbb{R}^{d+1} : \langle X_i - e, z \rangle = 0 \}, \hspace{0.5cm} i \in \{1, \ldots, n\}.$$ 

Note that $L_i$ passes through the origin of $\mathbb{R}^{d+1}$ and that $e \in L^+_i$. Consider the convex random polyhedral cone

$$C_n := \bigcap_{i=1}^{n} L^+_i \subset \mathbb{R}^{d+1},$$

see Figure 2.2. By definition, the $k$-dimensional faces of the spherical polytope $C_n \cap S^d$ (which is the right-hand side of (2.2)) are in bijective correspondence with the $(k+1)$-dimensional faces of the polyhedral cone $C_n$. Thus, we arrive at the distributional equality

$$\left( f_k(V_{n+1,d}) \right)_{k=0}^{d-1} = \left( f_k(C_n \cap S^d) \right)_{k=0}^{d-1} = \left( f_{k+1}(C_n) \right)_{k=0}^{d-1}. \hspace{1cm} (2.3)$$

The **dual** or **polar** of the convex cone $C_n$ is defined as

$$C^\circ_n := \{ x \in \mathbb{R}^{d+1} : \langle x,y \rangle \leq 0 \text{ for all } y \in C_n \}.$$
Since the \((k+1)\)-dimensional faces of \(C_n\) are in bijective correspondence with the \((d-k)\)-dimensional faces of \(C_n^\circ\), it follows from (2.3) that
\[
(f_k(Y_{n+1,d}))_{k=0}^{d} = (f_{d-k}(C_n^\circ))_{k=0}^{d-1}.
\]
(2.4)
Since \(C_n\) is defined as the intersection of the half-spaces \(L_1^+, \ldots, L_n^+\), the dual cone is the positive hull of the outward normal vectors of these half-spaces, that is
\[
C_n^\circ = \text{pos}(X_1 - e, \ldots, X_n - e).
\]

It follows from \(e \in C_n\) that \(C_n^\circ\) is contained in the lower half-space \(\{x_0 \leq 0\}\). Recall from (2.1) that \(X_i - e = e(\cos \theta_i - 1) + U_i \sin \theta_i\). We may ignore the case when some \(\theta_i = 0\) because it has probability 0. Normalizing the vectors spanning \(C_n^\circ\) in such a way that their 0-th coordinate becomes \(-1\), we get
\[
C_n^\circ = \text{pos}\left(-e + \frac{U_1}{R_1}, \ldots, -e + \frac{U_n}{R_n}\right),
\]
where
\[
R_i := \frac{1 - \cos \theta_i}{\sin \theta_i} = \tan\left(\frac{\theta_i}{2}\right), \quad i \in \{1, \ldots, n\}.
\]
(2.7)
It follows from \(C_n^\circ \subset \{x_0 \leq 0\}\) that the \((d-k)\)-dimensional faces of \(C_n^\circ\) are in one-to-one correspondence with the \((d-k-1)\)-dimensional faces of the polytope obtained by intersecting \(C_n^\circ\) with the tangent space to \(S^d\) at its south pole \(-e\). Define the polytope
\[
Q_n := (C_n^\circ \cap \{x_0 = -1\}) + e = \text{conv}\left(\frac{U_1}{R_1}, \ldots, \frac{U_n}{R_n}\right) \subset \{x_0 = 0\}.
\]
(2.5)
Recalling (2.4) we can write
\[
(f_k(Y_{n+1,d}))_{k=0}^{d} = (f_{d-k-1}(Q_n))_{k=0}^{d-1}.
\]
(2.6)
To complete the proof of Theorem 1.3 it remains to verify that the random polytope \(Q_n\) has the same distribution as the beta’ polytope \(\tilde{P}^d_{n,d}\) in \(\mathbb{R}^d\) with parameter \(\beta = d\). We claim that the random variables \(R_i\) enjoy the following distribution invariance property.

**Lemma 2.2.** For each \(i \in \{1, \ldots, n\}\) one has that \(R_i = \frac{d}{R_i}\).

*Proof.* The trigonometric identity \(\frac{1 - \cos \theta}{\sin \theta} = 1 + \cos \theta\) implies that
\[
\frac{1 - \cos \theta_i}{\sin \theta_i} = \frac{\sin \theta_i}{1 + \cos \theta_i}.
\]
(2.7)
Since the random point \(X_i\) has the same distribution as the reflection of \(X_i\) at the hyperplane \(\{x_0 = 0\} \subset \mathbb{R}^{d+1}\) we have that \(\theta_i = \pi - \theta_i\). Thus,
\[
R_i = \frac{1 - \cos(\pi - \theta_i)}{\sin(\pi - \theta_i)} = \frac{1 + \cos \theta_i}{\sin \theta_i} = \frac{1}{R_i}.
\]
This completes the argument. \(\square\)

In a next step we determine the probability density of the random variables \(R_i\).

**Lemma 2.3.** For each \(i \in \{1, \ldots, n\}\) the random variable \(R_i\) has density
\[
g(r) = \frac{2^d \Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} \frac{r^{d-1}}{(1 + r^2)^d}, \quad r \geq 0,
\]
(2.8)
with respect to the Lebesgue measure on \([0, \infty)\).
Proof. In view of (2.7) we have that
\[ R_i^2 = \frac{1 - \cos \theta_i}{\sin \theta_i} \cdot \frac{\sin \theta_i}{1 + \cos \theta_i} = \frac{1 - \cos \theta_i}{1 + \cos \theta_i} = 1 - h_i. \]
This implies that for each \( r \geq 0, \)
\[ \mathbb{P}[R_i \geq r] = \mathbb{P}\left[ \sqrt{\frac{1 - h_i}{1 + h_i}} \geq r \right] = \mathbb{P}[h_i \leq \frac{1 - r^2}{1 + r^2}] = \frac{\Gamma(d+1)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} \int_0^{\frac{1 - r^2}{1 + r^2}} (1 - h^2)^{\frac{d}{2} - 1} dh, \]
where the last identity comes from Lemma 2.1. Differentiation with respect to \( r \) thus proves that the density of \( R_i \) is
\[ g(r) = \frac{\Gamma(d+1)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} \left(1 - \left(\frac{1 - r^2}{1 + r^2}\right)^2\right)^{\frac{d}{2} - 1} \frac{4r}{(1 + r^2)^2} = \frac{2^d \Gamma(d+1)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} \frac{r^{d-1}}{(1 + r^2)^d}, \]
which completes the argument. \( \square \)

We are now in position to complete the proof of Theorem 1.3. It follows from Lemma 2.2 and (2.5) that
\[ Q_n \overset{d}{=} \text{conv} \left( U_1 R_1, \ldots, U_n R_n \right). \]
Recall that \( U_1, \ldots, U_n \) are i.i.d. and uniformly distributed on the unit sphere in \( \mathbb{R}^d \). Recall also that this family is independent of the collection \( R_1, \ldots, R_n \) of random variables which are also i.i.d. and have density \( g(r) \) given by (2.8). Altogether, it follows that \( U_1 R_1, \ldots, U_n R_n \) are independent random points in \( \mathbb{R}^d \) with density \( f_{d,\beta} \) given by (1.3), where \( \beta = d \). Hence, \( Q_n \) has the same distribution as the beta polytope \( \tilde{P}_{n,d}^{\beta} \), and the proof of Theorem 1.3 is complete. \( \square \)

3. Palm distribution of a binomial process on the sphere

We denote by \( \mathcal{N} \) the measurable space of finite counting measures on \( S^d \) supplied with the Borel \( \sigma \)-field induced by the evaluation mappings \( \eta \mapsto \eta(B) \), where \( \eta \in \mathcal{N} \) and \( B \) is a Borel subset of \( S^d \). Let \( \zeta \) be a point process on \( S^d \), that is, a measurable mapping from an underlying probability space \((\Omega, \mathcal{A}, \mathbb{P})\) taking values in \( \mathcal{N} \). In what follows it is convenient for us to identify a point process with the finite set of its atoms, counted with multiplicities. For a point process \( \zeta \) we call \( \mathbb{E}[\zeta(S^d)] \) the intensity of \( \zeta \). We also recall that a point process \( \zeta \) on \( S^d \) is called isotropic if for all \( O \in \text{SO}(d) \) the rotated point process \( O\zeta \) has the same law as \( \zeta \). The unit sphere \( S^d \) is a homogeneous space on which the rotation group \( \text{SO}(d+1) \) acts transitively, and \( \text{SO}(d) \) can be identified with the stabilizer of the north pole \( e := (1,0,\ldots,0) \). For \( v \in S^d \) we let \( \Theta_v \) denote the set of all orthogonal transformations \( O : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \) such that \( Ov = e \). Note that \( \Theta_v \) is a group which can be identified with \( \text{SO}(d) \). By \( \nu_v \) we denote the unique Haar probability measure on \( \Theta_v \) and define the image measure \( \nu_v(A) := \nu_v(\{OO_v^{-1} : O \in A\} \}, A \subset \Theta_v, \) on \( \Theta_v, \) where \( O_v \in \Theta_v \) is arbitrary. This definition is independent of the choice of \( O_v, \) see [32]. We are now prepared to define the Palm distribution of an isotropic point process \( \zeta \) with respect to the point \( e \) by
\[ P_\zeta^e(B) := \frac{1}{\mu} \mathbb{E} \sum_{v \in \zeta} \int_{\Theta_v} 1(O^{-1} \zeta \in B) \nu_v(dO), \]
where \( B \subset \mathcal{N} \) is a Borel set, \( \mu := \mathbb{E}[\zeta(S^d)] \) is the intensity of \( \zeta \) and \( P_\zeta \) stands for the distribution of \( \zeta \).

The next result identifies the Palm distribution \( P_{\zeta_n+1}^e \) of a binomial process \( \xi_{n+1} \) on \( S^d \) consisting of \( n + 1 \geq 2 \) independent random points distributed according to the normalized spherical
Together with (3.1) this implies the identity

Choosing now for heterogeneous spaces (see the Corollary after Theorem 1 in [32]) says that

On the other hand, a special case of the refined Campbell theorem for random measures on homogeneous spaces (see the Corollary after Theorem 1 in [32]) says that

\begin{equation}
E \sum_{v \in \xi_{n+1}} f(v; \xi_{n+1}) = (n + 1) \int_{S^d} \int_{\mathcal{N}} \int_{\Theta_v} f(v; O(\varphi \cup \{e\})) \nu_v(dO) P_{\xi_n}(d\varphi) \sigma_d(dv)
\end{equation}

Note that this relation can be rewritten in the form

\begin{equation}
E \sum_{v \in \xi_{n+1}} f(v; \xi_{n+1}) = (n + 1) \int_{S^d} \int_{\mathcal{N}} \int_{\Theta_v} f(v; O(\varphi \cup \{e\})) \nu_v(dO) P_{\xi_{n+1}}(d\varphi) \sigma_d(dv)
\end{equation}

On the other hand, a special case of the refined Campbell theorem for random measures on homogeneous spaces (see the Corollary after Theorem 1 in [32]) says that

\begin{equation}
E \sum_{v \in \xi_{n+1}} f(v; \xi_{n+1}) = (n + 1) \int_{S^d} \int_{\mathcal{N}} \int_{\Theta_v} f(v; O\varphi) \nu_v(dO) P_{\xi_n}(d\varphi) \sigma_d(dv)
\end{equation}

Together with (3.1) this implies the identity

\begin{equation}
\int_{S^d} \int_{\mathcal{N}} \int_{\Theta_v} f(v; O\varphi) \nu_v(dO) P_{\xi_{n+1}}(d\varphi) \sigma_d(dv)
\end{equation}

Choosing now for the function \( f(v, \varphi) = \int_{\Theta_v} 1\{O^{-1}\varphi \in A\} \nu_v(dO) \) for a Borel set \( A \subset \mathcal{N} \), the last identity simplifies to

\( P_{\xi_{n+1}}(A) = P(\xi_n \cup \{e\} \in A) \).

This proves the claim.

\begin{remark}
\text{The proof carries over almost verbatim to binomial point processes on general homogeneous spaces, i.e. the framework of [32].}
\end{remark}

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