Research Article

Padmakar-Ivan index of some types of perfect graphs

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Abstract

The Padmakar-Ivan (PI) index of a graph \( G \) is defined as \( PI(G) = \sum_{e \in E(G)} (|V(G)| - N_G(e)) \), where \( N_G(e) \) is the number of equidistant vertices for the edge \( e \). A graph is perfect if for every induced subgraph \( H \), the equation \( \chi(H) = \omega(H) \) holds, where \( \chi(H) \) is the chromatic number and \( \omega(H) \) is the size of a maximum clique of \( H \). In this paper, the PI index of some types of perfect graphs is obtained. These types include co-bipartite graphs, line graphs, and prismatic graphs.

Keywords: PI index; co-bipartite graphs; line graphs; prismatic graphs.

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1. Introduction

All graphs considered in this paper are finite, simple and connected. For a graph \( G \), the distance between two vertices \( x, y \) is denoted by \( d(x, y) \). A vertex \( w \) is equidistant for an edge \( e = xy \) if \( d(x, w) = d(y, w) \). For an edge \( e \in E(G) \), denote by \( D_G(e) \) the set of all equidistant vertices in \( G \). In particular, \( D_i(e) \) denotes the set of vertices at distance \( i \) for \( e \). Also, we denote \( |D_G(e)| = N_G(e) \).

The vertex Padmakar-Ivan (PI) index of a graph \( G \) is a topological index, defined as

\[
PI(G) = \sum_{e=uv \in E(G)} (n_u(e) + n_v(e)),
\]

where \( n_u(e) \) denotes the number of those vertices of \( G \) whose distance from the vertex \( u \) is smaller than the distance from the vertex \( v \) and \( n_v(e) \) denotes the number of those vertices of \( G \) whose distance from \( v \) is smaller than the distance from \( u \). Since \( n_u(e) + n_v(e) = |V(G)| - N_G(e) \), the PI index can be rewritten as

\[
PI(G) = \sum_{e \in E(G)} (|V(G)| - N_G(e)).
\]

The PI index was proposed by Khadikar [10] in 2000. Khadikar and his coauthors investigated the chemical and biological applications of this index in [11]. Khalifeh [12] introduced a vertex version of the PI index and using this notion, they computed exact expression for the PI index of Cartesian product of graphs. John and Khadikar established a method for calculating the PI index of benzenoid hydrocarbons using orthogonal cuts in [9]. Gutman and Ashrafi [6] obtained the PI index of phenylenes and their hexagonal squeezes. The PI index of bridge graphs and chain graphs was studied in [13]. Das and Gutman [3] obtained a lower bound on the PI index of a connected graph in terms of the number of vertices, edges, pendent vertices, and the clique number, and also they characterized the extremal graphs. There are different types of topological indices; for example distance-based topological indices, degree-based topological indices, etc. Topological indices has many applications in the field of mathematical chemistry. Trinajstic and Zhou introduced the sum-connectivity index and found several basic properties in [16]. Many topological indices and their applications are thoroughly explored in [15]. Ilić and Milosavljević introduced the weighted vertex PI index and established some of its basic properties in [7]. The weighted PI index of a graph \( G \) is given as

\[
P_I^w(G) = \sum_{e=uv \in E(G)} (d_G(u) + d_G(v)) (|V(G)| - N_G(e)).
\]

Gopika et al. [5] obtained the weighted PI index of the direct and strong product for certain types of graphs. Indulal et al. [8] studied the graphs satisfying the equation \( PI(G) = PI(G - e) \).
A graph is perfect if for every induced subgraph $H$, the equation $\chi(H) = \omega(H)$ holds, where $\chi(H)$ is the chromatic number and $\omega(H)$ is the size of a maximum clique of $H$. A claw-free graph is a graph in which no vertex has three pairwise non-adjacent neighbours. Every claw-free graph is a perfect graph. A survey on claw-free graphs is given in [4]. Chudnovsky and Seymour studied the structure of claw-free graphs thoroughly in a series of seven papers from 2007 to 2012. For example, in the first paper [1] of this series, they studied the orientable prismatic graphs and in the second paper [2], they studied non-orientable prismatic graphs. In this paper, we obtain the PI index of some classes of perfect graphs, including co-bipartite graphs, line graphs, and prismatic graphs.

2. Co-bipartite graphs

An edge $e = xy$ of a graph $G$ is said to be an equidistant edge for a vertex $a \in V(G)$ if $d(a, x) = d(a, y)$. The edge $e$ is at distance $r$ for a vertex $a$ if $d(a, x) = d(a, y) = r$. The set of all equidistant edges of $a$ is $D_G(a) = \{e = xy \in E(G) : d(a, x) = d(a, y)\}$ and we take $N_G(a) = |D_G(a)|$. It is easy to see that $\sum_{e \in E(G)} N_G(e) = \sum_{a \in V(G)} N_G(a)$.

**Lemma 2.1.** Let $G$ be a graph with $n$ vertices and $m$ edges. Then, $PI(G) = mn - \sum_{a \in G} N_G(a)$.

**Proof.**

\[
PI(G) = \sum_{e \in E(G)} (|V(G)| - N_G(e)) = \sum_{e \in E(G)} |V(G)| - \sum_{e \in E(G)} N_G(e) = mn - \sum_{a \in G} N_G(a).
\]

Let $G(U, V)$ be a bipartite graph with partite sets $U$ and $V$. A co-bipartite graph is the complement of a bipartite graph $G(U, V)$ and it is denoted as $\overline{G}$. In $\overline{G}$, the vertices in $U$ and the vertices in $V$ forms two disjoint cliques. Every co-bipartite graph is a perfect graph. The diameter of a connected co-bipartite graph is either 2 or 3.

Consider a bipartite graph $G(U, V)$ with $|U| = n$ and $|V| = m$. Let $\Delta_1$ and $\Delta_2$ be the maximum degree in $U$ and $V$ respectively, where $\Delta_1 \leq m$ and $\Delta_2 \leq n$. Let $U_1 = \{u \in U : d(u) < m\}$ and $U_2 = \{u \in U : d(u) = m\}$. It is noted that $U = U_1 \cup U_2$. Similarly, $V = V_1 \cup V_2$, provided that the degree of every vertex in $V_1$ is less than $n$ and the degree of every vertex in $V_2$ is $n$. Let $U_1 = \{u_1, u_2, \ldots, u_p\}$, $U_2 = \{u_{p+1}, u_{p+2}, \ldots, u_n\}$, $V_1 = \{v_1, v_2, \ldots, v_q\}$ and $V_2 = \{v_{q+1}, v_{q+2}, \ldots, v_m\}$. Let $d(u_i) = f_i$ for $i = 1, 2, \ldots, p$ and $d(v_j) = g_j$ for $i = 1, 2, \ldots, q$. We denote $\sum_{i=1}^{p} f_i$ by $f^*$, $\sum_{j=1}^{q} g_j$ by $g^*$ and $\sum_{i=1}^{p} g_i^2$ by $g^*$.

**Theorem 2.1.** Let $G(U, V)$ be a bipartite graph. Then $PI(\overline{G}) = n(n-1) + m(m-1) + mn(2p+q) - mp(m+n-p-q) - mp(p-1) - nq(n+q-1) + f(2m-n-1) - 2(f^* + g^*) + g(3n-1)$.

**Proof.** Let $U = U_1 \cup U_2$ and $V = V_1 \cup V_2$, where $U_1 = \{u_1, u_2, \ldots, u_p\}$, $U_2 = \{u_{p+1}, u_{p+2}, \ldots, u_n\}$, $V_1 = \{v_1, v_2, \ldots, v_q\}$, $V_2 = \{v_{q+1}, v_{q+2}, \ldots, v_m\}$, $d(u_i) = f_i$ if $i \leq p$, $d(u_i) = m$ if $i > p$, $d(v_j) = g_j$ if $j \leq q$, and $d(v_j) = n$ if $j > q$. The degrees in $\overline{G}$ (see Figure 1) are given as

\[
d(u_i) = \begin{cases} (m - f_i) + (n-1) & \text{if } i = 1, 2, \ldots, p \\ (n-1) & \text{if } i > p \end{cases}
\]

and

\[
d(v_j) = \begin{cases} (n - g_j) + (m-1) & \text{if } j = 1, 2, \ldots, q \\ (m-1) & \text{if } j > q. \end{cases}
\]

We partition $E(\overline{G})$ with $E_1$, $E_2$ and $E_3$, where $E_1$ is the set of edges in the clique with vertices in $U$, $E_2$ is the set of edges in the clique with vertices in $V$ and $E_3 = \{(u, v) : u \in U, v \in V\}$.

\[
\begin{array}{cccccccc}
u_1 & u_2 & \cdots & u_p & u_{p+1} & u_{p+2} & \cdots & u_n \\
v_1 & v_2 & \cdots & v_q & v_{q+1} & v_{q+2} & \cdots & v_m \\
\end{array}
\]

**Figure 1:** The graph $\overline{G}$ used in the proof of Theorem 2.1.

For a vertex $u \in U$, it is easy to see that

\[
N_{E_1}(u) = \frac{(n-1)(n-2)}{2}.
\]
A vertex \( v_i \in V_1 \) has \( (n - g_i) \) neighbours in \( U \) and the remaining \( g_i \) vertices are at distance 2, which means that

\[
N_{E_1}(v_i) = \frac{(n - g_i)(n - g_i - 1)}{2} + \frac{g_i(g_i - 1)}{2}.
\]

Similarly, a vertex \( v \in V_2 \) has no neighbours in \( U \) and

\[
d(u_i, v) = \begin{cases} 
2 & \text{if } u_i \in U_1, \\
3 & \text{if } u_i \in U_2.
\end{cases}
\]

Also,

\[
N_{E_1}(v) = \frac{p(p - 1)}{2} + \frac{(n - p)(n - p - 1)}{2}
\]

and

\[
\sum_{e \in E_1} N_{G}(e) = \frac{n(n - 1)(n - 2)}{2} + \sum_{i=1}^{q} \left( \frac{(n - g_i)(n - g_i - 1)}{2} + \frac{g_i(g_i - 1)}{2} \right) + (m - q) \left( \frac{p(p - 1)}{2} + \frac{(n - p)(n - p - 1)}{2} \right).
\]

(1)

Similarly, for edges in \( E_2 \), one has

\[
\sum_{e \in E_2} N_{G}(e) = \frac{m(m - 1)(m - 2)}{2} + \sum_{j=1}^{p} \left( \frac{(m - f_j)(m - f_j - 1)}{2} + \frac{f_j(f_j - 1)}{2} \right)
\]

\[
+ (n - p) \left( \frac{q(q - 1)}{2} + \frac{(m - q)(m - q - 1)}{2} \right) + (m + n)(m - f_1) + (m - f_2) + \ldots + (m - f_p)
\]

\[
- \left( \sum_{j=1}^{p} (m - f_j)(m - f_j - 1) + \sum_{i=1}^{q} (n - g_i)(n - g_i - 1) \right)
\]

\[
= \frac{n(m - 1)(m + n)}{2} - \frac{n(n - 1)(n - 2)}{2} - \sum_{i=1}^{q} \left( \frac{(n - g_i)(n - g_i - 1)}{2} + \frac{g_i(g_i - 1)}{2} \right)
\]

\[
- (m - q) \left( \frac{p(p - 1)}{2} + \frac{(n - p)(n - p - 1)}{2} \right)
\]

\[
+ \frac{m(m - 1)(m + n)}{2} - \frac{m(m - 1)(m - 2)}{2} - \sum_{j=1}^{p} \left( \frac{(m - f_j)(m - f_j - 1)}{2} + \frac{f_j(f_j - 1)}{2} \right)
\]

\[
- (n - p) \left( \frac{q(q - 1)}{2} + \frac{(m - q)(m - q - 1)}{2} \right) + (m + n)(m - f_1) + (m - f_2) + \ldots + (m - f_p)
\]

\[
- \left( \sum_{j=1}^{p} (m - f_j)(m - f_j - 1) + \sum_{i=1}^{q} (n - g_i)(n - g_i - 1) \right)
\]

\[
= n(m - 1)(m + 2) - \sum_{i=1}^{q} (n - g_i)(n - g_i - 1) - \sum_{i=1}^{q} g_i(g_i - 1) - (m - q)(p(p - 1) + (n - p)(n - p - 1))
\]

\[
+ m(m - 1)(m + 2) - \sum_{j=1}^{p} (m - f_j)(m - f_j - 1) - \sum_{j=1}^{p} f_j(f_j - 1) - (n - p)(q(q - 1) + (m - q)(m - q - 1))
\]

\[
+ 2(m + n) \sum_{j=1}^{p} (m - f_j) - 2 \sum_{j=1}^{p} (m - f_j)(m - f_j - 1) - 2 \sum_{i=1}^{q} (n - g_i)(n - g_i - 1)
\]

\[
= 2n^2 - 2n + 4npm - 2p^2 m - 2npq + 2p^2 q + 2m^2 - 2m + 2mnq - 2nq^2 - 2mpq + 2pq^2
\]

\[
- 2n^2 q + 2nq + 2pm - 2n \sum_{j=1}^{p} f_j + 4m \sum_{j=1}^{p} f_j - 4 \sum_{j=1}^{p} f_j^2 - 2 \sum_{j=1}^{p} f_j + 6n \sum_{i=1}^{q} g_i - 4 \sum_{i=1}^{q} g_i^2 - 2 \sum_{i=1}^{q} g_i
\]

\[
= n(n - 1) + m(m - 1) + mn(2p + q) - pq(m + n - q - p) - mp(p - 1)
\]

\[
- nq(q + n - 1) + f(2m - n - 1) - 2(f^* + g^*) + g(3n - 1).
\]
A bipartite graph \( G(U, V) \) is \((x, y)\)-biregular if each vertex in \( U \) has degree \( x \) and each vertex in \( V \) has degree \( y \).

**Corollary 2.1.** If \( G(U, V) \) is a \((x, y)\)-biregular graph then \( PI(G) = (n + m)(n + m - 1) + 2my(n + m - (x + y + 1)) \). 

**Proof.** From Theorem 2.1, we have

\[
PI(G) = n(n - 1) + m(m - 1) + mn(2p + q) - pq(m + n - p - q) - mp(p - 1) - nq(q + n - 1) + f(2m - n - 1) - 2(f + g) + g(3n - 1).
\]

Here \( p = n, q = m, f = nx, g = my, f^* = nx^2 \), and \( g^* = my^2 \). Thus,

\[
PI(G) = n^2 - n + m^2 - m + 2mn + 2mnx - nx^2 - n(x - 2n)x - 2n^2 - 2my^2 + 3nmy - my
= (n + m)^2 - (n + m) + 2nym + 2(m^2y - 2mxy - 2my^2 + my
= (n + m)(n + m - 1) + 2my(n + m - (x + y + 1)).
\]

**Corollary 2.2.** If \( G \) is a \( k\)-regular bipartite graph with \( 2n \) vertices then \( PI(G) = 2n[2n(k + 1) - (2k^2 + k + 1)] \).

**Proof.** In Theorem 2.1, by taking \( n = m \) and \( x = y = k \), one gets

\[
PI(G) = 2n(2n - 1) + 2nk(2n - (2k + 1))
= 2n(2n(k + 1) - (2k^2 + k + 1)).
\]

Two particular examples of Corollary 2.2 are \( PI(G_{2n}) = 2n(6n - 11) \) and \( PI(K_{n,n}) = 2n(n - 1) \).

**Corollary 2.3.** If \( G \) is a \( k\)-regular bipartite graph with \( 2n \) vertices then

\[
PI_w(G) = 4n(2n - k - 1)(2n(k + 1) - (2k^2 + k + 1)).
\]

**Proof.** We know that the weighted PI index of a regular graph is a multiple of its PI index. Therefore,

\[
PI_w(G) = 2(2n - k - 1)PI(G) = 4n(2n - k - 1)(2n(k + 1) - (2k^2 + k + 1)).
\]

3. Line graphs of some classes of graphs

Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Its line graph denoted by \( L(G) \), is a simple graph whose vertices are the edges of \( G \) and two vertices are adjacent in \( L(G) \) if the corresponding edges are adjacent in \( G \). Let \( T \) be a tree with \( n \) vertices. Every vertex \( v \) in \( T \) with degree \( i \), \( i > 2 \), forms a star \( K_{1,i} \), in \( T \), we denote it by \( S_i \). Let \( S \) be the collection of all stars in \( T \). If we delete edges of all stars in \( T \), the remaining edges of \( T \) are parts of paths. Some paths have both of its end vertices common with the stars; we call them as central paths and the remaining have one end vertex shared with stars (paths) and the other end vertex is a pendent vertex; we call them leaf paths. We denote the central path with the length \( l \) by \( P_t \) and pendent path with the length \( l \) by \( P_p^* \). As we know that line graphs of stars are complete graphs and line graphs of paths are paths. Each star \( S_i \) in \( T \) is transformed to a clique with \( K_i \) in \( L(T) \). The central path \( P_t \) has \( l \) edges, so it is transformed to the path with \( l \) vertices having length \( l - 1 \) and each of its end vertices is connected with a vertex of a clique in \( L(T) \), so it has \( l - 1 + 2 = l + 1 \) edges. Each leaf path \( P_p^* \) is transformed to a path with \( l \) vertices and \( l - 1 \) edges, and it is connected with a vertex of \( L(T) \), so it has \( l \) edges.

**Theorem 3.1.** Let \( T \) be a tree with \( n \) vertices then

\[
PI(L(T)) = (n - 1)(n - 2).
\]

**Proof.** Let \( T \) be a tree with \( n \) vertices. Assume that the edge set \( E(T) \) is the union of \( m \) stars \( S_{k_1}, \ldots, S_{k_m} \), \( r \) central paths \( P_{t_1}, \ldots, P_{t_r} \), and \( s \) pendent paths \( P_{p_1}^*, \ldots, P_{p_s}^* \). Let us assume that

\[
S = \cup_{i=1}^{m} S_{k_i} \quad \text{and} \quad P' = (\cup_{i=1}^{r} P_{t_i}) \cup (\cup_{j=1}^{s} P_{p_j}^*)
\]
Then,

$$|E(T)| = n - 1 = \sum_{i=1}^{m} k_i + \sum_{i=1}^{r} f_i + \sum_{i=1}^{s} g_i.$$  

We claim that

$$\sum_{e \in E(K_{k_i})} (|V(T)| - N_T(e)) = (k_i - 1) (n - 1).$$

Let $e$ be an edge of $K_{k_i}$ in $L(T)$ and let $v \in V(K_{k_i})$ be equidistant to $e$. If we delete all the edges of $K_{k_i}$, then $L(T)$ has more than one component. All vertices in the component $W$ containing $v$ are also equidistant to $e$. If we consider all the edges and vertices of $K_{k_i}$, then

$$\sum_{e \in E(K_{k_i})} (|V(T)| - N_T(e)) = (k_i - 1) (n - 1).$$

Also, since each edge of a path is a cut edge, there is no equidistant vertex corresponding to those edges. Each $P_{f_i+1}$ contributes $(f_i + 1)(n - 1)$ and each $P_{g_i}$ contributes $g_i(n - 1)$ to the PI index of $L(T)$. Thus,

$$PI \left( L(T) \right) = \sum_{e \in E(L(T))} (|V(L(T))| - N_{L(T)}(e)) = \sum_{e \in E(\cup S_{k_i})} (|V(L(T))| - N_{L(T)}(e)) + \sum_{e \in E(\cup P_{f_i+1})} (|V(L(T))| - N_{L(T)}(e)) + \sum_{e \in E(\cup P_{g_i})} (|V(L(T))| - N_{L(T)}(e))$$

$$= \sum_{i=1}^{m} (k_i - 1) (n - 1) + \sum_{i=1}^{r} (f_i + 1) (n - 1) + \sum_{i=1}^{s} g_i (n - 1)$$

$$= (n - 1) \left( \sum_{i=1}^{m} k_i + \sum_{i=1}^{r} f_i + \sum_{i=1}^{s} g_i \right)$$

$$= (n - 1) (n - 1 - m + r).$$

Since each $P_f$ lies between two $S_{k_i}$, it holds that $r = m - 1$. Therefore,

$$PI \left( L(T) \right) = (n - 1) (n - 1 - 1) = (n - 1) (n - 2) = PI \left( T \right) - 2 (n - 1).$$

Let $K_n$ be the complete graph with $n$ vertices. The graph $L(K_n)$ is the edge disjoint union of $n$ cliques $A_1, A_2, A_3, ..., A_n$, each of which has order $n - 1$. Also, each vertex of $L(K_n)$ is a part of exactly two cliques and any two cliques in $L(K_n)$ have exactly one vertex in common.

**Theorem 3.2.** $PI \left( L(K_n) \right) = n (n - 1) (n - 2)^2$.

**Proof:** The edge set of $L(K_n)$ can be partitioned as

$$E(L(K_n)) = \bigcup_{i=1}^{n} E(A_i),$$

where $A_i$'s are cliques of order $n - 1$. Let $e = uv$ be an arbitrary edge in $L(K_n)$, then $e \in A_i$ for some $i$. All the remaining vertices in $A_i$ are at distance one, so $V(A_i) \setminus \{u, v\} \subseteq D_1(e)$. Since each vertex belongs to exactly two cliques, $u \in A_j$ and $v \in A_h$, for some $i \notin \{j, h\}$. Also, two cliques have exactly one vertex in common, say $w$, which is different from $u$ and $v$. So, $d(u, w) = d(v, w) = 1$ implies that $w \in D_1(e)$. Moreover, the number of vertices at distance 2 is

$$\frac{n(n - 1)}{2} - (d(u) + d(v) - D_1(e)) = \frac{n(n - 1)}{2} - (4(n - 2) - (n - 2)).$$

Therefore,

$$N_{L(K_n)}(e) = (n - 2) + \frac{n(n - 1)}{2} - 3(n - 2) = \frac{n(n - 1)}{2} - 2(n - 2).$$
and hence
\[
PI(L(K_n)) = \sum_{e \in E(L(K_n))} (|V(L(K_n))| - N_{L(K_n)}(e))
\]
\[
= \sum_{e \in E(L(K_n))} \frac{n(n-1)}{2} - \left( \frac{n(n-1)}{2} - 2(n-2) \right)
\]
\[
= 2 \left( \sum_{v \in V(K_n)} d^2(v) - m \right)(n-2)
\]
\[
= \left( \sum_{v \in V(K_n)} d^2(v) - 2m \right)(n-2)
\]
\[
= (n(n-1)^2 - n(n-1)) (n-2) = n(n-1)(n-2)^2 = PI(K_n)(n-2)^2.
\]

Next, we consider the complete bipartite graph \( K_{n,m} = G(U,V) \) with \(|U| = n \) and \(|V| = m \). Its line graph \( L(G) \) is the edge disjoint union of \( m+n \) cliques, where \( m \) cliques have order \( n \) and \( n \) cliques have order \( m \). Each vertex in \( L(G) \) belongs to exactly two cliques, one of which has order \( n \) and the other is of order \( m \). Two cliques of the same order have no vertex in common.

**Theorem 3.3.** \( PI(L(K_{n,m})) = mn(2mn - (m+n)). \)

**Proof.** Take \( G = K_{n,m} \). Its edge set can be partitioned as \( E(G) = E(\cup K_n) \cup E(\cup K_m) \). Take an arbitrary edge \( e \in E(L(G)) \). Then there are two possibilities.

**Case 1.** \( e = xy \) is an edge of a clique \( K_n \) of order \( n \).
All the vertices of \( K_n \) other than the end vertices of \( e \) are at distance 1. There is no other vertex at distance 1. (If there exists a vertex \( z \) at distance 1, then the edge \( xz \) belongs to a clique of order \( m \) and \( yz \) belongs to another clique of the same order. So, the vertex \( w \) belongs to exactly two cliques of order \( m \), it is not possible). Thus,
\[
mn - (d(x) + d(y) - (n-2)) = mn - (2(n + m - 2) - (n-2))
\]
are the number of vertices at distance 2. So,
\[
N_{L(G)}(e) = mn - (2(n + m - 2) - (n-2)) + (n-2) = mn - 2(n + m - 2) + 2(n-2).
\]

**Case 2.** \( e \) is an edge of a clique \( K_m \) of order \( m \).
In the same way as in Case 1, one gets
\[
N_{L(G)}(e) = mn - 2(n + m - 2) + 2(m-2).
\]

Therefore,
\[
PI(L(G)) = \sum_{e \in E(L(G))} (|V(L(G))| - N_{L(G)}(e))
\]
\[
= mn \left( \frac{mn(n + m - 2)}{2} \right) - \left( \sum_{e \in E(\cup K_n)} N_{L(G)}(e) + \sum_{e \in E(\cup K_m)} N_{L(G)}(e) \right)
\]
\[
= mn \left( \frac{mn(n + m - 2)}{2} \right) - (mn - 2(n + m - 2)) \left( \frac{mn(m + n - 2)}{2} \right) - mn[(n-1)(n-2) + (m-1)(m-2)]
\]
\[
= mn((m + n - 2)^2 - (n-1)(n-2) - (m-1)(m-2))
\]
\[
= mn(2mn - (m+n)).
\]
4. Prismatic graphs

Chudnovsky and Seymour studied different structural properties of claw-free graphs in a series of seven papers. In their first paper [1] of this series, they studied the orientable prismatic graphs and in the second paper [2] they studied non-orientable prismatic graphs. A graph $G$ is prismatic if for every triangle $T$ in $G$, every vertex not in $T$ has exactly one neighbour in $T$. Core of a prismatic graph is the union of all triangles in $G$. Total coloring of prismatic graphs are discussed in [14]. Here, we consider a particular class of prismatic graphs, namely rigid prismatic graphs. A prismatic graph $G$ with core $W$ is rigid if

- there does not exist two distinct vertices $u$ and $v$, not in the core, with the same neighbouring set in $W$,
- every two non-adjacent vertices have a common neighbour in the core.

**Theorem 4.1.** If $G$ is a rigid prismatic graph with $p$ triangles and $n$ vertices, then its PI index is

$$\text{PI}(G) = M_1(G) + 2np - \sum_{(u,v) \in E(W)} (d(u) + d(v)).$$

**Proof.** Let $G$ be a rigid prismatic graph with $p$ triangles, $n$ vertices, and $m$ edges. Since every two non-adjacent vertices of $G$ have a common neighbour in the core, its diameter is 2. The edge set of $G$ can be partitioned as $E(G) = E(W) \cup E_1 \cup E_2 \cup E_3$, where $E_1 = \{(u,v) / \not{\in} E(W) \mid u, v \in W\}$, $E_2 = \{(u,v) \mid \text{either } u \in W \text{ or } v \in W\}$, and $E_3 = \{(u,v) \mid u, v \not{\in} W\}$.

$$\text{PI}(G) = \sum_{e \in E(W)} (|V(G)| - N_G(e)) + \sum_{e \in E_1} (|V(G)| - N_G(e)) + \sum_{e \in E_2} (|V(G)| - N_G(e)) + \sum_{e \in E_3} (|V(G)| - N_G(e)).$$

Since each triangle contributes $2n$ to $\text{PI}(G)$, one has

$$\sum_{e \in E(W)} (|V(G)| - N_G(e)) = 2np.$$

Since each edge in $E_i$, $i = 1, 2, 3$, is not a part of a triangle, it holds that

$$\sum_{e \in E_i} (|V(G)| - N_G(e)) = \sum_{(u,v) \in E_i} (n - (n - (d(u) + d(v)))) = \sum_{(u,v) \in E_i} (d(u) + d(v))$$

and thus,

$$\text{PI}(G) = 2np + \sum_{(u,v) \in E_1 \cup E_2 \cup E_3} (d(u) + d(v)) = 2np + \sum_{(u,v) \in E(G) \setminus E(W)} (d(u) + d(v))$$

$$= 2np + \sum_{(u,v) \in E(G)} (d(u) + d(v)) - \sum_{(u,v) \in E(W)} (d(u) + d(v))$$

$$= M_1(G) + 2np - \sum_{(u,v) \in E(W)} (d(u) + d(v)).$$

\[\square\]

![Figure 2: The rotator.](image)

For illustration of Theorem 4.1, we consider two non-orientable prismatic graphs: rotator and twister. The rotator and twister are shown in 2 and 3, and their PI indices are 120 and 154, respectively.
Figure 3: The twister.

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