GLOBAL EXISTENCE AND BLOW-UP FOR A STOCHASTIC TRANSPORT EQUATION WITH NON-LOCAL VELOCITY

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Abstract. In this paper we investigate a non-linear and non-local one dimensional transport equation under random perturbations on the real line. We first establish a local-in-time theory, i.e., existence, uniqueness and blow-up criterion for pathwise solutions in Sobolev spaces $H^s$ with $s > 3$. Thereafter, we give a complete picture of the long time behavior of the solutions based on the type of noise we consider. On one hand, we identify a family of noises such that blow-up can be prevented with probability 1, guaranteeing the existence and uniqueness of global solutions almost surely. On the other hand, in the particular linear noise case, we show that singularities occur in finite time with positive probability, and we derive lower bounds of these probabilities. To conclude, we introduce the notion of stability of exiting times and show that one cannot improve the stability of the exiting time and simultaneously improve the continuity of the dependence on initial data.

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1. Introduction and main results

Consider the following one dimensional non-local transport equation
\[ u_t + \langle Hu \rangle u_x = 0, \quad (1.1) \]
where $H$ denotes the Hilbert transform. This equation first appears in the literature due to its analogy with the Birkhoff-Rott equations describing the evolution of vortex sheets with surface tension [4, 45]. Moreover, (1.1) can be also viewed as a toy-model of the two-dimensional surface quasi-geostrophic equation (SQG) which describes the evolution of the potential temperature in a rapidly rotating stratified fluid with uniform potential vorticity [25, 47]. A striking result showing the finite time blow-up of classical solutions to (1.1) for a generic class of smooth initial data was first obtained by Córdoba, Córdoba and Fontelos in [11] by means of complex analysis techniques. After that breakthrough, equation (1.1) is known as the CCF equation. Subsequent works have shown finite blow-up avoiding complex analysis approach (cf. [38, 53]). In particular, in the later Silvestre and Vicol provided four elegant and simple real analysis proofs of the blow-up phenomena.

In this paper, we are interested in stochastic variants of the CCF equation (1.1). Indeed, the introduction of stochasticity into ideal fluid dynamics has received special attention over the past two decades. The inclusion of stochastic noise can be a way of representing model uncertainty and turbulence. For example, in weather forecasting, phenomena as cloud formation is to this day poorly understood and the inclusion of stochastic noise has become an essential tool for gaining better understanding about it. Since the pioneering work of Holm in [31], where a variational approach for introducing noise in equations in a fashion that respects the geometry of the system is developed, the literature regarding the analysis of non-linear stochastic partial differential equations with transport type noise has increase substantially (cf. [1, 2, 9, 10, 20] and the references therein). To the best of the authors knowledge, there are very few results regarding the CCF model under random perturbations. Only recently, by applying an abstract framework for singular stochastic partial differential equations (SPDEs) derived by two of the authors, cf. [3], the local existence, uniqueness and blow-up criterion of pathwise solutions to (1.1) with transport noise has been addressed in the periodic setting, i.e for $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

To extend the theory developed in [3] to the real line case, i.e., $x \in \mathbb{R}$, and to study the noise effect, in this paper we will consider the following stochastic CCF model
\[
\begin{align*}
    du + (Hu) u_x \, dt &= h(t,u) \, d\mathcal{W}, & x \in \mathbb{R}, & t > 0, \\
    u(\omega,0,x) &= u_0(\omega,x), & x \in \mathbb{R},
\end{align*}
\]  
(1.2)

where $\mathcal{W}$ is a cylindrical Wiener process and $h$ is a non-linear function. In this work, under some natural assumptions collected in Assumption (A), we obtain the local existence, uniqueness and a blow-up criterion of pathwise solutions to (1.2). The detailed result is stated in Theorem 1.1. Here we notice that classical probabilistic compactness arguments cannot be directly applied to deal with the whole space $\mathbb{R}$ and new ideas are in order, see Remark 1.2 for a more precise explanation.

It is well-known that noise effect is one of the crucial subjects in the study of SPDEs. Indeed, regularization effects due to noise have been observed for various models and different settings. For example, it is known that the well-posedness of linear stochastic transport equation with noise can be established under weaker hypotheses than its deterministic counterpart and restore uniqueness of solutions, cf. [15, 18]. Regularization effects caused by the noise on flux for stochastic scalar conservation laws have been studied in [22]. Extensions of the previous works for the stochastic transport and continuity equations to $k$-forms has been recently addressed in [5]. In terms of numerical simulations, the regularization effects of noise can be found in [40]. Moreover, for different fluid models with linear multiplicative noise, we refer to [24, 37, 54] where noise provides a damping effect on the pathwise behavior of solutions.

Compared to the deterministic counterpart of (1.2), i.e., (1.1), in this article we focus on the following issues for the problem (1.2) regarding the noise effect:

- Noise versus finite time blow-up;
- Noise versus dependence on initial data.

1.1. **Noise versus finite time blow-up.** In this direction, we attempt to answer the following two important questions:

(Q-1) What kind of noise can prevent blow-up?
(Q-2) If blow-up may occur, what is the corresponding probability?

We remark here that most previous results in the literature on regularization by noise are restricted to linear equations or linear noises. For instance, we refer to [15, 18, 16, 37, 46] for linear transport equations, and to [19, 24, 50, 54] for linear noise. Therefore, for non-linear SPDEs, it is very natural to analyze the validity of the regularization effects by non-linear noise. Indeed, searching for nonlinear noise such that blow-up can be prevented is important because it helps us to understand the regularization mechanisms of noise, and this is the main motivation to study question (Q-1). Actually, even in the case of non-linear equations with linear
noise, the noise effects are complicated because there are both, examples in positive direction, i.e., noises can regularize singularities, and negative direction, i.e., noises cannot regularize singularities. For example, for the stochastic 2D Euler equations, coalescence of vortices disappears (see [19]) but noise cannot prevent the formation of shocks in the Burgers’ equation (see [1, 17]).

For simplicity, we set (Q-1) in the framework where \( h(t, u) \, dW = \alpha(t, u) \, dW \), with \( W \) a standard 1-D Brownian motion and \( \alpha \) a non-linear function. We then focus on the system

\[
\begin{align*}
\frac{du}{dt} + (\mathcal{H}u)_{x}dt &= \alpha(t, u) \, dW, \quad x \in \mathbb{R}, \ t > 0, \\
\left(u(\omega, 0, x) = u_{0}(\omega, x)\right)_{x} &= \mathcal{H}u_{x}, \quad x \in \mathbb{R}.
\end{align*}
\] (1.3)

We will show in Theorem 1.2 that if \( \alpha(t, \cdot) \) grows fast enough, then global existence of pathwise solutions holds true with probability 1. This is strongly in contrast with its deterministic counterpart where the breakdown of classical solutions to (1.1) with generic smooth initial data occurs, cf. [11]. Hence we justify the idea that fast growing non-linear noise (strong noise) has regularization effects on the solutions in terms of preventing singularities.

By Theorem 1.2, we have identified a family of noises that can prevent blow-up, and this partially answers (Q-1). Next, we will pay our attention to the case that blow-up may occur. Indeed, as a toy-model for the 2D surface quasi-geostrophic equation (SQG)(and hence for the 3D incompressible Euler equation), analyzing the possible blow-up of solutions is is one of the central questions in the study of non-local transport type equations, cf. [11, 14, 38, 43, 53]. In this paper we are also interested in identifying the possible formation of singularities in finite time and estimating its probability, hence given a partial answer to (Q-2). Since Theorem 1.2 shows that fast growing noises can prevent blow-up, it is natural to ponder that singularities can only occur when the noise is somehow weak. Indeed, in contrast to the fast growing noise, we will show that in the case of linear noise (weak noise) given by equation

\[
\begin{align*}
\frac{du}{dt} + (\mathcal{H}u)_{x}dt &= b(t)u \, dW, \quad x \in \mathbb{R}, \ t > 0, \\
\left(u(\omega, 0, x) = u_{0}(\omega, x)\right)_{x} &= \mathcal{H}u_{x}, \quad x \in \mathbb{R}.
\end{align*}
\] (1.4)

where \( b \) is some continuous function and \( W \) is a standard 1-D Brownian motion, finite time blow-up cannot be prevented and finite time singularities occur. The precise statement of this result is given in Theorem 1.3.

1.2. Noise versus dependence on initial data. Now we turn to the problem of noise effect on the initial-data dependence. There are very few results concerning the noise effect in the direction of dependence on initial data. In this work we will partially answer the following question:

(Q-3) Whether and how noise can affect the initial-data dependence?

The main motivation to consider question (Q-3) relies on the following observation. On the one hand, regularization provided by noise may look related to regularization effects induced by an additional dissipative term (a Laplacian). On the other hand, if one would add a real Laplacian to the governing equations, parabolic techniques may be used to improve the continuity of the initial-data dependence. For example, in the deterministic incompressible Euler equations, the solution map \( u_{0} \mapsto u \) cannot be better than continuous [29] but for the deterministic incompressible Navier-Stokes equations with sufficiently large viscosity, it is at least Lipschitz continuous in sufficiently high Sobolev spaces (see pp. 79–81 in [26]). In the deterministic setting, similar questions regarding the continuity map on the initial-data have been widely investigated for various non-linear dispersive and integrable equations of which we only mention a few related results. Koch and Tzvetkov [39] proved that the solution map of the Benjamin–Ono equation cannot be uniform continuous. For Camassa–Holm type equations, we refer to [27, 28] for the non-uniform dependence on initial data in Sobolev spaces \( H^{s} \) with \( s > 3/2 \). Similar results in Besov spaces \( B_{p,q}^{s} \) first appear in [57, 56], where the critical index \( s \) can be also covered. In the case of the SQG system, we refer the reader to [32]. In the stochastic setting, the interplay between regularization provided by noise and the dependence on initial conditions is first studied in [52, 55].

In this article we consider question (Q-3) for (1.2). More precisely, we first recall the concept of stability of the exiting time as in [52, 55]. Roughly speaking, this notion refers to the continuous changes of the point in time with respect to the initial condition, where such point is defined as the time when the solution leaves a certain range, see Definition 1.2 below. Later on, in Theorem 1.4, we show that when \( h(t, u) \) satisfies certain conditions (see Assumption (D)), the multiplicative noise cannot improve the stability of the exiting time, and, at the same time, improve the continuity of the map \( u_{0} \mapsto u \) defined by (1.2).
1.3. Notations, definitions and hypotheses. We now introduce some notations. \( L^2(\mathbb{R}) \) is the usual space of square-integrable functions on \( \mathbb{R} \). For \( s \in \mathbb{R} \), \( D^s = (1 - \partial_x^2)^{s/2} \) is defined by \( \hat{D}^s f(\xi) = \) \( (1 + \xi^2)^{s/2} \hat{f}(\xi) \), where \( \hat{g} \) is the Fourier transform of \( g \). The Sobolev space \( H^s(\mathbb{R}) \) is defined as

\[
H^s(\mathbb{R}) \triangleq \left\{ f \in L^2(\mathbb{R}) : \|f\|_{H^s(\mathbb{R})}^2 = \int_\mathbb{R} (1 + \xi^2)^s |\hat{f}(\xi)|^2 < +\infty \right\},
\]

in which the inner product \( (f,g)_{H^s} \) is given by

\[
(f,g)_{H^s} \triangleq \int_\mathbb{R} (1 + \xi^2)^s |\hat{f}(\xi) \cdot \overline{\hat{g}(\xi)} = (D^s f, D^s g)_{L^2}.
\]

When the function space refers to \( R \), we will drop \( R \) if there is no ambiguity. \( x \preceq y \) \((x \succeq y)\) means that \( x \leq cy \) \((x \geq cy)\) holds for some universal deterministic constant \( c \). Such constant may differ from line to line. For linear operators \( A \) and \( B \), the commutator \([A, B]\) is defined by \([A, B] = AB - BA\).

The triplet \((\Omega, \mathcal{F}, \mathbb{P})\) denotes a complete probability space, where \( \mathbb{P} \) is a probability measure on \( \Omega \) and \( \mathcal{F} \) is a \( \sigma\)-algebra. \( \mathbb{E} X \) is the mathematical expectation of \( X \) with respect to \( \mathbb{P} \). Let \( \mathcal{W}(t) = \mathcal{W}(\omega, t) \), \( \omega \in \Omega \) be a cylindrical Wiener process. More precisely, we consider a separable Hilbert space \( \mathcal{U} \) and let \( \{e_k\} \) be a complete orthonormal basis of \( \mathcal{U} \). Then we define

\[
\mathcal{W} \triangleq \sum_{k=1}^\infty W_k e_k,
\]

where \( \{W_k\}_{k>1} \) is a sequence of mutually independent standard one-dimensional Brownian motions. We call \( \mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W}) \) a stochastic basis, where \( \{\mathcal{F}_t\}_{t \geq 0} \) is a right-continuous filtration endowed on \((\Omega, \mathcal{F}, \mathbb{P})\) such that \( \{\mathcal{F}_0\} \) contains all the \( \mathbb{P}\)-negligible subsets.

\( \mathcal{L}_2(\mathcal{U}; \mathcal{X}) \) stands for the set of Hilbert-Schmidt operators from \( \mathcal{U} \) to another Hilbert space \( \mathcal{X} \). For a predictable process \( Z \in \mathcal{L}_2(\mathcal{U}; \mathcal{X}) \),

\[
\int_0^t Z d\mathcal{W} \triangleq \sum_{k=1}^\infty \int_0^t Z e_k dW_k
\]

is a well-defined \( \mathcal{X}\)-valued continuous square integrable martingale, see [12, 48] for more details. In the sequel of the paper, when a stopping time is defined, we set \( \inf \emptyset \triangleq \infty \) by convention.

We now give the precise notion of a pathwise solution to (1.2).

\textbf{Definition 1.1 (Pathwise solutions).} Let \( \mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W}) \) be a fixed stochastic basis. Let \( s > 3/2 \) and \( u_0 \) be an \( H^s\)-valued \( \mathcal{F}_0\)-measurable random variable.

(1) A local pathwise solution to (1.2) is a pair \((u, \tau)\), where \( \tau \) is a stopping time satisfying \( \mathbb{P}\{\tau > 0\} = 1 \) and \( u : \Omega \times [0,\infty] \rightarrow H^s \) is an \( \mathcal{F}_t\)-predictable \( H^s\)-valued process satisfying

\[
u(t \wedge \tau) \in C([0,\infty); H^s) \quad \mathbb{P} \text{-a.s.},
\]

and for all \( t > 0 \),

\[
u(t \wedge \tau) - \nu(0) + \int_0^{t \wedge \tau} (H \nu)_{u_x} u_t dt = \int_0^{t \wedge \tau} h(t', u) d\mathcal{W} \quad \mathbb{P} \text{-a.s.}
\]
(2) The local pathwise solutions are said to be pathwise unique, if given any two pairs of local pathwise solutions \((u_1, \tau_1)\) and \((u_2, \tau_2)\) with \( \mathbb{P}\{u_1(0) = u_2(0)\} = 1 \), we have

\[
\mathbb{P}\{u_1(t, x) = u_2(t, x), \forall (t, x) \in [0, \tau_1 \wedge \tau_2] \times \mathbb{R}\} = 1.
\]

(3) Additionally, \((u, \tau^*)\) is called a maximal pathwise solution to (1.2) if \( \tau^* > 0 \) almost surely and if there is an increasing sequence \( \tau_n \rightarrow \tau^* \) such that for any \( n \in \mathbb{N} \), \((u, \tau_n)\) is a pathwise solution to (1.2) and on the set \( \{\tau^* < \infty\} \),

\[
\sup_{t \in [0, \tau_n]} \|u\|_{H^s} \geq n.
\]

(4) If \((u, \tau^*)\) is a maximal pathwise solution and \( \tau^* = \infty \) almost surely, then we say that the pathwise solution exists globally.

Inspired by [52, 55], we introduce the concept on stability of exiting time in Sobolev spaces. Exiting time, as the name suggests, describes the first time that the solution leaves a given range. More precisely,
Definition 1.2 (Stability of exiting time). Let \( s > 3/2 \) and \( S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \mathbb{W}) \) be a fixed stochastic basis. Let \( u_0 \in L^2(\Omega; H^s) \) and \( \{u_{n0}\} \subset L^2(\Omega; H^s) \) be \( \mathcal{F}_0 \)-measurable. For each \( n \), let \( u_n \) and \( u_{n0} \) be the unique solutions to (1.2), as in Definition 1.1, with initial values \( u_0 \) and \( u_{0n} \), respectively. For any \( R > 0 \), define the \( R \)-exting times as
\[
\tau_n^R \triangleq \inf \{ t \geq 0 : \|u_n\|_{H^s} > R \}, \quad \tau^R \triangleq \inf \{ t \geq 0 : \|u\|_{H^s} > R \}. \tag{1.5}
\]
Then we define the following properties on stability:
\begin{enumerate}
\item If \( u_{0n} \to u_0 \) in \( H^s \) \( \mathbb{P} \)-a.s. implies that
\[
\lim_{n \to \infty} \tau_n^R = \tau^R \quad \text{\( \mathbb{P} \)-a.s.}, \tag{1.6}
\]
then the \( R \)-exting time of \( u \) is said to be stable.
\item If \( u_{0n} \to u_0 \) in \( H^{s'} \) for all \( s' < s \) almost surely, implies that (1.6) holds true, the \( R \)-exting time of \( u \) is said to be strongly stable.
\end{enumerate}

To study the existence of pathwise solutions to (1.2), we need the following assumptions on the \( h \):

**Assumption (A).** We assume that when \( s > 3/2 \), \( h : [0, \infty) \times H^s \ni (t, u) \mapsto h(t, u) \in L_2(\mathcal{U}; H^s) \) is continuous. Furthermore, we assume that there are two non-decreasing locally bounded functions \( f, q : [0, +\infty) \to [0, +\infty) \) such that
\begin{itemize}
\item For any \( t > 0 \) and \( u \in H^s \),
\[
\|h(t, u)\|_{L_2(\mathcal{U}; H^s)} \leq f(\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty}) (1 + \|u\|_{H^s}). \tag{1.7}
\]
\item For any \( t > 0 \),
\[
\sup_{\|u\|_{H^s}, \|v\|_{H^s} \leq N} \left\{ \frac{1_{\{u \neq v\}}\|h(t, u) - h(t, v)\|_{L_2(\mathcal{U}; H^s)}}{|u - v|_{H^s}} \right\} \leq q(N), \quad N \geq 1. \tag{1.8}
\]
\end{itemize}

**Example.** Now we give an example of noise coefficient satisfying Assumption (A). For simplicity, we only consider the 1-D case, i.e., \( \mathcal{W} = \mathbb{W} \), and \( h : (t, u) \mapsto h(t, u) \) is a map from \([0, \infty) \times H^s \) to \( H^s \). It is easy to verify that if that \( h \) can be constructed, the extension of this example to the general \( L_2(\mathcal{U}; H^s) \)-valued \( h \) can be carried out by considering \( h_0 \), where \( \{e_k\}_{k \geq 1} \) is a complete orthonormal basis of \( U \) (as 1-D case with suitable coefficient) such that \( \sum_{k \geq 1} \|h(e_k)\|_{H^s}^2 < \infty \). To that purpose, let \( G = \frac{1}{2} e^{-\|x\|} \). Then in 1-D case, \((1 - \partial_{xx}^2)^{-1}f\) can be understood as
\[
(1 - \partial_{xx}^2)^{-1}f(x) = (G * f(x)) \quad \text{for} \quad f \in L^2(\mathbb{R}), \tag{1.9}
\]
where \(*\) stands for the convolution. Now we let
\[
h(t, u) = q(t) (1 - \partial_{xx}^2)^{-1} \partial_x ([u_x]^k + (\mathcal{H}u_x)^n), \quad k, n \geq 1.
\]
If \( q(\cdot) \) is smooth with both upper and lower bounds, then it is easy to see that \( h \) satisfies Assumption (A).

To find global existence, we need some stronger condition on the noise coefficient \( h \) and we make the following assumption:

**Assumption (B).** We assume that when \( s > \frac{3}{2} \), \( \alpha : [0, \infty) \times H^s \ni (t, u) \mapsto \alpha(t, u) \in H^s \) is continuous. Moreover, we assume the following properties hold true:
\begin{itemize}
\item For any \( t > 0 \) and \( u \in H^s \) there is a non-decreasing locally bounded function \( l(\cdot) : [0, \infty) \to [0, \infty) \) such that for any \( t \geq 0 \),
\[
\sup_{\|u\|_{H^s}, \|v\|_{H^s} \leq N} \left\{ 1_{\{u \neq v\}} \frac{\|\alpha(t, u) - \alpha(t, v)\|_{H^s}}{|u - v|_{H^s}} \right\} \leq l(N), \quad N \geq 1, \quad s > 3/2. \tag{1.10}
\]
\item Define
\[
\mathfrak{G} = \left\{ G \in C^2([0, \infty); [0, \infty)) : G(0) = 0, \quad G'(x) > 0, \quad G''(x) \leq 0 \quad \text{and} \quad \lim_{x \to +\infty} G(x) = \infty \right\},
\]
and we assume that there is a function \( G \in \mathfrak{G} \) and constants \( K_1, K_2 > 0 \) such that for all \((t, u) \in [0, \infty) \times H^s \) with \( s > 5/2 \),
\[
G'(\|u\|_{H^{s-1}}^2) \mathcal{M}(t) + 2G''(\|u\|_{H^{s-1}}^2) \|\alpha(t, u)\|_{H^{s-1}}^2 \leq K_1 - K_2 \frac{G''(\|u\|_{H^{s-1}}^2) \|\alpha(t, u)\|_{H^{s-1}}^2}{1 + G(\|u\|_{H^{s-1}}^2)}, \tag{1.11}
\]
where
\[
\mathcal{M}(t) = 2Q(\|u_x\|_{L^\infty} + \|\mathcal{H}u_x\|_{L^\infty}) \|u\|_{H^{s-1}}^2 + \|\alpha(t, u)\|_{H^{s-1}}^2
\]
and \( Q \) is the constant given in Lemma 2.3.
**Example.** Let \( q(t) \) be a continuous function such that \( q_* < q^2(t) < q^* \) for all \( t \) and let \( Q \) be the constant given in Lemma 2.3 below. Then it is easy to check that

\[
\alpha(t, u) \doteq q(t) \left( 1 + \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty} \right)^	heta u
\]  
(1.12)

with

\[
either \theta > \frac{1}{2}, q^*> q_*> 0 or \theta = \frac{1}{2}, q^*> q_*> 2Q
\]  
(1.13)
satisfies Assumption (B) with \( G(x) = \log(1 + x) \in \mathcal{G} \). For simplicity we only prove that (1.11). Indeed, we observe that

\[
G'(\|u\|^2_{H^{-1}}, \mathcal{M}_t) + 2G''(\|u\|^2_{H^{-1}})(\alpha(t, u), u)_{H^{-1}} + K_2 \left\{ \frac{G'(\|u\|^2_{H^{-1}})(\alpha(t, u), u)_{H^{-1}}}{1 + G'(\|u\|^2_{H^{-1}})} \right\}^2
\]

\[
= \frac{2Q(\|u\|_{L^\infty} + \|Hu_x\|_{L^\infty})\|u\|^2_{H^{-1}} + q^2(t) (1 + \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty})^2 \|u\|^2_{H^{-1}}}{1 + \|u\|^2_{H^{-1}}}
\]

\[
- \frac{2q^2(t) (1 + \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty})^2 \|u\|^2_{H^{-1}}}{1 + \|u\|^2_{H^{-1}}}
\]

\[
\leq \frac{2Q(\|u\|_{L^\infty} + \|Hu_x\|_{L^\infty}) + q^2(t) (1 + \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty})^2}{1 + \|u\|^2_{H^{-1}}}
\]

\[
- \frac{2q^2(t) (1 + \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty})^2 \|u\|^2_{H^{-1}}}{1 + \|u\|^2_{H^{-1}}}
\]

\[
= \frac{q^2(t) (1 + \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty})^2 \|u\|^2_{H^{-1}}}{1 + \|u\|^2_{H^{-1}}}
\]

\[
: \mathfrak{A}
\]

If \( \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty} \) is bounded, then \( \mathfrak{A} \) is also bounded, and hence can be controlled by some constant \( K_1 > 0 \). To prove (1.11), we only need to check that \( \mathfrak{A} \) can also be controlled by \( K_1 \) when \( \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty} \to \infty \). Let \( \mathcal{P} = q^2(t) (1 + \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty})^2 \). Due to the embedding \( H^{s-1} \hookrightarrow W^{1, \infty} \), when \( \|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty} \to \infty \), \( \|u\|_{H^{-1}} \to \infty \). Hence, we have that

\[
\limsup_{\|u\|_{L^\infty} + \|Hu_x\|_{L^\infty} \to +\infty} \mathfrak{A} \leq \limsup_{\|u\|_{L^\infty} + \|Hu_x\|_{L^\infty} \to +\infty} \left\{ \frac{2Q(\|u\|_{L^\infty} + \|Hu_x\|_{L^\infty})}{\mathcal{P}} + 1 - \frac{2\|u\|^2_{H^{-1}}}{(1 + \|u\|^2_{H^{-1}})} \right\} \mathcal{P} + K_2 \frac{\|u\|^2_{H^{-1}}}{(1 + \|u\|^2_{H^{-1}})} \left( 1 + \log(1 + \|u\|^2_{H^{-1}}) \right)
\]

Moreover, if (1.13) is satisfied,

\[
\limsup_{\|u\|_{L^\infty} + \|Hu_x\|_{L^\infty} \to +\infty} \frac{2Q(\|u\|_{L^\infty} + \|Hu_x\|_{L^\infty})}{\mathcal{P}} < 1
\]

and consequently \( \limsup_{\|u\|_{L^\infty} + \|Hu_x\|_{L^\infty} \to +\infty} \mathfrak{A} \leq 0 \). Therefore \( \alpha \) given by (1.12) and \( G(\cdot) = \log(1 + \cdot) \) satisfies (1.11).

In the linear noise case (1.4), we make the following assumption on \( b \):

**Assumption (C).** We assume \( b(t) \) in (1.4) satisfies that \( b(t) \in C ([0, \infty); [0, \infty)) \) and there exists some \( b^* > 0 \) such that \( b^2(t) < b^* \) for all \( t \geq 0 \).

Finally, we assume the following conditions to study the question (Q-3):

**Assumption (D).** When considering (1.2) in Section 6, we assume that for \( s > 3/2 \), \( h : [0, \infty) \times H^s \ni (t, u) \mapsto h(t, u) \in L^2(\mathbb{T}; H^s) \) is continuous. Moreover, we assume the following:

- There exists a non-decreasing and locally bounded functions \( l(\cdot) : [0, +\infty) \to [0, +\infty) \) such that for any \( t \geq 0 \) and \( u \in H^s \) with \( s > 3 \), we have that

\[
\|h(t, u)\|_{L^2(\mathbb{T}; H^s)} \leq l(\|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty}) \|u\|_{H^s}.
\]

- There exists a non-decreasing and locally bounded functions \( g(\cdot) : [0, +\infty) \to [0, +\infty) \) and a real number \( \sigma_0 \in (3/2, 7/4) \) such that for all \( N \geq 1 \),

\[
\sup_{t \geq 0, \|u\|_{H^s} \leq N} \|h(t, u)\|_{L^2(\mathbb{T}; H^{\sigma_0})} \leq g(N) e^{-\frac{1}{N^{\sigma_0}}}.
\]

- Property (1.8) holds true.
Example. Let us here give an example of noise structure satisfying Assumption (D). As above, we only consider the case where \( b(t,u) \) is \( b(t,u) \) and \( q(\cdot) \) be a continuous and bounded function, then for any \( k, n \geq 1 \),
\[
b(t,u) = q(t)e^{-\frac{\tau - \tau_0}{\sigma^2}}(1 - \partial_x^2)^{-1}\partial_t[(u_x)_k + (Hu_x)_n]
\]
satisfies Assumption (D), where \((1 - \partial_x^2)^{-1} \) is defined in \((1.9)\).

1.4. Main results and remarks.
In this subsection, we present the precise statements of the different results shown in this article. The first result reads

**Theorem 1.1.** Let \( s > 3 \) and let \( b(t,u) \) satisfy Assumption (A). If \( u_0 \) is an \( H^s \) valued \( F_0 \)-measurable random variable satisfying \( \mathbb{E}[\|u_0\|_{H^s}] < \infty \), then there is a unique local pathwise solution \( (u,\tau) \) to \((1.2)\) in the sense of Definition 1.1 with
\[
u(\cdot \wedge \tau) \in L^2(\Omega; C([0,\infty);H^s)).
\]
Moreover, \((u,\tau)\) can be extended to a unique maximal pathwise solution \((u,\tau^*)\) with the following blow-up criterion:
\[
\mathbb{1}_{\{\limsup_{t\to\tau^*} \|u(t)\|_{H^s} = \infty\}} = \mathbb{1}_{\{\limsup_{t\to\tau^*} \|u_x(t)\|_{L^\infty} + \|Hu_x(t)\|_{L^\infty} = \infty\}} \mathbb{P} \text{- a.s.}
\]

**Remark 1.1.** Before we explain the ideas and difficulties regarding the existence of local pathwise solutions, let us first stress some important remarks on the blow-up criterion \((1.15)\).

- The blow-up criterion \((1.15)\) implies that the \( H^{s'} \) norm of the solution within the range \( s' \in (\frac{3}{2},s] \) blows up at the same time \( \tau^* \). For any fixed \( s \) and \( s' \in (\frac{3}{2},s) \), since
\[
\|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty} \lesssim \|u\|_{H^{s'}}
\]
we can conclude that \( \|u\|_{H^{s'}} \) blows up no later than the time \( \|u_x(t)\|_{L^\infty} + \|Hu_x(t)\|_{L^\infty} \) blows up but no earlier than the time \( \|u\|_{H^s} \) blows up. Therefore, equality \((1.15)\) shows that all of the \( H^{s'} \) norms have the same blow-up time. This fact will be used to prove Theorem 1.2 (see \((4.1)\)).

- Invoking the well-known logarithmic Sobolev inequality involving the Hilbert transform \( H \)
\[
\|Hu_x\|_{L^\infty} \lesssim (1 + \|u_x\|_{L^\infty} \log (e + \|u_x\|_{H^1}) + \|u_x\|_{L^2}),
\]
one can show in deterministic case, by using \((1.16)\), that the blow-up criterion \((1.15)\) can be improved into (cf. \([14]\))
\[
\limsup_{t\to\tau^*} \|u(t)\|_{H^s} = \infty \iff \limsup_{t\to\tau^*} \|u_x(t)\|_{L^\infty} = \infty.
\]

However, it is still not clear how to achieve this in the stochastic setting. Technically, because
\[\mathbb{E}((1 + \|u_x\|_{L^\infty} \log (e + \|u_x\|_{H^1}) + \|u_x\|_{L^2})\|u\|_{H^s}) \]
is involved, we have not been able to close the estimate for \( \mathbb{E}\|u\|_{H^s}^2 \).

**Remark 1.2.** Now we give a remark regarding the existence part in Theorem 1.1. Since we focus on a Cauchy problem defined on the whole space \( \mathbb{R} \), the classical probabilistic compactness argument for nonlinear SPDEs in bounded domain seems inapplicable in this work. Indeed, we have the following essential difficulties:

- For smooth \( u \), in the a priori \( L^2(\Omega;H^s) \) estimate for \( u \), after using the Itô formula for \( \|u\|_{H^s}^2 \), we will have to deal with \( \mathbb{E}(\int_0^T([H^x]_{L^\infty} + [u]_{H^s}) d\tau) \) and \( \mathbb{E}(\|u_x\|_{L^\infty} + \|Hu_x\|_{L^\infty})\|u\|_{H^s}^2 \), coming from \( b(t,u) \) and \((Hu)u_x \), respectively. To close the estimate, these two terms should be controlled in terms of \( \|u\|_{H^s}^2 \). Since we cannot split the mathematical expectation, we add a cut-off function to the original problem to cut the non-linear parts in terms of \( \|u\|_{H^s} \) (see \((3.1)\) below) with suitable \( r \) such that \( H^s \hookrightarrow H^r \hookrightarrow W^{1,\infty} \). This enables us to close the a priori estimate. In this work, \( s > 3 \) and we set \( r = s - 3/2 \).

- As a second step, we construct the approximation scheme and obtain certain uniform estimates. Our next aim, is to pass to the limit to obtain the existence of a solution. If the target problem is defined on bounded domain, usually one can find \( q \in (r,s) \) such that \( H^s \hookrightarrow H^q \hookrightarrow H^r \) (here \( \hookrightarrow \) means the embedding is compact). By the uniform estimates, one can follow the compactness argument (cf. Prokhorov’s Theorem and Skorokhod’s Theorem) to obtain the convergence in \( H^q \), which enables us to pass to the limit to find a martingale solution to the cut-off version of the target SPDE. We can a posteriori introduce a stopping time to remove the cut-off. We refer to \([6,8,13,30,24,54,55]\) for different examples. On the contrary, in unbounded domains, the compact embedding only holds true for the “local” space \( H^s_{loc} \hookrightarrow H^q_{loc} \). Because the cut-off \( \|u\|_{H^s} \) appears in the problem itself, even though we obtain the convergence in \( H^q_{loc} \) (cf. Prokhorov’s Theorem and Skorokhod’s Theorem), it is still not enough to guarantee the convergence of \( \|u\|_{H^s} \) because \( \|u\|_{H^s} \) is a global
object (for space variable), which cannot be controlled by a local condition. As mentioned above, because the cut-off is needed, we have to establish a convergence result in some topology no weaker than \(H^s\).

- In this paper, motivated by [42], by a careful analysis on the differences between any two approximate solutions, we will show that there exists a sub-sequence of the approximate solutions converging in \(C([0,T];H^s)\) almost surely. We also remark that our analysis is also available for the torus case, i.e. \(x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}\), since all the estimates can be perform in the same way (up to some obvious modifications) for \(x \in \mathbb{T}\). In the deterministic case where the cut-off is no longer needed, convergence in local Sobolev spaces is sufficient to take limit to find a solution.

Hereafter, we consider the noise effect versus finite time blow-up. Our second result gives a partial answer to (Q-1):

**Theorem 1.2** (Noise preventing blow-up). Let \(s > 3\) and \(u_0 \in H^s\) be an \(H^s\)-valued \(\mathcal{F}_0\)-measurable random variable with \(\mathbb{E}\|u_0\|_{H^s}^2 < \infty\). If Assumption (B) holds true, then the corresponding maximal pathwise solution \((u, \tau^*)\) to (1.3) satisfies

\[
\mathbb{P}\{\tau^* = \infty\} = 1.
\]

**Remark 1.3.** Let us first recall that in the deterministic counterpart of (1.3), the blow-up of regular enough solutions actually cannot be prevented, cf. [11, 53, 43]. Therefore, Theorem 1.2 justifies the idea that fast growing non-linear noise (strong noise) can regularize the solutions in terms of preventing singularities. Here we use the terminology “fast growing” condition, described by (1.11) in Assumption (B), to cancel (notice that \(\mathcal{G}'' < 0\) in (1.11)) the growth of the non-local transport term \((\mathcal{H} u) u\) such that \(E\mathcal{G}(\|u\|_{H^s})\) can be controlled with a Lyapunov type function \(\mathcal{G}\). The idea of using a Lyapunov type function is motivated by the works [7, 36, 49, 51].

Complementary, in the case of linear noise, we can show that singularities occur in finite time with positive probability which yields a partial answers to (Q-2). The precise statement reads

**Theorem 1.3** (Blow-up linear noise). Let \(0 < K < 1\), \(s > 3\) and \(b(t)\) satisfy Assumption (C). Assume that \(u_0 = u_0(\omega, x)\) is an \(H^s\) valued \(\mathcal{F}_0\) measurable random variable satisfying \(\mathbb{E}\|u_0\|_{H^s}^2 < \infty\) and attaining a global maximum at \(x_0\). If

\[
\Lambda u_0(\omega, x_0) > \frac{b^*}{K} \mathbb{P} - a.s.,
\]

where \(b^*\) is given in Assumption (C), then the unique maximal pathwise solution \((u, \tau^*)\) to (1.4) satisfies

\[
\mathbb{P}\{\tau^* < \infty\} \geq \mathbb{P}\{\lim_{t \to \tau^*} \max_{x \in \mathbb{R}} \Lambda u(t) = +\infty\} \geq \mathbb{P}\{e^{\int_0^t lb'(r)dr}dw(t) - K > 0\}.
\]

**Remark 1.4.** We make the following remarks regarding Theorem 1.3:

- Motivated by previous works [24, 50, 51], to study (1.4), we make use of Girsanov type transform to obtain a PDE with random coefficient instead of a SPDE to study blow-up. In this linear noise case, the blow-up criterion (1.15) also holds true. By direct computation, we have \(\mathcal{H} u_x = \Lambda u\) for the fractional Laplace operator \(\Lambda = (-\Delta)\frac{s}{2}\). Hence the blow-up criterion (1.15) becomes

\[
1\{\limsup_{t \to \tau^*} \|u(t)\|_{\mathcal{B}^s} = \infty\} = 1\{\limsup_{t \to \tau^*} \|u_x(t)\|_{L^\infty} + \|\Lambda u(t)\|_{L^\infty} = \infty\} \mathbb{P} - a.s.
\]

This motivates us to consider blow-up for the particular case \(\limsup_{t \to \tau^*} \|u(t)\|_{L^\infty} = \infty\). Actually the case we obtain, as shown in (1.18), is \(\limsup_{t \to \tau^*} \max_{x \in \mathbb{R}} \Lambda u(t) = +\infty\). So far it is still not clear if \(\limsup_{t \to \tau^*} \min_{x \in \mathbb{R}} \Lambda u(t) = -\infty\) holds for blow-up.

- The key identity used in our analysis on blow-up is originally introduced in [53, Proposition 3.5], where the authors show that \(C^1\) global solutions to (1.4) cannot exists. The precise statement of this identity in our stochastic context is given in (5.9). Although the idea towards the proof of (5.9) is similar to [53], we cannot assume without loss of generality (as in [53, Proposition 3.5]) that \(\tilde{v}(z_0) = 0\) which simplifies the proof of (5.9). The main reason relies on the fact that in the stochastic setting \(z_0\) depends on \(t\) and \(\omega \in \Omega\). As a consequence, for different \(\omega\), the time \(t\) such that \(\tilde{v}(z_0) = 0\) may be different, and therefore we cannot assume that \(\tilde{v}(z_0) = 0\). On the other hand, if we fix \(t\) such that \(\tilde{v}(z_0) = 0\), then \(\tilde{v}(z_0) = 0\) may not hold almost surely, which brings further obstacles in proving (5.9).

- For trivial \(b^* = 0\), which means \(b(t) \equiv 0\), Theorem 1.3 recovers the deterministic result [53, Theorem 3.7]. Due to the presence of the noise term \(b(t)udW\), the condition (1.17) is more restrictive compared to the deterministic case where \(\Lambda u_0(x_0) > 0\) suffices to show the finite time blow-up, [53,
Theorem 3.7], therefore, although finite time singularities can be shown in the stochastic case it is somehow restrictive and strongly depends on the choice of the coefficient \( b(t) \).

Regarding question (Q-3), we have the following partial (negative) answer:

**Theorem 1.4** (Weak instability). Let \( S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \mathcal{W}) \) be a fixed stochastic basis and \( s > 3 \). If \( h \) satisfies Assumption (D), then at least one of the following properties holds true.

1. For any \( R \gg 1 \), the R-exiting time is not strongly stable for the zero solution to (1.2) in the sense of Definition 1.2;
2. There is a \( T > 0 \) such that solution map \( u_0 \mapsto u \) defined by (1.2) is not uniformly continuous as a map from \( L^\infty(\Omega, H^s) \) into \( L^1(\Omega; \mathcal{C}([0, T], H^s)) \). More precisely, there exist two sequences of solutions \( u^{1,n}(t, x) \) and \( u^{2,n}(t, x) \), and two sequences of stopping time \( \tau_{1,n} \) and \( \tau_{2,n} \), such that
   - For \( i = 1, 2 \), \( \mathbb{P}\{\tau_{i,n} > 0\} = 1 \) for each \( n > 1 \). Besides,
     \[
     \lim_{n \to \infty} \tau_{1,n} = \lim_{n \to \infty} \tau_{2,n} = \infty \implies \mathbb{P} - a.s. \tag{1.19}
     \]
   - For \( i = 1, 2 \), \( u^{i,n} \in C([0, \tau_{i,n}], H^s) \) \( \mathbb{P} - a.s. \), and
     \[
     \mathbb{E}\left( \sup_{t \in [0, \tau_{1,n}]} \|u^{1,n}(t)\|_{H^s} + \sup_{t \in [0, \tau_{2,n}]} \|u^{2,n}(t)\|_{H^s} \right) \lesssim 1. \tag{1.20}
     \]
   - At initial time \( t = 0 \),
     \[
     \lim_{n \to \infty} \mathbb{E}\|u^{1,n}(0) - u^{2,n}(0)\|^2_{H^s} = 0. \tag{1.21}
     \]
   - When \( t > 0 \),
     \[
     \liminf_{n \to \infty} \mathbb{E}\sup_{t \in [0, T \wedge \tau_{1,n} \wedge \tau_{2,n}]} \|u^{1,n}(t) - u^{2,n}(t)\|_{H^s} \gtrsim \sup_{t \in [0, T]} |\sin(t)|. \tag{1.22}
     \]

**Remark 1.5.** The idea behind Theorem 1.4 can be understood as follows: one cannot improve the continuity of the map \( u_0 \mapsto u \), and simultaneously, the stability of the exiting time at \( u \equiv 0 \). Hereafter, we briefly outline the main difficulties encountered in the proof of Theorem 1.4 and the main strategies used to tackle them.

- For system (1.2), we do not know how to obtain any explicit expression of the solutions. Therefore, to establish (1.22), the idea relies on finding two sequences of approximate solutions \( \{u_{m,n}\} \) (\( m \in \{1, 2\} \)) such that the difference between \( u_{m,n} \) and the actual solution \( u^{m,n} \) tends to zero as \( n \to \infty \), from which one can prove (1.22) by using \( \{u_{m,n}\} \) rather than \( \{u^{m,n}\} \). In this way, the first difficulty lies in the construction of such approximate solutions \( \{u_{m,n}\} \). In this work, by some delicate calculation, we are able to construct two sequences of approximate solutions \( \{u_{m,n}\} \) such that the actual solutions \( \{u^{m,n}\} \) satisfy
   \[
   u^{m,n}(0) = u_{m,n}(0), \quad \lim_{n \to \infty} \mathbb{E}\sup_{t \in [0, \tau_{m,n}]} \|u^{m,n} - u_{m,n}\|_{H^s} = 0, \tag{1.23}
   \]
   where \( u^{m,n} \) exists at least on \([0, \tau_{m,n}]\). Technically, since the equation involves the non-local Hilbert transform \( \mathcal{H} \) and the problem is defined on \( \mathbb{R} \), the construction of approximate solutions \( u_{m,n} \) is much more elaborated than the constructions in [44, 52, 55].

- The second difficulty we have to surpass is that we need to guarantee that \( \inf_n \tau_{m,n} > 0 \) almost surely when dealing with (1.23). This obstacle comes from the lack of lifespan estimates which we believe is quite a common issue in SPDEs. Indeed, in deterministic cases, one can easily obtain the lifespan estimate, from which it is not difficult to find a common interval \([0, T]\) such that all actual solutions exist on \([0, T]\) (see for example Lemma 6.1). In the stochastic setting, we are not able to show the precise estimate. The key observation to surpass this difficulty is that, the property \( \inf_n \tau_{m,n} > 0 \) can be connected to the stability property of the exiting time (see Definition 1.2). The condition comprising the fact that the \( R_0 \)-existing time is strongly stable at the zero solution will be used to provide a common existence time \( T > 0 \) such that for all \( n \), \( u^{m,n} \) exists up to \( T \) (see Lemma 6.4 below). Therefore, to prove Theorem 1.4, we will show that, if the \( R_0 \)-existing time is strongly stable at the zero solution for some \( R_0 \gg 1 \), then the solution map \( u_0 \mapsto u \) defined by (1.2) cannot be uniformly continuous.
Plan of the paper. We outline the structure of the paper. In Section 2 we provide some relevant preliminaries and recall well-known estimates that will be employed throughout the paper. In Section 3 we show the existence and uniqueness of local pathwise solutions and derived a blow-up criterion proving Theorem 1.1. We will divide the proof into several subsections. First we use an approximation scheme and perform uniform bounds in Subsection 3.1. Next, in Subsection 3.2 we show the convergence of the approximated solutions and afterwards in Subsection 3.3 we prove the global pathwise solutions to the cut-off problem. We conclude the proof of Theorem 1.1 in Subsection 1.1. Section 4 is devoted to study the effect of strong noise and prove Theorem 1.2. In Section 5 we show that finite time singularity occur in the case of linear noise thus proving Theorem 1.3. To conclude the article, in Section 6, we prove Theorem 1.4. The proof is divided into Subsection 6.1 - Subsection 6.3.

2. Preliminaries

For any $\varepsilon \in (0, 1)$, $J_\varepsilon$ is the Friedrichs mollifier defined by $J_\varepsilon f(x) = J_\varepsilon \ast f(x)$, where $\ast$ stands for the convolution, $J_\varepsilon(x) = \frac{1}{\varepsilon} j(\frac{x}{\varepsilon})$ and $j(x)$ is a Schwartz function satisfying $0 \leq \hat{j}(\xi) \leq 1$ for all $\xi \in \mathbb{R}$ and $\hat{j}(\xi) = 1$ for any $|\xi| \leq 1$, where $\hat{f}$ denotes the Fourier transform of $f$. It is obvious that $\hat{j}_\varepsilon(\xi) = \hat{j}(\varepsilon\xi)$. Moreover, for any $u \in H^s$, we have, cf. [54, 55],

\begin{align}
\| u - J_\varepsilon u \|_{H^r} &\sim o(\varepsilon^{r-s}), \quad r \leq s, \\
\| J_\varepsilon u \|_{H^r} &\lesssim O(\varepsilon^{r-s})\|u\|_{H^s}, \quad r > s, \\
(\varepsilon J_\varepsilon f, g)_{L^2} &\sim (f, J_\varepsilon g)_{L^2}.
\end{align}

The Hilbert transform is defined as

$$
\mathcal{H}f(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{y-x} \, dy.
$$

The fractional differential operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ is defined as the following singular integral operator

$$
\Lambda^\alpha f(x) = c_\alpha \text{p.v.} \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x-y|^{1+\alpha}} \, dy,
$$

where the constant $c_\alpha = \frac{2^\alpha \Gamma(\frac{\alpha}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{\alpha}{2})}$ is a normalization constant and $\Gamma$ represents the classical gamma function. In particular when $\alpha = 1$, $\mathcal{H}f = -\Lambda f$. Since $J_\varepsilon$, $\mathcal{H}$ and $D^s$ can be characterized by their Fourier multipliers, it is easy to see

$$
[D^s, \mathcal{H}] = [D^s, J_\varepsilon] = [\partial_\varepsilon, \mathcal{H}] = [\partial_\varepsilon, J_\varepsilon] = [J_\varepsilon, \mathcal{H}] = 0,
$$

and for any $s \geq 0$,

$$
\| \mathcal{H} u \|_{H^s} \leq \| u \|_{H^s}, \quad \| J_\varepsilon u \|_{H^s} \leq \| u \|_{H^s}.
$$

Lemma 2.1 (Page 3 in [58]). Let $J_\varepsilon$ be defined as in the above. Assume $g \in W^{1,\infty}$ and $f \in L^2$. Then for some $C > 0$,

$$
\| [J_\varepsilon, g] \partial_\varepsilon f \|_{L^2} \leq C \| \partial_\varepsilon g \|_{L^\infty} \| f \|_{L^2}.
$$

We also recall the following well-known estimates.

**Lemma 2.2** ([34, 35]). If $f, g \in H^s \cap W^{1,\infty}$ with $s > 0$, then for $p, p_i \in (1, \infty)$ with $i = 2, 3$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, we have

$$
\| [D^s, f] g \|_{L^p} \leq C_{s,p}(\| \nabla f \|_{L^{p_1}} \| D^{s-1} g \|_{L^{p_2}} + \| D^s f \|_{L^{p_3}} \| g \|_{L^{p_4}}),
$$

and

$$
\| D^s(f g) \|_{L^p} \leq C_{s,p}(\| f \|_{L^{p_1}} \| D^s g \|_{L^{p_2}} + \| D^s f \|_{L^{p_3}} \| g \|_{L^{p_4}}).
$$

**Lemma 2.3.** Let $s > 3$. Let $J_\varepsilon$ be the Friedrichs mollifier defined before. There is a constant $Q = Q(s) > 0$ such that

$$
|[(\mathcal{H} u)_{x^2}, u]_{H^{-1}}| \leq Q(\| u_x \|_{L^\infty} + \| \mathcal{H} u_x \|_{L^\infty}) \| u \|_{H^{-1}},
$$

and for all $\varepsilon > 0$,

$$
|((J_\varepsilon u x^2)_{x^2}, J_\varepsilon u)_{H^{-1}}| \leq Q(\| u_x \|_{L^\infty} + \| \mathcal{H} u_x \|_{L^\infty}) \| u \|_{H^{-1}}.
$$
Proof. We first prove (2.9). Due to (2.6) and (2.3), we commute the operator to derive

\[
(D^*J_z((Hu)u_x), D^*J_zu)_{L^2} = \left((D^*Hu)x, D^*J_z^2u\right)_{L^2} + \left((J_zHu)D^*u_x, D^*J_zu\right)_{L^2} + \left((Hu)D^*J_zu_x, D^*J_zu\right)_{L^2}.
\]

Then it follows from Lemmas 2.1 and 2.2, integration by parts, (2.7) and \(H^* \hookrightarrow W^{1,\infty}\) that

\[
\left|(D^*Hu)x, D^*J_z^2u\right|_{L^2} \lesssim \left\|u_x\right\|_{L^\infty} + \left\|Hu_x\right\|_{L^\infty}\left\|u\right\|_{H^r}^2,
\]

\[
\left|(J_zHu)D^*u_x, D^*J_zu\right|_{L^2} \lesssim \left\|Hu_x\right\|_{L^\infty}\left\|u\right\|_{H^r}^2,
\]

and

\[
\left|(Hu)D^*J_zu_x, D^*J_zu\right|_{L^2} \lesssim \left\|Hu_x\right\|_{L^\infty}\left\|u\right\|_{H^r}^2.
\]

Combining the above inequalities gives rise to (2.9). For (2.8), since \((Hu)u_x, u)_{H^{r-1}}\) is well-defined in this case, by repeating the above analysis, one can easily obtain the (2.8) and the proof is therefore completed. \(\square\)

Remark 2.1. We remark that (2.9) will be used in the proof of blow-up criterion (1.15) (see (3.26) below) and (2.8) will be used in the proof of Theorem 1.2 (see (4.3) below). But it is worthwhile pointing out that (2.8) is also used implicitly in the proof of Theorem 1.2. We refer to Remark 4.1 for more details.

Let us collect some identities and formulas regarding the Hilbert transform and the fractional Laplacian operator, cf. [53, 11]. The first one is the so-called Cotlar’s identity

\[
2\mathcal{H}(f(hf)) = (hf)^2 - f^2. \tag{2.10}
\]

The second one is the following equality:

Lemma 2.4 (Corollary 3.3, [53]). For any \(f \in L^1(\mathbb{R}; \mathbb{R})\), we have that for \(g(x) = xf(x)\),

\[
\Lambda g(x) = x\Lambda f(x) - \mathcal{H}f(x)
\]

Finally, we recall the following estimate on the product of a Schwartz function and a trigonometric function.

Lemma 2.5 ([27]). Let \(\delta > 0\) and \(\alpha \in \mathbb{R}\). Then for any \(r \geq 0\) and any Schwartz function \(\psi\), the following equation holds true:

\[
\lim_{n \to \infty} n^{-\frac{\delta}{2} + r} \left\|\psi\left(\frac{x}{n}\right)\cos(nx - \alpha)\right\|_{H^r} \leq \frac{1}{\sqrt{2}}\left\|\psi\right\|_{L^2}. \tag{2.11}
\]

Moreover, (2.11) also holds true when \(\cos\) is replaced by \(\sin\).

3. Proof of Theorem 1.1

For the sake of clarity, the proof is divided into several subsections.

3.1. Approximation scheme and uniform estimates. The first step is to construct a suitable approximation scheme. For any \(R > 1\), we let \(\chi_R(x) : [0, \infty) \to [0, 1]\) be a \(C^\infty\) function such that \(\chi_R(x) = 1\) for \(x \in [0, R]\) and \(\chi_R(x) = 0\) for \(x > 2R\). Then we consider the following cut-off problem on \(\mathbb{R}\),

\[
\begin{cases}
    du + \chi_R(|u|_{H^{r-3/2}})(Hu)u_xdt = \chi_R(|u|_{H^{r-3/2}})h(t, u)d\mathcal{W}, \\
    u(\omega, 0, x) = u_0(\omega, x) \in H^s, \quad s > 3.
\end{cases} \tag{3.1}
\]

To apply the theory of SDEs in Hilbert space to (3.1), we will have to mollify the transport term \((Hu)u_x\) since the product \((Hu)u_x\) loses regularity. To this end, we consider the following approximation scheme:

\[
\begin{cases}
    du + G_{1,\varepsilon}(u)dt = G_2(t, u)d\mathcal{W}, \\
    G_{1,\varepsilon}(u) = \chi_R(|u|_{H^{r-3/2}})J_z(\mathcal{H}J_zu)\partial_x J_zu, \\
    G_2(t, u) = \chi_R(|u|_{H^{r-3/2}})h(t, u), \\
    u(0, x) = u_0(x) \in H^s,
\end{cases} \tag{3.2}
\]

where \(J_z\) is the Friedrichs mollifier defined in Section 2.

After mollifying the non-local transport term \((Hu)u_x\), we see that \(G_{1,\varepsilon}(\cdot)\) and \(G_2(\cdot, \cdot)\) are locally Lipschitz continuous in \(H^r\). Moreover, the cut-off function \(\chi_R(|u|_{H^{r-3/2}})\) gives the linear growth condition (cf. Lemma 2.3 and (1.7)), i.e., there are constants \(l_1 = l_1(\varepsilon, R) > 0\) and \(l_2 = l_2(R) > 0\) such that for all \(t \geq 0\) and \(s > 3\),

\[
\|G_{1,\varepsilon}(u)\|_{H^r} \leq l_1(1 + \|u\|_{H^r}), \quad \|G_2(t, u)\|_{L^2(0, T; H^r)} \leq l_2(1 + \|u\|_{H^r}), \quad t \in [0, T]. \tag{3.3}
\]
Therefore, for a fixed stochastic basis $S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ and for $u_0 \in L^2(\Omega; H^*)$ with $s > 3$, the existence theory of SDEs in Hilbert space (see for example [48, Theorem 4.2.4 with Example 4.1.3 and [33]), (3.2) admits a unique solution $u_\varepsilon \in C([0, \infty), H^*) \mathbb{P} - a.s.$

We have the following uniform-in-$\varepsilon$ estimate:

**Proposition 3.1.** Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ be a fixed stochastic basis. Let $s > 3$, $R > 1$ and $\varepsilon \in (0, 1)$. Assume $h$ satisfies Assumption (A) and $u_0 \in L^2(\Omega; H^*)$ is an $H^s$-valued $\mathcal{F}_0$-measurable random variable. Let $u_\varepsilon \in C([0, \infty); H^*)$ solve (3.2) $\mathbb{P} - a.s.$, then for any $T > 0$, there are $C = C(R, T, u_0) > 0$ ($i = 0, 1, 2, 3, 4$) such that

$$
\sup_{t \in [0, T]} \mathbb{E} \sup_{\varepsilon > 0} \|u_\varepsilon(t)\|_{H^s}^2 \leq C. \tag{3.4}
$$

**Proof.** Using the Itô formula for $\|u_\varepsilon\|_{H^s}^2$, we have that for any $t > 0$,

$$
\begin{align*}
\frac{d}{dt}\|u_\varepsilon(t)\|_{H^s}^2 &= 2\chi_R(\|u_\varepsilon\|_{H^{s-3/2}})(h(t, u_\varepsilon)d\mathcal{W}, u_\varepsilon)_{H^s} \\
&\quad - 2\chi_R(\|u_\varepsilon\|_{H^{s-3/2}})(D^s J_\varepsilon [J_\varepsilon(H u_\varepsilon) \partial_x J_\varepsilon u_\varepsilon], D^s u_\varepsilon)_{L^2} dt \\
&\quad + \chi_R^2(\|u_\varepsilon\|_{H^{s-3/2}})\|h(t, u_\varepsilon)\|_{E_2(\varepsilon, H^s)}^2 dt.
\end{align*}
$$

Then, by means of the BDG inequality, (1.7), $H^{s-3/2} \hookrightarrow W^{1, \infty}$ and (2.7), we find that for some constant $C = C(R) > 0$,

$$
\begin{align*}
\mathbb{E} \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{H^s}^2 &\leq \mathbb{E}\left(\int_0^T \chi_R^2(\|u_\varepsilon\|_{H^{s-3/2}}) f^2(2\|u_\varepsilon\|_{H^{s-3/2}})(1 + \|u_\varepsilon\|_{H^s}) \|u_\varepsilon\|_{H^s}^2 dt\right)^{\frac{1}{2}} \\
&\quad + 2\mathbb{E}\int_0^T \chi_R(\|u_\varepsilon\|_{H^{s-3/2}})(D^s J_\varepsilon [J_\varepsilon(H u_\varepsilon) \partial_x J_\varepsilon u_\varepsilon], D^s u_\varepsilon)_{L^2} dt \\
&\quad + \mathbb{E}\int_0^T \chi_R^2(\|u_\varepsilon\|_{H^{s-3/2}}) f^2(2\|u_\varepsilon\|_{H^{s-3/2}})(1 + \|u_\varepsilon\|_{H^s})^2 dt \\
&\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, T]} \|u_\varepsilon\|_{H^s}^2 + C(R)\mathbb{E} \int_0^T (1 + \|u_\varepsilon\|_{H^s}) dt \\
&\quad + 2\mathbb{E}\int_0^T \chi_R(\|u_\varepsilon\|_{H^{s-3/2}})(D^s J_\varepsilon [J_\varepsilon(H u_\varepsilon) \partial_x J_\varepsilon u_\varepsilon], D^s u_\varepsilon)_{L^2} dt.
\end{align*}
$$

Let $J_\varepsilon u_\varepsilon = v$. It follows from (2.3), Lemma 2.2, integration by parts and (2.7) that

$$
|D^s J_\varepsilon [J_\varepsilon(H u_\varepsilon) \partial_x J_\varepsilon u_\varepsilon], D^s u_\varepsilon|_{L^2} \leq |(D^s H v)_{L^2} v, D^s v|_{L^2} + |(H v) D^s v, D^s v|_{L^2} \\
\leq C(\|H v\|_{H^s}, \|v\|_{L^\infty} + \|H v\|_{L^\infty}, \|v\|_{H^s}) \|v\|_{H^s} \\
\leq C\|v\|_{H^{s-3/2}}^2 \|v\|_{H^s},
$$

which implies

$$
\mathbb{E}\int_0^T \chi_R^2(\|u_\varepsilon\|_{H^{s-3/2}}) f^2(2\|u_\varepsilon\|_{H^{s-3/2}})(1 + \|u_\varepsilon\|_{H^s}) \|u_\varepsilon\|_{H^s}^2 dt \leq C(R)\mathbb{E} \int_0^T \|u_\varepsilon\|_{H^s}^2 dt.
$$

Therefore we obtain

$$
\mathbb{E} \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{H^s}^2 \leq 2\mathbb{E}\|u_0\|_{H^s}^2 + C(R)\int_0^T \left(1 + \mathbb{E} \sup_{t' \in [0, t]} \|u(t')\|_{H^s}^2\right) dt.
$$

Using Grönwall’s inequality to the above estimate implies that for some $C = C(R, T, u_0) > 0$,

$$
\mathbb{E} \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{H^s}^2 \leq C(R, T, u_0),
$$

which is (3.4).
3.2. Convergence of approximate solutions. Now we are going to show that, there is a subsequence of \( u_e \) converging in \( C([0,T], H^{s-3/2}) \) almost surely. To this end, for the solutions \( u_e \) and \( u_\eta \) to (3.2), we consider the following problem for \( v_{e,\eta} = u_e - u_\eta \),

\[
\begin{align*}
dv_{e,\eta} + [G_{1,e}(u_e) - G_{1,\eta}(u_\eta)] \ dt &= [G_2(t,u_e) - G_2(t,u_\eta)] \ dW, \quad v_{e,\eta}(0) = 0. 
\end{align*}
\]

We notice that

\[
G_{1,e}(u_e) - G_{1,\eta}(u_\eta) = \chi_R(\|u_e\|_{H^{s-3/2}}) J_e - \chi_R(\|u_\eta\|_{H^{s-3/2}}) J_\eta + \chi_R(\|u_e\|_{H^{s-3/2}}) J_e - \chi_R(\|u_\eta\|_{H^{s-3/2}}) J_\eta + \chi_R(\|u_e\|_{H^{s-3/2}}) J_e - \chi_R(\|u_\eta\|_{H^{s-3/2}}) J_\eta
\]

\[
\triangleq \sum_{i=1}^{6} q_i.
\]

and

\[
G_2(t,u_e) - G_2(t,u_\eta) = \chi_R(\|u_e\|_{H^{s-3/2}}) h(t,u_e) - \chi_R(\|u_\eta\|_{H^{s-3/2}}) h(t,u_\eta) + \chi_R(\|u_e\|_{H^{s-3/2}}) h(t,u_e) - \chi_R(\|u_\eta\|_{H^{s-3/2}}) \|h(t,u_e) - h(t,u_\eta)\|
\]

\[
\triangleq \sum_{i=1}^{8} q_i.
\]

Invoking Itō’s formula in (3.5) and recalling (3.6), (3.7) we find that for any \( t > 0 \),

\[
\|v_{e,\eta}(t)\|_{H^{s-3/2}}^2 \leq Q_1 - \int_0^t Q_2 \ dt' + \int_0^t Q_3 \ dt',
\]

where

\[
Q_1 = 2 \int_0^t \left( \sum_{i=1}^{6} q_i \ dW, v_{e,\eta} \right)_{H^{s-3/2}}, \quad Q_2 = 2 \sum_{i=1}^{6} (q_i, v_{e,\eta})_{H^{s-3/2}}, \quad Q_3 = \sum_{i=7}^{8} q_i \left\| v_{e,\eta} \right\|_{L_2(U,H^{s-3/2})}.
\]

Lemma 3.1. Let \( s > 3 \). For any \( \varepsilon, \eta \in (0,1) \), there is a constant \( C > 0 \) such that \( Q_2 \) given by (3.9) satisfies

\[
|Q_2| \leq C(1 + \|u_e\|_{H^s}^2 + \|u_\eta\|_{H^s}^2) \|v_{e,\eta}\|_{H^{s-3/2}} + C(\|u_e\|_{H^s}^4 + \|u_\eta\|_{H^s}^4) \max\{\varepsilon, \eta\}.
\]

Proof. Using the mean value theorem for \( \chi_R(\cdot) \) and (2.7), we have

\[
\|q_i\|_{H^{s-3/2}} \leq \|v_{e,\eta}\|_{H^{s-3/2}} \|u_e\|_{H^s}^2.
\]

Using (2.1) and (2.7), we see that

\[
\|q_i\|_{H^{s-3/2}} \leq \max\{\varepsilon^{1/2}, \eta^{1/2}\} \|u_e\|_{H^s}^2, \quad i = 2, 3,
\]

\[
\|q_4\|_{H^{s-3/2}} \leq \|v_{e,\eta}\|_{H^{s-3/2}} \|u_e\|_{H^s},
\]

\[
\|q_5\|_{H^{s-3/2}} \leq \max\{\varepsilon^{1/2}, \eta^{1/2}\} \|u_e\|_{H^s} \|u_\eta\|_{H^s}.
\]

For \( q_6 \), using (2.3), (2.6) and then integrating by part, we have

\[
\begin{align*}
(q_6, v_{e,\eta})_{H^{s-3/2}} &= \chi_R(\|u_e\|_{H^{s-3/2}}) \int_R \int R^{s-3/2} J_e(\varepsilon, \eta) J_\eta v_{e,\eta} \ dx \\
&= \chi_R(\|u_e\|_{H^{s-3/2}}) \int_R \int R^{s-3/2} J_\eta(\varepsilon, \eta) J_\eta v_{e,\eta} \ dx \\
&+ \chi_R(\|u_\eta\|_{H^{s-3/2}}) \int_R J_e(\varepsilon, \eta) J_\eta(\varepsilon, \eta) \ dx \\
&= \chi_R(\|u_e\|_{H^{s-3/2}}) \int_R \int R^{s-3/2} J_\eta(\varepsilon, \eta) J_\eta v_{e,\eta} \ dx \\
&- \frac{1}{2} \chi_R(\|u_\eta\|_{H^{s-3/2}}) \int_R J_\eta(\varepsilon, \eta) J_\eta(\varepsilon, \eta) \ dx.
\end{align*}
\]
Using \( \chi_R(\cdot) \leq 1 \), Lemma 2.2, (2.7) and the embedding \( H^{s-3/2} \hookrightarrow W^{1,\infty} \), we have

\[
(q_6, v_{\varepsilon, \eta})_{H^{s-3/2}} \lesssim \|u_0\|_{H^{s-3/2}} \|\partial_x J_h v_{\varepsilon, \eta}\|_{L^\infty} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} + \|\partial_x J_h (H u_0)\|_{L^\infty} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} ^2 \lesssim \|u_0\|_{H^s} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} ^2.
\]

Therefore we can put these all together to see that there is a constant \( C > 0 \) such that

\[
|Q_2| \leq C(1 + \|u_0\|_{H^s} ^2 + \|u_0\|_{H^s} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} ^2 + C(\|u_0\|_{H^s} ^2 + \|u_0\|_{H^s} ^2) \max\{\varepsilon, \eta\},
\]

which is the desired estimate. \( \square \)

**Lemma 3.2.** Let \( S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W}) \) be a fixed stochastic basis. Let \( s > 3, R > 1 \) and \( \varepsilon \in (0, 1) \). Let \( u_\varepsilon \in C([0, \infty); H^s) \) solve (3.2) \( \mathbb{P} - a.s. \). For any \( T > 0 \) and \( K > 1 \), we define

\[
\tau_{\varepsilon, K}^T = \inf \{t \geq 0 : \|u_\varepsilon(t)\|_{H^s} \geq K \} \wedge T,
\]

and

\[
\tau_{\varepsilon, \eta, K}^T = \tau_{\varepsilon, K}^T \wedge \tau_{\eta, K}^T.
\]

Then we have

\[
\limsup_{\varepsilon \to 0} \sup_{\eta \leq \varepsilon} \mathbb{E} \sup_{t \in [0, \tau_{\varepsilon, \eta, K}^T]} \|u_\varepsilon - u_\eta\|_{H^{s-3/2}} = 0, \quad K > 1.
\]

**Proof.** Recalling (3.8) and (3.9), we have

\[
\|v_{\varepsilon, \eta}(t)\|_{H^{s-3/2}} ^2 \leq |Q_1| + \int_0^t |Q_2| dt + \int_0^t |Q_3| dt.
\]

The mean value theorem for \( \chi_R(\cdot) \) and Assumption (A) yield that

\[
\|Q_1\|_{L^2(t_1, t_2)} \lesssim C\|v_{\varepsilon, \eta}\|_{H^{s-3/2}} \max\{\varepsilon, \eta\} \|u_\varepsilon\|_{H^s} \int_{t_1}^{t_2} \|u_\varepsilon\|_{H^s} dt.
\]

By (3.11) and Assumption (A), we see that

\[
\|q_8\|_{L^2(t_1, t_2)} \leq C\|v_{\varepsilon, \eta}\|_{H^{s-3/2}} q(K), \quad t \in [0, \tau_{\varepsilon, \eta, K}^T] \mathbb{P} - a.s.,
\]

where \( q(\cdot) \) is given in Assumption (A). Therefore we find a constant \( C = C(K) > 0 \) such that

\[
\mathbb{E} \int_0^{\tau_{\varepsilon, \eta, K}^T} |Q_3| dt \leq C(K) \mathbb{E} \int_0^{\tau_{\varepsilon, \eta, K}^T} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} ^2 dt \leq C(K) \int_0^T \mathbb{E} \sup_{t' \in [0, \tau_{\varepsilon, \eta, K}^T]} \|v_{\varepsilon, \eta}(t')\|_{H^{s-3/2}} ^2 dt.
\]

Then we employ the BDG inequality to (3.8) to find

\[
\mathbb{E} \sup_{t \in [0, \tau_{\varepsilon, \eta, K}^T]} \|v_{\varepsilon, \eta}(t)\|_{H^{s-3/2}} ^2 \leq C(K) \mathbb{E} \left( \int_0^{\tau_{\varepsilon, \eta, K}^T} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} ^2 dt \right) ^\frac{1}{2} + \frac{3}{2} \mathbb{E} \int_0^{\tau_{\varepsilon, \eta, K}^T} |Q_3| dt
\]

\[
\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, \tau_{\varepsilon, \eta, K}^T]} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} ^2 + C(K) \mathbb{E} \left( \int_0^{\tau_{\varepsilon, \eta, K}^T} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} ^2 dt \right) + \frac{3}{2} \mathbb{E} \int_0^{\tau_{\varepsilon, \eta, K}^T} |Q_3| dt
\]

\[
\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, \tau_{\varepsilon, \eta, K}^T]} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} ^2 + C(K) \int_0^T \mathbb{E} \sup_{t' \in [0, \tau_{\varepsilon, \eta, K}^T]} \|v_{\varepsilon, \eta}(t')\|_{H^{s-3/2}} ^2 dt + \mathbb{E} \int_0^{\tau_{\varepsilon, \eta, K}^T} |Q_3| dt.
\]

On account of Lemma 3.1, we arrive at

\[
\mathbb{E} \int_0^{\tau_{\varepsilon, \eta, K}^T} |Q_2| dt \leq C(K) \mathbb{E} \int_0^{\tau_{\varepsilon, \eta, K}^T} \|v_{\varepsilon, \eta}\|_{H^{s-3/2}} ^2 dt + C(K) T \max\{\varepsilon, \eta\}
\]

\[
\leq C(K) \int_0^T \mathbb{E} \sup_{t' \in [0, \tau_{\varepsilon, \eta, K}^T]} \|v_{\varepsilon, \eta}(t')\|_{H^{s-3/2}} ^2 dt + C(K) T \max\{\varepsilon, \eta\}.
\]

Now we can put these all together to obtain

\[
\mathbb{E} \sup_{t \in [0, \tau_{\varepsilon, \eta, K}^T]} \|v_{\varepsilon, \eta}(t)\|_{H^{s-3/2}} ^2 \leq C(K) \int_0^T \mathbb{E} \sup_{t' \in [0, \tau_{\varepsilon, \eta, K}^T]} \|v_{\varepsilon, \eta}(t')\|_{H^{s-3/2}} ^2 dt + C(K) T \max\{\varepsilon, \eta\},
\]
which means that
\[
\mathbb{E} \sup_{t \in [0, T]} \| u_{\varepsilon, \eta}(t) \|_{H^{s-3/2}}^2 \leq C(K, T) \max \{ \varepsilon, \eta \},
\]
and hence (3.12) holds true. \( \square \)

**Lemma 3.3.** For any fixed \( s > 3 \) and \( T > 0 \), there is an \( \{ F_t \}_{t \geq 0} \) progressive measurable \( H^s \)-valued process
\[
u \in L^2(\Omega; L^\infty(0, T; H^s))
\]
and a countable subsequence of \( \{ u_{\varepsilon} \} \) (still denoted as \( \{ u_{\varepsilon} \} \)) such that
\[ u_{\varepsilon} \xrightarrow{\varepsilon \to 0} u \text{ in } C([0, T]; H^{s-3/2}) \ \mathbb{P} - \text{a.s.} \tag{3.19} \]

**Proof.** We notice that for each \( \varepsilon \in (0, 1) \), (3.2) has solution \( u_{\varepsilon} \) almost surely. Now we first take \( \varepsilon \) to be discrete such that for all \( \varepsilon, \ v_{\varepsilon} \) can be defined on the same set \( \Omega \) with \( \mathbb{P} \{ \Omega \} = 1 \) (Actually, one can pick discrete \( \varepsilon = \frac{1}{n} \) from the beginning in (3.2)). Recall (3.10) and (3.11). For any \( \varepsilon > 0 \), by using Proposition 3.1 and Chebyshev’s inequality, we see that
\[
\mathbb{P} \left\{ \sup_{t \in [0, T]} \| u_{\varepsilon} - u_\eta \|_{H^{s-3/2}} > \varepsilon \right\} \\
= \mathbb{P} \left\{ \{ T_{\varepsilon, \eta, K} < T \} \cup \{ T_{\eta, K} = T \} \right\} \cap \left\{ \sup_{t \in [0, T]} \| u_{\varepsilon} - u_\eta \|_{H^{s-3/2}} > \varepsilon \right\} \\
\leq \mathbb{P} \{ T_{\varepsilon, \eta, K} < T \} + \mathbb{P} \{ T_{\eta, K} < T \} + \mathbb{P} \left\{ \sup_{t \in [0, T]} \| u_{\varepsilon} - u_\eta \|_{H^{s-3/2}} > \varepsilon \right\} \\
\leq \frac{2C(R, T, u_0)}{K^2} + \mathbb{P} \left\{ \sup_{t \in [0, T]} \| u_{\varepsilon} - u_\eta \|_{H^{s-3/2}} > \varepsilon \right\}. \]

Now (3.12) clearly forces
\[
\lim_{\varepsilon \to 0} \sup_{\eta \leq \varepsilon} \mathbb{P} \left\{ \sup_{t \in [0, T]} \| u_{\varepsilon} - u_\eta \|_{H^{s-3/2}} > \varepsilon \right\} \leq \frac{2C(R, T, u_0)}{K^2}, \quad K > 1.
\]

Letting \( K \to \infty \), we see that \( u_{\varepsilon} \) converges in probability in \( C([0, T]; H^{s-3/2}) \). Therefore, up to a further subsequence, (3.19) holds true.

Now we prove (3.18). Indeed, since \( H^s \hookrightarrow H^{s-3/2} \) is continuous, there exist continuous maps \( \pi_m : H^{s-3/2} \to H^s \), \( m \geq 1 \) such that
\[
\| \pi_m u \|_{H^s} \leq \| u \|_{H^s}, \quad \lim_{m \to \infty} \| \pi_m u \|_{H^s} = \| u \|_{H^s}, \quad u \in H^{s-3/2},
\]
where \( \| u \|_{H^s} \triangleq \infty \) if \( u \notin H^s \). For example, one may take \( \pi_m \) as the standard mollifier. Then it follows from Proposition 3.1 and Fatou’s lemma that
\[
\mathbb{E} \sup_{t \in [0, T]} \| u(t) \|_{H^s}^2 \leq \liminf_{m \to \infty} \mathbb{E} \sup_{t \in [0, T]} \| \pi_m u(t) \|_{H^s}^2 \\
\leq \liminf_{m \to \infty} \liminf_{\varepsilon \to 0} \mathbb{E} \sup_{t \in [0, T]} \| \pi_m u_{\varepsilon}(t) \|_{H^s}^2 \\
\leq \liminf_{m \to \infty} \liminf_{\varepsilon \to 0} \mathbb{E} \sup_{t \in [0, T]} \| u_{\varepsilon}(t) \|_{H^s}^2 < C(R, u_0, T).
\]

Hence (3.18) holds true. \( \square \)

### 3.3. Global pathwise solution to the cut-off problem.

**Proposition 3.2.** Let \( S = (\Omega, F, \mathbb{P}, \{ F_t \}_{t \geq 0}, W) \) be a stochastic basis fixed in advance. Let \( s > 3, R > 1 \) and \( \varepsilon \in (0, 1) \). Assume Assumption (A) is satisfied. Let \( u_0 \in L^2(\Omega; H^s) \) be an \( H^s \)-valued \( F_0 \) measurable random variable. Then for any \( T > 0 \), (3.1) has a solution \( u \in L^2(\Omega; C([0, T]; H^s)) \). Moreover, there is a constant \( C(R, T, u_0) > 0 \) such that
\[
\mathbb{E} \sup_{t \in [0, T]} \| u(t) \|_{H^s}^2 \leq C(R, T, u_0).
\]
Proof. Since for each $\varepsilon \in (0, 1)$, $u_{\varepsilon} \in \{F_t\}_{t \geq 0}$ progressive measurable, so is $u$. By Lemma 3.3 and the embedding $H^{s-3/2} \hookrightarrow W^{1, \infty}$, we can send $\varepsilon \to 0$ in (3.2) to conclude that $u$ solves (3.1). Now we only need to prove (1.14). Due to Lemma 3.3, $u \in C([0, T]; H^{s-3/2}) \cap L^\infty (0, T; H^s)$ almost surely. Since $H^s$ is dense in $H^{s-3/2}$, we see that ([59, page 263, Lemma 1.4]) $u \in C_{w} ([0, T]; H^s)$, where $C_{w} ([0, T]; H^s)$ is the space of weakly continuous functions with values in $H^s$. Therefore to prove (1.14), we only need to prove the continuity of $[0, T] \ni t \mapsto \|u(t)\|_{H^s}$.

However, we cannot directly apply the Itô formula for $\|u\|_{H^s}$, to get control of $E\|u(t)\|_{H^s}^2$, because we only have $u \in H^s$ and $(Hu)u_x \notin H^{s-1}$. Indeed, the Itô formula in a Hilbert space ([12, Theorem 4.32] or [21, Theorem 2.10]) requires $((Hu)u_x, u)_H$ to be well-defined and the Itô formula under a Gelfand triplet ([41, Theorem I.3.1] or [48, Theorem 4.2.5]) requires the dual product $H^{-1}((Hu)u_x, u)_{H'^{-1}}$ to be well-defined. In our case neither of them is satisfied. To this end, we recall the mollifier $J_e$ defined in Section 2 and apply the Itô formula to $\|J_e u\|_{H^s}^2$, to obtain

$$
\|J_e u(t)\|_{H^s}^2 = 2\chi_R(\|u\|_{W^{1, \infty}}) (J_e h(t, u) dW_t, J_e u)_{H^s} + 2\chi_R(\|u\|_{W^{1, \infty}}) (J_e ((Hu)u_x), J_e u)_{H^s} dt + \chi_R^2(\|u\|_{W^{1, \infty}}) \|J_e h(t, u)\|_{L^2([t, t_0]; H^0)}^2 dt. \tag{3.20}
$$

By (3.18),

$$
\tau_N = \inf \{ t \geq 0 : \|u(t)\|_{H^s} > N \} \to \infty \text{ as } N \to \infty \quad \mathbb{P} \text{-a.s.} \tag{3.21}
$$

Then we only need to prove the continuity up to time $\tau_N \wedge T$ for each $N \geq 1$. We first notice that $J_e$ satisfies (2.3), (2.6) and (2.7). Therefore for any $[t_2, t_1] \subset [0, T]$ with $t_1 - t_2 < 1$, we use Lemma 2.3, the BDG inequality and Assumption (A) and (3.21) to find

$$
\mathbb{E} \left[ \left( \|J_e u(t_1 \wedge \tau_N)\|_{H^s}^2 - \|J_e u(t_2 \wedge \tau_N)\|_{H^s}^2 \right)^4 \right] \leq C(N, T) |t_1 - t_2|^2.
$$

Using Fatou’s lemma, we arrive at

$$
\mathbb{E} \left[ \left( \|u(t_1 \wedge \tau_N)\|_{H^s}^2 - \|u(t_2 \wedge \tau_N)\|_{H^s}^2 \right)^4 \right] \leq C(N, T) |t_1 - t_2|^2.
$$

This and Kolmogorov’s continuity theorem ensure the continuity of $t \mapsto \|u(t \wedge \tau_N)\|_{H^s}$, completing the proof.

3.4. Concluding the proof of Theorem 1.1. Finally, we are in the position to finish the proof of Theorem 1.1. For the sake of clarity, we split the proof into three steps.

Step 1: Existence. For $u_0(\omega, x) \in L^2(\Omega; H^s)$, we let

$$
\Omega_k = \{ k - 1 \leq \|u_0\|_{H^s} < k \}, \quad k \in \mathbb{N}, \quad k \geq 1.
$$

Since $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, we have

$$
u_0(\omega, x) = \sum_{k \geq 1} u_{0,k}(\omega, x) = \sum_{k \geq 1} u_0(\omega, x) \mathbf{1}_{\{ k - 1 \leq \|u_0\|_{H^s} < k \}} \quad \mathbb{P} \text{-a.s.}
$$

On account of Proposition 3.2, we let $u_{k,R}$ be the pathwise global solution to the cut-off problem (3.1) with initial value $u_{0,k}$ and cut-off function $\chi_R(\cdot)$. Define

$$
\tau_{k,R} = \inf \left\{ t > 0 : \sup_{t' \in [0, t]} \|u_{k,R}(t')\|_{H^s}^2 > \|u_{0,k}\|_{H^s}^2 + 2 \right\}. \tag{3.22}
$$

Then for any $R > 0$ and $k \in \mathbb{N}$, we have $\mathbb{P}\{\tau_{k,R} > 0\} = 1$. The difficulty here is that we have to take $R$ to be deterministic. Otherwise Proposition 3.1 will fail. To overcome this difficulty, we let $R = R_k$ be discrete and then denote $(u_k, \tau_k) = (u_{k,R_k}, \tau_{k,R_k})$. It is clear that $\mathbb{P}\{\tau_k > 0, \forall k \geq 1\} = 1$. Let $E > 0$ be the embedding constant such that $\|\cdot\|_{W^{1, \infty}} \leq E \|\cdot\|_{H^s}$ for $s > 3$. Particularly, we take $R_k^2 > E^2 \|u_{0,k}\|_{H^s}^2 + 2 E^2$, and then we have

$$
\mathbb{P} \{ \|u_{0,k}\|_{W^{1, \infty}} \leq E^2 \|u_{k,R}\|_{H^s} \leq E^2 \|u_{0,k}\|_{H^s} + 2 E^2 < R_k^2, \forall t \in [0, \tau_k], \forall k \geq 1 \} = 1,
$$

which means

$$
\mathbb{P}\{\chi_{R_k}(\|u_k\|_{W^{1, \infty}}) = 1, \forall t \in [0, \tau_k], \forall k \geq 1\} = 1.
$$

Therefore $(u_k, \tau_k)$ is the pathwise solution to (1.2) with initial value $u_{0,k}$. Notice that

$$
\begin{align*}
1_{\Omega_k} u_k(t \wedge \tau_k) - 1_{\Omega_k} u_{0,k} &= \int_0^{t \wedge \tau_k} 1_{\Omega_k} [(Hu_k) \partial_x u_k] dt + \int_0^{t \wedge \tau_k} 1_{\Omega_k} h(t, u_k) dW_t, \\
1_{\Omega_k} h(t, u_k) &= h(t, 1_{\Omega_k} u_k) - 1_{\Omega_k^c} h(t, 0)
\end{align*}
$$
and
\[ 1_{\Omega_k}[(H u_k) \partial_x u_k] = (H 1_{\Omega_k} u_k) \partial_x 1_{\Omega_k} u_k. \]
By Assumption (A), we have \(||h(t, 0)||_{L^2(\Omega; H^s)} < \infty.\) Then we have
\[
1_{\Omega_k} u_k (t \wedge \tau_k) - 1_{\Omega_k} u_0, k
= -\int_0^{t \wedge \tau_k} (H 1_{\Omega_k} u_k) \partial_x (1_{\Omega_k} u_k) \, dt' + \int_0^{t \wedge \tau_k} h(t, 1_{\Omega_k} u_k) \, dW.
\]
Therefore \((1_{\Omega_k} u_k, 1_{\Omega_k} \tau_k)\) is a solution to (1.2) with initial data \(u_{0,k}\). Since \(\Omega_k \cap \Omega_{k'} = \emptyset\) for \(k \neq k'\) and \(\bigcup_{k \geq 1} \Omega_k\) is a set of full measure, we see that
\[
\left( u = \sum_{k \geq 1} 1_{\{k-1 \leq ||u_0||_{H^s} < k\}} u_k, \quad \tau = \sum_{k \geq 1} 1_{\{k-1 \leq ||u_0||_{H^s} < k\}} \tau_k \right)
\]
is a pathwise solution to (1.2) corresponding to the initial condition \(u_0\). Besides, using (3.22), we have
\[
\sup_{t \in [0, T]} \|u\|_{H^s}^2 = \sum_{k \geq 1} 1_{\{k-1 \leq ||u_0||_{H^s} < k\}} \sup_{t \in [0, \tau_k]} \|u_k\|_{H^s}^2 \leq \sum_{k \geq 1} 1_{\{k-1 \leq ||u_0||_{H^s} < k\}} (\|u_{0,k}\|_{H^s}^2 + 2) \leq 2 \|u_0\|_{H^s}^2 + 4.
\]
Taking expectation gives rise to (1.14).

**Step 2: Uniqueness and maximal pathwise solution.** With \((u, \tau)\) in hand, we can extend \((u, \tau)\) to a maximal pathwise solution in the sense of Definition 1.1 by following the techniques as in [9, 24, 23, 50]. For uniqueness, we let \((u_1, \tau_1)\) and \((u_2, \tau_2)\) be two solutions to (1.2) such that \(u_j(0) = u_0\) almost surely and \(u_j(\cdot \wedge \tau_j) \in L^2(\Omega; C([0, \infty); H^s))\) with \(s > 3\) for \(j = 1, 2\). Let \(\frac{1}{2} < \delta < s - 1\) and define
\[ \tau_k^T = \inf \{ t \geq 0 : ||u_1(t)||_{H^s} + ||u_2(t)||_{H^s} > K \} \wedge T, \quad K \in \mathbb{N}, \quad T > 0. \]
Using (1.8) and the definition of \(\tau_k^T\), then the estimate of \(\mathbb{E} \sup_{t \in [0, \tau_k^T]} \|u_1(t) - u_2(t)\|_{H^s}^2\) is essential as in the derivation of (3.12) and we have
\[
\mathbb{E} \sup_{t \in [0, \tau_k^T]} \|u_1(t) - u_2(t)\|_{H^s}^2 = 0.
\]
If necessary, to guarantee \(\tau_k^T > 0\) almost surely, we can first assume \(u_0 \in L^\infty(\Omega; H^s)\) and then remove this restriction by using the techniques as in Step 1. Hence we obtain uniqueness and the details are omitted here for brevity.

**Step 3: Blow-up criterion.** We first define
\[ \tau_{1,m} = \inf \{ t \geq 0 : ||u(t)||_{H^s} \geq m \}, \quad \tau_{2,n} = \inf \{ t \geq 0 : ||u_x(t)||_{L^\infty} + ||Hu_x(t)||_{L^\infty} \geq n \}, \]
and then let \(\tau_1 = \lim_{m \to \infty} \tau_{1,m}\) and \(\tau_2 = \lim_{n \to \infty} \tau_{2,n}\). We notice that for fixed \(m, n > 0\), even if \(\mathbb{P}[\tau_{1,m} = 0]\) or \(\mathbb{P}[\tau_{2,n} = 0]\) is larger than 0, for a.e. \(\omega \in \Omega\), there is \(m > 0\) or \(n > 0\) such that \(\tau_{1,m}, \tau_{2,n} > 0\). By continuity of \(||u(t)||_{H^s}\) and the uniqueness of \(u\), it is easy to check that \(\tau_1 = \tau_2\) is actually the maximal existence time \(\tau^*\) of \(u\) in the sense of Definition 1.1. Therefore to prove (1.15), we only need to verify that
\[
\tau_1 = \tau_2 \quad \mathbb{P} - a.s. \tag{3.23}
\]
The approach here is motivated by [9, 3]. Since \(H^s \hookrightarrow W^{1,\infty}\) and \(H\) is continuous in \(H^s\) (cf. (2.7)), there exists a constant \(M > 0\) such that,
\[
\sup_{t \in [0, \tau_{1,m}]} (||u_x(t)||_{L^\infty} + ||Hu_x(t)||_{L^\infty}) \leq M \sup_{t \in [0, \tau_{1,m}]} \|u(t)||_{H^s} \leq (|M| + 1)m,
\]
where \([M]\) denotes the integer part of \(M\). Therefore we have \(\tau_{1,m} \leq \tau_{2,([M]+1)m} \leq \tau_2 \quad \mathbb{P} - a.s\), which means that \(\tau_1 \leq \tau_2 \quad \mathbb{P} - a.s\). Now we only need to prove \(\tau_2 \leq \tau_1 \quad \mathbb{P} - a.s\). We do the following claim:

**Claim:**
\[
\mathbb{P} \left\{ \sup_{t \in [0, \tau_{2,n_1} \wedge n_2]} \|u(t)||_{H^s} < \infty \right\} = 1, \quad \forall n_1, n_2 \in \mathbb{N}. \tag{3.24}
\]
As is explained before, we cannot directly apply the Itô formula for $\|u\|_{H^s}^2$ to get control of $\mathbb{E}\|u(t)\|_{H^s}^2$. Similar to (3.20), by applying $J_\varepsilon$ to (1.2) and using the Itô formula for $\|J_\varepsilon u\|_{H^s}^2$, we have that for any $t > 0$, 

$$d\|J_\varepsilon u(t)\|_{H^s}^2 = (J_\varepsilon h(t, u)dW, J_\varepsilon u)_{H^s} - 2(D^*J_\varepsilon [(H(u)u), D^*J_\varepsilon u]_{L^2} dt + \|J_\varepsilon h(t, u)\|_{L^2(\mathcal{U}, H^s)}^2 dt. \quad (3.25)$$

By the BDG inequality, we have

$$\mathbb{E}\sup_{t \in [0, t_\varepsilon]} \|J_\varepsilon u(t)\|_{H^s}^2 \leq \mathbb{E}\|J_\varepsilon u_0\|_{H^s}^2 + C\mathbb{E}\left(\int_0^{t_\varepsilon} \|J_\varepsilon h(t, u)\|_{L^2(\mathcal{U}, H^s)}^2 \|J_\varepsilon u\|_{H^s}^2 dt \right)^{1/2} + 2\mathbb{E}\int_0^{t_\varepsilon} \|J_\varepsilon h(t, u)\|_{L^2(\mathcal{U}, H^s)}^2 \|J_\varepsilon u\|_{H^s}^2 dt \quad \text{and} \quad 2\mathbb{E}\int_0^{t_\varepsilon} \|J_\varepsilon h(t, u)\|_{L^2(\mathcal{U}, H^s)}^2 \|J_\varepsilon u\|_{H^s}^2 dt \leq Cn_1 \int_0^{t_\varepsilon} (1 + \mathbb{E}\|u\|_{H^s}^2) dt. \quad (3.26)$$

It follows from (1.7) that for some constant $C > 0$,

$$\mathbb{E}\int_0^{t_\varepsilon} \|J_\varepsilon h(t, u)\|_{L^2(\mathcal{U}, H^s)}^2 dt \leq Cf^2(2n_1) \int_0^{t_\varepsilon} (1 + \mathbb{E}\|u\|_{H^s}^2) dt,$$

Therefore we combine the above estimates, use (2.7), and then send $\varepsilon \to 0$ in the resulting inequality to obtain

$$\mathbb{E}\sup_{t \in [0, t_\varepsilon]} \|u(t)\|_{H^s}^2 \leq C\mathbb{E}\|u_0\|_{H^s}^2 + C\int_0^{t_\varepsilon} \left(1 + \mathbb{E}\sup_{t' \in [0, t \wedge t_\varepsilon]} \|u(t')\|_{H^s}^2 \right) dt. \quad (3.27)$$

Then Grönwall’s inequality shows that for each $n_1, n_2 \in \mathbb{N}$, there is a constant $C = C(n_1, n_2, u_0) > 0$ such that

$$\mathbb{E}\sup_{t \in [0, t_\varepsilon]} \|u(t)\|_{H^s}^2 < C(n_1, n_2, u_0),$$

which gives (3.24) and concludes the claim.

Hence (3.24) implies that for all $n_1, n_2 \in \mathbb{N}$, $\mathbb{P}\left\{ \sup_{t \in [0, t_\varepsilon \wedge \tau_{n_2}]} \|u(t)\|_{H^s} < \infty \right\} = 1$. On the other hand, it is easy to see that for all $n_1, n_2 \in \mathbb{N}$,

$$\left\{ \sup_{t \in [0, t_\varepsilon \wedge \tau_{n_2}]} \|u(t)\|_{H^s} < \infty \right\} \subset \bigcup_{m \in \mathbb{N}} \{ \tau_{2, n_2} \wedge n_2 \leq \tau_{1, m} \} \subset \{ \tau_{2, n_2} \wedge n_2 \leq \tau_{1} \}.$$

Consequently,

$$\mathbb{P}\{ \tau_2 \leq \tau_1 \} = \mathbb{P}\left\{ \bigcap_{n_1 \in \mathbb{N}} \{ \tau_{2, n_1} \leq \tau_1 \} \right\} = \mathbb{P}\left\{ \bigcap_{n_1, n_2 \in \mathbb{N}} \{ \tau_{2, n_1} \wedge n_2 \leq \tau_1 \} \right\} = 1. \quad (3.28)$$

Combining the above three steps, we complete the proof of Theorem 1.1.

4. PROOF OF THEOREM 1.2

To begin with, we can follow the steps as in the proof of Theorem 1.1 to obtain that, if $u_0$ is an $H^s$-valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$ with $s > 3$, then (1.3) has a unique pathwise solution $u \in H^s$ with maximal existence time $\tau^*$. Now the target is to show that $\mathbb{P}\{ \tau^* = \infty \} = 1$. To this end, we define

$$\hat{\tau}_k = \inf \{ t \geq 0 : \|u(t)\|_{H^{s-1}} \geq k \}, \quad k \geq 1 \quad \text{and} \quad \hat{\tau}^* = \lim_{k \to \infty} \hat{\tau}_k.$$

Recalling Remark 1.1, we have

$$\hat{\tau}^* = \tau^* \quad \mathbb{P} - \text{a.s.} \quad (4.1)$$
Therefore we only need to show \( \mathbb{P}\{\hat{\tau} > \alpha\} = 1 \). Applying the Itô formula to \( \|u(t)\|^2_{H^{-1}} \) gives

\[
d\|u\|^2_{H^{-1}} = 2 \left( \alpha(t, u) \right) \, dt + \|u(t)\|^2_{H^{-1}} \, dt.
\]

(4.2)

Let \( G \in \mathcal{G} \). On account of the Itô formula, we derive

\[
dG(\|u\|^2_{H^{-1}}) = 2G'(\|u\|^2_{H^{-1}}) \left( \alpha(t, u) \right) \, dt + 2G''(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}} \, dt.
\]

Recall that in Assumption (B),

\[
\mathcal{M}(t) = 2\mathbb{Q}(\|u_s(t)\|_{L^\infty} + \|\mathcal{H}u_s(t)\|_{L^\infty}) \|u(t)\|^2_{H^{-1}} + \|\alpha(t, u)\|^2_{H^{-1}}.
\]

Hence taking expectation, using inequality (2.7), Lemma 2.3 and Assumption (B) we find that for any \( t > 0 \),

\[
\mathbb{E}G(\|u(t)\|^2_{H^{-1}}) - \mathbb{E}G(\|u_0\|^2_{H^{-1}}) + \mathbb{E} \int_0^t G'(\|u\|^2_{H^{-1}}) \left\{ -2 \left( \mathcal{H}u(t, u)_{H^{-1}} + \|\alpha(t, u)\|^2_{H^{-1}} \right) \right\} \, dt
\]

\[
+ \mathbb{E} \int_0^t 2G''(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}} \, dt
\]

\[
\leq \mathbb{E} \left\{ G(\|u_0\|^2_{H^{-1}}) + \mathbb{E} \int_0^t G'(\|u\|^2_{H^{-1}}) \mathcal{M}(t) + 2G''(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}} \right\} \, dt
\]

\[
\leq \mathbb{E} G(\|u_0\|^2_{H^{-1}}) + K_1 t + \mathbb{E} \int_0^t K_2 \left\{ G'(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}} \right\} \, dt,
\]

(4.3)

which shows that there exists a constant \( C = C(u_0, K_1, K_2, t) > 0 \) such that

\[
\mathbb{E} \int_0^t \frac{G'(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}}}{1 + G(\|u\|^2_{H^{-1}})} \, dt \leq C(u_0, K_1, K_2, t).
\]

(4.4)

Moreover, for any \( T > 0 \), it follows from Assumption (B) and the BDG inequality that

\[
\mathbb{E} \sup_{t \leq T} G(\|u\|^2_{H^{-1}})
\]

\[
\leq \mathbb{E} G(\|u_0\|^2_{H^{-1}}) + C \mathbb{E} \left\{ \int_0^T \left\{ G'(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}} \right\} \, dt \right\}^\frac{1}{2}
\]

\[
+ K_1 T + K_2 \mathbb{E} \int_0^T \frac{G'(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}}}{1 + G(\|u\|^2_{H^{-1}})} \, dt
\]

\[
\leq \mathbb{E} G(\|u_0\|^2_{H^{-1}}) + \frac{1}{2} \mathbb{E} \sup_{t \leq T} (1 + G(\|u\|^2_{H^{-1}})) + C \mathbb{E} \int_0^T \frac{G'(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}}}{1 + G(\|u\|^2_{H^{-1}})} \, dt
\]

\[
+ K_1 T + K_2 \mathbb{E} \int_0^T \frac{G'(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}}}{1 + G(\|u\|^2_{H^{-1}})} \, dt.
\]

Thus, using (4.4) we obtain

\[
\mathbb{E} \sup_{t \leq T} G(\|u\|^2_{H^{-1}}) \leq C(u_0, K_1, T) + C(K_2) \mathbb{E} \int_0^T \frac{G'(\|u\|^2_{H^{-1}}) \|\alpha(t, u)\|^2_{H^{-1}}}{1 + G(\|u\|^2_{H^{-1}})} \, dt
\]

\[
\leq C(u_0, K_1, K_2, T).
\]

We can infer from the above estimate that

\[
\mathbb{P}\{\hat{\tau}^* < T\} \leq \mathbb{P}\{\tau_k < T\} \leq \mathbb{P}\left\{ \sup_{t \leq T} G(\|u\|^2_{H^{-1}}) \geq G(\hat{k}^2) \right\} \leq \frac{C(u_0, K_1, K_2, T)}{G(\hat{k}^2)}.
\]

Therefore, since \( \lim_{x \to x_0} G(x) = \infty \), one can send \( k \to \infty \) to identify that \( \mathbb{P}\{\hat{\tau}^* < T\} = 0 \). Since \( T > 0 \) is arbitrary, we have that \( \mathbb{P}\{\hat{\tau}^* = \infty\} = 1 \) which shows the desired assertion.
**Remark 4.1.** We remark that the using of Lyapunov is motivated by the non-explosion text [36], see also [7, 49, 51]. In the above proof, (2.8) is used to obtain (4.3), but we remark that (2.9) is also used implicitly. Indeed, as in the proof of (3.23), (2.9) is used to obtain (3.26), and here (4.1) also requires (2.9) because it is a consequence of (3.23). In this work we estimate $H^s$ norm (i.e., $\sup_{t \in [0,T]} \mathbb{E}[\|u\|_{H^s-1}^2]$) and use the fact (4.1) to prove global existence. Let us stress that this is different from the recent work [51], where the authors estimate $H^s$ norm of the solution to show global existence.

5. PROOF OF THEOREM 1.3

Due to the linear nature of the noise we use the Girsanov type transformation

$$v = \frac{1}{\beta(\omega, t)} u, \quad \beta(\omega, t) = e^{\int_0^t b'(\omega, t, x) dW_t - \frac{1}{2} \int_0^t b''(\omega, t, x) dt}. \tag{5.1}$$

In the following lemma we show that the defined process $v$ is the solution to a random PDE enjoying desired regularity properties.

**Lemma 5.1.** Let $s > 3$, $b(t)$ satisfies assumption (C) and fix $S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W)$. Assume that $u_0(\omega, x)$ is an $H^s$-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}[\|u_0\|_{H^s}^2] < \infty$ and $(u, \tau^*)$ is the corresponding unique maximal pathwise solution to (1.4). Then for $t \in [0, \tau^*)$, the process $v$ as in (5.1) is a solution $\mathbb{P}$-a.s. to

$$
\begin{align*}
\frac{dv}{\beta} + \beta(\mathcal{H}v) dx &= 0, \\
v(\omega, 0, x) &= u_0(\omega, x), \quad x \in \mathbb{R},
\end{align*}
$$

with $v \in C([0, \tau^*]; H^s) \cap C^1([0, \tau^*]; H^{s-1}) \mathbb{P}$-a.s.

**Proof.** Applying Theorem 1.1 for the particular case $b(t, u) = b(t)u$ and noticing that $b(t)$ satisfies Assumption (C) (and therefore $b(t, u) = b(t)u$ satisfies Assumption (A)), we infer that equation (1.4) has a unique maximal pathwise solution $(u, \tau^*)$. Itô’s formula yields

$$
\frac{d}{\beta} = -b(t)\frac{1}{\beta} dW + b'(t)\frac{1}{\beta} dt,
$$

and hence straightforward computation shows that

$$
\frac{dv}{\beta} = \beta(\mathcal{H}v) dx dt,
$$

yielding the first equation in (5.2). At time $t = 0$, $v(\omega, 0, x) = u_0(\omega, x)$ since $\beta(\omega, 0) = 1$ almost surely, so $v$ satisfies (5.2) almost surely. Furthermore, Theorem 1.1 shows that $u \in C([0, \tau^*]; H^s) \mathbb{P}$-a.s., hence $v \in C([0, \tau^*]; H^s)$ and $v \in C^1([0, \tau^*]; H^{s-1}) \mathbb{P}$-a.s. \hfill \square

Invoking Lemma 5.1 we have that a.e. $\omega \in \Omega$ the process $v(\omega, t, x)$ solves (5.2) on $[0, \tau^*)$ and $v \in C([0, \tau^*]; H^s) \cap C^1([0, \tau^*]; H^{s-1})$ for $s > 3$. In particular, by the Sobolev embedding, $v \in C([0, \tau^*]; C^1)$, therefore for a.e. $\omega \in \Omega$, the particle trajectory mapping related to the process $v$ given by

$$
\begin{align*}
\frac{d(\phi(\omega, t, x))}{dt} &= \beta(\omega, t)\mathcal{H}v(\omega, t, \phi(\omega, t, x)), \quad t \in [0, \tau^*), \\
\phi(\omega, 0, x) &= x, \quad x \in \mathbb{R},
\end{align*}
$$

has a unique solution $\phi(\omega, t, x) \in C^1([0, \tau^*) \times \mathbb{R})$. Now for a.e. $\omega \in \Omega$, we let $x_0 = x_0(\omega) \in \mathbb{R}$ be the point that $u_0$ attains it global maximum, i.e.,

$$
v(\omega, 0, x_0(\omega)) = u_0(\omega, x_0(\omega)) = \max_{x \in \mathbb{R}} u_0(\omega, x), \quad \text{for a.e. } \omega \in \Omega.
$$

Then we focus on the particle trajectory mapping from $x_0$ in (5.4), i.e.,

$$
\begin{align*}
\frac{d(\phi(\omega, t, x))}{dt} &= \beta(\omega, t)\mathcal{H}v(\omega, t, \phi(\omega, t, x_0)), \quad t \in [0, \tau^*), \\
\phi(\omega, 0, x_0) &= x_0.
\end{align*}
$$

On the other hand, by the transport nature of equation (5.2) the value of $v$ is constant along characteristics and since $v(\omega, 0, x_0)$ attains a global maximum, we have that

$$
\partial_x v(\omega, t, \phi(\omega, t, x_0)) = 0, \quad t \in [0, \tau^*) \mathbb{P}$-a.s. \tag{5.6}
$$
Computing the quantity $\Lambda v(\omega, t, \phi(\omega, t, x_0))$, i.e., the evolution of the $\Lambda$ operator of $v$ along the trajectory, we see that for a.e. $\omega \in \Omega$ and $t \in [0, \tau^*)$, there holds
\[
\frac{d\Lambda v(\omega, t, \phi(\omega, t, x_0))}{dt} = \Lambda v(t, \phi(\omega, t, x_0)) + \frac{d\Lambda v(\omega, t, \phi(\omega, t, x_0))}{d\phi(\omega, t, x_0)} \frac{d\phi(\omega, t, x_0)}{dt}
\]
\[
= - \beta(\omega, t) \{ [\mathcal{H}v](\omega, t, \phi(\omega, t, x_0)) - \mathcal{H}v(\omega, t, \phi(\omega, t, x_0)) \} \Lambda v(\omega, t, \phi(\omega, t, x_0)),
\]
(5.7)
where chain rule is used in the former equality and the fact that $v$ solves (5.2) and the particle trajectory equation (5.5) in the latter. Let $\tilde{v} = \mathcal{H}v$. Then we have $v_x = \Lambda \tilde{v}$ and $\Lambda v_x = -\tilde{v}_{xx}$ and then we can rewrite equation (5.7) as
\[
\frac{d\Lambda v(\omega, t, \phi(\omega, t, x_0))}{dt} = - \beta(\omega, t) \{ [\mathcal{H}v](\omega, t, \phi(\omega, t, x_0)) + \tilde{v}(\omega, t, \phi(\omega, t, x_0)) \tilde{v}_{xx}(\omega, t, \phi(\omega, t, x_0)) \}.
\]
(5.8)
Now we denote $z_0 \triangleq \phi(\omega, t, x_0)$ and omit the dependence of $\omega$ and $t$ in (5.8) for simplicity if there is no ambiguity.

5.1. An identity for the fractional Laplacian $\Lambda$.

**Lemma 5.2.** For a.e. $\omega \in \Omega$ and $t \in [0, \tau^*)$, there holds the following equation
\[
\Lambda(\tilde{v}\Lambda \tilde{v})(z_0) + \tilde{v}(z_0)\tilde{v}_{xx}(z_0) = -\frac{1}{2} (\Lambda v(z_0))^2 - \frac{1}{\pi} \left\| \mathcal{H}v(z_0) - \mathcal{H}v(z) \right\|^2_{L^2}.
\]
(5.9)

**Proof.** We remark that (5.9) has been obtained in [53, Proposition 3.5] in the deterministic case. However, notice that one cannot assume without loss of generality (as in the deterministic case) that $\tilde{v}(z_0) = 0$ which simplifies the proof of (5.9) (see Remark 1.4) and hence we present also the complete proof here.

Recalling (5.6) and $v_x = \Lambda \tilde{v}$, we have $\Lambda \tilde{v}(z_0) = 0$ for $t \in [0, \tau^*)$ almost surely. Then, invoking the integral representation (2.5) for $\alpha = 1$, we arrive at
\[
\Lambda(\tilde{v}\Lambda \tilde{v})(z_0) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\tilde{v}(y)\tilde{v}(z_0) - \tilde{v}(y)\Lambda \tilde{v}(y)}{|z_0 - y|^2} dy
\]
\[
= \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{-\tilde{v}(y)\Lambda \tilde{v}(y)}{|z_0 - y|^2} dy, \quad t \in [0, \tau^*) \quad \mathbb{P} - \text{a.s.}
\]
(5.10)
On the other hand, we have that
\[
\tilde{v}(z_0)\tilde{v}_{xx}(z_0) = -\tilde{v}(z_0)\Lambda(\Lambda \tilde{v})(z_0) = \tilde{v}(z_0) - \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\Lambda \tilde{v}(y)}{|z_0 - y|^2} dy, \quad t \in [0, \tau^*) \quad \mathbb{P} - \text{a.s.}
\]
where we have used in the first equality the fact that $\partial_{xx} = -\Lambda^2$ and the semigroup property of the fractional Laplacian operator $\Lambda^2 = \Lambda \Lambda$. Therefore, we have that
\[
\Lambda(\tilde{v}\Lambda \tilde{v})(z_0) + \tilde{v}(z_0)\tilde{v}_{xx}(z_0) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\tilde{v}(z_0) - \tilde{v}(y)) \Lambda \tilde{v}(y)}{|z_0 - y|^2} dy, \quad t \in [0, \tau^*) \quad \mathbb{P} - \text{a.s.}
\]
(5.11)
Recalling the notation $\tilde{v}(z_0) = \tilde{v}(\omega, t, z_0)$, then for a.e. $\omega \in \Omega$ and $t \in [0, \tau^*)$ we define
\[
\tilde{\eta}(\omega, t, y) \triangleq \tilde{v}(\omega, t, z_0) - \tilde{v}(\omega, t, y).
\]
(5.12)
Since $\tilde{\eta}(\omega, t, z_0) = 0$, factorizing the root implies that there exists a process $\eta(\omega, t, y)$ such that
\[
\tilde{\eta}(\omega, t, y) = (z_0 - y)\eta(\omega, t, y), \quad t \in [0, \tau^*) \quad \mathbb{P} - \text{a.s.}
\]
(5.13)
Therefore we can observe that
\[
\eta(\omega, t, z_0) = -\tilde{\eta}(\omega, t, z_0), \quad \mathcal{H}\eta(\omega, t, z_0) = \Lambda \tilde{\eta}(\omega, t, z_0), \quad t \in [0, \tau^*) \quad \mathbb{P} - \text{a.s.}
\]
(5.14)
Again, we drop $\omega$ and $t$ if there is no ambiguity. Then (5.11) reduces to
\[
\Lambda(\tilde{v}\Lambda \tilde{v})(z_0) + \tilde{v}(z_0)\tilde{v}_{xx}(z_0) = -\frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\tilde{\eta}(y)\Lambda \tilde{\eta}(y)}{|z_0 - y|^2} dy
\]
\[
= -\frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{(z_0 - y)\eta(y)\Lambda((z_0 - \cdot)\eta(\cdot))}{|z_0 - y|^2} dy, \quad t \in [0, \tau^*) \quad \mathbb{P} - \text{a.s.}
\]
Using the linearity of the fractional Laplacian operator and Lemma 2.4, we have
\[
\Lambda((z_0 - \cdot)\eta(\cdot))(y) = z_0\Lambda\eta(y) - y\Lambda\eta(y) + \mathcal{H}\eta(y) = (z_0 - y)\Lambda\eta(y) + \mathcal{H}\eta(y)
\]
\[
\Lambda((z_0 - \cdot)\eta(\cdot))(y) = z_0\Lambda\eta(y) - y\Lambda\eta(y) + \mathcal{H}\eta(y) = (z_0 - y)\Lambda\eta(y) + \mathcal{H}\eta(y)
\]
and hence
\[
A(\tilde{v}_x)(z_0) + \tilde{v}(z_0)\tilde{v}_{xx}(z_0) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{-(z_0 - y)\eta(y) \{ (z_0 - y)\Lambda\eta(y) + \mathcal{H}\eta(y) \}}{|z_0 - y|^2} \, dy \\
= \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{(z_0 - y)^2\eta(y)\Lambda\eta(y)}{|z_0 - y|^2} \, dy + \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \eta(y)\Lambda\eta(y) \, dy \\
= \frac{1}{\pi} ||\Lambda^{1/2}\eta||^2 + [\mathcal{H}(\eta\Lambda\eta)](z_0), \quad t \in [0, \tau^*) \quad \mathbb{P} - a.s.
\]

Therefore, applying the identity (2.10) we can rewrite the above equation as
\[
A(\tilde{v}_x)(z_0) + \tilde{v}(z_0)\tilde{v}_{xx}(z_0) = \frac{1}{\pi} (\eta(z_0))^2 + \frac{1}{\pi} (\Lambda\eta(z_0))^2 - \frac{1}{\pi} ||\Lambda^{1/2}\eta||^2, \quad t \in [0, \tau^*) \quad \mathbb{P} - a.s.
\]

Using (5.12), (5.13), (5.14) and noticing \( \tilde{v}(z_0) = 0 \), we arrive at
\[
A(\tilde{v}_x)(z_0) + \tilde{v}(z_0)\tilde{v}_{xx}(z_0) = -\frac{1}{\pi} \beta(\eta(z_0))^2 + \frac{1}{\pi} (\Lambda\eta(z_0))^2 - \frac{1}{\pi} ||\eta(z_0)||^2 \\
= \frac{1}{\pi} \left( \tilde{v}(z_0) - \tilde{v}(z_0) \right) \tilde{v}(z_0) - \frac{1}{\pi} ||\tilde{v}(z_0) - \tilde{v}(z_0)||^2, \quad t \in [0, \tau^*) \quad \mathbb{P} - a.s.
\]

Then (5.9) is a direct consequence of the above equation and the fact \( \tilde{v} = \mathcal{H}v \).

5.2. Proof of Theorem 1.3. Now we are in the position to prove Theorem 1.3. To begin with, we can infer from Lemma 5.2 and (5.8) that
\[
\frac{dA\tilde{v}(z_0)}{dt} = \beta \left[ \frac{1}{2} (A\tilde{v}(z_0))^2 + \frac{1}{\pi} \left| \frac{\mathcal{H}(v(z_0) - \mathcal{H}(v))}{z_0 - \cdot} \right|^2 \right] \geq \frac{1}{2} (A\tilde{v}(z_0))^2, \quad t \in [0, \tau^*) \quad \mathbb{P} - a.s.
\]

Let \( F(\omega, t) \triangleq A\tilde{v}(\omega, t, z_0) \). Then the above estimate becomes
\[
\frac{dF(\omega, t)}{dt} \geq \frac{1}{2} \beta(\omega, t)F^2(\omega, t), \quad t \in [0, \tau^*) \quad \mathbb{P} - a.s.
\]

Let \( 0 < K < 1 \) and define \( \Omega^* \triangleq \{ \omega : \beta(t) \geq K e^{-\frac{K}{K}} \text{ for all } t \} \). If \( F(\omega, 0) > \frac{K}{K} \) almost surely, then \( \tau^* < \infty \) for a.e. \( \omega \in \Omega^* \). Indeed, integrating (5.15) leads to
\[
-\frac{1}{F(\omega, t)} + \frac{1}{F(\omega, 0)} \geq \frac{1}{2} \int_0^t \beta(\omega, t') \, dt', \quad t \in [0, \tau^*) \quad \mathbb{P} - a.s.
\]

Since \( F(\omega, 0) > \frac{K}{K} \) almost surely, (5.15) means that \( F \) is increasing almost surely. Therefore we restrict the above inequality to \( \omega \in \Omega^* \) and we arrive at
\[
\frac{1}{F(\omega, 0)} \geq \frac{1}{2} K \int_0^{\tau^*} e^{-\frac{K}{K} t'} \, dt' = \frac{K}{b^*} \left( 1 - e^{-\frac{K}{K} \tau^*} \right)
\]

and hence
\[
\frac{1}{F(\omega, 0)} - \frac{K}{b^*} \left( 1 - e^{-\frac{K}{K} \tau^*} \right) \geq 0.
\]

By the assumption \( F(\omega, 0) > \frac{K}{K} \) almost surely, we arrive at
\[
\frac{K}{b^*} e^{-\frac{K}{K} \tau^*} \geq \frac{K}{b^*} - \frac{1}{F(0)} > 0, \quad \text{a.e. } \omega \in \Omega^*.
\]

Thus \( \tau^* < \infty \) a.e. on \( \Omega^* \) as desired. Recalling that \( \beta(t) = e^{\int_0^t b(t')dW_t} - \int_0^t e^{-\frac{K}{K} t'} \, dt' \), we have shown that
\[
\mathbb{P}\{ \tau^* < \infty \} \geq \mathbb{P}\{ \beta(t) \geq K e^{-\frac{K}{K}} \text{ for all } t \}
\]

which implies that
\[
\mathbb{P}\{ \tau^* < \infty \} \geq \mathbb{P}\{ e^{\int_0^t b(t')dW_t} > K \text{ for all } t \}
\]

since \( b^2(t) < b^* \) for all \( t > 0 \). The proof is now complete.
6. Proof of Theorem 1.4

In this section, we provide the proof of Theorem 1.4. As is mentioned in Remark 1.5, since we cannot get an explicit expression of the solution to (1.2), we start with constructing some approximate solutions from which (1.22) can be established. Similarly as before we divide the proof into several subsections.

6.1. Approximate solutions and actual solutions. Following [44], we construct the approximate solution as follows. First, we fix two functions \( \phi, \tilde{\phi} \in C_c^\infty \) such that
\[
\phi(x) = \begin{cases} 
1, & \text{if } |x| < 1, \\
0, & \text{if } |x| \geq 2,
\end{cases} \quad \text{and} \quad \tilde{\phi}(x) = 1 \text{ if } x \in \text{supp } \phi. \tag{6.1}
\]
Next, we construct the following sequence of approximate solutions
\[
u_{m,n} = u_h + u_l, \quad m \in \{-1, 1\}, \tag{6.2}
\]
where
- \( u_h = u_{h,m,n} \) is the high-frequency part defined by
  \[
u_h = u_{h,m,n}(t, x) = n^{-\frac{1}{2}-s} \phi \left( \frac{x}{n^s} \right) \cos(nx - nt), \quad n \in \mathbb{N}. \tag{6.3}
\]
- \( u_l = u_{l,m,n} \) is the low-frequency part defined as the solution to the following problem:
  \[
\begin{aligned}
&\partial_t u_l + (Hu_l) \partial_x u_l = 0, \quad x \in \mathbb{R}, \quad t > 0, \\
u_l(0, x) &= -H \left( mn^{-1} \tilde{\phi} \left( \frac{x}{n^s} \right) \right), \quad x \in \mathbb{R}. \tag{6.4}
\end{aligned}
\]
In (6.3)-(6.4), \( \delta > 0 \) is a parameter that will be determined later in the proof.

Let us consider the problem (1.2) with the initial data \( u_{m,n}(0, x) \), i.e.,
\[
\begin{cases}
du + (Hu) du = h(t, u) dW, & x \in \mathbb{R}, \quad t > 0, \\
u(0, x) = -H \left( mn^{-1} \tilde{\phi} \left( \frac{x}{n^s} \right) \right) + n^{-\frac{1}{2}-s} \phi \left( \frac{x}{n^s} \right) \cos(nx), & x \in \mathbb{R}. \tag{6.5}
\end{cases}
\]
Since Assumption (D) implies Assumption (A), Theorem 1.1 immediately yields that for each fixed \( n \in \mathbb{N} \), (6.5) has a unique pathwise solution \( u^{n,m,n} \) such that \( u^{n,m,n} \in C([0, \tau^{m,n}]; H^s) \) \( \mathbb{P} \) a.s. with \( s > 3 \).

6.2. Estimates on the errors. Substituting (6.2) into (1.2), we define the error \( \mathcal{E}(\omega, t, x) \) as
\[
\mathcal{E}(\omega, t, x) = \nu_{m,n}(t, x) - u_{m,n}(0, x) + \int_0^t (Hu_{m,n}) \partial_x u_{m,n} \, dt' - \int_0^t h(t', u_{m,n}) \, dW \quad \mathbb{P} \text{ a.s.}
\]
By using (6.2) and (6.4), we reformulate \( \mathcal{E}(\omega, t, x) \) as
\[
\begin{aligned}
\mathcal{E}(\omega, t, x) &= u_l(t, x) - u_l(0, x) + \int_0^t (Hu_l) \partial_x u_l \, dt' + u_h(t, x) - u_h(0, x) \\
&\quad + \int_0^t (Hu_l) \partial_x u_h + (Hu_h)(\partial_x u_l + \partial_x u_h) \, dt' - \int_0^t h(t', u_{m,n}) \, dW \\
&= u_h(t, x) - u_h(0, x) + \int_0^t (Hu_l) \partial_x u_h + (Hu_h)(\partial_x u_l + \partial_x u_h) \, dt' - \int_0^t h(t', u_{m,n}) \, dW \quad \mathbb{P} \text{ a.s.} \tag{6.6}
\end{aligned}
\]
The following lemma shows the decay estimate for the low-frequency part of \( u_{m,n} \).

Lemma 6.1. Let \( |m| = 1, s > 3, \delta \in (0, 2) \) and \( n \gg 1 \). Then there exists a \( T_1 > 0 \) such that for all \( n \gg 1 \), the initial value problem (6.4) has a unique smooth solution \( u_l = u_{l,m,n} \in C([0, T_1]; H^s) \) such that \( T_1 \) does not depend on \( n \). Besides, for all fixed \( r > 0 \), there exists a constant \( C = C(r, \tilde{\phi}, T_1) > 0 \) such that \( u_l \) satisfies
\[
\| u_l(t) \|_{H^s} \leq C|m|^{-\frac{1}{2}} n^{-1}, \quad t \in [0, T_1]. \tag{6.7}
\]
Proof. For \( |m| = 1 \) and any fixed \( n \geq 1 \), since \( u_l(0, x) \in H^\infty \), by applying Theorem 1.1 with \( h = 0 \) and deterministic initial data, we see that for any \( s > 3 \), (6.4) has a unique (deterministic) solution \( u_l = u_{l,m,n} \in C([0, T_1]; H^s) \). We will show that there exists a lower bound of the existence time, i.e., there is a \( T_1 > 0 \) such that for all \( n \gg 1 \), \( u_l = u_{l,m,n} \) exists on \([0, T_1]\) and satisfies (6.7). The proof of Lemma 6.1 consists of three main steps.
Step 1: Estimate \( \|u_t(0,x)\|_{H^r} \). Let \( g(x) = mn^{-1}(\frac{x}{n^r}) \). For \( n \gg 1 \), by using (2.7), we have that

\[
\|u_t(0,x)\|^2_{H^r} = \left\| -H\left(mn^{-1}(\frac{x}{n^r})\right) \right\|^2_{H^r} \leq \left\| mn^{-1}(\frac{x}{n^r}) \right\|^2_{H^r} = \int_{\mathbb{R}} \left(1 + |\xi|^2\right)|\hat{\phi}(\xi)|^2 d\xi = mn^{2r-2} \int_{\mathbb{R}} \left(1 + \frac{z}{n^r}\right)^r |\hat{\phi}(z)|^2 dz \leq mn^{2r-2} \int_{\mathbb{R}} \left(1 + |z|^2\right)^r |\hat{\phi}(z)|^2 dz \leq Cn^{2r-2},
\]

for some constant \( C = C_{r,\phi} > 0 \). Therefore we find that

\[
\|u_t(0,x)\|_{H^r} \leq C|n|^{\frac{4}{4}-1}.
\]

Step 2: Proof of (6.7) for \( r > 3/2 \). In this case, we apply Lemma 2.2, (2.7), \( H^r \hookrightarrow W^{1,\infty} \) and integration by parts to find

\[
\frac{1}{2} \frac{d}{dt}\|u_t\|^2_{H^r} = - \int_{\mathbb{R}} D^r u_t D^r ((Hu_t) \partial_x u_t) \, dx \leq |D^r u_t, D^r ((Hu_t) \partial_x u_t)|_{L^2} + |(Hu_t) D^r \partial_x u_t, D^r u_t|_{L^2} \leq \|D^r (Hu_t)\|_{L^2} \|\partial_x u_t\|_{H^r} + \|D^r (Hu_t)\|_{L^2} \|D^r-1 \partial_x u_t\|_{L^2} \|u_t\|_{H^r} + \frac{1}{2} \int_{\mathbb{R}} (Hu_t) \partial_x (D^r u_t)^2 \, dx \leq \|\partial_x u_t\|_{L^2} \|u_t\|_{H^r} + \|D^r (Hu_t)\|_{L^2} \|u_t\|_{H^r} \leq C\|u_t\|_{H^r}, \quad C = C_r > 0.
\]

Solving the above inequality gives

\[
\|u_t\|_{H^r} \leq \frac{\|u_t(0)\|_{H^r}}{1 - Ct\|u_t(0)\|_{H^r}}, \quad 0 \leq t < \frac{1}{C\|u_t(0)\|_{H^r}}.
\]

Remember that \( u_t = u_{t,m,n} \). Then we define the time interval \([0, T_{l,m,n}]\) such that

\[
\|u_t\|_{H^r} \leq \|u_t(0)\|_{H^r}, \quad t \in [0, T_{l,m,n}], \quad T_{l,m,n} = \frac{1}{2C\|u_t(0)\|_{H^r}}.
\]

By Step 1, we have that for \( |m| = 1 \), \( T_{l,m,n} \gg \frac{1}{2C_{n^{\frac{4}{4}}} \to \infty, \quad n \to \infty. \) Therefore we can find a common time interval \([0, T_l]\) such that

\[
\|u_t\|_{H^r} \leq \|u_t(0)\|_{H^r} \leq C|n|^{\frac{4}{4}-1}, \quad t \in [0, T_l], \quad C = C_{r,\phi} > 0,
\]

which is (6.7).

Step 3: Proof of (6.7) for \( 0 < r \leq 3/2 \). Applying Lemma 2.2, (2.7), we have

\[
\frac{1}{2} \frac{d}{dt}\|u_t\|^2_{H^r} = - \int_{\mathbb{R}} D^r u_t D^r ((Hu_t) \partial_x u_t) \, dx \leq |D^r u_t, D^r ((Hu_t) \partial_x u_t)|_{L^2} + |(Hu_t) D^r \partial_x u_t, D^r u_t|_{L^2} \leq \|D^r (Hu_t)\|_{L^2} \|\partial_x u_t\|_{H^r} + \|D^r (Hu_t)\|_{L^2} \|D^r-1 \partial_x u_t\|_{L^2} \|u_t\|_{H^r} + \frac{1}{2} \int_{\mathbb{R}} (Hu_t) \partial_x (D^r u_t)^2 \, dx \leq \|\partial_x u_t\|_{L^2} \|u_t\|_{H^r} + \|D^r (Hu_t)\|_{L^2} \|u_t\|_{H^r} \leq C\|u_t\|_{H^r}, \quad C = C_r > 0.
\]
It follows from the embedding $H^{r+\frac{3}{2}} \hookrightarrow W^{1,\infty}$ that

$$\frac{1}{2} \frac{d}{dt} \| u_t \|_{H^r}^2 \lesssim \| \partial_x u_t \|_{L^\infty} \| u_t \|_{H^r}^2 + \| \partial_x (H u_t) \|_{L^\infty} \| u_t \|_{H^r}^2$$

$$\lesssim \| u_t \|_{W^{1,\infty}} \| u_t \|_{H^r}^2 + \| H u_t \|_{W^{1,\infty}} \| u_t \|_{H^{r+\frac{3}{2}}}^2$$

$$\lesssim \| u_t \|_{H^{r+\frac{3}{2}}}^2 \| u_t \|_{H^r}^2 + \| H u_t \|_{H^{r+\frac{3}{2}}} \| u_t \|_{H^r}^2$$

$$\lesssim \| u_t \|_{H^{r+\frac{3}{2}}} \| u_t \|_{H^r}^2.$$

Using the conclusion of Step 2 for $r + \frac{3}{2} > \frac{5}{2}$, we have

$$\frac{d}{dt} \| u_t \|_{H^r} \lesssim \| u_t \|_{H^r} \| u_t(0) \|_{H^{r+\frac{3}{2}}}, \quad t \in [0, T),$n$ and hence

$$\| u_t(t) \|_{H^r} \lesssim \| u_t(0) \|_{H^r} + \int_0^t \| u_t \|_{H^r} \| u_t(0) \|_{H^{r+\frac{3}{2}}} \, dt, \quad t \in [0, T].$$

Applying Grönwall’s inequality to the above inequality, we have

$$\| u_t \|_{H^r} \lesssim \| u_t(0) \|_{H^r} \exp \left\{ \| u_t(0) \|_{H^{r+\frac{3}{2}}} T \right\}, \quad t \in [0, T].$$

Since $\delta \in (0, 2)$, we can infer from Step 1 that $\exp \left\{ \| u_t(0) \|_{H^{r+\frac{3}{2}}} T \right\} \leq C(\delta, T)$ for some constant $C(\delta, T) > 0$. Therefore we see that there exists a constant $C = C(\delta, T) > 0$ such that

$$\| u_t \|_{H^r} \leq C(x) \| u_t \|_{H^r}, \quad t \in [0, T],$$

concluding the desired estimate (6.7).

The above result implies that the $H^s$-norm of $u_n$, the low-frequency part of the approximate solution defined by (6.2), is decaying. For the high-frequency part $u_h$, due to Lemma 2.5, its $H^s$-norm is bounded. To sum up, let $T_h$ be given in Lemma 6.1, for any $s > 0$, there is a constant $M = M_s(\delta, T_h) > 0$ such that

$$\| u_{m,n}(t) \|_{H^r} \lesssim M, \quad t \in [0, T_h].$$

(6.10)

Moreover, although not strictly necessary, we can infer from (6.9) that $M$ can be independent of $T_h$ when $s > 3/2$.

6.2.1. Estimating the error $E$. Recall (6.6). By using (6.1), we have that $\phi = \tilde{\phi} \cdot \phi$. Then by (6.3) and $u_t(0, x)$ in (6.4), we see that for $m \in \{-1, 1\}$,

$$u_h(t, x) - u_0(t, x)$$

$$= n^{-\frac{1}{2} + s} \left( \frac{x}{n^s} \right) \cos(nx - mt) - n^{-\frac{1}{2} - s} \left( \frac{x}{n^s} \right) \cos(nx)$$

$$= m^{-1} n^{-\frac{1}{2} + s} \left( \frac{x}{n^s} \right) \cos(nx - mt) - m^{-1} n^{-\frac{1}{2} - s} \left( \frac{x}{n^s} \right) \cos(nx)$$

$$= m^{-1} (H u_0(t, x)) n^{-\frac{1}{2} + s} \left( \frac{x}{n^s} \right) \cos(nx - mt) - m^{-1} (H u_0(t, x)) n^{-\frac{1}{2} - s} \left( \frac{x}{n^s} \right) \cos(nx)$$

$$= \int_0^t (H u_t(0, x)) n^{-\frac{1}{2} + s} \left( \frac{x}{n^s} \right) \sin(nx - m't) \, dt'.$$

Furthermore,

$$\int_0^t (H u_t) \partial_x u_h \, dt' = - \int_0^t (H u_t) n^{-\frac{1}{2} - s} \left( \frac{x}{n^s} \right) \sin(nx - m't) \, dt'$$

$$+ \int_0^t (H u_t) n^{-\frac{1}{2} - s} \partial_x \phi \left( \frac{x}{n^s} \right) \cos(nx - m't) \, dt'.$$
Thus, (6.6) becomes

\[
E(\omega, t, x) = \int_0^t \left[ (H u_t)(0) - (H u_t)(t') \right] n^{1-s} \phi \left( \frac{x}{nt} \right) \sin(nx - mt') dt'
\]

\[
+ \int_0^t (H u_t)(t') n^{-\frac{2s}{n^2}} \partial_x \phi \left( \frac{x}{nt} \right) \cos(nx - mt') dt'
\]

\[
+ \int_0^t (H u_t)(t') \partial_x u_t(t') dt' - \int_0^t h(t', u_{m,n}) dW
\]

\[
= \int_0^t E dt' - \int_0^t h(t', u_{m,n}) dW \quad \mathbb{P} \text{- a.s.}, \tag{6.12}
\]

where

\[
E = E(\omega, t, x) = \left[ (H u_t)(0) - (H u_t)(t) \right] n^{1-s} \phi \left( \frac{x}{nt} \right) \sin(nx - mt)
\]

\[
+ (H u_t)(t) n^{-\frac{2s}{n^2}} \partial_x \phi \left( \frac{x}{nt} \right) \cos(nx - mt) + (H u_t)(\partial_x u_t + \partial_x u_h).
\]

Now we shall estimate the $H^{s_0}$-norm of the error $E$, where $s_0$ is given in Assumption (D). Actually, we will show that the $H^{s_0}$-norm of $E$ is decaying.

**Lemma 6.2.** Let $n \gg 1$, $s > 3$, $\frac{3}{4} < \delta < 1$. Let $T_1$ be given in Lemma 6.1, and $s_0$ be given in Assumption (D). Let

\[
r_s = -s - 1 + s_0 + \delta. \tag{6.14}
\]

Then $r_s < 0$ and the error $E$ given by (6.12) satisfies

\[
\mathbb{E} \sup_{t \in [0, T_1]} \| E(t) \|^2_{H^{s_0}} \leq C n^{2r_s},
\]

where $C = C(s_0, \tilde{\phi}, \phi, T_1) > 0$ is a constant independent of $n$.

**Proof.** It is obvious by construction that $r_s$ given by (6.14) is negative. Combining (6.13), the embedding $H^{s_0} \hookrightarrow L^\infty$ and Lemmas 6.1 and 2.5, we find that for $t \in [0, T_1],$

\[
\| E \|_{H^{s_0}} \leq \left\| (H u_t)(0) - (H u_t)(t) n^{1-s} \phi \left( \frac{x}{nt} \right) \sin(nx - mt) \right\|_{H^{s_0}}
\]

\[
+ \| (H u_t)(t) n^{-\frac{2s}{n^2}} \partial_x \phi \left( \frac{x}{nt} \right) \cos(nx - mt) \|_{H^{s_0}}
\]

\[
+ \| (H u_t) \partial_x u_l \|_{H^{s_0}} + \| (H u_t) \partial_x u_h \|_{H^{s_0}}
\]

\[
\leq n^{1-s} \left\| (H u_t)(0) - (H u_t)(t) \phi \left( \frac{x}{nt} \right) \sin(nx - mt) \right\|_{H^{s_0}}
\]

\[
+ n^{-\frac{2s}{n^2}} \left\| (H u_t)(t) \partial_x \phi \left( \frac{x}{nt} \right) \cos(nx - mt) \right\|_{H^{s_0}}
\]

\[
+ \| (H u_t) \partial_x u_l \|_{H^{s_0}} + \| (H u_t) \partial_x u_h \|_{H^{s_0}}
\]

\[
\leq n^{1-s+s_0} \left\| (H u_t)(0) - (H u_t)(t) \right\|_{H^{s_0}} + n^{s_0-1} \left\| (H u_t) \partial_x u_l \right\|_{H^{s_0}} + \| (H u_t) \partial_x u_h \|_{H^{s_0}}
\]

\[
\leq n^{1-s+s_0} \left\| (H u_t)(0) - (H u_t)(t) \right\|_{H^{s_0}} + n^{s_0} + \| (H u_t) \partial_x u_l \|_{H^{s_0}} + \| (H u_t) \partial_x u_h \|_{H^{s_0}}. \tag{6.15}
\]

For $\| (H u_t)(0) - (H u_t)(t) \|_{H^{s_0}},$ it follows from the fundamental theorem of calculus and $H^{s_0} \hookrightarrow L^\infty$ that for $t \in [0, T_1],$

\[
\| (H u_t)(0) - (H u_t)(t) \|_{H^{s_0}} \leq \| u_l(0) - u_l(t) \|_{H^{s_0}}
\]

\[
= \left\| \int_0^t \partial_t u_l(t') dt' \right\|_{H^{s_0}}
\]

\[
\leq \int_0^t \| (H u_t) \partial_x u_l \|_{H^{s_0}} dt'
\]

\[
\leq \int_0^t \| u_l \|_{H^{s_0+1}} ^2 dt' \leq n^{s_0} T_1,
\]

where we used (6.4) with $t \in [0, T_1]$, Lemma 6.1 and the embedding $H^{s_0+1} \hookrightarrow W^{1,\infty}$. Therefore,

\[
n^{1-s+s_0} \| (H u_t)(0) - (H u_t)(t) \|_{H^{s_0}} \leq n^{1-s+s_0-s-2} T_1 = n^{r_s} T_1, \quad t \in [0, T_1]. \tag{6.16}
\]
Next, applying Lemma 6.1 and 2.5, we have for $t \in [0,T]$,\[
\| (\mathcal{H} u_h) \partial_x u_t \|_{H^{r_0}} \lesssim \| u_h \|_{H^s} \| u_t \|_{H^{s+1}} \lesssim n^{-s+\sigma_0 + \frac{4}{3} - 1} \lesssim n^{r_*}, \tag{6.17}
\]
and
\[
\| (\mathcal{H} u_h) \partial_x u_h \|_{H^{r_0}} \lesssim \| u_h \|_{H^s} \| u_h \|_{H^{s+1}} \lesssim n^{-s+\sigma_0 + \frac{4}{3} + 1} \lesssim n^{r_*}. \tag{6.18}
\]
Here in (6.18) we used the assumption $\sigma_0 \in (3/2, 7/4)$ to guarantee $n^{-2s+2\sigma_0 + 1} \lesssim n^{r_*}$. Inserting (6.16), (6.17) and (6.18) into (6.15), we finally obtain
\[
\| E \|_{H^{r_0}} \lesssim n^{r_*}, \quad t \in [0,T]. \tag{6.19}
\]
With (6.19) at hand, we are in the position to estimate $\mathbb{E} \sup_{t \in [0,T]} \| \mathcal{E}(t) \|^2_{H^{r_0}}$. Invoking Itô formula in (6.12) leads to
\[
\| \mathcal{E}(t,x) \|^2_{H^{r_0}} \leq -2 \int_0^t (h(t', u_{m,n}) \, dW, \mathcal{E})_{H^{r_0}} + 2 \int_0^t \| (E, \mathcal{E})_{H^{r_0}} \| \, dt' + \int_0^t \| h(t', u_{m,n}) \|^2_{L^2(L_t,H^{r_0})} \, dt'.
\]
Taking supremum with respect to $t \in [0,T]$ and using the BDG inequality, we can find some $C > 0$ such that
\[
\mathbb{E} \sup_{t \in [0,T]} \| \mathcal{E}(t) \|^2_{H^{r_0}} \leq \frac{1}{2} \mathbb{E} \sup_{t \in [0,T]} \| \mathcal{E}(t) \|^2_{H^{r_0}} + C \mathbb{E} \int_0^T \| h(t, u_{m,n}) \|^2_{L^2(L_t,H^{r_0})} \, dt + C \int_0^T \mathbb{E} \| E \|^2_{H^{r_0}} + \mathbb{E} \| \mathcal{E}(t) \|^2_{H^{r_0}} \, dt.
\]
By virtue of (6.19), we arrive at
\[
\mathbb{E} \sup_{t \in [0,T]} \| \mathcal{E}(t) \|^2_{H^{r_0}} \lesssim T \mathbb{E} \| h(t, u_{m,n}) \|^2_{L^2(L_t,H^{r_0})} \, dt + \int_0^T \mathbb{E} \sup_{t' \in [0,t]} \| \mathcal{E}(t') \|^2_{H^{r_0}} \, dt.
\]
Now, we estimate $\| h(t, u_{m,n}) \|^2_{L^2(L_t,H^{r_0})}$. For any fixed $s > 3$, on account of Assumption (D), Lemmas 2.5 and 6.1, we can pick $\kappa > 2(1 + \frac{4-\sigma_0}{2-\delta})$ such that
\[
\| h(t, u_{m,n}) \|^2_{L^2(L_t,H^{r_0})} \lesssim \left( \frac{1}{e^{s+\sigma_0}} \right)^2 \lesssim \left( n^{-s+\sigma_0 + \frac{4}{3} - 1} \right)^{2\kappa} \lesssim n^{2r_*},
\]
which gives
\[
\mathbb{E} \sup_{t \in [0,T]} \| \mathcal{E}(t) \|^2_{H^{r_0}} \lesssim T n^{2r_*} + \int_0^T \mathbb{E} \sup_{t' \in [0,t]} \| \mathcal{E}(t') \|^2_{H^{r_0}} \, dt.
\]
Obviously, for each $n \geq 1$, $\mathbb{E} \sup_{t \in [0,T]} \| \mathcal{E}(t) \|^2_{H^{r_0}}$ is finite and $T_1 > 0$ is fixed. Then by the Grönwall inequality, we have
\[
\mathbb{E} \sup_{t \in [0,T]} \| \mathcal{E}(t) \|^2_{H^{r_0}} \leq C n^{2r_*}.
\]
The proof is completed. \(\square\)

6.2.2. Estimating $u_{m,n} - u_{m,n}$. Recall the approximate solutions $u_{m,n}$ given by (6.2). Then we have the following estimates on the difference between the actual solutions and the approximate solutions.

Lemma 6.3. Let $s > 3$, $\frac{4}{3} < \delta < 1$, $\sigma_0$ be given in Assumption (D) and $r_0 < 0$ be given in (6.14). For any $R > 1$, we define
\[
\tau_R^{m,n} := \inf \{ t > 0 : \| u_{m,n} \|_{H^s} > R \}.
\]
Then for $n \gg 1$ and $T_1 > 0$ given in Lemma 6.1,
\[
\mathbb{E} \sup_{t \in [0,T] \wedge \tau_R^{m,n}} \| u_{m,n} - u_{m,n} \|^2_{H^{r_0}} \leq C n^{2r_*}, \tag{6.21}
\]
and
\[
\mathbb{E} \sup_{t \in [0,T] \wedge \tau_R^{m,n}} \| u_{m,n} - u_{m,n} \|^2_{H^{2s-\sigma_0}} \leq C n^{2s-2\sigma_0}, \tag{6.22}
\]
where $C = C(s, \sigma_0, \tilde{\phi}, \phi, T_1, R) > 0$ is a constant independent of $n$. 

Proof. Let \( v = v_{m,n} = u_{m,n} - u_{m,n} \). Then \( v \) satisfies \( v(0) = 0 \) and

\[
v(t) + \int_0^t \left( (Hv) \partial_x u_{m,n} + (Hu_{m,n}) \partial_x v \right) dt' = -\int_0^t h(t',u_{m,n}) dW + \int_0^t E dt'.
\]

where (6.12) is used. For \( T_1 > 0 \), we use the Itô formula to find

\[
\|v(t)\|_{H^{s,0}}^2 = -2 \int_0^t \left( h(t',u_{m,n}) dW, v \right)_{H^{s,0}} + 2 \int_0^t \left( (E,v)_{H^{s,0}} dt' - 2 \int_0^t ((Hu_{m,n}) \partial_x v, v)_{H^{s,0}} dt'
\]

Taking supremum with respect to \( t \in [0, T_1 \land \tau_{R, m}^n] \), and then using the BDG inequality yield that for some \( C > 0 \),

\[
E \sup_{t \in [0, T_1 \land \tau_{R, m}^n]} \|v(t)\|_{H^{s,0}}^2 \\
\leq C E \left( \int_0^{T_1 \land \tau_{R, m}^n} \|h(t,u_{m,n})\|_{L_2(\mathcal{U},H^{s,0})}^2 dt \right)^{1/2} + 2E \int_0^{T_1 \land \tau_{R, m}^n} |(E,v)_{H^{s,0}}| dt
\]

Recall that (6.10) gives \( \|u_{m,n}\|_{H^s} \leq M_{s,\bar{\phi},\tilde{\phi},T_1} \) on \( t \in [0, T_1 \land \tau_{R, m}^n] \). Hence we can infer from Assumption (D) that for some \( C > 0 \)

\[
\|h(t,u_{m,n})\|_{L_2(\mathcal{U},H^{s,0})} \leq \|h(t,u_{m,n})\|_{L_2(\mathcal{U},H^{s,0})}^2 + \|h(t,u_{m,n}) - h(t,u_{m,n})\|_{L_2(\mathcal{U},H^{s,0})}^2
\]

where \( q(\cdot) \) is given in (1.8). As a result, for any fixed \( s > 3 \), by applying Lemmas 2.5 and 6.1 again, we can pick \( \lambda > 2(1 + \frac{2s-\sigma}{2s}) \) to derive

\[
\|h(t,u_{m,n})\|_{L_2(\mathcal{U},H^{s,0})} \lesssim \|u_{m,n}\|_{H^{s,0}}^{2\lambda} + \|v\|_{H^{s,0}}^2
\]

Via (6.19), we have

\[
2 \|(E,v)_{H^{s,0}}\| \leq 2E \|E\|_{H^{s,0}} \|v\|_{H^{s,0}} \lesssim \|E\|_{H^{s,0}}^2 + \|v\|_{H^{s,0}}^2 \lesssim n^{2s} + \|v\|_{H^{s,0}}^2.
\]

Using Lemma 2.2, (6.10), (2.7), integration by parts, and the embedding \( H^s \hookrightarrow H^0 \hookrightarrow W^{1,\infty} \), we obtain that the almost surely

\[
|((Hv) \partial_x u_{m,n}, v)_{H^{s,0}}| \lesssim \|Hv\|_{H^{s,0}} \|u_{m,n}\|_{H^{s+1}} \|v\|_{H^{s,0}} \lesssim \|u_{m,n}\|_{H^s} \|v\|_{H^{s,0}} \lesssim n^{2s} + \|v\|_{H^{s,0}}^2, \ t \in [0, T_1 \land \tau_{R, m}^n],
\]

and

\[
|((Hu_{m,n}) \partial_x v, v)_{H^{s,0}}| = \|D^{s,0} Hu_{m,n} \partial_x v, D^{s,0} v\|_{L^2} + \|Hv \partial_x u_{m,n}, D^{s,0} v\|_{L^2}
\]

\[
\lesssim \|D^{s,0} Hu_{m,n}\|_{L^2} \|\partial_x u_{m,n}\|_{L^\infty} + \|\partial_x u_{m,n}\|_{L^\infty} \|D^{s,0-1} \partial_x v\|_{L^2} \|v\|_{H^{s,0}} + \|\partial_x H u_{m,n}\|_{L^\infty} \|v\|_{H^{s,0}}^2
\]

\[
\lesssim \|H u_{m,n}\|_{H^s} \|v\|_{H^{s,0}}^2 \lesssim \|u_{m,n}\|_{H^s} \|v\|_{H^{s,0}}^2 \lesssim \|v\|_{H^{s,0}}, \ t \in [0, T_1 \land \tau_{R, m}^n].
\]
To sum up, we obtain that
\[
E \sup_{t \in [0, T_1 \wedge \tau_{R_n}^m]} \|v(t)\|^2_{H^{2s-\sigma_0}} \lesssim T \|u_{m,n}(0)\|_{H^{2s-\sigma_0}}^2 + \int_0^{T_1} E \sup_{t' \in [0, T \wedge \tau_{R_n}^m]} \|v(t')\|^2_{H^{2s-\sigma_0}} dt.
\]
Using the Grönwall inequality, we obtain (6.21).

Now we prove (6.22). To this end, we first notice that \(2s - \sigma_0 > 0\) and \(r_{m,n}\) is the unique solution to (6.5). Then, similar to (3.27), we can use (6.20) and Assumption (D) to find for each fixed \(n \in \mathbb{N}\) that
\[
E \sup_{t \in [0, T_1 \wedge \tau_{R_n}^m]} \|u_{m,n}\|^2_{H^{2s-\sigma_0}} \lesssim E\|u_{m,n}(0)\|^2_{H^{2s-\sigma_0}} + \int_0^{T_1} E \sup_{t' \in [0, T \wedge \tau_{R_n}^m]} \|u_{m,n}\|^2_{H^{2s-\sigma_0}} dt.
\]
Using the Grönwall inequality and Lemmas 6.1 and 2.5, we have
\[
E \sup_{t \in [0, T_1 \wedge \tau_{R_n}^m]} \|u_{m,n}\|^2_{H^{2s-\sigma_0}} \lesssim E\|u_{m,n}(0)\|^2_{H^{2s-\sigma_0}} \lesssim (n^{\frac{s}{2}+1} + n^{s-\sigma_0})^2 \lesssim n^{2s-2\sigma_0}, \quad n \geq 1.
\]
Hence, by Lemmas 6.1 and 2.5 again, we arrive at
\[
E \sup_{t \in [0, T_1 \wedge \tau_{R_n}^m]} \|u_{m,n}\|^2_{H^{2s-\sigma_0}} \lesssim 2E \sup_{t \in [0, T_1 \wedge \tau_{R_n}^m]} \|u_{m,n}\|^2_{H^{2s-\sigma_0}} \leq C n^{2s-2\sigma_0}, \quad n \geq 1.
\]
Therefore, we complete the proof. \(\square\)

6.3. Concluding the proof of Theorem 1.4. To begin with, we have the following property:

**Lemma 6.4.** Let Assumption (D) hold true. Suppose that for some \(R_0 \gg 1\), the \(R_0\)-exiting time of the zero solution to (1.2) is strongly stable. Then we have
\[
\lim_{n \to \infty} \tau_{R_{0},n}^m = \infty \quad \mathbb{P} - a.s.
\]

**Proof.** By Assumption (D), the unique solution with zero initial data to (1.2) is zero. Now we notice that for all \(s' < s\), \(\lim_{n \to \infty} \|u_{m,n}(0)\|_{H^{s'}} = \lim_{n \to \infty} \|u_{m,n}(0) - 0\|_{H^{s'}} = 0\) and the \(R_0\)-exiting time of the zero solution is infinite. Then the assumption that \(R_0\)-exiting time of the zero solution to (1.2) is strongly stable immediately implies (6.23). \(\square\)

**Proof of Theorem 1.4.** We only need to show that if the \(R_0\)-exiting time is strongly stable at the zero solution for some \(R_0 \gg 1\), then \(\{u^{-1,n}\}\) and \(\{u^{1,n}\}\) are two sequences of pathwise solutions such that (1.19), (1.20), (1.21) and (1.22) are satisfied.

For each \(n > 1\) and for fixed \(R_0 \gg 1\), Lemmas 6.1, 2.5 and (6.20) give \(\mathbb{P}\{\tau_{R_0}^{m,n} > 0\} = 1\), and Lemma 6.4 implies (1.19). Then, it follows from Theorem 1.1 and (6.20) that \(u_{m,n} \in C([0, \tau_{R_0}^{m,n}]; H^s) \quad \mathbb{P} - a.s.\) and (1.20) holds true. Next, we check (1.21). By interpolation, we have
\[
E \sup_{t \in [0, T_1 \wedge \tau_{R_{0},n}^m]} \|u_{m,n} - u_{m,n}^{m,n}\|_{H^s}
\leq \left( E \sup_{t \in [0, T_1 \wedge \tau_{R_{0},n}^m]} \|u_{m,n} - u_{m,n}^{m,n}\|_{H^s}^2 \right)^{\frac{1}{2}} \left( E \sup_{t \in [0, T_1 \wedge \tau_{R_{0},n}^m]} \|u_{m,n} - u_{m,n}^{m,n}\|_{H^{2s-\sigma_0}}^2 \right)^{\frac{1}{2}}.
\]
Combining Lemma 6.3 and the above estimate yields
\[
E \sup_{t \in [0, T_1 \wedge \tau_{R_{0},n}^m]} \|u_{m,n} - u_{m,n}^{m,n}\|_{H^s} \lesssim n^{\frac{s}{2} + 2s - \sigma_0} \cdot n^{\frac{s}{2} + (2s - 2\sigma_0)} = n^{r_s'},
\]
where \(r_s'\) is defined by (6.14) and
\[
0 > r_s' = r_s \cdot \frac{1}{2} + (s - \sigma_0) \cdot \frac{1}{2} = \frac{\delta - 1}{2}.
\]
Since \(r_s' < 0\), we can deduce that
\[
\lim_{n \to \infty} E \sup_{t \in [0, T_1 \wedge \tau_{R_{0},n}^m]} \|u_{m,n} - u_{m,n}^{m,n}\|_{H^s} = 0.
\]
Since $\delta < 1$, we have

$$\|u^{-1,n}(0) - u^{1,n}(0)\|_{H^s}^2 = \|u_{-1,n}(0) - u_{1,n}(0)\|_{H^s}^2,$$

which implies that (1.21) holds true.

Now we prove (1.22). Let $T_l > 0$ be given in Lemma 6.1. We can infer from (6.25) that

$$\liminf_{n \to \infty} \mathbb{E} \sup_{t \in [0,T_l] \cap \tau^{-1,n}_R \wedge \tau^{1,n}_R} \|u^{-1,n}(t) - u^{1,n}(t)\|_{H^s} \geq \liminf_{n \to \infty} \mathbb{E} \sup_{t \in [0,T_l] \cap \tau^{-1,n}_R \wedge \tau^{1,n}_R} \|u_{-1,n}(t) - u_{1,n}(t)\|_{H^s} - \liminf_{n \to \infty} \mathbb{E} \sup_{t \in [0,T_l] \cap \tau^{-1,n}_R \wedge \tau^{1,n}_R} \|u_{1,n}(t) - u^{-1,n}(t)\|_{H^s} \geq \liminf_{n \to \infty} \mathbb{E} \sup_{t \in [0,T_l] \cap \tau^{-1,n}_R \wedge \tau^{1,n}_R} \|u_{-1,n}(t) - u_{1,n}(t)\|_{H^s}.$$
[13] A. Debussche, N. E. Glatt-Holtz, and R. Temam. Local martingale and pathwise solutions for an abstract fluids model. *Physica D: Nonlinear Phenomena*, 240(14-15):1123–1144, 2011.

[14] H. Dong. Well-posedness for a transport equation with nonlocal velocity. *Journal of Functional Analysis*, 255(11):3070–3097, 2008.

[15] E. Fedrizzi and F. Flandoli. Noise prevents singularities in linear transport equations. *Journal of Functional Analysis*, 264(6):1329–1354, 2013.

[16] E. Fedrizzi, W. Neves, C. Oliveira, On a class of stochastic transport equations for $L^2_{loc}$ vector fields, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V (5) Vol. XVIII, 397–419, 2018.

[17] F. Flandoli. *Random perturbation of PDEs and fluid dynamic models*, volume 2015 of Lecture Notes in Mathematics. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010, École d’Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[18] F. Flandoli, M. Gubinelli, and E. Priola. Well-posedness of the transport equation by stochastic perturbation. *Inventiones Mathematicae*, 180(1):1–53, 2010.

[19] F. Flandoli, M. Gubinelli, and E. Priola. Full well-posedness of point vortex dynamics corresponding to stochastic 2D Euler equations, 121(7):1445–1463, 2011.

[20] F. Flandoli and D. Luo. Euler-Lagrangian approach to 3-D stochastic Euler equations. *Journal of Geometric Mechanics*, 11 (2):153–165, 2019.

[21] L. Gawarecki and V. Mandrekar. *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*. Probability and its Applications (New York). Springer, Heidelberg, 2011.

[22] B. Gess and P. E. Souganidis. Long-time behavior, invariant measures, and regularizing effects for stochastic scalar conservation laws. *Communications on Pure and Applied Mathematics*, 70(8):1562–1597, 2017.

[23] N. Glatt-Holtz and M. Ziane. Strong pathwise solutions of the stochastic Navier-Stokes system. *Advances in Differential Equations*, 14(5-6):567–600, 2009.

[24] N. E. Glatt-Holtz and V. C. Vicol. Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise. *Annals of Probability*, 42(1):80–145, 2014.

[25] A. I. Held, R. T. Pierrehumbert, S. T. Garner and K. L. Swanson. Surface quasi-geostrophic dynamics. *Journal of Fluid Mechanics*, 282: pp. 1720, 1995.

[26] D. Henry. Geometric theory of semilinear parabolic equations. *Lecture Notes in Mathematics*, 840, 1981.

[27] A. A. Himonas and C. Kenig. Non-uniform dependence on initial data for the CH equation on the line. *Differential Integral Equations*, 22(3-4):201–224, 2009.

[28] A. A. Himonas, C. Kenig, and G. Misiołek. Non-uniform dependence for the periodic CH equation. *Communications in Partial Differential Equations*, 35(6):1145–1162, 2010.

[29] A. A. Himonas and G. Misiołek. Non-uniform dependence on initial data of solutions to the Euler equations of hydrodynamics. *Communications in Mathematical Physics*, 296(1):285–301, 2010.

[30] M. Hofmanová. Degenerate parabolic stochastic partial differential equations. *Stochastic Processes and their Applications*, 123(12):4294–4336, 2013.

[31] D. D. Holm. Variational principles for stochastic fluid dynamics. *Proceedings of the Royal Society A*, 471(2176):20140963, 19, 2015.

[32] Inci H. On the well-posedness of the inviscid SQG equation. *Journal of Differential Equations*, 2018, 264(4): 2660-2683.

[33] G. Kallianpur and J. Xiong. Stochastic differential equations in infinite-dimensional spaces. 26:vi+342, 1995. Expanded version of the lectures delivered as part of the 1993 Barrett Lectures at the University of Tennessee, Knoxville, TN, March 25–27, 1993, With a foreword by Balram S. Rajput and Jan Rosinski.

[34] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Communications on Pure and Applied Mathematics*, 41(7):891–907, 1988.

[35] C. E. Kenig, G. Ponce, and L. Vega. Well-posedness of the initial value problem for the Korteweg-de Vries equation. *Journal of the American Mathematical Society*, 4(2):323–347, 1991.

[36] R. Z. Khas’minskii. *Stability of systems of differential equations under random perturbations of their parameters*. (Russian). Izdat. “Nauka”, Moscow, 1969.

[37] J. U. Kim. On the Cauchy problem for the transport equation with random noise. *Journal of Functional Analysis*, 259(12):3328–3359, 2010.
[38] A. Kiselev. Regularity and blow up for active scalars. Mathematical Modelling of Natural Phenomena, 5 (2010), no. 4, 2257-255.

[39] H. Koch and N. Tzvetkov. Non-linear wave interactions for the Benjamin-Ono equation. International Mathematics Research Notices, (30):1833–1847, 2005.

[40] I. Kröker and C. Rohde. Finite volume schemes for hyperbolic balance laws with multiplicative noise. Applied Numerical Math., 62(4):441–456, 2012.

[41] N. V. Krylov and B. L. Rozovskii. Stochastic evolution equations. In Current problems in mathematics, Vol. 14 (Russian), pages 71–147, 256. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979.

[42] J. Li, H. Liu, and H. Tang. Stochastic MHD equations with fractional kinematic dissipation and partial magnetic diffusion in \( \mathbb{R}^2 \). Stochastic Processes and their Applications, 135:139–182, 2021.

[43] D. Li and J. Rodrigo. Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation Advances in Mathematics, 217 (6), 2563–2568, 2008.

[44] Y. Miao, C. Rohde and H. Tang. Well-posedness for a stochastic Camassa-Holm type equation with higher order non-linearities. arXiv:2105.08607, 2021.

[45] A. C. Morlet. Further properties of a continuum of model equations with globally defined flux. Journal of mathematical analysis and applications, 221(1): pp. 132? 160,1998

[46] W. Neves, C. Olivera, Wellposedness for stochastic continuity equations with Ladyzhenskaya-Prodi-Serrin condition, NoDEA non-linear Differential Equations Appl. 22 (2015), 1247–1258.

[47] P. Pedlosky Geophysical Fluid Dynamics. Springer Verlag, 1982.

[48] C. Prévôt and M. Röckner. A concise course on stochastic partial differential equations, volume 1905 of Lecture Notes in Mathematics. Springer, Berlin, 2007.

[49] P. Ren, H. Tang, and F.-Y. Wang. Distribution-path dependent non-linear SPDEs with application to stochastic transport type equations. arXiv:2007.09188, 2020.

[50] M. Röckner, R. Zhu, and X. Zhu. Local existence and non-explosion of solutions for stochastic fractional partial differential equations driven by multiplicative noise. Stochastic Process. Appl., 124(5):1974–2002, 2014.

[51] C. Rohde and H. Tang. On the stochastic Dullin-Gottwald-Holm equation: global existence and wave-breaking phenomena. NoDEA non-linear Differential Equations Appl., 28(1):Paper No. 5, 34, 2021.

[52] C. Rohde and H. Tang. On a Stochastic Camassa–Holm Type Equation with Higher Order non-linearities. J. Dyn. Diff. Equat. 33, 1823–1852, 2021.

[53] L. Silvestre and V. Vicol. On a transport equation with nonlocal drift. Transactions of the American Mathematical Society, 368. no. 9, 6159-6188, 2016.

[54] H. Tang. On the pathwise solutions to the Camassa-Holm equation with multiplicative noise. SIAM J. Math. Anal., 50(1):1322–1366, 2018.

[55] H. Tang. Noise effects on dependence on initial data and blow-up for stochastic Euler–Poincaré equations. arXiv:2002.08719, 2020.

[56] H. Tang, S. Shi, and Z. Liu. The dependences on initial data for the b-family equation in critical Besov space. Monatsh. Math., 177(3):471–492, 2015.

[57] H. Tang, Y. Zhao, and Z. Liu. A note on the solution map for the periodic Camassa-Holm equation. Appl. Anal., 93(8):1745–1760, 2014.

[58] M. E. Taylor. Partial Differential Equations III, volume 117 of Applied Mathematical Sciences. Springer New York, New York, NY, 2011.

[59] R. Temam. Navier-Stokes equations. Theory and numerical analysis. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Studies in Mathematics and its Applications, Vol. 2.