Compton polarimetry revisited

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Abstract

We compute the average polarisation asymmetry from the Klein-Nishina differential cross section on free electrons at rest. As expected from the expression for the asymmetry, the average asymmetry is found to decrease like the inverse of the incident photon energy asymptotically at high energy. We then compute a simple estimator of the polarisation fraction that makes optimal use of all the kinematic information present in an event final state, by the use of “moments” method, and we compare its statistical power to that of a simple fit of the azimuthal distribution. In contrast to polarimetry with pair creation, for which we obtained an improvement by a factor of larger than two in a previous work, here for Compton scattering the improvement is only of 10–20%.

Key words: Hard X-ray, gamma-ray, Compton scattering, polarimeter, polarisation asymmetry, optimal variable

1. Cosmic-source polarimetry: the high-energy frontier

Polarimetry is a powerful diagnostic of specific phenomena at work in cosmic sources in the radio-wave and optical energy bands, but very few results are available at high photon energies: the only significant observation in the X-gamma energy range, to date, is the measurement of a linear polarisation fraction of $P = 19 \pm 1\%$ of the 2.6 keV emission of the Crab nebula by a Bragg polarimeter on board OSO-8 [1]. At higher energies, hard-X-ray and soft-gamma-ray telescopes that have flown to space in the past (COMPTEL...
were not optimized for polarimetry, and their sensitivity to polarisation was poor. Presently active missions (Integral IBIS\cite{1,2} and SPI \cite{3}) have provided some improvement, with, in particular, mildly significant measurements of $P = 28 \pm 6\%$ (130 to 440 keV \cite{4}) and $P = 47_{-13}^{+19}\%$ (200 to 800 keV \cite{4}) for the Crab Nebula. A number of Compton polarimeter/telescope projects have been developed, some of which also propose to record photon conversions to $e^+e^-$ pairs. A variety of technologies have been considered, such as scintillator arrays (POGO \cite{7}, GRAPE \cite{8}, POLAR \cite{9}), Si or Ge microstrip detectors (MEGA \cite{10}, ASTROGAM \cite{11}) or combinations of these (Si + LaBr$_3$ for GRIPS \cite{12}, Si + CsI(Tl) for TIGRE \cite{13}), semiconductor pixel detectors (CIPHER \cite{14}) and liquid xenon (LXeGRIT \cite{15}) time projection chambers (TPC). In most Compton telescopes the reconstruction of the direction of the incident photon provides an uncertainty area which has the shape of a thin cone arc. The tracking of the recoil electron from the first Compton interaction with a measurement of the direction of the recoil momentum, as is within reach with a gas TPC, allows to decrease the length of the arc and therefore to improve dramatically the sensitivity of the detector (\cite{16} and references therein).

Some of these telescopes are sensitive to photon energies up to tens of MeV in the Compton mode, but their sensitivity to polarisation above a few MeV is either nonexistent or undocumented.

2. Polarisation asymmetry and average polarisation asymmetry

As is well known, the sensitivity to polarisation of Compton scattering is excellent at low energies (Thomson scattering), as the polarisation asymmetry $A$, also known as the modulation factor and defined by the phase-space dependence of the differential cross section,

$$\frac{d\sigma}{d\Omega} \propto [1 + A P \cos (2(\phi - \phi_0))], \quad (1)$$

reaches $-1$ at a polar angle $\theta$ of 90$^\circ$ (Fig. 1 of Ref. \cite{17}). In this expression, $\phi$ is the azimuthal angle, that is the angle between the scattering plane and the
direction of polarisation of the incident photon. Unfortunately, $A$ is decreasing
with energy, and as the precision of the measurement scales as $\sigma_P \propto \frac{1}{(A\sqrt{N})}$
when the background noise is negligible and where $N$ is the number of signal
event, the sensitivity of Compton polarimetry decreases at high energies. With
the goal of a quantitative assessment of this sensitivity, in this paper we compute
the average polarisation asymmetry $\langle A \rangle$ from the Klein-Nishina differential cross
section on free electrons at rest \[18, 19\]. $\langle A \rangle$ is defined from the differential cross
section in $\phi$, that is after the full differential cross section (eq. (1)) has been
integrated over the other variables that describe the final state:

$$\frac{d\sigma}{d\phi} \propto \left[1 + \langle A \rangle P \cos \left(2(\phi - \phi_0)\right)\right].$$

(2)

Following Heitler \[20\], the doubly differential cross section for linear polarised
radiation reads:

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2} x^2 \left[x + \frac{1}{x} - 2 \sin^2 \theta \cos^2 \phi\right],$$

(3)

where $x = k/k_0$, $k_0$ and $k$ are the energy of the incident and scattered $\gamma$s,
respectively; $\theta$ is the scattering angle, that is the polar angle of the direction of
the scattered $\gamma$ with respect to the direction of the incident $\gamma$. The differential
element $d\Omega$ is $\sin \theta d\theta d\phi$ as usual. In the case of partially polarised emission
with polarisation fraction $P$, the differential cross section becomes:

$$d\sigma = \frac{r_0^2}{2} x^2 \left[x + \frac{1}{x} - \sin^2 \theta (\cos (2\phi)P + 1)\right] \sin \theta d\theta d\phi.$$

(4)

The minus sign reflects the fact that photons Compton scatter preferentially
into the direction perpendicular to the orientation of the electric field of the
incoming radiation. The energy of the scattered $\gamma$ is related to $\theta$ from energy-
momentum conservation: $k = k_0/[1 + k_0(1 - \cos \theta)]$, $x = 1/[1 + k_0(1 -
\cos \theta)]$, $\cos \theta = 1 - (1/x - 1)/k_0$, $\sin \theta d\theta = -dx/(k_0 x^2)$ and $\sin^2 \theta =
2[1/(xk_0) - 1/k_0] - [1/(xk_0) - 1/k_0]^2$. We then obtain \[20\] [21]:

$$d\sigma = \frac{r_0^2}{2k_0} \left[x + \frac{1}{x} - \left(2 \left(\frac{1}{xk_0} - \frac{1}{k_0}\right) - \left(\frac{1}{xk_0} - \frac{1}{k_0}\right)^2\right) (\cos (2\phi)P + 1)\right] dx d\phi.$$

(5)
\( k \) varies in a range such that \(-1 \leq \cos \theta \leq 1\), that is \(1/(1 + 2k_0) \leq x \leq 1\).

The distributions of these kinematic variables are shown in Fig. 1. After an elementary integration over \( x \), we obtain:

\[
\begin{align*}
\mathrm{d}\sigma &= r_0^2 \left[ \frac{1 + k_0}{k_0^3} \left( \frac{2k_0(1 + k_0)}{1 + 2k_0} - \ln (1 + 2k_0) \right) + \frac{\ln(1 + 2k_0)}{2k_0} \right. \\
&\quad \left. - \frac{1 + 3k_0}{(1 + 2k_0)^2} + \frac{(2k_0 - (k_0 + 1) \ln(2k_0 + 1))}{k_0^3} \cos (2\phi) P \right] \mathrm{d}\phi,
\end{align*}
\]

that is a total cross section of [15] [20] [21]:

\[
\sigma = 2\pi r_0^2 \times \left[ \frac{1 + k_0}{k_0^3} \left( \frac{2k_0(1 + k_0)}{1 + 2k_0} - \ln (1 + 2k_0) \right) + \frac{\ln(1 + 2k_0)}{2k_0} - \frac{1 + 3k_0}{(1 + 2k_0)^2} \right].
\]

Equating the constant term and the term proportional to \( \cos (2\phi) P \) in eqs. [2] and [6], we obtain for the average polarisation asymmetry:

\[
\langle A \rangle = \frac{(2k_0 - (k_0 + 1) \ln(2k_0 + 1))}{(1 + k_0) \left( \frac{2k_0(1 + k_0)}{1 + 2k_0} - \ln (1 + 2k_0) \right) + \frac{k_0^2 \ln(1 + 2k_0)}{2} - \frac{(1 + 3k_0)k_0^3}{(1 + 2k_0)^2}}.
\]

Figure 1: Spectra of the azimuthal and polar angles \( \phi \) and \( \theta \), of \( \cos \theta \), of the fraction \( x \) of the incident photon energy carried away by the scattered photon, and of the 1D and 2D weights \( w \) and \( w_{opt} \), for incident photon energies \( 0.1mc^2 \), \( mc^2 \), and \( 10mc^2 \), all for a fully polarized beam.

We now examine two limiting cases:
• At low energies, $k_0 \approx 0$, eq. (6) reduces to:

$$d\sigma = \frac{r_0^2}{2} \left[ 1 - \cos(2\phi) \frac{P}{2} \right] \frac{k_0}{3} d\phi,$$

which results in a total cross section of $\sigma = 8r_0^2 \pi/3$, i.e., the Thomson cross section. The low-energy average asymmetry is $\langle A \rangle = -1/2$.

• At high energies,

$$d\sigma = \frac{r_0^2}{2k_0} \left[ \log(2k_0) + \frac{1}{2} + P \frac{4 - 2 \log(2k_0)}{k_0} \cos(2\phi) \right] d\phi,$$

which results in a total cross section of $\sigma = \pi r_0^2 \left( \log(2k_0) + \frac{1}{2} \right)/k_0$. The high-energy average asymmetry is

$$\langle A \rangle = -\frac{4(\log 2k_0 - 2)}{k_0(2\log 2k_0 + 1)}. \quad (11)$$

The average asymmetry decreases at high energies, asymptotically approaching $\langle A \rangle \approx -2/k_0$. The variation of the average polarisation asymmetry of photon Compton scattering on free electrons at rest (eq. (8)) is compared to its high-energy approximation (eq. (11)) in Fig. 2. The absence of sensitivity of
Compton polarimeters at high energies \cite{22} is due to this strong decrease of $|\langle A \rangle|$.

3. Optimal variable for polarisation measurement

The value of the polarisation fraction $P$ is classically obtained by a fit to the $\phi$ distribution. A way to improve the polarisation sensitivity is to make an optimal use of the information contained in the multi-dimensional probability density function (pdf) through the use of an optimal variable (\cite{23} and references therein), that is, of a weight $w(\Omega)$ such that the $P$ dependence of the expectation value $E(w)$ of $w$ allows a measurement of $P$, and that the variance of such a measurement is minimal. The solution, up to a multiplicative factor, is (eg. \cite{24}):

$$w_{\text{opt}} = \frac{\partial \ln p(\Omega)}{\partial P}.$$ (12)

In the present case of a polarisation measurement:

$$p(\Omega) \equiv f(\Omega) + P \times g(\Omega),$$ (13)

with $\int f(\Omega) d\Omega = 1$ and $\int g(\Omega) d\Omega = 0$, we obtain:

$$w_{\text{opt}} = \frac{g(\Omega)}{f(\Omega) + P \times g(\Omega)},$$ (14)

that is, if $|P \times g(\Omega)|$ is small compared to $f(\Omega)$,

$$w_{\text{opt}} = \frac{g(\Omega)}{f(\Omega)}.$$ (15)

The 1st moment of $w_{\text{opt}}$ is $\langle w_{\text{opt}} \rangle = \int \frac{g(\Omega)}{f(\Omega)} \times [f(\Omega) + P \times g(\Omega)] d\Omega = P \int \frac{g^2(\Omega)}{f(\Omega)} d\Omega$, which is proportional to $P$. The expressions for $f$ and $g$ are obtained from the measured values of $\phi$ and $\theta$ (and therefore of $x$) by equating the constant term and the term proportional to $\cos(2\phi)P$ in eqs. (13) and (1). The spectrum of $w_{\text{opt}}$ is shown in Fig. 1 for a fully polarised beam. We can see that $|w_{\text{opt}}|$ is most often much smaller than unity (beware the vertical log scale), so that our neglecting $|P \times g(\Omega)|$ in the expression of $f(\Omega) + P \times g(\Omega)$ was
legitimate. The asymmetry, the non-evenness of the \( w_{\text{opt}} \) distribution makes the non-zero average due to the beam polarisation explicit.

Moment’s methods are equivalent to a likelihood analysis in the case where the pdf is a linear function of the variables that one aims to measure, as is the case here, but they are much simpler to instantiate as one just has to compute \( w_{\text{opt}}(\theta, \phi, x) \), and average it over the whole statistics. Although the analysis of experimental data is beyond the scope of this paper, the following considerations apply:

- Background subtraction reduces to a simple subtraction in computing the average of \( w_{\text{opt}} \). Their \( n \)-dimensional parametrization is not needed.
- Likelihood methods need the use of a \( n \)-dimensional parametrization of the acceptance, or efficiency, for correction. This is pretty simple in the case of Compton scattering for which the final state is described by only two variables, but for higher-dimensional systems, producing enough Monte Carlo (MC) statistics and determining a parametrization becomes a nightmare: in that case the use of a moments-based efficiency correction becomes mandatory (for a real-case presentation see eg., section IV.A, eqs. (18)-(24) and VI.B eqs. (47)-(49) of [25]).

In the “reduced” 1D case of eq. (2), \( w_{\text{opt}} \) becomes \( w = 2 \cos 2\phi \) and the estimator for \( \langle A \rangle P \) is \( \langle w \rangle [23] \). The uncertainty then reads:

\[
\sigma_P = \frac{1}{\langle A \rangle \sqrt{N}} \sqrt{2 - (\langle A \rangle P)^2},
\]

that is, in the case of Thomson scattering (\( \langle A \rangle = -1/2 \)), \( \sigma_P \approx \sqrt{(8 - P^2)/N} \).

Needless to say, in the case where the direction of the polarisation of the emission of a particular cosmic source “on the sky” is unknown, a combined use of \( \langle 2 \cos 2\phi \rangle \) and of \( \langle 2 \sin 2\phi \rangle \) should be used.

The performance of the 2D estimator \( \langle w_{\text{opt}} \rangle \) is compared to that of the 1D \( \langle w \rangle \) by the comparison of the ratios of the RMS width normalized to the mean value:

\[
r = \frac{\text{RMS}_{w_{\text{opt}}} \langle w_{\text{opt}} \rangle}{\text{RMS}_w \langle w \rangle}. \tag{17}
\]
In contrast with polarimetry performed with $e^+e^-$ telescopes, for which an improvement in the precision of the measurement of the linear polarisation fraction by a factor of larger than two is at hand (Fig. 21 right of [23]), in the case of Compton polarimeters the improvement is found to be marginal, varying from $\approx 20\%$ at low energy to $\approx 10\%$ at high energy (Fig. 3). These results are in qualitative agreement with those obtained at 100 keV by a likelihood analysis of the doubly differential cross section [17].

In summary, we have obtained the expression for the average polarisation asymmetry, or modulation factor, of Compton scattering on free electrons at rest, eq. (8), Fig. 2. We have then obtained a simple optimal estimator of the polarisation fraction $P$ that makes use of all the information (azimuthal and polar angles of the scatter), avoiding the technicalities of a maximum likelihood analysis but with the same performance.

4. Acknowledgments

It a pleasure to acknowledge the support by the French National Research Agency (ANR-13-BS05-0002) and the scrutiny and the suggestions of referee #1 of Nuclear Instruments and Methods in Physics Research A.
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