The Physics Inside
Topological Quantum Field Theories\(^1\)

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Abstract

We show that the equations of motion defined over a specific field space are realizable as operator conditions in the physical sector of a generalized Floer theory defined over that field space. The ghosts associated with such a construction are found not to be dynamical. This construction is applied to gravity on a four dimensional manifold, \(M\); whereupon, we obtain Einstein’s equations via surgery, along \(M\), in a five-dimensional topological quantum field theory.

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Within the last year, a number of authors have espoused the view that at least a subset of the physics of four-dimensional quantum gravity may be realized in a topological quantum field theory (TQFT) [1, 2, 3]. However, only partial success in establishing this statement has been obtained thus far. Unlike previous work, we will focus our attention on a five-dimensional TQFT in order to realize four-dimensional gravity. The clue which we will exploit is that for a TQFT on a manifold with boundary, the “classical” fields are dynamical on the boundary whereas the ghosts are not. Furthermore and of equal importance, is the absence of dynamical degrees of freedom in the bulk (compliment of the boundary) of the five-dimensional manifold. With Atiyah’s axioms and Floer cohomology theory in mind, we are then lead to the examination of the quantum mechanics of a five-dimensional TQFT.

Our note is organized as follows. First, we will derive some results which follow as essentially straightforward exercises/extrapolations from the existing literature [4]. Secondly, we realize the connection of these results with Einstein’s equations. Finally, we discuss the covariant version of the five-dimensional TQFT. Four dimensional space-time will be denoted as $M$ while the five-dimensional manifold will be written as $X$.

To start, we assume that we are given the equations of motion of a physical field theory (or any field theory for that matter) for a set of fields, $\Phi^I$, and that these are obtained by extremizing the action $S_{ft}[\Phi^I]$. We now show that:

Formally, a TQFT on a $(d+1)$-dimensional manifold, $X$ exists such that the equations of motion for the $\Phi^I$ on a $d$-dimensional manifold, $M$, arise as operator conditions in the physical sector of Hilbert space. Furthermore, in this Hilbert subspace, the only dynamical degrees of freedom on $M$ are those of the $\Phi^I$.

As was the case with Floer’s treatment of holomorphic maps [3] and flat connections [5], we identify $S_{ft}$ as the Morse functional on the space, $U$, of these fields. From this we introduce a second set of fields, $\psi^I$, which carry the same spin, but opposite statistics to the $\Phi^I$ and define the exterior derivative, $Q$, and its adjoint, $Q^*$, on $U$:

\[
Q = \langle (\frac{\delta}{\delta \Phi^I} + \frac{\delta S_{ft}[\Phi]}{\delta \Phi^I}), \psi^I \rangle,
\]

\[
Q^* = -\langle \chi^I, (\frac{\delta}{\delta \Phi^I} - \frac{\delta S_{ft}[\Phi]}{\delta \Phi^I}) \rangle,
\]

where the bracket represents the inner products on $M$ and $U$, while $\chi^I$ is the
canonical momenta (dual) of $\psi$. Lastly, define the Hamiltonian of the TQFT to be

$$H_{TQFT} = \frac{1}{2}\{Q^*, Q\} . \tag{2}$$

With these definitions, we can now proceed to establish our statement.

A choice of hermitian conjugation exists for which $H_{TQFT}$ is hermitian. By standard arguments it follows that the physical Hilbert space of the TQFT is $H_{TQFT} \in \text{Ker}(Q)/\text{Im}(Q)$ and are ground states of the Hamiltonian. Let $|\text{phys}_i\rangle$ denote a basis element for this space so that $\langle\text{phys}_i|\text{phys}_j\rangle = \delta_{ij}$. It then follows that for any operator $O$, all physical state matrix elements of the commutator $[H_{TQFT}, O]$, vanish. In particular, the matrix elements of the momenta conjugate to the $\Phi^I$ fields vanish:

$$\langle\text{phys}_i|\pi_I|\text{phys}_j\rangle = 0 , \tag{3}$$

The vanishing of the momenta is consistent with the absence of dynamical degrees of freedom in the TQFT. Furthermore, this means that as a composite operator, $\frac{\delta S_{ft}}{\delta \Phi^I}$ is represented by 0 on the physical subspace since its matrix elements vanish:

$$\langle\text{phys}_i|\frac{\delta S_{ft}[\Phi]}{\delta \Phi^I}|\text{phys}_j\rangle = 0 \implies \frac{\delta S_{ft}}{\delta \Phi^I} = 0 . \tag{4}$$

Finally, given that $\psi^I$ is $Q$-exact, its matrix elements between any physical states vanish:

$$\langle\text{phys}_i|\psi^I|\text{phys}_j\rangle = 0 . \tag{5}$$

We have established the statement.

The exterior derivative defined in Eqn. (1) actually follows as the BRST charge from the action

$$I_0[\Phi] = \int_X dL , \tag{6}$$

where $L$ is what we might normally write down (but lifted to $X$) as a Lagrangian density for $\Phi$: $S_{ft} \equiv \int_M L$. In this regard, the four-dimensional manifold $M$ is the boundary of some $X$ if its signature vanishes: $\sigma(M) = 0$ \[8\]. If we take $X$ to be the semi-infinite cylinder with time normal to the boundary $M$, we then find that the momentum, $\pi_I$, conjugate to $\Phi^I$ is given by the constraint

$$\pi_I - \frac{\delta}{\delta \Phi^I} \int_M L \approx 0 . \tag{7}$$
Then $Q$ follows by BFV-quantization with this constraint. Thus our definitions are not absolutely necessary. However, we prefer to start with $Q$ rather than $I_0$ as we have seen that all we need is the canonical theory to establish our results. The construction of Hamiltonia for TQFT’s wherein the constraints in the BFV charges were taken to be the ADM-constraints of quantum gravity theories, has been discussed.

Let us now specialize to gravity. The functional, $S_{ft}$, in this case is the Einstein-Cartan action which we treat as a gauge theory with gauge group $SO(3,1)$ (indices $a,b,...$). The fields of this theory are one-forms $e^a$ and $\omega^{ab}$ on $M$; these are the vierbien and Lorentz spin-connection of the four-dimensional theory. The space of these fields will be denoted as $U$. We lift the fields to $X$ where they continue to carry the same gauge group representations but vastly different physical interpretations; we will adorn the lifted fields with $\hat{\cdot}$: $(\hat{e}^a, \hat{\omega}^{ab})$. The vierbien becomes $\hat{e}^a$ which is a peculiar matter field on $X$ that, as we will see below, is coupled to the curvature of a $SO(3,1)$ connection, $\hat{\omega}^{ab}$. The space of these lifted fields will be denoted as $\hat{U}$. In addition to these fields, we also introduce a pair of Grassmann-odd one-forms $\zeta^a$ and $\xi^{ab}$ along with their lifts $\hat{\zeta}^a$ and $\hat{\xi}^{ab}$. These are in $T^*U$ and $T^*\hat{U}$, respectively.

The exterior derivative on the space of fields which we will use is

$$Q = e^{-S_{EC}} \int_M Tr[\zeta \wedge \frac{\delta}{\delta \omega} + \xi \wedge \frac{\delta}{\delta e}] e^{S_{EC}},$$

$$S_{EC} = \int_M e^a \wedge e^b \wedge F^{cd}(\omega) \epsilon_{abcd},$$

where $S_{EC}$ is the Einstein-Cartan action so that a critical point is specified by Einstein’s equations. Then a wavefunctional which satisfies the functional equations

$$\frac{\delta}{\delta \omega^{ab}} - *(e^c \wedge De^d) \epsilon_{abcd} = 0 \quad \text{and} \quad \frac{\delta}{\delta e^a} + *(e^b \wedge F^{cd}) \epsilon_{abcd} = 0,$$

is an element of ker $Q$. By construction, the $1$ in the cohomology of $Q$ is

$$\Psi[e,\omega] = e^{-S_{EC}[e,\omega]}.$$

The Einstein-Cartan (hence Einstein’s) equations of motion on $M$ follow as per our discussion above:

$$e^a \wedge De^b \epsilon_{abcd} = 0 \quad \text{and} \quad e^a \wedge F^{bc} \epsilon_{abcd} = 0.$$
Although a metric on $X$ was needed in the construction of $Q$ it drops out when Eqn. (4) is implemented. Consequently, in focusing only on the equations in (11) we are free to identify $e^a$ with the vierbien just as we do in the usual discussion of the Einstein-Cartan action. Of course, this is a residue of the topological nature of the theory on $X$ which we now turn to.

We can deduce Eqns. (9) and (11) by starting with the classical five-dimensional action:

$$I_0[\hat{e}, \hat{\omega}] = \int_X \hat{e}^a \wedge \hat{T}^b \wedge \hat{F}^{cd} \epsilon_{abcd}, \quad \hat{T} \equiv \hat{D} \hat{e}.$$  \hspace{1cm} (12)

Writing this action on $X = M \times R$, we see that Eqn. (9) follows from the computation of the canonical momenta of the fields $\omega$ and $e$. The covariant quantum theory on arbitrary $X$, follows by imposing the forty gauge fixing conditions

$$\hat{T}_d = \ast \hat{F}^{ab} \wedge \hat{e}^c \epsilon_{abcd},$$  \hspace{1cm} (13)

on the fifty fields in $\mathcal{U}$. We have introduced the symbol $\hat{T}$ as the restriction, $T$, of this 2-form to four dimensions is in fact the torsion of $M$. Notice that the four-dimensional Einstein tensor and torsion are not related by this equation. Rather, equation (13) equates the Einstein tensor with the derivative of the would be vierbien with respect to the extra fifth coordinate. Much as in the previous section, we can write down the purely bosonic part of the gauge fixed TQFT action as

$$I_{bos} = \int_X Tr \left[ \frac{1}{2} (T - \ast F \wedge e)^2 \right].$$  \hspace{1cm} (14)

This functional plus its ghost completion and $I_0$ is the five-dimensional quantum action for four dimensional classical gravity.

We are assured that the TQFT whose moduli space is defined by Eqn. (13) is non-trivial by virtue of the fact that we can identify some of its subspaces. One subspace which is immediate is given by $X = M \times S^1$ and using the five-dimensional diffeomorphism and gauge invariance of Eqn. (13) to set $\hat{e}^a$ and $\hat{\omega}^{ab}$ in the $S^1$ direction to zero. Then we see that $M$ is generically an Einstein space as the $S^1$ harmonics of the $M$ frame $e^a$ result in a (quantized) cosmological constant.

In summary, we have discovered that the equations of motion of a $d$-dimensional field theory follow from a $(d + 1)$-dimensional TQFT via surgery. Importantly, the ghosts of the TQFT vanish on the $d$-dimensional manifold.
The explicit example of four dimensional gravity was given. Another, and perhaps more immediate, example of our construction is the realization of the three dimensional Chern-Simons equations of motion, from the four dimensional Pontryagin density via surgery. Recall that this is the root of the flows of flat connections as described by Floer [5].

We infer from these results that classical physics appears in the midst of pure topology. Alternatively, we can arrange for the equations of motion of a field theory on a manifold $M$ to arise as boundary conditions for a TQFT on a manifold $X$ with boundary $\partial X = M$. Thus the classical physics on a manifold follow from a functor from the category of those manifolds into the category of physical Hilbert spaces of the TQFT. We can take this further by using results from the axiomatization of quantum field theories [10] to realize such processes as spacetime (as opposed to spatial) topology change by taking $\partial X = M \cup (-M')$. As the $(d + 1)$-dimensional theory whose $d$-dimensional “spaces” (which we in turn realize as spacetimes) are undergoing topology change is topological, we are free to relax the isochronous condition [11] as causality is not an issue here. Degenerate metrics on $X$ are allowed.

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References

[1] R. Brooks and G. Lifschytz, *Nucl. Phys.* B438 (1995) 211.
[2] L. Crane, *J. Math. Phys.* 36 (1995) 6180.
[3] L. Smolin, *J. Math. Phys.* 36 (1995) 6417.
[4] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Reps.* 209 (1991) 129 and references therein.
[5] A. Floer, *Bull. Am. Math. Soc.* 16 (1987) 279.
[6] A. Floer, *Commun. Math. Phys.* 118 (1988) 215.
[7] E. Witten, *Commun. Math. Phys.* **117** (1988) 353.

[8] For a contemporary treatment of this theorem (originally due to V. A. Rohlin) see P. Melvin, *Contemp. Math.* **35** (1984) 399.

[9] R. Brooks, *Mod. Phys. Letts.* **8** (1993) 2277.

[10] G. Segal, *Two-dimensional conformal field theories and modular functors*, in the proceedings of the IX**th** International Congress on Mathematical Physics, July 17 - 27, 1988, Swansea, Wales; B. Simon, A. Truman and I. M. Davies, eds.; Adam Hilger, Bristol, 1989.

[11] R. Geroch, *J. Math. Phys.* **8** (1967) 782