Quantum Mechanical Hamiltonians with Large Ground-State Degeneracy

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Abstract

Nonrelativistic Hamiltonians with large, even infinite, ground-state degeneracy are studied by connecting the degeneracy to the property of a Dirac operator. We then identify a special class of Hamiltonians, for which the full space of degenerate ground states in any spatial dimension can be exhibited explicitly. The two-dimensional version of the latter coincides with the Pauli Hamiltonian, and recently-discussed models leading to higher-dimensional Landau levels are obtained as special cases of the higher-dimensional version of this Hamiltonian. But, in our framework, it is only the asymptotic behavior of the background ‘potential’ that matters for the ground-state degeneracy. We work out in detail the ground states of the three-dimensional model in the presence of a uniform magnetic field and such potential. In the latter case one can see degenerate stacking of all 2d Landau levels along the magnetic field axis.

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1 Introduction

Quantum mechanical Hamiltonians admitting highly degenerate energy eigenstates are rather special, and they can thus serve as useful theoretical models to explain some novel properties exhibited by physical systems under certain circumstances. The most notable example is provided by the integer quantum Hall effect in two-dimensional (2d) electron gas, for which the quantized, and infinitely degenerate, energy eigenspace structure of the 2d Landau Hamiltonian \[1\] is largely responsible. The study of the 2d quantum Hall effect, including both integer and fractional ones, has been greatly benefitted by the elegant analytic properties of the Landau level wave functions (not only in Euclidean space, but also in curved or topologically different backgrounds \[2\]). In this 2d Landau Hamiltonian, the vector potential describing the background magnetic field is Abelian.

Recently, it has been noticed by various authors that there also exist higher-dimensional, especially three-dimensional (3d), Hamiltonians the eigenstate structure of which exhibit Landau-level-like degeneracy \[3–6\]. A distinctive feature from the 3d Euclidean-space Landau systems of Refs. \[4–6\] is that the related Hamiltonian typically contains spin-orbit coupling term \(E \times p \cdot \sigma\) (for a radial electric field \(E\) for instance), which can be accommodated within the standard ‘magnetic Hamiltonian’ by having the spin-\(\frac{1}{2}\) particle couple to a non-Abelian SU(2) gauge potential of the form \(A = E \times \sigma\). One may expect that some nontrivial physical applications utilizing the latter Hamiltonians come out in the future.

It would certainly be useful to know something on the Hamiltonian ‘form’ that can give rise to such highly degenerate eigenstate structure, especially for the ground state of the system. In the case of a nonrelativistic particle described by a spinor wave function in \(n\)-dimensional Euclidean (configuration) space, we may take the Hamiltonian to have the general form

\[
H = \frac{1}{2M} \left( p \cdot p + \mathcal{P}(x, S) \cdot p + \mathcal{Q}(x, S) \right),
\]

(1)

where \(x = (x^1, \cdots, x^n)\) denote position coordinates, \(p = -i\hbar \nabla\) the momentum differential operators, and \(S = \{S_{ij}; 1 \leq i < j \leq n\}\) represent related spin generators. (If the Hamiltonian involves functions \(\mathcal{P}_i(x, S)\) which are linear in \(S_{ij}\), one may describe the related interaction by defining appropriate background gauge fields associated with the group SO(\(n\)); but in our work, we will not follow this line.) We are here particularly keen on the ground state degeneracy, and to facilitate our discussion it will be assumed that the function \(\mathcal{Q}(x, S)\) in (1) has been chosen so that the lowest eigenvalue of \(H\) be equal to zero. Now our problem is: what kind of structures (and behaviors) for the functions \(\mathcal{P}_i(x, S)\) and \(\mathcal{Q}(x, S)\) should be assumed to make our Hamiltonian to have a large, even infinite, zero-energy eigenspace?

In the 2d case, Aharonov and Casher \[7\] made an interesting observation. If one
chooses

\[ P_i = -\frac{2e}{c} A_i(x), \quad (i = 1, 2; \ x = (x, y)) \]

\[ Q = \left( \frac{i\hbar e}{c} \nabla \cdot A + \frac{e^2}{c^2} A \cdot A \right) - \frac{\hbar e}{c} \sigma_3 B \]  

(2)

in (1), with arbitrary vector potential \( A(r) \) and its magnetic field \( B = \partial_1 A_2 - \partial_2 A_1 \), one is led to the Pauli Hamiltonian

\[ H = \frac{1}{2M} H_D^2 \]  

(3)

with

\[ H_D = \sigma \cdot (p - \frac{e}{c} A) . \]  

(4)

The \( \sigma \)'s are usual \( 2 \times 2 \) Pauli matrices. Then, using the property of the ‘Dirac Hamiltonian’ \( H_D \), they showed that ground state wave functions of \( H \) – the states with zero energy – can always be found analytically, with the degree of degeneracy determined by the total magnetic flux in accordance with the index theorem \[8, 9\]. Hence, if there exists a magnetic field approaching a nonzero constant value asymptotically, infinitely degenerate ground states follow always.

In this work we will push this idea to higher dimensional setting to get a unified understanding on the problem posited above (in the Euclidean space only). It is found that there exists a special family of Hamiltonians that enjoys the explicit ground-state integrability and encompasses all previously known models with the same property as special cases. New 3d models with interesting ground-state structure emerge along the way. In our 3d model with a uniform magnetic field, it is shown that entire 2d Landau-level wave functions, separated along the field direction, make up degenerate ground states.

The rest of this work is organized as follows. In Sec. 2, a rather general discussion is offered on the structure of nonrelativistic Hamiltonians which can exhibit large ground-state degeneracy. In Sec. 3 we consider two classes of 3d Hamiltonians for which the full space of degenerate ground states can be exhibited explicitly as in the case of the 2d Pauli Hamiltonian. One is without any magnetic field and the other is with a magnetic field. In Sec. 4 we conclude with some remarks. In Appendix, we summarize the ideas in arbitrary spatial dimension.

2 Hamiltonians with large ground-state degeneracy

Our general strategy is as follows: large ground-state degeneracy results if a given nonrelativistic Hamiltonian \( H \) can be written in a square form \[3\], with the related Dirac Hamiltonian \( H_D \) having a structure that admits a large number of zero-energy eigenstates as required by the index analysis in open Euclidean spaces \[9\]. Here we
restrict to hermitian $H_D$ so that the spectrum of the nonrelativistic Hamiltonian $\mathcal{H}$ is nonnegative.

Being interested ultimately in nonrelativistic theory, we will write our Dirac Hamiltonian in the form

$$H_D = \alpha \cdot p + K(x, \alpha, \beta), \quad (5)$$

by using hermitian Dirac matrices $\alpha = (\gamma^0, \gamma^1, \cdots, \gamma^n)$ and $\beta = \gamma^0$ which satisfy the relations $\{\alpha^i, \alpha^j\} = 2\delta^{ij}I$ and $\{\beta, \alpha^i\} = 0$ (with the matrices $\gamma^\mu (\mu = 0, 1, \cdots, n)$ satisfying the Dirac-Clifford algebra relation $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$, $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, \cdots)$). The matrix $\beta$ is taken to have the diagonal form

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (6)$$

and the function $K(x, \alpha, \beta)$ in (5), a Dirac matrix polynomial, may be constrained by the condition $\{\beta, K(x, \alpha, \beta)\} = 0$ so that we have $[\beta, H_D] = 0$ and $[\beta, \mathcal{H}] = 0$. As $\beta$ is diagonal, the Hamiltonian (3) can be written as

$$\mathcal{H} = \frac{1}{2M} H_D^2 = \begin{pmatrix} \mathcal{H}_+ & 0 \\ 0 & \mathcal{H}_- \end{pmatrix}. \quad (7)$$

With our choice of the Dirac Hamiltonian (5), eigenstates of the nonrelativistic Hamiltonian $\mathcal{H}$ of (3) can always be chosen to have a definite $\beta$-parity, $\eta = \pm 1$. For a given energy eigenstate $\Psi$ with the properties $\beta \Psi = \eta \Psi$ and $\mathcal{H} \Psi = E \Psi$ where $E > 0$, another state $\Psi' \equiv H_D \Psi$ will satisfy

$$\beta \Psi' = -\eta \Psi', \quad \mathcal{H} \Psi' = \frac{1}{2M} (H_D)^3 \Psi = E \Psi'. \quad (8)$$

Hence all nonzero energy eigenstates of $\mathcal{H}$ appear pairwise, one with $\eta = +1$ and the other with $\eta = -1$; this means that, when we leave out zero energy states, the two nonrelativistic Hamiltonians given in (7) are isospectral. For zero energy states, which must be the ground states, if exist, the situation can be different.

Subsequent developments depend on the spatial dimension, and we shall study 2d problem in this section and 3d problems in the next section. The case of higher-dimensional spaces will be considered in the context of specific models in the Appendix.

In 2d case, Dirac matrices are assumed by $\alpha^i = \sigma_i (i = 1, 2)$ and $\beta = \sigma_3$. Our Dirac Hamiltonian will then have the general form

$$H_D^{(2)} = \sum_{i=1}^2 \left\{ \sigma_i p_i + \sigma_i U_i(x) + i \sigma_i \beta V_i(x) \right\}. \quad (9)$$
But we here have a 2d-specific identity $\sigma_i \beta = -ic_{ij} \sigma_j$, and so the $U_i$ and $V_i$ terms in (3) are not independent. The form (4), used to define the 2d Pauli Hamiltonian, is essentially the unique possibility; but, one may express this by an alternative form

$$H_D^{(2)} = \sum_{i=1}^{2} \sigma_i \left( p_i - \frac{ic}{e} \beta \epsilon_{ij} A_j(x) \right). \quad (10)$$

In the gauge $\nabla \cdot A = 0$, we can find a scalar ‘potential’ $\phi(x)$ such that

$$A_1 = \frac{hc}{e} \partial_2 \phi, \quad A_2 = \frac{hc}{e} \partial_1 \phi \quad (11)$$

and so the 2d magnetic field $B = \frac{hc}{e} \nabla^2 \phi$. The alternative form (10) becomes

$$H_D^{(2)} = -ih \sum_{i=1}^{2} \sigma_i \left[ \partial_i + \beta \partial_i \phi(x) \right]. \quad (12)$$

If we here take a background of the form $\phi = \frac{e}{\sqrt{2}mc} B_0 (x^2 + y^2)$ as appropriate for a uniform magnetic field $B(x) = B_0$, the above Hamiltonian $H_D^{(2)}$ reduces to the 2D version of a so-called Dirac oscillator Hamiltonian (but without mass term). (This was noticed also in Ref. [5]). If $\Psi_0$ corresponds to a zero energy state of $\mathcal{H}^{(2)} = (H_D^{(2)})^2 / 2M$, it should satisfy the equation

$$H_D^{(2)} \Psi_0 = -ih \sum_{i=1}^{2} \sigma_i \cdot (\partial_i + \beta \partial_i \phi) \Psi_0 = 0. \quad (13)$$

With the $\beta$ parity chosen such that $\beta \Psi_0 = \eta \Psi_0$ ($\eta = \pm 1$), we may set

$$\Psi_0(\vec{x}) = e^{-\eta \phi(x)} F_\eta(x) \quad (14)$$

to recast (13) into the following equation on the 2-component spinor $F_\eta$:

$$\left( \sum_{i=1}^{2} \sigma_i \partial_i \right) F_\eta = 0. \quad (15)$$

If we here write

$$F_{\eta=+1} = \begin{pmatrix} f_{+1} \\ 0 \end{pmatrix}, \quad F_{\eta=-1} = \begin{pmatrix} 0 \\ f_{-1} \end{pmatrix}, \quad (16)$$

we then see from (15) that the scalar functions $f_\eta(x)$ should satisfy

$$(\partial_x + i\eta \partial_y) f_\eta(x, y) = 0, \quad (17)$$

that is, $f_\eta(x, y)$ should be a function of the variable $\zeta = x + i\eta y$ only. This was noted already in Ref. [7], and one may use this result together with (14) to produce ground state wave functions completely.
The number of independent ground states is decided by the asymptotic behavior of the scalar $\phi$ which is in turn dictated by total magnetic flux on the 2d plane. If the magnetic field is asymptotically uniform, i.e., $B(\mathbf{x}) \to B_0(>0)$ as $r = \sqrt{x^2 + y^2} \to \infty$, $\phi(\mathbf{x}) \to \frac{\phi_0}{r^2} B_0 r^2$ asymptotically in the rotationally symmetric gauge. In this case, one finds infinitely degenerate ground states (all with the eigenvalue $\eta = +1$), and the explicit ground state wave functions take the form

$$\left\{ \Psi_{0(n)}(\mathbf{x}) = e^{-\phi(\mathbf{x})}(x + iy)^n \;; \; n = 0, 1, 2, \ldots \right\}. \quad (18)$$

Note that all wave functions in (18) remain normalizable as long as $(\ln r)^{-1}|\phi(\mathbf{x})| \to \infty$; with $|\phi(\mathbf{x})| \to c \ln r$ ($c > 0$), only a finite number of zero-energy modes are allowed. In the asymptotically uniform magnetic field, one could have chosen the asymptotically Landau gauge potential with $\phi(\mathbf{x}) \to \frac{c}{2} B_0 y^2$ as $r \to \infty$. Then, the ground state wave function would take the form of continuum states

$$\left\{ \Psi_{0(k)}(\mathbf{x}) = e^{-\phi(\mathbf{x})} e^{ik(x+iy)} \;; \; k \text{ real} \right\}. \quad (19)$$

### 3 Models based on 3d Dirac Hamiltonians

In 3d case, Dirac matrices $\alpha^i$ ($i = 1, 2, 3$) and $\beta$ are provided by following $4 \times 4$ matrices

$$\alpha^1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (20)$$

Now our hermitian Dirac Hamiltonian $H_D$ satisfying the condition $\{\beta, H_D\} = 0$ may have the form

$$H_D^{(3)} = \alpha \cdot \left[ p + U + i\beta V \right] + \gamma^5 W(\mathbf{x}) + i\gamma^5 \beta X(\mathbf{x}) , \quad (21)$$

where

$$\gamma^5 = -i\alpha^1 \alpha^2 \alpha^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (22)$$

and all terms exhibited above are independent. Notice that our Dirac operator in (21) is entirely off-diagonal. Here the situation differs from the 2D case as Dirac spinors are defined in a 4-column space in contrast to nonrelativistic spinors defined in a 2-column space. Because of this, we are led to consider a pair of nonrelativistic Hamiltonians $H_{\pm}^{(3)}$ (defined in 2-column spaces), according to

$$H_{\pm}^{(3)} = \begin{pmatrix} H_+^{(3)} & 0 \\ 0 & H_-^{(3)} \end{pmatrix} = \frac{1}{2M} \left( H_D^{(3)} \right)^2. \quad (23)$$

Note that the subscripts $\pm$ denote the parity under 3d $\beta$. In the form (23) we have natural candidate 3d Hamiltonians which can result in large ground state degeneracy.
with suitably chosen background potentials $U_i(x)$, $V_i(x)$, $W(x)$ and $V(x)$. Various potentials here may be considered in connection with their specific effects: $U_i(x)$ for standard magnetic vector potentials, and $V_i(x)$ for spin-orbit-coupling-like terms (see below), etc.

In quantum-field-theoretic investigations the Dirac Hamiltonians similar to our form (21) have been discussed previously \[11\]-\[13\]; there, potentials are typically those related to localized solitonic backgrounds (e.g., magnetic monopoles) and as such they usually involve internal space generators also. They also appear in some condensed matter literature discussing topological defects in insulators and superconductors; see Ref. \[14\] for instance. Zero modes of those Hamiltonians have been studied, together with responsible topological invariants. But, with the backgrounds of less restricted asymptotic behaviors (e.g. with unbounded potentials), and especially with an eye on their physical significance in nonrelativistic Hamiltonian contexts, the analysis is not so simple and up to our knowledge no systematic study has been made. Hence, leaving such to our future study, we shall below concentrate on some special class of Hamiltonians, which enjoy explicit ground-state integrability and so can be used to exhibit some of the expected features. As we shall see, our model Hamiltonians, which have not been seriously considered in a relativistic setting, turn out to have some interesting nonrelativistic contents.

Our first model in 3d is obtained from (21) by keeping only the potential $V(x)$ in the form $V(x) = -\hbar \nabla \phi(x)$, $\phi(x)$ being arbitrary. That is, we consider the Dirac Hamiltonian

$$H_D^{(3)} = \alpha \cdot \left[ p - i\hbar \beta \nabla \phi(x) \right],$$

(24)

and this model may in fact be considered in arbitrary spatial dimensions. In 3d case, a simple calculation using the Dirac matrices in (20) yields the 3d Hamiltonian (23) which consists of a pair of nonrelativistic Hamiltonians

$$H_\pm^{(3)} = \frac{1}{2M} \left( p^2 \mp 2\hbar (\nabla \phi \times p) \cdot \sigma + \hbar^2 \nabla \phi \cdot \nabla \phi \mp \hbar^2 \nabla^2 \phi \right).$$

(25)

If we here define the ‘electric’ field by $E(x) = -\nabla \phi(x)$, the second term in the right hand side of (25) obviously describes a spin-orbit coupling. Some recently discussed models with three-dimensional Landau-level-like structures correspond to special cases of this model, i.e., $\phi(x) = (\text{const.})(x^2 + y^2 + z^2)$ in Refs. \[4\],\[5\], and $\phi(x) = (\text{const.})z^2$ in the model of Ref. \[6\]. In our discussion, however, the detailed profile of the potential $\phi(x)$ will be left largely arbitrary. As far as ground-state degeneracy structure is concerned, what matters is the asymptotic behavior of the potential – in our framework, a particular symmetry in the background potential is irrelevant. But our approach cannot say anything as regards possible excited-state degeneracy.

The zero-energy eigenfunctions $\Psi_0$ of the Dirac Hamiltonian (24), as needed for
the ground states of the Hamiltonian \( (25) \), can be written as

\[
\Psi_0(\vec{x}) = e^{-\eta \phi(\vec{x})} F_{\eta}(\vec{x})
\]

(26)

with the \( \beta \) parity \( \eta \). Then the wave function \( F_{\eta}(\vec{x}) \) satisfies

\[
\alpha \cdot \nabla F_{\eta}(\vec{x}) = 0 . \tag{27}
\]

For a given large-\( |\vec{x}| \) behavior of \( \phi(\vec{x}) \), we usually obtain a physically acceptable wave function from (27) only with a particular sign for \( \eta \). Aside from this, finding ground state wave function is now down to solving the background-free equation (27). In this case, we again write \( F_{\eta=+1} = \begin{pmatrix} f_+ \\ 0 \end{pmatrix} \) and \( F_{\eta=-1} = \begin{pmatrix} 0 \\ f_- \end{pmatrix} \), but this time \( f_\eta \) make 2-component spinors. Then the Dirac equation (27), with the Dirac matrices in (20) used, reduces to the 2-component spinor equation

\[
\sigma \cdot \nabla f_{\eta} = 0 . \tag{28}
\]

Since

\[
(\sigma \cdot \nabla)^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \equiv \nabla^2 , \tag{29}
\]

the 2-component spinor \( f \) should have harmonic functions as its components. Now, in view of the identity

\[
\nabla^2(z + ix \cos u + iy \sin u)^p = 0 , \quad (u \text{ real}) \tag{30}
\]

one may write the general solution to (28) in the form [15]

\[
f(\vec{x}) = \int_{-\pi}^{\pi} h(z + ix \cos u + iy \sin u, u) \chi(u) du . \tag{31}
\]

Here \( h \) is an arbitrary function, and \( \chi(u) \) represents a 2-spinor satisfying

\[
(\sigma_3 + i\sigma_1 \cos u + i\sigma_2 \sin u)\chi(u) = 0 , \tag{32}
\]

whose solution has the following form

\[
\chi(u) = \begin{pmatrix} 1 \\ ie^{iu} \end{pmatrix} . \tag{33}
\]

In (31) any function \( h \) is allowed as long as, when used in (26), it gives rise to a physically acceptable ground-state wave function \( \Psi_0(\vec{x}) \). Given the potential with the asymptotic behavior \( \phi(\vec{x}) \to cr^2 \ (c > 0) \) as \( r = \sqrt{x^2 + y^2 + z^2} \to \infty \) (as in the model of Refs. [4,5]), we must choose \( \eta = +1 \) and then take for \( h \) following polynomial forms

\[
h = (z + ix \cos u + iy \sin u)^l e^{imu} , \ (l = 0, 1, 2, \cdots ; m = -l, -l+1, \cdots l) . \tag{34}
\]
Then, using (33) and performing the integration over \( u \), we obtain the results for \( f(x) \) written using spherical harmonics:

\[
f_{+1}(x) = \left( \sqrt{\frac{l+m+1}{2l+1}} r^l Y_{l,m}(\theta, \phi) \right)
\]

We thus have infinitely degenerate ground states in any such background. Actually all forms in (35) leads to normalizable wave functions as long as the asymptotics of the background potential is such that \((\ln r)^{-1}|q\phi(x)| \to \infty \) as \( r \to \infty \), and, if \( |q\phi(x)| \to c \ln r \) \((c > 0) \) as \( r \to \infty \), only a finite number of zero-energy modes survive. The detailed form of the background potential at finite \( r \) is not important for our discussion.

On the other hand, with the background potential having the asymptotic behavior \( \phi(x) \to cz^2 \) \((c > 0) \) as \( r \to \infty \) (as in the model considered in Ref. [6]), we obtain bounded ground-state wave functions with \( f_+(x) \) given by the continuum

\[
f_{+1}(x) = e^{k(z+ix \cos u+y \sin u)} \left( \frac{1}{ie^{iu}} \right), \quad (k, u \text{ real}) .
\]

Note that, integrating this form over \( u \) with weight \( e^{imu} \), one may also take for \( f_{+1} \) the following expression:

\[
f_{+1}(x) = e^{kz+im\varphi} \left( \begin{array}{c} J_m(k\rho) \\ iJ_{m+1}(k\rho) \end{array} \right), \quad (k \text{ real } ; m = 0, \pm 1, \pm 2, \ldots)
\]

where \( x + iy = re^{i\varphi} \) and \( J_m(k\rho) \) denotes Bessel functions.

For further discussions on these wave functions including possible physical applications, see Ref. [4–6]. In 3d, the two expressions we have chosen above for the asymptotic form of the scalar \( \phi \) are not gauge equivalent as the scalar field is not connected to the gauge field in 3d and so the physics for these two cases are different.

Let us now consider a little more complicated 3d Dirac Hamiltonian which has both gauge field and the scalar gradient as in

\[
H_D = \alpha \cdot \left( [p - \frac{e}{c} A(x)] - i\hbar \beta \nabla \phi(x) \right)
\]

Under this choice, the related nonrelativistic Hamiltonians (23) read as

\[
\mathcal{H}_\pm = \frac{1}{2M} \left( [p - \frac{e}{c} A]^2 \mp 2\hbar \nabla \phi \times [p - \frac{e}{c} A] \cdot \sigma - \frac{\hbar e}{c} \sigma \cdot B + \hbar^2 \nabla \phi \cdot \nabla \phi \mp \hbar^2 \nabla^2 \phi \right).
\]

where \( B = \nabla \times A \). If \( \Psi_0 \) corresponds to a zero-energy ground state of \( \mathcal{H} \), it should satisfy the equation

\[
H_D \Psi_0 = \alpha \cdot \left( [ - i\hbar \nabla - \frac{e}{c} A(x)] - i\hbar \beta \nabla \phi \right) \Psi_0 = 0 .
\]
With the $\beta$ parity $\eta$ chosen so that $\beta\Psi_0 = \eta\Psi_0$ ($\eta = \pm 1$), we may set
\[ \Psi_0(\vec{x}) = e^{-\eta\phi(x)}F_\eta(x) \] (41)
to recast (40) into the following equation on $F_\eta(x)$:
\[ \alpha \cdot \left[ -i\hbar\nabla - \frac{e}{c}A(x) \right] F_\eta(x) = 0 . \] (42)

For a given large-$|x|$ behavior of $\phi(x)$, we usually obtain a physically acceptable wave function from (41) only with a particular sign for $\eta$. Aside from this, finding ground state wave function is now down to solving the simpler equation (42). Writing $F_{+1} = \begin{pmatrix} f_{+1} \\ 0 \end{pmatrix}$ and $F_{-1} = \begin{pmatrix} 0 \\ f_{-1} \end{pmatrix}$, (42) reduces to the 2-component spinor equations
\[ \sigma \cdot (\nabla - \frac{ie}{\hbar c}A) f_\eta = 0 \] (43)

For some magnetic flux, one could find normalizable zero-energy ground states if the scalar function $\phi(x)$ and the $\beta$ parity $\eta$ were chosen suitably.

Since a direct analysis of (43) with an arbitrary 3d vector potential $A(x)$ is impossible, let us restrict our attention to the case of a strictly 2d vector potential
\[ A = (A_1(x,y), A_2(x,y), A_3 = 0) \] (44)
and rewrite (43) as
\[ (\tilde{H}_D^{(2)} + ip_3)f(x,y,z) = 0, \quad \tilde{H}_D^{(2)} = i\sigma_3 \sum_{i=1}^{2} \sigma_i(p_i - \frac{e}{c}A_i) \] (45)

As $[\tilde{H}_D^{(2)}, p_3] = 0$, we choose the eigenfunctions of $p_3$ as the solution of (45). Also note that $\tilde{H}_D^{(2)}$ is another hermitian Dirac Hamiltonian related to a 2d Pauli Hamiltonian, and so all normalizable energy eigenfunctions of this operator can be found in principle. For the solutions of (45) with $(p_3)' = 0$ we need zero-energy eigenfunctions of $\tilde{H}_D^{(2)}$: they were found in Sec.2. But, for the solutions with $(p_3)' \neq 0$, we here need also nonzero-energy eigenfunctions of $\tilde{H}_D^{(2)}$. As nonzero eigenvalues of this Dirac operator should appear pairwise with opposite signs, we may express the related eigenvalue equations schematically as
\[ \tilde{H}_D^{(2)} h_\ell = k_\ell h_\ell , \quad \tilde{H}_D^{(2)} \sigma_3 h_\ell = -k_\ell \sigma_3 h_\ell . \] (46)

where $\ell$ labels all independent nonzero-energy eigenstates. Then we can represent the $(p_3)' = \pm ik_\ell(\neq 0)$ solutions to (45) by the form
\[ f = e^{-\frac{ik_\ell}{2}}\hat{h}_\ell(x,y), \quad f = e^{\frac{ik_\ell}{2}}\sigma_3\hat{h}_\ell(x,y), \] (47)
Note that wave functions with imaginary eigenvalue of \( p_3 = -i\hbar \partial_3 \) would be out of consideration usually. However, in our case, there is additional factor \( e^{-\eta \phi} \) entering the wave function \( (41) \), and an appropriate choice of \( \eta \) and \( \phi \) (imagine the scalar \( \phi \) having the asymptotic behavior \( \phi \sim cz^2 \)) would provide sufficient fall off for finite \( k_\ell \) at large \( |z| \), making the ground wave function \( \Psi_0 \) in \( (41) \) normalizable.

Especially, if the vector potential \( (44) \) is chosen to be that of the uniform magnetic field \( B = B \hat{z} \), we can represent the eigenstates of \( \tilde{H}_D^{(2)} \) in terms of the well-known Landau-level wave functions. Indeed, we then have \( \tilde{H}_D^{(2)} \) (in the symmetric gauge) expressed as

\[
\tilde{H}_D^{(2)} = -\sqrt{\frac{2\hbar eB}{c}} \begin{pmatrix} 0 & \bar{a} \\ a & 0 \end{pmatrix}
\]

where \( a, \bar{a} \) denote the creation and annihilation operators

\[
a = \sqrt{\frac{2\hbar c}{\epsilon B}} (\partial \zeta + \partial \chi), \quad \bar{a} = -\sqrt{\frac{2\hbar c}{\epsilon B}} (\partial \zeta - \partial \chi),
\]

with \( \zeta = x + iy, A_i = -\frac{\hbar e}{\epsilon} \epsilon_i \partial_j \chi, \) and \( \chi = \frac{\epsilon B}{4\hbar c} \zeta \bar{\zeta} \). The eigenstates and eigenvalues of \( \tilde{H}_D^{(2)} \) are, respectively,

\[
h_{\ell,n}(x) = \begin{pmatrix} \bar{a}_\ell \Phi_n \\ -\ell \bar{a}_\ell^{-1} \Phi_n \end{pmatrix}, \quad k_\ell = \ell \sqrt{\frac{2\hbar eB}{c}}
\]

where \( \ell = 0, \pm 1, \pm 2, \cdots \), and \( \Phi_n = \zeta^n e^{-\chi} \) with \( n = 0, 1, 2, \cdots \). Especially, a choice \( \phi = cz^2 \) would make all 2d Landau levels be available for ground states of the 3d problem; but, as they are to be multiplied by the factor \( e^{\pm \frac{\hbar e}{4\hbar c} \zeta \bar{\zeta}^2} \), the excited 2d Landau levels appear separated along the z-axis. As such degenerate stacking of all 2d landau levels is possible, ground state degeneracy of this 3d system is infinite times larger than that of the 2d Landau system. The 2d Landau level functions in \( (50) \) can also be used to find the ground state wave functions when the scalar field \( \phi \) has somewhat different asymptotic behaviors. In the case that we have \( \phi \sim \gamma |z| (\gamma \geq 0) \) asymptotically, only a partial set of the above 2d Landau levels, that is, those with the Landau level index \( \ell \) satisfying the condition \( |k_\ell/\hbar| < \gamma \), would be acceptable for the ground states of the 3d system.

### 4 Concluding Remarks

In this paper we have shown that a fruitful way to study quantum systems exhibiting large ground-state degeneracy is to look for a connection to a Dirac operator. We are then led to a particular class of Hamiltonians, which exhibit explicit ground-state integrability (for any background potential with the given asymptotic behavior) and at the same time serve as a unified framework for some of the recently proposed
Hamiltonians in condensed matter physics. Needless to say, if the models considered in this work turn out to have some direct experimental relevance, it will be most welcome. To gain further insight, it should be desirable to have our analysis extended to the 3d model based on the full 3d Dirac Hamiltonian [21]. Considering our model in a curved space will be another interesting future problem.

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A Arbitrary Dimension

We shall comment on the case with a higher-dimensional-space version of the Hamiltonian $\mathcal{H}$ obtained from the $n$-dimensional generalization of the Dirac Hamiltonian [22]. For this, following Ref. [5], it is convenient to introduce rank-$k$ $\Gamma$-matrices, $\Gamma_i^{(k)}$, $i = 1, 2, \ldots, 2k + 1$ which are $2^k$ by $2^k$ and satisfy $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$; these matrices can be constructed iteratively using the recursive formulas

\begin{align}
\Gamma_i^{(k)} &= \begin{pmatrix} 0 & \Gamma_i^{(k-1)} \\ \Gamma_i^{(k-1)} & 0 \end{pmatrix} (i \leq k-1), \\
\Gamma^{(k)}_{2k} &= \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \\
\Gamma^{(k)}_{2k+1} &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\end{align}

\begin{equation}
(\text{A.1})
\end{equation}

starting from rank-1 $\Gamma$-matrices $\Gamma_i^{(1)} = \sigma_i$ ($i = 1, 2, 3$). Using these $\Gamma$-matrices, the SO(d) generators in the fundamental spinor representation can be given. In $d = 2k + 1$-dimensional space, one can take

\begin{equation}
S_{ij} = -S_{ji} = -\frac{i}{4} \left[ \Gamma_i^{(k)}, \Gamma_j^{(k)} \right] (1 \leq i < j \leq 2k + 1)
\end{equation}

\begin{equation}
(\text{A.2})
\end{equation}

for the rotation generators. On the other hand, in $d = 2k$-dimensional space, we have two inequivalent set of SO(d) generators, $\{S_{ij}\}$ and $\{S'_{ij}\}$, which can be identified with

Set 1 : $S_{ij} = -S_{ji} = \begin{cases} -\frac{i}{4} \left[ \Gamma_i^{(k-1)}; \Gamma_j^{(k-1)} \right], & \text{for } 1 \leq i < j \leq 2k - 1 \\ \frac{i}{2} \Gamma_i^{(k-1)}, & \text{for } j = 2k \text{ and } 1 \leq i \leq 2k - 1 \end{cases}$, \begin{equation}
(\text{A.3})
\end{equation}

Set 2 : $S'_{ij} = -S'_{ji} = \begin{cases} -\frac{i}{4} [ \Gamma_i^{(k-1)}; \Gamma_j^{(k-1)} ], & \text{for } 1 \leq i < j \leq 2k - 1 \\ \frac{i}{2} \Gamma_i^{(k-1)}, & \text{for } j = 2k \text{ and } 1 \leq i \leq 2k - 1 \end{cases}$, \begin{equation}
(\text{A.4})
\end{equation}
We can also specify our Dirac matrices $\alpha^i$ and $\beta$ in even- or odd-dimensional space as follows: in $2k$-dimensional space, take the $2^k \times 2^k$ matrices

$$\alpha^i = \Gamma^{(k)}_{i} \quad (i = 1, \cdots, 2k), \quad \beta = \Gamma^{(k)}_{2k+1}, \quad (A.5)$$

and, in $(2k+1)$-dimensional space, use the expressions (which are $2^{k+1}$ by $2^{k+1}$)

$$\alpha^i = \begin{pmatrix} 0 & \Gamma^{(k)}_{i} \\ \Gamma^{(k)}_{i} & 0 \end{pmatrix} \quad (i = 1, \cdots, 2k+1), \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (A.6)$$

Inserting these Dirac matrices in the definition for $H_D = \alpha \cdot \left[ p - i\hbar \beta \nabla \phi(x) \right]$ (A.7)

we come up with the Hamiltonian $H = \frac{1}{2M} H_D^2$ in arbitrary spatial dimensions. In term of two components $H_{\pm}$ in (7), we have explicitly

$$d = 2k$$

$$2M H_+ = p^2 - 4\hbar S_{ij}(\partial_i\phi)p_j + h^2 \nabla \phi \cdot \nabla \phi - h^2 \nabla^2 \phi$$

$$2M H_- = p^2 + 4\hbar S_{ij}'(\partial_i\phi)p_j + h^2 \nabla \phi \cdot \nabla \phi + h^2 \nabla^2 \phi, \quad (A.8)$$

$$d = 2k + 1$$

$$2M H_+ = p^2 - 4\hbar S_{ij}(\partial_i\phi)p_j + h^2 \nabla \phi \cdot \nabla \phi - h^2 \nabla^2 \phi$$

$$2M H_- = p^2 + 4\hbar S_{ij}'(\partial_i\phi)p_j + h^2 \nabla \phi \cdot \nabla \phi + h^2 \nabla^2 \phi. \quad (A.9)$$

For $k = 1$ with (A.8), setting $S_{12} = -S_{12}' = \frac{1}{2}$ produces the 2D Pauli Hamiltonian with $A_i = -\epsilon^{ij} \partial_j \phi$; taking $k = 1$ in (A.9) and setting $S_{ij} = \frac{1}{2} \epsilon^{ijk} \sigma_k$ leads to our earlier 3D expression.

For the ground-state wave functions one may solve the zero-energy Dirac equation $H_D \Psi_0 = 0$ by setting $\Psi_0 = e^{-\beta \phi} F$. Then, with the definite choice of the $\beta$ parity $\eta = \pm 1$, we again obtain the equation (27) for $F_\eta$. If we here write $F_{\eta = 1} = \begin{pmatrix} f_{+1} \\ 0 \end{pmatrix}$ and $F_{\eta = -1} = \begin{pmatrix} 0 \\ f_{-1} \end{pmatrix}$, $f_{\pm 1}$ in $d = 2k$-dimensional ($d = 2k + 1$-dimensional) space will have $2^{k-1}$ columns ($2^k$ columns) and must satisfy the following equations:

$$d = 2k : \quad \sum_{i=1}^{2k-1} \Gamma_{i}^{(k-1)} \partial_i + i\eta \partial_{2k} f_{\eta} = 0, \quad (A.10)$$

$$d = 2k + 1 : \quad \sum_{i=1}^{2k+1} \Gamma_{i}^{(k)} \partial_i f_{\eta} = 0 \quad \text{(for both } \eta = \pm 1). \quad (A.11)$$
Note that $f_\eta$, in both even- and odd-dimensional spaces, should have harmonic functions as its components. Hence, generalizing (31), we may express the column function $f_\eta$ in higher dimensional space by an integral

$$f_\eta = \int_{u \in S^{d-2}} h(x_d + i u \cdot x) \chi_\eta(u) \, d\Omega_{d-2}$$

(A.12)

where $u \cdot x \equiv u^1 x^1 + \cdots + u^{d-1} x^{d-1}$, $u$ denoting a vector which can take values on the sphere $S^{d-2} : (u^1)^2 + \cdots + (u^{D-1})^2 = 1$, and $\chi_\eta(u)$ represents a column vector satisfying the condition

$$d = 2k : \left( \sum_{i=1}^{2k-1} u^i \Gamma_i^{(k-1)} \right) \chi_\eta(u) = -\eta \chi_\eta(u) ,$$

(A.13)

$$d = 2k + 1 : \left( \sum_{i=1}^{2k} i u^i \Gamma_i^{(k)} + \Gamma^{(k)}_{2k+1} \right) \chi_\eta(u) = 0 \text{ (for both } \eta = \pm 1)$$

(A.14)

in even-and odd-dimensional spaces, respectively. Note that (A.13) and (A.14) may be written as eigenvector equations involving SO(D) generator matrices, viz.,

$$d = 2k : \left( \sum_{i=1}^{2k-1} u^i S_{i,2k} \right) \chi_{+1}(u) = -\frac{1}{2} \chi_{+1}(u) ,$$

$$d = 2k : \left( \sum_{i=1}^{2k-1} u^i S_{i,2k}^t \right) \chi_{-1}(u) = -\frac{1}{2} \chi_{-1}(u) ,$$

(A.15)

$$d = 2k + 1 : \left( \sum_{i=1}^{2k} u^i S_{i,2k+1} \right) \chi_{\pm 1}(u) = -\frac{1}{2} \chi_{\pm 1}(u) .$$

(A.16)

So one can use group theory to fix $\chi_\eta(u)$ here [16]. Then, depending on the asymptotic behaviors of the background potential $\phi(x)$, one may consider polynomial or/and exponential types for the function $h$ in (A.12), just as in 2d and 3d cases treated earlier. This way, spinor wave functions analogous to the form (35) (but now involving hyper-spherical harmonics [16]) or to the continuum expression (36) can be obtained ; using these in (26) will lead to normalizable (or at least bounded) ground-state wave functions only if the background potential $\phi(x)$ has ‘right’ asymptotic behaviors. For some of the explicit expressions regarding these wave functions, readers may consult Refs. [4–6] where they are discussed in the context of specially chosen background configurations.
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