LATTICES WITH MANY CONGRUENCES ARE PLANAR

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Dedicated to the memory of Ivan Rival

ABSTRACT. Let $L$ be an $n$-element finite lattice. We prove that if $L$ has strictly more than $2^{n-5}$ congruences, then $L$ is planar. This result is sharp, since for each natural number $n \geq 8$, there exists a non-planar lattice with exactly $2^{n-5}$ congruences.

1. Aim and introduction

Our goal is to prove the following statement.

Theorem 1.1. Let $L$ be an $n$-element finite lattice. If $L$ has strictly more than $2^{n-5}$ congruences, then it is a planar lattice.

In order to point out that this result is sharp, we will also prove the following easy remark. An $n$-element finite lattice $L$ is dismantlable if there is a sequence $L_1 \subset L_2 \subset \cdots \subset L_n = L$ of its sublattices such that $|L_i| = i$ for every $i \in \{1, \ldots, n\}$; see Rival [14]. We know from Kelly and Rival [10] that every finite planar lattice is dismantlable.

Remark 1.2. For each natural number $n \geq 8$, there exists an $n$-element non-dismantlable lattice $L(n)$ with exactly $2^{n-5}$ congruences; this $L(n)$ is non-planar.

We know from Freese [4] that an $n$-element lattice $L$ has at most $2^{n-1} = 16 \cdot 2^{n-5}$ congruences. In other words, denoting the lattice of congruences of $L$ by $\text{Con}(L)$, we have that $|\text{Con}(L)| \leq 2^{n-1}$. For $n \geq 5$, the second largest number of the set

$$\text{ConSizes}(n) := \{|\text{Con}(L)| : L \text{ is a lattice with } |L| = n\}$$

is $8 \cdot 2^{n-5}$ by Czédli [2], while Kulin and Mureșan [11] proved that the third, fourth, and fifth largest numbers of $\text{ConSizes}(n)$ are $5 \cdot 2^{n-5}$, $4 \cdot 2^{n-5}$, and $\frac{7}{2} \cdot 2^{n-5}$, respectively. Since both [2] and Kulin and Mureșan [11] described the lattices witnessing these numbers, it follows from these two papers that $|\text{Con}(L)| \geq \frac{7}{2} \cdot 2^{n-5}$ implies the planarity of $L$. So, [2], [11], and even their precursor, Mureșan [12] have naturally lead to the conjecture that if an
An $n$-element lattice $L$ has many congruences with respect to $n$, then $L$ is necessarily planar. However, the present paper needs a technique different from Kulin and Mureşan [11], because a [11]-like description of the lattices witnessing the sixth, seventh, eighth, \ldots, $k$-th largest numbers in ConSizes$(n)$ seems to be hard to find and prove; we do not even know how large is $k$. Fortunately, we can rely on the powerful description of planar lattices given by Kelly and Rival [10].

Note that although an $n$-element finite lattice with “many” (that is, more than $2^{n-5}$) congruences is necessarily planar by Theorem 1.1, an $n$-element planar lattice may have only very few congruences even for large $n$. For example, the $n$-element modular lattice of length 2, denoted usually by $M_{n-2}$, has only two congruences if $n \geq 5$. On the other hand, we know, say, from Kulin and Mureşan [11] that there are a lot of lattices $L$ with many congruences, whereby a lot of lattices belong to the scope of Theorem 1.1.

Outline and prerequisites. Section 2 recalls some known facts from the literature and, based on these facts, proves Remark 1.2 in three lines. The rest of the paper is devoted to the proof of Theorem 1.1.

Due to Section 2, the reader is assumed to have only little familiarity with lattices. Apart from some figures from Kelly and Rival [10], which should be at hand while reading, the present paper is more or less self-contained modulo the above-mentioned familiarity. Note that [10] is an open access paper at the time of this writing; see \url{http://dx.doi.org/10.4153/CJM-1975-074-0}.

2. Some known facts about lattices and their congruences

In the whole paper, all lattices are assumed to be finite even if this is not repeated all the time. For a finite lattice $L$, the set of nonzero join-irreducible elements, that of nonunit meet-irreducible elements, and that of doubly irreducible (neither 0, not 1) elements will be denoted by $J(L)$, $M(L)$, and $\text{Irr}(L) = J(L) \cap M(L)$, respectively. For $a \in J(L)$ and $b \in M(L)$, the unique lower cover of $a$ and the unique (upper) cover of $b$ will be denoted by $a^-$ and $b^+$, respectively. For $a, b \in L$, let $\text{con}(a, b)$ stand for the smallest congruence of $L$ such that $\langle a, b \rangle \in \text{con}(a, b)$. For $x, y \in J(L)$, let $x \equiv_{\text{con}} y$ mean that $\text{con}(x^-, x) = \text{con}(y^-, y)$. Then $\equiv_{\text{con}}$ is an equivalence relation on $J(L)$, and the corresponding quotient set will be denoted by

$$Q(L) := J(L)/\equiv_{\text{con}}.$$  \hfill (2.1)

As an obvious consequence of Freese, Ježek and Nation [5, Theorem 2.35] or Nation [13, Corollary to Theorem 10.5], for every finite lattice $L$,

$$|\text{Con}(L)| \leq 2^{|Q(L)|} \leq 2^{|J(L)|};$$  \hfill (2.2)

more explanation will be given later. The situation simplifies for distributive lattices; it is well known that

if $L$ is a finite distributive lattice, then $|\text{Con}(L)| = 2^{|J(L)|}$.  \hfill (2.3)
In order to explain how to extract (2.2) and (2.3) from the literature, we recall some facts. A quasiordered set is a structure \((A; \leq)\), where \(\leq\) is a quasiordering, that is, a reflexive and symmetric relation on \(A\). For example, if we let \(a \leq_{\text{con}} b\) mean \(\text{con}(a\sim, a) \leq \text{con}(b\sim, b)\), then \((J(L); \leq_{\text{con}})\) is a quasiordered set. A subset \(X\) of \((A; \leq)\) is hereditary, if \((\forall x \in X)(\forall y \in A)(y \leq x \Rightarrow y \in X)\). The set of all hereditary subsets of \((A; \leq)\) with respect to set inclusion forms a lattice \(\text{Hered}((A; \leq))\). Freese, Ježek and Nation [5, Theorem 2.35] can be reworded as \(\langle\text{Con}(L); \subseteq\rangle \cong \text{Hered}(\langle J(L); \leq_{\text{con}}\rangle)\). To recall this theorem in a form closer to [5, Theorem 2.35], for the \(\equiv_{\text{con}}\)-blocks of \(a,b \in J(L)\), we define the meaning of \(a/\equiv_{\text{con}} \leq_{\text{con}} a/\equiv_{\text{con}}\) as \(a \leq_{\text{con}} b\). In this way, we obtain a poset \(\langle Q(L); \leq_{\text{con}} /\equiv_{\text{con}}\rangle\). With this notation, the original form of [5, Theorem 2.35] states that \(\text{Con}(L) \cong \text{Hered}(Q(L); \leq_{\text{con}} /\equiv_{\text{con}})\).

Since \(\equiv_{\text{con}}\) will play an important role later, recall that for intervals \([a, b]\) and \([c, d]\) in a lattice \(L\), \([a, b]\) transposes up to \([c, d]\) if \(b \land c = a\) and \(b \lor c = d\). This relation between the two intervals will be denoted by \([a, b] \equiv_{\text{up}} [c, d]\). We say that \([a, b]\) transposes down to \([c, d]\), in notation \([a, b] \equiv_{\text{down}} [c, d]\) if \([c, d] \equiv_{\text{up}} [a, b]\). We call \([a, b]\) and \([c, d]\) transposed intervals if \([a, b] \equiv_{\text{up}} [c, d]\) or \([a, b] \equiv_{\text{down}} [c, d]\). It is well known and easy to see that

\[
\text{if } [a, b] \text{ and } [c, d]\text{ are transposed intervals, then } \text{con}(a, b) = \text{con}(c, d). \quad (2.4)
\]

Next, we note that \(\text{Con}(L)\) in (2.3) is a Boolean lattice. Oddly enough, we could find neither this fact, nor (2.3) in the literature explicitly. Hence, in this paragraph, we outline briefly a possible way of deriving these statements from explicitly available and well-known other facts; the reader may skip over this paragraph. So let \(L\) be a finite distributive lattice. Since \(L\) is modular, \(\text{Con}(L)\) is a Boolean lattice by Grätzer [6, Corollary 3.12 in page 41]. Pick a maximal chain \(0 < a_1 < \cdots < a_t = 1\) in \(L\). Here \(t\) is the length of \(L\), and it is well known that \(t = |J(L)|\); see Grätzer [7, Corollary 112 in page 114]. If \(\text{con}(a_{i-1}, a_i) = \text{con}(a_{j-1}, a_j)\), then it follows from Grätzer [9] and distributivity (in fact, modularity) that there is a sequence of prime intervals (edges in the diagram) from \([a_{i-1}, a_i]\) to \([a_{j-1}, a_j]\) such that any two neighboring intervals in this sequence are transposed. In the terminology of Adaricheva and Czédli [1], the prime intervals \([a_{i-1}, a_i]\) and \([a_{j-1}, a_j]\) belong to the same trajectory. Since no two distinct comparable prime intervals of \(L\) can belong to the same trajectory by [1, Proposition 6.1], it follows that \(i = j\). Hence, the congruences \(\text{con}(a_{i-1}, a_i), i \in \{1, \ldots, t\}\), are pairwise distinct. They are join-irreducible congruences by Grätzer [7, page 213], whereby they are atoms in \(\text{Con}(L)\) since \(\text{Con}(L)\) is Boolean. Clearly, \(\bigwedge_{i=1}^{t} \text{con}(a_{i-1}, a_i) = 1_{\text{Con}(L)}\), which implies that \(|\text{Con}(L)| = 2^t\). This proves (2.3) since \(t = |J(L)|\).

Next, a lattice is called planar if it is finite and has a Hasse diagram that is a planar representation of a graph in the usual sense that any two edges can intersect only at vertices. Let \(\mathbb{N}_0\) and \(\mathbb{N}_+\) denote the set \(\{0, 1, 2, \ldots\}\) of nonnegative integers and the set \(\{1, 2, 3, \ldots\}\) of positive integers, respectively. In their fundamental paper on planar lattices, Kelly and Rival [10]
gave a set
\[ \mathcal{L}_{KR} = \{A_n, E_n, F_n, G_n, H_n : n \in \mathbb{N}_0\} \cup \{B, C, D\} \]

of finite lattices such that the following statement holds.

**Proposition 2.1** (A part of Kelly and Rival [10, Theorem 1]). A finite lattice \( L \) is planar if and only if neither \( L \), nor its dual contains some lattice of \( \mathcal{L}_{KR} \) as a subposet.

Note that the lattices \( A_n \) are selfdual and Kelly and Rival [10] proved the minimality of \( \mathcal{L}_{KR} \), but we do not need these facts.

Next, we prove Remark 1.2. The *ordinal sum* of lattices \( L' \) and \( L'' \) is their disjoint union \( L' \cup L'' \) such that for \( x, y \in L' \cup L'' \), we have that \( x \leq y \) if and only if \( x \in L' \) or \( x \leq_{L''} y \) or \( x \in L' \) and \( y \in L'' \).

**Proof of Remark 1.2.** Let \( L(8) \) be the eight-element Boolean lattice. Next, for \( n > 8 \), let \( L(n) \) be the ordinal sum of \( L(8) \) and an \((n-8)\)-element chain. Since \( |J(L_n)| = n - 5 \), Remark 1.2 follows from (2.3). \( \square \)

Note that \( L(n) \) above occurs also in page 93 of Rival [14].

3. A lemma on subposets that are lattices

While \( \mathcal{L}_{KR} \) consists of lattices, they appear in Proposition 2.1 as subposets. This fact causes some difficulties in proving our theorem; this section serves as a preparation to overcome these difficulties. The set of *join-reducible elements* of a lattice \( L \) will be denoted by \( \text{JRed}(L) \). Note that
\[ \text{JRed}(L) = L \setminus (\{0\} \cup J(L)) = \{a \lor b : a, b \in L \text{ and } a \parallel b\}, \tag{3.1} \]

where \( \parallel \) stands for incomparability, that is, \( a \parallel b \) is the conjunction of \( a \nleq b \) and \( b \nleq a \). Similarly, \( \text{MRed}(L) = L \setminus (\{1\} \cup M(L)) \) denotes the set of *meet-reducible elements* of \( L \).

**Lemma 3.1.** Let \( L \) and \( K \) be finite lattices such that \( K \) is a subposet of \( L \). Then the following four statements and their duals hold.

(i) If \( a_1, \ldots, a_t \in K \) and \( t \in \mathbb{N}^+ \), then \( a_1 \lor \ldots \lor_L a_t \leq a_1 \lor_K \ldots \lor_K a_t \).

(ii) If \( t, s \in \mathbb{N}^+, a_1, \ldots, a_t, b_1, \ldots, b_s \in K \), and \( a_1 \lor_K \ldots \lor_K a_t \) is distinct from \( b_1 \lor_K \ldots \lor_K b_s \), then \( a_1 \lor_L \ldots \lor_L a_t \neq b_1 \lor_L \ldots \lor_L b_s \).

(iii) \( |\text{JRed}(L)| \geq |\text{JRed}(K)| \text{ and, dually, } |\text{MRed}(L)| \geq |\text{MRed}(K)| \).

(iv) If \( |\text{JRed}(L)| = |\text{JRed}(K)| \), \( u_1, u_2, v_1, v_2 \in K \), \( u_1 \parallel u_2, v_1 \parallel v_2 \), and \( u_1 \lor_K v_2 = v_1 \lor_K v_2 \), then \( u_1 \lor_L v_2 = v_1 \lor_L v_2 \).

Note that, according to (ii) and (iv), the distinctness of joins is generally preserved when passing from \( K \) to \( L \), but equalities are preserved only under additional assumptions. The dual of a condition \((X)\) will be denoted by \((X)^d\); for example, the dual of Lemma 3.1(i) is denoted by Lemma 3.1(i)^d or simply by 3.1(i)^d.
Proof of Lemma 3.1. Part (i) is a trivial consequence of the concept of joins as least upper bounds.

In order to prove (ii), assume that \( a_1 \lor \ldots \lor a_t = b_1 \lor \ldots \lor b_s \). Part (i) gives that \( a_i \leq \bigvee a \lor \ldots \lor \bigvee b \), for all \( i \in \{1, \ldots, t\} \). Since \( K \) is a subposet of \( L \), \( a_i \leq \bigvee \bigvee K \). But \( i \in \{1, \ldots, t\} \) is arbitrary, whereby \( a_1 \lor \ldots \lor a_t \leq \bigvee b \). We have equality here, since the converse inequality follows in the same way. Thus, we conclude (ii) by contraposition.

Next, let \( \{c_1, \ldots, c_t\} \) be a repetition-free list of \( \text{JRed}(K) \). For each \( i \) in \( \{1, \ldots, t\} \), pick \( a_i, b_i \in K \) such that \( a_i \parallel b_i \) and \( c_i = a_i \lor K b_i \). That is,

\[
\text{JRed}(K) = \{c_1 = a_1 \lor K b_1, \ldots, c_t = a_t \lor K b_t\}. \tag{3.2}
\]

Since \( a_i \parallel b_i \) holds also in \( L \),

\[
\{a_1 \lor L b_1, \ldots, a_t \lor L b_t\} \subseteq \text{JRed}(L). \tag{3.3}
\]

The elements listed in (3.3) are pairwise distinct by part (ii). Therefore, \( |\text{JRed}(K)| = t \leq |\text{JRed}(L)| \), proving part (iii).

Finally, to prove part (iv), we assume the premise of part (iv), and we let \( t := |\text{JRed}(K)| = |\text{JRed}(L)| \). Choose \( c_i, a_i, b_i \in K \) as in (3.2). Since \( t = |\text{JRed}(L)| \), part (ii) and (3.3) give that

\[
\text{JRed}(L) = \{a_1 \lor L b_1, \ldots, a_t \lor L b_t\}. \tag{3.4}
\]

As a part of the premise of (iv), \( u_1 \parallel u_2 \) has been assumed. Hence, (3.2) yields a unique subscript \( i \in \{1, \ldots, t\} \) such that \( c_i = u_1 \lor K u_2 = v_1 \lor K v_2 \). Since \( c_1, \ldots, c_t \) is a repetition-free list of the elements of \( \text{JRed}(K) \), we have that \( u_1 \lor v_2 \neq c_j = a_j \lor K b_j \) for every \( j \in \{1, \ldots, t\} \setminus \{i\} \). So, for all \( j \neq i \), part (ii) gives that \( u_1 \lor v_2 \neq a_j \lor L b_j \). But \( u_1 \lor v_2 \in \text{JRed}(L) \), whence (3.4) gives that \( u_1 \lor v_2 = a_i \lor L b_i \). Since the equality \( v_1 \lor v_2 = a_i \lor L b_i \) follows in the same way, we conclude that \( u_1 \lor v_2 = v_1 \lor v_2 \), as required. This yields part (iv) and completes the proof of Lemma 3.1. \( \square \)

Note that \( a_i \lor L b_i \) in the proof above can be distinct from \( c_i \); this will be exemplified by Figures 1 and 2.

4. THE REST OF THE PROOF

In this section, to ease our terminology, let us agree on the following convention. We say that a finite lattice \( L \) has many congruences if \( |\text{Con}(L)| > 2^{|L| - 5} \). Otherwise, if \( |\text{Con}(L)| \leq 2^{|L| - 5} \), then we say that \( L \) has few congruences.

Lemma 4.1. For every finite lattice \( L \), the following two assertions holds.

(i) If \( |\text{JRed}(L)| \geq 4 \) or \( |\text{MRed}(L)| \geq 4 \), then \( L \) has few congruences.

(ii) If \( |\text{JRed}(L)| = 3 \) and there are \( p, q \in J(L) \) such that \( p \neq q \) and \( \text{con}(p^-, p) = \text{con}(q^-, q) \), then \( L \) has few congruences.
Proof. Let $n := |L|$. If $|\text{JRed}(L)| \geq 4$, then (3.1) leads to $|\text{J}(L)| \leq n - 5$, and it follows by (2.2) that $L$ has few congruences. By duality, this proves part (i). Under the assumptions of (ii), $p \equiv_\text{con} q$, and we obtain from (2.1) that $|Q(L)| \leq |\text{J}(L)| - 1 = n - 4 - 1 = n - 5$, and (2.2) implies again that $L$ has few congruences. This proves the lemma. □

![Figure 1. $K = F_0$ and an example for $L$ containing $K$ as a subposet](image)

Lemma 4.2. Let $K := F_0 \in \mathcal{L}_{KR}$, see on the left in Figure 1. If $K$ is a subposet of a finite lattice $L$, then $L$ has few congruences.

Proof. Label the elements of $K = F_0$ as shown in Figure 1. A possible $L$ is given on the right in Figure 1; the elements of $K$ are black-filled. The diagram of $L$ is understood as follows: for $y_1, y_2 \in L$, a thick solid edge, a thin solid edge, and a thin dotted edge ascending from $y_1$ to $y_2$ mean that, in the general case, we know that $y_1 \prec y_2$, $y_1 < y_2$, and $y_1 \leq y_2$, respectively. In a concrete situation, further relations can be fulfilled; for example, a thin dotted edge can happen to denote that $y_1 = y_2$. The two dashed edges and the element $x$ as well as similar edges and elements can be present but they can also be missing. Note that $y_1 \leq y_2$ is understood as $y_1 \leq_K y_2$; for $y_1, y_2 \in K$, this is the same as $y_1 \leq_K y_2$ since $K$ is a subposet of $L$. Since $L$ in the figure carries a lot of information on the general case, the reader may choose to inspect Figure 1 instead of checking some of our computations that will come later. Note also that the convention above applies only for $L$; for $K$, every edge stands for covering.

Clearly, $|\text{JRed}(K)| = |\text{MRed}(K)| = 3$. Hence, Lemma 3.1(iii) gives that $|\text{JRed}(L)| \geq 3$ and $|\text{MRed}(L)| \geq 3$. We can assume that none of $|\text{JRed}(L)| \geq 4$ and $|\text{MRed}(L)| \geq 4$ holds, because otherwise Lemma 4.1(i) would immediately complete the proof. Hence,

$$|\text{JRed}(L)| = 3 \quad \text{and} \quad |\text{MRed}(L)| = 3. \quad (4.1)$$
Suppose, for a contradiction, that $u$ is a previous lower cover, etc., and it follows from (4.6) that this chain contains $H$. Hence, we obtain from $u$ that $H$ is a contradiction. Similarly, $q := b \land d$ and let $u_2 \in [q, d]_L$ be a cover of $q$. Finally, let $r := b \land g$, and let $u_3 \in [r, g]_L$ be a cover of $r$. Since we have formed the joins and the meets of incomparable elements in $L$ such that the corresponding joins are pairwise distinct in $K$ and the same holds for the meets, (4.1) and Lemma 3.1 imply that

$$\text{JRed}(L) = \{p, e \lor d, e \lor g\} \quad \text{and} \quad \text{MRed}(L) = \{q, r, b \land c\}. \quad (4.2)$$

Since $u_1 \not\ll_L p$, $u_1 \neq p$. If we had that $u_1 = e \lor d$, then

$$b \leq u_1 = e \lor d \leq e \lor_K d = a$$

would contradict $b \not\ll_K a$. Replacing $\langle d, a \rangle$ by $\langle g, c \rangle$, we obtain similarly that $u_1 \neq e \lor_L g$. Hence, (4.2) gives that $u_1 \in J(L)$. If we had that $u_2 = p$, then $b \leq p = u_2 \leq d$ would be a contradiction. Similarly, $u_2 = e \lor_L d$ would lead to $e \leq e \lor_L d = u_2 \leq d$ while $u_2 = e \lor_L g$ again to $e \leq e \lor_L g = u_2 \leq d$, which are contradictions. Hence, $u_2 \not\in \text{JRed}(L)$ and so $0_L \leq q \not\ll_L u_2$ gives that $u_2 \in J(L)$. We have that $u_3 \neq p$, because otherwise $b \leq p = u_3 \leq g$ would be a contradiction. Similarly, $u_3 = e \lor_L d$ and $u_3 \leq e \lor_L g$ would lead to the contradictions $e \leq e \lor_L d = u_3 \leq g$ and $e \leq e \lor_L g = u_3 \leq g$, respectively. So, $u_3 \not\in \text{JRed}(L)$ by (4.2). Since $r \not\ll_L u_3$ excludes that $u_3 = 0$, we obtain that $u_3 \in J(L)$. Since $u_3 = u_2$ would lead to

$$f \leq q \leq u_2 = u_3 \leq g, \quad (4.3)$$

which is a contradiction, we have that

$$u_1, u_2, u_3 \in J(L) \quad \text{and} \quad u_2 \neq u_3. \quad (4.4)$$

Next, we claim that

$$[q, u_2] \setminus [u_1, p] \quad \text{and} \quad [u_1, p] \setminus [r, u_3]. \quad (4.5)$$

Since $b \not\ll a$, $b \not\ll c$, and $b \not\ll d$, none of $e \lor_L d$, $e \lor_L g$, and $u_2$ belongs to $[b, i]_L$. In particular, we obtain from $u_1 \not\ll_L p$ and (4.2) that

$$[b, u_1]_L \subseteq J(L) \quad \text{and} \quad b \not\ll u_2. \quad (4.6)$$

Suppose, for a contradiction, that $u_2 \leq u_1$, and pick a maximal chain in the interval $[u_2, u_1]$. So we pick a lower cover of $u_1$, then a lower cover of the previous lower cover, etc., and it follows from (4.6) that this chain contains $b$. Hence, $u_2 \leq b$, and we obtain that $q \not\ll_L u_2 \leq b \land d = q$, a contradiction. Hence, $u_2 \not\ll u_1$. This means that $u_1 \land u_2 < u_2$. But $q \leq u_2 \leq u_1$, so we have that $q \leq u_1 \land u_2 < u_2$. Since $q \not\ll u_2$, we obtain that $u_1 \land u_2 = q$. Similarly, $u_2 \leq d \leq p$ and $u_2 \not\ll u_1$ give that $u_1 < u_1 \land u_2 \leq p$, whereby $u_1 \not\ll_L p$ yields that $u_1 \land u_2 = p$. The last two equalities imply the first half of (4.5). The second half follows basically in the same way, so we give less details. Based on (4.6), $u_3 \leq u_1$ would lead to $u_3 \leq b$ and $r \not\ll_L u_3 \leq b \land_L g = r$, whence
Since \( u_3 \not\leq u_1 \). Since \( u_3 \leq g \leq c \leq b = L g \leq b \leq u_1 \), we obtain that \( r \leq u_1 \land_L u_3 < u_3 \) and \( u_1 < u_1 \land_L u_3 \leq p \). Hence the covering relations \( r \triangleleft_L u_3 \) and \( u_1 \triangleleft_L p \) imply the second half of (4.5).

Finally, (2.4) and (4.5) give that \( \text{con}(q, u_2) = \text{con}(u_1, p) = \text{con}(r, u_3) \).

We still need another lemma.

**Figure 2.** \( K = E_0 \) and an example for \( L \) containing \( K \) as a subposet

**Lemma 4.3.** Let \( K := E_0 \in \mathcal{L}_{KR} \), see on the left in Figure 2. If \( K \) is a subposet of a finite lattice \( L \), then \( L \) has few congruences.

**Proof.** This proof shows a lot of similarities with the earlier proof. In particular, the same convention applies for the diagram of \( L \) in Figure 2 and, again, there can be several elements of \( L \) not indicated in the diagram. We have already noted that (4.1) holds in the present situation. Figure 2 shows how to pick \( u_1, u_2 \in L \); they are covers of \( a \land_L b \) in \( [a \land_L b, a] \) and \( b \land_L c \) in \( [b \land_L c, c] \), respectively. As a counterpart of (4.2), now we obtain in the same way from (4.1) and Lemma 3.1 that

\[
\begin{align*}
\text{JRed}(L) &= \{p := a \lor_L b = b \lor_L c, e \lor_L f, e \lor_L g = f \lor_L g\} \text{ and } \\
\text{MRed}(L) &= \{a \land_L b = a \land_L d, b \land_L c = d \land_L c, a \land_L c = e \land_L f\}. \quad (4.7)
\end{align*}
\]

Not all the equalities above will be used but they justify Figure 2. In particular, even if there can be more elements, the comparabilities and the incomparabilities in the figure are correctly indicated. Using (4.7) in the same way as we used (4.2) in the proof of Lemma 4.2 and the above-mentioned correctness of Figure 2, it follows that

\[
[a \land_L b, a] \setminus \{a \land_L b\} \subseteq \text{Irr}(L), \text{ and } [a, p] \setminus \{p\} \subseteq \text{Irr}(L). \quad (4.8)
\]
In particular, \( u_1 \in J(L) \). Similarly to the argument verifying (4.5), now (4.8) implies that \([u_1^- , u_1] = [a \land_L b, u_1] \uparrow [b, p]\). Since \( \langle a, u_1 \rangle \) and \( \langle c, u_2 \rangle \) play symmetric roles, we obtain that \( u_2 \in J(L) \) and \([u_2^- , u_2] = [b \land_L c, u_2] \uparrow [b, p]\). Since \( u_1 \) and \( u_2 \) are distinct by Figure 2 and they belong to \( J(L) \) by (4.8) and the \( \langle a, u_1 \rangle \)–\( \langle c, u_2 \rangle \)-symmetry, (4.1) and Lemma 4.1(ii) imply that \( L \) has few congruences. This completes the proof of Lemma 4.3.

Now, we are in the position to prove our theorem.

**Proof of Theorem 1.1.** Let \( L \) be an arbitrary non-planar lattice; it suffices to show that \( L \) has few congruences. By Proposition 2.1, there is a lattice \( K \) in Kelly and Rival’s list \( L_{KR} \) such that \( K \) is a subposet of \( L \) or the dual \( L_{dual} \) of \( L \). Since \( \text{Con}(L_{dual}) = \text{Con}(L) \), we can assume that \( K \) is a subposet of \( L \). A quick glance at the lattices of \( L_{KR} \), see their diagrams in Kelly and Rival [10], shows that if \( K \in L_{KR} \setminus \{E_0, F_0\} \), then \(|J(K)| \geq 4\) or \(|M(K)| \geq 4\). Hence, if \( K \in L_{KR} \setminus \{E_0, F_0\} \), then Lemma 4.1(i) implies that \( L \) has few congruences, as required. If \( K \in \{E_0, F_0\} \), then the same conclusion is obtained by Lemmas 4.2 and 4.3. This completes the proof of Theorem 1.1.

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