LOCALISATION OF BOCHNER RIESZ MEANS CORRESPONDING TO THE SUB-LAPLACIAN ON THE HEISENBERG GROUP

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Dedicated to Prof. Sundaram Thangavelu on the occasion of his 60th birthday.

Abstract. In this article we prove localisation of Bochner Riesz means \( S_R \) of order 0 for the sub-Laplacian \( L \) on the Heisenberg Group \( H^d \). More precisely, we show that for any \( 0 < \eta < 1 \) and \( \beta > \eta \), \( \lim_{R \to \infty} R^{-\beta/2} S_R f \) goes to 0 a.e. on the set \( \| (z,t) \| \leq 1 \) for \( f \in L^2(\mathbb{H}^d \setminus \{ \| (z,t) \| \leq 3 \}, \| (z,t) \|^{-\eta} \, dz \, dt) \). We generalise the method of Carbery and Soria [3] in the context of \( \mathbb{H}^d \).

1. Introduction

In this article, we study the localisation of Bochner Riesz means corresponding to the sub-Laplacian on the Heisenberg group. Let us first define the meaning of localisation of Bochner Riesz means. The Bochner Riesz means \( T^\alpha_R f \), of order \( \alpha \geq 0 \), for the standard Laplacian \( \Delta \) on the Euclidean space \( \mathbb{R}^d \) are defined as:

\[
T^\alpha_R f(x) := \int_{\mathbb{R}^d} \left( 1 - \frac{|\xi|^2}{R^2} \right)_+^\alpha \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi,
\]

where \( \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx \) and \( \left( 1 - \frac{|\xi|^2}{R^2} \right)_+^\alpha = \left( 1 - \frac{|\xi|^2}{R^2} \right)^\alpha \) when \( |\xi| \leq R \) and 0 elsewhere. In terms of the functional calculus of the standard Laplacian \( \Delta \) on \( \mathbb{R}^d \), \( T^\alpha_R = (I + \frac{1}{R^2} \Delta)^\alpha \).

Let us consider a measurable function \( f \) from a suitable function space so that \( T^\alpha_R f(x) \) makes sense. Let us further assume that \( f \) vanishes identically on an open and bounded subset \( B \) of \( \mathbb{R}^d \). When \( d = 1 \), and \( \alpha \geq 0 \), it is well known that \( T^\alpha_R f \) converges to 0 uniformly on every compact subset of \( B \) for any \( f \in L^p(\mathbb{R}) \) with \( 1 \leq p \leq 2 \). This is referred to as the Riemann Localisation principle (see [1]). For \( d \geq 2 \), and \( \alpha \geq \frac{d-1}{2} \), \( T^\alpha_R f \) converges to 0 uniformly on every compact subset of \( B \) for any \( f \in L^p(\mathbb{R}^d) \) with \( 1 \leq p \leq 2 \) (see [1] and Chapter 7 of [13]). In the case when \( \alpha = 0 \), and for \( f \in L^p(\mathbb{R}^d) \) with \( 2 \leq p < \frac{2d}{d-1} \), A. Carbery and F. Soria [3] proved that \( T^0_R f \to 0 \) a.e. on compact subsets of \( B \). In fact, they proved their result by...
showing that for any $0 < \eta < 1$, the following weighted-norm inequality holds:

$$
\int_{|x| \leq 1} \sup_{R > 1} |T_R^\eta f(x)|^2 \, dx \leq C_Q \int_{|x| \geq 3} \frac{|f(x)|^2}{|x|^\eta} \, dx,
$$

One could observe that the above inequality is stronger in the sense that $L^p(\mathbb{R}^d \setminus \{|x| < 3\}, dx) \subseteq L^2(|x|^{-\eta} \, dx)$, for each $2 \leq p < \frac{2d}{d-1}$, and for a suitable $0 < \eta < 1$.

In the present article, we study an analogue of the above mentioned result of Carbery and Soria for the Bochner Riesz means on the Heisenberg Group $\mathbb{H}^d$ corresponding to the sub-Laplacian $L$.

Before going to the Bochner Riesz means for $L$ on $\mathbb{H}^d$, let us first recall some of the basics of $\mathbb{H}^d$. The Heisenberg Group $\mathbb{H}^d$ can be identified with $\mathbb{C}^d \times \mathbb{R}$. The Haar measure on $\mathbb{H}^d$ is given by the Lebesgue measure on $\mathbb{C}^d \times \mathbb{R}$. We denote a point in $\mathbb{H}^d$ by $(z, t), z = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d, t \in \mathbb{R}$, and $z_j = x_j + iy_j; x_j, y_j \in \mathbb{R}$. It is well known that $\mathbb{H}^d$ is a simply connected, step one, unimodular nilpotent Lie group under the group operation

$$(z, t)(w, s) = (z + w, t + s + \frac{1}{2}Im(z \cdot \bar{w})),$$

where $z \cdot \bar{w} = \sum_{j=1}^d z_j \bar{w_j}$. The convolution of functions on $\mathbb{H}^d$ is given by

$$f \ast g(z, t) = \int_{\mathbb{H}^d} f((z, t)(-w, -s))g(w, s) \, dw \, ds.$$

We now define the vector fields on $\mathbb{H}^d$ by

$$X_j := \partial_{x_j} + \frac{1}{2}y_j \partial_t,$$

$$Y_j := \partial_{y_j} - \frac{1}{2}x_j \partial_t,$$

$$T := \partial_t,$$

for $1 \leq j \leq d$. These form a basis of the vector space of left invariant vector fields on $\mathbb{H}^d$. In fact, the Lie algebra of left invariant vector fields on $\mathbb{H}^d$ is generated by taking the Lie brackets and finite linear combinations of $\{X_j, Y_j\}_{j=1}^d$. We define

$$L := -\sum_{j=1}^d (X_j^2 + Y_j^2),$$

the positive sub-Laplacian (or Kohn-Laplacian) on $\mathbb{H}^d$. It is known that $L$ is a hypoelliptic operator, it has a self-adjoint extension on $L^2(\mathbb{H}^d)$, and that it also commutes with both left and right translations. The spectrum of $L$ is well known (see, for example, Section 2.1 of [16]). More explicitly, for each $\lambda \neq 0$,

$$L(e^{i\lambda} \varphi_k(\sqrt{|\lambda|}z)) = |\lambda|(2k + d)e^{i\lambda} \varphi_k(\sqrt{|\lambda|}z).$$

Here $\varphi_k$’s are Laguerre functions of order $d - 1$. We write $e^\lambda_k(z, t) = e^{i\lambda} \varphi_k(\sqrt{|\lambda|}z) = e^{i\lambda} \varphi_k(\sqrt{|\lambda|}z)$. It is easy to check that for $f \in S(\mathbb{H}^d)$

$$L(f \ast e^\lambda_k) = |\lambda|(2k + d)f \ast e^\lambda_k.$$

When $\lambda = 0$, i.e., $t$ dependence is not there, $L$ reduces to the standard Laplacian on $\mathbb{R}^{2d}$ whose spectral decomposition is well known. For our spectral analysis, it suffices to work on the spectrum

$$\{|\lambda|(2k + d) : \lambda \in \mathbb{R} \setminus \{0\}, k \in \mathbb{N} \cup \{0\}\}.$$
Using the functional calculus of $\mathcal{L}$, we define the Bochner Riesz means of order $\alpha \geq 0$, corresponding to $\mathcal{L}$ as

$$S^\alpha_R(f) := \left(I - \frac{1}{R^2}\mathcal{L}\right)^\alpha f.$$ 

In terms of spectral decomposition of $\mathcal{L}$, it can be expressed as

$$S^\alpha_R(f)(z, t) = (2\pi)^{-d-1} \int_\mathbb{R} \sum_{k \geq 0} \left(1 - \frac{(2k + d)|\lambda|}{R^2}\right)^\alpha f \ast e^\lambda(z, t)|\lambda|^d d\lambda.$$ 

Now, for a nice function $F$ on $\mathbb{C}^d$, and $\lambda \neq 0$, let us write

$$P^\lambda_k(F)(z) = \int_{\mathbb{C}^d} F(w)\varphi_k^\lambda(z - w)e^{i\lambda m(z - w)} dw.$$ 

It is known that $|\lambda|^dP^\lambda_k$'s are orthonormal projections on $L^2(\mathbb{C}^d)$, and for $F \in L^2(\mathbb{C}^d)$, we have $F = \sum_{k \geq 0} |\lambda|^dP^\lambda_k(F)$ in $L^2$-sense. Therefore, one could also express $S^\alpha_R$ as follows:

$$S^\alpha_R(f)(z, t) = (2\pi)^{-d-1} \int_\mathbb{R} \sum_{k \geq 0} \left(1 - \frac{(2k + d)|\lambda|}{R^2}\right)^\alpha P^\lambda_k(f^{\lambda})(z)e^{i\lambda t}|\lambda|^d d\lambda,$$

where $f^{\lambda}$ denotes the inverse Euclidean Fourier transform in the last variable which is defined by $f^{\lambda}(z) = \int_\mathbb{R} f(z, t)e^{i\lambda t} dt$.

By abuse of notation, we write $S^0_R$ as $S_R$. It is clear from the definition above that

$$S_R(f)(z, t) = (2\pi)^{-d-1} \int_\mathbb{R} \sum_{k \geq 0} \chi_{\{k, |\lambda|\}}(F_k^\lambda)(z)e^{i\lambda t}|\lambda|^d d\lambda,$$

Recall that the homogeneous norm (Cygan-Kóvary norm) on $\mathbb{H}^d$ is given by $\|z, t\| = (|z|^4 + |t|^2)^{1/4}$. With this, our main result is:

**Theorem 1.1.** For any $0 < \eta < 1$, and $\beta > \eta$,

$$\int \sup_{R > 1} R^{-\beta} |S_R f(z, t)|^2 dz dt \lesssim_{\beta, \eta} \int \|z, t\|^{1/3} \|f(z, t)\|^{\eta} dz dt.$$ 

As a consequence of the fact that the space $L^p(\mathbb{H}^d \setminus \{\|z, t\| < 3\}, dz dt) \subseteq L^2(\mathbb{H}^d \setminus \{\|z, t\| < 3\}, dz dt)$, for any $\eta > 1/2$, with $Q = 2d + 2$ the homogeneous dimension of $\mathbb{H}^d$, we get following localisation of the Bochner Riesz means of order $0$ as a corollary of Theorem 1.1.

**Theorem 1.2.** Let $2 \leq p < 2Q/(Q - 1)$ and $f \in L^p(\mathbb{H}^d)$. Then, for any $\beta > Q\left(1 - \frac{2}{p}\right)$, we have $R^{-\beta/2}S_R f(z, t) \to 0$ as $R \to \infty$, almost everywhere off $\text{supp}(f)$.

**Remark 1.1.** When $\alpha > 0$, Gorges and Müller [5] proved that $S^\alpha_R f \to f$ a.e. for $f \in L^p(\mathbb{H}^d)$, $Q^{-\alpha} < \frac{1}{2} - \frac{\alpha}{Q} < \frac{1}{2}$, where $Q = 2d + 2$ and $D = 2d + 1$. In this article, we mainly deal with the case $\alpha = 0$, considering functions vanishing in a neighbourhood of $0 \in \mathbb{H}^d$. Let us also remark here that our method is different from that of Gorges and Müller [6]. Our method also works in proving localisation of the Bochner Riesz means of any order $\alpha > 0$ for the sub-Laplacian on $\mathbb{H}^d$. In fact, when $\alpha > 0$, we will not need extra $R^{-\beta}$ in the corresponding inequality for $S^\alpha_R$ in Theorem 1.1. See Remarks 2.1 and 2.3.
In \cite{8}, the authors have proved a.e. convergence of spectral sums for the right invariant sub-Laplacian $\mathcal{L}$ on a connected Lie Group. In fact, there they proved an analogue of Rademacher- Menshov theorem \cite{12} for general Lie groups. More precisely, they showed that the spectral sums $S_R f \to f$ a.e. as $R \to \infty$ for any $f$ such that $\log(2+\mathcal{L}) f \in L^2(G)$, where $S_R f := \int_0^R dE_\lambda f$ with $\int_0^\infty dE_\lambda$ being the spectral resolution of identity corresponding to $\mathcal{L}$. See also \cite{5} for a similar result for the partial sums for the standard Laplacian on $\mathbb{R}^d$ as well.

For proving Theorem 1.1 we generalise the main ideas of \cite{3} in the context of the Heisenberg group. For that we need to prove the key Lemma 2.3. We will see in the next section that the exponents we get in Lemma 2.3 are not sufficient to get rid of $R^{-\beta/2}$ in $\lim_{R \to \infty} R^{-\beta/2} S_R f(z,t)$ to converge to 0 a.e. However, we will also show that the exponents that we obtain in Lemma 2.3 are sharp. We believe that the conclusion of Theorem 1.1 should hold without the extra decay $R^{-\beta/2}$, but currently we don’t know how to prove that.

The organisation of the paper is as follows: In Section 2 we state and prove the key estimates that are required in establishing the main result (Theorem 1.1). In the proof of Lemma 2.3 we make use of interpolation of certain Besov-type spaces, and we prove it in the Appendix (Section 3).

2. Proof of Theorem 1.1

Without loss of generality, we may assume that $f$ is identically 0 in the open ball $\{||z,t|| < 3\}$. We shall show that for an appropriate $\beta > 0$, $R^{-\beta/2} S_R f \to 0$ a.e. on the open unit ball $\{||z,t|| < 1\}$. The proof of this a.e. convergence on any other open ball $\{||z,t|| \leq r\}$, $0 < r < 3$, is similar. Let $0 \leq \psi \leq 1$ be an even and smooth function on $\mathbb{R}$ such that $\psi \equiv 1$ when $|\rho| \leq \frac{\pi}{2}$ and support of $\psi$ is contained in the set $|\rho| \leq \epsilon$, for some $0 < \epsilon < 1$. Denote the characteristic function of the interval $[-R, R]$ on $\mathbb{R}$ by $\chi_R$. Then, in $L^2$-sense, we have

$$\chi_R(\sqrt{\eta}) = 2 \int_0^\infty \frac{J_\frac{\rho}{2}(R\rho)}{(R\rho)^{1/2}} \cos(\sqrt{\eta}\rho) \, d\rho$$

$$= 2 \int_0^\infty \psi(\rho) R \frac{J_\frac{\rho}{2}(R\rho)}{(R\rho)^{1/2}} \cos(\sqrt{\eta}\rho) \, d\rho$$

$$+ 2 \int_0^\infty (1 - \psi(\rho)) R \frac{J_\frac{\rho}{2}(R\rho)}{(R\rho)^{1/2}} \cos(\sqrt{\eta}\rho) \, d\rho.$$

As done in Section 2 in \cite{7}, for any $f \in S(\mathbb{H}^d)$, one can write

$$S_R f(z,t) = 2 \int_0^\infty \psi(\rho) R \frac{J_\frac{\rho}{2}(R\rho)}{(R\rho)^{1/2}} \cos(\rho \sqrt{\eta}) f(z,t) \, d\rho$$

$$+ 2 \int_0^\infty (1 - \psi(\rho)) R \frac{J_\frac{\rho}{2}(R\rho)}{(R\rho)^{1/2}} \cos(\rho \sqrt{\eta}) f(z,t) \, d\rho.$$

Note that $\cos(\rho \sqrt{\eta}) f$ is the solution to the wave equation for the sub-Laplacian $\mathcal{L}$ with initial data $f$ and initial velocity 0. By the finite speed of propagation of the wave equation for $\mathcal{L}$ on $\mathbb{H}^d$ (see equation (3.16), and appendix of \cite{11}), we know that there exists a constant $\kappa > 0$ such that when $f$ is identically 0 on an open, bounded subset $B$ of $\mathbb{H}^d$, then for any compact subset $K$ of $B$ with $d(K, \partial B) > \epsilon$ and for $(z,t) \in K$, $\cos(\rho \sqrt{\eta}) f(z,t) = 0$ for any $\rho \leq \kappa^{-1} \epsilon$. Here $d((z,t),(w,s))$
stands for \( \| (z, t)(w, s)^{-1}\| \). Hence, for \( f \equiv 0 \) on the set \( \{\| (z, t)\| < 3\} \) and \( (z, t) \) such that \( \| (z, t)\| < 1 \), by choosing suitable \( \epsilon > 0 \), it is enough to study \[
B_R f(z, t) = \int_0^\infty (1 - \psi(\rho)) \frac{J_{\pm}(R\rho)}{(R\rho)^{1/2}} \cos(\rho \sqrt{L}) f(z, t) \, d\rho.
\]

In order to prove our result, we further break \( B_R \) into simpler operators and analyse each piece separately. Let \( \Psi \) be an even and smooth function on \( \mathbb{R} \) supported in \([-2, 2]\], further with the property that \( 0 \leq \Psi \leq 1 \) and that it is identically 1 on the interval \([-1, 1]\). It is easy to verify that on \([0, \infty)\), we can write \[
1 = \sum_{j=1}^\infty \tilde{\psi}_j(\rho) + \Psi(\rho),
\]
where \( \tilde{\psi}_j(\rho) = \tilde{\psi}(2^{-j} \rho) - \tilde{\psi}(2^{-j+1} \rho) \), for \( j \geq 1 \). Clearly, each \( \tilde{\psi}_j \in C_c^\infty([0, \infty)) \) with support in \([2^{j-1}, 2^{j+1}]\). From this, we have \[
1 - \psi(\rho) = \sum_{j=1}^\infty (1 - \psi(\rho)) \tilde{\psi}_j(\rho) + (1 - \psi(\rho)) \Psi(\rho).
\]

Again, for a Schwartz class function \( f \) on \( \mathbb{H}_d \), it is easy to check that \[
B_R f(z, t) = \sum_{j \geq 1} \int_0^\infty (1 - \psi(\rho)) \tilde{\psi}_j(\rho) R \frac{J_{\pm}(R\rho)}{(R\rho)^{1/2}} \cos(\rho \sqrt{L}) f(z, t) \, d\rho
\]
\[
+ \int_0^\infty (1 - \psi(\rho)) \Psi(\rho) R \frac{J_{\pm}(R\rho)}{(R\rho)^{1/2}} \cos(\rho \sqrt{L}) f(z, t) \, d\rho.
\]

Now, since \( 1 - \psi(\rho) \equiv 1 \) on \([\epsilon, \infty)\), and is identically 0 on \((0, \epsilon/2]\), it suffices to analyse \( \sum_{j \geq 1} B_{R,j} f(z, t) \), where \[
B_{R,j} f(z, t) = \int_0^\infty (1 - \psi(\rho)) \tilde{\psi}_j(\rho) R \frac{J_{\pm}(R\rho)}{(R\rho)^{1/2}} \cos(\rho \sqrt{L}) f(z, t) \, d\rho.
\]

We can handle \[
\int_0^\infty (1 - \psi(\rho)) \Psi(\rho) R \frac{J_{\pm}(R\rho)}{(R\rho)^{1/2}} \cos(\rho \sqrt{L}) f(z, t) \, d\rho
\]
in a similar manner since \( (1 - \psi(\rho)) \Psi(\rho) \) is compactly supported away from 0.

For each \( j \geq 1 \), we define \[
m_{R,j}(\eta) = \int_0^\infty (1 - \psi(\rho)) \tilde{\psi}_j(\rho) R \frac{J_{\pm}(R\rho)}{(R\rho)^{1/2}} \cos(\rho \eta) \, d\rho.
\]

Using the definition of \( \cos(\rho \sqrt{L}) \) and the above expression for \( m_{R,j} \), we can rewrite \( B_{R,j} f(z, t) \) as \[
B_{R,j} f(z, t) = \int_{\mathbb{R}} \sum_{k \geq 0} m_{R,j} \left( \sqrt{(2k + d) |\lambda|} \right) P_k(f^{-\lambda})(z)e^{\mathbf{i} \lambda t} \, d\lambda.
\]

As discussed earlier, the proof of Theorem 1.1 will follow if we could prove the following weighted bound estimates for \( \sum_{j \geq 1} B_{R,j} \).
Lemma 2.1. For any $0 < \eta < 1$, and $\beta > \eta$,

$$
\int \sup_{\|z,t\| \leq 1} R^{-\beta} \left| \sum_{j \geq 1} B_{R,j} f(z,t) \right|^2 \, dz \, dt \lesssim_{\beta, \eta} \int \frac{|f(z,t)|^2}{\|z,t\|^\eta} \, dz \, dt.
$$

To prove Lemma 2.1, we need to first estimate $m_{R,j}$'s and their derivatives.

Lemma 2.2. For any $l, q \in \mathbb{N} \cup \{0\}$, there exists a constant $C_{l,q} > 0$ such that

$$
\left| \int \frac{d^l}{dR^l} m_{R,j}(\eta) \right| \leq C_{l,q} \frac{2^j}{(1 + 2^j |R - \eta|)^q},
$$

for all $\eta \geq 0$ and $j, R \geq 1$.

Proof. We know that $R_{\lambda}^{J_1(\rho)} = \sqrt{\frac{2 \sin(\rho)}{\rho}}$, see [13]. Using this in the expression of $m_{R,j}(\eta)$, we get that

$$
m_{R,j}(\eta) = \int_0^\infty (1 - \psi(\rho)) \tilde{\psi}_j(\rho) \sqrt{\frac{2 \sin(\rho)}{\rho}} \cos(\rho \eta) \, d\rho.
$$

Now, since $1 - \psi(\rho) \equiv 1$ on $[\epsilon, \infty)$, we have $(1 - \psi(\rho)) \tilde{\psi}_j(\rho) = \tilde{\psi}_j(\rho)$, for every $j \geq 1$ (for sufficiently small $\epsilon > 0$). A further simplification gives us that

$$
m_{R,j}(\eta) = \sqrt{\frac{1}{2\pi}} \int_0^\infty \tilde{\psi}_j(\rho) \frac{\sin((R + \eta)\rho) + \sin((R - \eta)\rho)}{\rho} \, d\rho
= \sqrt{\frac{1}{2\pi}} \int_0^\infty (\Psi(\rho) - \Psi(2\rho)) \frac{\sin((R + \eta)2\rho) + \sin((R - \eta)2\rho)}{\rho} \, d\rho.
$$

Recalling that $\Psi(\rho) - \Psi(2\rho)$ is supported in $[\frac{1}{2}, 2]$, the usual integration by parts gives us the desired estimates for $m_{R,j}$ and it’s derivatives.

□

Remark 2.1. For $\alpha > 0$, if we decompose $S^\alpha_R f$ for $f$ vanishing in the set $\|z,t\| < 3$, the corresponding $m_{R,j}$’s will have extra $R^{-\alpha}$ decay. See Proposition 2.1 in [7].

We next move to a key estimate, which in essence is the analogue of Lemma 2.3 of [3], and this will be used further in the proof of Lemma 2.1. For this, we define

$$
T_\varepsilon f(z,t) = \int R \sum_{k \geq 0} \chi_{[1 - \sqrt{\varepsilon}(2k+1) \leq |\lambda| < 1]} P_k^\lambda f(z)e^{i\lambda t}|\lambda|^d \, d\lambda.
$$

We have the following estimate for the operator $T_\varepsilon$.

Lemma 2.3. For $0 \leq \beta < 1$, and $\epsilon > 0$ small enough, we have

$$
\int_{\mathbb{R}^d} |T_\varepsilon f(z,t)|^2 \, dz \, dt \lesssim_{\beta} \epsilon^{\beta} \int_{\mathbb{R}^d} |f(z,t)|^2 |t|^\beta \, dz \, dt
$$

and the weight $|t|^\beta$ cannot be replaced by $\|z,t\|^\gamma$ for any $\gamma < 2\beta$.
Proof. Let us assume that $0 < \epsilon < 1/4$. Using the Plancherel Theorem for the Euclidean space in the last variable and orthogonality of $P_k^\lambda$'s in $L^2(\mathbb{C}^d)$, we get

\[
\int_{\mathbb{R}^d} |T_\epsilon f(z,t)|^2 \, dz \, dt = \int_{\mathbb{R}^d} \sum_{k \geq 0} \chi_{[1-\sqrt{\lambda}(2k+d)\leq \epsilon]} \|P_k^\lambda (f^{-\lambda})\|^2_{L^2(\mathbb{C}^d)} |\lambda|^{2d} \, d\lambda
\]

\[
\leq \int_{\mathbb{R}^d} \sum_{k \geq 0} \chi_{[1-\sqrt{\lambda}(2k+d)\leq 3\epsilon]} \|P_k^\lambda (f^{-\lambda})\|^2_{L^2(\mathbb{C}^d)} |\lambda|^{2d} \, d\lambda
\]

\[
= \int_{\mathbb{R}} \sum_{1-\epsilon/k < 2k+d < 1+\epsilon/k \atop \|\lambda\| \geq 1} \|P_k^\lambda (f^{-\lambda})\|^2_{L^2(\mathbb{C}^d)} |\lambda|^{2d} \, d\lambda
\]

\[
= I_{1,\epsilon} + I_{2,\epsilon},
\]

where,

\[
I_{1,\epsilon} = \int_{\mathbb{R}} \sum_{1-\epsilon/k < 2k+d < 1+\epsilon/k \atop \|\lambda\| \geq 1} \|P_k^\lambda (f^{-\lambda})\|^2_{L^2(\mathbb{C}^d)} |\lambda|^{2d} \, d\lambda,
\]

\[
I_{2,\epsilon} = \int_{\mathbb{R}} \sum_{1-\epsilon/k < 2k+d < 1+\epsilon/k \atop \|\lambda\| = 0} \|P_k^\lambda (f^{-\lambda})\|^2_{L^2(\mathbb{C}^d)} |\lambda|^{2d} \, d\lambda.
\]

Now, for any $0 \leq \beta < 1$,

\[
I_{2,\epsilon} = \int_{\mathbb{R}} \sum_{1-\epsilon/k < 2k+d < 1+\epsilon/k \atop \|\lambda\| = 0} \|P_k^\lambda (f^{-\lambda})\|^2_{L^2(\mathbb{C}^d)} |\lambda|^{2d} \, d\lambda
\]

\[
= \int_{\mathbb{R}^d} \int_{1-\epsilon/k < 2k+d < 1+\epsilon/k \atop \|\lambda\| = 0} |(f * e_k)^\lambda(z)|^2 \, d\lambda \, dz
\]

\[
\lesssim \int_{\mathbb{R}^d} \sum_{2k+d \leq \frac{1}{1+\epsilon/k}} \int_{1-\epsilon/k < 2k+d < 1+\epsilon/k} |(f * e_k)^\lambda(z)|^2 \, d\lambda \, dz
\]

\[
\lesssim \beta \epsilon^\theta \sum_{k \geq 0} \int_{\mathbb{R}^d} |(f * e_k)(z,t)|^2 (2k+d)^{-\beta} |t|^\beta \, dz \, dt
\]

\[
\lesssim \beta \epsilon^\theta \int_{\mathbb{R}^d} |(f * e_k)(z,t)|^2 (2k+d)^{-\beta} |t|^\beta \, dz \, dt,
\]

where the second last inequality follows from Lemma 2.3 of [3].

On the other hand, using the fact that

\[
\sum_{k \geq 0} \|P_k^\lambda (f^{-\lambda})\|^2_{L^2(\mathbb{C}^d)} |\lambda|^{2d} = C \|f^\lambda\|^2_{L^2(\mathbb{C}^d)},
\]

we get

\[
I_{1,\epsilon} \lesssim \epsilon^\theta \int_{\mathbb{R}} \frac{1}{|\lambda|^\theta} \|f^\lambda\|^2_{L^2(\mathbb{C}^d)} \, d\lambda,
\]
for every $\beta \geq 0$. Restricting $0 \leq \beta < 1$, we once again invoke Hardy’s inequality for fractional Laplacian in $t$-variable to obtain
\[
I_{t,e} \lesssim_\beta e^{\beta t} \int_{\mathbb{R}^d} |f(z,t)|^2 |t|^\beta \, dz \, dt.
\]
Writing $e_k(z,t) = \int_{\mathbb{R}} e^{i\lambda t} \varphi_k(z) |\lambda|^d \, d\lambda$, it is easy to see that
\[
|\lambda|^d P_k^\lambda (f^{-\lambda}) = (f \ast e_k)^{-\lambda}.
\]
We will now show that for any $0 \leq \beta < 1$,
\[
\sum_{k \geq 0} \int_{\mathbb{R}^d} |(f \ast e_k)(z,t)|^2(2k + d)^{-\beta} |t|^\beta \, dz \, dt \lesssim_\beta \int_{\mathbb{R}^d} |f(z,t)|^2 |t|^\beta \, dz \, dt.
\]
We will prove the above estimate by analysing each summand of the left hand side in the case of $\beta = 0$ and $\beta = 2$, and then by invoking interpolation between parameters $\beta = 0$ and $\beta = 2$. Note that for $\beta = 0$, the above estimate reduces to
\[
\sum_{k \geq 0} \int_{\mathbb{R}^d} |(f \ast e_k)(z,t)|^2 \, dz \lesssim \int_{\mathbb{R}^d} |f(z,t)|^2 \, dz.
\]
which is true in view of the Plancherel Theorem for the Heisenberg group, and in fact the above estimate holds with equality. We now turn our focus to the case when $\beta = 2$. For this, note that the following identity holds:
\[
w(f \ast e_k) = (wf) \ast e_k + f \ast (we_k),
\]
where $w(z,t) = t$. Therefore,
\[
\int_{\mathbb{R}^d} |(f \ast e_k)(z,t)|^2(2k + d)^{-2} |t|^2 \, dz \, dt \leq 2 \int_{\mathbb{R}^d} |((wf) \ast e_k)(z,t)|^2(2k + d)^{-2} |t|^2 \, dz \, dt + 2 \int_{\mathbb{R}^d} |(f \ast (we_k))(z,t)|^2(2k + d)^{-2} |t|^2 \, dz \, dt.
\]
It is easy to handle the sum (over $k \geq 0$) of the first term on the right hand side of above inequality. In fact,
\[
\sum_{k \geq 0} \int_{\mathbb{R}^d} |((wf) \ast e_k)(z,t)|^2(2k + d)^{-2} |t|^2 \, dz \, dt \leq \sum_{k \geq 0} \int_{\mathbb{R}^d} |((wf) \ast e_k)(z,t)|^2 |t|^2 \, dz \, dt = C \int_{\mathbb{R}^d} |(wf)(z,t)|^2 \, dz \, dt = C \int_{\mathbb{R}^d} |f(z,t)|^2 |t|^2 \, dz \, dt.
\]
To estimate the sum (over $k \geq 0$) of the second term of the inequality, we make use of the recurrence identities for Laguerre polynomials and Laguerre functions (see Page 92 of [16], and equation (5.1.10) on Page 101 of [15]) to verify that
\[
\frac{d}{d\lambda} (\varphi_k^\lambda(z)|\lambda|^d) = \left\{ \frac{d}{2} \varphi_k^\lambda(z) - \frac{k + d - 1}{2} \varphi_{k-1}^\lambda(z) + \frac{k + 1}{2} \varphi_{k+1}^\lambda(z) \right\} \frac{\lambda}{|\lambda|} |\lambda|^{d-1}.
\]
From this we get
\[
(we_k)(z,t) = t \int_{\mathbb{R}} e^{i\lambda t} \varphi_k^\lambda(z) |\lambda|^d \, d\lambda = i \int_{\mathbb{R}} e^{i\lambda t} \frac{d}{d\lambda} (\varphi_k^\lambda(z)|\lambda|^d) \, d\lambda
\]
\[
= i \int_{\mathbb{R}} \left\{ \frac{d}{2} \varphi_k^\lambda(z) - \frac{k + d - 1}{2} \varphi_{k-1}^\lambda(z) + \frac{k + 1}{2} \varphi_{k+1}^\lambda(z) \right\} \frac{\lambda}{|\lambda|} |\lambda|^{d-1} e^{i\lambda t} \, d\lambda.
\]
And then, using the Euclidean Plancherel Theorem in $t$-variable, we have
\[
\int_{\mathbb{R}} |(f * (we_k))(z,t)|^2 \, dt \\
= C \int_{\mathbb{R}} \left| f^{-\lambda} *_\lambda \left\{ \frac{d}{2} \varphi^\lambda_k(z) - \frac{k + d - 1}{2} \varphi^\lambda_{k-1}(z) + \frac{k + 1}{2} \varphi^\lambda_{k+1}(z) \right\} \right|^2 |\lambda|^{2d} \, d\lambda.
\]
Using orthogonality of $\{\varphi^\lambda_k\}_{k \geq 0}$ in $L^2(\mathbb{C}^d)$, the above identity implies that
\[
\int_{\mathbb{R}^d} |(f * (we_k))(z,t)|^2 \, dz \, dt \\
= C \int_{\mathbb{C}^d} \int_{\mathbb{R}} \frac{d^2}{4|\lambda|^2} \left| f^{-\lambda} *_\lambda \varphi^\lambda_k(z) \right|^2 |\lambda|^{2d} \, d\lambda \, dz \\
+ C \int_{\mathbb{C}^d} \int_{\mathbb{R}} \frac{(k + d - 1)^2}{4|\lambda|^2} \left| f^{-\lambda} *_\lambda \varphi^\lambda_{k-1}(z) \right|^2 |\lambda|^{2d} \, d\lambda \, dz \\
+ C \int_{\mathbb{C}^d} \int_{\mathbb{R}} \frac{(k + 1)^2}{4|\lambda|^2} \left| f^{-\lambda} *_\lambda \varphi^\lambda_{k+1}(z) \right|^2 |\lambda|^{2d} \, d\lambda \, dz.
\]
Hence, we have that
\[
\sum_{k \geq 0} \int_{\mathbb{R}^d} |(f * (we_k))(z,t)|^2 (2k + d)^{-2} \, dz \, dt \lesssim \sum_{k \geq 0} \int_{\mathbb{C}^d} \int_{\mathbb{R}} \frac{1}{|\lambda|^2} \left| f^{-\lambda} *_\lambda \varphi^\lambda_k(z) \right|^2 \, d\lambda \, dz = C \int_{\mathbb{C}^d} \int_{\mathbb{R}} \frac{1}{|\lambda|^2} \left| f^\lambda(z) \right|^2 \, d\lambda \, dz.
\]
Putting together these two estimates, we have
\[
(2.0.3) \quad \sum_{k \geq 0} \int_{\mathbb{R}^d} |(f * e_k)(z,t)|^2 (2k + d)^{-2} |t|^2 \, dz \, dt \\
\lesssim \int_{\mathbb{R}^d} |f(z,t)|^2 |t|^2 \, dz \, dt + \int_{\mathbb{C}^d} \int_{\mathbb{R}} \frac{1}{|\lambda|^2} \left| f^\lambda(z) \right|^2 \, d\lambda \, dz.
\]
Interpolating inequalities \eqref{eq:2.0.2} and \eqref{eq:2.0.3}, one can prove that for all $0 \leq \beta \leq 2$,
\[
(2.0.4) \quad \sum_{k \geq 0} \int_{\mathbb{R}^d} |(f * e_k)(z,t)|^2 (2k + d)^{-\beta} |t|^\beta \, dz \, dt \\
\lesssim_\beta \int_{\mathbb{R}^d} |f(z,t)|^2 |t|^\beta \, dz \, dt + \int_{\mathbb{C}^d} \int_{\mathbb{R}} \frac{1}{|\lambda|^2} \left| f^\lambda(z) \right|^2 \, d\lambda \, dz.
\]
We postpone proof of above mentioned interpolation to the next section (appendix). Now, restricting $\beta$ in the open interval $(0,1)$, we can apply Hardy’s inequality for fractional derivatives in $\lambda$-variable to have
\[
\int_{\mathbb{R}} \frac{1}{|\lambda|^\beta} \left| f^\lambda(z) \right|^2 \, d\lambda \lesssim_\beta \int_{\mathbb{R}} \left| \frac{d^{\beta/2}}{d\lambda^{\beta/2}} f^\lambda(z) \right|^2 \, d\lambda = C_\beta \int_{\mathbb{R}} |f(z,t)|^2 |t|^\beta \, dt,
\]
where $\frac{d^\delta}{d\lambda^\delta}$ denotes the fractional derivative defined as
\[
\frac{d^\delta}{d\lambda^\delta} g(t) = |t|^\delta \hat{g}(t).
\]
Hence, we have proved that for any $0 \leq \beta < 1$,
\[
\sum_{k \geq 0} \int_{\mathbb{R}^d} |(f \ast e_k)(z, t)|^2 (2k + d)^{-\beta} |t|^\beta \, dz \, dt \lesssim \int_{\mathbb{R}^d} |f(z, t)|^2 |t|^\beta \, dz \, dt.
\]
This completes the proof of the inequality (2.0.1) as stated in Lemma 2.3.

Finally, we shall show that the weight which appears on the R.H.S. of the inequality cannot be improved. For this, let us assume that
\[
(2.0.5) \quad \int_{\mathbb{R}^d} |T_\epsilon f(z, t)|^2 \, dz \, dt \lesssim_{\beta, \gamma} \epsilon^\beta \int_{\mathbb{R}^d} |f(z, t)|^2 \|\nu(z, t)\|^{\gamma} \, dz \, dt
\]
holds for some $\gamma < 2\beta$, and all $0 < \epsilon < 1$, and for all $f$ such that the R.H.S. of the above inequality is finite.

Consider a function $f$ such that $f^\lambda(z) = \hat{g}(\lambda) \chi_{|z| \leq \eta}(z)$, where $g$ is a Schwartz class function on $\mathbb{R}$ and $\eta > 0$, a fixed positive number.

By definition of the operator $T_\epsilon$, for any $0 < \epsilon < 1$,
\[
\int_{\mathbb{R}^d} |T_\epsilon f(z, t)|^2 \, dz \, dt = \int_{\mathbb{R}^d} \sum_{k \geq 0} \chi_{|1 - \sqrt{|\lambda|d(2k + d)|z|} < \epsilon} \|P_k^\lambda(f^{-\lambda})\|_{L^2(C^\beta)}^2 \, d\lambda
\]
\[
\geq \int_{\mathbb{R}^d} \sum_{k \geq 0} \chi_{|1 - |\lambda|d(2k + d)|z| < \epsilon} \|P_k^\lambda(f^{-\lambda})\|_{L^2(C^\beta)}^2 \, d\lambda
\]
\[
\geq \int_{\mathbb{R}} \chi_{|1 - |\lambda|| < \epsilon} \|P_0^\lambda(f^{-\lambda})\|_{L^2(C^\beta)}^2 |\lambda|^{-2d} \, d\lambda
\]
\[
= C \int_{|1 - |\lambda|| < \epsilon} \left| \int_{\mathbb{C}} f^\lambda(z)e^{-\frac{|z|^2}{2}} \, dz \right|^2 |\lambda|^d \, d\lambda.
\]
In particular, for our choice of $f$, the following holds:
\[
\int_{|1 - |\lambda|| < \epsilon} \left| \int_{\mathbb{C}} \chi_{|z| \leq \eta}(z)e^{-\frac{|z|^2}{2}} \, dz \right|^2 |\hat{g}(\lambda)|^2 |\lambda|^d \, d\lambda
\]
\[
\lesssim_{\beta, \gamma} \epsilon^\beta \int_{\mathbb{R}^d} |g(t)|^2 \chi_{|z| \leq \eta}(z)\|\nu(z, t)\|^{\gamma} \, dz \, dt.
\]

For $\epsilon > 0$ small, $|\lambda| \sim d^{-1}$ in the integral on the L.H.S. Using this fact, we get that
\[
\int_{|1 - |\lambda|| < \epsilon} |\hat{g}(\lambda)|^2 \, d\lambda \lesssim_{\beta, \eta} \epsilon^\beta \left( \eta^{\gamma} \int_{|t| \leq \eta^2} |g(t)|^2 \, dt + \int_{|t| \geq \eta^2} |g(t)|^2 |t|^{\frac{\gamma}{2}} \, dt \right).
\]

By change of variables, it is easy to see that for $R > 1$, the last inequality implies that
\[
(2.0.6) \quad \int_{|\lambda| - R < \epsilon} |\hat{g}(\lambda)|^2 \, d\lambda
\]
\[
\lesssim_{\beta, \eta} \epsilon^\beta \frac{R^\beta}{\eta^\gamma} \left( \eta^{\gamma} \int_{|t| \leq \frac{\eta^2}{R^2}} |g(t)|^2 \, dt + R^\frac{\gamma}{2} \int_{|t| \geq \frac{\eta^2}{R^2}} |g(t)|^2 |t|^{\frac{\gamma}{2}} \, dt \right).
\]

Let us now choose $g$ such that $\hat{g}(\lambda) = \phi(\lambda - R)$, for some $\phi \in C_c^\infty(\mathbb{R})$ with $\phi \equiv 1$ on $|\lambda| \leq \frac{1}{2}$. It is easy to see that $g(t) = e^{itR}\hat{\phi}(-t)$. For the above choice of $g$, it is
straightforward to verify that
\[ \int_{|\lambda| < R \leq \varepsilon} |\hat{g}(\lambda)|^2 d\lambda = 2\varepsilon. \]

Therefore, inequality (2.0.6) implies that
\[ \varepsilon \lesssim \varepsilon \frac{C \varepsilon^{\beta}}{R^{2\beta}} \left( \eta \int_{|t| \leq \varepsilon^\eta} |\hat{\phi}(t)|^2 dt + R^{2\beta} \int_{|t| \geq \varepsilon^\eta} |\hat{\phi}(t)|^2 dt \right). \]

As we take \( R \to \infty \), the R.H.S. of the above inequality goes to 0 because by assumption \( \frac{1}{2} < \beta \). But \( \varepsilon > 0 \), and therefore we arrive at a contradiction. This proves the sharpness of the exponent of \( \varepsilon \) in the inequality (2.0.6) as stated in Lemma 2.3 and completes the proof of the lemma. \( \square \)

Before starting the proof of Lemma 2.1, let us also remark about estimates of \( T_\varepsilon \) for \( \varepsilon > 0 \) away from 0.

**Remark 2.2.** When \( \varepsilon \) is away from 0, say \( \varepsilon \geq 1/4 \), then \( 1 - \sqrt{|\lambda|(2k + d)} \) implies \( |\lambda|(2k + d) < 25\varepsilon^2 \). In this case one could use Plancherel Theorem for the Euclidean space in last variable and the orthogonality of \( P_\lambda \)'s in \( L^2(\mathbb{C}^d) \) to have
\[ \left| \int_{\mathbb{H}^d} \left| T_\varepsilon f(z,t) \right|^2 dz dt \right| \leq C \varepsilon^{4\beta} \int_{\mathbb{R}} \sum_{k \geq 0} \frac{1}{(|\lambda|(2k + d))^{2\beta}} \| P_\lambda (f - \lambda) \|_{L^2(\mathbb{C}^d)}^2 d\lambda \]
\[ = C \varepsilon^{4\beta} \int_{\mathbb{H}^d} \| \mathcal{L}^{-\beta} f(z,t) \|^2 dz dt \]
\[ \lesssim \varepsilon^{4\beta} \int_{\mathbb{H}^d} \| f(z,t) \|^2 2^{2\beta} dz dt, \]
where last inequality follows from the Hardy-Sobolev inequality on the Heisenberg group, see [1] (Section 3) for details.

We are now in a position to prove Lemma 2.1.

**Proof of Lemma 2.1.** Let \( \omega \in C^\infty(\mathbb{R}) \) be such that \( 0 \leq \omega \leq 1 \) on \( \mathbb{R}, \omega(R) \equiv 1 \) when \( R \geq 1 \) and \( \omega(R) \equiv 0 \) when \( R \leq \frac{1}{2} \). For convenience, we work with the operator \( \omega(R)B_{R,j} \). We know by Sobolev inequality
\[ \sup_{R} |\omega(R)B_{R,j} f(z,t)|^2 \lesssim \gamma \left| \int_{\mathbb{R}} \frac{d^\gamma}{dR^\gamma} \omega(R)B_{R,j} f(z,t) dR \right|^2, \]
for each \( \frac{1}{2} < \gamma < 1 \). Therefore,
\[ \sup_{R \geq 1} R^{-\beta} \left| \sum_{j \geq 1} B_{R,j} f(z,t) \right|^2 \]
\[ \leq \sup_{R} R^{-\beta} |\omega(R) \sum_{j \geq 1} B_{R,j} f(z,t)|^2 \]
\[ \lesssim \gamma \left| \int_{\mathbb{R}} \sum_{j \geq 1} \frac{d^\gamma}{dR^\gamma} \left( R^{-\beta/2} \omega(R)B_{R,j} f(z,t) \right) dR \right|^2, \]
\[ \lesssim \gamma \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{d^\gamma}{dR^\gamma} \left( R^{-\beta/2} \omega(R)B_{R,j} f(z,t) \right)^2 dR \right)^{1/2} \right)^2. \]
where, as also mentioned in the proof of Lemma 2.3 \( \frac{d^{\gamma}}{dR^\gamma} \) (for \( \gamma < 1 \)) denotes the fractional derivative defined as 
\[
\hat{\frac{d^{\gamma}}{dR^\gamma}(\xi)} = |\xi|^\gamma \hat{g}(\xi).
\]
Therefore, by Minkowski’s inequality (2.0.7)
\[
\left( \int_{\mathbb{R}^d} R^{-\beta} |\omega(R)\sum_{j \geq 0} B_{R,j} f(z,t)|^2 \, dz \, dt \right)^{1/2} \leq \gamma \sum_{j \geq 1} \int_{\|z,t\| \leq 1} \int_0^\infty \frac{\rho \sin(R\rho)}{\rho} \cos(\rho \sqrt{E}) \delta(z,t) \, d\rho,
\]
where \( \delta \) denotes the Dirac distribution at origin in \( \mathbb{H}^d \). We claim that \( K_{R,j} \) is compactly supported in \( \{\|z,t\| \leq 2^j\} \). This claim follows from the fact that \( \cos(\rho \sqrt{E}) \delta \) is compactly supported in \( \{\|z,t\| \leq \kappa \rho\} \) and \( \tilde{\psi}_j \) is supported in the interval \( [2^{j-1}, 2^{j+1}] \). Here \( \kappa \) is same as mentioned in the description of the finite speed of propagation of the wave equation (in the beginning of Section 2). This further implies that
\[
\int_{\|z,t\| \leq 1} \int_0^\infty R^{-\beta} \omega(R)^2 |K_{R,j} * f(z,t)|^2 \, dz \, dt \, dR
\]
is zero when \( f \) is supported outside the ball \( \|z,t\| \leq c2^j \). As also done in [3], it is then enough to show that
\[
\int_{\mathbb{H}^d} \int_0^\infty R^{-\beta} \omega(R)^2 |K_{R,j} * f(z,t)|^2 \|z,t\|^{-2\beta} \, dz \, dt \, dR \leq 2^{-j \beta} 2^{-j} \int_{\mathbb{H}^d} |f(z,t)|^2 \, dz \, dt.
\]
By duality, the above is equivalent to proving
\[
\int_{\mathbb{H}^d} \int_0^\infty \left| K_{R,j} * f_R(z,t) \omega(R) \right|^2 \, dz \, dt \leq 2^{-j \beta} 2^{-j} \int_{\mathbb{H}^d} \int_0^\infty |f_R(z,t)|^2 \|z,t\|^{2\beta} \, dz \, dt \, dR.
\]
After applying Euclidean Plancherel theorem in the \( t \)-variable and orthogonality of the special Hermite projections, we can write
\[
\int_{\mathbb{H}^d} \int_0^\infty \left| K_{R,j} * f_R(z,t) \omega(R) \right|^2 \, dz \, dt
\]
\[
= \int \sum_k \int_0^\infty m_{R,j} \left( (2k + d)|\lambda| \right) P_k^d f_k^j(\lambda) \left| \lambda \right| |d\omega(R) | d\lambda \, dz.
\]
By Lemma 2.2 it is easy to see that
\[ |m_{R,j}(\sqrt{(2k + d)|\lambda|})| \lesssim \sum_{n} 2^{-nM} \chi(2^j \sqrt{(2k + d)|\lambda|} - R \leq 2^n), \]
for \( M > 0 \) large enough. So, it is enough to look at
\[ \int_{R^{2d+1}} \sum_{k} \int_{0}^{\infty} \chi(2^j \sqrt{(2k + d)|\lambda|} - R \leq 2^n) P_{k}^\lambda(f_{R}^j)(z) \omega(R) \, dR \left| \lambda \right|^2 d\lambda \, dz. \]

Now, we apply Cauchy Schwarz inequality to get
\[
\left| \int_{0}^{\infty} \chi(2^j \sqrt{(2k + d)|\lambda|} - R \leq 2^n) P_{k}^\lambda(f_{R}^j)(z) \omega(R) \, dR \right|^2 \leq 2^{n-j} \int_{0}^{\infty} \chi(2^j \sqrt{(2k + d)|\lambda|} - R \leq 2^n) |P_{k}^\lambda(f_{R}^j)(z)|^2 \omega(R)^2 \, dR.
\]

Using the inequality obtained in Lemma 2.3 (or Remark 2.2, as the case may be) and a non homogeneous dilation \((z, t) \rightarrow (Rz, R^2t)\), it is easy to see that
\[
\int_{R^{2d+1}} \sum_{k} \int_{0}^{\infty} \chi(2^j \sqrt{(2k + d)|\lambda|} - R \leq 2^n) P_{k}^\lambda(f_{R}^j)(z) \omega(R) \, dR \left| \lambda \right|^2 d\lambda \, dz
\leq 2^{n-j} \int_{0}^{\infty} \left( \int_{R^{2d+1}} \chi(2^j \sqrt{(2k + d)|\lambda|} - R \leq 2^n) |P_{k}^\lambda(f_{R}^j)(z)|^2 \, dz \, dt \right) \omega(R)^2 \, dR
\leq 2^{n-j} 2^{2(n-j)} \int_{\mathbb{H}^d} \int_{0}^{\infty} |f_{R}(z, t)|^2 \|(z, t)\|^{2\beta} R^\delta \, dz \, dR,
\]
for any \( 0 \leq \beta < 1 \). In obtaining the above estimate, we have used the fact that \( 0 \leq \omega \leq 1 \) and \( \omega \equiv 0 \) for \( R \leq 1/2 \), and therefore the estimates through Remark 2.2 (for large \( n \)) are even better in view of \( R \geq 1/2 \).

And therefore, we have
\[
\int_{\mathbb{H}^d} \int_{0}^{\infty} R^{-\beta} \omega(R)^2 |B_{R,j} f(z, t)|^2 \|(z, t)\|^{-2\beta} \, dz \, dR
\leq_\beta 2^{-j-\beta j} \int_{\mathbb{H}^d} |f(z, t)|^2 \, dz \, dt
\sim_\beta 2^{-j-\beta j} \int_{\|(z, t)\| \leq 2^j} |f(z, t)|^2 \, dz \, dt
\leq_{\beta, \delta} 2^{-j-\delta j} \int_{\mathbb{H}^d} |f(z, t)|^2 \|(z, t)\|^{-(\beta-\delta)} \, dz \, dt
\leq_{\beta, \delta} 2^{-j-\delta j} \int_{\mathbb{H}^d} |f(z, t)|^2 \|(z, t)\|^{-(\beta-\delta)} \, dz \, dt
\]
for any \( 0 < \beta < 1 \), and \( \delta > 0 \) such that \( \beta - \delta > 0 \).

When \( \gamma = 1 \), using the similar method as above we can obtain
\[
\int_{\mathbb{H}^d} \int_{0}^{\infty} \frac{d}{dR}(R^{-\beta/2} \omega(R) B_{R,j} f(z, t))^2 \|(z, t)\|^{-2\beta} \, dz \, dR
\leq_{\beta, \delta} 2^{2(j-\delta j)} \int_{\mathbb{H}^d} |f(z, t)|^2 \|(z, t)\|^{-(\beta-\delta)} \, dz \, dt.
\]
And then for $0 < \gamma < 1$, one could apply interpolation theorem for weighted $L^p$ spaces (see 5.3 of Section 5, Chapter 5 of [14] with $p_0 = p_1 = 2$). In particular, we have

$$
\int_{\mathbb{R}^d} \int_0^\infty \left| \frac{d^\gamma}{dR^\gamma} \left( R^{-\beta/2} \omega(R) B_{R,j} f(z,t) \right) \right|^2 \| \omega(R) B_{R,j} f(z,t) \|^{-2\beta} dz \, dR \\
\lesssim_{\beta,\delta,\gamma} 2^{-j-\delta j+2\gamma j} \int_{\mathbb{R}^d} |f(z,t)|^2 \| \omega(R) B_{R,j} f(z,t) \|^{-(\beta-\delta)} dz dt,
$$

for $\frac{1}{2} < \gamma \leq 1$.

Finally, given $\delta > 0$, one could choose $\frac{1}{2} < \gamma < 1$ such that $1 + \delta - 2\gamma > 0$. Using these estimates in inequality (2.0.7), we have

$$
\int_{\| \omega(R) B_{R,j} f(z,t) \| \leq 1} \sup_{R > 1} R^{-\beta} \| \sum_{j \geq 1} B_{R,j} f(z,t) \|^2 dz \, dt \\
\lesssim_{\beta,\delta,\gamma} \left( \sum_{j \geq 1} \left( \int_{\| \omega(R) B_{R,j} f(z,t) \| \leq 1} \left| \frac{d^\gamma}{dR^\gamma} \left( R^{-\beta/2} \omega(R) B_{R,j} f(z,t) \right) \right|^2 \, dR \, dz \, dt \right)^{\frac{1}{2}} \right)^2 \\
\lesssim_{\beta,\delta,\gamma} \left( \sum_{j \geq 1} \left( 2^{-j-\delta j+2\gamma j} \int_{\mathbb{R}^d} |f(z,t)|^2 \| \omega(R) B_{R,j} f(z,t) \|^{-(\beta-\delta)} dz \, dt \right)^{\frac{1}{2}} \right)^2 \\
\lesssim_{\beta,\delta,\gamma} \int_{\mathbb{R}^d} |f(z,t)|^2 \| \omega(R) B_{R,j} f(z,t) \|^{-(\beta-\delta)} dz \, dt.
$$

Since $0 < \delta < \beta$ is arbitrary, this completes proof of Lemma 2.1.

\begin{proof}
\end{proof}

**Remark 2.3.** If we decompose $S_R^\alpha$ as done for $S_R$ in Lemma 2.1, we can prove the corresponding inequality without the need of $R^{-\beta}$ by simply choosing $\beta < 2\alpha$ because of the extra decay of $R^{-\beta}$ in the pointwise estimate of $m_R^\alpha$ as mentioned in Remark 2.1. We leave the details to the reader.

3. **Appendix**

We made use of an interpolation of certain Hilbert spaces while claiming inequality (2.0.4) in the proof of Lemma 2.1. We shall prove the same here. Consider following Hilbert spaces:

$$
A_0 = L^2(\mathbb{R}^{2d+1}) = \left\{ f : \| f \|_0^2 = \int_{\mathbb{R}^{2d+1}} |f(z,t)|^2 \, dz \, dt < \infty \right\},
$$

and

$$
A_s = \left\{ f \in D'(\mathbb{R}^{2d+1}) : \| f \|_s^2 < \infty \right\},
$$

for each $0 < s \leq 1$, where $D'(\mathbb{R}^{2d+1})$ is the space of tempered distributions on $\mathbb{R}^{2d+1}$ and

$$
\| f \|_s^2 = \int_{\mathbb{R}^{2d+1}} |f(z,t)|^2 \left( \frac{1}{|t|^{2s}} \right) dz \, dt + \int_{\mathbb{R}^{2d+1}} |D^s f(z,t)|^2 \, dz \, dt.
$$

Here $D^s$, for $0 < s < 1$, denotes the fractional derivative in the last variable defined by the Euclidean Fourier multiplier as

$$
(D^s)^\lambda f(z) = |\lambda|^s \hat{f}^\lambda(z),
$$

whereas $D$ denotes the distributional derivative in the last variable.
Let $M_k = L^2(\mathbb{C}^d \times \mathbb{R})$ and $N_k = L^2(\mathbb{C}^d \times \mathbb{R}, \omega_k)$, with $\omega_k(z, t) = (2k + d)^{-2}|t|^2$, for $k = 0, 1, 2, 3, \ldots$, and consider following vector valued Hilbert Spaces

\[ l_2(M_k) = \left\{ a = (a_k)_{k=0}^{\infty} \mid a_k \in M_k, \|a\|^2_{l_2(M_k)} = \sum_{k=0}^{\infty} \|a_k\|^2_{M_k} < \infty \right\}, \]

\[ l_2(N_k) = \left\{ b = (b_k)_{k=0}^{\infty} \mid b_k \in N_k, \|b\|^2_{l_2(N_k)} = \sum_{k=0}^{\infty} \|b_k\|^2_{N_k} < \infty \right\}. \]

Define the linear operator $T$ on $A_0 + A_1$ by

\[ Tf = (f^t * e_k)_{k=0}^{\infty}. \]

Then, as shown in inequalities [2.0.2] & [2.0.3] in the proof of Lemma 2.3, $T$ maps $A_0$ and $A_1$ boundedly into $l_2(M_k)$ and $l_2(N_k)$ respectively, that is,

\[ \|Tf\|_{l_2(M_k)} \lesssim \|f\|_{A_0}, \]

\[ \|Tf\|_{l_2(N_k)} \lesssim \|f\|_{A_1}. \]

For each $0 < s < 1$, the complex interpolation of the spaces $l_2(M_k)$ and $l_2(N_k)$ is (see Page 121, Section 1.18.1 of [18]):

\[ [l_2(M_k), l_2(N_k)]_s = l_2([M_k, N_k]_s). \]

Now, the interpolation of the spaces $M_k$ and $N_k$ is well known and follows from general theory of interpolation of weighted $L^p$ spaces (see 5.4 of Section 5, Chapter 5 of [13]), and for each $0 < s < 1$, we have

\[ [M_k, N_k]_s = L^2(\mathbb{C}^d \times \mathbb{R}, \omega_k^s). \]

Finally, if we could show that $A_s$, for $0 < s < 1$, is the complex interpolation space of $A_0$ and $A_1$, then we will have that $T$ is bounded from $A_s$ into $l_2(L^2(\mathbb{C}^d \times \mathbb{R}, \omega_k^s))$. And, this will complete the proof of the inequality [2.0.4]. We now explain the complex interpolation of $A_0$ and $A_1$.

**Theorem 3.1.** Let the Hilbert spaces $A_s$, $0 \leq s \leq 1$, be as defined above. For each $0 < s < 1$, the complex interpolation of the pair $(A_0, A_1)$ is $A_s$.

**Proof.** We closely follow the proof of [17], and rewrite it to suit to our context. For each $j \in \mathbb{Z}$, consider intervals $\Omega_j^+ = (2^{-j-3}, 2^{-j+3})$, and $\Omega_j^- = (-2^{-j+3}, -2^{-j-3})$. Clearly, these intervals have finite overlap, $(0, \infty) = \bigcup_{j \in \mathbb{Z}} \Omega_j^+$, and $(-\infty, 0) = \bigcup_{j \in \mathbb{Z}} \Omega_j^-$. Choose and fix $\phi \in C_c^\infty(\Omega_j^+)$ with the property that $\phi \equiv 1$ on the interval $[\frac{1}{2}, 2]$, $0 < \phi \leq 1$ on the interval $[\frac{1}{2}, 4]$, and $0$ elsewhere. Now, define $\phi_j(t) = \phi(2^j t)$ for $j \in \mathbb{Z} \setminus \{0\}$. Finally, for each $j \in \mathbb{Z}$, define

\[ \tilde{\phi}_j(t) = \begin{cases} \frac{\phi_j(-t)}{\sum_{k \in \mathbb{Z}} \phi_k(-t)} & \text{for } t \in (-\infty, 0) \\ \frac{\phi_j(t)}{\sum_{k \in \mathbb{Z}} \phi_k(t)} & \text{for } t \in (0, \infty). \end{cases} \]
Clearly, \( \tilde{\phi}_j \in C_c^\infty (\Omega^+_j \cup \Omega^-_j) \), and \( \sum_j \tilde{\phi}_j \equiv 1 \). Moreover, for any \( t \in (0, \infty) \),

\[
\tilde{\phi}_j(t) = \phi_j(t) - \phi_j(t) \sum_k \phi_k(t)
\]

\[
= \phi_j(t) \sum_{k=j-3}^{j+3} \phi_k(t) - \phi_j(t) \sum_{k=j-3}^{j+3} \phi_k(t)
\]

\[
\lesssim 2^j.
\]

In the last inequality, we used the fact that \( \sum_{k \in \mathbb{Z}} \phi_k \) is uniformly bounded from below on \( (0, \infty) \). In fact, \( \sum_{k \in \mathbb{Z}} \phi_k(t) \geq 1 \). Similar estimate holds true on \( (-\infty, 0) \).

Now, for each \( 0 \leq s \leq 1 \), let us consider the spaces

\[
W^s_2 = \left\{ f \in D'(\mathbb{R}^{2d+1}) : \|f\|_{s,*}^2 < \infty \right\},
\]

where

\[
\|f\|_{s,*}^2 = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{2d+1}} \left( 2^{2js} \left| \left( \tilde{\phi}_j f \right) (z,t) \right|^2 + \left| D^s \left( \tilde{\phi}_j f \right) (z,t) \right|^2 \right) dz dt.
\]

We shall show that

1. \( W^s_2 \) is a Banach space.
2. \( C_c^\infty (\mathbb{R}^{2d} \setminus \mathbb{R} \setminus \{0\}) \) is dense in \( W^s_2 \) in the \( \| \cdot \|_{s,*} \) norm.
3. \( W^s_2 \) in the completion of \( C_c^\infty (\mathbb{R}^{2d} \setminus \mathbb{R} \setminus \{0\}) \) in the \( \| \cdot \|_s \) norm. In other words, the spaces \( A_s \) and \( W^s_2 \) are identical with norm equivalence.

It is easy to note that \( W^0_2 \) is not being but \( A_0 \) with norm equivalence. So, we only need to study the case when \( 0 < s \leq 1 \).

**Step 1:** \( W^s_2 \) is a Banach space.

It could be proved using standard arguments, so we omit the proof.

**Step 2:** \( C_c^\infty (\mathbb{R}^{2d} \setminus \mathbb{R} \setminus \{0\}) \) is dense in \( W^s_2 \) in the \( \| \cdot \|_{s,*} \) norm.

Take any \( f \in W^s_2 \). Note that

\[
\|f - f \sum_{|k| \leq M+1} \tilde{\phi}_k f\|_{s,*}^2 = \| \sum_{|k| > M+1} \tilde{\phi}_k f\|_{s,*}^2
\]

\[
= \sum_{|j| > M} 2^{2js} \left\| \sum_{|k| > M+3} \tilde{\phi}_k \tilde{\phi}_j f \right\|_{L^2(\mathbb{R}^{2d+1})}^2
\]

\[
+ \sum_{|j| > M} \left\| D^s \left( \sum_{|k| > M+3} \tilde{\phi}_k \tilde{\phi}_j f \right) \right\|_{L^2(\mathbb{R}^{2d+1})}^2
\]

\[
= \sum_{|j| > M} 2^{2js} \left\| \sum_{|k| > M+3} \tilde{\phi}_k \tilde{\phi}_j f \right\|_{L^2(\mathbb{R}^{2d+1})}^2
\]

\[
+ \sum_{|j| > M} \left\| D^s \left( \sum_{|k| > M+3} \tilde{\phi}_k \tilde{\phi}_j f \right) \right\|_{L^2(\mathbb{R}^{2d+1})}^2
\]

\[
\lesssim \sum_{|j| > M} \sum_{|k-j| \leq 3} 2^{2js} \left\| \tilde{\phi}_j f \right\|_{L^2(\mathbb{R}^{2d+1})}^2
\]

\[
+ \sum_{|j| > M} \sum_{|k-j| \leq 3} \left\| D^s \left( \tilde{\phi}_j f \right) \right\|_{L^2(\mathbb{R}^{2d+1})}^2,
\]

using the fact that \( \tilde{\phi}_j \tilde{\phi}_k = 0 \) for any \( j \) and \( k \) with \( |j - k| > 3 \).
From above estimation we get
\[ \|f - f\|_{s, \ast}^2 \lesssim \sum_{|j| > M} 2^{2js} \|D_j f\|_{L^2(R^{2d+1})}^2 + \sum_{|j| > M} \sum_{|k-j| \leq 3} \|D^s(\overline{\phi}_k f)\|_{L^2(R^{2d+1})}^2. \]

Clearly, the first term on the right hand side of the last inequality tends to 0 as \( M \to \infty \). For the second term, if we could show that
\[ \|D^s(\overline{\phi}_j f)\|_{L^2(R^{2d+1})}^2 \lesssim 2^{2ks} \|f\|_{L^2(R^{2d+1})}^2 + \|D^s f\|_{L^2(R^{2d+1})}^2, \]
then, using this estimate with \( \overline{\phi}_j f \) in place of \( f \), we will have that
\[ \lim_{M \to \infty} \|f - f\|_{s, \ast} = 0. \]

The claimed estimate for \( D^s(\overline{\phi}_k f) \), for \( s = 1 \), follows once we apply Leibniz rule for differentiation and then using estimates of derivatives of \( \overline{\phi}_k \)'s. For \( 0 < s < 1 \), we assume the claimed estimate for now. In fact, in the next step, we estimate the full sum \( \sum_k \|D^s(\overline{\phi}_k f)\|_{L^2(R^{2d+1})}^2 \). In those detailed calculations, one could easily verify that the claimed estimate for each fixed \( k \) also holds. Thus, we have so far shown that any function \( f \) in \( W^s \) could be approximated by sequence of functions supported in sets of the form \( \mathbb{R}^{2d} \times K \) with \( K \) compact in \( \mathbb{R} \setminus \{0\} \). Finally, one could use standard methods of approximation to complete the claim that \( C^\infty_c(\mathbb{R}^{2d} \times \mathbb{R} \setminus \{0\}) \) is dense in \( W^s \) in the \( \| \cdot \|_s \) norm.

**Step 3: \( W^s \) is the completion of \( C^\infty_c(\mathbb{R}^{2d} \times \mathbb{R} \setminus \{0\}) \) in the \( \| \cdot \|_s \) norm.**

It is straightforward to verify that
\[ \int_{\mathbb{R}^{2d} \times \mathbb{R} \setminus \{0\}} |f(z, t)|^2 \frac{1}{|t|^2s} \, dz \, dt \sim \sum_j 2^{2js} \int_{\mathbb{R}^{2d+1}} |(\phi_j f)(z, t)|^2 \, dz \, dt. \]

Next, writing \( \Omega_j = \Omega_j^+ \cup \Omega_j^- \), we perform following calculations to estimate \( \sum_j \|D^s(\overline{\phi}_j f)\|_{L^2(R^{2d+1})}^2 \). For \( 0 < s < 1 \),
\[ \sum_j \|D^s(\overline{\phi}_j f)\|_{L^2(R^{2d+1})}^2 = C \sum_j \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} |(\overline{\phi}_j f)(z, t_1) - (\overline{\phi}_j f)(z, t_2)|^2 \frac{1}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2 \]
\[ \lesssim \sum_j \int_{\mathbb{R}^{2d}} \int_{\mathbb{R} \times \Omega_j} \frac{|(\overline{\phi}_j f)(z, t_1) - (\overline{\phi}_j f)(z, t_2)|^2}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2 \]
\[ \lesssim \sum_j \int_{\mathbb{R}^{2d} \times \Omega_j} \int_{\mathbb{R} \times \Omega_j} |\overline{\phi}_j(t_1)|^2 |f(z, t_1) - f(z, t_2)|^2 \frac{1}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2 \]
\[ + \sum_j \int_{\mathbb{R}^{2d} \times \Omega_j} \int_{\mathbb{R} \times \Omega_j} |\overline{\phi}_j(t_2)|^2 |f(z, t_2) - f(z, t_1)|^2 \frac{1}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2 \]
\[ + \sum_j \int_{\mathbb{R}^{2d} \times \Omega_j} \int_{\mathbb{R} \times \Omega_j} |\overline{\phi}_j(t_1) - \overline{\phi}_j(t_2)|^2 |f(z, t_1) - f(z, t_2)|^2 \frac{1}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2 \]
\[ =: I + II + III. \]
We now estimate sums $I, II,$ and $III$ as follows.

\[ I \leq \sum_{j} \int_{\mathbb{R}^{2d}} \int_{\Omega_{j} \times \Omega_{j}} \frac{|f(z, t_{1}) - f(z, t_{2})|^{2}}{|t_{1} - t_{2}|^{1+2s}} \, dz \, dt_{1} \, dt_{2} \]

\[ \lesssim \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2}} |f(z, t_{1}) - f(z, t_{2})|^{2} \, dz \, dt_{1} \, dt_{2} \]

\[ = C \|D^{s}f\|_{L^{2}([\mathbb{R}^{2d} + 1])}^{2} \]

\[ II = \sum_{j} \int_{\mathbb{R}^{2d} \times \Omega_{j}} \left( \int_{\Omega_{j}} \frac{|\tilde{\phi}_{j}(t_{1}) - \tilde{\phi}_{j}(t_{2})|^{2}}{|t_{1} - t_{2}|^{1+2s}} \, dt_{1} \right) |f(z, t_{2})|^{2} \, dz \, dt_{2} \]

\[ \lesssim \sum_{j} 2^{2js} \int_{\mathbb{R}^{2d} \times \Omega_{j}} |f(z, t_{2})|^{2} \, dz \, dt_{2} \]

\[ \lesssim \int_{\mathbb{R}^{2d} \times \Omega \{0\}} |f(z, t)|^{2} \frac{1}{|t|^{2s}} \, dz \, dt. \]

Here we have used the following estimate

\[ \int_{\Omega_{j}} \frac{|\tilde{\phi}_{j}(t_{1}) - \tilde{\phi}_{j}(t_{2})|^{2}}{|t_{1} - t_{2}|^{1+2s}} \, dt_{1} \lesssim 2^{2js} \]

uniformly for $t_{2} \in \Omega_{j}$, and the proof of this estimate is

\[ \int_{\Omega_{j}} \frac{|\tilde{\phi}_{j}(t_{1}) - \tilde{\phi}_{j}(t_{2})|^{2}}{|t_{1} - t_{2}|^{1+2s}} \, dt_{1} = \int_{\Omega_{j}} \left| \frac{\tilde{\phi}_{j}(t_{1}) - \tilde{\phi}_{j}(t_{2})}{t_{1} - t_{2}} \right|^{2} \, dt_{1} \]

\[ \lesssim \|\tilde{\phi}_{j}\|_{\infty}^{2} \int_{\Omega_{j}} |t_{1} - t_{2}|^{-2s} \, dt_{1} \]

\[ \sim 2^{2js}. \]

Recalling that $\Omega_{j} = (-2^{-j+3}, -2^{-j-3}) \cup (2^{-j-3}, 2^{-j+3})$, and $\tilde{\phi}_{j}$ are supported on $[2^{-j-2}, 2^{-j+2}] \cup [-2^{-j-2}, -2^{-j+2}]$, one can perform calculations similar to the above ones to verify that

\[ \int_{\Omega_{j}} \frac{1}{|t_{1} - t_{2}|^{1+2s}} \, dt_{1} \lesssim 2^{2js}, \]

and thus,

\[ III \lesssim \sum_{j} 2^{2js} \int_{\mathbb{R}^{2d} \times \Omega_{j}} |f(z, t_{2})|^{2} \, dz \, dt_{2} \]

\[ \lesssim \int_{\mathbb{R}^{2d} \times \Omega \{0\}} |f(z, t)|^{2} \frac{1}{|t|^{2s}} \, dz \, dt. \]

So, we have have shown that for any $0 < s < 1$,

\[ \|f\|_{s, \star} \lesssim \|f\|_{s}. \]

For $s = 1$, one could simply apply the Leibniz formula for differentiation and then make use of the estimated of $\phi_{j}$’s together with the fact that these are supported in $\Omega_{j}$’s to easily verify that

\[ \|f\|_{1, \star} \lesssim \|f\|_{1}. \]
On the other hand, for $0 < s < 1$,
\[
\int_{\mathbb{R}^{2d+1}} |D^s f(z,t)|^2 \, dz \, dt
\]
\[
= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \frac{|f(z,t_1) - f(z,t_2)|^2}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2
\]
\[
\leq \sum_m \sum_{l \geq m} \int_{\Omega_m \times \Omega_l} \frac{|\sum_j (f(\tilde{\phi}_j))(z,t_1) - \sum_k (f(\tilde{\phi}_k))(z,t_2)|^2}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2
\]
\[
= \sum_m \sum_{l \geq m} \int_{\Omega_m \times \Omega_l} \frac{|\sum_{j=m+3}^{m+3} (f(\tilde{\phi}_j))(z,t_1) - \sum_{k=l+3}^{l+3} (f(\tilde{\phi}_k))(z,t_2)|^2}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2.
\]

For each $m$, one could arrange the summand in the above expression in the following manner. For $m \leq l \leq m + 9$, write each pair of terms with same index together and the remaining terms separately. For $l > m + 9$, there is no common index, and we write each term separately. Finally, one could apply Cauchy Schwarz inequality to verify that the above summation is dominated by
\[
\sum_m \int_{\Omega_m \times \Omega_m} \frac{|(f(\tilde{\phi}_m))(z,t_1) - (f(\tilde{\phi}_m))(z,t_2)|^2}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2
\]
\[
+ \sum_m \int_{\Omega_m \times \Omega_m} \frac{|(f(\tilde{\phi}_m))(z,t_1)|^2}{|t_1 - t_2|^{1+2s}} \, dz \, dt_1 \, dt_2
\]
which is further bounded from above by
\[
\sum_m \|D^s(f(\tilde{\phi}_m))\|^2_{L^2(\mathbb{R}^{2d+1})} + \sum_m 2^{2m} \|\tilde{\phi}_m\|^2_{L^2(\mathbb{R}^{2d+1})}.
\]

This completes the proof of the fact that for $0 < s < 1$,
\[
\|f\|_s \lesssim \|f\|_{s,*}.
\]

Once again, when $s = 1$, one could directly estimate $Df$ in terms of $D(f(\tilde{\phi}_m))$ as follows:
\[
\left\|\frac{\partial f}{\partial t}\right\|^2_{L^2(\mathbb{R}^{2d+1})} = \int_{\mathbb{R}^{2d+1}} \left| \sum_m \frac{\partial (f(\tilde{\phi}_m))}{\partial t}(z,t) \right|^2 \, dz \, dt
\]
\[
\lesssim \sum_m \int_{\Omega_m} \left| \frac{\partial (f(\tilde{\phi}_m))}{\partial t}(z,t) \right|^2 \, dz \, dt
\]
\[
\lesssim \sum_m \left\| \frac{\partial (f(\tilde{\phi}_m))}{\partial t} \right\|^2_{L^2(\mathbb{R}^{2d+1})}.
\]

Hence, for all $0 \leq s \leq 1$, the two spaces $(A_s, \| \cdot \|_s)$ and $(W^s_2, \| \cdot \|_{s,*})$ are identical with norm equivalence. For any $0 < s < 1$, since $W^s_2$ is the complex interpolation of $W^0_2$ and $W^1_2$ (see Page 121, Section 1.18.1 of [LS]), it follows that $A_s$ is the complex interpolation of $A_0$ and $A_1$. This completes the proof of Theorem 3.1. \qed
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