ON ENDOMORPHISMS OF QUANTUM TENSOR SPACE

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ABSTRACT. We give a presentation of the endomorphism algebra \( \text{End}_{U_q(\mathfrak{sl}_2)}(V^\otimes r) \), where \( V \) is the 3-dimensional irreducible module for quantum \( \mathfrak{sl}_2 \) over the function field \( \mathbb{C}(q^{\frac{1}{2}}) \). This will be as a quotient of the Birman-Wenzl-Murakami algebra \( BMW_r(q) := BMW_r(q^{-4}, q^2 - q^{-2}) \) by an ideal generated by a single idempotent \( \Phi_q \). Our presentation is in analogy with the case where \( V \) is replaced by the 2-dimensional irreducible \( U_q(\mathfrak{sl}_2) \)-module, the BMW algebra is replaced by the Hecke algebra \( H_r(q) \) of type \( A_{r-1} \), \( \Phi_q \) is replaced by the quantum alternator in \( H_3(q) \), and the endomorphism algebra is the classical realisation of the Temperley-Lieb algebra on tensor space. In particular, we show that all relations among the endomorphisms defined by the \( R \)-matrices on \( V^\otimes r \) are consequences of relations among the three \( R \)-matrices acting on \( V^\otimes 4 \). The proof makes extensive use of the theory of cellular algebras. Potential applications include the decomposition of tensor powers when \( q \) is a root of unity.

1. Introduction and statement of results

Let \( q^{\frac{1}{2}} \) be an indeterminate over \( \mathbb{C} \). Our objective is to give a presentation of \( \text{End}_{U_q(\mathfrak{sl}_2)}(V^\otimes r) \), where \( V_q \) is the irreducible 3-dimensional representation of quantum \( \mathfrak{sl}_2 \). This presentation will be as a quotient of the Birman-Murakami-Wenzl algebra \( BMW_r(q) := BMW_r(q^{-4}, q^2 - q^{-2}) \) by an ideal generated by a single quasi-idempotent \( \Phi_q \). The endomorphism algebra therefore has the same presentation as \( BMW_r(q) \), with the additional relation \( \Phi_q = 0 \). This paper is a sequel to [LZ2], where the main results were conjectured, and by and large we maintain the notation of that work. Our results may be stated integrally, i.e. in terms of algebras over the ring \( \mathbb{C}[q^{\frac{1}{2}}] \), and they therefore have the potential to generalise to the situation where \( q \) is a root of unity. We intend to address that issue in a future work.

1.1. General notation. Denote the function field \( \mathbb{C}(q^{\frac{1}{2}}) \) by \( K \). Let \( U_q = U_q(\mathfrak{sl}_2) \) be the quantised universal enveloping algebra of \( \mathfrak{sl}_2 \) over \( K \), and write \( V_q \) for the 3-dimensional irreducible \( U_q \) module. More generally, write \( V(d)_q \) for the irreducible \( U_q \)-module with highest weight \( d \in \mathbb{Z}_{\geq 0} \), so that \( V_q = V(2)_q \). It is well known that there is a homomorphism

\[
\eta : KB_r \longrightarrow \text{End}_{U_q}(V(d)^\otimes r)_q := E_q(d, r),
\]

where \( B_r \) is the \( r \)-string braid group, and \( \eta \) maps the \( i \)-th generator of \( B_r \) to the \( R \)-matrix acting on the \( i \)-th and \( (i+1) \)-st factors of the tensor power (cf. [LZ2 §3.4]).

It was shown in [LZ1] that \( \eta \) is surjective for all \( d \) and \( r \). It is our purpose to use this to give an explicit presentation of \( E_q(2, r) \), analogous to the celebrated presentation of the Temperley-Lieb algebra \( E_q(1, r) \) as a quotient of the Hecke algebra of type \( A \)

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The first step is to identify a finite dimensional quotient of $K\mathcal{B}$ through which $\eta$ factors.

1.2. The algebra $BMW_r(\mathcal{K})$. It was shown in [LZ2, Theorem 4.4] that $\eta$ factors through the finite dimensional algebra $BMW_r(\mathcal{K})$, which we now proceed to define.

Let $\mathcal{A}_{y,z}$ be the ring $\mathbb{C}[y^{\pm 1}, z]$, where $y, z$ are indeterminates. The BMW algebra $BMW_r(y, z)$ over $\mathcal{A}_{y,z}$ is the associative $\mathcal{A}_{y,z}$-algebra with generators $g_i^{\pm 1}, \ldots, g_{r-1}^{\pm 1}$ and $e_1, \ldots, e_{r-1}$, subject to the following relations:

The braid relations for the $g_i$:

\begin{align}
g_i g_j &= g_j g_i \text{ if } |i - j| \geq 2 \\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \text{ for } 1 \leq i \leq r - 1;
\end{align}

The Kauffman skein relations:

\begin{align}
g_i - g_i^{-1} &= z(1 - e_i) \text{ for all } i;
\end{align}

The de-looping relations:

\begin{align}
g_i e_i &= e_i g_i = y e_i; \\
ee_i g_{i-1}^\pm e_i &= y^{\mp 1} e_i; \\
ee_i g_{i+1}^\pm e_i &= y^{\mp 1} e_i.
\end{align}

The next five relations are easy consequences of the previous three.

\begin{align}
ee_i e_{i+\pm 1} e_i &= e_i; \\
(g_i - y)(g_i^2 - z g_i - 1) &= 0; \\
z e_i^2 &= (z + y^{-1} - y) e_i, \\
-y z e_i &= g_i^2 - z g_i - 1; \\
y z g_i e_{i+1} e_i &= z g_{i+1}^{-1} e_i.
\end{align}

It is easy to show that $BMW_r(y, z)$ may be defined using the relations (1.2), (1.4), (1.6) and (1.8) instead of (1.2), (1.3) and (1.4), i.e. that (1.3) is a consequence of (1.6) and (1.8).

We shall require a particular specialisation of $BMW_r(y, z)$ to a subring $\mathcal{A}_q$ of $\mathcal{K}$, which is defined as follows. Let $\mathcal{S}$ be the multiplicative subset of $\mathbb{C}[q, q^{-1}]$ generated by $[2]_q$, $[3]_q$ and $[3]_q^{-1} - 1$. Let $\mathcal{A}_q := \mathbb{C}[q, q^{-1}]S := \mathbb{C}[q, q^{-1}, [2]_q^{-1}, [3]_q^{-1}, (q^2 + q^{-2})^{-1}]$ be the localisation of $\mathbb{C}[q, q^{-1}]$ at $\mathcal{S}$.

Now let $\psi : \mathbb{C}[y^{\pm 1}, z] \rightarrow \mathcal{A}_q$ be the homomorphism defined by $y \mapsto q^{-4}$, $z \mapsto q^2 - q^{-2}$. Then $\psi$ makes $\mathcal{A}_q$ into an $\mathcal{A}_{y,z}$-module, and the specialisation $BMW_r(q) := \mathcal{A}_q \otimes_{\mathcal{A}_{y,z}} BMW_r(y, z)$ is the $\mathcal{A}_q$-algebra with generators which we denote, by abuse of notation, $g_i^{\pm 1}, e_i$ ($i = 1, \ldots, r - 1$) and relations (1.10) below, with the relations (1.11) being consequences of (1.10).
\[ g_i g_j = g_j g_i \text{ if } |i - j| \geq 2 \]

\[ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \text{ for } 1 \leq i \leq r - 1 \]

\[ g_i - g_i^{-1} = (q^2 - q^{-2})(1 - e_i) \text{ for all } i \]

\[ g_i e_i = e_i g_i = q^{-4} e_i \]

\[ e_i g_{i+1} e_i = q^{4} e_i \]

\[ e_i g_{i+1} e_i = q^{4} e_i. \]

\[ (g_i - q^2)(g_i + q^{-2}) = -q^{-4}(q^2 - q^{-2})e_i \]

\[ (g_i - q^{-4})(g_i - q^2)(g_i + q^{-2}) = 0 \]

\[ e_i^2 = (q^2 + 1 + q^{-2})e_i \]

\[ g_i^{-1} e_i = q^{-4} g_i e_{i+1} e_i. \]

We shall be concerned with the following two specialisations of \( BMW_r(q) \).

**Definition 1.1.** Let \( \phi_q : A_q \longrightarrow \mathbb{K} = \mathbb{C}(q^{\frac{1}{2}}) \) be the inclusion map, and let \( \phi_1 : A_q \longrightarrow \mathbb{C} \) be the \( \mathbb{C} \)-algebra homomorphism defined by \( q \mapsto 1 \). Define the specialisations \( BMW_r(\mathbb{K}) := \mathbb{K} \otimes_{\phi_q} BMW_r(q) \), and \( BMW_r(1) := \mathbb{C} \otimes_{\phi_1} BMW_r(q) \).

The following facts are proved in [LZ2, Lemma 4.2, Theorem 4.4].

**Proposition 1.2.**

1. The surjective homomorphism \( \eta \) of \( \mathbb{K} \) factors through \( BMW_r(\mathbb{K}) \). That is, there is a surjective homomorphism

\[ \eta_q : BMW_r(\mathbb{K}) \longrightarrow E_r(2, q), \]

in which the generators \( g_i \) are mapped to the \( R \) matrices on factors \( i, i + 1 \).

2. The specialisation \( BMW_r(1) \) is isomorphic to the Brauer algebra \( B_r(3) \).

**1.3. The quasi-idempotent \( \Phi_q \).** Following [LZ2], we introduce an element \( \Phi_q \) of \( BMW_r(q) \), and state its principal properties. Our objective will be to show that \( \Phi_q \in BMW(\mathbb{K}) \) generates the kernel of \( \eta_q \). We begin with the following elements of \( BMW_r(q) \). Note that we regard \( BMW_s(q) \subset BMW_{s+1}(q) \) in the usual way; it is well known that the algebra \( BMW_s(q) \) generated by the \( e_i \) and \( g_i \) with \( 1 \leq i \leq s - 1 \) and the relations among those generators is a subalgebra of \( BMW_{s+1}(q) \). In terms of diagrams, this subalgebra is spanned by the diagrams spanning \( BMW_s(q) \), with an additional string joining the rightmost nodes.

Let \( f_i = -g_i - (1 - q^{-2})e_i + q^2 \), and set

\[ F_q = f_1 f_3. \]

We also define \( e_{14} = g_3^{-1} g_1 e_2 g_1^{-1} g_3 \) and \( e_{1234} = e_{29} g_3^{-1} g_2 g_1^{-1} g_3 \).

**Definition 1.3.** Maintaining the above notation, define the following element of \( BMW_4(q) \subset BMW_r(q) \):

\[ \Phi_q = a F_q e_2 F_q - b F_q - c F_q e_2 e_{14} F_q + d F_q e_{1234} F_q, \]
where
\[
\begin{align*}
    a &= 1 + (1 - q^{-2})^2, \\
    b &= 1 + (1 - q^{-2})^2 + (1 - q^{-2})^2, \\
    c &= \frac{1 + 2 - q^{-2}(1 - q^{-2})^2 + (1 + q^2)(1 - q^{-2})^4}{([3]_q - 1)^2}, \\
    d &= (q - q^{-1})^2 = q^2(a - 1).
\end{align*}
\]

The principal properties of these elements are summarised in the following statement, which is \cite[Prop. 7.3]{LZ2}.

**Proposition 1.4.** The elements \( F_q, \Phi_q \) have the following properties:

1. \( F_q^2 = (q^2 + q^{-2})^2 F_q \).
2. \( e_i \Phi_q = \Phi_q e_i = 0 \) for \( i = 1, 2, 3 \).
3. \( \Phi_q^2 = -(q^2 + q^{-2})^2(1 + (1 - q^2)^2 + (1 - q^{-2})^2) \Phi_q \).
4. \( \Phi_q \) acts as 0 on \( V_q^\otimes 4 \).

1.5. The classical limit \( q \to 1 \); the Brauer algebra. Let \( B_r(\delta) \) be the Brauer algebra over a commutative ring \( A \), with \( \delta \in A \); this may be defined as follows. It has generators \( \{ s_1, \ldots, s_{r-1}; e_1, \ldots, e_{r-1} \} \), with relations \( s_i^2 = 1, \ e_i^2 = \delta e_i, \ s_i e_i = e_i s_i = e_i \) for all \( i \), \( s_i s_j = s_j s_i \), \( s_i e_j = e_j s_i \), \( e_i e_j = e_j e_i \) if \( |i - j| \geq 2 \), and \( s_i s_{i+1} s_i = s_i s_{i+1} s_{i+1} \), \( e_i e_{i+1} e_i = e_i \) and \( s_i e_{i+1} e_i = s_{i+1} e_i \), \( e_{i+1} e_i s_{i+1} = e_{i+1} s_i \) for all applicable \( i \). We shall assume the reader is familiar with the diagrammatic representation of a basis of \( B_r(\delta) \), and how basis elements are multiplied by concatenation of diagrams. In particular, the group ring \( A\text{Sym}_r \) is the subalgebra of \( B_r(\delta) \) spanned by the diagrams with \( r \) “through strings”, and the algebra contains elements \( w \in \text{Sym}_r \) which are appropriate products of the \( s_i \).

In this work we shall take \( A = \mathbb{C} \) and \( \delta = 3 \); the corresponding Brauer algebra will be denoted by \( B_r(3) \). In the identification given by Proposition 1.2 of the specialisation \( BWM_r(q) \otimes \delta q \mathbb{C} \) with \( B_r(3) \), \( g_i \otimes 1 \) corresponds to \( s_i \) and \( e_i \otimes 1 \) corresponds to \( e_i \in B_r(3) \). Accordingly, \( F_q \) and \( \Phi_q \) specialise respectively to \( F = (1 - s_1)(1 - s_3) \) and \( \Phi = F e_2 F - F - F e_2 e_{1,4} F \), where \( e_{1,4} = s_1 s_2 e_3 s_1 \). From the relations in Proposition 1.4 we see that \( F^2 = 4F \), and \( \Phi^2 = -4\Phi \).

Let \( V_1 \) be the irreducible \( U(\mathfrak{sl}_2) \)-module. The following statement may be found in \cite[§6]{LZ2}.

**Proposition 1.5.**

1. There is a surjective homomorphism
\[
\eta : B_r(3) \to \text{End}_{U(\mathfrak{sl}_2)}(V_1^\otimes r) := E(r).
\]

2. The kernel of \( \eta \) contains \( \Phi \).

1.5. **Statements.** We are now able to state our main results.

**Theorem 1.6.** Let \( U_q \) be the quantised enveloping algebra of \( \mathfrak{sl}_2 \) over the field \( \mathbb{K} = \mathbb{C}(q^2) \). Let \( V_q \) be the irreducible three-dimensional \( U_q \)-module (with highest weight 2), and let \( \eta_q : BWM_r(\mathbb{K}) \to E_r(2, q) := \text{End}_{U_q}(V_q^\otimes r) \) be the surjective homomorphism defined in Proposition 1.2 (1). Then the kernel of \( \eta_q \) is the two sided ideal of \( BWM(\mathbb{K}) \) generated by the quasi-idempotent \( \Phi_q \), defined in Definition 1.3.

In particular, \( E_r(2, q) \) has a presentation given as follows. \( E_r(2, q) \) has generators \( g_i, e_i, i = 1, \ldots, r - 1 \), with relations (1.10), plus the relation \( \Phi_q = 0 \).
The classical ($q = 1$) analogue of the above statement is as follows.

**Theorem 1.7.** Let $U$ be the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, and let $V_1$ be the irreducible $U$-module of dimension three. If $B_r(3)$ is the complex $r$-string Brauer algebra with parameter 3 (§1.4) and $\eta : B_r(3) \rightarrow \text{End}_{U(\mathfrak{sl}_2)}(V_1^{\otimes r}) := E(r)$ be the surjective homomorphism of Prop. 1.5. Then the kernel of $\eta$ is the two sided ideal of $B_r(3)$ which is generated by the quasi-idempotent $\Phi$ of §1.4.

It should be noted that both BMW$_r(K)$ and $B_r(3)$ are non-semisimple if $r \geq 5$. This is part of the significance of these results.

We shall prove both theorems together, making use of the following fact.

**Proposition 1.8.** Theorem 1.6 is a consequence of Theorem 1.7.

**Sketch of Proof.** This is Corollary 7.12 (1) of [LZ2]. For the convenience of the reader, we give a brief sketch of the proof here.

The algebra BMW$_{r}(q)$ has a well known cellular structure ([X]), with the cells being indexed by the partially ordered set

$$\Lambda_r := \{\text{partitions } \lambda \mid |\lambda| = t, \ 0 \leq t \leq r, \ t \equiv r(\text{ mod 2})\},$$

where for any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0)$, we write $|\lambda| = \sum \lambda_i$. The partial order on $\Lambda_r$ is given by $\mu > \lambda$ if $|\mu| > |\lambda|$ or $|\mu| = |\lambda|$ and $\mu > \lambda$ in the dominance order on partitions of $t$. This cellular structure is inherited by the specialisations BMW$_{r}(K)$ and $B_r(3)$ (cf. [GL96]), and by Prop. 7.1 of [LZ2]. The cell modules, their radicals, and the irreducible modules for any specialisation of BMW$_{r}(q)$ are obtained by specialising the respective BMW$_{r}(q)$-modules.

Using this, one shows, using the general criteria established in [LZ2] §5 that if $J_q$ is the ideal of BMW$_{r}(q)$ generated by $\Phi_q$, and $J_K$ and $J$ are its respective specialisations to BMW$_{r}(K)$ and $B_r(3)$, then $J_K$ contains the radical of BMW$_{r}(K)$ if and only if $J$ contains the radical of $B_r(3)$. But the latter are precisely the respective conditions for $J_K$ (resp. $J$) to be the whole kernel of $\eta_q$ (resp. $\eta$), whence the assertion. \square

We therefore turn to the proof of Theorem 1.7, which will require detail concerning the cellular structure of the Brauer algebras. Before discussing the details in §3, we shall recall some basic facts concerning cellular algebras.

2. **Cellular algebras and their radicals.**

In this section we summarise the main properties of cellular algebras of which we make use below. The principal references are [GL03] [GL04] and [LZ2] §5.

2.1. **Cellular algebras and cell modules.** Let $\mathbb{F}$ be any commutative ring, which later will be taken to be a field. The $\mathbb{F}$-algebra $B$ is cellular if it has a cell datum $(\Lambda, M, C, \ast)$ with the following properties:

- $\Lambda$ is a finite partially ordered set
- $M : \Lambda \rightarrow \text{Sets}$ is a function which associates a finite set $M(\lambda)$ to each $\lambda \in \Lambda$
- $C$ is a function $C : \prod_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \rightarrow B$,
  whose image is an $\mathbb{F}$-basis of $B$; for $S, T \in M(\lambda)$, write $C(S, T) := C^\Lambda_{S,T}$. 

For any element \(a \in B\) and \(S, S' \in M(\lambda)\) there are scalars \(r_a(S', S) \in F\) such that for \(S, T \in M(\lambda)\), we have

\[
aC_{S,T}^\lambda \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C_{S',T}^\lambda \mod B(< \lambda),
\]

where \(B(< \lambda)\) is the two-sided ideal of \(B\) spanned by \(\{C_{U,V}^\mu | \mu < \lambda, U, V \in M(\mu)\}\).

The linear map defined by \(* : C_{S,T}^\lambda \mapsto C_{T,S}^\lambda := C_{T,S}^\lambda\) is an involutory anti-automorphism of \(B\).

The cell module \(W(\lambda) (\lambda \in \Lambda)\) is the free \(F\)-module spanned by symbols \(C(S) (S \in M(\lambda))\), with the \(B\)-action defined by (2.1), i.e.

\[
a \cdot C(S) = \sum_{S' \in M(\lambda)} r_a(S', S) C(S'), \quad \text{for } a \in B.
\]

Define the bilinear form \(\phi_\lambda\) on the basis \(\{C(S)\}\) of \(W(\lambda)\) by

\[
C_{S,T}^2 = \phi_\lambda(C(S), C(T)) C_{S,T}^\lambda \mod B(< \lambda).
\]

The principal properties of \(\phi_\lambda\) are:

- \(\phi_\lambda\) is well-defined and is symmetric.
- \(\phi_\lambda\) is invariant, in the sense that for \(u, v \in W(\lambda)\) and \(a \in B\), we have \(\phi_\lambda(a \cdot u, v) = \phi_\lambda(u, a^* \cdot v)\).
- It follows that \(\text{Rad}(\lambda) := \{w \in W(\lambda) | \phi_\lambda(w, v) = 0 \text{ for all } v \in W(\lambda)\}\) is a submodule of \(W(\lambda)\).
- If \(F\) is a field, then for any element \(w \in W(\lambda) \setminus \text{Rad}(\lambda)\), we have \(W(\lambda) = B \cdot w\).
  Hence \(\text{Rad}(\lambda)\) is the unique maximal submodule of \(W(\lambda)\).

The last statement above arises in the following context. In addition to its structure as a left \(B\)-module defined above, the \(F\)-module \(W(\lambda)\) also has a right \(B\)-module structure, where the action is as above, composed with \(*\). We define the \(F\)-module monomorphism \(C^\lambda\) by

\[
C^\lambda : W(\lambda) \otimes_F W(\lambda) \longrightarrow B
\]

\[
C(S) \otimes C(T) \mapsto C_{S,T}^\lambda.
\]

The properties of this map are summarised as follows.

- There is an \(F\)-module isomorphism \(B \longrightarrow \bigoplus_{\lambda \in \Lambda} \text{Im}(C^\lambda)\).
- If we compose \(C^\lambda\) with the natural map \(B \rightarrow B/B(< \lambda), \text{where } B(< \lambda)\) is the two-sided ideal defined above, we obtain a monomorphism of \((B, B)\) bimodules, whose image is, in the obvious notation, \(B(\leq \lambda)/B(< \lambda)\).
- For \(u, v\) and \(w\) in \(W(\lambda)\), we have

\[
C^\lambda(u \otimes v) \cdot w = \phi_\lambda(v, w) u.
\]

If \(F\) is a field, the cyclic nature of \(W(\lambda)\) if \(\phi_\lambda \neq 0\) follows from the last statement.
2.2. Simple modules and composition factors. It follows from Proposition 2.1 that for each \( \lambda \in \Lambda \), either \( \phi_\lambda = 0 \) or \( L(\lambda) := W(\lambda) / \text{Rad}(\lambda) \) is a simple \( B \)-module. One of the keys to understanding these modules, and more generally the composition factors of the cell modules \( W(\lambda) \), is the following fact.

**Proposition 2.1.** Let \( \lambda, \mu \in \Lambda \), and suppose we have a non-zero homomorphism \( \psi : W(\mu) \to W(\lambda) / M \), where \( M \) is some submodule of \( W(\lambda) \). Then

1. \( \mu \geq \lambda \).
2. If \( \mu = \lambda \) and \( \mathbb{F} \) is a field then \( \psi \) is realised as multiplication by a scalar.

Although neither is essential, we shall now make two simplifying assumptions. First, we assume that henceforth \( \mathbb{F} \) is a field, and secondly, that for each \( \lambda \in \Lambda \), \( \phi_\lambda \neq 0 \). This says that \( B \) is quasi-hereditary. The algebras to which we shall apply the theory satisfy both these assumptions, but it is possible to develop the theory without them.

Given the assumptions, we have the following summary of the main points concerning the representation theory of \( B \).

- The modules \( L(\lambda) = W(\lambda) / \text{Rad}(\lambda) \) (\( \lambda \in \Lambda \)) form a complete set of non-isomorphic simple \( B \)-modules.
- If \( L(\mu) \) is a composition factor of \( W(\lambda) \), then \( \mu \geq \lambda \).
- The following statements are equivalent: (a) \( B \) is semisimple; (b) Each cell module \( W(\lambda) \) is simple; (c) For each \( \mu \neq \lambda \), \( \text{Hom}_B(W(\mu), W(\lambda)) = 0 \).

2.3. The radical. Suppose \( B \) is a cellular algebra over the field \( \mathbb{F} \) as above, and denote the radical of \( B \) by \( \mathcal{R} \). This is a two sided ideal of \( B \) which may be defined by the property that \( B / \mathcal{R} \) is semisimple, and any homomorphism \( B \to A \), where \( A \) is a semisimple \( \mathbb{F} \)-algebra, factors through \( B / \mathcal{R} \). In particular, we have an exact sequence of \( \mathbb{F} \)-algebras

\[
0 \to \mathcal{R} \to B \xrightarrow{\pi} \bigoplus_{\lambda \in \Lambda} \text{End}_\mathbb{F}(L(\lambda)) \to 0,
\]

so that \( \mathcal{R} \) may be characterised as the set of elements of \( B \) which act trivially on each of the irreducible \( B \)-modules. Note that in the exact sequence above, for each \( \lambda \in \Lambda \), the subspace \( \text{Im}(C^\lambda) \) of \( B \) is mapped by \( \pi \) (surjectively) onto \( \text{End}_\mathbb{F}(L(\lambda)) \).

Now the key point of the proof of Theorem 1.6 is to prove that a certain two-sided ideal of \( B \) contains \( \mathcal{R} \). With this in mind we state the next result, which may be found in [LZ2, Lemma 5.3].

**Proposition 2.2.** The radical \( \mathcal{R} \) has a filtration \( (\mathcal{R}(\lambda) := \mathcal{R} \cap B(\leq \lambda)_{\lambda \in \Lambda}) \) with the property that there is a \((B, B)\) bimodule isomorphism

\[
\mathcal{R}(\lambda) / (\mathcal{R} \cap B(< \lambda)) \to W(\lambda) \otimes_\mathbb{F} \text{Rad}(\lambda) + \text{Rad}(\lambda) \otimes_\mathbb{F} W(\lambda) \subseteq W(\lambda) \otimes_\mathbb{F} W(\lambda).
\]

This leads to the next result, which is easily deduced from [LZ2, Theorem 5.4], and which is crucial in the proof of our main theorem.

**Theorem 2.3.** Maintaining the notation above, let \( J \) be a two-sided ideal of \( B \) which satisfies \( J^* = J \). Define complementary subsets \( \Lambda^0 \) and \( \Lambda^1 \) of \( \Lambda \) by \( \Lambda^0 := \{ \lambda \in \Lambda \mid J \cdot L(\lambda) = 0 \} \) and \( \Lambda^1 := \Lambda \setminus \Lambda^0 \). Then \( J \supseteq \mathcal{R} \) if and only if, for each \( \lambda \in \Lambda^0 \), all composition factors of the cell module \( W(\lambda) \) other than \( L(\lambda) \) are isomorphic to some \( L(\mu) \) for \( \mu \in \Lambda^1 \).
Proof. It is a straightforward consequence of Proposition 2.2 above and the general characterisation of $\mathcal{R}$ as the set of elements of $B$ which annihilate each $L(\lambda)$, that $J \supseteq \mathcal{R}$ if and only if, for each $\lambda \in \Lambda$, $J \cdot W(\lambda) \supseteq \text{Rad}(\lambda)$. But for $\lambda \in \Lambda^1$, since $J \cdot W(\lambda) \not\subseteq \text{Rad}(\lambda)$, it follows from the cyclic nature of $W(\lambda)$ (see above) that $J \cdot W(\lambda) = W(\lambda) \supseteq \text{Rad}(\lambda)$.

It is therefore evident (cf. [LZ2 Theorem 5.4(3)]), that $J \supseteq \mathcal{R}$ if and only if for each $\lambda \in \Lambda^0$, $J \cdot W(\lambda) \supseteq \text{Rad}(\lambda)$. We shall show that the condition in our statement is equivalent to this.

Take $\lambda \in \Lambda^0$. Let

$$W(\lambda) = W_0 \supset W_1 = \text{Rad}(\lambda) \supset W_2 \supset \cdots \supset W_s = 0$$

be a composition series of $W(\lambda)$, and write $L_i$ for the simple quotient $L_i := W_{i-1}/W_i$ ($i = 1, \ldots, s$). Then $L_1 \cong L(\lambda)$ and since the $L(\mu)$ ($\mu \in \Lambda$) form a complete set of isomorphism classes of simple $B$-modules, it follows from Proposition 2.1 that for

$$J \cdot W(\lambda) = J \cdot W_0 \supseteq J \cdot W_1 = J \cdot \text{Rad}(\lambda) \supseteq J \cdot W_2 \supseteq \cdots \supseteq J \cdot W_s = 0,$$

and the quotient $J \cdot W_{i-1}/J \cdot W_i \cong J \cdot L(\mu_i)$.

It is therefore clear that $J \cdot W(\lambda) \supseteq \text{Rad}(\lambda)$ if and only if, for each $i > 1$, $J \cdot L(\mu_i) \neq 0$, i.e. $\mu_i \in \Lambda^1$. \hfill $\Box$

3. On the representation theory of the Brauer algebras.

We shall require some detailed analysis of the cell modules of the Brauer algebras and their radicals. Most of the material in this section is adapted from [GL96, DHW, HW, LZ2]. Since there is no difference in the arguments, for the purposes of the current section, $K$ could be any field of characteristic zero, and we shall consider the Brauer algebras $B_r(\delta)$, where $\delta$ is any non-zero element of $K$. This restriction is not essential, but enables us to simplify the exposition; moreover our application is to this situation ($\delta = 3$). We shall write $B_r$ for $B_r(\delta)$ for this section only.

Notation will be similar to that in [LZ2]. For $\lambda \in \Lambda_r$, we have a cell module $W_r(\lambda)$ which has a canonical symmetric bilinear invariant form $(-,-)_{\lambda}$, whose radical $\text{Rad}_r(\lambda)$ is the unique maximal submodule of $W_r(\lambda)$. In our situation ($\delta \neq 0$) we have $(-,-)_{\lambda} \neq 0$ for all $\lambda \in \Lambda_r$ (the “quasi-hereditary” case). The irreducible quotients $L_r(\lambda) := W_r(\lambda)/\text{Rad}_r(\lambda)$ are pairwise non-isomorphic, and provide a complete set of irreducible $B_r$-modules.

For the proof of the next Lemma, we shall require the following explicit description of the cell modules. Recall that for any $t$, the group algebra $\mathbb{C}\text{Sym}_t$ is a subalgebra of $B_t$, realised as the span of the diagrams with $t$ through strings (or generated by the $s_i$). Let $\lambda \in \Lambda_r$ with $|\lambda| = t$, and $r = t+2k$. Since $B_r$ is a $(B_r, B_r)$ bimodule, the subspace $I^t_r := B_r e_{t+1} e_{t+3} \cdots e_{t+2k-1}$ is a $(B_r, B_t)$-bimodule, and hence a foriori a $(B_r, \mathbb{C}\text{Sym}_t)$-bimodule. Clearly the subspace $I^t_r$ spanned by diagrams with at most $t - 2$ through strings is a sub-bimodule, to be interpreted as 0 if $t < 2$. Define $I^t_r$ as the quotient $(B_r, \mathbb{C}\text{Sym}_t)$-bimodule $I^t_r/\bar{I}^t_r$.

It is easy to see from the description in [GL96 Def. (4.9)] (cf. also [DHW]) that

$$W_r(\lambda) \cong I^t_r \otimes_{\mathbb{C}\text{Sym}_t} S(\lambda)$$

(3.1)
Lemma 3.1. Let $\mu_1, \mu_2 \in \Lambda_r$. If $L_r(\mu_1)$ is a composition factor of $W_r(\mu_2)$, then $|\mu_1| > |\mu_2|$ or $\mu_1 = \mu_2$.

Proof. Suppose $\phi : W_r(\mu_1) \rightarrow W_r(\mu_2)/M$ is a $B_r$-homomorphism, where $\mu_1, \mu_2 \in \Lambda_r$ and $|\mu_1| = |\mu_2| = t$. We shall show that if $\mu_1 \neq \mu_2$, $\phi = 0$.

Consider the subspaces $W^0_r(\mu_i)$ of $W_r(\mu_i)$ ($i = 1, 2$) defined by

$$W^0_r(\mu_i) = e_{t+1}e_{t+3} \ldots e_{r-1}W_r(\mu_i).$$

Since $e_{t+1}e_{t+3} \ldots e_{r-1} \in B^t_r$ and for $w \in e_{t+1}e_{t+3} \ldots e_{r-1}W_r(\mu_i)$ we have

$$e_{t+1}e_{t+3} \ldots e_{r-1}w = \delta^kw\ (r = t + 2k),$$

it follows that $W^0_r(\mu_i)$ generates $W_r(\mu_i)$ as $B_r$-module. Moreover it is evident that $W^0_r(\mu_i)$ is stable under $\mathbb{C}\text{Sym}_t \subset B_t \subset B_r$, and is isomorphic as $\mathbb{C}\text{Sym}_t$-module to $S(\mu_i)$.

Since $\phi$ respects the $B_r$-action, we have $\phi(W^0_r(\mu_1)) \subseteq e_{t+1}e_{t+3} \ldots e_{r-1}(W_r(\mu_2)/M)$. It follows that the restriction to $W^0_r(\mu_1)$ of $\phi$ defines a $\mathbb{C}\text{Sym}_t$-homomorphism $\phi^0 : S(\mu_1) \rightarrow S(\mu_2)$. But since $K$ has characteristic zero, the cell modules $S(\nu)$ of $\mathbb{C}\text{Sym}_t$ are simple, whence if $\mu_1 \neq \mu_2$, $\phi^0 = 0$. It follows that $\phi = 0$ on a generating subspace of $W_r(\mu_1)$, whence $\phi = 0$. \hfill $\square$

Note that Lemma 3.1 is false if the restriction on the characteristic of $K$ is lifted, since then there may be non-trivial homomorphisms between the cell (Specht) modules $S(\lambda)$ for $\text{Sym}_r$.

This leads to the following alternative characterisation of the radical $\text{Rad}_r(\lambda)$ of a cell module of $B_r$.

Corollary 3.2. Let $\lambda \in \Lambda_r$, with $|\lambda| = t$. For any integer $m$, let $B^m_r$ be the ideal of $B_r$ spanned by diagrams with at most $m$ through strings. Then

$$\text{Rad}_r(\lambda) = \{w \in W_r(\lambda) \mid bw = 0 \text{ for all } b \in B^t_r\} = \text{Ann}_{W_r(\lambda)}(B^t_r).$$

Proof. By the previous Lemma, any composition factor of $W_r(\lambda)$ other than $L_r(\lambda)$ is of the form $L_r(\mu)$ with $|\mu| > t$. In particular, any composition factor of $\text{Rad}_r(\lambda)$ is of this form. But if $|\mu| > t$, then $B^t_r$ acts as zero on $W_r(\mu)$, since the number of through strings of any diagram $bD$ ($b \in B^t_r$) has at most $t(> r - |\mu|)$ through strings. It follows that $B^t_rL_r(\mu) = 0$. Hence $B^t_r$ acts as zero on each composition factor of $\text{Rad}_r(\lambda)$, and hence on $\text{Rad}_r(\lambda)$. So $\text{Rad}_r(\lambda) \subseteq \text{Ann}_{W_r(\lambda)}(B^t_r)$.

But $\text{Rad}_r(\lambda)$ the unique maximal submodule of $W_r(\lambda)$; in fact any element of $W_r(\lambda)$ which is not in $\text{Rad}_r(\lambda)$ generates $W_r(\lambda)$ ([GL96 Prop. 2.5]). Hence we have equality. \hfill $\square$

Next, we require the following strengthening of [DHW Thm. 5.4]. In the statement, we use the fact that for any positive integer $r$, we have $\Lambda_r \subset \Lambda_{r+2}$.

Lemma 3.3. Let $\mu, \lambda \in \Lambda_r$. Then $L_r(\mu)$ is a composition factor of $W_r(\lambda)$ if and only if $L_{r+2}(\mu)$ is a composition factor of $W_{r+2}(\lambda)$.
Proof. The proof will use the functors $F,G$ introduced in [G] in the context of Schur algebras, and used in the current context in [DHW, M]. They are defined as follows. Let $B_r - \text{mod}$ denote the category of left $B_r$-modules. Recalling that $B_{r-2} \subset B_r \subset B_{r+2}$ as described above, define $F : B_r - \text{mod} \rightarrow B_{r-2} - \text{mod}$ and $G : B_r - \text{mod} \rightarrow B_{r+2} - \text{mod}$ by

\[
F(M) := e_{r-1}M \quad G(M) := B_{r+2}e_{r+1} \otimes_{B_r} M,
\]

for any $B_r$-module $M$. The main properties of $F,G$ are that $FG = \text{id}$, $F$ is exact, and $G$ is right exact (cf. [HW, Cor 4.4, p. 143]).

To prove the lemma, first note that $L_r(\mu)$ is a composition factor of $W_r(\lambda)$ if and only if there is a non-trivial homomorphism $\xi : W_r(\mu) \rightarrow W_r(\lambda)/N$ for some submodule $N$ of $W_r(\lambda)$. Applying the functor $G$, we obtain $G(\xi) : G(W_r(\mu)) \rightarrow G(W_r(\lambda)/N)$, and by [DHW, Lemma 5.1], $G(\xi) \neq 0$. Moreover by [loc. cit., Prop. 5.2], $G(W_r(\mu)) = W_{r+2}(\mu)$. To prove that $L_{r+2}(\mu)$ is a composition factor of $W_{r+2}(\lambda)$, it will therefore suffice to show that $G(W_r(\lambda)/N) = W_{r+2}(\lambda)/M$, for some submodule $M$ of $W_{r+2}(\lambda)$. But the exact sequence $W_r(\lambda) \rightarrow W_r(\lambda)/N \rightarrow 0$ is taken by the (right exact) functor $G$ to an exact sequence of $B_{r+2}$-modules, whence $G(W_r(\lambda)/N)$ is isomorphic to a quotient of $G(W_r(\lambda)) = W_{r+2}(\lambda)$. Hence if $L_r(\mu)$ is a composition factor of $W_r(\lambda)$, then $L_{r+2}(\mu)$ is a composition factor of $W_{r+2}(\lambda)$.

Conversely, if $L_{r+2}(\mu)$ is a composition factor of $W_{r+2}(\lambda)$, there is a non-trivial homomorphism $\psi : W_{r+2}(\mu) \rightarrow W_{r+2}(\lambda)/M$, for some submodule $M$ of $W_{r+2}(\lambda)$. Applying the functor $F$, we obtain a homomorphism

\[
F(\psi) : F(W_{r+2}(\mu)) \rightarrow F(W_{r+2}(\lambda)/M).
\]

But by the exact nature of $F$, we have $F(W_{r+2}(\lambda)/M) \cong F(W_{r+2}(\lambda))/F(M)$, and since $F(W_{r+2}(\nu)) = W_r(\nu)$ for $\nu \in \Lambda_r$ [DHW, Prop. 5.3], we obtain $F(\psi) : W_r(\mu) \rightarrow W_r(\lambda)/F(M)$. It remains only to prove that $F(\psi) \neq 0$. For this, consider the exact sequence

\[
0 \rightarrow \text{Ker} \psi \rightarrow W_{r+2}(\mu) \xrightarrow{\psi} \text{Im} \psi \rightarrow 0.
\]

Applying $F$, we obtain an exact sequence

\[
0 \rightarrow F(\text{Ker} \psi) \rightarrow W_r(\mu) \xrightarrow{F(\psi)} F(\text{Im} \psi) \rightarrow 0,
\]

and hence it suffices to show that $F(\text{Ker} \psi) \neq W_r(\mu)$. But $\text{Ker} \psi \subseteq \text{Rad}_{r+2}(\mu) = \text{Ann}_{W_{r+2}(\mu)}(B^*_r)$, where $|\mu| = s$ (by Cor [3.2]). It follows that $e_{s+1}e_{s+3} \cdots e_{r-1}\text{Ker} \psi = 0 = e_{s+1}e_{s+3} \cdots e_{r-1}F(\text{Ker} \psi)$. Moreover since $e_{s+1}e_{s+3} \cdots e_{r-1}$ generates the two-sided ideal $B^*_r$ of $B_r$, it follows that $F(\text{Ker} \psi) \subseteq \text{Ann}_{W_r(\mu)}(B^*_r) = \text{Rad}_r(\mu)$. \hfill $\square$

**Corollary 3.4.** (cf. [DHW, Thm. 5.4]) Let $\lambda, \mu \in \Lambda_r$, with $|\mu| = s$. Then $L_r(\mu)$ is a composition factor of $W_r(\lambda)$ if and only if $L_s(\mu)$ is a composition factor of $W_s(\lambda)$.

This is immediate from repeated application of Lemma 3.3.

We shall require a combinatorial consequence of Corollary 3.4 which is proved in [DHW]. The statement involves the following details concerning partitions. Associated to a partition $\lambda := (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0)$ we have a “Young diagram” which is the set of plane lattice points $Y(\lambda) := \{(i,j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid 1 \leq j \leq \lambda_i\}$. For any point $P = (i,j) \in Y(\lambda)$, define the content $c(P)$ of $P$ by $c(P) = j - i$. Write $c(\lambda) = \sum_{P \in Y(\lambda)} c(P)$. 


Proposition 3.5. ([DHW] Thms. 3.1 and 3.3) Let $\mu, \lambda \in \Lambda_r$ with $|\mu| = s$. If $L_s(\mu)$ is a composition factor of $W_s(\lambda)$, then

1. $Y(\lambda) \subseteq Y(\mu)$ (as sets), and
2. $|\mu| - |\lambda| + \sum_{P \in Y(\mu) \setminus Y(\lambda)} c(P) = 0$.

Corollary 3.6. Let $\mu, \lambda \in \Lambda_r$ with $|\mu| = s$. If $L_r(\mu)$ is a composition factor of $W_r(\lambda)$, then the conditions (1) and (2) of Proposition 3.5 hold.

This is immediate from Corollary 3.4.

4. Proof of the main theorems.

Recall that we have the surjective homomorphism \( \eta : B_r(3) \longrightarrow E(r) \), where $E(r)$ is the semisimple $\mathbb{C}$-algebra $\text{End}_{U(sl_2)}(V_1^{\otimes r})$. Write $N = \text{Ker}\, \eta$; then $\Phi \in N$, so that if $J$ is the two sided ideal of $B_r(3)$ generated by $\Phi$, we have $J \subseteq N$. Define the subsets $\Lambda^0_r, \Lambda^1_r$ of $\Lambda_r$ by $\Lambda^0_r = \{(t), (t-1, 1), 1^3 \mid 0 \leq t \leq r; \ t \equiv r(\text{mod } 2)\}$, and $\Lambda^1_r : = \Lambda \setminus \Lambda^0_r$.  

Proposition 4.1. The following are equivalent conditions on an element $\lambda \in \Lambda_r$.

1. $\lambda \in \Lambda^0_r$.
2. $NL_r(\lambda) = 0$.
3. $JL_r(\lambda) = 0$.

Sketch of Proof. This statement is contained in [LZ2] Thm. 6.8; we give a brief sketch of the argument for the reader’s convenience. One first shows that if $\lambda \not\in \Lambda^0_r$ (i.e. $\lambda \in \Lambda^1_r$) then $\Phi L_r(\lambda) \neq 0$; this involves explicit computation, and a knowledge of the cases $r = 4, 5$. It follows that $N$ acts non-trivially on $L_r(\lambda)$ for $\lambda \in \Lambda^1_r$, since $\Phi \in N$.

But the semisimple algebra $E(r)$ has $r + 1$ simple components, whence $N$ acts as zero on precisely $r + 1$ of the modules $L_r(\lambda)$. Since $|\Lambda^0_r| = r + 1$, it follows that $N$ acts trivially on $L_r(\lambda)$ for $\lambda \in \Lambda^0_r$. The statement is now clear. □

Corollary 4.2. Let $\mathcal{R}$ be the radical of $B_r$. Then $N = J$ if and only if $J$ contains $\mathcal{R}$.

Proof. We have $E(r) \cong B_r/N$, and by Prop. 4.1 $B_r/N$ and $B_r/J$ have the same maximal semisimple quotient (viz. $E(r)$). Hence $N = J + \mathcal{R}$. □

Proof of Theorem 1.6. It remains only to show that $J \supseteq \mathcal{R}$. Since the element $\Phi$ satisfies $\Phi^* = \Phi$, we may apply Theorem 2.3 above, which implies that it suffices to show that for $\lambda \in \Lambda^0_r$, $JW_r(\lambda) = \text{Rad}_r(\lambda)$. Note that by Prop. 4.1 we have for $\lambda \in \Lambda^0_r$, $JL_r(\lambda) = 0$, whence $JW_r(\lambda) \subseteq \text{Rad}_r(\lambda)$. We shall prove that we have equality by showing, for $\lambda \in \Lambda^0_r$,

\[(4.1) \quad \text{If } L_r(\mu) \text{ is a composition factor of } W_r(\lambda) \text{ and } \mu \neq \lambda, \text{ then } \mu \in \Lambda^1_r.\]

Given (4.1), it follows that for $\lambda \in \Lambda^0_r$, all composition factors $L_r(\mu)$ of $\text{Rad}_r(\lambda)$ satisfy $\mu \in \Lambda^1_r$, whence $JL_r(\mu) = L_r(\mu)$. It follows that $J\text{Rad}_r(\lambda) = \text{Rad}_r(\lambda)$, whence by Theorem 2.3 $J \supseteq \mathcal{R}$. □
It therefore remains only to prove 1.11. We do this by invoking the criteria in Prop. 3.5 for \( L_r(\mu) \) to be a composition factor of \( W_r(\lambda) \). There are three cases.

First, if \( \lambda = (t) \) and \( \mu \in \Lambda_r^0 \) with \( Y(\mu) \supset Y(\lambda) \), then there are three possibilities for \( \mu \): (a) \( \mu = (s) \), \( s > t \); (b) \( \mu = (s-1,1) \), \( s > t \); (c) \( t = 1 \) and \( \mu = (1^3) \). In cases (a) and (b), \( |\mu| - |\lambda| + \sum_{P \in Y(\mu) \setminus Y(\lambda)} c(P) > 0 \), while in case (c), \( |\mu| - |\lambda| + \sum_{P \in Y(\mu) \setminus Y(\lambda)} c(P) = -1 \neq 0 \).

Secondly, if \( \lambda = (t-1,1) \), the only possibility for \( \mu \in \Lambda_r^1 \) with \( Y(\mu) \supset Y(\lambda) \) is \( \mu = (s-1,1) \) with \( s > t \). In this case, again \( |\mu| - |\lambda| + \sum_{P \in Y(\mu) \setminus Y(\lambda)} c(P) > 0 \).

Finally, if \( \lambda = (1^3) \), there is no \( \mu \in \Lambda_r^0 \) with \( Y(\mu) \supset Y(\lambda) \).

This proves Theorem 1.7, and hence by Prop. 1.8 completes the proof of Theorem 1.6.

5. An integral form for the endomorphism algebra.

Let \( A \) be the ring \( \mathbb{C}[q^{\pm 1}] \). Lusztig [L1, L2] has defined an \( A \)-form \( U_A \) of the quantised universal enveloping algebra \( U_q(\mathfrak{g}) \) of a Lie algebra \( \mathfrak{g} \), which involves the divided powers of the generators \( e_i, f_i \) of \( U_q \). He also showed how to construct \( A \)-forms of higest weight modules for \( U_A \) by applying these divided powers to highest weight vectors of the corresponding \( U_q \)-modules. Define \( \phi_1 : A \to \mathbb{C} \) by \( \phi_1(q) = 1 \) and let \( \phi_q : A \to K \) be the inclusion map.

We shall define an \( A \)-form of the \( K \)-algebra \( E_r(2, q) \cong BMW_r(K)/\langle \Phi_q \rangle \) which specialises to the endomorphism algebras \( E_r(2) \) and \( E_r(2, q) \) respectively when we map \( A \) to \( \mathbb{C} \) and to \( K \) via \( \phi_1 \) and \( \phi_q \) respectively.

The definition is as follows. Let \( BMW_r(A) \) be the \( A \)-algebra generated by the set \( \{ q_1^{\pm 1}, \ldots, q_r^{\pm 1}; e_1, \ldots, e_{r-1} \} \) subject to the relations (1.10). It is explained in [X] p.285 that \( BMW_r(A) \) is free as an \( A \)-module, with basis a set of “tangle diagrams”.

Let \( a, b, c \) and \( d \) be the elements of \( K \) defined in (1.14), and write \( \tilde{a} = (q^2 + q^{-2})a, \tilde{b} = (q^2 + q^{-2})b, \tilde{c} = (q^2 + q^{-2})c, \tilde{d} = (q^2 + q^{-2})d \). Then \( \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in A \), and we define \( \tilde{\Phi}_q \) as \( \tilde{\Phi}_q = (q^2 + q^{-2})\Phi_q \), i.e. (cf. (1.13))

\[
\tilde{\Phi}_q = \tilde{a}F_qe_2F_q - \tilde{b}F_q - \tilde{c}F_qe_2e_1F_q + \tilde{d}F_qe_{1234}F_q.
\]

Then \( \tilde{\Phi}_q \in BMW_r(A) \).

**Definition 5.1.** Define the \( A \)-algebra \( \mathcal{E}_r(A) \) by \( \mathcal{E}_r(A) := BMW_r(A)/\langle \tilde{\Phi}_q \rangle \).

**Proposition 5.2.** Let \( \mathcal{E}_r(A) \) be the \( A \)-algebra of (5.1), and let \( \phi_1 : A \to \mathbb{C} \) and \( \phi_q : A \to K \) be the homomorphisms defined above, which make \( \mathbb{C} \) and \( K \) into \( A \)-modules. Then we have isomorphisms

\[
\mathcal{E}_r(A) \otimes_A \mathbb{C} \cong \text{End}_U(\mathfrak{sl}_2) V_1^{\otimes r},
\]

and

\[
\mathcal{E}_r(A) \otimes_A K \cong \text{End}_U(\mathfrak{sl}_2) V_q^{\otimes r},
\]

where \( V_1 \) is the irreducible \( \mathfrak{sl}_2(\mathbb{C}) \) module of highest weight 2 (dimension 3), and \( V_q \) is its \( U_q \)-analogue.

**Proof.** We prove the second statement. the proof of the first is similar. We have an exact sequence of \( A \)-algebras

\[
0 \longrightarrow \langle \tilde{\Phi}_q \rangle \longrightarrow BMW_r(A) \longrightarrow \mathcal{E}_r(A) \longrightarrow 0.
\]
Moreover we know that $BMW_r(A)$ is free as $A$-module, whence so is the kernel $\langle \tilde{\Phi}_q \rangle$. Applying the functor $- \otimes_A K$, which by (cf. [CWu Cor 4.4, p. 143]) is right exact, we obtain an exact sequence

\begin{equation}
BMW_r(K) \longrightarrow E_r(A) \otimes_A K \longrightarrow 0,
\end{equation}

and the kernel of the first map contains $\Phi_q$.

Now since $\tilde{\Phi}_q$ acts trivially on Lusztig’s $A$-form of $V^{\otimes r}$, it follows that there is a homomorphism $E_r(A) \longrightarrow \text{End}_{U(A)}(V^{\otimes r})$, whose image is the algebra of endomorphisms which are generated by the $R$-matrices, and whose kernel is, by Theorem [L6] a torsion submodule. The statement follows.

6. INTERPRETATION IN TERMS OF ORTHOGONAL LIE ALGEBRAS.

It is well known that the Lie algebras $sl_2(C)$ and $so_3(C)$ are abstractly isomorphic. In this section we shall make explicit the interpretation of our main theorem in terms of $so_3$. We shall discuss the classical ($q = 1$) situation; the quantum case may be discussed entirely similarly.

The three-dimensional representation of $sl_2(C)$ may be realised as follows. Let $V = C^3$ be the subspace of the polynomial ring $C[x,y]$ with basis $x^2, xy, y^2$. Then $sl_2(C)$ acts via $e = x \frac{\partial}{\partial y}$, $f = y \frac{\partial}{\partial x}$ and $h = [e, f] = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$. With respect to the given basis, $e, f$ and $h$ are represented by the matrices $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

Now $so_3(C)$ may be identified with the space of skew symmetric matrices. If we write $(a,b,c)$ for the matrix $\begin{bmatrix} 0 & -b & a \\ b & 0 & c \\ -a & -c & 0 \end{bmatrix}$, the Lie product $[(a_1, a_2, a_3), (b_1, b_2, b_3)] = (c_1, c_2, c_3)$, where $c_i = (-1)^{i-1} \det M_i$, where $M_i$ is the $2 \times 2$ matrix obtained from $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ by deleting the $i$th column. Writing $E = (i, 0, 1), F = (i, 0, -1)$ and $H = (0, 2i, 0)$, where $i = \sqrt{-1}$, it is easily verified that $E, F$ and $H$ satisfy the $sl_2$ relations, and that $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ conjugates $e, f, h$ into $E, F, H$ respectively, i.e. that $T e T^{-1} = E$ etc.

Inverting $T$, it follows that with respect to the basis $w_1 = x^2 + y^2, w_2 = -2 i xy, w_3 = -ix^2 + iy^2, e, f$ and $h$ have skew symmetric matrices. This identifies the 3-dimensional representation of $sl_2(C)$ explicitly with the natural representation of $so_3(C)$. The quantum case is similar.

It follows that our main theorems [L7] and [L6] may be stated in terms of the natural representation of $so_3$. Now the natural representation $V_n$ of quantum $so_{2n+1}$ shares with the case $n = 1$ the properties that it is strongly multiplicity free, and that the surjection $C(q)B_r \rightarrow \text{End}_{U_r(so_{2n+1}^+)}(V_n^{\otimes r})$, where $B_r$ is the $r$-string braid group, factors through a specialisation of the algebra $BMW_r(y, z)$. It may be reasonable to speculate that a result similar to ours holds in this generality.

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