Generalized Jaynes-Cummings Model with Intensity-Dependent and Non-resonant Coupling

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Abstract

We study the intensity-dependent and nonresonant Jaynes-Cummings Hamiltonian when the field is described by an arbitrary shape-invariant system. We determine the eigenstates, eigenvalues, time evolution matrix and the population inversion matrix factor.

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I. INTRODUCTION

In the preceding paper [1] we extended our earlier work [2] describing interactions between a two-level system and a shape-invariant [3–5] system. Here we present another generalization.

The model studied in Ref. [2] is a generalization of the Jaynes-Cummings model [6]. In the standard Jaynes-Cummings model the “field” is described by a harmonic oscillator. In our generalization it can be described by any shape-invariant system. In developing this model we made extensive use of the algebraic approach [4,5] to the supersymmetric quantum mechanics [7]. In this paper we further generalize the model to one with intensity-dependent interactions.

The standard Jaynes-Cummings model is an idealized model describing the interaction of matter with electromagnetic radiation. A variant of the Jaynes-Cummings model takes the coupling between matter and the radiation to depend on the intensity of the electromagnetic field [8–11]. This model has great relevance since this kind of interaction means that the coupling is proportional to the amplitude of the field which is a very simple case of a nonlinear interaction corresponding to a more realistic physical situation. The results of this model can also give insight into the behavior of other quantum systems with strong nonlinear interactions.

II. THE GENERALIZED INTENSITY-DEPENDENT AND NON-RESONANT JAYNES-CUMMINGS HAMILTONIAN

The expression of the intensity-dependent and non-resonant Jaynes-Cummings Hamiltonian can be written as

$$\hat{H} = \hat{A}^\dagger \hat{A} + \frac{1}{2} [\hat{A}, \hat{A}^\dagger] (\hat{\sigma}_3 + 1) + \alpha \left( \hat{\sigma}_+ \sqrt{\hat{A}^\dagger \hat{A}} + \hat{\sigma}_- \sqrt{\hat{A} \hat{A}^\dagger} \right) + \hbar \Delta \hat{\sigma}_3,$$

where $\alpha$ is a constant related with the coupling strength, $\Delta$ is a constant related with the detuning of the system and $\hat{\sigma}_i$, with $i = 1, 2,$ and 3, are the Pauli matrices.

However, the harmonic oscillator systems, used in this context, is only the simplest example of supersymmetric and shape-invariant potential. Our goal at this point is to generalize that Hamiltonian for all supersymmetric and shape-invariant systems. With this purpose we introduce the operators

$$\hat{S} = \hat{\sigma}_+ \hat{A} + \hat{\sigma}_- \hat{A}^\dagger \quad (2.2a)$$

$$\hat{S}_i = \hat{\sigma}_+ \sqrt{\hat{A}^\dagger \hat{A}} + \hat{\sigma}_- \sqrt{\hat{A} \hat{A}^\dagger} \hat{A}^\dagger,$$

where

$$\hat{\sigma}_\pm = \frac{1}{2} (\hat{\sigma}_1 \pm i \hat{\sigma}_2) \quad (2.3)$$

and, now, the operators $\hat{A}$ and $\hat{A}^\dagger$ satisfy the shape invariance condition [2]. Using this definition we can decompose the Jaynes-Cummings Hamiltonian in the form.
\[ \hat{H} = \hat{H}_o + \hat{H}_{\text{int}}, \quad (2.4) \]

where

\[ \hat{H}_o = \hat{S}^2, \quad (2.5a) \]
\[ \hat{H}_{\text{int}} = \alpha \hat{S}_z + \hbar \Delta \hat{\sigma}_3. \quad (2.5b) \]

We search for the eigenstates of \( \hat{H} \) and, in this case, it is more convenient to work with its \( B \)-operator expressions, which can be written as [2]

\[ \hat{S}^2 = \begin{bmatrix} \hat{T} \hat{B}_- \hat{B}_+ \hat{T}^\dagger & 0 \\ 0 & \hat{B}_+ \hat{B}_- \end{bmatrix} \equiv \begin{bmatrix} \hat{H}_2 & 0 \\ 0 & \hat{H}_1 \end{bmatrix} \quad (2.6a) \]
\[ \hat{H}_{\text{int}} = \alpha \begin{bmatrix} \beta & \sqrt{\hat{B}_+ \hat{B}_-} \\ \sqrt{\hat{B}_+ \hat{B}_-} & -\beta \end{bmatrix} = \alpha \begin{bmatrix} \beta & \hat{T} \hat{B}_- \sqrt{\hat{H}_1} \\ \sqrt{\hat{H}_1} \hat{B}_+ \hat{T}^\dagger & -\beta \end{bmatrix}, \quad (2.6b) \]

where \( \beta = \hbar \Delta / \alpha \). We use the same notation as the preceding paper [1]. There we show that the states

\[ | \Psi_m^{(\pm)} \rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} | m \rangle \\ C_{m+1}^{(\pm)} | m+1 \rangle \end{bmatrix}, \quad m = 0, 1, 2, \ldots \quad (2.7) \]

are the eigenstates of the operator \( \hat{S}^2 \)

\[ \hat{S}^2 | \Psi_m^{(\pm)} \rangle = \mathcal{E}_{m+1} | \Psi_m^{(\pm)} \rangle. \quad (2.8) \]

In this case \( C_{m,m+1}^{(\pm)} \equiv C_{m,m+1}^{(\pm)} [R(a_1), R(a_2), R(a_3), \ldots] \) are auxiliary coefficients and, \( | m \rangle \) and \( | m+1 \rangle \) are the abbreviated notation for the states \( | \psi_m \rangle \) and \( | \psi_{m+1} \rangle \) [1].

At this point, we observe that the wave-state orthonormalization conditions imply in the following relations among the \( C \)'s real coefficients

\[ \langle \Psi_m^{(\pm)} | \Psi_m^{(\pm)} \rangle = \left[ C_m^{(\pm)} \right]^2 + \left[ C_{m+1}^{(\pm)} \right]^2 = 1 \quad (2.9a) \]
\[ \langle \Psi_m^{(\pm)} | \Psi_m^{(\pm)} \rangle = C_m^{(\pm)} C_m^{(\mp)} - C_{m+1}^{(\pm)} C_{m+1}^{(\mp)} = 0. \quad (2.9b) \]

Now, considering that \( \hat{S}^2 \) and \( \hat{H}_{\text{int}} \) commute then it is possible to find a common set of eigenstates. We can use this fact to determine the eigenvalues of \( \hat{H}_{\text{int}} \) and the relations among the \( C \)'s coefficients. For that we need to calculate

\[ \hat{H}_{\text{int}} | \Psi_m^{(\pm)} \rangle = \lambda_m^{(\pm)} | \Psi_m^{(\pm)} \rangle, \quad (2.10) \]

where \( \lambda_m^{(\pm)} \) are the eigenvalues to be determined. Using the Eqs. (2.4), (2.5) and (2.7), the last eigenvalue equation can be rewritten in a matrix form as

\[ \alpha \begin{bmatrix} \beta & \sqrt{\hat{H}_1} \\ \sqrt{\hat{H}_1} & -\beta \end{bmatrix} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} | m \rangle \\ C_{m+1}^{(\pm)} | m+1 \rangle \end{bmatrix} = \lambda_m^{(\pm)} \begin{bmatrix} C_m^{(\pm)} | m \rangle \\ C_{m+1}^{(\pm)} | m+1 \rangle \end{bmatrix}, \quad (2.11) \]

Since the \( C \)'s coefficients commute with the \( \hat{A} \) or \( \hat{A}^\dagger \) operators, then the last matrix equation permits to obtain the following equations
\[
\left[ \alpha \beta - \lambda_m^{(\pm)} \right] \left( \hat{T} C_m^{(\pm)} \hat{T}^\dagger \right) \hat{T} \mid m \rangle \pm \alpha C_m^{(\pm)} \hat{T} \hat{B}_- \sqrt{\hat{H}_1} \mid m + 1 \rangle = 0
\] (2.12a)
\[
\alpha \left( \hat{T} C_m^{(\pm)} \hat{T}^\dagger \right) \sqrt{\hat{H}_1} \hat{B}_+ \mid m \rangle \mp \left[ \alpha \beta + \lambda_m^{(\pm)} \right] C_m^{(\pm)} \mid m + 1 \rangle = 0.
\] (2.12b)

Introducing the operator \[12\]
\[
\hat{Q}^\dagger = \left( \hat{B}_+ \hat{B}_- \right)^{-1/2} \hat{B}_+ = \left( \hat{H}_1 \right)^{-1/2} \hat{B}_+
\] (2.13)
one can write the normalized eigenstate of \( \hat{H}_1 \) as
\[
\mid m \rangle = \left( \hat{Q}^\dagger \right)^m \mid 0 \rangle,
\] (2.14)
and, with Eqs. (2.13) and (2.14) we can show that \[2\]
\[
\hat{B}_+ \mid m \rangle = \sqrt{\mathcal{E}_{m+1}} \mid m + 1 \rangle,
\] (2.15a)
\[
\hat{T} \hat{B}_- \mid m + 1 \rangle = \sqrt{\mathcal{E}_{m+1}} \hat{T} \mid m \rangle.
\] (2.15b)

Therefore, we have that
\[
\hat{T} \hat{B}_- \sqrt{\hat{H}_1} \mid m + 1 \rangle = \hat{T} \hat{B}_- \sqrt{\mathcal{E}_{m+1}} \mid m + 1 \rangle
\]
\[
= \sqrt{\mathcal{E}_{m+1}} \hat{T} \hat{B}_- \mid m + 1 \rangle
\]
\[
= \mathcal{E}_{m+1} \hat{T} \mid m \rangle,
\] (2.16)
and
\[
\sqrt{\hat{H}_1} \hat{B}_+ \mid m \rangle = \sqrt{\hat{H}_1} \sqrt{\mathcal{E}_{m+1}} \mid m + 1 \rangle
\]
\[
= \sqrt{\mathcal{E}_{m+1} \hat{H}_1} \mid m + 1 \rangle
\]
\[
= \mathcal{E}_{m+1} \hat{H}_1 \mid m + 1 \rangle,
\] (2.17)

Using Eqs. (2.16) and (2.17), then Eqs. (2.12) take the form
\[
\left\{ \left[ \alpha \beta - \lambda_m^{(\pm)} \right] \left( \hat{T} C_m^{(\pm)} \hat{T}^\dagger \right) \hat{T} \mid m \rangle \pm \alpha \mathcal{E}_{m+1} C_m^{(\pm)} \right\} \mid m + 1 \rangle = 0
\] (2.18a)
\[
\left\{ \alpha \mathcal{E}_{m+1} \left( \hat{T} C_m^{(\pm)} \hat{T}^\dagger \right) \mp \left[ \alpha \beta + \lambda_m^{(\pm)} \right] C_m^{(\pm)} \right\} \mid m + 1 \rangle = 0.
\] (2.18b)

From Eqs. (2.18) it follows that
\[
\lambda_m^{(\pm)} = \pm \alpha \sqrt{\mathcal{E}_{m+1}^2 + \beta^2},
\] (2.19)
and
\[
C_m^{(\pm)} = \left( \frac{\sqrt{\mathcal{E}_{m+1}^2 + \beta^2} \mp \beta}{\mathcal{E}_{m+1}} \right) \left( \hat{T} C_m^{(\pm)} \hat{T}^\dagger \right).
\] (2.20)

Eqs. (2.9) and (2.20) imply that
\[ C_{m+1}^{(\pm)} = C_m^{(\mp)} , \quad (2.21) \]

and the eigenstates and eigenvalues of the generalized intensity-dependent and non-resonant Jaynes-Cummings Hamiltonian can be written as

\[ E_m^{(\pm)} = \mathcal{E}_{m+1} \pm \sqrt{\alpha^2 \mathcal{E}_{m+1}^2 + \hbar^2 \Delta^2} , \quad (2.22) \]

and

\[ | \Psi_m^{(\pm)} \rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)} | m \rangle \\ C_m^{(\mp)} | m + 1 \rangle \end{bmatrix} , \quad m = 0, 1, 2, \ldots \quad (2.23) \]

**a) The Intensity-Dependent Resonant Limit**

From these general results we can verify two simple limiting cases. The first one corresponds to the resonant situation, which is for \( \Delta = 0 \) (\( \beta = 0 \)). Using these conditions into Eqs. (2.20) and (2.22) and Eqs. (2.9) we can promptly conclude that

\[ E_m^{(\pm)} = (1 \pm \alpha) \mathcal{E}_{m+1} , \quad (2.24) \]

and

\[ C_{m+1}^{(\pm)} = \hat{T} C_m^{(\pm)} \hat{T}^{\dagger} = C_m^{(\pm)} = \frac{1}{\sqrt{2}} . \quad (2.25) \]

Therefore the intensity-dependent Jaynes-Cummings resonant eigenstate is given by

\[ | \Psi_m^{(\pm)} \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} | m \rangle \\ | m + 1 \rangle \end{bmatrix} , \quad m = 0, 1, 2, \ldots \quad (2.26) \]

If we compare this last particular result with that one found in the reference [2], we conclude that the intensity-dependent and intensity-independent generalized Jaynes-Cummings Hamiltonians have the same eigenstates in the resonant situation.

**b) The Standard Intensity-Dependent Jaynes-Cummings Limit**

The second important limit corresponds to the standard intensity-dependent Jaynes-Cummings case, related with the harmonic oscillator system. In this limit we have that \( \hat{T} = \hat{T}^{\dagger} \rightarrow 1, \quad \hat{B}_- \rightarrow \hat{a}, \quad \hat{B}_+ \rightarrow \hat{a}^{\dagger}, \quad \Delta = \omega - \omega_o \) and \( \mathcal{E}_{m+1} = (m+1)\hbar \omega \). Using these conditions in Eqs. (2.20), (2.22) and Eqs. (2.9) we can promptly conclude that

\[ E_m^{(\pm)} = (m+1)\hbar \omega \pm \hbar \sqrt{\alpha^2 \omega^2 (m+1)^2 + (\omega - \omega_o)^2} , \quad (2.27) \]

and

\[ C_{m+1}^{(\pm)} = \gamma_m^{(\pm)} C_m^{(\mp)} = C_m^{(\mp)} = \frac{1}{\sqrt{1 + \gamma_m^{(\mp)}}} , \quad (2.28) \]

where
\[
\gamma_m^{(\pm)} = \sqrt{1 + \delta_m^2 \pm \delta_m}, \quad (2.29a)
\]
\[
\delta_m = \frac{\omega - \omega_o}{\alpha \omega (m+1)}. \quad (2.29b)
\]

Therefore the standard intensity-dependent Jaynes-Cummings eigenstate, written in a matrix form, is given by
\[
|\Psi_m^{(\pm)}\rangle = \frac{1}{\sqrt{1 + (\gamma_m^{(\pm)})^2}} \begin{bmatrix} 1 & 0 & m \rangle & m+1 \rangle, \quad m = 0, 1, 2, \ldots \end{bmatrix} \quad (2.30)
\]

### III. TIME EVOLUTION OF THE SYSTEM

To resolve the time-dependent Schrödinger equation for intensity-dependent and non-resonant Jaynes-Cummings systems
\[
i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = (\hat{H}_o + \hat{H}_{\text{int}}) |\Psi(t)\rangle \quad (3.1)
\]
we can write the state as
\[
|\Psi(t)\rangle = \exp \left(-i\hat{H}_o t/\hbar\right) |\Psi_i(t)\rangle, \quad (3.2)
\]
and, by substituting this into Schrödinger equation and taking into account the commutation property between \(\hat{H}_o\) and \(\hat{H}_{\text{int}}\), we obtain
\[
i\hbar \frac{\partial}{\partial t} |\Psi_i(t)\rangle = \hat{H}_{\text{int}} |\Psi_i(t)\rangle, \quad (3.3)
\]
Now, we can introduce the evolution matrix \(\hat{U}_i(t,0)\), related with the interaction Hamiltonian, by
\[
|\Psi_i(t)\rangle = \hat{U}_i(t,0) |\Psi_i(0)\rangle. \quad (3.4)
\]
with
\[
i\hbar \frac{\partial}{\partial t} \hat{U}_i(t,0) = \hat{H}_{\text{int}} \hat{U}_i(t,0), \quad (3.5)
\]
that is, in matrix form, written as
\[
i\hbar \begin{bmatrix} \hat{U}^{(\prime)}_{11} & \hat{U}^{(\prime)}_{12} \\ \hat{U}^{(\prime)}_{21} & \hat{U}^{(\prime)}_{22} \end{bmatrix} = \alpha \begin{bmatrix} \beta & \hat{T}\hat{B}_- \sqrt{\hat{H}_1} \\ \sqrt{H_1} \hat{B}_+ \hat{T}^\dagger & -\beta \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix}, \quad (3.6)
\]
where the primes denote the time derivative. One fast way to diagonalize the evolution matrix differential equation is by differentiating Eq. (3.3) with respect to time. After that, if we use again the same Eq. (3.3), we find
\[ i\hbar \frac{\partial^2}{\partial t^2} \hat{U}_i(t,0) = \hat{H}_{\text{int}} \frac{\partial}{\partial t} \hat{U}_i(t,0) = \frac{1}{i\hbar} \hat{H}_{\text{int}} \hat{U}_i(t,0), \]  

which can be written as

\[
\begin{bmatrix}
\hat{U}''_{11} & \hat{U}''_{12} \\
\hat{U}''_{21} & \hat{U}''_{22}
\end{bmatrix} = -\begin{bmatrix}
\hat{\omega}_1 & 0 \\
0 & \hat{\omega}_2
\end{bmatrix} \begin{bmatrix}
\hat{U}_{11} & \hat{U}_{12} \\
\hat{U}_{21} & \hat{U}_{22}
\end{bmatrix},
\]

where

\[
\hat{\omega}_1 = \alpha \sqrt{(\hat{T}\hat{B}_- + \hat{B}_+ \hat{T}^\dagger)^2 + \beta^2} = \sqrt{\alpha^2 \hat{H}_1^2 + (\hbar\Delta)^2},
\]

\[
\hat{\omega}_2 = \alpha \sqrt{(\hat{B}_+ \hat{T}^\dagger - \hat{B}_-)^2 + \beta^2} = \sqrt{\alpha^2 \hat{H}_2^2 + (\hbar\Delta)^2}.
\]

Now, since by initial conditions \(\hat{U}_i(0,0) = \hat{I}\), then we can write the solution of the evolution matrix differential equation (3.7) as

\[
\hat{U}_i(t,0) = \begin{bmatrix}
\cos (\hat{\omega}_1 t) & \sin (\hat{\omega}_1 t) \\
\sin (\hat{\omega}_2 t) \hat{D} & \cos (\hat{\omega}_2 t)
\end{bmatrix},
\]

and the \(\hat{C}\) and \(\hat{D}\) operators can be determined by the unitary transformation conditions

\[
\hat{U}^\dagger_i(t,0) \hat{U}_i(t,0) = \hat{U}_i(t,0) \hat{U}^\dagger_i(t,0) = \hat{I}.
\]

Following the same steps used in the appendix A of the reference [1] we can conclude that to satisfy the unitary conditions (3.11) these operators must have the form

\[
\hat{C} = -\hat{D}^\dagger = i \frac{\alpha}{\hat{H}_2^{1/4}} \sqrt{\hat{T}\hat{B}_-}
\]

\[
\hat{D} = -\hat{C}^\dagger = \sqrt{\hat{B}_+ \hat{T}^\dagger} i \frac{\alpha}{\hat{H}_2^{1/4}}.
\]

Therefore, we can write the final expression of the time evolution matrix of the system as

\[
\hat{U}_i(t,0) = \begin{bmatrix}
\cos (\hat{\omega}_1 t) & \sin (\hat{\omega}_1 t) \hat{C}^\dagger \\
-\sin (\hat{\omega}_2 t) \hat{C}^\dagger & \cos (\hat{\omega}_2 t)
\end{bmatrix}.
\]

For Jaynes-Cummings systems an important physical quantity to see how the system under consideration evolves in time is the population inversion factor [9,13,14], defined by

\[
\hat{W}(t) \equiv \hat{\sigma}_+(t) \hat{\sigma}_-(t) - \hat{\sigma}_-(t) \hat{\sigma}_+(t) = \hat{\sigma}_3(t),
\]

where the time dependence of the operators is related with the Heisenberg picture. In this case, the time evolution of the population inversion factor will be given by

\[
\frac{d\hat{\sigma}_3(t)}{dt} = \frac{1}{i\hbar} \hat{U}^\dagger_i(t,0) \left[ \hat{\sigma}_3, \hat{H} \right] \hat{U}_i(t,0),
\]

and since we have

\[
\left[ \hat{\sigma}_3, \hat{H} \right] = \alpha \left[ \hat{\sigma}_3, \hat{S}_i \right] = -2\alpha \hat{S}_i \hat{\sigma}_3,
\]
then Eq. (3.13) can be written as
\[ \frac{d\hat{\sigma}_3(t)}{dt} = \frac{2i\alpha}{\hbar} \hat{S}_i(t) \hat{\sigma}_3(t). \] (3.17)

We can obtain a differential equation with constant coefficients for \( \hat{\sigma}_3(t) \) by taking the time derivative of Eq. (3.17)
\[ \frac{d^2\hat{\sigma}_3(t)}{dt^2} = \frac{2i\alpha}{\hbar} \left\{ \frac{d\hat{S}_i(t)}{dt} \hat{\sigma}_3(t) + \hat{S}_i(t) \frac{d\hat{\sigma}_3(t)}{dt} \right\}. \] (3.18)

Having in mind that
\[ \frac{d\hat{S}_i(t)}{dt} = \frac{1}{i\hbar} \hat{U}_i^\dagger(t,0) \left[ \hat{S}_i, \hat{H} \right] \hat{U}_i(t,0), \] (3.19)

and,
\[ \left[ \hat{S}_i, \hat{H} \right] = \alpha\beta \left[ \hat{S}_i, \hat{\sigma}_3 \right] = 2\alpha\beta \hat{S}_i \hat{\sigma}_3, \] (3.20)

we can conclude that
\[ \frac{d\hat{S}_i(t)}{dt} = -\frac{2i\alpha\beta}{\hbar} \hat{S}_i(t) \hat{\sigma}_3(t). \] (3.21)

Now using Eqs. (3.17) and (3.21) into Eq. (3.18) we obtain
\[ \frac{d^2\hat{\sigma}_3(t)}{dt^2} + \hat{\Theta}^2 \hat{\sigma}_3(t) = \hat{F}(t) \] (3.22)

where
\[ \hat{\Theta}^2 = \frac{4\alpha^2}{\hbar^2} \hat{S}_i \] (3.23a)
\[ \hat{F}(t) = \frac{4\alpha^2\beta}{\hbar^2} \hat{U}_i^\dagger(t,0) \hat{S}_i \hat{U}_i(t,0). \] (3.23b)

The Eq. (3.21) corresponds to a non-homogeneous linear differential equation for \( \hat{\sigma}_3(t) \) with constant coefficients since \( \hat{S}_i \) and \( \hat{H} \) commute and, therefore, \( \hat{\Theta} \) is a constant of the motion. The general solution of this differential equation can be written as
\[ \hat{\sigma}_3(t) = \hat{\sigma}^H(t) + \hat{\sigma}^P(t), \] (3.24)

and each matrix element of the homogeneous solution, satisfies the differential equation
\[ \frac{d^2\hat{\sigma}^H_{jk}(t)}{dt^2} + \hat{\nu}^2_{j,k} \hat{\sigma}^H(t) = 0, \] (3.25)

with
\[ \hbar \hat{\nu}_1 = 2\alpha \hat{T} \hat{B}_- \hat{B}_+ \hat{T}^\dagger = 2\alpha \hat{H}_2, \] (3.26a)
\[ \hbar \hat{\nu}_2 = 2\alpha \hat{B}_+ \hat{B}_- = 2\alpha \hat{H}_1. \] (3.26b)
The solution of Eq. (3.23) is given by

\[ \hat{\sigma}_{jk}^H(t) = \hat{y}_j(t) \hat{c}_{jk} + \hat{z}_j(t) \hat{d}_{jk}, \]  

(3.27)

where

\[ \hat{y}_j(t) = \cos(\hat{\nu}_j t) \]  

(3.28a)

\[ \hat{z}_j(t) = \sin(\hat{\nu}_j t), \]  

(3.28b)

and the coefficients \( \hat{c}_{jk} \) and \( \hat{d}_{jk} \) can be determined by the initial conditions.

The matrix elements of the particular solution of the \( \hat{\sigma}_3(t) \) differential equation needs to satisfy

\[ \frac{d^2 \hat{\sigma}_{jk}^P(t)}{dt^2} + \hat{\nu}_j^2 \hat{\sigma}_{jk}^P(t) = \hat{F}_{jk}(t), \quad j, k = 1, \text{ or } 2, \]  

(3.29)

and can be obtained by the variation of parameter or by Green function methods, giving

\[ \hat{\sigma}_{jk}^P(t) = \hat{\nu}_j^{-1} \left\{ \hat{z}_j(t) \int_0^t \xi \hat{y}_j(\xi) \hat{F}_{jk}(\xi) - \hat{y}_j(t) \int_0^t d\xi \hat{z}_j(\xi) \hat{F}_{jk}(\xi) \right\}, \]  

(3.30)

where we used that the Wronskian of the system of solutions \( \hat{y}_j(t) \) and \( \hat{z}_j(t) \) is given by \( \hat{\nu}_j \).

After we determine the elements of the \( \hat{F}(t) \)-matrix, it is necessary to resolve the integrals in Eq. (3.30) to obtain the explicit expression of the particular solution. In the appendix we show that, using Eqs. (2.2), (3.13), and (3.23), it is possible to conclude that these matrix elements can be written as

\[ \hat{\sigma}_{11}^P(t) = i \gamma \hat{\nu}_1^{-1} \sqrt{\hat{B}_+ \hat{T}_-} \left\{ \hat{z}_1(t) \mathcal{G}_{CS}^{(+)}(t; \hat{\nu}_1, \hat{\omega}_1, \hat{\omega}_2) - \hat{y}_1(t) \mathcal{G}_{CS}^{(+)}(t; \hat{\nu}_1, \hat{\omega}_1, \hat{\omega}_2) \right\} \hat{H}_2^{3/4} \]

(3.31a)

\[ \hat{\sigma}_{12}^P(t) = \frac{\gamma}{2} \hat{\nu}_1^{-1} \sqrt{\hat{B}_+ \hat{T}_-} \left\{ \hat{z}_1(t) \mathcal{G}_{CC}^{(+)}(t; \hat{\nu}_1, \hat{\omega}_1, \hat{\omega}_2) - \hat{y}_1(t) \mathcal{G}_{CC}^{(+)}(t; \hat{\nu}_1, \hat{\omega}_1, \hat{\omega}_2) \right\} \sqrt{\hat{B}_+ \hat{T}_-} \]  

(3.31b)

\[ \hat{\sigma}_{21}^P(t) = \frac{\gamma}{2} \hat{\nu}_1^{-1} \sqrt{\hat{B}_+ \hat{T}_+} \left\{ \hat{z}_1(t) \mathcal{G}_{CS}^{(+)}(t; \hat{\nu}_1, \hat{\omega}_1, \hat{\omega}_2) - \hat{y}_1(t) \mathcal{G}_{CS}^{(+)}(t; \hat{\nu}_1, \hat{\omega}_1, \hat{\omega}_2) \right\} \hat{H}_2^{3/4} \]  

(3.31c)

\[ \hat{\sigma}_{22}^P(t) = i \gamma \hat{\nu}_2^{-1} \sqrt{\hat{B}_- \hat{T}_+ \hat{H}_2} \left\{ \hat{z}_2(t) \mathcal{G}_{CS}^{(+)}(t; \hat{\nu}_1, \hat{\omega}_1, \hat{\omega}_2) - \hat{y}_1(t) \mathcal{G}_{CS}^{(+)}(t; \hat{\nu}_1, \hat{\omega}_1, \hat{\omega}_2) \right\} \hat{H}_2^{3/4} \]

(3.31d)

where \( \gamma = 4\alpha^2 \beta/\hbar^2 \), and the auxiliary functions are given by

\[ \mathcal{G}_{XY}^{(+)}(t; \hat{p}, \hat{q}, \hat{r}) = \mathcal{F}_{XY}(t; \hat{p} - \hat{q}, \hat{r}) \pm \mathcal{F}_{XY}(t; \hat{p} + \hat{q}, \hat{r}), \quad X, Y = C \text{ or } S, \]  

(3.32)

with
\[ F_{CC}(t; \hat{x}, \hat{w}) \equiv \int_0^t d\xi \, \cos(\hat{x}\xi) \cos(\hat{w}\xi) \]
\[ = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m+\hat{w}^{2n}} \xi^{2m+2n+1}}{(2m)! (2n)! (2m+2n+1)} \]  
(3.33a)

\[ F_{CS}(t; \hat{x}, \hat{w}) \equiv \int_0^t d\xi \, \cos(\hat{x}\xi) \sin(\hat{w}\xi) \]
\[ = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m+\hat{w}^{2n}} \xi^{2m+2n+2}}{(2m)! (2n+1)! (2m+2n+2)} \]  
(3.33b)

\[ F_{SC}(t; \hat{x}, \hat{w}) \equiv \int_0^t d\xi \, \sin(\hat{x}\xi) \cos(\hat{w}\xi) \]
\[ = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m+1}\hat{w}^{2n+1} \xi^{2m+2n+3}}{(2m+1)! (2n+1)! (2m+2n+3)} \]  
(3.33c)

\[ F_{SS}(t; \hat{x}, \hat{w}) \equiv \int_0^t d\xi \, \sin(\hat{x}\xi) \sin(\hat{w}\xi) \]
\[ = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m+1}\hat{w}^{2n+1} \xi^{2m+2n+3}}{(2m+1)! (2n+1)! (2m+2n+3)} \]  
(3.33d)

With these results for the particular solution we can conclude that
\[ \hat{\sigma}_{ij}^P(0) = 0 = \frac{d\hat{\sigma}_{ij}^P(0)}{dt}. \]  
(3.34)

Now, using Eqs. (3.17), (3.24), (3.27) and the initial conditions, we have
\[ [\hat{\sigma}_3(0)]_{ij} = \hat{c}_{ij} \]  
(3.35a)
\[ \begin{bmatrix} \frac{d\hat{\sigma}_3(0)}{dt} \end{bmatrix}_{ij} = \frac{2i\alpha}{\hbar} [\hat{S}_i(0) \hat{\sigma}_3(0)]_{ij} = \hat{\nu}_i \hat{d}_{ij}. \]  
(3.35b)

Therefore, the final expression for the elements of the population inversion matrix of the system can be written as
\[ [\hat{\sigma}_3(t)]_{ij} = \cos(\hat{\nu}_t) [\hat{\sigma}_3(0)]_{ij} + \frac{2i\alpha}{\hbar} \sin(\hat{\nu}_t) \hat{\nu}_i^{-1} [\hat{S}(0) \hat{\sigma}_3(0)]_{ij} + \hat{\sigma}_{ij}^P(t). \]  
(3.36)

Again, using these final results we can verify two limiting cases.

a) The Intensity-Dependent Resonant Limit

The first one corresponds to the intensity-dependent resonant (\( \Delta = 0 \)). Eqs. (3.9), (3.13), (3.26) and (3.31) allow us to conclude that, in this case, the evolution matrix of the system is given by
\[ \hat{U}_i(t, 0) = \begin{bmatrix} \cos(\frac{1}{2}\hat{\nu}_1 t) & \sin(\frac{1}{2}\hat{\nu}_1 t) & \hat{C} \cos(\frac{1}{2}\hat{\nu}_2 t) \\ -\sin(\frac{1}{2}\hat{\nu}_2 t) \hat{C}^\dagger & \cos(\frac{1}{2}\hat{\nu}_2 t) \end{bmatrix} \]  
(3.37)
and the elements of the population inversion of the system is
\[ [\hat{\sigma}_3(t)]_{ij} = \cos (\hat{\nu}_i t) \ [\hat{\sigma}_3(0)]_{ij} + \frac{2i\alpha}{\hbar} \sin (\hat{\nu}_i t) \ \hat{\nu}_i^{-1} [\dot{\hat{S}}_i(0) \hat{\sigma}_3(0)]_{ij}. \] (3.38)

b) The Standard Intensity-Dependent Jaynes-Cummings Limit

This second important limit corresponds to the case of the harmonic oscillator system, and in this limit we have that \( \hat{T} = \hat{T}^\dagger \rightarrow 1, \ \hat{B}_- \rightarrow \hat{a}, \ \hat{B}_+ \rightarrow \hat{a}^\dagger \) and \([\hat{a}, \hat{a}^\dagger] = \hbar \omega.\) With these conditions the operators \(\hat{\omega}_1\) and \(\hat{\omega}_2\) commute, and this fact permits to evaluate the integrals related with the particular solution of the population inversion factors using trigonometric product relations. Using that and the expressions obtained in the appendix, after a considerable amount of algebra and trigonometric product relations we can show that it is possible to write the expressions for the \(\hat{\sigma}^{P}_{ij}(t)\)-matrix elements as

\[
\hat{\sigma}^{P}_{11}(t) = i\frac{\gamma}{2} \hat{\nu}_1^{-1} \sqrt{a} \left\{ \mathcal{K}_S(t; \hat{\omega}_2, \hat{\omega}_1, \hat{\nu}_2) - \mathcal{K}_S(t; \hat{\omega}_2, -\hat{\omega}_1, \hat{\nu}_2) \right\} \left(\hat{a} \hat{a}^\dagger\right)^{3/4}
- i\frac{\gamma}{2} \hat{\nu}_1^{-1} \left(\hat{a} \hat{a}^\dagger\right)^{3/4} \left\{ \mathcal{K}_S(t; \hat{\omega}_2, \hat{\omega}_1, \hat{\nu}_1) - \mathcal{K}_S(t; \hat{\omega}_2, -\hat{\omega}_1, \hat{\nu}_1) \right\} \sqrt{\hat{a} \hat{a}^\dagger} \] (3.39a)

\[
\hat{\sigma}^{P}_{12}(t) = \frac{\gamma}{2} \hat{\nu}_1^{-1} \sqrt{\hat{a} \hat{a}^\dagger} \left\{ \mathcal{K}_C(t; \hat{\omega}_2, \hat{\omega}_1, \hat{\nu}_2) - \mathcal{K}_C(t; \hat{\omega}_2, -\hat{\omega}_1, \hat{\nu}_2) \right\} \] (3.39b)

\[
\hat{\sigma}^{P}_{21}(t) = \frac{\gamma}{2} \hat{\nu}_1^{-1} \sqrt{\hat{a} \hat{a}^\dagger} \left\{ \mathcal{K}_C(t; \hat{\omega}_2, \hat{\omega}_1, \hat{\nu}_1) + \mathcal{K}_C(t; \hat{\omega}_2, -\hat{\omega}_1, \hat{\nu}_1) \right\} \sqrt{\hat{a} \hat{a}^\dagger} \] (3.39c)

\[
\hat{\sigma}^{P}_{22}(t) = i\frac{\gamma}{2} \hat{\nu}_1^{-1} \sqrt{\hat{a} \hat{a}^\dagger} \left\{ \mathcal{K}_S(t; \hat{\omega}_2, \hat{\omega}_1, \hat{\nu}_2) + \mathcal{K}_S(t; \hat{\omega}_2, -\hat{\omega}_1, \hat{\nu}_2) \right\} \left(\hat{a} \hat{a}^\dagger\right)^{3/4}
- i\frac{\gamma}{2} \hat{\nu}_1^{-1} \left(\hat{a} \hat{a}^\dagger\right)^{3/4} \left\{ \mathcal{K}_S(t; \hat{\omega}_2, \hat{\omega}_1, \hat{\nu}_1) + \mathcal{K}_S(t; \hat{\omega}_2, -\hat{\omega}_1, \hat{\nu}_1) \right\} \sqrt{\hat{a} \hat{a}^\dagger}, \] (3.39d)

where, now, the auxiliary functions are given by

\[
\mathcal{K}_S(t; \hat{p}, \hat{q}, \hat{r}) = \frac{\hat{r} \sin [(\hat{p} + \hat{q}) t] - (\hat{p} + \hat{q}) \sin (\hat{r} t)}{\hat{r}^2 - (\hat{p} + \hat{q})^2} \] (3.40a)

\[
\mathcal{K}_C(t; \hat{p}, \hat{q}, \hat{r}) = \frac{\hat{r} \cos [(\hat{p} + \hat{q}) t] - \hat{r} \cos (\hat{r} t)}{\hat{r}^2 - (\hat{p} + \hat{q})^2}. \] (3.40b)

Considering the expressions above we may easily verify that the particular solution for the population inversion factor must still satisfy the initial conditions (3.34). Therefore, in this case the final expression of the population inversion factor has the same form given by Eq. (3.36), with

\[
\hbar \hat{\nu}_1 = 2\alpha \hat{a} \hat{a}^\dagger, \quad \hbar \hat{\nu}_2 = 2\alpha \hat{a}^\dagger \hat{a}, \] (3.41a)

\[
\hbar \hat{\omega}_1 = \alpha \sqrt{(\hat{a} \hat{a}^\dagger)^2 + \beta^2}, \quad \hbar \hat{\omega}_2 = \alpha \sqrt{(\hat{a}^\dagger \hat{a})^2 + \beta^2}. \] (3.41b)
IV. CONCLUSIONS

In this article we introduced a class of shape-invariant bound-state problems which represent two-level systems. The corresponding coupled-channel Hamiltonians generalize the intensity-dependent and non-resonant Jaynes-Cummings Hamiltonian. These models are not only interesting on their own account. Being exactly solvable coupled-channels problems they may help to assess the validity and accuracy of various approximate approaches to the coupled-channel problems \(^{17}\).

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Appendix

In this appendix we show the steps necessary to obtain the explicit expression of the elements of the population inversion particular solution. To resolve the integrals in the Eq. (3.30), first we need to determine the elements of the \( \hat{F}(t) \)-matrix. To do that we can use Eqs. (2.2), (3.23), and (3.13) to conclude that

\[
\begin{align*}
\hat{F}_{11}(t) &= -\gamma \left\{ \cos(\hat{\omega}_1 t) \hat{T} \hat{B}_- \sin(\hat{\omega}_2 t) \hat{C} + \hat{C} \sin(\hat{\omega}_2 t) \hat{B}_- \hat{T} \cos(\hat{\omega}_1 t) \right\} \\
&= i\gamma \left\{ \sqrt{\hat{T} \hat{B}_-} \cos(\hat{\omega}_2 t) \sin(\hat{\omega}_1 t) \hat{H}_2^{1/4} - \hat{H}_2^{1/4} \sin(\hat{\omega}_1 t) \cos(\hat{\omega}_2 t) \sqrt{\hat{B}_+ \hat{T}} \right\} \quad (1a) \\
\hat{F}_{12}(t) &= \gamma \left\{ \cos(\hat{\omega}_1 t) \hat{T} \hat{B}_- \cos(\hat{\omega}_2 t) - \hat{C} \sin(\hat{\omega}_2 t) \hat{B}_+ \hat{T} \sin(\hat{\omega}_1 t) \hat{C} \right\} \\
&= \gamma \left\{ \sqrt{\hat{T} \hat{B}_-} \cos(\hat{\omega}_2 t) \cos(\hat{\omega}_1 t) \sqrt{\hat{T} \hat{B}_-} + \hat{H}_2^{1/4} \sin(\hat{\omega}_1 t) \sin(\hat{\omega}_2 t) \hat{H}_1^{1/4} \right\} \quad (1b) \\
\hat{F}_{21}(t) &= \gamma \left\{ \cos(\hat{\omega}_2 t) \hat{B}_+ \hat{T} \cos(\hat{\omega}_1 t) - \hat{C} \sin(\hat{\omega}_1 t) \hat{T} \hat{B}_- \sin(\hat{\omega}_2 t) \hat{C} \right\} \\
&= \gamma \left\{ \sqrt{\hat{B}_+ \hat{T}} \cos(\hat{\omega}_1 t) \cos(\hat{\omega}_2 t) \sqrt{\hat{B}_+ \hat{T}} + \hat{H}_1^{1/4} \sin(\hat{\omega}_2 t) \sin(\hat{\omega}_1 t) \hat{H}_2^{1/4} \right\} \quad (1c) \\
\hat{F}_{22}(t) &= \gamma \left\{ \hat{C} \sin(\hat{\omega}_1 t) \hat{T} \hat{B}_- \cos(\hat{\omega}_2 t) + \cos(\hat{\omega}_2 t) \hat{B}_+ \hat{T} \sin(\hat{\omega}_1 t) \hat{C} \right\} \\
&= i\gamma \left\{ \sqrt{\hat{B}_+ \hat{T}} \cos(\hat{\omega}_1 t) \sin(\hat{\omega}_2 t) \hat{H}_1^{1/4} - \hat{H}_1^{1/4} \sin(\hat{\omega}_2 t) \cos(\hat{\omega}_1 t) \sqrt{\hat{T} \hat{B}_-} \right\}, \quad (1d)
\end{align*}
\]
where \( \gamma = 4\alpha^2 \beta / h^2 \), and we used the properties

\[
\hat{C} \hat{C}^\dagger = \hat{C}^\dagger \hat{C} = 1 \tag{2a}
\]

\[
\hat{C} \sin (\hat{\omega}_2 t) = \sin (\hat{\omega}_1 t) \hat{C} \tag{2b}
\]

\[
\hat{C}^\dagger \cos (\hat{\omega}_1 t) = \cos (\hat{\omega}_2 t) \hat{C} \tag{2c}
\]

\[
\sqrt{T B_- \hat{\omega}_2^a} = \hat{\omega}_1^a \sqrt{T B_-} \tag{2d}
\]

\[
\sqrt{B_+ \hat{T}^\dagger \hat{\omega}_1^a} = \hat{\omega}_2^a \sqrt{B_+ \hat{T}^\dagger}, \tag{2e}
\]

proved in the appendix A of the Ref. [1], together with the operators relations

\[
\hat{C} \sqrt{B_+ \hat{T}^\dagger} = -\sqrt{T B_-} \hat{C} \dagger = i \hat{H}_2^{1/4} \tag{3a}
\]

\[
\sqrt{B_+ \hat{T}^\dagger} \hat{C} = -\hat{C} \dagger \sqrt{T B_-} = i \hat{H}_1^{1/4}. \tag{3b}
\]

At this point, if we remember that \( [\hat{\nu}_j, \hat{\omega}_j] = 0, (j = 1, \text{ or } 2) \), then we conclude that we can use the trigonometric relations involving the product of trigonometric function with arguments \( \hat{\nu}_j t \) and \( \hat{\omega}_j t \) because, in this case, we know that \( \exp (\hat{\nu}_j t) \exp (\pm \hat{\omega}_j t) = \exp [ (\hat{\nu}_j \pm \hat{\omega}_j) t] \). Now, using this fact, the commutators

\[
[\hat{\nu}_1, \hat{H}_2] = [\hat{\omega}_1, \hat{H}_2] = [\hat{\nu}_2, \hat{H}_1] = [\hat{\omega}_2, \hat{H}_1] = 0, \tag{4}
\]

and the properties (2), we can show that

\[
\hat{y}_1(t) \hat{F}_{11}(t) = \frac{i \gamma}{2} \sqrt{T B_-} \left\{ \cos [(\hat{\nu}_2 - \hat{\omega}_2) t] \sin (\hat{\omega}_1 t) + \cos [(\hat{\nu}_2 + \hat{\omega}_2) t] \sin (\hat{\omega}_1 t) \right\} \hat{H}_2^{1/4}
\]

\[
+ \frac{i \gamma}{2} \hat{H}_2^{1/4} \left\{ \sin [(\hat{\nu}_1 - \hat{\omega}_1) t] \cos (\hat{\omega}_2 t) - \sin [(\hat{\nu}_1 + \hat{\omega}_1) t] \cos (\hat{\omega}_2 t) \right\} \sqrt{B_+ \hat{T}^\dagger} \tag{5a}
\]

\[
\hat{y}_1(t) \hat{F}_{12}(t) = \frac{\gamma}{2} \sqrt{T B_-} \left\{ \cos [(\hat{\nu}_2 - \hat{\omega}_2) t] \cos (\hat{\omega}_1 t) + \cos [(\hat{\nu}_2 + \hat{\omega}_2) t] \cos (\hat{\omega}_1 t) \right\} \sqrt{T B_-}
\]

\[
+ \frac{\gamma}{2} \hat{H}_2^{1/4} \left\{ \sin [(\hat{\nu}_1 + \hat{\omega}_1) t] \sin (\hat{\omega}_2 t) - \sin [(\hat{\nu}_1 - \hat{\omega}_1) t] \sin (\hat{\omega}_2 t) \right\} \hat{H}_1^{1/4} \tag{5b}
\]

\[
\hat{y}_2(t) \hat{F}_{21}(t) = \frac{\gamma}{2} \sqrt{B_+ \hat{T}^\dagger} \left\{ \cos [(\hat{\nu}_1 - \hat{\omega}_1) t] \cos (\hat{\omega}_2 t) + \cos [(\hat{\nu}_1 + \hat{\omega}_1) t] \cos (\hat{\omega}_2 t) \right\} \sqrt{B_+ \hat{T}^\dagger}
\]

\[
+ \frac{\gamma}{2} \hat{H}_1^{1/4} \left\{ \sin [(\hat{\nu}_2 + \hat{\omega}_2) t] \sin (\hat{\omega}_1 t) - \sin [(\hat{\nu}_2 - \hat{\omega}_2) t] \sin (\hat{\omega}_1 t) \right\} \hat{H}_2^{1/4} \tag{5c}
\]

\[
\hat{y}_2(t) \hat{F}_{22}(t) = \frac{\gamma}{2} \sqrt{B_+ \hat{T}^\dagger} \left\{ \cos [(\hat{\nu}_1 - \hat{\omega}_1) t] \sin (\hat{\omega}_2 t) + \cos [(\hat{\nu}_1 + \hat{\omega}_1) t] \sin (\hat{\omega}_2 t) \right\} \hat{H}_1^{1/4}
\]

\[
+ \frac{\gamma}{2} \hat{H}_2^{1/4} \left\{ \sin [(\hat{\nu}_2 - \hat{\omega}_2) t] \cos (\hat{\omega}_1 t) - \sin [(\hat{\nu}_2 + \hat{\omega}_2) t] \cos (\hat{\omega}_1 t) \right\} \sqrt{T B_-}. \tag{5d}
\]

In the same way, we can show that

\[
\hat{\varepsilon}_1(t) \hat{F}_{11}(t) = \frac{i \gamma}{2} \sqrt{T B_-} \left\{ \sin [(\hat{\nu}_2 - \hat{\omega}_2) t] \sin (\hat{\omega}_1 t) + \sin [(\hat{\nu}_2 + \hat{\omega}_2) t] \sin (\hat{\omega}_1 t) \right\} \hat{H}_2^{1/4}
\]

\[
- \frac{i \gamma}{2} \hat{H}_2^{1/4} \left\{ \cos [(\hat{\nu}_1 - \hat{\omega}_1) t] \cos (\hat{\omega}_2 t) - \cos [(\hat{\nu}_1 + \hat{\omega}_1) t] \cos (\hat{\omega}_2 t) \right\} \sqrt{B_+ \hat{T}^\dagger} \tag{6a}
\]

\[
\hat{\varepsilon}_1(t) \hat{F}_{12}(t) = \frac{\gamma}{2} \sqrt{T B_-} \left\{ \sin [(\hat{\nu}_2 - \hat{\omega}_2) t] \cos (\hat{\omega}_1 t) + \sin [(\hat{\nu}_2 + \hat{\omega}_2) t] \cos (\hat{\omega}_1 t) \right\} \sqrt{T B_-}
\]

13
\[
- \frac{\gamma}{2} \hat{H}_2^{1/4} \left\{ \cos [(\hat{\nu}_1 + \hat{\omega}_1)t] \sin (\hat{\omega}_2 t) - \cos [(\hat{\nu}_1 - \hat{\omega}_1)t] \sin (\hat{\omega}_2 t) \right\} \hat{H}_1^{1/4} \tag{6b}
\]

\[
\hat{z}_2(t) \hat{F}_{21}(t) = \gamma \frac{2}{\sqrt{\hat{B}_+ \hat{T}^+}} \{ \sin [(\hat{\nu}_1 - \hat{\omega}_1)t] \cos (\hat{\omega}_2 t) + \sin [(\hat{\nu}_1 + \hat{\omega}_1)t] \cos (\hat{\omega}_2 t) \} \sqrt{\hat{B}_+ \hat{T}^+}
- \frac{\gamma}{2} \hat{H}_1^{1/4} \left\{ \cos [(\hat{\nu}_2 + \hat{\omega}_2)t] \sin (\hat{\omega}_1 t) - \cos [(\hat{\nu}_2 - \hat{\omega}_2)t] \sin (\hat{\omega}_1 t) \right\} \hat{H}_2^{1/4} \tag{6c}
\]

\[
\hat{z}_2(t) \hat{F}_{22}(t) = i \gamma \frac{2}{\sqrt{\hat{B}_+ \hat{T}^+}} \{ \sin [(\hat{\nu}_1 - \hat{\omega}_1)t] \sin (\hat{\omega}_2 t) + \sin [(\hat{\nu}_1 + \hat{\omega}_1)t] \sin (\hat{\omega}_2 t) \} \hat{H}_1^{1/4}
- i \frac{\gamma}{2} \hat{H}_1^{1/4} \left\{ \cos [(\hat{\nu}_2 - \hat{\omega}_2)t] \cos (\hat{\omega}_1 t) - \cos [(\hat{\nu}_2 + \hat{\omega}_2)t] \cos (\hat{\omega}_1 t) \right\} \sqrt{\hat{T} \hat{B}_-} . \tag{6d}
\]

Now, the non-commutativity between the operators \( \hat{\omega}_1 \) and \( \hat{\omega}_2 \) imply that to calculate the integrals involving the terms given by the Eqs. (5) and (6) we need to use the series expansion of the trigonometric functions. In this case the integrals can be easily done because the time variable can be considered as a parameter factor. Finally, using these results into Eq. (3.30) is trivial to find the expression (3.31) for the matrix elements of the particular solution.
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