DEHN-SOMMERVILLE FROM GAUSS-BONNET

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ABSTRACT. We give a “zero curvature” proof of Dehn-Sommerville for finite simple graphs. It uses a parametrized Gauss-Bonnet formula telling that the curvature of the valuation $G \to f_G(t) = 1 + f_0 t + \cdots + f_d t^{d+1}$ defined by the $f$-vector of $G$ is the anti-derivative $F$ of $f$ evaluated on the unit sphere $S(x)$. Gauss Bonnet is then parametrized, $f_G(t) = \sum_x F_S(x)(t)$, and holds for all Whitney simplicial complexes $G$. The Gauss-Bonnet formula $\chi(G) = \sum_x K(x)$ for Euler characteristic $\chi(G)$ is the special case $t = -1$. Dehn-Sommerville is equivalent to the reflection symmetry $f_G(t) + (-1)^d f_G(-1 - t) = 0$ which is equivalent to the same symmetry for $F$. Gauss-Bonnet therefore relates Dehn-Sommerville for $G$ with Dehn-Sommerville for the unit spheres $S(x)$, where it is a zero curvature condition. A class $\mathcal{X}_d$ of complexes for which Dehn-Sommerville holds is defined inductively by requiring $\chi(G) = 1 + (-1)^d$ and $S(x) \in \mathcal{X}_{d-1}$ for all $x$. It starts with $\mathcal{X}_{-1} = \{\emptyset\}$. Examples are simplicial spheres, including homology spheres, any odd-dimensional discrete manifold, any even-dimensional discrete manifold with $\chi(G) = 2$. It also contains non-orientable ones for which Poincaré-duality fails or stranger spaces like spaces where the unit spheres allow for two disjoint copies of manifolds with $\chi(G) = 1$. Dehn-Sommerville is present in the Barycentric limit. It is a symmetry for the Perron-Frobenius eigenvector of the Barycentric refinement operator $A$. The even eigenfunctions of $A^T$, the Barycentric Dehn-Sommerville functionals, vanish on $\mathcal{X}$ like $22f_1 - 33f_2 + 40f_3 - 45f_4 = 0$ for all 4-manifolds.

1. Gauss Bonnet

1.1. The category of finite abstract simplicial complexes $G$ requires only one axiom: $G$ is a set of non-empty sets closed under the operation of taking finite non-empty subsets. The $f$-vector of $G$ is $f = (f_0, f_1, \ldots, f_d)$ of $G$, where $f_k$ is the set of sets in $G$ with $k + 1$ elements. The $f$-function of $G$ is $f_G(t) = 1 + \sum_{k=0}^d f_k(G) t^k$. If $G$ is the Whitney complex of a graph $(V, E)$ the unit sphere $S(x)$ for $x \in V(G)$ is the unit sphere in that graph. For any $G$, let $F_G(t) = \int_0^t f_G(s) \, ds$ denote the anti-derivative of $f_G$. The curvature valuation to $f$ is the anti-derivative of $f$ evaluated on the unit sphere:

Theorem 1 (Gauss-Bonnet). $f_G(t) = \sum_{x \in G} F_S(x)(t)$.

Proof. If every $k$-simplex $y$ in $S(x)$ carries a charge $t^{k+1}$, then $f_G(t)$ is the total charge. Because every $k$-simplex $y$ in $S(x)$ defines a $(k + 1)$-simplex $z$ in $G$, the simplex $z$ in $S(x)$ carries a charge $t^{k+2}$. It contains $(k + 2)$ zero-dimensional points, which were simplices in $G$. Distributing the charge equally to the points, gives each a charge $t^{k+2}/(k + 2)$. The curvature $F_S(x)(t)$ at $x$ adds up all the charges of the simplices attached to $x$. There is code at the end allowing to experiment. □

Date: 5/12/2019.

1991 Mathematics Subject Classification. 05Cxx, 05Exx, 68Rxx, 55U10, 57Mxx.
1.2. For $t = -1$, we get a classical Gauss-Bonnet statement
\[ \chi(G) = \sum_{x \in G} K(x), \]
where $K(x) = F_{S(x)}(-1)$ is the Levitt curvature, the discrete analogue of the Gauss-Bonnet-Chern curvature in the continuum. An explicit formula for $K(x)$ with $f_{-1} = 1$ is
\[ K(x) = \sum_{k=-1}^{d} (-1)^k f_k(S(x)) \frac{1}{k+2}. \]
It appeared first in [21]. For the continuum proof, see [5].

1.3. By differentiation of Gauss-Bonnet with respect to $t$ we get: $f'_G(t) = \sum_x f'_{S(x)}(t)$. For $t = -1$ in particular, this gives an identity seen in [18]. $f'_G(-1) = \sum_{x \in G} 1 - \chi(S(x))$ which is a trace of the Green function operator $g = L^{-1}$ with $L_{xy} = 1$ if $x \cap y \neq \emptyset$ and $L_{xy} = 0$ else, where the diagonal entries are $g(x,x) = 1 - \chi(S(x))$.

2. Dehn-Sommerville symmetry

2.1. Given a complex $G$, the $h$-function $h_G(x) = (x-1)^d f_G(1/(x-1))$ is a polynomial $h_G(x) = h_0 + h_1 x + \cdots + h_d x^d + h_{d+1} x^{d+1}$ defining a $h$-vector $(h_0, h_1, \ldots, h_{d+1})$. The Dehn-Sommerville relations assert that the $h$-vector is palindromic, meaning that $h_i = h_{d+1-i}$ for all $i = 0, \ldots, d + 1$. Let us call a complex $G$ Dehn-Sommerville if the Dehn-Sommerville relations hold for $G$.

2.2. For example, for the icosahedron complex $G$ generated by the triangles \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,4,6\}, \{2,5,9\}, \{2,6,1\}, \{2,9,10\}, \{3,4,8\}, \{3,5,11\}, \{3,8,11\}, \{4,6,12\}, \{4,8,12\}, \{5,9,11\}, \{6,10,12\}, \{7,8,11\}, \{7,8,12\}, \{7,9,10\}, \{7,9,11\}, \{7,10,12\}, with $d = 2$ and $f_G(t) = 1 + 12t + 30t^2 + 20t^3$ we have $h_G(t) = 1 + 9t + 9t^2 + t^3$. The f-vector (12,30,20) defined the h-vector (1,9,9,1). The graph is Dehn-Sommerville. The Whitney complex $G_1$ of the icosahedron has the $f$-vector (62,180,120) and $h$-vector (1,59,59,1). As an other example, the Möbius strip $G$ generated by \{1,2,5\}, \{1,5,8\}, \{2,3,6\}, \{2,5,6\}, \{3,4,7\}, \{3,6,7\}, \{4,5,8\}, \{4,7,8\}, with f-vector (8,16,8) gets the h-vector $(-1,3,5,1)$ which is not palindromic. The Möbius strip is not Dehn-Sommerville. Manifolds with boundary in general are not Dehn-Sommerville. We will see that cohomology, orientability etc are irrelevant. The only thing which matters is the Euler characteristic of the complex as well as whether the unit spheres and unit spheres of unit spheres etc are Dehn-Sommerville.

2.3. The following result holds for any simplicial complex:

**Theorem 2.** The simplex generating function $f_G(t)$ of $G$ satisfies the symmetry $f(t) + (-1)^d f(-1-t)$ if and only if $G$ is Dehn-Sommerville.

**Proof.** The palindromic condition can be rephrased that the roots of the $h$-function $h(t) = 1 + h_0 t + \cdots + h_d t^{d+1}$ are invariant under the involution $x \to 1/x$. This is equivalent that the roots of $f$ are invariant under the involution $x \to -1-x$ and so to the symmetry $f(-1-t) = \pm f(t)$ for the $f$-function. \qed
2.4. Dehn-Sommerville complexes must have the Euler characteristic of a $d$-sphere:

**Corollary 1.** If $G$ is a complex with maximal dimension $d$ and $G$ satisfies Dehn-Sommerville, then $\chi(G) = 1 + (-1)^d$.

**Proof.** We have $f_G(0) = 1$. The symmetry tells $f(-1) = (-1)^d f(0) = (-1)^d$. But $f(-1) = 1 - \chi(G)$. □

2.5. For the zero-sphere $S^0 = \{\{1\}, \{2\}\}$ we have $f_G(t) = 1 + 2t$ which satisfies $f_G(-t - 1) = -f_G(t)$. Since $f_{G+H}(t) = f_G(t)f_H(t)$, we immediately see that the suspension $S_0 + G$ of a Dehn-Sommerville complex is Dehn-Sommerville. More generally:

**Corollary 2.** If $G$ and $H$ are Dehn-Sommerville, then the join $G + H$ is Dehn-Sommerville.

2.6. While the join of a $k$-sphere and a $l$-sphere is always a $k + l + 1$-sphere, we in general do not get discrete manifolds, if we take the join of two discrete manifolds. The join can produce lots of examples of simplicial Dehn-Sommerville complexes which are not manifolds.

2.7. The **Barycentric refinement** $G_1$ of a complex $G$ is the order complex of $G$. It is more intuitive to think of $G_1$ as the Whitney complex of the graph $\Gamma(G)$ defined by $G$. Barycentric refinements are always Whitney complexes of graphs. The sets in $G_1$ are the vertex sets of the complete sub-graphs of $\Gamma(G)$. The following statement can be reformulated algebraically as a commutation between two operations, the Barycentric refinement operation and the Dehn-Sommerville involution. But it is also a geometric statement:

**Proposition 1.** If $G$ is Dehn-Sommerville then its Barycentric refinement $G_1$ is Dehn-Sommerville.

**Proof.** This can be proven by induction with respect to dimension. Gauss-Bonnet implies that $G$ satisfies Dehn-Sommerville if and only it has the right Euler characteristic $1 + (-1)^d$ and all unit spheres satisfy Dehn-Sommerville. The unit spheres of $G$ are either spheres or Barycentric refinements of unit spheres of $G$. Both cases satisfy Dehn-Sommerville by induction. □

2.8. Let $A$ be the Barycentric refinement operator defined by $f(G_1) = Af(G)$. The matrix $A$ is a $(d + 1) \times (d + 1)$ upper triangular matrix and explicitly been given as $A_{ij} = \text{Stirling}(i, j)!$. Since all eigenvalues $\lambda_k = k!$ are distinct, the eigenvalues of $A$ are an eigenbasis of $A$ on the vector space $V_d = \mathbb{R}^{d+1}$ which is isomorphic to the space $P_d$ of polynomials of degree less or equal to $d$. The isomorphism is given by $[a_0, a_1, \ldots, a_d] \to a_0 + a_1 t + \cdots + a_d t^d$. As an affine space, it is isomorphic to $1 + a_0 t + a_1 t^2 + \cdots + a_d t^{d+1}$ which is the form of an $t$-function of a complex.

2.9. The linear unitary reflection $T(f)(x) = f(-1 - x)$ on polynomials defines an involution on $\mathbb{R}^{d+1}$. As an unitary reflection $T^2 = \text{Id}$, it has the eigenvalues $1$ and $-1$ which by the spectral theorem of normal operators define an eigenbasis even so the algebraic multiplicities are larger than $1$ for $d > 1$. In analogy to $\tilde{T}(f)(x) = f(-x)$, we can call eigenfunctions of $1$ even functions and eigenfunctions of $-1$ odd functions.
2.10. Any eigenvector $V$ of $A^T$ defines a functional $\phi_V$ on complexes. Most functionals $\phi_V(G_n)$ explode when looking at Barycentric refinements $G_n$ of $G$. There is just one functional $\phi_1$ which stays invariant and this is the Euler characteristic. Since the matrix $A$ is upper triangular and $A^T$ lower triangular, the eigenbasis diagonalizing $A$ is triangular too. We say an eigenvector is even if it has an odd number of non-zero entries.

**Lemma 1.** The eigenbasis of $A$ is also an eigenbasis of $T$: even eigenvectors are eigenfunctions of $T$ to the eigenvalue $1$ and odd eigenvectors of $A$ are eigenfunctions of $T$ to the eigenvalue $-1$.

**Proof.** As the linear operators $T$ and the Barycentric operation $A$ commute. They therefore have the same eigenbasis. □

2.11. We know from linear algebra that the eigenvectors $V_k$ of $A^T$ and the eigenvectors $W_k$ of $A$ have the property that $V_k W_l = c_{kl} \delta_{k,l}$ meaning that if they are normalized, then they define dual coordinate systems. We think about eigenvectors of $A^T$ as functionals. Functionals in the even eigenspace of $T$ are zero on even functions etc. This gives us convenient Dehn-Sommerville invariants:

**Corollary 3.** For even $d$, the even eigenvectors of $A^T$ define functionals which are zero on the class $\mathcal{X}_d$. For odd $d$, the odd eigenvectors of $A^T$ define functionals which are zero on the class $\mathcal{X}_d$.

2.12. This was Theorem (1) in [17], where already the idea of proving Dehn-Sommerville via curvature has appeared and multi-variate versions of Dehn-Sommerville were given, answering a open problem of Gruenbaum [6] from 1970. The current approach is much simpler. In multi-dimensions, the Dehn-Sommerville symmetry just has to hold for each of the variables appearing in the simplex generating function $f(t_1, \cdots, t_m)$. The proof in higher dimensions is identical using Gauss-Bonnet.

2.13. For the next part, we assume that $G$ can be realized as a graph like if $G$ is the Barycentric refinement of an arbitrary complex. An edge refinement of a graph cuts an edge $e = (a, b)$ into two by adding a new vertex $c$ in the middle and connecting the new vertex to the intersection of spheres at $a$ and $b$. More formally, we remove the edge $(a, b)$, and adding new edges $(a, c), (c, b)$ as well as $\{(c, z) \mid z \in S(a) \cap S(b)\}$. Edge refinements preserve discrete manifolds. More generally:

**Proposition 2.** If $G$ is in $\mathcal{X}_d$ and $e$ is an edge in $G$, then the edge refinement is in $\mathcal{X}_d$.

**Proof.** The effect of the operation on the $f$-vector can be split into two parts. The first one is to increase $f_0$ and $f_1$ by $1$ (which means adding $t + t^2$ to $f_G(t)$). Then we add $t f_{S(a) \cap S(b)} + 2t^2 f_{S(a) \cap S(b)}$, because every $k$-simplex in $S(a) \cap S(b)$ defines a new $(k+1)$-simplex connecting to $c$ and two new $(k+2)$-simplices connecting $S(a) \cap S(b)$ to $(a, c)$ and $(b, c)$. Now, the set of functions satisfying the Dehn-Sommerville symmetry form a linear space. The claim follows as the added part $t + t^2 + t f_{S(a) \cap S(b)} + 2t^2 f_{S(a) \cap S(b)}$ satisfy the Dehn-Sommerville symmetry by induction because the space $S(a) \cap S(b)$ is in $\mathcal{X}_{d-2}$ if $G \in \mathcal{X}_d$. □
3. Remarks

3.1. For Dehn-Sommerville, see chapter nine in [7] for convex polytopes. It is also covered in [2] where we read: these relations had already been found by Dehn by 1905 for the dimensions 3, 4, 5; they were known in all dimensions by Sommerville by 1927 but were then forgotten until they were rediscovered by Klee in 1963. The relations have been extended to larger classes of polytopes. An example of recent work is [1]. More literature is [10, 24, 23, 22, 4, 8, 9].

3.2. The Levitt curvature for Euler characteristic Formula (1) appeared in [21]. We have rediscovered that formula \( \chi(G) = \sum_x K(x) \) in the introduction to [11], an article which focused on geometric graphs (discrete manifolds). It surprises that higher dimensional curvature in the discrete is so elegant, especially if one compares to the continuum, where one has to refer to Pfaffians of curvature expressions to get to the general Gauss-Bonnet-Chern theorem (see [5]). In the continuum, the Euler curvature is not even defined for odd-dimensional manifolds. In the continuum, it is zero for odd-dimensional manifolds as we see here again as it is then a special case of the Dehn-Sommerville equations.

3.3. The Taylor expansion of the parametrized Gauss-Bonnet formula at \( t = 0 \) gives a generalized handshake formula \( f_k(G) = \sum_{x \in G} V_{k-1}(S(x))/(k+1) \) which by linearity produces Gauss-Bonnet formulas for any discrete valuation \( X(G) = \sum_k X_k f_k(G) \) and especially for Euler characteristic \( \chi(G) = \sum_k (-1)^k f_k(G) \). One can also just define \( f(t) = 1 + \sum_{k=0}^d X_k f_k t^{k+1} \) and its anti derivative. The Gauss-Bonnet formula is the same. For example, for \( X(G) = v_1(G) \), where \( f(t) = 1 + v_1 t^2 \), the curvature is \( K(x) = \deg(v)/2 \). Gauss-Bonnet is then Euler handshake formula, the fundamental theorem of graph theory. More generally, we have for \( v_k(G) \) the curvature is \( K(x) = f_{k-1}(S(x))/(k+1) \).

3.4. In [11] we first noticed experimentally that the curvature is zero for odd-dimensional geometric complexes but we could not prove it yet at that time. These zero curvature relations were later proven with discrete integro-geometric methods in [13, 14] by seeing curvature as an average of Poincaré-Hopf indices when integrating over all functions or colorings. The connection to Dehn-Sommerville emerged especially in the work about Wu characteristic [17]. So, Dehn-Sommerville conditions appeared for us three times independently: first as a zero curvature condition for odd-dimensional discrete manifolds, then as Barycentric invariants (eigenvectors of \( A^T \)), then as a symmetry for the roots of simplex generating function \( t \rightarrow f_G(t) \). In each case, we were unaware of the Dehn-sommerville connection at first. We hope that this note makes clear how all these concepts (curvature, Barycentric refinement and root symmetry) are related.

3.5. The classical Dehn-Sommerville valuations are

\[
X_{k,d} = \sum_{j=k}^{d-1} (-1)^{j+d} \binom{j+1}{k+1} v_j(G) + v_k(G)
\]
If the vectors $X_0, d \ldots, X_{d-2}, d$ are written as row vectors in a matrix $X_d$, we have

$$X_2 = \begin{bmatrix} 2 & -2 \\ 0 & 2 & -3 & 0 & 2 & -3 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 2 & -4 \end{bmatrix}, X_3 = \begin{bmatrix} 2 & -2 & 3 & -4 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 2 & -4 \end{bmatrix}.$$ 

We have mentioned before (like [19] that the curvature of $X_{k,d}$ is $K(x) = X_{k-1,d-1}(S(x))$. But it is less obvious there. The reason is the combinatorial identity

$$X_{k+1,d+1}(l+1)/(l+1) = X(k,d)(l)/(k+2).$$

But it also implies that the Dehn-Sommerville curvatures are all zero for a geometric graph. Use Gauss-Bonnet and induction using the fact that the unit sphere of a geometric graph is geometric and that for $d = 1$, a geometric graph is a cyclic graph $C_n$ with $n \geq 4$. For such a graph, the Dehn-Sommerville valuations are zero.

### 3.6. Gauss-Bonnet and Dehn-Sommerville

Gauss-Bonnet and Dehn-Sommerville can be generalized to multi-valuate valuations like Wu characteristic $\omega(G) = \sum_{x \sim y} \omega(x)\omega(y)$ with $\omega(x) = (-1)^{\dim(x)}$. The Wu characteristic is then $1 - f(-1, -1)$ where

$$f(t,s) = 1 + \sum_{k,l} f_{kl}(G)t^{k+1}s^{l+1}$$

is the **multivariate simplex generating function**. Here, $f_{kl}(G)$ is the $f$-matrix, counting the number of intersecting $k$-dimensional and $l$-dimensional simplices.

### 3.7. The curvature of Wu characteristic is then $F_G(t,s) = \int_0^t f(r,s) \, dr$. Gauss-Bonnet reads

$$f_G(t,s) = \sum_{x \in G} F_{S(x)}(t,s)$$

and especially $\omega(G) = \sum_{x \in G} K(x)$, where $K(x)$ is the Wu curvature.

### 3.8. While investigating Barycentric limits [16, 15], an other angle to Dehn-Sommerville appeared. We first did not see the connection between Barycentric invariants and Dehn-Sommerville. The Barycentric refinement operator $A_d$ was first explored empirically by looking at the best linear operator implementing the map $f(G) \rightarrow f(G_1)$ (brute force data fitting with hundreds of random graphs) and were surprised that the fitting would lead to an exact formula. After getting the formula for $A$ and proving it, we learned that it is “well known”. It appears in [25, 22, 8].

### 3.9. The value $g(x,x) = 1 - \chi((x))$ is the **Green function**, the diagonal entries of the inverse $g = L^{-1}$ of the unimodular connection matrix $L$ defined as $L(x,y) = 1$ if $x \cap y \neq \emptyset$ and $L(x,y) = 0$ else. The Green function entries $g(x,y)$ are potential energy values between two simplices $x, y$. We called $f'_G(-1) = \sum_{x \in G}(1 - \chi(S(x))) = \text{tr}(L - L^{-1})$ the **Hydrogen functional**.
3.10. The energy theorem assures that the total potential energy \( \sum_{x,y} g(x, y) \) is the Euler characteristic \( \chi(G) \), which is defined as the super trace \( \text{str}(L) = \sum_x \omega(x)L(x, x) \) and agrees with the super trace \( \sum_x \omega(x)g(x, x) \) of \( g = L^{-1} \). The entries \( \omega(x)L(x, x) = (-1)^{\dim(x)+1} \) and \( \omega(x)g(x, x) \) are integers, the Poincaré-Hopf indices \([12]\) of the function \( h(x) = \dim(x) \) or \( h(x) = -\dim(x) \) which are colorings of the graph \( \Gamma(G) \).

3.11. From the energy theorem and Gauss Bonnet we can express \( d/dt \log(f_G(t)) \) at \( t = -1 \) through the connection operator \( L \). Let \( E \) be the matrix which has everywhere 1. \( \frac{d}{dt} \log(f(t))_{t=-1} = \text{tr}(L^{-1})/\text{Tr}(L^{-1})E \). Proof: From Gauss Bonnet, we have \( \frac{d}{dt} \log(f_G(t)) = \frac{L}{f_G} = \sum_x \frac{\delta_{S(x)}}{f_G(x)} \). For \( t = -1 \), we have \( \sum_x \chi(S(x))/\chi(G) = \text{tr}(L^{-1})/\chi(G) = \text{tr}(L^{-1})/\text{Tr}(L^{-1})E \).

3.12. The involutive symmetry \( T(f) = \pm f(-1-t) \) given by the Dehn-Sommerville condition implies root pairing for \( f \). This article has started with such an observation. We noticed that for even-dimensional spheres, there is always root with \( \text{Re}(t) = -1/2 \) and that the roots are reflection symmetric with respect to \( t = -1/2 \). A simplicial complex is defined to be a \( d \)-sphere if every unit sphere is a \((d-1)\)-sphere and removing one vertex renders the complex contractible. This inductive definition is primed by the empty complex 0 being the \((-1)\)-sphere. There are various operations which preserve \( d \)-spheres. We observe that for spheres the roots of \( f \) pair up to \(-1\) in the odd-dimensional case and do so also in the even-dimensional case with the remaining roots.

3.13. There are simplicial complexes outside \( \mathcal{X} \) which are Dehn-Sommerville. Similarly as zero Euler curvature implies zero Euler characteristic for even-dimensional manifolds, zero Euler characteristic does not necessarily mean zero curvature. The zero Dehn-Sommerville curvature condition is sufficient for the complex to be Dehn-Sommerville, but it is not necessary. There are complexes which are Dehn-Sommerville, but which are not Dehn-Sommerville flat. We give an example in the illustration section.

4. Examples

4.1. Examples. For \( d = 3 \), we have \( A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & 6 & 36 \\ 0 & 0 & 0 & 24 \end{bmatrix} \). The eigenvectors are \([0, 0, 0, 1]^T, [0, 0, -1, 2]^T, [0, 22, -33, 40]^T, [-1, 1, -1, 1]^T\). The eigenvector \([-1, 1, -1, 1]\) to the eigenvalue 1 is the Euler characteristic. The second and last eigenvector leads to the Dehn-Somerville invariants \( f_2 - 2f_3 = 0 \) and \( f_0 - f_1 + f_2 - f_3 = 0 \).

For \( d = 4 \), we have \( A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & 6 & 36 \\ 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). From the eigenvectors \([0, 0, 0, 0, 1]\),
For $d = 0$, where $h = -1 + t + v_0$, the condition is $v_0 = 2$ implying that the zero-dimensional graph must have 2 vertices.

For $d = 1$, where $f = (v_0, v_1)$ gives the number of vertices, edges and triangles, then $f = 1 + v_0t + v_1t^2$ and $h = (1 - v_0 + v_1) + (v_0 - 2)t + t^2$. The Dehn-Sommerville condition is $v_0 = v_1$ meaning $\chi(G) = 0$. Note that we can attach hairs and even arbitrary trees to a circular graph and still have $v_0 = v_1$ satisfied. This shows, that at least in one dimension, the Dehn-Sommerville relations can hold for a larger class of complexes than $X$ to be defined below.

For $d = 2$, the conditions give $v_1 = 3v_0 - 6, v_2 = -4 + 2v_0$. This is equivalent to $v_0 - v_1 + v_2 = 2, 2v_1 = 3v_2$. This means that Euler characteristic is 2 and that every edge meets two triangles.

For $d = 3$, the condition is equivalent to $\chi(G) = v_0 - v_1 + v_2 - v_3 = 0$ and $22v_1 - 33v_2 + 40v_3 = 0$. We will see in a moment how to get the more intuitive Barycentric expressions through eigenvectors of the Barycentric refinement operators.

### 4.3. The root pairing property

The root pairing property was already mentioned in [20]. We found this while investigating the statistics of the simplex cardinality distribution in simplicial complexes. The **root pairing statement** is obviously true for 0-dimensional spheres. If a zero dimensional complex has $n$ points, then the generating function of the $f$-vector is $1 + nt$. This has a root $-1/2$ if and only the complex has exactly $n = 2$ point, which means that $G$ has to be a 0-sphere. Let us also mention the $(-1)$-dimensional complex which is the empty complex. In that case, the function is $f = 1$ which has no roots. Root pairing still works, there are just no pairs.

### 4.4. For 1-dimensional complexes

For 1-dimensional complexes with $n$ vertices and $m$ edges, we have the generating function $1 + nt + mt^2$. The Euler characteristic is $n - m = \chi(G)$. The roots are $-n \pm \sqrt{-4m + n^2)/(2m)}$. The sum of the roots is $-n/m$. This is $-1$ if and only if $n = m$, meaning that we need $\chi(G) = 0$. Beside circular complexes, there are many complexes like **sun graphs** for which $n = m$. We can attach arbitrary trees to the circular graph for example and still have the property. There is a sphere complex which is not the Whitney complex of a graph, which is $G = \{(1,2),(2,3),(3,1),(1,2,3)\}$ where $n = 3, m = 3$ and where the roots become complex. We see confirmed here that roots are not real if and only if the complex is not the Whitney complex of a graph.

### 4.5. For 2-dimensional complexes

For 2-dimensional complexes the simplex generating function is $1 + nt + mt^2 + lt^3$, in order to have a root $-1/2$, we need $n = (8 - l + 2m)/4$. For a 2-manifold, we have $2m = 3l$. The two equations give $m = 3n - 6, l = 2n - 4$ implying $\chi(G) = n - m + l = 2$. Actually, for two dimensional complexes, the two equations $2m = 3l, \chi(G) = n - m + l = 2$ imply that $f(-1/2) = 0$. This in particular holds for two disjoint copies of the projective plane.
Figure 1. The functions $f_G$ for the smallest spheres $S^1 = C_4 = S^0 + S^0$, $S^2 = C_4 + S^0 = S^0 + S^0 + S^0$ (the octahedron), $S^3 = S^2 + S^0 = S^1 + S^2$ (the three sphere), $S^4 = S^4 + S^0 = 5 \times S^0$ (the four sphere), which are all cross polytopes. The index generating function $f_G(t)$ of $G = S^0$ is $1 + 2t$. So that $f_{S^d}(t) = (1 + 2t)^{d+1}$. We then observed experimentally that all spheres satisfy the symmetry $f(x) + (-1)^d f(-1 - x) = 0$, then linked it to Dehn-Sommerville.
Figure 2. If the faces are included to a cube graph we get a CW-complex which models a discrete 2-sphere. Its generating function is $f_G(x) = 1 + 8x + 12x^2 + 6x^3$. It does not satisfy Dehn-Sommerville. It also has non-real roots. After Barycentric refinements however, the roots become real. We see $f_{G_1}(x) = 1 + 26x + 60x^2 + 36x^3$ and $f_{G_2}(x) = 1 + 122x + 336x^2 + 216x^3$ (we plotted $f_{G_2}/2$). The coefficients $[122, 336, 216]$ are already aligned quite well with the Perron-Frobenius eigenvector $[1, 3, 2]^T$ to the Barycentric refinement operator $A_2$ in dimension 2 which defines a function having only real roots. In general we see that the Perron-Frobenius functions $a_1 t + \cdots + a_n t^{d+1}$ for the Perron-Frobenius eigenvector to the $(d+1) \times (d+1)$ matrix $A_d$ always has only real roots. It looks like a simple calculus/linear algebra problem, but we can not prove this yet. It would imply that for sufficiently large Barycentric refinement of any CW complex and especially simplicial complexes, the roots of $f_{G_n}$ are real for large enough $n$. 
Figure 3. Also the dodecahedron (when seen as a CW-sphere and not a 1-dimensional graph, which it is when seen as a simplicial complex), has non-real roots for $f_G(t) = 1 + 20t + 30t^2 + 12t^3$. But here also $f_{G_1}$ has non-real roots. Only $f_{G_2}$ for the second Barycentric refinement $G_2$ starts to have real roots. As the Perron-Frobenius eigenvector produces a function $f$ which satisfies the Dehn-Sommerville symmetry, we get roots for $f_{G_n}$ which are more and more symmetric. Also this was just observed experimentally at first. The linear algebra of the eigenvectors of the Barycentric refinement operators $A_d$ explains this. Indeed, as we show here, Dehn-Sommerville for complexes of the type $X_d$ is a manifestation for a symmetry one has in the Barycentric limit. Since the involutary symmetry (duality) in the limit has eigenvalues 1 or $-1$, only half of the Barycentric invariants are Dehn-Sommerville invariants.
Figure 4. **Sun graphs** are 1-dimensional complexes which satisfy Dehn-Sommerville, even-so they are only varieties, not manifolds. As they have the same number of vertices than edges, we have \( f(t) = 1 + nt + nt^2 \) which satisfies \( f(-1 - t) = f(t) \). It is an example of a 1-variety. A \( d \)-variety is a complex \( G \) for which all unit spheres are \( d - 1 \) varieties. Like manifolds, it starts with the induction that the empty complex is a \(-1\) variety but unlike for manifolds, we do not insist that unit spheres are \((d - 1)\)-spheres. In this example, \( f_G(t) = 1 + 29t + 29t^2 \). The roots \(-1/2 \pm \sqrt{25/116}\) are symmetric with respect to \( \text{Re}(t) = -1/2 \). We have Dehn-Sommerville symmetry.
Figure 5. When adding hairs to a 2-sphere, the Dehn-Sommerville property gets destroyed. The complex $G$ shown here is a simplicial complex but it is not pure. Its inductive dimension is $47/30 = 1.56667$. Its average simplex cardinality $f'_G(1)/f_G(1)$ is $156/79 = 1.97468...$ for the function $f_G(t) = 1 + 20t + 38t^2 + 20t^3$. The function $f_G$ does not honor the Dehn-Sommerville symmetry $f(t) = \pm f(-1 - t)$. We have $f(-1 - t) = -1 - 4t - 22t^2 - 20t^3$. What happens is that the unit spheres do not satisfy Dehn-Sommerville. There are unit spheres which are a disjoint union of a 1-sphere and a point which does not satisfy Dehn-Sommerville. This means that the Dehn-Sommerville curvatures are not zero. The complex is not Dehn-Sommerville flat.
Figure 6. The Dehn-Sommerville property gets destroyed with disjoint sums as well as most connected sums. We see here the connected sum $G$ of a 2-sphere $O$ (an octahedron graph) and a 4-sphere $C_4 + O$ joined at a vertex $v$. The unit sphere $S(v)$ is a disjoint union of a 1-sphere $C_4$ and a 3-sphere $C_4 + C_4$. This disjoint union does not satisfy Dehn-Sommerville. By Gauss-Bonnet (since all other points are Dehn-Sommerville flat), also $G$ is not Dehn-Sommerville. It is almost, $f + 1/2$ would satisfy the Dehn-Sommerville symmetry.
Figure 7. We see a random four sphere $G$. It is Dehn-Sommerville of course. As for any even dimensional sphere, there is a root $t = -1/2$ for the simplex generating function $f_G(t)$.
Figure 8. We see the disjoint union of two projective planes. Any even dimensional manifold of Euler characteristic 2 satisfies the Dehn-Sommerville condition. So also $G = \mathbb{P}^2 \cup \mathbb{P}^2$. We have $f_G(t) = (1 + 2x) \ast (1 + 28x + 28x^2)$. The Betti-vector is $(b_0, b_1, b_2) = (2, 0, 0)$. Obviously, Poincaré duality is failing for $G$ as $G$ is non-orientable. Still, Dehn-Sommerville is intact.
Figure 9. We see the graph of $f_G(t) = 1 + 16t + 106t^2 + 180t^3 + 90t^4$, where $G$ is a Poincaré sphere complex with 16 zero-dimensional simplices, found in [3]. All 4 roots of $f_G$ are complex. As a 3-manifold with zero Euler characteristic, $G$ must be Dehn-Sommerville.
Figure 10. We see a graph obtained by poking around randomly in Erdős-Rényi spaces looking for Dehn-Sommerville graphs. This example has the simplex generating function $f_G(t) = 1 + 9t + 21t^2 + 14t^3$ but it is not a 2-sphere. It is not a manifold but has the Betti numbers $(b_0, b_1, b_2) = (1, 0, 1)$ of the 2-sphere. The graph has inductive dimension 2. It is interesting for us because it is an example which is not Dehn-Sommerville flat. It shows that there are Dehn-Sommerville complexes for which some unit spheres are not Dehn-Sommerville. The complex is **Dehn-Sommerville non-flat**. In other words, having zero Dehn-Sommerville curvatures (unit spheres are Dehn-Sommerville) is only sufficient and not necessary for $G$ to be Dehn-Sommerville.
6. Code

6.1. Here is Mathematica code (see ArXiv version to copy paste) which computes $F_{S(x)}(t)$, then adds up to $f_G(t)$.

```mathematica
UnitSphere [s_, a_] := Module[{b = NeighborhoodGraph [s, a]}, 
   If [Length [VertexList [b]] < 2, Graph [], VertexDelete [b, a]]];
UnitSpheres [s_] := Module[{v = VertexList [s]}, 
   Table [UnitSphere [s, v[[k]]], {k, Length [v]}] ];
ErdoesRenyi [M_, p_] := Module[{q, e, a, v = Range [M]}, 
   e = EdgeRules [CompleteGraph [M]]; q = {}; 
   Do [If [Random[] < p, q = Append [q, e[[j]]], {j, Length [e]}]; 
   UndirectedGraph [Graph [V, q]] ];
CliqueNumber [s_] := Length [First [FindClique [s]] ];
ListClique [s_, k_] := Module[{n, t, m, u, r, V, W, U, l = {}, L}, 
   L = Length; 
   VL = VertexList; EL = EdgeList ; V = VL [s]; W = EL [s]; 
   m = L [W]; n = L [V]; 
   r = Subsets [V, {k, k}]; 
   U = Table [{W[[j, 1]], W[[j, 2]]}, {j, L [W]}]; 
   If [k == 1, V, If [k == 2, l = Append [l, VL [t]], 
   If [L [EL [t]] == k (k - 1)/2, l = Append [l, VL [t]]], 
   {j, L [r]}]]] ];
Whitney [s_] := Module [{F, a, u, v, d, V, LC, L = Length}, 
   V = VertexList [s]; 
   d = If [L [V] == 0, 1, CliqueNumber [s]]; 
   LC = ListClique; 
   If [d == 0, a [x_] := Table [{x[[k]]}, {k, L [x]}]; 
   (t_, 1_): = If [1 == 1, a [LC [t, 1]], If [1 == 0, {}, LC [t, 1]]]; 
   u = Delete [Union [Table [F [s, 1], {1, 0, d}]], 1]; v = {}; 
   Do [v = Append [v, u[[m]], {1, L [u[[m]]]}], {m, L [u]}]; v = {}; ];
Fvector [s_] := Delete [BinCounts [Map [Length, Whitney [s]]], 1];
Ffunction [s_, x_] := Module [{f = Fvector [s], n}, 
   n = Length [f]; 
   If [Length [VertexList [s]] == 0, 1, 1 + Sum [f [[k]] x^k, {k, n}]] ];
DehnSommerville [s_] := Module [{f}, 
   Clear [x]; f = Ffunction [s, x]; 
   Simplify [f] ];
Curvature [s_, x_] := Module [{g = Ffunction [s, y]}, 
   Integrate [g, {y, 0, x}]]; EulerChi [s_] := Module [{f = Fvector [s]}, 
  -Sum [f [[k]] (-1)^(k - 1) x^k, {k, Length [f]}] ];
Curvatures [s_, x_] := Module [{S = UnitSpheres [s]}, 
   Table [Curvature [S[[k]], x], {k, Length [S]}]]; s = ErdoesRenyi [16, 0.4]; 
{EulerChi [s], -Total [Curvatures [s, x]]} /. x -> -1 ];
```
6.2. And here are the Barycentric invariants

\[
A[n] := \text{Table}[\text{StirlingS2}[j, i]*i!, \{i, n+1\}, \{j, n+1\}];
\]

\[
\text{Invariants}[n] := \text{Eigenvectors[Transpose[A[n]]]};
\]

\[
\text{MatrixForm[Transpose[Reverse[Invariants[4]]]]}.
\]

\[
\text{MatrixForm[Binvariants]}.
\]

\[
\text{Binvariants}[4] . f
\]

7. Questions

7.1. (A) One open question is: for which complexes are there non-real roots of \(f\)? We measure that for all simplicial \(G\), the roots of \(f_{G_n}\) for Barycentric refinements \(G_n\) of \(G = G_0\) are real and contained in the open interval \((-1, 0)\) if \(n\) is large enough. This is a still unsolved concrete calculus problem as it requires to find roots of explicitly given polynomials defined by Perron-Frobenius eigenvectors of the Barycentric refinement operator.

7.2. We have never seen that non-real roots from \(f\) appear after a few Barycentric refinements. Non-real roots can appear for Whitney complexes of random graphs, for non-Whitney complexes like homology spheres \(G\) like the one with \(f = 1 + 16t + 106t^2 + 180t^3 + 90t^4\) or the Barnette 3-sphere with \(f = 1 + 8t + 27t^2 + 38t^3 + 19t^4\) or then the boundary sphere of a simplex like the tetrahedral sphere with \(f = 1 + 4t + 6t^2 + 4t^3\) or sphere-CW-complexes like the cube \(f = 1 + 8t + 27t^2 + 6t^3\), the roots of \(f\) can become complex.

7.3. (B) An other question is to see how large the set of Dehn-Sommville complexes are if we look at all graphs with \(n\) vertices. For \(n = 1, 2, 3\) there are none, for \(n = 4\), there is only the cyclic graph, for \(n = 5\), we have only cyclic \(C_5\) of \(C_4\) with a hair. When fishing randomly in the pool of Erdős-Rényi graphs, we come up empty in general. The probability of having a Dehn-Sommville complex must be very small. One can wonder whether the probability is exponentially small in \(n\) or super exponentially small in \(n\).

References

[1] M. Bayer and L.J. Billera. Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets. Invent. Math., 79(1):143–157, 1985.
[2] M. Berger. Jacob’s Ladder of Differential Geometry. Springer Verlag, Berlin, 2009.
[3] A. Björner and F.H. Lutz. A 16-vertex triangulation of the poincaré homology 3-sphere and non-pl spheres with few vertices. http://www.eg-models.de/models/Simplicial_Mani_folds/2003.04.001 2003.
[4] F. Brenti and V. Welker. \(f\)-vectors of barycentric subdivisions. Math. Z., 259(4):849–865, 2008.
[5] H.L. Cycon, R.G.Froese, W.Kirsch, and B.Simon. Schrödinger Operators—with Application to Quantum Mechanics and Global Geometry. Springer-Verlag, 1987.
[6] B. Grünbaum. Polytopes, graphs, and complexes. *Bull. Amer. Math. Soc.*, 76:1131–1201, 1970.
[7] B. Grünbaum. *Convex Polytopes*. Springer, 2003.
[8] G. Hetyei. The Stirling polynomial of a simplicial complex. *Discrete and Computational Geometry*, 35:437–455, 2006.
[9] D. Klain. Dehn-Sommerville relations for triangulated manifolds. [http://faculty.uml.edu/dklain/ds.pdf](http://faculty.uml.edu/dklain/ds.pdf), 2002.
[10] V. Klee. A combinatorial analogue of Poincaré’s duality theorem. *Canadian J. Math.*, 16:517–531, 1964.
[11] O. Knill. A graph theoretical Gauss-Bonnet-Chern theorem. [http://arxiv.org/abs/1111.5395](http://arxiv.org/abs/1111.5395), 2011.
[12] O. Knill. A graph theoretical Poincaré-Hopf theorem. [http://arxiv.org/abs/1201.1162](http://arxiv.org/abs/1201.1162), 2012.
[13] O. Knill. On index expectation and curvature for networks. [http://arxiv.org/abs/1202.4514](http://arxiv.org/abs/1202.4514), 2012.
[14] O. Knill. Curvature from graph colorings. [http://arxiv.org/abs/1410.1217](http://arxiv.org/abs/1410.1217), 2014.
[15] O. Knill. The graph spectrum of barycentric refinements. [http://arxiv.org/abs/1508.02027](http://arxiv.org/abs/1508.02027), 2015.
[16] O. Knill. Universality for Barycentric subdivision. [http://arxiv.org/abs/1509.06092](http://arxiv.org/abs/1509.06092), 2015.
[17] O. Knill. Gauss-Bonnet for multi-linear valuations. [http://arxiv.org/abs/1601.04533](http://arxiv.org/abs/1601.04533), 2016.
[18] O. Knill. On a Dehn-Sommerville functional for simplicial complexes. [https://arxiv.org/abs/1705.10439](https://arxiv.org/abs/1705.10439), 2017.
[19] O. Knill. The amazing world of simplicial complexes. [https://arxiv.org/abs/1804.08211](https://arxiv.org/abs/1804.08211), 2018.
[20] O. Knill. The average simplex cardinality of a finite abstract simplicial complex. [https://arxiv.org/abs/1905.02118](https://arxiv.org/abs/1905.02118), 2019.
[21] N. Levitt. The Euler characteristic is the unique locally determined numerical homotopy invariant of finite complexes. *Discrete Comput. Geom.*, 7:59–67, 1992.
[22] A. Luzon and M.A. Morón. Pascal triangle, Stirling numbers and the unique invariance of the euler characteristic. arxiv.1202.0663, 2012.
[23] S. Murai and I. Novik. Face numbers of manifolds with boundary. [http://arxiv.org/abs/1509.05115](http://arxiv.org/abs/1509.05115), 2015.
[24] I. Novik and E. Swartz. Applications of Klee’s Dehn-Sommerville relations. *Discrete Comput. Geom.*, 42(2):261–276, 2009.
[25] R. Stanley. *Enumerative Combinatorics, Vol. I*. Wadsworth and Brooks/Cole, 1986.