A Parametric Study of the Mechanical Properties of Open-Cell Kelvin Structures

D A Şerban1,*, S Sărăndan1, R Negru1, G Belgiu2 and L Marşavina1

1Department of Mechanics and Strength of Materials, Politehnica University of Timisoara, 1 Mihai Viteazu Avenue, 300 222 Timisoara, Romania
2 Department of Management of Production and Transportation, Politehnica University Timişoara, Romania

*Corresponding author: dan.serban@upt.ro

Abstract. This study investigates the elastic response of a tessellated structure based on cylindrical struts generated along the wireframe of a truncated octahedron (the Kelvin cell). The structure has two variable dimensions (strut thickness and strut length), the relative density being plotted as a function of the given parameters and ranged between $6.7 \cdot 10^{-4}$ and 0.217. The variation of the relative stiffness of the structure was determined as a function of relative density. The size effect was also investigated by determining the elastic response for different cell numbers.

1. Introduction

Cellular solids represent a class of materials which consist of interconnected matrices of solid layers having a well-defined shape of the edges and facets [1]. Due to their good mechanical properties (good impact resistance and high energy absorption) relative to their specific mass [2], cellular structures are used in a large variety of applications (from aerospace and automotive to personal protection or packaging [1]), either as stand-alone materials [3, 4] or as components in composite materials, especially in sandwich structures [5, 6].

Inspired by natural cellular structures such as honeycombs, xylem tissue or trabecular bone, a new class of materials was developed, called metamaterials (beyond materials) [7]. They represent human developed structures with increased mechanical, thermal or optical properties, that have no equivalent in nature [7]. This paper focuses on a class of mechanical metamaterials consisting of a Kelvin cell tessellation.

Considering the deformation of cellular structures in uniaxial loadings, Maxwell proposed two types of microtruss lattice structures: stretch dominated structures and bending dominated structures [8]. Bending dominated structures were considered as locked joints connecting bar frames that underwent bending when the structure is loaded. On the other hand, for the stretch dominated structures, the bars would be subjected primarily to tension or compression. In order to determine the behaviour of such lattices, Maxwell came up with a simple equation which takes into account the number of bars $b$ and the number of joints $j$. For a three-dimensional space, the relation is presented in Equation (1).

$$ M = b - 3j + 6 $$

If $M < 0$, the structure is bending dominated. Otherwise, the structure is stretch dominated.
Considering the slenderness of thin bars that make up such structures, it was observed that stretch dominated structures present higher efficiency than bending dominated structures (as bars are generally stiffer when stretched than when bent) [9]. The variation of relative stiffness with relative density is presented in Equation (2a) for the ideal stretched dominated structure and in Equation (2b) for the ideal bending dominated structure.

\[
\frac{\bar{E}}{E_s} \approx \frac{\bar{\rho}}{\rho_s} \quad (2a)
\]

\[
\frac{\bar{E}}{E_s} \approx \left(\frac{\bar{\rho}}{\rho_s}\right)^2 \quad (2b)
\]

where \(\bar{E}\) and \(\bar{\rho}\) are the structure stiffness and density while \(E_s\) and \(\rho_s\) represent the solid stiffness and density.

For the Kelvin structure, \(M = 36 - 3 \cdot 24 + 6 < 0\), determining a bending dominated behaviour. Though less efficient than stretch dominated structures, The Kelvin structure is one of the most efficient foams with equal volume cells (a structure with the least area of surface between cells), surpassed only by the Weaire–Phelan structure with a margin of 0.3% [10]. The advantage that the Kelvin structure holds over the Weaire–Phelan structure lays in its simplicity. The Kelvin structure consists of tessellated truncated octahedrons while the Weaire–Phelan structure consists of two types of cells: an irregular dodecahedron and a truncated hexagonal trapezohedron [10].

The advantages of parametric structures consist in the ease of designing the required mechanical characteristics: the relative density/stiffness needed for a specific application can be obtained through the input of parameter values (such as strut thickness, strut length or chamfer radius) [11]. In addition, the regular structure avoids defects that occur in natural cellular material; the behaviour of structures with reduced number of cells can be accurately extrapolated to structures with large and very large number of cells, significantly reducing computation times in predicting the mechanical response.

The aim of this study is to determine the stiffness variation of the kelvin structure with the cell wall thickness and cell wall length as well as with the relative density of the structure, for small deformations. Future work will investigate the yielding and post-yielding behaviour of the lattice. Considering that such structures will be manufactured with rapid prototyping technology, the mechanical behaviour of the polymers must be taken into account for complex loading scenarios such as dynamic loadings, high deformations fatigue or fracture. In such cases, the viscoelastic nature of the polymers [12-14] will play an important role in the behaviour of the structure [15].

2. Structure development
The Kelvin cell represents a regular geometrical entity obtained by truncating the tips of an octahedron. The design of the cell began with the generation of a 3D sketch, resulting in a wireframe structure (Figure 1 a). Considering that all the sides of the cell have equal length, the dimension was considered the first structure variable. The solid structure was obtained through the generation of cylinders of equal radius (the second structure variable) along the wireframe (Figure 1 b).

In order for the cell to be tessellated into a structure, the cylinders (cell struts) had to be cut, as seen in Figure 2 a. The Kelvin structure was obtained by multiplying and positioning the cells accordingly, merging the solids and cropping the structure in the shape of a prism with a square base. The resulting 3 x 3 x 2.5 cell structure is presented in Figure 2 b.
The relative density $\rho_{rel}$ of a cellular material is defined as:

$$\rho_{rel} = \frac{\rho_{cell}}{\rho_s} \quad [-]$$  \hspace{1cm} (3)

where $\rho_{cell}$ represents the density of the cellular material and $\rho_s$ represents the density of the solid material. Considering that the densities are calculated for the same unit volume, the relative density can be expressed as the cellular structure mass $m_{cell}$ divided by the solid mass $m_s$.

$$\rho_{rel} = \frac{m_{cell}}{m_s} \quad [-]$$  \hspace{1cm} (4)
As both the cellular structure and the solid are made of the same material (with the same density $\rho_s$), the relative density of the structure can be expressed as:

$$\rho_{rel} = \frac{V_{cell}}{V_s} \quad [-]$$

(5)

where $V_{cell}$ represents the cell volume and $V_s$ represents the volume of the circumscribed prism.

For the study of the variation in relative density with the two structural parameters, five values were chosen for the cell wall length $l$ and 7 values for the cell wall thickness $t$. The cellular structure volume and the circumscribed prism value were evaluated by SolidWorks™. The variation of the relative density with the two parameters is presented in Table 1. The graphical representation is presented in Figure 3.

**Table 1. Variation of relative density with cell length and cell wall thickness**

| Cell wall thickness [mm] | 15   | 20   | 25   | 30   | 35   |
|-------------------------|------|------|------|------|------|
| 1                       | 0.0036 | 0.002 | 0.0013 | 0.00091 | 0.00067 |
| 2                       | 0.014 | 0.0079 | 0.0051 | 0.0036 | 0.0026 |
| 3                       | 0.0306 | 0.0176 | 0.0114 | 0.0079 | 0.0059 |
| 4                       | 0.0529 | 0.0306 | 0.0199 | 0.014 | 0.0103 |
| 6                       | 0.1119 | 0.0659 | 0.0433 | 0.0306 | 0.0228 |
| 8                       | 0.1863 | 0.1119 | 0.0743 | 0.0529 | 0.0395 |
| 10                      | 0.2714 | 0.1665 | 0.1119 | 0.0802 | 0.0602 |

**Figure 3.** Relative density variation with cell wall thickness and cell wall length
A surface fitting was performed for the data presented in Table 1, resulting in the relation presented in Equation (6), having an error $r^2 = 0.9993$:

$$\rho_{rel}(l, t) = e^{9.32 + 1.8 \ln(l) + \frac{15.01}{\sqrt{t}}}$$ (6)

### 3. Numerical analyses

The structures presented in chapter 3 were subjected to numerical analyses in the commercial software **Abaqus™**. The geometries were meshed using second order tetrahedral elements (C3D10), the average element size varying with the cell wall thickness, with an average of 20 elements per cross section. As boundary conditions, a symmetry was imposed on the bottom surface (to avoid stress concentration that would occur in a fixed condition) and a displacement on the top surface, along the normal direction. The material model chosen was isotropic linear elastic. A linear correspondence was observed between the stiffness of the solid material and the structure, so the actual value of the Young’s modulus is irrelevant when considering the relative stiffness. The simulation results for the relative stiffness variation with cell wall length and thickness is presented in Table 2.

| Cell length [mm] | 15       | 20       | 25       | 30       | 35       |
|------------------|----------|----------|----------|----------|----------|
| 1                | 1.075·10^{-5} | 3.34·10^{-6} | 1.35·10^{-6} | 6.48·10^{-7} | 9.67·10^{-7} |
| 2                | 1.82·10^{-4}    | 5.61·10^{-5} | 2.26·10^{-5} | 1.07·10^{-5} | 5.74·10^{-6} |
| 3                | 9.51·10^{-4}    | 2.94·10^{-4} | 1.18·10^{-4} | 5.6·10^{-5}  | 2.98·10^{-5} |
| 4                | 0.0031          | 9.53·10^{-4} | 3.83·10^{-4} | 1.82·10^{-4} | 9.65·10^{-5} |
| 6                | 0.0151          | 0.0048     | 0.0019     | 9.53·10^{-4} | 5.03·10^{-4} |
| 8                | 0.0431          | 0.0148     | 0.0062     | 0.003      | 0.00162   |
| 10               | 0.0914          | 0.0342     | 0.0147     | 0.0073     | 0.00399   |

The graphical representation of the variation of the relative stiffness with cell wall thickness and length is presented in Figure 4.

Similar to the case of relative density, a surface fitting was performed, yielding the relation presented in Equation (7), with an error $r^2 = 0.9994$.

$$\rho_{rel}(l, t) = e^{13.56 - 3.35 \ln(l) + \frac{-19.98}{\sqrt{t}}}$$ (7)

All the values for the determined relative stiffness were plotted as a function of the relative density, along with the ideal behaviour for the stretch dominated and the bending dominated structures, the resulting graph in double logarithmic scale being presented in Figure 5.
Figure 4. Relative stiffness variation with cell wall thickness and cell wall length

Figure 5. Relative stiffness variation with relative density

The curve fitting procedure resulted in the relation presented in Equation (8), with an error $r^2 = 0.9999$. 

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book2.xls : (1)Sheet1, Cell Wall Thickness [mm], Cell Length [mm]
4. Discussions and conclusions
This work investigated the elastic response of a parametric Kelvin structure with two variables: cell wall length and cell wall thickness. It was observed that the variation in relative stiffness with relative density is very close to that of an ideal bending dominated structure, which recommends the use of the structure in engineering applications.

The development of the three equations (relative density and stiffness variation with cell wall thickness and length and relative stiffness variation with relative density) will aid in designing Kelvin structures for specific applications that require particular mechanical properties.

Future work on this topic will focus on the large deformation behaviour of the structure. Experiments will be performed on prototyped structures and the results will be compared to the numerical ones in order to validate the designed models.

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\[ E_{\text{rel}} = 1.43 \cdot \left( \frac{\hat{\rho}}{\rho_s} \right)^{2.096} \] (8)
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