Universal exit probabilities in the TASEP

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Abstract. We study the joint exit probabilities of particles in the totally asymmetric simple exclusion process (TASEP) from space–time sets of a given form. We extend previous results on the space–time correlation functions of the TASEP, which correspond to exits from the sets bounded by straight vertical or horizontal lines. In particular, our approach allows us to remove ordering of time moments used in previous studies so that only a natural space-like ordering of particle coordinates remains. We consider sequences of general staircase-like boundaries going from the northeast to southwest in the space–time plane. The exit probabilities from the given sets are derived in the form of a Fredholm determinant defined on the boundaries of the sets. In the scaling limit, the staircase-like boundaries are treated as approximations of continuous differentiable curves. The exit probabilities with respect to points of these curves belonging to an arbitrary space-like path are shown to converge to the universal Airy$_2$ process.

Keywords: correlation functions, stochastic particle dynamics (theory)

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1. Introduction

Consider a system of particles on the 1D integer lattice. At any time moment a configuration of particles is specified by a set of $N$ strictly increasing integers, $(x_1 > x_2 > \cdots)$, denoting particle coordinates. They evolve in a discrete time $t \in \mathbb{Z}$ according to the TASEP [1] dynamical rules.

(I) A particle takes a step forward, $(x_i \to x_i + 1)$, with probability $p$ and stays at the same site, $(x_i \to x_i)$, with probability $q \equiv 1 - p$ provided that the target site is empty, $(x_i + 1 \neq x_{i-1})$.

(II) If the next site is occupied, $(x_i + 1 = x_{i-1})$, the particle stays with probability 1.

(III) The backward sequential update is used [2]: at each time step the positions $x_i$ of all particles are updated one by one, in the order of increasing particle index: $i = 1, 2, 3, \ldots$.

These dynamical rules define transition probabilities for a Markov chain constructed on the set of particle configurations. Given the initial conditions, one can inquire for probabilities of different events in course of the Markov evolution. In present paper, we are interested in the correlation functions which are the probabilities for events associated with a few specified particles and given space–time positions.

1.1. Spatial correlation functions of the TASEP

The first exact result on correlation functions in TASEP goes back to Johansson’s work [3], where he considered the evolution of TASEP with parallel update and step
initial conditions,
\[ x^0 = Z_{\leq 0}, \]  \hspace{1cm} (1.1)
and obtained the distribution, \( P_t(x_N > M - N) \), of the distance \( M \) traveled by \( N \)th particle up to time \( t \). This result was later generalized to backward sequential update \([4]\) and flat initial conditions \([5]\). The connection of the TASEP with the theory of determinantal point processes revealed in \([6, 7]\) allowed also calculation of the multi-particle correlation functions, i.e. distribution \( P_t(x_{n_1} > a_1, \ldots, x_{n_m} > a_m) \), of positions of \( m \) selected particles at fixed time \( t \), where \( 1 \leq n_1 < \cdots < n_m \) are \( m \) integers numbering the selected particles. The multi-particle correlation functions were extensively studied for different initial conditions in a series of papers \([7\)–\[9\]. The result can generally be represented in the form of the Fredholm determinant of the operator with some integral kernel. An asymptotic analysis of the kernel is of special interest as it allows one to study the scaling limit of the correlation functions, which is believed to yield universal scaling functions of the Kardar–Parisi–Zhang (KPZ) universality class \([10]\).

There is a law of large numbers, which implies that the stochastic evolution converges to a deterministic limit \([11, 12]\). Specifically, in the TASEP, if we measure coordinate \( x_n \) of the \( n \)th particle at time \( t \), the deterministic relation between rescaled variables
\[ \nu \equiv n/L, \quad \omega \equiv t/L, \quad \gamma \equiv (x_n + n)/L \]  \hspace{1cm} (1.2)
holds with probability one as \( L \to \infty \). An explicit form of this relation can be found from the hydrodynamic conservation law
\[ \partial_t \rho + \partial_x j = 0 \]  \hspace{1cm} (1.3)
for the density of particles \( \rho \). Here \( j \equiv j(\rho) \) is the stationary current of particles, which is a model-dependent function of the density. In the case of backward update the current is
\[ j(\rho) = \frac{p \rho (1 - \rho)}{1 - p \rho} \]  \hspace{1cm} (1.4)
Then, the solution of (1.3) with initial conditions (1.1), yields relation
\[ \sqrt{p \omega} - \sqrt{q \nu} - \sqrt{\gamma} = 0, \]  \hspace{1cm} (1.5)
which holds in the range \(-p/q \leq (\gamma - \nu)/\omega \leq p\). For the formula (1.4) and its relation to (1.5) we address the reader to \([3, 4]\).

An exact calculation of the correlation functions allows one to study fluctuations of the random variables near their value on the deterministic scale. Given \( \nu \) and \( \omega \), let \( \gamma(\omega, \nu) \) be the rescaled particle coordinate. The deviation \( \delta x_n \equiv x_n - L(\gamma(\omega, \nu) - \nu) \) of the particle coordinate \( x_n \) develops on the KPZ characteristic scale fluctuations
\[ \delta x_n \sim L^\alpha, \quad \alpha = 1/3. \]  \hspace{1cm} (1.6)
The distribution of the rescaled variable
\[ s = \kappa_x^{-1} \lim_{L \to \infty} \delta x_n L^{-\alpha}, \]  \hspace{1cm} (1.7)
is a universal scaling function of the KPZ class, dependent only on the form of the initial macroscopic density profile. Note that the model dependence is incorporated into a single non-universal constant \( \kappa_x \). Examples of distributions obtained from the asymptotic analysis of the one-point correlation function are the Tracy–Widom functions \( F_1 \) and \( F_2 \).
for flat and step initial conditions respectively. These functions are well known to appear in the theory of random matrices as the distributions of the largest eigenvalue in the orthogonal and unitary Gaussian ensembles [13, 14]. Their presence turns out to be a universal feature of the KPZ class. Furthermore, the study of multipoint distributions shows that the fluctuations of coordinates of different particles, say $x_{n_1}$ and $x_{n_2}$, remain non-trivially correlated random variables on the scale

$$|n_1 - n_2| \sim L^\beta, \quad \beta = 2/3.$$  \hfill (1.8)

This is the second power law characterizing the KPZ class. The critical exponents $\alpha = 1/3$ and $\beta = 2/3$ are called fluctuation and correlation exponents respectively. After corresponding rescaling of particle numbers, one arrives at the one-parameter family of correlated random variables:

$$s(u) = \lim_{L \to \infty} \frac{x_{[\nu N + uN^\beta \kappa_n]} - L(\gamma(\omega, \nu + uL^{\beta-1}\kappa_n) - \nu - uL^{\beta-1}\kappa_n)}{L^\alpha \kappa_x},$$  \hfill (1.9)

where $\kappa_n$ is another non-universal constant. For the cases of flat and step initial conditions, the joint distributions of these variables define universal Airy$_1$ [6] and Airy$_2$ [15] ensembles, whose one-point distributions are $F_1$ and $F_2$.

### 1.2. Space–time correlations and mapping to the last passage percolation

So far we have been discussing only the spatial correlations between positions of different particles at a fixed time moment. However, generally, one can consider joint probability distributions of events associated with different particles, positions and time moments, which happen in the course of the TASEP evolution. We will refer to these distributions as the space–time correlation functions. An example of such a function, the distribution of positions of a tagged particle at different moments of time, has been calculated in [16]. A more general correlation function, the distribution $P(x_{n_1}(t_1) > a_1, \ldots, x_{n_m}(t_m) > a_m)$ of positions $x_{n_1}, \ldots, x_{n_m}$ of selected particles with numbers

$$n_1 \leq \cdots \leq n_m$$  \hfill (1.10)

at time moments $t_1, \ldots, t_m$, was studied in [17, 18]. A method was used that restricted the analysis to sets of space–time points, such that the time coordinates decreased weakly with the particle number and vice versa:

$$t_i \geq t_{i+1}, \quad \text{if } n_i < n_{i+1},$$  \hfill (1.11)

$$t_i > t_{i+1}, \quad \text{if } n_i = n_{i+1}.$$  \hfill (1.12)

This arrangement of time moments was named space-like by the authors of [17, 18]. Another example of the space–time correlation function, the current correlation function, was recently obtained in [19]. This was the probability distribution $P(t_{n_1} < a_1, \ldots, t_{n_m} < a_m)$ of time moments $t_{n_1}, \ldots, t_{n_m}$ at which $m$ selected particles with numbers

$$n_1 < \cdots < n_m$$  \hfill (1.13)

jump from the respective sites $x_{n_1}, \ldots, x_{n_m}$ selected from the set

$$\{x_i = x - i + N : i = 1, \ldots, n_m\},$$  \hfill (1.14)

given $x \in \mathbb{Z}$, $N \geq n_m$ and the initial configuration $x_i^0 = 1 - i, i \in \mathbb{N}$. Due to non-crossing of space–time particle trajectories, the range of time moments accessible for the
dynamics is
\[ t_{n_1} \leq \cdots \leq t_{n_m}. \]  

The time orderings \((1.11), (1.12)\) and \((1.13), (1.15)\) are opposite to each other. These orderings, however, have different origins. In \([16]–[18]\), numbers of particles \(n_1, n_2, \ldots\) and time moments \(t_{n_1}, t_{n_2}, \ldots\) are fixed, and particle coordinates \(x_{n_1}, x_{n_2}, \ldots\) are random variables. In the case of current correlations \([19]\), time moments \(t_n\) are random, while particle coordinates \(x_n\) and numbers \(n_i\) are related fixed parameters. Therefore, unlike \((1.11), (1.12)\) in \([17, 18]\), \((1.15)\) from \([19]\) is not an external constraint, but is the consequence of dynamics: it shows domains which can be reached in the random process with nonzero probability.

Which variable is chosen to be random is, however, not important in the scaling limit, when the three variables, time and space coordinate and the number of a particle, acquire equivalent significance due to separation of fluctuation and correlation scales. Indeed, once we have fixed the values of any two of the parameters \(n, x, t\) on the large scale, the value of the third one is uniquely fixed to the same order by the deterministic relation \((1.5)\). Then, the random fluctuations of any of these quantities characterize the degree of violation of this relation. In other words, we fix a point on the 2D surface defined by the relation \((1.5)\) in 3D space of parameters \(\gamma, \omega, \nu\). Then, the small fluctuations in the vicinity of this point are represented by an infinitesimal vector normal to the surface, which can be projected to one of three directions \(\gamma, \omega, \nu\) or any other direction in 3D space. A choice of the direction affects only the angle-dependent constants defining the fluctuation scale, while the functional form of the distributions is universal. Furthermore, the correlations between fluctuations associated with different points of the surface are also universal, as far as the points are separated by a distance of order of correlation scale, \(N^3\). The universality holds as the mutual positions of the points vary in a wide range. Indeed, the limiting correlation functions of both positions \([16]–[18]\) and times \([19]\) chosen within the domains \((1.11), (1.12)\) and \((1.13), (1.15)\), respectively, yield Airy2 correlations for the case of step initial conditions.

How rigid is the universality with respect to the choice of points within the correlation function was clarified by Ferrari in \([20]\), using arguments based on the observed slow decorrelation phenomena. He explained that the limiting correlations can be of two types, depending on whether the point configurations under consideration are space-like or time-like. The correlations for the space-like configurations are, up to a non-universal scaling factor, of the same form as the purely spatial correlations. Specifically, when the distance between points is of order \(N^3\), the fluctuations at these points are described by the Airy1, Airy2 etc ensembles, depending on the initial conditions, like in the purely spatial case. However, if the point configuration is time-like, the fluctuations, measured at the characteristic fluctuation scale \(N^\alpha\), remain fully correlated, i.e. identical, until the distance between the points becomes of the order of \(N\), which is much larger than \(N^3\).

The definitions of space-like and time-like point configurations used in \([20]\) for the polynuclear growth (PNG) model and extended by Corwin, Ferrari and Peche (CFP), \([21]\), to a wide range of other models including TASEP were, however, different from the one accepted in \([17, 18]\). To classify our results correctly, we recap here the main idea of CFP. Their formulation used the language of the last passage percolation \([3]\), which can be directly mapped to the TASEP as well as to many different models \([21]\). Let \(\mathbb{R}_+^2\) be the
first quadrant of \( \mathbb{R}^2 \). Each point of \( \mathbb{R}^2 \) with positive integer coordinates \((i, j) \in \mathbb{N}^2 \cap \mathbb{R}^2_+ \), is assigned a geometrically distributed random variable \( T_{i,j} \),

\[
P(T_{i,j} = t) = q^t(1 - q).
\]

(1.16)

A particular realization of the TASEP evolution is recorded in the values of \( T_{i,j} \). Namely, \( T_{i,j} \) is the time the \( i \)th particle waits for before making the \( j \)th step after it has been allowed to move. A directed lattice path, \( \Pi_{(x_1,y_1) \rightarrow (x_2,y_2)} \), is the path which starts at the point \((x_1, y_1)\) and, making only unit steps either upward, \((i, j) \rightarrow (i, j + 1)\), or rightward, \((i, j) \rightarrow (i + 1, j)\), ends at the point \((x_2, y_2)\). The sum of \( T_{i,j} \) over the path is referred to as the last passage time. As was shown by Johansson for the TASEP with parallel update [3], the last passage time, maximized over the set of all paths from \((1, 1)\) to \((n, m)\),

\[
T_{n,m} = \max_{\{\Pi_{(0,0)\rightarrow(n,m)}\} \sum_{(i,j) \in \Pi} T_{i,j}
\]

(1.17)

is related to time \( t_\circ(m) \) the \( n \)th particle takes to make \( m \) steps, \( t_\circ(m) = T_{n,m} + n \). For the TASEP with backward sequential update these two times are simply equal, \( t_\circ(m) = T_{n,m} \).

Other models can be obtained as limiting cases. In the limit \( q \to 1 \) with rescaling of time \( t \to t(1 - q) \) we obtain the exponential distribution of waiting times, which defines the continuous time TASEP. In the opposite limit \( q \to 0 \) the first quadrant is filled mainly by zeros, while ‘one’ appears rarely having concentration \( q \). After going to the continuous limit with rescaled coordinates \((x, y) \to (qx, qy)\), the distribution of ‘ones’ on the background of zeros becomes the Poisson process in the first quadrant, which in turn can be used to define the PNG [15, 22]. Given \((n, m)\), the probability distribution of waiting times (1.16) induces the distribution \( P(T_{n,m} < a) \) of the last passage time \( T_{n,m} \).

The joint distributions \( P(T_{n_1,m_1} < a_1, \ldots, T_{n_k,m_k} < a_k) \) of the last passage times for \( k \) different points \((n_1, m_1), \ldots, (n_k, m_k)\) are referred to as \( k \)-point correlation functions.

According to CFP, two-point configuration \(((n_1, m_1), (n_2, m_2))\) is time-like if the points can be connected by a directed path \( \Pi_{(n_1,m_1)\rightarrow(n_2,m_2)} \) and is space-like otherwise. Suppose that \( n_1 \leq n_2 \). Obviously, the time-like conditions are

\[
m_1 \leq m_2 \quad \text{when} \ n_1 < n_2,
\]

\[
m_1 < m_2 \quad \text{when} \ n_1 = n_2.
\]

(1.18)

Recall that in the TASEP with step initial conditions a particle with the number \( n \) starts at initial position \( x_n^0 = -n + 1 \). Therefore, the spatial coordinate of the particle, which has traveled for the distance \( m \), is \( x_n = m - n + 1 \). Then, the space-like condition opposite to (1.18) can be translated to the one for the space coordinates:

\[
x_i < x_j \quad \text{when} \ i \geq j.
\]

(1.19)

This is the condition that slow decorrelation does not occur, and, correspondingly, universality holds. One can see that the points (1.14) of final configurations within the current correlation functions satisfy this condition. Also, due to non-crossing of particle trajectories, these conditions hold automatically when the time moments are chosen in the domain \(((1.11), (1.12))\). Therefore, the point configurations studied in [16]–[19] are space-like according to the CFP classification. However, in the complementary domain, both types of scaling behavior are present. Thus, the division to time-like and space-like configurations proposed by CFP is more adequate if one wants to distinguish between
different types of universal behavior of correlation functions. For this reason, we retain their terminology, where the space-like configurations in TASEP are defined by the condition (1.19) and time-like by the opposite one. The current correlation functions calculated in [19] were just an example of space-like correlations beyond the domain studied in [17, 18]. In fact, the earliest result on space-like correlations was obtained in [22], where the universality of the scaling limit was shown in the context of the PNG model in the whole space-like domain. However the microscopic consideration in context of the TASEP was limited to ((1.11), (1.12)) in [16][19] and to ((1.13), (1.15)) in [19], where the spatial coordinates were fixed by (1.14).

In this paper we extend the microscopic derivation of the TASEP correlation functions to the rest of the space-like domain, which has not been covered by previous analysis.

1.3. General overview and aim of the present work

We conclude the introductory part with an informal outline of the recent development of the theory of multipoint correlation functions described above and a formulation of the tasks we are going to fulfil below.

Though the previous results were formulated in terms of distributions of various quantities, they can be considered in a similar fashion if we look at the TASEP as the probability measure over collections of interacting lattice paths (the space–time trajectories of particles), which can go one step down (particle stays) or down-right (particle makes a step) in the space–time plane. Then the correlation functions give marginal probabilities of certain points or bonds of the underlying lattice belonging to paths corresponding to selected particles. Specifically the development can be roughly divided into the three stages depicted in figure 1. At the first stage the points were fixed at the same moment of time, e.g. those encircled in figure 1(a). The basic achievement of this stage, mentioned in section 1.1, is revealing the structure of determinantal process in the TASEP [6, 7].

The second stage described in section 1.2 is characterized by an extension of the range of point configurations to the space–time domain shown in figure 1(b). The condition crucial for the solution is the possibility to cut off the part of particle trajectory following the selected point without affecting the remaining part. In the first case we just stopped

Figure 1. (a) Equal-time correlation functions; (b) the first extension of the space–time domain; (c) current correlation functions.
Universal exit probabilities in the TASEP

at the moment of interest and the independence from the future was a trivial consequence of the fact that the TASEP is a Markov process. In the second case similar independence follows from another Markov property specific for the TASEP dynamics [17, 18]: the particles in the TASEP do not affect an evolution of other particles to the right of them. Therefore, one can drop a part of a particle trajectory if there are no points fixed to the left of it at later time, see figure 1(b), so that the time corresponding to the selected points increases weakly from left to right. Finally one again arrives at the determinantal process, though more elaborate than the one in the first case.

The third stage, referred to as current correlation functions, is depicted in figure 1(c). Here the particle trajectories propagate equal distances in the spatial direction and the selected points are fixed at different moments of time, which, as seen from the picture, must increase weakly from right to left. At the first glance this situation is in contradiction with the above ‘trajectory cutting’ ideology. However it is not difficult to convince oneself that if we require that the trajectory makes a step forward after the selected point, it has no chance to interact with the trajectory that ends one step to the left of it at later time. Therefore the part of the trajectory after this step can be dropped. This is a Markov property analogous to the previous one, which lies behind the solution. Technically, the reduction of the number of particles continuing evolution can be performed by use of so called generalized Green functions introduced in [23] and applied in [19], which in turn can be reduced to the determinantal process again. On the language of lattice paths this solution yields the probability of having fixed bonds within the trajectories selected particles.

Our goal here is to unify all the previous achievements. Below we calculate the probabilities of trajectories of selected particles to contain given points or bonds, as shown in figure 2. The range of point configurations we consider is wider than in the earlier solutions. Combination of two above Markov properties and use of the generalized Green function allow us to remove time ordering completely. The tools we use, however, are applicable only when the spatial positions of the endpoints are strictly ordered in space. This is the only major constraint, which is nothing but the space-like condition described above (1.19).

Though the ensemble of lattice paths gives a good pictorial representation of the problem, this language is not suitable for real calculations and presentation of the results, because the whole set of lattice paths is too big. To quantify the results we need a suitable probability space where we could enumerate all our possible random outcomes. In the solutions mentioned above this was the set of particle coordinates (Z), i.e. the lower horizontal line in figure 1(a), product of several such sets, i.e. subsequent horizontal lines in figure 1(b), or the set of exit times enumerating the points at the vertical lines in figure 1(c), respectively. Let us think about these lines as the boundaries dividing the space–time plane into two parts. In all cases the space–time trajectories of particles go from one part to another right at the points we select. Therefore we can think of the probabilities under consideration as the probabilities for particle trajectories to go from the boundary at specified points. Known as exit probabilities, such quantities are important in the extremal statistics [24]. Exit probability is a convenient language to represent most general correlation functions. To extend the range of space–time configurations, we consider boundaries of a more general form: a broken line going from northeast to southwest by unit steps, either vertical or horizontal, which divides the space–time plane.
into two parts. Consider now the space–time trajectory of a single particle starting at
the northwest part. Obviously, going from the northwest to southeast, this trajectory will
finally traverse the boundary. The question is, where will it happen? We can enumerate the
sites of the plane belonging to the boundary by a single generalized coordinate \( \tau = t - x \),
which runs over \( \mathbb{Z} \). The value of \( \tau \) corresponding to the site where the trajectory exits the
boundary is a random variable, and its distribution \( P(\tau < a) \) is the quantity of interest.
The probability distribution of particle coordinate at specified time moment and of the
time the particle jumps from a specified site are particular cases of this general quantity.
Note that the exit occurs in two ways (down and down-right) from horizontal parts of the
boundary and only down-right from vertical parts, in the same way as above.

The problem we address below is a direct generalization of the one-particle picture
described. We consider a collection of \( m \) arbitrary boundaries, each with its own
space–time coordinate \( \tau_i \) running in \( \mathbb{Z} \), and inquire about the joint distribution \( P(\tau_1 < a_1, \ldots, \tau_m < a_m) \) of the coordinates of sites at which specified particles go from given
boundaries, see figure 3. This construction allows one to remove any time ordering
constraints and include into the scheme a possibility to consider both the probability of a
particle being at a site and jumping from it. The geometric constraints on the boundaries
from which the constraint on the accessible point configuration follow will be detailed in
section 2.

After obtaining the results on exit probabilities we perform the scaling analysis of
the formulas obtained. The lattice boundaries can be used to approximate smooth curves
in the plane, and the selected points are considered in the vicinity of a smooth path
traversing these curves. The main claim stemming from this analysis is that the large
scale behavior of the exit probabilities is universal as far as the path under consideration does not violate the space-like constraint: the fluctuations of generalized exit coordinates of particles starting from step initial conditions are described by a Airy$_2$ ensemble in the same way as in purely spatial case.

The article is organized as follows. In the section 2 we give definitions and formulate the main results of the paper: the exit probability distribution for trajectories of a finite number of particles at the lattice (theorem 2.4) and its scaling limit (theorem 2.5). In section 3 we reformulate the TASEP in terms of a signed determinantal process and prove theorem 2.4 about the exact form of the correlation function. In section 4 we devote ourselves to an asymptotic analysis of the results of the previous sections, where we prove theorem 2.5.

2. Method and results

2.1. Exit probabilities for particle trajectories on the space–time lattice

To define exit probability for a single particle performing a 1D asymmetric random walk, consider a decomposition of the space–time 2D lattice into two complementary subsets $\Omega \cup \bar{\Omega} = \mathbb{Z}^2$. Given the random walk having started at point $(x^0, t^0) \in \Omega$, the exit probability referring to $\Omega$ is a probability distribution of subsets of the boundary of $\Omega$ from which the particle exits $\Omega$. We will consider only sets having a property that once the particle has exited $\Omega$, it never returns there again. Then the probability of exit from a given point of the boundary does not depend on the global form of the boundary of $\Omega$. Rather it is simply a product of the probability for the particle trajectory to reach this point and the probability that the step from this points results an exit from $\Omega$. This is the case if the boundary of $\Omega$ is defined in the following way.

\textbf{Definition 2.1.} The boundary $\mathcal{B}$ is an infinite countable subset of $\mathbb{Z}^2$

\begin{equation}
\mathcal{B} = \{b(\tau) \in \mathbb{Z}^2 \mid \tau \in \mathbb{Z}\},
\end{equation}

Figure 3. Exit probabilities. The broken lines are the boundaries. An obligatory step forward must be added for exit to happen from vertical parts of the boundaries. The trajectory exiting from the horizontal part can continue in any of two ways.
Universal exit probabilities in the TASEP

with the following staircase-like structure. Let \( b(\tau) = (x, t) \). Then the next point of the boundary will be either

\[
b(\tau + 1) = (x - 1, t)
\]

or

\[
b(\tau + 1) = (x, t + 1),
\]

for any \( \tau \in \mathbb{Z} \). A natural integer variable \( \tau \) increasing along the boundary from northeast to southwest can be chosen as \( \tau = t - x, (x, t) \in \mathcal{B} \).

Note that this construction ensures that the trajectory of a particle started in \( \Omega \) eventually leaves \( \Omega \) through the points of the boundary \( \mathcal{B} \) with probability one and never returns there again. The probability distribution of the sets of these points is the simplest example of the problem we address here. More generally one can consider a collection of embedded sets \( \Omega_1 \subset \Omega_2 \subset \ldots \), with boundaries \( \mathcal{B}_1, \mathcal{B}_2, \ldots \) and look for the joint distribution of successive exits from these boundaries.

The idea of exit probabilities for \( N \) particles undergoing the TASEP evolution on a 1D lattice generalizes the single-particle picture. Now we are interested in how the trajectories of collection of interacting particles exit given sets. The quantity of interest is the joint distribution of subsets of their boundaries at which exits occur. Again, great simplification takes place (i) for such boundaries, that once the trajectories exited them they never return there again. On the other hand we would like that for many particles (ii) all possible configurations of exit points on the collection of boundaries would be assigned a probability measure in the same way as the points of the boundary in single-particle case. The main tool which allows us to work with exit probabilities is the Generalized Green Function (GGF). Unlike the purely spatial Green function used by other authors, the GGF allows us to work directly with space–time point configurations \( (x, t) = ((x_1, t_1), \ldots, (x_N, t_N)) \) belonging to the set of admissible configurations defined by constraints

\[
x_1 > x_2 > \cdots > x_N
\]

\[
t_1 \leq t_2 \leq \cdots \leq t_N.
\]

For \( N \) particles the concept of the boundary can be generalized to \( N \)-boundary, which allows us meet (i) as well as (ii).

**Definition 2.2.** Given boundary \( \mathcal{B} \), the \( N \)-boundary \( \mathcal{B}_N \subset \{1, \ldots, N\} \times \mathbb{Z} \), is defined as a disjoint union of \( N \) copies of \( \mathcal{B} \),

\[
\mathcal{B}_N = \bigcup_{k=1}^{N} \mathcal{B}_k,
\]

where the copy \( \mathcal{B}_k = \{b_k(i)\}_{i \in \mathbb{Z}} \) associated with \( k \)th particle is shifted by \((k - 1)\) steps back with respect to the first one in the horizontal (spatial) direction of space–time plane,

\[
b_k(i) = (x(i) - k + 1, t(i)),
\]

\( k = 1, \ldots, N \).

The \( N \)-boundary is a generalization of the line with fixed time coordinate and of the set of lines with fixed space coordinates, which were the probability spaces used in [17, 18] and in [19] respectively. Having started from an admissible point configuration, \( N \) particle
trajectories will reach the given $N$-boundary after some evolution, traverse it and go from some points of the $N$-boundary to continue the evolution. Then, the non-crossing of the trajectories ensures that the configuration of the departure points at the $N$-boundary is admissible as well.

To specify from which to which point sets the system can pass in the course of the TASEP evolution, we also need a relation between subsets of $\{1, \ldots, N\} \times \mathbb{Z}^2$.

**Definition 2.3.** Let $\Omega, \Omega' \subset \{1, \ldots, N\} \times \mathbb{Z}^2$. We say that relation

$$
\Omega \prec \Omega'
$$

holds, if for any $(x_k, t_k) \in \Omega$ and any $(x_k', t_k') \in \Omega'$

$$(x_k', t_k') \in \{(x, t) : t \geq t_k\} \cup \{(x, t) : x > x_k\}.
$$

Note that the subindices denote the variable from the set $\{1, \ldots, N\}$ and are associated with the number of a particle.

As it was explained in [19], a space–time trajectory of a particle starting from a point preceding to a given boundary, eventually transverses the boundary with probability one. The question we address is: what is the probability for the trajectory to go from a given subset of the boundary? More generally we address the same question to a collection of particles and a set of points at several boundaries.

To be specific, consider the TASEP evolution of $N$ particles governed by the dynamical rules I–III. Let the initial configuration $x^0$ be defined by

$$
x^0_i = -i + 1, \quad i = 1, \ldots, N.
$$

Let us fix a collection of $N$-boundaries, $B^1, \ldots, B^m$, $m > 0$, such that

$$
x^0 \prec B^1 \prec \cdots \prec B^m
$$

and fix the one-particle boundaries $B_{N_1}^k, \ldots, B_{N_l}^k$ within the $N$-boundaries. Here the upper indices $1 = k_1 \leq \cdots \leq k_l = m$ refer to the number of the $N$-boundary, the lower indices, $N_1 \leq \cdots \leq N_l \equiv N$, to the particle number, and $l \geq m$. We suggest that at least one particle is fixed at each $N$-boundary, i.e. either $k_{i+1} = k_i$ or $k_{i+1} = k_i + 1$. We also require that equality $N_i = N_{i+1}$ for some $i$ suggests that $k_{i+1} = k_i + 1$, i.e. two subsequent space–time points chosen for one particle should be put onto subsequent $N$-boundaries, and no other particles with number less than $N_i$ can be fixed at the $N$-boundary $k_i$. Let space–time positions of points $b^k_n(i)$ within the corresponding boundary $B^k_n$ be indexed by index $i \in \mathbb{Z}$ in the same way as in definitions 2.1 and 2.2. The quantity of interest is the joint probability distribution $P(i_1 < a_1, \ldots, i_l < a_l)$ of the points $(b_{N_1}(i_1), \ldots, b_{N_l}(i_l))$ from which the space–time trajectories of particles $N_1, \ldots, N_l$ make steps when leaving the boundaries $(B_{N_1}^1, \ldots, B_{N_l}^1)$ respectively.

The first main result of the present paper can be stated as the following theorem.

**Theorem 2.4.** Under the above conditions the joint probability distribution of exit points is given by the Fredholm determinant

$$
P(i_1 < a_1, \ldots, i_l < a_l) = \det(1 - \eta_a K \eta_a)_{\varphi((b_{N_1}^1, \ldots, b_{N_l}^1))}
$$

$
doi:10.1088/1742-5468/2012/08/P08013
$

12
with the kernel
\[
K(b_{N_i}^{k_i}, b_{N_j}^{k_j}) = \oint_{\Gamma_1} \frac{dw}{2\pi i v} \oint_{\Gamma_0} \frac{dw}{2\pi i w} \
\times \left( \frac{(1-p((w-1)/v))^{k_i}}{(1-p((v-1)/v))^{k_j}}((w-1)^{N_i}/(v-1)^{N_j})(w^{x_{N_i}}/v^{x_{N_j}}) \right) \
\times \left( \frac{1}{w-v}(1/v+1/\pi_2-1) \right)
\]
\[
- \text{I}(N_2 > N_1) \oint_{\Gamma_{0,1}} \frac{dw}{2\pi i w^2} \left( \frac{1}{w-1} \right)^{N_j-N_i}(1/v+1/\pi_2-1).
\]
where \( \eta_a = \prod(i_1 \geq a_1) \times \cdots \times \prod(i_m \geq a_m) \), \( b_{N_i}^{k_i} = (x_{N_i}, t_{N_i}) \in \mathcal{B}_{N_i}^{k_i} \), \( i, j = 1, \ldots, l \) and \( \pi_2 = 1, p \) is the probability of step from the boundary \( \mathcal{B}_{N_j}^{k_j} \) at point \( b_{N_j}^{k_j} \).

2.2. Scaling limit of correlation functions

In the large scale the boundaries can be treated as approximations of continuous differentiable paths in the space–time plane. Consider a scaling limit associated with sending to infinity a large parameter \( L \to \infty \), as the time–space coordinates and particle numbers measured at \( L \)-scale are fixed: \( x/L, n/L, t/L = \text{const} \). Let us introduce the variable change \((x, t) \to (\chi, \theta)\):
\[
\tau = t - x = L \chi \quad \text{(2.14)}
\]
\[
t + x = L \zeta(\chi, \theta). \quad \text{(2.15)}
\]
As was noted earlier, the variable (2.14) naturally enumerates points at the boundary. Correspondingly, the function \( \zeta(\chi, \theta) \) defines a one-parameter family of curves spanning the whole space–time plane as \( \theta \) varies in \( \mathbb{R} \). As the parameter \( \chi \) runs in \( \mathbb{R} \), it defines a point at a particular curve corresponding to some fixed value of \( \theta \). The properties of \( \zeta(\chi, \theta) \) follow from the properties of boundaries. Specifically, we suggest that
\[
\left| \frac{\partial \zeta(\chi, \theta)}{\partial \chi} \right| \leq 1 \quad \text{(2.16)}
\]
and
\[
\left( \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \chi} \right) \zeta(\chi, \theta) \geq 1. \quad \text{(2.17)}
\]
We now suppose that for \( k = 1, \ldots, m \) the boundaries \( \mathcal{B}_k^1 \) approximate the curves corresponding to fixed set \( (\theta_1, \ldots, \theta_m) \):
\[
b_k^1([L\chi]) = L \left( \frac{\zeta(\chi, \theta_k) - \chi}{2} \right) + o(L^\sigma), \quad \text{(2.18)}
\]
where the notation \([ \cdot ]\) is for integer part of a real number and the correction term should not contribute on a characteristic fluctuation scale, i.e. \( \sigma = 1/3 \). For technical purposes we will suggest that the correction term is uniform over the boundary. These boundaries correspond to the first particle. For a general particle with number \( n = [L \nu] \) we have to consider the boundary \( \mathcal{B}_n^k \) shifting the spatial coordinate by \( n - 1 \) steps backward:
\[
b_n^k([L\chi]) = L \left( \frac{\zeta(\chi, \theta_k) - \chi}{2} - \nu \right) + o(L^\sigma). \quad \text{(2.19)}
\]
Recall that on the large scale, \( x \sim n \sim t \sim L \to \infty \), the trajectories of particles are deterministic, defined by the relation (1.5). In terms of the new variables the relation turns into

\[
\sqrt{p(\zeta(\chi, \theta) + \chi)} - \sqrt{\zeta(\chi, \theta) - \chi} - \sqrt{2q}\nu = 0,
\]

which uniquely fixes the value of \( \chi \) given those of \( \theta \) and \( \nu \), provided that the corresponding curve passes through the rarefaction fan defined by

\[
\chi \leq \zeta(\chi, \theta) \leq 1 + p - p\chi.
\]

Let us consider a path in the \( \theta - \nu \) plane:

\[
\theta = \theta(r), \quad \nu = \nu(r), \quad r \in \mathbb{R},
\]

with differentiable functions \( \theta(r) \) and \( \nu(r) \), such that

\[
\frac{\partial \theta}{\partial r} \geq 0, \quad \frac{\partial \nu}{\partial r} \leq 0
\]

and

\[
\frac{\partial \theta}{\partial r} - \frac{\partial \nu}{\partial r} \geq 1.
\]

We select \( m \) points at the path, \( r = r_1, \ldots, r_m \), so that the integers \( N_1, \ldots, N_m \) from theorem 2.4 are given by \( N_i = [L\nu(r_i)] \), and \( \theta_i = \theta(r_i) \). The inequalities (2.23) and (2.24) then guarantee that the constraints on \( k_1, \ldots, k_m \) and \( N_1, \ldots, N_m \) from theorem 1.1 are satisfied and together with non-crossing of particle trajectories ensure that points of this path accessible for particle trajectories with nonzero probability form space-like configurations.

Substituting functions \( \theta(r) \) and \( \nu(r) \) into (2.20) we obtain an equation, which, given \( r \), can be resolved with respect to \( \chi \). For a given path a unique solution exists for any \( r \) within the range in which the boundary corresponding to \( \theta(r) \) passes through the rarefaction fan (2.21). This solution is a monotonous function of \( r \), which we denote \( \chi(r) \). It defines the macroscopic deterministic location of the point, from where a given particle exits given boundary, see figure 5. We now turn to the fluctuations of these points referred to the boundaries and particle numbers separated by distances of the order of the correlation length from each other. Suppose that

\[
r_i = r_0 + u_iL^{-1/3}.
\]

The corresponding values of \( \chi \) are given by their deterministic parts \( \chi(r_i) \) plus a random variable of the order of the fluctuation scale

\[
\chi_i = \chi(r_i) + \xi_L(u_i)L^{-2/3}.
\]

In what follows we show that the random variable \( \xi_L(u) \) converges to the universal Airy \( 2 \) process for a class of boundaries, which can be approximated by (2.18) and (2.19).

**Theorem 2.5.** The following limit holds in a sense of finite-dimensional distributions:

\[
\lim_{L \to \infty} \xi_L = \kappa_L A_2(\kappa_L u),
\]

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where $A_2$ is the Airy$_2$ process characterized by multipoint distributions:

$$\text{Prob}(A_2(u_1) < s_1, \ldots, A_2(u_m) < s_m) = \det(\mathbb{1} - \eta K_{\text{Airy}_2, \eta})_{L^2(n_1, \ldots, n_m) \times \mathbb{R}}$$

(2.28)

where in the rhs we have the extended Airy kernel,

$$K_{\text{Airy}_2}(\xi_1, \xi_2; \xi_2, \xi_2) = \begin{cases} 
\int_0^\infty d\eta \exp(\eta \xi_2 - \xi_1) A(\lambda + \xi_1) A(\lambda + \xi_2), & \xi_2 \leq \xi_1 \\
- \int_{-\infty}^0 d\eta \exp(\eta \xi_2 - \xi_1) A(\lambda + \xi_1) A(\lambda + \xi_2), & \xi_2 > \xi_1 
\end{cases}$$

(2.29)

The model-dependent constants $\kappa_c$ and $\kappa_f$ defining the correlation and fluctuation scales respectively are given by

$$\kappa_c = \frac{p^{1/6}}{2\gamma^{1/6} \omega^{1/6}} \left( \sqrt{\omega - \sqrt{\gamma}} \right)^{-1/3} \left( \sqrt{p\omega - \sqrt{\gamma}} \right)^{-1/3} \times \left[ g + p'(r_0) (\zeta(r_0) - \chi(r_0)) \zeta(0,1)(r_0) \right] (\sqrt{p\omega - \sqrt{\gamma}})^{-1}$$

$$\kappa_f = \frac{(\sqrt{p\omega - \sqrt{\gamma}})^{1/3} \sqrt{p\omega - \sqrt{\gamma}}}{2p^{1/6} \omega^{1/3} \gamma^{1/3}} \times \left( \sqrt{\omega - \sqrt{\gamma}} \right)^{2/3}$$

(2.30)

$$\zeta(r) \equiv \zeta(\theta, \chi), \zeta(1,0)(r_0) \equiv \zeta(0,1)(r_0)$$

(2.31)

where we denote $\zeta(r) \equiv \zeta(\theta(r), \chi(r))$, $\zeta(1,0)(r_0)$ as the derivative of the function $\zeta(\theta, \chi)$ with respect to the first (second) argument at the point $(\theta(r_0), \chi(r_0))$ and parameters $\gamma$ and $\omega$ are those defined in (1.2), $\gamma = (\zeta(r_0) - \chi(r_0))/2$ and $\omega = (\zeta(r_0) + \chi(r_0))/2$.

The non-universal constants $\kappa_f$ and $\kappa_c$ are the most general ones for the TASEP with backward update. They depend not only on the macroscopic space–time location defined by $\zeta(r_0)$ and $\chi(r_0)$, but also on the local slope and local density of the boundaries at this point via the derivatives $\zeta(0,1)(r_0)$ and $\zeta(1,0)(r_0)$ respectively. Particular cases studied before can easily be restored from the expressions obtained. For example, for a purely spatial boundary used for measuring particle coordinates at fixed time we can take $\zeta(\theta, \chi) = 2t - \chi$, while the case of current correlation functions $[19]$ corresponds to $\zeta(\theta, \chi) = 2x + \chi$. For the space-like correlation functions of particle coordinates studied in $[17, 18]$ we take $\zeta(\theta, \chi) = 2\theta - \chi$, and the tagged particle case $[16]$ corresponds to $\nu'(r) = 0$.

3. Determinantal point processes on the boundaries

3.1. Single N-boundary

We first introduce the Generalized Green Function (GGF) using the determinantal formula proposed in $[23]$ and proved in $[19]$, which generalizes the formulas of simple Green function obtained in $[26]$ for continuous time TASEP and generalized to the backward sequential update in $[25]$. 

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Given two admissible configurations
\[ b^0 \equiv (x^0, t^0) = ((x_1^0, t_1^0), \ldots, (x_N^0, t_N^0)) \]
and
\[ b \equiv (x, t) = ((x_1, t_1), \ldots, (x_N, t_N)) \]
we define
\[ G(b|b^0) = \det [F_{j-i}(b_i - b_j^0)]_{1 \leq i,j \leq N}. \tag{3.1} \]
where \((b_i - b_j^0) = (x_i - x_j^0, t_i - t_j^0)\) is componentwise extraction and
\[ F_n(x, t) = \begin{cases} \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{dw}{w} \left( q + \frac{p}{w} \right)^t (1 - w)^{-n} w^x, & t \geq 0 \\ 0, & t < 0. \end{cases} \tag{3.2} \]
For point \(b(i) = (x, t) \in \mathcal{B}\) at the boundary, we introduce an exit probability
\[ \pi^\mathcal{B}(b(i)) = \begin{cases} p, & \text{if } b(i+1) = (x, t+1) \\ 1, & \text{if } b(i+1) = (x+1, t), \end{cases} \tag{3.3} \]
and for \(N\)-point configuration \(b \in \mathcal{B}\)
\[ \pi^\mathcal{B}(b) = \prod_{k=1}^{N} \pi^\mathcal{B}_k(b_k), \tag{3.4} \]
where the subscript \(k\) specifies a boundary within the \(N\)-boundary, or the associated particle. The function
\[ G(b|b^0) \equiv \pi^\mathcal{B}(b)G(b|b^0) \tag{3.5} \]
gives the probability for the space–time trajectories of particles to go away from the boundary via the points of \(b\), given they started from \(b^0\).

We now show that this probability can be reinterpreted in terms of an auxiliary signed determinantal point process on \(\mathcal{B}\). Consider a signed measure on \(\mathbb{Z}_{\geq \tau_0} \times \{1, \ldots, N\}\),
\[ \mathcal{M}(\mathcal{T}) = \frac{1}{Z_N} \prod_{n=0}^{N-1} \det [\phi_n(\tau^{n+1}_i, \tau^{n+1}_j)]_{i,j=1}^{n+1} \det [\Psi^N_{N-1}(\tau^N_j)]_{i,j=1}^{N}, \tag{3.6} \]
assigned to the sets of the form
\[ \mathcal{T} = \bigcup_{1 \leq n \leq N} \{ \tau^n_1, < \tau^n_{n-1}, < \cdots, < \tau^n_1 \} \subset \mathbb{Z}_{\geq \tau_0} \times \{1, \ldots, N\}. \tag{3.7} \]
Here we define the function
\[ \phi_n(z, y) = \begin{cases} \pi^\mathcal{B}_{n+1}(b_{n+1}(y)), & y \geq z \\ 0, & y < z \end{cases} \tag{3.8} \]
for \(x, y \in \mathbb{Z}_{\geq \tau_0}\) and \(b_k(y) \in \mathcal{B}_k, k = 1, \ldots, N\) and the function
\[ \Psi^N_k(t) = (-1)^k \tilde{F}_{-k}(b_N(t) - b_{N-k}^0), \tag{3.9} \]
where
\[
\tilde{F}_n(x, t) = \frac{1}{2\pi i} \oint_{\tau_0} \frac{dw}{w} (q + \frac{p}{w})^t (1 - w)^{-n} w^\tau.
\]

The integral representation holds for \( t \in \mathbb{Z} \). This is unlike \( F_n(x, t) \), which coincides with \( \tilde{F}_n(x, t) \) when \( t \geq 0 \) and vanishes at \( t < 0 \), see (3.2). The numbers \( \tau_i \) are integers bounded by number \( \tau_0 \) from below. The number \( \tau_0 \) is chosen so that \( \Psi_k^N(\tau_0) = 0 \), which is always possible by construction of the boundaries.

We also introduce fictitious variables \( \tau_{n-1}^n, 1 \leq n \leq N \), which are fixed to \( \tau_{n-1}^n = \tau_0 \). Thus for any \( \tau_i^j \in \mathbb{Z}_{\geq \tau_0} \) we have
\[
\phi_n(\tau_{n+1}^n, \tau_j^{n+1}) \equiv \pi^{B_{n+1}}(b_{n+1}(\tau_j^{n+1}))
\]
for \( j = 1, \ldots, n + 1 \). The numbers \( \tau_i^n, i = 1, \ldots, n \) are mapped to the sites \( b_n(\tau_i^n) \) on \( B_n \). Therefore the measure on the \( N \)-boundary \( B \) is naturally defined as pushforward of \( \mathcal{M}(T) \) under this mapping.

One can consider \( b_n(\tau_i^n), 1 \leq j \leq n \leq N \), as coordinates of auxiliary fictitious particles indexed by \( j \) leaving at the boundaries \( B_n \). These particles evolve as shown in figure 6. First, \( N \) particles arrive from their initial state encoded in the functions \( \Psi_k^N(t) \) at the points of boundary \( B_N \) with numbers \( \tau_1^N, \ldots, \tau_N^N > \tau_0 \). Then, they jump to the sites of the boundary \( B_{N-1} \) with the same numbers, and go up along the boundary \( B_{N-1} \) (from southwest to northeast, so that the number \( \tau \) indexing position at the boundary decreases) any distance respecting mutual non-crossing of particle trajectories. The weight of the jump between the boundaries, outgoing from a site \( b \), is \( \pi^{B_N}(b) \) and the weight of going along the boundary is 1 independently of the distance. The last (\( N \))th particle is forced to go to the reservoir (\( \tau_{N-1}^N = \tau_0 \)) and disappear. The final positions of the other particles at the boundary \( B_{N-1} \) are denoted \( \tau_1^{N-1}, \ldots, \tau_{N-1}^{N-1} \), from which they jump to the boundary \( B_{N-2} \) with the weights \( \pi^{B_{N-1}}(\tau_1^{N-1}), \ldots, \pi^{B_{N-1}}(\tau_{N-1}^{N-1}) \), etc. The process is repeated until the particle number 1 jumps from the point \( b_1(\tau_1^1) \) of \( B_1 \) and disappears. This picture generalizes the auxiliary processes described for the cases of constant time \([6, 7, 17, 18]\) and fixed spatial coordinates \([19]\) to the case of general boundaries.

The fictitious particles are similar to vicious walkers (or free fermions), which can be seen from the Karlin–McGregor–Lindström–Gessel–Viennot \([27]–[29]\) determinantal form of the transition weights entering the product (3.6) that ensure nonintersection of their space–time trajectories. The last determinant can be treated as integrated with given initial distribution. Such a free fermionic structure allows the calculation of the correlation functions for fictitious particles, which turns out to be determinantal. On the other hand, below we show that the joint distribution of \( N \) positions of the first fictitious particle obtained by integration of the measure (3.6) over the positions of the other particles coincides with the Green function of TASEP. Thus the problem of correlations in TASEP can be reduced to calculation of correlations between noninteracting mutually avoiding fictitious particles.

To show that the GGF can be interpreted in terms of the measure \( \mathcal{M}(T) \) we prove proposition 3.1.

**Proposition 3.1.** Given \( N \)-boundary \( B \), initial and final configurations \( b^0 \prec B \) and \( b \subset B \) respectively, GGF \( \mathcal{G}^B(b|b^0) \) associated with the boundary \( B \) is a marginal of the
measure $\mathcal{M}$:

$$
\mathcal{G}^B(b|b^0) = \mathcal{M}\left(\bigcup_{k=1}^{N} \{\tau^b_k = i_k\}\right),
$$

(3.12)

where $i_1, \ldots, i_N$ determine the location of the points of $b = (b(i_1), \ldots, b_N(i_N))$ at corresponding boundaries within the $N$-boundary.

To prove this statement, one represents the GGF as a sum over the boundary points in a way similar to that used for space variables in [6, 7, 17, 18] and for time variables in [5, 19]. The proof of the summation uses contiguous relations for the values of the function $\tilde{F}_n(b(\tau))$ at adjacent points of the boundary, which unify similar relations for space and time variables.

Lemma 3.2. Let $b_k(\tau)$ be the point at the boundary $B_k$ within the $N$-boundary $B$. Then the contiguous relations hold for the function $\tilde{F}_n(b_k(\tau))$

$$
\pi^B(b_k(\tau))\tilde{F}_n(b_k(\tau)) = \tilde{F}_{n+1}(b_{k-1}(\tau + 1)) - \tilde{F}_{n+1}(b_{k-1}(\tau)).
$$

(3.13)

Proof. The relation to be proved is in fact two contiguous relations for the function $\tilde{F}_n(x,t)$,

$$
\tilde{F}_n(x,t) = \tilde{F}_{n+1}(x,t) - \tilde{F}_{n+1}(x + 1,t),
$$

(3.14)

$$
p\tilde{F}_n(x,t) = \tilde{F}_{n+1}(x + 1,t + 1) - \tilde{F}_{n+1}(x + 1,t),
$$

(3.15)

as one relation. The two latter relations follow from the integral representation of the function $\tilde{F}_n(x,t)$. □

Then we have:

Lemma 3.3. Given $N$-boundary $B$, initial and final configurations $b^0 \prec B$ and $b \subset B$ respectively, the function $\mathcal{G}(b|b^0)$ can be represented as a sum:

$$
\mathcal{G}(b|b^0) = \sum_D \prod_{1 \leq n \leq N} \pi^B_n(b_n(\tau^n))
\times (-1)^{N(N-1)/2} \det[\tilde{F}_{-N+1+i}(b_N(\tau^N_{j+1}) - b^0_{N-j})]_{i,j=0}^{N-1}
$$

(3.16)

where $b_j(\tau^j_i) \equiv b_j, j = 1, \ldots, N$, and the summation variables take their values in the domain

$$
D = \{\tau^j_i \in \mathbb{Z}_{\geq \tau_0}, 2 \leq i \leq j \leq N | \tau^j_i \geq \tau^{j-1}_i, \tau^j_i > \tau^{j+1}_{i+1}\}.
$$

(3.17)

Proof. Using the contiguous relation (3.13) the proof just follows the similar proofs in [5–7, 17–19]. Note that the lower summation bound $\tau_0$ is chosen such that the functions $\tilde{F}$ under the determinant vanish at this point. Indeed this is true for $\tilde{F}_n(x,t)$ when $x > t$. By construction of boundaries it is always possible to find suitable $\tau_0$ to ensure this inequality. To be specific we choose the maximal of these numbers. □

To complete the proof of proposition (3.1) we need to show that the summation over the domain $D$ can be replaced by the summation over the sets of the form (3.7).

Lemma 3.4. The domain of summation in (3.16) can be replaced by

$$
\tilde{D} = \{\tau^j_i \in \mathbb{Z}_{\geq \tau_0}, 2 \leq i \leq j \leq N | \tau^j_i < \tau^{j}_{i-1}\}.
$$

(3.18)

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Proof. Apparently the inequalities in (3.17) suggest those in (3.18). We also need to show the converse: the measure (3.6) is zero everywhere in \( D \) unless the inequalities from (3.17) are satisfied. The statement can be proved by reproducing the arguments from [19]. \( \square \)

To find the correlation functions of the TASEP we first calculate the correlation functions of the measure \( \mathcal{M}(T) \). The functional form of \( \mathcal{M}(T) \) suggests that the correlation functions are determinantal. Derivation of the correlation kernel was explained in great detail in [7]. To proceed with the calculation, we introduce convolution

\[
\phi^{(n_1,n_2)}(x,y) = \begin{cases} 
(\phi_{n_1} \ast \phi_{n_1+1} \ast \cdots \ast \phi_{n_2-1})(x,y), & n_1 < n_2 \\
0, & n_1 \geq n_2,
\end{cases}
\]

(3.19)

where \((a \ast b)(x,y) = \sum_{z \in \mathbb{Z} \geq 0} a(x,z)b(z,y)\), and

\[
\Psi_{n-j}^{n}(\tau) = (\phi^{n,N} \ast \Psi_{N-j}^{n})(\tau).
\]

(3.20)

Note that in terms of the coordinates of fictitious particles function \( \phi^{(n_1,n_2)}(x,y) \) is the transition weight between points at the boundaries \( \mathcal{B}_{n_1} \) and \( \mathcal{B}_{n_2} \). Hence, the points parameterized by the variables \( x \) and \( y \) in (3.19) live at \( \mathcal{B}_{n_1} \) and \( \mathcal{B}_{n_2} \), respectively, while the argument of \( \Psi_{n-j}^{n}(\tau) \) in (3.20) lives on the boundary \( \mathcal{B}_{n} \).

Consider functions

\[
\{ (\phi_0 \ast \phi^{(1,n)})(\tau_0^0, \tau), \ldots, (\phi_{n-2} \ast \phi^{n-1,n})(\tau_{n-1}^{n-2}, \tau), \phi_{n-1}(\tau_{n-1}^{n-1}, \tau) \}.
\]

(3.21)

They are linearly independent and hence can serve as a basis of an \( n \)-dimensional linear space \( V_n \). We construct another basis of \( V_n \), \( \{ \Phi_j^{n}(\tau), j = 0, \ldots, n - 1 \} \), which is fixed by the orthogonality relations

\[
\sum_{\tau \in \mathbb{Z}} \Phi_i^{n}(\tau)\Phi_j^{n}(\tau) = \delta_{i,j}.
\]

(3.22)

Then, under the

Assumption (A). \( \phi_n(\tau_{n+1}^n, \tau) = c_n \Phi_0^n(\tau) \) with some \( c_n \neq 0, n = 1, \ldots, N \),

the kernel has the form

\[
K(n_1, \tau_1; n_2, \tau_2) = -\phi^{(n_1,n_2)}(\tau_1, \tau_2) + \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1}(\tau_1)\Psi_{n_2-k}^{n_2}(\tau_2).
\]

(3.23)

Applying repeatedly the convolution with \( \phi_{N-1}, \ldots, \phi_j \) to \( \Psi_{N-j}^{n}(\tau) \) we obtain.

Lemma 3.5. Given \( N \)-boundary \( \mathcal{B} \), the functions \( \Psi_j^{n}(\tau) \) have the following integral representation.

\[
\Psi_j^{n}(\tau) = \oint_{\Gamma_{0,1}} \frac{dw}{2\pi i} \left( 1 - p \frac{w-1}{w} \right)^{t_{n}(\tau)-t_{n-k}^{0}} (w-1)^{k} w^{x_{n}(\tau)-x_{n-k}^{0}-1}.
\]

(3.24)

\( ^{3} \) For a single \( N \)-boundary this comment is not essential as the points of different boundaries within the same \( N \)-boundary have the same dependence on the index \( \tau \) (specifically the value of \( \pi(b(\tau)) \)). Therefore, one could stick to, for example, the boundary \( \mathcal{B}_{1} \), shifting the spatial coordinate accordingly. Later, however, when we consider a sequence of \( N \)-boundaries, the information on what boundary the functions under consideration refer to becomes important.
The contour of integration $\Gamma_{0,1}$ encircles the poles $w = 0, 1$, leaving all the other singularities outside.

To find basis $V_n$, we have to specify initial conditions. For usual step initial conditions, the orthogonalization can easily be performed.

**Lemma 3.6.** Given step initial conditions, $b_k^0 = (-k+1,0)$ for $k = 1, \ldots, N$, the functions $\Psi_k^n(\tau)$ and $\Phi_j^n(\tau)$ satisfying (3.22) are given by

$$
\Psi_k^n(\tau) = \int_{\Gamma_{0,1}} \frac{dw}{2\pi i} \left( 1 - p \frac{w - 1}{w} \right)^{t_n(\tau)} \left( \frac{w - 1}{w} \right)^{k} w^{x_n(\tau) + n - 2},
$$

$$
\Phi_j^n(\tau) = \int_{\Gamma_{0,1}} \frac{dv}{2\pi i} \left( 1 - p \frac{v - 1}{v} \right)^{-t_n(\tau)} \frac{(v - 1)^{-j} 1_{t_n(\tau) - n}}{(1/\pi^B(b(\tau)) - 1)v + 1)},
$$

where the contour of integration $\Gamma_1$ encircles the pole $v = 1$ anticlockwise.

**Proof.** The function $\Psi_k^n(\tau)$ is obtained from (3.24) by an explicit substitution of the step initial conditions. To prove the orthogonality conditions (3.22) one must evaluate the sum $\sum_{\tau \in \mathbb{Z}} \Psi_k^n(\tau) \Phi_j^n(\tau)$. This is done by an interchange of summation and integration. After successively summing the geometric progressions for space-like and time-like parts of the boundary and taking into account the pole at $v = w$, we obtain the desired result. To provide the convergence of the resulting sum we note that the choice of contours ensures convergence of the sum for $\tau \to \infty$, while at the lower limit the sum is truncated at $\tau = s$, so that $x_n(s) - t_n(s) - 1 = 0$. Obviously $\Psi_k^n(\tau) = 0$ for $\tau < s$ because no poles remain inside the integration contour for $k \geq 0$. □

Note that the form of $\Phi_j^n(\tau)$ depends on whether the site $b(\tau)$ belongs to a time-like or space-like part of the boundary, which is reflected in the term containing the exit probability in the denominator. Now we note that the assumption $A$ is fulfilled,

$$
\Phi_0^n(\tau) = \pi^B_n(b(\tau)) = \phi_n(x_{n+1}, \tau),
$$

and we can write the kernel. The summation in (3.23) yields

$$
\sum_{k=1}^{\infty} \Psi_{n_1-k}(\tau_1) \Phi_{n_2-k}(\tau_2) = \int_{\Gamma_1} \frac{dv}{2\pi i} \int_{\Gamma_0} \frac{dw}{2\pi i w} \left( 1 - p \frac{(w - 1)/w}{(v - 1)/v} \right)^{t_n(\tau_1)+t_n(\tau_2)} \left( \frac{(w - 1)^{n_1}(w - 1)^{n_2}}{v^{x_{n_1}(\tau_1) + x_{n_2}(\tau_1)}} \right) \left( \frac{(w - 1)}{v + 1/\pi_2 - 1} \right)
$$

$$
(3.28)
$$

where $\pi^B_2 = \pi^B(b(\tau_2))$.

Observe that the function $\phi_n(x, y)$ can be written in the form

$$
\phi_n(\tau_1, \tau_2) = \int_{\Gamma_{0,1}} \frac{dw}{2\pi i w^2} \frac{(w - 1) (1 - p ((w - 1)/w))^{t_{n_1}(\tau_1) - t_{n_1+1}(\tau_2) + x_{n_1}(\tau_1) - x_{n_1+1}(\tau_2)}}{(w - 1)(1/v + 1/\pi_2 - 1)}. \quad (3.29)
$$

After a few convolutions we have

$$
\phi^{(n_1, n_2)}(\tau_1, \tau_2) = 1 (n_2 > n_1)
$$

$$
\times \int_{\Gamma_{0,1}} \frac{dw}{2\pi i w^2} \frac{(w - 1)^{n_2-n_1}(1/v + 1/\pi_2 - 1)}{(w - 1)^{n_1}(1/v + 1/\pi_2 - 1)}. \quad (3.30)
$$
Then we obtain:

**Proposition 3.7.** The correlation kernel of the measure $\mathcal{M}$, (3.6), is

$$K(n_1, \tau_1; n_2, \tau_2) = \int_{\Gamma_0} \frac{dv}{2\pi iv} \int_{\Gamma_0, 2\pi iv} \frac{dw}{2\pi iv} \times$$

$$\frac{(1-p((w-1)/w))^{\ell_1((\tau_1))}((w-1)^{\ell_2((\tau_2))})(w-1)^{n_1}/(v-1)^{n_2}((w-1)^{n_1}/(v-1)^{n_2})(w^{x_{\tau_1}(\tau_1)}/v^{x_{\tau_2}(\tau_1)})}{(w-v)(1/v + 1/\pi_2 - 1)}$$

$$- \mathbb{1}(n_2 > n_1) \oint_{\Gamma_0, 2\pi iv} \frac{dw}{2\pi iv} \frac{(1-p((w-1)/w))^{\ell_1((\tau_1))}^{\ell_2((\tau_2))}w^{x_{\tau_1}(\tau_1)-x_{\tau_2}(\tau_2)}}{(w-1)^{n_2-n_1}(1/w + 1/\pi_2 - 1)}$$

(3.31)

where $\pi_2 \equiv \pi^B(b(\tau_2))$ and $x_n(\tau) = x(\tau) + n - 1$.

Determinants of the above correlation kernel yield the correlation functions of the measure $\mathcal{M}$, i.e. probabilities of point sets $T$, (3.7), having any given subsets. Then, using the inclusion–exclusion principle, we can write down the joint distribution

$$P = \mathcal{M}(T \supset \{\tau_1^{n_1} \leq a_1\} \cap \cdots \cap \{\tau_m^{n_m} \leq a_m\})$$

(3.32)

of sequences $\{\tau_1^{n_1}, \ldots, \tau_k^{n_k}\}$ for any fixed collection $1 \leq n_1 \leq \cdots \leq n_m \leq N$, where $1 \leq m \leq N$, in the form

$$P = \det(1 - \chi_n K \chi_n)_{t \in \{n_1, \ldots, n_m\} \times \mathbb{Z}}$$

(3.33)

where the Fredholm determinant is defined as a sum

$$P = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1, \ldots, i_n=1}^{\ell_1} \sum_{a_1}^{\ell_2} \cdots \sum_{a_n}^{\ell_m} \det\{K(n_{i_k}, \tau_{k}; n_{i_j}, \tau_{j})\}_{k,j=1}^{n}$$

(3.34)

and $\chi_n(\tau_i)(t) = \mathbb{1}(t > a_i)$. This distribution is the TASEP correlation function of interest,

$$P \equiv \text{Prob}\left(\{\tau_1 \leq a_1\} \cap \{\tau_2 \leq a_2\} \cap \cdots \cap \{\tau_m \leq a_m\}\right)$$

(3.35)

and (3.33) is a particular case of theorem 2.4 applied to the case of single $N$-boundary. Remarkably, the GGF allowed us to treat very wide range of space–time point configurations ‘in one go’, in the same way as the fixed time and space cases were treated in [7] and [19], respectively. Any admissible point configuration can be processed in this way, when put to a suitable boundary. The set of admissible configurations, however, does not exhaust all the possibilities. It turns out that the time ordering constraint (2.5) can also be removed. To this end we apply a multicascade procedure, similar to that used in [17], to a sequence of $N$-boundaries.

### 3.2. Multiple $N$-boundary case

Consider $m$ mutually distinct $N$-boundaries, $\mathcal{B}^1 < \cdots < \mathcal{B}^m$. In this section we derive the joint $l$-point probability distribution for positions $b_{N_i}^{k_i}$ at which the trajectories of particles $N_i$ depart from boundaries $\mathcal{B}_{N_i}^{k_i}$, where $i = 1, 2, \ldots, l$, and the indices $N_i$ and $k_i$ satisfy the assumptions of theorem 2.4. We suggest that the space–time points within each $N$-boundary are indexed independently by the indices $\tau_i(k)$, where the subindex $i = 1, \ldots, N$ stands for the number of the boundary $\mathcal{B}_i^k$ within the $N$-boundary $\mathcal{B}_i^k$ and...
the argument \( k = 1, \ldots, m \) indexes the \( N \)-boundaries. According to definitions 2.1 and 2.2, we first independently define an indexing order \( b^k(\tau(k)) \) for the first boundaries within each \( N \)-boundary and then translate it to the other \( N-1 \)-boundaries by the corresponding left shifts.

Given a fixed collection of integers \( a_1, \ldots, a_l \), we are looking for the joint probability

\[
P \equiv \text{Prob} \left( \bigcap_{i=1}^{l} \{ b_{N_i}^{k_i} = b_{N_i}^{k_i}(\tau_{N_i}(k_i)) \} : \{ \tau_{N_i}(k_i) \leq a_i \} \right) \]

for trajectories of particles \( N_1, \ldots, N_l \) to leave corresponding boundaries via points \( b_{N_i}^{k_i} \), located above (in terms of the corresponding indices \( \tau_{N_i}(k_i) \)) the sites \( b_{N_i}^{k_i}(a_i) \).

Similarly to the case of single \( N \)-boundary, our strategy is to represent this probability distribution as a marginal of a signed determinantal measure on a larger set. Suppose that the set \( (k_1, \ldots, k_l) \) is of the form

\[
k_{p_1} = \cdots = k_{p_2-1} = 1, \\
k_{p_2} = \cdots = k_{p_3-1} = 2, \\
\vdots \\
k_{p_m} = \cdots = k_l = m,
\]

which defines a collection of \( m \) integers \( 1 \equiv p_1 \leq \cdots \leq p_m \leq l \). The quantity of interest can be given in terms of measure \( \mathcal{M}(\cdot) \) on point sets

\[
\mathcal{T} = \bigcup_{1 \leq k \leq m} \mathcal{T}(k),
\]

where

\[
\mathcal{T}(k) = \bigcup_{N_{p_k} \leq n \leq N_{p_{k+1}}} \mathcal{X}_n(k)
\]

and

\[
\mathcal{X}_n(k) = \{ \{ \tau_{n}(k), \tau_{n-1}(k), \ldots, \tau_{1}(k) \} \subset \mathbb{Z}_{\geq \tau_0(k)} \}.
\]

Then for \( 1 \leq k \leq m \) the collections of integers \( (\tau_{n}(k), \ldots, \tau_{1}(k)) \) define point configurations \( (b_{N_1}^{k_1}(\tau_{n}(k)), \ldots, b_{N_l}^{k_l}(\tau_{n}(k))) \subset \mathcal{B}_n \), which can be treated as coordinates of fictitious particles similarly to the single \( N \)-boundary case. The pushforward of the measure \( \mathcal{M} \) under this mapping is a measure on the collection of \( N \)-boundaries \( \mathcal{B}_1, \ldots, \mathcal{B}_m \). An explicit form of this measure is as follows.

\[
\mathcal{M}(\mathcal{T}) = N^{-1} \det[\psi_{N_{1}^{-1}}(\tau_N(1))]_{1 \leq k, l \leq N_1} \\
\times \prod_{i=2}^{m} \det[\mathcal{F}_{i-1}(\tau_{N_{i}}(i), N_{i})_{1 \leq k, l \leq N_{i}}] \\
\times \prod_{n=N_{P_{i-1}+1}}^{N_{p_i}} \det[\phi_{n}(\tau_{N_{i}}(i-1), \tau_{N_{i}}(i))]_{1 \leq k, l \leq n},
\]

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where we define functions
\[ \Psi_{N_i}^{N_i-l}(\tau_{N_i}^{N_i}(1)) = (-1)^{N_i-l}\tilde{F}_{-N_i+l}(b_{N_i}^0(\tau_{N_i}^{N_i}(1)) - b_0^0) \]
\[ \mathcal{F}_{i,i-1}(\tau_i^{N_i}(i), \tau_k^{N_i}(i-1)) = \pi^{B_{i-1}}(b_{i-1}^0(\tau_k^{N_i}(i-1))) \]
\[ \times \tilde{F}_0(b_i^0(\tau_i^{N_i}(i)) - b_0^0(\tau_k^{N_i}(i-1))), \]
\[ \phi_n(\tau_i^n(i), \tau_k^{n+1}(i)) = \begin{cases} \pi^{B_{i+1}}(b_{i+1}^0(\tau_k^{n+1}(i))), & \tau_k^{n+1}(i) \geq \tau_i^n(i), \\ 0, & \tau_k^{n+1}(i) < \tau_i^n(i), \end{cases} \]
\[ \phi_n(\tau_{n+1}^n(i), \tau_k^{n+1}(i)) = \pi^{B_{i+1}}(b_{i+1}^0(\tau_k^{n+1}(i))), \]
\[ N \text{ is a normalization constant and for } 1 \leq i \leq m \]
\[ b_k^0(\tau_k(i-1)) = \left( x_k(i-1) + \frac{1 - \pi^{B_{k-1}}(b_{k-1}^0)}{1 - p}, t_k(i-1) + \frac{1 - \pi^{B_{k-1}}(b_{k-1}^0)}{1 - p} \right). \]  

(3.43)

Lower cutoff \( \tau_0(k) \) is separately chosen for every \( N \)-boundary \( B_k \) in such a way, that any transitions to these points have zero measure. Specifically, as in the single \( N \)-boundary case considered in section 3.1 \( \Psi_0^1(\tau) = 0 \) for any \( \tau \leq \tau_0 \). In addition \( \mathcal{F}_{i,i-1}(\tau(i), \tau_0(i-1)) = 0 \) for any \( \tau(i) < \tau_0(i) \). Correspondingly, the auxiliary variables \( \tau_n^{n-1}(k) \) are fixed to \( \tau_n^{n-1}(k) = \tau_0(k) \).

The relation between the correlation functions in TASEP and the measure \( \mathcal{M} \) is given by the following proposition.

**Proposition 3.8.** Consider the TASEP evolution starting with the initial conditions \( b^0 = ((x_1^0, t_1^0), \ldots, (x_N^0, t_N^0)) \), where \( b^0 \) is admissible configuration. Consider also \( m \) mutually distinct \( N \)-boundaries, \( b^0 \prec B^1 \prec \cdots \prec B^m \). Let \( (N_1, \ldots, N_l) \) and \( (k_1, \ldots, k_l) \) be collections of integers satisfying assumptions of the theorem 2.4. Then, the joint probability for space–time trajectories of particles \( N_1, \ldots, N_l \) to go from \( N \)-boundaries \( B^{k_1}, \ldots, B^{k_l} \) via points \( b_{N_1}^{k_1}, \ldots, b_{N_l}^{k_l} \), respectively, given the trajectories of all particles started from the position \( b^0 \), is a marginal of the measure \( \mathcal{M}(\mathcal{T}) \) of the form
\[ P \left( \bigcup_{i=1}^l \{ b_{N_i}^{k_i} = b_{N_i}^{k_i}(\tau_i(k)) \} \big| b^0 \right) = \mathcal{M} \left( \mathcal{T} \supset \bigcup_{i=1}^l \{ \tau_i^{N_i}(k_i) = \tau_i(k), 1 \leq i \leq l \} \right). \]  

(3.44)

**Proof.** We first note that instead of the \( N \)-boundaries \( B^1, \ldots, B^m \) we can consider auxiliary \( N_{p_i} \)-boundaries \( B^i_{N_{p_i}} = (B^1_{N_{p_i}}, \ldots, B^m_{N_{p_i}}) \) for \( i = 1, \ldots, m \). This is possible because in TASEP the trajectories of particles \( p_i + 1, \ldots, N \) do not influence the trajectories \( 1, \ldots, p_i + 1 \), and no point of the former group is fixed after (and on) \( B^i \) within the correlation function (3.44). Given trajectories \( 1, \ldots, p_i \), the sum over all realizations of the trajectories \( p_i + 1, \ldots, N \) amounts to one. Therefore, after \( B^i \) has been passed we can drop the former evolution and consider only the latter. Thus, we first consider the transition of \( N_1 \) particles from \( b^0 \) to \( B^1_{N_{p_1}} \), then the transition of \( N_{p_2} \) particles from \( B^1_{N_{p_2}} \) to \( B^2_{N_{p_2}} \), etc (see figure 4). The probability of each transition is given by corresponding \( N_{p_i} \)-particle Green function. To ensure the admissibility of particle configurations within the Green function and keep its probabilistic meaning we require that after each transition the particles do leave the boundaries. This suggests that we insert a compulsory step forward at the points belonging to vertical parts of the boundaries. To this end, we supply

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Universal exit probabilities in the TASEP

Figure 4. Trajectories of five TASEP particles traversing three 5-boundaries. Black segments emphasize that particles make a compulsory step forward at the sites belonging to vertical parts of boundaries from which the exits occur. The exits included into correlation function with $N_1 = N_2 = 5$, $N_3 = 4$, $N_4 = 3$, $N_5 = 2$, $N_6 = 1$ and $k_1 = 1, k_2 = k_3 = 2, k_4 = k_5 = k_6 = 3$ are shown in circles.

Each step of this kind by the factor of $p$ and define the starting points for every transition to be of the form (3.43). Finally, the probability of interest, $P((b_{N_i}^k(\tau_{N_i}(k_i)))_{i=1}^l|b^0)$, is the following:

$$
P\left(\bigcup_{i=1}^l\{b_{N_i}^k(\tau_{N_i}(k))\}|b^0\right) = \sum_\Delta \prod_{j=1}^m G(\{b_i^j(\tau_j(j))\}_{i=1}^{N_{p_j}}|\{b_i^{0,j}(\tau_j(j-1))\}_{i=1}^{N_{p_j}}) \times \pi^{b^0,j}(b^0)$$ (3.45)



where $\{b_i^{0,1,(-1)}\}_{i=1}^N \equiv b^0$ and the summation is over domain

$\Delta = \{\tau_j(i) \in Z_{\geq n_0(i)}, \tau_j(i) > \tau_{j-1}(i), 1 \leq j \leq N_{p_i}, 1 \leq i \leq m\} \setminus \{\tau_{N_i}(k_i), 1 \leq i \leq l\}$.

Using the determinantal formula of the GGF (3.1), we have

$$P(\{\hat{b}_{N_i}^k(\tau_{N_i}(k_i))\}_{i=1}^l|b^0) = \sum_{\Delta} \prod_{k=1}^m \pi^{b^k}(b^k) \det[F_{i,j}(b_j^k(\tau_j(k))) - b_j^{0,k}(\tau_j(k))]|_{1 \leq i,j \leq N_{p_k}}.$$ (3.46)

In what follows we are going to introduce auxiliary variables $\tau_i^j(k)$ in the same way as we did for the case of the single $N$-boundary, with the only difference that there is a separate set for every $N$-boundary, indexed by an extra argument $k$. To proceed further we define several domains of summation in these variables:

$$D_i = \{\tau_i^j(k) \in Z_{\geq n_0(i)}, 1 \leq k \leq j \leq N_{p_i}, \tau_i^j(k) > \tau_i^j(k+1)\}$$ (3.47)

$$\hat{D}_i = \{\tau_i^j(k) \in Z_{\geq n_0(i)}, 1 \leq k \leq j \leq N_{p_i+1} - 1, \tau_i^j(k) > \tau_i^j(k+1)\}$$ (3.48)
Figure 5. Exit probabilities on a space-like path in the $x$–$t$ plane. The wedge bounded by black straight lines is the rarefaction fan area. The deterministic trajectories of particles with numbers $N_1, N_2, N_3$ are shown in red. The green lines are boundaries with coordinates $x = L((\zeta(\theta_i, \chi) - \chi)/2 - \nu_i)$ and $t = L((\zeta(\theta_i, \chi) + \chi)/2$, where $i = 1, 2, 3$, corresponding to three fixed values of $\theta$: $\theta_1 < \theta_2 < \theta_3$. Dashed line is the projection of the path $(\nu(r), \theta(r))$ to the $x$–$t$ plane; $x = L((\zeta(r) - \chi(r))/2 - \nu(r))$, $t = L((\zeta(r) + \chi(r))/2$. The black dots are the points where exits occur.

\[ D^*_i = D_i \setminus \{ \tau_{l-1}^i(i), N_{p_i} \leq j \leq N_{p_i+1} \}; \quad \hat{D}^*_i = D^*_i \setminus \hat{D}_i \]  
\[ D = \bigcup_{i=1}^{m} \hat{D}^*_i \]  

(3.49) (3.50)

where we set $p_{m+1} \equiv l + 1$ and $\tau_{l-1}^i(i) \equiv \tau_l(i)$ for $k = 1, \ldots, N_{p_i}$ and $i = 1, \ldots, m$.

Now we apply lemmas 3.3 and 3.4 to each determinant under the product in the rhs of (3.46) to represent it as a sum over the auxiliary variables:

\[
\det[F_{k-l}(b_{l}^j(\tau(i)) - b_{k}^{0,i}(\tau(i-1))))]_{1 \leq k, l \leq N_{p_i}} = \sum_{D^*_i} (-1)^{\frac{N_{p_i}}{2}} \det[F_{-N_{p_i}+k}(b_{N_{p_i}}^k(\tau_{N_{p_i}}^i(i)) - b_{k}^{0,i}(\tau_{l}^i(i-1))))]_{1 \leq k, l \leq N_{p_i}} \times \prod_{n=0}^{N_{p_i}-1} \det[\phi_n(\tau_k^n(i), \tau_k^{n+1}(i))]_{1 \leq k, l \leq n+1}.
\]  

(3.51)

The endpoints, $b_{j}^k(\tau)$, of part of the trajectories within a transition between two $N_{p_i}$-boundaries are related to the starting points, $b_{j}^{0,k+1}(\tau)$, of the trajectories within the next transition by (3.43). The sums over the range of these positions can be evaluated along...
Universal exit probabilities in the TASEP

Figure 6. Evolution of fictitious particles for $N = 3$. The first stage from initial points to the boundary $B_3$ is described by the functions $\Psi_i^N(\tau_{N_i}^N)$. The following stages from $B_3$ to $B_2$, from $B_2$ to $B_1$, and from $B_1$ to $B_0$ are encoded in $\phi_2, \phi_1$, and $\phi_0$ respectively. The coordinates of real TASEP particles are shown as black dots.

with a few sums in auxiliary variables coupled to them (see figure 7):

$$
\sum_{B_{i-1}} \det [\tilde{F}_{-N_i, +k} (b_i^{N_i} (\tau_i^{N_i} (i)) - b_i^{0,i} (\tau_i^0 (i - 1)))_{1 \leq k,l \leq N_i} = (-1)^{\frac{N_i}{2}} \det [\tilde{F}_0 (b_i^{N_i} (\tau_i^{N_i} (i)) - b_i^{0,i} (\tau_i^0 (i - 1)))_{1 \leq k,l \leq N_i}. 
$$

(3.52)

The last identity can be proved by repeatedly applying formula

$$
\sum_{i=1}^{i_2} \pi^R (b_k(i)) \tilde{F}_n (c_0 - b_k(i)) = \tilde{F}_{n+1} (c_0 - b_{k-1}(i_2 + 1)) - \tilde{F}_{n+1} (c_0 - b_{k-1}(i_1)),
$$

(3.53)

which is another form of lemma 3.2, where $(c_0, t_0)$ is a pair of arbitrary constants.

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Figure 7. Example of summation over the auxiliary variables $\tau_k^i(i)$ for two $N$-boundaries in the case $N_1 = 5$ and $N_{p_2} = N_2 = 3$. The dotted lines show the variables over which the summation can be explicitly performed. The points shown by circles are fixed within the correlation function.

The resulting expression for the joint distribution is

$$P(\{b_{N_i}^k(\tau_{N_i}(k_i))\}) = \text{const} \sum_{D} \det[\tilde{F}_{-N_i+l}(b_{N_i}^k(\tau_{N_i}^N(1)) - b_{N_i}^0)]_{1 \leq k,l \leq N_1}$$

$$\times \prod_{i=2}^{m} \det[\tilde{F}_{0}(b_{N_i}^i(\tau_{N_i}^{N_i}(i)) - b_{N_i}^{0,i}(\tau_{N_i}^{N_i}(i-1)))_{1 \leq k,l \leq N_{p_i}}]$$

$$\times \prod_{n=N_{p_{i+1}}}^{\prod_{n=N_{p_{i+1}}}^{N_{p_{i+1}}}} \det[\phi_n(\tau_{N_i}^n(i), \tau_{N_i}^{n+1}(i))_{1 \leq k,l \leq n+1},$$

where we put $N_{p_{m+1}} = 0$, $\tilde{D}_m = 0$ and therefore $\tilde{D}_m = D_m$. □

If we again appeal to the correspondence with coordinates of fictitious particles, we see that the indices $\tau_n^k(i)$ define coordinates of particles at the boundary $B_n^i$. The functions $F_0()$ under the product in (3.54) describe the transitions between two subsequent $N$-boundaries, while the functions $\phi(\tau^n,\tau^{n+1})$ are responsible for transitions between subsequent $n$th and $(n+1)$th boundaries within the same $N$-boundary. Note that after we summed out part of coordinates, some boundaries fell out of the consideration and only the following remained:

$$B_{N_1}^1, \ldots, B_{N_{p_2}}^1, B_{N_{p_2}}^2, \ldots, B_{N_{p_{m+1}}}^{m-1}, B_{N_{p_{m+1}}}^m, \ldots, B_1^m.$$
Therefore, it is convenient to develop another enumeration, which counts only these boundaries. As one can see, either the upper index decreases or the lower one increases when going through the sequence. Now we introduce a new pair of indices, which distinguishes these two situations. Each group within which the lower index does not change, such that for some $i$ we have $N_{p_i−1} > N_{p_i} = N_{p_i+1} = \cdots = N_{p_i+c(n)−1} \equiv n > N_{p_i+c(n)}$, is uniquely characterized by number $n$, $1 \leq n \leq N$ and cardinality $c(n) \in \{0, \ldots, m+1\}$. This means that the particle number $n$ appears $c(n)$ times in the correlation function. It is convenient to introduce a pair of indices $(n, a)$, where index $n$ is the number of particles arriving at given boundary and index $a$, $0 \leq a \leq n−1$, labels the position of boundary within the group. Then, instead of the notation $\tau_i(k) \in B^r_j$ we use $\tau_i^{[n,a]} \in B^{(n,a)}$, implying that for each transition between two $N_p$-boundaries, in which the particle number does not change, the second index $a$ increases by 1, while in each transition within single $N_p$-boundary, which effectively reduces the number of fictitious particles by one, index $n$ decreases by one. As a result, the rhs of (3.41) can be rewritten in a more uniform way

\[
\text{const} \times \prod_{n=1}^{N_1} \text{det}[\Psi_{N_{1−l}}^{N_1}](\tau_{k}^{N_1,0})]_{1 \leq k,l \leq N_1} \\
\times \prod_{n=1}^{N_1} \text{det}[\phi_n(\tau_l^{n−1,0}, \tau_k^{n,c(n)})]_{1 \leq k,l \leq n} \\
\times \prod_{a=1}^{c(n)} \text{det}[\mathcal{F}_{n,a},(n,a−1)(\tau_l^{n,a}, \tau_k^{n,a−1})]_{1 \leq k,l \leq n}.
\]  

(3.55)

We are in position to apply theorem 4.2 from [17]. It states that the measure (3.55) is determinantal and gives a recipe of construction of the correlation kernel for given initial conditions. Specifically, let us define function $\phi^{(n_1,a_1),(n_2,a_2)}$ of transition between the boundaries $B^{(n_1,a_1)}$ and $B^{(n_2,a_2)}$.

\[
\phi^{(n_1,a_1),(n_2,a_2)} = \mathcal{F}_{(n_1,a_1),(n_1,0)} * \phi_{n_1+1} * \mathcal{F}_{(n_1+1,c(n_1+1),(n_1+1,0)} * \cdots * \phi_{n_2} * \mathcal{F}_{(n_2,c(n_2),(n_2,a_2)},
\]

(3.56)

where we used a definition of convolution

\[
(a * b)(x, y) = \sum_{z \in \mathbb{Z}^{(n,a)}} a(x, z)b(z, y),
\]

(3.57)

with the summation in $z$ performed over the points of the boundary $B^{(n,a)}$, which is between the boundaries where the indices $x$ and $y$ live, and

\[
\Psi^{(n,a)}_{N−l} = \phi^{(n,a),(N_1,0)} \ast \Psi^{(N_1,0)}_{N_{1−l}},
\]

(3.58)

where $\Psi^{(N_1,0)}_{N_{1−l}} \equiv \Psi^{N_{1−l}}_{N_{1−l}}(\tau^{N_1,0})$. The argument $\tau^{(n,a)}$ of $\Psi^{(n,a)}_{n−l}$ lives on $B^{(n,a)}$ due to the convolution with the function $\phi^{(n,a),(N_1,0)}$. For the cases when $c(n) = 0$ we formally define $\mathcal{F}_{(n,0),(n,0)}(x, y) = \delta_{x,y}$.

Consider matrix $M$ with matrix elements

\[
M_{k,l} = (\phi_k * \phi^{(k,c(k)),(N_1,0)} \ast \Psi^{(N_1,0)}_{N_{1−l}}(\tau^{k−1})
\]

(3.59)

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28
where we can omit the dependence of $\tau_{k-1}^i(i)$ on the label $i$ of the $N$-boundary $B_i$. If the matrix $M$ is invertible, the normalizing constant of the measure (3.55) is equal to $(\det M)^{-1}$. According to the theorem 4.2 from [17], the correlation kernel of (3.55) is as follows

\[ K(b^{(n_1,a_1)}(\tau_1), b^{(n_2,a_2)}(\tau_2)) = \sum_{k=1}^{N_1} \sum_{l=1}^{n_2} \Psi_{n_1-k}^{(n_1,a_1)}(\tau_1)[M^{-1}]_{k,l}(\phi_l * \phi((l,c(l)),(n_2,a_2)))(\tau_{l-1}^1, \tau_2) \]

\[ - \phi^{(n_1,a_1),(n_2,a_2)}(\tau_1, \tau_2). \]  

(3.60)

Furthermore, if the matrix $M$ is upper triangular, the derivation of the kernel is significantly simplified. In this case we construct the set of functions $\{\Phi_k^{(n,a)}\}$, which form a basis of the linear span of the set

\[ \{(\phi_1 * \phi^{(1,c(1)),(n,a)})(\tau_{1}^0, \tau), \ldots, (\phi_n * \phi^{(n,c(n)),(n,a)})(\tau_{n-1}^n, \tau)\}, \]  

(3.61)

fixed by orthogonality condition

\[ \sum_{\tau \in \mathbb{Z}} \Psi_i^{(n,a)}(\tau)\Phi_j^{(n,a)}(\tau) = \delta_{i,j}, \quad i, j = 0, \ldots, n - 1. \]  

(3.62)

Then the kernel takes the following form

\[ K(b^{(n_1,a_1)}(\tau_1), b^{(n_2,a_2)}(\tau_2)) = \sum_{k=1}^{n_2} \Psi_{n_1-k}^{(n_1,a_1)}(\tau_1)\Phi_{n_2-k}^{(n_2,a_2)}(\tau_2) - \phi^{(n_1,a_1),(n_2,a_2)}. \]  

(3.63)

As a result we have:

**Proposition 3.9.** Given densely packed initial conditions

\[ b^0 = (0, -1, \ldots, -N + 1) \]  

(3.64)

the correlation kernel of the determinantal measure (3.55) has the form

\[ K(b_1^{(n_1,a_1)}, b_2^{(n_2,a_2)}) = \int \frac{dv}{\Gamma_1} \int \frac{dw}{\Gamma_{o,v}} \int \frac{dw}{\Gamma_{o,w}} \]

\[ \times \frac{(1 - p((w - 1)/w))^{l_1}((w - 1)^{n_1}w^{x_1}/(1 - p((w - 1)/w)))^{l_2}((v - 1)^{n_2}v^{x_2})}{(w - v)(1/v + 1/\pi_2 - 1)} \]

\[ - 1(n_2 > n_1) \int \frac{dw}{\Gamma_{o,w}^2} \frac{(1 - p((w - 1)/w))^{l_1}w^{x_1} - x_2^{l_2}v^{x_2}}{(w - 1)^{n_2-n_1}(1/w + 1/\pi_2 - 1)}, \]  

(3.65)

where $b_i^{(n_1,a_1)} \equiv (x_i, t_i) \in B^{(n_1,a_1)}$, $i = 1, 2$ and $\pi_2 \equiv \pi_{B_{o,w}^2}(b_{o,w}^2)$.

**Proof.** We first introduce function $\hat{F}_n(b)$ defined by an integral representation, similar to the one of $\hat{F}_n(b)$, with different integration contour.

\[ \hat{F}_n(x, t) = \frac{1}{2\pi i} \int_{\Gamma_{o,1}} \frac{dw}{w} \left(q + \frac{p}{w}\right)^t (1 - w)^{-n}w^x. \]  

(3.66)

One can check that this function has the following properties:

\[ \phi_n * \hat{F}_k = -\hat{F}_{k+1}, \]  

(3.67)

\[ \mathcal{F}_{(n,a),(n,a-1)} * \hat{F}_k = \hat{F}_k, \]  

(3.68)

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and

\[ \hat{F}_k(x, t) = 0, \quad \text{when } k \leq 0 \quad \text{and} \quad -x > t. \]  \tag{3.69}

Note that the choice of the contour \( \Gamma_{0,1} \) ensures uniform convergence of convolution sums, which may extend to \( x = -\infty \) and \( t = \infty \). Therefore one can interchange summation and integration, from where the formulas (3.67) and (3.68) follow. The choice of the contour becomes relevant for \( \hat{F}_k \) with positive \( k \) as in this case there is a pole at \( w = 1 \), which must be placed inside the contour. One also must keep in mind that the convolution with \( \phi_n \) applied to the function of a point at \( B^{(n,c(n))} \) results in a function of a point at \( B^{(n-1,0)} \), while the convolution with \( \mathcal{F}_{(n,a),(n,a-1)} \) yields the transition from \( B^{(n,a-1)} \) to \( B^{(n,a)} \). Since \( \hat{F}_k = \hat{F}_k \) for \( k < 0 \), \( \Psi_{N_1-l}^{(N_1,0)}(\tau) = (-1)^{N_1-l} \hat{F}_{-N_1+l}(b^{(N_1,0)}(\tau) - b_l^0) \), and hence, using (3.67) and (3.68), we have

\[ \Psi_{-n-l}^{(n,a)}(\tau) = (-1)^{n-l} \hat{F}_{-n+l}(b^{(n,a)}(\tau) - b_l^0). \]  \tag{3.70}

Then, the elements of the matrix \( M \) defined in (3.59) are

\[ M_{k,l} = \phi_n \ast \Psi_{k-l}^{(k,c(k))}(\tau_k^{k-1}) = \hat{F}_{-k+l+1}(b^{(k-1,0)}(\tau_k^{k-1}) - b_l^0). \]  \tag{3.71}

It follows from the definition of \( \tau_k^{k-1} \) and formula (3.69) that \( M_{k,l} = 0 \) when \( k > l \) and \( M_{l,l} = 1 \). Therefore the matrix \( M \) is invertible and upper triangular and we can straightforwardly go to the orthogonalization procedure.

Substituting the initial conditions (3.64) we obtain

\[ \Psi_k^{(n,a)}(\tau) = \frac{1}{2\pi i} \oint_{\Gamma_{0,1}} dw \left( 1 - \frac{w-1}{w} \right)^{t^{(n,a)}} (w - 1)^k w x^{(n,a)} + n-k-2, \]  \tag{3.72}

where \((x^{(n,a)}, t^{(n,a)}) = b^{(n,a)}(\tau) \). It is not a surprise that this is the same function as the one obtained in the case of single \( N \)-boundary. Its argument lives on single boundary \( B^{(n,a)} \), and the orthogonalization procedure referring to this boundary shows no difference from section 3.1:

\[ \Phi_j^{(n,a)}(\tau) = \frac{1}{2\pi i} \oint_{\Gamma_1} dv \left( 1 - \frac{v}{w} \right)^{-t^{(n,a)}} (v - 1)^{-j-1} w^j w x^{(n,a)} \frac{1}{(1/\pi b(b(\tau)) - 1) + 1}. \]  \tag{3.73}

Apparently, the double integral part of the kernel coincides with the one obtained in section 3.1 as well. We only need to derive an explicit expression for \( \phi_{n_1,a_1},(n_2,a_2) \). To this end we note that we start the series of convolutions in (3.56) with applying them either to \( \mathcal{F}_{(n_2,c(n_2)),(n_2,a_2)} \) or, if \( c(n_2) = 0 \), to \( \phi_{n_2} \). These functions can also be expressed in terms of \( \hat{F}_k(x, t) \). Specifically, the expression for \( \phi_n \) obtained in section 3.1 is

\[ \phi_n(\tau_1, \tau_2) = -\pi (b^{(n+1,c(n))},0(\tau_2)) \hat{F}_1(t^{(n,0)}(\tau_1) - b^{(n+1,c(n)),0}(\tau_2)) \]  \tag{3.74}

and from (3.42)

\[ \mathcal{F}_{(n_2,c(n_2)),(n_2,a_2)}(\tau_1, \tau_2) = \pi (b^{(n_2,a_2)},0) \hat{F}_0(t^{(n_2,a_2)}(\tau_1) - b^{(n_1,a_1),0}(\tau_2)). \]  \tag{3.75}

Therefore we can use formulas (3.67) and (3.68) for convolutions, which show that the lower index of the function \( F_k \) increases by one and the function itself picks up a minus...
sign every time the number \( n \) decreases by one. Finally we have
\[
\phi^{(n_1,a_1),(n_2,a_2)} = (-1)^{n_1 - n_2} \pi(b_2^{(n_2,a_2),0}) \hat{F}_{n_2 - n_1}(b_2^{(n_2,c(n_2))} - b_1^{(n_2,a_2),0}),
\]
which again coincides with the expression obtained in the single \( N \)-boundary case. As a result we arrive at the kernel expression (3.65). \( \square \)

Finally adopting the arguments from the end of section 3.1 for the collection of the boundaries \( \mathcal{B}_{N_1}^{k_1}, \ldots, \mathcal{B}_{N_m}^{k_m} \) we arrive at the Fredholm determinant expression, stated in the theorem 2.4. For the sake of mathematical rigor one would have to analyze the convergence of the series obtained (i.e. the properties of the operator \( K \)). Similar analysis however has been done in many papers and we address the reader to them [3, 7, 8, 17].

4. Asymptotic analysis of the correlation kernel

Now we use the parametrization of the space–time plane discussed in section 2.2. Below we evaluate the scaling limit of the correlation kernel, suggesting that the arguments of the kernel are associated with a pair of boundaries and particle numbers fixed by choosing two points at the path (2.22)–(2.24) in the plane from each other.

Lemma 4.1. Let us fix two points at the path (2.22)–(2.24) in the \( \theta \)–\( \nu \) plane
\[
r_i = r_0 + u_i L^{-1/3}, \tag{4.1}
\]
where \( i = 1, 2 \) and correspondingly set \( n_i = [L \nu(r_i)] \) and \( \theta_i = \theta(r_i) \). Let us consider two boundaries \( \mathcal{B}^1 \equiv \mathcal{B}_1^1, \mathcal{B}^2 \equiv \mathcal{B}_2^1 \), which approximate smooth curves according to (2.18) with the parameters \( \theta_1, \theta_2 \) fixed above. Then (2.19) define the curves approximated by boundaries \( \mathcal{B}_{n_1}^1 \) and \( \mathcal{B}_{n_2}^2 \) corresponding to particles \( n_1, n_2 \), respectively. For the coordinates \( \tau_i \) of points on the boundary we also suggest the scaling
\[
\tau_i / L = \chi_i = \chi(r_i) + s_i L^{-2/3}, \tag{4.2}
\]
with \( u_i, s_i \) fixed as \( L \to \infty \) and the function \( \chi(r) \) defined in the section 2.2 as a deterministic part of the random variable \( \chi \), obtained as a solution of the equation (2.20) given \( \theta(r) \) and \( \nu(r) \). Then
\[
\lim_{L \to \infty} L^{1/3} K(b_1^{1}(\chi_1);b_2^{2}(\chi_2)) \sim \kappa_1 \Upsilon_2(\pi_2) K_{\text{Airy}_2}(\kappa_c u_1, \kappa_f s_1; \kappa_c u_2, \kappa_f s_2), \tag{4.3}
\]
where in the rhs we have the extended Airy kernel (2.29), \( \kappa_c \) and \( \kappa_f \) are the model-dependent constants (2.30) and (2.31) and
\[
\Upsilon_2(\pi_2) = \frac{2}{\left( \sqrt{\pi}(1 + \zeta^{(0,1)}(r_0)) + \sqrt{\omega}(1 - \zeta^{(0,1)}(r_0)) \right)} \times \begin{cases} \sqrt{\omega}, & \pi^B(b(\tau_2)) = 1 \\ \sqrt{\pi}, & \pi^B(b(\tau_2)) = p. \end{cases} \tag{4.4}
\]
The sign ‘\( \sim \)’ means the equality up to the matrix conjugation, which does not affect matrix minors.

Proof. We introduce the following functions
\[
f(w; \theta, \chi) = \frac{\zeta(\theta, \chi) + \chi}{2} \ln(q + p/w) + \nu(r) \ln \left( \frac{w - 1}{w} \right) + \frac{\zeta(\theta, \chi) - \chi}{2} \ln(w), \tag{4.5}
\]
\[
h(w) = \ln(1 - 1/w). \tag{4.6}
\]
doi:10.1088/1742-5468/2012/08/P08013 31
To analyze the double integral part of the kernel $K_0$, we represent it as a sum

$$K_0(b_{n_1}(\chi_1); b_{n_2}(\chi_2)) = \sum_{k=1}^{\infty} \Psi_{n_1-k}^{B_1}(\chi_1) \Phi_{n_2-k}^{B_2}(\chi_2)$$

(4.7)

where the functions $\Psi_{n}^{B_1}$, $\Phi_{n}^{B_2}$ are given in (3.72) and (3.73). Note that, instead of the index in the superscript characterizing the number of the boundary, we placed the notation for the boundary explicitly, to reflect the dependence of the functions on the form of this boundary and not of the others (here $B$ means the first particle boundary, while the index $n$ shows that we have to shift it $n - 1$ steps back in horizontal direction). In terms of above notations the integrals entering the summands become

$$\Psi_{n_1-k}^{B_1}(\tau_1) = \int_{\Gamma_{1,0}} \frac{dw}{2\pi i w^2} e^{L_f(w;\theta_1,\chi_1) + L^{1/3}zh(w)},$$

(4.8)

$$\Phi_{n_2-k}^{B_2}(\tau_2) = \int_{\Gamma_{1,-1}} \frac{p\,dw}{2\pi i} \frac{e^{-L_f(w;\theta_2,\chi_2) - L^{1/3}zh(w)}}{(w-1)((1/\pi_2-1)v+1)},$$

(4.9)

where $z = kL^{-1/3}$ and $\pi_2 \equiv \pi^{B_2}(b(\tau_2))$. To obtain the asymptotics of $K_0$, we first evaluate the integrals for $\Psi_{n_1-k}^{B_1}(\tau_1)$ and $\Phi_{n_2-k}^{B_2}(\tau_2)$ asymptotically as $L \to \infty$ and then perform the summation.

Taking into account (4.2) one can approximate the function $f(w;\theta_i,\chi_i)$ up to the terms of constant order by

$$f(w;\theta_i,\chi_i) = f_r(w) + L^{-2/3}s_i g(w),$$

(4.10)

where we introduce the notations

$$f_r(w) \equiv f(w;\theta(r),\chi(r))$$

(4.11)

and

$$g_r(w) = \frac{1}{2} \left[ \left( \frac{\partial \zeta(r)}{\partial \chi} + 1 \right) \ln(q + p/w) + \left( \frac{\partial \zeta(r)}{\partial \chi} - 1 \right) \ln w \right],$$

(4.12)

where $\zeta(r) \equiv \zeta(\theta(r),\chi(r))$. The position of the double critical point of function $f_r(w)$, which satisfies $f'_r(w_0) = f''_r(w_0) = 0$ is

$$w_0(r) = 1 + \sqrt{\frac{2\nu(r)}{q(\zeta(r) - \chi(r))}}.$$  

(4.13)

Instead of the exponentiated functions we use their Taylor expansion at the points $w_i \equiv w_0(r_i)$, with $i = 1,2$ for $\Psi_{n_1-k}^{B_1}(\chi_1)$ and $\Phi_{n_2-k}^{B_2}(\chi_2)$ respectively.

$$f_r(w) \approx f_r(w_i) + \frac{1}{2} f'''_r(w_i)(w - w_i)^3$$

(4.14)

$$g_r(w) \approx g_r(w_i) + g''_r(w_i)(w - w_i)$$

(4.15)

$$h(w) \approx h(w_i) + h'(w_i)(w - w_i)$$

(4.16)

where in the coefficients of $w$-dependent terms we, without loss of accuracy, replace $r_i$ and $w_i$ by $r_0$ and $w_0 \equiv w_0(r_0)$ respectively. We substitute these expansion into the integrals, and choose steep descent contours such that they approach the horizontal axis at the points $w_i$ respectively.
w_1 and w_2 at the angles ±π/3 and ±2π/3 respectively. Changing the integration variables to ξ_i = (w - w_i)L^{1/3}f'''(w_0)/2 we arrive at the integrals defining the Airy functions:

$$Ai(a) = \int_{-\infty}^{\infty} \frac{dx}{2\pi i} \exp\left(\frac{x^3}{3} - xa\right).$$

(4.17)

As a result we have

$$\Psi_{n_1-k}^{B^1}(\tau_1) \approx \frac{\exp(Lf_{\tau_1}(w_1) + L^{1/3}(s_1 f_{\tau_1}(w_1) + z h(w_1)))}{w_0^2(Lf''_{\tau_0}(w_0)/2)^{1/3}}$$

$$\times Ai\left(\frac{zh'(w_0) - s_1 g'_{\tau_0}(w_0)}{(f''_{\tau_0}(w_0)/2)^{1/3}}\right),$$

(4.18)

$$\Phi_{n_2-k}^{B^2}(\tau_2) \approx \frac{\exp(-Lf_{\tau_2}(w_2) - L^{1/3}(s_2 g_{\tau_2}(w_2) + z h(w_2)))}{(w_0 - 1)((1/\pi B)(b(\gamma_2)) - 1)w_0 + 1(Lf''_{\tau_0}(w_0)/2)^{1/3}}$$

$$\times Ai\left(\frac{zh'(w_0) - s_2 g'_{\tau_0}(w_0)}{(f''_{\tau_0}(w_0)/2)^{1/3}}\right).$$

(4.19)

The summation over k can be replaced by an integration over z. To perform the summations we use one more expansion:

$$h(w_1) = h(w_0) - h'(w_0)w_0 r_0 L^{-1/3} + O(L^{-2/3}).$$

(4.20)

Finally, taking into account that h'(w_0) = 1/(w_0(w_0 - 1)), we obtain

$$\sum_{k=1}^{\infty} \Psi_{n_1-k}^{B^1}(\tau_1)\Phi_{n_2-k}^{B^2}(\tau_2)$$

$$\approx Y_2(\pi_2) L^{-1/3} \kappa_0 e^{(L(f_{\tau_2}(w_1) - f_{\tau_2}(w_2)) + L^{1/3}(s_1 f_{\tau_1}(w_1) - s_2 g_{\tau_2}(w_2)))}$$

$$\times \int_0^{\infty} d\lambda e^{\lambda e_{\pi B}(u_2 - u_1)} Ai(\lambda + \kappa_t s_1) Ai(\lambda + \kappa_t s_2),$$

(4.21)

where

$$\kappa_c = \frac{w_0'(\theta)f'''_{\tau_0}(w_0)^{1/3}}{2^{1/3}}$$

(4.22)

$$\kappa_t = \frac{-2^{1/3} g'_{\tau_0}(w_0)}{f'''_{\tau_0}(w_0)^{1/3}}$$

(4.23)

and

$$Y_2(\pi_2) = -\left[g'(w_0)w_0((1/\pi B)(b(\tau_2)) - 1)w_0 + 1]\right]^{-1}.$$  

(4.24)

Substituting

$$w_0'(r_0) = \frac{\sqrt{p}}{2\gamma q\sqrt{\gamma(1 + \zeta^{(0,1)}(r_0)) + \sqrt{\omega(1 - \zeta^{(0,1)}(r_0))}}},$$

$$\times \left[q'\frac{1}{\sqrt{\alpha}}(\zeta(r_0) - \chi(r_0)\zeta^{(0,1)}(r_0))(\sqrt{\omega} - \sqrt{\gamma})^{-1}$$

$$- \theta'(r_0) \zeta^{(1,0)}(r_0)(\sqrt{\omega} - \sqrt{\gamma})\right]$$

(4.25)
Let us now evaluate the second part of the kernel given by the single integral, which can be written as

\[ I = \int \frac{dz \exp[L(f_{r_1}(z) - f_{r_2}(z)) + L^{1/3}(s_1 g_{r_1}(z) - s_2 g_{r_2})]}{z((1/\pi^b(b(\tau)) - 1)z + 1).} \]

The critical point of the exponentiated function is found to be

\[ z_c = w_0 \equiv w(r_0). \]  \hspace{1cm} (4.28)

Then using Taylor expansions we show that

\[ f_{r_1}(w) - f_{r_2}(w) \approx f_{r_1}(w_1) - f_{r_2}(w_2) + \frac{f''_{r_0}(w_0)(u_2^2 - u_1^2)w'_0(r_0)^2}{6L} + \frac{f''_{r_0}(w_0)(u_2^2 - u_1^2)w'_0(r_0)^2}{2L^{2/3}}(z - w_0) + \frac{f''_{r_0}(w_0)(u_2 - u_1)w'_0(r_0)}{2L^{1/3}}(z - w_0)^2 \]  \hspace{1cm} (4.29)

and

\[ (s_1 g_{r_1}(z) - s_2 g_{r_2}(z)) \approx s_1 g_{r_1}(w_1) - s_2 g_{r_2}(w_2) + (u_2 s_2 - u_1 s_1)L^{-1/3}g'_{r_0}(w_0)w'(\nu) + (s_1 - s_2)g'_{r_0}(w_0)(z - w_0). \]  \hspace{1cm} (4.30)

Substituting these expansions into the integral and integrating along the vertical line crossing the horizontal axis at \( w_0 \) we obtain:

\[ I = L^{-1/3} \frac{\pi_2(\pi_2) \kappa_{c} \rho L(f_{r_1}(w_1) - f_{r_2}(w_2)) + L^{1/3}(s_1 g_1(w_1) - s_2 g_2(w_2))}{\kappa_{c}(\kappa_{c}(u_2^2 - u_1^2)/3) - ((\kappa_{c}(u_2^2 - u_1^2) - \kappa_\ell(s_1 - s_2))^2/4\kappa_\ell(u_2 - u_1)) - \kappa_\ell u_2 - s_1 u_1)} \]

One can see that the first line of this expression exactly coincides with the factor before the integral in (4.21). Furthermore, its exponential part does not change the value of the determinants, so that it can be omitted. The second part can be rewritten using the formula from [30]

\[ \frac{1}{\sqrt{4\pi(\tau' - \tau)}} e^{-\frac{(\xi - \xi')^2}{4(\tau' - \tau)} - (\tau' - \tau)(\xi + \xi')/2 + (\tau' - \tau)^3/12} = \int_{-\infty}^{\infty} e^{-\lambda(\tau - \tau')} Ai(\xi + \lambda) Ai(\xi' + \lambda) d\lambda, \]  \hspace{1cm} (4.32)
where we should set $\tau = \kappa_c u_1, \tau' = \kappa_c u_2, \xi = \kappa_f s_1, \xi' = \kappa_f s_2$. As a result we obtain the Airy extended kernel

$$L^{-1/3} \Upsilon_2(\pi_2) \kappa_1 \mathcal{K}_{\text{Airy}}(\kappa_c u_1, \kappa_f s_1; \kappa_c u_2, \kappa_f s_2).$$

\(\Box\)

To finish the proof of the theorem 2.5 one has to prove the uniform convergence of the kernel in bounded sets and that the part of the sum (3.34) coming from the complement to these sets is negligible while the bound is uniform in $L$. For similar proofs we address the reader to [3, 7, 8, 17]. After that interchange of the sum and the limit is allowed. However we note that the limiting expression for the kernel still depends on which site $b_B(\tau_2)$ is via the value of $\Upsilon_2(\pi_2)$, which in turn depends on $\pi_B(b(\tau_2))$. To go from the sums (3.34) to integrals we note that within every summation in an index $\tau$ running through a boundary $B$ the function $\Upsilon_B(\pi_B(b(\tau)))$ will enter linearly as a coefficient. It turns out that this coefficient amounts exactly to unit. This happens because the boundaries defined in (2.18) are locally straight and the $\tau$-dependent coefficient is averaged out on a smaller scale than the fluctuational one, which affects the resulting integral. The following lemma shows how the averaging works. After we apply it the statement of theorem 2.5 follows.

**Lemma 4.2.** Suppose that $b(\tau) \in B$, $\tau = \tau_0 + [sL^{1/3}]$, $f_{\text{lim}}(s)$ is a differentiable function and the following limit

$$\lim_{L \to \infty} L^{1/3} f(b(\tau)) = \Upsilon_B(\pi) f_{\text{lim}}(s)$$

holds uniformly in bounded sets $s \in [a, b]$. Then, if the boundary is close to a continuous differentiable path in a sense (2.18) and (2.19), we have

$$\lim_{L \to \infty} \sum_{\tau = \tau_0 + [sL^{1/3}]}^b f(b(\tau)) = \int_a^b f_{\text{lim}}(s) \, ds. \quad (4.34)$$

**Proof.** The proof is based on the fact that the order of the correction term accounting for the difference between the boundary on the lattice and its continuous differentiable counterpart allows one to consider the boundary as locally straight at scales up to the fluctuation scale. This in particular means that in such a small scale, where the site-independent part of the limiting function can be considered as constant, the site-dependent part can be summed separately. It turns out that under this summation the site dependence exactly cancels with the slope dependence defined at the macroscopic scale, so that the remaining expression converges to the integral of the site-independent part only.

To be specific, let us divide the range of summation into bins of size $\varepsilon L^{1/3}$, where $\varepsilon$ is small, and perform the summation in two stages: first within each bin and second over all the bins. The first summation yields

$$\sum_{\tau = \tau_0 + [\varepsilon n L^{1/3}]}^{\tau_0 + [\varepsilon (n+1) L^{1/3}]} f(b(\tau)) \simeq L^{-1/3} (N^c_\varepsilon \Upsilon_B(p) + N^v_\varepsilon \Upsilon_B(1)) (f_{\text{lim}}(\varepsilon n) + O(\varepsilon)), \quad (4.35)$$

where $N^c_\varepsilon$ and $N^v_\varepsilon$ are the numbers of horizontal and vertical segments of the boundary within the summation range. Note that the fraction of these numbers, which corresponds
to the slope of the boundary, being defined on the macroscopic scale persists up to the fluctuation scale, i.e. depends only on the value of $\chi = \lim_{L \to \infty} \tau_0 / L$:

$$N_v^\varepsilon = \delta t \simeq \varepsilon L^{1/3} \left( \frac{\partial \zeta(\theta, \chi)}{\partial \chi} + 1 \right),$$ (4.36)

$$N_h^\varepsilon = -\delta x \simeq -\varepsilon L^{1/3} \left( \frac{\partial \zeta(\theta, \chi)}{\partial \chi} - 1 \right).$$ (4.37)

From the explicit form of $\Upsilon_B(\pi)$, (4.4), we have

$$\frac{\partial \zeta(\theta, \chi)}{\partial \chi} + 1 \Upsilon_B(p) - \frac{\partial \zeta(\theta, \chi)}{\partial \chi} - 1 \Upsilon_B(1) = 1,$$ (4.38)

i.e. $(N_v^\varepsilon \Upsilon(p) + N_h^\varepsilon \Upsilon(1)) \simeq \varepsilon L^{1/3}$. Finally, after taking limit $L \to \infty$, performing the second summation $\sum_{1 \leq n \leq \lfloor (b-a)/\varepsilon \rfloor} \varepsilon f_{\lim}(\varepsilon n)$ and taking limit $\varepsilon \to 0$ we arrive at the desired result (4.34). $\square$

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