GROTHENDIECK’S PAIRING ON NÉRON COMPONENT GROUPS: GALOIS DESCENT FROM THE SEMISTABLE CASE

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Abstract. In our previous study of duality for complete discrete valuation fields with perfect residue field, we treated coefficients in finite flat group schemes. In this paper, we treat abelian varieties. This in particular implies Grothendieck’s conjecture on the perfectness of his pairing between the Néron component groups of an abelian variety and its dual. The point is that our formulation is well-suited with Galois descent. From the known case of semistable abelian varieties, we deduce the perfectness in full generality. We also treat coefficients in tori and, more generally, 1-motives.

Contents

1. Introduction
1.1. Main results
1.2. General picture
1.3. Organization
2. Site-theoretic preliminaries
2.1. Sites and algebraic groups: setup and first properties
2.2. Generalities on derived categories of ind-procategories
2.3. The derived categories of ind-proalgebraic groups and of sheaves
2.4. Serre duality and P-acyclicity
3. Local fields with ind-rational base
3.1. Basic notions and properties
3.2. Topology on rational points of varieties
3.3. The relative fppf site of a local field
3.4. Cohomology as sheaves on the residue field
4. Statement of the duality theorem
4.1. Formulation
4.2. Reduction to components groups and the first cohomology
5. Relation to Grothendieck’s and Šafarevič’s conjectures
5.1. Grothendieck’s pairing
5.2. Bester-Bertapelle’s isomorphism
5.2.1. Bester’s finite flat duality
5.2.2. Proof of the finite flat duality
5.2.3. Bertapelle’s isomorphism
5.3. Bégueré’s isomorphism
6. Galois descent

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1. Introduction

1.1. Main results. Let $K$ be a complete discrete valuation field with ring of integers $\mathcal{O}_K$ and perfect residue field $k$ of characteristic $p > 0$. Let $A$ be an abelian variety over $K$, $\mathcal{A}$ the Néron model of $A$ over $\mathcal{O}_K$ and $A_x$ the special fiber of $A$ over $x = \text{Spec} \ k$. The component group $\pi_0(A_x)$ of $A_x$ is a finite étale group scheme over $k$. From the dual abelian variety $A^\vee$, we have corresponding objects $A^\vee, A^\vee_x$ and $\pi_0(A^\vee_x)$. In [Gro72, IX, 1.2.1], Grothendieck constructed a canonical pairing

$$\pi_0(A^\vee_x) \times \pi_0(A_x) \to \mathbb{Q}/\mathbb{Z},$$

which appears as the obstruction to extending the Poincaré biextension on $A^\vee \times A$ to the Néron models $A^\vee \times A$. In this paper, we prove the following conjecture of Grothendieck [Gro72, IX, Conj. 1.3]:

Grothendieck’s conjecture. The pairing $\pi_0(A^\vee_x) \times \pi_0(A_x) \to \mathbb{Q}/\mathbb{Z}$ above is perfect.

Some previously known cases include: $k$ is finite [McC86]; $K$ has mixed characteristic [Bég81]; $A$ has semistable reduction [Wer97]; the prime-to-$p$ part [Ber01]; $A$ is the Jacobian of a curve with a rational point [BL02]; $A$ has potentially multiplicative reduction [Bos97]. McCallum used (the usual) local class field theory and local Tate duality. Bégneri instead used Serre’s local class field theory [Ser61], which is applicable for general perfect residue fields $k$. To deduce Grothendieck’s conjecture, Bégneri used some dimension counting argument, which does not work in equal characteristic since the corresponding objects are infinite-dimensional. Werner, on the other hand, used rigid analytic uniformization for semistable abelian varieties. It does not seem possible to deduce the general case from her result by a simple application of the semistable reduction theorem, since Néron models are known to behave very badly under base change.

The method we take in this paper is first to formulate a local Tate duality type statement for the general case. This is a version for abelian varieties of the author’s reformulation of a known duality theorem, where coefficients were finite flat group schemes ([Suz13]). Our formulation crucially relies on the techniques of sheaves on the category of fields developed in [Suz13]. We prove that the duality we formulate here is equivalent to the conjunction of Grothendieck’s conjecture and Šafarevič’s conjecture, the latter of which was posed in [Šaf61] and solved by Bégneri [Bég81] (mixed characteristic case), Bester [Bes78] (equal characteristic, good reduction) and Bertapelle [Ber03] (equal characteristic, general reduction). Our duality is

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1 The conjecture may fail when $k$ is imperfect [BB00]. The results on the semistable case and the prime-to-$p$ part are valid for any (possibly imperfect or zero characteristic) residue field $k$. In this paper, we only consider the case that $k$ is perfect of positive characteristic.
very functorial in an appropriate derived category that is strong enough to treat the
infinite-dimensional groups involved, and well-suited with Galois descent. Therefore
we are able to prove that if the duality is true over a finite Galois extension of $K$,
then it is true over $K$. This and the semistable case cited above together imply
Grothendieck’s conjecture.

Here we sketch our formulation of the duality with coefficients in abelian vari-
eties. Recall from [Suz13] that a $k$-algebra is said to be rational if it is a finite direct
product of the perfections (direct limit along Frobenii) of finitely generated fields
over $k$, and ind-rational if it is a filtered union of rational $k$-subalgebras. We denote
the category of ind-rational $k$-algebras by $k_{\text{indrat}}$. We can endow this category with
the pro-étale topology ([BS15]; see §2.1 below for the details about the pro-étale
topology in this ind-rational setting). We denote the resulting site by Spec $k_{\text{indrat}}$ proet.
The derived category of sheaves on Spec $k_{\text{indrat}}$ proet is denoted by $D(k_{\text{indrat}})$ and the
derived sheaf-Hom functor by $R\text{Hom}_{k_{\text{indrat}}}$.

For an object $C \in D(k_{\text{indrat}})$, we define
$$C^{SD} = R\text{Hom}_{k_{\text{indrat}}}(C, \mathbb{Z})$$
and call it the Serre dual of $C$ (cf. [Ser60, 8.4, Remarque], [Mil06, III, Thm. 0.14]).
The double dual SDSD sends the perfection (inverse limit along Frobenii) of a pro-
nipotent group to itself. It sends the perfection of a semi-abelian variety $G$ to its
profinite Tate module placed in degree $-1$. Hence the universal covering of $G$
placed in degree $-1$ gives a mapping cone of $G \rightarrow G^{SDSD}$, which is uniquely
divisible.

As in the beginning of the paper, let $K$ be a complete discrete valuation field
with perfect residue field $k$ of characteristic $p > 0$ and $A$ an abelian variety over
$K$. We can regard the cohomology complex $R\Gamma(A) = R\Gamma(K, A)$ of $K$ as a com-
plex of sheaves on Spec $k_{\text{indrat}}$ proet in a suitable way. We denote the resulting complex
by $R\Gamma_{k_{\text{indrat}}}$ and its $n$-th cohomology by $H^n_{k_{\text{indrat}}}$.

The sheaf $\Gamma(A) = H^0(A)$ is represented by the perfection of the Greenberg transform of the Néron model of $A$.
This is a proalgebraic group over $k$, and the reduction map induces an isomorphism $\pi_0(\Gamma(A)) \rightarrow \pi_0(A_{\text{et}})$. The sheaf $H^1(A)$ is ind-algebraic, and all higher cohomology
vanishes. We have
$$R\Gamma(A)^{SD} = R\text{Hom}_{k_{\text{indrat}}}(R\Gamma(A), \mathbb{Q}/\mathbb{Z})[-1].$$

The trace morphism $R\Gamma(G_m) \rightarrow \mathbb{Z}$ given in [Suz13, Prop. 2.4.4] and the cup product formalism give a pairing
$$R\Gamma(A^\vee) \times R\Gamma(A) \rightarrow \mathbb{Z}[1]$$
in $D(k_{\text{indrat}})$. This induces morphisms
$$R\Gamma(A) \rightarrow R\Gamma(A^\vee)^{SD}[1] \quad \text{and} \quad R\Gamma(A^\vee)^{SDSD} \rightarrow R\Gamma(A)^{SD}[1].$$

We will prove the following.

**Theorem A.** The above defined morphism
$$R\Gamma(A^\vee)^{SDSD} \rightarrow R\Gamma(A)^{SD}[1]$$
in $D(k_{\text{indrat}})$ is an isomorphism.

We recall Šafarevič’s conjecture, or a theorem of Bégueri and Bester-Bertapelle.
In our terminology, Šafarevič’s conjecture can be stated as the existence of a canonical isomorphism
$$H^1(A^\vee) \cong \text{Ext}^1_{k_{\text{indrat}}}(\Gamma(A), \mathbb{Q}/\mathbb{Z})$$
for $k$ algebraically closed. The right-hand side is the Pontryagin dual the fundamental group $\pi_1(\Gamma(A))$ of the proalgebraic group $\Gamma(A)$ in the sense of Serre [Ser60]. Hence Šafarevič’s conjecture claims the existence of a perfect pairing

$$H^1(A^\vee) \times \pi_1(\Gamma(A)) \to \mathbb{Q}/\mathbb{Z}$$

between the torsion group and the profinite group. Note that $\pi_1$ is indifferent to component groups and observe how this conjecture is complementary to Grothendieck’s conjecture.

We will prove the following more precise form of the above theorem.

**Theorem B.**

(a) Theorem A is equivalent to the conjunction of Grothendieck’s conjecture and Šafarevič’s conjecture.

(b) If Theorem A is true for $A \times_K L$ for a finite Galois extension $L$ of $K$

$$R\Gamma(L, A^\vee)^{SDSD} \cong R\Gamma(L, A)^{SD}[1],$$

then it is true for the original $A$.

Since Šafarevič’s conjecture is already a theorem, these results together with the semistable reduction theorem and Werner’s result imply:

**Theorem C.** Grothendieck’s conjecture is true.

In the next logically optional subsection, we explain the general picture of our formulation and proof. The organization of the paper will be explained in the subsection after next.

1.2. General picture. We explain the general picture of our proof of Grothendieck’s conjecture without getting into the details or giving precise definitions, in order to motivate our very peculiar constructions. For the convenience of the reader, we repeat the main ideas in our previous paper [Suz13]. Notation is as in the previous subsection. Assume that $k$ is algebraically closed for simplicity.

Suppose for the moment that we have a nice complex $R\Gamma(K, A) = R\Gamma(A)$ of ind- or proalgebraic groups over $k$ for an abelian variety $A$ over $K$, so that its $k$-points is $R\Gamma(A)$ (hence $H^n(A) = 0$ for $n \geq 2$ by a cohomological dimension reason). The group $\Gamma(A)$ is proalgebraic and $H^1(A)$ ind-algebraic. What is nice about $R\Gamma(A)$ is that it can capture the Néron component group $\pi_0(A_x)$ and behaves well under base change.

To explain this, let $p_K$ be the maximal ideal of $O_K$. We have a surjection

$$\Gamma(K, A) = \Gamma(O_K, A) = \lim_{\substack{\text{\small n} \to \infty}} \Gamma(O_K/p_K^n, A) \to A_x$$

of proalgebraic groups with connected kernel, so $\pi_0(\Gamma(A)) = \pi_0(A_x)$. This explains how $\pi_0(A_x)$ can be functorially recovered from $\Gamma(A)$. Here it is essential to put a proalgebraic group structure on $\Gamma(A)$ to make sense of its $\pi_0$. On the other hand, the Hochschild-Serre spectral sequence

$$R\Gamma(\text{Gal}(L/K), R\Gamma(L, A)) = R\Gamma(K, A)$$

for a finite Galois extension $L/K$ shows that $R\Gamma(K, A)$ can be recovered from $R\Gamma(L, A)$ very cheaply, while there is no such simple relation for Néron models.

We expect some duality between the proalgebraic group $\Gamma(A)$ and the ind-algebraic group $H^1(A^\vee)$ that takes care of $\pi_0(\Gamma(A))$. This should be analogous
to the usual local Tate duality in the finite residue field case. A little more precisely, let’s expect that there should be an isomorphism

\[(1.2.1) \quad R\Gamma(A^\vee) \sim R\text{Hom}_k(R\Gamma(A), \mathbb{Q}/\mathbb{Z})\]

up to some completion of the left-hand side. Here \(R\text{Hom}_k\) is some internal \(R\text{Hom}\) functor for the category of ind- or proalgebraic groups over \(k\). The actual duality statement needs the completion or double-dual as in previous subsection, since semiabelian varieties (appearing in \(A_\mathbb{A}\)) are not double-dual invariant (while unipotent groups are so). Let’s ignore the double-dual. This duality statement is robust for Galois descent: if it is true for \(A\) over a finite Galois extension \(L/K\), then it is true for \(A\) over \(K\), basically by the Hochschild-Serre spectral sequence above.

The statement is equivalent to the conjunction of Grothendieck’s and Šafarevič’s conjectures as follows. The isomorphism (ignoring the completion) gives a hyperext spectral sequence

\[E_2^{ij} = \text{Ext}^i_k(H^{-j}(A), \mathbb{Q}/\mathbb{Z}) \implies H^{i+j}(A^\vee).\]

The functor \(\text{Ext}^i_k(\cdot, \mathbb{Q}/\mathbb{Z})\) is dual to the \(i\)-th homotopy group functor by \([\text{Ser}60] \S 5, \text{Cor. to Prop. 7}\), hence zero for \(i \geq 2\) by \([\text{Ser}60] \S 10, \text{Thm. 2}\). Therefore this spectral sequence degenerates at \(E_2\) and becomes (ignoring the term \(\text{Hom}_k(H^1(A), \mathbb{Q}/\mathbb{Z})\)) an exact sequence and an isomorphism

\[0 \to \text{Ext}^1_k(H^1(A), \mathbb{Q}/\mathbb{Z}) \to \Gamma(A^\vee) \to \text{Hom}_k(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \to 0,
\]

\[H^1(A^\vee) \cong \text{Ext}^1_k(\Gamma(A), \mathbb{Q}/\mathbb{Z})\]

(with a completion to \(\Gamma(A^\vee)\)). The isomorphism gives Šafarevič’s conjecture. As \([\text{Mil}06] \S 11, \text{Lem. 0.13 (c)}\) suggests, the group \(\text{Ext}^1_k(H^1(A), \mathbb{Q}/\mathbb{Z})\) should be connected. Hence the exact sequence should be a connected-étale sequence. The connected part gives the statement that the identity component of \(\Gamma(A^\vee)\) up to completion is \(\text{Ext}^1_k(H^1(A), \mathbb{Q}/\mathbb{Z})\). This is dual to Šafarevič’s conjecture. The étale part gives an isomorphism

\[\pi_0(\Gamma(A^\vee)) \cong \text{Hom}_k(\pi_0(\Gamma(A)), \mathbb{Q}/\mathbb{Z}).\]

Writing the groups as Néron component groups, we get the isomorphism predicted by Grothendieck’s conjecture.

Here it is very important to have a complex, \(R\Gamma(A)\), of ind- or proalgebraic groups, and work in a derived category. Looking at the terms individually seriously breaks the good behavior under base change.

How can we define such a complex? A naive approach is the following. For a perfect \(k\)-algebra \(R\), we define its canonical lift “\(R \otimes_k K\)” to \(K\) to be

\[K(R) = (W(R) \otimes_{W(k)} \mathcal{O}_K) \otimes_{\mathcal{O}_K} K,\]

where \(W\) is the affine scheme of Witt vectors of infinite length. More explicitly, if \(K = W(k)[1/p][x]/(f(x))\) with some Eisenstein polynomial \(f(x) \in W(k)[x]\), then \(K(R) = W(R)[1/p][x]/(f(x))\), and if \(K = k((T))\), then \(K(R) = R[[T]][1/T]\). If \(R = k'\) is a perfect field, the ring \(K(k')\) is a complete discrete valuation field obtained from \(K\) by extending the residue field from \(k\) to \(k'\). Take an injective resolution \(I\) of \(A\) over the fppf site of \(K\). Consider the complex of presheaves

\[R \mapsto \Gamma(K(R), I)\]
on the category of perfect \( k \)-algebras. Its (pro-)étale sheafification is the candidate of our complex. Its \( n \)-th cohomology is the sheafification of the presheaf
\[
R \mapsto H^n(K(R), A).
\]
But this fppf cohomology group is very difficult to calculate. It is a classical object only if we restrict the \( R \)’s to be perfect fields. This means that we can only obtain the generic behavior of the sheaf from our classical knowledge.

But generic behavior is sufficient to describe algebraic groups in view of Weil’s theory of birational groups. This means that there is no information lost by treating proalgebraic groups as functors on the category of ind-rational \( k \)-algebras, which is essentially the category of perfect fields over \( k \). Now the sheafification of the complex of presheaves
\[
k' \mapsto \Gamma(K(k'), \Gamma)
\]
on our ind-rational pro-étale site \( \text{Spec} \ k_{\text{indrat proet}} \) yields the sought-for object \( R\Gamma(A) \). This explain why this strange site \( \text{Spec} \ k_{\text{indrat proet}} \) is needed in this paper. Moreover, this type of complexes and cohomology groups looks very similar to the well-known general description of higher pushforward sheaves. Therefore we are tempted to define a morphism of sites
\[
\pi: \text{Spec} \ K_{\text{fppf}} \to \text{Spec} \ k_{\text{indrat proet}}
\]
corresponding to the functor \( k' \mapsto K(k') \) on the underlying categories, so that we have a more systematic definition \( R\Gamma(A) = R\pi_*A \). In fact, a little care and modifications are needed for continuity and exactness of pullback.

Now the picture is the following. The duality we want should be a relative duality for the morphism \( \pi \). The cup product formalism in site theory will give us the duality morphism \( \mathbb{L}2.1 \). The derived sheaf-Hom functor \( R\text{Hom}_{k_{\text{indrat proet}}} \) on the site \( \text{Spec} \ k_{\text{indrat proet}} \) should give the internal \( R\text{Hom} \) functor on the category of ind-proalgebraic groups.

All we do in this paper is to carry out these ideas rigorously making necessary corrections to imprecise ideas. There are three difficulties to overcome. One is to show that \( R\text{Hom}_{k_{\text{indrat proet}}} (G, H) \) behaves well for ind-proalgebraic groups (i.e., ind-objects of proalgebraic groups) \( G, H \). We already know this for proalgebraic \( G \) and algebraic \( H \) by [Suz13 Thm. 2.1.5]. To extend this result for ind-proalgebraic groups, we will need heavy derived limit arguments following Kashiwara-Schapira [KS06, Chap. 15]. Second, cohomology groups of the form \( H^n(K(k'), A) \) are not completely classical when \( k' \) has infinitely many direct factors and \( \text{Spec} k' \) as a topological space is profinite. The ring \( K(k') \) is a finite direct product of complete discrete valuation fields if \( k' \) has only finitely many direct factors, but otherwise it is quite complicated. We study some site-theoretic properties of the ring \( K(k') \) by approximation by complete discrete valuation subfields. In [Suz13 §2.5], we calculated \( H^n(K(k'), G_m) \). We need to further develop the techniques used there in order to calculate \( H^n(K(k'), A) \) for an abelian variety \( A \). The third point is that we need to redo Bester’s work [Bes78] within the style of this paper. This is lengthy, but there is essentially no new idea needed in this part.

1.3. Organization. This paper is organized as follows. In §2 we study how to treat ind-proalgebraic groups over \( k \) as sheaves on \( \text{Spec} k_{\text{indrat proet}} \). In §3 we view cohomology of \( K \) with various coefficients as sheaves on \( \text{Spec} k_{\text{indrat proet}} \) and compute
it. In §4, we construct the duality morphism of Theorem A and show that it induces two pairings
\[ \pi_0(A^\vee) \times \pi_0(A_x) \to \mathbb{Q}/\mathbb{Z}, \quad H^1(A^\vee) \times \pi_1(\Gamma(A)) \to \mathbb{Q}/\mathbb{Z}. \]
The perfectness of these pairings is equivalent to Theorem A. In §5, we show that the first pairing agrees with Grothendieck’s pairing and the second with Bégueri-Bester-Bertapelle’s pairing. This proves Theorem B (a) and the semistable case of Theorem A. In §6, we prove that Galois descent works for Theorem A, namely Theorem B (b). In §7, we summarize the results of the preceding sections to conclude the proof of Theorems A, B, and C, which finishes the proof of Grothendieck’s conjecture.

In §8, we formulate and prove an analogue of Theorem A for tori. Most of the results of this section has already been obtained by Bégueri (loc.cit.), Xarles [Xar93] and Bertapelle-González-Avilés [BGA15]. In §9, the dualities for abelian varieties and tori are combined into a duality for 1-motives. This extends the finite residue field case of the duality shown by Harari-Szamuely [HS05]. In §10, we show, in the finite residue field case, how to pass from the above sheaf setting to the classical setting. In Appendix A, we explain how to treat proalgebraic groups that are proschemes but not schemes. We do this by extending the pro-fppf topology introduced in [Suz13] for affine schemes to proalgebraic proschemes.

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Notation. We fix a perfect field \( k \) of characteristic \( p > 0 \). A perfect field over \( k \) is said to be finitely generated if it is the perfection (direct limit along Frobenii) of a finitely generated field over \( k \). The same convention is applied to morphisms of perfect \( k \)-algebras or \( k \)-schemes being finite type, finite presentation, etc. The categories of sets and abelian groups are denoted by Set and Ab, respectively. Set theoretic issues are omitted for simplicity as the main results hold independent of the choice of universes. The opposite category of a category \( C \) is denoted by \( C^{op} \). The procategory and indcategory of \( C \) are denoted by \( P\mathcal{C} \) and \( I\mathcal{C} \), respectively, so that \( I\mathcal{C} = I(P\mathcal{C}) \) is the ind-procategory. All group schemes (except for Galois groups) are assumed to be commutative. We say that a (commutative) étale group scheme over \( k \) is finitely generated if its group of geometric points is finitely generated as an abelian group. A lattice is a finitely generated étale group with no torsion or, equivalently, a free abelian group with a Galois action. For an abelian category \( C \), we denote by \( D^b(C) \), \( D^+(C) \), \( D^-(C) \), \( D(C) \) its bounded, bounded below, bounded above and unbounded derived categories, respectively. If we say \( A \to B \to C \) is a distinguished triangle that
in a triangulated category, we implicitly assume that a morphism $C \to A[1]$ to the shift of $A$ is given and $A \to B \to C \to A[1]$ is distinguished. For a Grothendieck site $S$ and a category $C$, we denote by $\mathcal{C}(S)$ the category of sheaves on $S$ with values in $C$. For an object $X$ of $S$, the category $S/X$ of objects of $S$ over $X$ is equipped with the induced topology ([AGV72, III, §3]), which is the localization of $S$ at $X$ ([AGV72, III, §5]). The category $D^*(\text{Ab}(S))$ for $* = b, +, -$ or (blank) is also denoted by $D^*(S)$. By a continuous map $f: S' \to S$ between sites $S'$ and $S$, we mean a continuous functor from the underlying category of $S$ to that of $S'$, i.e., the right composition (or the pushforward $f_*$) sends sheaves on $S'$ to sheaves on $S$. By a morphism $f: S' \to S$ of sites we mean a continuous map whose pullback functor $\text{Set}(S) \to \text{Set}(S')$ is exact. For an abelian category $C$, we denote by $\text{Ext}^i_C$ the $i$-th Ext functor for $C$. If $C = \text{Ab}(S)$, we also write $\text{Ext}^i_S = \text{Ext}^i_C$. The sheaf-Hom and sheaf-Ext functors are denoted by $\text{Hom}_S$ and $\text{Ext}^n_S$, respectively. For sites such as $\text{Spec} k_{\text{indrat proet}}$, we also use $\text{Ext}^k_{\text{indrat proet}}$, $\text{Ab}(k_{\text{indrat proet}})$ etc. omitting Spec from the notation. Similarly, cohomology of the site $\text{Spec} k_{\text{indrat proet}}$ for example is denoted by $R\Gamma(k_{\text{indrat proet}}, A)$, where $A$ is a sheaf of abelian groups.

2. Site-theoretic preliminaries

Let $k$ be a perfect field of characteristic $p > 0$. In this section, we introduce the ind-rational pro-étale site $\text{Spec} k_{\text{proet}}$. We also recall the perfect pro-fppf site $\text{Spec} k_{\text{profppf}}$ from [Suz13]. We call a help from $\text{Spec} k_{\text{profppf}}$ to establish a basic method of treating ind-proalgebraic groups as sheaves on $\text{Spec} k_{\text{proet}}$. This needs to extend the result [Suz13, Thm. 2.1.5] on Ext groups of proalgebraic groups to ind-proalgebraic groups. For this, we develop a general study on the derived Hom functor $R\text{Hom}$ on ind-procategories. Then we can embed the derived category of ind-proalgebraic groups into the derived category of sheaves on $\text{Spec} k_{\text{proet}}$. We also extend Serre duality ([Ser60, 8.4, Remarque], [Mil06, III, Thm. 0.14]) on perfect unipotent groups (= quasi-algebraic unipotent groups) to proalgebraic and ind-algebraic groups.

At the end of this section, we introduce the notion of $P$-acyclicity. In the next section, we will view the cohomology of complete discrete valuation fields with residue field $k$ as sheaves on the ind-rational étale site $\text{Spec} k_{\text{et}}$ first and then as sheaves on $\text{Spec} k_{\text{proet}}$ by sheafification. The cohomology of the pro-étale sheafification of an étale sheaf is difficult to calculate in general. In the situations we are interested in, however, we will see that most of these étale sheaves are $P$-acyclic. This means that the associated pro-étale sheaves still remember the original étale sheaves and hence the cohomology of the original complete discrete valuation field. This notion is also useful when we want to obtain some information specific to a non-closed residue field $k$. See (7.3) and (8.9).
2.1. Sites and algebraic groups: setup and first properties. As in \cite{BS15} Def. 2.1.1, we say that a perfect $k$-algebra $k'$ is rational if it is a finite direct product of finitely generated perfect fields over $k$, and ind-rational if it is a filtered union of rational $k$-subalgebras. Since any perfect field over $k$ is ind-rational, we know that a $k$-algebra is ind-rational if and only if it can be written as a filtered union of finite products of (not necessarily finitely generated) perfect fields over $k$. The rational (resp. ind-rational) $k$-algebras form a full subcategory of the category of perfect $k$-algebras, which we denote by $k^{\text{rat}}$ (resp. $k^{\text{indrat}}$). Since a $k$-algebra homomorphism from $k' \in k^{\text{rat}}$ to $k'' = \bigcup k''_i \in k^{\text{indrat}}$ factors through some $k''_i \in k^{\text{rat}}$, we know that $k^{\text{indrat}}$ is equivalent to the indcategory of $k^{\text{rat}}$\footnote{Do not confuse $k^{\text{indrat}}$ with another indcategory $\mathcal{IC}$, where $\mathcal{C}$ is the category of finite products of not necessarily finitely generated perfect fields over $k$ with $k$-algebra homomorphisms. There are natural functors $k^{\text{indrat}} \to \mathcal{IC} \to k^{\text{indrat}}$. The first one is fully faithful and the second one is essentially surjective, with composite the identity functor. The object of $\mathcal{C}$ given by the perfection of the field $k(x_1, x_2, \ldots)$ and the ind-object in $\mathcal{C}$ consisting of the increasing family of the perfections of the fields $k(x_1, \ldots, x_n)$ are not isomorphic in $\mathcal{IC}$, but become isomorphic in $k^{\text{indrat}}$.}. We define the rational (resp. ind-rational) étale site of $k$ to be the étale site on $k^{\text{rat}}$ (resp. $k^{\text{indrat}}$) \cite{Suz13} Def. 2.1.3). These sites are denoted by $\text{Spec } k^{\text{rat}}_\text{et}$ and $\text{Spec } k^{\text{indrat}}_\text{et}$, respectively.

We also introduce the pro-étale topology on $k^{\text{indrat}}$. The pro-étale topology for schemes is introduced in \cite{BS15}. We throughout use the affine variant of the pro-étale site \cite{BS15} Def. 4.2.1, Rmk. 4.2.5], which behaves simpler for limit arguments. Note that any $k$-algebra étale over an ind-rational $k$-algebra is ind-rational \cite{Suz13} Prop. 2.1.2]. Hence if $k' \in k^{\text{indrat}}$ and $R$ is a perfect $k'$-algebra, then $R$ is ind-étale \cite{BS15} Def. 2.2.1.5] over $k'$ if and only if $R \cong \lim_{\lambda} k'_\lambda$ for some filtered direct system $\{k'_\lambda\}$ of étale $k'$-algebras ind-rational over $k$. In particular, such an $R$ itself is ind-rational over $k$. Therefore we can introduce the pro-étale topology on the category $k^{\text{indrat}}$. That is, a covering of an ind-rational $k$-algebra $k'$ is a finite family $\{k'_i\}$ of ind-étale $k'$-algebras such that $\prod k'_i$ is faithfully flat over $k'$. We call the resulting site the ind-rational pro-étale site and denote it by $\text{Spec } k^{\text{indrat}}_{\text{proet}}$.

Some care is needed for localizations (see Notation) of $\text{Spec } k^{\text{rat}}_\text{et}$, $\text{Spec } k^{\text{indrat}}_\text{et}$ and $\text{Spec } k^{\text{indrat}}_{\text{proet}}$, which comes from subtleties of the underlying categories $k^{\text{rat}}$ and $k^{\text{indrat}}$. See \cite{Suz13} the paragraphs after Def. 2.1.3 for the details. We quickly recall the facts and notation there. As in Notation in this paper, for $k' \in k^{\text{indrat}}$, the category of objects over $k'$ in $k^{\text{indrat}}$ (i.e. the category of $k'$-algebras ind-rational over $k$) is denoted by $k^{\text{indrat}}/k'$, and the localization of $\text{Spec } k^{\text{proet}}$ at $k'$ is denoted by $\text{Spec } k^{\text{proet}}_{\text{et}}/k'$. If $k'$ is a field that is in $k^{\text{rat}}$ or algebraic over $k$, then $k^{\text{indrat}} = k^{\text{rat}}/k'$ \cite{Suz13} loc. cit.], hence

\[
\text{Spec } k^{\text{indrat}}_{\text{et}} = \text{Spec } k^{\text{indeat}}_{\text{et}}/k', \quad \text{Spec } k^{\text{indrat}}_{\text{proet}} = \text{Spec } k^{\text{proet}}_{\text{et}}/k'.
\]

For any $k' \in k^{\text{rat}}$, we define

\[
k^{\text{indrat}} := k^{\text{indrat}}/k', \quad \text{Spec } k^{\text{indrat}}_{\text{et}} := \text{Spec } k^{\text{indeat}}_{\text{et}}/k', \quad \text{Spec } k^{\text{indrat}}_{\text{proet}} := \text{Spec } k^{\text{proet}}_{\text{et}}/k'.
\]
Spec $k_{\text{perf}}^{\text{prof}}$, Spec $k_{\text{prof}}^{\text{proet}}$, respectively. A covering in Spec $k_{\text{prof}}^{\text{proet}}$ is a finite family $\{R \to S_i\}$ with each $S_i$ ind-étale over $R$ and $\prod S_i$ faithfully flat over $R$. For Spec $k_{\text{prof}}^{\text{proppf}}$, recall from [Suz13, §3.1] that a homomorphism $R \to S$ in $k_{\text{perf}}^{\text{prof}}$ is said to be flat of finite presentation if $S$ is the perfection of a flat $R$-algebra of finite presentation. Also, a homomorphism $R \to S$ in $k_{\text{perf}}^{\text{prof}}$ is said to be flat of ind-finite presentation if it can be written as a filtered direct limit of flat homomorphisms $R \to S_i$ of finite presentation ([Suz13, Def. 3.1.1]). A covering in Spec $k_{\text{prof}}^{\text{proppf}}$ is a finite family $\{R \to S_i\}$ with each $S_i$ flat of ind-finite presentation and $\prod S_i$ faithfully flat over $R$. For a perfect $k$-algebra $R$, the category of objects over $R$ in $k_{\text{perf}}^{\text{prof}}$ is nothing but the category of perfect $R$-algebras, in contrast to the case of the category of ind-rational $k$-algebras $k_{\text{indrat}}^{\text{proet}}$. Hence we will write the localization of Spec $k_{\text{perf}}^{\text{prof}}$ at $R$ by Spec $k_{\text{perf}}^{\text{profppf}}/R = \text{Spec } R_{\text{perf}}^{\text{proppf}}$. Similar notation applies to Spec $k_{\text{prof}}^{\text{et}}$ and Spec $k_{\text{proet}}^{\text{et}}$.

For $R \in k_{\text{perf}}^{\text{prof}}$, we define the small pro-étale site Spec $R_{\text{proet}}$ of $R$ to be the category $R_{\text{proet}}$ of ind-étale $R$-algebras ([BS15, Def. 2.2.1.5]) with the topology induced from Spec $k_{\text{perf}}^{\text{proet}}$. That is, a covering $\{R'_i\}$ of an object $R' \in R_{\text{proet}}$ is a finite family of ind-étale $R'_i$-algebras such that $\prod R'_i$ is faithfully flat over $R'$.

We have the following commutative diagram of continuous maps of sites:

$$
\begin{array}{ccc}
\text{Spec } k_{\text{prof}}^{\text{proppf}} & \longrightarrow & \text{Spec } k_{\text{prof}}^{\text{proet}} \\
\downarrow & & \downarrow \\
\text{Spec } k_{\text{proet}}^{\text{proppf}} & \longrightarrow & \text{Spec } k_{\text{et}}^{\text{profppf}} \\
\end{array}
$$

(2.1.1)

All the maps are defined by the identity. The horizontal maps are morphisms of sites (i.e., have exact pullback functors), but the vertical ones are not [Suz13, Prop. 3.2.3].

Let $\text{Alg}/k$ be the category of quasi-algebraic groups (commutative, as assumed throughout the paper) over $k$ in the sense of Serre [Ser60]. Recall that a quasi-algebraic group is the perfection (inverse limit along Frobenii) of an algebraic group [Ser60, §1.2 Déf. 2; §1.4, Prop. 10], [Pep14, Prop. 1.2.10]. The category $\text{Alg}/k$ is an abelian category [Ser60, §1.2, Prop. 5]. We simply call an object of the procategory $\text{PAlg}/k$ a proalgebraic group following [Ser60, §2.1, Déf. 1; §2.6, Prop. 12]. Similarly, we call an object of the indcategory $\text{IAlg}/k$ an ind-algebraic group and an object of the ind-procategory $\text{IPAlg}/k = \text{I}(\text{PAlg}/k)$ an ind-proalgebraic group. (Therefore we will not say $\mathbb{Z}$ ind-algebraic or ind-proalgebraic in this paper.) Note that the indcategory (and hence also the procategory) of an abelian category is an abelian category [KS06, Thm. 8.6.5 (i)]. Following [Ser60, §1.3], we say that a quasi-algebraic group is a unipotent group, a torus or an abelian variety if it is the perfection of a unipotent group, a torus or an abelian variety, respectively. Let $\text{Alg}_{\text{uc}}/k \subset \text{Alg}/k$ be the full subcategory consisting of those whose identity component is unipotent. This is the direct product of the category of (not necessarily connected) unipotent quasi-algebraic groups and the category of finite étale groups of order prime to $p$. We have fully faithful exact embeddings of abelian categories

$$
\begin{array}{ccc}
\text{Alg}/k & \longrightarrow & \text{PAlg}/k \\
\cap & & \cap \\
\text{IAlg}/k & \longrightarrow & \text{IPAlg}/k
\end{array}
$$
by [KS06] Thm. 8.6.5 (ii)]. Let \( \text{LAlg}/k \) be the category of perfects of smooth group schemes over \( k \) (which contains \( \mathbb{Z} \) but is not abelian). Let \( \text{Et}/k \) be the category of étale group schemes over \( k \). Let \( \text{FEt}/k \) be its full subcategory consisting of finitely generated étale groups, namely those whose group of geometric points is a finitely generated abelian group. Let \( \text{FEt}/k \) be the full subcategory consisting of finite étale group schemes. The categories \( \text{FEt}/k \), \( \text{FGEt}/k \) and \( \text{Et}/k \) are abelian. We have fully faithful embeddings \( \text{FEt}/k \subset \text{FGEt}/k \subset \text{Et}/k \subset \text{LAlg}/k \). The identity component of \( A \in \text{LAlg}/k \) is denoted by \( A_0 \). The formation of \( A_0 \) is functorial, so the notion and notation of identity component extend to any \( A \in \text{LAlg}/k \), \( \text{PAlg}/k \) or \( \text{IPAlg}/k \). We define \( \pi_0(A) = A/A_0 \), which is in \( \text{IFEt}/k \), \( \text{PFEt}/k \) or \( \text{IPFe}/k \) in each case. We say that \( A \in \text{IPAlg}/k \) is connected if \( \pi_0(A) = 0 \). We have natural additive functors \( \text{IPAlg}/k, \text{LAlg}/k/k \otimes \text{Ab}(\text{kindrat}) \).

We say that a sheaf \( A \) of abelian groups on a site \( S \) is acyclic if \( H^n(X, A) = 0 \) for any object \( X \) of \( S \) and any \( n \geq 1 \). If \( S, S' \) are sites defined by pretopologies, and if \( u \) is a functor from the underlying category of \( S \) to that of \( S' \) that sends coverings to coverings and \( u(Y \times_X Z) = u(Y) \times_{u(X)} u(Z) \) whenever \( Y \to X \) appears in a covering family, then \( u \) defines a continuous map \( f: S' \to S \), and \( f_* \) sends acyclic sheaves to acyclic sheaves and hence induces the Leray spectral sequence \( R\Gamma(X, Rf_*A') = R\Gamma(u(X), A') \) for any \( X \in S \) and \( A' \in \text{Ab}(S') \) [Art62, §II.4].

**Proposition (2.1.2).**

(a) On any of the sites in the diagram, cohomology of any object of the site commutes with filtered direct limits of coefficient sheaves. Products and filtered direct limits of acyclic sheaves are again acyclic.

(b) The sites \( \text{Spec} k_{\text{profppf}} \), \( \text{Spec} k_{\text{proet}} \) and \( \text{Spec} k_{\text{perf}} \) are coherent, i.e., the objects of their underlying categories are quasi-compact and stable under finite inverse limits.

(c) Let \( f: \text{Spec} k_{\text{proet}}^\text{perf} \to \text{Spec} k_{\text{proet}}^\text{indrat} \) be the continuous map defined by the identity. Then \( f \) induces isomorphisms on cohomology. More precisely, \( f_* \) is exact, \( f_* f^* = \text{id} \) on \( \text{Ab}(k_{\text{proet}}^\text{indrat}) \) (hence \( f^* \) is a fully faithful embedding), and

\[
R\Gamma(k_{\text{proet}}^\text{indrat}/k', f_* A) = R\Gamma(k_{\text{proet}}^\text{perf}, A)
\]

for any \( k' \in k_{\text{indrat}}^\text{indrat} \) and \( A \in \text{Ab}(k_{\text{proet}}^\text{indrat}) \), where the left-hand side is the cohomology of the site \( \text{Spec} k_{\text{proet}}^\text{indrat} \) at \( k' \). The same is true for the étale version \( \text{Spec} k_{\text{et}}^\text{perf} \to \text{Spec} k_{\text{et}}^\text{indrat} \).

(d) In the categories \( \text{Set}(k_{\text{profppf}}^\text{perf}) \), \( \text{Set}(k_{\text{proet}}^\text{perf}) \) and \( \text{Set}(k_{\text{proet}}^\text{indrat}) \), the product of any family of surjections is a surjection. This implies that \( \text{Ab}(k_{\text{proet}}^\text{profppf}) \), \( \text{Ab}(k_{\text{profppf}}^\text{perf}) \) and \( \text{Ab}(k_{\text{proet}}^\text{indrat}) \) are AB4* categories.

(e) The natural functor from \( \text{PAlg}/k \) to any of \( \text{Ab}(k_{\text{profppf}}^\text{perf}), \text{Ab}(k_{\text{profppf}}^\text{perf}), \text{Ab}(k_{\text{proet}}^\text{indrat}) \) is exact.

---

More customary notation is \( A^0 \) or \( A^c \). We prefer the subscript 0 in this paper, however, since we extensively use derived categories and \( A^0 \) may be ambiguous to the zeroth term of a complex \( \cdots \to A^0 \to A^1 \to \cdots \). A circle \( \circ \) in the script size looks too similar to a zero \( 0 \) and \( A^0 \) may still be confusing in this paper. Moreover, we use below tons of duality operations such as the Serre dual \( A^{SD} \), the Pontryagin dual \( A^{PD} \), the dual abelian variety \( A^\vee \) etc. We want to distinguish such contravariant operations from the covariant operation \( A \to A_0 \).
(f) Let $A = \varprojlim_{\lambda} A_{\lambda} \in P\text{Alg}/k$ with $A_{\lambda} \in \text{Alg}/k$. Let $R\varprojlim_{\lambda} A_{\lambda}$ be the derived functor of $\varprojlim_{\lambda} A_{\lambda}$ considered in either $D(k_{\text{profpf}}^\text{perf})$, $D(k_{\text{proet}}^\text{perf})$ or $D(k_{\text{indrat}}^\text{proet})$. Then we have $R\varprojlim_{\lambda} A_{\lambda} = A$ in each case.

(g) Let $f: \text{Spec} k_{\text{profpf}}^\text{perf} \to \text{Spec} k_{\text{proet}}^\text{perf}$ be the morphism defined by the identity. Let $g$ be either the morphism $\text{Spec} k_{\text{proet}}^\text{perf} \to \text{Spec} k_{\text{proet}}^\text{et}$ or $\text{Spec} k_{\text{proet}}^\text{indrat} \to \text{Spec} k_{\text{proet}}^\text{et}$ defined by the identity. If $A \in I\text{Alg}/k$, the pro-fppf cohomology with coefficients in $A$ agrees with the pro-étale cohomology: $Rf_* A = A$. If $A \in I\text{Alg}/k$ or $L\text{Alg}/k$, the pro-fppf cohomology with coefficients in $A$ agrees with the étale cohomology: $Rf_* A = A$ and $Rg_* A = A$.

Proof. [3] This is true for any site defined by finite coverings. See [Mil80, III, Rmk. 3.6] for the statement that cohomology commutes with filtered direct limits. It follows that filtered direct limits of acyclics are acyclic. If a family of sheaves has vanishing higher Čech cohomology, then so is the product. We can deduce the corresponding statement for the derived functor cohomology by [Mil80, III, Prop. 2.12].

[5] Obvious.

[6] The exactness of $f_*$ is obvious. We know that $f_*$ commutes with sheafification and $f_* f^{-1} = \text{id}$, where $f^{-1}$ is the pullback for presheaves. Hence $f_* f^* = \text{id}$. The stated equality is the Leray spectral sequence for $f$, which is available by the remark in the paragraph before the proposition. The same proof works for the étale version $\text{Spec} k_{\text{proet}}^\text{et} \to \text{Spec} k_{\text{proet}}^\text{indrat}$.

[7] Let $\{G_{\lambda} \to F_{\lambda}\}$ be a family of surjections in $\text{Set}(k_{\text{profpf}}^\text{perf})$. Let $R \in k_{\text{proet}}^\text{perf}$ and $\{s_{\lambda}\} \in \prod F_{\lambda}(R)$. For each $\lambda$, there are an object $S_{\lambda} \in k_{\text{proet}}^\text{perf}$ faithfully flat of ind-finite presentation over $R$ and a section $t_{\lambda} \in G_{\lambda}(S_{\lambda})$ that maps to the natural image $s_{\lambda} \in F_{\lambda}(S_{\lambda})$. The tensor product $S$ of all the $S_{\lambda}$ over $R$ (which is the filtered direct limit of the finite tensor products) is again faithfully flat of ind-finite presentation over $R$. Since $\{t_{\lambda}\} \in \prod G_{\lambda}(S)$ maps to $\{s_{\lambda}\} \in \prod F_{\lambda}(S)$, the morphism $\prod G_{\lambda} \to \prod F_{\lambda}$ is surjective. The same proof works for $\text{Set}(k_{\text{proet}}^\text{et})$ and $\text{Set}(k_{\text{proet}}^\text{indrat})$.

[8] In either case, the left exactness is obvious.

For the right exactness of $P\text{Alg}/k \to \text{Ab}(k_{\text{profpf}}^\text{perf})$, let $0 \to A \to B \to C \to 0$ be an exact sequence in $P\text{Alg}/k$. To show that $B \to C$ is a surjection in $\text{Ab}(k_{\text{profpf}}^\text{perf})$, it is enough to show that $B \to C$ is faithfully flat of profinite presentation. Suppose first that $A \in \text{Alg}/k$. Let $\{B_{\mu}\}$ be a filtered decreasing family of proalgebraic subgroups of $B$ such that $B \cong \varprojlim B_{B_{\mu}}$ (i.e., $\bigcap B_{\mu} = 0$: [Ser60, §2.5, Cor. 3 to Prop. 10]) and $B_{B_{\mu}} \in \text{Alg}/k$ for all $\mu$. Then $A \cap B_{\mu} = 0$ for some $\mu$ since $\text{Alg}/k$ is artinian [Ser60, §1.3, Prop. 6]. This implies $B = (B/B_{\mu}) \times_{C/B_{\mu}} C$. We have an exact sequence $0 \to A \to B/B_{\mu} \to C/B_{\mu} \to 0$ in $\text{Alg}/k$. Hence $B/B_{\mu} \to C/B_{\mu}$ is faithfully flat of finite presentation. Its base change $B \to C$ is thus faithfully flat of finite presentation. Suppose next that $A \in P\text{Alg}/k$. Write $A = \varprojlim A_{A_{\lambda}}$ with $A_{A_{\lambda}} \in \text{Alg}/k$ as above. Then $B \to C$ can be written as the filtered inverse limit of the morphisms $B_{A_{\lambda}} \to C$. The kernel of $B_{A_{\lambda}} \to C$ is $A_{A_{\lambda}} \in \text{Alg}/k$. Hence the previous case shows that $B_{A_{\lambda}} \to C$ is faithfully flat of finite presentation.

By (3) (exactness of pushforward), the exactness of $P\text{Alg}/k \to \text{Ab}(k_{\text{proet}}^\text{indrat})$ is reduced to that of $P\text{Alg}/k \to \text{Ab}(k_{\text{proet}}^\text{proet})$. To show this final statement, let $0 \to A \to B \to C \to 0$ be an exact sequence in $P\text{Alg}/k$. Suppose that we are given an $X$-valued point of $C$, where $X = \text{Spec} R$ with $R \in k_{\text{proet}}^\text{perf}$. By the exactness of
PAlg/k → Ab(k_
{proet}^{perf}), the fiber Y of the morphism B → C over the X-valued point X → C is a pro-fppf torsor for A over X. We want to show that Y → X admits a section pro-étale locally. By [BST15, Cor. 2.2.14], there exists a pro-étale cover X' → X such that X' is w-strictly local [BST15, Def. 2.2.1]. In particular, every étale cover of X' admits a section. Therefore we may assume that X itself is w-strictly local. We show that Y → X then admits a section. Let \{A_\lambda\} be a filtered decreasing family of proalgebraic subgroups of A such that A \cong \varprojlim A/A_\lambda (i.e., \bigcap A_\lambda = 0) and A/A_\lambda ∈ Alg/k for all \lambda. Then we have Y \cong \varprojlim Y/A_\lambda and Y/A_\lambda is a pro-fppf torsor for A/A_\lambda over X. Since A/A_\lambda is quasi-algebraic, we know that H^I(X_{\text{propp}}, A/A_\lambda) = H^I(X_{\text{et}}, A/A_\lambda) by [Suz13, Cor. 3.3.3]. Therefore Y/A_\lambda is an étale torsor for A/A_\lambda over X. For each \lambda, this torsor is trivial since X is w-strictly local. In this situation, if R is an algebraically closed field, then [Ser60, 2.3, Prop. 2] or [SY12, Lem. 3.7] says that Y → X = Spec R admits a section. The arguments there actually work for any w-strictly local X.

(1) The higher R\varprojlim vanishes on PAlg/k. The functors from PAlg/k to each of the categories preserves R\varprojlim by the previous two assertions and [Nee01, Lem. A.3.2].

(2) The statement for the morphism g on the ind-rational sites follows from that for the morphism g on the perfect sites by (3). The cases A ∈ IPAlg/k, LAlg/k are reduced to the cases A ∈ PAlg/k, Alg/k, respectively, by (2). Note that R\varprojlim commutes with Rf_* with this and (1), the case A ∈ PAlg/k is reduced to the case A ∈ Alg/k. Summing all up, we are reduced to the case A ∈ LAlg/k. This case is [Suz13, Cor. 3.3.3].

For the rest of the paper, we denote the object R\Gamma(k_{\text{infrat}}^{\text{proet}} / k', f_*A) appearing in (2) simply by 

R\Gamma(k_{\text{proet}}^{\text{infrat}}, A).

This is the same as the cohomology of the small pro-étale site Spec k_{\text{proet}} of coefficients given by the restriction of A to Spec k_{\text{proet}}^{\text{infrat}} with big sites (Spec k_{\text{proet}}^{\text{perf}}) and small sites (Spec k_{\text{proet}}^{\text{infrat}}) have the same cohomology theory ([Mil80, III, Rmk. 3.2]). Its étale version R\Gamma(k_{\text{et}}^{\text{infrat}}, A) is used in a similar sense.

Note that the sites Spec k_{\text{infrat}}^{\text{proet}} and Spec k_{\text{infrat}}^{\text{proet}} are not coherent. If x and y are the generic points of irreducible varieties over k, then their fiber product as a sheaf on these sites is given by the disjoint union of the points (identified with the Spec of their residue fields) of the underlying set of the usual fiber product x ×_k y. This is infinite unless x or y is finite over k.

2.2. Generals on derived categories of ind-procategories. A general reference on derived categories of indcategories is [KS06, Chap. 15]. We need to develop some more here. For a certain abelian category A, we describe the derived Hom functor RHom_{IP,A} on the ind-procategory IP,A = I(P,A) (i.e. the ind-category of the pro-category P,A) in terms of RHom_A. Recall that Hom_{IP,A} is defined as the inverse limit of the direct limit of the inverse limit of the direct limit of Hom_A. Roughly speaking, the two inverse limits and the one Hom functor should be derived to get the required description of RHom_{IP,A}. This turns out to be quite complicated both notationally and mathematically. We organize it by introducing a relatively reasonable notation. In the next subsection, we will apply these results and notation to A = Alg/k and Ab(k_{\text{infrat}}^{\text{proet}}). To clarify the argument, we generalize
the situation and treat small abelian categories and Grothendieck categories in this subsection.

We need notation. Let \( \mathcal{A} \) be an additive category. A filtered direct system \( \{ A_\lambda \}_{\lambda \in \Lambda} \in \mathcal{L} \) will occasionally be written as \( \varprojlim_{\lambda \in \Lambda} A_\lambda \). This is the direct limit of the \( A_\lambda \) in \( \mathcal{L} \). If \( \{ A_\lambda \}_{\lambda \in \Lambda} \) is a family of objects of \( \mathcal{A} \), then its direct sum in \( \mathcal{L} \) (which is the filtered direct limit of finite sums) is denoted by \( \bigoplus_{\lambda \in \Lambda} A_\lambda \). If \( \mathcal{A}' \) is a full additive subcategory of \( \mathcal{A} \), then we denote by \( \bigoplus_{\lambda \in \Lambda} A_\lambda \) with \( A_\lambda \in \mathcal{A}' \). A similar notation is applied to \( \mathcal{P} \mathcal{A} \), for example, objects \( \varprojlim_{\lambda \in \Lambda} A_\lambda \), \( \prod_{\lambda \in \Lambda} A_\lambda \) of \( \mathcal{P} \mathcal{A} \) and full additive subcategories \( \prod_{\lambda \in \Lambda} A_\lambda \) of \( \mathcal{P} \mathcal{A} \).

Now let \( \mathcal{A} \) be an abelian category. Assume that \( \mathcal{A} \) is small. Then \( \mathcal{P} \mathcal{A} \) has exact filtered inverse limits and a set of cogenerators given by objects of \( \mathcal{A} \). Hence \( \mathcal{P} \mathcal{A} \) is a co-Grothendieck category (i.e. the opposite of a Grothendieck category) and, in particular, has enough projectives. Therefore the Hom functor on \( \mathcal{P} \mathcal{A} \) admits a derived functor \( R \hom_{\mathcal{P} \mathcal{A}} \). Its restriction to the bounded derived category of \( \mathcal{A} \) is denoted by \( R \hom_{\mathcal{A}} \), namely we define

\[
R \hom_{\mathcal{A}} (A, B) := R \hom_{\mathcal{P} \mathcal{A}} (A, B) \quad \text{for } A, B \in \mathcal{D}^b(\mathcal{A}).
\]

We do not claim that this \( R \hom_{\mathcal{A}} \) is the derived functor of \( \hom_{\mathcal{A}} \). By \[KS06\] Thm. 15.3.1, (i)], the natural functor \( \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{P} \mathcal{A}) \) is fully faithful, hence

\[
H^n R \hom_{\mathcal{A}} (A, B) = \ext^n_{\mathcal{A}}(A, B) \quad \text{for } A, B \in \mathcal{A},
\]

where the right-hand side is the usual Ext functor for the abelian category \( \mathcal{A} \). Therefore if the derived functor of \( \hom_{\mathcal{A}} \) on the bounded derived category exists, then the morphism of functors from this derived functor to the above \( R \hom_{\mathcal{A}} \) defined by universality is an isomorphism. In general, we do not assume this. If \( A = \varprojlim_{\lambda} A_\lambda \in \mathcal{P} \mathcal{A} \) and \( B \in \mathcal{A} \), then

\[
\ext^n_{\mathcal{P} \mathcal{A}} (A, B) = \lim_{\lambda} \ext^n_{\mathcal{A}}(A_\lambda, B)
\]

for all \( n \) by \[KS06\] Cor. 15.3.9]. We express this result by the equality

\[
R \hom_{\mathcal{P} \mathcal{A}} (A, B) = \lim_{\lambda} R \hom_{\mathcal{A}} (A_\lambda, B),
\]

which is intuitive and convenient, but may be confusing since the \( \varprojlim_{\lambda} \) in the right-hand side is not a direct limit in the triangulated category \( \mathcal{D}(\mathcal{A}) \).

Next we want to treat the case that \( B \) also is a pro-object and, more generally, the case \( A, B \in \mathcal{P} \mathcal{A} \). An object \( A \in \mathcal{P} \mathcal{A} \) can be written as \( A = \varprojlim_{\lambda \in \Lambda} A_\lambda \), where each \( A_\lambda \in \mathcal{P} \mathcal{A} \) can be written as \( A_\lambda = \varprojlim_{\lambda' \in \Lambda'} A_{\lambda \lambda'} \) with \( A_{\lambda \lambda'} \in \mathcal{A} \). Note that the morphisms \( A_{\lambda_1} \to A_{\lambda_2} \in \mathcal{P} \mathcal{A} \) for \( \lambda_1 \leq \lambda_2 \in \Lambda \) are assumed as given, but no maps \( \Lambda_1 \to \Lambda_2 \) between the index sets are assumed. The Hom functor for \( \mathcal{P} \mathcal{A} \) is given by definition as

\[
\hom_{\mathcal{P} \mathcal{A}} (A, B) = \lim_{\lambda} \lim_{\mu} \lim_{\mu'} \hom_{\mathcal{A}} (A_{\lambda \lambda'}, B_{\mu \mu'})
\]

for \( A = \varprojlim_{\lambda} \varprojlim_{\lambda'} A_{\lambda \lambda'} \in \mathcal{P} \mathcal{A} \) and \( B = \varprojlim_{\mu} \varprojlim_{\mu'} B_{\mu \mu'} \in \mathcal{P} \mathcal{A} \), which is of the form

\[
\hom_{\mathcal{P} \mathcal{A}} : \mathcal{P}(\mathcal{A}^{op}) \times \mathcal{P} \mathcal{A} \to \mathcal{A}.
\]

A general method to derive this type of functors is the following.
Proposition (2.2.1). Let $A, B, C$ be abelian categories and $F : A \times B \to C$ an additive bifunctor that is left exact in both variables. Let $A', B', C'$ be full additive subcategories of $A, B, C$, respectively, such that $F(A' \times B') \subset C'$. Let $PF : P A \times P B \to P C$, $IPF : IP A \times IP B \to IP C$, $PIPF : PIP A \times PIP B \to PIP C$ etc. be the natural extensions of $F$.

(a) Assume that the pair $(A', B')$ is $F$-injective in the sense of [KS06] Def. 13.4.2 (which implies the existence of the derived functor of $F$ on the bounded below derived categories). Then $(LA', IB')$ is $IF$-injective, and $(P A', PB')$ and $(\prod A', \prod B')$ are both $PF$-injective. In particular, $IF$ and $PF$ admit derived functors on the bounded below derived categories. The diagrams

\[ D^+(A) \times D^+(B) \xrightarrow{RF} D^+(C) \quad D^+(A) \times D^+(B) \xrightarrow{RF} D^+(C) \]

\[ D^+(IA) \times D^+(IB) \xrightarrow{RF} D^+(IC), \quad D^+(PA) \times D^+(PB) \xrightarrow{RF} D^+(PC) \]

commute. We have natural isomorphisms $R^nIF \cong 1R^nF$ and $R^nPF \cong PR^nF$ of functors for all $n$.

(b) Assume that: $C$ has products and exact filtered directed limits; $C'$ is closed by products and filtered directed limits; the subcategory $C' \subset C$ is cogenerating, i.e., any object of $C$ has an injection into an object of $C'$; and for any family of exact sequences $0 \to C^1_\lambda \to C^2_\lambda \to C^3_\lambda \to 0$ in $C$ with $C^1_\lambda, C^2_\lambda \in C'$ for any $\lambda$, we have $C^3_\lambda \in C'$ for any $\lambda$ and the sequence $0 \to \prod C^1_\lambda \to \prod C^2_\lambda \to \prod C^3_\lambda \to 0$ is exact. Then the sequence

\[ \lim P \lim \xrightarrow{PIPF} \lim P \lim \xrightarrow{PIPF} \lim P \lim \xrightarrow{PIPF} \lim P \lim \xrightarrow{PIPF} C \]

of functors restricts to the sequence

\[ \prod I \prod I' \to \prod I \prod I' \to \prod I \prod I' \to \prod I \prod I' \to \prod I \prod I' \to C' \]

on the subcategories. Each category in the latter sequence is injective with respect to the functor that follows in the former sequence (i.e. $\prod I \prod I' C'$ is $PIPF$-injective etc.). In particular, the composite of the derived functors of the functors in the former sequence gives the derived functor of the “take-all-the-limits” functor

\[ \lim P \lim \xrightarrow{PIPF} \lim P \lim \xrightarrow{PIPF} \lim P \lim \xrightarrow{PIPF} \lim P \lim \xrightarrow{PIPF} C \]

(c) Under the assumptions of the previous two assertions, the composite

\[ PIP A \times PIP B \xrightarrow{PIPF} PIP C \xrightarrow{\lim P \lim \xrightarrow{PIPF} \lim P \lim \xrightarrow{PIPF} \lim P \lim \xrightarrow{PIPF} C} \]

admits a derived functor on the bounded below derived categories, which is given by the composite of the derived functors.

Recall that $(A', B')$ being $F$-injective means that: for any bounded below complex $A$ in $A$, there is a quasi-isomorphism $A \to A'$ to a bounded below complex $A'$ in $A'$; for any bounded below complex $B$ in $B$, there is a quasi-isomorphism $B \to B'$ to a bounded below complex $B'$ in $B'$; and $F(A', B')$ is an exact complex if $A'$ (resp. $B'$) is a bounded below complex in $A'$ (resp. $B'$) such that either $A'$ or $B'$ is exact. In this situation, according to [KS06 §13.4], the derived functor

\[ RF : D^+(A) \times D^+(B) \to D^+(C) \]
of the two-variable functor $F$ is defined by $RF(A, B) = F(A', B')$, where $A \xrightarrow{\sim} A'$ is a quasi-isomorphism to a bounded below complex in $A'$ and $B \xrightarrow{\sim} B'$ is a quasi-isomorphism to a bounded below complex in $B'$. We need to replace the both variables at the same time. In the proposition, we chose a pro-ind-pro-indcategory as an example. There is nothing special about this choice. There is a corresponding statement for any finite sequence of P's and I's.

Proof. (a) This is nothing but a two-variable version of [KS06, Prop. 15.3.2, 15.3.7]. We merely indicate what should be modified from the original single-variable version. Let $\mathcal{A}'$ be the full subcategory of $I\mathcal{A}$ consisting of objects $A \in I\mathcal{A}$ such that $1R^nF(A, B) = 0$ for any $B \in I\mathcal{B}'$ and $n \geq 1$. Let $\mathcal{B}'$ be the full subcategory of $I\mathcal{B}$ consisting of objects $B \in I\mathcal{B}$ such that $1R^nF(A, B) = 0$ for any $A \in \mathcal{A}'$ and $n \geq 1$. Then exactly in the same manner as [KS06] loc.cit., we can show that $\mathcal{A}' \times \mathcal{B}'$ contains $I\mathcal{A}' \times I\mathcal{B}'$ and that $(\mathcal{A}', \mathcal{B}')$ is IF-injective. We can deduce from this that $(I\mathcal{A}', I\mathcal{B}')$ is IF-injective. A similar argument for the pro version implies that $(P\mathcal{A}', P\mathcal{B}')$ is PF-injective. Since $\prod \mathcal{A}'$ is cogenerating in $P\mathcal{A}'$, we know that $(\prod \mathcal{A}', \prod \mathcal{B}')$ is PF-injective. We omit the details.

(b) The assumptions imply that the subcategory $\prod \mathcal{C} \subset P\mathcal{C}$ is injective with respect to the functor $\lim: P\mathcal{C} \to \mathcal{C}$ by [KS06, Prop. 13.3.15]. The subcategory $I\mathcal{C}' \subset I\mathcal{C}$ is cogenerating by [KS06, Thm. 15.2.5], hence injective with respect to the exact functor $\lim: I\mathcal{C} \to \mathcal{C}$. These imply the rest of the statement by an iterated usage of [KS06, Prop. 15.3.2, 15.3.7] and the theorem on derived functors of composition [KS06, Prop. 13.3.13].

(c) This follows from the previous two assertions and [KS06, Prop. 13.3.13]. □

Hence, in the situation of the proposition, we have
\[
R(\lim \leftarrow P \lim \leftarrow PI \lim \leftarrow PIP \lim \leftarrow PIPIF)(A, B)
\]
\[
= R(\lim \leftarrow P \lim \leftarrow RPI \lim \leftarrow PIP \lim \leftarrow RPIPIF)(A, B) \in D^+(\mathcal{C})
\]
for $A \in D^+(P\mathcal{I}\mathcal{A})$, $B \in D^+(P\mathcal{I}\mathcal{B})$. Again, an intuitive and convenient but less rigorous way to denote the painful right-hand side (when $A \in P\mathcal{I}\mathcal{A}$, $B \in P\mathcal{I}\mathcal{B}$) is
\[
R \lim_{\lambda_1, \mu_1} \lim_{\lambda_2, \mu_2} R \lim_{\lambda_3, \mu_3} \lim_{\lambda_4, \mu_4} RF(A_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}, B_{\mu_1, \mu_2, \mu_3, \mu_4}),
\]
where
\[
A = \left(\prod \leftarrow \right)_{\lambda_1} \left(\prod \leftarrow \right)_{\lambda_2} \left(\prod \leftarrow \right)_{\lambda_3} \left(\prod \leftarrow \right)_{\lambda_4} A_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}, \quad \text{with} \quad A_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \in \mathcal{A},
\]
\[
B = \left(\prod \leftarrow \right)_{\mu_1} \left(\prod \leftarrow \right)_{\mu_2} \left(\prod \leftarrow \right)_{\mu_3} \left(\prod \leftarrow \right)_{\mu_4} B_{\mu_1, \mu_2, \mu_3, \mu_4}, \quad \text{with} \quad B_{\mu_1, \mu_2, \mu_3, \mu_4} \in \mathcal{B}.
\]
What is rigorously true about this notation is that there are two spectral sequences. One (for fixed $\lambda_1, \lambda_2, \mu_1, \mu_2$) has $E^2_{ij}$-terms given by
\[
R^i \lim_{\lambda_3, \mu_3} \lim_{\lambda_4, \mu_4} R^i F(A_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}, B_{\mu_1, \mu_2, \mu_3, \mu_4}),
\]
converging to \( R^{i+j} \tilde{F}(A_{\lambda_1, \lambda_2}, B_{\mu_1, \mu_2}) \), where

\[
A_{\lambda_1, \lambda_2} = \lim_{\lambda_3} \lim_{\lambda_4} A_{\lambda_2, \lambda_3, \lambda_4}, \quad B_{\mu_1, \mu_2} = \lim_{\mu_3} \lim_{\mu_4} B_{\mu_1, \mu_2, \mu_3, \mu_4},
\]

\[
\tilde{F} = \lim_{\lambda_3} P \lim_{\lambda_4} PIF, \quad \text{i.e.,} \quad \tilde{F}(A_{\lambda_1, \lambda_2}, B_{\mu_1, \mu_2}) = \lim_{\lambda_3, \lambda_4} \lim_{\lambda_5, \lambda_6} F(A_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8}).
\]

Varying \( \lambda_1, \lambda_2, \mu_1, \mu_2 \), this spectral sequence takes values in PIC. The other has \( E_2^{ij} \)-terms given by

\[
R^i \lim_{\lambda_1, \lambda_2} R^j \tilde{F}(A_{\lambda_1, \lambda_2}, B_{\mu_1, \mu_2}),
\]

converging to \( R^{i+j} \tilde{F}(A, B) \), where

\[
\tilde{F} = \lim_{\lambda_1} P \lim_{\lambda_2} P \lim_{\mu_1} P \lim_{\mu_2} P \lim_{\lambda_3} P \lim_{\mu_4} PIF, \quad \text{i.e.,} \quad \tilde{F}(A, B) = \lim_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \lim_{\mu_1, \mu_2, \mu_3, \mu_4} F(A_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu_1, \mu_2, \mu_3, \mu_4}).
\]

A special case we use below is where \( A \in \text{PLA} \) and \( B \in \text{IPB} \). Here we embed \( \text{PLA} \) into \( \text{PIPLA} \) by adding \( I \) and \( P \) in the middle and \( \text{IPB} \) into \( \text{PIPIB} \) by adding \( P \) from the left and \( I \) from the right. These embeddings are exact functors. The first one takes the subcategory \( \prod I \) of \( LA' \) into \( \prod P \) of \( LA' \). The second takes the subcategory \( \prod I B' \) of \( I \prod B' \) into \( \prod I \) of \( B' \). Hence the derived functor of the restriction of \( \tilde{F} \) to \( \text{PLA} \times \text{IPA} \) is the restriction of \( R \tilde{F} \) to \( D^+(\text{PLA}) \times D^+(\text{IPA}) \), and we have

\[
R \tilde{F}(A, B) = R \lim_{\lambda} \lim_{\mu} R \lim_{\lambda'} \lim_{\mu'} RF(A_{\lambda, \lambda'}, B_{\mu, \mu'}),
\]

where \( A = \lim_{\lambda} A_{\lambda} \in \text{PLA} \), \( B = \lim_{\mu} B_{\mu} \in \text{IPB} \). Similar observations apply to adding more or less \( P \)'s and/or \( I \)'s in different places.

**Proposition (2.2.2).** Let \( A \) be a small abelian category. Then we have natural fully faithful embeddings

\[
D^b(A) \rightarrow_{\subseteq} D^b(\text{PA}) \quad \text{and} \quad D^b(\text{LA}) \rightarrow_{\subseteq} D^b(\text{IPA})
\]

of triangulated categories. We have

\[
R \text{Hom}_{\text{IPA}}(A, B) = R \lim_{\lambda} \lim_{\mu} R \text{Hom}_{\text{A}}(A_{\lambda}, B_{\mu}),
\]

for \( A = \lim_{\lambda} A_{\lambda} \in \text{PA} \) and \( B = \lim_{\mu} B_{\mu} \in \text{PA} \),

\[
R \text{Hom}_{\text{LA}}(A, B) = R \lim_{\lambda} \lim_{\mu} R \text{Hom}_{\text{A}}(A_{\lambda}, B_{\mu}),
\]

for \( A = \lim_{\lambda} A_{\lambda} \in \text{LA} \) and \( B = \lim_{\mu} B_{\mu} \in \text{LA} \), and

\[
R \text{Hom}_{\text{IPLA}}(A, B) = R \lim_{\lambda} \lim_{\mu} R \lim_{\lambda'} \lim_{\mu'} R \text{Hom}_{\text{A}}(A_{\lambda, \lambda'}, B_{\mu, \mu'}),
\]

for \( A = \lim_{\lambda} A_{\lambda} \in \text{IPB} \) and \( B = \lim_{\lambda} B_{\lambda} \in \text{IPB} \).
Proof. The fully faithful embedding $D^b(\mathcal{A}) \hookrightarrow D^b(P, \mathcal{A})$ has already been mentioned ([KS06, Thm. 15.3.1 (i)]). The same implies the fully faithfulness of $D^b(\mathcal{A}) \hookrightarrow D^b(L\mathcal{A})$ and $D^b(P, \mathcal{A}) \hookrightarrow D^b(IP, \mathcal{A})$. The exactness of the embedding $L\mathcal{A} \hookrightarrow IP, \mathcal{A}$ yields a morphism of functors from $R\text{Hom}_{L\mathcal{A}}$ to $R\text{Hom}_{IP, \mathcal{A}}$ restricted to $D^b(L\mathcal{A})$. This will turn out to be an isomorphism and hence the fully faithfulness of $D^b(L\mathcal{A}) \hookrightarrow D^b(IP, \mathcal{A})$ will follow once we compute these $R\text{Hom}$ functors and verify the stated formulas.

We compute $R\text{Hom}_{IP, \mathcal{A}}$. Let $\mathcal{B} = (P, \mathcal{A})^{\text{opp}}$, $\mathcal{B}'$ its full subcategory of injectives (equality of the opposite of projectives of $P, \mathcal{A}$) and $\mathcal{C} = \mathcal{C}' = \text{Ab}$. We will apply the observations we made before the proposition for $F := \text{Hom}_{P, \mathcal{A}}: (P, \mathcal{A})^{\text{opp}} \times \mathcal{A} =: \mathcal{B} \times \mathcal{A} \to \mathcal{C} = \text{Ab}$ and

$$\lim_{\leftarrow} P \lim_{\rightarrow} P\text{Ilim}_{\leftarrow} \text{IPF} = \text{Hom}_{IP, \mathcal{A}}: \text{P}\mathcal{B} \times IP, \mathcal{A} \to \text{P}\mathcal{IPC} \to C$$

(the P in P\mathcal{B} corresponds to the left P in P\mathcal{IPC} and the IP in P\mathcal{IPC}). The pair $(\mathcal{B}', \mathcal{A})$ is $F$-injective. Obviously $\mathcal{C} = \mathcal{C}' = \text{Ab}$ satisfies the conditions of (2.2.1) (b). Hence we have

$$R\text{Hom}_{IP, \mathcal{A}}(A, B) = R\lim_{\lambda} R\lim_{\mu} R\lim_{\mu'} \text{Hom}_{A, A_{\lambda\lambda'}, B_{\mu\mu'}}$$

$$= R\lim_{\lambda} R\lim_{\mu} R\lim_{\mu'} \text{Hom}_{A, A_{\lambda\lambda'}, B_{\mu\mu'}}$$

for $A = \left(\lim_{\lambda} \lim_{\mu} A_{\lambda\lambda'}\right) \in IP, \mathcal{A}$ and $B = \left(\lim_{\lambda} \lim_{\mu} B_{\mu\mu'}\right) \in IP, \mathcal{A}$.

Since $D^b(P, \mathcal{A}) \hookrightarrow D^b(IP, \mathcal{A})$ is fully faithful, this also verifies, by restriction, the stated formula for $R\text{Hom}_{IP, \mathcal{A}}$. Dualizing, this in turn verifies the stated formula for $R\text{Hom}_{L\mathcal{A}}$. This completes the proof. □

Note that this proposition in particular implies that the restriction of $R\text{Hom}_{L\mathcal{A}}$ to $D^b(\mathcal{A})$ agrees with $R\text{Hom}_{\mathcal{A}}$, which was originally defined as the restriction of $R\text{Hom}_{IP, \mathcal{A}}$. The proposition is also true for a Grothendieck category $\mathcal{A}$. We omit the proof as we do not need this case.

What we need to know about Grothendieck categories is when direct and derived inverse limits commute with $R\text{Hom}$ in the both variables. The following shows what is true in general and what should be checked in specific cases.

**Proposition (2.2.3).** Let $\mathcal{A}, \mathcal{C}$ be Grothendieck categories and $F: \mathcal{A}^{\text{opp}} \times \mathcal{A} \to \mathcal{C}$ an additive bifunctor that is left exact in both variables. Let $\mathcal{C}'$ be a full additive subcategory of $\mathcal{C}$. Assume that the functor $F(\cdot, I): \mathcal{A}^{\text{opp}} \to \mathcal{C}$ for any injective $I \in \mathcal{A}$ is exact with image contained in $\mathcal{C}'$. Assume also that $F$ commutes with filtered inverse limits. Assume finally that $\mathcal{C}' \subset \mathcal{C}$ satisfies the conditions of (2.2.1).
There we have canonical morphisms and isomorphisms

\[
R \lim \! \lim_{\mu} R \lim \! \lim_{\mu'} RF(A_{\lambda' \mu'}, B_{\mu''})
\]

\[
\rightarrow R \lim \! \lim_{\lambda} R \lim \! \lim_{\mu'}, RF\left(\lim_{\lambda'} A_{\lambda' \mu'}, B_{\mu''}\right)
\]

\[
= R \lim \! \lim_{\lambda} RF\left(\lim_{\lambda'} A_{\lambda' \mu'}, \lim_{\mu'} B_{\mu''}\right)
\]

\[
\rightarrow R \lim \! \lim_{\lambda} RF\left(\lim_{\lambda'} A_{\lambda' \mu'}, \lim_{\mu'} B_{\mu''}\right)
\]

\[
= RF\left(\lim_{\lambda'} \lim_{\lambda'} A_{\lambda' \mu'}, \lim_{\mu'} \lim_{\mu'} B_{\mu''}\right)
\]

in \(D^+(C)\) for any \(A = \lim_{\lambda'} \lim_{\lambda'} A_{\lambda' \mu'}, A_{\lambda' \mu'} \in \text{IPA}\) and \(B = \lim_{\mu'} \lim_{\mu'} B_{\mu''}, B_{\mu''} \in \text{IPA}\).

Note that the \(R \lim_{\lambda'}\) in the displayed equations are the derived ones but the \(\lim_{\lambda'}\) are not, so that all the variables lie in \(D^-(A)^{op} \times D^+(A)\) where \(RF\) is defined.

**Proof.** In the rigorous terms, the morphisms and isomorphisms to be constructed are

\[
R \lim \! \lim_{\lambda} R \lim \! \lim_{\mu'} RPI\lim \! \lim_{\mu} RPI\lim \! \lim_{\mu'} (A, B)
\]

\[
\rightarrow R \lim \! \lim_{\lambda} R \lim \! \lim_{\mu'} RPI\lim \! \lim_{\mu} RPI\lim \! \lim_{\mu'} (\lim_{\lambda'} A, B)
\]

\[
= R \lim \! \lim_{\lambda} \lim_{\lambda'} RPI\lim \! \lim_{\mu} \lim_{\mu'} (\lim_{\lambda'} A, \lim_{\mu'} B)
\]

\[
\rightarrow R \lim \! \lim_{\lambda} \lim_{\lambda'} \lim_{\mu} \lim_{\mu'} (\lim_{\lambda'} A, \lim_{\mu'} B)
\]

\[
= RF\left(\lim_{\lambda'} \lim_{\lambda'} A_{\lambda' \mu'}, \lim_{\mu'} \lim_{\mu'} B_{\mu''}\right)
\]

for \(A, B \in \text{IPA}\). It suffices to construct a morphism

\[
\lim_{\lambda'} RPI\lim (A, B) \rightarrow RPI\lim (\lim_{\lambda'} A, B) \quad \text{in} \quad D^+(\text{PIC})
\]

for \(A \in \text{IPA}\) and \(B \in D^+(\text{IPA})\), an isomorphism

\[
RPI\lim (A, B) = RPI\lim (\lim_{\lambda'} A, \lim_{\lambda'} B) \quad \text{in} \quad D^+(\text{PC})
\]

for \(A \in D^-(A), B \in D^+(\text{IPA})\), a morphism

\[
\lim_{\lambda'} RPI\lim (A, B) \rightarrow RPI\lim (A, \lim_{\lambda'} B) \quad \text{in} \quad D^+(\text{PC})
\]

for \(A \in D^-(A), B \in D^+(\text{LA})\), and an isomorphism

\[
R \lim \! \lim_{\lambda} RPI\lim (A, B) = RF\left(\lim_{\lambda'} A, B\right) \quad \text{in} \quad D^+(\text{C})
\]

for \(A \in D^-(A), B \in D^+(A)\).

We construct the first morphism. We fix \(A \in \text{IPA}\). Let \(\mathcal{A}' \subset \mathcal{A}\) be the full subcategory of injectives. Then \((\mathcal{A}'^{op}, \mathcal{A}')\) is \(F\)-injective by assumption. Hence \(((\text{IPA})^{op}, \mathcal{A}')\) is \(\text{PIPIF}\)-injective by (2.2.1) (m). This implies that \(\text{PIPIF}(A, B)\) (which is a priori calculated by resolving \(A\) and \(B\) at the same time) is the value at \(B\) of the derived functor of the single-variable functor \(\text{PIPIF}(A, \cdot)\) ([Klo01] Cor. 13.4.5)). Similarly \(\text{PIPIF}(\lim_{\lambda'} A, B)\) is the value at \(B\) of the derived functor of \(\text{PIF}(\lim_{\lambda'} A, \cdot)\). Hence we need to construct a morphism

\[
\lim_{\lambda'} \text{PIPIF}(A, \cdot) \rightarrow \text{PIPIF}(\lim_{\lambda'} A, \cdot)
\]
of functors $D^+(\text{IP} \mathcal{A}) \to D^+(\text{PI}\mathcal{P})$. Since $\text{PIP lim}_{\leftarrow} : \text{PI}\mathcal{P} \to \text{PI}\mathcal{P}$ is exact, we have

$$\text{PIP lim}_{\leftarrow} R\text{PIPIF}(A, \cdot) = R(\text{PIP lim}_{\leftarrow} \text{PIPIF})(A, \cdot).$$

Deriving the natural morphism

$$\text{PIP lim}_{\leftarrow} \text{PIPIF}(A, \cdot) \to \text{PIPF}(\text{I lim}_{\leftarrow} A, \cdot)$$

of functors $\text{IP} \mathcal{A} \to \text{PI}\mathcal{P}$, we get the required morphism of derived functors.

We construct the second isomorphism. Let $\mathcal{A}' \subset \mathcal{A}$ be the full subcategory of injectives. Since $F$ commute with filtered inverse limits, we have a commutative diagram

$$\begin{array}{ccc}
(LA)^{op} \times \text{IP} \mathcal{A} & \xrightarrow{\text{PIPF}} & \text{PI}\mathcal{P} \\
\downarrow \text{id} \times \text{I lim} & & \downarrow \text{PI lim} \\
(LA)^{op} \times LA & \xrightarrow{\text{PIF}} & \text{PI}\mathcal{C}.
\end{array}$$

This diagram restricts to the diagram

$$\begin{array}{ccc}
(\text{"} \bigoplus \text{"} \mathcal{A})^{op} \times I \text{"} \prod \text{"} \mathcal{A}' & \longrightarrow & \text{"} \prod \text{"} I \text{"} \prod \text{"} \mathcal{C}' \\
\downarrow & & \downarrow \\
(\text{"} \bigoplus \text{"} \mathcal{A})^{op} \times LA' & \longrightarrow & \text{"} \prod \text{"} \mathcal{C}'.
\end{array}$$

of full subcategories since $F((\mathcal{A})^{op} \times \mathcal{A}') \subset \mathcal{C}'$ and products of injectives are injective. The category $(\text{"} \bigoplus \text{"} \mathcal{A})^{op} \times I \text{"} \prod \text{"} \mathcal{A}'$ is injective with respect to both $\text{PIPF}$ and $\text{id} \times \text{I lim}$ by (2.2.1). Therefore the above commutative diagram derives to a commutative diagram of the derived functors on the derived categories. This gives the required isomorphism.

We construct the third morphism. Consider the following (non-commutative) diagram:

$$\begin{array}{ccc}
(LA)^{op} \times LA & \xrightarrow{\text{PIF}} & \text{PI}\mathcal{C} \\
\downarrow \text{id} \times \text{lim} & & \downarrow \text{lim} \\
(LA)^{op} \times \mathcal{A} & \xrightarrow{\text{PF}} & \mathcal{C}.
\end{array}$$

This gives two functors from the left upper term to the right lower term. There is a natural morphism of functors from the one factoring through the right upper term to the one factoring through the left lower term. Since the right vertical arrow is an exact functor, this morphism of functors induces a morphism of functors in the derived categories, which is the required morphism.

The fourth isomorphism can be constructed in the same way as the second. □

One might expect that the $\lim_{\leftarrow} \lambda'$ in the proposition may be replaced by $R\lim_{\leftarrow} \lambda'$ to make the statement more natural. This would need to extend the results of [KS06, Chap. 15] on derivation of indcategories cited here several times to unbounded derived categories. It is not clear whether such an extension is possible or not.

The following should be a well-known fact on the commutation of the derived pushforward and direct/derived inverse limits. We note it here since we need to treat continuous maps of sites without exact pullbacks and non-coherent sites, which do not frequently appear in the literature.

**Proposition (2.2.4).**
(a) Let $S$ be a site defined by finite coverings. Let $C' \subset Ab(S)$ be the full subcategory of acyclic sheaves. Then $C'$ satisfies the conditions of (2.2.1).

(b) Let $f : S_2 \to S_1$ be a continuous map of sites. Assume that both $S_1$ and $S_2$ are defined by finite coverings. Assume also that $f_*$ sends acyclic sheaves to acyclic sheaves. Then $Rf_*$ commutes with lim and $R\lim$ on bounded below derived categories. That is, the diagrams

\[
\begin{array}{ccc}
D^+(I Ab(S_2)) & \xrightarrow{Rf_*} & D^+(I Ab(S_1)) \\
\downarrow \lim & & \downarrow \lim \\
D^+(Ab(S_2)) & \xrightarrow{Rf_*} & D^+(Ab(S_1))
\end{array}
\]

commute.

Proof. (a) The category $C'$ is closed by products and filtered direct limits by the comments in the proof of (2.1.2) (a). All the other conditions are immediate.

(b) Let $C'_i \subset Ab(S_i)$ be the full subcategory of acyclic sheaves, $i = 1, 2$. The commutative diagrams

\[
\begin{array}{ccc}
I Ab(S_2) & \xrightarrow{f_*} & I Ab(S_1) \\
\downarrow \lim & & \downarrow \lim \\
Ab(S_2) & \xrightarrow{f_*} & Ab(S_1)
\end{array}
\]

restrict to their subcategories

\[
\begin{array}{ccc}
I C'_2 & \longrightarrow & I C'_1 \\
\downarrow & & \downarrow \\
C'_2 & \longrightarrow & C'_1
\end{array}
\]

Hence the diagrams derive to commutative diagrams of the derived categories by (2.2.1).

2.3. The derived categories of ind-proalgebraic groups and of sheaves. We apply the results of the previous subsection to $\text{Alg}/k$ and $\text{Ab}(\text{indrat})$. We continue using the notation in the previous subsection about filtered direct/derived inverse limits in derived categories of ind/pro-objects. We use, however, the usual $\lim$ and $\lim$ instead of “$\lim$” and “$\lim$” for simplicity when they are considered in $\text{IAlg}/k$, $\text{PAlg}/k$ or $\text{IPAlg}/k$. The $\lim$ and $\lim$ for sheaves mean limits as sheaves, not as ind-sheaves or pro-sheaves.

We will use the four operations formalism (about pushforward, pullback, sheaf-Hom and tensor products) for derived categories of sites throughout the paper. A good reference (for arbitrary morphisms of sites and unbounded derived categories) is [KS06, Chap. 18].

The following theorem is a summary of the site-theoretic results proved in [Suz13]. We clarify below which results of [Suz13] imply each statement.

Theorem (2.3.1). Let

\[
\begin{array}{ccc}
\text{Spec } k^{\text{perf}}_{\text{proppf}} & \xrightarrow{f} & \text{Spec } k^{\text{perf}}_{\text{et}} \\
\downarrow & & \downarrow \\
\text{Spec } k^{\text{indrat}}_{\text{et}}
\end{array}
\]
be the continuous maps defined by the identity and \( S \) either one of these sites. Let \( A = \lim_{\lambda} A_\lambda \in \text{PAlg}/k \) with \( A_\lambda \in \text{Alg}/k \).

(a) If \( B \in \text{Ab}(k_{\text{perf}}^{\text{proppf}}) \), then

\[
g_* Rf_* R\text{Hom}_{k_{\text{perf}}^{\text{proppf}}}(A, B) = g_* R\text{Hom}_{k_{\text{perf}}}^{\text{et}}(A, Rf_* B) = R\text{Hom}_{k_{\text{indrat}}}^{\text{et}}(A, g_* Rf_* B).
\]

(Note that \( g_* \) is an exact functor by \((2.1.2) \) (c).)

(b) If \( B = \lim_{\mu} \to B_\mu \) with \( B_\mu \in \text{Ab}(S_{\text{et}}) \), then

\[
R\text{Hom}_S(A, B) = \lim_{\mu} R\text{Hom}_S(A, B_\mu)
\]

for \( S = \text{Spec } k_{\text{perf}}^{\text{proppf}} \) or \( \text{Spec } k_{\text{et}}^{\text{proppf}} \).

(c) If \( B \in \text{LAlg}/k \), then \( Rf_* B = B \) (and \( g_* B = B \) as notation), and we have

\[
R\text{Hom}_S(A, B) = \lim_{\lambda} R\text{Hom}_S(A_\lambda, B)
\]

(with \( S = \text{Spec } k_{\text{indrat}}^{\text{et}} \) allowed). Its \( R\Gamma(k', \cdot) \) for \( k' = \bigcup k'_\nu \subset k_{\text{indrat}}^{\text{et}} \) is

\[
\lim_{\lambda, \nu} R\text{Hom}_{S/k'_\nu}(A_\lambda, B),
\]

where \( S/k'_\nu \) is the localization of \( S \) at \( k'_\nu \) (see the notation section of \[L^{am}\]). If \( k' \) is a field and \( S = \text{Spec } k_{\text{indrat}}^{\text{et}} \), then this is further isomorphic to

\[
R\text{Hom}_{k_{\text{indrat}}^{\text{et}}}(A, B).
\]

(d) If \( B \in \text{Et}/k \) (\( \subset \text{LAlg}/k \)) is uniquely divisible, then

\[
R\text{Hom}_{k_{\text{indrat}}^{\text{et}}}(A, B) = 0.
\]

(e) If \( B \in \text{Alg}/k \), then

\[
R\text{Hom}_S(A, B) = R\text{Hom}_{\text{PAlg}/k}(A, B).
\]

The corresponding results in \[Suz13\] were proved for affine \( A \). (The symbol \( \text{Alg}/k \) in \[Suz13\] denoted the category of affine quasi-algebraic groups.) The same proof works for abelian varieties as remarked after \[Suz13\] Thm. 2.1.5. It even works for arbitrary proalgebraic groups; see Appendix \[A\].

**Proof.** Note that \( f^* \) is the pro-fppf sheafification functor and hence \( f^* A = A \). (\( A \) is a sheaf even for the fpqc topology by the fpqc descent and the fact that inverse limits of sheaves are sheaves.) Therefore the sheafified adjunction between \( f^* \) and \( Rf_* \) \[KS06\] Thm. 18.6.9 (iii) gives an isomorphism

\[
Rf_* R\text{Hom}_{k_{\text{perf}}^{\text{proppf}}}(A, B) = R\text{Hom}_{k_{\text{et}}^{\text{per}}}(A, Rf_* B)
\]

in \( D(k_{\text{et}}^{\text{per}}) \). This proves the first isomorphism. For the second, if \( Rf_* B \to J \) is an injective resolution over \( \text{Spec } k_{\text{et}}^{\text{per}} \), then we have natural isomorphisms and the
functoriality morphism of $g_*$:

$$g_* R \text{Hom}_{\text{perf}}(A, Rf_* B) = g_* \text{Hom}_{\text{perf}}(A, J)$$

$$\rightarrow \text{Hom}_{\text{indrat}}(g_* A, g_* J)$$

$$\rightarrow R \text{Hom}_{\text{indrat}}(g_* A, g_* J)$$

$$= R \text{Hom}_{\text{indrat}}(A, g_* Rf_* B)$$

in $D(k_{\text{indrat}})$, where the middle $\text{Hom}$'s are the total complexes of the sheaf-Hom double complexes and $g_* A = A$ as notation. Applying $R \Gamma(k'_{\text{et}}, \cdot)$ for any $k' \in k_{\text{indrat}}$ and using the Leray spectral sequence for $g_*$, we have a morphism

$$R \text{Hom}_{\text{perf}}(A, Rf_* B) \rightarrow R \text{Hom}_{k_{\text{indrat}}}(A, g_* Rf_* B)$$

in $D(\text{Ab})$. This is an isomorphism by [Suz13, Prop. 3.7.4]. Therefore we obtain the second isomorphism.

Let $S = \text{Spec } k'_{\text{et}}$. We have a morphism

$$\lim_{\mu} R \text{Hom}_{k_{\text{et}}}(A, B_\mu) \rightarrow R \text{Hom}_{k_{\text{et}}}(A, B)$$

by (2.2.3). (The construction of this morphism given in the proof does not involve the subcategory $C'$.) To see this is an isomorphism, we apply $R \Gamma(R_{\text{et}}, \cdot)$ for any $R \in k_{\text{perf}}$ and take cohomology in degree $n \geq 0$:

$$\lim_{\mu} \text{Ext}^n_{R_{\text{et}}}(A, B_\mu) \rightarrow \text{Ext}^n_{R_{\text{et}}}(A, B).$$

This is an isomorphism by [Suz13, Lem. 3.8.2 (2)]. Hence the result follows.

The same proof (including the proof of [Suz13, loc. cit.]) works for $k'_{\text{profppf}}$ in the same manner.

We have a morphism

$$\lim_{\lambda} R \text{Hom}_{\xi}(A, B) \rightarrow R \text{Hom}_{\xi}(A, B)$$

by (2.2.3) (with the same remark about $C'$ as above). Again, by taking cohomology at each object of $S$ and cohomology at each degree, we are reduced to showing that the following four induced morphisms are isomorphisms:

$$\lim_{\lambda, \mu} \text{Ext}^n_{(R_{\text{et}})_{\text{perf}}}(A, B) \rightarrow \text{Ext}^n_{R_{\text{et}}}(A, B),$$

$$\lim_{\lambda, \mu} \text{Ext}^n_{(R_{\text{et}})_{\text{profppf}}}(A, B) \rightarrow \text{Ext}^n_{R_{\text{et}}}(A, B),$$

for a filtered direct system $\{R_\nu\}$ in $k_{\text{perf}}$ with limit $R$,

$$\lim_{\lambda, \mu} \text{Ext}^n_{(k'_{\text{et}})_{\text{indrat}}}(A, B) \rightarrow \text{Ext}^n_{k_{\text{et}}}(A, B),$$

for a filtered direct system $\{k'_\nu\}$ in $k_{\text{et}}$ with limit $k'$, and

$$\text{Ext}^n_{k_{\text{et}}}(A, B) \rightarrow \text{Ext}^n_{k_{\text{et}}}(A, B)$$

for a field $k' \in k_{\text{indrat}}$. The first isomorphism is [Suz13, Lem. 3.8.2 (1)]. This lemma also works for the pro-fppf topology, so we get the second isomorphism. Since $Rf_* B = B$ by (2.1.2) [Suz13, Prop. 3.7.4] implies that

$$\text{Ext}^n_{k_{\text{profppf}}}(A, B) = \text{Ext}^n_{k_{\text{et}}}(A, B) = \text{Ext}^n_{k_{\text{et}}}(A, B).$$
for any $k' \in k^\text{indrat}$. Hence the third isomorphism follows. The fourth isomorphism is \cite[Thm. 2.1.5]{Suz13}.

Such a group $B$ can be written as a filtered direct limit of étale groups over $k$ whose groups of geometric points are finite-dimensional $Q$-vector spaces. Hence by \cite[b], we may assume that $B$ is an étale group over $k$ with $\dim_Q B(k) < \infty$. Let $k_1$ be a finite Galois extension of $k$ with Galois group $G$ such that $B$ becomes a constant group over $k_1$. Let $k' \in k^\text{indrat}$ and $k'_1 = k' \otimes_k k_1$. The Hochschild-Serre spectral sequence for the coefficients in $R\text{Hom}_{(k_1)\text{et}}(A, B) = R\text{Hom}_{k_1^\text{indrat}/k_1}(A, B)$ yields

$$R\Gamma(G, R\text{Hom}_{k^\text{indrat}/k_1}(A, B)) = R\text{Hom}_{k_1^\text{indrat}/k_1}(A, B).$$

Therefore we may assume that $B$ is constant and, moreover, $B = Q$. The statement to prove is thus

$$\text{Ext}^n_{k^\text{indrat}/k}(A, Q) = 0$$

for $k' \in k^\text{indrat}$ and $n \geq 0$. This is \cite[Thm. 2.1.5]{Suz13}.

By \cite[Thm. 2.1.5, Prop. 3.7.4 and Prop. 3.8.1]{Suz13}, we have

$$R\text{Hom}_{k^\text{perf}/k^\text{proet}}(A, B) = R\text{Hom}_{k^\text{et}}(A, B) = R\text{Hom}_{k^\text{indrat}}(A, B) = R\text{Hom}_{\mathrm{PAlg}/k}(A, B).$$

\[\square\]

We generalize this theorem to the case where both $A$ and $B$ are ind-proalgebraic groups.

**Proposition (2.3.2).** (2.3.1) remains true when all the subscripts $\text{et}$ are replaced by $\text{proet}$.

**Proof.** This is a straightforward consequence of the fact that $A \in \mathrm{PAlg}/k$ is a sheaf for the pro-fppf topology (as commented in the proof of the theorem) and (2.1.2) \[\square\]. We only explain this for $\square$ and $\square$.

We need to show that the morphisms

$$R\text{Hom}_{k^\text{perf}/k^\text{proet}}(A, B) \to R\text{Hom}_{k^\text{et}}(A, R\tilde{f}_*B) \to R\text{Hom}_{k^\text{indrat}/k'}(A, \tilde{g}_*R\tilde{f}_*B)$$

are isomorphisms for $k' \in k^\text{indrat}$, where

$$\text{Spec} k^\text{perf}_{\text{proppf}} \overset{f}{\longrightarrow} \text{Spec} k^\text{perf}_{\text{proet}} \overset{g}{\longrightarrow} \text{Spec} k^\text{indrat}_{\text{proet}} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \frac{\lambda}{\lambda}
(a) Let $B = \varprojlim B_\mu = \varprojlim \varprojlim B_{\mu'} \in \text{IPAlg}/k$ similarly. Then

$$R\text{Hom}_S(A, B) = \varprojlim \varprojlim \varprojlim R\text{Hom}_S(A_{\lambda'}, B_{\mu'})$$

(b) Its $R\Gamma(k, \cdot)$ is

$$R\text{Hom}_{\text{IPAlg}/k}(A, B) = \varprojlim \varprojlim \varprojlim R\text{Hom}_{\text{IPAlg}/k}(A_\lambda, B_\mu)$$

$$= \varprojlim \varprojlim R\text{Hom}_{\text{Alg}/k}(A_{\lambda'}, B_{\mu'})$$

When already $A_\lambda, B_\mu \in \text{Alg}/k$ for all $\lambda, \mu$ so that $A, B \in \text{IPAlg}/k$, then this further equals to $R\text{Hom}_{\text{Alg}/k}(A, B)$.

(c) If $B \in \text{LAAlg}/k$, then

$$R\text{Hom}_S(A, B) = \varprojlim \varprojlim R\text{Hom}_S(A_{\lambda'}, B)$$

(d) If $B \in \text{Et}/k$ is uniquely divisible, then

$$R\text{Hom}_{\text{indrat}_{\text{proet}}}(A, B) = 0.$$
by (2.3.1) (e) and (2.3.2), since $B \in \text{IPA}_{/k}$ and hence $Rf_* B = B$ by (2.1.2) (g).

For the right-hand side, we have

\[ g_* R \lim_{\lambda} R \lim_{\mu'} R \text{Hom}_{\text{proet}}^\text{perf} (A_{\lambda \lambda'}, B_{\mu \mu'}) \]

\[ = R \lim_{\lambda} R \lim_{\mu'} R \text{Hom}_{\text{proet}}^\text{perf} (A_{\lambda \lambda'}, B_{\mu \mu'}) \]

\[ = R \lim_{\lambda} R \lim_{\mu'} R \text{Hom}_{\text{Alg}_{/k}} (A_{\lambda \lambda'}, B_{\mu \mu'}) \]

by (2.2.4) and again by (2.3.1) (a).

(b). We have

\[ R \Gamma (k, R \lim_{\lambda} R \lim_{\mu'} R \text{Hom}_{\text{proet}}^\text{perf} (A_{\lambda \lambda'}, B_{\mu \mu'})) \]

\[ = R \lim_{\lambda} R \lim_{\mu'} R \text{Hom}_{\text{proet}}^\text{perf} (A_{\lambda \lambda'}, B_{\mu \mu'}). \]

\[ = R \lim_{\lambda} R \lim_{\mu'} R \text{Hom}_{\text{Alg}_{/k}} (A_{\lambda \lambda'}, B_{\mu \mu'}) \]

by (2.3.1) (e) and (2.3.2).

(c) The proof of (a) also applies.

(d) This follows from (c) here, (2.3.1) (d) and (2.3.2).

□

Proposition (2.3.4). We have fully faithful embeddings

\[ D^b(\text{Alg}_{/k}) \rightarrow D^b(\text{IPA}_{/k}) \]

\[ D^b(\text{IAlg}_{/k}) \rightarrow \subset \rightarrow D^b(\text{IPA}_{/k}) \rightarrow \subset \rightarrow D^b(k_{\text{proet}}), \]

\[ D^b(\text{FGE}_{/k}) \subset D^b(\text{Et}_{/k}) = D^b(k_{\text{et}}) \subset D^b(k_{\text{proet}}), \]

of triangulated categories. The same is true with $D^b(k_{\text{proet}})$ replaced by $D^b(k_{\text{proet}}^\text{perf})$ or $D^b(k_{\text{proet}}^\text{profppf})$.

Proof. This follows from (2.2.2) and (2.3.3). □

2.4. Serre duality and P-acyclicity. In this subsection, we mainly focus on the site $\text{Spec} k_{\text{indrat}}^{\text{proet}}$. From the next section, it is important to use $\text{Spec} k_{\text{et}}^{\text{proet}}$ and $\text{Spec} k_{\text{indrat}}^{\text{proet}}$ in order to view cohomology of local fields as sheaves over the residue field and compute it. (Using $\text{Spec} k_{\text{proet}}^\text{perf}$ instead might be slightly problematic for the treatment of Serre duality for semi-abelian varieties. The group $\text{Ext}^n_{\text{perf}} (\mathbb{G}_m, \mathbb{G}_m)$ for example might not be torsion for $n \geq 2$ if $R$ is not the perfection of a regular scheme and the computation done in [Bre70] does not literally apply. This implies that the sheaf-$\text{Ext}^n_{\text{proet}} (\mathbb{G}_m, \mathbb{G}_m)$ might not be torsion. The same problem exists for $\text{Spec} k_{\text{proet}}^{\text{profppf}}$.)

For a complex $A \in D(k_{\text{proet}}^{\text{indrat}})$, we define

\[ A^{SD} = R \text{Hom}_{k_{\text{proet}}^{\text{indrat}}} (A, \mathbb{Z}). \]
We call $A^{SD}$ the Serre dual of $A^{SD}$ It defines a contravariant triangulated endofunctor SD on $D(\mathcal{K}_{\text{proet}})$. If $A \in \text{PAlg}/k$, then
\[ A^{SD} = R \text{Hom}_{\text{proet}}(A, Q/Z)[-1] \]
by (2.3.3) (d). Note that this is not true for $A \in D(\mathcal{K}_{\text{proet}})$, we have a natural morphism $A \to (A^{SD})^{SD} = A^{SDSD}$. If this is an isomorphism, we say that $A$ is Serre reflexive. Serre reflexive complexes form a full triangulated subcategory of $D(\mathcal{K}_{\text{proet}})$, where the functor SD is a contravariant auto-equivalence with inverse itself.

**Proposition (2.4.1).**

(a) If $A \in \text{PAlg}/k$, then
\[ \text{Ext}_{\text{proet}}^n(A, Q/Z) = 0 \]
for $n \neq 0, 1$. We have
\[ \text{Hom}_{\text{proet}}(A, Q/Z) = \text{Hom}_{\text{proet}}(\pi_0(A), Q/Z). \]

(b) The objects of the categories $D^b(\text{PAlg}_{uc}/k), D^b(\text{IAlg}_{uc}/k), D^b(\text{FEt}/k)$
are Serre reflexive. The functor SD gives an auto-equivalence on $D^b(\text{Alg}_{uc}/k)$, which agrees with the Breen-Serre duality on perfect unipotent groups ([Ser60, 8.4, Remarque], [Mil06, III, Thm. 0.14]) shifted by $-1$. It also gives an auto-equivalence on $D^b(\text{FEt}/k)$, which agrees with the Pontryagin duality $PD = \text{Hom}_k(\cdot, Q/Z)$ shifted by $-1$. It further gives an auto-equivalence on $D^b(\text{FEt}/k)$, and an equivalence between $D^b(\text{PAlg}_{uc}/k)$ and $D^b(\text{IAlg}_{uc}/k)$. For $A \in \text{PAlg}/k$ (resp. $A \in \text{IAlg}/k$), we have: $H^1 A^{SD} = \pi_0(A)^{PD}$;
\[ H^2 A^{SD} = \text{Ext}_{\text{proet}}^1(A, Q/Z) \]
is a connected ind-unipotent (resp. pro-unipotent) group; and $H^n A^{SD} = 0$ for $n \neq 1, 2$.

(c) Let $A$ be (the perfection of) a semi-abelian variety over $k$. Let $TA$ be the Tate module of $A$, i.e. the pro-finite-étale group over $k$ given by the inverse limit of the kernel of multiplication by $n \geq 1$ on $A^n$. Let $\tilde{A}$ be the pro-étale universal covering group of $A$, i.e. the inverse limit of multiplication by $n \geq 1$ on $A$. Then the morphism $A \to TA[1]$ (where $[1]$ is the shift) coming from the natural exact sequence
\[ 0 \to TA \to \tilde{A} \to A \to 0 \]

\[ 4 \text{ Milne [Mil06, III, 0] used the term “Breen-Serre duality”. Breen’s work [Bre78] brought Serre’s observation [Ser60, 8.4, Remarque] to the setting of the perfect étale site and provided Artin-Milne [AM72] a basis of their “second context” of global duality. Just to make it notationally parallel to and easily distinguishable with Cartier duality CD and Pontryagin duality PD, we omit Breen’s name and use the symbol SD in this paper. Of course our treatment of the functor SD heavily relies on the work of both Breen and Serre.} \]

\[ 5 \text{ Strictly speaking, the objects of } \text{Alg}_{uc}/k \text{ are not exactly unipotent since their component groups can be any objects of } \text{FEt}/k \text{ not necessarily } p\text{-primary. The Breen-Serre duality for finite étale } p\text{-primary groups agrees with the Pontryagin duality [Mil06, III, Lem. 0.13 (b)].} \]

\[ 6 \text{ This is a reduced scheme due to the perfection. A perfect scheme is reduced. Hence } TA \text{ is the Pontryagin dual of the torsion part of the discrete Galois module } A(\mathbb{R}). \]
in \(\text{Ab}(k^\text{indrat}_{\text{proet}})\) and \([\mathfrak{I}]\) gives isomorphisms
\[
A^{SDSD} \simeq TA[1]^{SDSD} = TA[1].
\]
We also have \(A^{SD} = (TA)^{\text{PD}}[-2]\).

(d) For any \(A \in \text{PAAlg}/k\), there is a canonical morphism from \(A^{SDSD}\) to the inverse limit \(\lim_{n} A\) of multiplication by \(n \geq 1\) placed in degree \(-1\), so that we have
\[
A^{SDSD} = \left[\lim_{n} A \to A\right],
\]
where the morphism \(\lim_{n} A \to A\) is the projection to the \(n = 1\) term and \([ \cdot ]\) denotes the mapping cone. To give a more explicit description, let \(A_0\) be the identity component of \(A\), \(A_u\) the maximal semi-unipotent subgroup of \(A_0\), \(A_{k\text{Ab}} = A_0/A_u\) the maximal semi-abelian quotient of \(A_0\). Then we have \(\lim_{n} A = A_{k\text{Ab}}\), whose Serre dual is zero. The commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \longrightarrow & TA_{k\text{Ab}} & \longrightarrow & \lim_{n} A & \longrightarrow & A_{k\text{Ab}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_u & \longrightarrow & A & \longrightarrow & A/A_u & \longrightarrow & 0
\end{array}
\]
gives a distinguished triangle\(\footnote{Perhaps a more helpful description of \(A^{SDSD} = [\lim A \to A]\) is that its \(H^0\) is the quotient of \(A\) by the maximal semi-abelian subgroup \(A'_{k\text{Ab}}\) of \(A\) (so \(H^0\) is the maximal quotient of \(A\) that belongs to \(\text{PAAlg}_{\text{et}}\)) and \(H^{-1}\) the Tate module of \(A'_{k\text{Ab}}\). The composite \(A'_{k\text{Ab}} \twoheadrightarrow A_0 \to A_{k\text{Ab}}\) is an isogeny.}
\[
[TA_{k\text{Ab}} \to A_u] \to \left[\lim_{n} A \to A\right] \to \pi_0(A).
\]
In particular, we have \(A^{SDSD} \in D^b(\text{PAAlg}_{\text{et}}/k)\). We also have \(A^{SD} \in D^b(\text{IAlg}_{\text{et}}/k)\), which is Serre reflexive.

Proof. (i) For any \(n\), the sheaf in question is the pro-étale sheafification of the presheaf
\[
k' \in k^\text{indrat} \mapsto \text{Ext}^{n}_{k^\text{proet}}(A, Q/Z).
\]
Let \(k' = \bigcup_{\nu} k'_\nu \in k^\text{indrat}\) with \(k'_\nu \in k^\text{rat}\). Then the group on the right can also be written as
\[
\text{Ext}^{n}_{k^\text{proet}}(A, Q/Z) = \text{Ext}^{n}_{k^\text{proet}}(A, Q/Z) = \lim_{\nu} \text{Ext}^{n}_{k'_\nu}(A, Q/Z)
\]
by \((2.3.1) (a), (c) and (2.3.2)\).

For the statement for \(n \neq 0,1\), it is enough, replacing \(k'_\nu\) with \(k\), to show that the étale sheafification of
\[
k' \in k^\text{et} \mapsto \text{Ext}^{n}_{k^\text{et}}(A, Q/Z)
\]
is zero (for any perfect field \(k\), where \(k^\text{et}\) is the category of étale \(k\)-algebras. Equivalently, it is enough to show that the group
\[
\text{Ext}^{n}_{k^\text{et}}(A, Q/Z) = \text{Ext}^{n}_{k^\text{et}}(A, Q/Z) = \lim_{k'/k \text{ finite}} \text{Ext}^{n}_{k^\text{et}}(A, Q/Z)
\]
is zero for \( n \neq 0,1 \), where \( \overline{k} \) is an algebraic closure of \( k \). For this, we may assume that \( k = \overline{k} \). We have

\[
\text{Ext}_{n}^{\text{indrat}}(A, \mathbb{Q}/\mathbb{Z}) = \lim_{m} \text{Ext}_{n}^{\text{indrat}}(A, \mathbb{Z}/m\mathbb{Z}) = \lim_{m} \text{Ext}_{n}^{\text{Alg}/k}(A, \mathbb{Z}/m\mathbb{Z})
\]

by (2.3.1) (b) and (c). This final group is the Pontryagin dual of the \( n \)-th homotopy group of \( A \) over \( k \) in the sense of Serre [Ser60, §5, Cor. to Prop. 7]. It is zero for \( n \neq 0,1 \) by [Ser60 §10, Thm. 2].

The statement for \( n = 0 \) is proven similarly.

(3) First assume that \( A \in \text{Alg}_{\text{uc}}/k \). Let \( g: \text{Spec}_{\text{proet}} \rightarrow \text{Spec}_{\text{indrat}} \) be the continuous map defined by the identity. Then

\[
A^{\text{SD}} = R\text{Hom}_{\text{proet}}^{\text{indrat}}(A, \mathbb{Q}/\mathbb{Z})[-1] = g_* R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z})[-1]
\]

by (2.1.2) (g) and (2.3.2) (a). Let \( P: \text{Spec}_{\text{proet}}^{\text{perf}} \rightarrow \text{Spec}_{\text{et}}^{\text{perf}} \) be the morphism defined by the identity. Then \( R\text{P}_{\text{et}}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \) by (2.1.2) (g) and \( P^* A = A \). Hence

\[
R\text{P}_{\text{et}} R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}) = R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}),
\]

the \( p \)-primary part of the right-hand side is the Breen-Serre dual of \( A \) by definition [Mil06, III, Thm. 0.14], which is in \( D^b(\text{Alg}_{\text{uc}}/k) \). The prime-to-\( p \) part is the Pontryagin dual of the prime-to-\( p \) part of \( A \) (or \( \pi_0(A) \)). With \( P^* R\text{P}_{\text{et}} = \text{id} \), we have

\[
R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}) = P^* R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}),
\]

hence

\[
A^{\text{SD}} = g_* P^* R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z})[-1].
\]

Note that the objects in \( \text{Alg}/k \) are already pro-\( \text{étale} \) sheaves and hence invariant under the pro-\( \text{étale} \) sheafification \( P^* \). Hence we know that \( A^{\text{SD}} \) is the Breen-Serre dual of \( A \) shifted by \(-1\) in the \( p \)-primary part and the Pontryagin dual of \( A \) in the prime-to-\( p \) part shifted by \(-1\). In particular, we have by [Mil06, III, Lem. 0.13 (b), (c), (d)]; \( H^1 A^{\text{SD}} = \pi_0(A)^{\text{PD}} \); \( H^2 A^{\text{SD}} \in \text{Alg}/k \) is connected unipotent; and \( H^n = 0 \) for \( n \neq 1,2 \).

We show that \( A^{\text{SDSD}} = A \). The Breen-Serre duality [Mil06, III, Thm. 0.14] and the Pontryagin duality show that

\[
A = R\text{Hom}_{\text{proet}}^{\text{perf}}(R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}).
\]

By \( R\text{P}_{\text{et}}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \), \( P^* R\text{P}_{\text{et}} = \text{id} \) and \( P^* A = A \), we have

\[
A = R\text{Hom}_{\text{proet}}^{\text{perf}}(P^* R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}).
\]

Applying \( g_* \), we have a morphism

\[
A = g_* R\text{Hom}_{\text{proet}}^{\text{perf}}(P^* R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})
\rightarrow R\text{Hom}_{\text{proet}}^{\text{indrat}}(g_* P^* R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})
\rightarrow A^{\text{SDSD}}.
\]

Since \( R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}) \in D^b(\text{Alg}_{\text{uc}}/k) \), this morphism is an isomorphism by (2.3.1) (a) and (2.3.2) (applied to the cohomologies of the complex \( R\text{Hom}_{\text{proet}}^{\text{perf}}(A, \mathbb{Q}/\mathbb{Z}) \)). Thus \( A^{\text{SDSD}} = A \).

If \( A = \lim_{\rightarrow} A_n \in \text{PA}_{\text{uc}}/k \), then \( A^{\text{SD}} = \lim_{\rightarrow} A_n^{\text{SD}} \) by (2.3.3) (a). Since the direct limit of sheaves is exact, we have \( H^n A^{\text{SD}} = \lim_{\rightarrow} H^n A_n^{\text{SD}} \) for any \( n \). Hence \( A^{\text{SD}} \in \text{PA}_{\text{uc}}/k \).
$D^b(IAlg_{uc}/k)$. On the other hand, if $A = \lim_n A_\lambda \in IAlg_{uc}/k$, then $A^{SD} = R\lim A_\lambda^{SD}$ by the same assertion. We have $H^n A^{SD} = \lim H^n A_\lambda^{SD}$ for any $n$ by (2.1.2)\,\cite{1}. Hence $A^{SD} \in D^b(IAlg_{uc}/k)$. Therefore the statements about $D^b(IAlg_{uc}/k)$ and $D^b(IAlg_{uc}/k)$ are reduced to the statements about $D^b(IAlg_{uc}/k)$, which have already been proved above.

The statement for $D^b(FGet/k)$ follows from the easy computation $\mathbb{Z}^{SD} = \mathbb{Z}$.

By (2.3.3)\,\cite{1}, we have

$$R\text{Hom}_{\text{indrat}}^1(\overline{A}, \mathbb{Q}/\mathbb{Z}) = \lim_{\rightarrow} R\text{Hom}_{\text{proet}}^1(\overline{A}, \mathbb{Z}/n\mathbb{Z}) = 0$$

since $\overline{A} \in \text{PAlg}/k$ is uniquely divisible. With the vanishing result \cite{1} above, we have

$$\text{Ext}_{\text{proet}}^1(A, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{\text{proet}}^1(TA, \mathbb{Q}/\mathbb{Z})$$

by the universal covering exact sequence. Hence

$$A^{SD} = R\text{Hom}_{\text{indrat}}^1(A, \mathbb{Q}/\mathbb{Z})[-1] = \lim_{\rightarrow} R\text{Hom}_{\text{proet}}^1(A, \mathbb{Z}/m\mathbb{Z})[-1]$$

Therefore $A^{SD\text{SD}} = TA[1]$.

We have

$$A^{SD} = R\text{Hom}_{\text{indrat}}^1(A, \mathbb{Q}/\mathbb{Z})[-1] = \lim_{\rightarrow} R\text{Hom}_{\text{proet}}^1(A, \mathbb{Z}/m\mathbb{Z})[-1]$$

by (2.3.3)\,\cite{1}, which is killed after tensoring with $\mathbb{Q}$. Hence the same assertion implies that

$$\left(\lim_{\rightarrow} A\right)^{SD} = \lim_{\rightarrow} (A^{SD}) = 0.$$ 

Therefore we have a natural morphism and an isomorphism

$$\left[\lim_{\rightarrow} A \to A\right] \to \left[\lim_{\rightarrow} A \to A\right]^{SD\text{SD}} = A^{SD\text{SD}}.$$ 

We will know that the first morphism is an isomorphism once we show that $\left[\lim_{\rightarrow} A \to A\right] \in D^b(PAlg_{uc}/k)$ (whose objects are Serre reflexive by (1))

Note that the exact endofunctor $\lim_{\rightarrow}$ on $\text{PAlg}/k$ kills protorsion groups and hence profinite groups and pro-unipotent groups. This implies that $\lim_{\rightarrow} A = \lim_{\rightarrow} A_{sAb} = \overline{A}_{sAb}$. This yields the stated commutative diagram. In particular, the mapping cone $[A_{sAb} \to A/A_0]$ is quasi-isomorphic to the mapping cone of the morphism of complexes $[TA_{sAb} \to A_0] \to \left[\lim_{\rightarrow} A \to A\right]$. This induces the mapping cone distinguished triangle

$$[TA_{sAb} \to A_0] \to \left[\lim_{\rightarrow} A \to A\right] \to [A_{sAb} \to A/A_0].$$

Since the kernel of $A_{sAb} \to A/A_0$ is zero and the cokernel is $\pi_0(A)$, we obtain the stated distinguished triangle. This implies that $\left[\lim_{\rightarrow} A \to A\right] \in D^b(PAlg_{uc}/k)$ since $[TA_{sAb} \to A_0], \pi_0(A) \in D^b(PAlg_{uc}/k)$. Therefore $A^{SD\text{SD}} = \left[\lim_{\rightarrow} A \to A\right] \in D^b(PAlg_{uc}/k)$.

Applying the Serre dual to the stated distinguished triangle and using $(\lim_{\rightarrow} A)^{SD} = 0$, we have a distinguished triangle

$$\pi_0(A)^{SD} \to A^{SD} \to [TA_{sAb} \to A_0]^{SD}.$$ 

Hence $A^{SD} \in D^b(IAlg_{uc}/k)$. \qed
As examples, we have

\[ W_n^{SD} = W_n[-2] \quad \text{[Mil06 III, Lem. 0.13 (c)]}, \]

\[ W^{SD} = \lim_{\to} W_n^{SD} = \lim_{\to} W_n[-2], \]

\[ (\mathbb{Z}/n\mathbb{Z})^{SD} = \mathbb{Z}/n\mathbb{Z}[-1], \quad \mathbb{Z}^{SD} = \mathbb{Z}, \]

\[ G_n^{SD} = \bigoplus_{l \neq p}(\mathbb{Q}_l/\mathbb{Z}_l)(-1)[-2], \quad A^{SD} = (TA)^{PD}[-2], \]

where \( n \geq 1 \), \( W_n \) the group of Witt vectors of length \( n \), \( W = \lim_{\to} W_n, (\mathbb{Q}_l/\mathbb{Z}_l)(-1) \) the negative Tate twist and \( A \) an abelian variety over \( k \). (Recall that all the groups here are the perfections of the corresponding group schemes. In particular, the \( p \)-th power map on the quasi-algebraic group \( G_n \) is an isomorphism.)

Before introducing the notion of \( P \)-acyclicity, we discuss sheaves locally of finite presentation on \( \text{Spec} l^{\text{indrat}}_\text{et} \). A sheaf \( F \in \text{Set}(l^{\text{indrat}}_\text{et}) \) is said to be locally of finite presentation if it commutes with filtered direct limits as a functor on \( l^{\text{indrat}}_\text{proet} \). Such a sheaf is automatically a pro-\( \acute{e}tale \) sheaf: \( F \in \text{Set}(l^{\text{proet}}_\text{proet}) \). For \( k' = \bigcup_{\lambda} k'_{\lambda} \in l^{\text{indrat}}_\text{et} \) with \( k'_{\lambda} \in l^{\text{rat}}_\text{et} \), we define

\[ F^{fp}(k') = \lim_{\lambda} F(k'_{\lambda}). \]

Clearly \( F^{fp} \) is locally of finite presentation. We see that \( F^{fp} \in \text{Set}(l^{\text{indrat}}_\text{et}) \) since an \( \acute{e}tale \) covering \( k''/k' \) in \( l^{\text{indrat}}_\text{et} \) is a filtered direct limit of \( \acute{e}tale \) coverings in \( l^{\text{rat}}_\text{et} \). We have a natural morphism \( F^{fp} \to F \), which is an isomorphism if and only if \( F \) is locally of finite presentation. If \( \{F_{\lambda}\} \) is a filtered direct system of sheaves locally of finite presentation, then \( \lim_{\to} F_{\lambda} \) is locally of finite presentation. Since objects of \( L\text{Alg}/k \) and \( \text{Alg}/k \) in particular are locally of finite presentation, it follows that objects of \( I\text{Alg}/k \) and \( \text{Alg}/k \) are also locally of finite presentation. If \( A \in P\text{Alg}/k \) and \( B \in L\text{Alg}/k \), then \( \text{Ext}^n_{l^{\text{indrat}}_\text{et}}(A, B) \) for any \( n \geq 0 \) is locally of finite presentation by (2.3.1) \( \text{(a)} \). In particular, \( \text{Ext}^n_{l^{\text{indrat}}_\text{et}}(A, B) \) and \( \text{Ext}^n_{l^{\text{proet}}_\text{proet}}(A, B) \) are equal as functors on \( l^{\text{indrat}}_\text{et} \). If \( f: E \to F \in \text{Set}(k^{\text{indrat}}_\text{et}) \) is a morphism between sheaves locally of finite presentation, then the property of \( f \) being injective, surjective or invertible can be tested on geometric points (i.e. if \( E(k') \to F(k') \) for any algebraically closed field \( k' \) over \( k \), then \( E \to F \) etc.). Similarly, if \( A \in \text{Ab}(l^{\text{indrat}}_\text{et}) \) is locally of finite presentation with vanishing geometric points (i.e. \( A(k') = 0 \) for any algebraically closed field \( k' \) over \( k \)), then \( A = 0 \). This argument has already been used in the proof of (2.4.1) \( \text{(a)} \). A sheaf \( F \in \text{Set}(k^{\text{indrat}}_\text{et}) \) is representable by an \( \acute{e}tale \) scheme over \( k \) if and only if it is locally of finite presentation and \( F(k') \) does not depend on algebraically closed \( k' \) over \( k \), in which case \( F = F(k) \) as \( \text{Gal}(k/k) \)-sets.

Now let \( P: \text{Spec} k^{\text{indrat}}_\text{et} \to \text{Spec} k^{\text{indrat}}_\text{proet} \) be the morphism defined by the identity. We say that a sheaf \( A \in \text{Ab}(k^{\text{indrat}}_\text{et}) \) is \( P \)-\textit{acyclic} if the natural morphism

\[ A \to RP_\ast P^\ast A \]
is an isomorphism\footnote{The “P” in “P-acyclicity” is just the initial letter of “pro” and not (necessarily) a variable. We just avoid naming it as “pro-acyclicity”, since it is not a pro-object version of some acyclicity property.} This is equivalent that \( A \in \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{proet}}) \) and \( H^n(k'_{\operatorname{proet}}, A) \cong H^n(k'_{\operatorname{proet}}, A) \) for all \( k' \in k^{\text{indrat}} \). We say that a complex \( A \in D^+(k^{\text{indrat}}_{\operatorname{et}}) \) is P-acyclic if its cohomology is P-acyclic in each degree. This implies that 

\[
A \cong R_P P^* A \quad \text{in} \quad D^+(k^{\text{indrat}}_{\operatorname{et}})
\]

by writing down the hypercohomology spectral sequence for \( R_P P^* A \). If \( 0 \to A \to B \to C \to 0 \) is an exact sequence in \( \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{et}}) \) and any two of the terms are P-acyclic, then so is the other. P-acyclicity of a sheaf is preserved under filtered direct limits. If \( f : \operatorname{Spec} k' \to \operatorname{Spec} k \) is a finite étale morphism and \( A' \in \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{et}}) \) is P-acyclic, then the Weil restriction \( f_* A' \in \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{et}}) \) is P-acyclic. When \( A \in D^+(k^{\text{indrat}}_{\operatorname{et}}) \) is P-acyclic, we will use the same letter \( A \) to mean its inverse image \( P^* A \in D^+(k^{\text{indrat}}_{\operatorname{proet}}) \). Here is a criterion of P-acyclicity.

**Proposition (2.4.2).**

(a) If \( A \in \operatorname{LAlg}/k \) or \( \operatorname{IAlg}/k \), then \( A \) is P-acyclic. More generally, any sheaf \( A \in \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{et}}) \) locally of finite presentation is P-acyclic.

(b) If \( A \in \operatorname{PAlg}/k \) can be written as \( A = \lim_{\longleftarrow} \bigwedge_{n \geq 1} A_n \) with \( A_n \in \operatorname{Alg}/k \) such that all the transition morphisms \( \varphi_n : A_{n+1} \to A_n \) are surjective with connected unipotent kernel, then \( A \) is P-acyclic.

**Proof.**\footnote{We already saw above that objects of \( \operatorname{LAlg}/k \) and \( \operatorname{IAlg}/k \) are locally of finite presentation and a sheaf locally of finite presentation is in \( \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{proet}}) \). Let \( A \in \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{et}}) \) be locally of finite presentation (so \( A \in \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{proet}}) \)). We need to show that \( H^0(k'_{\operatorname{proet}}, A) = H^0(k'_{\operatorname{et}}, P_* A) \) for \( n \geq 1 \) and \( k' \in k^{\text{indrat}} \). In [Suz13 Prop. 3.3.1], a statement corresponding to pro-fppf and fppf cohomology instead of pro-étale and étale cohomology is proved. The same proof works.} We already saw above that objects of \( \operatorname{LAlg}/k \) and \( \operatorname{IAlg}/k \) are locally of finite presentation and a sheaf locally of finite presentation is in \( \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{proet}}) \). Let \( A \in \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{et}}) \) be locally of finite presentation (so \( A \in \operatorname{Ab}(k^{\text{indrat}}_{\operatorname{proet}}) \)). We need to show that \( H^0(k'_{\operatorname{proet}}, A) = H^0(k'_{\operatorname{et}}, P_* A) \) for \( n \geq 1 \) and \( k' \in k^{\text{indrat}} \). In [Suz13 Prop. 3.3.1], a statement corresponding to pro-fppf and fppf cohomology instead of pro-étale and étale cohomology is proved. The same proof works. \footnote{Let \( B_n \) be the maximal connected unipotent subgroup of \( A_n \). Then \( C := A_n/B_n \in \operatorname{Alg}/k \) does not depend on \( n \) by assumption and \( B = \lim_{\longleftarrow} B_n \) satisfies the same assumption as \( A \). If \( B \) is P-acyclic, then the sequence \( 0 \to B \to A \to C \to 0 \) is exact in \( \operatorname{Ab}(k^{\text{indrat}}) \), and \( A \) is P-acyclic. Therefore we may assume that all \( A_n \) are connected unipotent. Then \( H^m(k'_{\operatorname{et}}, A_n) = 0 \) for all \( m \geq 1 \). Therefore we have \( R^m \prod_n A_n = 0 \) for all \( m \geq 1 \) by [Roo06 Prop. 1.6]. Therefore by [Roo06 Thm. 2.1, Lem. 2.2 and Rmk. 2.3], the object \( R \lim_n A_n \in D(k^{\text{indrat}}_{\operatorname{et}}) \) is represented by the complex 

\[
\prod_n A_n \to \prod_n A_n,
\]

concentrated in degrees 0 and 1, where the morphism is given by \( (a_n) \mapsto (a_n - \varphi_n(a_{n+1})) \). Note that the morphisms \( \varphi_n : A_{n+1}(k') \to A_n(k') \) are surjective for all \( k' \in k^{\text{indrat}} \) and \( n \) since ker(\( \varphi_n \)) is connected unipotent. Therefore the evaluated complex

\[
\prod_n A_n(k') \to \prod_n A_n(k'),
\]

which is \( R \lim_n A_n(k') \in D(\operatorname{Ab}) \), is concentrated in degree 0. Hence \( R \lim_n A_n = A \) in \( D(k^{\text{indrat}}_{\operatorname{et}}) \). Therefore

\[
R \Gamma(k'_{\operatorname{et}}, A) = R \Gamma(k'_{\operatorname{et}}, R \lim_n A_n) = R \lim_n R \Gamma(k'_{\operatorname{et}}, A_n) = R \lim_n A_n(k') = A(k').
\]
Since $R\lim A_n = A$ in $D(k_{\text{indrat}})$ follows from (2.1.2) [10], a similar calculation shows that $R\Gamma(k_{\text{proet}}, A) = A(k')$. Hence $A$ is $P$-acyclic.

For example, the proalgebraic groups $G_a^n$, $W$ and $W^n$ are $P$-acyclic. The groups $\mathbb{Z}_l$ for a prime $l$ and $G_a^n_m$ are not.

3. Local fields with ind-rational base

Let $K$ be a complete discrete valuation field with perfect residue field $k$ of characteristic $p > 0$. In this section, we view cohomology of $K$ as a complex of sheaves on $\text{Spec} k_{\text{indrat}}$. As explained in [12], the $n$-th cohomology of this complex is the pro-étale sheafification of the presheaf

$$k' \mapsto H^n(K(k'), A),$$

where $A$ is an fppf sheaf on $K$. To study cohomology of this form, we introduce a topological space structure on the sets of rational points of schemes over the ring $K(k')$ following [Con12]. We give some density and openness results using some Artin type approximation methods and results, which reduce the problems to classical results on complete discrete valuation fields.

Then we recall the relative fppf site $\text{Spec} K_{\text{fppf}}/k_{\text{et}}\text{indrat}$ and the structure morphism

$$\pi: \text{Spec} K_{\text{fppf}}/k_{\text{et}}\text{indrat} \to \text{Spec} K_{\text{et}}\text{indrat}$$

from [Suz13]. The complex we want is defined to be the pro-étale sheafification of the derived pushforward by $\pi$.

3.1. Basic notions and properties. We denote the ring of integers of $K$ by $\mathcal{O}_K$. The maximal ideal of $\mathcal{O}_K$ is denoted by $p_K$. We denote by $W$ the affine ring scheme of Witt vectors of infinite length. Since $k$ is a perfect field of characteristic $p > 0$, the ring $\mathcal{O}_K = \lim_{\leftarrow} \mathcal{O}_K/p^n_K$ has a canonical structure of a $W(k)$-algebra of pro-finite-length [Ser79, II, §5, Thm. 4] (which is actually finite free over $W(k)$ in the mixed characteristic case and factors as $W(k) \to k \to \mathcal{O}_K$ in the equal characteristic case). We define sheaves of rings on the site $\text{Spec} k_{\text{et}}\text{indrat}$ by assigning to each $k' \in k_{\text{indrat}}$,

$$\mathcal{O}_K(k') = W(k') \hat{\otimes}_{W(k)} \mathcal{O}_K = \lim_n \left( W_n(k') \otimes_{W_n(k)} \mathcal{O}_K/p^n_K \right),$$

$$K(k') = \mathcal{O}_K(k') \otimes_{\mathcal{O}_K} K.$$
(a) For any \( m \in \text{Spec} \, k' \), the ideal \( K(m) \) (resp. \( p_K(k') + O_K(m) \)) is a maximal ideal of \( K(k') \) (resp. \( O_K(k') \)). The map \( m \mapsto K(m) \) (resp. \( m \mapsto p_K(k') + O_K(m) \)) gives a homeomorphism from \( \text{Spec} \, k' \) onto the maximal spectrum of \( K(k') \) (resp. \( O_K(k') \)).

(b) A neighborhood base of the maximal ideal \( K(m) \) of \( K(k') \) (resp. \( p_K(k') + O_K(m) \) of \( O_K(k') \)) is given by the family of open and closed sets

\[
\text{Spec} \, K(k')[1/\omega(e)] = \text{Spec} \, K(k'[1/e])
\]

for idempotents \( e \in k' \setminus m \). In particular, any Zariski covering of \( \text{Spec} \, K(k') \) (resp. \( \text{Spec} \, O_K(k') \)) can be refined by a disjoint Zariski covering.

(c) Let \( \pi \) be a prime element of \( O_K \). An element \( \sum_{n \geq 0} \omega(a_n)\pi^n \) of \( O_K(k') \) (with \( a_n \in k' \)) is invertible if and only if \( a_0 = 0 \). An element \( \sum_{n \geq 0} \omega(a_n)\pi^n \) of \( K(k') \) (with \( a_n = 0 \) for sufficiently small \( n \) \( < \) \( 0 \)) is invertible if and only if some finitely many of the \( a_n \) generate the unit ideal of \( k' \).

Proof. \([\text{Suz13}]\) for \( K(k') \) were proved in [Suz13] Lem. 2.5.1-2. (To see that the map \( m \mapsto K(m) \) is not only a bijection but also a homeomorphism, use (b).) The same proof works for \( O_K(k') \).

The statement for \( O_K(k') \) is obvious. If \( \sum \omega(a_n)\pi^n \in K(k') \) is invertible, then so is its image in the complete discrete valuation field \( K(k'/m) \) for any \( m \in \text{Spec} \, k' \), hence \( a_n \notin m \) for some \( n \). Therefore the \( a_n \) generate the unit ideal of \( k' \). Some finite linear combination of them gives 1 in \( k' \).

For the converse, denote by \( D(\omega(a)) \) and \( V(\omega(a)) \) for \( a \in k' \) the open and closed set \( \text{Spec} \, K(k')[1/\omega(a)] \) and its complementary closed set \( \text{Spec} \, K(k'/a) \) of \( \text{Spec} \, K(k') \), respectively. They are both open and closed since \( k' \in k^{\text{indrat}} \) and hence \( a \) is a unit times an idempotent. For any \( m \in \mathbb{Z} \), on the open and closed set \( U_m = \bigcap_{a \in k'} V(\omega(a)) \cap D(\omega(a_m)) \), the element

\[
\sum_{n \geq m} \omega(a_n)\pi^n \equiv \sum_{n \geq m} \omega(a_n)\pi^n = \pi^m \sum_{n \geq 0} \omega(a_{n+m})\pi^n
\]

becomes a unit. Some finitely many of these disjoint open and closed sets \( \{ U_m \} \) cover \( \text{Spec} \, K(k') \) by assumption. Hence \( \sum \omega(a_n)\pi^n \) is a unit in \( K(k') \). \(\square\)

Some properties of schemes over the rings \( O_K(k'), K(k') \):

**Proposition (3.1.3).** Let \( k' \in k^{\text{indrat}} \).

(a) If \( X \) is an affine \( O_K(k') \)-scheme and \( x \in X(K(k')) \), then the subset

\[
\{ m \in \text{Spec} \, k' \mid x(m) \in X(O_K(k'/m)) \}
\]

of \( \text{Spec} \, k' \) is closed. It is open if \( X \) is of finite type.

(b) Let \( X \) be a separated \( O_K(k') \)-scheme locally of finite type and \( x \in X(K(k')) \). If \( x(m) \in X(O_K(k'/m)) \) for all \( m \in \text{Spec} \, k' \), then \( x \in X(O_K(k')) \).

(c) Let \( X \) be either a proper \( O_K(k') \)-scheme or written as \( G \times_{O_K} O_K(k') \) with \( G \) the Néron model of a smooth group scheme over \( K \). Then we have

\[
X(O_K(k')) = X(K(k')).
\]

(d) If \( X \) is any \( O_K(k') \)-scheme, then

\[
X(O_K(k')) = \varprojlim_n X(O_K/p^n_K(k'))
\]
Proof. Write $X = \text{Spec } S$. Let $\varphi : S \to K(k')$ be the $O_K(k')$-algebra homomorphism corresponding to $x \in X(K(k'))$. For $f \in S$, write $\varphi(f) = \sum \omega(a_{n,f}) \pi^n$ with $a_{n,f} \in k'$, where $\pi$ is a prime element of $O_K$. Then a maximal ideal $m \in \text{Spec } k'$ is in the given subset if and only if the composite $S \to K(k') \to K(k'/m)$ factors through $O_K(k'/m)$. This is equivalent that $a_{n,f} \in m$ for all $n < 0$ and $f \in S$. Therefore the given subset is the closed subset defined by the ideal generated by the elements $a_{n,f} \in k'$ for all $n < 0$ and $f \in S$. We may consider only those $f$'s in any fixed set of generators of the $O_K(k')$-algebra $S$. Hence the ideal is finitely generated if $S$ is finitely generated. A finitely generated ideal of $k' \in \text{indrat}$ is generated by a single idempotent. Hence the given set is open in this case.

Note that the separatedness of $X$ and the injectivity of $O_K(k') \to K(k')$ implies that the natural map $X(O_K(k')) \to X(K(k'))$ is injective. We first treat the case where $X$ is affine. Using the notation in the proof of the previous assertion, the assumption implies that $a_{n,f} = 0$ for all $n < 0$ and $f \in S$. Hence $\varphi$ factors through $O_K(k')$ and therefore $x \in X(O_K(k'))$ in this case.

Next we treat the general case. Let $\{U_\lambda\}$ be an affine open cover of $X$. For any $m \in \text{Spec } k'$, the sets $U_\lambda(O_K(k'/m))$ indexed by $\lambda$ cover $X(O_K(k'/m))$ since $O_K(k'/m)$ is local. Hence the assumption implies that the sets

$$\{m \in \text{Spec } k' \mid x(m) \in U_\lambda(O_K(k'/m))\}$$

indexed by $\lambda$ cover $\text{Spec } k'$. For each $\lambda$, this set is open since the affine $O_K(k')$-scheme $U_\lambda$ is of finite type and by 2. Note this open covering of $\text{Spec } k'$ by a disjoint open covering $\{\text{Spec } k'[1/e_\lambda]\}$ with $e_\lambda$ idempotents. Note that all but finitely many $e_\lambda$ are zero. By considering $k'[1/e_\lambda]$ for each $\lambda$ instead of $k'$, we are reduced to the affine case.

By the previous assertion, it is reduced to showing $X(O_K(k'/m)) = X(K(k'/m))$ for all $m \in \text{Spec } k'$. The proper case is the valuative criterion. For the Néron case, note that $k'/m$ is a separable (possibly transcendental) field extension of $k$ in the sense of [Bou03, Chap. V, §15, no. 2-3] by [loc. cit., no. 5, Thm. 3 b)]. Therefore [BLR00, 10.1, Prop. 3] shows that $G \times_{O_K} O(k'/m)$ is the Néron model, over the discrete valuation ring $O(k'/m)$, of its generic fiber. Hence the Néron mapping property implies that $X(O_K(k'/m)) = X(K(k'/m))$.

This is trivial if $X$ is affine. In general, let $\{U_\lambda\}$ be an affine open cover of $X$. Let $(x_n)_n$ be an element of the limit in question. The pullback of $\{U_\lambda\}$ by the morphism $x_1 : \text{Spec } k' \to X$ gives an open covering of $\text{Spec } k'$. Refine it by a disjoint covering $\{\text{Spec } k'[1/e_\lambda]\}$ with idempotents $e_\lambda$. Then $x_1$ restricted to $\text{Spec } k'[1/e_\lambda]$ factors through $U_\lambda$. Since the surjection $O_K/p^n_K(k') \to k'$ has a nilpotent kernel, we know that $x_n$ restricted to $\text{Spec } O_K/p^n_K(k'[1/e_\lambda])$ factors through $U_\lambda$ for all $n$. The affine case then implies that $(x_n)_n$ comes from $X(O_K(k'))$. 

3.2. Topology on rational points of varieties. We give a topology on the set $X(K(k'))$ for any $K(k')$-scheme $X$ locally of finite type with $k' \in \text{indrat}$ and on the set $Y(O_K(k'))$ for any $O_K(k')$-scheme $Y$ locally of finite type. We follow [Con12, §2-3].

First for each $k' \in \text{indrat}$, the ring $O_K(k')$ is a topological ring by the ideals $p^n_K(k')$, $n \geq 0$. We give a topological ring structure on $K(k')$ so that the subring $O_K(k')$ is open. Recall from [Con12, Prop. 2.1] that the set $X(K(k'))$ for an affine $K(k')$-scheme $X$ of finite type has a canonical structure of a topological space. Explicitly, choose an embedding $X \hookrightarrow \mathbb{A}^n$ for some $n$, give the product topology
on $K^n(K(k')) = K(k')^n$ and give the subspace topology on $X(K(k'))$. This is independent of the choice, and relies only on the fact that $K(k')$ is a topological ring. Similarly we have a canonical topological space structure on $Y(O_K(k'))$ for an affine $O_K(k')$-scheme $Y$ of finite type.

To proceed to the non-affine case, note that the subsets $K^n(k') \subset K(k')$ and $U_K(k') \subset O_K(k')$ are open, and the inverse maps on them are continuous, by (3.1.1) and (3.1.2). Therefore if $X$ is an affine $K(k')$-scheme and $U$ is a basic open affine subset of $X$ (i.e., a localization by one element), then $U(K(k')) \subset X(K(k'))$ is an open immersion of topological spaces as explained in the proof of [Con12, Prop. 3.1]. If we try to apply the whole cited proposition, we will need the equality

$$X(K(k'))_\lambda = \bigcup_{\lambda} U_\lambda(K(k'))$$

for a $K(k')$-scheme $X$ locally of finite type and an arbitrary affine open cover $\{U_\lambda\}$ of $X$, which requires that $K(k')$ be local. When $k' \in k^{\text{indrat}}$ is a field, then $K(k')$ is a complete discrete valuation field, so the equality above is true. If $k' \in k^{\text{indrat}}$ is a finite product of fields $\prod k_i'$, then the above equality is not true in general, but the equality $X(K(k')) = \prod X(K(k'_i))$ gives a product topology on $X(K(k'))$. For a general $k' = \bigcup k'_\lambda \in k^{\text{indrat}}$ with $k_i' \in k^{\text{rat}}$, it is not true that $K(k') = \bigcup K(k'_\lambda)$, and hence it is not immediately clear how to use the topologies on the sets $U_\lambda(K(k'))$ to topologize $X(K(k'))$. The situation is the same for $O_K(k')$. To topologize $X(K(k'))$ for a general $k' \in k^{\text{indrat}}$, we will use the following.

**Proposition (3.2.1).** Let $k' \in k^{\text{indrat}}$.

(a) Let $X$ be a $K(k')$-scheme locally of finite type and $\{U_\lambda\}_{\lambda \in \Lambda}$ any affine open cover of $X$. Given a family $\{e_{\lambda}\}$ of disjoint idempotents of $k'$ indexed by $\Lambda$ such that $\sum e_{\lambda} = 1$ and a family of $K$-morphisms $\text{Spec} K(k'[1/e_{\lambda}]) \to U_\lambda$ ($\leftrightarrow X$), a trivial patching gives a $K$-morphism $\text{Spec} K(k') \to X$. The map

$$X(K(k')) = \bigcup_{\{e_{\lambda}\}} \prod_{\lambda} U_\lambda(K(k'[1/e_{\lambda}]),$$

thus obtained is bijective.

(b) Let $X$ be an affine $K(k')$-scheme of finite type and $\{U_i\}$ any basic affine open finite cover of $X$. Let $\{e_i\}$ be a family of disjoint idempotents of $k'$ such that $\sum e_i = 1$. Then the bijection and the natural map

$$X(K(k')) = \prod_{X} X(K(k'[1/e_i])) \leftrightarrow \prod_{U_i} U_i(K(k'[1/e_i]))$$

is a homeomorphism and an open immersion, respectively.

There are statements for $O_K(k')$ instead of $K(k')$ in the obvious manner.

**Proof.** The first assertion follows from (3.1.1). To check that the bijection in the second assertion is a homeomorphism, we can use the fact that $K(k') = \prod K(k'[1/e_i])$ is a homeomorphism. The final map is an open immersion as seen before. The statements for $O_K(k')$ can be proven similarly.

Given this, we introduce a topology on $X(K(k'))$ by declaring that a subset of $X(K(k'))$ is open if its intersection with the product topological space $\prod \Lambda U_\lambda(K(k'[1/e_{\lambda}]))$ is open for any $\{e_{\lambda}\}$. This is independent of the choice of the affine open covering $\{U_\lambda\}$ as in [Con12, Prop. 3.1]. Similarly we have a topology on $Y(O_K(k'))$ for an $O_K(k')$-scheme $Y$ locally of finite type. Some properties
of the topology given in [Con12] also hold for our topology. We list them. A $K(k')$-morphism $X_1 \to X_2$ induces a continuous map $X_1(K(k')) \to X_2(K(k'))$ for any $k' \in k^{\text{indrat}}$. We have homeomorphisms $\mathbb{A}^n(K(k')) = K(k')^n$ and $(X_2 \times X_1)X_3(K(k')) = X_2(K(k')) \times X_1X_3(K(k'))$. A closed (resp. open) immersion $X_1 \hookrightarrow X_2$ corresponds to a closed (resp. open) immersion $X_1(K(k')) \hookrightarrow X_2(K(k'))$. If $X$ is separated, then $X(K(k'))$ is Hausdorff. Similar for $O_K(k')$.

We need the following three topological propositions.

**Proposition (3.2.2).** Let $k' \in k^{\text{indrat}}$. Let $X$ be a separated $O_K(k')$-scheme locally of finite type. Then $X(O_K(k'))$ is an open subset of $X(K(k'))$.

**Proof.** We may assume that $X$ is affine. Let $X \hookrightarrow \mathbb{A}^n$ be a closed immersion. Then $X(O_K(k')) = X(K(k')) \cap \mathbb{A}^n(O_K(k'))$ in $\mathbb{A}^n(K(k'))$. Since $O_K(k')$ is open in $K(k')$ by definition, it follows that $X(O_K(k'))$ is open in $X(K(k'))$.

Before the next proposition, recall from [2.3] that

$$K^{\text{fp}}(k') = \lim_{\rightarrow} K(k'_\lambda), \quad O_K^{\text{fp}}(k') = \lim_{\rightarrow} O_K(k'_\lambda)$$

for $k' = \bigcup k'_\lambda \in k^{\text{indrat}}$ with $k'_\lambda \in k^{\text{rat}}$. The first (resp. second) ring is a filtered union of finite products of complete discrete valuation fields (resp. rings). When $k' = \overline{k}$ (and hence the $k'_\lambda$ are finite extensions of $k$), the $K$-algebra $K^{\text{fp}}(k')$ is the maximal unramified extension $K^{\text{ur}}$ of $K$, whose completion is $\overline{K^{\text{ur}}} = K(\overline{k})$.

**Proposition (3.2.3).** Let $X$ be a $K$-scheme locally of finite type and $k' \in k^{\text{indrat}}$. Then $X(K^{\text{fp}}(k'))$ is a dense subset of $X(K(k'))$. There is a similar statement for $O_K$ and $O_K$ in place of $K$ and $K$, respectively.

**Proof.** The ring $K(k')$ is faithfully flat over $K^{\text{fp}}(k')$ and the map $X(K^{\text{fp}}(k')) \to X(K(k'))$ is injective. We may assume that $X$ is affine and defined over $O_K$. The map $X(O_K^{\text{fp}}(k')) \to X(O_K(k'))$ is injective by the same reason. By scaling, it is enough to show that $X(O_K^{\text{fp}}(k'))$ is dense in $X(O_K(k'))$.

Recall Greenberg’s approximation theorem [Gre64, Cor. 1 to Thm. 1]: there are integers $N \geq 1$, $c \geq 0$, $s \geq 0$ such that for any $n \geq N$ and any $x \in (O_K/p^n_K)^{[n/c]-s}$, the image of $x$ in $X(O_K/p^n_K^{[n/c]-s})$ lifts to $X(O_K)$.

The proof of this theorem works for the following slightly stronger statement: there are integers $N \geq 1$, $c \geq 0$, $s \geq 0$ such that for any perfect field $k''$ over $k$, any $n \geq N$ and any $x \in (O_K(k''))^{[n/c]-s}$, the image of $x$ in $X(O_K(k''))^{[n/c]-s}$ lifts to $X(O_K(k''))$. (Note that $O_K(k'')$ is a complete discrete valuation ring in which a prime element of $O_K$ remains prime.)

Now let $n \geq N$ and $x \in X(O_K(k'))$. Write $k' = \bigcup k'_\lambda$ with $k'_\lambda \in k^{\text{rat}}$. Since $(O_K/p^n_K)^{[n/c]-s}(k'_\lambda) = O_K/p^n_K(k'_\lambda) = O_K(k'_\lambda)^{[n/c]-s}$, the image of $x$ in $X(O_K/p^n_K^{[n/c]-s}(k'_\lambda))$ lifts to $X(O_K/p^n_K(k'_\lambda))$. Since $k'_\lambda$ is a finite product of perfect fields over $k$, the image of $x$ in $X(O_K/p^n_K^{[n/c]-s}(k'_\lambda)) \subset X(O_K/p^n_K^{[n/c]-s}(k'))$ lifts to $X(O_K(k'_\lambda)) \subset X(O_K^{\text{fp}}(k'))$.

---

9 Do not confuse $O_\mathbb{K}^{\text{fp}}(k')$ with the uncompleted tensor product $W(k') \hat{\otimes}_{W(k)} O_K$. If $K$ has mixed characteristic (and hence $O_K$ is finite free over $W(k)$) and $k' = \overline{k}$, then the former is $O_\mathbb{K}^{\text{ur}}$, which is smaller than the latter $O_\mathbb{K}^{\text{fp}}$. If $K$ has equal characteristic and $k'$ is (the perfection of) $k(x)$, then the former is $k(x)[[T]]$, which is bigger than latter $k(x) \otimes_k (k[[T]])$.
This shows that $X(O_K(k')^n)$ is dense in $X(O_K(k'))$.

\[\square\]

**Proposition (3.2.4).** Let $k' \in \k^{\text{indrat}}$ and $X, Y$ $k'$-schemes locally of finite presentation. If $f: Y \to X$ is a smooth $k'$-morphism, then the image of $Y(k')$ under $f$ is an open subset of $X(k(k'))$.

**Proof.** This is obvious if $Y$ is an affine space over $X$. Hence we may assume that $f$ is étale. We may further assume that $X$ and $Y$ are affine. Let $y \in Y(k(k'))$ and $x = f(y)$. We want to show that any elements $x'$ of $X(k(k'))$ sufficiently close to $x$ come from $Y(k(k'))$. For this, we are going to apply Togëron’s lemma [Art69, Lem. 5.10] to solve the equation $f(y') = x'$ for an unknown $y' \in Y(k(k'))$. This lemma is true for any Henselian pair, and the pair $(O_K(k'), p_K)$ is complete and hence Henselian. To apply the lemma, we need to write down all the conditions as polynomial equations and approximate solutions.

Let $f': Y' \to X'$ be a morphism between affine $O_K(k')$-schemes of finite presentation whose base change to $K(k')$ is $f$. Embed $X'$ and $Y'$ to affine spaces $A^n_{O_K(k')}$ and $A^n_{O_K(k')}$ over $O_K(k')$, respectively, and extend $f'$ to a morphism $f': Y' \to A^n_{O_K(k')}$. By scaling, we may assume that $y \in Y'(O_K(k'))$.

Let $P = (P_1, \ldots, P_l)$ be a polynomial system defining $Y' \subset A^n_{O_K(k')}$. Let $\pi$ be a prime element of $O_K$. Since $f: Y \to X$ is étale, the differential module $\Omega^1_{Y'/X'}$ is killed by a power $\pi^n$ of $\pi$. We want to show that any element $x' \in X'(O_K(k')) \subset O_K(k')^n$ with term-wise difference

\[(3.2.5) \quad x' - x \in p_k^{2rn+1}(k')^m\]

comes from $Y'(O_K(k'))$, where $p_k^{2rn+1}(k')^m \subset O_K(k')^m$ is the set-theoretic product of $m$ copies of the $(2rn+1)$-st power of the ideal $p_k$. Let $f'^{-1}(x')$ be the fiber of $f': Y' \to X'$ over $x' \in X'(O_K(k'))$. Then $\Omega^1_{f'^{-1}(x')/O_K(k')}$ is killed by $\pi^n$. We have a commutative diagram with cartesian squares of finitely presented $O_K(k')$-schemes

\[
\begin{array}{c}
f'^{-1}(x') \quad \text{incl} \quad Y' \quad \text{incl} \quad \mathcal{A}^n_{O_K(k')/p} \quad \text{incl} \quad \mathcal{A}^m_{O_K(k')} \\
\downarrow \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
x' \quad \text{incl} \quad X' \quad \text{incl} \quad \mathcal{A}^m_{O_K(k')}.
\end{array}
\]

Let $y' = (y_1', \ldots, y_n')$ be the coordinates of $A^n_{O_K(k')}$. We view $f': \mathcal{A}^n_{O_K(k')} \to \mathcal{A}^m_{O_K(k')}$ as a system of $m$ polynomials in the $n$ variables $(y_1', \ldots, y_n')$ and $x'$ as an element of $O_K(k')^m$ (or $m$ constant polynomials). Define

\[S = O_K(k')[y']/(f'(y') - x', P(y'))\]

where the ideal on the right is generated by all the $m + l$ polynomials in the polynomial system $(f'(y') - x', P(y'))$. We have $f'^{-1}(x') = \text{Spec } S$. Let $J$ be the Jacobian matrix of the system $(f'(y') - x', P(y'))$, which has entries in $O_K(k')[y']$. Then $\Omega^1_{f'^{-1}(x')/O_K(k')}$ is given by the cokernel of the $S$-module homomorphism $S^{m+l} \to S^n$ corresponding to $J$. Let $F$ be the ideal of $O_K(k')[y']$ generated by all minors of size $n$ of $J$. Then the image $\bar{F}$ of $F$ in $S$ is the zeroth Fitting ideal of $\Omega^1_{f'^{-1}(x')/O_K(k')}$. [Lan02, XIX, \S2]. Therefore $\bar{F}$ contains the $n$-th power of the annihilator of $\Omega^1_{f'^{-1}(x')/O_K(k')}$ by [Lan02, XIX, \S2, Prop. 2.5] and hence contains $\pi^{rn}$. Hence

\[\pi^{rn} \in F + (f'(y') - x', P(y')) \subset O_K(k')[y']\].
Evaluate this at \( y' = y \). Since \( y \in Y'(O_K(k')) \), we have \( P(y) = 0 \). Also \( f'(y) = x \).

Hence

\[
\pi^{rn} \in F(y) + (x - x') \subset O_K(k'),
\]

where \( F(y) \) is the image of \( F \) under the evaluation map \( O_K(k')\langle y'\rangle \to O_K(k') \) at \( y \). With this and using (3.2.5), we know that \( \pi^{rn} \in F(y) \). By (3.2.7) below, we may assume that the value \( \delta(y) \) of a single minor \( \delta \) divides \( \pi^{rn} \) in \( O_K(k') \), so \( p_K^{rn}(k') \subset \delta(y)O_K(k') \). Hence

(3.2.6) \[ p_K^{2rn+1}(k') \subset \delta(y)^2p_K(k'). \]

Now we have

\[
f'(y) - x' \in \delta(y)^2p_K(k')^m, \quad P(y) = 0
\]

by (3.2.6) and (3.2.10). Hence the system \( (f'(y') - x', P) \) has a root \( y' \in O_K(k')^n \) by Tougeron’s lemma [Art69, Lem. 5.10]. For this \( y' \), we have \( f'(y') = x' \) and \( y' \in Y'(O_K(k')) \).

\[ \square \]

**Lemma (3.2.7).** If the ideal of \( O_K(k') \), generated by \( q \) elements \( \delta_1, \ldots, \delta_q \in O_K(k') \), contains \( \pi^s \), then there exists a disjoint Zariski covering

\[
\text{Spec } k' = \bigsqcup_{i=1}^q \text{Spec } k'[1/e_i]
\]

with idempotents \( e_i \) such that the image of \( \delta_i \) in \( O_K(k'[1/e_i]) \) divides \( \pi^s \) for each \( i \).

**Proof.** Let \( v \) be the normalized valuation of \( O_K \). For each \( m \in \text{Spec } k' \), the ideal of \( O_K(k'/m) \) generated by \( \delta_1(m), \ldots, \delta_q(m) \) contains \( \pi^s \). Therefore the sets

\[
U_i = \{ m \in \text{Spec } k' \mid v(\delta_i(m)) \leq s \}
\]

for \( i = 1, \ldots, q \) cover \( \text{Spec } k' \). For each \( i \), if \( \delta_i = \sum_{n} \omega(a_{i,n})\pi^n \), then \( U_i \) is the union of the open sets \( \text{Spec } k'[1/a_{i,1}], \text{Spec } k'[1/a_{i,2}], \ldots \). Hence \( \{ U_i \} \) is an open covering of \( \text{Spec } k' \). Refine it by a disjoint Zariski covering \( \text{Spec } k' = \sqcup_{i=1}^q \text{Spec } k'[1/e_i] \) with idempotents \( e_i \). This choice does the job.

\[ \square \]

Now we can prove the following proposition. This is useful especially to prove that the first cohomology of an abelian variety over \( K \) is ind-algebraic and all of the higher cohomology is zero as sheaves over \( K \). The proof below and the use of the above topological statements are inspired by the proof of [Ces15, Prop. 3.5 (a) and Lem. 5.3].

**Proposition (3.2.8).** Let \( A \) be a smooth group scheme over \( K \) and \( k' \in k^{\text{indrat}} \). Then we have

\[
H^n(K(k')_{\text{et}}, A) = H^n(K^{\text{fp}}(k')_{\text{et}}, A)
\]

for any \( n \geq 1 \). This group is torsion.

**Proof.** Let \( f : \text{Spec } K(k') \to \text{Spec } K^{\text{fp}}(k') \) be the natural morphism. Then

\[
f_* : \text{Ab}(K(k')_{\text{et}}) \to \text{Ab}(K^{\text{fp}}(k')_{\text{et}})
\]

is exact by [Suz13, Lem. 2.5.5], hence \( H^n(K(k')_{\text{et}}, A) = H^n(K^{\text{fp}}(k')_{\text{et}}, f_* A) \). Note that \( K^{\text{fp}}(k') \) is a filtered union of finite products of complete discrete valuation fields. The étale cohomology of a field can be calculated by Galois cohomology.
or Čech cohomology. Hence the cohomology of $K^{fp}(k')$ can be calculated by Čech cohomology. The isomorphism to be proved is thus

$$\lim_{L'/K^{fp}(k')} \tilde{H}^n(L'/K^{fp}(k'), f_*A) = \lim_{L'/K^{fp}(k')} \tilde{H}^n(L'/K^{fp}(k'), A),$$

where $L'$ runs through faithfully flat étale $K^{fp}(k')$-algebras and $\tilde{H}$ denotes Čech cohomology. It is enough to prove the isomorphism for each $L'$ before passing to the limit:

$$\tilde{H}^n(L'/K^{fp}(k'), f_*A) = \tilde{H}^n(L'/K^{fp}(k'), A).$$

Write $k' = \bigcup k'_{\lambda}$ with $k'_{\lambda} \in k_{\text{rat}}$. For some $\lambda$, such $L'$ can be written as $L \otimes_{K(k'_{\lambda})} K^{fp}(k')$ for some faithfully flat étale $K(k'_{\lambda})$-algebra $L$. Let $g: \text{Spec } K(k') \to \text{Spec } K(k'_{\lambda})$ and $h: \text{Spec } K^{fp}(k') \to \text{Spec } K(k'_{\lambda})$ be the natural morphisms. Then the Čech complex of $L'/K^{fp}(k')$ with values in $f_*A$ (resp. $A$) is the Čech complex of $L/K(k'_{\lambda})$ with values in $g_*A$ (resp. $h_*A$). Hence the isomorphism to be proven is

$$\tilde{H}^n(L/K(k'_{\lambda}), g_*A) = \tilde{H}^n(L/K(k'_{\lambda}), h_*A).$$

By replacing $k'_{\lambda}$ with $k$, it is enough to show that

$$\tilde{H}^n(L/K, g_*A) = \tilde{H}^n(L/K, h_*A)$$

(for any perfect field $k$), where $L$ is a finite Galois extension of $K$ and $g: \text{Spec } K(k') \to \text{Spec } K$, $h: \text{Spec } K^{fp}(k') \to \text{Spec } K$ the natural morphisms.

Let $C^n$ be the Weil restriction of $A$ from the $(n + 1)$-fold tensor product $L \otimes_K \cdots \otimes_K L$ to $K$, which is representable by a smooth $K$-scheme. The Čech complex of $L/K$ with coefficients in $g_*A$ (resp. $h_*A$) is the $K(k')$-valued (resp. $K^{fp}(k')$-valued) points of the complex $\{C^n\}$ of group schemes over $K$ with the usual coboundary maps $\{d^n\}$. Let $Z^n$ be the kernel of $d^n: C^n \to C^{n+1}$, which is a $K$-scheme locally of finite type. We know that $d^{n-1}: C^{n-1} \to Z^n$ is a smooth morphism as shown in the proof of [Mil80] III, Thm. 3.9. Consider the commutative diagram

$$C^{n-1}(K^{fp}(k')) \xrightarrow{d^{n-1}} Z^n(K^{fp}(k')) \xrightarrow{d^{n-1}} Z^n(K(k')).$$

The vertical maps are injective with dense image by [3.2.3]. The horizontal ones have open image by [3.2.4]. Therefore the map $\tilde{H}^n(L/K, h_*A) \to \tilde{H}^n(L/K, g_*A)$ induced on the cokernels of the horizontal maps is an isomorphism.

Galois cohomology is torsion in positive degrees. Hence $\tilde{H}^n(L/K, h_*A)$ is torsion (killed by $[L : K]$), and $H^n(K^{fp}(k')_{\text{et}}, A)$ is torsion.

The quotient in the following proposition will appear in the next subsection as the first cohomology of $\mathcal{O}_K$ with support on the closed point.

**Proposition (3.2.9).** Let $A$ be a separated group scheme locally of finite type over $\mathcal{O}_K$ and $k' \in k_{\text{indrat}}$. Then we have

$$A(K(k'))/A(\mathcal{O}_K(k')) = A(K^{fp}(k'))/A(\mathcal{O}^{fp}_K(k')).$$
Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
A(O_K^\text{fp}(k')) & \longrightarrow & A(K^\text{fp}(k')) \\
\downarrow & & \downarrow \\
A(O_K(k')) & \longrightarrow & A(K(k')).
\end{array}
\]

As in the proof of the previous proposition, the vertical maps are injective with dense image. The horizontal inclusions are open immersions by \(3.2.2\). Hence the map induced on the cokernels of the horizontal maps is an isomorphism. \(\square\)

We will use the following to see that the group \(N(O_K)\) for a finite flat group scheme \(N\) over \(O_K\) can be viewed as an étale group over \(k\). A direct algebraic proof is not difficult, but a proof using topology is clearer.

**Proposition (3.2.10).** Let \(X\) be a locally quasi-finite separated \(O_K\)-scheme with locally finite part \(X^f\). Let \(K^\text{ur}\) be a maximal unramified extension of \(K\). Let \(x\) and \(x^f\) be the sets \(X(K^\text{ur})\) and \(X^f(K^\text{ur})\), respectively, viewed as étale schemes over \(k\). Then \(X(O_K(k')) = x^f(k')\) and \(X(K(k')) = x(k')\) for any \(k' \in k^\text{intrad}\).

**Proof.** Note that \(X\) is a disjoint union of the \(\text{Spec}'s\) of finite local \(O_K\)-algebras and/or finite local \(K\)-algebras. Write \(X = X^f \sqcup Y\), where \(Y\) is a locally finite \(K\)-scheme. We first show that for any \(k' \in k^\text{intrad}\), any \(O_K\)-scheme morphism \(\text{Spec} O_K(k') \to X\) factors through \(X^f\). By \(3.2.1\) \(\text{(a)}\), we can write \(k' = k'_1 \times k'_2\) with \(k'_1, k'_2 \in k^\text{intrad}\) such that the morphism \(\text{Spec} O_K(k') \to X\) is the disjoint union of morphisms \(\text{Spec} O_K(k'_1) \to X^f\) and \(\text{Spec} O_K(k'_2) \to Y\). Then \(O_K(k'_2)\) has to be a \(K\)-algebra. Therefore if \(k'_2\) were non-zero, then the discrete valuation ring \(O_K(k'_2/m)\) for \(m \in \text{Spec} k'_2\) would have to be a \(K\)-algebra, which is absurd. Hence \(k'_2 = 0\), and \(\text{Spec} O_K(k') \to X\) factors through \(X^f\).

Hence \(X(O_K(k')) = X^f(O_K(k'))\). This is further equal to \(X^f(K(k'))\) by \(3.1.3\) \(\text{(a)}\). Hence the first statement \(X(O_K(k')) = X^f(K(k')) = x^f(k')\) is reduced to the second \(X(K(k')) = x(k')\).

To prove \(X(K(k')) = x(k')\), we may assume that \(X\) is connected by \(3.2.1\) \(\text{(a)}\). We may further assume that \(X\) is reduced, so \(X = \text{Spec} L\) for some finite extension \(L/K\). Let \(k_L\) be the residue field of \(L\). Then \(x = \text{Spec} k_L\) if \(L/K\) is unramified and \(x = \emptyset\) otherwise. We also have \(\text{Hom}_K(L, K^\text{fp}(k')) = \text{Hom}_k(k_L, k')\) if \(L/K\) is unramified and \(\text{Hom}_K(L, K^\text{fp}(k')) = \text{Hom}_k(0, k')\) otherwise. Hence \(X(K^\text{fp}(k')) = x(k')\), which is finite discrete. Since \(X\) is separated, the topological space \(X(K(k'))\) is Hausdorff. By \(3.2.3\), we know that \(X(K^\text{fp}(k'))\) is dense in \(X(K(k'))\). Therefore \(X(K(k')) = X(K^\text{fp}(k')) = x(k')\). \(\square\)

### 3.3. The relative fpff site of a local field

We recall from \[\text{Suz13} \S2.3-4\] the relative fpff site of \(K\) over \(k\), the fpff structure morphism and the cup product formalism in this setting. We provide more on cohomology of \(O_K\) with or without support. The site used in \[\text{Suz13} \text{loc. cit.}\] was \(\text{Spec} k^\text{intrad} \text{et} \). We will apply pro-étale sheafification to obtain a similar formalism over \(\text{Spec} k^\text{proet} \text{et} \).

Let \(K/k^\text{intrad}\) be the category of pairs \((S, k_S)\), where \(k_S \in k^\text{intrad}\) and \(S\) a finitely presented \(K(k_S)\)-algebra. A morphism \((S, k_S) \to (S', k_{S'})\) consists of a morphism
$k_S \to k_{S'}$ in $k_{\text{indrat}}$ and a ring homomorphism $S \to S'$ such that the diagram

$$
\begin{array}{ccc}
K(k_S) & \longrightarrow & K(k_{S'}) \\
\downarrow & & \downarrow \\
S & \longrightarrow & S'
\end{array}
$$

commutes. The composite of two morphisms is defined in the obvious way. We say that a morphism $(S, k_S) \to (S', k_{S'})$ is flat/étale if $k_S \to k_{S'}$ is étale and $S \to S'$ is flat. We endow the category $K/k_{\text{indrat}}$ with the topology where a covering of an object $(S, k_S)$ is a finite family $\{(S_i, k_i)\}$ of objects flat/étale over $(S, k_S)$ such that $\prod S_i$ is faithfully flat over $S$. The resulting site is the relative fppf site $\text{Spec} K_{\text{fppf}}/k_{\text{indrat}}$ of $K$ over $k$. \cite{Suzuki13} Def. 2.3.2. The cohomology of an object $(S, k_S)$ with coefficients in $A \in \text{Ab}(K_{\text{fppf}}/k_{\text{et}})$ is given by the fppf cohomology of $S$:

$$R\Gamma((S, k_S), A) = R\Gamma(S_{\text{fppf}}, f_*A),$$

where $f:\text{Spec} K_{\text{fppf}}/k_{\text{indrat}} \to \text{Spec} S_{\text{fppf}}$ is given by the functor sending a finitely presented $S$-algebra $S'$ to the object $(S', k_{S'})$ \cite{Suzuki13} Prop. 2.3.4). We also define the category $\mathcal{O}_K/k_{\text{indrat}}$ and the relative fppf site $\text{Spec} \mathcal{O}_K/\text{fppf} k_{\text{indrat}}$ of $\mathcal{O}_K$ over $k$ in a similar way, using $\mathcal{O}_K$ instead of $K$. Its cohomology theory is similarly described.

The functors

$$k_{\text{indrat}} \to \mathcal{O}_K/k_{\text{indrat}}, \quad k' \mapsto (\mathcal{O}_K(k'), k'),$$

$$\mathcal{O}_K/k_{\text{indrat}} \to K/k_{\text{indrat}}, \quad (S, k_S) \mapsto (S \otimes_{\mathcal{O}_K} K, k_S)$$

and their composite

$$k_{\text{indrat}} \to K/k_{\text{indrat}}, \quad k' \mapsto (K(k'), k'),$$

define morphisms

$$\pi_{K/k} : \text{Spec} K_{\text{fppf}}/k_{\text{et}} \to \text{Spec} \mathcal{O}_K/\text{fppf} k_{\text{et}}, \quad \pi_{\mathcal{O}_K/k} : \text{Spec} \mathcal{O}_K/\text{fppf} k_{\text{et}} \to \text{Spec} \mathcal{O}_K/\text{fppf} k_{\text{et}}$$

of sites (see \cite{Suzuki13} Def. 2.4.2] for $\pi_{K/k}$). We call $\pi_{K/k}$ (resp. $\pi_{\mathcal{O}_K/k}$) the fppf structure morphism of $K$ (resp. $\mathcal{O}_K$) over $k$. We denote

$$\Gamma(K, \cdot) = (\pi_{K/k})_*, \quad \Omega^n(K, \cdot) = R^n(\pi_{K/k})_* : \text{Ab}(K_{\text{fppf}}/k_{\text{indrat}}) \to \text{Ab}(k_{\text{indrat}}),$$

and similarly for $\mathcal{O}_K$. Again, see \cite{KS06} Chap. 18] for the formalism of pushforward, pullback, sheaf-Hom and tensor products in unbounded derived categories and arbitrary morphisms of sites.\footnote{We work with unbounded derived categories when the general theory in \cite{KS06} Chap. 18] allows us to do so without extra effort. This eliminates some unnecessary assumptions on cohomological or Tor dimensions traditionally needed for morphisms of functoriality of derived pushforward and derived cup products. However, we do not emphasize the use of unbounded derived categories. Most of the complexes we actually need turn out to be bounded after non-trivial (but essentially classical) calculations, as seen in the next subsection. Therefore, with some care, it is possible avoid unbounded derived categories in order to prove the main theorems of this paper.}

For any $A \in \text{Ab}(K_{\text{fppf}}/k_{\text{indrat}})$ and $k' \in k_{\text{indrat}}$, we have

$$(3.3.1) \quad R\Gamma(k', R\Gamma(K, A)) = R\Gamma(K(k'), A),$$

where
where we view $A$ as an fppf sheaf on $K(k')$ by identifying it with the functor that sends a $K(k')$-algebra $S$ of finite presentation to $A(S, k')$. If $A$ is a group scheme locally of finite type over $K$, then this identification is consistent with the identification $A(S, k') = A(S)$. The sheaf $H^n(K, A)$ for any $A \in \text{Ab}(K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}})$ is the étale sheafification of the presheaf
\[ k' \mapsto H^i(K(k')_{\text{fppf}}, A), \]
A similar description exists over $O_K$.

The sheaf-Hom functor for Spec $K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}$ is simply denoted by $\text{Hom}_K$. A morphism of fppf sheaves over $K$ (in the usual sense) induces a morphism of sheaves over Spec $K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}$. The derived versions are similarly denoted: $\text{Ext}_K^n$ and $R\text{Hom}_K$. There are versions $\text{Hom}_{O_K}$, $\text{Ext}_{O_K}^n$ and $R\text{Hom}_{O_K}$ over Spec $O_K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}$.

We define a left exact functor
\[
\Gamma_x(O_K, \cdot): \text{Ab}(O_K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}) \to \text{Ab}(k_{\text{et}}^{\text{indrat}}),
\]
\[ A \mapsto \text{Ker}(\Gamma(O_K, A) \to \Gamma(K, j^* A)). \]
This is the version of $\Gamma(O_K, \cdot)$ with support on the closed point $x = \text{Spec } k \hookrightarrow \text{Spec } O_K$. Its derived functor is denoted by $R\Gamma_x(O_K, \cdot)$ with cohomology $\text{H}^n_x(O_K, \cdot)$. As in [AGV72 §2.2], [Mil06] III, §0. Cohomology with support on closed subscheme, this fits in the following localization triangle:

**Proposition (3.3.3).** We have a distinguished triangle
\[
R\Gamma_x(O_K, A) \to R\Gamma(O_K, A) \to R\Gamma(K, A)
\]
in $D(k_{\text{et}}^{\text{indrat}})$ for any $A \in D(O_K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}})$. (We frequently omit $j^*$ from the notation.)

**Proof.** The site Spec $K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}$ is the localization [AGV72] III, §5 of the site Spec $O_K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}$ at the object $(K, k)$. Hence the functor $j^*$ admits an exact left adjoint functor $j_!$ by [AGV72] IV, Prop. 11.3.1. The sheaf $Z_K := j_! Z$ is represented by the usual étale $O_K$-scheme of extension by zero of $Z$, i.e. Spec $O_K \cup \bigcup_{n \in \mathbb{Z}\setminus\{0\}}$ Spec $K$. Recall from [Mil06] III, §0, Lem. 0.2 that there is an exact sequence
\[
0 \to Z_K \to Z \to Z_x \to 0
\]
of étale group schemes over $O_K$, where $Z_x$ is supported on the closed point $x$ with special fiber $Z$, i.e. the non-separated scheme obtained by gluing infinitely many copies of Spec $O_K$ by the common open subscheme Spec $K$. For any $A \in \text{Ab}(O_K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}})$, the induced exact sequence
\[
0 \to \Gamma(O_K, \text{Hom}_{O_K}(Z_z, A)) \to \Gamma(O_K, \text{Hom}_{O_K}(Z, A)) \to \Gamma(O_K, \text{Hom}_{O_K}(Z_K, A))
\]
can be identified with
\[
0 \to \Gamma_x(O_K, A) \to \Gamma(O_K, A) \to \Gamma(K, j^* A).
\]
This extends to term-wise exact sequences of complexes when $A$ is a complex in $\text{Ab}(O_K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}})$.

We want to pass to the derived category. Let $A \xrightarrow{j} I$ be a quasi-isomorphism to a $K$-injective complex (called a homotopically injective complex in [KS06] Def. 14.1.4 (i)); for the existence of such $I$, see [KS06] Thm. 14.3.1 (ii)]. The existence of the exact left adjoint $j_!$ of $j^*$ implies that $j^*$ preserves $K$-injectives by the definition
of K-injectivity. Since K-injectives calculate any right derived functors by [KS06 Thm. 14.3.1 (vi)], the terms in the sequence of complexes
\[ 0 \to \Gamma_x(O_K, I) \to \Gamma(O_K, I) \to \Gamma(K, j^* I) \]
represents
\[ R\Gamma_x(O_K, A), \quad R\Gamma(O_K, A), \quad R\Gamma(K, j^* A), \]
respectively.

On the other hand, the object \( R\text{Hom}_{O_K}(B, A) \) for any complex \( B \) can be represented by the total complex \( \text{Hom}_{O_K}(B, I) \) of the sheaf-Hom double complex by [KS06 Prop. 18.4.5]. This complex is K-limp in the sense of [Spa88 Def. 5.11 (a)] by [Spa88 Prop. 5.14 and §5.12]. Hence its derived pushforward \( R\Gamma(O_K, \cdot) \) can be calculated by applying \( \Gamma(O_K, \cdot) \) term-wise by [Spa88 Prop. 6.7 (a)] (generalized to sites). Therefore the sequence of complexes
\[ 0 \to \Gamma(O_K, \text{Hom}_{O_K}(\mathbb{Z}_x, I)) \to \Gamma(O_K, \text{Hom}_{O_K}(\mathbb{Z}, I)) \to \Gamma(O_K, \text{Hom}_{O_K}(\mathbb{Z}_K, I)) \]
represents the distinguished triangle
\[ R\Gamma(O_K, R\text{Hom}_{O_K}(\mathbb{Z}_x, A)) \to R\Gamma(O_K, R\text{Hom}_{O_K}(\mathbb{Z}, A)) \]
\[ \to R\Gamma(O_K, R\text{Hom}_{O_K}(\mathbb{Z}_K, A)) \]
coming from \( 0 \to \mathbb{Z}_K \to \mathbb{Z} \to \mathbb{Z}_x \to 0 \).

The above identification and distinguished triangle yield the required distinguished triangle. \( \square \)

Relative fppf sites have étale counterparts ([Suz13 Introduction]). The category \( K_{\text{et}}/k^{\text{indrat}} \) is the full subcategory of \( K/k^{\text{indrat}} \) consisting of objects \((L, k_L)\) with \( L \) étale over \( K(k_L) \). An étale morphism \((L, k_L) \to (L', k_{L'})\) is a morphism with \( k_L \to k_{L'} \) (hence also \( L \to L' \) this case) is étale. This defines étale coverings in the category \( K_{\text{et}}/k^{\text{indrat}} \) and hence the relative étale site \( \text{Spec } K_{\text{et}}/k^{\text{indrat}} \) of \( K \) over \( k \).

The relative étale site \( \text{Spec } O_{K, \text{et}}/k^{\text{indrat}}_{\text{et}} \) of \( O_K \) over \( k \) is defined similarly. We have the étale versions
\[ \text{Spec } K_{\text{et}}/k^{\text{indrat}}_{\text{et}} \hookrightarrow \text{Spec } O_{K, \text{et}}/k^{\text{indrat}}_{\text{et}} \to \text{Spec } k^{\text{indrat}}_{\text{et}}. \]

of the structure morphisms. The notation for their pushforward functors are \( R\Gamma(K_{\text{et}}, \cdot) \) and \( R\Gamma(O_{K, \text{et}}, \cdot) \). We mostly consider fppf cohomology for \( K \) and \( O_K \), so when there is no subscript, it means fppf cohomology.

Recall from [Suz13 Prop. 2.4.3] that we have a natural morphism
\[ R\Gamma(K, R\text{Hom}_K(A, B)) \to R\text{Hom}_{k^{\text{indrat}}_{\text{et}}}(R\Gamma(K, A), R\Gamma(K, B)) \]
of functoriality of \( R\Gamma \) in \( D(k^{\text{indrat}}_{\text{et}}) \) for any \( A, B \in D(K_{\text{fppf}}/k^{\text{indrat}}_{\text{et}}) \). Here \( R\text{Hom}_K \) is the derived sheaf-Hom for the site \( K_{\text{fppf}}/k^{\text{indrat}}_{\text{et}} \). As we saw in loc.cit. the morphism of functoriality is equivalent to the cup-product pairing
\[ R\Gamma(K, A) \otimes^L_k R\Gamma(K, C) \to R\Gamma(K, A \otimes^L_K C) \]

[11] The setting in [Spa88] is over topological spaces, but can be generalized to sites in the style of [KS06 Chap. 18]. As a quick definition (cf. [Spa88 Cor. 5.17]), we say that a complex of sheaves of abelian groups \( C \) on a site \( S \) is K-limp if the natural morphism \( \Gamma(X, C) \to R\Gamma(X, C) \) in \( D(S) \) is an isomorphism for any object \( X \) of \( S \), where \( \Gamma(X, C) \) is the complex \( C \) with \( \Gamma(X, \cdot) \) applied term-wise. A bounded below complex of acyclic sheaves (see the paragraph before [4.12]) is K-limp.
(where \( \otimes^L_k \) and \( \otimes^R_k \) denote the derived tensor products over \( \text{Spec} \, k_{\text{et}}^{\text{indrat}} \) and \( \text{Spec} \, K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}} \), respectively) by the derived tensor-hom adjunction [KS06, Thm. 18.6.4 (vii)] via the change of variables \( R\text{Hom}_K(A, B) \to C \) and \( A \otimes^L_k C \to B \).

We need a version for \( R\Gamma_x \). The proof of its existence is slightly different from [Suz13, Prop. 2.4.3], so we prove it here.

**Proposition (3.3.4).** The functoriality of \( R\Gamma_x \) induces natural morphisms

\[
(3.3.5) \quad R\Gamma_x(O_K, R\text{Hom}_{O_K}(A, B)) \to R\text{Hom}_{k_{\text{et}}^{\text{indrat}}}(R\Gamma_x(O_K, A), R\Gamma_x(O_K, B)),
\]

\[
(3.3.6) \quad R\Gamma_x(O_K, R\text{Hom}_{O_K}(A, B)) \to R\text{Hom}_{k_{\text{et}}^{\text{indrat}}}(R\Gamma_x(O_K, A), R\Gamma_x(O_K, B)),
\]

for any \( A, B \in D(O_K, k_{\text{et}}^{\text{indrat}}) \), where \( R\text{Hom}_{O_K} \) denotes derived sheaf-Hom for the site \( \text{Spec} \, O_K, \text{fppf} \).

**Proof.** For (3.3.5), let \( A \xrightarrow{i} I \) and \( B \xrightarrow{j} J \) be quasi-isomorphisms to \( K \)-injective complexes. As we saw in the proof of (3.3.3), the object \( R\Gamma(O_K, R\text{Hom}_{O_K}(A, B)) \) is represented by \( \Gamma(O_K, R\text{Hom}_{O_K}(I, J)) \). The functoriality of \( \Gamma_x \) induces a natural morphism

\[
\Gamma(O_K, R\text{Hom}_{O_K}(I, J)) \to \text{Hom}_{k_{\text{et}}^{\text{indrat}}}(\Gamma_x(O_K, I), \Gamma_x(O_K, J))
\]

of double complexes in \( \text{Ab}(k_{\text{et}}^{\text{indrat}}) \). In \( D(k_{\text{et}}^{\text{indrat}}) \), there is a natural morphism from the right-hand side to \( R\text{Hom}_{k_{\text{et}}^{\text{indrat}}}(\Gamma_x(O_K, I), \Gamma_x(O_K, J)) \to R\text{Hom}_{k_{\text{et}}^{\text{indrat}}}(R\Gamma_x(O_K, A), R\Gamma_x(O_K, B)) \).

By composing, we obtain the required morphism.

For (3.3.6), we have

\[
R\Gamma(O_K, A) \to R\Gamma(O_K, R\text{Hom}_{k_{\text{et}}^{\text{indrat}}}(A, B))
\]

\[
\quad \to R\text{Hom}_{k_{\text{et}}^{\text{indrat}}}(R\Gamma_x(O_K, A), R\Gamma_x(O_K, B))
\]

where the first morphism is the natural evaluation morphism and the second morphism is (3.3.5). In general, morphisms of the form \( C \to R\text{Hom}_{k_{\text{et}}^{\text{indrat}}}(D, E) \) and \( D \to R\text{Hom}_{k_{\text{et}}^{\text{indrat}}}(C, E) \) are both equivalent to \( C \otimes^L D \to E \) by the derived tensor-hom adjunction [KS06, Thm. 18.6.4 (vii)]. Hence the above yields the desired morphism.

These morphisms of functoriality are compatible in the following sense.

**Proposition (3.3.7).** Let \( A, B \in D(O_K, k_{\text{et}}^{\text{indrat}}) \). To simplify the notation, we denote

\[
[\cdot, \cdot]_{O_K} = R\text{Hom}_{O_K}, \quad [\cdot, \cdot]_K = R\text{Hom}_K, \quad [\cdot, \cdot]_k = R\text{Hom}_{k_{\text{et}}^{\text{indrat}}},
\]

\[
R\Gamma_x = R\Gamma_x(O_K, \cdot), \quad R\Gamma_{O_K} = R\Gamma(O_K, \cdot), \quad R\Gamma_K = R\Gamma(K, \cdot).
\]

Then we have a morphism of distinguished triangles

\[
\begin{array}{cccc}
R\Gamma_x[A, B]_{O_K} & \longrightarrow & R\Gamma_{O_K}[A, B]_{O_K} & \longrightarrow & R\Gamma_K[A, B]_K \\
\downarrow & & \downarrow & & \downarrow \\
[R\Gamma_{O_K}A, R\Gamma_xB]_k & \longrightarrow & [R\Gamma_xA, R\Gamma_xB]_k & \longrightarrow & [R\Gamma_KA, R\Gamma_xB]_k[1]
\end{array}
\]

in \( D(k_{\text{et}}^{\text{indrat}}) \), where the horizontal triangles are the localization triangles in (3.3.3), the left two vertical morphisms are the morphisms of functoriality of \( R\Gamma_x \), and the
right vertical morphism is the morphisms of functoriality of $R\Gamma_K$ composed with the connecting morphism $R\Gamma_K B \to R\Gamma_X B[1]$ of the localization triangle.

Proof. For complexes $C, D$ in $\text{Ab}(\mathcal{O}_K, \text{fppt}/k_{\text{indrat}})$, we denote the total complex of the sheaf-Hom double complex $\text{Hom}_{\mathcal{O}_K}(C, D)$ by $[C, D]_{\mathcal{O}_K}^c$. Note that there is a natural morphism $[C, D]_{\mathcal{O}_K}^c \to [C, D]_{\mathcal{O}_K}$ in $D(k_{\text{indrat}})$. We use the notation $\cdot, \cdot, \cdot, \cdot)_{\mathcal{O}_K}$, $\cdot, \cdot, \cdot, \cdot)_{\mathcal{O}_K}$ similarly. We denote by $\Gamma\mathcal{O}_K C$ the complex $\Gamma(\mathcal{O}_K, C)$, where $\Gamma(\mathcal{O}_K, \cdot, \cdot)$ is applied term-wise. We use the notation $\Gamma_K$ similarly. We denote the mapping fiber of the morphism $\Gamma\mathcal{O}_K C \to \Gamma_K C$ of complexes in $\text{Ab}(k_{\text{indrat}})$ by $\Gamma^c_{\mathcal{O}_K}.

Let $A \to I$ and $B \to J$ be $K$-injective replacements. Then $[A, B]_{\mathcal{O}_K}$ and $[A, B]_K$ can be represented by $[I, J]_{\mathcal{O}_K}^c$ and $[I, J]_K^c$, respectively. Hence $R\Gamma\mathcal{O}_K[A, B]_{\mathcal{O}_K}$, $R\Gamma_K[A, B]_K$ and therefore $R\Gamma^c_{\mathcal{O}_K}[A, B]_{\mathcal{O}_K}$ can be represented by $\Gamma\mathcal{O}_K[I, J]_{\mathcal{O}_K}^c$, $\Gamma_K[I, J]_K^c$, and $\Gamma^c_{\mathcal{O}_K}[I, J]_{\mathcal{O}_K}^c$, respectively. Hence, if we show that the diagram

$$
\begin{array}{ccc}
\Gamma^c_{\mathcal{O}_K}[I, J]_{\mathcal{O}_K}^c & \longrightarrow & \Gamma\mathcal{O}_K[I, J]_{\mathcal{O}_K}^c \\
\downarrow & & \downarrow \\
[\Gamma\mathcal{O}_K[I, \Gamma^c_{\mathcal{O}_K}J]_{\mathcal{O}_K}^c] & \longrightarrow & [\Gamma^c_{\mathcal{O}_K}[I, \Gamma^c_{\mathcal{O}_K}J]_{\mathcal{O}_K}^c]_{\mathcal{O}_K}[1] \\
\end{array}
$$

of complexes in $\text{Ab}(k_{\text{indrat}})$ is commutative up to homotopy, then we get the result by passing to the derived category and using the morphism $\cdot, \cdot, \cdot, \cdot)_{\mathcal{O}_K} \to \cdot, \cdot, \cdot, \cdot)_{\mathcal{O}_K}$. On each square, it is routine to check that the square is commutative, or to construct a homotopy up to which the square is commutative. \qed

Now we pass to the pro-étale topology. Let $P : \text{Spec} k_{\text{proet}} \to \text{Spec} k_{\text{indrat}}$ be the morphism defined by the identity. The pullback $P^*$ is the pro-étale sheafification functor. We denote the composite functor $P^* R\Gamma(K, \cdot, \cdot)$ by $\tilde{R}\Gamma(K, \cdot, \cdot)$\footnote{We cannot define this functor as a certain pushforward functor without sheafification. The functor $k' \to (\mathcal{K}(k'), k')$ does not define a continuous map $\text{Spec} k_{fppf}/k_{\text{indrat}} \to \text{Spec} k_{\text{indrat}}/k_{\text{proet}}$, and it is not clear whether there is a nice definition of the fppf site of $K$ relative to $\text{Spec} k_{\text{indrat}}/k_{\text{proet}}$. For a faithfully flat ind-étale morphism $k' \to k''$ in $k_{\text{indrat}}$, the corresponding morphism $\mathcal{O}_K(k') \to \mathcal{O}_K(k'')$ is not finitely presented, not even ind-étale, unless $k' \to k''$ is étale.}. If $A$ is an object of $\text{Ab}(K_{fppf}/k_{\text{indrat}})$ such that $R\Gamma(K, A)$ is $P$-acyclic, then we have

$$
R\Gamma(k'_{\text{proet}}, \tilde{R}\Gamma(K, A)) = R\Gamma(K(k'), A)
$$

for any $k' \in k_{\text{proet}}$, the sheaf $\tilde{\mathcal{H}}^n(K, A)$ is the étale (not pro-étale) sheafification of the presheaf

$$
k' \mapsto H^n(K(k'), A),
$$

and we have

$$
\Gamma(k', \tilde{\mathcal{H}}^n(K, A)) = H^n(K(k'), A)
$$

for any algebraically closed $k' \in k_{\text{indrat}}$. With this in mind, if $R\Gamma(K, A)$ for $A \in D^+(K_{fppf}/k_{\text{indrat}})$ is $P$-acyclic, then we simply write $R\tilde{\Gamma}(K, A) = R\Gamma(K, A)$ following the convention made in \[2,3\]. Similar notation and convention will be applied to $R\tilde{\Gamma}(\mathcal{O}_K, \cdot, \cdot)$ and $R\tilde{\Gamma}_x(\mathcal{O}_K, \cdot, \cdot)$.

**Proposition (3.3.8).** There is a localization distinguished triangle

$$
R\tilde{\Gamma}_x(\mathcal{O}_K, A) \to R\tilde{\Gamma}(\mathcal{O}_K, A) \to R\tilde{\Gamma}(K, A)
$$


in \( D(k_{\text{indrat}}) \) for \( A \in D(O_{K, \text{proet}}/k_{\text{indrat}}) \). The functoriality of \( R\hat{\Gamma} \) and \( R\hat{\Gamma}_x \) induce morphisms
\[
R\hat{\Gamma}(K, R\text{Hom}_K(A, B)) \rightarrow R\text{Hom}_{k_{\text{proet}}}^{k_{\text{indrat}}}(R\hat{\Gamma}(K, A), R\hat{\Gamma}(K, B))
\]
in \( D(k_{\text{proet}}) \) for \( A, B \in D(O_{K, \text{proet}}/k_{\text{et}}) \) and
\[
R\hat{\Gamma}(O_K, R\text{Hom}_{O_K}(A, B)) \rightarrow R\text{Hom}_{k_{\text{proet}}}^{k_{\text{indrat}}}(R\hat{\Gamma}_x(O_K, A), R\hat{\Gamma}_x(O_K, B)),
\]
\[
R\hat{\Gamma}_x(O_K, R\text{Hom}_{O_K}(A, B)) \rightarrow R\text{Hom}_{k_{\text{proet}}}^{k_{\text{indrat}}}(R\hat{\Gamma}(O_K, A), R\hat{\Gamma}_x(O_K, B)),
\]
in \( D(k_{\text{proet}}) \) for \( A, B \in D(O_{K, \text{proet}}/k_{\text{indrat}}) \). With a similar set of notation to \([3.3.7]\), we have a morphism of distinguished triangles
\[
R\hat{\Gamma}_x[A, B]_{O_K} \rightarrow R\hat{\Gamma}_{O_K}[A, B]_{O_K} \rightarrow R\hat{\Gamma}_K[A, B]_{K}
\]
in \( D(k_{\text{proet}}) \) for \( A, B \in D(O_{K, \text{proet}}/k_{\text{et}}) \) (with \([ \cdot, \cdot ]_k \) this times being \( R\text{Hom}_{k_{\text{proet}}}^{k_{\text{indrat}}} \)),
where the horizontal triangles are localization triangles and the vertical morphisms are the functoriality morphisms together with the connecting morphism \( R\hat{\Gamma}_K B \rightarrow R\hat{\Gamma}_x B[1] \) on the right lower term.

Proof. We only prove the existence of the functoriality morphism of \( R\hat{\Gamma} \). The others are treated similarly. By adjunction, we have
\[
R\hat{\Gamma}(K, R\text{Hom}_K(A, B)) \rightarrow R\text{Hom}_{k_{\text{proet}}}^{k_{\text{indrat}}}(R\hat{\Gamma}(K, A), R\hat{\Gamma}(K, B))
\]
\[
\rightarrow R\text{Hom}_{k_{\text{proet}}}^{k_{\text{indrat}}}(R\hat{\Gamma}(K, A), RP_*P^* R\hat{\Gamma}(K, B))
\]
\[
= RP_* R\text{Hom}_{k_{\text{proet}}}^{k_{\text{indrat}}}(P^* R\hat{\Gamma}(K, A), P^* R\hat{\Gamma}(K, B)),
\]
in \( D(k_{\text{proet}}) \). Adjoining again gives the result. \( \square \)

3.4. Cohomology as sheaves on the residue field. We compute \( R\Gamma \) of several group schemes over \( K \) and \( O_K \). In most cases below, we obtain \( P \)-acyclic ind-pro-algebraic groups. In some cases, the groups are in \( \text{PAlg} \) or \( \text{IAlg} \) (having unipotent connected part), so that they are Serre reflexive. In this subsection, all sheaves over \( k \), their exact sequences and distinguished triangles are considered in \( \text{Ab}(k_{\text{et}}) \), \( D(k_{\text{proet}}) \) unless the pro-étale topology is explicitly mentioned. As soon as objects are proved to be \( P \)-acyclic, we can regard them as objects on \( \text{Spec} k_{\text{proet}} \) without losing any information and apply the results of the previous section.

Proposition (3.4.1). The pushforward functor \( \text{Ab}(O_{K, \text{et}}/k_{\text{et}}) \rightarrow \text{Ab}(k_{\text{et}}) \) for the morphism \( \text{Spec} O_{K, \text{et}}/k_{\text{et}} \rightarrow \text{Spec} k_{\text{et}} \) is exact. Hence \( H^n(\text{Spec} O_{K, \text{et}}, \cdot ) = 0 \) as functors for all \( n \geq 1 \).

Proof. We need to show that an étale covering \((S, k_S)\) of an object of the form \((O_K(k'), k')\) with \( k' \in k_{\text{proet}} \) can be refined by a covering of the form \((O_K(k''), k'')\) with \( k'' \) faithfully flat étale over \( k' \). By the \( O_K(k') \)-version of \([\text{Sz13}] \) Lem. 2.5.3], the étale covering \( S \) of \( O_K(k') \) can be refined by a covering of the form \( O'' \otimes O_K(k') \)
\( O_K(k') \), where \( O'' \) is a faithfully flat étale \( O_K(k') \)-algebra. We can write \( O'' = O''_0 \otimes O_K(k'_0) \) with \( k'_0 \) a rational \( k \)-subalgebra of \( k' \) and \( O''_0 \) a faithfully flat étale
\(O_K(k'_0)\)-algebra. Since \(O_K(k'_0)\) is a finite product of complete discrete valuation rings with residue ring \(k'_0\), we may assume (by refining) that \(O'_0\) is finite over \(O_K(k'_0)\) and hence written as \(O_K(k'_0)\) with \(k'_0\) faithfully flat \(\acute{e}tale\) over \(k'_0\). Then we have \(O'' = O'_K(k'')\) with \(k'' = k'_0 \otimes_{k'_0} k'\) a faithfully flat étale \(k'\)-algebra. Hence we have a refinement \((S, k_S) \rightarrow (O'_K(k'), k_S)\) of coverings of \((O_K(k'), k')\). The \(k'\)-algebra homomorphism \(k_S \rightarrow k''\) is étale since both \(k_S\) and \(k''\) are étale over \(k'\). Therefore the morphism \((O_K(k''), k_S) \rightarrow (O_K(k''), k'')\) is an étale covering, even though \(k_S \rightarrow k''\) is not necessarily faithfully flat (\cite{Suz13} Prop. 2.3.3). The composite \((S, k_S) \rightarrow (O_K(k''), k_S) \rightarrow (O_K(k''), k'')\) gives a desired refinement. □

Proposition (3.4.2).

(a) Let \(A\) be a smooth group scheme over \(O_K\) and \(A_x\) its special fiber. Let \(\Gamma(p_K, A)\) be the kernel of the reduction morphism \(\Gamma(O_K, A) \rightarrow A_x\). Then the sequence

\[0 \rightarrow \Gamma(p_K, A) \rightarrow \Gamma(O_K, A) \rightarrow A_x \rightarrow 0\]

in \(\text{Ab}(k^\text{indrat})\) is exact. All the terms are \(P\)-acyclic. The group \(\Gamma(p_K, A)\) is connected pro-unipotent. In particular, we have \(\pi_0(\Gamma(O_K, A)) = \pi_0(A_x)\), and if \(A_x\) is of finite type, then \(\Gamma(O_K, A) \in \text{PAlg}/k\). We have \(H^n(O_K, A) = 0\) for all \(n \geq 1\). In particular, \(R\Gamma(O_K, A)\) is \(P\)-acyclic.

(b) Let \(N\) be a finite flat group scheme over \(O_K\). Then \(\Gamma(O_K, N) = \Gamma(K, N) \in \text{Fet}/k\), \(H^1(O_K, N)\) is connected pro-unipotent, and \(H^n(O_K, N) = 0\) for all \(n \geq 2\). The complex \(R\Gamma(O_K, N)\) is \(P\)-acyclic and Serre reflexive.

Proof. (a). For each \(m \geq 1\), the functor \(\Gamma(O_K/p^m_K, A)\) is represented by the perfection of the Greenberg transform of \(A\) of level \(m\) (\cite{Gre61}).\(^{13}\) Implies that \(\Gamma(O_K, A) = \varprojlim_m \Gamma(O_K/p^m_K, A)\). The reduction map \(A(O_K(k')) \rightarrow A_x(k')\) is surjective for any \(k' \in k^\text{indrat}\) by smoothness. Hence the reduction morphism \(\Gamma(O_K, A) \rightarrow A_x\) is surjective. The kernel of the surjection \(\Gamma(O_K/p^m_K, A) \rightarrow \Gamma(O_K/p^{m+1}_K, A)\) is the perfection of a vector group by \cite{Beg81} Lem. 4.1.1, 2\(^{13}\) and the proof of \cite{Bes78} §1.1, Lem. 1.1 (ii). Therefore

\[\Gamma(p_K, A) = \varprojlim_m \ker(\Gamma(O_K/p^m_K, A) \rightarrow A_x)\]

is connected pro-unipotent and \(P\)-acyclic by (3.4.2). The quasi-algebraic group \(A_x\) is \(P\)-acyclic by the same proposition. Being an extension of \(P\)-acyclics, the group \(\Gamma(O_K, A)\) is \(P\)-acyclic. We have \(H^n(O_K, A) = H^n(O_K, A)\) for all \(n \geq 1\) since \(A\) is smooth and the fpf cohomology with smooth group scheme coefficients agrees with the étale cohomology by \cite{Mil80} III, Rmk. 3.11 (b)]. We have \(H^n(O_K, A) = 0\) for \(n \geq 1\) by the previous proposition.

(b) The functors \(\Gamma(O_K, N) : k' \mapsto N(O_K(k'))\) and \(\Gamma(K, N) : k' \mapsto N(K(k'))\) for finite flat \(N\) are represented by the same finite étale \(k\)-scheme by (3.4.10).

For cohomology of degree \(\geq 1\), recall from \cite{Beg81} Prop. 2.2.1 that there is an exact sequence \(0 \rightarrow N \rightarrow G \rightarrow H \rightarrow 0\) of group schemes over \(O_K\) with \(G, H\) smooth affine with connected fibers (more specifically, \(G\) is the Weil restriction of \(G_m\) from

\(^{13}\) Note that this lemma by Bégner is true only after perfection. See \cite{BGA13} Rmk. 14.22, 15.9). In our case, it is enough to check this lemma for perfect-field-valued points only by the following reason. The sheaf \(\Gamma(O_K/p_K, A)\) on \(\text{Spec} k^\text{indrat}\) is locally of finite presentation and the kernel of the surjection mentioned here is also locally of finite presentation. Hence it is enough to treat them as sheaves on \(\text{Spec} k^\text{indrat}\). For perfect-field-valued points, the lemma is classical.
the Cartier dual $N^{CD}$ to Spec $O_K$ and $N \mapsto G$ is the natural embedding). The long exact sequence shows that $H^n(O_K, N) = 0$ for $n \geq 2$ and yields an exact sequence

$$0 \to \Gamma(O_K, N) \to \Gamma(O_K, G) \to \Gamma(O_K, H) \to H^1(O_K, N) \to 0$$

(in $\text{Ab}(k^{\text{indrat}})$ a priori). The first three terms are P-acyclic. Hence so is the fourth. Therefore the sequence is exact also in $\text{Ab}(k^{\text{proet}})$. Since the first term is finite and the second and third terms are connected proalgebraic, the fourth term is connected proalgebraic. Since $H^1(O_K, N)$ is killed by the order of $N$, it does not have semi-abelian part. Hence $H^1(O_K, N)$ is pro-unipotent. Therefore $R\Gamma(O_K, N)$ is Serre reflexive by (3.4.1) □

**Proposition (3.4.3).**

(a) Let $A$ be a smooth group scheme over $K$. Then $H^n(A, K)$ for any $n \geq 1$ is torsion and locally of finite presentation as a functor on $k^{\text{indrat}}$. In particular, it is P-acyclic.

(b) Let $N$ be a finite flat group scheme over $K$. Then $\Gamma(K, N) \in \text{Fet}/k$, $H^1(K, N) \in \text{IPA}_{\text{uc}}/k$, $H^n(K, N) = 0$ for all $n \geq 2$, and $R\Gamma(K, N)$ is P-acyclic and Serre reflexive. The group $H^1(K, N)$ is in $\text{IAlg}_{\text{uc}}/k$ if $N$ is étale and in $\text{PA}_{\text{uc}}/k$ if $N$ is multiplicative. If $K$ has mixed characteristic, then $H^1(K, N) \in \text{Alg}_{\text{uc}}/k$.

(c) Let $Y$ be a lattice over $K$ (i.e., a finite free abelian group with a Galois action of $K$). Then $\Gamma(K, Y)$ is a lattice over $k$, $H^1(K, Y) \in \text{Fet}/k$, $H^2(K, Y) \in \text{IAlg}_{\text{uc}}/k$, $H^n(K, Y) = 0$ for $n \geq 3$, and $R\Gamma(K, Y)$ is P-acyclic and Serre reflexive.

(d) Let $A$ be an abelian variety over $K$ with Néron model $A$. Then $\Gamma(A, K) = \Gamma(O_K, A) \in \text{PAlg}/k$, which is described by the previous proposition, $H^1(A, K) \in \text{IAlg}_{\text{uc}}/k$, $H^n(A, K) = 0$ for all $n \geq 2$, and $R\Gamma(A, K)$ is P-acyclic.

(e) Let $T$ be a torus over $K$ with Néron model $T$. Then $\Gamma(T, K) = \Gamma(O_K, T)$ is an extension of an étale group by a P-acyclic proalgebraic group described by the previous proposition. We have $H^n(T, K) = 0$ for all $n \geq 1$, and $R\Gamma(T, K)$ is P-acyclic.

Proof. (a) [3.2.2] shows that the functor $k' \mapsto H^n(A(k')_{\text{et}}, A)$ is locally of finite presentation for $n \geq 1$ and takes values in torsion groups. Hence so is its étale sheafification $H^n(A, K)$. Therefore $H^n(A, K)$ is torsion and P-acyclic by (2.4.2).

(b) By [3.1.10], the sheaf $\Gamma(K, N)$ is a finite étale group over $k$ whose $\overline{k}$-points is given by $\Gamma(K^{\text{ur}}, N)$. (Note that $N$ does not have to extend to $O_K$ and hence we are not using [3.4.2] here.)

Suppose that $N$ has order prime to $p$. Then it is classical to see that $H^n(K^{\text{ur}}, N)$ is zero for $n \geq 2$ and $H^1(K^{\text{ur}}, N)$ is finite. These groups do not change under residue field extensions, i.e., for any algebraically closed field $k'$ over $k$, we have $H^n(K(k'), N) = H^n(K^{\text{ur}}, N)$ for any $n \geq 1$. Since $N$ is étale, the sheaf $H^n(K, N)$ is locally of finite presentation for any $n \geq 1$ by the previous assertion. Hence it is determined by $k'$-points for various algebraically closed fields $k'$. Therefore $H^n(K, N) = 0$ for $n \geq 2$ and $H^1(K, N)$ is the finite étale group given by $H^1(K^{\text{ur}}, N)$.

Suppose that $N$ is $p$-primary. Suppose also that $K$ has equal characteristic. We first show the statements for $N = \mathbb{Z}/p\mathbb{Z}$, $\mu_p$ and $\alpha_p$. If $N = \mathbb{Z}/p\mathbb{Z}$, then
**H**^n(K, ℤ/pℤ) = 0 for n ≥ 2 and

\[ \text{H}^1(K, ℤ/pℤ) = K/ρ(K) \cong G_a^\oplus N \in \text{IAlg}_{uc}/k \]

by the Artin-Schreier sequence 0 → ℤ/pℤ → G_a → G_a → 0 and that H^n(K, G_a) = 0 for all n ≥ 1. In particular, H^1(K, ℤ/pℤ) is P-acyclic and Serre reflexive by (2.4.1) \( \text{[b]} \) and (2.4.2) \( \text{[c]} \). For N = μ_p, the paragraph before [Suz13, Lem. 2.7.4] and the proof of the cited lemma show that H^n(K, μ_p) = 0 for n ≥ 2 and

\[ \text{H}^1(K, μ_p) = K^\times/(K^\times)^p \cong ℤ/pℤ \times G_a^N \in \text{PAlg}_{uc}/k. \]

In particular, H^1(K, μ_p) is P-acyclic by (2.4.2) and Serre reflexive by (2.4.1) \( \text{[b]} \). For N = α_p, a calculation similar to the case of ℤ/pℤ shows that H^n(K, α_p) = 0 for n ≥ 2 and

\[ \text{H}^1(K, α_p) = K/K^p \cong G_a^\oplus N \times G_a^N \in \text{IPAlg}_{uc}/k. \]

Each factor is P-acyclic and Serre reflexive by the same propositions. Hence so is the product.

Next, if 0 → N_1 → N_2 → N_3 → 0 is an exact sequence of finite flat group schemes over K, and if we know that N_1 and N_3 satisfy the statements, then the long exact sequence

\[ 0 \rightarrow \Gamma(K, N_1) \rightarrow \Gamma(K, N_2) \rightarrow \Gamma(K, N_3) \]

\[ \rightarrow \text{H}^1(K, N_1) \rightarrow \text{H}^1(K, N_2) \rightarrow \text{H}^1(K, N_3) \rightarrow 0 \]

and the finiteness of \( \Gamma(K, N_i) \) (i = 1, 2, 3) implies that N_3 satisfies the statements. Therefore if N has a filtration whose successive subquotients are ℤ/pℤ, μ_p or α_p, then N satisfies the statements.

Now, let N be any p-primary finite flat group scheme over K. Then N has a filtration whose successive subquotients are étale, multiplicative or α_p by [DG70 IV, §3, 5.8, 5.9]. Let L be a finite Galois extension of K that trivializes the Galois actions on the étale part and the Cartier dual of the multiplicative part. It is enough to show the statements after a finite unramified extension, so we may assume that L/K is totally ramified. Let M be the intermediate field of L/K that corresponds to a (or the) p-Sylow subgroup of Gal(L/K). If a p-group acts on a non-zero \( \mathbb{F}_p \)-vector space, then it has a non-zero fixed part by [Ser70 IX, Lem. 4]. Hence the base-changed group N ×_K M over M has a filtration whose successive subquotients are ℤ/pℤ, μ_p or α_p. Therefore N ×_K M satisfies the statements, i.e. \( \text{H}^1(M, N) \in \text{IPAlg}_k/\text{IAlg}_k/k \) if N is étale, PAlg_k/k if multiplicative, \( \text{H}^n(M, N) = 0 \) for n ≥ 2 and \( \mathbf{R} \Gamma(M, N) \) is P-acyclic and Serre reflexive. We have \( \mathbf{R} \Gamma(M, N) = \mathbf{R} \Gamma(K, \text{Res}_{M/K} N) \), where Res_{M/K} denotes the Weil restriction functor. The composite of the inclusion map N → Res_{M/K} N and the norm map Res_{M/K} N → N is the multiplication by \([M : K]\), which is an isomorphism since N is p-primary and \([M : K]\) is prime to p. Hence N is a direct summand of Res_{M/K} N. Therefore N (over K) satisfies the statements.

Suppose next that K has mixed characteristic. If N = μ_p, then \( \text{H}^1(K, μ_p) = K^\times/(K^\times)^p \) similarly. We have \( K^\times = ℤ \times U_K \) by [Suz13, the paragraph before Prop. 2.4.4]. The group \( U_K = O_K^\times = \Gamma(O_K, G_m) \) is P-acyclic by (3.1.2) \( \text{[m]} \). Since \( \Gamma(K, μ_p) = 0 \) or ℤ/pℤ, The exact sequence

\[ 0 \rightarrow \Gamma(K, μ_p) \rightarrow K^\times \rightarrow K^\times \rightarrow K^\times/(K^\times)^p \rightarrow 0 \]
shows that $\mathbf{K}^\times/(\mathbf{K}^\times)^p$ and $(\mathbf{K}^\times)^p$ are P-acyclic. The logarithm map shows that $(\mathbf{K}^\times)^p$ contains the group $U^p_{\mathbf{K}} = 1 + p\mathbf{R}_\mathbf{K}$ of $m$-th principal units for some $m$. Since $U_{\mathbf{K}}/U^p_{\mathbf{K}}$ is an $m$-dimensional quasi-algebraic group, its quotient $U_{\mathbf{K}}/(U^p_{\mathbf{K}})^p = U_{\mathbf{K}}^1/(U^p_{\mathbf{K}})^p$ is quasi-algebraic unipotent. Hence $H^1(K, \mu_p) \in \operatorname{Alg}_{\text{ac}}/k$. Then the same process as the equal characteristic case shows that $H^1(K, \mathbb{N}) \in \operatorname{Alg}_{\text{ac}}/k$ for any finite flat (hence étale) $N$ and $N$ satisfies the statements.

We have an exact sequence $0 \to Y \to Y \otimes \mathbb{Q} \to Y \otimes \mathbb{Q}/\mathbb{Z} \to 0$. We have $H^n(K^\text{ur}, Y \otimes \mathbb{Q}) = 0$ for $n \geq 1$. From this, since $Y \otimes \mathbb{Q}$ is étale and hence smooth, we know that $H^n(K, Y \otimes \mathbb{Q}) = 0$ for $n \geq 1$ by the same method as the second paragraph of the proof of the previous assertion. Therefore

$$H^n(K, Y) = H^{n-1}(K, Y \otimes \mathbb{Q}/\mathbb{Z}) = \lim_{\rightarrow} H^{n-1}(K, Y \otimes \mathbb{Z}/m\mathbb{Z})$$

for $n \geq 2$. This sheaf is zero for $n \geq 3$ and in $\operatorname{IAlg}_{\text{ac}}/k$ for $n = 2$ by the previous assertion.

By (2.2.10), we know that $\Gamma(K, Y)$ is a lattice over $k$. Let $I$ be the inertia group of a finite Galois extension of $K$ over which $Y$ becomes trivial. The same proposition shows that the exact sequence

$$0 \to \Gamma(K, Y) \to \Gamma(K, Y \otimes \mathbb{Q}) \to \Gamma(K, Y \otimes \mathbb{Q}/\mathbb{Z}) \to H^1(K, Y) \to 0$$

is identified with the exact sequence

$$0 \to H^0(I, Y) \to H^0(I, Y \otimes \mathbb{Q}) \to H^0(I, Y \otimes \mathbb{Q}/\mathbb{Z}) \to H^1(I, Y) \to 0$$

of group cohomology groups. Since $I$ is finite and $Y$ finitely generated, we know that $H^1(I, Y)$ is finite \textit{Ser79} VIII, §2, Cor. 2]. Hence $H^1(K, Y)$ is finite étale.

Therefore $H\Gamma(K, Y)$ has Serre reflexive cohomologies by (2.4.1) \textit{b} and hence itself is Serre reflexive.

1 We have $\Gamma(K, A) = \Gamma(O_K, A)$ by (3.1.3) \textit{a}. This is in $P\operatorname{Alg}/k$ by the previous proposition. We know that $H^n(K, A)$ is torsion for any $n \geq 1$ by (m). For each $m \geq 1$, the exact sequence $0 \to A[m] \to A \to A \to 0$ yields an exact sequence

$$0 \to \Gamma(K, A)/m\Gamma(K, A) \to H^1(K, A[m]) \to H^1(K, A)[m] \to 0.$$

The first term is $P$-acyclic since $\Gamma(K, A)$ and its $m$-torsion part $\Gamma(K, A[m])$ are so. It is in $P\operatorname{Alg}_{\text{ac}}/k$, since its semi-abelian part is divisible and killed by $m$, hence zero. The middle term is in $IP\operatorname{Alg}_{\text{ac}}/k$ and $P$-acyclic by the finite flat case. Therefore the right term is in $IP\operatorname{Alg}_{\text{ac}}/k$ and $P$-acyclic. So is the filtered union $H^1(K, A)$. We know that $H^1(K, A)$ is locally of finite presentation by (m). Hence (3.4.4) below implies that $H^1(K, A) \in \operatorname{IAlg}_{\text{ac}}/k$. Similarly the vanishing $H^n(K, A[m]) = 0$ of the finite flat case implies $H^n(K, A) = 0$ for $n \geq 2$.

2 For $\Gamma$, it is similar to abelian varieties (though in this case $\mathcal{T}$ is only locally of finite type). To show $H^n(K, T) = 0$ for $n \geq 1$, it is enough to see that $H^n(K(k'), T) = 0$ for any algebraically closed field $k'$ over $k$ by (m). This is given in \textit{Ser79} X §7, Application].

\textbf{Lemma (3.4.4).} If $A \in P\operatorname{Alg}/k$ is locally of finite presentation as a functor on $k^{\text{indrat}}$, then we have $A \in \operatorname{IAlg}/k$.

\textbf{Proof.} Let $A = \varprojlim A_m$ with $A_m \in P\operatorname{Alg}/k$. We need to show that any morphism $B \to A$ from an object $B \in P\operatorname{Alg}/k$ factors through an object of $\operatorname{Alg}/k$. Let $B = \varprojlim B_m$ with $B_m \in \operatorname{Alg}/k$. We may assume that the transition morphisms
Let $\xi_B$ be the generic point of $B$ in the sense of [Suz13 Def.
3.2.1], namely $\xi_B = \lim_{n \to \infty} \xi_{B_n}$ and $\xi_{B_n}$ is the disjoint union of the generic points of the irreducible components of $B_n$. As a scheme, $\xi_B$ is the Spec of an ind-rational $k$-algebra $k'_B$. For any $C \in \text{Ab}(k_{\text{proet}})$, we denote $C(\xi_B) = C(k'_B)$. We define a group homomorphism $\sigma : C(\xi_B) \to C(\xi_{B \times B})$ by sending a morphism $f : \xi_B \to C$ to the morphism $\xi_{B \times B} \to C$ given by $(b_1, b_2) \mapsto f(b_1) + f(b_2) - f(b_1 + b_2)$. Then for $C \in \text{PAlg}/k$, we have

\begin{equation}
\text{Hom}(B, C) = \text{Ker}(C(\xi_B) \to C(\xi_{B \times B})),
\end{equation}

which means that a homomorphism of birational groups extends to a everywhere regular group homomorphism ([Ser88, V, §1.5, Lem.
6] plus a limit argument). Now let $C = A$. Then using the assumption, we have

\begin{align*}
\text{Hom}(B, A) &= \lim_{\lambda, \mu} \text{Hom}(B, A_{\lambda, \mu}) \\
&= \lim_{\lambda, \mu} \text{Ker}(A_{\lambda}(\xi_B) \to A_{\lambda}(\xi_{B \times B})) \\
&= \lim_{\lambda, \mu} \text{Ker}(A_{\lambda, \mu}(\xi_B) \to A_{\lambda, \mu}(\xi_{B \times B})) \\
&= \lim_{\lambda, \mu} \text{Hom}(B_{\mu}, A_{\lambda}) \\
&= \lim_{\mu} \text{Hom}(B_{\mu}, A).
\end{align*}

Hence a morphism $B \to A$ factors through some $B_{\mu} \in \text{Alg}/k$.  

**Proposition (3.4.6).** Let $N$ be a finite flat group scheme over $\mathcal{O}_K$. Then $\mathbf{H}^n_{\text{et}}(\mathcal{O}_K, N) = 0$ for $n \neq 2$. We have $\mathbf{H}^2_{\text{et}}(\mathcal{O}_K, N) \in \text{IAlg}_{\text{uc}}/k$, and $R\Gamma_{\text{et}}(\mathcal{O}_K, N)$ is $\text{P}$-acyclic and Serre reflexive. If $K$ has mixed characteristic, then $\mathbf{H}^2_{\text{et}}(\mathcal{O}_K, N) \in \text{Alg}_{\text{uc}}/k$.

**Proof.** We have $\Gamma(\mathcal{O}_K, N) = \Gamma(K, N)$ and $\mathbf{H}^n(\mathcal{O}_K, N) = \mathbf{H}^n(K, N) = 0$ for $n \geq 2$ by the previous two propositions. We show that the morphism $\mathbf{H}^1(\mathcal{O}_K, N) \to \mathbf{H}^1(K, N)$ is injective. Let $k' \in \text{k}_{\text{indet}}$ and $X$ an fppf $N$-torsor over $\text{Spec} \mathcal{O}_K(k')$. Since $N$ is finite, we have $X(\mathcal{O}_K(k')) = X(K(k'))$ by [3.1.3] (c). Hence if $X$ maps to zero under $H^1(\mathcal{O}_K(k'), N) \to H^1(K(k'), N)$, then it is zero. Therefore $H^1(\mathcal{O}_K(k'), N) \to H^1(K(k'), N)$ is injective and $\mathbf{H}^1(\mathcal{O}_K, N) \to \mathbf{H}^1(K, N)$ is injective. Hence the localization triangle [3.3.3]

$$R\Gamma_{\text{et}}(\mathcal{O}_K, N) \to R\Gamma(\mathcal{O}_K, N) \to R\Gamma(K, N)$$

in $D(k_{\text{indet}})$ reduces to an exact sequence

$$0 \to \mathbf{H}^1(\mathcal{O}_K, N) \to \mathbf{H}^1(K, N) \to \mathbf{H}^2_{\text{et}}(\mathcal{O}_K, N) \to 0$$

in $\text{Ab}(k_{\text{indet}})$. Since the first two terms are $\text{P}$-acyclic, Serre reflexive and in $\text{IPAlg}_{\text{uc}}/k$ by [3.4.2] (d) and [3.4.3] (d), so is the third $\mathbf{H}^2_{\text{et}}(\mathcal{O}_K, N)$.

To deduce $\mathbf{H}^2_{\text{et}}(\mathcal{O}_K, N) \in \text{IAlg}_{\text{uc}}/k$, let $0 \to N \to G \to H \to 0$ be an exact sequence of group schemes over $\mathcal{O}_K$ with $G, H$ smooth affine with connected fibers, as we took in the proof of [3.4.2] (d). Since $R\Gamma(\mathcal{O}_K, G) = R\Gamma(\mathcal{O}_{K, \text{et}}, G)$ is concentrated in degree 0 by [3.4.1] and the morphism $\Gamma(\mathcal{O}_K, G) \to \Gamma(K, G)$ is injective, we know that

$$\Gamma_{\text{et}}(\mathcal{O}_K, G) = 0, \quad \mathbf{H}^n(\mathcal{O}_K, G) = \Gamma(K, G)/\Gamma(\mathcal{O}_K, G),$$

$$\mathbf{H}^n_{\text{et}}(\mathcal{O}_K, G) = \mathbf{H}^{n-1}(K, G), \quad n \geq 2.$$
These are locally of finite presentation by \((3.2.8)\) and \((3.2.9)\). Similarly, \(H^m_x(O_K, H)\) is locally of finite presentation for any \(n\). The distinguished triangle \(R\Gamma_x(O_K, N) \to R\Gamma_x(O_K, G) \to R\Gamma_x(O_K, H)\) then shows that \(H^2_x(O_K, N)\) is locally of finite presentation. We saw above that \(H^2_x(O_K, N) \in I\text{Alg}_{\text{uc}}/k\). Hence \((3.4.4)\) implies that \(H^2_x(O_K, N) \in I\text{Alg}_{\text{uc}}/k\).

If \(K\) has mixed characteristic, then \(H^1(K, N) \in \text{Alg}_{\text{uc}}/k\) implies \(H^1(O_K, N) \in \text{Alg}_{\text{uc}}/k\), so \(H^2_x(O_K, N) \in \text{Alg}_{\text{uc}}/k\).

\(\square\)

\textbf{Remark} (3.4.7). Another method to compute \(R\Gamma(O_K, N), R\Gamma(K, N)\) and \(R\Gamma_x(O_K, N)\) for finite flat \(N\) over equal characteristic \(K\) is to use the two exact sequences of \(\text{[Mil06, III, \S 5]}\) (see also \(5.2.2\) of this paper). Then these cohomology complexes can be calculated by the cohomology with coefficients in vector groups in the case \(N\) or the Cartier dual \(N^{\text{CD}}\) has height 1. The general case follows by d\&eacute;vissage. This method is due to Artin-Milne [AM76] in the global situation and Bester [Bes78] in the local situation.

\section{Statement of the Duality Theorem}

From now on throughout the paper, all sheaves over \(k\), their exact sequences and distinguished triangles are considered in \(\text{Ab}(\text{IAlg}_{\text{proet}})\), \(D(\text{k}_{\text{proet}})\) unless otherwise noted. We denote \(R\Gamma(\cdot, \cdot) = R\Gamma(K, \cdot, \cdot)\) when there is no confusion.

\subsection{Formulation}

We formulate the duality theorem with coefficients in abelian varieties. First, \(R\Gamma(G_m)\) is P-acyclic by \((3.4.3)\) \((\S)\), so we write \(R\Gamma(G_m) = R\Gamma(G_m) = R\Gamma(G_m) = K^x\). The same assertion or \([\text{Suz13, Prop. 2.4.4}]\) shows that

\[ R\Gamma(G_m) = \Gamma(G_m) = K^x. \]

Recall from the paragraph before \([\text{Suz13, Prop. 2.4.4}]\) that there are the valuation map \(K^x \to \mathbb{Z}\) as a morphism of sheaves and a split exact sequence

\[ 0 \to U_K \to K^x \to \mathbb{Z} \to 0, \]

where \(U_K = O_K^x\). An alternative definition of this sequence and the valuation map is the exact sequence

\[ 0 \to \Gamma(O_K, G_m) \to \Gamma(O_K, G_m) = \Gamma(K, G_m) \to \Gamma(O_K, \mathbb{Z}_x) \to 0 \]

coming from the exact sequence \(0 \to G_m \to G_m \to \mathbb{Z}_x \to 0\) of group schemes over \(O_K\), where \(G_m\) is the Néron (lift) model of \(G_m\) and \(\mathbb{Z}_x\) the étale group with support on \(x = \text{Spec} k\) and special fiber \(\mathbb{Z}\), and we used \((3.1.3)\) \((\S)\) for the middle isomorphism and \((3.4.1)\) for the exactness. On \(k\)-points, it is the usual sequence \(0 \to U_K \to K^x \to \mathbb{Z} \to 0\), where \(U_K = O_K^x\). We call the composite

\[(4.1.1) \quad R\Gamma(K, G_m) = K^x \to \mathbb{Z}\]

the \textit{trace morphism}.

Let \(A\) be an abelian variety over \(K\) with dual \(A^\vee\). Recall from \((3.4.3)\) \((\S)\) that \(\Gamma(A) \in \text{PAlg}/k, \ H^1(A) \in \text{IAlg}_{\text{uc}}/k, \ H^n(A) = 0 \text{ for } n \geq 2, \) and \(R\Gamma(A)\) is P-acyclic. In particular, we write \(R\Gamma(A) = R\Gamma(A)\). With the morphism of functoriality of
Then the following are equivalent:

For any algebraically closed field

Therefore the closed field

Ext (2.4.1). In particular, morphism id

We have

H 4.2. Reduction to components groups and the first cohomology. First,

the sheaf \( H^1(A) \in \text{IAlg}/k \) is P-acyclic as we saw. Since \( \Gamma(A) \in \text{PAlg}/k \), the sheaf \( \operatorname{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \) is locally of finite presentation by what we saw after (2.4.1). In particular, \( \operatorname{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \) is P-acyclic by (2.4.2) \( \text{iii} \), and \( \operatorname{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \) and \( \operatorname{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \) are equal as functors on \( k^{\text{indrat}} \).

Therefore the \( k' \)-points of \( H^1(A) \) and \( \operatorname{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \) for any algebraically closed field \( k' \in k^{\text{indrat}} \) are \( H^1(\mathbb{K}(k'), A) \) and \( \operatorname{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \), respectively.

Proposition (4.2.1). The morphism \( \theta_A \) induces two morphisms

\[
\theta_A^+: \pi_0(A^\vee) \to \pi_0(A^\vee)^{\text{PD}} \quad \text{in} \quad \text{Fet}/k,
\]

\[
\theta_A^{-1}: H^1(A^\vee) \to \operatorname{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \quad \text{in} \quad \text{IAlg}_{\text{ac}}/k.
\]

For any algebraically closed field \( k' \in k^{\text{indrat}} \), denote by \( \theta_A^+(k') \) the morphism \( \theta_A^+ \) induced on the \( k' \)-points:

\[
\theta_A^+(k'): H^1(\mathbb{K}(k'), A^\vee) \to \operatorname{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}).
\]

Then the following are equivalent:

- \( \theta_A \) is an isomorphism.
- \( \theta_A^+, \theta_A^{-1}, \theta_A^+ \) are isomorphisms.

We start proving this from the next subsection. The proof finishes at §7.
• \(\theta^+_A, \theta^{+1}_A(k'), \theta^{+1}_A(k')\) are isomorphisms for any algebraically closed field \(k' \in k^{\text{indrat}}\).

We prove this in this subsection. Basically the morphisms will be obtained by writing down the spectral sequence associated with \(\theta_A\), as we did in \([4.2]\). More precisely, \(\theta^+_A\) is the morphism induced on the \(\pi_0\) of \(H^0\) of the both sides of \(\theta_A\) and \(\theta^{+1}_A\) is the morphism induced on the \(H^1\). The part for \(H^{-1}\) and the identity component of \(H^0\) contain no additional information by symmetry, and \(H^n = 0\) for the both sides for \(n \neq -1, 0, 1\). To clarify the symmetry mentioned and the treatment of the double-dual, we split the construction and the proof into several steps.

Consider the morphisms

\[
\varprojlim_n \Gamma(A) \rightarrow \Gamma(A) \rightarrow R\Gamma(A),
\]

where the (non-derived) limit on the left is over multiplication by \(n \geq 1\). Note that \(\varprojlim_n \Gamma(A)\) is no longer P-acyclic. There is a canonical choice of a mapping cone of the composite \(\varprojlim_n \Gamma(A) \rightarrow R\Gamma(A)\) in \(D(k^{\text{indrat}}_{\text{proet}})\) since the former is concentrated in degree 0 and the latter is concentrated in non-negative degrees. We denote this mapping cone by \([\varprojlim_n \Gamma(A) \rightarrow R\Gamma(A)]\).

**Proposition (4.2.2).** There is a canonical isomorphism

\[
[\varprojlim_n \Gamma(A) \rightarrow R\Gamma(A)] = R\Gamma(A)^{\text{SDSD}}.
\]

The induced morphism \(R\Gamma(A) \rightarrow R\Gamma(A)^{\text{SDSD}}\) is the natural evaluation morphism.

**Proof.** Since \(\Gamma(A) \in \text{PAlg}/k\), we have \((\varprojlim_n \Gamma(A))^\text{SD} = 0\) as seen in the proof of \([2.4.1]\) \((\text{ii})\). Therefore we have a natural morphism and an isomorphism

\[
[\varprojlim_n \Gamma(A) \rightarrow R\Gamma(A)] \rightarrow [\varprojlim_n \Gamma(A) \rightarrow R\Gamma(A)]^{\text{SDSD}} = R\Gamma(A)^{\text{SDSD}}.
\]

We need to show that the left term is Serre reflexive. Since \(H^n(A) = 0\) for \(n \geq 2\), we have a distinguished triangle

\[
[\varprojlim_n \Gamma(A) \rightarrow \Gamma(A)] \rightarrow [\varprojlim_n \Gamma(A) \rightarrow R\Gamma(A)] \rightarrow H^1(A)[-1].
\]

The left mapping cone is isomorphic to \(\Gamma(A)^{\text{SDSD}} \in D^f(\text{PAlg}_{\text{unc}}/k)\) by \([2.4.1]\) \((\text{iii})\), which is Serre reflexive. We have \(H^1(A) \in \text{IAlg}_{\text{unc}}/k\), which is Serre reflexive. Therefore the middle term is Serre reflexive. Thus we get the required isomorphism. \(\square\)

Let \(\Gamma(A^\vee)_0\) be the identity component of \(\Gamma(A^\vee)\). We have two distinguished triangles

\[
\begin{array}{ccc}
\Gamma(A^\vee)_0 & \longrightarrow & R\Gamma(A^\vee) \\
H^1(A)^{\text{SD}}[2] & \longrightarrow & R\Gamma(A)^{\text{SD}}[1] \\
\end{array}
\rightarrow [\Gamma(A^\vee)_0 \rightarrow R\Gamma(A^\vee)]
\]

and the morphism \(\vartheta_A: R\Gamma(A^\vee) \rightarrow R\Gamma(A)^{\text{SD}}[1]\) between the middle terms.
Proposition (4.2.3). There is a unique way to extend $\vartheta_A$ to a morphism of triangles

$$
\begin{array}{c}
\Gamma(A^\vee)_0 \longrightarrow \mathcal{R}\Gamma(A^\vee) \longrightarrow \left[ \Gamma(A^\vee)_0 \to \mathcal{R}\Gamma(A^\vee) \right] \\
\downarrow \vartheta_A \quad \downarrow \vartheta_A \quad \downarrow \vartheta_A \\
H^1(A)^{SD}[2] \longrightarrow \mathcal{R}\Gamma(A)^{SD}[1] \longrightarrow \Gamma(A)^{SD}[1]
\end{array}
$$

(which in particular means that $\vartheta_A^+$ and $\vartheta_A^-$ are compatible with the connecting morphisms of the triangles). This diagram can further be extended uniquely to a morphism of triangles

$$
\begin{array}{c}
\Gamma(A^\vee)_0^{SDSD} \longrightarrow \mathcal{R}\Gamma(A^\vee)^{SDSD} \longrightarrow \left[ \Gamma(A^\vee)_0 \to \mathcal{R}\Gamma(A^\vee) \right] \\
\downarrow \vartheta_A^- \quad \downarrow \vartheta_A \quad \downarrow \vartheta_A^+ \\
H^1(A)^{SD}[2] \longrightarrow \mathcal{R}\Gamma(A)^{SD}[1] \longrightarrow \Gamma(A)^{SD}[1]
\end{array}
$$

(where the middle $\vartheta_A$ has been defined earlier.)

Proof. Since $\Gamma(A) \in \text{PAlg}/k$, the complex $\Gamma(A)^{SD}[1] = \mathcal{R}\text{Hom}^{\text{indrat}}_\text{proet}(\Gamma(A), Q/Z)$ is concentrated in degrees 0 and 1 with $H^0 = \pi_0(\Gamma(A))^{PD} = \pi_0(\mathcal{A}_{\text{et}})^{PD} \in \text{Fet}/k$ by (2.4.1) [a], [b] and [c]. In particular, its $(-1)$-shift $\Gamma(A)^{SD}$ is concentrated in degrees 1 and 2. Therefore

$$
\text{Hom}_{D(\mathcal{K}_{\text{proet}}^{\text{indrat}})}(\Gamma(A^\vee)_0, \Gamma(A)^{SD}[1]) = \text{Hom}_{D(\mathcal{K}_{\text{proet}}^{\text{indrat}})}(\Gamma(A^\vee)_0, \Gamma(A)^{SD}) = 0,
$$

since a morphism in $D(\mathcal{K}_{\text{proet}}^{\text{indrat}})$ from an object of $\text{Ab}(k_{\text{proet}}^{\text{indrat}})$ to a complex concentrated in non-negative degrees factors through $H^0$. With this, we can uniquely extend $\vartheta_A$ to the first diagram by the general lemma below. For the second diagram, it is sufficient to note that the right upper term (whose $H^0 = \pi_0(\Gamma(A^\vee))$ is finite), the left bottom term, the middle bottom term and the right lower term are all Serre reflexive by (2.4.1) [b], [c].

Lemma (4.2.5). Let $\mathcal{D}$ be a triangulated category. Let $X \to Y \to Z$ and $X' \to Y' \to Z'$ be two distinguished triangles in $\mathcal{D}$. Assume that the homomorphisms $\text{Hom}(X, X') \to \text{Hom}(X, Y')$ and $\text{Hom}(Z, Z') \to \text{Hom}(Y, Z')$ are isomorphisms. Then any morphism $f : Y \to Y'$ can uniquely be extended to a morphism of the triangles. The assumption is satisfied if $\text{Hom}(X, Z') = \text{Hom}(X, Z'[-1]) = 0$.

Proof. This is elementary and well-known. (See also [KS06] Prop. 10.1.17.) We recall its proof. Let $g : X \to X'$ be the morphism corresponding to the composite of $X \to Y$ and $f : Y \to Y'$ under the isomorphism $\text{Hom}(X, X') \cong \text{Hom}(X, Y')$. Then by an axiom of triangulated categories, there is a morphism $h : Z \to Z'$ such that the triple $(g, f, h)$ is a morphism of the triangles. This in particular means that the image of $h$ under the isomorphism $\text{Hom}(Z, Z') \cong \text{Hom}(Y, Z')$ is the composite of $f : Y \to Y'$ and $Y' \to Z'$. Hence such a morphism $h$ is unique. This shows that $f$ is uniquely extended to a morphism of the triangles. If $\text{Hom}(X, Z') = \text{Hom}(X, Z'[-1]) = 0$, then the assumption follows from the long exact sequence of $\text{Hom}$. 


Proposition (4.2.6). The morphism $\theta_A^+$ in (4.2.4) induces morphisms
\[
\begin{align*}
\theta_A^{+0}: \pi_0(A_{\geq 1}) &\twoheadrightarrow \pi_0(A_{\leq 1}) \quad \text{in } \text{FEt}/k, \\
\theta_A^{+1}: H^1(A_{\geq 1}) &\rightarrow \text{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \quad \text{in } \text{IAlg}_{\text{proet}}/k,
\end{align*}
\]
on $H^0$, $H^1$, respectively.

Proof. This follows from $\Gamma(A)^{SD}[1] = R\text{Hom}_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z})$, $\pi_0(\Gamma(A)) = \pi_0(A_{\leq 1})$, and (2.4.1) [1].

Proposition (4.2.7). The morphism $\theta_A$ is an isomorphism if and only if $\theta_A^+$ and $\theta_A^-$ are so.

Proof. The “if” part is trivial. We show the converse. We have $\lim\lim\Gamma(A^\vee) = \lim\Gamma(A^\vee)_0$ since finite groups are killed by $\lim\lim$. By (2.4.1) [4] and (4.2.2), we have
\[
\begin{align*}
\Gamma(A^\vee)^{SD}_0 &= \lim\Gamma(A^\vee) \rightarrow \Gamma(A^\vee)_0, \\
R\Gamma(A)^{SD} &= \lim\Gamma(A) \rightarrow R\Gamma(A).
\end{align*}
\]
From these, we can see that the upper triangle in the diagram (4.2.4) is of the form $X \rightarrow Y \rightarrow Z$ with: $X$ concentrated in degrees $-1, 0$; $Y$ in $-1, 0, 1$; and $Z$ in $0, 1$.

For such a distinguished triangle, we have isomorphisms and an exact sequence
\[
H^{-1}(X) = H^{-1}(Y), \quad H^1(Y) = H^1(Z), \quad 0 \rightarrow H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z) \rightarrow 0.
\]
The exact sequence in the second line is a connected-étale sequence in $\text{PAlg}_{\text{proet}}$.

We show a similar statement for the lower triangle in the diagram (4.2.4): it has cohomologies in the same ranges with the same property (i.e. $H^0$ is a connected-étale sequence). Since $H^1(A) \in \text{IAlg}_{\text{proet}}/k$, we know that $H^1(A)^{SD}[2]$ is concentrated in degrees $-1, 0$ whose $H^0$ is connected pro-unipotent by (2.4.1) [4]. We already saw that $\Gamma(A)^{SD}[1]$ is concentrated in degrees $0, 1$ whose $H^0$ is finite étale. Therefore $R\Gamma(A)^{SD}[1]$ is concentrated in degrees $-1, 0, 1$ whose $H^0$ is in $\text{PAlg}_{\text{proet}}$. Thus the lower triangle in (4.2.4) has the expected properties.

Now if the morphism on the $Y$’s of the two triangles in (4.2.4) is an isomorphism, then so are the morphisms on the $X$’s and $Z$’s.

Before the next proposition, note that, by (2.4.1) [3],
\[
\begin{align*}
\Gamma(A)^{SD}_0 &= R\text{Hom}_{\text{proet}}(\Gamma(A)_0, \mathbb{Q}/\mathbb{Z})[-1] \\
&= \text{Ext}^1_{\text{proet}}(\Gamma(A)_0, \mathbb{Q}/\mathbb{Z})[-2].
\end{align*}
\]
Hence the Serre dual (up to shift) of $\theta_A^{+1}$ in (4.2.0) can be written as a morphism $\Gamma(A)^{SD}_0 \rightarrow H^1(A^\vee)^{SD}[2]$. Since $\theta_A^{-1}$ is also a morphism $\Gamma(A)^{SD}_0 \rightarrow H^1(A^\vee)^{SD}[2]$, it is meaningful to compare $\theta_A^{-1}$ and $(\theta_A^+)^{SD}$.

Proposition (4.2.8). We have $\theta_A^\vee = (\theta_A)^{SD}$, $\theta_A^{-1} = (\theta_A^+)^{SD}$ and $\theta_A^{+0} = (\theta_A^0)^{PD}$.
Proof. We first show $\theta_A^\vee = (\theta_A)^{SD}$, or $\theta_A = (\theta_A^\vee)^{SD}$. The morphisms $A^\vee \otimes^L A \to G_m[1]$ and $A^\vee \otimes^L A^\vee \to G_m[1]$ in $D(K_{ppf})$ coming from the Poincaré bundle (or $G_m$-bixension) are compatible with the biduality isomorphism $A \sim A^\vee$. Hence we have a commutative diagram

$$
\begin{array}{ccc}
A^\vee \otimes^L A & \longrightarrow & G_m[1] \\
\downarrow \iota & & \\
A^\vee \otimes^L A^\vee & \longrightarrow & G_m[1]
\end{array}
$$

in $D(K_{ppf}/k^{indrat})$. Recall from the second paragraph after (3.3.3) that the morphism of functoriality and the cup product pairing are equivalent under the derived tensor-Hom adjunction, which is $\delta_D(G)$ Lemma (4.2.9).

For $\theta_A^\vee = (\theta_A)^{SD}$, or $\theta_A = (\theta_A^\vee)^{SD}$, we have a commutative diagram

$$
\begin{array}{ccc}
\Gamma(A^\vee) \otimes^L \Gamma(A) & \longrightarrow & \Gamma(G_m)[1] \\
\downarrow \iota & & \\
\Gamma(A^\vee) \otimes^L \Gamma(A^\vee) & \longrightarrow & \Gamma(G_m)[1]
\end{array}
$$

in $D(K_{indrat})$. The upper morphisms give a morphism $\Gamma(A^\vee)^{SDSD} \to \Gamma(A)^{SD}$ via the derived tensor-Hom adjunction, which is $\theta_A$. Similarly, the lower morphisms give a morphism $\Gamma(A^\vee) \to \Gamma(A^\vee)^{SD}$, which is $\theta_A^\vee$, and a morphism $\Gamma(A^\vee)^{SDSD} \to \Gamma(A^\vee)^{SD}$, which is $(\theta_A^\vee)^{SD}$. The commutativity of the diagram implies that $\theta_A = (\theta_A^\vee)^{SD}$.

The morphism $(\theta_A^\vee)^{SD}$ is a morphism between complexes concentrated in degree 1. The complex $\Gamma(A)^{SDSD}$ is concentrated in degrees $-1, 0$ by (4.2.1) (4). Consider the distinguished triangle

$$
\begin{array}{ccc}
\Gamma(A^\vee)^{SDSD}[2] & \longrightarrow & \Gamma(A^\vee)_0 \to \Gamma(A^\vee)[1] \\
\downarrow & & \\
\Gamma(A^\vee)^{SDSD}[1] & \longrightarrow & \Gamma(A^\vee)^{SDSD}[1]
\end{array}
$$

By $H^1(A^\vee) \in I_{AAlg/k}$ and (2.2.1) (3), we know that $H^1(A^\vee)^{SDSD}[2]$ is concentrated in degrees $-1, 0$. Hence the middle term $\Gamma(A^\vee)_0 \to \Gamma(A^\vee)[1]$ is concentrated in degrees $-1, 0$. Therefore $H^1((\theta_A)^{SD}) = (\theta_A)^{SD}$. By a similar observation, we have $H^1(\theta_A^\vee) = (\theta_A^\vee)^{SD}$. Hence $(\theta_A)^{SD} = \theta_A^\vee$ proved above implies $(\theta_A^\vee)^{SD} = \theta_A^\vee$.

Finally, the equality $\theta_A^\vee = (\theta_A)^{SD}$ induces the equality $\theta_A^\vee = (\theta_A^\vee)^{PD}$ on $\pi_0 H^P$. □

The previous two propositions show that the first two assertions in (4.2.1) are equivalent. To connect them to the third, it is enough to show the following general lemma.

Lemma (4.2.9). Let $\varphi \colon B \to C$ be a morphism in $I_{AAlg/k}$. Assume that $\varphi \colon B(k') \to C(k')$ is an isomorphism for any algebraically closed field $k'$ over $k$. Then $\varphi \colon B \to C$ is an isomorphism.
Proof. By considering the kernel and cokernel, it is enough to show that if $D \in \text{IAlg}/k$ satisfies $D(k') = 0$ for any algebraically closed field $k'$ over $k$, then $D = 0$. For this, it is enough to show that $\text{Hom}(E, D) = 0$ for any $E \in \text{Alg}/k$. Let $\xi_E$ be the generic point of $E$. Then by (3.4.5), we have

$$\text{Hom}(E, D) = \text{Ker}(D(\xi_E) \to D(\xi_{E \times_k E}))$$

(with the same notation as the cited proof). But the assumption implies that $D(\xi_E) = 0$. Hence $\text{Hom}(E, D) = 0$ and $D = 0$. □

Remark (4.2.10). Here is another method to obtain (4.2.10) or, more precisely, (4.2.3). The Poincaré biextension $A^\vee \times A \to \mathbf{G}_m[1]$ canonically extends to a biextension $A_0^\vee \times A \to \mathbf{G}_m[1]$ by [Gro72, IX, 1.4.3], where $A_0^\vee$ is the maximal open subgroup scheme of $A^\vee$ with connected special fiber. Hence we have a morphism $A_0^\vee \to R\text{Hom}_{\text{proet}}(A, \mathbf{G}_m[1])$. We have the trace isomorphism $R\Gamma_x(\mathcal{O}_K, \mathbf{G}_m[1]) = \mathbb{Z}$; see the second paragraph of (5.2.1). Hence we have a morphism

$$R\Gamma_x(\mathcal{O}_K, A_0^\vee) \to R\text{Hom}_{\text{proet}}(R\Gamma(\mathcal{O}_K, A), R\Gamma_x(\mathcal{O}_K, \mathbf{G}_m[1]))$$

$$= R\text{Hom}_{\text{proet}}(R\Gamma(\mathcal{O}_K, A), \mathbb{Z}) = R\Gamma(\mathcal{O}_K, A)^{\text{SD}}.$$
Néron lft models). Then the sequence \( 0 \to \mathcal{G}_m \to \mathcal{X} \to \mathcal{A} \to 0 \) in \( \text{Ab}(\mathcal{O}_{K,\text{sm}}) \) is exact since \( R^1 j_* \mathbb{G}_m = 0 \) (see the proof of \cite{Mil06} loc.cit., Lem.C.10). We can view this as an exact sequence of group schemes over \( \mathcal{O}_K \). Hence we have an exact sequence \( 0 \to \mathcal{G}_m \to \mathcal{X} \to \mathcal{A} \to 0 \) in \( \text{Ab}(\mathcal{O}_{K,\text{fppf}}) \) (though these no longer represent the pushforward sheaves of the original algebraic groups over \( K_{\text{fppf}} \)). We also have an exact sequence \( 0 \to \mathbb{G}_m \to i_* \mathbb{Z} \to 0 \) in \( \text{Ab}(\mathcal{O}_{K,\text{fppf}}) \), where \( i : \text{Spec} k_{\text{fppf}} \to \text{Spec} \mathcal{O}_{K,\text{fppf}} \) is the natural morphism. By pushing out the extension \( 0 \to \mathbb{G}_m \to \mathcal{X} \to \mathcal{A} \to 0 \) by the morphism \( \mathcal{G}_m \to i_* \mathbb{Z} \), we get an extension \( 0 \to i_* \mathbb{Z} \to \mathcal{X}/\mathcal{G}_m \to \mathcal{A} \to 0 \) in \( \text{Ab}(\mathcal{O}_{K,\text{fppf}}) \). Pulling back by \( i \), we have an exact sequence \( 0 \to \mathbb{Z} \to \mathcal{X}_x/\mathcal{G}_m \to \mathcal{A}_x \to 0 \) in \( \text{Ab}(k_{\text{fppf}}) \), where the subscript \( x \) denotes the special fiber. Therefore we get an element of

\[
\text{Ext}^1_{k_{\text{fppf}}} (\mathcal{A}_x, \mathbb{Z}) = \text{Hom}_{k_{\text{fppf}}} (\mathcal{A}_x, \mathbb{Q}/\mathbb{Z}) = \pi_0 (\mathcal{A}_x)_{\text{PD}}.
\]

This defines a homomorphism \( \Gamma(K, \mathcal{A}^\vee) = \Gamma(\mathcal{O}_K, \mathcal{A}^\vee) \to \pi_0 (\mathcal{A}_x)_{\text{PD}} \), which turns out to factor through \( \pi_0 (\mathcal{A}_x^\vee) \) \((\text{Mil06} \text{ loc.cit., Lem. C.11})\). The homomorphism \( \pi_0 (\mathcal{A}_x^\vee) \to \pi_0 (\mathcal{A}_x)_{\text{PD}} \) thus obtained is Grothendieck’s pairing.

We next describe \( \theta_A^{+0} \). By definition, the morphism \( \vartheta_A \) after taking \( R\Gamma(k, \cdot) \) and \( H^0 \) is

\[
\Gamma(K, \mathcal{A}^\vee) \to \text{Hom}_{D(k_{\text{fppf}}/_{\text{proet}})} (\mathcal{A}, \mathbb{G}_m[1]) \to \text{Hom}_{D(k_{\text{proet}})} (R\Gamma(A), R\Gamma(\mathbb{G}_m)[1]) \to \text{Hom}_{D(k_{\text{proet}})} (\Gamma(A), \mathbb{Z}[1]).
\]

Since \( R \text{Hom}_{k_{\text{proet}}} (\Gamma(A), \mathbb{Q}) = 0 \) by \((2.3.3)\) \((\text{d})\), the final group in the above displayed equation is further isomorphic to

\[
\text{Hom}_{k_{\text{proet}}} (\Gamma(A), \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{k_{\text{proet}}} (\pi_0 (\mathcal{A}_x), \mathbb{Q}/\mathbb{Z}).
\]

The composite \( \Gamma(K, \mathcal{A}^\vee) \to \pi_0 (\mathcal{A}_x)_{\text{PD}} \) factors through \( \pi_0 (\mathcal{A}_x^\vee) \) and the resulting homomorphism \( \pi_0 (\mathcal{A}_x^\vee) \to \pi_0 (\mathcal{A}_x)_{\text{PD}} \) is the definition of \( \theta_A^{+0} \). Therefore an explicit description of \( \theta_A^{+0} \) is given as follows. Let \( 0 \to \mathbb{G}_m \to \mathcal{X} \to \mathcal{A} \to 0 \) be as above (which corresponds to a morphism \( \mathcal{A} \to \mathbb{G}_m[1] \) in \( D(k_{\text{fppf}}/_{\text{proet}}) \)). We apply \( R\Gamma \).

We have \( H^1(\mathbb{G}_m) = 0 \) by \((3.4.3)\) \((\text{c})\). Hence we have an exact sequence

\[
0 \to \mathbb{K}^\times \to \Gamma(\mathcal{X}) \to \Gamma(\mathcal{A}) \to 0
\]

in \( \text{Ab}(k_{\text{proet}}) \), which gives an element of \( \text{Ext}_{k_{\text{proet}}}^1 (\Gamma(A), \mathbb{K}^\times) \). By pushing it out by the valuation map \( \mathbb{K}^\times \to \mathbb{Q}/\mathbb{Z} \), we have an exact sequence \( 0 \to \mathbb{Q}/\mathbb{Z} \to \Gamma(\mathcal{X})/U_K \to \Gamma(\mathcal{A}) \to 0 \). This gives a morphism \( \Gamma(\mathcal{A}) \to \mathbb{Q}/\mathbb{Z}[1] \) in \( D(k_{\text{proet}}) \) and hence a morphism \( \pi_0 (\mathcal{A}_x) \to \mathbb{Q}/\mathbb{Z} \), which is the value of \( \theta_A^{+0} \).

Now we compare the two constructions. The sequence \( 0 \to \mathcal{X} \to \mathcal{X}_x \to \mathcal{A}_x \to 0 \) of Néron models by \((3.1.3)\) \((\text{c})\). We have a commutative diagram with exact rows

\[
\begin{array}{c}
0 \longrightarrow \mathbb{K}^\times \longrightarrow \Gamma(\mathcal{X}) \longrightarrow \Gamma(\mathcal{A}) \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow (\mathcal{G}_m)_x \longrightarrow \mathcal{X}_x \longrightarrow \mathcal{A}_x \longrightarrow 0
\end{array}
\]
in $\text{Ab}(k_{\text{indrat}}^{\text{proet}})$, where the vertical morphisms are the reduction maps of the Néron models. The valuation map $K^\times \to \mathbb{Z}$ and the morphism $(G_m)_x \to \mathbb{Z}$ are compatible. Hence the pushouts by them yield the same morphism $\pi_0(\Gamma(A)) = \pi_0(A_x) \to \mathbb{Q}/\mathbb{Z}$.

Thus our $\theta^+_A$ coincides with Grothendieck’s pairing. □

Remark (5.1.2). In the paragraph before the cited lemma [Mil06, loc.cit., Lem. C.11], the cases $r = 0, 1$ of the equality

$$\text{Ext}_{O_K}(A, i_n \mathbb{Z}) = i_n \text{Ext}^r_k(i^*_A \mathcal{A}, \mathbb{Z})$$

are stated and used. In this remark, we show, as an illustration of our methods in [Suz13], that this equality is true for any $r$. The tricky point is that the topologies on $O_K$ and $k$ here should be the smooth topologies and the $i$ here should be the natural continuous map $i_{sm}$: Spec $k_{sm} \to$ Spec $O_{K, sm}$. Hence, if $r \geq 2$, the equality above might look as if it required that the pullback $i^*_{sm}$ be exact or, equivalently, the pushforward $i_{sm,*}$ send injectives to injectives. However, $i^*_{sm}$ is not exact, since the category of smooth schemes over $O_K$ is not closed under finite inverse limits. In fact, the morphism $G_a \to G_a$ over $O_K$ given by multiplication by a prime element is injective in $\text{Ab}(O_{K, sm})$, while it becomes the zero map $G_a \to G_a$ after applying $i^*_{sm}$. In the above proof of (5.1.1), we used the fact that $j_* X$ is representable (by the Néron model $A'$) to pass from the smooth site to the fppf site. Without the exactness of $i^*_{sm}$, we can still prove the above equality for general $r$ as follows.

The last paragraph before (2.1.2) tells us that $i_{sm,*}$ sends acyclic sheaves to acyclic sheaves. By [Suz13, Lem. 3.7.2], we know that $i^*_{sm}: \text{Ab}(O_{K, sm}) \to \text{Ab}(k_{sm})$ admits a left derived functor $L i^*_{sm}: D(O_{K, sm}) \to D(k_{sm})$, which is left adjoint to $R i_* = i_*$. It is enough to show that $L_n i^* \mathcal{A} = 0$ for any $n \geq 1$. Let $M(\mathcal{A})$ be Mac Lane’s resolution of $\mathcal{A}$ in $\text{Ab}(O_{K, sm})$ ([ML57], [Suz13 §3.4]). Its $n$-th term is a direct summand of a direct sum of sheaves of the form $\mathbb{Z}[A^m]$ for various $m \geq 0$, where $\mathbb{Z}[A^m]$ is the sheafification of the presheaf that sends a smooth $O_K$-algebra $S$ to the free abelian group generated by the set $A^m(S)$ ([Suz13 §3.4, 3.5]). We have $L i^*_{sm} A = i^*_{sm} M(\mathcal{A})$, where $i^*_{sm}$ on the right is applied termwise. We have $i^*_{sm} \mathbb{Z}[A^m] = \mathbb{Z}[i^*_{sm} A^m]$, where $i^*_{sm}: \text{Set}(O_{K, sm}) \to \text{Set}(k_{sm})$ is the pullback for sheaves of sets. Since $A^m$ is in the underlying category of Spec $O_{K, sm}$, we know that $i^*_{sm}(A^m)$ is the special fiber $A^m_{x} = A^m \times_{O_{K, sm}} k$. Therefore $i^*_{sm} M(\mathcal{A}) = M(A_x) = M(i^* \mathcal{A})$ is Mac Lane’s resolution of $A_x$ in $\text{Ab}(k_{sm})$, which is acyclic outside degree zero. Therefore $L_n i^* \mathcal{A} = 0$ for any $n \geq 1$.

In the same way, we can prove that if $X$ is a scheme and $f$ is the continuous map from the big étale site of $X$ to the smooth site of $X$ defined by the identity, then $f^*$ admits a left derived functor $L f^*$, and we have $L_n f^* T = 0$ for any smooth group scheme $T$ over $X$ and $n \geq 1$. This implies that [Mil06 Thm. 4.11] is true with the big étale site replaced by the smooth site by the same argument as above. This answers Milne’s question made right after the cited theorem.

5.2. Bester-Bertapelle’s isomorphism. First note that

$$\text{Ext}^1_{k_{\text{indrat}}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) = \lim_n \text{Ext}^1_{\text{PAlg}/k}(\Gamma(A), \mathbb{Z}/n\mathbb{Z})$$

by (2.3.3) [B].

Proposition (5.2.1). Assume that $k$ is algebraically closed. The morphism

$$\theta_A^{-1}(k): H^1(K, A^\vee) \to \text{Ext}^1_{k_{\text{indrat}}}(\Gamma(A), \mathbb{Q}/\mathbb{Z})$$
of (4.2.1) coincides with Bester-Bertapelle’s isomorphism \cite[§2.7, Thm. 7.1]{Bes78}, \cite[Thm. 3]{Ber03} when \( K \) has equal characteristic and \( A \) has semistable reduction.

We prove this in this subsection. In \cite[§2.3, Thm. 3]{Bes78}, Bester proved a duality theorem for cohomology of \( \mathcal{O}_K \) with coefficients in finite flat group schemes. Based on this result, Bertapelle proved its generalization for coefficients in the quasi-finite flat group scheme \( A[[n]] \) of torsion points of the Néron model of any semistable abelian variety \( A \) (\cite[Thm. 1]{Ber03}). Then she deduced the existence of the above isomorphism for such \( A \) (\cite[Thm. 2]{Ber03}), and deduced the existence of the above isomorphism for a general abelian variety.

In this subsection, we first give another proof of Bester’s finite flat duality, by giving another construction of the duality morphism and directly showing that it is an isomorphism. The idea of proof is the same as Bester’s (which is the local version of \cite{AM76}), but we need to make it work within the formulation of this paper. Then we use our finite flat duality isomorphism (instead of Bester’s isomorphism) as the input for Bertapelle’s constructions. This outputs an isomorphism

\[ \psi_A: H^1(K, A^\vee) = \text{Ext}^1_{\text{fin flat}}(\Gamma(A), \mathbb{Q}/\mathbb{Z}) \]

as in the proof of \cite[Thm. 2]{Ber03}. Finally we show that \( \theta_A^{-1}(k) \) and \( \psi_A \) are equal. Hence we do not actually compare our isomorphism and Bester’s isomorphism (see (5.2.1.5) below for the reason). Therefore the statement of (5.2.1) is slightly imprecise in this sense. See (5.2.3.1) below for the precise statement. We do not have to compare Bertapelle’s isomorphism with \( \theta_A^{-1}(k) \) for non-semistable abelian varieties for the purpose of proving our main theorems. But this can be done; see (6.13) (b) below.

Throughout this subsection, we assume that \( K \) has equal characteristic. Some statements can be formulated without assuming the residue field \( k \) to be algebraically closed. Therefore we assume that \( k \) is a general perfect field.

### 5.2.1. Bester’s finite flat duality

Let \( N \) be a finite flat group scheme over \( \mathcal{O}_K \) with Cartier dual \( N^{\text{CD}} \). In this subsection, we first formulate a duality theorem that relates the two complexes

\[ R\Gamma_x(\mathcal{O}_K, N^{\text{CD}}), \quad R\Gamma(\mathcal{O}_K, N) \]

each other. This is written in \cite[Rmk. 2.7.6]{Suz13} without proof. We show that this implies Bester’s duality. The proof of the duality theorem itself will be given in the next subsection.

We have \( R\Gamma(K, G_m) = K^\times \) as recalled in the previous section, and \( R\Gamma(\mathcal{O}_K, G_m) = \Gamma(\mathcal{O}_K, G_m) = U_K \) by (3.4.2) (a). Hence we have

\[ R\Gamma_x(\mathcal{O}_K, G_m) = \mathbb{Z}[-1], \]

which we call the trace isomorphism.

Let \( N \) be a finite flat group scheme over \( \mathcal{O}_K \) with Cartier dual \( N^{\text{CD}} \). Recall from (3.4.2) (b) and (3.4.6) that \( R\Gamma(\mathcal{O}_K, N) \in D^b(\text{PAlg}_{\text{uc}}/k), \quad R\Gamma_x(\mathcal{O}_K, N) \in D^b(\text{IAlg}_{\text{uc}}/k) \) are P-acyclic and Serre reflexive, \( \Gamma(\mathcal{O}_K, N) \) is finite, \( H^1(\mathcal{O}_K, N) \) is connected pro-unipotent, \( H^2_x(\mathcal{O}_K, N) \in \text{IAlg}_{\text{uc}}/k \), and the cohomology of \( R\Gamma(\mathcal{O}_K, N) \) and \( R\Gamma_x(\mathcal{O}_K, N) \) vanishes at all other degrees. The morphism of functoriality of
We saw in §3.3 that \( R\Gamma \) and \( R\tilde{\Gamma} \) in \([3.3.3]\) and the trace isomorphism give morphisms

\[
R\Gamma_x(\mathcal{O}_K, N^{CD}) \to R\tilde{\Gamma}_x(\mathcal{O}_K, R\text{Hom}_{\mathcal{O}_K}(N, G_m))
\]

\[
\to R\text{Hom}_{\text{proet}^{\text{indrat}}}^{\text{proet}}(R\Gamma(\mathcal{O}_K, N), \mathbb{Z}[-1]) = R\Gamma(\mathcal{O}_K, N)^{SD}[−1]
\]

**Theorem (5.2.1.2).** The morphism

\[
R\Gamma_x(\mathcal{O}_K, N^{CD}) \to R\Gamma(\mathcal{O}_K, N)^{SD}[−1]
\]

in \( D(k^{\text{indrat}}) \) defined above is an isomorphism.

This is stated in [Suz13, Rmk. 2.7.6] without proof. We will prove this in the next subsection. We show here that this implies Bester’s duality.

The both sides are in \( D^b(\text{Alg}_k) \). The functor \( R\Gamma(\mathcal{O}_K, N) \) restricted to \( D^b(\text{Alg}_k) \) takes values in the derived category of torsion discrete \( \text{Gal}(\overline{k}/k) \)-modules, since unipotent quasi-algebraic groups in positive characteristic are torsion. Applying this functor to the morphism in the theorem, we obtain a morphism

\[
R\Gamma_x(\mathcal{O}_K^{\text{ur}}, N^{CD}) \to R\text{Hom}_{\text{proet}^{\text{indrat}}}^{\text{proet}}(R\Gamma(\mathcal{O}_K, N), \mathbb{Q}/\mathbb{Z})[-2]
\]

in the derived category of torsion discrete \( \text{Gal}(\overline{k}/k) \)-modules, where \( \mathcal{O}_K^{\text{ur}} \) is the maximal unramified extension of \( \mathcal{O}_K \). We have

\[
R\text{Hom}_{\text{proet}^{\text{indrat}}}(R\Gamma(\mathcal{O}_K, N), \mathbb{Q}/\mathbb{Z}) = R\text{Hom}_{\text{PAAlg}/\overline{k}}(R\Gamma(\mathcal{O}_K, N), \mathbb{Q}/\mathbb{Z})
\]

\[
(\lim_n \to R\text{Hom}_{\text{PAAlg}/\overline{k}}(R\Gamma(\mathcal{O}_K, N), \mathbb{Z}/n\mathbb{Z}))
\]

by \([2.3.3] (b)\). Let \( \text{PFET}/k \) be the category of pro-finite-étale group schemes over \( k \) and \( \pi_0 : \text{PAAlg}/k \to \text{PFET}/k \) the maximal pro-finite-étale quotient functor as before. We saw in \([2.2]\) that \( \text{PAAlg}/k \) (and also \( \text{PFET}/k \)) is the opposite of a Grothendieck category and hence admits enough \((K-)\)projectives. Hence \( \pi_0 \) admits a left derived functor \( L\pi_0 : D(\text{PAAlg}/k) \to D(\text{PFET}/k) \). The \( n \)-th functor \( \pi_n := L_n\pi_0 \) is the \( n \)-th homotopy group functor defined by Serre [Ser60, §5.3]. The functor \( \pi_0 \) is left adjoint to the inclusion functor \( \text{PFET}/k \to \text{PAAlg}/k \), which is exact. Hence \( \pi_0 \) sends projectives to projectives. By [KS06 Prop. 13.3.13 (ii)], we have

\[
R\text{Hom}_{\text{PAAlg}/\overline{k}}(R\Gamma(\mathcal{O}_K, N), \mathbb{Q}/\mathbb{Z}) = R\text{Hom}_{\text{PFET}/\overline{k}}(L\pi_0 R\Gamma(\mathcal{O}_K, N), \mathbb{Q}/\mathbb{Z}).
\]

We have a distinguished triangle

\[
L\pi_0 \Gamma(\mathcal{O}_K, N) \to L\pi_0 R\Gamma(\mathcal{O}_K, N) \to L\pi_0 H^1(\mathcal{O}_K, N)[−1].
\]

Recall from [Ser60 §10.2, Thm. 2] that \( \pi_n = L_n\pi_0 = 0 \) for all \( n \geq 2 \). The finiteness of \( \Gamma(\mathcal{O}_K, N) \) implies that \( \pi_1 \Gamma(\mathcal{O}_K, N) = 0 \) and hence \( L\pi_0 \Gamma(\mathcal{O}_K, N) = \Gamma(\mathcal{O}_K, N) \). The connectedness of \( H^1(\mathcal{O}_K, N) \) implies that \( L\pi_0 H^1(\mathcal{O}_K, N) = \pi_1 H^1(\mathcal{O}_K, N)[1] \). Therefore \( L\pi_0 R\Gamma(\mathcal{O}_K, N) \) is concentrated in degree 0 and the above triangle reduces to an exact sequence

\[
0 \to \Gamma(\mathcal{O}_K, N) \to H^0 L\pi_0 R\Gamma(\mathcal{O}_K, N) \to \pi_1 H^1(\mathcal{O}_K, N) \to 0.
\]

As in the proof of \([3.4.2] (b)\), let \( 0 \to N \to G \to H \to 0 \) be an exact sequence of group schemes over \( \mathcal{O}_K \) with \( G, H \) smooth affine with connected fibers. Then \([3.4.2] (a)\) shows that \( R\Gamma(\mathcal{O}_K, G) \) is concentrated in degree zero and \( \Gamma(\mathcal{O}_K, G) \in \text{PAAlg}/k \) is connected. Hence \( L\pi_0 R\Gamma(\mathcal{O}_K, G) = \pi_1 \Gamma(\mathcal{O}_K, G)[1] \) and \( L\pi_0 R\Gamma(\mathcal{O}_K, H) = \pi_1 \Gamma(\mathcal{O}_K, H)[1] \). Therefore the distinguished triangle

\[
L\pi_0 R\Gamma(\mathcal{O}_K, N) \to L\pi_0 \Gamma(\mathcal{O}_K, G) \to L\pi_0 \Gamma(\mathcal{O}_K, H)
\]
whose cohomology is canonically identified with Bester’s group $O_{K, ff}$ finite flat site $\text{Spec } K$.

The existence of a trace isomorphism is stated in [Bes78 §1.3].

Therefore our morphism after $R\Gamma(\mathcal{F}_{\text{proet, } \cdot})$ takes the form

$$R\Gamma_x(O_{K, ff}^\text{ur}, N^{\text{CD}}) \to \text{Hom}_{\text{proet}}(\mathcal{F}(N), \Bbb{Q}/\Bbb{Z})[-2].$$

Recall again that the left-hand side is concentrated in degree 2.

**Theorem (5.2.1.4).** The morphism

$$H^2_z(O_{K, ff}^\text{ur}, N^{\text{CD}}) \to \mathcal{F}(N)^{\text{PD}}$$

of torsion discrete $\text{Gal}(\overline{k}/k)$-modules thus obtained is an isomorphism.

This is stated in [Bes78 §2.3, Thm. 3.1]. We will directly prove this by proving (5.2.1.2) in the next subsubsection.

**Remark (5.2.1.5).** Comparing our finite flat duality isomorphism above and Bester’s isomorphism, which we omit here, is highly technical and complicated. We have to bring many constructions in Bester’s work into derived categories of sheaves. Several technical fixes that we do not discuss here in detail are also necessary. In this paper, we avoid the comparison and fixes to save pages and prove Bester’s duality independently.

Below we merely indicate one issue that needs a fix. Assume for simplicity that $k$ is algebraically closed. Recall that the finite flat site $\text{Spec } O_{K, ff}$ of $O_K$ is the category of finite flat $O_K$-algebras where a cover of an object $S$ is a finite family $\{S_i\}$ of finite flat $S$-algebras such that $S \to \prod S_i$ is faithfully flat. For a finite flat group scheme $N$ over $O_K$, Bester defined his group $\mathcal{F}(N)$ in [Bes78 §1.3, Def. 3.3] not only as a pro-finite-étale group over $k$ but also as a sheaf on the finite flat site $\text{Spec } O_{K, ff}$. Let us denote this sheaf by $\mathcal{F}_{\text{sh}}(N)$ for clarity. He stated the existence of a trace isomorphism

$$H^2_z(O_K, \mathcal{F}_{\text{sh}}(\mu_{p^n})) = \Bbb{Z}/p^n\Bbb{Z}$$

for any $n \geq 1$ in [Bes78 §2.6, Lem. 6.2] and its proof. Here $\mathcal{F}_{\text{sh}}(\mu_{p^n})$ is pulled back to $\text{Spec } O_{K, \text{ff}}$ [Bes78 §1.2, Rmk. 2.2]) and then taken the fppf cohomology with support ([Bes78 §2.2]). However, we can show that $\mathcal{F}_{\text{sh}}(\mu_{p^n}) = 0$ as a sheaf on $\text{Spec } O_{K, ff}$, which contradicts to his statement.

To prove $\mathcal{F}_{\text{sh}}(\mu_{p^n}) = 0$, we first recall the definition of this sheaf. For a flat group scheme $G$ over $O_K$, recall from [Bes78 §1.2, i)] that another sheaf denoted by $\pi_1(G)$ on $\text{Spec } O_{K, ff}$ is defined as follows. Assume for simplicity that $G$ is affine. For a finite flat local $O_K$-algebra $S$, let $\text{Res}_{S/O_K}$ be the Weil restriction functor. Consider the Greenberg transform $\Gamma(O_K, \text{Res}_{S/O_K} G)$ of $\text{Res}_{S/O_K} G$ and denote it by $\Gamma(S, G)$. This is a proalgebraic group over $k$ representing the functor

$$R \mapsto \Gamma(R[[T]], \text{Res}_{S/O_K} G) = \Gamma(R[[T]] \otimes_{k[[T]]} S, G)$$

on the category of $k$-algebras, where we identified $O_K = k[[T]]$. We have

$$\Gamma(S, G) = \lim_{\rightarrow} \text{Res}_{(S/T^n S)/k} G.$$
Define $\pi_1(\mathcal{G})$ to be the presheaf that assigns to each a finite flat $\mathcal{O}_K$-algebra $S$ and its decomposition $\prod S_i$ into local rings the group

$$\prod_i \pi_1(\Gamma(S_i, G)).$$

This is in fact a sheaf by the faithfully flat descent and the left exactness of $\pi_1$ (Bes78 §1.2, Lem. 2.1). Here we treat $\pi_1(\mathcal{G})$ as a single set of notation and do not decompose it into the $\pi_1$ of anything in the usual sense. Let $N$ be a finite flat group scheme over $\mathcal{O}_K$. Take an exact sequence $0 \to N \to G \to H \to 0$ of group schemes over $\mathcal{O}_K$ with $G, H$ smooth affine with connected fibers, as we took in the proof of (3.4.2) (3). Then the sheaf $\mathcal{F}_{\text{sh}}(N) \in \text{Ab}(\mathcal{O}_K, \mathcal{O}_K)$, as defined in [Bes78 §1.3, Def. 3.3], is the cokernel of the morphism of sheaves

$$\pi_1(\mathcal{G}) \to \pi_1(H) \in \text{Ab}(\mathcal{O}_K, \mathcal{O}_K).$$

We show that $\mathcal{F}_{\text{sh}}(\mu_{p^n}) = 0$ for any $n \geq 1$ as a sheaf on $\text{Spec} \mathcal{O}_K$. For this, it is enough to show that for any finite flat local $\mathcal{O}_K$-algebra $S$, there exists a finite faithfully flat local $S$-algebra $S'$ such that the homomorphism from the cokernel of

$$\pi_1(\Gamma(S, G_m)) \xrightarrow{p^n} \pi_1(\Gamma(S, G_m))$$

to the cokernel of

$$\pi_1(\Gamma(S', G_m)) \xrightarrow{p^n} \pi_1(\Gamma(S', G_m))$$

is zero. The group $\pi_1(\Gamma(S, G_m))$ is connected. Since $\text{Res}_{R/k} \mu_{p^n}$ is connected for any finite $k$-algebra $R$, it follows that $\Gamma(S, G_m)$ is also connected (cf. [Bes78 §1.1, Lem. 1.2]). This implies

$$\pi_1(\Gamma(S, G_m)/p^n) = \pi_1(\Gamma(S, G_m)/p^n),$$

where $/p^n$ denotes the cokernel of multiplication by $p^n$. Similar for $\pi_1(\Gamma(S', G_m))$. Hence it is enough to show that the morphism $\pi_1(\Gamma(S, G_m)) \to \pi_1(\Gamma(S', G_m))$ factors through the image $D$ of the $p^n$-th power map $\pi_1(S, G_m) \to \pi_1(S', G_m)$. The groups $\pi_1(S', G_m)$, $\pi_1(S', G_m)$ and $D$ are inverse limits of smooth algebraic groups over $k$. Therefore it is enough to check the statement for $k$-points ($k$ being algebraically closed). The requirement for $S'$ is now that every element of $\pi_1(S', G_m)$ should become a $p^n$th power in $\pi_1(S', G_m)$. If $S = k[[T]][x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some polynomials $f_1, \ldots, f_m$ of the polynomial ring, then $S' = k[[T]][x_1/p^n, \ldots, x_n/p^n]/(f_1, \ldots, f_m)$ does the job. This choice of $S'$ was suggested by Bertapelle.

We should instead consider the mapping cone of the morphism (5.2.1.6) in $D(\mathcal{O}_K)$. This is the first step for a fix, but we should stop here.

5.2.2. Proof of the finite flat duality. Let $N$ be a finite flat group scheme over $K$. The morphism of functoriality of $R \Gamma$ (3.3.8) and the trace morphism (4.1.1) give morphisms

$$R \Gamma(K, N^{\text{CD}}) \to R \Gamma(K, R \text{Hom}_K(N, G_m))$$

$$\to R \text{Hom}_{\text{proet}}(R \Gamma(K, N), \mathbb{Z}) = R \Gamma(K, N)^{\text{SD}}$$

To see this, it is enough to show that the scheme-theoretically isomorphic group $\text{Res}_{R/k} \alpha_{p^n}$ is connected. Let $\{r_i\}_{i=1}^d$ be a $k$-basis of $R$. Write $r_i^{p^n} = \sum c_{ij} r_j$ with $c_{ij} \in k$. Let $C: G^{d} \to G^{d}$ be the $k$-linear morphism given by the matrix with entries $c_{ij}$. Then $\text{Res}_{R/k} \alpha_{p^n} = \text{Ker}(C F^n)$ as a subgroup of $G^{d} = \text{Res}_{R/k} G^{m}$, where $F: G^{d} \to G^{d}$ is the Frobenius over $k$. Hence $\text{Res}_{R/k} \alpha_{p^n}$ is an extension of the vector group $\text{Ker} C$ by $\alpha_{p^n}$, thus connected.
Theorem (5.2.2.1). The morphism
\[ R\Gamma(K, N^{CD}) \to R\Gamma(K, N)^{SD} \]
in \( D(k_{\text{proet}}) \) defined above is an isomorphism.

This is proved in [Suz13, Thm. 2.7.1] for the case \( N \) does not have connected unipotent part. We prove [5.2.2.1] for general \( N \), in addition to and at the same time proving [5.2.1.2]. Note that Pépin [Pép14, §7.6.1] also proved a result equivalent to [5.2.2.1] for general \( N \).

Proposition (5.2.2.2). For a finite flat group scheme \( N \) over \( O_K \) with Cartier dual \( N^{CD} \), we have a morphism of distinguished triangles
\[
\begin{align*}
R\Gamma(O_K, N^{CD}) &\longrightarrow R\Gamma(K, N^{CD}) \longrightarrow R\Gamma_x(O_K, N^{CD})[1] \\
\downarrow &\quad \downarrow &\quad \downarrow \\
R\Gamma_x(O_K, N)^{SD}[−1] &\longrightarrow R\Gamma(K, N)^{SD} \longrightarrow R\Gamma(O_K, N)^{SD}
\end{align*}
\]
in \( D(k_{\text{proet}}) \).

Proof. Apply the diagram in (3.3.8) for \( A = N \) and \( B = G_m \) and note that the trace (iso)morphisms \( R\Gamma(K, G_m) \to \mathbb{Z} \) and \( R\Gamma_x(O_K, G_m)[1] \to \mathbb{Z} \) are compatible with the morphism \( R\Gamma(K, G_m) \to R\Gamma_x(O_K, G_m)[1] \) by construction.

Proposition (5.2.2.3). To prove [5.2.1.2] and [5.2.2.1], it is enough to show [5.2.1.2] for the case that the finite flat group scheme \( N \) or \( N^{CD} \) has height 1, i.e. has zero Frobenius.

Proof. First note that if \( 0 \to N_1 \to N_2 \to N_3 \to 0 \) is an exact sequence of finite flat group schemes (over \( O_K \) or \( K \)) and if the statements are true for any two of them, then so is for the other by the five lemma. The statements are classical and elementary if \( N \) has order prime to \( p \) (cf. [Ber03, §3.2]). Assume \( N \) (over \( O_K \) or \( K \)) has \( p \)-power order. Then it has a filtration whose successive subquotients are either height 1 or have Cartier dual of height 1 ([Bes78, §2.6, Lem. 6.1]). A finite flat group scheme over \( K \) of height 1 can be extended to a finite flat group scheme over \( O_K \) of height 1 by [Mil06, III, Prop. B.4].

Let \( \Omega^1_{O_K} = \Omega^1_{O_K/k} \) and \( \Omega^1_K = \Omega^1_{K/k} \) be the first differential modules. We identify them as group schemes isomorphic to \( G_a \). We have \( R\Gamma(O_K, G_a) = O_K \), \( R\Gamma(K, G_a) = K \), and hence \( R\Gamma_x(O_K, G_a) = K/O_K[−1] \). The residue map \( \sum a_n T^n dT/T \mapsto a_0 \) may be viewed as a morphism
\[
\text{Res}: R\Gamma(K, \Omega^1_K) \to R\Gamma_x(O_K, \Omega^1_{O_K})[1] \to G_a
\]
in \( D(k_{\text{proet}}) \). Together with the coboundary map \( G_a \to \mathbb{Z}/p\mathbb{Z}[1] \) of the Artin-Schreier sequence \( 0 \to \mathbb{Z}/p\mathbb{Z} \to G_a \to G_a \to 0 \) and the coboundary map \( \mathbb{Z}/p\mathbb{Z}[1] \to \mathbb{Z}[2] \) of the sequence \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0 \), we have morphisms
\[
R\Gamma(K, \Omega^1_K) \to R\Gamma_x(O_K, \Omega^1_{O_K})[1] \to \mathbb{Z}[2]
\]
in \( D(k_{\text{proet}}) \), which we call the additive trace morphisms (as opposed to the multiplicative trace morphisms \( R\Gamma(K, G_m) \to R\Gamma_x(O_K, G_m)[1] = \mathbb{Z} \)).
Proposition (5.2.2.4). Let $M \times L \to \Omega^1_{\mathcal{O}_K}$ (resp. $M \times L \to \Omega^1_{\mathcal{O}_K}$) be a perfect pairing of finite free $\mathcal{O}_K$-modules (resp. finite free $K$-modules). Consider the morphisms

$$R\Gamma_x(\mathcal{O}_K, M) \to R\Gamma_x(\mathcal{O}_K, R\text{Hom}_{\mathcal{O}_K}(L, \Omega^1_{\mathcal{O}_K}))$$

$$\to R\text{Hom}^\text{indrat}_{\text{proét}}(R\Gamma(\mathcal{O}_K, L), R\Gamma_x(\mathcal{O}_K, \Omega^1_{\mathcal{O}_K}))$$

$$\to R\Gamma(\mathcal{O}_K, L)^{\text{SD}}[1]$$

and similarly

$$R\Gamma(\mathcal{O}_K, M) \to R\Gamma_x(\mathcal{O}_K, L)^{\text{SD}}[1]$$

(resp.

$$R\Gamma(K, M) \to R\Gamma(K, R\text{Hom}_K(L, \Omega^1_K))$$

$$\to R\text{Hom}^\text{indrat}_{\text{proét}}(R\Gamma(K, L), R\Gamma(K, \Omega^1_K))$$

$$\to R\Gamma(K, L)^{\text{SD}}[2]$$

in $D(k^{\text{indrat}})$ defined by the functoriality of $R\Gamma_x(\mathcal{O}_K, \cdot)$ (resp. $R\Gamma(K, \cdot)$) and the additive trace morphism. These are isomorphisms.

Proof. The latter isomorphism (for cohomology of $K$) follows from the former two isomorphisms (for cohomology of $\mathcal{O}_K$) by a morphism of distinguished triangles similar to the one in [5.2.2.2]. For the former two, it is enough to treat the morphism $R\Gamma_x(\mathcal{O}_K, M) \to R\Gamma(\mathcal{O}_K, L)^{\text{SD}}[1]$ since $\mathcal{O}_K \cong G^N_a$, $K/\mathcal{O}_K \cong G^N_{\mathcal{O}_K}$ and $K \cong G^N_a \oplus G^N_{\mathcal{O}_K}$ are all Serre reflexive by (2.3.1) [3]. We may assume that $M = \mathcal{O}_K$, $L = \Omega^1_{\mathcal{O}_K}$ and the pairing $M \times L \to \Omega^1_{\mathcal{O}_K}$ is the multiplication. Then the morphism can be written as the composite

$$K/\mathcal{O}_K \to \text{Hom}^\text{indrat}_{\text{proét}}(\Gamma(\mathcal{O}_K, \Omega^1_{\mathcal{O}_K}), G_a) \to \Gamma(\mathcal{O}_K, \Omega^1_{\mathcal{O}_K})^{\text{SD}}[2],$$

where the first morphism comes from the $k$-linear map $K/\mathcal{O}_K \to \text{Hom}_{k\text{-mod}}(\Omega^1_{\mathcal{O}_K}, k)$, the residue map. We have

$$\text{Hom}^\text{indrat}_{\text{proét}}(\Gamma(\mathcal{O}_K, \Omega^1_{\mathcal{O}_K}), G_a) = \lim_{n \geq 1} \text{Hom}^\text{indrat}_{\text{proét}}(\Gamma(\mathcal{O}_K/p^n_{K}, \Omega^1_{\mathcal{O}_K}), G_a)$$

by (2.3.3) [4] and (3.1.3) [5]. Hence the mentioned morphism can also be written as the direct limit in $n \geq 1$ of the morphisms

$$p_{K,n}/\mathcal{O}_K \to \text{Hom}^\text{indrat}_{\text{proét}}(\Gamma(\mathcal{O}_K/p^n_{K}, \Omega^1_{\mathcal{O}_K}), G_a),$$

where $p_{K,n}(k') = p^n_{K,n} \otimes_{\mathcal{O}_K} \mathcal{O}_K(k')$ for $k' \in k^{\text{indrat}}$. For each $n$, this morphism comes from the $k$-linear morphism

$$\text{Res}: p_{K,n}/\mathcal{O}_K \to \text{Hom}_{k\text{-mod}}(\Gamma(\mathcal{O}_K/p^n_{K}, \Omega^1_{\mathcal{O}_K}), k),$$

which is well-known (and easily seen) to be an isomorphism. Hence the Breen-Serre duality [MII00, III, Lem. 0.13 (c)] (where the perfect étale site is used) and (2.3.1) [3] shows that

$$p_{K,n}/\mathcal{O}_K \cong \Gamma(\mathcal{O}_K/p^n_{K}, \Omega^1_{\mathcal{O}_K})^{\text{SD}}[2].$$

The direct limit in $n$ gives the result.

We denote $K' = K^{1/p}$ and $\mathcal{O}'_K = \mathcal{O}_{K'}^{1/p}$, which are viewed as a $K$-algebra and an $\mathcal{O}_K$-algebra, respectively, via inclusions $K \hookrightarrow K'$ and $\mathcal{O}_K \hookrightarrow \mathcal{O}'_K$. Let $F: \text{Spec } K' \to \text{Spec } K$ and $F: \text{Spec } \mathcal{O}'_K \to \text{Spec } \mathcal{O}_K$ be the natural morphisms.
Let $N$ be a finite flat group scheme over $O_K$ of height 1. Recall from [AM76, Prop. 1.1, Lem. 2.2] that there exist canonical exact sequences
\[ 0 \to N^{CD} \to V^0(N^{CD}) \to V^1(N^{CD}) \to 0, \]
\[ 0 \to N \to F_*N \to U^0(N) \to U^1(N) \to 0 \]
of group schemes over $O_K$. The terms $U^i(N)$, $V^i(N^{CD})$ are vector groups. The second exact sequence above for $N = \mu_p$ is explicitly given by
\[ 0 \to \mu_p \to F_*\mu_p \to F_*\Omega^1_{O_K} \to \Omega^1_{O_K} \to 0, \]
where the middle morphism is $F_*\log$ and the right morphism is the $O$-linear Cartier operator $C$ minus the formal $p$-th power $W$ (AM76, Lem. 2.1, 2.2). Let $U(N)$ be the complex $U^i(N) \to U^1(N)$ in degrees 0 and 1 and $V(N^{CD})$ the complex $V^0(N^{CD}) \to V^1(N^{CD})$ in degrees 0 and 1. The morphism $U(N) \to N[1]$ induces isomorphisms
\[ R\Gamma(O_K(k'), U(N)) \simeq R\Gamma(O_K(k'), N)[1], \]
\[ R\Gamma(K(k'), U(N)) \simeq R\Gamma(K(k'), N)[1], \]
\[ R\Gamma_x(O_K, U(N)) \simeq R\Gamma_x(O_K, N)[1]. \]
for any $k' \in k^{\text{indrat}}$ by [AM76, Prop. 2.4]. (One checks that the proof there also works for our situation Spec $O_K(k') \to \text{Spec } k'$ and Spec $K(k') \to \text{Spec } k'$.) Therefore we have
\[ R\Gamma(O_K, U(N)) \simeq R\Gamma(O_K, N)[1], \]
\[ R\Gamma(K, U(N)) \simeq R\Gamma(K, N)[1], \]
\[ R\Gamma_x(O_K, U(N)) \simeq R\Gamma_x(O_K, N)[1]. \]
(5.2.2.5)
Also
\[ R\Gamma(O_K, V(N^{CD})) \cong R\Gamma(O_K, N^{CD}), \]
\[ R\Gamma(K, V(N^{CD})) \cong R\Gamma(K, N^{CD}), \]
\[ R\Gamma_x(O_K, V(N^{CD})) \cong R\Gamma_x(O_K, N^{CD}). \]
(5.2.2.6)
By [AM76, Prop. 3.4], there exists a canonical pairing
\[ V(N^{CD}) \times U(N) \to U(\mu_p) \]
of complexes of group schemes over $O_K$. The parts
\[ V^1(N^{CD}) \times U^0(N) \to U^1(\mu_p) = \Omega^1_{O_K}, \quad V^0(N^{CD}) \times U^1(N) \to U^1(\mu_p) = \Omega^1_{O_K} \]
are given by perfect pairings of finite free $O_K$-modules. Let $Z(N)$ be the complex $F_*N \to U^0(N) \to U^1(N)$ of group schemes in degrees 0, 1, 2, which is a resolution of $N$. By [AM76, (4.7)], the diagram
\[ \begin{array}{ccc}
N \times N^{CD} & \to & \mu_p \\
\downarrow & & \\
Z(N) \times N^{CD} & \to & Z(\mu_p) \\
\downarrow & & \downarrow \\
U(N)[-1] \times V(N^{CD}) & \to & U(\mu_p)[-1]
\end{array} \]
(5.2.2.7)
of pairings of complexes of group schemes over $O_K$ is commutative.
The complete discrete valuation field $K' = K^{1/p}$ has residue field $k$. We can define a sheaf of rings $K' = \Gamma(K', G_a) = \Gamma(K, F_* G_a)$ on $\text{Spec} \, k^{\text{proet}}$ in a way similar to $K$. We also have a residue map $\Gamma(K, F_* \Omega^1_{K'}) = \Gamma(K', \Omega^1_{K'}) \to G_a$ for $K'$.

**Proposition (5.2.2.8).** The residue map gives a morphism of complexes from

$$R \Gamma_x(\mathcal{O}_K, U(\mu_p)) = \left[ \frac{\Gamma(K, F_* \Omega^1_{K'})}{\Gamma(\mathcal{O}_K, \Omega^1_{\mathcal{O}_K})} \right] [-2]$$

Thus obtained and the morphism

$$R \Gamma_x(\mathcal{O}_K, U(\mu_p)) \to \mathbb{Z}/p\mathbb{Z}[-1]$$

coming from $R \Gamma(\mathcal{O}_K, \mu_p) \cong U_K/(U_K)^p[-1]$ and $R \Gamma(K, \mu_p) \cong K^\times/(K^\times)^p[-1]$ \cite{AM76}. $\mathbb{Z}/p\mathbb{Z}[-1]$ are compatible with the identification \cite{5.2.2.5}.

**Proof.** By \cite{AM76} Lem. 2.1, we have an exact sequence

$$0 \to G_m \to F_* G_m \xrightarrow{d\log} F_* \Omega^1_K \xrightarrow{C - W^*} \Omega^1_K \to 0$$

of group schemes over $K$. Since $H^1(K, G_m) = 0$, applying $R \Gamma(K, \cdot)$ gives the exact sequence in the top row of the following diagram:

$$0 \to K^\times \xrightarrow{\text{incl}} K'^\times \xrightarrow{\text{dlog}} \Gamma(K', \Omega^1_{K'}) \xrightarrow{C - W^*} \Gamma(K, \Omega^1_K)$$

$$\downarrow v_{K'} \quad \downarrow \text{Res} \quad \downarrow \text{Res}$$

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{1 - F} G_a \xrightarrow{\text{Res}} G_a \to 0.$$ 

Here $v_{K'}$ is the normalized valuation for $K'$ and $\Gamma(K', \Omega^1_{K'}) \to G_a$ is the residue map for $K'$. The result follows if we check that the squares are commutative. The left two squares are easily seen to be commutative. For the right square, let $T$ be a prime element of $K$ and write

$$F_* \Omega^1_K = \bigoplus_{i=0}^{p-1} G_a T^{1/p} T^{1/p} / T^{1/p} \cong G_a^p$$

as group schemes over $K$. The morphism $C - W^*$ sends an element $\sum f_i T^{i/p} T^{1/p} / T^{1/p}$ to

$$f_0 dT / T - \sum (f_i^p T^i) dT / T.$$ 

Its residue is $a_0 - a_0^p$, where $a_0$ is the constant term of $f_0$, or the residue of $f_0$. This proves the commutativity. \hfill \square

We also call the morphism

$$R \Gamma_x(\mathcal{O}_K, U(\mu_p)) \to \mathbb{Z}/p\mathbb{Z}[-1] \to \mathbb{Z}$$

thus obtained the additive trace morphism.
Proposition (5.2.2.9). Let \( N \) be a finite flat group scheme of height 1 over \( \mathcal{O}_K \). Then the morphisms
\[
R\Gamma_x(\mathcal{O}_K, V(N^{CD})) \to R\text{Hom}_{k^{\text{proet}}} (R\Gamma(\mathcal{O}_K, U(N)), R\Gamma_x(\mathcal{O}_K, U(\mu_p))) \\
\to R\Gamma(\mathcal{O}_K, U(N))^{\text{SD}}
\]
and
\[
R\Gamma(\mathcal{O}_K, V(N^{CD})) \to R\text{Hom}_{k^{\text{proet}}} (R\Gamma_x(\mathcal{O}_K, U(N)), R\Gamma_x(\mathcal{O}_K, U(\mu_p))) \\
\to R\Gamma(\mathcal{O}_K, U(N))^{\text{SD}}
\]
in \( D(k^{\text{proet}}) \) defined by the pairing \( V(N^{CD}) \times U(N) \to U(\mu_p) \), the functoriality of \( R\Gamma_x \) and the additive trace morphism are isomorphisms.

Proof. We only treat the first morphism as the second morphism can be treated similarly. We have a morphism between distinguished triangles
\[
R\Gamma_x(\mathcal{O}_K, V(N^{CD})) \longrightarrow R\Gamma_x(\mathcal{O}_K, V^0(N^{CD})) \longrightarrow R\Gamma_x(\mathcal{O}_K, V^1(N^{CD})) \\
\downarrow \quad \downarrow \quad \downarrow \\
R\Gamma(\mathcal{O}_K, U(N))^{\text{SD}} \longrightarrow R\Gamma(\mathcal{O}_K, U^0(N))^{\text{SD}}[1] \longrightarrow R\Gamma(\mathcal{O}_K, U^1(N))^{\text{SD}}[1]
\]
The right two morphisms are isomorphisms by (5.2.2.6). So is the left one. \( \square \)

Proposition (5.2.2.10). Let \( N \) be a finite flat group scheme of height 1 over \( \mathcal{O}_K \).

The isomorphisms
\[
R\Gamma_x(\mathcal{O}_K, V(N^{CD})) \to R\Gamma(\mathcal{O}_K, U(N))^{\text{SD}}, \quad R\Gamma(\mathcal{O}_K, V(N^{CD})) \to R\Gamma(\mathcal{O}_K, U(N))^{\text{SD}}
\]
in the previous proposition and the morphisms
\[
R\Gamma_x(\mathcal{O}_K, N^{CD}) \to R\Gamma(\mathcal{O}_K, N)^{\text{SD}}[-1], \quad R\Gamma(\mathcal{O}_K, N^{CD}) \to R\Gamma(\mathcal{O}_K, N)^{\text{SD}}[-1]
\]
given in (5.2.12) are compatible under the identifications (5.2.2.6) and (5.2.2.8).

Proof. Applying the functoriality of \( R\hat{\Gamma}_x \) to the diagram (5.2.2.7), we have commutative diagrams
\[
R\Gamma_x(\mathcal{O}_K, N^{CD}) \longrightarrow R\text{Hom}_k (R\Gamma(\mathcal{O}_K, N), R\Gamma_x(\mathcal{O}_K, \mu_p)) \\
\| \quad \quad \| \\
R\Gamma_x(\mathcal{O}_K, V(N^{CD})) \longrightarrow R\text{Hom}_k (R\Gamma(\mathcal{O}_K, U(N)), R\Gamma_x(\mathcal{O}_K, U(\mu_p)))
\]
and
\[
R\Gamma(\mathcal{O}_K, N^{CD}) \longrightarrow R\text{Hom}_k (R\Gamma_x(\mathcal{O}_K, N), R\Gamma_x(\mathcal{O}_K, \mu_p)) \\
\| \quad \quad \| \\
R\Gamma(\mathcal{O}_K, V(N^{CD})) \longrightarrow R\text{Hom}_k (R\Gamma_x(\mathcal{O}_K, U(N)), R\Gamma_x(\mathcal{O}_K, U(\mu_p)))
\]
in \( D(k^{\text{proet}}) \). The morphism in (5.2.12) is given by applying the morphism
\[
R\Gamma_x(\mathcal{O}_K, \mu_p)[1] \to \mathbb{Z}/p\mathbb{Z}[-1].
\]
The morphism in (5.2.2.9) is given by applying the morphism
\[
R\Gamma_x(\mathcal{O}_K, U(\mu_p)) \to \mathbb{Z}/p\mathbb{Z}[-1].
\]
As they are compatible by (5.2.2.8), the result follows. \( \square \)
This finishes the proof of [5.2.1.2] and [5.2.2.1].

5.2.3. Bertapelle’s isomorphism.

**Proposition (5.2.3.1).** Assume that $K$ has equal characteristic and $k$ is algebraically closed. Let $A$ be a semistable abelian variety over $K$. Suppose that we take the morphism constructed in [5.2.1.1] as the definition of Bertapelle’s isomorphism that Bertapelle used in [Ber03, Thm. 1]. Then the resulting isomorphism

$$\psi_A : H^1(K, A^\vee) \cong \text{Ext}^1_{\text{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z})$$

Bertapelle constructed in [Ber03, Thm. 2] coincides with our morphism $\theta^+_{A, k}$.

We prove this below. For the moment, $k$ is assumed to be a general perfect field. First we need a preparation about the finite flat site of $\mathcal{O}_K$.

**Proposition (5.2.3.2).** Let $k' \in k^{\text{indrat}}$. The local ring (for the Zariski topology) of $\mathcal{O}_K(k')$ at any maximal ideal is Henselian.

**Proof.** By [3.1.1.1] (m), a maximal ideal $\mathfrak{n}$ of $\mathcal{O}_K(k')$ is of the form $\mathfrak{p}_K(k') + \mathcal{O}_K(\mathfrak{m})$ for some $\mathfrak{m} \in \text{Spec } k'$, with residue field $k'/\mathfrak{m}$. By [3.1.1.1 (3)], the local ring $\mathcal{O}_K(k')_{\mathfrak{n}}$ is given by the filtered direct limit of $\mathcal{O}_K(k'[1/e]) = \mathcal{O}_K(k'/(1 - e))$ for the idempotents $e \in k' \setminus \mathfrak{m}$.

Let $f_1, \ldots, f_n \in \mathcal{O}_K(k')[x_1, \ldots, x_n]$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_K(k')^n$. Suppose that the images $\bar{f}_i(\alpha) \in k'/\mathfrak{m}$ are zero for all $i$ and $\det(\partial f_i/\partial x_j)(\alpha) \neq 0$. We need to show that the polynomial system $(f_1, \ldots, f_n)$ has a root in $\mathcal{O}_K(k')_{\mathfrak{n}}$ whose reduction is $\bar{\alpha} \in k'/\mathfrak{m}$. It is enough to show the existence of a root in $\mathcal{O}_K(k')$ after replacing $k'$ by $k'[1/e]$ for an idempotent $e \in k' \setminus \mathfrak{m}$. Since the element $\det(\partial f_i/\partial x_j)(\alpha) \in \mathcal{O}_K(k')$ has non-zero image in $k'/\mathfrak{m}$, [3.1.1 (c)] shows that $\det(\partial f_i/\partial x_j)(\alpha) \in \mathcal{U}_K(k')$ after replacing $k'$. Since $\mathcal{O}_K(k')$ is $\mathcal{p}_K(k')$-adically complete, Hensel’s lemma shows the existence of a desired root.

Recall that a morphism of schemes is finite locally free if and only if it is finite flat locally of finite presentation ([Bor98, Chap. II, §5, No. 2, Cor. 2 to Thm. 1], [Gro64, Prop. 1.4.7]). For a commutative ring $S$, we denote by $\text{Spec } S_{\text{ffl}}$ the finite flat site of $S$, namely the category of finite locally free $S$-algebras where a covering of an object $S'$ is a finite family $\{S'_i\}$ of finite locally free $S'$-algebras such that $\prod S'_i$ is faithfully flat over $S'$.

**Proposition (5.2.3.3).** Let $k' \in k^{\text{indrat}}$ and $S$ a finite $\mathcal{O}_K(k')$-algebra. Then any fppf covering of $\text{Spec } S$ can be refined by a finite locally free covering. That is, for any faithfully flat $S$-algebra $S'$ of finite presentation, there exist a faithfully flat finite locally free $S$-algebra $S''$ and an $S$-algebra homomorphism $S' \to S''$. In particular, the continuous map $g : \text{Spec } S_{\text{fppf}} \to \text{Spec } S_{\text{fl}}$ of sites defined by the identity induces an exact pushforward functor $g_* : \text{Set}(S_{\text{fppf}}) \to \text{Set}(S_{\text{fl}})$.

**Proof.** Refine $\text{Spec } S'$ by a quasi-finite flat covering $\text{Spec } S''$ of finite presentation ([Gro64, 17.16.2]). Since $S$ is finite, the previous proposition shows that the local ring of $S$ at any maximal ideal of $S$ is Henselian. Hence $\text{Spec } S''$ can be refined by a finite locally free covering.

The following is stated in [Suz13, Rmk. 2.7.6 (3)].
Proposition (5.2.3.4). Let $j$: Spec $K_{\text{fppf}}/k^{\text{instr}} \hookrightarrow \text{Spec} O_K$, be the morphism induced by the open immersion $\text{Spec} K \hookrightarrow \text{Spec} O_K$. Denote by $j_!$ the zero extension functor by $j$ (AGV72 Exp. III, 5.3, 3]). Then $R\Gamma(O_K, j_! A) = 0$ for any $A \in \text{Ab}(K_{\text{fppf}}/k^{\text{instr}})$.

Proof. It is enough to show that $R\Gamma(O_K(k'), j_! A) = 0$ for any $k' \in k^{\text{instr}}$. Let $g$: Spec $O_K(k')_{\text{fppf}} \rightarrow \text{Spec} O_K(k')_{\text{fppf}}$ be the continuous map defined by the identity. We have $R\Gamma(O_K(k'), j_! A) = R\Gamma(O_K(k')_{\text{fppf}}, g_! j_! A)$ by the previous proposition. The $j$ here should be understood as the open immersion $\text{Spec} K(k') \hookrightarrow \text{Spec} O_K(k')$. Let $S$ be a non-zero finite locally free $O_K(k')$-algebra of (locally constant) rank $r \geq 1$. Then the norm $N_{S/O_K(k')}(\pi) = \pi^r$ does not belong to $U_K(k')$. Hence the homomorphism $O_K(k') \rightarrow S$ cannot factor through $K(k')$. Therefore $g_! j_! = 0$ by the construction of the zero-extension functor $j_!$.

Bertapelle extended Bester’s group $\mathcal{F}(N)$ for quasi-finite flat separated group schemes $N$ over $O_K$. For such a group $N$, we denote its finite part by $N^f$. By [Ber03 §3.1 Example], we have $\mathcal{F}(N) = \mathcal{F}(N^f)$.

Proposition (5.2.3.5). Let $N$ be a quasi-finite flat separated group scheme over $O_K$ with finite part $N^f$. Then we have $R\Gamma(O_K, N) = R\Gamma(O_K, N^f) \in \text{D} \text{PAg}/k$ and it is $P$-acyclic. In particular, there exists a canonical isomorphism $L\pi_0 R\Gamma(O_K, N) \cong \mathcal{F}(N)$ compatible with the isomorphism $L\pi_0 R\Gamma(O_K, N^f) \cong \mathcal{F}(N^f)$ of (5.2.1.3).

Proof. Let $Y = N \setminus N^f$ be the part of $N$ finite over $K$. Let $k' \in k^{\text{instr}}$ and $S$ a finite locally free $O_K(k')$-algebra. If there is an $O_K(k')$-scheme morphism $\text{Spec} S \rightarrow Y$, then $S$ must be a $K(k')$-algebra, which happens only when $S = 0$ by the same reasoning as the proof of the previous proposition. Hence $\Gamma(S, N^f) = \Gamma(S, N)$. Therefore $N^f = N$ as sheaves on $\text{Spec} O_K(k')$. Hence (5.2.3.3) implies that $R\Gamma(O_K(k'), N^f) = R\Gamma(O_K(k'), N)$, and we have $R\Gamma(O_K, N^f) = R\Gamma(O_K, N)$.

Now we recall the definition of Bertapelle’s isomorphism in the case the abelian variety $A$ has semistable reduction. Assume that $k$ is algebraically closed. Let $A$ be the Néron model of $A$ and $A_0$ the maximal open subgroup scheme with connected special fiber. Let $n \geq 1$ be an integer that kills $\pi_0(A_0)$. Let $A[n]^f$ be the finite part of the $n$-torsion part $A[n]$ and $(A[n]^f)_K = A[n]^f \times_{O_K} K$. The inclusion $(A[n]^f)_K \hookrightarrow A[n]$ defines a surjection $A[n]^{\text{CD}} \rightarrow (A[n]^f)_K^{\text{CD}}$. By [Ber03 Lem. 14], this canonically extends to a morphism

$$A[n]^f \rightarrow (A[n]^f)^{\text{CD}}$$

of group schemes over $O_K$, which induces an isomorphism

$$H_2^2(O_K, A[n]^f) \cong H_2^2(O_K, (A[n]^f)^{\text{CD}}).$$

We denote by $\theta_{A[n]^f}$ the isomorphism

$$H_2^2(O_K, (A[n]^f)^{\text{CD}}) \cong \mathcal{F}(A[n]^f)^{\text{PD}}$$

in (5.2.1.4) for $N = A[n]^f$. Combining the above two isomorphisms and $\mathcal{F}(A[n]^f) \cong \mathcal{F}(A[n])$, we have an isomorphism

$$H_2^2(O_K, A[n]) \cong \mathcal{F}(A[n])^{\text{PD}}.$$

We denote this isomorphism by $\psi_{A[n]}$. Note that

$$H_2^2(O_K, A[n]_0) = H_2^2(O_K, A[n]) = H^1(K, A[n]).$$
Hence the exact sequence $0 \to \mathcal{A}^\vee[n] \to \mathcal{A}^\vee \xrightarrow{\theta} \mathcal{A}^\vee_0 \to 0$ induces a surjection
\[ H^2_\varepsilon(\mathcal{O}_K, \mathcal{A}^\vee[n]) \to H^1(K, \mathcal{A}^\vee)[n]. \]
By [Ber03, §3.1, Examples], the map $\mathcal{F}(\mathcal{A}_0[n]) \to \mathcal{F}(\mathcal{A}[n])$ is injective, and we have
\[
\mathcal{F}(\mathcal{A}_0[n]) = \text{Coker}(\pi_1\Gamma(\mathcal{O}_K, \mathcal{A}_0) \xrightarrow{\theta} \pi_1\Gamma(\mathcal{O}_K, \mathcal{A}_0))
= \text{Coker}(\pi_1\Gamma(\mathcal{O}_K, \mathcal{A}) \xrightarrow{\theta} \pi_1\Gamma(\mathcal{O}_K, \mathcal{A}))
= \text{Coker}(\pi_1\Gamma(K,A)) \xrightarrow{\theta} \pi_1\Gamma(K,A)).
\]
By the proof of [Ber03, Thm. 2], the isomorphism $\psi_A[n]: H^2_\varepsilon(\mathcal{O}_K, \mathcal{A}^\vee[n]) \isom \mathcal{F}(\mathcal{A}[n])^{PD}$ induces an isomorphism
\[ H^1(K, \mathcal{A}^\vee)[n] \isom (\pi_1\Gamma(K,A))^{PD}[n] \]
on the quotients. The resulting isomorphism
\[ H^1(K, \mathcal{A}^\vee) \isom (\pi_1\Gamma(K,A))^{PD} \]
is the definition of Bertapelle's isomorphism in the semistable case. We denote it by $\psi_A$.

Hence, to prove (5.2.3.1), it is enough to show the following.

**Proposition (5.2.3.6).** Let $A$ and $n$ be as above. There exists a diagram (commutativity to be mentioned below)
\[
\begin{array}{cccc}
H^1(K, \mathcal{A}^\vee[n]) & \longrightarrow & H^2_\varepsilon(\mathcal{O}_K, \mathcal{A}^\vee[n]) & \longrightarrow & H^1(K, \mathcal{A}^\vee)[n] \\
\downarrow^{\theta^1_A[n]} & & \downarrow^{\psi_A[n]} & & \downarrow^{\theta^1_A \text{ or } \psi_A} \\
H^0(\mathcal{F}(A[n])^{PD}) & \longrightarrow & \mathcal{F}(A[n])^{PD} & \longrightarrow & (\pi_1\Gamma(K,A))^{PD}[n].
\end{array}
\]
Here we define $\mathcal{F}(A[n])^{PD} = R\text{Hom}_A(\mathcal{F}(K,A[n]), \mathbb{Q}/\mathbb{Z})$.

The left vertical morphism $\theta^1_A[n]$ is the isomorphism induced on $H^1$ by the isomorphism given in (5.2.2.1). We denoted $\theta^1_A = \theta^1_A(k)$. The horizontal homomorphisms in the left square are to be constructed below, and those in the right square are already mentioned. The upper horizontal homomorphisms are surjective.

The left square is commutative. If we use Bertapelle’s isomorphism $\psi_A$ for the right vertical arrow, then the right square is commutative (by definition). If we use our morphism $\theta^1_A$, then the total square (omitting the middle $\psi_A[n]$) is commutative.

As a consequence, we have $\theta^1_A = \psi_A$ and the right square is commutative.

**Proof.** The upper horizontal homomorphism in the left square is defined as the coboundary map of the localization sequence for $\mathcal{A}^\vee[n]$. It is surjective since $R\Gamma(\mathcal{O}_K, \mathcal{A}^\vee[n]) = R\Gamma(\mathcal{O}_K, \mathcal{A}^\vee[n])^f$ by (5.2.3.2) and this is concentrated in degrees 0 and 1. The composite of the upper two horizontal homomorphisms is induced by the inclusion $A^\vee[n] \inj A^\vee$. By
Applying the morphism of functoriality of \( R \) to mixed characteristic. Assume that Proposition (5.3.1).

This comes from the \( H(5.2.3.5) \) and by the same calculation as the second paragraph after \( (5.2.1.2) \), we have

\[
\mathcal{F}(A[n])^{PD} = R\text{Hom}_{\text{proét}}(L\pi_0 R\Gamma(O_K, A[n]), \mathbb{Q}/\mathbb{Z}) = R\text{Hom}_A(R\Gamma(O_K, A[n]), \mathbb{Q}/\mathbb{Z}).
\]

The lower horizontal homomorphism in the left square is defined by dualizing \( R\Gamma(O_K, A[n]) \to R\Gamma(K, A[n]) \). The left square is decomposed into

\[
\begin{array}{cccc}
H^1(K, A^\vee[n]) & \longrightarrow & H^1(K, (A[n])^{CD}) & \longrightarrow & H^2_\ast(O_K, (A[n])^{CD}) \\
\downarrow \theta_{A[n]} & & \downarrow \theta_{(A[n])^{CD}} & & \downarrow \theta_{(A[n])^{CD}} \\
\mathcal{F}(A[n])^{PD} & \longrightarrow & \mathcal{F}((A[n])^{CD})^{PD} & \longrightarrow & \mathcal{F}(A[n])^{PD}.
\end{array}
\]

The commutativity of the left square is the functoriality of the isomorphisms \( \theta_A \) for finite flat group schemes \( N \) over \( K \), which is easy to check. That of the right square follows from \( (5.2.2.2) \).

We show the commutativity of the total square with \( \theta_A^{-1} \) used for the right vertical arrow. With the identification \( \mathcal{F}(A_0[n])^{PD} = \pi_1 \Gamma(K, A)^{PD}[n] \) used earlier, the lower two horizontal homomorphisms can be written as the \( H^0 \) of the Pontryagin dual of the morphisms

\[
L\pi_0 R\Gamma(O_K, A_0[n]) \to L\pi_0 R\Gamma(O_K, A[n]) \to L\pi_0 R\Gamma(K, A[n]).
\]

The \( H^0 \) of the composite of these morphisms is

\[
\text{Coker}(\pi_1 \Gamma(K, A) \to \pi_1 \Gamma(K, A)) \to H^0 L\pi_0 R\Gamma(K, A[n]).
\]

This comes from the \( H^0 \) of the distinguished triangle

\[
L\pi_0 R\Gamma(K, A)[-1] \to L\pi_0 R\Gamma(K, A)[-1] \to L\pi_0 R\Gamma(K, A[n]).
\]

Therefore it is enough to construct a morphism of distinguished triangles

\[
\begin{array}{ccc}
R\Gamma(K, A^\vee[n]) & \longrightarrow & R\Gamma(K, A^\vee) \\
\downarrow & & \downarrow \\
R\Gamma(K, A[n])^{SD} & \longrightarrow & R\Gamma(K, A)^{SD}[1].
\end{array}
\]

(The left square gives the total square in the statement.) For this, note that we have a morphism of triangles from the short exact sequence \( 0 \to A^\vee[n] \to A^\vee \to A^\vee \to 0 \) to

\[
R\text{Hom}_K(A[n], G_m) \to R\text{Hom}_K(A, G_m)[1] \to R\text{Hom}_K(A, G_m)[1].
\]

Applying the morphism of functoriality of \( R\Gamma \) to it and using the trace morphism \( R\Gamma(K, G_m) \to \mathbb{Z} \), we get the desired morphism of the distinguished triangles. \( \square \)

This completes the proof of (5.2.3.1) and hence (5.2.1).

5.3. Bégueri’s isomorphism.

**Proposition (5.3.1).** Assume that \( k \) is algebraically closed. The morphism

\[
\theta_A^{-1}(k): H^1(K, A^\vee) \to \text{Ext}^1_{\text{proét}}(\Gamma(A), \mathbb{Q}/\mathbb{Z})
\]

of (5.2.1) coincides with Bégueri’s isomorphism \([\text{Bégu81}]\) Thm. 8.3.6] when \( K \) has mixed characteristic.
Proposition (6.1). Let available literature seems to be either less general or more topological (in the sense of Thm. 8.3.6). This completes the comparison.

This homomorphism fits in the following commutative diagram:

\[
\begin{array}{ccc}
R\Hom_K(\mathbb{Z}/n\mathbb{Z}, A^\vee) & \longrightarrow & R\Hom_K(\mathbb{Z}/n\mathbb{Z}, G_m) \\
\downarrow & & \downarrow \\
R\Gamma(A^\vee) & \longrightarrow & R\Hom_k(\mathbb{Z}/n\mathbb{Z}, G_m)
\end{array}
\]

in \(D(Ab)\), where the last morphism comes from the Kummer sequence and the trace morphism \(R\Gamma(G_m) = K^\times \rightarrow \mathbb{Z}\). Using the sequence \(0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0\), this and our duality morphism fit in the following commutative diagram:

\[
\begin{array}{ccc}
R\Hom_K(\mathbb{Z}/n\mathbb{Z}, A^\vee) & \longrightarrow & R\Hom_k(\mathbb{Z}/n\mathbb{Z}, G_m) \\
\downarrow & & \downarrow \\
R\Gamma(A^\vee) & \longrightarrow & R\Hom_k(\mathbb{Z}/n\mathbb{Z}, G_m)
\end{array}
\]

By construction, the top horizontal morphism in degree 1 is given as follows. First, for \(B, C \in \text{Ab}(K_{\text{proet}}/k_{\text{et}})\), the sheaf \(\text{Hom}_K(B, C) := \Gamma(\text{Hom}_K(B, C))\) on Spec \(k_{\text{et}}\) is the pro-étale sheafification of the étale sheaf \(k' \rightarrow \text{Hom}_K(k'(k'), k'(B, C))\), where \(\text{Hom}_K(k'(k'), k')\) is the Hom functor for the localization of \(\text{Spec} K_{\text{proet}}/k_{\text{et}}\) at the object \((K(k'), k')\). For any \(n\), the \(n\)-th cohomology \(\text{Ext}^n_K(B, C)\) of the derived functor \(R\text{Hom}_K(B, C) = R\Gamma(\text{Hom}_K(B, C))\) is the pro-étale sheafification of the presheaf \(k' \rightarrow \text{Ext}^n_k(k'(k'), k'(B, C))\). Now let \(0 \rightarrow A^\vee \rightarrow X \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0\) be an extension over \(K\). The long exact sequence for \(R\text{Hom}_K\) gives

\[
0 \rightarrow \text{Ext}_K(\mathbb{Z}/n\mathbb{Z}, G_m) \rightarrow \text{Ext}_K(X, G_m) \rightarrow \text{Ext}_K(A^\vee, G_m) \rightarrow 0,
\]

or

\[
0 \rightarrow H^1(\mu_n) \rightarrow \text{Ext}_K^1(X, G_m) \rightarrow \Gamma(A) \rightarrow 0,
\]

hence an element of \(\text{Ext}_K^1(\text{Hom}_K(\Gamma(A), H^1(\mu_n)))\), hence an element of \(\text{Ext}_k^1(\text{Hom}_K(\Gamma(A), \mathbb{Z}/n\mathbb{Z}))\).

This homomorphism fits in the following commutative diagram:

\[
\begin{array}{ccc}
\text{Ext}_K(\mathbb{Z}/n\mathbb{Z}, A^\vee) & \longrightarrow & \text{Ext}_k^1(\text{Hom}_K(\Gamma(A), \mathbb{Z}/n\mathbb{Z})) \\
\downarrow & & \downarrow \\
H^1(A^\vee) & \longrightarrow & \text{Ext}_k^1(\text{Hom}_K(\Gamma(A), \mathbb{Q}/\mathbb{Z}))
\end{array}
\]

where the bottom arrow is our homomorphism \(\theta_A^1(k)\). The description above shows that the top arrow is the same as Bégueri’s homomorphism ([Bégu81, Lem. 8.2.2, Thm. 8.3.6]). This completes the comparison. \(\square\)

6. Galois descent

Proposition (6.1). Let \(A\) be an abelian variety over \(K\) and \(L/K\) a finite Galois extension. If \(A\) is for \(A\) over \(L\), then so is for \(A\) over \(K\).

We prove this below. We need notation and lemmas on group (co)homology. The basic reference is [Ser79, VII, VIII, IX]. We need Tate cohomology in the setting of derived categories of sheaves of \(G\)-modules on sites. For the purpose of this section, available literature seems to be either less general or more topological (in the sense
of equivariant stable homotopy theory) than what is needed here. Hence we include some basics for the convenience of the reader.

Let $G$ be a finite (abstract) group and $S$ a site. We denote the category of (left) $G$-modules (or $\mathbb{Z}[G]$-modules) by $G\text{-Mod}$, so that $G\text{-Mod}(S)$ is the (abelian) category of sheaves of $G$-modules on $S$. For a complex $M \in D(G\text{-Mod}(S))$, we define its group cohomology, group homology by

$$R\Gamma(G, M) = R\text{Hom}_{G\text{-Mod}(S)}(\mathbb{Z}, M), \quad L\Delta(G, M) = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M,$$

respectively. They are objects of $D(S) = D(\text{Ab}(S))$. Let $C(G)$ be the standard resolution

$$\cdots \rightarrow \mathbb{Z}[G^3] \rightarrow \mathbb{Z}[G^2] \rightarrow \mathbb{Z}[G]$$

of the trivial $G$-module $\mathbb{Z}$ (Serre [Ser79, VII, §3]). This is viewed as a complex $\{C(G)^i\}_{i \leq 0}$ concentrated in non-positive degrees (in cohomological grading, as for all the complexes in this paper). Let $C^\vee(G) = \text{Hom}_{\text{Ab}}(C(G), \mathbb{Z})$. Define the standard complete resolution $\hat{C}(G)$ of $\mathbb{Z}$ [Mil06, I, §0, “Tate (modified) cohomology groups”] to be the complex

$$\cdots \rightarrow \mathbb{Z}[G^2] \rightarrow \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \rightarrow \mathbb{Z}[G^2] \rightarrow \cdots,$$

where the map $N$ from the degree 0 term to the degree 1 term is the norm map $\sum_{\sigma \in G} \sigma$, and the non-positive degree part of the complex is $C(G)$ and the positive degree part $C^\vee(G)[-1]$. The morphism of complexes $C(G) \rightarrow C^\vee(G)$ induced by $N : \mathbb{Z}[G] \rightarrow \mathbb{Z}[G]$ in degree zero is also denoted by $N$. Then $\hat{C}(G)$ is the mapping fiber of $N : C(G) \rightarrow C^\vee(G)$. For a bounded complex $M$ of sheaves of $G$-modules on $S$, we define the Tate cohomology as the sheaf-$\text{Hom}$ (total) complex

$$R\hat{\Gamma}(G, M) = \text{Hom}_{G\text{-Mod}(S)}(\hat{C}(G), M)$$

viewed as an object of $D(S)$. Note that $R\text{Hom}_{G\text{-Mod}(S)}(\hat{C}(G), M) = 0$ since $\hat{C}(G)$ is an exact complex.

We relate the three functors $R\Gamma, L\Delta$ and $R\hat{\Gamma}$ and show that $R\hat{\Gamma}$ factors through the derived category. We have $R\Gamma(G, M) = R\text{Hom}_{G\text{-Mod}(S)}(C(G), M)$. Consider the hyperext spectral sequence

$$E_1^{ij} = \prod_{i' + i'' = i} \text{Ext}^j_{G\text{-Mod}(S)}(C(G)^{i'}, M^{i''}) \Rightarrow H^{i+j}R\text{Hom}_{G\text{-Mod}(S)}(C(G), M),$$

constructed in the same way as [ML63, Thm. 12.2]. This is convergent since $C(G)$ is bounded above and $M$ bounded below. Since each term of $C(G)$ is finite free over $\mathbb{Z}[G]$, we have $E_1^{ij} = 0$ for any $i, j$ with $j \geq 1$. Hence $R\text{Hom}_{G\text{-Mod}(S)}(C(G), M)$ is represented by the total complex $\text{Hom}_{G\text{-Mod}(S)}(C(G), M)$. Thus

$$R\Gamma(G, M) = \text{Hom}_{G\text{-Mod}(S)}(C(G), M).$$

\footnote{Using $\Delta$, the Greek letter next to $\Gamma$, to denote homology is non-standard. There seems to be no widely used notation for homology in derived categories that is parallel to cohomology $R\Gamma$. Perhaps $\Lambda$ instead, in accordance with the definition of the homology $X \mapsto E \wedge X$ of a spectrum $E$.}
Similarly we have $L\Delta(G, M) = C(G) \otimes \mathbb{Z}[G] M$. The $n$-th term of $C(G) \otimes \mathbb{Z}[G] M$ for each $n$ is
\[
\bigoplus_{i+j=n} C(G)^i \otimes \mathbb{Z}[G] M^j = \bigoplus_{i+j=n} \text{Hom}_{G-\text{Mod}(S)}(C^{\vee}(G)^{-i}, M^j) = \prod_{i+j=n} \text{Hom}_{G-\text{Mod}(S)}(C^{\vee}(G)^{-i}, M^j),
\]
where the second equality comes from the property that $C^{\vee}(G)$ is bounded below and $M$ bounded above. The last term is the $n$-th term of $\text{Hom}_{G-\text{Mod}(S)}(C^{\vee}(G), M)$. Thus
\[
(6.4) \quad L\Delta(G, M) = \text{Hom}_{G-\text{Mod}(S)}(C^{\vee}(G), M).
\]
In particular, the norm map $N : C(G) \to C^{\vee}(G)$ induces a morphism $L\Delta(G, M) \to R\Gamma(G, M)$ in $D(S)$, which we denote by the same symbol $N$. Combining (6.2), (6.3), (6.4) and the mapping fiber distinguished triangle
\[
\check{\mathcal{C}}(G) \to C(G) \xrightarrow{N} C^{\vee}(G),
\]
we have a distinguished triangle
\[
(6.5) \quad L\Delta(G, M) \overset{N}{\longrightarrow} R\Gamma(G, M) \to R\check{\Gamma}(G, M).
\]
In particular, if $M$ is an exact complex, then $L\Delta(G, M)$, $R\Gamma(G, M)$ and hence $R\check{\Gamma}(G, M)$ are all zero in $D(S)$. Therefore the assignment $M \mapsto R\check{\Gamma}(G, M)$ defines a well-defined triangulated functor $D^b(G-\text{Mod}(S)) \to D(S)$ by [KS06, Prop. 10.3.3].

We need to know how dual of group cohomology and group cohomology of dual are related. Let $M \in D^b(G-\text{Mod}(S))$ and $P \in D(S)$. The complex $R\text{Hom}_S(M, P)$ can be viewed as an object of $D(G-\text{Mod}(S))$ by giving $M$ a right $G$-action by $mg := g^{-1}m$ and $P$ a trivial $G$-action. We have
\[
R\text{Hom}_{G-\text{Mod}(S)}(\mathbb{Z}, R\text{Hom}_S(M, P)) = R\text{Hom}_S(M \otimes \mathbb{Z}[G] \mathbb{Z}, P)
\]
\[
= R\text{Hom}_S(\mathbb{Z} \otimes \mathbb{Z}[G], M, P)
\]
by the derived tensor-hom adjunction [KS06, Rmk. 18.6.11] and interchanging the tensor factors. In our notation, this means
\[
(6.6) \quad R\Gamma(G, R\text{Hom}_S(M, P)) = R\text{Hom}_S(L\Delta(G, M), P)
\]
(cf. the universal coefficient theorem). Hence the triangle (6.5) induces a distinguished triangle
\[
(6.7) \quad R\text{Hom}_S(R\check{\Gamma}(G, M), P) \to R\text{Hom}_S(R\Gamma(G, M), P)
\]
\[
\longrightarrow R\Gamma(G, R\text{Hom}_S(M, P)).
\]

**Proposition (6.8).** Let $G$ be a finite group, $S$ a site and $M \in D^b(G-\text{Mod}(S))$. Assume that $G$ is cyclic and $R\check{\Gamma}(G, M)$ is bounded. Then we have $R\check{\Gamma}(G, M) = 0$, and the norm map gives an isomorphism $L\Delta(G, M) \xrightarrow{N} R\Gamma(G, M)$ between homology.

\[16\] If $M$ is unbounded, the above definition of $R\check{\Gamma}(G, M)$ does not factor through the derived category $D(G-\text{Mod}(S))$ and hence is not “correct”. In this case, we need to define $R\Gamma(G, M)$ to be the mapping cone of the norm map $C(G) \otimes \mathbb{Z}[G] I \to \text{Hom}_{G-\text{Mod}(S)}(C(G), I)$, where $M \sim I$ is a $K$-injective replacement in $G-\text{Mod}(S)$. Below we use bounded $M$ only.
and cohomology. In particular, the triangle \([6,7]\) for any \(P \in D(S)\) reduces to an isomorphism

\[
R\text{Hom}_S(R\Gamma(G, M), P) = R\Gamma(G, R\text{Hom}_S(M, P))
\]

in \(D(S)\).

**Proof.** Let \(\sigma\) be a generator of the cyclic group \(G\). Then \(\hat{\mathcal{C}}(G)\) is chain homotopic to the periodic complex \(\cdots \rightarrow \mathbb{Z}[G] \xrightarrow{\mathcal{N}} \mathbb{Z}[G] \xrightarrow{\mathcal{N}} \mathbb{Z}[G] \xrightarrow{\mathcal{N}} \cdots\) by [Ser79] VIII, Prop. 6] (see also [Bro82, VI, Prop. 3.3]). Therefore we have \(R\hat{\Gamma}(G, M) = R\Gamma(G, M)[2]\). On the other hand, the boundedness of \(M\) and \(R\Gamma(G, M)\) implies that \(R\hat{\Gamma}(G, M)\) is acyclic in large degrees. Therefore \(R\hat{\Gamma}(G, M) = 0\). Hence \(L\Delta(G, M) \sim R\Gamma(G, M)\).

\(\square\)

When \(S = \text{Spec} k_{\text{indrat}}^{\text{proet}}\), the last isomorphism in the above proposition for \(P = Z\) may be written as \(R\Gamma(G, M)_{\text{SD}} = R\Gamma(G, M^{\text{SD}})\).

Next we give a variant of the Hochschild-Serre spectral sequence. Let \(L\) be a totally ramified (for simplicity) finite Galois extension of \(K\) with Galois group \(G\). For any \(A \in \text{Ab}(K_{fppf}/k_{\text{indrat}}^{\text{et}})\) and \(k' \in k_{\text{indrat}}^{\text{et}}\), the \(G\)-action on \(L\) induces a \(G\)-action on \(\Gamma(L, k')\). Hence the functor \(\Gamma(L, \cdot)\) factors through \(\text{G-mod}(k_{\text{indrat}}^{\text{et}})\) and the functor \(R\Gamma(L, \cdot)\) factors through \(D(G, \text{G-mod}(k_{\text{indrat}}^{\text{et}}))\). An object of \(\text{Ab}(k_{\text{indrat}}^{\text{et}})\) can be viewed as an object of \(\text{G-mod}(k_{\text{indrat}}^{\text{et}})\) by putting the trivial \(G\)-action. The resulting functor \(D(k_{\text{indrat}}^{\text{et}}) \rightarrow D(G, \text{G-mod}(k_{\text{indrat}}^{\text{et}}))\) is left adjoint to the derived \(G\)-invariants \(R\Gamma(G, \cdot)\). For any \(A \in D(K_{fppf}/k_{\text{indrat}}^{\text{et}})\), the natural morphism \(R\Gamma(K, A) \rightarrow R\Gamma(G, A)\) in \(D(k_{\text{indrat}}^{\text{et}})\) factors through \(R\Gamma(G, \text{G-mod}(L, A))\) by adjunction. Similarly, the inclusion \(R\Gamma(K, A) \rightarrow R\Gamma(L, A)\) in \(D(k_{\text{indrat}}^{\text{et}})\) factors through \(R\Gamma(G, \text{G-mod}(L, A))\).

**Proposition (6.9).** Let \(L\) be a totally ramified finite Galois extension of \(K\) with Galois group \(G\). Let \(A \in D(K_{fppf}/k_{\text{indrat}}^{\text{et}})\). The morphism

\[
R\Gamma(G, R\Gamma(L, A)) \leftarrow R\Gamma(K, A)
\]

in \(D(k_{\text{indrat}}^{\text{et}})\) defined above is an isomorphism. The morphism

\[
R\Gamma(G, R\hat{\Gamma}(L, A)) \leftarrow R\hat{\Gamma}(K, A)
\]

in \(D(k_{\text{proet}}^{\text{et}})\) defined above is an isomorphism if \(A\) is bounded below.

**Proof.** We first treat the first morphism. For any \(k' \in k_{\text{indrat}}^{\text{et}}\), the right-hand side after applying \(R\Gamma(k_{\text{et}}^{\text{et}}, \cdot)\) is \(R\Gamma(K(k'), A)\), where we view \(A\) as an fppf sheaf on \(\text{Spec} K(k')\) as in [5.3.1]. The left-hand side after applying \(R\Gamma(k_{\text{et}}^{\text{et}}, \cdot)\) is

\[
R\Gamma\left(k_{\text{et}}^{\text{et}}, R\Gamma(G, R\Gamma(L, A))\right) = R\Gamma\left(G, R\Gamma(k_{\text{et}}^{\text{et}}, R\Gamma(L, A))\right) = R\Gamma(G, R\Gamma(L(k'), A)),
\]

where the \(R\Gamma(G, \cdot)\) in the second and third terms are the usual group cohomology. Hence the first morphism in the statement after applying \(R\Gamma(k_{\text{et}}^{\text{et}}, \cdot)\) is

\[
R\Gamma(G, R\Gamma(L(k'), A)) \leftarrow R\Gamma(K(k'), A)
\]

in \(D(\text{Ab})\). This is an isomorphism by the usual Hochschild-Serre spectral sequence [Mil80] III, Rmk. 2.21 (a)] since the morphism \(\text{Spec} L(k') \rightarrow \text{Spec} K(k')\) is a \(G\)-covering. This implies that the first morphism in the proposition is an isomorphism.
For the second, it is enough to show that \( R\Gamma(G, M) = R\Gamma(G, \tilde{M}) \) if \( M \in D^+(G, \text{Mod}(k_{\text{indrat}})) \), where \( \sim \) denotes pro-\( \acute{e} \text{tale} \) sheafication. By (6.3) (which does not require \( M \) to be bounded above), the \( n \)-th term of \( R\Gamma(G, M) \) is given by

\[
\prod_{i+j=n} \text{Hom}_{k_{\text{indrat}}}^\text{proet}(C(G)^{-i}, M^j) = \prod_{i+j=n} C^\vee(G)^i \otimes_{\mathbb{Z}[G]} M^j.
\]

The final product is a finite product since \( C^\vee(G) \) and \( M \) are bounded below. Hence the \( n \)-th term of \( R\Gamma(G, M) \) is

\[
\prod_{i+j=n} \text{Hom}_{k_{\text{indrat}}}^{\text{proet}}(C(G)^{-i}, \tilde{M}^j),
\]

which is the \( n \)-th term of \( R\Gamma(G, \tilde{M}) \).

\( \square \)

**Proof of (8.1).** Let \( A \) be an abelian variety over \( K \). Assume that (4.1.2) is true for \( A \) over a finite Galois extension \( L \) of \( K \):

\[
\theta_{A \times K, L} : R\Gamma(L, A^\vee)^{SDSD} \rightarrow R\Gamma(L, A)^{SD},
\]

in \( D(k_{\text{indrat}}) \), where \( k' \) is the residue field of \( L \). We want to deduce the corresponding statement for \( A \) over \( K \). If \( L/K \) is unramified, then the morphism above for \( A \) over \( K \) is nothing but the morphism for \( A \) over \( K \) restricted from \( D(k_{\text{proet}}) \) to \( D(k_{\text{indrat}}) \). Therefore the invertibility of these morphisms is equivalent. Hence we may assume that \( L/K \) is totally ramified. Since it is a solvable extension [Ser79, IV, Cor. 5 to Prop. 7], we may further assume that \( L/K \) is cyclic.

Let \( G = \text{Gal}(L/K) \). As we saw, the complexes \( R\Gamma(L, A) \) and \( R\Gamma(L, A^\vee) \) may be viewed as objects of \( D(G, \text{Mod}(k_{\text{proet}})) \). We show that the morphism \( \theta_{A \times K, L} \) is \( G \)-equivariant, i.e. a morphism in \( D(G, \text{Mod}(k_{\text{proet}})) \). From the construction of the (normalized) functorial valuation map \( v_L : L^\times \rightarrow \mathbb{Z} \) for \( L \) given in the paragraph before [Suz13, Prop. 2.4.4], we see that \( v_L \) is \( G \)-equivariant, where we put a trivial \( G \)-action on \( \mathbb{Z} \). Recall from (4.1) that the morphism \( \theta_{A \times K, L} \) is defined as the Serre dual of the composite

\[
R\Gamma(L, A) \rightarrow R\Gamma(L, R\text{Hom}_{k_{\text{proet}}}^\text{proet}(A^\vee, G_m))[1] \\
\rightarrow R\text{Hom}_{k_{\text{proet}}}^\text{proet}(R\Gamma(L, A^\vee), R\Gamma(L, G_m))[1] \\
\rightarrow R\text{Hom}_{k_{\text{proet}}}^\text{proet}(R\Gamma(L, A^\vee), \mathbb{Z})[1] = R\Gamma(L, A^\vee)^{SD}[1].
\]

All these morphisms, including the trace morphism \( R\Gamma(L, G_m) = L^\times \otimes_{\mathbb{Z}} \mathbb{Z} \) over \( L \), are \( G \)-equivariant.

We apply \( R\Gamma(G, \cdot) \) to \( \theta_{A \times K, L} \). We have

\[
R\Gamma(G, R\Gamma(L, A)) = R\Gamma(K, A)
\]

in \( D(k_{\text{indrat}}) \) by the Hochschild-Serre spectral sequence (6.3). The both complexes \( R\Gamma(L, A) \) and \( R\Gamma(K, A) \) are concentrated in degrees 0 and 1. Hence we can apply (6.8) for \( M = R\Gamma(L, A) \) and \( P = \mathbb{Z} \). This yields

\[
R\Gamma(G, R\Gamma(L, A)^{SD}) = R\Gamma(K, A)^{SD}.
\]

The both complexes \( R\Gamma(L, A)^{SD} \) and \( R\Gamma(K, A)^{SD} \) are concentrated in degrees 0 to 2. Applying the same proposition again, we have

\[
R\Gamma(G, R\Gamma(L, A)^{SDSD}) = R\Gamma(K, A)^{SDSD}.
\]
Now we apply $R\Gamma(G, \cdot)$ to the both sides of $\vartheta_{A_{X,K,L}}$ to get an isomorphism
\[ \vartheta_{A_{X,K,L}}^G : R\Gamma(K, A^\vee)^{\text{SDSD}} \cong R\Gamma(K, A)^{\text{SD}}. \]

We show that this isomorphism $\vartheta_{A_{X,K,L}}^G$ is equal to $\vartheta_A$. Consider the morphism
\[ (6.10) \quad R\Gamma(L, A^\vee) \otimes^L R\Gamma(L, A) \rightarrow R\Gamma(L, G_m)[1] \]
in $D(G\text{-Mod}(\text{indrat}^\proet))$. Apply $R\Gamma(G, \cdot)$. The same proof as [Suz12, Prop. 2.4.3 and the paragraph after] shows that there is a cup product pairing
\[ (6.11) \quad R\Gamma(G, R\Gamma(L, A^\vee)) \otimes^L R\Gamma(G, R\Gamma(L, A)) \rightarrow R\Gamma(G, R\Gamma(L, G_m))[1], \]
which can be identified with
\[ R\Gamma(K, A^\vee) \otimes^L R\Gamma(K, A) \rightarrow R\Gamma(K, G_m)[1]. \]

This and the trace morphism $R\Gamma(K, G_m)[1] = K^\times \xrightarrow{\theta_K} \mathbb{Z}[1]$ for $K$ leads to the morphism $\vartheta_A$ via the derived tensor-hom adjunction (used before). On the other hand, (6.10) gives morphisms
\[ R\Gamma(L, A^\vee) \rightarrow R\text{Hom}_{\text{indrat}^\proet}(R\Gamma(L, A), R\Gamma(L, G_m))[1] \]
\[ \quad \rightarrow R\text{Hom}_{\text{indrat}^\proet}(R\Gamma(L, A), L\Delta(G, R\Gamma(L, G_m)))[1] \]
in $D(G\text{-Mod}(\text{indrat}^\proet))$, where the second morphism is the natural morphism (i.e. $M \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$ given by $m \mapsto 1 \otimes m$) and $L\Delta(G, R\Gamma(L, G_m))$ here is given the trivial $G$-action. Applying $R\Gamma(G, \cdot)$ and using (6.6) for $P = L\Delta(G, R\Gamma(L, G_m))$, we have a morphism
\[ R\Gamma(G, R\Gamma(L, A^\vee)) \]
\[ \rightarrow R\text{Hom}_{\text{indrat}^\proet}(L\Delta(R\Gamma(L, A)), L\Delta(G, R\Gamma(L, G_m)))[1]. \]

By the derived tensor-hom adjunction, we have a morphism
\[ (6.12) \quad R\Gamma(G, R\Gamma(L, A^\vee)) \otimes^L L\Delta(G, R\Gamma(L, A)) \rightarrow L\Delta(G, R\Gamma(L, G_m))[1]. \]

This and the trace morphism $L\Delta(G, R\Gamma(L, G_m)) = L\Delta(G, L^X) \xrightarrow{\tau_L} \mathbb{Z}$ for $L$ leads to the morphism $\vartheta_{A_{X,K,L}}^G$. The two morphisms (6.11) and (6.12) and the two trace morphisms are compatible:
\[ R\Gamma(G, R\Gamma(L, A^\vee)) \otimes^L L\Delta(G, R\Gamma(L, A)) \quad \xrightarrow{\text{id} \otimes N} \quad L\Delta(G, R\Gamma(L, G_m))[1] \quad \xrightarrow{N} \quad \mathbb{Z} \]
\[ R\Gamma(G, R\Gamma(L, A^\vee)) \otimes^L R\Gamma(G, R\Gamma(L, A)) \quad \rightarrow \quad R\Gamma(G, R\Gamma(L, G_m))[1] \quad \rightarrow \quad \mathbb{Z}. \]

The commutativity of the left square is trivial. That of the right comes from the equality $v_L = v_K \circ N : L^X \rightarrow K^X \rightarrow \mathbb{Z}$ of the normalized valuations of $K$ and $L$, which is true as $L/K$ is totally ramified. This implies $\vartheta_A = \vartheta_{A_{X,K,L}}^G$. Hence the invertibility of $\vartheta_{A_{X,K,L}}^G$ implies that of $\vartheta_A$. \hfill \Box

Remark (6.13).

(a) The assumptions in (6.8) can be weakened to make it parallel to Nakayama’s theorem ([Ser79, IX, §5, Thm. 8]). The group $G$ may be any finite group. The complex $M$ may just be bounded, with no restrictions on the number of non-zero terms or their positions. The assumption for $R\Gamma(G, M)$ should now be that for any prime number $l$ and an $l$-Sylow subgroup $G_l$ of $G$, the
complex $R\hat{\Gamma}(G_1, M)$ be acyclic in two consecutive degrees. The conclusion is that $R\hat{\Gamma}(H, M) = 0$ for any subgroup $H$ of $G$. To prove this, first notice that the statement is reduced to the corresponding statement for a complex of usual $G$-modules (not sheaves of $G$-modules), by considering the Hom-complex $\text{Hom}_{k}\text{proet}(M, I)$ for any injective sheaf $I \in \text{Ab}(k\text{proet})$. Then the statement was proved by Koya [Koy93, Prop. 4]. Alternatively, we may assume that $M$ is concentrated in degree zero, since the bounded derived category of $\mathbb{Z}[G]$-modules modulo bounded perfect complexes and the stable category of $\mathbb{Z}[G]$-modules are equivalent ([BIK13 Prop. 8.2]). Then the statement is equivalent to Nakayama’s theorem.

(b) (5.2.1) is true without assuming $A$ to be semistable. We can see this by the following Galois descent argument. Assume that $k$ is algebraically closed and $A$ has semistable reduction over a finite Galois extension $L$ of $K$. Let $G = \text{Gal}(L/K)$. Let $A_L$ be the Néron model of $A \times_K L$. The group $A_L$ has a $G$-equivariant sheaf structure coming from the descent data of $A_L$. Therefore, for an integer $n \geq 1$ that kills $\pi_0((A_L)_L)$, the finite part $A_L[n]$ of the $n$-torsion part of $A_L$ also has a $G$-equivariant sheaf structure. Since the pairing $(A_L[n])^{\text{CD}} \times_{O_L} A_L[n] \rightarrow G_m$ is $G$-equivariant and the trace isomorphism $R\Gamma_x(O_L, G_m) = \mathbb{Z}[-1]$ is $G$-invariant, the isomorphism

$$R\Gamma_x(O_K, (A_L[n])^{\text{CD}}) \rightarrow R\Gamma(O_K, A_L[n])^{\text{SD}[-1]}$$

in (5.2.1.2) is an isomorphism in $D(G\text{-Mod}(k\text{proet}))$. From this, we can see that all the steps in the construction of Bertapelle’s isomorphism

$$\psi_{A_L}: H^1(L, A^\vee) \approx \text{Ext}_k^1(\Gamma(L, A), \mathbb{Q}/\mathbb{Z})$$

is $G$-equivariant. Hence $\psi_{A_L}$ itself is $G$-equivariant. For $n \geq 0$, let $L^{\otimes n}$ be the tensor product of $n$ copies of $L$ over $K$ and $\text{Res}_{L^{\otimes n}/K}$ be the Weil restriction functor. Then we have an exact sequence

$$0 \rightarrow A \rightarrow \text{Res}_{L/K} A \rightarrow \text{Res}_{L^{\otimes 2}/K} A \rightarrow \text{Res}_{L^{\otimes 3}/K} A \rightarrow \cdots$$

of abelian varieties over $K$ coming from the Čech complex (cf. the proof of [Mil80 III, Thm. 3.9]). The cokernel of any morphism in the sequence is an abelian variety. Dualizing, we have an exact sequence

$$\cdots \rightarrow \text{Res}_{L^{\otimes 3}/K} A^\vee \rightarrow \text{Res}_{L^{\otimes 2}/K} A^\vee \rightarrow \text{Res}_{L/K} A^\vee \rightarrow A^\vee \rightarrow 0$$

of abelian varieties over $K$. Since $H^2(K, \text{abelian variety}) = 0$, this induces an exact sequence

$$H^1(K, \text{Res}_{L^{\otimes 2}/K} A^\vee) \rightarrow H^1(K, \text{Res}_{L/K} A^\vee) \rightarrow H^1(K, A^\vee) \rightarrow 0,$$

or

$$\prod_{\sigma \in G} H^1(L, A^\vee) \underbrace{\text{Corres}} H^1(K, A^\vee) \rightarrow 0.$$

Hence the corestriction identifies $H^1(K, A^\vee)$ as the $G$-coinvariants of $H^1(L, A^\vee)$. Similarly, by (2.4.1) (ii), the inclusion $\Gamma(K, A) \rightarrow \Gamma(L, A)$ identifies $\text{Ext}_k^1(\Gamma(K, A), \mathbb{Q}/\mathbb{Z})$ as the $G$-coinvariants of $\text{Ext}_k^1(\Gamma(L, A), \mathbb{Q}/\mathbb{Z})$. The construction of Bertapelle’s isomorphism

$$\psi_A: H^1(K, A^\vee) \approx \text{Ext}_k^1(\Gamma(K, A), \mathbb{Q}/\mathbb{Z})$$

GROTHENDIECK’S PAIRING ON NÉRON COMPONENT GROUPS 81
given in [Ber03, Thm. 3] is the same as to take the $G$-coinvariants of $\psi_{A_L}$.
The same is true for our morphisms $\theta_{A_L}^+(k)$ and $\theta_{A_L}^-(k)$ by the functoriality of our constructions. Hence the equality $\theta_{A_L}^+(k) = \psi_{A_L}$ implies the equality $\theta_{A_L}^-(k) = \psi_{A_L}$.

7. END OF PROOF: GROTHENDIECK’S CONJECTURE

Now we prove (4.1.2) and Grothendieck’s conjecture by summarizing what we have done so far. Recall from Introduction that Šafarevič’s conjecture, proved by Bégueri [Bégu81, Thm. 8.3.6], Bester [Bes78, §2.7, Thm. 7.1] and Bertapelle [Ber03, Thm. 3], states that there exists a canonical isomorphism

$$H^1(A^\vee) \to \text{Ext}^1_\mathbb{Z}(\Gamma(A), \mathbb{Q}/\mathbb{Z})$$

for an abelian variety $A$ over $K$ when $k$ is algebraically closed. To be clear, in the proposition, the theorem and their proofs below, we mean by “Šafarevič’s conjecture for $A$” the statement that the morphism $\theta_{A_L}^1$ in (4.2.1) is an isomorphism.

**Proposition (7.1).** Let $A$ be an abelian variety over $K$. Then (4.1.2) for $A$ is equivalent to the conjunction of the following three statements:

- Grothendieck’s conjecture for $A$,
- Šafarevič’s conjecture for $A \times_K K(k')$ for any algebraically closed field $k' \in k^{\text{indrat}}$, and
- Šafarevič’s conjecture for $A^\vee \times_K K(k')$ for any algebraically closed field $k' \in k^{\text{indrat}}$.

**Proof.** Immediate from (4.2.1) and (5.1.1). □

**Theorem (7.2).** (4.1.2) and Grothendieck’s conjecture are both true for any abelian variety $A$ over $K$. That is, the morphism

$$\theta_A : R\Gamma(A^\vee)_{\text{SD}} \to R\Gamma(A)^{\text{SD}}$$

in $D(k)$ is an isomorphism, and Grothendieck’s pairing

$$\pi_0(A^\vee_L) \times \pi_0(A_L) \to \mathbb{Q}/\mathbb{Z}$$

is perfect.

**Proof.** By the semistable reduction theorem, there exists a finite Galois extension $L/K$ such that $A$ has semistable reduction over $L$. Then Grothendieck’s conjecture is true for $A$ over $L$ by Werner’s result [Wer97]. Also Šafarevič’s conjecture is true for $A$ over $L$ by (5.2.1) and (5.3.1) and the results of Bégueri, Bester and Bertapelle cited above. Hence (4.1.2) is true for $A$ over $L$ by the previous proposition. Hence it is true for $A$ over $K$ by (6.1). Hence Grothendieck’s conjecture is true for $A$ over $K$ by the previous proposition. □

This completes the proof of Theorems A, B and C.

**Remark (7.3).** Since $R\Gamma(A^\vee)$ is $P$-acyclic, we know that its $R\Gamma(k^{\text{proet}}, \cdot)$ is the usual cohomology complex $R\Gamma(A^\vee) = R\Gamma(K, A^\vee)$. Therefore the duality

$$\left[\lim_{\leftarrow} \Gamma(A^\vee) \to R\Gamma(A^\vee)\right] \overset{\sim}{\to} R\text{Hom}^{\text{indrat}}_{k^{\text{proet}}}(R\Gamma(A), \mathbb{Q}/\mathbb{Z}),$$

true in $D(k^{\text{indrat}})$, gives a statement about $R\Gamma(A^\vee)$ by taking the $R\Gamma(k^{\text{proet}}, \cdot)$ of the both sides, even if $k$ is not algebraically closed.
To make this explicit, let $(A^\vee_x)_{sAb}$ be the maximal semi-abelian quotient of $(A^\vee_x)_{0}$ and $(A^\vee_x)_{sAb}$ its universal covering. Then

$$
\lim_{\leftarrow n} \Gamma(A^\vee) = \lim_{\leftarrow n} (A^\vee_x)_{sAb} = (A^\vee_x)_{sAb}
$$

by (2.4.1) (d), where $\lim_{\leftarrow n}$ denotes the inverse limit for multiplication by $n \geq 1$. These groups do not change under replacing $\lim_{\leftarrow n}$ by $R\lim_{\leftarrow n}$ by (2.1.2) (d). We have

$$
R\Gamma(k_{proet}, R\lim_{\leftarrow n} (A^\vee_x)_{sAb}) = R\lim_{\leftarrow n} R\Gamma(k_{proet}, (A^\vee_x)_{sAb}) = R\lim_{\leftarrow n} R\Gamma(k_{et}, (A^\vee_x)_{sAb})
$$

by (2.4.1) (d) and (2.1.2) (g). Hence

$$
\Gamma(k_{proet}, \lim_{\leftarrow n} \Gamma(A^\vee)) = \Gamma(k_{proet}, (A^\vee_x)_{sAb}) = \Gamma(k_{et}, (A^\vee_x)_{sAb}).
$$

Denote these isomorphic objects by $Q_{k,A^\vee}$. It is a complex of $\mathbb{Q}$-vector spaces. We have

$$
R\Gamma(k_{proet}, R\text{Hom}_{k_{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z})) = R\text{Hom}_{k_{proet}}(\Gamma(A), \mathbb{Q}/\mathbb{Z})
$$

by (2.3.3). Therefore the duality isomorphism in $D(k_{proet}^{\text{indrat}})$ above gives an isomorphism

$$
\left[ Q_{k,A^\vee} \to R\Gamma(A^\vee) \right] \sim R\text{Hom}_{IPAlg/k}(R\Gamma(A), \mathbb{Q}/\mathbb{Z})
$$

in $D(Ab)$.

### 8. Duality with Coefficients in Tori

In this section, we give an analogue of (4.1.2) for tori and describe it. In this and the next sections, we write $\text{Hom}_k = \text{Hom}_{k_{proet}}$ and similarly for $\text{Ext}$ and $R\text{Hom}$, and $R\Gamma(\cdot) = R\Gamma(K, \cdot)$, when there is no confusion.

Let $T$ be a torus over $K$ with Néron model $\mathcal{T}$ and Cartier dual $T^{\text{CD}}$. Let $\mathcal{T}_x$ be the special fiber of $\mathcal{T}$. By (3.4.3) (g), we know that $\Gamma(T)$ is an extension of an étale group by a P-acyclic proalgebraic group, $H^\vee(T) = 0$ for $n \geq 1$, and $R\Gamma(T) = \Gamma(T)$ is P-acyclic. Also by (3.4.3) (g), $\Gamma(T^{\text{CD}})$ is a lattice over $k$, $H^1(T^{\text{CD}}) \in \text{FEt}/k$, $H^2(T^{\text{CD}}) \in \text{IALg}_{et}/k$, $H^nu(T^{\text{CD}}) = 0$ for $n \geq 3$, and $R\Gamma(T^{\text{CD}})$ is P-acyclic and Serre reflexive. Therefore we will write $R\Gamma(T) = R\Gamma(T)$ and $R\Gamma(T^{\text{CD}}) = R\Gamma(T^{\text{CD}})$. The morphism of functoriality of $R\Gamma$ (3.3.8) and the trace morphism $R\Gamma(G_m) \to \mathbb{Z}$ (4.1.1) yield morphisms

$$
R\Gamma(T^{\text{CD}}) \to R\Gamma R\text{Hom}_{k}(T, G_m)
$$

$$
\to R\text{Hom}_k(R\Gamma(T), R\Gamma(G_m))
$$

$$
\to R\text{Hom}_k(R\Gamma(T), \mathbb{Z}) = R\Gamma(T)^{\text{SD}}
$$

in $D(k_{proet}^{\text{indrat}})$. The following is the duality with coefficients in tori. See also [Bég81 7.2] in the case of mixed characteristic $K$ with algebraically closed $k$.

**Theorem (8.1).** The morphism

$$
\theta_T : R\Gamma(T^{\text{CD}}) \to R\Gamma(T)^{\text{SD}}
$$
in $D(k_{\text{indrat}}^{\text{proet}})$ defined above is an isomorphism\footnote{Since $R\Gamma(T^{\text{CD}})$ is Serre reflexive as we saw above, there is no point putting SDSD to it and using a different letter $\theta_T$ to denote this morphism, as opposed to the case of abelian varieties.} Its Serre dual yields isomorphisms

$$\theta_{T^{\text{CD}}} : \left[ \lim_n \Gamma(T) \to R\Gamma(T) \right] = R\Gamma(T)^{\text{SDSD}} = R\Gamma(T^{\text{CD}})^{\text{SD}}.$$

Proof. We first treat $\theta_T$. If $T = G_m$, then it is an isomorphism by \cite[Thm. 2.6.1]{Suzuki} (which is essentially Serre’s local class field theory \cite{Serre}). For general $T$, let $L/K$ be a finite Galois extension with Galois group $G$ that splits $T$, so that we have

$$\theta_{T \times_K L} : R\Gamma(L, T^{\text{CD}}) \to R\Gamma(L, T)^{\text{SD}}.$$

We may assume that $L/K$ is totally ramified and cyclic as in the proof of \cite[(6.1)]{Suzuki}. The complexes $R\Gamma(L, T)$ and

$$R\Gamma(G, R\Gamma(L, T)) = R\Gamma(K, T)$$

are both concentrated in degree zero. Therefore we can apply \cite[(6.8)]{Suzuki}, obtaining

$$R\Gamma(G, R\Gamma(L, T)^{\text{SD}}) = R\Gamma(K, T)^{\text{SD}}.$$

Applying $R\Gamma(G, \cdot)$ to $\theta_{T \times_K L}$, we obtain an isomorphism $R\Gamma(K, T^{\text{CD}}) \to R\Gamma(K, T)^{\text{SD}}$. We can see that this isomorphism is equal to $\theta_T$ by the same method as the last part of the proof of \cite[(6.1)]{Suzuki}.

Before treating $\theta_{T^{\text{CD}}}$, we show that $\pi_0(\mathcal{T}_x)$ is a finitely generated étale group over $k$. This is well-known (see \cite[Prop. 3.5]{HN11} for example), but can be deduced as a corollary of the method of proof in the previous paragraph, as follows. By \cite[(6.8)]{Suzuki}, we know that the norm map gives an isomorphism

$$L\Delta(G, \Gamma(L, T)^N) \sim R\Gamma(G, \Gamma(L, T)) = \Gamma(K, T)$$

if $L/K$ is finite Galois totally ramified and cyclic. The cyclicity can be removed by dévissage. Since the zeroth group homology is the coinvariants, we know that the $G$-coinvariants of $\Gamma(L, T)$ is $\Gamma(K, T)$. The group $\pi_0(\Gamma(K, T))$ is the Néron component group of $T$. Hence the $G$-coinvariants of the Néron component group of $T \times_K L$ is the Néron component group of $T$. Since the Néron component group of a split torus is obviously finite free, this implies that $\pi_0(\mathcal{T}_x)$ is finitely generated.

For $\theta_{T^{\text{CD}}}$, recall again that $R\Gamma(T) = \Gamma(T)$. Hence the only non-trivial part (after the invertibility of $\theta_T$ shown above) is

$$\Gamma(T)^{\text{SDSD}} = \left[ \lim_n \Gamma(T) \to \Gamma(T) \right].$$

Let $\Gamma(T)_0 \in \text{PAlg}/k$ be the identity component of $\Gamma(T)$. By \cite[(1.12)]{Suzuki}, we have

$$\Gamma(T)_0^{\text{SDSD}} = \left[ \lim_n \Gamma(T)_0 \to \Gamma(T)_0 \right] \in D^b(\text{PAlg}_{\text{sm}}).$$

Since $\pi_0(\Gamma(T))$ is finitely generated, we have $\lim_n \Gamma(T) = \lim_n \Gamma(T)_0$, which is a uniquely divisible proalgebraic group. Hence

$$\left( \lim_n \Gamma(T) \right)^{\text{SD}} = \left( \lim_n \Gamma(T)_0 \right)^{\text{SD}} = 0$$

as seen in the proof of \cite[(1.12)]{Suzuki}. Therefore

$$\Gamma(T)^{\text{SDSD}} = \left[ \lim_n \Gamma(T) \to \Gamma(T) \right]^{\text{SDSD}}.$$
We do not need SDSD on the right-hand side, since the distinguished triangle
\[
\lim_n \Gamma(T)_0 \to \Gamma(T)_0 \to \lim_n \Gamma(T) \to \pi_0(\Gamma(T))
\]
shows that the middle term is Serre reflexive.

We deduce concrete consequences from this theorem, as in (4.2.1). Note that if 
\(G\) is a connected proalgebraic group over \(k\), then \(\text{Ext}_n^k(G, \mathbb{Z}) = 0\) for \(n = 0, 1\) since \(\text{RHom}_k(G, \mathbb{Z}[1])\) and \(\text{Hom}_k(G, \mathbb{Z}[1])\) are locally just abelian groups and \(\text{Ext}^n_{\mathbb{Z}}(G, \mathbb{Z}) = 0\) by (2.4.1) (\(\mathbb{Z}\)). This implies that
\[
(8.2) \quad \text{Ext}_n^k(\Gamma(T), \mathbb{Z}) = \text{Ext}_n^k(\pi_0(\Gamma(T)), \mathbb{Z}) = \text{Ext}_n^1(\pi_0(T_x), \mathbb{Z})
\]
for \(n = 0, 1\). Hence the isomorphism \(\theta_T\) in the above theorem induces, in degrees 0 and 1, isomorphisms
\[
\theta_T^0 : \Gamma(T)_{\text{CD}} \cong \text{Hom}_k(\pi_0(T_x), \mathbb{Z}),
\]
\[
\theta_T^1 : H^1(T)_{\text{CD}} \cong \text{Ext}_k^1(\pi_0(T_x), \mathbb{Z})
\]
in \(\text{FGEt}/k\). We will put them together, and make explicit the isomorphism in degree 2. We need notation to do this.

Let \(I_K\) be the inertia group of the absolute Galois group of \(K\), \(X^*(T) = T_{\text{CD}}\)
the character lattice, \(X_*(T) = \text{Hom}_K(G_m, T)\) the cocharacter lattice and \(\tilde{T}_{\text{tor}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}\) the torsion part of the dual torus. For an étale group \(X\) over \(K\), the \(I_K\)-coinvariants \(X(K_{\text{sep}})/I_K\) is a discrete \(\text{Gal}(\mathbb{F}/k)\)-module. We denote the corresponding étale group over \(k\) by \(X/I_K\) by abuse of notation.

**Proposition (8.3).** From \(H^0\) and \(H^1\) of the isomorphism \(\theta_T\), we can deduce an isomorphism
\[
\theta_T^0 : \pi_0(T_x) \cong X_*(T) / I_K
\]
in \(\text{FGEt}/k\), and from \(H^2\), an isomorphism
\[
\theta_T^2 : H^2(\tilde{T}_{\text{tor}}) \cong \text{Ext}_k^1(\Gamma(T), \mathbb{Q}/\mathbb{Z})
\]
in \(\text{IAlg}_{\text{uc}}/k\).

We need a lemma.

**Lemma (8.4).** For any finitely generated étale group \(A\) over \(K\), we have a natural isomorphism
\[
\tau_{\leq 1} R\Gamma R\text{Hom}_K(A, \mathbb{Z}) = R\text{Hom}_k(A/I_K, \mathbb{Z})
\]
in \(D(k)\), where \(\tau\) denotes the truncation functor.

**Proof.** First note that \(R\text{Hom}_K\) between étale groups over \(K\) is concentrated in degrees 0 and 1 since étale groups are locally just abelian groups and \(\text{Ext}_{\mathbb{Z}}^2_{\text{Ab}} = 0\). The same is true for \(R\text{Hom}_k\). Let \(0 \to B \to F \to A \to 0\) be an exact sequence with \(F\) a finite free \(\mathbb{Z}[G]\)-module for some finite Galois extension \(L/K\) and \(G = \text{Gal}(L/K)\). Let \(I_G\) be the inertia subgroup of \(G\).

First we show that the distinguished triangle
\[
R\Gamma R\text{Hom}_K(A, \mathbb{Z}) \to R\Gamma R\text{Hom}_K(F, \mathbb{Z}) \to R\Gamma R\text{Hom}_K(B, \mathbb{Z})
\]
\footnote{Here we treat the torsion part of the dual torus algebraically without using \(\mathbb{C}\) or \(\overline{\mathbb{Q}}\). Therefore, we ignore the Tate twist \(\mathbb{Q}/\mathbb{Z}(1)\), which has nothing to do with the Galois action of \(K\).}
induces a canonical morphism
\[ \Gamma \text{Hom}_K(F, Z) \to \Gamma \text{Hom}_K(B, Z)[-1] \to R\Gamma R \text{Hom}_K(A, Z), \]
where \([\cdot]\) denotes the mapping cone. Let \( Z \to J \) be an injective resolution in \( \text{Ab}(K_{\text{fppf}}/k^{\text{indrat}}) \). Then in the category of complexes in \( \text{Ab}(k^{\text{proet}}) \), we have natural morphism and isomorphism
\[ \Gamma \text{Hom}_K(F, Z) \to \Gamma \text{Hom}_K(B, Z)[-1] \to \Gamma \text{Hom}_K(F, J) \to \Gamma \text{Hom}_K(B, J)[-1] \]
\[ = \Gamma \text{Hom}_K([B \to F], J). \]

For another injective resolution \( Z \to J \) and a homotopy equivalence \( J \to J' \) over \( Z \), we have a commutative diagram
\[
\begin{array}{ccc}
\Gamma \text{Hom}_K(F, Z) & \to & \Gamma \text{Hom}_K(B, Z)[-1] \\
\downarrow & & \downarrow \\
\Gamma \text{Hom}_K(F, J) & \to & \Gamma \text{Hom}_K(B, J)[-1] \\
\end{array}
\]
where the right vertical arrow is a homotopy equivalence. All homotopy equivalences \( J \to J' \) over \( Z \) are homotopic to each other, so they induce the same homotopy class for the right vertical arrow. We saw in the proof of \( \text{Lemma 3.3.3} \) that the complex \( \Gamma \text{Hom}_K([B \to F], J) \) represents \( R\Gamma R \text{Hom}_K([B \to F], Z) \), or \( R\Gamma R \text{Hom}_K(A, Z) \).

Hence in \( D(k^{\text{proet}}) \), we have a canonical morphism
\[ \Gamma \text{Hom}_K(F, Z) \to \Gamma \text{Hom}_K(B, Z)[-1] \to R\Gamma R \text{Hom}_K(A, Z). \]

Since the left-hand side of this morphism is concentrated in degrees \( \leq 1 \), the morphism factors through the truncation \( \tau_{\leq 1} \) of the right-hand side:
\[ \Gamma \text{Hom}_K(F, Z) \to \Gamma \text{Hom}_K(B, Z)[-1] \to \tau_{\leq 1} R\Gamma R \text{Hom}_K(A, Z). \]

This is an isomorphism since \( F \) is finite free over \( \mathbb{Z}[G] \), \( H^1(K, \mathbb{Z}[G]) = H^1(L, \mathbb{Z}) = 0 \) and hence \( R\Gamma R \text{Hom}_K(F, Z) = R\Gamma \text{Hom}_K(F, Z) \) has trivial \( H^1 \). The sheaf \( \Gamma \text{Hom}_K(B, Z) \) is a lattice over \( k \) by \( \text{Lemma 3.3.3} \). Its \( k \)-points is \( \text{Hom}_K(B, Z) = \text{Hom}_K(B/I_G, Z) \) by \( \text{Lemma 3.2.10} \). Therefore \( \Gamma \text{Hom}_K(B, Z) = \text{Hom}_k(B/I_G, Z) \).

Similarly we have \( \Gamma \text{Hom}_K(F, Z) = \text{Hom}_k(F/I_G, Z) \) and hence we have
\[ \tau_{\leq 1} R\Gamma R \text{Hom}_K(A, Z) = [\Gamma \text{Hom}_K(F, Z) \to \Gamma \text{Hom}_K(B, Z)[-1] \to \text{Hom}_k(F/I_G, Z) \to \text{Hom}_k(B/I_G, Z)[-1]]. \]

Consider the following part of the long exact sequence of group homology:
\[ H_1(I_G, A) \to B/I_G \to F/I_G \to A/I_G \to 0. \]

This is an exact sequence of étale groups over \( k \). Let \( X \) be the kernel of \( F/I_G \to A/I_G \). In the short exact sequence \( 0 \to X \to F/I_G \to A/I_G \to 0 \), the middle term \( F/I_G \) is a lattice over \( k \). Hence the same argument as above yields an isomorphism
\[ \text{Hom}_k(F/I_G, Z) \to \text{Hom}_k(X, Z)[-1] = R\text{Hom}_k(A/I_G, Z). \]

(No truncation \( \tau_{\leq 1} \) is necessary on the right.) Consider the exact sequence
\[ 0 \to \text{Hom}_k(X, Z) \to \text{Hom}_k(B/I_G, Z) \to \text{Hom}_k(H_1(I_G, A), Z). \]
Since $G$ is finite, the group $H_1(I_G, A)$ is torsion \((\text{Ser}	ext{\textsuperscript{79}} \text{, VIII, Cor. 1 to Prop. 4})\), so the final term is zero. With $A/I_G = A/I_K$, we get

\[
[\text{Hom}_k(F/I_G, \mathbb{Z}) \to \text{Hom}_k(B/I_G, \mathbb{Z})] [-1] = R\text{Hom}_k(A/I_K, \mathbb{Z}).
\]

Combining all the above, we get the required isomorphism. This isomorphism is independent of the choice of the sequence $0 \to B \to F \to A \to 0$.

\[\square\]

**Proof of (8.3).** We construct $\theta_T^0$. By (8.2), the natural morphism

\[
R\text{Hom}_k(\pi_0(\Gamma(T)), \mathbb{Z}) \to \tau_{\leq 1} R\text{Hom}_k(\Gamma(T), \mathbb{Z})
\]
is an isomorphism. Hence we have

\[
\tau_{\leq 1}(R\Gamma(T)^{\text{SD}}) = R\text{Hom}_k(\pi_0(\Gamma(T)), \mathbb{Z}) = R\text{Hom}_k(\pi_0(T_x), \mathbb{Z}).
\]

Since $T^{\text{CD}} = X^*(T) = \text{Hom}_K(X_*(T), \mathbb{Z})$, we have

\[
\tau_{\leq 1} R\Gamma(T^{\text{CD}}) = \tau_{\leq 1} R\Gamma(X_*(T), \mathbb{Z}) = R\text{Hom}_K(X_*(T)/I_K, \mathbb{Z})
\]

using the lemma. Hence the isomorphism $\theta_T$ induces an isomorphism

\[
R\text{Hom}_K(X_*(T)/I_K, \mathbb{Z}) \sim \tau_{\leq 1} R\text{Hom}_K(\pi_0(T_x), \mathbb{Z})
\]
in $D^b(\text{FGET}/k)$. Apply $\text{SD}$ to the both sides. Then \((2.4.1)\) \((\text{b})\) gives an isomorphism $\theta_T^0 : \tau_{\leq 1}(\pi_0(T_x)) \sim X_*(T)/I_K$.

We construct $\theta_T^f$. Applying the direct limit by multiplication by $n \geq 1$ for $\theta_T$, we have

\[
\theta_T \otimes \mathbb{Q} : \lim_n R\Gamma(T^{\text{CD}}) \sim \lim_n R\text{Hom}_K(R\Gamma(T), \mathbb{Z}).
\]

For any $k' \in k^{\text{indrat}}$, we have

\[
\lim_n R\Gamma(K(k'), T^{\text{CD}}) = R\Gamma(K(k'), \lim_n T^{\text{CD}}) = R\Gamma(K(k'), T^{\text{CD}} \otimes \mathbb{Q})
\]

by \((\text{Mil80}) \text{ III, Rmk. 3.6})\), so $\lim_n R\Gamma(T^{\text{CD}}) = R\Gamma(T^{\text{CD}} \otimes \mathbb{Q})$. Also

\[
\lim_n R\text{Hom}_K(\Gamma(T)_0, \mathbb{Z}) = R\text{Hom}_K(\Gamma(T)_0, \mathbb{Q}) = 0
\]

by \((2.3.1)\) \((\text{b})\), \((\text{d})\) and \((2.3.2)\) since $\Gamma(T)_0 \in \text{PAlg}/k$, and

\[
\lim_n R\text{Hom}_K(\pi_0(\Gamma(T)), \mathbb{Z}) = R\text{Hom}_K(\pi_0(\Gamma(T)), \mathbb{Q})
\]

since $\pi_0(\Gamma(T)) \in \text{FGET}/k$ and $\text{Ext}_{\text{Ab}}^n(G, \cdot )$ for finitely generated $G$ commutes with filtered direct limits. Hence $\lim_n R\text{Hom}_K(\Gamma(T), \mathbb{Z}) = R\text{Hom}_K(\Gamma(T), \mathbb{Q})$.

Therefore

\[
\theta_T \otimes \mathbb{Q} : R\Gamma(T^{\text{CD}} \otimes \mathbb{Q}) \sim R\text{Hom}_K(R\Gamma(T), \mathbb{Q}).
\]

Together with $\theta_T$, we have a morphism

\[
\begin{array}{c}
R\Gamma(T^{\text{CD}}) \\
\downarrow \\
R\text{Hom}_K(\Gamma(T), \mathbb{Z})
\end{array} \quad \longrightarrow \quad \begin{array}{c}
R\Gamma(T^{\text{CD}} \otimes \mathbb{Q}) \\
\downarrow \\
R\text{Hom}_K(\Gamma(T), \mathbb{Q})
\end{array} \quad \longrightarrow \quad \begin{array}{c}
R\Gamma(T^{\text{CD}} \otimes \mathbb{Q}/\mathbb{Z}) \\
\downarrow \\
R\text{Hom}_K(\Gamma(T), \mathbb{Q}/\mathbb{Z})
\end{array}
\]

of distinguished triangles, whose left two vertical arrows are isomorphisms. Thus we get an isomorphism

\[
R\Gamma(T_{\text{tor}}) \sim R\text{Hom}_K(\Gamma(T), \mathbb{Q}/\mathbb{Z}).
\]

In degree 1, it gives $\theta_T^2$. \[\square\]
The existence of an isomorphism $\pi_0(T_x) \cong X_*(T)/I_K$ was also obtained by Bertapelle and González-Avilés [BGA15] in another method. Note that the result of Xarles [Xar93] has been recovered in the course of the above proof.

We describe $\theta^0_{T_x}$. First we need a preparation about not necessarily totally ramified extensions (cf. [SY12 §4.2]). For a finite Galois extension $L/K$ with residue extension $k'/k$, let $L^\times \in \text{Ab}(L_{\text{proet}})$ be the sheaf defined in the same way as $K^\times$:

$$ L^\times(k'') = (\{(W(k'')\hat{\otimes}_{W(k')}O_L) \otimes_{O_L} L\})^\times $$

for $k'' \in k'^{\text{indrat}}$. On the other hand, we denote $L^\times_k = \Gamma(K, \text{Res}_{L/K}G_m)$, where $\text{Res}_{L/K}$ is the Weil restriction. These sheaves are related in the following way: for any $k'' \in k'^{\text{indrat}}$, we have

$$ L^\times_k(k'') = (K(k'') \otimes_K L)^\times = L^\times(k'' \otimes_k k') $$

(see [SY12 §4.2]; the notation there for $L^\times$ was $L^\times_k$). Hence $L^\times_k = \text{Res}_{k'/k}L^\times$. Let $\text{Res}_{k'/k}Z = Z[\text{Gal}(k'/k)]$ be the group ring viewed as an étale group over $k$. Applying $\text{Res}_{k'/k}$ to the split exact sequence $0 \to U_L \to L^\times \to Z \to 0$ in $\text{Ab}(k'^{\text{indrat}})$ (where $U_L$ is defined similarly as $U_K$), we have a split exact sequence

$$ 0 \to \text{Res}_{k'/k}U_L \to L^\times_k \to Z[\text{Gal}(k'/k)] \to 0 $$

in $\text{Ab}(k'^{\text{indrat}})$. The group $\text{Res}_{k'/k}U_L$ is connected, so $\pi_0(L^\times_k) = Z[\text{Gal}(k'/k)]$.

Now we define a map that is going to be inverse to $\theta^0_{T_x}$. Let $\lambda: G_m \to T$ be a morphism over $L$. The duality for the finite étale morphism Spec $L \to \text{Spec} K$ gives a morphism $\text{Res}_{L/K}G_m \to T$ over $K$. In other words, this is the composite of $\lambda: \text{Res}_{L/K}G_m \to \text{Res}_{L/K}T$ with the norm map $\text{Res}_{L/K}T \to T$ for $T$. Applying $\Gamma$, we have a morphism $L^\times_k \to \Gamma(K, T)$ over $k$, Applying $\pi_0$, we have a morphism $Z[\text{Gal}(k'/k)] \to \pi_0(\Gamma(T)) = \pi_0(T_x)$ over $k$. The image of $1 \in Z[\text{Gal}(k'/k)]$ corresponds to a $k'$-point of $\pi_0(T_x)$. Thus we have a homomorphism $\text{Hom}_L(G_m, T) \to \Gamma(k', \pi_0(T_x))$. This is compatible with the action of $\text{Gal}(L/K)$ (which factors through $\text{Gal}(k'/k)$ on the right). It is also compatible with extending $L$. Therefore we obtain a homomorphism $X_*(T)/I_K \to \pi_0(T_x)$ of étale groups over $k$, which we denote by $\phi$.

**Proposition (8.6).** The homomorphism $\varphi: X_*(T)/I_K \to \pi_0(T_x)$ just defined is the inverse map of the isomorphism $\theta^0_{T_x}$.

**Proof.** We may assume that $k$ is algebraically closed, so $G_K := I_K$ is the absolute Galois group of $K$. It is enough to compare the homomorphisms induced on the $\mathbb{Q}/\mathbb{Z}$-duals $\text{Hom}_{\text{Ab}}(X_*(T)/I_K, \mathbb{Q}/\mathbb{Z})$ and $\text{Hom}_{\text{Ab}}(\pi_0(T_x), \mathbb{Q}/\mathbb{Z})$.

By (8.5), we have

$$ R\Gamma(\tilde{T}_{\text{tor}}) \xrightarrow{\sim} R\text{Hom}_K(\Gamma(T), \mathbb{Q}/\mathbb{Z}). $$

Therefore

$$ \Gamma(\tilde{T}_{\text{tor}}) \xrightarrow{\sim} \text{Hom}_K(\Gamma(T), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \text{Hom}_{\text{Ab}}(\pi_0(\Gamma(T)), \mathbb{Q}/\mathbb{Z}). $$

Also, since $\tilde{T}_{\text{tor}} = X^*(T) \otimes_{\mathbb{Q}/\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = \text{Hom}_K(X_*(T), \mathbb{Q}/\mathbb{Z})$, we have a natural isomorphism

$$ \Gamma(\tilde{T}_{\text{tor}}) \xrightarrow{\sim} \text{Hom}_K(X_*(T), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \text{Hom}_{\text{Ab}}(X_*(T)/G_K, \mathbb{Q}/\mathbb{Z}). $$


By construction, the $\mathbb{Q}/\mathbb{Z}$-dual of $\theta^2_T$ is given by the composite
\[
\text{Hom}_{\text{Ab}}(X_*(T)/G_K, \mathbb{Q}/\mathbb{Z}) \overset{\sim}{\rightarrow} \Gamma(\hat{T}_{\text{tor}}) \overset{\sim}{\rightarrow} \text{Hom}_{\text{Ab}}(\pi_0(\Gamma(T)), \mathbb{Q}/\mathbb{Z}).
\]
It is enough to show that the morphism
\[
\Gamma(\hat{T}_{\text{tor}}) \overset{\sim}{\rightarrow} \text{Hom}_{\text{Ab}}(\pi_0(\Gamma(T)), \mathbb{Q}/\mathbb{Z}) \overset{\sim}{\rightarrow} \text{Hom}_{\text{Ab}}(X_*(T)/G_K, \mathbb{Q}/\mathbb{Z})
\]
coincides with $\langle 8.3 \rangle$. Let $\chi \otimes r \in \hat{T}_{\text{tor}} = X^*(T) \otimes \mathbb{Q}/\mathbb{Z}$ be $G_K$-invariant, where $\chi: T \to G_m$ is a morphism over a finite Galois extension $L/K$ and $r \in \mathbb{Q}/\mathbb{Z}$. Then we have a morphism
\[
\Gamma(L, T) \xrightarrow{\chi} L^\times \overset{\psi}{\rightarrow} \mathbb{Z} \overset{r}{\rightarrow} \mathbb{Q}/\mathbb{Z}
\]
over $k$. The norm map $N: \Gamma(L, T) \to \Gamma(K, T)$ identifies $\Gamma(K, T)$ as the $G_K$-coinvariants of $\Gamma(L, T)$ as we saw in the second paragraph of the proof of $\langle 8.1 \rangle$. The $G_K$-invariance of $\chi \otimes r$ implies that the above $r\psi \circ \chi$ factors through $\Gamma(T) = \Gamma(K, T)$ and hence through its $\pi_0$. This defines a homomorphism
\[
\pi_0(\hat{T}_{\text{tor}}) \circ \chi \circ N^{-1}: \pi_0(\hat{T}_{\text{tor}}) \to \mathbb{Q}/\mathbb{Z}.
\]
This is the element given by $\langle 8.3 \rangle$.

We calculate the image (or pullback) of this element by $\varphi$. If $\lambda: G_m \to T$ is a morphism over $L$, then $\varphi(\lambda) = N\lambda(\pi_L)$ by the construction of $\varphi$, where $\pi_L$ is a prime element of $L$. Hence the pullback by $\varphi$ of $\pi_0(\hat{T}_{\text{tor}}) \circ \chi \circ N^{-1}$ sends $\lambda$ to
\[
\pi_0(\hat{T}_{\text{tor}}) \circ \chi \circ N^{-1} \circ N \circ \lambda = \pi_0(\hat{T}_{\text{tor}}) \circ \chi \circ \lambda = \pi_0(\hat{T}_{\text{tor}}) \circ \chi \circ \lambda(\pi_L) = r\psi \circ \chi \circ \lambda(\pi_L) = r\psi(\pi_L(\chi, \lambda)) = r(\chi, \lambda) \in \mathbb{Q}/\mathbb{Z},
\]
where $\langle \chi, \lambda \rangle \in \mathbb{Z}$ is the natural pairing between a character and a cocharacter. The obtained assignment $\lambda \mapsto r(\chi, \lambda)$ agrees with the image of $\chi \otimes r$ under the homomorphism $\langle 8.3 \rangle$. □

We describe $\theta^2_T$. One method to do this is to go back to the proof of $\langle 8.1 \rangle$, namely use Galois descent to reduce to Serre’s local class field theory; cf. $\langle 6.13 \rangle$. Another method is to use the sequence $0 \to T[n] \to T \overset{n}{\rightarrow} T \to 0$ for each $n \geq 1$ to reduce it to the description of the duality with coefficients in finite flat group schemes $T[n]$. We give here yet another, more direct method. It is enough to describe the morphism induced on $k$-points since the groups involved with $\theta^2_T$ are ind-algebraic and $k$ can be any perfect field.

An element of
\[
H^1(\hat{T}_{\text{tor}}) = H^2(T^\text{CD}) = \text{Ext}^2_T(\mathbb{Z}, T^\text{CD})
\]
corresponds to an extension class
\[
E: 0 \to T^\text{CD} \to X_2 \to X_1 \to \mathbb{Z} \to 0.
\]
The terms $X_i$ can be chosen more explicitly as follow. If the element lives in $H^2(G, \Gamma(L, T^\text{CD}))$ with $G = \text{Gal}(L/K)$ for a finite Galois extension $L/K$, then it gives a morphism of complexes of $G$-modules from the standard resolution of $\mathbb{Z}$ to $\Gamma(L, T^\text{CD})[2]$. By pushing out the standard resolution
\[
\cdots \to \mathbb{Z}[G^4] \to \mathbb{Z}[G^3] \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0
\]
(of homogeneous chains) at the degree $-2$ term $\mathbb{Z}[G^3]$ to $\Gamma(L, T^{CD})$, we get a choice of (the $L$-valued points of) the extension class $E$:
\[ 0 \to \Gamma(L, T^{CD}) \to X \to \mathbb{Z}[G] \to \mathbb{Z} \to 0. \]
All these terms are lattices over $K$. The image of $E$ in $\text{Ext}^2_K(T, G_m)$ is given by taking its Cartier dual
\[ 0 \to G_m \to \text{Res}_{L/K} G_m \to X^{CD} \to T \to 0. \]
Apply $R\tilde{\Gamma}$. Since the terms are all tori, we have an exact sequence
\[ 0 \to K^\times \to L^\times_k \to \Gamma(X^{CD}) \to \Gamma(T) \to 0 \]
in $\text{Ab}(\text{indrat}_{\text{proet}})$ by (8.3). By pushing it out by the valuation map $K^\times \to \mathbb{Z}$, we get an extension
\[ 0 \to \mathbb{Z} \to L_k^\times /U_K \to \Gamma(X^{CD}) \to \Gamma(T) \to 0. \]
This is the element of $\text{Ext}^2_k(\Gamma(T), \mathbb{Z})$ corresponding to $\theta_T$. Let $\psi$ be the middle morphism $L_k^\times \to \Gamma(X^{CD})$. Recall the morphism $L_k^\times \to \mathbb{Z}[[\text{Gal}(k'/k)]]$ defined before (8.0). The maximal constant quotient of the étale group $\mathbb{Z}[[\text{Gal}(k'/k)]]$ is $\mathbb{Z}$. Let $U_{L, k'/k}$ be the kernel of the composite $L_k^\times \to \mathbb{Z}[[\text{Gal}(k'/k)]] \to \mathbb{Z}$. Let $n = [L : K]$. Then the above extension class comes from the extension
\[ 0 \to \mathbb{Z}/n\mathbb{Z} \to \Gamma(X^{CD})/\psi(U_{L, k'/k}) \to \Gamma(T) \to 0 \]
in $\text{Ext}^1_k(\Gamma(T), \mathbb{Z}/n\mathbb{Z})$. By pushing it out by $\mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$, we get an corresponding extension class in $\text{Ext}^1_k(\Gamma(T), \mathbb{Q}/\mathbb{Z})$. This is the element given by $\theta_T^\ast$.

**Remark (8.9).** As in (7.3), we can give a statement on $R\Gamma(T)$ and $R\Gamma(T^{CD})$ even if $k$ is not algebraically closed, thanks to the P-acyclicity of $R\Gamma(T)$ and $R\Gamma(T^{CD})$. Let $(T_x)_l$ be the maximal torus of the special fiber $T_x$ and set $Q_{k, T} = R\lim_{\rightarrow} R\Gamma(k_{\text{et}}, (T_x)_l)$. Then by the same reasoning as the cited remark, we have
\[ R\Gamma(T^{CD}) = R\text{Hom}_{k_{\text{et}}} (\Gamma(T), \mathbb{Z}), \]
\[ [Q_{k, T} \to R\Gamma(T)] = R\text{Hom}_{k_{\text{et}}} (R\Gamma(T^{CD}), \mathbb{Z}) \]
in $D(\text{Ab})$. By the isomorphism (8.3), we have
\[ R\Gamma(\hat{T}_{\text{cor}}) = R\text{Hom}_{k_{\text{et}}} (\Gamma(T), \mathbb{Q}/\mathbb{Z}). \]
If we look at the maximal subgroups of $\Gamma(T)$ and $\Gamma(T^{CD})$ whose quotients are lattices, then the corresponding parts of the right-hand sides of these isomorphisms may be written by $R\text{Hom}_{\text{PA}k_{\text{et}}} M$ and $R\text{Hom}_{\text{PA}k_{\text{et}}} M$.

9. **Duality with coefficients in 1-motives**

Next we treat the case of coefficients in 1-motives. We only formulate and prove the duality, without giving its full description. Let $M = [Y \to G]$ be a 1-motive over $K$ in the sense of Deligne, where $Y$ is a lattice placed in degree $-1$ and $G$ a semi-abelian variety. We can naturally regard $M$ as an object in $D(K_{\text{fppf}})$ and hence in $D(K_{\text{fppf}}^{\text{proet}})$. We calculate $R\Gamma(M)$ by (3.4.3). Applying $R\Gamma$ on $M$, we have a long exact sequence
\[ 0 \to H^{-1}(M) \to \Gamma(Y) \to \Gamma(G) \to H^0(M) \]
\[ \to H^1(Y) \to H^1(G) \to H^1(M) \to H^2(Y) \to 0 \]
in \(\text{Ab}(k_{\text{indrat}}^{\text{proet}})\) and \(H^0(M) = 0\) for \(n \neq -1, 0, 1\). We know that \(\Gamma(Y)\) is a lattice over \(k\), \(\Gamma(G)\) is an extension of a lattice by a P-acyclic proalgebraic group (which can be checked by reducing it to the case of an abelian variety and a torus, using the fact that an extension of a proalgebraic group by \(\mathbb{Q}\) splits by \((2.3.3)\), \(H^1(Y) \in \text{Fet}/k\), and \(H^2(Y) \in \text{IAlg}_{\text{fuc}}/k\). Hence \(H^{-1}(M)\) is a lattice over \(k\) and \(H^1(M) \in \text{IAlg}_{\text{fuc}}/k\). Since all the terms but \(H^0(M)\) are P-acyclic, we know that \(H^0(M)\) is P-acyclic, Therefore \(R\Gamma(M)\) is P-acyclic, and we will write \(R\Gamma(M) = R\Gamma(M) \in D(k_{\text{indrat}}^{\text{proet}})\).

Let \(M^\vee\) be the dual 1-motive of \(M\). In \(D(K_{\text{fppf}})\), the complex \(M^\vee\) is identified with \(\tau_{\leq 0}R\text{Hom}_{K}(M, G_m[1])\). Note the shift: if \(M\) is a torus \(T\), then \(M^\vee\) is the character lattice \(T^{\text{CD}}\) placed in degree \(-1\). We have a morphism

\[
R\Gamma(M^\vee) \rightarrow R\hat{\Gamma}R\text{Hom}_{K}(M, G_m[1]) \rightarrow R\text{Hom}_K(R\Gamma(M), \mathbb{Z})[1] = R\Gamma(M)^{\text{SD}}[1]
\]

in \(D(k_{\text{indrat}}^{\text{proet}})\) and hence a morphism

\[
R\Gamma(M^\vee)^{\text{SD}} \rightarrow R\Gamma(M)^{\text{SD}}[1]
\]

by taking SD of the both sides and replacing \(M\) with \(M^\vee\).

**Theorem (9.1).** The above defined morphism

\[
R\Gamma(M^\vee)^{\text{SD}} \rightarrow R\Gamma(M)^{\text{SD}}[1]
\]

in \(D(k_{\text{indrat}}^{\text{proet}})\) is an isomorphism.

**Proof.** Note that the both sides are triangulated functors on \(M\). Therefore if \(M_1 \rightarrow M_2 \rightarrow M_3\) is a distinguished triangle of \(1\)-motives and the theorem is true for any two of them, then so is the third. Let \(M = [Y \rightarrow G]\) with \(Y\) a lattice over \(K\), \(G\) semi-abelian with torus part \(T\) and \(A = G/T\). We have the weight filtration \(W_n M\) of \(M\), where \(W_0 M = M, W_1 M = [0 \rightarrow G], W_2 M = [0 \rightarrow T]\) and \(W_3 M = 0\). The distinguished triangles

\[
W_{-1} M \rightarrow M \rightarrow [Y \rightarrow 0], \quad W_{-2} M \rightarrow W_{-1} M \rightarrow [0 \rightarrow A]
\]

show that the statement for \(M\) is reduced to that of \(A, T\) and \(Y\). These are \((4.1.2)\) and \((8.1)\).

**Remark (9.2).** In the proof of \((4.1.2)\), we have used the case of semistable abelian varieties. The above formulation for 1-motives allows us to deduce this case from the case of good reduction abelian varieties, as follows.

Let \(A\) be a semistable abelian variety over \(K\). Then by rigid analytic uniformization, we have an exact sequence \(0 \rightarrow Y \rightarrow G \rightarrow A \rightarrow 0\) of rigid \(K\)-groups, where \(Y\) is a lattice with unramified Galois action and \(G\) an extension of a good reduction abelian variety by an unramified torus. Set \(M = [Y \rightarrow G]\). Let \(K_{\text{et}}^{\text{rp}}/k_{\text{indrat}}\) be the modification of the category \(K_{\text{et}}/k_{\text{indrat}}\) where objects are pairs \((S, k_S)\) with \(k_S \in k_{\text{indrat}}\) and \(S\) an étale \(K_{\text{et}}^{\text{rp}}(k_S)\)-algebra. Denote \(\hat{S} = S \otimes_{K_{\text{et}}^{\text{rp}}(k_S)} K(k_S)\). Let \(\text{Spec} K_{\text{et}}^{\text{rp}}/k_{\text{indrat}}\) be the étale site on \(K_{\text{et}}^{\text{rp}}/k_{\text{indrat}}\). The functors \(k' \mapsto (K^{\text{rp}}(k'), k')\) and \((S, k_S) \mapsto (\hat{S}, k_S)\) define morphisms

\[
\text{Spec} K_{\text{et}}/k_{\text{indrat}}^{\text{proet}} \xrightarrow{f} \text{Spec} K_{\text{et}}^{\text{rp}}/k_{\text{indrat}}^{\text{proet}} \xrightarrow{\pi} \text{Spec} k_{\text{indrat}}^{\text{proet}}
\]

of sites. The pushforward \(f_*\) is exact by \([\text{Suz13}, \text{Lem. 2.5.5}]\). Note that the ring \(\hat{S}\) for any étale \(K_{\text{et}}^{\text{rp}}(k_S)\)-algebra \(S\) is a complete topological \(K\)-algebra. We can
show that the sequence \(0 \to Y \to G \to A \to 0\) of rigid \(k\)-groups induces an exact sequence \(0 \to f_*Y \to f_*G \to f_*A \to 0\) in \(\text{Ab}(K^{\text{indrat}}_{et}/k_{et})\), by a rigid analytic version of the approximation argument used in the proof of the cited lemma. Hence we have \(f_*M \cong f_*A\) in \(\text{Ab}(K^{\text{indrat}}_{et}/k_{et})\). Applying \(R\pi^f\) and using the exactness of \(f_*\), we have an isomorphism
\[
R\Gamma(M) \cong R\Gamma(A).
\]
Hence (4.1.2) for \(A\) is reduced to (4.1) for \(M\). The cases of \(G_m\) and \(\mathbb{Z}\) were explained in [BS15, Ex. 4.1.10], which was essentially Serre’s local class field theory. Hence we are reduced to the case of abelian varieties with good reduction.

Note that the case of abelian varieties with good reduction is further reduced to the case of finite flat group schemes over \(\mathcal{O}_K\) obtained as the torsion part, as explained in Bégueri [Bégu81, §7.1] and Bester [Best78, §2.7]. With this argument, we can avoid Bertapelle’s and Werner’s results, and prove Grothendieck’s and Šafarevič’s conjectures directly from the Bégueri-Bester duality for \(\mathcal{O}_K\) with coefficients in finite flat group schemes.

10. Connection with classical statements for finite residue fields

We show that the duality theorems in this paper induce classical duality statements when the residue field \(k\) is finite. Basically this can be done as follows. Take the derived global section \(R\Gamma(k_{\text{proet}}, \cdot)\) of the both sides of our duality to translate \(R\Gamma(K, A) \in D(k^{\text{indrat}}_{proet})\) into \(R\Gamma(K, A) \in D(\text{Ab})\). Then use Lang’s theorem (see below)
\[
\text{Ext}_k^1(C, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{\text{Ab}}(C(k), \mathbb{Q}/\mathbb{Z})
\]
for a connected quasi-algebraic group \(C\) over \(k\) to translated the Serre dual to the Pontryagin dual. We can actually treat not only Galois group cohomology but also profinite group structures on cohomology groups into account.

We need to define and study several notions for this. Assume that \(k = \mathbb{F}_q\). Let \(\overline{k}\) be an algebraic closure of \(k\). We denote by \(F\) the \(q\)-th power Frobenius homomorphism on any \(k\)-algebra. It induces an action on any object of \(\text{Ab}(k^{\text{indrat}}_{proet})\).

For \(A \in D(k^{\text{indrat}}_{proet})\), we denote the mapping cone of \(F - 1: A \to A\) by \(A^F\).

With a shift, we have a triangulated functor
\[
D(k^{\text{indrat}}_{proet}) \to D(k^{\text{indrat}}_{proet}), \quad A \mapsto [A^F -1].
\]
This is the mapping fiber of \(F - 1: A \to A\).

Next, as in [2.1] let \(k_{\text{proet}}\) be the category of ind-étale \(k\)-algebras and \(\text{Spec} k_{\text{proet}}\) the pro-étale site on \(k_{\text{proet}}\). Let \(k_{\text{prozar}}\) be the full subcategory of \(k_{\text{proet}}\) consisting of filtered unions \(\bigcup k'_i\) where each \(k'_i\) is a finite product of copies of \(k\). For \(k' \in k_{\text{prozar}}\), we say that a finite family \(\{k'_i\}\) of objects of \(k_{\text{prozar}}\) over \(k'\) is a covering if \(k' \to \prod k'_i\) is faithfully flat. This defines a site, which we call the pro-Zariski site of \(k\) and denote by \(\text{Spec} k_{\text{prozar}}\). This is equivalent to the pro-étale site \(\text{Spec} k_{\text{proet}}\) of \(\overline{k}\) via the functors \(k' \in k_{\text{prozar}} \mapsto k' \otimes_k \overline{k} \in k_{\text{proet}}\). It is also equivalent to the site of profinite sets given in [BS15, Ex. 4.1.10]. By composing the continuous map of sites \(\text{Spec} k^{\text{indrat}}_{proet} \to \text{Spec} \overline{k}_{proet}\) defined by the identity with the identification
Spec $k_{\text{proet}} \cong \text{Spec } k_{\text{prozar}},$ we have a continuous map of sites

$$f : \text{Spec } k_{\text{prozar}} \rightarrow \text{Spec } k_{\text{proet}},$$

which is defined by the functor $k_{\text{prozar}} \rightarrow k_{\text{proet}}$ given by $k' \mapsto k' \otimes_k \overline{k}.$ Its pushforward functor $f_* : \text{Ab}(k_{\text{prozar}}) \rightarrow \text{Ab}(k_{\text{proet}})$ is exact, which induces a triangulated functor

$$(10.2) \quad f_* : D(k_{\text{prozar}}^{\text{inind}}) \rightarrow D(k_{\text{proet}}^{\text{inind}}).$$

The functor $f_*$ is a sort of a geometric points functor in the following sense. For any $A \in \text{Ab}(k_{\text{prozar}})$ we have $(f_*A)(k) = A(\overline{k}).$ If $A$ is locally of finite presentation, we may view the abelian group $A(\overline{k})$ as a constant sheaf on Spec $k_{\text{prozar}},$ and for any $k' = \bigcup k'_\lambda \in k_{\text{prozar}}$ with $k_\lambda$ a finite product of copies of $k$, we have

$$(f_*A)(k') = \lim_{\lambda} A(k'_\lambda \otimes_k \overline{k}) = \lim_{\lambda} A(\overline{k})(k'_\lambda) = A(\overline{k})(k').$$

Thus $f_*A = A(\overline{k})$ in $\text{Ab}(k_{\text{prozar}})$ if $A$ is locally of finite presentation. Note that $f_*$ commutes with filtered direct limits and arbitrary inverse limits. Hence if $A = \lim_{\lambda} A_{\lambda \lambda'} \in \text{IPAlg}/k$ with $A_{\lambda \lambda'} \in \text{Alg}/k$, then $f_*A = \lim_{\lambda} \lim_{\lambda'} (A_{\lambda \lambda'}(\overline{k}))$ in $\text{Ab}(k_{\text{prozar}}).

Now we define a functor $R\Gamma(k_W, \cdot)$ to be the composite of $\text{IFin}$ and $\text{IPFin}$:

$$R\Gamma(k_W, \cdot) : D(k_{\text{prozar}}^{\text{inind}}) \rightarrow D(k_{\text{proet}}^{\text{inind}}),$$

$$R\Gamma(k_W, A) = f_*[A \overset{F^{-1}}{\rightarrow} A].$$

For $A \in \text{Ab}(k_{\text{prozar}})$, we define $H^n(k_W, A) \in \text{Ab}(k_{\text{prozar}})$ to be the $n$-th cohomology of $R\Gamma(k_W, A),$ and set $\Gamma(k_W, A) = H^0(k_W, A).$ Obviously $H^n(k_W, A) = 0$ for $n \neq 0, 1.$ We have an exact sequence

$$0 \rightarrow \Gamma(k_W, A)(k) \rightarrow A(\overline{k}) \overset{F^{-1}}{\rightarrow} A(\overline{k}) \rightarrow H^1(k_W, A)(k) \rightarrow 0,$$

so that the global section of $R\Gamma(k_W, A) \in D(k_{\text{prozar}})$ is $R\Gamma(W_k, A(\overline{k})) \in D(\text{Ab}),$ where $W_k \cong \mathbb{Z}$ is the Weil group of $k.$ Thus $R\Gamma(k_W, \cdot)$ is an enhancement of the Weil group cohomology of $k$ as a functor valued in $D(k_{\text{prozar}}).

Let Fin be the category of finite abelian groups. Let PFin, IFin, IPFin its procategory, indcategory, ind-procategory, respectively. Note that IFin is equivalent to the category of torsion abelian groups. A finite set $X$ can be identified with a finite $k$-scheme $\bigsqcup_{x \in X} \text{Spec } k$ and hence a finite product $\prod_{x \in X} k$ of copies of $k.$ This extends to a functor from the category of profinite sets to $k_{\text{prozar}} \subseteq k_{\text{inind}}.$ Hence we have an additive functor $\text{IPFin} \rightarrow \text{Ab}(k_{\text{prozar}}) \subseteq \text{Ab}(k_{\text{prozar}}^{\text{inind}}).$ There is also an obvious functor $\text{Ab} \rightarrow \text{Ab}(k_{\text{prozar}})$ that sends an abelian group $A$ to $\bigsqcup_{a \in A} \text{Spec } k.$

**Proposition (10.3).**

(a) We have fully faithful embeddings

$$D^b(\text{Fin}) \rightarrow D^b(\text{Fin}) \rightarrow D^b(\text{Ab}),$$

$$D^b(\text{PFin}) \rightarrow D^b(\text{PFin}) \rightarrow D^b(k_{\text{prozar}}),$$

of triangulated categories.
It follows that it is enough to show the corresponding statement that replaces $\kappa$ with the identity. Then $\bar{A}$ is a (finite) étale $\bar{k}$-scheme. Therefore it is enough to show the corresponding statement that replaces $\kappa_{\text{proet}}$ with $\kappa_{\text{proet}}$, $\Fin$ with $\Et$ and $\Ab$ with $\Et$. Let $f: \Spec \kappa_{\text{proet}} \to \Spec \kappa_{\text{proet}}$ be the morphism defined by the identity. Then $f_* f^* = \text{id}$ on $D(\kappa_{\text{proet}})$, so $f^*: D(\kappa_{\text{proet}}) \to D(\kappa_{\text{proet}})$ is fully faithful. The fully faithful embeddings

$$D^b(\IPAlg, D^b(\Et/\bar{k}) \to D^b(\kappa_{\text{proet}}),$$

in \(\ref{2.3.4}\) then restrict to fully faithful embeddings

$$D^b(\IPF\Et, D^b(\Et/\bar{k}) \to D^b(\kappa_{\text{proet}}).$$

\(\square\) It is enough to show the statements for $A \in \Alg/k$. By Lang’s theorem [Ser88 VI, Prop. 3], we have an exact sequence

$$0 \to A(k) \to A \xrightarrow{F^{-1}} A \to \pi_0(A)_F \to 0,$$

where $\pi_0(A)_F$ is the coinvariants under $F$. The groups $A(k), \pi_0(A)_F$ are finite constant groups and, in particular, locally of finite presentation. Hence $\Gamma(k_W, A) = A(k) \in \Fin$ and $H^1(k_W, A) = \pi_0(A)_F \in \Fin$. \(\square\)

Since $F = 1$ on $\Z$, we have $R\Gamma(k_W, \Z) = \Z \oplus \Z[-1]$. Hence for any $A \in D(\kappa_{\text{proet}})$, we have natural morphisms

$$R\Gamma(k_W, A^{SD}) = R\Gamma(k_W, R\Hom_{k_{\text{proet}}} (A, \Z)) \quad \rightarrow R\Hom_{k_{\text{proet}}} (R\Gamma(k_W, A), R\Gamma(k_W, \Z))$$

$$\rightarrow R\Hom_{k_{\text{proet}}} (R\Gamma(k_W, A), \Z)[-1]$$

$$= R\Gamma(k_W, A)^{\text{PD}}[-2],$$

where we denote $B^{\text{LD}} = R\Hom_{k_{\text{proet}}} (B, \Z)$ (linear dual) and $B^{\text{PD}} = R\Hom_{k_{\text{proet}}} (B, \Q/\Z)$ (Pontryagin dual) for $B \in D(k_{\text{proet}})$.

**Proposition (10.4).** If $A \in \IPA/k$, then the above defined morphisms

$$R\Gamma(k_W, A^{SD}) \to R\Gamma(k_W, A)^{\text{LD}}[-1] \leftrightarrow R\Gamma(k_W, A)^{\text{PD}}[-2]$$

are isomorphisms. If $A \in \F\Et/k$, then the morphism

$$R\Gamma(k_W, A^{SD}) \to R\Gamma(k_W, A)^{\text{LD}}[-1]$$

is an isomorphism.
Proof. For the moment, let \( A \in \text{Ab}(k^\text{indrat}) \) be arbitrary. We use the identification \( \text{Spec} \, \overline{k} \text{proet} \cong \text{Spec} \, k \text{prozar} \). Hence we view the map \( f \) used in \( \text{(10.2)} \) as a continuous map of sites \( \text{Spec} \, k^\text{indrat} \rightarrow \text{Spec} \, \overline{k} \text{proet} \) defined by the identity. Let \( g : \text{Spec} \, k^\text{indrat} \rightarrow \text{Spec} \, k \text{proet} \) be the morphism defined by the identity. Let \( j : \text{Spec} \, \overline{k} \text{proet} \rightarrow \text{Spec} \, k \text{proet} \) be the natural morphism (defined by the functor \( k' \rightarrow k' \otimes \overline{k} \in \text{Spec} \, \overline{k} \text{proet} \)). The pullback \( j^* \) is the restriction functor |\( \text{Spec} \, \overline{k} \). Hence \( f_* = j^* g_* \) as functors \( D(k^\text{indrat}) \rightarrow D(\overline{k} \text{proet}) \).

We have
\[
R\Gamma(k_W, A^{SD}) = j^* g_* R\text{Hom}_{k^\text{indrat}}\left([A \xrightarrow{f} A], Z\right).
\]

On the other hand, the restriction \( j^* \) commutes with derived sheaf-Hom's:
\[
j^* R\text{Hom}(B, C) = R\text{Hom}_{k^\text{proet}}(j^* B, j^* C)
\]

(\cite[Prop. 18.4.6]{KS05}). Together with the adjunction \( g^* \leftrightarrow g_* \) and \( j^* \mathbb{Z} = \mathbb{Z} \) and \( g_* \mathbb{Z} = \mathbb{Z} \), we have
\[
R\Gamma(k_W, A)^{LD}[-1] = R\text{Hom}_{\overline{k} \text{proet}}(j^* g_* [A \xrightarrow{F^{-1}} A], Z) = j^* g_* R\text{Hom}_{k^\text{indrat}}\left(g_* [A \xrightarrow{F^{-1}} A], Z\right).
\]

Hence the morphism \( R\Gamma(k_W, A^{SD}) \rightarrow R\Gamma(k_W, A)^{LD}[-1] \) is an isomorphism if \( g^* g_* [A \rightarrow A] = [A \rightarrow A] \). For a profinite set \( X \) viewed as an object of \( k^\text{indrat} \), we have \( g^* g_* X = g^* X = X \) since \( X \) is representable in \( k \text{proet} \). The same is true if \( X \) is an ind-object of profinite sets (in particular, a set) since both \( g^* \) and \( g_* \) commute with filtered direct limits. We have \( [A \rightarrow A] \in D^b(\text{IPFin}) \) if \( A \in \text{IPAlg}/k \) by \( \text{(10.3)} \) \cite{B}, and \( [A \rightarrow A] \in D^b(\text{Ab}) \) if \( A \in \text{FGeT}/k \). Hence \( g^* g_* [A \rightarrow A] = [A \rightarrow A] \) in either case. (Note that \( g^* g_* \mathbb{G}_a \) is not \( \mathbb{G}_a \) but the étale group whose group of geometric points is the discrete \( \text{Gal}(\overline{k}/k) \)-modules of the additive group \( \mathbb{G}_a \).)

For the isomorphism \( R\Gamma(k_W, A)^{PD}[-2] = R\Gamma(k_W, A)^{LD}[-1] \) for \( A \in \text{IPAlg}/k \), it is enough to show that \( R\text{Hom}_{\overline{k} \text{proet}}(B, \mathbb{Q}) = 0 \) for any \( B \in \text{IPFin} \). We can see this by the same proof as \( \text{(2.3.1)} \) \cite{B} or by reducing to it by \( R\text{Hom}_{k^\text{indrat}}\left(\overline{k} \text{proet}, \mathbb{Q}\right) = \overline{j}_* R\text{Hom}_{k^\text{proet}}(B, \mathbb{Q}) = 0 \), where \( \overline{j} : \text{Spec} \, k^{\text{indrat}} \rightarrow \text{Spec} \, k \text{proet} \) is the morphism defined by the identity. \( \square \)

Let \( A \in \text{Ab}(K_\text{ffpf}/k^\text{indrat}) \). Assume that \( R\Gamma(K, A) \) is P-acyclic. Then
\[
R\Gamma(k_W, R\Gamma(K, A))(k) = [R\Gamma(\hat{K}^{ur}, A) \xrightarrow{F^{-1}} R\Gamma(\hat{K}^{ur}, A)][-1].
\]

This is equal to \( R\Gamma(W_K, A(L)) \) when the fppf cohomology \( R\Gamma(\hat{K}^{ur}, A) \) agrees with the étale cohomology \( R\Gamma(K^{et}, A) \) (such as when \( A \) is a smooth group scheme over \( K \)), where \( W_K \) is the Weil group of \( K \) and \( L \) a separable closure of \( K^{ur} \). For this reason, we denote \( \text{et} \)
\[
R\Gamma(K_W, A) = R\Gamma(k_W, R\Gamma(K, A)) \in D(k \text{prozar}),
\]
\[
H^n(K_W, A) = H^n R\Gamma(K_W, A) \in \text{Ab}(k \text{prozar})
\]

\(19\) The notation \( R\Gamma(K_W, \cdot) \) reads “Weil-fppf cohomology of the scheme Spec \( K \),” while \( R\Gamma(W_K, \cdot) \) is group cohomology of the topological group \( W_K \). It is not good to choose the latter as general notation for the cohomology theory defined here since fppf cohomology is not always described as group cohomology.
for any \( A \in \text{Ab}(\mathcal{K}_{\text{proff}}/\mathcal{K}_{\text{indrat}}) \). Whenever \( R\Gamma(K, A) \) is \( \mathcal{P} \)-acyclic and in \( D^b(\text{IPAlg}/k) \) (resp. \( D^b(\text{Et}/k) \)), which happens in many cases by (3.3), the complex \( R\Gamma(K, A) \) is thus an enhancement of the Weil group cohomology of \( K \) as a complex of ind-profinite groups (resp. finitely generated abelian groups). The above proposition says that the Serre duality for \( R\hat{\Gamma}(K, A) \) is translated into the linear duality or Pontryagin duality for \( R\Gamma(K, A) \) up to shift.

For example, let \( A \) be an abelian variety over \( K \). Then our duality theorem (4.1.2) and (4.2.2) show that
\[
\left[ \lim_{n} \Gamma(K, A^\vee) \to R\Gamma(K, A^\vee) \right] = R\Gamma(K, A)^{SD}[1].
\]

Apply \( R\Gamma(k_W, \cdot) \). If \( \Gamma(K, A^\vee)_{\text{sAb}} \) denotes the semi-abelian quotient of \( \Gamma(K, A^\vee)_0 \), which is quasi-algebraic, then \( R\Gamma(k_W, \Gamma(K, A^\vee)_{\text{sAb}}) \) is finite by (10.3). Therefore

\[
R\Gamma(k_W, \lim_n \Gamma(K, A^\vee)) = R\lim_{n} R\Gamma(k_W, \Gamma(K, A^\vee)_{\text{sAb}}) = 0.
\]

Hence (10.3) shows that we have an isomorphism
\[
R\Gamma(K_W, A^\vee) = R\Gamma(K_W, A)^{PD}[-1]
\]
in \( D^b(\text{IPFin}) \). The description of \( H^n(K, A) \) in (3.4.3) (4) shows that \( H^0(K_W, A) = A(K) \) is a profinite group, \( H^1(K_W, A) = H^1(K, A) \) an indfinite (i.e., torsion) group, and \( H^n(K_W, A) = 0 \) for \( n \neq 0, 1 \). This recovers Tate-Milne’s duality (Tat58 Mil70 Mil72) (see also Mil06 I, Cor. 3.4 and III, Thm. 7.8).

If \( T \) is a torus over \( K \), then the same procedure gives isomorphisms
\[
R\Gamma(K_W, T^{\text{CD}}) = R\Gamma(K_W, T)^{\text{LD}}[-1]
\]
\[
R\Gamma(K_W, T) = R\Gamma(K_W, T^{\text{CD}})^{\text{LD}}[-1]
\]
in \( D^b(\mathcal{K}_{\text{prozar}}) \). We see that \( H^0(K_W, T) = T(K) \) is an extension of a finite free abelian group by a profinite group, \( H^1(K_W, T) \) a finitely generated abelian group and \( H^n(K_W, T) = 0 \) for \( n \neq 0, 1 \). We also see that \( H^0(K_W, T^{\text{CD}}) \) and \( H^1(K_W, T^{\text{CD}}) \) are finitely generated abelian groups, \( H^2(K_W, T^{\text{CD}}) = H^1(K, \tilde{T}_{\text{tor}}) \) an indfinite (i.e., torsion) group and \( H^n = 0 \) for \( n \neq 0, 1, 2 \). Let \( C \) be an algebraically closed field of characteristic zero. We view \( C^\times \) as a constant group scheme over \( k \) and let \( \tilde{T}(C) = T^{\text{CD}} \otimes_{\mathbb{Z}} C^\times \) be the \( C \)-value points of the dual torus. Taking \( \otimes_{\mathbb{Z}} C^\times \) of the above isomorphism, we have
\[
R\Gamma(K_W, \tilde{T}(C)) = R\text{Hom}_{\mathcal{K}_{\text{prozar}}}(R\Gamma(K_W, T), C^\times)[-1]
\]
by the same method as the proof of (8.3). From \( H^1 \), we get
\[
H^1(K_W, \tilde{T}(C)) = \text{Hom}_{\mathcal{K}_{\text{prozar}}}(\Gamma(K, T), C^\times).
\]
This recovers the local Langlands correspondence for tori (Lan97).

If \( M \) is a 1-motive over \( K \), then we have
\[
R\Gamma(k_W, R\Gamma(K, M)^{SDSD}) = R\Gamma(k_W, R\Gamma(K, M))
\]
by reducing it to the previous cases of abelian varieties, tori and lattices by the weight filtration, and an isomorphism
\[
R\Gamma(K_W, M^\vee) = R\Gamma(K_W, M)^{LD}
\]
in $D^b(k_{prozar})$. This case, it is a little bit complicated to describe each $H^n(K_W, M)$ due to its possible uniquely divisible part. This is essentially a result of Harari-Szamuely [HS05 Thm. 2.3].

If $N$ is a finite flat group scheme over $K$, then the Bégneri-Bester-Kato duality

$$RG(K, N^{CD}) = RG(K, N)^{SD}$$

(see Thm. [Suz13 Thm. 2.7.1] for this form of the statement) implies

$$RG(K_2, N^{CD}) = RG(K_2, N)^{PD}[-2]$$

in $D^b(IPFin)$. We see that $H^0(K_W, N) = H^0(K, N)$ and $H^2(K_W, N) = H^2(K, N)$ are finite, $H^1(K_W, N)$ the group $H^1(K, N)$ endowed with an ind-profinite group structure, and $H^n = 0$ for $n \neq 0, 1, 2$. If $K$ has mixed characteristic, then $H^1(K_W, N)$ is finite. This recovers Tate-Shatz’s duality [Ser02 II, §5, Thm. 2], [Sha64].

Remark (10.5). As in [Sha64], the group $H^1(K, N)$ for equal characteristic $K$ is not only ind-profinite but locally compact (see also [Mil06 III, §6] and [ˇCes15]), i.e., it can be written as a filtered union $\bigcup B_\lambda$ of profinite groups such that $B_\mu/B_\lambda$ is finite for any $\mu \geq \lambda$. In our setting, we can deduce this from the fact (true for general perfect $k$) that $H^1(K, N) \in IPAlg/k$ can be written as a filtered union $\bigcup C_\lambda$ of proalgebraic groups such that $C_\mu/C_\lambda$ is quasi-algebraic for any $\mu \geq \lambda$. For the moment, let us say such an ind-proalgebraic group algebraically growing. To see that $H^1(K, N)$ is algebraically growing, note that this is true for $N = \alpha_p, \mu_p, \mathbb{Z}/n\mathbb{Z}$ as seen from the proof of [3.13]. Let $0 \to C_1 \to C_2 \to C_3 \to 0$ be an exact sequence in IPAlg/k. If $C_2$ is algebraically growing, so is $C_1$. If $C_1$ and $C_3$ are both algebraically growing, so is $C_2$. If $C_2$ is algebraically growing and $C_1 \in PAlg/k$, then $C_3$ is algebraically growing. Then the general finite flat group case follows from the proof of the same proposition.

### Appendix A. The pro-fppf topology for proalgebraic proschemes

In this appendix, we briefly explain how (2.3.1), originally proved in [Suz13 §3] for affine proalgebraic groups $A$, can be extended for arbitrary proalgebraic groups. For abelian varieties, the same proof as [Suz13] works. Although the general case including infinite products of abelian varieties is not needed in the main body of this paper, we treat the general case in this appendix for completeness. The problem is that infinite products of abelian varieties are not schemes, but proschemes. Let $k_{proalg}$ be the procategory of the category of quasi-algebraic $k$-schemes (i.e., perfections of $k$-schemes of finite type). We call an object of $k_{proalg}$ a proalgebraic $k$-proscheme. Examples are infinite products of perfections of proalgebraic groups. Proalgebraic groups are group objects in $k_{proalg}$. In [Suz13 §3.1], we introduced the pro-fppf topology on the category of perfect affine $k$-schemes. Below we extend it to the category $k_{proalg}$ of proalgebraic $k$-proschemes in the paragraph after (A.3). We also show in (A.4) that the pro-fppf site of proalgebraic $k$-proschemes has an equivalent topos to the pro-fppf site of perfect affine $k$-schemes. Then most part of [Suz13 §3] goes without large modifications.

Generalizing [Suz13 §3.1] from the affine case, we say that a morphism of quasi-compact quasi-separated perfect $k$-schemes $Y \to X$ is of finite presentation if it is the perfection of a $k$-scheme morphism $Y_0 \to X$ of finite presentation in the usual sense. If moreover $Y_0 \to X$ can be chosen to be (faithfully) flat, we say that
$Y \to X$ is (faithfully) flat of finite presentation (in the perfect sense). A quasi-compact quasi-separated perfect $k$-scheme can naturally be viewed as an object of $k^{\text{proalg}}$ since it is a filtered inverse limit of quasi-algebraic $k$-schemes with affine transition morphisms (use the absolute noetherian approximation [TT90 Thm. C.9] and apply the perfection). We say that a morphism $Y \to X$ in $k^{\text{proalg}}$ is of finite presentation if it can be written as the base change of a morphism of finite presentation $Y_0 \to X_0$ in the above perfect sense between quasi-compact quasi-separated perfect $k$-schemes by some morphism $X \to X_0$ in $k^{\text{proalg}}$. If $Y_0 \to X_0$ can be chosen to be (faithfully) flat of finite presentation (in the above perfect sense), then we say that $Y \to X$ is (faithfully) flat of finite presentation. We say that a morphism $Y \to X$ in $k^{\text{proalg}}$ is (faithfully) flat of profinite presentation if there exists a filtered inverse system $\{Y_\lambda\}$ in $k^{\text{proalg}}$ such that each morphism $Y_\lambda \to X$ is (faithfully) flat of finite presentation and $Y$ is isomorphic to $\varprojlim_\lambda Y_\lambda$ over $X$ in $k^{\text{proalg}}$. Here are two basic permanence properties of morphisms (faithfully) flat of finite presentation.

**Proposition (A.1).** If a morphism $Y \to X$ in $k^{\text{proalg}}$ is (faithfully) flat of finite presentation, then it can be written as the base change of a morphism of finite presentation $Y_0 \to X_0$ in $k^{\text{proalg}}$.

**Proof.** We may assume that $X$ and $Y$ are quasi-compact quasi-separated perfect $k$-schemes. Write $Y \to X$ as the perfection of a morphism $Y'_0 \to X$ (faithfully) flat of finite presentation in the usual sense. By [TT90 Thm. C.9], we can write $X$ as a filtered inverse limit of algebraic $k$-schemes with affine transition morphisms. Hence the permanence property of flatness (resp. surjectivity) under passage to limits [Gro66 Thm. 11.2.6] (resp. [Gro66 Thm. 8.10.5]) shows that $Y'_0 \to X$ can be written as the base change of a morphism $Y_{00} \to X_{00}$ (faithfully) flat of finite presentation in the usual sense between algebraic $k$-schemes by some morphism $X \to X_{00}$. Let $Y_0 \to X_0$ be the perfection of $Y_{00} \to X_{00}$, which is (faithfully) flat of finite presentation (in the perfect sense) between quasi-algebraic $k$-schemes. Its base change by $X \to X_0$ is $Y \to X$. □

**Proposition (A.2).** Let $\{X_\lambda\}$ be a filtered inverse system in $k^{\text{proalg}}$ with inverse limit $X$. Let $Y \to X$ be a (faithfully) flat morphism of finite presentation in $k^{\text{proalg}}$. Then there exist an index $\lambda$, an object $Y_\lambda$ of $k^{\text{proalg}}$ and a (faithfully) flat morphism of finite presentation $Y_\lambda \to X_\lambda$ in $k^{\text{proalg}}$ such that the base change of $Y_\lambda \to X_\lambda$ by $X \to X_\lambda$ gives the original morphism $Y \to X$.

**Proof.** As in the previous proposition, let $Y_0 \to X_0$ be a (faithfully) flat morphism of finite presentation between quasi-algebraic $k$-schemes whose base change to $X$ by some morphism $X \to X_0$ gives the morphism $Y \to X$. Then the morphism $X \to X_0$ factors through some $X_\lambda$. Let $Y_\lambda \to X_\lambda$ be the base change of $Y_0 \to X_0$ by the morphism $X_\lambda \to X_0$. Then $Y_\lambda \to X_\lambda$ is (faithfully) flat of finite presentation, and its base change to $X$ is the morphism $Y \to X$. □

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20 The notion of morphisms of finite presentation in the sense here is Zariski local in the source and the target, as shown in [Ser69 §1, Cor. to Prop. 9] when $X = \text{Spec} \, k$ and $Y$ separated, and in [BS17 Prop. 3.13] in general. But it is not clear whether the notion of flat morphisms of finite presentation is Zariski local or not. The definition given here is sufficient for defining a nice enough category of sheaves.
A fact already used above is that a base change of a morphism (faithfully) flat of finite presentation is also. The same is true for a morphism (faithfully) flat of profinite presentation. For composites, we have the following.

**Proposition (A.3).** Let $Z \to Y \to X$ be morphisms in $\mathcal{k}_{\text{proalg}}$. If $Z \to Y$ and $Y \to X$ are (faithfully) flat of finite presentation, then so is $Z \to X$. If $Z \to Y$ and $Y \to X$ are (faithfully) flat of profinite presentation, then so is $Z \to X$.

**Proof.** Assume that $Z \to Y$ and $Y \to X$ are (faithfully) flat of finite presentation. We first treat the case that $X, Y, Z$ are quasi-compact quasi-separated perfect schemes. Write $Z \to Y$ (resp. $Y \to X$) as the perfection of a morphism $Z_0 \to Y$ (resp. $Y_0 \to X$) (faithfully) flat of finite presentation in the usual sense. For any $n \geq 0$, let $Y_0^{(n)}$ be the scheme $Y_0$ with a new $k$-scheme structure given by the composite $Y_0 \to \text{Spec } k \to \text{Spec } k$, where the first morphism is the original $k$-scheme structure morphism of $Y_0$ and the second morphism is the $p^n$-th power absolute Frobenius of Spec $k$. The scheme $Y_0^{(n)}$ is (faithfully) flat of finite presentation over $X$ (= $X^{(n)}$) in the usual sense. Since $Y = \lim_{\rightarrow} Y_0^{(n)}$, the permanence property of flatness (resp. surjectivity) under passage to limits [Gro66, Thm. 11.2.6] (resp. [Gro66, Thm. 8.10.5]) shows that we can write $Z_0 \to Y$ as the base change of some (faithfully) flat morphism of finite presentation $Z_0 \to Y_0^{(n)}$ in the usual sense by the morphism $Y \to Y_0^{(n)}$. Then $Z \to X$ is the perfection of the composite $Z_0 \to Y_0^{(n)} \to X$, which is (faithfully) flat of finite presentation in the usual sense. Hence $Z \to X$ is (faithfully) flat of finite presentation (in the perfect sense).

We treat the general case. Write $X$ as a filtered inverse limit of quasi-algebraic $k$-schemes $X_\lambda$. Write $Y \to X$ as the base change of a morphism $Y_\lambda \to X_\lambda$ (faithfully) flat of finite presentation (in the perfect sense) as in the previous proposition. Let $Y_\mu = Y_\lambda \times_{X_\lambda} X_\mu$ for $\mu \geq \lambda$. Then $Y = \lim_{\mu \geq \lambda} Y_\mu$. Write $Z \to Y$ as the base change of a morphism $Z_\mu \to Y_\mu$ (faithfully) flat of finite presentation as in the previous proposition. Then $Z \to X$ is the base change of the composite $Z_\mu \to Y_\mu \to X_\mu$, which is (faithfully) flat of finite presentation by the quasi-compact quasi-separated case treated earlier.

Next assume that $Z \to Y$ and $Y \to X$ are (faithfully) flat of profinite presentation. Write $Z$ (resp. $Y$) as a filtered inverse limit of morphisms $Z_\mu \to Y$ (resp. $Y_\lambda \to X$) (faithfully) flat of finite presentation. By a similar argument as [Swa98, Lem. 1.5], it is enough to show that any morphism $Z \to Z'$ over $X$ to an object $Z'$ over $X$ of finite presentation factors through an object over $X$ (faithfully) flat of finite presentation. The $X$-morphism $Z \to Z'$ factors through some $Z_\mu$. Write the morphism $Z_\mu \to Y$ as the base change of some morphism $Z_\lambda \to Y_\lambda$ (faithfully) flat of finite presentation as in the previous proposition. Let $Z_{\lambda'} = Z_\lambda \times_{Y_\lambda} Y_{\lambda'}$ for $\lambda' \geq \lambda$. Then $Z_\mu = \lim_{\lambda' \geq \lambda} Z_{\lambda'}$. Then the $X$-morphism $Z \to Z'$ factors through $Z_{\lambda'}$ for some $\lambda' \geq \lambda$. The composite $Z_{\lambda'} \to Y_{\lambda'} \to X$ is (faithfully) flat of finite presentation by what we have proven above. Thus $Z \to X$ is (faithfully) flat of profinite presentation.

With this, we can define the perfect fpf site of proalgebraic $k$-proschemes, $\text{Spec } k_{\text{fpf}}$, to be the category $k_{\text{proalg}}$ endowed with the topology where a covering $\{Y_i \to X\}$ is a finite family of morphisms such that $\bigcup Y_i \to X$ is faithfully
flat of finite presentation (which implies that each \( Y_i \to X \) is flat of finite presentation). Replacing “faithfully flat of finite presentation” by “faithfully flat of profinite presentation”, we can define the perfect pro-fppf site of proalgebraic \( k \)-proschemes \( \text{Spec} \, k_{\text{proalg}} \) in the same way.

As in the main text, let \( k_{\text{perf}} \) be the category of perfect \( k \)-algebras or perfect affine \( k \)-schemes. The identity functor defines a morphism of sites \( \text{Spec} \, k_{\text{proalg}} \to \text{Spec} \, k_{\text{perf}} \).

**Proposition (A.4).** The morphism of sites \( \text{Spec} \, k_{\text{proalg}} \to \text{Spec} \, k_{\text{perf}} \) defined by the identity induces an equivalence on the topoi.

In particular, Ext groups and Ext sheaves between proalgebraic groups can be calculated using either \( \text{Spec} \, k_{\text{proalg}} \) or \( \text{Spec} \, k_{\text{perf}} \).

To prove the proposition, the key is that a proalgebraic \( k \)-proscheme is pro-Zariski locally affine \((\text{A.6}) \) below). This is trivial for a proalgebraic \( k \)-proscheme \( X = \lim_{\leftarrow} X_i \) indexed by integers \( i \geq 1 \): we can find an affine “pro-Zariski” covering \( Y = \lim_{\leftarrow} Y_i \) of \( X \) by successively taking affine Zariski coverings \( Y_1 \to X_1, Y_2 \to Y_1 \times X_1, Y_3 \to Y_2 \times X_2, X_3 \) and so on. Here we say that a surjective morphism of schemes \( Y \to Z \) is an (affine) Zariski covering if there exists a decomposition \( Y = \bigsqcup Y_\lambda \) into disjoint (affine) open subschemes such that the composite \( Y_\lambda \to Y \to Z \) is an open immersion for any \( \lambda \). The difficulties thus come from the complexity of the index set.

Recall that a poset (partially ordered set) is a pre-ordered set such that \( \lambda \leq \lambda' \leq \lambda \) implies \( \lambda = \lambda' \). A pre-ordered set \( \Lambda \) is said to be cofinite if for every \( \lambda \in \Lambda \), there are only finitely many elements \( \leq \lambda \). An element \( \lambda \) of a pre-ordered set is said to be greatest if \( \lambda' \leq \lambda \) for any \( \lambda' \). If \( \Lambda \) is a directed set without a greatest element and \( \Lambda \) is the set of all finite subsets \( \Lambda' \subset \Lambda \) having a unique greatest element \( \max(\Lambda') \), then \( \Lambda \) is a cofinite poset by inclusion, and the map \( \Lambda' \mapsto \max(\Lambda') \) from \( \Lambda \) to \( \Lambda \) is cofinal. Therefore every filtered inverse system in a category is isomorphic (in the procategory) to a system indexed by a cofinite directed poset (again implicit in the proof of \([AGV72\text{ I, Prop. 8.1.6}] \)). If \( \Lambda \) has a greatest element \( \lambda \), then \( \{\lambda\} \to \Lambda \) is cofinal. Therefore every filtered inverse system in a category is isomorphic (in the procategory) to a system indexed by a cofinite directed poset (again implicit in the proof of \([AGV72\text{ loc. cit.}] \)). For a cofinite poset \( \Lambda \) and an element \( \lambda \in \Lambda \), we define \( n(\lambda) \) to be the largest integer \( n \geq 1 \) such that there exists a chain \( \lambda_1 < \cdots < \lambda_{n-1} < \lambda \) of elements of \( \Lambda \) (so \( n(\lambda) = 1 \) if \( \lambda \) is minimal). This function satisfies the property that if \( \lambda \leq \lambda' \), then \( \lambda = \lambda' \) or \( n(\lambda) < n(\lambda') \).

**Proposition (A.5).** Let \( \{X_\lambda\}_{\lambda \in \Lambda} \) be an inverse system of quasi-compact quasi-separated \( k \)-schemes. Assume that the index set \( \Lambda \) is a cofinite poset \( ^2 \). Then there exists an inverse system \( \{Y_\lambda\} \) of affine \( k \)-schemes and a system \( \{Y_\lambda \to X_\lambda\} \) of \( k \)-scheme morphisms such that each \( Y_\lambda \to X_\lambda \) is a Zariski covering.

**Proof.** We will construct a system

\[
\{ \cdots \to Y_\lambda^{n+1} \to Y_\lambda^n \to \cdots \to Y_\lambda^0 = X_\lambda \}
\]

of sequences of Zariski coverings of quasi-compact quasi-separated \( k \)-schemes such that for each \( n \) and \( \lambda \) with \( n(\lambda) \leq n \), the scheme \( Y_\lambda^n \) is affine and the morphisms \( \cdots \to Y_\lambda^{n+1} \to Y_\lambda^n \) after the \( n \)-th term are isomorphisms. Then, letting \( Y_\lambda = Y_\lambda^n \) for \( n(\lambda) \leq n \), we will obtain a required system \( \{Y_\lambda\} \) of coverings.

\(^2\)We do not need to assume that \( \Lambda \) is directed, though we only need the directed case below.
Fix an integer \( n \geq 0 \). Suppose that we have such a sequence \( \{ Y^n_\lambda \to \cdots \to Y^0_\lambda \}_\lambda \) up to \( n \). For each \( \lambda \) with \( n(\lambda) = n + 1 \), we choose an affine Zariski covering \( Y^{n+1}_\lambda \to Y^n_\lambda \). For general \( \lambda \), we define \( Y^{n+1}_\lambda = \prod_{\lambda' \leq \lambda, \ n(\lambda') = n+1} (Y^{n+1}_{\lambda'} \times_{Y^n_{\lambda'}} Y^n_\lambda) \), where \( \prod_{Y^n_\lambda} \) denotes the fiber product over \( Y^n_\lambda \). This is a finite product since \( \Lambda \) is cofinite. The property of the function \( n(\lambda) \) above shows that this definition is consistent with \( Y^{n+1}_\lambda \) taken early for \( n(\lambda) = n + 1 \), and the natural morphism \( Y^{n+1}_\lambda \to Y^n_\lambda \) is an isomorphism for \( n(\lambda) \leq n \). The morphism \( Y^{n+1}_\lambda \to Y^n_\lambda \) for general \( \lambda \) is a Zariski covering since it is a finite fiber product of Zariski coverings. Each \( Y^{n+1}_\lambda \) is a quasi-compact quasi-separated \( k \)-scheme since it is a fiber product of such. The morphisms \( Y^{n+1}_\lambda \to Y^n_\lambda \) form a system. By induction, we obtain a required sequence. □

We also need to compare the topology on \( k^{\text{perf}} \) induced from \( \text{Spec} k^{\text{proalg}} \) and the topology of \( \text{Spec} k^{\text{proppf}} \) defined in \( \text{(2.1)} \). The following shows that they agree up to refinement.

**Proposition (A.6).** If a morphism \( \text{Spec} S \to \text{Spec} R \) between perfect affine (hence quasi-compact quasi-separated) \( k \)-schemes is (faithfully) flat of profinite presentation in the sense of this appendix, then there exists a homomorphism \( S \to T \) faithfully flat of ind-finite presentation in the sense of \( \text{(2.1)} \) between perfect \( k \)-algebras,\(^\text{22}\) such that \( R \to T \) is (faithfully) flat of ind-finite presentation in the sense of \( \text{(2.1)} \).

**Proof.** Let \( X = \text{Spec} R \) and \( Y = \text{Spec} S \). Write \( Y \to X \) as a filtered inverse limit of morphisms \( Y_\lambda \to X \) (faithfully) flat of finite presentation in the sense of this appendix with \( Y_\lambda \) quasi-compact quasi-separated \( k \)-schemes. We may assume that the index set \( \Lambda \) is a cofinite directed poset. Take a system of affine Zariski coverings \( \{ Z_\lambda = \text{Spec} T_\lambda \to Y_\lambda \} \) as in the previous proposition. The scheme \( Z_\lambda \) for any \( \lambda \) is perfect since \( Y_\lambda \) is so. For each \( \lambda \), the composite \( Z_\lambda \to Y_\lambda \to X \) is (faithfully) flat of finite presentation in the sense of this appendix by \( \text{(A.3)} \). Therefore \( Z_\lambda \to X \) can be written as the perfection of a morphism \( Z_{\lambda 0} \to X \) (faithfully) flat of finite presentation in the usual sense, where \( Z_{\lambda 0} \) is a priori a quasi-compact quasi-separated \( k \)-scheme. Since \( Z_\lambda \) is affine, we know that \( Z_{\lambda 0} \) is actually affine by \( \text{(IT90) Prop. C.6} \). Hence \( R \to T_\lambda \) is (faithfully) flat of finite presentation in the sense of \( \text{(2.1)} \). Thus \( T = \varprojlim T_\lambda \) is (faithfully) flat of ind-finite presentation over \( R \) in the sense of \( \text{(2.1)} \). Let \( Z = \text{Spec} T = \varprojlim Z_\lambda \). Then \( Z = \varprojlim Z_\lambda \times_{Y_\lambda} Y \) and \( S \to T_\lambda \otimes_{S_\lambda} S \) is faithfully flat of finite presentation in the sense of \( \text{(2.1)} \). Therefore \( S \to T \) is faithfully flat of ind-finite presentation in the sense of \( \text{(2.1)} \). (Of course this proof shows that \( Z \to Y \) is actually a pro-Zariski covering.) □

**Proof of (A.4).** Let \( S = \text{Spec} k^{\text{proppf}} \) and \( S' = \text{Spec} k^{\text{proalg}} \). Let \( S'' \) be the site on \( k^{\text{perf}} \) with topology induced from \( \text{Spec} k^{\text{proalg}} \) \( [AGV72] \) Exp. II, \( \S 3 \). We have morphisms \( S' \to S'' \to S \) of sites defined by the identity. A sieve on an object of \( k^{\text{perf}} \) is a covering for \( S \) if and only if it is for \( S'' \) by the previous proposition. Hence

\(^{22}\) This means that \( S \to T \) is flat of ind-finite presentation in the sense of \( \text{(2.1)} \) and faithfully flat.
$S'' \to S$ as sites. Any object of $S'$ is covered by an object of $S''$ by \(\text{(A.5)}.\) Hence $S' \to S''$ induces an equivalence on the topoi by \[AGV72\] Exp. II, Thm. 4.1. \qed

We can define the étale site $\text{Spec}_{\text{proalg}}$ and the pro-étale site $\text{Spec}_{\text{proalg}}$ on the category $[\text{proalg}]$ in the same way. See \[Gro67\] Prop. 17.7.8 for the necessary permanence property of étaleness under passage to limits. We can show that the morphism of sites $\text{Spec}_{\text{proalg}} \to \text{Spec}_{\text{proalg}}$ defined by the identity induces an equivalence on the topoi by the same proof as above.

We generalize some more definitions in \[Suz13\] §3. A proalgebraic $k$-scheme can be viewed as a locally ringed space since the category of locally ringed spaces has all inverse limits (\[Tem11\] Rmk. 2.1.1, \[Chi11\] Cor. 5). If $\{X_\lambda\}$ is a filtered inverse system of quasi-compact quasi-separated perfect $k$-schemes, then the underlying topological space of $X = \varprojlim X_\lambda$ is the inverse limit of the underlying topological spaces of $X_\lambda$. The structure sheaf of $X$ is the direct limit of the pullbacks by $X \to X_\lambda$ of the structure sheaves of $X_\lambda$. A morphism $Y \to X$ in $[\text{proalg}]$ is faithfully flat of (pro)finite presentation if and only if it is flat of (pro)finite presentation and surjective on the underlying topological spaces. We say that $Y \to X$ is dominant if its image on the underlying topological spaces is dense in $X$. (This definition will be needed to generalize \[Suz13\] §3.6.) Taking $X_\lambda$ to be quasi-algebraic, we can define profinite sets of points of $X = \varprojlim X_\lambda$ and, if the transition morphisms $X_\mu \to X_\lambda$ are flat (or more generally, send irreducible components of $X_\mu$ onto dense subsets of irreducible components of $X_\lambda$), the generic point of $X$, in the same way as \[Suz13\] Def. 3.2.1. They are Spec’s of ind-rational $k$-algebras.

Then all of the discussions and results in \[Suz13\] §3 work without modifications with perfect affine $k$-schemes generalized to proalgebraic $k$-proschemes and with affine proalgebraic groups generalized to arbitrary proalgebraic groups. This proves \[2.3.1\] in full generality.

\section*{References}

\[AGV72\] Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J.-L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

\[AM76\] M. Artin and J. S. Milne. Duality in the flat cohomology of curves. \textit{Invent. Math.}, 35:111–129, 1976.

\[Art62\] M. Artin. Grothendieck topologies. Cambridge, Mass.: Harvard University. 133 p. (1962), 1962.

\[Art69\] M. Artin. Algebraic approximation of structures over complete local rings. \textit{Inst. Hautes Études Sci. Publ. Math.}, (36):23–58, 1969.

\[BB00\] Alessandra Bertapelle and Siegfried Bosch. Weil restriction and Grothendieck’s duality conjecture. \textit{J. Algebraic Geom.}, 9(1):155–164, 2000.

\[Bég81\] Lucile Bégneri. Duality sur un corps local à corps résiduel algébriquement clos. \textit{Mém. Soc. Math. France (N.S.)}, (4):121, 1980/81.

\[Ber01\] Alessandra Bertapelle. On perfectness of Grothendieck’s pairing for the $l$-parts of component groups. \textit{J. Reine Angew. Math.}, 538:223–236, 2001.

\[Ber03\] Alessandra Bertapelle. Local flat duality of abelian varieties. \textit{Manuscripta Math.}, 111(2):141–161, 2003.

\[Bes78\] Michal Bester. Local flat duality of abelian varieties. \textit{Math. Ann.}, 235(2):149–174, 1978.

\[BGA13\] Alessandra Bertapelle and Cristian D. González-Aviles. The Greenberg functor revisited. Preprint, \texttt{arXiv:1311.0351v4}, 2013.

\[BGA15\] Alessandra Bertapelle and Cristian D. González-Aviles. On the cohomology of tori over local fields with perfect residue field. \textit{Israel J. Math.}, 206(1):431–455, 2015.
GROTHENDIECK’S PAIRING ON NÉRON COMPONENT GROUPS

[BGA17] Alessandra Bertapelle and Cristian D. González-Avilés. On the perfection of schemes. *Expositiones Mathematicae*, 2017. DOI: 10.1016/j.exmath.2017.08.001.

[BIK13] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Module categories for group algebras over commutative rings. *J. K-Theory*, 11(2):297–329, 2013. With an appendix by Greg Stevenson.

[BL02] Siegfried Bosch and Dino Lorenzini. Grothendieck’s pairing on component groups of Jacobians. *Invent. Math.* 148(2):353–396, 2002.

[BLR90] Siegfried Bosch, Werner Lükebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990.

[Bos97] Siegfried Bosch. Component groups of abelian varieties and Grothendieck’s duality conjecture. *Ann. Inst. Fourier (Grenoble)*, 47(5):1257–1287, 1997.

[Bou03] Nicolas Bourbaki. *Algebra II. Chapters 4–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2003. Translated from the 1990 English edition by P. M. Cohn and J. Howie, Reprint of the 1998 English translation.

[BS15] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *Astérisque*, (369):99–201, 2015.

[BS17] Bhargav Bhatt and Peter Scholze. Projectivity of the Witt vector affine Grassmannian. *Inventiones mathematicae*, 209(2):329–423, Aug 2017.

[Čes15] Kestutis Česnavičius. Topology on cohomology of local fields. *Forum Math. Sigma*, 3:e16, 55, 2015.

[Con12] Brian Conrad. Weil and Grothendieck approaches to adelic points. *Enseign. Math. (2)*, 58(1-2):61–97, 2012.

[CR15] Clifton Cunningham and David Roe. From the function-sheaf dictionary to quasicharacters of p-adic tori. *Journal of the Institute of Mathematics of Jussieu*, pages 1–37, 2015.

[DG70] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice *Corps de classes local* par Michiel Hazewinkel.

[Gil11] William D. Gillam. Localization of ringed spaces. *Advances in Pure Mathematics*, 1(5):259–263, 2011.

[Gre61] Marvin J. Greenberg. Schemata over local rings. *Ann. of Math.* (2), 73:624–648, 1961.

[Gre66] Marvin J. Greenberg. Rational points in Henselian discrete valuation rings. *Inst. Hautes Études Sci. Publ. Math.*, (31):59–64, 1966.

[Gro64] A. Grothendieck. Éléments de géométrie algébrique (rédigé avec la collaboration de Jean Dieudonné). IV. Étude locale des schémas et des morphismes de schémas. I. *Inst. Hautes Études Sci. Publ. Math.*, (20):259, 1964.

[Gro66] A. Grothendieck. Éléments de géométrie algébrique (rédigé avec la collaboration de Jean Dieudonné). IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.

[Gro67] A. Grothendieck. Éléments de géométrie algébrique (rédigé avec la collaboration de Jean Dieudonné). IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.

[Gro72] *Groupes de monodromie en géométrie algébrique. I*. Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim.

[HN11] Lars Halvard Halle and Johannes Nicaise. Motivic zeta functions of abelian varieties, and the monodromy conjecture. *Adv. Math.*, 227(1):610–653, 2011.
[HS05] David Harari and Tamás Szamuely. Arithmetic duality theorems for 1-motives. *J. Reine Angew. Math.*, 578:93–128, 2005.

[Kat86] Kazuya Kato. Duality theories for the $p$-primary étale cohomology. I. In *Algebraic and topological theories (Kinosaki, 1984)*, pages 127–148. Kinokuniya, Tokyo, 1986.

[Kat91] Kazuya Kato. Generalized class field theory. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 419–428. Math. Soc. Japan, Tokyo, 1991.

[Koy93] Yoshihiro Koya. A generalization of Tate-Nakayama theorem by using hypercohomology. *Proc. Japan Acad. Ser. A Math. Sci.*, 69(3):53–57, 1993.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[Lan97] R. P. Langlands. Representations of abelian algebraic groups. *Pacific J. Math.*, (Special Issue):231–250, 1997. Olga Taussky-Todd: in memoriam.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[McC86] William G. McCallum. Duality theorems for Néron models. *Duke Math. J.*, 53(4):1093–1124, 1986.

[Mil70] J. S. Milne. Weil-Châtelet groups over local fields. *Ann. Sci. École Norm. Sup.* (4), 3:273–284, 1970.

[Mil72] J. S. Milne. Addendum: “Weil-Châtelet groups over local fields” (Ann. Sci. École Norm. Sup. (4) 3 (1970), 273–284). *Ann. Sci. École Norm. Sup.* (4), 5:261–264, 1972.

[Mil80] J. S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[Mil06] J. S. Milne. *Arithmetic duality theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.

[ML57] Saunders Mac Lane. Homologie des anneaux et des modules. In *Colloque de topologie algébrique, Louvain, 1956*, pages 55–80. Georges Thone, Liège, 1957.

[ML63] Saunders Mac Lane. *Homology*. Die Grundlehren der mathematischen Wissenschaften, Bd. 114. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[Nee01] Amnon Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[Pép14] Cédric Pépin. Dualité sur un corps local de caractéristique positive à corps résiduel algébriquement clos. Preprint, arXiv:1411.0742v1, 2014.

[Roo06] Jan-Erik Roos. Derived functors of inverse limits revisited. *J. London Math. Soc. (2)*, 73(1):65–83, 2006.

[Sha64] Stephen S. Shatz. Cohomology of artinian group schemes over local fields. *Ann. of Math.* (2), 79:411–449, 1964.

[Spa88] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.

[Suz13] Takashi Suzuki. Duality for local fields and sheaves on the category of fields. Preprint, arXiv:1310.4941v5, 2013.
[Swa98] Richard G. Swan. Néron-Popescu desingularization. In Algebra and geometry (Taipei, 1995), volume 2 of Lect. Algebra Geom., pages 135–192. Int. Press, Cambridge, MA, 1998.

[SY12] Takashi Suzuki and Manabu Yoshida. A refinement of the local class field theory of Serre and Hazewinkel. In Algebraic number theory and related topics 2010, RIMS Kôkyûroku Bessatsu, B33, pages 163–191. Res. Inst. Math. Sci. (RIMS), Kyoto, 2012.

[Tat58] J. Tate. WC-groups over p-adic fields, volume 13 of Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 156. Secrétariat mathématique, Paris, 1958.

[Tem11] Michael Temkin. Relative Riemann-Zariski spaces. Israel Journal of Mathematics, 185(1):1–42, 2011.

[TT90] Robert Wayne Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In The Grothendieck Festschrift, Vol. III, volume 88 of Progr. Math., pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Wer97] Annette Werner. On Grothendieck’s pairing of component groups in the semistable reduction case. J. Reine Angew. Math., 486:205–215, 1997.

[Xar93] Xavier Xarles. The scheme of connected components of the Néron model of an algebraic torus. J. Reine Angew. Math., 437:167–179, 1993.

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