INDEPENDENCE PROPERTIES
IN SUBALGEBRAS OF ULTRAPRODUCT II₁ FACTORS

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Abstract. Let $M_n$ be a sequence of finite factors with $\dim(M_n) \to \infty$ and denote $M = \Pi_\omega M_n$ their ultraproduct over a free ultrafilter $\omega$. We prove that if $Q \subset M$ is either an ultraproduct $Q = \Pi_\omega Q_n$ of subalgebras $Q_n \subset M_n$, with $Q_n \not\prec_{M_n} Q'_n \cap M_n$, $\forall n$, or the centralizer $Q = B' \cap M$ of a separable amenable $*$-subalgebra $B \subset M$, then for any separable subspace $X \subset M \ominus (Q' \cap M)$, there exists a diffuse abelian von Neumann subalgebra in $Q$ which is free independent to $X$, relative to $Q' \cap M$. Some related independence properties for subalgebras in ultraproduct $\text{II}_1$ factors are also discussed.

0. Introduction

We continue in this paper the investigation of independence properties in subalgebras of ultraproduct $\text{II}_1$ factors, from [P6], [P12]. The main result we prove along these lines is the following:

0.1. Theorem. Let $M_n$ be a sequence of finite factors with $\dim(M_n) \to \infty$ and denote by $M$ the ultraproduct $\Pi_\omega M_n$, over a a free ultrafilter $\omega$ on $\mathbb{N}$. Let $Q \subset M$ be a von Neumann subalgebra satisfying one of the following:

(a) $Q = \Pi_\omega Q_n$, for some von Neumann subalgebras $Q_n \subset M_n$ satisfying the condition $Q_n \not\prec_{M_n} Q'_n \cap M_n$, $\forall n$ (in the sense of [P10]);

(b) $Q = B' \cap M$, for some separable amenable von Neumann subalgebra $B \subset M$.

Then given any separable subspace $X \subset M \ominus (Q' \cap M)$, there exists a diffuse abelian von Neumann subalgebra $A \subset Q$ such that $A$ is free independent to $X$, relative to $Q' \cap M$, i.e. $E_{Q' \cap M}(x_0 \Pi_{i=1}^n a_i x_i) = 0$, for all $n \geq 1$, $x_0 \in X \cup \{1\}$, $x_i \in X$, $a_i \in A \ominus \mathbb{C}1$, $1 \leq i \leq n$.

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Note that the particular case when $Q_n \subset M_n$ are II$_1$ factors with atomic relative commutant, for which one clearly has $Q_n \not\prec_{M_n} Q'_n \cap M_n$, recovers (2.1 in [P6]).

The conclusion in 0.1 above can alternatively be interpreted as follows: given any separable von Neumann subalgebra $P$ of $M$ that makes a commuting square with $Q' \cap M$ (in the sense of 1.2 in [P2]; see Sec. 1.2 below for the definition) and we let $B_1 = P \cap (Q' \cap M)$, there exists a separable von Neumann subalgebra $B_0 \subset Q$, such that $P \vee B_0 \simeq P \ast_{B_1} (B_1 \overline{\otimes} B_0)$ (amalgamated free product of finite von Neumann algebras over a common subalgebra, see [V], [P4]). Since in the case (b) of 0.1 we have $Q' \cap M = B$ (see 2.1 below) and all embeddings into an ultraproduct II$_1$ factor $M$ of an amenable separable von Neumann algebra $B$ are conjugate by unitaries in $M$, Theorem 0.1 shows in particular that if two separable finite von Neumann algebras $N_1, N_2$ containing copies of $B$ are embeddable into $M$, then $N_1 \ast_B N_2$ is embeddable into $M$ as well. Note that the case $B$ atomic of this result already appears in [P6], while the case $B$ arbitrary but with $M = R^\omega$ was shown in [BDJ].

More precisely, 0.1 implies the following strengthening of these results:

0.2. Corollary. Let $M = \Pi_\omega M_n$ be an ultraproduct II$_1$ factor as in 0.1. Let $N_i \subset M$ be separable finite von Neumann subalgebras with amenable von Neumann subalgebras $B_i \subset N_i$, $i = 1, 2$, such that $(B_1, \tau|_{B_1}) \simeq (B_2, \tau|_{B_2})$. Then there exists a unitary element $u \in M$ so that $uB_1u^* = B_2$ and so that, after identifying $B = B_1 \simeq B_2$ this way, we have $uN_1u^* \vee N_2 \simeq N_1 \ast_B N_2$.

To prove Theorem 0.1, we first construct unitaries $u \in Q$ that are approximately $n$-independent with respect to given finite sets $X \perp Q' \cap M$. Taking larger and larger $n$, larger and larger finite sets $X$ and better and better approximations, and combining with a diagonalization procedure, one can then get unitaries that are free independent to a given countable set, due to the ultraproduct framework.

The approximately independent unitary $u$ is constructed by patching together incremental pieces of it, while controlling the trace of alternating words involving $u$ and a given set $X$. This technique was initiated in [P3], being then fully developed in [P6], where it has been used to prove a particular case of 0.1(a). More recently, it has been used in [P12] to establish existence of free independence in ultraproducts of maximal abelian *-subalgebras (abbreviated hereafter MASA) $A_n \subset M_n$ that are singular in the sense of [D1] (i.e., any unitary element in $M_n$ that normalizes $A_n$ must lie in $A_n$), thus settling the Kadison-Singer problem for the corresponding ultrapower inclusion $\Pi_\omega A_n \subset \Pi_\omega M_n$.

If in turn the normalizers of the MASAs $A_n \subset M_n$ are large, then one can still detect certain independence properties inside $A$, by using the same type of techniques. Thus, it was shown in [P12] that 3-independence always occurs in $A$, and we prove here that given any countable group of unitaries $\Gamma$ in $M$, that
normalizes $A$ and acts freely on it, there exists a diffuse subalgebra $B_0$ in $A$ such that any word $\Pi_{i=1}^n u_i b_i u_i^*$ with $b_i \in B_0 \ominus \mathbb{C}I$ and distinct $u_i \in \Gamma$, has trace 0. This actually amounts to $B_0$ being the base of a Bernoulli $\Gamma$-action, more precisely:

**0.3. Theorem.** Let $A_n \subset M_n$ be MASAs in finite factors, as before, and denote $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$. Assume $\Gamma \subset M$ is a countable group of unitaries normalizing $A$ and acting freely on it, and let $H \subset \Gamma$ be an amenable subgroup. Given any separable abelian von Neumann subalgebra $B \subset A$, there exists a $\Gamma$-invariant subalgebra $\mathcal{A} \subset A$ such that $\mathcal{A}, B$ are $\tau$-independent and $\Gamma \actson A$ is isomorphic to the generalized Bernoulli action $\Gamma \actson L^\infty([0,1]^{\Gamma/H})$.

Note that if the above ultraproduct inclusion $A \subset M$ comes from a sequence of finite dimensional diagonal inclusions $D_n \subset M_{n \times n}(\mathbb{C})$, or is of the form $R^\omega \subset R^\omega$, where $D \subset R$ is the unique (up to conjugacy by an automorphism, by [CFW]) Cartan subalgebra of the hyperfinite $\Pi_1$ factor, then a countable group $\Gamma$ can be embedded into the normalizer $N_M(A)$ of $A$ in $M$, in a way that it acts freely on $A$, iff it is sofic (in the sense of [W]; see the expository paper [Pe]). Thus, with the terminology in [EL], where an action of a sofic group $\Gamma \actson X$ is called sofic if the inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ admits a commuting square embedding into $A \subset M$, with $\Gamma$ embedding into $N_M(A)$, it follows from 0.3 that if $\Gamma \actson X$ is sofic then any product action $\Gamma \actson X \times Y$, with $\Gamma \actson Y = [0,1]^I$ a generalized Bernoulli action corresponding to the left action of $\Gamma$ on a set $I = \bigoplus_i \Gamma/H_i$, for some countable family of amenable subgroups $H_i \subset \Gamma$, is sofic. This generalizes a result in [EL].

The paper is organized as follows. In Section 1 we recall some basic facts needed in the paper, such as the local quantization lemma from [P1], [P5] and the criterion for (non-)conjugacy of subalgebras from [P10]. We also prove a general fact about centralizers (or commutants) of countable sets in ultraproduct $\Pi_1$ factors (see Theorem 1.5). In Section 2 we prove some bicentralizer results concerning amenable algebras and groups, in ultrapower framework, that we need in the proofs of 0.1 and respectively 0.3. We conjecture that, in fact, the bicentralizer property characterizes amenability (see 2.3.1).

In Section 3 we prove the main technical result needed in the proof of Theorem 0.1, by using incremental patching techniques. This result, stated as Lemma 3.2, actually amounts to an “approximate version” of the free independence result in 0.1. In Section 4 we derive Theorem 0.1 (in fact a strengthening of it, stated as Theorem 4.3), by using Lemma 3.2 and an appropriate diagonalization procedure.

In Section 5 we prove Theorem 0.3 (stated as Theorem 5.1). Also, we use the incremental patching technique to show (see 5.3) that if $A_n \subset M_n$ are Cartan subalgebras in finite factors, with $\dim M_n \to \infty$, and $\Gamma_i$ are countable subgroups of the normalizer $\mathcal{N}$ of $A = \Pi_\omega A_n$ in $M = \Pi_\omega M_n$, acting freely on $A$, with $H_i \subset \Gamma_i$...
isomorphic amenable subgroups, then there exists $u \in \mathcal{N}$ such that $uH_1u^* = H_2$ and such that the group generated by $u\Gamma_1u^*$ and $\Gamma_2$ is the amalgamated free product $\Gamma_1 *_H \Gamma_2$, where $H$ is the identification of $H_1, H_2$ via $\text{Ad}(u)$. Taking $M_n$ finite dimensional, this recovers a result from [ES], [Pa], on the soficity of amalgamated free products of sofic groups over amenable subgroups and on the uniqueness of sofic embeddings of an amenable group.

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1. Preliminaries

1.1. Some generalities. All von Neumann algebras $M$ considered in this paper are finite (in the sense of [MvN1]) and come equipped with a fixed faithful normal trace state, generically denoted $\tau$. We denote by $\mathcal{U}(M)$ the group of unitary elements of $M$ and by $\mathcal{P}(M)$ the set of projections of $M$. Recall that a von Neumann algebra is a factor if its center is reduced to the scalars. Recall that there exists a unique trace state on a finite factor ([D2]). A finite factor $M$ is either finite dimensional (in which case $M \simeq M_{n \times n}(\mathbb{C})$ for some $n \geq 1$ with its unique trace state $\tau$ given by the normalized trace $\text{tr} = Tr/n$) or infinite dimensional. In this latter case, it is called a $\Pi_1$ factor, and is characterized by the fact that the range of the trace on the set of projections satisfies $\tau(\mathcal{P}(M)) = [0, 1]$.

More generally, a finite von Neumann algebra splits as a direct sum $M = M_1 \oplus M_2$ with $M_1$ of type I (i.e. $M_1 \simeq \bigoplus_{n \geq 1} M_{n \times n}(\mathbb{C}) \otimes A_n$, where $A_n$ are abelian von Neumann algebras, possibly equal to 0) and $M_2$ of type $\Pi_1$ (which by definition means $M_2$ has no type I summand).

We denote by $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$, the $L^2$ Hilbert-norm given by the trace. We denote by $L^2M$ the completion of $M$ in this norm. We often view $M$ in its standard representation, acting on $L^2M$ by left multiplication.

We will also use the $L^1$ norm $\|\|_1$ on $M$, defined by $\|x\|_1 := \tau(|x|) = \sup\{\tau(xy) \mid y \in M, \|y\| \leq 1\}$. We denote by $L^1M$ the completion of $M$ in the norm $\|\|_1$. Note that by the Cauchy-Schwartz inequality we have $\|x\|_1 \leq \|x\|_2$, while by the inequality $x^*x \leq \|x\|_1 |x|$ we have $\|x\|_2^2 \leq \|x\|_1 \|x\|$. If $B \subset M$ is a von Neumann subalgebra, then $E_B : M \to B$ denotes the (unique) $\tau$-preserving conditional expectation of $M$ onto $B$, which is contractible in both the operatorial norm $\|\|$ and the above $L^p$-norms, $p = 1, 2$. If we view $M$ in its standard representation on $L^2M$, then the expectation $E_B$ is implemented by the orthogonal projection $e_B$ of $L^2M$ onto $L^2B \subset L^2M$ (viewed as the closure in the norm $\|\|_2$ of $B \subset M$), by $e_Bxe_B = E_B(x)e_B$, $x \in M$. 


A finite von Neumann algebra \((M, \tau)\) is \emph{separable} if it is separable with respect to the Hilbert norm \(\| \cdot \|_2\). Note that this condition is equivalent to the fact that \(M\) is countably generated as a von Neumann algebra. More generally, if \(X \subset M\) is a subspace, then \(X\) is separable if it is separable with respect to the norm \(\| \cdot \|_2\).

The von Neumann algebra \(M\) is \emph{atomic} if \(1_M = \sum_i e_i\) with \(e_i \in M\) a family of mutually orthogonal minimal projections \(e_i \in M\) (or equivalently, atomic projections, i.e. with the property that \(e_i Me_i = C e_i\)). \(M\) is \emph{diffuse} if it has no minimal (non zero) projection. Any abelian von Neumann algebra \(A\) which is diffuse and separable is isomorphic to \(L^\infty([0,1])\) (or to \(L^\infty(\mathbb{T})\)). Moreover, if \(A\) is endowed with a faithful normal state \(\tau\), then the isomorphism \(A \cong L^\infty([0,1])\) can be taken so that to carry \(\tau\) onto the integral \(\int \cdot \, d\mu\), where \(\mu\) is the Lebesgue measure on \([0,1]\).

We will often consider maximal abelian \(*\)-subalgebras (MASA) \(A\) in a finite von Neumann algebra \(M\), i.e. abelian \(*\)-subalgebras \(A \subset M\) with \(A' \cap M = A\). In such a case, we denote \(N_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}\), the \emph{normalizer} of \(A\) in \(M\). Following [FM], if the normalizer generates \(M\) as a von Neumann algebra, we call \(A\) a \emph{Cartan subalgebra} in \(M\). An isomorphism of Cartan inclusions \((A_0 \subset M_0; \tau) \cong (A_1 \subset M_1; \tau)\) is a trace preserving isomorphism of \(M_0\) onto \(M_1\) in such a way that \(A_0\) is carried onto \(A_1\) such that \(M_0 \cap A_1 = A_0\), with the commuting square condition \(E_{A_1} E_{M_0} = E_{A_0}\) satisfied (see 1.2 below for more on this condition), and such that \(N_{M_0}(A_0) \subset N_{M_1}(A_1)\).

For various other general facts about finite von Neumann algebras, we refer the reader to the classic book [D2].

### 1.2. Commuting squares of subalgebras.

Two von Neumann subalgebras \(B_1, B_2 \subset M\) are in \emph{commuting square} position if the expectations \(E_{B_1}, E_{B_2}\) commute (see Sec. 1.2 in [P2]). Note that if this is the case then we in fact have \(E_{B_1} E_{B_2} = E_{B_2} E_{B_1} = E_{B_1 \cap B_2}\). Also, for this to happen it is sufficient that \(E_{B_1}(B_2) \subset B_1 \cap B_2\).

A typical example when the commuting square condition is satisfied is the following: let \(Q \subset P \subset M\) be von Neumann algebras; then \(P\) and \(Q' \cap M\) are in commuting square position (see 1.2.2 in [P2]).

We notice here an observation showing that in the statement of Theorem 0.1, we may equivalently take the space \(X\) to be a separable von Neumann algebra making a commuting square with \(Q' \cap M\), a fact that we will not use in the sequel but is good to keep in mind. See also (3.8 in [P12]) for a similar statement.
Lemma. Let $N \subset M$ be a von Neumann subalgebra in the finite von Neumann algebra $M$. If $X \subset M$ is a separable subspace, then there exists a separable von Neumann subalgebra $P \subset M$ that contains $X$ and makes a commuting square with $N$.

Proof. We let $P_0 \subset M$ be the (separable) von Neumann algebra generated by $X$ and then construct recursively an increasing sequence of inclusions of separable von Neumann algebra $B_n \subset P_n$, $n \geq 1$, by letting $B_n$ be the von Neumann algebra generated by $E_N(P_{n-1})$ and $P_n$ be the von Neumann algebra generated by $B_n$ and $P_{n-1}$.

If we now define $B = \bigcup_n B_n$ and $P = \bigcup_n P_n$, then both algebras are separable and $B \subset P \cap N$, by construction. Moreover, we have $E_N(P_n) \subset B_{n+1} \subset P_n$, implying that $E_N(P) \subset P \cap N$, i.e. $N, P$ make a commuting square with $B = N \cap P$. □

1.3. Amenable algebras. An important example of a (separable) II$_1$ factor is the hyperfinite II$_1$ factor $R$ of Murray and von Neumann ([MvN2]), defined as the infinite tensor product $(R, \tau) = \bigotimes_k (M_{2 \times 2}(\mathbb{C}), \text{tr})_k$. By [MvN2], $R$ is the unique approximately finite dimensional (AFD) separable II$_1$ factor (a separable finite von Neumann algebra algebra $(M, \tau)$ is AFD if there exists an increasing sequence of finite dimensional von Neumann subalgebras $M_n \subset M$ such that $\bigcup_n M_n$ is dense in $M$ in the norm $\|\cdot\|_2$).

By Connes’ results in [C1], $R$ is in fact the unique amenable separable II$_1$ factor. Recall in this respect that a finite von Neumann algebra $(M, \tau)$ is called amenable if there exists a state $\varphi$ on $B(L^2 M)$ that has $M$ (when viewed in its standard representation on $L^2 M$) in its centralizer, $\varphi(xT) = \varphi(Tx)$, $\forall x \in M$, $\forall T \in B(L^2 M)$, and such that $\varphi|_M = \tau$. Note that the latter condition is redundant in case $M$ is a factor, because $\varphi|_M$ is a trace and because of the uniqueness of the trace on factors. Connes Fundamental Theorem in [C1] actually shows that amenability is equivalent to the AFD property, for any finite von Neumann algebra.

From all this, it follows that $R$ can be represented in many different ways, for instance as the group measure space II$_1$ factor $L^\infty(X) \rtimes \Gamma$, associated with a free ergodic measure preserving action of a countable amenable group $\Gamma$ on a probability space $(X, \mu)$ ([MvN2]). When viewed this way, $R$ has $D = L^\infty(X)$ as a natural Cartan subalgebra. By [CFW], [OW] the Cartan subalgebra of $R$ is in fact unique, up to conjugacy by an automorphism of $R$. We may thus represent $D \subset R$ as the infinite tensor product $\bigotimes_k (D_2)_k \subset \bigotimes_k (M_{2 \times 2}(\mathbb{C}))_k$, where $D_2$ is the diagonal subalgebra in $M_{2 \times 2}(\mathbb{C})$.

More generally, by [CFW], if $A_0 \subset R_0$ is a Cartan subalgebra in an amenable separable finite von Neumann algebra $R_0$, then there exists an increasing sequence of finite dimensional Cartan inclusions $(A_{0,n} \subset R_{0,n}) \subset (A_0 \subset R_0)$ (with Cartan
embeddings, as defined before) such that $\bigcup_n A_{0,n} = A_0 \subset R_0 = \bigcup_n R_{0,n}$.

1.4. Local quantization relative to subalgebras. We recall here a result from [P1], [P5], showing that if $Q \subset M$ are $\Pi_1$ von Neumann algebras, then one can “simulate” the expectation onto the commutant $Q' \cap M$ by “squeezing” with appropriate projections in $Q$, a phenomenon called “local quantization” in [P5]:

**Theorem.** 1° Let $M$ be a finite von Neumann algebra and $Q \subset M$ a von Neumann subalgebra. Given any finite set $F \subset M \ominus Q \cup (Q' \cap M)$ and any $\varepsilon > 0$, there exists a projection $q \in Q$ such that $\|qxq\|_1 < \varepsilon \tau(q)$, $\forall x \in F$.

2° Let $Q \subset M$ be an inclusion of $\Pi_1$ von Neumann algebras. Given any finite set $X \subset M$ and any $\varepsilon > 0$, there exists a projection $q \in Q$ such that $\|qxq - E_{Q' \cap M}(x)q\|_1 < \varepsilon \tau(q)$, $\forall x \in X$. Moreover, $q$ can be taken so that to have scalar central trace in $Q$.

**Proof.** Part 1° is already proved in [P1] (see also Theorem 3.6 in [P12]), while part 2° is (Theorem A.1.4 in [P5]).

1.5. A criterion for non-conjugacy of subalgebras. Let $Q, P \subset M$ be von Neumann subalgebras of the finite von Neumann algebra $M$. Following [P10], we say that a corner of $Q$ can be embedded into $P$ inside $M$ and write $Q \prec_M P$ if the following condition holds true: there exist non-zero projections $p \in P$, $q \in Q$, a unital isomorphism $\psi : qQq \to pPp$ (not necessarily onto) and a partial isometry $v \in M$ such that $vv^* \in qQq \cap qMq$, $v^*v \in \psi(qQq)^\prime \cap pMp$, $xv = v\psi(x)$, $\forall x \in qQq$, and $x \in qQq$, $xvv^* = 0$, implies $x = 0$.

In this paper we will actually consider cases when the above condition is not satisfied. We recall from (2.1 in [P10]) a useful necessary and sufficient criterion for this to happen:

**Theorem.** Let $M$ be a finite von Neumann algebra and $P, Q \subset M$ von Neumann subalgebras. For each $q \in \mathcal{P}(Q)$, fix $U_q \subset U(qQq)$ a subgroup generating $qQq$ as a von Neumann algebra. Then $Q \not\prec_M P$ if and only if the following condition holds true:

(1.5.1) Given any $q \in \mathcal{P}(Q)$ and any separable subspace $X \subset M$ there exists a sequence of unitary elements $u_n \in U_q$ such that $\lim_n \|E_{\mathcal{P}}(xu_ny)\|_2 = 0$, $\forall x, y \in X$.

1.6. Ultraproducts of algebras. We fix once for all an (arbitrary) free ultrafilter $\omega$ on $\mathbb{N}$. If $M_n$, $n \geq 1$, is a sequence of finite von Neumann algebras then, we denote by $\Pi_\omega M_n$ their $\omega$-ultraproduct, i.e., the finite von Neumann algebra obtained as the quotient of $\bigoplus_n M_n$ by its ideal $\mathcal{I}_\omega = \{(x_n) \mid \lim_\omega \tau(x_n^* x_n) = 0\}$, endowed with the trace $\tau(y) = \lim_\omega \tau(y_n)$, where $(y_n)_n \in \bigoplus_n M_n$ is in the class $y \in \bigoplus_n M_n/\mathcal{I}_\omega ([W])$. 

Recall that if $M_n$ are factors and $\dim M_n \to \infty$, then $\Pi_\omega M_n$ is a $\Pi_1$ factor ([Wr]) and it is non-separable ([F]).

If $Q_n \subset M_n$ are von Neumann subalgebras, $n \geq 1$, then the ultraproduct $\Pi_\omega Q_n$ identifies naturally to a von Neumann subalgebra in $\Pi_\omega M_n$ and its centralizer (or commutant) in $\Pi_\omega M_n$ is given by the formula $(\Pi_\omega Q_n)' \cap \Pi_\omega M_n = \Pi_\omega (Q'_n \cap M_n)$ (see e.g. [P1]).

If $M$ is a finite von Neumann algebra, then $M^\omega$ denotes its $\omega$-ultrapower, i.e. the ultraproduct of infinitely many copies of $M$. Note that $M$ naturally embeds into $M^\omega$, as the von Neumann subalgebra of constant sequences, and that if $M$ is a $\Pi_1$ factor then $M^\omega$ is a (non-separable by [F]) $\Pi_1$ factor.

1.7. Centralizers of countable sets in ultraproducts. Let $S = \{b_n\}_n$ be a countable subset in the ultrapower $R^\omega$ of the hyperfinite $\Pi_1$ factor $R$ and let $b_n = (b_{n,m})_m$ be representations of each of its elements with $b_{n,m} \in R = \otimes_k (M_{2 \times 2}(\mathbb{C})) = \bigcup_n M_n^{\omega n}$, where $M_n$ is the tensor product of the first $n$ copies of $M_{2 \times 2}(\mathbb{C})$. Thus, we may assume that for each $m$, $\{b_{n,m}\}_{n \leq m} \subset M_{k_m}$, for a large enough $k_m$. Then we have $b_n \in \Pi_\omega M_{k_m} \subset R^\omega$, $\forall n$, viewed as a subalgebra of $R^\omega$.

But then the ultraproduct subalgebra $\Pi_\omega (M_{k_m} \cap R) \simeq R^\omega$ commutes with the set $\{b_n\}_n$. This shows that the centralizer of any separable von Neumann subalgebra of $R^\omega$ is a type $\Pi_1$ von Neumann algebra without separable direct summands.

However, for general ultraproducts $\Pi_\omega M_n$ and ultrapowers $M^\omega$, we may have countable (or even finite) subsets $S$ that have trivial centralizer: For instance, if $M$ is a non-Gamma $\Pi_1$ factor ([MvN2]), such as the group $\Pi_1$ factor $M = L(\Gamma)$ associated with an infinite conjugacy (ICC) countable group $\Gamma$ with the property (T) of Kazhdan (for example, $\Gamma = PSL(n, \mathbb{Z})$, $n \geq 3$), then $M$ is finitely generated and $M' \cap M^\omega = \mathbb{C}$. Similarly, by results in [Be], it follows that if for some fixed $n \geq 3$ we take $(\pi_m, \mathcal{H}_m)$ to be any sequence of finite dimensional irreducible representations of $\Gamma = PSL(n, \mathbb{Z})$ so that $k_m = \dim \mathcal{H}_m \to \infty$, then the the von Neumann subalgebra $M$ generated by $\{(\pi_m(g))_m \mid g \in \Gamma\}$ in the ultraproduct $\Pi_\omega M_{k_m \times k_m}(\mathbb{C})$ is isomorphic to the group factor $L(\Gamma)$ and has trivial relative commutant.

The following result shows that in fact the centralizer of a any separable von Neumann subalgebra $P$ of an arbitrary ultraproduct $\Pi_1$ factor $M := \Pi_\omega M_n$, coming from a sequence of finite factors $M_n$ with $\dim M_n \to \infty$, splits as the direct sum of an atomic von Neumann algebra and a diffuse von Neumann algebra with only non-separable direct summands.

**Theorem.** If $P$ is a separable von Neumann subalgebra of $M$ then $P' \cap M = B_0 \oplus B_1$, with $B_0$ atomic and $B_1$ diffuse and having no separable direct summand (even more: any MASA of $B_1$ has only non-separable direct summands).

**Proof.** Denote $Q = P' \cap M$ and let $z \in Z(Q)$ be the maximal central projection
with the property that $Qz$ is diffuse. We have to prove that $Qz'$ is non-separable for any central projection $z' \in Z(Q)z$. By replacing $P \subset M$ by $Pz \subset z'Mz$, we may clearly assume $z = 1$.

Assuming by contradiction that $Q$ has separable direct summands, we may further reduce with the maximal central projection $z_0$ in $Q$ with the property that $Qz_0$ is separable to actually assume, by contradiction, that $P \subset M$ is separable with $Q = P' \cap M$ diffuse and separable.

Let $\{b_n\}_n \subset P$ be a countable subset of the unit ball of $P$, dense in in the Hilbert norm $\| \|_2$. Let $b_n = (b_n,m)$ be representations of $b_n$ with $b_n,m \in M_m$, $\|b_n,m\| \leq \|b_n\|$, $\forall n,m$. Let also $u \in Q$ be a Haar unitary generating a maximal abelian $*$-subalgebra $A_0$ of $Q$, and let $u = (u_m)_m$ be a representation of $u$ with $u_m \in U(M_m), \forall m$.

The fact that $u$ belongs to $Q = \{b_n\}_n \cap M$ translates into the condition

$$\lim_{m \to \omega} \| [b_k,m, u_m] \|_2 = 0, \forall k \geq 1,$$

while the fact that $u$ is a Haar unitary amounts to the condition

$$\lim_{m \to \omega} \tau(u^j_m) = 0, \forall j \neq 0.$$

Let $V_n$ denote the set of all $m \in \mathbb{N}$ with the property that

$$\| [b_k,m, u_m] \|_2 < 2^{-n}, |\tau(u^j_m)| < 2^{-n}, \forall 1 \leq k, |j| \leq 2n.$$

If we identify $\ell^\infty \mathbb{N}$ with the algebra $C(\Omega)$ of continuous functions on its spectrum $\Omega$ (via the GNS representation), and we view $\omega$ as a point in $\Omega$, then by (1) and (2) it follows that $V_n$ correspond to an open-closed neighborhoods of $\omega \in \Omega$. Let now $W_n, n \geq 0$, be defined recursively as follows: $W_0 = \mathbb{N}$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} | n > \min W_n\}$. Note that, with the same identification as before, $W_n$ correspond to a strictly decreasing sequence of neighborhoods of $\omega$.

Noticing that the sets $\{W_n \setminus W_{n-1}\}_{n \geq 1}$ form a partition of $\mathbb{N}$, we define $v = (v_m)_m$ by letting $v_m = u^n_m$ for $m \in W_{n-1} \setminus W_n$. Since $v_m \in U(M_m)$, it follows that $v$ is a unitary element in $M$. By the first relation in (3), if $m \in W_n \setminus W_{n-1}$ then

$$\| [b_k,m, v_m] \|_2 = \| [b_k,m, u^n_m] \|_2 \leq \sum_{j=0}^{n-1} \| u^j_m [b_k,m, u_n] u^{n-j-1}_m \|_2 \leq n 2^{-n},$$

for all $1 \leq k \leq n$, while by the second relation in (3) we have

$$|\tau(v_m u^j_m)| < 2^{-n}$$

for all $1 \leq |j| \leq n$.

But then (4) implies $v \in \{b_n\}_n \cap M = P' \cap M = Q$, while by (5) we have $\tau(vu^j) = 0$, for all $j \neq 0$, i.e. $v \in Q$ is perpendicular to the maximal abelian $*$-subalgebra $A_0 = \{u\}''$ of $Q$ generated by $u \in Q$. Since by construction we have $uv = vu$, this shows that at the same time we have $v \in \{u\}' \cap Q = A_0$ and $v \perp A_0$, a contradiction. This also shows the stronger form of the statement. \qed
2. Bicentralizer characterizations of amenability

2.1. Theorem. 1° Let $M_n$ be a sequence of finite factors with $\dim M_n \to \infty$ and denote $M = \prod_n M_n$. If $B \subset M$ is a separable amenable von Neumann subalgebra, then $(B' \cap M)' \cap M = B$. Moreover, $B' \cap M$ is of type $\Pi_1$ and has only non-separable direct summands.

2° If $R$ denotes the hyperfinite $\Pi_1$ factor then $(R' \cap R^\omega)' \cap R^\omega = R$.

Proof. Part 2° is just a particular case of part 1°, so we only need to prove 1°. By Connes’ Theorem ([C1]), since $B$ is amenable and separable, it is approximately finite dimensional, so $B = \overline{\cup_n B_n^\omega}$, for some increasing sequence of finite dimensional von Neumann subalgebras $B_n \subset B$. Note that $B' \cap M = \cap_n (B_n' \cap M)$ and that for each $n$ we have $(B_n' \cap M)' \cap M = B_n$ (in fact, it is trivial to see that given any inclusion of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ with $\dim \mathcal{N} < \infty$ and $\mathcal{M}$ a factor, we have $(\mathcal{N}' \cap \mathcal{M})' \cap \mathcal{M} = \mathcal{N}$). We first need to prove the following:

Fact. Let $P \subset M$ be an inclusion of finite von Neumann algebras. Let $x \in M \ominus (P' \cap M)$ and $\varepsilon > 0$. There exists a unitary element $u \in P$ such that $\Re \tau(x^*uxu^*) \leq \varepsilon \|x\|_2^2$.

To prove this, let $K_x$ denote the weak closure of the convex set $\co\{uxu^* \mid u \in U(P)\}$ and note right away that $\|y\| \leq \|x\|$ and $\|y\|_2 \leq \|x\|_2$, $\forall y \in K_x$. Thus, $K_x$ is a uniquely closed bounded subspace in both $M$ and $L^2 M$. In particular, there exists a unique element $y_0 \in K_x$ of minimal Hilbert-norm: $\|y_0\|_2 = \min\{\|y\|_2 \mid y \in K_x\}$. Since $K_x$ is Ad$(U(P))$-invariant (because it is the weak closure of the Ad$(U(P))$-invariant set $\co\{uxu^* \mid u \in U(P)\}$) and since $\|uy_0u^*\|_2 = \|y_0\|_2$, by the uniqueness of $y_0$ it follows that $uy_0u^* = y_0$, $\forall u \in U(P)$. Thus, $uy_0 = y_0u$, $\forall u \in U(P)$. By taking linear combinations of $u$, this implies $y_0 \in P' \cap M$. By its construction, the entire $K_x$ lies in $M \ominus (P' \cap M)$. Thus, $y_0$ is both in $P' \cap M$ and perpendicular to it, implying that $y_0 = 0$, i.e. $0 \in K_x$.

Assuming now that we have $\Re \tau(x^*uxu^*) \geq \varepsilon \|x\|_2^2$, for all $u \in U(P)$, by taking convex combinations over $u \in U(P)$ and then weak closure, it follows that $\Re \tau(x^*y) \geq \varepsilon \|x\|_2^2$, for all $y \in P$. In particular, $0 = \Re \tau(x^*y_0) \geq \varepsilon \|x\|_2^2$, forcing $x = 0$. This ends the proof of the above Fact.

Denote for simplicity $Q = B' \cap M$ and note that $B \subset Q' \cap M$. Assume there exists $x \in Q' \cap M$ with $x \perp B$. In particular $x \perp B_n = (B_n' \cap M)' \cap M$. By applying the Fact to the inclusion $B_n' \cap M \subset M$ and the element $x$, it follows that there exists a unitary element $u_n \in B_n' \cap M$ such that $\Re \tau(x^*u_nxu_n^*) < 2^{-n}$, $\forall n$.

Let $\{e_{k,1}^n\}_{k \in \mathbb{N}} \subset B_n$ denote the (finite) pseudogroup of all partial isometries in $B_n$ that can be obtained as a sum of elements from a given matrix unit of $B_n$, and which we take so that $\{e_i^n\}_i$ is a subset of $\{e_{j+1}^n\}_j$, $\forall n$. Let $e_k^n = (e_{k,1}^n)_m$, with
$e_{k,m}^n \in M_m$ chosen so that $\|e_{k,m}^n\| \leq \|e_k^n\|$ and $\{e_{i,m}^n\}_i \subset \{e_{j,m}^{n+1}\}_j$ for all $n, m$. Let also $u_n = (u_{n,m})_m$, with $u_{n,m} \in \mathcal{U}(M_m)$. Then the above properties translate into

$$\lim_{m \to \omega} \|[u_{n,m}, e_{k,m}^n]\|_2 = 0, \lim_{m \to \omega} \mathcal{R}\tau(x_m^* u_{n,m} x_m u_{n,m}^*) < 2^{-n},$$

for all $k$ and all $n$, where $x = (x_m)_m$ with $x_m \in M_m$.

Let $V_n$ denote the set of all $m \in \mathbb{N}$ with the property that

$$\|[u_{n,m}, e_{k,m}^n]\|_2 \leq 2^{-n}, \mathcal{R}\tau(x_m^* u_{n,m} x_m u_{n,m}^*) < \|x\|_2^2/2, \forall k.$$

By (1), it follows that $V_n$ corresponds to an open-closed neighborhood of $\omega$ in the spectrum $\Omega$ of $\ell^\infty \mathbb{N}$, under the identification $\ell^\infty \mathbb{N} = C(\Omega)$. Let now $W_n$, $n \geq 0$, be defined recursively as follows: $W_0 = \mathbb{N}$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} \mid n > \min W_n\}$. Note that, with the same identification as before, $W_n$ correspond to a strictly decreasing sequence of neighborhoods of $\omega$. Define $v = (v_{n,m})_m$ by letting $v_m = u_{n,m}$ for $m \in W_{n-1} \setminus W_n$. Since $v_m \in \mathcal{U}(M_m)$, it follows that $v$ is a unitary element in $M$, while by the first relation in (2) and the fact that $\{e_{i,m}^n\}_i \subset \{e_{j,m}^{n+1}\}_j$ it follows that $v \in \cap_n B'_n \cap M = B' \cap M = Q$. By the second relation in (2), we also have $\mathcal{R}\tau(x^* v x v^*) \leq \|x\|_2^2/2$. But $x \in Q' \cap M$ by our assumption, thus $v x v^* = x$, giving $\tau(x^* v x v^*) = \|x\|_2^2$, a contradiction.

If $Q = Qz + Q(1 - z)$ with $z$ a non-zero central projection of $Q$ and $Qz$ separable, then by the bi-commutant property we have $z \in B$ and by Proposition 1.4 $Qz$ is atomic. Thus, $Bz = (Qz)' \cap zMz$ would follow non-separable, a contradiction.

Assume now that $Q = Qz + Q(1 - z)$ with $z \in \mathcal{P}(\mathcal{Z}(Q))$ such that $Qz$ is type I. By the bi-commutation relation, it follows again that $z \in B$ and that $Bz = (Qz)' \cap zMz$ is non-separable (because the commutant of any abelian von Neumann subalgebra of $M$ is non-separable, by 4.3 in [P1], or 2.3 in [P12]).

2.2. Theorem. Let $A_n \subset M_n$ be a sequence of MASAs in finite factors and denote $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$, $N' = N_M(A)$.

1° If $H \subset N'$ is a countable amenable subgroup, then $(H' \cap A)' \cap M = A \vee H$.

2° Assume the MASAs $A_n \subset M_n$ are Cartan. Let $R_0 \subset M$ be a separable amenable von Neumann subalgebra such that $D_0 = R_0 \cap A$ is a Cartan subalgebra in $R_0$ with $N_{R_0}(D_0) \subset N'$ (i.e. $(D_0 \subset R_0) \subset (A \subset M)$ is a Cartan embedding, in the sense of 1.3). Then $(N_{R_0}(D_0)' \cap N')' \cap N = N_{R_0}(D_0)$.

Moreover, if $D_1 \subset R_1$ is another Cartan inclusion embedded in the same way into $A \subset M$, then given any isomorphism $\rho : (D_0 \subset R_0; \tau) \to (D_1 \subset R_1; \tau)$, there exists $u \in N'$ such that $Ad(u) = \rho$ on $R_0$.

Proof. 1° Let first $\{e_{j,n}^n\}_j$ be an increasing sequence of finite partitions in $\mathcal{P}(A)$ such that $\lim_n \|\Sigma_j e_{j,n}^n u e_{j,n}^n - E_A(u)\|_2 = 0$, $\forall u \in H$ (e.g., by [P1], or 3.3 in [P12]). If we
denote by $A_0$ the von Neumann subalgebra of $A$ generated by $\cup_{u \in H} u \{e^n_j \mid j, n\} u^*$ and $R_0 = A_0 \vee H$, then $H$ normalizes $A_0$, $A_0$ is a Cartan subalgebra of $R_0$ and $A \vee H = A \vee R_0$. In particular, $H' \cap A = R'_0 \cap A$. Moreover, since $H$ is amenable, $R_0$ follows amenable so by ([CFW], [OW]) there exists an increasing sequence of finite pseudogroups of partial isometries $G_n = \{e^n_j\}_{j}$, normalizing $A_0$ (and $A$ as well), with source and targets either equal or mutually orthogonal, for each $n \geq 1$, and such that $\{e^n_j \mid j, n\}$ generate $R_0$.

It is then trivial to see that $H' \cap A = \cap_n (G'_n \cap A)$ and $(G'_n \cap A)' \cap A = G_n \vee A$, $\forall n$. Then the rest of the proof proceeds with a “diagonalization” argument, exactly as at the end of the proof of Theorem 2.1.

2° The proof of this part is similar to the one of 2.1° and of 2.2.1° above, so we leave it as an exercise. □

2.3. Some remarks and open problems. 1° It is well known (and trivial to show) that if $M_n$ is a sequence of finite factors with $\dim M_n \to \infty$ and $(B, \tau)$ is a finite separable AFD von Neumann algebra, then there exists a trace preserving embedding $\theta_0 : B \hookrightarrow M := \Pi_\omega M_n$ and that given any other such trace preserving embedding $\theta : B \hookrightarrow M$, there exists a unitary element $u \in M$ such that $\theta_1(b) = u\theta_0(b)u^*$, $\forall b \in B$. In particular, any two copies of $(B, \tau)$ in $M$ are unitary conjugate. By Connes’ theorem [C1], this means that the same holds true for any finite, separable, amenable $B$.

Moreover, by a result of K. Jung in [J], the converse is also true: if a finite separable von Neumann algebra $(B, \tau)$ has a unique (up to unitary conjugacy) embedding into either an ultraproduct $\Pi_\omega M_{n \times n}(\mathbb{C})$ or in $R^\omega$, then $B$ is amenable (see [J]). In fact, by a result of N. Brown in [B] (see also Ozawa’s Appendix 8.1 in that paper), if $B \subset R^\omega$ is non-amenable, then there exist uncountably many non-conjugate copies of $B$ in $R^\omega$.

Since given any ultraproduct $\Pi_1$ factors $M = \Pi_\omega M_n$, all embeddings $B \hookrightarrow M$ of a given separable amenable finite von Neumann algebra are unitary conjugate in $M$, it seems interesting to investigate the converse in this general setting: is it true that if $B \subset M$ is a separable non-amenable von Neumann algebra of an arbitrary ultraproduct $\Pi_1$ factor, then there exist “many” non-conjugate copies of $B$ in $M$? (I am grateful to N. Ozawa for pointing out to me that the answer to this problem is not known; see [FHS] for related considerations.)

On the other hand, related to Theorem 2.1 above, we propose the following new characterization of amenability for separable finite von Neumann algebras:

(2.3.1) Conjecture: Let $P$ be a separable finite von Neumann subalgebra of an ultraproduct $\Pi_1$ factor $M$ (notably, of $M = R^\omega$, or of $M = \Pi_\omega M_{n \times n}(\mathbb{C})$). If the bicentralizer condition $(P' \cap M)' \cap M = P$ is satisfied, then $P$ is amenable. In
particular, if \( M \) is a separable II_1 factor such that \((M' \cap M^\omega)' \cap M^\omega = M\), then \( M \simeq R \).

Note that for a separable von Neumann subalgebra of an ultraproduct II_1 factor, conjecture (2.3.1) is equivalent to the following statement:

\[(2.3.1') \text{Conjecture: Let } P \text{ be a separable von Neumann subalgebra of an ultraproduct II}_1 \text{ factor } M. \text{ If } P \text{ is the centralizer of a von Neumann subalgebra } Q \subset M, \text{ i.e., } P = Q' \cap M, \text{ then } P \text{ is necessarily amenable.}\]

Indeed, one clearly has that (2.3.1') implies (2.3.1). Assume in turn that (2.3.1) holds true. Let \( Q \subset M \) be so that \( P = Q' \cap M \) is separable and denote \( \tilde{Q} = P' \cap M \).

Then we still have \( \tilde{Q}' \cap M = P \), so \( P \) satisfies the bicentralizer condition and it is separable, thus \( P \) is amenable.

Note also that the bicentraliser condition \((M' \cap M^\omega)' \cap M^\omega = M\) for a separable II_1 factor, implies that \( M \) must be McDuff ([McD]), i.e., it splits off the hyperfinite II_1 factor (or else \( M' \cap M^\omega \) is abelian, implying that the bicentralizer is non-separable), but that it cannot be of the form \( N \otimes R \), with \( N \) non-Gamma ([MvN2]). In fact, if \( M \) has a II_1 von Neumann subalgebra \( N \subset M \) satisfying the spectral gap condition \( N' \cap M^\omega = (N' \cap M)^\omega \) ([P11]), then \( M \) cannot satisfy the bicentralizer condition \((M' \cap M^\omega)' \cap M^\omega = M\).

Indeed, this is because taking bicentralizer is an operation preserving inclusions of algebras, and thus the bicentralizer of \( M \) in \( M^\omega \) contains the bicentralizer of \( N \) in \( M^\omega \), which is equal to \((N' \cap M)^\omega \). But the latter is non-separable, so it cannot be contained in \( M \), which is separable.

\[2^o\] Since by ([CFW]), any Cartan inclusion \( A_0 \subset M_0 \) with \( M_0 \) separable amenable finite von Neumann algebra is a limit of an increasing sequence of finite dimensional Cartan inclusions (see 1.3), it follows that any isomorphism between two embeddings of \( A_0 \subset M_0 \) into an ultraproduct inclusion \( A \subset M \) is implemented by a unitary element in \( \mathcal{N}_M(A) \). Indeed, this is clear for finite dimensional \( A_0 \subset M_0 \), and the general case follows by a diagonalisation procedure.

If in turn \( A_0 \subset M_0 \) is a Cartan subalgebra with \( M_0 \) non-amenable, and \( A_0 \subset M_0 \) is embeddable into an ultraproduct \( A \subset M \) which is either of the form \( \Pi_\omega D_n \subset \Pi_\omega M_{n \times n}(\mathbb{C}) \), or of the form \( D^\omega \subset R^\omega \), then any two copies of \( A_0 \subset M_0 \) into \( A \subset M \) that are conjugate by a unitary in \( \mathcal{N}_M(A) \), will have the corresponding copies of \( M_0 \) unitary conjugate in \( M \). The procedure of constructing “many” non-conjugate embeddings of a non-amenable \( M_0 \subset M \) in the proof of (8.1 of [B]), is easily seen to actually give embeddings of \( A_0 \subset M_0 \) into \( A \subset M \). Thus, (8.1 in [B]) also implies that there exist uncountably many non-conjugate embeddings of \( A_0 \subset M_0 \) into \( A \subset M \). Altogether, this gives an analogue for Cartan inclusions (equivalently, for
countable equivalence relations [FM]), of K. Jung’s characterization of amenability in [J], by a “unique embedding” - type property.

Part 2° of Theorem 2.2 above suggests that, for a separable Cartan inclusion \( A_0 \subset M_0 \) embedded into an ultraproduct of Cartan inclusions \( A \subset M \), the bicentralizer property of the inclusion of full groups \( N_{M_0}(A_0) \subset N_M(A) \) characterizes the amenability of \( A_0 \subset M_0 \).

3° G. Elek and G. Szabo proved in [ES] the following “unique embedding” type characterization of the amenability property for a countable group \( H \), analogue to the one for finite separable von Neumann algebras in [J]: if \( H \) is amenable then any two embeddings of \( H \) into the normalizer \( N = \Pi_\omega D_n \subset \Pi_\omega M_{n \times n}(\mathbb{C}) = \mathcal{M} \), acting freely on \( A \), are conjugate by a unitary in \( N \) (this easily implies the same thing for \( A = D_\omega \subset R_\omega = \mathcal{M} \); note that by Corollary 5.2 below, the same “unique embedding” result actually holds true for ANY ultraproduct inclusion \( A \subset M \)); and that if \( H \) is sofic and non-amenable, then there exist at least two embeddings of \( H \) into \( N \) acting freely on \( A \), non-conjugate by unitaries in \( N \). In fact, as we mentioned in 2.3.2° above, by (8.1 in [B]) there even exist uncountably many non-conjugate such embeddings.

Part 1° of Theorem 2.2 suggests the following alternative “bicentralizer” characterization of amenability for countable groups:

\[
(2.3.3) \text{Conjecture: } \text{Let } H \text{ be a countable group embeddable into the normalizer of an ultraproduct MASA } A \subset M \text{ (notably } D_\omega \subset R_\omega = \mathcal{M}), \text{ such that } H \text{ acts freely on } A \text{ and such that it satisfies the bicentralizer condition } (H' \cap A)' \cap M = A \lor H. \text{ Then } H \text{ is amenable.}
\]

3. Approximate free independence in subalgebras

3.1. Notation. Let \( \mathcal{M} \) be a von Neumann algebra. If \( v \in \mathcal{M} \) is a partial isometry with \( v^*v = vv^* \), \( X \subset \mathcal{M} \) is a subset and \( k \) a nonnegative integer, then denote \( X_v^0 \overset{\text{def}}{=} X \) and \( X_v^k \overset{\text{def}}{=} \{ x_0 \prod_{i=1}^k v_i x_i \mid x_i \in X, 1 \leq i \leq k-1, x_0, x_k \in X \cup \{1\}, v_i \in \{v, v^*\} \} \).

3.2. Lemma. Let \( Q \subset M \) be an inclusion of \( \Pi_1 \) von Neumann algebras and assume \( Q \not\prec Q' \cap M \). Let \( f \in Q \) be a non-zero projection. For any \( n \geq 1 \) and any \( \varepsilon > 0 \), there exists a partial isometry \( v \) in \( fQf \) such that \( vv^* = v^*v \), \( \tau(vv^*) > \tau(f)/4 \) and \( \|E_{Q' \cap M}(x)\|_1 \leq \varepsilon, \forall x \in \bigcup_{k=1}^n F_v^k \).

Proof. It is clearly sufficient to prove the statement in case \( F = F^* \) and \( \|x\| \leq 1, \forall x \in F \). Let \( \delta > 0 \). Denote \( \varepsilon_0 = \delta, \varepsilon_k = 2^{k+1} \varepsilon_{k-1}, k \geq 1 \). Denote \( \mathcal{W} = \{ v \in fQf \mid vv^* = v^*v \in P(Q), \|E_{Q' \cap M}(x)\|_1 \leq \varepsilon_k \tau(vv^*), \forall 1 \leq k \leq n, \forall x \in F_v^k \} \).
Endow \( \mathcal{W} \) with the order \( \leq \) in which \( w_1 \leq w_2 \) iff \( w_1 = w_2 w_1^* w_1 \). \((\mathcal{W}, \leq)\) is then clearly inductively ordered. Let \( v \) be a maximal element in \( \mathcal{W} \). Assume \( \tau(v^* v) \leq \tau(f)/4 \) and denote \( p = f - v^* v \). Note that this implies \( \tau(vv^*)/\tau(p) \leq 1/3 \).

If \( w \) is a partial isometry in \( pQp \) with \( q = ww^* = w^* w \) and we let \( u = v + w \), then for \( x = x_0 \prod_{i=1}^k u_i x_i \in F_u^k \) we have

\[
x = x_0 \prod_{i=1}^k v_i x_i + \sum_\ell \sum_i z_{0,i} \prod_{j=1}^\ell w_{i,j} z_{j,i},
\]

where the sum is taken over all \( \ell = 1, 2, \ldots, k \) and all \( i = (i_1, \ldots, i_\ell) \), with \( 1 \leq i_1 < \cdots < i_\ell \leq k \), and where \( w_{i,j} = w \) (resp. \( w_{i,j} = w^* \)) whenever \( v_{i,j} = v \) (resp. \( v_{i,j} = v^* \)), \( z_{0,i} = x_0 v_{1,i} x_1 \cdots x_{i-1} p, \) \( z_{j,i} = px_{i,j} v_{i,j+1} \cdots v_{i+1,j} x_{i+1} \cdots x_{j-1} p, \) for \( 1 \leq j < \ell \), and \( z_{\ell,i} = px_{i,\ell} v_{i+1} x_{i+1} \cdots v_k x_k \).

By applying \( E_{Q' \cap M} \) to the above equation, then taking \( || \cdot ||_1 \) and applying triangle inequality, we then get:

\[
(1') \quad \| E_{Q' \cap M} (x) \|_1 \leq \| x_0 \prod_{i=1}^k v_i x_i \|_1 + \sum_\ell \sum_i z_{0,i} \prod_{j=1}^\ell w_{i,j} z_{j,i} \|_1
\]

Since \( v \in \mathcal{W} \), the first term on the right side in \((1')\) is majorized by \( \varepsilon_k \tau(vv^*) \), so we are left with estimating the terms \( z = z_{0,i} \prod_{j=1}^\ell w_{i,j} z_{j,i} \) in the double summation on the right hand side, which all have \( \ell \geq 1 \) number of appearances of powers of \( w \). We first deal with the terms where \( \ell \geq 2 \).

Since for \( y_1, y_2, y \in M \) with \( \| y_1 \| \leq 1, \| y_2 \| \leq 1 \) we have \( \| E_{Q' \cap M} (y_1 y_2) \|_1 \leq \| y_1 y_2 \|_1 \leq \| y_1 \|_1 \), it follows that for any \( \ell \geq 2 \) we have:

\[
(2) \quad \| E_{Q' \cap M} (z) \|_1 = \| E_{Q' \cap M} (z_{0,i} w_{i,1} z_{1,i} w_{i,2} z_{2,i} \cdots w_{i,\ell} z_{\ell,i}) \|_1
\]

\[
\leq \| w_{i,1} z_{1,i} w_{i,2} \|_1 = \| q z_{1,i} q \|_1 = \| q z_{1,i} q \|_{1, pMp} \tau(p),
\]

where \( \tau_{pMp} = \tau(p)^{-1} \tau_M \) and \( \| \cdot \|_{1, pMp} \) denotes the \( L^1 \)-norm on \( pM p \) associated with this trace.

By applying Theorem 1.4 to the inclusion \( pQp \subset pM p \) (with its trace \( \tau_{pMp} \)) and to the finite set \( X \subset pM p \) of all elements of the form \( z_{1,i} - E_{(Q' \cap M)p} (z_{1,i}) \in pM p \oplus (Q' \cap M)p \), for some \( i = (i_1, \ldots, i_\ell), \ \ell \geq 2 \), we obtain that for any \( \alpha > 0 \), there exists \( q \in \mathcal{P}(pQp) \) such that

\[
(3) \quad \| q z_{1,i} q - E_{(Q' \cap M)p} (z_{1,i}) q \|_{1, pMp} < \alpha \tau_{pMp} (q).
\]
Thus, by combining (2) and (3) we get

\[
\|E_{Q \cap M}(z)\|_1 \leq \|qz_1, q\|_{1,pMp} \tau(p) \\
\leq (\|E_{(Q \cap M)p}(z_1, q)\|_{1,pMp} + \alpha \tau_{pMp}(q)) \tau(p) \\
= \|E_{(Q \cap M)p}(z_1, q)\|_{1,pMp} \tau_{pMp}(q) + \alpha \tau(q) \\
= \|E_{(Q \cap M)p}(z_1, q)\|_{1,pMp} \tau(q) + \alpha \tau(q).
\]

We now take into account that by the definition of the norm \(\| \|_1\), we have

\[
\|E_{(Q \cap M)p}(z_1, q)\|_{1,pMp} = \sup\{\|y \tau(z_1, q)\|/\tau(p) \mid y \in (Q' \cap M)p, \|y\| \leq 1\}
\]

\[
= \sup\{\|y(1 - vv^*)x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1}(1 - vv^*)\|/\tau(p) \mid y \in Q' \cap M, \|y\| \leq 1\}.
\]

But since \(y \in Q' \cap M\) commutes with \(v, 1 - vv^* \in Q\) and \(\tau\) is a trace, we actually have \(\tau(y(1 - vv^*)x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1}(1 - vv^*)) = \tau(1 - vv^*)x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1}v\), so the last term in (5) is further majorized by

\[
\sup\{\|y(1 - vv^*)x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1}\|/\tau(p) \mid y \in Q' \cap M, \|y\| \leq 1\}
\]

\[
+ \sup\{\|y(1 - vv^*)x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1}\|/\tau(p) \mid y \in Q' \cap M, \|y\| \leq 1\}
\]

\[
= (\|E_{Q' \cap M}(x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1})\|_1
\]

\[
+ \|E_{Q' \cap M}(v^*x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1}v)\|_1) / \tau(p).
\]

Note at this point that \(x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1}\) lies in \(F_{i_2-i_1-1,n}^v\) and \(v^*x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1}v\) lies in \(F_{i_2-i_1+1,n}^v\). Also, \(i_2 - i_1 + 1 \leq k\), with the only case when \(i_2 - i_1 + 1 = k\) corresponding to the case \(i_1 = 1, i_2 = k, l = 2\), i.e., to the (single) term \(z = x_0w_i(px_1v_2x_2 \cdots v_{k-1}x_{k-1}p)x_kx_k\) of the double summation in (1). Thus, by combining (4) and (6) and using that \(\tau(v^*)/\tau(p) \leq 1/3\) and choosing \(\alpha = \delta/3\) (which is less than \((\varepsilon_j - \varepsilon_{j-2})/3, \forall j\)), for this particular \(z\) we get

\[
\|E_{Q' \cap M}(z)\|_1 \leq \varepsilon_{k-2}(\tau(v^*)/\tau(p))\tau(q) + \varepsilon_k(\tau(v^*)/\tau(p))\tau(q) + \alpha \tau(q)
\]

\[
\leq (\varepsilon_{k-2}/3 + \varepsilon_k/3 + \alpha) \tau(q) \leq (2\varepsilon_k/3) \tau(q),
\]

while for any \(z\) with \(i_2 - i_1 + 1 \leq k - 1, \) we get

\[
\|E_{Q' \cap M}(z)\|_1 \leq \varepsilon_{k-3}(\tau(v^*)/\tau(p))\tau(q) + \varepsilon_{k-1}(\tau(v^*)/\tau(p))\tau(q) + \alpha \tau(q)
\]
\[\leq (\varepsilon_{k-3}/3 + \varepsilon_{k-1}/3 + \alpha)\tau(q) \leq (2\varepsilon_{k-1}/3)\tau(q).\]

Since \(2^{k+1}\varepsilon_{k-1} = \varepsilon_k\) and since there are \(\sum_{i=2}^{k} \binom{k}{i} = 2^k - k\) elements in the double sum in (1) for which \(\ell \geq 2\), of which exactly one has \(i_2 - i_1 + 1 = k\) and the rest satisfy \(i_2 - i_1 + 1 \leq k - 1\), by summing up (8) and (9), we get

\[
\sum_{\ell \geq 2} \sum_i \|z_{0,i} \prod_{j=1}^{\ell} w_{i,j} z_{j,i}\|_1
\leq (2^k - k - 1)(2\varepsilon_{k-1}/3)\tau(q) + (2\varepsilon_k/3)\tau(q)

= \varepsilon_k \tau(q) - (2k + 2)(\varepsilon_{k-1}/3)\tau(q).
\]

Finally, from the double sum on the right hand side of (1′) we will now estimate the terms with \(\ell = 1\). These are terms which are obtained from \(x_0v_1x_1v_2x_2\ldots v_kx_k\) by replacing exactly one \(v_i\) by \(w_i\), so they are of the form \(z = z_{0,i} w_i z_{1,i}\), where \(i = 1, 2, \ldots, k\), \(z_{0,i} = x_0v_1x_1\ldots v_{i-1}x_{i-1}p\), \(z_{1,i} = px_i v_{i+1}\ldots v_kx_k\) and \(w_i = w^s\) if \(v_i = v^s\). Note that there are \(k\) of them.

One should notice at this point that in the above estimates we only used the fact that \(w^*w = ww^* = q \in \mathcal{P}(Q)\) and that it satisfies (3) for appropriate \(\alpha\). But we did not use so far the actual form of \(w\). We will make the appropriate choice for \(w\) now, by making use of the condition \(Q \not\prec Q' \cap M\). Indeed, by Theorem 1.4 (2.1 in [P10]), this latter condition implies that for all \(\beta > 0\) and all finite sets \(Y_1 = Y_1^* \subset M \cap Q' \cap M, Y_2 \subset M\), there exists a unitary element \(w \in qQq\) such that

\[
\|E_{Q' \cap M}(y_1wy_2)\|_1 < \beta, \|E_{Q' \cap M}(y_2wy_1)\|_1 < \beta, \forall y_1 \in Y_1, y_2 \in Y_2.
\]

Note that since \(Y_1, Y_2\) are selfadjoint sets, by taking adjoints in (11), from these estimates we also get:

\[
\|E_{Q' \cap M}(y_2w^*y_1)\|_1 < \beta, \|E_{Q' \cap M}(y_1w^*y_2)\|_1 < \beta, \forall y_1 \in Y_1, y_2 \in Y_2.
\]

Denote by \(Z\) the set of elements of the form \(x_0v_1x_1\ldots v_{i-1}x_{i-1}p\), or \(px_i v_{i+1}\ldots v_kx_k\), for all possible choices arising from elements in \(\bigcup_{k=1}^{n} F^k_{\psi}\). By applying (11), (11′) to \(\beta = \varepsilon_{k-1}\tau(q)/2k\), \(n \geq 1\) and \(Y_2 = Z \cup Z^* \cup \{E_{Q' \cap M}(z) \mid z \in Z \cup Z^*\}\), \(Y_1 = \{y_2 - E_{Q' \cap M}(y_2) \mid y_2 \in Y_2\}\), it follows that there exists \(w \in U(qQq)\) such that

\[
\|E_{Q' \cap M}((x_0v_1x_1\ldots v_{j-1}x_{j-1})\).
\]
\[-E_{Q'\cap M}(x_0v_1x_1\ldots v_{j-1}x_{j-1}p))w_jx_jv_{j+1}\ldots v_kx_k\|_1 \leq \varepsilon_{k-1}\tau(q)/2k,\]

\[(12') \quad \|E_{Q'\cap M}(x_0v_1x_1\ldots v_{j-1}x_{j-1}w_j(x_jv_{j+1}\ldots v_kx_k)\|_1 \leq \varepsilon_{k-1}\tau(q)/2k.\]

Thus, for each element with \(\ell = 1\) in the double summation \(\sum_{\ell, \tau} z_{0, \ell} \Pi_{j=1}^{\ell} w_{j, z_{j, \ell}}\) in (1), i.e., of the form \(x_0v_1x_1\ldots v_{j-1}x_{j-1}w_jx_jv_{j+1}\ldots v_kx_k\), we have the estimate:

\[(13) \quad \|E_{Q'\cap M}(x_0v_1x_1\ldots v_{j-1}x_{j-1})q\|_1 = \tau(q)\|E_{Q'\cap M}(x_0v_1x_1\ldots v_{j-1}x_{j-1})\|_1\]

and

\[\|qE_{Q'\cap M}(x_jv_{j+1}\ldots v_kx_k)\|_1 = \tau(q)\|E_{Q'\cap M}(x_jv_{j+1}\ldots v_kx_k)\|_1\]

Both elements \(x_0v_1x_1\ldots v_{j-1}x_{j-1}, x_jv_{j+1}\ldots v_kx_k\) belong to some \(F_{v}^{j, n}\) with \(j \leq k - 1\), and at least one of them with \(j \neq 0\). Thus, by the properties of \(v \in \mathcal{W}\) and the assumption \(\tau(vv^*) \leq \tau(f)/4\), we have \(\gamma \leq \varepsilon_{k-1}\tau(vv^*)\tau(q) \leq \varepsilon_{k-1}\tau(q)/4\). Hence, the last term in (13) is majorized by \(\varepsilon_{k-1}\tau(q)/k + \varepsilon_{k-1}\tau(q)/4\). Since there are \(k\) terms with \(\ell = 1\), obtained by taking \(j = 1, \ldots, k\), by summing up over \(j\) in (13) and combining with (10), we deduce from (1') the following final estimate:

\[(14) \quad \|E_{Q'\cap M}(x)\|_1 \leq \|E_{Q'\cap M}(x_0\Pi_{i=1}^{k} v_i x_i)\|_1 + \Sigma_{\ell, \tau} \|E_{Q'\cap M}(z_0, \ell \Pi_{j=1}^{\ell} w_{j, z_{j, \ell}})\|_1\]

\[\leq \varepsilon_k \tau(vv^*) + (2^k - k - 1)\varepsilon_{k-1}\tau(q) + (k + 1)\varepsilon_{k-1}\tau(q)/4\]

\[\leq \varepsilon_k \tau(vv^*) + \varepsilon_k \tau(wv^*) = \varepsilon_k \tau((v + w)(v + w)^*).\]

Since \(u = v + w\) has also the property that \(uw^* = u^*u\), it follows from (14) that \(u \in \mathcal{W}\). But this contradicts the maximality of \(v \in \mathcal{W}\).

We conclude that \(\tau(v^*v) > \tau(f)/4\). If we now take \(\delta \leq \varepsilon/2^{n+1}\), then \(\varepsilon_n = 2^{(n+1)(n+2)/2\delta} < 2^{n+1}\delta \leq \varepsilon\) and the statement follows.

\(\square\)
4. Free independence in ultraproduct framework

4.1. Notation. Let $M_n$ be a sequence of finite factors with $\dim(M_n) \to \infty$. Let $\omega$ be a free ultrafilter on $\mathbb{N}$ and denote $M = \prod_\omega M_n$. We consider the following two special classes of subalgebras of $M$:

(4.1.1) We denote by $\mathcal{Q}_u$ the class of von Neumann subalgebras $Q \subset M$ which are of the form $Q = \prod_\omega Q_n$, for some subalgebras $Q_n \subset M_n$, and have the property that $Q \nprec_M Q'$.

(4.1.2) We denote by $\mathcal{Q}_b$ the class of von Neumann subalgebras $Q \subset M$ with the property that $Q' \cap M$ is separable and $(Q' \cap M)' \cap M = Q$.

The next result provides some properties and examples of algebras in these two classes.

Proposition 4.2. 1° If $Q \in \mathcal{Q}_u$, then $Q$ is of type $\Pi_1$.

2° If $Q_n \subset M_n$ are von Neumann subalgebras such that $Q_n \nprec_{M_n} Q_n' \cap M_n$, $\forall n$, then $Q = \prod_\omega Q_n$ satisfies $Q \nprec_M Q' \cap M$, and thus $Q \in \mathcal{Q}_u$.

3° Assume $m_n$ is an increasing sequence of positive integers of the form $m_n = d_n \cdot k_n$, with $d_n, k_n \in \mathbb{N}$. Let $M_n = M_{m_n \times m_n}(\mathbb{C})$, with $P_n = M_{d_n \times d_n}(\mathbb{C})$, $Q_n = M_k, k_n(\mathbb{C})$, viewed as subalgebras of $M_n$ that commute and generate $M_n$. Then $Q = \prod_\omega Q_n$, $P = \prod_\omega P_n$ satisfy the following properties: $Q' \cap M = P$, $P' \cap M = Q$; $Q \nprec_M P$ (and thus $Q \in \mathcal{Q}_u$) if and only if $\lim \omega d_n/k_n = 0$.

4° If $B \subset M$ is a separable amenable von Neumann subalgebra, then $Q := B' \cap M$ satisfies $Q' \cap M = B$. Thus $Q \in \mathcal{Q}_b$.

5° If $Q \in \mathcal{Q}_b$ then $Q$ is of type $\Pi_1$, has no separable direct summand, and $Q \nprec_M Q' \cap M$ (the latter being separable).

Proof. 1° If an inclusion of finite von Neumann algebras $B \subset M$ is so that $B$ is type I, then there exists a non-zero projection $e \in B$ such that $eBe$ is abelian, implying that $eBe \subset (eBe)' \cap eMe$, thus $B \nprec_M B' \cap M$. Since in our case we have $Q \nprec_M Q' \cap M$, this shows that $Q$ cannot have type I summands, thus $Q$ is $\Pi_1$.

Part 2° is an immediate consequence of Theorem 1.5 and of the fact that $Q' \cap M = \prod_\omega (Q_n' \cap M_n)$ with $E_{Q' \cap M}(x) = (E_{Q_n' \cap M_n}(x_n))_n$, for $x = (x_n)_n \in M = \prod_\omega M_n$.

Part 3° is an easy exercise (using Theorem 1.5) while part 4° is a direct consequence of Theorem 2.1.

To prove part 5°, note that if $Q \in \mathcal{Q}_b$ then $Q$ has no separable direct summand, by the same observation we have used in the proof of part 1°.
Note that conjecture (2.3.1) predicts that the class $\mathcal{Q}_b$ only consists of centralizers of separable amenable subalgebras of $M$, i.e., of the examples $4.2.4^\circ$ above.

4.3. Theorem. Assume $Q \subset M$ is either in the class $\mathcal{Q}_u$, or $\mathcal{Q}_b$. If $X \in M \ominus (Q' \cap M)$ is a separable subspace, then there exists a diffuse von Neumann subalgebra $B_0 \subset Q$ such that $B_0$ is free independent to $X$, relative to $Q' \cap M$, more precisely $E_{Q' \cap M}(x_0 \prod_{i=1}^{k} y_i x_i) = 0$, for all $k \geq 1$ and all $x_i \in X$, $1 \leq i \leq k - 1$, $x_0, x_k \in X \cup \{1\}$, $y_i \in B_0 \ominus \mathbb{C}$, $1 \leq i \leq k$.

4.4. Corollary. With the same assumptions and notations as in 4.3 above, we have:

1° Let $P \subset M$ be a von Neumann subalgebra making a commuting square with $Q' \cap M$ and denote $B_1 = P \cap (Q' \cap M)$. Assume that $L^2P$ is countably generated both as a left and as a right $B_1$ Hilbert module (equivalently, there exists a separable space $X \subset P$ such that $X \perp B_1$, and $\text{sp}XB_1$ and $\text{sp}B_1X$ are both $\| \|_2$-dense in $P \ominus B_1$). Then there exists a diffuse von Neumann subalgebra $B_0 \subset Q$ such that $P \vee B_0 \simeq P *_{B_1} (B_1 \varnothing B_0)$.

2° Let $N_i \subset M$ be separable von Neumann algebras, with amenable subalgebras $B_i$, $i = 1, 2$, such that $(B_1, \tau) \simeq (B_2, \tau)$. Then there exists a unitary element $u \in M$ such that $uB_1u^* = B_2$ and such that, after identifying $B = B_1 \simeq B_2$ via $\text{Ad}(u)$, we have $N_1 \vee uN_2u^* \simeq N_1 *_B N_2$.

Note that the case $B$ atomic of $4.4.2^\circ$ above has already been shown in [P6], while the case $B$ arbitrary but $M = R^n$ was shown in [BDJ] (see also [FGR] for more recent related considerations).

A particular case when the assumptions in $4.4.1^\circ$ are satisfied, is when the sub-algebra $P \subset M$ making a commuting square with $Q' \cap M$ is itself separable. But there are interesting non-separable examples as well, that may even allow obtaining free product with amalgamation over the entire $Q' \cap M$ (which is non-separable in case $Q \in \mathcal{Q}_u$). For instance, if $\mathcal{U} \subset \mathcal{U}(M)$ is a countable group of unitaries normalizing $Q' \cap M$, then the von Neumann algebra $P$ generated by $\mathcal{U}$ and $Q' \cap M$ satisfies all the conditions in $4.4.1^\circ$ with $B_1 = Q' \cap M$.

Note in this respect that one can alternatively take in the statement of Theorem 4.3 the separable space $X$ to be of the form $X = P \ominus (P \cap Q' \cap M)$, for some separable von Neumann algebra $P$ making a commuting square with $Q' \cap M$. Indeed, due to Lemma 1.2, the two versions follow equivalent.

Lemma 4.5. Let $Q \subset M$ be a von Neumann subalgebra lying in either the class $\mathcal{Q}_u$ or the class $\mathcal{Q}_b$. Let $f \in Q$ be a projection and $X \subset M \ominus Q' \cap M$ a countable set.
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Then there exists a partial isometry $v$ in $fQf$ such that $vv^* = v^*v$, $\tau(vv^*) \geq \tau(f)/4$ and $E_{Q'\cap M}(x) = 0$, $\forall x \in X_k$, $\forall k \geq 1$.

Proof. Let $X = \{x_k\}_{k \geq 1}$ be an enumeration of $X$ and denote $x_0 = 1$. By applying Lemma 3.2 to the inclusion of $\Pi_1$ von Neumann algebras $Q \subset M$, the projection $f \in Q$, the positive constant $\varepsilon = 2^{-n}$ and the finite set $X_n = \{x_k \mid k \leq n\}$, we get a partial isometry $w_n$ in $fQf$ with the property that $w_nw_n^* = w_n^*w_n$, $\tau(w_n^*w_n) \geq \tau(f)/4$ and

\begin{equation}
(1) \quad \|E_{Q'\cap M}(x)\|_1 < 2^{-n}, \forall x \in \bigcup_{k \leq n} (X_n)_n.
\end{equation}

Let $f = (f_m)_m$ be a representation of $f$ with $f_m$ projections. Let also $x_k = (x_{k,m})_m$ be a representation of $x_k$, with $x_{k,m} \in M_m$, $\|x_{k,m}\| \leq \|x_k\|$, $\forall k,m$, and $w_k = (w_{k,m})_m \in Q$ a representation of $w_k$ with $w_{k,m}$ partial isometries satisfying $w_{k,m}w_{k,m}^* = w_{k,m}^*w_{k,m} \leq f_m$.

Assume first that $Q = \Pi_1Q \subset Q$, in which case we may clearly also assume $f_m \in P(Q_m)$ and $w_{k,m} \in f_mQ_mf_m$, $\forall k,m$. Noticing that if $y = (y_n)_n \in M$ then $E_{Q'\cap M}(y) = (E_{Q'\cap M}(y_n))_n$, it follows from (1) that

\begin{equation}
(2) \quad \lim_{m \to \omega} \|E_{Q'\cap M}(x_{j_0,m}\Pi_{i=1}^k w_{n,i,m}^* x_{j_i,m})\|_1 < 2^{-n},
\end{equation}

for all $1 \leq k \leq n$, $x_{j_0} \in X_n \cup \{1\}$, $x_{j_i} \in X_n$, $\gamma_i \in \{\pm 1\}$.

Let $V_n$ be the set of all $m \in \mathbb{N}$ with the property that

\begin{equation}
(3) \quad \|E_{Q'\cap M}(x_{j_0,m}\Pi_{i=1}^k w_{n,i,m}^* x_{j_i,m})\|_1 < 2^{-n},
\end{equation}

for all $1 \leq k \leq n$, $1 \leq j_i \leq n$ for $i \geq 1$, $0 \leq j_0 \leq n$, $\gamma_i \in \{\pm 1\}$. By (2) it follows that $V_n$ corresponds to an open-closed neighborhood of $\omega$ in $\Omega$, under the identification $\ell_\omega \mathbb{N} = C(\Omega)$. Let now $W_n$, $n \geq 0$, be defined recursively as follows: $W_0 = \mathbb{N}$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} \mid n > \min W_n\}$. Note that, with the same identification as before, $W_n$ is a strictly decreasing sequence of neighborhoods of $\omega$.

Define $v = (v_m)_m$ by letting $v_m = w_{n,m}$ for $m \in W_{n-1} \setminus W_n$. It is then easy to check that $v$ is a partial isometry in $fQf$ satisfying all the required conditions.

Assume now that $Q \subset Q$. Let $\{y_\ell\}_\ell \subset Q' \cap M$ be a countable set dense in the unit ball of $Q' \cap M$ in the norm $\|\cdot\|_2$. Note that if $y_\ell = (y_\ell)_m$ then $x = (x_n)_n \in M$ satisfies $x \in Q$ iff $\lim_{m \to \omega} \|[x_m, y_\ell]_m\|_2 = 0$, $\forall \ell$. Also, $x \perp Q' \cap M$ iff $\lim_{m \to \omega} \tau(x_m y_\ell, m) = 0$, $\forall \ell$. Moreover, if $\delta > 0$, then $\|E_{Q'\cap M}(x)\|_1 < \delta$ iff $\lim_{m \to \omega} \tau(x_m y_\ell, m) < \delta$, $\forall \ell$. 


With this in mind, from (1) it follows that the partial isometries \( w_n = (w_{n,m})_m \in Q \) satisfy

\[
\lim_{m \to \omega} |\tau((x_{j_0,m} \Pi_{i=1}^k w_{n,m}^{\gamma_i}(x_{j_i,m})y_{\ell,m})| < 2^{-n} ,
\]

for all \( 1 \leq k \leq n \), \( x_{j_0} \in X_n \cup \{1\}, x_{j_i} \in X_n, \gamma_i \in \{\pm 1\} \), and for all \( \ell \geq 1 \). Also, the fact that \( w_n \) belongs to \( fQf \) is equivalent to

\[
\lim_{m \to \omega} \|w_{n,m}, y_{\ell,m}\|_2 = 0, \forall \ell; \lim_{m \to \omega} \|f_m w_{n,m} f_m - w_{n,m}\|_1 = 0
\]

Let \( V_n \) be the neighborhood of \( \omega \) consisting of all \( m \in \mathbb{N} \) with the property that

\[
|\tau((x_{j_0,m} \Pi_{i=1}^k w_{n,m}^{\gamma_i}(x_{j_i,m})y_{\ell,m})| < 2^{-n} ;
\]

\[
\|w_{n,m}, y_{\ell,m}\|_2 < 2^{-n} ; \|f_m w_{n,m} f_m - w_{n,m}\|_1 < 2^{-n} ;
\]

for all \( \ell = 1, 2, \ldots, n \) as well as for all \( 1 \leq k \leq n \), \( x_{j_0} \in X_n \cup \{1\}, x_{j_i} \in X_n, \gamma_i \in \{\pm 1\} \). Let further \( W_n \subset \mathbb{N}, n \geq 0 \), be defined recursively as follows: \( W_0 = \mathbb{N} \) and \( W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} | n \geq \min W_n\} \). It follows that \( W_n \) are all neighborhoods of \( \omega \), that \( W_n \subset \cap_{j \leq n} V_j, W_{n+1} \subset W_n \), and \( W_{n+1} \neq W_n \).

We now define \( v = (v_m)_m \), by letting \( v_m = w_{n,m} \) if \( m \in W_{n-1} \setminus W_n \). By the way \( w_{n,m} \) have been taken, \( v \) follows a partial isometry with \( vv^* = v^* v \), while by the second relation in (6) we have \( v \in fQf \) and by the first relation in (6) we have \( E_{Q \cap M}(x) = 0, \forall x \in X^k_v, \forall k \geq 1 \).

\[\square\]

**Proof of 4.3.** We construct recursively a sequence of partial isometries \( v_1, v_2, \ldots \in Q \) such that

(i) \( v_{j+1} v_j^* v_j = v_j, v_j v_j^* = v_j^* v_j \) and \( \tau(v_j v_j^*) \geq 1 - 1/2^j, \forall j \geq 1 \).

(ii) \( E_{Q \cap M}(x) = 0, \forall x \in X^k_v, \forall k \geq 1 \).

Assume we have constructed \( v_j \) for \( j = 1, \ldots, m \). If \( v_m \) is a unitary element, then we let \( v_j = v_m \) for all \( j \geq m \). If \( v_m \) is not a unitary element, then let \( f = 1 - v_m^* v_m \in Q \). Note that \( E_{Q \cap M}(x') = 0, \forall x' \in X' \). If we apply Lemma 4.5 to \( Q \subset M \), the projection \( f \in Q \) and the countable set \( X' \subset M \cap (Q' \cap M) \), then we get a partial isometry \( u \in fQf \), with \( uu^* = u^* u \) satisfying \( \tau(uu^*) \geq \tau(f)/2 \) and \( E_{Q \cap M}(x) = 0 \) for all \( x \in \cup_k (X')_u^k \). But then \( v_{m+1} = v_m + u \) will satisfy both (i) and (ii) for \( j = m + 1 \).
It follows now from $(i)$ that the sequence $v_j$ converges in the norm $\|\cdot\|_2$ to a unitary element $v \in Q$, which due to $(ii)$ will satisfy the condition $E_{Q' \cap M}(x)$, \forall x \in \bigcup_n X_n^\omega$. Now, since $Q$ is a $\Pi_1$ von Neumann algebra, $Q$ contains a copy of the hyperfinite $\Pi_1$ factor, which in turn contains a Haar unitary $u_0 \in R$. But then $u = vu_0v^*$ clearly satisfies the conditions required in part $(a)$ of 4.3. \hfill $\Box$

**Proof of 4.4.** 1° Let $X_0 \subset P \ominus B_1$ be a separable subspace such that $\text{sp}X_0B_1$ and $\text{sp}B_1X_0$ are $\|\cdot\|_2$-dense in $P \ominus B_1$. By Theorem 4.3, there exists a diffuse von Neumann subalgebra $B_0 \subset Q$ such that $B_0$ is free independent to $X_0$ relative to $Q' \cap M$. It is sufficient to prove that $E_{Q' \cap M}(x_0\Pi_iy_ix_i) = 0$, for any $x_0 \in X_0B_1 \cup \{1\}$, $x_i \in X_0B_1$, $y_i \in B_0 \ominus C1$, $1 \leq i \leq n$. But any element in $X_0B_1$ can be approximated arbitrarily well by a linear combination of elements in $B_1X_0$. The “coefficient” in $B_1$ of each one of these elements commutes with $y_{i-1}$, so we can “move it to the left”, being “swollen” by the $x_{i1} \in X_0B_1$. Thus, in the end, it follows that it is sufficient to have $E_{Q' \cap M}(x_{00}\Pi_iy_ix_i) = 0$ for $x_{00} \in X_0 \cup \{1\}$, $x_{0i} \in X_0$, $y_i \in B_0 \ominus C1$, which is indeed the case because $B_1$ is free independent to $X_0$ relative to $Q' \cap M$.

2° By the first part of Remark 2.3, after possibly conjugating with a unitary $u_0 \in M$, we may assume the subalgebras $B_1$, $B_2$ coincide. Denote $B$ this common algebra and let $Q = B' \cap M$, which by 2.1 satisfies $Q' \cap M = B$ and by 4.2.4° it belongs to $\mathcal{Q}_b$. Now apply 4.3 to $Q$ and to the separable space $X = N_1 \ominus B + N_2 \ominus B$, to conclude that there exists a unitary element $u \in Q$ such that $uN_2u^*$ and $N_1$ generate the free amalgamated product $\simeq N_1 *_B N_2$. \hfill $\Box$

## 5. More on the incremental patching method

The crucial step in proving Theorem 4.3 is Lemma 3.2. The technique used in its proof consists of building unitaries $u$ that are approximately $n$-independent with respect to certain finite sets, by “patching” together infinitesimal pieces of $u$. This technique was first considered in (2.1 of [P3]), to show that given any countable set $X$ in a finite von Neumann algebra $M$ and any diffuse abelian von Neumann subalgebra $A \subset M$, there exists a Haar unitary $u \in A^\omega$ such that any word that alternates letters from $X$ and $\{u^n \mid n \geq 1\}$, has 0-trace. This result was a key tool in proving that any derivation of a $\Pi_1$ factor into the ideal of compact operators is inner, in [P3].

The technique was substantially refined in [P6], to prove a particular case of the case $Q \in \mathcal{Q}_u$ of Theorem 4.3, in which $Q = \Pi_{\omega}Q_n \in \mathcal{Q}_u$ is so that $Q_n \subset M_n$ are $\Pi_1$ subfactors with atomic relative commutant $Q'_n \cap M_n$ (which thus clearly satisfy $Q_n \not\prec_{M_n} Q'_n \cap M_n$). The result in [P6] had several applications over the years: Thus, it played an important role in developing reconstruction methods in Jones theory of subfactors in ([P4], [P7], [P9]) and it led, in combination with ([V]),
to the definition of amalgamated free product of inclusions of finite von Neumann algebras in [P4]. It was also used to prove key technical results in ([IPeP], [Va]) and to show that the free product of standard invariants of subfactors defined in ([BiJ]) can be realized in the hyperfinite II$_1$ factor $R$ (see A.3 in [IPeP] and [Va]).

More recently, the same incremental patching method was used in [P12] to prove that if $A_n \subset M_n$ is a sequence of MASAs in II$_1$ factors, then the abelian von Neumann algebra $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$ contains diffuse subalgebras $B_0$ that are $\tau$-independent to any given separable subalgebra $B \subset A$ and 3-independent to any given countable set $X \subset M \otimes A$, i.e. any alternating word with at most 3 letters from $X$ and 3 letters from $B_0 \otimes \mathbb{C}1$ has trace 0 (see 0.2 in [P12]). Moreover, if $A_n$ are all singular (in the sense of [D1], i.e. any unitary normalizing $A_n$ is contained in $A_n$), then $B_0$ can be chosen to be free independent to $X$, relative to $A$, a fact that allowed settling the Kadison-Singer problem for ultraproducts of singular MASAs $A \subset M$ (see 0.1 in [P12]).

A concrete example of a diffuse subalgebra $B_0$ in an ultraproduct MASA $A$ satisfying the 3-independence property is the following: Let $\Gamma \acts X$ be an ergodic (but not necessarily free) measure preserving action of a discrete group $\Gamma$ on a probability space $(X,\mu)$ and $\Gamma \acts Y = [0,1]^{\Gamma}$ be the Bernoulli $\Gamma$-action with diffuse base. Let $A = L^\infty(X) \otimes L^\infty(Y)$ with $\Gamma \acts A$ the product action. Let $M = A \rtimes \Gamma$ and $A = A^\omega \subset M^\omega = M$. If we take $B = L^\infty(X)$ and let $B_0 = 1 \otimes L^\infty([0,1]) \otimes 1 \subset L^\infty(Y)$ be the base of the Bernoulli action, viewed as a tensor component of the infinite tensor product $L^\infty(Y) = \otimes_{g \in \Gamma} (L^\infty([0,1]))_g$, then it is easy to see that $B_0$ is $\tau$-independent to $B$ and 3-independent with respect to $X = \{u_g \mid g \in \Gamma\}$.

This construction can actually be recovered “asymptotically” inside any group measure space von Neumann algebra. Indeed, using the incremental patching technique, we will now prove that (generalized) Bernoulli $\Gamma$-actions can be retrieved inside any free action of $\Gamma$ on an ultrapower of measure spaces. More generally we have:

5.1. Theorem. Let $A_n \subset M_n$ be a sequence of MASAs in finite factors, with $\dim M_n \to \infty$, and denote $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$. Assume $\Gamma \subset N_M(A)$ is a countable group of unitaries acting freely on $A$ and let $H \subset \Gamma$ be an amenable subgroup. Given any separable abelian von Neumann subalgebra $B \subset A$, there exists a separable diffuse abelian subalgebra $A \subset A$ such that: $A, B$ are $\tau$-independent, $\Gamma$ normalizes $A$, and the action of $\Gamma$ on $A$ is isomorphic to the generalized Bernoulli action $\Gamma \acts L^\infty([0,1]^{\Gamma/H})$.

Proof. Let $\{u_g \mid g \in \Gamma\}$ be the unitaries in $\Gamma$. Denote by $g_0 = 1, g_1, g_2, \ldots \in \Gamma$ a set of representatives of $\Gamma/H$. It is clearly sufficient to construct a Haar unitary $w$ in $A$ such that $w$ commutes with $u_h, \forall h \in H$, and such that $B$ and $u_g, \{w^n \mid n \in \mathbb{N}\}$...
\[\mathbb{Z}\}u^n_{g_i}, \ i = 0, 1, 2, \ldots,\] are all multi-independent, in the sense that for any \(k,\) any non-zero integers \(n_j,\) distinct non-negative integers \(m_j,\) and any \(b \in B,\) we have
\[
\tau(b \Pi_{j=0}^k u_{g_{m_j}} w^{n_j} u^*_{g_{m_j}}) = 0.
\]

We need some notations. Thus, we let \(\mathbf{A}_0\) be the subalgebra of all elements in \(\mathbf{A}\) that are fixed by \(H\) and let \(\{a_n\}_n\) be a \(\|\|_2\)-dense subset of the unit ball of \(B.\) If \(v\) is a partial isometry in \(\mathbf{A}_0,\) then we denote by \(F_{v, n}\) the set of all elements of the form \(b_i \Pi_{j=0}^k u_{g_{m_j}} w^{n_j} u^*_{g_{m_j}},\) where \(1 \leq i \leq n, 1 \leq k \leq n, m_j\) are distinct integers between 0 and \(n,\) and \(1 \leq |n_j| \leq n.\) We first prove the following:

**Fact.** Given any \(n \geq 1\) and any \(\delta > 0,\) there exists a Haar unitary \(v \in \mathbf{A}_0\) such that \(|\tau(x)| \leq \delta, \ \forall x \in F_{v, n}.\)

To prove this, let \(\mathcal{W} := \{v \in \mathbf{A}_0 \mid |\tau(x)| \leq \delta \tau(v^* v), \ \forall x \in F_{v, n}, \tau(v^m) = 0, \ \forall m \neq 0\}.\) Endow \(\mathcal{W}\) with the order \(\leq\) in which \(w_1 \leq w_2\) iff \(w_1 = w_2 w_1 w_1.\) \(\mathcal{W}, \leq\) is then clearly inductively ordered. Let \(v\) be a maximal element in \(\mathcal{W}.\) Assume \(\tau(v^* v) < 1\) and denote \(p = 1 - v^* v.\) If \(w \in \mathbf{A}_0 p\) is a partial isometry satifying \(\tau(w^m) = 0, \ \forall m \neq 0,\) and we denote \(u = v + w,\) then we have

\[
(1) \quad b_i \Pi_{j=0}^k u_{g_{m_j}} w^{n_j} u^*_{g_{m_j}} = b_i \Pi_{j=0}^k u_{g_{m_j}} v^{n_j} u^*_{g_{m_j}} + \Sigma b_i \Pi_{j=0}^k u_{g_{m_j}} z_j^{n_j} u^*_{g_{m_j}}
\]

where \(z_j \in \{v, w\}\) and the sum is taken over all possible choices for \(z_j = v\) or \(z_j = w,\) with at least one occurrence of \(z_j = w\) (thus, there are \(2^k + 1 - 1\) many terms in the summation). We thus get the estimate

\[
(2) \quad |\tau(b_i \Pi_{j=0}^k u_{g_{m_j}} w^{n_j} u^*_{g_{m_j}})|
\]

\[
\leq |\tau(b_i \Pi_{j=0}^k u_{g_{m_j}} v^{n_j} u^*_{g_{m_j}})| + \Sigma |\tau(b_i \Pi_{j=0}^k u_{g_{m_j}} z_j^{n_j} u^*_{g_{m_j}})|
\]

\[
\leq \delta \tau(v v^*) + \Sigma' |\tau(b_i \Pi_{j=0}^k u_{g_{m_j}} z_j^{n_j} u^*_{g_{m_j}})| + \Sigma'' |\tau(b_i \Pi_{j=0}^k u_{g_{m_j}} z_j^{n_j} u^*_{g_{m_j}})|
\]

where the summation \(\Sigma'\) contains the terms with just one occurrence of \(z_j = w\) and \(\Sigma''\) is the summation of the terms that have at least 2 occurrences of \(z_j = w.\) Since \(\mathbf{A}\) is abelian, the terms \(u_{g_{m_j}} z_j^{n_j} u^*_{g_{m_j}}\) in a product can be permuted arbitrarily.

Thus, in each summand of \(\Sigma''\) we can bring two of the occurrences of \(w\) so that to be adjacent, i.e., of the form \(y_1 u_{g_{m_j}} w^{n_j} u^*_{g_{m_j}} u_{g_{m_i}} w^{n_i} u^*_{g_{m_j}} y_2.\) Since \(g_{m_i} \neq g_{m_j}\) for all \(i \neq j,\) by applying part 1° of Theorem 1.4 to \(Q = \mathbf{A}_0\) and the finite set \(F = \{u_{g_{m_j}} u_{g_{m_i}} \mid i \neq j\} \perp \mathbf{A} = \mathbf{A}_0' \cap \mathbf{M},\) it follows that for any \(\alpha > 0,\) there exists a non-zero \(q \in \mathcal{P}(\mathbf{A}_0 p)\) such that
\[
(3) \quad \|qu_{g_{m,j}}u_{g_m, q}\|_1 < \alpha \tau(q), \forall 0 \leq m_i \neq m_j \leq n.
\]

Since there are \(2^{k+1} - (k + 1) - 1\) terms in the summation \(\Sigma''\), this shows that \(\Sigma'' < (2^{k+1} - (k + 1) - 1)\alpha \tau(q)\), for any choice of \(w\) that has support \(q\) satisfying condition (3). Thus, if we choose \(\alpha \leq 2^{-n-2}\delta\), then by (3) we get \(\Sigma'' \leq \delta \tau(q)/2\).

So we are left with estimating the terms in the summation \(\Sigma'\), which have just one occurrence of \(w^j, j \neq 0\), i.e are of the form \(|\tau(y_1 w^j y_2)| = |\tau(w^j E_A(q y_2 y_1 q))|\), for some \(y_1, y_2 \in M, 1 \leq |j| \leq n\). There are \(k + 1\) many such terms for each \(k = 1, \ldots, n\). Let’s denote by \(Y_0\) the set of all \(y_1, y_2\) which appear this way, and note that this is a finite set in \(q M q\). Thus \(Y = E_A(q Y_0 \cdot Y_0 q)\) is finite as well.

It is sufficient now to find a Haar unitary \(w \in A_0 q\) such that \(|\tau(w^j y)| \leq \delta \tau(q)/2(n+1), \forall y \in Y, 1 \leq |j| \leq n\), because then the sum of the \(k + 1\) terms in \(\Sigma'\) will be majorized by \(\delta \tau(q)/2\), altogether showing that for all \(x \in F_{u,n}\), we have \(|\tau(x)| \leq \delta \tau(u^\dagger)\). Since \(A_0 q\) is diffuse, it contains a separable diffuse von Neumann subalgebra \(A_0\), which is isomorphic to \(L^\infty(T)\) with the Lebesgue measure corresponding to \(\tau(q)^{-1} \tau_{A_0}\). Let then \(w_0 \in A_0\) be a Haar unitary generating \(A_0\). Since \(\{w_0^n\}_n\) tends to 0 in the weak operator topology and \(Y \subset A_0 q\) is a finite set, there exists \(n_0 \geq n\) such that \(|\tau(w_0^n y)| \leq \delta \tau(q)/2(n+1), \forall y \in Y\) and \(|m| \geq n_0\). But then \(w = w_0^{n_0}\) is still a Haar unitary and it satisfies all the required conditions.

This ends the proof of the Fact.

By using this Fact, it follows that for each \(n\) there exists a unitary element \(v_n \in A_0\) such that

\[
(4) \quad |\tau(x)| < 2^{-n}, \forall x \in F_{v_n,n}.
\]

For each \(g \in \Gamma\), let \(u_g = (u_{g, m})_m\) be a representation of \(u_g\) with \(u_{g, m} \in \mathcal{N}_{M_n}(A_n)\). Let also \(b_i = (b_{i,m})_m\) and \(v_n = (v_{n,m})_m \in A_0\), with \(b_{i,m}, v_{n,m} \in A_m, \forall m\). Then (4) becomes

\[
(5) \quad \lim_{m \to \omega} |\tau(b_{i,m} \Pi_{j=0}^{k} u_{g_{j,m}} v_{n,m} u_{g_{j,m}}^{*})| < 2^{-n}
\]

for all \(1 \leq i, k \leq n, 0 \leq j_0 < j_1 \ldots < j_k \leq n\). Also, the fact that \(v_n\) lies in \(A_0\) translates into

\[
(6) \quad \lim_{m \to \omega} \|u_{h,m}, v_{n,m}\|_1 = 0, \forall h \in H, n \geq 1
\]
Let then $V_n$ be the set of all $m \in \mathbb{N}$ satisfying the following properties:

\begin{equation}
\tau(b_{i,m}\Pi_{j=0}^ku_{g_j,m}v_{n,m}u_{g_j,m}^*) < 2^{-n}
\end{equation}

\begin{equation}
\|([u_{h_i,m}, v_{n,m}]|_1 < 2^{-n}
\end{equation}

for all $1 \leq i, k \leq n, 0 \leq j_0 < j_1 \ldots < j_k \leq n$, where $\{h_i\}_i = H$ is an enumeration of $H$. Note that by (5) and (6), $V_n$ corresponds to an open-closed neighborhood of $\omega$ in $\Omega$, under the identification $\ell^\infty(\mathbb{N}) = C(\Omega)$. Define now recursively $W_0 = \mathbb{N}$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} | n > \min W_n\}$. Then $W_n$ is a strictly decreasing sequence of neighborhoods of $\omega$ (under the same identification as before) with $W_n \subset \cap_{j \leq n} V_j$.

We now define $w = (w_m)_m$, by letting $w_m = v_{n,m}$ if $m \in W_{n-1} \setminus W_n$. By the way $v_{n,m}$ have been taken, $w$ follows unitary element in $A$, while by the second relation in (7) we have $w \in A^H = A_0$. Also, by the first relation in (7) it follows that $B$ and $u_{g_i}\{w^n | n \in \mathbb{Z}\}u_{g_i}^*$, $i = 0, 1, 2, \ldots$, are all multi-independent. Thus, if we denote by $A \subset A$ the von Neumann algebra generated by $u_{g_i}\{w^n | n \in \mathbb{Z}\}u_{g_i}^*$, $i \geq 0$, then $A$ and $B$ are $\tau$-independent and $\Gamma \curvearrowright A$ is isomorphic to the generalized Bernoulli action $\Gamma \curvearrowright L^\infty([0,1]^{\Gamma/H})$, as desired.

\[\square\]

5.2. Corollary. As in 5.1, let $A_n \subset M_n$ be a sequence of MASAs in finite factors, with $\dim M_n \to \infty$, and denote $A = \prod_\omega A_n \subset \prod_\omega M_n = M$. Let $G \curvearrowright X$ be a measure preserving (but not necessarily free) action of a countable amenable group $G$ on a probability space $(X, \mu)$. Let $\rho_i : L^\infty(X) \times G \leftrightarrow M$ be trace preserving embeddings taking $L^\infty(X)$ into $A$, with commuting squares, and $G$ in the normalizer $\mathcal{N}$ of $A$ in $M$, such that $\rho_i(G)$ acts freely on $A$, $i = 1, 2$. Then there exists $u \in \mathcal{N}$ such that $u\rho_1(x)u^* = \rho_2(x), \forall x \in L^\infty(X) \times G$. In particular, any two embeddings of $G$ into $\mathcal{N}$ acting freely on $A$, are conjugate by a unitary in $\mathcal{N}$.

Proof. By Theorem 5.1 applied to $\Gamma = G$ and $H = \{1\}$, each one of the embeddings $\rho_i$ can be extended to embeddings, still denoted by $\rho_i$, of $A = L^\infty(X \times [0,1]^G) \subset L^\infty(X \times [0,1]^G) \times H = M$ into $A \subset M$, satisfying the same properties, where $G \curvearrowright X \times [0,1]^G$ is the product action. This action is free, so the corresponding inclusion is Cartan, with $M$ AFD. Thus, by observation 2.3.2\textdegree, the specific isomorphism $\rho_2 \circ \rho_1^{-1} : \rho_1(M) \simeq \rho_2(M)$ is implemented by a unitary in $\mathcal{N}$.

Finally, let us mention that a slight adaption of the proof of 4.3 allows showing that given any two countable groups $\Gamma_1, \Gamma_2$ normalizing $D^\omega$ in $R^\omega$ (where as before $D \subset R$ is the Cartan subalgebra of the hyperfinite $\Pi_1$ factor), there exists a unitary element $u \in \mathcal{N}_{R^\omega}(D^\omega)$ that conjugates $\Gamma_1$ in free position with $\Gamma_2$. Moreover, if
$H \subset \Gamma_1 \cap \Gamma_2$ is a common amenable group, then $u$ can be taken so that to commute with $H$ and so that the group $\Gamma$ generated by $u \Gamma_1 u^*$ and $\Gamma_2$ satisfies $\Gamma \simeq \Gamma_1 \ast_H \Gamma_2$, with $\Gamma$ acting freely if $\Gamma_1, \Gamma_2$ act freely. This recovers a result from [Pa], [ES]. We'll actually state and prove only the case $\Gamma_i$ act freely of such a statement, for clarity:

**5.3. Theorem.** Let $A_n \subset M_n$ be a sequence of Cartan MASAs in finite factors, with $\dim M_n \to \infty$, and denote $A = \Pi_o A_n \subset \Pi_o M_n = M$, as before. Assume $\Gamma_i \subset \mathcal{N}_M(A)$ are countable groups of unitaries acting freely on $A$, with amenable subgroups $H_i \subset \Gamma_i$, $i = 1, 2$, such that $H_1 \simeq H_2$. Then there exists a unitary element $u \in \mathcal{N}_M(A)$ such that $uH_1u^* = H_2$ and such that the group generated by $u \Gamma_1 u^*$ and $\Gamma_2$ is isomorphic to $\Gamma_1 \ast_H \Gamma_2$ and acts freely on $A$, where $H$ is the identification $H_1 \simeq H_2$ under $\text{Ad}(u)$.

**Proof.** By 5.2 above, there exists a unitary element $u_0 \in \mathcal{N} := \mathcal{N}_M(A)$ such that $u_0H_1u_0^* = H_2$. We may thus assume $H_1 = H_2$, a common subgroup we will denote by $H$.

Denote $A_0 = H' \cap A$. Let also $\mathcal{N}_0 = H' \cap \mathcal{N}$ and note that $\mathcal{N}_0$ normalizes $A_0$. Since by Theorem 2.2 we have $A_0' \cap M = A$, it follows that $A_0$ is a MASA in $M_0 = A_0 \lor \mathcal{N}_0$ and that $\mathcal{N}_0$ is the normalizer of $A_0$ in $M_0$. We denote by $\mathcal{G}_0 = \{ uv \mid u \in \mathcal{N}_0, p \in \mathcal{P}(A_0) \}$ the set of partial isometries in $M_0$ normalizing $A_0$.

With this mind, the proof becomes very similar to the proof of Theorem 4.3. We will only show what the analogue of Lemma 3.2 becomes, and leave all other details for the reader to complete.

Thus, for each finite subset $F \subset \Gamma_1 \cup \Gamma_2 \setminus \{1\}$, $n \geq 1$, a non-zero projection $f \in A_0$ and $v \in \mathcal{G}_0$ satisfying $vv^* = v^*v \leq 1$, we denote by $F_{v,n}$ the set of all elements of the form $x = u_0 \Pi_{i=1}^k v_{\gamma_i} u_{\gamma_i}$, where $u_0 \in F \cup \{1\}$, $u_{\gamma_i} \in F$, $\gamma_i = \pm 1$, $1 \leq k \leq n$. We need to prove that given any $\varepsilon > 0$, there exists $u \in \mathcal{G}_0$ such that $\|E_A(x)\|_1 \leq \varepsilon$, $\forall x \in F_{v,n}$, and $\tau(uu^*) > \tau(f)/4$.

To do this, let $\delta = 2^{-n^2-1}\varepsilon$ and denote $\varepsilon_0 = \delta$, $\varepsilon_k = 2^{k+1}\varepsilon_{k-1}$, $k \geq 1$. Note that $\varepsilon_n < \varepsilon$. Let $\mathcal{W}$ denote the set of partial isometries $v \in \mathcal{G}_0$ with $vv^* = v^*v \leq f$ such that $\|E_A(x)\|_1 \leq \varepsilon_k \tau(vv^*), \forall x \in F_{v,k}$, for all $1 \leq k \leq n$, and endow $\mathcal{W}$ with the order given by $w_1 \leq w_2$ if $w_1 = w_2w_1^*w_1$. Noticing that $\mathcal{W}$ is well ordered with respect to $\leq$, we let $v \in \mathcal{W}$ be a maximal element. Assume that $\tau(vv^*) \leq \tau(f)/4$ and note that $p = f - vv^* \in \mathcal{P}(A_0)$ will then satisfy $\tau(vv^*)/\tau(p) \leq 1/3$.

If $w \in \mathcal{G}_0$ satisfies $ww^* = w^*w \leq p$, then $u = v + w$ belongs to $\mathcal{G}_0$ and satisfies $uu^* = u^*u$. When we develop $u_0 \Pi_{i=1}^k (v + w)^{\gamma_i} u_i$ binomially, we get

$$\|E_A(u_0 \Pi_{i=1}^k u_{\gamma_i} u_{\gamma_i})\|_1 \leq \|E_A(v_0 \Pi_{i=1}^k v_{\gamma_i} u_i)\|_1 + \Sigma' + \Sigma'',$$

where $\Sigma''$ is the sum of the $L^1$-norm of terms that contain at least two occurrences...
of $w^{\pm 1}$, while $\Sigma'$ is the sum the $L^1$-norm of terms containing exactly one occurrence of $w^{\pm 1}$.

To estimate $\Sigma''$ we use $1.2.1^\circ$, exactly the same way $1.2.2^\circ$ is used in the estimates $(2)-(10)$ in the proof of 3.2, to get that $\Sigma'' \leq \varepsilon_k \tau(q) - (2k+2)(\varepsilon_k-1/3)\tau(q)$. Note that in order to do that, we only use the properties of the support $q$ of $w$, namely the fact that given any finite set $Y \subset M \ominus A$ and any $\alpha > 0$, one can take $q \in \mathcal{P}(A_0)$ such that $\|qyq\|_1 < \alpha \tau(q), \forall y \in Y$ (by applying $1.2.1^\circ$ to $Q = A_0$ and using the fact that $A'_0 \cap M = A$).

Now, in order to estimate $\Sigma'$, we denote by $\mathcal{U}_q$ the set of partial isometries in $G_0$ that have left and right support equal to $q$, which we view as a subgroup of unitaries in $qM_0q$. Notice that $\mathcal{U}_q$ generate $qM_0q$ and that $M_0 \nsubseteq M_0' \cap M$ (because this centralizer is separable and amenable, and by applying 2.1 and 4.2). Thus, given any finite set $Y \subset M$ and any $\alpha > 0$, there exists by $1.5$ unitary elements $w \in \mathcal{U}_q$ such that $\|E_A(y_1wy_2)\|_1 < \alpha \tau(q), \forall y_1, y_2 \in Y$.

Then the same estimates as the ones in $(11)-(14)$ in the proof of 3.2, show that $u = v + w \in \mathcal{W}$, contradicting the maximality of $v$. Thus, we do have indeed $\tau(vv^*) > \tau(f)/4$. With this technical fact in hand, the rest of the proof proceeds exactly as the proof of $4.3$ in Section 4.

\[\Box\]

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INDEPENDENCE PROPERTIES
IN SUBALGEBRAS OF ULTRAPRODUCT II_1 FACTORS

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Abstract. Let \( M_n \) be a sequence of finite factors with \( \dim(M_n) \to \infty \) and denote \( M = \Pi_\omega M_n \) their ultraproduct over a free ultrafilter \( \omega \). We prove that if \( Q \subset M \) is either an ultraproduct \( Q = \Pi_\omega Q_n \) of subalgebras \( Q_n \subset M_n \), with \( Q_n \not\prec M_n \), \( Q' \cap M_n \), \( \forall n \), or the centralizer \( Q = B' \cap M \) of a separable amenable *-subalgebra \( B \subset M \), then for any separable subspace \( X \subset M \oplus (Q' \cap M) \), there exists a diffuse abelian von Neumann subalgebra in \( Q \) which is free independent to \( X \), relative to \( Q' \cap M \). Some related independence properties for subalgebras in ultraproduct II_1 factors are also discussed.

0. Introduction

We continue in this paper the investigation of independence properties in subalgebras of ultraproduct II_1 factors, from [P6], [P12]. The main result we prove along these lines is the following:

0.1. Theorem. Let \( M_n \) be a sequence of finite factors with \( \dim M_n \to \infty \) and denote by \( M \) the ultraproduct \( \Pi_\omega M_n \), over a free ultrafilter \( \omega \) on \( \mathbb{N} \). Let \( Q \subset M \) be a von Neumann subalgebra satisfying one of the following:

(a) \( Q = \Pi_\omega Q_n \), for some von Neumann subalgebras \( Q_n \subset M_n \) satisfying the condition \( Q_n \not\prec M_n \), \( Q' \cap M_n \), \( \forall n \) (in the sense of [P10]);

(b) \( Q = B' \cap M \), for some separable amenable von Neumann subalgebra \( B \subset M \).

Then given any separable subspace \( X \subset M \oplus (Q' \cap M) \), there exists a diffuse abelian von Neumann subalgebra \( A \subset Q \) such that \( A \) is free independent to \( X \), relative to \( Q' \cap M \), i.e. \( E_{Q' \cap M}(x_0 \Pi_{i=1}^n a_i x_i) = 0 \), for all \( n \geq 1 \), \( x_0, x_n \in X \cup \{1\} \), \( x_i \in X \), \( 1 \leq i \leq n - 1 \), \( a_i \in A \oplus \mathbb{C}1 \), \( 1 \leq i \leq n \).

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Note that the particular case when $Q_n \subset M_n$ are $\text{II}_1$ factors with atomic relative commutant, for which one clearly has $Q_n \not\prec_{M_n} Q_n' \cap M_n$, recovers (2.1 in [P6]).

The conclusion in 0.1 above can alternatively be interpreted as follows: given any separable von Neumann subalgebra $P$ of $M$ that makes a commuting square with $Q' \cap M$ (in the sense of 1.2 in [P2]; see Sec. 1.2 below for the definition) and we let $B_1 = P \cap (Q' \cap M)$, there exists a separable von Neumann subalgebra $B_0 \subset Q$, such that $P \vee B_0 \simeq P \ast_{B_1} (B_1 \otimes B_0)$ (amalgamated free product of finite von Neumann algebras over a common subalgebra, see [V], [P4]). Since in the case (b) of 0.1 we have $Q' \cap M = B$ (see 2.1 below) and all embeddings into an ultraproduct II$_1$ factor $M$ of an amenable separable von Neumann algebra $B$ are conjugate by unitaries in $M$, Theorem 0.1 shows in particular that if two separable finite von Neumann algebras $N_1, N_2$ containing copies of $B$ are embeddable into $M$, then $N_1 \ast_B N_2$ is embeddable into $M$ as well. Note that the case $B$ atomic of this result already appears in [P6], while the case $B$ arbitrary but with $M = R^\omega$ was shown in [BDJ].

More precisely, 0.1 implies the following strengthening of these results:

0.2. Corollary. Let $M = \Pi_\omega M_n$ be an ultraproduct II$_1$ factor as in 0.1. Let $N_i \subset M$ be separable finite von Neumann subalgebras with amenable von Neumann subalgebras $B_i \subset N_i$, $i = 1, 2$, such that $(B_1, \tau_{|B_1}) \simeq (B_2, \tau_{|B_2})$. Then there exists a unitary element $u \in M$ so that $uB_1u^* = B_2$ and so that, after identifying $B = B_1 \simeq B_2$ this way, we have $uN_1u^* \vee N_2 \simeq N_1 \ast_B N_2$.

To prove Theorem 0.1, we first construct unitaries $u \in Q$ that are approximately $n$-independent with respect to given finite sets $X \perp Q' \cap M$. Taking larger and larger $n$, larger and larger finite sets $X$ and better and better approximations, and combining with a diagonalization procedure, one can then get unitaries that are free independent to a given countable set, due to the ultraproduct framework.

The approximately independent unitary $u$ is constructed by patching together incremental pieces of it, while controlling the trace of alternating words involving $u$ and a given set $X$. This technique was initiated in [P3], being then fully developed in [P6], where it has been used to prove a particular case of 0.1(a). More recently, it has been used in [P12] to establish existence of free independence in ultraproducts of maximal abelian *-subalgebras (abbreviated hereafter MASA) $A_n \subset M_n$ that are singular in the sense of [D1] (i.e., any unitary element in $M_n$ that normalizes $A_n$ must lie in $A_n$), thus settling the Kadison-Singer problem for the corresponding ultrapower inclusion $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$.

If in turn the normalizers of the MASAs $A_n \subset M_n$ are large, then one can still detect certain independence properties inside $A$, by using the same type of techniques. Thus, it was shown in [P12] that 3-independence always occurs in $A$, and we prove here that given any countable group of unitaries $\Gamma$ in $M$, that
normalizes $A$ and acts freely on it, there exists a diffuse subalgebra $B_0$ in $A$ such
that any word $\Pi_{i=1}^n u_i b_i u_i^*$ with $b_i \in B_0 \ominus \mathbb{C}1$ and distinct $u_i \in \Gamma$, has trace 0. This
actually amounts to $B_0$ being the base of a Bernoulli $\Gamma$-action. We in fact prove
the following stronger result:

0.3. Theorem. Let $A_n \subset M_n$ be MASAs in finite factors, as before, and denote
$A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$. Assume $\Gamma \subset M$ is a countable group of unitaries nor-
malizing $A$ and acting freely on it, and let $H \subset \Gamma$ be an amenable subgroup. Given
any separable abelian von Neumann subalgebra $B \subset A$, there exists a $\Gamma$-invariant
subalgebra $A \subset A$ such that $A, B$ are $\tau$-independent and $\Gamma \curvearrowright A$ is isomorphic to
the generalized Bernoulli action $\Gamma \curvearrowright L^\infty([0, 1]^{\Gamma/H})$.

Note that if the above ultraproduct inclusion $A \subset M$ comes from a sequence of
finite dimensional diagonal inclusions $D_n \subset M_{n \times n}(\mathbb{C})$, or is of the form $D^\omega \subset R^\omega$,
where $D \subset R$ is the unique (up to conjugacy by an automorphism, by [CFW])
Cartan subalgebra of the hyperfinite $\Pi_1$ factor, then a countable group $\Gamma$ can be
embedded into the normalizer $\mathcal{N}_M(A)$ of $A$ in $M$, in a way that it acts freely on
$A$, iff it is sofic (in the sense of [W]; see the expository paper [Pe] and [Pa]). Thus,
with the terminology in [EL], where an action of a sofic group $\Gamma \curvearrowright X$ is called
sofic if the inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ admits a commuting square embedding
into $A \subset M$, with $\Gamma$ embedded into $\mathcal{N}_M(A)$, it follows from 0.3 that if $\Gamma \curvearrowright X$
is sofic then any product action $\Gamma \curvearrowright X \times Y$, with $\Gamma \curvearrowright Y = [0, 1]^I$ a generalized
Bernoulli action corresponding to the left action of $\Gamma$ on a set $I = \oplus_i \Gamma/H_i$, for some
countable family of amenable subgroups $H_i \subset \Gamma$, is sofic. This generalizes a result
in [EL].

The paper is organized as follows. In Section 1 we recall some basic facts needed
in the paper, such as the local quantization lemma from [P1], [P5] and the criterion
for (non-)conjugacy of subalgebras from [P10]. We also prove a general fact about
centralizers (or commutants) of countable sets in ultraproduct $\Pi_1$ factors (see The-
orem 1.7). In Section 2 we prove some bicentralizer results concerning amenable
algebras and groups, in ultrapower framework, that we need in the proofs of 0.1
and respectively 0.3. We conjecture that, in fact, the bicentralizer property char-
erizes amenability (see 2.3.1).

In Section 3 we prove the main technical result needed in the proof of Theorem
0.1, by using incremental patching techniques. This result, stated as Lemma 3.2,
actually amounts to an “approximate version” of the free independence result in
0.1. In Section 4 we derive Theorem 0.1 (in fact a strengthening of it, stated as
Theorem 4.3), by using Lemma 3.2 and an appropriate diagonalization procedure.

In Section 5 we prove Theorem 0.3 (stated as Theorem 5.1). Also, we use the
incremental patching technique to show (see 5.3) that if $A_n \subset M_n$ are Cartan
subalgebras in finite factors, with \( \dim M_n \to \infty \), and \( \Gamma_i \) are countable subgroups of the normalizer \( N \) of \( A = \prod \omega A_n \) in \( M = \prod \omega M_n \), acting freely on \( A \), with \( H_i \subset \Gamma_i \) isomorphic amenable subgroups, then there exists \( u \in N \) such that \( uH_1u^* = H_2 \) and such that the group generated by \( u\Gamma_1u^* \) and \( \Gamma_2 \) is the amalgamated free product \( \Gamma_1 \ast_H \Gamma_2 \), where \( H \) is the identification of \( H_1, H_2 \) via \( \text{Ad}(u) \). Taking \( M_n \) finite dimensional, this recovers a result from [ES], [Pa], on the soficity of amalgamated free products of sofic groups over amenable subgroups and on the uniqueness of sofic embeddings of an amenable group.

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1. Preliminaries

1.1. Some generalities. All von Neumann algebras \( M \) considered in this paper are finite (in the sense of [MvN1]) and come equipped with a fixed faithful normal trace state, generically denoted \( \tau \). We denote by \( U(M) \) the group of unitary elements of \( M \) and by \( P(M) \) the set of projections of \( M \). Recall that a von Neumann algebra is a factor if its center is reduced to the scalars. Recall that there exists a unique trace state on a finite factor ([D2]). A finite factor \( M \) is either finite dimensional (in which case \( M \cong M_{n \times n}(\mathbb{C}) \) for some \( n \geq 1 \) with its unique trace state \( \tau \) given by the normalized trace \( \text{tr} = \frac{\text{Tr}}{n} \)) or infinite dimensional. In this latter case, it is called a \( \Pi_1 \) factor, and is characterized by the fact that the range of the trace on the set of projections satisfies \( \tau(P(M)) = [0, 1] \).

More generally, a finite von Neumann algebra splits as a direct sum \( M = M_1 \oplus M_2 \) with \( M_1 \) of type I (i.e. \( M_1 \cong \bigoplus_{n \geq 1} M_{n \times n}(\mathbb{C}) \otimes A_n \), where \( A_n \) are abelian von Neumann algebras, possibly equal to 0) and \( M_2 \) of type \( \Pi_1 \) (which by definition means \( M_2 \) has no type I summand).

We denote by \( \|x\|_2 = \tau(x^*x)^{1/2}, x \in M \), the \( L^2 \) Hilbert-norm given by the trace. We denote by \( L^2M \) the completion of \( M \) in this norm. We often view \( M \) in its standard representation, acting on \( L^2M \) by left multiplication.

We will also use the \( L^1 \) norm \( \| \|_1 \) on \( M \), defined by \( \|x\|_1 := \tau(|x|) = \sup \{ |\tau(xy)| : y \in M, \|y\| \leq 1 \} \). We denote by \( L^1M \) the completion of \( M \) in the norm \( \| \|_1 \). Note that by the Cauchy-Schwarz inequality we have \( \|x\|_1 \leq \|x\|_2 \), while by the inequality \( x^*x \leq \|x\||x| \) we have \( \|x\|_2 \leq \|x\|_1 \). If \( B \subset M \) is a von Neumann subalgebra, then \( E_B : M \to B \) denotes the (unique) \( \tau \)-preserving conditional expectation of \( M \) onto \( B \), which is contractible in both the operatorial norm \( \| \| \) and the above \( L^p \)-norms, \( p = 1, 2 \). If we view \( M \) in its standard representation on \( L^2M \), then the expectation \( E_B \) is implemented by the orthogonal
projection $e_B$ of $L^2M$ onto $L^2B \subset L^2M$ (viewed as the closure in the norm $\| \|_2$ of $B \subset M$), by $e_Bxe_B = E_B(x)e_B$, $x \in M$.

Given a von Neumann subalgebra $B \subset M$ and a set $X \subset M$, we say that $X$ is perpendicular to $B$ and write $X \perp B$ if $\tau(x^*b) = 0$, $\forall x \in X$ and $b \in B$.

A finite von Neumann algebra $(M, \tau)$ is separable if it is separable with respect to the Hilbert norm $\| \|_2$. Note that this condition is equivalent to the fact that $M$ is countably generated as a von Neumann algebra. More generally, if $X \subset M$ is a subspace, then $X$ is separable if it is separable with respect to the norm $\| \|_2$.

The von Neumann algebra $M$ is atomic if $1_M = \Sigma_i e_i$ with $e_i \in M$ a family of mutually orthogonal minimal (or atomic) projections $e_i \in M$, i.e. with the property that $e_iMe_i = \mathbb{C}e_i$. $M$ is diffuse if it has no minimal (non zero) projection. Any abelian von Neumann algebra $M$ which is diffuse and separable is isomorphic to $L^\infty([0, 1])$ (or equivalently, to $L^\infty(\mathbb{T})$). Moreover, if $A$ is endowed with a faithful normal state $\tau$, then the isomorphism $A \simeq L^\infty([0, 1])$ can be taken so that to carry $\tau$ onto the integral $f \cdot d\mu$, where $\mu$ is the Lebesgue measure on $[0, 1]$.

We will often consider maximal abelian *-subalgebras (MASA) $A$ in a finite von Neumann algebra $M$, i.e. abelian *-subalgebras $A \subset M$ with $A' \cap M = A$. In such a case, we denote $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$, the normalizer of $A$ in $M$. Following [FM], if the normalizer generates $M$ as a von Neumann algebra, we call $A$ a Cartan subalgebra in $M$. An isomorphism of Cartan inclusions $(A_0 \subset M_0; \tau) \simeq (A_1 \subset M_1; \tau)$ is a trace preserving isomorphism of $M_0$ onto $M_1$ carrying $A_0$ onto $A_1$.

If $A_0 \subset M_0$ is Cartan and $A_1 \subset M_1$ is an arbitrary MASA, then a Cartan embedding (or simply an embedding) of $A_0 \subset M_0$ into $A_1 \subset M_1$ is a trace preserving embedding of $M_0$ into $M_1$ that carries $A_0$ into $A_1$ such that $M_0 \cap A_1 = A_0$, with the commuting square condition $E_{A_1}E_{M_0} = E_{A_0}$ satisfied (see 1.2 below for more on this condition), and such that $\mathcal{N}_{M_0}(A_0) \subset \mathcal{N}_{M_1}(A_1)$.

For various other general facts about finite von Neumann algebras, we refer the reader to the classic book [D2].

1.2. Commuting squares of subalgebras. Two von Neumann subalgebras $B_1, B_2 \subset M$ are in commuting square position if the expectations $E_{B_1}, E_{B_2}$ commute (see Sec. 1.2 in [P2]). Note that if this is the case then we in fact have $E_{B_1}E_{B_2} = E_{B_2}E_{B_1} = E_{B_1 \cap B_2}$. Also, for this to happen it is sufficient that $E_{B_1}(B_2) \subset B_1 \cap B_2$.

A typical example when the commuting square condition is satisfied is the following: let $Q \subset P \subset M$ be von Neumann algebras; then $P$ and $Q' \cap M$ are in commuting square position (see 1.2.2 in [P2]).

We notice here an observation showing that in the statement of Theorem 0.1, we may equivalently take the space $X$ to be a separable von Neumann algebra making
a commuting square with $Q' \cap M$, a fact that we will not use in the sequel but is
good to keep in mind. See also (3.8 in [P12]) for a similar statement.

**Lemma.** Let $N \subset M$ be a von Neumann subalgebra in the finite von Neumann
algebra $M$. If $X \subset M$ is a separable subspace, then there exists a separable von
Neumann subalgebra $P \subset M$ that contains $X$ and makes a commuting square with $N$.

**Proof.** Let $P_0 \subset M$ be the (separable) von Neumann algebra generated by $X$. We
then denote by $B_1$ the von Neumann algebra generated by $E_N(P_0)$ and by $P_1$ the
von Neumann algebra generated by $B_1$ and $P_0$. Note that $B_1 \subset P_1$ are separable,
with $B_1 \subset N$ and $X \subset P_1$. More generally, we construct recursively an increasing
sequence of inclusions of separable von Neumann algebras $B_n \subset P_n$, $n \geq 1$, by
letting $B_n$ be the von Neumann algebra generated by $E_N(P_{n-1})$ and $P_n$ be the von
Neumann algebra generated by $B_n$ and $P_{n-1}$.

If we now define $B = \bigcup_n B_n$ and $P = \bigcup_n P_n$ , then both algebras are separable
and $B \subset P \cap N$, by construction. Moreover, we have $E_N(P_n) \subset B_{n+1} \subset P_{n+1} \subset P$,
implying that $E_N(P) \subset B \subset P \cap N$. Thus, we actually have $E_N(P) = B = N \cap P$,
i.e., $N,P$ make a commuting square with $B = N \cap P$. □

1.3. **Amenable algebras.** An important example of a (separable) II$_1$ factor is
the hyperfinite II$_1$ factor $R$ of Murray and von Neumann ([MvN2]), defined as the
infinite tensor product $(R, \tau) = \bigotimes_{k}(M_{2 \times 2}(\mathbb{C}),tr)_k$. By [MvN2], $R$
is the unique approximately finite dimensional (AFD) separable II$_1$ factor (a separable finite von
Neumann algebra algebra $(M,\tau)$ is AFD if there exists an increasing sequence of
finite dimensional von Neumann subalgebras $M_n \subset M$ such that $\cup_n M_n$ is dense in
$M$ in the norm $\| \|$).

By Connes’ results in [C1], $R$ is in fact the unique amenable separable II$_1$ factor.
Recall in this respect that a finite von Neumann algebra $(M,\tau)$ is called amenable
if there exists a state $\varphi$ on $B(L^2M)$ that has $M$ (when viewed in its standard
representation on $L^2M$) in its centralizer, $\varphi(xT) = \varphi(Tx), \forall x \in M, \forall T \in B(L^2M)$,
and such that $\varphi|_M = \tau$. Note that the latter condition is redundant in case $M$ is a
factor, because $\varphi|_M$ is a trace and because of the uniqueness of the trace on factors.
Connes Fundamental Theorem in [C1] actually shows that amenability is equivalent
to the AFD property, for any finite von Neumann algebra.

From all this, it follows that $R$ can be represented in many different ways, for
instance as the group measure space II$_1$ factor $L^\infty(X) \rtimes \Gamma$, associated with a free
ergodic measure preserving action of a countable amenable group $\Gamma$ on a probability
space $(X,\mu)$ ([MvN2]). When viewed this way, $R$ has $D = L^\infty(X)$ as a natural
Cartan subalgebra. By [CFW], [OW] the Cartan subalgebra of $R$ is in fact unique,
up to conjugacy by an automorphism of $R$. We may thus represent $D \subset R$ as
the infinite tensor product $⊗_k(D_2)_k \subset ⊗(M_{2×2}(\mathbb{C}))_k$, where $D_2$ is the diagonal subalgebra in $M_{2×2}(\mathbb{C})$.

More generally, by [CFW], if $A_0 \subset R_0$ is a Cartan subalgebra in an amenable separable finite von Neumann algebra $R_0$, then there exists an increasing sequence of finite dimensional Cartan inclusions $(A_{0,n} \subset R_{0,n}) \subset (A_0 \subset R_0)$ (with Cartan embeddings, as defined before) such that $\bigcup_n A_{0,n} = A_0 \subset R_0 = \bigcup_n R_{0,n}$.

1.4. Local quantization relative to subalgebras. We recall here a result from [P1], [P5], showing that if $Q \subset M$ are II$_1$ von Neumann algebras, then one can “simulate” the expectation onto the commutant $Q' \cap M$ by “squeezing” with appropriate projections in $Q$, a phenomenon called “local quantization” in [P5]:

**Theorem.** 1° Let $M$ be a finite von Neumann algebra and $Q \subset M$ a von Neumann subalgebra. Given any finite set $F \subset M \ominus Q \mathbin{\lor} (Q' \cap M)$ and any $\varepsilon > 0$, there exists a projection $q \in Q$ such that $\|qxq\|_1 < \varepsilon \tau(q)$, $\forall x \in F$.

2° Let $Q \subset M$ be an inclusion of II$_1$ von Neumann algebras. Given any finite set $X \subset M$ and any $\varepsilon > 0$, there exists a projection $q \in Q$ such that $\|qxq - E_{Q' \cap M}(x)q\|_1 < \varepsilon \tau(q)$, $\forall x \in X$. Moreover, $q$ can be taken so that to have scalar central trace in $Q$.

**Proof.** Part 1° is already proved in [P1] (see also Theorem 3.6 in [P12]), while part 2° is (Theorem A.1.4 in [P5]).

1.5. A criterion for non-conjugacy of subalgebras. Let $Q, P \subset M$ be von Neumann subalgebras of the finite von Neumann algebra $M$. Following [P10], we say that a corner of $Q$ can be embedded into $P$ inside $M$ and write $Q \prec_M P$ if the following condition holds true: there exist non-zero projections $p \in P$, $q \in Q$, a unital isomorphism $\psi : qQq \rightarrow pPp$ (not necessarily onto) and a partial isometry $v \in M$ such that $vv^* \in (qQq)' \cap qMq$, $v^*v \in \psi(qQq)' \cap pMp$, $xv = xv(x)\psi(x), \forall x \in qQq$, and $x \in qQq$, $xvv^* = 0$, implies $x = 0$.

In this paper we will actually consider cases when the above condition is not satisfied. We recall from (2.1 in [P10]) a useful necessary and sufficient criterion for this to happen:

**Theorem.** Let $M$ be a finite von Neumann algebra and $P, Q \subset M$ von Neumann subalgebras. For each $q \in \mathcal{P}(Q)$, fix $U_q \subset U(qQq)$ a subgroup generating $qQq$ as a von Neumann algebra. Then $Q \not\prec_M P$ if and only if the following condition holds true:

(1.5.1) Given any $q \in \mathcal{P}(Q)$ and any separable subspace $X \subset M$ there exists a sequence of unitary elements $u_n \in U_q$ such that $\lim_n \|E_P(xu_ny)\|_2 = 0$, $\forall x, y \in X$. 
1.6. Ultraproducts of algebras. We fix once for all an (arbitrary) free ultrafilter \( \omega \) on \( \mathbb{N} \). If \( M_n, n \geq 1 \), is a sequence of finite von Neumann algebras then, we denote by \( \Pi_\omega M_n \) their \( \omega \)-ultra product, i.e., the finite von Neumann algebra obtained as the quotient of \( \bigoplus_n M_n \) by its ideal \( \mathcal{I}_\omega = \{(x_n) | \lim_\omega \tau(x_n^* x_n) = 0\} \), endowed with the trace \( \tau(y) = \lim_\omega \tau(y_n) \), where \( (y_n)_n \in \bigoplus_n M_n \) is in the class \( y \in \bigoplus_n M_n / \mathcal{I}_\omega \) ([Wr]).

Recall that if \( M_n \) are factors and \( \dim M_n \to \infty \), then \( \Pi_\omega M_n \) is a \( \Pi_1 \) factor ([Wr]) and it is non-separable ([F]).

If \( Q_n \subset M_n \) are von Neumann subalgebras, \( n \geq 1 \), then the ultraproduct \( \Pi_\omega Q_n \) identifies naturally to a von Neumann subalgebra in \( \Pi_\omega M_n \) and its centralizer (or commutant) in \( \Pi_\omega M_n \) is given by the formula \( (\Pi_\omega Q_n)' \cap \Pi_\omega M_n = \Pi_\omega (Q_n' \cap M_n) \) (see e.g. [P1]).

If \( M \) is a finite von Neumann algebra, then \( M^\omega \) denotes its \( \omega \)-ultrapower, i.e. the ultrapower of infinitely many copies of \( M \). Note that \( M \) naturally embeds into \( M^\omega \), as the von Neumann subalgebra of constant sequences, and that if \( M \) is a \( \Pi_1 \) factor then \( M^\omega \) is a (non-separable by [F]) \( \Pi_1 \) factor.

1.7. Centralizers of countable sets in ultraproducts. Let \( S = \{b_n\}_n \) be a countable subset in the ultrapower \( R^\omega \) of the hyperfinite \( \Pi_1 \) factor \( R \) and let \( b_n = (b_{n,m})_m \) be representations of each of its elements with \( b_{n,m} \in R = \bigotimes_k (M_{2 \times 2}(\mathbb{C}))_k = \bigcup_n M_n^\omega \), where \( M_n \) is the tensor product of the first \( n \) copies of \( M_{2 \times 2}(\mathbb{C}) \). Thus, we may assume that for each \( m \), \( \{b_{n,m}\}_{n\leq m} \subset M_k \), for a large enough \( k \). Then we have \( b_n \in \Pi_\omega M_k \subset R^\omega \), \( \forall n \), viewed as a subalgebra of \( R^\omega \). But then the ultraproduct subalgebra \( \Pi_\omega (M_k^r \cap R) \simeq R^\omega \) commutes with the set \( \{b_n\}_n \). This shows that the centralizer of any separable von Neumann subalgebra \( B \) of \( R^\omega \) is a type \( \Pi_1 \) von Neumann algebra without separable direct summands.

More generally, the same argument shows that if \( M = \Pi_\omega M_n \) is an ultraproduct of arbitrary McDuff \( \Pi_1 \) factors \( M_n \) (i.e., for which we have \( M_n \simeq M_n \bigotimes R \), see [McD]), then the centralizer of any separable subalgebra \( B \subset M \) is of type \( \Pi_1 \) with no separable direct summands.

However, for general ultraproducts \( \Pi_\omega M_n \) and ultrapowers \( M^\omega \), we may have countable (or even finite) subsets \( S \) that have trivial centralizer: for instance, if \( M \) is a separable non-Gamma \( \Pi_1 \) factor ([MvN2]), then \( M \) is countably generated and \( M' \cap M^\omega = \mathbb{C}1 \) (by [McD]). This is the case if \( M = L(\mathbb{F}_n) \), with \( \mathbb{F}_n \) the free group with \( 2 \leq n \leq \infty \) generators (cf. [MvN2]), or if \( M = L(\Gamma) \) with \( \Gamma \) an ICC group with the property (T) of Kazhdan (for example, \( \Gamma = \text{PSL}(n, \mathbb{Z}) \), \( n \geq 3 \)). Similarly, by results in [Be], it follows that if for some fixed \( n \geq 3 \) we take \( (\pi_m, \mathcal{H}_m) \) to be any sequence of finite dimensional irreducible representations of \( \Gamma = \text{PSL}(n, \mathbb{Z}) \) so that \( k_m = \dim \mathcal{H}_m \to \infty \), then the von Neumann subalgebra \( M \) generated by \( \{((\pi_m(g))_m | g \in \Gamma \} \) in the ultraproduct \( \Pi_\omega M_{k \times k} (\mathbb{C}) \) is isomorphic to
the group factor $L(\Gamma)$ and has trivial relative commutant.

The following result shows that in fact the centralizer of any separable von Neumann subalgebra $P$ of an arbitrary ultraproduct $\Pi_1$ factor $M := \Pi_\omega M_n$, coming from a sequence of finite factors $M_n$ with $\dim M_n \to \infty$, splits as the direct sum of an atomic von Neumann algebra and a diffuse von Neumann algebra with only non-separable direct summands.

**Theorem.** If $P$ is a separable von Neumann subalgebra of $M$ then $P' \cap M = B_0 \oplus B_1$, with $B_0$ atomic and $B_1$ diffuse and having no separable direct summand (even more: any MASA of $B_1$ has only non-separable direct summands).

**Proof.** Denote $Q = P' \cap M$ and let $z \in Z(Q)$ be the maximal central projection with the property that $Qz$ is diffuse. We have to prove that $Qz'$ is non-separable for any central projection $z' \in Z(Q)z$. By replacing $P \subset M$ by $Pz \subset zMz$, we may clearly assume $z = 1$.

Assuming by contradiction that $Q$ has separable direct summands, we may further reduce with the maximal central projection $z_0$ in $Q$ with the property that $Qz_0$ is separable to actually assume, by contradiction, that $P \subset M$ is separable with $Q = P' \cap M$ diffuse and separable.

Let $\{b_n\}_n \subset P$ be a countable subset of the unit ball of $P$, dense in the Hilbert norm $\| \|_2$. Let $b_n = (b_{n,m})_m$ be representations of $b_n$ with $b_{n,m} \in M_m$, $\|b_{n,m}\| \leq \|b_n\|$, $\forall n, m$. Let also $u \in Q$ be a Haar unitary generating a maximal abelian $^*$-subalgebra $A_0$ of $Q$, and let $u = (u_m)_m$ be a representation of $u$ with $u_m \in U(M_m)$, $\forall m$.

The fact that $u$ belongs to $Q = \{b_n\}_n' \cap M$ translates into the condition

$$\lim_{m \to \omega} \| [b_{k,m}, u_m] \|_2 = 0, \forall k \geq 1,$$

while the fact that $u$ is a Haar unitary amounts to the condition

$$\lim_{m \to \omega} \tau(u_m^j) = 0, \forall j \neq 0.$$

Let $V_n$ denote the set of all $m \in \mathbb{N}$ with the property that

$$\| [b_{k,m}, u_m] \|_2 < 2^{-n}, |\tau(u_m^j)| < 2^{-n}, \forall 1 \leq k \leq n, 1 \leq |j| \leq 2n.$$

If we identify $\ell^\infty \mathbb{N}$ with the algebra $C(\Omega)$ of continuous functions on its spectrum $\Omega$ (via the GNS representation), and we view $\omega$ as a point in $\Omega$, then by (1) and (2) it follows that $V_n$ correspond to an open-closed neighborhoods of $\omega \in \Omega$. Let now $W_n, n \geq 0$, be defined recursively as follows: $W_0 = \mathbb{N}$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{ n \in \mathbb{N} : V_n \}$. 

Let now $W_n, n \geq 0$, be defined recursively as follows: $W_0 = \mathbb{N}$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{ n \in \mathbb{N} : V_n \}$. 


\[ n \mid n > \min W_n \}. \text{ Note that, with the same identification as before, } W_n \text{ correspond to a strictly decreasing sequence of neighborhoods of } \omega.

Noticing that the sets \( \{ W_{n-1} \setminus W_n \}_{n \geq 1} \) form a partition of \( \mathbb{N} \), we define \( v = (v_m)_m \) by letting \( v_m = u_m^n \) for \( m \in W_{n-1} \setminus W_n \). Since \( v_m \in \mathcal{U}(M_m) \), it follows that \( v \) is a unitary element in \( M \). By the first relation in (3), if \( m \in W_{n-1} \setminus W_n \) then

\[
\| [b_{k,m}, v_m] \|_2 = \| [b_{k,m}, u_m^n] \|_2 \leq \sum_{j=0}^{n-1} \| u_m^j [b_{k,m}, u_m] u_m^{n-j-1} \|_2 \leq n 2^{-n},
\]

for all \( 1 \leq k \leq n \), while by the second relation in (3) we have

\[
|\tau(v_m u_m^j)| < 2^{-n}
\]

for all \( 1 \leq |j| \leq n \).

But then (4) implies \( v \in \{ b_n \}_n \cap M = P' \cap M = Q \), while by (5) we have \( \tau(vu^j) = 0 \), for all \( j \neq 0 \), i.e. \( v \in Q \) is perpendicular to the maximal abelian *-subalgebra \( A_0 = \{ u \}'' \) of \( Q \) generated by \( u \in Q \). Since by construction we have \( uv = vu \), this shows that at the same time we have \( v \in \{ u \}' \cap Q = A_0 \) and \( v \perp A_0 \), a contradiction. This also shows the stronger form of the statement. \( \Box \)

2. Bicentralizer characterizations of amenability

2.1. Theorem. 1° Let \( M_n \) be a sequence of finite factors with \( \dim M_n \to \infty \) and denote \( M = \Pi_\omega M_n \). If \( B \subset M \) is a separable amenable von Neumann subalgebra, then \( (B' \cap M)' \cap M = B \). Moreover, \( B' \cap M \) is of type \( \Pi_1 \) and has only non-separable direct summands.

2° If \( R \) denotes the hyperfinite \( \Pi_1 \) factor then \( (R' \cap R^\omega)' \cap R^\omega = R \).

Proof. Part 2° is just a particular case of part 1°, so we only need to prove 1°. By Connes' Theorem ([C1]), since \( B \) is amenable and separable, it is approximately finite dimensional, so \( B = \bigcup_n B_n^{\omega} \), for some increasing sequence of finite dimensional von Neumann subalgebras \( B_n \subset B \). Note that \( B' \cap M = \cap_n (B_n' \cap M) \) and that for each \( n \) we have \( (B_n' \cap M)' \cap M = B_n \) (in fact, it is trivial to see that given any inclusion of von Neumann algebras \( N \subset M \) with \( \dim N < \infty \) and \( M \) a factor, we have \( (N' \cap M)' \cap M = N \)). We first need to prove the following:

Fact. Let \( P \subset M \) be an inclusion of finite von Neumann algebras. Let \( x \in M \ominus (P' \cap M) \), \( x \neq 0 \), and \( \varepsilon > 0 \). There exists a unitary element \( u \in P \) such that \( \Re \tau(x^*uxu^*) < \varepsilon \| x \|^2 \).

To prove this, let \( K_x \) denote the weak closure of the convex set \( \text{co}\{uxu^* \mid u \in \mathcal{U}(P)\} \) and note right away that \( \| y \| \leq \| x \| \) and \( \| y \|_2 \leq \| x \|_2 \), \( \forall y \in K_x \). Thus, \( K_x \) is a weakly closed bounded subspace in both \( M \) and \( L^2 M \). In particular, there exists
a unique element \( y_0 \in K_x \) of minimal Hilbert-norm: \( \|y_0\|_2 = \min\{\|y\|_2 \mid y \in K_x\} \).

Since \( K_x \) is Ad\((U(P))\)-invariant (because it is the weak closure of the Ad\((U(P))\)-invariant set co\(\{uxu^* \mid u \in U(P)\}\)) and since \( \|uy_0u^*\|_2 = \|y_0\|_2 \), by the uniqueness of \( y_0 \) it follows that \( uy_0u^* = y_0 \), \( \forall u \in U(P) \). Thus, \( uy_0 = y_0u, \forall u \in U(P) \). By taking linear combinations of \( u \), this implies \( y_0 \in P' \cap M \). But by its construction, the entire \( K_x \) lies in \( M \cap (P' \cap M) \). Thus, \( y_0 \) is both in \( P' \cap M \) and perpendicular to it, implying that \( y_0 = 0 \), i.e. \( 0 \in K_x \).

Assuming now that we have \( \Re \tau(x^*uxu^*) \geq \varepsilon \|x\|_2^2 \), for all \( u \in U(P) \), by taking convex combinations over \( u \in U(P) \) and then weak closure, it follows that \( \Re \tau(x^*y) \geq \varepsilon \|x\|_2^2 \), for all \( y \in P \). In particular, \( 0 = \Re \tau(x^*y_0) \geq \varepsilon \|x\|_2^2 \), forcing \( x = 0 \), a contradiction. This ends the proof of the above Fact.

Denote for simplicity \( Q = B' \cap M \) and note that \( B \subset Q' \cap M \). Assume there exists \( x \in Q' \cap M \) with \( x \perp B \). In particular \( x \perp B_n = (B'_n \cap M)' \cap M \). By applying the Fact to the inclusion \( B_n' \cap M \subset M \) and the element \( x \), it follows that there exists a unitary element \( u_n \in B_n' \cap M \) such that \( \Re \tau(x^*u_nxu_n^*) < 2^{-n} \), \( \forall n \).

Let \( \{e_{i,k}^n\}_k \subset B_n \) denote the (finite) pseudogroup of all partial isometries in \( B_n \) that can be obtained as a sum of elements from a given matrix unit of \( B_n \), and which we take so that \( \{e_{i,j}^n\}_i \) is a subset of \( \{e_{i,j}^{n+1}\}_j \), \( \forall n \). Let \( e_{k,m}^n = (e_{k,m}^n)_m \), with \( e_{k,m}^n \in M_m \) chosen so that \( \|e_{k,m}^n\| \leq \|e_{k}^n\| \) and \( \{e_{i,j}^n\}_i \subset \{e_{i,j}^{n+1}\}_j \) for all \( n, m \). Let also \( u_n = (u_{n,m})_m \), with \( u_{n,m} \in U(M_m) \). Then the above properties translate into

\[
\lim_{m \to \omega} \|(u_{n,m}, e_{k,m}^n)\|_2 = 0, \lim_{m \to \omega} \Re \tau(x^*_n u_n, x_n u_n^*) < 2^{-n},
\]

for all \( k \) and all \( n \), where \( x_n = (x_m)_m \) with \( x_m \in M_m \).

Let \( V_n \) denote the set of all \( m \in \mathbb{N} \) with the property that

\[
\|(u_{n,m}, e_{k,m}^n)\|_2 < 2^{-n}, \Re \tau(x^*_n u_n, x_n u_n^*) < \|x\|_2^2/2, \forall k.
\]

By (1), it follows that \( V_n \) corresponds to an open-closed neighborhood of \( \omega \) in the spectrum \( \Omega \) of \( \ell^\infty \mathbb{N} \), under the identification \( \ell^\infty \mathbb{N} = C(\Omega) \). Let now \( W_n, n \geq 0 \), be defined recursively as follows: \( W_0 = \mathbb{N} \) and \( W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} \mid n > \min W_n\} \). Note that, with the same identification as before, \( W_n \) correspond to a strictly decreasing sequence of neighborhoods of \( \omega \). Define \( v = (v_m)_m \) by letting \( v_m = u_{n,m} \) for \( m \in W_{n-1} \setminus W_n \). Since \( v_m \in U(M_m) \), it follows that \( v \) is a unitary element in \( M \), while by the first relation in (2) and the fact that \( \{e_{i,m}^n\}_i \subset \{e_{i,m}^{n+1}\}_j \) it follows that \( v \in \cap B'_n \cap M = B' \cap M = Q \). By the second relation in (2), we also have \( \Re \tau(x^*v^*xv^*) \leq \|x\|_2^2/2 \). But \( x \in Q' \cap M \) by our assumption, thus \( vxv^* = x \), giving \( \tau(x^*v^*xv^*) = \|x\|_2^2 \), a contradiction.
If \( Q = Qz + Q(1-z) \) with \( z \) a non-zero central projection of \( Q \) and \( Qz \) separable, then by the bi-commutant property we have \( z \in B \) and by Theorem 1.7, \( Qz \) is atomic. Thus, \( Bz = (Qz)' \cap zzMz \) would follow non-separable, a contradiction.

Assume now that \( Q = Qz + Q(1-z) \) with \( z \in \mathcal{P}(\mathcal{Z}(Q)) \) such that \( Qz \) is type I. By the bi-commutant relation, it follows again that \( z \in B \) and that \( Bz = (Qz)' \cap zzMz \) is non-separable (because the commutant of any abelian von Neumann subalgebra of \( M \) is non-separable, by 4.3 in [P1], or 2.3 in [P12]). \( \square \)

2.2. Theorem. Let \( A_n \subset M_n \) be a sequence of MASAs in finite factors and denote \( A = \Pi \omega A_n \subset \Pi \omega M_n = M, N = N_M(A) \).

1° If \( H \subset N \) is a countable amenable subgroup, then \( (H' \cap A)' \cap M = A \vee H \).

2° Assume the MASAs \( A_n \subset M_n \) are Cartan. Let \( R_0 \subset M \) be a separable amenable von Neumann subalgebra such that \( D_0 = R_0 \cap A \) is a Cartan subalgebra in \( R_0 \) and such that \( (D_0 \subset R_0) \) is Cartan embedded into \( (A \subset M) \), in the sense of 1.1. Then \( (N_{R_0}(D_0)' \cap N)' \cap N = N_{R_0}(D_0) \). Moreover, if \( D_1 \subset R_1 \) is another Cartan inclusion which is Cartan embedded into \( A \subset M \), then given any isomorphism \( \rho : (D_0 \subset R_0; \tau) \to (D_1 \subset R_1; \tau) \), there exists \( u \in N \) such that \( A\text{d}(u) = \rho \) on \( R_0 \).

3° With the same assumptions and notations as in 2° above, let \( A_0 = R_0' \cap A \) and \( N_0 = N_{R_0}(D_0)' \cap N \). Then \( A_0 \) is maximal abelian in \( R_0' \cap M \), \( N_0 \) coincides with the normalizer of \( A_0 \) in \( R_0' \cap M \) and \( M_0 = A_0 \vee N_0 \) satisfies \( M_0' \cap M = R_0 \).

Proof. 1° Let first \( \{e_j^n\}_j \) be an increasing sequence of finite partitions in \( \mathcal{P}(A) \) such that \( \lim_n \| \sum_j e_j^n u e_j^n - E_A(u) \|_2 = 0, \forall u \in H \) (e.g., by [P1], or 3.6 in [P12]). If we denote by \( A_0 \) the von Neumann subalgebra of \( A \) generated by \( \bigcup_{u \in H} \{e_j^n \mid j, n\} u^* \) and \( R_0 = A_0 \vee H \), then \( H \) normalizes \( A_0 \), \( A_0 \) is a Cartan subalgebra of \( R_0 \) and \( A \vee H = A \vee R_0 \). In particular, \( H' \cap A = R_0' \cap A \). Moreover, since \( H \) is amenable, \( R_0 \) follows amenable so by ([CFW], [OW]) there exists an increasing sequence of finite pseudogroups of partial isometries \( G_n = \{e_j^n\}_j \), normalizing \( A_0 \) (and \( A \) as well), with source and targets either equal or mutually orthogonal, for each \( n \geq 1 \), and such that \( \{e_j^n \mid j, n\} \) generate \( R_0 \).

It is then trivial to see that \( H' \cap A = \cap_n (G_n' \cap A) \) and \( (G_n' \cap A)' \cap A = G_n \vee A \), \( \forall n \). Then the rest of the proof proceeds with a “diagonalization” argument, exactly as at the end of the proof of Theorem 2.1.

2° The proof of this part is similar to the one of 2.1.1° and of 2.2.1° above. Indeed, the statement obviously holds true for \( D_0 \subset R_0 \) finite dimensional. Then for general \( D_0 \subset R_0 \) one takes \( G_n \) as in the proof of 2.2.1° and one denotes by \( D(n) \subset R(n) \) the associated (finite dimensional) Cartan inclusion. Noticing that \( N_{R_0}(D_0)' \cap N = \cap_n N_{R(n)}(D(n))' \cap N \), one then combines the finite dimensional case with a diagonalization argument, as in the proof of 2.1.

3° Note first that \( N_0 \) normalizes \( A_0 \). Indeed, if \( a_0 \in A \) commutes with \( R_0 \) and
$u \in \mathcal{N}_0$, then $ua_0u^* \in A$ and it commutes with $R_0$ (because both $a_0$ and $u$ commute with $R_0$).

To see that $A_0$ is a MASA in $R'_0 \cap M$, note that by part 2° above we have $A'_0 \cap M = A \vee \mathcal{N}_{R_0}(D_0)$. Thus

$$A'_0 \cap (R'_0 \cap M) = R'_0 \cap (A \vee R_0) = R'_0 \cap (D'_0 \cap (A \vee R_0)) = R'_0 \cap A = A_0,$$

where we have used the fact that $E_{D'_0 \cap M}(R_0) = D_0 \subset A$. This also shows that $(A_0 \vee D_0)' \cap M = A$. If now $u \in R'_0 \cap M$ is a unitary that normalizes $A_0$, then $u$ commutes with $D_0$ so it normalizes $A_0 \vee D_0$, and thus also its commutant $A$, i.e., $u \in \mathcal{N} \cap R'_0 = N_0$.

Finally, by part 2° above, we have $M'_0 \cap M = (A \vee R_0) \cap N'_0 = R_0$. □

2.3. Some remarks and open problems. 1° It is well known (and trivial to show) that if $M_n$ is a sequence of finite factors with $\dim M_n \to \infty$ and $(B, \tau)$ is a finite separable AFD von Neumann algebra, then there exists a trace preserving embedding $\theta_0 : B \leftrightarrow M := \Pi_\omega M_n$ and that given any other such trace preserving embedding $\theta_1 : B \leftrightarrow M$, there exists a unitary element $u \in M$ such that $\theta_1(b) = u\theta_0(b)u^*, \forall b \in B$. In particular, any two copies of $(B, \tau)$ in $M$ are unitary conjugate. By Connes’ theorem [C1], this means that the same holds true for any finite, separable, amenable $B$.

Moreover, by a result of K. Jung in [J], the converse is also true: if a finite separable von Neumann algebra $(B, \tau)$ has a unique (up to unitary conjugacy) embedding into either an ultraproduct $\Pi_\omega M_{n \times n}(\mathbb{C})$ or in $R^\omega$, then $B$ is amenable (see [J]). In fact, by a result of N. Brown in [B] (see also Ozawa’s Appendix 8.1 in that paper), if $B \subset R^\omega$ is non-amenable, then there exist uncountably many non-conjugate copies of $B$ in $R^\omega$.

Since given any ultraproduct $\Pi_1$ factors $M = \Pi_\omega M_n$, all embeddings $B \leftrightarrow M$ of a given separable amenable finite von Neumann algebra are unitary conjugate in $M$, it seems interesting to investigate the converse in this general setting: is it true that if $B \subset M$ is a separable non-amenable von Neumann algebra of an arbitrary ultraproduct $\Pi_1$ factor, then there exist “many” non-conjugate copies of $B$ in $M$? (I am grateful to N. Ozawa for pointing out to me that the answer to this problem is not known; see [FHS] for related considerations.)

On the other hand, related to Theorem 2.1 above, we propose the following new characterization of amenability for separable finite von Neumann algebras:

(2.3.1) Conjecture: Let $P$ be a separable von Neumann subalgebra of an ultraproduct $\Pi_1$ factor $M$ (notably, of $M = R^\omega$, or of $M = \Pi_\omega M_{n \times n}(\mathbb{C})$). If the bicentralizer condition $(P' \cap M)' \cap M = P$ is satisfied, then $P$ is amenable. In particular, if $M$ is a separable $\Pi_1$ factor such that $(M' \cap M^\omega)' \cap M^\omega = M$, then $M \simeq R$. 
Note that for a separable von Neumann subalgebra $P$ of an ultraproduct $\Pi_1$ factor $M$, being equal to its bicentralizer is equivalent to being equal to the centralizer of some $^*$-subalgebra of $M$. Thus, conjecture (2.3.1) is equivalent to the following: (2.3.1’) Conjecture: Let $P$ be a separable von Neumann subalgebra of an ultraproduct $\Pi_1$ factor $M$. If $P$ is the centralizer of a von Neumann subalgebra $Q \subset M$, i.e., $P = Q' \cap M$, then $P$ is necessarily amenable.

Indeed, one clearly has that (2.3.1’) implies (2.3.1). Assume in turn that (2.3.1) holds true. Let $Q \subset M$ be so that $P = Q' \cap M$ is separable and denote $\tilde{Q} = P' \cap M$. Then we still have $\tilde{Q} \cap M = P$, so $P$ satisfies the bicentralizer condition and it is separable, thus $P$ is amenable.

Note also that the bicentraliser condition $(M' \cap M^\omega)' \cap M^\omega = M$ for a separable $\Pi_1$ factor $M$, implies that $M$ must be McDuff ([McD]), i.e., it splits off the hyperfinite $\Pi_1$ factor (or else $M' \cap M^\omega$ is abelian, implying that the bicentralizer is non-separable), but that it cannot be of the form $N \otimes R$, with $N$ non-Gamma ([MvN2]). More generally, if $M$ has a $\Pi_1$ von Neumann subalgebra $N \subset M$ satisfying the spectral gap condition $N' \cap M^\omega = (N' \cap M)^\omega$ ([P11]), then $M$ cannot satisfy the bicentralizer condition $(M' \cap M^\omega)' \cap M^\omega = M$. Indeed, this is because taking bicentralizer is an operation preserving inclusions of algebras, and thus the bicentralizer of $M$ in $M^\omega$ contains the bicentralizer of $N$ in $M^\omega$, which is equal to $((N' \cap M)^\omega)' \cap M^\omega = N^\omega$. But the latter is non-separable, so it cannot be contained in $M$, which is separable. Finally, note that if $M$ is McDuff, then given any separable $^*$-subalgebra $B \subset M^\omega$, its centralizer $B' \cap M^\omega$ is of type $\Pi_1$. More precisely, if $R = \otimes_n (M_{2 \times 2}(\mathbb{C}))_n$, then there exists a sufficiently fast growing $k_n \to \infty$ such that if we denote $R_n = \otimes_{m \geq k_n} (M_{2 \times 2}(\mathbb{C}))_m$, then $B' \cap M^\omega$ contains $\Pi_\omega R_n$.

2° Since by ([CFW]), any Cartan inclusion $A_0 \subset M_0$ with $M_0$ separable amenable finite von Neumann algebra is a limit of an increasing sequence of finite dimensional Cartan inclusions (see 1.3), it follows that any isomorphism between two embeddings of $A_0 \subset M_0$ into an ultraproduct inclusion $A \subset M$ is implemented by a unitary element in $N_M(A)$. Indeed, this is clear for finite dimensional $A_0 \subset M_0$, and the general case follows by a diagonalization procedure.

If in turn $A_0 \subset M_0$ is a Cartan subalgebra with $M_0$ non-amenable, and $A_0 \subset M_0$ is embeddable into an ultraproduct $A \subset M$ which is either of the form $\Pi_\omega D_n \subset \Pi_\omega M_{n \times n}(\mathbb{C})$, or of the form $D^\omega \subset R^\omega$, then any two copies of $A_0 \subset M_0$ into $A \subset M$ that are conjugate by a unitary in $N_M(A)$ will in particular have the corresponding copies of $M_0$ unitary conjugate in $M$. The procedure of constructing “many” non-conjugate embeddings of a non-amenable $M_0 \subset M$ starting from an initial embedding of $M_0$ in the proof of (8.1 of [B]), is easily seen to actually give embeddings of $A_0 \subset M_0$ into $A \subset M$, once the initial embedding of $M_0$ is in fact
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a Cartan embedding of \(A_0 \subset M_0\) into \(A \subset M\). Thus, (8.1 in [B]) also implies that there exist uncountably many non-conjugate embeddings of \(A_0 \subset M_0\) into \(A \subset M\). Altogether, this gives an analogue for Cartan inclusions (equivalently, for countable equivalence relations [FM]), of K. Jung’s characterization of amenability in [J], by a “unique embedding” - type property.

Part 2° of Theorem 2.2 above suggests that, for a separable Cartan inclusion \(A_0 \subset M_0\) embedded into an ultraproduct of Cartan inclusions \(A \subset M\), the bicentralizer property of the inclusion of full groups \(N_{M_0}(A_0) \subset N_{M}(A)\) characterizes the amenability of \(A_0 \subset M_0\).

3° G. Elek and G. Szabo proved in [ES] the following “unique embedding” type characterization of the amenability property for a countable group \(H\), analogue to the one for finite separable von Neumann algebras in [J]: if \(H\) is amenable then any two embeddings of \(H\) into the normalizer \(N\) of \(A = \Pi_\omega D_n \subset \Pi_\omega M_{n \times n}(\mathbb{C}) = M\), acting freely on \(A\), are conjugate by a unitary in \(N\) (this easily implies the same thing for \(A = D_\omega \subset R_\omega = M\); note that by Corollary 5.2 below, the same “unique embedding” result actually holds true for ANY ultraproduct inclusion \(A \subset M\); and that if \(H\) is sofic and non-amenable, then there exist at least two embeddings of \(H\) into \(N\) acting freely on \(A\), non-conjugate by unitaries in \(N\). In fact, as we mentioned in 2.3.2° above, by (8.1 in [B]) there even exist uncountably many non-conjugate such embeddings.

Part 1° of Theorem 2.2 suggests the following alternative “bicentralizer” characterization of amenability for countable groups:

(2.3.3) **Conjecture.** Let \(H\) be a countable group embeddable into the normalizer of an ultraproduct MASA \(A \subset M\) (notably \(D_\omega \subset R_\omega = M\), or \(\Pi_\omega D_n \subset \Pi_\omega M_{n \times n}(\mathbb{C}) = M\)), such that \(H\) acts freely on \(A\) and such that it satisfies the bicentralizer condition \((H' \cap A)' \cap M = A \lor H\). Then \(H\) is amenable.

3. APPROXIMATE FREE INDEPENDENCE IN SUBALGEBRAS

3.1. **Notation.** Let \(M\) be a von Neumann algebra. If \(v \in M\) is a partial isometry with \(v^* v = vv^*\), \(X \subset M\) is a subset and \(k\) a nonnegative integer, then denote \(X_0 \overset{\text{def}}{=} X\) and \(X_v^k \overset{\text{def}}{=} \{x_0 \prod_{i=1}^{k} v_i x_i \mid x_i \in X, \ 1 \leq i \leq k - 1, \ x_0, \ x_k \in X \cup \{1\}, v_i \in \{v, v^*\}\}\).

3.2. **Lemma.** Let \(Q \subset M\) be an inclusion of \(\Pi_1\) von Neumann algebras and assume \(Q \not\subset \_ M Q' \cap M\). Let \(f \in Q\) be a non-zero projection. For any finite set \(F \subset M \ominus (Q' \cap M)\), any \(n \geq 1\) and any \(\varepsilon > 0\), there exists a partial isometry \(v\) in \(fQf\) such that \(vv^* = v^* v\), \(\tau(vv^*) > \tau(f)/4\) and \(\|E_{Q' \cap M}(x)\|_1 \leq \varepsilon, \ \forall x \in \bigcup_{k=1}^{n} F_v^k\).
Proof. It is clearly sufficient to prove the statement in case $F = F^*$ and $\|x\| \leq 1$, $\forall x \in F$. Let $\delta > 0$. Denote $\varepsilon_0 = \delta$, $\varepsilon_k = 2^{k+1}\varepsilon_{k-1}$, $k \geq 1$. Denote $\mathcal{W}' = \{ v \in fQf \mid vv^* = v^*v \in \mathcal{P}(Q), \|E_{Q' \cap M}(x)\|_1 \leq \varepsilon_k \tau(v^*v), \forall 1 \leq k \leq n, \forall x \in F^k \}$.

Endow $\mathcal{W}'$ with the order in which $w_1 \leq w_2$ iff $w_1 = w_2 w_1^* w_1$. $(\mathcal{W}', \leq)$ is then clearly inductively ordered. Let $v$ be a maximal element in $\mathcal{W}'$. Assume $\tau(v^*v) \leq \tau(f)/4$ and denote $p = f - v^*v$. Note that this implies $\tau(vv^*)/\tau(p) \leq 1/3$.

If $w$ is a partial isometry in $pQp$ with $q = ww^* = w^*w$ and we let $u = v + w$, then for $x = x_0 \prod_{i=1}^k u_i x_i \in F_u$ we have

\[
(1) \quad x = x_0 \prod_{i=1}^k v_i x_i + \sum_{\ell} \sum_i z_{0,i} \prod_{j=1}^{\ell} w_{i,j} z_{j,i},
\]

where the sum is taken over all $\ell = 1, 2, \ldots, k$ and all $i = (i_1, \ldots, i_{\ell})$, with $1 \leq i_1 < \cdots < i_{\ell} \leq k$, and where $w_{i,j} = w$ (resp. $w_{i,j} = w^*$) whenever $v_{i,j} = v$ (resp. $v_{i,j} = v^*$), $z_{0,i} = x_0 v_1 x_1 \cdots v_{i-1} x_{i-1}$, $z_{j,i} = px_{i,j} v_{i,j} \cdots v_{i,j+1} x_{i,j+1} \cdots v_{i,k} x_{i,k}$.

By applying $E_{Q' \cap M}$ to the above equation, then taking $\| \|_1$ and applying triangle inequality, we then get:

\[
(1') \quad \|E_{Q' \cap M}(x)\|_1 \leq \|E_{Q' \cap M}(x_0 \prod_{i=1}^k v_i x_i)\|_1 + \sum_{\ell} \sum_i \|E_{Q' \cap M}(z_{0,i} \prod_{j=1}^{\ell} w_{i,j} z_{j,i})\|_1
\]

Since $v \in \mathcal{W}'$, the first term on the right side in $(1')$ is majorized by $\varepsilon_k \tau(vv^*)$, so we are left with estimating the terms $z = z_{0,i} \prod_{j=1}^{\ell} w_{i,j} z_{j,i}$ in the double summation on the right hand side, which all have $\ell \geq 1$ number of appearances of $w$ or $w^*$.

The case $\ell \geq 2$. Since for $y_1, y_2, y \in M$ with $\|y_1\| \leq 1, \|y_2\| \leq 1$ we have $\|E_{Q' \cap M}(y_1 y y_2)\|_1 \leq \|y_1 y y_2\|_1 \leq \|y\|_1$, it follows that for any $\ell \geq 2$ we have:

\[
(2) \quad \|E_{Q' \cap M}(z)\|_1 = \|E_{Q' \cap M}(z_{0,i} w_{i,1} z_{1,i} w_{i,2} z_{2,i} \cdots w_{i,\ell} z_{\ell,i})\|_1
\]

\[
\leq \|w_{i,1} z_{1,i} w_{i,2}\|_1 = \|q z_{i,1} q\|_1 = \|q z_{i,1} q\|_{1,pMp} \tau(p),
\]

where $\tau_{pMp} = (\tau(p))^{-1} \tau_M$ and $\| \|_{1,pMp}$ denotes the $L^1$-norm on $pMp$ associated with this trace.

By applying Theorem 1.4 to the inclusion $pQp \subset pMp$ (with its trace $\tau_{pMp}$) and to the finite set $X \subset pMp$ of all elements of the form $z_{1,i} - E_{(Q' \cap M)p}(z_{1,i}) \in pMp \ominus (Q' \cap M)p$, for some $i = (i_1, \ldots, i_{\ell})$, $\ell \geq 2$, we obtain that for any $\alpha > 0$, there exists $q \in \mathcal{P}(pQp)$ such that

\[
(3) \quad \|q z_{i,1} q - E_{(Q' \cap M)p}(z_{1,i}) q\|_{1,pMp} < \alpha \tau_{pMp}(q).
\]
Thus, by combining (2) and (3) we get

\begin{equation}
\|E_{Q' \cap M}(z)\|_1 \leq \|qz_1, iq\|_{1, pMp} \tau(p)
\end{equation}

\begin{equation}
\leq (\|E(Q' \cap M)p(z_1, iq\|_{1, pMp} + \alpha \tau_{pMp}(q)) \tau(p)
\end{equation}

\begin{equation}
= \|E(Q' \cap M)p(z_1, iq\|_{1, pMp} \tau_{pMp}(q) + \alpha \tau(q)
\end{equation}

\begin{equation}
= \|E(Q' \cap M)p(z_1, iq\|_{1, pMp} \tau(q) + \alpha \tau(q).
\end{equation}

We now take into account that by the definition of the norm \( \| \cdot \|_1 \), we have

\begin{equation}
\|E(Q' \cap M)p(z_1, i)\|_{1, pMp} = \sup\{\|y(1 - vv^*)x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1}(1 - vv^*)\|/\tau(p) \mid y \in (Q' \cap M)p, \|y\| \leq 1\}
\end{equation}

But since \( y \in Q' \cap M \) commutes with \( v, 1 - vv^* \in Q \) and \( \tau \) is a trace, we actually have

\begin{equation}
\tau(y(1 - vv^*)x_{i_1} \cdots v_{i_2-1}(1 - vv^*)) = \tau(yx_{i_1} \cdots v_{i_2-1}) - \tau(vv^*x_{i_1} \cdots v_{i_2-1}),
\end{equation}

so the last term in (5) is further majorized by

\begin{equation}
\sup\{\|\tau(yx_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1})\|/\tau(p) \mid y \in (Q' \cap M), \|y\| \leq 1\}
+ \sup\{\|\tau(vv^*x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1})\|/\tau(p) \mid y \in (Q' \cap M), \|y\| \leq 1\}
\end{equation}

\begin{equation}
= (\|E_{Q' \cap M}(x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1})\|_1
+ \|E_{Q' \cap M}(v^*x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1})\|_1)/\tau(p).
\end{equation}

Note at this point that \( x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1} \) lies in \( F^{i_2-i_1-1}_{i_1} \) and \( v^*x_{i_1}v_{i_1+1} \cdots v_{i_2-1}x_{i_2-1} \) lies in \( F^{i_2-i_1+1}_{i_1} \). Also, \( i_2 - i_1 + 1 \leq k \), with the only case when \( i_2 - i_1 + 1 = k \) corresponding to the case \( i_1 = 1, i_2 = k, l = 2 \), i.e., to the (single) term \( z = x_0w_1(px_{1}v_{2} \cdots v_{k-1}x_{k-1}p)w_kx_k \) of the double summation in \( (1') \). Thus, by combining (4) and (6) and using that \( \tau(vv^*)/\tau(p) \leq 1/3 \) and choosing \( \alpha \leq \delta/3 \) (which is less than \((\varepsilon_j - \varepsilon_{j-2})/3, \forall j\)), for this particular \( z \) we get

\begin{equation}
\|E_{Q' \cap M}(z)\|_1 \leq \varepsilon_{k-2}(\tau(vv^*)/\tau(p))\tau(q) + \varepsilon_k(\tau(vv^*)/\tau(p))\tau(q) + \alpha \tau(q)
\end{equation}

\begin{equation}
\leq (\varepsilon_{k-2}/3 + \varepsilon_k/3 + \alpha)\tau(q) \leq (2\varepsilon_k/3)\tau(q),
\end{equation}

while for any \( z \) with \( i_2 - i_1 + 1 \leq k - 1 \), we get

\begin{equation}
\|E_{Q' \cap M}(z)\|_1 \leq \varepsilon_{k-3}(\tau(vv^*)/\tau(p))\tau(q) + \varepsilon_{k-1}(\tau(vv^*)/\tau(p))\tau(q) + \alpha \tau(q)
\end{equation}
\[ \leq (\varepsilon_{k-3}/3 + \varepsilon_{k-1}/3 + \alpha)\tau(q) \leq (2\varepsilon_{k-1}/3)\tau(q). \]

Since \(2^{k+1}\varepsilon_{k-1} = \varepsilon_k\) and since there are \(\sum_{i=2}^{k} \binom{k}{i} = 2^k - k - 1\) elements in the double sum in (1) for which \(\ell \geq 2\), of which exactly one has \(i_2 - i_1 + 1 = k\) and the rest satisfy \(i_2 - i_1 + 1 \leq k - 1\), by summing up (7) and (8), we get

\[ \sum_{\ell \geq 2} \Sigma_i \|z_{0,i}^j \Pi_j - w_i \|_1 \]

\[ \leq (2^k - k - 2)(2\varepsilon_{k-1}/3)\tau(q) + (2\varepsilon_k/3)\tau(q) \]

\[ = \varepsilon_k\tau(q) - (2k + 4)(\varepsilon_{k-1}/3)\tau(q). \]

The case \(\ell = 1\). From the double sum on the right hand side of (1') we will now estimate the terms with \(\ell = 1\). These are terms which are obtained from \(x_0v_1x_1v_2x_2\ldots v_kx_k\) by replacing exactly one \(v_i\) by \(w_i\), so they are of the form \(z = z_0,w_iz_{1,i}\), where \(i = 1,2,\ldots,k\), \(z_0,i = x_0v_1x_1\ldots v_i-1x_{i-1}p\), \(z_{1,i} = px_iv_{i+1}\ldots v_kx_k\) and \(w_i = w^s\) if \(v_i = v^s\), \(s \in \{\pm 1\}\) (with the convention that \(v^{-1} = v^*\), \(w^{-1} = w^*\)). Note that there are \(k\) such terms.

One should notice at this point that in the above estimates we only used the fact that \(w^*w = ww^* = q \in P(Q)\) and that it satisfies (3) for appropriate \(\alpha\). But we did not use so far the actual form of \(w\). We will make the appropriate choice for \(w\) now, by making use of the condition \(Q \not\subseteq Q' \cap M\). Indeed, by Theorem 1.5 (2.1 in [P10]), this latter condition implies that for all \(\beta > 0\) and all finite sets \(Y_1 = Y_1^* \subset M \ominus Q' \cap M\), \(Y_2 = Y_2^* \subset M\), there exists a unitary element \(w \in qQq\) such that

\[ \|E_{Q' \cap M}(y_1wy_2)\|_1 < \beta, \|E_{Q' \cap M}(y_2wy_1)\|_1 < \beta, \forall y_1 \in Y_1, y_2 \in Y_2. \]

Note that since \(Y_1, Y_2\) are selfadjoint sets, by taking adjoints in (10), from these estimates we also get:

\[ \|E_{Q' \cap M}(y_2w^*y_1)\|_1 < \beta, \|E_{Q' \cap M}(y_1w^*y_2)\|_1 < \beta, \forall y_1 \in Y_1, y_2 \in Y_2. \]

Denote by \(Z\) the set of elements of the form \(x_0v_1x_1\ldots v_i-1x_{i-1}\), or \(x_iv_i+1\ldots v_kx_k\), for all possible choices arising from elements in \(\bigcup_{k=1}^{n} F_k\). By applying (10), (10') to \(\beta = \varepsilon_{k-1}\tau(q)/2k\), \(n \geq 1\) and \(Y_2 = Z \cup Z^* \cup \{E_{Q' \cap M}(z) \mid z \in Z \cup Z^*\}\), \(Y_1 = \{y_2 - E_{Q' \cap M}(y_2) \mid y_2 \in Y_2\}\), it follows that there exists \(w \in U(qQq)\) such that
\[\|E_{Q \cap M}(x_0v_1x_1 \ldots v_jx_{j-1}x_{j-1}) - E_{Q \cap M}(x_0v_1x_1 \ldots v_jx_{j-1})w_jx_jv_{j+1} \ldots v_kx_k)\|_1 \leq \varepsilon_{k-1}\tau(q)/2k,\]

\[\|E_{Q \cap M}((x_0v_1x_1 \ldots v_jx_{j-1})w_j(x_j\ldots v_kx_k) - E_{Q \cap M}(px_jv_{j+1} \ldots v_kx_k))\|_1 \leq \varepsilon_{k-1}\tau(q)/2k.\]

Thus, for each element with \(\ell = 1\) in the double summation \(\Sigma_\ell \Sigma_i z_{0,i}^j w_{i,j} \varepsilon_{j,i}\) in (1), i.e., of the form \(x_0v_1x_1 \ldots v_jx_{j-1}w_jx_jv_{j+1} \ldots v_kx_k\), we have the estimate:

\[\|E_{Q \cap M}(x_0v_1x_1 \ldots v_jx_{j-1}x_{j-1})q\|_1 = \tau(q)\|E_{Q \cap M}(x_0v_1x_1 \ldots v_jx_{j-1})\|_1,\]

and

\[\|qE_{Q \cap M}(x_jv_{j+1} \ldots v_kx_k)\|_1 = \tau(q)\|E_{Q \cap M}(x_jv_{j+1} \ldots v_kx_k)\|_1\]

Both elements \(x_0v_1x_1 \ldots v_jx_{j-1}, x_jv_{j+1} \ldots v_kx_k\) belong to some \(F_j^i\) with \(j \leq k - 1\), and at least one of them with \(j \neq 0\). Thus, by the properties of \(v \in \mathcal{W}\) and the assumption \(\tau(vv^*) \leq \tau(f)/4\), we have \(\gamma \leq \varepsilon_{k-1}\tau(vv^*)\tau(q) \leq \varepsilon_{k-1}\tau(q)/4\). Hence, the last term in (12) is majorized by

\[\varepsilon_{k-1}\tau(q)/k + \varepsilon_{k-1}\tau(q)/4 = (k/4+1)\varepsilon_{k-1}\tau(q).\]

**Summing up the cases \(\ell \geq 2\) and \(\ell = 1\).** Since there are \(k\) terms with \(\ell = 1\), obtained by taking \(j = 1, \ldots, k\), by summing up over \(j\) in (12) – (13) and combining with the estimate (9), obtained in the case \(\ell \geq 2\), we deduce from (1') the following final estimate:

\[\|E_{Q \cap M}(x)\|_1 \leq \|E_{Q \cap M}(x_0\Pi_{i=1}^k v_i^x_i)\|_1 + \Sigma_\ell \Sigma_i \|E_{Q \cap M}(z_0, i^j w_i^j z_{j,i})\|_1\]
\[ \leq \varepsilon_k \tau(vv^*) + (\varepsilon_k - (2k + 4)\varepsilon_{k-1}/3)\tau(q) + (k/4 + 1)\varepsilon_{k-1}\tau(q) \]
\[ \leq \varepsilon_k \tau(vv^*) + \varepsilon_k \tau(ww^*) = \varepsilon_k \tau((v + w)(v + w)^*). \]

Since \( u = v + w \) has also the property that \( uu^* = u^*u \), it follows from (13) that \( u \in \mathcal{W} \). But this contradicts the maximality of \( v \in \mathcal{W} \).

We conclude that \( \tau(v^*v) > \tau(f)/4 \). If we now take \( \delta \leq \varepsilon/2n^2 + 1 \), then \( \varepsilon_n = 2^{(n+1)(n+2)/2} \delta < 2n^2 + 1 \delta \leq \varepsilon \) and the statement follows.

\[ \square \]

### 4. Free Independence in Ultraproduct Framework

#### 4.1. Notation

Let \( M_n \) be a sequence of finite factors with \( \dim(M_n) \to \infty \). Let \( \omega \) be a free ultrafilter on \( \mathbb{N} \) and denote \( M = \Pi_\omega M_n \). We consider the following two special classes of subalgebras of \( M \):

1. **(4.1.1) We denote by \( \mathcal{Q}_u \) the class of von Neumann subalgebras \( Q \subset M \) which are of the form \( Q = \Pi_\omega Q_n \), for some subalgebras \( Q_n \subset M_n \), and have the property that \( Q \not\prec_M Q' \cap M \).**

2. **(4.1.2) We denote by \( \mathcal{Q}_b \) the class of von Neumann subalgebras \( Q \subset M \) with the property that \( Q' \cap M \) is separable and \( (Q' \cap M)' \cap M = Q \).**

The next result provides some properties and examples of algebras in these two classes.

#### 4.2. Proposition

1° If \( Q \in \mathcal{Q}_u \), then \( Q \) is of type II\(_1\).

2° If \( Q_n \subset M_n \) are von Neumann subalgebras such that \( Q_n \not\prec_M Q' \cap M_n \), \( \forall n \), then \( Q = \Pi_\omega Q_n \) satisfies \( Q \not\prec_M Q' \cap M \), and thus \( Q \in \mathcal{Q}_u \).

3° Assume \( m_n \) is an increasing sequence of positive integers of the form \( m_n = d_n \cdot k_n \), with \( d_n, k_n \in \mathbb{N} \). Let \( M_n = M_{m_n \times m_n}(\mathbb{C}) \), with \( P_n = M_{d_n \times d_n}(\mathbb{C}) \), \( Q_n = M_{k_n \times k_n}(\mathbb{C}) \), viewed as subalgebras of \( M_n \) that commute and generate \( M_n \). Then \( Q = \Pi_\omega Q_n \), \( P = \Pi_\omega P_n \) satisfy the following properties: \( Q' \cap M = P \), \( P' \cap M = Q \), \( Q \not\prec_M P \) (and thus \( Q \in \mathcal{Q}_u \)) if and only if \( \lim_{n} d_n/k_n = 0 \).

4° If \( B \subset M \) is a separable amenable von Neumann subalgebra, then \( Q := B' \cap M \) satisfies \( Q' \cap M = B \). Thus \( Q \in \mathcal{Q}_b \).

5° If \( Q \in \mathcal{Q}_b \) then \( Q \) is of type II\(_1\), has no separable direct summand, and \( Q \not\prec_M Q' \cap M \) (the latter being separable).

**Proof.**

1° If an inclusion of finite von Neumann algebras \( B \subset M \) is so that \( B \) is type I, then there exists a non-zero projection \( e \in B \) such that \( eBe \) is abelian, implying that \( eBe \subset (eBe)' \cap eMe \), thus \( B \prec_M B' \cap M \). Since in our case we have \( Q \not\prec_M Q' \cap M \), this shows that \( Q \) cannot have type I summands, thus \( Q \) is II\(_1\).
Part 2° is an immediate consequence of Theorem 1.5 and of the fact that \( Q' \cap M = \Pi_\omega (Q'_n \cap M_n) \) with \( E_{Q' \cap M} (x) = (E_{Q'_n \cap M_n} (x_n))_n \), for \( x = (x_n)_n \in M = \Pi_\omega M_n \).

Part 3° is an easy exercise (using Theorem 1.5) while part 4° is a direct consequence of Theorem 2.1.

To prove part 5°, note that if \( Q \in \mathcal{D}_b \) then \( Q \) has no separable direct summand, by the same observation we have used in the proof of part 1°.

\( \square \)

Note that conjecture (2.3.1) predicts that in fact the class \( \mathcal{D}_b \) only consists of centralizers of separable amenable subalgebras of \( M \), i.e., that any subalgebra in \( \mathcal{D}_b \) is of the form 4.2.4° above.

4.3. Theorem. Assume \( Q \subset M \) is either in the class \( \mathcal{D}_u \), or \( \mathcal{D}_b \). If \( X \subset M \ominus (Q' \cap M) \) is a separable subspace, then there exists a diffuse abelian von Neumann subalgebra \( A \subset Q \) such that \( A \) is free independent to \( X \), relative to \( Q' \cap M \), more precisely \( E_{Q' \cap M} (x_0 \prod a_i x_i) = 0 \), for all \( n \geq 1 \) and all \( x_i \in X, 1 \leq i \leq n - 1 \), \( x_0, x_n \in X \cup \{1\} \), \( a_i \in A \ominus \mathbb{C} \), \( 1 \leq i \leq n \).

4.4. Corollary. With the same assumptions and notations as in 4.3 above, we have:

1° Let \( P \subset M \) be a von Neumann subalgebra making a commuting square with \( Q' \cap M \) and denote \( B_1 = P \cap (Q' \cap M) \). Assume that \( L^2 P \) is countably generated both as a left and as a right \( B_1 \) Hilbert module (equivalently, there exists a separable space \( X \subset P \) such that \( X \perp B_1 \), and \( \text{sp} X B_1 \) and \( \text{sp} B_1 X \) are both \( \| \|_2 \)-dense in \( P \ominus B_1 \)). Then there exists a diffuse von Neumann subalgebra \( B_0 \subset Q \) such that \( P \vee B_0 \simeq P \ast_{B_1} (B_1 \bar{\otimes} B_0) \).

2° Let \( N_i \subset M \) be separable von Neumann algebras, with amenable subalgebras \( B_i, i = 1, 2 \), such that \( (B_1, \tau) \simeq (B_2, \tau) \). Then there exists a unitary element \( u \in M \) such that \( u B_1 u^* = B_2 \) and such that, after identifying \( B = B_1 \simeq B_2 \) via \( \text{Ad}(u) \), we have \( N_1 \vee u N_2 u^* \simeq N_1 \ast_B N_2 \).

Note that the case \( B \) atomic of 4.4.2° above has already been shown in [P6]. The case \( M = R^\omega \) of 4.4.2° shows in particular that if \( N_1, N_2 \) are two separable finite von Neumann algebras with a common amenable subalgebra \( B \subset N_i \) and \( N_1, N_2 \) are both embeddable into \( R^\omega \), then so is \( N_1 \ast_B N_2 \). This recovers a result in [BDJ] (see also [FGR] for more recent related considerations).

A particular case when the assumptions in 4.4.1° are satisfied, is when the subalgebra \( P \subset M \) making a commuting square with \( Q' \cap M \) is itself separable. But there are interesting non-separable examples as well, that can even allow obtaining
free product with amalgamation over the entire $Q' \cap M$ (which is non-separable in case $Q \in \mathcal{Q}_u$). For instance, if $\mathcal{U} \subset \mathcal{U}(M)$ is a countable group of unitaries normalizing $Q' \cap M$, then the von Neumann algebra $P$ generated by $\mathcal{U}$ and $Q' \cap M$ satisfies all the conditions in 4.4.1 with $B_1 = Q' \cap M$.

Note in this respect that one can alternatively take in the statement of Theorem 4.3 the separable space $X$ to be of the form $X = P \oplus (P \cap Q' \cap M)$, for some separable von Neumann algebra $P$ making a commuting square with $Q' \cap M$. Indeed, due to Lemma 1.2, the two versions follow equivalent.

**Lemma 4.5.** Let $Q \subset M$ be a von Neumann subalgebra lying in either the class $\mathcal{Q}_u$ or the class $\mathcal{Q}_b$. Let $f \in Q$ be a non-zero projection and $X \subset M \ominus Q' \cap M$ a countable set. Then there exists a partial isometry $v$ in $fQf$ such that $vv^* = v^*v$, $\tau(vv^*) \geq \tau(f)/4$ and $E_{Q' \cap M}(x) = 0, \forall x \in X_k, \forall k \geq 1$.

**Proof.** Let $X = \{x_k\}_{k \geq 1}$ be an enumeration of $X$ and denote $x_0 = 1$. By applying Lemma 3.2 to the inclusion of $\Pi_1$ von Neumann algebras $Q \subset M$, the projection $f \in Q$, the positive constant $\varepsilon = 2^{-n}$ and the finite set $X_n = \{x_k \mid k \leq n\}$, we get a partial isometry $w_n$ in $fQf$ with the property that $w_nw_n^* = w_n^*w_n$, $\tau(w_n^*w_n) \geq \tau(f)/4$ and

$$
(1) \quad \|E_{Q' \cap M}(x)\|_1 < 2^{-n}, \forall x \in \bigcup_{k \leq n} (X_n)_{w_n}.
$$

Let $f = (f_m)_m$ be a representation of $f$ with $f_m$ projections. Let also $x_k = (x_{k,m})_m$ be a representation of $x_k$, with $x_{k,m} \in M_m$, $\|x_{k,m}\| \leq \|x_k\|, \forall k, m$, and $w_k = (w_{k,m})_m \in Q$ a representation of $w_k$ with $w_{k,m}$ partial isometries satisfying $w_{k,m}w_{k,m}^* = w_{k,m}^*w_{k,m} \leq f_m$.

Assume first that $Q = \Pi \omega Q_0 \subset \mathcal{Q}_u$, in which case we may clearly also assume $f_m \in \mathcal{P}(Q_m)$ and $w_{k,m} \in f_m Q_m f_m, \forall k, m$. Noticing that if $y = (y_n)_n \in M$ then $E_{Q' \cap M}(y) = (E_{Q' \cap M_n}(y_n))_n$, it follows from (1) that

$$
(2) \quad \lim_{m \to \omega} \|E_{Q' \cap M_m}(x_{j_0,m}\Pi_{i=1}^k w_{n,i}^*x_{j_i,m})\|_1 < 2^{-n},
$$

for all $1 \leq k \leq n, x_{j_0}, x_{j_k} \in X_n \cup \{1\}, x_{j_i} \in X_n, 1 \leq i \leq k - 1, \gamma_i \in \{\pm 1\}, \forall i$ (as before, for partial isometries $y \in M$ with $yy^* = y^*y$, we use the convention $y^{-1} = y^*$).

Let $V_n$ be the set of all $m \in \mathbb{N}$ with the property that

$$
(3) \quad \|E_{Q' \cap M_m}(x_{j_0,m}\Pi_{i=1}^k w_{n,i}^*x_{j_i,m})\|_1 < 2^{-n},
$$
for all $1 \leq k \leq n$, $1 \leq j_i \leq n$ for $i \geq 1$, $0 \leq j_0 \leq n$, $\gamma_i \in \{\pm 1\}$. By (2) it
follows that $V_n$ corresponds to an open-closed neighborhood of $\omega$ in $\Omega$, under the
identification $\ell^\infty \mathbb{N} = C(\Omega)$. Let now $W_n$, $n \geq 0$, be defined recursively as follows:
$W_0 = \mathbb{N}$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} \mid n > \min W_n\}$. Note that, with the same
identification as before, $W_n$ is a strictly decreasing sequence of neighborhoods of $\omega$.

Define $v = (v_m)_m$ by letting $v_m = w_{n,m}$ for $m \in W_{n-1} \setminus W_n$. It is then easy to
check that $v$ is a partial isometry in $fQf$ satisfying all the required conditions.

Assume now that $Q \in \mathcal{Q}_b$. Let $(y_\ell)_\ell \subset Q' \cap M$ be a countable set dense in the
unit ball of $Q' \cap M$ in the norm $\| \|_2$. Note that if $y_\ell = (y_{\ell,m})_m$ then $x = (x_n)_n \in M$
satisfies $x \in Q$ iff $\lim_{m \to \omega} \| [x_m, y_{\ell,m}]_2 = 0$, $\forall \ell$. Also, $x \perp Q' \cap M$ iff $\lim_{m \to \omega} \tau(x_m y_{\ell,m}) = 0$, $\forall \ell$. Moreover, if $\delta > 0$, then $\| E_{Q' \cap M}(x) \|_1 \leq \delta$ iff
$\lim_{m \to \omega} \| \tau(x_m y_{\ell,m}) \|_1 \leq \delta$, $\forall \ell$.

With this in mind, from (1) it follows that the partial isometries $w_n = (w_{n,m})_m \in Q$ satisfy

\begin{equation}
\lim_{m \to \omega} \| \tau((x_{j_0,m} \Pi_{i=1}^k \psi_{\gamma_i}^{\gamma_i} x_{j_i,m}) y_{\ell,m}) \|_1 < 2^{-n},
\end{equation}

for all $1 \leq k \leq n$, $x_{j_0}, x_{j_k} \in X_n \cup \{1\}$, $x_{j_i} \in X_n$, $1 \leq i \leq k - 1$, $\gamma_i \in \{\pm 1\}$, $\forall i$, and
for all $\ell \geq 1$. Also, the fact that $w_n$ belongs to $fQf$ is equivalent to

\begin{equation}
\lim_{m \to \omega} \| [w_{n,m}, y_{\ell,m}]_2 = 0, \forall \ell; \lim_{m \to \omega} \| f_m w_{n,m} f_m - w_{n,m} \|_1 = 0
\end{equation}

Let $V_n$ be the neighborhood of $\omega$ consisting of all $m \in \mathbb{N}$ with the property that

\begin{equation}
\| [w_{n,m}, y_{\ell,m}]_2 < 2^{-n}; \| f_m w_{n,m} f_m - w_{n,m} \|_1 < 2^{-n};
\end{equation}

for all $\ell = 1, 2, ..., n$ as well as for all $1 \leq k \leq n$, $x_{j_0} \in X_n \cup \{1\}$, $x_{j_i} \in X_n$, $\gamma_i \in \{\pm 1\}$. Let further $W_n \subseteq \mathbb{N}$, $n \geq 0$, be defined recursively as follows: $W_0 = \mathbb{N}$
and $W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} \mid n > \min W_n\}$. It follows that $W_n$ are all
neighborhoods of $\omega$, that $W_n \subseteq \cap_{j \leq n} V_j$, $W_{n+1} \subseteq W_n$, and $W_{n+1} \not= W_n$.

We now define $v = (v_m)_m$, by letting $v_m = w_{n,m}$ if $m \in W_{n-1} \setminus W_n$. By the
way $w_{n,m}$ have been taken, $v$ follows a partial isometry with $vv^* = v^*v$, while by
the second relation in (6) we have $v \in fQf$ and by the first relation in (6) we have
$E_{Q' \cap M}(x) = 0$, $\forall x \in X_k^k$, $\forall k \geq 1$.

\begin{proof}
Proof of 4.3. Recall that by 4.2.1° and 4.2.5°, $Q$ is of type $\Pi_1$. Thus it contains a
copy $R$ of the hyperfinite $\Pi_1$ factor $R$ and any element $y \in R$ of trace 0 satisfies

generate the free amalgamated product

Note that

\[ \forall B \text{ and } sp \]

then we let

\[ B \]

its proof consists of building unitaries

\[ u \]

sequence of partial isometries \( v_1, v_2, \ldots \in Q \) such that

(i) \( v_{j+1}v_j^*v_j = v_j, \quad v_jv_j^* = v_j^*v_j \) and \( \tau(v_jv_j^*) \geq 1 - 1/2^j, \forall j \geq 1. \)

(ii) \( E_Q \cap M(x) = 0, \forall x \in Y_{v_j}, \forall k \geq 1. \)

Assume we have constructed \( v_j \) for \( j = 1, \ldots, m \). If \( v_m \) is a unitary element, then we let \( v_j = v_m \) for all \( j \geq m \). If \( v_m \) is not a unitary element, then let \( f = 1 - v_m^*v_m \in Q \). Note that \( E_Q \cap M(x') = 0, \forall x' \in X' \defeq \cup_k Y_{v_m}. \) Thus, if we apply Lemma 4.5 to \( Q \subset M \), the projection \( f \in Q \) and the countable set \( X' \subset M \cap (Q' \cap M) \), then we get a partial isometry \( u \in fQf \), with \( uu^* = u^*u \) satisfying \( \tau(uu^*) \geq \tau(f)/4 \) and \( E_Q \cap M(x) = 0, \forall x \in \cup_k (X')_{v_m}^k \). But then \( v_{m+1} = v_m + u \) will satisfy both (i) and (ii) for \( j = m. \)

It follows now from (i) that the sequence \( v_j \) converges in the norm \( \| \| \) to a unitary element \( v \in Q \), which due to (ii) will satisfy the condition \( E_Q \cap M(x) = 0, \forall x \in \cup_n Y_v^n \). But then the von Neumann algebra \( A \) generated by the Haar unitary \( u = vu_0v^* \in Q \) clearly satisfies the conditions required in 4.3.

\[ \square \]

Proof of 4.4. 1° Let \( X_0 \subset P \ominus B_1 \) be a separable subspace such that \( spX_0B_1 \) and \( spB_1X_0 \) are \( \| \| \)-dense in \( P \ominus B_1. \) By Theorem 4.3, there exists a diffuse von Neumann subalgebra \( B_0 \subset Q \) such that \( B_0 \) is free independent to \( X_0 \) relative to \( Q' \cap M. \) It is sufficient to prove that \( E_Q \cap M(x_0\Pi_i y_i x_i) = 0, \forall x_0 \in X_0B_1 \cup \{1\}, \)

\[ x_i \subset X_0B_1, y_i \subset B_0 \ominus C1, 1 \leq i \leq n. \]

But any element in \( X_0B_1 \) can be approximated arbitrarily well by a linear combination of elements in \( B_1X_0. \) The “coefficient” in \( B_1 \) of each one of these elements commutes with \( y_{i-1} \), so we can “move it to the left”, being “swollen” by the \( x_i \in X_0B_1. \) Thus, in the end, it follows that it is sufficient to have \( E_Q \cap M(x_00\Pi_i y_i x_i,0) = 0, \forall x_0,0 \in X_0 \cup \{1\}, x_0,0, i \in X_0, y_i \in B_0 \ominus C1, \) which is indeed the case because \( B_0 \) is free independent to \( X_0 \) relative to \( Q' \cap M. \)

2° By the first part of Remark 2.3, after possibly conjugating with a unitary \( u_0 \in M, \) we may assume the subalgebras \( B_1, B_2 \) coincide. Denote \( B \) this common algebra and let \( Q = B' \cap M, \) which by 2.1 satisfies \( Q' \cap M = B \) and by 4.2.4° it belongs to \( \mathcal{O}_b. \) Now apply 4.3 to \( Q \) and to the separable space \( X = N_1 \ominus B + N_2 \ominus B, \) to conclude that there exists a unitary element \( u \in Q \) such that \( un_2u^* \) and \( N_1 \) generate the free amalgamated product \( \simeq N_1 *_B N_2. \) \[ \square \]

5. More on the incremental patching method

The crucial step in proving Theorem 4.3 is Lemma 3.2. The technique used in its proof consists of building unitaries \( u \) that are approximately \( n \)-independent with respect to certain finite sets, by “patching” together infinitesimal pieces of \( u. \) This
technique was first considered in (2.1 of [P3]), to show that given any countable set \( X \) in a finite von Neumann algebra \( M \) and any diffuse abelian von Neumann subalgebra \( A \subset M \), there exists a Haar unitary \( u \in A^{\omega} \) such that any word that alternates letters from \( X \) and \( \{ u^n \mid n \geq 1 \} \), has 0-trace. This result was a key tool in proving that any derivation of a II\(_1\) factor into the ideal of compact operators is inner, in [P3].

The technique was substantially refined in [P6], to prove a particular case of the case \( Q \in \mathcal{D}_u \) of Theorem 4.3, in which \( Q = \Pi_\omega Q_n \in \mathcal{D}_u \) is so that \( Q_n \subset M_n \) are II\(_1\) subfactors with atomic relative commutant \( Q'_n \cap M_n \) (which thus clearly satisfy \( Q_n \not\prec M_n, Q'_n \cap M_n \)). The result in [P6] had several applications over the years. For instance, it played an important role in developing reconstruction methods in Jones theory of subfactors in ([P4], [P7], [P9]) and it led, in combination with ([V]), to the definition of amalgamated free product of inclusions of finite von Neumann algebras in [P4]. It was also used to prove key technical results in ([IPP], [Va]) and to show that the free product of standard invariants of subfactors defined in ([BiJ]) can be realized in the hyperfinite II\(_1\) factor \( R \) (see A.2 in [IPP] and [Va]).

More recently, the same incremental patching method was used in [P12] to prove that if \( A_n \subset M_n \) is a sequence of MASAs in II\(_1\) factors, then the abelian von Neumann algebra \( A = \Pi_\omega A_n \subset \Pi_\omega M_n = M \) contains diffuse subalgebras \( B_0 \) that are \( \tau \)-independent to any given separable subalgebra \( B \subset A \) and 3-independent to any given countable set \( X \subset M \otimes A \), i.e. any alternating word with at most 3 letters from \( X \) and 3 letters from \( B_0 \cap C_1 \) has trace 0 (see 0.2 in [P12]). Moreover, if \( A_n \) are all singular (in the sense of [D1], i.e. any unitary normalizing \( A_n \) is contained in \( A_n \)), then \( B_0 \) can be chosen to be free independent to \( X \), relative to \( A \), a fact that allowed settling the Kadison-Singer problem for ultraproducts of singular MASAs \( A \subset M \) (see 0.1 in [P12]).

A concrete example of a diffuse subalgebra \( B_0 \) in an ultraproduct MASA \( A \) satisfying the 3-independence property is the following:

Let \( \Gamma \acts X \) be an ergodic (but not necessarily free) measure preserving action of a discrete group \( \Gamma \) on a probability space \((X, \mu)\) and \( \Gamma \acts Y = [0, 1]^\Gamma \) be the Bernoulli \( \Gamma \)-action with diffuse base. Let \( A = L^\infty(X) \otimes L^\infty(Y) \) with \( \Gamma \acts A \) the product action. Let \( M = A \rtimes \Gamma \) and \( A = A^{\omega} \subset M^{\omega} = M \). If we take \( B = L^\infty(X) \) and let \( B_0 = 1 \otimes L^\infty([0, 1]) \otimes 1 \subset L^\infty(Y) \) be the base of the Bernoulli action, viewed as a tensor component of the infinite tensor product \( L^\infty(Y) = \otimes_{g \in \Gamma} (L^\infty([0, 1]))_g \), then it is easy to see that \( B_0 \) is \( \tau \)-independent to \( B \) and 3-independent with respect to \( X = \{ u_g \mid g \in \Gamma \} \).

This construction can actually be recovered “asymptotically” inside any group measure space von Neumann algebra. Indeed, using the incremental patching technique, we will now prove that (generalized) Bernoulli \( \Gamma \)-actions can be retrieved
inside any free action of $\Gamma$ on an ultrapower of measure spaces. More generally we have:

**5.1. Theorem.** Let $A_n \subset M_n$ be a sequence of MASAs in finite factors, with $\dim M_n \to \infty$, and denote $A = \prod_\omega A_n \subset \prod_\omega M_n = M$. Assume $\Gamma \subset \mathcal{N}_M(A)$ is a countable group of unitaries acting freely on $A$ and let $H \subset \Gamma$ be an amenable subgroup. Given any separable abelian von Neumann subalgebra $B \subset A$, there exists a separable diffuse abelian subalgebra $A \subset A$ such that: $A, B$ are $\tau$-independent, $\Gamma$ normalizes $A$, and the action of $\Gamma$ on $A$ is isomorphic to the generalized Bernoulli action $\Gamma \curvearrowright L^\infty([0, 1]^{F/H})$.

**Proof.** Let $\{u_g \mid g \in \Gamma\}$ be the unitaries in $\Gamma$. Denote by $g_0 = 1, g_1, g_2, \ldots \in \Gamma$ a set of representants of $\Gamma/H$. It is clearly sufficient to construct a Haar unitary $w$ in $A$ such that $w$ commutes with $u_h, \forall h \in H$, and such that $B$ and $u_g, \{w^n \mid n \in \mathbb{Z}\}u_{g_i}, i = 0, 1, 2, \ldots$, are all multi-independent, in the sense that for any $k$, any non-zero integers $n_j$, distinct non-negative integers $m_j$, and any $b \in B$, we have $\tau(b\Pi_{j=0}^k u_{g_{m_j}} w^{n_j} u_{g_{m_j}}^*) = 0$.

We need some notations. Thus, we let $A_0$ be the subalgebra of all elements in $A$ that are fixed by $H$ and let $\{b_n\}_n$ be a $\|\|_2$-dense subset of the unit ball of $B$. If $v$ is a partial isometry in $A_0$, then we denote by $F_{v,n}$ the set of all elements of the form $b_i \Pi_{j=0}^k u_{g_{m_j}} v^{n_j} u_{g_{m_j}}^*$, where $1 \leq i \leq n$, $1 \leq k \leq n$, $m_j$ are distinct integers between 0 and $n$, and $1 \leq |n_j| \leq n$. We first prove the following:

**Fact.** Given any $n \geq 1$ and any $\delta > 0$, there exists a Haar unitary $v \in A_0$ such that $|\tau(x)| \leq \delta$, $\forall x \in F_{v,n}$.

To prove this, let $\mathcal{W} := \{v \in A_0 \mid |\tau(x)| \leq \delta \tau(v^*v), \forall x \in F_{v,n}, \tau(v^m) = 0, \forall m \neq 0\}$. Endow $\mathcal{W}$ with the order $\leq$ in which $w_1 \leq w_2$ iff $w_1 = w_2 w_1^* w_1$. ($\mathcal{W}, \leq$) is then clearly inductively ordered. Let $v$ be a maximal element in $\mathcal{W}$. Assume $\tau(v^*v) < 1$ and denote $p = 1 - v^*v$. If $w \in A_{0p}$ is a partial isometry satisfying $ww^* = w^*w$, $\tau(w^m) = 0, \forall m \neq 0$, and we denote $u = v + w$, then by noticing that $(v + w)^{n_j} = v^{n_j} + w^{n_j}$, we obtain:

\begin{equation}
(1) \quad b_i \Pi_{j=0}^k u_{g_{m_j}} v^{n_j} u_{g_{j}}^* = b_i \Pi_{j=0}^k u_{g_{m_j}} v^{n_j} u_{g_{m_j}}^* + \Sigma b_i \Pi_{j=0}^k u_{g_{m_j}} z_j^{n_j} u_{g_{m_j}}^*,
\end{equation}

where $z_j \in \{v, w\}$ and the sum is taken over all possible choices for $z_j = v$ or $z_j = w$, with at least one occurrence of $z_j = w$ (thus, there are $2^{k+1} - 1$ many terms in the summation). We thus get the estimate

\begin{equation}
(2) \quad |\tau(b_i \Pi_{j=0}^k u_{g_{m_j}} v^{n_j} u_{g_{m_j}}^*)|
\end{equation}
\[
\delta \tau \left( (w^*)^j \right) + \Sigma \left| \tau (b_i \prod_{j=0}^{k} u_{g_{m_j}} w_{n_j} u_{g_{m_j}}^*) \right| + \Sigma^\prime \left| \tau (b_i \prod_{j=0}^{k} u_{g_{m_j}} z_{j} u_{g_{m_j}}^*) \right|
\]

where the summation \( \Sigma' \) contains the terms with just one occurrence of \( z_j = w \)
and \( \Sigma'' \) is the summation of the terms that have at least 2 occurrences of \( z_j = w \).

Since \( A \) is abelian, the terms \( u_{g_{m_j}} z_{j} u_{g_{m_j}}^* \) in a product can be permuted arbitrarily.

Thus, in each summand of \( \Sigma'' \) we can bring two of the occurrences of \( w \) so that to be adjacent, i.e., of the form \( y_1 u_{g_{m_j}} w_{n_j} u_{g_{m_j}}^* u_{g_{m_j}} y_2 \).

Recall at this point that by Theorem 2.2.1° we have \( A_0^\prime \cap M = A \vee H \).

By applying part 1° of Theorem 1.4 to \( Q = A_0 p \) and the finite set \( F = \{ u_{g_{m_i}}^* | i \neq j \} \), it follows that for any \( \alpha > 0 \), there exists a non-zero \( q \in P(A_0 p) \) such that

\[
(3) \quad \| q u_{g_{m_j}} u_{g_{m_i}}, q \|_1 < \alpha \tau (q), \forall 0 \leq m_i \neq m_j \leq n.
\]

Since there are \( 2^{k+1} - (k + 1) - 1 \) terms in the summation \( \Sigma'' \), this shows that \( \Sigma'' < (2^{k+1} - (k + 1) - 1) \alpha \tau (q) \), for any choice of \( w \) that has support \( q \) satisfying condition (3).

Thus, if we choose \( \alpha \leq 2^{-n-2} \delta \), then by (3) we get \( \Sigma'' \leq \delta \tau (q)/2 \).

So we are left with estimating the terms in the summation \( \Sigma' \), which have just one occurrence of \( w^i, j \neq 0 \), i.e. are of the form \( |\tau (y_1 w^i y_2)| = |\tau (w^i E_A (q y_2 y_1 q))| \),
for some \( y_1, y_2 \in M, 1 \leq |j| \leq n \). There are \( k + 1 \) many such terms for each \( k = 1, ..., n \).

Let’s denote by \( Y_0 \) the set of all \( y_1, y_2 \) which appear this way, and note that this is a finite set in \( q M q \).

Thus \( Y = E_A (q Y_0 \cdot Y_0 q) \) is finite as well.

It is sufficient now to find a Haar unitary \( w \in A_0 q \) such that \( |\tau (w^i y)| \leq \delta \tau (q)/2 (n + 1), \forall y \in Y, 1 \leq |j| \leq n \), because then the sum of the \( k + 1 \) terms in \( \Sigma' \) will be majorized by \( \delta \tau (q)/2 \), altogether showing that for all \( x \in F_{u,n} \), we have \( |\tau (x)| \leq \delta \tau (w^* u) \). Since \( A_0 q \) is diffuse, it contains a separable diffuse von Neumann subalgebra \( A_0 \), which is isomorphic to \( L^\infty (\mathbb{T}) \) with the Lebesgue measure corresponding to \( \tau (q)^{-1} \tau_{A_0} \).
Let then \( w_0 \in A_0 \) be a Haar unitary generating \( A_0 \). Since \( \{ w_0^m \}_{m} \) tends to 0 in the weak operator topology and \( Y \subset A_q \) is a finite set, there exists \( n_0 \geq n \) such that \( |\tau (w_0^m y)| \leq \delta \tau (q)/2 (n + 1), \forall y \in Y \) and \( |m| \geq n_0 \). But then \( w = w_0^{n_0} \) is still a Haar unitary and it satisfies all the required conditions.

This ends the proof of the Fact.

By using this Fact, it follows that for each \( n \) there exists a unitary element \( u_n \in A_0 \) such that

\[
(4) \quad |\tau (x)| < 2^{-n}, \forall x \in F_{v,n}.
\]
For each $g \in \Gamma$, let $u_g = (u_{g,m})_m$ be a representation of $u_g$ with $u_{g,m} \in \mathcal{N}_{M_n}(A_m)$. Let also $b_i = (b_{i,m})_m$ and $v_n = (v_{n,m})_m \in \mathbf{A}_0$, with $b_{i,m}, v_{n,m} \in A_m$, $\forall m$. Then (4) becomes

$$\lim_{m \to \infty} \|\tau(b_{i,m} \Pi_{j=0}^k u_{g_{m_j,m}} v_{n,m} u_{g_{m_j,m}}^*)\| < 2^{-n}$$

for all $1 \leq i, k \leq n$, $0 \leq m_0 < m_1 \ldots < m_k \leq n$. Also, the fact that $v_n$ lies in $\mathbf{A}_0$ translates into

$$\lim_{m \to \infty} \|[u_{h,m}, v_{n,m}]\|_1 = 0, \forall h \in H, n \geq 1$$

Let then $V_n$ be the set of all $m \in \mathbb{N}$ satisfying the following properties:

$$\|\tau(b_{i,m} \Pi_{j=0}^k u_{g_{m_j,m}} v_{n,m} u_{g_{m_j,m}}^*)\| < 2^{-n}$$

$$\|[u_{h,m}, v_{n,m}]\|_1 < 2^{-n}$$

for all $1 \leq i, k \leq n$, $0 \leq m_0 < m_1 \ldots < m_k \leq n$, where $\{h_i\}_i = H$ is an enumeration of $H$. Note that by (5) and (6), $V_n$ corresponds to an open-closed neighborhood of $\omega$ in $\Omega$, under the identification $\ell^\infty(\mathbb{N}) = C(\Omega)$. Define now recursively $W_0 = \mathbb{N}$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} | n > \min W_n\}$. Then $W_n$ is a strictly decreasing sequence of neighborhoods of $\omega$ (under the same identification as before) with $W_n \subset \cap_{j \leq n} V_j$.

We now define $w = (w_m)_m$, by letting $w_m = v_{n,m}$ if $m \in W_{n-1} \setminus W_n$. By the way $v_{n,m}$ have been taken, $w$ follows unitary element in $\mathbf{A}$, while by the second relation in (7) we have $w \in \mathbf{A}^H = \mathbf{A}_0$. Also, by the first relation in (7) it follows that $B$ and $u_{g_i}\{w^n | n \in \mathbb{Z}\}u_{g_i}^*$, $i = 0, 1, 2, \ldots$, are all multi-independent. Thus, if we denote by $A \subset \mathbf{A}$ the von Neumann algebra generated by $u_{g_i}\{w^n | n \in \mathbb{Z}\}u_{g_i}^*$, $i \geq 0$, then $A$ and $B$ are $\tau$-independent and $\Gamma \curvearrowright A$ is isomorphic to the generalized Bernoulli action $\Gamma \curvearrowright L^\infty([0, 1]^{\Gamma/H})$, as desired.

\[ \square \]

5.2. Corollary. As in 5.1, let $A_n \subset M_n$ be a sequence of MASAs in finite factors, with $\dim M_n \to \infty$, and denote $\mathbf{A} = \Pi_\omega A_n \subset \Pi_\omega M_n = \mathbf{M}$. Let $G \curvearrowright X$ be a measure preserving (but not necessarily free) action of a countable amenable group $G$ on a probability space $(X, \mu)$. Let $\rho_i : L^\infty(X) \times G \hookrightarrow \mathbf{M}$ be trace preserving embeddings taking $L^\infty(X)$ into $\mathbf{A}$, with commuting squares, and $G$ in the normalizer $\mathcal{N}$ of $\mathbf{A}$ in $\mathbf{M}$, such that $\rho_i(G)$ acts freely on $\mathbf{A}$, $i = 1, 2$. Then there exists $u \in \mathcal{N}$
such that \( u \rho_1(x) u^* = \rho_2(x), \forall x \in L^\infty(X) \rtimes G \). In particular, any two embeddings of \( G \) into \( \mathcal{N} \) acting freely on \( A \), are conjugate by a unitary in \( \mathcal{N} \).

**Proof.** By Theorem 5.1 applied to \( \Gamma = G \) and \( H = \{1\} \), each one of the embeddings \( \rho_i \) can be extended to embeddings, still denoted by \( \rho_i \), of \( A = L^\infty(X \times [0, 1]^G) \subset L^\infty(X \times [0, 1]^G) \rtimes G = M \) into \( A \subset M \), satisfying the same properties, where \( G \curvearrowright X \times [0, 1]^G \) is the product action. This action is free, so the corresponding inclusion is Cartan, with \( M \) AFD. Thus, by observation 2.3.2°, the specific isomorphism \( \rho_2 \circ \rho_1^{-1} : \rho_1(M) \simeq \rho_2(M) \) is implemented by a unitary in \( \mathcal{N} \).

Finally, let us mention that a slight adaption of the proof of 4.3 allows showing that given any two countable groups \( \Gamma_1, \Gamma_2 \) normalizing \( D^\omega \) in \( R^\omega \) (where as before \( D \subset R \) is the Cartan subalgebra of the hyperfinite II\(_1\) factor), there exists a unitary element \( u \in \mathcal{N}_{R^\omega}(D^\omega) \) that conjugates \( \Gamma_1 \) in free position with \( \Gamma_2 \). Moreover, if \( H \subset \Gamma_1 \cap \Gamma_2 \) is a common amenable group, then \( u \) can be taken so that to commute with \( H \) and so that the group \( \Gamma \) generated by \( u \Gamma_1 u^* \) and \( \Gamma_2 \) satisfies \( \Gamma \simeq \Gamma_1 * H \Gamma_2 \), with \( \Gamma \) acting freely if \( \Gamma_1, \Gamma_2 \) act freely. This recovers a result from [Pa], [ES]. We’ll actually state and prove only the case \( \Gamma_i \) act freely of such a statement, for clarity:

**5.3. Theorem.** Let \( A_n \subset M_n \) be a sequence of Cartan MASAs in finite factors, with \( \dim M_n \to \infty \), and denote \( A = \Pi_n A_n \subset \Pi_n M_n = M \), as before. Assume \( \Gamma_i \subset \mathcal{N}_M(A) \) are countable groups of unitaries acting freely on \( A \), with amenable subgroups \( H_i \subset \Gamma_i, \), such that \( H_1 \simeq H_2 \). Then there exists a unitary element \( u \in \mathcal{N}_M(A) \) such that \( uH_1 u^* = H_2 \) and such that the group generated by \( u \Gamma_1 u^* \) and \( \Gamma_2 \) is isomorphic to \( \Gamma_1 * H \Gamma_2 \) and acts freely on \( A \), where \( H \) is the identification \( H_1 \simeq H_2 \) under \( \text{Ad}(u) \).

**Proof.** By 5.2 above, there exists a unitary element \( u_0 \in \mathcal{N} := \mathcal{N}_M(A) \) such that \( u_0 H_1 u_0^* = H_2 \). We may thus assume \( H_1 = H_2 \), a common subgroup we will denote by \( H \).

Denote \( A_0 = H' \cap A \). By Corollary 5.2 there exists an \( H \)-invariant separable von Neumann subalgebra \( D_0 \subset A \) such that the action of \( H \) on \( D_0 \) is a Bernoulli \( H \)-action. Denote \( R_0 = D_0 \rtimes H \) and notice that \( R_0 \) is a separable, amenable von Neumann algebra, \( D_0 \) is a Cartan subalgebra in \( R_0 \), and \( D_0 \subset R_0 \) is Cartan embedded into \( A \subset M \). Denote \( \mathcal{N}_0 = \mathcal{N}_{R_0}(D_0)' \cap \mathcal{N} \).

By 2.2.1°, we have \( A_0' \cap M = A \vee H = A \vee R_0 \), while by 2.2.2°, 2.2.3°, we see that \( \mathcal{N}_0' \cap \mathcal{N} = \mathcal{N}_{R_0}(D_0) \). \( A_0 \) is maximal abelian in \( R_0' \cap M \) and \( \mathcal{N}_0 \) coincides with the normalizer of \( A_0 \) in \( R_0' \cap M \).

With this mind, the proof is very similar to the proof of Theorem 4.3. We will only show what the analogue of Lemma 3.2 becomes, and leave all other details for the reader to complete.
Denote $G_0 = \{up \mid u \in N_0, p \in P(A_0)\}$, which by 2.2.3° and [Dy] coincides with the set of partial isometries in $M_0 = A_0 \vee N_0$ that normalize $A_0$. For each finite subset $F \subset \Gamma_1 \cup \Gamma_2 \setminus \{1\}$, $n \geq 1$, a non-zero projection $f \in A_0$ and $v \in G_0$ satisfying $vv^* = v^*v \leq f$, we denote by $F_{v,n}$ the set of all elements of the form $x = u_0^{i=1} v^{\gamma_i} u_i$, where $u_0 \in F \cup \{1\}$, $u_i \in F$, $\gamma_i = \pm 1$, $1 \leq k \leq n$. We need to prove that given any $\varepsilon > 0$, there exists $u \in G_0$ such that $uu^* = u^*u$, $\|E_A(x)\|_1 \varepsilon$, $\forall x \in F_{u,n}$, and $\tau(uu^*) > \tau(f)/4$.

To do this, let $\delta = 2^{-n^2-1}\varepsilon$ and denote $\varepsilon_0 = \delta$, $\varepsilon_k = 2^{k+1}\varepsilon_{k-1}$, $k \geq 1$. Note that $\varepsilon_n < \varepsilon$. Let $W$ denote the set of partial isometries $v \in G_0$ with $vv^* = v^*v \leq f$ such that $\|E_A(x)\|_1 \varepsilon_k \tau(vv^*)$, $\forall x \in F_{v,k}$, for all $1 \leq k \leq n$, and endow $W$ with the order given by $w_1 \leq w_2$ if $w_1 = w_2 w_1^* w_1$. Noticing that $W$ is well ordered with respect to $\leq$, we let $v \in W$ be a maximal element. Assume that $\tau(vv^*) \leq \tau(f)/4$ and note that $p = f - vv^* \in P(A_0)$ will then satisfy $\tau(vv^*)/\tau(p) \leq 1/3$.

If $w \in G_0$ satisfies $ww^* = w^*w \leq p$, then $u = v + w$ belongs to $G_0$ and satisfies $uu^* = u^*u$. When we develop $u_0^{i=1} (v + w)^{\gamma_i} u_i$ binomially, we get

$$\|E_A(u_0^{i=1} v^{\gamma_i} u_i)\|_1 \leq \|E_A(u_0^{i=1} v^{\gamma_i} u_i)\|_1 + \Sigma' + \Sigma'',$$

where $\Sigma''$ is the sum of the $L^1$-norm of terms that contain at least two occurrences of $w^\pm 1$, while $\Sigma'$ is the sum the $L^1$-norm of terms containing exactly one occurrence of $w^\pm 1$ (as before, we use the notation $w^{-1}$ for $w^*$).

To estimate $\Sigma''$ we use 1.4.1°, exactly the same way 1.4.2° is used in the estimates (2) − (9) in the proof of 3.2, to get that $\Sigma'' \leq \varepsilon_k \tau(q) - (2k + 4)(\varepsilon_k - 1/3) \tau(q)$, where $q = ww^*$. Note that in order to do that, we only use the properties of the support $q$ of $w$, namely the fact that given any finite set $Y \subset M \oplus (A \vee H)$ and any $\alpha > 0$, one can take $q \in P(A_0)$ such that $\|qyq\|_1 < \alpha \tau(q)$, $\forall y \in Y$ (by applying 1.4.1° to $Q = A_0$ and using the fact that $A'_0 \cap M = A \vee H$).

Now, in order to estimate $\Sigma'$, we denote by $U_q$ the set of partial isometries in $G_0$ that have left and right support equal to $q$, which we view as a subgroup of unitaries in $qM_0 q$. Notice that $U_q$ generate $qM_0 q$ (by 2.2.3°) and that $M_0 \not\subset M_0' \cap M = R_0$ (because this centralizer is separable and amenable, and by applying 2.1 and 4.2). Thus, given any finite set $Y \subset M$ and any $\alpha > 0$, there exists by 1.5 unitary elements $w \in U_q$ such that $\|E_A(y_1 wy_2)\|_1 < \alpha \tau(q)$, $\forall y_1, y_2 \in Y$.

Then the same estimates as the ones in (10) − (14) in the proof of 3.2, show that $u = v + w \in W$, contradicting the maximality of $v$. Thus, we do have indeed $\tau(vv^*) > \tau(f)/4$. With this technical fact in hand, the rest of the proof proceeds exactly as the proof of 4.3 in Section 4.

□
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