Bounds on variable-length compound jumps

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Abstract
In Euclidean space there is a trivial upper bound on the maximum length of a compound “walk” built up of variable-length jumps, and a considerably less trivial lower bound on its minimum length. The existence of this non-trivial lower bound is intimately connected to the triangle inequalities, and the more general “polygon inequalities”. Moving beyond Euclidean space, when a modified version of these bounds is applied in “rapidity space” they provide upper and lower bounds on the relativistic composition of velocities. Similarly, when applied to “transfer matrices” these bounds place constraints either (in a scattering context) on transmission and reflection coefficients, or (in a parametric excitation context) on particle production. Physically these are very different contexts, but mathematically there are intimate relations between these superficially very distinct systems.

Keywords: Variable-length jumps, rapidity space, composition of velocities, transfer matrices, scattering, transmission and reflection, parametric excitation, particle production.
1 Background

One is often confronted with physical or mathematical situations where some complicated process can be built up by compounding (that is, chaining together) a number of simpler but not necessarily equal individual steps. Examples (by no means an exhaustive list) include compounding a series of variable-length jumps in physical space, the relativistic composition of multiple velocities, and the composition of transfer matrices for scattering from multiple distinct (non-overlapping) barriers.

An interesting and pragmatically useful question is whether information concerning the individual steps can be used to place useful bounds on the overall compound process. Herein, we present examples of several such phenomena. From a purely technical perspective, this discussion is largely based on the analysis of compound scattering processes presented in reference [1], but the applications will be completely different:

1. There is a simplification of the upper and lower bounds of that article to variable-length compound jumps in ordinary Euclidean physical space.

2. There is a modification of the upper and lower bounds of that article to the special relativistic composition of velocities.

Mathematically, the intimate relationship between special relativity and quantum scattering is due to the fact that the Lorentz group can be represented by $SO(3,1)$, which is locally isomorphic to $SL(2,\mathbb{C})$, whereas the set of transfer matrices form a representation of $SU(1,1)$, which is locally isomorphic to $SL(2,\mathbb{R})$. See, for example, the recent review article [2] and references therein. (For other relevant background material see for instance [3, 4, 5, 6, 7, 8] on composition of velocities in special relativity and [9, 10, 11, 12, 13, 14, 15, 16, 17, 18] on quantum scattering.)

It is the structural similarity between the Lie groups $SL(2,\mathbb{C})$ and $SL(2,\mathbb{R})$ that underlies the close mathematical similarities between relativistic composition of velocities and the compounding of transfer matrices.

Furthermore the Lie algebras of both of these Lie groups can be mapped homeomorphically to the Euclidean translations, which ultimately underlies the close connection to compound jumps in ordinary Euclidean space. Indeed, working with the Euclidean space formulation in some sense “trivializes” the bounds and makes clear the close connection between the lower bound and the triangle inequalities (or more generally the polygon inequalities).
2 Variable length random walks in physical space

Suppose we have a compound “walk” in physical space where the individual step sizes ("jumps") are fixed but variable, $\ell_1, \ell_2, \ell_3, \ldots, \ell_n$, but the directions $n_i$ are arbitrary. What if anything can we say about upper and lower bounds on the net displacement

$$x_{12\cdots n} = \sum_{i=1}^{n} n_i \ell_i.$$  \hfill (1)

Consider the two step case

$$x_{12} = n_1 \ell_1 + n_2 \ell_2,$$  \hfill (2)

then it is elementary that

$$|\ell_1 - \ell_2| \leq |x_{12}| \leq \ell_1 + \ell_2.$$  \hfill (3)

Furthermore it is also clear that for $n$ steps

$$|x_{12\cdots n}| \leq M_{12\cdots n} \equiv \sum_{i=1}^{n} \ell_i.$$  \hfill (4)

But can one place a lower bound on $|x_{12\cdots n}|$? Yes, by a straightforward modification (and simplification) of the analysis of reference [1], for a three-step walk we assert (and shall soon prove):

$$|x_{123}| \geq \max\{\ell_1 - \ell_2 - \ell_3, \ \ell_2 - \ell_3 - \ell_1, \ \ell_3 - \ell_1 - \ell_2, \ 0\}.$$  \hfill (5)

More generally, for an $n$-step walk we assert (and shall soon prove)

$$|x_{12\cdots n}| \geq \max\left\{\ell_i - \sum_{j \neq i}^{n} \ell_j, \ 0\right\},$$  \hfill (6)

or equivalently

$$|x_{12\cdots n}| \geq \max\left\{2\ell_i - \sum_{j=1}^{n} \ell_j, \ 0\right\}.$$  \hfill (7)

We can also write this as

$$|x_{12\cdots n}| \geq m_{12\cdots n} \equiv \max\{2\ell_i - M_{12\cdots n}, \ 0\}.$$  \hfill (8)

(So, as is reasonably common notation, we use $M$ to denote the maximum, and $m$ to denote the minimum.)
3 Triangle and polygon inequalities

To first see why these lower bounds have any hope of working, it is useful to consider the triangle inequalities.

3.1 3 steps

A key observation is this: The 3-step lower bound is non-trivial if and only if the three step-lengths, $\ell_1, \ell_2,$ and $\ell_3$, violate the triangle inequalities. To see this, recall that for a three-step compound walk in physical space we asserted:

$$|x_{123}| \geq \max \{\ell_1 - \ell_2 - \ell_3, \ell_2 - \ell_3 - \ell_1, \ell_3 - \ell_1 - \ell_2, 0\}.$$ \hspace{1cm} (9)

Why this odd combination? This is related to the triangle inequalities in a quite elementary manner. If $\ell_1, \ell_2,$ and $\ell_3$ are the lengths of the sides of a physical triangle in Euclidean space then they must satisfy the triangle inequalities: The length of any one side of the triangle must be less than or equal to the sum of the lengths of the other two sides. That is:

$$\ell_1 \leq \ell_2 + \ell_3; \quad \ell_2 \leq \ell_3 + \ell_1; \quad \ell_3 \leq \ell_1 + \ell_2.$$ \hspace{1cm} (10)

This implies:

$$\ell_1 - \ell_2 - \ell_3 \leq 0; \quad \ell_2 - \ell_3 - \ell_1 \leq 0; \quad \ell_3 - \ell_1 - \ell_2 \leq 0.$$ \hspace{1cm} (11)

Therefore in this situation:

$$\max\{\ell_1 - \ell_2 - \ell_3, \ell_2 - \ell_3 - \ell_1, \ell_3 - \ell_1 - \ell_2, 0\} = 0.$$ \hspace{1cm} (12)

That is, if the quantities $\ell_1, \ell_2,$ and $\ell_3$ are the lengths of the sides of a physical triangle in Euclidean space, then there is no constraint on $|x_{123}|$ apart from the trivial one: $|x_{123}| \geq 0$. Therefore, the lower bound on $|x_{123}|$ is non-trivial if and only if $\ell_1, \ell_2,$ and $\ell_3$ cannot be interpreted as the lengths of the sides of a physical triangle in Euclidean space. Furthermore, if the triangle inequalities are violated, then the non-trivial lower bound specifies the extent to which the 3 edges of the “would-be triangle” fail to close.
3.2  $n$ steps

Generalizing the above observation: *For $n$ steps the lower bound is non-trivial if and only if the polygon inequalities are violated.* To see this, observe that for an $n$-step random walk the lengths $\ell_i$ can be interpreted as the physical lengths of an $n$-sided polygon if and only if all $n$ polygon inequalities are satisfied:

$$\forall i \quad \ell_i \leq \sum_{j \neq i} \ell_j. \quad (13)$$

These polygon inequalities are the natural generalization of the triangle inequalities. They can be built up iteratively by subdividing any polygon into triangles, and then applying the triangle inequalities step-by-step. That is

$$\forall i \quad \ell_i - \sum_{j \neq i} \ell_j \leq 0. \quad (14)$$

But then

$$\max \left\{ \ell_i - \sum_{j \neq i} \ell_j, 0 \right\} = 0. \quad (15)$$

So if the lengths $\ell_i$ can be interpreted as the physical lengths of an $n$-sided polygon then there is no constraint on $|x_{12\ldots n}|$ apart from the trivial one: $|x_{12\ldots n}| \geq 0$. Therefore, the lower bound on $|x_{12\ldots n}|$ is non-trivial if and only if the $\ell_i$ cannot be interpreted as the lengths of the sides of a physical $n$-sided polygon in Euclidean space. Furthermore, if the polygon inequalities are violated, then the non-trivial lower bound specifies the extent to which the $n$ edges of the “would-be polygon” fail to close.

These observations, though mathematically rather straightforward, and possibly even trivial, make it much clearer why the lower bounds take the form they do, why there is any realistic hope of obtaining any non-trivial lower bound, and also why there is no realistic hope of a lower bound more stringent than the one we have enunciated.
4 Proof of the lower bound

Start by defining the sums \( j \in \{1, 2, 3, \ldots, n\} \)

\[
M_{123\ldots j} = \sum_{i=1}^{j} \ell_i. \tag{16}
\]

Then it is elementary that

\[
|x_{123\ldots j}| \leq M_{123\ldots j} \tag{17}
\]

for all \( j \in \{1, 2, 3, \ldots, n\} \).

4.1 Iterative version of the lower bound

Now take

\[
m_1 = \ell_1, \tag{18}
\]

and, for \( j \in \{1, 2, 3, \ldots, n-1\} \), iteratively define the quantities \( m_{123\ldots(j+1)} \) by

\[
m_{123\ldots(j+1)} = (\ell_{j+1} - M_{123\ldots j}) H(\ell_{j+1} - M_{123\ldots j})
+ (m_{123\ldots j} - \ell_{j+1}) H(m_{123\ldots j} - \ell_{j+1}), \tag{19}
\]

where \( H(\cdot) \) is the Heaviside step function. We can equivalently re-write this iterative definition as

\[
m_{123\ldots(j+1)} = \max \{\ell_{j+1} - M_{123\ldots j}, m_{123\ldots j} - \ell_{j+1}, 0\}. \tag{20}
\]

**Theorem:** By iterating the 2-step bounds one has

\[
\forall n : \quad m_{123\ldots n} \leq |x_{12\ldots n}| \leq M_{123\ldots n}. \tag{21}
\]

**Proof by induction:** When we iterate the definitions for \( M_{123\ldots j} \) and \( m_{123\ldots j} \), then the first two times we obtain

\[
M_1 = \ell_1; \quad m_1 = \ell_1; \tag{22}
\]

\[
M_{12} = \ell_1 + \ell_2; \quad m_{12} = |\ell_1 - \ell_2|. \tag{23}
\]

Thus the claimed theorem is certainly true for \( n = 2 \). Now apply mathematical induction: Assume that at each stage the interval \([m_{123\ldots j}, M_{123\ldots j}]\) characterizes the
highest possible and lowest possible values of $|x_{12\ldots j}|$. Applying the 2-step bound to the pair $|x_{12\ldots j}|$ and $\ell_{j+1}$ leads trivially to $|x_{12\ldots(j+1)}|$ being bounded from above by

$$M_{123\ldots(j+1)} = M_{123\ldots j} + \ell_{j+1},$$

(24)

and less trivially to being bounded from below by

$$m_{123\ldots(j+1)} = \max \{\ell_{j+1} - M_{123\ldots j}, m_{123\ldots j} - \ell_{j+1}, 0\}.$$  

(25)

This completes the inductive step. That is:

$$|x_{12\ldots(j+1)}| \in [m_{123\ldots(j+1)}, M_{123\ldots(j+1)}],$$

(26)

as claimed.

However these bounds are currently defined in a relatively messy iterative manner. Can this be usefully simplified? Can we make the bounds explicit?

### 4.2 Symmetry properties for the lower bound

When we iterate the definitions of $M_{123\ldots j}$ and $m_{123\ldots j}$ a third time we see

$$M_{123} = \ell_1 + \ell_2 + \ell_3; \quad m_{123} = \max\{\ell_3 - (\ell_1 + \ell_2), |\ell_1 - \ell_2| - \ell_3, 0\}. \tag{27}$$

We can further simplify this by rewriting $m_{123}$ as

$$m_{123} = \max\{\ell_1 - \ell_2 - \ell_3, \ell_2 - \ell_3 - \ell_1, \ell_3 - \ell_1 - \ell_2, 0\}. \tag{28}$$

Note that this form of $m_{123}$ is manifestly symmetric under arbitrary permutations of the labels 123. One suspects that there is a good reason for this. In fact there is.

**Theorem:** The quantity $m_{123\ldots j}(\ell_i)$ is a totally symmetric function of the $j$ parameters $\ell_i$, where $i \in \{1, 2, 3, \ldots, j\}$.

**Proof:** By inspection the result is true for $m_1$, $m_{12}$, and $m_{123}$. But this argument now generalizes. In fact, the easiest way of completing the argument is to provide an explicit formula, which we shall do in the next section.

### 4.3 Non-iterative formula for the lower bound

**Theorem:**

$$\forall n : m_{123\ldots n} = \max_{i \in \{1, 2, \ldots, n\}} \{2\ell_i - M_{123\ldots n}, 0\} = \max_{i \in \{1, 2, \ldots, n\}} \left\{\ell_i - \sum_{k=1, k\neq i}^{n} \ell_k, 0\right\}. \tag{29}$$
Proof by induction: We have already seen that the iterative definition of $m_{123\ldots j}$ can be written as

$$m_{123\ldots (j+1)} = \max\{\ell_{j+1} - M_{123\ldots j}, m_{123\ldots j} - \ell_{j+1}, 0\},$$  

(30)

which we can also rewrite as

$$m_{123\ldots (j+1)} = \max\{2\ell_{j+1} - M_{123\ldots (j+1)}, m_{123\ldots j} - \ell_{j+1}, 0\}.$$  

(31)

Now apply induction. The assertion of the theorem is certainly true for $n = 1$ and $n = 2$, and has even been explicitly verified for $n = 3$. Now assume it holds up to some $j$, then

$$m_{123\ldots (j+1)} = \max\{2\ell_{j+1} - M_{123\ldots (j+1)}, m_{123\ldots j} - \ell_{j+1}, 0\} = \max\{2\ell_{i+1} - M_{123\ldots (j+1)}, \max_{i\in\{1,2,\ldots,j\}}\{2\ell_i - M_{123\ldots j}, 0\} - \ell_{j+1}, 0\} = \max_{i\in\{1,2,\ldots,j+1\}}\{2\ell_i - M_{123\ldots (j+1)}, 0\}.$$  

(32)

This proves the inductive step. Consequently

$$\forall n : m_{123\ldots n} = \max_{i\in\{1,2,\ldots,n\}}\{2\ell_i - M_{123\ldots n}, 0\},$$  

(33)

as claimed. \hfill \Box

To simplify the formalism even further, let us now define

$$\ell_{\text{peak}} = \max_{i\in\{1,2,\ldots,n\}}\ell_i.$$  

(34)

(We shall use the subscript “peak” for the maximum of the individual $\ell_i$’s; the words “max” and “min” will be reserved for bounds on the $n$-fold composition of the $\ell_i$.)

Then we can simply write

$$\forall n : m_{123\ldots n} = \max\{2\ell_{\text{peak}} - M_{123\ldots n}, 0\}.$$  

(35)

This is perhaps the simplest way of presenting the lower bound.
5 Relativistic composition of velocities

Let us now apply the Euclidean space result derived above to a more subtle situation; the relativistic composition of velocities. (For general background see references [3, 4, 5, 6, 7, 8].)

5.1 Collinear velocities

When it comes to the relativistic composition of velocities the key thing is to note that for a pair of collinear (parallel or anti-parallel) velocities we have

\[ v_{12} = \frac{v_1 + v_2}{1 + v_1 v_2}, \]  

(36)

which implies

\[ \frac{|v_1| - |v_2|}{1 - |v_1||v_2|} \leq |v_{12}| \leq \frac{|v_1| + |v_2|}{1 + |v_1||v_2|}. \]  

(37)

If we work with the (non-negative) rapidities \( \zeta_i \) defined by

\[ |v_i| = \tanh \zeta_i, \]  

(38)

then

\[ \tanh |\zeta_1 - \zeta_2| \leq |v_{12}| \leq \tanh(\zeta_1 + \zeta_2). \]  

(39)

That is

\[ \tanh |\zeta_1 - \zeta_2| \leq \tanh(\zeta_{12}) \leq \tanh(\zeta_1 + \zeta_2). \]  

(40)

which implies

\[ |\zeta_1 - \zeta_2| \leq \zeta_{12} \leq \zeta_1 + \zeta_2. \]  

(41)

It is this version that is closest in spirit to the Euclidean result, and this version that is more likely to lead to a suitable constraint on the composition of \( n \) relative velocities. We could also write the 2-velocity constraint as

\[ \tanh \left| \tanh^{-1} |v_1| - \tanh^{-1} |v_2| \right| \leq |v_{12}| \leq \tanh \left( \tanh^{-1} |v_1| + \tanh^{-1} |v_2| \right). \]  

(42)
5.2 Non-collinear velocities

If the velocities are not collinear there is a more complicated rule for combining velocities:

\[ \vec{v}_{12} = \vec{v}_1 \oplus \vec{v}_2. \]  

(43)

Fortunately we will not need to be explicit about the details. (For more details see for instance almost any medium-level technical book on special relativity \[3, 4\], or for example references \[5, 6, 7, 8\].) If we further define a rapidity vector

\[ \vec{\zeta} = \{ \tanh^{-1} |v| \} \hat{v}, \]  

(44)

there will be an analogous vectorial composition rule in rapidity space

\[ \vec{\zeta}_{12} = \vec{\zeta}_1 \boxplus \vec{\zeta}_2. \]  

(45)

Fortunately we do not need the full power of the non-collinear composition rule, we only need to know the simple result obtained by looking at the extreme case of collinear (parallel/anti-parallel) motion:

\[ |\vec{\zeta}_1 - \vec{\zeta}_2| \leq |\vec{\zeta}_1 \boxplus \vec{\zeta}_2| \leq |\vec{\zeta}_1| + |\vec{\zeta}_2|. \]  

(46)

That is:

\[ |\vec{\zeta}_1| - |\vec{\zeta}_2| \leq |\vec{\zeta}_{12}| \leq |\vec{\zeta}_1| + |\vec{\zeta}_2|. \]  

(47)

So even for non-collinear motion we still have

\[ |\zeta_1 - \zeta_2| \leq \zeta_{12} \leq \zeta_1 + \zeta_2. \]  

(48)

We can now immediately apply the bound we have already derived for compound walks in physical Euclidean space.
5.3 Bounds on the composition of velocities

5.3.1 Upper bounds

For $n$ velocities the upper bound is straightforward, we just iterate the two-step result to obtain

$$\zeta_{12...n} \leq \sum_{i=1}^{n} \zeta_i,$$  \hspace{1cm} (49)

whence

$$|v_{12...n}| \leq \tanh \left[ \sum_{i=1}^{n} \zeta_i \right].$$  \hspace{1cm} (50)

We can also write this as

$$|v_{12...n}| \leq \tanh \left[ \sum_{i=1}^{n} \tanh^{-1} |v_i| \right].$$  \hspace{1cm} (51)

Here are some explicit special cases obtained by straightforward manipulation of hyperbolic trig identities. Relativistically combining three velocities one has:

$$|v_{123}| \leq \frac{|v_1| + |v_2| + |v_3| + |v_1||v_2||v_3|}{1 + |v_1||v_2| + |v_2||v_3| + |v_3||v_1|},$$  \hspace{1cm} (52)

Similarly, relativistically combining four velocities one has:

$$|v_{1234}| \leq \frac{|v_1| + |v_2| + |v_3| + |v_4| + |v_1||v_2||v_3| + |v_2||v_3||v_4| + |v_3||v_4||v_1| + |v_4||v_1||v_2|}{1 + |v_1||v_2| + |v_2||v_3| + |v_3||v_4| + |v_4||v_1| + |v_1||v_2||v_3||v_4|}.$$  \hspace{1cm} (53)

If one additionally knows that all velocities are collinear, then instead of bounds one has the related equalities

$$v_{123} = \frac{v_1 + v_2 + v_3 + v_1v_2v_3}{1 + v_1v_2 + v_2v_3 + v_3v_1},$$  \hspace{1cm} (54)

and

$$v_{1234} = \frac{v_1 + v_2 + v_3 + v_4 + v_1v_2v_3 + v_2v_3v_4 + v_3v_4v_1 + v_4v_1v_2}{1 + v_1v_2 + v_2v_3 + v_3v_4 + v_4v_1 + v_1v_3 + v_2v_4 + v_1v_2v_3v_4}.$$  \hspace{1cm} (55)

(There does not seem to be any more pleasant reformulation of these results, and in the completely general $n$-velocity case the general the “tanh” formula above seems to be the best one can do.)
5.3.2 Lower bounds

Obtaining an explicit lower bound is again a lot trickier than the upper bound. When relativistically combining three velocities then, (because of the monotonicity of the tanh function), one has

$$|v_{123}| \geq \tanh \left[ \max \{ \zeta_1 - \zeta_2 - \zeta_3, \; \zeta_2 - \zeta_3 - \zeta_1, \; \zeta_3 - \zeta_1 - \zeta_2, \; 0 \} \right].$$  (56)

When relativistically combining \(n\) velocities the best one can do is this:

$$|v_{12...n}| \geq \tanh \left[ \max \left\{ \zeta_i - \sum_{j \neq i} \zeta_j, \; 0 \right\} \right].$$  (57)

We can also write this as

$$|v_{12...n}| \geq \tanh \left[ \max \left\{ 2\zeta_i - \sum_{j=1}^{n} \zeta_j, \; 0 \right\} \right].$$  (58)

Now defining

$$M_{12...n} \equiv \tanh \left[ \sum_{j=1}^{n} \zeta_j \right],$$  (59)

and

$$v_{\text{peak}} = \max_i \{|v_i|\},$$  (60)

and setting

$$m_{12...n} \equiv \tanh \left[ \max \left\{ 2 \tanh^{-1} v_{\text{peak}} - \tanh^{-1} M_{12...n}, \; 0 \right\} \right],$$  (61)

we can also write this as

$$m_{12...n} \leq |v_{12...n}| \leq M_{12...n}.$$  (62)

So there certainly are quite non-trivial constraints one can place on the relativistic combination of velocities, but they are a little less obvious than one might at first suspect.
6 Scattering

Compound scattering processes were extensively discussed in reference [1]. (For additional background see [2, 12, 13, 14, 15, 16, 17, 18]; for various explicit bounds on transmission and reflection probabilities for scattering processes see references [19, 20, 21, 22, 23, 24, 25, 26, 27]; for a survey of exact results see reference [28].) Rather than unnecessarily repeating the results of reference [1], we shall here content ourselves with a few explicit comments regarding 2-barrier, 3-barrier, and 4-barrier systems. For two non-overlapping barriers the transmission and reflection probabilities are bounded by

\[
\frac{T_1 T_2}{\left(1 + \sqrt{1 - T_1} \sqrt{1 - T_2}\right)^2} \leq T_{12} \leq \frac{T_1 T_2}{\left(1 - \sqrt{1 - T_1} \sqrt{1 - T_2}\right)^2};
\]

(63)

and

\[
\left\{\frac{\sqrt{R_1} - \sqrt{R_2}}{1 - \sqrt{R_1} \sqrt{R_2}}\right\}^2 \leq R_{12} \leq \left\{\frac{\sqrt{R_1} + \sqrt{R_2}}{1 + \sqrt{R_1} \sqrt{R_2}}\right\}^2.
\]

(64)

For three non-overlapping barriers, the results of reference [1], combined with a little work using hyperbolic trigonometric identities, lead to

\[
T_{123} \geq \frac{T_1 T_2 T_3}{\left\{1 + \sqrt{(1 - T_2)(1 - T_3)} + \sqrt{(1 - T_3)(1 - T_1)} + \sqrt{(1 - T_1)(1 - T_2)}\right\}^2};
\]

(65)

and

\[
R_{123} \leq \left\{\frac{\sqrt{R_1 R_2 R_3} + \sqrt{R_1} + \sqrt{R_2} + \sqrt{R_3}}{1 + \sqrt{R_2 R_3} + \sqrt{R_3 R_1} + \sqrt{R_1 R_2}}\right\}^2.
\]

(66)

For four non-overlapping barriers, a completely analogous calculation straightforwardly yields

\[
T_{1234} \geq \frac{T_1 T_2 T_3 T_4}{\left\{1 + \sum_{i<j} \sqrt{(1 - T_i)(1 - T_j)} + \sqrt{(1 - T_1)(1 - T_2)(1 - T_3)(1 - T_4)}\right\}^2};
\]

(67)

and

\[
R_{1234} \leq \left\{\frac{\sqrt{R_1} + \sqrt{R_2 R_3 R_4} + (\text{cyclic permutations})}{1 + \sum_{i<j} \sqrt{R_i R_j} + \sqrt{R_1 R_2 R_3 R_4}}\right\}^2.
\]

(68)
That is, explicitly,

\[ R_{1234} \leq \left\{ \frac{\sqrt{R_1} + \sqrt{R_2} + \sqrt{R_3} + \sqrt{R_4} + \sqrt{R_2 R_3 R_4} + \sqrt{R_3 R_4 R_1} + \sqrt{R_4 R_1 R_2} + \sqrt{R_1 R_2 R_3}}{1 + \sum_{i<j} \sqrt{R_i R_j} + \sqrt{R_1 R_2 R_3 R_4}} \right\}^2. \]  

(69)

Upper bounds on \( T \), and lower bounds on \( R \), are less algebraically tractable, (at least in explicit closed form), and we refer the reader to reference \[1\] for more details.

\section{Parametric excitations}

By working in the temporal rather than spatial domain, particle scattering processes can be re-phrased in terms of particle production via parametric excitation. (See reference \[1\] for details). In this context, the net particle production due to two non-overlapping excitation events is bounded by

\[ \left\{ \sqrt{N_1(N_2 + 1)} - \sqrt{N_2(N_1 + 1)} \right\}^2 \leq N_{12} \leq \left\{ \sqrt{N_1(N_2 + 1)} + \sqrt{N_2(N_1 + 1)} \right\}^2. \]  

(70)

For three non-overlapping excitation events one obtains

\[ N_{123} \leq \left\{ \sqrt{N_1(1 + N_2)(1 + N_3)} + \sqrt{N_2(1 + N_3)(1 + N_1)} \right\}^2. \]  

(71)

For four non-overlapping excitation events a straightforward (but rather tedious) calculation yields

\[ N_{1234} \leq \left\{ \sqrt{N_1(1 + N_2)(1 + N_3)(1 + N_4)} + \sqrt{N_1 N_2 N_3(1 + N_4)} \right\}^2. \]  

(72)

Further “explicit” algebraic formulae would be rather unwieldy, and for all practical purposes one is better off using the somewhat less “explicit” formulae in presented terms of hyperbolic functions in reference \[1\]. Similarly lower bounds on \( N \) are less algebraically tractable, (at least in explicit closed form), and we again refer the reader to reference \[1\] for more details.
8 Discussion

That particle scattering in the spatial domain is mathematically intimately related to particle production in the temporal domain is a very standard result, ultimately going back to the relationship between scattering and transmission amplitudes and the Bogoliubov coefficients. (See for instance references [1, 2, 13, 28] for more details on this specific point.) The intimate mathematical relationship between particle scattering and relativistic composition of velocities is less well-known, but is quite standard. The $SO(3,1)$ Lorentz group is locally isomorphic to $SL(2,\mathbb{C})$, while the group of transfer matrices $SU(1,1)$ is locally isomorphic to $SL(2,\mathbb{R})$. Ultimately it is the fact that their Lie algebras are both isomorphic to Euclidean space that ties the three problems (physical Euclidean space, relativistic composition of velocities, and composition of scattering processes) together. The overall result of the current article is to rigorously establish several clearly motivated and robust mathematical bounds on these three closely inter-related physical problems.

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References

[1] Petarpa Boonserm and Matt Visser,
“Compound transfer matrices: Constructive and destructive interference”,
J. Math. Phys. 53 (2012) 012104 [arXiv:1101.4014 [math-ph]].

[2] L. L. Sanchez-Soto, J. J. Monzon, A. G. Barriuso, J. F. Cariñena,
“The transfer matrix: A geometrical perspective”,
Physics Reports 513 (2012) 191–227.

[3] C. Møller. The Theory of Relativity. (Oxford University Press, London, 1952).
[4] J. D. Jackson. *Classical Electrodynamics, 3rd Ed.* (Wiley, New York, 1998).

[5] Kane O’Donnell and Matt Visser, “Elementary analysis of the special relativistic combination of velocities, Wigner rotation, and Thomas precession”, Eur. J. Phys. 32 (2011) 1033 [arXiv:1102.2001 [gr-qc]].

[6] H. P. Stapp, “Relativistic theory of polarization phenomena”, *Physical Review*, 103 (1956) 425–434.

[7] G. P. Fisher, “Thomas precession”, *American Journal of Physics*, 40 (1972) 1772.

[8] M. Ferraro and R. Thibault, “Generic composition of boosts: an elementary derivation of the Wigner rotation”, *European Journal of Physics*, 20 (1999) 143.

[9] E. Merzbacher, *Quantum Mechanics*, (Wiley, New York, 1965).

[10] P. M. Mathews and K. Venkatesan, *A textbook of Quantum Mechanics*, (McGraw-Hill, New York, 1978).

[11] J. Singh, *Quantum Mechanics: Fundamentals and applications to technology*, (Wiley, New York, 1997).

[12] L. L. Sanchez-Soto, J. F. Cariñena, A. G. Barriuso, J. J. Monzón, “Vectorlike representation of one-dimensional scattering”, European Journal of Physics 26 (2005) 469, [arXiv:quant-ph/0411081].

[13] Petarpa Boonserm and Matt Visser, “One dimensional scattering problems: A pedagogical presentation of the relationship between reflection and transmission amplitudes”, Thai Journal of Mathematics, Special Issue (Annual Meeting in Mathematics, 2010), pages 83–97. Online ISSN 1686-0209. www.math.science.cmu.ac.th/thaijournal

[14] Asher Peres, “Transfer matrices for one-dimensional potentials”, J. Math. Phys. 24 (1983) 1110–1119.

[15] Jacek M. Kowalski and John L. Fry, “Tunneling in one-dimensional ideal barriers”, J. Math. Phys. 28 (1987) 2407–2415.
[16] S. Korasani and A. Adibi, “Analytical solution of linear ordinary differential equations by a differential transfer matrix method”, Electronic Journal of Differential Equations 79 (2003) 1–18.

[17] A. G. Barriuso, J. J. Monzon, L. L. Sanchez-Soto, J. F. Cariñena, “A vectorlike representation of multilayers”, Journal of the Optical Society of America A 21 (2004) 2386-2391, doi:10.1364/JOSAA.21.002386, [arXiv:physics/0403140].

[18] A. G. Barriuso, J. J. Monzon, L. L. Sanchez-Soto, J. F. Cariñena, “Geometrical aspects of first-order optical systems”, Journal of Optics A: Pure and Applied Optics 7 (2005) 451–456, doi: 10.1088/1464-4258/7/9/002, [arXiv:physics/0506112 [physics.optics]].

[19] Matt Visser, “Some general bounds for 1-D scattering”, Phys. Rev. A 59 (1999) 427–438 [arXiv: quant-ph/9901030].

[20] Petarpa Boonserm and Matt Visser, “Bounding the Bogoliubov coefficients”, Annals of Physics 323 (2008) 27792798 [arXiv: quant-ph/0801.0610].

[21] Petarpa Boonserm and Matt Visser, “Bounding the greybody factors for Schwarzschild black holes”, Phys. Rev. D 78 (2008) 101502 [arXiv:0806.2209 [gr-qc]].

[22] Petarpa Boonserm and Matt Visser, “Transmission probabilities and the Miller-Good transformation”, J. Phys. A 42 (2009) 045301 [arXiv:0808.2516 [math-ph]].

Petarpa Boonserm and Matt Visser, “Constraining transmission and reflection probabilities by using the Miller-Good transformation”, Presented at RCAEM-II 2012 — the 2nd Regional Conference on Applied and Engineering Mathematics (Penang).

[23] Petarpa Boonserm and Matt Visser, “Analytic bounds on transmission probabilities”, Annals of Physics 325 (2010) 1328–1339 [arXiv:0901.0944 [gr-qc]]. doi: 10.1016/j.aop.2010.02.005

[24] Petarpa Boonserm and Matt Visser, “Reformulating the Schrödinger equation as a Shabat–Zakharov system”, Journal of Mathematical Physics 51 (2010) 022105 [arXiv:0910.2600 [math-ph]].
[25] Petarpa Boonserm,
*Rigorous bounds on Transmission, Reflection, and Bogoliubov coefficients*,
(PhD thesis, Victoria University of Wellington, July 2009),
[arXiv:0907.0045](https://arxiv.org/abs/0907.0045) [math-ph].

[26] Tritos Ngampitipan and Petarpa Boonserm,
“Bounding the greybody factors for non-rotating black holes”,
[arXiv:1211.4070](https://arxiv.org/abs/1211.4070) [math-ph].

[27] Tritos Ngampitipan and Petarpa Boonserm,
“Bounding the greybody factors for the Reissner–Nordström black holes”,
Presented at ICAST 2012 — International Conference on Advancement in Science and Technology (Kuantan, Malaysia.),
Journal of Physics Conference Series (in press).

[28] Petarpa Boonserm and Matt Visser,
“Quasi-normal frequencies: Key analytic results”,
Journal of High Energy Physics **1103** (2011) 073 [arXiv:1005.4483v2 [math-ph]].