SINGULARITY CONTENT

MOHAMMAD E. AKHTAR AND ALEXANDER M. KASPRZYK

ABSTRACT. We show that a cyclic quotient surface singularity $\sigma$ can be decomposed, in a precise sense, into a number of elementary $T$-singularities together with a cyclic quotient surface singularity called the residue of $\sigma$. A normal surface $X$ with isolated cyclic quotient singularities $\{\sigma_i\}$ admits a $Q$-Gorenstein partial smoothing to a surface with singularities given by the residues of the $\sigma_i$. We define the singularity content of a Fano lattice polygon $P$: this records the total number of elementary $T$-singularities and the residues of the corresponding toric Fano surface $X_P$. We express the degree of $X_P$ in terms of the singularity content of $P$; give a formula for the Hilbert series of $X_P$ in terms of singularity content; and show that singularity content is an invariant of $P$ under mutation.

1. Introduction

Let $C$ be a two-dimensional rational cone and let $X_C$ denote the corresponding affine toric surface singularity. Let $u$, $v$ be primitive lattice points on the rays of $C$. Let $\ell$, the local index of $C$, denote the lattice height of the line segment $uv$ above the origin and let $w$, the width of $C$, denote the lattice length of $v-u$. Write $w = n\ell + \rho$ for $n, \rho \in \mathbb{Z}_{\geq 0}$ with $0 \leq \rho < \ell$. Then $X_C$ is a $T$-singularity if and only if $\rho = 0$, and we say that $X_C$ is an elementary $T$-singularity if $n = 1$ and $\rho = 0$ (so $w = \ell$); these correspond to singularities of the form $\frac{1}{n\ell^2}(1, n\ell c - 1)$ and $\frac{1}{\ell}(1, \ell c - 1)$, respectively. Choose a decomposition of $C$ into a cone $R$, of width $\rho$ and local index $\ell$, and $n$ other cones, each of width and local index $\ell$. Then, up to lattice isomorphism, $R$ depends only on $C$ and not on the decomposition chosen (Proposition 2.3) and we give explicit formula for $R$ in terms of $C$. There is a $Q$-Gorenstein deformation of $X_C$ such that the general fibre is the affine toric surface singularity $X_R$ (Proposition 2.7). We call $X_R$ the residue of $C$, and write it as $\text{res}(C)$. Given a normal surface $X$ with isolated cyclic quotient singularities $\{X_{C_i} : i \in I\}$ there exists a $Q$-Gorenstein deformation of $X$ such that the general fibre is a surface with isolated singularities $\{\text{res}(C_i) : i \in I\}$ (Corollary 2.8).

Let $P$ be a Fano polygon and let $X_P$ denote the corresponding toric Fano surface defined by the spanning fan $\Sigma$ of $P$. For a cone $C_i$ of $\Sigma$ with width $w_i$ and local index $\ell_i$, write $w_i = n_i\ell_i + \rho_i$ with $0 \leq \rho_i < \ell_i$. The singularity content of $P$ is the pair $(n, B)$ where $n = \sum n_i$ and $B$ is the cyclically-ordered list $\{\text{res}(C_i)\}_i$ with empty residues omitted. We compute the degree of $X_P$ in terms of the singularity content of $P$ (Proposition 3.3) and express the Hilbert series of $X_P$, in the style of [Rei87], as the sum of a leading term controlling the order of growth followed by contributions from the elements of $B$ (Corollary 3.4). The singularity content of $P$ is invariant under mutation [ACGK12].

2. Singularity content of a cone

Let $N$ be a lattice of rank two, and consider a (strictly convex) two-dimensional cone $C \subset N_Q := N \otimes \mathbb{Z} \mathbb{Q}$. Let $u$ and $v$ be the primitive lattice vectors in $N$ defined by the rays

2010 Mathematics Subject Classification: 14M25 (Primary); 14B05, 14J45 (Secondary).
of \( C \). Define the width \( w \in \mathbb{Z}_{>0} \) of \( C \) to be the lattice length of \( v - u \), and the local index \( \ell \in \mathbb{Z}_{>0} \) of \( C \) to be the lattice height of the line segment \( uv \) above the origin.

**Notation 2.1.** Given \( C, u, v \) as above, and a non-negative integer \( m \) such that \( m \leq w/\ell \), we define a sequence of lattice points \( v_0, v_1, \ldots, v_{n+1} \) on \( uv \) as follows:

(i) \( v_0 = u \) and \( v_{n+1} = v \);

(ii) \( v_{i+1} - v_i \) is a non-negative scalar multiple of \( v - u \), for \( i \in \{0, 1, \ldots, n\} \);

(iii) \( v_{i+1} - v_i \) has lattice length \( \ell \) for \( i \in \{0, \ldots, m, \ldots, n\} \);

(iv) \( v_{m+1} - v_m \) has lattice length \( \rho \), with \( 0 \leq \rho < \ell \).

The sequence \( v_0, \ldots, v_{n+1} \) is uniquely determined by \( m \) and the choice of \( u \). Note that \( w = n\ell + \rho \). We consider the partition of \( C \) into subcones \( C_i := \text{cone}\{v_i, v_{i+1}\}, 0 \leq i \leq n \).

**Lemma 2.2** ([AK13] Proof of Proposition 3.9]). If the cone \( C \subset \mathbb{N}_Q \) has singularity type \( \frac{1}{2}(a, b) \) then \( w = \gcd\{r, a, b\} \) and \( \ell = r/\gcd\{r, a + b\} \).

**Proposition 2.3.** Let \( C \subset \mathbb{N}_Q \) be a two-dimensional cone of singularity type \( \frac{1}{2}(1, a - 1) \). Let \( u, v \) be the primitive lattice vectors defined by the rays of \( C \), ordered such that \( u, v, \) and \( w^{-1}u + \frac{1}{r}v \) generate \( N \). Let \( v_0, \ldots, v_{n+1} \) be as in Notation 2.1. Then:

(i) The lattice points \( v_0, \ldots, v_{n+1} \) are primitive;

(ii) The subcones \( C_i, 0 \leq i < m \), are of singularity type \( \frac{1}{2}(1, \frac{\alpha}{w} - 1) \);

(iii) If \( \rho \neq 0 \) then the subcone \( C_m \) is of singularity type \( \frac{1}{2}\rho(1, \frac{\alpha}{w} - 1) \);

(iv) The subcones \( C_i, m < i \leq n \), are of singularity type \( \frac{1}{2}(1, \frac{a}{w} - 1) \).

Here \( \bar{a} \) is an integer satisfying \((a - 1)(\bar{a} - 1) \equiv 1 \pmod{r} \), and so exchanging the roles of \( u \) and \( v \) exchanges \( a \) and \( \bar{a} \). Note that the singularity type of \( C_m \) depends only on \( C \).

**Proof.** Without loss of generality we may assume that \( u = (0, 1) \), that \( v = (r, 1 - a) \), and that \( m \neq 0 \). The primitive vector in the direction \( v - u \) is \( (\alpha, \beta) := (\ell, -a/w) \). Thus \( v_1 = (\alpha^2, 1 + \alpha\beta) \), and so \( v_1 \) is primitive. There exists a change of basis sending \( v_1 \) to \((0, 1) \) and leaving \( (\alpha, \beta) \) unchanged. This change of basis sends \( v_i \) to \( v_{i-1} \) for each \( 1 \leq i \leq m \). It follows that the lattice points \( v_i, 1 \leq i \leq m \), are primitive, and that the cones \( C_i, 1 \leq i \leq m \), are isomorphic. Since

\[
\frac{1}{\alpha^2}(\alpha^2, 1 + \alpha\beta) - \frac{1 + \alpha\beta}{\alpha^2}(0, 1) = (1, 0),
\]

we have that \( C_1 \) has singularity type \( \frac{1}{\alpha^2}(1, -1 - \alpha\beta) = \frac{1}{\alpha}(1, \frac{\alpha}{w} - 1) \). Since \( r = w\ell \) (Lemma 2.2) we have that \( \frac{\ell(a + kr)}{w} - 1 \equiv \frac{a}{w} - 1 \pmod{\ell^2} \) for any integer \( k \), so that the singularity only depends on the equivalence class of \( a \) modulo \( r \). This proves (i). Switching the roles of \( u \) and \( v \) proves (ii) and (iii).

It remains to prove (iv). As before, we may assume that \( u = (0, 1) \) and \( v = (r, 1 - a) \). Consider the change of basis described above. After applying this \( m \) times, the cone \( C_{m+1} \) has primitive generators \((0, 1)\) and \((\rho\alpha, 1 + \rho\beta)\). Since

\[
\frac{1}{\rho^2}(\rho\alpha, 1 + \rho\beta) - \frac{1 + \rho\beta}{\rho^2}(0, 1) = (1, 0),
\]

we see that \( C_{m+1} \) has singularity type \( \frac{1}{\rho^2}(1, -1 - \rho\beta) = \frac{1}{\rho}(1, \frac{\alpha}{w} - 1) \). Since \( r/w = \ell \) (again by Lemma 2.2) we see that \( \frac{\rho(a + kr)}{w} - 1 \equiv \frac{a}{w} - 1 \pmod{\rho\ell} \) for any integer \( k \), hence the singularity only depends on \( a \) modulo \( r \).
Next, we need to show that (iii) is well-defined: that is, that the quotient singularities \( \frac{1}{\rho}(1, \frac{a}{w} - 1) \) and \( \frac{1}{\rho}(1, \frac{\bar{a}}{w} - 1) \) are equivalent. It is sufficient to show that

\[
\left( \frac{\rho a}{w} - 1 \right) \left( \frac{\rho\bar{a}}{w} - 1 \right) \equiv 1 \pmod{\rho \ell}.
\]

Let \( k, c \in \mathbb{Z}_{\geq 0}, 0 \leq c < \rho \ell \) be such that

\[
\left( \frac{\rho a}{w} - 1 \right) \left( \frac{\rho\bar{a}}{w} - 1 \right) = k\rho\ell + c.
\]

From Lemma 2.2 we see that \( 0 \leq c < \rho \ell \), and (2.1) becomes

\[
\left( a - 1 - \frac{nra}{d} \right) \left( \bar{a} - 1 - \frac{n\bar{a}}{d} \right) = kr - \frac{knr^2}{d} + c, \quad \text{where} \quad d := \gcd\{r, a\} \cdot \gcd\{r, \bar{a}\}.
\]

Multiplying through by \( d \) and reducing modulo \( r \) we obtain \( d(a - 1)(\bar{a} - 1) \equiv dc \pmod{r} \).

Suppose that \( d \not\equiv 0 \pmod{r} \). Since \( (a - 1)(\bar{a} - 1) \equiv 1 \pmod{r} \), we conclude that \( c = 1 \).

Finally, suppose that \( d \equiv 0 \pmod{r} \). Writing \( a = a' \cdot \gcd\{r, a\} \) and \( \bar{a} = \bar{a}' \cdot \gcd\{r, \bar{a}\}, \) we obtain

\[
1 \equiv (a' \cdot \gcd\{r, a\} - 1)(\bar{a}' \cdot \gcd\{r, \bar{a}\} - 1) \equiv 1 - a - \bar{a} \pmod{r},
\]

and hence \( a \equiv -\bar{a} \pmod{r} \). But this implies that \( 1 \equiv (a - 1)(-a - 1) \equiv 1 - a^2 \pmod{r} \) and so \( a \mid r \). Hence \( w = r, \ell = 1 \), and the singularity in (iii) is equivalent to \( \frac{1}{\rho}(1, \rho - 1) \).

Notice that the quantities \( a/w \) and \( \bar{a}/w \) appearing in Proposition 2.3 are integers by Lemma 2.2.

**Definition 2.4.** Let \( C \subset \mathbb{N} \) be a cone of singularity type \( \frac{1}{r}(1, a - 1) \). Let \( \ell \) and \( w \) be as above, and write \( w = n\ell + \rho \) with \( 0 \leq \rho < \ell \). The **residue** of \( C \) is given by

\[
\text{res}(C) := \begin{cases} \frac{1}{\rho}(1, \frac{a}{w} - 1) & \text{if} \ \rho \neq 0, \\ \emptyset & \text{if} \ \rho = 0. \end{cases}
\]

The **singularity content** of \( C \) is the pair \( \text{SC}(C) := (n, \text{res}(C)) \).

**Example 2.5.** Let \( C \) be a cone corresponding to the singularity \( \frac{1}{100}(1, 23) \). Then \( w = 12, \ell = 5, \) and \( \rho = 2 \). Setting \( m = 1 \) we obtain a decomposition of \( C \) into three subcones: \( C_0 \), singularity type \( \frac{1}{25}(1, 9) \), \( C_1 \) of singularity type \( \frac{1}{10}(1, 3) \), and \( C_2 \) of singularity type \( \frac{1}{25}(1, 4) \). In particular, \( \text{res}(C) = \frac{1}{100}(1, 3) \).

Recall that a \( T \)-singularity is a quotient surface singularity which admits a \( \mathbb{Q} \)-Gorenstein one-parameter smoothing; \( T \)-singularities correspond to cyclic quotient singularities of the form \( \frac{1}{ndc}(1, ndc - 1) \), where \( \gcd\{d, c\} = 1 \) [KSBSS Proposition 3.10]. We now show that \( T \)-singularities are precisely the cyclic quotient singularities with empty residue.

**Corollary 2.6.** Let \( C \subset \mathbb{N} \) be a cone and let \( w, \ell \) be as above. The following are equivalent:

1. \( \text{res}(C) = \emptyset; \)
2. There exists an integer \( n \) such that \( w = n\ell; \)
3. There exists a crepant subdivision of \( C \) into \( n \) cones of singularity type \( \frac{1}{\ell}(1, \ell c - 1), \gcd\{\ell, c\} = 1; \)
4. \( C \) corresponds to a \( T \)-singularity of type \( \frac{1}{nd}(1, n\ell c - 1), \gcd\{\ell, c\} = 1. \)
Proof. (i) and (ii) are equivalent by definition. (iii) follows from (i) by Proposition 2.3 and (i) follows from (iii) by Lemma 2.2. Assume (iii) and let the singularity type of $C$ be $\frac{1}{R}(1, A-1)$. The width of $C$ is $n$ times the width of a given subcone. Since $\gcd\{\ell, c\} = 1$, Lemma 2.2 implies that
$$\gcd\{R, A\} = w = n \cdot \gcd\{\ell^2, \ell c\} = n\ell.$$
The local index of a given subcone coincides, by construction, with the local index of $C$. By Lemma 2.2 we see that
$$R = \ell \cdot \gcd\{R, A\} = n\ell^2.$$Finally, Proposition 2.3 gives that $\ell A/w = \ell c$, hence $A = n\ell c$, and so (iii) implies (iv). □

2.1. Residue and deformation. Define the residue of a cyclic quotient singularity $\sigma$ to be the residue of $C$, where $C$ is any cone of singularity type $\sigma$. The residue encodes information about $\mathbb{Q}$-Gorenstein deformations of $\sigma$.

Proposition 2.7. A cyclic quotient singularity $\sigma$ admits a $\mathbb{Q}$-Gorenstein smoothing if and only if $\text{res}(\sigma) = \emptyset$. Otherwise there exists a $\mathbb{Q}$-Gorenstein deformation of $\sigma$ such that the general fibre is a cyclic quotient singularity of type $\text{res}(\sigma)$.

Proof. By definition, $\sigma$ admits a $\mathbb{Q}$-Gorenstein smoothing if and only if it is a $T$-singularity. Thus the first statement follows from Corollary 2.6. Assume $\sigma$ is not a $T$-singularity and let $\omega, \ell$, and $\rho$ be as above. By Corollary 2.6 we must have $\rho > 0$. Now $\sigma = \frac{1}{R}(1, a-1)$ has index $\ell$ and canonical cover
$$\frac{1}{\omega}(1, -1) = (xy - z^\omega) \subset \mathbb{A}^3_{x,y,z}.$$Taking the quotient by the cyclic group $\mu_\ell$, and noting that $\omega \equiv \rho \pmod{\ell}$, we have:
$$\frac{1}{\ell}(1, a - 1) = (xy - z^\omega) \subset \frac{1}{\ell}(1, \frac{\rho}{\omega} - 1, \frac{a}{\omega}).$$A $\mathbb{Q}$-Gorenstein deformation is given by
$$(xy - z^\omega + tz^\rho) \subset \frac{1}{\ell}(1, \frac{\rho}{\omega} - 1, \frac{a}{\omega}) \times \mathbb{A}^1_t,$$and the general fibre of this family is the cyclic quotient singularity $\frac{1}{\ell t}(1, \frac{\rho}{\omega} - 1)$. □

By combining Proposition 2.7 above with the proof of Proposition 3.4 and the Remark immediately following it from [Tzi09], which tells us that there are no local-to-global obstructions, we obtain:

Corollary 2.8. Let $H$ be a normal surface over $\mathbb{C}$ with isolated cyclic quotient singularities. There exists a global $\mathbb{Q}$-Gorenstein smoothing of $H$ to a surface $H^{\text{res}}$ with isolated singularities such that $\text{Sing}(H^{\text{res}}) = \{\text{res}(\sigma) \mid \sigma \in \text{Sing}(H), \text{res}(\sigma) \neq \emptyset\}$.

3. Singularity content of a complete toric surface

Definition 3.1. Let $\Sigma$ be a complete fan in $N_\mathbb{Q}$ with two-dimensional cones $C_1, \ldots, C_m$, numbered cyclically, with $\text{SC}(C_i) = (n_i, \text{res}(C_i))$. The singularity content of the corresponding toric surface $X_\Sigma$ is
$$\text{SC}(X_\Sigma) := (n, B),$$where $n := \sum_{i=0}^m n_i$ and $B$ is the cyclically ordered list $\{\text{res}(C_1), \ldots, \text{res}(C_m)\}$, with the empty residues $\text{res}(C_i) = \emptyset$ omitted. We call $B$ the residual basket of $X_\Sigma$. 
Notation 3.2. We recall some standard facts about toric surfaces; see for instance [Ful93]. Let \( X \) be a toric surface with singularity \( \frac{1}{r}(1, a-1) \). Let \([b_1, \ldots, b_k]\) denote the Hirzebuch–Jung continued fraction expansion of \( r/(a-1) \), having length \( k \in \mathbb{Z}_{>0} \). For \( i \in \{1, \ldots, k\} \), define \( \alpha_i, \beta_i \in \mathbb{Z}_{>0} \) as follows: Set \( \alpha_1 = \beta_k = 1 \) and set
\[
\alpha_i/\alpha_{i-1} := [b_{i-1}, \ldots, b_1], \quad 2 \leq i \leq k, \\
\beta_i/\beta_{i+1} := [b_{i+1}, \ldots, b_k], \quad 1 \leq i \leq k-1.
\]
If \( \pi : \tilde{X} \to X \) is a minimal resolution then
\[
K_{\tilde{X}} = \pi^*K_X + \sum_{i=1}^{k} d_i E_i,
\]
where \( E_i = -b_i \) and \( d_i = -1 + (\alpha_i + \beta_i)/r \) is the discrepancy.

Proposition 3.3. Let \( X \) be a complete toric surface with singularity content \((n, B)\). Then
\[
K_X^2 = 12 - n - \sum_{\sigma \in B} A(\sigma), \quad \text{where } A(\sigma) := k_\sigma + 1 - \sum_{i=1}^{k_\sigma} d_i b_i + 2 \sum_{i=1}^{k_\sigma-1} d_i d_{i+1}.
\]

Proof. Let \( \Sigma \) be the fan in \( N_\mathbb{Q} \) of \( X \). If \( C \in \Sigma \) is a two-dimensional cone whose rays are generated by the primitive lattice vectors \( u \) and \( v \) then, possibly by adding an extra ray through a primitive lattice vector on the line segment \( uv \), we can partition \( C = S \cup R_C \), where \( S \) is a (possibly smooth) \( T \)-singularity or \( S = \emptyset \), and \( R_C = \text{res}(C) \). Repeating this construction for all two-dimensional cones of \( \Sigma \) gives a new fan \( \tilde{\Sigma} \) in \( N_\mathbb{Q} \). If \( \tilde{X} \) is the toric variety corresponding to \( \tilde{\Sigma} \) then the natural morphism \( \tilde{X} \to X \) is crepant. In particular \( K_{\tilde{X}}^2 = K_X^2 \). Notice that \( \text{SC}(X) = (n, B) = \text{SC}(\tilde{X}) \).

By resolving singularities on all the nonempty cones \( R_C \), we obtain a morphism \( Y \to \tilde{X} \) where the toric surface \( Y \) (whose fan we denote \( \Sigma_Y \)) has only \( T \)-singularities. Thus by Noether’s formula [HP10, Proposition 2.6]
\[
(3.1) \quad K_Y^2 + \rho_Y + \sum_{\sigma \in \text{Sing}(Y)} \mu_{\sigma} = 10,
\]
where \( \rho_Y \) is the Picard rank of \( Y \), and \( \mu_{\sigma} \) denotes the Milnor number of \( \sigma \). But \( \rho_Y + 2 \) is equal to the number of two-dimensional cones in \( \Sigma_Y \), and the Milnor number of a \( T \)-singularity \( \frac{1}{ndc}(1, ndc - 1) \) equals \( n - 1 \), hence
\[
(3.2) \quad \rho_Y + \sum_{\sigma \in \text{Sing}(Y)} \mu_{\sigma} = -2 + n + \sum_{\sigma \in B} (k_\sigma + 1),
\]
where \( k_\sigma \) denotes the length of the Hirzebuch–Jung continued fraction expansion \([b_1, \ldots, b_{k_\sigma}]\) of \( \sigma \in B \). With notation as in Notation 3.2
\[
(3.3) \quad K_Y^2 = K_{\tilde{X}}^2 + \sum_{\sigma \in B} \left( -\sum_{i=1}^{k_\sigma} d_i^2 b_i + 2 \sum_{i=1}^{k_\sigma-1} d_i d_{i+1} \right).
\]
Substituting (3.2) and (3.3) into (3.1) gives the desired formula. \( \square \)

Remark 3.4. If \( X \) has only \( T \)-singularities, or equivalently if \( B = \emptyset \), then Proposition 3.3 gives \( K_X^2 = 12 - n \).
The $m$-th Dedekind sum, $m \in \mathbb{Z}_{\geq 0}$, of the cyclic quotient singularity $\frac{1}{r}(a, b)$ is
\[ \delta_m := \frac{1}{r} \sum (1 - \varepsilon(a))(1 - \varepsilon(b)), \]
where the summation is taken over those $\varepsilon \in \mu_r$ satisfying $\varepsilon^a \neq 1$ and $\varepsilon^b \neq 1$. By Proposition 3.3 and [Rei87, §8] we obtain an expression for the Hilbert series of $X$ in terms of its singularity content:

**Corollary 3.5.** Let $X$ be a complete toric surface with singularity content $(n, B)$. Then the Hilbert series of $X$ admits a decomposition
\[ \text{Hilb}(X, -K_X) = \frac{1 + (K_X^2 - 2)t + t^2}{(1 - t)^3} + \sum_{\sigma \in B} Q_\sigma(t), \]where $Q_\sigma(a, b) := \sum_{i=0}^{r-1} (\delta(a+b)i - \delta_0)t^i$. 

### 3.1. Singularity content and mutation

A lattice polygon in $\mathbb{Q}^n$ is called Fano if $0$ lies its strict interior, and all its vertices are primitive; see [KN12] for an overview. The *singularity content* of a Fano polygon $P$ is $\text{SC}(P) := \text{SC}(X_\Sigma)$, where $\Sigma$ is the spanning fan of $P$; that is, $\Sigma$ is the complete fan in $\mathbb{Q}^n$ with cones spanned by the faces of $P$.

Under certain conditions, one can construct a Fano polygon $Q := \text{mut}_h(P, F) \subset \mathbb{Q}^n$ called a (combinatorial) mutation of $P$. Here $h \in M := \text{Hom}(N, \mathbb{Z})$ is a primitive vector in the dual lattice, and $F \subset \mathbb{Q}^n$ is a point or line segment satisfying $h(F) = 0$. For the details of this construction see [ACGK12].

**Proposition 3.6.** Let $Q := \text{mut}_h(P, F)$. Then $\text{SC}(P) = \text{SC}(Q)$. In particular, singularity content is an invariant of Fano polygons under mutation.

**Proof.** The dual polygon $P^\vee \subset M_\mathbb{Q}$ is an intersection of cones
\[ P^\vee = \bigcap (C_L^\vee - v_L), \]where the intersection ranges over all facets $L$ of $P$. Here $C_L \subset \mathbb{Q}^n$ is the cone over the facet $L$ and $v_L$ is the vertex of $P^\vee$ corresponding to $L$.

If $P$ is a point then $P \cong Q$ and we are done. Let $F$ be a line segment and let $P_{\text{max}}$ and $P_{\text{min}}$ (resp. $Q_{\text{max}}$ and $Q_{\text{min}}$) denote the faces of $P$ (resp. $Q$) at maximum and minimum height with respect to $h$. By assumption the mutation $Q$ exists, hence $P_{\text{min}}$ must be a facet, and so there exists a corresponding vertex $v_0 \in M$ of $P^\vee$. $P_{\text{max}}$ can be either facet or a vertex. The argument is similar in either case, so we will assume that $P_{\text{max}}$ is a facet with corresponding vertex $v_1 \in M$ of $P^\vee$. 

The inner normal fan of $F$, denoted $\Sigma$, defines a decomposition of $M_\mathbb{Q}$ into half-spaces $\Sigma^+$ and $\Sigma^-$. The vertices $v_0$ and $v_1$ of $P^\vee$ lie on the rays of $\Sigma$; any other vertex lies in exactly one of $\Sigma^+$ or $\Sigma^-$. Mutation acts as an automorphism in both half-spaces. Thus the contribution to $\text{SC}(Q)$ from cones over all facets excluding $Q_{\text{max}}$ and $Q_{\text{min}}$ is equal to the contribution to $\text{SC}(P)$ from cones over all facets excluding $P_{\text{max}}$ and $P_{\text{min}}$. Finally, mutation acts by exchanging $T$-singular subcones between the facets $P_{\text{max}}$ and $P_{\text{min}}$, leaving the residue unchanged. Hence the contribution to $\text{SC}(Q)$ from $Q_{\text{max}}$ and $Q_{\text{min}}$ is equal to the contribution to $\text{SC}(P)$ from $P_{\text{max}}$ and $P_{\text{min}}$. 

**Example 3.7.** If two Fano polygons are related by a sequence of mutations then the corresponding toric surfaces have the same anti-canonical degree [ACGK12, Proposition 4]. The Fano polygons $P_1 := \text{conv}\{(0, 1), (5, 4), (-7, -8)\}$ and $P_2 := \text{conv}\{(0, 1), (3, 1), (-112, -79)\}$
correspond to \( \mathbb{P}(5,7,12) \) and \( \mathbb{P}(3,112,125) \), respectively. These both have degree \( 48/35 \), however their singularity contents differ:

\[
\text{SC}(P_1) = \left( 12, \left\{ \frac{1}{3}(1,1), \frac{1}{7}(1,1) \right\} \right), \quad \text{SC}(P_2) = \left( 5, \left\{ \frac{1}{11}(1,9), \frac{1}{125}(1,79) \right\} \right).
\]

Hence they are not related by a sequence of mutations.

**Lemma 3.8.** Let \( P \) be a Fano polygon with \( \text{SC}(P) = (n, \mathcal{B}) \), and let \( \rho_X \) denote the Picard rank of the corresponding toric surface. Then \( \rho_X \leq n + |\mathcal{B}| - 2 \).

**Proof.** The cone over any facet of \( P \) admits a subdivision (in the sense of Notation 2.1) into at least one subcone. Therefore we must have that \( |\text{vert}(P)| \leq n + |\mathcal{B}| \). Recalling that \( \rho_X = |\text{vert}(P)| - 2 \) we obtain the result. \( \square \)

Since singularity content is preserved under mutation, Lemma 3.8 gives an upper bound on the rank of the resulting toric varieties.

**Example 3.9.** In [AK13] we classified one-step mutations of (fake) weighted projective planes. It is natural to ask how much of the graph of mutations of a given (fake) weighted projective plane is captured by the graph of one-step mutations. Lemma 3.8 shows that the two graphs coincide if the singularity content of the (fake) weighted projective plane in question satisfies \( n + |\mathcal{B}| = 3 \). For example the full mutation graph of \( \mathbb{P}^2 \) is isomorphic to the graph of solutions of the Markov equation \( 3xyz = x^2 + y^2 + z^2 \) [AK13, Example 3.14]. More interestingly, the weighted projective plane \( \mathbb{P}(3,5,11) \) does not admit any mutations [AK13, Example 3.5].

**Acknowledgements.** We thank Tom Coates, Alessio Corti, and Diletta Martinelli for many useful conversations. This research is supported by EPSRC grant EP/I008128/1.