Fixed-time stabilisation of boundary controlled linear parabolic distributed parameter systems with space-dependent reactivity

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Abstract
The problems of state feedback and output feedback for fixed-time stabilisation of a linear parabolic distributed parameter system with space-dependent reactivity are considered by means of continuous backstepping approach and Lyapunov method. First, the invertible Volterra integral transformation with time dependent gain kernel is adopted to convert the original system into a fixed-time stable target system with time-dependent coefficient. The well-posedness of the resulting kernel equations is proved by the method of successive approximation. Then a state feedback controller is designed to guarantee fixed-time stabilisation of the closed-loop system. Moreover, a fixed-time observer is considered to estimate the state of the original system on the basis of the measurement signal at the boundary. Based on this observer, a observer-based output feedback controller is established to fixed-time stabilise the closed-loop system in a prescribed time by using separation principle. Finally, a numerical simulation is provided to verify the feasibility of the proposed theoretical results by using the modified Ablowitz-Kruskal-Ladik scheme.

1 | INTRODUCTION

Time constraints and precision are required in many practical applications [1, 2]. A considerable amount of research results have been obtained on the problems of the fixed-time stabilisation and the fixed-time observation of the lumped parameter systems, which are modeled by the ordinary differential equations in [3–10]. In the scope of parabolic distributed parameter systems, the finite-time control has become a hot research area [11–14]. In contrast to the lumped parameter systems, the problems of the fixed-time control and the fixed-time observer design of parabolic distributed parameter systems have not achieved a sufficient level of maturity [15–18].

From an engineering point of view, for distributed parameter systems, the actuators and the sensors are installed in the positions either at the interior [19, 20] or the boundary [21] of the spatial domain. Due to the difficulty that the state of distributed parameter systems is infinite dimensional, it is impossible to reach measurement and actuation in the whole spatial domain. In addition, the economical and technical constraints, the sensor and actuator at the interior of spatial domain is not realisable in some applications such as viscous fluid, high temperature and chemical reactor. Therefore, the boundary control of parabolic distributed parameter systems has received an increased attention due to the more realistic problem in many applications [22–28]. It is worth noting that the backstepping technique has been become one of the popular research methods to the boundary feedback control of the parabolic distributed parameter systems [29–35]. For infinite dimensional systems, the main idea of backstepping method is to convert the unstable system into a suitable target system by invertible Volterra integral transformation. Then, the stability problem of the system is reduced to a well-posedness problem of the resulting kernel partial differential equations (PDEs). In [29], Liu first presented the continuous backstepping method to handle the problem of the boundary control of an unstable parabolic distributed parameter system. The problem of the boundary control of a partial integro differential equation is considered by backstepping method in [30], where the explicit controller and analytical solution of resulting kernel equations are given by Bessel function. For dis-
distributed parameter systems with spatially-varying diffusivity and temporal-varying reactivity, Smyshlyaev and Krstic [31] first discussed the problem of boundary control by applying the continuous backstepping method, and the established explicit solution of the resulting time-varying kernel. The tracking control for parabolic distributed parameter systems with time dependent and space dependent coefficient was studied by backstepping method and flatness in [32]. The resulting kernel system with spatially varying and time-varying coefficient by invertible Volterra integral transformation with time varying kernel were provided and the well-posedness of kernel PDEs with spatially and temporally varying parameters were proved by successive approximations methods and Gevery class. In addition, in most cases, the state of system may not be measurable due to the various reasons. Therefore, the observer is designed to estimate the state of the distributed parameter system by the backstepping method [36–40]. The observer-based output feedback controller combining of observer and state feedback controller was designed to stabilise the unstable partial integro differential equation based on backstepping approach [36]. In [37], Jadachowski et al. designed the observer-based output feedback controller for time-varying parabolic distributed parameter systems by using backstepping method. The systematic backstepping method of linear parabolic distributed parameter systems was presented by Krstic and Smyshlyaev [41].

As mentioned in the above, the existing literatures focus on boundary feedback for exponential stabilisation of the parabolic distributed parameter systems by backstepping method. Recently, the fixed-time concept of lumped parameter systems was extended to the distributed parameter systems which are modeled by partial differential equations. For example, in Espitia et al. [16], the problem of the boundary state feedback for fixed-time stabilisation of a distributed parameter system with constants coefficient by backstepping methods is presented based on Song et al. [7] and Coron and Nguyen [13]. On the basis of Smyshlyaev and Krstic [31], the explicit representation of solution of the kernel equations is carried out by employing the generalised Laguerre polynomials and the modified Bessel function. In [15], Espitia et al. explored the state feedback for the fixed-time stabilisation of a coupled parabolic distributed parameter system by means of the backstepping method. Moreover, the observation with time constraints is an significant issue in control field. Therefore, the problems of the observer-based output feedback fixed-time stabilisation of a linear parabolic distributed parameter system with constants coefficient was investigated in Hilbert space based on backstepping approaches by Steeves et al. [18]. The problem of the observer-based output feedback for prescribed-time stabilisation of reaction-diffusion equations with constants coefficients was studied in $H^1$ Sobolev space by backstepping methods in Steeves et al. [17]. However, previous research efforts are scarce with regard to boundary feedback for the fixed-time stabilisation and the fixed-time observation of the distributed parameter system. Especially, the problem of the fixed-time stabilisation of a distributed parameter system with spatially-varying reactivity by the state feedback and the output feedback have not been considered. The parabolic distributed parameter system with space-dependent coefficients resulting from the heat conduction process of rod with nonhomogenous materials and irregularly shaped domains [31].

In this paper, the problems of the state feedback and the observer-based output feedback for fixed-time stabilisation of a linear parabolic distributed parameter system with space-dependent thermal conductivity are studied by using the backstepping method, the Lyapunov method and the separation principle which are inspired by the existing results [16, 18, 31, 32, 36]. The sensor and actuator are disposed at the different boundaries to measure state and to control the system respectively. Firstly, by using the backstepping method, the voltaire integral transformation is introduced to convert the original system into a fixed-time stable target system. The well-posedness of the resulting kernel PDEs is presented by the method of successive approximation. Secondly, a state feedback boundary controller is designed to fixed-time stabilise the parabolic distributed parameter system with space dependent reactivity in a prescribed time. Thirdly, the fixed-time observer is designed to estimate the state of the boundary controlled system in the prescribed time. Fourthly, the observer-based output feedback controller is designed by the separation principle to prove the fixed-time stabilisation of the original system within the prescribed time. Finally, the solutions of the kernel PDEs are numerically solved by using the modified Ablowitz-Kruskal-Ladik method to obtain the gain kernel of the controller and the observer gains.

The structure of this paper is organised as follows. A parabolic distributed parameter system is introduced in Section 2. The state feedback controller is established in Section 3. Fixed-time observer is designed by backstepping method in Section 4. The observer-based output feedback controller is achieved by making use of separation principle in Section 5. In Section 6, a numerical simulation is established to testify feasibility of the theoretical results. Finally, a brief conclusion is showed in Section 7.

The notation used throughout this paper is fairly standard. $\frac{\partial^2 z(s,t)}{\partial s^2} = z_{ss}(s,t), \frac{\partial z(s,t)}{\partial r} = z_{s}(s,t), \frac{\partial z(s,t)}{\partial r^l} = z_{s r^l}(s,t), \frac{\partial^l}{\partial r^l}. \mathbb{R}$ will stands for the set of real numbers. The $L^2$ stands for the Hilbert space consisting of square integrable functions $g(s,t)$ in the $s \in [0, 1], t \in [0, T]$ with the corresponding norm $\|g(s,t)\|_{L^2} = \left(\int_0^1 g^2(s,t) \, ds\right)^{\frac{1}{2}} < \infty$.

2 | PROBLEM DESCRIPTION

The heat conduction process of one dimensional rod can be modeled by the following linear parabolic distributed parameter system with space-dependent reactivity

$$z_{s}(s,t) = ε z_{ss}(s,t) + \lambda(s) z_{s}(s,t), (s,t) \in (0,1) \times [0, T), \quad (1)$$

$$z_0(s,t) = 0, \quad t \in [0, T), \quad (2)$$

$$z(1,t) = u(t), \quad t \in [0, T), \quad (3)$$
with initial condition

\( z(s, 0) = \zeta_0(s), \quad s \in [0, 1], \) \hspace{1cm} (4)

where \( z(s, t) \) is the value of the temperature field of the plant at location \( s \in [0, 1] \) and time moment \( t \in [0, T] \) along the rod, \( T > 0 \) is a given constant which will be called prescribed time. \( z(s, t) \) is defined on the domain \( \mathbb{H} = \{ (s, t) \in [0, 1] \times [0, T] \} \). The diffusion coefficient \( \varepsilon \in (0, +\infty) \) and the reaction coefficient \( \lambda(s) \in C^1 [0, 1] \) stands for the heat conduction coefficient and the coefficient of the heat exchange with the surrounding, respectively. In addition, \( u(t) \) is the control input acting on the system boundary and \( \zeta_0(s) \in L^2 \) is the initial value. The function \( \lambda(s) \) results in the instability of the system (1)–(3) when the control input \( u(t) = 0 \). In what follows, we introduce the definition of fixed-time stabilisation.

**Definition 1.** Based on [7], the system (1)–(3) is said to be fixed-time stable in prescribed time \( T > 0 \) and converges to zero, if for arbitrary initial value \( \zeta_0(s) \in L^2 \), there exists a class \( \mathcal{K} \) function \( \lambda(s) \) such that

\[ \| z(s, t) \|_{L^2} \leq \lambda \left( \| \zeta_0(s) \|_{L^2}, \lambda(t) \right), \quad s \in [0, 1], t \in [0, T]. \]

According to [16], where \( \lambda(t) \) is defined by

\[ \lambda(t) = \frac{\rho^2 \tau^2}{(T - t)^2}; \quad \lambda(0) = \rho_0^2, \quad \rho_0 > 0, \quad t \in [0, T]. \] \hspace{1cm} (5)

Note that \( \lambda(t) \) is monotonically increasing function with the property \( \lambda(T) = +\infty \).

### 3 | STATE FEEDBACK CONTROLLER DESIGN

In this section, the fixed-time stability of the original system (1)–(3) with state feedback boundary controller by means of the backstepping integral transformation [31] is as follows

\[ w(s, t) = \zeta(s, t) - \int_0^t k(s, y, t) \zeta(y, t) \, dy, \quad (s, t) \in \mathbb{H}, \] \hspace{1cm} (6)

where time dependent gain kernel \( k(s, y, t) \) is defined on the domain \( \mathbb{D} = \{ (s, y, t) \in \mathbb{R}^2 \times [0, T] : 0 \leq y \leq s \leq 1 \} \). Using the integral transformation (6), the system (1)–(3) can be transformed into a target system with time-dependent reactivity

\[ w(s, t) = \epsilon w(s, t) - \epsilon(t) w(s, t), \quad (s, t) \in (0, 1) \times [0, T), \] \hspace{1cm} (7)

\[ w(0, t) = 0, \quad t \in [0, T), \] \hspace{1cm} (8)

\[ w(1, t) = 0, \quad t \in [0, T), \] \hspace{1cm} (9)

equipped with initial condition

\[ w(s, 0) = w_0(s), \quad s \in [0, 1], \] \hspace{1cm} (10)

where \( w(s, t) \) is state of the target system (7)–(9), which is defined on the domain \( \mathbb{H} \), the reaction parameter \( c(t) \in C^\infty ([0, T]) \) is a real analytic function and \( c(t) > 0 \), initial value \( w_0(s) \in L^2 \). The coefficient \( c(t) \) will be designed a suitable analytic function to derive fixed-time stabilisation of the system (7)–(9).

Design the function \( c(t) \) in (7) as follows

\[ c(t) = \rho(t) - \frac{2\rho_0^2}{T - t}; \quad c(0) = \rho_0 > 0, \quad t \in [0, T]. \] \hspace{1cm} (11)

The fundamental idea of backstepping method [41] is to transform the system (1)–(3) into a fixed-time stable target system (7)–(9) by means of integral transformation (6). As long as the existence and uniqueness of the gain kernel \( k(s, y, t) \) is proved, then the transformation (6) is reasonable. In what follows, kernel equations with gain kernel \( k(\infty, y, t) \) will be obtained by the following calculation.

Differentiating the integral transformation (6) with regard to \( t \) and employing the integration by parts, one can obtain

\[ w(s, t) = \epsilon w(s, t) - \epsilon(t) w(s, t) - \int_0^t k(s, y, t) \zeta(y, t) \, dy \]

\[ - \int_0^t \lambda(y)k(s, y, t) \zeta(y, t) \, dy - \epsilon \int_0^t k_y(s, y, t) \zeta(y, t) \, dy \]

\[ - \epsilon k(s, y, t) \zeta(s, t) + \epsilon k(0, y, t) \zeta(0, t) \]

\[ + \epsilon k_y(s, y, t) \zeta(s, t) - \epsilon k_y(0, y, t) \zeta(0, t). \] \hspace{1cm} (12)

Taking second spatial derivative with respect to the integral backstepping transformation (6) by the Leibnitz differentiation rule yields

\[ w_{ss}(s, t) = \zeta_{ss}(s, t) - \frac{d}{dt} k(s, s, t) \zeta(s, t) - k(s, s, t) \zeta_y(s, t) \]

\[ - k(s, s, t) \zeta(s, t) - \int_0^t k_y(s, y, t) \zeta(y, t) \, dy \]

\[ - k(s, s, t) \zeta(s, t) - \int_0^t k_y(s, y, t) \zeta(y, t) \, dy \] \hspace{1cm} (13)

where \( k(s, s, t) = k(s, s, t)|_{y=s}, \quad k(s, s, t) = k(s, y, t)|_{y=s}, \) and \( \frac{d}{dt} k(s, s, t) = k(s, s, t) + k_y(s, s, t) \). Substituting (12) and (13) into the target system (7)–(9), then in the light of equalities (12), (13)
and boundary condition (2), one can easily get
\[ w_{t}(s, t) - \varepsilon w_{ss}(s, t) + c(t)w(s, t) \]
\[ = \left[ \lambda(s) + \varepsilon k_{s}(s, t) + \varepsilon \frac{d}{ds} k(s, s, t) + \varepsilon k_{s}(s, s, t) + c(t) \right] \]
\[ + \varepsilon k_{ss}(s, t) - \varepsilon c(t)k(s, s, t) \]
\[ = 0. \quad (14) \]

Due to \( w_{t}(0, t) = 0 \) and \( z_{0}(0, t) = 0 \), the gain kernel has to fulfill the additional restriction \( k(0, 0, t) = 0 \). Obviously, based on (14), the so-called kernel PDEs with the time-varying gain kernel \( k(s, y, t) \) are derived as follows
\[ k_{s}(s, y, t) = \varepsilon k_{ss}(s, s, t) - \varepsilon k_{ss}(s, y, t) - \gamma(s, t)k(s, s, t), \quad (s, y) \in D, \quad (15) \]
\[ k_{y}(s, 0, t) = 0, \quad s \in [0, 1], t \in [0, T], \quad (16) \]
\[ k_{y}(0, 0, t) = 0, \quad t \in [0, T], \quad (17) \]
\[ k(0, 0, t) = 0, \quad t \in [0, T], \quad (18) \]

where \( \gamma(s, t) = \lambda(s) + c(t) \) is a smooth scalar function with respect to \( t \) defined on the domain \( H \). The initial condition satisfies \( \int_{0}^{s} k_{y}(s, y) \psi_{0}(y) \, dy = \psi_{0}(s), \quad s \in [0, 1] \). Therefore, the boundary control problem of the original system (1)–(3) is reduced to prove the well-posedness of the system (15)–(18). Using the transformation (6), the state feedback controller \( u(t) \) is devised as follows
\[ u(t) = \int_{0}^{t} k(1, y, t)z(y, t) \, dy, \quad t \in [0, T]. \]

The state feedback controller gain \( k(1, y, t) \) is given by solving the kernel PDEs (15)–(18).

**Remark 1.** It is noticeable that the kernel equations (15)–(18) differ from the kernel equations [32] in that (15) and (16) contain coefficients \( \varepsilon \) and \( c(t) \in C^{\infty}[0, T] \) with property \( c(t) \to +\infty \) as \( t \to T \).

Inspired by Si et al. [22] and Meurer et al. [32], the following assumption is provided to prove the well-posedness of the kernel PDEs (15)–(18).

**Assumption 1.** The function \( \gamma(s, t) \) is real analytic with respect to time in \([0, T]\) if \( \gamma(s, t) \in C^{2}([0, 1]) \times C^{\infty}([0, T]) \) and for every compact subset \([0, T^{*}]\) of \([0, T]\) there exists a constant \( D > 0 \) such that for every integer \( I \geq 0 \),
\[ \left| \frac{\partial^{I}}{\partial s^{I}} \gamma(s, t) \right| \leq D^{I+1} / \pi, \quad s \in [0, 1], t \in [0, T^{*}], \]
where \( T^{*} < T \), \( T^{*} \) is arbitrary close to \( T \).

Based on the Assumption 1 and existing method in [32], we present the following lemma.

**Lemma 1.** If the function \( \gamma(s, t) \) satisfies Assumption 1, then the kernel PDEs \((15)–(18)\) has a unique solution \( k(s, y, t) \) defined on the domain \( D \), which is infinitely continuously differentiable with regard to time. The solution \( k(s, y, t) \) is bounded by
\[ |k(s, y, t)| \leq \frac{D_{I}}{2e} \left( \gamma^{2} - y^{2} \right), \quad 0 \leq y \leq s \leq 1, 0 \leq t \leq T^{*}. \]
Moreover, for all \( 0 \leq y \leq s \leq 1, t \in [0, T^{*}] \), there exists a positive constant \( M_{1} \) such that \( |k(s, y, t)| \leq M_{1} \).

The proof of Lemma 1 is shown in the Appendix according to [32, Theorem 9].

**Remark 2.** If the reactivity \( \lambda(s) \) of the system (1) is an arbitrary constant \( \lambda \) and \( \gamma(s, t) = \rho(s) \) and \( k(s, 0, t) = 0 \) holds, then the kernel PDEs (15)–(18) can be reduced to a particular form in [16, Theorem 1]. The analytical solution of the special kernel PDEs can be represented as
\[ k(s, y, t) = -\frac{\rho(s)}{\varepsilon} \frac{I_{1}\left( \sqrt{\frac{\rho(s)}{\varepsilon}} (\varepsilon^{2} - y^{2}) \right)}{\sqrt{\rho(s)} \varepsilon}, \]
where \( I_{1}(\xi) = \sqrt{-1}J_{1}(\sqrt{-1} \xi) \), \( J_{1}(\xi) \) denotes a first order Bessel function [38]
\[ J_{1}(\xi) = \sum_{n=0}^{\infty} (-1)^{n} \frac{\left( \frac{\pi}{2} \right)^{2n+1}}{n! (n + 1)!}. \]

**Remark 3.** If the reactivity \( \lambda(s) \) of the system (1) is an arbitrary constant \( \lambda \) and \( \gamma(s, t) = \gamma \) then the kernel PDEs (15)–(18) can be reduced to a special form in [30]. The analytical solution of the special kernel PDEs can be represented as
\[ k(s, y) = -\frac{\gamma}{\varepsilon} \frac{I_{1}\left( \sqrt{\frac{\rho(s)}{\varepsilon}} (\varepsilon^{2} - y^{2}) \right)}{\sqrt{\rho(s)} \varepsilon}, \quad (19) \]

The premise of proving the fixed-time stabilisation of the original system (1)–(3) is to testify the invertibility of the integral transformation (6).
The inverse transformation of the integral transformation (6) is presented by

\[ \zeta(s, t) = w(s, t) + \int_0^t l(s, y, t)w(y, t) \, dy, \quad (s, t) \in \mathbb{H}, \]  

where time dependent gain kernel \( l(s, y, t) \) is confined on the domain \( \mathbb{D} \). In virtue of integral transformation (6) and inverse transformation (20), we can deduce that the gain kernel functions \( k(s, y, t) \) and \( l(s, y, t) \) satisfy the following relationship

\[ l(s, y, t) = k(s, y, t) + \int_{y}^{s} k(s, \zeta, t)l(\zeta, y, t) \, d\zeta, \quad (s, y, t) \in \mathbb{D}. \]

The proof of this result can be found in [41].

The inverse transformation (20) is to transform the target system (7)–(9) into the original system (1)–(3). Substituting the inverse transformation (20) into the original system (1)–(3), and applying the target system (7)–(9), the integrating by parts and Leibnitz differentiation rule, the resulting kernel PDEs have the form

\[ l(s, y, t) = \varepsilon I_{l}(s, y, t) - \varepsilon I_{y}(s, y, t) + \gamma(s, t)l(s, y, t), \]

\[ (s, y, t) \in \mathbb{D} \]  

(21)

\[ \frac{d}{ds}l(s, y, t) = -\frac{1}{2}x^{2} \gamma(s, t), \quad s \in [0, 1], t \in [0, T], \]  

(22)

\[ l(s, 0, t) = 0, \quad s \in [0, 1], t \in [0, T], \]  

(23)

\[ l(0, 0, t) = 0, \quad t \in [0, T]. \]  

(24)

The well-posedness of the inverse kernel PDEs (21)–(24) is given by the following lemma.

**Lemma 2.** If the function \( \gamma(s, t) \in C_{s}([0, 1]) \times C_{s}^{\infty}([0, T]) \) satisfies Assumption 1, then the kernel PDEs (21)–(24) has a unique solution \( l(s, y, t) \) defined on the domain \( \mathbb{D} \), which is infinitely continuously differentiable with respect to time. The solution \( l(s, y, t) \) is bounded by

\[ |l(s, y, t)| \leq \frac{D_{e}}{2} \varepsilon^{2}(\gamma^{2} - \gamma^{2}), \quad 0 \leq y \leq s \leq 1, 0 \leq t \leq T^{*}, \]  

Furthermore, for all \( 0 \leq y \leq s \leq 1, 0 \leq t \leq T^{*}, \) \( |l(s, y, t)| \leq M_{1} \) holds.

Similarly to the proof of Lemma 1, the proof of Lemma 2 is omitted.

**Remark 4.** If the reactivity \( \lambda(t) \) of the system (1) is an arbitrary constant that is \( \gamma(s, t) = \rho(t) \) and \( k(s, 0, t) = 0 \) holds, then the kernel PDEs (21)–(24) can be reduced to a special form in [16, Theorem 2]. The analytical solution of the special kernel PDEs can be indicated as

\[ l(s, y, t) = \varepsilon \frac{\rho(t)}{e^{-\gamma t} \rho(t)} \int_{0}^{\gamma t} \frac{\rho^{2}(s)}{2(\rho^{2} - 1)} \, ds. \]  

**Remark 5.** If the reactivity \( \lambda(t) \) of the system (1) is a arbitrary constant \( \lambda \) and the reactivity \( c(t) \) of the target system (7) is a constant \( c > 0 \) that is \( \gamma(s, t) = \gamma \), then the kernel PDEs (21)–(24) can be reduced to a special form in [30]. The explicit solution of the special kernel PDEs can be represented as

\[ l(s, y) = -\frac{\gamma}{\varepsilon} \frac{J_{1}\left(\sqrt{\frac{2}{\varepsilon}}(s^{2} - y^{2})\right)}{\sqrt{2}\left(s^{2} - y^{2}\right)}. \]

Note that the fixed-time stabilisation of the target system (7)–(9) indicates the fixed-time stabilisation of the original system (1)–(3) based on backstepping method.

### 3.2 State feedback for fixed-time stabilisation

Let us discuss the fixed-time stability of the target system (7)–(9) by using Lyapunov method and bump-like function [7].

**Theorem 1.** For a given analytic function \( c(t) > 0 \) in (11), if there exists a monotonically increasing function \( \rho(t) = \frac{c^{2}}{4} (t - 1)^{2} \) with properties \( \rho(0) = \rho_{0}^{2} \) and \( \rho(T) = +\infty \) such that for all \( t \in [0, T] \),

\[ \|w(s, t)\|_{L^{2}} \leq e^{-\gamma T} \xi_{2}(s, 0) \|w(s, 0)\|_{L^{2}}, \]  

(25)

where \( \xi_{2}(t) = e^{2\rho_{0} T} \sqrt{\rho(0)} \sqrt{\rho(T)} \) is a monotonically decreasing function, then for arbitrary initial value \( w_{0}(s) \in L^{2} \), the target system (7)–(9) is fixed-time stable in the prescribed time \( T \).

**Proof.** Consider a Lyapunov function \( V^{*} : [0, T] \rightarrow [0, +\infty) \),

\[ V^{*}(t) = \frac{1}{2} \int_{0}^{1} w^{2}(s, t) \, ds = \frac{1}{2} \|w(s, t)\|_{L^{2}}^{2}, \quad t \in [0, T]. \]  

(26)

Computing the time derivative for the Lyapunov function \( V^{*}(t) \) along the trajectory of the target system (7)–(9), applying integration by parts and equalities (8), (9), for all \( t \in [0, T] \), one can obtain

\[ V^{*}(t) = -\varepsilon \int_{0}^{1} w^{2}(s, t) \, ds - c(t) \int_{0}^{1} w^{2}(s, t) \, ds. \]
Thus, employing the Wirtinger's inequality [41], we get that
\[ V'(t) \leq -\frac{\varepsilon T^2}{4} \int_0^1 w^2(s,t) \, ds - \varepsilon(t) \int_0^1 w^2(s,t) \, ds \]
\[ = \left( -\frac{\varepsilon T^2}{2} - 2\varepsilon(t) \right) V(t). \] (27)

Integrating the above inequality (27) with respect to \( t \) from 0 to \( t \) and using (11), we obtain that
\[ V'(t) \leq e^{-\int_0^t \left( \frac{\varepsilon T^2}{2} + 2\varepsilon(s) \right) \, ds} V(0) \]
\[ = e^{-\int_0^t \rho(s) \, ds + 2\varepsilon T^2 \int_0^s \, ds} V(0), \]
\[ \leq e^{-\int_0^t \rho(s) \, ds + 2\varepsilon T} V(0). \]

In addition, denoting \( \xi(t) = e^{-\int_0^t \rho(s) \, ds} \) and employing (5), \( \xi(t) \) is easily rewritten as follows
\[ \xi(t) = e^{\int_0^t \rho(s) \, ds + 2\varepsilon T}. \] (28)
The time dependent function \( \xi(t) \) is a monotonically decreasing function with the properties \( \xi(0) = 1 \) and \( \xi(T) = 0 \). It is named as a smooth bump-like function. Hence, for all \( t \in [0, T] \), we have
\[ V'(t) \leq e^{2\varepsilon T} \xi(t) V(0). \] (29)

Applying (29) and the Lyapunov function (26), one can state that the following inequality holds, for all \( t \in [0, T] \),
\[ \|w(t,t)\|_{L^2} \leq e^{2\varepsilon T} \sqrt{\xi(t)} \|w(s,0)\|_{L^2}. \]

Combining with the bump-like function (28), we get that \( \|w(t,t)\|_{L^2} \rightarrow 0 \) as \( t \rightarrow T \). In virtue of Definition 1, the target system (7)–(9) is fixed-time stable.

As is illustrated in the above analysis, we will prove the main result on fixed-time stability of the considered system (1)–(4) with state feedback controller.

**Theorem 2.** If there exists a monotonically increasing function \( \rho(t) \) in (5) with properties \( \rho(0) = \rho_0^\circ \) and \( \rho(T) = +\infty \) such that for all \( t \in [0, T] \),
\[ \|z(s,t)\|_{L^2} \leq (1 + M_1) e^{2\varepsilon T} \sqrt{\xi(t)} \|z(s,0)\|_{L^2}, \] (30)
where \( \xi(t) = e^{2\varepsilon T} \sqrt{\rho(0)} \int_0^t e^{-2\rho(s) \sqrt{\rho(0)}} \, ds \), then for arbitrary initial value \( z_0(t) \in L^2 \), the closed loop system (1)–(3) with state feedback controller
\[ u(t) = \int_0^1 k(s,y,t)z(y,t) \, dy, \quad t \in [0, T], \] (31)
is fixed-time stable in the prescribed time \( T \). Moreover, \( |u(t)| \rightarrow 0 \) as \( t \rightarrow T \).

**Proof.** In terms of Lemma 1 and the transformation (6), we can easily deduce
\[ \|w(s,0)\|_{L^2} \leq (1 + M_1) \|z(s,0)\|_{L^2}, \quad t \in [0, T]. \] (32)
Similarly, by means of Lemma 2 and transformation (20), we have
\[ \|z(s,t)\|_{L^2} \leq (1 + M_1) \|w(s,t)\|_{L^2}, \quad t \in [0, T]. \] (33)
By the Theorem 1, we can obtain that the target system (7)–(9) is fixed-time stable, and
\[ \|w(s,t)\|_{L^2} \leq e^{\varepsilon T} \sqrt{\xi(t)} \|w(s,0)\|_{L^2}, \quad t \in [0, T]. \]

Hence, combining the (33) with the (32), we derive
\[ \|z(s,t)\|_{L^2} \leq (1 + M_1) e^{\varepsilon T} \sqrt{\xi(t)} \|z(s,0)\|_{L^2}, \quad t \in [0, T]. \]
We can obtain that \( \|z(s,t)\|_{L^2} \rightarrow 0 \) as \( t \rightarrow T \) by means of (28) and (30). Then, the system (1)–(3) is fixed time stable based on Definition 1. Moreover, the state feedback controller \( u(t) \) converges to zero in prescribed time \( T \). Because of the inverse backstepping transformation (20), the state feedback controller (31) can equivalently be rewritten as
\[ u(t) = \int_0^1 l(s,y,t)w(y,t) \, dy, \quad t \in [0, T]. \]

Applying Lemma 2, the inequalities (25) and (32), one can derive the following estimate
\[ |u(t)| \leq M_1 (1 + M_1) e^{\varepsilon T} \sqrt{\xi(t)} \|z(s,0)\|_{L^2}, \quad t \in [0, T]. \]
In view of (28), we can deduce that \( |u(t)| \rightarrow 0 \) as \( t \rightarrow T \).

**4 | FIXED-TIME OBSERVER DESIGN**

Since the state of the system is usually not completely measurable, we design an observer to estimate the state of the original system (1)–(3) in a prescribed time. To this end, we consider anti-collocated case that the sensor and actuator are deployed at the opposite boundary of the spatial domain, then the following observer is established
\[ \hat{z}(s,t) = \varepsilon \hat{z}_0(s,t) + \lambda(t) \hat{z}(s,t) + \eta(s,t)[z(s,0) - \hat{z}_0(t)], \]
\[ \hat{z}_0(s,t) \in (0, 1) \times [0, T], \] (34)
\[ \hat{z}_0(t) = p_1(t)[z(s,0) - \hat{z}(0,t)], \quad t \in [0, T], \] (35)
\[ z_1(t, \tau) = u(t), \quad t \in [0, T), \]  

with initial condition
\[ \hat{z}_1(t, 0) = \hat{z}_0(\tau), \quad s \in [0, 1], \]  

where \( \hat{z}_1(s, t) \) is state of the observer system \((34)–(36)\) and \( z_1(t, 0) \) is measurement output signal. The sensor and actuator are deployed at the opposite boundary of the spatial domain. The observer gains \( \hat{p}(s, t) \) and \( \hat{p}_1(\tau) \) will be determined by backstepping method. The observer error \( \hat{z}_1(s, t) = z_1(s, t) - \hat{z}_1(s, t) \) satisfies the following error system
\[ \dot{\hat{z}}_1(s, t) = \xi \hat{z}_1(s, t) + \mathcal{A}(s) \hat{z}_1(s, t) - \hat{p}(s, t) \hat{z}_1(0, t), \quad (s, t) \in (0, 1) \times [0, T), \]  

\[ \hat{z}_1(0, t) = -p_1(\tau) \hat{z}_1(0, t), \quad t \in [0, T), \]  

\[ \hat{z}_1(1, t) = 0, \quad t \in [0, T), \]  

with initial condition
\[ \hat{z}_1(t, 0) = \hat{z}_0(\tau), \quad s \in [0, 1]. \]

In what follows, we introduce the following backstepping transformation
\[ \bar{z}_1(s, t) = \bar{u}(s, t) - \int_0^t q(s, y, t) \bar{u}(y, t) \, dy, \quad (s, t) \in \mathbb{H}, \]  

with time varying gain kernel \( q(s, y, t) \) to maps the error system \((38)–(40)\) into a suitable target system
\[ \bar{z}_1(s, t) = \xi \bar{u}_1(s, t) - \bar{c}(t) \bar{u}(s, t), \quad (s, t) \in (0, 1) \times [0, T), \]  

\[ \bar{u}_1(0, t) = 0, \quad t \in [0, T), \]  

\[ \bar{u}(1, t) = 0, \quad t \in [0, T), \]  

with initial condition
\[ \bar{u}(s, 0) = \bar{u}_0(\tau), \quad s \in [0, 1], \]  

where positive definite real analytic function \( \bar{c}(t) \in C^\infty([0, 1]) \) is designed as follows
\[ \bar{c}(t) = \bar{p}(t) - \frac{\tilde{c}_{\alpha_0} T}{T - t} > 0, \tilde{c}_{\alpha_0} > 0, \quad t \in [0, T), \]  

where \( \bar{p}(t) \) is defined by
\[ \bar{p}(t) = \frac{\bar{p}_0^2 T^2}{(T - t)^2}, \quad \bar{p}_0 > 0, \quad t \in [0, T) \]  

with \( \bar{p}(0) = \bar{p}_0^2 = \tilde{c}_{\alpha_0} \) is monotonically increasing function and having the property that \( \bar{p}(T) = +\infty \). To calculate the observer gains \( \hat{p}(s, t) \) and \( \hat{p}_1(\tau) \), the gain kernel \( q(s, y, t) \) needs to be solved. In order to realise this, taking the time derivative and the space derivative with respect to the integral transformation \((42)\) and using the observer error system \((38)–(40)\) and the target system \((43)–(45)\), then, kernel PDEs are given by
\[ \frac{d}{ds} q(s, y, t) = \frac{1}{2\xi} \gamma_1(s, t), \quad s \in [0, 1], t \in [0, T), \]  

\[ q(1, y, t) = 0, \quad y \in [0, 1], t \in [0, T). \]  

where \( \gamma_1(s, t) = \mathcal{A}(s) + \bar{c}(t) \) is a smooth scalar function defined on the domain \( \mathbb{H} \), and the observer gains are given by
\[ p_1(\tau) = q(0, 0, t), \quad t \in [0, T), \]  

\[ p(s, t) = \xi q(s, 0, t), \quad s \in [0, 1], t \in [0, T). \]  

The gain kernel \( q(s, y, t) \) is defined on the domain \( \mathbb{D} \). The well-posedness of the kernel PDEs \((49)–(51)\) is presented in the following lemma.

**Lemma 3.** If the function \( \gamma_1(s, t) \in C^2([0, 1]) \times C^\infty([0, T]) \) satisfies Assumption 1, then the kernel PDEs \((49)–(51)\) has a unique solution \( q(s, y, t) \) defined on the domain \( \mathbb{H} \), which is infinitely continuously differentiable with respect to time \( t \). For all \( 0 \leq y \leq s \leq 1, 0 \leq t \leq T^*, \) the solution \( q(s, y, t) \) is bounded by
\[ |q(s, y, t)| \leq \frac{D(1 - s)}{2\xi} \frac{\mathcal{M}_2}{e^{\psi((1-x)^2-(1-y)^2)}}. \]  

Furthermore, for all \( 0 \leq y \leq s \leq 1, 0 \leq t \leq T^* \), there exists a constant \( M_2 > 0 \) such that \( |q(s, y, t)| \leq M_2 \).

The proof of Lemma 3 can be referred to the proof method of Lemma 1 in the Appendix and [32, Theorem 8]. In terms of Lemma 3 and transformation \((42)\), we can get
\[ \|\bar{u}(0, t)\|_{L^2} \leq (1 + M_2) \|\bar{z}_1(0, t)\|_{L^2}, \quad t \in [0, T). \]  

The inverse transformation of the integral transformation \((42)\) has the following form
\[ \bar{u}(s, t) = \bar{z}_1(s, t) + \int_0^t r(s, y, t) \bar{z}_1(y, t) \, dy, \quad (s, t) \in \mathbb{H}, \]  

with time dependent gain kernel \( r(s, y, t) \) is defined on the domain \( \mathbb{D} \). The gain kernel \( q(s, y, t) \) of the integral transformation \((42)\) and gain kernel \( r(s, y, t) \) of the inverse transformation
Theorem 3. For a given real analytic function \( \bar{z}(t) \) in (47), if there exists a monotonically increasing function \( \bar{p}(t) \) with properties \( \bar{p}(0) = \bar{p}_0 \) and \( \bar{p}(T) = +\infty \) in (48) that satisfies

\[
\|\bar{w}(s,t)\|_{L^2} \leq \bar{p}(T) \|\bar{z}(s,0)\|_{L^2}, \quad t \in [0, T),
\]

where \( \bar{z}(t) = e^{\bar{z}(t)^T \sqrt{\bar{z}(t)^T \bar{z}(t)}} \) is a monotonically decreasing bump-like function, then for arbitrary initial value \( \bar{z}_0(s) \in L^2 \), the target system (43)–(45) is fixed-time stable in the prescribed time \( T \).

Since it is similar to Theorem 1, the proof of this theorem is omitted.

Theorem 4. If there exists a monotonically increasing function \( \bar{p}(t) \) with properties \( \bar{p}(0) = \bar{p}_0 \) and \( \bar{p}(T) = +\infty \) that satisfies

\[
\|\bar{w}(s,t)\|_{L^2} \leq (1 + M_2)^{\bar{p}(T)} \sqrt{\bar{z}(t)^T \bar{z}(s,0)\|_{L^2}}, \quad t \in [0, T),
\]

where \( \bar{z}(t) = e^{\bar{z}(t)^T \sqrt{\bar{z}(t)^T \bar{z}(t)}} \) is a monotonically decreasing bump-like function, then for arbitrary initial value \( \bar{z}_0(s) \in L^2 \), the error system (38)–(40) is fixed-time stable in the prescribed time \( T \).

Similarly to the proof of Theorem 2, this theorem is proved by conclusion of Theorem 3 combining with Lemma 3 and Lemma 4. This theorem implies that the observation state converges to the actual state in the prescribed time \( T \).

5 | OBSERVER-BASED OUTPUT FEEDBACK CONTROLLER DESIGN

The state feedback controller require full state of the system (1)–(3), which is usually unavailable. Therefore, the observer-based output feedback controller is established by using separation principle combining with the observer (34)–(36) and state feedback controller. In this section, the fixed-time stability of the system (1)–(3) is proved based on the observer-based output feedback controller. In order to prove the above result, one can introduce invertible integral transformations

\[
\hat{w}(s,t) = \bar{z}(s,t) - \int_0^s k(s,y,t) \bar{z}(y,t) dy, \quad (s,t) \in \mathbb{H},
\]

(63)

\[
\hat{z}(s,t) = \hat{w}(s,t) + \int_0^s l(s,y,t) \bar{z}(y,t) dy, \quad (s,t) \in \mathbb{H}.
\]

(64)

The invertible integral transformation (63) is to convert the observer (34)–(36) into the suitable target system

\[
\tilde{w}_r(s,t) = \tilde{e} \tilde{w}_r(s,t) - \tilde{c}(t) \tilde{z}(s,t) + \{ \tilde{p}(s,t) - \int_0^s \tilde{l}(s,y,t) \}
\]

\[
\bar{p}(y,t) dy \tilde{w}(0,t), \quad (s,t) \in (0,1) \times [0,T),
\]

(65)

\[
\hat{w}_r(0,t) = \hat{p}(t) \tilde{w}(0,t), \quad t \in [0, T),
\]

(66)
\[ \hat{w}(1, t) = 0, \quad t \in [0, T), \]  
\[ \hat{w}(s, 0) = \hat{w}_0(s), \quad s \in [0, 1]. \]  

with initial condition

Consider a Lyapunov function

\[ V(t) = \frac{1}{2} \| \hat{w}(s, t) \|_{L^2}^2 + B \| \tilde{w}(s, t) \|_{L^2}^2 \]

\[ = \frac{1}{2} \int_0^1 \hat{w}^2(s, t) \, ds + B \int_0^1 \tilde{w}^2(s, t) \, ds, \quad t \in [0, T), \]  

where \( B = 2(\varepsilon^2 \rho_{10}^2 + A^2) \) with

\[ A = \sup_{(s, t) \in [0, 1] \times [0, T)} \left( p(s, t) - \int_0^t k(s, y, t) p(y, t) \, dy \right), \]  

\[ \rho_{10} = \sup_{t \in [0, T]} p_1(t). \]  

Calculating the time derivative for the Lyapunov function \( V(t) \) along the trajectory of the cascade systems (43)–(45) and (65)–(67) yields

\[ \dot{V}(t) = \varepsilon \int_0^1 \hat{w}(s, t) \hat{w}_0(s, t) \, ds - \varepsilon(t) \int_0^1 \hat{w}^2(s, t) \, ds \]

\[ + \int_0^1 \hat{w}(s, t) \left( p(s, t) - \int_0^t k(s, y, t) p(y, t) \, dy \right) \tilde{w}(0, t) \, ds \]

\[ + \varepsilon B \int_0^1 \tilde{w}(s, t) \tilde{w}_0(s, t) \, ds - B \varepsilon(t) \int_0^1 \tilde{w}^2(s, t) \, ds. \]  

Applying the integration by parts and the boundary conditions (44)–(45), (66)–(67), we have

\[ \dot{V}(t) \leq -\varepsilon p_1(t) \hat{w}(0, t) \bar{w}(0, t) - \varepsilon \int_0^1 \hat{w}_0^2(s, t) \, ds \]

\[ - \varepsilon(t) \int_0^1 \hat{w}^2(s, t) \, ds - B \varepsilon(t) \int_0^1 \tilde{w}^2(s, t) \, ds \]

\[ + \int_0^1 \hat{w}(s, t) \left( p(s, t) - \int_0^t k(s, y, t) p(y, t) \, dy \right) \bar{w}(0, t) \, ds \]

\[ - \varepsilon B \int_0^1 \tilde{w}^2(s, t) \, ds. \]  

We can deduce that \( p_1(t) \) and \( p(s, t) \) are bounded by means of (52) and (53) due to the boundedness of unique solution \( p(s, y, t) \) of the kernel PDEs (49)–(51). Therefore, the first term and the fourth term of inequality (74) are estimated by utilising the Young’s inequality [41] and the Cauchy-Schwarz inequality [41], which derive

\[ -\varepsilon p_1(t) \hat{w}(0, t) \bar{w}(0, t) \]

\[ \leq \frac{1}{4} \int_0^1 \hat{w}_0^2(s, t) \, ds + \varepsilon^2 \rho_{10}^2 \int_0^1 \tilde{w}_0^2(s, t) \, ds, \]  

and

\[ \hat{w}(0, t) \int_0^1 \hat{w}(s, t) \left( p(s, t) - \int_0^t k(s, y, t) p(y, t) \, dy \right) \, ds \]

\[ \leq \frac{1}{4} \int_0^1 \hat{w}_0^2(s, t) \, ds + A^2 \int_0^1 \tilde{w}_0^2(s, t) \, ds, \]  

where \( p_{10} \) and \( A \) are shown by (72) and (73), respectively. Substituting the inequalities (75) and (76) into (74), we have

\[ \dot{V}(t) \leq \left( \frac{1}{2} - \varepsilon \right) \int_0^1 \hat{w}_0^2(s, t) \, ds - \varepsilon(t) \int_0^1 \hat{w}^2(s, t) \, ds \]

\[ + (\varepsilon^2 \rho_{10}^2 + A^2 - \varepsilon B) \int_0^1 \tilde{w}_0^2(s, t) \, ds \]

\[ - B \varepsilon(t) \int_0^1 \tilde{w}^2(s, t) \, ds. \]  

Using the Wirtinger’s inequality [41] and \( B = 2(\varepsilon^2 \rho_{10}^2 + A^2) \), we
obtain
\[ V'(t) \leq -\frac{1}{4} \left( \varepsilon \pi^2 - \frac{1}{2} \pi^2 + 4\varepsilon(t) \right) \int_0^1 \hat{w}^2(s, t) \, ds - \frac{1}{4} B \left( \varepsilon \pi^2 - \frac{1}{2} \pi^2 + 4\varepsilon(t) \right) \int_0^1 \hat{w}^2(s, t) \, ds \]
\[ = -\frac{1}{2} \left( \varepsilon \pi^2 - \frac{1}{2} \pi^2 + 4\varepsilon(t) \right) V'(t), \]  
where
\[ \hat{\gamma}(t) = \hat{\gamma}_0 \frac{T^2}{(T-t)^2} - \frac{2 \hat{\gamma}_0 t}{T} = \hat{\rho}(t) - \frac{2 \hat{\gamma}_0 t}{T}, \quad t \in [0, T), \]  
with \( \hat{\gamma}_0 = \min\{\gamma_0, \tilde{\gamma}_0\} \). The condition \( \hat{\gamma}_0 > \left( \frac{1}{8} - \frac{1}{4} \varepsilon \right) \pi^2 \) implies \( \varepsilon \pi^2 - \frac{1}{2} \pi^2 + 4\varepsilon(t) > 0 \).

Next, by integrating (78) with respect to \( t \) from 0 to \( t \) and exploiting (79), we have
\[ V'(t) \leq e^{-\frac{1}{2} \int_0^t \left( \frac{\varepsilon \pi^2 - \frac{1}{2} \pi^2 + 4\varepsilon(s)}{s} \right) \, ds} V(0) \]
\[ \leq e^{-\frac{1}{2} \int_0^t \left( \frac{\hat{\rho}(s)}{s} \right) \, ds} \left( \frac{\hat{\gamma}_0 + \frac{\varepsilon^2}{4}}{t} \right) V(0). \]

In addition, denoting \( \hat{\xi}(t) = e^{-\frac{1}{2} \int_0^t \left( \frac{\hat{\rho}(s)}{s} \right) \, ds} \), we get a monotonically decreasing function
\[ \hat{\xi}(t) = \frac{\hat{\gamma}_0 + \frac{\varepsilon^2}{4}}{t} e^{\int_0^t \left( \frac{\hat{\rho}(s)}{s} \right) \, ds}, \]  
which possess the properties \( \hat{\xi}(0) = 1 \) and \( \hat{\xi}(T) = 0 \). It is called as smooth bump-like function. Then, for all \( t \in [0, T) \), we have
\[ V'(t) \leq \hat{\xi}(t) e^{\int_0^t \left( \frac{\hat{\gamma}_0 + \frac{\varepsilon^2}{4}}{s} \right) \, ds} V(0). \]

Therefore, in the light of above inequality and (71), the following inequality is derived by
\[ \| \hat{w}(s, t) \|_{L^2} + \| \tilde{w}(s, t) \|_{L^2} \]
\[ \leq \sqrt{\hat{\xi}(t) e^{\int_0^t \left( \frac{\hat{\gamma}_0 + \frac{\varepsilon^2}{4}}{s} \right) \, ds} \left( \| \hat{w}(s, 0) \|_{L^2} + \| \tilde{w}(s, 0) \|_{L^2} \right). \]

which can deduce that \( \| \hat{w}(s, t) \|_{L^2} + \| \tilde{w}(s, t) \|_{L^2} \to 0 \) as \( t \to T \) by means of (80). Then based on Definition 1, we can claim that the cascade target system \((\hat{w}, \tilde{w})\) is fixed-time stable.

\[ \boxdot \]

**Theorem 6.** If there exists a monotonically increasing function \( \hat{\rho}(t) = \frac{\gamma_0 T^2}{(T-t)^2} \) with initial value \( \hat{\gamma}_0 = \min\{\gamma_0, \tilde{\gamma}_0\} \) that satisfies the condition
\[ \hat{\gamma}_0 > \left( \frac{1}{8} - \frac{1}{4} \varepsilon \right) \pi^2 \] such that for all \( t \in [0, T) \),
\[ \| \hat{\xi}(s, t) \|_{L^2} + \| \tilde{\xi}(s, t) \|_{L^2} \]
\[ \leq (1 + M^2) \left( \frac{\hat{\gamma}_0 + \frac{\varepsilon^2}{8}}{8} \right)^T \left( \| \hat{\xi}(s, 0) \|_{L^2} + \| \tilde{\xi}(s, 0) \|_{L^2} \right), \]  
with \( M = \max\{M_1, M_2\} \), \( \hat{\xi}_0 = \min\{\hat{\gamma}_0, \tilde{\gamma}_0\} \) and \( \hat{\xi}(t) = \frac{\gamma_0 T^2}{(T-t)^2} e^{\int_0^t \left( \frac{\hat{\gamma}_0 + \frac{\varepsilon^2}{8}}{8} \right) \, ds} \), then for arbitrary initial values \( \hat{\xi}_0(t) \in L^2 \) and \( \tilde{\xi}_0(t) \in L^2 \), the closed-loop system (1)–(3) with observer-based output feedback controller
\[ u(t) = \int_0^1 k(z, y, t, z) \, dz, \quad t \in [0, T), \]  
is fixed-time stable, where \( \hat{\xi}(s, t) \) is state of the observer system (34)–(36). Furthermore, \( |u(t)| \to 0 \) as \( t \to T \).

**Proof.** The cascade target system \((\hat{w}, \tilde{w})\) is fixed-time stable such that
\[ \| \hat{w}(s, t) \|_{L^2} + \| \tilde{w}(s, t) \|_{L^2} \]
\[ \leq \sqrt{\hat{\xi}(t) e^{\int_0^t \left( \frac{\hat{\gamma}_0 + \frac{\varepsilon^2}{8}}{8} \right) \, ds} \left( \| \hat{w}(s, 0) \|_{L^2} + \| \tilde{w}(s, 0) \|_{L^2} \right). \]  
Employing Lemma 1, Lemma 2 and the transformations (63), (64), we have that
\[ \| \hat{w}(0, t) \|_{L^2} \leq (1 + M_1) \| \hat{\xi}(0, t) \|_{L^2}, \quad t \in [0, T). \]  
\[ \| \tilde{\xi}(s, t) \|_{L^2} \leq (1 + M_1) \| \tilde{\xi}(s, t) \|_{L^2}, \quad t \in [0, T). \]  
Applying the inequalities (85), (62) and (83), one can get that
\[ \| \hat{\xi}(s, t) \|_{L^2} + \| \tilde{\xi}(s, t) \|_{L^2} \]
\[ \leq (1 + M) \| \hat{w}(s, t) \|_{L^2} + (1 + M_2) \| \tilde{w}(s, t) \|_{L^2} \]
\[ \leq (1 + M) \sqrt{\hat{\xi}(t) e^{\int_0^t \left( \frac{\hat{\gamma}_0 + \frac{\varepsilon^2}{8}}{8} \right) \, ds} \left( \| \hat{w}(s, 0) \|_{L^2} + \| \tilde{w}(s, 0) \|_{L^2} \right), \]
which can deduce that \( \| \hat{w}(s, t) \|_{L^2} + \| \tilde{w}(s, t) \|_{L^2} \to 0 \) as \( t \to T \) by means of (80). Then based on Definition 1, we can claim that the cascade target system \((\hat{w}, \tilde{w})\) is fixed-time stable.

In terms of (80) and (81), we can compute that \( \| \hat{\xi}(s, t) \|_{L^2} + \| \tilde{\xi}(s, t) \|_{L^2} \to 0 \) as \( t \to T \). Then based on the Definition 1, it is easily obtain that the cascade system \((\hat{\xi}, \tilde{\xi})\) consisting of the observer (34)–(37) and the error system (38)–(41) is fixed-time stable. In other words, under the action of the observer-based output feedback controller, the system is fixed-time stable.
within the prescribed time $T$. Moreover, the output feedback controller $u(t)$ converges to zero in fixed-time $T$. Due to the inverse backstepping transformation (64), the observer-based output feedback controller (82) can equivalently be indicated as follows,

$$u(t) = \int_{0}^{1} l(s, y(t)) \hat{w}(y(t)) \, dy, \quad t \in [0, T).$$

In view of Lemma 2 and the inequalities (83), (84), we obtain that

$$|u(t)| \leq M_1 \| \hat{w}(s, t) \|_{L^2} \leq M_1 (1 + M_1) \sqrt{\xi(t) \epsilon_0 + \frac{\epsilon^2}{\tau}} \| \hat{z}(s, 0) \|_{L^2}.$$

In virtue of above inequality and properties of the bump-like function, we deduce that $|u(t)| \to 0$ as $t \to T$. Therefore, the control input $u(t)$ is bounded in the domain $t \in [0, T)$. □

6 | NUMERICAL SIMULATION

A numerical example is exhibited to verify the validity of the obtained results. Consider a thermal conduction process of one dimensional inhomogeneous rod can be modeled by the following parabolic distributed parameter system with space-dependent reactivity

$$\frac{\partial \zeta}{\partial t} = \zeta_{ss} + (9 + 2t(2 - s)) \zeta, \quad \zeta(0, s) = 0, \quad \zeta(1, t) = u(t), \quad t \in [0, 2), \quad s \in (0, 1) \times [0, 2),$$

(86)

with initial condition

$$\zeta(s, 0) = 3 + 2 \cos(2\pi s), \quad s \in [0, 1].$$

6.1 | Case of state feedback controller

The parameter of the target system (7)–(9) is taken to be $\epsilon(t) = \frac{\tau^2}{(T - t)^2} - \frac{2\nu}{T}, \epsilon(0) = 1, T = 2$. Then, by Theorem 2, the closed-loop system with state feedback controller is fixed-time stable within the prescribed time $T = 2$. The analytic solution of the kernel PDEs (15)–(18) cannot be solved, therefore, we use the numerical method to calculate the numerical solution of the time varying gain kernel function $\hat{k}(s, s, t)$. The Ablowitz-Kruskal-Ladik [30] scheme is modified to solve the numerical solution of the kernel PDEs (15)–(18). Firstly, the numerical stationary solution $k(s, s, 0)$ of the kernel PDEs (15)–(18) at $t = 0$ is solved. The space $0 \leq y \leq s \leq 1$ is discretised with $s_i = (i - 1)h, i = 1, \ldots, M + 1, y_j = (j - 1)h, j = 1, 2, \ldots, i, b = 1/M, M$ is the number of steps for the space variable. Denoting $y_i = y_i(0, 0), k_{i,j} = k(y_i, y_j)$. Then, for $t = 0$, the numerical stationary solution of the kernel PDEs (15)–(18) is given by

$$k_{i+1,j} = \left(\frac{\nu^2}{\epsilon}y_j + 2\right) \hat{k}_{i,j} - k_{i-1,j}, \quad k_{i,0} = k_{i,2}, \quad k_{i,1} = 0,$$

(90)

$$k_{i+1,i+1} = \frac{b}{4\epsilon} (y_{i+1} + y_i),$$

(91)

$$k_{i,0} = k_{i,2},$$

(92)

where $\hat{k}_{i,j} = (k_{i+1,j} + k_{i,j-1})/2$ and $\hat{k}_{i,j} \approx k_{i,j}, i = 2, \ldots, M + 1, j = 2, \ldots, i$. Next, the time $0 \leq t \leq T$ is discretised with $t_n = \frac{T}{n_1}, n = 1, \ldots, N + 1, N$ is the number of steps for the time variable. Denoting $\hat{k}^n_{i,j} = k(y_i, y_j, t_n), y^0_i = y(y_i, t_n)$. Thus, the kernel PDEs (15)–(18) are approximated as follows

$$k^\gamma_{i+1,j} = \frac{\epsilon}{\nu^2} \left(1 + \frac{2\nu}{\epsilon^2} \xi^\gamma_j \right) \hat{k}^\gamma_{i,j} - k_{i-1,j},$$

(93)

$$k^\gamma_{i+1,i+1} = \frac{b}{4\epsilon} (y^\gamma_{i+1} + y^\gamma_i),$$

(94)

$$k^\gamma_{i,0} = k_{i,2},$$

(95)

where $\hat{k}^\gamma_{i,j} = (k^\gamma_{i+1,j} + k^\gamma_{i,j-1})/2, k^\gamma_{i,j} \approx k^\gamma_{i,j}, i = 2, \ldots, M + 1, j = 2, \ldots, i, n = 1, \ldots, N + 1$. Then, we can obtain the numerical solution of the kernel PDEs (15)–(18). Thus, the gain of the state feedback controller is given by the numerical solution for time varying kernel function $k(s, s, t)$. The state evolution of the open-loop system (86)–(88) ($u(t) = 0$) is depicted in Figure 1, which is unstable. Figure 2 describes the state evolution of the closed-loop system with state feedback controller which is fixed-time stable. Figure 3 shows the time evolution.
of $L^2$-norm of state of the closed-loop system (86)–(88) with state feedback controller $u(t) = \int_0^1 k(1, y, t) z(y, t) \, dy$ drawn in logarithmic scale to better illuminate that the state of closed-loop system converges to zero in a prescribed time $T = 2$.\[ \| z(s, t) \|_{L^2} = 0.0008 \] as $t = 1.5$ for fixed-time stabilisation and\[ \| z(s, t) \|_{L^2} = 0.0135 \] as $t = 1.5$ for exponential stabilisation. It is observed that the convergence time of fixed-time stabilisation of system (blue solid line) is faster than the exponential stabilisation (dashed red line) and state of the system converges to zero within the prescribed time $T = 2$. Figure 4 depicts the state feedback controller $u(t) = \int_0^1 k(1, y, t) \hat{z}(y, t) \, dy$ and it indicates that the control input $u(t)$ is bounded in the domain $t \in [0, T)$.

### 6.2 Case of output feedback controller

The initial condition of the state observer is $\tilde{z}(s, 0) = 0$. The observer gains $p_1(t)$ and $p(s, t)$ are derived as given in (52) and (53). The parameter of the target error system (43)–(45) is taken to be $\hat{e}(t) = \frac{1.2 T^2}{(T-t)^2} - \frac{2.4}{T}$, $\hat{e}(0) = 1.2$. Therefore, by Theorem 4 and Theorem 6, the error system (38)–(40) and the closed-loop system (86)–(87) with observer-based output feedback controller are fixed-time stable within the prescribed time $T = 2$, respectively. The numerical solution of the kernel PDEs (49)–(51) is obtained by the numerical method similar to the kernel PDEs (15)–(18). Then, in the light of (52) and (53), the observer gains are determined by the numerical solution of the time varying gain kernel function $q(s, y, t)$. Figure 5 depicts the state evolution process of the closed-loop system (86)–(87) equipped with observer-based output feedback controller $u(t) = \int_0^1 k(1, y, t) \tilde{z}(y, t) \, dy$. It is observed that the system is fixed-time stable. Figure 6 shows the $L^2$-norms of the system (86)–(88), the observer and the error system. It can be observed that $\| \tilde{z}(s, t) \|_{L^2}$ of the observer converges to $\| z(s, t) \|_{L^2}$ of the system (86)–(88) along with $\| \tilde{z}(s, t) \|_{L^2}$ of the observer error system converges to zero within the prescribed time $T = 2$. Figure 7 shows the time evolution process of $L^2$-norms of states of the closed-loop system and observer error system drawn in logarithmic scale to better illustrate that the fixed-time...
stabilisation of the closed-loop system within the prescribed time $T = 2$. $\|z(s, t)\|_{L^2} = 0.0001$ and $\|\tilde{z}(s, t)\|_{L^2} = 0.0064$ as $t = 1.5$ for fixed-time stabilisation and $\|z(s, t)\|_{L^2} = 0.0099$ and $\|\tilde{z}(s, t)\|_{L^2} = 0.0171$ as $t = 1.5$ for exponential stabilisation. It is seen that the convergence time of $\|z(s, t)\|_{L^2}$ (blue solid line) and $\|\tilde{z}(s, t)\|_{L^2}$ (cyan dashed line) for fixed-time stabilisation is faster than the exponential stabilisation (violet solid line and red dashed line), respectively. Figure 8 shows the observer-based output feedback controller $u(t) = \int_0^1 k(1, y, t)\hat{z}(y, t)\,dy$ and it demonstrates that the control input $u(t)$ is bounded in the domain $t \in [0, T]$.

## 7 CONCLUSION

In this article, the fixed-time stability problem of the linear parabolic distributed parameter system with space-dependent reactivity based on boundary feedback control has been analyzed. The observer has been established to estimate the state of the system (1)–(3) in the light of obtained measurement information at the boundary. By utilising the backstepping approach, the invertible Volterra integral transformation with time-varying gain kernel has been considered to convert the boundary controlled system into a fixed-time stable target system. The well-posedness of the resulting kernel PDEs has been proved by the method of successive approximation. On the basis of the state feedback controller and the observer-based output feedback controller, the fixed-time stability of the system (1)–(3) within the prescribed time $T$ has been analyzed by using the Lyapunov method. In addition, the output feedback controller has been established by the combining of the observer and state feedback controller based on separation principle. Then, the numerical solutions of the kernel PDEs have been solved by applying the modified Ablowitz-Kruskal-Ladik scheme to derive the gain of the controller and the observer gains. Finally, numerical simulation has been provided to illustrate effectiveness of the obtained results.

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APPENDIX A

A.1 | Proof of Lemma 1

According to [29], the well-posedness of the kernel PDEs (15)–(18) is proved by the method of successive approximation. First, we introduce a change of variables $\zeta = s + \gamma, \eta = s - \gamma$ and denote $k(s, y, t) = G(\zeta, \eta, t)$. Thus, the kernel PDEs (15)–(18) are transformed to the following PDEs

$$4\varepsilon G_{\eta\eta}(\zeta, \eta, t) = G_s(\zeta, \eta, t) + \gamma \left( \frac{\zeta - \eta}{2} \right) G(\zeta, \eta, t), \quad (A.1)$$

$$G_s(\zeta, 0, t) = -\frac{1}{4\varepsilon} \gamma \left( \frac{\zeta}{2} \right)^2, \quad (A.2)$$

$$G_{ss}(\zeta, \zeta, t) - G(\zeta, \zeta, t) = 0, \quad (A.3)$$

$$G(0, 0, t) = 0, \quad t \in [0, T), \quad (A.4)$$

where $G(\zeta, \eta, t)$ is defined on the domain $[\eta, 2 - \eta] \times [0, 1] \times [0, T)$. Next, the equations (A.1)–(A.4) are transformed to an integral equation to determine the solution $G(\zeta, \eta, t)$. Integrating (A.1) with respect to $\eta$ and $\zeta$, using (A.2)–(A.4), we obtain...
an implicit solution
\[ G(\xi, \eta, t) = \]
\[ = -\frac{1}{2\epsilon} \int_0^\eta \gamma(\frac{t}{2}, t) \, dt + \frac{1}{2\epsilon} \int_0^\eta \int_0^t G_r(\sigma, \tau, t) \, d\sigma \, d\tau \]
\[ + \frac{1}{2\epsilon} \int_0^\eta \int_0^t \gamma(\frac{t-\sigma}{2}, t) G_r(\tau, \sigma, t) \, d\sigma \, d\tau \]
\[ - \frac{1}{4\epsilon} \int_\eta^\xi \gamma(\frac{t}{2}, t) \, dt + \frac{1}{4\epsilon} \int_\eta^\xi \int_0^\eta G_r(\tau, \sigma, t) \, d\sigma \, d\tau \]
\[ + \frac{1}{4\epsilon} \int_\eta^\xi \int_0^\eta \gamma(\frac{t-\sigma}{2}, t) G_r(\tau, \sigma, t) \, d\sigma \, d\tau. \]  
(A.5)

To prove the solvability of the integral equation (A.5), the method of successive approximation is adopted. Let us assume \( G^n(\xi, \eta, t) = 0 \) and construct the recursive formula for integral equation (A.5) as follows
\[ G^n(\xi, \eta, t) = \]
\[ = -\frac{1}{2\epsilon} \int_0^\eta \gamma(\frac{t}{2}, t) \, dt + \frac{1}{2\epsilon} \int_0^\eta \int_0^t G_{n-1}(\tau, \sigma, t) \, d\sigma \, d\tau \]
\[ + \frac{1}{2\epsilon} \int_0^\eta \int_0^t \gamma(\frac{t-\sigma}{2}, t) G_{n-1}(\tau, \sigma, t) \, d\sigma \, d\tau \]
\[ - \frac{1}{4\epsilon} \int_\eta^\xi \gamma(\frac{t}{2}, t) \, dt + \frac{1}{4\epsilon} \int_\eta^\xi \int_0^\eta G_{n-1}(\tau, \sigma, t) \, d\sigma \, d\tau \]
\[ + \frac{1}{4\epsilon} \int_\eta^\xi \int_0^\eta \gamma(\frac{t-\sigma}{2}, t) G_{n-1}(\tau, \sigma, t) \, d\sigma \, d\tau. \]  
(A.6)

If the function sequence \( \{ G^n(\xi, \eta, t) \} \) is uniformly and absolutely convergent, then, for \( n \in \mathbb{N} \), there exist a limit \( G(\xi, \eta, t) \) such that
\[ G(\xi, \eta, t) = \lim_{n \to \infty} G^n(\xi, \eta, t). \]  
(A.6)

To prove the function sequence \( \{ G^n(\xi, \eta, t) \} \) is uniformly and absolutely convergent, the following infinite series is introduced by
\[ \sum_{i=1}^{\infty} \Delta G^n(\xi, \eta, t), \quad n \in \mathbb{N}. \]  
(A.7)

where \( \Delta G^n(\xi, \eta, t) = G^n(\xi, \eta, t) - G^{n-1}(\xi, \eta, t) \). Thus, for all \( n \in \mathbb{N}, n \geq 1 \), we have
\[ \Delta G^1(\xi, \eta, t) = -\frac{1}{2\epsilon} \int_0^\eta \gamma(\frac{t}{2}, t) \, dt - \frac{1}{4\epsilon} \int_\eta^\xi \gamma(\frac{t}{2}, t) \, dt. \]  
(A.8)

\[ \Delta G^n(\xi, \eta, t) = \frac{1}{2\epsilon} \int_0^\eta \int_0^t \Delta G^{n-1}(\tau, \sigma, t) \, d\tau \, d\sigma \]
\[ + \frac{1}{2\epsilon} \int_0^\eta \int_0^t \gamma(\frac{t-\sigma}{2}, t) \Delta G^{n-1}(\tau, \sigma, t) \, d\tau \, d\sigma \]
\[ + \frac{1}{4\epsilon} \int_\eta^\xi \int_0^\eta \Delta G^{n-1}(\tau, \sigma, t) \, d\tau \, d\sigma \]
\[ + \frac{1}{4\epsilon} \int_\eta^\xi \int_0^\eta \gamma(\frac{t-\sigma}{2}, t) \Delta G^{n-1}(\tau, \sigma, t) \, d\tau \, d\sigma. \]  
(A.9)

In the light of Assumption 1 and (A.8), for all \( (\xi, \eta, t) \in \Omega = [\eta, 2-\eta] \times [0, 1] \times [0, T^*] \), we have the following estimate
\[ \left| \frac{\partial^{i+j}}{\partial \tau^i \eta^j} \Delta G^n(\xi, \eta, t) \right| \leq \frac{\eta}{2\epsilon} D^{i+j}/l! + \frac{1}{4\epsilon} D^{i+j}/l! (\xi - \eta) \]
\[ \leq \frac{1}{4\epsilon} D^{i+j}/l! (\xi + \eta). \]  
(A.10)

We now prove that the series (A.7) converges uniformly and absolutely. For arbitrary \( n \in \mathbb{N} \), we have the following assumption
\[ \left| \frac{\partial^{i+j}}{\partial \tau^i \eta^j} \Delta G^n(\xi, \eta, t) \right| \leq D^{i+j} (\frac{\xi + \eta}{(n+1)!}). \]  
(A.11)

In order to show that (A.11) holds for any positive integer \( n \), using (A.9), we have
\[ \frac{\partial^{i+j}}{\partial \tau^i \eta^j} \Delta G^{n+1}(\xi, \eta, t) \]
\[ \leq \frac{1}{2\epsilon} \left( \int_0^\eta \int_0^t \left| \frac{\partial^{i+j}}{\partial \tau^i \eta^j} \Delta G^n(\tau, \sigma, t) \right| \, d\tau \, d\sigma \right) \]
\[ + \int_0^\eta \int_0^t \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^{i-j}}{\partial \tau^{i-j}} \gamma(\frac{t-\sigma}{2}, t) \left| \frac{\partial^j}{\partial \eta^j} \Delta G^n(\tau, \sigma, t) \right| \, d\tau \, d\sigma \]
\[ + \frac{1}{4\epsilon} \left( \int_\eta^\xi \int_0^\eta \left| \frac{\partial^{i-j}}{\partial \tau^{i-j}} \Delta G^n(\tau, \sigma, t) \right| \, d\tau \, d\sigma \right) \]
\[ + \int_\eta^\xi \int_0^\eta \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^{i-j}}{\partial \tau^{i-j}} \gamma(\frac{t-\sigma}{2}, t) \left| \frac{\partial^j}{\partial \eta^j} \Delta G^n(\tau, \sigma, t) \right| \, d\tau \, d\sigma. \]  
(A.12)

Next, applying (A.11), (A.12) and [29, Lemma 15], we can estimate
\[ \left| \frac{\partial^{i+j}}{\partial \tau^i \eta^j} \Delta G^{n+1}(\xi, \eta, t) \right| \]
\[ \leq \frac{D^{i+j+1}}{(n+1)!} \left( i+j + \sum_{j=0}^{i} \binom{i}{j} j! (l-j+n-1)! \right) \]
Next, we will prove the uniqueness of solution $G(\zeta, \eta, t)$ of the integral equation (A.5). Suppose that $G'(\zeta, \eta, t)$ and $G''(\zeta, \eta, t)$ are two different solutions of the kernel integral equations (A.5). Then $|\delta G(\zeta, \eta, t)| = |G'(\zeta, \eta, t) - G''(\zeta, \eta, t)|$. Applying (A.15), we have

$$|\delta G(\zeta, \eta, t)| \leq 2 \frac{D}{4\varepsilon} (\zeta + \eta)^{1+\varepsilon}. \quad (A.16)$$

Following the same estimates as in (A.13), we further obtain

$$|\delta G(\zeta, \eta, t)| \leq 2 \frac{D}{4\varepsilon} (\zeta + \eta)^{1+\varepsilon} \to 0, \quad n \in \mathbb{N}, n \geq 1, n \to \infty. \quad (A.17)$$

Therefore, we claim that the integral equations (A.5) has a unique solution in the domain $\Omega$. Due to the continuation theorem of the solution, the unique solution $G(\zeta, \eta, t)$ exists in the domain $[\eta, 2-\eta] \times [0,1] \times [0, T]$. This means that the equations (A.1)–(A.4) has a uniqueness solution in the domain $[\eta, 2-\eta] \times [0,1] \times [0, T^*]$. In summary, the kernel PDEs (15)–(18) has a unique solution which is bounded in the domain $[0,1] \times [0, a] \times [0, T^*]$. In the light of (A.15) and $k(s, y, \tau) = G(\zeta, \eta, t)$, for all $(s, y, \tau) \in [0,1] \times [0, a] \times [0, T^*]$, there exist a positive constant $M_1$ such that $|k(s, y, \tau)| \leq \frac{D}{2\varepsilon} e^{\tau} (s^2 - \tau^2) \leq M_1$. The well-posedness of the kernel equations (15)–(18) is only valid up to times arbitrarily close to $T$. In other words, The well-posedness of the kernel equations (15)–(18) is valid in the domain $\mathcal{D}$. 

\[
\frac{1}{2\varepsilon} \int_0^\eta \int_0^\tau (s\tau)^{n-1}(s + \tau) \, ds \, d\tau + \frac{1}{4\varepsilon} \int_0^\eta \int_0^\tau r^n(s\tau)^{n-1} (s + \tau) \, ds \, d\tau
\]

\[
(\sigma + \tau) \, ds \, d\tau
\]

\[
\leq D^{\varepsilon+1} \frac{1}{4\varepsilon} (\zeta + \eta)^{1+\varepsilon} (\zeta + \eta) (\frac{(n+1)!}{(n-1)!})^2.
\]

(A.13)

Therefore, in view of mathematical induction, we can deduce that the assumption (A.11) applies for all $n \in \mathbb{N}$. Further, for all $\zeta \in [\eta, 2-\eta]$ and $\eta \in [0,1]$, the infinite positive series

$$\sum_{n=1}^{\infty} D^{\varepsilon+1} \frac{(\zeta + \eta)^{n-1}}{(4\varepsilon)^n} (\zeta + \eta) (\frac{(n+1)!}{(n-1)!})^2, \quad n \in \mathbb{N}, \quad (A.14)$$

is uniformly and absolutely convergent, therefore the infinite series $\frac{\delta}{\partial \tau} G^{n+1}(\zeta, \eta, t)$ is uniformly and absolutely convergent. By means of (A.11) for $l = 0$, the series (A.7) converges absolutely and uniformly in the domain $\Omega$. Then $G(\zeta, \eta, t)$ is infinitely continuously differentiable with respect to $t$ and bounded in the domain $\Omega$, and for all $(\zeta, \eta, t) \in \Omega$, the upper bound on $G(\zeta, \eta, t)$ is

$$|G(\zeta, \eta, t)| \leq \sum_{n=1}^{\infty} \frac{D^n}{(4\varepsilon)^n} (\zeta + \eta)^{n-1} \frac{1}{(n-1)!} \to 0, \quad n \in \mathbb{N}, n \geq 1, n \to \infty. \quad (A.15)$$