An attempt to study cnoidal and solitary waves in the bloodstream using computer mathematics Maple

Gennady Chuiko, Olga Dvornik, Yevhen Darnapuk*

*Department of Computer Engineering, Petro Mohyla Black Sea National University, 68 Desantnikov St., 10, Mykolayiv, Ukraine, 54003

Abstract

Korteweg-de Vries equation and its modified shape were studied with Maple, a system of computer mathematics. We derived and dealt with their dimensionless forms. The traveling wave type solutions were found in both cases. These waves based on different Jacobi’s elliptic functions. Conditions, formulated for both models from bloodstream description in vessels, are fulfilled regarding these waves. Note, that the traveling waves within both models are similar enough, despite vital diversities found with Maple. First, they have the same periodicity, which depends on the elliptic module \(0 \leq m \leq 1\). Second, they have similar behavior in the harmonic and soliton limits \((m = 0\) and \(m = 1\)). Finally, they have similar dispersion relations.

Keywords: Korteweg-de Vries equation; solitons; bloodstream modeling; traveling waves; Jacobi’s elliptic functions; computer mathematics.

1. INTRODUCTION

The Korteweg-de Vries equation (KdV) [1, 2], as well as its modified form (mKdV), are the matter of this paper. Miura transform relates them [3].

Both are widely used for the 1D-modeling of waves in fluid [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In particular, it may be the waves in blood [4, 6, 7, 10]. They are handy because designed for long waves in nonlinear media with weak dispersion [6, 10].

---

*Corresponding author

Email addresses: gennchuiko@gmail.com (Gennady Chuiko), olga.dvornik@chmnu.edu.ua (Olga Dvornik), yevhen.darnapuk@chmnu.edu.ua (Yevhen Darnapuk)

ORCID: 0000-0001-5590-9404 (Gennady Chuiko), 0000-0002-4545-1599 (Olga Dvornik), 0000-0002-7099-5344 (Yevhen Darnapuk)

1doi: http://dx.doi.org/10.26693/cse2020.01.020
Speaking more exact, we will be dealing here with their exact solutions to traveling wave type. These involve cnoidal as well as solitary waves in the bloodstream [7]. We will be using a system of computer mathematics, we mean Maple [12], to do it.

It has to note, these equations are correct mathematical models only under some additional conditions. Especially, nonlinear Hooke’s law for the blood vessel walls was in use [4, 7]. The count of quadratic correction leads to the KdV [4, 5], while cubic ones in add - to the mKdv model [4].

The neglecting of medium resistance forces and dissipation processes are the necessary conditions for either KdV or mKdv models. The determining factor here is the elastic properties of vessel walls [4, 7].

Periodicity, let even irregular one, is another necessary demand for the pulse waves [7, 8, 9]. Then the cnoidal waves have to be first in the view field. However, they are closely similar to the solitary waves if the elliptic module tends to the maximal value ($m \to 1$) [7]. So, we will select only those cnoidal waves, which ensures the right limit (a soliton) in this case.

Besides, authors [4, 5] have pointed out the limited time ranges within which both models are correct. These ranges correspond to one of the early stages of the cardiac cycle [4]. We believe that it means the systole [7].

On the other hand, people are looking for new exact solutions for both equations. Now among them are virtually all direct and reverse Jacobi elliptic functions as well as its combinations [11]. Not all of them are suitable for bloodstream modeling. Besides, they are not always stable and those are so-called rogue waves [13, 14, 15].

Thus, the main goal of this paper is applying Maple maths to the study of exact solutions of KdV and mKdv equations. We are going to pick up suitable, stable solutions with the right limits at maximal/minimal elliptic module value. One of these limits has to be bell-shaped soliton of KdV, of course.

2. Theory and Methods

2.1. Simplifying and rescaling to the dimensionless forms

Consider the initial equations as:

$$u_t + au_x + \mu_2 u(x, t) u_x + \mu_3 u_{xxx} = 0 \quad (1)$$
$$u_t + au_x + \mu_2 u(x, t)^2 u_x + \mu_3 u_{xxx} = 0 \quad (2)$$

Here:

- $\mu_2, \mu_3$ are constants which defined the nonlinear and dispersive traits of the media;
- lower indexes denote the derivatives of the appropriate orders.
- (1) is KdV equation, (2) is mKdv equation;
- $u(x, t)$ is a relevant unknown function (a solution);
- $a$ is the wave speed (Korteweg-Moens velocity).

Equations (1) and (2) often are simplifying via an elementary transform:

$$\begin{cases}
  t = \tau \\
  x = z + a\tau \\
  u(x, t) = \nu(z, t)
\end{cases} \quad (3)$$
Then equations (1), (2) take the simpler shapes with the new variables:

\begin{align*}
\nu_t + \mu_2 \nu(z, \tau) \nu_z + \mu_3 \nu_{xxx} &= 0 \\
\nu_t + \mu_2 \nu(z, \tau)^2 \nu_z + \mu_3 \nu_{xxx} &= 0
\end{align*}

Nondimensionality is a useful way to present a math model in the most universal shape. Such a model is the same does no matter which system of units is in use. Another utility outcome is a minimal number of variables and/or parameters of the model. Let now apply a bit more complex rescaling to (1), following [10]:

\begin{align*}
\begin{cases}
  t = \sqrt{\frac{\mu_3}{a^3}} \cdot \tau \\
  x = \sqrt{\frac{\mu_3}{a}} \cdot z + a \sqrt{\frac{\mu_3}{a^3}} \cdot \tau \\
  u(x, t) = \frac{6 \cdot \nu(z, t)}{\mu^2}
\end{cases}
\end{align*}

The coefficient 6 in the last relation of (6) is a matter of taste. This is often chosen as one [10]. We recommend the same new variables with a little vary the third relation for the equation (2) too:

\begin{align*}
\begin{cases}
  t = \sqrt{\frac{\mu_3}{a^3}} \cdot \tau \\
  x = \sqrt{\frac{\mu_3}{a}} \cdot z + a \sqrt{\frac{\mu_3}{a^3}} \cdot \tau \\
  u(x, t) = \sqrt{\frac{6}{\mu^2}} \cdot \nu(z, t)
\end{cases}
\end{align*}

Note, the transforms (3), (6), (7), or the ones reversed to them, are easy executable with the command ‘dchange’ from the program package ‘PDEtools’ of Maple. This way one can find dimensionless forms of equations (1) and (2). They are the following:

\begin{align*}
\nu_{zzz} + 6\nu(z, \tau) \cdot \nu_z + \nu_z = 0
\end{align*}

This equation presents KdV model in dimensionless form, and

\begin{align*}
\nu_{zzz} + 6\nu(z, \tau)^2 \cdot \nu_z + \nu_z = 0
\end{align*}

presenting the mKdV model.

2.2. Using of dimensionless forms

Dimensionless forms (8), (9) both are in use in the hemodynamics. Analogous rescaling and the same form for dimensionless KdV (see (8)) were used in [5, 10]. In particular, the spread of KdV soliton in a tube with thin elastic walls after the momentary injection a stroke volume of blood has been considered in [5]. The dimensionless KdV equation, identical to (8), was derived from the momentum conservation law in the bloodstream.
3. Results and Discussions

3.1. KdV equation and its traveling waves solutions

We will search only certain solutions of canonical equations (8) and (9). Our main requires are the following:

- solutions should be non-trivial, with real arguments and parameters;
- solutions should not have a singularity;
- solutions should be traveling waves periodical in time and space;
- solutions should have the real non-zero limits at minimal and maximal elliptic module value \((m \to 0 \text{ and } m \to 1)\).

The looking for the solutions of the traveling wave (TWS) type means the transition to the united phase variable \((\varphi)\), which linear depends on both prior variables \((z, t)\):

\[
\varphi = kz + \omega \tau + \varphi_0
\]

Constants \(k, \omega, \varphi_0\) have here the usual meaning for the traveling wave: dimensionless wavenumber, cyclic frequency, and initial phase.

One can get several possible TWS within Maple, using the special command 'TWSolutions' of the 'PDEtools' package. These are the traveling waves with the phase (10) which built on the base of the following set of Jacobi’s elliptic functions:

\[
\begin{align*}
\{ & \text{sn}(\varphi, m), \text{cn}(\varphi, m), \text{dn}(\varphi, m) \\
& \text{ns}(\varphi, m), \text{nc}(\varphi, m), \text{nd}(\varphi, m) \}
\end{align*}
\]

Where \(0 \leq m \leq 1\) is the elliptic module.

Two functions of this list have singularities (\(\text{nc}(\varphi, m)\) and \(\text{ns}(\varphi, m)\)). One more of them (\(\text{nd}(\varphi, m)\)) has no required limit value under condition \((m \to 1)\). Suitable solutions have the following forms:

\[
\begin{align*}
w_1 &= 2m^2k^2 \text{cn}(\varphi, m)^2 + \frac{4k^3(1 - 2m^2) - \omega}{6k} \\
w_2 &= -2m^2k^2 \text{sn}(\varphi, m)^2 + \frac{4k^3(m^2 - 2) - \omega}{6k} \\
w_3 &= 2k^2 \text{dn}(\varphi, m)^2 + \frac{4k^3(m^2 + 1) - \omega}{6k}
\end{align*}
\]

These fit solutions seem to be different at a glance. However, there are relations well-known for squares of these three elliptic functions [16]:

\[
\begin{align*}
\text{cn}(\varphi, m)^2 &= 1 - \text{sn}(\varphi, m)^2 \\
\text{dn}(\varphi, m)^2 &= 1 - m^2 \text{sn}(\varphi, m)^2
\end{align*}
\]

The applying of these relations to the waves (12) allows proving their equivalency. Thus, that is one and the same traveling wave and the lower index may be omitted below.

\[
w(\varphi, m) = 2m^2k^2 \text{cn}(\varphi, m)^2 - \frac{4k^3(2m^2 - 1) - \omega}{6k}
\]
Consider the soliton limit, setting \( m = 1 \) for this in (14). Jacobi cosine transforms to the hyperbolic secant and we have for the soliton solution:

\[
w(\varphi, 1) = 2k^2 \text{sech}(\varphi)^2 - \frac{4k^3 - \omega}{6k}
\] (15)

The constant part of (15) obviously has to be equal to zero, because of solitons should vanish far away from their maxima (these maxima are located at \( \varphi = 0 \)). Hence we have the determined relation between dimensionless frequency and wavenumber in this case:

\[
\omega = 4k^3
\] (16)

The soliton (15) has the amplitude equal to \( A = 2k^2 \) and its phase velocity \( \frac{\omega}{k} \sim k^2 \sim A \) proportional to the amplitude. That is typical for KdV solitons [2].

Let the soliton amplitude is equal to one: \( A = 1 \). Then \( k = \frac{1}{\sqrt{2}} \) and the cnoidal wave (14) is the function only of the phase variable and the elliptic module taking into mind (16) of course:

\[
w(\varphi, 1) = m^2 \text{cn}(\varphi, m)^2 - \frac{2}{3}(m^2 - 1)
\] (17)

The magnitude of the cnoidal wave (17) depends on the elliptic module parabolically, increasing with them (Fig. 1).

![Fig. 1. The dependence of the dimensionless magnitude of the KdV cnoidal wave (17) on the elliptic module.](image)

One can see the amplitude of the soliton solution \( (m = 1) \) is one and a half times larger than the magnitude of the harmonic solution \( (m = 0) \). The dependence looks fairly weak.

More interesting are profiles of cnoidal waves for different elliptic modules (Fig. 2).

The expression (14) shows the periodic part of the cnoidal wave is equal to zero in the harmonic limit \( (m = 0) \), That is why the harmonic solution looks like the constant on the Fig. 2. The cnoidal waves \( (0 < m < 1) \) are periodic with periodicity \( 2K(m) \) [7]. There \( 2K(m) \) is a well-known elliptic integral of the first kind [16]. The soliton solution, or solitary wave \( (m = 1) \), formally has the infinity period [2, 7, 16].
Fig. 2. Profiles of cnoidal waves for different elliptic modules. Curve 1: \( m = 0 \) (dotted line); curve 2: \( m = 0.75 \) (dashed line); curve 3: \( m = 0.95 \) (dash-dotted line); curve 4: \( m = 1 \) (solid line). The KdV soliton profile is shadowed.

3.2. mKdV equation and its traveling waves solutions

The sole traveling wave, satisfying the conditions of the previous section, in mKdB model has the form:

\[
w'(z, \tau) = k \cdot \text{dn}(kz + k^3(m^2 - 2)\tau + \varphi_0m)
\] (18)

Speaking more precise, there are two such solutions with the same absolute values, which differ only by its sign.

One can see, comparing the waves (18) and (14), the cubic dispersion law having the other coefficient at the mKdV model case (compare with (16)):

\[
\omega = (m^2 - 2) \cdot k^3
\] (19)

Moreover, now the phase of the traveling wave depends on the elliptic module in contrast to the wave (14).

The solitonic limit \((m = 1)\) of the wave (18) also has the specificities (compare with (15)):

\[
w(\varphi, 1) = k \cdot \text{sech}(kz - k^3\tau + \phi_0)
\] (20)

The amplitude of the soliton is proportional to the dimensionless wavenumber \((A = k)\) as shows the formula (20). The shape of mKdV soliton (20) differs from KdV one (see(15)).

We can present the profiles of cnoidal waves as the functions of the dimensionless time \((\tau)\) under conditions \((z = 0, \varphi_0 = 0)\), taking \(k = A = 1\). The computing results for the mKdV model are in Fig. 3. The periodicity of cnoidal waves is unchanged and equal to \(2K(m)\).

4. Conclusions

The dimensionless equations in the framework of the KdV and mKdV models have one solution that is stable to long-wave perturbations (in the sense of [13, 14, 15]). These solutions satisfy
Fig. 3. Profiles of cnoidal waves for different elliptic modules. Curve 1: \( m = 0 \) (dotted line); curve 2: \( m = 0.75 \) (dashed line); curve 3: \( m = 0.95 \) (dash-dotted line); curve 4: \( m = 1 \) (solid line). The mKdV soliton profile is shadowed.

the established conditions: they are periodic, nonsingular, have real arguments and parameters, and nonzero real values for harmonics and solitons limits.

Consider the case of the Korteweg-de Vries model. The traveling wave solution has the form (14) on the base of the quadrat of elliptic Jacobis cosine. The phase velocity of the KdV soliton is proportional to its amplitude that is proportional to the quadrat of dimensionless wavenumber.

Somewhat other situation is within the modified Korteweg-de Vries model. First, the traveling wave is based on another Jacobi elliptic function (\( \text{dn}(\Phi, m) \)) as one can see from expression (17)). Second, the amplitude of soliton is proportional to the wavenumber, not to its quadrat. Besides, the dimensionless dispersion law in this case directly includes the elliptic module(see expression (19)).

Profiles of the solutions of the traveling wave are similar enough for both models despite the diversities, pointed above. It is evident from the comparisons Figures 2 and 3.

5. Authors’ contributions

Gennady Chuiko, Olga Dvornik and Yevhen Darnapuk contributed equally to this study. No patient data or interventions were utilized in this analysis, and thus no ethical issues related to confidentiality of data or related matters were addressed. This work required no subjects to review and utilized no external funding.

All co-authors completed work towards the production of this manuscript, either through writing, the conduct of research, or support/leadership of the programmatic interventions described and evaluated in the manuscript.

6. Conflict of interest statement

No part of this investigation has competing interests.
7. **Acknowledgements**

The authors are grateful for the access to a licensed copy of Maple 18, that was given courtesy in the needful moment by the Associate Professor of Lviv Commercial Academy Dr. Markyan Girnyk.

**References**

[1] D. J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 39 (240) 1895. 422–443. doi: 10.1080/14786449508620739.

[2] K. Brauer, The Korteweg-de Vries Equation: History, exact Solutions, and graphical Representation, 2000, pp. 1–15 [cited 30 Jan 2020]. url: https://www.researchgate.net/publication/2806104_The_Korteweg-de_Vries_Equation_History_exact_Solutions_and_graphical_Representation.

[3] R. M. Miura, Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation, Journal of Mathematical Physics 9 (8) 1968. 1202–1204. doi: 10.1063/1.1664700.

[4] N. A. Kudryashov, I. L. Chernyavskii, Nonlinear waves in fluid flow through a viscoelastic tube, Fluid Dynamics 41 (1) 2006. 49–62. doi: 10.1007/s10697-006-0021-3.

[5] A. N. Volobuev, Fluid flow in tubes with elastic walls, Physics-Uspekhi 38 (2) 1995. 169–178. doi: 10.1070/pu1995v038n02abeh000069.

[6] G. P. Chuiko, O. V. Dvornik, S. I. Shyian, Validity of Korteweg-de Vries equation for arterial pulse waves, Electronic Journal of Theoretical Physics 13 (36) [cited 30 Jan 2020]. url: http://www.ejtp.com/articles/ ejtpv13i36p99.pdf.

[7] G. P. Chuiko, O. V. Dvornik, S. I. Shyian, Y. A. Baganov, A new age-related model for blood stroke volume, Computers in Biology and Medicine 79 2016. 144–148. doi: 10.1016/j.compbiomed.2016.10.013.

[8] H. Demiray, On some nonlinear waves in fluid-filled viscoelastic tubes: Weakly dispersive case, Communications in Nonlinear Science and Numerical Simulation 10 (4) 2005. 425–440. doi: 10.1016/j.cnsns.2003.08.005.

[9] H. Demiray, Solitary waves in fluid-filled elastic tubes: Weakly dispersive case, International Journal of Engineering Science 39 (4) 2001. 439–451. doi: 10.1016/S0020-7225(00)00048-3.

[10] Solitary waves in fluids, in: R. H. J. Grimshaw (Ed.), Advances in Fluid Mechanics, Vol. 47.

[11] M. C. Abdel-Latif, Lie symmetry analysis and some new exact solutions for a variable coefficient modified Korteweg-de Vries equation arising in arterial mechanics, Herald of Saratov Univ. New series 11 (2) 2011. 42–49, (in Russian) [cited 30 Jan 2020]. url: http://www.mathnet.ru/links/630d88f1b5074862afa9f5ef6b087c16/isu217.pdf.

[12] What is maple: Product features - math and engineering software - maplesoft, 2019 [cited 30 Jan 2020]. url: https://www.maplesoft.com/products/Maple/features/.
[13] M. A. Johnson, Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation, SIAM Journal on Mathematical Analysis 41 (5) 2009. 1921–1947. doi: 10.1137/090752249.

[14] B. Deconinck, M. Nivala, The stability analysis of the periodic traveling wave solutions of the mKdV equation, Studies in Applied Mathematics 126 (1) 2011. 17–48. doi: 10.1111/j.1467-9590.2010.00496.x.

[15] J. Chen, D. E. Pelinovsky, Rogue periodic waves of the modified KdV equation, Nonlinearity 31 (5) 2018. 1955–1980. doi: 10.1088/1361-6544/aaa2da.

[16] P. F. Byrd, M. D. Friedman, Handbook of Elliptic Integrals for Engineers and Scientists, Springer Berlin Heidelberg, Palo Alto, 1971. doi: 10.1007/978-3-642-65138-0.