ON THE TORUS BIFURCATION IN AVERAGING THEORY

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ABSTRACT. In this paper, we take advantage of the averaging theory to investigate a torus bifurcation in two-parameter families of 2D nonautonomous differential equations. Our strategy consists in looking for generic conditions on the averaged functions that ensure the existence of a curve in the parameter space characterized by a Neimark-Sacker bifurcation in the corresponding Poincaré map. A Neimark-Sacker bifurcation for planar maps consists in the birth of an invariant closed curve from a fixed point, as the fixed point changes stability. In addition, we apply our results to study a torus bifurcation in a family of 3D vector fields.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULT

In the present study, we consider two-parameter families of nonautonomous differential equations given by

\[ \dot{x} = \varepsilon F_1(t, x; \mu) + \varepsilon^2 \tilde{F}(t, x; \mu, \varepsilon). \]  

(1)

Here, \( F_1 \) and \( \tilde{F} \) are \( C^1 \) functions \( T \)-periodic in the variable \( t \in \mathbb{R}, x = (x, y) \in \Omega \) with \( \Omega \) an open bounded subset of \( \mathbb{R}^2 \), \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \) for some \( \varepsilon_0 > 0 \) small, and \( \mu \in \mathbb{R} \). Throughout in this paper, we shall consider the differential equation (1) defined in the extended phase space \( S^1 \times \Omega \) by taking \( \dot{t} = 1 \), where \( S^1 = \mathbb{R}/T\mathbb{Z} \).

Detecting limit sets of differential equations is a major problem in the qualitative theory of dynamical systems. In particular, there are several research pieces dealing with the existence of limit cycles for differential equations of kind (1). In this direction, the averaging theory (see [11] and [12, Chapter 11]) is one of the most used methods. In short, this theory provides a sequence of functions \( g_i, i = 1, 2, \ldots, k \), each one called \( i \)-th order averaged function, which “control” the bifurcation of isolated periodic solutions of (1) (see Appendix). The first averaged function can be defined as

\[ g_1(x; \mu) = (g_1^1(x; \mu), g_1^2(x; \mu)) = \int_0^T F_1(t, x; \mu) dt, \]

which provides the so-called averaged system

\[ \dot{x} = \varepsilon g_1(x; \mu). \]  

(2)

When dealing with invariant sets of differential equations, the Poincaré map \( P : \Sigma \rightarrow \Sigma \) is a classic tool in understanding their properties. It is defined in

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a transversal section of the orbits \(\Sigma\), which leads to a dimensional reduction of the problem. In turn, this provides conceptual clarity for many notions that are somewhat cumbersome to state for differential equations. For instance, a periodic orbit of a differential equation corresponds to a fixed point of a Poincaré map and, consequently, the notion of orbital stability is reduced to the stability of a fixed point (see [13, Chapter 10]). As another example, an invariant torus of a differential equation corresponds to an invariant closed curve \(\gamma \subset \Sigma\) of a Poincaré map \(P\), that is, \(P(\gamma) = \gamma\).

No general method exists for constructing Poincaré maps of arbitrary differential equations. Nevertheless, if a nonautonomous \(T\)-periodic differential equation \(\dot{x} = F(t, x)\) admits, for each initial condition \(x \in \Omega\), a unique solution \(\phi(t, x)\) defined for every \(t \in [0, T]\) and satisfying \(\phi(0, x) = x\), then the Poincaré map defined in the transversal section \(\Sigma = \{0\} \times \Omega\) is given by \(P(x) = \phi(T, x)\).

Indeed, since \(F\) is \(T\)-periodic in the time variable \(t\), we can take \(\dot{t} = 1\) and consider the differential equation defined in the extended phase space \(S^1 \times \Omega\). In this way, \(P\) maps \(\Sigma\) onto itself, which is identified with \(\Omega\).

When \(F\) is given as (1), Lemma 5 from Appendix (see [6, Lemma 1]) provides the Taylor expansion of the Poincaré map \(P(x) = P(x; \mu, \varepsilon)\) around \(\varepsilon = 0\). The coefficients of this expansion determine the averaged functions \(g_i\)'s.

One can find results in research literature correlating the existence of invariant tori of the differential equation (1) with Hopf bifurcation of the averaged system (2). This fact is briefly commented on [11, Appendix C.5]. Similar results can be found in [3, Section 4C] and [1, Chapter 2].

The main goal of this paper is to provide generic conditions on the averaged functions \(g_i\)'s to guarantee the existence of a codimension-one bifurcation curve \(\mu(\varepsilon)\) in the parameter space \((\mu, \varepsilon)\) characterized by the birth of an invariant torus of (1) from a periodic solution.

Our strategy consists in looking for conditions that ensure the existence of a Neimark-Sacker Bifurcation (see [7, 9, 10]) in the Poincaré map of (1). In discrete dynamical system theory, a Neimark-Sacker bifurcation in a one-parameter family of planar maps is characterized by the birth of an invariant closed curve from a fixed point, as the fixed point changes stability. Since invariant closed curves of Poincaré maps correspond to invariant tori of differential equations, a Neimark-Sacker bifurcation in Poincaré maps is called Torus Bifurcation. We shall discuss this bifurcation in Section 2.

1.1. Setting of the problem. In what follows, we shall assume that \((x_{\mu_0}; \mu_0)\) is a Hopf point of the averaged system (2), that is, \(g_1(x_{\mu_0}; \mu_0) = 0\) and the Jacobian matrix \(D_xg_1(x_{\mu_0}; \mu_0)\) has a pair of conjugated purely imaginary eigenvalues \(\pm i\omega_0\) \((\omega_0 > 0)\). By the Implicit Function Theorem, there exists a continuous curve \(\mu \mapsto x_\mu \in \Omega\), defined in an interval \(J \ni \mu_0\), such that \(g_1(x_\mu; \mu) = 0\) for every \(\mu \in J\). Clearly, the pair of complex conjugated eigenvalues \(\alpha(\mu) \pm i\beta(\mu)\) of the Jacobian matrix \(D_xg_1(x_\mu; \mu)\) satisfies \(\alpha(\mu_0) = 0\) and \(\beta(\mu_0) = \omega_0 > 0\). Through a linear change of variables, we can assume that \(D_xg_1(x_{\mu_0}; 0)\) is in its real Jordan
normal form. Thus, the existence of a Hopf point \((x_{\mu_0}; \mu_0)\) is equivalent to the following assumption.

**A1.** There exists a continuous curve \(\mu \in J \mapsto x_{\mu} \in \Omega\), defined in an interval \(J \ni \mu_0\), such that \(g_1(x_{\mu}; \mu) = 0\) for every \(\mu \in J \subset \mathbb{R}\), the pair of complex conjugated eigenvalues \(\alpha(\mu) \pm i\beta(\mu)\) of \(D_{x}g_1(x_{\mu}; \mu)\) satisfies \(\alpha(\mu_0) = 0\) and \(\beta(\mu_0) = \omega_0\), and \(D_{x}g_1(x_{\mu_0}; 0)\) is in its real Jordan normal form.

As a first consequence of hypothesis **A1**, the next lemma ensures the existence of a periodic solution of the differential equation (1).

**Lemma 1.** Assume that hypothesis **A1** holds. Then, there exist a neighborhood \(J_0 \subset J\) of \(\mu_0\) and \(\varepsilon_1, 0 < \varepsilon_1 < \varepsilon_0\) such that, for every \((\mu, \varepsilon) \in J_0 \times (-\varepsilon_1, \varepsilon_1)\), the differential equation (1) admits a unique \(T\)-periodic solution \(\varphi(t; \mu, \varepsilon)\) satisfying \(\varphi(0; \mu, \varepsilon) = x_{\mu}\) as \(\varepsilon \to 0\).

Lemma 1 is proven in Section 3. We notice that, when the differential equation (1) is defined in the extended phase space \(S^1 \times \Omega\), such a periodic solution is given by \(\Phi(t; \mu, \varepsilon) = (t, \varphi(t; \mu, \varepsilon))\).

We also assume the following traversal hypothesis.

**A2.** Let \(\alpha(\mu) \pm i\beta(\mu)\) be the pair of complex conjugated eigenvalues of \(D_{x}g_1(x_{\mu}; \mu)\) such that \(\alpha(\mu_0) = 0\), \(\beta(\mu_0) = \omega_0 > 0\). Assume that \(\alpha'(\mu_0) \neq 0\).

Finally, define the number
\[
\ell_{1,1} = \frac{1}{8} \left( \frac{\partial^3 g_1^1(x_{\mu_0}; \mu_0)}{\partial x^3} + \frac{\partial^3 g_1^1(x_{\mu_0}; \mu_0)}{\partial x \partial y^2} + \frac{\partial^3 g_1^2(x_{\mu_0}; \mu_0)}{\partial x^2} + \frac{\partial^3 g_1^2(x_{\mu_0}; \mu_0)}{\partial y^3} \right) + \frac{1}{8\omega_0} \left( \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial x \partial y} + \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial x^2} + \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial y^2} \right)
\]
\[
+ \frac{1}{8\omega_0} \left( \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial x \partial y} + \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial x^2} + \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial y^2} \right).
\]

It is worth mentioning that \(\varepsilon \ell_{1,1}\) is the first Lyapunov coefficient of the averaged system (2) at \((x_{\mu_0}; \mu_0)\). Thus, **A1**, **A2**, and \(\ell_{1,1} \neq 0\) characterize a Hopf Bifurcation in the averaged system (2) (see [2, 3]).

### 1.2. First order torus bifurcation

As our first main result, we establish the relation between a Hopf Bifurcation in the averaged system (2) with a Torus Bifurcation in the differential equation (1).

**Theorem A.** In addition to hypotheses **A1** and **A2**, assume that \(\ell_{1,1} \neq 0\). Then, for each \(\varepsilon > 0\) sufficiently small, there exist a \(C^1\) curve \(\mu(\varepsilon) \in J_0\), with \(\mu(0) = \mu_0\), and neighborhoods \(U_\varepsilon \subset S^1 \times \Omega\) of the periodic solution \(\Phi(t; \mu(\varepsilon), \varepsilon)\) and \(J_\varepsilon \subset J_0\) of \(\mu(\varepsilon)\) for which the following statements hold.

(i) For \(\mu \in J_\varepsilon\) such that \(\ell_{1,1}(\mu - \mu(\varepsilon)) > 0\), the periodic orbit \(\Phi(t; \mu(\varepsilon), \varepsilon)\) is unstable (resp. asymptotically stable), provided that \(\ell_{1,1} > 0\) (resp. \(\ell_{1,1} < 0\)), and the differential equation (1) does not admit any invariant tori in \(U_\varepsilon\).
(ii) For $\mu \in J_\varepsilon$ such that $\ell_{1,1}(\mu - \mu(\varepsilon)) < 0$, the differential equation (1) admits a unique invariant torus $T_{\mu,\varepsilon}$ in $U_\varepsilon$ surrounding the periodic orbit $\Phi(t;\mu,\varepsilon)$. Moreover, $T_{\mu,\varepsilon}$ is unstable (resp. asymptotically stable), whereas the periodic orbit $\Phi(t;\mu,\varepsilon)$ is asymptotically stable (resp. unstable), provided that $\ell_{1,1} > 0$ (resp. $\ell_{1,1} < 0$).

(iii) $T_{\mu,\varepsilon}$ is the unique invariant torus of the differential equation (1) bifurcating from the periodic orbit $\Phi(t;\mu(\varepsilon),\varepsilon)$ in $U_\varepsilon$ as $\mu$ passes through $\mu(\varepsilon)$.

Remark 1. The $C^1$ differentiability of the functions $F_1$ and $\tilde{F}$ was the very first assumption on the differential equation (1). It is worth mentioning that this hypothesis is not strictly necessary in order to apply Theorem A. In fact, we shall see that it is sufficient to have the differentiability of the “Poincaré Map” of the differential equation (1). This implies that Theorem A can be applied to a wider class of differential equations, in particular for the class of piecewise smooth differential equation introduced in [5].

1.3. Structure of the paper. In Section 2, we discuss the Neimark-Sacker bifurcation, which plays a key role in the proof of Theorem A. Section 3 is devoted to the proof of Theorem A. Afterward, in Section 4 we state Theorem B which generalizes Theorem A by establishing weaker conditions on the higher order averaged functions $g_i$ still ensuring a torus bifurcation. In Section 4.4 we relate Hopf bifurcation in the higher order averaged system to a torus bifurcation in the corresponding differential equation. Finally, in Section 5 the obtained results are applied to study a torus bifurcation in a family of 3D vector fields. An Appendix is provided with the formulae of the averaged functions.

2. Neimark-Sacker Bifurcation

The proof of our main result is mainly based on the classical Neimark-Sacker Bifurcation, which is a version of Hopf Bifurcation for maps. In what follows we shall briefly discuss this bifurcation.

Consider the following one parameter family of maps

$$x \mapsto F(x;\sigma), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \quad \sigma \in \mathbb{R}^1. \tag{4}$$

Assume that $x = 0$ is a fixed point of the map (4), for every $|\sigma|$ sufficiently small. Denote by $r(\sigma)e^{\pm i\varphi(\sigma)}$ the pair of complex conjugated eigenvalues of the Jacobian matrix $D_x F(0,\sigma)$. We shall assume that $r(0) = 1$ and $\varphi(0) = \theta$, with $0 < \theta < \pi$. Also, consider the Taylor expansion of $F(x;0)$ around $x = 0$ as

$$F(x;0) = Ax + \frac{1}{2} B(x,x) + \frac{1}{6} C(x,x,x) + O(||x||^4),$$
where $B(x, y) = (B^1(x, y), B^2(x, y))$ and $C(x, y, z) = (C^1(x, y, z), C^2(x, y, z))$ are multilinear functions with the following components

\[
B^i(x, y) = \sum_{j,k=1}^{2} \frac{\partial^2 F_i}{\partial x_j \partial x_k}(0;0) x_j y_k, \\
C^i(x, y, z) = \sum_{j,k,l=1}^{2} \frac{\partial^2 F_i}{\partial x_j \partial x_k \partial x_l}(0;0) x_j y_k z_l,
\]

for $i = 1, 2$, and $A = D_x F(0, 0)$.

We use the elements above to construct the Lyapunov coefficient $\ell_1$ of the map (4) at $(x; \sigma) = (0;0)$. Accordingly, let $p, q \in \mathbb{C}^2$ be, respectively, complex eigenvectors of $A^T$ and $A$ satisfying $A^T p = e^{-i\theta} p$, $A q = e^{i\theta} q$, and $\langle p, q \rangle = 1$. Here, for $u, v \in \mathbb{C}^2$, we are considering the inner product $\langle u, v \rangle = \bar{u}^T \cdot v$. Thus, we define

\[
\ell_1 := \text{Re} \left( \frac{e^{-i\theta} g_{21}}{2} \right) - \text{Re} \left( \frac{(1 - 2e^{i\theta})e^{-2i\theta}}{2(1 - e^{i\theta})} g_{20} g_{11} \right) - \frac{1}{2} |g_{11}|^2 - \frac{1}{4} |g_{02}|^2 \neq 0,
\]

where

\[
g_{21} = \langle p, C(q, q, \bar{q}) \rangle, \quad g_{20} = \langle p, B(q, q) \rangle, \\
g_{11} = \langle p, B(q, \bar{q}) \rangle, \quad \text{and} \quad g_{02} = \langle p, B(\bar{q}, \bar{q}) \rangle.
\]

Under generic conditions, a Neimark-Sacker bifurcation is characterized by the existence of a neighborhood of the fixed point $x = 0$ in which a unique invariant closed curve bifurcates from $x = 0$ (see [4] Theorem 4.6). The next theorem provides generic conditions ensuring a Neimark-Sacker bifurcation in (4).

**Theorem 1.** Suppose that for $|\sigma|$ sufficiently small $x = 0$ is a fixed point of the map (4) with complex eigenvalues $r(\sigma)e^{\pm i\varphi(\sigma)}$ satisfying $r(0) = 1$ and $\varphi(0) = \theta$, $0 < \theta < \pi$. In addition, assume that

\begin{itemize}
  \item[(C.1)] $r'(0) \neq 0$, \\
  \item[(C.2)] $e^{ik\theta} \neq 1$, for $k = 1, 2, 3, 4$, and \\
  \item[(C.3)] $\ell_1 \neq 0$.
\end{itemize}

Then, there exists neighborhoods $U \subset \mathbb{R}^2$ of $x = 0$ and $I \subset \mathbb{R}$ of $\sigma = 0$ for which the following statements hold.

\begin{itemize}
  \item[(i)] For $\sigma \in I$ such that $\ell_1 \sigma \geq 0$, the fixed point $x = 0$ is unstable (resp. asymptotically stable), provided that $\ell_1 > 0$ (resp. $\ell_1 < 0$), and the map (4) does not admit any invariant closed curve in $U$.
  \item[(ii)] For $\sigma \in I$ such that $\ell_1 \sigma < 0$, the map (4) admits a unique invariant closed curve $S_\mu$ in $U$ surrounding the fixed point $x = 0$. Moreover, $S_\mu$ is unstable (resp. asymptotically stable), whereas the fixed point $x = 0$ is asymptotically stable (resp. unstable), provided that $\ell_1 > 0$ (resp. $\ell_1 < 0$).
  \item[(iii)] $S_\mu$ is the unique invariant closed curve of the map (4) bifurcating from the fixed point $x = 0$ in $U$ as $\sigma$ pass through 0.
\end{itemize}
3. Proofs of Lemma 1 and Theorem A

The Poincaré Map of the differential equation (1), defined on the section $\Sigma = \{0\} \times \Omega$, writes

$$x \mapsto P(x; \mu, \epsilon) = x + \epsilon g_1(x; \mu) + \epsilon^2 \tilde{G}(x; \mu, \epsilon).$$

(8)

In what follows, we shall prove Lemma 1 by showing the existence of fixed points $\xi(\mu, \epsilon)$ for the Poincaré Map.

**Proof of Lemma 1.** Define

$$f(x, \mu, \epsilon) := \frac{P(x; \mu, \epsilon) - x}{\epsilon} = g_1(x; \mu) + \epsilon \tilde{G}(x; \mu, \epsilon).$$

Notice that $f(x_{\mu_0}, 0, 0) = (0, 0)$ and

$$\frac{\partial f}{\partial x}(x_{\mu}, \mu, 0) = \partial_x g_1(x_{\mu}; \mu).$$

From the hypothesis A1, $\alpha(\mu_0) = 0$ and $\beta(\mu_0) = \omega_0 \neq 0$, where $\alpha(\mu) = i \beta(\mu)$ are the complex conjugated eigenvalues of $\partial_x g_1(x_{\mu}; \mu)$. Therefore, there exists a neighborhood $J_0 \subset J$ of $\mu_0$ such that $\beta(\mu) \neq 0$ for every $\mu \in J_0$. Consequently,

$$\left| \frac{\partial f}{\partial x}(x_{\mu}, \mu, 0) \right| \neq 0,$$

for every $\mu \in J_0$. Hence, from the Implicit Function Theorem and from the compactness of $J_0$, there exists $\epsilon_1, 0 < \epsilon_1 < \epsilon_0$, and a unique function $\xi(\mu, \epsilon)$, defined on $J_0 \times (-\epsilon_1, \epsilon_1)$, such that $\xi(\mu, 0) = x_{\mu}$ and $f(\xi(\mu, \epsilon), \epsilon) = 0$ for every $\epsilon \in (-\epsilon_1, \epsilon_1)$ and $\mu \in J_0$. \hfill \Box

The next result provides a curve $\mu(\epsilon)$ of critical values for the parameter $\mu$ regarding the fixed point $\xi(\mu, \epsilon)$ of the map (8) for which the conditions of Theorem A hold.

**Lemma 2.** For each $(\mu, \epsilon) \in J_0 \times (-\epsilon_1, \epsilon_1)$, let $\lambda(\mu, \epsilon)$ and $\overline{\lambda(\mu, \epsilon)}$ be the pair of complex conjugated eigenvalues of $D_x P(\xi(\mu, \epsilon); \mu, \epsilon)$ and assume that hypotheses A1 and A2 hold. Then, there exists $\epsilon_2, 0 < \epsilon_2 < \epsilon_1$, and a unique smooth function $\mu : (-\epsilon_2, \epsilon_2) \to J_0$, with $\mu(0) = \mu_0$, satisfying

1. $|\lambda(\mu(\epsilon), \epsilon)| = 1$,

2. $(\lambda(\mu(\epsilon), \epsilon))^k \neq 1$, for $k \in \{1, 2, 3, 4\}$, and

3. $\frac{d}{d\mu}|\lambda(\mu, \epsilon)||_{\mu = \mu(\epsilon)} \neq 0$,

for every $\epsilon \in (-\epsilon_2, \epsilon_2) \setminus \{0\}$.

**Proof.** For each $(\mu, \epsilon) \in J_0 \times (-\epsilon_1, \epsilon_1)$, the Jacobian matrix of the first return map $P(x; \mu, \epsilon)$ at its fixed point $\xi(\mu, \epsilon)$ is given by

$$D_x P(\xi(\mu, \epsilon); \mu, \epsilon) = Id + \epsilon \frac{\partial g_1}{\partial x}(x_{\mu}; \mu) + O(\epsilon^2),$$

and

$$\frac{d}{d\mu}|\lambda(\mu, \epsilon)||_{\mu = \mu(\epsilon)} = \frac{d}{d\mu}\left|\frac{\partial f}{\partial x}(x_{\mu}, \mu, 0)\right|_{\mu = \mu(\epsilon)} = \frac{d}{d\mu}\left|\frac{\partial g_1}{\partial x}(x_{\mu}; \mu)\right|_{\mu = \mu(\epsilon)} + O(\epsilon^2),$$

for every $\epsilon \in (-\epsilon_2, \epsilon_2) \setminus \{0\}$.\hfill \Box
which has the following eigenvalues

\[ \lambda(\mu, \varepsilon) = 1 + \varepsilon (\alpha(\mu) + i\beta(\mu)) + \mathcal{O}(\varepsilon^2), \text{ and} \]
\[ \bar{\lambda}(\mu, \varepsilon) = 1 + \varepsilon (\alpha(\mu) - i\beta(\mu)) + \mathcal{O}(\varepsilon^2). \]

Notice that

\[ |\lambda(\mu, \varepsilon)|^2 = 1 + 2\varepsilon\alpha(\mu) + \mathcal{O}(\varepsilon^2) \]
\[ = 1 + \varepsilon \ell(\mu, \varepsilon), \tag{9} \]

where \( \ell(\mu, \varepsilon) = 2\alpha(\mu) + \mathcal{O}(\varepsilon). \) From hypothesis A2, we have

\[ \ell(\mu_0, 0) = 0 \quad \text{and} \quad \frac{\partial \ell}{\partial \mu}(\mu_0, 0) = 2\alpha'(\mu_0) \neq 0. \]

Thus, by the Implicit Function Theorem, there exist \( \varepsilon_2, 0 < \varepsilon_2 < \varepsilon_1, \) and a unique function \( \mu : (-\varepsilon_2, \varepsilon_2) \to J_0 = (\mu_0 - \delta_1, \mu_0 + \delta_1) \) such that \( \mu(0) = \mu_0 \) and \( \ell(\mu(\varepsilon), \varepsilon) = 0, \) for every \( \varepsilon \in (-\varepsilon_2, \varepsilon_2). \) This implies that \( |\lambda(\mu(\varepsilon), \varepsilon)|= 1, \) for every \( \varepsilon \in (-\varepsilon_2, \varepsilon_2). \) Hence, statement 1 is proved. Moreover, since

\[ \lambda(\mu(\varepsilon), \varepsilon) = 1 + \varepsilon (\alpha(\mu(\varepsilon)) + i\beta(\mu(\varepsilon))) + \mathcal{O}(\varepsilon^2) \]
\[ = 1 + \varepsilon (i\omega_0) + \mathcal{O}(\varepsilon^2), \tag{10} \]

and \( \omega_0 > 0, \) the parameter \( \varepsilon_2 > 0 \) can be made smaller, if necessary, in order that

\[ \lambda(\mu(\varepsilon), \varepsilon) \notin \left\{ \pm 1, \pm i, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right\}, \]

for every \( \varepsilon \in (-\varepsilon_2, \varepsilon_2) \setminus \{0\}. \) Consequently, for \( k \in \{1, 2, 3, 4\}, \) \( (\lambda(\mu(\varepsilon), \varepsilon))^k \neq 1 \)

for every \( \varepsilon \in (-\varepsilon_2, \varepsilon_2) \setminus \{0\}, \) which proves statement 2. Finally, computing the derivative of (9) at \( \mu = \mu_0 \) we implicitly obtain that

\[ \frac{\partial}{\partial \mu} |\lambda(\mu, \varepsilon)|_{\mu=\mu(\varepsilon)} = \alpha'(\mu(\varepsilon))\varepsilon + \mathcal{O}(\varepsilon^2) = \alpha'(\mu_0)\varepsilon + \mathcal{O}(\varepsilon^2). \]

Since \( \alpha'(\mu_0) \neq 0, \) the parameter \( \varepsilon_2 > 0 \) can be made smaller again, if necessary, in order that

\[ \frac{\partial}{\partial \mu} |\lambda(\mu, \varepsilon)|_{\mu=\mu(\varepsilon)} \neq 0, \]

for every \( \varepsilon \in (-\varepsilon_2, \varepsilon_2) \setminus \{0\}. \) This concludes the proof of statement 3. \( \square \)

Now, we are ready to prove Theorem A.

Proof of Theorem A. For each \((\mu, \varepsilon) \in J_0 \times (-\varepsilon_2, \varepsilon_2), \) let \( \xi(\mu, \varepsilon) \) be the fixed point of the Poincaré map (8) given by Lemma 1 and let \( \mu(\varepsilon) \) be the curve of critical values of the parameter \( \mu \) given by Lemma A2.

Changing the coordinates in (8) by setting \( x = y + \xi(\mu, \varepsilon) \) and taking \( \mu = \sigma + \mu(\varepsilon), \) we get the map

\[ y \to H_\varepsilon(y; \sigma) := P(y + \xi(\sigma + \mu(\varepsilon), \varepsilon); \sigma + \mu(\varepsilon), \varepsilon) - \xi(\sigma + \mu(\varepsilon), \varepsilon). \tag{11} \]

Notice that

\[ H_\varepsilon(y; \sigma) = y + \varepsilon g_1(y + \xi(\sigma + \mu(\varepsilon), \varepsilon), \sigma + \mu(\varepsilon)); \sigma + \mu(\varepsilon), \varepsilon) + \varepsilon^2 \tilde{G}(y + \xi(\sigma + \mu(\varepsilon), \varepsilon), \varepsilon; \sigma + \mu(\varepsilon)). \]
The proof of Theorem A will follow by showing that, for each $\varepsilon > 0$ sufficiently small, the map (11) satisfies the hypotheses of Theorem 1.

First of all, fix $\varepsilon_3$ (to be chosen later on) satisfying $0 < \varepsilon_3 < \varepsilon_2$. Denote by $I'_\varepsilon$ the set of $\sigma \in \mathbb{R}$ such that $\sigma + \mu(\varepsilon) \in J_0$. Since, from Lemma 2, $\mu(\varepsilon) \in J_0$, we get that $0 \in I'_\varepsilon$. Thus, for each $\varepsilon \in (0, \varepsilon_3)$, Lemma 1 implies that $y = 0$ is a fixed point of (11) for every $\sigma \in I'_\varepsilon$. Notice that $\eta_{\varepsilon}(\sigma) = \lambda(\sigma + \mu(\varepsilon), \varepsilon)$ and $\overline{\eta_{\varepsilon}(\sigma)} = \overline{\lambda(\sigma + \mu(\varepsilon), \varepsilon)}$ are the eigenvalues of $D_{\varepsilon}H_\varepsilon(0; \sigma)$. Denote $\eta_{\varepsilon}(\sigma) = r_{\varepsilon}(\sigma)e^{i\varphi(\sigma)}$ and $\varphi_{\varepsilon}(0) = \theta_{\varepsilon}$, $0 < \theta_{\varepsilon} < \pi$. Thus, from Lemma 2, we get

$$r_{\varepsilon}(0) = 1, \quad e^{i\theta_{\varepsilon}} \neq 1, \text{ for } k \in \{1, 2, 3, 4\}, \text{ and } r'_{\varepsilon}(0) \neq 0,$$

for every $\varepsilon \in (0, \varepsilon_3)$. Therefore, the map (11) satisfies all the conditions of Theorem 1 but C.3 for each $\varepsilon \in (0, \varepsilon_3)$.

In order to check condition C.3, we need to compute $\ell_1$ as defined in (6). Following the procedure of Section 2, we first compute the Taylor expansion of $H_\varepsilon(y; 0)$ around $y = 0$ as

$$H_\varepsilon(y; 0) = A_\varepsilon y + \frac{1}{2}B_\varepsilon(y, y) + \frac{1}{6}C_\varepsilon(y, y, y) + \mathcal{O}(||y||^4),$$

where $B_\varepsilon(u, v) = (B^1_\varepsilon(u, v), B^2_\varepsilon(u, v))$ and $C_\varepsilon(u, v, w) = (C^1_\varepsilon(u, v, w), C^2_\varepsilon(u, v, w))$ are multilinear functions with the following components

$$B^i_\varepsilon(u, v) = \varepsilon \sum_{j,k=1}^{2} \frac{\partial^2 g^i_\varepsilon}{\partial x_j \partial x_k} (\xi(\mu(\varepsilon), \varepsilon); \mu(\varepsilon)) u_j v_k + \mathcal{O}(\varepsilon^2),$$

$$C^i_\varepsilon(u, v, w) = \varepsilon \sum_{j,k,l=1}^{2} \frac{\partial^3 g^i_\varepsilon}{\partial x_j \partial x_k \partial x_l} (\xi(\mu(\varepsilon), \varepsilon); \mu(\varepsilon)) u_j v_k w_l + \mathcal{O}(\varepsilon^2),$$

for $i = 1, 2$, and

$$A_\varepsilon = D_xH_\varepsilon(0; 0) = \text{Id} + \varepsilon D_xg_1(\xi(\mu(\varepsilon), \varepsilon); \mu(\varepsilon)) + \mathcal{O}(\varepsilon^2).$$

Now, let $p_\varepsilon \in \mathbb{C}^2$ and $q_\varepsilon \in \mathbb{C}^2$ be, respectively, the eigenvectors of the matrices $A^\dagger_\varepsilon$ and $A_\varepsilon$ satisfying $A^\dagger_\varepsilon p_\varepsilon = e^{-i\theta_\varepsilon}p_\varepsilon$, $A_\varepsilon q_\varepsilon = e^{i\theta_\varepsilon}q_\varepsilon$, and $\langle p_\varepsilon, q_\varepsilon \rangle = 1$. We claim that $p_\varepsilon = p + \mathcal{O}(\varepsilon)$ and $q_\varepsilon = q + \mathcal{O}(\varepsilon)$, where $p, q \in \mathbb{C}^2$ satisfy $D_xg_1(x_{\mu_0}; \mu_0)p = -i\omega_0p$, $D_xg_1(x_{\mu_0}; \mu_0)q = i\omega_0q$, and $\langle p, q \rangle = 1$. Indeed, an eigenvector $y \in \mathbb{C}^2$ of $A_\varepsilon$ with respect to $\eta_{\varepsilon}(0)$ satisfies

$$[A_\varepsilon - \eta_{\varepsilon}(0)\text{Id}]y = 0.$$

From (10), (13), and taking $y = y_0 + \mathcal{O}(\varepsilon)$, equation (14) writes

$$[D_xg_1(x_{\mu_0}; \mu_0) - i\omega_0\text{Id}]y_0 + \mathcal{O}(\varepsilon) = 0.$$
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Matching the coefficients of \( \varepsilon \), we get that \( y_0 \) is an eigenvector of \( D_xg_1(x_{\mu_0}; \mu_0) \) with respect to the eigenvalue \( i\omega_0 \). We can do the same for the matrix \( A_1' \). Finally, since \( \langle p, q \rangle = 1 \) for every \( \varepsilon \), we conclude that \( \langle p, q \rangle = 1 \). Furthermore, from hypothesis A1, \( D_xg_1(x_{\mu_0}; \mu_0) \) is in its real normal Jordan form, thus we can take \( p = q = (1, -i)/\sqrt{2} \).

Defining

\[
B_0(u, v) = \frac{d}{dz} B_z(u, v) \bigg|_{z=0} \quad \text{and} \quad C_0(q, q, \bar{q}) = \frac{d}{dz} C_z(q, q, \bar{q}) \bigg|_{z=0} ,
\]

and denoting \( g_{00}^0 = \langle p, B_0(q, q) \rangle, g_{01}^0 = \langle p, B_0(q, \bar{q}) \rangle, g_{02}^0 = \langle p, B_0(\bar{q}, q) \rangle, \) and \( g_{21}^0 = \langle p, C_0(q, \bar{q}, q) \rangle \), we get, from (7) and (12), that

\[
g_{20} = \langle p, B_0(q_0, q_0) \rangle = \langle p, B_0(\bar{q}, \bar{q}) \rangle + \mathcal{O}(\varepsilon^2) = \varepsilon g_{20} + \mathcal{O}(\varepsilon^2).
\]

Analogously, \( g_{11} = \varepsilon g_{11}^0 + \mathcal{O}(\varepsilon^2) \), \( g_{02} = \varepsilon g_{02}^0 + \mathcal{O}(\varepsilon^2) \), and \( g_{21} = \varepsilon g_{21}^0 + \mathcal{O}(\varepsilon^2) \). From (10), \( e^{i\theta_l} = \eta_l(0) = \lambda(\mu(\varepsilon), \varepsilon) = 1 + \varepsilon(i\omega_0) + \mathcal{O}(\varepsilon^2) \), thus

\[
\frac{2}{1 - 2e^{i\theta_l}} e^{-2i\theta_l} = -\varepsilon - i\omega_0 \varepsilon g_{20} g_{11} + \mathcal{O}(\varepsilon^2).
\]

Hence, substituting the above expressions into (6), we obtain

\[
\ell_1 = \frac{\varepsilon}{2} \left( \text{Re}(g_{21}^0) - \text{Re}(i g_{20}^0 g_{11}) \right) + \mathcal{O}(\varepsilon^2) .
\]

Moreover, from (12) and (15), we compute

\[
\text{Re}(g_{21}^0) = \frac{1}{4} \left( \frac{\partial^3 g_1^1(x_{\mu_0}; \mu_0)}{\partial x^3} + \frac{\partial^3 g_1^1(x_{\mu_0}; \mu_0)}{\partial x^2 y} + \frac{\partial^3 g_1^1(x_{\mu_0}; \mu_0)}{\partial x y^2} + \frac{\partial^3 g_1^1(x_{\mu_0}; \mu_0)}{\partial y^3} \right),
\]

\[
\text{Re}(i g_{20}^0 g_{11}) = \frac{1}{4} \left( \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial x \partial y} \left( \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial x^2} + \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial y^2} \right) - \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial x^2 \partial y} \right.
\]

\[
\left. \cdot \left( \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial x^2} + \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial y^2} \right) + \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial x^2} \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial y^2} - \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial x \partial y} \right).
\]

Substituting the above expressions into (16) we conclude that

\[
\ell_1 = \varepsilon \ell_{1,1} + \mathcal{O}(\varepsilon^2),
\]

where \( \ell_{1,1} \) is given by (5). From hypothesis, \( \ell_{1,1} \neq 0 \). Therefore, we can choose \( \varepsilon_3 > 0, 0 < \varepsilon_3 < \varepsilon_2 \), in order that \( \text{sgn}(\ell_1) = \text{sgn}(\ell_{1,1}) \) for every \( \varepsilon \in (0, \varepsilon_3) \).

Then, for each \( \varepsilon \in (0, \varepsilon_3) \), applying Theorem 1 for the map \( H_\varepsilon \), we get the existence of neighborhoods \( U_\varepsilon \subset \mathbb{R}^2 \) of the fixed point \( y = 0 \) and \( I_\varepsilon \subset I_0 \) of the critical parameter \( \sigma = 0 \) for which items (i), (ii), and (iii) of Theorem 1 holds.

Going back through the change of variables and parameters we get the existence of neighborhoods \( U'_\varepsilon \subset \Omega \) of the fixed point \( x = \xi(\mu(\varepsilon), \varepsilon) \) and \( J_\varepsilon \subset J_0 \) of the critical parameter \( \mu = \mu(\varepsilon) \) for which the following statements hold.
(i) For $\mu \in J_\varepsilon$ such that $\ell_{1,1}(\mu - \mu(\varepsilon)) \geq 0$, the fixed point $x = \xi(\mu(\varepsilon), \varepsilon)$ is unstable (resp. asymptotically stable), provided that $\ell_1 > 0$ (resp. $\ell_1 < 0$), and the Poincaré map $\Phi$ does not admit any invariant closed curve in $U'_\varepsilon$.

(ii) For $\mu \in J_\varepsilon$ such that $\ell_{1,1}(\mu - \mu(\varepsilon)) < 0$, the Poincaré map $\Phi$ admits a unique invariant closed curve $S_{\mu, \varepsilon}$ in $U'_\varepsilon$ surrounding the fixed point $\xi(\mu, \varepsilon)$. Moreover, $S_{\mu, \varepsilon}$ is unstable (resp. asymptotically stable), whereas the fixed point $x = \xi(\mu, \varepsilon)$ is asymptotically stable (resp. unstable), provided that $\ell_1 > 0$ (resp. $\ell_1 < 0$).

(iii) $S_{\mu, \varepsilon}$ is the unique invariant closed curve of the Poincaré map $\Phi$ bifurcating from the fixed point $x = \xi(\mu(\varepsilon), \varepsilon)$ in $U'_\varepsilon$ as $\mu$ pass through $\mu(\varepsilon)$.

Finally, define $\mathcal{U}_\varepsilon$ as the saturation of $\{0\} \times U'_\varepsilon$ through $\Phi$, that is, $\mathcal{U}_\varepsilon = \{\Phi(t, x; \mu, \varepsilon) : t \in [0, T], x \in U'_\varepsilon\}$. Hence, the proof of Theorem A follows by noticing that the invariant closed curve $S_{\mu, \varepsilon}$ in $U'_\varepsilon$ of the Poincaré map $\Phi$ surrounding the fixed point $\xi(\mu, \varepsilon)$ corresponds to an invariant torus $T_{\mu, \varepsilon}$ in $\mathcal{U}_\varepsilon$ of the differential equation (17) (defined in the extended phase space $S^1 \times \Omega$) surrounding the periodic orbit $\Phi(t; \mu, \varepsilon)$.

\section{Higher order approach}

In this section, we consider two-parameter families of nonautonomous differential equations given by

$$\dot{x}(t) = \sum_{i=1}^{k} \varepsilon^i F_i(t, x; \mu) + \varepsilon^{k+1} \tilde{F}(t, x; \mu, \varepsilon). \quad (17)$$

Here, $F_i$, $i = 1, 2, \ldots, k$, and $\tilde{F}$ are sufficiently smooth functions and $T$-periodic in the variable $t \in \mathbb{R}$, $x = (x, y) \in \Omega$ with $\Omega$ an open bounded subset of $\mathbb{R}^2$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$ small, and $\mu \in \mathbb{R}$.

In what follows we shall apply the same ideas of the previous section for obtaining a higher order version of Theorem A.

\subsection{Setting of the problem}

Consider the averaged functions $g_i$, $i = 1, 2, \ldots, k$, as defined in Appendix. Let $l, 1 \leq l < k$, be the subindex of the first non-vanishing averaging function. From Lemma 5 of the Appendix, the Poincaré Map of the differential equation (17), defined on the transversal section $\Sigma = \{0\} \times \Omega$, writes

$$x \mapsto P(x; \mu, \varepsilon) = x + \varepsilon^l G(x, \mu, \varepsilon), \quad (18)$$

where

$$G(x, (\mu, \varepsilon)) = g_1(x; \mu) + \varepsilon^1 g_{l+1}(x; \mu) + \cdots + \varepsilon^{k-l} g_k(x; \mu) + \varepsilon^{k+1-l} \tilde{G}(x; \mu, \varepsilon).$$

As a first hypothesis we assume that the averaged system of order $l$

$$\dot{x} = \varepsilon^l g_1(x; \mu) \quad (19)$$

has a Hopf point at $(x_{\mu_0}; \mu_0)$. Equivalently, suppose that
B1. there exists a continuous curve \( \mu \in J \mapsto x_\mu \in \Omega \), defined in an interval \( J \ni \mu_0 \), such that \( g_j^i(x_\mu; \mu) = 0 \) for every \( \mu \in J \) and the pair of complex conjugated eigenvalues \( \alpha(\mu) \pm i\beta(\mu) \) of \( D_xg_j^i(x_\mu; \mu) \) satisfies \( \alpha(\mu_0) = 0 \) and \( \beta(\mu_0) = \omega_0 > 0 \).

Here, the proof of Lemma 1 can be followed straightforwardly in order to get a neighborhood \( J_0 \subset J \) of \( \mu_0 \), a parameter \( \varepsilon_1 \), \( 0 < \varepsilon_1 < \varepsilon_0 \), and a unique function \( \xi : J_0 \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^2 \) satisfying \( \xi(\mu, 0) = x_\mu \) and \( P(\xi(\mu, \varepsilon); \mu, \varepsilon) = \xi(\mu, \varepsilon) \), for every \( (\mu, \varepsilon) \in J_0 \times (-\varepsilon_1, \varepsilon_1) \). This provides the following lemma.

Lemma 3. Assume that hypothesis B1 holds. Then, there exists a neighborhood \( J_0 \subset J \) of \( \mu_0 \) and \( \varepsilon_1 \), \( 0 < \varepsilon_1 < \varepsilon_0 \) such that, for every \( (\mu, \varepsilon) \in J_0 \times (-\varepsilon_1, \varepsilon_1) \) the differential equation \( \xi = \xi(\mu, t; \mu, \varepsilon) \) admits a unique \( T \)-periodic orbit \( \varphi(t; \mu, \varepsilon) \) satisfying \( \varphi(0; \mu, \varepsilon) \to x_\mu \) as \( \varepsilon \to 0 \).

We notice that, when the differential equation \( \xi = \xi(\mu, t; \mu, \varepsilon) \) is defined in the extended phase space \( S^1 \times \Omega \), such a periodic solution is given by \( \Phi(t; \mu, \varepsilon) = (t, \varphi(t; \mu, \varepsilon)) \).

We also assume the following transversal hypothesis.

B2. Let \( \alpha(\mu) \pm i\beta(\mu) \) be the pair of complex conjugated eigenvalues of \( D_xg_j^i(x_\mu; \mu) \) such that \( \alpha(\mu_0) = 0 \), \( \beta(\mu_0) = \omega_0 > 0 \). Assume that

\[
\frac{d\alpha(\mu)}{d\mu} \bigg|_{\mu=\mu_0} = d \neq 0.
\]

The proof of Lemma 2 can also be followed directly in order to get the following result.

Lemma 4. For each \( (\mu, \varepsilon) \in J_0 \times (-\varepsilon_1, \varepsilon_1) \), let \( \lambda(\mu, \varepsilon) \) and \( \overline{\lambda(\mu, \varepsilon)} \) be the pair of complex conjugated eigenvalues of \( D_xP(\xi(\mu, \varepsilon); \mu, \varepsilon) \) and assume that hypotheses B1 and B2 hold. Then, there exists \( \varepsilon_2 \), \( 0 < \varepsilon_2 < \varepsilon_1 \), and a unique smooth function \( \mu : (-\varepsilon_2, \varepsilon_2) \to J_0 \), with \( \mu(0) = \mu_0 \), satisfying

1. \( |\lambda(\mu(\varepsilon), \varepsilon)| = 1 \),
2. \( (\lambda(\mu(\varepsilon), \varepsilon))^k \neq 1 \), for \( k \in \{1, 2, 3, 4\} \), and
3. \( \frac{d}{d\mu} |\lambda(\mu, \varepsilon)| \bigg|_{\mu=\mu(\varepsilon)} \neq 0 \).

We emphasize that the functions \( \xi(\mu, \varepsilon) \) and \( \mu(\varepsilon) \) can be both explicitly expanded in Taylor series around \( \varepsilon = 0 \) up to order \( \varepsilon^k \). Due to the complexity of the coefficients of these expansions, we shall omit them here.

Now, applying the change of variables \( x = y + \xi(\mu, \varepsilon) \) and taking \( \mu = \sigma + \mu(\varepsilon) \), the Poincaré map \( (18) \) writes

\[
y \mapsto H_\varepsilon(y; \sigma) := y + \varepsilon \mathbf{G}(y + \xi(\sigma + \mu(\varepsilon), \varepsilon), \sigma + \mu(\varepsilon), \varepsilon).
\]

Now, for each \( \varepsilon \in (0, \varepsilon_2) \), denote by \( I'_\varepsilon \) the set of \( \sigma \in \mathbb{R} \) such that \( \sigma + \mu(\varepsilon) \in J_0 \).

Since, from Lemma 1, \( \mu(\varepsilon) \in J_0 \), we get that \( 0 \in I'_\varepsilon \). Thus, for each \( \varepsilon \in (0, \varepsilon_2) \), Lemma 3 implies that \( y = 0 \) is a fixed point of \( (20) \) for every \( \sigma \in I'_\varepsilon \). Notice that \( \eta_\varepsilon(\sigma) = \lambda(\sigma + \mu(\varepsilon), \varepsilon) \) and \( \tilde{\eta}(\sigma) = \overline{\lambda(\sigma + \mu(\varepsilon), \varepsilon)} \) are the eigenvalues of
Denote $\eta_\varepsilon(\sigma) = r_\varepsilon(\sigma)e^{i\varphi_\varepsilon(\sigma)}$ and $\varphi_\varepsilon(0) = \theta_\varepsilon$, $0 < \theta_\varepsilon < \pi$. Thus, from Lemma 4 we get

$$r_\varepsilon(0) = 1, \quad e^{ik\theta_\varepsilon} \neq 1, \text{ for } k \in \{1, 2, 3, 4\}, \text{ and } r'_\varepsilon(0) \neq 0,$$

for every $\varepsilon \in (0, \varepsilon_2)$. Therefore, the map (20) satisfies all the conditions of Theorem 1 but C.3 for each $\varepsilon \in (0, \varepsilon_2)$.

Theorem 1 will be generalized by showing that the Poincaré map (20) admits a Neimark-Sacker bifurcation. From here, it remains to show that condition C.3 holds. Accordingly, we need to compute $\ell_1$ as defined in (6). Following the procedure of Section 2, we first compute the Taylor expansion of $H_\varepsilon(y; 0)$ around $y = 0$ as

$$H_\varepsilon(y, 0) = A_\varepsilon y + \frac{1}{2} B_\varepsilon(y, y) + \frac{1}{6} C_\varepsilon(y, y, y) + O(|y|^4),$$

where $B_\varepsilon(u, v)$ and $C_\varepsilon(u, v, w)$ are the multilinear functions defined in (12), and

$$A_\varepsilon = D_y H_\varepsilon(0, 0).$$

Moreover, $A_\varepsilon = Id + \varepsilon I \mathcal{A}_\varepsilon + O(\varepsilon^k)$, $B_\varepsilon = \varepsilon I \mathcal{B}_\varepsilon + O(\varepsilon^k)$, $C_\varepsilon = \varepsilon I \mathcal{C}_\varepsilon + O(\varepsilon^k)$, where

$$\mathcal{A}_\varepsilon = A_\varepsilon + \varepsilon A_{k+1} + \cdots + \varepsilon^{k-l} A_k,$$

$$\mathcal{B}_\varepsilon(u, v) = B_{k+1}(u, v) + \varepsilon B_{k+1}(u, v) + \cdots + \varepsilon^{k-l} B_{k}(u, v),$$

$$\mathcal{C}_\varepsilon(u, v, w) = C_{k+1}(u, v, w) + \varepsilon C_{k+1}(u, v, w) + \cdots + \varepsilon^{k-l} C_{k}(u, v, w).$$

We stress that $\mathcal{A}_\varepsilon$, $\mathcal{B}_\varepsilon$, and $\mathcal{C}_\varepsilon$ can be explicitly computed. Due to the complexity of these expressions, we shall omit them here.

Through a linear change of variables in (20), we can assume, without loss of generality, that

$$\text{B3. for each } \varepsilon \in (0, \varepsilon_3), \text{ the matrix } Id + \varepsilon I \mathcal{A}_\varepsilon \text{ is in its real Jordan normal form.}$$

This implies that

$$Id + \varepsilon I \mathcal{A}_\varepsilon = \begin{pmatrix} 1 + \tilde{\alpha}_\varepsilon & -\tilde{\beta}_\varepsilon \\ \tilde{\beta}_\varepsilon & 1 + \tilde{\alpha}_\varepsilon \end{pmatrix},$$

where $\tilde{\alpha}_\varepsilon + i \tilde{\beta}_\varepsilon$ and $\tilde{\alpha}_\varepsilon - i \tilde{\beta}_\varepsilon$ are the eigenvalues of $\mathcal{A}_\varepsilon$. Considering

$$e^{i\theta_\varepsilon} = \eta_\varepsilon(0) = 1 + \sum_{j=l}^{k} \varepsilon^j (\alpha_j + i \beta_j) + O(\varepsilon^{k+1}),$$

with $\alpha_l = 0$ and $\beta_l = \omega_0 > 0$, we have that

$$\tilde{\alpha}_\varepsilon = \sum_{j=l}^{k} \varepsilon^j \alpha_j, \quad \text{and} \quad \tilde{\beta}_\varepsilon = \sum_{j=l}^{k} \varepsilon^j \beta_j.$$

Finally, for $p = (1, -i)/\sqrt{2} \in \mathbb{C}^2$, define the number

$$\ell'_1 = - \text{Re} \left( \frac{1 - 2e^{i\theta_\varepsilon}}{2(1 - e^{i\theta_\varepsilon})} \langle p, \varepsilon I \mathcal{B}_\varepsilon(p, p) \rangle \langle p, \varepsilon I \mathcal{B}_\varepsilon(p, p) \rangle \right)$$

$$+ \text{Re} \left( e^{-i\theta_\varepsilon} \langle p, \varepsilon I \mathcal{C}_\varepsilon(p, p, \bar{p}) \rangle - \frac{1}{2} |\langle p, \varepsilon I \mathcal{B}_\varepsilon(p, p) \rangle|^2 - \frac{1}{4} |\langle p, \varepsilon I \mathcal{B}_\varepsilon(p, p) \rangle|^2 ,
$$
and consider its Taylor expansion around $\varepsilon = 0$, which can be explicitly computed as

$$\ell^*_l = \varepsilon^l \ell_{1,l} + \varepsilon^{l+1} \ell_{1,l+1} + \varepsilon^{l+2} \ell_{1,l+2} + \ldots + \varepsilon^k \ell_{1,k} + O(\varepsilon^{k+1}).$$

In Section 4.3, we provide the expressions of $\ell_{1,j}$, $l \leq j \leq k$.

4.2. Higher order torus bifurcation. As our second main result, Theorem A is generalized as follows.

**Theorem B.** Let $1 \leq l \leq k$, be the subindex of the first non-vanishing averaging function and $\ell_{1,j}$, $l \leq j \leq k$, as defined in (23). In addition to hypotheses B1, B2, and B3, assume that $\ell_{1,j} \neq 0$, for some $l \leq j \leq k$. Let $j^*$, $l \leq j^* \leq k$, be the first subindex such that $\ell_{1,j^*} \neq 0$. Then, for each $\varepsilon > 0$ sufficiently small there exist a $C^1$ curve $\mu(\varepsilon) \in J_0$, with $\mu(0) = \mu_0$, and neighborhoods $U_\varepsilon \subset S^1 \times \Omega$ of the periodic solution $\Phi(t; \mu(\varepsilon), \varepsilon)$ and $J_\varepsilon \subset J_0$ of $\mu(\varepsilon)$ for which the following statements hold.

(i) For $\mu \in J_\varepsilon$ such that $\ell_{1,j^*}(\mu - \mu(\varepsilon)) \geq 0$, the periodic orbit $\Phi(t; \mu(\varepsilon), \varepsilon)$ is unstable (resp. asymptotically stable), provided that $\ell_{1,j^*} > 0$ (resp. $\ell_{1,j^*} < 0$), and the differential equation (17) does not admit any invariant tori in $U_\varepsilon$.

(ii) For $\mu \in J_\varepsilon$ such that $\ell_{1,j^*}(\mu - \mu(\varepsilon)) < 0$, the differential equation (17) admits a unique invariant torus $T_{\mu,\varepsilon}$ in $U_\varepsilon$ surrounding the periodic orbit $\Phi(t; \mu, \varepsilon)$. Moreover, $T_{\mu,\varepsilon}$ is unstable (resp. asymptotically stable), whereas the periodic orbit $\Phi(t; \mu, \varepsilon)$ is asymptotically stable (resp. unstable), provided that $\ell_{1,j^*} > 0$ (resp. $\ell_{1,j^*} < 0$).

(iii) $T_{\mu,\varepsilon}$ is the unique invariant torus of the differential equation (17) bifurcating from the periodic orbit $\Phi(t; \mu(\varepsilon), \varepsilon)$ in $U_\varepsilon$ when $\mu$ passes through $\mu(\varepsilon)$.

**Proof.** As before, fix $\varepsilon_3$ (to be chosen later on) satisfying $0 < \varepsilon_3 < \varepsilon_2$. We already have that for each $\varepsilon \in (0, \varepsilon_3)$ the map (11) satisfies all the conditions of Theorem 1 but C.3.

In order to check condition C.3 we need to compute $\ell_1$ as defined in (6). Accordingly, let $p_\varepsilon, q_\varepsilon \in C^2$ be respectively, eigenvectors of $A_1^\varepsilon$ and $A_\varepsilon$ satisfying $A_1^\varepsilon p_\varepsilon = e^{i\theta} p_\varepsilon$, $A_1^\varepsilon q_\varepsilon = e^{-i\theta} p_\varepsilon$, and $\langle p_\varepsilon, q_\varepsilon \rangle = 1$. Analogous to the proof of Theorem A, $p_\varepsilon = \tilde{p}_\varepsilon + O(\varepsilon^{k-l+1})$ and $q_\varepsilon = \tilde{q}_\varepsilon + O(\varepsilon^{k-l+1})$, where $\tilde{p}_\varepsilon, \tilde{q}_\varepsilon \in C^2$ satisfy $A_1^\varepsilon \tilde{p}_\varepsilon = (\tilde{\alpha}_e - i\tilde{\beta}_e) \tilde{p}_\varepsilon$, $A_\varepsilon \tilde{q}_\varepsilon = (\tilde{\alpha}_e + i\tilde{\beta}_e) \tilde{q}_\varepsilon$, and $\langle \tilde{p}_\varepsilon, \tilde{q}_\varepsilon \rangle = 1$. Furthermore, from hypothesis B3, $A_\varepsilon$ is in its normal Jordan form, thus we can take $\tilde{p}_\varepsilon = q_\varepsilon = p := (1, -i)/\sqrt{2}$. Hence,

$$
\begin{align*}
g_{20} &= \langle p_\varepsilon, B_\varepsilon(q_\varepsilon, q_\varepsilon) \rangle = \langle p, e^i B_\varepsilon(p, p) \rangle + O(\varepsilon^{k+1}), \\
g_{11} &= \langle p_\varepsilon, B_\varepsilon(q_\varepsilon, q_\varepsilon) \rangle = \langle p, e^i B_\varepsilon(p, p) \rangle + O(\varepsilon^{k+1}), \\
g_{02} &= \langle p_\varepsilon, B_\varepsilon(q_\varepsilon, q_\varepsilon) \rangle = \langle p, e^i B_\varepsilon(p, p) \rangle + O(\varepsilon^{k+1}), \\
g_{21} &= \langle p_\varepsilon, C_\varepsilon(q_\varepsilon, q_\varepsilon, q_\varepsilon) \rangle = \langle p, e^i C_\varepsilon(p, p, p) \rangle + O(\varepsilon^{k+1}).
\end{align*}
$$

(24)

Now, from (6), we compute

$$\ell_1 = \ell^*_1 + O(\varepsilon^{k+1}),$$

(25)
where $\ell_1^*$ is given by (23). From hypothesis, $j^*, l \leq j^* \leq k$, is the first subindex such that $\ell_{1,j^*} \neq 0$. Therefore, we can choose $\varepsilon_3 > 0$, $0 < \varepsilon_3 < \varepsilon_2$, in order that $\text{sgn}(\ell_1) = \text{sgn}(\ell_{1,j^*})$ for every $\varepsilon \in (0, \varepsilon_3)$. From here, the proof follows analogously to the proof of Theorem A by applying Theorem 1.

### 4.3. Lyapunov Coefficient

Suppose that the expressions defined in (21) for $A_j$, $B_j$, and $C_j$, $l \leq j \leq k$, are known. If the Taylor expansion (22) for $e^{i\theta_x}$ is explicitly known, then we can also compute the following Taylor expansions around $\varepsilon = 0$,

\[
e^{-i\theta_x} = 1 + \varepsilon^l \sum_{n=0}^{k-2l} \varepsilon^n r_{n+l} + O(\varepsilon^{k-l+1}) \quad \text{and} \quad (26)
\]

\[
\varepsilon^l (1 - 2e^{i\theta_x})e^{-2i\theta_x} \frac{1}{1 - e^{-i\theta_x}} = \sum_{n=0}^{k-l} \varepsilon^n s_n + O(\varepsilon^{k-l+1}).
\]

Notice that $s_0 = -\frac{i}{\omega_0}$ and $r_l = -i\omega_0$.

In what follows, we provide the formulae for $\ell_{1,j}$, $l \leq j \leq k$. Thus, let $p = (1, -i)/\sqrt{2}$ and, for $m \leq k - l$, denote

\[
L_m = \langle p, C_{l+m}(p, p, p) \rangle - \sum_{n_1+n_2+n_3=m} s_{n_1} \langle p, B_{l+n_2}(p, p) \rangle \langle p, B_{l+n_3}(p, p) \rangle,
\]

\[
\tilde{L}_m = \langle p, C_{l+m}(p, p, p) \rangle - \sum_{n_1+n_2+n_3=m} s_{n_1} \langle p, B_{l+n_2}(p, p) \rangle \langle p, B_{l+n_3}(p, p) \rangle
\]

\[
+ \sum_{n_1+n_2+n_3=m} r_{n_1+n_2} \langle p, C_{l+n_2}(p, p, p) \rangle - \sum_{n_1+n_2=m} \langle p, B_{l+n_1}(p, p) \rangle \langle B_{l+n_1}(p, p), p \rangle
\]

\[
+ \frac{1}{2} \sum_{n_1+n_2=m} \langle p, B_{l+n_1}(\bar{p}, p) \rangle \langle B_{l+n_1}(\bar{p}, p), p \rangle.
\]

**Proposition 1.** Assume the hypotheses of Theorem B and consider the coefficients $\ell_{1,j}$, $l \leq j \leq k$, as defined in (23). Then,

\[
\ell_{1,j+m} = \begin{cases} 
\frac{1}{2} \text{Re} \left(L_m\right) & \text{for } 0 \leq m < l, \\
\frac{1}{2} \text{Re} \left(L_m + \tilde{L}_{m-l} - L_{m-l}\right) & \text{for } l \leq m \leq k - l.
\end{cases}
\]

**Proof.** Substituting (24) and (26) into (6) and collecting the coefficients of $\varepsilon^l$ and $\varepsilon^{2l}$, we have

\[
\ell_4 = \varepsilon^l \frac{\text{Re}(K_1^l)}{2} + \varepsilon^{2l} \frac{\text{Re}(K_2^l)}{2} + O(\varepsilon^{k+1}), \quad (27)
\]
Thus, from (21),

\[ K^1_\varepsilon = \langle p, C_\varepsilon(p, p, \bar{p}) \rangle - \langle p, B_\varepsilon(p, p) \rangle \langle p, B_\varepsilon(p, \bar{p}) \rangle \sum_{n=0}^{k-l} \varepsilon^n s_n \]

\[ K^2_\varepsilon = \langle p, C_\varepsilon(p, p, \bar{p}) \rangle \sum_{n=0}^{k-2l} \varepsilon^n r_{l+n} - \langle p, B_\varepsilon(p, p) \rangle \langle B_\varepsilon(p, \bar{p}) \rangle p \]

\[ - \frac{1}{2} \langle p, B_\varepsilon(p, p) \rangle \langle B_\varepsilon(p, p), p \rangle. \]

First, notice that

\[ \langle p, B_\varepsilon(q, q) \rangle \langle p, B_\varepsilon(p, \bar{p}) \rangle \sum_{n=0}^{k-l} \varepsilon^n s_n = \sum_{m=0}^{k-l} \sum_{n_1+n_2+n_3=m} s_{n_1} \langle p, B_{l+n_2}(p, p) \rangle \langle p, B_{l+n_3}(p, \bar{p}) \rangle + O(\varepsilon^{k-l+1}). \]

Thus, from (21),

\[ K^1_\varepsilon = \sum_{m=0}^{k-l} \varepsilon^m L_m. \]  \hspace{1cm} (28)

Now, since

\[ \langle p, C_\varepsilon(p, p, \bar{p}) \rangle \sum_{n=0}^{k-2l} \varepsilon^n r_{l+n} = \sum_{m=0}^{k-2l} \varepsilon^m \sum_{n_1+n_2=n} r_{l+n_1} \langle p, C_{l+n_2}(p, p, \bar{p}) \rangle + O(\varepsilon^{k-2l+1}) \]

\[ \langle p, B_\varepsilon(p, p) \rangle \langle B_\varepsilon(p, p), p \rangle = \sum_{m=0}^{k-2l} \varepsilon^m \sum_{n_1+n_2=n} \langle p, B_{l+n_1}(p, \bar{p}) \rangle \langle B_{l+n_1}(p, \bar{p}), p \rangle + O(\varepsilon^{k-2l+1}), \]

\[ \langle p, B_\varepsilon(p, p) \rangle \langle B_\varepsilon(p, p), p \rangle = \sum_{m=0}^{k-2l} \varepsilon^m \sum_{n_1+n_2=n} \langle p, B_{l+n_1}(p, \bar{p}) \rangle \langle B_{l+n_1}(p, \bar{p}), p \rangle + O(\varepsilon^{k-2l+1}), \]

we have that

\[ K^2_\varepsilon = \sum_{m=0}^{k-2l} \varepsilon^m (\bar{L}_m - L_m). \]  \hspace{1cm} (29)

Substituting (28) and (29) into (27), we get

\[ \ell_1 = \frac{\varepsilon^l}{2} \left( \sum_{m=0}^{k-l} \varepsilon^m \text{Re}(L_m) + \sum_{m=l}^{k-1} \varepsilon^m \text{Re}(\bar{L}_{m-l} - L_{m-l}) \right) + O(\varepsilon^{k+1}). \]

From here, we split the analysis in two cases, namely $k - l < l$ and $l \leq k - l$.

If $k - l < l$, we have

\[ \ell_1 = \sum_{m=0}^{k-l} \varepsilon^{m+l} \frac{\text{Re}(L_m)}{2} + O(\varepsilon^{k+1}). \]

Thus, in this case, $\ell_{1,l+m} = \frac{1}{2} \text{Re}(L_m)$, for $m = 0, 1, \ldots, k - l < l$. 
Otherwise, if \( l \leq k - l \), we have

\[
\ell_1 = \sum_{m=0}^{l-1} \varepsilon^{m+l} \frac{\text{Re}(L_m)}{2} + \sum_{m=l}^{k-l} \varepsilon^{m+l} \text{Re}(L_m + \tilde{L}_{m-l} - L_{m-l}) + \mathcal{O}(\varepsilon^{k+1}).
\]

Thus, in this case,

\[
\ell_{1,l+m} = \begin{cases} 
\frac{1}{2} \text{Re}(L_m) & \text{for } 0 \leq m < l, \\
\frac{1}{2} \text{Re}(L_m + \tilde{L}_{m-l} - L_{m-l}) & \text{for } l \leq m \leq k - l.
\end{cases}
\]

Hence, we have concluded this proof. \( \square \)

4.4. Hopf bifurcation in the \( l \)th averaged system. Let \( l, 1 \leq l < k \), be the subindex of the first non-vanishing averaging function. As set in hypothesis B1, \((x_{\mu_0}; \mu_0)\) is a Hopf point of the averaged system of order \( l \) \( \square \). It is easy to see that its first Lyapunov coefficient at \((x_{\mu_0}; \mu_0)\) is given by \( \varepsilon^l \ell_{1,l} \). In addition, from Proposition 1, we compute

\[
\ell_{1,l} = \frac{1}{8} \left( \frac{\partial^3 g^3_1(x_{\mu_0}; \mu_0)}{\partial x^3} + \frac{\partial^3 g^3_1(x_{\mu_0}; \mu_0)}{\partial x \partial y^2} + \frac{\partial^3 g^3_1(x_{\mu_0}; \mu_0)}{\partial x \partial y^2} + \frac{\partial^3 g^3_1(x_{\mu_0}; \mu_0)}{\partial x \partial y^2} + \frac{\partial^3 g^3_1(x_{\mu_0}; \mu_0)}{\partial x \partial y^2} \right) + \frac{1}{8 \omega_0} \left( \frac{\partial^2 g^2_1(x_{\mu_0}; \mu_0)}{\partial x \partial y} \left( \frac{\partial^2 g^2_1(x_{\mu_0}; \mu_0)}{\partial x^2} + \frac{\partial^2 g^2_1(x_{\mu_0}; \mu_0)}{\partial x^2} \right) - \frac{\partial^2 g^2_1(x_{\mu_0}; \mu_0)}{\partial x^2} \right)
\]

We notice that a Hopf bifurcation in the averaged system \( \square \) is characterized by hypotheses B1, B2, and \( \ell_{1,l} \neq 0 \). Hypothesis B3 is just a technical assumption with no loss of generality. Therefore, from Theorem 3, a Hopf bifurcation in the averaged system of order \( l \) \( \square \) implies in a Torus bifurcation in the differential equation \( \square \).

5. Invariant torus in a 3D vector field

In this section, as an example of application of the developed theory, we show how to use Theorems A and B for detecting an invariant torus in the following family of 3D vector fields,

\[
\begin{align*}
x' &= -y + \varepsilon P_1(x, y, z, \mu) + \varepsilon^2 P_2(x, y, z, \mu) + \mathcal{O}(\varepsilon^3), \\
y' &= x + \varepsilon^2 Q(x, y, z, \mu) + \mathcal{O}(\varepsilon^3), \\
z' &= \varepsilon R_1(x, y, z, \mu) + \varepsilon^2 R_2(x, y, z, \mu) + \mathcal{O}(\varepsilon^3),
\end{align*}
\]

(30)
there exists a smooth curve
unique limit cycle
Proof. Writing the vector field (30) in cylindrical coordinates \((r, \theta, z)\), of \(\epsilon\), we get the following nonautonomous

\[\rho(x, r) = \frac{1}{\sqrt{x^2 + y^2}}\]

Proposition 2. For \(\epsilon > 0\) and \(|\mu|\) sufficiently small the vector field (30) admits a unique limit cycle \(\varphi(t; \mu, \epsilon) = (x(t; \mu, \epsilon), y(t; \mu, \epsilon), z(t; \mu, \epsilon))\) satisfying \(x(t; \mu, 0)^2 + y(t; \mu, 0)^2 = 1\) and \(z(t; \mu, 0) = 0\), for every \(t \in \mathbb{R}\). Assume that \(a^2 + b^2 \neq 0\). Then, there exists a smooth curve \(\mu(\epsilon)\), defined for \(\epsilon > 0\) sufficiently small and satisfying \(\mu(\epsilon) = -\epsilon \pi/2 + O(\epsilon^2)\), and neighborhoods \(V_\epsilon \subset \mathbb{R}^3\) of the limit cycle \(\varphi(t; \mu(\epsilon), \epsilon)\) and \(J_\epsilon\) of \(\mu(\epsilon)\) for which the following statement holds.

(i) If \(a \neq 0\), then a unique invariant torus of the vector field (30) bifurcates in \(V_\epsilon\) from the limit cycle \(\varphi(t; \mu(\epsilon), \epsilon)\), as \(\mu\) passes through \(\mu(\epsilon)\). Such a torus exists whenever \(\mu \in J_\epsilon\) and \(a(\mu - \mu(\epsilon)) < 0\) and surrounds the limit cycle \(\varphi(t; \mu, \epsilon)\). In addition, if \(a > 0\) (resp. \(a < 0\)) the torus is unstable (resp. stable), whereas the limit cycle \(\varphi(t; \mu, \epsilon)\) is asymptotically stable (resp. unstable) (see Figure [1]).

(ii) If \(a = 0\) and \(b \neq 0\), then a unique invariant torus of the vector field (30) bifurcates in \(V_\epsilon\) from the limit cycle \(\varphi(t; \mu(\epsilon), \epsilon)\), as \(\mu\) passes through \(\mu(\epsilon)\). Such a torus exists whenever \(\mu \in J_\epsilon\) and \(b(\mu - \mu(\epsilon)) < 0\) and surrounds the limit cycle \(\varphi(t; \mu, \epsilon)\). In addition, if \(b > 0\) (resp. \(b < 0\)) the torus is unstable (resp. stable), whereas the limit cycle \(\varphi(t; \mu, \epsilon)\) is stable (resp. unstable) (see Figure [3]).

Proof. Writing the vector field (30) in cylindrical coordinates \((x, y, z) = (r \cos \theta, r \sin \theta, z)\) and taking \(\theta\) as the new independent variable we get the following nonautonomous

\[P_1(x, y, z; \mu) = 10x(3\mu + a(9 + 4x^2 + 3z^2)) - 30x\rho(x, y)(z + \mu + a(1 + 4x^2 + z^2)),\]

\[P_2(x, y, z; \mu) = 10b x(9 + 4x^2 + 3z^2) - 30b x\rho(x, y)(1 + 4x^2 + z^2),\]

\[Q(x, y, z; \mu) = -30y(15(\mu^2 - 1) + 20a(\mu(4y^2 + 3z^2 + 9) + 3z)) + 3a^2(24y^4 + 40y^2(z^2 + 5) + 15(z^4 + 6z^2 + 5)) + 150\pi y\rho(x, y)(3\mu^2 + 6\mu(2a(4y^2 + z^2 + 1) + z)) + a(3a(24y^4 + 8y^2(3z^2 + 5) + 3(z^2 + 1)^2) + 8y^2z + 6z^3 + 6z) - 3),\]

\[R_1(x, y, \mu) = 30x^2(1 - 2a z)\rho(x, y) + 15(a(2x^2z + z^3 + z) + \mu z - 1),\]

\[R_2(x, y, z; \mu) = -225\pi a^2 z(8y^4 + 12y^2(z^2 + 3) + 3(z^2 + 1)^2) - 450\pi a(y^2(4\mu z - 6) + (z^2 + 1)(2\mu z - 1)) + 15b(2x^2z + z^3 + z) - 225\pi((\mu^2 - 1)z - 2\mu) + 60\rho(x, y)(5\pi y^2(a((6a z - 1)(4y^2 + 3z^2) + 18a z + 12\mu z - 9) - 3\mu) - bx^2z),\]

and
differential equation

\[
\frac{\partial r}{\partial \theta} = \varepsilon \cos \theta \tilde{P}_1(\theta, r, z) + \varepsilon^2 \left( \frac{1}{r} \cos \theta \sin \theta \tilde{P}_1(\theta, r, z)^2 + \cos \theta \tilde{P}_2(\theta, r, z) + \sin \theta \tilde{Q}(\theta, r, z) \right),
\]

\[
\frac{\partial z}{\partial \theta} = \varepsilon \tilde{R}_1(\theta, r, z) + \varepsilon^2 \left( \frac{1}{r} \sin \theta \tilde{P}_1(\theta, r, z) \tilde{R}_1(\theta, r, z) + \tilde{R}_2(\theta, r, z) \right),
\]

where \( \tilde{P}_i(\theta, r, z) = P_i(r \cos \theta, r \sin \theta, z) \) and \( \tilde{R}_i(\theta, r, z) = R_i(r \cos \theta, r \sin \theta, z) \), for \( i = 1, 2 \), and \( \tilde{Q}(\theta, r, z) = Q(r \cos \theta, r \sin \theta, z) \). Computing the first and second averaging functions of (31) we get

\[
g_1(r, z) = 30\pi \left( (r - 1) \mu - z + a (r - 1)((r - 1)^2 + z^2) \right),
\]

\[
g_2(r, z) = 30\pi b \left( (r - 1)((r - 1)^2 + z^2), z((r - 1)^2 + z^2) \right).
\]

We notice that \( g_1 \) satisfies hypothesis B1 for \( x_\mu = (1, 0) \) and \( \mu_0 = 0 \).

Following the method described in the previous section we take \( y = x + \xi(\mu, \varepsilon) \) and \( \mu = \sigma + \mu(\varepsilon) \). It is easy to see that \( \xi(\mu, \varepsilon) = (1, 0) + \mathcal{O}(\varepsilon^2) \) and \( \mu(\varepsilon) = -\varepsilon \pi/2 + \mathcal{O}(\varepsilon^2) \). Thus, the transformed Poincaré map (20) for the differential equation (31) writes

\[
H_\varepsilon(y, 0) = A_\varepsilon y + \frac{1}{6} C_\varepsilon(y, y, y) + \mathcal{O}(|y|^4),
\]

with \( A_\varepsilon = I + \varepsilon^1 A_1 + \varepsilon^2 A_2 + \mathcal{O}(\varepsilon^3) \), \( B_\varepsilon(u, v) = \varepsilon^1 B_1(u, v) + \varepsilon^2 B_2(u, v) + \mathcal{O}(\varepsilon^3) \), and \( C_\varepsilon(u, v, w) = \varepsilon^1 C_1(u, v, w) + \varepsilon^2 C_2(u, v, w) + \mathcal{O}(\varepsilon^3) \), where

\[
A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad B_1(u, v) = B_2(u, v) = (0, 0),
\]

and

\[
C_1(u, v, w) = 2\ell_1^i \left( 3u_1v_1w_1 + u_2v_2w_1 + u_2v_1w_2 + u_1v_2w_2, u_2v_1w_1 + u_1v_2w_1 + u_1v_1w_2 + 3u_2v_2w_2 \right),
\]

for \( i = 1, 2 \).

Notice that \( I + \varepsilon A_1 + \varepsilon^2 A_2 \) satisfies hypotheses B3. Thus, we can define \( e^{i\theta_0} = 1 + \varepsilon i - \varepsilon^2 \frac{1}{2} + \mathcal{O}(\varepsilon^3) \). From Proposition (1) and (25) we get \( \ell_1 = \varepsilon 4a + \varepsilon^2 4b + \mathcal{O}(\varepsilon^3) \). In this case, following the notation (23), \( \ell_{1,1} = 4a \) and \( \ell_{1,2} = 4b \). Hence, applying Theorem 13 we conclude this proof.
Appendix: Higher order averaged functions

The averaging theory is one of the most classical analytical methods to study isolated periodic solutions of differential equations in the presence of a small parameter. Usually, this theory deals with differential equations in the following standard form

\[ \dot{x}(t) = \sum_{i=1}^{k} \varepsilon^i F_i(t, x) + \varepsilon^{k+1} \tilde{F}(t, x, \varepsilon), \]  

(32)
where $F_i$ and $\bar{F}$ are sufficiently smooth functions, $T$-periodic in the variable $t$, $x \in \Omega$ with $\Omega$ an open bounded subset of $\mathbb{R}^2$, $t \in \mathbb{R}$, and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. In [6,8] it has been established that the $i$-th order averaged function of (32) is given by

$$g_i(x) = \frac{y_i(T, x)}{i!},$$

where $y_i : \mathbb{R} \times D \to \mathbb{R}^n$, for $i = 1, 2, \ldots, k$, are defined recurrently as

$$y_1(t, x) = \int_0^t F_1(s, x) \, ds,$$

$$y_i(t, x) = \int_0^t \left( i! F_i(s, x) + \sum_{l=1}^{i-1} \sum_{m=1}^l \frac{i!}{l! m!} \partial^m F_{i-l}(s, x) \mathbb{B}_{l,m}(y_1(s, x), \ldots, y_{i-m+1}(s, x)) \right) \, ds.$$

Here, $\partial^L F(t, x)$ denotes the Frechet’s derivative with respect to the variable $x$, which is a $L$-multilinear map applied to a “product” of $L$ vectors of $\mathbb{R}^n$, $\bigotimes_{j=1}^L y_j \in \mathbb{R}^{n^L}$, where $y_j = (y_{j1}, \ldots, y_{jn}) \in \mathbb{R}^n$. Formally,

$$\partial^L F(t, x) \bigotimes_{j=1}^L y_j = \sum_{i_1, \ldots, i_L=1}^n \frac{\partial^L F(t, x)}{\partial x_{i_1} \ldots \partial x_{i_L}} y_{i1} \cdots y_{iL}.$$

Also, for $p$ and $q$ positive integers, $\mathbb{B}_{p,q}$ denotes the partial Bell polynomials,

$$\mathbb{B}_{p,q}(x_1, \ldots, x_{p+q+1}) = \sum_{\tilde{S}_{p,q}} \frac{p!}{b_1! b_2! \cdots b_{p+q+1}!} \prod_{j=1}^{p-q+1} \left( \frac{x_j}{j!} \right)^{b_j},$$

where now $\tilde{S}_{p,q}$ is the set of all $(p-q+1)$-tuple of nonnegative integers $(b_1, b_2, \ldots, b_{p+q+1})$ satisfying $b_1 + 2b_2 + \cdots + (p - q + 1)b_{p-q+1} = p$, and $b_1 + b_2 + \cdots + b_{p+q+1} = q$.

The next results were proved in [6]. Nonsmooth versions of these results can be found in [5].

**Lemma 5** ([6]). Let $\varphi(\cdot, x, \varepsilon) : [0, t_\varepsilon] \to \mathbb{R}^n$ be the solution of (17) with $\varphi(0, x, \varepsilon) = x$. Then, for $|\varepsilon|$ sufficiently small, $t_\varepsilon > T$ and

$$\varphi(t, x, \varepsilon) = x + \sum_{i=1}^k \varepsilon^i \frac{y_i(t, x)}{i!} + O(\varepsilon^{k+1}).$$

**Theorem 2** ([6]). Assume that, for some $l \in \{1, 2, \ldots, k\}$, $g_l = 0$ for $i = 1, 2, \ldots, l-1$, and $g_l \neq 0$. Then, for each $a^* \in D$ such that $g_l(a^*) = 0$ and $\det(\partial f_l(a^*)) \neq 0$, there exists, for $|\varepsilon| > 0$ sufficiently small, a $T$-periodic solution $\varphi(\cdot, \varepsilon)$ of (32) such that $\varphi(0, \varepsilon) \to a^*$ when $\varepsilon \to 0$. 
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