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A Hybrid Consensus Protocol for Pointwise Exponential Stability with Intermittent Information

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Abstract: We propose a solution to the problem of achieving consensus of the state of multiple systems connected over a network (over a directed graph) in which communication events are triggered stochastically. Our solution consists of a protocol design that, using intermittent information obtained over the network, asymptotically drive the values of their states to agreement, with stability, globally and with robustness to perturbations. More precisely, we propose a protocol with hybrid dynamics, namely, an algorithm with variables that jump at the communication events and evolve continuously in between such events. We design the protocols by recasting the consensus problem as a set stabilization problem and applying Lyapunov stability tools for hybrid systems. We provide sufficient conditions for exponential stability of the consensus set. Furthermore, we show that under additional conditions this set is also partially pointwise globally exponentially stable. Robustness of consensus to certain classes of perturbations is also established. Numerical examples confirm the main results.

Keywords: Consensus Protocol, Set Stability of Hybrid Systems, Networked Systems

1. INTRODUCTION

The topic of consensus has gained massive traction in recent years due to the wide range of applications science and engineering. A challenge to the design of consensus protocols is when information is only available at intermittent time instances. Different from consensus of continuous and discrete-time systems, which is thoroughly understood in Olfati-Saber and Murray (2004) and Cortés (2008), the introduction of a sampling period or impulsive information transfer for first and higher order systems has been studied in Jie and Zhong (2014); Liu et al. (2010); Guan et al. (2012); Hu et al. (2013); Wen et al. (2013). For such cases, the application of systems theory tools like Lyapunov functions, contraction theory, and incremental input-to-state stability have been proposed. Notably, recent research efforts on sample-data systems and event triggered control for the stabilization of sets provide results that can become useful for consensus, though some of the assumptions need to be carefully fit to the consensus problem under intermittent communication networks.

This article deals with the problem of consensus of first-order integrator systems communicating at stochastically determined time instances over a network. The consensus problem studied here consists of designing a protocol guaranteeing that the state of each agent converges to a common value by only using intermittent information from their neighbors. To solve this problem, we design hybrid state-feedback protocols that undergo an instantaneous change in their states when new information is available, and evolves continuously between such events. Due to the combination of continuous and impulsive dynamics, we use hybrid systems theory to model the interconnected systems, the controller, and the network topologies as well as to design the protocols, for which, we apply a Lyapunov theorem for asymptotic stability of sets for hybrid systems. Aside from asymptotic stability, we specify the point to which the consensus states converge to and, when the communications graph is strongly connected and weight balanced, write it as a function of their initial conditions. We show that a diagonal-like set is, in fact, partially pointwise globally exponentially stable, which is a stronger notion than typical notions of asymptotic stability due to the additional requirement that each point in the set is Lyapunov stable; see e.g. Goebel (2010) and Bhat and Bernstein (2003) for similar notions. Furthermore, we show that the consensus condition is robust to a class of perturbations on the information. Finally, we give some brief insight into modeling and an asymptotic stability result for the case when information may arrive at asynchronous events for each agent.

The remainder of this paper is organized as follows. Section 2 gives some preliminary background on graph theory and hybrid systems. Section 3 introduces the consensus problem, impulsive network model, and the control structure. In Section 4, we define a hybrid protocol and give the main results. The preliminary results for asynchronous update times are in Section 5.

Notation The set of natural numbers is denoted as \( \mathbb{N} \), i.e., \( \mathbb{N} := \{0, 1, 2, 3, \ldots\} \). The set \( \text{eig}(A) \) contains the eigenvalues of \( A \). Given two vectors \( u, v \in \mathbb{R}^n \), \( |u| := \sqrt{u^\top u} \), notation \( |u^\top v| \) is equivalent to \( (u, v) \). Given a function \( m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \), \( |m|_\infty := \sup_{t \geq 0} |m(t)| \). Given a symmetric matrix \( P \), \( \lambda(P) := \max(\lambda : \lambda \in \text{eig}(P)) \) and \( \lambda(P) := \min(\lambda : \lambda \in \text{eig}(P)) \). Given matrices \( A, B \) with proper dimensions, we define the operator \( \text{He}(A, B) := A^\top B + B^\top A \). \( 1_N \) is a vector of \( N \) ones.

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2. PRELIMINARIES ON GRAPH THEORY AND HYBRID SYSTEMS

2.1 Preliminaries on graph theory

A directed graph (digraph) is defined as $\Gamma = (V, \mathcal{E}, G)$. The set of nodes of the digraph are indexed by the elements of $V = \{1, 2, \ldots, N\}$ and the edges are pairs in the set $\mathcal{E} \subseteq V \times V$. Each edge directly links two different nodes, i.e., an edge from $i$ to $k$, denoted by $(i, k)$, implies that agent $i$ can send information to agent $k$. The adjacency matrix of the digraph $\Gamma$ is denoted by $G = (g_{ik}) \in \mathbb{R}^{N \times N}$, where $g_{ik} = 1$ if $(i, k) \in \mathcal{E}$, and $g_{ik} = 0$ otherwise. The in-degree and out-degree of agent $i$ are defined by $d^\text{in}(i) = \sum_{k=1}^{N} g_{ki}$ and $d^\text{out}(i) = \sum_{k=1}^{N} g_{ik}$. The largest (smallest) in-degree in the digraph is given by $d = \max_{i \in V} d^\text{in}(i)$ ($\underline{d} = \min_{i \in V} d^\text{in}(i)$). The in-degree matrix $D$ is the diagonal matrix with entries $D_{ii} = d^\text{in}(i)$ for all $i \in V$. The Laplacian matrix of the digraph $\Gamma$, denoted by $L$, is defined as $L = D - G$. The Laplacian has the property that $L_{ij} = 0$. The set of indices corresponding to the neighbors that can send information to the $i$-th agent is denoted by $\mathcal{N}(i) = \{k \in V : (k, i) \in \mathcal{E}\}$.

In this article, we will make varying assumptions on the complexity of the underlying graph structure corresponding to the network. For self-containment, we summarize the needed notions and results from the literature.

**Definition 2.1.** A directed graph is said to be

- **weight balanced** if, at each node $i \in V$, the out-degree and in-degree are equal; i.e., for each $i \in V$, $d^\text{out}(i) = d^\text{in}(i)$;
- **complete** if every pair of distinct vertices is connected by a unique edge; that is $g_{ik} = 1$ for each $i, k \in V$, $i \neq k$;
- **strongly connected** if and only if any two distinct nodes of the graph can be connected via a path that traverses the directed edges of the digraph.

**Lemma 2.2.** ([Olfati-Saber and Murray, 2004, Theorem 6], (Fax and Murray, 2004, Propositions 1, 3, and 4)) For an undirected graph, $\mathcal{L}$ is symmetric and positive semidefinite and each eigenvalue of $\mathcal{L}$ is real. For a directed graph, zero is a simple eigenvalue of $\mathcal{L}$ if the directed graph is strongly connected.

**Lemma 2.3.** (Godsil and Royle (2013)) Consider an $n \times n$ symmetric matrix $A = \{a_{ij}\}$ satisfying $\sum_{i=1}^{n} a_{ik} = 0$ for each $k \in \{1, 2, \ldots, n\}$. The following statements hold:

(i) There exists an orthogonal matrix $U$ such that

$$
U^T A U = \begin{bmatrix} 0 & \star \\
0 & 0 \end{bmatrix}
$$

where $\star$ represents any nonsingular matrix with an appropriate dimension and 0 represents any zero matrix with an appropriate dimension.

(ii) The matrix $A$ has a zero eigenvalue with eigenvector $1_{n} \in \mathbb{R}^{n}$.

2.2 Preliminary on Hybrid Systems

A hybrid system $\mathcal{H}$ has data $(C, f, D, G)$ and is defined by

$$
z = f(z) \quad z \in C, 
z^+ = G(z) \quad z \in D,
$$

where $z \in \mathbb{R}^n$ is the state, $f$ defines the flow map capturing the continuous dynamics and $C$ defines the flow set on which $f$ is effective. The set-valued map $G$ defines the jump map and models the discrete behavior, while $D$ defines the jump set, which is the set of points from where jumps are allowed. A solution $^1$ $\phi$ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where $t$ denotes ordinary time and $j$ denotes jump time. The domain $\phi$ is $C, f, D, G$ is a hybrid time domain if for every $(T, j) \in \phi$, the set $\text{dom} \phi \cap ([0, T] \times \{0, 1, \ldots, j\})$ can be written as the union of sets $\text{dom} \phi = \bigcup_{j=0}^{T} \phi(I_j \times \{j\})$, where $I_j := [(t, t+1)]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{j+1}$. The $t_i$'s with $j > 0$ define the time instants when the state of the hybrid system jumps and $j$ counts the number of jumps. The set $\mathcal{S}_\mathcal{H}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_\mathcal{H}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from $\xi$.

In this paper, we consider the following stability notions.

**Definition 2.4.** (global exponential stability) Let a hybrid system $\mathcal{H}$ be defined on $\mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ be closed. The set $A$ is said to be **globally exponentially stable** (GES) for $\mathcal{H}$ if there exist $\kappa, \alpha > 0$ such that every maximal solution $\phi$ to $\mathcal{H}$ is complete and satisfies

$|\phi(t, j)|_A \leq \kappa e^{-\alpha(t+j)}|\phi(0,0)|_A$

for each $(t, j) \in \text{dom} \phi$.

**Definition 2.5.** (partial pointwise global exponential stability) Consider a hybrid system $\mathcal{H}$ with state $z = (p, q) \in \mathbb{R}^n$. The closed set $A \subseteq \mathbb{R}^n$ is partially pointwise globally exponentially stable with respect to the state component $p$ for $\mathcal{H}$ if

1) every maximal solution $\phi$ to $\mathcal{H}$ is complete and has a limit belonging to $A$;
2) $A$ is exponentially attractive for $\mathcal{H}$, namely, for each $\phi \in \mathcal{S}_\mathcal{H}$ there exist $\kappa, \alpha > 0$ such that $|\phi(t, j)|_A \leq \kappa e^{-\alpha(t+j)}|\phi(0,0)|_A$ for all $(t, j) \in \text{dom} \phi$; and
3) for each $p^* \in \mathbb{R}^n$ such that there exists $q \in \mathbb{R}^n$ satisfying $(p^*, q) \in A$, it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $\phi = (\phi_p, \phi_q)$ to $\mathcal{H}$ with $\phi_p(0,0) \in p^* + \delta B$ satisfies $|\phi_p(t, j) - p^*| \leq \varepsilon$ for all $(t, j) \in \text{dom} \phi$.

A hybrid system is said to satisfy the hybrid basic conditions if (Goebel et al., 2012, Assumption 6.5) holds. We refer the reader to Goebel et al. (2012) for more details on these notions and the hybrid systems framework. The asymptotic version of the notion in Definition 2.5 can be found in Goebel and Sanfelice (2016).

3. CONSENSUS USING INTERMITTENT INFORMATION

3.1 Problem Description

Consider a group of $N$ agents with dynamics

$$
\dot{x}_i = u_i \quad i \in V := \{1, 2, \ldots, N\}
$$

that exchange information over a digraph $\Gamma = (V, \mathcal{E}, G)$, where $x_i \in \mathbb{R}$ is the state and $u_i \in \mathbb{R}$ is the control input of the $i$-th agent. Our goal is to design a control protocol (or feedback controller) assigning the input $u_i$ to drive the solutions of each agent to a common constant value. In particular, we are interested in the following asymptotic

$^1$ A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the $t$ direction. A solution is precompact if it is complete and bounded.
convergence property of the states $x_i$, known as static consensus; see Olafitî-Saiber and Murray (2004, 2002).

**Definition 3.1.** (static consensus). Given the agents in (3) over a digraph $\Gamma$, a control protocol $u_i$ is said to solve the consensus problem if every resulting maximal solution with $u = (u_1,u_2,\ldots,u_N)$ is complete and its $x$ component $t \mapsto (x_1(t),x_2(t),\ldots,x_N(t))$ satisfies
\[
\lim_{t \to \infty} |x_i(t) - x_k(t)| = 0
\]
for each $i, k \in \mathcal{V}, i \neq k$.

We consider the scenario where the state of each system is available to each other system only at isolated time instances. Namely, the $i$-th agent receives information from its neighbors at time instances $t_s$, where $s \in \mathbb{N} \setminus \{0\}$ is the communication event index; specifically, agent $i$ receives
\[y_{ki}(t_s) = x_k(t_s) \quad \forall k \in \mathcal{N}(i)\]
at each $t_s$. Given positive numbers $T_2 \geq T_1$, we assume that the time between these events is governed by a discrete random variable with some bounded distribution probability. Namely, for each $i \in \mathcal{V}$, the random variable $\Omega_s \in [T_1,T_2]$ determines the time elapsed between such communication events for each $i$-th system, i.e.,
\[t_{s+1} - t_s = \Omega_s \quad \forall s \in \mathbb{N} \setminus \{0\}.\]
(4)
The scalar values $T_1$ and $T_2$ define the lower and upper bounds, respectively, of the time allowed to elapse between consecutive transmission instances. In this way, the random variable $\Omega_s$ may take values only on the bounded interval $[T_1,T_2]$, while the probability density function governing its distribution can be arbitrary as long as it assigns values to $\Omega_s$ that are in $[T_1,T_2]$.

### 3.2 Proposed Controller Design

We propose a hybrid control protocol and design procedure for consensus of (3) over networks with intermittent transmission of information as defined in the previous section. For the $i$-th agent, the proposed control protocol assigns a value to $u_i$ based on the measured outputs of the neighboring agents obtained at communication events. In particular, the controller assigns $u_i$ to a control variable $\eta_i$ which is allowed to be impulsively updated via
\[\eta_i^+ := \sum_{k \in \mathcal{N}(i)} G_{ik}(x_i,y_{ki},\eta_k) \quad \text{at each communication event, where}\]
\[G_{ik} : \mathbb{R}^3 \to \mathbb{R}\]
is the update law using information from the $k$-th neighbor. Furthermore, the controller state $\eta_i$ is allowed to evolve continuously between such events. Due to the nonperiodic arrival of information and impulsive dynamics, classical analysis tools (for continuous-time or discrete-time systems) do not apply to the design of the proposed controller. This motivates us to design the proposed controller by recasting the interconnected systems, the impulsive network, and such a control protocol in a hybrid system framework; specifically, the one given in Goebel et al. (2012).

### 4. MAIN RESULTS

#### 4.1 Hybrid Modeling

A timer state $\tau$ is introduced to model the network communication times given by (4). We design its hybrid dynamics as follows: from positive values it decreases to zero as ordinary time increases and, whenever it reaches zero, it is reset to an arbitrary value in the interval $[T_1,T_2]$. Such dynamics lead to the hybrid system
\[
\begin{align*}
\dot{\tau} &= -1 \\
\tau^+ &\in [T_1,T_2] \\
\tau &= 0
\end{align*}
\]
(5)
Due to its set-valued jump map in particular, this system effectively generates any sequence of communication events at times satisfying (4) with $\Omega$ determined by any bounded probability distribution function that assigns $\Omega$ to a value in $[T_1,T_2]$.

At communication times, each system shares its state information to its neighboring agents. A control protocol using this impulsive information is proposed next. As we show in Section 5, multiple timers can be used to trigger communication events for each agent asynchronously.

**Protocol 4.1.** Given parameter $T_2$ of the network, the $i$-th hybrid controller has state $\eta_i$ with the following dynamics:
\[
\begin{align*}
u_i &= \eta_i \\
\dot{\eta}_i &= 0 \\
\eta_i^+ &= -\gamma \sum_{k \in \mathcal{N}(i)} (x_i - x_k) \\
\tau &= 0
\end{align*}
\]
(6)
where $\gamma > 0$ is the controller gain parameter.

Using Protocol 4.1, we employ the hybrid system framework outlined in Section 2 and the properties of the Laplacian matrix to build the interconnected state-feedback network system, which we denote by $\mathcal{H}$. The state of $\mathcal{H}$ is given by $x = (x,y,\tau) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T_2] := X$, where $x = (x_1,x_2,\ldots,x_N)$ and $y = (\eta_1,\eta_2,\ldots,\eta_N)$ comprise the agents' system states and controller states, respectively. By combining the agents' continuous dynamics in (3), the timer's hybrid dynamics in (5), and the protocol in (6), we arrive to the hybrid system $\mathcal{H}$ given by
\[
\begin{align*}
\xi &= \begin{bmatrix}
\eta \\
0 \\
-1
\end{bmatrix} := f(\xi) \\
\xi &\in C := X,
\end{align*}
\]
\[
\begin{align*}
\xi^+ \in \begin{bmatrix}
x \\
\frac{-\gamma L x}{T_1,T_2}
\end{bmatrix} := G(\xi) \\
\xi &\in D := \mathbb{R}^N \times \mathbb{R}^N \times [0,T_2].
\end{align*}
\]
(7)
Remark 4.2. Due to the fact that the timer variable being zero is the only trigger of the jumps, some properties of the domain of solutions can easily be characterized. In particular, a solution $\phi$ to the hybrid system $\mathcal{H}$ is such that, with $t_0 = 0$, the assumption that $T_1 \leq t_{j+1} - t_j \leq T_2$ for all $j \geq 2$, and $0 \leq t_1 \leq T_2$, leads to the hybrid time domain having the following property for the flow time $t$:
\[(j-1)T_1 \leq t \leq (j+1)T_2 \quad \forall j \geq 1\]
for all $(t,j) \in \text{dom} \phi$. Moreover, due to the assumption that $T_1 > 0$, every $\phi \in \mathcal{S}_\mathcal{H}$ is complete and the hybrid time domain is unbounded in both $t$ and $j$.

Our goal is to show that Protocol 4.1 only guarantees the static consensus property in Definition 3.1 with an exponential decay rate, but also renders Lyapunov stable the set of points such that $x_i = x_k$ for all $i,k \in \mathcal{V}$. To this end, we define the set to exponentially stabilize as
\[
\mathcal{A} := \{x \in X : x_i = x_k, \eta_i = 0 \forall i,k \in \mathcal{V}, \tau \in [0,T_2]\}.
\]
(8)
We establish exponential stability by changing to coordinates obtained through a key property of the Laplacian matrix. More precisely, let $\Gamma$ be a strongly connected digraph. Using Lemma 2.2 and Lemma 2.3, its associated Laplacian $\mathcal{L}$ is such that there exists a nonsingular matrix
\[
T = [I_N,T_1] \text{ such that } T^{-1} \mathcal{L} T = \begin{bmatrix}
0 & 0 \\
0 & \mathcal{L}
\end{bmatrix},
\]
which is a
diagonal matrix containing the eigenvalues of $L$, where $L$ is a diagonal matrix with diagonal elements $(\lambda_2, \ldots, \lambda_N)$ with $\lambda_1$ being the positive eigenvalues of $L$. Then, we change the coordinates $\xi$ of $H$ to the new coordinates $\chi$ defined using $\tilde{x} = T^{-1}x$ and $\tilde{\eta} = T^{-1}\eta$. By applying the transformation $T^{-1}$ to both sides of the continuous dynamics of the state $x$ and $\eta$ of $H$ in (7), we have $\tilde{x} = \eta$ and $\tilde{\eta} = 0$. During jumps, the difference equations of the states $x$ and $\eta$ of $H$ in (7) become $\dot{\tilde{x}}^T = \tilde{x}$ and $\dot{\tilde{\eta}}^T = -\gamma [0, 0, 0]^T \tilde{x}$. Then, the new coordinates denoted as $\chi$ are defined by collecting the states variables $x_1$ and $\eta_1$ into $\tilde{z}_1 = \tilde{x}_1, \tilde{\eta}_1$ and the remaining states of $x$ and $\eta$ into $\tilde{z}_2 = (\tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_N, \tilde{\eta}_2, \tilde{\eta}_3, \ldots, \tilde{\eta}_N)$, so as to write $\chi = (\tilde{z}_1, \tilde{z}_2, \tau) \in \mathcal{X}$. The new coordinates lead to a hybrid system denoted as $\bar{H}$ with the following data:

$$
\bar{f}(\chi) := \begin{bmatrix} A_{f1} & 0 \\ A_{f2} & 0 \\ \end{bmatrix} \quad \forall \chi \in \bar{C} := \mathcal{X}
$$

$$
\bar{g}(\chi) := \begin{bmatrix} A_{g1} \\ A_{g2} \\ \end{bmatrix} \quad \forall \chi \in \bar{D} := \{ \chi \in \mathcal{X} : \tau = 0 \}
$$

where

$$
A_{f1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{g1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{f2} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad A_{g2} = \begin{bmatrix} I \\ -\gamma \end{bmatrix}
$$

and $\gamma > 0$. Moreover, in the new coordinates, the set to stabilize for the hybrid system $\bar{H}$ in (9) is defined as

$$
\bar{A} := \{ (\tilde{z}_1, \tilde{z}_2, \tau) \in \mathcal{X} : \tilde{z}_1 = (x^*, 0), x^* \in \mathcal{R}, \tilde{z}_2 = 0 \}.
$$

### 4.2 Global Exponential Stability Results

Inspired by Ferrante et al. (2015), we have the following stability results for $H$.

**Proposition 4.3.** Let $T_1$ and $T_2$ be two positive scalars such that $T_1 \leq T_2$. The set $\bar{A}$ is GES for the hybrid system $H$ if either one of the following properties hold:

1. the digraph is strongly connected, and there exist a positive scalar $\gamma$ and a positive definite symmetric matrix $P$ satisfying

$$
A_{g2}^T e^{A_{g2} T} Pe^{A_{g2} T} A_{g2} - P < 0 \quad \forall \nu \in \{T_1, T_2\},
$$

where the matrices $A_{g2}$ and $A_{f2}$ are given in (10).

2. the digraph is completely connected, and there exist a positive scalar $\gamma$ and a positive definite symmetric matrix $P$ satisfying

$$
A_{g2}^T e^{A_{g2} T} Pe^{A_{g2} T} A_{g2} - P < 0 \quad \forall \nu \in \{T_1, T_2\},
$$

where $A_{g2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_{g2} = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma \end{bmatrix}$.

Furthermore, if the digraph $\Gamma$ is weight balanced, then every solution $\phi = (\phi_1, \phi_2, \phi_3, \ldots)$ satisfies

$$
limit_{\tau \to +\infty} \phi(t, j) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \phi(t, j) = 0
$$

where

$$
\rho(0, 0) := \left( \frac{1}{N} \left( 1_N \phi_1(0, 0) + 1_N \phi_2(0, 0) \right) \right) 1_N.
$$

### Example 4.7

Consider five agents with dynamics as in (3) over the strongly connected graph with adjacency matrix

$$
G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.
$$

Let $T_1 = 0.5$ and $T_2 = 1.5$. If $\gamma = 0.3$, then a matrix $P$ can be found such that condition (12) is satisfied. Figure 1 shows the $x_i$ components $i \in \{1, 2, 3, 4, 5\}$ of a
4.3 Robustness to Perturbations on Communication Noise

In a realistic setting, the information transmitted is affected by communication noise. In this section, we consider the systems under the effect of communication noise $m_i$ when agent $i$ sends out information. Specifically, if the $k$-th agent receives information of the $i$-th agent received information asynchronously. To model such events, we attach a local timer to each agent so that when it reaches zero it triggers the transmission of information of its connected neighbors.

Protocol 5.1. Given parameter $T_2$ of the network, the $i$-th hybrid controller has state $\eta_i$ with the following dynamics:

$$\begin{align*}
\dot{\eta}_i &= \eta_i \\
\eta_i^+ &= \gamma \sum_{k \in N(i)} (x_i - x_k) \\
\tau_i &= 0
\end{align*}$$

where $\gamma, \tau_i \in \mathbb{R}$ are the controller’s parameters.

Define $\bar{x}_i = x_i - \frac{1}{N} \sum_{k=1}^{N} x_k$ and $\theta_i = \gamma \sum_{k \in N(i)} (x_i - x_k) - \eta_i$. From the interconnection between (3), (17), and each $\tau_i$ with dynamics as in (5), the continuous dynamics of $\bar{x}_i$ and $\theta_i$ are given by $\dot{x}_i = \eta_i - \frac{1}{N} \sum_{k=1}^{N} \eta_k$ and $\dot{\theta}_i = \gamma \sum_{k \in N(i)} (\eta_i - \eta_k) - h\eta_i$. Denote $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$. From the definition of $\theta_i$ and noting that $\bar{x}_i - \bar{x}_i = x_i - x_k$, it follows that $\theta = \gamma L \bar{x} - \eta_i$.

With the definitions of $\bar{x}_i$ and $\theta_i$ and the assumption of $\Gamma$ being weight balanced\(^2\), we have that $\dot{\bar{x}} = \gamma L \bar{x} - (I - \frac{1}{N} 1_N 1_N^\top) \theta$ and $\dot{\theta} = \bar{x} - (\gamma L - hI) \theta$ for each $\tau_i \in [0, T_2]$. At each jump, say, there exists $i \in \mathcal{V}$ such that $\tau_i = 0$, $\bar{x}_i$ and $\theta_i$ are updated as $\bar{x}_i = \bar{x}_i$, and $\theta_i = 0$, while all other states are updated by the identity.

In the coordinates $\bar{x}$ and $\theta$, we define the hybrid system $\mathcal{H}_a$ with state $\xi_a = (\psi, \tau) \in \mathbb{R}^N \times \mathbb{R} \times [0, T_2]^N =: X_a$ where $\psi = (\bar{x}, \theta)$ and data $(C_a, f_a, D_a, G_a)$ given by

$$f_a(\xi_a) := \left[\begin{array}{c}
A \psi \\
-A \tau
\end{array}\right], A := \left[\begin{array}{c}
\gamma L \\
-\gamma L \bar{x} + (I - \frac{1}{N} 1_N 1_N^\top)
\end{array}\right]$$

for each $\xi_a \in C_a := X_a$, and $G_a(\xi_a) := \{G_i(\xi_a) : \xi_a \in D_i, i \in \mathcal{V}\}$ for each $\xi_a \in D_a := \bigcup_{i \in \mathcal{V}} D_i$, where

$$D_i := \{\xi_a \in C_a : \tau_i = 0\}$$

The definition of $G_i$ is such that the $i$-th component of $\theta$ and $\tau$ are updated only when $\tau_i = 0$.

It follows that the set to stabilize is given by

$$A_a := \{\xi_a \in X_a : \xi_a = (\bar{x}^* 1_N, 0, \nu), \bar{x}^* \in \mathbb{R} \times [0, T_2]^N\}$$

for the hybrid system $\mathcal{H}_a$ with data $(C_a, f_a, D_a, G_a)$. We have the following stability result for $\mathcal{H}_a$.

Proposition 5.2. Let $T_1$ and $T_2$ be two positive scalars such that $T_1 \leq T_2$ and a digraph $\Gamma$ be strongly connected and weight balanced. If there exist scalars $\gamma, h \in \mathbb{R}$, and $\sigma > 0$, positive definite diagonal matrices $P$ and $Q$ such that

$$\begin{align*}
\gamma L - hI &> 0 \\
\gamma L \bar{x} &< 0
\end{align*}$$

then $\mathcal{H}_m$ with input $\bar{m}$ is ISS with respect to $\mathcal{H}$ in (11).

5. ON ASYNCHRONOUS EVENT TIMES

In this section, we present results for the scenario where each agent receives information asynchronously. To model such events, we attach a local timer to each agent so that when it reaches zero it triggers the transmission of information of its connected neighbors.

For a weight balanced digraph, $1_N^\top L = 0$, 

\(^2\) Code at https://github.com/HybridSystemsLab/ConsSyncTme.

\(^3\) In this way, $m_i$ can account for errors local to the $i$-th agent as well as communication noise.

\(^4\) We use the ISS definition in Cai and Teel (2009).
to design large-scale networked systems that communicate at stochastic time instants over general communication graphs.

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