TALAGRAND’S INEQUALITY AT HIGHER ORDER AND APPLICATION TO BOOLEAN ANALYSIS

KEVIN TANGUY
UNIVERSITY OF ANGERS, FRANCE

Abstract. This note is concerned with an extension, at higher order, of an inequality on the discrete cube $C_n = \{-1, 1\}$ with the uniform measure due to Talagrand ([Tal94]). As an application, we provide a Theorem in the spirit of a famous result from Kahn, Kalai and Linial (cf. [KKL88]) concerning the influence of Boolean functions. We introduce the notion of influence of a couple of coordinate $(i, j) \in \{1, \ldots, n\}^2$ and we proved the following alternative: for any, centered, function $f : C_n \to \{0, 1\}$, either there exists a coordinate with influence at least of order $(1/n)^{1/(1+\eta)}$, with $0 < \eta < 1$ or there exists a couple of coordinate $(i, j) \in \{1, \ldots, n\}^2$, with $i \neq j$, with influence at least of order $(\log n/n)^2$. We also show that this extension of Talagrand’s inequality can be obtained for the standard Gaussian measure $\gamma_n$ on $\mathbb{R}^n$ with minor modifications. The obtained inequality can be of independent interest. The arguments rely on interpolation methods by semigroup together with hypercontractive estimates. At the end of this article, we present some related questions to our work and some variations of Kahn, Kalai and Linial’s Theorem at order two due to Oleszkiewicz.

1. Introduction

The notion of influences of variables on Boolean functions has been extensively studied over the last twenty years, with applications in various areas such as combinatorics, statistical physics and theoretical computer science, in particular in cryptography and computational lower bounds (cf. e.g. the survey [KM13]). Let us introduce the setting of our work. Consider the discrete cube $C_n = \{-1, 1\}^n$, $n \geq 1$, for any function $f : C_n \to \mathbb{R}$ we define the influence of coordinate $i \in \{1, \ldots, n\}$ by

$$I_i(f) = \mathbb{P}(f(X) \neq f(\tau_i(X)))$$

where $\mathcal{L}(X) = \mu^n$ stands for the uniform measure on $C_n$ and $\tau_i(x) = (x_1, \ldots, -x_i, \ldots, x_n)$ for any $x \in C_n$ (it corresponds to the point $x$ where its $i$-th coordinate has been flipped). For more details on the analysis of Boolean functions we refer the reader to the book of O’Donnel [O’D14] or the survey of [GS15].

In [BOL90], the authors showed that the so-called Tribes function (which will be defined in the sequel) has all its coordinates with influence of order $\log n/n$, they have conjectured that this result is optimal. More precisely, they studied the influence of the Tribes functions defined as follows, if $n = km$, the function

$$\text{Tribes}_{km}(x) = \{(x_1, \ldots, x_k), \ldots, (x_{m(k-1)+1}, \ldots, x_m)\} \in \{-1, 1\}^{km}$$

take the value 1 if and only if one of the tribes of length $k$ $(x_{j(k-1)+1}, \ldots, x_{kj})$ is the tribes where all the $x_i$ are equal to 1. In [BOL90], Ben-Or and Linial proved the following

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Proposition 1 (Ben-Or, Linial). With the preceding notations, let \( n \) sufficiently large and set \( k = \log n - \log \log n + \log \log 2 \). Then, for all \( i \in \{1, \ldots, n\} \), the following holds true

\[ I_i(\mathrm{Tribes}_n) = \frac{\log n}{n} (1 + o(1)) \]

where \( o(1) \) is a function \( g \) such that \( g(n) \to 0 \) as \( n \to +\infty \).

In [KKL88], they authors proved that the conjecture is indeed true and obtain the following result

Theorem 2 (Kahn-Kalai-Linial). For any \( f : \mathbb{C}_n \to \{-1, 1\} \), \( n \geq 1 \). Then, it exists \( i \in \{1, \ldots, n\} \) and \( c > 0 \) such that

\[ I_i(f) \geq c \Var_{\mu^n}(f) \log \frac{n}{n}. \]

As we will briefly explain below, Theorem 2 can be proved with the help of Talagrand’s inequality (which is a raffinement of the Poincaré inequality satisfied by the measure \( \mu^n \)). This inequality can be stated as follows.

Theorem 3 (Talagrand). For any \( f : \mathbb{C}_n \to \mathbb{R} \), the following inequality holds

\[ \Var_{\mu^n}(f) \leq C \sum_{i=1}^{n} \frac{\|D_i f\|^2_2}{1 + \log \left( \frac{\|D_i f\|_2}{\|D_i f\|_1} \right)}, \]

where \( 2D_i f = f(x) - f(\tau_i(x)) \), \( x \in \mathbb{C}_n \) stands for the discrete derivative along the i-th coordinate, \( \| \cdot \|_p \), \( p \geq 1 \) designs the \( L^p \) norm on \( \mathbb{C}_n \) with respect to the measure \( \mu^n \) and \( C > 0 \) is a numerical constant.

Remark. Talagrand inequality improves, by a logarithmic factor, upon classical Poincaré’s inequality (up to numerical constant) : 

\[ \Var_{\mu^n}(f) \leq \frac{1}{2} \sum_{i=1}^{n} \|D_i f\|^2_2. \]

As mentioned before, it also provides an alternative proof for Theorem 2. Indeed, consider \( f : \mathbb{C}_n \to \{-1, 1\} \) and notice that, for any \( p \geq 1 \), \( \|D_i f\|^p_2 = I_i(f) \) (again, up to numerical constants). Then, to deduce (2.4) from (1.2), assume that

\[ I_i(f) \leq \left( \frac{\Var_{\mu^n}(f)}{n} \right)^{1/2} \]

for any \( i = 1, \ldots, n \), since if not the results holds. Then, from (1.2), it exists \( i \in \{1, \ldots, n\} \) such that

\[ \frac{\Var_{\mu^n}(f)}{Cn} \leq \frac{2I_i(f)}{1 + \log \left( \frac{1}{\sqrt{2I_i(f)}} \right)} \leq \frac{8I_i(f)}{4 + \log \left( \frac{n}{4\Var_{\mu^n}(f)} \right)} \]

which easily leads to (2.4).

The aim of this note is to develop an interpolation method by semigroup together with hypercontractive arguments to reach Talagrand’s inequality at higher order (the new inequalities will be similar to (1.2) and involved derivatives of higher order). The following Theorem is the main result of this note.

Theorem 4. For any function \( f : \{-1,1\}^n \to \mathbb{R} \), \( n \geq 1 \), it exists \( 0 < s_0 < 1 \) (fixed) such that the following holds
\begin{equation}
\text{Var}_{\mu^*}(f) \leq C \left( \sum_{i=1}^{n} \|D_i f\|_1^2 e^{-2_0 n} + \sum_{i \neq j=1}^{n} \frac{\|D_{ij} f\|_2^2}{1 + \log \frac{\|D_{ij} f\|_2}{\|D_{ij} f\|_1}}\right),
\end{equation}

where $D_{ij} = D_i \circ D_j$ for any $i, j \in \{1, \ldots, n\}$ and $C > 0$ is a numerical constant.

As an application of this result, we propose a theorem in the spirit of Theorem 2 with the influence of a couple of coordinate $(i, j) \in \{1, \ldots, n\}^2$ (which will be defined in the sequel and is easily seen as an extension of the more classical notion of influence).

**Corollary 5.** Let $f : C_n \to \mathbb{R}$ be a centered function. Then, we obtain the following alternative: either it exists $i \in \{1, \ldots, n\}$ such that

$$I_i(f) \geq C \left( \frac{1}{n} \right)^{1/(1+\eta)}$$

or it exists $(i, j) \in \{1, \ldots, n\}^2$, $i \neq j$, such that

$$I_{(i,j)}(f) \geq C \left( \frac{\log n}{n} \right)^2.$$

**Remark.** As it will be explain in the sequel, the scheme of proof permits to easily obtain similar results for higher order. The numerical value $\frac{1}{2}$ is arbitrary, it would have been possible to choose $A$ such that $0 < \epsilon < \mu^n(A) \leq 1 - \epsilon < 1$, for any $0 < \epsilon < 1$ independent of $n$.

The remaining of this paper is organized as follows: section two will provide the semi-group tools and the framework of Boolean analysis needed to prove Theorem 4. The proof of the extension of Talagrand’s inequality at higher order will be given in section three. Section four will be devoted to Corollary 5. In section five, we will present how the same arguments can also be used in a Gaussian context. Finally, in the last section, we will give further remarks and comments about possible extension of our work together with some observations from Oleszkiewicz about Kahn, Kalai and Linial’s Theorem at order two.

## 2. Framework and tools

The discrete cube $C_n = \{-1, 1\}^n$ is an interesting example for which semigroup interpolation methods can be used to reach functional inequalities. Let us briefly collect some basic properties of this space embedded with the product measure $\mu^n$, where $\mu = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{1}$. The space $L^2(C_n, \mu^n)$ is an Hilbert space for the usual scalar product, it is possible to produce an orthonormal basis of polynomials: this is the so-called Walsh-Fourier basis. These polynomials are defined as follows: for any subset $S \subset \{1, \ldots, n\}$ set $W_S(x) = \prod_{i \in S} x_i$.

A lot of standard Fourier formulas holds with this orthonormal basis, for instance it is possible to expand a function $f \in L^2(C_n, \mu^n)$ along the Fourier-Walsh basis

$$f = \sum_{S \subset \{1, \ldots, n\}} \hat{f}(S) W_S,$$

with $\hat{f}(S) = (W_S, f)_{L^2(C_n, \mu^n)}$. A Plancherel formula can also be obtained for two functions in $f, g \in L^2(C_n, \mu^n)$,
\[ E_{\mu^n}[fg] = \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S)\hat{g}(S), \]
from which it is easy to see that \( \text{Var}_{\mu^n}(f) = \sum_{S \neq \emptyset} \hat{f}^2(S) \), since \( E_{\mu^n}[f] = \hat{f}(\emptyset) \).

There is a classical semi-group associated to \((C_n, \mu^n)\) (cf. \cite{OD14, GS15, CEL12}), this is the so-called Bonami-Beckner semigroup \((Q_t)_{t \geq 0}\), it admits the measure \(\mu^n\) as a invariant and reversible measure (more details will be given below). More precisely, if we define the discrete Laplacian by \(L = \frac{1}{2} \sum_{i=1}^{n} D_i\), where \(D_i\) stands for the (discrete) partial derivative along the \(i\)-th coordinate. That is to say, for any \(f : C_n \to \mathbb{R}\), \(D_i f = f(\tau_i(x)) - f(x)\) with \(\tau_i(x) = (x_1, \ldots, -x_i, \ldots, x_n) \in C_n\) for any \(i = 1, \ldots, n\) and any \(x = (x_1, \ldots, x_n) \in C_n\). With this differential operator, we can defined a Dirichlet form on \(C_n\) : for any \(f, g : C_n \to \mathbb{R}\)

\[(2.1) \quad \mathcal{E}_{\mu^n}(f, g) = \int_{C_n} f(-Lg) d\mu^n = \int_{C_n} Df \cdot Dg d\mu^n,\]
where \(Dh = (D_1 h, \ldots, D_n h)\) for any function \(h : C_n \to \mathbb{R}\). It is a simple matter to check that the polynomials from the Fourier-Walsh’s basis are eigenfunction of the operator \(L\). Indeed, for any \(S \subseteq \{1, \ldots, n\}\),

\[ LW_S(x) = -\frac{1}{2} \sum_{i=1}^{n} \left( W_S(x) - W_S(\tau_i(x)) \right) = -\text{Card}(S)W_S(x), \]

since \(W_S(x) - W_S(\tau_i(x)) = 21_{i \in S}W_S(x)\). The operator \(L\) gives rise to a semi-group, the so-called Bonami-Beckner semigroup, \((Q_t = e^{tL})_{t \geq 0}\). Let us recall some important properties of \((Q_t)_{t \geq 0}\).

**Proposition 6.** This semigroup can be represented along the Fourier-Walsh basis, for any \(f : C_n \to \mathbb{R}\) and any \(t \geq 0\) :

\[ Q_t(f) = \sum_{S \subseteq \{1, \ldots, n\}} e^{-t\text{Card}(S)} \hat{f}(S)W_S. \]

\((Q_t)_{t \geq 0}\) is Markovian and \(\mu^n\) is its invariant and reversible measure. Namely, for any \(t \geq 0\), \(Q_t 1 = 1\) and

\[ \int_{C_n} fQ_t g d\mu^n = \int_{C_n} gQ_t f d\mu^n, \]
for any \(f, g : C_n \to \mathbb{R}\). The Bonami-Beckner semi-group also admits a integral representation formula, for any \(t \geq 0\),

\[ Q_tf(x) = \int_{C_n} f(y) \prod_{i=1}^{n} (1 + e^{-t}x_iy_i) d\mu^n(y), \quad x \in C_n \]

**Remark.** This integral representation easily leads to the following commutation property \(Q_tD = DQ_t\), \(t \geq 0\) between the semigroup and the (discrete) gradient.

It has been proven, by Bonami and Beckner, that \((Q_t)_{t \geq 0}\) satisfies an hypercontractive property.

**Theorem 7.** (Bonami-Beckner) The semigroup \((Q_t)_{t \geq 0}\) is hypercontractive. Namely, for any \(f : C_n \to \mathbb{R}\), every \(t \geq 0\) and every \(p \geq 1\)
\begin{equation}
\|Q_tf\|_q \leq \|f\|_p,
\end{equation}
with \( p = p(t) = 1 + (q - 1)e^{-2t} \).

Remark. According to Gross’s Theorem (cf. [BGL14]), inequality (2.2) is equivalent to a Sobolev logarithmic inequality

\begin{equation}
\text{Ent}_{\mu^n}(f^2) \leq \int_{C_n} |Df|^2 d\mu^n,
\end{equation}
where \( \text{Ent}_{\mu^n}(f) = \int_{C_n} f \log fd\mu^n - \left( \int_{C_n} f d\mu^n \right) \log \left( \int_{C_n} f d\mu^n \right) \) and \( |\cdot| \) stands for the Euclidean norm. Although it is also possible to deduce the Poincaré’s inequality satisfied by \( \mu^n \) with the spectral decomposition along the Fourier-Walsh basis; it can be deduced, with a Taylor expansion, from (2.3). More precisely, for any \( f : C_n \to \mathbb{R} \), the following inequality holds true

\[ \text{Var}_{\mu^n}(f) \leq \frac{1}{2} \int_{C_n} |Df|^2 d\mu^n. \]

The exponential decay of the variance, of a function \( f \), along the semi-group induces by the Poincaré’s inequality (cf. (3.1)) will also be used in the sequel.

2.1. Influences. In this section we remind the reader of the basics notions of influence in Boolean Analysis. We refer to [O’D14, GS15] for more details.

For any \( f : C_n \to \{-1, 1\}, n \geq 1 \), it is possible to see \( f \) as an election rule with two candidates. Indeed, a point \( x \in \{-1, 1\}^n \) designs the votes of a population of \( n \) individuals for either the candidate \(-1\) or the candidate \( 1 \); then the value of \( f(x) \) will give the elected candidate. It is natural to ask about the influence of the \( i \)-th vote for any \( i \in \{1, \ldots, n\} \): is it possible that the flip of the \( i \)-th vote can lead to a different outcome? This is formalized in the following definition

**Definition 2.1.** For any \( f : C_n \to \{-1, 1\}, n \geq 1 \). The \( i \)-th coordinate, for any \( i \in \{1, \ldots, n\} \), is said to be pivotal if \( f(x) \neq f(\tau_i(x)) \), \( x \in C \) where \( \tau_i(x) = (x_1, \ldots, -x_i, \ldots, x_n) \) stands for the vector \( x \) where \( x_i \) has been flipped into \(-x_i\).

This leads to the notion of influence of a coordinate of a Boolean function.

**Definition 2.2.** For any \( f : C_n \to \{-1, 1\}, n \geq 1 \). The influence of the \( i \)-th coordinate of the function \( f \) is given by the probability that the \( i \)-th coordinate is pivotal (for input \( X \))

\[ I_i(f) = \mathbb{P}\left(f(X) \neq f(\tau_i(X))\right) \]

with \( \mathcal{L}(X) = \mu^n \) the uniform measure on \( C_n \).

It is worthwhile noticing that influence of a coordinate can be expressed in term of \( L^p \), \( p \geq 1 \), norms of the partial derivative of \( f \). Indeed, up to numerical constants,

\[ I_i(f) = \|D_i f\|_1 = \|D_i f\|_p^p \quad i \in \{1, \ldots, n\} \]

We refer, again, the reader to [O’D14, GS15] for more details and examples. As we have mentionned it in the introduction, a major results on influence of Boolean functions is the celebrated Theorem of Kahn, Kalai et Linial [KKL88].
Theorem 8 (Kahn-Kalai-Linial). For any \( f : C_n \rightarrow \{-1, 1\}, n \geq 1 \). Then, it exists \( i \in \{1, \ldots, n\} \) and \( c > 0 \) such that

\[
I_i(f) \geq c \text{Var}_{\mu^n}(f) \frac{\log n}{n}.
\]

In what follows, we propose a definition for the influence of a couple \((i, j) \in \{1, \ldots, n\}^2\) which extend the classical notion of influence. As presented before, it is possible to express an influence in term of an \( L^p \) norm of a partial derivative. By analogy, we define similarly the notion of double influence \( I_{(i,j)}(f) \) from the \( L^2 \) norm of \( D_{ij}f \) where \( D_{ij} = D_i \circ D_j \). It is also possible to show that it is consistent with the fact that \((i, j)\) is pivotal.

It can be shown that, for any \((i, j) \in \{1, \ldots, n\}^2\) and any \( f : C_n \rightarrow \{-1, 1\}, \)

\[
D_{ij}f = \frac{1}{4} \left( f(x) + f(\tau_j(x)) - f(\tau_i(x)) - f(\tau(x)) \right),
\]

with \( \tau_j = \tau_i \circ \tau_j \). Notice that, if \( i = j \) we find out that \( D_{ii}f = D_i f \).

Definition 2.3. Without loss of generality, a Boolean function \( f : C_n \rightarrow \{0, 1\}, n \geq 1 \) can be expressed as \( f = 1_A \) with \( A \subset C_n \). For any \((i, j) \in \{1, \ldots, n\}^2\), the influence of the couple \((i, j)\) is given by

\[
I_{ij}(f) = \frac{1}{4} \left[ \mathbb{P}(X \in A, \tau_i(X) \notin A, \tau_j(X) \notin A) + \mathbb{P}(X \notin A, \tau_i(X) \in A, \tau_j(X) \in A) \right],
\]

where \( L(X) = \mu^n \).

Remark. This is consistent with the influence of a single coordinate, since, for any \( i \in \{1, \ldots, n\} \) and any \( f = 1_A, \)

\[
2I_i(f) = \mathbb{P}(X \in A, \tau_i(X) \notin A) + \mathbb{P}(X \notin A, \tau_i(X) \in A).
\]

Furthermore, as already mentionned, \( I_i(f) = ||D_i f||_2^2 \) (up to numerical constants). Elementary calculus show that the next equality holds true

\[
||D_{ij}f||_2^2 = 4 \left[ \mathbb{P}(X \in A, \tau_i(X) \notin A, \tau_j(X) \notin A) + \mathbb{P}(X \notin A, \tau_i(X) \in A, \tau_j(X) \in A) \right].
\]

As we will be see in the sequel, it will be essential to compare \( ||D_{ij}f||_2^2 \) with \( ||D_{ij}f||_1 \) in order to obtain, from Talagrand’s inequality at order two, a Theorem in the spirit of Theorem 2 from [KKL88]. By disjunction, (if \( x \in A, \tau_i(x) \in A, \tau_j(x) \in A, \tau_j(x) \in A \ldots \) it can be shown that

\[
||D_{ij}f||_1 \leq 4 ||D_{ij}f||_2^2 \leq 2 ||D_{ij}f||_1, (i, j) \in \{1, \ldots, n\}^2.
\]

From an heuristic point of view, this can be explained as follow : for any \( (i, j) \in \{1, \ldots, n\}^2, 4D_{ij}1_A(x) \in \{-2, -1, 0, 1, 2\}, x \in C_n \). Then, for any \( p \geq 1 \), it yields

\[
|4D_{ij}1_A(x)| \leq |4D_{ij}1_A(x)|^p \leq 2^p |4D_{ij}1_A(x)|, x \in C_n,
\]

allowing us to compare the \( L^p \) norms of \( D_{ij}1_A \) together.
3. Proof of the main result and its consequence

The proof will rest on interpolation method and hypercontractive estimates. Notice that the Poincaré’s inequality, satisfied by the measure \( \mu^n \), induces an exponential decay of the variance, of a function \( f \), along the Bonami-Beckner semi-group. That is to say, for any \( f : C_n \to \mathbb{R} \),

\[
\text{Var}_{\mu^n}(Q_t(f)) \leq e^{-2t\|f\|_2^2}, \quad t \geq 0,
\]

which is equivalent, when \( f \) is centered under \( \mu^n \), to

\[
(3.1) \quad \|Q_t f\|_2^2 \leq e^{-2t\|f\|_2^2}, \quad t \geq 0
\]
since \( \mu^n \) is the invariant measure of \( (Q_t)_{t \geq 0} \). During the proof, this argument will be used with the functions \( D_if, D_{ij}f, i, j = 1, \ldots, n \). Indeed, notice that

\[
\int_{C_n} f(x) d\mu^n = \int_{C_n} f(\tau_i(x)) d\mu^n
\]

therefore, \( D_if \) and \( D_{ij}f, i, j = 1, \ldots, n \) are centered under \( \mu^n \).

3.1. Proof of Theorem 4. The scheme of proof starts with the representation of the variance of \( f \) along the Bonami-Beckner’s semi group \( (Q_t)_{t \geq 0} \) (cf. [CEL12, BGH01]) :

\[
\text{Var}_{\mu^n}(f) = 2 \int_0^\infty \sum_{i=1}^n \int_{C_n} Q_t^2(D_if) d\mu^n dt.
\]

Then, set \( 2s = t \) and, for any \( i = 1, \ldots, n \), use the fact that

\[
\int_{C_n} Q_s^2(D_if) d\mu^n = \|Q_s \circ Q_s(D_if)\|_2^2 \leq e^{-2s\|Q_s D_if\|_2^2}
\]

This gives the following upper bound,

\[
(3.2) \quad \text{Var}_{\mu^n}(f) \leq 4 \int_0^\infty e^{-2s} \sum_{i=1}^n \int_{C_n} Q_s^2(D_if) d\mu^n ds.
\]

For any \( s \geq 0 \), set \( K(s) = \int_{C_n} |Q_s^2(D_if)|^2 d\mu^n \). Integration by parts (2.1) and the fundamental theorem of analysis leads to

\[
K(s) = K(\infty) - \int_s^\infty K'(u) du = K(\infty) + 2 \int_s^\infty \int_{C_n} |Q_s(D_if)|^2 d\mu^n du, \quad s \geq 0
\]

Besides, since Bonami-Beckner’s semi group is ergodic

\[
K(\infty) = \left| \int_{C_n} Df d\mu^n \right|^2 = \sum_{i=1}^n \left( \int_{C_n} D_if d\mu^n \right)^2 = 0.
\]

Recall, that for any \( i = 1, \ldots, n \), \( D_if \) is centered for the measure \( \mu^n \). In other words, we have shown that, for any \( s \geq 0 \),

\[
(3.3) \quad K(s) = 2 \sum_{i,j=1}^n \int_s^\infty \int_{C_n} Q_s^2(D_{ij}f) d\mu^n du.
\]

Substitute (3.3) in (3.2) and apply Fubini’s theorem to get
\begin{align*}
\text{Var}_{\mu^n}(f) & \leq 4 \sum_{i,j=1}^{n} \int_{0}^{\infty} (1 - e^{-2u}) \int_{C_n} Q_i^2(D_{ij}f) d\mu^n du. \\
\text{Again, set } 2s = u \text{ and use the exponential decay of } (Q_{t})_{t \geq 0} \text{ in } L^2(\mu^n), \text{ namely } \|Q_{2s}D_{ij}f\|_2^2 \leq e^{-2s}\|Q_{s}f\|_2^2 \forall i,j \in \{1, \ldots, n\}. \text{ This yields }
\text{Var}_{\mu^n}(f) & \leq 8 \sum_{i,j=1}^{n} \int_{0}^{\infty} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_i^2(D_{ij}f) d\mu^n ds. \\
\text{Now, cut the sum according to } i = j \text{ or not. We will profit from the fact that } D_{ii} = D, \text{ for any } i = 1, \ldots, n. \text{ The variance of } f \text{ is now bounded by two terms,}
8 \sum_{i=1}^{n} \int_{0}^{\infty} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_i^2(D_{i}f) d\mu^n ds + 8 \sum_{i \neq j} \int_{0}^{\infty} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_i^2(D_{ij}f) d\mu^n ds.
\text{Let } s_0 > 0 \text{ be a parameter to be choosen later. The first term of the sum is managed as follows}
8 \sum_{i=1}^{n} \int_{0}^{s_0} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_i^2(D_{i}f) d\mu^n ds = 8 \sum_{i=1}^{n} \int_{0}^{s_0} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_i^2(D_{i}f) d\mu^n ds + 8 \sum_{i=1}^{n} \int_{s_0}^{\infty} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_i^2(D_{ij}f) d\mu^n ds.
\text{It is straightforward to see that,}
8 \sum_{i=1}^{n} \int_{0}^{s_0} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_i^2(D_{i}f) d\mu^n ds & \leq 8 \sum_{i=1}^{n} \int_{0}^{s_0} 4s \int_{C_n} Q_i^2(D_{i}f) d\mu^n ds \\
& \leq 32s_0 \sum_{i=1}^{n} \int_{0}^{\infty} \int_{C_n} Q_i^2(D_{i}f) d\mu^n ds = 16s_0 \text{Var}_{\mu^n}(f)
\text{where last equality comes from the dynamical representation of the variance along the semi group. Choose } s_0 \text{ such that } s_016 \leq 1/2 \text{ to obtain}
\frac{1}{2} \text{Var}_{\mu^n}(f) & \leq 8 \int_{s_0}^{\infty} e^{-2s}(1 - e^{-4s}) \sum_{i=1}^{n} \int_{C_n} Q_i^2(D_{i}f) d\mu^n ds + 8 \sum_{i \neq j} \int_{s_0}^{\infty} e^{-2s}(1 - e^{-4s}) \int_{C_n} Q_i^2(D_{ij}f) d\mu^n ds.
\text{We will use the hypercontractive property of the Bonami-Beckner’s semi group to bound from above the following quantity}
8 \int_{s_0}^{\infty} e^{-2s}(1 - e^{-4s}) \sum_{i=1}^{n} \int_{C_n} Q_i^2(D_{i}f) d\mu^n ds.
\text{To this task, notice first that } (Q_{t})_{t \geq 0} \text{ is also a contraction of } L^2(\mu^n) : \text{ for any } i = 1, \ldots, n \text{ and every } s \geq s_0,
\|Q_{s}D_{i}f\|_2^2 = \|Q_{s-s_0} \circ Q_{s_0}D_{i}f\|_2^2 \leq \|Q_{s_0}(D_{i}f)\|_2^2.
\text{Therefore,}

at hand we can prove Corollary 5. Then, observe that
\[ \|Q_{s_i}D_{ij}f\|_2^2 \leq \sum_{i=1}^n \|Q_{s_i}D_{ij}f\|_2^2 \int_{s_0}^{\infty} e^{-2s_i(1 - e^{-4s})} ds \]
\[ \leq 4 \sum_{i=1}^n \|D_{ij}f\|_2^2 \]
\[ \leq 4 \sum_{i=1}^n \|D_{ij}f\|_2^2 e^{-2s_i} \]
where, in the last inequality, we used the hypercontractive property (2.2). To conclude the proof, we have to bound the sum when \( i \neq j, 1, \ldots, n \).

\[ I = 8 \sum_{i \neq j} \int_{0}^{\infty} e^{-2s_i(1 - e^{-4s})} \int_{C_n} Q_{s_i}^2(D_{ij}f) d\mu^n ds \]
\[ \leq 16 \sum_{i \neq j} \int_{0}^{\infty} e^{-2s_i(1 - e^{-2s})} \int_{C_n} Q_{s_i}^2(D_{ij}f) d\mu^n ds. \]

Recall that the hypercontractive property (2.2) of \((Q_t)_{t \geq 0}\) implies that, for any function \( g : C_n \rightarrow \mathbb{R} \),
\[ \|Q_t(g)\|_2^2 \leq \|g\|_{1 + e^{-2t}}^2, \quad t \geq 0. \]
Apply this to \( g = D_{ij}f \), for any \( i, j = 1, \ldots, n \). Then, set \( v = 1 + e^{-2t} \) to get
\[ I \leq \sum_{i \neq j} \int_{0}^{1} (2 - v)\|\partial_{ij}f\|_2^2 dv. \]
Furthermore, Hölder’s inequality yields
\[ \|D_{ij}f\|_v \leq \|D_{ij}f\|_2^\theta \|D_{ij}f\|_2^{1-\theta}, \quad \text{with } \theta = \theta(v) \text{ satisfying } \frac{1}{\theta} = \frac{2}{v} + \frac{1}{2}, \text{ for any } v \in [1, 2]. \]
All of this amounts of saying that
\[ I \leq \sum_{i \neq j} \|D_{ij}f\|_2^2 \int_{1}^{2} (2 - v) \left( \frac{\|D_{ij}f\|_1}{\|D_{ij}f\|_2} \right)^{2\theta} dv. \]
Now, set \( \alpha = \frac{\|D_{ij}\|_2^2}{\|D_{ij}f\|_2} \leq 1 \), after a change of variables, we easily obtain, for \( i \neq j \),
\[ \int_{1}^{2} (2 - v) \left( \frac{\|D_{ij}f\|_1}{\|D_{ij}f\|_2} \right)^{2\theta} dv = \int_{0}^{1} e^{-\frac{2\theta}{\alpha}} \log(1/\alpha) du. \]
Then, observe that
\[ \int_{0}^{1} e^{-\frac{2\theta}{\alpha}} \log(1/\alpha) du \leq C \frac{1}{1 + \log(1/\alpha)} \]
with \( C > 0 \) a numerical constant. Finally, we have
\[ 16 \sum_{i \neq j} \int_{0}^{\infty} e^{-2s_i(1 - e^{-2s})} \int_{C_n} Q_{s_i}^2(D_{ij}f) d\mu^n ds \leq C \sum_{i \neq j} \frac{\|D_{ij}f\|_2^2}{1 + \log \left( \frac{\|D_{ij}f\|_2}{\|D_{ij}f\|_1} \right)^2} \]

Remark. Notice that the scheme of proof can be extended to higher order. For instance, for the order three, cut the sum in three parts: the diagonal terms will give \( D_{ik} \), when two indexes are equal we obtain terms of the form \( D_{ik} \), the other terms will give discrete partials derivatives of order three. Then we can apply the same methodology as the preceding proof.

With Theorem 4 at hand we can prove Corollary 5.
3.2. Proof of Corollary 5. Let $A \subset C_n$ be and set $f = 1_A$. Then, apply to $f$ (1.3) from Theorem 4 together with the relation between influence and $L^p$ norm of the discrete derivative. First, notice that for such a function $f, \|D_i f\|_{1 + e^{-2s_0}}^2 = I_i(A)^{2/(1 + e^{-2s_0})}$ for any $i = 1, \ldots, n$. Besides, $s_0$ can be close as we want from zero, so $\frac{2}{1 + e^{-2s_0}} = 1 + s_0/2 + o(s_0)$. Therefore, for $s_0$ close to zero fixed, this can be rewritten as $1 + \eta$ with $0 < \eta < 1$.

As we already mentioned, $L^p$ norms of $\|D_{ij} f\|_2$ and $\|D_{ij} f\|_1$ with $f = 1_A$ are comparable, for every $i \neq j = 1, \ldots, n$, and correspond to influence of order two $I_{(i,j)}(A)$. More precisely, for any $i \neq j = 1, \ldots, n$

$$\frac{1}{2}\|D_{ij} f\|_2^{1/2} \leq \|D_{ij} f\|_1 \leq \frac{1}{\sqrt{2}}\|D_{ij} f\|_1^{1/2}$$

and $\|D_{ij} f\|_2 = 4I_{ij}(A)$. Then, it yields

$$\mu^n(A)(1 - \mu^n(A)) \leq C \sum_{i=1}^n I_i(A)^{1+\eta} + C \sum_{i\neq j} \frac{I_{(i,j)}(A)}{1 + \log \left( \frac{1}{16n I_{(i,j)}(A)} \right)}^2.$$ 

Again, we face the following alternative: either the second term is dominated by the other one or this is the other way around. In the first case scenario, the preceding inequality leads to

$$\mu^n(A)(1 - \mu^n(A)) \leq C \sum_{i=1}^n I_i(A)^{1+\eta}.$$ 

So, is exists $i \in \{1, \ldots, n\}$ such that $I_i(A)^{1+\eta} \geq \frac{\mu^n(A)(1-\mu^n(A))}{C n}$. For the alternative scenario, we obtain

$$\mu^n(A)(1 - \mu^n(A)) \leq C \sum_{i\neq j} \frac{I_{(i,j)}(A)}{1 + \log \left( \frac{1}{16n I_{(i,j)}(A)} \right)}^2.$$ 

Then, we proceed as follows: if it exists $(i, j) \in \{1, \ldots, n\}^2, i \neq j$ such that $I_{(i,j)}(A) \geq \frac{\mu^n(A)(1-\mu^n(A))}{16n}$ the Theorem holds. Otherwise, if for every $(i, j) \in \{1, \ldots, n\}^2, i \neq j$, we have $I_{(i,j)}(A) \leq \frac{\mu^n(A)(1-\mu^n(A))}{16n}$, we could obtain the following upper bound

$$\mu^n(A)(1 - \mu^n(A)) \leq C \sum_{i\neq j} \frac{I_{(i,j)}(A)}{1 + \log \left( \frac{n}{\mu^n(A)(1-\mu^n(A))} \right)}^2.$$ 

Then, it exists $(i, j) \in \{1, \ldots, n\}^2, i \neq j$ such that

$$\frac{\mu^n(A)(1 - \mu^n(A))}{C n (n-1)} \leq \frac{I_{ij}(A)}{1 + \log \left( \frac{n}{\mu^n(A)(1-\mu^n(A))} \right)}^2,$$ 

which conclude the proof.
Remark. The proof is similar, with minor modifications, if we consider a centered function \( f : C_n \to \{0, 1\} \) instead; we leave it to the reader. The fact that the influences \( I_{(i,j)}(\text{Tr} \kappa_m), \) for \( i \neq j, \) is precisely of order \( n^2 \log^2 n \) follows with minor and obvious variations of the proof of Proposition 1. Therefore, we will not present it.

3.3. Talagrand inequality at higher order. We find it relevant to state what we could be obtain if we iterate our argument at order \( p \geq 0 \) during the proof of Theorem 4.

Theorem 9. Let \( f : C_n \to \mathbb{R}, n \geq 1. \) For any \( p \geq 0, \) the following holds

\[
\text{Var}_{\mu_n}(f) \leq 2^{p+2+\sum_{k=0}^{p} k} \int_0^{\infty} e^{-2s}(1 - e^{-2s})^p \int_{C_n} \sum_{i_1, \ldots, i_{p+1}=1}^n Q_s^2(D_{i_1, \ldots, i_{p+1}}f) d\mu^n ds.
\]

In particular, by hypercontractive argument,

\[
\text{Var}_{\mu_n}(f) \leq C_p \sum_{i_1, \ldots, i_{p+1}=1}^n \left\| D_{i_1, \ldots, i_{p+1}} f \right\|_2^p \left[ 1 + \log \left( \left\| D_{i_1, \ldots, i_{p+1}} f \right\|_2 \right) \right]^{p+1},
\]

with \( C_p > 0. \)

Remark. Notice that the bounds thus obtained seems to be too rough. It would have been better to proceed as the proof of Theorem 4 and cut the sum according to different scenario (all the indexes are equal, all the indexes are equal but one, . . . ). However it would have lead to tedious combinatorics calculus.

4. Extension to a Gaussian setting

It is well known (cf. [Cha14]) that Talagrand’s inequality can be obtain for the Gaussian measure. We want to emphasize the fact that the interpolation method with the exact same arguments also work the standard Gaussian measure on \( \mathbb{R}^n \) with the Ornstein-Uhlenbeck semigroup. Therefore, we will briefly remind to the reader some properties of such semigroup (for more details we refer the reader to [BGL14]). Then, we present a variance representation formula, which already appeared in [HPAS98], which can be seen as a Taylor expansion with some remainder term. Finally we will briefly explain how the proof can be done with the help of the arguments used during the proof of Theorem 4.

4.1. Ornstein-Uhlenbeck semi-group. This section gather some essentials properties of Ornstein-Uhlenbeck semi-group \( (P_t)_{t \geq 0}. \) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function smooth enough, the Ornstein-Uhlenbeck’s semi-group admits the following representation formula

Proposition 10. \( (P_t)_{t \geq 0} \) is Markovian and \( \gamma_n \) is its invariant and reversible measure. Namely, for any \( t \geq 0, \) \( P_t 1 = 1 \) and

\[
\int_{\mathbb{R}^n} f P_t g d\gamma_n = \int_{\mathbb{R}^n} g P_t f d\gamma_n,
\]

for any \( f, g : \mathbb{R}^n \to \mathbb{R} \) smooth enough. The Ornstein-Uhlenbeck’s semigroup also admits a integral representation formula, for any \( t \geq 0, \)

\[
P_t(f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}} y) d\gamma_n(y), \quad x \in \mathbb{R}^n, \ t \geq 0
\]
Remark. This integral representation easily leads to the following commutation property between the semigroup and the gradient $\nabla$:

$$P_t \nabla = e^{-t} \nabla P_t, \quad t \geq 0$$

(4.1)

We will also need an integration by parts formula in the sequel. Denote by $P_{(4.1)}$ the infinitesimal generator of $(P_t)_{t \geq 0}$, then for $f, g$ smooth enough it holds

$$\int_{\mathbb{R}^n} f(-Lg)d\gamma_n = \int_{\mathbb{R}^n} \nabla f \cdot \nabla d\gamma_n.$$  

(4.2)

It has been proven (cf. [BGL14]) that $(P_t)_{t \geq 0}$ also satisfies an hypercontractive property.

**Theorem 11** (Nelson). The semi-group $(P_t)_{t \geq 0}$ is hypercontractive. Namely, for any $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough, every $t \geq 0$ and every $p \geq 1$

$$\|P_t f\|_q \leq \|f\|_p,$$

with $p = p(t) = 1 + (q - 1)e^{-2t}$.

4.2. **Variance representation.** We state one important result of this section that will be crucial to reach the version of Theorem 4 in a Gaussian setting.

**Theorem 12.** Let $f : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, be and assume that there exists $m \geq 1$ such that $f \in C^m(\mathbb{R}^n)$. Assume also that $f$ and its partial derivatives belong to $L^2(\gamma_n)$, where $\gamma_n$ stands for the standard Gaussian measure on $\mathbb{R}^n$. Then, for every $1 \leq p \leq m - 1$, we have the following representation formula of $f$, under the measure $\gamma_n$, along the Ornstein-Uhlenbeck semigroup $(P_t)_{t \geq 0}$.

$$\text{Var}_{\gamma_n}(f) = \frac{1}{2} \sum_{k=1}^{m} \left[ \int_{\mathbb{R}^n} \nabla^k f d\gamma_n \right]^2 + \frac{2}{p} \int_0^{\infty} \frac{1}{e^{2t} (1 - e^{-2t})^p} \int_{\mathbb{R}^n} |P_t(\nabla^{p+1} f)|^2 d\gamma_n dt,$$

(4.4)

Remark. Let us make a few comments. Our method is not new, in his article [Led95], Ledoux used similar interpolation arguments, between 0 and $t$ in order to obtain the following representation formula for the variance of a function $f$. Also, it seems that (4.4) is already present, in a more general setting, in [HPAS98].

**Proposition 13** (Ledoux). For $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough, $f \in C^p(\mathbb{R}^n) p \geq 1$ with all partial derivatives belonging to the space $L^2(\gamma_n)$, we have

$$\text{Var}_{\gamma_n}(f) = \sum_{k=1}^{p-1} \frac{(-1)^k}{k!} \int_{\mathbb{R}^n} |\nabla^k f|^2 d\gamma_n - \frac{(-1)^{p-1}}{(p-1)!} \int_0^{\infty} 2e^{-2pt} \int_{\mathbb{R}^n} |P_t(\nabla^p f)|^2 d\gamma_n dt.$$  

(4.5)

Also notice that, when $p \to \infty$, the formula (4.5) yields, up to integration by parts, the decomposition of a function of $L^2(\gamma_n)$ along the Hermite polynomial basis (cf. [BGL14]).

As in [Led95], we can perform the same proof for with the entropy instead of the variance. However, formula are not so easily handled. At the first iteration of the method we obtain, for $f : \mathbb{R}^n \to \mathbb{R}$ such that $f > 0$,
\[2\text{Ent}_{\gamma}(f) = \left| \int_{\mathbb{R}^n} \nabla f \, d\gamma_n \right|^2 + \int_0^\infty e^{-2u}(1 - e^{-2u}) \int_{\mathbb{R}^n} k_u d\gamma_n \, du\]

with \(k_u = (P_u f)^{-3}(P_u(\nabla f))^4P_u(\nabla f) - P_u f P_u(\nabla^2 f)^2\). Since \(k_u \geq 0\) for every \(u \geq 0\), this implies, for any \(f\) such that \(\int_{\mathbb{R}^n} f \, d\gamma_n = 1\),

\[2\text{Ent}_{\gamma}(f^2) \geq \int_{\mathbb{R}^n} \nabla f \, d\gamma_n \]

This lower bound correspond to the inverse Sobolev logarithmic inequality.

**Proof.** The starting point of the proof is the dynamical representation of the variance, of function \(f : \mathbb{R}^n \to \mathbb{R}\), along the Ornstein-Uhlenbeck’s semigroup.

\[
\text{Var}_{\gamma_n}(f) = 2 \int_0^\infty e^{-2t} \int_{\mathbb{R}^n} |P_t(\nabla f)|^2 \, d\gamma_n \, dt.
\]

Set, for any \(t \geq 0\)

\[K_1(t) = \int_{\mathbb{R}^n} |P_t(\nabla f)|^2 \, d\gamma_n.
\]

Then, according the fundamental Theorem of analysis, for any \(s \geq t \geq 0\)

\[K_1(t) = K_1(s) - \int_t^s K_1'(u) \, du
\]

using the fact that \(\nabla P_u f = e^{-u}P_u \nabla f\) and the integration by parts formula (4.2), we obtain

\[K_1'(u) = \frac{d}{du} \int_{\mathbb{R}^n} |P_u \nabla f|^2 \, d\gamma_n = -2 \int_{\mathbb{R}^n} e^{-2u} |P_u \nabla^2 f|^2 \, d\gamma_n
\]

Finally, for every \(s \geq t \geq 0\),

\[K_1(t) = K_1(s) + 2 \int_t^s e^{-2u} \int_{\mathbb{R}^n} |P_u \nabla^2 f|^2 \, d\gamma_n \, du,
\]

Thus, when \(s \to \infty\),

\[K_1(t) = \left| \int_{\mathbb{R}^n} \nabla f \, d\gamma_n \right|^2 + 2 \int_t^\infty e^{-2u} \int_{\mathbb{R}^n} |P_u \nabla^2 f|^2 \, d\gamma_n \, du
\]

by ergodicity of \((P_t)_{t \geq 0}\). Substitute \(K_1\) in the representation formula to get

\[
\text{Var}_{\gamma_n}(f) = \left| \int_{\mathbb{R}^n} \nabla f \, d\gamma_n \right|^2 + 4 \int_0^\infty e^{-2t} \int_t^\infty e^{-2u} \int_{\mathbb{R}^n} |P_u(\nabla^2 f)|^2 \, d\gamma_n \, dudt.
\]

Then, by Fubini’s Theorem,

\[
\text{Var}_{\gamma_n}(f) = \left| \int_{\mathbb{R}^n} \nabla f \, d\gamma_n \right|^2 + 2 \int_0^\infty e^{-2u}(1 - e^{-2u}) \int_{\mathbb{R}^n} |P_u(\nabla^2 f)|^2 \, d\gamma_n \, du.
\]

In order to obtain the general statement, iterate the scheme of proof: set similarly

\[K_2(u) = \int_{\mathbb{R}^n} |P_u(\nabla^3 f)|^2 \, d\gamma_n,
\]

then

\[K_2(u) = \left| \int_{\mathbb{R}^n} \nabla^2 f \, d\gamma_n \right|^2 + 2 \int_0^u e^{-2t} \int_{\mathbb{R}^n} |P_t \nabla^3 f|^2 \, d\gamma_n \, dt.
\]

After some substitution, it suffices to calculate
\[ \left| \int_{\mathbb{R}^n} \nabla^2 f d\gamma_n \right|^2 \left[ 2 \int_0^\infty e^{-2u}(1 - e^{-2u})du \right] = \frac{1}{2} \int_{\mathbb{R}^n} \nabla^2 f d\gamma_n \]

and

\[ 4 \int_0^\infty e^{-2t} \left( \int_{\mathbb{R}^n} |P_t \nabla^3 f|^2 d\gamma_n \right) \left[ \int_0^t e^{-2u}(1 - e^{-2u})du \right] dt. \]

A straightforward calculus yields

\[ 2 \int_0^t e^{-2u}(1 - e^{-2u})du = (1 - e^{-2t})^2 \]

Then, proceed by induction to conclude. Indeed, we can define by induction the coefficients that appeared at each iteration. To this task, set \( a_0(t) = 2e^{-2t} \) and \( a_1 = \int_0^\infty a_0(t)dt \). Then, for \( k \geq 1 \), \( a_k(t) = a_0(t) \int_0^t a_{k-1}(u)du \) and \( a_k = \int_0^\infty a_k(t)dt \).

It is not difficult to show that, for every \( k \geq 0 \) and every \( t \geq 0 \),

\[ a_k(t) = \frac{2}{k!}e^{-2t}(1 - e^{-2t})^k. \]

Thus, it yields, for every \( k \geq 0 \), \( a_k = \frac{1}{k!} \). \( \square \)

4.3. Taylor expansion of the variance with remainder term. We focus on the particular case when \( p = 1 \), to see what can be obtained from the representation formula (4.4)

4.3.1. Order 1. Notice that : for \( p = 1 \), the representation formula of the variance tells us that

\[ \text{Var}_{\gamma_n}(f) = \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2 + 2 \int_0^\infty e^{-2t}(1 - e^{-2t}) \int_{\mathbb{R}^n} |P_t \nabla^2 f|^2 d\gamma_n dt. \]

The second term is always strictly positive, so it implies inverse Poincaré’s inequality (cf. [BGL14])

\[ \text{Var}_{\gamma_n}(f) \geq \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2. \]

It is also possible to use the remainder term \( 2 \int_0^\infty e^{-2t}(1 - e^{-2t}) \int_{\mathbb{R}^n} |P_t \nabla^2 f|^2 d\gamma_n dt \) as follow :

\[ \text{Var}_{\gamma_n}(f) = \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2 + 2 \int_0^\infty e^{-2t}(1 - e^{-2t}) \int_{\mathbb{R}^n} |P_t \nabla^2 f|^2 d\gamma_n dt. \]

Based on the preceding formula, we can apply the hypercontractive scheme of proof (of Talagrand’s inequality). Indeed,

\[ 2 \int_0^\infty e^{-2t}(1 - e^{-2t}) \int_{\mathbb{R}^n} |P_t \nabla^2 f|^2 d\gamma_n dt = 2 \sum_{i,j=1}^n \int_0^\infty e^{-2t}(1 - e^{-2t}) \int_{\mathbb{R}^n} (P_t(\partial_{ij}^2 f))^2 d\gamma_n dt, \]

Otherwise saying, we have to bound

\[ I = 2 \sum_{i,j=1}^n \int_0^\infty e^{-2t}(1 - e^{-2t}) \| P_t(\partial_{ij} f) \|^2 dt \]

To this task, we can use the hypercontractive property (11) of \( (P_t)_{t \geq 0} \), for any function \( g : \mathbb{R}^n \to \mathbb{R} \) smooth enough,
\[ \|P_t(g)\|_2^2 \leq \|g\|_{1 + e^{-2t}}^2, \quad t \geq 0. \]

Apply this to \( g = \partial_{ij}f \), for any \( i, j = 1, \ldots, n \), then set \( v = 1 + e^{-2t} \). To sum up, we have

\[ I \leq \sum_{i,j=1}^{n} \int_{0}^{\infty} (2 - v)\|\partial_{ij}f\|_2^2 dv. \]

Besides, Hölder’s inequality implies that \( \|\partial_{ij}f\|_v \leq \|\partial_{ij}f\|_1^{\theta} \|\partial_{ij}f\|_2^{1-\theta} \), with \( \theta = \theta(v) \) satisfying the following relation \( \frac{1}{\theta} = \frac{1}{2} + \frac{1-\theta}{2} \), for every \( v \in [1,2] \). This yields the following upper bound

\[ I \leq \sum_{i,j=1}^{n} \|\partial_{ij}f\|_2^2 \int_{1}^{2} (2 - v) \left( \frac{\|\partial_{ij}f\|_1}{\|\partial_{ij}f\|_2} \right)^{2\theta} dv. \]

Set \( \alpha = \frac{\|\partial_{ij}f\|_v}{\|\partial_{ij}f\|_2} \leq 1 \), then it is not difficult to show (after a change of variable) that, for every \( i, j = 1, \ldots, n \),

\[ \int_{1}^{2} (2 - v) \left( \frac{\|\partial_{ij}f\|_1}{\|\partial_{ij}f\|_2} \right)^{2\theta} dv = \int_{0}^{1} e^{-\frac{2\alpha}{\theta} \log(1/\alpha)} du. \]

Then, observe that \( \int_{0}^{1} e^{-\frac{2\alpha}{\theta} \log(1/\alpha)} du \leq C \frac{1}{1+\log(1/\alpha)} \) with \( C > 0 \) a numerical constant. Finally, we have obtained

\[ \text{Var}_{\gamma_n}(f) \leq \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2 + C \sum_{i,j=1}^{n} \frac{\|\partial_{ij}^2 f\|_2^2}{1 + \log \left( \frac{\|\partial_{ij}f\|_2}{\|\partial_{ij}f\|_1} \right)^2}, \]

with \( C > 0 \), a numerical constant. As a conclusion, we have proved the following Theorem

**Theorem 14.** Under the preceding framework, for any function \( f : \mathbb{R}^n \to \mathbb{R} \) smooth enough, we have obtained the following upper bound on the variance of \( f \)

\[ \text{Var}_{\gamma_n}(f) \leq \left| \int_{\mathbb{R}^n} \nabla f d\gamma_n \right|^2 + C \sum_{i,j=1}^{n} \frac{\|\partial_{ij}^2 f\|_2^2}{1 + \log \left( \frac{\|\partial_{ij}f\|_2}{\|\partial_{ij}f\|_1} \right)^2}, \]

Similarly as the discrete case, by induction, we can deduce Talagrand inequality at higher order. Notice that the second term (with the logarithmic factor) can be seen as the remainder term of Taylor’s expansion of the variance.

**Theorem 15.** For \( f : \mathbb{R}^n \to \mathbb{R} \) smooth enough, \( f \in C^p(\mathbb{R}^n) \ p \geq 1 \) with all partial derivatives belonging to the space \( L^2(\gamma_n) \), for any \( p \geq 1 \), we have

\[ \text{Var}_{\gamma_n}(f) \leq \sum_{k=1}^{p} \frac{1}{k!} \left| \int_{\mathbb{R}^n} \nabla^k f d\gamma_n \right|^2 + C \sum_{i_1, \ldots, i_{p+1}} \frac{\|\partial_{i_1, \ldots, i_{p+1}} f\|_2^2}{1 + \log \left( \frac{\|\partial_{i_1, \ldots, i_{p+1}} f\|_2}{\|\partial_{i_1, \ldots, i_{p+1}} f\|_1} \right)^{p+1}}, \]
Remark. Again, observe that the sum $\sum_{k=1}^{p} \frac{1}{k!} \left| \int_{\mathbb{R}^n} \nabla^k f \, d\gamma_n \right|^2$ is precisely the beginning of the expansion of a function $f \in L^2(\gamma_n)$ along the Hermite’s polynomials basis.

5. Further comments and remarks

To conclude this note, we would like to make some remarks about possible extension of our work.

5.1. Potential extensions.

5.1.1. Biased cube. It is possible to embedded the discrete cube $\{-1, 1\}^n$ with a biased measure $\nu^p_n = (p\delta_1 + p\delta_{-1})^\otimes n$ with $p \in [0, 1]$ and $p + 1 = 1$. This measure also satisfied a Poincaré and Sobolev’s logarithmic inequalities (with different constants though) and gives rise to a hypercontractive and ergodic semi-group $(T^T_p)_{t \geq 0}$. It is then obvious that our results can be immediately extended to such setting. However, some care has to be taken with the constant, some of them will depend of the Sobolev’s logarithmic constant $pq \log p - \log q \neq p$ of $\nu^p_n$.

The study of the dependance in $p$ of the measure $\nu^p_n$ has been proven useful (cf. [BLM13, Tal94] for more details) concerning sharp threshold for monotone graph. For instance, in [FK96], the authors proved the following

**Theorem 16** (Friedgut-Kalai). For every symmetric monotone set $A$ and every $\epsilon > 0$, if $\nu^p_n(A) > \epsilon$ then $\nu^p_n(A) > 1 - \epsilon$ for $q = p + c_1 \log (1/2\epsilon) / \log n$ with $c_1$ is an absolute constant.

They also asked if the following conjecture holds true (cf. [FK96] for more details)

**Conjecture 1.** Let $P$ be any monotone property of graphs on $n$ vertices and $\epsilon > 0$. If $\nu^p_n(P) > \epsilon$, then $\nu^p_n(P) > 1 - \epsilon$ for $q = p + c_1 \log (1/2\epsilon) / \log^2 n$.

The proof fo Theorem 16 rests on the so-called Russo-Margulis’s Lemma (cf. [BLM13, FK96]) and Kahn-Kalai-Linial’s Theorem 2. It is then natural to ask if Talagrand’s inequalities at higher order (and its consequences in terms of influences) for the baiaised cube can be used to prove Conjecture 1?

Elementary calculus showed that Russo-Margulis’s Lemma can be extended ar order two. However it seems (cf. [Ros]) that the extension of Kahn-Kalai-Linial at order two is too rough to prove the conjecture. Maybe one has to add further arguments.

5.1.2. General setting. As another extension of our work, it is possible to consider the general framework of Cordero-Erausquin and Ledoux’s article [CEL12]. Indeed, as they proved, the crucial point of Talagrand’s inequality (1.2) is the decomposition of the Dirichlet’s energy along directions which commutes with the semi-group (cf. [CEL12] for more details) together with some hypercontractive estimates. Even if this extension is straightforward, we did not want to get to this level of generality for the sake of clarity of our exposition.

5.2. Link with concentration of measure. As far as we are concerned, it seems that our work has some connection with some recent results in refinement of concentration of measure. For more details on concentration of measure phenomenon, we refer the reader to [Led01, BLM13].
5.3. **Gaussian small deviations inequalities.** In a Gaussian setting, concentration of measure is usually stated as follows

**Theorem 17** (Borell-Sudakov-Tsirel’son-Ibragimov). *Let* \( f : \mathbb{R}^n \to \mathbb{R} \) *be a Lipschitz function and* \( X \) *a standard Gaussian vector in* \( \mathbb{R}^n \). *Then, the following holds*

\[
P \left( \left| f(X) - \mathbb{E}[f(X)] \right| \geq t \right) \leq 2e^{-t^2/2\|f\|_{Lip}^2}, \quad t \geq 0
\]

This result is known to be sharp for the large deviation (cf. \[LT11, PV16\]). However, it is not the case for the small deviation when one considers particular functions (\( f(x) = \max_{i=1,...,n} x_i \), for instance). In their article \[PV17\], Paouris and Valettas, proved that Talagrand’s inequality \((1.2)\) (with \( \gamma_n \) and continuous partial derivatives instead of \( \mu_n \) and discrete derivatives) can be used to precise inequality \((5.1)\) in the small deviation regime. More precisely, they proved the following

**Proposition 18** (Paouris-Valettas). *Let* \( f : \mathbb{R}^n \to \mathbb{R} \) *be a Lipschitz map with*

\[
|f(x) - f(y)| \leq b|x-y|_2, \quad |f(x) - f(y)| \leq a|x-y|_\infty, \quad x,y \in \mathbb{R}^n
\]

*and* \( \|\partial_i f\|_1 \leq A \) *for all* \( i \in \{1,\ldots,n\} \). *Then, if we set* \( F = f - \mathbb{E}_{\gamma_n}[f] \), *for all* \( \lambda > 0 \) *we have*

\[
\text{Var}_{\gamma_n}(e^{\lambda F}) \leq \frac{C\lambda^2b^2}{\log(e + \frac{\lambda^2}{aA})} \mathbb{E}_{\gamma_n}[e^{2\lambda F}]
\]

*Moreover, we obtain*

\[
P \left( \left| f(X) - \mathbb{E}[f(X)] \right| \geq t \right) \leq 4\exp \left( -c \max \left\{ \frac{t^2}{b^2}, \frac{t}{b} \sqrt{\log \left( e + \frac{b^2}{aA} \right)} \right\} \right), \quad t \geq 0
\]

*where* \( C, c > 0 \) *are universal constants.\*

**Remark.** It is a simple matter to check that equation \((5.2)\) is sharp for the function \( f(x) = \max_{i=1,...,n} x_i \). Such achievements is part of the superconcentration phenomenon introduced by Chatterjee in \[Cha14\]. We also refer to \[Tan15, Tan17a, Tan17b\] for recent results in this topic (in particular, the article \[Tan15\] gives some sort of extension of Proposition 18 for particular Gaussian measures).

Since Paouris and Valettas’s work rests on Talagrand’s inequality \((1.2)\) (at order one), we wonder if Theorem can be of any help to precise any further the concentration of measure for the Gaussian measure \( \gamma_n \)?

5.4. **Higher order of concentration of measure.** Recently, Bobkov, Gotze and Sambale wrote an article \[BGS17\] about higher order of concentration inequalities. In particular, they studied sharpened forms of the concentration of measure phenomenon typically centered at stochastic expansions (the so-called Hoeffding decomposition) of order \( d - 1 \) for any \( d \in \mathbb{N} \). The bounds are based on \( d \)-th order derivatives. They also considered deviations of functions of independent random variables and differentiable functions over probability measures satisfying a logarithmic Sobolev inequality. As a sample, we will present one important results of their paper.
First, we introduce some notations as it is presented in [BGS17]. Given a function $f \in C^d(\mathbb{R}^n)$ we define $f^{(d)}$ to be the (hyper-) matrix whose entries represent the $d$-fold (continuous) partial derivatives of $f$ at $x \in \mathbb{R}^n$. By considering $f^{(d)}(x)$ as a symmetric multilinear $d$-form, we define operator-type norms by

$$|f^{(d)}(x)|_{Op} = \sup \{ f^{(d)}(x)[v_1, \ldots, v_d] : |v_1| = \ldots = |v_d| = 1 \}$$

For instance, $|f^{(1)}(x)|_{Op}$ is the Euclidean norm of the gradient $\nabla f(x)$, and $|f^{(2)}(x)|_{Op}$ is the operator norm of the Hessian $f''(x)$. Furthermore, the following short-hand notation will be used

$$\|f^{(d)}\|_{Op,p} = \left( \int_{\mathbb{R}^n} |f^{(d)}|_{Op}^p d\mu \right)^{1/p}, \quad p \in (0, +\infty]$$

Now, we can state their result.

**Theorem 19 (Bobkov-Götze-Sambale).** Let $\mu$ be a probability measure on $\mathbb{R}^n$ satisfying a logarithmic Sobolev’s inequality with constant $\sigma^2$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be $C^d$-smooth function such that

$$\int_{\mathbb{R}^n} f d\mu = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \partial_{i_1 \ldots i_k} f d\mu = 0$$

for all $k = 1, \ldots, d-1$ and $1 \leq i_1 \leq \ldots \leq i_k \leq n$. Assume that

$$\|f^{(d)}\|_{HS,2} \leq 1 \quad \text{and} \quad \|f^{(d)}\|_{Op,\infty} \leq 1$$

Then, there exists some universal constant $c > 0$ such that

$$\int_{\mathbb{R}^n} \exp \left( \frac{c}{\sigma^2} |f|^{2/d} \right) d\mu \leq 2$$

Remark. A possible choice is $c = 1/(8\epsilon)$. Note that, by partial integration, if $f$ is the standard Gaussian measure $\gamma_n$, the conditions $\int_{\mathbb{R}^n} f d\mu = 0$ and $\int_{\mathbb{R}^n} \partial_{i_1 \ldots i_k} f d\mu = 0$ are satisfied, if $f$ is orthogonal to all polynomials of (total) degree at most $d - 1$. Such concentration’s results for non Lipschitz function (which are orthogonal to some part of an orthonormal basis) have been already obtained in various papers, we refer to the article [BGS17] and references therein for more details.

Their proof relies on the logarithmic Sobolev’s inequality together with some comparison of moments. Recall that logarithmic Sobolev’s inequality is equivalent to the hypercontractive property of the associated semi-group (cf. [BGL14]). We ask if is possible de recover their results with semi-groups arguments? In particular, is it possible to prove (and maybe improve by a dimension factor) Theorem 19 for $d = 2$ with Talagrand’s inequality at order two from Theorem ???

5.5. **Alternative proof.** Several months after this note was written, Oleszkiewicz kindly communicated to us another way to reach some extension of Kahn-Kalai and Linial’s Theorem at order two. We warmly thank him for the permission to reproduce his proof. Oleszkiewicz’s argument rests on a smart use of the logarithmic Sobolev’s inequality satisfied by the measure $\mu^n$. We state below Oleszkiewicz’s variant of Kahn-Kalai and Linial’s theorem at order two. During the proof, we will need some fact from the $L^2$ Fourier-Walsh theory (cf. section 2). Again, we refer to [O’D14] for more details on this matter.
Theorem 20 (Oleszkiewicz). Let \( f : \{-1,1\}^n \to \{-1,1\} \) be a centered Boolean function. Then, the following holds

\[
\max_{i \neq j} I_{i,j}(f) \geq c \left( \frac{\ln n}{n} \right)^2, \quad \text{for some universal constant} \quad c > 0
\]

unless there exists \( i \in \{1, \ldots, n\} \) such that

\[
I_i(f) = 1 - O\left( \frac{\ln^2 n}{n} \right) \quad \text{and} \quad \sup_{j \in \{1, \ldots, n\}, j \neq i} I_{j}(f) = O\left( \frac{\ln n}{n} \right)
\]

when \( n \to +\infty \).

Proof. Consider the Walsh-Fourier decomposition of the function \( f \):

\[
f = \sum_{A \subseteq \{1, \ldots, n\}} \hat{f}(A) W_A.
\]

It is easy to check that, for \( i, j \in \{1, \ldots, n\}, i \neq j \),

\[
D_i f = \sum_{A \subseteq \{1, \ldots, n\}, A \ni i} \hat{f}(A) W_A \quad \text{and} \quad D_{ij} f = \sum_{A \subseteq \{1, \ldots, n\}, A \ni i, j} \hat{f}(A) W_A.
\]

Similarly, we have

\[
I_i(f) = \sum_{A \ni i} \hat{f}(A)^2 \quad \text{and} \quad I_{i,j}(f) = \sum_{A \ni i, j} \hat{f}(A)^2
\]

Then, observe that \( \phi_i = W_{\{i\}} \cdot D_i f \) does not depend one the \( i \)-th coordinate, so that \( D_i \phi_i = 0 \). Moreover, for \( j \neq i \), \( D_j \phi_i = W_{\{i\}} \cdot D_{ij} f \). Also, note that \( \phi_i \) takes values in \( \{-1,0,1\} \). Thus, applying the logarithmic Sobolev’s inequality \((2.3)\) to \( \phi_i \), we arrive at

\[
I_i(f) \ln \left( 1/I_i(f) \right) = \operatorname{Ent}_{\mu^n} (|D_i f|^2) = \operatorname{Ent}_{\mu^n} (\phi_i^2) \leq 2 \sum_{j=1}^n \|D_j \phi_i\|_2^2
\]

Then, observe that

\[
2 \sum_{j=1}^n \|D_j \phi_i\|_2^2 = 2 \sum_{j \in \{1, \ldots, n\}, j \neq i} \|D_{ij} f\|_2^2 = 2 \sum_{j \in \{1, \ldots, n\}, j \neq i} I_{i,j}(f)
\]

Recall that, for \( f \) a centered Boolean function, Theorem 2 asserts that there exists \( i \in \{1, \ldots n\} \) such that \( I_i(f) \geq c n^{-1} \ln n \) where \( c \) is some universal positive constant. Note that \( x \mapsto x \ln(1/x) \) is increasing in \( x \) on \((0,1/e]\) and not less than \((1-x)/(e-1)\) for \( x \in [1/e,1] \).

So, unless \( I_i(f) = 1 - O(n^{-1}(\ln n)^2) \), this implies that \( \sum_{j \in \{1, \ldots, n\}, j \neq i} I_{i,j}(f) \geq c' n^{-1}(\ln n)^2 \) and thus

\[
\max_{i \neq j} I_{i,j}(f) \geq c'' n^{-2}(\ln n)^2,
\]

for some universal positive constants \( c' \) and \( c'' \).

Actually, it may happen that \( I_k(f) \) and \( I_l(f) \) are both close to one for some \( k \neq l \), but then it is easy to show that \( I_{k,l}(f) \geq I_k(f) + I_l(f) - 1 \) is also close to one. So, for large enough \( n \), the only case when we do not get \( \max_{i \neq j} I_{i,j}(f) \geq c'' n^{-2}(\ln n)^2 \)
is when exactly one of the influence is close to one and the remaining ones are close to zero.

The mean-zero assumption can be easily weakened to bounding $|E_{\mu^n}[f]|$ away from one, upon appropriate changes of constants. In some sense, this allows one to rule out the constant function $\pm 1$ from our setting.

**Theorem 21** (Oleszkiewicz). Let $f : \{-1, 1\}^n \to \{-1, 1\}$ and denote by $d = \max_{A \subseteq \{1, \ldots, n\}, \Card(A) \leq 1} |\hat{f}(A)|$. Then,

$$\max_{i \neq j} I_{(i,j)}(f) \geq c \left(\frac{\ln n}{n}\right)^2$$

for some universal constant $c > 0$ unless $d \geq \frac{1}{4}$.

**Proof.** Let $f : \{-1, 1\}^n \to \mathbb{R}$ and its associated $L^2$-decomposition $f = \sum_{A \subseteq \{1, \ldots, n\}} \hat{f}(A)W_A$. Since

$$\int_0^\infty (e^{2t} - 1)e^{-2kt} dt = \frac{1}{2} \left( \frac{1}{k - 1} - \frac{1}{k} \right) = \frac{1}{4} \left( \frac{k}{2} \right)^{-1},$$

by the spectral decomposition we easily get

$$\sum_{A \subseteq \{1, \ldots, n\}, \Card(A) \geq 2} |\hat{f}(A)|^2 = 4 \sum_{i < j} \int_0^\infty \|Q_t D_{ij} f\|^2_2 (e^{2t} - 1) dt,$$

where $(Q_t)_{t \geq 0}$ denotes the Bonami-Beckner semigroup on the discrete cube. We set $\rho^2 = \sum_{A \subseteq \{1, \ldots, n\}, \Card(A) \geq 2} |\hat{f}(A)|^2$ to ease the notation. Hence, if $f$ is a Boolean (with values in $\{-1, 1\}$) function, then

$$\rho^2 \leq C \sum_{i < j} \frac{I_{(i,j)}(f)}{1 + \ln^2 (1/I_{(i,j)}(f))}.$$

This inequality is obtained by similar means as Theorem 4. Indeed, it suffices to use the hypercontractive bound for small values of $t$, as well as the hypercontractive bound for $Q_{t_0}$ combined with the $L^2$-contractivity of $Q_{t-t_0}$ for large value of $t$. Note that $\|Q_t (D_{ij} f)\|^2_2 \leq e^{-4t} \|D_{ij} f\|^2_2 = e^{-4t} I_{(i,j)}(f)$ because $D_{ij} f$ has zero Fourier-Walsh coefficient on first two levels (i.e. for $\Card(A) \leq 1$). One uses also the fact that $D_{ij} f = \{-1, 1/2, 0, 1/2, 1\}$-valued, so that $\|D_{ij} f\|_q = \left[I_{(i,j)}(f)\right]^{1/q}$ for $q \in [1, 2]$. The Friedgut, Kalai and Naor’s Theorem (cf. [FKN02, O’D14]) implies that if $\rho \leq \rho_0$ for some universal positive constant $\rho_0$, then $d \geq \frac{1}{2}$. This proves that

$$\max_{i \neq j} I_{(i,j)}(f) \geq c \left(\frac{\ln n}{n}\right)^2$$

unless $d \geq 1/2$ (which implies that $|E_{\mu^n}[f]| = |\hat{f}(\emptyset)| \geq 1/2$ or $I_i(f) \geq \hat{f}(\{i\}) \geq 1/4$ for some $i \in \{1, \ldots, n\}$). The constant $1/2$ has no particular role and can be replaced by $1 - \epsilon, \epsilon > 0$, for a precise statement we refer to Theorem 5.3 in [JOW15].

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