The role of cosmological constant in f(R, G) gravity

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Abstract
Einstein–Hilbert action is supplemented by the Gauss–Bonnet squared term, its phase-space structure is constructed and canonical quantization is performed. Resolution of a contradiction that emerges in the process, requires the presence of other fields at least in the form of vacuum energy-density, commonly known as the cosmological constant. This reveals the very importance of the presence of other fields at least in the form of the cosmological constant, in the very early universe.

Keywords: Gauss–Bonnet squared term, phase-space structure, quantization and semiclassical approximation, the need for cosmological constant

1. Introduction

The general theory of relativity (GTR) or any of its modified versions treats all the fields on the same footing. Although all the fields are quantized much prior to the gravitational field—being described by geometry, it is not known which field should be treated as fundamental, say for the creation of the universe. It is well known that GTR is non-renormalizable and a renormalized theory of gravity has not been found as yet. However, all attempts to cast a renormalized theory of gravity (different versions of string theories) indicate that in the very early universe the Einstein–Hilbert action must be supplemented by higher order curvature invariant terms. The existence of vacuum de-Sitter (anti-de-Sitter) solution, being a primary check for the choice of particular combination of higher-order terms, does not require the presence of fields of any kind other than gravity. Further, lot of research had been oriented [1–7] regarding creation of particles from strong gravitational field. In this sense, it is a preconceived notion that geometry must be treated as fundamental. In this article, with the motivation to study the early universe, we construct the phase-space structure of a recently advocated (to explain early inflation by some authors [8, 9] and late-time cosmic evolution of the universe by some
others [10–13]) geometric theory associated with Einstein–Hilbert action in the presence of Gauss–Bonnet squared term alone, by analysing constrained dynamics following Dirac’s algorithm. In the process, we show that a quantized version of the theory, leads to some sort of inconsistency, viz: while classical inflationary solution admits a positive coupling parameter, extremization of the effective potential although admits inflationary solution, requires a negative coupling parameter. We also perform semiclassical approximation about the classical inflationary solution, following the same technique adopted earlier [14–19]. In the process, we find with surprise that the Hamilton–Jacobi equation so obtained, does not satisfy the Hamilton–Jacobi equation, although everything else appears to be consistent. These problems are resolved under the addition of a cosmological constant in the said action. The fact that the cosmological constant is the sum of zero point vacuum energy density of all possible fields existing in the very early universe, the resolution of the said contradiction reveals that geometry might not be treated as fundamental.

As mentioned, in the present manuscript we supplement Einstein–Hilbert action with Gauss–Bonnet squared term. Gauss–Bonnet term is topologically invariant in four dimension, as a result it does not contribute to the classical field equations. Therefore to ensure some contribution, it is usually coupled to a dilatonic scalar field, which arises naturally in the weak field limits of several effective string theories [20–26]. Such an action is well suited to explain late-time cosmic evolution, particularly the recently observed accelerated expansion of the universe, after a long matter dominated Friedmann-like evolution $a \propto t^2$, where, $a$ is the scale factor [27]. It also admits transient double crossing of the phantom divide line, which is not excluded from the recent observations [28]. Further, it has also been shown that the effective cosmological constant is comparable with the present value of the Hubble parameter [29]. The question arises regarding the presence of the scalar field itself. While there is no indication of the presence of a scalar field in any of its form in the late universe, a dilatonic coupled Gauss–Bonnet interaction seems to be unrealistic. It is therefore reasonable to discard the scalar field and try to explain the late-time cosmic evolution otherwise. This is possible under modification of the left hand side of Einstein’s equation, which corresponds to incorporating higher order curvature invariant terms in the Einstein–Hilbert action. Such modification as discussed, is essential for a renormalized theory of gravity. One, out of such many attempts has been successfully made by taking higher powers of Gauss–Bonnet term in the action, which does not require coupling to a scalar field even in 4-dimension, and leads to GTR as the low energy limit of an affective quantum theory [8]. Soon after, Neutron star solutions were presented [30], and also relativistic massive objects viz. compact star solutions and their dynamical stability were studied for a viable power law model of $f(G)$ [31]. Some attempt to unify early inflation and late-time cosmic evolution with non-minimal coupling, which appears to be unrealistic, has also been reported [32]. It is therefore tempting to test the viability of such an action in the very early universe.

In this paper we therefore couple Einstein–Hilbert sector of the gravitational action with the Gauss–Bonnet squared term and study its viability in the very early universe in regard of the quantum dynamics and inflationary scenario. It is important to mention that while Gauss–Bonnet term leads to second order field equations, its higher powers truly behaves as higher order theory. Canonical formulation of such higher-order theory is clearly non-trivial, but there are options to handle the situation, invoking additional degree of freedom. The oldest technique in this regard was developed long ago by Ostrogradski [33]. However, Ostrogradski’s technique does not work for degenerate (singular) Lagrangian, nevertheless, Dirac’s algorithm of constrained analysis works elegantly, in such situations [34, 35]. Interestingly enough, it is possible to bypass the constrained analysis following Horowitz formalism (HF) [36]. First let
us recall that the induced three-space metric $h_{ij}$ is the basic variable in the theory of gravity. For canonical formulation of higher order theory of gravity, the extrinsic curvature tensor $K_{ij}$ is treated as additional degree of freedom. But, in HF, instead of extrinsic curvature tensor, it is required to introduce an auxiliary variable at the beginning (which is the first derivative of the action with respect to the highest derivative of the field variable that appears in the action) judiciously into the action, so that the action turns out to be canonical. Phase-space structure is finally obtained upon translation to the basic variable $(K_{ij})$ through canonical transformation. whatsoever, HF suffers from some pathologies. For example, it allows introduction of auxiliary variable even in linear theory of gravity, resulting in wrong quantum equation, being different from the standard Wheeler–de-Witt equation [14, 37–42]. Further, if one starts from an action being expressed in terms of the scale factor, as proposed by Horowitz [36], there is a possibility of eliminating some additional total derivative terms which do not appear from the variational principle [41, 42]. In order to get rid of such uncanny situation yet another canonical approach, by the name ‘Modified Horowitz’ Formalism’ (MHF) has been proposed, which also bypasses constraint analysis [14–19, 37–43]. Essential features of MHF are, it is firstly preferred to set $\delta h_{ij}|_{SV} = 0 = \delta R|_{SV}$ at the boundary instead of the conventional choice, viz. setting $\delta h_{ij}|_{SV} = 0 = \delta K_{ij}|_{SV}$, and the action is supplemented by counter boundary terms. The reason being, under conformal transformation or following some appropriate redefinition, $F(R)$ theory of gravity, may be translated to equivalent scalar–tensor theories in Einstein’s or Jordan’s frames respectively. Euler–Lagrange equations may then be derived provided the scalar field vanishes at the end points [44, 45]. This is equivalent to the vanishing of the 4-dimensional Ricci scalar $(R)$ at the boundary. Next, the action is expressed in terms of $h_{ij}$, and under integration by parts the total derivative terms get cancelled with some (not all) of the supplementary boundary terms. Auxiliary variable is then introduced following Horowitz’ prescription [36] and total derivative term appearing under integration by parts again, cancels with the rest of the supplementary boundary terms. Horowitz’ technique is then followed to obtain a canonical Hamiltonian. Although, the resulting Hamiltonian is canonically equivalent to the one obtained following Horowitz’ prescription, in all the cases studied so far, they produce completely different quantum descriptions [19]. In fact, it has been shown that for ‘Modified Einstein–Gauss–Bonnet-dilatonic coupled action’, Horowitz’ prescription suffers from serious setback [19]. Nevertheless, it has also been proved that Dirac’s formalism (with conventional settings, $\delta h_{ij}|_{SV} = 0 = \delta K_{ij}|_{SV}$), after taking care of total derivative terms appearing in the action, yields identical Hamiltonian description and its quantum counterpart as in MHF [19]. It is therefore a matter of taste to apply one of the two prescriptions towards canonical formalism of higher-order theories of gravity. Here, we adopt Dirac’s formalism.

We organize the present manuscript as follows. In the following section, we take up Gauss–Bonnet squared term along with the linear gravitational sector, find the field equations in the background of isotropic and homogeneous space-time which have been found to admit vacuum de-Sitter solution. We study the behaviour of the action in the radiation dominated era and also the inflation. In section 3, we cast the action in canonical form and formulate the phase-space structure, following Dirac’s algorithm, as mentioned. Thereafter we follow the standard canonical quantization scheme and perform semiclassical approximation about the classical de-Sitter solution. In the process we observe that the theory suffers from certain pathologies. The situation changes dramatically under the influence of a cosmological constant, which we study next in section 4. We conclude in section 5.
2. The action, field equations, classical solutions and Inflation

We start with the following $f(R, \mathcal{G})$ action expressed in its simplest form as,

$$A = \int \left[ \alpha R + \gamma \mathcal{G}^2 \right] \sqrt{-g} d^4x, \quad (1)$$

where, $\alpha = \frac{1}{16\pi G}$ and $\gamma$ are coupling constants, while the Gauss–Bonnet term is $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. In homogeneous and isotropic Robertson–Walker metric

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \quad (2)$$

under the choice $h_{ij} = a^2 \delta_{ij}$, the Ricci scalar ($R$) and the Gauss–Bonnet ($\mathcal{G}$) terms take the following forms,

$$R = \frac{6}{N^2} \left( \frac{\dot{a}}{a} + \frac{\ddot{a}}{a^2} + N^2 \frac{k}{a^2} - \frac{\dot{N}\dot{a}}{Na} \right) = \frac{6}{N^2} \left[ \frac{\dot{z}}{2z} + N^2 \frac{k}{2} - \frac{\dot{N}\dot{z}}{2Nz} \right]$$

$$\mathcal{G} = \frac{24}{N^2 a^2} (N\ddot{a} - \dot{N}\dot{a}) \left( \frac{\ddot{a}}{N^2} + k \right) = \frac{12}{N^2} \left[ \frac{\dot{z}}{2z} - \frac{\dot{z}^2}{2z^2} - \frac{\dot{N}\dot{z}}{Nz} \left( \frac{\dot{z}^2}{4N^2 z^2} + \frac{k}{z} \right) \right]. \quad (3)$$

The action therefore reads as

$$A = \int \left[ \frac{6\alpha}{N^2} \left( \frac{\dot{z}}{2z} + N^2 \frac{k}{2} \right) - \frac{1}{2Nz} \right] + \frac{144\gamma}{N^4} \left( \frac{\dot{z}}{z} - \frac{\dot{z}^2}{2z^2} - \frac{\dot{N}\dot{z}}{Nz} \right)^2 \left( \frac{\dot{z}^2}{4N^2 z^2} + \frac{k}{z} \right) N z^2 d\theta \int d^3x. \quad (4)$$

Field equations ($k = 0$) under the inclusion of a barotropic fluid with pressure ($p$) and energy density ($\rho$) read as,

$$2\alpha \left( \frac{\dot{z}}{z} - \frac{\dot{z}^2}{4z^2} \right) + 12\gamma \left[ \frac{\dot{z}^2 z^4}{z^4} + \frac{8\dot{z}^2 z^4}{z^6} - \frac{9\dot{z}^2 z^4}{z^8} + \frac{6z^2 z^3}{z^6} - \frac{135\dot{z}^2 z^3}{4z^8} + \frac{159\dot{z}^2 z^3}{4z^8} - \frac{195z^5}{16z^8} \right] = -p. \quad (5)$$

$$3\alpha \dot{z}^2 - 18\gamma \left[ \frac{\dot{z}^2 z^6}{z^6} + \frac{3\dot{z}^2 z^6}{2z^8} - \frac{9\dot{z}^2 z^6}{2z^8} + \frac{15z^8}{8z^8} \right] = \rho. \quad (6)$$

together with the energy conservation equation,

$$\dot{\rho} + \frac{3}{a} (\rho + p) = \dot{\rho} + \frac{3z}{2z}(\rho + p) = 0. \quad (7)$$

2.1. Classical solution

One can clearly notice that the above field equations do not admit power law solution of any form in the radiation dominated era ($p = \frac{4}{3}\rho, \rho z^2 = \rho_m$, $\rho_m$ being a constant which measures the amount of radiation left in the present day universe). Thus, here we observe that, despite the claim that the theory can explain late-time cosmic evolution with appropriate accelerated expansion, it confronts with the standard model in the radiation dominated era. This tells upon the structure formation and the CMBR we observe. Therefore the theory in no way is viable to explain the evolution history of the universe. This is the first pathology appearing in connection with the theory under investigation. We would like to mention that in some models
the radiation era shows Friedmann-like cosmic evolution \((a \propto \sqrt{t})\) and therefore is well-behaved.

However, in the present manuscript, we are indeed interested in the very early vacuum dominated \((p = 0 = \rho)\) era. The above set of field equations admit the following vacuum de-Sitter inflationary solution in the form

\[
z = z_0 e^{2\lambda t} \Rightarrow a = a_0 e^{\lambda t}; \quad \text{where, } \lambda = \frac{1}{2} \left( \frac{2\alpha}{3\gamma} \right)^{\frac{1}{2}} = \frac{1}{2} \left( \frac{M_p^2}{3\gamma} \right)^{\frac{1}{2}},
\]

where, \(M_p\) is the Planck’s mass. We shall require the above vacuum de-Sitter solution later, to compute on-shell semiclassical approximation.

2.2. Inflation

Let us first describe the background evolution by a set of horizon flow functions (the behaviour of Hubble distance during inflation) starting from

\[
\epsilon_0 = \frac{dH}{dH_i},
\]

where, \(dH = H^{-1}\) is the Hubble distance, or more commonly the horizon, in the unit \(c = 1\). Suffix \(i\) stands for the era when inflation was initiated. Let us now define a hierarchy of functions in a systematic way by,

\[
\epsilon_{m+1} = \frac{d\ln |\epsilon_m|}{dN}, \quad m \geq 0, \quad N = \ln \frac{a}{a_i} = \int Hdt,
\]

where \(H\) is the Hubble parameter. One can compute \(\epsilon_1 = \frac{d\ln a}{dN}\), which is the logarithmic change of Hubble distance per e-fold expansion \(N\). It is in fact the first slow-roll parameter \(\epsilon_1 = \dot{H}/H^2\), which implies that the Hubble parameter almost remains constant during inflation. The above hierarchy allows one to compute \(\epsilon_2 = \frac{d\ln \epsilon_1}{dN} = \frac{1}{H} \frac{dH}{dN}\), leading to \(\epsilon_1 \epsilon_2 = dH \dot{H} = -\frac{1}{H} \left( \frac{\dot{H}^2}{H^2} - 2 \frac{H''}{H'} \right)\). In the same manner higher slow roll parameters may be computed. Equation (6) may now be expressed as,

\[
\alpha + 96\gamma \left[ 2H^6 \left( 1 + \frac{1}{H^2} \left( \frac{\dot{H}}{H} - \frac{2H^2}{H'} \right) \right) + 7H^6 \left( 1 + \frac{\dot{H}}{H^2} \right)^2 - 8H^6 \left( 1 + \frac{\dot{H}}{H^2} \right) - 2H^6 \right] = 0,
\]

or in terms of the slow-roll parameters,

\[
\alpha + 96\gamma \left[ 2H^6 (1 - \epsilon_1 \epsilon_2) + 7H^6 (1 - \epsilon_1)^2 - 8H^6 (1 - \epsilon_1) - 2H^6 \right] = 0.
\]

Now, under slow roll approximation \(\epsilon_m \ll 0\), one arrives at,

\[
\alpha = 96H^6 \gamma,
\]

which is solved to yield

\[
a(t) = a_0 \exp \left[ \left( \frac{\alpha}{96\gamma} \right)^{\frac{1}{2}} t \right].
\]
This slow roll inflationary solution resembles exactly with the one obtained in view of an effective Lagrangian \( \mathcal{L} = R + \beta R^4 + \gamma R^3 \), in the regime when \( 96\gamma H^4 \gg 6\beta \), i.e. in the regime when \( R^2 \) term is neglected. This implies that Gauss–Bonnet squared term \((G^2)\) effectively plays the same role as \( R^4 \) term in the modified theory of gravity, at least in the background of homogeneous and isotropic Robertson–Walker space-time \([8]\). It is important to mention for later reference that both the classical de-Sitter solution \((8)\) and the slow-roll inflationary solution \((14)\) are admissible for a positive coupling parameter \( \gamma \). It is to be noticed that while during slow-roll \( H \) is considered to be slowly varying, here, it turns out to be a constant, and as a result, the slow-roll parameters vanish. This fact clearly indicates that standard-slow roll technique originally introduced for scalar fields, must not be extrapolated in every situation (higher-order theories), unless the theory has explicit scalar-tensor equivalence.

3. Canonical formulation and quantization

In the introduction, we have mentioned that the Hamiltonian obtained following MHF is identical with Dirac’s algorithm of constrained dynamics if the action is made free from total derivative terms a-priori \([19]\). Here, we follow Dirac’s formalism. Therefore, performing some algebraic manipulation and integrating the appropriate terms appearing in the action \((4)\) by parts, following total derivative terms are found

\[
\Sigma = \int \left[ \frac{6\alpha \sqrt{\dot{z}^2}}{2N} - 24^2 \gamma \left( \frac{\dot{z}^2}{48N^2\dot{z}^2} + \frac{k^2\dot{z}^5}{40N^2\dot{z}^2} + \frac{k^2\dot{z}^3}{12N^3\dot{z}^3} \right) \right] \, d^3x
\]

which vanish identically in the Dirac’s formalism in which \( \delta h_{ij} \partial \nu = 0 = \delta K_{ij} \partial \nu \), where, \( h_{ij} \) is the induced three-metric and \( K_{ij} \) is the extrinsic curvature tensor. Therefore the action finally takes the form,

\[
A = \int \left[ 6\alpha \left( \frac{\dot{z}^2}{4N\sqrt{\dot{z}}} + kN\sqrt{\dot{z}} \right) + 24^2 \gamma \left\{ \frac{\dot{z}^2}{64N^2\dot{z}^2} + \frac{k^2\dot{z}^5}{8N^2\dot{z}^2} + \frac{k^2\dot{z}^3}{4N^3\dot{z}^3} \right\} \right] \int d^3x.
\]

It has already been mentioned that canonical formulation must be performed in terms of the pair of basic variables, viz. \( \{h_{ij}, K_{ij}\} \). In the present situation \( h_{ij} = a^2 \delta_{ij} = z \delta_{ij} \) and \( K_{ij} = -\frac{h_{ij}}{2N} = -\frac{a^2}{N} = -\frac{\dot{z}}{2N} \). This is the reason for expressing the action in terms of \( \dot{z} = a^2 \) from the very beginning.

3.1. Analysing the constraint

Now to follow Dirac’s algorithm, we substitute \( \dot{z} = Nx \), i.e. \( \ddot{z} = N\dot{x} + \dot{N}x \), and thus one can express the point Lagrangian in the form,

\[
L = 6\alpha \left( -\frac{N\dot{x}^2}{4\sqrt{\dot{z}}} + kN\sqrt{\dot{z}} \right) + 288 \left[ \left( \frac{x^4}{32N^2\dot{z}^2} + \frac{k^2\dot{z}^3}{24N^2\dot{z}^2} \right) \dot{x}^2 - \frac{15N\dot{x}^8}{896\dot{z}^2} - \frac{13kN\dot{x}^6}{80\dot{z}^2} - \frac{11k^2N\dot{x}^4}{24\dot{z}^2} \right] + a \left( \frac{\dot{z}}{N} - x \right),
\]

(17)
where we have treated the expression \((\frac{\dot{\chi}}{N} - x)\) as a constraint and introduced it through the Lagrange multiplier \(u\) in the above point Lagrangian. The canonical momenta are,

\[
p_x = \frac{288\gamma}{N} \left( \frac{x^4}{16\zeta^2} + \frac{k\alpha^2}{2\zeta^2} + \frac{k^2}{\zeta^2} \right); \quad p_z = \frac{u}{N}; \quad p_N = 0 = p_u.
\]  

(18)

The primary Hamiltonian therefore reads as

\[
H_{p1} = \frac{N^2 \bar{p}_x^2}{576\gamma} + \frac{u}{N} + 6\alpha \left( \frac{N\bar{p}^2}{4\sqrt{2}} - kN\sqrt{2} \right) + 36\gamma N\bar{p}^4 \left( \frac{15\alpha^4}{112\zeta^2} + \frac{13k\alpha^2}{10\zeta^2} + \frac{11k^2}{3\zeta^2} \right) - u \frac{\dot{z}}{N} + ux.
\]

(19)

Now introducing the constraints \(\phi_1 = Np_x - u \approx 0\) and \(\phi_2 = p_u \approx 0\) through the Lagrange multipliers \(u_1\) and \(u_2\) respectively, we get

\[
H_{p1} = N \left[ \frac{\bar{z}^2 \bar{p}_z^2}{36\gamma (x^2 + 4k\zeta)^2} + 6\alpha \left( \frac{x^2}{4\sqrt{2}} - k\sqrt{2} \right) + 36\gamma x^4 \left( \frac{15\alpha^4}{112\zeta^2} + \frac{13k\alpha^2}{10\zeta^2} + \frac{11k^2}{3\zeta^2} \right) \right] + u_1(Np_x - u) + u_2 p_u + ux.
\]

(20)

Note that the Poisson brackets \(\{x, p_x\} = \{z, p_z\} = \{u, p_u\} = 1\), hold. Now constraint should remain preserved in time, which are exhibited through the following Poisson brackets, viz.

\[
\dot{\phi}_1 = \{\phi_1, H_{p1}\} = -u_2 - N \frac{\partial H_{p1}}{\partial \bar{z}} \approx 0 \Rightarrow u_2 = -N \frac{\partial H_{p1}}{\partial \bar{z}}; \quad \dot{\phi}_2 = \{\phi_2, H_{p1}\} = -u_1 + x \approx 0 \Rightarrow u_1 = x.
\]

(21)

Therefore the primary Hamiltonian is modified to

\[
H_{p2} = N \left[ x \bar{p}_z + \frac{\bar{z}^2 \bar{p}_z^2}{36\gamma (x^2 + 4k\zeta)^2} + 6\alpha \left( \frac{x^2}{4\sqrt{2}} - k\sqrt{2} \right) + 36\gamma x^4 \left( \frac{15\alpha^4}{112\zeta^2} + \frac{13k\alpha^2}{10\zeta^2} + \frac{11k^2}{3\zeta^2} \right) \right] - Np_u \frac{\partial H_{p1}}{\partial \bar{z}}.
\]

(22)

As the constraint should remain preserved in time in the sense of Dirac, so

\[
\dot{\phi}_1 = \{\phi_1, H_{p2}\} = N \frac{\partial H_{p1}}{\partial \bar{z}} - N \left[ \frac{\partial H_{p1}}{\partial \bar{z}} - Np_u \frac{\partial^2 H_{p1}}{\partial \bar{z}^2} \right] \approx 0 \Rightarrow p_u = 0.
\]

(23)

Thus, finally the phase-space structure of the Hamiltonian, being free from constraints reads as

\[
H = N \left[ x \bar{p}_z + \frac{\bar{z}^2 \bar{p}_z^2}{36\gamma (x^2 + 4k\zeta)^2} + 6\alpha \left( \frac{x^2}{2\sqrt{2}} - k\sqrt{2} \right) + 36\gamma x^4 \left( \frac{15\alpha^4}{112\zeta^2} + \frac{13k\alpha^2}{10\zeta^2} + \frac{11k^2}{3\zeta^2} \right) \right] = NH.
\]

(24)

It is important to note that the Einstein–Hilbert sector appears only in the effective potential part of the above Hamiltonian. The action \((16)\) may now be expressed in canonical ADM form \((k = 0)\) as,

\[
A = \int \left( \bar{z} \bar{p}_z + x \bar{p}_x - NH \right) d^3x = \int \left( \bar{h}_{ij} \bar{p}^{ij} + \bar{K}_{ij} \bar{P}^{ij} - NH \right) d^3x.
\]

(25)

In the above equation, \(p^{ij}\) and \(P^{ij}\) are the momenta canonical to \(h_{ij}\) and \(K_{ij}\) respectively. The presence of \(x \bar{p}_z\) term in the Hamiltonian \((24)\) clearly indicates that as in the cases of higher
order theories studied earlier, it also leads to Schrödinger-like equation, and hence quantum mechanical probabilistic interpretation should be admissible. It is also required to check for the sake of consistency, if the same classical field equations (5) and (6) are arrived at from the above Hamiltonian (24). To avoid complication, we express the Hamilton’s equations (24) for $k = 0$, as,

$$
\mathcal{H} = xp_z + \frac{z^2 p_x^2}{36\gamma x^4} + \frac{135\gamma x^8}{28z^2} + \frac{3\alpha x^2}{2\sqrt{z}}.
$$

(26)

Hamilton’s equations are,

$$
\dot{x} = \frac{z^2 p_x}{18\gamma x^4}; \quad \dot{z} = x; \quad \dot{p}_x = -p_z + \frac{z^2 p_x^2}{9\gamma x^3} - \frac{270\gamma x^7}{7\sqrt{z}} - \frac{3\alpha x}{\sqrt{z}};
$$

$$
\dot{p}_z = -\frac{z^2 p_x^2}{8\gamma x^4} + \frac{1755\gamma x^8}{56z^2} + \frac{3\alpha x^2}{4z^2} = -18\gamma \left[ \frac{9z^4 x^2}{4z^2} - \frac{195z^8}{112z^2} \right] + \frac{3\alpha x^2}{4z^2}.
$$

(27)

From the first relation of the above set of Hamilton’s equations, we find,

$$
p_x = 18\gamma x^4 \frac{\dot{x}}{z^2} = \frac{18\gamma x^4}{z^2}; \quad p_z = 18\gamma \left[ \frac{z^2 x^2}{z^2} + \frac{12z^2 x^4}{z^2} - \frac{9z^6 x^4}{z^2} - \frac{12z^2 x^6}{z^2} - \frac{81z^4 x^2}{2z^2} + \frac{99z^6 x^4}{4z^2} \right],
$$

(28)

and so, $p_x = -18\gamma \left[ \frac{z^4 x^2}{z^2} + \frac{2z^2 x^4}{z^2} - \frac{9z^6 x^4}{2z^2} + \frac{15z^8}{7z^2} \right] - \frac{3\alpha x}{\sqrt{z}}.

Now, derivative of the second equation of the first set of equation yields,

$$
\dot{p}_x = 18\gamma \left[ \frac{4z^2 x^2}{z^2} + \frac{6z^2 x^4}{z^2} - \frac{27z^4 x^4}{z^2} - \frac{15z^6 x^4}{z^2} - \frac{195z^8}{16z^2} \right] - \frac{3\alpha x^2}{4z^2}.
$$

(29)

Equating relation $\dot{p}_x$ between relations (28) and (29), we obtain,

$$
\alpha \left( \frac{\dot{z}}{z} - \frac{\dot{x}^2}{4z^2} \right) + 6\gamma \left[ \frac{z^4 x^2}{z^2} + \frac{8z^2 x^4}{z^2} - \frac{9z^6 x^4}{z^2} + \frac{6z^2 x^6}{z^2} - \frac{135z^4 x^2}{4z^2} + \frac{159z^6 x^4}{4z^2} - \frac{195z^8}{16z^2} \right] = 0.
$$

(30)

This is equation (5). From the other set, one can also find the $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ equation of Einstein. However, the simplest way is to express the Hamiltonian in terms of configuration space variables and then setting it to zero, to find the energy equation, viz.

$$
E = -18\gamma \left[ \frac{z^4 x^2}{z^2} + \frac{3z^2 x^4}{2z^2} - \frac{9z^6 x^4}{2z^2} + \frac{15z^8}{8z^2} \right] - \frac{3\alpha x^2}{2\sqrt{z}} = 0.
$$

(31)

This is the $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ component (6) of the field equations. For a cross check, we take time derivative of the energy equation to find

$$
-\frac{\dot{E}}{3\sqrt{z}} = 6\gamma \left( \frac{z^4 x^2}{z^2} + \frac{8z^2 x^4}{z^2} - \frac{9z^6 x^4}{z^2} + \frac{6z^2 x^6}{z^2} - \frac{135z^4 x^2}{4z^2} + \frac{159z^6 x^4}{4z^2} - \frac{195z^8}{16z^2} \right) + \alpha \left( \frac{\dot{z}}{z} - \frac{\dot{x}^2}{4z^2} \right) = 0.
$$

(32)
which is equation (30). Thus, we have cross checked, and there is no inconsistency at the classical level.

3.2. Canonical quantization

Now canonical quantization leads to

\[ i\hbar \frac{\partial \Psi}{\partial z} = \frac{z^2 + \vec{p}^2}{36\gamma (x^2 + 4kz)} \Psi + \left[ \frac{\alpha}{2\sqrt{z}} - 6k\sqrt{z} \right] + 36\gamma z^4 \left( \frac{15x^4}{112z^2} + \frac{13kx^2}{10z^2} + \frac{11k^2}{3z^2} \right) \Psi. \]  

Operator ordering is required in the first term of the right hand side. Performing D'Alembertian operator ordering, \((n\) being the operator ordering index) one obtains,

\[ i\hbar \frac{\partial \Psi}{\partial z} = \frac{\hbar^2}{36\gamma z(x^2 + 4kz)^2} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) \Psi + \left[ \frac{3\alpha(x^2 - 4kz)}{2\pi^2} + 36\gamma x^3 \left( \frac{15x^4}{112z^2} + \frac{13kx^2}{10z^2} + \frac{11k^2}{3z^2} \right) \right] \Psi. \]  

This type of ordering was performed by Hartle–Hawking [49] and Hawking and Page [50] and also by Vilenkin [51, 52], in an attempt to interpret the wave function of the universe associated with GTR. Different choice of the operator ordering index \((n\) leads to different probability amplitudes which assumes that the transition amplitude for nucleation of a universe from ‘nothing’ is dominated by a Euclidean instanton. While no boundary proposal was put forward by the former [49, 50] under the choice \(n = 1\), tunnelling wave function was proposed by the later [51, 52] under the choice \(n = -1\). In all the cases studied so far with higher order theories of gravity, with different curvature squared terms even in the presence of Gauss–Bonnet-dilatonic coupling [14–19, 37–43], hermiticity of the effective Hamiltonian and the probability amplitude could be established only under the choice \(n = -1\). Now, finally, under a change of variable \(\sigma = z^2\), so that \(z^2 \frac{\partial}{\partial z} = \frac{1}{2} \frac{\partial}{\partial \sigma}\) one obtains

\[ i\hbar \frac{\partial \Psi}{\partial \sigma} = \frac{\hbar^2}{198\gamma x(x^2 + 4k\sigma \pi)^2} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) \Psi + \hat{V}_e(x, \sigma) \Psi = \hat{H}_e \Psi \]

where, \(\hat{V}_e(x, \sigma) = \frac{3\alpha}{11x} \left( \frac{x^2}{\sigma \pi} - \frac{4k}{\sigma \pi} \right) + 72\gamma x^3 \left( \frac{15x^4}{112\sigma^2} + \frac{13kx^2}{10\sigma^2} + \frac{11k^2}{3\sigma^2} \right) \),

is the effective potential, while \(\hat{H}_e\) is the effective Hamiltonian operator. In the above Schrödinger-like equation, \(\sigma = a^{11}\) acts as internal time-parameter of the theory. Note that in the curvature squared theories handled earlier, the internal parameter was simply the proper volume \(a^3\).

3.3. Hermiticity of the effective Hamiltonian, Probability interpretation and extremization of the effective potential

In the effective Hamiltonian operator \(\hat{H}_e\) (35), the effective potential \(\hat{V}_e(x, \sigma)\) being real is clearly hermitian. Now taking, \(\tilde{T} = \frac{i}{x(x^2 + 4k\sigma \pi)^2} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{n}{x} \frac{\partial \psi}{\partial x} \right)\), we compute
\[ \int (\hat{T})^* \psi^* dx = \int \left[ \frac{1}{x(x^2 + 4k\sigma \hat{\pi}^2)} \left( \frac{\partial^2 \psi^*}{\partial x^2} + \frac{n}{x} \frac{\partial \psi^*}{\partial x} \right) \right] \psi dx \]

\[ = - \int \left[ \frac{\partial \psi}{\partial x} \right]^* \left[ \frac{1}{x(x^2 + 4k\sigma \hat{\pi}^2)} \left( \frac{\partial \psi^*}{\partial x} - \frac{5x^2 + 4k\sigma \hat{\pi}^2}{x^2(x^2 + 4k\sigma \hat{\pi}^2)} \psi - \frac{n}{x^2} \frac{\psi}{x^2 + 4k\sigma \hat{\pi}^2} \right) \right] dx \]

where we have performed integration by parts and used fall-off condition. Repeating the same, and then rearranging, we finally arrive at

\[ \int (\hat{T})^* \psi^* dx = \int \psi^* \left[ \frac{1}{x(x^2 + 4k\sigma \hat{\pi}^2)} \left( \frac{\partial^2 \psi}{\partial x^2} - \frac{10x^2 + 8k\sigma \hat{\pi}^2}{x^2(x^2 + 4k\sigma \hat{\pi}^2)^2} + \frac{n}{x^2} \frac{\psi}{x^2 + 4k\sigma \hat{\pi}^2} \right) \right] \psi dx \]

\[ + \left( \frac{3(5x^2 + 4k\sigma \hat{\pi}^2)^2}{2x^2(x^2 + 4k\sigma \hat{\pi}^2)^4} - \frac{15x^2 - 4k\sigma \hat{\pi}^2}{2x^2(x^2 + 4k\sigma \hat{\pi}^2)^3} \right) \psi dx \]

\[ = \int \psi^* \left[ \frac{1}{x(x^2 + 4k\sigma \hat{\pi}^2)} \left( \frac{\partial^2 \psi}{\partial x^2} - \frac{(10 + n)x^2 + (2 + n)k\sigma \hat{\pi}^2}{x^2(x^2 + 4k\sigma \hat{\pi}^2)^3} \right) \right] \psi dx. \]

One can now clearly note that only under the condition \( n = -5 \) and \( k = 0 \),

\[ \int (\hat{T})^* \psi^* dx = \int \left[ \frac{1}{x^2} \left( \frac{\partial^2}{\partial x^2} - \frac{5}{x} \frac{\partial}{\partial x} \right) \psi \right]^* \psi dx = \int \psi^* \left[ \frac{1}{x^2} \left( \frac{\partial^2}{\partial x^2} - \frac{5}{x} \frac{\partial}{\partial x} \right) \psi \right] \psi dx = \int \psi^* \hat{T} \psi dx. \]

Thus, hermiticity of the effective Hamiltonian demands that apart from the choice \( n = -5 \), the universe must have to be flat. Now, the continuity equation reads as,

\[ \frac{dP}{d\sigma} + \nabla \cdot \mathbf{J} = 0, \]

where, \( P = \Psi^* \Psi \) is the probability density and \( \mathbf{J} \) is the current density. In view of the quantum equation (35), one can write,

\[ \Psi^* \frac{\partial \Psi}{\partial \sigma} = \frac{i\hbar \Psi^*}{198 \gamma x(x^2 + 4k\sigma \hat{\pi}^2)^2} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) \Psi - \frac{i}{\hbar} V_E \Psi \Psi^*. \]

\[ \Psi \frac{\partial \Psi^*}{\partial \sigma} = - \frac{i\hbar}{198 \gamma x(x^2 + 4k\sigma \hat{\pi}^2)^2} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) \Psi^* + \frac{i}{\hbar} V_E \Psi \Psi^*. \]

Therefore,

\[ \frac{dP}{d\sigma} = \Psi^* \frac{d\Psi}{d\sigma} + \Psi \frac{d\Psi^*}{d\sigma} = \frac{i\hbar (\Psi^* \Psi_{,x} - \Psi \Psi_{,x})}{198 \gamma x(x^2 + 4k\sigma \hat{\pi}^2)^2} + \frac{i\hbar n (\Psi^* \Psi - \Psi \Psi^*)}{198 \gamma x(x^2 + 4k\sigma \hat{\pi}^2)^2} \]

\[ = - \frac{\partial}{\partial x} \left[ \frac{i\hbar (\Psi^* \Psi_{,x} - \Psi \Psi_{,x})}{198 \gamma x(x^2 + 4k\sigma \hat{\pi}^2)^2} \right] - \frac{i\hbar n (\Psi^* \Psi - \Psi \Psi^*)}{198 \gamma x(x^2 + 4k\sigma \hat{\pi}^2)^2} \left[ (n + 5)k^2 + (n + 1)4k\sigma \hat{\pi}^2 \right]. \]

(42)

\[ \frac{dJ_x}{d\sigma} = \frac{dJ_x}{d\sigma} \]

provided \( k = 0 \) and \( n = -5 \), so that the second term on the right hand side of equation (42) vanishes. Thus as expected, continuity equation also holds only in the flat space, for \( n = -5 \).

In the above, \( \mathbf{J} = (J_x, 0, 0) \), and hence standard quantum mechanical probability interpretation
holds in a straightforward manner. Note that in the flat space, the kinetic part does not show up explicit dependence on the internal time parameter $\sigma$, although the effective potential does. This makes the quantum equation easily tractable. Although, operator ordering ambiguity is somewhat fixed ($n = -5$) from physical consideration, viz. for hermiticity of the effective Hamiltonian and probability interpretation to hold, nevertheless, the fact that it is different ($n = -1$) from the one obtained for curvature squared theories of gravity, even in the presence of non-minimal coupling and Gauss–Bonnet-dilatonic coupling, is a clear contradiction. It is also important to mention that, higher-order theories explored earlier do not need the space to be flat even. It is oversimplification to assume a flat space a-priori, which is ensured only due to inflation. The fact that probability interpretation holds in the flat space ($k = 0$), in the present situation, may only be interpreted in the following manner. Either, Gauss–Bonnet squared term should not be included in the action, or, probabilistic interpretation holds only after the inflationary regime. Since such a term cannot be incorporated all of a sudden in the action post inflation, clearly favours the first option, viz. higher powers of Gauss–Bonnet term appears to be unrealistic and should be disregarded.

Additionally, we can extremized the effective potential, to unveil some more information. The effective potential in the flat space reads as

$$V_e(x, \sigma) = \frac{2}{11} \left[ \frac{3\alpha x^2}{28\sigma^2} + \frac{135\gamma x^7}{28\sigma^{11}} \right].$$

The extremum of the effective potential with respect to $x$ then gives ($\sigma$ being the effective built-in time parameter of the theory),

$$\frac{\partial V_e}{\partial x} = \frac{2}{11} \left[ \frac{3\alpha x^2}{28\sigma^2} + \frac{135\gamma x^6}{4\sigma^{11}} \right] = 0 \Rightarrow \frac{3\alpha x^2}{28\sigma^2} + \frac{135\gamma x^6}{4\sigma^{11}} = 0.$$  (44)

Thus, de-Sitter solution is obtained fixing the Lapse function $N = 1$, i.e. using the relation $\dot{\tilde{z}} = x$ as,

$$z = z_0 \exp \left[ \left( \frac{2\alpha}{45\gamma_1} \right)^{\frac{1}{4}} t \right] \Rightarrow a = a_0 e^{\lambda_1 t}, \text{ where } \lambda_1 = \frac{1}{2} \left( \frac{2\alpha}{45\gamma_1} \right)^{\frac{1}{4}} = \frac{1}{2} \left( \frac{M_P^2}{45\gamma_1} \right)^{\frac{1}{4}},$$

provided $\gamma \to -\gamma_1$. The required choice that the coupling parameter has to be negative, of course results in a clear contradiction with both the classical solution (8) and the slow-roll inflationary solution (14), which hold for positive coupling parameter $\gamma$. Thus we encounter the second pathology associated with the theory under consideration.

3.4. Semiclassical approximation

Even though we have already made a harsh comment regarding Gauss–Bonnet term, let us compute a little beyond, to explore its role in semiclassical regime. We can only perform semiclassical approximation on-shell in view of the classical solution (8) or the Inflationary solution (14). For this purpose, we express the wave-equation (34) in the following form ($k = 0$)

$$- \frac{\hbar^2 \tilde{z}^2}{36\gamma x^3} \left( \frac{\partial^2}{\partial \tilde{z}^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) \Psi - i\hbar \frac{\partial \Psi}{\partial \tilde{z}} + \mathcal{V}(x, z) \Psi = 0.$$  (46)

where, $\mathcal{V}(x, z) = \left[ \frac{3\alpha x^2}{2\sqrt{\tilde{z}}} + \frac{135\gamma x^7}{28\tilde{z}^{11}} \right]$. 

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The above equation may be treated as time independent Schrödinger equation with two variables \(x\) and \(z\). Hence as usual, let us seek the solution of the wave-function \((46)\) as,
\[
\Psi = \Psi_0 e^{\frac{i}{\hbar} \hat{S}(x,z)}
\]
and expand \(S\) in power series of \(\hbar\) as,
\[
S = S_0(x,z) + h S_1(x,z) + h^2 S_2(x,z) + ....
\]
One can then find,
\[
\Psi_{,x} = \frac{i}{\hbar}[S_{0,x} + h S_{1,x} + h^2 S_{2,x} + \mathcal{O}(h)]\Psi; \quad \Psi_{,zz} = \frac{i}{\hbar}[S_{0,zz} + h S_{1,zz} + h^2 S_{2,zz} + \mathcal{O}(h)]\Psi
\]
\[
- \frac{1}{\hbar^2}[\frac{\hbar^2}{2} S_{0,xx} + h^2 S_{1,xx} + h^4 S_{2,xx} + 2h S_{0,xx} S_{1,xx} + 2h^2 S_{0,xx} S_{2,xx} + 2h^3 S_{1,xx} S_{2,xx} + \mathcal{O}(h)]\Psi;
\]
\[
\Psi_z = \frac{i}{\hbar}[S_{0,z} + h S_{1,z} + h^2 S_{2,z} + \mathcal{O}(h)]\Psi,
\]

etc, where ‘comma’ in the suffix stands for derivative. Now, inserting \(\Psi, \Psi_{,x}, \Psi_{,zz}, \Psi_z\) etc in view of \((47)\), and \((49)\) in equation \((46)\) and equating the coefficients of different powers of \(\hbar\) to zero, the following set of equations (upto second order) are obtained,
\[
\frac{z^3}{36\gamma x^5} S_{0,xx}^2 + S_{0,z} + \nabla = 0 \tag{50a}
\]
\[
i \left[ \frac{z^5}{36\gamma x^3} S_{0,xx} + \frac{nz^3}{36\gamma x^3} S_{0,zz} \right] - S_{1,z} - \frac{z^2 S_{0,xx} S_{1,zz}}{18\gamma x^5} = 0 \tag{50b}
\]
\[
i \left[ \frac{z^5}{36\gamma x^3} S_{1,xx} + \frac{nz^3}{36\gamma x^3} S_{1,zz} \right] - S_{2,z} - \frac{z^2 S_{0,xx} S_{2,zz}}{18\gamma x^5} - \frac{z^2 S_{1,xx}^2}{36\gamma x^3} = 0 \tag{50c}
\]

which are to be solved successively to find \(S_0(x,z), S_1(x,z)\) and \(S_2(x,z)\) and so on. Now, Identifying \(S_{0,x}\) with \(p_x\) and \(S_{0,z}\) with \(p_z\), the Hamilton constraint equation \((26)\) is retrieved.

Further, using the definitions of momenta \((27)\) and equation \((28)\) the \(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\) component of Einstein’s equation \((6)\) (for \(N = 1, k = 0\)) is also retrieved. So everything is consistent so far.

Now, in view of the classical solutions \((8)\), one can compute the momenta \((28)\) and their integrals as
\[
p_x = 18\gamma (2\lambda)^{\frac{2}{7}} \sqrt{x}; \quad p_z = -6\alpha \lambda \sqrt{z} - \frac{(2\lambda)^{\frac{3}{7}} \times 81\gamma \sqrt{z}}{7}
\]
\[
\int p_x dx = (2\lambda)^{\frac{7}{7}} \times 12\gamma z^{\frac{2}{7}}; \quad \int p_z dz = -4\alpha \lambda z^{\frac{5}{7}} - \frac{(2\lambda)^{\frac{7}{7}} \times 54 \gamma z^{\frac{5}{7}}}{7}.
\]

Thus \(S_0\), which when expressed in terms of the integrals of momenta yields
\[
S_0 = \int p_x dx + \int p_z dz = -4\alpha \lambda z^{\frac{5}{7}} + \frac{3840}{7} \gamma \lambda z^{\frac{5}{7}}.
\]

One can also compute the zeroth order on-shell action \((16)\) in view of the classical solution \((8)\) as
\[ A_0 = \int \left[ \frac{3\alpha z^2}{2\sqrt{\bar{z}}} + 24^2\gamma \left( \frac{z^4\dot{z}^2}{64z^2} - \frac{15z^8}{1792z^2} \right) \right] \, dt \]
\[ = \int \left[ \frac{3\alpha (2\lambda z^2)}{2\sqrt{\bar{z}}} + 9\gamma \left( \frac{(2\lambda)^8 z^6}{z^8} - \frac{15(2\lambda)^8 z^8}{28z^8} \right) \right] \, \frac{dz}{2\lambda z} = -2\alpha \lambda z^3 + \frac{2496}{7} - \gamma \lambda^3 \overline{z}^3. \] (53)

Substituting \( \gamma = \frac{\alpha}{63\bar{z}} \) in view of relation (8) both in (52) and (53), one finally finds \( A_0 = S_0 = \frac{1792}{28\sqrt{\bar{z}}} \), and the classical on-shell action matches exactly with the Hamilton--Jacobi function. Alas! Equation (50a) is not satisfied for the form of \( S_0 \) so obtained. In fact, one can easily check that the Hamilton--Jacobi equation (50a) is satisfied provided, either \( \alpha \) or \( \lambda \) be set to vanish. Setting \( \alpha \) to zero, implies that one cannot incorporate Einstein–Hilbert sector with Gauss–Bonnet squared term, while setting \( \lambda \) to zero implies that the universe is devoid of expansion. Both are clearly unphysical. Thus, we encounter yet another contradiction, the third pathology associated with the theory under consideration, which is clearly independent on the choice of the operator ordering parameter \( n \). It is therefore useless to compute semiclassical wavefunction any further.

4. Adding a cosmological constant

In an attempt to find the phase-space structure corresponding to action (16), we have faced at least two serious pathologies, that had never been encountered with the higher-order gravitational actions handled so far. Firstly, the classical as well as the slow roll inflationary solutions require a positive coupling parameter \( \gamma \), while the extremum of the effective potential requires negative. One can in no way expect that the classical solution should match the extremum of the potential. This is because, the effective potential does not exhibit all the information (being devoid of kinetic terms) associated in classical field equations. A viable quantum theory exhibits classical solution only under appropriate semiclassical approximation. But appearance of the coupling parameter with opposite signs really matters. Next, and even more notorious problem that we encounter is the fact that the Hamilton–Jacobi function \( S_0 \) does not satisfy the Hamilton–Jacobi equation (50a). We put a big question mark on this issue, since we are unable to explore the reason behind such uncanny behaviour. In order to overcome such pathologies, let us now modify the action (16) taking cosmological constant \( \Lambda \) into account, which is essentially the vacuum energy density of all possible fields that exist in the very early universe. The modified action (16) now reads as,

\[ A = \int \left[ \frac{R - 2\Lambda}{16\pi G} + \beta G^2 \right] \sqrt{-g} \, d^4x, \] (54)

which in the R-W metric under consideration takes the following form

\[ A = \int \left[ \frac{6\alpha}{8N\sqrt{\bar{z}}} + \frac{N\dot{z}^3}{2N\sqrt{\bar{z}}} - \frac{\Lambda}{3}N\bar{z}^3 \right] + 24^2\gamma \left( \frac{z^4}{64N^7z^2} + \frac{k^2}{8N^5z^2} + \frac{k^2}{4N^3z^4} \right) \bar{z}^2 \]
\[ - \left( \frac{N\dot{z}^5}{32N^9z^2} + \frac{k\dot{N}\dot{z}^3}{4N^6z^2} + \frac{k^2N^3\dot{z}^2}{2N^4z^2} \right) \bar{z} - \frac{15z^8}{1792N^2z^2} - \frac{13k^6}{160N^2z^2} - \frac{11k^6z^4}{48N^2z^2} \right] \] (55)

As before, in view of the new variable, \( \bar{z} = N\dot{x} \), i.e. \( \bar{z} = N\dot{x} + \dot{N}\dot{x} \), one can express the point Lagrangian in the form,
\[ L = 6\alpha N \left( -\frac{x^2}{4\sqrt{z}} + k\sqrt{z} - \frac{\Lambda}{3} z^3 \right) \]

\[ 288\gamma \left[ \left( \frac{x^4}{32Nz^4} + \frac{kx^2}{4Nz^2} + \frac{k^2}{2Nz^2} \right) x^2 - \frac{15N\alpha^8}{896z^7} - \frac{13kN\alpha^6}{80z^5} - \frac{11k^2N\alpha^4}{24z^3} \right], \tag{56} \]

after removing the total derivative terms from the action (55), following integration by parts. In the flat space \( k = 0 \), field equations read as,

\[ 2\alpha \left( \frac{\tilde{\varepsilon}}{\varepsilon} - \frac{\tilde{\varepsilon}^2}{4\varepsilon^2} - \Lambda \right) + 12\gamma \left[ \frac{4\varepsilon^2}{\varepsilon^5} + \frac{8\varepsilon^4}{\varepsilon^8} - \frac{9\varepsilon^4z}{\varepsilon^8} + \frac{6\varepsilon^2z^3}{\varepsilon^6} - \frac{135\varepsilon^4z^4}{4\varepsilon^8} + \frac{159\varepsilon^2z^6}{4\varepsilon^8} - \frac{195\varepsilon^4z^8}{16\varepsilon^8} \right] = 0. \tag{57} \]

\[ 2\alpha \left( \frac{3\varepsilon^2}{4\varepsilon^2} - \Lambda \right) + 18\gamma \left[ \frac{\varepsilon^6}{\varepsilon^5} + \frac{3\varepsilon^4z^2}{2\varepsilon^6} - \frac{9\varepsilon^2z^6}{2\varepsilon^7} + \frac{15\varepsilon^4z^8}{8\varepsilon^8} \right] = 0. \tag{58} \]

The above field equations admit the following exponential solution in vacuum

\[ \varepsilon = z_0 \epsilon^{2\lambda t}, \quad \text{under the condition} \quad \Lambda = 3\lambda^2 - 288\frac{2\lambda^8}{\alpha}. \tag{59} \]

To keep \( \Lambda > 0 \), the condition required is \( \alpha > 96\gamma\lambda^6 \). One can check that the solution reduces to (8) setting \( \Lambda = 0 \). One can further, perform slow-roll approximation, which finally ends up with

\[ 96\gamma H^3 - \alpha H^2 - \frac{4}{3}\alpha \Lambda = 0. \tag{60} \]

The above algebraic equation for the Hubble parameter \( H \), may be solved to obtain eight real roots, and as a result the scale factor admits exponential expansion. We do not present the solutions to avoid unnecessary complications. Nevertheless, it is important to mention that, instead of slow variation of the Hubble parameter, it remains absolutely constant here too as before. Thus, as mentioned earlier, slow-roll with higher powers of Gauss–Bonnet term is not viable. We can now perform the constraint analysis as before, to end up with the following Hamiltonian,

\[ H = N \left[ xp_z + \frac{z^2p_z^2}{36\gamma(x^2 + 4kz)^2} + \alpha \left( \frac{3x^2}{2\sqrt{z}} - 6k\sqrt{z} + 2\Lambda z^3 \right) \right. \]

\[ + 36\gamma x^4 \left( \frac{15x^4}{112z^2} + \frac{13kx^2}{10z^4} + \frac{11k^2}{3z^6} \right) \right] = NH. \tag{61} \]

The action (55) may again be expressed in canonical ADM form \((k = 0)\) as,

\[ A = \int \left( ip_z + \dot{p}_z - NH \right) \text{d}^3x = \int \left( \dot{h}_{ij}p^{ij} + \dot{K}_i \Pi^i - NH \right) \text{d}^3x. \tag{62} \]

One can follow the same procedure towards canonical quantization of the Hamiltonian (61). It does not make any considerable change in the quantum equation (35), rather it just modifies the effective potential appearing in equation (46) following an additive term involving cosmological constant \( \Lambda \) to,

\[ V_e(x, \sigma) = \left[ \frac{3x^2}{4\sigma^2} - \frac{4k}{3\sigma^3} + 4A \right] + 72\gamma x^3 \left( \frac{15x^4}{112 \sigma^2} + \frac{13kx^2}{10 \sigma^4} + \frac{11k^2}{3 \sigma^6} \right). \tag{63} \]
As a result the probability interpretation remains unaltered, i.e. it holds in flat space \( k = 0 \), for operator ordering index \( n = -5 \). Now, extremization of the effective potential \((k = 0)\) leads to

\[
z = z_0 e^{2\lambda t}, \quad \text{or equivalently} \quad a = a_0 e^{\lambda t}, \quad \text{for} \quad \Lambda = 3\lambda^2 + 4320\frac{\gamma}{\alpha} \lambda^8. \tag{64}
\]

Of course, the classical solution (59) differs from (64) to the extent of the condition required to impose on the cosmological constant. Nonetheless, classical solution is not expected to match at the extremum of the potential and therefore has nothing to do with consistency, as already mentioned earlier. Whatsoever, the important development is, it does not require a reverse sign of \( \gamma \) unlike the situation encountered in solution (45) without cosmological constant. One can again check that the above condition (64) reduces to the previous one (45) setting \( \Lambda = 0 \). Thus the pathology is removed, and cosmological constant saves the soul.

In the context of semiclassical approximation, we express the quantized equation in the form (46) just with the modified effective potential,

\[
\mathcal{V}_1(x, z) = \left[ \frac{3\alpha x}{2\sqrt{z}} + \frac{135\gamma x^2}{28z^2} + 2\alpha \Lambda z^2 \right], \tag{65}
\]

and proceed as before. The momenta remains unaltered and one can thus calculate the Hamilton–Jacobi function as,

\[
S_0 = -4\alpha \lambda z^2 + \frac{3840}{7}\gamma \lambda^2 z^2, \tag{66}
\]

which is the same as (52). However, zeroth order classical on-shell action is calculated from (55) in the following manner.

\[
A_0 = \int \left[ -6\alpha \lambda^2 z^2 - 6\alpha \lambda^2 z^2 + 576\gamma \lambda^8 z^2 + 9\gamma (16^2 - \frac{15 \times 256}{28}) \lambda z^2 \right] dt = \int \left[ -12\alpha \lambda^2 z^2 + \frac{11520}{7} \gamma \lambda^8 z^2 \right] dz = \int \left[ -6\alpha \lambda \sqrt{z} + \frac{5760}{7} \gamma \lambda^7 \sqrt{z} \right] dz = -4\alpha \lambda z^2 + \frac{3840}{7}\gamma \lambda^2 z^2. \tag{67}
\]

Since, Hamilton–Jacobi equation matches zeroth order on-shell action, so this part is well-behaved. One can check that the Hamilton–Jacobi function \( S_0 \) now satisfies the Hamilton–Jacobi equation (50a) under the condition,

\[
\frac{\alpha^2}{128\gamma \lambda^5} + \frac{63 \times 648}{49} \gamma \lambda^7 - \frac{15}{7} \alpha \lambda = 0, \tag{68}
\]

which simply restricts \( \lambda \) by the other two parameters (\( \alpha \) and \( \gamma \)) of the theory. In the process, the two notorious pathologies are alleviated simply in the presence of cosmological constant. The semiclassical wavefunction upto first order approximation now reads as,

\[
\Psi = \Psi_0 e^{\frac{i}{\hbar} \left( -4\alpha \lambda + \frac{3840}{7}\gamma \lambda^2 \right) z^2}. \tag{69}
\]

In order to compute the wavefunction upto first order of approximation, we note that in view of classical solution (8), the independent variables \( x \) and \( z \) are related as, \( x = 2\lambda z \). Thus one can compute \( S_{0,x}, S_{0,xx} \) and also express \( S_{1,1} \) in terms of \( S_{1,z} \) to finally obtain,
\[ S_{1,z} = -i \left( \frac{9}{4} \right) \left( \frac{7 \alpha - 960 \gamma \lambda^6}{7 \alpha - 3648 \gamma \lambda^6} \right) \frac{1}{z} \]  

which may be integrated to obtain

\[ S_1 = -i \left( \frac{9}{4} \right) \left( \frac{7 \alpha - 960 \gamma \lambda^6}{7 \alpha - 3648 \gamma \lambda^6} \right) \ln z, \]  

apart from a constant of integration. Hence, the wave function up to first order of approximation finally reads as,

\[ \Psi = \Psi_0 \times \left[ z^{\left(\frac{\frac{7 \alpha - 960 \gamma \lambda^6}{7 \alpha - 3648 \gamma \lambda^6}}{2}\right)} \right] e^{i \hbar \left( -4 \alpha \lambda + \frac{2 \alpha \gamma}{3} \lambda^3 \right) z^{\frac{3}{2}}}. \]  

One can clearly observe that the first order approximation, just modifies the pre-factor keeping the exponent unaltered. In this manner, it is possible to find the semiclassical wave function for even higher order approximations, which exhibits oscillatory behaviour around classical de-Sitter solution. The reason is: since comparison with the classical Hamiltonian constraint equation reveals \( p_x = \frac{\partial S}{\partial x} \), and \( p_z = \frac{\partial S}{\partial z} \), so the wavefunction shows a strong correlation between coordinates and momenta. Now using the relation between velocities and momenta and the fact that \( S_0 \) obeys Hamilton–Jacobi equation, it is apparent that the above relations define a set of trajectories in the \( x-z \) plane, which are solutions to the classical field equations. Thus the semiclassical wave function (72) is strongly peaked around classical inflationary solutions (59).

5. Conclusion

Modified theory of gravity had originally been envisaged as an alternative to quintessence, due to the absence of scalar field in the present universe. Further, while transient crossing of phantom divide line requires more than one scalar, modified theories can exhibit such phenomena. Therefore, Gauss–Bonnet term has also been chosen to be a candidate in this regard. However, such a term being topologically invariant, requires dilatonic coupling, which again invites a scalar. In the absence of a scalar field in the late stage of cosmic evolution, higher powers of Gauss–Bonnet term serves the purpose as well. It is therefore required to study the behaviour of a theory that modifies the Einstein–Hilbert action in the presence of Gauss–Bonnet squared term, at the first place. In the present manuscript, we study early universe in view of such an action. We have formulated the phase-space structure of such an action and followed standard canonical quantization scheme. In the process, the action has been found to suffer from several pathologies, enlisted below.

1. The classical equations of motion do not admit a power-law solution in the radiation era, which is necessary for structure formation and the formation of CMB.
2. Hermiticity of the effective Hamiltonian operator and the conservation of quantum probability require a flat space \( (k = 0) \), together with the ‘operator ordering index’ \( n = -5 \) for the theory, whereas other modifications of gravity considered earlier, require \( n = -1 \), and do not impose any constraint on the curvature parameter \( k \).
3. The extremum of the effective potential (43) (of the Schrodinger-alike equation (35)), corresponds to a de-Sitter inflationary solution, but only if \( \frac{\alpha}{\gamma} < 0 \), which requires opposite sign of \( \gamma \) compared to the classical de Sitter solution.
4. The form of the Hamilton–Jacobi function (52) obtained under the semi-classical approximation does not satisfy the Hamilton–Jacobi equation (50a).
Out of these four, the last two are alleviated by invoking a cosmological constant in addition. This proves the very importance of considering a cosmological constant, which is essentially the sum of zero point energies of all quantum fields, existing in the very early universe. On the other way round, it might be interpreted as: ‘the creation of the universe was initiated with non-trivial vacuum’. It is also important to mention that a vacuum energy density in the form of a cosmological constant leads us back to the domain of the naturalness problem. Also note that, with the seeds of perturbation developed in the Inflationary regime, a Friedmann-like radiation dominated era \( a(t) \propto \sqrt{t} \) exactly can formulate the structures of the universe, we presently observe. The CMBR is also an artefact of Friedmann-like radiation dominated era. The naturalness problem along with the first pathology, viz., the non-appearance of a power law solution of the scale factor in the radiation dominated era, may be circumvented if the action is modified by the inclusion of higher powers of curvature scalars or some form of matter fields in the form of a scalar (say). Standard slow-roll formulation in the inflationary regime is also possible under the inclusion such modification. This we pose in future. However, we do not find an option to alleviate the second pathology, viz., the conservation of probability in the presence of higher powers of Gauss–Bonnet term.

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