LONG MONOTONE PATHS ON SIMPLE 4-POLYTOPES

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Abstract. The Monotone Upper Bound Problem (Klee, 1965) asks if the number \( M(d, n) \) of vertices in a monotone path along edges of a \( d \)-dimensional polytope with \( n \) facets can be as large as conceivably possible: Is \( M(d, n) = M_{\text{ubt}}(d, n) \), the maximal number of vertices that a \( d \)-polytope with \( n \) facets can have according to the Upper Bound Theorem?

We show that in dimension \( d = 4 \), the answer is “yes”, despite the fact that it is “no” if we restrict ourselves to the dual-to-cyclic polytopes. For each \( n \geq 5 \), we exhibit a realization of a polar-to-neighborly 4-dimensional polytope with \( n \) facets and a Hamilton path through its vertices that is monotone with respect to a linear objective function.

This constrasts an earlier result, by which no polar-to-neighborly 6-dimensional polytope with 9 facets admits a monotone Hamilton path.

1. Introduction

While investigating the complexity of the simplex algorithm for linear programming, Klee [4] in 1965 posed the Monotone Upper Bound Problem: For \( n > d \geq 2 \), he asked for the maximal number \( M(d, n) \) of vertices of a \( d \)-dimensional polytope with \( n \) facets that can lie on a monotone path, i.e., on a path along edges that is strictly increasing with respect to a linear objective function.

McMullen’s 1971 Upper Bound Theorem [5] (claimed by Motzkin [6] in 1957) states that the maximal number \( M_{\text{ubt}}(d, n) \) of vertices that any \( d \)-dimensional polytope with \( n \) facets can have is achieved by the polars \( C_d(n)^\Delta \) of cyclic \( d \)-polytopes with \( n \) facets.

The Upper Bound Theorem yields, for all \( n > d \geq 2 \), the inequality

\[
M(d, n) \leq M_{\text{ubt}}(d, n),
\]

but from this it is not clear whether equality always holds, that is, if for all \( n > d \geq 2 \) one can construct a simple polar-to-neighborly \( d \)-polytope with \( n \) facets that admits a monotone Hamilton path with respect to a linear objective function. Equality in (1) is known in the cases \( d \leq 3 \) and \( n \leq d + 2 \).

Date: February 16, 2004.

2000 Mathematics Subject Classification. 52B12; 52B05.

The author was financed by the DFG Graduiertenkolleg Combinatorics, Geometry, and Computation (GRK 588-2), the GIF project Combinatorics of Polytopes in Euclidean Spaces (I-624-35.6/1999), and post-doctoral fellowships from MSRI and Institut de Matemàtica de la Universitat de Barcelona.
However, in [7] we show that in fact $M(6,9) < M_{\text{ubt}}(6,9)$: there exists no realization of the (combinatorially unique) polar-to-neighborly 6-polytope $C_6(9)^\Delta$ with 9 facets and 30 vertices that admits such a monotone Hamilton path.

For the parameters $d = 4$, $n = 8$, one can show using the same (basically combinatorial) methods that there is also no realization of $C_4(8)^\Delta$ with a monotone Hamilton path — but as we will show here, there are other dual-to-neighborly but not dual-to-cyclic 4-dimensional polytopes with 8 facets that admit a realization with a monotone path through all vertices.

In fact, in this paper we prove considerably more: we provide a geometric construction that shows that the inequality (1) is tight in dimension $d = 4$ for all $n \geq 5$.

**Main Theorem.** For each integer $m \geq 0$, there exists a simple polar-to-neighborly 4-dimensional polytope $Q_m$ with $n = m + 5$ facets and a linear objective function $f : \mathbb{R}^4 \to \mathbb{R}$, such that the orientation induced by $f$ on the 1-skeleton of $Q_m$ admits a monotone Hamilton path. Therefore,

$$M(4, n) = M_{\text{ubt}}(4, n) = \frac{1}{2} n(n - 3).$$

In other words, the maximal number $M(4, n)$ of vertices on a strictly monotone path in the graph of a 4-dimensional polytope with $n$ facets equals the maximal number of vertices that such a polytope can have according to the Upper Bound Theorem.

An interesting feature used in our proof is that for $m \geq 3$, the (polar-to-)neighborly polytopes $Q_m$ are not polar to cyclic ones. In fact, exhaustive enumeration shows that already the graph of $C_4(8)^\Delta$ does not satisfy a combinatorial condition necessary for the existence of a monotone path, namely, it does not admit a Hamilton AOF Holt-Klee orientation [2]. This is also true for the graphs of the polytopes $C_4(n)^\Delta$ for $8 \leq n \leq 12$; we conjecture that the graphs of $C_4(n)^\Delta$ for all $n \geq 8$ admit no Hamilton AOF Holt-Klee orientation.

The structure of the paper is as follows: We first give an explicit description, reminiscent of Gale’s Evenness Criterion for polar-to-cyclic polytopes, of the combinatorial structure of a family $\{Q^d_m : d \geq 4 \text{ even}, m \geq 0\}$ of simple (polar-to-)neighborly $d$-dimensional polytopes with $m + d + 1$ facets (Sections 2 and 3). For $d = 4$, we then use this description to specify a Hamilton path $\pi_m$ on each $Q_m := Q_4^m$ (Section 4). In Section 5, we start with a monotone path $\pi_0$ on a certain realization of the 4-simplex $Q_0$, and for $m \geq 0$ inductively realize the polytope $Q_{m+1}$ in such a way that the path $\pi_{m+1}$ is strictly monotone with respect to a suitable objective function (Theorem 2.5). We proceed in three steps: First, we position $Q_m$ in a suitable way with respect to the standard coordinates on $\mathbb{R}^4$ (Section 5.4). We then find a “cutting plane” $H_{m+1}$ such that the polytope $Q_m \cap H_{m+1}^{\geq 0}$ has the right combinatorial type (Section 5.5). Finally, we complete the construction in Section 5.6 by applying a projective transformation $\psi$ to $\mathbb{R}^4$ such that the path $\psi(\pi_m)$ on $Q_{m+1} := \psi(Q_m \cap H_{m+1}^{\geq 0})$ is strictly monotone with respect to the objective function $f : \mathbb{R}^4 \to \mathbb{R}$, $x \mapsto x_4$.  

2. Main results

**Theorem 2.1** (modified Gale’s Evenness Criterion). For each \( m \geq 0 \) and even \( d \geq 4 \), the following sets correspond to the vertices of a combinatorial type \( \tilde{Q}_m^d \) of a simple \( d \)-dimensional polar-to-neighborly polytope with \( n = m + d + 1 \) facets.

- **Type 1.** The union of one “triplet with a hole” and \( d/2 - 1 \) pairs of indices
  \[
  \{j_1, j_1 + 2\} \cup \{j_2, j_2 + 1\} \cup \cdots \cup \{j_{d/2}, j_{d/2} + 1\},
  \]
  where \( 1 \leq j_1 < n - d + 1, j_1 + 3 \leq j_2, j_k + 2 \leq j_{k+1} \) for \( 2 \leq k \leq d/2 - 1 \), and \( j_{d/2} < n \).

- **Type 2a.** The union of one triplet, the singleton \( \{n\} \), and \( d/2 - 2 \) pairs of indices
  \[
  \{j_1, j_1 + 1, j_1 + 2\} \cup \{j_2, j_2 + 1\} \cup \cdots \cup \{j_{d/2-1}, j_{d/2-1} + 1\} \cup \{n\},
  \]
  where \( 1 \leq j_1 < n - d + 1, j_1 + 3 \leq j_2, j_k + 2 \leq j_{k+1} \) for \( 2 \leq k \leq d/2 - 2 \), and \( j_{d/2-1} < n - 1 \).

- **Type 2b.** The union of \( d/2 \) pairs of indices
  \[
  \{1, 2\} \cup \{j_1, j_1 + 1\} \cup \cdots \cup \{j_{d/2-1}, j_{d/2-1} + 1\},
  \]
  where \( 3 \leq j_1, j_k + 2 \leq j_{k+1} \) for \( 2 \leq k \leq d/2 - 2 \), and \( j_{d/2-1} < n \).

**Figure 1:** The vertex-facet incidences of the polytopes \( \tilde{Q}_m^d \) are obtained from these patterns by fixing the dark boxes, and sliding the lighter boxes between 1 and \( n \) without overlap. For Type 1, the box \( \{i, i + 2\} \) must be regarded as one rigid unit.

**Remark 2.2.** If we accept for the moment the existence of the polytopes \( \tilde{Q}_m^d \), it is easy to verify that they are polar-to-neighborly by counting the number of vertices using Figure 1:

\[
\begin{align*}
    f_0(\tilde{Q}_{n-d-1}^d) &= \left( \frac{n - 2 - (d/2 - 1)}{d/2} \right) + \left( \frac{n - 2 - (d/2 - 2) - 1}{d/2 - 1} \right) + \left( \frac{n - 2 - (d/2 - 1)}{d/2 - 1} \right) \\
    &= \left( \frac{n - 1 - d/2}{d/2} \right) + 2 \left( \frac{n - 1 - d/2}{d/2 - 1} \right) \\
    &= \left( \frac{n - d/2}{d/2} \right) + \left( \frac{n - 1 - d/2}{d/2 - 1} \right),
\end{align*}
\]
which is the number of vertices of a simple polar-to-neighborly $d$-polytope with $n = m + d + 1$ facets, since $d$ is assumed even. By [9, Chapter 8], any polytope with that many vertices is polar-to-neighborly. □

From now on, we will always write $\tilde{Q}_m := \tilde{Q}_4^m$.

**Figure 2:** Graph of the 4-polytope $\tilde{Q}_4$ with $n = 9$ facets. Vertices of type 1, 2a, and 2b are drawn in gray, white, and black, respectively. Each vertex is labelled with the facets it is incident to.

**Proposition 2.3.** Each polytope $\tilde{Q}_m$ admits a Hamilton path $\tilde{\pi}_m$ in its graph that induces an AOF-orientation (cf. Figure 3 and Definition 4.1 below).

**Figure 3:** Left: Graph of $\tilde{Q}_4^1$. The partition of the vertices into the tips $T_0^1$, $T_1^1$, ..., $T_4^1$ is shown, along with the Hamilton path $\tilde{\pi}_4$ (bold). The source $\alpha_m$ is labeled $\{n - 3, n - 2, n - 1, n\}$, and the sink $\omega_m = \{n - 5, n - 3, n - 1, n\}$. See Convention 5.1 for the labels of the other marked vertices. Right: The facet $F_3^3$ with the restriction of $\tilde{\pi}_4$ to it.

**Remark 2.4.** The crucial property for our realization construction is that the path $\tilde{\pi}_m$ begins in a certain facet $F_m^3$ of the polytope $Q_m$ (defined below), traverses the rest of $Q_m$,
and then returns to $F^3_m$ (cf. Figure 3). This permits us to add new vertices to the beginning and end of $\tilde{\pi}_m$ by modifying only the facet $F^3_m$.

**Theorem 2.5.** There exists a family $\{Q_m : m \geq 0\}$ of special realizations of the combinatorial types $\tilde{Q}_m$, in which each Hamilton path $\pi_m$ visits the vertices of $Q_m$ in the order given by increasing $x_4$-coordinate. This family may be realized inductively starting from the 4-simplex $Q_0$ in such a way that for all $m \geq 0$, a realization of $Q_{m+1}$ with a monotone Hamilton path $\pi_{m+1}$ may be obtained from any realization of $Q_m$ with such a path $\pi_m$.

### 3. Constructing the combinatorial types $\tilde{Q}^d_m$

#### 3.1. Facet splitting

We will prove Theorem 2.1 using Barnette’s technique of facet splitting [1]. Put briefly, for each even $d \geq 4$ we will inductively construct a family $\{(\tilde{Q}^d_m, \mathcal{F}_m) : m \geq 0\}$, where each $\tilde{Q}^d_m$ is the combinatorial type of a simple $d$-dimensional polytope with $m + d + 1$ facets, and $\mathcal{F}_m$ is a flag of faces on $\tilde{Q}^d_m$ (to be defined shortly). We then use $\mathcal{F}_m$ to find a “good” oriented hyperplane $H_{m+1}$ in general position with respect to the vertices of $\tilde{Q}^d_m$, and set $\tilde{Q}^d_{m+1} := \tilde{Q}^d_m \cap H_{m+1}^\geq$.

**Definition 3.1.** Let $P$ be a $d$-dimensional simple polytope. A flag of faces on $P$ is a chain

$$\mathcal{F} : \emptyset = F^{-1} \subset F^0 \subset F^1 \subset \cdots \subset F^d = P$$

of faces of $P$ such that $\dim F^i = i$ for $i = 0, 1, \ldots, d$. The $i$-th tip of a flag $\mathcal{F}$ is $T^i := \text{vert } F^i \setminus \text{vert } F^{i-1}$, for $0 \leq i \leq d$. We say that the tip $T^i$ is even resp. odd according to the parity of $i$. Moreover, for $0 \leq k \leq d$ we set

$$T^k_{\mathrm{even}} = \bigcup_{0 \leq k \leq d \text{ even}} T^k \quad \text{and} \quad T^k_{\mathrm{odd}} = \bigcup_{1 \leq k \leq d \text{ odd}} T^k.$$

**Lemma 3.2.** Let $P$ be a simple $d$-dimensional polytope with $n$ facets, and $\mathcal{F}$ a flag of faces as in (2). Then there exists an affine oriented hyperplane $H$ in general position with respect to $P$ such that $T^k_{\mathrm{even}} \subset H^+$ and $T^k_{\mathrm{odd}} \subset H^-$. In particular, $P \cap H^{\geq 0}$ is a simple $d$-polytope with $n + 1$ facets.

**Proof.** Pick an oriented point $\{v\} = H^0 \subset \text{relint } F^1$ such that $T^0 \in (H^0)^+$. Inductively, for $1 \leq k \leq d - 1$, if we have already chosen an oriented $(k-1)$-dimensional affine subspace $H^{k-1}$ in aff $F^k$ such that

$$T^k_{\mathrm{even}} \subset (H^{k-1})^+ \quad \text{and} \quad T^k_{\mathrm{odd}} \subset (H^{k-1})^-,$$

we take a $k$-plane $H^k$ that initially coincides with aff $F^k$, and orient it in such a way that $T^{k+1}$ lies in $(H^k)^+$ if $k + 1$ is even, respectively in $(H^k)^-$ if $k + 1$ is odd. Now we rotate $H^k$ by a sufficiently small amount around $H^{k-1}$ in such a way that $T^k_{\mathrm{even}} \subset (H^k)^+$. Then (3) even holds with $k$ replaced by $k + 1$. By construction, the hyperplane $H := H^{d-1}$ is in general position with respect to $P$. \(\square\)
**Definition 3.3.** The family \( \{ (\tilde{Q}^d_m, \mathcal{F}_m) : m \geq 0 \} \) of \( d \)-dimensional polytopes \( \tilde{Q}^d_m \) equipped with flags \( \mathcal{F}_m \) of faces is defined in the following way:

(a) \( \tilde{Q}^d_0 \) is the combinatorial type of the \( d \)-simplex \( \text{conv}\{v_1, v_2, \ldots, v_{d+1}\} \). The flag \( \mathcal{F}_0 \) is defined by setting \( F^i_0 := \text{conv}\{v_1, v_2, \ldots, v_{i+1}\} \) for \( i = 0, 1, \ldots, d \). Then

\[
T^i_0 := \text{vert} \left( F^i_0 \right) \setminus \text{vert} \left( F^{i-1}_0 \right) = \{v_{i+1}\} \quad \text{for} \quad i = 0, 1, \ldots, d.
\]

(b) For \( m \geq 0 \), let \( H = H_{m+1} \) be the oriented hyperplane given by applying Lemma 3.2 to \( P = \tilde{Q}^d_m \) and \( \mathcal{F} = \mathcal{F}_m \), and set \( \tilde{Q}^d_{m+1} := \tilde{Q}^d_m \cap H_{m+1} \) and (cf. Figure 4)

\[
T^0_{m+1} := \text{vert} \left( \text{conv}(T^1_m \cup T^2_m) \cap H_{m+1} \right),
\]
\[
T^1_{m+1} := \text{vert} \left( \text{conv}(T^0_m \cup T^1_m) \cap H_{m+1} \right),
\]
\[
T^j_{m+1} := \text{vert} \left( \text{conv} \left( T^{j+1}_m \cup \bigcup_{0 \leq k < j \atop k+j=0 \mod 2} T^k_m \right) \cap H_{m+1} \right) \quad \text{for} \quad j = 2, 3, \ldots, d - 1,
\]
\[
T^d_{m+1} := \bigcup_{0 \leq k \leq d/2} T^{2k}_m.
\]

**Figure 4:** New tips in the case \( d = 4 \).

The flag \( \mathcal{F}_{m+1} \) is now defined by \( F^j_{m+1} := \bigcup_{i=0}^j T^i_{m+1} \) for \( j = 0, 1, \ldots, d \). Moreover, put

\[
T^{\leq k}_{\text{even}}(m) = \bigcup_{0 \leq \epsilon \leq k \atop \epsilon \text{ even}} T^\epsilon_m \quad \text{and} \quad T^{\leq k}_{\text{odd}}(m) = \bigcup_{1 \leq \epsilon \leq k \atop \epsilon \text{ odd}} T^\epsilon_m.
\]
Remark 3.4.
(a) The polytopes $C_d(n)^E$ arise by exchanging the definitions of $T_{m+1}^0$ and $T_{m+1}^1$.
(b) All new vertices arise as the intersection of $H_{m+1}$ with some edge $\text{conv}\{v, w\}$ of $\tilde{Q}_m^d$, where $v$ and $w$ lie in tips of different parity. Furthermore, all vertices of $Q_m^d$ belonging to even tips are also vertices of $\tilde{Q}_m^{d+1}$, and vertices in odd tips disappear.

**Proposition 3.5.** For each $m \geq 0$, the following is true for the pair $(\tilde{Q}_m^d, F_m)$:
(a) For all $i, j \in \mathbb{N}$ with $0 \leq i < j \leq d$ and $i + j = 1 \mod 2$ and all $v \in T_m^j$, there is exactly one $w \in T_m^j$ such that $\text{conv}\{v, w\} \in \text{sk}^1(\tilde{Q}_m^d)$. This gives rise to bijections $T_{m+1}^{<k}(m) \cong T_{m+1}^k$ for odd $0 < k < d$ resp. $T_{m+1}^{\leq k}(m) \cong T_{m+1}^k$ for even $0 \leq k \leq d$.

**Proof.** (a) This follows because $v$ lies in $F_m^{j-1} = \bigcup_{i=0}^{j-1} T_m^i$, and $\text{conv}(F_m^{j-1})$ is a $(j-1)$-dimensional face of the simple polytope $\text{conv}(F_m^j) = \text{conv}(F_m^{j-1} \cup T_m^j)$.
(b) We proceed by induction, and can assume that the assertion holds for $m \geq 0$. From the bijections in part (a), we conclude for all even $e = 0, 2, \ldots, d - 2$ that

$$|T_{m+1}^e| = |T_m^e| = \sum_{i = 0}^{e/2} |T_m^i| = \sum_{k=0}^{e/2} |T_m^{2k}| = \sum_{k=0}^{e/2} \binom{k + m}{m} = \binom{e/2 + m + 1}{m + 1}.$$ 

The calculation for $|T_{m+1}^d|$ is similar. The fact that $\tilde{Q}_m^d$ is polar-to-neighborly follows by the same argument as in Remark 2.2, since

$$f_0(\tilde{Q}_m^d) = \sum_{k=0}^{d/2} \binom{k + m}{m} + \sum_{k=0}^{\lfloor (d-1)/2 \rfloor} \binom{k + m}{m} = \left( m + \frac{d+1}{2} \right) + \left( m + \frac{\lfloor (d-1)/2 \rfloor + 1}{\lfloor (d-1)/2 \rfloor} \right).$$

$$= \left( n - \frac{d/2}{2} \right) + \left( n - \frac{\lfloor (d-1)/2 \rfloor - 1}{\lfloor (d-1)/2 \rfloor} \right).$$

\[\square\]

### 3.2. Combinatorics of the family $\tilde{Q}_m^d$.

**Convention 3.6.** We introduce labelings to make the combinatorics of the $\tilde{Q}_m$ explicit:
(a) For any labeling of the facets of a simple $d$-polytope $P$ with labels in $[n] := \{1, 2, \ldots, n\}$, let $\lambda : \text{vert} P \to \binom{[n]}{d}$ assign to each vertex $v$ of $P$ the set of labels of all facets that $v$ is incident to. We identify a vertex $v$ with its label $\lambda(v)$.
(b) The facets of the $d$-simplex $\tilde{Q}_d^0$ on the vertex set $\{v_1, v_2, \ldots, v_{d+1}\}$ are labeled in such a way that $v_1 \equiv \lambda(v_1) = [d + 1] \setminus \{2\}$, $v_2 = [d + 1] \setminus \{1\}$, and $v_j = [d + 1] \setminus \{j\}$ for $j = 3, 4, \ldots, d + 1$ (cf. Figure 5).
(c) The “new” facet $\tilde{Q}_m^d \cap H_{m+1}$ of $\tilde{Q}_{m+1}^d$ is labelled $m + d + 2$.

$$\{v_5\} = T_0^3 \ 2b \ 12|34$$
$$\{v_3\} = T_0^2 \ 2b \ 12|45$$
$$\{v_1\} = T_0^0 \ 1 \ 13|45$$

**Figure 5:** The labeling of the vertices of the 4-simplex $\tilde{Q}_0$ according to Convention 3.6(b). Also shown is the classification of the vertices into types 1, 2a, 2b as in Proposition 3.7.

**Proposition 3.7.** Let $m \geq 0$ and $n = m + d + 1$. A vertex $v$ of $\tilde{Q}_m^d$ lies in $T_i^m$ exactly if

$$\max_n v := \max ([n] \setminus v) = \begin{cases} m + 2 & \text{for } i = 0, \\ m + 1 & \text{for } i = 1, \\ m + i + 1 & \text{for } 2 \leq i \leq d. \end{cases}$$

**Proof.** This is true for $m = 0$ by (4) and Convention 3.6, see also Figures 5 and 6. For $m > 0$ and $2 \leq i \leq d - 1$, the statement follows because any vertex $\tilde{v} \in T_i^m$ is of the form $\tilde{v} = \text{conv}\{v, w\} \cap H_m \equiv (v \cap w) \cup \{n\}$ for some $v \in T_{m-1}^k$ and $w \in T_{m-1}^{i+1}$ with $k \leq i$. But then by induction,

$$\max_n \tilde{v} < \max_n \tilde{w} = (m - 1) + (i + 1) + 1 = m + i + 1,$$

so $\max ([n] \setminus \tilde{v}) = m + i + 1$ as required. The case $i = d$ follows directly from Definition 3.3, and the cases $i = 0, 1$ are checked similarly. \hfill \square

**Proof of Theorem 2.1.** The existence of the family $\{(\tilde{Q}_m^d, F_m) : m \geq 0\}$ follows from Lemma 3.2. Using Propositions 3.5 and 3.7, it is somewhat tedious but elementary to verify that for all $m \geq 0$, the vertices of $\tilde{Q}_m^d$ are of the given types. More precisely, all vertices of $T_{\text{even}}^d(m)$ are of type 1 or 2b, and $T_{\text{odd}}^{d-1}(m)$ is made up entirely of vertices of type 2a, cf. Figure 6. \hfill \square

4. A Hamilton path $\tilde{\pi}_m$ that induces an AOF-orientation on $\tilde{Q}_m$

**Definition 4.1.** Let $P$ be a simple $d$-polytope. An acyclic orientation of the graph of $P$ that has a unique sink in each face (including $P$ itself) is called an AOF-orientation on $P$. For any orientation $O$ of the graph of $P$ and $0 \leq k \leq d$, denote by $h_k(O)$ the number of vertices of in-degree $k$ in $O$.

**Proposition 4.2.** (see e.g. [9, Chap. 8.3] and [3]) An acyclic orientation $O$ of the graph of a simple $d$-polytope $P$ is an AOF-orientation if and only if the $h$-vector of $P$ coincides with the vector $(h_0(O), h_1(O), \ldots, h_d(O))$. \hfill \square
Figure 6: Vertex labels in the polytopes $\tilde{Q}_1$ (left) and $\tilde{Q}_2$ (right). Also shown are the type (outside) of each vertex $v$ and the value of $\max_n \bar{v}$ (inside).

Proof of Proposition 2.3. By inspection of Figures 2 and 3, the algorithm of Figure 7 yields a Hamilton path $\tilde{\pi}_m$ in the graph of $\tilde{Q}_m$. Note that $\tilde{\pi}_m$ passes through $T^1_m, T^3_m, T^4_m, T^0_m,$ and $T^2_m$, in this order (cf. Remark 2.4).

(1) “Odd stage”.  

for $i$ from $n - 3$ to $1$ do 

visit $\{i, i + 1, i + 2, n\}$;

(2) “Even stage”.  

for $j$ from $3$ to $n - 1$ do 

$i := j - 3$;  

while $i \geq 1$ do  

“down” phase 

visit $\{i, i + 2, j, j + 1\}$;  

$i := i - 2$;  

visit $\{1, 2, j, j + 1\}$;  

if $j$ is even then $i := 2$; else $i := 1$;  

while $i \leq j - 4$ do  

“up” phase 

visit $\{i, i + 2, j, j + 1\}$;  

$i := i + 2$;

Figure 7: A Hamilton path $\tilde{\pi}_m$ on the graph of $\tilde{Q}_m$ that induces an AOF-orientation ($n := m+5$).

We now verify that $\tilde{\pi}_m$ induces an AOF orientation on the graph of $\tilde{Q}_m$. The $h$-vector of a simple polar-to-neighborly $d$-dimensional polytope with $n = m + d + 1$ facets is given
by $h_k = \binom{n-d-1+k}{k} = \binom{m+k}{k}$ for $k = 0, 1, \ldots, d$. Therefore, by Proposition 3.5,
\[
(|T^1_m|, |T^3_m|, |T^4_{m}|, |T^0_m|) = (h_0, h_1, h_2, h_3, h_4).
\]
By Proposition 4.2, it suffices to verify using Figure 3 that if the orientation of each edge of the graph of $\tilde{Q}_m$ is consistent with the total ordering induced by $\tilde{\pi}_m$, then the vertices of $T^1$, $T^3$, resp. $T^4$ all have in-degree 0, 1 resp. 2, furthermore $T^0$ and all but one of the vertices of $T^2$ have in-degree 3, and this vertex, the sink, has in-degree 4. \hfill \square

5. Realizing the monotone Hamilton paths

In this section we prove Theorem 2.5, and therefore our Main Theorem.

5.1. Outline of the inductive construction. For all $m \geq 0$, we first find an oriented hyperplane $H_{m+1}$ that separates the odd part $T^\leq (m) = T^1_m \cup T^3_m$ from the even part $T^\geq (m) = T^0_m \cup T^2_m \cup T^4_m$ of $\pi_m$. We then create an intermediate pair $(Q_{m+1}', F_{m+1}')$ as in Proposition 3.5: $Q_{m+1}' := Q_m \cap H^\geq_{m+1}$ is a simple polar-to-neighborly polytope of the same combinatorial type as $\tilde{Q}_{m+1}$, and the flag $F_{m+1}'$ of faces is defined as in Definition 3.3(b).

Our combinatorial model $\tilde{Q}_{m+1}$ provides us with a Hamilton path $\pi_{m+1}$ on $Q_{m+1}'$ that is not yet monotone with respect to the objective function $f : x \mapsto x_4$. However, we will choose $H_{m+1}$ in such a way that there exists a pencil
\[
\mathcal{H} = \{H_t : t \in \mathbb{P}^1(\mathbb{R}) \cong \mathbb{R} \cup \{\infty\}\}
\]
of hyperplanes in $\mathbb{R}^4$ with the following properties:

(S1) The common intersection of all hyperplanes in $\mathcal{H}$ is a 2-flat $R = \bigcap_{t \in \mathbb{P}^1(\mathbb{R})} H_t$ (the axis of $\mathcal{H}$), and vert $Q'_{m+1} \cap R = \emptyset$.

(S2) The pencil $\mathcal{H}$ “sorts the vertices of $Q'_{m+1}$ correctly”: If $p \in H_r$ and $q \in H_s$ are vertices of $Q'_{m+1}$ with $r, s \neq \infty$ and $p$ precedes $q$ in $\pi_{m+1}$, then $r < s$.

We then apply a projective transformation $\psi$ to $\mathbb{R}^4 \subset \mathbb{P}^4(\mathbb{R})$ that maps $H_\infty$ to the hyperplane at infinity. Because the common intersection $R$ of all hyperplanes in $\mathcal{H}$ is also mapped to infinity, the image $\psi(\mathcal{H}^0) = \psi(\mathcal{H} \setminus H_\infty) = \{\psi(H_t) : t \in \mathbb{R}\}$ is a family of parallel affine hyperplanes in $\mathbb{R}^4$. The new objective function $f$ is then defined by the common normal vector to the hyperplanes in $\psi(\mathcal{H}^0)$, and the Hamilton path $\psi(\pi_{m+1})$ on $Q_{m+1} := \psi(Q_{m+1}')$ is strictly monotone with respect to $f_{m+1}$ by (S2).

5.2. Properties of the family of polytopes.

Notation 5.1. We use the following names for some special vertices of $\tilde{Q}_m$:

- The source $\{n-3, n-2, n-1, n\}$ of $\tilde{\pi}_m$ is called $\alpha_m$ (so that $T^1_m = \{\alpha_m\}$).
- The sink is $\omega_m := \{n-5, n-3, n-1, n\} \in T^2_m$.
- $\beta_m := \{n-4, n-2, n-1, n\}$ (so that $T^0_m = \{\beta_m\}$).
- $\tau_m := \{n-5, n-3, n-2, n-1\} \in T^4_m$. 
**Proposition 5.2.**

(a) The induced subgraph of $sk^1(Q_m)$ on $T_m^1 \cup T_m^3$ is a path of length $m + 1$ on the $m + 2$ vertices $v_0^m = \alpha_m$, $v_1^m$, \ldots, $v_{m+1}^m$, and the induced subgraph on $T_m^2$ is a path $w_1^m$, $w_2^m$, \ldots, $w_{m+1}^m$.

(b) For $0 \leq i \leq m$, the edge $e_i = \text{conv}\{v_i^m, v_{i+1}^m\}$ in $T_m^3$ is incident to a 2-face $G_i$ of $Q_m$ such that the vertices of $G_i \setminus e_i$ are consecutive in $\pi_m \cap T_m^3$.

(c) For $1 \leq i \leq m$, the edge $f_i$ of $Q_m$ that connects $w_i^m$ and $w_{i+1}^m$ in $T_m^2 \cap \pi_m$ is incident to a quadrilateral $R_i$ whose other two vertices are consecutive in $T_m^4 \cap \pi_m$.

(d) Set $G(m) = \text{vert} \bigcup_{i=0}^{m} G_i \setminus e_i$ and $R(m) = \text{vert} \bigcup_{i=1}^{m} R_i \setminus f_i$. Then $G(m) \cup R(m) = T_m^4$, and $G(m) \cap R(m) = \tau_m$.

**Proof.** (a) All vertices of $T_m^3$ are of the form $\{i, i+1, i+2, n\}$ for $1 \leq i \leq n - 3$, and the only way for two such vertices $v_i^m$ and $v_j^m$ to be adjacent for $i < j$ is to have $j = i + 1$. The statement about the $w_i^m$ follows in a similar way. (b) For $1 \leq i \leq m+1$, the 2-face incident to $v_{m+2-i} = \{i, i+1, i+2, n\}$ and $v_{m+1-i} = \{i+1, i+2, i+3, n\}$ that is the intersection of the facets $i+1$ and $i+2$ consists of the vertices of Figure 8. The claim (b) follows because these vertices form a contiguous segment of $\pi_m$, and (c) and (d) from Figure 9 (left). □

**Observation 5.3.** The new start vertex $\alpha_{m+1}$ of $\pi_{m+1}$ lies on $\text{conv}\{\alpha_m, \beta_m\}$, the new end vertex $\omega_{m+1}$ on $\text{conv}\{v_1^m, \beta_m\}$, and $\beta_{m+1}$ on $\text{conv}\{\alpha_m, \omega_m\}$; see Figure 9 (right).

**5.3. Start of the induction and inductive invariant.** We work in $\mathbb{R}^4$ with standard coordinate vectors $e_1, e_2, e_3, e_4$. An essential tool will be shear transformations: these are linear maps $\sigma_{i,j}^a : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, and $a \in \mathbb{R}$ whose matrix is $I_4 + a\delta_{i,j}$ with respect to the standard basis of $\mathbb{R}^4$. Here $I_4$ is the $4 \times 4$ unit matrix and $\delta_{i,j}$ is the $4 \times 4$ matrix whose only nonzero entry is a 1 in position $(i, j)$. In particular, $\sigma_{i,j}^a$ maps $e_i$ to $e_i + ae_j$, and the standard basis vectors $e_k$, $k \neq i$, to themselves.

![Figure 8](image-url)  

**Figure 8:** Vertices of a 2-face incident to $v_i = \{i, i+1, i+2, n\}$ and $v_{i+1} = \{i+1, i+2, i+3, n\}$ (dark) in $T_m^1 \cup T_m^3$. The light vertices lie in $T_m^4$ and form a subpath of $\pi_m$. 

![Figure 9](image-url)
Figure 9: Left: More details about the graph of $Q_m$. We have highlighted the graphs of the 2-faces $G_0$ and $G_1$ that correspond to the edges $e_0$ and $e_1$ by Proposition 5.2 (b), and the 2-face $R_1$ that corresponds to the edge $f_1$ according to Proposition 5.2 (c). Right: The portion of the new Hamilton path $\tilde{\pi}_{m+1}$ in the facet $F_m^3$.

The start of the induction is the pair $(Q_0, F_0)$, where $Q_0$ is the 4-simplex whose vertices $v_1, v_2, v_3, v_4, v_5$ are given by the columns of the matrix

$$
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-3 & -1 & 3 & 2 & 1 \\
-2 & -1/2 & 0 & 1/4 & 2
\end{pmatrix},
$$

and $F$ is the flag $F_0 : F_0^0 \subset F_0^1 \subset \cdots \subset F_0^4 = Q_0^4$ of faces labeled as in Definition 3.3. In particular, the vertices $v_i$ lie in the following tips,

$$
\begin{array}{ccccccc}
& v_1 & v_2 & v_3 & v_4 & v_5 \\
\hline
T_0 & 0 & 0 & 1 & 0 & 0 \\
T_0^3 & 0 & 1 & 0 & 0 & 0 \\
T_0^4 & -3 & -1 & 3 & 2 & 1 \\
T_0^5 & -2 & -1/2 & 0 & 1/4 & 2
\end{array}
$$

and $F^2 = \text{conv}\{v_1, v_4, v_5\}$, $F^3 = \text{conv}\{v_1, v_2, v_4, v_5\}$, and $\pi_0 = (v_1, v_2, v_3, v_4, v_5)$.

For all $m \geq 0$ the polytopes $Q_m$ will maintain the following property:

(M1) The Hamilton path $\pi_m$ in the 1-skeleton of $Q_m$ is strictly monotone with respect to the objective function $f : \mathbb{R}^4 \to \mathbb{R}$, $x \mapsto x_4$.

5.4. Induction step I: Positioning the polytope. In this and the following section, we will position the polytope $Q_m$ in such a way that the coordinate subspaces of $\mathbb{R}^4$ are compatible with the flag $F_m$. More precisely,

- $F_m^3 = Q_m \cap \{x \in \mathbb{R}^4 : x_1 = 0\}$, and $T_m^4 \subset \{x \in \mathbb{R}^4 : x_1 > 0\}$; and
- the hyperplane $H_3 = \{x \in \mathbb{R}^4 : x_3 = 0\}$ will separate $T_{\text{even}}^4(m)$ from $T_{\text{odd}}^4(m)$. 
Lemma 5.4. Let \( \pi \) be the linear projection \( \pi : \mathbb{R}^4 \rightarrow \mathbb{R} \langle e_3, e_4 \rangle \), and use the notation of Convention 5.1 and Proposition 5.2(a). Then there exists a non-singular affine transformation \( \sigma \) of \( \mathbb{R}^4 \) such that \( Q_m \equiv \sigma(Q_m) \) satisfies the following additional conditions, while \( \tau_m \equiv \sigma(\tau_m) \) still satisfies (M1):

(M2) \( F_m^2 \subset \{ x \in \mathbb{R}^4 : x_1 = 0 \} \).

(M3) \( \text{aff } F_m^3 \equiv \{ x \in \mathbb{R}^4 : x_1 = 0 \} \) and \( Q_m \subset \{ x \in \mathbb{R}^4 : x_1 \geq 0 \} \).

(M4) \((\alpha_m)_2 = 0, r_2 < 0\) for all \( r \in F_m^2 \setminus \{ \alpha_m \} \), and \((\beta_m)_2 < (v_m^3)_2\).

(M5) \( \text{The image of } F_m^2 \text{ under } \pi \text{ is full-dimensional: dim } \text{aff } \pi(F_m^2) = 2 \).

(M6) \( \text{The 3-flat } H_S = \{ x \in \mathbb{R}^4 : x_3 = 0 \} \) strictly separates \( T_{\text{even}}^4(m) \) from \( T_{\text{odd}}^4(m) \).

Moreover, we may choose the point of \( H_S \cap F_m^3 \) of lowest 4-coordinate to be \( \alpha_m+1 = \text{conv}\{\alpha_m, \beta_m\} \cap H_S \), where \( \alpha_m+1 = \tau_4 := (\tau_m)_4 \).

Proof. Properties (M2) and (M3) are a matter of trivial affine transforms that can be chosen to leave the 4-coordinates invariant, thereby maintaining (M1), and property (M4) can be achieved via a translation and a shear \( \sigma_{2,4}^a : x_2 \mapsto x_2 + ax_4 \).

For (M5), choose \( t \in F_m^2 \) with \( t_4 = q_4 \) for some \( q \in T_m^3 \); such a point exists, since \( \alpha_m \in F_m^2 \), and \((\alpha_m)_4 < q_4 < \max\{s_4 : s \in F_m^2\}\) for all \( q \in T_m^3 \) by (M1) and Remark 2.4. Translate \( t \) such that \( t = (0, t_2, 0, 0) \) with \( t_2 < 0 \), and apply a shear transform \( \sigma_{3,2}^b : x_3 \mapsto x_3 + bx_2 \) to \( \mathbb{R}^3 \), where \( b \in \mathbb{R} \) is chosen such that \( \pi(\sigma_{3,2}^b(q)) = \pi(\sigma_{3,2}^b(t)) \). This can be done because \( \pi(t) - \pi(q) \in \mathbb{R}\langle e_3 \rangle \). Then (M5) is fulfilled because \( \dim \text{aff } F_m^3 = 3 \); supposing that \( \dim \text{aff } (\pi(F_m^2)) = 1 \) would imply via \( t \in F_m^2 \) and \( q \in F_m^3 \) that \( q \in \text{aff } F_m^2 \); however, this is absurd by the choice \( q \in T_m^3 \). Note that none of the maps we used affects (M2)–(M4).

For (M6), define \( \bar{b} \) to be the point of greatest 3-coordinate of \( F_m^2 \cap \{ x \in \mathbb{R}^4 : x_4 = \tau_4 \} \). In particular, \( \bar{b}_4 > \max_{x \in T_m^3} z_4 \) by (M1), and \( \bar{b} \) lies either on the edge \( \text{conv}\{\alpha_m, \beta_m\} \) or on the edge \( \text{conv}\{\alpha_m, \omega_m\} \) of \( F_m^2 \subset Q_m \) (cf. Figure 10).

Possibly using the transform \( x_3 \mapsto -x_3 \), we can achieve \( \bar{b} \in \text{conv}\{\alpha_m, \beta_m\} \), and \( \bar{b} = \alpha_{m+1} \) after a translation along the 3-axis. Now choose a non-horizontal line \( \ell \) through \( \alpha_{m+1} \) such that \( \pi(\ell) \) separates \( \pi(T_m^1 \cup T_m^3) \) from \( \pi(F_m^2 \setminus T_m^1) \) (for example, perturb \( \ell = \alpha_{m+1} + Re_3 \)), translate \( Q_m \) again such that \( \alpha_{m+1} = 0 \), and apply a shear \( \sigma_{3,4}^a : x_3 \mapsto x_3 + ax_4 \) to \( \mathbb{R}^4 \) such that \( \ell := \sigma_{3,4}^a(\ell) \equiv \{ x \in \mathbb{R}^4 : x_1 = x_3 = 0 \} \cap \text{aff } F_m^2 \) is vertical, and \( x_3 < 0 < y_3 \) for all \( x \in T_m^1 \cup T_m^3 \) and \( y \in F_m^2 \setminus T_m^1 \) (cf. Figure 10). If the hyperplane \( \pi^{-1}(\pi(\ell')) \) does not yet separate \( T_m^1 \) from \( T_m^1 \), apply another shear \( \sigma_{3,1}^d : x_3 \mapsto x_3 + dx_1 \) with \( d > 0 \) until it does (note that (M3) already holds), and then define \( H_S := \pi^{-1}(\pi(\ell')) \). This hyperplane then separates the odd and even parts of \( \tau_m \) by construction, and \( \alpha_{m+1} = \tau_4 \) also by construction and because the shears \( \sigma_{3,4}^a \) and \( \sigma_{3,1}^d \) do not affect 4-coordinates. Neither do they affect conditions (M1)–(M5), so we define \( \sigma \) as the composition of all these maps. □

Remark 5.5. The conditions (M1)–(M6) are satisfied by the coordinates (5) for \( Q_0 \).

5.5. Induction step II: Finding the cutting plane. In this section, we will find a hyperplane \( H_{m+1} \) that gives rise to a polytope \( Q_{m+1} = Q_m \cap H_{m+1}^\geq 0 \) of the same combinatorial
Figure 10: Positioning the polytope, step (M6). The map \( \sigma_{3,4}^h \) shears the polytope until (the preimage under \( \pi \) of) a vertical line \( \ell' \) separates the odd from the even tips. On the right, the approximate position of \( \omega' \) is marked; cf. Lemma 5.7.

type as \( \tilde{Q}_{m+1} \). Namely, assume that (M1)–(M6) hold, define \( H_{m+1} \) to be the hyperplane \( \{ x \in \mathbb{R}^4 : n^T x = 0 \} \) with \( n = (0, -\delta, 1, \varepsilon)^T \) for some small \( \varepsilon \gg \delta > 0 \), and assign the label \( n_{m+1} = m_d + 2 \) to \( H_{m+1} \). Note that \( H_{m+1} \) converges to \( H_S \) as \( \varepsilon, \delta \to 0 \).

Remark 5.6. Up to now, we have put the facet \( F_m^3 \) into the 3-plane \( \{ x \in \mathbb{R}^4 : x_1 = 0 \} \) and the tip \( T_m^4 \) into the half-space \( \{ x \in \mathbb{R}^4 : x_1 > 0 \} \). This allows us to move “almost all” of the vertices of \( \pi_m \) (namely, the portion inside \( T_m^4 \)) “out of the way”, via a shear \( \sigma_{3,1}^a \) that only affects 3-coordinates. These “old” vertices will be dealt with in Lemma 5.8 below.

We still need to arrange for the first and last part of \( \pi_{m+1} \) to be traversed in the right order. We achieve this by adjusting the position of \( H_{m+1} \) via the parameters \( \varepsilon \) and \( \delta \) in the definition of \( n \) (note that we chose \( n_1 = 0 \), because we are already done with \( T_m^4 \)). If \( \delta = 0 \), then \( \pi(H_{m+1}) \) is a line whose slope is determined by \( \varepsilon \). We choose \( \varepsilon > 0 \) to ‘push out’ the first part \( T_{m+1}^0 \cup T_{m+1}^2 \) of the new path \( \pi_{m+1} \). However, if we left \( \delta = 0 \) we would not correctly sweep the last portion \( T_{m+1}^0 \cup T_{m+1}^2 \). Items (M8)–(M10) of Lemma 5.7 guarantee a correct sweep in Lemma 5.8 for sufficiently small \( 0 < \delta \ll \varepsilon \).

Lemma 5.7. Assume conditions (M1)–(M6) and \( (\alpha_{m+1})_3 = (\alpha_{m+1})_4 = 0 \), and fix vertices \( q \in T_{\text{odd}}^{\leq 4}(m) \) and \( s \in T_{\text{even}}^{\leq 4}(m) \). Let \( q' = \text{conv}\{q, s\} \cap H_{m+1} \) be the intersection with \( H_{m+1} \) of the line through \( q \) and \( s \) (which is not necessarily an edge of \( Q_m \)). Then, if \( a > 0 \) is sufficiently large and \( 0 < \delta \ll \varepsilon \) are sufficiently small, the image \( \sigma_{3,1}^a(Q_m) \) of \( Q_m \) under the shear \( \sigma_{3,1}^a \) satisfies the following conditions (M7)–(M10); cf. also Figure 12 below.

(M7) \( q'_3 > 0 \) for \( 0 < \delta \ll \varepsilon \), and \( q'_3 \searrow 0 \) as \( \delta, \varepsilon \searrow 0 \). In other words, all points in \( \sigma_{3,1}^a(Q_m) \cap H_{m+1} \) can be chosen to have positive 3-coordinate, but to lie arbitrarily close to \( H_S \).
(M8) Set \( u := (v^{m+1})' = \text{conv}\{\alpha_m, \tau_m\} \cap H_{m+1} \) and suppose that \( q' \neq u \). Then the image 
\( \pi(\text{aff}\{u, q'\}) \subset R(e_3, e_4) \) of the line through \( u \) and \( q' \) under \( \pi \) comes arbitrarily close to being vertical as \( a \to \infty \) and \( \varepsilon, \delta \to 0 \).

(M9) Set \( \alpha' := \alpha_{m+1}' = \text{conv}\{\alpha_m, \beta_m\} \cap H_{m+1} \). If \( q, \bar{q} \in T^3_m \) and \( q_4 < \bar{q}_4 \), so that 
\( q', \bar{q}' \in T^3_{m+1} \) and \( q'_4 < \bar{q}'_4 \), then the slope \( \sigma_{\alpha'q'} \) of the line \( \pi(\text{aff}\{\alpha', q'\}) \) is greater than the slope \( \sigma_{\alpha'\bar{q}'} \) of the line \( \pi(\text{aff}\{\alpha', \bar{q}'\}) \) (and both are negative).

(M10) Set \( \omega' := \omega_{m+1}' = \text{conv}\{\beta_m, v^m\} \cap H_{m+1} \). Then the slope \( \sigma_{\omega'\omega'} \) of \( \pi(\text{aff}\{\omega', \alpha'\}) \) is less than the slope \( \sigma_{\omega'u} \) of \( \pi(\text{aff}\{\omega', u\}) \).

Proof. We abbreviate \( \sigma = \sigma_{3,1}^a \). For (M7), we have \( \text{conv}\{q, s\} \cap H_{m+1} \neq \emptyset \) since \( q \) and \( s \) are separated by \( H_{m+1} \) for small enough \( \delta, \varepsilon \). We calculate the intersection point \( q' = \text{conv}\{q, s\} \cap H_{m+1} \) by solving 
\[
\begin{align*}
\mathbf{n}^T q + \mu (\mathbf{n}^T (s - q)) &= 0 \quad \text{for } \mu,
\end{align*}
\]

By (M2), the map \( \sigma \) leaves the points \( \alpha', q', \) and \( \omega' \) invariant, and maps \( s \) to \( \sigma(s) = s + as_1e_3 \); as a consequence, \( \mathbf{n}^T \sigma(s) = \mathbf{n}^T s + as_1 \). Using \( \mathbf{n}^T q = -\delta q_2 + q_3 + \varepsilon q_4 \), we obtain
\[
\sigma(q') = q + \frac{n^T q}{n^T (q - s) - as_1} (s - q + as_1 e_3)
\]

(6) \[
\lim_{a \to \infty} q + (0, 0, -n^T q, 0)^T = (0, q_2, \delta q_2 - \varepsilon q_4, q_4)^T.
\]

Because \( q_4 < (\alpha_{m+1})_4 = 0 \), we can choose \( 0 < \delta \ll \varepsilon \) so small that \( \sigma(q')_3 > 0 \) (note that \( q_2 \leq 0 \) by (M4)). In particular, we obtain \( \sigma(q')_3 \searrow 0 \) as \( \varepsilon, \delta \searrow 0 \).

Statement (M8) follows from (6) and the fact that
\[
\lim_{a \to \infty} \frac{\sigma(q')_4 - \sigma(u)_4}{\sigma(q')_3 - \sigma(u)_3} = \frac{q_4 - u_4}{\delta(q_2 - u_2) - \varepsilon(q_4 - u_4)}.
\]

For (M9), note that since \( \alpha' \) is invariant under \( \sigma \),
\[
\sigma_{\alpha'\bar{q}'} = \frac{\sigma(q')_4 - \alpha'_4}{\sigma(q')_3 - \alpha'_3} \lim_{a \to \infty} \frac{q_4 - \alpha'_4}{\delta q_2 - \alpha'_3 - \varepsilon q_4},
\]

and similarly for \( \bar{q} \); the statement now follows from \( q_4 < \bar{q}_4 \) and \( 0 < \delta \ll \varepsilon \).

To prove (M10), set \( \alpha := \alpha_m, \beta := \beta_m, v := v^m \) and \( \tau := \tau_m \). Then \( u = \text{conv}\{\alpha, \tau\} \cap H_{m+1}, \alpha' = \text{conv}\{\alpha, \beta\} \cap H_{m+1}, \) and \( \omega' = \text{conv}\{v, \beta\} \cap H_{m+1} \). We need to verify that
\[
\sigma_{\omega'\omega'} := \frac{\alpha'_4 - \omega'_4}{\omega_3} < \frac{u_4 - \omega'_4}{u_3 - \omega'_3} =: \sigma_{\omega'u}.
\]
From equation (6) and condition (M4), we deduce that \( \lim_{a \to \infty} u = (0, 0, -\varepsilon \alpha_4, \alpha_4)^T \). For \( \alpha' \) and \( \omega' \) we get the following expressions:

\[
\alpha' = \alpha + \frac{n^T\alpha}{n^T(\alpha - \beta)} (\beta - \alpha) = \begin{pmatrix}
0 \\
0 \\
\alpha_3 \\
\alpha_4
\end{pmatrix} + \frac{\alpha_3 + \varepsilon \alpha_4}{\delta \beta_2 + \alpha_3 - \beta_3 + \varepsilon (\alpha_4 - \beta_4)} \begin{pmatrix}
0 \\
\beta_2 \\
\beta_3 - \alpha_3 \\
\beta_4 - \alpha_4
\end{pmatrix},
\]

\[
\omega' = v + \frac{n^Tv}{n^T(v - \beta)} (\beta - v) = \begin{pmatrix}
v_2 \\
v_3 \\
v_4
\end{pmatrix} + \frac{-\delta v_2 + v_3 + \varepsilon v_4}{-\delta (v_2 - \beta_2) + v_3 - \beta_3 + \varepsilon (v_4 - \beta_4)} \begin{pmatrix}
0 \\
\beta_2 - v_2 \\
\beta_3 - v_3 \\
\beta_4 - v_4
\end{pmatrix}.
\]

For convenience, we will verify that \( 1/\sigma_{\omega'\alpha'} > 1/\sigma_{\omega'u} \). Indeed, expanding these expressions in terms of \( \delta, \varepsilon \), we obtain

\[
\frac{1}{\sigma_{\omega'\alpha'}} = \frac{\beta_3 v_2 - \beta_2 v_3 + \alpha_3 (\beta_2 - v_2)}{v_3 (\alpha_4 - \beta_4) + \beta_3 (v_4 - \alpha_4) + \alpha_3 (\beta_4 - v_4)} \delta - \varepsilon + p_1(\delta, \varepsilon),
\]

\[
\frac{1}{\sigma_{\omega'u}} = \frac{\beta_3 v_2 - \beta_2 v_3}{v_3 (\alpha_4 - \beta_4) + \beta_3 (v_4 - \alpha_4)} \delta - \varepsilon + p_2(\delta, \varepsilon),
\]

where \( p_1 \) and \( p_2 \) are power series in \( \delta, \varepsilon \) with min-degree at least 2. Notice that up to terms of degree at least 2 in \( \delta, \varepsilon \), the two formulas are equal except for the expressions \( t_1 \) resp. \( t_2 \) in the numerator resp. denominator of \( 1/\sigma_{\omega'\alpha'} \). Therefore, we can write the difference between the inverses of the slopes as

\[
\frac{1}{\sigma_{\omega'\alpha'}} - \frac{1}{\sigma_{\omega'u}} = \left( \frac{A + t_1}{B + t_2} - \frac{A}{B} \right) \delta + p_3(\delta, \varepsilon).
\]

Since \( \alpha_3 < (\alpha_{m+1})_3 < 0 \) by assumption and \( \beta_2 < v_2 \) by (M4), we obtain \( t_1 > 0 \); and the inductive assumption (M1) implies that \( \beta_4 > v_4 \) and therefore \( t_2 < 0 \). The claim follows.

5.6. Induction step III: The projective transformation. Finally, we construct a 1-parameter family \( \mathcal{H} = \{ H_t : t \in \mathbb{P}^1(\mathbb{R}) \} \) of hyperplanes that contains a 2-plane \( R \) as their common “axis”, as in Section 5.1. Let \( O = \pi(b + \varepsilon_1 (\omega - \alpha) - \varepsilon_2 e_4) \) for some small \( \varepsilon_1, \varepsilon_2 > 0 \), so that \( O \) lies outside but very close to the edge \( \text{conv}\{\alpha, \omega\} \) of \( \pi(F_{m+1}^2) \), and define the 2-plane \( R \subset \mathbb{P}^4 \) to be \( R = \pi^{-1}(O) \).

**Lemma 5.8.** Let \( \mathcal{H} \) be the pencil of hyperplanes in \( \mathbb{P}^4 \) sharing the 2-plane \( R \), and such that \( \pi(H_{\infty}) \) is the line through \( O \) parallel to \( \text{conv}\{\alpha, \omega\} \), and the slope of \( \pi(H_r) \) is smaller than the slope of \( \pi(H_s) \) exactly if \( r < s \). Then \( \mathcal{H} \) fulfills (S2), i.e., it sorts the vertices of \( Q_{m+1} \) in the order given by \( \pi_{m+1} \).
Proof. We examine the pieces of $\pi_{m+1}$ in order; cf. Figure 12.

$\triangleright T_{m+1}^1 = \{\alpha\}$ is the start of $\pi_{m+1}$: This follows for small enough $\varepsilon_3$ by (M10).

$\triangleright T_{m+1}^3$ is traversed next, in the right order, and before $T_{m+1}^4$. The first two statements follow from (M7), (M8) and (M9), and the last one because $z_3 \to \infty$ as $a \to \infty$ for any $z \in T_m^4$, while the 3-coordinates of $T_{m+1}^3$ remain bounded by (M7).

$\triangleright$ The correct order in $T_m^4 \subset T_{m+1}^4$. By Proposition 5.2(b), each of the edges $e_i = \conv\{v_{m+i}^i, v_{m+i+1}^i\}$, $0 \leq i \leq m$, of $T_m^1 \cup T_m^3$ is incident to an $(m+1)$-gonal 2-face $G_i$ (see Figure 9), and the edges $E_i$ of $G_i$ not incident to $e_i$ form a monotone subpath of $\pi_{m+1}$. This implies that for each $e_i \in T_m^3$, the slopes of the projection of each $E_i$ to $\mathbb{R}(e_3, e_4)$ are strictly positive (and, by convexity, monotonically decreasing; see Figure 11).

![Figure 11: Convexity of the $(m+1)$-gonal faces enforces the correct order in $T_m^4 \subset T_{m+1}^4$.](image)

Therefore, $\pi\left(\bigcup_{i=0}^m E_i\right)$ is a strictly increasing chain of edges, and this remains true after applying the linear map $\sigma = \sigma_{\varepsilon_3}$ by invariance of the $e_i$'s and all 4-coordinates under $\sigma$, and the convexity of the projections of 2-faces. The correct order up to $\tau$ in $T_m^4 \subset T_{m+1}^4$ follows from condition (M6): $\alpha_4 \geq s_4$ for all $s \in \bigcup_{i=0}^m \text{vert } G_i \setminus \text{vert } e_i$. Similarly, the 4-gonal 2-faces incident to $T_m^2$ of Proposition 5.2(c) enforce the right order between $\tau$ and $T_{m+1}^0$. $\triangleright T_{m+1}^2$ is traversed after $T_{m+1}^4$: Since $\beta$, the first vertex of $\pi_{m+1}$ to come after $T_{m+1}^4$, lies on $\conv\{\alpha_m, \omega_m\}$, this can be achieved by choosing $\varepsilon$ and $\varepsilon_1$ suitably small.

$\triangleright$ Correct order in $T_{m+1}^2$ and $T_{m+1}^0$. This follows because the convex polygon $\pi(F_{m+1}^2)$ is star-shaped with respect to any point on its boundary, and the choice of $O$ close to an edge of $\pi(F_{m+1}^2)$. This concludes the proof of Lemma 5.8.

Finally, we apply the projective transform $\psi : \mathbb{R}^4 \to \mathbb{R}^4$, $x \mapsto x/(ax-a_0)$ that sends the 3-plane $H_\infty = \{x \in \mathbb{R}^4 : ax = a_0\}$ to infinity, and set $Q_{m+1} := \psi(Q_{m+1}')$. Lemma 5.8 then implies the inductive condition (M1), namely that $Q_{m+1}$ admits an monotone Hamilton path $\pi_{m+1}$. The proof of Theorem 2.5, and so of the Main Theorem, is concluded.
Figure 12: The inductive step: We show the projection of the polytope $Q_4$ to the $\langle 3, 4 \rangle$-plane, and the vertices obtained by intersecting $Q_4$ with $H_5$. The arrows next to the labels 9 and 10 point to the lines about whose slope the corresponding condition in Lemma 5.7 makes an assertion. The line through $O$ is the projection of the 3-plane $H_\infty$. A sweep around $O$ encounters all vertices of $Q_m \cap H_{m+1}$ in the correct order $\pi_m$ prescribed by $\bar{\pi}_{m+1}$. 
6. Acknowledgements

It is a pleasure to thank Günter M. Ziegler for suggesting this problem, and Volker Kaibel for his careful reading of an earlier version of the paper.

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