SPARSITY OF CURVES AND ADDITIVE AND
MULTIPLICATIVE EXPANSION OF RATIONAL MAPS
OVER FINITE FIELDS

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ABSTRACT. For a prime $p$ and a polynomial $F(X,Y)$ over a finite
field $\mathbb{F}_p$ of $p$ elements, we give upper bounds on the number of
solutions

$$F(x,y) = 0, \quad x \in A, \ y \in B,$$

where $A$ and $B$ are very small intervals or subgroups. These bounds
can be considered as positive characteristic analogues of the result
of Bombieri and Pila (1989) on sparsity of integral points on curves.
As an application we prove that distinct consecutive elements in
sequences generated compositions of several rational functions are
not contained in any short intervals or small subgroups.

1. Introduction

1.1. Background. We study some geometric properties of polynomial
maps in finite fields. In particular, we continue investigating the intro-
duced in [11] question of expansion of dynamical systems generated
by polynomial and rational function maps in positive characteristic,
see [5–9, 14–18, 20] and the reference therein for recent results, methods
and applications. Here we consider both additive and multiplicative
expansion, and also study more general compositions of several maps.
This is based on new results of sparsity of rational points on algebraic
curves over finite fields, which can be considered as an analogue of the
celebrated result of Bombieri and Pila [2, Theorem 4] on sparsity of
integral points on curves in characteristic zero.

Let $p$ be a prime number and let $\mathbb{F}_p$ be the finite field of $p$ elements,
represented by the integers $\{0, 1, \ldots, p - 1\}$. For a polynomial $F \in
\mathbb{F}_p[X,Y]$ and sets $A, B \subset \mathbb{F}_p$ write

$$N_F(A, B) = \#\{(a,b) \in A \times B : F(a, b) = 0\}.\]
Our goal is to give bounds on $N_F(A, B)$ for some interesting sets such as intervals or subgroups and in particular, improve the trivial bound

$$N_F(A, B) = O\left(\min\{\#A, \#B\}\right).$$

We are especially interested in the case of the sets of small cardinalities $\#A$ and $\#B$, more specifically in the cases where traditional methods coming from algebraic geometry do not work anymore.

If both $A$ and $B$ are intervals of length $H$ and $F(X, Y)$ is absolutely irreducible, then it is known from the Bombieri bound [1] that

$$N_F(A, B) = \frac{H^2}{p} + O\left(p^{1/2}(\log p)^2\right),$$

where the implied constant depends only on $\deg F$, see [12]. The main term dominates the error term if $H \geq p^{3/2}\log p$ and for $H \leq p^{1/2}(\log p)^2$ the result becomes weaker than the trivial upper bound $N_F(A, B) = O(H)$. For smaller $H$ and for polynomials $F$ having a special form, this question have been studied intensively in the literature, see [5–9, 16, 20] and the reference therein. For example, when $F$ defines a modular hyperbola, $F(X, Y) = X \cdot Y - c$ for some $c \neq 0$, the problem is studied by Cilleruelo and Garaev [7], see also [20], for general quadratic forms. If $F$ defines the graph of a polynomial $F(X, Y) = Y - f(X)$ or an hyperelliptic curve $F(X, Y) = Y^2 - f(X)$, the problem was studied by Cilleruelo, Garaev, Ostafe and Shparlinski [8] and by Chang, Cilleruelo, Garaev, Hernández, Shparlinski, Zumalacárregui [6]. Finally, Chang [5] considered the function of the form $F(X, Y) = f(X) - g(Y)$.

Here we consider the general case of arbitrary bivariate polynomials $F(X, Y)$. Although some of our techniques have already been used, the main novelty of this paper is in investigating the conditions under which these ideas apply and also in exploiting the possible sparsity of the polynomial $F$, see the definition of $\delta(F)$ below. We also give new application to the dynamics of polynomial semigroups.

1.2. Notation. In order to state the result we denote by $\delta(F)$ the number of distinct divisors of the monomial terms of

$$F(X, Y) = \sum_{(i, j) \in \mathcal{F}} F_{i, j} X^i Y^j$$

where $\mathcal{F}$ is the support of the coefficients of $F$, that is, $F_{i, j} \neq 0$ if and only if $(i, j) \in \mathcal{F}$. Alternatively

$$\delta(F) = \#\{(k, \ell) : 0 \leq k \leq i, 0 \leq \ell \leq j \text{ for some } (i, j) \in \mathcal{F}\}.$$
Clearly, we have the following trivial bound
\[ \delta(F) \leq \binom{\deg F + 1}{2}, \]
which is attained for dense polynomials. However for sparse polynomials it can be significantly smaller; for example \( \delta(X^n + Y^n + XY) = 2n + 2 \) for any \( n \geq 1 \).

As usual, we use \( \deg_X F \) and \( \deg_Y F \) for the local degrees of \( F \) with respect to \( X \) and \( Y \), respectively.

We recall that the notations \( U \approx V \), \( U \ll V \) and \( V \asymp U \) are all equivalent to the statement that the inequality \( |U| \leq cV \) holds with some absolute constant \( c > 0 \). We also write \( U \approx d V \) or \( U \ll_{d, \nu} V \) if the implied constants may depend on \( d \) or on \( d \) and \( \nu \).

1.3. Main results. We start with the case of intervals

Theorem 1.1. Let \( F(X, Y) \in \mathbb{F}_p[X, Y] \) be an absolutely irreducible polynomial of degree \( d \geq 2 \) with \( \delta = \delta(F) \). Then for any positive integer
\[ H \leq p^{1/(d-1/2)\delta+1/2}, \]
uniformly over arbitrary intervals \( I = [K, K+H] \) and \( J = [L, L+H] \), we have
\[ N_F(I, J) \leq H^{1/d+o(1)}. \]

For a rational function \( \psi(X) = f(X)/g(X) \) with \( f(X), g(X) \in \mathbb{F}_p[X] \) and for a set \( A \subset \mathbb{F}_p \) write
\[ \psi(A) = \{ \psi(x) : x \in A, g(x) \neq 0 \}. \]

Applying the result to \( F(X, Y) = f(X) - Yg(X) \) with \( \delta(F) \leq 2 \deg F + 2 \), we have the following bound on values of the rational function \( \psi(X) \) in small intervals.

Corollary 1.2. Let \( \psi(X) \in \mathbb{F}_p(X) \) be a rational function of degree \( d \geq 2 \). Then for any positive integer
\[ H \leq p^{1/(2d^2+d-1/2)}, \]
uniformly over arbitrary intervals \( I = [K, K+H] \) and \( J = [L, L+H] \), we have
\[ \#(\psi(I) \cap J) \leq H^{1/d+o(1)}. \]

Here we also consider the case, when \( A \) or \( B \) is a subgroup. We recall that for subgroups \( \mathcal{G}, \mathcal{H} \subset \mathbb{F}_p^* \), Corvaja and Zannier [10] have given a nontrivial on \( N_F(\mathcal{G}, \mathcal{H}) \). Using their result we obtain a bound on \( N_F(I, \mathcal{G}) \) with an interval \( I \) and a subgroup \( \mathcal{G} \). It extends the result.
of Karpinski, Mérai and Shparlinski [15] who bound \( N_F(\mathcal{I}, \mathcal{G}) \) with \( F(X, Y) = f(X) - Yg(X) \), see also [11, 14, 17, 18].

**Theorem 1.3.** Let \( F(X, Y) \in \mathbb{F}_p[X, Y] \) be a polynomial of total degree \( d \) and local degrees \( d_X, d_Y \geq 1 \) such that \( F(X, Y^n) \) is irreducible for all \( n \geq 1 \). Then for any interval \( \mathcal{I} = [1, H] \) of length \( H < p \) and any subgroup \( \mathcal{G} \subset \mathbb{F}_p^* \) of order \( e \), we have

\[
N_F(\mathcal{I}, \mathcal{G}) \ll d_X^{1/2} H^{1/2} \max\{de^{-1/2}, d^{2/3}e^{1/3}, d_X^{1/2}d_Y^2\}.
\]

1.4. **Applications.** As an application of these results we study the geometric properties of the orbits of a transformation \( x \mapsto \psi(x) \) associated with a rational function \( \psi(X) \in \mathbb{F}_p(X) \), which have also being studied in [3, 4, 8, 13]. In fact we consider a much more general scenarios of semigroups generated by a system

\[
\Psi = (\psi_1, \ldots, \psi_s) \in \mathbb{F}_p(X)^s
\]

of \( s \) rational functions.

More precisely, for \( u \in \mathbb{F}_p \), we call any sequence

\[
(1.1) \quad u_0 = u, \quad u_{n+1} = \psi_{j_n}(u_n), \quad n = 0, 1, \ldots,
\]

with arbitrary \( j_n \in \{1, \ldots, s\} \) a path originating from \( u \). Let \( \Pi_{\Psi, u} \) be the set of possible paths originating from \( u \). We denote by \( T_{\Psi, u} \) the largest positive integer \( T \) for which in any path \( \{u_n\} \in \Pi_{\Psi, u} \) the first \( T \) elements are pairwise distinct. Clearly if \( s = 1 \) then \( T_{\Psi, u} \) is the total length of the pre-periodic and periodic parts of the orbit of \( u \) in the dynamical system generated by \( \Psi \).

Given \( u \in \mathbb{F}_p \), we consider the sequence \( \{u_n\} \) as a dynamical system on \( \mathbb{F}_p \) and study how far it propagates in \( N \) steps or how large is the group the first \( N \) elements generate along its paths. In fact, we study more generals quantities

\[
L_{\Psi, u}(N) = \min_{\{u_n\} \in \Pi_{\Psi, u}} \min_{v \in \mathbb{F}_p} \max_{0 \leq n \leq N} |u_n - v|
\]

and

\[
G_{\Psi, u}(N) = \min_{\{u_n\} \in \Pi_{\Psi, u}} \min_{v \in \mathbb{F}_p^*} \#\langle vu_n : n = 0, \ldots, N \rangle.
\]

Previously these quantities have been studied only for \( s = 1 \), here we show that our methods work as well for an arbitrary \( s \).

It has been shown by Gutierrez and Shparlinski [13], in the case of one function, that is, for \( \Psi = (\psi) \), that

\[
L_{\Psi, u}(N) = p^{1+o(1)}
\]
provided that $N \geq p^{1/2+\varepsilon}$ for some fixed $\varepsilon > 0$. Moreover, for linear fractional function $\psi(X) = (aX + b)/(cX + d)$ with $ad \neq bc$ and $c \neq 0$ they obtain nontrivial result for shorter orbit, namely

$$L_{\Psi,u} \gg N^{1+\delta}$$

provided $N \leq \min\{T_{\Psi,u}, p^{1-\varepsilon}\}$. In the case of one polynomial, Cilleruelo, Garaev, Ostafe and Shparlinski [8] obtain a lower bound for essentially arbitrary values of $N$ and $T_{\Psi,u}$.

Here, to exhibit the main ideas in the least technical way, we always assume that the functions $\psi_1, \ldots, \psi_s \in \mathbb{F}_p(X)$ are of the same degree. Note that this condition automatically holds in the classical case of one function.

**Theorem 1.4.** Let

$$\Psi = (\psi_1, \ldots, \psi_s) \in \mathbb{F}_p(X)^s$$

be rational functions of degree $d \geq 2$. Then for $N \leq T_{\Psi,u}$ and any integer $\nu \geq 1$ we have

$$L_{\Psi,u}(N) \gg s,d,\nu \min \left\{ N^{d\nu + o(1)}, p^{1/(2d\nu + d\nu - 1/2)} \right\}.$$

The group size $G_{\Psi,u}(N)$ generated by the first $N$ elements has been investigated [11, 17, 18]. For example, in [17, Theorem 1.1] it is proved that for a polynomial $f(X) \in \mathbb{F}_q[X]$ of degree $d$, satisfying some mild conditions, and for initial value $u \in \mathbb{F}_q$, we have for any $\nu \geq 1$ that

$$G_{\Psi,u}(N) \geq C_1(\nu, d)N^{1/\nu}q^{1-1/\nu}, \quad \text{for } T_{\Psi,u} \geq N \geq C_2(\nu, d)q^{1/2},$$

for some constants $C_1(\nu, d), C_2(\nu, d)$ which may depend only on $\nu$ and $d$. Shparlinski [17] has also given a lower bound on $G_{\Psi,u}(N)$ for much smaller $N$ under the condition that the coefficients of the polynomial are small. Namely, if the polynomial $f(X)$ is defined over $\mathbb{Z}$, then for any prime $p$ and any initial value $u \in \mathbb{F}_p$ we have

$$G_{\Psi,u}(N) \geq \sqrt{(N - 2d) \log p \log((d + 2)h)}, \quad \text{for } T_{\Psi,u} \geq N \geq 2d,$$

where $d$ is the degree and $h$ is the height of $f(X)$, respectively, that is

$$h = \max\{|a_0|, \ldots, |a_d|\} \text{ where } f(X) = a_nX^n + \ldots + a_1X + a_0.$$

Here we bound $G_{\Psi,u}(N)$ for rational functions $\Psi = (\psi_1, \ldots, \psi_s) \in \mathbb{F}_p(X)^s$. If any of the rational function $\psi_i$ is a power, that is $\psi_i(X) = \alpha X^m$, $m \in \mathbb{Z}$, then there are paths contained in small subgroup. Indeed, if $\alpha, u \in \mathcal{G}$ for a small subgroup $\mathcal{G}$, then the path $\{u_n\}$ is in $\mathcal{G}$ with the choice $f_n = i, n = 1, 2, \ldots$. However, if all the rational functions $\psi_i$ are not powers, we can give nontrivial bound on $G_{\Psi,u}$. Note that here we
do not need the simplifying assumption that all functions $\psi_1, \ldots, \psi_s$ are of the same degree.

**Theorem 1.5.** Let

$$\Psi = (\psi_1, \ldots, \psi_s) \in \mathbb{F}_p(X)^s$$

with

$$\psi_i(X) \not= \alpha X^m \quad \alpha \in \mathbb{F}_p, \ m \in \mathbb{Z}, \ i = 1, \ldots, s,$$

be rational functions of degree at most $d \geq 2$. Then for $N \leq T_{\psi,u}$

$$G_{\Psi,u} \gg \min \left\{ \frac{N^{1/2}p^{1/2}}{ds^{1/2}}, \frac{N^{3/2}}{d^2s^{3/2}} \right\}.$$

2. Proof of Theorem 1.1

2.1. Preliminaries. The proof is based on the following bound of Bombieri and Pila [2, Theorem 4].

**Lemma 2.1.** Let $C$ be an absolutely irreducible curve of degree $d \geq 2$ and $H \geq \exp(d^6)$. Then the number of integral points on $C$ and inside of a square $[K, K + H] \times [L, L + H]$ does not exceed

$$H^{1/d} \exp(12\sqrt{d \log H \log \log H}).$$

We recall, that a polynomial $F(X,Y) \in \mathbb{Z}[X,Y]$ is said to be primitive if the greatest common divisor of its coefficients is 1.

**Lemma 2.2.** Let $F(X,Y) \in \mathbb{Z}[X,Y]$ be a primitive polynomial of degree $d = \deg F$ and let $p$ be a prime number. Let $\tilde{F}(X,Y) \in \mathbb{F}_p[X,Y]$ with $F(X,Y) \equiv \tilde{F}(X,Y) \mod p$. Assume, that $\tilde{F}(X,Y)$ has degree $d$ and is absolutely irreducible. Then $F(X,Y)$ is absolutely irreducible.

**Proof.** Assume that $F(X,Y)$ is not absolutely irreducible, that is there is a number field $\mathbb{K}$ such that

$$F(X,Y) = f(X,Y) \cdot g(X,Y), \quad f(X,Y), g(X,Y) \in \mathbb{K}[X,Y].$$

By the multivariable Gauss lemma [19] we can assume, that $f, g \in \mathcal{O}_{\mathbb{K}}[X,Y]$ with $\deg f, \deg g < d$, where $\mathcal{O}_{\mathbb{K}}$ is the ring of integers of $\mathbb{K}$. Let $p$ be a prime ideal in $\mathcal{O}_{\mathbb{K}}$ over $p$. Then $\mathcal{O}_{\mathbb{K}}/p$ is a finite extension of $\mathbb{F}_p$ and $\tilde{F}(X,Y) \equiv f(X,Y) \cdot g(X,Y) \mod p.$ As the degrees of $f(X,Y)$ and $g(X,Y)$ modulo $p$ are strictly less than $d$, we get a nontrivial factorisation of $\tilde{F}(X,Y)$.

$\square$
2.2. Concluding the proof. Put $N = N_F(\mathcal{I}, \mathcal{J})$. Replacing $F(X, Y)$ by $F(X - K_0, Y - L_0)$ for some $K_0, L_0$, we can assume that the congruence

$$F(x, y) \equiv 0 \mod p, \quad |x|, |y| \leq H,$$

has at least $N$ solutions. Covering the square $[-H, H]^2$ by $O(N)$ squares with the side length $2dH/\sqrt{N}$ and shifting the variables again, we can also assume, that there are solutions

\begin{equation}
(x_1, y_1), \ldots, (x_{d^2+1}, y_{d^2+1}), \quad 0 < x_k, y_k \leq 2dH/\sqrt{N}.
\end{equation}

We remark, that the quantity $\delta(F)$ is invariant under linear changes of variable, that is $\delta(F) = \delta(F(X - K_0, Y - L_0))$.

Let $\Delta$ be the set of multiindices $(i, j)$ such that the coefficient of $X^iY^j$ in $F(X, Y)$ is nonzero. By definition we have $\#\Delta \leq \delta$.

Write

$$F(X, Y) = \sum_{(i,j) \in \Delta} F_{i,j}X^iY^j$$

and consider the linear congruence system

\begin{equation}
\sum_{(i,j) \in \Delta} F_{i,j}x_k^iy_k^j \equiv 0 \mod p, \quad k = 1, \ldots, d^2 + 1,
\end{equation}

for the coefficients $F_{i,j}$. This system determines $F(X, Y)$ up to a constant multiple. Indeed, if $G(X, Y)$ is a polynomial of degree at most $d$ whose non-zero coefficients satisfy the system of congruences (2.2), then $G(x_k, y_k) = 0$ for $k = 1, \ldots, d^2 + 1$, then

$$\#\{F(X, Y) = G(X, Y) = 0\} > d^2.$$

As $F(X, Y)$ is absolutely irreducible, by the Bézout theorem we have $F(X, Y) \mid G(X, Y)$. Thus the coefficient matrix of (2.2) of size $(d^2 + 1) \times \delta$ has rank $\delta - 1$ over $\mathbb{F}_p$. We can assume that the first $\delta - 1$ rows are linearly independent over $\mathbb{F}_p$ and thus over $\mathbb{Q}$.

Fix $(k, \ell) \in \Delta$ and let $V \in \mathbb{Z}^{(\delta-1)\times(\delta-1)}$ be a matrix whose columns are indexed by the elements $\Delta \setminus \{(k, \ell)\}$ and the $(m, n)$-th column is

$$V_{(m,n)} = \left( x_1^my_1^n, \ldots, x_{\delta-1}^my_{\delta-1}^n \right)^T.$$

Similarly, for $(i, j) \neq (k, \ell)$, let $U(i, j) \in \mathbb{Z}^{(\delta-1)\times(\delta-1)}$ be a matrix such that its $(m, n)$-th column is

\begin{equation}
U(i, j)_{(m,n)} = \begin{cases}
V_{r,s} & \text{if } (r, s) \neq (i, j), \\
-x_1^{k}y_1^{l}, \ldots, -x_{\delta-1}^{k}y_{\delta-1}^{l} & \text{if } (r, s) = (i, j).
\end{cases}
\end{equation}

Put

\begin{equation}
v = \det V \quad \text{and} \quad u_{i,j} = \det U_{i,j}, \quad (i, j) \neq (k, \ell).
\end{equation}
Then \( v \neq 0 \) and by the Cramer rule

\[
v \cdot F_{i,j} \equiv u_{i,j} \cdot F_{k,\ell} \pmod{p} \quad \text{for} \quad (i, j) \neq (k, \ell)
\]

and by (2.1) and (2.3) we have

\[
|v|, |u_{i,j}| \leq \delta! (2dH/\sqrt{N})^{d(d-1)}.
\]

Write

\[
\bar{F}(X, Y) = \sum_{(i,j) \in \Delta \setminus \{(k, \ell)\}} u_{i,j} X^i Y^j + v X^k Y^\ell \in \mathbb{Z}[X, Y],
\]
then the congruence \( F(x, y) \equiv 0 \pmod{p}, |x|, |y| \leq H \), is equivalent to the congruence

\[
\bar{F}(x, y) \equiv 0 \pmod{p}, \quad |x|, |y| \leq H.
\]

We can write it as the diophantine equation

\[
(2.4) \quad \bar{F}(x, y) = pt, \quad |x|, |y| \leq H, \quad t \in \mathbb{Z}.
\]

All possible values \( t \) satisfies

\[
|t| \leq \frac{1}{p} \left( \sum_{(i,j) \in \Delta \setminus \{(k, \ell)\}} |u_{i,j}| H^{i+j} + |v| H^{k+\ell} \right) \ll_d \frac{(H/\sqrt{N})^{d(d-1)} H^d}{p}
\]

\[
\leq_d \frac{H/\sqrt{N})^{d(d-1)} H^d}{H^{(d-1)/2} N^{d-1/2}} = \frac{H^{d-1/2}}{N^{d-1/2}}.
\]

For each value of \( t \) the polynomial \( \bar{F}(X, Y) - pt \) is absolutely irreducible by Lemma 2.2, thus by Lemma 2.1, the number of integral solutions of (2.4) inside the box \([-H, H] \times [-H, H]\) is at most \( H^{1/d+o(1)} \), thus

\[
N \leq \left( \frac{H^{(d-1)/2}}{N^{d-1/2}} \right) H^{1/d+o(1)}
\]

which implies \( N \leq H^{1/d+o(1)} \) as \( H \to \infty \).

3. Proof of Theorem 1.3

3.1. Preliminaries. The proof of the theorem is based on the result of Corvaja and Zannier [10, Corollary 2] in a form given by Karpinski, Mérai and Shparlinski [15].

Lemma 3.1. Assume that \( F(X, Y) \in \mathbb{F}_p[X, Y] \) is of degree \( \deg F = d \) and does not have the form

\[
(3.1) \quad \alpha X^m Y^n + \beta \quad \text{or} \quad \alpha X^m + \beta Y^n.
\]
For any multiplicative subgroups $\mathcal{G}, \mathcal{H} \subset \mathbb{F}_{p}^\ast$, we have

$$N_{F}(\mathcal{G}, \mathcal{H}) \ll \max \left\{ \frac{d^{2}W}{p}, d^{4/3}W^{1/3} \right\},$$

where

$$W = \#\mathcal{G}\#\mathcal{H}.$$  

We need the following result on the non-vanishing of some resultant.

**Lemma 3.2.** Let $F(X, Y) \in \mathbb{F}_{p}[X, Y]$ such that $F(X, Y^n)$ is irreducible for all $n \geq 1$ with local degrees $d_{X}, d_{Y} \geq 1$. Then there is a set $\mathcal{E} \subset \mathbb{F}_{p}^\ast$ of cardinality $\#\mathcal{E} \ll (d_{X}d_{Y})^{2}$ such that for $a \in \mathbb{F}_{p}^\ast \setminus \mathcal{E}$ the resultant

$$R_{a}(U, V) = \text{Res}_{X} (F(X, U), F(X + a, V))$$

with respect to $X$, is not divisible by a polynomial of the form (3.1).

**Proof.** If $R_{a}(U, V)$ is divisible by a polynomial of form (3.1), then the variety

$$(3.2) \quad F(X, Y^{m}) = 0 \quad \text{and} \quad F(X + a, bY^{n}) = 0$$

has positive dimension for some $b$ and some integers $m, n$.

If $mn = 0$ the result is trivial. Replacing $F(X, Y)$ by $Y^{d_{Y}}F(X, 1/Y)$, we can assume that $m > 0$.

First assume, that $n > 0$. As both polynomials are irreducible, they are equal up to a constant factor $\alpha \in \mathbb{F}_{p}^\ast$:

$$F(X, Y^{m}) = \alpha F(X + a, bY^{n})$$

and thus $m = n$. Write

$$F(X, Y) = \sum_{i=0}^{d_{Y}} f_{i}(X)Y^{i}.$$

Let $h$ be the maximal index such that $f_{h}(X)$ is not constant. Then

$$f_{h}(X)Y^{hm} = \alpha f_{d_{Y}}(X + a)b^{h}Y^{hn},$$

so there are at most $\deg f_{h}(X) \leq d_{X}$ choices for $a$ and $h \leq d_{Y}$ choices for $b$.

If $n < 0$, then (3.2) is equivalent to

$$F(X, Y^{m}) = 0 \quad \text{and} \quad (bY^{n})^{d_{Y}}F(X + a, bY^{n}) = 0,$$

and we get in the same way as before that there are at most $d_{X}$ choices for $a$ and $d_{Y}$ choices for $b$.

As $m \leq \deg_{U} R_{a} \ll d_{X}$ and $n \leq \deg_{V} R_{a} \ll d_{Y}$ we get the result. $\square$
3.2. **Concluding the proof.** We now closely follow the proof of [15, Lemma 5.1].

Denote \( N = N_F(I, G) \). Let \( T = [-H, H] \). Then the system of equations

\[
F(x, u) = 0, \quad F(x + y, v) = 0, \quad x \in I, \ y \in T, \ u, v \in G
\]

has at least \( N^2 \) solutions. Let \( N_y \) be the number of solutions with a fixed \( y \). Then

\[
\sum_{y \in T} N_y \geq N^2.
\]

Let

\[
L = \frac{N^2}{2(2H + 1)}
\]

and write

\[
Y = \{y \in T : N_y > L\}.
\]

Using that \( N_y \leq d_T^2 H \) we write

\[
d_T^2 H \# Y \geq \sum_{y \in T, N_y > L} N_y \geq N^2 - \sum_{y \in I, N_y \leq L} N_y \geq N^2 - (2H + 1)L \geq \frac{1}{2} N^2.
\]

Let \( E \) as in Lemma 3.2. If \( \# Y \leq \# E \), then

\[
N^2 \leq 2H \# Y \ll d_X d_Y^4 H
\]

so \( N \ll d_X d_Y^2 \sqrt{H} \). Thus we can assume, that \( \# Y \geq \# Y \). Then fix an \( a \in Y \setminus E \) and consider the system of equations

\[
F(x, u) = 0 \quad \text{and} \quad F(x + a, v) = 0, \quad x \in I, \ u, v \in G.
\]

Then for \( R_a(U, V) = \text{Res}_X (F(X, U), F(X + a, V)) \), we have \( R_a(u, v) = 0 \) for each solution \( u, v \). By Lemma 3.2, \( R_a(U, V) \) is not divisible by a polynomial of form (3.1), thus by Lemma 3.1 we have

\[
L < N_a \leq d_X F \# \{(u, v) \in G \times G : R_a(u, v) = 0\}
\]

\[
\ll d_X \max \left\{ \frac{d^2 c^2}{p}, d^{1/3} c^{2/3} \right\}
\]

as for a fixed \( u \) there are at most \( d_X \) possible values for \( x \). Recalling the definition of \( L \), we obtain

\[
N \ll d_X^{1/2} H^{1/2} \max\{d e p^{-1/2}, d^{2/3} e^{1/3}\},
\]

which concludes the proof.
4. Proof of Theorem 1.4

Consider a path \( \{u_n\} \in \Pi_{\psi, u} \) and assume that \( \psi_{j_1}, \ldots, \psi_{j_N} \) is the sequence of rational functions used to generate the first \( N \) elements \( u_1, \ldots, u_N \) as in (1.1). Let \( \psi_{i_1}, \ldots, \psi_{i_1} \) be the most frequent sequence of rational functions among \( \psi_{j_{n+\nu-1}}, \ldots, \psi_{j_n} \) for \( n = 0, \ldots, N - \nu \), and let \( \mathcal{N} \) be the set of corresponding values of \( n \).

Clearly, \( \mathcal{N} \) is of cardinality at least

\[
\#\mathcal{N} \geq \frac{N - \nu}{s^\nu}.
\]

Denote by \( \psi \) the composition

\[
\psi(X) = \psi_{i_1} (\ldots (\psi_{i_1}(X))).
\]

Since

\[
\mathcal{N} \subseteq [1, N - \nu] \subseteq [1, T_{\psi, u} - \nu],
\]

the pairs \( (u_n, u_{n+\nu}) = (u_n, \psi^{(\nu)}(u_n)) \) are all pairwise distinct for \( n \in \mathcal{N} \) and belong of the square

\[
[v - L_{\psi, u}(N), v + L_{\psi, u}(N)] \times [v - L_{\psi, u}(N), v + L_{\psi, u}(N)].
\]

Thus by Corollary 1.2 applied to the iteration \( \psi^{(\nu)} \) we have

\[
\mathcal{N} \leq L_{\psi, u}(N)^{1/d^\nu + o(1)}
\]

as \( N \to \infty \), and recalling (4.1) we conclude the proof.

5. Proof of Theorem 1.5

Let \( \mathcal{G} \) be the group generated by \( \{vu_n : n = 0, \ldots, N\} \). We now proceed as in the proof of Theorem 1.4 with \( \nu = 1 \). Namely, let \( \psi \) be the most frequent function among \( \psi_{j_n} \) for \( n = 1, \ldots, N - \nu \), and let \( \mathcal{N} \) be set of corresponding values of \( n \).

Then \( (vu_n, vu_{n+1}) = (vu_n, \psi(u_n)) \in \mathcal{G} \times \mathcal{G} \) for \( n \in \mathcal{N} \) thus by Lemma 3.1 and (4.1) we have

\[
N/s \ll \max \left\{ \frac{d^2 \#\mathcal{G}^2}{p}, d^{4/3} \#\mathcal{G}^{2/3} \right\}
\]

and the result follows.

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