Spectrum of the fixed point Dirac operator in the Schwinger model*

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Recently, properties of the fixed point action for fermion theories have been pointed out indicating realization of chiral symmetry on the lattice. We check these properties by numerical analysis of the spectrum of a parametrized fixed point Dirac operator investigating also microscopic fluctuations and fermion condensation.

1. INTRODUCTION

Ginsparg and Wilson [1] provided an explicit condition (GWC) for the realization of the chiral symmetry on the lattice,

\[ \frac{1}{2} \{ D, \gamma^5 \} = D \gamma^5 R D \ , \tag{1} \]

where \( D \) is the lattice Dirac operator and \( R \) is an operator acting on space-time and color indices; the \( O(a) \) violation of the symmetry is local since \( R \) is. The GWC ensures the restoration of the main properties of the continuum theory related to chirality [1–5].

Recently it has been realized [2] that the fixed point (FP) action of a block-spin transformation (BST) is a solution of the GWC. An independent solution comes from the overlap formulation of chiral fermions, Neuberger’s lattice Dirac operator [6], obtained by a projection of the original Wilson operator with negative quark mass.

Here we check the expected classical and quantum chiral features of the FP Dirac operator of the Schwinger model in a parametrized form [7]. The cut-off of the less localized couplings in the parametrization is expected to introduce deviations from the ideal behavior. Some of the results presented here appeared already (with lower statistics) in [8].

2. GINSPARG-WILSON FERMIONS

For a non-overlapping BST, the operator \( R \) in (1) is diagonal: \( R = \frac{1}{2} \), and (1) assumes the form (shared also by Neuberger’s operator)

\[ D + D^\dagger = D^\dagger D = D D^\dagger \ . \tag{2} \]

As a consequence of (2), \( D \) is a normal operator, its spectrum lies on the circle \(|\lambda - 1| = 1\) in the complex plane and all real modes have definite chirality. Moreover a ‘lattice’ Index Theorem (IT) holds [3]:

\[ Q_{FP} = \sum_{\{v_0\}} \langle v_0, \gamma^5 v_0 \rangle , \tag{3} \]

where \( Q_{FP} \) is the FP topological charge of the background gauge configuration and \( \{v_0\}\) the set of zero modes of \( D \).

The GWC allows the definition of a subtracted fermion condensate [4] representing (for the number of fermion species \( N_f > 1 \)) an order parameter for the spontaneous breaking of the symmetry,

\[ \langle \bar{\psi} \psi \rangle_{\text{sub}} = - \frac{1}{V} \left\langle \text{tr}(\tilde{D}^{-1}) \right\rangle_{\text{gauge}} . \tag{4} \]

Because of (2), the redefined fermion matrix \( \tilde{D} = D (1 - D/2)^{-1} \) is anti-hermitian and so its spectrum \( \{\lambda\} \) is purely imaginary. The corresponding spectral density \( \tilde{\rho}(\lambda) = 1/V(dN/d\lambda) \) complies with the Banks-Casher formula:

\[ \langle \bar{\psi} \psi \rangle_{\text{sub}} = - \pi \tilde{\rho}(0) . \tag{5} \]

\( \tilde{D} \) anti-commutes with \( \gamma_5 \) and so the microscopic fluctuations of its spectrum are expected

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to be described by the chiral Random Matrix Theory (chRMT) which in the case of irreducible matrices predicts just three classes of universality corresponding to orthogonal, unitary or symplectic ensembles distributed according to the Gaussian measure (chGOE, chGUE chGSE respectively). The universal behavior can be probed through the microscopic spectral density 
\[ \rho_m(z) = \lim_{V \to \infty} \frac{1}{\Sigma} \sum \tilde{\rho} \left( \frac{z}{\sqrt{\Sigma}} \right), \]
(with \( \Sigma = -\langle \bar{\psi} \psi \rangle \)) and the distribution of the smallest eigenvalue \( P(z_{\min}) \).

3. RESULTS

The parametrized (fermion) FP action considered here has the usual form in terms of paths (and loops) of link variables; in our case these are enclosed in a 7 × 7 lattice, with altogether 429 terms per site. It was determined for non-compact gauge fields. For the present context we studied the associated Dirac operator \( D_p \) for samples of 5000 configurations on a 16^2 lattice, generated according to both compact and non-compact standard Wilson gauge action (quenched generation) for \( \beta = 2, 4, 6 \) (for reason of space we report the results mostly for the compact case). The unquenching has been realized by properly including the fermion determinant in the observables.

3.1. Spectrum

The spectrum of \( D_p \) is close to circular, somewhat fuzzy at small values of \( \beta \leq 2 \) but excellently living up to the theoretical expectations at large gauge couplings \( \beta \geq 4 \). The dispersion \( \sigma \) of the eigenvalues around the circle follows the law
\[ \sigma \propto \frac{1}{\beta^{2.41}} \approx a^5 \]

Already at low values of \( \beta \) there is a clear distinction between real modes accumulating around 0 and around 2 (ideally 0 and 2 are the only possible real values). We checked their chirality finding a distribution sharply peaked around -1 and 1 for \( \beta \geq 4 \). Each (quasi) zero mode has either three (20% of cases at \( \beta = 6 \)) or one companion, the overall chirality of real modes being zero.

We checked the IT adopting the geometric definition for the topological charge of the background gauge configuration (in principle one

should use the FP topological charge: the two definitions agree for smooth configurations). We obtain 97% of successes already at \( \beta = 2 \) and 100% for \( \beta \geq 4 \).

3.2. Fermion condensate

The infinite volume fermion condensate \( \langle \bar{\psi} \psi \rangle \) can be obtained from the spectral density \( \tilde{\rho}(\lambda) \) through the Banks-Casher formula (5). In Fig. 1 we report our results for the spectral density \( \tilde{\rho}(\lambda) \) comparing to the value of \( \tilde{\rho}(0) \) expected according to \( \langle \bar{\psi} \psi \rangle \) of the continuum.

We also determined the finite volume condensate \( \langle \bar{\psi} \psi \rangle_V \) by use of the direct definition (4), obtaining (lattice units) 0.072(7) for \( \beta = 4 \) and 0.063(3) for \( \beta = 6 \) (continuum result for the corresponding physical volume [12]: 0.080 and 0.065).

3.3. Microscopic fluctuations

We report here results just for the quenched \( (N_f = 0) \) situation. In Fig. 2 we present our outcomes for \( P(z_{\min}) \) for the \( \nu = 0 \) and \( |\nu| = 1 \) topological sectors (the topological charge has been counted according to the number of zero modes of \( D_p \)). For the trivial sector we can compare with the three variants [13] predicted by the chRMT: the chGSE seems to give the best agreement.
Figure 2. Distribution of the smallest eigenvalue $P(z_{\text{min}})$: $\beta = 4$, lattice size $16^2$, $\nu = 0$ (thick full lines) and $|\nu| = 1$. We compare with the predictions of the chRMT: chGOE (dotted line), chGUE (dashed line), chGSE (full line).

We also checked different lattices and values of $\beta$ finding consistency with the expected universality. The results for the non-compact gauge ensemble, $\nu = 0$, agree within the statistical errors. The topological excitations seem to produce just a shift of the distribution by $|\nu|$.

Fig. 2 reports the microscopic spectral density $\rho_m(z)$ for the non-compact gauge field ensemble (in the $\nu = 0$ sector). Here we find disagreement with all three predictions of chRMT, indicating possible problems with some of the assumptions (as maybe irreducibility of the matrices).

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