Rank 4 vector bundles on the quintic threefold

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Abstract

By the results of the author and Chiantini in [3], on a general quintic threefold $X \subset \mathbb{P}^4$ the minimum integer $p$ for which there exists a positive dimensional family of irreducible rank $p$ vector bundles on $X$ without intermediate cohomology is at least three. In this paper we show that $p \leq 4$, by constructing series of positive dimensional families of rank 4 vector bundles on $X$ without intermediate cohomology. The general member of such family is an indecomposable bundle from the extension class $\text{Ext}^1(E, F)$, for a suitable choice of the rank 2 ACM bundles $E$ and $F$ on $X$. The existence of such bundles of rank $p = 3$ remains under question.

1 Introduction

Let $X \subset \mathbb{P}^4$ be a smooth quintic hypersurface and let $E$ be a rank 2 vector bundle without intermediate cohomology, i.e. such that

$$h^i(X, E(n)) = 0$$

(1.1)

for all $n \in \mathbb{Z}$ and $i = 1, 2$. In [6] we found all the possible Chern classes of an indecomposable rank 2 vector bundle satisfying condition (1.1). Moreover in [8] we showed, when $X$ is general, if such bundles exist then they are all infinitesimally rigid, i.e. $\text{Ext}^1(E, E) = 0$.

On the other hand it was showed in [2] the existence of infinitely many isomorphism classes of irreducible vector bundles without intermediate cohomology on any smooth hypersurface $X_r$ of degree $r \geq 3$ in $\mathbb{P}^4$. It can be checked that when the hypersurface is general then the rank of these bundles is $2^3$. Hence we introduced in [3] the number

$$\text{BGS}(X_r)$$

defined as the minimum positive integer $p$ for which there exists a positive dimensional family of irreducible rank $p$ vector bundles without intermediate cohomology on $X_r$.

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Then combining the above quoted results we get, on a general quintic $X$, that

$$3 \leq BGS(X) \leq 8. \quad (1.2)$$

In this paper we show the following:

**Theorem 1.1.** If $X$ is general then $BGS(X) \leq 4$.

We should then answer the following:

**Question 1.2.** Let $X$ be a general quintic hypersurface in $\mathbb{P}^4$. Could it be $BGS(X) = 3$?

To show our main result we give examples of rank 4 vector bundles without intermediate cohomology, which are not infinitesimally rigid.

The examples are constructed by means of extension classes

$$0 \to E_1 \to \mathcal{E} \to E_2 \to 0 \quad (1.3)$$

i.e. elements in $Ext^1(E_2, E_1)$, where $E_1$ and $E_2$ are rank 2 bundles on $X$. When the bundles $E_1$ and $E_2$ are not split then $\mathcal{E}$ has no trivial summand. Moreover for a suitable choice of bundles $E_1$ and $E_2$, there exists a non-trivial extension class such that the rank 4 bundle $\mathcal{E}$ which corresponds to this class does not split as a direct sum of two rank 2 bundles, for reason of Chern classes. Of course if $E_1$ and $E_2$ have no intermediate cohomology it is so also for $\mathcal{E}$. We then conclude by direct calculations to make the right choice of bundles $E_1$ and $E_2$.

## 2 Generalities

We work over the complex numbers $\mathbb{C}$ and we denote by $X \subset \mathbb{P}^4$ a smooth hypersurface of degree 5 in $\mathbb{P}^4$. Since $Pic(X) \cong \mathbb{Z}[H]$ is generated by the class of a hyperplane section, given the vector bundle $E$ we identify $c_1(E)$ with the integer number $c_1$ which corresponds to $c_1(E)$ under the above isomorphism. We identify $c_2$ with $\text{deg} c_2(E) = c_2(E) \cdot H$. If $E$ is a rank $k$ vector bundle on $X$ we denote by $E(n) = E \otimes \mathcal{O}_X(n)$.

**Definition 2.1.** A rank $k$ vector bundle $E$ is called arithmetically Cohen-Macaulay (ACM for short) if $E$ has no intermediate cohomology, i.e.

$$h^i(E(n)) = 0 \quad (2.1)$$

for all $i = 1, 2$, and $n \in \mathbb{Z}$.

Theorem will follow by:

**Proposition 2.2.** Let $X$ be a smooth quintic hypersurface in $\mathbb{P}^4$. Then, there exist indecomposable rank 2 vector bundles $E_1$ and $E_2$ on $X$ without intermediate cohomology such that there exists an open subset of a positive dimensional projective space parameterizing extension classes $Ext^1(E_2, E_1)$ which correspond to infinitely many isomorphism classes of irreducible rank 4 vector bundles $\mathcal{E}$ on $X$ without intermediate cohomology.
A proof of previous proposition will be given in the next section.

We will frequently use the following version of Riemann-Roch theorem for vector bundles:

**Theorem 2.3.** If $E$ is a rank 2 vector bundle on a smooth hypersurface $X \subset \mathbb{P}^4$ of degree 5 with Chern classes $c_i(E) = c_i \in \mathbb{Z}$ for $i = 1, 2$, then

$$
\chi(E) = \frac{5}{6} c_3^4 - \frac{1}{2} c_1 c_2 + \frac{25}{6} c_1
$$

(2.2)

3 The examples

In this section we will give a proof of Proposition 2.2 which is a direct consequence of Proposition 3.1 and Theorem 3.4 below. As in [3] given a rank 2 vector bundle $E$ we introduce the non negative integer

$$
b(E) = \max \{n \mid h^0(E(-n)) \neq 0\}.
$$

(3.1)

We say that the vector bundle $E$ is normalized if $b(E) = 0$. Notice that changing $E$ by $E(-b)$ we may always assume that $E$ is normalized. The rank two bundle $E$ is semistable if $2b - c_1 \leq 0$. If $2b - c_1 < 0$ then $E$ is stable.

All the possible Chern classes of irreducible rank 2 ACM bundles are listed in the following (see [6] and [3]):

**Proposition 3.1.** Let $E$ be a normalized and indecomposable rank 2 ACM bundle on a smooth quintic $X$. Then

$$(c_1, c_2) \in A \cup B$$

where

$$A = \{(−2, 1), (−1, 2), (0, 3), (0, 4), (0, 5), (1, 4), (1, 6), (1, 8), (4, 30)\}$$

and $B = \{(2, \alpha), (3, 20)\}$ with $\alpha = 11, 12, 13, 14$. When $X$ is general, all the case in $A$ arise on $X$ and moreover for all the pairs $(c_1, c_2) \in A \cup B$ the corresponding rank 2 ACM bundles are infinitesimally rigid i.e. $\text{Ext}^1(E, E) = 0$.

Below we shall construct examples of rank 4 bundles $G$ as extensions of type

$$0 \to F(m) \to G \to E \to 0, \quad (3.2)$$

where $m \leq 0$, and $F$ and $E$ are indecomposable and normalized rank 2 ACM bundles on $X$ with Chern classes as in Proposition 3.1. Such nontrivial extensions $G$ will exist whenever the extension space $\text{Ext}^1(E, F(m))$ has positive dimension, i.e. $h^1(F(m) \otimes E^*) > 0$. By the long exact sequence of cohomology of (3.2), any such extension $G$ has vanishing intermediate cohomology since $F$ and $E$ are ACM.
Lemma 3.2. Let $E$ and $F$ be two normalized and indecomposable rank 2 ACM bundles on the smooth quintic $X$, and suppose that $h^0(F^\vee(\chi_c(E) - m)) = 0$ (hence $\chi_c(E) - \chi_c(F) - m < 0$ since $F$ is normalized). Then for any zero-locus $C \subset X$ of a global section of $E$

$h^0(3C(c_1(E)) \otimes F^\vee(-m)) = 0$.

Moreover, if $h^0(F^\vee(-m)) = 0$ (hence $-m - \chi_c(F) < 0$ since $F$ is normalized) then

$h^3(F(m) \otimes E^\vee) = h^0(E \otimes F^\vee(-m)) = 0$.

Proof. From the tensored by $F^\vee(-m + \chi_c(E))$ ideal sheaf sequence of $C \subset X$:

$0 \to 3C(c_1(E)) \otimes F^\vee(-m) \to F^\vee(c_1(E) - m) \to \mathcal{O}_C(c_1(E)) \otimes F^\vee(-m) \to 0$

we get $h^0(3C(c_1(E)) \otimes F^\vee(-m)) \leq h^0(F^\vee(c_1(E) - m)) = 0$. The rank 2 bundle $E$ fits in the exact sequence

$0 \to \mathcal{O}_X \to E \to 3C(c_1(E)) \to 0; \quad (3.3)$

and after tensoring by $F^\vee(-m)$ we get

$0 \to F^\vee(-m) \to E \otimes F^\vee(-m) \to 3C(c_1(E)) \otimes F^\vee(-m) \to 0$.

Therefore, since $h^0(F^\vee(-m)) = h^0(3C(c_1(E)) \otimes F^\vee(-m)) = 0$ then

$h^0(E \otimes F^\vee(-m)) = 0$, and by duality $h^3(F(m) \otimes E^\vee) = 0$.

Remark 3.3. Let $E$ and $F$ be in $\chi_c$, and suppose that $\chi(F(m) \otimes E^\vee) < 0$. Then by the above lemma, the space of extensions $\chi_c$ will be no-empty since $h^1(F(m) \otimes E^\vee) = h^0(F(m) \otimes E^\vee) + h^2(F(m) \otimes E^\vee) - \chi(F(m) \otimes E^\vee) > 0$.

More generally the argument used here works whenever

$h^3(F(m) \otimes E^\vee) < -\chi(F(m) \otimes E^\vee)$.

In the following table we summarize the cases, which we are interested in, depending on the Chern classes of the bundles $E$ and $F$. To get the value of $\chi(F(m) \otimes E^\vee)$ we used Schubert package (see [5]), and then by the Lemma we derived the lower bound for $d$.

| Case | $(c_1(F), c_2(F))$ | $(c_1(E), c_2(E))$ | $\chi(F(m) \otimes E^\vee)$ | $m$ | $d$ |
|------|------------------|------------------|------------------|-----|-----|
| (1)  | (4,30)           | (1,8)            | -14              | 0   | >14 |
| (2)  | (4,30)           | (0,3)            | -6               | -1  | >6  |
| (3)  | (4,30)           | (0,4)            | -8               | -1  | >8  |
| (4)  | (4,30)           | (0,5)            | -10              | -1  | >10 |
| (5)  | (1,8)            | (0,3)            | -1               | 0   | >1  |
| (6)  | (1,8)            | (0,4)            | -2               | 0   | >2  |
| (7)  | (1,8)            | (0,5)            | -3               | 0   | >3  |
We are now ready to show the following:

**Theorem 3.4.** Let $X$ be a smooth quintic in $\mathbb{P}^4$, and let $E, F, m, d$ be as in the above table. Then in each of the cases (1) – (7) there exists a $d$-dimensional parameter space of extensions (3.2), with a general element $G$ an indecomposable rank 4 vector bundle on $X$ without intermediate cohomology.

**Proof.** For $F, E$ as in the above table, the dimension $d = \dim \text{Ext}^1(E, F(m))$ is always $d > 1$. Therefore for such $F, E$ there exist nontrivial extensions given by (3.2), and let $G$ be one of them.

Since $E$ and $F$ are ACM then by the cohomology sequence of (3.2) $G$ is without intermediate cohomology, and by Remark 3.3 we need only to show that $G$ is indecomposable.

Suppose the contrary, i.e. that $G$ splits. Then either

(i) $G = \mathcal{O}_X(a) \oplus G_1$ for $a \in \mathbb{Z}$ and $G_1$ a rank 3 bundle without intermediate cohomology, or

(ii) $G = G_1 \oplus G_2$ for two rank 2 ACM bundles $G_1$ and $G_2$.

We show that under the conditions of the theorem both cases (i) and (ii) are impossible.

Let us start with case (i). In this case the exact sequence (3.2) reads as

$$0 \longrightarrow F(m) \xrightarrow{f} \mathcal{O}_X(a) \oplus G_1 \xrightarrow{g} E \longrightarrow 0.$$  

(3.4)

We use the following (see below for a proof)

**Lemma.** Under the above conditions either $h^0(E(-a)) = 0$ or $h^0(F(-m-c_1(F)+a)) = 0$.

Suppose $h^0(E(-a)) = 0$. Then by the exact sequence (3.2) tensorized by $\mathcal{O}_X(-a)$ we have $h^0(F(m-a)) > 0$. Let $s$ be a non trivial global section of $F(m-a)$, then we have a map

$$s : \mathcal{O}_X(a) \rightarrow F(m).$$

Let $j : \mathcal{O}_X(a) \oplus G_1 \rightarrow \mathcal{O}_X(a)$ be the projection. Then we have the composition map

$$\varphi := j \circ f \circ s : \mathcal{O}_X(a) \rightarrow F(m) \rightarrow \mathcal{O}_X(a).$$

Then $\varphi \in H^0\mathcal{O}_X \cong \mathbb{C}$ and hence it is either the identity map or the zero map. If this map is the identity then $j \circ f$ is surjective and hence $\ker(j \circ f) \cong \mathcal{O}_X(b)$ for some $b \in \mathbb{Z}$. Then we have exact sequence

$$0 \rightarrow \mathcal{O}_X(b) \rightarrow F(m) \rightarrow \mathcal{O}_X(a) \rightarrow 0$$

and $F(m)$, and hence also $F$, splits since $\dim \text{Ext}^1(\mathcal{O}_X(a), \mathcal{O}_X(b)) = 0$, which is absurd.

Now suppose $\varphi$ is zero. Then $j \circ f$ is zero. Thus the image of $F(m)$ in exact sequence (3.4) is contained in $G_1$. Then the kernel of $g$ is contained in $G_1$, being
equal to the image of \( f \). Let \( i : \mathcal{O}_X(a) \to \mathcal{O}_X(a) \oplus \mathcal{G}_1 \) be the inclusion. By the assumption the map \( g \circ i : \mathcal{O}_X(a) \to E \) is the zero map, which means that \( \ker g \) is not contained in \( \mathcal{G}_1 \), which is absurd.

Suppose now that \( h^0 F(-m - c_1(F) + a) = 0 \) and consider the dual exact sequence of exact sequence (3.4)

\[
0 \to \mathcal{E}^\vee \to \mathcal{O}_X(-a) \oplus \mathcal{G}_1^\vee \to \mathcal{F}^\vee(-m) \to 0.
\]

(3.5)

Set \( c = c_1(F) \) and \( c' = c_1(E) \). Since \( \mathcal{E}^\vee \cong \mathcal{E}(c') \) and \( \mathcal{F}^\vee(-m) \cong \mathcal{F}(-c - m) \) the above exact sequence reads as

\[
0 \to \mathcal{E}(c') \to \mathcal{O}_X(-a) \oplus \mathcal{G}_1^\vee \to \mathcal{F}(-c - m) \to 0.
\]

This exact sequence tensorized by \( \mathcal{O}_X(a) \) reads as

\[
0 \to \mathcal{E}(c' + a) \to \mathcal{O}_X \oplus \mathcal{G}_1^\vee(a) \to \mathcal{F}(-c - m + a) \to 0.
\]

Then \( h^0 \mathcal{E}(c' + a) > 0 \) and a non trivial global section \( s \) of \( \mathcal{E}(c' + a) \) gives a non zero map

\[
s : \mathcal{O}_X(-a) \to \mathcal{E}(c').
\]

Arguing as above this implies that \( \mathcal{E} \) splits which is absurd.

Then to finish the proof that case (i) can not arise we have to show the lemma.

**Proof of the Lemma.** If \( h^0 \mathcal{E}(-a) > 0 \), since \( \mathcal{E} \) is normalized then \(-a \geq 0\) i.e. \( a \leq 0 \). Suppose that \( h^0 \mathcal{F}(-m - c + a) > 0 \). Since \( \mathcal{F} \) is normalized then \(-m - c + a \geq 0 \). Then from conditions \(-m - c + a \geq 0 \) and \(-a \geq 0 \) we derive condition \( c + m \leq 0 \) which is absurd since by hypotheses we have condition \( c + m > 0 \) (see the table).

To show the theorem it remains now to consider the case (ii) i.e. when \( \mathcal{G} \) has an indecomposable summand which is ACM of rank equal to 2, i.e. when

\[
\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2
\]

(3.6)

with both \( \mathcal{G}_i \) ACM of rank equal to 2. Of course, we may assume that \( \mathcal{G}_i \) are both indecomposable otherwise we reduce to the case (i) above. Then we have non trivial extension class

\[
0 \longrightarrow \mathcal{F}(m) \stackrel{f}{\longrightarrow} \mathcal{G}_2 \oplus \mathcal{G}_1 \stackrel{g}{\longrightarrow} \mathcal{E} \longrightarrow 0.
\]

(3.7)

The extension class \( \mathcal{E} \) is non trivial by assumption. Moreover one has

\[
c_1(\mathcal{G}_i) \notin \{c_1(\mathcal{F}(m)), c_1(\mathcal{E})\}
\]

(3.8)

for \( i = 1, 2 \), by the corollary to Lemma 1.2.8 in [7]. Indeed, suppose that \( c_1(\mathcal{G}_i) \in \{c_1(\mathcal{F}(m)), c_1(\mathcal{E})\} \) for at least on \( i = 1, 2 \). Here we note that at least one of the bundles \( \mathcal{F}(m) \) and \( \mathcal{E} \) is stable. Hence \( \mathcal{G}_i \)'s are semistable and one
of these is always stable. Then from the above exact sequence we have map between semistable bundles of the same rank with the same first Chern class where at least one is stable. Therefore this map is an isomorphism and hence the extension class is trivial, which is absurd.

Then to show that the splitting of $\mathcal{E}_0$ can not arise we will use Proposition 3.1 and a direct computation on the Chern classes. It will show that the only possibility is that the extension class $\mathcal{E}_0$ is trivial, which is absurd, since by assumption $\mathcal{G}$ is represented by a non trivial class in $\text{Ext}^1(E, F(m))$.

To start with, we notice that the bundle $\mathcal{G}$ of $\mathcal{E}_0$ is normalized since so are $F$ and $E$, and $m \leq 0$. In particular also $\mathcal{G}_1$ and $\mathcal{G}_2$ are normalized.

Then we consider all the possible splitting type of $\mathcal{G}$ under condition $\mathcal{E}_0$ in all the cases (1)-(7) of the table. Some of these decompositions are easy to show to be impossible, so we give here only the cases, which require some more computations.

Case (1). In this case (see the table)

$$(c_1(\mathcal{G}), c_2(\mathcal{G})) = (5, 58).$$

By Proposition 3.1 and by condition $\mathcal{E}_0$ if $\mathcal{G} \cong \mathcal{G}_1 \oplus \mathcal{G}_2$ splits then $c_2(\mathcal{G}_1) = 20$ and $c_2(\mathcal{G}_2) = \alpha$, with $\alpha = 11, 12, 13, 14$, are the only possible cases. A direct calculation on the Chern classes shows in these cases $(c_1(\mathcal{G}), c_2(\mathcal{G})) \neq (5, 58)$.

Case (2). In this case

$$(c_1(\mathcal{G}), c_2(\mathcal{G})) = (2, 18).$$

By Proposition 3.1 and by condition $\mathcal{E}_0$ if $\mathcal{G} \cong \mathcal{G}_1 \oplus \mathcal{G}_2$ then we have only two possibilities: either $(c_1(\mathcal{G}_1), c_2(\mathcal{G}_1)) = (2, 14)$ and $(c_1(\mathcal{G}_2), c_2(\mathcal{G}_2)) = (0, 4)$ or $(c_1(\mathcal{G}_1), c_2(\mathcal{G}_1)) = (2, 13)$ and $(c_1(\mathcal{G}_2), c_2(\mathcal{G}_2)) = (0, 5)$. The first case is impossible since by Riemann-Roch theorem we have $h^0F(-1) + h^0(E) = 1 < h^0(\mathcal{G}_1) + h^0(\mathcal{G}_2) = 2$. The second case is also impossible since one computes $h^0(\mathcal{G}_1) + h^0(\mathcal{G}_2) = 3$.

Case (3). In this case we have

$$(c_1(\mathcal{G}), c_2(\mathcal{G})) = (2, 19).$$

If $\mathcal{G} \cong \mathcal{G}_1 \oplus \mathcal{G}_2$ by Proposition 3.1 and by condition $\mathcal{E}_0$ we could have possible cases $(c_1(\mathcal{G}_1), c_2(\mathcal{G}_1)) = (1, 6)$ and $(c_1(\mathcal{G}_2), c_2(\mathcal{G}_2)) = (1, 8)$ or $(c_1(\mathcal{G}_1), c_2(\mathcal{G}_1)) = (2, 14)$ and $(c_1(\mathcal{G}_2), c_2(\mathcal{G}_2)) = (0, 5)$. In the first case by Riemann-Roch we compute $0 = h^0F(-1) + h^0E < h^0\mathcal{G}_1 + h^0\mathcal{G}_2 = 3$. In the second case one concludes in similar way since $h^0\mathcal{G}_1 + h^0\mathcal{G}_2 = 1$. One concludes in similar way for the other cases.

Case (4). In this case we have

$$(c_1(\mathcal{G}_1), c_2(\mathcal{G}_2)) = (2, 20).$$

If $\mathcal{G} \cong \mathcal{G}_1 \oplus \mathcal{G}_2$ by Proposition 3.1 and by condition $\mathcal{E}_0$ we could have possible cases $(c_1(\mathcal{G}_1), c_2(\mathcal{G}_1)) = (1, \alpha)$ and $(c_1(\mathcal{G}_2), c_2(\mathcal{G}_2)) = (1, \alpha')$ with $\alpha, \alpha' = 4, 6, 8$ and $\alpha + \alpha' = 15$ which is impossible.
Cases (5)–(7). In this case we have
\[(c_1(G_1), c_2(G_2)) = (1, \alpha + 8)\]
where \(\alpha = 3, 4, 5\). If \(G \cong G_1 \oplus G_2\) by Proposition 3.1 and by condition 3.3, the conclusion follows.

\[\square\]

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