VANISHING STRUCTURE SET OF 3-MANIFOLDS

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Abstract. In this short note we update a result proved in [16]. This will complete our program of [12] showing that the structure set vanishes for compact aspherical 3-manifolds.

1. Introduction

This paper is to note that the program we started in [12] is now complete.

Let us first recall that a compact manifold \( M \) with boundary is called topologically rigid if any homotopy equivalence \( f : (N, \partial N) \to (M, \partial M) \) from another compact manifold with boundary, so that \( f|_{\partial N} : \partial N \to \partial M \) is a homeomorphism is homotopic to a homeomorphism relative to boundary.

Let \( M \) be a compact connected 3-manifold whose fundamental group is torsion free.

We prove the following theorem.

Theorem 1.1. If \( M \) is aspherical then \( M \times \mathbb{D}^n \) is topologically rigid for \( n \geq 2 \). Here \( \mathbb{D}^n \) denotes the \( n \)-dimensional disc.

In [12] and [13] we proved Theorem 1.1 under various conditions. In [12] we proved it for the nonempty boundary case and for the situation when the manifold contains an incompressible square root closed torus. In [13] we assumed the manifold has positive first Betti number. Due to some recent developments in Geometric Topology (see [1], [2], [14], [15] and [16]) we are now able to deduce Theorem 1.1. Also the main ideas from [12] and [13] go behind the proof of this general case.

The first step to prove Theorem 1.1 is to show that the Whitehead group of \( \pi_1(M) \) is trivial. We deduce the following for this purpose.

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Theorem 1.2. Let $G$ be isomorphic to the fundamental group of $M$ then

$$Wh(G) = K_{-i}(G) = \tilde{K}_0(G) = 0$$

for all $i \geq 2$.

2. Proofs of Theorems 1.1 and 1.2

For terminologies on 3-manifolds used in the proofs see [7] or [12].

Proof of Theorem 1.2. By (Kneser-Milnor) prime decomposition theorem $G$ is isomorphic to the free product of a free group and finitely many groups $G_1, G_2, \ldots, G_n$ where for each $i$, $G_i$ is isomorphic to the fundamental group of an aspherical irreducible 3-manifold $M_i$ (see [[13], Lemma 3.1]). Since the Whitehead group of a free product is the direct sum of the Whitehead groups of the individual factors of the free product (see [18]) it is enough to prove that the Whitehead group vanishes for $G_i$. Now by the Geometrization Theorem (conjectured by Thurston and proved by Perelman) $M_i$ is either Seifert fibered, Haken or hyperbolic. The hyperbolic case follows from some more general result of Farrell and Jones in [4], for Haken case it follows from Waldhausen’s result in [19]. For non-Haken Seifert fibered space the vanishing result is due to Plotnick (see [9]). For the reduced projective class groups $\tilde{K}_0(-)$ and for the negative $K$-groups $K_{-i}(-)$ the same sequence of arguments and references work. For details see [3]. In fact, more generally it is shown in [3] that $G$ is $K$-flat, i.e., $Wh(G \times \mathbb{Z}^n) = 0$ for all non-negative integer $n$.

This completes the proof of Theorem 1.2. \hfill \Box

Below we recall the statement of the Fibered Isomorphism Conjecture of Farrell and Jones. For details about this conjecture see [5]. Here we follow the formulation given in [[6], Appendix].

Let $\mathcal{F}$ be one of the three functors from the category of topological spaces to the category of spectra: (a) the stable topological pseudoisotopy functor $\mathcal{P}()$; (b) the algebraic $K$-theory functor $\mathcal{K}()$; and (c) the $L$-theory functor $L^{(-\infty)}()$. The $L$-theory functor also includes an orientation data, that is a homomorphism $\omega : \pi_1(X) \to \mathbb{Z}_2$. If the topological space is an oriented manifold then this homomorphism is zero.

Let $\mathcal{M}$ be a category whose objects are continuous surjective maps $p : E \to B$ between topological spaces $E$ and $B$. And a morphism between two maps $p : E_1 \to B_1$ and $q : E_2 \to B_2$ is a pair of continuous maps $f : E_1 \to E_2$, $g : B_1 \to B_2$ such that the following diagram commutes.
There is a functor defined by Quinn in [10] from $\mathcal{M}$ to the category of $\Omega$-spectra which associates to the map $p$ a spectrum $\mathbb{H}(B, \mathcal{F}(p))$ with the property that $\mathbb{H}(B, \mathcal{F}(p)) = \mathcal{S}(E)$ if $B$ is a single point space. For an explanation of $\mathbb{H}(B, \mathcal{F}(p))$ see [5], Section 1.4. Also the map $\mathbb{H}(B, \mathcal{F}(p)) \to \mathcal{F}(E)$ induced by the morphism: $\text{id}: E \to E; B \to \ast$ in the category $\mathcal{M}$ is called the Quinn assembly map.

Let $\Gamma$ be a discrete group and $\mathcal{E}$ be a $\Gamma$-space which is universal for the class of all virtually cyclic subgroups of $\Gamma$ and denote $\mathcal{E}/\Gamma$ by $\mathcal{B}$. For definition and properties of universal space see [5], Appendix. Let $X$ be a space on which $\Gamma$ acts freely and properly discontinuously and $p : X \times_{\Gamma} \mathcal{E} \to \mathcal{E}/\Gamma = \mathcal{B}$ be the map induced by the projection onto the second factor of $X \times \mathcal{E}$.

The Fibered Isomorphism Conjecture for $\Gamma$ states that the map

$$\mathbb{H}(\mathcal{B}, \mathcal{F}(p)) \to \mathcal{F}(X \times_{\Gamma} \mathcal{E}) = \mathcal{F}(X/\Gamma)$$

is an (weak) equivalence of spectra. The equality in the above display is induced by the map $X \times_{\Gamma} \mathcal{E} \to X/\Gamma$ and using the fact that $\mathcal{F}$ is homotopy invariant. If $X$ is simply connected then this is called the Isomorphism Conjecture for $\Gamma$.

In this paper we consider the case when $\mathcal{F}() = L^{(-\infty)}()$. We have already mentioned that this $L$-theory functor contains the orientation data $\omega : \Gamma \to \mathbb{Z}_2$ so as to include the case of nonorientable manifolds.

Let us now deduce the following theorem which is an immediate consequence of [[16], 3(a) of Theorem 2.2] and some recent results from [1] and [2].

**Theorem 2.1.** Let $G$ be isomorphic to the fundamental group of a 3-manifold. Then the Farrell-Jones Fibered Isomorphism conjecture in $L^{(-\infty)}$-theory is true for $G \wr H$ where $H$ is some finite group.

**Proof.** The theorem follows from [[16], 3(a) of Theorem 2.2] provided we show that the conjecture is true for $\Gamma \wr H$ where $H$ is some finite group and $\Gamma$ belongs to the following classes of groups:

1). $\mathbb{Z}^2 \rtimes_{\sigma} \mathbb{Z}$ for all actions $\sigma$ of $\mathbb{Z}$ on $\mathbb{Z}^2$.

2). Fundamental groups of closed nonpositively curved Riemannian 3-manifolds.
3). $\Gamma \simeq \lim_{i \in I} \Gamma_i$ where $\{\Gamma_i\}$ is a directed system of groups so that for each $i \in I$ the conjecture is true for $\Gamma_i \wr K$ where $K$ is some finite group.

We now note the following to complete the proof of the Theorem.

(1) follows from [2] where the conjecture is proved for virtually polycyclic groups.

(2) follows from [1] where the conjecture is proved for finite dimensional CAT(0)-groups.

And (3) follows from [[6], Theorem 7.1].

□

Proof of Theorem 1.1. If $\partial M \neq \emptyset$ then the theorem follows from [[12], Theorem 1.1]. Therefore we can assume that $M$ is closed. Now recall that the combination of Theorems 1.2 and 2.1 imply the isomorphism of the classical assembly map in $L$-theory. Namely, the map $H_k(BG, \mathbb{L}_0) \to L_k(G)$ is an isomorphism for all $k$. Since $M$ aspherical it is a model of $BG$, thus we have the isomorphism $H_k(M, \mathbb{L}_0) \to L_k(G)$. See the proof of [[8], Theorem 1.28] or [[17], Corollary 5.3] for a detailed argument.

Next we recall the definition of structure set and the surgery exact sequence.

Let $M$ be a compact manifold with boundary (may be empty) so that $Wh(\pi_1(M)) = 0$. Consider all objects $(N, \partial N, f)$, where $N$ is a manifold with boundary $\partial N$ and $f : N \to M$ is a homotopy equivalence such that $f|_{\partial N} : \partial N \to \partial M$ is a homeomorphism. Two such objects $(N_1, \partial N_1, f_1)$ and $(N_2, \partial N_2, f_2)$ are equivalent if there is a homeomorphism $g : N_1 \to N_2$ such that the obvious diagram commutes up to homotopy relative to the boundary. The equivalence classes of these objects is the homotopy-topological structure set $S(M, \partial M)$.

In [11] Ranicki defined homotopy functors $S_k(X)$ from the category of topological spaces to the category of abelian groups which fit into the following exact sequence:

$$\cdots \to S_k(X) \to H_k(X, \mathbb{L}_0) \to L_k(\pi_1(X)) \to S_{k-1}(X) \to \cdots.$$ 

Also it is shown in [11] that there is a bijection between $S(M \times \mathbb{D}^k, \partial(M \times \mathbb{D}^k))$ and $S_{k+\dim M}(M)$ provided $\dim M + k \geq 5$.

The proof of the theorem is now complete since we have already proved the isomorphism $H_k(M, \mathbb{L}_0) \to L_k(G)$ for all $k$. □
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