Half line Titchmarsh–Weyl $m$ functions of vector-valued discrete Schrödinger operators

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Abstract
We show that the half-line $m$ functions associated with the vector-valued discrete Schrödinger operators are the elements in the Siegel upper half space. We introduce a metric on the space of $m$ functions associated to these operators. Then, we show that the action of transfer matrices on these $m$ functions is distance decreasing.

Keywords Vector-valued discrete Schrödinger operators · Titchmarsh–Weyl $m$ functions · Transfer matrices · Herglotz functions

Mathematics Subject Classification 39A70 · 47A05

1 Introduction

The spectral theory of one dimensional Schrödinger operator has been well studied. The extension of the theory to higher dimensions has yet to be developed and is of interest of many researchers. There are some literature which deal with the Schrödinger equations in higher dimensions with matrix-valued potentials, for example see [3, 4, 7–9]. In these articles, authors have considered the Schrödinger equations in continuous case. Likewise, as the theory of Titchmarsh–Weyl $m$ functions provide elegant methods in describing the spectrum of the associated operators, it is natural to study these functions for vector-valued discrete Schrödinger operators. Some basic theory has been...
developed in [1]. To study these \( m \) functions, we need to study matrix-valued Herglotz functions. In this regard, there has been some studies about the matrix-valued Herglotz functions, for example see [10]. Our goal in this paper is to discuss some basic properties of half-line Titchmarsh–Weyl \( m \) functions associated to the vector-valued discrete Schrödinger operators.

We begin by considering a \( d \)-dimensional discrete Schrödinger equation:

\[
y(n + 1) + y(n - 1) + B(n)y(n) = zy(n), \ z \in \mathbb{C}
\]

where for fixed \( n \), \( y(n) = (y_1(n), y_2(n), \ldots, y_d(n))^\top \in \mathbb{C}^d \), \( \top \) stands for transpose, and \( B(n) \in \mathbb{C}^{d \times d} \) is a Hermitian matrix, that is \( B(n)^* = B(n) \), where \( * \) stands for complex conjugate transpose. Hence, \( \{y(n)\} \) and \( \{B(n)\} \) are, respectively, vector-valued and matrix-valued sequences. Equation (1) is a particular case of \( d \)-dimensional Jacobi equation:

\[
A(n)y(n + 1) + A(n - 1)y(n - 1) + B(n)y(n) = zy(n), \ z \in \mathbb{C}
\]

with \( \{A(n)\} \), \( \{B(n)\} \) are \( d \times d \) matrix-valued sequences. It is also a special case of discrete canonical system, discrete symplectic systems and linear Hamiltonian difference systems, see [5, 6, 14, 15] for these systems and accociated Titchmarsh–Weyl theory.

In this paper, we give insight into half-line Titchmarsh–Weyl \( m \) functions associated to the Eq. (1) and discuss properties these functions enjoy. Particularly, we show that these \( m \) functions are matrix-valued Herglotz functions. Therefore, the set of all half-line \( m \) functions is a subset of a Siegel upper half space. Then we describe the action of transfer matrices on these functions. The Siegel upper half space is equipped with a metric defined in terms of Finsler norm. The metric, we consider, is an extension of the hyperbolic metric on complex upper half plane. We then introduce a metric on the space of \( m \) functions an extended analogous of hyperbolic metric in complex upper half plane. Finally we observe the distance properties. The paper is organized as follows: We present the preliminaries in Sect. 2. In Sect. 3, we define the half-line \( m \) functions following the definition in [1], present very important properties. Then we define a metric on the space of \( m \) functions and show that action of transfer matrices on \( m \) functions is distance decreasing.

\section{2 Preliminaries}

Similar to the one dimensional space, we consider the Eq. (1) in a Hilbert space. Let \( \ell^2(\mathbb{N}, \mathbb{C}^d) \) be the Hilbert space of square summable vector-valued sequences. That is

\[
\ell^2(\mathbb{N}, \mathbb{C}^d) = \left\{ y : \sum_{n=1}^{\infty} y(n)^*y(n) < \infty \right\}.
\]

Equation (1) induces a vector-valued discrete Schrödinger operator \( J \) on \( \ell^2(\mathbb{N}, \mathbb{C}^d) \) as
\[(Jy)(n) = \begin{cases} 
 y(n + 1) + y(n - 1) + B(n)y(n) & \text{if } n > 1 \\
 y(2) + B(1)y(1) & \text{if } n = 1.
\end{cases} \tag{2} \]

Notice from the definition that \(J\) is linear. It is also bounded if \(B(n)\) is bounded with an operator norm on \(\ell^2\). In addition, if \(B(n)\) is a Hermitian matrix then \(J\) is a self-adjoint operator.

**Lemma 2.1** If \(B(n) \in \mathbb{C}^{d \times d}\) in (2) is bounded and \(B(n)^* = B(n)\), then \(J\) is a self-adjoint bounded operator on \(\ell^2(\mathbb{N}, \mathbb{C}^d)\).

**Proof** Let \(x, y \in \ell^2(\mathbb{N}, \mathbb{C}^d)\) then

\[
\langle x, Jy \rangle = \sum_{n=1}^{\infty} x(n)^* Jy(n) = x(1)^* (J(y)(1)) \\
+ \sum_{n=2}^{\infty} x(n)^* (y(n + 1) + y(n - 1) + B(n)y(n))
\]

\[
= x(1)^* (y(2) + B(1)y(1)) + \sum_{n=2}^{\infty} x(n)^* y(n + 1) + \sum_{n=2}^{\infty} x(n)^* y(n - 1)
\]

\[
+ \sum_{n=2}^{\infty} x(n)^* B(n)y(n) = (x(2) + B(1)x(1))^* y(1)
\]

\[
- x(2)^* y(1) + x(1)^* y(2) + \sum_{n=2}^{\infty} x(n)^* y(n + 1)
\]

\[
+ \sum_{n=2}^{\infty} x(n)^* y(n - 1) + \sum_{n=2}^{\infty} x(n)^* B(n)y(n)
\]

\[
= (x(2) + B(1)x(1))^* y(1) + \sum_{n=2}^{\infty} x(n - 1)^* y(n) + \sum_{n=2}^{\infty} x(n + 1)^* y(n)
\]

\[
+ \sum_{n=2}^{\infty} x(n)^* B(n)y(n) = \left( x(2) + B(1)x(1) \right)^* y(1)
\]

\[
+ \sum_{n=2}^{\infty} \left( x(n + 1) + x(n - 1) + B(n)x(n) \right)^* y(n)\]

\[
= \sum_{n=1}^{\infty} (Jx(n))^* y(n) = \langle Jx, y \rangle
\]

This shows that \(J\) is self-adjoint. To show that \(J\) is bounded, since \(J\) is self-adjoint, we need only consider \(|\langle y, Jy \rangle|\) for \(y\) on the unit sphere in \(\ell^2(\mathbb{N}, \mathbb{C}^d)\). A direct calculation yields
Using the Cauchy–Schwartz inequality, we get
\[ |\langle y, Jy \rangle| \leq (2 + |||B|||)\|y\|^2, \]
where
\[ |||B||| = \sup_{n \in \mathbb{N}} \|B(n)\| \]
and \(\|B(n)\|\) is the operator norm for matrices acting on \(\mathbb{C}^d\). This shows that \(J\) is bounded and that \(Jy \in \ell^2(\mathbb{N}, \mathbb{C}^d)\) for all \(y \in \ell^2(\mathbb{N}, \mathbb{C}^d)\).

It should be noted that this lemma remains true if one keeps the coefficient matrix \(A(n)\) in the generalized version of Eq. (1) and assumes that \(A(n)\) is self-adjoint, nonzero and bounded. That is if
\[ (\tilde{J}y)(n) = \begin{cases} A(n)y(n + 1) + A(n - 1)y(n - 1) + B(n)y(n) & \text{if } n > 1 \\ A(1)y(2) + B(1)y(1) & \text{if } n = 1 \end{cases} \]
with \(A(n)\) and \(B(n)\), both self-adjoint, bounded and \(A(n) \neq 0\) for all \(n\), we get that \(J\) is a self-adjoint, bounded operator and
\[ \|\tilde{J}\| \leq 2|||A||| + |||B|||. \]

As already mentioned we only focus on the case when \(A(n) = I\) for all \(n\) for the remainder of the paper.

In this paper, we will try to obtain a spectral measure for the operator \(J\) through Titchmarsh–Weyl \(m\) function. Define the difference expression \(\tau\) by
\[ \tau(u(n)) = u(n + 1) + u(n - 1) + B(n)u(n) \quad (3) \]
which looks like \(J\) but acts on any vector-valued sequence. In fact, for any \(z \in \mathbb{C}\), a solution \(u(n)\) of \((\tau - z)u = 0\) is uniquely determined by the values \(u(n_0) \in \mathbb{C}^d\) and \(u(n_0 + 1) \in \mathbb{C}^d\); therefore, the solution space is a \(2d\) dimensional vector space of sequences. It is shown in [1] that there are precisely \(d\) linearly independent solutions to Eq. (1) that are in \(\ell^2(\mathbb{N}, \mathbb{C}^d)\).

One first observation is to connect Eq. (1) with a second-order differential expression: \(-\frac{d^2}{dx^2}(y) + V(x)y\), similar to the one in one dimensional space, see [16]. Indeed, define the following first order difference operator: \(\Delta : \ell^2(\mathbb{N}, \mathbb{C}^d) \to \ell^2(\mathbb{N}, \mathbb{C}^d)\) by
\[ (\Delta u)(n) = u(n + 1) - u(n). \]
An immediate calculation yields the adjoint to \(\Delta\), namely
\[ \Delta^* : \ell^2(\mathbb{N}\setminus\{1\}, \mathbb{C}^d) \to \ell^2(\mathbb{N}, \mathbb{C}^d) \]
via
Using the definition of \( \tau \), consider the following calculations:

\[-(\Delta^* \Delta u)(n) = -\Delta^* (u(n + 1) - u(n)) = -u(n) + u(n + 1) + u(n - 1) - u(n)\]

and

\[-(\Delta \Delta^* u)(n) = -(\Delta (u(n) - u(n))) = u(n + 1) + u(n - 1) - 2u(n)\]

Thus, \( (\tau u)(n) = -(\Delta \Delta^* u)(n) + (B(n) + 2I)u(n) \). Notice that this is similar to the vector-valued Sturm–Liouville differential expression:

\[-\frac{d^2}{dx^2}(y) + V(x)y\]

where \( y \) is in some properly chosen domain and \( V \) is a potential function.

For any two vector valued sequences \( \{x(n)\}, \{y(n)\} \), recall that the Wronskian is defined by

\[W_n(x(n), y(n)) = x(n + 1)^{\top} y(n) - x(n)^{\top} y(n + 1).\]  

(4)

It can be easily shown that if \( u(n) \) and \( v(n) \) are any two solutions to Eq. (1), then \( W_n(u(n), v(n)) \) is independent of \( n \). This definition can be easily generalized to \( d \times d \) matrix-valued sequences \( \{X(n)\} \) and \( \{Y(n)\} \) as follows:

\[W_n(X(n), Y(n)) = X(n + 1)^{\top} Y(n) - X(n)^{\top} Y(n + 1).\]

(5)

In this case \( W_n(X(n), Y(n)) \) is a \( d \times d \) matrix-valued function. It was shown in [1] that if \( X(n) \) and \( Y(n) \) are matrix valued solutions to Eq. (1), then is \( W_n(X(n), Y(n)) \) independent of \( n \). Using Abel summation we arrive at Green’s formula:

\[\sum_{j=n}^{m} (X(j)^{\ast}(\tau Y)(j) - (\tau Y)(j)^{\ast} Y(j))\]

(6)

\[= W_{n-1}(\bar{X}(n - 1), Y(n - 1)) - W_{m}(\bar{X}(m), Y(m)).\]

Please see [1] for a detailed computation of formula (6).

3 Half line Weyl \( m \) functions

A \( d \times 2d \) matrix-valued sequence \( Y(n) \) is said to be a fundamental matrix solution to Eq. (1) if it satisfies Eq. (1) in matrix form and columns of \( Y(n) \) form a set of linearly independent solutions to Eq. (1).

Write \( Y = (U, V) \), where \( U \) and \( V \) are \( d \times d \) matrix-valued solutions to Eq. (1). One of the goals of this paper is to discuss the half line \( m \) function. Let
ℕ_ = {1, ..., n} and ℕ_+ = {n + 1, n + 2, ...}. Suppose U and V satisfy the following initial conditions at n:

\[ U(n) = -I, \quad V(n) = 0 \]
\[ U(n + 1) = 0, \quad V(n + 1) = I. \]

For any \( z \in \mathbb{C}^+ \), define the Weyl m functions on \( \mathbb{N}_- \) and \( \mathbb{N}_+ \) by

\[ F_\pm(n, z) = U(n, z) \pm M_\pm(n, z)V(n, z) \]

where we require that \( F_+ \in \ell^2(\mathbb{N}, \mathbb{C}^{d\times d}) \) and we take \( F_-(0) = 0 \) for normalization of the matrix-valued solutions to Eq. (1). Notice that the columns of \( F_\pm(n, z) \) form a linearly independent set of solutions to Eq. (1), and therefore, for each \( n \), \( F_\pm(n) \) are invertible. For \( z \in \mathbb{C}^+ \), these half-line m functions are uniquely determined and are given by:

\[ M_+(n, z) = -F_+(n + 1, z)F_+(n, z)^{-1}, \quad M_-(n, z) = F_-(n + 1, z)F_-(n, z)^{-1}. \]

The theory of these m functions has been very useful tool in the study of spectral theory of Schrödinger operators. There are numerous articles in the literature dealing with these m functions for one dimensional Schrödinger operators, see [16] as an example, for Jacobi operators. There are also some articles which discuss about the Weyl m functions for matrix-valued Schrödinger operators in continuous case, see [3, 4, 7, 9, 10]. Our goal is to discuss some basic properties of the m functions defined by (8).

First we show that \( M_\pm(n) \) can be expressed in terms of a resolvent operator. As such we introduce the following notation. Let \( J_\pm \) be the restriction of \( J \) to the space \( \ell^2(\mathbb{N}_\pm, \mathbb{C}^d) \). More precisely, \( J_+ : \ell^2(\mathbb{N}_+, \mathbb{C}^d) \to \ell^2(\mathbb{N}_+, \mathbb{C}^d) \) is given by

\[ (J_+ y)(l) = \begin{cases} y(l + 1) + y(l - 1) + B(l)y(l) & \text{if } l > n + 1 \\ y(n + 2) + B(n + 1)y(n + 1) & \text{if } l = n + 1. \end{cases} \]

Similarly, \( J_- : \ell^2(\mathbb{N}_-, \mathbb{C}^d) \to \ell^2(\mathbb{N}_-, \mathbb{C}^d) \) is given by

\[ (J_- y)(l) = \begin{cases} y(l + 1) + y(l - 1) + B(l)y(l) & \text{if } l = 2, \ldots, n \\ y(2) + B(1)y(1) & \text{if } l = 1. \end{cases} \]

Denote the Dirac delta-type vector-valued sequences as follows:
Therefore,
\[ \delta_j^n(k) \]
is the \( k \)th term(vector) in the sequence \[ \delta_j^n \]

. Similarly for matrix-valued sequence denote
\[ _1 = \{ I, 0, \ldots \}, \ldots, _n = \{ 0, \ldots, I, \ldots \} \]
where \( I \) and 0 are the \( d \times d \) identity and null matrix, respectively. In the following proposition we express the \( M_+(n, z) \) in terms of resolvant operator. A similar idea can be found in [1].

**Proposition 3.1** Let \( z \in \mathbb{C}^+ \). Then

\[ M_+(n, z) = \langle \Lambda_{n+1}, (J_+ - zI)^{-1}\Lambda_{n+1} \rangle = \left( \langle \delta_{i}^{n+1}, (J_+ - zI)^{-1}\delta_{i}^{n+1} \rangle \right) \]

**Proof** Let

\[ g_j(k, z) = (J_+ - zI)^{-1}\delta_j^{n+1}(k) \]

. Since \( J_+ \) is self-adjoint, it has real spectrum and hence \( (J_+ - zI)^{-1} \) exists and is bounded for \( z \in \mathbb{C}^+ \). Therefore, \( g_j(k, z) \) is a well-defined element. Notice that \( g_j(k, z) \), for \( j = 1, \ldots, d \) and for \( k > n + 1 \) are solutions to Eq. (1) and belong to \( \ell^2(N_+, \mathbb{C}^d) \). Define \( G(k, z) \) by
From this definition it follows that $G(k, z)$ is a matrix-valued solution to Eq. (1) for $k > n + 1$ and $G(k, z) \in L^2(\mathbb{N}_+, \mathbb{C}^{d \times d})$. Thus, there exists an invertible constant matrix $C \in \mathbb{C}^{d \times d}$, such that

$$G(k, z) = F_+(k, z)C. \quad (9)$$

Since $F_+(n, z) = -I$ and $G(n, z) = -C$, it follows that

$$M_+(n, z) = -F_+(n + 1, z)F_+(n, z)^{-1} = -G(n + 1, z)G(n, z)^{-1} = G(n + 1, z)C^{-1}. \quad (10)$$

To find the matrix $C$, we need only to compare the values at $n + 2$ place. Notice that

$$(J_+ - zI)G(n + 1, z) = (\delta_1^1, \delta_2^1, \ldots, \delta_d^1)$$

and hence $(J_+ - zI)G(n + 1, z) = I$. Therefore

$$G(n + 2, z) = (z - B(n + 1))G(n + 1, z) + I. \quad (10)$$

Moreover, since $F_+(n, z)$ is a solution to Eq. (1), we have

$$F_+(n + 2, z) = (z - B(n + 1))F_+(n + 1, z) - F_+(n, z).$$

From Eq. (9), it follows that

$$G(n + 2, z) = (z - B(n + 1))F_+(n + 1, z)C - F_+(n, z)C. \quad (11)$$

Finally, comparing Eqs. (10) and (11), we deduce that $-F_+(n, z)C = I$. Since $F_+(n, z) = -I$, it follows that $C = I$. Therefore, we have that $M_+(n, z) = G(n + 1, z)$.

Thus, the matrix $M_+(n, z)$ by comparing the values at $k = n + 1$:

$$G(n + 1, z) = F_+(n + 1, z)C$$

Now, we find the entries of the matrix solution $G(n + 1, z)$. Indeed, for each $j = 1, \ldots, d$, the components $g_{ij}(n + 1, z)$ for

$$g_j(n + 1, z) = \begin{bmatrix} g_{1j}(n + 1, z) \\ g_{2j}(n + 1, z) \\ \vdots \\ g_{dj}(n + 1, z) \end{bmatrix}$$

are given by

$$g_{ij}(n + 1, z) = (\delta_i^{n+1}, (J_+ - zI)^{-1}\delta_j^{n+1})$$

for $i = 1, \ldots, d$. It follows that
This completes the proof.  

Equation (9) suggests that $M_+(n, \tilde{z}) = M_+(n, z)$. We now show that $M_\pm(n, z)$ are matrix-valued Herglotz functions.

**Lemma 3.2** For any $z \in \mathbb{C}^+$ and any $n$, $M_\pm(n, z)$ are symmetric and satisfy the relation:

$$M_+(n, z) + M_-(n \mp 1, z)^{-1} \mp (zI - B(n)) = 0. \tag{12}$$

**Proof** We first apply the Wronskian to show that $M_\pm(n)$ are symmetric. Indeed, for $M_-(n, z)$, consider the following calculation:

$$M_-(n, z)^T - M_-(n, z) = \left( F_-(n + 1, z)F_-(n, z)^{-1} \right)^T - F_-(n + 1, z)F_-(n, z)^{-1}$$

$$= \left( F_-(n, z)^{-1} \right)^T F_-(n + 1, z)^T - F_-(n + 1, z)F_-(n, z)^{-1}$$

$$= \left( F_-(n, z)^{-1} \right)^T \left( F_-(n + 1, z)^T F_-(n, z) - F_-(n, z)^T F_-(n + 1, z) \right) F_-(n, z)^{-1}$$

$$= \left( F_-(n, z)^{-1} \right)^T \mathcal{W}_0(F_-(n, z), F_-(n, z)) F_-(n, z)^{-1}.$$ 

Since $F_-(n, z)$ is a solution to Eq. (1), $\mathcal{W}_n(F_-(n, z), F_-(n, z))$ is independent of $n$. It, therefore, follows that

$$\mathcal{W}_n(F_-(n, z), F_-(n, z)) = \mathcal{W}_0(F_-(0, z), F_-(0, z)) = 0.$$

This in turn implies that $M_-(n, z)$ is symmetric. A similar calculation shows that $M_+(n, z)$ is also symmetric. All that remains is to show that $M_\pm(n, z)$ satisfies Eq. (12).

Indeed, from Eq. (1), we have

$$F_+(n + 1, z) + F_+(n - 1, z) + (B(n) - zI) F_+(n, z) = 0$$

and hence

$$F_+(n + 1, z)F_+(n, z)^{-1} + F_+(n - 1, z)F_+(n, z)^{-1} + (B(n) - zI) = 0.$$ 

This implies that

$$M_+(n, z) + M_+(n - 1, z)^{-1} + zI - B(n) = 0.$$

An analogous calculation shows that for $M_-(n, z)$, we have

$$M_-(n, z) + M_-(n - 1, z)^{-1} + B(n) - zI = 0.$$
This completes the proof. \( \square \)

Recall the imaginary part of \( M_{\pm}(n, z) \) is given by

\[
\text{Im} M_{\pm}(n, z) = \frac{1}{2i} (M_{\pm}(n, z) - M_{\pm}(n, z)^*) .
\]

We have the following proposition:

**Proposition 3.3** For any \( z \in \mathbb{C}^+ \), the matrices \( \text{Im} M_{\pm}(n, z) \) are positive.

**Proof** First, we show that \( \text{Im} M_{\pm}(n, z) \) is a positive matrix. Using Green’s formula, Eq. (6) with \( X(n) \) and \( Y(n) \) are replaced by \( F_-(n, z) \) we get

\[
\sum_{j=1}^{n} (F_-(j, z)^*(\tau F_-(j, z) - (\tau F_-(j, z)) F_-(j, z))
\]

\[
= W_0(F_-(0, z), F_-(0, z)) - W_n(F_-(n, z), F_-(n, z)) .
\]

Since \( F_-(0, z) = 0 \) and \( (\tau F_-(j, z) = z F_-(j, z) \), Eq. (13) simplifies to

\[
(z - \bar{z}) \sum_{j=1}^{n} F_-(j, z)^* F_-(j, z) = -W_n(F_-(n, z), F_-(n, z)) .
\]

Since \( F_-(n, z) \) can be decomposed into the fundamental solutions \( U(n, z) \) and \( V(n, z) \), that is \( F_-(n, z) = U(n, z) - M_-(n, z)V(n, z) \), the linearity of Wronskian implies that

\[
(z - \bar{z}) \sum_{j=1}^{n} F_-(j, z)^* F_-(j, z)
\]

\[
= -W_n(U(n, z) - M_-(n, z)V(n, z), U(n, z) - M_-(n, z)V(n, z))
\]

\[
= W_n(U(n, \bar{z}), U(n, z)) + W_n(U(n, \bar{z}), M_-(n, z)V(n, z))
\]

\[
+ W_n(M_-(n, \bar{z})V(n, \bar{z}), U(n, z)) - W_n(M_-(n, \bar{z})V(n, \bar{z}), M_-(n, z)V(n, z)) .
\]

Using definition of the Wronskian and the boundary conditions at \( n \) and \( n + 1 \), we obtain that

\[
W_n(U(n, \bar{z}), U(n, z)) = 0
\]

\[
W_n(M_-(n, \bar{z})V(n, \bar{z}), M_-(n, z)V(n, z)) = 0 .
\]

It follows that

\[
W_n(U(n, \bar{z}), M_-(n, z)V(n, z)) = M_-(n, z)
\]

\[
W_n(M_-(n, \bar{z})V(n, \bar{z}), U(n, z)) = -M_-(n, \bar{z}) .
\]

Hence, Eq. (15) becomes
\[ (z - \bar{z}) \sum_{j=1}^{n} F_-(j, z)^* F_-(j, z) = M_-(n, z) - M_-(n, \bar{z}). \]

Since \( z \in \mathbb{C}^+ \), we have \((z - \bar{z})/2i > 0 \) and thus it follows that \( \text{Im} M_-(n, z) \) is a positive matrix.

Similarly, to show \( \text{Im} M_+(n, z) \) is a positive matrix, we replace \( X(n) \) and \( Y(n) \) both with \( F_+(n, z) \) in Eq. (6). Since \( F_+(n, z) \) is defined on \( \mathbb{N}_+ \), for \( N \geq n + 1 \), we have

\[
\sum_{j=n+1}^{N} \left( F_+(j, z)^* (\tau F_+)(j, z) - (\tau F_+)(j, z)^* F_+(j, z) \right)
= W_n(F_+(n, z), F_+(n, z)) - W_n(F_+(N, z), F_+(N, z)).
\]

Since \( F_+(z) \in \ell^2(\mathbb{N}, \mathbb{C}^{d \times d}) \) for every \( z \in \mathbb{C}^+ \), the limit as \( N \to \infty \) on the left-hand side of Eq. (16) exists. Recall the definition of Wronskian

\[ W_N(F_+(N, z), F_+(N, z)) = F_+(N + 1, z)^\top F_+(N, z) - F_+(N, z)^\top F_+(N + 1, z) \]
\[ = F_+(N + 1, z)^\top F_+(N, z) - F_+(N, z)^\top F_+(N + 1, z). \]

Again, since \( F_+(z) \in \ell^2(\mathbb{N}, \mathbb{C}^{d \times d}) \) the right-hand side of this equation is 0. So \( \lim_{N \to \infty} W_N(F_+(N, z), F_+(N, z)) = 0 \). Now, taking the limit as \( N \to \infty \) on both sides of Eq. (16), we have

\[ (z - \bar{z}) \sum_{j=n+1}^{\infty} F_+(j, z)^* F_+(j, z) = W_n(F_+(n, z), F_+(n, z)). \]

Using the similar argument as above and the initial conditions at \( n \), we have

\[ W_n(F_+(n, z), F_+(n, z)) = M_+(n, z) - M_+(n, \bar{z}). \]

Thus, Eq. (16) becomes

\[ (z - \bar{z}) \sum_{j=n+1}^{\infty} F_+(j, z)^* F_+(j, z) = M_+(n, z) - M_+(n, \bar{z}). \]

Since

\[ ||F_+(z)||^2 = \sum_{j=1}^{\infty} F_+(j, z)^* F_+(j, z), \]

and \( z \in \mathbb{C}^+ \), it follows that the left-hand side of Eq. (17) is positive. Therefore, \( \text{Im} M_+(n, z) \) is a positive matrix completing the proof. \( \square \)

The fractional linear transformation have been used in analyzing the theory of \( m \) functions. More specifically, the transfer matrices associated to the Schrödinger equations are fractional linear transformation. For the matrix-valued Schrödinger
operators, the transfer matrices are considered as matrix-valued fractional transformation. For any \( T \in \mathbb{C}^{2d \times 2d} \) of the form:

\[
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

where are \( A, B, C, \) and \( D \) are \( d \times d \) matrices, a \textit{matrix-valued fractional transformation} is a map \( T : \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d} \) defined by

\[
T(Z) = (AZ + B)(CZ + D)^{-1},
\]

for all \( Z \in \mathbb{C}^{d \times d} \) for which \( T \) is well defined. We observe that the transfer matrices associated to the Eq. (1) are in fact matrix-valued fractional transformation of the form (18).

If \( F(n) \) is a matrix solution to Eq. (1), we have

\[
\begin{bmatrix} F(n+1) \\ \mp F(n) \end{bmatrix} = \begin{pmatrix} zI - B(n) & \pm I \\ \mp I & 0 \end{pmatrix} \begin{bmatrix} F(n) \\ \mp F(n-1) \end{bmatrix},
\]

where \( B(n) \) is a \( d \times d \) symmetric matrix. The matrices

\[
T_{\pm}(n, z) = \begin{pmatrix} zI - B(n) & \pm I \\ \mp I & 0 \end{pmatrix}
\]

given in Eq. (19) are called \textit{transfer matrices} which describe the evolution of the vectors:

\[
\begin{bmatrix} F(n+1) \\ \mp F(n) \end{bmatrix}
\]

under iteration of \( T_{\pm}(n, z) \). These transfer matrices \( T_{\pm}(n, z) \) can be considered as complex matrix-valued fractional transformations (18) acting on the space of \( m \) functions by using (20):

\[
T_{\pm}(n, z)M_{\pm}(n, z) = (zI - B(n))(M_{\pm}(n, z) \pm I)(\mp IM_{\pm}(n, z))^{-1}.
\]

The transfer matrices also relate the \( m \) functions at \( n \) and \( n - 1 \) as shown in the following lemma:

**Lemma 3.4** \textit{For any} \( z \in \mathbb{C}^+ \),

\[
M_{\pm}(n, z) = T_{\pm}(n, z)M_{\pm}(n - 1, z)
\]

\textit{Proof} \textit{Since} \( F_{\pm}(n, z) \) \textit{are solutions to Eq. (1), we have}

\[
F_{\pm}(n + 1, z) + F_{\pm}(n - 1, z) + B(n)F_{\pm}(n, z) = zF_{\pm}(n, z).
\]
This provides
\[ F_{\pm}(n + 1, z) + F_{\pm}(n - 1, z) = (z - B(n))F_{\pm}(n, z). \]

Equivalently,
\[ F_{\pm}(n + 1, z)F_{\pm}(n, z)^{-1} + F_{\pm}(n - 1, z)F_{\pm}(n, z)^{-1} = (z - B(n)). \]

Consider the following calculation:
\[
T_+(n, z)M_+(n - 1, z) = \begin{pmatrix} z - B(n) & I \\ -I & 0 \end{pmatrix} M_+(n - 1, z) \\
= \left( (z - B(n))M_+(n - 1, z) + I \right)(M_+(n - 1, z)^{-1}) \\
= \left( - (z - B(n))F_+(n, z)F_+(n - 1, z)^{-1} + I \right)(F_+(n, z)F_+(n - 1, z)^{-1})^{-1} \\
= - (z - B(n)) + F_+(n - 1, z)F_+(n, z)^{-1}
\]

Using (22), we obtain
\[ T_+(n, z)M_+(n - 1, z) = -F_+(n + 1, z)F_+(n, z)^{-1} = M_+(n, z). \]

This shows the lemma is true for \( M_+(n, z) \). A similar calculation shows \( M_-(n, z) = T_-(n, z)M_-(n - 1, z) \).

This completes the proof.

Moreover, these matrices \( T_{\pm}(n, z) \) satisfy the symplectic-type identity. Let
\[ J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \]

**Lemma 3.5** For all \( n \in \mathbb{N} \) and all Hermitian \( d \times d \) matrices \( B(n) \), \( T_{\pm}(n, z) \) satisfy the following identities:

1. \( T_+(n, z)^*JT_+(n, z) = J \).
2. \( J\overline{T}_+(n, z)J = T_-(n, z) \).

**Proof** These identities follow from matrix multiplication. Indeed, we have
This shows the first identity. Similarly, we have

\[
T_\pm(n, \bar{z})^* J T_\pm(n, z) = \begin{pmatrix} zI - B(n) & \mp I \\ \mp I & 0 \end{pmatrix}^* \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} zI - B(n) & \mp I \\ \mp I & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} zI - B(n) & \mp I \\ \mp I & 0 \end{pmatrix} \begin{pmatrix} -zI + B(n) & \mp I \\ \mp I & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]

\[
= J
\]

This shows the first identity. Similarly, we have

\[
J T_+(n, z) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} zI - B(n) & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]

\[
= \begin{pmatrix} zI - B(n) & -I \\ I & 0 \end{pmatrix}
\]

\[
= T_-(n, z)
\]

showing the second identity and completing the proof. \(\square\)

If \(B(n)\) is a real symmetric matrix, then the first identity in Lemma 3.5 reduces to the symplectic identity

\[
T_\pm(n, z)^T J T_\pm(n, z) = J
\]

and, therefore, \(T_\pm(n, z) \in \text{SL}(2d, \mathbb{C})\) which is the group of \(2d \times 2d\) complex symplectic matrices. These are matrix-valued functions mapping complex upper half plane \(\mathbb{C}^+\) to \(\text{SL}(2d, \mathbb{C})\). From now on we consider (1) with real and symmetric matrix \(B(n)\). From Lemma 3.2 and Proposition 3.3 that for any \(n\) and any \(z \in \mathbb{C}^+\), \(M_\pm(n, z)\) are symmetric and have positive definite imaginary part. Thus as a function of \(z \in \mathbb{C}^+\), these are matrix-valued Herglotz functions mapping complex upper half plane to Siegel upper half plane. This is a space of complex matrices with positive definite imaginary part:

\[
S_d = \{Z \in \mathbb{C}^{d \times d} : Z = X + iY, X^T = X, Y^T = Y, Y > 0\}
\]

If \(T \in \text{SL}(2d, \mathbb{R})\), the mapping \(Z \mapsto T(Z)\) in Eq. (18) is a well defined map and is a group action on \(S_d\), see [2] for details. The transfer matrices \(T_\pm(n, z)\), however, are complex symplectic matrices acting on the space of \(m\) functions. In fact, if \(T \in \text{SL}(2d, \mathbb{C})\) is a complex symplectic matrix, then [2] provided a necessary condition so that the map \(Z \mapsto T(Z)\) is well defined, namely, that \(i(T^* J T - J)\) is positive. Thus, we can extend the action of \(T_\pm(n, z)\) on \(S_d\) and show that the map \(Z \mapsto T_\pm(n, z)(Z)\) on \(S_d\) is well defined. More precisely, we have the following lemma.

**Lemma 3.6** For any \(z \in \mathbb{C}^+\), if \(B(n)\) is real and symmetric, then \(T_\pm(n, z)(Z) \in S_d\) for all \(Z \in S_d\).
Proof. For any \( z \in \mathbb{C}^+ \), the second identity in Lemma 3.5, implies \( T_+(n, z) \) and \( T_-(n, z) \) are conjugate to each other. Therefore, it is enough to show that \( T_-(n, z)(Z) \in S_d \).

Write the transfer matrix \( T_-(n, z) \) as the following product of matrices:

\[
T_-(n, z) = \begin{pmatrix} I & -B(n) \\ 0 & I \end{pmatrix} \begin{pmatrix} I & zJ \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} := T_BT_J.
\]

Since \( J \) is a symplectic matrix, so is \(-J\). So the matrix \( TB \in SL(2d, \mathbb{R}) \). So \( T_j(Z) \) and \( TB(W) \) are in \( S_d \) for any \( Z \) and \( W \) in \( S_d \). Claim: \( T_j(Z) \in S_d \) for any \( Z \in S_d \). But this follows easily since \( T_j(Z) = Z + zI \) which is clearly in \( S_d \). \( \square \)

Next, we observe the distance properties of Weyl \( m \) functions. In one dimensional space, hyperbolic metrics are commonly used to analyze the value distribution for solutions of Schrödinger equations.

Define the following map on \( S_d \), \( d_\infty : S_d \times S_d \to \mathbb{R} \) via

\[
d_\infty(Z_1, Z_2) = \inf_{Z(t)} \int_0^1 F_{Z(t)}(\dot{Z}(t)) \, dt, \quad Z_1, Z_2 \in S_d
\]

where

\[
F_Z(W) = \| Y^{-1/2}WY^{-1/2} \|,
\]

and where the infimum is taken over all differentiable paths \( Z(t) \) joining \( Z_1 \) to \( Z_2 \). Here \( Y \) is a positive definite matrix and possesses the square root and \( Y^{-1/2} = (Y^{1/2})^{-1} = (Y^{-1})^{1/2} \). The norm in Eq. (24) is the operator norm of matrices acting on \( \mathbb{C}^d \).

A calculation shows that \( d_\infty \) is a metric on \( S_d \) and hence \( (S_d, d_\infty) \) is a metric space.

Lemma 3.7 For any \( d \times d \) positive definite matrix with real entries and any positive \( \lambda \):

\[
A^{1/2}(A + \lambda I)^{-1/2} = (A(A + \lambda I)^{-1})^{1/2} = ((I + \lambda A^{-1})^{-1})^{1/2}.
\]

Moreover, if \( I - \lambda A \) is positive, then

\[
\| A^{1/2}(A + \lambda I)^{-1/2} \|^2 < \frac{1}{1 + \lambda^2}.
\]

Proof. The first equation follows from the continuous functional calculus with the square root function on \([0, \infty)\) and the fact that \( \lambda > 0 \). The norm inequality now follows from the first equation and the usual norm estimate on a Neumann series of operators. \( \square \)

For any \( T \in SL(2d, \mathbb{R}) \), when considering the map from (18) as a group action, [2] showed that this action, in fact, is distance preserving with respect to the metric \( d_\infty \). That is,
**Theorem 3.8** For any $T \in SL(2d, \mathbb{R})$ and any $W_1, W_2 \in S_d$ we have
\[
d_\infty(T(W_1), T(W_2)) = d_\infty(W_1, W_2).
\]

In general, if $T \in SL(2d, \mathbb{C})$, Theorem 3.8 need not be true. However, the map given in Eq. (18) with transfer matrices $T_\pm(n)$ is distance decreasing. We have the following theorem.

**Theorem 3.9** Let $z \in \mathbb{C}^+$ and suppose $W_j = T_-(0, z)Z_j$ for $Z_1$ and $Z_2$ in $S_d$. Then
\[
d_\infty(T_-(n, z)W_1, T_-(n, z)W_2) \leq \frac{1}{1 + y^2} d_\infty(W_1, W_2),
\]
for all $n \in \mathbb{N}$.

**Proof** As in the proof of Lemma 3.6 Write $T_-(n, z) = T_B T_z T_J$. Since $T_B$ and $T_J$ are in $SL(2d, \mathbb{R})$, and the fact that $T_z$ has range in $S_d$, by Theorem 3.8, we get
\[
d_\infty(T_-(n, z)(W_1), T_-(n, z)(W_2)) = d_\infty(T_B T_z T_J(W_1), T_B T_z T_J(W_2)) = d_\infty(T_z T_J(W_1), T_z T_J(W_2)).
\]

Moreover, since $T_J \in SL(2d, \mathbb{R})$, we have
\[
d_\infty(T_J(W_1), T_J(W_2)) = d_\infty(W_1, W_2).
\]
It, therefore, follows that $T_-(n, z)(W) \in S_d$.

Let $U_j = T_J(W_j) = JW_j$, for $j = 1, 2$. Then $T_z(U_j) = U_j + zI$. Let $U(t)$ be a length minimizing path between $U_1$ and $U_2$. Then $U(t) + zI$ is a path between $U_1 + zI$ and $U_2 + zI$. Moreover
\[
\frac{d}{dt}(U(t) + zI) = \dot{U}(t).
\]
Suppose $V(t) = \text{Im} U(t)$ then $\text{Im}(U(t) + zI) = V(t) + yI$. We have the following calculation:
\[
F_{(U(t) + zI)}(\dot{U}(t)) = \|(V(t) + yI)^{-1/2} \dot{U}(t)(V(t) + yI)^{-1/2}\|
\leq \|(V(t) + yI)^{-1/2} V(t)^{1/2} \dot{U}(t)V(t)^{-1/2}(V(t) + yI)^{-1/2}\|
\leq \|(V(t) + yI)^{-1/2} V(t)^{1/2} \| \cdot \| V(t)^{-1/2} \dot{U}(t)V(t)^{-1/2} \| \cdot \| V(t)^{1/2}(V(t) + yI)^{-1/2} \|.
\]
Since $V(t)^{1/2}$ and $(V(t) + yI)^{-1/2}$ are symmetric, it follows that
\[
\|(V(t) + yI)^{-1/2} V(t)^{1/2}\| = \|V(t)^{1/2}(V(t) + yI)^{-1/2}\|.
\]
Therefore, from the symmetry and the above inequality, we have
\[
F_{(U(t) + zI)}(\dot{U}(t)) \leq \|(V(t) + yI)^{-1/2} V(t)^{1/2}\|^2 \| V(t)^{-1/2} \dot{U}(t)V(t)^{-1/2} \|.
\]
Next, since $U_j = JW_j$, for $j = 1, 2$, $U(t)$ must be of the form $JW(t)$, where $W(t)$ is a path between $W_1$ and $W_2$. However, since $W_j = T_-(0, z)Z_j$, $W$ is of the form...
\[ W = T_-(0, z)Z, \] where \( Z(t) = X(t) + iY(t) \) is a path between \( Z_1 \) and \( Z_2 \). By viewing the multiplication as the group action, we have the following

\[
W(t) = T_-(0, z)Z(t) = \left( \begin{array}{cc} zI - B(0) & I \\ -I & 0 \end{array} \right) Z(t) = zI - B(0) - Z(t)^{-1},
\]

where \(|Z| = X^2 + Y^2\). Since \( Y \) is positive and \( \text{Im}W(t) = yI + Y(X^2 + Y^2)^{-1} \), it follows that \( \text{Im}W(t) - yI \) is also positive. Thus by applying the continuous functional calculus to \( \text{Im}W(t) - yI \) with the function \( f(x) = 1/x \) on \((0, \infty)\), we get

\[
(\text{Im}W(t))^{-1} - \frac{1}{y} I < 0. \tag{28}
\]

Since \( U(t) = JW(t) = -W(t)^{-1} \), by applying the continuous functional calculus again, we have

\[
U(t) = -W(t)^{-1} = -W(t)|W(t)|^{-1} = -W(t)(\text{Re}W(t))^2 + (\text{Im}W(t))^2)^{-1}.
\]

Therefore

\[
\text{Im}U(t) = \text{Im}W(t)((\text{Re}W(t))^2 + (\text{Im}W(t))^2)^{-1}.
\]

Since

\[
(\text{Im}W(t))^2 \leq (\text{Re}W(t))^2 + (\text{Im}W(t))^2,
\]

using the functional calculus again with the function \( f(x) = 1/x \) on \((0, \infty)\) and using inequality (28), we obtain

\[
\text{Im}W(t)((\text{Re}W(t))^2 + (\text{Im}W(t))^2)^{-1} \leq \text{Im}W(t)^{-1} < \frac{1}{y} I.
\]

This implies that \( I - y\text{Im}U(t) \) is positive and hence \( I - y\text{Im}V(t) \) is positive. By Lemma 3.7, we get

\[
\|(V + yI)^{-1/2}V^{1/2}\| < \frac{1}{1 + y^2}.
\]

Using Eq. (27) becomes

\[
F_{(U(t)+z)}(\dot{U}(t)) \leq \|(V(t) + yI)^{-1/2}V^{1/2}\|^2\|V(t)^{-1/2}\dot{U}(t)V(t)^{-1/2}\|
\]

\[
\leq \frac{1}{1 + y^2}F_{(U(t))}(\dot{U}(t)).
\]

Integrating the above inequality yields:

\[
\int_0^1 F_{(U(t)+z)}(\dot{U}(t)) \, dt \leq \frac{1}{1 + y^2} \int_0^1 F_{(U(t))}(\dot{U}(t)) \, dt.
\]
Then, taking the infimum over all such paths $U(t)$, we obtain the following:

$$d_\infty(T_\zeta U_1, T_\zeta U_2) \leq \frac{1}{1 + y^2} d_\infty(U_1, U_2).$$

Again, since $J \in \text{SL}(2d, \mathbb{R})$, we have

$$d_\infty(U_1, U_2) = d_\infty(T_J W_1, T_J W_2) = d_\infty(W_1, W_2).$$

Using the previous two equations in Eq. (26), we obtain:

$$d_\infty(T_{(n, z)} W_1, T_{(n, z)} W_2) \leq \frac{1}{1 + y^2} d_\infty(W_1, W_2)$$

which completes the proof. \hfill \Box

Let $P_{(n, z)} = T_{(n, z)} \cdots T_{(1, z)}$. Then, since $T_{(j, z)}$ are symplectic matrices for $j = 1, \ldots, n$ and $\text{SL}(2d, \mathbb{C})$ is a group, it follows that $P_{(n, z)} \in \text{SL}(2d, \mathbb{C})$. We have the following corollary.

**Corollary 3.10** For any $z \in \mathbb{C}^+$,

$$d_\infty(M_{(n, z)} P_{(n, z)} M_{(0, z)}) \leq \frac{1}{(1 + y^2)^n} d_\infty(M_{(0, z)} M_{(0, z)}).$$

**Proof** For any $z \in \mathbb{C}^+$ and $n \in \mathbb{N}$ by Lemma 3.4, we have

$$M_{(n, z)} = P_{(n, z)} M_{(0, z)}.$$

Hence, the result follows from Theorem 3.9. \hfill \Box

We define another $m$ function on the left half line $(-\infty, n)$. Assume that $F_{(n, z)}$ is a square summable matrix-valued solution to Eq. (1) defined by Eq. (7) on $(-\infty, n)$. Define $\tilde{M}_{(n, z)}$ by the following:

$$\tilde{M}_{(n, z)} = -F_{(n - 1, z)} F_{(n, z)}^{-1}.$$

Similar to the proofs in Lemma 3.2 and Proposition 3.3, we see that $\tilde{M}_{(n, z)}$ is symmetric and $\text{Im}\tilde{M}_{(n, z)}$ is a positive operator. In addition, $\tilde{M}_{(n, z)}$ and $M_{(n, z)}$ are related by the following equation:

$$\tilde{M}_{(n, z)} = M_{(n, z)} - zI + B(n).$$

For fixed $n \in \mathbb{N}$, the mappings $z \mapsto M_{\pm}(n, z)$ are matrix valued Herglotz functions. Set $M_{\pm}(z) = M_{\pm}(0, z)$. Since $M_{\pm}$ is a matrix-valued Herglotz function, by Riesz–Herglotz representation, there exists a matrix-valued measure $\mu_+$ on the bounded Borel subset of $\mathbb{R}$, so that
for some constant matrices $C$ and $D$. This integral representation of $M_+$ is a well-known result, see [3, 10, 17]. In addition, $M_+(z)$ has finite normal limits, that is

$$M_+(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} M_+(\lambda \pm i\varepsilon)$$

for almost all $\lambda \in \mathbb{R}$. More on matrix-valued Herglotz functions are explained in the same papers. By the Lebesgue Decomposition Theorem, the matrix-valued Borel measure $\mu_+$ can be decomposed as

$$\mu_+ = \mu_{+,ac} + \mu_{+,sc} + \mu_{+pp},$$

where $\mu_{+,ac}$ is the measure that is absolutely continuous with respect to $\mu_+$, $\mu_{+,sc}$ is the singular continuous part of $\mu_+$ and $\mu_{+pp}$ is the pure point part of $\mu_+$. Let $\Sigma$, $\Sigma_{ac}$, $\Sigma_{sc}$, and $\Sigma_{pp}$ be the topological support of $\mu_+$, $\mu_{+,ac}$, $\mu_{+,sc}$, and $\mu_{+pp}$, respectively.

The following theorem from [10] showed the connection between the matrix-valued Herglotz functions and the essential support of corresponding spectral measures. We state it without proof.

$$\Sigma_{ac} = \bigcup_{r=1}^{d} \left\{ \lambda \in \mathbb{R} : \lim_{\varepsilon \downarrow 0} \text{Im}(M_+(\lambda + i\varepsilon)) = r \right\}$$

$$\Sigma_{sc} = \bigcup_{r=1}^{d} \left\{ \lambda \in \mathbb{R} : \lim_{\varepsilon \downarrow 0} \text{Im}(\text{tr } M_+(\lambda + i\varepsilon)) = \infty, \lim_{\varepsilon \downarrow 0} \text{tr } M_+(\lambda + i\varepsilon) = 0 \right\}$$

$$\Sigma_{pp} = \bigcup_{r=1}^{d} \left\{ \lambda \in \mathbb{R} : \text{rank } \lim_{\varepsilon \downarrow 0} \varepsilon M_+(\lambda + i\varepsilon) = r \right\}.$$  

**Theorem 3.11**

Describing the absolutely continuous spectrum is one of the main goals in studying vector-valued discrete Schrödinger operators. Remling successfully did this for one dimensional Schrödinger operators, [13], and in his renowned paper [12], he describes it for Jacobi matrices. The techniques utilized there involve the concept of reflectionless. An operator is said to be reflectionless on a Borel set $B \subseteq \mathbb{R}$ if the associated Herglotz functions satisfy $m_+(x) = -\overline{m}_-(x)$ for almost all $x \in B$ in the Lebesgue sense. This definition naturally extends to our matrix-valued Herglotz functions $M_+$. The above theorem is also utilized in describing the absolutely continuous spectrum as yields precise limit conditions on the Herglotz functions. Describing the absolutely continuous spectrum is an extremely difficult question and this is one of our main goals in future work.

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References

1. Acharya, K.R.: Titchmarsh–Weyl theory for vector-valued discrete Schrödinger operators. Anal. Math. Phys. 9, 1831–1847 (2019)
2. Acharya, K.R., McBride, M.: Action of complex symplectic matrices on the Siegel upper half space. Linear Algebra Appl. 563, 47–62 (2019)
3. Clark, S.L., Gesztesy, F.: Weyl–Titchmarsh M-function asymptotics for matrix valued Schrödinger operators. Proc. Lond. Math. Soc. 82, 701–720 (2001)
4. Clark, S.L., Gesztesy, F., Holden, H., Levitan, B.M.: Borg-type theorems for matrix-valued Schrödinger operators. J. Differ. Equ. 167, 181–210 (2000)
5. Clark, S.L., Zemánek, P.: On a Weyl–Titchmarsh theory for discrete symplectic systems on a half line. Appl. Math. Comput. 217(7), 2952–2976 (2010)
6. Clark, S.L., Zemánek, P.: On discrete symplectic systems: Associated maximal and minimal linear relations and nonhomogeneous problems. J. Math. Anal. Appl. 421(1), 779–805 (2015)
7. Damanik, D., Pushnitski, A., Simon, B.: The analytic theory of matrix orthogonal polynomials. Surv. Approx. Theory 4, 1–85 (2008)
8. Eckhardt, J., Gesztesy, F., Nichols, R., Sakhnovich, A., Teschl, G.: Inverse spectral problems for Schrödinger-type operators with distributional matrix-valued potentials. Differ. Integral Equ. 28, 505–522 (2015)
9. Geronimo, J.S.: Scattering theory and matrix orthogonal polynomials on the real line. Circuit Syst. Signal Process. 1, 472–495 (1982)
10. Gesztesy, F., Rsekanovskii, E.: On matrix-valued Herglotz functions. Math. Nachr. 218, 61–138 (2000)
11. Kozhan, R.: Equivalence classes of block Jacobi matrices. Proc. Am. Math. Soc. 139, 799–805 (2011)
12. Remling, C.: The absolutely continuous spectrum of Jacobi matrices. Ann. Math. (2) 17(1), 125–171 (2011)
13. Remling, C.: The absolutely continuous spectrum of one-dimensional Schrödinger operators. Math. Phys. Anal. Geom. 10, 359–373 (2007)
14. Shi, Y.: Weyl–Titchmarsh theory for a class of discrete linear Hamiltonian systems. Linear Algebra Appl. 416(2–3), 452–519 (2006)
15. Šimon Hilscher, R., Zemánek, P.: Weyl-Titchmarsh theory for discrete symplectic systems with general linear dependence on spectral parameter. J. Differ. Equ. Appl. 20(1), 84–117 (2014)
16. Teschl, G.: Jacobi Operators and Completely Integrable Nonlinear Lattices, Mathematical Monographs and Surveys, vol. 72. American Mathematical Society, Providence (2000)
17. Zemánek, P.: Eigenfunctions expansion for discrete symplectic systems with general linear dependence on spectral parameter. J. Math. Anal. Appl. 499(2), 1–37 (2021). (Article no. 125054)