Relativistic two-dimensional hydrogen-like atom in a weak magnetic field

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Abstract

A two-dimensional (2D) hydrogen-like atom with a relativistic Dirac electron, placed in a weak, static, uniform magnetic field perpendicular to the atomic plane, is considered. Closed forms of the first- and second-order Zeeman corrections to energy levels are calculated analytically, within the framework of the Rayleigh–Schrödinger perturbation theory, for an arbitrary electronic bound state. The second-order calculations are carried out with the use of the Sturmian expansion of the two-dimensional generalized radial Dirac–Coulomb Green function derived in the paper. It is found that, in contrast to the case of the three-dimensional atom [P. Stefańska, Phys. Rev. A 92 (2015) 032504], in two spatial dimensions atomic magnetizabilities (magnetic susceptibilities) are expressible in terms of elementary algebraic functions of a nuclear charge and electron quantum numbers. The problem considered here is related to the Coulomb impurity problem for graphene in a weak magnetic field.

Key words: Two-dimensional (2D) atom; Dirac equation; Zeeman effect; Coulomb Green function; Sturmian functions; Magnetic susceptibility

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1 Introduction

Properties of model two-dimensional hydrogenic systems immersed in a magnetic field have been investigated for several decades within the frameworks of nonrelativistic [1–31] and relativistic [32–43] quantum mechanics. Besides of being interesting from a purely theoretical point of view, results of such studies are also important for understanding various aspects of physics of low-dimensional semiconductors [1–3,6–8,10,12,15,18,26] and of graphene [44–51]. The subject is still far from being exhausted, and further research in this area, especially the one based on the use of analytical methods, is certainly demanded.

The present paper meets that need. On the following pages, we shall consider the planar hydrogen-like atom subjected to the action of a static, uniform and weak magnetic field perpendicular to the atomic plane. The main assumptions about the system are: (i) the interaction potential between an electron and a nucleus, with the latter taken to be point-like and spinless, is the three-dimensional one-over-distance attractive Coulomb potential, (ii) the electron is relativistic in the sense that its constrained planar dynamics is governed by the two-dimensional Dirac equation. With these premises, within the framework of the Rayleigh–Schrödinger perturbation theory, we shall derive closed-form analytical expressions for the first- and second-order Zeeman corrections to an arbitrary atomic fine-structure energy level. The reader will see that while the first-order

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calculations, presented in Sec. 3, are straightforward and require the knowledge of unperturbed planar Dirac–Coulomb eigenfunctions only, the second-order analysis appears to be quite challenging. As the standard sum-over-eigenstates formula for $E^{(2)}$ is of no practical use in the present context (since the energy spectrum of the Dirac–Coulomb Hamiltonian is mixed and in addition to discrete eigenvalues it contains two scattering continua as well), in Sec. 4 we shall exploit an alternative representation of $E^{(2)}$ involving the radial Dirac–Coulomb Sturmian functions. This will lead us eventually to a relatively simple analytical formula for a magnetizability (magnetic susceptibility) of a relativistic two-dimensional hydrogen-like atom in an arbitrary discrete energy eigenstate.

2 Setting the problem

Consider a planar one-electron atom with a motionless, point-like and spinless nucleus of electric charge $Ze$, embedded in a static uniform magnetic field of induction $B$ perpendicular to the atomic plane. Stationary energy levels of the atomic electron in such a system are eigenvalues of the Dirac equation

$$\begin{aligned}
\left\{ c\alpha \cdot [-i\hbar \nabla + eA(r)] + \beta mc^2 - \frac{Ze^2}{(4\pi\varepsilon_0)r} - E \right\} \Psi(r) = 0 \quad (r \in \mathbb{R}^2),
\end{aligned}$$

(2.1a)

which is to be solved subject to the constraints that the wave function $\Psi(r)$ is single-valued and forced to satisfy the boundary conditions

$$\begin{aligned}
\sqrt{r} \Psi(r) \to 0 \quad \text{as} \quad r \to 0, \\
\sqrt{r} \Psi(r) \to 0 \quad \text{as} \quad r \to \infty.
\end{aligned}$$

(2.1b)

In Eq. (2.1a), $\alpha$ is the matrix vector defined as

$$\alpha = \alpha_1 n_x + \alpha_2 n_y$$

(2.2)

($n_x$ and $n_y$ are the unit vectors along axes of a Cartesian $\{x,y\}$ coordinate system in the atomic plane), with

$$\begin{aligned}
\alpha_1 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \\
\alpha_2 &= \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix},
\end{aligned}$$

(2.3)

where $\sigma_1$ and $\sigma_2$ are the Pauli matrices

$$\begin{aligned}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\end{aligned}$$

(2.4)

while $\beta$ is a $4 \times 4$ matrix of the form

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

(2.5)

where $I$ is the unit $2 \times 2$ matrix. In the symmetric gauge used in this work, the vector potential $A(r)$ is taken to be

$$A(r) = \frac{1}{2}B \times r.$$  

(2.6)

The Dirac equation (2.1a) is separable in the standard polar coordinates $r, \varphi$ (we choose the polar axis along the unit vector $n_x$), in the sense that it possesses particular solutions of the form

$$\Psi_{\nu \kappa \mu m}(r, \varphi) = \frac{1}{\sqrt{r}} \begin{pmatrix} P_{\nu \kappa \mu m}(r) \Phi_{\kappa \mu m}(\varphi) \\ iQ_{\nu \kappa \mu m}(r) \Phi_{-\kappa \mu m}(\varphi) \end{pmatrix},$$

(2.7)

where

$$\Phi_{\kappa \mu m}(\varphi) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \delta_{-\kappa,\mu} e^{i(m_{\kappa}-1/2)\varphi} \\ \delta_{\kappa,\mu} e^{i(m_{\kappa}+1/2)\varphi} \end{pmatrix} \quad (\kappa = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots; m_{\kappa} = \pm \kappa)$$

(2.8)
are the axial spinors introduced by Poszwa and Rutkowski [42] (for a summary of properties of these spinor functions, see Appendix A at the end of the present paper; the reader is warned that the quantum number $\kappa$ we use here has the opposite sign in relation to the one that appeared in Refs. [42, 43, 52]). If we insert Eq. (2.7) into Eq. (2.1a), and then exploit the identities (A.14), (A.11b) and (A.15), we find that the radial spinor

$$
\psi_{n\kappa m}(r) = \begin{pmatrix} P_{n\kappa m}(r) \\ Q_{n\kappa m}(r) \end{pmatrix}
$$

(2.9)
solves the equation

$$
[H_{\kappa m}(r) - E_{n\kappa m}](\psi_{n\kappa m}(r) = 0
$$

(2.10a)
subject to the boundary conditions

$$
\psi_{n\kappa m}(r) \rightarrow 0, \quad \psi_{n\kappa m}(r) \rightarrow 0,
$$

(2.10b)
with the radial Hamiltonian

$$
H_{\kappa m}(r) = \begin{pmatrix} mc^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} & -ch \left( -\frac{d}{dr} + \frac{\kappa}{r} \right) - \frac{1}{2}\epsilon ceBr \\
-ch \left( \frac{d}{dr} + \frac{\kappa}{r} \right) & -mc^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} \end{pmatrix}
$$

(2.11)
and with $E_{n\kappa m}$ being the energy eigenvalue. We label the eigensolutions with three quantum numbers $n$, $\kappa$, $m$. The latter two have been defined in Eq. (2.8) (cf. also Appendix A; the reader should observe that, in contrast to the counterpart three-dimensional problem, in the present case the quantum number $\kappa$ is a half-integer), while the first one — the principal quantum number $n$, is defined to be

$$
n = n_r + |\kappa| + \frac{1}{2}.
$$

(2.12)
The radial quantum number $n_r$ appearing in Eq. (2.12) is defined so that the number of nodes of $P_{n\kappa m}(r)$ in the open interval $(0, \infty)$ is $n_r$ for $\kappa \leq -\frac{1}{2}$ (in that case $n_r \in \mathbb{N}_0$) and $n_r - 1$ for $\kappa \geq \frac{1}{2}$ (in that case $n_r \in \mathbb{N}_+$).

For $Z \neq 0$ and $B \neq 0$, no general method of obtaining analytical solutions to the system (2.10) is known, and consequently one is relied on the use of approximations. If the external magnetic field is weak, as it will be assumed from now on, one may exploit the Rayleigh–Schrödinger perturbation theory. To this end, we split the radial Hamiltonian (2.11) in the following manner:

$$
H_{\kappa m}(r) = H_{\kappa}^{(0)}(r) + H_{\kappa m}^{(1)}(r),
$$

(2.13)
with the zeroth-order operator being the radial Dirac–Coulomb Hamiltonian

$$
H_{\kappa}^{(0)}(r) = \begin{pmatrix} mc^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} & -ch \left( -\frac{d}{dr} + \frac{\kappa}{r} \right) \\
-ch \left( \frac{d}{dr} + \frac{\kappa}{r} \right) & -mc^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} \end{pmatrix}
$$

(2.14)
and with the perturbing operator being

$$
H_{\kappa m}^{(1)}(r) = -\frac{1}{2}\epsilon ceBr \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

(2.15)
In concordance with the partition (2.13), we shall be seeking solutions to the eigensystem (2.10) in the form of the Rayleigh–Schrödinger series

$$
E_{n\kappa m} = E_{n \kappa m}^{(0)} + E_{n \kappa m}^{(1)} + E_{n \kappa m}^{(2)} + \cdots
$$

(2.16a)
and

$$
\psi_{n\kappa m}(r) = \psi_{n \kappa m}^{(0)}(r) + \psi_{n \kappa m}^{(1)}(r) + \psi_{n \kappa m}^{(2)}(r) + \cdots.
$$

(2.16b)
where
\[ \psi^{(k)}(r) = \begin{pmatrix} P^{(k)}(r) \\ Q^{(k)}(r) \end{pmatrix}; \] (2.17)

the superscripts indicate orders of individual terms with respect to the magnetic induction strength \( B \).

The zeroth-order terms in the series (2.16a) and (2.16b) are solutions to the radial bound-state Dirac–Coulomb problem

\[ [H^{(0)}_{\kappa}(r) - E_{n\kappa}^{(0)}] \psi_{n\kappa}^{(0)}(r) = 0, \] (2.18a)
\[ \psi_{n\kappa}^{(0)}(r) \xrightarrow{r \to 0} 0, \quad \psi_{n\kappa}^{(0)}(r) \xrightarrow{r \to \infty} 0. \] (2.18b)

Solving the system (2.18) as in the three-dimensional case, bound-state energy levels of the electron in an isolated planar atom are found to be

\[ E_{n\kappa}^{(0)} = \frac{mc^2 n_r + \gamma_\kappa}{N_{n\kappa}} = \frac{mc^2}{\sqrt{1 + (\alpha Z)^2}}, \] (2.19)

where
\[ N_{n\kappa} = \sqrt{n_r^2 + 2n_r \gamma_\kappa + \kappa^2} \] (2.20)
and
\[ \gamma_\kappa = \sqrt{\kappa^2 - (\alpha Z)^2}, \] (2.21)

with \( \alpha = e^2/(4\pi\epsilon_0)c \) being the Sommerfeld fine-structure constant. To ensure that \( \gamma_\kappa \) is real for all admitted values of \( \kappa \), we impose the constraint
\[ Z < \frac{1}{2} \alpha^{-1}. \] (2.22)

The corresponding radial wave functions, orthonormal in the sense of
\[ \int_0^\infty dr \psi_{n\kappa}^{(0)T}(r)\psi_{n'\kappa}^{(0)}(r) = \delta_{n'n}, \] (2.23)
(the superscript \( T \) denotes the matrix transpose), may be shown to have the components

\[ P_{n\kappa}^{(0)}(r) = \begin{pmatrix} Z(1 + \epsilon_{n\kappa}^{(0)})n_r!(n_r + 2\gamma_\kappa) \\ 2a_0N_{n\kappa}^2(N_{n\kappa} - \kappa)\Gamma(n_r + 2\gamma_\kappa) \end{pmatrix} \frac{2Zr}{N_{n\kappa}a_0} \gamma_\kappa \] \[ \times \left[ L_{n_r-1}^{(2\gamma_\kappa)} \left( \frac{2Zr}{N_{n\kappa}a_0} \right) - \frac{N_{n\kappa} - \kappa}{n_r + 2\gamma_\kappa} L_{n_r}^{(2\gamma_\kappa)} \left( \frac{2Zr}{N_{n\kappa}a_0} \right) \right], \] (2.24a)

and

\[ Q_{n\kappa}^{(0)}(r) = -\begin{pmatrix} Z(1 - \epsilon_{n\kappa}^{(0)})n_r!(n_r + 2\gamma_\kappa) \\ 2a_0N_{n\kappa}^2(N_{n\kappa} - \kappa)\Gamma(n_r + 2\gamma_\kappa) \end{pmatrix} \frac{2Zr}{N_{n\kappa}a_0} \gamma_\kappa \] \[ \times \left[ L_{n_r-1}^{(2\gamma_\kappa)} \left( \frac{2Zr}{N_{n\kappa}a_0} \right) + \frac{N_{n\kappa} - \kappa}{n_r + 2\gamma_\kappa} L_{n_r}^{(2\gamma_\kappa)} \left( \frac{2Zr}{N_{n\kappa}a_0} \right) \right], \] (2.24b)

where for brevity we have denoted
\[ \epsilon_{n\kappa}^{(0)} = \frac{E_{n\kappa}^{(0)}}{mc^2} = \frac{n_r + \gamma_\kappa}{N_{n\kappa}}, \] (2.25)

and where \( L_{n}^{(\alpha)}(\rho) \) stands for the generalized Laguerre polynomial \cite[Sec. 5.5]{53}; we define \( L_{n}^{(\alpha)}(0) \equiv 0 \). The reader should observe that for \( \kappa \geq \frac{1}{2} \) and \( n_r = 0 \) (i.e., for \( n = \kappa + \frac{1}{2} \)) the expressions in
the square braces in both Eqs. (2.24a) and (2.24b) do vanish. Consequently, there are no planar Dirac–Coulomb bound states in that case.

Each of the energy levels (2.19) associated with a given value of $n$ is seen to be fourfold degenerate (twice with respect to the sign of $\kappa$ and, after the latter is fixed, twice with respect to the sign of $m_\kappa$); the latter is only doubly degenerate (with respect to the sign of $m_\kappa$). The sum of degeneracies of all levels corresponding to a particular value of $n$ is $2(2n - 1)$.

It is possible to classify planar atomic states according to a quasi-spectroscopic scheme, proposed by Poszwa and Rutkowski [42] and resembling the one used for atoms in three dimensions. Within the framework of that scheme, which we shall adopt hereafter, an atomic state with given quantum numbers $n$ and $\kappa$ is labeled as $nl|\kappa|$, where

$$l = |\kappa + \frac{1}{2}| \quad (2.26)$$

(Poszwa and Rutkowski [42] defined $l = |\kappa - \frac{1}{2}|$, but we recall that their $\kappa$ had the opposite sign), with the usual letter designation $l = 0 \rightarrow s$, $l = 1 \rightarrow p$, $l = 2 \rightarrow d$, etc. (2.27)

Examples of the use of that classification scheme are given in Table I.

|Place for Table I|

3 The first-order Zeeman corrections to the Dirac–Coulomb energy levels

Since the radial zeroth-order wave functions $\psi^{(0)}_{nm}(r)$ are normalized to unity [cf. Eq. (2.23)], the first-order contribution $E^{(1)}_{nm\kappa}$ to $E_{nm\kappa}$ is simply given by

$$E^{(1)}_{nm\kappa} = \int_0^\infty dr \psi^{(0)T}_n(r)H^{(1)}_{nm\kappa}(r)\psi^{(0)}_{nm}(r). \quad (3.1)$$

With the use of Eqs. (2.15) and (2.17), Eq. (3.1) may be cast into the form

$$E^{(1)}_{nm\kappa} = -\frac{m_\kappa}{\kappa} eB \int_0^\infty dr rP^{(0)}_{nm}(r)Q^{(0)}_{nm}(r). \quad (3.2)$$

The radial integral in Eq. (3.2) may be taken with the aid of the identity

$$\int_0^\infty dx x^{\alpha+1}e^{-x}[F^{(\alpha)}_n(x)]^2 = \frac{(\alpha + 2n + 1)\Gamma(\alpha + n + 1)}{n!} \quad (Re \alpha > -2), \quad (3.3)$$

which results from the general formula (cf. Ref. [54], Eqs. (E54), (E56) and (E60)))

$$\int_0^\infty dx x^{\gamma}e^{-x}L^{(\alpha)}_n(x)L^{(\beta)}_n(x) = (-)^{n+n'} \sum_{k=0}^{\min(n,n')} \Gamma(k+\gamma+1) \Gamma(\gamma-\alpha) \Gamma(\gamma-\beta) \frac{\Gamma(n-k)}{k!} \frac{\Gamma(n' - k)}{n' - k} \quad (Re \gamma > -1). \quad (3.4)$$

Thus, one has

$$\int_0^\infty dr rP^{(0)}_{nm}(r)Q^{(0)}_{nm}(r) = \frac{1}{4}a_0 \left[ 1 - \frac{2\kappa(n_r + \gamma_\kappa)}{N_{n_\kappa}} \right], \quad (3.5)$$

and consequently $E^{(1)}_{nm\kappa}$ is found to be

$$E^{(1)}_{nm\kappa} = -\frac{m_\kappa}{4\kappa} \left[ 1 - \frac{2\kappa(n_r + \gamma_\kappa)}{N_{n_\kappa}} \right] \frac{B}{B_0} \frac{e^2}{(4\pi\epsilon_0)a_0}. \quad (3.6)$$
Here
\[ B_0 = \frac{\hbar}{ea_0} = \frac{m^2 c^3}{(4\pi\epsilon_0)^2 \hbar^3} \simeq 2.35 \times 10^5 \text{T} \] (3.7)
is the atomic unit of magnetic induction. For states with \( n_r = 0 \) (i.e., those with \( \kappa = -n + \frac{1}{2} \)), Eq. (3.6) simplifies and gives
\[ E_{n,-n+1/2,m_{-n+1/2}}^{(1)} = \frac{m_{-n+1/2}}{4(n - \frac{1}{2})} (2\gamma_{n-1/2} + 1) \frac{B}{B_0} \frac{e^2}{(4\pi\epsilon_0)a_0}. \] (3.8)

4 The second-order Zeeman corrections to the Dirac–Coulomb energy levels. Atomic magnetizabilities

The Rayleigh–Schrödinger perturbation theory gives the following expression for the second-order correction to energy:
\[ E_{n\kappa m}^{(2)} = \int_0^{\infty} \, dr \, \psi_{n\kappa}^{(0)T}(r) H_{n\kappa m}^{(1)}(r) \psi_{n\kappa m}(r). \] (4.1)

Here \( \psi_{n\kappa m}^{(1)}(r) \) is the first-order contribution to the radial spinor wave function. It solves the inhomogeneous equation
\[ [H_\kappa^{(0)}(r) - E_{n\kappa m}^{(0)}(r)] \psi_{n\kappa m}^{(1)}(r) = -[H_{n\kappa m}^{(1)}(r) - E_{n\kappa m}^{(1)}(r)] \psi_{n\kappa m}^{(0)}(r), \] (4.2a)
with \( E_{n\kappa m}^{(1)} \) determined in Sec. 3, subject to the boundary conditions
\[ \psi_{n\kappa m}^{(1)}(r) \xrightarrow{r \to 0} 0, \quad \psi_{n\kappa m}^{(1)}(r) \xrightarrow{r \to \infty} 0 \] (4.2b)
and subject to the further constraint
\[ \int_0^{\infty} \, dr \, \psi_{n\kappa m}^{(0)T}(r) \psi_{n\kappa m}^{(1)}(r) = 0. \] (4.2c)

A formal solution to the system (4.2) is
\[ \psi_{n\kappa m}^{(1)}(r) = -\int_0^{\infty} \, dr' \, \hat{G}_{n\kappa}^{(0)}(r, r') [H_{n\kappa m}^{(1)}(r') - E_{n\kappa m}^{(1)}(r')] \psi_{n\kappa m}^{(0)}(r'), \] (4.3)
where \( \hat{G}_{n\kappa}^{(0)}(r, r') \) is a generalized radial Dirac–Coulomb Green function associated with the unperturbed Coulomb energy level \( E_{n\kappa m}^{(0)} \). The function \( \hat{G}_{n\kappa}^{(0)}(r, r') \) is defined to be a solution to the inhomogeneous Dirac–Coulomb equation
\[ [H_\kappa^{(0)}(r) - E_{n\kappa m}^{(0)}] \hat{G}_{n\kappa}^{(0)}(r, r') = \delta(r - r') I - \psi_{n\kappa}^{(0)}(r) \psi_{n\kappa}^{(0)T}(r'), \] (4.4a)
subject to the boundary conditions
\[ \hat{G}_{n\kappa}^{(0)}(r, r') \xrightarrow{r \to 0} 0, \quad \hat{G}_{n\kappa}^{(0)}(r, r') \xrightarrow{r \to \infty} 0, \] (4.4b)
together with the orthogonality constraint
\[ \int_0^{\infty} \, dr \, \psi_{n\kappa}^{(0)T}(r) \hat{G}_{n\kappa}^{(0)}(r, r') = 0. \] (4.4c)

It is a \( 2 \times 2 \) matrix-valued function and since the Dirac–Coulomb operator is self-adjoint, it is symmetric in the sense of
\[ \hat{G}_{n\kappa}^{(0)T}(r, r') = \hat{G}_{n\kappa}^{(0)}(r', r). \] (4.5)

Application of Eq. (4.5) to Eq. (4.4c) implies the orthogonality relation
\[ \int_0^{\infty} \, dr' \, \hat{G}_{n\kappa}^{(0)}(r, r') \psi_{n\kappa}^{(0)}(r') = 0, \] (4.6)
which simplifies Eq. (4.3) to the form
\[
\psi_{n_\kappa nk}(r) = -\int_0^\infty dr' \hat{G}^{(0)}_{nk}(r, r') H^{(1)}_{nk}(r') \psi^{(0)}_{nk}(r').
\] (4.7)

Upon insertion of Eq. (4.7) into Eq. (4.11), we obtain the following formula for the second-order energy correction \(E^{(2)}_{nk\kappa}\):
\[
E^{(2)}_{nk\kappa} = -\int_0^\infty dr \int_0^\infty dr' \psi^{(0)T}_{nk}(r) H^{(1)}_{nk}(r) \hat{G}^{(0)}_{nk}(r, r') H^{(1)}_{nk}(r') \psi^{(0)}_{nk}(r').
\] (4.8)

Application of Eqs. (2.15) and (2.17) casts Eq. (4.8) into the form
\[
E^{(2)}_{nk\kappa} = -\frac{\alpha^2}{4} e^2 B^2 \int_0^\infty dr \int_0^\infty dr' \left( Q^{(0)}_{nk}(r) P^{(0)}_{nk}(r') \right) r \hat{G}^{(0)}_{nk}(r, r') r' \left( Q^{(0)}_{nk}(r') \right),
\] (4.9)
where we have made use of the fact that the ratio \(m_{\kappa}/\kappa\) is of unit modulus. Since the right-hand side of the above equation is evidently independent of \(m_{\kappa}\), the third subscript at \(E^{(2)}\) has been, and henceforth will be, dropped.

To evaluate the double integral in Eq. (4.9), one has to insert into the integrand some particular explicit representation of the generalized Green function \(\hat{G}^{(0)}_{nk}(r, r')\). The one we shall use here has a form of a series expansion in the radial Dirac–Coulomb Sturmian basis. We shall construct that expansion below, omitting most details since the procedure is very much analogous to the one we have developed for three-dimensional problems [54].

The discrete radial Dirac–Coulomb Sturmian functions for the problem at hand are defined to be solutions to the differential eigensystem
\[
\begin{pmatrix}
mc^2 - E - \mu^{(0)}_{n'\kappa}(E) \frac{Ze^2}{(4\pi \epsilon_0) r} & -\alpha \left( \frac{d}{dr} + \frac{\kappa}{r} \right) \\
-\alpha \left( \frac{d}{dr} + \frac{\kappa}{r} \right) & -mc^2 - E - \mu^{(0)-1}_{n'\kappa}(E) \frac{Ze^2}{(4\pi \epsilon_0) r}
\end{pmatrix}
\begin{pmatrix}
S^{(0)}_{n'\kappa}(E, r) \\
T^{(0)}_{n'\kappa}(E, r)
\end{pmatrix} = 0,
\] (4.10a)

\[
\begin{align}
S^{(0)}_{n'\kappa}(E, r) & \to 0, & T^{(0)}_{n'\kappa}(E, r) & \to 0, \\
S^{(0)}_{n'\kappa}(E, r) & \to 0, & T^{(0)}_{n'\kappa}(E, r) & \to 0.
\end{align}
\] (4.10b)

Here \(E\) is a fixed, real, energy-dimensional parameter from the interval \(-mc^2 < E < mc^2\) and \(\mu^{(0)}_{n'\kappa}(E)\) is a Sturmian eigenvalue; moreover, as we have previously assumed in Sec. 2 it holds that \(Z < \alpha^{-1}/2\). The reader should observe that this is the inverse of the Sturmian eigenvalue \(\mu^{(0)}_{n'\kappa}(E)\) which multiplies the Coulomb potential in the lower diagonal term of the differential operator in Eq. (4.10a). Proceeding as in Ref. 54, with some labor one finds that eigensolutions to the system (4.10) are
\[
\mu^{(0)}_{n'\kappa}(E) = \frac{\alpha}{\epsilon Z} (|n'| + \gamma_{nk} + N_{n'\kappa})
\] (4.11)
and
\[
S^{(0)}_{n'\kappa}(E, r) = \sqrt{\frac{4\pi \epsilon_0}{e^2} \frac{\alpha |n'|! (|n'| + 2\gamma_{nk})}{2\pi N_{n'\kappa}(N_{n'\kappa} - \kappa)! (|n'| + 2\gamma_{nk})} (2kr)^{\gamma_{nk}} e^{-kr}} \times \left[ L^{(2\gamma_{nk})}_{|n'|-1}(2kr) - \frac{N_{n'\kappa} - \kappa}{|n'| + 2\gamma_{nk}} L^{(2\gamma_{nk})}_{|n'|}(2kr) \right],
\] (4.12a)
Moreover, they obey the generalized closure relations
\[ (2kr)^\nu e^{-kr} \]
\[ \times \left[ L^{(2\gamma_r)}_{|n'_\nu| - 1}(2kr) + \frac{N'_{n'_\nu}}{|n'_\nu| + 2\gamma_r} L^{(2\gamma_r)}_{|n'_\nu|}(2kr) \right], \]
where
\[ \epsilon = \sqrt{\frac{mc^2 - E}{mc^2 + E}}, \quad k = \frac{\sqrt{(mc^2)^2 - E^2}}{\hbar} \]
and
\[ N_{n'_\nu} = \pm \sqrt{n'_\nu^2 + 2|n'_\nu|\gamma_r + \kappa^2}. \]
In contrast to the case of the energy-spectral problem discussed in Sec. 2, the Sturmian radial quantum number \( n'_\nu \) used here runs through all integers, i.e., \( n'_\nu \in \mathbb{Z} \). The following sign convention is adopted in Eq. (4.14): one chooses the positive sign for \( n'_\nu > 0 \) and the negative sign for \( n'_\nu < 0 \); if \( n'_\nu = 0 \), then the positive sign is to be chosen for \( \kappa \leq -\frac{1}{2} \) and the negative one for \( \kappa \geq \frac{1}{2} \), i.e., it holds that \( N_{n'_\nu} = -\kappa \).

The functions given in Eqs. (4.12a) and (4.12b) possess the following generalized orthogonality properties:
\[ \int_0^\infty \frac{Z e^2}{(4\pi\alpha_0)\kappa} \left[ \mu^{(0)}_{n'_\nu}(E) S^{(0)}_{n'_\nu}(E, r) S^{(0)}_{n''_\nu}(E, r) - \mu^{(0)}_{n''_\nu}(E) T^{(0)}_{n''_\nu}(E, r) T^{(0)}_{n'_\nu}(E, r) \right] \, dr = \delta_{n'_\nu n''_\nu} \]
and
\[ \text{c}\kappa \int_0^\infty \frac{Z e^2}{(4\pi\alpha_0)\kappa} \left[ \epsilon S^{(0)}_{n'_\nu}(E, r) S^{(0)}_{n''_\nu}(E, r) + \epsilon^{-1} T^{(0)}_{n'_\nu}(E, r) T^{(0)}_{n''_\nu}(E, r) \right] \, dr = \delta_{n'_\nu n''_\nu}. \]
Moreover, they obey the generalized closure relations
\[ \frac{Z e^2}{(4\pi\alpha_0)\kappa} \sum_{n'_\nu = -\infty}^{\infty} \left( \begin{array}{c} \mu^{(0)}_{n'_\nu}(E) S^{(0)}_{n'_\nu}(E, r) \\ T^{(0)}_{n'_\nu}(E, r) \end{array} \right) \left( \begin{array}{c} S^{(0)}_{n''_\nu}(E, r') \\ -\mu^{(0)}_{n''_\nu}(E) T^{(0)}_{n''_\nu}(E, r') \end{array} \right) = \delta(r - r') I \]
and
\[ \text{c}\kappa \sum_{n'_\nu = -\infty}^{\infty} \left( \begin{array}{c} S^{(0)}_{n'_\nu}(E, r) \\ T^{(0)}_{n'_\nu}(E, r) \end{array} \right) \left( \begin{array}{c} \epsilon S^{(0)}_{n''_\nu}(E, r') \\ \epsilon^{-1} T^{(0)}_{n''_\nu}(E, r') \end{array} \right) = \delta(r - r') I. \]
It follows from Eqs. (4.11) and (2.19) that in the limit \( E \to E_{\nu}^{(0)} \), the Sturmian eigenvalue \( \mu^{(0)}_{n'_\nu}(E) \), with nonnegative \( n'_\nu \equiv n_r = n - |\kappa| - 1/2 \) [cf. Eq. (2.12)], becomes equal to unity:
\[ \mu^{(0)}_{n'_\nu}(E_{\nu}^{(0)}) = 1 \quad \left( n_r = n - |\kappa| - \frac{1}{2} \right) \left\{ \begin{array}{ll} 0 & \text{for } \kappa \leq -\frac{1}{2} \\ 1 & \text{for } \kappa \geq \frac{1}{2} \end{array} \right. \]
In the same limit and under the same restraint on \( n'_\nu \), the Sturmian functions \( S^{(0)}_{n'_\nu}(E, r) \) and \( T^{(0)}_{n'_\nu}(E, r) \) become
\[ S^{(0)}_{n'_\nu}(E_{\nu}^{(0)}, r) = \frac{N_{n'_\nu}}{Z} \sqrt{\frac{4\pi\alpha_0}{e^2}} P^{(0)}_{n'_\nu}(r), \quad T^{(0)}_{n'_\nu}(E_{\nu}^{(0)}, r) = \frac{N_{n'_\nu}}{Z} \sqrt{\frac{4\pi\alpha_0}{e^2}} Q^{(0)}_{n'_\nu}(r) \]
\[ n_r = n - |\kappa| - \frac{1}{2} \left\{ \begin{array}{ll} 0 & \text{for } \kappa \leq -\frac{1}{2} \\ 1 & \text{for } \kappa \geq \frac{1}{2} \end{array} \right. \]
The radial Dirac–Coulomb Green function \( G^{(0)}_\kappa(E, r, r') \) is defined as that particular solution to the inhomogeneous equation
\[ [H^{(0)}_\kappa(r) - E] G^{(0)}_\kappa(E, r, r') = \delta(r - r') I \quad (-mc^2 < E < mc^2), \]
which obeys the boundary conditions
\[ G_{\kappa}^{(0)}(E, r, r') \xrightarrow{r \to 0} 0, \quad G_{\kappa}^{(0)}(E, r, r') \xrightarrow{r \to \infty} 0. \quad (4.19b) \]

One may seek \( G_{\kappa}^{(0)}(E, r, r') \) in the form of the Sturmian series
\[ G_{\kappa}^{(0)}(E, r, r') = \sum_{n'=1}^{\infty} \frac{1}{\mu_{n',\kappa}(E)} \left( S_{n',\kappa}^{(0)}(E, r) T_{n',\kappa}^{(0)}(E, r') - T_{n',\kappa}^{(0)}(E, r') S_{n',\kappa}^{(0)}(E, r) \right), \quad (4.20) \]

To determine the coefficients \( C_{n',\kappa}^{(0)}(E, r') \), we insert the expansion (4.20) into Eq. (4.24), premultiply the resulting equation with \( \left( \mu_{n',\kappa}^{(0)}(E) S_{n',\kappa}^{(0)}(E, r) \right) \left( T_{n',\kappa}^{(0)}(E, r') \right) \), and integrate with respect to \( r \) over the interval \([0, \infty)\). With the use of the orthogonality relation (4.15a), this eventually yields
\[ C_{n',\kappa}^{(0)}(E, r') = \frac{1}{\mu_{n',\kappa}^{(0)}(E)} R_{n'} \left( n_{\kappa}(E) S_{n',\kappa}^{(0)}(E, r) T_{n',\kappa}^{(0)}(E, r') - T_{n',\kappa}^{(0)}(E, r') S_{n',\kappa}^{(0)}(E, r) \right), \quad (4.21) \]

and consequently the explicit form of the Sturmian expansion of the radial Dirac–Coulomb Green function \( G_{\kappa}^{(0)}(E, r, r') \) is found to be
\[ G_{\kappa}^{(0)}(E, r, r') = \sum_{n'=1}^{\infty} \frac{1}{\mu_{n',\kappa}^{(0)}(E)} \left( S_{n',\kappa}^{(0)}(E, r) T_{n',\kappa}^{(0)}(E, r') - T_{n',\kappa}^{(0)}(E, r') S_{n',\kappa}^{(0)}(E, r) \right). \quad (4.22) \]

We are now ready to accomplish the task to determine the Sturmian series representation of the generalized radial Dirac–Coulomb Green function \( G_{\kappa}^{(0)}(r, r') \). It is evident from Eqs. (4.14) and (4.19) that the relationship between \( \hat{G}_{\kappa}^{(0)}(r, r') \) and \( G_{\kappa}^{(0)}(E, r, r') \) is
\[ \hat{G}_{\kappa}^{(0)}(r, r') = \lim_{{E \to E_{\kappa}}^{(0)}} \left[ G_{\kappa}^{(0)}(E, r, r') - \frac{\psi_{\kappa}^{(0)}(r) \psi_{\kappa}^{(0)T}(r')}{E_{\kappa} - E} \right]. \quad (4.23) \]

Upon exploiting the de l’Hospital rule, Eq. (4.23) may be rewritten as
\[ \hat{G}_{\kappa}^{(0)}(r, r') = \lim_{{E \to E_{\kappa}}^{(0)}} \left[ \frac{\partial}{\partial E} \left( E - E_{\kappa}^{(0)} \right) G_{\kappa}^{(0)}(E, r, r') \right]. \quad (4.24) \]

If the expansion (4.22) is plugged into the right-hand side of Eq. (4.24), with the aid of the identities
\[ \frac{\partial S_{n',\kappa}^{(0)}(E, r)}{\partial E} = -\frac{E}{(mc)^2 - E^2} \left[ \frac{dS_{n',\kappa}^{(0)}(E, r)}{dr} - \frac{mc^2}{2E} S_{n',\kappa}^{(0)}(E, r) \right] \quad (4.25a) \]

and
\[ \frac{\partial T_{n',\kappa}^{(0)}(E, r)}{\partial E} = -\frac{E}{(mc)^2 - E^2} \left[ \frac{dT_{n',\kappa}^{(0)}(E, r)}{dr} + \frac{mc^2}{2E} T_{n',\kappa}^{(0)}(E, r) \right], \quad (4.25b) \]

as well as the relations
\[ \frac{E - E_{\kappa}^{(0)}}{\mu_{n,\kappa}^{(0)}(E)} - 1 \approx \frac{\varepsilon_{\kappa}^{(0)}(E)}{\varepsilon - \varepsilon_{\kappa}^{(0)}} = \frac{\varepsilon_{\kappa}^{(0)}(\varepsilon + \varepsilon_{\kappa}^{(0)})(mc^2 + E)(mc^2 + E_{\kappa}^{(0)})}{2mc^2}, \quad (4.26a) \]

we find
\[ \lim_{{E \to E_{\kappa}^{(0)}}} \frac{E - E_{\kappa}^{(0)}}{\mu_{n,\kappa}^{(0)}(E)} - 1 = -\frac{(mc^2)^2 - (E_{\kappa}^{(0)})^2}{2mc^2}. \quad (4.26b) \]
\[
\lim_{E \to E_{n\kappa}^{(0)}} \frac{\partial}{\partial E} \frac{E - E_{n\kappa}^{(0)}}{\mu_{n\kappa}^{(0)}(E)} = 1 \quad (4.26c)
\]

and

\[
\lim_{E \to E_{n\kappa}^{(0)}} \frac{E - E_{n\kappa}^{(0)}}{\mu_{n\kappa}^{(0)}(E)} \frac{\partial \mu_{n\kappa}^{(0)}(E)}{\partial E} = 1, \quad (4.26d)
\]

where \( n_r \) is related to \( n \) through Eq. (2.12) and where

\[
\varepsilon_{n\kappa}^{(0)} = \sqrt{\frac{mc^2 - E_{n\kappa}^{(0)}}{mc^2 + E_{n\kappa}^{(0)}}} = \frac{\alpha Z}{n_r + \gamma_k + N_{n\kappa}}, \quad (4.27)
\]

we find that the sought Sturmian expansion of \( G_{n\kappa}^{(0)}(r, r') \) is

\[
G_{n\kappa}^{(0)}(r, r') = \sum_{n'_{\kappa} = -\infty}^{\infty} \frac{1}{\mu_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}) - 1} \left( \begin{array}{c} S_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r) \\ T_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r) \end{array} \right) \left( \begin{array}{c} \mu_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}) S_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r') \\ T_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r') \end{array} \right) \\
+ \frac{2E_{n\kappa}^{(0)} - mc^2}{2mc^2} \left( \begin{array}{c} S_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r) \\ T_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r) \end{array} \right) \left( \begin{array}{c} S_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r') \\ T_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r') \end{array} \right) \\
+ \left( \begin{array}{c} I_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r) \\ K_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r) \end{array} \right) \left( \begin{array}{c} S_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r') \\ T_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r') \end{array} \right), \quad (4.28)
\]

with

\[
I_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r) = \lim_{E \to E_{n\kappa}^{(0)}} \left[ \frac{E - E_{n\kappa}^{(0)}}{\mu_{n'\kappa}^{(0)}(E)} \frac{\partial S_{n'\kappa}^{(0)}(E, r)}{\partial E} \right] = \frac{E_{n\kappa}^{(0)}}{mc^2} \frac{dS_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r)}{dr} - \frac{mc^2}{2E_{n\kappa}^{(0)}} S_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r), \quad (4.29a)
\]

\[
J_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r) = \lim_{E \to E_{n\kappa}^{(0)}} \left[ \frac{E - E_{n\kappa}^{(0)}}{\mu_{n'\kappa}^{(0)}(E)} \frac{\partial S_{n'\kappa}^{(0)}(E, r)}{\partial E} \right] = \frac{E_{n\kappa}^{(0)}}{mc^2} \frac{dS_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r)}{dr} + \frac{mc^2}{2E_{n\kappa}^{(0)}} S_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r), \quad (4.29b)
\]

and

\[
K_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r) = \lim_{E \to E_{n\kappa}^{(0)}} \left[ \frac{E - E_{n\kappa}^{(0)}}{\mu_{n'\kappa}^{(0)}(E)} \frac{\partial T_{n'\kappa}^{(0)}(E, r)}{\partial E} \right] = \frac{E_{n\kappa}^{(0)}}{mc^2} \frac{dT_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r)}{dr} + \frac{mc^2}{2E_{n\kappa}^{(0)}} T_{n'\kappa}^{(0)}(E_{n\kappa}^{(0)}, r). \quad (4.29c)
\]
With the expansion (4.28) in hands, we may return to the problem of evaluation of the second-order energy correction $E^{(2)}_{n,k}$. Insertion of Eq. (4.28) into Eq. (4.30), followed by the use of Eqs. (4.29a)–(4.29c) and of Eq. (4.13), gives

$$E^{(2)}_{n,k} = -\frac{1}{4}e^2 c^2 B^2 \sum_{n' \neq n} \frac{1}{\mu_{n',k}^{(0)}(E_{n'}^{(0)})} - \int_0^\infty dr r\left[Q_{n,k}^{(0)}(r)S_{n,k}^{(0)}(E_{n,k}^{(0)}, r) + P_{n,k}^{(0)}(r)T_{n,k}^{(0)}(E_{n,k}^{(0)}, r)\right]$$

$$\times \int_0^\infty dr' r'\left[\mu_{n',k}^{(0)}(E_{n,k}^{(0)})Q_{n,k}^{(0)}(r')S_{n',k}^{(0)}(E_{n,k}^{(0)}, r') + P_{n,k}^{(0)}(r')T_{n,k}^{(0)}(E_{n,k}^{(0)}, r')\right] + \frac{(4\pi\epsilon_0)\alpha_0 B^2 N_{n,k}^{(0)}E_{n,k}^{(0)} Z^2}{m} \left[\int_0^\infty dr rP_{n,k}^{(0)}(r)Q_{n,k}^{(0)}(r)\right]^2.$$ (4.30)

The last integral on the right-hand side of Eq. (4.30) is the one displayed in Eq. (3.5). The first and the second integrals may be evaluated using Eqs. (2.24) and (4.12), with the aid of the formula

$$\int_0^\infty dx x^{\alpha+1} e^{-x} L_n^{(\alpha)}(x)L_n^{(\alpha)}(x) = -\frac{\Gamma(\alpha + n + 2)}{n!} \delta_{n',n} + \frac{\Gamma(\alpha + n + 1)\Gamma(\alpha + n + 1)}{n! (n-1)!} \delta_{n',n-1}$$ (4.31)

which generalizes the one in Eq. (3.3) and, similarly to the latter, may be inferred from Eq. (5.4). After much algebra, one finds that

$$\int_0^\infty dr r\left[Q_{n,k}^{(0)}(r)S_{n,k}^{(0)}(E_{n,k}^{(0)}, r) + P_{n,k}^{(0)}(r)T_{n,k}^{(0)}(E_{n,k}^{(0)}, r)\right]$$

$$= \alpha \sqrt{\frac{4\pi\epsilon_0}{{\epsilon}_0}} \frac{N_{n,k}}{2Ze} \left[\frac{n_r[(n_r+2\gamma_0)|n'_r|!(|n'_r|+2\gamma_0)]}{N_{n,k}(N_{n,k} - \kappa)\Gamma(n_r+2\gamma_0)N_{n',k}(N_{n',k} - \kappa)\Gamma(|n'_r|+2\gamma_0)}\right]$$

$$\times \left\{ (N_{n,k} - \kappa)(N_{n',k} - \kappa)\Gamma(n_r+2\gamma_0)\delta_{n|n_r, n_r+1} + \frac{4(n_r + \gamma_0)\Gamma(n_r+2\gamma_0)}{(n_r-1)!}\delta_{n',-n_r} + \frac{(N_{n',k} - \kappa)(N_{n,k} - N_{n',k} - 2\kappa)\Gamma(n_r+2\gamma_0 - 1)}{(n_r-1)!}\delta_{n|n_r, n_r-1} \right\}$$ (4.32)

and

$$\int_0^\infty dr r\left[\mu_{n',k}^{(0)}(E_{n,k}^{(0)})Q_{n,k}^{(0)}(r)S_{n,k}^{(0)}(E_{n,k}^{(0)}, r) + P_{n,k}^{(0)}(r)T_{n,k}^{(0)}(E_{n,k}^{(0)}, r)\right]$$

$$= -\alpha \sqrt{\frac{4\pi\epsilon_0}{{\epsilon}_0}} \frac{N_{n,k}}{8Ze} [P_{n,k}^{(0)}(E_{n,k}^{(0)}) - 1]$$

$$\times \left\{ \frac{n_r!(n_r+2\gamma_0)|n'_r|!(|n'_r|+2\gamma_0)}{N_{n,k}(N_{n,k} - \kappa)\Gamma(n_r+2\gamma_0)N_{n',k}(N_{n',k} - \kappa)\Gamma(|n'_r|+2\gamma_0)}\right\}$$

$$\times \left\{ -\frac{(N_{n,k} - \kappa)\Gamma(n_r+2\gamma_0 + 2)}{n_r!(n_r+2\gamma_0)}\delta_{n|n_r, n_r+2} + 2(N_{n,k} - \kappa)[2n_r + 2\gamma_0 + 1 - \kappa(N_{n,k} + N_{n',k})]\Gamma(n_r+2\gamma_0)\right\}$$

$$\times \left(\frac{N_{n,k}^2 + 2(n_r + \gamma_0)^2\Gamma(n_r+2\gamma_0)}{N_{n,k}(n_r-1)!}\delta_{n|n_r, n_r} - \frac{2(N_{n',k} - \kappa)[2n_r + 2\gamma_0 - 1 - \kappa(N_{n,k} + N_{n',k})]\Gamma(n_r+2\gamma_0 - 1)}{(n_r-1)!}\right.$$ (4.33)
the second-order energy correction which follows from the definitions (4.11) and (2.19). Combining Eqs. (4.30), (4.32), (4.33) and (4.39), the form of the induction vector involves irreducible generalized hypergeometric series is expressible in terms of elementary functions, while the one for an atom in three dimensions is a three-dimensional one-electron Dirac atom. It is interesting that the result for the planar atom characterizing the perturbing uniform magnetic field may be written in the form

\[ E^{(2)}_{\text{m}} = \frac{1}{2^4} \left[ \kappa \left( 3n_r^2 + 6n_r \gamma_n + 4\gamma_n^2 - \kappa^2 \right) + \frac{n_r + \gamma_n}{N_{n,\kappa}} \left( 5n_r^4 + 20n_r^3 \gamma_n + n_r^2 + 22n_r^2 \gamma_n^2 + 5n_r \gamma_n^2 + 2n_r \gamma_n^2 + 10n_r \gamma_n^2 + 4\gamma_n^2 \kappa^2 - 2\kappa^4 + \kappa^2 \right) Z^{-2} \frac{B^2 \epsilon^2}{B_0^2 (4\pi\epsilon_0) a_0}. \]  

The expression in Eq. (4.33) simplifies considerably for the states with \( n_r = 0 \) (i.e., those with \( \kappa = -n + \frac{1}{2} \)), for which it becomes

\[ E^{(2)}_{n,\kappa} = \frac{1}{2^4} \left( n - \frac{1}{2} \right) (2\gamma_n - 1+1) \left[ 2\gamma_{n-1/2} + \gamma_{n-1/2} - \left(n - \frac{1}{2}\right)^2 \right] Z^{-2} \frac{B^2 \epsilon^2}{B_0^2 (4\pi\epsilon_0) a_0}. \]  

In general, the relationship between the second-order energy correction \( E^{(2)} \) and the modulus of the induction vector \( B \) characterizing the perturbing uniform magnetic field may be written in the form

\[ E^{(2)} = -\frac{1}{2} \left( \frac{\mu_0}{4\pi} \right)^{-1} \chi B^2, \]  

where \( \mu_0 \) is the vacuum permeability. The factor of proportionality, \( \chi \), is the magnetizability (magnetic susceptibility) of the system. Comparison of Eqs. (4.35) and (4.37) shows that the magnetizability of the planar atom in the state characterized by the quantum numbers \( n \) and \( \kappa \)

\[ \chi_{n,\kappa} = \frac{1}{2^4} \left[ \kappa \left( 3n_r^2 + 6n_r \gamma_n + 4\gamma_n^2 - \kappa^2 \right) - \frac{n_r + \gamma_n}{N_{n,\kappa}} \left( 5n_r^4 + 20n_r^3 \gamma_n + n_r^2 + 22n_r^2 \gamma_n^2 + 5n_r \gamma_n^2 + 2n_r \gamma_n^2 + 10n_r \gamma_n^2 + 4\gamma_n^2 \kappa^2 - 2\kappa^4 + \kappa^2 \right) Z^{-2} \frac{\alpha^2 a_0^3}{Z^2}. \]  

For states with \( n_r = 0 \), Eq. (4.38) yields

\[ \chi_{n,\kappa} = -\frac{1}{2^4} \left( n - \frac{1}{2} \right) (2\gamma_n - 1+1) \left[ 2\gamma_{n-1/2} + \gamma_{n-1/2} - \left(n - \frac{1}{2}\right)^2 \right] Z^{-2} \frac{\alpha^2 a_0^3}{Z^2}. \]  

In particular, for the ground state, for which \( n = 1 \), one finds that

\[ \chi_{1,1/2} = -\frac{1}{2^6} (2\gamma_{1/2} + 1) (8\gamma_{1/2}^2 + 4\gamma_{1/2} - 1) \frac{\alpha^2 a_0^3}{Z^2}. \]  

The formula in Eq. (4.38) is a counterpart of the one derived recently by Štefánska [55, 56] for a three-dimensional one-electron Dirac atom. It is interesting that the result for the planar atom is expressible in terms of elementary functions, while the one for an atom in three dimensions involves irreducible generalized hypergeometric series \( 3F_2 \) with the unit argument.
5 Recapitulation and discussion

The purpose of the present paper has been to analyze the influence of a weak, static, uniform magnetic field on energy levels of a planar Dirac one-electron atom. In the preceding sections, with the use of the Rayleigh–Schrödinger perturbation theory, we have found that energy of the atomic state which evolves from the state $\Psi_{n\kappa m}(r)$ of the isolated atom is

$$E_{n\kappa m,\kappa} = E_{n\kappa}^{(0)} + E_{n\kappa m,\kappa}^{(1)} + E_{n\kappa}^{(2)} + O(B^3/B_0^3),$$  \hspace{1cm} (5.1)

where

$$E_{n\kappa}^{(0)} = mc^2 + \varepsilon_{n\kappa}^{(0)} Z^2 \frac{e^2}{(4\pi\epsilon_0)a_0},$$  \hspace{1cm} (5.2a)

$$E_{n\kappa m,\kappa}^{(1)} = \frac{\varepsilon_{n\kappa m,\kappa}^{(1)}}{B_0} \frac{e^2}{(4\pi\epsilon_0)a_0},$$  \hspace{1cm} (5.2b)

and

$$E_{n\kappa}^{(2)} = \varepsilon_{n\kappa}^{(2)} Z^2 \frac{B^2}{B_0^2} \frac{e^2}{(4\pi\epsilon_0)a_0},$$  \hspace{1cm} (5.2c)

[here $B_0$ is the atomic unit of magnetic induction defined in Eq. (3.7)], with the dimensionless coefficients $\varepsilon_{n\kappa}^{(k)}$ given by

$$\varepsilon_{n\kappa}^{(0)} = (\alpha Z)^{-2} \left( \frac{n_r + \gamma_\kappa}{N_{n_r,\kappa}} - 1 \right),$$  \hspace{1cm} (5.3a)

$$\varepsilon_{n\kappa m,\kappa}^{(1)} = \frac{m_\kappa}{4\kappa} \left[ 1 - \frac{2\kappa(n_r + \gamma_\kappa)}{N_{n_r,\kappa}} \right],$$  \hspace{1cm} (5.3b)

and

$$\varepsilon_{n\kappa}^{(2)} = \frac{1}{2\kappa} \left[ -\kappa \left( 3n_r^2 + 6n_r\gamma_\kappa + 4\gamma_\kappa^2 - \kappa^2 \right) + \frac{n_r + \gamma_\kappa}{N_{n_r,\kappa}} \left( 5n_r^4 + 20n_r^2\gamma_\kappa + n_r^2 + 22n_r^2\gamma_\kappa^2 + 5n_r^2\kappa^2 \right. \\
+ 4n_r\gamma_\kappa^3 + 2n_r\gamma_\kappa + 10n_r\gamma_\kappa \kappa^2 + 4\gamma_\kappa^2 \kappa^2 - 2\kappa^4 + \kappa^2 \right].$$  \hspace{1cm} (5.3c)

In Table III we display explicit expressions for the coefficient $\varepsilon_{n\kappa}^{(2)}$ for atomic states with the principal quantum numbers $1 \leq n \leq 3$.

The reader may wish to observe that with the use of the coefficient $\varepsilon_{n\kappa}^{(2)}$, the magnetizabilities (4.38) may be written as

$$\chi_{n\kappa} = -2\varepsilon_{n\kappa}^{(2)} \frac{\alpha^2 a_0^3}{Z^2}. $$  \hspace{1cm} (5.4)

For reference purposes, in Table III we compile values of the second-order perturbation-theory estimates of the reduced eigenenergies $(E_{n\kappa m,\kappa} - mc^2)/Z^2$ for the planar atom in states with the principal quantum numbers $n = 1$ and $n = 2$, derived from Eqs. (5.1)–(5.3) at the value of the induction of the perturbing magnetic field equal to $B = 10^{-4}Z^2B_0$. The value of the inverse of the fine-structure constant used in calculations has been $\alpha^{-1} = 137.035 999 139$ (from CODATA 2014 [57]). Results of parallel calculations carried out with the CODATA 2006 value of $\alpha^{-1} (= 137.035 999 76)$ have been found to be in an excellent agreement with corresponding numerically exact values obtained by A. Poszwa (private communication), who used the method presented in Ref. [43].

[Place for Table III]
Expanding the expressions in Eqs. (5.3a)–(5.3b) in the Maclaurin series in $\alpha Z$, and retaining terms of orders not higher than quadratic in that variable, one finds the following quasi-relativistic approximations to the coefficients $\varepsilon^{(n)}$:

\[ \varepsilon^{(0)}_{n\kappa} = -\frac{1}{2(n - \frac{1}{2})^2} \left[ 1 + (\alpha Z)^2 \frac{1}{(n - \frac{1}{2})^2} \left( n - \frac{1}{2} \right)^2 - \frac{3}{4} \right] + O((\alpha Z)^4), \]  

\[ \varepsilon^{(1)}_{n\kappa m\kappa} = \begin{cases} \frac{m_{\kappa}(2\kappa - 1)}{4\kappa} \left[ 1 - (\alpha Z)^2 \frac{\kappa}{(2\kappa - 1)(n - \frac{1}{2})^2} \right] + O((\alpha Z)^4) & \text{for } \kappa \neq \frac{1}{2} \\ -\frac{(\alpha Z)^2 m_{\kappa}}{4(n - \frac{1}{2})^2} + O((\alpha Z)^4) & \text{for } \kappa = \frac{1}{2} \end{cases} \]  

\[ \varepsilon^{(2)}_{n\kappa} = \frac{1}{2^n} (n - \frac{1}{2})^2 (20n^2 - 20n - 12\kappa^2 - 12\kappa + 9) \times \left[ 1 + (\alpha Z)^2 \frac{\beta^{(2)}_{n\kappa}}{2|\kappa| (n - \frac{1}{2})^2 (20n^2 - 20n - 12\kappa^2 - 12\kappa + 9)} \right] + O((\alpha Z)^4), \]  

where

\[ \beta^{(2)}_{n\kappa} = -80n^3 + 120n^2 + 44n^2|\kappa| - 68n + 24n\kappa^2 + 24n\kappa - 44n|\kappa| - 28\kappa^2|\kappa| + 8|\kappa| \]
\[ -12\kappa^2 - 12\kappa + 15|\kappa| + 14. \]  

In the purely nonrelativistic limit, i.e., for $\alpha \to 0$, Eqs. (5.5a)–(5.5c) yield

\[ \varepsilon^{(0)}_{n\kappa} \xrightarrow{c \to \infty} -\frac{1}{2(n - \frac{1}{2})^2}, \]  

\[ \varepsilon^{(1)}_{n\kappa m\kappa} \xrightarrow{c \to \infty} \begin{cases} \frac{m_{\kappa}(2\kappa - 1)}{4\kappa} & \text{for } \kappa \neq \frac{1}{2} \\ 0 & \text{for } \kappa = \frac{1}{2} \end{cases} \]  

and

\[ \varepsilon^{(2)}_{n\kappa} \xrightarrow{c \to \infty} \frac{1}{2^n} (n - \frac{1}{2})^2 (20n^2 - 20n - 12\kappa^2 - 12\kappa + 9). \]  

To facilitate comparison of the above limits with results of direct nonrelativistic calculations reported in Ref. [30] (cf. also Ref. [31]), Eqs. (5.7b) and (5.7c) should be transformed. To this end, in the case of Eq. (5.7b) we introduce two quantum numbers $m_l$ and $m_s$, relating them to $\kappa$ and $m_\kappa$ in the following way:

\[ m_l = m_\kappa + \frac{m_\kappa}{2\kappa}, \quad m_s = \frac{m_\kappa}{2\kappa}. \]  

\[ \text{[Place for Table IV]} \]

It is evident that $m_s = \pm 1/2$ and $m_l = m_\kappa \mp 1/2$, and that relations inverse to those in Eq. (5.8) are

\[ \kappa = \frac{1}{2} \left( 1 + \frac{m_l}{m_s} \right), \quad m_\kappa = m_l + m_s. \]  

Insertion of the latter into Eq. (5.7b) gives

\[ \varepsilon^{(1)}_{n\kappa m\kappa} \xrightarrow{c \to \infty} \frac{1}{2}(m_l + 2m_s). \]
To transform Eq. (5.7c), we observe that it holds that

$$\kappa (\kappa + 1) = l^2 - \frac{1}{4}, \quad (5.11)$$

where the nonnegative integer $l$ has been defined in Eq. (2.26); the reader may also wish to verify that $l = |m|$ . Plugging Eq. (5.11) into Eq. (5.7c) casts the latter into the form

$$\varepsilon^{(2)}_{\kappa m} \leftrightarrow \frac{1}{24} (n - \frac{1}{2})^2 (5m^2 - 5n - 3l^2 + 3). \quad (5.12)$$

The expressions on the right-hand sides of Eqs. (5.7a) and (5.12) are exactly the same as those in Eqs. (4.4) and (4.6) from Ref. [39], respectively, while the expression on the right-hand side of Eq. (5.11) is identical to the one which may be inferred from Eqs. (4.2) and (4.14) in Ref. [39].

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## A Appendix: The axial spinors

The axial (or cylindrical) spinors, introduced by Poszwa and Rutkowski [42], are two-component functions of the angular variable $\varphi \in [0, 2\pi]$ defined as

$$\Phi_{\kappa m_{\kappa}}(\varphi) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \delta_{-\kappa, m_{\kappa}} e^{i(m_{\kappa} - 1/2)\varphi} \\ \delta_{\kappa, m_{\kappa}} e^{i(m_{\kappa} + 1/2)\varphi} \end{pmatrix} \quad (\kappa = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots; m_{\kappa} = \pm \kappa), \quad (A.1)$$

or equivalently as

$$\Phi_{\kappa m_{\kappa}}(\varphi) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \delta_{-\kappa, m_{\kappa}} e^{-i(\kappa + 1/2)\varphi} \\ \delta_{\kappa, m_{\kappa}} e^{i(\kappa + 1/2)\varphi} \end{pmatrix} \quad (\kappa = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots; m_{\kappa} = \pm \kappa). \quad (A.2)$$

The quantum number $\kappa$ appearing in Eqs. (A.1) and (A.2), and in the rest of the present paper, is defined with the sign opposite in relation to the one used in Refs. [42][43][52]. Explicit forms of the spinors $\Phi_{\kappa m_{\kappa}}(\varphi)$ are thus

$$\Phi_{\kappa, -\kappa}(\varphi) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} e^{-i(\kappa + 1/2)\varphi} \\ 0 \end{pmatrix}, \quad \Phi_{\kappa, \kappa}(\varphi) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 0 \\ e^{i(\kappa + 1/2)\varphi} \end{pmatrix}. \quad (A.3)$$

These functions are orthonormal in the sense of

$$\int_0^{2\pi} d\varphi \Phi_{\kappa m_{\kappa}}(\varphi) \Phi_{\kappa', m_{\kappa}'}^\dagger(\varphi) = \delta_{\kappa \kappa'} \delta_{m_{\kappa} m_{\kappa}'} \quad (A.4)$$

and form a set which is complete in the space of square-integrable two-component spinor functions of $\varphi \in [0, 2\pi]$; the corresponding closure relation is

$$\sum_{\kappa=-\infty}^{+\infty} \sum_{m_{\kappa}=\pm \kappa} \Phi_{\kappa m_{\kappa}}(\varphi) \Phi_{\kappa m_{\kappa}}^\dagger(\varphi') = \delta(\varphi - \varphi') I, \quad (A.5)$$

where $I$ is the $2 \times 2$ unit matrix.

The products of $\Phi_{\kappa m_{\kappa}}(\varphi)$ with $\cos \varphi$ or $\sin \varphi$ have the expansions

$$\cos \varphi \Phi_{\kappa m_{\kappa}}(\varphi) = \frac{1}{2} \Phi_{\kappa + m_{\kappa}, \kappa, m_{\kappa} + 1}(\varphi) + \frac{1}{2} \Phi_{\kappa - m_{\kappa}, \kappa, m_{\kappa} - 1}(\varphi) \quad (A.6a)$$

$$\sin \varphi \Phi_{\kappa m_{\kappa}}(\varphi) = \frac{1}{2} \Phi_{\kappa + m_{\kappa}, \kappa, m_{\kappa} - 1}(\varphi) - \frac{1}{2} \Phi_{\kappa - m_{\kappa}, \kappa, m_{\kappa} + 1}(\varphi) \quad (A.6b)$$
This leads to the following integral formulas:

\[
\int_{0}^{2\pi} d\varphi \cos \varphi \Phi_{k,m}(\varphi) \Phi_{k',m'}(\varphi) = \frac{1}{2} \delta_{m,m'}/\delta_{k,k'}(\delta_{m,m'+1} + \delta_{m,m'-1})
\]

and

\[
\int_{0}^{2\pi} d\varphi \sin \varphi \Phi_{k,m}(\varphi) \Phi_{k',m'}(\varphi) = \frac{1}{2i} \delta_{m,m'}/\delta_{k,k'}(\delta_{m,m'+1} - \delta_{m,m'-1})
\]

If \(\sigma_1, \sigma_2, \sigma_3\) are the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

then it holds that

\[
\sigma_1 \Phi_{k,m}(\varphi) = \Phi_{-k,m_1+m_2/k}(\varphi),
\]

\[
\sigma_2 \Phi_{k,m}(\varphi) = -\frac{i m_2}{k} \Phi_{-k_1,m_1+m_2/k}(\varphi)
\]

and

\[
\sigma_3 \Phi_{k,m}(\varphi) = -\frac{m_1}{k} \Phi_{k,m}(\varphi).
\]

Let \(n_x\) and \(n_y\) be the unit vectors of a planar Cartesian coordinate system \(\{x, y\}\) and let

\[
n_r = n_x \cos \varphi + n_y \sin \varphi, \quad n_\varphi = -n_x \sin \varphi + n_y \cos \varphi
\]

be the unit vectors of a polar coordinate system \(\{r, \varphi\}\) with the same origin. It holds that

\[
n_r \cdot \sigma \Phi_{k,m}(\varphi) = \Phi_{-k,m}(\varphi)
\]

and

\[
n_\varphi \cdot \sigma \Phi_{k,m}(\varphi) = -\frac{i m_2}{k} \Phi_{-k,m}(\varphi),
\]

where

\[
\sigma = \sigma_1 n_x + \sigma_2 n_y.
\]

The reader may wish to observe that results for the expressions \((n_z \times n_\varphi) \cdot \sigma \Phi_{k,m}(\varphi)\) and \((n_z \times n_r) \cdot \sigma \Phi_{k,m}(\varphi)\), where

\[
n_z = n_x \times n_y,
\]

may be deduced immediately from Eqs. (A.11a) and (A.11b), respectively, since one has

\[
n_z \times n_\varphi = -n_r, \quad n_z \times n_r = n_\varphi.
\]

Equation (A.6b) expresses the fact that the axial spinors are eigenvectors of the Pauli matrix \(\sigma_3\). They also appear to be simultaneous eigenfunctions of the three operators

\[
\Lambda = -i \frac{\partial}{\partial \varphi}, \quad J = \Lambda + \frac{1}{2} \sigma_3, \quad K = -(\sigma_3 \Lambda + \frac{1}{2}) = -\sigma_3 J.
\]
as it holds that

\[ \Lambda \Phi_{\kappa m}(\phi) = \frac{m}{\kappa} \left( \kappa + \frac{1}{2} \right) \Phi_{\kappa m}(\phi), \]  
\[ (A.16a) \]

\[ J \Phi_{\kappa m}(\phi) = m \phi \Phi_{\kappa m}(\phi) \]  
\[ (A.16b) \]

and

\[ K \Phi_{\kappa m}(\phi) = \kappa \Phi_{\kappa m}(\phi). \]  
\[ (A.16c) \]

The result of the action of the operator

\[ \sigma \cdot \nabla = n_r \cdot \sigma \frac{\partial}{\partial r} + \frac{1}{r} n_\phi \cdot \sigma \frac{\partial}{\partial \phi} \]  
\[ (A.17) \]

on the product \( F(r) \Phi_{\kappa m}(\phi) \) is

\[ \sigma \cdot \nabla F(r) \Phi_{\kappa m}(\phi) = \left( \frac{\partial}{\partial r} + \frac{\kappa + \frac{1}{2}}{r} \right) F(r) \Phi_{-\kappa m}(\phi). \]  
\[ (A.18) \]

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Table I: Relativistic quantum numbers and the spectroscopic designation for selected states of the planar Dirac one-electron atom (after Ref. [42], except for the quantum number $\kappa$ which in the present paper is defined with the sign *opposite* in relation to the one used in Refs. [42],[52]).

| $n$ | $n_r$ | $\kappa$ | $l = |\kappa + \frac{1}{2}|$ | Spectroscopic notation $n\ell_{|\kappa|}$ |
|-----|-------|----------|-----------------|---------------------------------|
| 1   | 0     | $-\frac{1}{2}$ | 0               | $1s_{1/2}$                     |
| 2   | 1     | $-\frac{1}{2}$ | 0               | $2s_{1/2}$                     |
| 2   | 1     | $\frac{1}{2}$  | 1               | $2p_{1/2}$                     |
| 2   | 0     | $\frac{3}{2}$  | 1               | $2p_{3/2}$                     |
| 3   | 2     | $-\frac{1}{2}$ | 0               | $3s_{1/2}$                     |
| 3   | 2     | $\frac{1}{2}$  | 1               | $3p_{1/2}$                     |
| 3   | 1     | $-\frac{3}{2}$ | 1               | $3p_{3/2}$                     |
| 3   | 1     | $\frac{3}{2}$  | 2               | $3d_{3/2}$                     |
| 3   | 0     | $-\frac{5}{2}$ | 2               | $3d_{5/2}$                     |
Table II: Explicit forms of the coefficient $\varepsilon_{n\kappa}^{(2)}$, defined in Eq. (5.3c), for atomic states with the principal quantum numbers $1 \leq n \leq 3$. The symbols $N_{n\kappa}$ and $\gamma_{n}$ have been defined in Eqs. (2.20) and (2.21), respectively.

| Atomic state | $\varepsilon_{n\kappa}^{(2)}$ | Exact | Nonrelativistic limit |
|--------------|-------------------------------|-------|-----------------------|
| $1s_{1/2}$   | $\frac{1}{27}(2\gamma_{1/2} + 1)(8\gamma_{1/2}^2 + 4\gamma_{1/2} - 1)$ | $\frac{3}{64}$ | |
| $2s_{1/2}$   | $\frac{1}{27}\left[16\gamma_{1/2}^2 + 24\gamma_{1/2} + 11 + \frac{(\gamma_{1/2} + 1)(32\gamma_{1/2}^3 + 184\gamma_{1/2}^2 + 196\gamma_{1/2} + 59)}{N_{1,1/2}}\right]$ | $\frac{117}{64}$ | |
| $2p_{1/2}$   | $\frac{1}{27}\left[-16\gamma_{1/2}^2 - 24\gamma_{1/2} - 11 + \frac{(\gamma_{1/2} + 1)(32\gamma_{1/2}^3 + 184\gamma_{1/2}^2 + 196\gamma_{1/2} + 59)}{N_{1,1/2}}\right]$ | $\frac{45}{32}$ | |
| $2p_{3/2}$   | $\frac{3}{27}(2\gamma_{3/2} + 1)(8\gamma_{3/2}^2 + 4\gamma_{3/2} - 9)$ | $\frac{45}{32}$ | |
| $3s_{1/2}$   | $\frac{1}{27}\left[16\gamma_{1/2}^2 + 48\gamma_{1/2} + 47 + \frac{(\gamma_{1/2} + 2)(64\gamma_{1/2}^3 + 712\gamma_{1/2}^2 + 1352\gamma_{1/2} + 713)}{N_{2,1/2}}\right]$ | $\frac{825}{64}$ | |
| $3p_{1/2}$   | $\frac{1}{27}\left[-16\gamma_{1/2}^2 - 48\gamma_{1/2} - 47 + \frac{(\gamma_{1/2} + 2)(64\gamma_{1/2}^3 + 712\gamma_{1/2}^2 + 1352\gamma_{1/2} + 713)}{N_{2,1/2}}\right]$ | $\frac{375}{32}$ | |
| $3p_{3/2}$   | $\frac{1}{27}\left[3(16\gamma_{3/2}^2 + 24\gamma_{3/2} + 3) + \frac{(\gamma_{3/2} + 1)(32\gamma_{3/2}^3 + 248\gamma_{3/2}^2 + 356\gamma_{3/2} + 75)}{N_{1,3/2}}\right]$ | $\frac{375}{32}$ | |
| $3d_{3/2}$   | $\frac{1}{27}\left[-3(16\gamma_{3/2}^2 + 24\gamma_{3/2} + 3) + \frac{(\gamma_{3/2} + 1)(32\gamma_{3/2}^3 + 248\gamma_{3/2}^2 + 356\gamma_{3/2} + 75)}{N_{1,3/2}}\right]$ | $\frac{525}{64}$ | |
| $3d_{5/2}$   | $\frac{5}{27}(2\gamma_{5/2} + 1)(8\gamma_{5/2}^2 + 4\gamma_{5/2} - 25)$ | $\frac{525}{64}$ | |
Table III: Second-order perturbation-theory approximations to reduced energies \((E_{n\kappa m_e} - m\alpha^2)/Z^2\) for states with \(n = 1\) and \(n = 2\) of the planar Dirac one-electron atom in a perpendicular magnetic field of induction \(B = 10^{-4}Z^2\) au. The value of the inverse of the fine-structure constant used in calculations has been \(\alpha^{-1} = 137.035999139\) (from CODATA 2014 [57]).

| \(Z\) | \(1s_{1/2}\) | \(2s_{1/2}\) | \(2p_{1/2}\) | \(2p_{3/2}\) |
|-------|--------------|--------------|--------------|--------------|
| 1     | -2.000 156 510 9 | -2.000 056 516 2 | -0.222 234 042 7 | -0.222 323 522 1 |
| 10    | -2.010 814 964 0 | -2.010 715 499 3 | -0.222 317 679 2 | -0.222 453 760 1 |
| 20    | -2.044 567 648 9 | -2.044 469 826 3 | -0.227 157 713 7 | -0.222 850 296 2 |
| 30    | -2.106 361 188 1 | -2.106 266 235 5 | -0.233 972 211 9 | -0.223 517 542 5 |
| 40    | -2.207 665 355 6 | -2.207 574 760 3 | -0.245 051 771 0 | -0.224 655 333 2 |
| 50    | -2.375 720 169 5 | -2.375 635 983 0 | -0.263 177 194 3 | -0.225 708 076 2 |
| 60    | -2.697 476 735 7 | -2.697 402 591 3 | -0.296 998 786 0 | -0.227 265 470 5 |
| 68    | -3.562 768 640 8 | -3.562 712 504 2 | -0.382 221 851 2 | -0.228 755 183 2 |
Table IV: The quantum numbers $m_l$ and $m_s$ derived from Eq. 5.8 for selected values of $\kappa$ and $m_\kappa$.

| $\kappa$ | $m_\kappa$ | $m_l$ | $m_s$ |
|----------|-------------|-------|-------|
| $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |
| $-\frac{3}{2}$ | $\frac{3}{2}$ | 1 | $-\frac{3}{2}$ |
| $-\frac{3}{2}$ | $-\frac{3}{2}$ | 1 | $\frac{3}{2}$ |
| $-\frac{3}{2}$ | $\frac{3}{2}$ | 1 | $-\frac{3}{2}$ |
| $-\frac{5}{2}$ | $\frac{5}{2}$ | 2 | $-\frac{5}{2}$ |
| $-\frac{5}{2}$ | $-\frac{5}{2}$ | 2 | $\frac{5}{2}$ |
| $-\frac{5}{2}$ | $\frac{5}{2}$ | 2 | $-\frac{5}{2}$ |