Effect of rotation on stability of advective flow in horizontal liquid layer with a free upper boundary

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Abstract. A horizontal liquid film of infinite extent is bounded below by a rigid plane and above by free surface, rotating about a vertical axis. Advective flow is set by imposing a constant temperature gradient on both boundaries. In the framework of Boussinesq approximation, we obtain an exact stationary solution of Navier–Stokes equations. The linear theory of stability for such flow is numerically investigated by reducing the initial system to the boundary problem for linear one-dimensional partial differential equations. The neutral stability flow states for normal perturbations are carried out; then the critical flow parameters allow us to get the dependence between the critical Grashof number and the wave number for various Taylor ones at a fixed Prandtl number, Pr=6.7.

Introduction

Advection generated in the liquid layer is due to horizontal heterogeneity in density on the boundaries. The advection usually shows in the form of different flows, the difference from convective currents is perpendicularity of velocity to gravity and buoyancy [1,2]. The advective flows interest is closely related to the geophysical and technological applications. Depending on the values of the Prandtl number, the motion in ocean, the dynamics of the atmosphere and Earth’s crust and mantle, the movement of melt in installation for variant horizontal directions of crystallization, the transport processes in shallow basin are among the numerous application fields. When the linear dependence of temperature at the rigid boundary and at the undeformable free surface with the x-coordinate (horizontal) is prescribed, the advective flow can be described analytically.

The theoretical study of advective flow with constant pressure gradient on the boundaries in the absence of rotation was first performed by Birikh [1], considering only one horizontal component of velocity, which is perpendicular to buoyancy force. The buoyancy force is the main cause of motion. This advective flow property remains unchanged for different boundary conditions of velocity. [2–3] made the first investigations on stability of such advective flows. They studied stability of the plane–parallel stationary flows without rotation for either rigid or free upper boundary. Research of stability was conducted with usage of a Galerkin method. It was shown, that the horizontal parallel plate flow is unstable concerning two critical modes of disturbances. At small Prandtl numbers (Pr) the
instability has the hydrodynamic nature and is developmental of vortexes on border of counter currents. With growth of a Prandtle number in the field of formation of vortexes there is a steady temperature stratification handicapping development of instability of this type. At moderate and large Prandtle numbers the instability has the Rayleigh nature, it is connected to availability in a flow potentially of unstable zones of distribution of temperature. The stability limit depends on a direction of a wave vector of normal disturbances in relation to speed of the basic flow. Actually this surge represents convective Benard’s cells, which one communicates with a fluid flow. Moreover, a disturbance for lower and the high bounds practically do not influence against each other. The flat Rayleigh mode and helical (wave) Rayleigh modes (even and odd) is excreted, if the wave vector is directed bevel way to the basic flow, the arising spatial helical disturbances are more dangerous, than flat one. The researches of stability of advection flow in a layer with flat free high bound and firm lower have shown, that the crisis is produced by flat hydrodynamic mode (for \(Pr < 0.075\)), or by helical Rayleigh mode (for \(Pr > 0.075\)). The helical Rayleigh mode will be localized in an instability region near to lower boundary. As well as in case of both firm borders, the hydrodynamic mode with increase of a Prandtle number is stabilized. The flat and helical Rayleigh disturbances developing in the upper unstably stratified zone are less dangerous as contrasted to by localized disturbances near to lower boundary.

Smith M.K. and Davis S.H. [4–5], Hart [6] investigated the stability of the advective thermocapillary flow and the advective flow with thermal insulative boundaries in horizontal layer without rotation.

The other kind of flow, which was analytically described by Ekman [7–9], is generated by the rotation of a horizontal layer of isothermal liquid caused by the friction on the lower boundary and the tangential stress of an external force on the upper free boundary. Ekman flow, which is not an advective one, has the typical spiral, and it is the main boundary flow of the atmosphere and ocean. The stability problem of the Ekman is well studied and basically is resolved (see, for example [10]).

The analytical description of the advective flow in a rotating horizontal layer of liquid was first given by Aristov [11]. In this work, the velocity vector is perpendicular to the buoyancy force too, but it has two horizontal components. The advective nature of this flow is similar to Birikh’s one and the corresponding spiral is similar to Ekman’s flow. It is possible for the advective flow in rotating layer of liquid to play the same role in the atmosphere and ocean as in Ekman’s flow. A large class of advective currents in different media and at different boundary conditions is analytically described [12]. The rate of advective flow grows with an increase of the lateral temperature gradient, and it may become unstable. In such situation the hydrodynamic crisis is accompanied by the crisis of heat transfer. As a result of instability, a secondary flow arises, which, in turn, may lose its stability and the development of turbulence may follow.

The instability of rotating advective flow has been come under research only recently. The flows of this family are used to derive quasi-two-dimensional models of geophysical hydrodynamics [13–15], which are a development of “shallow water” theory for viscous non-isothermal liquids, and it is very important for such models to investigate the stability of advective flows. In the papers [16–19] it was studied the stability of advective flow in rotating liquid layer with rigid boundaries for Prandtl number \(Pr=6.7\) (water). This is the simplest situation. Instability has oscillatory character and it develops near the boundaries. Then, to follow the founders [1], we study the stability of advective flow in rotating liquid layer with a free surface and linear temperature on the boundaries. This assumption is used to study the main thermal mechanism in large and meso scale phenomena in atmosphere and ocean. This is one of the ways to investigate the complicated hydrothermodynamic phenomena with the help of mathematical methods of research. Such boundary conditions for temperature may be realized in laboratory experiment too. And these simple assumptions of boundary conditions are necessary to progress in understanding the complexity of such flows.
1. Mathematical model
In the present investigation, we consider a horizontal liquid layer of infinite extent limited from below and above respectively by a rigid plane and a free upper surface (Fig. 1). The rigid plane rotates with a constant angular velocity $\Omega_0$ about a vertical axis. The rotating rectangular frame of reference $Oxyz$ is chosen as follows: the Oz-axis coincides with the axis of rotation, $Oxy$ lies on the horizontal mid-plane of the liquid layer. We denote: $h$, the layer half-thickness of the fluid layer; $\vec{v} = (v_x, v_y, v_z)$, the velocity vector; $p$, the pressure; $t$, the time; $\rho_0$ a homogeneous density; $T$ is counted off a homogeneous temperature; $\beta$ is the coefficient of thermal expansion; $\vec{\Omega} = \Omega_0 \hat{e}_z$, where $\hat{e}_z$ is the unit vector directed vertically upwards; $\vec{r}$ is radius-vector, $g$ is the gravity acceleration, $\nu$ is the coefficient of kinematic viscosity and $\chi$ is the coefficient of heat diffusivity.

\[
\begin{align*}
\rho_0 \frac{\partial v_x}{\partial z} & = 0, \quad v_z = 0, \quad T = Ax \\
\vec{v} & = 0, \quad T = Ax
\end{align*}
\]

Figure 1. A sketch of the geometry of the infinite horizontal liquid layer.

We assume that $T$, along the two boundaries changes linearly as a function of $x$-coordinate. Our aim is to study the flow structure evolutions for an incompressible fluid and analyze its stability in the rotating Cartesian coordinate system $Oxyz$ on the basis of free convective equations in Boussinesq approximation [2, 16, 20]:

\[
\begin{align*}
\rho = \rho_0 (1 - \beta T), \\
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \nabla) \vec{v} &= -\frac{\nabla p}{\rho_0} + \nu \Delta \vec{v} + g\beta T \hat{e}_z - 2(\vec{\Omega} \times \vec{v}) + \beta T \vec{\Omega} \times (\vec{\Omega} \times \vec{r}),
\end{align*}
\]
\( \nabla \cdot \vec{v} = 0 \),  
\[ \frac{\partial T}{\partial t} + \vec{v} \nabla T = \chi \Delta T \cdot \]

We assume that the Froude number \([20, 21]\) \( Fr = \Omega_0^2 R / g << 1 \), i.e., the convective forces, which arises because of the non-homogeneous density in centrifugal field, are negligible with regards to the convective force in gravitational field. The deformations of the free surface caused by rotation and thermocapillary effects are ignored.

The characteristic length, time, velocity and pressure for scaling are \( h, h^2 / \nu, g \beta Ah^3 / \nu, Ah \), and \( \rho_0 g \beta h^3 \), respectively, where, \( A \) is a constant temperature gradient on both boundaries. Then, the dimensionless governing equations can be written in the form as follows:

\[
\frac{\partial \tilde{\vec{v}}}{\partial t} + Gr (\tilde{\vec{v}} \nabla) \tilde{\vec{v}} + \sqrt{Ta} (\tilde{\vec{e}}_z \times \tilde{\vec{v}}) = - \nabla p + \Delta \tilde{\vec{v}} + T \tilde{\vec{e}}_z ; \quad (1)
\]
\[
\nabla \cdot \tilde{\vec{v}} = 0 ; \quad (2)
\]
\[
\frac{\partial T}{\partial t} + Gr \tilde{v} \nabla T = \frac{1}{Pr} \Delta T \cdot (3)
\]

and the boundary conditions:

\( T = x, \tilde{\vec{v}} = 0 \) at \( z = -1 \); \( T = x, \tilde{v}_y = v_z = 0 \) at \( z = 1 \). \( (4a,b) \)

Here \( Gr = \frac{g \beta Ah^4}{\nu^2} \), is the Grashof number; \( Ta = \left( \frac{2 \Omega_0 h^2}{\nu} \right)^2 \), is the Taylor number; \( Pr = \frac{\nu}{\chi} \), is the Prandtl number; the Rayleigh number is \( Ra = Gr Pr \). The subscripts \( x, y \) in the equation \( (4,b) \) mean that the equality holds for both \( v_x \) and \( v_y \).

2. Basic flow

An exact steady flow state solution of the system \((2)-(4)\), subject to the boundary conditions \((4)\) can be found in the form:

\( v_x = u_0 (z), \quad v_y = v_0 (z), \quad v_z = 0, \quad T = T_0 (x, z) = x + \tau_0 (z), \quad p = p_0 (x, y, z). \) \( (5) \)

Notice in this system that the form of the velocity components is admissible, as the boundary conditions do not require lateral variations in \( v_x, v_y, v_z \).

Substituting formulae \((5)\) in the system \((1)-(3)\), we get the following partial differential equations for speed, temperature and pressure distributions:

\[
\frac{\partial p_0}{\partial z} = x + \tau_0 , \quad (6)
\]
\[
- \sqrt{Ta} v_0 = - \frac{\partial p_0}{\partial x} + u_0'' , \quad (7)
\]
\[
\sqrt{Ta} u_0 = - \frac{\partial p_0}{\partial y} + v_0'' , \quad (8)
\]
\[
Ra u_0 = \tau_0'' . \quad (9)
\]

The boundary conditions write:

\( u'_0 = v'_0 = 0, \quad \tau_0 = 0 \) at \( z = 1 \), \( (10) \)
\( u_0 = v_0 = 0, \quad \tau_0 = 0 \) at \( z = -1 \) \( (11) \)
and closed flow conditions read:
\[
\int_{-1}^{1} u_0 dz = 0, \quad \int_{-1}^{1} v_0 dz = 0. \tag{12}
\]

Solving the system (6)-(12) gives after some tedious manipulations, the exact solution:
\[
u_0(z) = R(M(z)), \quad v_0(z) = \mathcal{Z}(M(z)), \tag{14a,b}
\]
\[
\tau_0(z) = \frac{Ra}{\sqrt{Ta}} \left[ v_0(z) - 0.5v_0(1)(1 + z) + 0.5(1 - z^2)\mathcal{Z}\left(\frac{D_2}{D}\mu \sinh(\mu)\right)\right], \tag{14c}
\]
where,
\[
M(z) = \cosh\left(\frac{\mu(z - 1)}{\mu D}\right) - \frac{\sinh(\mu(z + 1))}{\mu D} - \frac{\sinh(2\mu)}{2\mu D} + \frac{\sinh(\mu)}{\mu^2 D}(\sinh(\mu z) + \sinh(\mu)) - \frac{z}{\mu^2},
\]
\[
\mu = \sqrt[4]{\frac{Ta}{4}} (1 + i), \quad i = \sqrt{-1}, \quad D = \frac{\sinh(2\mu)}{2\mu} - \cosh(2\mu), \quad D_2 = -\frac{\sinh(\mu)}{\mu^2} + \frac{\cosh(\mu)}{\mu^2}.
\]
Here, \(R\) and \(\mathcal{Z}\) represent the real and imaginary parts of the corresponding complex quantity, respectively.

For the results presented below, \(Pr\) was fixed: \(Pr = 6.7\) (water). In the manner of horizontal velocity profiles, plotted respectively in Figs. 2(a), (b), the variation of dimensionless temperature \(\tau_0\) is displayed versus \(z\) for different values of Taylor number in Figs. 2(c). In general, the flow structure is asymmetric about the horizontal mid-plane and can be divided into two currents moving in opposite directions as it can be checked from the figures. A dynamical boundary layer appears only close to the bottom, whereas a steep temperature gradient develops in the thermal boundary layer near the upper free surface too. Both thicknesses grow when the Taylor number decreases. In the central core, i.e., outside the boundary layers, the component \(u_0\) can be considered as almost constant, whereas \(v_0\) and \(\tau_0\), are approximately linear. The intensity of motion is much lower inner the central core of the fluid than at near the boundaries.

Figure 2. Dependence of velocity components (a) \(u_0\), (b) \(v_0\), and of (c) temperature \(\tau_0\) on vertical \(z\)-coordinate for various given Taylor numbers.
Figure 3. Minimal and maximal velocity components (a) $u_0$, (b) $v_0$, and (c) temperature $\tau_0$ versus Taylor number $Ta$.

In Figs. 3(a), (b), (c) are displayed the minima (resp. the maxima) of the stationary flow solution $(u_0,v_0,0)$, $\tau_0$ as a function of $Ta$. As it can be seen, the minima (resp. the maxima) of $\tau_0$ and $u_0$ increase monotonically (resp. decrease), as $Ta$ increases. Fig. 3(b) shows that above the horizontal mid-plane, the maxima of $v_0$, increase with $Ta$, up to the limit $Ta = 24.5$, then decrease almost proportionally to $1/\sqrt{Ta}$. Below the plane $z = 0$, about the minima of $v_0$, the tendency is inverted: they decrease with $Ta$ till the value $Ta = 27$, then, they monotonically increase.

3. Linear stability

The stability analysis dealt with herein aims to investigate the growth rate of small perturbations superimposed to a steady flow state solution:

$$
\begin{align*}
\tilde{v} &= \tilde{v}_0 + \tilde{V}, \quad \tilde{v}_0 = (u_0,v_0,0), \quad \tilde{V} = (u,v,w), \quad T = T_0 + \theta, \quad p = p_0 + \tilde{p}.
\end{align*}
$$

Here $\tilde{V}$, $\theta$ and $\tilde{p}$ are small perturbations. Linearizing in the usual manner, taking the flow equations for $(\tilde{v}_0, T_0, p_0)$ into account, the linearized system (1)-(3) takes the form:

$$
\begin{align*}
\partial_t \tilde{V} + Gr \left[ (\tilde{V} \nabla) \tilde{v}_0 + (\tilde{v}_0 \nabla) \tilde{V} + \sqrt{Ta} (\tilde{e}_z \times \tilde{V}) \right] &= - \nabla \tilde{p} + \Delta \tilde{V} + \theta \tilde{e}_z, \\
\nabla \cdot \tilde{V} &= 0, \\
\partial_t \theta + Gr \left[ \tilde{V} \nabla T_0 + \tilde{v}_0 \nabla \theta \right] &= \frac{1}{Pr} \Delta \theta, \\
\tilde{V} &= 0, \quad \theta = 0, \quad \text{at } z = -1, \\
\text{and} \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0, \quad \theta = 0, \quad \text{at } z = 1.
\end{align*}
$$

In the general case of spatial normal disturbances, all the variables are proportional to $\exp[-\lambda t + \i \lambda_2 x + \lambda_3 y]$, where $\lambda = \lambda_1 + \i \lambda_2$ is the characteristic decrement, $k_x$ and $k_y$ are the components of the wave vector along $x$– and $y$– directions. If the decrement $\lambda$ is complex, the disturbances oscillate with frequency $\lambda_2$ and propagate in the flow like the waves with a phase
velocity $c = \frac{\lambda}{k}$. Squire’s theorem does not apply due to the presence of longitudinal temperature gradient (Gershuni et al., 1989; Gershuni and Zhukovitskii, 1996). Thus, the disturbance equations cannot be transformed and reduced to the corresponding plane problem. Moreover, we shall take into consideration two more important limiting cases. They are spatial disturbances of the first type in the form of rolls with axes parallel to $x$-axis and spatial disturbances of the second type in form of rolls with axes perpendicular to $x$-direction.

3.1. The spatial disturbances in form of rolls with axes perpendicular to $x$-axis

The problem of linear stability for the advective flow in a horizontal rotating layer of liquid is here solved by reducing it to a boundary problem for a system of linear one-dimensional partial differential equations. The equations for the spatial disturbances of the second type are obtained for all the three components of velocity disturbance and temperature one, are functions of time $t$ and two spatial coordinates $x$ and $z$. We introduce the stream function $\psi(t, x, z) = -\frac{\partial \psi}{\partial x}, w = \frac{\partial \psi}{\partial x}$ and vorticity $\phi = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial t} = -\Delta \psi$. Looking for all unknown functions $F$ in the form: $F = [F_1(t, z) + iF_2(t, z)]\exp(ik_x x)$, the linearized system (16)-(20) is reduced to the one-dimensional system as follows:

$$\frac{\partial \phi_1}{\partial t} - k_x Gr \left[ u_0(z)\phi_2 + u_0^*(z)\phi_2^* \right] - \sqrt{Ta} \frac{\partial v_1}{\partial z} = \frac{\partial^2 \phi_1}{\partial z^2} - k_x^2 \phi_1 + k_x \theta_2,$$

$$\frac{\partial^2 \psi_1}{\partial z^2} - k_x^2 \psi_1 + \phi_1 = 0,$$

$$\frac{\partial \phi_2}{\partial t} + k_x Gr \left[ u_0(z)\phi_1 + u_0^*(z)\phi_1^* \right] - \sqrt{Ta} \frac{\partial v_2}{\partial z} = \frac{\partial^2 \phi_2}{\partial z^2} - k_x^2 \phi_2 - k_x \theta_1,$$

$$\frac{\partial^2 \psi_2}{\partial z^2} - k_x^2 \psi_2 + \phi_2 = 0,$$

$$\frac{\partial v_1}{\partial t} - k_x Gr \left[ u_0(z)v_2 + v_0^*(z)v_2^* \right] - \sqrt{Ta} \frac{\partial \psi_1}{\partial z} = \frac{\partial^2 v_1}{\partial z^2} - k_x^2 v_1,$$

$$\frac{\partial v_2}{\partial t} + k_x Gr \left[ u_0(z)v_1 + v_0^*(z)v_1^* \right] - \sqrt{Ta} \frac{\partial \psi_2}{\partial z} = \frac{\partial^2 v_2}{\partial z^2} - k_x^2 v_2,$$

$$\frac{\partial \theta_1}{\partial t} - k_x Gr \left[ u_0(z)\theta_2 + r_0^*(z)\theta_2^* \right] - Gr \frac{\partial \psi_1}{\partial z} = \frac{1}{Pr} \left[ \frac{\partial^2 \theta_1}{\partial z^2} - k_x^2 \theta_1 \right],$$

$$\frac{\partial \theta_2}{\partial t} + k_x Gr \left[ u_0(z)\theta_1 + r_0^*(z)\theta_1^* \right] - Gr \frac{\partial \psi_2}{\partial z} = \frac{1}{Pr} \left[ \frac{\partial^2 \theta_2}{\partial z^2} - k_x^2 \theta_2 \right],$$

subject to the boundary conditions:

$$\psi_{1, 2} = \frac{\partial \psi_{1, 2}}{\partial z} = v_{1, 2} = \theta_{1, 2} = 0 \text{ at } z = -1,$$

$$\psi_{1, 2} = \phi_{1, 2} = \frac{\partial v_{1, 2}}{\partial z} = \theta_{1, 2} = 0 \text{ at } z = 1.$$
The advective flow stability is characterized by neutral curves, which describe the dependence between the Grashof number and $k_x$, and separate the zones of stability and instability of this flow. The real part of the characteristic decrement $\lambda_i(Gr,k_x,Pr,Ta)$ is a function of several parameters. It vanishes on neutral curve. In the stability domain, $\lambda_i < 0$, and in the region of instability $\lambda_i > 0$. The numerical methodology of computing neutral curves for different values of Taylor number is represented in detail [17–18]. We define the function $\lambda_i$ discretely by repeated calculation of the system (21)-(29). This system is solved by finite difference method with the help of implicit scheme for numerical calculations. The implicit schemes for the system (21)–(29) is absolutely stable. The number of points $N$, used on the $z$– axis depends on the Taylor number value and the needed precision. It must be noticed that the boundary layer thickness decreases with the growth of $Ta$. The approximation of schemes is $O(\Delta t + \Delta z^2)$, where $\Delta t$ is a step of discretization along time axis and $\Delta z = 2/N$ is a step of discretization along $z$-axis. According to the theory of finite difference method the stability and approximation guaranties convergence. For the values of Taylor number ranged in the interval $0 \leq Ta \leq 10^6$, $N$ is varying between 101 and 201. It must be increased with the growth of $Ta$.

Figs. 4 show marginal curves for various values of Taylor number. Below each of these curves the flow is stable, and it is instable above. It is to be emphasized that the critical Grashof number increases with the increasing values of $Ta$. Stated otherwise, the rotation is a stabilizing factor against this kind of disturbance.

The wave numbers $k_x$ corresponding to the critical Grashof numbers are presented as function of Taylor numbers in Figs. 5. As one can see from the figures, $k_x$ varies with the growth of $Ta$. At small values of Taylor number: $0 < Ta < 500$, the wave number $k_x$ varies from 3.25 to 4.25. For $0 < Ta < 110$, it monotonically increases, while in the interval $110 < Ta < 500$ it monotonically decreases (Fig. 5a). For $Ta \geq 500$ with a further increase of Taylor number (Fig. 5b), a growth in $k_x$ is observed, that approximately follows:

$$k_x \approx 3.36 + 0.0154\sqrt{Ta - 500}.$$
Figure 5. The wave number $k_x$ (corresponding to a critical Grashof number $Gr_c$) versus the Taylor number, (a) for a small, and (b) for a large $Ta$.

As for the critical value of Grashof number $Gr_c$ which increases with the growth of Taylor number, and satisfactorily fits (see Fig. 6a).

$$Gr_c \approx 62.07 + 1.5617a^{0.675}$$  \hspace{1cm} (31)

The numerical calculations show that, like in the absence of rotation, a case that was considered in (Gershuni G.Z., 1989), instability phenomenon preserves the oscillating character for all Taylor numbers (Fig. 6b). The period $\lambda = 2\pi/\lambda_2$ grows with an increasing of Taylor number on the interval $0 \leq Ta \leq 2 \cdot 10^4$, and then, decreases for $2 \cdot 10^4 < Ta \leq 10^6$ (see Fig. 6b). Maximum of a dimensionless phase velocity $c = 0.841$ is reached at $Ta = 5000$.

Figure 6. Dependence (a) of the critical Grashof number (b) of the period $\lambda$ on $Ta$.

3.2. The spatial disturbances in the form of rolls with axes parallel to x-direction

The equations for the spatial disturbances of the first type are obtained for $k_y = 0$. Reducing this other limiting case to the boundary problem for the system of linear one-dimensional partial differential equations, similarly to part 3.1., allow to solve the problem of linear stability. The disturbances of the three velocity components and temperature are here functions of time $t$ and two spatial coordinates $y$.
and \( z \). Let us introduce the stream function \( \Psi(t,y,z) \): \( v = -\frac{\partial \Psi}{\partial z} \), \( w = \frac{\partial \Psi}{\partial y} \) and vorticity: \( \Phi = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = -\Delta \Psi \). All unknown functions \( F \) are introduced as \( F = [F_1(t,z) + iF_2(t,z)] \text{exp}(ik_{y,v}) \). Substituting into the system (16)-(20) reduces to the following system of one-dimensional partial differential equations:

\[
\frac{\partial \Phi_1}{\partial t} - k_y Gr \left[ v_0(z) \Phi_2 + v_0'(z) \Psi_2 \right] + \sqrt{Ta} \frac{\partial u_1}{\partial z} = \frac{\partial^2 \Phi_1}{\partial z^2} - k_y^2 \Phi_1 + k_y \theta_2, \tag{32}
\]

\[
\frac{\partial^2 \Psi_1}{\partial z^2} - k_y^2 \Psi_1 + \Phi_1 = 0, \tag{33}
\]

\[
\frac{\partial \Phi_2}{\partial t} + k_y Gr \left[ v_0(z) \Phi_1 + v_0'(z) \Psi_1 \right] + \sqrt{Ta} \frac{\partial u_2}{\partial z} = \frac{\partial^2 \Phi_2}{\partial z^2} - k_y^2 \Phi_2 - k_y \theta_1, \tag{34}
\]

\[
\frac{\partial^2 \Psi_2}{\partial z^2} - k_y^2 \Psi_2 + \Phi_2 = 0, \tag{35}
\]

\[
\frac{\partial u_1}{\partial t} - k_y Gr \left[ v_0(z)u_2 + u_0'(z) \Psi_2 \right] + \sqrt{Ta} \frac{\partial u_1}{\partial z} = \frac{\partial^2 u_1}{\partial z^2} - k_y^2 u_1, \tag{36}
\]

\[
\frac{\partial u_2}{\partial t} + k_y Gr \left[ v_0(z)u_1 + u_0'(z) \Psi_1 \right] + \sqrt{Ta} \frac{\partial u_2}{\partial z} = \frac{\partial^2 u_2}{\partial z^2} - k_y^2 u_2, \tag{37}
\]

\[
\frac{\partial \theta_1}{\partial t} - k_y Gr \left[ v_0(z)\theta_2 + \tau_0(z) \Psi_2 \right] + Gr u_1 = \frac{1}{Pr} \left[ \frac{\partial^2 \theta_1}{\partial z^2} - k_y^2 \theta_1 \right], \tag{38}
\]

\[
\frac{\partial \theta_2}{\partial t} + k_y Gr \left[ v_0(z)\theta_1 + \tau_0(z) \Psi_1 \right] + Gr u_1 = \frac{1}{Pr} \left[ \frac{\partial^2 \theta_2}{\partial z^2} - k_y^2 \theta_2 \right], \tag{39}
\]

with the boundary conditions

\[ \Psi_{1,2} = \frac{\partial \Psi_{1,2}}{\partial z} = u_{1,2} = \theta_{1,2} = 0 \text{ at } z = -1, \]

\[ \Psi_{1,2} = \Phi_{1,2} = \frac{\partial u_{1,2}}{\partial z} = \theta_{1,2} = 0 \text{ at } z = 1. \tag{40} \]

The stability is numerically investigated by the normal disturbance methodology, similar to that applied in part 4.1 [17–18]. The system (32)-(40) is here solved by finite difference method with an implicit scheme for numerical calculations too. The number of points is between 101 and 201. The Taylor number ranges in the interval \( 0 \leq Ta \leq 10^6 \). The Prandtl number \( Pr=6.7 \) (water).
In the absence of rotation, i.e. $Ta=0$, the instability of advective flow in a horizontal layer of a liquid has a monotonous character, $\lambda_2 = 0$ [2]. At $Ta > 0$, it becomes an oscillatory one as we can see in Fig. 7a. For small values of Taylor number, $0 < Ta < 42$, rotation stabilizes the advective flow in a similar way to the case $k_y = 0$, considered in part 4.1. The zones below the neutral curves in Fig. 8a are the regions of stability, and above these curves the flow is unstable. In this interval, the critical Grashof number grows quasi linearly, as this is well shown in the left part of curve 1 of Fig. 7b:

$$Gr_c \approx 55.27 + 1.772Ta.$$  

The corresponding numerical values of the wave number $k_y$ belong to the narrow interval $[3.4, 3.74]$. As for the period $\lambda$, it is proportional to $\sqrt{Ta}$ in these cases (see curve 1 of Fig. 7a).

For $Ta=42$, to which correspond $k_y = 0.12$ and $\lambda = 11.9$, the long-wave oscillatory instability starts to develop (see curves 2 of Figs. 7a, b). It sharply reduces the critical Grashof number $Gr_c$, and the advective steady flow becomes less stable even when the rotation is absent (see Fig. 7b). From $42 \leq Ta \leq 120$, $Gr_c$ decreases. After the value $Ta = 120$, it starts again to increase with the growth of Taylor number; Stated otherwise, sufficiently high rotation is a stabilizing factor for the advective flow.

The evolution of the neutral curves for different values of the Taylor number is presented in Figs. 8. Until the value $Ta = 42$, only one instability region, which is situated above any neutral curve, is observed (see Figs. 8a). Above this value of Taylor number, contrary to the case $k_y = 0$ of part 4.1, a second zone of instability starts, which, with the growth of $Ta$ increasingly enlarges. The example of Fig. 8b, drawn for $Ta = 70$, illustrates well this situation. With the further increase of $Ta$, the two neutral curves unify in a solely one with two minima. That may happen approximately for $100 < Ta < 250$. For $Ta \geq 250$, the neutral curves find again their initial tendency with one minimum (see Fig. 8c, d).
Figure 8. The neutral curves, (a) for the small Taylor numbers, (b) for \( Ta=70 \), (c) in a transitional situation, (d) for high \( Ta \).

In Figures 9 are presented the dependence of the critical Grashof number, \( Gr_c \), and the corresponding wave number \( k_y \), and period \( \lambda \) upon the Taylor number in the interval \( 120 \leq Ta \leq 10^6 \). As one can see from the figures the evolution of these functions approximately follows the forms:

\[
Gr_k \approx 28.86 + 0.539(Ta - 120)^{0.6},
\]

\[
k_y \approx 0.12 + 0.268(Ta - 42)^{0.162},
\]

and

\[
\lambda \propto Ta^{1/8}.
\]
Figure 9. Dependence of (a) the critical Grashof number, and the corresponding (b) wave number \( k_y \), and (c) the period \( \lambda \) upon the Taylor number \( Ta \).

4. Conclusion

The advective flow that may develop in a horizontal layer of a rotating liquid with a free surface, bordered below by a horizontal solid plane subject to a constant temperature gradient on the two boundaries has been studied.

Starting from Navier-Stokes equations, an exact stationary solution that describes analytically this type of flow has been derived. The resulting calculations for the horizontal velocity components have shown that the structure of the advective flow is spiral and that boundary layers form at both the bottom and free surface. As for the temperature, only close to the solid plane that the boundary layer has been observed. The profiles of velocity components and temperature are depended on the Taylor number.

With an increase of Taylor number a decrease in thickness of both boundary layers have been shown. In the core of the fluid layer, when \( Ta \) increases the profile of the \( x \)- component of velocity becomes almost constant, while that of the \( y \)- one remains approximately linear. As for the temperature profile, with an increase of Taylor number the parabolic shape is closely achieved. The intensity of the movement has been shown much lower inner the core of the flow than outer.

Using small perturbation technique, linear one-dimensional systems of partial differential equations have been obtained. Numerical computations performed at fixed Prandtl number, \( Pr=6.7 \) (water) have shown that an oscillatory instability of the advective flow for any value of Taylor number can be developed. The study has revealed that the rotation is a stabilizing factor for the advective flow at spatial disturbances of the second type in form of rolls with axes perpendicular to \( x \)-direction. As for spatial disturbances of the first type in the form of rolls with axes parallel to \( x \)-direction, at low and high values of \( Ta \), the rotation has also a stabilizing effect on the advective flow. At moderate Taylor numbers: \( 42 \leq Ta \leq 1000 \), the inverted trend has been observed.
Helical Rayleigh instability is developed. One critical helical Rayleigh mode take place for spatial disturbances of the first type and two modes take place for spatial disturbances of the second type. Finally, as a main result of rotational effect on the stability we can retain that the first type of spatial disturbance is more dangerous than the second one.

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