On Time Variations of Gravitational and Yang-Mills Constants in a Cosmological Model of Superstring Origin

V. D. Ivashchuk 1 and V. N. Melnikov 2,

Center for Gravitation and Fundamental Metrology, VNIIMS, Ozyornaya 46, Moscow 119361, Russia and
Institute of Gravitation and Cosmology, Peoples’ Friendship University of Russia, Miklukho-Maklaya 6, Moscow 117198, Russia

Abstract

In the framework of 10-dimensional “Friedmann-Calabi-Yau” cosmology of superstring origin we show that the time variation of either Newton’s gravitational constant or Yang-Mills one is unavoidable in the present epoch.

1 Introduction

The idea of time variations of Newton’s gravitational constant originally proposed by Dirac [1] (for a review see [2]) acquired more importance with the appearance of superstring theories [3, 4].

The present experimental constraints on time variations of the gravitational constant have the form

\[ |\dot{G}/(GH)| < 0.001, \]  

where \( H \) is the Hubble parameter, see [5] and references therein.

Here we consider the “Friedmann-Calabi-Yau” (FCY) cosmology based on the ten-dimensional \( SO(32) \) or \( E_8 \times E_8 \) Yang-Mills supergravity theory [6] with a Lorentz Chern-Simons three-form, introduced by Green and Schwarz [3] for anomaly cancellation, and with the Gauss-Bonnet term, introduced in [7, 8]. These additional terms are of superstring origin [9]: they appear as the next to leading terms in the \( \alpha' \) decomposition (\( \alpha' \) is the string parameter) of the Fradkin-Tseytlin effective action [10] for a heterotic string [4] (see [11]). The supergravity action is a leading term in this decomposition.

In the open-universe case of FCY cosmology with dust matter \( \rho > 0, p_3 = p_6 = 0 \) (\( \rho > 0 \) is the density and \( p_3, p_6 \) are pressures, see (12) below) and a constant dilaton field, \( \varphi(t) = \text{const} \), Wu and Wang calculated the present value of \( \dot{G}/G \) [12] and obtained the estimate

\[ (\dot{G}/G)_0 \approx -10^{-11 \pm 1} \text{ yr}^{-1}. \]  

This asymptotic relation was explained and generalized in [13], by simply using the equations of motion for a multidimensional cosmological model with anisotropic fluid, which describe the asymptotical behavior of FCY model with \( \varphi = \text{const} \).

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1 e-mail: ivashchuk@mail.ru
2 e-mail: melnikov@phys.msu.ru
Here we present a corrected and updated version of our old paper [14], which was devoted to the problem of variation of $G$ in FCY cosmological model with the restrictions on the density $\rho > 0$ and 3-dimensional pressure $p_3 \geq 0$ imposed [12]. At that times such restrictions looked physical. Now, we consider the problem of simultaneous stability of the effective gravitational constant $G$ and the effective Yang-Mills constant $g_{GUT}$ of Grand Unification Theory (GUT), which arise in this 10D model. In what follows we do not put any restrictions on $\rho$, $p_3$ and $p_6$. We prove that in the FCY cosmology there are no solutions to equations of motion with a constant radius of internal space and a constant dilation field, which do not contradict the present accelerated expansion of our Universe [15, 16]. Hence in FCY cosmology a simultaneous stability of $G$ and $g_{GUT}$ is impossible in the present epoch.

2 The model

We take the action of the model as in [12]

$$S = \int d^{10}x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{3}{4} \kappa^2 \varphi^{-3/2} H_{MNP}^2 - \frac{9}{16\kappa^2} (\varphi^{-1} \partial_M \varphi)^2 ight. $$

$$\left. - \frac{1}{4} \varphi^{-1/4} \left[ \frac{1}{30} \text{tr} F_{MN}^2 + (R_{MNPQ}^2 - 4R_{MN}^2 + R^2) \right] \right\} + S_F, \quad (3)$$

where $g_{MN}$ and $\varphi$ are the metric and dilation fields, $F_{MN}$ and $H_{MNP}$ are the Yang-Mills and Kolb-Ramond field strengths

$$F = \frac{1}{2} F_{MN} dx^M \wedge dx^N = dA + A \wedge A,$$

where $A = A_M dx^M$ is the one-form with the value in the Lie algebra ad $\hat{g}$, $\hat{g} = so(32), e_8 \oplus e_8$ ($\text{ad} \hat{g}$ is the image of the adjoint representation of $\hat{g}$, $\text{ad} \hat{g} \approx \hat{g}$ for any semi-simple Lie algebra $\hat{g}$);

$$H = \frac{1}{3!} H_{MNP} dx^M \wedge dx^N \wedge dx^P = dB - \omega_{3Y} + \omega_{3L}, \quad (4)$$

where $B = \frac{1}{2} B_{MN} dx^M \wedge dx^N$ is a two-form, $\omega_{3Y}$ is the Yang-Mills Chern-Simons three-form,

$$\omega_{3Y} = \frac{1}{30} \text{tr} (A \wedge F - \frac{1}{3} A \wedge A \wedge A), \quad (5)$$

and $\omega_{3L}$ is the Lorentz Chern-Simons three-form,

$$\omega_{3L} = \text{tr} (\omega \wedge \Omega - \frac{1}{3} \omega \wedge \omega \wedge \omega). \quad (6)$$

In (6) $\omega = \omega_M dx^M$ is the spin connection, which is a one-form with the value in $so(1, 9)$:

$$\omega_M = ||\omega_{BM}|| = ||e^A_N \nabla_M e^N_B|| \subset so(1, 9),$$

$e^A_N$ is the basis (zehnbein) of vector fields which diagonalizes the metric

$$g_{MN} = e^A_M e^B_N \eta_{AB},$$

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\( \eta_{AB} = \text{diag}(-1, +1, \ldots, +1) \), \( \Omega \) is the curvature two-form:

\[
\Omega = d\omega + \omega \wedge \omega.
\]

In (3) \( \kappa^2 \) is the 10-dimensional gravitational constant and \( S_F \) is the Fermi part of the action [6]. Here we deal with pure bosonic solutions to equations of motion (i.e., with zero Fermi fields) and hence an explicit relation for \( S_F \) is irrelevant for our consideration.

The action (3) and the energy-momentum tensor \( T_{MN} \) lead to the following equations of motions (see [12] for \( \varphi = \text{const} \)):

\[
R_{MN} - \frac{1}{2} g_{MN} R = \frac{9}{2} \kappa^4 \varphi^{-3/2} \left( H_{MPS} H_{NPQ} - \frac{1}{6} g_{MN} H^{PQS} \right) + 9\kappa^4 \nabla^S (\varphi^{-3/2} H_{MPS} R_{SN}^{PQ}) + \frac{9}{8} \varphi^{-2} \left[ \partial_M \varphi \partial_N \varphi - \frac{1}{2} g_{MN} (\partial_P \varphi)^2 \right] + \frac{1}{30} \kappa^2 \varphi^{-3/4} \left( \text{tr} F_{MP} F_{N}^{P} - \frac{1}{4} g_{MN} \text{tr} F_{PQ}^2 \right)
\]

\[
- \frac{1}{2} \kappa^2 \varphi^{-3/4} \left[ \frac{1}{2} g_{MN} (R_{PQST}^{2} - 4 R_{PQ}^2 + R^2) - 2 R R_{MN}^{2} \right]
\]

\[
+ 4 R_{MP} R_{N}^{P} + 4 R_{MPNQ} R^{PQ} - 2 R_{M}^{PQ} R_{NPQS}^{S} \right] + \kappa^2 T_{MN} + D_{MN},
\]

\[
\nabla_M (\varphi^{-3/2} H^{MNP}) = 0,
\]

\[
D_M (\varphi^{3/4} F^{MP}) + 9\kappa^2 (\varphi^{-3/2} F_{MN} H^{MNP}) = 0,
\]

\[
6 \nabla_M (\varphi^{-2} \partial^M \varphi) + 6 \varphi^{-3} (\partial_M \varphi)^2 + 6 \kappa^4 \varphi^{-5/2} H_{MNP}^{2} + \kappa^2 \varphi^{-7/4} \left[ \frac{1}{30} \text{tr} F_{MN}^2 + (R_{MNPQ}^2 - 4 R_{MN}^2 + R^2) \right] = 0.
\]

In (7) \( D_{MN} \) is a term with derivatives of the dilaton field which appears from the variation of the Gauss-Bonnet term. This \( D \)-term vanishes for \( \varphi = \text{const} \).

Let us consider the ten-dimensional manifold

\[
M^{10} = R \times M^3_k \times K,
\]

where \( M^3_k = S^3, \ R^3, \ L^3 \) for \( k = +1, 0, -1 \), respectively, and \( K \) is a Calaby-Yau manifold, i.e., a compact 6-dimensional Kähler Ricci-flat manifold with the SU(3) holonomy group.

Let the energy-momentum tensor be

\[
T = T_{MN} dx^M \otimes dx^N = \rho(t) dt \otimes dt + p_3(t) a_3^2(t) g^{(3)} + p_6(t) a_6^2(t) g^{(6)},
\]

where \( g^{(3)} \) and \( g^{(6)} \) are metrics on \( M^3_k \) and \( K \), \( \rho(t) \) is the energy density, \( p_3(t) \) and \( p_6(t) \) are pressures corresponding to \( M^3_k \) and \( K \).

The system (7)-(10) on the manifold (11) with the source (12) and the following ansatz

\[
g^{(10)} = -dt \otimes dt + a_3^2(t) g^{(3)} + a_6^2(t) g^{(6)},
\]

\[
H = 0,
\]

\[
\varphi = \varphi(t),
\]

\[
A = \text{ad}(i(\omega^{(6)})),
\]

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leads to a cosmology model, which we call the “Friedman-Calaby-Yau” (FCY) cosmology. In (16) \( \omega(6) \) is the spin connection on \( K \) corresponding to the basis \( e^{(6)\alpha} \), which diagonalizes \( g^{(6)} ; \ i : so(6) \to \hat{g} \) is the enclosure of the Lie algebra \( so(6) \) (in the case \( \hat{g} = e_8 \oplus e_8 \), \( i \) may be defined, for example, with the aid of the decomposition [4]: \( e_8 = so(16) \oplus V_{128} \). It follows from (16) that

\[
F = \text{ad}(i(\Omega(6))),
\]

where \( \Omega(6) = d\omega(6) + \omega(6) \wedge \omega(6) \).

From (16) and the trace identity

\[
\frac{1}{30} \text{tr}(\text{ad}(i(X))\text{ad}(i(Y))) = \text{tr}(XY)
\]

for all \( X, Y \in so(6) \), we have

\[
\omega_{3Y} = \frac{1}{30} \text{tr} \left( \frac{2}{3} A \wedge F + \frac{1}{3} A \wedge dA \right) = \text{tr} \left( \frac{2}{3} \omega(6) \wedge \Omega(6) + \frac{1}{3} \omega(6) \wedge d\omega(6) \right) = \omega_{3L}(6).
\]

It can be verified that in the basis \( (e^{(10)A}) = (dt, a_3(t)e^{(3)a}, a_6(t)e^{(6)\alpha}) \), where \( e^{(3)a} \) is the basis on \( M^3_k \) diagonalizing \( g^{(3)} \), we obtain

\[
\omega_{3L} = \omega_{3L}^{(3)} + \omega_{3L}^{(6)} + f_3,
\]

where \( df_3 = 0 \) and

\[
\omega_{3L}^{(3)} = \text{tr}(\omega^{(3)} \wedge \Omega^{(3)} - \frac{1}{3} \omega^{(3)} \wedge \omega^{(3)} \wedge \omega^{(3)}),
\]

\( \omega^{(3)} \) is the spin connection on \( M^3_k \) corresponding to \( e^{(3)a} \). From (11), (19) and (20) we have

\[
H = dB + \omega_{3L}^{(3)} + f_3.
\]

It follows from (21) and \( d\omega_{3L}^{(3)} = df_3 = 0 \) that for every domain \( \Omega \subset M^1_{10} \) with \( H^3(\Omega, R) = 0 \) there is some \( B \) such that \( H = 0 \).

The spin connection \( \omega(6) \) on \( K \) obeys the identity

\[
D_m(\omega^{(6)}(6))\Omega^{(6)mn} = 0,
\]

where \( D_m(\omega) = \nabla_m + [\omega_m, \cdot] \). The identity (22) is equivalent to

\[
\nabla^{(6)}_m R^{(6)mnp} = 0
\]

and is valid for any Kähler Ricci-flat manifold [17]. Equation (9) is satisfied identically due to (22), (13)-(16) and (17) \( D_M = D_M(A) = \nabla_M + [A_M, .] \); (8) is satisfied owing to (14).

Now we put

\[
a_6(t) = \text{const}, \quad \varphi(t) = \text{const}.
\]

What is the reason for the restrictions (23)? For the cosmology under consideration the effective 4D gravitational constant reads
\[ G = \text{const} \cdot a_6^{-6}(t). \] (24)

while the effective 4D Yang-Mills constant has the following form
\[ g_{\text{GUT}} = \text{const} \cdot a_6^{-3}(t)\varphi^{3/8}(t). \] (25)

This follows from the action (3) and the ansatz (13) for the metric. It is obvious that the stability of \( G \) and \( g_{\text{GUT}} \) is equivalent to the condition (23). Thus we are interested in solutions with stable effective constants \( G \) and \( g_{\text{GUT}} \).

Then, equations (7) and (10) in the ansatz (13)-(16) may be rewritten as follows [14]:
\[ 3\dot{a}_3^2(k + \dot{a}_3^2) = \kappa^2 \rho, \] (26)
\[ a_3^{-2}(k + \dot{a}_3^2 + 2a_3\dot{a}_3) = -\kappa^2 p_3, \] (27)
\[ a_3^{-2}(k + \dot{a}_3^2 + a_3\ddot{a}_3) = -(1/3)p_6 \]
\[ + 2\kappa^2 \varphi^{-3/4} a_3^{-3}\ddot{a}_3(k + \dot{a}_3^2), \] (28)
\[ 4\kappa^2 \varphi^{1/4} a_3^{-3}\ddot{a}_3(k + \dot{a}_3^2) = 0. \] (29)

Here \( 24a_3^{-3}\dot{a}_3(k + \dot{a}_3^2) = GB^{(4)} \) is the Gauss-Bonnet term corresponding to the 4-dimensional part of the metric.

Equations (26)-(28) and (29) are obtained from (7) and (10), respectively, using the Ricci flatness of \( K \) and the equality
\[ R^{(6)}_{pqmn} R^{(6)qp'mn} = \frac{1}{30} \text{tr} \ F_{mn} F_{pq} g^{(6)mpq} g^{(6)mn}, \]
which follows from (17), (18) and the relation
\[ R^{(6)\mu}_{\nu qmn} = e^{(6)\mu}_\alpha e^{(6)\beta}_q \Omega^{(6)\beta}_{\nu \beta mn}. \]

Due to (7)-(10) we get
\[ \nabla_M T^{MN} = 0. \] (30)

Relation (30) with the substitution of (12), (13) and (23) is equivalent to
\[ \dot{\rho} + 3a_3^{-1}\dot{a}_3(\rho + p_3) = 0. \] (31)

3 Solutions with constant \( a_6 \) and \( \varphi \).

It follows from (29) that
\[ \ddot{a}_3(k + \dot{a}_3^2) = 0. \] (32)

(A.) Consider the case \( \ddot{a}_3 \neq 0 \). Then we get from (32)
\[ k + \dot{a}_3^2 = 0 \] (33)

\[ 5 \]
which implies either $\dot{a}_3 = 0$ for $k = 0$ or $\dot{a}_3 = \pm 1$ for $k = -1$. In both cases we get $\ddot{a}_3 = 0$, which contradicts our suggestion $\ddot{a}_3 \neq 0$.

(B.) Now we put $\ddot{a}_3 = 0$. Then we obtain $\dot{a}_3 = C$ and $a_3 = Ct + a_0$, where $C$ and $a_0$ are constants. First we consider the case (B.3) which implies either $C = 0$ for $k = 0$ or $C = \pm 1$ for $k = -1$. For the density and pressures we have

$$\rho = 0, \quad p_3 = 0, \quad p_6 = 0.$$  \hfill (34)

For $k = 0, C = 0$, we get a static configuration [8]. For $k = -1$, we obtain $C = \pm 1$, the 4-dimensional part of the metric is flat. It is the well-known Milne solution which is isomorphic to the upper light cone (for $t > t_0, a_0 = -Ct_0$) in the Minkowski space. Thus we also obtain a part of a static configuration in this case.

Consider the case $k + C^2 \neq 0$. We obtain

$$\rho \neq 0, \quad p_3 = -\rho/3, \quad p_6 = -\rho.$$  \hfill (35)

From (26) we get

$$\rho = \rho_0 a_3^{-2},$$  \hfill (36)

where $\kappa^2 \rho_0 = 3(k + C^2) \neq 0$.

Thus, in the FCY cosmology there are no solutions to the equations of motion obeying (23), or equivalently, with stable constants $G$ and $g_{GUT}$, which are compatible with the modern cosmological data [15, 16]

$$\dot{a}_3 > 0, \quad \ddot{a}_3 > 0.$$  \hfill (37)

4 Conclusions

We have shown that in the framework of 10D “Friedmann-Calabi-Yau” cosmology of superstring origin the simultaneous stability of Newton’s gravitational constant $G$ and Yang-Mills constant $g_{GUT}$ is impossible in the present epoch. Thus one of the constants should be varying.

We note that the Yang-Mills running constant $g_{GUT}(\mu)$ ($\mu^2 = q^2$, where $q$ is 4D momentum) is used for generating the running constants for electromagnetic, weak and strong interactions, i.e., $\alpha(\mu), \alpha_w(\mu), \alpha_s(\mu)$, respectively [18, 19]. The temporal stability of $g_{GUT}(\mu)$ implies the temporal stability of the constants $\alpha(\mu), \alpha_w(\mu)$ and $\alpha_s(\mu)$ for fixed $\mu$.

An open problem here is to find exact solutions for a FCY cosmological model with small enough variations of $G$ and $\alpha$ obeying modern observational restrictions. (Multidimensional cosmological models with small enough variations of $G$ were considered previously in [20, 21, 22, 23] while variations of $\alpha$ and $G$ were obtained recently in [24, 25, 26] in the framework of nonlinear multidimensional gravity.) This can be a subject of a separate study.

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