THE SINGULARITIES OF YANG-MILLS CONNECTIONS FOR
BUNDLES ON A SURFACE. I. THE LOCAL MODEL

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ABSTRACT. Let Σ be a closed surface, \( G \) a compact Lie group, not necessarily connected, with Lie algebra \( g \), endowed with an adjoint action invariant scalar product, let \( \xi: P \to \Sigma \) be a principal \( G \)-bundle, and pick a Riemannian metric and orientation on \( \Sigma \), so that the corresponding Yang-Mills equations

\[
d_A \ast K_A = 0
\]

are defined, where \( K_A \) refers to the curvature of a connection \( A \). For every central Yang-Mills connection \( A \), the data induce a structure of unitary representation of the stabilizer \( Z_A \) on the first cohomology group \( H^1_A(\Sigma, \text{ad}(\xi)) \) with coefficients in the adjoint bundle \( \text{ad}(\xi) \), with reference to \( A \), with momentum mapping \( \Theta_A \) from \( H^1_A(\Sigma, \text{ad}(\xi)) \) to the dual \( z^*_A \) of the Lie algebra \( z_A \) of \( Z_A \). We show that, for every central Yang-Mills connection \( A \), a suitable Kuranishi map identifies a neighborhood of zero in the Marsden-Weinstein reduced space \( H_A \) for \( \Theta_A \) with a neighborhood of the point \([A]\) in the moduli space of central Yang-Mills connections on \( \xi \).

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Introduction

Let $\Sigma$ be a closed surface, $G$ a compact Lie group, not necessarily connected, with Lie algebra $g$, and $\xi: P \to \Sigma$ a principal $G$-bundle. Further, pick a Riemannian metric on $\Sigma$ and an orthogonal structure on $g$, that is, an adjoint action invariant positive definite inner product. These data then determine a Yang-Mills theory on the space $\mathcal{A}(\xi)$ of connections studied for connected $G$ extensively by Atiyah-Bott in [4] to which we refer for background and notation. In particular, the corresponding Yang-Mills equations look like

\[(0.1) \quad d_A * K_A = 0\]

where $K_A$ refers to the curvature of a connection $A$, and the solutions $A$ of (0.1) are referred to as Yang-Mills connections. Let $\mathcal{N}(\xi)$ be the space of central Yang-Mills connections on $\xi$, $\mathcal{G}(\xi)$ the group of gauge transformations, and $\mathcal{N}(\xi) = \mathcal{N}(\xi)/\mathcal{G}(\xi)$ the corresponding moduli space; see Section 1 below for a precise definition of a central connection. In the present paper we describe the singularities of $\mathcal{N}(\xi)$ explicitly. It turns out that in a suitable sense they are as simple as possible. For reasons that will become clear below we do not assume $G$ connected; we have extended the requisite results of Atiyah-Bott [4] to bundles with non-connected structure group in [9]. Thoughwe shall assume that solutions of (0.1) exist, so that the space $\mathcal{N}(\xi)$ is non-empty. For connected structure group this will always be so, cf. [4].

Recall that the adjoint bundle $\text{ad}(\xi)$ is associated with $\xi$ via the adjoint representation of $G$ on its Lie algebra $g$; its sections constitute the Lie algebra $g(\xi)$ of infinitesimal gauge transformations of $\xi$. For a central connection $A$, not necessarily a Yang-Mills one, the operator $d_A: \Omega^*(\Sigma, \text{ad}(\xi)) \to \Omega^*(\Sigma, \text{ad}(\xi))$ of covariant derivative is a differential, that is, satisfies $d_A d_A = 0$, even though in general $A$ is not flat; see (1.2) below for details. Hence the cohomology groups $H_A^*(\Sigma, \text{ad}(\xi))$ are defined. Moreover the given orthogonal structure on $g$ induces a symplectic structure $\sigma_A$ on the (finite dimensional) vector space $H_A^1(\Sigma, \text{ad}(\xi))$, and the Lie bracket on $g$ induces a graded Lie bracket $[\cdot, \cdot]_A$ on $H_A^*(\Sigma, \text{ad}(\xi))$ which, for degree reasons, is symmetric on $H_A^1(\Sigma, \text{ad}(\xi))$. Let $Z_A \subseteq G(\xi)$ be the stabilizer of $A$. It is a compact Lie group which acts canonically on $H_A^*(\Sigma, \text{ad}(\xi))$, preserving $\sigma_A$ and $[\cdot, \cdot]_A$, and its Lie algebra $z_A$ equals $H_A^0(\Sigma, \text{ad}(\xi))$. The orthogonal structure on $g$ induces a canonical isomorphism between $H_A^2(\Sigma, \text{ad}(\xi))$ and the dual $z_A^*$ of $z_A$ preserving the $Z_A$-actions. Furthermore, cf. (1.2.5) below, the assignment to $\eta \in H_A^1(\Sigma, \text{ad}(\xi))$ of $\Theta_A(\eta) = \frac{1}{2}[\eta, \eta]_A$ yields a momentum mapping $\Theta_A$ for the $Z_A$-action on the symplectic vector space $H_A^*(\Sigma, \text{ad}(\xi))$, see Section 1 below. Marsden-Weinstein reduction [15] yields the space $H_A = \Theta_A^{-1}(0)/Z_A$, and we have the following, cf. (2.32) below for a more precise statement.

Theorem A. For every central Yang-Mills connection $A$, a suitable Kuranishi map identifies a neighborhood of the class $[A]$ in $N(\xi)$ with a neighborhood of the class of zero in $H_A$.

We shall say that a point $[A]$ of $N(\xi)$ is non-singular provided $Z_A$ acts trivially on $H_A^1(\Sigma, \text{ad}(\xi))$. The Theorem entails that $N(\xi)$ is smooth near a non-singular point $[A]$ and we have the following immediate consequence.
Corollary B. The non-singular part of $N(\xi)$ inherits from $\sigma$ a structure of symplectic manifold.

In fact, for a central Yang-Mills connection $A$ representing a non-singular point of $N(\xi)$, Theorem A furnishes Darboux coordinates on $N(\xi)$ near the class of $A$; in a sense, Theorem A or rather (2.32) below yields “Darboux coordinates” near an arbitrary point of $N(\xi)$ where Darboux coordinates now means the whole structure of momentum mapping $\Theta_A$ for the $Z_A$-action on the symplectic vector space $H^1_A(\Sigma, \text{ad}(\xi))$. Theorem A makes precise the above remark that the singularities of $N(\xi)$ are “as simple as possible”. In fact, Theorem A reduces the study of the singularities of $N(\xi)$ to the standard example (2.4) on p. 52 of Arms-Gotay-Jennings [2]. Combining it with results of Sjamaar-Lerman [21] we shall show in [7] that the decomposition of $N(\xi)$ into connected components of orbit types of central Yang-Mills connections is a stratification in the strong sense in such a way that each stratum, being a smooth manifold, inherits a finite volume symplectic structure from the given data. This will in general refine the Atiyah-Bott-decomposition of the moduli space of all Yang-Mills connections. In particular, for $G = U(n)$, the unitary group, in the “coprime case”, cf. Atiyah-Bott [4], the component of the absolute minimum of the Yang-Mills functional has no singularities, that is, $N(\xi)$ is smooth, and our stratification then consists of a single piece.

The statement of Corollary B was obtained by Atiyah-Bott by means of symplectic reduction involving infinite dimensional spaces, see p. 587 of [4]. Our Corollary B avoids this infinite dimensional symplectic reduction, at the cost of exploiting the implicit function theorem for Banach manifolds. Another proof of the closedness of the symplectic structure at the non-singular points relying on finite dimensional techniques has recently been given by Weinstein [22]. Corollary B also paves the way to handle arbitrary central Yang-Mills connections $A$, not just those yielding non-singular points of $N(\xi)$. The details are given in the follow up paper [11] where we construct a stratified symplectic structure in the sense of Sjamaar [20] and Sjamaar-Lerman [21]; this is a Poisson structure defined at every point of $N(\xi)$; on each stratum, it restricts to the corresponding symplectic Poisson structure. As a stratified symplectic space, for every central Yang-Mills connection $A$, the space $H_A$ will then be a local model for $N(\xi)$ near the point represented by $A$. Thus additional geometric information not spelled out here is lurking behind our Theorem A, cf. e. g. our paper [12].

In another follow up paper [8] we identify the strata of $N(\xi)$ with reductions to suitable subbundles. Thereby we cannot avoid running into principal bundles with non-connected structure groups, even when the structure group of the bundle $\xi$ we started with is connected. This is the reason why the present theory has been set up for general compact not necessarily connected structure groups.

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1. Preliminaries

1.1. The space of connections as a Kähler manifold

Write $\Omega^* = \Omega^*(\Sigma, \text{ad}(\xi))$. The data we shall use throughout are the chosen orthogonal structure on $g$, the Riemannian metric on $\Sigma$, and a fixed orientation on $\Sigma$, with unique length one volume form $\text{vol}_\Sigma$ in this orientation. We recall from [4] that the data induce

\begin{align*}
(1.1.1) \quad [\cdot, \cdot]: & \Omega^* \otimes \Omega^* \to \Omega^*, \quad \text{a graded Lie bracket;} \\
(1.1.2) \quad \wedge: & \Omega^* \otimes \Omega^* \to \Omega^*(\Sigma, \mathbf{R}), \quad \text{a graded commutative pairing;} \\
(1.1.3) \quad (\cdot, \cdot): & \Omega^* \otimes \Omega^{2-*} \to \mathbf{R}, \quad \text{a weakly non-degenerate bilinear pairing}, \\
(1.1.4) \quad \cdot: & \Omega^* \otimes \Omega^* \to \mathbf{R}, \quad \text{a weak inner product;} \\
(1.1.5) \quad *: & \Omega^* \to \Omega^{2-*}, \quad \text{a duality operator}.
\end{align*}

In degree one, the pairing (1.1.3) and duality operator (1.1.5) amount to a weakly symplectic structure $\sigma = (\cdot, \cdot)$ and a complex structure $*$ on $\Omega^1$, respectively.

The space $\mathcal{A}(\xi)$ of connections on $\xi$ is affine, having $\Omega^1(\Sigma, \text{ad}(\xi))$ as its group of translations, and hence the three pieces of structure $\sigma, \star, \cdot$ extend to the space of connections $\mathcal{A}(\xi)$; moreover they fit together so that $\zeta \cdot \lambda = (\zeta, \star \lambda)$ whence, in particular, they turn $\mathcal{A}(\xi)$ into a Kähler manifold in the appropriate sense; the Kähler structure is manifestly preserved by the action of the group $G(\xi)$ of gauge transformations on $\mathcal{A}(\xi)$.

We recall from p. 546 of [4] that any three elements $u, v, w$ in $\Omega^*(\Sigma, \text{ad}(\xi))$ satisfy

\begin{align*}
(1.1.6) \quad [u, v] \wedge w = u \wedge [v, w].
\end{align*}

This implies that, for $|u| + |v| + |w| = 2$,

\begin{align*}
(1.1.7) \quad ([u, v], w) = (u, [v, w]).
\end{align*}

Next, for every connection $A$, when $p + q = 1$, $\phi \in \Omega^p(\Sigma, \text{ad}(\xi))$ and $\psi \in \Omega^q(\Sigma, \text{ad}(\xi))$ satisfy

\begin{align*}
(1.1.8) \quad (\phi, d_A \psi) = (-1)^{|\phi|} (d_A \phi, \psi).
\end{align*}

Finally, given $\alpha$ and $\beta$ in $\Omega^1(\Sigma, \text{ad}(\xi))$, we have the identity

\begin{align*}
(1.1.9) \quad \alpha \wedge \beta = \star \alpha \wedge \star \beta.
\end{align*}

1.2. Central connections

We denote the centre of $G$ by $Z$, and we write $z$ for its Lie algebra. It is a sub Lie algebra of the centre of $g$, stable under the adjoint representation. A central connection $A$ on $\xi$ is one whose curvature $K_A \in \Omega^2(\Sigma, \text{ad}(\xi))$ is a 2-form with values
in \( z \). Thus in particular a flat connection is central. We recall that the topology of \( \xi \) determines an element \( X_\xi \in z \) so that, given an arbitrary central Yang-Mills connection \( A \), its curvature \( K_A \) equals the image \( K_\xi \in \Omega^2(\Sigma, ad(\xi)) \) of the constant 2-form \( X_\xi \otimes \text{vol}_\Sigma \in \Omega^2(\Sigma, z) \) under the canonical map from \( \Omega^2(\Sigma, z) \) to \( \Omega^2(\Sigma, ad(\xi)) \).

We established this fact in (1.1) of our paper [9] for an arbitrary compact structure group. Henceforth we denote by \( d_A: \Omega^* \rightarrow \Omega^{*+1} \) the operator of covariant derivative.

Let now \( A \) be a central connection. It is clear that its operator of covariant derivative \( d_A \) is a differential, that is, satisfies \( d_Ad_A = 0 \), since \( d_Ad_A = [K_A, \cdot] \).

Hence the cohomology groups \( H^*_A = H^*_A(\Sigma, ad(\xi)) \) are defined. Since the operator of covariant derivative behaves as a derivation under both the graded Lie bracket (1.1.1) and the wedge product (1.1.2), (1.1.1) – (1.1.3) induce

\[
\begin{align*}
&[\cdot, \cdot]_A: H^*_A \otimes H^*_A \rightarrow H^*_A, \quad \text{a graded Lie bracket,} \\
&\wedge_A: H^*_A \otimes H^*_A \rightarrow H^*(\Sigma, \mathbb{R}), \quad \text{a graded commutative pairing,} \\
&\langle \cdot, \cdot \rangle_A: H^*_A \otimes H^{2-*}_A \rightarrow \mathbb{R}, \quad \text{a non-degenerate bilinear pairing,}
\end{align*}
\]

which, in particular, yields the symplectic structure \( \sigma_A = \langle \cdot, \cdot \rangle_A \) on \( H^*_A \) mentioned already in the Introduction.

The kernel of the operator \( d_A: \Omega^0(\Sigma, ad(\xi)) \rightarrow \Omega^1(\Sigma, ad(\xi)) \) of covariant derivative is the Lie algebra \( z_A \) of \( Z_A \). Hence \( H^0_A(\Sigma, ad(\xi)) \) is the Lie algebra \( z_A \) of \( Z_A \). Furthermore, (1.2.3) identifies \( H^2_A(\Sigma, ad(\xi)) \) with the dual \( z_A^* \) of \( z_A \), and it is clear that the \( G(\xi) \)-action on \( A(\xi) \) induces an action of \( Z_A \) on \( H^*_A(\Sigma, ad(\xi)) \).

Moreover, the corresponding infinitesimal \( z_A \)-action on \( H^*_A(\Sigma, ad(\xi)) \) is given by the restriction of the graded Lie bracket (1.2.1) to \( \Omega^0(\Sigma, ad(\xi)) \otimes H^*_A(\Sigma, ad(\xi)) \), that is, by the assignment to \( \phi \in \Omega^0(\Sigma, ad(\xi)) \) of the operation \( X_\phi: H^*_A \rightarrow H^*_A \) given by

\[
X_\phi(\eta) = [\phi, \eta]_A, \quad \eta \in H^*_A(\Sigma, ad(\xi)).
\]

Since the \( G(\xi) \)-action on \( A(\xi) \) preserves \( \sigma \), it is clear that the \( Z_A \)-action on \( H^*_A(\Sigma, ad(\xi)) \) preserves \( \sigma_A \), and we have the following, the proof of which we leave to the reader, cf. [6].

**Lemma 1.2.5.** For an arbitrary central connection \( A \), the assignment to \( \eta \in H^1_A(\Sigma, ad(\xi)) \) of \( \Theta_A(\eta) = \frac{1}{2}[\eta, \eta]_A \) yields a momentum mapping \( \Theta_A \) from \( H^1_A(\Sigma, ad(\xi)) \) to \( z_A^* \) for the action of \( Z_A \) on the symplectic vector space \( H^1_A(\Sigma, ad(\xi))\).

### 1.3. Hodge decomposition

Let \( A \) be a connection on \( \xi \). As usual we write \( d_A^*: \Omega^* \rightarrow \Omega^{*-1} \) for the adjoint of \( d_A \) with respect to the weak inner product (1.1.4) on \( \Omega^*(\Sigma, ad(\xi)) \), cf. p. 552 of [4]. Since the inner product is only weak the existence of the adjoint relies on a suitable version of the Hodge decomposition theorem.

For \( j = 1, 2 \), we denote by \( B^j_A(\Sigma, ad(\xi)) \) the subspace \( d_A(\Omega^{j-1}(\Sigma, ad(\xi))) \) of coboundaries in \( \Omega^j(\Sigma, ad(\xi)) \) and by \( \mathcal{P}_A \) the orthogonal projection from \( \Omega^j(\Sigma, ad(\xi)) \) to \( B^j_A(\Sigma, ad(\xi)) \) and, for \( j = 0, 1, 2 \), we denote by \( H^j_A(\Sigma, ad(\xi)) \) the vector space of harmonic \( j \)-forms in \( \Omega^j(\Sigma, ad(\xi)) \). The Laplace operator

\[
\Delta_A = d_Ad_A^* + d_A^*d_A: \Omega^*(\Sigma, ad(\xi)) \rightarrow \Omega^*(\Sigma, ad(\xi))
\]

is manifestly \( Z_A \)-equivariant. We reproduce the following well known facts which rely on the properties of the corresponding Green’s operator.
Proposition 1.3.2. For a central connection $A$ on $\xi$, for $j = 1, 2$, the restriction

\[(1.3.3) \quad \Delta_A = d_A d_A^* : B^j_A(\Sigma, \text{ad}(\xi)) \to B^j_A(\Sigma, \text{ad}(\xi))\]

is an $Z_A$-equivariant isomorphism of real vector spaces.

Let $A$ be a central connection on $\xi$. For $j = 0, 1, 2$, we write

\[(1.3.4) \quad \alpha_A : \Omega^j(\Sigma, \text{ad}(\xi)) \to \mathcal{H}^j_A(\Sigma, \text{ad}(\xi)), \quad \iota_A : \mathcal{H}^j_A(\Sigma, \text{ad}(\xi)) \to \Omega^j(\Sigma, \text{ad}(\xi)) \]

for the orthogonal projection and canonical injection, respectively. They are manifestly $Z_A$-equivariant. For $j = 1, 2$, we then consider the operator

\[(1.3.5) \quad h_A = d_A^* \Delta_A^{-1} \mathcal{P}_A : \Omega^j(\Sigma, \text{ad}(\xi)) \to \Omega^{j-1}(\Sigma, \text{ad}(\xi)).\]

Clearly it also looks like $h_A = G_1 d_A^* \mathcal{P}_A = d_A^* G_2 \mathcal{P}_A$ where $G_1$ and $G_2$ refer to the corresponding Green’s operators. We spell out some of its properties.

(1.3.6) It is $Z_A$-equivariant.

(1.3.7) It satisfies $d_A^* h_A = 0$ and $* h_A = d_A * \Delta_A^{-1} \mathcal{P}_A$.

(1.3.8) $\ker(h_A) = \ker(d_A^*)$.

(1.3.9) For $j = 1, 2$, we have $\mathcal{P}_A = d_A h_A : \Omega^j(\Sigma, \text{ad}(\xi)) \to B^j_A(\Sigma, \text{ad}(\xi))$.

The proofs of these properties are straightforward and left to the reader. Finally we spell out the following version of the Hodge decomposition theorem.

Lemma 1.3.10. For a central connection $A$ on $\xi$, the operators $h_A$ furnish a chain homotopy $h_A : \text{Id} \simeq \iota_A \alpha_A$, that is, we have

\[(1.3.11) \quad d_A h_A + h_A d_A = \text{Id} - \iota_A \alpha_A.\]

Let now $A$ be a central connection. Then (1.3.10) implies that the obvious map

\[(1.3.12) \quad \kappa_A : \mathcal{H}^j_A(\Sigma, \text{ad}(\xi)) \to \mathcal{H}^j_A(\Sigma, \text{ad}(\xi))\]

is an isomorphism of vector spaces; indeed, each cohomology class in $\mathcal{H}^j_A(\Sigma, \text{ad}(\xi))$ has a unique harmonic representative. Furthermore, the duality operator (1.1.5) passes to a duality operator “$*$” from $\mathcal{H}^*_A(\Sigma, \text{ad}(\xi))$ to $\mathcal{H}^{2-*}_A(\Sigma, \text{ad}(\xi))$, and the weak inner product (1.1.4) induces an inner product on each $\mathcal{H}^q_A(\Sigma, \text{ad}(\xi))$ which we denote by the same symbol “$\cdot$”; since these spaces are finite dimensional there is no difference here between “weak” and “strong”. It is clear that these pieces of structure together with the restriction of the symplectic structure $\sigma$ constitute a hermitian structure on $\mathcal{H}^j_A(\Sigma, \text{ad}(\xi))$, having * as its complex structure. By means of the isomorphism (1.3.12), we also have this structure on $\mathcal{H}^j_A(\Sigma, \text{ad}(\xi))$; its symplectic structure is just $\sigma_A$. Furthermore, the $Z_A$-action on $\mathcal{H}^j_A(\Sigma, \text{ad}(\xi))$ is in fact a unitary representation since it is compatible with all the structure, and the momentum mapping $\Theta_A$ is the unique one for this representation having the value zero at the origin.
2. The description of the singularities

In this Section we spell out and prove a more precise version of Theorem A in the Introduction. Following the arguments in [3] we use the Kuranishi theory of deformations to describe the exact structure of the singularities of our spaces of interest. Technically it may be worthwhile remembering that, for any connection $A$, the operator $d_A + d_A^*$ is elliptic, and it will be convenient to work with suitable Sobolev spaces.

The assignment to a connection $A$ of its curvature $K_A$ is a smooth map $J$ from $\mathcal{A}(\xi)$ to $\Omega^2(\Sigma, \text{ad}(\xi))$. It is well known to be given by the formula

$$J(A + \eta) = K_{A+\eta} = K_A + d_A \eta + \frac{1}{2} [\eta, \eta], \quad \eta \in \Omega^1(\Sigma, \text{ad}(\xi)),$$

and its tangent map $dJ(A)$ at $A$ amounts to the covariant derivative $d_A$ from $\Omega^1(\Sigma, \text{ad}(\xi))$ to $\Omega^2(\Sigma, \text{ad}(\xi))$ where the tangent space $T_A \mathcal{A}(\xi)$ is identified with $\Omega^1(\Sigma, \text{ad}(\xi))$ as usual. Moreover, cf. what is said in (1.2) above, on the subspace $\mathcal{N}(\xi)$ of central Yang-Mills connections, the map $J$ has a constant value $K_\xi \in \mathcal{H}^2_A(\Sigma, \text{ad}(\xi))$, determined by the topology of $\xi$.

Let $A \in \mathcal{N}(\xi)$ be a smooth central Yang-Mills connection, fixed throughout. Since $A$ is a solution of the Yang-Mills equations (0.1) the value of its curvature $K_A = K_\xi$ lies in the space $\mathcal{H}^2_A(\Sigma, \text{ad}(\xi))$ of harmonic 2-forms. Consider the Hodge decomposition

$$\Omega^2(\Sigma, \text{ad}(\xi)) = B^2_A(\Sigma, \text{ad}(\xi)) \oplus \mathcal{H}^2_A(\Sigma, \text{ad}(\xi)).$$

Clearly the point $A$ of $\mathcal{N}(\xi)$ is regular for $J$ if and only if $d_A$ is surjective, that is, if and only if $\mathcal{H}^2_A(\Sigma, \text{ad}(\xi))$ is zero. In a neighborhood of $A$, the pre-image $J^{-1}(K_\xi) = J^{-1}(0)$ is then a smooth manifold, and the tangent space to $\mathcal{N}(\xi)$ at $A$ equals the space

$$T_A \mathcal{N}(\xi) = \ker(d_A) = Z^1_A(\Sigma, \text{ad}(\xi))$$

of 1-cocycles at $A$.

Suppose now that $\mathcal{H}^2_A(\Sigma, \text{ad}(\xi))$ is non-zero. As before, let $\mathcal{P}_A$ be the orthogonal projection from $\Omega^2(\Sigma, \text{ad}(\xi))$ onto $B^2_A(\Sigma, \text{ad}(\xi))$, and let

$$\mathcal{A}_A = (\mathcal{P}_A J)^{-1}(0) = J^{-1}(\mathcal{H}^2_A(\Sigma, \text{ad}(\xi))).$$

Notice that $A + \eta \in \mathcal{A}_A$ if and only if $d_A^*(d_A \eta + \frac{1}{2} [\eta, \eta]) = 0$. By construction, the space $\mathcal{A}_A$ contains $\mathcal{N}(\xi)$. Further, since the composite map $\mathcal{P}_A J$ from $\mathcal{A}(\xi)$ to $B^2_A(\Sigma, \text{ad}(\xi))$ is a submersion at $A$, in a neighborhood of $A, \mathcal{A}_A$ is a smooth manifold in such a way that, for $A + \eta \in \mathcal{A}_A$,

$$T_{A+\eta} \mathcal{A}_A = \{ \psi; \mathcal{P}_A d_{A+\eta} \psi = 0 \} = \{ \psi; d_{A+\eta} \psi \in \mathcal{H}^2_A(\Sigma, \text{ad}(\xi)) \} \subseteq \Omega^1(\Sigma, \text{ad}(\xi)).$$

In particular, $T_A \mathcal{A}_A = \ker(d_A) = Z^1_A(\Sigma, \text{ad}(\xi))$, and $Z^1_{A+\eta}(\Sigma, \text{ad}(\xi)) \subseteq T_{A+\eta} \mathcal{A}_A$. Define the smooth map

$$J^\sharp: \mathcal{A}_A \to \mathcal{H}^2_A(\Sigma, \text{ad}(\xi))$$
as the restriction of \( J \) so that, for \( A + \eta \in \mathcal{A}_A \),

\[(2.6) \quad J^\sharp(A + \eta) = K_\xi + d_A \eta + \frac{1}{2}[\eta, \eta] \in \mathcal{H}^2_A(\Sigma, \text{ad}(\xi)).\]

Note that \( J^\sharp(A) = K_\xi \) and that, for \( A + \eta \in \mathcal{A}_A \), the tangent map \( dJ^\sharp(A + \eta) \) from \( T_{A+\eta}\mathcal{A}_A \) to \( \mathcal{H}^2_A(\Sigma, \text{ad}(\xi)) \) is given by the restriction of the operator \( d_{A+\eta} \); in particular, the derivative \( dJ^\sharp(A): Z_A^1(\Sigma, \text{ad}(\xi)) \rightarrow \mathcal{H}^2_A(\Sigma, \text{ad}(\xi)) \) is zero since so is the restriction of \( d_A \) to \( Z_A^1(\Sigma, \text{ad}(\xi)) \). It is clear that the space of central Yang-Mills connections \( \mathcal{N}(\xi) \) is smooth near \( A \) and coincides with \( \mathcal{A}_A \) near \( A \) if and only if the map \( J^\sharp \) is constant, having constant value \( K_\xi \). Hence:

**Lemma 2.7.** For a 1-form \( \eta \in \Omega^1(\Sigma, \text{ad}(\xi)) \) having the property that \( A + \eta \in \mathcal{A}_A \), the following are equivalent.

1. The connection \( A + \eta \in \mathcal{A}_A \) is a central Yang-Mills connection;
2. \( d_A \eta + \frac{1}{2}[\eta, \eta] = 0 \in \mathcal{H}^2_A(\Sigma, \text{ad}(\xi)) \);
3. \( [\eta, \eta]_A = 0 \in \mathcal{H}^2_A(\Sigma, \text{ad}(\xi)). \)

At this stage we thus obtain already the following well known.

**Theorem 2.8.** The space \( \mathcal{N}(\xi) \) of central Yang-Mills connections coincides with \( \mathcal{A}_A \) near \( A \) and hence is smooth near \( A \in \mathcal{N}(\xi) \), having the space of 1-cocycles \( Z_A^1(\Sigma, \text{ad}(\xi)) \) as tangent space for every \( A + \eta \in \mathcal{N}(\xi) \) near \( A \), if and only if the symmetric bilinear pairing \( [\cdot, \cdot]_A \) on \( \mathcal{H}^1_A(\Sigma, \text{ad}(\xi)) \) is zero. \( \square \)

Recall that after a choice \( Q \in \Sigma \) of base point, the normal subgroup \( G^Q(\xi) \) of \( Q \) based gauge transformations acts freely on \( \mathcal{A}(\xi) \), and, after a choice \( Q \in P \) of pre-image of \( Q \) has been made, evaluation of gauge transformations at \( Q \) furnishes a surjective homomorphism from \( G(\xi) \) onto \( G \) whose kernel equals \( G^Q(\xi) \). Consequently this surjection maps the stabilizer of an arbitrary connection isomorphically onto a closed subgroup of \( G \). Under the present circumstances, the method of Kuranishi consists of parametrizing a neighborhood of \( A \) in \( \mathcal{A}_A \) equivariantly with respect to its stabilizer \( Z_A \) by a neighborhood of the tangent space of \( \mathcal{A}_A \). Here are the details:

Recall that, in view of the Hodge decomposition theorem, an infinitesimal slice for the \( G(\xi) \)-action on \( \mathcal{A}(\xi) \) is given by the transverse gauge, that is, by the affine subspace

\[ S_A = A + \ker \left( d_A^* : \Omega^1(\Sigma, \text{ad}(\xi)) \rightarrow \Omega^0(\Sigma, \text{ad}(\xi)) \right). \]

Standard analytic arguments involving Sobolev spaces then show that this infinitesimal slice generates a local slice but for the moment this is not important for us. We only note that \( T_A S_A = \ker(d_A^* \Sigma, \text{ad}(\xi)) \) and that, with respect to (1.1.4), the Hodge decomposition

\[ T_A \mathcal{A}(\xi) = d_A \left( \Omega^0(\Sigma, \text{ad}(\xi)) \right) \oplus \ker(d_A^*) \]

is an orthogonal decomposition.

By means of the operator \( h_A \), cf. (1.3.5), the corresponding Kuranishi map \( F_A \) from \( \mathcal{A}(\xi) \) to itself is defined by

\[(2.9) \quad F_A(A + \eta) = A + \eta + \frac{1}{2} h_A [\eta, \eta], \quad \eta \in \Omega^1(\Sigma, \text{ad}(\xi)). \]
We shall use its properties spelled out below, cf. Lemmata 9 – 12 in [3].

(2.10) It is $Z_A$-equivariant.

(2.11) It is a local diffeomorphism of a neighborhood of $A$ to a neighborhood of $A$.

(2.12) It satisfies the formula

$$d_A(F_A(A + \eta) - A) = \mathcal{P}_A(J(A + \eta)), \quad \eta \in \Omega^1(\Sigma, \text{ad}(\xi)).$$

(2.13) Hence it identifies a neighborhood of $A$ in $\mathcal{A}_A$ with a neighborhood of $A$ in $A + Z_A^1(\Sigma, \text{ad}(\xi))$.

(2.14) For every $\eta \in \Omega^1(\Sigma, \text{ad}(\xi))$, we have $d_A^*(F_A(A + \eta) - A) = d_A^*(\eta)$. Consequently $F_A$ maps $\mathcal{S}_A$ to itself in such a way that $F_A(A + \eta) \in \mathcal{S}_A$ implies $A + \eta \in \mathcal{S}_A$.

Properties (2.13) and (2.14) above imply:

(2.15) Near $A$, the intersection $\mathcal{A}_A \cap \mathcal{S}_A$ is a smooth finite dimensional manifold, and the Kuranishi map $F_A$ restricts to a local diffeomorphism of $\mathcal{A}_A \cap \mathcal{S}_A$ onto $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$.

Smoothness of $\mathcal{A}_A \cap \mathcal{S}_A$ near $A$ follows also from the fact that $\mathcal{A}_A$ and $\mathcal{S}_A$ intersect transversely near $A$. With respect to (1.1.4), we now explicitly choose a ball $\mathcal{B}_A$ around $A$ in $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ in such a way that (i) the space

$$\mathcal{M}_A = F_A^{-1}(\mathcal{B}_A) \subseteq \mathcal{A}_A \cap \mathcal{S}_A,$$

is a smooth finite dimensional $Z_A$-manifold, and (ii) the Kuranishi map $F_A$ restricts to a diffeomorphism

$$f_A: \mathcal{M}_A \to \mathcal{B}_A,$$

necessarily $Z_A$-equivariant. In fact, the space $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ being of finite dimension, the restriction of the inner product (1.1.4) yields a norm on $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ in the usual (strong) sense. Moreover, the action of the group $\mathcal{G}(\xi)$ of gauge transformations restricts to an action of $Z_A$ on $\mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ and hence on the affine subspace $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$, and the weak inner product (1.1.4) is invariant under gauge transformations whence any ball therein is an $Z_A$-invariant subspace; in particular, the ball $\mathcal{B}_A$ in $A + \mathcal{H}_A^1(\Sigma, \text{ad}(\xi))$ is $Z_A$-invariant. Since $F_A$ is $Z_A$-equivariant, $\mathcal{M}_A$ inherits a $Z_A$-action, and $f_A$ is $Z_A$-equivariant.

We now have the machinery in place to show that the diffeomorphism (2.17) yields a symplectic change of coordinates: Let $\omega_A$ be the restriction of the (weakly) symplectic structure $\sigma$ on $\mathcal{A}(\xi)$ to the smooth submanifold $\mathcal{M}_A$; it is necessarily closed.

**Lemma 2.18.** The diffeomorphism $f_A$ is compatible with the 2-forms $\omega_A$ and $\sigma$ in the sense that, for $A + \eta \in \mathcal{M}_A$, given $\psi, \vartheta \in T_{A + \eta} \mathcal{M}_A$, we have

$$\sigma(\psi, \vartheta) = \omega_A(\psi, \vartheta) = \sigma(f_A'(A + \eta) \psi, f_A'(A + \eta) \vartheta).$$

Consequently $\omega_A$ is non-degenerate, that is, $\sigma$ passes to a symplectic structure on $\mathcal{M}_A$.

**Proof.** Let $A + \eta \in \mathcal{M}_A$. In a neighborhood of $A + \eta$, the map $f_A$ looks like

$$f_A(A + \eta + \psi) = A + \eta + \frac{1}{2} h_A[\eta, \eta] + (\psi + h_A[\eta, \psi]) + \frac{1}{2} h_A[\psi, \psi],$$
for suitable $\psi \in \Omega^1(\Sigma, \text{ad}(\xi))$, cf. (2.9). Consequently its derivative
\[ f'_{A}(A + \eta): T_{A + \eta}M_A \to T_{F_{A}(A + \eta)}H^1_A(\Sigma, \text{ad}(\xi)) = H^1_A(\Sigma, \text{ad}(\xi)) \]
at $A + \eta \in M_A$ is given by the assignment to $\psi \in T_{A + \eta}M_A$ of $\psi + h_A[\eta, \psi]$. Thus the statement of the Lemma will be a consequence of the following.

**Proposition 2.19.** Given $\psi, \vartheta \in T_{A + \eta}S_A = \ker (d^*_A: \Omega^1(\Sigma, \text{ad}(\xi)) \to \Omega^0(\Sigma, \text{ad}(\xi)))$, we have
\[ \sigma(\psi, \vartheta) = \sigma(\psi + h_A[\eta, \psi], \vartheta + h_A[\eta, \vartheta]). \]

**Proof.** Clearly it will suffice to show that $\sigma(h_A[\eta, \psi], \vartheta)$, $\sigma(\psi, h_A[\eta, \vartheta])$, and $\sigma(h_A[\eta, \psi], h_A[\eta, \vartheta])$ are zero. In order to see this we recall that the duality operator $*$ is symplectic whence $\sigma(u, v) = \sigma(*u, *v)$, whatever $u, v \in \Omega^1(\Sigma, \text{ad}(\xi))$. By virtue of (1.3.7), letting $\tau = *\Delta^1_{A}T_A[\eta, \psi] \in \Omega^0(\Sigma, \text{ad}(\xi))$, we have, cf. (1.1.8),
\[ \sigma(*h_A[\eta, \psi], *\vartheta) = (d_A\tau, *\vartheta) = (\tau, d_A * \vartheta) = 0 \]
whence
\[ \sigma(h_A[\eta, \psi], \vartheta) = \sigma(*h_A[\eta, \psi], *\vartheta) = 0. \]
The same kind of argument shows that $\sigma(\psi, h_A[\eta, \vartheta])$ is zero. Finally, to see that $\sigma(*h_A[\eta, \psi], *h_A[\eta, \vartheta])$ is zero, we proceed as above and obtain
\[ \sigma(*h_A[\eta, \psi], *h_A[\eta, \vartheta]) = (\tau, d_A * h_A[\eta, \vartheta]). \]
However this is zero since $d_A * h_A$ is zero, cf. (1.3.7). \[ \square \]

**Corollary 2.20.** The local diffeomorphism $f_A$ from $M_A$ to $A + H^1_A(\Sigma, \text{ad}(\xi))$ is a symplectic change of coordinates, that is, it yields Darboux coordinates on $M_A$. \[ \square \]

Our next aim is to examine the restriction $J_A$ of $J^*$ to $M_A$, cf. (2.5). We recall that (1.2.3) identifies $H^2_A(\Sigma, \text{ad}(\xi))$ with the dual $z^*_A$ of $z_A$ and, cf. [4], that $J$, cf. (2.1), is a momentum mapping for $\sigma$ and the $\tilde{G}(\xi)$-action on $A(\xi)$, the space $\Omega^2(\Sigma, \text{ad}(\xi))$ being identified with the dual of the Lie algebra $\mathfrak{g}(\xi) = \Omega^0(\Sigma, \text{ad}(\xi))$ of infinitesimal gauge transformations via (1.1.3). This implies at once the following.

**Lemma 2.21.** The composition of $J_A$ with the canonical isomorphism $\kappa_A$ from $H^2_A(\Sigma, \text{ad}(\xi))$ to $H^0_A(\Sigma, \text{ad}(\xi))$, cf. (1.3.12), is a momentum mapping for the action of $Z_A$ on the symplectic manifold $M_A$. \[ \square \]

For $\phi \in z_A = H^0_A(\Sigma, \text{ad}(\xi))$, let $X_{\phi}$ denote the vector field on $A + H^1_A(\Sigma, \text{ad}(\xi))$ induced by the infinitesimal $z_A$-action so that , cf. (1.2.4), for $\eta \in H^1_A(\Sigma, \text{ad}(\xi))$,
\[ (2.22) \quad X_{\phi}(A + \eta) = [\phi, \eta] \in H^1_A(\Sigma, \text{ad}(\xi)) = T_{A + \eta} (A + H^1_A(\Sigma, \text{ad}(\xi))). \]

**Lemma 2.23.** The assignment to $A + \eta \in A + H^1_A(\Sigma, \text{ad}(\xi))$ of
\[ j_A(A + \eta) = \kappa_A(K_\xi) + \frac{1}{2}[\eta, \eta]_A \]
yields a momentum mapping $j_A$ for the action of $Z_A$ on the affine symplectic space $A + H^1_A(\Sigma, \text{ad}(\xi))$, in fact, the unique one having the value $\kappa_A(K_\xi)$ at the point $A$.

**Proof.** This is established in much the same way as (1.2.5), by means of the canonical isomorphism (1.3.12), combined with the canonical symplectomorphism of affine symplectic manifolds from $A + H^1_A(\Sigma, \text{ad}(\xi))$ to $H^1_A(\Sigma, \text{ad}(\xi))$. \[ \square \]

As far as the statement of (2.23) is concerned, the term involving $K_\xi$ may safely be ignored. It has been included to have a consistent result in (2.24) below.
Theorem 2.24. The symplectomorphism $f_A$ preserves the momentum mappings in the sense that the diagram

$$\begin{array}{ccc}
\mathcal{M}_A & \xrightarrow{J_A} & \mathcal{H}^2_A(\Sigma, \text{ad}(\xi)) \\
\downarrow f_A & & \downarrow \kappa_A \\
A + \mathcal{H}^1_A(\Sigma, \text{ad}(\xi)) & \xrightarrow{J_A} & \mathcal{H}^2_A(\Sigma, \text{ad}(\xi))
\end{array}$$

is commutative where $\kappa_A$ refers to the canonical isomorphism from $\mathcal{H}^2_A(\Sigma, \text{ad}(\xi))$ to $H^2_A(\Sigma, \text{ad}(\xi))$, cf. (1.3.12).

In order to prove this we need some preparations.

Lemma 2.25. For $A + \eta \in A_A \cap S_A$, the elements $[\eta, h_A[\eta, \eta]]$ and $[h_A[\eta, \eta], h_A[\eta, \eta]]$ are coboundaries, that is, pass to zero in $H^2_A(\Sigma, \text{ad}(\xi))$.

To see this we proceed as follows: Since $A + \eta \in A_A$ the element $d_A\eta + \frac{1}{2}[\eta, \eta]$ lies in $\mathcal{H}^2_A(\Sigma, \text{ad}(\xi))$, and hence $h_A^2\eta + \frac{1}{2}h_A[\eta, \eta] = 0$, since $\mathcal{H}^2_A(\Sigma, \text{ad}(\xi)) = \ker(h_A)$, cf. (1.3.8). Hence it will suffice to show that $[\eta, h_A^2\eta]$ and $[h_A^2\eta, h_A^2\eta]$ pass to zero in $H^2_A(\Sigma, \text{ad}(\xi))$.

Lemma 2.26. For $\phi \in H^0_A(\Sigma, \text{ad}(\xi))$ and $\eta \in \Omega^1(\Sigma, \text{ad}(\xi))$, $[\phi, *\eta]$ equals $*[\phi, \eta]$.

Proof. The action of the group $G(\xi)$ of gauge transformations on $A(\xi)$ preserves $*$ in the sense that, given a gauge transformation $\gamma$, we have

$$(T_A\gamma) \circ * = * \circ (T_A\gamma) : T_A A(\xi) = \Omega^1(\Sigma, \text{ad}(\xi)) \to \Omega^1(\Sigma, \text{ad}(\xi)) = T_{\gamma A} A(\xi).$$

Consequently the action of $Z_A$ on $\Omega^1(\Sigma, \text{ad}(\xi)) = T_A A(\xi)$ preserves $*$. Given $\psi \in \Omega^0(\Sigma, \text{ad}(\xi)) = g(\xi)$, let $X_\psi : A(\xi) \to \Omega^1(\Sigma, \text{ad}(\xi))$ be the vector field on $A(\xi)$ coming from the infinitesimal $g(\xi)$-action on $A(\xi)$; for a connection $\tilde{A}$, it is given by $X_\psi(\tilde{A}) = -d_A(\psi)$, and its derivative $dX_\psi(\tilde{A})$ looks like

$$dX_\psi(\tilde{A}) : \Omega^1(\Sigma, \text{ad}(\xi)) \to \Omega^1(\Sigma, \text{ad}(\xi)).$$

Since the action of the isotropy subgroup $Z_A$ on $\Omega^1(\Sigma, \text{ad}(\xi)) = T_A A(\xi)$ preserves the duality operator $*$, for $\phi \in H^0_A(\Sigma, \text{ad}(\xi)) = z_A$, we have

$$(T^* \circ dX_\phi(A) = dX_\phi(A) \circ *.$$
**Corollary 2.27.** For \( \phi \in H^0_A(\Sigma, \text{ad}(\xi)) \) and \( \eta, \vartheta \in \Omega^1(\Sigma, \text{ad}(\xi)) \), as real valued 2-forms,

\[
(\eta, \vartheta) \wedge \phi = [\ast \eta, \ast \vartheta] \wedge \phi.
\]

Consequently, given \( \eta, \vartheta \in \Omega^1(\Sigma, \text{ad}(\xi)) \), the 2-forms \( [\eta, \vartheta] \) and \( [\ast \eta, \ast \vartheta] \) represent the same class in \( H^2_A(\Sigma, \text{ad}(\xi)) \).

**Proof.** Applying the identity (1.1.9) with \( \alpha = \eta \) and \( \beta = [\vartheta, \phi] \) and keeping in mind that, in view of (2.26), \( [\vartheta, \phi] = [\ast \vartheta, \phi] \), we obtain, cf. (1.1.6),

\[
[\eta, \vartheta] \wedge \phi = \eta \wedge [\vartheta, \phi] = \ast \eta \wedge \ast [\vartheta, \phi] = \ast \eta \wedge [\ast \vartheta, \phi] = [\ast \eta, \ast \vartheta] \wedge \phi.
\]

To verify the other statement, let \( \phi \in H^0_A(\Sigma, \text{ad}(\xi)) \). With reference to (1.2.3), in view of (2.27.1), we then have

\[
(\phi, [\eta, \vartheta])_A = (\phi, [\ast \eta, \ast \vartheta])_A.
\]

This implies the assertion, since the bilinear pairing (1.2.3) is non-degenerate. \( \square \)

We can now complete the proof of Lemma 2.25. In view of (2.27) it will suffice to show that \( [\ast \eta, \ast h_A d_A \eta] \) and \( [h_A d_A \eta, \ast h_A d_A \eta] \) pass to zero in \( H^2_A(\Sigma, \text{ad}(\xi)) \). However, cf. (1.3.7),

\[
\ast h_A d_A \eta = d_A \ast \Delta_A^{-1} d_A \eta = d_A \psi
\]

where \( \psi = \ast \Delta_A^{-1} d_A \eta \in \Omega^0(\Sigma, \text{ad}(\xi)) \). Consequently

\[
[\ast \eta, \ast h_A d_A \eta] = [\ast \eta, d_A \psi] = d_A [\ast \eta, \psi] - [d_A \ast \eta, \psi].
\]

However, since also \( \eta \in S_A \), \( d_A \ast \eta = 0 \) whence \( d_A \ast \eta = 0 \), that is, \( [\ast \eta, \ast h_A d_A \eta] \) equals \( d_A [\ast \eta, \psi] \). Likewise,

\[
[\ast h_A d_A \eta, \ast h_A d_A \eta] = [d_A \psi, d_A \psi] = d_A [d_A \psi, \psi].
\]

Consequently \( [\ast \eta, \ast h_A d_A \eta] \) and \( [h_A d_A \eta, h_A d_A \eta] \) both pass to zero in \( H^2_A(\Sigma, \text{ad}(\xi)) \). This completes the proof of (2.25). \( \square \)

**Proof of (2.24).** Let \( A + \eta \in M_A \). In view of (2.25), the elements \( [h_A [\eta, \eta], \eta] \) and \( [h_A [\eta, \eta], h_A [\eta, \eta]] \) pass to zero in \( H^2_A(\Sigma, \text{ad}(\xi)) \). Consequently we have

\[
j_A F_A (A + \eta) = j_A \left( A + \eta + \frac{1}{2} h_A [\eta, \eta] \right)
\]

\[
= K_{\xi} + \frac{1}{2} \left[ \eta + \frac{1}{2} h_A [\eta, \eta], \eta + \frac{1}{2} h_A [\eta, \eta] \right]_A
\]

\[
= K_{\xi} + \frac{1}{2} [\eta, \eta]_A = \kappa_A f_A (A + \eta). \quad \square
\]

Henceforth we write \( N_A = N(\xi) \cap M_A \).
Corollary 2.28. The symplectomorphism $f_A$ maps $\mathcal{N}_A$ locally 1-1 onto the cone

$$C_A = A + \{\eta \in H^1_A(\Sigma, \text{ad}(\xi)); [\eta, \eta]_A = 0 \in H^2_A(\Sigma, \text{ad}(\xi))\}.$$ 

Finally, let

$$(2.29) \quad \Phi_A: \mathcal{M}_A \to H^1_A(\Sigma, \text{ad}(\xi))$$

be the smooth map defined by

$$\Phi_A(A + \eta) = \kappa_A(F_A(A + \eta) - A) = \kappa_A(\eta + \frac{1}{2} h_A[\eta, \eta]), \quad A + \eta \in \mathcal{M}_A,$$

where $\kappa_A$ refers to the isomorphism (1.3.12). In view of (2.15), this is an injective immersion, in fact, cf. (2.20), a local symplectomorphism. Moreover, let

$$(2.30) \quad \vartheta_A: H^2_A(\Sigma, ad(\xi)) = z^*_A$$

be the smooth map defined by $\vartheta_A(x) = \kappa_A J_A(x) - \kappa_A(K_\xi)$, for $x \in \mathcal{M}_A$. Since $K_\xi$ remains invariant under the $Z_A$-action, Lemma 2.21 implies that $\vartheta_A$ is a momentum mapping for the action of $Z_A$ on the symplectic manifold $\mathcal{M}_A$; in fact, it is the unique one having the value zero at the point $A$.

Corollary 2.31. The local diffeomorphism $\Phi_A$ maps $\mathcal{M}_A$ locally 1-1 symplectically and $Z_A$-equivariantly onto $H^1_A(\Sigma, \text{ad}(\xi))$ and, furthermore, preserves the momentum mappings in the sense that $\vartheta_A = \Theta_A \Phi_A: \mathcal{M}_A \to z^*_A$.

From this we deduce a more precise version of Theorem A in the Introduction.

Theorem 2.32. The local symplectomorphism $\Phi_A$ induces a homeomorphism of a neighborhood of $[A]$ in $N(\xi)$ onto a neighborhood of zero in the Marsden-Weinstein reduced space $H_A = \Theta_A^{-1}(0)/Z_A$.

Proof. Since $Z_A$ is a compact group, a suitable Sobolev completion of the space $\ker(d^*_A)$ may be endowed with a $Z_A$-invariant inner product. Any ball with respect to this inner product will then inherit a $Z_A$-action. By the slice theorem, cf. e.g. [17], [5], the map from $\mathcal{M}_A/Z_A$ to $\mathcal{A}(\xi)/\mathcal{G}(\xi)$ induced by the injection of $\mathcal{M}_A$ into $\mathcal{A}(\xi)$ is itself injective provided the ball $B_\delta$ around $A + \mathcal{H}^1_A(\Sigma, \text{ad}(\xi))$ coming into play in (2.16) is chosen sufficiently small; this map restricts to a homeomorphism of $N_A = \mathcal{N}_A/Z_A$ onto a neighborhood of $[A]$ in $N(\xi)$. On the other hand, in view of (2.31), the local symplectomorphism $\Phi_A$ identifies $N_A$ locally 1-1 with $H_A$. $\square$

We shall say that a central Yang-Mills connection $A$ is non-singular if its stabilizer $Z_A$ acts trivially on $H^1_A(\Sigma, \text{ad}(\xi))$; the point $[A]$ of $N(\xi)$ will then be said to be non-singular.

Theorem 2.33. Near a non-singular point $[A]$ the space $N(\xi)$ is smooth.

Proof. In fact, for a non-singular central Yang-Mills connection $A$ the momentum mapping $\Theta_A$ is zero and $H_A$ coincides with $H^1_A(\Sigma, \text{ad}(\xi))$. $\square$

It may happen that the subspace of smooth points of $N(\xi)$ is larger than that of its non-singular ones; for example this occurs for $G = \text{SU}(2)$ over a surface of genus 2, see [12]. However the subspace of non-singular points is exactly that where the symplectic structure is defined, that is, the symplectic structure on the subspace of non-singular points cannot be extended to other smooth points.
References

1. R. Abraham and J. E. Marsden, *Foundations of Mechanics*, Benjamin-Cummings Publishing Company, 1978.
2. J. M. Arms, M. J. Gotay, and G. Jennings, *Geometric and algebraic reduction for singular momentum mappings*, Advances in Mathematics 79 (1990), 43–103.
3. J. M. Arms, J. E. Marsden, and V. Moncrief, *Symmetry and bifurcation of moment mappings*, Comm. Math. Phys. 78 (1981), 455–478.
4. M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. London A 308 (1982), 523–615.
5. D. Ebin, *The manifold of Riemannian metrics*, Proceedings of symposia in Pure Mathematics 15 (1970), American Math. Soc., Providence, R. I, 11–40.
6. W. M. Goldman and J. Millson, *Differential graded Lie algebras and singularities of level set momentum mappings*, Commun. Math. Phys. 131 (1990), 495–515.
7. J. Huebschmann, *The singularities of Yang-Mills connections for bundles on a surface. II. The stratification*, Math. Z. (to appear).
8. J. Huebschmann, *The singularities of Yang-Mills connections for bundles on a surface. III. The identification of the strata*, In preparation.
9. J. Huebschmann, *Holonomies of Yang-Mills connections for bundles on a surface with disconnected structure group*, Math. Proc. Cambr. Phil. Soc. 116 (1994), 375–384.
10. J. Huebschmann, *Smooth structures on certain moduli spaces for bundles on a surface*, preprint 1992.
11. J. Huebschmann, *Poisson structures on certain moduli spaces for bundles on a surface*, Annales de l’Institut Fourier (to appear).
12. J. Huebschmann, *Poisson geometry of flat connections for SU(2)-bundles on surfaces*, Math. Z. (to appear), hep-th/9312113.
13. J. Huebschmann, *Symplectic and Poisson structures of certain moduli spaces*, Preprint 1993, hep-th/9312112.
14. J. Huebschmann and L. Jeffrey, *Group cohomology construction of symplectic forms on certain moduli spaces*, Int. Math. Research Notices 6 (1994), 245–249.
15. J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetries*, Rep. on Math. Phys. 5 (1974), 121–130.
16. P. K. Mitter and C. M. Viallet, *On the bundle of connections and the gauge orbit manifold in Yang-Mills theory*, Comm. Math. Phys. 79 (1981), 457–472.
17. M. S. Narasimhan and T. R. Ramadas, *Geometry of SU(2)-gauge fields*, Comm. in Math. Phys. 67 (1979), 121–136.
18. M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. 82 (1965), 540–567.
19. I. M. Singer, *Some remarks about the Gribov ambiguity*, Comm. in Math. Phys. 60 (1978), 7–12.
20. R. Sjamaar, *Singular orbit spaces in Riemannian and Symplectic geometry*, Thesis, University of Utrecht, 1990.
21. R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. 134 (1991), 375–422.
22. A. Weinstein, *On the symplectic structure of moduli space*, A. Floer memorial (to appear).