On the spectral radius of graphs without a star forest

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Abstract

In this paper, we determine the maximum spectral radius and all extremal graphs for (bi-
partite) graphs of order $n$ without a star forest, extending Theorem 1.4 (iii) and Theorem 1.5
for large $n$. As a corollary, we determine the minimum least eigenvalue of $A(G)$ and all extremal
graphs for graphs of order $n$ without a star forest, extending Corollary 1.6 for large $n$.

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1 Introduction

Let $G$ be an undirected simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$,
where $n$ is called the order of $G$. The adjacency matrix $A(G)$ of $G$ is the $n \times n$ matrix $(a_{ij})$, where
$a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and 0 otherwise. The spectral radius of $G$ is the largest eigenvalue
of $A(G)$, denoted by $\rho(G)$. The least eigenvalue of $A(G)$ is denoted by $\rho_n(G)$. For $v \in V(G)$, the
neighborhood $N_G(v)$ of $v$ is $\{u : uv \in E(G)\}$ and the degree $d_G(v)$ of $v$ is $|N_G(v)|$. We write $N(v)$
and $d(v)$ for $N_G(v)$ and $d_G(v)$ respectively if there is no ambiguity. Denote by $\Delta(G)$ the maximum
degree of $G$. Let $S_{n-1}$ be a star of order $n$. The center of a star is the vertex of maximum degree in
the star. The centers of a star forest are the centers of the stars in the star forest. A graph $G$ is
$H$-free if it does not contain $H$ as a subgraph. For two vertex disjoint graphs $G$ and $H$, we denote
by $G \cup H$ and $G \nabla H$ the union of $G$ and $H$, and the join of $G$ and $H$ which is obtained by joining
every vertex of $G$ to every vertex of $H$, respectively. Denote by $kG$ the the union of $k$ disjoint copies
of $G$. For graph notation and terminology undefined here, readers are referred to [2].

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Recall that the problem of maximizing the number of edges over all graphs without fixed subgraphs is one of the cornerstones of graph theory.

**Problem 1.1** Given a graph $H$, what is the maximum number of edges of a graph $G$ of order $n$ without $H$?

Many instances of Problems 1.1 have been solved. For example, Lidický, Liu, and Palmer determined the maximum number of edges of graphs without a forest if the order of a graph is sufficiently large.

**Theorem 1.2** [3] Let $F = \bigcup_{i=1}^{k} S_{d_i}$ be a star forest with $k \geq 2$ and $d_1 \geq \cdots \geq d_k \geq 2$. If $G$ is an $F$-free graph of sufficiently large order $n$, then

$$e(G) \leq \max_{1 \leq i \leq k} \left\{ (i-1)(n-i+1) + \left(\frac{i-1}{2}\right) + \left\lfloor \frac{(d_i-1)(n-i+1)}{2} \right\rfloor \right\}. $$

In spectral extremal graph theory, a similar central problem is of the following type:

**Problem 1.3** Given a graph $H$, what is the maximum $\rho(G)$ of a graph $G$ of order $n$ without $H$?

Many instances of Problem 1.3 have been solved, for example, see [4, 6, 8, 12, 13, 14, 16]. In addition, if $H$ is a linear forest, Problem 1.3 was solved in [3]. For $H = kP_3$, the bipartite version of Problem 1.3 was also proved in [3]. In order to state these results, we need some symbols for given graphs.

Let $S_{n,h} = K_h \nabla K_{n-h}$. Furthermore, $S_{n,h}^+ = K_h \nabla (K_2 \cup K_{n-h-2})$. Let $F_{n,k} = K_{k-1} \nabla ((pK_2) \cup K_s)$, where $n = (k-1) = 2p + s$ and $0 \leq s < 2$. In addition, for $k \geq 2$ and $d_1 \geq \cdots \geq d_k \geq 1$, define

$$f(k,d_1,\ldots,d_k) = \frac{k^2(\sum_{i=1}^{k} d_i + k - 2)^2(\sum_{i=1}^{k} 2d_i + 5k - 4)^{k-2} + 2(k-2)(\sum_{i=1}^{k} d_i)}{k-2}. $$

**Theorem 1.4** [3] Let $F = \bigcup_{i=1}^{k} P_{a_i}$ be a linear forest with $k \geq 2$ and $a_1 \geq \cdots \geq a_k \geq 2$. Denote $h = \sum_{i=1}^{k} \left\lfloor \frac{a_i}{2} \right\rfloor - 1$ and suppose that $G$ is an $F$-free graph of sufficiently large order $n$.

(i) If there exists an even $a_i$, then $\rho(G) \leq \rho(S_{n,h})$ with equality if and only if $G = S_{n,h}$;

(ii) If all $a_i$ are odd and there exists at least one $a_i > 3$, then $\rho(G) \leq \rho(S_{n,h}^+)$ with equality if and only if $G = S_{n,h}^+$.

(iii) If all $a_i$ are $3$, i.e., $F = kP_3$, then $\rho(G) \leq \rho(F_{n,k})$ with equality if and only if $G = F_{n,k}$.

**Theorem 1.5** [3] Let $G$ be a $kP_3$-free bipartite graph of order $n \geq 11k - 4$ with $k \geq 2$. Then

$$\rho(G) \leq \sqrt{(k-1)(n-k+1)}$$

with equality if and only if $G = K_{k-1,n-k+1}$.

**Corollary 1.6** [3] Let $G$ be a $kP_3$-free graph of order $n \geq 11k - 4$ with $k \geq 2$. Then

$$\rho_n(G) \geq -\sqrt{(k-1)(n-k+1)}$$

with equality if and only if $G = K_{k-1,n-k+1}$.
In Theorem 1.4, the extremal graph for \( kP_3 \) varies from other linear forests. Note that \( kP_3 \) is also a star forest \( kS_2 \). Motivated by Problem 1.3, Theorems 1.2, 1.4 and 1.5, we determine the maximum spectral radius and all extremal graphs for all (bipartite) graphs of order \( n \) without a star forest. As a corollary, we determine the minimum least eigenvalue of \( A(G) \) and all extremal graphs for graphs of order \( n \) without a star forest, extending Corollary 1.6 for large \( n \). The main results of this paper are stated as follows.

**Theorem 1.7** Let \( F = \bigcup_{i=1}^{k} S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k \geq 1 \). If \( G \) be an \( F \)-free graph of order \( n \geq \frac{(\sum_{i=1}^{k} 2d_i + 5k - 8)^2 + (\sum_{i=1}^{k} d_i + k - 2)^2}{4k - 8} \), then

\[
\rho(G) \leq \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n - k + 1)}}{2}
\]

with equality if and only if \( G = K_{k-1} \bigtriangleup H \), where \( H \) is a \((d_k - 1)\)-regular graph of order \( n - k + 1 \). In particular, if \( d_k = 2 \), then

\[
\rho(G) \leq \rho(F_{n,k})
\]

with equality if and only if \( G = F_{n,k} \).

**Remark 1.** The extremal graph in Theorem 1.7 only depends on the number of the components of \( F \) and the minimum order of the stars in \( F \).

**Theorem 1.8** Let \( F = \bigcup_{i=1}^{k} S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k \geq 1 \). If \( G \) is an \( F \)-free bipartite graph of order \( n \geq \frac{f^2(k,d_1,\ldots,d_k)}{4k - 8} \), then

\[
\rho(G) \leq \sqrt{(k - 1)(n - k + 1)}
\]

with equality if and only if \( G = K_{k-1,n-k+1} \).

**Corollary 1.9** Let \( F = \bigcup_{i=1}^{k} S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k \geq 1 \). If \( G \) is an \( F \)-free graph of order \( n \geq \frac{f^2(k,d_1,\ldots,d_k)}{4k - 8} \), then

\[
\rho_n(G) \geq -\sqrt{(k - 1)(n - k + 1)}
\]

with equality if and only if \( G = K_{k-1,n-k+1} \).

**Remark 2.** For sufficiently large \( n \), the extremal graphs in Theorem 1.8 and Corollary 1.9 only depend on the number of the components of \( F \).

### 2 Preliminary

We first give a very rough estimation on the number of edges for a graph of order \( n \geq \sum_{i=1}^{k} d_i + k \) without a star forest.

**Lemma 2.1** Let \( F = \bigcup_{i=1}^{k} S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k \geq 1 \). If \( G \) is an \( F \)-free graph of order \( n \geq \sum_{i=1}^{k} d_i + k \), then

\[
e(G) \leq \left( \sum_{i=1}^{k} d_i + 2k - 3 \right) n - (k - 1) \left( \sum_{i=1}^{k} d_i + k - 1 \right).
\]
Proof. Let $C = \{ v \in V(G) : d(v) \geq \sum_{i=1}^{k} d_i + k - 1 \}$. Since $G$ is $F$-free, $|C| \leq k - 1$, otherwise we can embed an $F$ in $G$ by the definition of $C$. Hence

$$e(G) = \sum_{v \in C} d(v) + \sum_{v \in V(G) \setminus C} d(v)$$

$$\leq (n - 1)|C| + (n - |C|) \left( \sum_{i=1}^{k} d_i + k - 2 \right)$$

$$= \left( n - \sum_{i=1}^{k} d_i - k + 1 \right)|C| + \left( \sum_{i=1}^{k} d_i + k - 2 \right)n$$

$$\leq (k - 1) \left( n - \sum_{i=1}^{k} d_i - k + 1 \right) + \left( \sum_{i=1}^{k} d_i + k - 2 \right)n$$

$$= \left( \sum_{i=1}^{k} d_i + 2k - 3 \right)n - (k - 1) \left( \sum_{i=1}^{k} d_i + k - 1 \right)$$

\[ \square \]

Lemma 2.2 Let $F = \bigcup_{i=1}^{k} S_{d_i}$ be a star forest with $k \geq 2$ and $d_1 \geq \cdots \geq d_k \geq 1$. Let $G$ be an $F$-free connected bipartite graph of order $n \geq \frac{d_1^2}{2} + k - 1$ with the maximum spectral radius $\rho(G)$ and $x = (x_u)_{u \in V(G)}$ be a positive eigenvector of $\rho(G)$ such that $\max \{ x_u : u \in V(G) \} = 1$. Then $x_u \geq \frac{1}{\rho(G)}$ for all $u \in V(G)$.

Proof. Set for short $\rho = \rho(G)$. Choose a vertex $w \in V(G)$ such that $x_w = 1$. Since $K_{k-1, n-k+1}$ is $F$-free, we have

$$\rho \geq \rho(K_{k-1, n-k+1}) = \sqrt{(k-1)(n-k+1)}.$$

If $u = w$, then $x_u = 1 \geq \frac{1}{\rho}$. So we next suppose that $u \neq w$. We consider the following two cases.

Case 1. $u$ is adjacent to $w$. By eigenequation of $A(G)$ on $u$,

$$\rho x_u = \sum_{u \in E(G)} x_v \geq x_w = 1,$$

which implies that

$$x_u \geq \frac{1}{\rho}.$$

Case 2. $u$ is not adjacent to $w$. Let $G_1$ be a graph obtained from $G$ by deleting all edges incident with $u$ and adding an edge $uw$. Note that $uw$ is a pendant edge in $G_1$.

Claim. $G_1$ is also $F$-free.

Suppose that $G_1$ contains an $F$ as a subgraph. Since $G$ is $F$-free and $G_1$ contains an $F$ as a subgraph, we have $uw \in E(F)$. Since $uw$ is a pendant edge in $G_1$, $w$ is a center of $F$ with $d_F(w) = d_j$, where $1 \leq j \leq k$. Let $G_2$ be the subgraph of $G_1$ by deleting $w$ and all its neighbors in $F$. Note that $G_2$ is also a subgraph of $G$. Since $G_1$ contains an $F$ as a subgraph, $G_2$ contains $\bigcup_{i \neq j} S_{d_i}$ as a subgraph. By eigenequation of $G$ on $w$,

$$d(w) \geq \sum_{v \in E(G)} x_v = \rho x_w = \rho \geq \sqrt{(k-1)(n-k+1)} \geq d_1 \geq d_j.$$

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This implies that $G$ contains an $F$ as a subgraph, a contradiction.

By Claim, $G_1$ is $F$-free. Then

$$0 \geq \rho(G_1) - \rho \geq \frac{x^T A(G_1)x}{x^T x} - \frac{x^T A(G)x}{x^T x}$$

$$= \frac{2}{x^T x} \left( x_u x_w - x_u \sum_{uv \in E(G)} x_v \right)$$

$$= \frac{2x_u}{x^T x} \left( 1 - \rho x_u \right),$$

which implies that

$$x_u \geq \frac{1}{\rho}.$$

This completes the proof. □

**Lemma 2.3** Let $d \geq 1$, $k \geq 1$, $n \geq \frac{(d-1)^2+(k-1)^2}{k-1}$, and $H$ be a graph of order $n - k + 1$. If $G = K_{k-1} \nabla H$ and $\Delta(H) \leq d - 1$, then

$$\rho(G) \leq \frac{k + d - 3 + \sqrt{(k-d-1)^2 + 4(k-1)(n-k+1)}}{2}$$

with equality if and only if $H$ is a $(d - 1)$-regular graph.

**Proof.** If $d = 1$, then $G = K_{k-1} \nabla K_{n-k+1}$. It is easy to calculate that

$$\rho(K_{k-1} \nabla K_{n-k+1}) = \frac{k - 2 + \sqrt{(k-2)^2 + 4(k-1)(n-k+1)}}{2}.$$

Next suppose that $d \geq 2$. Let $u_1, u_2, \ldots, u_{k-1}$ be the vertex of $G$ corresponding to $K_{k-1}$ in the representation $G := K_{k-1} \nabla H$. Set for short $\rho = \rho(G)$ and let $x = (x_v)_{v \in E(G)}$ be a positive eigenvector of $\rho$. By symmetry, $x_{u_1} = \cdots = x_{u_{k-1}}$. Choose a vertex $v \in V(H)$ such that

$$x_v = \max_{w \in V(H)} x_w.$$

By eigenequation of $A(G)$ on $u_1$ and $v$, we have

$$\rho x_{u_1} = (k-2)x_{u_1} + \sum_{uv \in E(H)} x_u \leq (k-2)x_{u_1} + (n-k+1)x_v$$

$$\rho x_v \leq (k-1)x_{u_1} + \sum_{uv \in E(H)} x_u \leq (k-1)x_{u_1} + (d-1)x_v,$$

which implies that

$$(\rho - k + 2)x_{u_1} \leq (n- k + 1)x_v$$

$$(\rho - d + 1)x_v \leq (k-1)x_{u_1}.$$

Since

$$\rho > \rho(K_{k-1}) = k - 2,$$

and

$$\rho > \rho(K_{k-1,n-k+1}) = \sqrt{(k-1)(n-k+1)} \geq d - 1,$$
we have
\[ \rho^2 - (k + d - 3)\rho + (k - 2)(d - 1) - (k - 1)(n - k + 1) \leq 0. \]

Hence
\[ \rho \leq \frac{k + d - 3 + \sqrt{(k - d - 1)^2 + 4(k - 1)(n - k + 1)}}{2}. \]

If equality holds, then all equalities in (1) and (2) hold. So \( d(v) = k + d - 2 \) and \( x_u = x_v \) for any vertex \( u \in V(H) \). Since for any \( u \in V(H) \),
\[ \rho x_u = (k - 1)x_{u_1} + \sum_{u \in E(H)} x_2 \leq (k - 1)x_{u_1} + (d - 1)x_v = \rho x_v, \]
we have \( d(u) = d(v) = d + k - 2 \). So \( H \) is \((d - 1)\)-regular. \( \square \)

3 Proof of Theorem 1.7

Before proving Theorem 1.7, we first prove the following important result for connected graphs without a star forest.

Theorem 3.1 Let \( F = \bigcup_{i=1}^{k} S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k \geq 1 \). If \( G \) is an \( F \)-free connected graph of order \( n \geq (\sum_{i=1}^{k} 2d_i + 5k - 7)^2(\sum_{i=1}^{k} d_i + k - 2)^2 \), then
\[ \rho(G) \leq \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n - k + 1)}}{2} \]
with equality if and only if \( G = K_{k-1} \backslash H \), where \( H \) is a \((d_k - 1)\)-regular graph of order \( n - k + 1 \). In particular, if \( d_k = 2 \), then
\[ \rho(G) \leq \rho(F_{n,k}) \]
with equality if and only if \( G = F_{n,k} \).

Proof. Let \( G \) be an \( F \)-free connected graph of order \( n \) with the maximum spectral radius. Set for short \( V = V(G) \), \( E = E(G) \), \( A = A(G) \), and \( \rho = \rho(G) \). Let \( x = (x_v)_{v \in V(G)} \) be a positive eigenvector of \( \rho \) such that
\[ x_v = \max \{ x_u : u \in V(G) \} = 1. \]

Since \( K_{k-1,n-k+1} \) is \( F \)-free, we have
\[ \rho \geq \rho(K_{k-1,n-k+1}) = \sqrt{(k - 1)(n - k + 1)}. \]

Let \( L = \{ v \in V : x_v > \epsilon \} \) and \( S = \{ v \in V : x_v \leq \epsilon \} \), where \( \epsilon = \frac{1}{\sum_{i=1}^{k} 2d_i + 5k - 7} \).

Claim. \( |L| = k - 1 \).

If \( |L| \neq k - 1 \), then \( |L| \geq k \) or \( |L| \leq k - 2 \).

First suppose that \( |L| \geq k \). By eigenequation of \( A \) on any vertex \( u \in L \), we have
\[ \sum_{i=1}^{k} d_i + k - 2 \leq \frac{\sqrt{(k - 1)(n - k + 1)}}{\sum_{i=1}^{k} 2d_i + 5k - 7} = \sqrt{(k - 1)(n - k + 1)} \epsilon < \rho x_u = \sum_{uv \in E} x_v \leq d(u), \]
where the first inequality holds because \( n \geq (\sum_{i=1}^{k} 2d_i + 5k - 7)^2(\sum_{i=1}^{k} d_i + k - 2)^2 \). Hence

\[
d(u) \geq \sum_{i=1}^{k} d_i + k - 1.
\]

Then we can embed an \( F \) with all centers in \( L \) in \( G \), a contradiction.

Next suppose that \( |L| \leq k - 2 \). Then

\[
e(L) \leq \left( \frac{|L|}{2} \right) \leq \frac{1}{2}(k - 2)(k - 3)
\]

and

\[
e(L, S) \leq (k - 2)(n - k + 2).
\]

In addition, by Lemma 2.1,

\[
e(S) \leq e(G) \leq \left( \sum_{i=1}^{k} d_i + 2k - 3 \right)n.
\]

By eigenequation of \( A^2 \) on \( w \), we have

\[
(k - 1)(n - k + 1) \leq \rho^2 = \rho^2x_w = \sum_{uv \in E} \sum_{uv \in E} x_u \leq \sum_{uv \in E} (x_u + x_v)
\]

\[
= \sum_{uv \in E(L,S)} (x_u + x_v) + \sum_{uv \in E(S)} (x_u + x_v) + \sum_{uv \in E(L)} (x_u + x_v)
\]

\[
\leq \sum_{uv \in E(L,S)} (x_u + x_v) + 2e(S) + 2e(L)
\]

\[
\leq \sum_{uv \in E(L,S)} (x_u + x_v) + 2e\left( \sum_{i=1}^{k} d_i + 2k - 3 \right)n + (k - 2)(k - 3)
\]

Hence

\[
\sum_{uv \in E(L,S)} (x_u + x_v) \geq (k - 1)(n - k + 1) - 2e\left( \sum_{i=1}^{k} d_i + 2k - 3 \right)n - (k - 2)(k - 3).
\]

On the other hand, by the definition of \( L \) and \( S \), we have

\[
\sum_{uv \in E(L,S)} (x_u + x_v) \leq (1 + \epsilon)e(L, S) \leq (1 + \epsilon)(k - 2)(n - k + 2).
\]

Thus

\[
(1 + \epsilon)(k - 2)(n - k + 2) \geq (k - 1)(n - k + 1) - 2e\left( \sum_{i=1}^{k} d_i + 2k - 3 \right)n - (k - 2)(k - 3),
\]

which implies that

\[
\left( \left( \sum_{i=1}^{k} 2d_i + 5k - 8 \right) \epsilon - 1 \right)n \geq \epsilon(k - 2)^2 - (k^2 - 3k + 3).
\]
Since \( \epsilon = \frac{1}{\sum_{i=1}^{k} 2d_i + 5k - 7} \), we have

\[
n \leq (k^2 - 3k + 3) \left( \sum_{i=1}^{k} 2d_i + 5k - 8 \right) \left( \sum_{i=1}^{k} 2d_i + 5k - 7 \right) -
\]

\[
(k - 2)^2 \left( \sum_{i=1}^{k} 2d_i + 5k - 8 \right)
\]

\[
\leq \left( \sum_{i=1}^{k} 2d_i + 5k - 7 \right) \left( \sum_{i=1}^{k} d_i + k - 2 \right)^2,
\]

a contradiction. This proves the Claim.

By Claim, \(|L| = k - 1\) and thus \(|S| = n - k + 1\). Then the subgraph \(H\) induced by \(S\) in \(G\) is \(S_{d_k}\)-free. Otherwise, we can embed \(F\) in \(G\) with \(k - 1\) centers in \(L\) and a center in \(S\) as \(d(u) \geq \sum_{i=1}^{k} d_i + k - 1\) for any \(u \in L\), a contradiction. Now \(\Delta(H) \leq d_k - 1\). Note that the resulting graph obtained from \(G\) by adding all edges in \(L\) and all edges with one end in \(L\) and the other in \(S\) is also \(F\)-free and its spectral radius increases strictly. By the extremality of \(G\), we have \(G = K_{k-1} \nabla H\). By Lemma 2.3 and the extremality of \(G\), it follows that \(H\) is a \((d_k - 1)\)-regular graph and

\[
\rho = \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n - k + 1)}}{2}.
\]

In particular, if \(d_k = 2\) then \(\Delta(H) \leq 1\), i.e., \(H = pK_2 \cup qK_1\), where \(2p + q = n - k + 1\). By the extremality of \(G\), \(G = F_{n,k}\). This completes the proof. \(\square\)

**Proof of Theorems 1.7.** Let \(G\) be an \(F\)-free graph of order \(n\) with the maximum spectral radius.

If \(G\) is connected, then the result follows directly from Theorem 3.1. Next we suppose that \(G\) is not connected. Since \(K_{k-1,n-k+1}\) is \(F\)-free, we have

\[
\rho(G) \geq \sqrt{(k - 1)(n - k + 1)}.
\]

Let \(G_1\) be a component of \(G\) such that \(\rho(G_1) = \rho(G)\) and \(n_1 = |V(G_1)|\). Then

\[
n_1 - 1 \geq \rho(G_1) = \rho(G) \geq \sqrt{(k - 1)(n - k + 1)} \geq (k - 2)n
\]

\[
\geq \left( \sum_{i=1}^{k} 2d_i + 5k - 8 \right) \left( \sum_{i=1}^{k} d_i + k - 2 \right)^2,
\]

which implies that

\[
n_1 \geq \left( \sum_{i=1}^{k} 2d_i + 5k - 8 \right) \left( \sum_{i=1}^{k} d_i + k - 2 \right)^2 + 1.
\]

By Theorem 3.1 again,

\[
\rho(G) = \rho(G_1) \leq \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n_1 - k + 1)}}{2}
\]

\[
< \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n - k + 1)}}{2}.
\]

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In particular, if \( d_k = 2 \) then it follows from By Theorem 3.1 again,

\[
\rho(G) = \rho(G_1) = \rho(F_{n_1,k}) < \rho(F_{n,k}).
\]

Hence the result follows. \( \square \)

Note that extremal graph in Theorem 1.4 (iii) also holds for signless Laplacian special radius \( q(G) \) \([5]\). We conjecture the extremal graph in Theorem 1.7 also holds for signless Laplacian spectral radius \( q(G) \).

**Conjecture 3.2** Let \( F = \cup_{i=1}^{k} S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k \geq 1 \). If \( G \) be an \( F \)-free graph of large order \( n \), then

\[
q(G) \leq \frac{n + 2k + 2d_k - 6 + \sqrt{(n + 2k - 2d_k - 2)^2 - 8(k - 1)(k - d_k - 1)}}{2}
\]

with equality if and only if \( G = K_{k-1} \Delta H \), where \( H \) is a \((d_k - 1)\)-regular graph of order \( n - k + 1 \).

In particular, if \( d_k = 2 \), then

\[
q(G) \leq q(F_{n,k})
\]

with equality if and only if \( G = F_{n,k} \).

4 Proofs of Theorem 1.8 and Corollary 1.9

Before proving Theorem 1.8 and Corollary 1.9 we first prove the following important result for bipartite connected graphs without a star forest.

**Theorem 4.1** Let \( F = \cup_{i=1}^{k} S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k \geq 1 \). If \( G \) is an \( F \)-free connected bipartite graph of order \( n \), then

\[
\rho(G) \leq \sqrt{(k - 1)(n - k + 1)}
\]

with equality if and only if \( G = K_{k-1,n-k+1} \).

**Proof.** Let \( G \) be an \( F \)-free connected bipartite graph of order \( n \) with the maximum spectral radius. Set for short \( V = V(G) \), \( E = E(G) \), \( A = A(G) \), and \( \rho = \rho(G) \). Let \( x = (x_v)_{v \in V(G)} \) be a positive eigenvector of \( \rho \) such that

\[
x_w = \max \{ x_u : u \in V(G) \} = 1.
\]

Since \( K_{k-1,n-k+1} \) is \( F \)-free, we have

\[
\rho \geq \rho(K_{k-1,n-k+1}) = \sqrt{(k - 1)(n - k + 1)}.
\]

Let \( L = \{ v \in V : x_v > \epsilon \} \) and \( S = \{ v \in V : x_v \leq \epsilon \} \), where

\[
\frac{1}{\sqrt{(k - 1)(n - k + 1)}} \leq \epsilon \leq \frac{1}{k \sum_{i=1}^{k} (2d_i + 5k - 4)^{2k-1}} \left( 1 - \frac{\sum_{i=1}^{k} d_i}{n} \right)^{k-1}.
\]

**Claim 1.** \( |L| \leq k - 1 \).

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Suppose that $|L| \geq k$. By eigenequation of $A$ on any vertex $u \in L$, we have
\[
\sum_{i=1}^{k} d_i + k - 2 \leq \sqrt{(k-1)(n-k+1)}\epsilon < \rho x_u = \sum_{uv \in E} x_v \leq d(u).
\]
Hence
\[
d(u) \geq \sum_{i=1}^{k} d_i + k - 1.
\]
Then we can embed an $F$ in $G$ with all centers in $L$, a contradiction. This proves Claim 1.

Since $|L| \leq k - 1$, we have
\[
\epsilon(L) \leq \binom{|L|}{2} \leq \frac{1}{2}(k-1)(k-2)
\]
and
\[
\epsilon(L, S) \leq (k-1)(n-k+1).
\]

In addition, by Lemma 2.1
\[
\epsilon(S) \leq \epsilon(G) \leq \left( \sum_{i=1}^{k} d_i + 2k - 3 \right) n.
\]

We next show that for any vertex in $L$ has large degree.

Claim 2. Let $u \in L$ and $x_u = 1 - \delta$. Then
\[
d(u) \geq \left( 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 5 \right) (\delta + \epsilon) \right) n.
\]

Let $B_u = \{ v \in V : uv \notin E \}$. We first sum of eigenvector over all vertices of $G$.
\[
\rho \sum_{v \in V} x_v = \sum_{v \in V} \rho x_v = \sum_{v \in V} \sum_{vz \in E} x_z = \sum_{v \in V} d(v)x_v \leq \sum_{v \in L} d(v)x_v + \sum_{v \in S} d(v)x_v
\]
\[
\leq \sum_{v \in L} d(v) + \epsilon \sum_{v \in S} d(v) = 2\epsilon(L) + \epsilon(L, S) + \epsilon(2\epsilon(S) + \epsilon(L, S))
\]
\[
= 2\epsilon(L) + 2\epsilon(S) + (1 + \epsilon)\epsilon(L, S),
\]
which implies that
\[
\sum_{v \in V} x_v \leq \frac{2\epsilon(L) + 2\epsilon(S) + (1 + \epsilon)\epsilon(L, S)}{\rho}.
\]

Next we sum of eigenvector over all vertices in $B_u$ by E.q. (4) and Lemma 2.2. Since
\[
\frac{1}{\rho} |B_u| \leq \sum_{v \in B_u} x_v \leq \sum_{v \in V(G)} x_v - \sum_{uv \in E(G)} x_v = \sum_{v \in V(G)} x_v - \rho x_u
\]
\[
\leq \frac{2\epsilon(L) + 2\epsilon(S) + (1 + \epsilon)\epsilon(L, S)}{\rho} - \rho x_u,
\]
we have
\[
|B_u| \leq 2\epsilon(L) + 2\epsilon(S) + (1 + \epsilon)\epsilon(L, S) - \rho^2 x_u
\]
\[
\begin{align*}
\leq & \ 2\epsilon(L) + 2\epsilon e(S) + (1 + \epsilon) e(L, S) - (k - 1)(n - k + 1)(1 - \delta) \\
\leq & \ (k - 1)(k - 2) + 2\epsilon\left(\sum_{i=1}^{k} d_i + 2k - 3\right)n + (1 + \epsilon)(k - 1)(n - k + 1) - \\
& (k - 1)(n - k + 1)(1 - \delta) \\
= & \ \left(2\epsilon\left(\sum_{i=1}^{k} d_i + 2k - 3\right) + (\delta + \epsilon)(k - 1)\right)n + (k - 1)(k - 2) - (\delta + \epsilon)(k - 1)^2 \\
\leq & \ \left(\sum_{i=1}^{k} 2d_i + 4k - 6 + (k - 1) + 1\right)(\delta + \epsilon)n \\
= & \ \left(\sum_{i=1}^{k} 2d_i + 5k - 6\right)(\delta + \epsilon)n,
\end{align*}
\]

where the last second inequality holds since \((k - 1)(k - 2) \leq \epsilon n < (\delta + \epsilon)n\) by the definition of \(\epsilon\) and \(n\). Hence

\[
d(u) \geq n - 1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 6\right)(\delta + \epsilon)n \geq \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 5\right)(\delta + \epsilon)\right)n.
\]

This completes Claim 2.

**Claim 3.** Let \(1 \leq s < k - 1\). Suppose that there is a set \(X\) of \(s\) vertices such that \(X = \{v \in V : x_v \geq 1 - \eta\) and \(d(v) \geq (1 - \eta)n\}. Then there exists a vertex \(u \in L \setminus X\) such that

\[x_u \geq 1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 5\right)^2(\eta + \epsilon)\]

and

\[d(u) \geq \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 5\right)^2(\eta + \epsilon)\right)n.\]

By eigenequation of \(A^2\) on \(w\), we have

\[
\rho^2 = \rho^2 x_w = \sum_{uv \in E} \sum_{uv \in E} x_u \leq \sum_{uv \in E} (x_u + x_v)
\]

\[
= \sum_{uv \in E(S)} (x_u + x_v) + \sum_{uv \in E(L)} (x_u + x_v) \sum_{uv \in E(L, S)} (x_u + x_v)
\]

\[
\leq 2\epsilon e(S) + 2\epsilon e(L) + \sum_{uv \in E(L, S)} (x_u + x_v)
\]

\[
\leq 2\epsilon e(S) + 2\epsilon e(L) + \epsilon e(L, S) + \sum_{uv \in E(L \setminus X, S)} x_u + \sum_{uv \in E(L \cap X)} x_u,
\]

which implies that

\[
\sum_{uv \in E(L \setminus X, S) \atop u \in L \setminus X} x_u
\]

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\[ \geq \rho^2 - 2\epsilon e(S) - 2\epsilon (L) - \epsilon e(L, S) - \sum_{u \in E(L \setminus X, S)} x_u \]
\[ \geq (k - 1)(n - k + 1) - 2\epsilon \left( \sum_{i=1}^{k} d_i + 2k - 3 \right) n - (k - 1)(k - 2) - \epsilon(k - 1)(n - k + 1) - sn \]
\[ = \left( k - 1 - s - \epsilon \left( \sum_{i=1}^{k} 2d_i + 5k - 7 \right) \right) n - (k - 1)(2k - 3) + \epsilon(k - 1)^2 \]
\[ \geq \left( k - 1 - s - \epsilon \left( \sum_{i=1}^{k} 2d_i + 5k - 7 \right) \right) n - \epsilon n \]
\[ = \left( k - 1 - s - \epsilon \left( \sum_{i=1}^{k} 2d_i + 5k - 6 \right) \right) n, \]

where the last third inequality holds since \((k - 1)(2k - 3) \leq \epsilon n\) by the definition of \(\epsilon\) and \(n\). In addition,

\[ e(L \setminus X, S) = e(L, S) - e(L \cap X, S) \]
\[ \leq (k - 1)(n - k + 1) - s(1 - \eta)n + \left( \frac{s}{2} \right) \]
\[ \leq (k - 1 - s(1 - \eta))n - \left( (k - 1)^2 - \left( \frac{k - 2}{2} \right) \right) \]
\[ \leq (k - 1 - s(1 - \eta))n. \]

Let

\[ g(s) = \frac{k - 1 - s - \epsilon \left( \sum_{i=1}^{k} 2d_i + 5k - 6 \right)}{k - 1 - s(1 - \eta)}. \]

It is easy to see that \(g(s)\) is decreasing with respect to \(1 \leq s \leq k - 2\). Then

\[ \frac{\sum_{u \in \{E(L \setminus X, S)\}} x_u}{e(L \setminus X, S)} \geq g(s) \geq g(k - 2) = \frac{1 - \epsilon \left( \sum_{i=1}^{k} 2d_i + 5k - 6 \right)}{1 + (k - 2)\eta} \]
\[ \geq 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 6 \right)(\eta + \epsilon). \]

Hence there exists a vertex \(u \in L \setminus X\) such that

\[ x_u \geq 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 6 \right)(\eta + \epsilon). \]

By Claim 2,

\[ d(u) \geq \left( 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 5 \right) \left( \left( \sum_{i=1}^{k} 2d_i + 5k - 6 \right)(\eta + \epsilon) + \epsilon \right) \right) n \]
\[ \geq \left( 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 5 \right)^2 (\eta + \epsilon) \right) n \]

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This completes Claim 3.

**Claim 4.** $|L| = k - 1$. Furthermore, for all $u \in L$,

$$x_u \geq 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 4 \right) \epsilon$$

and

$$d(u) \geq \left( 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 4 \right) \epsilon \right) n.$$

Note that $w \in L$ and $x_w = 1$. By Claim 2,

$$d(w) \geq \left( 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 5 \right) \epsilon \right) n.$$ 

Applying Claim 5 iteratively for $k - 2$ times, we can find a set $X \subseteq L \setminus \{w\}$ of $k - 2$ vertices such that for any $u \in X$,

$$x_u \geq 1 - \left( \sum_{j=1}^{k-2} \left( \sum_{i=1}^{k} 2d_i + 5k - 5 \right)^{2j} \right) \epsilon$$

and

$$d(u) \geq \left( 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 4 \right) \epsilon \right) n.$$ 

Noting $|L| \leq k - 1$, we have $L = X \cup \{w\}$. Hence $|L| = k - 1$. This proves Claim 4.

Let $T$ be the common neighborhood of $L$ and $R = S \setminus T$. By Claim 4,

$$|L| = k - 1$$

and

$$|T| \geq \left( 1 - k \left( \sum_{i=1}^{k} 2d_i + 5k - 4 \right) \epsilon \right) n \geq \sum_{i=1}^{k} d_i.$$ 

Since $G$ is bipartite, $L$ and $T$ are both independent sets of $G$.

**Claim 5.** $R$ is empty.

Suppose that $R$ is not empty, i.e., there is a vertex $v \in R$. Then $v$ has at most $d_k - 1$ neighbors in $S$, otherwise we can embed an $F$ in $G$. Let $H$ be a graph obtained from $G$ by removing all edges incident with $v$ and then connecting $v$ to each vertex in $L$. Clearly, $H$ is still $F$-free. By the definition of $R$, $v$ can be adjacent to at most $k - 2$ vertices in $L$. Let $u \in L$ be the vertex not adjacent to $v$. Then By Claims 4 and 5, we have

$$\rho(H) - \rho \geq \frac{x^T A(H)x}{x^T x} - \frac{x^T A x}{x^T x}.$$ 

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\[
\begin{align*}
&\geq \frac{2x_v}{x^T x} \left( x_u - \sum_{z \in S} x_z \right) \\
&\geq \frac{2x_v}{x^T x} \left( 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 4 \right)^{2k-1} \epsilon - (d_k - 1)\epsilon \right) \\
&= \frac{2x_v}{x^T x} \left( 1 - \left( \sum_{i=1}^{k} 2d_i + 5k - 4 \right)^{2k-1} + d_k - 1 \right) \epsilon \\
&> 0,
\end{align*}
\]

Hence \(\rho(H) > \rho\), a contradiction. This proves Claim 5.

By Claim 5, \(S = T\). By the definition of \(T\), we have \(G = K_{k-1, n-k+1}\). This completes the proof.

\[\square\]

**Proof of Theorem 1.8.** Let \(G\) be an \(F\)-free bipartite graph of order \(n\) with the maximum spectral radius.

If \(G\) is connected, then the result follows directly from Theorem 4.1. Next we suppose that \(G\) is not connected. Since \(K_{k-1, n-k+1}\) is \(F\)-free,

\[\rho(G) \geq \sqrt{(k-1)(n-k+1)}.\]

Let \(G_1\) be a component of \(G\) such that \(\rho(G_1) = \rho(G)\) and \(n_1 = |V(G_1)|\). Note that \(G\) is triangle-free. By Wilf theorem [15, Theorem 2], we have

\[\frac{n_1^2}{4} \geq \rho^2(G_1) = \rho(G)^2 \geq (k-1)(n-k+1) \geq (k-2)n \geq \frac{f^2(k, d_1, \ldots, d_k)}{4},\]

which implies that

\[n_1 \geq f(k, d_1, \ldots, d_k).\]

By Theorem 4.1 again,

\[\rho(G_1) \leq \sqrt{(k-1)(n_1-k+1)} < \sqrt{(k-1)(n-k+1)},\]

a contradiction. This completes the proof. \[\square\]

**Proof of Corollary 1.9.** By a result of Favaron et al. [7], \(\rho_n(G) \geq \rho_n(H)\) for some spanning bipartite subgraph \(H\). Moreover, the equality holds if and only if \(G = H\), which can be deduced by its original proof. By Theorem 1.8

\[\rho(H) \leq \sqrt{(k-1)(n-k+1)}\]

with equality if and only if \(H = K_{k-1, n-k+1}\). Since the spectrum of a bipartite graph is symmetric [10],

\[\rho_n(H) \geq -\sqrt{(k-1)(n-k+1)}\]

with equality if and only if \(H = K_{k-1, n-k+1}\). Thus we have

\[\rho_n(G) \geq -\sqrt{(k-1)(n-k+1)}\]

with equality if and only if \(G = K_{k-1, n-k+1}\). \[\square\]
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