Stability criterion for solitons of the ZK-type equations

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Early results concerning the linear stability of the solitons in equation of the KDV-type are generalized to solitons describing by the ZK-type equation. The linear stability criterion for ground solitons in the Vakhitov-Kolokolov form is derived for such equations with arbitrary nonlinearity. For the power nonlinearity the instability criterion coincides with the condition of the Hamiltonian unboundedness from below. The latter represents the main feature for appearance of collapse in such systems.

INTRODUCTION

There are well known two the most popular multidimensional generalizations of the KDV equation: the Kadomtsev-Petvishvili (KP) equation and the so-called Zakharov-Kuznetsov (ZK) equation. The KP equation belongs to the universal models, it describes the nonlinear behavior of weakly nonlinear waves of the acoustic type. In comparison with the KDV equation the KP equation takes into account the diffraction of the waves. In the two-dimensional (2D) case, this equation admits integration by the inverse scattering transform (IST) and therefore solitons in this case play very essential role in the wave dynamics. For negative dispersion (KP-I) solitons are one-dimensional KDV solitons propagating in any direction. They are stable relative to the KP instability. For positive dispersion (KP-II), in the 2D case, solitons are localized in both directions and have the form of the so-called lump solutions. These solitons realize minimums of the Hamiltonian for the fixed momentum and by this reason are stable in the Lyapunov sense. In the 3D case, however, 3D solitons are unstable realizing saddle points of the Hamiltonian. In this case the Hamiltonian turns out to be unbounded from below. As it was shown numerically due to the Hamiltonian unboundedness collapse becomes possible in this case.

As far as the ZK equation concerns it describes ion-acoustic waves in magnetized plasma with low β, which is the ratio between thermal pressure and magnetic pressure. Among all MHD waves the ion-acoustic waves at β << 1 are the waves with the lowest frequencies and therefore they can not excite another MHD waves while the nonlinear interaction between waves. Their group velocity is mainly directed along the magnetic field. Dispersion of these waves is defined by the Debye radius along the magnetic field direction and in the transverse direction by the ion Larmor radius. The ZK equation in the dimensionless variables has the following form

\[ u_t + \frac{\partial}{\partial x} [\Delta u + 3u^2] = 0. \]

This equation is written for waves propagating in one direction along the magnetic field (||x||) with the sound velocity, the Laplace operator is responsible for the wave dispersion, the nonlinear term describes the nonlinear correction to the sound velocity. This equation belongs to the Hamiltonian type

\[ u_t = \frac{\partial}{\partial x} \frac{\delta H}{\delta u}, \quad H = \int \left[ \frac{1}{2} \left( \nabla u \right)^2 - u^3 \right] dx. \]

Besides H this equation conserves also the momentum P which x-component is positive definite quantity, P = 1/2 \int u^2 dx > 0. The simplest soliton solutions are stationary localized waves propagating with velocity V > 0 along the magnetic field, u = u(x - Vt, r⊥). These solutions are stationary points of the Hamiltonian for fixed P,

\[ \delta (H + EP) = 0. \]

As it was shown first time in [3] (see also [4]) ground (spherical symmetrical, without nodes) solitons realize minimum of the Hamiltonian for fixed P and therefore are stable in the Lyapunov sense. This stability proof is based on the application of the Gagliardo-Nirenberg inequalities following from the Sobolev embedding theorems. This result can be easily generalized to the case of arbitrary power nonlinearity (see, e.g. the review [5]) when instead \( u^3 \) in the Hamiltonian stands \( u^p \) with \( p > 2 \). The simple analysis, however, shows that the boundedness of H for fixed P takes place at \( p - 2 < 4/d \) where d is the space dimension. For \( p - 2 > 4/d \) the Hamiltonian becomes unbounded from below that is one of the criteria for the wave collapse (see [10, 11]) and therefore we should expect instability of solitons in this case. When instead \( u^p \) in the Hamiltonian H we have arbitrary function \( f(u) \) then the approach developed in [3] is not so effective. In this case, one needs to consider the linearized problem.

The main aim of this paper is to generalize the results about linear soliton stability in the equations of the KDV type to the equation of the ZK type with arbitrary nonlinearity. We show that the linear stability analysis for solitons gives the Vakhitov-Kolokolov-type criterion well known for the NLS equations. For the power nonlinearity the instability criterion coincides with the condition of the Hamiltonian unboundedness from below.
LINEAR STABILITY PROBLEM

Let us consider the equation of motion in the system of coordinates moving with the velocity \( V \) along \( x \)-axis which Hamiltonian is written as

\[
\tilde{H} = \int \left[ \frac{1}{2} (\nabla u)^2 - f(u) \right] \, dr + VP.
\]

With respect to the function \( f(u) \) we suppose that it vanishes for \( u \to 0 \) as \( au^{2+\epsilon} \) \( (a, \epsilon > 0) \) and increases faster than \( u^2 \) as \( u \to \infty \). Such a behavior guarantees the existence of the soliton solutions \( u = u_s(x - Vt, r_\perp) \) determined from the variational problem:

\[
- Vu_s + \Delta u_s + f'(u_s) = 0.
\] (3)

We will consider only ground soliton solution of this equation which is spherical symmetric and without nodes. The most important point is that this solution is symmetric with respect to change \( x \to -x \). Now let us perform linearization of the equation of motion on the background on the soliton solution putting \( u = u_s(x, r_\perp) + w(x, r_\perp) \) where \( w \) is a small perturbation. This results in the following linear equation

\[
w = \frac{\partial}{\partial x} \frac{\delta H'}{\delta w},
\]

where \( H' \) is the second variation of the Hamiltonian \( \tilde{H} \),

\[
H' = \frac{1}{2} \int wLwdr \equiv \frac{1}{2} \langle w|L|w \rangle
\]

and \( L = -\Delta + V - f''(u_s) \) is the Schroedinger operator. Because \( u_s \) is an even function relative to \( x \) we decompose the perturbation \( w \) by even \((\varphi)\) and odd \((\psi)\) parts, \( w = \varphi + \psi \). As the result expansion of \( H' \) will contain two terms:

\[
H' = \frac{1}{2} \left( \langle \varphi|L|\varphi \rangle + \langle \psi|L|\psi \rangle \right).
\]

Besides, these two functions \( \varphi \) and \( \psi \) are canonically conjugated variables with the Hamiltonian \( H' \):

\[
\varphi_t = \frac{\partial}{\partial x} \frac{\delta H'}{\delta \psi}, \quad \psi_t = \frac{\partial}{\partial x} \frac{\delta H'}{\delta \varphi}.
\]

These linear equations should be implemented by the solvability condition

\[
\langle \varphi|u_s \rangle = 0 \quad (4)
\]

which is consequence of the conservation of \( P \).

The soliton solution, evidently, is stable if both quadratic forms \( \langle \varphi|L|\varphi \rangle \) and \( \langle \psi|L|\psi \rangle \) are of the same sign. It can be easily seen that the second quadratic part \( \langle \psi|L|\psi \rangle \) is not negative definite because \( L \frac{\partial}{\partial x} u_s = 0 \), i.e. function \( \frac{\partial}{\partial x} u_s \) is a neutral eigen function of the operator \( L \) corresponding to a shift (along \( x \)) of the soliton as a whole. Because \( u_s \) is symmetric relative to \( x \) and has no nodes, the function \( \frac{\partial}{\partial x} u_s \) corresponds to \( p \)-state. According to the oscillatory theorem for the Schroedinger operators the \( L \) operator will have the ground eigen function with negative energy, symmetric and without nodes. Among odd functions the function \( \frac{\partial}{\partial x} u_s \) has the minimal energy \( E = 0 \). Because any (small) shift of the soliton as a whole can not influence on its stability we can consider the second quadratic form as positive definite.

Now we turn to the question about a sign of the first quadratic part \( \langle \varphi|L|\varphi \rangle \). If it will be positive we will get stability, and instability in the opposite case. Consider the eigen value problem for the \( L \) operator,

\[
L \varphi = E \varphi + Cu_s.
\] (5)

Here we add the second term with constant \( C \) which is Lagrange multiplier because of condition. Let us expand \( \varphi \) through the eigen (even) functions of this operator \( \langle L \varphi_n = E_n \varphi_n \rangle \), \( \varphi = \sum_n C_n \varphi_n \). Hence using the compatibility condition we arrive at the following dispersion relation (compare with [11])

\[
F(E) \equiv \sum_n \frac{\langle u_s|\varphi_n \rangle \langle \varphi_n|u_s \rangle}{E_n - E} = 0.
\]

Because of orthogonality \( \langle u_s|\nabla_L u_s \rangle = 0 \) in this sum there is absent a term with \( E = 0 \). Consider now behavior of the function \( F(E) \) in the interval \( E_0 < E < E_2 \) between the energy of the ground state \( E_0 (< 0) \) and the first positive energy \( E_2 (> 0) \). If the function will intersect the abscissa axis at \( E > 0 \) then the quadratic form will be positive definite. In this case evidently

\[
\sum_n \frac{\langle u_s|\varphi_n \rangle \langle \varphi_n|u_s \rangle}{E_n} < 0.
\]

In the opposite case we have

\[
\sum_n \frac{\langle u_s|\varphi_n \rangle \langle \varphi_n|u_s \rangle}{E_n} > 0
\]

when the quadratic form can take negative values. These sums can be expressed in terms of the soliton solution if one differentiates equation relative to \( V \):

\[
L \frac{\partial u_s}{\partial V} = -u_s.
\]

Hence on the class of functions orthogonal to \( u_s \) we can see that

\[
\sum_n \frac{\langle u_s|\varphi_n \rangle \langle \varphi_n|u_s \rangle}{E_n} = \langle u_s|L^{-1}|u_s \rangle = - \frac{1}{2} \frac{\partial P}{\partial V}.
\]

Thus, if

\[
\frac{\partial P}{\partial V} > 0
\]
soliton will be stable and unstable in the opposite case. This criterion represents the analog of the Vakhitov-Kolokolov criterion\textsuperscript{[1]} for the NLS equations. In the case of power nonlinearity $f(u) = u^p$, the dependence of momentum $P$ on $V$ turns out to be powerful: $P \propto V^\gamma$ where \[ \gamma = \frac{2}{p-2} - \frac{d}{2}. \]

Hence one can see that the instability criterion for solitons $p-2 > 4/d$ coincides with the unboundedness condition of the Hamiltonian. Like for the NLS-type equations we can state that the nonlinear stage of this instability should result in the wave collapse.

**CONCLUSION**

Thus, we have found the linear stability criterion for ground soliton solutions in the ZK-type equation. This criterion is necessary and sufficient: if $\partial P/\partial V > 0$ the solitons are stable and unstable in the opposite case. This criterion is analogous to the Kolokolov-Vakhitov criterion for soliton stability in the NLS-type equations. For power nonlinearity this criterion demonstrates different behavior of the system. In the stable region solitons realize minimum of the Hamiltonian with fixed momentum $P$, i.e. they are stable in the Lyapunov sense. But it does not mean that scattering of solitons will be elastic. While scattering of such solitons it is energetically favorable to form solitons with higher amplitude. This process will be accompanied by radiation of small amplitude waves which play the role of friction in the system. For the systems with Hamiltonians unbounded from below the nonlinear stage of the soliton instability should result in the formation of singularity, probably, in a finite time.

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