Renormalization of the leading-order chiral nucleon-nucleon interaction and bulk properties of nuclear matter

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We renormalize the two-nucleon interaction at leading order (LO) in chiral perturbation theory using the scheme proposed by Nogga, Timmermans, and van Kolck—also known as modified Weinberg counting. With this interaction, we calculate the energy per nucleon of symmetric nuclear matter in the Brueckner pair approximation and obtain a converged, cutoff-independent result that shows saturation, but also substantial underbinding. We find that the renormalized LO interaction is characterized by an extraordinarily strong tensor force (from one-pion exchange), which is the major cause for the lack of binding. The huge tensor force also leads to the unusually large wound integral of 40% in nuclear matter, which implies a very slow convergence of the hole-line or coupled-cluster expansion, rendering this interaction impractical for many-body calculations. In view of the unusual properties of the renormalized LO interaction and in view of the poor convergence of the nuclear many-body problem with this interaction, there is doubt if this interaction and its predictions can serve as a reasonable and efficient starting point that is improved by perturbative corrections.

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I. INTRODUCTION

The problem of a proper derivation of nuclear forces is as old as nuclear physics itself, namely, almost 80 years. The modern view is that, since the nuclear force is a manifestation of strong interactions, any serious derivation has to start from quantum chromodynamics (QCD). However, the well-known problem with QCD is that it is non-perturbative in the low-energy regime characteristic for nuclear physics. For many years this fact was perceived as the great obstacle for a derivation of nuclear forces from QCD—impossible to overcome except by lattice QCD. The effective field theory (EFT) concept has shown the way out of this dilemma. One has to realize that the scenario of low-energy QCD is characterized by pions and nucleons interacting via a force governed by spontaneously broken approximate chiral symmetry. This chiral EFT allows for a systematic low-momentum expansion known as chiral perturbation theory (ChPT) [1]. Contributions are analyzed in terms of powers of small momenta over the large scale: \((Q/\Lambda_\chi)^\nu\), where \(Q\) is generic for a momentum (nucleon three-momentum or pion four-momentum) or pion mass and \(\Lambda_\chi \approx 1\) GeV is the chiral symmetry breaking scale. The early applications of ChPT focused on systems like \(\pi\pi\) [2] and \(\pi N\) [3], where the Goldstone-boson character of the pion guarantees that the expansion converges. The past 15 years have also seen great progress in applying ChPT to nuclear forces [4][20].

However, there is a difference between the purely-pionic and the one-nucleon sector, on the one hand, and two- and multi-nucleon systems, on the other hand. Nuclear physics is characterized by bound states that are nonperturbative in nature. Weinberg showed [4] that the strong enhancement of the amplitude arises from purely nucleonic intermediate states (“infrared enhancement”). He therefore suggested a two-step procedure: In step one, ChPT and naive dimensional analysis is used to calculate a “potential” which consists of only irreducible diagrams and, in step two, this potential is iterated to all orders by inserting it into a Schrödinger or Lippmann-Schwinger (LS) equation to generate the amplitude.

At leading order (LO), the potential consists of static one-pion exchange (1PE) and two non-derivative contact terms. At next-to-leading order (NLO), multi-pion exchange starts which involves divergent loop integrals that need to be regularized. An elegant way of doing this is dimensional regularization which (besides the main nonpolynomial
result) typically generates polynomial terms with coefficients that are, in part, infinite or scale dependent [9]. One reason why so-called contact terms are introduced in the EFT is to absorb all infinities and scale dependencies and make sure that the final result is finite and scale independent. This is the renormalization of the perturbatively calculated $NN$ amplitude (which, by definition, is the “$NN$ potential”). It is very similar to what is done in the ChPT calculations of $\pi\pi$ and $\pi N$ scattering, namely, a renormalization order by order, which is the method of choice for any EFT. Thus, up to this point, the $NN$ calculation fully meets the standards of an EFT and there are no problems. The perturbative $NN$ amplitude can be used to make model independent predictions for peripheral partial waves [9,10,15].

For calculations of the structure of nuclear few and many-body systems, the lower partial waves are the most important ones. The fact that in $S$ waves we have large scattering lengths and shallow (quasi) bound states indicates that these waves need to be treated nonperturbatively. Following Weinberg’s prescription [4], this is accomplished by inserting the potential $V$ into the LS equation:

$$T(p',\hat{p}) = V(p',\hat{p}) + \int \frac{d^3p''}{(2\pi)^3} V(p',\hat{p}'') \frac{M_N}{p^2 - p'^2 + i\epsilon} T(p'',\hat{p}),$$ (1)

where $M_N$ denotes the nucleon mass.

In general, the integral in the LS equation is divergent and needs to be regularized. One way to achieve this is by multiplying $V$ with a regulator function, e.g.,

$$V(p',\hat{p}) \rightarrow V(p',\hat{p}) e^{-(p'/\Lambda)^{2n}} e^{-(p/\Lambda)^{2n}}.$$ (2)

Typical choices for the cutoff parameter $\Lambda$ that appears in the regulator are $\Lambda \approx 0.5$ GeV $\ll \Lambda_c \approx 1$ GeV [16,17].

It is pretty obvious that results for the $T$-matrix may depend sensitively on the regulator and its cutoff parameter. This is acceptable if one wishes to build models. For example, the meson models of the past [21,22] always depended sensitively on the choices for the cutoff parameters which were, in fact, welcome fit-parameters for achieving a good reproduction of the $NN$ data. However, the EFT approach wishes to be fundamental in nature and not just another model.

In field theories, divergent integrals are not uncommon and methods have been developed for how to deal with them. One regulates the integrals and then removes the dependence on the regularization parameters (scales, cutoffs) by renormalization. In the end, the theory and its predictions do not depend on cutoffs or renormalization scales.

So-called renormalizable quantum field theories, like QED, have essentially one set of prescriptions that takes care of renormalization through all orders. In contrast, EFTs are renormalized order by order, in which case the number of adjustable parameters increases.

As discussed, the renormalization of perturbative EFT calculations is not a problem. The problem is nonperturbative renormalization. This problem typically occurs in nuclear EFT because nuclear physics is characterized by bound states which are nonperturbative in nature.

Weinberg’s implicit assumption was that the counterterms introduced to renormalize the perturbatively calculated potential, based upon naive dimensional analysis (“Weinberg counting”), are also sufficient to renormalize the nonperturbative resummation of the potential in the LS equation. Unfortunately, it has turned out that this assumption is not quite correct, as pointed out by Kaplan, Savage, and Wise (KSW) [23], and others. The criticism of the Weinberg counting scheme resulted in a flurry of publications on the renormalization of the nonperturbative $NN$ problem [24–26]. The literature is too comprehensive to discuss all contributions in detail. Let us just mention some of the work that has particular relevance to our present paper.

If the potential $V$ consists of contact terms only (a.k.a. pion-less theory), then the nonperturbative summation, Eq. (1), can be performed analytically, which makes it easier to deal with the renormalization issue. However, when pion exchange is included, then Eq. (1) can be solved only numerically and the renormalization problem is less transparent. Perturbative ladder diagrams of arbitrarily high order, where the rungs of the ladder represent a potential made up from irreducible pion exchange, suggest that an infinite number of counterterms is needed to achieve cutoff independence for all the terms of increasing order generated by the iterations. For that reason, Kaplan, Savage, and Wise (KSW) [23] proposed to sum the contact interaction to all orders (analytically) and to add pion exchange perturbatively up to the given order. Unfortunately, it turned out that the order by order convergence of 1PE is poor in the $S_1$-$D_1$ state [24]. The failure was triggered by the $1/r^3$ singularity of the 1PE tensor force when iterated to second order. Therefore, KSW counting is no longer taken into consideration (see, however, Ref. [39]). A balanced discussion of possible solutions can be found in Ref. [29].

Some researchers decided to take a second look at Weinberg’s original proposal. A systematic investigation and re-analysis of Weinberg counting in leading order has been conducted by Nogga, Timmermans, and van Kolck [31] in momentum space, and by Pavón Valderrama and one of the present authors (ERA) at LO and higher orders in configuration space [32,33]. A comprehensive discussion of both approaches and their equivalence can be found.
in Refs. 36 40. The LO NN potential consists of 1PE plus two nonderivative contact terms that contribute only
in $S$ waves. By numerical calculations\(^1\), Nogga et al. find that the given counterterms renormalize the $S$ waves, i.e.,
the naively expected infinite number of counterterms is not needed. This means that Weinberg power counting does
actually work in $S$ waves at LO (ignoring the $m_{\pi}$ dependence of the contact interaction discussed in Refs. 28 29).
However, there are problems with a particular class of higher partial waves, namely those in which the tensor force
from 1PE is attractive. The first few cases of this kind of low angular momentum are $3P_0$, $3P_2$, and $3D_2$. The leading order
(nonderivative) counterterms do not contribute in $P$ and higher waves, which is the reason for the problem. But
the second order contact potential provides counterterms for $P$ waves. Therefore, the promotion of, particularly, the
$3P_0$ and $3P_2$ contacts from NLO to LO would fix the problem in $P$ waves. To take care of the $3D_2$ problem, a fourth
order contact needs to be promoted to LO. In this way, one arrives at a scheme of 'modified Weinberg counting' 31
for the leading order two-nucleon interaction.

Once cutoff independence of the on-shell NN $T$-matrix (and NN phase shifts and observables) has been achieved,
it is of interest to know if cutoff independent results are also obtained when this interaction is applied in nuclear
few- and many-body systems. Nogga et al. 31 investigated the lightest such system, namely, the three-nucleon
bound state, and found cutoff independence of the triton binding energy. It is the purpose of this note to conduct a
similar investigation in heavier nuclear systems. Since finite nuclei are difficult to calculate, we choose nuclear matter
(infinitely many nucleons). We will show that the renormalized LO two-nucleon interaction leads to converged results
for the energy per nucleon in nuclear matter.

Section II briefly describes and repeats the LO renormalization procedure with modified Weinberg counting intro-
duced in Ref. 31. In Sec. III, we present the novel point of this paper, namely, the calculation of the energy per
nucleon in symmetric nuclear matter as a function of density and, in Sec. IV, we compare our results with the work
by other authors. Conclusions are drawn in Sec. V.

II. RENORMALIZING THE NN POTENTIAL IN LEADING ORDER

In naive dimensional analysis ("Weinberg counting"), the order by order expansion of the chiral NN potential is
given as:

\[ V_{\text{LO}} = V_{\text{ct}}^{(0)} + V_{\pi}^{(0)} \]
\[ V_{\text{NLO}} = V_{\text{LO}} + V_{\text{ct}}^{(2)} + V_{\pi}^{(2)} + V_{2\pi}^{(2)} \]
\[ V_{\text{NNLO}} = V_{\text{NLO}} + V_{\pi}^{(3)} + V_{2\pi}^{(3)} \]
\[ V_{\text{NNLO}} = V_{\text{NNLO}} + V_{\pi}^{(4)} + V_{1\pi}^{(4)} + V_{2\pi}^{(4)} + V_{3\pi}^{(4)} \]

where the superscript denotes the order $\nu$ of the low-momentum expansion. LO stands for leading order, NLO for
next-to-leading order, etc.. Contact potentials carry the subscript "ct" and pion-exchange potentials can be identified
by an obvious subscript.

The charge-independent one-pion exchange (1PE) potential reads

\[ V_{1\pi}(\vec{p}', \vec{p}) = -\frac{g_A^2}{4f_\pi^2} \tau_1 \cdot \tau_2 \frac{\vec{\sigma}_1 \cdot \vec{q} \: \vec{\sigma}_2 \cdot \vec{q}}{q^2 + m_\pi^2}, \]

where $\vec{p}'$ and $\vec{p}$ designate the final and initial nucleon momenta in the center-of-mass system (CMS) and $\vec{q} = \vec{p}' - \vec{p}$
is the momentum transfer; $\vec{\sigma}_{1,2}$ and $\tau_{1,2}$ are the spin and isospin operators of nucleon 1 and 2; $g_A$, $f_\pi$, and $m_\pi$ denote
axial-vector coupling constant, the pion decay constant, and the pion mass, respectively. We use $f_\pi = 92.4$ MeV and
$g_A = 1.29$ to correct for the Goldberger-Treiman discrepancy. Since higher order corrections contribute only to mass
and coupling constant renormalizations and since, on shell, there are no relativistic corrections, the on-shell 1PE has the form Eq. (7) in all orders.

Here, we will specifically calculate LO neutron-proton ($np$) scattering and take charge-dependence (isospin violation)
into account. Thus, the 1PE potential reads

\[ V_{1\pi}^{(np)}(\vec{p}', \vec{p}) = -V_{1\pi}(m_{\pi^0}) + (-1)^{I+1} 2 V_{1\pi}(m_{\pi^\pm}), \]

---

\(^1\) For the purposes of the present paper, we conduct the discussion in momentum space. We note however, that all information on the
necessary number of counterterms can be determined a priori and analytically by inspecting the potential in configuration space at
short distances 30 32 33 36 30.
where \( I \) denotes the isospin of the two-nucleon system and

\[
V_{1\pi}(m_{\pi}) = -\frac{\bar{g}_A^2}{4f_{\pi}^2} \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}.
\]  

(9)

We use \( m_{\pi^0} = 134.9766 \text{ MeV} \) and \( m_{\pi^\pm} = 139.5702 \text{ MeV} \) [43]. In the LS equation, Eq. (1), we apply

\[
M_N = \frac{2M_pM_n}{M_p + M_n} = 938.9182 \text{ MeV},
\]  

(10)

\[
p^2 = \frac{M_p^2T_{\text{lab}}(T_{\text{lab}} + 2M_n)}{(M_p + M_n)^2 + 2T_{\text{lab}}M_p},
\]  

(11)

where \( M_p = 938.2720 \text{ MeV} \) and \( M_n = 939.5653 \text{ MeV} \) are the proton and neutron masses [43], respectively, and \( T_{\text{lab}} \) is the kinetic energy of the incident neutron in the laboratory system (“Lab. Energy”). The relationship between \( p^2 \) and \( T_{\text{lab}} \) is based upon relativistic kinematics.

Besides the 1PE potential, Eq. (8), the EFT includes contact terms which represent short-range interactions that cannot be resolved at the low-energy scale. Furthermore, the contacts are needed for renormalization. Stating the contact potentials in partial-wave decomposition, we have one zero-order (\( \nu = 0 \)) contact in each \( S \) wave:

\[
V^{(0)}_{ct}(1S_0) = c_{1S_0},
\]  

(12)

\[
V^{(0)}_{ct}(3S_1) = c_{3S_1}.
\]  

(13)

Up to this point, we are still applying Weinberg counting. However, as discussed in the Introduction, higher partial waves in which the pion’s tensor force is attractive need counter terms to achieve cutoff independence—which leads
us to modified Weinberg counting. To be specific, two \( P \) waves receive counter terms of second order,

\[
V^{(2)}_{\text{ct}}(3P_0) = C_{3P_0} p'p,
\]

\[
V^{(2)}_{\text{ct}}(3P_2) = C_{3P_2} p'p,
\]

and one \( D \) wave needs a fourth-order counter term,

\[
V^{(4)}_{\text{ct}}(3D_2) = D_{3D_2} p'^2 p^2.
\]

For the solution of the LS equation, Eq. (1), a regulator function is necessary, for which we choose the one given in Eq. (2) with \( n = 2 \). The regulator depends on the cutoff mass \( \Lambda \), which we vary over a wide range from 0.5 GeV to 10 GeV. In \( S \)-waves, we readjust the contact parameter for each choice of \( \Lambda \) such that the empirical scattering lengths \( a_s = -23.748 \) fm for \( ^1S_0 \) and \( a_t = 5.4170 \) fm for \( ^3S_1 \) are reproduced. In those \( P \) and \( D \) waves which carry a contact in modified Weinberg counting, the contact parameter is used to fit—for the various choices of \( \Lambda \)—the empirical phase shift at 50 MeV as given in Ref. [44]. In all cases, we then calculate the phase shifts for all energies below 350 MeV.

The resulting phase shifts and mixing parameters for total angular momentum \( J \leq 2 \) are shown in Fig. 1. The curves refer to \( \Lambda = 0.5 \) GeV (dotted), 1 GeV (dash-dotted), 5 GeV (dashed), and 10 GeV (solid). The curves for \( \Lambda = 5 \) GeV and 10 GeV are, in general, indistinguishable on the scale of the figure, which demonstrates that cutoff independence (nonperturbative renormalization) has been achieved. It is known from the work of Nogga et al. [31] in momentum space\(^2\) that this is possible. Our results represent an independent confirmation.

\(^2\) Again, the analysis is easier in configuration space [30, 32, 33]; see, in particular, Table II and the thorough convergence analysis of phase shifts with total angular momentum \( J \leq 5 \) in Ref. [34].
TABLE I: Partial-wave contact parameters as a function of the cut-off $\Lambda$ for the leading-order $NN$ potential in modified Weinberg counting. The parameters are defined in Eqs. (12)-(16).

| Partial-wave contact parameter | Cutoff parameter $\Lambda$ in units of GeV |
|--------------------------------|------------------------------------------|
| $\tilde{C}_{1S_0}$ ($10^4$ GeV$^{-2}$) | -0.109966 -0.087189 -0.06739623 -0.064460345 |
| $\tilde{C}_{3S_1}$ ($10^4$ GeV$^{-2}$) | -0.076005 1.349900 -0.02692560 0.021786000 |
| $C_{3P_0}$ ($10^4$ GeV$^{-4}$) | 0.840321 -0.1722517 0.001856514 0.000384981 |
| $C_{3P_2}$ ($10^4$ GeV$^{-4}$) | -0.2316105 -0.0700665 -0.00251447 0.001251038 |
| $D_{3D_2}$ ($10^4$ GeV$^{-6}$) | -0.3347880 0.3899800 -0.00020581 -0.00001055 |

III. NUCLEAR MATTER

As discussed in the Introduction, once cutoff independence has been achieved for the two-nucleon system, a good question to ask is if cutoff-independent predictions are also obtained in the nuclear many-body problem when this renormalized $NN$ potential is applied. Nogga et al. [31] addressed this question for the three-nucleon system where they confirmed the cutoff independence of the triton binding energy at LO. We wish to turn to heavier nuclear systems and choose nuclear matter as the representative sample.

By definition, nuclear matter refers to an infinite uniform system of nucleons interacting via the strong force without electromagnetic interactions. This hypothetical system is believed to approximate conditions in the interior of heavy nuclei. We shall assume equal neutron and proton densities, that is, we will consider symmetric nuclear matter. This many-body system is characterized by its energy per nucleon as a function of the particle density.

We will use the well-established Brueckner-Bethe-Goldstone method (in short: Brueckner theory) [22, 46–48] to calculate the nuclear matter energy. In this theory, a central role is played by the Brueckner $G$-matrix which is a solution of the Bethe-Goldstone integral equation

$$G(w) = V - V \frac{Q}{H_0 - w} G(w),$$

where $w$ denotes the starting energy, $H_0$ the unperturbed Hamiltonian, and the Pauli operator $Q$ projects onto unoccupied states. In the pair approximation, the energy per nucleon is given by

$$\frac{E}{A} = \frac{1}{A} \sum_{m \leq k_F} \langle m | t | m \rangle + \frac{1}{2A} \sum_{m,n \leq k_F} \langle mn | G(w) | mn - nm \rangle$$

where $A$ denotes the number of nucleons, $t$ the kinetic energy operator, and $k_F$ the Fermi momentum, which is related to the density $\rho$ of symmetric nuclear matter by

$$\rho = \frac{2}{3\pi^2} k_F^3.$$

The starting energy is chosen on-shell, i.e.

$$w = e(m) + e(n)$$

with single-particle energy

$$e(m) = t(m) + U(m)$$

and single-particle potential

$$U(m) = \begin{cases} \sum_{n \leq k_F} \langle mn | G(w) | mn - nm \rangle, & m \leq k_F \\ 0, & m > k_F \end{cases}$$

also known as the “gap” choice for the single-particle potential, since a gap will obviously occur at the Fermi surface. The calculations are conducted in partial-wave decomposition and the Brueckner integral-equation is solved by matrix inversion, see Ref. [48] for details.

The Bethe-Goldstone method was originally devised to handle the short distance hard core in nuclear systems. In chiral EFT, an expansion both in $1/f_\pi$ and $1/M_N$ is carried out. For short distances $r \ll 1/m_\pi$ pion mass effects
can be neglected and on purely dimensional grounds an inverse power short distance singularity should be expected: to LO $V(r) \sim 1/(r^3)$, to NLO $V(r) \sim 1/(r^5)$, to NNLO, $V(r) \sim 1/(r^7)$ etc. It is natural to ask whether the $G-$matrix result converges in the limit $\Lambda \to \infty$. In the appendix we show that this is indeed the case for the gap choice Eq. (22) [48]. The proof rests on the finiteness of the off-shell $K$-matrix [49]. This gives some confidence on the stability of the numerics for increasing cut-off values.

In Fig. 2, we display our results for the energy per nucleon in symmetric nuclear matter as a function of density (measured by the Fermi momentum $k_F$) applying the LO $NN$ potentials with the various cutoffs used in the phase-shift calculations of Sec. II. The same curve patterns in Fig. 1 and 2 indicate the same cutoffs. The nuclear matter curves for $\Lambda = 5$ GeV (green dashed) and $\Lambda = 10$ GeV (red solid) cannot be distinguished on the scale of Fig. 2, demonstrating that cutoff independence of the predictions is achieved; or in other words, the red solid curve represents the renormalized result which, as expected, is convergent. This curve shows saturation at a Fermi momentum $k_F \approx 1.0$ fm$^{-1}$ and an energy per nucleon $E/A = -12.6$ MeV.

Based upon various pieces of circumstantial evidence, it is generally believed that the “empirical” saturation properties of symmetric nuclear matter are $k_F = 1.35 \pm 0.05$ fm$^{-1}$ and $E/A = -16 \pm 1$ MeV [22]. Thus, our renormalized LO result shows considerable underbinding. In Ref. [31], a triton energy of $-3.6$ MeV was found for the converged LO result which also deviates considerably from the empirical value of $-8.5$ MeV.

The chief reason for this lack of attraction is the fact that the tensor force of the renormalized LO interaction is unusually strong, as we will explain now.

A simple indicator for the strength of the tensor force component contained in a given $NN$ potential is the predicted $D$-state probability of the deuteron, $P_D$, because the transition from $S$ to $D$ states can only proceed via the tensor force. For the LO interaction at $\Lambda = 5$ GeV and 10 GeV, the $P_D$ comes out to be 7.2% (it’s converged). Conventional potentials predict $P_D$ typically lower, namely, between 4 and 6%; for example, the AV18 [51], CD-Bonn [50], and N3LO [16] potentials predict 5.76%, 4.85%, and 4.51%, respectively. Historically, the largest $P_D$ ever predicted by a “realistic” $NN$ potential was 7.0% by the Hamada-Johnston potential [52] of 1962.

In nuclear matter, the so-called wound integral $\kappa$ is known to depend sensitively on the strength of the tensor force [22, 48]. The wound integral is defined as

$$\kappa = \rho \int |\phi - \psi|^2 d\tau,$$

where $\phi$ denotes the uncorrelated two-nucleon wave function and $\psi$ the correlated one, which are related by

$$G\phi = V\psi,$$

implying

$$\psi = \phi - \frac{Q}{H_0 - \omega} G\phi.$$

FIG. 2: (color online). Energy per nucleon, $E/A$, in symmetric nuclear matter as a function of the Fermi momentum $k_F$ applying the LO $NN$ potentials with the various cutoffs used in the phase-shift calculations of Fig. 1. The curve patterns represent the same cutoffs as in Fig. 1.
TABLE II: Partial-wave contributions for S and P waves and total contributions to symmetric nuclear matter at a density equivalent to a Fermi momentum \( k_F = 1.35 \text{ fm}^{-1} \). Unless denoted otherwise; numbers without parentheses or brackets state contributions to the potential energy as obtained from using the Brueckner \( G \) matrix; numbers in parentheses are corresponding results obtained in Born approximation, i.e., for \( G = V \); finally, figures in brackets state contributions to the wound integral.

| Partial wave | LO \(^a\) | N3LO \(^b\) | CD-Bonn \(^5\) | AV18 \(^6\) |
|-------------|-----------|-----------|---------|---------|
| \(^1\)S\(_0\) | -13.42    | -16.74    | -16.76  | -16.07  |
|             | (-5.98)   | (-14.73)  | (-12.64)| (-2.77) |
|             | 0.056     | 0.008     | 0.005   | 0.017   |
| \(^3\)S\(_1\) | -13.54    | -19.42    | -18.96  | -17.10  |
|             | (+10.65)  | (-12.51)  | (-8.63) | (+5.99) |
|             | [0.090, 0.112]\(^c\) | [0.017, 0.017]\(^c\) | [0.004, 0.034]\(^c\) | [0.015, 0.053]\(^c\) |
| \(^1\)P\(_1\) | 3.24      | 3.90      | 3.91    | 3.88    |
|             | (3.27)    | (4.06)    | (4.24)  | (4.23)  |
|             | 0.000     | 0.001     | 0.002   | 0.001   |
| \(^3\)P\(_0\) | -1.01     | -3.14     | -3.08   | -3.15   |
|             | (-6.30)   | (-3.03)   | (-2.41) | (-2.48) |
|             | [0.109]   | [0.001]   | [0.002] | [0.002] |
| \(^3\)P\(_1\) | 10.17     | 9.68      | 9.81    | 9.74    |
|             | (11.09)   | (10.29)   | (11.74) | (12.08) |
|             | [0.004]   | [0.003]   | [0.006] | [0.007] |
| \(^3\)P\(_2\) | -5.37     | -7.27     | -7.05   | -6.96   |
|             | (-1.49)   | (-6.96)   | (-6.25) | (-6.09) |
|             | [0.015, 0.015]\(^d\) | [0.001, 0.001]\(^d\) | [0.003, 0.001]\(^d\) | [0.002, 0.002]\(^d\) |
| Total potential | -22.44   | -37.02    | -36.35  | -33.96  |
| energy      | (+9.07)   | (-26.72)  | (-17.92)| (+6.96) |
| Kinetic energy | 22.67   | 22.67     | 22.67   | 22.67   |
| energy      | (+0.23)   | (-14.35)  | (-13.67)| (-11.29)|
| Total wound | 0.405     | 0.050     | 0.058   | 0.101   |

\(^a\)The renormalized LO \( NN \) potential of this work with \( \Lambda = 10 \text{ GeV} \).
\(^b\)Quantitative N3LO \( NN \) potential regularized by a Gaussian with cutoff parameter \( \Lambda = 0.5 \text{ GeV} \) \(^7\).
\(^c\)The \(^3\)S\(_1\)-\(^3\)D\(_1\) and the \(^3\)S\(_1\)-\(^3\)D\(_1\) contributions to the wound integral are given.
\(^d\)The \(^3\)P\(_2\)-\(^3\)P\(_2\) and the \(^3\)P\(_2\)-\(^3\)P\(_2\) contributions to the wound integral are given.

The physical significance of the wound integral is that it measures the probability for exiting two nucleons to states above the Fermi surface. This probability is large for “hard” and strong-tensor force potentials. According to arguments conveyed by Brandow \(^53\), a \( n \)-hole line diagram is proportional to \( \kappa^{n-1} \), and, hence, the convergence of the hole-line expansion depends on the size of \( \kappa \), with large \( \kappa \) suggesting slow convergence.

As shown in the bottom row of Table II, the renormalized LO interaction produces a total \( \kappa \) of 40.5%, while the corresponding numbers are 10.1%, 5.8%, and 5.0% for AV18, CD-Bonn, and N3LO. The Hamada-Johnston potential generated a total \( \kappa \) of 21.1%. The partial-wave contributions to \( \kappa \) listed in Table II (numbers in square brackets) confirm that the strong tensor force of the LO interaction is the main reason for the extraordinarily high \( \kappa \). The \(^3\)S\(_1\)-\(^3\)D\(_1\) transition, which depends entirely on the tensor force, contributes 11.2% to the LO \( \kappa \), while it’s 5.3%, 3.4%, and 1.7% for AV18, CD-Bonn, and N3LO, respectively. In Fig. 3, we show the \(^3\)S\(_1\)-\(^3\)D\(_1\) transition potential of the LO interaction with \( \Lambda = 10 \text{ GeV} \) and of conventional potentials revealing dramatic differences, particularly, for high momenta. An unusual difference occurs also in the \(^3\)P\(_0\) state where LO generates a contribution to \( \kappa \) of 10.9%, whereas conventional potentials have at most 0.2%. The tensor operator is known to have a large matrix element in the \(^3\)P\(_0\) state. The \(^3\)P\(_0\) potentials are included in Fig. 3.

The fact that a strong tensor force (and a large \( \kappa \)) leads to less binding energy in nuclear matter and finite nuclei can be understood as follows \(^22\). For the purpose of discussion, let’s approximate the Brueckner \( G \) by

\[
G(w) \approx V_C - V_T \frac{Q}{H_0 - w} V_T,
\]

(26)
where $V_C$ denotes the central force and $V_T = v_T S_{12}$ the tensor force component of a given $NN$ potential (with $S_{12}$ the usual tensor operator). Note now that all quantitative nuclear potentials are fit to the same $NN$ Sec. II for $\Lambda = 10$ GeV (red solid). The on-shell point at $p = 153$ MeV/c (equivalent to $T_{\text{lab}} = 50$ MeV) is marked by a solid dot.

$$K(w_f) \approx V_C - \mathcal{P} V_T \frac{1}{t - w_f} V_T,$$

(27)

where $w_f$ is the free (purely kinetic) starting energy and $\mathcal{P}$ denotes the principal value.

A potential with a strong $V_T$ (implying a large, attractive second order tensor term) will have a less attractive central force $V_C$ to arrive at the same on-shell $K$-matrix as compared to a potential with a weak tensor force. Now, when we enter nuclear matter and calculate the $G$-matrix Eq. (26), the Pauli operator $Q$ (which is absent in the free-space Eq. (27)) and a larger energy denominator (due to the single particle potential in the many-body environment) reduces the magnitude of the second term of the $G$-matrix equation. These two medium effects are known as the Pauli and dispersion effects. The larger the attractive second order tensor term in Eq. (26), the larger the reduction of the attraction through the medium effects. Therefore, potentials which produce large integral terms in the $G$-matrix equation will predict less attraction in the many-body system. When the central force is very strong ("hard" potential), the above mechanism applies also to the iterations of the central term. This happens obviously in the $1S_0$ state where no tensor force is involved, but never-the-less a large $\kappa$ occurs for the LO interaction. This is also part of the reason why the $3S_1-3S_1$ contribution to the wound is large for LO, namely 9.0% (cf. the very hard LO central force seen in the $3S_1$ frame of Fig. 3).

An idea of the size of the integral term in the Brueckner equation, Eq. (17), is also obtained by comparing the Born approximation (i.e., $G = V$) with the full $G$. We therefore provide in Table II also the Born approximation results (numbers in parentheses) for the various partial-wave contributions.

As explained in length in Ref. [54], arguments similar to the above also apply to Faddeev calculations of the three-particle energy. Thus, the substantial underbinding of the triton found in Ref. [31] is most likely also related to the huge tensor force of the renormalized LO interaction.

We note that the discussion of the severely reduced attraction in nuclear matter due to "hard" central potentials and large tensor forces applies, of course, only to a calculation conducted in the Brueckner pair approximation. The huge wound integral suggests that there will be large three, four, and higher hole-line contributions which may provide additional binding. However, the evaluation of multi-hole line contributions is extremely involved and cumbersome. Similar arguments apply when the coupled-cluster expansion is used to deal with the nuclear many-body problem [53]. Thus, an extraordinarily strong tensor force makes it very difficult to obtain converged results in the many-body system, which is one reason why a large tensor-force potential is inconvenient, to say the least.

The best studied phenomenology of nuclear forces is the one-boson-exchange model. This model includes a $\rho$ meson, which produces a tensor force of opposite sign as compared to the pion (cf. Fig. 3.7 of Ref. [22]). Careful studies have shown that the reduction of the pion's tensor force at short range by the $\rho$-meson is crucial to arrive at a realistic strength for the nuclear tensor force [56].

In ChPT theory, contributions from heavy mesons, like the $\rho$, are too short-ranged to be dissolved, but instead contact terms are added to the theory. The set of contacts which appears at NLO ($\sim Q^2$) includes a tensor term that
May be perceived as simulating $\rho$ exchange. Therefore, there is a chance that, at NLO or higher order, the problem of the extraordinarily large tensor force encountered at LO will be resolved.

IV. COMPARISON WITH OTHER WORKS AND THE BROADER PERSPECTIVE

Our results clearly show that the saturation mechanism in nuclear matter is compatible with a non-perturbative renormalization of the venerable 1PE potential. However, the binding energy turns out to be rather small and the approach looks discouraging from a coupled-cluster expansion point of view. Let us therefore analyze our results in the light of other chiral approaches to nuclear matter, based in spirit on the EFT concept, where mainly perturbative schemes but also low cut-offs have been employed. We will also provide some perspective on future work.

If the cut-off parameter takes sufficiently small values, perturbation theory becomes applicable, since in such a case high momentum components are suppressed. For the smallest cut-off value presented in Fig. 2, $\Lambda = 0.5$ GeV, we see a clear attraction which is strongly dependent, in fact linearly, on the Fermi momentum. Actually, chiral symmetry based approaches for nuclear matter have also been pursued in Ref. [67] in a purely perturbative scheme. The main source of attraction stems from once iterated 1PE, is proportional to the density, $\sim k_F^4$, and depends linearly also on the cut-off parameter, which needs to be fine-tuned to a value $\Lambda = 0.4 - 0.5$ GeV to achieve saturation. The origin of the divergence is related to vacuum amplitudes which to n-th order with the 1PE potential yield a contribution to the $T$-matrix, Eq. (1), scaling by naive power counting as $\sim \Lambda^{-n}$ (pion mass neglected). The present calculation contains iterated 1PE with additional counterterms to all orders for any value of the cut-off parameter and, as we see, it does not exhibit this very strong $k_F^4$ dependent attraction. Thus, in Fig. 2 only some residual cut-off effect is displayed after low energy $NN$ physics has been fixed. Clearly, the separately large perturbative contributions which scale with positive powers of $\Lambda$ would not converge without inclusion of counterterms to all orders.

A NLO $G$-matrix calculation with a finite cut-off of $\Lambda = 500$ MeV and counterterms was also undertaken in Ref. [58] and saturation was found. As noted in [32], there would be a fundamental problem of removing the cut-off at that order, since the deuteron becomes unbound due to the strong $1/r^3$ repulsive interaction in the triplet channels. More recently, a perturbative approach has also been proposed where a power-counting scheme is introduced with extremely low cut-off values $\Lambda \sim m_\pi$ for which saturation is achieved [59, 60].

Finally, $V_{\text{low}-k}$ approaches represent a coarse graining of the interaction in the physically accessible $NN$ elastic region for CM momenta $k \leq \Lambda \approx 400$ MeV with the result that all high precision potentials fitting data with $\chi^2/DOF \approx 1$ including the 1PE tail collapse into a unique $V_{\text{low}-k}$ potential. It has been suggested [62] that low-momentum interactions in general and chiral $N^3$LO interactions (having their own cut-off $\Lambda \approx 500 - 700$ MeV [16, 17]), in particular, can be treated perturbatively in nuclear matter calculations, as the corresponding Weinberg eigenvalues lie inside the unit circle. Given the universality of the $V_{\text{low}-k}$ approach and the fact that all interactions contain 1PE while successfully describing the phase shifts in vacuum, there seems to be no fingerprint left of chiral dynamics from the two-body sector. When three-body chiral forces determined from few-body data are included, realistic saturation properties with more controlled uncertainties are obtained [63]. A trading between two and three body forces is observed and the role played by $3N$ forces as the $V_{\text{low}-k}$ cut-off is increased from $\Lambda = 400$ MeV to $\Lambda = 560$ MeV becomes less important although they still produce saturation at realistic densities.

Clearly, explicit chiral dynamics is enhanced for larger cut-off values but also the theoretical difficulties increase. The spirit of the original proposal of Weinberg’s was that the potential could be perturbatively defined according to a prescribed counting. This is clearly possible at large impact parameters where neither the strength of the nuclear force nor our lack of knowledge of its short distance components prevent us from using a fuzzy but sensibly large momentum cut-off value. In a sense, the $G$-matrix approach is close in spirit to this idea, where the strength of the nucleon force in the nuclear medium is characterized by an effective interaction. As we have shown, the LO 1PE interaction yields cut-off independent results, is entirely parameterized by vacuum properties, and has the nice feature of saturation. The problem is to define what is meant by NLO, and in nuclear matter a scheme tightly close to the standard perturbative EFT idea is expected to face the same problems already found in the vacuum sector and described in the Introduction.

Several possible and perturbatively motivated schemes for the renormalization of the chiral $NN$ interaction suggested already in Ref. [31] include considering higher pion exchanges as perturbations around the LO calculation using distorted waves. This proposal was analyzed in Refs. [32, 33], where it was shown to be feasible and cut-off independent, but with little gain from a practical viewpoint: many more counter terms were needed and a worse description was obtained in the $^1S_0$ channel and the deuteron. Actually, non-perturbative calculations with singular potentials behave non-analytically in the coupling constant, i.e. $1/f_\pi^\alpha$ with non-integer $\alpha$ [32]. A RG-based program for pion-full theories was advanced in Ref. [27], with further detailed results provided in Ref. [28]. This RG analysis yields one counter term in each $S$-wave of order $Q^{-1}$, which must be iterated, and one in each $^3P_J$ and $^3D_J$ wave of order $Q^{-1/2}$, which may be iterated. This is very similar to the power counting suggested by Nogga et al. [31].
Subleading counterterms occur in $^{1}S_0$ at $Q^0$ and in $^{3}S_1$ at $Q^{1/2}$, while the subleading $^{3}P_J$ and $^{3}D_J$ terms are of order $Q^{3/2}$. Subleading and higher terms ought to be treated perturbatively. Finally each $S$ wave receives an additional term at order $Q^2$. The number of terms at $Q^3$ in the RG scheme is larger than in Weinberg counting at NNLO; however, except for the subleading $D$-wave interactions, it is essentially the same as Weinberg counting at N$^3$LO ($Q^4$).

Toy models where a $1/r^2$ singular potential is perturbed by $1/r^4$ interactions provide useful insight, but the consequences for more realistic cases have not yet been worked out [38]. A follow-up KSW scheme has been pursued in Ref. [39], where the short distance singularity is tamed by the introduction of a Pauli-Villars like pion-mass of about the $\rho$-meson mass. An important prerequisite for specific calculations is the perturbative renormalizability described in Ref. [42]. However, as we discussed in Ref. [39] the huge change needed from the simple LO $^{1}S_0$ phase to the real observed one does not suggest any sort of small effect in the large momentum region as non-perturbatively renormalized calculations suggest. As already mentioned, this partial wave provides an important contribution to the energy in nuclear matter.

Turning to non-perturbative schemes, it was noted in Ref. [49] that off-shell properties of renormalized chiral potentials do not look much different from more conventional ones as the singularities are effectively removed. Furthermore, we know that the inclusion of $\Delta$ degrees of freedom in chiral potentials [10] lead to a pattern of better convergence, where e.g. the NLO-$\Delta$ and NNLO-$\Delta$ deuteron does exist [61], with an acceptable phenomenological success and where much larger and naively more natural momentum cut-off values display a better convergence (see also the second Ref. [11] where $\Lambda \sim 1$GeV is taken). Actually, this is a case where the discussion on renormalizability becomes pointless as the renormalized and the natural sized cut-offs are not too far apart, since finite cut-off effects are less important the more singular the potential [35]. Indeed, at order $\nu$ in the chiral counting the potential scales as $1/(\Lambda^{-\nu/2}+r_{\chi}^{-\nu})$ and then the finite cut-off correction, $\delta_{\nu}(k)$, to the renormalized phase shifts, $\delta_{\nu}(k)$, behaves as $\delta_{\nu}(k) = O(\Lambda^{-1/2}\nu/2)$ [35]. Moreover, unlike the $\Delta$-less renormalized scheme, the deuteron $D$-state probability becomes $P_D \sim 5.8\%$, a comparable value to conventional and phenomenologically successful potentials. According to our discussion, the corresponding wound integral would be small enough as to guarantee a good converging pattern of few body correlations in the nuclear many body problem. Thus, a $G$-matrix approach applied to chiral interactions including $\Delta$-isobar degrees of freedom has some chance of furnishing the multiple theoretical requirements of renormalizability, power counting for the potential (a la Weinberg), and presumably convergence in the nuclear medium. At present it is unclear whether such a scheme will be phenomenologically successful as the potentials do not contain the important spin-orbit contributions and it remains to be seen if those can be represented by appropriate counterterms.

V. CONCLUSIONS

In this work, we have renormalized the two-nucleon interaction at leading order (LO) in chiral perturbation theory using the scheme proposed by Nogga, Timmermans, and van Kolck [31]—also known as modified Weinberg counting. With this interaction, we have calculated the energy of symmetric nuclear matter in the Brueckner pair approximation. We find that the energy per nucleon as a function of nuclear matter density converges to a cutoff-independent (i.e., renormalized) result and shows saturation. The predicted value for the energy per nucleon at saturation shows $P_D \sim 5.8\%$, a comparable value to conventional and phenomenologically successful potentials. According to our discussion, the corresponding wound integral would be small enough as to guarantee a good converging pattern of few body correlations in the nuclear many body problem. Thus, a $G$-matrix approach applied to chiral interactions including $\Delta$-isobar degrees of freedom has some chance of furnishing the multiple theoretical requirements of renormalizability, power counting for the potential (a la Weinberg), and presumably convergence in the nuclear medium. At present it is unclear whether such a scheme will be phenomenologically successful as the potentials do not contain the important spin-orbit contributions and it remains to be seen if those can be represented by appropriate counterterms.

Several possible schemes for the renormalization of the chiral $NN$ interaction that have recently been given serious thought are designed to renormalize the LO interaction non-perturbatively (as done in the present work) and to add higher order corrections in perturbation theory. However, in view of the usual properties of the renormalized LO interaction and in view of the poor convergence of the nuclear many-body problem with this interaction, there is doubt if this interaction and its predictions can serve as a reasonable and efficient starting point that is improved by perturbative corrections. To make the interaction more suitable for many-body calculations, one may consider to apply renormalization group methods [64] to construct a soft two-nucleon force (2NF) plus a three-nucleon force.
(3NF) equivalent to the original LO interaction. Then there will be no problem with the application of this \( V_{\text{low-k}} \) 2NF in the many-body system. However, it may turn out that the 3NF (needed for proper equivalence with the original LO interaction) is so strong that the issue is just shifted towards a problem with the convergence of the 3NF contribution.

Appendix A: Finiteness of \( G \)-matrix for singular interactions

In this appendix, we show the finiteness of the \( G \)-matrix for singular interactions such as the 1PE potential for the gap choice Eq. (22) [48], which implies that the potential energy is suppressed as compared to the kinetic energy in the large momentum region. The idea underlying the proof is that short distance physics much below the healing distance [46] does not depend on the Fermi momentum. It is convenient to start with the extension of the \( G \)-matrix to any arbitrary energy \( z \),

\[
G(z) = V + V \frac{Q}{z - H_0} G(z),
\]

and re-write this equation as

\[
G(z)^{-1} = V^{-1} - \frac{Q}{z - H_0}.
\]

The \( G \)-matrix used in the main text, Eq. (17) corresponds to taking \( z = w = e(m) + e(n) \). We now introduce the extended \( K \)-matrix defined as

\[
K(z) = V + V \mathcal{P} \frac{1}{z - t} K(z),
\]

where \( \mathcal{P} \) denotes the principal value and \( t \) is the kinetic energy operator. The \( K \)-matrix used in the main text, Eq. (27), corresponds to using \( z = w_f = t(m) + t(n) \). Below threshold, the principal value prescription can be removed as the pole is never hit. This equation can likewise be written as

\[
K(z)^{-1} = V^{-1} - \mathcal{P} \frac{1}{z - t}.
\]

In Ref. [49] the finiteness of the \( K \)-matrix off-shell for short distance singular interactions was established solely on the basis of on-shell renormalization conditions. This means that if \( K \) is finite on shell then the off-shell extension remains finite as well. Note that the renormalization conditions are basically equivalent to a fixed energy, e.g. zero energy. Thus, \( K \) will remain finite also below threshold. Taking \( z = w \) and subtracting \( K(w)^{-1} \) from \( G(w)^{-1} \) we have (principal value where ever necessary)

\[
G(w) = K(w) + K(w) \left[ \frac{Q - 1}{w - H_0} - \frac{1}{w - t} + \frac{1}{w - H_0} \right] G(w)
\]

(A5)

In this equation, the regulator may be effectively removed from the \( K \)-matrix at the operator level even if the energy \( w \neq w_f \) does not correspond to the scattering energy [49]. If the interaction is attractive the single particle potential \( U(n) < 0 \) and \( w < w_f \) and the (finite cutoff) \( K(w) \) involves states below threshold. As we see, the first integral only involves states below the Fermi surface and is thus bound and cut-off independent provided \( k_F \ll \Lambda \). The only remaining piece where the cut-off enters explicitly is in the last two terms involving \( (w - H_0)^{-1} - (w - t)^{-1} \), which for the gap choice, see Eq. (22) [48], vanishes for states above the Fermi surface, making the integral convergent as well. Thus, the finiteness of the \( G \)-matrix is reduced to the finiteness of the \( K \)-matrix off-shell.

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