Spin-entanglement wave in a coarse-grained optical lattice

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In the present work we explore a suitable coarse graining channel as a tool to describe entanglement spreading in a coarse-grained spin-chain with different degrees of resolution. Comparing with the experimental realizations performed with ultracold atoms, our results suggest that even if we are not able to fully resolve the system, entanglement can still be detected for some coarse graining levels. Furthermore, we show that it is possible to have some information about the “microscopic” entanglement, even if we have access only to the system’s coarse graining description. We show that the amount of entanglement decays exponentially with the lack of system resolution. The lack of experimental resolution might thus lead to a classical effective description.

I. INTRODUCTION

Quantum entanglement is a special kind of correlation present in quantum many-body systems, and absent in their classical counterparts. Especially in strong correlated regimes, entanglement plays a crucial role in the quantitive description of quantum many-body systems [1]. In addition, entanglement is a resource for quantum information processing, so that its microscopic control is needed for various applications [2]. Since entanglement is intrinsically fragile against environmental action [3], the microscopic control of quantum entangled systems becomes a great experimental challenge. Given that, currently there are few well-controlled physical systems that offer the possibility of exploring many-body entanglement [4,8]. Among these, a prominent one is that of spin systems realized with ultracold atoms in optical lattices [9–13]. Within this platform, recent works accomplished a single site detection, which enabled the measurement of spin-entanglement waves in a few qubits Bose-Hubbard chain [14].

However, for many-body systems, the experimental control and characterization of entangled states usually spends expensive resources, rapidly becoming infeasible for increasing system size. Brute force quantum state tomography, for instance, requires a challenging individual particle access, and a time consuming experiment and data post-processing [15]. Single site detection in optical lattices requires highly accurate equipment, like high-resolution quantum gas microscope [16,17], which then becomes an obstacle. In order to make the experimental procedure simpler and more feasible, a question arises: Given an entangled spin-system, is it possible to have a satisfactory description of the spin-entanglement if we are not able to fully resolve the system?

In order to address this question, here we develop a suitable effective description of a Bose-Hubbard spin chain via a physically motivated coarse graining map. Such a coarse graining map simulates a detector that can not resolve a single site, and thus it takes a number of neighboring sites of the spin chain as an effective single site. This idea can be thought as reminiscent of Kadanoff’s blocking procedure [18] that initiated the renormalization theory [19,21], and is also related to the more recent entanglement renormalization procedure proposed by Vidal [22]. More recently, in [23], an abstract type of coarse graining scheme was used to provide an entanglement criteria for high dimensional systems. Different from these previous works, our coarse graining map models an incoherent loss of information: a pure microscopic state is seen as a mixed effective macroscopic state. In a sense, the use of coarse graining maps is a generalization of decoherence theory [24] which allows us to describe the entanglement behavior taking into account different ranges of resolution. By applying the coarse graining map, early proposed in [25,26], we will be able to compare the loss of information about the entanglement dynamics in different degrees of the lattice sharpness and see to what extent its detection is still achievable.

This paper is organized as follows: In Sec. II, we present a brief description of the entanglement spreading during single spin-impurity dynamics in one-dimensional Bose-Hubbard system. After that, in Sec. III we define a general coarse graining operation as a CPTP map. Then, inspired by fluorescence imaging of atoms in an optical lattice, we construct a coarse graining map that describes a spin detection with insufficient resolution. In Sec. IV we use this coarse graining map to describe the entanglement due to spin-impurity dynamics in a coarse-grained Bose-Hubbard spin-chain. We explore how entanglement behaves when we take into account different ranges of resolutions of the spin-chain. Finally in Sec. V, we summarize our results and discuss some implications of this coarse graining approach.

II. ENTANGLEMENT IN A BOSE-HUBBARD SPIN-CHAIN

In this section we describe the entanglement generation and spreading during spin-impurity dynamics in a 1D spin-1/2 XX-chain [27,28]. In the experimental work [14,29], it was produced a ferromagnetic Heisenberg spin-chain with ultracold bosonic atoms in an optical lattice. In these experiments, individuals atoms are trapped in each potential minimum of a periodic potential associated with a stationary wave created by counter-propagating laser beams [9,30]. Deep in the Mott-insulator regime with unity filling, two hyperfine levels of each atom act as a spin-1/2 (qubit), and neighboring spins interact with each other via isotropic spin-1/2 Heisenberg...
Hamiltonian \[\hat{H} = -J_{ex} \sum_j \hat{\sigma}_j \cdot \hat{\sigma}_{j+1}\]. In this Hamiltonian, \(\hat{\sigma}_j = (\hat{\sigma}_j^x, \hat{\sigma}_j^y, \hat{\sigma}_j^z)\) denotes the spin-1/2 vector of Pauli matrices at site \(j\), and \(J_{ex}\) is the exchange coupling which is constant for homogeneous chains (see the supplementary information of [29]). In the case of a single spin-impurity in a 1D lattice (single excitation subspace), the Hamiltonian can be written in a simplified form:

\[
\hat{H} = -J_{ex} \sum_j (\hat{\sigma}_j^+ \hat{\sigma}_{j+1}^- + \hat{\sigma}_j^- \hat{\sigma}_{j+1}^+),
\]

where \(\hat{\sigma}_j^\pm = (\hat{\sigma}_j^x \pm i\hat{\sigma}_j^y)/2\) are the spin-1/2 raising (lowering) operators. The term \(J_{ex} \sum_j \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z\) was dropped, since it gives rise only to an energy offset within the single excitation subspace [29].

A infinite spin-up chain with a single spin-down (spin-impurity) on site \(j\) can be represented by the state

\[
|j\rangle \equiv |\cdots, 0_{j-1}, 1_j, 0_{j+1}, \cdots\rangle,
\]

where \(|1\rangle\) and \(|0\rangle\) refers to spin down and spin up states respectively, in the \(z\)-basis. As initial state it is considered a single spin-down at the “center” of the chain \((j = 0)\). The spin-impurity spreading is given by the time evolution generated by the Hamiltonian in equation (1), and it can be described by

\[
|\psi(t)\rangle = \sum_j \phi_j(t)|j\rangle,
\]

where \(\phi_j(t) = e^{iJ_j(J_{ex}t/\hbar)}\), with \(J_j(x)\) the Bessel function of the first kind [32]. For simplicity, from now on we will consider the time evolution in dimensionless time unit \(J_{ex}t/\hbar\).

### A. Concurrence between two sites

The next step is to quantify the entanglement between spins in two different sites \(A\) and \(B\) in the chain. The reduced density operator related to a pair of different arbitrary sites \(A\) and \(B\) is given by:

\[
\rho_{AB}(t) = \text{Tr}_{\overline{AB}}[|\psi(t)\rangle\langle\psi(t)|],
\]

where \(\overline{AB}\) means the complementary sites to \(AB\). Using (3) and the basis states \(|00\rangle, |01\rangle, |10\rangle\) and \(|11\rangle\) for sites \(A\) and \(B\), we get:

\[
|\psi_{AB}\rangle = \begin{pmatrix}
1 - |\phi_A|^2 & -|\phi_B|^2 & 0 & 0 & 0
0 & |\phi_B|^2 & \phi_A \phi_B^* & 0 & 0
0 & 0 & |\phi_A|^2 & \phi_A \phi_B^* & 0
0 & 0 & 0 & |\phi_B|^2 & \phi_A \phi_B^*
0 & 0 & 0 & 0 & |\phi_A|^2
\end{pmatrix}.
\]

Explicit time dependence is suppressed whenever obvious, to avoid cluttered notation. The matrix representation [5] of \(|\psi_{AB}\rangle\) can be identified as a \(X\)-matrix, whose concurrence can be easily calculated [33] to give:

\[
C(\psi_{AB}) = 2|\phi_A \phi_B^*|,
\]

Figure 1 illustrates the expected entanglement dynamics between symmetric sites with respect to the spin at position \(j = 0\). Such an entanglement wave – see Fig. 1(b) – was observed experimentally showing a good agreement with the theoretical prediction [14].

### III. COARSE-GRAINED DESCRIPTION OF A SPIN QUANTUM SYSTEM

Following a scheme proposed in [34] and supported by an experimental setup with single-site-resolved fluorescence imaging [16, 17], researchers have detected a lower bound for the concurrence of two sites. Their results are consistent with those presented in Fig. 1. The single-site-resolved fluorescence imaging is roughly described as follows: each atom of the chain is illuminated with a laser in way that if light is scattered, the atom was in the state \(|0\rangle\); whereas if no light is scattered, the atom was in the state \(|1\rangle\). To detect the light coming from each single atom, a high resolution quantum gas microscope is required (single-site-resolution scheme [16]). In the case of truly many-body system such equipment might no longer be able to resolve a single atom, what might become an obstacle in order to observe the entanglement dynamics. With that in mind, we propose a coarse-grained scheme to simulate a detector with insufficient resolution and we show how entanglement behaves in that situation.
In general, coarse-grained models aim at describing the behavior of complex systems using a simplified representation which preserves the properties of the system that we have access to. In other words, complex systems are represented by an adequate simplified representation with less degrees of freedom that facilitates its study. The coarse graining operation is thus a quantum channel that can be defined as a completely positive trace preserving (CPTP) map \( \Lambda_{\text{CG}} : \mathcal{L}(\mathcal{H}_D) \to \mathcal{L}(\mathcal{H}_D) \) for which \( \dim(\mathcal{H}_D) := D > d := \dim(\mathcal{H}_d) \), and \( \mathcal{L}(\mathcal{H}) \) represents the set of linear operators acting on \( \mathcal{H} \) [25][26].

A. A blurred and saturated detector coarse graining channel

Back to the experimental situation, imagine that we want to measure the light that comes from a number of neighboring atoms in a lattice by fluorescence imaging [16], but our detecting device does not have enough resolution to identify the light coming from each individual atom. To describe this situation we can construct a coarse graining model in such a way that we take the information of these multiple unresolved signals as effectively coming from one single atom in a coarse-grained level (such situation is pictorially illustrated in Fig. 2).

![FIG. 2. A pictorial scheme of a coarse graining model. In the left the two spheres represent a composite system of two atoms. In the middle we have a illustration suggesting the blurred view of the detection system. Then in the third picture we approximate the blurred signal as a single effective atom in a coarse-grained level.](image)

Bringing this situation to the scenario of quantum information, we can identify each atom with a qubit (a spin-1/2). In this way a system of \( N \) spins is described by a \( D \)-dimensional state \( \psi \in \mathcal{L}(\mathcal{H}_D) \), with \( D = 2^N \). So we want to construct a coarse graining map \( \Lambda_{\text{CG}}^{N \to 1} : \mathcal{L}(\mathcal{H}_D) \to \mathcal{L}(\mathcal{H}_D) \) such that it takes a system of \( N \) spins to an effective coarse-grained level of a single spin.

Starting from the simplest case, where our detector cannot resolve two neighboring atoms and the amount of light coming from a single atom is already sufficient to saturate the detector, the resulting signal can be related to a single atom in a coarse-grained level. Such situation suggests the following coarse graining map \( \Lambda_{\text{CG}}^{2 \to 1} : \mathcal{L}(\mathcal{H}_4) \to \mathcal{L}(\mathcal{H}_2) \), introduced in [25][26]:

\[
\begin{align*}
\Lambda_{\text{CG}}^{2 \to 1}(00|00) &= 0|0\rangle 0|0\rangle \quad &\Lambda_{\text{CG}}^{2 \to 1}(10|00) &= \frac{1}{\sqrt{3}} 1|0\rangle \langle 1|0\rangle \\
\Lambda_{\text{CG}}^{2 \to 1}(00|01) &= \frac{1}{\sqrt{3}} 0|1\rangle \langle 0|1\rangle \quad &\Lambda_{\text{CG}}^{2 \to 1}(10|01) &= 0 \\
\Lambda_{\text{CG}}^{2 \to 1}(00|10) &= \frac{1}{\sqrt{3}} 0|1\rangle \langle 0|1\rangle \quad &\Lambda_{\text{CG}}^{2 \to 1}(10|10) &= |1\rangle |1\rangle \\
\Lambda_{\text{CG}}^{2 \to 1}(01|00) &= \frac{1}{\sqrt{3}} 1|0\rangle \langle 1|0\rangle \quad &\Lambda_{\text{CG}}^{2 \to 1}(11|00) &= \frac{1}{\sqrt{3}} 1|0\rangle |0\rangle \\
\Lambda_{\text{CG}}^{2 \to 1}(01|01) &= |1\rangle |1\rangle \quad &\Lambda_{\text{CG}}^{2 \to 1}(11|01) &= 0 \\
\Lambda_{\text{CG}}^{2 \to 1}(01|10) &= 0 \quad &\Lambda_{\text{CG}}^{2 \to 1}(11|10) &= 0 \\
\Lambda_{\text{CG}}^{2 \to 1}(01|11) &= |1\rangle |1\rangle \quad &\Lambda_{\text{CG}}^{2 \to 1}(11|11) &= |1\rangle |1\rangle 
\end{align*}
\]

The heuristics that lead us to construct such a map are as follows: as the detector cannot resolve if the fluorescence light comes from one atom or the other, then both states \( |01\rangle |01\rangle \) and \( |10\rangle |10\rangle \) lead to an effective single excitation \( |1\rangle |1\rangle \). Moreover, we assume that the detector makes no distinction between one or two excitations, i.e. that it saturates already with a single excitation signal. As such, the state \( |11\rangle |11\rangle \) is also mapped to the effective state \( |1\rangle |1\rangle \). The coherence between the excited subspace, span\{\( |01\rangle, |10\rangle, |11\rangle \}\), with the no-excitation subspace \( |00\rangle \), maps to the effective coherence \( 1/\sqrt{3}|1\rangle |0\rangle \). The factor \( 1/\sqrt{3} \) comes from the dimensionality of the subspaces, and it ensures the complete positiveness of the coarse graining map. Lastly, since we cannot distinguish the states \( |01\rangle, |10\rangle \) and \( |11\rangle \), there can be no coherence within this subspace, so the above null terms appear. The Hermitian-conjugated instances follow immediately from the Hermiticity preserving property of quantum channels. It is also important to notice that this map is not simply a partial trace, and as such it might be seen as a generalization of a decoherence process.

In the case where the detector cannot resolve more atoms, we can compose the above map as follows:

\[
\begin{align*}
\Lambda_{\text{CG}}^{4 \to 1} &= \Lambda_{\text{CG}}^{2 \to 1} \circ (\Lambda_{\text{CG}}^{2 \to 1} \otimes \Lambda_{\text{CG}}^{2 \to 1}), \\
\Lambda_{\text{CG}}^{8 \to 1} &= \Lambda_{\text{CG}}^{4 \to 1} \circ (\Lambda_{\text{CG}}^{2 \to 1} \otimes \Lambda_{\text{CG}}^{2 \to 1}) \circ \\
&\circ (\Lambda_{\text{CG}}^{2 \to 1} \otimes \Lambda_{\text{CG}}^{2 \to 1} \otimes \Lambda_{\text{CG}}^{2 \to 1} \otimes \Lambda_{\text{CG}}^{2 \to 1}), \\
\Lambda_{\text{CG}}^{N \to 1} &= \Lambda_{\text{CG}}^{2 \to 1} \circ (\Lambda_{\text{CG}}^{2 \to 1} \otimes \Lambda_{\text{CG}}^{2 \to 1}) \circ \cdots \circ \\
&\circ \left( \bigotimes_{k=1}^{N/4} \Lambda_{\text{CG}}^{2 \to 1} \right) \circ \left( \bigotimes_{k=1}^{N/2} \Lambda_{\text{CG}}^{2 \to 1} \right),
\end{align*}
\]

where "\( \circ \)" denotes composition of maps. For later convenience, we define the coarse graining level \( l \) as the number of times a layer of coarse graining maps is applied: for the microscopic level, where no coarse graining operation is applied we have \( l = 0 \); starting from \( N \) qubits in the microscopic level a single effective qubit will be obtained at level \( l = \log N \) (we use logarithms in base 2 throughout the article) after the application of successive layers of coarse graining maps. This process is schematically represented in Fig. 5.
and using \( \Lambda^{4}_{CG} \), and \( \Lambda^{3}_{CG} \) can be defined.

### IV. ENTANGLEMENT SPREADING IN A COARSE-GRANDED SPIN CHAIN

Now that we have already constructed the coarse graining map that plays the role of a blurred detector, Eq. (7), let’s analyze how entanglement evolves under this coarse graining view. In the same way as the entanglement detection was studied in [14], we will calculate the entanglement behavior between two symmetric sites around the center of a spin-chain, but now considering coarse-grained sites, that is, entanglement between two coarse-grained blocks of spins.

Before we start, in order to properly write the microscopic (\( l = 0 \)) reduced state on the blocks \( A = \{ A_1, \ldots, A_N \} \) and \( B = \{ B_1, \ldots, B_N \} \), \( \psi^{0}_{AB} \), it is convenient to define the following family of vectors:

\[
|k_N\rangle_A \equiv |0_{A_1}, \ldots, 0_{A_{k-1}}, 1_{A_k}, 0_{A_{k+1}}, \ldots, 0_{A_N}\rangle, \\
|0_N\rangle_A \equiv |0_{A_1}, \ldots, 0_{A_N}\rangle,
\]

with \( k \in \{1, \ldots, N\} \), and similarly for \( B \). It is clear that \( |k_N\rangle_{A(B)} \) is a state vector with a single spin impurity at the \( k \)-th site of block \( A(B) \), and \( |0_N\rangle_{A(B)} \) is the non-impurity state.

#### A. \( l = 1 \) : CG Entanglement (2 \( \rightarrow \) 1)

In the first situation, \( l = 1 \), we want to compute the entanglement between two coarse-grained sites, each one coming from a block of two neighboring sites (\( N = 2 \)) at the “microscopic” level (\( l = 0 \)), \( A = \{ A_1, A_2 \} \) and \( B = \{ B_1, B_2 \} \), as is schematically represented in Fig. 4.

\[
\begin{array}{c}
\cdots \quad A_1 \quad A_2 \quad B_1 \quad B_2 \quad \cdots
\end{array}
\]

FIG. 4. Schematic representation of a block of spins where the coarse graining map \( \Lambda^{4}_{CG} \) is going to be applied.

Remembering that we are in the single excitation subspace, and using \( \{ |k_2\rangle_{A(B)} \} \cup \{ |0_2\rangle_{A(B)} \} \) as basis, the reduced state \( \psi^{l=0}_{AB} \) can be written as

\[
\psi^{l=0}_{AB} = \text{tr}_{\mathcal{H}}^{\mathcal{H}}(\psi(t)|\psi(t)\rangle) = \zeta |0_2\rangle_{A} |0_2\rangle_{B} + \sum_{k=1}^{2} |\phi_{A_k}\rangle_{A} |\phi_{B_k}\rangle_{B} \]

\[
+ \sum_{k \neq k'}^{2} (|\phi_{A_k}\rangle_{A} |\phi_{B_{k'}}\rangle_{B} + |\phi_{B_k}\rangle_{B} |\phi_{A_{k'}}\rangle_{A}) + \sum_{k, k'}^{2} (|\phi_{A_{k'}}\rangle_{A} |\phi_{B_{k'}}\rangle_{B} + |\phi_{B_{k'}}\rangle_{B} |\phi_{A_{k'}}\rangle_{A})
\]

where \( \zeta = 1 - \sum_{k=1}^{2} (|\phi_{A_k}|^2 + |\phi_{B_k}|^2) \). In this form, it is now simple to apply the coarse graining map defined in Eq. (7). Observe that the internal coherences \( \phi_{A_k} \phi_{B_{k'}}^{*} \) and \( \phi_{B_k} \phi_{B_{k'}}^{*} \) (for \( k \neq k' \)) related to each spin-block present in state (12) no longer survive in the coarse grained state, as \( \Lambda^{4}_{CG} |01_{AB}\rangle = |01_{AB}\rangle \) = 0. The effective two-qubit is then:

\[
\psi^{l=1}_{AB} = (\Lambda^{2}_{CG} \otimes \Lambda^{2}_{CG}) (\psi^{l=0}_{AB})
\]

\[
= \left( \begin{array}{cccc}
\zeta & 0 & 0 & 0 \\
0 & \sum_{k=1}^{2} |\phi_{B_k}|^2 & \frac{1}{2} \sum_{k, k'}^{2} \phi_{A_k} \phi_{B_{k'}}^{*} & 0 \\
0 & \frac{1}{2} \sum_{k, k'}^{2} \phi_{A_k} \phi_{B_{k'}}^{*} & \sum_{k=1}^{2} |\phi_{A_k}|^2 & 0 \\
0 & 0 & 0 & 0
\end{array} \right),
\]

where we used the basis states \( \{ |k_1\rangle_{A(B)} \} \cup \{ |0_1\rangle_{A(B)} \} \) for each effective site.

The entanglement between two coarse-grained sites is derived analogously to that found before for two (non-coarse-grained) sites in [14]. Since the state (13) takes the X-form, its concurrence is given by

\[
C(\psi^{l=1}_{AB}) = \frac{2}{3} \sum_{k, k'=1}^{2} |\phi_{A_k} \phi_{B_{k'}}^{*}|.
\]

For the purpose of giving a concrete view of the consequences of this result, we numerically investigated the entanglement evolution in a seventeen sites chain with the spin-impurity beginning at its center. In Fig. 5(a) we plot in blue the concurrence between symmetric sites around the center of the spin-chain in the "microscopic" level, and in red the concurrence considering their related coarse-grained sites. In the microscopic level we use Eq. (10) to calculate the concurrence between the first two pairs of sites before the coarse graining: \( C(\psi^{l=0}_{1\rightarrow 1}) = 2 |\phi_1 \phi_{2}^{*}| \) and \( C(\psi^{l=0}_{2\rightarrow 4}) = 2 |\phi_{2} \phi_{2}^{*}| \).
Then, we calculate the concurrence of the resulting pair of coarse-grained sites using Eq. (14): $C(\psi_{l=1}^{j\rightarrow1}) = (2/3)[(\phi_1 + \phi_2)(\phi_{-1} + \phi_{-2})^*]$. In the same way we calculate the concurrence between other symmetric sites around the center both at the “microscopic” level and at coarse-grained level with $l = 1$.

As expected, we observe that concurrence decays in the coarse graining level. Despite of this, a significant amount of entanglement still survives. So even in the coarse-grained description we observe a propagating entanglement “wave” (See Fig. 5(b)), with values above the error bar (black dashed line) observed by the experiment realization in [14]. It is interesting to notice that the maximum value of concurrence values in coarse graining level $l = 1$ are in better agreement with the experimental results showed in [14] than when compared with the theoretical results in “microscopic” level $l = 0$. One can thus speculate that the experimental procedure in [14] does take into account some coarse graining as the one introduced here.

A complementary way of studying the differences in entanglement between the “microscopic” level and the coarse-grained level is through negativity [35]. Using concurrence we are somewhat restricted to calculate only bipartite entanglement for systems with two spins (two sites in the spin-chain). Negativity, on the other hand, can be evaluated for bipartite systems of any local dimensions. Here we restrict our analysis to the coarse-grained state $\psi_{l=1,1}$ and its related “microscopic” state $\psi_{l=0,1}^{j\rightarrow1}(1,2)$. Results are shown in Fig. 5(b). Differently from the concurrence approach, with negativity we can properly compare the entanglement within blocks in different ranges of description. We observe that entanglement in the coarse graining level is smaller than the total entanglement in the “microscopic” state.

**B. $l = 2$: CG Entanglement ($4 \rightarrow 1$)**

For the next level of resolution ($l = 2$), we map four sites ($N = 4$) as a single effective one. Given an arbitrary pair of spin-blocks composed by four neighboring atoms $A = \{A_1, A_2, A_3, A_4\}$ and $B = \{B_1, B_2, B_3, B_4\}$—as shown in Fig. 6.

We proceed in the same way as above to find the coarse-grained state:

$$\psi_{AB}^{l=2} = (\Lambda_{CG}^{4\rightarrow1} \otimes \Lambda_{CG}^{4\rightarrow1})(\psi_{AB}^{l=0})$$

$$= \left( \begin{array}{cccc}
\zeta & 0 & 0 & 0 \\
0 & \frac{4}{3} \sum_{k} |\phi_{B_k}|^2 & \frac{4}{3} \sum_{k,k'} \phi_{A_k}^* \phi_{B_{k'}} & 0 \\
0 & \frac{4}{3} \sum_{k,k'} \phi_{A_k}^* \phi_{B_{k'}} & \frac{4}{3} \sum_{k} |\phi_{A_k}|^2 & 0 \\
0 & 0 & 0 & 0
\end{array} \right).$$

As observed in the previous section [IV A], the internal coherence related to spin-blocks, $\phi_{A_k}^* \phi_{A_{k'}}$ and $\phi_{B_k}^* \phi_{B_{k'}}$, with $k \neq k'$, do not survive in the effective state (15). As before, this happens because $\Lambda_{CG}^{4\rightarrow1}(|00\rangle|01\rangle) = \Lambda_{CG}^{4\rightarrow1}(|01\rangle|01\rangle) = 0$.

and due to the fact that $\Lambda_{CG}^{4\rightarrow1} = \Lambda_{CG}^{2\rightarrow1} \circ (\Lambda_{CG}^{2\rightarrow1} \otimes \Lambda_{CG}^{2\rightarrow1})$. The square power on the $1/3$ factor reflects the two layers of coarse graining which were applied.

For such a state the concurrence can be easily evaluated to

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Coarse-grained entanglement in $l = 1$. (a) Effective spin chain with a single layer of coarse graining. (b) Comparison between concurrence evolution of the first four symmetric pairs of coarse-grained sites, $C(\psi_{l=0,A}^{j\rightarrow1})$ (in red), with respect to the concurrence between their relative “microscopic” sites (in blue). (c) Entanglement wave in the coarse-grained level. The black dashed line represents the error in experimental detection (taken from [14]). (d) Comparison between negativity evolution of the first symmetric pairs of coarse-grained sites, $N(\psi_{l=1,1}^{j\rightarrow1})$ (in red), and the negativity $N(\psi_{l=0,1}^{j\rightarrow1}(1,2))$ among their relative “microscopic” sites (in blue).}
\end{figure}
give:

\[ C(\psi_{AB}^{l=2}) = \frac{2}{3\sqrt{3}} \left| \sum_{k,k' = 1}^4 2\phi_{A_k}\phi_{B_{k'}} \right|. \] (16)

Again we analyze the entanglement evolution by looking at the seventeen centralized sites of the “microscopic” chain \( l = 0 \) whose spin-impurity begins at its center. Results are shown in Fig. 7 (a) We use equation (10) to calculate the concurrence between the first four pair of sites before the coarse graining (in blue): \( C(\psi_{-1,1}^{l=1}) = 2|\phi_{-1}\phi_{1}|, C(\psi_{2,2}^{l=2}) = 2|\phi_{-2}\phi_{2}|, C(\psi_{-2,2}^{l=2}) = 2|\phi_{-3}\phi_{3}| \) and \( C(\psi_{4,4}^{l=4}) = 2|\phi_{-4}\phi_{4}| \). Then, we calculate the concurrence of resulting pair of coarse-grained states using the equation (16) (in green): \( C(\psi_{1,1}^{l=1}) = (2/9)(|\phi_{-1} + \phi_{-2} + \phi_{-3} + \phi_{-4}|(\phi_{1} + \phi_{2} + \phi_{3} + \phi_{4})^* \). We proceed in the same way to calculate the concurrence between the other symmetric sites.

As expected, we observe that the concurrence becomes weaker when compared to the concurrence in the “microscopic” level or even with the one in the first coarse grained level \( l = 2 \). Consequently we observe an very weak entanglement “wave”, in the limit of the experimental error detection (see Fig. 7 (c)). Therefore in this scheme, if we consider the error bars of the experimental work \( l = 4 \), our results suggest that entanglement would no longer be detectable.

Analogously to the last section, here we calculate the negativity for the different description levels of the system \( l = 0, 1, 2 \). We consider the coarse-grained states \( \psi_{-2,1}^{l=2} \) and \( \psi_{-1,1}^{l=1} \). and their correspondent “microscopic” state \( \psi_{-4,-3,-2,-1}^{l=4} \). Results are shown in Fig. 7 (c). We observe how drastic it is the effect of the coarse graining on the entanglement among different levels. This result is in agreement with the ones for concurrence, indicating a weak entanglement signal after two layers of coarse graining (level \( l = 2 \)). That is, if the detector cannot resolve the signal coming for a ensemble of four neighboring sites in a spin-chain, entanglement will be hardly detectable in the experimental situation. Again we observe that the effective entanglement is smaller than the total entanglement in the super-cell.

C. \( l = \log N \): CG Entanglement \((N \rightarrow 1)\)

In this section we generalize the previous concurrence results to any level of coarse graining. At the microscopic level, we consider two blocks of sites \( A = \{A_1, \cdots, A_2, A_N\} \) and \( B = \{B_1, B_2, \cdots, B_N\} \), in which we will apply \( \log N \) layers of the coarse graining map:

\[
\Lambda_{CG}^{N-1} = \Lambda_{CG}^{2-1} \circ (\Lambda_{CG}^{2-1} \otimes \Lambda_{CG}^{2-1}) \circ \cdots \circ \\
\circ \left( \times \Lambda_{CG}^{N/4} \right) \circ \left( \times \Lambda_{CG}^{N/2} \right). \] (17)

Using the total pure state in equation (15), and definition (11), we can write the reduced state over the blocks \( A \) and
$B$ as:

$$\psi_{AB}^l = \zeta \left| 0_N \right> A \left( 0_N \otimes | 0_N \right> B (0_N) +$$

$$+ \sum_{k=1}^{N} (| \phi_{Ak} |^2 | kN \rangle A \left( kN \right) \otimes | 0_N \rangle B (0_N) +$$

$$+ | \phi_{Bk} |^2 | 0_N \rangle A (0_N) \otimes | kN \rangle B (kN) \right) +$$

$$+ \sum_{k \neq k'}^{N} \left( \phi_{Ak} \phi_{Ak'}^* | kN \rangle A \left( kN \right) \otimes | 0_N \rangle B (0_N) +$$

$$+ \phi_{Bk} \phi_{Bk'}^* | 0_N \rangle A (0_N) \otimes | kN \rangle B (kN') \right) +$$

$$+ \sum_{k,k'}^{N} \left( \phi_{Ak} \phi_{Bk'}^* | 0_N \rangle A (0_N) \otimes | kN \rangle B (kN') +$$

$$+ \phi_{Ak}^* \phi_{Bk}^* | 0_N \rangle A (0_N) \otimes | kN \rangle B (kN') \right),$$

(18)

where $\zeta = 1 - \sum_{k=1}^{N} (| \phi_{Ak} |^2 + | \phi_{Bk} |^2)$.

The rational to apply the coarse graining map in both supercells, $A_{CG}^{N+1} \otimes A_{CG}^{N-1}$, is as before. The first term contains no excitations, and as such goes to $\zeta | 0_N \rangle A | 0_N \rangle B | 0_N \rangle$. The second term contains the diagonal/population elements where the excitation is either in part $A$ or in part $B$. As the coarse graining maps states in the single-excitation subspace into (smaller dimensional) single-excitation subspace, this term goes to

$$\sum_{k=1}^{N} (| \phi_{Ak} |^2 | 1 \rangle A | 0 \rangle B | 0 \rangle + | \phi_{Bk} |^2 | 0 \rangle A | 0 \rangle B | 1 \rangle).$$

The third term contains off-diagonal/coherence elements where the excitation is either in the part $A$ or in the part $B$. As $k \neq k'$, these elements represent coherences between states that the detector cannot distinguish, and as such, at some level, these elements vanish. Lastly we have the term with coherences between the single-excitation subspace and the no-excitation subspace within a supercell. At a given layer there will always be per block one element of the form $| 01 \rangle A | 00 \rangle$ or $| 10 \rangle A | 00 \rangle$, with all others being as $| 00 \rangle A | 00 \rangle$. The action of the coarse graining map on the first type of elements gives $| 1 \rangle A | 0 \rangle B / \sqrt{3}$, while on the second gives $| 0 \rangle A | 0 \rangle B$. The same reasoning applies for the Hermitian conjugate elements and for the block $B$. Since we start with $N$ particles per block, we can repeat this process $\log N$ times, and as such, at level $l = \log N$, this term goes to $(1 / 3)^{\log N} \sum_{k,k'=1}^{N} \phi_{Ak} \phi_{Bk'}^* | 1 \rangle A | 0 \rangle B | 1 \rangle + \phi_{Ak}^* \phi_{Bk}^* | 0 \rangle A | 0 \rangle B | 1 \rangle$. In summary, after applying the coarse graining map $\log N$ times on the pure state $^{3}$, we get:

$$\psi_{AB}^{l=\log N} = (A_{CG}^{N+1} \otimes A_{CG}^{N-1})^{l} (\psi_{AB}^{0})$$

$$= \begin{pmatrix}
\zeta & 0 & \cdots & 0 \\
0 & \frac{1}{3^\log N} \sum_{k=1}^{N} | \phi_{Ak} |^2 & \cdots & 0 \\
. & . & . & . \\
0 & \frac{1}{3^\log N} \sum_{k,k'=1}^{N} \phi_{Ak} \phi_{Bk'} & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}.$$  (19)

As before, now it is simple to evaluate the effective concurrence:

$$C(\psi_{AB}^{l=\log N}) = \frac{2}{3^\log N} \sum_{k,k'=1}^{N} \phi_{Ak} \phi_{Bk'}^*.$$  (20)

Now we can evaluate the limit of effective concurrence detection. To do this we compare the maximum value attained for concurrence $C(\psi_{AB}^{l=\log N})$ at each coarse graining level. Results are shown in Fig. 8. We note that from $l = 3$ onward the values became smaller than the experimental detection error observed in [14], which suggests that no entanglement could be detected in such an experimental resolution.

![FIG. 8. Concurrence behavior at each coarse graining level.](image)

From Eq. (20) it is simple to established that the effective concurrence at a given coarse graining level $l$ is smaller than the sum of all bipartite microscopic concurrences:

$$C(\psi_{AB}^{l=\log N}) \leq \frac{1}{3^\log N} \sum_{k,k'=1}^{N} C(\psi_{Ak} B_{k'}).$$  (21)

We observe an exponential entanglement decay with respect to the coarse graining level $l = \log N$.

Moreover, now we can address the following physically motivated question: From the coarse-grained concurrence can we say something about entanglement at the “microscopic” level? Let $C_{max}^{l=0} = \max_{k,k'} C(\psi_{Ak} B_{k'})$. Then from the above inequality we have:

$$C_{max}^{l \geq 3} \geq \left( \frac{3}{4} \right)^{\log N} C(\psi_{AB}^{l=\log N}).$$  (22)

Therefore with this procedure we can construct a lower bound for the maximal concurrence between a pair of spins in the “microscopic” level. Naturally, we are not able to identify the pair of spins which shares the maximum entanglement at a given time, but we can say that there is a pair of spins in the “microscopic” chain with entanglement at least $(3/4)^{\log N}$ of the concurrence measured in the “macroscopic” level.

V. SUMMARY AND CONCLUSIONS

In the present work we have explored a suitable coarse graining channel, proposed in [25, 26], as a tool to describe...
entanglement spreading in a coarse-grained spin-chain with different degrees of resolution. Comparing with the experimental realizations performed with ultracold atoms in optical lattices [14][29], our results suggest that even if we are not able to fully resolve the system, entanglement can be still detected at some coarse graining levels. With our approach we showed that even if the blurred detector cannot resolve the signal that comes from two or four neighboring spins an entanglement wave can be still detected. Furthermore, we showed that it is possible to have some information about the “microscopic” entanglement even if we have access only to the coarse-graining description [22].

Beyond that experimental situation, ultracold atoms in optical lattice represent an excellent platform for simulation of many other quantum many body problems [17]. Part of this success is due to the development of high precision detection devices like quantum gas microscopes [16, 17]. Since our coarse graining is constructed to model this kind of detection, we hope that our coarse graining model will be relevant for other ultracold atom-based quantum simulations. Some possibilities are: to probe quantum magnetism, to realize and detect topological matter and the study of quantum systems with long-range interactions [17].

Other coarse graining situations can also be explored. Beyond modeling a blurred detection, the coarse graining approach can be a tool to explore other problems in physics. From our results and other recent works, like [25], we expect that coarse graining maps will play an important role in exploring foundations of statistical mechanics and emergent phenomenons in quantum physics.

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