The homotopy types of $Sp(n)$-gauge groups over $S^{4m}$

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Abstract

Let $m$ and $n$ be two positive integers such that $m < n$. Denote by $P_{n,k}$ the principal $Sp(n)$-bundle over $S^{4m}$ and $G_{k,m}(Sp(n))$ be the gauge group of $P_{n,k}$ classified by $k\varepsilon'$, where $\varepsilon'$ is a generator of $\pi_{4m}(B(Sp(n))) \cong \mathbb{Z}$. In this article, we will partially classify the homotopy types of $G_{k,m}(Sp(n))$ by giving a lower bound for the number of homotopy types of $G_{k,m}(Sp(n))$. Also, in special cases $Sp(3)$-gauge groups over $S^8$ and $Sp(4)$-gauge groups over $S^{12}$ we give an upper bound for the number of homotopy types of these gauge groups.

Keywords: gauge group; homotopy type; $Sp(n)$.

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1 Introduction

Let $G$ be a topological group and let $P \to B$ be a principal $G$-bundle over connected finite complex $B$. The gauge group $G(P)$ of $P$ is the group of $G$-equivariant automorphisms of $P$ which fix $B$. The topology of gauge groups plays an important role in the physics and identification of four-dimensional manifolds. Crabb and Sutherland [3] have shown that if $B$ and $G$ are as above, then the number of homotopy types amongst all the gauge groups of principal $G$-bundles over $B$ is finite. This is in spite of the fact that the number of isomorphism classes of principal $G$-bundles over $B$ is often infinite.

Let $G$ be a simply-connected, simple compact Lie group. For integers $a$ and $b$ let $(a, b)$ be the greatest common divisor of $|a|$ and $|b|$. There is a one-to-one correspondence between the number of equivalence classes of principal $G$-bundles over $B = S^4$ and homotopy classes of maps $[S^4, BG] \cong \mathbb{Z}$. The first classification was done by Kono [11] for $G = SU(2)$. Kono showed that there is a homotopy equivalence $G_k \simeq G_{k'}$ if and only if $(12, k) = (12, k')$. It thus follows that there are precisely six homotopy types of $SU(2)$-gauge groups over $S^4$. In recent years, the determination of the precise number of homotopy types of gauge groups for some Lie groups on manifolds with different dimensions interested topologists. For principal bundles over $S^4$ with different structure groups, they obtained some useful results that we in here mention some of their results. $U(n)$-gauge groups (see [2]); $PU(3)$-and $PSp(2)$-gauge groups (see [7]); The gauge groups of exceptional Lie groups (see [8]); $SO(4)$-gauge groups (see [13]); $SU(n)$-gauge groups (see [21]). Also, there are also several classification results for
gauge groups of principal bundles with base spaces other than $S^4$, such as the gauge groups of quaternionic line bundles over spheres (see [4]); The gauge groups over high-dimensional manifolds (see [5]); $SU(n)$-gauge groups over $S^{2m}$ (see [14]); $PU(5)$-and $PU(3)$-gauge groups over $S^4$ and $S^6$, respectively (see [19]).

In [12] Kishimoto and Kono classify the homotopy types of $Sp(n)$ gauge groups over $S^4$, by showing that if there is a homotopy equivalence $G_{k,1}(Sp(n)) \simeq G_{k',1}(Sp(n))$ then $(4n(2n+1),k) = (4n(2n+1),k')$. In this paper, in general case, we study the classification of the homotopy types of the gauge groups of principal $Sp(n)$-bundles over $S^{4m}$ and we give only lower bounds for the number of homotopy types and does not prove the converse, which is very hard and realistically out of reach. Also, in special cases $Sp(3)$-gauge groups over $S^8$ and $Sp(4)$-gauge groups over $S^{12}$ we obtain an upper bound for the number of homotopy types of $G_{k,2}(Sp(3))$ and $G_{k,3}(Sp(4))$, respectively. Put $t = (2n+1)2n(2n−1)\cdots(2n−2m+2)$. Let $P_{n,k}$ be the principal $Sp(n)$-bundles over $S^{4m}$ and $G_{k,m}(Sp(n))$ be the gauge group of $P_{n,k}$ classified by $k\varepsilon'$, where $\varepsilon'$ is a generator of $\pi_{4m}(B(\text{Sp}(n))) \simeq \mathbb{Z}$. We will prove the following theorems.

**Theorem 1.1.** If $G_{k,m}(Sp(n))$ is homotopy equivalent to $G_{k',m}(Sp(n))$ then the following hold:

**Case one:** if $m$ is even integer, then

$$(k,4t) = (k',4t) \quad \text{if } n \text{ is even},$$

$$(k,t) = (k',t) \quad \text{if } n \text{ is odd}.$$  

**Case two:** if $m$ is odd integer, then $(k,2t) = (k',2t)$.

**Theorem 1.2.** Localized at odd primes, the following hold:

(a): if $(3^3 \cdot 5 \cdot 7,k) = (3^3 \cdot 5 \cdot 7,k')$ then $G_{k,2}(Sp(3))$ is homotopy equivalent to $G_{k',2}(Sp(3))$.

(b): if $(3^4 \cdot 5^2 \cdot 7 \cdot 11,k) = (3^4 \cdot 5^2 \cdot 7 \cdot 11,k')$ then $G_{k,3}(Sp(4))$ is homotopy equivalent to $G_{k',3}(Sp(4))$.

## 2 Preliminaries

Let $Map_k(S^{4m},BSp(n))$ be the component of the space of continuous unbased maps from $S^{4m}$ to $BSp(n)$ which contains the map inducing $P$, similarly let $Map_k(S^{4m},BSp(n))$ be the spaces of pointed continuous maps from $S^{4m}$ and $BSp(n)$ which contains the map inducing $P$. We know that there is a fibration

$$Map_k(S^{4m},BSp(n)) \rightarrow Map_k(S^{4m},BSp(n)) \xrightarrow{ev} BSp(n),$$

where the map $ev$ is evaluation map at the basepoint of $S^{4m}$. Let $BG_{k,m}(Sp(n))$ be the classifying spaces of $G_{k,m}(Sp(n))$. By Atiyah-Bott [1], there is a homotopy equivalence $BG_{k,m}(Sp(n)) \simeq Map_k(S^{4m},BSp(n))$.

The evaluation fibration therefore determines a homotopy fibration sequence

$$Sp(n) \xrightarrow{\alpha} Map_k(S^{4m},BSp(n)) \rightarrow BG_{k,m}(Sp(n)) \xrightarrow{ev} BSp(n). \quad (2.1)$$
It is well known that there is a homotopy equivalence

\[ \text{Map}_0^*(S^{4m}, BSp(n)) \simeq \text{Map}_0^*(S^{4m}, BSp(n)), \]

we writing \( \Omega_0^{4m-1}Sp(n) \) for \( \text{Map}_0^*(S^{4m}, BSp(n)) \). Let \( G \) be the topological group and let \( c: G \times G \to G \) denote the commutator of \( G \). The Samelson product of two maps \( \alpha: X \to G \) and \( \beta: Y \to G \) denoted by \( \langle \alpha, \beta \rangle \) and defined to be the composition \( \langle \alpha, \beta \rangle: X \wedge Y \xrightarrow{\alpha \wedge \beta} G \times G \xrightarrow{c} G. \) Let \( \varepsilon_{m,n}: S^{4m-1} \to Sp(n) \) be the represent the generator of \( \pi_{4m-1}(Sp(n)) \). For an \( H \)-space \( X \), let \( k: X \to X \) be the \( k \)-th power map. By [14] we have the following lemma.

**Lemma 2.1.** The adjoint of the connecting map \( Sp(n) \xrightarrow{\alpha_k} \Omega_0^{4m-1}Sp(n) \) is homotopic to the Samelson product \( S^{4m-1} \wedge Sp(n) \xrightarrow{(k \varepsilon_{m,n}, 1)} Sp(n) \), where \( 1: Sp(n) \to Sp(n) \) is the identity map on \( Sp(n) \). \( \Box \)

The linearity of the Samelson product implies that \( \langle k\varepsilon_{m,n}, 1 \rangle \simeq k \langle \varepsilon_{m,n}, 1 \rangle \). Thaking adjoints therefore implies the following.

**Corollary 2.2.** The connecting map \( \alpha_k \) satisfies \( \alpha_k \simeq k \circ \alpha_1 \). \( \Box \)

The organization of this article is as follows. In Section 3 we will study the group \( [\Sigma^{4n-5}Q_2, Sp(n)] \). In Section 4 we will study the group \( [\Sigma^{4n-4(m+1)}Q_2, B\mathcal{G}_{k,m}(Sp(n))] \) and will prove Theorem 1.1. Also, in Section 5 we will prove Theorem 1.2.

### 3 The group \([\Sigma^{4n-5}Q_2, Sp(n)]\)

Our main goal in this section is to study the group \([\Sigma^{4n-5}Q_2, Sp(n)]\). We denote \( Sp(\infty)/Sp(n) \) by \( X_n \) and \([X, Sp(n)]\) by \( Sp_n(X) \). We recall that the symplectic quasi projective space \( Q_2 \) has the cellular structure \( Q_2 = S^3 \cup_{v_1} e^7 \), where \( v_1 \in \pi_0(S^3) \cong \mathbb{Z}_{12} \). Recall that the cohomologies of \( Sp(n) \), \( Sp(\infty) \) and \( BSp(\infty) \) as an algebra are given by

\[
\begin{align*}
H^*(Sp(n); \mathbb{Z}) &= \bigwedge \langle y_3, y_7, \cdots, y_{4n-1} \rangle, \\
H^*(Sp(\infty); \mathbb{Z}) &= \bigwedge \langle y_3, y_7, \cdots \rangle, \\
H^*(BSp(\infty); \mathbb{Z}) &= \mathbb{Z}[q_1, q_2, \cdots],
\end{align*}
\]

where \( y_{4i-1} = \sigma q_i \), \( \sigma \) is the cohomology suspension and \( q_i \) is the \( i \)-th universal symplectic Pontrjagin class. Consider the projection map \( \pi: Sp(\infty) \to X_n \), also the cohomologies of \( X_n \) and \( \Omega X_n \) as an algebra are given by

\[
\begin{align*}
H^*(X_n; \mathbb{Z}) &= \bigwedge \langle \bar{y}_{4n+3}, \bar{y}_{4n+7}, \cdots \rangle, \\
H^*(\Omega X_n; \mathbb{Z}) &= \mathbb{Z}\{b_{4n+2}, b_{4n+6}, \cdots, b_{8n+2}\} \quad (* \leq 8n + 2),
\end{align*}
\]

where \( \pi^*(\bar{y}_{4i+3}) = y_{4i+3} \) and \( b_{4n+4j-2} = \sigma(\bar{y}_{4n+4j-1}) \). Put \( X = \Sigma^{4n-5}Q_2 \). Note that \( X \) has a cellular structure as \( X \simeq S^{4n-2} \cup e^{4n+2} \). Consider the following fibre sequence

\[
\Omega Sp(\infty) \xrightarrow{\Omega \pi} \Omega X_n \xrightarrow{\delta} Sp(n) \xrightarrow{j} Sp(\infty) \xrightarrow{\pi} X_n.
\]

(3.1)
Now applying the functor \([X, ]\) to fibration sequence (3.1), there is an exact sequence
\[
[X, \Omega Sp(\infty)] \xrightarrow{(\Omega \eta)} [X, \Omega X_n] \xrightarrow{\delta} Sp_n(X) \xrightarrow{j} [X, Sp(\infty)] \xrightarrow{\pi_n} [X, X_n].
\]
Note that \(X_n\) has a cellular structure as following
\[
X_n \simeq S^{4n+3} \cup \eta' e^{4n+7} \cup e^{4n+11} \cup \ldots,
\]
where \(\eta'\) is the generator of \(\pi_{4n+6}(S^{4n+3})\). Also, we have
\[
\Omega X_n \simeq S^{4n+2} \cup e^{4n+6} \cup e^{4n+10} \cup \ldots.
\]
According to the CW-structure of \(X_n\) we have the following isomorphisms
\[
\pi_i(X_n) = 0 \quad (\text{for} \quad i \leq 4n + 2), \quad \pi_{4n+3}(X_n) \cong \mathbb{Z}.
\]
We use the isomorphism \([\Sigma^i \mathbb{Z}, BS\!p(\infty)] \cong \widehat{KS\!p}(\mathbb{Z})\). Observe that
\[
[\Sigma^{4n-5} Q_2, Sp(\infty)] \cong [\Sigma^{4n-4} Q_2, BS\!p(\infty)] \cong \widehat{KS\!p}(\Sigma^{4n-5} Q_2).
\]
Consider the cofibration sequence
\[
S^{4n-2} \to \Sigma^{4n-5} Q_2 \to S^{4n+2}. \quad (\ast)
\]
Applying \(\widehat{KS\!p}(\Sigma^{4n-5} Q_2)\) to the cofibration sequence (\(\ast\)), we know that \(\widehat{KS\!p}(S^{4n+2}) \cong 0\) for \(i \geq 1\). Thus we can conclude that \(\widehat{KS\!p}(\Sigma^{4n-5} Q_2) \cong 0\).

On the other hand we know that \(\Omega X_n\) is \((4n+1)\)-connected and \(H^{4n+2}(\Omega X_n) \cong \mathbb{Z}\) which is generated by \(b_{4n+2} = \sigma(\bar{y}_{4n+3})\). The map \(b_{4n+2}: \Omega X_n \to K(\mathbb{Z}, 4n + 2)\) is a loop map and is a \((4n+3)\)-equivalence. Since \(\dim \bar{X} \leq 4n + 2\), the map \((b_{4n+2}): [X, \Omega X_n] \to H^{4n+2}(X)\) is an isomorphism of groups. Thus we get the following exact sequence
\[
\widehat{KS\!p}(\Sigma^{4n-5} Q_2) \xrightarrow{\vartheta} H^{4n+2}(X) \to Sp_n(X) \to 0.
\]
Therefore we get the following lemma.

**Lemma 3.1.** There is an isomorphism \(Sp_n(X) \cong \text{Coker}\vartheta\). \(\square\)

In what follows this section, we will calculate the image of map \(\vartheta\).

Let \(Y\) be a CW-complex with \(\dim Y \leq 4n + 2\), we denote \([Y, U(2n + 1)]\) by \(U_{2n+1}(Y)\). By Theorem 1.1 in [6], there is an exact sequence
\[
\hat{K}^{-2}(Y) \xrightarrow{\varphi} H^{4n+2}(Y) \to U_{2n+1}(Y) \to \hat{K}^{-1}(Y) \to 0,
\]
for any \(f \in \hat{K}^{-2}(Y)\) the map \(\varphi\) is define as \(\varphi(f) = (2n + 1)!ch_{2n+1}(f)\), where \(ch_{4n+2}(f)\) is the \((4n+2)\)-th part of \(ch(f)\). Also, we use the isomorphism \(\hat{K}^{-i}(Y) \cong [\Sigma^i Y, BU(\infty)]\). In
this paper, we denote both the canonical inclusion $Sp(n) \hookrightarrow U(2n)$ and the induced map $\widetilde{K}Sp^*(X) \to \tilde{K}^*(X)$ by $c'$. By Theorem 1.3 in [13], there is a following commutative diagram

$$
\begin{array}{ccc}
\tilde{K}Sp^{-2}(\Sigma^{4n-5}Q_2) & \xrightarrow{c'} & H^{4n+2}(\Sigma^{4n-5}Q_2) \\
\downarrow \phi & & \downarrow (-1)^{n+1} \\
\tilde{K}^{-2}(\Sigma^{4n-5}Q_2) & \xrightarrow{\phi} & H^{4n+2}(\Sigma^{4n-5}Q_2)
\end{array}
$$

(3.2)

Therefore to calculate the image of $\vartheta$, we first calculate the image of $\phi$. We denote the free abelian group with a basis $e_1, e_2, \ldots$, by $\mathbb{Z}\{e_1, e_2, \ldots\}$. We need the following lemma.

**Lemma 3.2.** The following hold:

(a): $\widetilde{K}Sp^{-2}(\Sigma^{4n-5}Q_2) = \mathbb{Z}\{\alpha, \beta\}$, where $\alpha \in \widetilde{K}Sp^{-2}(S^{4n-2})$ and $\beta \in \widetilde{K}Sp^{-2}(S^{4n+2})$,

(b): $\tilde{K}^{-2}(\Sigma^{4n-5}Q_2) = \mathbb{Z}\{\alpha', \beta'\}$, where $\alpha' \in \tilde{K}^{-2}(S^{4n-2})$ and $\beta' \in \tilde{K}^{-2}(S^{4n+2})$,

(c): we have

$c'(\alpha) = \begin{cases} 
2\alpha' & n \text{ is even}, \\
\alpha' & n \text{ is odd}.
\end{cases}$

$c'(\beta) = \begin{cases} 
\beta' & n \text{ is even}, \\
2\beta' & n \text{ is odd}.
\end{cases}$

**Proof.** Put $L = \widetilde{K}Sp^{-2}(S^{4n-1})$ and $L' = \widetilde{K}Sp^{-2}(S^{4n+1})$. The cofibration sequence (*) induces the following commutative diagram of exact sequences

$$
\begin{array}{cccc}
L & \longrightarrow & \widetilde{K}Sp^{-2}(S^{4n+2}) & \longrightarrow & \widetilde{K}Sp^{-2}(\Sigma^{4n-5}Q_2) & \longrightarrow & \widetilde{K}Sp^{-2}(S^{4n-2}) & \longrightarrow & L' \\
& & \downarrow c' & & \downarrow c' & & \downarrow c' & & \\
& & 0 & \longrightarrow & \tilde{K}^{-2}(S^{4n+2}) & \longrightarrow & \tilde{K}^{-2}(\Sigma^{4n-5}Q_2) & \longrightarrow & \tilde{K}^{-2}(S^{4n-2}) & \longrightarrow & 0,
\end{array}
$$

where $L'$ is zero and $L$ is $\mathbb{Z}/2\mathbb{Z}$ when $n$ is even and is zero when $n$ is odd. Since $\widetilde{K}Sp^{-2}(S^{4i+2})$ is isomorphic to $\mathbb{Z}$, thus we have

$$
\widetilde{K}Sp^{-2}(\Sigma^{4n-5}Q_2) = \mathbb{Z}\{\alpha, \beta\}, \quad \tilde{K}^{-2}(\Sigma^{4n-5}Q_2) = \mathbb{Z}\{\alpha', \beta'\},
$$

where $\alpha \in \widetilde{K}Sp^{-2}(S^{4n-2})$, $\beta \in \widetilde{K}Sp^{-2}(S^{4n+2})$, also $\alpha' \in \tilde{K}^{-2}(S^{4n-2})$ and $\beta' \in \tilde{K}^{-2}(S^{4n+2})$. We can choose $\alpha, \alpha', \beta, \beta'$ such that when $n$ is even then $c'(\alpha) = 2\alpha'$, $c'(\beta) = \beta'$ and when $n$ is odd then $c'(\alpha) = \alpha'$, $c'(\beta) = 2\beta'$.

Consider the map $c': Sp(n) \to SU(2n)$, restriction the map $c'$ to their quasi-projective spaces we get to map $c': Q_2 \to \Sigma\mathbb{C}P^3$. The cohomologies of $Q_2$ and $\Sigma\mathbb{C}P^3$ are given by

$$
H^*(Q_2) = \mathbb{Z}\{\bar{y}_3, \bar{y}_7\}, \quad H^*(\Sigma\mathbb{C}P^3) = \mathbb{Z}\{\bar{x}_3, \bar{x}_5, \bar{x}_7\},
$$

such that $c'(\bar{x}_3) = \bar{y}_3$, $c'(\bar{x}_5) = 0$ and $c'(\bar{x}_7) = \bar{y}_7$. Note that $\tilde{K}^{-2}(\Sigma^{4n-4}\mathbb{C}P^3) \cong \tilde{K}^0(\Sigma^{4n-4}\mathbb{C}P^3)$ is a free abelian group generated by $\zeta_{2n-1} \otimes x$, $\zeta_{2n-1} \otimes x^2$ and $\zeta_{2n-1} \otimes x^3$. According to the map $c': \tilde{K}^{-2}(\Sigma^{4n-4}\mathbb{C}P^3) \to \tilde{K}^{-2}(\Sigma^{4n-5}Q_2)$, we can put $\alpha', \beta'$ such that $\alpha' = c'(\zeta_{2n-2} \otimes x)$ and $\beta' = c'(\zeta_{2n-2} \otimes x^3)$. We have the following proposition.
Proposition 3.3. The image of map $\tilde{K}\Sigma^{n-2}(\Sigma^{4n-5}Q_2) \xrightarrow{\phi} H^{4n+2}(\Sigma^{4n-5}Q_2)$ is isomorphic to

$$\begin{cases} \mathbb{Z}\{\frac{1}{3}(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7}\} & \text{if } n \text{ is even ,} \\ \mathbb{Z}\{\frac{1}{6}(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7}\} & \text{if } n \text{ is odd .} \end{cases}$$

Proof. Consider the following commutative diagram

$$\begin{CD} \tilde{K}^{-2}(\Sigma^{4n-4}CP^3) @>{\phi'}>> H^{4n+2}(\Sigma^{4n-4}CP^3) \\
@V{c'}VV \cong V @V{\phi}VV \tilde{K}^{-2}(\Sigma^{4n-5}Q_2) \longrightarrow H^{4n+2}(\Sigma^{4n-5}Q_2), \end{CD} \tag{3.3}$$

where the map $\phi'$ is defined similarly to the map $\phi$. By definition of the map of $\phi'$ we have

$$\begin{align*}
\phi'(\zeta_{2n-2} \otimes x) &= (2n+1)!ch_{2n+1}(\zeta_{2n-2} \otimes x) = \frac{1}{3!}(2n+1)!\sigma^{4n-4}t^3, \\
\phi'(\zeta_{2n-2} \otimes x^2) &= (2n+1)!ch_{2n+1}(\zeta_{2n-2} \otimes x^2) = \frac{1}{2}(2n+1)!\sigma^{4n-4}t^3, \\
\phi'(\zeta_{2n-2} \otimes x^3) &= (2n+1)!ch_{2n+1}(\zeta_{2n-2} \otimes x^3) = 2(2n+1)!\sigma^{4n-5}t^3.
\end{align*}$$

Now according to the commutativity of diagram (3.3) we have

$$\begin{align*}
\vartheta(\alpha) &= \vartheta(c'(\zeta_{2n-2} \otimes x)) = \vartheta'(\zeta_{2n-2} \otimes x) = \frac{1}{3!}(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7}; \\
\vartheta(\beta) &= \vartheta(c'(\zeta_{2n-2} \otimes x^3)) = \vartheta'(\zeta_{2n-2} \otimes x^3) = 2(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7}.
\end{align*}$$

Thus by the commutativity of diagram (3.2) we obtain the following two cases

$$\vartheta(\alpha) = \vartheta(c'(\alpha)) = \begin{cases} -\frac{1}{3}(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7} & \text{if } n \text{ is even ,} \\
\frac{1}{6}(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7} & \text{if } n \text{ is odd ,} \end{cases}$$

and

$$\vartheta(\beta) = \vartheta(c'(\beta)) = \begin{cases} -2(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7} & \text{if } n \text{ is even ,} \\
4(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7} & \text{if } n \text{ is odd .} \end{cases}$$

Therefore we can conclude that

$$\text{Im}\vartheta \cong \begin{cases} \mathbb{Z}\{\frac{1}{3}(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7}\} & \text{if } n \text{ is even ,} \\
\mathbb{Z}\{\frac{1}{6}(2n+1)!\sigma^{4n-5} \otimes \bar{y}_{7}\} & \text{if } n \text{ is odd .} \end{cases}$$

Thus by Lemma 3.1 and Proposition 3.3 we get the following theorem.

Theorem 3.4. There is an isomorphism

$$[\Sigma^{4n-5}Q_2, Sp(n)] \cong \begin{cases} \mathbb{Z}\frac{1}{3!}(2n+1)! & \text{if } n \text{ is even ,} \\
\mathbb{Z}\frac{1}{6!}(2n+1)! & \text{if } n \text{ is odd .} \end{cases} \quad \square$$
4 Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Put \( A = \Sigma^{4n-4m-5} Q_2 \) and \( X' = S^{4m} \wedge A \). First, we need the following lemmas.

**Lemma 4.1.** Let \( \mu : \Sigma A \to Sp(n) \) be an element of \( \overline{KSp}(\Sigma^2 A) \) and \( \mu' : S^{4m} \wedge A \to Sp(n) \) be the adjoint of the following composition

\[
S^{4m} \wedge \Sigma A \xrightarrow{\Sigma \varepsilon_{m,n} \wedge \mu} \Sigma Sp(n) \wedge Sp(n) \xrightarrow{[ev, ev]} BSp(n).
\]

Then there is a lift \( \tilde{\mu} : S^{4m} \wedge A \to \Omega X_n \) of \( \mu' \) such that \( \tilde{\mu}^*(a_{4n+2}) = \sigma^{4m} \otimes \Sigma^{-1} \mu^*(y_{4(n-m+1)-1}) \).

**Proof.** By Lemma 3.1 in [13], we know that there is a lift \( \lambda : \Sigma Sp(n) \wedge Sp(n) \to X_n \) of \([ev, ev]\) such that \( \lambda^*(\overline{\bar{\gamma}_{4n+3}}) = \sum_{i+j=n+1} \gamma_{4i-1} \otimes \gamma_{4j-1} \). Let \( \check{\lambda} \) be the composition \( \check{\lambda} : S^{4m} \wedge \Sigma A \xrightarrow{\Sigma \varepsilon_{m,n} \wedge \mu} \Sigma Sp(n) \wedge Sp(n) \xrightarrow{\lambda} X_n \). We have

\[
\check{\lambda}^*(\overline{\bar{\gamma}_{4n+3}}) = (\Sigma \varepsilon_{m,n} \wedge \mu)^* \lambda^*(\overline{\bar{\gamma}_{4n+3}}) = (\Sigma \varepsilon_{m,n} \wedge \mu)^* \left( \sum_{i+j=n+1} \gamma_{4i-1} \otimes \gamma_{4j-1} \right) = \sigma^{4m} \otimes \mu^*(y_{4(n-m+1)-1}).
\]

Now we take \( \check{\mu} : S^{4m} \wedge A \to \Omega X_n \) to be the adjoint of the following composition

\[
\Sigma S^{4m} \wedge A \xrightarrow{S} S^{4m} \wedge \Sigma A \xrightarrow{\check{\lambda}} X_n,
\]

that is \( \check{\mu} : ad(\check{\lambda} \circ S) \), where the map \( S : \Sigma S^{4m} \wedge A \to S^{4m} \wedge \Sigma A \) is the swapping map and the map \( ad : [\Sigma S^{4m} \wedge A, \Omega X_n] \to [S^{4m} \wedge A, \Omega X_n] \) is the adjunction. Then \( \check{\mu} \) is a lift of \( \mu \). Note that

\[
(\check{\lambda} \circ S)^*(\overline{\bar{\gamma}_{4n+3}}) = S^* \circ \check{\lambda}^*(\overline{\bar{\gamma}_{4n+3}}) = S^*(\sigma^{4m} \otimes \mu^*(y_{4(n-m+1)-1})) = \sigma^{4m+1} \otimes \Sigma^{-1} \mu^*(y_{4(n-m+1)-1}),
\]

thus we get \( \check{\mu}^*(a_{4n+2}) = \sigma^{4m} \otimes \Sigma^{-1} \mu^*(y_{4(n-m+1)-1}) \).

Apply the functor \([\Sigma^{4n-4(m+1)} Q_2, \ ]\) to fibration sequence (2.1), we get the following exact sequence

\[
[\Sigma^{4n-4(m+1)} Q_2, Sp(n)] \xrightarrow{(\alpha_{k})} [\Sigma^{4n-5} Q_2, Sp(n)] \to [\Sigma^{4n-4(m+1)} Q_2, B\mathcal{G}_{k,m}(Sp(n))] \to [\Sigma^{4n-4(m+1)} Q_2, BSp(n)]. \quad (**)
\]

In the following lemma, we show that \([\Sigma^{4n-4(m+1)} Q_2, BSp(n)]\) is isomorphic to zero.

**Lemma 4.2.** There is an isomorphism \([\Sigma^{4n-4(m+1)} Q_2, BSp(n)] \cong 0\).
Proof. Since the dimension of $\Sigma^{4n-4(m+1)} Q_2$ is equal to $4(n-m)+3$, we have
\[
[\Sigma^{4n-4(m+1)} Q_2, BSp(n)] \cong [\Sigma^{4n-4(m+1)} Q_2, BSp(\infty)] \cong \widetilde{K}Sp(\Sigma^{4n-4(m+1)} Q_2).
\]
The cofibration sequence $S^{4(n-m)-1} \to \Sigma^{4n-4(m+1)} Q_2 \to S^{4n-4(m+1)+3}$ induces the following exact sequence
\[
\cdots \to \widetilde{K}Sp(S^{4(n-m)+3}) \to \widetilde{K}Sp(\Sigma^{4n-4(m+1)} Q_2) \to \widetilde{K}Sp(S^{4(n-m)-1}) \to \cdots.
\]
Since $\widetilde{K}Sp(S^{4i-1}) \cong 0$ for $i \geq 1$, we conclude $\widetilde{K}Sp(\Sigma^{4n-4(m+1)} Q_2) \cong 0$.

Therefore by Lemma 4.3 we get the following lemma.

**Lemma 4.3.** There is an isomorphism $[\Sigma^{4n-4(m+1)} Q_2, BG_{k,m}(Sp(n))] \cong \text{Coker}(\alpha_k)_*$. □

In what follows this section, we will compute the Im $(\alpha_k)_*$ and then will prove Theorem 1.1. Consider the map $\theta : \Sigma A \to Sp(n)$, then by Lemma 4.1 there is a lift $\tilde{\theta}$ of $(\alpha_k)_*(\theta)$ such that we have
\[
\tilde{\theta}^*(a_{4n+2}) = \sigma^{4n} \otimes \Sigma^{-1} \theta^*(y_{4(n-m+1)-1}).
\]
We define the map $\beta_k : \widetilde{K}Sp(\Sigma^2 A) \to H^{4n+2}(X')$ by $\beta_k(\theta) = \tilde{\theta}^*(a_{4n+2}) = a_{4n+2} \circ \tilde{\theta}$. Now, Consider the following commutative diagram
\[
\begin{array}{ccc}
[\Sigma^{4n-4(m+1)} Q_2, G_k] & \longrightarrow & [\Sigma^{4n-4(m+1)} Q_2, Sp(n)] \\
\beta_k & \longrightarrow & \widetilde{K}Sp(\Sigma^{4n-5} Q_2) \rightarrow H^{4n+2}(\Sigma^{4n-5} Q_2) \\
& & \longrightarrow [\Sigma^{4n-5} Q_2, Sp(n)] \longrightarrow 0,
\end{array}
\]
where $[\Sigma^{4n-4(m+1)} Q_2, Sp(n)] \cong \widetilde{K}Sp(\Sigma^2 A)$. We have $\text{Im}(\alpha_k)_* = \text{Im}\beta_k / \text{Im}\tilde{\theta}$. Note that in the Proposition 3.3 we calculated the $\text{Im}\tilde{\theta}$, thus we need to calculate $\text{Im}\beta_k$.

Let $a : \Sigma Q_2 \to BSp(\infty)$ be the adjoint of the composition of the inclusions $Q_2 \to Sp(2) \to Sp(\infty)$ and also $b : \Sigma Q_2 \to BSp(\infty)$ be the pinch map of the bottom cell $q : \Sigma Q_2 \to S^8$ followed by a generator of $\pi_8(BSp(\infty)) \cong \mathbb{Z}$. Note that $\widetilde{K}Sp(\Sigma Q_2)$ is a free abelian group with a basis $a, b$. Then we have
\[
ch(c'(a)) = \Sigma y_3 - \frac{1}{6} \Sigma y_7, \quad ch(c'(b)) = -2 \Sigma y_7.
\]

Let $\theta_1 = q(\zeta_{2(n+m+1)-4} c'(a)) \in \widetilde{K}Sp(\Sigma^{4(n-m+1)-7} Q_2)$, where $q : K \to KSp$ is the quaternionization. Also, let $\theta_2 : \Sigma^{4(n-m+1)-7} Q_2 \to BSp(\infty)$ be the composite of the pinch map to the top cell $\Sigma^{4(n-m+1)-7} Q_2 \to S^{4(n-m+1)}$ and a generator of $\pi_{4(n-m+1)}(BSp(\infty)) \cong \mathbb{Z}$. Then $\widetilde{K}Sp(\Sigma^2 A)$ is a free abelian group with a basis $\theta_1, \theta_2$. We have
\[
ch(c'(\theta_1)) = \begin{cases} 
\Sigma^{4(n-m+1)-7} y_3 - \frac{1}{6} \Sigma^{4(n-m+1)-7} y_7 & \text{if } n - m + 1 \text{ is even}, \\
2 \Sigma^{4(n-m+1)-7} y_3 + \frac{1}{3} \Sigma^{4(n-m+1)-7} y_7 & \text{if } n - m + 1 \text{ is odd},
\end{cases}
\]
and
\[
ch(c'(\theta_2)) = \begin{cases} 
-2 \Sigma^{4(n-m+1)-7} y_7 & \text{if } n - m + 1 \text{ is even}, \\
\Sigma^{4(n-m+1)-7} y_7 & \text{if } n - m + 1 \text{ is odd}.
\end{cases}
\]
Lemma 4.4. $\text{Im} \beta_k$ is isomorphic to
\[
\begin{cases}
\mathbb{Z} \{ \frac{1}{6}(2(n-m+1)-1)!k\sigma^{4n-5} \otimes y_7 \} & \text{if } n-m+1 \text{ is even,} \\
\mathbb{Z} \{ \frac{1}{12}(2(n-m+1)-1)!k\sigma^{4n-5} \otimes y_7 \} & \text{if } n-m+1 \text{ is odd.}
\end{cases}
\]

Proof. For $i = 1, 2$, by Lemma 4.1, $(\beta_k)_*(\theta_i)$ has a lift $\tilde{\theta}_{i,k} : S^{4m} \wedge A \rightarrow \Omega X_n$ such that
\[
\tilde{\theta}_{i,k}^*(a_{4n+2}) = k\sigma^{4m} \otimes \Sigma^{-1} \theta_i^*(y_{4(n-m+1)-1}).
\]
Since $y_{4i-1} = \sigma(q_i)$ and $q_i = c^*(c_2)$, so we get
\[
\tilde{\theta}_{i,k}^*(a_{4n+2}) = k\sigma^{4m} \otimes \Sigma^{-2} q_{(n-m+1)}(\theta_i) = k\sigma^{4m} \otimes \Sigma^{-2} c_{2(n-m+1)}(c'(\theta_i)).
\]
On the other hand we have
\[
c_{2(n-m+1)}(c'(\theta_1)) = \begin{cases}
\frac{1}{6}(2(n-m+1)-1)!\Sigma^{4(n-m+1)-7} y_7 & \text{if } n-m+1 \text{ is even,} \\
\frac{1}{3}(2(n-m+1)-1)!\Sigma^{4(n-m+1)-7} y_7 & \text{if } n-m+1 \text{ is odd,}
\end{cases}
\]
\[
c_{2(n-m+1)}(c'(\theta_2)) = \begin{cases}
2(2(n-m+1)-1)!\Sigma^{4(n-m+1)-7} y_7 & \text{if } n-m+1 \text{ is even,} \\
(2(n-m+1)-1)!\Sigma^{4(n-m+1)-7} y_7 & \text{if } n-m+1 \text{ is odd.}
\end{cases}
\]
Therefore we get
\[
\tilde{\theta}_{1,k}^*(a_{4n+2}) = \begin{cases}
k\sigma^{4m} \otimes \frac{1}{6}(2(n-m+1)-1)!\Sigma^{4(n-m+1)-9} y_7 & \text{if } n-m+1 \text{ is even,} \\
k\sigma^{4m} \otimes \frac{1}{3}(2(n-m+1)-1)!\Sigma^{4(n-m+1)-9} y_7 & \text{if } n-m+1 \text{ is odd,}
\end{cases}
\]
and
\[
\tilde{\theta}_{2,k}^*(a_{4n+2}) = \begin{cases}
k\sigma^{4m} \otimes 2(2(n-m+1)-1)!\Sigma^{4(n-m+1)-9} y_7 & \text{if } n-m+1 \text{ is even,} \\
k\sigma^{4m} \otimes (2(n-m+1)-1)!\Sigma^{4(n-m+1)-9} y_7 & \text{if } n-m+1 \text{ is odd.}
\end{cases}
\]
If $n-m+1$ is even, then $\text{Im} \beta_k \cong \mathbb{Z}\{\alpha_1, \beta_1\}$, where
\[
\alpha_1 = \frac{1}{6}(2(n-m+1)-1)!k\sigma^{4n-5} \otimes y_7,
\]
\[
\beta_1 = 2(2(n-m+1)-1)!k\sigma^{4n-5} \otimes y_7.
\]
Also, $\text{Im} \beta_k$ is generated by $2\alpha_1 - \frac{1}{12}\beta_1 = \frac{1}{6}(2(n-m+1)-1)!k\sigma^{4n-5} \otimes y_7$. If $n-m+1$ is odd, then $\text{Im} \beta_k \cong \mathbb{Z}\{\alpha_2, \beta_2\}$, where
\[
\alpha_2 = \frac{1}{3}(2(n-m+1)-1)!k\sigma^{4n-5} \otimes y_7,
\]
\[
\beta_2 = (2(n-m+1)-1)!k\sigma^{4n-5} \otimes y_7.
\]
Also, $\text{Im} \beta_k$ is generated by $\alpha_2 - \frac{1}{4} \beta_2 = \frac{1}{12} (2(n - m + 1) - 1) ! k \sigma^{4n-5} \otimes y_7$. Therefore we can conclude that

$$\text{Im} \beta_k \cong \begin{cases} 
\mathbb{Z} \left\{ \frac{1}{6} (2(n - m + 1) - 1) ! k \sigma^{4n-5} \otimes y_7 \right\} & \text{if } n - m + 1 \text{ is even}, \\
\mathbb{Z} \left\{ \frac{1}{12} (2(n - m + 1) - 1) ! k \sigma^{4n-5} \otimes y_7 \right\} & \text{if } n - m + 1 \text{ is odd}.
\end{cases}$$

Therefore we can conclude that $\text{Im} \beta_k \sim \begin{cases} \mathbb{Z} \{ 1 \} & \text{if } n - m + 1 \text{ is even}, \\
\mathbb{Z} \{ \frac{1}{12} (2(n - m + 1) - 1) ! k \sigma^{4n-5} \otimes y_7 \} & \text{if } n - m + 1 \text{ is odd}. \end{cases}$

Therefore by Proposition 3.3 and Lemma 4.4, we obtain the following proposition.

**Proposition 4.5.** The following hold:

**Case one:** if $m$ is even integer then

$$\text{Im}(\alpha_k) \cong \begin{cases} 
\mathbb{Z} / ((2n + 1)! / (3(k, 4t))) & \text{if } n \text{ is even}, \\
\mathbb{Z} / ((2n + 1)! / (6(k, t))) & \text{if } n \text{ is odd}.
\end{cases}$$

**Case two:** if $m$ is odd integer then

$$\text{Im}(\alpha_k) \cong \begin{cases} 
\mathbb{Z} / ((2n + 1)! / (3(k, 2t))) & \text{if } n \text{ is even}, \\
\mathbb{Z} / ((2n + 1)! / (6(k, 2t))) & \text{if } n \text{ is odd}. \end{cases}$$

Therefore by Theorem 3.4, Lemma 4.3 and Proposition 4.5, we get the following theorem.

**Theorem 4.6.** There is an isomorphism

**Case one:** if $m$ is even integer then

$$[\Sigma^{4n-4(m+1)}Q_2, \mathcal{B}_k,m(Sp(n))] \cong \begin{cases} \mathbb{Z} (k, u) & \text{if } n \text{ is even}, \\
\mathbb{Z} (k, t) & \text{if } n \text{ is odd}. \end{cases}$$

**Case two:** if $m$ is odd integer then

$$[\Sigma^{4n-4(m+1)}Q_2, \mathcal{B}_k,m(Sp(n))] \cong \mathbb{Z} (k, 2t). \quad \square$$

Here we prove Theorem 1.1

**Proof of Theorem 1.1**

We write $|H|$ for the order of a group $H$. Consider the exact sequence (**). Suppose that $\mathcal{G}_k,m(Sp(n)) \simeq \mathcal{G}_{k,m}(Sp(n))$, then there is an isomorphism of groups

$$[\Sigma^{4n-4(m+1)}Q_2, \mathcal{B}_k,m(Sp(n))] \cong [\Sigma^{4n-4(m+1)}Q_2, \mathcal{G}_{k,m}(Sp(n))].$$

Thus we have

$$|[\Sigma^{4n-4(m+1)}Q_2, \mathcal{B}_k,m(Sp(n))]| = |[\Sigma^{4n-4(m+1)}Q_2, \mathcal{B}_k,m(Sp(n))]|.$$

Therefore by Theorem 4.6, we can conclude Theorem 1.1. \quad \square
5 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. First, we need to the following lemma that is very important in determining the upper bound of the number of homotopy types of gauge groups. Let $Y$ be an $H$-space with a homotopy inverse, and let $k : Y \to Y$ be the $k^{th}$-power map. By [20] we have the following lemma.

**Lemma 5.1.** Let $X$ be a space and $Y$ be an $H$-space with a homotopy inverse. Suppose there is a map $f : X \to Y$ of order $m$, where $m$ is finite. Let $F_k$ be the homotopy fiber of map $k \circ f$. If $(m, k) = (m, k')$ then $F_k$ and $F_{k'}$ are homotopy equivalent when localized rationally or at any prime.

Consider the cofibration sequence
\[ S^{4(n+m)−3} \to \Sigma^{4m−1}Q_{n−1} \to \Sigma^{4m−1}Q_n. \]
Since $\Sigma^{4m−1}Q_{n−1}$ is $(4m + 1)$-connected, by [22] the suspension map
\[ \Sigma^\infty : [S^{4(n+m)−3}, \Sigma^{4m−1}Q_{n−1}] \to \{S^{4(n+m)−3}, \Sigma^{4m−1}Q_{n−1}\} \]
is isomorphic for $n \leq m + 1$. On the other hand, by [10] we have $\Sigma^\infty(\theta) = 0$, so $\theta = 0$. Therefore we get the homotopy equivalent
\[ \Sigma^{4m−1}Q_n \simeq \Sigma^{4m−1}Q_{n−1} \lor S^{4(n+m)−2}. \quad (\star) \]

Here, localized at odd primes, we will study the order of Samelson products $S^{4m−1} \land Sp(n) \to Sp(n)$ for cases $m = 2, n = 3$ and $m = 3, n = 4$, respectively. Then we will obtain an upper bound to the number of homotopy types of $Sp(3)$-gauge group over $S^8$ and $Sp(4)$-gauge group over $S^{12}$, respectively.

**case m=2, n=3**

By relation $(\star)$, we have the isomorphism
\[ [\Sigma^7Q_3, Sp(3)] \cong [\Sigma^7Q_2 \lor S^{18}, Sp(3)] \cong [\Sigma^7Q_2, Sp(3)] \oplus \pi_{18}(Sp(3)). \]

By [16], we have $[\Sigma^7Q_2, Sp(3)] \cong \mathbb{Z}_{3,5,7}$ and also by [17], we have $\pi_{18}(Sp(3)) \cong \mathbb{Z}_{3,5,7}$. Thus the order of Samelson product $S^7 \land Q_3 \to Sp(3)$ is at most $3^3 \cdot 5 \cdot 7 = 945$. Therefore localized at odd primes, we can conclude that the order of Samelson product $S^7 \land Sp(3) \to Sp(3)$ is at most $3^3 \cdot 5 \cdot 7 = 945$. Therefore by Lemma 5.1 we get the following theorem.

**Theorem 5.2.** Localized at odd primes, if $(945, k) = (945, k')$ then $G_{k,2}(Sp(3))$ is homotopy equivalent to $G_{k',2}(Sp(3))$. \qed

**case m=3, n=4**

By relation $(\star)$, we have the isomorphism
\[ [\Sigma^{11}Q_4, Sp(4)] \cong [\Sigma^{11}Q_3 \lor S^{26}, Sp(4)] \cong [\Sigma^{11}Q_3, Sp(4)] \oplus \pi_{26}(Sp(4)), \]

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where by [9], we have $\pi_{26}(Sp(4)) \cong \mathbb{Z}_{3^4 \cdot 5^2 \cdot 7 \cdot 11}$. On the other hand, similarly, we have $[\Sigma^{11}Q_3, Sp(4)] \cong [\Sigma^{11}Q_2 \vee S^{22}, Sp(4)]$
\[
\cong [\Sigma^{11}Q_2, Sp(4)] \oplus \pi_{22}(Sp(4)).
\]
Now, by [10], we have $[\Sigma^{11}Q_2, Sp(4)] \cong \mathbb{Z}_{3^4 \cdot 5^2 \cdot 7 \cdot 11}$ and also by [11], we know that $\pi_{22}(Sp(4)) \cong \mathbb{Z}_{3^4 \cdot 5^2 \cdot 7 \cdot 11}$. Thus the order of Samelson product $S^{11} \wedge Q_4 \rightarrow Sp(4)$ is at most $3^4 \cdot 5^2 \cdot 7 \cdot 11$. Therefore localized at odd primes, we can conclude that the order of Samelson product $S^{11} \wedge Sp(4) \rightarrow Sp(4)$ is at most $3^4 \cdot 5^2 \cdot 7 \cdot 11$. Therefore by Lemma 5.1, we get the following theorem.

**Theorem 5.3.** Localized at odd primes, if $(3^4 \cdot 5^2 \cdot 7 \cdot 11, k) = (3^4 \cdot 5^2 \cdot 7 \cdot 11, k')$ then $G_{k,3}(Sp(4))$ is homotopy equivalent to $G_{k',3}(Sp(4))$. \(\square\)

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