STRATEGIC SENSORS AND ASYMPTOTIC REGIONAL BOUNDARY GRADIENT OBSERVATION

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Abstract: In this research paper an extension of asymptotical regional gradient observability concept to the case of boundary region of ∂Ω has been discussed and analyzed together with different types of strategic sensors. Our purpose is to show that, the number and location of sensor may be some interest in the existence of asymptotical regional gradient observation state in the boundary region for a parabolic system type.

Keywords: Γε-strategic sensor, Γε-asymptotical detectability, Γε-asymptotical observability.

1. Introduction

The controllability and observability are two important major concepts of modern control system theory. The observability measures means the ability of the particular sensor configuration to supply all necessary information to estimate or reconstruct all states of the system [1-2]. Notions of regional observability concepts are provided and developed by El Jai et al. [3-4] and extended to the case where the subregion is a part of the boundary of the system evolution domain in [5,6]. The concept of regional gradient observability was developed in many real world problem, where one is cares about the knowledge of the gradient state only in a critical subregion of the system domain without the knowledge of the state itself. This concept was introduced for parabolic systems but little has been done for hyperbolic by Zerrik et al., Al-Saphory et al. and Boutoulout et al. [7,8,9,10].

The concept of asymptotic regional analysis was submitted recently by Al-Saphory, El Jai and et al. in [11,12,13,14,15,16], consists in studying the behavior of systems not in all the domain Ω but only on particular regions ω or Γ of the domain. Our interest in this study, is to establish some results on asymptotically regional boundary gradient observability in connection with sensors structure, their numbers, and locations on a subregion Γ of the boundary ∂Ω. The established results are also applied to the case of diffusion systems in two dimensions. The important reasons behind the study is to bring the light on the link between the asymptotic regional boundary gradient observability and sensor structure (location and number (Figure 1)). This paper is organized as follows:
Section 2 focuses on the system considerations and the problem statement of asymptotic regional boundary gradient observability. In the third section, we will introduce and characterize regional boundary gradient strategic sensors. In section four, an approach introduced which allows the determination of regional boundary gradient state on $\Gamma_G$. In section five, the relation between asymptotic regional boundary gradient observability and strategic sensor and also, we show that a gradient state which is not asymptotically observable in the usual sense may be asymptotically observable on $\Gamma_G$. Finally, the last section illustrates applications to a two-dimensional diffusion process, in addition to various situations which will be examined.

**Figure 1:** The domain of $\Omega$, sub regions $\omega$ and $\Gamma_G$, various sensors locations

2. Asymptotic Regional Boundary Gradient Observability

In this subsection we give firstly, the statement of the problem with the hypothesis of considered system, and then the concept of asymptotic regional gradient observability is explained, and we provide a theorem, which gives the approach observed the current gradient state $\nabla x(\xi, t)$ of the original system (1) asymptotically.

2.1 Problem Statement

The considered system is represented by the parabolic equations:

\[
\begin{align*}
\frac{\partial x}{\partial t}(\xi, t) &= Ax(\xi, t) + Bu(t) & Q \\
x(\xi, 0) &= x_0(\xi) & \Omega \\
x(\eta, t) &= 0 & \Sigma \\
\end{align*}
\]

(1)

Where $\Omega$ is the domain when the above system is defined as bounded open subset of $\mathbb{R}^n$ with boundary $\partial \Omega$, $[0, T]$ is the time interval for $T > 0$, $A$ is a self adjoint linear differential operator of order two with compact resolvent, and which generates a strongly continuous semi-group $(S_A(t))_{t \geq 0}$ on the state space $X = H^1(\Omega)$ which is Sobolev space of order one. The operators $B \in L(R^p, X)$ and $C \in L(X, R^q)$, where $p$ given number of actuators and $q$ given the number of sensors. The initial gradient state $\nabla x_0(\xi)$ is supposed to be unknown and located in $H^1(\Omega)$. The measurements of system (1) are obtained through internal or boundary zone or pointwise sensors which characterize the output function

\[
y(\cdot, t) = Cx(\xi, t)
\]

(2)

Under these hypotheses, the above system has a unique solution [1, 17]

\[
x(\xi, t) = S_A(t)x_0(\xi) + \int_0^t S_A(t - \tau)Bu(\tau)d\tau
\]

(3)

Now, we define the following operators
\[ K: x \in X \rightarrow Kx = CS_A(.)x \in \mathcal{O} \]
with an adjoint \( K^*: \mathcal{O} \rightarrow X \)
given by
\[ K^*y^* = \int_0^t S_A^*(s)Cy^*(s)ds \]
The operator \( \nabla \) denotes the gradient is given by
\[ \nabla \xi: H^1(\Omega) \rightarrow (H^1(\Omega))^n \]
and the adjoint of \( \nabla \) is given by
\[ \nabla^*: ((H^1(\Omega))^n) \rightarrow H^1(\Omega) \]
where \( \nu \) is a solution of the Dirichlet problem
\[ \{ \begin{align*}
\Delta \nu &= -div(x) & \text{in } \Omega \\
\nu &= 0 & \text{in } \partial \Omega
\end{align*} \]
The operator defined as \( \gamma^*_0: H^1(\Omega) \rightarrow H^{1/2}(\partial \Omega) \) is trace operator with order zero which is a linear, subjective and continuous. And, the extension of \( \gamma^*_0 \) denoted by \( \gamma \) defined as
\[ \gamma: (H^1(\Omega))^n \rightarrow (H^{1/2}(\partial \Omega))^n \]
and the adjoints are respectively given by \( \gamma^*_0, \gamma^* \).
For a sub-boundary region of \( \partial \Omega \) the function \( \chi_\Gamma \) defined by
\[ \{ \begin{align*}
\chi_\Gamma: H^{1/2}(\partial \Omega) &\rightarrow H^{1/2}(\Gamma) \\
x &\rightarrow \chi_\Gamma x = x |_\Gamma
\end{align*} \]
\( x |_\Gamma \) means the restriction of the gradient state \( \nabla x \) in a boundary subregion, and
\[ \chi_\Gamma: (H^{1/2}(\partial \Omega))^n \rightarrow (H^{1/2}(\Gamma))^n \]
Where the adjoints are respectively given by \( \chi^*_\Gamma, \chi^*_\Gamma \).
Finally introduce the operator \( H_{\Gamma_G} = \chi_\gamma \nabla K^* \) from \( \mathcal{O} \) into \((H^{1/2}(\Gamma))^n \) with adjoint \( H_{\Gamma_G}^* = K^* \gamma^* \chi^*_\Gamma \).

**Definition 2.1:** The autonomous system associated with system (1) and the output function (2) is called exactly regional boundary gradient observable (or exactly \( \Gamma_G \)-observable) if
\[ ImH_{\Gamma_G} = (H^{1/2}(\Gamma))^n \]

**Definition 2.2:** The autonomous system associated with system (1) and the output function (2) is called weakly regional boundary gradient observable (or weakly \( \Gamma_G \)-observable) if
\[ \text{Im} H_{\Gamma_G} = (H^{1/2}(\Gamma))^n \]

3. Regional Boundary Gradient Strategic Sensor

This subsection involve a bout a characterization of strategic sensor as in ref. [18] in order for the system to be weakly \( \Gamma_G \)-observable.

**Definition 2.3:** Sensors are any couple \((D_i, f_i)\) where \(D_i\) form closed subsets of \(\bar{\Omega}\), which are spatial supports of sensors and \(f_i \in L^2(D_i)\) define the spatial distribution of sensors on \(D_i\).

In case, when the measurements of system (1) are given by \(t\) sensors \((1 \leq i \leq q)\), then the output function (2) is given by;
\[
y(t, t) = y_i(t), \ldots, y_q(t)
\]
with
\[
y_i(t) = x(b_i, t), b_i \in \bar{\Omega} \text{ for } 1 \leq i \leq q
\]
in the pointwise case, and
\[
y_i(t) = \int_{D_i} x(\xi, t)f_i(\xi)d\xi, D_i \in \bar{\Omega} \text{ for } 1 \leq i \leq q
\]
in the zone case.

**Definition 2.4:** A sequence of sensors \((D_i, f_i)\) is called boundary gradient strategic boundary subregion \(\Gamma\) \((\Gamma_G\)-strategic) when the corresponding system is weakly \(\Gamma_G\)-observable.

Now, assume that there exist a complete set of eigenfunctions \((\varphi_{mj})_{m \in I, j = 1, \ldots, r_m}\) of \(A\) in state space \((H^1(\Omega))^n\) associated with the eigenvalues \(\lambda_m\) of multiplicities \(r_m\). Let \(r = \sup_{m \in I} r_m\) is finite and for \(x = (x_1, \ldots, x_n) \in \Omega\) with \(m = (m_1, \ldots, m_n) \in I\), suppose that \(\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{n-1})\) and \(\bar{m} = (m_1, \ldots, m_{n-1})\).

Suppose that the functions \(\psi_{mj}(\bar{x}) = \chi_{\bar{x} \Gamma} \varphi_{mj}(x)\) form a complete set in \((H^{1/2}(\Gamma))^n\).

Hence, we get the following result.

**Proposition 2.5.** The sensors \((D_i, f_i)\) where \(i = 1, \ldots, q\) is \(\Gamma_G\)-strategic iff:
1- \(q \geq r\)
2- \(\text{rank } G_m = r_m, \forall m, m = 1, \ldots, J\) with
\[
G_m = (G_m)_{ij} = \begin{cases} 
\langle \varphi_{mj}, f_i(x) \rangle_{L^2(D_i)} & \text{in the zone case} \\
\varphi_{mj}(b_i) & \text{in the pointwise case}
\end{cases}
\]
where \(\sup r_m = r\) and \(J = 1, \ldots, r_m\).

**Proof:** The steps of the proof is depend on the rank condition in [19,20,5], the main difference step is that the rank condition
\[
\text{rank } G_m = r_m, \forall m
\]
for the proposition 2.5 need only to hold for \(\text{rank } G_m = r_m, \forall m, m = 1, \ldots, J\).

4. Asymptotic Regional Boundary Gradient State Reconstruction Method

Our goal of this section is to give an approach way to determination of regional boundary gradient state on \(\Gamma_G\) based on internal gradient observability this approach is derived from the method (HUM) developed by Lions and generalized by El Jai and Zerrik and Al-Saphory in [5,6,21-23]. Thus, the characterization of asymptotic regional boundary gradient observability requirements some notions which are related to the asymptotic behavior that are stability, detectability, and observer. In this subsection we
define the concepts which are related to $\Gamma_{AG}$-observability, and we provide an important theorem which gives an asymptotic observer for the original system in critical subregion $\Gamma_G$.

The asymptotic regional boundary gradient observer in $\Gamma_G$ may be treat as internal asymptotic regional gradient observer in $\omega_G$ if we assume he following:

- Let $\mathcal{R}$ be an operator which is continuous linear extension [21]
  
  $\mathcal{R}: (H^{1/2}(\partial \Omega))^n \rightarrow (H^1(\Omega))^n$, such that
  
  $\gamma^* \mathcal{R} h(\xi, t) = h(\xi, t), \forall h(\xi, t) \in (H^{1/2}(\partial \Omega))^n$

Let $r > 0$ is an sufficiently small arbitrary optional real value and let the sets

$E = \bigcup_{t \in \Gamma_G} B(x, r)$ and $\bar{\omega}_r = E \cap \Omega$. Where $r$ is the radius of the ball $B(x, r)$ with centered in $x(\xi, r)$ and $\Gamma_G$ is a part of $\bar{\omega}_r$ (see Figure 2).

![Figure 2: $\Omega$ is The Domain, Subregion $\bar{\omega}_r$ and $\Gamma_G$ is the Region.](image)

**Definition 2.6:** The semi-group $(S_A(t))_{t \geq 0}$ is asymptotic regional boundary gradient stable in($(H^{1/2}(\Gamma))^n$ or ($\Gamma_{AG}$-stable) if, for some positive constants $F_{\Gamma G}$ and $\sigma_{\Gamma G}$, then

$$\|x(t)\|_{(H^{1/2}(\Gamma))^n} \leq F_{\Gamma G} e^{-\sigma_{\Gamma G} t}, \ t \geq 0$$

If $(S_A(t))_{t \geq 0}$ is $\Gamma_{AG}$-stable, then for every $\forall x_{\bar{\omega}}(\cdot) \in (H^1(\bar{\Omega}))^n$ the solution of autonomous system related with (1) satisfies

$$\lim_{t \to \infty} \|x(t)\|_{(H^{1/2}(\Gamma))^n} = \lim_{t \to \infty} \|x(t)\|_{(H^{1/2}(\Gamma))^n} = 0$$

**Definition 2.7:** The system (1) is asymptotic regional boundary gradient stable on $\Gamma_G$ (or $\Gamma_{AG}$-stable) if, the operator $A$ generates a strongly continuous semi-group $(S_A(t))_{t \geq 0}$ which is $\Gamma_{AG}$-stable.

**Remark 2.8:** If the system (1) is $\Gamma_{AG}$-stable, that is the solution of autonomous system related with (1) converges asymptotically to zero when $t$ tends to $\infty$.

**Definition 2.9:** The systems (1) - (2) are asymptotic regional boundary gradient detectable on $\Gamma_G$ (or $\Gamma_{AG}$-detectable) if there exist an operator $H_{\Gamma_{AG}}: R^q \rightarrow (H^{1/2}(\Gamma))^n$ in order that the operator $(A - H_{\Gamma_{AG}})$ generates a strongly continuous semigroup $(S_{H_{\Gamma_{AG}}}(t))_{t \geq 0}$ which is $\Gamma_{AG}$-stable.

Now, consider the dynamical system

$$\begin{align*}
\frac{\partial w}{\partial t}(\xi, t) &= L_{\Gamma_{AG}} w(\xi, t) + G_{\Gamma_{AG}} u(t) + H_{\Gamma_{AG}} y(t) \quad \Omega \times (0, \infty) \\
w(\xi, 0) &= w_0(\xi) \quad \bar{\Omega} \\
w(\eta, t) &= 0 \quad \partial \Omega \times (0, \infty)
\end{align*}$$

(4)
where $L_{rAg}$ generates a strongly continuous semi-group $(S_{tL_{rAg}}(t))_{t \geq 0}$ which is stable on the $W$, $G_{rAg} \in L(R^p,W)$ and $H_{rAg} \in L(W,R^q)$. The system (4) defines an $\Gamma_{Ag}$ observer for the original system.

Definition 2.10: The system (4) is asymptotic regional boundary gradient observer on $\Gamma_g$ if there exist a dynamical system (4) which is observable by the gradient state of original system.

Definition 2.11: The system (4) is asymptotic regional boundary gradient identity observer (or identity $\Gamma_{Ag}$-observer) for the systems (1)-(2) if $X = W$ and $\alpha_{rAg} = I_{rAg}$. In the case, we have

$$L_{rAg} = A - H_{rAg}C \quad \text{and} \quad G_{rAg} = B$$

and then, the dynamical system (4) becomes

$$\begin{align*}
\frac{d}{dt}(\xi, t) &= Aw(\xi, t) + Bu(t) + H_{rAg}(Cw(\xi, t) - y(\xi, t)) \\
w(\xi, 0) &= 0 \\
w(\eta, t) &= 0
\end{align*}$$

Definition 2.12: We say that the systems (1)-(2) are asymptotic regional boundary gradient observable on $\Gamma_g$ if there exist a dynamical system (4) which is $\Gamma_{Ag}$-observer for the original system.

In the following, we present an approach which is observed in the current state $x(\xi, t)$ of the original system (1) asymptotically.

5. $\Gamma_{Ag}$-Observability and Strategic Sensor

The problem of $\Gamma_{Ag}$-observability is consists of estimation of the current gradient state asymptotically in a given boundary gradient sub-region of the boundary $\partial \Omega$. This approach is introduced by the following main theorem.

Theorem 5.1: Assume that the sequence of sensors $(\mathcal{D}_i, f_i)_{i \in \mathbb{N}}$ for $i = 1, \ldots, q$ are $\Gamma_{Ag}$-strategic and the spectrum of $A$ includes $J$ eigenvalues with positive real parts. Then (1)-(2) are $\Gamma_{Ag}$-observable by the dynamical system

$$\begin{align*}
\frac{d\xi}{dt}(\xi, t) &= Aw(\xi, t) + Bu(t) + H_{rAg}(Cw(\xi, t) - y(\xi, t)) \\
w(\xi, 0) &= w_0(\xi) \\
w(\eta, t) &= 0
\end{align*}$$

Proof: The demonstration has two parts:

Part 1. Under the hypothesis of problem (Section 2.1), the system (1) can be decomposed by the projections $P$ and $I - P$ on two parts, unstable and stable. The state vector may be given by

$$x(t) = [x_1(t) + x_2(t)]^\mathcal{T}$$

For unstable part the state component $x_1(\xi, t)$ is of the system (1) and may be written in the form

$$\begin{align*}
\frac{dx_1}{dt}(\xi, t) &= A_1x_1(\xi, t) + P Bu(t) \\
x_1(\xi, 0) &= x_0(\xi) \\
x_1(\eta, t) &= 0
\end{align*}$$

where $\Gamma_g$ generates a strongly continuous semi-group $(S_{tL_{rAg}}(t))_{t \geq 0}$ which is stable on the $W$, $G_{rAg} \in L(R^p,W)$ and $H_{rAg} \in L(W,R^q)$. The system (4) defines an $\Gamma_{Ag}$ — estimator for $\alpha_{rAg}x(\xi, t)$ if

$$\begin{align*}
(1) \lim_{t \to \infty} \|w(\xi, t) - \alpha_{rAg}x(\xi, t)\|_{(H^{1/2}(\Omega))^n} = 0 \\
(2) \alpha_{rAg} - \chi_{\Omega} \nabla T \quad \text{and} \quad w(\xi, t) \text{ is the solution of system (4)}.
\end{align*}$$

Theorem 5.1: Assume that the sequence of sensors $(\mathcal{D}_i, f_i)_{i \in \mathbb{N}}$ for $i = 1, \ldots, q$ are $\Gamma_{Ag}$-strategic and the spectrum of $A$ includes $J$ eigenvalues with positive real parts. Then (1)-(2) are $\Gamma_{Ag}$-observable by the dynamical system

$$\begin{align*}
\frac{d\xi}{dt}(\xi, t) &= Aw(\xi, t) + Bu(t) + H_{rAg}(Cw(\xi, t) - y(\xi, t)) \\
w(\xi, 0) &= w_0(\xi) \\
w(\eta, t) &= 0
\end{align*}$$

Proof: The demonstration has two parts:

Part 1. Under the hypothesis of problem (Section 2.1), the system (1) can be decomposed by the projections $P$ and $I - P$ on two parts, unstable and stable. The state vector may be given by

$$x(t) = [x_1(t) + x_2(t)]^\mathcal{T}$$

For unstable part the state component $x_1(\xi, t)$ is of the system (1) and may be written in the form

$$\begin{align*}
\frac{dx_1}{dt}(\xi, t) &= A_1x_1(\xi, t) + P Bu(t) \\
x_1(\xi, 0) &= x_0(\xi) \\
x_1(\eta, t) &= 0
\end{align*}$$
and \( x_2(\xi, t) \) is the state component of the stable part of (1) given by

\[
\begin{align*}
\frac{\partial x_2}{\partial t} (\xi, t) &= A_2 x_2(\xi, t) + (I - P) Bu(t) \quad \Omega \times (0, \infty) \\
x_2(\xi, 0) &= x_2^0(\xi) \\
x_2(n, t) &= 0 \quad \partial \Omega \times (0, \infty)
\end{align*}
\]  
(12)

The operator \( A_2 \) is represented by matrix of order \((\sum /g1870/g301 /g310 /g301 /g2880/g286 ->, \sum /g1870/g301 /g310 /g301 /g2880/g286 ->)\) is given by

\[
A_2 = diag \{ \lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \lambda_j, \ldots, \lambda_j \},
\]

\[ PB = [G_{1r}, G_{2r}, \ldots, G_{jr}] \]  
(13)

Part 2. For unstable part of the system (1) the subsystem (12) is weakly \( \Gamma_{AG} \)-observable [21] because the sensors \((D_i, f_i)_{\text{sys}}\) are \( \Gamma_{AG} \)-strategic as long it is finite dimensional, then it is exactly \( \Gamma_{AG} \)-observable hence, it is \( \Gamma_{AG} \)-detectable [24], theses, there exists an operator \( H_{1AG}^2 \) in order that \( (A_1 - H_{1AG}^2 C) \), satisfies the following:

\[
\exists F_{\Gamma_{AG}}^1 \text{ and } \sigma_{\Gamma_{AG}}^2 > 0 \text{ such that } \| e^{(A_1 - H_{1AG}^2 C)t} \| \leq F_{\Gamma_{AG}}^1 e^{-\sigma_{\Gamma_{AG}}^2 t} \text{ and, then, we have }
\]

\[
\| x_1(\xi, t) \|_{(H^{1/2}(\Gamma))^n} \leq F_{\Gamma_{AG}}^1 e^{-\sigma_{\Gamma_{AG}}^2 t} \| P x_0^1 \|_{(H^{1/2}(\Gamma))^n}
\]  
(14)

Now, since \( A_2 \) generates semigroup which is \( \Gamma_{AG} \)-stable, then, there is appositive constants \( F_{\Gamma_{AG}}^2 \) and \( \sigma_{\Gamma_{AG}}^2 \) such that

\[
\| x_2(\xi, t) \|_{(H^{1/2}(\Gamma))^n} \leq F_{\Gamma_{AG}}^2 e^{-\sigma_{\Gamma_{AG}}^2 t} \| I - P x_0^2 \|_{(H^{1/2}(\Gamma))^n} + \\
\int_0^t F_{\Gamma_{AG}}^2 e^{-\sigma_{\Gamma_{AG}}^2 s} \| I - P x_0^2 \|_{(H^{1/2}(\Gamma))^n} \| Bu(s) \|_{(H^{1/2}(\Gamma))^n} \text{ ds}
\]  
(15)

and therefore \( \| x(\xi, t) \|_{(H^{1/2}(\Gamma))^n} \) converges to zero when \( t \) tends to \( \infty \). Finally, the systems (1)-(2) are \( \Gamma_{AG} \)-detectable.

Now, Let \( e(\xi, t) = x(\xi, t) - w(\xi, t) \) where \( x(\xi, t) \) is solution of the systems (1)-(2) and \( w(\xi, t) \) is the solution of (9). Driving the above equation and using (1) and (9), we obtain

\[
\frac{\partial e}{\partial t} (\xi, t) = \frac{\partial x}{\partial t} (\xi, t) - \frac{\partial w}{\partial t} (\xi, t) \\
= A x(\xi, t) + B u(t) - A w(\xi, t) - B u(t) - H_{\Gamma_{AG}}(C w(\xi, t) - y(\xi, t)) \\
= A(x(\xi, t) - w(\xi, t)) - H_{\Gamma_{AG}}(C(w(\xi, t) - x(\xi, t)) \\
= (A - H_{\Gamma_{AG}} C)e(\xi, t)
\]  
(16)

Since the systems (1)-(2) are \( \Gamma_{AG} \)-detectable, there exists an operator \( H_{\Gamma_{AG}} \in L \left( \mathcal{O}, (H^{1/2}(\Gamma))^n \right) \) in order that the operator \( (A - H_{\Gamma_{AG}} C) \) generates \( \Gamma_{AG} \)-stable semigroup \( (S_{H_{\Gamma_{AG}}}(t)) \) satisfies the following relation:

\[
\exists F_{\Gamma_{AG}} \text{ and } \sigma_{\Gamma_{AG}} > 0 \text{ such that }
\]

\[
\| \chi_t y \nabla S_{H_{\Gamma_{AG}}}(t) \|_{(H^{1/2}(\Gamma))^n} \leq F_{\Gamma_{AG}} e^{-\sigma_{\Gamma_{AG}} t} 
\]  
(17)

Consequently, we get
\[\|e(\xi, t)\|_{H^{1/2}((\Omega))}^n = \|x(\xi, t) - w(\xi, t)\|_{H^{1/2}((\Omega))}^n\]
\[\leq \|\chi_T \nabla S_{H_{\Gamma G}}(t)e_0\|_{H^{1/2}((\Omega))}^n\]
\[\leq F_{\Gamma G} e^{-\sigma_{\Gamma G} t}\|e_0\|_{H^{1/2}((\Omega))}^n\]  
(18)

that is \(e(\xi, t)\) converges asymptotically to zero as \(t\) tends to \(\infty\). Thus, the dynamical system (9) observes asymptotically the regional gradient state \(\nabla \xi(\xi, t)\) of the original systems (1)-(2), and therefore it is \(\Gamma_{\Gamma G}\) observable.  

Now, we define a new type of strategic sensors.

**Definition 5.2:** The suite of sensors \(\{D_l, f_l\}_{l \in \mathcal{L}_G}\) are said to be asymptotic regional boundary gradient strategic (or \(\Gamma_{\Gamma G}\)-strategic) if, the corresponding system is \(\Gamma_{\Gamma G}\)-observable.

**Remark 5.3:** We can get that:

(1) Asymptotically observable system (on \(\Omega\)) is \(\Gamma_{\Gamma G}\)-observable,

(2) An exactly \(\Gamma_{\Gamma G}\)-observable system is \(\Gamma_{\Gamma G}\)-observable,

(3) A \(\Gamma_{\Gamma G}\)-observable system is \(\Gamma_{\Gamma G}\)-observable, for every subset \(\Gamma_{\Gamma G}\) subset of \(\Gamma_G\).

**Remark 5.4:** The usefulness of this section is that a gradient state which is not asymptotically observable in the usual sense may be asymptotically observable on \(\Gamma_{\Gamma G}\), this is illustrated by the following counter example.

**Example 5.5:** (counter example)

Consider the system represented by the following diffusion equation

\[
\begin{align*}
\frac{\partial x}{\partial t} (\xi_1, \xi_2, t) &= \frac{\partial^2 x}{\partial \xi_1^2} (\xi_1, \xi_2, t) + \frac{\partial^2 x}{\partial \xi_2^2} (\xi_1, \xi_2, t) + x(\xi_1, \xi_2, t) &\Omega \times (0, T) \\
x(\xi_1, \xi_2, 0) &= x_0 (\xi_1, \xi_2) &\bar{\Omega} \\
x(\eta_1, \eta_2, t) &= 0 &\partial \Omega \times (0, T)
\end{align*}
\]
(19)

With output function

\[y(t) = \int_\Omega x(\xi_1, \xi_2, t) \delta(\xi_1 - b_1, \xi_2 - b_2) d\xi_1 d\xi_2\]  
(20)

where \(\Omega = (0, \alpha) \times (0, \beta)\) and \((b_1, b_2) \in \Omega\) is the location of sensor \((b, \delta_b)\) (as in Figure 3). The operator \(A = \left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + 1\right)\) generates a strongly continuous semigroup on the state space \((H^1(\Omega))^n\).

![Figure 3](image)

**Figure 3:** domain as Rectangular region, \(\Gamma_{\Gamma G}\) and location \(b\) of internal pointwise sensor.

Now, we assume that \(\Omega \times (0, \infty) = \mathcal{Q}, \partial \Omega \times (0, \infty) = \Sigma\) and consider the dynamical system
where $H \in L(O,W)$, $W$ is the state space of above system. Here if $(\frac{b_1}{a}, \frac{b_2}{\beta}) \in \mathbb{Q}$, on unstable subsystem (19), the sensor $(\eta_1, \eta_2)$ has abad location so it is not strategic [17] and therefore the systems (19)-(20) are not asymptotically detectable in $\Omega$, and then, the dynamical system (21) is not asymptotically observer for (19)-(20), finally, the dynamical state $\nabla \phi_1(\xi_1, \xi_2)$ is not asymptotically observable in $\mathbb{R}^3$.

Now, we consider the boundary gradient region $\Gamma_g = (0, \alpha) \times \{\beta\} \subset \partial \Omega$, (figure.2) and the dynamical system

$$\begin{align*}
\frac{\partial w}{\partial t}(\xi_1, \xi_2, t) &= \frac{\partial^2 w}{\partial \xi_1^2}(\xi_1, \xi_2, t) + \frac{\partial^2 w}{\partial \xi_2^2}(\xi_1, \xi_2, t) + Q \\
w(\xi_1, \xi_2, t) &= H \xi_c (w(\xi_1, \xi_2, t) - x(\xi_1, \xi_2, t)) \\
w(\xi_1, \xi_2, 0) &= w_0(\xi_1, \xi_2) \\
w(\eta_1, \eta_2, t) &= 0
\end{align*}$$

(22)

where $H_{\xi_c} \in L(\xi_c, (H^{1/2}(\Gamma))^n)$. If there exist $i, j \in \{1, \ldots, f\}$ such that $\frac{\partial \alpha_i}{\alpha} \text{ and } \frac{\partial \beta_j}{\beta} \notin \mathbb{I}$ then the sensor $(\eta_1, \eta_2)$ is strategic, and this implies to the system (19)-(20) is $\Gamma_g$-detectable, therefore the may be $\Gamma_{\xi_c}$-observable. ■

6. Relations Between $\Gamma_{\xi_c}$-Observability and the Sensors Position

In this section, we apply the previous results to a two dimensional system defined on the rectangular domain once and on the disk domain twice.

6.1 Rectangular Domain

Consider the diffusion system

$$\begin{align*}
\frac{\partial x}{\partial t}(\xi_1, \xi_2, t) &= \Delta x(\xi_1, \xi_2, t) + x(\xi_1, \xi_2, t) + B u(t) \\
x(\xi_1, \xi_2, 0) &= x_0(\xi_1, \xi_2) \\
x(\eta_1, \eta_2, t) &= 0
\end{align*}$$

(23)

with measurements get it by output function given as (2), where $\Omega = (0, 1) \times (0, 1)$, $\Gamma_g = (0, 1) \times \{1\}$. In this case the eigenfunctions of the dynamic system (23) for Derichlet boundary gradient conditions are given by:

$$\varphi_{ij}(\xi_1, \xi_2) = 2 \cos(\pi \xi_1) \cos(\pi \xi_2)$$

(24)

associated with eigenvalues

$$\lambda_{ij} = -(i^2 + j^2)\pi^2$$

(25)

The results are below provide us with information about the internal location of zone or pointwise $\Gamma_{\xi_c}$-strategic sensors.

1. Internal Pointwise Sensor

Here we define a sensor as pointwise located inside of $\Omega$. Then the output function of the system (23) is defined as:

$$y(t) = \int_{\Omega} x(\xi_1, \xi_2, t) \delta(\xi_1 - b_1, \xi_2 - b_2) d\xi_1 d\xi_2$$

(26)

such that the sensor $b = (b_1, b_2) \in \Omega$, (see figure 4).
If there exist \(i, j \in \{1, \ldots, f\}\), such that \(ib_1, ib_2 \not\in I, r_m = 1\), then the sensor \(b = (b_1, b_2)\) can be sufficient for \(\Gamma_G\)-observability, with the dynamical system

\[
\begin{cases}
\frac{dw}{dt}(\xi_1, \xi_2, t) = \Delta w(\xi_1, \xi_2, t) + w(\xi_1, \xi_2, t) + Bu(t) + Q \\
H_{I_G}(w(\xi_1, \xi_2, t) - y(t)) = \Omega \\
w(\xi_1, \xi_2, 0) = w_0(\xi_1, \xi_2) \\
w(\eta_1, \eta_2, t) = 0
\end{cases}
\]

(27)

Forms an \(I_{AG}\)-observer for (23), thus we obtain the following result:

![Figure 4: Domain as rectangular region \(\Gamma_G\) and location \(b\) of internal pointwise sensor.](image)

**Corollary 6.1:** The systems (41)-(44) are \(I_{AG}\)-observable by the dynamical system (27), If there exist \(i, j \in \{1, \ldots, f\}\), such that \(ib_1, jb_2 \not\in I\).

2. (Internal Zone Sensor)

Here we have the sensor support \(D\) region is located in \(\Omega\). Hence, the output function (2) of the system (23) can be written as

\[
y(t) = \int_D x(\xi_1, \xi_2, t)f(\xi_1, \xi_2)d\xi_1d\xi_2
\]

(28)

where \(D = [\xi_{10} - l_1, \xi_{10} + l_1] \times [\xi_{20} - l_2, \xi_{20} + l_2] \subset \Omega\) is the location of sensor \((D, f)\) and \(f \in L^2(D)\)

(see figure 5), if \(f\) is not symmetric with respect to \(\xi_k = \xi_{k'}\), \(k = 1, 2\), there exist \(i, j \in \{1, \ldots, f\}\), such that \(i\xi_1, j\xi_2 \not\in I\) and \(r_m = 1\), then the sensor \((D, f)\) may be sufficient for \(\Gamma_{AG}\)-observability, with the dynamical system:

\[
\begin{cases}
\frac{dw}{dt}(\xi_1, \xi_2, t) = \Delta w(\xi_1, \xi_2, t) + w(\xi_1, \xi_2, t) + Bu(t) + Q \\
H_{I_G}(w(\xi_1, \xi_2, t), f(\xi_1, \xi_2), t) - y(t)) = \Omega \\
w(\xi_1, \xi_2, 0) = w_0(\xi_1, \xi_2) \\
w(\eta_1, \eta_2, t) = 0
\end{cases}
\]

(29)

thus we obtain the following result:

![Figure 5: Rectangular domain, region \(\Gamma_G\) and location \(D\) of internal zone sensor.](image)
Corollary 6.2: The systems (23)-(28) are $\Gamma_{AG}$-observable by the dynamical system (29), if there exist $i, j \in \{1, \ldots, J\}$, such that $i\xi_{1_0}, j\xi_{2_0} \notin I$ and $f$ is not symmetric with respect to $\xi_k = \xi_{k_0}, k = 1, 2$.

3. (Internal Filament Sensor)

Here the observation on the curve $\sigma = I(\gamma)$ with $\gamma \in C^1(0, 1)$ (see Figure 6), hence, we have the following.

Figure 6: Domain as a rectangular region $\Gamma_G$ and location $D$ of internal filament sensor.

Corollary 6.3: If the observation recovered by filament sensor $(\sigma, \delta\sigma)$ in order that it is symmetric about the line $\xi = \xi_0$, then the system (23) with output given by:

$$y(t) = \int_0^1 x(\xi t_1, \xi t_2, t)\delta(\xi t_1 - bi_1, \xi t_2 - bi_2)d\xi t_1 d\xi t_2$$

Is not $\Gamma_{AG}$-observable if there exist $i, j \in \{1, \ldots, J\}$, such that $i\xi_{1_0}, j\xi_{2_0} \in I$.

6.2 Disc Domain: In this case, we assume the system

$$\frac{\partial^2 x(r, \theta, t)}{\partial t^2} = \Delta x(r, \theta, t) + x(r, \theta, t) + Bu(t) \quad Q$$

$$x(r, \theta, 0) = x_0(r, \theta) \quad \Omega$$

$$x(r, \theta, t) = 0 \quad \Sigma$$

where $\Omega = D(0, 1), \theta \in [0, 2\pi]$, $\Gamma_G = D(1, \theta_1)_{2\pi}$. (31)

Remark 6.4: We can extended this case into different type of sensors (internal or boundary, zonal or pointwise) as in [24].

7. Conclusion

The goal of this paper is related to the asymptotic regional boundary gradient observability provides with strategic sensors. In the sense, It permits us to avoid some “bad” sensor locations in order to guarantee asymptotic observability achievement for the system. Many benefit results related with choice of sensors structure are given and illustrated in specific situations. For future research, an interesting direction would be the extension of these results to the Neumann condition problem. Also, many questions are still opened and need to study the possibility to develop to the boundary gradient case, for example, the problems of gradient detectability [25] and stabilizability [26] may be in quasi Banach space [27].

Acknowledgments: Our thanks in advance to the editors and experts for considering this paper to publish in this estimated journal and for their efforts.

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