Abstract

It is shown that, classically, the W-algebras are directly related to the extrinsic geometry of the embedding of two-dimensional manifolds with chiral parametrization (W-surfaces) into higher dimensional Kähler manifolds. We study the local and the global geometries of such embeddings, and connect them to Toda equations. The additional variables of the related KP hierarchy are shown to yield a specific coordinate system of the target-manifold, and this allows us to prove that W-transformations are simply particular diffeomorphisms of this target space. The W-surfaces are shown to be instantons of the corresponding non-linear σ-models.
1 Introduction

In many ways, W-algebras are natural generalisations of the Virasoro algebra. They were first introduced as consistent operator-algebras involving operators of spins higher than two[1]. Moreover, the Virasoro algebra is intrinsically related with the Liouville theory which is the Toda theory associated with the Lie algebra $A_1$, and this relationship extends to W-algebras which are in correspondence with the family of conformal Toda systems associated with arbitrary simple Lie algebras[2]. Another point is that the deep connection between Virasoro algebra and KdV hierarchy has a natural extension[3] to W-algebras and higher KdV (KP) hierarchies[4][5].

On the other hand, W-symmetries exhibit strikingly novel features. First, they are basically non-linear algebras. Since the transformation laws of primary fields contain higher derivatives, product of primaries are not primaries at the classical level. Naive tensor-products of commuting representations do not form representations. A related novel feature is that W-algebras generalise the diffeomorphisms of the circle by including derivatives of degree higher than one. Going beyond linear approximation (tangent space ) is a highly non-trivial step. Taking higher order derivatives changes the shape of the world-sheet in the target-space, thus W-geometry should be related to the extrinsic geometry of the embedding. Finally, Virasoro algebras are notoriously related to Riemann surfaces. The W-generalisation of the latter notion is a fascinating problem.

In the present letter, we show that 2D surfaces $\Sigma$ with chiral parametrisations, embedded into Kähler manifolds, are the geometrical framework where the features just recalled are naturally realised, at the classical level. This idea which hopefully will become clear below was supported at the beginning by several hints. First, the extrinsic geometry of abstract Riemann surfaces was beautifully discussed[6], using the complex analogue of Frenet’s formalism, by lifting the curve to Grassmannian manifolds with the Fubini-Study metric which is a Kähler metric associated with the standard metric on $CP^n$. Second, one of us has just shown[7] that the free fermion formalism of the KP hierarchy, and its related Grassmannian manifold — which is directly related to the one just mentionned — has a close connection with W-algebras, the additional variables of the KP systems, allowing to linearise the W-transformations.

This letter briefly summarizes our results. Details will be given elsewhere.
In section 2 we study the local structure of the embedding at generic regular points where Taylor expansion gives a local frame. Section 3 is concerned with singular points and with the global Plücker formula that describes the global topology of W-surfaces. Section 4 discusses the search for the action principle that would underly the picture just derived. We show that the W-surfaces just introduced are instantons of the non-linear \( \sigma \)-models with the same Kähler potential.

2 Local structure of the embedding at regular points

Our basic strategy is to study embeddings of two-dimensional surfaces \( \Sigma \) given by equations of the form

\[
X^A = f^A(z), \quad A = 1, \ldots, n, \quad \bar{X}^{\bar{A}} = \bar{f}^{\bar{A}}(\bar{z}), \quad \bar{A} = 1, \ldots, n, \tag{1}
\]

in a 2n-dimensional target manifold \( \mathcal{M} \) with a Kähler metric

\[
G_{AB} = G_{BA} = \partial_A \partial_{\bar{B}} \mathcal{K}(X, \bar{X}). \tag{2}
\]

One may either work with Minkowski coordinates where \( z = \sigma + \tau \), and \( \bar{z} = \sigma - \tau \), are light-cone coordinates, or perform a Wick rotation and work with the complex \( z \) variable (isothermal coordinates). The first description makes it clear that \( f^A \) and \( \bar{f}^{\bar{A}} \) are independent functions, so that we are not dealing with a single complex curve as is often done in this context. In the present section, we shall be concerned with regular points of \( \Sigma \) where Taylor expansion gives a local frame, so that a discussion of the Frenet type applies.

Our first result is that Toda fields naturally arise from the Gauss-Codazzi equations of this embedding. These equations are integrability conditions for derivatives of the tangent and of the normals to the surface. The latter are introduced by extending Frenet-Serre formula as follows. Define

\[
g_{ij} \equiv G_{A\bar{B}}(f, \bar{f}) \partial^{(i)} f^A(z) \bar{\partial}^{(j)} \bar{f}^{\bar{A}}(\bar{z}), \tag{3}
\]

In this first general part, the Kähler condition is not really needed. The same discussion applies to a general Hermitean manifold. The Kähler condition will play an essential rôle in the free-fermion approach, and in sections 3 and 4.
\[ \Delta_l \equiv \begin{vmatrix} g_{11} & \cdots & g_{1l} \\ \vdots & \ddots & \vdots \\ g_{l1} & \cdots & g_{ll} \end{vmatrix} \equiv e^{-\phi_i}. \quad (4) \]

\( \partial \) and \( \bar{\partial} \) are short hands for \( \partial/\partial z \) and \( \partial/\partial \bar{z} \) respectively. \( \partial^{(i)} \) stands for \( (\partial)^i \). Vectors in \( \mathcal{M} \) are denoted by boldface letters. It is straightforward to check that a set of \( 2n \) orthonormal vectors is given by (upper indices in between parenthesis denote derivatives)

\[ e_a = \frac{1}{\sqrt{\Delta_a \Delta_{a-1}}} \begin{vmatrix} g_{11} & \cdots & g_{1a} \\ \vdots & \ddots & \vdots \\ g_{a1} & \cdots & g_{aa-1} \\ f^{(1)} & \cdots & f^{(a)} \end{vmatrix}, \quad a = 1, \ldots, n, \quad (5) \]

\[ \bar{e}_a = \frac{1}{\sqrt{\Delta_a \Delta_{a-1}}} \begin{vmatrix} g_{11} & \cdots & g_{1a} \\ \vdots & \ddots & \vdots \\ g_{a1} & \cdots & g_{aa-1} \\ f^{(1)} & \cdots & f^{(a)} \end{vmatrix}, \quad a = 1, \ldots, n, \quad (6) \]

where determinants are to be computed for each component of the last line. Indeed, if we denote by \((X, Y)\) the inner product \( G_{AB}(X^AY^B + Y^AX^B) \) in \( \mathcal{M} \), we have \((e_a, e_b) = (\bar{e}_a, \bar{e}_b) = 0, (e_a, \bar{e}_b) = \delta_{a,b}\). It will be convenient at some point to treat bar and unbar indices together. This is done by using underlined letters. In this way, \( e_a \) and \( \bar{e}_a \) are denoted collectively as \( \underline{e}_a \). One has

\[ (\underline{e}_a, \underline{e}_b) = \eta_{ab}, \quad \eta_{ab} = \delta_{a,b}, \quad \eta_{a\bar{b}} = \eta_{\bar{b}\bar{a}} = 0 \quad (7) \]

\( e_1 \) and \( \bar{e}_1 \) are tangents to the surface, while the other vectors are clearly normals. One may show that the \( e_a \)'s satisfy equations of the form

\[ \nabla e_a = \omega^b_{za} e_b, \quad \bar{\partial} e_a = \omega^b_{z\bar{a}} e_b, \quad (8) \]

\[ \omega_z = -\frac{1}{2} \sum_{i=1}^n H_i (\partial \phi_i - \partial \phi_{i-1}) + \sum_{i=1}^{n-1} \exp \left( \sum_{j=1}^n K_{ij} \phi_j / 2 \right) E_{-i} \]

\[ \omega_{\bar{z}} = \frac{1}{2} \sum_{i=1}^n H_i \left( \partial \phi_i - \bar{\partial} \phi_{i-1} \right) - \sum_{i=1}^{n-1} \exp \left( \sum_{j=1}^n K_{ij} \phi_j / 2 \right) E_i, \quad (9) \]
where \( \phi_0 \equiv 0 \). \( \nabla \) stands for \( f^{(1)} f_A \), \( \nabla_A \) being the target-space covariant derivative, and \( K \) is the Cartan matrix of \( A_n \). \( H_i, E_{\pm i} \) are infinitesimal generators of \( U_n \) in the Chevalley basis, for the fundamental representation. \( H_i \) generate the Cartan subalgebra, and \( E_{\pm i} \) are associated with the usual set of primitive roots of \( A_n \). Remarkably, one sees that the right member is just the Toda Lax pair\(^4\). Toda equations are equivalent to the zero-curvature condition on \( \omega \). In the language of Riemannian geometry\(^5\), \( \omega^a_b \) and \( \omega^\bar{a}_b \) \( a > 1 \) give the second and third fundamental forms respectively. \( \bar{e}_a \) satisfy similar equations. The corresponding Toda Lax pair is in the adjoint of the fundamental representation\(^6\).

Going to the second derivatives, one deduces from the above that

\[
[\nabla, \bar{\partial}] e_a = F^b\bar{a}_b e_b
\]

\[
F^a_{\bar{a}b} = \sum_{i=1}^n H_i \bar{\partial}(\phi_i - \phi_{i-1}) + \sum_{i=1}^{n-1} (H_i - H_{i+1}) \exp \left( \sum_{j=1}^n K_{ij} \phi_j \right)
\]

On the other hand, \([\nabla, \bar{\partial}] e_a = \mathcal{R}(f^{(1)}, \bar{f}^{(1)})_a^b e_b \) where \( \mathcal{R} \) is the target-space curvature tensor. Thus we get

\[
F^b\bar{a}_b = \mathcal{R}(f^{(1)}, \bar{f}^{(1)})_a^b e_b,
\]

which are the Gauss-Codazzi equations\(^3\) for the embedding considered\(^4\). Geometrically, the vectors \( e_a \) span a moving frame, where the metric is constant and equal to \( \eta_{ab} \). Thus the \( e_a \) are 2n-beins, \( \omega_z, \bar{\omega}_z \) are the two components of the spin connection \( \omega_A \) along the surface, and the meaning of Eq. \(^{11}\) is clear.

Let us now turn to W-transformations. They are defined, as usual, to be of the form \( \delta_W f^A = D_W f^A \) where \( D_W \) is a differential operator in \( z \). Higher derivatives may be eliminated using the fact that the functions \( f^A \) are automatically solutions of the following differential equation

\[
\begin{vmatrix}
  f^1 & \ldots & f^n & f \\
  \vdots & \vdots & \vdots & \vdots \\
  f^{(n)1} & \ldots & f^{(n)n} & f^{(n)} \\
\end{vmatrix} \equiv \left( \sum_{k=0}^n U_{n-k} \partial^{(k)} \right) f = 0.
\]

\(^4\) A connection between Toda Lax pairs and Gauss-Codazzi equations already appears in ref.\(^{10}\). There, however, the former is defined in an abstract space whose geometrical meaning seems unclear.

\(^5\) Ideas, that are somewhat related to ours, have already been put forward in ref.\(^{11}\).
This allows us first to rewrite the W-transformation as \( \delta_W f = \sum_{a=1}^{n} w_a f^{(a)} \).
Moreover, taking the derivative of Eq.\( \text{12} \), one derives an equation of the form
\[
\sum_{a=1}^{n} \lambda_a f^{(a)}.
\]
Then one easily verifies that the transformation laws of \( f^{(a)} \) may be put under the matrix form
\[
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & \cdots & \lambda_1 \\
0 & 0 & 0 & \cdots & \lambda_2 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \lambda_{n-1} \\
0 & \cdots & 0 & \cdots & \lambda_n
\end{pmatrix}
\]
According to Eqs. \( \text{13} \), this leads to a matrix transformation for the \( e_a \) which is equivalent to a gauge transformation of \( \omega_z \) and \( \omega_{\bar{z}} \). By construction it is such that Eqs. \( \text{9} \) and \( \text{10} \) keep the same form. Thus it is the same as a W-transformation \( \phi_l \) in Toda theory\( \text{1} \). Geometrically, and since the \( e_a \) are 2n-beins, W-transformations appear as particular transformations of the local Lorentz group, which leave the form of the Toda Lax pair invariant.

An important point is that W-transformations in general mix the tangent and the normals at a given point of \( \Sigma \). Thus they move the surface and the distinction between intrinsic and extrinsic geometry is not W-invariant.

Next we show that the geometry of W-transformations becomes transparent if we introduce the additional variables \( z^k, \bar{z}^k, k = 2, \cdots n \), that play the rô\( \text{le} \) of the additional times of the KdV hierarchy\( \text{4} \). This was already shown in the free-fermion approach of \( \text{7} \). Inspired by this analogy we introduce these additional variables by extending \( f^A \) (or \( \bar{f}^A \)) to functions of \( z, z^2, \cdots, z^n \) (or \( \bar{z}, \bar{z}^2, \cdots, \bar{z}^n \)) that satisfy the differential equations
\[
\begin{align*}
\left( \frac{\partial}{\partial z^k} - \frac{\partial^k}{(\partial z)^k} \right) f^A(z, z^2, \cdots, z^n) &= 0, & f^A(z, 0, \cdots, 0) &= f^A(z), \\
\left( \frac{\partial}{\partial \bar{z}^k} - \frac{\partial^k}{(\partial \bar{z})^k} \right) \bar{f}^A(\bar{z}, \bar{z}^2, \cdots, \bar{z}^n) &= 0, & \bar{f}^A(\bar{z}, 0, \cdots, 0) &= \bar{f}^A(\bar{z}).
\end{align*}
\]

\( \text{14} \)

Note that these are defined in \( \text{3} \), through canonical Poisson brackets that do not assume that Toda equations hold.
With these additional variables, the W-transformations may be rewritten as (from now on on $z$ and $\bar{z}$ are replaced by $z^1$, and $\bar{z}^1$ if needed for compactness)

$$\delta_W f^A \equiv \sum_{a=1}^{n} w^a \partial(a) f^A = \sum_{a=1}^{n} w^a \partial_a f^A$$  \hspace{1cm} (15)

where $\partial_a \equiv \partial/\partial z_a$. By using the differential equations [13], one extends this last relation to the whole manifold $M$. Then the W-transformations become

$$\delta_W f^A \sim f^A(z + w^1, z^2 + w^2, \ldots, z^n + w^n) - f^A(z, z^2, \ldots, z^n).$$

They thus reduce to changes of variables

$$\delta_W z^k = u^k(z, z^2, \ldots, z^n).$$  \hspace{1cm} (16)

Eq. 3 may now be rewritten as $g_{ij} \equiv G_{AB}(f, \bar{f}) \partial_i f^A \bar{\partial}_j \bar{f}^A$, which simply means that $g_{ij}dz^i d\bar{z}^j = G_{AB} dX^A d\bar{X}^B$. Thus one sees that the variables $z^k$, and $\bar{z}^k$ are a set of coordinates on $M$ such that the embedded surface has equations $z^k = 0, \bar{z}^k = 0$ for $k \neq 1$. Thus W-transformations are particular reparametrisations of the target space $M$. Concerning the moving frame, and generalising the above discussion, one defines 2n-beins $e_a$ on $M$, by using the additional coordinates. Eqs. 8-11 are now a part of the standard equations relating the spin connection $\omega_{AB}^A$, its curvature $F_{AB}^A$ and the metric tensor $R$. From this viewpoint, the W-transformations appear as local Lorentz transformations that leave the form of Eq.9 invariant on $\Sigma$.

As it is evident from Eqs. 4, 5, 6, the nature of the chiral embedding is deeply connected with free fermions. Indeed, when the target space is simple, we can get neat expressions for Eqs. 4, 5, 6 as expectation values between quantum states of non-relativistic fermions. This is the non-chiral version of the method developed in [7] and we shall use the same notation.

The simplest situation is when the target space is a flat space, i.e. $C^n$. The key observation is that the metric $g_{ij}$ can be written as expectation values between one-fermion states,

$$g_{ij} = <\emptyset|\psi^{*}_{j-1} G_{C^n}(1) \psi_{i-1}|\emptyset>,$$

$$G_{C^n}^{(1)} = \sum_{A=1}^{n} e^{J_1 z} \psi^{*}_{\bar{A}} \bar{f}_A |\emptyset> <\emptyset|\psi^{*}_{A} e^{J_1 z}.  \hspace{1cm} (17)$$

\footnote{Of course there may be global obstructions, so that they only cover the component of $M$ which is connected to the surface.}
Once one recognizes this fact, Wick’s theorem tells us that the determinant Eq. 4 is given by the expectation value in the $a$-particle state, $\Delta_a = \langle a | G^{(a)}_C | a \rangle$, where

$$G^{(a)}_C = \sum_{A_1 < \cdots < A_a} e^{\bar{J}_1 \bar{z}} \psi \partial f_{A_1} \cdots \psi \partial f_{A_a} | \emptyset \rangle \langle \emptyset | \psi^* \partial f_{A_1} \cdots \psi^* \partial f_{A_a} e^{J_1 z}.$$ \(18\)

In this language, the orthogonality condition 7 or the extended Frenet formula 8 is the direct consequence of Hirota’s equations,

$$\sum_{i=0}^{\infty} \psi_i^* G^{(i)} \otimes \psi_i G^{(s)} = \sum_{i=0}^{\infty} G^{(i+1)} \psi_i^* \otimes G^{(s)} \psi_i.$$ \(19\)

( This is the re-interpretation of Hirota’s equations 12 in the free-fermion language). On the other hand, one is in a much more interesting situation when the target space has a non-trivial topology. The first such example is $CP^n$. Introduce the inhomogeneous coordinates $[X^0 \equiv 1, X^1, \cdots, X^n]$ and the Fubini-Study metric, $G_{AB} = \partial_A \partial_B \ln \sum_{C=0}^{n} X^C \bar{X}^C$. The induced world-sheet Kähler potential is given by,

$$K_{CP^n} = \ln \tau_1 \equiv \ln \sum_{A=0}^{n} f^A(z) \bar{f}^A(\bar{z}).$$ \(20\)

We should first remark that the $\tau$-function can be written as an expectation value $\tau_1 = \langle \ell | G^{(1)}_{CP^n} | \ell \rangle$, with

$$G^{(1)}_{CP^n} = \sum_{A=0}^{n} e^{\bar{J}_1 \bar{z}} \psi \bar{f}_A | \emptyset \rangle \langle \emptyset | \psi^* e^{J_1 z}.$$ \(21\)

The corresponding state is a one-particle state, in contrast with the previous case. Then it is easy to check that the metric $g_{ij}$ can be written as the two-particle expectation value,

$$g_{ij} = \frac{1}{\tau_1^2} (\tau_1 \delta^{(i) \bar{(j)}} \tau_1 - \delta^{(i) \bar{(j)}} \tau_1 \bar{\delta}^{(j) \bar{(i)}} \tau_1) = \frac{1}{\tau_1^2} \langle \ell | \psi^* G^{(2)}_{CP^n} \psi | \ell \rangle.$$ \(22\)

Using again Wick’s theorem, the determinant Eq. 4 can be expressed as a fermion expectation value: $\Delta_{\ell} = \tau_{\ell+1} / \tau_{\ell}^\ell$, $\tau_\ell = \langle \ell | G^{(\ell)}_{CP^n} | \ell \rangle$. Frenet’s theorem follows again from Hirota’s equations.
In $C^n$ and $CP^n$, the target-space has constant sectional curvature. Toda equations follow from Eq.11. This fact again illustrates the essential link between the embedding problem and Toda equations. One can, in addition, make contact with the conformally reduced WZNW theory[13]. This will be explained somewhere else.

For $C^n$ and $CP^n$, the Kähler potentials take the same form but the matrix element is between zero- and one- particle states respectively. More generally, when the target space is the Grassmannian manifold $Gr_{n+k,k}$, the Kähler potential can be related to the $k$-particle fermion expectation value. For these simple target-spaces, the change of the Kähler potential can be reduced to changes of the vacuum and of the operator $\hat{G}$. It natural to conjecture that there exists similar free-fermion expressions for the embedding in any Kähler manifold. It will be very important to spell out this correspondence because the free-fermion formalism is a very natural tool to understand the W-geometries. For example, as it was observed previously, the dependence on the higher coordinates can be easily introduced by the simple change of the Hamiltonian, $\exp(J_1z) \to \exp(\sum_{s=1}^{n} J_s z^s)$. Furthermore, the W-transformations of the $\tau$-functions can be written linearly through the bosonization of this free fermion.

For $n \to \infty$, very interesting phenomena may take place. This fermion formalism, with a non-trivial vacuum structure becomes closely related to the fermions of matrix-models.

3 Global structure of the embedding

The Grassmannian manifold $G_{n+k,k}$ is the set of $(n+k) \times k$ matrices $W$ with the equivalence relation $W \sim aW$, where $a$ is a $k \times k$ matrix. A chiral embedding in $G_{n+k,k}$ is thus defined by

$$W(z) = \begin{pmatrix} f^{1,1}(z) & \cdots & f^{n+k,1}(z) \\ \vdots & \ddots & \vdots \\ f^{1,k}(z) & \cdots & f^{n+k,k}(z) \end{pmatrix}$$

(23)

Following Ref.[14], we introduce the Kähler potential $K_k \equiv \ln(\det WW^T)$, that is,

$$e^{K_k} =$$
The difference is that, in our case, $\bar{f}$ will not the complex conjugate of $f$. From the embedding $\Sigma \to CP^n$, we can canonically construct the kth associated embedding $\Sigma \to Gr_{n+1,k+1} \hookrightarrow P(\Lambda^{k+1}C^n)$. We let $f^{l,s} = \delta(s-1)f^l(z)$ in Eq.23, for $s = 1, \cdots, k+1, l = 0, \cdots, n$. The induced metric on the corresponding Riemann surface is simply

$$g^{(k)}_{z\bar{z}} = \partial \bar{\partial} \ln \tau_{k+1}$$

so that the Toda field appears naturally. The infinitesimal Plücker formula is derived by computing the curvature

$$R^{(k)}_{z\bar{z}} = -\partial \bar{\partial} \ln g^{(k)}_{z\bar{z}} = -\partial \bar{\partial} \ln \left(\frac{\tau_{k+2}\tau_k}{\tau_{k+1}^2}\right),$$

and this gives

$$R^{(k)}_{z\bar{z}} = -g_{z\bar{z}}^{(k+1)} + 2g_{z\bar{z}}^{(k)} - g_{z\bar{z}}^{(k-1)}.$$  

The global Plücker formula contains the k-th instanton number

$$d_k \equiv \frac{i}{2\pi} \int_{\Sigma} dz d\bar{z} g_{z\bar{z}}^{(k)},$$

and follows from Gauss-Bonnet theorem for $g_{z\bar{z}}^{(k)}$:

$$\frac{i}{2\pi} \int_{\Sigma} dz d\bar{z} R^{(k)}_{z\bar{z}} = 2 - 2g + \beta_k.$$  

$\beta_k$ is defined as a sum of the k-th ramification indices. Near a singular point where there is an obstruction to the construction of the moving frame, the behaviour is of the form

$$g_{z\bar{z}}^{(k)} \sim (z - z_0)^{\beta_k(z_0)} (\bar{z} - \bar{z}_0)^{\bar{\beta}_k(\bar{z}_0)} \bar{g}_{z\bar{z}}^{(k)},$$

where $\bar{g}_{z\bar{z}}^{(k)}$ is regular at $z_0, \bar{z}_0$. Since we do not assume that $\bar{f}(z) = \bar{f}(\bar{z})$, $\beta_k(z_0)$ and $\bar{\beta}_k(\bar{z}_0)$ may be different. $\beta_k$ is defined by

$$\beta_k \equiv \frac{1}{2} \sum_{(z_0, \bar{z}_0) \in \Sigma} \left(\beta_k(z_0) + \bar{\beta}_k(\bar{z}_0)\right).$$
Finally we arrive at the global Plücker formula

\[ 2 - 2g + \beta_k = 2d_k - d_{k+1} - d_{k-1}, \quad k = 0, \cdots, n - 1 \]
\[ d_n \equiv 0, \quad d_{-1} \equiv 0 \]  
(32)

Using this formula, we find that there are \( n \) independent topological numbers \((d_0, \cdots, d_{n-1})\), which characterize the global topology of \( W \)-surfaces. A direct consequence of this observation is that \( W_{n+1} \)-strings have \( n \) coupling constants which play the same role as the genus for the usual string theories. Eq. (32) may be understood as the index theorem for \( W \)-surfaces.

4 Searching for an action principle

In the preceding sections, we have exhibited the geometry of \( W \)-surfaces. They are characterized by their chiral parametrization \( X^A = f^A(z) \), \( \bar{X}^\bar{A} = \bar{f}^{\bar{A}}(\bar{z}) \). This is of course a very restricted set since a generic 2D manifold embedded in \( M \) has equations \( X^A = u^A(z, \bar{z}) \), \( \bar{X}^{\bar{A}} = \bar{u}^{\bar{A}}(z, \bar{z}) \). For \( W \)-surfaces, \( u^A \) and \( \bar{u}^{\bar{A}} \) satisfy the Cauchy-Riemann equations which are self-duality equations. Thus \( W \)-surfaces are instantons in \( M \). At this point one should remember the exciting developments of the seventies concerning instantons in \( CP^n \) models[15], and more generally in Kähler manifolds[16]. Instanton solutions minimize the action of the associated non-linear \( \sigma \)-model[16]. Let us recall briefly the general argument[16] for completeness. Consider a general Kähler manifold \( M \) with coordinates \( \xi^\mu \) and \( \bar{\xi}^{\bar{\mu}} \) and metric \( h_{\mu\bar{\mu}} \). In the non-linear \( \sigma \)-model, the action associated with an arbitrary 2D manifold of \( M \) with equations \( \xi^\mu = \phi^\mu(z, \bar{z}) \), \( \bar{\xi}^{\bar{\mu}} = \bar{\phi}^{\bar{\mu}}(z, \bar{z}) \) is given by

\[ S = \frac{1}{2} \int d^2x \, h_{\mu\bar{\nu}} \partial_j \phi^\mu \partial_{\bar{j}} \bar{\phi}^{\bar{\nu}}. \]  
(33)

In this part, we let \( z = x_1 + ix_2 \), \( \partial_j = \partial/\partial x_j \). It is easy to see that

\[ S \geq cQ, \quad Q = \frac{i}{2c} \int d^2x \, \epsilon_{jk} h_{\mu\bar{\nu}} \partial_j \phi^\mu \partial_k \bar{\phi}^{\bar{\nu}}. \]  
(34)

(\( c \) is a suitably chosen constant) Clearly the equality is achieved if \( \partial_j \phi^\mu = \pm i \epsilon_{jk} \partial_k \phi^\mu \). These solutions are the standard instantons of Kähler manifolds.

At this point a general remark is in order. In the problems considered earlier[13][16], the natural reality condition is \( (\phi^\mu(z))^* = \bar{\phi}^{\bar{\mu}}(z^*) \). Such is
not the case for W-surfaces as we show next, since left- and right-moving modes are not correlated (apart from the zero-modes) for conformal systems without boundaries. The physical requirement is that the Toda fields be real in Minkowski space. Thus $g_{ij} \equiv G_{AB} (f, \bar{f}) \partial^{(i)} f^A (z) \bar{\partial}^{(j)} \bar{f}^\bar{A} (\bar{z})$ must be real for $z = \sigma + \tau$, and $\bar{z} = \sigma - \tau$, real. This is achieved by conditions of the form

$$\left( f^A (z) \right)^* = C_B^A f^A (z^*), \quad \left( \bar{f}^\bar{A} (\bar{z}) \right)^* = \bar{C}_B^\bar{A} \bar{f}^\bar{A} (\bar{z}^*),$$

(35)

where the conjugation matrices $C_\bar{C}$ must be such that

$$(G_{AA})^* C_B^A \bar{C}_B^\bar{A} = G_{BB}. \quad (36)$$

With the usual reality condition the self-duality equations for $\phi^\mu$ and $\bar{\phi}^\bar{\mu}$ are equivalent. This is no more true in our problem. Note that the lower bound $S = Q$ is reached as soon as the self duality equation holds for $\phi$ or $\bar{\phi}$. Thus there seems to exist more general instantons than the W-surfaces we have considered so far. We leave this problem for further studies.

Going back to our main line, we note that the analysis just recalled may be carried out not only on $\mathcal{M}$ but also on the Grassmannian manifolds whose construction was recalled in section 3 for $\mathbb{C}P^n$. For each $k$, Eq.25 precisely means that the action is topological, and that the lower bound $S = Q$ is reached. Considering these higher Grassmannians may shed new lights on the instanton problems of the seventies.

This non-linear $\sigma$-model action is quite interesting but is not the final answer. Since W-symmetries are so important, one should introduce a W-invariant action. We have seen that the W-symmetry is a part of general covariance so that one should start from an action invariant under diffeomorphisms of the target space. On the other hand, W-transformations applied to a W-surface, do not generically leave it invariant as a geometrical object. Thus the action must be topological, since it does not depend upon the shape of the W-surface. Moreover, since the position of the W-surface is irrelevant, there is no real distinction between intrinsic and extrinsic geometry. This is why the extrinsic curvature-components become part of the conformal theory on the world-sheet. A possible candidate for the W-action is a topological Yang-Mills action in $2n$ dimensions, involving the spin connection $\omega$. We leave this exciting problem for further investigations.

Acknowledgements One of us (J.-L. Gervais) is grateful to M. Saveliev for stimulating conversations and for pointing out his inspiring article[10].

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