Equivalence and Isomorphism for Boolean Constraint Satisfaction*

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Abstract. A Boolean constraint satisfaction instance is a conjunction of constraint applications, where the allowed constraints are drawn from a fixed set $C$ of Boolean functions. We consider the problem of determining whether two given constraint satisfaction instances are equivalent and prove a Dichotomy Theorem by showing that for all sets $C$ of allowed constraints, this problem is either polynomial-time solvable or coNP-complete, and we give a simple criterion to determine which case holds. A more general problem addressed in this paper is the isomorphism problem, the problem of determining whether there exists a renaming of the variables that makes two given constraint satisfaction instances equivalent in the above sense. We prove that this problem is coNP-hard if the corresponding equivalence problem is coNP-hard, and polynomial-time many-one reducible to the graph isomorphism problem in all other cases.

Keywords: computational complexity, propositional logic, constraint satisfaction problems, logic in computer science

Track: A

1 Introduction

In 1978, Thomas J. Schaefer proved a remarkable result. He examined the satisfiability of propositional formulas for certain syntactically restricted formula classes. Each such class is given by the finite set $C$ of Boolean functions allowed when constructing formulas. A $C$-formula in his sense now is a conjunction of clauses, where each clause consists of a Boolean function from $C$ applied to some propositional variables. Such a Boolean function can be interpreted as a constraint that has to be fulfilled by a given assignment; the satisfiability problem for $C$-formulas hence provides a mathematical model for the examination of the complexity of constraint satisfaction problems, studied in artificial intelligence.
and database theory. Let CSP($\mathcal{C}$) denote the problem of deciding whether a given $\mathcal{C}$-formula is satisfiable. Schaefer [Sch78] showed that, depending on $\mathcal{C}$, the problem CSP($\mathcal{C}$) is either (1) efficiently (i.e., in polynomial time) solvable or (2) NP-complete (see also [GJ79, Problem LO6]); and he gave a simple criterion that allows one to determine whether (1) or (2) holds. Since the complexity of CSP($\mathcal{C}$) is either easy or hard (and not located in one of the—under the assumption P $\neq$ NP—infinity many intermediate degrees between P and the NP-complete sets [Lad75]), Schaefer called this a “dichotomy theorem for satisfiability.”

In recent years, there has been renewed interest in Schaefer’s result and constraint satisfaction problems. N. Creignou examined in [Cre95] how difficult it is to find assignments to constraint satisfaction problems that do not necessarily satisfy all clauses but as much as possible. Together with Hermann she studied the difficulty of determining the number of satisfying assignments of a given constraint satisfaction problem in [CH96]. In [CH97] Creignou and Hébrard discussed algorithms that generate all satisfying assignments, turning their attention to the question whether such an algorithm, given the fact that it has already found a satisfying assignment, can find another one in polynomial time. Kirousis and Kolaitis researched the complexity of finding minimal satisfying assignments for constraint satisfaction problems in [KK01] and Khanna, Sudan and Trevisan examined the approximability of these problems [KST97, KSW97]. Reith and Vollmer had a look at lexicographical minimal or maximal satisfying assignments constraint satisfaction problems [RW00]. In [RW00] Reith and Wagner examined various problems in the vicinity of constraint satisfaction such as the circuit value problem, counting and threshold problems for restricted classes of Boolean circuits. The PhD thesis of S. Reith [Rei01] contains a wealth of results about problems dealing with restricted Boolean circuits, formulas, and constraint satisfaction.

As mentioned above, constraint satisfaction problems are used as a programming or query language in fields such as artificial intelligence and database theory, and the above complexity results shed light on the difficulty of design of systems in that areas. A problem of immense importance from a practical perspective is that of determining whether two sets of constraints express the same state of affairs (that is, are equivalent), for example, in the applications, if two programs or queries are equivalent, or if a program matches a given specification. Surprisingly, this problem has not yet been looked at from a complexity point of view. In the case of unrestricted propositional formulas, the equivalence problem is easily seen to be complete for coNP. The main result of the present paper (Theorem 6) is a complete classification of the complexity of determining if two given constraint satisfaction instances are equivalent. We consider constraints drawn from a fixed arbitrary finite set $\mathcal{C}$ of Boolean functions and show that for all such $\mathcal{C}$, the considered problem is either (1) solvable in polynomial time, or (2) complete for coNP. As in Schaefer’s result, our proof is constructive in the sense that it allows us to easily determine, given $\mathcal{C}$, if (1) or (2) holds.
Besides the immediate practical relevance of the equivalence problem, we also see our results as contributions to the study of two other decision problems: First, the equivalence problem is a “sub-problem” of the minimization problem, i.e., the problem to find out, given a set of constraints, if it can equivalently be expressed with a fewer number of constraints. Secondly, equivalence relates to the isomorphism problem, which has been studied from a theoretical perspective for various mathematical structures. Most prominently, the question if two given (directed or undirected) graphs are isomorphic is one of the few problems in NP neither known to be in P nor known to be NP-complete \[\text{KST93}\]. The most recent news about graph isomorphism are a number of hardness results (e.g., for NL, PL, and DET) given in \[\text{Torr00}\]. Related to our study are the papers \[\text{AT00, BRS98}\] presenting a number of results concerning isomorphism of propositional formulas. In Section 4 we show (Theorem 18) that the isomorphism problem for constraint applications is coNP-hard if the corresponding equivalence problem is coNP-hard, and polynomial-time many-one reducible to the just mentioned graph isomorphism problem in all other cases. We also show that for a number of these cases, the isomorphism problem is in fact polynomial-time many-one equivalent to graph isomorphism (Theorems 25 and 26). The same proof technique can be used to prove a general, non-trivial P||NP (parallel access to NP) upper bound for the isomorphism problems for constraint satisfaction (Theorem 24).

2 Boolean Constraint Satisfaction Problems

We start by formally introducing constraint satisfaction problems. The definitions necessary for the equivalence and isomorphism problems will be given in the upcoming sections.

**Definition 1.** 1. A constraint \(C\) (of arity \(k\)) is a Boolean function from \(\{0, 1\}^k\) to \(\{0, 1\}\).
2. If \(C\) is a constraint of arity \(k\), and \(x_1, x_2, \ldots, x_k\) are (not necessarily distinct) variables, then \(C(x_1, x_2, \ldots, x_k)\) is a constraint application of \(C\).
3. If \(C\) is a constraint of arity \(k\), and for \(1 \leq i \leq k\), \(x_i\) is a variable or a constant (0 or 1), then \(C(x_1, x_2, \ldots, x_k)\) is a constraint application of \(C\) with constants.

The decision problems examined by Schaefer are the following.

**Definition 2.** Let \(C\) be a finite set of constraints.

1. \(\text{CSP}(C)\) is the problem of, given a set \(S\) of constraint applications of \(C\), to decide whether \(S\) is satisfiable, i.e., whether there exists an assignment to the variables of \(S\) that satisfies every constraint application in \(S\).
2. \(\text{CSP}_c(C)\) is the problem of, given a set \(S\) of constraint applications of \(C\) with constants, to decide whether \(S\) is satisfiable.
Clearly, there are an infinite number of CSP($\mathcal{C}$) problems. In 1978, Schaefer proved the surprising result that constraint satisfiability problems are either in P or NP-complete. He also completely characterized for which sets of constraints the problem is in P and for which it is NP-complete. Consult the excellent monograph [CKS00] for an almost completely up-to-date overview of further results and dichotomy theorems for constraint satisfaction problems.

The question of whether satisfiability for CSPs is in P or NP-complete depends on those properties of the involved Boolean functions that we define next.

**Definition 3.** Let $C$ be a constraint.

- $C$ is 0-valid if $C(0) = 1$.
- $C$ is 1-valid if $C(1) = 1$.
- $C$ is Horn (a.k.a. weakly negative) if $C$ is equivalent to a CNF formula where each clause has at most one positive variable.
- $C$ is anti-Horn (a.k.a. weakly positive) if $C$ is equivalent to a CNF formula where each clause has at most one negative variable.
- $C$ is bijunctive if $C$ is equivalent to a 2CNF formula.
- $C$ is affine if $C$ is equivalent to an XOR-CNF formula.
- $C$ is complementive (a.k.a. C-closed) if for every $s \in \{0, 1\}^k$, $C(s) = C(\overline{s})$, where $k$ is the arity of $C$ and $\overline{s} = \text{def} (1 - s_1)(1 - s_2)\cdots(1 - s_k)$ for $s = s_1s_2\cdots s_k$.

Let $\mathcal{C}$ be a finite set of constraints. We say $\mathcal{C}$ is 0-valid, 1-valid, Horn, anti-Horn, bijunctive, affine, or complementive if every constraint $C \in \mathcal{C}$ is 0-valid, 1-valid, Horn, anti-Horn, bijunctive, affine, or complementive, respectively. Finally, we say that $\mathcal{C}$ is Schaefer if $\mathcal{C}$ is Horn or anti-Horn or affine or bijunctive.

Schaefer’s theorem can now be stated as follows.

**Theorem 4 (Schaefer [Sch78]).** Let $\mathcal{C}$ be a finite set of constraints.

1. If $\mathcal{C}$ is 0-valid, 1-valid, or Schaefer, then CSP($\mathcal{C}$) is in P; otherwise, CSP($\mathcal{C}$) is NP-complete.
2. If $\mathcal{C}$ is Schaefer, then CSP$_c(\mathcal{C})$ is in P; otherwise, CSP$_c(\mathcal{C})$ is NP-complete.

In this paper, we will study two other decision problems for constraint satisfaction problems. In the next section, we will look at the question of whether two given CSPs are equivalent. In Section 4, we address the isomorphism problem for CSPs. In both cases, we will prove dichotomy theorems.

### 3 The Equivalence Problem for Constraint Satisfaction

The decision problems studied in this section are the following:

**Definition 5.** Let $\mathcal{C}$ be a finite set of constraints.
1. EQUIV($C$) is the problem of, given two sets $S$ and $U$ of constraint applications of $C$, to decide whether $S$ and $U$ are equivalent, i.e., whether for every assignment to the variables, $S$ is satisfied if and only if $U$ is satisfied.

2. EQUIV$_c(C)$ is the problem of, given two sets $S$ and $U$ of constraint applications of $C$ with constants, to decide whether $S$ and $U$ are equivalent.

It is immediate that all equivalence problems are in coNP. Note that, in some sense, equivalence is at least as hard as non-satisfiability, since $S$ is not satisfiable if and only if $S$ is equivalent to 0. Thus, we obtain immediately that if $C$ is not Schaefer, then EQUIV$_c(C)$ is coNP-complete.

On the other hand, equivalence can be harder than satisfiability. For example, equivalence between Boolean formulas with $\land$ and $\lor$ (i.e., without negation) is coNP-complete [EG95] while non-satisfiability for these formulas is clearly in P. We will prove following dichotomy theorem.

**Theorem 6.** Let $C$ be a set of constraints. If $C$ is Schaefer, then EQUIV($C$) and EQUIV$_c(C)$ are in P; otherwise, EQUIV($C$) and EQUIV$_c(C)$ are coNP-complete.

The cases of constraints with polynomial-time equivalence problems are easy to identify, using the following theorem:

**Theorem 7.** EQUIV$_c(C)$ is truth-table reducible to CSP$_c(C)$.

*Proof.* Let $S$ and $U$ be two sets of constraint applications of $C$ with constants. Note that $S$ and $U$ are equivalent if and only if $U \rightarrow A$ for every constraint application $A \in S$, and $S \rightarrow B$ for every constraint application $B \in U$ (see, e.g., [HK92]). Here and in the following, when we write a set of constraint applications $S$ in a Boolean formula, we take this to be a shorthand for $\bigwedge_{A \in S} \hat{A}$.

Given a constraint application $\hat{A}$ with constants and a set $\hat{S}$ of constraint applications of $C$ with constants, it is easy to check whether $\hat{S} \rightarrow \hat{A}$ with at most $2^k$ truth-table queries to CSP$_c(C)$, where $k$ is the maximum arity of $C$: For every assignment to the variables in $\hat{A}$ that does not satisfy $\hat{A}$, substitute this partial truth assignment in $\hat{S}$, $\hat{S} \rightarrow \hat{A}$ if and only if none of these substitutions results in a satisfiable set of constraint applications.

If $C$ is Schaefer, then CSP$_c(C)$ is in P by Schaefer’s theorem and we immediately obtain the following corollary.

**Corollary 8.** If $C$ is Schaefer, then EQUIV$_c(C)$ is in P.

Having identified the easy equivalence cases, the following theorem proves the second half of our Dichotomy Theorem:

**Theorem 9.** If $C$ is not Schaefer, then EQUIV($C$) is coNP-hard.

First of all, note that this would be easy to prove if we had constants in the language, since for all sets $S$ of constraint applications of $C$ the following holds: $S$ is not satisfiable if and only if $S$ is equivalent to the constraint 0. Still, we can use this simple observation in the case where $C$ is not 0-valid and not 1-valid.
Claim 10 If \( C \) is not Schaefer, not 0-valid, and not 1-valid, then \( \text{EQUIV}(C) \) is coNP-hard.

Proof. We will reduce \( \text{CSP}(C) \) to \( \text{EQUIV}(C) \). Let \( S \) be a set of constraint applications of \( C \). As noted above, \( S \) is not satisfiable if and only if \( S \) is equivalent to 0. Let \( C_0 \in C \) be a constraint that is not 0-valid, and let \( C_1 \in C \) be a constraint that is not 1-valid. Note that, for any variable \( x \), \( \{C_0(x, \ldots, x), C_1(x, \ldots, x)\} \) is equivalent to 0, and thus \( S \in \text{CSP}(C) \) if and only if \( S \) is equivalent to \( \{C_0(y, \ldots, y), C_1(y, \ldots, y)\} \).

If \( C \) is not Schaefer, but is 0-valid or 1-valid, then every set of constraint applications of \( C \) is trivially satisfiable (by 0 or 1). In these cases, a reduction from \( \text{CSP}(C) \) will not help, since \( \text{CSP}(C) \) is in P. However, we will show that in these cases the problem of determining whether there exists a non-trivial satisfying assignment is NP-complete and we will use the complements of these satisfiability problems to reduce from.

Creignou and Hébrard prove the following result, concerning the existence of non-trivial satisfying assignments ([CH97, Proposition 4.7], their notation for our \( \text{CSP}_{\neq 0,1}(C) \) is SAT∗):

**Proposition 11** ([CH97]). If \( C \) is not Schaefer, then \( \text{CSP}_{\neq 0,1}(C) \) is NP-complete, where \( \text{CSP}_{\neq 0,1}(C) \) is the problem of, given a set \( S \) of constraint applications of \( C \), to decide whether there is a satisfying assignment for \( S \) other than 0 and 1.

\( \text{CSP}_{\neq 0,1}(C) \) corresponds to the notion of “having a non-trivial satisfying assignment” in the case that \( C \) is 0-valid and 1-valid. We will reduce \( \text{CSP}_{\neq 0,1}(C) \) to \( \text{EQUIV}(C) \) in this case in the proof of Claim 15 to follow.

For the cases that \( C \) is not 1-valid or not 0-valid, we obtain the following analogues of Proposition 11. A proof can be found in the appendix.

**Theorem 12.** 1. If \( C \) is not Schaefer and not 0-valid then \( \text{CSP}_{\neq 1}(C) \) is NP-complete, where \( \text{CSP}_{\neq 1}(C) \) is the problem of, given a set \( S \) of constraint applications of \( C \), to decide whether there is a satisfying assignment for \( S \) other than 0 and 1.

2. If \( C \) is not Schaefer and not 1-valid then \( \text{CSP}_{\neq 0}(C) \) is NP-complete, where \( \text{CSP}_{\neq 0}(C) \) is the problem of, given a set \( S \) of constraint applications of \( C \), to decide whether there is a satisfying assignment for \( S \) other than 0.

Proof. Careful inspection of Creignou and Hébrard’s proof of Proposition 11 shows that following holds if \( C \) is not Schaefer:

1. If \( C \) is not 0-valid and not 1-valid, then \( L = \{S \mid S \in \text{CSP}_{\neq 0,1}(C) \text{ and not } S(0) \text{ and not } S(1)\} \) is NP-complete (this is case 1 of Creignou and Hébrard’s proof).

2. If \( C \) is 0-valid and not 1-valid, then \( L_0 = \{S \mid S \in \text{CSP}_{\neq 0,1}(C) \text{ and not } S(1)\} \) is NP-complete (this is case 2b of Creignou and Hébrard’s proof).

3. If \( C \) is 1-valid and not 0-valid, then \( L_1 = \{S \mid S \in \text{CSP}_{\neq 0,1}(C) \text{ and not } S(0)\} \) is NP-complete (this is case 3b of Creignou and Hébrard’s proof).
This almost immediately implies Theorem 12. Let $C$ be not Schaefer and not 0-valid. If $C$ is not 1-valid, then $L$ trivially many-one reduces to CSP$_{\neq 1}(C)$, since, for $S$ a set of constraint applications of $C$, $S \in L$ if and only if not $S(0)$, not $S(1)$, and $S \in$ CSP$_{\neq 1}(C)$. Similarly, if $C$ is 1-valid, then $L_1$ trivially many-one reduces to CSP$_{\neq 1}(C)$. This proves part (1) of Theorem 12. Part (2) follows by symmetry.

\[\square\]

Claim 13 Let $C$ be a finite set of constraints.

1. If $C$ is 1-valid, not Schaefer, and not 0-valid, then EQUIV($C$) is coNP-hard.
2. If $C$ is 0-valid, not Schaefer and not 1-valid, then EQUIV($C$) is coNP-hard.

Proof. We will prove the first case; the proof of the second case is similar. We will reduce CSP$_{\neq 1}(C)$ to EQUIV($C$) as follows. Let $S$ be a set of constraint applications of $C$ and let $x_1, \ldots, x_n$ be the variables occurring in $S$. Note that 1 satisfies $S$, since every constraint in $S$ is 1-valid. Therefore, $S \notin$ CSP$_{\neq 1}(C)$ if and only if $S$ is equivalent to $\bigwedge_{i=1}^n x_i$. Let $C \in L$ be not 0-valid. Since $C$ is 1-valid, $x_i$ is equivalent to $C(x_1, \ldots, x_i)$. It follows that $S \notin$ CSP$_{\neq 1}(C)$ if and only if $S$ is equivalent to $\{C(x_1, \ldots, x_i) \mid 1 \leq i \leq n\}$.

The final case is where $C$ is both 0-valid and 1-valid. We need the following key lemma from Creignou and Hébrard which is used in their proof of Proposition 13.

Lemma 14 ([CH97], Lemma 4.9(1)). Let $C$ be a set of constraints that is not Horn, not anti-Horn, not affine, and 0-valid. Then either

1. There exists a set $V_0$ of constraint applications of $C$ with variables $x$ and $y$ and constant 0 such that $V_0$ is equivalent to $x \rightarrow y$, or
2. There exists a set $V_0$ of constraint applications of $C$ with variables $x, y, z$ and constant 0 such that $V_0$ is equivalent to $(\overline{x} \wedge \overline{y} \wedge \overline{z}) \wedge (x \wedge y \wedge z)$.

Claim 15 Let $C$ be a finite set of constraints. If $C$ is not Schaefer but both 0-valid and 1-valid, then EQUIV($C$) is coNP-hard.

Proof. We will reduce CSP$_{\neq 0,1}(C)$ to EQUIV($C$). Let $S$ be a set of constraint applications of $C$ and let $x_1, \ldots, x_n$ be the variables occurring in $S$. Note that 0 and 1 satisfy $S$, since every constraint in $S$ is 0-valid and 1-valid. Therefore, $S \notin$ CSP$_{\neq 0,1}(C)$ if and only if $S$ is equivalent to $\bigwedge_{i=1}^n x_i \vee \bigwedge_{i=1}^n \overline{x_i}$.

First, suppose there is a constraint $C \in L$ that is non-complementive. (This case is similar to Creignou and Hébrard’s case 2a). Let $k$ be the arity of $C$ and let $s \in \{0, 1\}^k$ be an assignment such that $C(s) = 1$ and $C(\overline{s}) = 0$. Let $A(x, y)$ be the constraint application $C(a_1, \ldots, a_k)$, where $a_i = y$ if $s_i = 1$ and $a_i = x$ if $s_i = 0$. Then $A(0, 0) = A(1, 1) = 1$, since $A$ is 0-valid and 1-valid; $A(0, 1) = 1$, since $C(s) = 1$; and $A(1, 0) = 0$, since $C(\overline{s}) = 0$. Thus, $A(x, y)$ is equivalent to $x \rightarrow y$. Since $\bigwedge_{i=1}^n x_i \vee \bigwedge_{i=1}^n \overline{x_i}$ is equivalent to $\bigwedge_{1 \leq i, j \leq n} (x_i \rightarrow x_j)$, it follows that $S \notin$ CSP$_{\neq 0,1}(C)$ if and only if $S$ is equivalent to $\bigwedge_{1 \leq i, j \leq n} A(x_i, x_j)$.

It remains to consider the case where every constraint in $C$ is complementive. Let $V_0$ be the set of constraint applications of $C$ with constant 0 from Lemma 14.
Let $V_f$ be the set of constraint applications of $\mathcal{C}$ that results when we replace each occurrence of 0 in $V_0$ by $f$, where $f$ is a new variable. Note that the following holds. There are two cases to consider, depending on the form of $V_0$.

**Case 1:** $V_0(x, y)$ is equivalent to $(x \rightarrow y)$. In this case, consider $V_f(f, x, y)$. Since $V_f(0, x, y)$ is equivalent to $x \rightarrow y$, and every constraint in $S$ is complementive, it follows that $V_f(f, x, y)$ is equivalent to $(\overline{f} \land (x \rightarrow y)) \lor (f \land (y \rightarrow x))$. Thus, $\bigwedge_{i=1}^n x_i \lor \bigwedge_{i=1}^n \overline{x_i}$ is equivalent to $\bigwedge_{1 \leq i, j \leq n} V_f(f, x_i, x_j)$, and it follows that $S \not\in \text{CSP}_{0,1}(\mathcal{C})$ if and only if $S$ is equivalent to $\bigwedge_{1 \leq i, j \leq n} V_f(f, x_i, x_j)$.

**Case 2:** $V_0(x, y, z)$ is equivalent to $(\overline{x} \land \overline{y} \land \overline{z}) \lor (x \land y \land z)$. Since all constraints in $V_0$ are complementive, $V_f(f, x, y, z)$ behaves as follows: $V_f(0, 0, 0, 0) = V_f(0, 1, 0, 1) = V_f(0, 0, 1, 1) = V_f(1, 1, 1, 1) = V_f(1, 0, 1, 0) = V_f(1, 1, 0, 0) = 1$, and $V_f$ is 0 for all other assignments. Note that $V_f(f, f, x_i, x_j)$ is equivalent to $(x_i \leftrightarrow x_j)$, and thus that $\bigwedge_{i=1}^n x_i \lor \bigwedge_{i=1}^n \overline{x_i}$ is equivalent to $\bigwedge_{1 \leq i, j \leq n} V_f(f, f, x_i, x_j)$. It follows that $S \not\in \text{CSP}_{0,1}(\mathcal{C})$ if and only if $S$ is equivalent to $\bigwedge_{1 \leq i, j \leq n} V_f(f, f, x_i, x_j)$.

\[\blacksquare\]

### 4 The Isomorphism Problem for Constraint Satisfaction

In this section, we study a more general problem: The question of whether a set of constraint applications can be made equivalent to a second set of constraint applications using a suitable renaming of its variables. We need some definitions.

**Definition 16.**

1. Let $X = \{x_1, \ldots, x_n\}$ be a set of variables. By $\pi: \{x_1, \ldots, x_n\} \rightarrow \{x_1, \ldots, x_n\}$ we denote a permutation of $X$.
2. Let $S$ be a set of constraint applications over variables $X$ and $\pi$ a permutation of $X$. By $\pi(S)$ we denote the set of constraint applications that results when we replace simultaneously all variables $x_i$ of $S$ by $\pi(x_i)$.
3. Let $S$ be a set of constraint applications over variables $X$. The number of satisfying assignments of $S$ is $\#_1(S) = \text{def} |\{I \mid I$ is an assignment to all variables in $X$ that satisfies every constraint application in $S\}|$.

The isomorphism problem now is formally defined as follows:

**Definition 17.**

1. ISO($\mathcal{C}$) is the problem of, given two sets $S$ and $U$ of constraint applications of $\mathcal{C}$ over variables $X$, to decide whether $S$ and $U$ are isomorphic, i.e., there exists a permutation $\pi$ of $X$ such that $\pi(S)$ is equivalent to $U$.
2. ISO$_c(\mathcal{C})$ is the problem of, given two sets $S$ and $U$ of constraint applications of $\mathcal{C}$ with constants over variables $X$, to decide whether $S$ and $U$ are isomorphic.

We remark that for $S$ and $U$ to be isomorphic, we require that formally they are defined over the same set of variables. Of course, this does not mean that all these variables actually have to occur textually in both formulas.
As in the case for equivalence, isomorphism is in some sense at least as hard as non-satisfiability, since $S$ is not satisfiable if and only if $S$ is isomorphic to $0$. Thus, we immediately obtain that if $C$ is not Schaefer, then $\text{ISO}_c(C)$ is coNP-hard. Unlike the equivalence case however, we do not have a trivial coNP upper bound for isomorphism problems. In fact, there is some evidence [AT00] that the isomorphism problem for Boolean formulas is not in coNP. Note that determining whether two formulas or two sets of constraint applications are isomorphic is trivially in $\Sigma^p_2$. However, the isomorphism problem for formulas is not $\Sigma^p_2$-complete unless the polynomial hierarchy collapses [AT00]. In the sequel (Theorem 24) we will prove a stronger result for the isomorphism problem for Boolean constraints: We will prove a $P^{NP}$ upper bound for these problems, where $P^{NP}$ is the class of problems that can be solved via parallel access to NP. This class has many different characterizations, see, for example, Hemaspaandra [Hem89], Papadimitriou and Zachos [PZ83], Wagner [Wag90].

For equivalence, we obtained a polynomial-time upper bound for sets of constraints that are Schaefer. In contrast, as we will show in the sequel, it is easy to see that, for example, isomorphism for positive 2CNF formulas (i.e., isomorphism between two sets of constraint applications of $\{(0,1), (1,0), (1,1)\}$) is polynomial-time many-one equivalent to the graph isomorphism problem (GI).

The main result of this section is the following theorem.

**Theorem 18.** Let $C$ be a set of constraints. If $C$ is Schaefer, then $\text{ISO}(C)$ and $\text{ISO}_c(C)$ are polynomial-time many-one reducible to GI, otherwise, $\text{ISO}(C)$ and $\text{ISO}_c(C)$ are coNP-hard.

Note that if $C$ is Schaefer, then the isomorphism problem $\text{ISO}(C)$ cannot be coNP-hard, unless the polynomial-time hierarchy collapses. (This follows since, by our theorem, if $\text{ISO}(C)$ is coNP-hard then GI is coNP-hard and, since GI $\in$ NP, coNP would be a subset of NP and thus NP=coNP which implies the mentioned collapse.) Under the assumption that the polynomial-time hierarchy does not collapse, Theorem 18 thus distinguishes an easy case (reducible to GI) and a hard case. In this sense, Theorem 18 is again a dichotomy theorem.

We will first have a look at the lower bound part of Theorem 18. For that we need the following property:

**Lemma 19.** Let $S$ and $U$ be sets of constraint applications of $C$ with constants. If $S$ is isomorphic to $U$ then $\#_1(U) = \#_1(S)$.

**Proof.** First note that every permutation of the variables of $S$ induces a permutation of the rows of the truth-table of $S$. Now let $\pi$ be a permutation such that $\pi(S) \equiv U$. Then $\#_1(S) = \#_1(\pi(S))$ and $\#_1(\pi(S)) = \#_1(U)$. □

**Theorem 20.** If $C$ is not Schaefer, then $\text{ISO}(C)$ is coNP-hard.

**Proof.** We first note that a claim analogous to Claim 10 also holds for isomorphism, i.e., if $C$ is not Schaefer, not $0$-valid, and not $1$-valid, then $\text{ISO}(C)$ is coNP-hard. For the proof, we use the same reduction as in
Theorem 23. The following.

Suppose $S, \{C_0(y, \ldots, y), C_1(y, \ldots, y)\} \in \text{EQUIV}(\mathcal{C})$ then also $(S, \{C_0(y, \ldots, y), C_1(y, \ldots, y)\}) \in \text{ISO}(\mathcal{C})$ by the identity permutation $\pi = \text{id}$. For the other direction note that $S \notin \text{CSP}(\mathcal{C})$ iff $\#_1(S) > 0$. Now suppose $(S, \{C_0(y, \ldots, y), C_1(y, \ldots, y)\}) \in \text{ISO}(\mathcal{C})$, then by Lemma 19 $\#_1(S) = \#_1(\{C_0(y, \ldots, y), C_1(y, \ldots, y)\})$, which is clearly a contradiction since $\#_1(\{C_0(y, \ldots, y), C_1(y, \ldots, y)\}) = 0$.

Next, we claim, analogously to Claim 13, that

1. If $C$ is 1-valid, not Schaefer, and not 0-valid, then $\text{ISO}(\mathcal{C})$ is coNP-hard; and
2. If $C$ is 0-valid, not Schaefer, and not 1-valid, then $\text{ISO}(\mathcal{C})$ is coNP-hard.

For the first case, we use the same reduction as in the proof of Claim 13. Note that if constraint set $S$ is equivalent to $\{C(x_1, \ldots, x_n) \mid 1 \leq i \leq n\}$, then $(S, \{C(x_1, \ldots, x_n) \mid 1 \leq i \leq n\}) \in \text{ISO}(\mathcal{C})$ via $\pi = \text{id}$. For the other direction note that $S \notin \text{CSP}_{\neq 1}(\mathcal{C})$ iff $\#_1(S) \geq 2$, but $\#_1(\{C(x_1, \ldots, x_n) \mid 1 \leq i \leq n\}) = 1$. By Lemma 19 the result follows. The proof of the second case is similar.

The remaining case is that of a set $C$ not Schaefer, but both 0-valid and 1-valid. We use the same reduction as in Claim 15. Clearly if $(S, U) \in \text{EQUIV}(\mathcal{C})$ then also $(S, U) \in \text{ISO}(\mathcal{C})$ via $\pi = \text{id}$. To show the other direction note that if $S \notin \text{CSP}_{\neq 0,1}(\mathcal{C})$ then $\#_1(S) \geq 3$, but $\#_1(\bigwedge_{i=1}^{n} x_i \lor \bigwedge_{i=1}^{n} \pi x_i) = 2$. Now use Lemma 13 to show that $S$ is not isomorphic to one of the formulas constructed in the cases examined in Claim 15. This completes the proof of the theorem. □

To complete the proof of Theorem 18 it remains to show that if $C$ is Schaefer, then $\text{ISO}(\mathcal{C})$ and $\text{ISO}_e(\mathcal{C})$ are polynomial-time many-one reducible to GI. We will reduce $\text{ISO}_e(\mathcal{C})$ to graph isomorphism for vertex-colored graphs, a GI variation that is polynomial-time many-one equivalent to GI.

Definition 21. VCGI is the problem of, given two vertex-colored graphs $G_1 = (V_1, E_1, \chi_1)$, $i \in \{1, 2\}$, $\chi_i : V \rightarrow \mathbb{N}$, to determine whether there exists an isomorphism between $G_1$ and $G_2$ that preserves colors, i.e., whether there exists a bijection $\pi : V_1 \rightarrow V_2$ such that $(v, w) \in E_1$ iff $(\pi(v), \pi(w)) \in E_2$ and $\chi(v) = \chi(\pi(v))$.

Proposition 22 ([Fon76, BC79]). VCGI is polynomial-time many-one equivalent to GI.

By Proposition 22, to complete the proof of Theorem 18 it suffices to show the following.

Theorem 23. Let $C$ be a set of constraints. If $C$ is Schaefer, then $\text{ISO}_e(\mathcal{C}) \leq_m \text{VCGI}$.

Proof. Suppose $C$ is Schaefer, and let $S$ and $U$ be sets of constraint applications of $C$ with constants over variables $X$. We will first bring $S$ and $U$ into normal form.
Let \( \hat{S} \) be the set of all constraint applications \( A \) of \( \mathcal{C} \) with constants such that all of \( A \)'s variables occur in \( X \) and such that \( S \to A \). Similarly, let \( \hat{U} \) be the set of all constraint applications \( B \) of \( \mathcal{C} \) with constants such that all of \( B \)'s variables occur in \( X \) and such that \( U \to B \). It is clear that \( \hat{S} \equiv \hat{U}, \) since \( S \subseteq \hat{S} \) and \( S \to \hat{S} \). Likewise, \( \hat{U} \equiv \hat{U} \). Note that \( \hat{S} \) and \( \hat{U} \) are polynomial-time computable (in \( \|(S, U)\| \)), since

1. there exist at most \( ||\mathcal{C}||(||X|| + 2)^k \) constraint applications \( A \) of \( \mathcal{C} \) with constants such that all of \( A \)'s variables occur in \( X \), where \( k \) is the maximum arity of constraints in \( \mathcal{C} \); and
2. since \( \mathcal{C} \) is Schaefer, determining whether \( S \to A \) or \( U \to A \) takes polynomial time, using the same argument as in the proof of Theorem \[3\].

Note that we have indeed brought \( S \) and \( U \) into normal form, since if \( S \equiv U \) then \( \hat{S} = \hat{U} \), so for any permutation \( \pi \) of \( X \), if \( \pi(S) \equiv U \), then \( \pi(\hat{S}) = \hat{U} \). We remark that this approach of first bringing the sets of constraint applications into normal form is also followed in the coIP[P]1[1][NP] upper bound proof for isomorphism between Boolean formulas \[4\].

It remains to show that we can in polynomial time encode \( \hat{S} \) and \( \hat{U} \) as vertex-colored graphs \( G(\hat{S}) \) and \( G(\hat{U}) \) such that there exists a permutation \( \pi \) of \( X \) with \( \pi(\hat{S}) = \hat{U} \) if and only if \( (G(\hat{S}), G(\hat{U})) \in \text{VCGI} \).

Let \( \mathcal{C} = \{C_1, \ldots, C_m\} \), let \( \mathcal{P} = \{C_{i_1}(x_{11}, x_{12}, \ldots, x_{1k_1}), C_{i_2}(x_{21}, x_{22}, \ldots, x_{2k_2}), \ldots, C_{i_\ell}(x_{\ell1}, x_{\ell2}, \ldots, x_{\ell k_\ell})\} \) be a set of constraint applications of \( \mathcal{C} \) with constants over variables \( X \) such that \( i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_\ell \).

Define \( G(P) = (V, E, \chi) \) as the following vertex-colored graph:

- \( V = \{0, 1\} \cup \{x \mid x \in X\} \cup \{a_{ij} \mid 1 \leq i \leq \ell, 1 \leq j \leq k_i\} \cup \{A_i \mid 1 \leq i \leq \ell\} \). That is, the set of vertices corresponds to the Boolean constants, the variables in \( X \), the arguments of the constraint applications in \( P \), and the constraint applications in \( P \).
- \( E = \{(x, a_{ij}) \mid x = x_{ij}\} \cup \{(a_{ij}, A_i) \mid 1 \leq i \leq \ell, 1 \leq j \leq k_i\} \).
- The vertex coloring \( \chi \) will distinguish the different categories. Of course, we want to allow any permutation of the variables, so we will give all elements of \( X \) the same color. In addition, we also need to allow a permutation of constraint applications of the same constraint.
  - \( \chi(0) = 0, \chi(1) = 1, \)
  - \( \chi(x) = 2 \) for all \( x \in X, \)
  - \( \chi(A_r) = 2 + j \) if \( i_r = j, \) and
  - \( \chi(a_{ij}) = 2 + m + j. \) (This will ensure that we do not permute the order of the arguments.)

If there is a permutation \( \pi \) of \( X \) such that \( \pi(\hat{S}) = \hat{U} \), it is straightforward to see that \( (G(\hat{S}), G(\hat{U})) \in \text{VCGI} \). On the other hand, if \( (G(\hat{S}), G(\hat{U})) \in \text{VCGI} \) via a permutation \( \pi \) of the vertices of \( G(\hat{S}) \), then note that vertices corresponding to constraint applications can only be permuted together with those vertices corresponding to the arguments of that constraint application. In addition, because of the coloring, the order of arguments is preserved. Thus, if \( \pi(A_i) = A_j \) then
Theorem 25. GI is polynomial-time many-one equivalent to $\text{ISO}(\{(0,1),(1,0),(1,1)\})$ and to $\text{ISO}_c(\{(0,1),(1,0),(1,1)\})$.

Proof. It suffices to show that GI $\leq^p_m$ $\text{ISO}(\{(0,1),(1,0),(1,1)\})$, since, by Theorem 23, $\text{ISO}_c(\{(0,1),(1,0),(1,1)\}) \leq^p_m$ GI.

Let $G = (V,E)$ be a graph and let $V = \{1, 2, \ldots, n\}$. We encode $G$ in the obvious way as a set of constraint applications: $S(G) = \{x_i \lor x_j \mid \{i,j\} \in E\}$. It is immediate that if $G$ and $H$ are two graphs with vertex set $\{1, 2, \ldots, n\}$, then $G$ is isomorphic to $H$ if and only if $S(G)$ is isomorphic to $S(H)$. \hfill \Box

Note that the constraint $\{(0,1),(1,0),(1,1)\}$ is the binary constraint $x \lor y$, denoted by OR$_0$ in $\text{CKS00}$. Theorem 23 can alternatively be formulated as: GI is polynomial-time many-one equivalent the isomorphism problem for positive 2CNF formulas (with or without constants). Also, from $\text{Tor00}$, we conclude that this isomorphism problem thus is hard for NL, PL, and DET.

Theorem 26. GI is polynomial-time many-one equivalent to $\text{ISO}(\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\})$ and to $\text{ISO}_c(\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\})$.

Proof. It suffices to show that GI $\leq^p_m$ $\text{ISO}(\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\})$, since, by Theorem 23, $\text{ISO}_c(\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}) \leq^p_m$ GI.

Let $G = (V,E)$ be a graph, let $V = \{1, 2, \ldots, n\}$, and enumerate the edges as $E = \{e_1, e_2, \ldots, e_m\}$. We encode $G$ as a set of XOR$_3$ constraint applications in which propositional variable $x_i$ will correspond to vertex $i$ and propositional variable $y_i$ will correspond to edge $e_i$. We encode $G$ as $S(G) = S_1(G) \cup S_2(G) \cup S_3(G)$ where

necessarily $\pi(a_{ir}) = a_{jr}$, for all $1 \leq r \leq k_i$ and $A_i$, and, because coloring is preserved, $A_i$ and $A_j$ are instances of the same constraint. This part of the permutation corresponds to a permutation of the constraint applications in the set $\hat{S}$. The remaining part of the permutation in $G(\hat{S})$ is one that solely permutes vertices corresponding to variables in $\hat{S}$, so $\pi(\hat{S}) = \hat{U}$. \hfill \Box

Note that the construction used in proof of the previous theorem can be used to provide a general upper bound on $\text{ISO}_c(\mathcal{C})$: Given sets $S$ and $U$ of constraint applications of $\mathcal{C}$ with constants, first bring $S$ and $U$ into the normal form ($\hat{S}$ and $\hat{U}$) described in the proof of the previous theorem (this can be done in polynomial time with parallel access to an NP oracle), and then determine if there exists a permutation $\pi$ such that $\pi(\hat{S}) = \hat{U}$ (this takes one query to an NP oracle). The whole algorithm takes polynomial time with two rounds of parallel queries to NP, which is equal to $P^{NP}$ (Buss and Hay $\text{BH91}$). Thus, we have the following upper bound on the isomorphism problem for constraint satisfaction:

**Theorem 24.** Let $\mathcal{C}$ be a finite set of constraints. $\text{ISO}(\mathcal{C})$ and $\text{ISO}_c(\mathcal{C})$ are in $P^{NP}$.

Finally, we show that for some simple instances of Horn, bijunctive, and affine constraints, the isomorphism problem is in fact polynomial-time many-one equivalent to the graph isomorphism problem. Proofs of these results will be given in the appendix.
- $S_1(G) = \{ x_i \oplus x_j \oplus y_k \mid e_k = \{i,j\} \}$ ($S_1(G)$ encodes the graph),
- $S_2(G) = \{ x_i \oplus z_i \oplus z_i' \mid i \in V \}$ ($S_2(G)$ will be used to distinguish $x$ variables from $y$ variables), and
- $S_3(G) = \{ y_i \oplus y_j \oplus y_k \mid e_i, e_j, e_k \text{ form a triangle in } G \}$. Note that for every $A \in S_3(G)$, $S_1(G) \rightarrow A$. We add these constraint applications to $S(G)$ to ensure that $S(G)$ is a maximum set of XOR_3 formulas.

We will show later that if $G$ and $H$ are two graphs with vertex set $\{1, 2, \ldots, n\}$ without isolated vertices, then $G$ is isomorphic to $H$ if and only if $S(G)$ is isomorphic to $S(H)$.

The proof of the theorem relies on the following lemma, which shows that $S(G)$ is a maximum set of XOR_3 formulas. This is an important property, since checking whether two maximum sets of functions are equivalent basically amounts to checking whether the sets are equal, as will be explained in detail after the proof of the lemma.

**Lemma 27.** Let $G = (V, E)$ be a graph such that $V = \{1, 2, \ldots, n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Then for every triple of distinct propositional variables $a, b, c$ in $S(G)$, the following holds: If $S(G) \rightarrow a \oplus b \oplus c$, then $a \oplus b \oplus c \in S(G)$. Note: we view $a \oplus b \oplus c$ as a function, and thus, for example, $a \oplus b \oplus c = c \oplus a \oplus b$.

**Proof.** Suppose that there exists a triple of distinct propositional variables $a$, $b$, and $c$ in $S(G)$ such that $S(G) \rightarrow a \oplus b \oplus c$ and $a \oplus b \oplus c \notin S(G)$. Let $X = \{x_i \mid i \in V\}$, $Y = \{y_i \mid e_i \in E\}$, and $Z = \{z_i, z_i' \mid i \in V\}$. Without loss of generality, assume that $a \leq b \leq c$, where $\leq$ is the following order on $X \cup Y \cup Z$:

$$x_1 < \cdots < x_n < y_1 < \cdots < y_m < z_1 < \cdots < z_n < z_1' < \cdots < z_n'.$$

The proof consists of a careful analysis of different sub-cases. We will show that in each case, there exists an assignment on $X \cup Y \cup Z$ such that that satisfies $S(G)$ but not $(a \oplus b \oplus c)$, which contradicts the assumption that $S(G) \rightarrow a \oplus b \oplus c$.

It is important to note that any assignment to $X$ can be extended to a satisfying assignment of $S(G)$.

1. If exactly three or exactly one of the variables in $\{a, b, c\}$ are in $Z$, then consider the assignment that assigns 0 to every variable in $Z$ and 1 to every variable in $X \cup Y$. Clearly, this assignment satisfies $S(G)$ but not $(a \oplus b \oplus c)$.
2. If exactly two of the variables in $\{a, b, c\}$ are in $Z$, then, since $a < b < c$, $b$ and $c$ are in $Z$. We consider the following two sub-cases, depending on whether $a \in X$ or $a \in Y$.
   (a) If $a \in X$, then set $b$ and $c$ to 1 and $a$ to 0. Since by assumption $a \oplus b \oplus c \notin S(G)$, it is easy to see that this assignment can be extended to an assignment on $X \cup Z$ that satisfies $x_i \oplus z_i \oplus z_i'$ for all $i \in V$. This assignment in turn can be extended to an assignment on $X \cup Y \cup Z$ that also satisfies every constraint application of the form $x_i \oplus x_j \oplus y_k$ for $e_k = \{i,j\}$. So, we now have an assignment that satisfies $S_1(G)$ and $S_2(G)$. Since $S_1(G) \rightarrow A$ for every constraint application $A \in S_3(G)$, it follows that this assignment also satisfies $S(G)$ while it does not satisfy $(a \oplus b \oplus c)$. 
   
   (b) If $a \in Y$, then set $b$ to 1 and $c$ to 0. Since by assumption $a \oplus b \oplus c \notin S(G)$, it is easy to see that this assignment can be extended to an assignment on $X \cup Z$ that satisfies $y_i \oplus z_i \oplus z_i'$ for all $i \in V$. This assignment in turn can be extended to an assignment on $X \cup Y \cup Z$ that also satisfies every constraint application of the form $x_i \oplus x_j \oplus y_k$ for $e_k = \{i,j\}$. So, we now have an assignment that satisfies $S_1(G)$ and $S_2(G)$. Since $S_1(G) \rightarrow A$ for every constraint application $A \in S_3(G)$, it follows that this assignment also satisfies $S(G)$ while it does not satisfy $(a \oplus b \oplus c)$. 

(b) If \( a \in Y \), let \( a = y_k \) where \( e_k = \{i, j\} \). If \( \{b, c\} = \{z_\ell, z'_\ell\} \) for some \( \ell \), then we set \( x_\ell \) to 1. In all cases, set exactly one of \( \{x_i, x_j\} \) to 1 (this could be \( x_\ell \)). Set all other elements of \( X \) to 0. We can extend this to a satisfying assignment of \( S(G) \) that does not satisfy \( (a \oplus b \oplus c) \).

3. If \( a, b, \) and \( c \) are in \( X \), then set \( a, b, \) and \( c \) to 0. It is easy to see that this assignment can be extended to an assignment on \( X \cup Z \) that satisfies \( x_i \oplus z_i \oplus z'_i \) for all \( i \in V \). This assignment in turn can be extended to an assignment on \( X \cup Y \cup Z \) that also satisfies \( x_i \oplus x_j \oplus y_k \) for \( e_k = \{i, j\} \). So, we now have an assignment that satisfies \( S(G) \) but does not satisfy \( (a \oplus b \oplus c) \).

4. If \( c \in Y \) and \( a \) and \( b \) are in \( X \), suppose that \( c = y_k \) and let \( e_k = \{i, j\} \). By the assumption that \( a \oplus b \oplus c \) is not in \( S(G) \), at least one of \( a \) and \( b \) is not in \( \{x_i, x_j\} \).

Without loss of generality, let \( a \notin \{x_i, x_j\} \). Set \( a \) to 0 and set \( X \setminus \{a\} \) to 1. This assignment can be extended to a satisfying assignment for \( S(G) \). Note that such an assignment will set \( y_k \) to 1. It follows that this assignment does not satisfy \( a \oplus b \oplus c \).

5. If \( a \in X \) and \( b \) and \( c \) are in \( Y \), then set \( a \) to 0 and \( b \) and \( c \) to 1. It is easy to see that this can be extended to a satisfying assignment for \( S(G) \).

6. If \( a, b, \) and \( c \) are in \( Y \), let \( a = y_{k_1}, b = y_{k_2}, c = y_{k_3} \) such that \( e_{k_\ell} = \{i_\ell, j_\ell\} \) for \( \ell \in \{1, 2, 3\} \). First suppose that for every \( \ell \in \{1, 2, 3\} \), for every \( x \in \{x_{i_\ell}, x_{j_\ell}\} \), there exists an \( \ell' \in \{1, 2, 3\} \) with \( \ell' \neq \ell \) and a constraint application \( A \) in \( S(G) \) such that \( x \) and \( y_{k_{\ell'}} \) occur in \( A \). This implies that every vertex in \( \{i_1, j_1, i_2, j_2, i_3, j_3\} \) is incident with at least 2 of the edges in \( e_{k_1}, e_{k_2}, e_{k_3} \). Since these three edges are distinct, it follows that the edges \( e_{k_1}, e_{k_2}, e_{k_3} \) form a triangle in \( G \), and thus \( y_{k_1} \oplus y_{k_2} \oplus y_{k_3} \in S(G) \). This is a contradiction.

So, let \( \ell \in \{1, 2, 3\}, x \in \{x_{i_\ell}, x_{j_\ell}\} \) be such that for all \( \ell' \in \{1, 2, 3\} \) with \( \ell' \neq \ell \), \( x \) and \( y_{k_{\ell'}} \) do not occur in the same constraint application in \( S(G) \). Set \( x \) to 0 and set \( X \setminus \{x\} \) to 1. This can be extended to a satisfying assignment of \( S(G) \) and such a satisfying assignment must have the property that \( y_{k_\ell} \) is 0 and \( y_{k_{\ell'}} \) is 1 for all \( \ell' \in \{1, 2, 3\} \) such that \( \ell' \neq \ell \).

How can Lemma 27 help us in the proof of Theorem 26? Note that if \( S \) and \( T \) are maximum sets of \( C \) constraint applications, then \( S \equiv T \) if and only if \( S = T \). Here equality should be seen as equality between sets of functions, i.e., \( a \oplus b \oplus c = b \oplus c \oplus a \) etc. So \( S \) is isomorphic to \( T \) if and only if there exists a permutation \( \rho \) of the variables of \( S \) such that \( \rho(S) = T \).

We will now prove Theorem 26. Let \( G \) and \( H \) be two graphs. Remove the isolated vertices from \( G \) and \( H \). If \( G \) and \( H \) thus modified do not have the same number of vertices or they do not have the same number of edges, then \( G \) and \( H \) are clearly not isomorphic. If \( G \) and \( H \) have the same number of vertices and the same number of edges, then rename the vertices in such a way that the vertex set of both graphs is \( V = \{1, 2, \ldots, n\} \). Let \( \{e_1, \ldots, e_m\} \) be an enumeration of the edges of \( G \) and let \( \{e'_1, \ldots, e'_m\} \) be an enumeration of the edges of \( H \).

We will show that \( G \) is isomorphic to \( H \) if and only if \( S(G) \) is isomorphic to \( S(H) \).
The left-to-right direction is trivial, since an isomorphism between the graphs induces an isomorphism between sets of constraint applications as follows. If \( \pi : V \to V \) is an isomorphism from \( G \) to \( H \), then \( \rho \) is an isomorphism from \( S(G) \) to \( S(H) \) defined as follows:

- \( \rho(x_i) = x_{\pi(i)} \), \( \rho(z_i) = z_{\pi(i)} \), \( \rho(z'_i) = z'_{\pi(i)} \), for \( i \in V \).
- For \( e_k = \{i, j\} \), \( \rho(y_k) = y'_k \) where \( e'_k = \{\pi(i), \pi(j)\} \).

For the converse, suppose that \( \rho \) is an isomorphism from \( S(G) \) to \( S(H) \).

By the observation above, \( \rho(S(G)) = S(H) \). Now look at the properties of the different classes of variables.

1. Elements from \( X \) are exactly those variables that occur at least twice and that also occur in an element of \( S(G) \) together with two variables that occur exactly once. So, \( \rho \) will map \( X \) onto \( X \).
2. Elements of \( Z \) are those variables that occur exactly once and that occur together with an element from \( X \) and another element that occurs exactly one. So \( \rho \) will map \( Z \) to \( Z \).
3. Everything else is an element of \( Y \). So, \( \rho \) will map \( Y \) onto \( Y \).

For \( i \in V \), define \( \pi(i) = j \) iff \( \rho(x_i) = x_j \). \( \pi \) is 1-1 onto by observation (1) above. It remains to show that \( \{i, j\} \in E \) iff \( \{\pi(i), \pi(j)\} \in E' \). Let \( e_k = \{i, j\} \). Then \( x_i \oplus x_j \oplus e_k \in S(G) \). Thus, \( \rho(x_i) \oplus \rho(x_j) \oplus \rho(y_k) \in S(H) \). That is, \( x_{\pi(i)} \oplus x_{\pi(j)} \oplus \rho(y_k) \in S(H) \). But that implies that \( \rho(y_k) = y'_k \) where \( e'_k = \{\pi(i), \pi(j)\} \). This implies that \( \{\pi(i), \pi(j)\} \in E' \). For the converse, suppose that \( \{\pi(i), \pi(j)\} \in E' \). Then \( x_{\pi(i)} \oplus x_{\pi(j)} \oplus y'_k \in S(H) \) for \( e'_k = \{\pi(i), \pi(j)\} \). It follows that \( x_i \oplus x_j \oplus \rho^{-1}(y'_k) = S(G) \). By the form of \( S(G) \), it follows that \( \{i, j\} \in E \).

Note that the constraint \( \{(1, 0, 0), (0, 1, 0), (0, 0, 0), (1, 1, 1)\} \) is the constraint \( x \oplus y \oplus z \), denoted by \( \text{NXOR}_3 \) in \([\text{CKS}00]\).

Note that the previous proof shows ISO(\( \text{NXOR}_3 \)) and ISO(\( \text{c}(\text{NXOR}_3) \)) are many-one equivalent to GI (just negate all variables). And we can replace the 3 by any \( k \geq 3 \) (just use duplicate copies of variables).

From Theorems 23 and 24, we conclude that, if we could show that isomorphism for bijunctive, anti-Horn (and, by symmetry, Horn) or affine CSPs is in \( P \), then the graph isomorphism problem is in \( P \), settling a long standing open question.

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