A Taylor-like Expansion of a Commutator with a Function of Self-adjoint, Pairwise Commuting Operators

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Abstract

Let $A$ be a $\nu$-vector of self-adjoint, pairwise commuting operators and $B$ a bounded operator of class $C^m_0(A)$. We prove a Taylor-like expansion of the commutator $[B, f(A)]$ for a large class of functions $f: \mathbb{R}^\nu \to \mathbb{R}$, generalising the one-dimensional result where $A$ is just a self-adjoint operator. This is done using almost analytic extensions and the higher-dimensional Helffer-Sjöstrand formula.

Keywords: Commutator expansions, functional calculus, almost analytic extensions, Helffer-Sjöstrand formula.

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1 Introduction

It is well-known that if $A$ is a self-adjoint operator, $B$ is a bounded operator of class $C^{n_0}(A)$ in the sense of [11] and $f$ satisfies $|f^{(n)}(x)| \leq C_n(x)^{s-n}$ for all $n$, then for $0 \leq t_1 \leq n_0$, $0 \leq t_2 \leq 1$ with $s + t_1 + t_2 < n_0$,

$$[B, f(A)] = \sum_{k=1}^{n_0-1} \frac{1}{k!} f^{(k)}(A) \text{ad}^k_A(B) + R_{n_0}(A, B)$$

where $\text{ad}^k_B(B)$ is the $k$'th iterated commutator, $R_{n_0}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}; \mathcal{H}_A^t)$ and $\mathcal{H}_A^t$ is defined as $\mathcal{D}((A)^t)$ equipped with the graph-norm $\|v\|_t = \| \langle A|^t v \rangle \|$ for $t \geq 0$ and $\mathcal{H}_A^{-t}$ is the dual space of $\mathcal{H}_A^t$. This follows relatively easily from using the (one-dimensional) Helffer-Sjöstrand formula

$$f(A) = \frac{1}{\pi} \int_C \tilde{\partial} f(z)(A - z)^{-1}dz,$$

where $\tilde{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\tilde{f}$ is an almost analytic extension of $f$, and the identity

$$[B, f(A)] = \sum_{k=1}^{n_0-1} \frac{1}{k!} \frac{1}{\pi} \int_C \tilde{\partial} f(z)(-1)^k(A - z)^{-k-1}dz$$

$$+ \frac{(-1)^{n_0}}{\pi} \int_C \tilde{\partial} f(z)(A - z)^{-n_0} \text{ad}^{n_0}_A(B)(A - z)^{-1}dz$$

when $\frac{d}{dz} \int_C \tilde{\partial} f(z)(-1)^k(A - z)^{-k-1}dz$ is recognised as $f^{(k)}(A)$ using [11]. Such commutator expansions where first proved in [7]. See e.g. [4] for details. Due to the higher complexity of the general Helffer-Sjöstrand formula, these calculations do not lead directly to the generalised result where $A$ is a vector of self-adjoint, pairwise commuting operators. However, we will follow the same idea.

The theorem may be viewed as an abstract analogue of pseudo-differential calculus. The one-dimensional version is an often used result, see e.g. [2] and [4]. Apart from the obvious interest in generalising the result to higher dimensions, our improvement has proven useful in the treatment of models in quantum field theory, see [3]. In particular, a lemma in [4] whose proof depends on our result, extends the results of [5] to a larger class of models.

2 The setting and result

In the following, $A = (A_1, \ldots, A_\nu)$ is a vector of self-adjoint, pairwise commuting operators acting on a Hilbert space $\mathcal{H}$, and $B \in \mathcal{B}(\mathcal{H})$ is a bounded operator on $\mathcal{H}$. We shall use the notion of $B$ being of class $C^{n_0}(A)$ introduced in [11]. For notational convenience, we adopt the following convention: If $0 \leq j \leq \nu$, then $\delta_j$ denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the $j$'th entry.

Definition 1. Let $n_0 \in \mathbb{N} \cup \{\infty\}$. Assume that the multi-commutator form defined iteratively by $\text{ad}^0_A(B) = B$ and $\text{ad}^\alpha_A(B) = [\text{ad}^{\alpha-\delta_j}_A(B), A_j]$ as a form on $\mathcal{D}(A_j)$, where
\(\alpha \geq \delta_j\) is a multi-index and \(1 \leq j \leq \nu\), can be represented by a bounded operator also denoted by \(\text{ad}_A^\alpha(B)\), for all multi-indices \(\alpha, |\alpha| < n_0 + 1\). Then \(B\) is said to be of class \(C^{n_0}(A)\) and we write \(B \in C^{n_0}(A)\).

**Remark 2.** The definition of \(\text{ad}_A^\alpha(B)\) does not depend on the order of the iteration since the \(A_j\) are pairwise commuting. We call \(|\alpha|\) the *degree* of \(\text{ad}_A^\alpha(B)\).

In the following, \(\mathcal{H}_A^s := D(|H|^s)\) for \(s \geq 0\) will be used to denote the scale of spaces associated to \(A\). For negative \(s\), we define \(\mathcal{H}_A^{-s} := (\mathcal{H}_A^s)^*\).

**Theorem 3.** Assume that \(B \in C^{n_0}(A)\) for some \(n_0 \geq n + 1 \geq 1\), \(0 \leq t_1 \leq n + 1\), \(0 \leq t_2 \leq 1\) and that \(\{f_\lambda\}_{\lambda \in I}\) satisfies

\[
\forall \alpha \exists C_\alpha: |\partial^\alpha f_\lambda(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}
\]

uniformly in \(\lambda\) for some \(s \in \mathbb{R}\) such that \(t_1 + t_2 + s < n + 1\). Then

\[
[B, f_\lambda(A)] = \sum_{|\alpha| = 1}^{n} \frac{1}{\alpha!} \partial^\alpha f_\lambda(A) \text{ad}_A^\alpha(B) + R_{\lambda, n}(A, B)
\]

as an identity on \(D((A)^s)\), where \(R_{\lambda, n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{s_1})\) and there exist a constant \(C\) independent of \(A, B\) and \(\lambda\) such that

\[
\|R_{\lambda, n}(A, B)\|_{\mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{s_1})} \leq C \sum_{|\alpha| = n + 1} \|\text{ad}_A^\alpha(B)\|.
\]

**Remark 4.** A similar statement holds with the \(\text{ad}_A^\alpha(B)\) and \(\partial^\alpha f_\lambda(A)\) interchanged at the cost of a sign correction given by \((-1)^{|\alpha|-1}\), and the corresponding remainder term \(R_{\lambda, n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{s_1})\). This can be seen either by proving it analogously or by taking the adjoint equation and replacing \(B\) by \(-B\).

**Remark 5.** If \(k \leq t_1\) and \(n_0 \geq n + 1 + k\), then \(R_{\lambda, n}(A, B)\) can be replaced by \(R_{\lambda, n}^k(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2+k}, \mathcal{H}_A^{s_1-k})\). This can be seen by commuting \(|A - z|^{-2}\) and \(\text{ad}_A^\alpha(B)\) in the terms of the remainder, see page \([\text{ref}].\)

### 3 The Proof

Let \(z \in \mathbb{C}^\nu, \text{ Im } z \neq 0, 1 \leq \ell \leq \nu\) and \(g, g_\ell: \mathbb{R}^\nu \to \mathbb{C}\) be given as \(g(t) = |t - z|^{-2}\) and \(g_\ell(t) = t_\ell - \bar{z}_\ell\). Write for \(2\beta \leq \alpha\)

\[
T_\alpha^\beta(t, z) := \frac{(-2)^{|\alpha - \beta|\alpha - \beta|}}{2^{\beta|\beta|}|\alpha - \beta|!} (t - \text{Re } z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|}.
\]

**Lemma 6.** Let \(g\) be as above and \(\alpha\) be any multi-index. Then

\[
\partial^\alpha g(t) = \sum_{2\beta \leq \alpha} \frac{\alpha!}{2^{\beta|\beta|}|\alpha - \beta|!} T_\alpha^\beta(t, z)|t - z|^{-2}.
\]
Proof. For brevity, we will write \( \alpha^i \) or \( \beta^i \) for \( \alpha + \delta_i \) or \( \beta + \delta_i \), respectively. The formula is obviously true for \( |\alpha| \leq 1 \). Now assume that we have proven the formula for \( |\alpha| \leq k \). Let \( |\alpha| = k \) and \( 0 \leq i \leq \nu \) be arbitrary. It suffices to prove the formula for \( \alpha^i \). One easily verifies using the chain rule that
\[
(\partial^\delta g^n)(t) = -2n(t_i - \text{Re} z_i)|t - z|^{-2n-2}.
\] (2)

Now by the induction hypothesis, we see that
\[
\partial^{\alpha+\delta_i} g(t) = \partial^\delta_i \sum_{2\beta \leq \alpha} \frac{(-2)^{[\alpha - \beta][\alpha - \beta]}(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2}}{2^{[\alpha - \beta][\alpha - 2\beta]}}(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2}
\]
\[
= \sum_{2\beta \leq \alpha} \frac{(-2)^{[\alpha - \beta][\alpha - \beta]}(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2}}{2^{[\alpha - \beta][\alpha - 2\beta]}}(\partial^\delta_i(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2})
\]
\[
+ \sum_{2\beta \leq \alpha} \frac{(-2)^{[\alpha - \beta][\alpha - \beta]}(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2}}{2^{[\alpha - \beta][\alpha - 2\beta]}}(t - \text{Re} z)^{\alpha - 2\beta}(\partial^\delta_i|t - z|^{-2|\alpha - \beta|-2}).
\] (3)

For the sake of clarity, we will now consider each sum independently.
\[
\sum_{2\beta \leq \alpha} \frac{(-2)^{[\alpha - \beta][\alpha - \beta]}(t - \text{Re} z)^{\alpha - 2\beta}(-2)(|\alpha - \beta| + 1)(t_i - \text{Re} z_i)|t - z|^{-2|\alpha - \beta|-4}}{2^{[\alpha - \beta][\alpha - 2\beta]}}
\]
\[
= \sum_{2\beta \leq \alpha} (\alpha_i + 1 - 2\beta_i)(-2)^{[\alpha - \beta][\alpha - \beta]}(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2}
\]
\[
= \sum_{2\beta \leq \alpha} \frac{(-2)^{[\alpha - \beta][\alpha - \beta]}(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2}}{2^{[\alpha - \beta][\alpha - 2\beta]}}
\]
\[
- \sum_{2\beta \leq \alpha} 2\beta_i \frac{(-2)^{[\alpha - \beta][\alpha - \beta]}(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2}}{2^{[\alpha - \beta][\alpha - 2\beta]}}.
\] (6)

Now (7) cancels (5) except for possible terms with \( 2\beta = \alpha + \delta_i \):
\[
\sum_{2\beta = \alpha + \delta_i} \frac{(-2)^{[\alpha - \beta][\alpha - \beta]}(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2}}{2^{[\alpha - \beta][\alpha - 2\beta]}}
\]
\[
= \sum_{2\beta = \alpha + \delta_i} \frac{(-2)^{[\alpha - \beta][\alpha - \beta]}(t - \text{Re} z)^{\alpha - 2\beta}|t - z|^{-2|\alpha - \beta|-2}}{2^{[\alpha - \beta][\alpha - 2\beta]}}.
\] (8)

Adding (5) and (8) finishes the induction. \( \square \)

Lemma 7. Let \( B \in C^{n_0}(A) \) for some \( n_0 \geq 1 \) and let \( n \in \mathbb{N}_0 \) and \( \alpha_0 \) be a multi-index satisfying \( |\alpha_0| + n + 1 \leq n_0 \). Then
\[
[\text{ad}^{\alpha_0}_A(B), g(A)] = \sum_{|\alpha|=1} \frac{1}{\alpha!} \partial^\alpha g(A) \text{ad}^{\alpha_0+\alpha}_A(B) + R_n^\alpha(A, \text{ad}^{\alpha_0}_A(B)),
\] (9)
where

\[
R^\beta_n (A, \text{ad}_A^{\alpha} (B)) = \sum_{|\alpha|=n-1}^{\nu} \sum_{2\beta \leq \alpha}^{\nu} \frac{\beta_i + 1}{|\alpha+\delta_i - \beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i} (A, z) \text{ad}_A^{\alpha+\alpha+2\delta_i} (B) |A - z|^{-2}
\]

(10)

\[+ \sum_{|\alpha|=n}^{\nu} \sum_{2\beta \leq \alpha}^{\nu} \frac{\beta_i + 1}{|\alpha+\delta_i - \beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i} (A, z) (A_i - \bar{z}_i) \text{ad}_A^{\alpha+\alpha+\delta_i} (B) |A - z|^{-2}
\]

(11)

\[+ \sum_{|\alpha|=n}^{\nu} \sum_{2\beta \leq \alpha}^{\nu} \frac{\beta_i + 1}{|\alpha+\delta_i - \beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i} (A, z) \text{ad}_A^{\alpha+\alpha+\delta_i} (B) (A_i - z_i) |A - z|^{-2}.
\]

(12)

**Proof.** The proof goes by induction. One may check by inspection of the following identity that the statement is true for \(n = 0\).

\[
[\text{ad}_A^{\alpha_0} (B), |A - z|^{-2}] = \sum_{i=1}^{\nu} |A - z|^{-2} (A_i - \bar{z}_i) \text{ad}_A^{\alpha_0+\delta_i} (B) |A - z|^{-2}
\]

(13)

\[- \sum_{i=1}^{\nu} |A - z|^{-2} \text{ad}_A^{\alpha_0+\delta_i} (B) (A_i - z_i) |A - z|^{-2}.
\]

Now assume that we have proven the formula for \(k \leq n, |\alpha_0| + n + 2 \leq n_0\). We will now show that this implies that the formula holds for \(n = k + 1\). We begin by noting two useful identities.

\[
T^{\beta}_{\alpha} (t, z) |t - z|^{-2} = -\frac{\beta_i + 1}{|\alpha+\delta_i - \beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i} (t, z).
\]

(14)

\[(\beta_i + 1) T^{\beta+\delta_i}_{\alpha+2\delta_i} (t, z) 2(t_i - \text{Re} z_i) = (\alpha_i + 1 - 2\beta_i) T^{\beta}_{\alpha+\delta_i} (t, z).
\]

(15)

Now using (13) and (14) we see that

\[
(10) = \sum_{|\alpha|=n}^{\nu} \sum_{2\beta \leq \alpha}^{\nu} \frac{\beta_i + 1}{|\alpha+\delta_i - \beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i} (A, z) |A - z|^{-2} \text{ad}_A^{\alpha+\alpha+2\delta_i} (B)
\]

(16)

\[+ \sum_{|\alpha|=n-1}^{\nu} \sum_{2\beta \leq \alpha}^{\nu} \sum_{j=1}^{\nu} \frac{\beta_i + 1}{|\alpha+\delta_i - \beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i} (A, z)
\]

\[\times (A_j - \bar{z}_j) \text{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j} (B) |A - z|^{-2}
\]

(17)

\[+ \sum_{|\alpha|=n-1}^{\nu} \sum_{2\beta \leq \alpha}^{\nu} \sum_{j=1}^{\nu} \frac{\beta_i + 1}{|\alpha+\delta_i - \beta|} T^{\beta+\delta_i+\delta_j}_{\alpha+2\delta_i+2\delta_j} (A, z)
\]

\[\times \text{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j} (B) (A_j - z_j) |A - z|^{-2},
\]

(18)

and by reordering and reindexing the sum in (10), (17) and (18), we get

\[
(16) = \sum_{i=1}^{\nu} \sum_{|\alpha|=n+1}^{\nu} \frac{\beta_i}{\alpha_i \geq 2 \beta_i \geq 1} T^{\beta}_{\alpha} (A, z) |A - z|^{-2} \text{ad}_A^{\alpha_0+\alpha} (B),
\]

(19)
and (17) equals
\[
\sum_{|\alpha|=n+1}^{\nu} \sum_{2\beta \leq \alpha, \alpha_i \geq 1}^{\nu} \sum_{\beta_i \geq 1}^{\nu} \beta_i \frac{\beta_{i+1}}{\alpha - \beta_{i+1}} T_{\alpha + \beta_j}^\beta (A, z) (A_j - z_j) \text{ad}_A^{\alpha + \alpha_0 + \delta_j} (B) |A - z|^{-2}
\]
(20)

and similarly for (18) with the factor \( (A_j - z_j) \text{ad}_A^{\alpha + \alpha_0 + \delta_j} (B) \) replaced by the factor \( \text{ad}_A^{\alpha + \alpha_0 + \delta_j} (B) (A_j - z_j) \). Note that we may relax the extra conditions on \( \alpha \) and \( \beta \) in the above statements, as a term with \( \beta_i = 0 \) contributes nothing.

Instead of continuing in the same fashion with (11) and (12), we note using (15) that
\[
(11) + (12) = \sum_{|\alpha|=n}^{\nu} \sum_{2\beta \leq \alpha, \alpha_i \geq 1}^{\nu} \sum_{\beta_i \geq 1}^{\nu} \beta_i \frac{\beta_{i+1}}{\alpha - \beta_{i+1}} T_{\alpha + \beta_j}^\beta (A, z) \text{ad}_A^{\alpha + \alpha_0 + \delta_j} (B) |A - z|^{-2}
\]
(21)

so we may focus our attention on (22):
\[
(22) = \sum_{|\alpha|=n+1}^{\nu} \sum_{2\beta \leq \alpha, \alpha_i \geq 1}^{\nu} \sum_{\beta_i \geq 1}^{\nu} \beta_i \frac{\beta_{i+1}}{\alpha - \beta_{i+1}} T_{\alpha + \beta_j}^\beta (A, z) |A - z|^{-2} \text{ad}_A^{\alpha + \alpha_0 + \alpha} (B)
\]
(23)

\[
+ \sum_{|\alpha|=n+1}^{\nu} \sum_{2\beta \leq \alpha, \alpha_i \geq 1}^{\nu} \sum_{\beta_i \geq 1}^{\nu} \beta_i \frac{\beta_{i+1}}{\alpha - \beta_{i+1}} T_{\alpha + \beta_j}^\beta (A, z)
\]
\[
\times (A_j - z_j) \text{ad}_A^{\alpha + \alpha_0 + \delta_j} (B) |A - z|^{-2}.
\]
(24)

\[
+ \sum_{|\alpha|=n+1}^{\nu} \sum_{2\beta \leq \alpha, \alpha_i \geq 1}^{\nu} \sum_{\beta_i \geq 1}^{\nu} \beta_i \frac{\beta_{i+1}}{\alpha - \beta_{i+1}} T_{\alpha + \beta_j}^\beta (A, z)
\]
\[
\times \text{ad}_A^{\alpha + \alpha_0 + \alpha_0 + \delta_j} (B) (A_j - z_j) |A - z|^{-2}
\]
(25)

We note again that the additional conditions on \( \alpha \) and \( \beta \) are superfluous.

We may now recollect the terms. First we see using Lemma 3
\[
\sum_{|\alpha|=n}^{\nu} \frac{1}{\alpha} \partial^\alpha g(A) \text{ad}_A^{\alpha + \alpha_0} (B) + (19) + (23) = \sum_{|\alpha|=n}^{\nu} \frac{1}{\alpha} \partial^\alpha g(A) \text{ad}_A^{\alpha + \alpha_0} (B),
\]
(26)

then
\[
(20) + (24) = \sum_{|\alpha|=n+1}^{\nu} \sum_{2\beta \leq \alpha}^{\nu} \beta_i \frac{\beta_{i+1}}{\alpha + \beta_j} T_{\alpha + \beta_j}^\beta (A, z) (A_j - z_j) \text{ad}_A^{\alpha + \alpha_0 + \delta_j} (B) |A - z|^{-2},
\]
(27)

and
\[
(18) + (25) = \sum_{|\alpha|=n+1}^{\nu} \sum_{2\beta \leq \alpha}^{\nu} \beta_i \frac{\beta_{i+1}}{\alpha + \beta_j} T_{\alpha + \beta_j}^\beta (A, z) \text{ad}_A^{\alpha + \alpha_0 + \delta_j} (B) (A_j - z_j) |A - z|^{-2},
\]
(28)
so adding up, we have proved that \( g_\ell \) equals the sum of \((26), (21), (27) \) and \((28)\) as stated.

The following lemma plays the same role for \( g_\ell \) as Lemma \( 7 \) plays for \( g \), but contrary to Lemma \( 7 \), the proof is trivial.

**Lemma 8.** Let \( B \in C^{n_0}(A) \) for some \( n_0 \geq 1 \) and let \( n \in \mathbb{N}_0 \) and \( \alpha_0 \) be a multi-index satisfying \(|\alpha_0| + n + 1 \leq n_0\). Then

\[
[\text{ad}_{\partial}^{\alpha_0}(B), g_\ell(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^{\alpha} g_\ell(A) \text{ad}_{\partial}^{\alpha_0+\alpha}(B) + R^\ell(A, \text{ad}_{\partial}^{\alpha_0}(B)),
\]

where \( R^\ell_n(A, \text{ad}_{\partial}^{\alpha_0}(B)) = 0 \) for \( n \geq 1 \), \( R^\ell_0(A, \text{ad}_{\partial}^{\alpha_0}(B)) = \text{ad}_{\partial}^{\alpha_0+\delta}(B) \).

The following lemma also follows by induction.

**Lemma 9.** Let \( B \in C^{n_0}(A) \) for some \( n_0 \geq 1 \). Assume that \( h_i \in C^\infty(\mathbb{R}^\nu), 1 \leq i \leq k \), satisfies

\[
[\text{ad}_{\partial}^{\alpha_0}(B), h_i(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^{\alpha} h_i(A) \text{ad}_{\partial}^{\alpha_0+\alpha}(B) + R^i_n(A, \text{ad}_{\partial}^{\alpha_0}(B)),
\]

where \( R^i_n(A, \text{ad}_{\partial}^{\alpha_0}(B)) \) is bounded for all \( n \in \mathbb{N}_0 \) and multi-indices \( \alpha_0 \) satisfying \(|\alpha_0| + n + 1 \leq n_0 \) and \( \partial^{\alpha} h_i(A) \) is bounded for all \( 1 \leq |\alpha| \leq n_0 - 1 \). Then

\[
[B, \prod_{i=1}^{k} h_i(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^{\alpha} \left( \prod_{i=1}^{k} h_i \right)(A) \text{ad}_{\partial}^{\alpha_0}(B)
\]

\[
+ \sum_{j=1}^{k} \sum_{|\alpha|=0}^{n} \frac{1}{\alpha!} \partial^{\alpha} \left( \prod_{i=1}^{j-1} h_i \right)(A) R^i_{n-|\alpha|}(A, \text{ad}_{\partial}^{\alpha_0}(B)) \prod_{i=j+1}^{k} h_i(A).
\]

Let \( n + 1 \leq n_0 \). If we put \( k = \nu + 1 \), \( h_i = g \) for \( i \neq \nu \), \( h_\nu = g_\ell \) and apply Lemma \( 7, 8 \) and \( 9 \) we see that

\[
[B, |A - z|^{-2\nu}(A_\ell - \bar{z}_\ell)]
\]

\[
= \sum_{|\alpha|=1}^{\nu-1} \frac{1}{\alpha!} \partial^{\alpha} \left( |\cdot - z|^{-2\nu}(\cdot - \bar{z}_\ell) \right)(A) \text{ad}_{\partial}^{\alpha}(B) + R_{\ell,n}(A, B)
\]

where

\[
R_{\ell,n}(A, B) \]

\[
= \sum_{j=1}^{\nu-1} \sum_{|\alpha|=0}^{n} \frac{1}{\alpha!} \partial^{\alpha} (g^{j-1})(A) R^j_{n-|\alpha|}(A, \text{ad}_{\partial}^{\alpha}(B)) |A - z|^{-2(\nu-j)}(A_\ell - \bar{z}_\ell)
\]

\[
+ \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^{\alpha} (g^{\nu-1})(A) \text{ad}_{\partial}^{\alpha+\delta}(B) |A - z|^{-2}
\]

\[
+ \sum_{|\alpha|=0}^{n} \frac{1}{\alpha!} \partial^{\alpha} (g^{\nu-1}g_\ell)(A) R^\ell_{n-|\alpha|}(A, \text{ad}_{\partial}^{\alpha}(B))
\]

\[
(30)
\]

\[
(31)
\]

\[
(32)
\]
In the following, we will refer to the terms of $R_{\ell,n}(A, B)$ as the remainder terms. Let $0 \leq t_1 \leq n+1$ and $0 \leq t_2 \leq 1$. By Hadamard’s three-line lemma and using (10) [12], Lemma 6 and the identity

$$\partial^\alpha \left( \prod_{i=1}^j f_i \right) = \sum_{\alpha_i=0} a! \prod_{i=1}^j \partial^\alpha_i f_i,$$

we may inspect that each remainder term (with $R_{\ell,n}(A, B)$ replaced by the remainder term) and hence $R_{\ell,n}(A, B)$ satisfies the inequality

$$\| \langle A \rangle^{t_1} R_{\ell,n}(A, B) \langle A \rangle^{t_2} \| \leq C \langle z \rangle^{t_1+t_2} |\text{Im} z|^{-n-2\nu}.$$  \hfill (33)

We will now use the functional calculus of almost analytic extensions. See e.g. [3] for details. In the following, we write $\partial = (\partial_1, \ldots, \partial_\nu)$ where $\partial_j = \frac{1}{2}(\partial_{u_j} + i\partial_{v_j})$ and $u_j + v_j = z_j \in \mathbb{C}$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^\nu$. The following proposition is inspired by [4] and [8 Chap. X.2].

**Proposition 10.** Let $s \in \mathbb{R}$ and $\{f_\lambda\}_{\lambda \in I} \subset C^\infty(\mathbb{R}^\nu)$ satisfy

$$\forall \alpha \exists C_\alpha: |\partial^\alpha f_\lambda(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}.$$

There exists a family of almost analytic extensions $\{\tilde{f}_\lambda\}_{\lambda \in I} \subset C^\infty(\mathbb{C}^\nu)$ satisfying

(i) $\text{supp}(\tilde{f}_\lambda) \subset \{u + iv \mid u \in \text{supp}(f_\lambda), |v| \leq C\langle u \rangle\}.$

(ii) $\forall \ell \geq 0 \exists C_\ell: |\partial^\ell \tilde{f}_\lambda(z)| \leq C_\ell \langle z \rangle^{s-\ell-1}|\text{Im} z|^\ell.$

**Proof.** We define a mapping $C^\infty(\mathbb{R}^\nu) \ni f \mapsto \tilde{f} \in C^\infty(\mathbb{C}^\nu)$ in the following way. Choose a function $\kappa \in C_0^\infty(\mathbb{R})$ which equals 1 in a neighboorhood of 0 and put $\lambda_0 = C_0$, $\lambda_k = \max\{\max_{|\alpha|=k} C_\alpha, \lambda_{k-1} + 1\}$ for $k \geq 1$. Writing $z = u + iv \in \mathbb{R}^\nu + i\mathbb{R}^\nu$, we now define

$$\tilde{f}(z) = \sum_\alpha \frac{\partial^\alpha f(u)}{\alpha!} (iu)^\alpha \prod_{j=1}^\nu \kappa \left( \frac{\lambda_{\text{max}_j} v_j}{\langle u \rangle} \right).$$

One can now check that the properties hold. \hfill \Box

**Remark 11.** Note that if we for a $\chi \in C_0^\infty([0, 1]; \mathbb{R}^\nu)$ with $\chi(0) = 1$ define a sequence of functions by $f_{k,\lambda}(x) = \chi(\frac{x}{k}) f_\lambda(x)$, then

$$[B, f_\lambda(A)] = \lim_{k \to \infty} [B, f_{k,\lambda}(A)]$$

as a form identity on $\mathcal{D}((A)^s)$ and we have the dominated pointwise convergence

$$\partial \tilde{f}_{k,\lambda}(x) \to \partial \tilde{f}_\lambda(x)$$

for $k \to \infty$.

Let $\{f_\lambda\}_{\lambda \in I}$ satisfy the assumption of Proposition 10 with $s < 0$. Then the almost analytic extensions provide a functional calculus via the formula

$$f_\lambda(A) = C_\nu \sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \partial_\ell \tilde{f}_\lambda(z) (A_\ell - \bar{z}_\ell) |A - z|^{-2\nu} dz,$$  \hfill (34)
where \( C_\nu \) is a positive constant (again we refer to [3] for details). Note that the integrals are absolutely convergent by Proposition 10(ii).

Multiplying \( \langle A \rangle^{t_1} R_{\ell,n}(A,B) \langle A \rangle^{t_2} \) with \( \bar{\partial} \tilde{f}_\lambda(z) \), we get from (33) and Proposition 10(ii) that

\[
\| \langle A \rangle^{t_1} \bar{\partial} \tilde{f}_\lambda(z) R_{\ell,n}(A,B) \langle A \rangle^{t_2} \| \leq C(z)^{t_1 + t_2 + s - n - 1 - 2\nu}.
\]

(35)

Hence, if \( t_1 + t_2 + s < n + 1 \), \( \langle A \rangle^{t_1} \bar{\partial} \tilde{f}_\lambda(z) R_{\ell,n}(A,B) \langle A \rangle^{t_2} \) is integrable over \( \mathbb{C}^\nu \). Using (29), (34) and (35), we see that

\[
[B, f_\lambda(A)] = C_\nu \sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) [B, (A_\ell - \bar{z}_\ell)] |A - z|^{-2\nu} \, dz \\
= C_\nu \sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha (|t - z|^{-2\nu} (\cdot - \bar{z}_\ell))(A) \, dz \, \text{ad}_{A_\ell}^\alpha(B) \\
+ C_\nu \sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) R_{\ell,n}(A,B) \, dz.
\]

(36)

We denote (36) by \( R_{\lambda,n}(A,B) \). Note that

\[
\sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) \frac{1}{\alpha!} \partial^\alpha (|t - z|^{-2\nu} (t_\ell - \bar{z}_\ell)) \, dz \\
= \frac{1}{\alpha!} \partial^\alpha \sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) |t - z|^{-2\nu} (t_\ell - \bar{z}_\ell) \, dz = \frac{1}{\alpha!} \partial^\alpha f_\lambda(t),
\]

which implies

\[
[B, f_\lambda(A)] = \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha f_\lambda(A) \, \text{ad}_{A_\ell}^\alpha(B) + R_{\lambda,n}(A,B).
\]

We have now proved Theorem \ref{thm:main} in the case \( s < 0 \). For the general case, we use Remark 11 to see that \( [B, f_\lambda(A)] = \lim_{k \to \infty} [B, f_{k,\lambda}(A)] \) and clearly, \( f_{k,\lambda} \) satisfies the assumption of Proposition 10 with the same \( s \), so the estimate corresponding to (35) is now uniform in \( k \) and \( \lambda \). The pointwise convergence and Lebesgue’s theorem on dominated convergence now finishes the argument.

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References

[1] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu. $C_0$-Groups, Commutator Methods and Spectral Theory of N-body Hamiltonians. Birkhäuser, 1996.

[2] J. Dereziński and C. Gérard. Scattering Theory of Classical and Quantum N-Particle systems. Texts and Monographs in Physics. Springer, Berlin, 1997.

[3] M. Dimassi and J. Sjöstrand. Spectral Asymptotics in the Semi-Classical Limit, volume 268 of London Mathematical Society Lecture Note Series. Cambridge University Press, 1999.

[4] J. S. Møller. An abstract radiation condition and application to N-body systems. Rev. Math. Phys., 12(5):767–803, 2000.

[5] J. S. Møller. The translation invariant massive Nelson model: I. the bottom of the spectrum. Ann. Henri Poincaré, 6:1091–1135, 2005.

[6] J. S. Møller and M. G. Rasmussen. The translation invariant massive Nelson model: II. The continuous spectrum below the two-boson threshold. Submitted.

[7] I. M. Sigal and A. Soffer, The N-particle scattering problem: Asymptotic completeness for short-range quantum systems, Ann. Math. 125 (1987), 35–108.

[8] F. Treves. Introduction to Pseudodifferential Operators and Fourier Integral Operators, volume 2. Plenum Press, 1980.