A NORMAL FORM FOR HNN-EXTENSION OF DIALGEBRAS

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Abstract. We consider a new version of Composition-Diamond Lemma for dialgebras in order to obtain an explicit Groebner-Shirshov basis for HNN-extension of dialgebras and determine a normal form for that.

Introduction

Associative dialgebras or dialgebras were introduced by Loday [11] and defined as $K$-vector spaces equipped with two associative $K$-linear products $⊣,⊢ : D \times D \to D$, called respectively, the left product and the right product, which satisfy the associativity laws:

\[
x ⊣ (y ⊢ z) = x ⊣ (y ⊢ z),
\]

\[
(x ⊣ y) ⊢ z = x ⊢ (y ⊢ z),
\]

\[
x ⊢ (y ⊣ z) = (x ⊢ y) ⊣ z,
\]

for all $x, y, z \in D$. Dialgebras are closely connected to the notion of Leibniz algebras in the same way as the associative algebras are connected to Lie algebras. Indeed, Loday in [12] showed that any dialgebra $(D, ⊣, ⊢)$ becomes a Leibniz algebra $D_{Leib}$ under the Leibniz bracket $[x, y] := x ⊣ y - y ⊢ x$ and the universal enveloping algebra of a Leibniz algebra has the structure of a dialgebra. Interesting properties of dialgebras allow to extend classical results. For instance, Bremner et al. in [4] obtained a new variety of nonassociative triple systems and provided a generalized statement of the BSO algorithm called Jordan triple disystems. Also, the concept of digroups as a generalization of continuous groups and dialgebra digroup have been studied by Salazar-Díaz et al. [13]. Some combinatorial studies of dialgebras can be found in [9], [20] and [21] as well.

The concept of HNN-extension is an important construction in combinatorial group theory and it was originally introduced by Higman, Neumann and Neumann in [6] stating that if $A_1$ and $A_2$ are isomorphic subgroups of a group $S$, then it is possible to find a group $H$ containing $S$ such that $A_1$ and $A_2$ are conjugate to each other in $H$ and $S$ is embeddable in $H$. The concept of HNN-extension was constructed for Lie algebras in independent works by Lichtman and Shirvani [10] and Wasserman [17], and it has recently been spread to generalized versions of Lie algebras, namely, Leibniz algebras, Lie superalgebras, Hom-setting of Lie algebras in [8] and [8], [9] and [10], respectively. The HNN-extension of dialgebras was firstly introduced in [9] as an effective approach for the construction of HNN-extensions of Leibniz algebras. Ladra et al. in [9] used Shirshov’s algorithm [14] [15] in order to
show that every dialgebra embeds inside its HNN-extension and then proved analogous embedding theorem for the case of Leibniz algebras. In this note, we intend to obtain an explicit Groebner-Shirshov basis for the HNN-extension of dialgebras by using a new version of Composition-Diamond Lemma. The Composition-Diamond Lemma (CD-Lemma, for short) is an essential concept in combinatorial algebra and the key ingredient of Groebner-Shirshov bases theory. It is used to solve various problems such as normal form, word problem, extensions and embedding theorems. The theory of Groebner-Shirshov bases is the parallel theory to Groebner bases [5] introduced for ideals of free (commutative, anti-commutative) nonassociative algebras, free Lie algebras and simplicity free associative algebras by Shirshov (see [1], [14]), and it has been actively developed to different algebraic structure since two decades ago. The first version of Groebner-Shirshov bases theory for associative dialgebras was introduced by Bokut et al., in [3], and they gave Groebner-Shirshov bases for the universal enveloping algebra of a Leibniz algebra, the bar extension of a dialgebra, the free product of two dialgebras and Clifford dialgebra. The new version of Composition-Diamond Lemma was introduced by Zhang and Chen in [19] based on an arbitrary monomial-center ordering. Zhang and Chen compared their version with the corresponding results in Bokut’s paper [3] and showed that it is useful and convenient in calculation of Groebner-Shirshov bases for free dialgebras. Moreover, they provided a method to find normal forms of elements of an arbitrary disemigroup. Interested reader in Groebner-Shirshov bases and applications is encouraged to study the recently published book by Bokut et al., [2].

The remainder of this note is organized as follows. In the first section, we recall the new version of CD-Lemma. In the second section, we construct HNN-extension of dialgebras and employ CD-Lemma in order to obtain an explicit Groebner-Shirshov basis and normal form for that.

1. New CD-Lemma for dialgebras

Let \( D(X) \) be the free dialgebra over a field \( K \) generated by a well-ordered set \( X \) and \( X^+ \) the free semigroup generated by \( X \) without the unit. For any \( u = x_1 \ldots x_m \ldots x_n \in X^+ \) with \( x_i \in X \), a normal diword is written as

\[
[u]_m = x_1 \ldots x_{m-1} x_m x_{m+1} \ldots x_n = x_1 \vdash \cdots \vdash x_{m-1} \vdash x_m \dashv x_{m+1} \dashv \cdots \dashv x_n.
\]

Write

\[
[X^+]_\omega = \{ [u]_m \mid u \in X^+, m \in \mathbb{Z}^+, 1 \leq m \leq |u| \},
\]

the set of all normal diwords on \( X \) which is a linear basis of free dialgebra \( D(X) \), and \( |u| \) is the number of letters in \( u \). Therefore, any polynomial \( f \in D(X) \) has the form

\[
f = \alpha[f]_n + \sum_{[u]_m \in [X^+]} \alpha_i [u_i]_m,
\]

where \([f]_n\) and \([u_i]_m\) are normal diwords in \( X \), \([f]_n > [u_i]_m\). Polynomial \( f \) is called left (right) normed if the position of the center letter is in the most right(left) side of the normal form. For any \( h = [u]_m \in [X^+] \), we call \( u \) the associative word of \( h \), and \( m \), the position of center of \( h \), is denoted by \( p(h) \). For example, if \( u: = x_1 x_2 \ldots x_n \in X^+, x_i \in X, h = [u]_m, 1 \leq m \leq n \), then \( p(h) = m \), and with the notation as in [3], \([u]_m: = x_1 \vdash \cdots \vdash x_{m-1} \vdash x_m \dashv x_{m+1} \dashv \cdots \dashv x_n\).
Let $X$ be a totally ordered set. The monomial ordering according to the Bokut et al.’s approach (lexicographic-weight) $<\text{lex}$ on normal diwords is defined as follows.

$$[u] < [v] \Leftrightarrow \text{wt}([u]) <_{\text{lex}} \text{wt}([v]) \text{ (lexicographically)},$$

where $\text{wt}([u]) = (n + m + 1, m, x_{-m}, \ldots, x_0, \ldots, x_n)$ with $[u] = x_{-m} \cdots x_0 \cdots x_n$.

Let $S \subset D(X)$ be a monic subset of polynomials such that $Id(S)$ is the ideal of $D(X)$ generated by $S$. An $S$-diword is a diword in $X \cup S$ with only one occurrence of $s \in S$. A normal diword $[u]_m$ is said to be $S$-irreducible if $[u]_m$ is not equal to the leading monomial of any normal $S$-diword. Let $Irr(S)$ be the set of all $S$-irreducible diwords. Consider the following statements:

(i) $S$ is a Groebner-Shirshov basis in $D(X)$.

(ii) $Irr(S)$ is a $k$-basis of $D(X|S) = D(X)/Id(S)$.

It is shown in [3] that (i) $\Rightarrow$ (ii) but (ii) $\not\Rightarrow$ (i). The main difference between the above result with the new version of Composition-Diamond Lemma in [13] is the ordering defined on $[X^+]$. In fact, Bokut et al., in [3] considered a fixed ordering and special definition of composition trivial modulo $S$, whereas Zhang et al., in [13] introduced monomial-center ordering on $[X^+])$ which makes the two above conditions equivalent. In the sequel, we recall the new version of Composition-Diamond Lemma in conformity with [13].

**Definition 1.1.** Let $>$ be a deg-lex ordering on $X^+$. The deg-lex-center ordering $>_d$ on $[X^+]_\omega$ is defined as follows. For any $[u]_m, [v]_n \in [X^+]_\omega$,

$$[u]_m >_d [v]_n \text{ if } (u, m) > (v, n) \text{ lexicographically.}$$

For any nonzero polynomial $f \in D(X)$, let us denote by $\bar{f}$ the leading monomial of $f$ with respect to the ordering $>$, $lt(f)$ the leading term of $f$, $lc(f)$ the coefficient of $\bar{f}$ and $\tilde{f}$ the associative word of $\bar{f}$. Polynomial $f$ is called monic if $lc(f) = 1$. A nonempty subset $S$ of $D(X)$ is called monic if $s$ is monic for all $s \in S$.

**Definition 1.2.** A nonzero polynomial $f \in D(X)$ is strong if $\bar{f} > r_f$, where $r_f : = f - lt(f)$.

An $S$-diword $g$ is a normal diword on $X \cup S$ with only one occurrence of $s \in S$. If this is the case and

$$g = [x_{i_1} \cdots x_{i_k} \cdots x_{i_n}]_m |_{x_{i_k} \cdots x_{i_n}},$$

where $1 \leq k \leq n$, $x_{i_l} \in X$, $1 \leq j \leq n$, then we also call $g$ an $s$-diword.

**Definition 1.3.** An $S$-diword is called a normal $S$-diword if either $k = m$ or $s$ is strong.

Let $(asb)$ be a normal $S$-diword. Then $(asb) = [a\tilde{s}b]_l$ for some $l \in P((asb))$, where

$$P((asb)) : = \{n \in \mathbb{Z}^+ \mid 1 \leq n \leq |a| \} \cup \{|a| + p(\tilde{s})\}$$

$$\cup \{n \in \mathbb{Z}^+ \mid |a\tilde{s}| < n \leq |a\tilde{s}b|\},$$

if $s$ is strong, otherwise,

$$P((asb)) : = \{|a| + p(\tilde{s})\}.$$  

In the following, we recall the available compositions between monic polynomials in $D(X)$.

**Definition 1.4.** Let $f$ and $g$ be monic polynomials in $D(X)$.
(i) If $f$ is not strong, then $x + f$ is called the composition of left multiplication of $f$ for all $x \in X$ and $f \vdash [u]_w$ is called the composition of right multiplication of $f$ for all $u \in X^+$.

(ii) Suppose that $w = \hat{f} = a\tilde{g}b$ for some $a, b \in X^*$ and $(agb)$ is a normal $g$-diword.

(a) If $p(\hat{f}) \in P([agb])$, then the composition of inclusion of $f$ and $g$ is defined as

$$(f, g)_f = f - [agb]_{p(f)}.$$ 

(b) If $p(\hat{f}) \notin P([agb])$ and both $f$ and $g$ are strong, then for any $x \in X$ the composition of left multiplication inclusion is defined as

$$(f, g)_{[wx]_1} = [xf]_1 - [xagb]_1$$

and

$$(f, g)_{[wx]_{wx}} = [fx]_{wx} - [agbx]_{wx}$$

is called the right multiplicative inclusion of $f$ and $g$.

(iii) Suppose that there exists $w = \hat{fb} - a\tilde{g}$ for some $a, b \in X^*$ such that $|\hat{f}| + |\tilde{g}| > |w|$, $(fb)$ is a normal $f$-diword and $(ag)$ is a normal $g$-diword.

(a) If $P([f\hat{b}]) \cap P([ag]) \neq 0$, then for any $m \in P([f\hat{b}]) \cap P([ag])$ we call

$$(f, g)_{[w]_m} = [fb]_m - [ag]_m$$

doing the composition of intersection of $f$ and $g$.

(b) If $P([f\hat{b}]) \cap P([ag]) = 0$ and both $f$ and $g$ are strong, then for any $x \in X$ we call

$$(f, g)_{[wx]_1} = [xf\hat{b}]_1 - [xag]_1$$

doing the composition of left multiplicative intersection of $f$ and $g$, and

$$(f, g)_{[wx]_{wx}} = [f\hat{bx}]_{wx} - [agx]_{wx}$$

doing the composition of right multiplicative intersection of $f$ and $g$.

**Triviality criteria.** Let $S$ be a monic subset of $D\langle X \rangle$. A polynomial $h \in D\langle X \rangle$ is trivial modulo $S$ and denoted by

$$h \equiv 0 \mod (S)$$

if $h = \sum \alpha_i [a_i s_i b_i]_{m_i}$, where $\alpha_i \in K$, $a_i, b_i \in X^*$, $s_i \in S$ and $[a_i s_i b_i]_{m_i} \leq \tilde{h}$.

A monic set $S$ is called Groebner-Shirshov basis in $D\langle X \rangle$ if any composition of polynomials in $S$ is trivial modulo $S$. The next theorem is the new version of Composition-Diamond Lemma (CD-Lemma) for the case of dialgebras with respect to monomial-center ordering and new triviality criteria.

**Theorem 1.5.** Let $S$ be a monic subset of $D\langle X \rangle$, $> a$ deglex-center ordering on $[X^+]$ and $Id(S)$ the ideal of $D\langle X \rangle$ generated by $S$. Then the following statements are equivalent:

(i) $S$ is a Groebner-Shirshov basis in $D\langle X \rangle$.

(ii) $f \in Id(S)$ implies $\hat{f} = a\tilde{b}b_{m}$ for some normal $S$-diword $[asb]_{m}$.

(iii) $Irr(S) = \{[u]_n \in [X^+] | [u]_n \neq [asb]_{m} \text{ for any normal } S \text{-diword } [asb]_{m} \}$ is a $K$-basis of $D\langle X \rangle|S = D\langle X \rangle/Id(S)$. 

2. HNN-EXTENSION OF DIALGEBRAS

Definition 2.1. For a dialgebra \( D \), a derivation is a map \( d: D \to D \), which is linear and satisfies: 
\[
d(x \cdot y) = d(x) \cdot y + x \cdot d(y) \quad \text{and} \quad d(x \cdot y) = d(x) \cdot y + x \cdot d(y),
\]
for all \( x, y \in D \).

Let \( D \) be a dialgebra and \( A \) be a subalgebra of \( D \). Let \( d: A \to D \) be a derivation defined on the subalgebra \( A \). Then the corresponding HNN-extension is defined as
\[
D'_d = \langle D, t \mid a \cdot t - t \cdot a = d(a), \ a \in A \rangle.
\]
Here \( t \) is a new symbol not belonging to \( D \). Let assume that \( X' = X \cup \{ t \} \), where \( X \) is a well-ordered basis of \( D \) and \( t \) \( \prec \) \( X \). Let also denote by \( Y \) the basis of \( A \). We consider the following polynomials
\[
\begin{align*}
& f(x, y) = [xy]_1 - \sum_v \alpha_{xy}^v v, \\
& g(x, y) = [xy]_2 - \sum_v \beta_{xy}^v v, \\
& h_z = [zt]_1 - [tz]_2 - \sum_v \delta_z^v v,
\end{align*}
\]
where \( v \) is an arbitrary element in \( X \), and \( x, y \in X \) such that \( x \prec y \) and \( z \in Y \). Let us consider \( D(X' \mid S) \) as a presentation of HNN-extension \( D'_d \) through structural constants, where \( S = \{ f, g, h_z \} \). We consider \( x \cdot y = \sum_v \alpha_{xy}^v v \), for some \( \alpha_{xy}^v \in K \). Similarly, we have \( x \cdot y = \sum_v \beta_{xy}^v v \), for some scalars \( \beta_{xy}^v \). Note that these scalars satisfy some relations according to the associativity laws; i.e. we have
\[
\begin{align*}
\sum_{v,u} \beta_{yz}^v \alpha_{xy}^u &= \sum_{v,u} \alpha_{yz}^v \alpha_{xy}^u, \\
\sum_{v,u} \alpha_{xy}^v \beta_{yz}^u &= \sum_{v,u} \beta_{xy}^v \beta_{yz}^u, \\
\sum_{v,u} \alpha_{xy}^v \beta_{xz}^u &= \sum_{v,u} \beta_{xy}^u \beta_{xz}^v.
\end{align*}
\]
Note also that, since \( A \) is a subalgebra, so for \( x, y \in Y \) and \( v \in X \setminus Y \), we have \( \alpha_{xy}^v = \beta_{xy}^v = 0 \). Consider the derivation \( d: A \to D \). For any \( x \in Y \), there are scalars \( \delta_z^v \) such that
\[
d(x) = \sum_v \delta_z^v v,
\]
and the Definition 2.1 implies that
\[
d(\sum_v \alpha_{xy}^v v) = \sum_v \delta_z^v v + y + \sum_v \delta_y^v x + v.
\]

Proposition 1. Let \( X = \{ x_i | i \in I \} \) be a well-ordered set and \( D'_d \langle X \mid S \rangle \) be the presentation of HNN-extension of dialgebra \( D \), where \( S = \{ f(x, y), g(x, y), h_z \} \) are the polynomials through structural constants. Then
\begin{itemize}
\item [(i)] The relations \( S = \{ f, g, h \} \) form a Groebner-Shirshov basis for HNN-extension of dialgebras with respect to deg-lex-center ordering.
\item [(ii)] The set \[
\begin{align*}
& \{ [x_{i_1} \ldots x_{i_l}]_1 | x_{i_1} \leq \cdots \leq x_{i_{l-1}}, \ x_{i_l} \in X, 1 \leq l \leq n, \ n \in \mathbb{Z}^+ \} \\
& \cup \{ [x_{j_1} \ldots x_{j_m}]_m | x_{j_1} \leq \cdots \leq x_{j_m}, \ x_{j_k} \in X, 1 \leq k \leq m, \ m \in \mathbb{Z}^+ \} \\
& \cup \{ [tx_{j_1} \ldots x_{j_m}]_m | x_{j_1} \leq \cdots \leq x_{j_m}, x_{j_k} \in X \} \\
& \cup \{ [tx_{j_1} \ldots x_{j_m}]_{m+1} | x_{j_1} \leq \cdots \leq x_{j_m}, x_{j_k} \in X \}
\end{align*}
\end{itemize}
Proof. (i) We compute all possible compositions between elements of S. Let us denote by, for example, \( f \land g \) the composition of the polynomials \( f \) and \( g \). We note that \( f, g \) and \( h \) are strong polynomial. Let assume that \( x > y > z \). We firstly check the intersection compositions. We put \( f \land g \). The intersection composition \( (f, g) \), \( v, w \) and \( \{v, w\} = \{1\} \). Therefore,

\[
(f_1, f_2)[xyz]_1 = [xyz]_1 - \sum_v \alpha^v_{xy} v + z - [xyz]_1 + \sum_u \alpha^{uv}_{yz} x + v
\]

\[
= \sum_v \alpha^{vz}_{yz} x + v - \sum_v \alpha^v_{xy} v + z
= \sum_u \alpha^{vz}_{yz} x + u - \sum_u \alpha^v_{xy} v + w
= \sum_u \beta^v_{yz} u + v
\]

Also, we put \( g_1 = [xyz]_2 - \sum_v \beta^v_{xy} v \) and \( g_2 = [xyz]_2 - \sum_v \beta^v_{yz} v \), then for the intersection composition \( g_1 \land g_2 \) we have \( w = xyz \) and \( P[g_1z] \cap P[xg_2] = \{1\} \). Therefore,

\[
(g_1, g_2)[xyz]_3 = [xyz]_3 - \sum_v \beta^v_{xy} v + z - [xyz]_3 + \sum_u \beta^v_{yz} u + v
\]

\[
= \sum_v \beta^{vz}_{yz} v + w - \sum_v \beta^v_{xy} v + z
= \sum_u \beta^{vz}_{yz} x + v - \sum_u \beta^v_{xy} v + w
= \sum_u \alpha^v_{xy} u + v
\]

Now we compute intersection composition of \( f \) and \( g \). Let us consider \( f = [xyz]_1 - \sum_v \alpha^v_{xy} v \) and \( g = [xyz]_2 - \sum_v \beta^v_{yz} v \), then we have \( w = xyz \) and \( P[g_1z] \cap P[xg_2] = \{1, 3\} \). Therefore,

\[
(f, g)[xyz]_1 = [xyz]_1 - \sum_v \alpha^v_{xy} v + z - [xyz]_1 + \sum_u \beta^v_{yz} x + v
\]

\[
= \sum_v \beta^v_{yz} x + v - \sum_v \alpha^v_{xy} v + z
= \sum_u \beta^v_{yz} x + u - \sum_u \alpha^v_{xy} v + w
= \sum_u \alpha^v_{yz} x + v - \sum_u \alpha^v_{xy} v + w
\]

The intersection composition \( (f, g)[xyz]_3 \) is calculated similarly and it is also trivial modulo \( S \). There is no intersection composition between \( g \) and \( f \). Let check the intersection composition of \( f \land h \). We put \( f = [xyz]_1 - \sum_v \alpha^v_{xy} v \) and \( h = [yt]_1 - [ty]_2 - \sum_v \beta^v_{xy} v \), so for the intersection composition \( f \land h \) we have \( w = xyt \).
and $P[ft] \cap P[xh] = \{1\}$. Therefore,
\[
(f, h)|_{[xyt]_1} = [xyt]_1 - \sum_v \alpha_{xy}^v v \cdot t - [xty]_1 + \sum_v \delta_y^v x \cdot v
\]
\[
= [xty]_1 - \sum_v \alpha_{xy}^v v + t + \sum_v \delta_y^v x + v
\]
\[
= ([xt]_1 - [tx]_2 - \sum_v \delta_x^v v \cdot y + [tx]_2 \cdot y
\]
\[
+ \sum_v \delta_x^v v \cdot y - \sum_v \alpha_{xy}^v v \cdot t + \sum_v \delta_y^v x \cdot v
\]
\[
= ([xt]_1 - [tx]_2 - \sum_v \delta_x^v v \cdot y \quad \text{(by relation 2.5)}
\]
\[
= h_x \cdot y.
\]
We have $\overline{(f, h)|_{[xyt]_1}} = [xt]_1 \cdot y < [w] = [xyt]_1$. This shows that $(f, h)|_{[xyt]_1}$ is trivial modulo $S$. There is no intersection composition $g \land h$ and $h_x \land h_y$. Therefore, $S = \{f, g, h\}$ is a Groebner-Shirshov basis for $D^*_d$.

(ii) This part follows from Theorem 1.5.

\[\square\]

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