CORRECTION TO: ON THE ARITHMETICALLY COHEN-MACaulay PROPERTY FOR SETS OF POINTS IN MULTIPROJECTIVE SPACES. PROC. AMER. MATH. SOC. 146 (2018), 2811-2825

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Abstract. We correct a mistake in the cited paper. It introduced a combinatorial property, the $(\ast_n)$-property, for a finite set of points $X$ in $(\mathbb{P}^1)^n$ and claimed that this property holds if and only if $X$ is ACM. In fact $X$ being ACM is a sufficient condition for the $(\ast_n)$-property, but we only prove that it is necessary when $n = 3$, and we give a counterexample when $n = 4$.

The purpose of this corrigendum is to correct a mistake in [1, Theorem 3.16], which claimed that a set of points $X$ in $(\mathbb{P}^1)^n$ is arithmetically Cohen-Macaulay (ACM) if and only if a certain property called $(\ast_n)$ holds. All other results remain true as they were stated. We thank Gunnar Fløystad for pointing out a gap in our proof and providing a counterexample.

In [1, Theorem 3.16], part $(\ast_1)$, first bullet we claimed that $\hat{Y}_1 \cap Y_2$ contains no subset of type (i). In particular, in the third paragraph we wrote:

Then by applying [1, Lemma 3.15] to the points $P_{(1,v_2,\ldots,v_n)}$ and $P_{2-1}$, we get a contradiction by forcing a point of $\hat{Y}_1 \cap Y_2$ to lie in the complete intersection.

This fact doesn’t lead necessarily to a contradiction since the mentioned point of $\hat{Y}_1 \cap Y_2$ forced to lie in the complete intersection could be $P_{2-1}$.

However, the gap in the proof of [1, Theorem 3.16] cannot be fixed. Indeed, G. Fløystad provided us the following example.

Example 1. Let $R = K[x_1,y_1,x_2,y_2,\ldots,x_4,y_4]$ be the coordinate ring of $(\mathbb{P}^1)^4$. Let $A = [0 : 1], B = [1 : 0] \in \mathbb{P}^1$ and consider the two sets of four points in $(\mathbb{P}^1)^4$

$Y_1 = \{(A,A,A,A),(B,A,A,A),(B,B,A,A),(B,B,B,A)\}$

and

$Y_2 = \{(B,B,B,B),(A,B,B,B),(A,A,B,B),(A,A,A,B)\}$.

Set $X = Y_1 \cup Y_2$. Then $\hat{Y}_1 \cap Y_2 = \{(B,B,B,B),(A,A,B,B)\}$ which is not ACM.

Moreover, one can check that $X$ has the $(\ast_4)$ property but $R/I_X$ is not ACM. In particular

$I_X = (x_1y_1,x_2y_2,x_3y_3,x_4y_4,x_1x_3y_2,x_1x_4y_2,x_1x_4y_3,x_2x_4y_3,x_2y_1y_4,x_3y_1y_4,x_3y_2y_4)$

and the Betti table of $R/I_X$ is

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & : & 1 & . & . & . & . \\
1 & : & . & 4 & . & . & . \\
2 & : & . & 8 & 38 & 48 & 28 & 8 & 1
\end{array}
\]

Thus we need to weaken the statement of Theorem 3.16, and we replace it with the following result.

Theorem 2. Let $X \subset (\mathbb{P}^1)^n$ be a finite set. If $X$ is ACM then $X$ has the $(\ast_n)$-property.

The original proof was correct for this part. For the reverse implication, Theorem 3.16 is still true for points in $(\mathbb{P}^1)^3$. For the convenience of the reader, we write here the original proof adapted for the case $(\mathbb{P}^1)^3$. However, only the item $(\ast_1)$ needed to be fixed; we believe that the rest is correct.

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Theorem 3. Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set. Then $X$ has the $(\ast_X)$-property if and only if $X$ is ACM.

Proof. As noted above, if $X$ is ACM, it was shown in [1 Corollary 3.11] that even more generally in $(\mathbb{P}^1)^n$, the $(\ast_n)$ property holds. For the converse, we assume $n = 3$ and we proceed by induction on $t$, the number of level sets with respect to some projection (say $\pi_1$, i.e., the projection omitting the first component, see [1 Definition 2.1]). If $t$ is equal to 1 the result follows from [1 Corollary 2.9]. Let $X = X_1 \cup X_2 \cup \cdots \cup X_t$ be the natural stratification of the points of $X$ as the union of 1-level sets (see [1 Definition 2.5]). Let $Y_1 = X_1$ be one of these 1-level sets (this could be any of them up to re-indexing) and let $Y_2 = X_2 \cup \cdots \cup X_t$ be the union of the remaining level sets. To shorten the notation, in the sequel of the proof, we denote by $A_i \in R_{(1,0,0)}$, $B_j \in R_{(0,1,0)}$ and $C_k \in R_{(0,0,1)}$ the linear forms defining the multihomogeneous hyperplanes. Moreover we denote by $P_{j,k}$ the point whose ideal is generated by $(A_i, B_j, C_k)$ and by $L_{j,k}$ the line of type $(0,1,1)$ whose ideal is generated by $(B_j, C_k)$, for some $A_i \in R_{(1,0,0)}$, $B_j \in R_{(0,1,0)}$ and $C_k \in R_{(0,0,1)}$. We assume $A_1 \in R_{(1,0,0)}$ is the linear form defining the hyperplane containing $Y_1$. We denote by $\hat{Y}_1$ the set of lines $L_{j,k}$ passing through points of $Y_1$ (one for each line), i.e., $\hat{Y}_1 = \pi_1^{-1}(\pi_1(X_1))$. (Viewed in $\mathbb{P}^5$, $\hat{Y}_1$ is a union of codimension 2 linear spaces.)

We know by induction that both $Y_1$ and $Y_2$ are ACM. In particular, this means that $\hat{Y}_1$ is also ACM. Hence we have an equality of saturated ideals $I_{\hat{Y}_1} = (A_1) + I_{\hat{Y}_1}$. Then it follows from the following exact sequence

$$0 \rightarrow I_{Y_1} \cap I_{Y_2} \rightarrow I_{Y_1} \oplus I_{Y_2} \rightarrow (A_1) + I_{Y_1} + I_{Y_2} \rightarrow 0$$

that it is enough to show that $I_{Y_1} + I_{Y_2}$ is an ACM ideal (hence clearly of height 3) and $A_1$ is a regular form in $R/(I_{\hat{Y}_1} + I_{Y_2})$. We proceed by steps.

($\sigma_1$) We show $\hat{Y}_1 \cap Y_2$ is an ACM set of points.

By the inductive hypothesis, it suffices to show that it has the $(\ast_X)$-property, i.e. $\hat{Y}_1 \cap Y_2$ does not contain any of the configurations (a), (b), (c) in [1 Remark 3.10].

- $\hat{Y}_1 \cap Y_2$ contains no subset of type (a). First consider another level set with respect to $\pi_1$; without loss of generality assume it is the second one. If $P_{211}, P_{222} \in \hat{Y}_1 \cap Y_2$, then since $X$ does not have a subset of type (a), it follows that either $P_{212} \notin X$ or $P_{221} \notin X$. Assume without loss of generality that $P_{212} \notin X$. If $P_{212} \notin X$, then since $P_{111}, P_{222} \notin Y_1$, which is ACM, it follows that $P_{221} \in X$. But $X$ does not contain a subset of type (c), so $P_{221} \notin X$, and therefore $P_{221} \notin \hat{Y}_1 \cap Y_2$.

Moreover suppose that a level set of $\hat{Y}_1 \cap Y_2$ with respect to another direction contains another subset of type (a); say $\{P_{211}, P_{321}\} \subseteq \hat{Y}_1 \cap Y_2$ and $\{P_{221}, P_{311}\} \not\subseteq \hat{Y}_1 \cap Y_2$. Then also $X$ must contain the same configuration, giving a contradiction.

- $\hat{Y}_1 \cap Y_2$ contains no subset of type (b); Indeed, suppose that $\hat{Y}_1 \cap Y_2$ contains a subset of type (b); say this subset is

\begin{equation}
(1.1) \quad P_{211} \in \hat{Y}_1 \cap Y_2, \quad P_{322} \in \hat{Y}_1 \cap Y_2
\end{equation}

and

\begin{equation}
(1.2) \quad P_{212}, P_{221}, P_{321}, P_{312} \not\subseteq \hat{Y}_1 \cap Y_2.
\end{equation}

Moreover, from \[1\], we get

\begin{equation}
(1.3) \quad P_{111} \in X \quad \text{and} \quad P_{122} \in X.
\end{equation}

Since $X$ has the $(\ast_X)$-property, from \[1\] either $P_{121}$, or $P_{112} \notin Y_1$; and there is a path (obtained by changing one coordinate at a time) in $Y_2$ joining $P_{212}$ and $P_{322}$. We note that if both $P_{212}, P_{122} \in X$ then we get a contradiction since then a path joining $P_{211}$ and $P_{322}$ in $X$ is also contained in $\hat{Y}_1 \cap Y_2$. So, we assume (without loss of generality)

$$P_{121} \in X \quad \text{and} \quad P_{112} \notin X.$$
The possible paths in $X$ connecting $P_{211}$ and $P_{322}$ are listed below. We show that each case leads to a contradiction.

1. If $P_{221}, P_{321} \in X$, since $P_{121} \in Y_1$, then these two points also are in $\hat{Y}_1 \cap Y_2$ contradicting the condition in (1.2).
2. If $P_{221}, P_{222} \in X$, since $P_{121}, P_{122} \in Y_1$ then these two points also are in $\hat{Y}_1 \cap Y_2$ contradicting the condition in (1.2);
3. If $P_{311}, P_{321} \in X$ since $P_{111}, P_{121} \in Y_1$ then these two points also are in $\hat{Y}_1 \cap Y_2$ contradicting the condition in (1.2);
4. If $P_{311}, P_{312} \in X$. The set $\{P_{111}, P_{212}, P_{311}, P_{312}, P_{322}\} \subseteq X$ is of type (c). Since $X$ has the ($\ast$)-property and $P_{112} \notin X$ we get $P_{321} \in X$. Then we are in the case of item 3.
5. If $P_{212}, P_{222} \in X$. The set $\{P_{111}, P_{212}, P_{221}, P_{212}, P_{222}\} \subseteq X$ is of type (c). Since $X$ has the ($\ast$)-property and $P_{112} \notin X$ we get $P_{221} \in X$. Then we are in the case of item 2.
6. If $P_{212}, P_{312} \in X$. The set $\{P_{222}, P_{312}\} \subseteq X$ is of type (a). Since $X$ has the ($\ast$)-property and $P_{112} \notin X$ we get $P_{222} \in X$. Then we are in the case of item 5.

- $\hat{Y}_1 \cap Y_2$ contains no subset of type (c). Indeed, if it did then this subset is contained in $X$, contradicting the ($\ast$)-property of $X$.

$(\sigma_2)$ We make a technical observation concerning the “outlier” points.

We denote by $Y'_1$ the set of points $P_{1jk} \in Y_1$ such that the line $L_{jk}$ has empty intersection with $Y_2$, and we denote by $Y'_2 := Y_2 \setminus (Y_1 \cap Y_2)$. Let $F \in I_{Y'_1 \cap Y'_2}$ be a product of linear forms of type $A_i, B_j$ and $C_k$. Taking the ideal of the empty set to be $\hat{R}$, we claim that

$$F \in (I_{Y'_1}) \cup (I_{Y'_2}).$$

We assume that both $Y'_1$ and $Y'_2$ are non-empty; otherwise the statement is trivial. Assume by contradiction that $F \notin (I_{Y'_1}) \cup (I_{Y'_2})$. Then there exist $P_{111} \in Y'_1$ (since $X$ has the ($\ast$)-property and by the definition of these two sets, this implies $P_{212}, P_{221} \notin Y'_2$) and $P_{222} \in Y'_2$, such that $F \notin (A_1, B_1, C_1)$ and $F \notin (A_2, B_2, C_2)$. Now, $P_{111} \in Y'_1$ implies $P_{211} \notin Y'_2$; moreover, since $P_{222} \in Y'_2$ we have $P_{122} \notin Y'_1$. (Here we use the fact that since $X$ has the ($\ast$)-property, this excludes either $B_1 = B_2$ or $C_1 = C_2$.) But $X$ does not contain subsets of type (a) or (b) so, without loss of generality, we can assume $P_{221}, P_{221} \in X$ i.e. $P_{221} \in Y_1 \cap Y_2$. Since $F \in (A_2, B_2, C_1)$ and it is a product of linear forms, we get either $F \in (A_2, B_2, C_2)$ or $F \in (A_1, B_1, C_1)$, which contradicts the assumption.

$(\sigma_3)$ We show that $I_{Y'_1 \cap Y'_2} \subseteq I_{Y'_1} + I_{Y'_2}$.

From $(\sigma_1)$ we know that $\hat{Y}_1 \cap Y_2$ is ACM, so $I_{Y'_1 \cap Y'_2}$ is minimally generated by products of linear forms (from \([1]\) Corollary 3.4). Let $F \in I_{Y'_1 \cap Y'_2}$ be such a generator. From the minimality of $F$ we note that $F \notin (A_1)$. From $(\sigma_2)$ we have $F \in I_{Y'_1} \cup I_{Y'_2}$. Assume first that $F \in I_{Y'_1}$. Then, from the definition of $Y'_2$, trivially we have $I_{Y_2} \subseteq I_{Y'_1} + I_{Y'_2}$.

Assume now that $F \notin I_{Y'_1}$ i.e. there exists a point, say $P_{222} \in Y'_2$, such that $F \notin (A_2, B_2, C_2)$. We collect the relevant facts:

$(f_1)$ $P_{222} \notin Y_1$ by definition of $Y'_2$;
$(f_2)$ $F \in I_{Y'_1}$ by $(\sigma_2)$;
$(f_3)$ $F \in I_{Y'_1 \cap Y'_2}$ but $F \notin (A_2, B_2, C_2)$ and $F \notin (A_1)$ by assumption.

We want to show that $F \in I_{Y'_1}$. Choose any point $P \in Y_1$. Suppose first that $P = P_{121}$. Then in order to avoid $X$ having a subset of type (a), from $(f_1)$ we must have $P_{221} \in X$. This means that $P_{221} \notin Y'_1$, so $P_{221}$ lies in $Y_1 \cap Y_2$. Thus, we have $F(P_{221}) = 0$ but $F(P_{222}) \neq 0$. It follows that $F \in (B_2, C_1) \subseteq I_{Y'_1}$.

The case $P = P_{112}$ is entirely analogous. Finally, suppose that $P$ is any other point of $Y_1$; without loss of generality say it is $P_{111}$. We consider two cases.
• If \( P_{111} \in Y'_1 \) then from \((f_2)\) we have \( F \in (A_1, B_1, C_1) \) and, since \( F \notin (A_1), F \in (B_1, C_1) \subseteq \hat{I}_{Y_1} \).

• If \( P_{111} \in Y_1 \setminus Y'_1 \) then since \( X \) has the \((\star_3)\)-property, to avoid a subset of type (a) or (b), at least one among \( P_{211}, P_{212} \) and \( P_{221} \) belongs to \( \hat{Y}_1 \cap Y_2 \). Therefore, since \( F \in \hat{I}_{\hat{Y}_1 \cap Y_2} \) and \( F \notin (A_2, B_2, C_2) \) we get \( F \in (B_1, C_1) \subseteq \hat{I}_{Y_1} \).

This shows that \( F \in \hat{I}_{Y_1} \) as desired, and concludes the proof of \((\sigma_3)\).

To complete the proof of our theorem, note that on the other hand we always have \( \hat{I}_{\hat{Y}_1 \cap Y_2} = \sqrt{I_{\hat{Y}_1} + I_{Y_2}} \supseteq I_{\hat{Y}_1} + I_{Y_2} \). Thus, \( I_{\hat{Y}_1} + I_{Y_2} \) is the ideal of an ACM set of reduced points \( (\hat{Y}_1 \cap Y_2) \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \), as desired. Moreover, this implies that \( A_1 \) is a regular form in \( R/(I_{\hat{Y}_1} + I_{Y_2}) \) since no point of \( \hat{Y}_1 \cap Y_2 \) belongs to the plane defined by \( A_1 \).

\[ \square \]

References

[1] Favacchio, G., Guardo, E. and Migliore, J., *On the arithmetically Cohen-Macaulay property for sets of points in multiprojective spaces*. Proceedings of the American Mathematical Society, 2018, 146(7), pp. 2811–2825. DOI: https://doi.org/10.1090/proc/13981