Weak field approximation in a model of de Sitter gravity: Schwarzschild solutions and galactic rotation curves

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Abstract

Weak field approximate solutions in the $\Lambda \to 0$ limit of a model of de Sitter gravity have been presented in the static and spherically symmetric case. Although the model looks different from general relativity, among those solutions, there still exist the weak Schwarzschild fields with the smooth connection to regular internal solutions obeying the Newtonian gravitational law. The existence of such solutions would determine the value of the coupling constant, which is different from that of the previous literature. Moreover, there also exist solutions that could deduce the galactic rotation curves without invoking dark matter.

Keywords: de Sitter gravity, Schwarzschild solutions, torsion, dark matter

1 Introduction

In the 1970s a model of de Sitter (dS) gravity had been proposed [1–4]. In this model the Einstein–Hilbert action with a cosmological term could be deduced from a gauge-like action besides two quadratic terms of the curvature and torsion. The astronomical observation [5, 6] on the asymptotically dS behavior of our universe has increased interest in the model as it may offer a way to deal with the dark energy problem [7]. If the Einstein–Hilbert term is required to be the main part of the gauge-like action, the cosmological constant should be large. The large cosmological constant may be canceled out by the vacuum energy density, leaving a small cosmological constant [4]. But it is difficult to explain why the large cosmological constant and the vacuum energy density are so close, but not exactly equal, to each other. On the other hand, if the cosmological constant is required to be small [1–3, 8], the quadratic curvature term would become the main part of the action. Note that in one of the field equations, the quadratic curvature term only contributes to the symmetric trace-free part. It is worth checking carefully whether the

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model under this case could explain the experimental observations. Actually it has been shown [9, 10] that the model with a small cosmological constant may explain the accelerating expansion of the universe and supply a natural transit from decelerating expansion to accelerating expansion without the help of introducing matter fields in addition to dust. It has also been shown [11, 12] that all torsion-free vacuum solutions of this model are the vacuum solutions of Einstein’s field equation with the same cosmological constant, and vice versa. Therefore, one may expect that the model with a small cosmological constant may pass all solar-system-scale experimental tests for general relativity (GR).

However, it has been pointed out [9] that the energy-momentum-stress tensor of a spinless fluid in the torsion-free case of this model should be with a constant trace. Questions would then appear such as, could the torsion-free Schwarzschild-dS (S-dS) solution be smoothly connected to internal solutions with nonzero torsion, or are there any S-dS solutions with nonzero torsion? In fact, the different dS spacetimes with nonzero torsion in this model have been obtained in [13, 14], but they are still not the S-dS solutions. On the other hand, S-dS solutions with long-range spherically symmetric torsion have been given [15] in some special cases (not necessarily under the double duality ansatz [16]) of quadratic models of Poincaré gauge theory of gravity, but our model does not fall into those special cases.

We would like to firstly check the existence of the S-dS solutions with nonzero torsion in the weak field approximation. The Newtonian limit of general quadratic models in Poincaré gauge theory of gravity has been calculated [17, 18] in the 1980’s. In those calculations, quadratic terms in the field equations have been thrown away as usual. However, in the weak field approximation of our model, the quadratic curvature terms could not be easily thrown away, for the reason that they are the main parts of one of the field equations. In fact, we may let the cosmological constant be \( \Lambda \to 0 \), then only those quadratic curvature terms would appear in the limit of the field equation which contains the energy-momentum-stress tensor of the matter field. On the other hand, as those quadratic curvature terms are symmetric and trace-free, they would not appear in the trace part and the antisymmetric part of the field equations. The Newtonian limit of the trace equations has been recently analyzed [19], but a more complete analysis of all components of the field equations is needed.

As was well known, Newton’s theory of gravity meets great difficulties in the explanation of the flat rotation curves [20] of spiral galaxies. The most widely adopted way to resolve this problem is the dark matter hypothesis. But up to now, all of the possible candidates of dark matter (such as neutralino, axion, etc.) are either undetected or unsatisfactory. In the meanwhile, there also exist some models [21–24] which could deduce the galactic rotation curves without involving dark matter. We would like to explore the possibility of a new explanation for the galactic rotation curves from the dS gravity model.

The paper is arranged as follows. We first briefly review the model of the dS gravity in section 2. In the third section, after dividing a field equation into its trace part, symmetric trace-free part and antisymmetric part, we attain the \( \Lambda \to 0 \) limit of the model and calculate its weak field approximation in the static and spherically symmetric case. The weak field approximate solutions contain the weak Schwarzschild fields with nonzero torsion, which could be smoothly linked to regular internal solutions obeying the Newtonian gravitational law. The coupling constant is determined by the existence of such solutions. Moreover, solutions that could deduce the galactic rotation curves without invoking dark matter are also attained. Finally we end with some remarks in the
A model of dS gravity

A model of dS gravity has been constructed with a gauge-like action

\[ S_G = \int L_G = \int \kappa [ - \text{tr}(F_{ab}F^{ab}) ] \]

\[ = \int \kappa [ R_{abcd}R^{abcd} - \frac{4}{l^2} (R - \frac{6}{l^2}) + \frac{2}{l^2} S_{abc}S^{abc} ] \]

(1)

in the units of \( \hbar = c = 1 \), where \( \kappa \) is a dimensionless coupling constant to be determined, and

\[ F_{ab} = (dA + \frac{1}{2}[A, A])_{ab} \]

(2)

or explicitly

\[ F^A_{ab} = (dA^A_{B})_{ab} + A^{A}_{Ca} \wedge A^{C}_{Bb} \]

\[ = \begin{pmatrix} R_{ab}^{\alpha \beta} - l^{-2}e_{a}^{\alpha} \wedge e_{b}^{\beta} & l^{-1}S_{ab}^{\alpha} \\ -l^{-1}S_{ab}^{\beta} & 0 \end{pmatrix} \]

(3)

is a dS algebra-valued 2-form and

\[ A^A_{Ba} = \begin{pmatrix} \Gamma_{\beta a}^{\alpha} & l^{-1}e_{a}^{\alpha} \\ -l^{-1}e_{\beta a} & 0 \end{pmatrix} \]

(4)

is a dS algebra-valued 1-form. Here \( A, B \ldots = 0, 1, 2, 3, 4 \) stand for matrix indices (internal indices) and the trace in Eq. (1) is taken for those indices. In addition, \( \{e_{a}^{\alpha}\} \) is some local orthonormal frame field on the spacetime manifold and \( \Gamma_{\beta a}^{\alpha} \) is the connection 1-form in this frame field, where \( a, b \ldots \) stand for abstract indices [25, 26] and \( \alpha, \beta \ldots = 0, 1, 2, 3 \) are concrete indices related to the frame field mentioned above. The curvature 2-form \( R_{ab}^{\alpha \beta} \) and torsion 2-form \( S_{ab}^{\alpha} \) are related to the connection 1-form \( \Gamma_{\beta a}^{\alpha} \) as follows:

\[ R_{ab}^{\alpha \beta} = (d\Gamma_{\beta a}^{\alpha})_{ab} + \Gamma_{\gamma a}^{\alpha} \wedge \Gamma_{\beta b}^{\gamma} \]

(5)

\[ S_{ab}^{\alpha} = (de_{a}^{\alpha})_{ab} + \Gamma_{\beta a}^{\alpha} \wedge e_{b}^{\beta} \]

(6)

Moreover,

\[ R_{abc}^{d} = R_{aba}^{c} e_{c}^{\alpha} e_{\alpha}^{d} \]

\[ S_{ab}^{c} = S_{ab}^{\alpha} e_{c}^{\alpha} \]

\[ R_{ab} = R_{acb}^{c} \]

\[ R = g^{ab} R_{ab} \]

In fact, if spacetime is an umbilical submanifold of some \((1+4)\)-dimensional ambient manifold and with positive normal curvature, then \( A_{a} \) and \( F_{ab} \) could be viewed \[8\] as the connection 1-form and curvature 2-form (in the dS-Lorentz frame) of the ambient manifold restricted to spacetime. Here, an umbilical submanifold means a submanifold with constant normal curvature, such as the dS spacetime which could be seen as an umbilical submanifold of a 5d Minkowski spacetime with positive normal curvature. \( A_{a} \) could also be seen \[27\] as the Cartan connection of a Cartan geometry modeled on the dS spacetime and based on the spacetime manifold, with \( F_{ab} \) the corresponding curvature.
2-form. The Cartan geometry is a generalization of homogenous spaces with fibre bundle language, and one may refer to [27] for more details. In addition, it should be noted that \(3/l^2\) is identified [8] with a small cosmological constant \(\Lambda\) here, which is very different from the viewpoint of [4] where \(l\) is identified with the Planck length. The signature is chosen such that the metric coefficients are \(\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)\).

The total action is \(S = S_M + S_G\), where \(S_M\) is the action of the matter fields and the field equations can be given via the variational principle with respect to \(e^\alpha_a, \Gamma^\alpha_{\beta a}\):

\[
\frac{8}{l^2} (G_{ab} + \Lambda g_{ab}) | |R| g_{ab} - 4 R_{acde} R_b^{cde} + \frac{2}{l^2} |S|^2 g_{ab} 
- \frac{8}{l^2} S_{cda} S_{cd}^b + \frac{8}{l^2} \nabla_c S_{ab}^c + \frac{4}{l^2} S_{acdq} T_{b}^{cd} + \frac{1}{\kappa} \Sigma_{ab} = 0, \tag{7}
\]

\[
- \frac{4}{l^2} T_{bc}^a - 4 \nabla_d R_{da}^{bc} + 2 T_{de}^a R_{bc}^{de} - \frac{8}{l^2} S_{[bc]}^a + \frac{1}{\kappa} \tau_{bc}^a = 0, \tag{8}
\]

where

\[
G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \quad T_{bc}^a = S_{bc}^a + 2 \delta_{[a}^c d \delta_{b]}^d,

|R|^2 = R_{abcd} R^{abcd}, \quad |S|^2 = S_{abcd} S^{abcd},

\Sigma_{\alpha}^a = \delta S_M / \delta e^\alpha_a, \quad \Sigma_{b}^a = \Sigma_{\alpha}^a e^\alpha_b,

\tau_{\alpha}^{\beta a} = \delta S_M / \delta \Gamma^\alpha_{\beta a}, \quad \tau_{b}^{\alpha a} = \tau_{\alpha}^{\beta a} e^\alpha_b e^\beta_c,
\]

and the variational derivatives are defined as follows: if

\[
\delta S_M = \int \left( X_a^a \delta e^\alpha_a + Y_{\alpha\beta}^a \delta \Gamma^\alpha_{\beta a} \right),
\]

then

\[
\delta S_M / \delta e^\alpha_a = X_a^a, \quad \delta S_M / \delta \Gamma^\alpha_{\beta a} = Y_{[\alpha\beta]}^a.
\]

### 3 Weak field approximation in the case with \(\Lambda \to 0\)

As \(\Lambda\) is very small, it is interesting to see the case with \(\Lambda \to 0\) \((l \to \infty)\). If \(\Lambda \to 0\) is directly set in the first field equation, then only the quadratic curvature terms are left, which are symmetric and trace-free. Thus, we would like to perform the following procedure. Divide the first field equation into its symmetric trace-free part, trace part, and antisymmetric part, then let \(l\) tend to infinity \((l \to \infty)\) in the above three parts and in the second field equation. The limiting equations are:

\[
|R|^2 g_{ab} - 4 R_{acde} R_b^{cde} = 0, \tag{9}
\]

\[
- R - \nabla_c S_{bc}^a + \frac{1}{2} S_{bcd} T_{b}^{bcd} + \frac{1}{8} \kappa \Sigma = 0, \tag{10}
\]

\[
R_{[ab]} + \nabla_c S_{[ab]}^c + \frac{1}{2} S_{[ac]d} T_{b]cd} + \frac{1}{8} \kappa \Sigma_{[ab]} = 0, \tag{11}
\]

\[
- 2 \nabla_d R_{da}^{bc} + T_{de}^a R_{bc}^{de} = 0. \tag{12}
\]

When \(l \to \infty\), \(l^2 / \kappa\) should tend to a finite value, otherwise Eqs. (10) and (11) would give \(\Sigma = 0\) and \(\Sigma_{[ab]} = 0\), which are unreasonable. In the torsion-free case, the scalar...
Curvature would be a constant from Eq. (12), and, therefore, \( \Sigma = \text{const} \) from Eq. (10). This property has been pointed out by [9]. Now we are going to consider the weak field approximation of the above equations. It would be assumed that

\[
\begin{align*}
g_{ab} &= \eta_{ab} + \gamma_{ab}, \quad \gamma_{ab} = O(s), \quad S^c_{\ ab} = O(s), \\
\sigma_{ab} &= O(s), \quad \tau_{ab}^c = O(s),
\end{align*}
\]

(13)

(14)

where \( s \) is a dimensionless parameter, called the weak field parameter. We will restrict ourselves to the static and \( O(3) \)-symmetric case, with the static spherical coordinate system \( \{ t, r, \theta, \varphi \} \). \( \eta_{ab} \) could be defined by its components \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) in the approximate inertial coordinate system \( \{ x^\mu \} \). \( \{ x^\mu \} \) is related to \( \{ t, r, \theta, \varphi \} \) as usual:

\[
\begin{align*}
x^0 &= t, \quad x^1 = r \sin \theta \cos \varphi, \quad x^2 = r \sin \theta \sin \varphi, \quad x^3 = r \cos \theta.
\end{align*}
\]

It could be proved [15, 25, 28] that \( \gamma_{ab} \) and \( S^c_{\ ab} \) have only these dependent components in the static spherical coordinate system:

\[
\begin{align*}
\gamma_{00} &= -2\phi(r), \quad \gamma_{rr} = -2\psi(r), \\
\begin{cases}
S^0_{\ 0r} = f(r), & S^r_{\ 0r} = h(r), \\
S^0_{\ r\theta} = g(r), & S^\theta_{\ r\theta} = -k(r), \\
S^\varphi_{\ r\varphi} = g(r), & S^\varphi_{\ 0\varphi} = -k(r)
\end{cases}
\end{align*}
\]

(15)

(16)

where \( \phi \) plays the role of the Newtonian gravitational potential [1–3, 25, 29] and \( \psi \) is an unknown function. It can be shown that components of \( \gamma_{ab} \) and \( S^c_{\ ab} \) in \( \{ x^\mu \} \) are as follows:

\[
\begin{align*}
\gamma_{00} &= -2\phi, \quad \gamma_{0i} = 0, \quad \gamma_{ij} = (-2\psi)x_i x_j / r^2, \\
\begin{cases}
S^0_{\ 0i} = f x_i / r, & S^0_{\ ij} = 0, \\
S^i_{\ 0j} = (h + k)x_i x_j / r^2 - k\delta_j^i, \\
S^j_{\ ik} = -(g/r)(\delta_j^i x_k - \delta_k^i x_j)
\end{cases}
\end{align*}
\]

(17)

(18)

For this case the contorsion tensor is related to the torsion tensor by

\[
K_{abc} = S_{cba}.
\]

(19)

Let \( \Gamma^c_{\ ab} = \Gamma^\sigma_{\ \mu\nu} \partial_\sigma (dx^\mu)_a (dx^\nu)_b \), where \( \Gamma^\sigma_{\ \mu\nu} \) is the connection coefficient in \( \{ x^\mu \} \). \( \Gamma^c_{\ ab} \) and the curvature tensor have the following first order approximate expressions:

\[
\Gamma^c_{\ ab} = \frac{1}{2} (\partial_a \gamma^c_{\ b} + \partial_b \gamma^c_{\ a} - \partial^c \gamma_{ab}) - K^c_{\ ab},
\]

(20)

\[
R^d_{\ abc} = - (\partial_d \partial_{[a} \gamma_{b]}^c - \partial^d \partial_{[a} \gamma_{b]}_{c}) + 2 \partial_{[a} K^d_{\ c]b},
\]

(21)

where

\[
K^c_{\ ab} = \frac{1}{2} (S^c_{\ ab} + S^a_{\ bc} + S^b_{\ ac})
\]

(22)

is the contorsion tensor. In the case with \( \Sigma_{ab} = \rho(r)(dt)_a (dt)_b \), the equations for the first order approximate \( g_{ab} \) and \( S^c_{\ ab} \) are as follows:

\[
|R|^2 \eta_{ab} - 4R_{acde} R^c_{\ bde} = 0,
\]

(23)
\[ R + \partial_c S^{bc}_b + (l^2/8\kappa)\rho = 0, \quad (24) \]
\[ R_{[ab]} + \partial_c S_{[ab]}^c = 0, \quad (25) \]
\[ \partial_d R^{da_{bc}} = 0. \quad (26) \]

The first order approximation of Eq. (9) is an identity. Here Eq. (23) is the second order approximation of Eq. (9). It should be considered for the reason that it is still an equation for the first order approximate \( g_{ab} \) and \( S^{c}_{ab} \).

Now we are going to solve the above weak field equations. Applying Eq. (21) to Eq. (26), we have
\[ \partial_d (\partial^d \partial_\gamma_{\gamma b} + \partial^d K_{eb}) = 0. \quad (27) \]
The 00i (abc) component of Eq. (27) is
\[ \partial_i (\partial^i \partial_\gamma_{00} + 2\partial^i K_{i00}) = 0, \]
which gives
\[ \Delta[(\phi' + f)x_i/r] = 0, \]
i.e. \( (\phi' + f)''/r + 2(\phi' + f)'/r^2 - 2(\phi' + f)/r^3 = 0 \).
Solving this equation we get that
\[ \phi' + f = Cr + D/r^2. \quad (28) \]
The 0ij component of Eq. (27) is an identity. The i0j component of Eq. (27) is
\[ \partial^d \partial_d K_{j0i} - \partial^d \partial_i K_{j0d} = 0, \]
which gives
\[ \delta_{ij}(-\Delta k - h'/r) + (x_ix_j/r^2)[\Delta k + h'/r - 2k'/r - 2(h + k)/r^2] = 0, \]
i.e. \( \Delta k + h'/r = 0, \quad \Delta k + h'/r - 2k'/r - 2(h + k)/r^2 = 0, \)
or equivalently
\[ h + k + r k' = 0. \quad (29) \]
The ijk component of Eq. (27) is
\[ \partial^d \partial_d \partial_k \gamma_{ij} - \partial^d \partial_i \partial_j \gamma_{ki} - \partial^d \partial_j \partial_k \gamma_{id} + \partial^d \partial_i \partial_j \gamma_{kd} + 2\partial^d \partial_d K_{kji} - 2\partial^d \partial_i K_{kjd} = 0, \]
which gives
\[ \Delta[(2\psi/r^2 - 2g/r)(\delta_{ik}x_j - \delta_{ij}x_k)] = 0, \]
i.e. \( (2\psi/r^2 - 2g/r)'' + (4/r)(2\psi/r^2 - 2g/r)' = 0. \)
Solving this equation we get that
\[ \psi/r^2 - g/r = B/r^3 + A. \quad (30) \]

From Eqs. (21), (17) and (18) the components of \( R_{abcd} \) in \( \{x^\mu\} \) could be attained as follows:
\[
\begin{align*}
R_{0i0j} &= (\phi'' + f')x_i x_j/r^2 + (\phi' + f)(\delta_{ij}r^2 - x_i x_j)/r^3, \\
R_{0ijk} &= 0, \\
R_{ijk0} &= (2/r^2)(h + k + r k')x_i \delta_{jk}, \\
R_{ijkl} &= (\psi/r^2 - g/r)'(2/r) x_i (\delta_{jk} x_l - \delta_{jl} x_k) - 4(\psi/r^2 - g/r)\delta_{ik} \delta_{lj}. \\
\end{align*}
\]
Substituting Eqs. (28), (29) and (30) into the above equation, we get that

\[
\begin{align*}
R_{00ij} &= (C + D/r^2)x_i x_j/r^2 + (C + D/r^2)(\delta_{ij} r^2 - x_i x_j)/r^3, \\
R_{0ijk} &= 0, \\
R_{ijkl} &= 0, \\
R_{ijkl} &= (B/r^3 + A)(2/r) x_i(\delta_{jk} x_l - \delta_{jl} x_k) - 4(B/r^3 + A) \delta_{ik} \delta_{lj}.
\end{align*}
\]

Then

\[
|R|^2 \eta_{0i} - 4 R_{0cde} R^{cde} = 12(C^2 - 4A^2) + 24(D^2 - B^2)/r^6, \\
|R|^2 \eta_{0i} - 4 R_{0cde} R^{cde} = 0, \\
|R|^2 \eta_{ij} - 4 R_{icde} R^{cde} = (4C^2 - 16A^2 - 16CD/r^3 - 32AB/r^3 + 16D^2/r^6 - 16B^2/r^6) \delta_{ij} + (48CD/r^3 + 96AB/r^3 - 24D^2/r^6 + 24B^2/r^6)x_i x_j/r^2,
\]

Applying Eq. (33), the symmetric trace-free equation (23) could be solved, resulting in the following relations:

\[
C = \pm 2A, \quad B = \pm D, \quad CD + 2AB = 0.
\]

From Eq. (34) one could see that \(R_{[ab]}\) = 0 and thus the antisymmetric equation (25) gives \(\partial_c S_{[ab]}^c = 0\). Components of \(\partial_c S_{ab}^c\) are as follows:

\[
\begin{align*}
\partial_c S_{00}^c &= -f' - 2f/r, \\
\partial_c S_{0k}^c &= 0, \\
\partial_c S_{0i}^c &= x_i[h'/r + 2(h + k)/r^2], \\
\partial_c S_{ij}^c &= [-2(g/r) - r(g/r)'] \delta_{ij} + (g/r)' x_i x_j/r.
\end{align*}
\]

Therefore, \(\partial_c S_{[ab]}^c = 0\) results in

\[
h'/r + 2(h + k)/r^2 = 0.
\]

Combining Eqs. (29) and (37) we have

\[
(h - 2k)' = 0, \quad h + k = -r(h + k)'/3, \\
h = 2k + C_1, \quad h + k = C_2/r^3.
\]

Now we turn to the trace equation (24). From Eqs. (36), (28) and (30) we have

\[
\begin{align*}
\partial_c S^{bc}_{b} &= -f' - 2f/r + 2(r(g/r)') + 6(g/r) \\
&= 2\psi'/r + 2\psi/r^2 + \Delta \phi - 3(2A + C).
\end{align*}
\]

Substituting Eqs. (34) and (39) into Eq. (24) results in

\[
-(\Delta \phi + 2\psi'/r + 2\psi/r^2) + 9(2A + C) = (\ell^2/8\kappa)\rho.
\]
Equation. \((40)\) is an underdetermined equation for \(\phi\) and \(\psi\). To solve it, more conditions are needed. Suppose that the external solution is the weak Schwarzschild field, i.e., \(\phi = \psi = -GM/r\) and the internal solution is regular and satisfies \(\triangle \phi = 4\pi G \rho\). Also, it is assumed that the internal solution could be linked to the external solution smoothly at \(r = R_S\), resulting in a complete solution. For this complete solution, there would be \(C = -2A\),

\[
\phi = \int_0^r \left[ \frac{G m(r)}{r^2} \right] dr + \phi(0) \quad (41)
\]

with

\[
m(r) = \int_{\mathbb{B}^3(r)} \rho = \int_0^r 4\pi \rho r^2 dr,
\]

\[
\phi(0) = -GM/R_S - \int_0^{R_S} \left[ \frac{G m(r)}{r^2} \right] dr,
\]

and

\[
\psi = \frac{-Gm(r)}{r} \left( \frac{l^2}{64\pi G \kappa} + 1/2 \right). \quad (42)
\]

Note that \(\psi(R_S) = -Gm(R_S)/R_S\), thus,

\[
\psi = -Gm(r)/r, \quad \kappa = l^2/32\pi G. \quad (43)
\]

Equation. \((42)\) is just the same as the corresponding case in GR. The result \(C = -2A\) is in accordance with Eq. \((35)\). The torsion solutions will be given later in the more general case.

Actually, Eq. \((40)\) can also be solved with other supplementary conditions. For example, instead of assuming \(\triangle \phi = 4\pi G \rho\), we may let

\[
\triangle \phi = 4\pi G (\rho + \tilde{\rho}), \quad (44)
\]

where \(\tilde{\rho}\) is an arbitrarily given function and does not contribute to the energy-momentum-stress tensors of matter fields.Fixing \(\kappa\) by Eq. \((43)\), then from Eqs. \((35)\), \((40)\) and \((44)\) we have

\[
\phi = G \int_0^r \left\{ [m(r) + \tilde{m}(r)]/r^2 \right\} dr + \phi(0), \quad (45)
\]

\[
\psi = -\frac{G}{r} \left[ \frac{m(r) + \tilde{m}(r)}{2} + \frac{3}{2} (2A + C) r^2 \right] \quad (46)
\]

with

\[
\tilde{m}(r) = \int_{\mathbb{B}^3(r)} \tilde{\rho} = \int_0^r 4\pi \tilde{\rho} r^2 dr, \quad C = \pm 2A.
\]

As the internal solutions are regular, from Eqs. \((28)\), \((30)\) and \((38)\), there should be \(B = D = 0, C_2 = 0\). Therefore, the torsion solutions corresponding to Eqs. \((45)\) and \((46)\) are as follows:

\[
f = Cr - G[m(r) + \tilde{m}(r)]/r^2, \quad (47)
\]

\[
g = (4A + 3C)r/2 - G[m(r) + \tilde{m}(r)]/r^2, \quad (48)
\]

\[
h = -k = \frac{1}{3} C_1. \quad (49)
\]
The former case which is compatible with the Schwarzschild solutions corresponds to the special choice with $C = -2A$ and $\tilde{\rho} = 0$. In fact, another choice of $\tilde{\rho}$ could deduce the galactic rotation curves without invoking dark matter. To fit the galactic rotation curves, the dark matter density profile $\rho_{DM}$ has been given for many spiral galaxies, for example, see Refs. [30, 31], where the following choice is made:

$$\rho_{DM} = \frac{\sigma^2}{2\pi G(r^2 + a^2)}.$$  \hfill (50)

In Refs. [30, 31], the mass distribution of the galaxies is modeled as the sum of the bulge and disk stellar components and a halo of dark matter. The rotation curves are used to determine the two halo parameters $\sigma$ and $a$. In our model, we may just let the gravitational contribution $\tilde{\rho}$ to take the same form as the dark matter density profile and suitably choose the integration constants, i.e., $\tilde{\rho} = \rho_{DM}$ and $C = -2A$, without the inclusion of any real dark matter. For this case, Eqs. (45) and (46) is almost equivalent to the corresponding case in GR with dark matter. The parameters $\sigma$ and $a$ can be determined in the same way as that in Refs. [30, 31] and, therefore, with the same values. For example, for the galaxy NGC 2841, $\sigma = 232$ km/s and $a = 11.6$ kpc fit the rotation curve well, for the galaxy NGC 3031, $\sigma = 86$ km/s and $a = 2.0$ kpc fit the rotation curve well, and so on. To say ‘almost equivalent’ but not ‘equivalent’, is because the term $\tilde{m}/2$ in Eq. (46) is different from $\tilde{m}$, which should be the case in GR with dark matter. Fortunately, $\psi$ would not affect the rotation curves in the leading order of approximation, since the rotation velocity $v_c$ in a galaxy at a radius $r$ in the approximation is given by

$$v_c^2(r) = r(\partial \phi / \partial r).$$  \hfill (51)

For higher order approximations, more work is needed to be done and experiments with higher accuracy are needed to check the corresponding results.

One may argue that $\tilde{\rho}$ can not be specified a priori and could only be determined from observation. Actually, such kinds of functions also appear in other models which attempt to explain the galactic rotation curves without involving dark matter, such as Milgrom’s modification of Newtonian dynamics (MOND) [21, 22] and the modified Newton’s gravity in Finsler space [24]. What we can do now is to point out the geometrical meaning of $\tilde{\rho}$. From Eqs. (47) and (48), there is the relation:

$$4\pi G\tilde{\rho} = -\frac{2}{r^2}[r^2(f - g)]' - 3(4A + C),$$  \hfill (52)

which shows that the dark matter density could be directly related to the spacetime torsion.

When $l \to \infty$, $l^2/\kappa$ should tend to a finite value, as was mentioned before. Obviously Eq. (43) satisfies this requirement. In [1-3, 8], the coupling constant is chosen to be $-l^2/64\pi G$, which is different from Eq. (43). For this case, there exists no regular internal solution which satisfies $\Delta \phi = 4\pi G\rho$ and has a smooth junction to the Schwarzschild solutions. In fact, with this choice, the ratio of the coefficient of the Einstein term $G_{ab}$ to that of the matter term $\Sigma_{ab}$ in Eq. (7) would be 1 : $(-8\pi G)$, just like the case in GR. But the role of the Einstein term in our model is different from that of GR. Actually, from Eq. (34), the Einstein term only contributes a constant term to Eq. (24), while the torsion term plays an important role in that equation.
Generally, the torsion tensor can be decomposed \([32, 33]\) into three irreducible parts with respect to the Lorentz group: the tensor part, trace-vector part, and the axial vector part. For static and \(O(3)\)-symmetric torsion, the axial vector part vanishes automatically, the tensor part satisfies \(f = 2g, h = 2k\) and the trace-vector part satisfies \(f = -g, h = -k\). By Eqs. (47) and (48), if the torsion field only contains the tensor part, the matter density should be a constant:

\[
\rho = 3(2A + C)/2\pi G. \tag{53}
\]

If the torsion field only contains the trace-vector part, then

\[
4\rho + 3\tilde{\rho} = 3(4A + 5C)/4\pi G. \tag{54}
\]

When \(\tilde{\rho} = 0\), the matter density has to be a constant, too.

### 4 Remarks

The weak field approximation of the \(\Lambda \to 0\) limit of the dS gravity model has been calculated in the static and spherically symmetric case. The matter field is assumed to be a smoothly distributed dust sphere with finite radius. It comes out that if and only if \(\kappa = l^2/32\pi G\), there exist regular internal solutions which satisfy \(\Delta \phi = 4\pi G \rho\) and have a smooth junction to the weak Schwarzschild fields. Recall that the main part of the action is the quadratic curvature term, not the Einstein–Hilbert term. The existence of the above solutions is of significance. The choice of the coupling constant here is different from that of \([1–3, 8]\), where \(\kappa = -l^2/64\pi G\). The choice in \([1–3, 8]\) may due to a comparison between the dS gravity model and GR. But the Einstein term in the dS gravity model plays a different role from that of GR, as it only contributes to a constant term in the weak field approximate equations. Actually, the metric components and torsion components are closely related by Eqs. (28) and (30), such that the curvature tensor has to take a special form (32). Moreover, one may let \(C = A = 0\) and \(B = D = 0\), then \(R_{abcd} = 0\), i.e., the Weizenböck spacetime \([34, 35]\) would be attained. On the other hand, the torsion tensor plays an important role in this model. In the weak field approximate solutions, if the torsion tensor only contains the tensor part, the matter density should be a constant; if it only contains the trace-vector part, Eq. (54) should be upheld. In particular, the matter density should be a constant in the torsion-free case.

The trace equation (24) results in an underdetermined equation for the metric components \(\phi\) and \(\psi\) in the weak field approximation. Solutions with \(\Delta \phi = 4\pi G(\rho + \tilde{\rho})\) could be attained, which can explain the galactic rotation curves without the help of introducing dark matter. The geometrical meaning of \(\tilde{\rho}\) has been given by Eq. (52). It is a geometric quantity related to the spacetime torsion and can play the role of dark matter density though it is irrelevant to the energy-momentum-stress tensors of matter fields. To see the higher order behavior of the model, more work is needed to be done and experiments with higher accuracy are needed to check the corresponding results.

Finally, it should be remarked that all the above results need to be reexamined in the case with \(\Lambda \neq 0\). But we can conclude, at least, that if there exist regular internal solutions which are in accordance with the Newtonian gravitational law and could be smoothly extended to the weak S-dS fields, the coupling constant should be chosen as \(\kappa = l^2/32\pi G\).
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