THE EULER–POINCARÉ CHARACTERISTIC
AND MIXED MULTIPlicITIES

Duong Quoc VIET and Truong Thi Hong THANH
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Abstract. This paper defines mixed multiplicity systems; the Euler–Poincaré characteristic and the mixed multiplicity symbol of $\mathbb{N}^d$-graded modules with respect to a mixed multiplicity system, and proves that the Euler–Poincaré characteristic and the mixed multiplicity symbol of any mixed multiplicity system of the type $(k_1, \ldots, k_d)$ and the $(k_1, \ldots, k_d)$-difference of the Hilbert polynomial are the same. As an application, we obtain results for mixed multiplicities.

1. Introduction

Mixed multiplicity is an important invariant of algebraic geometry and commutative algebra. Risler–Teissier [19] in 1973 showed that each mixed multiplicity of ideals of dimension 0 in a Noetherian local ring is the multiplicity of an ideal generated by a superficial sequence. Katz and Verma in 1989 [9] started the investigation of mixed multiplicities of ideals of positive height. For the case of arbitrary ideals, Viet [23] in 2000 described mixed multiplicities as the Hilbert–Samuel multiplicity via (FC)-sequences. Trung and Verma in 2007 [21] interpreted mixed volumes of polytopes as mixed multiplicities of ideals. Moreover, by using filter-regular sequences, Manh and Viet [29] in 2013 characterized mixed multiplicities of multi-graded modules in terms of the length of modules. In past years, the theory of mixed multiplicities has attracted much attention and has been continually developed (see, for example, [4, 5, 11–15, 18–32]).

Kirby and Rees [10] in 1994 studied a kind of mixed multiplicities of multi-graded modules and proved that these mixed multiplicities can be expressed as the Euler–Poincaré characteristic of a certain sequence. However, how to find mixed multiplicity formulas, which are analogous to Serre’s formula (see, for example, [16] or [3, Theorem 4.7.6]) and Auslander–Buchsbaum’s formula (see, for example, [1] or [3, Theorem 4.7.4]) in the Hilbert–Samuel multiplicity theory, is not yet known. This problem became an open question of the mixed multiplicity theory. This paper gives a different approach from the one in [10]. Our approach begins with defining mixed multiplicity systems and related invariants, which is similar to the approaches in [1, 3]. We need to choose objects exactly suitable for the goal of the paper and give mixed multiplicity formulas, which are analogous to formulas in Hilbert–Samuel multiplicity theory.

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Let \((A, m)\) be an Artinian local ring with maximal ideal \(m\) and infinite residue field \(A/m\). Denote by \(\mathbb{N}\) the set of all non-negative integers. Let \(d\) be a positive integer. Put \(e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^d\) for each \(1 \leq i \leq d\) and \(e! = k_1! \cdots k_d!\); \(|e| = k_1 + \cdots + k_d\) for any \(e = (k_1, \ldots, k_d) \in \mathbb{N}^d\). Moreover, set \(0 = (0, \ldots, 0) \in \mathbb{N}^d\); \(1 = (1, \ldots, 1) \in \mathbb{N}^d\) and \(n^k = n_1^{k_1} \cdots n_d^{k_d}\) for each \(n, k \in \mathbb{N}^d\) and \(n \geq 1\). Let \(S = \bigoplus_{n \in \mathbb{N}^d} S_n\) be a finitely generated standard \(\mathbb{N}^d\)-graded algebra over \(A\) (i.e., \(S\) is generated over \(A\) by elements of total degree 1) and let \(M = \bigoplus_{n \geq 0} M_n\) be a finitely generated \(\mathbb{N}^d\)-graded \(S\)-module. For each subset \(x\) of \(S\), we assign \(xM = 0\) if \(x = \emptyset\), and \(xM = (x)M\) if \(x \neq \emptyset\). Set \(S_{++} = \bigoplus_{n \geq 1} S_n\) and \(S_i = S_{ei}\) for \(1 \leq i \leq d\). Denote by \(\Proj S\) the set of the homogeneous prime ideals of \(S\) which do not contain \(S_{++}\). Put \[
abla_{++} M = \{P \in \Proj S \mid M_P \neq 0\}.
\]
Assume that \(S_1 = S_{\{1, \ldots, 1\}}\) is not contained in \(\sqrt{\Ann_S M}\) and \(\dim S_{++} M = s\), then by [7, Theorem 4.1], \(\ell_A(M_n)\) is a polynomial of degree \(s\) for all large \(n\). Denote by \(P_M(n)\) the Hilbert polynomial of the Hilbert function \(\ell_A(M_n)\). The terms of total degree \(s\) in the polynomial \(P_M(n)\) have the form \(\sum_{|k|=s} e(M; k)n^k/k!\). Then \(e(M; k)\) are non-negative integers not all zero, called the \textit{mixed multiplicity of} \(M\) \textit{of the type} \(k\) [7]. Now for each \(k \in \mathbb{N}^d\) such that \(|k| \geq \dim S_{++} M\), we put \[
eq_{++} M = \{P \in \Proj \mathbb{N}_{++} M\}.
\]
Next, we define mixed multiplicity systems: the Euler–Poincaré characteristic and the mixed multiplicity symbol of finitely generated \(\mathbb{N}^d\)-graded \(S\)-modules with respect to a mixed multiplicity system.

\textbf{Definition 1.1.} (Definition 2.5) Let \(x = x_1, \ldots, x_n\) be a sequence in \(\bigcup_{j=1}^{d} S_j\) consisting of \(m_1\) elements of \(S_1\), \(\ldots\), \(m_d\) elements of \(S_d\). Then \(x\) is called a \textit{mixed multiplicity system of} \(M\) \textit{of the type} \(m = (m_1, \ldots, m_d)\) if \(\dim S_{++} (M/xM) \leq 0\).

In the case that \(x\) is a \textit{mixed multiplicity system of the type} \(m\) and \(|m| = n\), denote by \(H_s(x, M)\) the homology of the Koszul complex of \(M\) with respect to \(x\). Then \(\sum_{i=0}^{n} (-1)^i \ell_A H_i(x, M, n)\) is a constant for \(n \gg 0\) (see Remark 3.1(ii)). Denote this constant by \(\chi(x, M)\). Then \(\chi(x, M)\) is called the \textit{Euler–Poincaré characteristic of} \(M\) \textit{with respect to} \(x\) \textit{for brevity}.

Another invariant related to mixed multiplicity systems is defined as follows.

\textbf{Definition 1.2.} (Definition 3.4) Let \(x = x_1, \ldots, x_n\) be a \textit{mixed multiplicity system of} \(M\). If \(n = 0\), then \(\ell_A(M_n) = c\) (const.) for all \(n \gg 0\) and we set \(\tilde{\chi}(x, M) = c\). If \(n > 0\), we set \(\tilde{\chi}(x, M) = \tilde{\chi}(x', M/x_1) - \tilde{\chi}(x', 0 : M/x_1);\) here \(x' = x_2, \ldots, x_n\). We call \(\tilde{\chi}(x, M)\) the \textit{mixed multiplicity symbol of} \(M\) \textit{with respect to} \(x\).

As one might expect, we first obtain the following theorem.

\textbf{Main Theorem.} (Theorem 3.7) \textit{Let} \(S\) \textit{be a finitely generated standard} \(\mathbb{N}^d\)-\textit{graded algebra over an Artinian local ring} \(A\) \textit{and let} \(M\) \textit{be a finitely generated standard} \(\mathbb{N}^d\)-\textit{graded} \(S\)-\textit{module. Then for any} \textit{mixed multiplicity system} \(x\) \textit{of} \(M\) \textit{of the type} \(k\), \(\chi(x, M) = \tilde{\chi}(x, M) = \Delta^k P_M(n)\).
From this theorem, it follows that the mixed multiplicity of the type \( k \), the Euler–Poincaré characteristic and the mixed multiplicity symbol of any mixed multiplicity system of the type \( k \) are the same by the following result.

**Theorem 1.3.** (Theorem 3.9) Let \( k \in \mathbb{N}^d \) such that \( \dim \text{Supp}_{++} M \leq |k| \). Then for any mixed multiplicity system \( x \) of \( M \) of the type \( k \), we have

\[
E(M; k) = \chi(x, M) = \tilde{e}(x, M).
\]

Theorem 1.3 not only yields interesting consequences in the case of graded modules (see, for example, Theorem 3.10, Corollary 3.11, Corollary 3.12, Corollary 3.13 and Corollary 3.14) but also gives applications to mixed multiplicities of ideals (see Corollary 4.8, Theorem 4.9, Corollary 4.10, Corollary 4.11 and Corollary 4.13).

This paper is divided into four sections. Section 2 is devoted to the discussion of filter-regular sequences and mixed multiplicity systems. In Section 3, we define the Euler–Poincaré characteristic; the mixed multiplicity symbol and obtain the Main Theorem (Theorem 3.7) and consequences. Section 4 gives applications of Section 3 to mixed multiplicities of ideals.

### 2. Filter-regular sequences and mixed multiplicity systems

This section defines mixed multiplicities, filter-regular sequences and mixed multiplicity systems of finitely generated \( \mathbb{N}^d \)-graded modules.

Assume that \( S_1 \not\subseteq \sqrt{\text{Ann}_S M} \) and \( \dim \text{Supp}_{++} M = s \), then by [7, Theorem 4.1], \( \ell_A(M_n) \) is a polynomial of degree \( s \) for all large \( n \). Denote by \( P_M(n) \) the Hilbert polynomial of the Hilbert function \( \ell_A(M_n) \). The terms of total degree \( s \) in the polynomial \( P_M(n) \) have the form \( \sum_{|k|=s} e(M; k)n^k/k! \). Then \( e(M; k) \) are non-negative integers, which are not all zero, called the mixed multiplicity of \( M \) of the type \( k \) [7].

Denote by \( \Delta^k f(n) \) the \( k \)-difference of the polynomial \( f(n) \) for each \( k \in \mathbb{N}^d \). Then we have the following comments.

**Remark 2.1.** If \( \dim \text{Supp}_{++} M = s \geq 0 \), then \( P_M(n) = \sum_{|k|=s} e(M; k)n^k/k! + Q_M(n) \) with \( \deg Q_M(n) < s \). Hence \( \Delta^k P_M(n) = e(M; k) \) for all \( k \in \mathbb{N}^d \) satisfying \( |k| = s \). Now we assign \( \dim \text{Supp}_{++} M = -\infty \) to the case that \( \text{Supp}_{++} M = \emptyset \) and the degree \( -\infty \) to the zero polynomial. Then by [7, Theorem 4.1] and [29, Proposition 2.7], we always have \( \deg P_M(n) = \dim \text{Supp}_{++} M \).

The notion of mixed multiplicities \( e(M; k) \) of the type \( k \) of a module \( M \) always requires the condition \( |k| = \dim \text{Supp}_{++} M \). This sometimes becomes an obstruction in describing the relationship between mixed multiplicities and the Euler–Poincaré characteristic and in expressing mixed multiplicity formulas. Consequently, we need the following extension.

**Definition 2.2.** For each \( k \in \mathbb{N}^d \) such that \( |k| \geq \dim \text{Supp}_{++} M \), we put

\[
E(M; k) = \begin{cases} 
  e(M; k) & \text{if } |k| = \dim \text{Supp}_{++} M, \\
  0 & \text{if } |k| > \dim \text{Supp}_{++} M.
\end{cases}
\]
Remark 2.4. We turn now to filter-regular sequences of multi-graded modules. The notion of filter-regular sequences was introduced by Stuckrad and Vogel in [17] (see [2]). The theory of filter-regular sequences became an important tool for studying some classes of singular rings and has been continually developed (see, for example, [2, 8, 20, 29, 31]).

Definition 2.3. A homogeneous element $a$ of $S$ is called an $S_{++}$-filter-regular element with respect to $M$ if $(0_M : a)_n = 0$ for all large $n$. A homogeneous sequence $x_1, \ldots, x_t$ in $S$ is called an $S_{++}$-filter-regular sequence with respect to $M$ if $x_i$ is an $S_{++}$-filter-regular element with respect to $M/\langle x_1, \ldots, x_{i-1} \rangle M$ for all $i = 1, \ldots, t$.

Remark 2.4. We have the following comments for filter-regular sequences of multi-graded modules.

(i) By [29, Note (i)], a homogeneous element $a \in S$ is an $S_{++}$-filter-regular element with respect to $M$ if and only if $0_M : a \subseteq 0_M : S^\infty_{++}$. Moreover, for each $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, there exists an $S_{++}$-filter-regular sequence $x$ in $\bigcup_{i=1}^d S_i$ with respect to $M$ consisting of $k_1$ elements of $S_1$, $\ldots$, $k_d$ elements of $S_d$ by [29, Proposition 2.2 and Note (ii)]. In this case, $x$ is called an $S_{++}$-filter-regular sequence of the type $k$.

(ii) Let $a \in S_i$. Since the following exact sequence

$$0 \longrightarrow (0_M : a)^{n-e_i} \longrightarrow M_{n-e_i} \xrightarrow{a} M_n \longrightarrow \frac{M_n}{aM_{n-e_i}} \longrightarrow 0,$$

it follows that $\Delta^e P_M(n) = P_{M/aM}(n) - P_{(0_M:a)}(n-e_i)$. Note that

$$\deg \Delta^e P_M(n) \leq \deg P_M(n) - 1 = \dim \text{Supp}_{++} M - 1.$$

Hence $\deg P_{M/aM}(n) \leq \dim \text{Supp}_{++} M - 1$ if and only if

$$\deg P_{(0_M:a)}(n) \leq \dim \text{Supp}_{++} M - 1.$$

So $\dim \text{Supp}_{++} (M/aM) \leq \dim \text{Supp}_{++} M - 1$ if and only if

$$\dim \text{Supp}_{++} (0_M : a) \leq \dim \text{Supp}_{++} M - 1.$$

If $a$ is an $S_{++}$-filter-regular element, then $P_{(0_M:a)}(n) = 0$. In this case, we have

$$\dim \text{Supp}_{++} (0_M : a) = -\infty \text{ and } \Delta^e P_M(n) = P_{M/aM}(n).$$

Therefore

$$\dim \text{Supp}_{++} \left( \frac{M}{aM} \right) \leq \dim \text{Supp}_{++} M - 1.$$

And if there exists an $e(M; k) \neq 0$ with $k_i > 0$, then $\dim \text{Supp}_{++} (M/aM) = \dim \text{Supp}_{++} M - 1$ by [29, Proposition 3.3(i)].

(iii) A homogeneous element $x \in S$ is an $S_{++}$-filter-regular element with respect to $M$ if $x \notin P$ for any $P \in \text{Ass}_S M$ not containing $S_{++}$. This means $x \notin \bigcup_{S_{++} \not\subseteq P, P \in \text{Ass}_S M} P$ by [29, Definition 2.1 and Proposition 2.5].

Now, we will give the following concept.

Definition 2.5. Let $x = x_1, \ldots, x_n$ be a sequence of elements in $\bigcup_{j=1}^d S_j$ consisting of $m_1$ elements of $S_1$, $\ldots$, $m_d$ elements of $S_d$. Then $x$ is called a mixed multiplicity system of $M$ of the type $m = (m_1, \ldots, m_d)$ if $\dim \text{Supp}_{++} (M/xM) \leq 0$. 

Remember that in the case of modules over local rings, the multiplicity systems that generate the ideals of definition have been previously given. The results of this paper will show the usefulness of mixed multiplicity systems.

**Remark 2.6.** By Remark 2.4(i), for each \( k \in \mathbb{N}^d \), there exists an \( S_+ \)-filter-regular sequence \( x = x_1, \ldots, x_s \) of the type \( k \). Then for any \( 0 \leq i \leq s \), we have

\[
\dim \operatorname{Supp}_+ (M/(x_1, \ldots, x_i)M) \leq \dim \operatorname{Supp}_+ M - i
\]

by Remark 2.4(ii). Now we choose \( k \) such that \(|k| \geq \dim \operatorname{Supp}_+ M\). Then by Remark 2.4(ii), we obtain \( \dim \operatorname{Supp}_+ (M/xM) \leq \dim \operatorname{Supp}_+ M - |k| \leq 0 \). Hence \( x \) is a mixed multiplicity system of \( M \).

We end this section with the following property of mixed multiplicity systems.

**Lemma 2.7.** Let \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \) be an exact sequence of \( \mathbb{N}^d \)-graded \( S \)-modules. And let \( x \) be a sequence of elements in \( \bigcup_{i=1}^{d} S_i \). Then \( x \) is a mixed multiplicity system of \( M \) if and only if \( x \) is a mixed multiplicity system of both \( M' \) and \( M'' \).

**Proof.** Note that \( \sqrt{\text{Ann}(F/cF)} = \sqrt{\text{Ann} F} + c \) for each ideal \( c \) and each module \( F \). Hence since \( \text{Ann}_S M \subseteq \text{Ann}_S M' \), it follows that

\[
\dim \operatorname{Supp}_+ \left( \frac{M'}{xM'} \right) \leq \dim \operatorname{Supp}_+ \left( \frac{M}{xM} \right).
\]

On the other hand from the exactness of

\[
M'/xM' \xrightarrow{g} M/xM \rightarrow M''/xM'' \rightarrow 0,
\]

it implies that \( \dim \operatorname{Supp}_+ (M'/xM') \geq \dim \operatorname{Supp}_+ \text{Im } g \) and

\[
\dim \operatorname{Supp}_+ \left( \frac{M}{xM} \right) = \max \left\{ \dim \operatorname{Supp}_+ \text{Im } g, \dim \operatorname{Supp}_+ \left( \frac{M''}{xM''} \right) \right\}.
\]

From the above facts, we immediately obtain that \( \dim \operatorname{Supp}_+ (M/xM) \leq 0 \) if and only if \( \dim \operatorname{Supp}_+ (M'/xM') \leq 0 \) and \( \dim \operatorname{Supp}_+ (M''/xM'') \leq 0 \). Thus, \( x \) is a mixed multiplicity system of \( M \) if and only if \( x \) is a mixed multiplicity system of both \( M' \) and \( M'' \).

## 3. The Euler–Poincaré characteristic and multiplicities

In this section, we define the Euler–Poincaré characteristic and the mixed multiplicity symbol of finitely generated standard \( \mathbb{N}^d \)-graded \( S \)-modules with respect to a mixed multiplicity system. Next we prove that the Euler–Poincaré characteristic and the mixed multiplicity symbol of any mixed multiplicity system of the type \( k = (k_1, \ldots, k_d) \in \mathbb{N}^d \) and the \( k \)-difference of the Hilbert polynomial are the same, and give interesting consequences for mixed multiplicities of \( \mathbb{N}^d \)-graded modules.

Let \( x \) be a system of \( n \) homogeneous elements in \( S \). Then by [3, Remark 1.6.15], we may consider the following Koszul complex \( K_*(x, M) \) of \( M \) with respect to \( x \) as an \( \mathbb{N}^d \)-graded
complex with a differential of degree 0:

\[ 0 \longrightarrow K_n(x, M) \longrightarrow K_{n-1}(x, M) \longrightarrow \cdots \longrightarrow K_1(x, M) \longrightarrow K_0(x, M) \longrightarrow 0. \]

Denote by \( H_i(x, M) \) the homology of the Koszul complex of \( M \) with respect to \( x \). We obtain the following sequence of the homology modules:

\[ H_0(x, M) : \ldots \longrightarrow H_0(x, M), H_1(x, M), \ldots, H_n(x, M). \]

The Koszul complex theory became an important tool for studying several different theories of commutative algebra and algebraic geometry.

**Remark 3.1.** Note that \( K_i(x, M) \) and \( H_i(x, M) \) are finitely generated \( \mathbb{N}^d \)-graded \( S \)-modules, and \( xH_i(x, M) = 0 \) for all \( 0 \leq i \leq n \) (see [3, Proposition 1.6.5(b)]). Moreover, we have the following notes.

(i) Since \( \sqrt{\text{Ann}_S M/xM} = \sqrt{\text{Ann}_S M + (x)} \) and \( \text{Ann}_S M + (x) \subseteq \text{Ann}_S H_i(x, M) \) for \( 0 \leq i \leq n \), it follows that \( \text{Ann}_S(M/xM) \subseteq \sqrt{\text{Ann}_S H_i(x, M)} \) for \( 0 \leq i \leq n \). Consequently, if \( \dim \text{Supp}^{++}(M/xM) \leq 0 \), then \( \dim \text{Supp}^{++} H_i(x, M) \leq 0 \) for all \( 0 \leq i \leq n \).

(ii) Let \( x = x_1, \ldots, x_n \) be a mixed multiplicity system of \( M \) of the type \( m \). Then for each \( 0 \leq i \leq n \) we have \( \dim \text{Supp}^{++} H_i(x, M) \leq 0 \) by (i). Therefore by Remark 2.1, we obtain \( \ell_A[H_i(x, M)_n] = a_i(\text{const.}) \) for all \( n \geq 0 \) and for each \( 0 \leq i \leq n \). Hence there exists \( v \in \mathbb{N}^d \) such that for each \( 0 \leq i \leq n \), \( \ell_A[H_i(x, M)_n] = a_i \) for all \( n \geq v \). So we obtain the constant \( \chi(x, M) = \sum_{i=0}^{n}(-1)^i \ell_A[H_i(x, M)_n] \) for all \( n \geq v \). In this case, \( \chi(x, M) \) is called the Euler–Poincaré characteristic of \( M \) with respect to \( x \) for brevity.

Some basic properties of the Euler–Poincaré characteristic with respect to mixed multiplicity systems are stated in the following lemma.

**Lemma 3.2.** Let \( x = x_1, \ldots, x_n \) be a mixed multiplicity system of \( M \). Then the following statements hold.

(i) \( \chi(x, -) \) is additive on short exact sequences, i.e. if

\[ 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \]

is a short exact sequence, then \( \chi(x, M) = \chi(x, M') + \chi(x, M'') \).

(ii) If \( x_1 \in \sqrt{\text{Ann}_S M} \), then \( \chi(x, M) = 0 \).

(iii) If \( x_1 \) is \( M \)-regular and \( x' = x_2, \ldots, x_n \), then \( \chi(x, M) = \chi(x', M/x_1M) \).

**Proof.** Without loss of generality, we may assume that \( \mathbb{N}^d \)-graded \( S \)-homomorphisms in this proof are \( \mathbb{N}^d \)-graded \( S \)-homomorphisms of degree 0.

The proof of (i). Since the Koszul complex is an exact functor, the exact sequence of \( \mathbb{N}^d \)-graded \( S \)-modules

\[ 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \]

yields a long exact sequence

\[ \cdots \longrightarrow H_i(x, M') \longrightarrow H_i(x, M) \longrightarrow H_i(x, M'') \longrightarrow H_{i-1}(x, M') \longrightarrow \cdots \]

of \( \mathbb{N}^d \)-graded homology \( S \)-modules. Hence for each \( n \in \mathbb{N}^d \) we have an exact sequence of \( A \)-modules

\[ \cdots \longrightarrow H_i(x, M')_n \longrightarrow H_i(x, M)_n \longrightarrow H_i(x, M'')_n \longrightarrow H_{i-1}(x, M')_n \longrightarrow \cdots \]
Recall that $\mathbf{x}$ is also a mixed multiplicity system of both $M'$ and $M''$ by Lemma 2.7. Hence by the additivity of length, the alternating sum of the lengths of the modules in this exact sequence is zero. From this fact, we obtain (i).

The proof of (ii). First we consider the case that $x_1 \in \operatorname{Ann}_S M$. Recall that $x' = x_2, \ldots, x_n$. By [3, Corollary 1.6.13(a)], we have the following exact sequence

$$
\cdots \longrightarrow H_i(x', M) \longrightarrow H_i(x, M) \longrightarrow H_{i-1}(x', M) \longrightarrow H_{i-1}(x, M) \longrightarrow \cdots
$$

(3.1)

Since $x_1M = 0$, $x'$ is also a mixed multiplicity system of $M$. Moreover, $x_1$ annihilates $H_i(x', M)$ for all $i$. Hence the exact sequence (3.1) breaks up into exact sequences

$$
0 \longrightarrow H_i(x', M) \longrightarrow H_i(x, M) \longrightarrow H_{i-1}(x', M) \longrightarrow 0
$$

for all $i$. Therefore

$$
\ell_A[H_i(x, M)_n] = \ell_A[H_i(x', M)_n] + \ell_A[H_{i-1}(x', M)_n]
$$

for all $i$ and for all large $n$. Note that $H_{n+1}(x, M) = H_{-1}(x, M) = 0$, so

$$
\sum_{i=0}^{n+1} (-1)^i \ell_A[H_i(x, M)_n] = \sum_{i=0}^{n+1} (-1)^i[\ell_A[H_i(x', M)_n] + \ell_A[H_{i-1}(x', M)_n]] = 0
$$

(3.2)

for all large $n$. Since $\ell_A[H_{n+1}(x, M)_n] = 0$ and

$$
\chi(x, M) = \sum_{i=0}^{n} (-1)^i \ell_A[H_i(x, M)_n]
$$

for all $n \gg 0$, $\chi(x, M) = 0$ by (3.2). So if $x_1M = 0$, then $\chi(x, M) = 0$. From this it follows that $\chi(x, M/aM) = 0$ for all $a \in x$.

We now turn to the case that $x_1 \in \sqrt{\operatorname{Ann}_S M}$. Then there exists $u > 0$ such that $x_1^uM = 0$. On the other hand since $\chi(x, M/x_1M) = 0$ and the exact sequence

$$
0 \longrightarrow x_1M \longrightarrow M \longrightarrow M/x_1M \longrightarrow 0,
$$

it follows from (i) that $\chi(x, M) = \chi(x, x_1M)$. Therefore, $\chi(x, M) = \chi(x, x_1^jM)$ for all $j \geq 1$. Consequently, $\chi(x, M) = \chi(x, x_1^uM) = \chi(x, 0M) = 0$. We obtain (ii).

The proof of (iii). Since $x_1$ is $M$-regular, we have

$$
H_i(x, M) \cong H_i(x', M/x_1M)
$$

by [3, Corollary 1.6.13(b)]. Thus

$$
\ell_A[H_i(x, M)_n] = \ell_A[H_i(x', M/x_1M)_n]
$$

for all large $n$ and for all $i$. From this, (iii) follows. □

Using Lemma 3.2, we prove the following recursion formula for the Euler–Poincaré characteristic of a graded module with respect to a mixed multiplicity system.

**Lemma 3.3.** Let $x = x_1, \ldots, x_n$ be a mixed multiplicity system of the module $M$. Set $x' = x_2, \ldots, x_n$. Then we have

$$
\chi(x, M) = \chi(x', M/x_1M) - \chi(x', 0_M : x_1).
$$
Proof. Consider the following cases.

If \( x_1 \in \sqrt{\text{Ann}_S M} \), then \( \chi(x, M) = 0 \) by Lemma 3.2(ii). In this case,

\[
\text{Supp}_+(M/xM) = \text{Supp}_+(M/x'M).
\]

Hence \( x' \) is also a mixed multiplicity system of \( M \). Therefore since the exact sequence

\[
0 \longrightarrow (0_M : x_1) \longrightarrow M \xrightarrow{x_1} M \xrightarrow{x_1M} M/x_1M \longrightarrow 0,
\]
we have \( \chi(x', M/x_1M) = \chi(x', 0_M : x_1) = 0 \) by Lemma 3.2(i). So

\[
\chi(x, M) = \chi(x', M/x_1M) - \chi(x', 0_M : x_1).
\]

In the case where \( x_1 \notin \sqrt{\text{Ann}_S M} \) and \( D = 0_M : x_1^\infty \), then \( x_1 \) is \((M/D)\)-regular. Since \( x_1M \cap D = x_1D \), we have the exact sequence

\[
0 \longrightarrow D \xrightarrow{x_1D} M \xrightarrow{x_1M} M \xrightarrow{x_1M + D} D \longrightarrow 0.
\]

Hence

\[
\chi(x', \frac{M}{x_1M + D}) = \chi(x', \frac{M}{x_1M}) - \chi(x', \frac{D}{x_1D})
\]
by Lemma 3.2(i). Since \( M \) is Noetherian, \( D = 0_M : x_1^v \) for some \( v > 0 \). Hence \( x_1 \in \sqrt{\text{Ann}_S D} \). So \( x' \) is also a mixed multiplicity system of \( D \). On the other hand \( \chi(x', D/x_1D) = \chi(x', 0_M : x_1) \) since the exact sequence

\[
0 \longrightarrow (0_M : x_1) \longrightarrow D \xrightarrow{x_1D} D \xrightarrow{x_1D} 0.
\]

Thus

\[
\chi(x', \frac{M}{x_1M + D}) = \chi(x', \frac{M}{x_1M}) - \chi(x', 0_M : x_1).
\] (3.3)

Recall that \( x_1 \) is \((M/D)\)-regular. So \( \chi(x', M/(x_1M + D)) = \chi(x, M/D) \) by Lemma 3.2(iii).

Now since \( \chi(x, M/D) = \chi(x, M) - \chi(x, D) \) by Lemma 3.2(i), we obtain

\[
\chi(x', \frac{M}{x_1M + D}) = \chi(x, M) - \chi(x, D).
\]

Note that \( x_1 \in \sqrt{\text{Ann}_S D} \), hence \( \chi(x, D) = 0 \) by Lemma 3.2(ii). So

\[
\chi(x', \frac{M}{x_1M + D}) = \chi(x, M).
\] (3.4)

From equations (3.3) and (3.4) we obtain

\[
\chi(x, M) = \chi(x', M/x_1M) - \chi(x', 0_M : x_1).
\]

The lemma has been proved. \( \square \)

Next we construct an invariant that is called the mixed multiplicity symbol with respect to a mixed multiplicity system.
Definition 3.4. Let $x = x_1, \ldots, x_n$ be a mixed multiplicity system of $M$. If $n = 0$, then $\ell_A(M_n) = c(\text{const.})$ for all $n \gg 0$ and we set $\widetilde{e}(x, M) = \widetilde{e}(\emptyset, M) = c$. If $n > 0$, we set

$$\widetilde{e}(x, M) = \widetilde{e}(x', M/x_1 M) - \widetilde{e}(x', 0_M : x_1);$$

here $x' = x_2, \ldots, x_n$. We call $\widetilde{e}(x, M)$ the mixed multiplicity symbol of $M$ with respect to $x$.

Then the relationship between the Euler–Poincaré characteristic and the mixed multiplicity symbol with respect to a mixed multiplicity system of $M$ is given by the following proposition.

**Proposition 3.5.** Let $x$ be a mixed multiplicity system of $M$. Then we have

$$\chi(x, M) = \widetilde{e}(x, M).$$

*Proof.* Note that if the length of $x$ is equal to 0, then we have

$$\chi(x, M) = \ell_A[H_0(x, M)_n] = \ell_A(M_n) = \widetilde{e}(x, M)$$

for all large $n$. Consequently the assertion follows from Lemma 3.3 and the definition of the mixed multiplicity symbol of $M$ with respect to $x$ (see Definition 3.4).

The Euler–Poincaré characteristic of a mixed multiplicity system of $M$ of the type $k$ and the $k$-difference of the Hilbert polynomial $P_M(n)$ are directly linked by the following proposition.

**Proposition 3.6.** Let $P_M(n)$ be the Hilbert polynomial of the Hilbert function $\ell_A(M_n)$. Then for any mixed multiplicity system $x$ of $M$ of the type $k \in \mathbb{N}^d$, we have

$$\chi(x, M) = \Delta^k P_M(n).$$

*Proof.* Set $|k| = s$. Denote by $r$ the number of all non-zero elements in $x$. We will prove that $\chi(x, M) = \Delta^k P_M(n)$ by induction on $r$.

Consider the case that $r = 0$. Then $\dim \text{Supp}_{++} M \leq 0$, so $\deg P_M(n) \leq 0$. Now if $s = 0$, then we have $\chi(x, M) = \ell_A[H_0(x, M)_n] = \ell_A(M_n)$ for all $n \gg 0$ and

$$\Delta^k P_M(n) = \Delta^0 P_M(n) = \ell_A(M_n)$$

for all $n \gg 0$. Hence $\chi(x, M) = \Delta^k P_M(n)$. If $s > 0$, then $\chi(x, M) = 0$ by Lemma 3.2(ii). Since $|k| = s > 0$ and $\deg P_M(n) \leq 0$, it follows that $\Delta^k P_M(n) = 0$. Thus

$$\chi(x, M) = \Delta^k P_M(n).$$

Therefore, if $r = 0$, then $\chi(x, M) = \Delta^k P_M(n)$.

Next assume that $r > 0$, $x = x_1, \ldots, x_s$ and $x_1 \neq 0$. Set $x' = x_2, \ldots, x_s$. Let

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_u = M$$

be a prime filtration of $M$, i.e. $M_{j+1}/M_j \cong S/P_j$ where $P_j$ is a homogeneous prime ideal for all $0 \leq j \leq u - 1$ and $\{P_0, P_1, \ldots, P_{u-1}\} \subseteq \text{Supp}M$. Then we obtain

$$P_M(n) = \sum_{j=0}^{u-1} P_{S/P_j}(n).$$
By Lemma 2.7, $x$ is a mixed multiplicity system of $S/P_j$ for each $0 \leq j \leq u - 1$, and by Lemma 3.2(i), it follows that $\chi(x, M) = \sum_{j=0}^{u-1} \chi(x, S/P_j)$.

Now if $x_1 \in P_j$ and denote by $\overline{x}$ the image of $x$ in $S/P_j$ and consider $\chi(\overline{x}, S/P_j)$ as the Euler–Poincaré characteristic of the $S/P_j$-module $S/P_j$ with respect to $\overline{x}$, then $\chi(\overline{x}, S/P_j) = \chi(x, S/P_j)$ and $\overline{x}_1 = 0$ in $S/P_j$. Hence if we let $v$ be the number of all non-zero elements in $\overline{x}$, then $v < r$. Then by the inductive assumption we have $\Delta^k P_{S/P_j}(n) = \chi(\overline{x}, S/P_j)$. Therefore $\Delta^k P_{S/P_j}(n) = \chi(x, S/P_j)$.

In the case that $x_1 \notin P_j$, then $x_1$ is $S/P_j$-regular of $S$-module $S/P_j$. Now assume that $x_1 \in S_I$. Then by [29, Remark 2.6], it follows that $\Delta^e P_{S/P_j}(n) = P_{S/(x_1, P_j)}(n)$. Since the number of all non-zero elements in $x'$ is less than $r$ and $x'$ is a mixed multiplicity system of the type $k - e_i \in \mathbb{N}^d$ of $S/(x_1, P_j)$, we obtain

$$\Delta^{k-e_i} P_{S/(x_1, P_j)}(n) = \chi(x', S/(x_1, P_j))$$

by the inductive assumption. Note that

$$\Delta^{k-e_i} P_{S/(x_1, P_j)}(n) = \Delta^{k-e_i} [\Delta^e P_{S/P_j}(n)] = \Delta^k P_{S/P_j}(n).$$

Since $x_1$ is $S/P_j$-regular of $S$-module $S/P_j$, we have $\chi(x', S/(x_1, P_j)) = \chi(x, S/P_j)$ by Lemma 3.2(iii). So $\Delta^k P_{S/P_j}(n) = \chi(x, S/P_j)$ for each $0 \leq j \leq u - 1$. Consequently

$$\Delta^k P_M(n) = \sum_{j=0}^{u-1} \Delta^k P_{S/P_j}(n) = \sum_{j=0}^{u-1} \chi(x, S/P_j) = \chi(x, M).$$

Thus $\chi(x, M) = \Delta^k P_M(n)$. The proof of induction is complete.

Proposition 3.5 and Proposition 3.6 yield the following result.

**THEOREM 3.7.** Let $S$ be a finitely generated standard $\mathbb{N}^d$-graded algebra over an Artinian local ring $A$ and let $M$ be a finitely generated standard $\mathbb{N}^d$-graded $S$-module. Then for any mixed multiplicity system $x$ of $M$ of the type $k$, we have

$$\chi(x, M) = \tilde{\eta}(x, M) = \Delta^k P_M(n).$$

**Remark 3.8.** From Theorem 3.7, it follows that if $|k| > \dim \text{Supp}_{++} M$, then $\chi(x, M) = \tilde{\eta}(x, M) = 0$ since $\Delta^k P_M(n) = 0$. Moreover, the mixed multiplicity symbol $\tilde{\eta}(x, M)$ does not depend on the order of the elements of $x$ and the Euler–Poincaré characteristic $\chi(x, M)$ depends only on the type of $x$.

Now if $|k| \geq \dim \text{Supp}_{++} M$, then we have $\Delta^k P_M(n) = E(M; k)$. Hence by Theorem 3.7, we obtain the following result.

**THEOREM 3.9.** Let $k \in \mathbb{N}^d$ such that $\dim \text{Supp}_{++} M \leq |k|$. Then for any mixed multiplicity system $x$ of $M$ of the type $k$, we have $E(M; k) = \chi(x, M) = \tilde{\eta}(x, M)$.

Recall that $S_1 = S_{\{1, \ldots, 1\}}$. Now if $S_1 \not\subseteq \sqrt{\text{Ann}_S M}$, then $\dim \text{Supp}_{++} M \geq 0$. Set $\dim \text{Supp}_{++} M = s$. Then we have $E(M; k) = e(M; k)$ for each $k \in \mathbb{N}^d$ with $|k| = s$. Hence from Theorem 3.9 we immediately obtain the following strong result.
**Theorem 3.10.** Assume that $S_1 \not\subseteq \sqrt{\Ann_S M}$. Set $\dim \text{Supp}_{++} M = s$. Let $x$ be a mixed multiplicity system of $M$ of the type $k \in \mathbb{N}^d$ with $|k| = s$. Then we have

$$e(M; k) = \chi(x; M) = \tilde{e}(x, M).$$

For any $k \in \mathbb{N}^d$ with $|k| = \dim \text{Supp}_{++} M$, there exists a mixed multiplicity system $x = x_1, \ldots, x_s$ of $M$ of the type $k$ such that for any $0 \leq i \leq s$,

$$\dim \text{Supp}_{++} \left[ \frac{M}{(x_1, \ldots, x_i)M} \right] \leq \dim \text{Supp}_{++} M - i$$

by Remark 2.6. By Theorem 3.10, we have $e(M; k) = \tilde{e}(x, M)$. Set $x' = x_2, \ldots, x_s$, and assume that $x_1 \in S_1$. Since $\tilde{e}(x, M) = \tilde{e}(x', M/x_1M) - \tilde{e}(x', 0_M : x_1)$, it follows that $\tilde{e}(x', M/x_1M) \geq e(M; k)$. Note that $x'$ is a mixed multiplicity system of the type $k - e_i$ of both $M/x_1M$ and $0_M : x_1$. Now assume that $e(M; k) > 0$. Then we have $\tilde{e}(x', M/x_1M) > 0$. Hence, since $\dim \text{Supp}_{++}(M/x_1M) \leq s - 1$, we obtain $\dim \text{Supp}_{++}(M/x_1M) = s - 1$ by Remark 3.8. So in this case, we have

$$\dim \text{Supp}_{++} \left[ \frac{M}{(x_1, \ldots, x_i)M} \right] = \dim \text{Supp}_{++} M - i.$$
Corollary 3.11. Let \( x = x_1, \ldots, x_s \) be a mixed multiplicity system of \( M \) of the type \( k \) with \( |k| = \dim \text{Supp}_+ M \) and
\[
\dim \text{Supp}_+ \left[ \frac{M}{(x_1, \ldots, x_i)M} \right] \leq \dim \text{Supp}_+ M - i
\]
for any \( 0 \leq i \leq s \). Denote by \( h_i = (h_{i1}, \ldots, h_{id}) \) the type of the subsequence \( x_1, \ldots, x_i \) of \( x \) for each \( 1 \leq i \leq s \). Assume that \( e(M; k) \neq 0 \). Then the following statements hold:

(i) for each \( 1 \leq i \leq s \), we have
\[
\dim \text{Supp}_+ \left[ \frac{M}{(x_1, \ldots, x_i)M} \right] = \dim \text{Supp}_+ M - i;
\]

(ii) \( e(M; k) = \ell_A \left( \left( \frac{M}{xM} \right)_n \right) - \sum_{i=1}^s E \left( \frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M} ; k - h_i \right) \) for all large \( n \).

Now assume that \( x \) is a mixed multiplicity system of \( M \) and \( a \in \mathfrak{x} \) is an \( S_{++} \)-filter-regular element with respect to \( M \). Set \( x' = x \setminus \{a\} \). Since \( a \) is an \( S_{++} \)-filter-regular element, we obtain \( \dim \text{Supp}_+(0_M : a) < 0 \) by Remark 2.4(ii). Consequently \( \tilde{e}(x', 0_M : a) = 0 \) by Remark 3.8. Therefore
\[
\tilde{e}(x, M) = \tilde{e}(x', M/aM) - \tilde{e}(x', 0_M : a) = \tilde{e}(x', M/aM).
\]

From this it follows that \( \chi(x, M) = \chi(x', M/aM) \) by Proposition 3.5.

We obtain the following corollary.

Corollary 3.12. Let \( x \) be a mixed multiplicity system of \( M \). Let \( a \in x \) be an \( S_{++} \)-filter-regular element with respect to \( M \). Set \( x' = x \setminus \{a\} \). Then:

(i) \( \chi(x, M) = \chi(x', M/aM) \);

(ii) \( \tilde{e}(x, M) = \tilde{e}(x', M/aM) \).

Let \( x = x_1, \ldots, x_s \) be an \( S_{++} \)-filter-regular sequence of \( M \) of the type \( k \) with \( |k| \geq \dim \text{Supp}_+ M \). Then \( x \) is a mixed multiplicity system of \( M \) by Remark 2.6, and we have
\[
\tilde{e}(x, M) = \tilde{e}(x_{i+1}, \ldots, x_s, M/(x_1, \ldots, x_i)M)
\]
for each \( 1 \leq i \leq s \) by Corollary 3.12(ii). Consequently,
\[
\tilde{e}(x, M) = \tilde{e}(\emptyset, M/xM) = \ell_A \left( \left( \frac{M}{xM} \right)_n \right)
\]
for \( n \gg 0 \). Note that \( \ell_A[(M/xM)_n] \neq 0 \) for \( n \gg 0 \) if and only if \( \dim \text{Supp}_+(M/xM) = 0 \).

So in this case, \( |k| = \dim \text{Supp}_+ M \) by Remark 2.6.

Hence we have the following result.

Corollary 3.13. (See [29, Theorem 3.4]) Let \( k \in \mathbb{N}^d \) with \( |k| \geq \dim \text{Supp}_+ M \). Assume that \( x \) is an \( S_{++} \)-filter-regular sequence with respect to \( M \) of the type \( k \). Then we have
\[
E(M; k) = \chi(x, M) = \tilde{e}(x, M) = \ell_A \left( \left( \frac{M}{xM} \right)_n \right)
\]
for all large \( n \), and \( E(M; k) \neq 0 \) if and only if \( \dim \text{Supp}_+(M/xM) = 0 \). In this case, \( |k| = \dim \text{Supp}_+ M \) and \( E(M; k) = e(M; k) \).
Let \( \Lambda \) be the set of all homogeneous prime ideals \( P \) of \( S \) such that \( P \in \text{Supp}_{++} M \) and \( \dim \text{Proj}(S/P) = \dim \text{Supp}_{++} M \).

Then by [31, Theorem 3.1], we have

\[
e(M; k) = \sum_{P \in \Lambda} \ell(M_P) e(S/P; k).
\]

Recall that if \( 0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_u = M \) is a prime filtration of \( M \), then for any \( P \in \Lambda \), there exists \( 0 \leq i \leq u - 1 \) such that \( S/P \cong M_{i+1}/M_i \) by the proof of [31, Theorem 3.1]. Therefore, if \( x \) is a mixed multiplicity system of \( M \), then \( x \) is also a mixed multiplicity system of \( S/P \) for any \( P \in \Lambda \) by Lemma 2.7.

Hence by Theorem 3.10, we immediately obtain the following corollary.

**Corollary 3.14.** Assume that \( S_1 \) is not contained in \( \sqrt{\text{Ann}_S M} \). Denote by \( \Lambda \) the set of all homogeneous prime ideals \( P \) of \( S \) such that \( P \in \text{Supp}_{++} M \) and \( \dim \text{Proj}(S/P) = \dim \text{Supp}_{++} M \). Set \( \dim \text{Supp}_{++} M = s \). Assume that \( x \) is a mixed multiplicity system of \( M \) of the type \( k \) with \( |k| = s \). Then

(i) \( \chi(x, M) = \sum_{P \in \Lambda} \ell(M_P) \chi(x, S/P) \);
(ii) \( \tilde{\varepsilon}(x, M) = \sum_{P \in \Lambda} \ell(M_P) \tilde{\varepsilon}(x, S/P) \).

### 4. Applications to mixed multiplicities of ideals

In this section, we will give some applications of Section 3 to mixed multiplicities of modules over local rings with respect to ideals.

Let \( (R, n) \) be a Noetherian local ring with maximal ideal \( n \) and infinite residue field \( R/n \). Let \( N \) be a finitely generated \( R \)-module. Let \( J, I_1, \ldots, I_d \) be ideals of \( R \) with \( J \) being \( n \)-primary and \( I_1 \cdots I_d \not\subseteq \sqrt{\text{Ann}_R N} \). Recall that \( k = (k_1, \ldots, k_d) \in \mathbb{N}^d \), \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \), \( I = I_1 \cdots I_d \), \( I[k] = I_1^{[k_1]} \cdots I_d^{[k_d]} \) and \( \mathbb{n} = I_1^{n_1} \cdots I_d^{n_d} \). We obtain an \( \mathbb{N}^{(d+1)} \)-graded algebra and an \( \mathbb{N}^{(d+1)} \)-graded module:

\[
T = \bigoplus_{n \geq 0, \mathbb{n} \geq 0} \frac{J^n \mathbb{n}}{J^{n+1} \mathbb{n}} \quad \text{and} \quad N^\prime = \bigoplus_{n \geq 0, \mathbb{n} \geq 0} \frac{J^n \mathbb{n} N}{J^{n+1} \mathbb{n} N}.
\]

Then \( T \) is a finitely generated standard \( \mathbb{N}^{(d+1)} \)-graded algebra over an Artinian local ring \( R/J \) and \( N^\prime \) is a finitely generated standard \( \mathbb{N}^{(d+1)} \)-graded \( T \)-module. The mixed multiplicity of \( N^\prime \) of the type \( (k_0, k) \) is denoted by \( e(J^{[k_0+1]}, I[k]; N) \), i.e.

\[
e(J^{[k_0+1]}, I[k]; N) := e(N^\prime; k_0, k)
\]

and which is called the **mixed multiplicity of \( N \) with respect to ideals \( J, I \) of the type \( (k_0 + 1, k) \)** (see [7, 13, 22]). Set \( I = J I_1 \cdots I_d, I_0 = J; T_i = I_i/J I_i \) for all \( 0 \leq i \leq d \).

**Remark 4.1.** On the one hand \( \dim P_N(n) = \dim \text{Supp}_{++} N \) by [7, Theorem 4.1], and on the other hand \( \dim P_N(n) = \dim N/(0_N : I^\infty) - 1 \) by [23, Proposition 3.1(i)] (see [13]). Hence \( \dim \text{Supp}_{++} N = \dim N/(0_N : I^\infty) - 1 \).

Defining mixed multiplicity systems of \( N^\prime \) is the reason for using the following Rees superficial sequences.
Definition 4.6. An element \( a \in R \) is called a Rees superficial element of \( N \) with respect to \( I \) if there exists \( i \in \{1, \ldots, d\} \) such that \( a \in I_i \) and \( aN \cap \mathfrak{p}^nI_iN = a\mathfrak{p}^nN \) for all \( n \gg 0 \).

Let \( x_1, \ldots, x_t \) be a sequence in \( R \). Then \( x_1, \ldots, x_t \) is called a Rees superficial sequence of \( N \) with respect to \( I \) if \( x_{j+1} \) is a Rees superficial element of \( N/(x_1, \ldots, x_j)N \) with respect to \( I \) for all \( j = 0, 1, \ldots, t-1 \). A Rees superficial sequence of \( N \) consisting of \( k_1 \) elements of \( I_1, \ldots, k_d \) elements of \( I_d \) is called a Rees superficial sequence of \( N \) of the type \( \mathbf{k} = (k_1, \ldots, k_d) \).

Remark 4.3. By [13, Lemma 2.2] which is a generalized version of [14, Lemma 1.2], for any set of ideals \( I_1, \ldots, I_d \) and each \( \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d \), there exists a Rees superficial sequence of \( N \) of the type \( \mathbf{k} \).

Let \( \mathbf{x} \) be a Rees superficial sequence of \( N \) with respect to ideals \( J, I \) and let \( \mathbf{x}^* \) be the image of \( \mathbf{x} \) in \( \bigcup_{i=0}^d T_i \). It can be verified (or see [29, Remark 4.1]) that

\[
(\mathcal{N}/\mathbf{x}^*\mathcal{N})_{(m,m)} \cong \bigoplus_{n \geq 0, m \geq 0} J^n\mathfrak{p}^n(N/\mathbf{x}N) \big/ J^{n+1}\mathfrak{p}^n(N/\mathbf{x}N) (m,m)
\]

for all \( m \gg 0, \ m \gg 0 \). Hence by Remark 4.1, we have

\[
\dim \text{Supp}_{++} \frac{\mathcal{N}}{\mathbf{x}^*\mathcal{N}} = \dim \frac{N}{\mathbf{x}N : I^\infty} - 1.
\]

So \( \mathbf{x}^* \) is a mixed multiplicity system of \( \mathcal{N} \) if and only if \( \dim N/(\mathbf{x}N : I^\infty) \leq 1 \).

These comments lead us to define the following system.

Definition 4.4. Let \( \mathbf{x} \) be a Rees superficial sequence of \( N \) with respect to ideals \( J, I \) of the type \( (k_0, \mathbf{k}) \in \mathbb{N}^{d+1} \). Then \( \mathbf{x} \) is called a mixed multiplicity system of \( N \) with respect to ideals \( J, I \) of the type \( (k_0, \mathbf{k}) \) if \( \dim N/(\mathbf{x}N : I^\infty) \leq 1 \).

From our point of view in this paper, both the above conditions of mixed multiplicity systems of \( \mathcal{N} \) are necessary to characterize mixed multiplicities of ideals in terms of the Euler–Poincaré characteristic and the mixed multiplicity symbol of \( \mathcal{N} \) with respect to a mixed multiplicity system of \( \mathcal{N} \).

Remark 4.5. Remark 4.1 proves that if \( \mathbf{x} \) is a Rees superficial sequence of \( N \) with respect to \( J, I \) and \( \mathbf{x}^* \) is the image of \( \mathbf{x} \) in \( \bigcup_{i=0}^d T_i \), then \( \mathbf{x}^* \) is a mixed multiplicity system of \( \mathcal{N} \) if and only if \( \mathbf{x} \) is a mixed multiplicity system of \( N \) with respect to \( J, I \).

In order to describe mixed multiplicity formulas of ideals we need to extend the notion of mixed multiplicities.

Definition 4.6. Let \( k_0 \in \mathbb{N} \) and \( \mathbf{k} \in \mathbb{N}^d \) such that \( k_0 + |\mathbf{k}| \geq \dim N/(0_N : I^\infty) - 1 \). We assign

\[
E(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}, N) = \begin{cases} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}, N) & \text{if } k_0 + |\mathbf{k}| = \dim \frac{N}{0_N : I^\infty} - 1, \\ 0 & \text{if } k_0 + |\mathbf{k}| > \dim \frac{N}{0_N : I^\infty} - 1. \end{cases}
\]
Put \( I_0 = J \). Next, assume that \( a \in I_1 \) is a Rees superficial element of \( N \) with respect to \( J, I \) and \( a^* \) is the image of \( a \) in \( T_i \). Then we have

\[
(J^{n+1}I/I^n : a) \bigcap J^nI = \left( \left( J^{n+1}I/I^n \bigcap aN \right) : a \right) \bigcap J^nI = (aJ^{n+1}I/I^n : a) \bigcap J^nI = (J^{n+1}I/I^n + (0 : a)) \bigcap J^nI = J^{n+1}I/I^n + (0 : a) \bigcap J^nI
\]

for all \( n \gg 0, n \gg 0 \). Consequently

\[
(J^{n+1}I/I^n : a) \bigcap J^nI = J^{n+1}I/I^n + (0 : a) \bigcap J^nI \tag{4.2}
\]

for all \( n \gg 0, n \gg 0 \). From (4.2) it follows that

\[
(0_N : a^*)_{(n, n)} = \frac{(J^{n+1}I/I^n : a) \bigcap J^nI}{J^{n+1}I/I^n} = \frac{J^{n+1}I/I^n + (0 : a) \bigcap J^nI}{J^{n+1}I/I^n} = \frac{(0 : a) \bigcap J^nI}{(0 : a) \bigcap J^{n+1}I/I^n}
\]

for all \( n \gg 0, n \gg 0 \). Remember that \( I = I_1 \cdots I_d \). By the Artin–Rees lemma, there exists \( u \gg 0 \) such that

\[
(0_N : a^*)_{(n+u, n+u+1)} = \frac{(0 : a) \bigcap I^uN}{(0 : a) \bigcap I^{u+1}N/I^n}
\]

for all \( n \geq 0, n \geq 0 \). Fix the \( u \) and set \( W = (0 : a) \bigcap I^uN, U = (0 : a)/W \). Since \( W : I^\infty = 0_N : aI^\infty \supseteq 0_N : a \), it follows that \( U/0_U : I^\infty = 0 \). So \( \dim U/0_U : I^\infty < 0 \). Hence, since the exact sequence \( 0 \rightarrow W \rightarrow (0 : a) \rightarrow U = (0 : a)/W \rightarrow 0 \), we obtain that the mixed multiplicities of \( (0 : a) \) and the mixed multiplicities of \( W \) with respect to ideals \( J, I \) are the same by [31, Corollary 3.9(ii)]. So

\[
E(J^{[k]_0+1}, I^{[k]}; W) = E(J^{[k]_0+1}, I^{[k]}; 0_N : a).
\]

On the other hand by (4.3), we have \( E(0_N : a^*; k_0, k) = E(J^{[k]_0+1}, I^{[k]}; 0_N : a) \). Hence \( E(0_N : a^*; k_0, k) = E(J^{[k]_0+1}, I^{[k]}; 0_N : a) \). By equation (4.1), we obtain

\[
E(N/a^*N; k_0, k) = E(J^{[k]_0+1}, I^{[k]}; N/aN).
\]

The above facts yield the following.

Remark 4.7. We have \( E(N; k_0, k) = E(J^{[k]_0+1}, I^{[k]}; N) \), and if \( a \in I_1 \) is a Rees superficial element of \( N \) with respect to \( J, I \) and \( a^* \) is the image of \( a \) in \( T_i \), then

\[
E(0_N : a^*; k_0, k) = E(J^{[k]_0+1}, I^{[k]}; 0_N : a),
\]

\[
E(N/a^*N; k_0, k) = E(J^{[k]_0+1}, I^{[k]}; N/aN).
\]

Moreover, if \( \dim N/(0_N : I^\infty) = 1 \), then by [23, Proposition 3.2] it can easily be seen that \( E(J^{[k]_1}, I^{[k]}; N) = e(J; N/(0_N : I^\infty)) \).
Now, let \( x \) be a mixed multiplicity system of \( N \) with respect to \( J, I \) of the type \( (k_0, k) \) and let \( x^* \) be the image of \( x \) in \( \bigcup_{i=0}^d T_i \). Then \( x^* \) is a mixed multiplicity system of \( N \) of the type \( (k_0, k) \) by Proposition 3.5, and \( \tilde{\nu}(x^*, N) = \nu(x^*, N) \) by Proposition 3.5. Hence \( \tilde{\nu}(x^*, N) = \nu(x^*, N) \) by Theorem 3.9. On the one hand \( E(N; k_0, k) = E(J[k_0+1], I[k]; N) \) by Remark 4.7. On the other hand \( \dim \text{Supp}_+ N = \dim N/(0_N : I^\infty) - 1 \) by Remark 4.1. Therefore, \( \tilde{\nu}(x^*, N) = \nu(x^*, N) = \tilde{\nu}(x^*, N) \).

Corollary 4.8. Let \( x \) be a mixed multiplicity system of \( N \) with respect to ideals \( J, I \) of the type \( (k_0, k) \) and let \( x^* \) be the image of \( x \) in \( \bigcup_{i=0}^d T_i \). Then the following statements hold:

(i) \( x^* \) is a mixed multiplicity system of \( N \) of the type \( (k_0, k) \);
(ii) \( \nu(x^*, N) = \tilde{\nu}(x^*, N) \);
(iii) if \( k_0 + |k| \geq \dim N/(0_N : I^\infty) - 1 \), then \( E(J[k_0+1], I[k]; N) = \nu(x^*, N) = \tilde{\nu}(x^*, N) \).

In the case that \( JJ_1 \cdots I_d \) is not contained in \( \sqrt{\text{Ann}_R N} \), by Corollary 4.8 we immediately obtain the main result of this section.

Theorem 4.9. Let \( J, I_1, \ldots, I_d \) be ideals of \( R \) with \( J \) being \( n \)-primary. Assume that \( I = JJ_1 \cdots I_d \) is not contained in \( \sqrt{\text{Ann}_R N} \) and \( (k_0, k) \in \mathbb{N}^{d+1} \) such that \( k_0 + |k| = \dim N/(0_N : I^\infty) - 1 \). Let \( x = x_1, \ldots, x_s \) be a mixed multiplicity system of \( N \) with respect to ideals \( J, I \) of the type \( (k_0, k) \) and \( x^* \) the image of \( x \) in \( \bigcup_{i=0}^d T_i \). Then we have

\[
e(J[k_0+1], I[k]; N) = \nu(x^*, N) = \tilde{\nu}(x^*, N).
\]

Now by Corollary 3.11 and Remark 4.7, we obtain the following corollary.

Corollary 4.10. Assume that \( I = JJ_1 \cdots I_d \) is not contained in \( \sqrt{\text{Ann}_R N} \) and \( e(J[k_0+1], I[k], N) \neq 0 \). Let \( x = x_1, \ldots, x_s \) be a mixed multiplicity system of \( N \) with respect to ideals \( J, I \) of the type \( (k_0, k) \) such that for each \( 1 \leq i \leq s \),

\[
\dim_{(x_1, \ldots, x_i)N : I^\infty} N \leq \dim_{0_N : I^\infty} N - i.
\]

Denote by \( (m_i, h_i) = (m_i, h_{i1}, \ldots, h_{id}) \) the type of the subsequence \( x_1, \ldots, x_i \) of \( x \) for each \( 1 \leq i \leq s \), and for each \( 1 \leq i \leq s \), set

\[
N_i = \frac{(x_1, \ldots, x_{i-1})N : x_i}{(x_1, \ldots, x_{i-1})N}.
\]

Then the following statements hold:

(i) \( \dim_{(x_1, \ldots, x_i)N : I^\infty} N = \dim_{0_N : I^\infty} N - i \) for each \( 1 \leq i \leq s \);

(ii) \( e(J[k_0+1], I[k], N) = e(J; \frac{N}{KN : I^\infty}) - \sum_{i=1}^s E(J[k_0-m_i+1], I[k-h_i], N_i)).
\]

Proof. Since \( x \) is a mixed multiplicity system of \( N \) with respect to ideals \( J, I \) of the type \( (k_0, k) \) such that

\[
\dim_{(x_1, \ldots, x_i)N : I^\infty} N \leq \dim_{0_N : I^\infty} N - i
\]
for each $1 \leq i \leq s$, $x^*$ is a mixed multiplicity system of the type $(k_0, k)$ of $N$ such that for each $1 \leq i \leq s$,

$$\dim \operatorname{Supp} \left[ \frac{N}{(x_1^*, \ldots, x_i^*)N} \right] \leq \dim \operatorname{Supp} \left[ \frac{N}{(x_1^*, \ldots, x_i^*)N} \right]$$

by Remark 4.1 and Remark 4.5. Recall that $e(N; k_0, k) = e(J^{[k_0+1]}, I^{[k]}; N)$. Hence $e(N; k_0, k) \neq 0$ since $e(J^{[k_0+1]}, I^{[k]}; N) \neq 0$. Consequently by Corollary 3.11(ii), we have

$$\dim \operatorname{Supp} \left[ \frac{N}{(x_1^*, \ldots, x_i^*)N} \right] = \dim \operatorname{Supp} \left[ \frac{N}{(x_1^*, \ldots, x_i^*)N} \right]$$

for each $1 \leq i \leq s$. Therefore

$$\dim \frac{N}{(x_1, \ldots, x_i)N : I^\infty} = \dim \frac{N}{0_N : I^\infty - i}$$

for each $1 \leq i \leq s$ by Remark 4.1. We obtain (i). Now, it is easily seen by Remark 4.7 and equation (4.1) that $e(N/x^*N; 0, 0) = e(J; N/(xN : I^\infty))$ and for each $1 \leq i \leq s$,

$$E \left( \frac{(x_1^*, \ldots, x_{i-1}^*)N : x_i^*}{(x_1^*, \ldots, x_{i-1}^*)N} ; k_0 - m_i, k - h_i \right)$$

$$= E \left( J^{[k_0-m_i+1]}, I^{[k-h_i]}; \frac{(x_1, \ldots, x_{i-1})N : x_i}{(x_1, \ldots, x_{i-1})N} \right).$$

Consequently, by Corollary 3.11(ii) we obtain (ii).

In particular, let $I = I_1, \ldots, I_d$ be $n$-primary ideals. Then $e(J^{[k_0+1]}, I^{[k]}; N) \neq 0$ by [19]. Moreover, $\dim N/xN = \dim N/(xN : I^\infty)$ and $e(J; N/(xN : I^\infty)) = e(J; N/xN)$ by [23]. Hence, since $\dim N/(xN : I^\infty) \leq 1$, it follows that $\dim N/xN \leq \dim N - s$. So $x$ is a part of a parameter system for $N$.

Then as an immediate consequence of Corollary 4.10, we have the following result.

**Corollary 4.11.** Let $J; I$ be $n$-primary ideals and $\dim N > 0$. Let $x = x_1, \ldots, x_s$ be a mixed multiplicity system of $N$ with respect to ideals $J, I$ of the type $(k_0, k)$. Denote by $(m_i, h_i) = (m_j, h_{i_1}, \ldots, h_{i_d})$ the type of the subsequence $x_1, \ldots, x_i$ of $x$ for each $1 \leq i \leq s$, and put $N_i = [(x_1, \ldots, x_{i-1})N : x_i]/(x_1, \ldots, x_{i-1})N$ for each $1 \leq i \leq s$. Then we have the following statements:

(i) $x$ is a part of a parameter system for $N$;

(ii) $e(J^{[k_0+1]}, I^{[k]}; N) = e \left( J; \frac{N}{xN} \right) - \sum_{i=1}^s E \left( J^{[k_0-m_i+1]}, I^{[k-h_i]}; N_i \right)$.

Using different sequences, one can express mixed multiplicities of ideals in terms of the Hilbert–Samuel multiplicity. For instance, in the case of $n$-primary ideals, Risler–Teissier [19] in 1973 showed that each mixed multiplicity is the multiplicity of an ideal generated by a superficial sequence and Rees [14] in 1984 proved that mixed multiplicities are multiplicities of ideals generated by joint reductions; for the case of arbitrary ideals, Viet [23] in 2000 characterized mixed multiplicities as the Hilbert–Samuel multiplicity via (FC)-sequences.

**Definition 4.12.** [23] Let $I = I_1, \ldots, I_d$ be ideals of $R$. Set $\mathcal{J} = I_1 \cdots I_d$. An element $a \in R$ is called a weak-(FC)-element of $N$ with respect to $I$ if there exists $i \in \{1, \ldots, d\}$ such that $a \in I_i$ and the following conditions are satisfied:
(i) $a$ is an $\mathfrak{I}$-filter-regular element with respect to $N$, i.e. $0_N : a \subseteq 0_N : \mathfrak{I}^\infty$;
(ii) $a$ is a Rees superficial element of $N$ with respect to $I$.

Note that [23] defines weak-(FC)-sequences in the condition $\mathfrak{I} \not\subseteq \sqrt{\text{Ann}_R N}$ (see, for example, [5, 6, 13, 24, 25, 26, 28, 30, 31]). In Definition 4.12, we omitted this condition.

We end this paper with the following result of mixed multiplicities of ideals.

**Corollary 4.13.** Let $k_0 + |k| = \dim N/(0_N : I^\infty) - 1$. Let $x$ be a weak-(FC)-sequence of $N$ with respect to $J, I$ of the type $(k_0, k)$ and let $x^*$ be the image of $x$ in $\bigcup_{i=0}^d T_i$. Then:

(i) \( \tilde{e}(x^*, N) = e(J^{[k_0+1]}, I^{[k]}; N) = E(J^{[1]}, I^{[0]}; N/xN); \)

(ii) \( e(J^{[k_0+1]}, I^{[k]}; N) \neq 0 \) if and only if \( \dim N/(xN : I^\infty) = 1 \). In this case,

\[ e(J^{[k_0+1]}, I^{[k]}; N) = e \left( J; \frac{N}{xN : I^\infty} \right). \]

**Proof.** Since $x$ is a weak-(FC)-sequence of $N$ with respect to $J, I$ of the type $(k_0, k)$, $x^*$ is a $T_+-$filter-regular sequence with respect to $N^*$ of the type $(k_0, k)$ by [31, Proposition 4.5]. Since $k_0 + |k| = \dim 0_N : I^\infty - 1$, $\dim \text{Supp}_{++} N^* = k_0 + |k|$ by Remark 4.1. So $E(N^*; k_0, k) = e(N^*; k_0, k)$. Hence by Corollary 3.13, we have

\[ e(N^*; k_0, k) = E(N^*/x^* N^*; 0, 0). \]

Remember that $E(N^*/x^* N^*; 0, 0) = E(J^{[1]}, I^{[0]}; N/xN)$ by Remark 4.7 and

\[ e(N^*; k_0, k) = e(J^{[k_0+1]}, I^{[k]}; N). \]

Consequently, by Theorem 4.9 we obtain (i) that

\[ \tilde{e}(x^*, N) = e(J^{[k_0+1]}, I^{[k]}; N) = E \left( J^{[1]}, I^{[0]}; \frac{N}{xN} \right). \]

So $e(J^{[k_0+1]}, I^{[k]}; N) \neq 0$ if and only if $E(J^{[1]}, I^{[0]}; N/xN) \neq 0$. This is equivalent to $\dim N/(xN : I^\infty) = 1$ by Corollary 3.13. In this case, $E(J^{[1]}, I^{[0]}; N/xN) = e(J; N/(xN : I^\infty))$ by Remark 4.7. Hence $e(J^{[k_0+1]}, I^{[k]}; N) = e(J; N/(xN : I^\infty))$ by (i).

Note that Corollary 4.13(ii) is also an immediate consequence of [23, Theorem 3.4] (see, for example, [6, 13, 25, 28, 30]).

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Duong Quoc Viet
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy street
Hanoi, Vietnam
(E-mail: duongquocviet@fmail.vnn.vn)

Truong Thi Hong Thanh
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy street
Hanoi, Vietnam
(E-mail: thanhtth@hnue.edu.vn)