First observations on Prefab posets’ Whitney numbers

A. Krzysztof Kwaśniewski

the Dissident - relegated by Bialystok University authorities
from the Institute of Computer Science to Faculty of Physics
ul. Lipowa 41, 15 424 Bialystok, Poland
e-mail: kwandr@gmail.com

Summary
We introduce a natural partial order \( \leq \) in structurally natural finite subsets of the cobweb prefabs sets recently constructed by the present author. Whitney numbers of the second kind of the corresponding subposet which constitute Stirling-like numbers’ triangular array - are then calculated and the explicit formula for them is provided. Next - in the second construction - we endow the set sums of prefabiants with such an another partial order that their their Bell-like numbers include Fibonacci triad sequences introduced recently by the present author in order to extend famous relation between binomial Newton coefficients and Fibonacci numbers onto the infinity of their relatives among which there are also the Fibonacci triad sequences and binomial-like coefficients (incidence coefficients included). The first partial order is \( F \)-sequence independent while the second partial order is \( F \)-sequence dependent where \( F \) is the so called admissible sequence determining cobweb poset by construction. An \( F \)-determined cobweb poset’s Hasse diagram becomes Fibonacci tree sheathed with specific cobweb if the sequence \( F \) is chosen to be just the Fibonacci sequence.

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1 Introduction

The clue algebraic concept of combinatorics - the so called prefab (with associative and commutative composition) was introduced in [1], see also [2,3]. The elements of prefabs are called since now on - prefabiants. In [4] the present author had constructed a new broader class of prefab’s extending combinatorial structure based on the so called cobweb posets (see Section 1. [4] for the definition of a cobweb poset as well as a combinatorial interpretation of its characteristic binomial-type coefficients - for example- fibonomial ones [5,6]).

Here we introduce two natural partial orders: one \( \leq \) in grading-natural subsets of cobweb’s prefabs sets [4] and in the second proposal we endow the set sums of prefabiants with such another partial order that one may extend the Bell numbers to sequences of Bell-like numbers encompassing among infinity of others the Fibonacci triad sequences introduced by the present author in [7].
2 Prefab based posets and their Whitney numbers.

Let the family $S$ of combinatorial objects (prefabiants) consists of all layers $⟨Φ_k → Φ_n⟩$, $k < n$, $k, n ∈ N = 0, 1, 2, ...$ and an empty prefabiant $i$.

The set $ϕ$ of prime objects consists of all sub-posets $⟨Φ_0 → Φ_m⟩$ i.e. all $P_m$’s $m ∈ N$ constitute from now on a family of prime prefabiants which we define after[4] in two steps. Namely accompanying the set $E$ of edges to the set $V$ of vertices - one obtains the Hasse diagram where here down $p, q, s ∈ N$. (Convention: Edges stay for arrows directed - say - upwards - see examples below).

Definition 1

$P = (V, E), V = \bigcup_{0 \leq p} Φ_p, \quad E = \{⟨⟨j, p⟩, ⟨q, (p+1)⟩⟩\} \bigcup\{⟨⟨1, 0⟩, ⟨1, 1⟩⟩\}, \quad$ where $1 \leq j \leq p_F, 1 \leq q \leq (p + 1)_F$.

. The finite cobweb sub-poset $P_m$ is then defined accordingly.

Definition 2 $P_m = (V_m, E_m)$, where $V_m = \bigcup_{0 \leq s \leq m} Φ_s$ and $E_m$ is defined as $E$ restricted to $V_m$ by $1 \leq p \leq m-1$ is called the prime cobweb poset.

Layer

$⟨Φ_k → Φ_n⟩$

is considered here to be the set of all max-disjoint isomorphic copies (iso-copies) of $P_m, m = n - k$ [4]. As a matter of illustration we quote after [4] examples of cobweb posets’ Hasse Diagrams [9] so that the layers become visualized.

and so on up

\[\bullet \quad \bullet \quad \bullet \quad \bullet\]

\[n_6 = 6\]

\[n_5 = 5\]

\[n_4 = 4\]

\[n_3 = 3\]

\[n_2 = 2\]

\[n_1 = 1\]
Fig. 1. Display of Natural numbers' cobweb poset.

Fig. 2. Display of Even Natural numbers $\cup \{1\}$-cobweb poset.
Fig.3. Display of Odd natural numbers' cobweb poset.

Fig.4. Display of divisible by 3 natural numbers $\cup\{1\}$ - cobweb poset.
3 Cobweb posets’ combinatorial interpretation

As seen above - for example the Fig. 5. displays the rule of the construction of the Fibonacci "cobweb" poset. It is being visualized clearly while defining this [non-lattice!] cobweb poset $P$ with help of its incidence matrix $[8]$. The incidence $\zeta$ function matrix representing uniquely just this cobweb poset $P$ has the staircase structure correspondent with "cobwebed" Fibonacci Tree i.e. a Hasse diagram $[9]$ of the particular partial order relation under consideration.

Fig. 5. Display of Fibonacci numbers' cobweb poset.
Figure 6. The incidence matrix $\zeta$ for the Fibonacci cobweb poset

Note: The knowledge of $\zeta$ matrix explicit form enables one to construct (count) via standard algorithms [8] the M"obius matrix $\mu = \zeta^{-1}$ and other typical elements of incidence algebra perfectly suitable for calculating number of chains, of maximal chains etc. in finite sub-posets of $P$. All elements of the corresponding incidence algebra are then given by a matrix of the Fig.6 with 1's replaced by any reals (or ring elements in more general cases).

We have a natural combinatorial object characterizing the cobweb posets Hasse directed graphs.

Namely - in general ([4-7], [10], [12], [13]) - given any sequence $\{F_n\}_{n \geq 0}$ of nonzero reals one may define its corresponding binomial-like $F$-binomial coefficients in the spirit of Ward's Calculus of sequences [13] as follows

Definition 3

$$\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!} \equiv \frac{n!_F}{k!_F}, \quad n_F \equiv F_n \neq 0, n \geq 0$$

where we make an analogy driven identifications in the spirit of Ward’s Calculus of sequences ($0_F \equiv 0$):

$$n!_F \equiv n_F(n-1)_F(n-2)_F(n-3)_F \ldots 2_F 1_F;$$

$$1!_F = 1; \quad n!_F = n_F(n-1)_F \ldots (n-k+1)_F.$$

This is just the adaptation of the notation for the purpose Fibonomial Calculus case (see references in [4-7], [10], [12]).

The crucial and elementary observation now is that the cobweb poset combinatorial interpretation of $F$-binomial coefficients [4-7,10,12,14,16] makes sense not for arbitrary $F$ sequences as $F$ — nominal coefficients should be nonnegative integers.

Definition 4 A sequence $F = \{n_F\}_{n \geq 0}$ is called cobweb-admissible iff

$$\binom{n}{k}_F \in \mathbb{N} \quad \text{for} \quad k, n \in \mathbb{N}.$$
Right from the definition of $P$ via its Hasse diagram here now follow quite obvious and important observations. They lead us to a combinatorial interpretation of cobweb poset’s characteristic binomial-like coefficients (for example - fibonomial ones [6,16]). Here they are with the first obvious observation at the start.

14. Observation 1

The number of maximal chains starting from The Root (level 0) to reach any point at the $n$ - th level with $n_F$ vertices is equal to $n_F!$.

Observation 2 ($k > 0$)

The number of all maximal chains in-between $(k+1)$-th level $\Phi_{k+1}$ and the $n$-th level $\Phi_n$ with $n_F$ vertices is equal to $\frac{n_m}{m+k}$, where $m + k = n$.

Indeed. Denote the number of ways to get along maximal chains from any fixed point (the leftist for example) in $\Phi_k$ to $\Phi_n$, $n > k$ with the symbol

$[\Phi_k \rightarrow \Phi_n]$ then obviously we have:

$[\Phi_0 \rightarrow \Phi_n] = n_F!$

and

$[\Phi_0 \rightarrow \Phi_k] \times [\Phi_k \rightarrow \Phi_n] = [\Phi_0 \rightarrow \Phi_n]$.

In order to formulate the combinatorial interpretation of $F$-sequence – nominal coefficients ($F$-nomial - in short) let us consider all finite "max-disjoint" sub-posets rooted at the $k$-th level at any fixed vertex $(r,k)$, $1 \leq r \leq k_F$ and ending at corresponding number of vertices at the $n$-th level ($n = k + m$) where the "max-disjoint" sub-posets are defined below.

Definition 5 Two families of maximal chains including two equipotent copies of $P_m$ are said to be max-disjoint if considered as sets of maximal chains they are disjoint i.e they have no maximal chain in common. (All $P_m$’s constitute from now on a family of the so called prime [4,10] prefabians). An equipotent copy of $P_m$ ["equip-copy"] is defined as such a family of maximal chains equinumerous with $P_m$ set of maximal chains that the it constitutes a sub-poset with one minimal element.

Definition 6 We denote the number of all max-disjoint equipotent copies of $P_m$ rooted at any vertex $(j,k)$, $1 \leq j \leq k_F$ of $k$ - th level with the symbol

$$\binom{n}{k}_F.$$

One uses the customary convention: $\binom{0}{0}_F = 1$.

Naturally- let us recall- the above definition makes sense not for arbitrary $F$ sequences as $F$ – nominal coefficients should be nonnegative integers i.e. the sequence $F = \{n_F\}_{n \geq 0}$ must be cobweb-admissible.

Problem 0. The partition or tiling problem. Suppose now that $F$ is a cobweb admissible sequence. Let us introduce

$$\sigma P_m = C_m[F; \sigma < F_1, F_2, \ldots, F_m >].$$
the equipotent sub-poset obtained from $P_m$ with help of a permutation $\sigma$ of the sequence $<F_1,F_2,\ldots,F_m>$. Then

$$P_m = C_m[F; <F_1,F_2,\ldots,F_m>].$$

Consider the layer $(\Phi_k \to \Phi_n), \ k < n, \ k,n \in N$. Layer is considered here to be the set of all max-disjoint equipotent copies of $P_{n-k}$. The question then arises, whether and under which conditions the layer may be partitioned with help of max-disjoint blocks of the form $\sigma P_m$. At first - this main question answer is in affirmative. Some computer experiments done by student Maciej Dziemiañczuk [17] are encouraging. However, problems: "how many?" or "find it all" are still opened.

Recall now that the number of ways to reach an upper level from a lower one along any of maximal chains i.e. the number of all maximal chains from the level $\Phi_{k+1}$ to $\Phi_n, \ n > k$ is equal to

$$[\Phi_k \to \Phi_n] = \frac{n}{k}.\$$

Naturally then we have

(1) $$(\begin{array}{c} n \\ k \end{array})_F \times [\Phi_0 \to \Phi_m] = [\Phi_k \to \Phi_n] = \frac{n}{k}$$

where $[\Phi_0 \to \Phi_m] = mF!$ counts the number of maximal chains in any equip-copy of $P_m$. With this in mind we see that the following holds.

Observation 3 ($n,k \geq 0$)

Let $n = k + m$. Let $F$ be any cobweb admissible sequence. Then the number of max-disjoint equip-copies i.e. sub-posets equipotent to $P_m$, rooted at the same fixed vertex of $k$-th level and ending at the $n$-th level is equal to

$$\frac{n}{k} \frac{m}{mF!} = \frac{n}{k} \frac{m}{mF!} = \frac{(k+1)(n+1)}{k}$$

Note The above Observation 3 provides us with the new combinatorial interpretation of the class of all classical $F$-nomial coefficients including distinguished binomial or distinguished Gauss $q$-binomial ones or Konvalina generalized binomial coefficients of the first and of the second kind [11,12]- which include Stirling numbers two. The vast family of Ward-like [13] admissible by $\psi = (\frac{1}{n+1})_{n>0}$-extensions $F$-sequences [12,14,16] includes also those desired here which shall be called "GCD-morphic" sequences. This means that $GCD[F_n,F_m] = F_{GCD[n,m]}$ where $GCD$ stays for Greatest Common Divisor operator. The Fibonacci sequence is a much on trivial [16,6] and guiding famous example of GCD-morphic sequence. Naturally incidence coefficients of any reduced incidence algebra of full binomial type [8] are GCD-morphic sequences therefore they are now independently given a new cobweb combinatorial interpretation via Observation 3. More on that - see the next section where prefab combinatorial description is being served. Before that - on the way - let us formulate the following problem (opened?).

Problem 1 Find effective characterizations of the cobweb admissible sequence i.e. find all examples.

Note on admissibility. Observation 3 from [16] provides us with the new combinatorial interpretation of the class of all classical $F$-nomial coefficients including
distinguished binomial or distinguished Gauss $q$-binomial ones or Konvalina generalized binomial coefficients of the first and of the second kind [11] - which include Stirling numbers too. This vast family of Ward-like [13] cobweb admissible $F$-sequences - admissible at first by the so called $\psi = \langle \frac{1}{F^n} \rangle_{n\geq 0}$ umbral extensions[9] - includes also those desired here which shall be called "GCD-morphic" sequences.

**Definition 7** The sequence of integers $F = \{n_F\}_{n\geq 0}$ is called the GCD-morphic sequence if $\text{GCD}[F_n,F_m] = F_{\text{GCD}[n,m]}$ where $\text{GCD}$ stays for Greatest Common Divisor operator.

Recall again: the Fibonacci sequence is a much nontrivial [6] and guiding example of GCD-morphic sequence. Naturally incidence coefficients of any reduced incidence algebra of full binomial type [8] are cobweb-admissible. Question: which of these above are GCD-morphic sequences?

In view of the Note on admissibility the following problems are apparently interesting also on their own.

**Characterization Problem** Find effective characterizations of the cobweb admissible sequence i.e. find all examples.

**GCD-morphism Problem** Find effective characterizations i.e. find all examples.

4 Prefabs’ Whitney numbers

Consider then now the partially ordered family $S$ of these layers considered to be sets of all max-disjoint isomorphic copies (iso-copies) of prime prefabants $P_m = P_{n,m}$ as displayed by Fig 1. - Fig.5. examples above. For any $F$-sequence determining cobweb poset let us define in $S$ the same partial order relation as follows.

**Definition 8**

$$\langle \Phi_k \rightarrow \Phi_n \rangle \leq \langle \Phi_{k'} \rightarrow \Phi_{n'} \rangle \equiv k \leq k' \quad \land \quad n \leq n'.$$

For convenience reasons we shall also adopt and use the following notation:

$$\langle \Phi_k \rightarrow \Phi_n \rangle = p_{k,n}.$$

The interval $[p_{k,n}, p_{k',n'}]$ is of course a subposet of $(S, \leq)$. We shall consider in what follows the subposet $(P_{k,n}, \leq)$ where

$$P_{k,n} = [p_{0,0}, p_{k,n}].$$

**Observation 1.** The size $|P_{k,n}|$ of $P_{k,n} = \{(l,m), \quad 0 \leq l \leq k \quad \land \quad 0 \leq m \leq n \quad \land \quad k \leq n\} = (n-k)(k+1) + \frac{k(k+1)}{2}$.

Proof: Obvious. Just draw the picture $\{(l,m), \quad 0 \leq l \leq k \quad \land \quad 0 \leq m \leq n \quad \land \quad k \leq n\}$ of $P_{k,n}$’ grid.

**Observation 2.** The number of maximal chains in $(P_{k,n}, \leq)$ is equal to the number $d(k,n)$ of 0-dominated strings of binary i.e. 0’s and 1’s sequences

$$d(k,n) = \frac{n + 1 - k}{n} \binom{k+n}{n}.$$

Proof. The number we are looking for equals to the number of minimal walk-paths in $[k \times n]$ Manhattan grid [15] - paths restricted by the condition $k \leq n$ i.e. it equals to the number of 0-dominated strings of 0’s and 1’s sequences.
Recall that \((d(k,n))\) infinite matrix’s diagonal elements are equal to the **Catalan** numbers \(C(n)\)

\[
C(n) = \frac{1}{n} \binom{2n}{n},
\]

as the Catalan numbers count the number of 0 - dominated strings of 0’s and 1’s with equal number of 0’s and 1’s. Recall that a 0 - dominated string of length \(n\) is such a string that the first \(k\) digits of the string contain at least as many 0’s as 1’s for \(k = 1, \ldots, n\) i.e. 0’s prevail in appearance, dominate 1’s from the left to the right end of the string. 0 - dominated strings correspond bijectively to minimal bottom - left corner to the right upper corner paths in an integer grid \(\mathbb{Z}_+ \times \mathbb{Z}_+\) rectangle part called Manhattan [15] with the restriction imposed on those minimal paths to obey the "safety" condition \(k \leq n\).

**Comment 1.** Observation 2. equips the poset \(\langle P_{k,n}, \leq \rangle\) with clear cut combinatorial meaning. The poset \(\langle P_{k,n}, \leq \rangle\) is naturally graded. \(\langle P_{k,n}, \leq \rangle\) poset’s maximal chains are all of equal size (Dedekind property) therefore the rang function is defined. **Observation 3.** The rang \(r(P_{k,n})\) of \(P_{k,n} = \) number of elements in maximal chains \(P_{k,n}\) minus one \(= k+n-1\). The rang \(r(p_{l,m})\) of \(\pi = p_{l,m} \in P_{k,n}\) is defined accordingly: \(r(p_{l,m}) = l+m-1\).

Proof: obvious. Just draw the picture \(\{(l, m), \ 0 \leq l \leq k \ \land \ 0 \leq m \leq n \ \land \ k \leq n\}\) of \(P_{k,n}\) grid and note that maximal means paths without at a slant edges.

Accordingly Whitney numbers \(W_k(P_{l,m})\) of the second kind are defined as follows (association: \(n \leftrightarrow (l, m)\))

**Definition 9**

\[
W_k(P_{l,m}) = \sum_{\pi \in P_{l,m}, r(\pi) = k} 1 \equiv S(k, (l, m)).
\]

Here now and afterwards we identify \(W_k(P_{l,m})\) with \(S(k, (l, m))\) called and viewed at as Stirling - like numbers of the second kind of the naturally graded poset \(\langle P_{k,n}, \leq \rangle\) - note the association: \(n \leftrightarrow (l, m)\).

**Right now challenge problems. I.**

I. Let us define now Whitney numbers \(w_k(P_{l,m})\) of the first kind as follows (association: \(n \leftrightarrow (l, m)\)). Note the text-book notation for Möbius function \(\mu\)

**Definition 10**

\[
w_k(P_{l,m}) = \sum_{\pi \in P_{l,m}, r(\pi) = k} \mu(0, \pi) \equiv s(k, (l, m)).
\]

Here now and afterwards we identify \(w_k(P_{l,m})\) with \(s(k, (l, m))\) called and viewed at as Stirling - like numbers of the first kind of the poset \(\langle P_{k,n}, \leq \rangle\) - note the association: \(n \leftrightarrow (l, m)\).

**Problem 1** Find an explicit expression for

\[
w_k(P_{l,m}) \equiv s(k, (l, m)) =?
\]

and

\[
W_k(P_{l,m}) \equiv S(k, (l, m)) =?
\]
Occasionally note that $S(k, (l, m))$ equals to the number of the grid points counted at a slant (from the up-left to the right-down) accordingly to the $l + m = k$ requirement.

**Problem 2** Find the recurrence relations for

$$w_k(P_{l,m}) \equiv s(k, (l, m)) \quad \text{and} \quad W_k(P_{l,m}) \equiv S(k, (l, m)).$$

We define now (note the association: $n \leftrightarrow (l, m)$) the corresponding Bell-like numbers $B((l, m))$ of the naturally graded poset $\langle P_{k,n}, \leq \rangle$ as follows.

**Definition 11**

$$B((l, m)) = \sum_{k=0}^{l+m} S(k, (l, m)).$$

**Observation 4.**

$$B((l, m)) = |P_{l,m}| = \frac{k(k+1)}{2} + (n-k)(k+1).$$

Proof: Just draw the picture $\{(l, m), \ 0 \leq l \leq k \ \land \ 0 \leq m \leq n \ \land \ k \leq n\}$ of $P_{k,n}$’ grid and note that $S(k, (l, m))$ equals to the number of the grid points counted at a slant (from the up-left to the right-down) accordingly to the $l + m = k$ requirement. Summing them up over all gives the size of $P_{k,n}$.

**Comment 2.** Observation 4. equips the poset’s $\langle P_{k,n}, \leq \rangle$ Bell-like numbers $B((l, m))$ with clear cut combinatorial meaning.

**5 Set Sums of prefabiants’ posets and their Whitney numbers.**

In this part we consider prefabiants’ set sums with an appropriate another partial order so as to arrive at Bell-like numbers including Fibonacci triad sequences introduced recently by the present author in [16] - see also [7,6].

Let $F$ be any “GCD-morphic” sequence. This means that $\text{GCD}[F_n, F_m] = \text{F}_{\text{GCD}(n,m)}$ where $\text{GCD}$ stays for Greatest Common Divisor mapping. We define the $F$-dependent finite partial ordered set $P(n, F)$ as the set of prime prefabiants $P_l$ given by the sum below.

**Definition 12**

$$P(n, F) = \bigcup_{0 \leq p} (\Phi_p \to \Phi_{n-p}) = \bigcup_{0 \leq l} P_{n-l}$$

with the partial order relation defined for $n - 2l \leq 0$ accordingly to

**Definition 13**

$$P_l \leq P_{\bar{l}} \quad \text{if} \quad l \leq \bar{l}, \quad P_l, P_{\bar{l}} \in (\Phi_l \to \Phi_{n-\bar{l}}).$$

Recall that rang of $P_l$ is $l$. Note that $(\Phi_l \to \Phi_{n-\bar{l}}) = \emptyset$ for $n - 2l \leq 0$. The Whitney numbers of the second kind are introduce accordingly.

**Definition 14**

$$W_k(P_{n,F}) = \sum_{\pi \in P_{n,F}, \pi(\pi)=k} \equiv S(n, k, F).$$
Right from the definitions above we infer that: (recall that \textbf{rang of } P_l \textit{ is } l.)

\textbf{Observation 5.}
\[ W_k(P_n,F) = \sum_{\pi \in P_n,F, r(\pi)=k} S(k, n-k, F) = \binom{n-k}{k} P. \]

Here now and afterwards we identify \( W_k(P_n,F) = S(n,k,F) \) viewed at and called as Stirling-like numbers of the second kind of the \( P \) defined in [10]. \( P \) by construction (see Figures above) displays self-similarity property with respect to its prime prefabriants sub- posets \( P_n = P(n,F) \).

\textbf{Right now challenge problems. II.}
We repeat with obvious replacements of corresponding symbols, names and definitions the same problems as in "Right now challenge problems. I".

Here now consequently - for any GCD-morphic sequence \( F \) (see: [10]) we define the corresponding Bell-like numbers \( B_n(F) \) of the poset \( P(n,F) \) as follows.

\textbf{Definition 15}
\[ B_n(F) = \sum_{k \geq 0} S(n,k,F). \]

Due to the investigation in [7,16] we have right now at our disposal all corresponding results of [16,7] as the following identification with special case of \( \langle \alpha, \beta, \gamma \rangle \) - Fibonacci sequence \( \langle F_{n}^{(\alpha,\beta,\gamma)} \rangle_{n \geq 0} \) defined in [7] holds.

\textbf{Observation 6.}
\[ B_n(F) \equiv F_{n+1}^{(\alpha=0,\beta=0,\gamma=0)}. \]

\textbf{Proof:} See the Definition 2.2. from [7]. Compare also with the special case of formula (6) in [16].

\textbf{Recurrence relations.} Recurrence relations for \( \langle \alpha, \beta, \gamma \rangle \) - Fibonacci sequences \( F_{n}^{(\alpha,\beta,\gamma)} \) are to be found in [7] - formula (9). Compare also with the special case formula (7) in [16].

\textbf{Closing-Opening Remark.} The study of further properties of these Bell-like numbers as well as the study of consequences of these identifications for the domain of the widespread data types [7] and perhaps for eventual new dynamical data types we leave for the possibly coming future. Examples of special cases - a bunch of them - one finds in [7] containing [16] as a special case. As seen from the identification Observation 6, the special cases of \( \langle \alpha, \beta, \gamma \rangle \) - Fibonacci sequences \( F_{n}^{(\alpha,\beta,\gamma)} \) gain additional with respect to [16,7] combinatorial interpretation in terms Bell-like numbers as sums over rang = \( k \) parts of the poset i.e. just sums of Whitney numbers of the poset \( P(n,F) \). This adjective "additional" shines brightly over Newton binomial connection constants between bases \( \langle x-1 \rangle^k \) and \( \langle x \rangle^k \) as these are Whitney numbers of the numbers from \( [n] \) chain i.e. Whitney numbers of the poset \( \langle [n], \leq \rangle \). For other elementary "shining brightly" examples see Joni, Rota and Sagan excellent presentation in [18].

\section{On applications of new cobweb posets' originated Whitney numbers}

Applications of new cobweb posets' originated Whitney numbers such as extended Stirling or Bell numbers are expected to be of at least such a significance in applications to linear algebra of formal series [linear algebra of generating functions [19]] as Stirling and Bell numbers or their \( q \)-extended correspondent already are in the so called coherent physics [20] (see [20] also for abundant references on the subject). Also
straightforward applications of prefabs to coherent physics [20] are on line. [Quantum coherent states physics is of course a linear theory with its principle of states’ superposition].

In order to say more on the subject of this section and give some examples let us remind the equivalence of exponential structures by Stanley [21] with corresponding exponential prefabs [1].

In this context the let us indicate the crucial "Ward'ian - prefab'ian" example we owe to Gessel [22] with his q-analog of the exponential formula as expressed by the Theorem 5.2 from [22].

We also recall that the q-analog of the Stirling numbers of the second kind investigated by Morrison in Section 3 of [23] constitute the same example of Ward'ian - prefab'ian extension as in the Bender - Goldman - Wagner Ward - prefab example. As noticed there by Morrison the (γ - e.g.f.) prefab exponential formula may equally well be derived from the corresponding Stanley’s exponential formula in [21]. Let us then now come over to these exponential structures of Stanley with an expected impact on the current considerations (for definitions, theorems etc. see [21]). In this connection we recall quoting (notation from [21]) an important class of Stanley’s Stirling - like numbers Sn,k to be the number of π ∈ Q_n of degree equal to k ≥ 1 i.e.

\[ S_n,k = \sum_{\pi \in Q_n, |\pi| = k} 1. \]

Define \( S_{n,k} \) - generating characteristic polynomials (vide exponential polynomials) in standard way

\[ W_n(x) = \sum_{\pi \in Q_n} x^{\pi} = \sum_{k=1}^{n} S_{n,k} x^k. \]

Then the exponential formula \( (W_0(x) = 1 = M(0)) \) becomes

\[ \sum_{n=0}^{\infty} \frac{W_n(x)y^n}{M(n)n!} = exp\{xq^{-1}(y)\}, \]

where

\[ q^{-1}(y) = \sum_{n=1}^{\infty} \frac{y^n}{M(n)n!} \equiv exp_q - 1, \]
with the obvious identification of \( \psi \)-extension choice here. Hence the polynomial sequence \( (p_n(x) = \frac{w_n(x)}{M(n)})_{n \geq 0} \) constitutes the sequence of binomial polynomials i.e. the basic sequence of the corresponding delta operator \( \hat{Q} = q(D) \). We observe then that

\[
p_n(x) = \sum_{k=0}^{n} S_{n,k} x^k \equiv \sum_{k=0}^{n} [0, 1, 2, ..., k; b_n] x^k
\]

are just exponential polynomials' sequence for the equidistant nodes case i.e. Newton-Stirling numbers of the second kind \( S_{n,k} \equiv \frac{S_{n,k}}{M(n)} \). Both numbers and the exponential sequence are being bi-univocally determined by the exponential structure \( Q \). This is a special case of the one considered in [20] and we have the - what we call- Newton-Stirling-Dobinski formula (notation, history and details- see [20])

\[
p_n(x) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k) x^k}{k!} = \sum_{k=0}^{n} [0, 1, 2, ..., k; b_n] x^k, \quad (N - S - Dob)
\]

where \( (b_n)_{n \geq 0} \) is defined by

\[
b_n(x) = \sum_{k=0}^{n} S_{n,k} x^k.
\]

**Note** the identification \( b_n(x) = \frac{w_n(x)}{M(n)} \), where

\[
w_n(x) = - \sum_{\pi \in Q_n} \mu(\hat{0}, \pi) \lambda|\pi|.
\]

\( \mu \) is Möbius function and \( \hat{0} \) is unique minimal element adjoined to \( Q_n \).

Corresponding Bell-like numbers [20] are then given by

\[
p_n(1) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k) x^k}{k!} = \sum_{k=0}^{n} [0, 1, 2, ..., k; b_n], \quad (N - S - Bell).
\]

Besides those above - in Stanley’s paper [21] there are implicitly present also inverse-dual "Whitney-Stanley" numbers \( s_{n,k} \) of the first kind i.e.

\[
s_{n,k} = - \sum_{\pi \in Q_n, |\pi| = k} \mu(\hat{0}, \pi).
\]

On this occasion and to the end of considerations on exponential structures and Stirling like numbers let us make few remarks. \( q \)-extension of exponential formula applied to enumeration of permutations by inversions is to be find in Gessel’s paper [22] (see there Theorem 5.2.) where among others he naturally arrives at the \( q \)-Stirling numbers of the first kind giving to them combinatorial interpretation. Recent extensions of the exponential formula in the prefab language [1] are to be find in [4]. Then **note**: exponential structures, prefab exponential structures (extended ones - included) i.e. schemas where exponential formula holds-imply the existence of Stirling like and Bell like numbers. As for the Dobinski-like formulas one needs binomial or extended binomial coefficients' convolution as it is the case with \( \psi \)-extensions of umbral calculus in its operator form.

**Other Generalizations in brief.** We indicate here **three** kinds of extensions of Stirling and Bell numbers - including those which appear in coherent states’ applications in quantum optics on one side or in the extended rook theory on the other side. In the supplement for this brief account to follow on this topics let us note that apart from applications to extended coherent states’ physics of quantum oscillators or strings [6 - 11, 24, 25] and related Feymann diagrams’ description [26] where we
face the spectacular and inevitable emergence of extended Stirling and Bell numbers (consult also [27]) there exists a good deal of work done on discretization of space-time [28] and/or Schrödinger equation using umbral methods [29] and GHW algebra representations in particular (see: [28, 29] for references).

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