Asymptotic Analysis of Perturbed Dust Cosmologies to Second Order

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Abstract

Nonlinear perturbations of Friedmann-Lemaître cosmologies with dust and a cosmological constant $\Lambda > 0$ have recently attracted considerable attention. In this paper our first goal is to compare the evolution of the first and second order perturbations by determining their asymptotic behaviour at late times in ever-expanding models. We show that in the presence of spatial curvature $K$ or a cosmological constant, the density perturbation approaches a finite limit both to first and second order, but the rate of approach depends on the model, being power law in the scale factor if $\Lambda > 0$ but logarithmic if $\Lambda = 0$ and $K < 0$. Scalar perturbations in general contain a growing and a decaying mode. We find, somewhat surprisingly, that if $\Lambda > 0$ the decaying mode does not die away, i.e. it contributes on an equal footing as the growing mode to the asymptotic expression for the density perturbation. On the other hand, the future asymptotic regime of the Einstein-de Sitter universe ($K = \Lambda = 0$) is completely different, as exemplified by the density perturbation which diverges; moreover, the second order perturbation diverges faster than the first order perturbation, which suggests that the Einstein-de Sitter universe is unstable to perturbations, and that the perturbation series do not converge towards the future. We conclude that the presence of spatial curvature or a cosmological constant stabilizes the perturbations. Our second goal is to derive an explicit expression for the second order density perturbation that can be used to study the effects of including a cosmological constant and spatial curvature.

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1 Introduction

In a recent paper Uggla and Wainwright (2013) (referred to as UW3) we gave a simplified structure for the system of equations that govern second order cosmological perturbations. Because of their generality these equations provide a starting point for determining the behaviour of nonlinear perturbations of Friedmann-Lemaître (FL) cosmologies with any given stress-energy content, using either the Poisson gauge or the uniform curvature gauge. In the present paper we use this system of equations, specialized to the Poisson gauge, to analyze the behaviour of linear and nonlinear perturbations of FL cosmologies with dust and possibly a cosmological constant as the matter-energy content. This problem has a lengthy history starting with Tomita (1967), and has received considerable attention over the past fifteen years.

Our first goal is to compare the evolution in time of the nonlinear (second order) perturbations to the linear (first order) perturbations by determining the asymptotic behaviour of the perturbations at late times in ever-expanding models, which has received little attention. Since we do not wish to make the customary assumption that the spatial geometry of the background is flat, there are three cases to consider:

i) $\Lambda > 0$, $K$ arbitrary, which we refer to as the de Sitter asymptotic regime,

ii) $\Lambda = 0$, $K < 0$, which we refer to as the Milne asymptotic regime,

iii) $\Lambda = 0$, $K = 0$, which we refer to as the Einstein-de Sitter asymptotic regime.

We show that in the de Sitter and Milne asymptotic regimes, the second order perturbations have the same asymptotic time dependence as the first order perturbations. For example, in both cases the density perturbation "freezes in", i.e. it approaches a finite limit as $x \rightarrow \infty$, where $x = a/a_0$ is the dimensionless background scale factor:

$$
\lim_{x \rightarrow \infty} (r)\delta = (r)\delta_\infty,
$$

The present paper also relies heavily on Uggla and Wainwright (2011) and Uggla and Wainwright (2012), which we shall refer to as UW1 and UW2, respectively.

See for example, Matarrese et al (1998), Bartolo et al (2005), Tomita (2005), Boubekeur et al (2008), Bartolo et al (2010) and Hwang et al (2012).

The only papers of which we are aware are Bruni et al (2002) and Mena et al (2002). They consider non-linear perturbations of a flat FL universe with dust and a positive cosmological constant. Their approach is heuristic in that they solve a truncated asymptotic version of the evolution equation for the perturbations in the synchronous gauge, whereas we solve the general evolution equation in the Poisson gauge. In addition we focus on the matter density perturbation at second order whereas they deal with the metric perturbations.

Current observations restrict $\Omega_k$ to be close to zero. See, for example, Okouma et al (2013). However, since a flat FL model is unstable to perturbations of $\Omega_k$ away from zero it is unlikely that $\Omega_k$ will remain close to zero. Thus we are interested in how the presence of spatial curvature affects the future asymptotic behaviour of perturbations.
where \( r = 1, 2 \) labels the first order and second order perturbations, respectively. There are two significant differences, however. First, in the de Sitter regime, the rate of approach to the asymptotic state is power law in \( x \), while in the Milne regime it is logarithmic:

\[
^{(r)}\delta_{\text{deSitter}} = ^{(r)}\delta_{\infty} + O(x^{-1}), \quad ^{(r)}\delta_{\text{Milne}} = ^{(r)}\delta_{\infty} + O(x^{-1} \ln x),
\]

as \( x \to \infty \). Second, the limiting expression \(^{(r)}\delta_{\infty}\) depends on the regime: in the de Sitter regime it depends on both the growing and decaying modes of the linear perturbation, while in the Milne regime it depends only on the growing mode at first order. In other words, in the de Sitter asymptotic regime the decaying mode does not die away.

Finally, we show that the Einstein-de Sitter asymptotic regime is totally different from the other two regimes. This is typified by the density perturbation, which diverges:

\[
^{(1)}\delta_{\text{EdS}} = O(x), \quad ^{(2)}\delta_{\text{EdS}} = O(x^2), \quad \text{as} \quad x \to \infty.
\]

In particular, the unbounded growth of these physical perturbations in the Einstein-de Sitter asymptotic regime suggests that the Einstein-de Sitter cosmology is unstable to perturbations, and that the perturbation series will not converge for \( x \) sufficiently large. On the other hand our analysis of the de Sitter and Milne asymptotic regimes as described above, shows that the presence of spatial curvature or of a cosmological constant stabilizes the perturbations.

Our second goal is to derive an explicit expression for the second order density perturbation that can be used to study the effects of including a cosmological constant and spatial curvature. In order to obtain a relatively simple expression which nevertheless illustrates various interesting phenomena, we consider the special case in which the decreasing mode of the linear perturbation is assumed to be zero, as is usually done in cosmological perturbation theory. When the spatial curvature is set to zero our expression is related to that given recently by Bartolo et al (2010).

The outline of the paper is as follows. In section 2 we give a new and concise derivation of the solutions of the equations that govern first order and second order perturbations, expressing them in integral form. Our derivation is made simple by our use of the factorization property of the linear differential operator that governs the evolution.\[^5\] We show that if the decaying mode of the scalar perturbation is set to zero the general solution can be given in explicit form, which leads to the expression for the density perturbation referred to above. In section 3 we determine the asymptotic behaviour of the perturbations in the three asymptotic regimes, and draw the conclusions described above. Section 4 contains a brief summary and discussion. The details of the analysis of the source terms and of the asymptotic behaviour are given in the appendix.

2 Perturbed dust cosmologies to second order

In this section we solve the governing equations for linear and second order perturbations of dust cosmologies, using the Poisson gauge. We make the simplifying

\[^5\text{See } \text{UW1 equation (55) and UW2 equation (41).} \]
assumption that the perturbations at the linear order are purely scalar. This implies that the matter has zero vorticity.  

2.1 Background

We consider perturbations of a FL cosmology containing pressure-free matter (dust) and a cosmological constant $\Lambda \geq 0$. The background Robertson-Walker metric is given by

$$ds^2 = a^2(-d\eta^2 + \gamma_{ij}dx^i dx^j),$$

(4)

where $\eta$ is conformal time, $a$ is the scale factor and $\gamma_{ij}$ a 3-metric of constant curvature with curvature index $K$. The scale factor $a$ determines the dimensionless background Hubble scalar $H$ according to

$$H = \frac{a'}{a} = aH,$$

(5)

where $H$ is the true background Hubble scalar and a prime denotes a derivative with respect to $\eta$. As in UW3 we introduce the scalars

$$A_G = 2(H' - H^2 + K), \quad A_T = a^2(0)^{p + (0)p}),$$

(6)

where

$$(0)^p = (0)^{p_m + \Lambda}, \quad (0)^p = (0)^{p_m - \Lambda},$$

(7)

where the subscript $m$ refers to the matter component. We assume that the Einstein equations in the background are satisfied, which imply that $A_G = A_T$. We thus drop the subscripts $G$ and $T$, and for pressure-free matter we have

$$A = 2(H' - H^2 + K) = a^2(0)^{p_m}.$$  

(8)

We introduce a reference time $\eta_0$ and a normalized scale factor $x$ defined by

$$x := a/a_0, \quad \text{where} \quad a_0 = a(\eta_0).$$

(9)

For a dust source the matter conservation equation in the background yields $(x^3(0)p)' = 0$, which implies that $(Ax)' = 0$. We define a constant $m$ by

$$m^2 := \frac{1}{3}Ax.$$  

(10)

We also introduce the usual density parameters:

$$\Omega_m := \frac{(0)^{p_m}}{3H^2}, \quad \Omega_k := \frac{-K}{H^2}, \quad \Omega_\Lambda := \frac{\Lambda}{3H^2},$$

(11)

and note that for dust

$$A = 3H^2\Omega_m, \quad m^2 = H_0^2\Omega_{m,0}.$$  

(12)

The Friedmann equation in the background, $\Omega_m + \Omega_k + \Omega_\Lambda = 1$, can now be used to express $H$ explicitly as a function of $x$:

$$H^2 = H_0^2 \left(\Omega_{\Lambda,0}x^2 + \Omega_{k,0} + \Omega_{m,0}x^{-1}\right),$$

(13)

with the subscript 0 indicating evaluation at $\eta_0$.  

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6See equation (B.41d) in UW1.
2.2 The governing equations

In the Poisson gauge the gauge invariants that describe the metric perturbation are the Bardeen curvature \(^{(r)}\Psi\) and the Bardeen potential \(^{(r)}\Phi\), and those that describe the matter perturbation in the case of pressure-free matter (dust) and a cosmological constant are \(^{(r)}\delta\), the density perturbation, and \(^{(r)}\nu\), the velocity perturbation, where \(r = 1\) at first order and \(r = 2\) at second order. We refer to UW3 for the definition of \(^{(r)}\Psi\) and \(^{(r)}\Phi\) (see equation (25)), and to equations (102) and (104) in the appendix of this paper for the definitions of \(^{(r)}\delta\) and \(^{(r)}\nu\). In UW3, in order to write the governing equations for first and second order perturbations in a so-called minimal form we also defined matter gauge invariants in terms of the components of the stress-energy tensor, denoted by \(^{(r)}\mathcal{D}\) and \(^{(r)}\mathcal{V}\), which we give in equations (106) in the appendix.

We now specialize the governing equations derived in UW3 to the case of dust. The Bardeen curvature \(^{(r)}\Psi\) plays a central role in determining the dynamics of the perturbations. At each order it satisfies a second order linear differential equation, which we write in operator notation in the following form:

\[
L^{(1)}\Psi = 0, \quad L^{(2)}\Psi = S, \tag{14a}
\]

where the differential operator \(L\) is given by

\[
L = \partial^2_\eta + 3\mathcal{H}\partial_\eta + (2\mathcal{H}' + \mathcal{H}^2 - K), \tag{14b}
\]

and the source term \(S\) depends quadratically on \(^{(r)}\Psi\) and its derivatives. We thus have to solve these equations successively, with the solution of the first equation determining the source term on the right side of the second equation. The remaining perturbation variables \(^{(r)}\mathcal{D}\), \(^{(r)}\mathcal{V}\) and \(^{(r)}\Phi\), with \(r = 1, 2\), are determined by \(^{(1)}\Psi\) and \(^{(2)}\Psi\) either algebraically or by differentiation, as follows:

\[
\begin{align*}
^{(1)}\mathcal{D} &= 2A^{-1}(D^2 + 3K)^{(1)}\Psi, \\
^{(2)}\mathcal{D} &= 2A^{-1}(D^2 + 3K)^{(2)}\Psi + \mathcal{S}_\mathcal{D}, \\
^{(1)}\mathcal{V} &= -2A^{-1}\mathcal{H}(\partial_\eta + \mathcal{H})^{(1)}\Psi, \\
^{(2)}\mathcal{V} &= -2A^{-1}\mathcal{H}(\partial_\eta + \mathcal{H})^{(2)}\Psi + \mathcal{S}_\mathcal{V}, \\
^{(1)}\Phi &= (^{(1)}\Psi), \\
^{(2)}\Phi &= (^{(2)}\Psi + \mathcal{S}_\Phi),
\end{align*} \tag{15a-c}
\]

where \(D^2 = \gamma^{ij}\mathcal{D}_i\mathcal{D}_j\) and \(\mathcal{D}_i\) is the spatial covariant derivative associated with \(\gamma_{ij}\). The complete expressions for the source terms \(\mathcal{S}, \mathcal{S}_\mathcal{D}, \mathcal{S}_\mathcal{V}\) and \(\mathcal{S}_\Phi\), which depend quadratically on \(^{(r)}\Psi\) and its derivatives, are given in equations (106) in the appendix. Once \(^{(r)}\mathcal{D}\) and \(^{(r)}\mathcal{V}\) have been calculated one can obtain \(^{(r)}\delta\) and \(^{(r)}\nu\) by using the following relations, that are a special case of equations (110) in the appendix:

\[
\begin{align*}
^{(1)}\delta &= (^{(1)}\mathcal{D} + 3\mathcal{H}^{(1)}\mathcal{V}, \\
^{(2)}\delta &= (^{(2)}\mathcal{D} + 3\mathcal{H}^{(2)}\mathcal{V} - 2(D^{(1)}\mathcal{V})^2, \\
^{(1)}\nu &= (^{(1)}\mathcal{V}, \\
^{(2)}\nu &= (^{(2)}\mathcal{V} - 2\mathcal{S}^i \left[(^{(1)}\delta - (^{(1)}\Psi)\mathcal{D}_i(^{(1)}\mathcal{V})\right].
\end{align*} \tag{16a-b}
\]

For our purposes it is important that the operator \(L\) in equations (14) can be written as the product of two first order differential operators:

\[
L(\bullet) = \mathcal{H}L_A L_B \left(\frac{\bullet}{\mathcal{H}}\right), \tag{17a}
\]

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\^See equations (61).
where
\[ H \mathcal{L}_A := \mathcal{H} \partial_\eta + 2 \mathcal{H}' + \mathcal{H}^2, \quad \mathcal{L}_B := \partial_\eta + 2 \mathcal{H}, \] (17b)
(see UW1, equation (55) and UW2, equation (39)). Replacing conformal time \( \eta \) by the normalized scale factor \( x \), noting that \( \partial_\eta = \mathcal{H} x \partial_x \), we can now write \( \mathcal{L} \) in the form\(^8\)
\[ \mathcal{L}(\bullet) = \partial_x \left( \mathcal{H}^3 \partial_x \left( \frac{x^2}{\mathcal{H}} \mathcal{H} \bullet \right) \right). \] (18)

### 2.3 Linear perturbations

The first order Bardeen curvature \((1)\Psi\) is obtained by solving the first of equations (14a), \( \mathcal{L}\^{(1)}\Psi = 0 \). We can use (18) to integrate this equation twice, obtaining the general solution in the form\(^9\)
\[ \Psi(x, x^i) = \frac{\mathcal{H}}{x^2} \left( C_+(x^i) I(x) + C_-(x^i) \right), \] (19a)
where
\[ I(x) := \int_0^x \frac{d\bar{x}}{\mathcal{H}(\bar{x})^3}. \] (19b)

The spatial functions \( C_+(x^i) \) and \( C_-(x^i) \) describe the growing and decaying modes of the perturbation, respectively. The function \( C_+(x^i) \) is of particular significance, for the following reason. Two "conserved quantities" that are associated with scalar perturbations of FL have been defined in the literature. These quantities, often denoted by the symbol \( \zeta \) with a subscript, satisfy an evolution equation of the form
\[ \partial_\eta \zeta_\bullet = D^2 C_\bullet, \] (20)
where \( C_\bullet \) is an expression involving the primary gauge invariants such as \( \Psi \) or \( \nabla \) and the background variables. This equation is referred to as a "conservation law", since if spatial derivatives are negligible ("perturbations outside the horizon") in some epoch, then (20) is approximated by \( \partial_\eta \zeta_\bullet = 0 \), i.e. \( \zeta_\bullet \) is approximately constant in time during that epoch. We refer to UW2, section 4, for a unified discussion of these conserved quantities. In particular the conserved quantity that we denote by \( \zeta_v \) can be expressed in terms of \( \Psi \) and \( \nabla \) according to\(^10\)
\[ \zeta_v = \left( 1 - \frac{2 K}{\mathcal{A}} \right) \Psi - \mathcal{H}\nabla. \] (21)
On substituting for \( \nabla \) from the first of equations (15b) we can rearrange (21) to obtain
\[ \zeta_v = \frac{2 \mathcal{H}^3}{A x} \partial_x \left( \frac{x^2}{\mathcal{H}} \Psi \right). \] (22)
\(^8\)Write the operators \( \mathcal{L}_A, \mathcal{L}_B \) in the form \( \mathcal{H} \mathcal{L}_A(\bullet) = \partial_\xi (\mathcal{H}^2 x \bullet) \) and \( \mathcal{L}_B(\bullet) = \frac{\mathcal{H}}{x} \partial_x (x^2 \bullet). \)
\(^9\)We will henceforth drop the superscript \((1)\) on first order quantities whenever there is no danger of confusion.
\(^10\)See UW2, equation (72). This quantity is sometimes referred to as "the curvature perturbation in the comoving gauge". See Malik and Wands (2009), equation (7.46), who use the symbol \( \mathcal{R} \) for \( \zeta_v \).
Substituting for $\Psi$ from (19a) and using (10) leads to

$$\zeta_v = \frac{2}{3} m^{-2} C_. \quad (23)$$

This result shows firstly that for any perturbation with dust, the conserved quantity $\zeta_v$ is exactly constant in time, and secondly it relates the arbitrary spatial function $C_+ (x^i)$ to $\zeta_v$. We can thus write the general solution (19a) in the form

$$\Psi (x, x^i) = \frac{2}{3} m^2 \frac{H}{x^2} \left( I(x) \zeta(x^i) + C_-(x^i) \right), \quad (24)$$
on rescaling $C_-(x^i)$. Here and in the rest of the paper, for notational convenience we drop the subscript $v$ on $\zeta_v$. We note in passing that in other discussions of scalar perturbations of dust, another conserved quantity, the so-called curvature perturbation on constant density hypersurfaces\footnote{This quantity is denoted by $\zeta$ in Malik and Wands (2009) (see equation (7.61)), and by $\zeta_\rho$ in UW2 (see equations (65)-(67)). In the long wavelength regime $\zeta_\rho$ and $\zeta_v$ are approximately equal, but unlike $\zeta_v$, $\zeta_\rho$ is not exactly constant at first order.} plays an important role (see, for example, Bartolo \textit{et al} (2006), equation (2.13)).

The other gauge invariants can be expressed in terms of $\Psi$. First, the matter gauge invariant $D$ is given by the first of equations (15a). Second, the gauge invariant $V$ is given by the first of equations (15b), or can be calculated using (21) since $\zeta$ is constant in time. Finally we have $\Phi = \Psi$ as follows from the first of equations (15c).

## 2.4 Second order perturbations

The second order Bardeen curvature $^{(2)}\Psi$ is obtained by solving the second of equations (14a), $L^{(2)}\Psi = S(\Psi)$. The solution of this equation that satisfies the initial conditions

$$(^{(2)}\Psi (x_{\text{init}}, x^i) = 0, \quad \partial_x {^{(2)}\Psi (x_{\text{init}}, x^i) = 0}, \quad (25)$$

where $x_{\text{init}} > 0$, can be obtained by using (18) and integrating twice. This leads to the following formula:

$$^{(2)}\Psi (x, x^i) = \frac{\mathcal{H}}{x^2} \int_{x_{\text{init}}}^x \frac{S(\bar{x}, x^i)}{\mathcal{H}(\bar{x})^3} d\bar{x}, \quad (26a)$$

where we have defined

$$S(x, x^i) := \int_{x_{\text{init}}}^x S(\bar{x}, x^i) d\bar{x}. \quad (26b)$$

Here

$$S(x, x^i) \equiv S(\Psi(x, x^i)), \quad (26c)$$

with $\Psi (x, x^i)$ given by (24). The general solution can be written in the form

$$(^{(2)}\Psi_{\text{gen}} = {^{(2)}\Psi} + {^{(2)}\Psi}_{\text{homog}}, \quad (27)$$

where $^{(2)}\Psi_{\text{homog}}$ is the general solution of the homogeneous equation $L^{(2)}\Psi = 0$ and hence is of the form (19a), with $C_+ (x^i)$ and $C_- (x^i)$ being determined by the values of $^{(2)}\Psi_{\text{gen}}$ and $\partial_x {^{(2)}\Psi_{\text{gen}}}$ at the initial time $x_{\text{init}}$.\footnote{This quantity is denoted by $\zeta$ in Malik and Wands (2009) (see equation (7.61)), and by $\zeta_\rho$ in UW2 (see equations (65)-(67)). In the long wavelength regime $\zeta_\rho$ and $\zeta_v$ are approximately equal, but unlike $\zeta_v$, $\zeta_\rho$ is not exactly constant at first order.}
The other perturbed quantities at second order, \( (2)D, (2)V \) and \( (2)\Phi \), are determined explicitly in terms of \( (2)\Psi \) and \( (1)\Psi \) by the constraint equations (15). We now give an alternative expression for \( (2)V \). By differentiating (26a) with respect to \( \eta \), using \( \frac{\partial}{\partial \eta} = H x \frac{\partial}{\partial x} \), one can express \((\partial_\eta + \mathcal{H})(2)\Psi\) in terms of \(S(x, x^i)\) and \((2)\Psi\). This leads to the following expression

\[
\mathcal{H} (2)V = -\frac{2}{3}m^{-2} S(x, x^i) + \left(1 - \frac{2K}{A}\right) (2)\Psi + S_\Psi, \tag{28}
\]

which is useful for analyzing the asymptotic behaviour of \((2)V\).

Finally we note that the solution (26a) can be written in an alternate form by changing the order of integration, yielding

\[
(2)\Psi(x, x^i) = \frac{\mathcal{H}}{x^2} \int_{x_{\text{init}}}^x [I(x) - I(\bar{x})] S(\bar{x}, x^i) d\bar{x}. \tag{29}
\]

### 2.5 Zero decaying mode at linear order

The solution \((2)\Psi\), given by (26a) or (29), depends on the two arbitrary spatial functions \(\zeta\) and \(C_-\) through the source term \(S\) which depends on \((1)\Psi\). We can obtain detailed information about the time and spatial dependence of \((2)\Psi\) by considering the special case of a linear perturbation with zero decaying mode\(^{12}\), i.e. \(C_- (x^i) = 0\).

When the decaying mode is zero, equation (24) and the first of equations (15b) reduce to

\[
\Psi(x, x^i) = g(x) \zeta(x^i), \quad \mathcal{H}V = -\frac{2}{3} \Omega_m^{-1} fg \zeta, \tag{30}
\]

\[
\delta = \frac{2}{3} g \left[ x \frac{D^2}{m^2} - 3 \Omega_m^{-1} (f + \Omega_k) \right] \zeta, \tag{31}
\]

where\(^{13}\)

\[
f(x) := 1 + \frac{xg'}{g}, \quad g(x) := \frac{3}{2} m^2 \frac{\mathcal{H}I}{x^2}. \tag{32}
\]

We use (29) to obtain a particular solution for \((2)\Psi\). In this case one can if desired choose \(x_{\text{init}} = 0\) in (29), since \(S = O(1)\) as \(x \to \infty\), as follows from (74) and (75). We now substitute the expression (74) for the source term \(S\) into (29), which leads to

\[
(2)\Psi(x, x^i) = \frac{1}{x} \left( B_1(x) \zeta^2 + B_2(x) D(\zeta) + m^{-2}[B_3(x)(D\zeta)^2 + B_4(x)D^2D(\zeta)] \right), \tag{33a}
\]

\(^{12}\)This simplifying assumption is often made, usually because it is claimed that the decaying mode will become negligible compared to the growing mode in the future. However, as we show in Section 3.1.1, this is not the case if \(\Lambda > 0\): the decaying mode contributes to the perturbations at both linear and second order. If one assumes that the decaying mode is zero one is essentially considering perturbations in a universe with an isotropic singularity (Goode and Wainwright (1985)).

\(^{13}\)We note in passing that the function \(D_+ := xg(x)\), defined up to a constant factor, is sometimes referred to as the growth suppression factor. See for example Bartolo et al (2006), in the text following equation (2.3). In our analysis the related function \(I(x)\) plays a central role.
where

\[ B_A(x) := \frac{\mathcal{H}}{x} \int_{x_{init}}^{x} [I(x) - I(\bar{x})] T_A(\bar{x}) d\bar{x}, \quad A = 1, \ldots, 4, \quad (33b) \]

and the functions \( T_A(x) \) are given by (7). The notation \((Df)^2\) and the operator \(D\) are defined in (72).

The density perturbation \( (2)\delta \) is given by (16a). We calculate the expression \( (2)D + 3\mathcal{H}(2)\mathcal{V} \) that is required by substituting \( (2)\Psi \), as given by (33), into the second of equations (15a) and (15b). After using (10), (12) and \( \partial_y = \mathcal{H} x \partial_x \) we obtain

\[ (2)D + 3\mathcal{H}(2)\mathcal{V} = -2\Omega^{-1}\partial_x (x(2)\Psi) + 2\frac{2}{3}m^{-2}D^2(x(2)\Psi) + S_\mathcal{D} + 3S_\mathcal{V}. \quad (34) \]

The source terms \( S_\mathcal{D} \) and \( S_\mathcal{V} \) are given by (70). The final result is

\[ (2)\delta = A_1 \zeta^2 + A_2 D(\zeta) + m^{-2} \left[ A_3 (D\zeta)^2 + A_4 (D^2\zeta) + A_5 D^2 \zeta^2 \right] \]

\[ + \frac{2}{3} m^{-4} \left[ B_3 (D\zeta)^2 + B_4 D^4 \zeta \right], \quad (35) \]

where

\[ A_1 = -2\Omega^{-1} \left[ \partial_x B_1 - g^2 ((1 - f)^2 - 4\Omega_k) \right], \quad (36a) \]
\[ A_2 = -2\Omega^{-1} \left[ \partial_x B_2 - 4g^2 (1 + \frac{2}{3} \Omega^{-1} f^2) \right], \quad (36b) \]
\[ A_3 = -2\Omega^{-1} \left[ \partial_x B_3 + \frac{2}{3} x g^2 (5\Omega_m + \frac{4}{3} f^2) \right], \quad (36c) \]
\[ A_4 = -2\Omega^{-1} \left[ \partial_x B_4 - \frac{2}{3} \Omega_m B_2 \right], \quad A_5 = \frac{2}{3} (B_1 + 4xg^2), \quad (36d) \]

with

\[ \partial_x B_A := (\partial_x + \Omega_k x^{-1}) B_A. \quad (36e) \]

Here the time-dependent functions \( A_1 \) and \( A_2 \) identify the Newtonian terms that dominate at late times, \( A_3 - A_5 \) identify the post-Newtonian terms, while \( B_3 \) and \( B_4 \) identify the super-horizon terms that describe the perturbations on the largest scales. These time-dependent functions depend on \( \Lambda \) and \( K \) through the functions \( \mathcal{H}(x) \) and \( I(x) \), and when \( K = 0 = \Lambda \) and we choose \( x_{init} = 0 \) they reduce to powers of \( x \), as given in section 2.5.1. Observe that the spatial dependence is described by expressions such as \((D\zeta)^2\) that are quadratic in \( \zeta \) and its spatial derivatives. In the super-horizon, post-Newtonian and Newtonian terms these expressions are of degree zero, two and four in the spatial derivative operator \( D \), respectively.

Equation (33) gives a particular solution for \((2)\Psi\) that satisfies \( \lim_{x \rightarrow 0} (2)\Psi = 0 \), and (35) gives the corresponding expression for the density perturbation \((2)\delta\). The general solution for \((2)\Psi\) subject to the condition that the decaying mode at linear order is zero is given by

\[ (2)\Psi_{gen} = (2)\Psi + C(x^i)g(x), \quad (37) \]

where \((2)\Psi\) is given by (33). The corresponding density perturbation is given by

\[ (2)\delta_{gen} = (2)\delta + \frac{2}{3} g \left[ x \frac{D^2 C}{m^2} - 3\Omega^{-1}_m (f + \Omega_k) C \right], \quad (38) \]

\[ ^{14}\text{The term } (2)\Psi_{homog} \text{ in (27) is of the form } (10a) \text{ with } C_- = 0. \]
where \( \delta \) is given by (35).

The expression for \( \delta_{\text{gen}} \) given by (35), (36) and (38) is new and represents one of the main results of this paper. We have shown that the expression for \( \delta \) given by Bartolo et al (2010) when \( K = 0 \) and \( \Lambda > 0 \) can be written in our form provided that the spatial function \( C(x^i) \) is chosen suitably. Specifically, in deriving their result they use the level of primordial non-Gaussianity at the end of inflation (ibid, section 3) to determine the function \( C(x^i) \) in (37), which has the form

\[
C(x^i) = \frac{4}{3} g_{\text{in}} [2D(\zeta) + (1 - \frac{5}{2} a_{nl}) \zeta^2],
\]

where \( a_{nl} \) is a constant that parametrizes the primordial non-Gaussianity and \( g_{\text{in}} \) is the value of \( g \) at some initial time. Finally we note that Tomita (2005) has also given an expression for \( \delta \) when \( K = 0 \) and \( \Lambda > 0 \), in which the time dependence is expressed in a completely different way.

### 2.5.1 A special case: Einstein-de Sitter

When \( K = 0 \) and \( \Lambda = 0 \), it follows from (13) and (19b) that \( H \) and \( I(x) \) are given by

\[
H^2 = m^2 x^{-1}, \quad m^3 I(x) = \frac{2}{5} x^{5/2},
\]

and hence that the time dependence functions \( f \) and \( g \) are

\[
g(x) = \frac{3}{5}, \quad f(x) = 1.
\]

One can now use (75) to calculate the functions \( T_A(x) \), which when substituted in (33b) with \( x_{\text{init}} = 0 \) leads to

\[
B_1 = 0 = B_2, \quad B_3 = \frac{2}{175} x^2, \quad B_4 = 20 B_3.
\]

Equations (33), (35) and (36) now lead to

\[
(2)\Psi = F(\zeta) x, \quad \text{with} \quad F(\zeta) := \frac{2}{175} m^{-2} [D(\zeta)^2 + 20 D^2 D(\zeta)],
\]

\[
(2)\delta = \frac{24}{5} D(\zeta) + \frac{2}{175} m^{-2} \left( 137 (D(\zeta)^2 - 80 D^2 D(\zeta) + 84 D^2 \zeta^2) x + \frac{2}{3} m^{-2} \right) F(\zeta) x^2.
\]

The corresponding general solution is given by (37) and (38). The second order density perturbation in the Poisson gauge for the Einstein-de Sitter universe has been given by a number of authors, including Matarrese et al (1998), (equation (6.10)), Bartolo et al (2005) (equation (8)), Hwang et al (2012), (equation (43)). We have shown that the expressions for \( (2)\delta \) given by these authors can be written in our form with the arbitrary function \( C(x^i) \) given by (39) with \( g_{\text{in}} = \frac{3}{5} \) and \( a_{nl} = 0 \) in the first and third papers.

---

15See their equation (29). Note that their \( g(\eta) \) equals our \( g(x) \) up to a constant multiple: \( g(x) = \frac{2}{5} g(\eta) \), and our \( f(x) \) equals their \( f(\eta) \). Some rearrangement of the spatial dependence terms has to be done to relate their functions \( B_A(x^i) \) to ours.

16See Tomita’s equation (4.16).
3 Asymptotic behaviour at late times

We now turn to the problem of determining the asymptotic behaviour of the second order perturbations as \( x \to \infty \), that arise from a general (scalar) linear perturbation (26a), i.e. one that has both a growing mode \( \zeta \) and a decaying mode \( C_- \). It is thus necessary to use the general solution (26a) for \( (2) \Psi \). We consider the three cases listed in the introduction.

3.1 The de Sitter asymptotic regime

In this section we determine the form of the nonlinear perturbation in the de Sitter asymptotic regime: \( x \to \infty \) with \( \Lambda > 0, K \) arbitrary (assuming background solutions that are forever expanding when \( K > 0 \)). The first step is to use the asymptotic form of \( (1) \Psi \) to determine the asymptotic form of the source term \( S(\Psi) \) in (26c). Then the asymptotic behaviour of \( (2) \Psi \) can be determined using equation (26a).

3.1.1 Linear perturbations

The asymptotic form of \( \Psi \) is determined by the functions \( \mathcal{H}(x) \) and \( I(x) \) through equation (24). It follows from (13) and (19b) that as \( x \to \infty \)

\[
\mathcal{H} = \lambda x \left( 1 + \frac{1}{2} k_\lambda x^{-2} + O(x^{-3}) \right), \quad \text{where} \quad \lambda := \sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{\Lambda,0}}}, \quad (45a)
\]

\[
\lambda^3 I(x) = I_\infty - \frac{1}{2} x^{-2} + O(x^{-4}), \quad \text{where} \quad I_\infty := \int_0^\infty \frac{1}{\mathcal{H}(x)^3} dx. \quad (45b)
\]

Equation (24) then leads to

\[
\Psi(x, x^i) = \frac{3}{2} m_\lambda (1)(G_\Lambda(x^i))x^{-1} + O(x^{-3}), \quad \text{as} \quad x \to \infty, \quad (46a)
\]

where

\[
(1)G_\Lambda(x^i) := I_\infty \zeta(x^i) + C_-(x^i). \quad (46b)
\]

Here we have introduced the scaling parameters

\[
m_\lambda := m^2 \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}, \quad k_\lambda := -K \frac{\Omega_{k,0}}{\Omega_{\Lambda,0}}. \quad (47)
\]

It follows from (21), (15a), (46a) and (16) that the asymptotic behaviour as \( x \to \infty \) of the matter gauge invariants is given by

\[
(1)\delta = m_\lambda \frac{D^2}{m^2} (1)G_\Lambda - 3\zeta + O(x^{-1}), \quad (48a)
\]

\[
(1)v = -(\zeta - k_\lambda (1)G_\Lambda) + O(x^{-1}). \quad (48b)
\]
3.1.2 Second order perturbations

A particular solution for \((2)\Psi\) is given by [26a]. In order to determine the asymptotic behaviour of \((2)\Psi\) as \(x \to \infty\) we need to determine the asymptotic behaviour of the source term \(S(x, x')\). The key result is that \(S(x, x') = f(x^3) + O(x^{-2})\). We now derive this result, obtaining an explicit expression for the leading order spatial term \(f(x^3)\).

It follows from (70a) and the asymptotic forms of \((1)\Psi\) and \(\mathcal{H}^{(1)}\Psi\) (details in the appendix) that

\[
S = \mathcal{H}^2 \left[ \Psi^2 + 4\mathcal{D}(\Psi) \right] + O(x^{-2}).
\]

Since \(S(x, x')\) is quadratic in \((1)\Psi\) and its derivatives it will have a multiplicative factor of \(m_\lambda^2\). We thus rescale \(S(x, x')\) and \(\bar{S}(x, x')\) by defining

\[
S(x, x') = \frac{3}{2} \lambda^2 m_\lambda^2 \tilde{S}(x, x'), \quad \bar{S}(x, x') = \frac{3}{2} \lambda^2 m_\lambda^2 \bar{S}(x, x').\]

It now follows from (46a), (49) and (50) that

\[
\tilde{S}(x, x') = \mathcal{G}_\Lambda(x^i) + O(x^{-2}),
\]

where

\[
\mathcal{G}_\Lambda(x^i) := \frac{3}{2} \left( (1)\mathcal{G}_\Lambda^2 + 4\mathcal{D}(1)\mathcal{G}_\Lambda \right) .
\]

This in turn implies that

\[
\bar{S}(x, x') = \mathcal{G}_\Lambda(x^i) x + O(1).
\]

We now use (26a) and (53) to obtain the following asymptotic expansion for \((2)\Psi\) as \(x \to \infty\) (details in the appendix):

\[
(2)\Psi(x, x') = \frac{3}{2} m_\lambda^2 \left( (2)G_\Lambda(x^i) x^{-1} - \mathcal{G}_\Lambda(x^i) x^{-2} + O(x^{-3}) \right),
\]

where

\[
(2)G_\Lambda(x^i) := \int_{x_{\text{init}}}^{\infty} \bar{S}(\bar{x}, x') \left( \frac{1}{\mathcal{H}(\bar{x})} \right)^3 d\bar{x}.
\]

This improper integral converges since \(\mathcal{H}(x) = O(x)\) and \(\bar{S}(x, x') = O(x)\) as \(x \to \infty\).

The asymptotic behaviour as \(x \to \infty\) of the remaining second order perturbation variables is determined by equations (15), (70) and (16):

\[
(2)\Phi = (2)\Psi + O(x^{-2}),
\]

\[
(2)\delta = m_\lambda^2 \frac{D^2}{m^2} (2)G_\Lambda - 3m_\lambda (2)\zeta + O(x^{-1}),
\]

\[
\mathcal{H}(2)\nu = -m_\lambda \left( (2)\zeta - k_\lambda (2)\mathcal{G}_\Lambda \right) + S_\nu + O(x^{-1}),
\]

where

\[
S_\nu = 2S^i \left[ \left( m_\lambda \frac{D^2}{m^2} (1)G_\Lambda - 3\zeta \right) D_i \left( \zeta - k_\lambda (1)\mathcal{G}_\Lambda \right) \right].
\]

Here \((2)\zeta\), as defined by [33], is a quantity that depends quadratically on the first order function \((1)\mathcal{G}_\Lambda(x^i)\) given by [16b]. If the decaying mode is zero \((C_\eta(x^i) = 0)\) then \((2)\zeta\) depends quadratically on \(\zeta\), as does \(S_\nu\). In this special case one can also express \((2)G_\Lambda\) and \((2)\zeta\) as a linear combination of the terms quadratic in \(\zeta\) and its spatial derivatives that appear in (33) and (35).
3.2 The Milne asymptotic regime

In this section we determine the form of the nonlinear perturbation in the asymptotic to Milne regime: \( x \to \infty \) with \( \Lambda = 0, K < 0 \).

3.2.1 Linear perturbations

It follows from (13) and (19b) that the asymptotic behaviour of \( H \) and \( I(x) \) as \( x \to \infty \) is given by

\[
H = \sqrt{-K} \left( 1 + \frac{1}{2} m_k x^{-1} + O(x^{-2}) \right), \quad \text{where} \quad m_k := \frac{\Omega_{m,0}}{\Omega_{k,0}} = -\frac{m^2}{K},
\]

(56a)

\[
(-K)^{\frac{3}{2}} I(x) = x \left( 1 - \frac{3}{2} m_k \ln x + O(x^{-1}) \right).
\]

(56b)

Equation (24) then leads to

\[
\Psi(x, x^i) = \frac{3}{2} m_k \zeta(x^i) x^{-1} \left( 1 - \frac{3}{2} m_k \ln x \right) + O(x^{-2}), \quad \text{as} \quad x \to \infty.
\]

(57)

It follows from (21), (15a), (57) and (16) that the asymptotic behaviour as \( x \to \infty \) of the matter gauge invariants is given by

\[
(1) \delta = (m_k \frac{D^2}{m^2} - 3) \zeta + O(x^{-1} \ln x),
\]

(58a)

\[
H(1) v = -\frac{3}{2} m_k \zeta \ln x + O(x^{-1}).
\]

(58b)

3.2.2 Second order perturbations

A particular solution for \(^{(2)}\Psi\) is given by (26a). In order to determine the asymptotic behaviour of \(^{(2)}\Psi\) as \( x \to \infty \) we need to determine the asymptotic behaviour of the source term \( S(x, x^i) \). The key result is that \( S(x, x^i) = f(x^i)x^{-2} + O(x^{-3} \ln x) \). We now derive this result, obtaining an explicit expression for the leading order spatial term \( f(x^i) \).

It follows from (70a) and the asymptotic forms of \( \Psi \) and \( H(1) v \) (details in the appendix) that

\[
S = K[3\Psi^2 + 4D(\Psi)] + \frac{1}{3}(-(D\Psi)^2 + 4D^2D(\Psi)] + O(x^{-3}(\ln x)^2).
\]

(59)

Since \( S(x, x^i) \) is quadratic in \(^{(1)}\Psi\) and its derivatives it will have a multiplicative factor of \( m_k^2 \). We thus rescale \( S(x, x^i) \) and \( S(x_i) \) by defining

\[
S(x, x^i) = \frac{3}{2} (-K) m_k^2 S(x, x^i), \quad S(x_i) = \frac{3}{2} (-K) m_k^2 S(x_i).
\]

(60)

It now follows from (57), (59) and (60) that

\[
S(x, x^i) = G_k(x^i)x^{-2} + O(x^{-3}(\ln x)^2),
\]

(61)

where

\[
G_k(x^i) := -\frac{3}{2} \left[ 3 \zeta^2 + 4D(\zeta) + \frac{5}{2} m_k m^{-2} \left( (D\zeta)^2 - 4D^2D(\zeta) \right) \right].
\]

(62)
4 DISCUSSION

This in turn implies that
\[ \int_{x_{\text{init}}}^{\infty} \mathcal{S}(\bar{x}, x^i) d\bar{x} = \lim_{x \to \infty} \mathcal{S}(x, x^i), \quad \text{exists and is non-zero}. \]  

(63)

We now use the general solution (26a) in conjunction with (63) to obtain the following asymptotic expansion for \( (2)\Psi \) as \( x \to \infty \) (details in the appendix):
\[ (2)\Psi(x, x^i) = \frac{3}{2} m_k^2 x^{-1} \left[ (2)G_k - \left( \frac{3}{2} m_k (2)G_k + G_k \right) \ln x \right] + O(x^{-2}), \]  

(64a)

where
\[ (2)G_k(x^i) := \int_{x_{\text{init}}}^{\infty} \mathcal{S}(\bar{x}, x^i) d\bar{x}. \]  

(64b)

In the limit \( x \to \infty \) the remaining second order perturbation variables are determined in terms of \((2)\Psi\) and \((1)\Psi\) by equations (15), (70), (28) and (16):
\[ (2)\Phi = (2)\Psi + O(x^{-2}), \]  

(65a)

\[ (2)\delta = m_k(m_k \frac{D^2}{m^2} - 3)(2)G_k + O(x^{-1} \ln x), \]  

(65b)

\[ \mathcal{H}^{(2)} = -\frac{3}{2} m_k^2 (2)G_k \frac{\ln x}{x} + O(x^{-1}). \]  

(65c)

We note that if the decaying mode is zero one can express \((2)G_k\) as a linear combination of the terms quadratic in \( \zeta \) and its spatial derivatives that appear in (33) and (35).

4 Discussion

We have used the minimal system of governing equations for second order perturbations that we developed in UW3 to investigate the behaviour of nonlinear scalar perturbations of FL cosmologies with dust and cosmological constant as matter-energy content. Although we have given a new formulation of the solutions of the governing equations, our main contribution in this paper has been to analyze the properties of the solutions, and this has been facilitated by the new form of the solutions. In particular we have shown that there are significant differences in the asymptotic behaviour at late times in three cases, characterized by \( K \) and \( \Lambda \), namely, the Einstein-de Sitter regime, the Milne regime and the de Sitter regime, as described in the introduction. These differences are best illustrated by considering the behaviour of the linear and nonlinear density perturbations \((1)\delta\) and \((2)\delta\). For ease of comparison we repeat the three asymptotic expressions for \((1)\delta\) as \( x \to \infty \):
\[ (1)\delta_{\text{deSitter}} = m_\lambda \frac{D^2}{m^2} (I_{\infty} \zeta + C_-) - 3\zeta + O(x^{-1}), \]  

(66a)

\[ (1)\delta_{\text{Milne}} = m_k \frac{D^2}{m^2} \zeta - 3\zeta + O(x^{-1} \ln x), \]  

(66b)

\[ (1)\delta_{\text{EdS}} = \frac{2}{5} x \frac{D^2}{m^2} \zeta + O(1), \]  

(66c)
where \( I_\infty \) is given by (45b). Note that the expressions for the density perturbation \( ^{(2)}\delta \) in the three asymptotic regimes are given by equations (55b), (65b) and (44).

An arbitrary scalar perturbation depends on two spatial functions \( \zeta(x^i) \) and \( C_-(x^i) \), that correspond to the growing and decaying modes, and on the state of the background model at some initial time, which determines the following constants:

\[
m^2 = H_0^2 \Omega_{m,0}, \quad m_\lambda = \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}, \quad k_\lambda = \frac{\Omega_{k,0}}{\Omega_{\Lambda,0}}, \quad m_k = \frac{\Omega_{m,0}}{\Omega_{k,0}}.
\]  

(67)

It follows from (45b) that

\[
I_\infty = I_\infty(m_\lambda, k_\lambda).
\]  

(68)

By inspection of equations (66) we see the following dependencies of the asymptotic states for the first order density perturbation \( ^{(1)}\delta \):

\[
\begin{align*}
\text{Einstein-de Sitter} : & \quad m, \zeta, \quad (69a) \\
\text{Milne} : & \quad m, m_k, \zeta, \quad (69b) \\
\text{de Sitter} : & \quad m, m_\lambda, k_\lambda, \zeta, C_-.
\end{align*}
\]

(69c)

We see that \( ^{(1)}\delta \) has the most complicated structure in the de Sitter asymptotic regime, depending on the ratio of \( \Omega_{m,0} \) and \( \Omega_{k,0} \) to \( \Omega_{\Lambda,0} \) and on the decaying mode \( C_- \). The situation is different as regards the second order density perturbation. Since \( ^{(2)}\Psi \) and hence \( ^{(2)}\delta \) has an integral dependence on the source term \( S \) (see equation (29)), the decaying mode enters into the leading term in all three asymptotic regimes.

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A Analysis for second order perturbations

A.1 Source terms for second order perturbations

In this section we give the full expressions for the source terms and then illustrate their structure in the special case when the decaying mode is zero.

The source terms for a perfect fluid model with a purely scalar linear perturbation are given in UW3 (see equations (61)). After specializing to the case of dust and
rarranging the terms we obtain\footnote{In UW3 the source terms are not given a label $S_\phi$, but can be identified by comparing equations (14a) and (15) with equations (61) in UW3, noting the difference in choice of variables as described in equation (85a). We have also used the first equation in (15b) in obtaining these expressions.}

$$S = (\mathcal{H}x \partial_i \Psi)^2 + 4K \Psi^2 - \frac{1}{3} (D \Psi)^2 - A(D \nabla)^2$$

$$+ 2 \left[ \mathcal{H}^2 (x \partial_x + 3\Omega + \Omega_k) + \frac{3}{2} D^2 \right] (2D(\Psi) + AD(\nabla)),$$

$$S_D = 2A^{-1} \left[ 4(D^2 + 3K) \Psi^2 - 5(D \Psi)^2 \right] + 6S^i (\Psi D_i \nabla \Psi) + \frac{3}{A} \nabla^2 ,$$

$$S_V = 2A^{-1} \mathcal{H}^2 \left[ \Psi^2 + 4D(\Psi) \right] + 2 \left[ S^i (\nabla \nabla D_i \Psi) + 2D(\nabla \nabla) \right] ,$$

$$S_\phi = 4[\Psi^2 - D(\Psi)] - 2AD(\nabla) .$$

(70a)

Here and elsewhere we use the following notation. First, the second order spatial differential operators are defined by

$$D^2 := \gamma^{ij} D_i D_j , \quad D_{ij} := D_i D_j - \frac{1}{3} \gamma_{ij} D^2 ,$$

(71)

where $D_i$ denotes covariant differentiation with respect to the spatial metric $\gamma_{ij}$. Second, we use the shorthand notation

$$(DA)^2 := (D^k A)(D_k A) , \quad D(A) := S^{ij} (D_i A)(D_j A) ,$$

(72)

where $A$ is a scalar field. Finally, we use the scalar mode extraction operators

$$S^i = D^{-2} D^i , \quad S^{ij} = \frac{3}{2} D^{-2} (D^2 + 3K)^{-1} D^{ij} ,$$

(73)

as defined in UW3 (see equation (85a)).

### A.1.1 Zero decaying mode at linear order

It follows from (70a) that $S(x, x^i)$ can be written in the form

$$S(x, x^i) = T_1(x) \zeta^2 + T_2(x) D(\zeta) + m^2 \left( T_3(x) (D \zeta)^2 + T_4(x) D^2 D(\zeta) \right) ,$$

(74)

where

$$T_1 = (\mathcal{H} g)^2 ((f - 1)^2 - 4\Omega_k) ,$$

(75a)

$$T_2 = 8(\mathcal{H} g)^2 \left( (f - 1)^2 - \frac{1}{3} (3\Omega + \Omega_k) + \frac{2\Omega_k}{3\Omega_m} f^2 \right) ,$$

(75b)

$$T_3 = -\frac{1}{3} m^2 g^2 \left( 1 - \frac{4}{3\Omega_m} f^2 \right) ,$$

(75c)

$$T_4 = \frac{4}{3} m^2 g^2 \left( 1 + \frac{2}{3\Omega_m} f^2 \right) ,$$

(75d)

and $f$ and $g$ are defined in (32). The remaining source terms are given by

$$S_D = \frac{2}{2} g^2 \left[ x m^{-2} (4D^2 \zeta^2 - 5(D \zeta)^2) + \frac{3}{\Omega_m} (f(f - 1) - 4\Omega_k) \zeta^2 \right] ,$$

(76a)

$$S_V = \frac{2}{3} \Omega^{-1} m^2 g^2 \left[ -f \zeta^2 + 4 \left( 1 + \frac{2}{3\Omega_m} f^2 \right) D(\zeta) \right] ,$$

(76b)

$$S_\phi = 4g^2 \left[ \zeta^2 - \left( 1 + \frac{2}{3\Omega_m} f^2 \right) D(\zeta) \right] .$$

(76c)
A.2 Asymptotic expressions for the source terms

A.2.1 Asymptotic to de Sitter

In order to derive (49) we have to examine (70a) in detail. We begin by using (13), (46a) and (48b) to conclude that $D(\Psi)$ and $AD(V)$ have the asymptotic form

$$D(\Psi) = A(x^i)x^{-2} + O(x^{-4}), \quad AD(V) = B(x^i)x^{-3} + O(x^{-4}),$$

and

$$H^2(x\partial_x + 3\Omega + \Omega_k) + \frac{1}{3}D^2 = \lambda^2 x^2(x\partial_x + 3) + O(1).$$

It then follows that

$$x^2(x\partial_x + 3)D(\Psi) = x^2D(\Psi) + O(x^{-2}), \quad x^2(x\partial_x + 3)AD(V) = O(x^{-2}).$$

The desired result (49) follows when (78) and (79) are substituted in (70a).

The asymptotic forms as $x \to \infty$ of the other source terms given by (70) are:

$$S_\Phi = 9m_A^2 \left[ G_A^2 - D(G_A) \right] x^{-2} + O(x^{-3}),$$

$$S_D = O(x^{-1}),$$

$$S_V = m_A G_A x + 4D(\zeta - k_A G_A) + O(x^{-1}).$$

The leading order term in $S_D$ is quite complicated, and is not needed for our purposes.

A.2.2 Asymptotic to Milne

In order to derive (59) we have to examine (70a) in detail. We begin by using (13), (57) and (58b) to conclude that that $D(\Psi)$ and $AD(V)$ have the asymptotic form

$$D(\Psi) = f(x^i)x^{-2} + O(x^{-4}), \quad AD(V) = g(x^i)x^{-3}(\ln x)^2 + O(x^{-4}),$$

and

$$H^2(x\partial_x + \Omega_k) + \frac{1}{3}D^2 = (-K)(x\partial_x + 1) + \frac{1}{3}D^2 + O(x^{-2}).$$

It then follows that

$$(x\partial_x + 1)D(\Psi) = -D(\Psi) + O(x^{-4}), \quad (x\partial_x + 1)AD(V) = O(x^{-3}(\ln x)^2).$$

The desired result (59) follows when (82) and (83) are substituted in (70a).

The asymptotic forms as $x \to \infty$ of the other source terms given by (70) are:

$$S_\Phi = O(x^{-2}),$$

$$S_D = \frac{3}{2}m_k^2 m^{-2} \left[ 4(D^2 + 3K)\zeta^2 - 5(D\zeta)^2 \right] + O(x^{-2}\ln x),$$

$$S_V = \frac{3}{2}m_k \left[ \zeta^2 + 4D(\zeta) \right] x^{-1} + O(x^{-2}(\ln x)^2).$$

---

18 The terms in the first line of (70a) are $H^2\Psi^2 + O(x^{-2}).$

19 The terms in the first line of (70a) are $3K\Psi^2 - \frac{1}{3}(D\Psi)^2 + O(x^{-3}\ln x).$
A.2.3 Einstein-de Sitter

The exact expressions for the source terms given by (70) are

\[ S = \frac{1}{9}[(D\Psi)^2 + 20D^2\mathcal{D}(\Psi)], \]
\[ S_\Phi = 4[\Psi^2 - \frac{5}{3}D(\Psi)], \]
\[ S_\mathcal{D} = 2A^{-1}[4D^2\Psi^2 - 5(D\Psi)^2], \]
\[ S_V = \frac{40}{9}D(\Psi). \]

(85a, 85b, 85c, 85d)

A.3 Asymptotic expansions for second order perturbations

A.3.1 Asymptotic to de Sitter

We outline the derivation of equation (54a) for \((2)\Psi\). It follows from (26a), (47) and (50) that

\[ x^{(2)}\Psi(x, x^i) = \frac{3}{2}m^2_\Lambda \frac{\mathcal{H}}{\lambda x} \int_{x_{init}}^{x} \frac{\bar{S}(\bar{x}, x^i)}{(\frac{1}{\lambda} \mathcal{H}(\bar{x}))^3} d\bar{x}. \]

(86)

Since

\[ \bar{S}(x, x^i) = xG_\Lambda(x^i) + O(1), \quad \frac{\mathcal{H}}{\lambda x} = 1 + O(x^{-2}), \]

(see equations (13) and (53)) it follows that \( \lim_{x \to \infty} x^{(2)}\Psi(x, x^i) \) exists. With \((2)G_\Lambda(x^i)\) defined as in (54b) we obtain

\[ x^{(2)}\Psi(x, x^i) = \frac{3}{2}m^2_\Lambda (2)G_\Lambda(x^i) = -\int_{x}^{\infty} \frac{\bar{S}(\bar{x}, x^i)}{(\frac{1}{\lambda} \mathcal{H}(\bar{x}))^3} d\bar{x}. \]

(88)

The integral on the right can be expanded using (87), leading to the desired equation (54a).

To calculate \((2)\mathcal{V}\) we need to extend the asymptotic expansion of \(\bar{S}\) given by (53). We write

\[ \bar{S}(x, x^i) = xG_\Lambda(x^i) + \bar{S}_0 + O(x^{-1}). \]

(89)

When (89) and (80c) are substituted into (28), after making use of (50), we obtain the desired equation (55a), with

\[ (2)\zeta := \bar{S}_0 + 4D(\zeta - k_{\lambda}G_\Lambda). \]

(90)

A.3.2 Asymptotic to Milne

We outline the derivation of equation (64a) for \((2)\Psi\). We begin by using (50) to write (26a) in the form

\[ x^{(2)}\Psi(x, x^i) = \frac{3}{2}m^2_k \bar{H} \int_{x_{init}}^{x} \frac{\bar{S}(\bar{x}, x^i)}{\mathcal{H}(\bar{x})^3} d\bar{x}; \]

(91)

where

\[ \bar{H}(x) := (-K)^{-1/2} \mathcal{H}(x) = (1 + m_k x^{-1})^{1/2}. \]

(92)
We next derive an asymptotic expansion for $\bar{S}(x, x^i)$. It follows from equations (26b), (60) and (64b) that
\[
\bar{S}(x, x^i) - (2)G_k(x^i) = - \int_x^\infty \bar{S}(\tilde{x}, x^i) d\tilde{x}. \tag{93}
\]
Substituting the expansion (61) of $\bar{S}(\tilde{x}, x^i)$ and evaluating the integral gives
\[
\bar{S}(x, x^i) = (2)G_k(x^i) - G_k(x^i)x^{-1} + O(x^{-2} \ln x), \tag{94}
\]
which in conjunction with (92) leads to
\[
\frac{\bar{S}(x, x^i)}{\mathcal{H}(x)^3} = (2)G_k - \left(\frac{3}{2}m_k (2)G_k + G_k\right)x^{-1} + O(x^{-2} \ln x). \tag{95}
\]
On substituting this expression in (91) and evaluating the integral we obtain (64a).

### B The matter gauge invariants

In this appendix we define the two types of matter gauge invariants that we use in this paper. We consider a perfect fluid with stress-energy tensor
\[
T^a_b = (\rho + p)u^a u_b + p\delta^a_b, \tag{96}
\]
with a linear equation of state $p = w\rho$ with $w$ constant. We begin by reformulating the Replacement Principle for the stress-energy tensor of a perfect fluid, as given in UW3 (see equations (84)), by viewing it as a function of the variables $F = (\rho, v_a, \bar{g}_{ab})$, where $v_a = a^{-1}u_a$ is the conformal fluid velocity and $\bar{g}_{ab} = a^{-2}g_{ab}$ is the conformal metric tensor. The perturbations of the stress-energy tensor can be written symbolically in the form:
\[
\mathcal{M}^2 (1)T^a_b = T^a_b (1)F, \quad \mathcal{M}^2 (2)T^a_b = T^a_b (2)F + T^a_b (1)F, \tag{97}
\]
where $T^a_b$ is the linear leading order operator and $T^a_b$ is the quadratic source term operator, and
\[
\overset{(r)}{F} = (\overset{(r)}{\mathcal{M}} \overset{(r)}{\rho}, \overset{(r)}{v}_a, \overset{(r)}{\bar{g}}_{ab}), \tag{98}
\]
with $r = 1, 2$, denotes the perturbations of $\rho, v_a$ and $\bar{g}_{ab}$, while $\mathcal{M}$ is defined by
\[
\overset{(r)}{\mathcal{M}} := (\overset{(0)}{\rho} + \overset{(0)}{p})^{-1/2}. \tag{99}
\]
For a linear equation of state the perturbations in the pressure are given by
\[
\overset{(r)}{p} = w \overset{(r)}{\rho}, \quad \text{for} \quad r = 1, 2; \tag{100}
\]

We associate gauge invariants with $\mathcal{M}^2 (r)T^a_b$ and the variables $(r)F$ as follows. The gauge invariants $(r)A[X]$ associated with the perturbations $(r)A$ of an arbitrary tensor $A$ are defined by
\[
(1)A[X] := (1)A - \mathcal{L}_{(1)X} (0)A, \tag{101a}
\]
\[
(2)A[X] := (2)A - \mathcal{L}_{(2)X} (0)A - \mathcal{L}_{(1)X} (2A (1) - \mathcal{L}_{(1)X} (0)A). \tag{101b}
\]

\footnote{Note that the same operator $T^a_b$ acts on both $(1)F$ and $(2)F$.}
\footnote{See Nakamura (2007), equations (2.26)-(2.27), and UW3, equations (81). Here we are omitting the factor of $a^n$ in the latter equations.
In terms of this definition the gauge invariants associated with $\mathcal{M}^2(r)T^a_b$ and $\mathcal{M}^2(r)\rho$ are defined by replacing the tensor $A$ in (101) by $T^a_b$ and by $\rho$:

$$^{(r)}T^a_b[X] := \mathcal{M}^2(r)T^a_b[X], \quad ^{(r)}\delta[X] := \mathcal{M}^2(r)\rho[X], \quad (102a)$$

while the gauge invariants associated with $^{(r)}u_a = a^{-1(r)}u_a$ and $^{(r)}g_{ab}$ are defined by replacing the tensor $A$ in (101) by $u_a$ and by $g_{ab}$:

$$^{(r)}\mathcal{V}_a[X] := a^{-1(r)}u_a[X], \quad ^{(r)}F_{ab}[X] := a^{-2(r)}g_{ab}[X]. \quad (102b)$$

The Replacement Principle states that the gauge invariants associated with $\mathcal{M}^2(r)T^a_b$ and with $^{(r)}F$ are related by the same operators as in (97):

$$^{(1)}T^a_b[X] = T^a_b[(1)^{(1)}F), \quad ^{(2)}T^a_b[X] = T^a_b[(2)^{(2)}F) + T^a_b[(1)^{(1)}F), \quad (103a)$$

where $^{(r)}F$ is shorthand for

$$^{(r)}F[X] = (^{(r)}\delta[X], ^{(r)}\mathcal{V}_a[X], ^{(r)}F_{ab}[X]). \quad (103b)$$

Finally the scalar velocity perturbations are defined by

$$^{(r)}v[X] = S^i(r)\mathcal{V}_i[X], \quad \text{for} \quad r = 1, 2, \quad (104)$$

where the scalar mode extraction operator $S^i$ was given in (73). We can now state that the first set of matter gauge invariants for linear and second order scalar perturbations are $^{(r)}\delta[X]$ and $^{(r)}v[X]$, with $r = 1, 2$, relative to an arbitrary choice of gauge.

Before continuing we briefly digress to comment on the relation between our variables and $^{(r)}\delta[X]$ and $^{(r)}v[X]$ and the variables used by other authors. First, our $v$ equals up to sign the variable $\nu$ used by others. Second, as follows from (99) and (102a), our $\delta$ is related to the usual relative density perturbation $\delta =^{(r)}\rho/^{(0)}\rho$ according to $\delta = (1 + w)\delta$, where $w$ is defined by $^{(0)}\rho = w^{(0)}\rho$.

We also define matter gauge invariants directly in terms of the stress-energy tensor:

$$^{(r)}\mathcal{V}[X] := S^i(r)T^0_0[X], \quad (106a)$$

$$^{(r)}\mathcal{D}[X] := -S^i(r)\{D^i(T^0_0[X] + 3\mathcal{H}^{(r)}T^0_0[X])\}. \quad (106b)$$

In order to relate these gauge invariants to $^{(r)}\delta[X]$ and $^{(r)}v[X]$ we need to obtain explicit expressions for the components of $^{(r)}T^a_b[X]$. Referring to (96) and (97) we see that

$$^{(r)}\Delta[X] = A^{(r)}\mathcal{D}[X], \quad ^{(r)}V[X] = A^{(r)}\mathcal{V}[X]. \quad (105)$$

For $\Delta$ see equations (45b) and (41c), and for $V$, see (45c).
perform an $\epsilon$-expansion for $T^a_b(\epsilon), g_{ab}(\epsilon), u_a(\epsilon)$ and $\rho(\epsilon)$ and obtain

\begin{align}
T^0_0(\epsilon)F &= -\mathcal{M}^2(\epsilon)\rho, \\
T^0_i(\epsilon)F &= (\epsilon)v_i, \\
T^i_i(\epsilon)F &= 3w\mathcal{M}^2(\epsilon)\rho, \\
\tilde{T}^{ij}(\epsilon)F &= 0,
\end{align}

\begin{align}
T^0_0(\epsilon)F &= -\gamma^{ij}v_{ij}, \\
T^0_i(\epsilon)F &= (2(1+w)\mathcal{M}^2(\epsilon)\rho + \rho_{00})v_i, \\
T^i_i(\epsilon)F &= \gamma^{ij}v_{ij}, \\
\tilde{T}^{ij}(\epsilon)F &= v_{ij},
\end{align}

where

$$v_{ij} := 2v_i(v_j - f_{0j}),$$

and $r = 1, 2$ in the first column. We now apply the Replacement Principle obtaining

\begin{align}
T^0_0(\epsilon)F &= -\gamma^{ij}v_{ij}, \\
T^0_i(\epsilon)F &= (2(1+w)\delta + \rho_{00})v_i, \\
T^i_i(\epsilon)F &= \gamma^{ij}v_{ij}, \\
\tilde{T}^{ij}(\epsilon)F &= v_{ij},
\end{align}

where

$$v_{ij} := 2v_i(v_j - f_{0j}).$$

When we specialize to linear perturbations that are purely scalar we have

\begin{align}
(1)v_i[X] &= D_i(1)v[X], \\
(1)v_{ij}[X] := 2D_i v[X] D_j(\nabla[X] - B[X])
\end{align}

In this case it follows from $(103)$, $(106)$ and $(108)$ that the two types of gauge invariants are related as follows:

\begin{align}
(1)v[X] &= (1)v[X], \\
(2)v[X] &= (2)v[X] - 2S^i[(1+w)\delta[X] - \Phi[X])D_i\nabla[X]], \\
(1)\delta[X] &= (1)\mathcal{D} + 3\mathcal{H}(1)v[X], \\
(2)\delta[X] &= (2)\mathcal{D}[X] + 3\mathcal{H}(2)v[X] - 2D^i\nabla[X]D_i(\nabla[X] - B[X]).
\end{align}

Finally in the Poisson gauge, which is given by $B[X_p] = 0$, equations $(110)$, when specialized to dust simplify to give equations $(113)$.

References

Bartolo, N., Matarrese, S., Pantano, O., and Riotto, A. (2010) Second-order matter perturbations in a $\Lambda$CDM cosmology and non-Gaussianity, Class. Quant. Grav. 27, 124009.

\textsuperscript{24}For brevity, we omit the $[X]$ associated with the terms on the right side of these equations.

\textsuperscript{25}We recall that $f_{00}[X] = -2\Phi[X]$ and $f_{0i}[X] = D_iB[X] + B_i$. See equations (8) in UW2.
Bartolo, N., Matarrese, S. and Riotto, A. (2006) The full second-order radiation transfer function for large-scale CMB anisotropies, *JCAP* **0605**, 010.

Bartolo, N., Matarrese, S. and Riotto, A. (2005) Signatures of Primordial Non-Gaussianity in the Large-Scale Structure of the Universe, *JCAP* **0510**, 010.

Boubekeur, L., Creminelli, P., Norena, J. and Vernizzi, F. (2008) Action approach to cosmological perturbations: the 2nd order metric in matter dominance, *JCAP* **0808**, 028.

Bruni, M, Mena, F.C. and Tavakol, R. (2002) Cosmic no-hair: non-linear asymptotic stability of de Sitter universe, *Class. Quant. Grav.* **19**, L23-L29.

Goode, S.W. and Wainwright, J. (1985), Isotropic singularities in cosmological models, *Class. Quant. Grav.* **2**, 99-115.

Hwang, J-C., Noh, H. and Gong, J-O. (2012), Second order solutions of cosmological perturbation in the matter dominated era, *Astrophys. J.* **752**, 50, doi:10.1088/0004-637X/752/1/50.

Malik, K. A. and Wands, D. (2009) Cosmological perturbations, *Physics Reports* **475**, 1-51.

Matarrese, S., Mollerach, S. and Bruni, M. (1998) Relativistic second-order perturbations of the Einstein-de Sitter universe, *Phys. Rev. D* **58**, 043504 (1-22).

Mena, F.C., Tavakol, R. and Bruni, M. (2002) Second order perturbations of flat dust FLRW universes with a cosmological constant, *International Journal of Modern Physics A* **17**, 4239-4244.

Okouma, P. M., Fantaye, Y. and Bassett, B. A. (2013) How flat is our Universe really? *Phys. Lett. B* **719** 1-4.

Noh, H. and Hwang, J. (2004) Second order perturbations of the Friedmann world model, *Phys. Rev. D* **69**, 104011(1-52).

Tomita, K. (1967) Non-linear theory of gravitational instability in an expanding universe, *Prog. Theor. Phys.* **37**, 831-846.

Tomita, K. (2005) Relativistic second-order perturbations of nonzero-Λ flat cosmological models and CMB anisotropies, *Phys. Rev. D* **71**, 083504(1-11).

Uggla, C. and Wainwright, J. (2011) Cosmological perturbation theory revisited, *Class. Quant. Grav.* **28**, 175017, arXiv:1102.5039

Uggla, C. and Wainwright, J. (2012) Dynamics of cosmological scalar perturbations,
Uggla, C. and Wainwright, J. (2013) A simplified structure for the second order cosmological perturbation equations, *Gen. Rel. Grav.* 45, 643-674, DOI 10.1007/s10714-012-1492-7, [arXiv:1203.4790](https://arxiv.org/abs/1203.4790).