On the characterization of drilling rotation in the 6–parameter resultant shell theory

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Outline of the contribution

- We present existence results for the 2D theory of (geometrically non-linear) 6-parameter elastic shells.
- We prove the existence of minimizers for the minimization problem of the total potential energy.
- To this aim, we employ the direct methods of the calculus of variations adapted to the case of general shells.
- The existence result is valid for general elastic shells with arbitrary geometry of the reference placement.
- The result is applicable for: isotropic shells, orthotropic shells, and composite layered shells.
- For shells without drilling rotations we present a representation theorem for the strain energy function.
We consider the 6-parameter resultant shell theory

two independent kinematic fields: the translation vector and the rotation tensor

This theory of shells is very general, and also effective

originally proposed by Reissner (1974) and developed subsequently by Libai & Simmonds (1998), Stumpf (1999), Chróścielewski, Makowski & Pietraszkiewicz (2004), Pietraszkiewicz and Eremeyev (2006, 2011) et al.

The 2D equilibrium equations of the shell are derived by direct through-the-thickness integration of 3D balance laws

The kinematic fields are then constructed on the 2D level using the integral identity of the virtual work principle

The 2D model is expressed in terms of stress resultants and work-averaged deformation fields on the base surface.
Relations to Cosserat shells

- The model refers to shells made of a **simple (classical) elastic material**, not a generalized continuum.
- The **rotation tensor field appears naturally** in this theory.
- The kinematical structure of 6-parameter shells is identical to the **kinematical structure of Cosserat shells**.
- It is related to the shell model proposed by the Cosserat brothers (1909) and developed by many authors, such as Zhilin (1976), Altenbach and Zhilin (1988), Zubov (1997), Eremeyev and Zubov (2008), Birsan and Altenbach (2010).
- Using the derivation approach, Neff (2004, 2007) has established a very similar **Cosserat–type model** for plates starting from a 3D micropolar continuum.
About existence results

- The **existence of solutions** for 2D equations of elastic shells has been investigated using different techniques.
- The **method of formal asymptotic expansions**: derivation and justification of plate and shell models (Ciarlet, 2000).
- The **method of \( \Gamma \)-convergence analysis** of thin structures (Paroni, 2006; Neff, 2007, 2010).
- Until now, there was no existence theorem in the literature for geometrically **non-linear** 6-parameter shells: we employ the direct methods of the calculus of variations.
  - M. Bîrsan & P. Neff (2013): *Existence of minimizers in the geometrically non-linear 6-parameter resultant shell theory with drilling rotations*. Mathematics Mechanics of Solids.
- For **linear** micropolar shells, the existence of weak solutions is well known, e.g. Eremeyev and Lebedev (ZAMM, 2011).
Kinematical model of 6–parameter resultant shells

- The reference (initial) configuration of the shell is given by the position vector \( y^0 \) plus the structure tensor \( Q^0 \):

\[
y^0 : \mathcal{W} \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3, \\
Q^0 : \mathcal{W} \subset \mathbb{R}^2 \rightarrow SO(3),
\]

\[
y^0 = y^0(x_1, x_2), \\
Q^0 = d^0_i(x_1, x_2) \otimes e_i,
\]

- The orthonormal triad of initial directors \( \{d^0_1, d^0_2, d^0_3\} \) describes the orthogonal structure tensor \( Q^0 \).

- The deformed configuration is characterized by

\[
y = \chi(y^0), \\
Q^e = d_i \otimes d^0_i \in SO(3),
\]

where \( \chi \) is deformation of the base surface and the orthogonal tensor \( Q^e \) is the (effective) elastic rotation.

- \( y \) denote the position vector and \( \{d_1, d_2, d_3\} \) the orthonormal triad of directors in the deformed configuration.
Figure 1: The base surface of the shell

\[ \begin{align*}
Q^0(y_0) & \quad Q^e(y_0) \\
\chi(y_0) & \quad R(x_1, x_2)
\end{align*} \]
The deformed configuration can alternatively be described by the functions defined on the flat domain \( \omega \):

\[
y = y(x_1, x_2), \quad R(x_1, x_2) = Q^e Q^0 = d_i(x_1, x_2) \otimes e_i \in SO(3),
\]

where \( R \) describes the total rotation from the fictitious planar reference configuration \( \omega \) to the deformed one. (\( R \) is the structure tensor of the deformed configuration)

After that we move the problem on the planar domain \( \omega \), we can apply similar methods as in the case of plates:

M. Birsan, P. Neff. *Existence theorems in the geometrically non-linear 6-parameter theory of elastic plates*. Journal of Elasticity (2013) vol. 112 : 185–198.

P. Neff. *Contin. Mech. Thermodyn.* (2004) 16 : 577-628.
Equilibrium equations and boundary conditions

- Let $\text{Grad}_S$ and $\text{Div}_S$ be the surface gradient and surface divergence operators and $F = \text{Grad}_S y = \partial_\alpha y \otimes a^\alpha$ denote the shell deformation gradient tensor.

- Equations of equilibrium for 6-parameter shells:

  \[
  \text{Div}_S N + f = 0, \quad \text{Div}_S M + \text{axl}(NF^T - FN^T) + c = 0,
  \]

  where $N$ and $M$ are the internal surface stress resultant and stress couple tensors, while $f$ and $c$ are the external surface resultant force and couple vectors.

- We consider boundary conditions of the type

  \[
  N\nu = n^*, \quad M\nu = m^* \quad \text{along } \partial S_f^0, \\
  y = y^*, \quad R = R^* \quad \text{along } \partial S_d^0.
  \]
Non-linear elastic strain and bending-curvature measures

- The elastic shell strain tensor $E^e$ in material representation

\[ E^e = Q^{e,T} \text{Grad}_s y - \text{Grad}_s y^0 , \]

or equivalently,

\[ E^e = (Q^T \partial_\alpha y - a_\alpha) \otimes a^\alpha , \]

where $a_\alpha = \partial_\alpha y^0$, and $a^\alpha$ is given by $a^\alpha \cdot a_\beta = \delta_\beta^\alpha$.

- It is useful in the proof to express the elastic shell strain tensor in terms of the total rotation $R$ and initial rotation $Q^0$:

\[ E^e = Q^0 (R^T \partial_\alpha y - Q^{0,T} \partial_\alpha y^0) \otimes a^\alpha . \]
The elastic shell bending-curvature tensor $K^e$:

$$K^e = \left[ Q^{e,T} \text{axl}(\partial_\alpha RR^T) - \text{axl}(\partial_\alpha Q^0 Q^{0,T}) \right] \otimes a^\alpha .$$

or, equivalently,

$$K^e = \text{axl}(Q^{e,T} \partial_\alpha Q^e) \otimes a^\alpha .$$

We write $K^e$ in terms of the total and initial rotations $R, Q^0$

$$K^e = K - K^0 , \quad K = Q^0 \text{axl}(R^T \partial_\alpha R) \otimes a^\alpha ,$$

$$K^0 = \text{axl}(\partial_\alpha Q^0 Q^{0,T}) \otimes a^\alpha ,$$

where $K$ is the total bending–curvature tensor, while $K^0$ is the initial bending-curvature tensor (or structure curvature).

In the case of plates these strain and curvature measures coincide with those defined for the Cosserat model of plates by Neff (CMT 2004, M3AS 2007).
Variational formulation for elastic shells

Let $W = W(E^e, K^e)$ denote the strain energy density.

The constitutive equations (hyperelasticity assumption):

$$N = Q^e \frac{\partial W}{\partial E^e}, \quad M = Q^e \frac{\partial W}{\partial K^e}.$$  

where $N$ is the internal surface stress resultant and $M$ is stress couple tensor.

$W$ is assumed a quadratic function of $E^e$ and $K^e$ : the model is physically linear and geometrically non-linear.

Two–field minimization problem: find the pair $(\hat{y}, \hat{R})$ in the admissible set $\mathcal{A}$ which realizes minimum of the functional

$$I(y, R) = \int_{S^0} W(E^e, K^e) \, dS - \Lambda(y, R) \quad \text{for} \quad (y, R) \in \mathcal{A}.$$
The admissible set $\mathcal{A}$ is defined by
\[
\mathcal{A} = \{(y, R) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)) \mid y|_{\partial S_d^0} = y^*, \ R|_{\partial S_d^0} = R^* \}.
\]

The function $\Lambda(y, R)$ represents the potential of external surface loads $f$, $c$, and boundary loads $n^*$, $m^*$.

Assume that the external loads satisfy the conditions
\[
f \in L^2(\omega, \mathbb{R}^3), \quad n^* \in L^2(\partial \omega_f, \mathbb{R}^3),
\]
and the boundary data satisfy the regularity conditions
\[
y^* \in H^1(\omega, \mathbb{R}^3), \quad R^* \in H^1(\omega, SO(3)).
\]
Main result: Existence of minimizers

Theorem (Birsan & Neff, *Journal of Elasticity*, 2013)

Assume that the initial configuration satisfies: \( y^0 : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is a continuous injective mapping and

\[
\begin{align*}
  y^0 & \in H^1(\omega, \mathbb{R}^3), & \quad & Q^0 \in H^1(\omega, SO(3)), \\
  \partial_\alpha y^0 & \in L^\infty(\omega, \mathbb{R}^3), & \quad & \det \left( a_{\alpha\beta}(x_1, x_2) \right) \geq a_0^2 > 0.
\end{align*}
\]

The strain energy density \( W(E^e, K^e) \) is assumed to be a quadratic, convex and coercive function of \((E^e, K^e)\):

\[
W(E^e, K^e) \geq C_0 \left( \|E^e\|^2 + \|K^e\|^2 \right).
\]

Then, the minimization problem admits at least one minimizing solution pair \((\hat{y}, \hat{R}) \in \mathcal{A}\).

Note: The problem is non-convex in \((y, Q)\)!
Sketch of the Proof

- We employ the direct methods of the calculus of variations.
- We show first that there exists $C > 0$ such that
  \[ |\Lambda(y, R)| \leq C \left( \|y\|_{H^1(\omega)} + 1 \right), \quad \forall (y, R) \in \mathcal{A}. \]
- Using this relation and the coercivity of $W$ we obtain
  \[ I(y, R) \geq C_0 \| \nabla y \|^2_{L^2(\omega)} - C_1 \|y\|_{H^1(\omega)} - C_2. \]
- Applying the Poincaré–inequality we infer
  \[ I(y, R) \geq c_p \|y - y^*\|^2_{H^1(\omega)} - C_3 \|y - y^*\|_{H^1(\omega)} + C_4, \quad \forall (y, R) \in \mathcal{A} \]
  so that the functional $I(y, R)$ is bounded from below over $\mathcal{A}$. 
There exists an infimizing sequence \((y_n, R_n)\) such that

\[
\lim_{n \to \infty} I(y_n, R_n) = \inf \{ I(y, R) \mid (y, R) \in A \}.
\]

The sequences \(\{y_n\}\) and \(\{R_n\}\) are bounded in \(H^1(\omega)\).

We can extract subsequences (not relabeled) such that

\[
y_n \rightharpoonup \hat{y} \quad \text{in} \quad H^1(\omega, \mathbb{R}^3) \quad \text{and} \quad y_n \to \hat{y} \quad \text{in} \quad L^2(\omega, \mathbb{R}^3),
\]

\[
R_n \rightharpoonup \hat{R} \quad \text{in} \quad H^1(\omega, \mathbb{R}^3 \times 3) \quad \text{and} \quad R_n \to \hat{R} \quad \text{in} \quad L^2(\omega, \mathbb{R}^3 \times 3).
\]

Next we show the weak convergence (on subsequences)

\[
E^e_n \rightharpoonup \hat{E}^e \quad \text{in} \quad L^2(\omega, \mathbb{R}^3 \times 3) \quad \text{and} \quad K^e_n \to \hat{K}^e \quad \text{in} \quad L^2(\omega, \mathbb{R}^3 \times 3).
\]
Finally, we use the convexity of the strain energy \( W \):

\[
\int_{\omega} W(\hat{E}^e, \hat{K}^e) \, a \, dx_1 \, dx_2 \leq \liminf_{n \to \infty} \int_{\omega} W(E_n^e, K_n^e) \, a \, dx_1 \, dx_2,
\]

which means that:
\[
I(\hat{y}, \hat{R}) \leq \liminf_{n \to \infty} I(y_n, R_n).
\]

Since \((y_n, R_n)\) is an infimizing sequence, we obtain:
\[
I(\hat{y}, \hat{R}) = \inf \{ I(y, R) \mid (y, R) \in \mathcal{A} \},
\]
i.e., \((\hat{y}, \hat{R})\) is a minimizing solution pair. \(\square\)

Remark. The boundary conditions on \( R \) can be relaxed: the Theorem holds for the larger admissible set

\[
\tilde{\mathcal{A}} = \{(y, R) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, SO(3)) \mid y|_{\partial\omega_\delta} = y^*\}.
\]
Isotropic shells

- We introduce the notations $E^e_\parallel$ and $K^e_\parallel$ by

$$E^e_\parallel = E^e - (n^0 \otimes n^0)E^e, \quad K^e_\parallel = K^e - (n^0 \otimes n^0)K^e.$$

- A general expression of the strain energy density for isotropic shells is (Eremeyev & Pietraszkiewicz, 2006)

$$2W(E^e, K^e) = \alpha_1 (\operatorname{tr} E^e_\parallel)^2 + \alpha_2 \operatorname{tr}(E^e_\parallel)^2 + \alpha_3 \operatorname{tr}(K^e_\parallel)^2 + \alpha_4 n^0 E^e E^{e, T} n^0$$

$$+ \beta_1 (\operatorname{tr} K^e_\parallel)^2 + \beta_2 \operatorname{tr}(K^e_\parallel)^2 + \beta_3 \operatorname{tr}(K^{e, T} K^e_\parallel) + \beta_4 n^0 K^e K^{e, T} n^0,$$

where $\alpha_k$ and $\beta_k$ are constant (in general depend on $K^0$).

- Applying our Theorem, there exists a minimizer provided

$$2\alpha_1 + \alpha_2 + \alpha_3 > 0, \quad \alpha_2 + \alpha_3 > 0, \quad \alpha_3 - \alpha_2 > 0, \quad \alpha_4 > 0,$$

$$2\beta_1 + \beta_2 + \beta_3 > 0, \quad \beta_2 + \beta_3 > 0, \quad \beta_3 - \beta_2 > 0, \quad \beta_4 > 0,$$

- These conditions ensure the coercivity and convexity of $W$. 
The strain energy density for isotropic shells $W(E^e, K^e)$ in its simplest form is (Pietraszkiewicz et al. 2004, 2010)

$$2W(E^e, K^e) = C \left[ \nu (\text{tr}E^e_\parallel)^2 + (1 - \nu) \text{tr}(E^e_\parallel, T E^e_\parallel) \right] + \alpha_s C (1 - \nu) n^0 E^e E^{e,T} n^0$$

$$+ D \left[ \nu (\text{tr}K^e_\parallel)^2 + (1 - \nu) \text{tr}(K^e_\parallel, T K^e_\parallel) \right] + \alpha_t D (1 - \nu) n^0 K^e K^{e,T} n^0,$$

where $C = \frac{E h}{1 - \nu^2}$ is the stretching (in-plane) stiffness, $D = \frac{E h^3}{12(1 - \nu^2)}$ is the bending stiffness, and $\alpha_s, \alpha_t$ are two shear correction factors.

The strain energy $W$ is convex and coercive provided

$$E > 0, \quad -1 < \nu < \frac{1}{2} \quad \iff \quad \mu > 0, \quad 2\mu + 3\lambda > 0.$$

Using our Theorem we deduce the existence of minimizers.
Orthotropic shells

- The expression of $W$ in terms of the tensor components

$$2W(E^e, K^e) = C^E_{\alpha\beta\gamma\delta}\tilde{E}_{\alpha\beta}\tilde{E}_{\gamma\delta} + D^E_{\alpha\beta}\tilde{E}_{3\alpha}\tilde{E}_{3\beta} + C^K_{\alpha\beta\gamma\delta}\tilde{K}_{\alpha\beta}\tilde{K}_{\gamma\delta} + D^K_{\alpha\beta}\tilde{K}_{3\alpha}\tilde{K}_{3\beta}$$

- where $C^E_{\alpha\beta\gamma\delta}$, $C^K_{\alpha\beta\gamma\delta}$, $D^E_{\alpha\beta}$ and $D^K_{\alpha\beta}$ are material constants.

- This function $W$ is coercive if and only if the matrices

$$
\begin{bmatrix}
C^E_{1111} & C^E_{1122} & C^E_{1112} & C^E_{1121} \\
C^E_{1122} & C^E_{2222} & C^E_{2212} & C^E_{2221} \\
C^E_{1112} & C^E_{2212} & C^E_{1212} & C^E_{1221} \\
C^E_{1121} & C^E_{2221} & C^E_{1221} & C^E_{2121}
\end{bmatrix},
\begin{bmatrix}
C^K_{1111} & C^K_{1122} & C^K_{1112} & C^K_{1121} \\
C^K_{1122} & C^K_{2222} & C^K_{2212} & C^K_{2221} \\
C^K_{1112} & C^K_{2212} & C^K_{1212} & C^K_{1221} \\
C^K_{1121} & C^K_{2221} & C^K_{1221} & C^K_{2121}
\end{bmatrix},
\begin{bmatrix}
D^E_{11} & D^E_{12} \\
D^E_{12} & D^E_{22}
\end{bmatrix},
\begin{bmatrix}
D^K_{11} & D^K_{12} \\
D^K_{12} & D^K_{22}
\end{bmatrix}
$$

- are positive definite.

- We show the existence of minimizers for orthotropic shells

- The Theorem is applicable for composite layered shells.
Shells without drilling rotations
Reissner–Type (5–parameter) theory
Shells without drilling rotations

- In this case, the strain energy density $W$ must be invariant under superposition of rotations $R_\theta$ of angle $\theta$ about $d_3$, i.e.
- $W(E^e, K^e)$ remains invariant under the transformations
  \[ Q \rightarrow R_\theta Q. \] (1)

- In view of the definitions of $E^e, K^e$, this is equivalent to
  \[ W(E^e, K^e) = W\left( [Q^T R_\theta^T \partial_\alpha y - a_\alpha] \otimes a^\alpha, \text{axl}[Q^T R_\theta^T \partial_\alpha (R_\theta Q)] \otimes a^\alpha \right) \]
  for any angle $\theta(x_1, x_2)$.
- The following result gives a characterization of shells without drilling rotation.
Theorem (Characterization)

Assume that the strain energy function $W$ is invariant under the transformation (1). Then, $W$ can be represented as a function of the arguments

$$W = \tilde{W}(F^TF, d_3F, F^T\text{Grad}_3d_3). \quad (2)$$

Conversely, any function $W$ of the form (2) is invariant under the superposition of drilling rotations (1).
Sketch of the Proof

- If $W$ is invariant under the transformation (1), then the derivative of $W$ with respect to $\theta$ and $\partial_\alpha \theta$ are zero.
- We obtain the equations

\[
\frac{\partial W}{\partial E^e} \cdot c(E^e + a) + \frac{\partial W}{\partial K^e} \cdot c(K^e + K^0) = 0
\]

and

\[
\frac{\partial W}{\partial (n^0 K^e)} = 0,
\]

where $a = d_0^\alpha \otimes a_\alpha^0$ and $c = d_1^0 \otimes d_2^0 - d_2^0 \otimes d_1^0$.

- $W(E^e, K^e)$ depends on 12 independent scalar arguments: the components of $E^e$ and $K^e$ in the tensor basis $\{d_i^0 \otimes a^\alpha\}$.

- We determine 11 first integrals of the system of ordinary differential equations associated to (3): 

\[
\frac{dE^e}{ds} = c(E^e + a), \quad \frac{dK^e}{ds} = c(K^e + K^0).
\]
The 11 first integrals are the components of

\[ U_1 = F^T F = (E^e + a)^T (E^e + a), \]
\[ U_2 = d_3 F = n^0 E^e, \]
\[ U_3 = n^0 K^e, \]
\[ U_4 = F^T \text{Grad}_s d_3 = (E^e + a)^T c (K^e + K^0). \]

It follows that \( W \) is a function of

\[ W = \tilde{W}(F^T F, d_3 F, n^0 K^e, F^T \text{Grad}_s d_3). \]

By virtue of (3)_2, \( W \) is independent of \( n^0 K^e \), so that

\[ W = \tilde{W}(F^T F, d_3 F, F^T \text{Grad}_s d_3). \]
Remarks

The strain energy $W$ can be alternatively expressed as

$$ W = \hat{W}(\varepsilon, \gamma, \Psi), \quad \gamma = d_3 F, \quad \varepsilon = \frac{1}{2} (F^T F - a), \quad \Psi = (F^T \text{Grad}_s d_3 - \text{Grad}_s n^0) - \varepsilon \text{Grad}_s n^0, $$

$\varepsilon$ is a symmetric tensor for extensional and in-plane shear strains, $\gamma$ is the vector of transverse shear deformation, and $\Psi$ is a tensor for bending and twist strains.

Zhilin (2006) presented similar results for shells without drilling rotations, but with a different bending–twist tensor

$$ \Phi = [F^T (d_3 \times \text{Grad}_s d_3) - n^0 \times \text{Grad}_s n^0] - \varepsilon (n^0 \times \text{Grad}_s n^0). $$

This representation is more complicated and it introduces an additional rotation of $\text{Grad}_s d_3$ in the plane $\{d_1, d_2\}$. 
In the linear theory these deformation tensors reduce to:

\[ \mathbf{E} = \text{sym}(a \text{Grad}_s u), \]
\[ \mathbf{\gamma} = n^0 \text{Grad}_s u + c \psi, \]
\[ \Psi = c \Phi = c \text{Grad}_s (a \psi) + [\text{skew}(a \text{Grad}_s u)]b, \]

where \( \psi \) is the vector of small rotations.

One can easily see that the strain measures \( \mathbf{E}, \mathbf{\gamma} \) and \( \Psi \) are independent of the drilling rotations \( (\psi \cdot n^0) \).

In this case one gets the Reissner–type kinematics of shells with 5 degrees of freedom (Wiśniewski 2010, Neff et al. 2010).
For shells without drilling rotations, we consider $W$ as a quadratic function of its arguments $(\mathbf{E}, \gamma, \Psi)$ (cf. Zhilin)

$$2\hat{W}(\mathbf{E}, \gamma, \Psi) = C[(1-\nu)\|\mathbf{E}\|^2 + \nu(\text{tr} \mathbf{E})^2] + \frac{1}{2} C(1-\nu)\kappa \gamma^2$$

$$+D\left[\frac{1}{2} (1-\nu)\|\Psi\|^2 + \frac{1}{2} (1-\nu) \text{tr}(\Psi^2) + \nu (\text{tr} \Psi)^2\right],$$

We express $W$ in terms of the strain tensors $(\mathbf{E}^e, \mathbf{K}^e)$ and keep only the quadratic terms in $(\mathbf{E}^e, \mathbf{K}^e)$:

$$2W(\mathbf{E}^e, \mathbf{K}^e) = C[\nu(\text{tr} \mathbf{E}^e)^2 + \frac{1-\nu}{2} \text{tr}(\mathbf{E}^e)^2 + \frac{1-\nu}{2} \text{tr}(\mathbf{E}^e,^T \mathbf{E}^e)]$$

$$+C \frac{1-\nu}{2} \kappa \|n^0 \mathbf{E}^e\|^2 + D[\text{tr}(\mathbf{K}^e,^T \mathbf{K}^e) - \frac{1-\nu}{2} (\text{tr} \mathbf{K}^e)^2 - \nu (\text{tr} \mathbf{K}^e)^2].$$

We compare this with the general form of $W$ for isotropic 6–parameter shells and identify the coefficients $\alpha_k$ and $\beta_k$: 
\[
\begin{align*}
\alpha_1 &= C \nu, & \alpha_2 &= \alpha_3 = C \frac{1-\nu}{2}, & \alpha_4 &= C \frac{1-\nu}{2} \kappa, \\
\beta_1 &= D \frac{\nu-1}{2}, & \beta_2 &= -D \nu, & \beta_3 &= D, & \beta_4 &= 0.
\end{align*}
\]

(5)

The coefficients given in (5) for shells without drilling rotations are different from the values of \(\alpha_k\) and \(\beta_k\) corresponding to shells with drilling rotations.

The conditions for the existence of minimizers are not satisfied by the coefficients (5) since

\[
\alpha_3 - \alpha_2 = 0, \quad 2\beta_1 + \beta_2 + \beta_3 = 0, \quad \beta_4 = 0.
\]

(This corresponds to \(\mu_c = 0\) in Neff, 2007)

In this case, \(W\) is not uniformly positive definite, and the proof of existence of minimizers is possible but much more difficult (Neff 2004, Neff 2007).
Conclusions

- We have proved the existence of minimizers for the geometrically non-linear model of 6–parameter elastic shells.
- For the case of shells without drilling rotations we present a representation theorem for the strain energy function.

Future plans:
- To extend the analysis to the case of 6–parameter shells with physically non-linear elastic behavior.
- To establish existence results for the equations of non-linear elasto-plastic shells.
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