COHOMOLOGY OF IDEALS IN ELLIPTIC SURFACE SINGULARITIES

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Abstract. We introduce the normal reduction number of two-dimensional normal singularities and prove that elliptic singularity has normal reduction number two. We also prove that for a two-dimensional normal singularity which is not rational, it is Gorenstein and its maximal ideal is a \( p_g \)-ideal if and only if it is a maximally elliptic singularity of degree 1.

1. Introduction

Let \((A, \mathfrak{m})\) be an excellent two-dimensional normal local domain containing an algebraically closed field isomorphic to the residue field. In this paper, we simply call such a local ring a normal surface singularity. Lipman [12] proved that if \((A, \mathfrak{m})\) is a rational singularity, then for any integrally closed \( \mathfrak{m} \)-primary ideals \( I \) and \( I' \) we have that the product \( II' \) is also integrally closed and that \( I^2 = QI \) for any minimal reduction \( Q \) of \( I \). Cutkosky [3] showed that the first property characterizes the two-dimensional rational singularities. In [17], [18], [16], we introduced the notion of \( p_g \)-ideals, which satisfy the properties above, and proved many nice properties. For any normal surface singularity, \( p_g \)-ideals exist plentifully and form a semigroup with respect to the product. It is easy to see that \( A \) is a rational singularity if and only if every integrally closed \( \mathfrak{m} \)-primary ideal is a \( p_g \)-ideal (see Remark 2.11). So it is natural to ask how the semigroup of the \( p_g \)-ideals encodes the properties of the singularity.

Let \( X \to \text{Spec} \, A \) be a resolution of singularity. Suppose that an integrally closed \( \mathfrak{m} \)-primary ideal \( I \) is represented by a cycle \( Z \) on \( X \) (see 2.2). Then \( I = H^0(X, \mathcal{O}_X(-Z)) \). We define an invariant \( q(I) \) to be \( \ell_A(H^1(X, \mathcal{O}_X(-Z))) \), where \( \ell_A \) denotes the length of \( A \)-modules. Then \( I \) is called the \( p_g \)-ideal if \( q(I) = p_g(A) \), where \( p_g \) denotes the geometric genus (see Definition 2.8). In general, we have \( p_g(A) \geq q(T^n) \geq q(T^{n+1}) \) (see Proposition 2.9), where \( T^n \) denotes the integral closure of \( I^n \), and we know that there exist ideals with \( q = 0 \) and \( q = p_g(A) \); however, the range of \( q \) is still unknown. We are

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interested in obtaining the range of \( q \) and also the minimal integer \( n_0 \) such that \( q(I^n) = q(I^{n_0}) \) for \( n \geq n_0 \). This integer connects with the normal reduction number \( \bar{r}(I) \) (see \( \S 3 \)). The results of Lipman and Cutkosky above implies that \( \bar{r}(A) = 1 \) if and only if \( A \) is a rational singularity (Theorem 3.2). Then a very simple question arises: can we characterize normal surface singularities with \( \bar{r}(A) = 2? \)

In this paper, we give partial answers to the questions above. We will prove the following (see Theorem 3.3, Corollary 3.13, Theorem 4.3).

**Theorem 1.** (1) If \( A \) is an elliptic singularity, then \( \bar{r}(A) = 2 \), and for any \( 0 \leq q \leq p_g(A) \) there exists an integrally closed \( \mathfrak{m} \)-primary ideal \( I \) with \( q(I) = q \).

(2) Assume that \( A \) is not rational. Then \( A \) is Gorenstein and \( \mathfrak{m} \) is a \( p_g \)-ideal if and only if \( A \) is a maximally elliptic singularity with \( -Z_E^2 = 1 \), where \( Z_E \) is the fundamental cycle on a resolution.

Throughout this paper, we assume the following.

**Assumption 1.1.** For any integrally closed \( \mathfrak{m} \)-primary ideal \( I \subset A \) represented on a resolution \( X \to \text{Spec} \ A \) with exceptional set \( E \), and for a general element \( h \in I \), if \( H \) denotes the the strict transform of \( \text{div}_{\text{Spec} \ A}(h) \) on \( X \), then \( H \) is a reduced divisor which is a disjoint union of nonsingular curves and each component of \( H \) intersects the exceptional set transversally, namely, the local equations of \( H \) and \( E \) generate the maximal ideal at the intersection point. (This condition holds in case the singularity is defined over a field of characteristic zero.)

This paper is organized as follows. In Section 2 we recall the definitions and several properties of elliptic singularities and \( p_g \)-ideals in normal surface singularities which are needed later. In Section 3 we introduce the normal reduction number and study the invariant \( q \), and then prove (1) of Theorem 1. In the last section, we prove (2) of Theorem 1 and give an example of non-Gorenstein elliptic singularity with \( -Z_E^2 = 1 \) of which the maximal ideal is a \( p_g \)-ideal.

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2. Preliminaries

Throughout this paper, let \((A, m)\) denote a normal surface singularity, namely, an excellent two-dimensional normal local domain containing an algebraically closed field isomorphic to the residue field and \(f: X \to \text{Spec } A\) a resolution of singularity with exceptional set \(E := f^{-1}(m)\). Let \(E = \bigcup_{i=1}^{r} E_i\) be the decomposition into irreducible components of \(E\). A divisor on \(X\) supported in \(E\) is called a cycle. A divisor \(D\) on \(X\) is said to be nef if \(DE_i \geq 0\) for all \(E_i \subset E\), where \(DE_i\) denotes the intersection number. A divisor \(D\) is said to be anti-nef if \(-D\) is nef. Since the intersection matrix is negative definite, there exists an anti-nef cycle \(Z \neq 0\) and it satisfies \(Z \geq E\).

For a cycle \(B > 0\), we denote by \(\chi(B)\) the Euler characteristic \(\chi(O_B)\). We have \(\chi(D) + \chi(F) - DF = \chi(D + F)\). By definition, \(p_a(B) = 1 - \chi(B)\). The fundamental cycle on \(\text{Supp}(B)\) is denoted by \(Z_B\); by definition, \(Z_B\) is the minimal cycle such that \(\text{Supp}(Z_B) = \text{Supp}(B)\) and \(Z_B E_i \leq 0\) for all \(E_i \leq B\).

For any function \(h \in H^0(\mathcal{O}_X) \setminus \{0\}\), which has zero of order \(a_i\) at \(E_i\), we put \((h)_{E_i} = \sum a_i E_i\). Clearly the cycle \((h)_{E_i}\) is anti-nef.

2.1. Elliptic singularities.

Definition 2.1 (Wagreich [25, p. 428]). A normal surface singularity \((A, m)\) is called an elliptic singularity if one of the following equivalent conditions holds:

1. \(\chi(D) \geq 0\) for all cycles \(D > 0\) and \(\chi(F) = 0\) for some cycle \(F > 0\);
2. \(\chi(Z_E) = 0\).

Remark 2.2. The proof of the implication \((2) \Rightarrow (1)\) is given by several authors: e.g., Laufer [10, Corollary 4.2], Tomari [23, Theorem (6.4)]. See also [23, Remark (6.5)].

Definition 2.3 (Laufer [10 Definition 3.1 and 3.2]). Suppose that \((A, m)\) is an elliptic singularity. Then there exists a unique cycle \(E_{\text{min}}\) such that \(\chi(E_{\text{min}}) = 0\) and \(\chi(D) > 0\) for all cycles \(D\) such that \(0 < D < E_{\text{min}}\). The cycle \(E_{\text{min}}\) is called a minimally elliptic cycle. The singularity \((A, m)\) is said to be minimally elliptic if the fundamental cycle is minimally elliptic on the minimal resolution.

The next proposition follows from [10 Proposition 3.2].

Proposition 2.4. Assume that \(A\) is an elliptic singularity. Let \(D > 0\) be a cycle with \(\chi(D) = 0\). Then we have the following.

1. \(D \geq E_{\text{min}}\). Consequently, \(D\) is connected (i.e., \(\text{Supp}(D)\) is connected).
2. Any connected reduced cycle \(F\) not containing any component of \(D\) is the exceptional set of a rational singularity and satisfies \(DF \leq 1\).
The notion of elliptic sequence was introduced by S. S.-T. Yau [26], [27] for elliptic singularities.

**Definition 2.5.** Assume that \((A, m)\) is an elliptic singularity. Let \(B\) be a connected reduced cycle such that \(\text{Supp}(E_{\min}) \subset B\). We define the elliptic sequence on \(B\) as follows: Let \(B_0 = B\). If \(Z_{B_0}E_{\min} < 0\), then the elliptic sequence is \(\{Z_{B_0}\}\). If \(Z_{B_0}E_{\min} = 0\), then define \(B_{i+1} \leq B_i\) to be the maximal reduced connected cycle containing \(\text{Supp}(E_{\min})\) such that \(Z_{B_i}B_{i+1} = 0\). If we have \(Z_{B_m}E_{\min} < 0\), then the elliptic sequence is \(\{Z_{B_0}, \ldots, Z_{B_m}\}\).

**Proposition 2.6** (Tomari [23 Theorem (6.4)]). Let \(\{Z_{B_0}, \ldots, Z_{B_m}\}\) be the elliptic sequence on \(B\). For an integer \(0 \leq t \leq m\), we define a cycle \(C_t\) by

\[
C_t = \sum_{i=0}^{t} Z_{E_i}.
\]

Then the set \(\{C_k \mid 0 \leq k \leq m\}\) coincides with the set of cycles \(C > 0\) supported on \(B\) such that \(C\) is anti-nef on \(B\) and \(\chi(C) = 0\).

**Lemma 2.7** (Röhr [21 1.7], cf. [19 Lemma 3.2]). Assume that \(A\) is an elliptic singularity. Let \(D\) be a nef divisor on \(X\) such that \(DE_{\min} > 0\). Then \(H^1(O_X(D)) = 0\).

### 2.2. \(p_g\)-ideals

Let \(I \subset A\) be an integrally closed \(m\)-primary ideal. Then there exists a resolution \(X \to \text{Spec} A\) and a cycle \(Z > 0\) on \(X\) such that \(I\text{O}_X = O_X(-Z)\). In this case, we denote the ideal \(I\) by \(I_Z\), and we say that \(I\) is represented on \(X\) by \(Z\). Note that \(I_Z = H^0(X, O_X(-Z))\).

When we write \(I_Z\), we always assume that \(O_X(-Z)\) is generated by global sections, namely, \(I\text{O}_X = O_X(-Z)\).

We denote by \(h^1(O_X(-Z))\) the length \(\ell_A(H^1(X, O_X(-Z)))\).

**Definition 2.8.** The geometric genus \(p_g(A)\) of \(A\) is defined by \(p_g(A) = h^1(O_X)\). We define an invariant \(q(I)\) by \(q(I) = h^1(O_X(-Z))\); this does not depend on the choice of representations of the ideal (see [17 Lemma 3.4]).

Kato’s Riemann-Roch formula [9] shows a relation between the colength \(\ell_A(A/I)\) and the invariant \(q(I)\) of \(I = I_Z\):

\[
\ell_A(A/I) + q(I) = -\frac{Z^2 + K_XZ}{2} + p_g(A).
\]

In particular, \(\ell_A(A/I)\) can be computed from the resolution graph if \(I\) is a \(p_g\)-ideal (see Definition 2.10). However, the computation of the invariant \(q(I)\) (or \(\ell_A(A/I)\)) is very difficult for non-rational singularities, and it seems to be given only for very special cases (e.g., [17 §7]).

We say that \(O_X(-Z)\) has no fixed component if \(H^0(O_X(-Z)) \neq H^0(O_X(-Z - E_i))\) for every \(E_i \subset E\); this is equivalent to the existence of an element \(h \in H^0(O_X(-Z))\) such that \((h)_E = Z\). It is clear that \(O_X(-Z)\) has no fixed component when \(I\) is represented by \(Z\).
Proposition 2.9 ([17] 2.5, 3.1). Let $Z'$ and $Z$ be cycles on $X$ and assume that $\mathcal{O}_X(-Z)$ has no fixed components. Then we have

$$h^1(\mathcal{O}_X(-Z')) \leq h^1(\mathcal{O}_X(-Z)).$$

In particular, $h^1(\mathcal{O}_X(-Z)) \leq p_g(A)$, if the equality holds, then $\mathcal{O}_X(-Z)$ is generated by global sections.

Definition 2.10. (1) We call $I$ a $p_g$-ideal if $q(I) = p_g(A)$.

(2) A cycle $Z > 0$ is called a $p_g$-cycle if $\mathcal{O}_X(-Z)$ is generated by global sections and $h^1(\mathcal{O}_X(-Z)) = p_g(A)$.

Remark 2.11. If $A$ is rational, namely $p_g(A) = 0$, every integrally closed $m$-primary ideal is a $p_g$-ideal by [12] 12.1. Conversely, this property characterizes a rational singularity because we always have integrally closed $m$-primary ideal $I$ with $q(I) = 0$ (see e.g. [17] 4.5).

In [17] and [18], we obtained many good properties and characterizations of $p_g$-ideals. Let us review some of these results.

Next we recall a characterization of $p_g$-ideals by cohomological cycle. Let $K_X$ denote the canonical divisor on $X$. Let $Z_{K_X}$ denote the canonical cycle, i.e., the $\mathbb{Q}$-divisor supported in $E$ such that $(K_X + Z_{K_X})E_i = 0$ for every $E_i \subset E$. By [20] §4.8, if $p_g(A) > 0$, there exists the smallest cycle $C_X > 0$ on $X$ such that $h^1(\mathcal{O}_{C_X}) = p_g(A)$; if $A$ is Gorenstein and the resolution $f: X \to \text{Spec} A$ is minimal, then $C_X = Z_{K_X}$. The cycle $C_X$ is called the cohomological cycle on $X$. We put $C_X = 0$ if $A$ is a rational singularity.

Proposition 2.13 (cf. [16] Proposition 2.6). Let $C \geq 0$ be the minimal cycle such that $H^0(X \setminus E, \mathcal{O}_X(K_X)) = H^0(X, \mathcal{O}_X(K_X + C))$. Then $C$ is the cohomological cycle. Therefore if $g: X' \to X$ is the blowing-up at a point in $\text{Supp}(C_X)$ and $E_0$ the exceptional set of $g$, then $C_{X'} = g^*C_X - E_0$. For any cycle $D > 0$ without common components with $C_X$, we have $h^1(\mathcal{O}_D) = 0$.

Proposition 2.14 ([17] 3.10]). Assume that $p_g(A) > 0$. Let $Z > 0$ be a cycle such that $\mathcal{O}_X(-Z)$ has no fixed component. Then $Z$ is a $p_g$-cycle if and only if $\mathcal{O}_{C_X}(-Z) \cong \mathcal{O}_{C_X}$. 
Proposition 2.15 ([18]). Let $I$ be an integrally closed $m$-primary ideal. Then $I$ is a $p_g$-ideal if and only if the Rees algebra $\bigoplus_{n \geq 0} I^n t^n \subset A[t]$ is a Cohen-Macaulay normal domain.

The following theorem shows that the $p_g$-ideals exist plentifully.

Theorem 2.16 (cf. [16, Theorem 5.1]). Let $I$ be an integrally closed $m$-primary ideal and $g$ an arbitrary element of $I$. Then there exists $h \in I$ such that the integral closure of the ideal $(g, h)$ is a $p_g$-ideal.

3. The normal reduction number

Definition 3.1. Let $I$ be an integrally closed $m$-primary ideal and $Q$ a minimal reduction of $I$. We define the normal reduction number $\bar{r}$ of $I$ by

$$\bar{r}(I) = \min \left\{ r \in \mathbb{Z}_{\geq 0} \mid \frac{I^n}{I^{n+1}} = \frac{Q^n}{Q^{n+1}} \text{ for all } n \geq r \right\}.$$

We shall see that $\bar{r}(I)$ is independent of the choice of minimal reductions by Corollary 3.9.

Let

$$\bar{r}(A) = \max \{ \bar{r}(I) \mid I \text{ is an integrally closed } m\text{-primary ideal of } A \}.$$

The normal reduction number has been studied by many authors implicitly or explicitly in the context of the Hilbert function and the Hilbert polynomial associated with $\{ \frac{I^n}{I^{n+1}} \}_{n \geq 0}$ (e.g., [14], [8], [6]). We study this invariant in terms of cohomology of ideal sheaves of cycles toward a geometric understanding of the normal reduction number.

If $A$ is rational, then by Lipman [12] (cf. Proposition 2.12), we have $\frac{I^n}{I^{n+1}} = I^2 = QI$ for any integrally closed $m$-primary ideal $I$. On the other hand, Cutkosky [3] proved that the converse holds too. Hence we have the following.

Theorem 3.2. $\bar{r}(A) = 1$ if and only if $A$ is a rational singularity.

Note that the rationality is determined by the resolution graph (see [1]).

The main result of this section is the following.

Theorem 3.3. If $A$ is an elliptic singularity, then $\bar{r}(A) = 2$.

Definition 3.4. Let $D \geq 0$ be an effective cycle and let

$$h(D) = \max \{ h^1(O_{D'}) \mid D' \geq 0, \text{Supp}(D') \subset \text{Supp}(D) \},$$

where we put $h^1(O_{D'}) = 0$ if $D' = 0$. There exists a unique minimal cycle $C$ such that $h^1(O_C) = h(D)$ (cf. [20, §4.8]). We call $C$ the cohomological cycle on $D$. We define a reduced cycle $D^\perp$ to be the sum of the components $E_i \subset E$ such that $DE_i = 0$.

Remark 3.5. Suppose that $O_X(-Z)$ has no fixed component. Then there exists a function $h \in H^0(O_X(-Z))$ such that $\text{div}_X(h) = Z + H$, where $H$ is the strict transform of $\text{div}_{\text{Spec } A}(h)$. Since $ZE_i = -HE_i$ for any $E_i \subset E$, it
follows that $\text{Supp}(Z^\perp)$ and $\text{Supp}(H)$ have no intersection. Thus for any cycle $F > 0$ supported in $Z^\perp$, we have $\mathcal{O}_F(-Z) = \mathcal{O}_F(-\text{div}_X(h)) \cong \mathcal{O}_F$.

Let $Z > 0$ be a cycle on $X$ and let $\mathcal{L}(n) = \mathcal{O}_X(-nZ)$.

**Lemma 3.6 (See [18] 3.1 and 3.4).** Suppose that $\mathcal{O}_X(-Z)$ has no fixed component. Let $C$ denote the cohomological cycle on $Z^\perp$. Then we have the following.

1. $h^1(\mathcal{L}(n)) \geq h^1(\mathcal{L}(n + 1))$ for $n \geq 0$.
2. Let $n_0(Z) = \min \{ n \in \mathbb{Z}_{>0} \mid h^1(\mathcal{L}(n)) = h^1(\mathcal{L}(n + 1)) \}$. Then $h^1(\mathcal{L}(n)) = h^1(\mathcal{L}(n_0(Z)))$ for $n \geq n_0(Z)$. If $Z$ is a $p_y$-cycle, then $n_0(Z) = 0$.
3. Let $n_0 = n_0(Z)$. Then $\mathcal{O}_C(-n_0 Z) = \mathcal{O}_C$ and $h^1(\mathcal{L}(n_0(Z))) = h^1(\mathcal{O}_C)$.
4. $\mathcal{L}(n)$ is generated by global sections for $n > n_0$.

**Proof.** The claims (1)–(3) are proved in [18]. Let $h \in I_Z$ be a general element and consider the exact sequence

\[ 0 \to \mathcal{L}(n - 1) \xrightarrow{h} \mathcal{L}(n) \to \mathcal{L}(n) \to 0, \]

where $\mathcal{L}(n)$ is supported on the divisor $\text{div}_X(h) - (h)_E$. If $n > n_0(Z)$, then $H^0(\mathcal{L}(n)) \to H^0(\mathcal{L}(n))$ is surjective since $H^1(\mathcal{L}(n)) = 0$. This shows that $H^0(\mathcal{L}(n))$ has no base points. □

**Definition 3.7.** For an integrally closed $m$-primary ideal $I$ represented by $Z$, let $n_0(I) = n_0(Z)$; this is independent of the choice of representations since so is $q(I)$.

**Remark 3.8.** Let us explain the invariant $q(I_{n_0Z})$ in terms of “partial resolution.” Suppose that $I$ is represented by a cycle $Z > 0$ on $X$. Let $Y$ be the normalization of the blowing-up of Spec $A$ by $I$, namely, $Y = \text{Proj} \bigoplus_{n \geq 0} I_{nZ} t^n$. Let $\phi : X \to Y$ be the natural morphism and let $Z' = \phi_* Z$. Then $\mathcal{O}_Y^\perp = \mathcal{O}_Y(-Z')$. Since $\phi_* \mathcal{O}_X = \mathcal{O}_Y$, from Leray’s spectral sequence, we obtain the following exact sequence for $n \geq 0$.

\[(3.1) \quad 0 \to H^1(\mathcal{O}_Y(-nZ')) \to H^1(\mathcal{O}_X(-nZ)) \to H^0(R^1\phi_* \mathcal{O}_X \otimes \mathcal{O}_Y(-nZ')) \to 0.\]

Let $\text{Sing}(Y)$ denote the set of singular points of $Y$. Since the support of $R^1\phi_* \mathcal{O}_X \otimes \mathcal{O}_Y(-nZ')$ is contained in $\text{Sing}(Y)$, we obtain that $R^1\phi_* \mathcal{O}_X \otimes \mathcal{O}_Y(-nZ') \cong R^1 \phi_* \mathcal{O}_X$. It follows from Lemma [3.6](#) (3) that

\[ \ell_A(R^1 \phi_* \mathcal{O}_X) = \sum_{y \in \text{Sing}(Y)} p_y(Y, y) = q(I_{n_0Z}). \]

The sequence (3.1) implies the following equalities.

\[ q(I_{n_0Z}) = p_y(A) - h^1(\mathcal{O}_Y) = h^1(\mathcal{O}_X(-nZ)) \quad \text{for } n \geq n_0(I), \]

\[ q(I_{nZ}) - q(I_{n_0}) = h^1(\mathcal{O}_Y(-nZ')). \]

In particular, $h^1(\mathcal{O}_Y(-nZ')) = 0$ if and only if $n \geq n_0$. 
Corollary 3.9. Let I be an integrally closed \( m \)-primary ideal represented by \( Z \). Then \( \bar{r}(I) = n_0(I) + 1 \).

Proof. Let \( Q = (f_1, f_2) \subset I_Z \) a minimal reduction of \( I_Z \). Then for any integer \( n \), we have the following exact sequence.

\[
0 \to \mathcal{L}(n-1) \overset{(f_1 : f_2)}{\to} \mathcal{L}(n) \overset{\leftarrow_{f_2}}{\to} \mathcal{L}(n+1) \to 0.
\]

From Lemma 3.6 (1), (2) and the sequence (3.2), for an arbitrary integer \( r \geq 0 \), we have that \( QI_{nZ} = I_{(n+1)Z} \) for all \( n \geq r \) if and only if \( h^1(L(n)) = h^1(L(r-1)) \) for all \( n \geq r \). \( \square \)

Remark 3.10. In [8, Corollary 14], Ito proved that if \( p_g(A) = 1 \), then \( m^3 = qm^2 \), where \( q \) is a minimal reduction of the maximal ideal \( m \). This fact is also obtained as follows. If \( p_g(A) = 1 \), then \( A \) is elliptic (e.g. [23, p. 425]). Therefore, \( \overline{m^2} = qm^2 \) by Theorem 3.3. Suppose that \( m = I_Z \) and \( m^2 \neq qm \). Then \( m \) is not a \( p_g \)-ideal by Proposition 2.7 (2), namely, \( h^1(O_X(-Z)) = 0 \). From the exact sequence (3.2) with \( n = 1 \), we have \( q_A(\overline{m^2}/qm) = 1 \). Since \( m^2 \neq qm \), we obtain \( \overline{m^2} = m^2 \). Hence the following ideals coincide:

\[
qm^2 = qm^2 \subset m^3 \subset \overline{m^2}.
\]

Lemma 3.11. Assume that \( A \) is an elliptic singularity, \( O_X(-Z) \) has no fixed component, and \( ZE_{\text{min}} = 0 \), where \( E_{\text{min}} \) is the minimally elliptic cycle. Let \( B \) be the maximal reduced connected cycle such that \( ZB = 0 \) and \( \text{Supp}(E_{\text{min}}) \subset B \). Then \( h^1(O_X(-Z)) = h(B) \).

Proof. Let \( \{Z_{B_0}, \ldots, Z_{B_m}\} \) be the elliptic sequence on \( B_0 = B \) and let \( C = \bigcup_{i=0}^m Z_{B_i} \). By Proposition 2.4 \( C \) is anti-nef on \( B \) and \( \chi(C) = 0 \). Suppose \( E_i \nsubseteq B \) and \( E_i \cap B \neq \emptyset \). By Proposition 2.3 (2), we have that \( CE_i \leq C \) and that the cohomological cycle on \( Z^+ \) has support in \( B \), so \( h(B) = h(Z^+) \).

Since \( ZE_i < 0 \) by the definition of \( B \), it follows that \( Z + C \) is anti-nef on \( E \). By Lemma 2.7 we have \( H^1(O_X(-Z - C)) = 0 \). Therefore, by Remark 3.5 \( h^1(O_X(-Z)) = h^1(O_C(-Z)) = h^1(O_C) \leq h(B) \). On the other hand, by Lemma 3.6 (1)–(3), we have \( h^1(O_X(-Z)) \geq h^1(O_X(-n_0Z)) = h(B) \). \( \square \)

Proof of Theorem 3.3. By Lemma 3.11 for any integrally closed \( m \)-primary ideal \( I \) represented by \( Z \), we have \( q(I_{nZ}) = q(I_Z) \) for \( n \geq 1 \). By Corollary 3.9 we obtain \( \bar{r}(A) \leq 2 \). \( \square \)

The invariant \( q \) is a function on the set of integrally closed \( m \)-primary ideals in \( A \). So we define a set \( \text{Im}_A(q) \subset Z \) by

\[
\text{Im}_A(q) = \{ q(I) \mid I \subset A \text{ is an integrally closed } m\text{-primary ideal} \}.
\]

By Proposition 2.9 we have

\[
\text{Im}_A(q) \subset \{0, 1, \ldots, p_g(A)\}.
\]
Proposition 3.12. If \( r(A) = 2 \), then \( \Im A(q) = \{0, 1, \ldots, p_g(A)\} \).

Proof. Let \( Z > 0 \) be a cycle on \( X \) such that \( O_X(-Z) \) is generated by global sections and \( q(I_Z) = 0 \) (e.g. [17, 4.5]). Take a general element \( h \in I_Z \) (see Assumption 1.1) and \( H := \text{div}_{\text{Spec} A(h)} \). Let \( X_0 = X \) and let \( \phi_i : X_i \to X_{i-1} \) be the blowing-up at a point in the intersection of \( \text{Supp}(C_{X_{i-1}}) \) and the strict transform of \( H \) on \( X_{i-1} \). Let \( F_i \) denote the exceptional set of \( \phi_i \) and \( Z_i := \phi_i^* Z_{i-1} + F_i \), where \( Z_0 = Z \). By Proposition 2.13 and Proposition 2.14 the sequence of blowing-ups \( \{\phi_i\} \) ends in a finite number of steps. If \( \phi_n \) is the last one, then \( Z_n \) is a \( p_g \)-cycle. From the exact sequence

\[
0 \to O_{X_i}(-Z_i) \to O_{X_i}(-\phi_i^* Z_{i-1}) \to O_{F_i} \to 0,
\]

we obtain that

\[
0 \leq h^1(O_{X_i}(-Z_i)) - h^1(O_{X_{i-1}}(-Z_{i-1})) \leq 1.
\]

Therefore, there exists a sequence \( \{i_0, \ldots, i_{p_g(A)}\} \subset \{0, 1, \ldots, n\} \) such that \( h^1(O_{X_{i_k}}(-Z_{i_k})) = k \). By the definition of the cycle \( Z_i \), \( O_X(-Z_i) \) has no fixed component. Therefore, for each \( i \), \( h^1(O_X(-nZ_i)) \) is stable for \( n \geq 1 \) since \( n_0(Z_i) \leq 1 \). By Lemma 3.11 (4), \( O_{X_{i_k}}(-2Z_{i_k}) \) is generated by global sections and thus \( q(I_{2Z_{i_k}}) = k \) by the proof of Theorem 3.3.

Theorem 3.3 and Proposition 3.12 implies the following.

Corollary 3.13. If \( A \) is an elliptic singularity, then

\[
\Im A(q) = \{0, 1, \ldots, p_g(A)\}.
\]

Remark 3.14. Assume that \( A \) is an elliptic singularity and \( Z > 0 \) is a \( p_g \)-cycle. Let \( B \) be the maximal reduced connected cycle such that \( ZB = 0 \) and \( \text{Supp}(E_{\text{min}}) \subset B \) and let \( \{Z_{B_0}, \ldots, Z_{B_m}\} \) be the elliptic sequence on \( B_0 = B \). Let \( Z_{B_{-1}} = Z \). Then it follows from Lemma 3.11 that \( h^1(O_X(-Z_{B_{i-1}})) = h^1(B_i) = p_g(A) - i \) for \( 0 \leq i \leq m \). In particular, \( p_g(A) = m + 1 \). Therefore \( \Im A(q) = \{h^1(B_i) \mid i = 0, 1, \ldots, m\} \cup \{0\} \).

The property 3.3 does not imply that \( A \) is an elliptic singularity. In fact, we have the following.

Example 3.15 (cf. [17, Example 4.6]). Let \( C \) be a nonsingular curve of genus \( g = 2 \) and put

\[
R = \bigoplus_{n \geq 0} H^0(O_C(nK_C)).
\]

Suppose that \( A \) is the localization of \( R \) at \( R_+ = \bigoplus_{n \geq 1} H^0(O_C(nK_C)) \) and let \( f : X \to \text{Spec} A \) be the minimal resolution. Then \( p_g(A) = 3 \), \( E \cong C \), \( O_E(-E) \cong O_E(K_E) \), \( -E^2 = 2 \), \( K_X = -2E = -C_X \), and \( O_X(-E) \) is generated by global sections. In particular, \( m = I_E \). It follows that \( H^1(O_X(-2E)) = 0 \) by the Grauert-Riemenschneider vanishing theorem.
We show that $\text{Im}_A(q) = \{0, 1, 2, 3\}$. From the exact sequence
\[0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0,\]
we have $h^1(\mathcal{O}_X(-E)) = p_g(A) - 2 = 1$. Hence $1 = q(m) \in \text{Im}_A(q)$. Let $h \in m$ be a general element and suppose $\text{div}_X(h) = E + H_1 + H_2$. Let $\phi : X' \to X$ be the blowing-up at $E \cap (H_1 \cup H_2)$, and let $E_i = \phi^{-1}(E \cap H_i)$ and $Z = \phi^*E + E_1 + E_2$. If $E_0$ denote the strict transform of $E$, then $\mathcal{O}_{E_0}(-Z) \cong \mathcal{O}_{E_0}$ (cf. Remark 3.5), and hence $h^1(\mathcal{O}_{X'}(-nZ)) \geq h^1(\mathcal{O}_{E_0}) = 2$ for $n \geq 1$. Since $C_{X'} = \phi^*(2E) - E_1 - E_2$ by Proposition 2.13 we have $ZC_{X'} = -2$. By Proposition 2.14, $h^1(\mathcal{O}_{X'}(-nZ)) \neq 3$. Hence $h^1(\mathcal{O}_{X'}(-nZ)) = 2$ for $n \geq 1$. By Lemma 3.6 (4), $\mathcal{O}_{X'}(-2Z)$ is generated by global sections and $2 = q(I_{2Z}) \in \text{Im}_A(q)$.

**Problem 3.16.** For any normal surface singularity $(A, m)$, does the equality $\text{Im}_A(q) = \{0, 1, \ldots, p_g(A)\}$ holds?

4. **When is the maximal ideal a $p_g$-ideal?**

From Example 3.15 we see that in general the maximal ideal is not a $p_g$-ideal. It is natural to ask for a characterization of normal surface singularities $(A, m)$ with $q(m) = p_g(A)$. In [13 Example 4.3], it is shown that for a complete Gorenstein local ring $A$ with $p_g(A) > 0$, $m$ is a $p_g$-ideal if and only if $A \cong k[[x, y, z]]/(x^2 + g(y, z))$, where $k$ is the residue field of $A$ and $g \in (y, z)^3 \setminus (y, z)^4$. In this section, we give a geometric characterization of such singularities. So we work on the resolution space. We assume that $p_g(A) > 0$.

Let us recall that for a function $h \in m$, which has zero of order $a_i$ at $E_i$, $(h)_E$ denotes a cycle such that $(h)_E = \sum a_iE_i$.

**Definition 4.1.** The **maximal ideal cycle** on $X$ is the minimum of $\{(h)_E \mid h \in m\}$.

A cycle $M > 0$ on $X$ is the maximal ideal cycle if and only if $\mathcal{O}_X(-M)$ has no fixed component and $m = H^0(X, \mathcal{O}_X(-M))$.

**Lemma 4.2.** Let $M$ be the maximal ideal cycle on $X$. Then $m$ is a $p_g$-ideal represented by $M$ if and only if $p_g(M) = 0$.

**Proof.** From the exact sequence
\[0 \to \mathcal{O}_Y(-M) \to \mathcal{O}_Y \to \mathcal{O}_M \to 0,\]
we have $p_g(M) = p_g(A) - h^1(\mathcal{O}_X(-M))$. Since $\mathcal{O}_X(-M)$ has no fixed component, the assertion follows from Proposition 2.9.

The following theorem is proved by Tomari (see [23 Corollary 3.12 and Theorem 4.3]). Let us give a proof from our point of view.
Theorem 4.3 (Tomari). Let $M$ be the maximal ideal cycle on $X$ and $f': X' \to \text{Spec } A$ be the blowing-up by $m$. Then $p_a(M) = 0$ if and only if the following three conditions are satisfied.

1. $\text{embdim } A = \text{mult } A + 1$.
2. $X'$ is normal.
3. $\mathcal{O}_X(-M)$ is generated by global sections.

Proof. Assume that $p_a(M) = 0$. By Lemma 4.2, $m$ is a $p_g$-ideal and $\mathcal{O}_X(-M)$ is generated by global sections. By [17, 6.2], (1) holds. Proposition 2.15 implies (2).

Conversely assume that the conditions (1)–(3) are satisfied. By (1) and Goto–Shimoda [5, 1.1 and 1.4], $G := \bigoplus_{n \geq 0} m^n/m^{n+1}$ is a Cohen-Macaulay ring with $a(G) < 0$, where $a(G)$ denote the $a$-invariant of Goto–Watanabe [4]. Then $h^1(\mathcal{O}_{X'}) = 0$ by [24 (1.18)]. By (2) and (3), $X'$ is obtained by contracting the cycle $M^\perp$ on $X$, and there exists the following exact sequence:

$$0 \to H^1(\mathcal{O}_{X'}) \to H^1(\mathcal{O}_X) \to H^0(R^1\phi_*\mathcal{O}_X) \to 0.$$ 

This shows that $p_g(A) = \ell_A(R^1\phi_*\mathcal{O}_X) = h(M^\perp)$. Since $h(M^\perp) \leq h^1(\mathcal{O}_X(-M))$ by Lemma 3.6, we obtain $h^1(\mathcal{O}_X(-M)) = p_g(A)$.

Corollary 4.4. If $A$ is Gorenstein and $m$ is a $p_g$-ideal, then $\text{mult } A = 2$.

Proof. It follows from Lemma 4.2 and Theorem 4.3 that $\text{embdim } A = \text{mult } A + 1$. Since $A$ is Gorenstein, $\text{mult } A = 2$ by [22, 3.1].

Remark 4.5. If $m$ is a $p_g$-ideal, then for any general element $h \in m$, $\text{Spec } A/(h)$ is a partition curve (see [2, §3]), because $\delta(A/(h)) = \text{embdim } A/(h) - 1$ by the formula of Morales [13, 2.1.4]. Note that if $m$ is represented on a resolution $X$, the strict transform of $\text{div } \text{Spec } A(h)$ on $X$ is nonsingular by Assumption 1.1.

Definition 4.6. A normal surface singularity $A$ is said to be numerically Gorenstein if $Z_{K_X} \in \sum_i ZE_i$. The definition is independent of the choice of the resolution.

It is known that $(A, m)$ is Gorenstein if and only if $(A, m)$ is numerically Gorenstein and $-K_X \sim Z_{K_X}$.

Definition 4.7 (Yau [28, §3]). Assume that $A$ is elliptic and numerically Gorenstein. Let $Z_0 \geq \cdots \geq Z_m$ be the elliptic sequence on $E$. Then $p_g(A) \leq m + 1$. If $p_g(A) = m + 1$, $A$ is called a maximally elliptic singularity.

Theorem 4.8 (Yau [28, Theorem 3.11]). A maximally elliptic singularity is Gorenstein.

Let $Z_E$ be the fundamental cycle. The number $-Z_E^2 > 0$ is called the degree of $A$. It is known that the degree is independent of the choice of the resolution.
The following result (even more general results) can be recovered from 2.15, 3.10 and 5.10 of [19] (cf. [15]). However we put a proof for readers’ convenience.

**Lemma 4.9.** Assume that $A$ is a numerically Gorenstein elliptic singularity and that $X \to \text{Spec} A$ is the minimal resolution. Moreover, assume that $-Z_E^2 = 1$. Then we have the following.

1. Let $E_{\min}$ be the minimally elliptic cycle. Then $E$ can be expressed as $E = \text{Supp}(E_{\min}) \cup (\bigcup_{i=0}^{m-1} E_i)$ with the following dual graph:

   ![Graph](image)

   Note that $E_{\min} E_{m-1} = 1$ by Proposition [2.4](2).

2. $A$ is Gorenstein and $Z_E$ coincides with the maximal ideal cycle if and only if $A$ is a maximally elliptic singularity.

**Proof.** (1) follows from Corollary 2.3 and Table 1 in [27]. We prove (2).

Let $Z_0 = \cdots = Z_m$ be the elliptic sequence on $E$. Then $p_g(A) \leq m + 1$. It is easy to see that $Z_i = E_{\min} + E_{m-1} + \cdots + E_i$. Let $C'_j := \sum_{i=j}^{m} Z_i$. Note that $O_{C'_{j+1}}(-Z_j) = O_{C'_{j+1}}(-Z_i)$ for $l \leq j$.

Assume that $A$ is Gorenstein and $Z_0 = Z_E$ is the maximal ideal cycle. By Remark 4.4 we have $O_{C'_{j+1}}(-C_j) \cong O_{C'_{j+2}}$ for $0 \leq j \leq m - 1$. It follows from Grauert-Riemenschneider vanishing theorem (or Lemma [2.7](2.13) and [19] Lemma 2.13) that $h^1(O_X(-Z_0)) = h^1(O_X(-C_m)) + m = m$. As in the proof of Lemma 4.2 we obtain $p_g(A) = h^1(O_X(-Z_0)) + 1 = m + 1$.

Conversely, assume that $A$ is a maximally elliptic singularity. Then $A$ is Gorenstein by Theorem 3.8 and $h^1(O_X(-Z_0)) = m$. By Proposition [2.4](2) we easily see that $Z_j$ is 1-connected (cf. [20] 3.9) for $0 \leq j \leq m$. Since $\chi(O_{Z_{j+1}}(-C_j)) = \chi(Z_{j+1}) - C_j Z_{j+1} = 0$, we have

$$h^1(O_{Z_{j+1}}(-C_j)) = h^0(O_{Z_{j+1}}(-C_j)) \leq 1$$

by [20] 3.11. From the exact sequence

$$0 \to O_X(-C_{j+1}) \to O_X(-C_j) \to O_{Z_{j+1}}(-C_j) \to 0,$$

we obtain that $0 \leq h^1(O_X(-C_j)) - h^1(O_X(-C_{j+1})) \leq 1$ for $0 \leq j \leq m - 1$. Thus $h^1(O_X(-C_j)) = h^1(O_X(-C_{j+1})) + 1$ for $0 \leq j \leq m - 1$. Therefore, by [19] Lemma 2.13 again, there exists $h \in H^0(O_X(-Z_0))$ which maps to the generator of $H^0(O_{Z_i}(-Z_0)) \cong H^0(O_{Z_1})$. Then the cycles $(h)_E$ and $Z_0$ coincide on $\text{Supp}(Z_1)$. Since $(h)_E$ is anti-nef, we must have $(h)_E = Z_0$. This shows that $Z_0$ is the maximal ideal cycle.

**Theorem 4.10.** Assume that $A$ is not a rational singularity, namely, $p_g(A) > 0$. Then the singularity $A$ is Gorenstein and $m$ is a $p_g$-ideal if and only if $A$ is
a maximally elliptic singularity with $-Z^2_E = 1$, where $Z_E$ is the fundamental cycle on $E$.

Proof. Let $Y \to \text{Spec } A$ be the resolution which is obtained by taking the minimal resolution of the blowing-up of $m$, and let $M$ be the maximal ideal cycle on $Y$. Let $X_0 \to \text{Spec } A$ be the minimal resolution and $\phi : Y \to X_0$ the natural morphism.

Assume that $A$ is Gorenstein and $m$ is a $p_g$-ideal. By Corollary 4.4 $\text{mult } A = -M^2 = 2$. Since $A$ is Gorenstein, there does not exist a $p_g$-cycle on the minimal resolution $X_0$ by Proposition 2.14. Thus $\phi : Y \to X_0$ is not an isomorphism. Let $N = \phi_\ast M$; this is also the maximal ideal cycle on $X_0$. Since $N$ is not a $p_g$-cycle, $m$ is not represented by $N$, namely, $O_{X_0}(-N)$ is not generated by global sections. Therefore $-N^2 \leq \text{mult } A = -M^2 = 2$. This implies that $-N^2 = 1$, and that $\phi$ is the blowing-up at the unique base point of $O_{X_0}(-N)$ and $M = \phi_\ast N + E_0$, where $E_0$ is the exceptional cycle on $Y$. Let $Z_0$ be the fundamental cycle on $X_0$. Since $Z_0 \leq N$ and $0 < -Z^2_0 \leq -N^2 = 1$, we have $Z_0 = N$, namely, $N$ is the fundamental cycle. Since $p_0(M) = (M^2 + K_Y M)/2 + 1 = 0$ by Lemma 4.12 and $K_Y M = (\phi^\ast K_{X_0} + E_0)(\phi^\ast N + E_0) = K_{X_0}N - 1$, we obtain that $K_{X_0}N = 1$. Thus $p_0(N) = (N^2 + K_{X_0}N)/2 + 1 = 1$. Hence $A$ is an elliptic singularity. If $g \geq 1$, $\text{there exists } h \in H^0(O_{X_0}(-Z_0))$ such that $\text{div}_{X_0}(h) = Z_0 + H$, where $H$ has no component of $E$. Since $-Z^2_0 = 1$, we have $HZ_0 = 1$ and that $O_{X_0}(-Z_0)$ has just one base point on $\text{Supp}(Z_0) \setminus \text{Supp}(Z_1)$ which is resolved by the blowing-up at this point (cf. [19, 4.5]). Then $M = \phi^\ast Z_0 + E_0$ and $C_Y = \phi^\ast (\sum_{i=0}^n Z_i) - E_0$ since $K_{X_0} = -\sum_{i=0}^n Z_i$ ([28 Theorem 3.7, 23. 1.8]). Since $Z_0 - Z_1$ is reduced (cf. Lemma 4.9), we have $E_0 \not\subseteq C_Y$ and thus $\text{COHOMOLOGY OF IDEALS IN SURFACE SINGULARITIES 13}$

Conversely, assume that $A$ is a maximally elliptic singularity with $-Z^2_E = 1$. Then $A$ is Gorenstein and $Z_0$ is the maximal ideal cycle by Lemma 4.4. There exists $h \in H^0(O_{X_0}(-Z_0))$ such that $\text{div}_{X_0}(h) = Z_0 + H$, where $H$ has no component of $E$. Since $-Z^2_0 = 1$, we have $HZ_0 = 1$ and that $O_{X_0}(-Z_0)$ has just one base point on $\text{Supp}(Z_0) \setminus \text{Supp}(Z_1)$ which is resolved by the blowing-up at this point (cf. [19, 4.5]). Then $M = \phi^\ast Z_0 + E_0$ and $C_Y = \phi^\ast (\sum_{i=0}^n Z_i) - E_0$ since $K_{X_0} = -\sum_{i=0}^n Z_i$ ([28 Theorem 3.7, 23. 6.8]). Since $Z_0 - Z_1$ is reduced (cf. Lemma 4.9), we have $E_0 \not\subseteq C_Y$ and thus $\text{COHOMOLOGY OF IDEALS IN SURFACE SINGULARITIES 13}$

Example 4.11 (Laufer [11 V, cf. [15 2.23]). Let $A_1 = \mathbb{C}\{x, y, z\}/(x^2 + y^3 + z^{18})$ and $A_2 = \mathbb{C}\{x, y, z\}/(z^2 - y(x^4 + y^3))$. Then the exceptional set $E$ of the minimal resolution $X$ of both these singularities consists of an elliptic curve $E_2$ and $-2$-curves $E_0$ and $E_1$, and $E = E_2 + E_1 + E_0$ is a chain of curves such that $E_2 E_1 = E_1 E_0 = 1$ (the dual graph of $E$ is similar to that in Lemma 4.9). We have $p_g(A_1) = 3$ and $p_g(A_2) = 2$. So $A_1$ is a maximally elliptic singularity. For $A_2$, we have that the maximal ideal cycle on $X$ is $M = 2E_2 + 2E_1 + E_0$, $O_X(-M)$ is generated by global sections since $\text{mult } A_2 = 2 = -M^2$ (cf. [20 4.6]), and $h^1(O_X(-M)) = 1 = p_g(A_2) - 1$ (cf. Lemma 3.11).
Example 4.12. By [19, 4.5, 6.3], for any positive integer $m$, there exists a numerically Gorenstein elliptic singularity $A$ with elliptic sequence $\{Z_0, \ldots, Z_m\}$ on the minimal resolution $X$ such that $-Z_0^2 = 1$, $C_X = Z_1 + \cdots + Z_m$, $p_g(A) = m$, $M_X = Z_0 + Z_1$, $\text{embdim} A - 1 = \text{mult} A = -M_X^2 + 1 = 3$, where $M_X$ denotes the maximal ideal cycle on $X$. This singularity is not $\mathbb{Q}$-Gorenstein by [19, 6.1]. We claim that $m$ is a $p_g$-ideal. The base point of $O_X(-M_X)$ is a nonsingular point of $C_X$, which is a point in $\text{Supp}(Z_1) \setminus \text{Supp}(Z_2)$ by [19, 3.1]. Let $\phi: Y \to X$ be the blowing-up at the base point of $O_X(-M_X)$ and $F$ the exceptional set of $\phi$. Then the maximal ideal cycle $M_Y$ on $Y$ is $\phi^*M_X + F$, and the cohomological cycle on $Y$ is $C_Y = \phi^*C_X - F$. Since $M_Y C_Y = M_X C_X - F^2 = Z_1^2 - F^2 = 0$, $M_Y$ is a $p_g$-cycle.

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