Asymptotic analysis of parameter estimation for the Ewens–Pitman Partition

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May 16, 2023

Abstract

We derive the exact asymptotic distribution of the maximum likelihood estimator (ˆαn, ˆθn) of (α, θ) for the Ewens–Pitman partition in the regime of 0 < α < 1 and θ > −α: we show that ˆαn is nα/2-consistent and converges to a variance mixture of normal distributions, i.e., ˆαn is asymptotically mixed normal, while ˆθn is not consistent and converges to a transformation of the generalized Mittag-Leffler distribution. As an application, we derive a confidence interval of α and propose a hypothesis testing of sparsity for network data. In our proof, we define an empirical measure induced by the Ewens–Pitman partition and prove a suitable convergence of the measure in some test functions, aiming to derive asymptotic behavior of the log likelihood.

Keyword: Ewens–Pitman partition, Asymptotic mixed normality, Mittag-Leffler distribution, Pitman–Yor process

1 Introduction

For a given positive integer n ∈ N, a partition of [n] := {1, 2, . . . , n} into k blocks is an unordered collection of nonempty disjoint sets whose union is [n]. Let Pn k be the set of all partitions of n into k blocks, and denote ∪k=1 Pn k by Pn, i.e., Pn is the set of all partitions of [n]. We denote the elements in Pn by

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Figure 1: Asymptotic behavior of the Ewens–Pitman partition when $0 < \alpha < 1, \theta > -\alpha$.

\{U_i : i \geq 1\}. Then, the Ewens–Pitman partition with parameter $(\alpha, \theta)$ is the distribution on $\mathcal{P}_n$ with the following density:

$$P_{\alpha\theta}(\{U_i : i \geq 1\}) = \frac{\prod_{i=1}^{K_n-1} (\theta + i\alpha)}{\prod_{i=1}^{n-1} (\theta + i)} \prod_{j=2}^{n} \left\{ \prod_{i=1}^{j-1} (-\alpha + i) \right\}^{S_{n,j}}$$

with $S_{n,j} := \sum_{i \geq 1} \mathbb{1}\{|U_i| = j\}$ and $K_n = \sum_{j \geq 1} S_{n,j}$.

Note that $S_{n,j}$ is the number of blocks with size $j$, and $K_n = \sum_{j \geq 1} S_{n,j}$ is the number of nonempty blocks. We can observe from (1.1) that $(S_{n,j})_{j=1}^{n}$ is a sufficient statistic. Now, we can consider the parameter spaces of $(\alpha, \theta)$: \{\alpha = 0, \theta > 0\}, \{\alpha < 0, \exists k \in \mathbb{N} \text{ such that } \theta = -k\alpha\} and \{0 < \alpha < 1, \theta > -\alpha\}. This paper focuses on the regime \{0 < \alpha < 1, \theta > -\alpha\}, where $K_n$ and $S_{n,j}$ have the following nonstandard asymptotics as $n \to \infty$:

$$\frac{K_n}{n^\alpha} \xrightarrow{a.s.} M_{\alpha\theta} \text{ (a positive random variable)}$$

$$\forall j \in \mathbb{N}, \frac{S_{n,j}}{K_n} \xrightarrow{a.s.} p_\alpha(j) := \frac{\alpha \prod_{i=1}^{j-1} (i - \alpha)}{j!}.$$ (1.2)

See Figure 1 for illustrations of these asymptotics. Here, we emphasize that the limit $M_{\alpha\theta}$ of $K_n/n^\alpha$ is not a constant but a nondegenerate positive random variable, and $p_\alpha(j)$ is a probability mass function of a heavy-tailed distribution on $\mathbb{N}$ (see Section 2.1 for details).

Owing to the nonstandard asymptotics, the Ewens–Pitman partition can be observed in several fields such as ecology Balocchi et al. [2022], Favaro and Naulet [2021], Favaro et al. [2009], Sibuya [2014], nonparametric Bayesian inference Caron et al. [2017], Dahl et al. [2017], disclosure risk assessment Favaro et al. [2021], Hoshino [2001], and network analysis Crane and Dempsey [2018], Naulet et al. [2021], as well as forensic fields Cereda et al. [2022]. In related studies, the estimation of $\alpha$ is of more interest than the estimation of $\theta$ because $\alpha$ controls the asymptotic behavior, mainly, as illustrated in Figure 1. Notice that the naive estimator $\hat{\alpha}_{naive}^n := \log K_n / \log n$ is log $n$-consistent by $n^{-\alpha} K_n = O_p(1)$, but it is not rate-optimal, owing to information loss from the sufficient statistic.
For the maximum likelihood estimator \( (\hat{\alpha}_n, \hat{\theta}_n) \), Favaro and Naulet [2021] show that it is given by the form of \( \sqrt{V_n}(\hat{\alpha}_n - \hat{\alpha}_{n,0}) \to N(0, 1) \), where \( V_n \) involves unknown quantities, and \( \hat{\alpha}_n \) is centered at \( \hat{\alpha}_{n,0} \) whose variance could be larger than \( V_n^{-1} \). Therefore, we cannot construct confidence intervals of \( \alpha \) from their results.

In this paper, we derive the exact asymptotics of the MLE \( (\hat{\alpha}_n, \hat{\theta}_n) \), which leads to a valid confidence interval for \( \alpha \). Furthermore, we apply the confidence interval to a hypothesis testing of sparsity for network data.

Before introducing the main result, we discuss an asymptotic analysis of the Fisher information to acquire insights into parameter estimation. Here, we denote the limit \( \lim_{n \to \infty} n^{-\alpha} K_n \) by \( M_{\alpha\theta} \) and define \( I_{\alpha} \) by

\[
\forall \alpha \in (0, 1), \ I_{\alpha} := -\sum_{j=1}^{\infty} p_{\alpha}(j) \log p_{\alpha}(j) \text{ with } p_{\alpha}(j) := \frac{\alpha \prod_{i=1}^{j-1} (i - \alpha)}{j!},
\]

(1.4)

i.e., \( I_{\alpha} \) is the Fisher information of the discrete distribution with a density \( p_{\alpha}(j) \).

Recall that \( M_{\alpha\theta} \) is a nondegenerate random variable, and \( p_{\alpha}(j) \) is the almost sure limit of \( S_{n,j}/K_n \) (see (1.2) and (1.3)). We claim that the leading terms of the Fisher information \( \{I_{\alpha\alpha}^{(n)}, I_{\theta\alpha}^{(n)}, I_{\theta\theta}^{(n)}\} \) for the Ewens–Pitman partition \((\alpha, \theta)\) are given as

\[
I_{\alpha\alpha}^{(n)} \sim n^{\alpha} E[M_{\alpha\theta}] I_{\alpha}, \quad I_{\theta\alpha}^{(n)} \sim \alpha^{-1} \log n, \quad I_{\theta\theta}^{(n)} \to \alpha^{-2} \log f'_{\alpha}(\theta/\alpha) < +\infty,
\]

(1.5)

where \( f'_{\alpha} \) is the derivative of a function \( f_{\alpha} : (-1, \infty) \to \infty \) that is strictly increasing, concave, and bijective. \( (1.5) \) implies the absence of identifiability of \( \theta \), and the optimal convergence rate of estimators of \( \alpha \) is at most \( n^{-\alpha/2} \), which is slow when \( \alpha \in (0, 1) \) is close to 0. Furthermore, \( (\alpha, \theta) \) are asymptotically orthogonal, i.e., information regarding \( \theta \) has less effect on the estimation of \( \alpha \) as \( n \) increases (see Figure 2).

Based on the above intuition, we claim our main theorem: the asymptotic distribution of the MLE \( (\hat{\alpha}_n, \hat{\theta}_n) \) is given as

\[
n^{\alpha/2}(\hat{\alpha}_n - \alpha) \to N/\sqrt{\alpha M_{\alpha\theta}} (\mathcal{F}_\infty\text{-stable}),
\]

(1.6)

\[
\hat{\theta}_n \to \alpha \cdot \log f'_{\alpha}(\log M_{\alpha\theta}),
\]

(1.7)

where \( N \sim N(0, 1) \) is independent of \( \mathcal{F}_\infty = \sigma(\cup_{n=1}^{\infty} \mathcal{F}_n) \) with \( \mathcal{F}_n \) being the \( \sigma \)-field generated by the partitions of \( [n] \), and \( M_{\alpha\theta} = \lim_{n \to \infty} n^{-\alpha} K_n \) is an \( \mathcal{F}_\infty \)-measurable nondegenerate random variable. Here, \( \mathcal{F}_\infty\text{-stable} \) convergence in (1.6) is a stronger notion of stochastic convergence than weak convergence (see Section 2.2). Interestingly, \( (1.5) \) and (1.6) imply that the error of \( \hat{\alpha}_n \) normalized by the square root of the Fisher information \( I_{\alpha\alpha}^{(n)} \) converges to the variance
mixtures of normals:
\[
\sqrt{I_{\alpha\alpha}(\hat{\alpha}_n - \alpha)} = \sqrt{\frac{I(\alpha)}{n^\alpha} \cdot \frac{n^\alpha}{2}(\hat{\alpha}_n - \alpha)} \to \sqrt{\mathbb{E}[M_{\alpha\theta}]} / M_{\alpha\theta} \cdot N.
\]

This type of asymptotics is referred to as \textit{asymptotic mixed normality}, which is often observed in “nonergodic” or “explosive” stochastic processes (c.f. Häusler and Luschgy [2015]). By contrast, the error normalized by the random statistic \[
\sqrt{K_n I_{\alpha}}
\]
converges to the standard normal:
\[
\sqrt{K_n I_{\alpha}}(\hat{\alpha}_n - \alpha) = \sqrt{K_n / n^\alpha \cdot \frac{n^\alpha}{2}(\hat{\alpha}_n - \alpha)} \overset{(*)}{\to} \sqrt{\mathbb{E}[M_{\alpha\theta}]} / \sqrt{M_{\alpha\theta}} = N,
\]
where \((*)\) follows from Slutsky’s lemma with \(M_{\alpha\theta} = \lim_{n \to \infty} K_n / n^\alpha\) and (1.6).

The above discussion implies that \(K_n\) (the number of blocks) corresponds to the sample size in typical parametric independent and identically distributed (i.i.d.) cases, and \(I_{\alpha}\), which is defined by (1.4), quantifies information regarding \(\alpha\) per block. Evidently, \([\hat{\alpha}_n \pm 1.96 / \sqrt{K_n I_{\alpha}}]\) is a 95% confidence interval. Using this confidence interval, we will propose a hypothesis testing of sparsity for network data.

The asymptotic law of \(\hat{\theta}_n\) itself is interesting. We demonstrate that \(\hat{\theta}_n\) has a positive bias, while we confirm via numerical simulations that the limit law \(\alpha f^{-1}_\theta(\log M_{\alpha\theta})\) converges to the normal distribution \(\mathcal{N}(\theta, \alpha^2 / f^{-1}_\theta(\theta/\alpha))\) as \(\alpha \to 0\) or \(\theta \to \infty\), where the variance \(\alpha^2 / f^{-1}_\theta(\theta/\alpha)\) of the normal is the inverse of the Fisher information regarding \(\theta\) given by (1.5). These observations imply that \(\hat{\theta}_n\) behaves like the MLE in typical i.i.d. cases in the limit of \(\alpha \to 0\) or \(\theta \to \infty\).

In our proof, we define an empirical measure induced by the Ewens–Pitman partition and demonstrate its convergence in some test functions. This convergence proves the asymptotic behavior of log-likelihood (see Section 5).

The remainder of this paper is organized as follows: Section 2 reviews the parameter dependency of the Ewens–Pitman partition and introduces stable convergence. In Section 3, we introduce the main theorem and its application. Numerical simulations supporting the main theorem follow in Section 4. In Section 5, we present the proof strategy. Section 6 compares our results with those reported in related papers. All proofs have been provided in the appendix.
2 Notation and preliminaries

2.1 Parameter dependency of the Ewens–Pitman partition

In Section 1, we introduced the Ewens–Pitman partition as a model on the set of partitions $\mathcal{P}_n$. Now, we redefine it as a stochastic process over $(\mathcal{P}_n)_{n \geq 1}$ that randomly assigns integers (balls) into blocks (urns) in the following sequential manners: The first ball belongs to urn $U_1$ with probability one. Now, we assume that $n(\geq 1)$ balls are partitioned into $K_n$ occupied urns $\{U_1, U_2, \ldots, U_{K_n}\}$, and we let $|U_i|$ be the number of balls in $U_i$. Then, the $(n + 1)$ th ball is assigned as

$$(n + 1)\text{th ball} \in \begin{cases} U_i & \text{with prob. } \frac{|U_i| - \alpha}{\theta + K_n \alpha} (\forall i = 1, 2, \ldots, K_n) \\ \text{A new urn} & \text{with prob. } \frac{\alpha \theta}{\theta + K_n \alpha}. \end{cases}$$

It easily follows that the probability of obtaining the partition $\{U_1, U_2, \ldots, U_{K_n}\}$ of $[n]$ coincides with the likelihood formula (1.1).

As mentioned in Section 1, the Ewens–Pitman partition has three parameter spaces: (i) $\alpha = 0, \theta > 0$, (ii) $\alpha < 0, \exists k \in \mathbb{N}$ s.t. $\theta = -k \alpha$, and (iii) $0 < \alpha < 1, \theta > -\alpha$. We briefly explain the parameter dependency below.

(i) $\alpha = 0, \theta > 0$: This is referred to as the Ewens partition or the standard Chinese restaurant process. By substituting $\alpha = 0$ into the likelihood formula (1.1), we observe that the likelihood is proportional to $\frac{\theta K_n}{\prod_{i=0}^{n-1} (\theta + i)}$, and hence, $K_n$ is a sufficient statistic for $\theta$. Here, we define $\theta^*_n := K_n / \log n$. Then, we claim that $\theta^*_n$ is $\sqrt{\log n}$-consistent and asymptotically normal:

$$(\theta^*_n - \theta) \rightarrow N(0, 1).$$

This result follows based on the following arguments: By the sequential definition, $K_n$ can be expressed as independent sum of the Bernoulli random variables

$$K_n = \sum_{i=1}^{n} \zeta_i, \quad \zeta_i \sim \text{Bernoulli} \left( \frac{\theta}{\theta + i - 1} \right)$$

Then, the Lindeberg-Feller theorem (c.f.[Durrett, 2019, p. 128]) leads to the asymptotic normality. Furthermore, $\theta^*_n$ is asymptotically efficient because the leading term of the Fisher information $I^{(n)}_{\theta\theta}$ is given by

$$I^{(n)}_{\theta\theta} = \frac{\mathbb{E}[K_n]}{\theta^2} - \sum_{i=0}^{n-1} \frac{1}{(\theta + i)^2} = \frac{1}{\theta^2} \sum_{i=1}^{n} \frac{\theta}{\theta + i - 1} - \sum_{i=0}^{n-1} \frac{1}{(\theta + i)^2} \sim \log n / \theta.$$

Note that the MLE for $\theta$ is also asymptotically efficient in this regime. Readers may refer to [Carlton, 1999, p. 80] for the MLE of $\theta$ and Tsukuda [2017] for the case where $\theta$ diverges as $n \rightarrow \infty$.

(ii) $\alpha < 0, \exists k \in \mathbb{N}$ s.t. $\theta = -k \alpha$: In this case, the number of occupied urns $K_n$ is finite, i.e., $K_n \rightarrow k$ (a.s.), as the probability of observing a new urn is proportional to $(-\alpha)(k - K_n)$ that is strictly positive until $K_n$ reaches $k$. 

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(iii) $0 < \alpha < 1, \theta > -\alpha$: In this regime, nonstandard asymptotics hold. Before its introduction, let us define some distributions appearing in the asymptotics.

**Definition 2.1 (Sibuya distribution).** The Sibuya distribution of parameter $\alpha \in (0, 1)$ Sibuya [1979], which is also called the Karlin–Rouault distribution Karlin [1967], Rouault [1978], is a discrete distribution on $\mathbb{N}$ with its density $p_\alpha(j)$ defined by

$$\forall j \in \mathbb{N}, p_\alpha(j) := \frac{\alpha \prod_{i=1}^{j-1}(i - \alpha)}{j!}.$$  \hspace{1cm} (2.1)

Here, Stirling’s formula $\Gamma(z) \sim \sqrt{2\pi/z} (z/e)^z$ implies $p_\alpha(j) \sim \alpha / \Gamma(1 - \alpha) \cdot j^{-(1 + \alpha)}$, i.e., the Sibuya distribution is heavy-tailed. Readers may refer to Resnick [2007] for its important role in the extreme value theory.

**Definition 2.2 (Generalized Mittag-Leffler distribution).** Let $S_\alpha$ be the positive random variable of parameter $\alpha \in (0, 1)$ with its Laplace transform given by $E[e^{-\lambda S_\alpha}] = e^{-\lambda \alpha} (\lambda \geq 0)$. Then, the law of $M_\alpha = S_\alpha^{\alpha}$ is referred to as the Mittag-Leffler distribution $(\alpha)$. Moreover, for each $\theta > -\alpha$, the generalized Mittag-Leffler distribution $(\alpha, \theta)$, denoted by $GMtLf(\alpha, \theta)$, is a tilted distribution with its density $g_{\alpha \theta}(x)$ proportional to $x^{\theta/\alpha} g_\alpha(x)$, where $g_\alpha(x)$ is the density of the Mittag-Leffler distribution $(\alpha)$.

**Remark 2.1.** The density $g_\alpha(x)$ of the Mittag-Leffler distribution is characterized by

$$\forall p > -1, \int_0^\infty x^p g_\alpha(x) \, dx = \frac{\Gamma(p + 1)}{\Gamma(p + 1)}.$$  \hspace{1cm} (2.2)

Then, it easily follows from $g_{\alpha \theta}(x) \propto x^{\theta/\alpha} g_\alpha(x)$ that the moments of $M_{\alpha \theta} \sim GMtLf(\alpha, \theta)$ are given by

$$\forall p > -(1 + \theta/\alpha), \ E[(M_{\alpha \theta})^p] = \frac{\Gamma(\theta + 1)}{\Gamma(\theta + 1)} \frac{\Gamma(\theta/\alpha + p + 1)}{\Gamma(\theta/\alpha + 1)}.$$  \hspace{1cm} (2.3)

Finally, we introduce the nonstandard asymptotics of the Ewens–Pitman partition when $0 < \alpha < 1, \theta > -\alpha$.

**Lemma 2.2.** We assume $0 < \alpha < 1, \theta > -\alpha$. Let $S_{n,j}$ be the number of blocks with size $j$, and let $K_n = \sum_{j \geq 1} S_{n,j}$ be the number of nonempty blocks. Then, we have

(A) $K_n / n^\alpha \rightarrow M_{\alpha \theta}$ a.s. and in the $p$-th moment for all $p > 0$, where $M_{\alpha \theta} \sim GMtLf(\alpha, \theta)$.

(B) $S_{n,j} / K_n \rightarrow p_\alpha(j)$ a.s. for all $j \in \mathbb{N}$, where $p_\alpha(j)$ is the density of the Sibuya distribution.
Sketch of proof. Let \( P_{\alpha \theta} \) denote the law of the Ewens–Pitman partition with parameter \((\alpha, \theta)\). Then, (A) can be proved by applying the martingale convergence theorem to the likelihood ratio \((dP_{\alpha \theta} / dP_{\alpha 0})|_{\mathcal{F}_n}\) under \( P_{\alpha 0} \), where \( \mathcal{F}_n \) is the \( \sigma \)-field generated by the partition of \( n \) balls. For (B), Kingman’s representation theorem implies that the Ewens–Pitman partition can be expressed as the tied observation of conditional i.i.d. samples from the Pitman–Yor process (see Section 6 or [Ghosal and van der Vaart, 2017, p. 440]). Then, we can analyze \( S_{n,j}/K_n \) in the setting of a classical occupancy problem. Readers may refer to [Pitman, 2006, Theorem 3.8] for the detailed proof of (A) and [Pitman, 2006, Lemma 3.11] and Gnedin et al. [2007] for that of (B).

Remark 2.3 ([Pitman, 2006, p. 71]). \( P_{\alpha 0} \) are absolutely mutual continuous for each \( \theta(> -\alpha) \): the Radon–Nikodym density is given by

\[
\frac{dP_{\alpha \theta}}{dP_{\alpha 0}} = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} M^{\theta/\alpha}_{\alpha} P_{\alpha 0} \text{-a.s.,}
\]

where \( M_{\alpha} \) is the almost sure limit of \( n^{-\alpha} K_n \) under \( P_{\alpha 0} \). This is consistent with Proposition 3.3, in the sense that the Fisher information about \( \theta \) is bounded as \( n \) increases. This result basically implies that we cannot consistently estimate \( \theta \) from data.

2.2 Stable convergence

Our main theorem on the asymptotic law of the MLE involves stable convergence, which is a notion of stochastic convergence. In this section, we introduce it in a general format. Let \((\Omega, \mathcal{F}, P)\) denote a probability space, and let \( \mathcal{X} \) be a separable metrizable topological space equipped with its Borel \( \sigma \)-field \( \mathcal{B}(\mathcal{X}) \). Furthermore, let \( L^1(\Omega, \mathcal{F}, P) = \mathcal{L}^1 \) denote the set of \( \mathcal{F} \)-measurable functions that satisfy \( \int |f| dP < +\infty \), and we denote the set of continuous bounded functions on \( \mathcal{X} \) by \( C_b(\mathcal{X}) \). Then, stable convergence is defined as follows:

**Definition 2.3.** For a sub \( \sigma \)-field \( \mathcal{G} \subset \mathcal{F} \), a sequence of \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\)-valued random variables \((X_n)_{n \geq 1}\) is said to converge \( \mathcal{G} \)-stably to \( X \), denoted by \( X_n \to X \mathcal{G} \)-stably, if

\[
\forall f \in \mathcal{L}^1, \forall h \in C_b(\mathcal{X}), \lim_{n \to \infty} E[f E[h(X_n)|\mathcal{G}]] = E[f E[h(X)|\mathcal{G}]]. \tag{2.4}
\]

If \( X \) is independent of \( \mathcal{G} \), \((X_n)_{n \geq 1}\) is said to converge \( \mathcal{G} \)-mixing, denoted by \( X_n \to X \mathcal{G} \)-mixing.

Note that stable convergence implies weak convergence, as the condition (2.4) with \( f = 1 \) is identical to the definition of weak convergence. By contrast, if \( \mathcal{G} \) is a trivial \( \sigma \)-field \( \{\emptyset, \Omega\} \), we claim \( E[f E[h(X_n)|\mathcal{G}]] = \int f dP \cdot E[h(X_n)] \) for all \( f \in \mathcal{L}^1 \). Thus, \( X_n \to X \) \( \mathcal{G} \)-stably coincides with \( X_n \to^d X \) in this trivial case.

The next lemma states that the well-known theorem for weak convergence holds for stable convergence. More precisely, Slutsky’s lemma holds in a stronger sense.
Lemma 2.4 ([Häusler and Luschgy, 2015, p. 34]). For \((\mathcal{X}, \mathcal{B}(\mathcal{X})), (\mathcal{Y}, \mathcal{B}(\mathcal{Y}))\), a pair of separable metrizable spaces with metric \(d\), let \((X_n)_{n \geq 1}\) be a sequence of \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\)-valued random variables, and let \((Y_n)_{n \geq 1}\) be a sequence of \((\mathcal{Y}, \mathcal{B}(\mathcal{Y}))\)-valued random variables. Assuming that a certain random variable \(X\) exists s.t. \(X_n \xrightarrow{G} X\) \(G\)-stably, the following statements hold.

(A) Let \(X = Y\). If \(d(X_n, Y_n) \xrightarrow{p} 0\) in probability, \(Y_n \xrightarrow{G} X\) \(G\)-stably.

(B) If \(Y_n \xrightarrow{p} Y\) in probability, and \(Y\) is \(G\)-measurable, \((X_n, Y_n) \xrightarrow{G} (X, Y)\) \(G\)-stably.

(C) If \(g : \mathcal{X} \to \mathcal{Y}\) is \((\mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{Y}))\)-measurable and continuous \(P_X\)-a.s., \(g(X_n) \xrightarrow{G} g(X)\) \(G\)-stably.

If \(G\) is a tribal \(\sigma\)-field, the above assertions are the well-known results for weak convergence. Importantly, (B) allows \(Y\) to be any \(G\)-measurable random variable and not just a constant. In this sense, Slutsky’s lemma holds strongly for stable convergence. In our theorem, we will set \(G\) as the limit of the sigma fields generated by the sequential partition and apply (B) to obtain a confidence interval.

3 Main result

3.1 Fisher Information

Henceforth, we will always assume \(0 < \alpha < 1, \theta > -\alpha\). Before introducing our main theorem, let us discuss the asymptotic analysis of the Fisher information to acquire insights into the parameter estimation of the Ewens–Pitman partition. All proofs in this section are given in Appendix D.

We define the Fisher information of the Sibuya distribution \(I_\alpha\) by
\[
\forall \alpha \in (0, 1), \ I_\alpha := -\sum_{j=1}^{\infty} p_\alpha(j) \cdot \partial_\alpha^2 \log p_\alpha(j) \quad \text{with} \quad p_\alpha(j) = \frac{\alpha}{j!} \prod_{i=1}^{j-1} (i - \alpha).
\]
(3.1)

The next proposition provides two formulas for \(I_\alpha\).

**Proposition 3.1.** \(I_\alpha\) is continuous in \(\alpha \in (0, 1)\) and can be written by
\[
I_\alpha = \frac{1}{\alpha^2} + \sum_{j=1}^{\infty} p_\alpha(j) \sum_{i=1}^{j-1} \frac{1}{(i - \alpha)^2} \overset{(B)}{=} \frac{1}{\alpha^2} + \sum_{j=1}^{\infty} \frac{p_\alpha(j)}{\alpha(j - \alpha)} > 0.
\]

We will use the two formulas in the proof of our main results. In particular, (A) will appear in the limit of the second derivative of the log-likelihood while (B) will be used for the limit of the variance of the first derivative of the log-likelihood. By contrast, our confidence interval of \(\alpha\) to be introduced below requires the computation of \(I_\alpha\). In this situation, we recommend (B) in terms of numerical errors: \(p_\alpha(j) = O(j^{-\alpha-1})\) implies that the numerical error caused by
truncating the infinite series of (A) at \( n \) is
\[
\sum_{j=n}^{\infty} p_{\alpha}(j) \sum_{i=1}^{j-1} (i-\alpha)^{-2} = O(n^{-\alpha}),
\]
while \( \sum_{j=n}^{\infty} p_{\alpha}(j)(\alpha(j-\alpha))^{-1} = O(n^{-\alpha-1}) \) for (B), where the error in (B) decays faster than that in (A). We plot \( I_{\alpha} \) in Figure 3 using (B) with \( j \) truncated at 10 for each \( \alpha \). We observe that \( I_{\alpha} \) looks like a log-convex function of \( \alpha \).

For the asymptotic analysis of the Fisher information of the Ewens–Pitman partition ahead, we define the function \( f_{\alpha} : (-1, \infty) \to \mathbb{R} \) for each \( \alpha \in (0, 1) \) by
\[
\forall z \in (-1, \infty), \quad f_{\alpha}(z) := \psi(1 + z) - \alpha \psi(1 + \alpha z), \quad (3.2)
\]
where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) is the digamma function. The next lemma claims the basic properties of \( f_{\alpha} \).

**Lemma 3.2.** The map \( f_{\alpha} : (-1, \infty) \to \mathbb{R} \) defined by (3.2) is bijective and satisfies \( f'_{\alpha}(z) > 0 \) and \( f''_{\alpha}(z) < 0 \) for all \( z \in (-1, \infty) \).

Here, it is important to emphasize that \( f_{\alpha} \) is bijective. We will prove later that \( f_{\alpha} \) also appears in the asymptotics of the MLE for \( \theta \) through the inverse function.

Finally, we discuss the Fisher information of the Ewens–Pitman partition. We denote the logarithm of the likelihood (1.1) by \( \ell_n(\alpha, \theta) \), and we define \( I_{\alpha\alpha}^{(n)} \), \( I_{\alpha\theta}^{(n)} \), and \( I_{\theta\theta}^{(n)} \) by
\[
I_{\alpha\alpha}^{(n)} := \mathbb{E}[(\partial_{\alpha} \ell_n(\alpha, \theta))^2], \quad I_{\alpha\theta}^{(n)} := \mathbb{E}[\partial_{\alpha} \ell_n(\alpha, \theta) \cdot \partial_{\theta} \ell_n(\alpha, \theta)], \quad I_{\theta\theta}^{(n)} := \mathbb{E}[(\partial_{\theta} \ell_n(\alpha, \theta))^2], \quad (3.3)
\]
i.e., they are the Fisher information obtained after \( n \) balls are partitioned according to the Ewens–Pitman partition \( (\alpha, \theta) \). The next proposition derives the leading terms as \( n \to \infty \).

**Proposition 3.3.** Let \( I_{\alpha} \) be the Fisher information of the Sibuya distribution (3.1), and let \( f'_{\alpha} \) be the derivative of \( f_{\alpha} \) defined by (3.2). Then, the leading terms of the Fisher information are given by
\[
I_{\alpha\alpha}^{(n)} \sim n^\alpha \mathbb{E}[M_{\alpha\theta}], \quad I_{\alpha\theta}^{(n)} \sim \alpha^{-1} \log n, \quad I_{\theta\theta}^{(n)} \to \alpha^{-2} f'_{\alpha}(\theta/\alpha) < +\infty,
\]
where \( \mathbb{E}[M_{\alpha\theta}] \) is the moment of \( \text{GMTL}_{\alpha}(\alpha, \theta) \) given by (2.3).
Observe that the Fisher information of \( \theta \) is finite, i.e., we cannot identify \( \theta \) consistently no matter how large \( n \) is. This agrees with the absolute mutual continuity given by Remark 2.3. Furthermore, the optimal convergence rate of estimators for \( \alpha \) is at most \( n^{-\alpha/2} \), which is slower than the typical rate \( n^{-1/2} \) in typical i.i.d. cases. Moreover, the cross term of the Fisher information matrix is negligible as \( n \to \infty \), which implies that \( \alpha \) and \( \theta \) are asymptotically orthogonal. This supports the well-known fact that the inference of \( \theta \) has less effect on \( \alpha \) as \( n \) increases Balocchi et al. [2022], Franssen and van der Vaart [2022b].

### 3.2 Maximum Likelihood Estimator

In this section, we derive the exact asymptotic distribution of the maximum likelihood estimator. Here, we denote the log-likelihood with \((\alpha, \theta) = (x, y)\) by \( \ell_n(x, y) \), i.e., we define the random function \( \ell_n \) by

\[
\forall x \in (0, 1), \forall y > -x, \quad \ell_n(x, y) := \sum_{i=1}^{K_n-1} \log(y + ix) - \sum_{i=1}^{n-1} \log(y + i) + \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \log(i - x). \tag{3.4}
\]

Then, we define the MLE \((\hat{\alpha}_n, \hat{\theta}_n)\) as the maxima of \( \ell_n \):

**Definition 3.1.** For \((\underline{\alpha}, \bar{\alpha}) \) s.t. \( \alpha \in [\underline{\alpha}, \bar{\alpha}] \subset (0, 1) \), we define \((\hat{\alpha}_n, \hat{\theta}_n)\) by

\[
(\hat{\alpha}_n, \hat{\theta}_n) \in \arg \max_{x \in [\underline{\alpha}, \bar{\alpha}], y > -x} \ell_n(x, y).
\]

**Remark 3.4.** In practice, it suffices to set \((\underline{\alpha}, \bar{\alpha}) = (\epsilon, 1 - \epsilon)\) for a sufficiently small \( \epsilon > 0 \).

As the parameter space \( \{(x, y) : x \in [\underline{\alpha}, \bar{\alpha}], y > -x\} \) is not compact, its existence and uniqueness are not obvious. The next theorem verifies these aspects rigorously.

**Theorem 3.5.** The solution of \( \partial_x \ell_n(x, y) = \partial_y \ell_n(x, y) = 0 \) on \( \{x \in [\underline{\alpha}, \bar{\alpha}], y > -x\} \) uniquely exists and equals the MLE \((\hat{\alpha}_n, \hat{\theta}_n)\) with probability 1\( - o(1) \).

Herein, we define the \( \sigma \)-field \( \mathcal{F}_\infty := \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n) \), with \( \mathcal{F}_n \) being the \( \sigma \)-field generated by the partition of \( n \) balls. Then, the exact asymptotic distribution of the MLE \((\hat{\alpha}_n, \hat{\theta}_n)\) is characterized as follows:

**Theorem 3.6.** Let \( I_\alpha \) be the Fisher information of the Sibuya distribution defined by (3.1), and let \( f_\alpha^{-1} \) be the inverse of the bijective function \( f_\alpha \) defined by (3.2). Then, the asymptotics of \((\hat{\alpha}_n, \hat{\theta}_n)\) is given by

\[
\begin{align*}
n^{\alpha/2}(\hat{\alpha}_n - \alpha) &\to (I_\alpha M_{\alpha\theta})^{-1/2} \cdot N \ (\mathcal{F}_\infty\text{-stable}), \tag{3.5} \\
\hat{\theta}_n &\to \alpha \cdot f_\alpha^{-1}(\log M_{\alpha\theta}) \ (\text{in probability}), \tag{3.6}
\end{align*}
\]

where \( N \sim N(0, 1) \) is independent of \( \mathcal{F}_\infty \), and \( M_{\alpha\theta} = \lim_{n \to \infty} n^{-\alpha} K_n \) is a nondegenerate positive random variable following GMtLf(\( \alpha, \theta \)).
Therefore, the convergence holds: partitioned. Then, in the same setting as Theorem 3.6, the following mixing α 

\[ \alpha \text{ is convex. Then, Jensen’s inequality implies} \]

\[ \lim_{\epsilon \to 0} \epsilon^{-1}((M_{\alpha \theta})^\epsilon - 1) = \lim_{\epsilon \to 0} \epsilon^{-1}(\mathbb{E}[(M_{\alpha \theta})^\epsilon] - \mathbb{E}[(M_{\alpha \theta})^0]) = \partial_{t=0} \mathbb{E}[(M_{\alpha \theta})^\epsilon] \]

\[ = \partial_{t=0} \left( \frac{\Gamma(\theta + 1)}{\Gamma(\theta + \alpha + 1)} - \frac{\Gamma'((\theta/\alpha + 1)}{\Gamma(\theta + 1)} - \alpha \Gamma((\theta/\alpha + 1)} \right) \]

\[ = \psi(\theta/\alpha + 1) - \alpha \psi(\theta + 1) = f_\alpha(\theta/\alpha), \]

\[ \sqrt{I_{\alpha n}(\hat{\alpha}_n - \alpha)} = \sqrt{I_{\alpha n}^{(2)} / n^\alpha \cdot n^{\alpha/2}(\hat{\alpha}_n - \alpha)} \to \sqrt{\mathbb{E}[M_{\alpha \theta}]/M_{\alpha \theta} \cdot N}. \quad (3.7) \]

As \( M_{\alpha \theta} \) is a nondegenerate random variable, (3.7) implies that the error of the MLE normalized by the Fisher information does not converge to the standard normal but a variance mixture of centered normals. This type of asymptotics is referred to as asymptotic mixed normality, which is often observed in “non-ergodic” or “explosive” stochastic processes (c.f. Häusler and Luschgy [2015]).

By contrast, Slutzky’s lemma for stable convergence (more precisely, (B) of Lemma 2.4 with \( K \) limit is the standard normal. Here, the number of blocks \( K_n \) corresponds to the sample size in typical i.i.d. cases, and \( I_\alpha \) plays the role of the Fisher information per block. Furthermore, it immediately follows that \( [\hat{\alpha}_n \pm 1.96/\sqrt{I_{\alpha n}K_n}] \) is a valid 95% confidence interval for \( \alpha \). We reiterate these observations as a corollary below.

**Corollary 3.6.1.** Let \( K_n \) be the number of blocks generated after \( n \) balls are partitioned. Then, in the same setting as Theorem 3.6, the following mixing convergence holds:

\[ \sqrt{I_\alpha K_n} \cdot (\hat{\alpha}_n - \alpha) \to N \mathcal{F}_\infty\text{-mixing}. \quad (3.8) \]

Therefore, \( [\hat{\alpha}_n \pm 1.96/\sqrt{I_{\alpha n}K_n}] \) is an approximate 95% interval for \( \alpha \).

Finally, we discuss the limit law \( \alpha f_\alpha^{-1}(\log M_{\alpha \theta}) \) of \( \hat{\theta}_n \). Here, we are interested in the mean of \( \alpha f_\alpha^{-1}(\log M_{\alpha \theta}) \). As \( f_\alpha \) is concave and bijective (see Lemma 3.2), \( f_\alpha^{-1} \) is convex. Then, Jensen’s inequality implies

\[ \mathbb{E}[\alpha f_\alpha^{-1}(\log M_{\alpha \theta})] \geq \alpha f_\alpha^{-1}(\mathbb{E}[\log M_{\alpha \theta}]). \]

Now it is difficult to calculate the exact value of \( \mathbb{E}[\log M_{\alpha \theta}] \). Therefore, we instead approximate \( \mathbb{E}[\log M_{\alpha \theta}] \) by

\[ \mathbb{E}[\log M_{\alpha \theta}] = \mathbb{E}[\lim_{\epsilon \to 0} \epsilon^{-1}((M_{\alpha \theta})^\epsilon - 1)] \approx \lim_{\epsilon \to 0} \mathbb{E}[\epsilon^{-1}((M_{\alpha \theta})^\epsilon - 1)] \]

(3.9)
where \( \psi \) is the digamma function, and we used the definition \( f_\alpha(z) = \psi(1 + z) - \alpha \psi(1 + \alpha z) \) in (\( \star 2 \)). In summary, \( \mathbb{E}[\alpha f^{-1}_\alpha(\log M_{\alpha \theta})] \) is approximately lower bounded by \( \theta \):

\[
\mathbb{E}[\alpha f^{-1}_\alpha(\log M_{\alpha \theta})] \geq \alpha f^{-1}_\alpha(\mathbb{E}[\log M_{\alpha \theta}]) \approx \alpha f^{-1}_\alpha(\lim_{\epsilon \to 0} \mathbb{E}[\epsilon^{-1}((M_{\alpha \theta})^\epsilon - 1)])
\]

which implies that \( \hat{\theta}_n \) is asymptotically positively biased. However, we approximated \( \mathbb{E}[\log M_{\alpha \theta}] \) by \( \lim_{\epsilon \to 0} \mathbb{E}[\epsilon^{-1}((M_{\alpha \theta})^\epsilon - 1)] \), which is not mathematically rigorous. In Section 4, we will plot the histogram \( \alpha f^{-1}_\alpha(\log M_{\alpha \theta}) \) by directly sampling \( M_{\alpha \theta} \sim \text{GMtL}(\alpha, \theta) \) and confirm \( \mathbb{E}[\alpha f^{-1}_\alpha(\log M_{\alpha \theta})] \geq \theta \). Moreover, the numerical simulation indicates that \( \alpha f^{-1}_\alpha(\log M_{\alpha \theta}) \) is distributed around \( \theta \) with positive skewness.

### 3.3 Quasi Maximum Likelihood Estimator

Thus far, we considered the case where \( \alpha \) and \( \theta \) are jointly estimated. However, the estimation of \( \alpha \) is of more interest than that of \( \theta \), and sometimes, \( \theta \) is regarded as a nuisance. Here, we consider the MLE of \( \alpha \) with \( \theta \) mis-specified. Considering the asymptotic orthogonality of \((\alpha, \theta)\) (recall Proposition 3.3), the MLE of \( \alpha \) with \( \theta \) mis-specified is expected to have the same asymptotic law as the MLE with \( \theta \) jointly estimated. In this section, we make these arguments more rigorous: we claim that they are identical up to terms of \( n^{-\alpha/2} \), but they differ in higher order terms. Furthermore, we demonstrate that the MLE with \( \theta \) jointly estimated is adaptive to the scale of the nuisance \( \theta \).

First, we define the quasi maximum likelihood estimator (QMLE) as the MLE of \( \alpha \) with \( \theta \) mis-specified.

**Definition 3.2 (QMLE).** For each \( \theta^* \in (-\alpha, \infty) \), we define the QMLE with plug-in \( \theta^* \), denoted by \( \hat{\alpha}_{n, \theta^*} \), and expressed as follows:

\[
\hat{\alpha}_{n, \theta^*} \in \arg \max_{x \in (-\theta^*, \vee 0,1)} \ell_n(x, \theta^*), \quad (3.10)
\]

where \( \ell_n \) is the function defined by (3.4).

In particular, the QMLE \( \hat{\alpha}_{n, \theta} = \hat{\alpha}_{n, \theta^* = \theta} \) with \( \theta^* \) being the true \( \theta \) is just the MLE of \( \alpha \) with \( \theta \) known. From now on, we regard this as an oracle estimator of \( \alpha \), and we will compare the QMLE \( \hat{\alpha}_{n, \theta^*} \) (with \( \theta^* \neq \theta \)) and the MLE \( \hat{\alpha}_n \) in Definition 3.1 (where \( \theta \) is jointly estimated) based on their error to the oracle \( \hat{\alpha}_{n, \theta} \).

The following propositions claim that \( \hat{\alpha}_{n, \theta^*} \) uniquely exists and has the same asymptotic distribution as \( \hat{\alpha}_n \).

**Proposition 3.7.** \( \partial_x \ell_n(x, \theta^*) \) is strictly decreasing in \( x \), and QMLE \( \hat{\alpha}_{n, \theta^*} \) is the unique solution of \( \partial_x \ell_n(\cdot, \theta^*) = 0 \) with probability \( 1 - o(1) \).
Proposition 3.8. (3.5) of Theorem 3.6 holds for QMLE \( \hat{\alpha}_{n,\theta} \), i.e.,
\[
n^{\alpha/2}(\hat{\alpha}_{n,\theta} - \alpha) \rightarrow (I_{\alpha} M_{\alpha\theta})^{-1/2} \cdot N \quad (F_{\infty}\text{-stable}).
\]

Proposition 3.8 implies that the QMLE \( \hat{\alpha}_{n,\theta} \) and \( \hat{\alpha}_n \) are asymptotically equivalent on the scale of \( n^{-\alpha/2} \). With this, it appears that jointly estimating \( \alpha \) and \( \theta \) is of no use. However, the next proposition implies that they differ on the order of \( n^{-\alpha} \log n \), and \( \hat{\alpha}_n \) is close to the oracle \( \hat{\alpha}_{n,\theta} \) regardless of the scale of \( \theta \).

Proposition 3.9. For the QMLE \( \hat{\alpha}_{n,\theta} \) and the MLE \( \hat{\alpha}_n \), their asymptotic errors to the oracle \( \hat{\alpha}_{n,\theta} = \theta \) are given by
\[
\frac{n^{\alpha}}{\log n} (\hat{\alpha}_{n,\theta} - \hat{\alpha}_{n,\theta}) \rightarrow^p - \frac{\theta^* - \theta}{\alpha I_{\alpha} M_{\alpha\theta}} \quad \text{for all } \theta^* \in (-\alpha, \infty) \quad (3.11)
\]
\[
\frac{n^{\alpha}}{\log n} (\hat{\alpha}_n - \hat{\alpha}_{n,\theta}) \rightarrow^p - \frac{\alpha f_{\alpha}^{-1}(\log M_{\alpha\theta}) - \theta}{\alpha I_{\alpha} M_{\alpha\theta}}, \quad (3.12)
\]
where \( M_{\alpha\theta} = \lim_{n \to \infty} n^{-\alpha}K_n \), \( I_{\alpha} \) is defined by (3.1), and \( f_{\alpha}^{-1} \) is the inverse of \( f_{\alpha} \) defined by (3.2).

We observe that the limit error in (3.11) depends on the “misspecification error” \( \theta^* - \theta \), while the corresponding term in (3.12) is replaced by \( \alpha f_{\alpha}^{-1}(\log M_{\alpha\theta}) - \theta \). Considering that \( \alpha f_{\alpha}^{-1}(\log M_{\alpha\theta}) \) is distributed around \( \theta \), we expect that the error of \( \hat{\alpha}_{n,\theta} \) is larger than \( \hat{\alpha}_n \) if the plug-in \( \theta^* \) is taken far away from the true value \( \theta \) by the users. We prove in Section 4 that these errors significantly affect coverage and efficiency.

3.4 Application to network data analysis

Here, we discuss the application of Corollary 3.6.1 to network data analysis. In Crane [2018], Crane and Dempsey [2018], the authors propose the “Hollywood process”, a statistical model for network data. This is a stochastic process over growing networks that sequentially attaches edges to vertices in the same manner as the Ewens–Pitman partition, where \( n \) is the total degree, \( K_n \) is the number of vertices, and \( S_{n,j} \) is the number of vertices with degree \( j \). They define that a growing network has sparsity if and only if
\[
nK_n^{-\mu} \rightarrow 0 \quad \text{as } n \rightarrow \infty,
\]
with \( \mu \) being the degree per vertex, e.g., \( \mu = 2 \) when the network is bivariate. Based on \( n^{-\alpha}K_n \rightarrow M_{\alpha\theta} > 0 \) (a.s.) by Lemma 2.2, the authors claim that the Hollywood process has sparsity if and only if
\[
\mu^{-1} < \alpha < 1.
\]

Now, we construct a hypothesis testing of the sparsity based on Corollary 3.6.1. We define the null hypothesis \( H_0 \) and the alternative hypothesis \( H_1 \) by
\[
\text{(not sparse) } H_0 : 0 < \alpha \leq \mu^{-1}, \quad \text{(sparse) } H_1 : \mu^{-1} < \alpha < 1.
\]
For a constant $\delta \in (0, 1)$, we reject the null $H_0$ if

$$\sqrt{I_{\hat{\alpha}_n} K_n (\hat{\alpha}_n - \mu^{-1})} > \Phi^{-1}(1 - \delta),$$

where $\Phi$ is the CDF of the standard normal. Then, the significance level of this testing is $\delta$, since the probability of rejecting the null when $\alpha \leq \mu^{-1}$ is upper bounded as

$$\Pr\left(\sqrt{I_{\hat{\alpha}_n} K_n (\hat{\alpha}_n - \mu - 1)} > \Phi^{-1}(1 - \delta)\right) \leq \Pr\left(\sqrt{I_{\hat{\alpha}_n} K_n (\hat{\alpha}_n - \alpha)} > \Phi^{-1}(1 - \delta)\right),$$

where the right hand side converges to $\delta$ from Corollary 3.6.1.

### 4 Numerical simulation

We visualize the limit law $\alpha f^{-1}_\alpha (\log M_{\alpha \theta})$ of the MLE $\hat{\theta}_n$. We sample $M_{\alpha \theta} \sim \text{GMtL}(\alpha, \theta)$ using the rejection algorithm proposed by Devroye [2009] with a sample size $10^6$ and plot the histogram of $\alpha f^{-1}_\alpha (\log M_{\alpha \theta})$ in Figure 4. We observe that the mean of $\alpha f^{-1}_\alpha (\log M_{\alpha \theta})$ is larger than $\theta$, which supports our discussion on the limit law in Section 3.2. We also plot the probability density function (pdf) of $\mathcal{N}(\theta, \alpha^2 / f'_{\alpha}(\theta/\alpha))$, where the variance $\alpha^2 / f'_{\alpha}(\theta/\alpha)$ is the inverse of $\lim_{n \to \infty} I_{\alpha \theta}^{(n)}$, i.e., the limit of the Fisher information about $\theta$ (see Proposition 3.3). Note that this normal distribution is a naive guess by standard asymptotic statistics. Surprisingly, the limit law is close to the normal distribution when $\alpha$ is small, or $\theta$ is large. As a related paper, note that Tsukuda [2017] tackles the case of $\theta \to \infty$ with $\alpha$ fixed to 0 and proves the asymptotic normality of $\hat{\theta}_n$. Our numerical simulation yields similar results, but we need more theoretical analyses of the limit law $\alpha f^{-1}_\alpha (\log M_{\alpha \theta})$. However, we leave that for future research.

Next, we compare $\hat{\alpha}_{n, \theta}$ (the MLE with $\theta$ known), $\hat{\alpha}_{n, 0}$ (the QMLE with plug-in $\theta^* = 0$), and $\hat{\alpha}_n$ (the MLE with $\theta$ jointly estimated) based on efficiency and coverage. Here, we sequentially generate the random partition and compute these estimators as $n$ increases from $n = 2^7$ to $2^{17}$. We replicate this random path $10^4$ times, and we calculate the efficiency by $1/\{\text{MSE of } \hat{\alpha}_n \times K_n I_{\hat{\alpha}_n}\}$ and the coverage of $[\hat{\alpha}_n + 1.96/\sqrt{K_n I_{\hat{\alpha}_n}}]$ for each $\alpha_n = \hat{\alpha}_{n, \theta}, \hat{\alpha}_{n, 0}, \hat{\alpha}_n$. We plot them in Figure 5 and Figure 6. Note that they converge to the theoretical limits as $n$ increases. However, for small $n$, the efficiency and the coverage of $\hat{\alpha}_{n, 0}$ are significantly small when the plugin error $|0 - \theta| = |\theta|$ is large, whereas $\hat{\alpha}_n$ is robust to the scale of $\theta$. These observations support our argument after Proposition 3.9.

### 5 Proof highlights

In this section, we outline the fundamental idea behind the proofs of theorems in Section 3. First, we consider the QMLE $\hat{\alpha}_{n, 0}$ with $\theta^* = 0$ for simplicity. (3.4)
Figure 4: Histogram of $\alpha f^{-1}(\log M_{\alpha \theta})$ with sample size $10^6$. The solid line is the pdf of $N(\theta, \alpha^2 / f_\alpha'(\theta/\alpha))$.

with $y = 0$ implies that the log-likelihood is given by

$$\forall x \in (0, 1), \quad \ell_n(x, 0) = (K_n - 1) \log x + \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j-1} \log(i - x),$$

where $S_{n,j}$ is the number of blocks of size $j$, and $K_n$ is the number of nonempty blocks. Then, the score function $\partial_x \ell_n(x, 0)$ is given by

$$\partial_x \ell_n(x, 0) = \frac{K_n - 1}{x} - \sum_{j=1}^{n} \frac{1}{j} \sum_{i=1}^{j-1} \frac{1}{i - x}.$$

Here, we define the random measure $P_n$ on $N$ as the ratio of blocks of size $j$:

$$\forall j \in \mathbb{N}, \quad P_n(j) := \frac{S_{n,j}}{\sum_{j'=1}^{\infty} S_{n,j'}} = \frac{S_{n,j}}{K_n}.$$

Note that $P_n(j) = 0$ for all $j > n$, as the total number of partitioned balls is $n$. In our proof, we denote $\sum_{j=1}^{\infty} P_n(j)f(j)$ by $P_n f$ for any function $f$ on $N$. Now, we define the random function $\Psi_{n,0}(x) = K_n^{-1} \partial_x \ell_n(x, 0)$, which is the
Figure 5: Efficiency of the MLE with $\theta$ known, MLE with $\theta$ unknown, and QMLE with $\theta^* = 0$. We fixed $\alpha$ to 0.6. Note that the QMLE with $\theta^* = 0$ coincides with the MLE with $\theta$ known when $\theta = 0$.

score function normalized by $K_n$. Then, the above discussion lead to

$$\hat{\Psi}_{n,0}(x) = \frac{1}{x} - \frac{1}{xK_n} - \sum_{j=1}^{n} \frac{S_{n,j}}{K_n} \sum_{i=1}^{j-1} \frac{1}{i - x} = \frac{1}{x} - \frac{1}{xK_n} - \sum_{j=1}^{n} \mathbb{P}_n(j) \sum_{i=1}^{j-1} \frac{1}{i - x}$$

$$= \frac{1}{x} - \frac{1}{xK_n} - \sum_{j=1}^{\infty} \mathbb{P}_n(j) g_x(j) = \frac{1}{x} - \frac{1}{xK_n} - \mathbb{P}_n g_x,$$

where $g_x$ is the function on $\mathbb{N}$ defined by $g_x(j) = \sum_{i=1}^{j-1} (i - x)^{-1}$. We observe that $\hat{\Psi}_{n,0}(x)$ is an expectation with respect to the empirical measure $\mathbb{P}_n$, and hence, the asymptotics of $\hat{\Psi}_{n,0}$ is boiled down to a suitable convergence of $\mathbb{P}_n$.

Here, the convergence $S_{n,j}/K_n \to p_\alpha(j) (a.s.)$ by Lemma 2.2 implies

$$\forall j \in \mathbb{N}, \mathbb{P}_n(j) \overset{a.s.}{\to} \mathbb{P}(j) := p_\alpha(j) = \frac{\alpha \prod_{i=1}^{j-1} (i - \alpha)}{j!},$$

i.e., the empirical measure $\mathbb{P}_n$ converges to the deterministic measure $\mathbb{P}$ in a point-wise manner. Collectively, we expect that $\hat{\Psi}_{n,0}$ converges to the deterministic function $\Psi$ as follows:

$$\forall x \in (0,1), \hat{\Psi}_{n,0}(x) = \frac{1}{x} - \frac{1}{xK_n} - \mathbb{P}_n g_x \overset{p}{\to} \frac{1}{x} - \mathbb{P} g_x : = \Psi(x).$$

Here, we emphasize that the above convergence does not follow directly from the pointwise convergence $\mathbb{P}_n(j) \to \mathbb{P}(j)$ as $g_x(j) = \sum_{i=1}^{j-1} (i - x)^{-1}$ diverges as $j \to \infty$. To make the arguments more rigorous, we prove the convergence of $\mathbb{P}_n$ for a suitable set of function $\mathcal{F}$:

$$\forall f \in \mathcal{F}, |\mathbb{P}_n f - \mathbb{P} f| \to 0.$$
Figure 6: Coverage of MLE with \( \theta \) known, MLE with \( \theta \) unknown, and QMLE with \( \theta^* = 0 \). We fixed \( \alpha \) to 0.6. Note that the QMLE with \( \theta^* = 0 \) coincides with the MLE with \( \theta \) known when \( \theta = 0 \).

Using this lemma, we show suitable convergences of \( \hat{\Psi}_{n,0} \to^p \Psi \) and \( \hat{\Psi}' \to^p \Psi' \). Now, we claim \( \Psi(\alpha) = 0 \) and \( \Psi'(\alpha) = -I_\alpha \) with \( I_\alpha \) being the Fisher information of the Sibuya distribution. Combining all of this, we obtain the consistency of \( \hat{\alpha}_n \). For the asymptotic mixed normality, we use martingale CLT for the score function. See Table 1 for rough comparisons with typical i.i.d. cases.

| Table 1: Comparison with typical i.i.d. parametric models | \((X_i)_{i=1}^n \sim \text{i.i.d. Pr}(X; \alpha)\) | Ewens–Pitman partition |
|---|---|---|
| Score function | Empirical CDF | \( F_n(x) = n^{-1} \sum_{i=1}^n 1\{X_i \leq x\} \) |
| | CDF | \( F(x) = \text{Pr}(X \leq x) \) |
| | Fisher Information | \( nI_\alpha \) |
| | MLE | \( \sqrt{n}(\hat{\alpha}_n - \alpha) \to \mathcal{N}(0, I_\alpha^{-1}) \) |

We have discussed the QMLE so far, where the unknown \( \theta \) is fixed. Now, we consider the MLE \((\hat{\alpha}_n, \hat{\theta}_n)\) that simultaneously estimates \((\alpha, \theta)\). The main difficulty here is that \( \hat{\theta}_n \) does not converge to a fixed value, which requires more technical argument than before. The first step is to reduce the dimension of the parameters that we have to consider; we define the function \( \hat{y}_n : (0, 1) \to (-1, \infty) \) by

\[
\forall x \in (0, 1), \quad \hat{y}_n(x) := \arg \max_{y > -x} \ell_n(x, y).
\]

where \( \ell_n(x, y) \) is the log likelihood given by (3.4). Here we claim that \( \hat{y}_n \) is well-defined with a high probability. The gain of introducing \( \hat{y}_n \) is that the MLE \((\hat{\alpha}_n, \hat{\theta}_n)\) can be rewritten as the solution of the following one-dimensional
maximization problem:

\[ \hat{\alpha}_n \in \arg \max_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \ell_n(x, \hat{y}_n(x)), \quad \hat{\theta}_n = \hat{y}_n(\hat{\alpha}_n). \]

Here, similarly to \( \hat{\Psi}_{n, \theta^*} \), we define the random function \( \hat{\Psi}_n \) by

\[ \forall x \in (0, 1), \quad \hat{\Psi}_n(x) := \frac{1}{K_n} \cdot \frac{d}{dx} \ell_n(x, \hat{y}_n(x)). \]

Then, for the deterministic function

\[ \Psi(x) = x^{-1} - P g_x, \]

we show

\[ \sup_{x \in I} |\hat{\Psi}_n(x) - \Psi(x)| \to^P 0 \text{ for all closed subset } I \subseteq (0, 1), \]

\[ \sup_{x \in [\alpha \pm \delta_n]} |\hat{\Psi}_n'(x) - \Psi'(x)| \to^P 0 \text{ for all } \delta_n = o(1/\log n). \]

When compared with \( \hat{\Psi}_{n, 0} \), the uniform convergence of \( \hat{\Psi}_n' \) is only shown for a local neighborhood of \( \alpha \). We will see later that it is sufficient for the proof of our main results.

6 Prior literature

6.1 Construction of the Ewens–Pitman partition by the Pitman–Yor process

Related papers consider the estimation of \( \alpha \) under some misspecified settings. To compare them with our work, we introduce an alternative representation of the Ewens–Pitman partition. Here, for a nonatomic measure \( G \), e.g., \( N(0, 1) \), the Pitman–Yor process \((\alpha, \theta; G)\) is the discrete random measure represented by

\[ P := \sum_{i=1}^{\infty} p_i \delta_{y_i}, \]

where \((y_i)_{i=1}^{\infty} \overset{i.i.d.}{\sim} G, p_i = v_i \prod_{j=1}^{i-1} (1 - v_j), v_i \sim \text{Beta}(1 - \alpha, \theta + j \alpha), \) and \( y_i \perp \perp v_i \).

Since \( P \) is discrete with probability 1, conditional i.i.d. samples \((X_i)_{i \geq 1} | P \overset{i.i.d.}{\sim} P \) induces a partition of \([n]\) by the equivalence relation \( i \sim j \) iff \( X_i = X_j \). Then, with the highly nontrivial calculation, we claim that the random partition induced by the Pitman–Yor process\((\alpha, \theta; G)\) has the same density as (1.1) of the Ewens–Pitman partition \((\alpha, \theta)\). In fact, any exchangeable partitions can be induced by a discrete measure (see Kingman [1982]). Now we define the function \( L_P : (1, \infty) \to \mathbb{N} \) for a discrete measure \( P \) by

\[ L_P(x) := \# \{ y : P(y) > x^{-1} \}. \]
Note that $L_P$ is an increasing function, and the order of $L_P(x)$ as $x \to +\infty$ characterizes the tail behavior of $P$. For the random discrete measure $P$ following Pitman–Yor process $(\alpha, \theta, G)$, $L_P(x) = O(x^\alpha)$ (a.e.), i.e., the Pitman–Yor process has a heavy tail of index $\alpha$. More precisely, $x^{-\alpha}L_P(x)$ has a limit as $x \to \infty$ with probability 1 and the limit is given by:

$$
\lim_{x \to \infty} x^{-\alpha}L_P(x) = \frac{M_{\alpha \theta}}{\Gamma(1 - \alpha)} \quad \text{(with probability 1),}
$$

(6.1)

where $M_{\alpha \theta}$ is a random variable following GMtLf$(\alpha, \theta)$ (see Pitman [2006]).

### 6.2 Model misspecifications

Previous research has focused on estimating the tail index $\alpha$ of the unknown discrete measure $P$, using the Pitman–Yor process $(\alpha, \theta, G)$ as a prior. Namely, they estimate the tail index $\alpha$ by fitting the Ewens–Pitman partition $(\alpha, \theta)$ to partition data induced by i.i.d. samples from $P$, with $\theta$ regarded as a nuisance parameter. Here, $P$ is allowed to be misspecified and the assumption on $P$ takes the following form:

$$
\exists L(x) \text{ slowly varying}, \exists r(x) = o(x^\alpha) \text{ s.t. } |L_P(x) - L(x)x^\alpha| \leq r(x),
$$

(6.2)

Observe that (6.2) is motivated by (6.1): the Pitman–Yor process satisfies (6.2) with $L(x) = M_{\alpha \theta} / \Gamma(1 - \alpha)$, where $M_{\alpha \theta}$ is a positive random variable following GMtLf$(\alpha, \theta)$.

Recently, several papers discuss the asymptotics of the MLE $\hat{\alpha}_n$ by imposing a strong assumption on $r(x)$ in (6.2); Favaro and Naulet [2021] show $\hat{\alpha}_n = \alpha + O_p(n^{-\alpha/2}/\log n)$ and $\hat{\alpha}_n$ is minimax near optimal, under the assumption of (6.2) with $L$ being constant and $r(x) = O(x^{\alpha/2} \log x)$. Comparing it with our rate $\hat{\alpha}_n - \alpha = O_p(n^{-\alpha/2})$ in Theorem 3.6, we observe that the price for such a misspecification is just $\log n$ factors. However, it is highly nontrivial that the Pitman–Yor process satisfies their assumption: what is known for the Pitman–Yor process is that it satisfies (6.2) with $r(x) = o(x^\alpha)$, which is strictly weaker than their assumption that $r(x) = O(x^{\alpha/2} \log x)$. For these reasons, their problem settings do not necessarily include the Pitman–Yor process, and hence we cannot obtain our rate from their results.

In contrast, Balocchi et al. [2022] discuss the asymptotics of $(\hat{\alpha}_n, \hat{\theta}_n)$ under the assumption of (6.2) with $L(x) = L$ (a constant) and $r(x) = o(x^\alpha/ \log x)$, which is weaker than $r(x) = O(x^{\alpha/2} \log x)$ in the previous assumption by Favaro and Naulet [2021]. They show $(\hat{\alpha}_n, \hat{\theta}_n) \rightarrow^p (\alpha, \theta^*)$ where $\theta^*$ is determined by

$$
L = \frac{\exp(\psi(\theta^*/\alpha + 1) - \alpha \psi(\theta^* + 1))}{\Gamma(1 - \alpha)}
$$

Recall $L = M_{\alpha \theta} / \Gamma(1 - \alpha)$ if $P$ follows the Pitman–Yor process. Combining it and the definition of $f_\alpha$ by (3.2), we can recover the limit law of $\theta_n$ in Theorem 3.6:

$$
\psi(\theta^*/\alpha + 1) - \alpha \psi(\theta^* + 1) = \log M_{\alpha \theta} \Leftrightarrow f_\alpha(\theta^*/\alpha) = \log M_{\alpha \theta} \Leftrightarrow \theta^* = \alpha f_\alpha^{-1}(\log M_{\alpha \theta}).
$$
Note that $\theta^*$ is a nondegenerate random variable owing to the randomness of $M_{\alpha\theta}$. However, there is again no guarantee that the Pitman–Yor process satisfies their condition since their assumption $r(x) = O(x^\beta / \log x)$ is stronger than $r(x) = o(x^\alpha)$ of the Pitman–Yor process.

After our paper was originally posted, Franssen and van der Vaart [2022b] derived the asymptotic distribution of $\hat{\alpha}_n$ under the assumption (6.2) with $r(x) = O(x^\beta)$ for some $\beta < \alpha/2$, which is again stronger than $r(x) = o(x^\alpha)$. They show that

$$\sqrt{L_p(n)(\hat{\alpha}_n - \alpha)} \to N(0, \tau_1^2/\tau_2^4)$$

as $n \to \infty$, where $\tau_1$ and $\tau_2$ are positive constants. Using our notation, $\tau_1$ can be written by $\tau_2^2 = \Gamma(1 - \alpha) I_\alpha$ with $I_\alpha$ being the Fisher Information of the discrete distribution defined by (3.1), so $\tau_1$ is interpretable. Combining this with $L_p(n) \sim M_{\alpha\theta} n^\alpha / \Gamma(1 - \alpha) \sim K_n / \Gamma(1 - \alpha)$ when $P$ follows the Pitman–Yor process, we obtain

$$\sqrt{K_n I_\alpha(\hat{\alpha}_n - \alpha)} = \sqrt{\frac{K_n I_\alpha}{L_p(n)}} \cdot \sqrt{L_p(n)(\hat{\alpha}_n - \alpha)} \to \sqrt{I_\alpha \Gamma(1 - \alpha)} \cdot N\left(0, \frac{\tau_1^2}{\tau_2^4}\right) = N\left(0, \tau_1^2 \cdot (I_\alpha \Gamma(1 - \alpha))^{-1}\right).$$

Thus, if $\tau_2^2 = \Gamma(1 - \alpha) I_\alpha$ holds, the above display coincides with Corollary 3.6.1. However, $\tau_2$ is a involved quantity so that we couldn’t check $\tau_2^2 = \Gamma(1 - \alpha) I_\alpha$. Furthermore, we stress again that their assumption $r(x) = o(x^\beta)$ with $\beta < \alpha/2$ is stronger than $r(x) = o(x^\alpha)$ for the Pitman–Yor process, and hence their result does not imply ours.

In summary, previous papers discuss the estimation of the tail index of the underlying discrete measure by using the Pitman–Yor process as a prior, but it is not clear whether the Pitman–Yor process satisfies their assumption. Furthermore, even if the Pitman–Yor process satisfies their assumption, we cannot obtain the exact asymptotic distribution of $\hat{\alpha}_n$ from their result. To the best of our knowledge, this problem was open until our paper was initially posted.

### 6.3 Our contribution

Considering the arguments in the previous section, the novelty of this paper is the exact asymptotic distribution of the MLE and the confidence interval of $\alpha$. In our proof, as highlighted in Section 5, we avoid the representation of the Ewens–Pitman partition by the Pitman–Yor process. Instead, we exploit the sequential definition of the Ewens–Pitman partition and its martingale property. Although our proof technique does not apply to misspecified settings, the main ingredients consist of the two asymptotics:

$$\exists M_{\alpha\theta} > 0 \text{ s.t. } \frac{K_n}{n^\alpha} \to M_{\alpha\theta}, \quad \frac{S_{n,j}}{K_n} \to \frac{\alpha \prod_{i=1}^{j-1} (i - \alpha)}{j!} (\text{a.s.}),$$
and they are universal properties of partitions induced by heavy-tailed discrete measures Gnedin et al. [2007]. Thus, extending our results to misspecified settings may be possible by imposing additional conditions on the behavior of $S_{n,j}$ and $K_n$. Furthermore, as a minor contribution, we make rigorous arguments on the simultaneous estimation of $(\alpha, \theta)$: Franssen and van der Vaart [2022a] consider the simultaneous estimation, but they still restrict the parameter space of $(\alpha, \theta)$ to be compact, which is not suitable since $\hat{\theta}_n$ does not degenerate as we have seen in Theorem 3.6.

7 Future topics

In this paper, we discussed the asymptotic analysis of the parameter estimation of $(\alpha, \theta)$ for the Ewens–Pitman partition in the regime of $0 < \alpha < 1$ and $\theta > -\alpha$. We derived the nonstandard asymptotic distribution, and we provide the confidence interval $[\hat{\alpha}_n \pm 1.96/\sqrt{K_n I(\hat{\alpha}_n)}]$ for $\alpha$. As an application, we can construct a hypothesis testing whether $0 < \alpha < \hat{\alpha}$ or $\hat{\alpha} \leq \alpha < 1$ for a given $\hat{\alpha} \in (0,1)$. However, considering that the asymptotics of the Ewens–Pitman partition change at $\alpha = 0$ (see Table 2 and Section 2.1), the testing of $\alpha = 0$ or $0 < \alpha < 1$ is of more statistical interest. Toward that, it is necessary to extend our results from the parameter space $\{0 < \alpha < 1, \theta > -\alpha\}$ to $\{0 \leq \alpha < 1, \theta > -\alpha\}$ that includes $\alpha = 0$. We expect that there is an increasing series $a_n \to +\infty$ and a law $F$ such that $a_n \hat{\alpha}_n \to F$ and $(\theta^{-1} \log n)^{1/2} \cdot (\hat{\theta}_n - \theta) \to N(0,1)$ weakly when $\alpha = 0$ and $\theta > 0$.

| Table 2: Phase transition at $\alpha = 0$ | $\alpha = 0$ | $0 < \alpha < 1$ |
|----------------------------------|-------------|-----------------|
| $K_n$                            | $O_p(\log n)$ | $O_p(n^{\alpha})$ |
| MLE $\hat{\theta}_n$            | Consistent   | Inconsistent    |
| Underlying discrete measure      | Dirichlet process | Pitman–Yor process |
| Network data                     | Dense       | Sparse         |

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**Acknowledgments**

The authors would like to thank Koji Tsukuda, Nobuaki Hoshino, and Stefano Favaro for helpful comments on our research.

**Funding**

Takeru Matsuda was supported by JSPS KAKENHI Grant Numbers 19K20220, 21H05205, 22K17865 and JST Moonshot Grant Number JPMJMS2024. Fumiyasu Komaki was supported by MEXT KAKENHI Grant Number 16H06533, JST CREST Grant Number JPMJCR1763, and AMED Grant Numbers JP21dm0207001 and JP21dm0307009.

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A Convergence of empirical measure

We fix $\alpha \in (0, 1)$ and $\theta > -\alpha$. We define the random measure and deterministic measure by

$$\forall j \in \mathbb{N}, \ P_n(j) := \frac{S_{n,j}}{K_n}, \ \mathbb{P}(j) = p_\alpha(j) = \frac{\alpha \prod_{i=1}^{j-1}(i-\alpha)}{j!}.$$  

Moreover, for any function $f$ on $\mathbb{N}$, we write $P_n f = \sum_{j=1}^\infty P_n(j) f(j)$ and $\mathbb{P} f = \sum_{j=1}^\infty \mathbb{P}(j) f(j)$. This section aims to show a suitable convergence of $P_n$ to $\mathbb{P}$.

**Lemma A.1.** $\sum_{j=1}^\infty |P_n(j) - \mathbb{P}(j)| \to 0$ a.s.

**Sketch of proof.** $S_{n,j}/K_n \to p_\alpha(j)$ by (B) of Lemma 2.2 and the dominated convergence theorem implies the claim. See the supplementary material for details.

**Corollary A.1.1.** $P_n f \to \mathbb{P} f$ a.s. for any bounded function $f$ on $\mathbb{N}$.

**Proof.** Take a positive constant $C$ s.t. $\|f\|_\infty \leq C$. Then, $|P_n f - \mathbb{P} f| \leq \|f\|_\infty \sum_{j=1}^\infty |P_n(j) - \mathbb{P}(j)| \leq C \sum_{j=1}^\infty |P_n(j) - \mathbb{P}(j)|$, where the upper bound converges to 0 a.s. by Lemma A.1. \qed

For each $x \in [0, 1)$, define the function $g_x$ on $\mathbb{N}$ by

$$\forall x \in [0, 1), \ \forall j \in \mathbb{N}, \ g_x(j) := \sum_{i=1}^{j-1} \frac{1}{i-x} \quad (A.1)$$

We claim that the leading term of the score function $\partial_\alpha \ell_n(\alpha, \theta)$ can be written as the expectation of $g_x$ with respect to $P_n$: (3.4) implies

$$\partial_\alpha \ell_n(\alpha, \theta) = \sum_{i=1}^{K_n-1} \frac{i}{\theta + i\alpha} - \sum_{j=1}^n S_{n,j} \sum_{i=1}^{j-1} \frac{1}{i}.$$  

Then, $K_n^{-1} \partial_\alpha \ell_n(\alpha, \theta)$ can be written as

$$\frac{\partial_\alpha \ell_n(\alpha, \theta)}{K_n} = \frac{1}{K_n} \left( \sum_{i=1}^{K_n-1} \frac{i}{\theta + i\alpha} - \sum_{j=1}^n S_{n,j} \sum_{i=1}^{j-1} \frac{1}{i} \right)$$

$$= \frac{K_n-1}{\alpha K_n} \frac{\theta}{\alpha K_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta + i\alpha} - \sum_{j=1}^n \frac{S_{n,j} K_n^{-1} \sum_{i=1}^{j-1} \frac{1}{i}}{K_n}.$$  

$$= \frac{1}{\alpha} - \mathbb{P}_n g_\alpha - \frac{1}{\alpha K_n} - \frac{\theta}{\alpha K_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta + i\alpha} \quad (A.2)$$

The next Lemma shows the convergence of $\mathbb{P}_n g_x$.  

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Lemma A.2. $P_n g_x \to^p P g_x < +\infty$ for all $x \in [0, 1]$.

It is important to emphasize that this Lemma does not directly follow from Corollary A.1.1 since $g_x$ is not bounded. For the proof, we define the deterministic function $\Psi : (0, 1) \to \mathbb{R}$ by

$$\forall x \in (0, 1), \Psi(x) := \frac{1}{x} - P g_x = \frac{1}{x} - \sum_{j=1}^{\infty} p_\alpha(j) \sum_{i=1}^{j-1} \frac{1}{i - x}.$$  \hspace{1cm} (A.3)

The next Lemma shows the basic properties of $\Psi$.

Lemma A.3. Let $I_\alpha$ be the Fisher information defined by (3.1). Then, $\Psi$ satisfies the following:

(A) $\Psi$ is of class $C^1$ on $(0, 1)$.

(B) $\Psi'(x) = -x^{-2} - \sum_{j=2}^{\infty} p_\alpha(j) \sum_{i=1}^{j-1} (i - x)^{-2} < 0$ and $\Psi'(\alpha) = -I_\alpha$.

(C) $\Psi(\alpha) = 0$, i.e., $P g_\alpha = \alpha^{-1}$

Sketch of proof. (A) and (B) easily follow from a simple calculation. (C) holds by $\partial_\alpha \sum_{i=1}^{\infty} p_\alpha(j) = \partial_\alpha 1 = 0$ and the change of differential and summation. See the supplementary material for details.

Proof of Lemma A.2. $P g_x < +\infty$, i.e., the convergence of the infinite sum $\sum_{j=1}^{\infty} p_\alpha(j) \sum_{i=1}^{j-1} (i - x)^{-1}$, is shown in the proof of Lemma A.3, so we assume it. Suppose $P_n g_x \to^p P g_x$ holds for $x = \alpha$. Then, the triangle inequality implies

$$\forall x \in [0, 1], \ |P_n g_x - P g_x| \leq |P_n g_\alpha - P g_\alpha| + |P_n (g_x - g_\alpha) - P (g_x - g_\alpha)|,$$

where the first term is $o_p(1)$ by the assumption. For the second term, we observe that $g_x - g_\alpha$ is a bounded function:

$$\forall j \in \mathbb{N}, \ |g_x(j) - g_\alpha(j)| = \left| \sum_{i=1}^{j-1} \left( \frac{1}{i - x} - \frac{1}{i - \alpha} \right) \right| \leq \sum_{i=1}^{\infty} \frac{|x - \alpha|}{(i - x)(i - \alpha)} < +\infty.$$

Then, Corollary A.1.1 applied with $f = g_x - g_\alpha$ implies $|P_n (g_x - g_\alpha) - P (g_x - g_\alpha)| = o_p(1)$, which concludes the proof. Thus, it suffices to show the claim for $x = \alpha$.

Now (A.2) and $\alpha^{-1} = P g_\alpha$ by (C) of Lemma A.3 implies

$$P_n g_\alpha - P g_\alpha = P_n g_\alpha - \frac{1}{\alpha} = -\frac{1}{\alpha K_n} - \frac{\theta}{\alpha K_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta + i\alpha} - \frac{\partial_\alpha \ell_n(\alpha, \theta)}{K_n},$$

$$= O(K_n^{-1}) + O(K_n^{-1} \log K_n) - K_n^{-1} \partial_\alpha \ell_n(\alpha, \theta)$$

$$\stackrel{(s)}{=} o_p(1) - K_n^{-1} \partial_\alpha \ell_n(\alpha, \theta)$$

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where $(\star)$ follows from $n^{-\alpha}K_n \to M_{\alpha \theta} > 0$ (a.s.) by Lemma 2.2. Now we claim
\[
\partial_\alpha \ell_n(\alpha, \theta) = o_p(n^\alpha)
\]
by the following reason:
\[
\forall \epsilon > 0, \Pr \left( \left| n^{-\alpha} \cdot \partial_\alpha \ell_n(\alpha, \theta) \right| > \epsilon \right) \leq n^{-2\alpha} \epsilon^{-2} \mathbb{E}[(\partial_\alpha \ell_n(\alpha, \theta))^2] = n^{-2\alpha} \epsilon^{-2} I_{\alpha \alpha}^{(n)} \\
(\star)\implies n^{-2\alpha} \epsilon^{-2} O(n^\alpha) = O(n^{-\alpha}) = o(1),
\]
where $(\star)$ is from Chebyshev’s inequality, and in $(\star)$, we used $I_{\alpha \alpha}^{(n)} = O(n^\alpha)$ by
Proposition 3.3 (see Appendix D.3 for the proof). Putting this and $K_n / n^{\alpha} \to M_{\alpha \theta} > 0$ (a.s.) together, we obtain
$K_n^{-1} \partial_\alpha \ell_n(\alpha, \theta) = o_p(1)$, which concludes the
proof.

B Proofs for QMLE

In this section, we prove the asymptotic properties of the QMLE.

B.1 Proof of Proposition 3.7

Here, we prove the existence and uniqueness of QMLE. Assume $n \geq 2$. We
observe \( \forall x \in (-\theta^* \lor 0, 1), \partial_x^2 \ell_n(x, \theta^*) = - \sum_{i=1}^{K_n-1} \frac{i^2}{(\theta^* + ix)^2} - \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{(i-x)^2}. \)
(B.1)

Now we claim that $\partial_x^2 \ell_n(x, \theta^*)$ is strictly negative with probability 1: The first
term is negative if $K_n > 1$, and otherwise $S_{n,n} = 1$ so $K_n = 1$ and hence the
second term is $- \sum_{j=1}^{n-1} (i-x)^{-2} < 0$. Thus, QMLE $\hat{\alpha}_{n,\theta}$ exists and equals the unique solution of $\partial_x \ell_n(\cdot, \theta^*) = 0$ if and only if $\lim_{x \to (-\theta^* \lor 0)^+} \partial_x \ell_n(x, \theta^*) > 0 > \lim_{x \to 1-} \partial_x \ell_n(x, \theta^*)$. The necessary and sufficient condition is given by
[Carlton, 1999, Lemma 5.1]
\[
1 < K_n < n \text{ and } \theta^* < \Theta_n := \frac{K_n(K_n-1)}{\sum_{j=2}^{n} 2S_{n,j} \sum_{i=1}^{j-1} i^{-1}}.
\]
(B.2)

See the supplementary material for the derivation of (B.2). We observe that the first condition is satisfied with a high probability since $K_n / n^\alpha \to M_{\alpha \theta} > 0$ (a.s.) holds with $\alpha \in (0, 1)$. For the second condition, we claim the followings:
\[
\Theta_n = \frac{1}{2 \sum_{j=2}^{n} S_{n,j} K_n \cdot \sum_{i=1}^{j-1} i^{-1}} \xrightarrow{(\star)} \frac{1}{2 \sum_{j=2}^{n} S_{n,j} K_n \cdot \sum_{i=1}^{j-1} i^{-1}} \xrightarrow{\mathbb{P}} \frac{1}{2 \sum_{j=2}^{n} S_{n,j} K_n} < +\infty.
\]

For $(\star)$, we used the definition of $g_x$ (A.1) with $x = 0$. $(\star)$ follows from
Lemma A.2 applied with $x = 0$. This result and $K_n \to +\infty$ (a.s.) lead to
$\Pr(\theta^* \geq \Theta_n) = o(1)$, which concludes that the second condition in (B.2) is satisfied with a high probability.
B.2 Proof of (3.11) in Proposition 3.9

Here, we derive the asymptotic error between the QMLE $\hat{\alpha}_{n,\theta}$ and the MLE $\hat{\alpha}_{n,\theta}$ with $\theta$ being well specified. For each $\theta^* \in (-\alpha, \infty)$, we define the random function $\hat{\Psi}_{n,\theta}$ by

$$\forall x \in (-\theta^* \lor 0, 1), \; \hat{\Psi}_{n,\theta}(x) := K_n^{-1} \cdot \partial_x \ell_n(x, \theta^*).$$

From (A.2) with $\theta = \theta^*$, we can write $\hat{\Psi}_{n,\theta}$ by

$$\hat{\Psi}_{n,\theta}(x) = -\frac{1}{xK_n} - \frac{\theta^*}{xK_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta^* + ix} + \frac{1}{x} - \mathbb{P}_n g_x, \quad (B.3)$$

where $g_x(j) = \sum_{i=1}^{j-1} (i - x), \forall j \in \mathbb{N}$. First we prove some convergence of $\hat{\Psi}_{n,\theta}$ to $\Psi$, where $\Psi$ was defined by $\Psi(x) = x^{-1} - \mathbb{P} g_x$ in (A.3).

**Lemma B.1.** For each $\theta^* \in (-\alpha, \infty)$, it holds that

(A) $\hat{\Psi}_{n,\theta}(x) \xrightarrow{p} \Psi(x)$ for all $x \in (-\theta^* \lor 0, 1)$.

(B) $\sup_{x \in I} |\hat{\Psi}_{n,\theta}(x) - \Psi'(x)| \to 0 \ a.s.$ for any closed subsets $I \subset (-\theta^* \lor 0, 1)$.

**Sketch of proof.** Observe

$$\hat{\Psi}_{n,\theta}(x) - \Psi(x) = -\frac{1}{xK_n} - \frac{\theta^*}{xK_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta^* + ix} - \mathbb{P}_n g_x + \mathbb{P} g_x \quad (B.4)$$

Then, (A) is obvious from this and Lemma A.2. For (B), define the bounded function $h_x$ by $h_x(j) = \partial_x g_x(j) = \sum_{i=1}^{j-1} (i - x)^2$. Then, it easily follows that

$$|\hat{\Psi}_{n,\theta}'(x) - \Psi'(x)| \leq \frac{1}{K_n x^2} + \frac{|\theta^*|}{x^2 K_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta^* + ix} + \frac{|\theta^*|}{xK_n} \sum_{i=1}^{K_n-1} \frac{i}{(\theta^* + ix)^2} + |\mathbb{P}_n h_x - \mathbb{P} h_x|,$$

for all $x \in (-\theta^* \lor 0, 1)$, and we can show that each term is uniformly $o_p(1)$. See the supplementary material for details.

Applying (B) above, we obtain the following Lemma:

**Lemma B.2.** Let $I_\alpha$ be the Fisher information defined by (3.1). Then for any sequence of random variables $(\tilde{\alpha}_n)_{n \geq 1}$ s.t. $\tilde{\alpha}_n \xrightarrow{p} \alpha$, and for all $\theta^* \in (-\alpha, \infty)$, $\hat{\Psi}_{n,\theta}'(\tilde{\alpha}_n) \xrightarrow{p} -I_\alpha < 0$ holds.

**Proof.** $\Psi'(\alpha) = -I_\alpha$ by (B) of Lemma A.3 and the triangle inequality imply

$$|\hat{\Psi}_{n,\theta}'(\tilde{\alpha}_n) + I_\alpha| \leq |\Psi'(\tilde{\alpha}_n) - \Psi'(\alpha)| + |\hat{\Psi}_{n,\theta}'(\tilde{\alpha}_n) - \Psi'(\tilde{\alpha}_n)|,$$
where the first term is $o_p(1)$ from the continuity of $\Psi'$ on $(0,1)$ (see (A) of Lemma A.3) and $\hat{\alpha}_n - \alpha = o_p(1)$ by the assumption. For the second term, take a sufficiently small $\delta$ s.t. $B_\delta(\alpha) : = [\alpha \pm \delta] \subset (-\theta \lor 0,1)$. Then,

$$\forall \epsilon > 0, \text{ Pr}(|\hat{\Psi}'_{n,\theta^*}(\hat{\alpha}_n) - \Psi'(\hat{\alpha}_n)| > \epsilon) \leq \text{ Pr}(\sup_{x \in B_\delta(\alpha)} |\hat{\Psi}'_{n,\theta^*}(x) - \Psi'(x)| > \epsilon) + \text{ Pr}(\hat{\alpha} \notin B_\delta),$$

where the second term is $o(1)$ by $\hat{\alpha}_n \to^p \alpha$ and the first term is also $o(1)$ from (B) of Lemma B.1 applied with $I = B_\delta(\alpha)$. This concludes the proof.

The next Lemma shows the consistency of QMLE.

**Lemma B.3.** $\hat{\alpha}_{n,\theta^*} \to^p \alpha$ for each $\theta^* \in (-\alpha, \infty)$.

**Proof.** Proposition 3.7 implies $\hat{\Psi}'_{n,\theta^*}(x)$ is strictly decreasing in $x \in (-\theta^* \lor 0,1)$ and $\hat{\Psi}'_{n,\theta^*}(\hat{\alpha}_{n,\theta^*}) = 0$ with a high probability. Under this event, for sufficiently small $\epsilon > 0$ s.t. $\epsilon < (\alpha - (-\theta^* \lor 0)) \land (1 - \alpha)$, it holds that

$$\text{ Pr}(|\hat{\alpha}_{n,\theta^*} - \alpha| > \epsilon) \leq \text{ Pr}(\hat{\alpha}_{n,\theta^*} < \alpha - \epsilon) + \text{ Pr}(\hat{\alpha}_{n,\theta^*} > \alpha + \epsilon) \leq \text{ Pr}(0 > \hat{\Psi}_{n,\theta^*}(\alpha - \epsilon)) + \text{ Pr}(0 < \hat{\Psi}_{n,\theta^*}(\alpha + \epsilon)) = \text{ Pr}(\Psi(\alpha - \epsilon) - \hat{\Psi}_{n,\theta^*}(\alpha - \epsilon) > \Psi(\alpha - \epsilon)) + \text{ Pr}(\hat{\Psi}_{n,\theta^*}(\alpha + \epsilon) - \Psi(\alpha + \epsilon) > -\Psi(\alpha + \epsilon)).$$

Note $\Psi(\alpha - \epsilon) > 0 > \Psi(\alpha + \epsilon)$ by (B) and (C) of Lemma A.3. Then, (A) of Lemma B.1 implies that the upper bounds are $o(1)$. This concludes the proof.

The next Lemma derives the asymptotic error between the two QMLE $(\hat{\alpha}_{n,\theta^*}, \hat{\alpha}_{n,\theta^*})$, where $\hat{\alpha}_{n,0} = \hat{\alpha}_{n,\theta^*} = 0$.

**Lemma B.4.** For each $\theta^* \in (-\alpha, \infty)$, it holds that

$$\frac{n^{\alpha}}{\log n} (\hat{\alpha}_{n,\theta^*} - \hat{\alpha}_{n,0}) \to^p \frac{-\theta^*}{I_\alpha M_{\alpha\theta}}.$$ 

If we take $\theta^* = \theta$, $\theta$ in Lemma B.4, we obtain

$$(n^{\alpha}/\log n) \cdot (\hat{\alpha}_{n,\theta} - \hat{\alpha}_{n,0}) = (n^{\alpha}/\log n) \cdot (\hat{\alpha}_{n,\theta^*} - \hat{\alpha}_{n,0}) - (n^{\alpha}/\log n) \cdot (\hat{\alpha}_{n,\theta} - \hat{\alpha}_{n,0}) \to -\theta^*/(\alpha I_\alpha M_{\alpha\theta}) + \theta/(\alpha I_\alpha M_{\alpha\theta}) = -(-\theta^* - \theta)/(\alpha I_\alpha M_{\alpha\theta}),$$

Which leads to (3.11) in Proposition 3.9 which is what we want to prove in this section. Thus, in the rest of this section, we show Lemma B.4.

Recall that Proposition 3.7 implies $\hat{\Psi}_{n,0}(\hat{\alpha}_{n,0}) = 0$ and $\hat{\Psi}_{n,\theta^*}(\hat{\alpha}_{n,\theta^*}) = 0$ with a high probability. Under this event, Taylor’s theorem implies that there exists $\alpha$ between $\hat{\alpha}_{n,\theta}$ and $\hat{\alpha}_{n,\theta^*}$ s.t.

$$0 = \hat{\Psi}_{n,0}(\hat{\alpha}_{n,0}) = \hat{\Psi}_{n,0}(\hat{\alpha}_{n,\theta^*}) + \hat{\Psi}'_{n,0}(\hat{\alpha}_n)(\hat{\alpha}_{n,0} - \hat{\alpha}_{n,\theta^*}).$$

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Lemma B.5. For the digamma function we introduce the basic property of the digamma function. It remains to be proven that \( \ln(1 + (\hat{\alpha}_n - \alpha)) \rightarrow_p 1 \). Then, Lemma B.2 applied with \( \theta^* = 0 \) implies \( -\hat{\Psi}'_{n,0}(\hat{\alpha}_n) \rightarrow_p I_\alpha \). In contrast, the expression of \( \hat{\Psi}_{n,\theta^*} \) given by (B.3) leads to

\[
\hat{\Psi}_{n,0}(\hat{\alpha}_{n,\theta^*}) = \hat{\Psi}_{n,0}(\hat{\alpha}_{n,\theta^*}) - \hat{\Psi}_{n,\theta^*}(\hat{\alpha}_{n,\theta^*}) = \frac{1}{K_n \hat{\alpha}_{n,\theta^*}} \sum_{i=1}^{K_n-1} \frac{\theta^*}{\theta^* + i \hat{\alpha}_{n,\theta^*}}.
\]

Thus, \( n^\alpha/\ln n \cdot (\hat{\alpha}_{n,\theta^*} - \hat{\alpha}_{n,0}) \) is expressed as

\[
n^\alpha/\ln n (\hat{\alpha}_{n,\theta^*} - \hat{\alpha}_{n,0}) = n^\alpha \frac{\hat{\Psi}_{n,0}(\hat{\alpha}_{n,\theta^*})}{\hat{\Psi}_{n,0}(\hat{\alpha}_n)} = n^\alpha \frac{1}{\ln n} \sum_{i=1}^{K_n-1} \frac{\theta^*}{\theta^* + i \hat{\alpha}_{n,\theta^*}}.
\]

Applying this lemma to \( (\theta^*/\hat{\alpha}_{n,\theta^*})/K_n \rightarrow_p 1 \) and the basic equation \( \sum_{i=1}^{m-1} (i + c) = \psi(m + c) - \psi(1 + c) \), we obtain

\[
\frac{1}{i + \theta^*/\hat{\alpha}_{n,\theta^*}} - \ln K_n = \psi(K_n + \theta^*/\hat{\alpha}_{n,\theta^*}) - \psi(1 + \theta^*/\hat{\alpha}_{n,\theta^*}) - \ln K_n
\]

\[
= o_P(1) - \psi(1 + \theta^*/\hat{\alpha}_{n,\theta^*}) = O_p(1),
\]

where \((*)\) follows from \( \hat{\alpha}_{n,\theta^*} \rightarrow_p 1 \) and \( \psi(1 + \theta/\alpha) < +\infty \) by \( \theta > -\alpha \). Combining this and \( \ln K_n \rightarrow_\alpha \infty \) a.s., we obtain \( (\ln K_n)^{-1} \sum_{i=1}^{K_n-1} (i + \theta^*/\hat{\alpha}_{n,\theta^*})^{-1} \rightarrow_p 1 \), thereby completing the proof.

### B.3 Proof of Proposition 3.8

Here, we prove the asymptotic mixed normality of QMLE. The next Lemma gives stable Martingale CLT for general settings.
Lemma B.6 ([Häusler and Luschgy, 2015, p. 109]). Let \( (X_n)_{n \geq 1} \) be a martingale difference sequence with respect to a filtration \( \mathcal{F} = (\mathcal{F}_n)_{n \geq 1} \) and define \( \mathcal{F}_\infty := \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n) \). For a sequence of positive real number \( (a_n)_{n \geq 1} \) with \( a_n \to \infty \), we assume the following two conditions:

(i) \( a_n^{-2} \sum_{m=1}^n \mathbb{E}[X_m^2 | \mathcal{F}_{m-1}] \toP \eta^2 \) for some random variable \( \eta \geq 0 \).

(ii) \( a_n^{-2} \sum_{m=1}^n \mathbb{E}[X_m^2 \mathbb{1}\{|X_m| \geq \epsilon a_n\} | \mathcal{F}_{m-1}] \toP 0 \) for all \( \epsilon > 0 \).

Then, \( a_n^{-1} \sum_{m=1}^n X_m \to \eta N \) \( \mathcal{F}_\infty \)-stably holds, where \( N \sim \mathcal{N}(0,1) \) is independent of \( \mathcal{F}_\infty \).

Corollary B.6.1. In the setting of Lemma B.6, if \( X_n \) is a bounded random variable for all \( n \in \mathbb{N} \), (i) is sufficient for \( a_n^{-1} \sum_{m=1}^n X_m \to \eta N \) \( \mathcal{F}_\infty \)-stable to hold.

Proof. It suffices to check that (ii) in Lemma B.6 is satisfied. Take a constant \( C \) s.t. \( |X_n| \leq C \) a.e. Then, \( a_n \to \infty \) implies that \( \forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \) s.t. \( n \geq N_\epsilon \Rightarrow a_n > C/\epsilon \). Observe that \( n > N_\epsilon \Rightarrow |X_n| < C < a_n \epsilon \). Then, for all \( n > N_\epsilon \),

\[
\sum_{m=1}^n \mathbb{E}[X_m^2 \mathbb{1}\{|X_m| \geq \epsilon a_n\} | \mathcal{F}_{m-1}] = \sum_{m=1}^{N_\epsilon} \mathbb{E}[X_m^2 \mathbb{1}\{|X_m| \geq \epsilon a_n\} | \mathcal{F}_{m-1}] \leq N_\epsilon C^2,
\]

which implies that \( a_n^{-2} \sum_{m=1}^n \mathbb{E}[X_m^2 \mathbb{1}\{|X_m| \geq \epsilon a_n\} | \mathcal{F}_{m-1}] \leq \alpha_n^{-2} N_\epsilon C^2 \to 0 \) (a.s.), especially in probability. This completes the proof.

Applying Corollary B.6.1 with \( X_n \) taken to be the increment of the score function, we obtain the asymptotic mixed normality:

Lemma B.7. For \( I_\alpha \) and \( \ell_n(\alpha, \theta) \) defined by (3.1) and (3.4) respectively, we have

\[
n^{-\alpha/2} \cdot \partial_\ell \ell_n(\alpha, \theta) \to \sqrt{M_{\alpha \theta}} I_\alpha \cdot N \text{ \( \mathcal{F}_\infty \)-stably as } n \to \infty,
\]

where \( M_{\alpha \theta} = \lim_{n \to \infty} n^{-\alpha} K_n \) and \( N \sim \mathcal{N}(0,1) \) is independent of \( \mathcal{F}_\infty \).

Proof. We take \( X_n \) to be the increment of \( \partial_\ell \ell_n(\alpha, \theta) \) with \( a_n = n^{\alpha/2} \) and \( \eta^2 = M_{\alpha \theta} I_\alpha \) in Corollary B.6.1. Since the score functions are \( \mathcal{F}_n \)-martingale, \( (X_n)_{n=1}^\infty \) is an \( \mathcal{F}_n \)-martingale difference sequence. Then, it remains to show that \( X_n \) is a bounded random variable and \( n^{-\alpha} \sum_{m=1}^n \mathbb{E}[X_m^2 | \mathcal{F}_{m-1}] \toP I_\alpha M_{\alpha \theta} \).

First, we check that \( X_n \) is a bounded random variable. Recall

\[
\partial_\ell \ell_n(\alpha, \theta) = \sum_{i=1}^{K_n-1} \frac{i}{\theta + i \alpha} - \sum_{j=1}^{n-1} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{i - \alpha}.
\]

The sequential definition given by Section 2.1 implies that the \((m+1)\)-th ball belongs to a new urn with probability \( (\theta + K_m \alpha)/(\theta + m) \), or one of the urns of size \( l \) with probability \( S_{m,l}(l-\alpha)/(\theta + m) \) for each \( l = 1, 2, \ldots, m \). In the
Thus, the second term in (B.6) converges to 0 a.s. The first term, \( Y_{m,l} \), can be written as

\[
Y_m = \frac{m}{m + \theta} \frac{K_m}{m^\alpha} \left\{ \frac{K_m}{\alpha^2(\theta/\alpha + K_m)} + \sum_{j=1}^{m} \frac{S_{m,j}}{K_m(\alpha(j - \alpha))} \right\}
\]

The triangle inequality implies that

\[
|n^{-\alpha} \sigma_n^2 - I_{\alpha} M_{\alpha\theta}| = \left| n^{-\alpha} \sum_{m=1}^{n-1} c_m Y_m - I_{\alpha} M_{\alpha\theta} \right|
\]

\[
\leq n^{-\alpha} \left| \sum_{m=1}^{n-1} c_m Y_m - I_{\alpha} M_{\alpha\theta} \right| + I_{\alpha} M_{\alpha\theta} \left| n^{-\alpha} \sum_{m=1}^{n-1} c_m - 1 \right| \quad \text{.} \quad \text{(B.6)}
\]

Now we claim that \( n^{-\alpha} \sum_{m=1}^{n-1} c_m \to 1 \) by

\[
(n-1)^\alpha = \int_0^{n-1} \alpha x^{\alpha-1} \, dx < \int_1^{n} \alpha x^{\alpha-1} \, dx = n^{\alpha} - 1.
\]

Thus, the second term in (B.6) converges to 0 a.s. The first term, \( Y_m \), can be written as

\[
Y_m = \frac{m}{m + \theta} \frac{K_m}{m^\alpha} \left\{ \frac{K_m}{\alpha^2(\theta/\alpha + K_m)} + \sum_{j=1}^{m} \frac{S_{m,j}}{K_m(\alpha(j - \alpha))} \right\}
\]

\[
= \frac{m}{m + \theta} \frac{K_m}{m^\alpha} \left\{ \frac{K_m}{\alpha^2(\theta/\alpha + K_m)} + \mathbb{P}(h) \right\}
\]
where \( h \) is the bounded function defined by \( h(j) = 1/(\alpha(j-\alpha)) \). Corollary A.1.1 implies \( \lim_{m \to \infty} \mathbb{P}_m h \to \mathbb{P} h \) (a.s.). Combining this and \( K_m/m^\alpha \to M_{\alpha \theta} \) (a.s.), we have

\[
Y_m \xrightarrow{a.s.} 1 \cdot M_{\alpha \theta} \cdot (\alpha^{-2} + \mathbb{P} h) \xrightarrow{(*)} M_{\alpha \theta} I_\alpha,
\]  

(B.7)

where \( (*) \) follows from the formula (B) of Proposition 3.1 and the definition \( \mathbb{P} h := \sum_j p_\alpha(j) h(j) \). (B.7) implies that \( \forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \) s.t. \( m > N_\epsilon \Rightarrow |Y_m - M_{\alpha \theta} I_\alpha| < \epsilon \) (a.e.). Then, the first term in (B.6) is bounded by

\[
\left| n^{-\alpha} \sum_{m=1}^{n-1} c_m (Y_m - M_{\alpha \theta} I_\alpha) \right| \leq n^{-\alpha} \sum_{m=1}^{n-1} c_m |Y_m - M_{\alpha \theta} I_\alpha|
\]

\[
\leq n^{-\alpha} \sum_{m=1}^{N_\epsilon} c_m |Y_m - M_{\alpha \theta} I_\alpha| + \epsilon \cdot n^{-\alpha} \sum_{m=1}^{n-1} c_m = n^{-\alpha} O_p(1) + \epsilon \cdot n^{-\alpha} \sum_{m=1}^{n-1} c_m
\]

\[
\to P 0 + \epsilon \cdot 1 = \epsilon,
\]

where we used \( n^{-\alpha} \sum_{m=1}^{n-1} c_m \to 1 \) for the last convergence. Since \( \epsilon > 0 \) is taken arbitrarily, \( n^{-\alpha} \sum_{m=1}^{n-1} c_m (Y_m - M_{\alpha \theta} I_\alpha) \to P 0 \) holds, thereby completing the proof of \( \alpha^{-n} \sum_{m=1}^{n} E [X_m^2 | F_{m-1}] \to P I_\alpha M_{\alpha \theta} \).

Finally, we prove the asymptotic mixed normality of QMLE.

**Proof of Proposition 3.8.** \( \hat{\alpha}_{n, \theta^*} - \hat{\alpha}_{n, \theta} = O_p(n^{-\alpha} \log n) \) by (3.11) in Proposition 3.9 implies

\[
n^{\alpha/2}(\hat{\alpha}_{n, \theta^*} - \alpha) = n^{\alpha/2}(\hat{\alpha}_{n, \theta^*} - \hat{\alpha}_{n, \theta}) + n^{\alpha/2}(\hat{\alpha}_{n, \theta} - \alpha)
\]

\[
= O_p(n^{-\alpha/2} \cdot \log n) + n^{\alpha/2}(\hat{\alpha}_{n, \theta} - \alpha) = o_p(1) + n^{\alpha/2}(\hat{\alpha}_{n, \theta} - \alpha),
\]

for all \( \theta^* \in (-\alpha, \infty) \). Therefore, it suffices to prove the claim for \( \theta^* = \theta \). Let \( \Phi_{n, \theta}(\alpha) = K_n^{-1} \partial_\alpha \ell_n(\alpha, \theta) \). Then \( \Phi_{n, \theta}(x) < 0 \) for all \( x \in (-\theta \lor 0, 1) \) and \( \Phi_{n, \theta}(\alpha_n) = 0 \) with a high probability by Proposition 3.7. Under this event, Taylor’s theorem implies that there exists \( \alpha_n \) between \( \alpha \) and \( \alpha_{n, \theta} \) (hence \( \alpha_n - \alpha = o_p(1) \) by Lemma B.3 with \( \theta^* = \theta \)) s.t. \( \Phi_{n, \theta}(\alpha) + \Phi'_{n, \theta}(\alpha_n)(\hat{\alpha}_{n, \theta} - \alpha) = 0 \). Combining this and \( \Phi(\alpha) = K_n^{-1} \partial_\alpha \ell_n(\alpha, \theta) \), we obtain

\[
\sqrt{n^{\alpha}}(\hat{\alpha}_{n, \theta} - \alpha) = \sqrt{I_\alpha n^{\alpha}} \cdot \frac{\hat{\Phi}_{n, \theta}(\alpha_n)}{-\Phi'_{n, \theta}(\alpha_n)} = \frac{I_\alpha}{\sqrt{n^{\alpha} I_\alpha}} \cdot \frac{n^{\alpha}}{-\Phi'_{n, \theta}(\alpha_n) K_n} \]

\[
= X_n \cdot Y_n \rightarrow \sqrt{M_{\alpha \theta} N} \cdot M_{\alpha \theta}^{-1} = N / \sqrt{M_{\alpha \theta}} \) (\( F_\infty \)-stable),
\]

which concludes the proof. \( \square \)
C Proofs for MLE

We have discussed QMLE so far, where the unknown \( \theta \) is fixed. In this section, we consider the MLE (\( \hat{\alpha}_n, \hat{\theta}_n \)) that simultaneously estimates (\( \alpha, \theta \)). The main difficulty here is that \( \hat{\theta}_n \) does not converge to a fixed value, so we need more technical arguments than before.

First, we reduce the dimension of the parameter that we have to consider: we define the function \( \hat{y}_n : (0, 1) \rightarrow (-1, \infty) \) by

\[
\forall x \in (0, 1), \quad \hat{y}_n(x) := \arg \max_{y > -x} \ell_n(x, y),
\]

where \( \ell_n(x, y) = \ell_n(\alpha, \theta) \mid (\alpha, \theta) = (x, y) \) is the log likelihood. The next Lemma shows that \( \hat{y}_n \) is well-defined.

**Lemma C.1.** Suppose \( 1 < K_n < n \). Then, for each \( x \in (0, 1) \), the solution \( \partial_y \ell_n(x, \cdot) = 0 \) uniquely exists and \( \partial_y^2 \ell_n(x, \cdot) \) takes a negative value at the solution.

**Proof.** This lemma immediately follows from [Carlton, 1999, Lemma 5.3] with \( \alpha \) replaced by \( x \). See Appendix D. \( \square \)

Observe that the condition \( 1 < K_n < n \) holds with a high probability since \( n^{-\alpha} K_n M_{\alpha \theta} \rightarrow M_{\alpha \theta} > 0 \) (a.s.). From now on, we assume \( 1 < K_n < n \).

Lemma C.1 claims that the log-likelihood is a unimodal function of \( y \) if we fix \( x \). Therefore \( \hat{y}_n(x) \), defined as the maxima, is well-defined and uniquely characterized by

\[
\forall x \in (0, 1), \quad \partial_y \ell_n(x, \hat{y}_n(x)) = 0.
\] (C.1)

Furthermore, the implicit function theorem with \( \partial_y^2 \ell_n(x, \hat{y}_n(x)) < 0 \) implies that \( \hat{y}_n \) is differentiable.

The gain of introducing \( \hat{y}_n \) is that the MLE (\( \hat{\alpha}_n, \hat{\theta}_n \)) defined as the maxima of the log-likelihood with respect to (\( \alpha, \theta \)) can be formulated as the solution of the one-dimensional maximization problem:

\[
\hat{\alpha}_n = \arg \max_{x \in [\alpha, \bar{\alpha}]} \ell_n(x, \hat{y}_n(x)), \quad \hat{\theta}_n = \hat{y}_n(\hat{\alpha}_n).
\] (C.2)

Here, in a manner similar to \( \hat{\Psi}_{n, \theta} \), (B.3), we define the random function \( \hat{\Psi}_n \) by

\[
\forall x \in (0, 1), \quad \hat{\Psi}_n(x) := \frac{1}{K_n} \cdot \frac{d}{dx} \ell_n(x, \hat{y}_n(x)).
\] (C.3)

Now the differentiability of \( \hat{y}_n \) and \( \partial_y \ell_n(x, \hat{y}_n(x)) = 0 \) lead to

\[
\hat{\Psi}_n(x) = K_n^{-1} \partial_x \ell_n(x, \hat{y}_n(x)) + K_n^{-1} \partial_y \ell_n(x, \hat{y}_n(x)) \cdot \hat{y}_n'(x) = K_n^{-1} \partial_x \ell_n(x, \hat{y}_n(x)).
\]

Thus, if the root of \( \hat{\Psi}_n \) uniquely exists in \([\alpha, \bar{\alpha}]\) and its derivative takes a negative at the points, the solution of \( \partial_x \ell_n(x, y) = \partial_y \ell_n(x, y) = 0 \) uniquely exists and equals (\( \hat{\alpha}_n, \hat{\theta}_n \)).
The next lemma shows a suitable convergence of $\hat{\Psi}_n$ to $\Psi$ that is similar to the convergence of $\hat{\Psi}_{n, \theta}^\star$ to $\Psi$ in Lemma B.1. The main difference is that $\hat{\Psi}'_n$ converges to $\Psi'$ in a local neighborhood of $\alpha$. We will observe later that it is sufficient for our arguments.

**Lemma C.2.** For $\hat{\Psi}_n$ and $\Psi$ defined by (C.3) and (A.3) respectively, the following holds.

(A) $\sup_{x \in I} |\hat{\Psi}_n(x) - \Psi(x)| = o_p(1)$ for all closed subset $I \subset (0, 1)$.

(B) $\sup_{x \in [\alpha \pm \delta_n]} |\hat{\Psi}_n'(x) - \Psi'(x)| = o_p(1)$ for any positive value $\delta_n = o(1/\log n)$.

Toward the proof, we introduce the two Lemmas.

**Lemma C.3.** Define the function $\hat{\gamma}_n : (0, 1) \to (-1, \infty)$ by

\[ \hat{\gamma}_n : x \mapsto \hat{\gamma}_n(x) = \frac{\hat{\Psi}_n(x)}{x} \quad (C.4) \]

where $\hat{\gamma}_n$ is the function characterized by (C.1). Then, $\hat{\Psi}_n$ can be written by

\[ \hat{\Psi}_n(x) = \frac{1}{K_n} - \frac{1}{K_n} \sum_{i=1}^{n-1} \frac{\hat{\gamma}_n(x)}{x} \hat{\gamma}_n(x) + i - \frac{1}{x} - P_n g_x, \]

where $g_x$ is the function s.t. $g_x(j) = \sum_{i=1}^{j-1} (i - x)^{-1}$.

Then, $\hat{\Psi}_n - \Psi$ is now simplified to

\[ \hat{\Psi}_n(x) - \Psi(x) = \frac{1}{K_n} - \frac{1}{K_n} \sum_{i=1}^{n-1} \frac{\hat{\gamma}_n(x)}{x} \hat{\gamma}_n(x) + i - (P_n g_x - g_x), \quad (C.5) \]

where the first and the third term have been well studied. This simplification motivates us to derive the stochastic order of $\hat{\gamma}_n$.

**Lemma C.4.** $\hat{\gamma}_n$ satisfies the followings:

(A) $\hat{\gamma}_n(x)$ is strictly decreasing in $x$, and its derivatives are given by

\[ \hat{\gamma}_n'(x) = \frac{\sum_{i=1}^{n-1} (x \hat{\gamma}_n(x) + i)^{-2}}{-\sum_{i=1}^{n-1} (\hat{\gamma}_n(x) + i)^{-2} + \sum_{i=1}^{n-1} x^2 \cdot (x \hat{\gamma}_n(x) + i)^{-2}.} \]

(B) $|\hat{\gamma}_n(x)| = \begin{cases} O_p(n^{-2}) & (0 < x < \alpha) \\ O_p(1) & (\alpha \leq x < 1) \end{cases}$

(C) For all $\delta_n = o(1/\log n)$,

\[ \sup_{x \in [\alpha \pm \delta_n]} |\hat{\gamma}_n(x)| = O_p(1) \quad \text{and} \quad \sup_{x \in [\alpha \pm \delta_n]} |f_\alpha(\hat{\gamma}_n(x)) - \log(K_n/n^\alpha)| = o_p(1), \]

where $f_\alpha$ is the function defined by (3.2).
Proof of Lemma C.2. First, we prove (A). We write $I = [s, t]$ with $0 < s < t < 1$. Then, (C.5) and the triangle inequality implies

$$\sup_{x \in I} |\hat{\Psi}(x) - \Psi(x)| \leq \sup_{x \in I} \left\{ \frac{1}{K_n} \sum_{i=1}^{n-1} \left| \frac{\hat{z}_n(x)}{x \hat{z}_n(x) + i} \right| + |\mathbb{P}_n g_x - \mathbb{P} g_x| \right\}$$

$$\leq \frac{1}{K_n} s + \sup_{x \in I} \frac{1}{K_n} \sum_{i=1}^{n-1} \left| \frac{\hat{z}_n(x)}{x \hat{z}_n(x) + i} \right| + \sup_{x \in I} |\mathbb{P}_n g_x - \mathbb{P} g_x|$$

$$\leq \frac{1}{K_n} s + \sup_{x \in I} \frac{1}{K_n} \sum_{i=1}^{n-1} \left| \frac{\hat{z}_n(x)}{x \hat{z}_n(x) + i} \right| + |\mathbb{P}_n g_x - \mathbb{P} g_x|$$

$$\sup_{x \in I} |\hat{z}_n(x)| = \sup_{x \in I} \left| \frac{\hat{z}_n(x)}{x \hat{z}_n(x) + i} \right| \leq \sup_{x \in I} \frac{1}{K_n} \sum_{i=1}^{n-1} \left| \frac{\hat{z}_n(x)}{x \hat{z}_n(x) + i} \right|$$

$$(\ast) \quad o_p(1) + B_n + o_p(1) + C_n = B_n + C_n + o_p(1),$$

where $(\ast)$ follows from $K_n \rightarrow \infty$ a.s. and Lemma A.2 applied with $x = \alpha$. For $C_n$, letting $h_x = g_x - g_\alpha$, we have

$$C_n = \sup_{x \in I} |\mathbb{P}_n h_x - \mathbb{P} h_x| = \sup_{x \in I} \sum_{j=1}^{\infty} |(\mathbb{P}_n(j) - \mathbb{P}(j)) \cdot h_x(j)|$$

$$\leq \sum_{j=1}^{\infty} |\mathbb{P}_n(j) - \mathbb{P}(j)| \cdot \sup_{x \in I} \sup_{j \in \mathbb{N}} |h_x(j)| \overset{(\ast)\text{2}}{=} o_p(1) \cdot \sup_{x \in I} \sup_{j \in \mathbb{N}} |h_x(j)|,$$

where $(\ast)\text{2}$ follows from Lemma A.1. Now we claim $\sup_{x \in I} \sup_{j \in \mathbb{N}} |h_x(j)|$ is finite:

$$\sup_{x \in I} \sup_{j \in \mathbb{N}} |h_x(j)| \leq \sup_{x \in I} \sup_{j \in \mathbb{N}} \left| \frac{\alpha - x}{(i - x)(i - \alpha)} \right| \leq \sum_{i=1}^{\infty} \frac{2}{(i - t)(i - \alpha)} < +\infty.$$

Therefore, $C_n = o_p(1)$. It remains to show that $B_n = o_p(1)$. Now $x \hat{z}_n(x) > -x > -t > -1$ for all $x \in [s, t]$ and $n^{-\alpha} K_n \rightarrow M_{\alpha g} > 0$ (a.s.) imply that

$$B_n \leq \frac{1}{K_n} \sum_{i=1}^{n-1} \sup_{x \in [s, t]} \left| \frac{\hat{z}_n(x)}{x \hat{z}_n(x) + i} \right| = O_p \left( \frac{\log n}{n^\alpha} \right) \sup_{x \in [s, t]} |\hat{z}_n(x)|.$$

(C.6)

Here the monotonicity of $z_n$ on $(0, 1)$ by (A) of Lemma C.4 and the stochastic order of $|z_n(x)|$ given by (B) of Lemma C.4 lead to

$$\sup_{x \in [s, t]} |\hat{z}_n(x)| \leq |\hat{z}_n(s)| + |\hat{z}_n(t)| = \begin{cases} O_p(1) + O_p(1) & (\alpha \leq s < t < 1) \\ O_p(n^{-\frac{\alpha}{1-\alpha}}) + O_p(1) & (0 < s < \alpha \leq t) \\ O_p(n^{-\frac{\alpha}{1-\alpha}}) + O_p(n^{-\frac{\alpha-1}{1-\alpha}}) & (0 < s < t < \alpha) \end{cases}$$

$$(\ast) \quad o_p(n^\alpha / \log n),$$

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Lemma C.4. For the second term, the formula of $\hat{z}_0$ that implies $B_n = [0 \pm \delta_n]$. Since $\delta_n \to 0$, we can take $[s, t]$ such that $B_n \subset [s, t] \subset (0, 1)$ for sufficiently large $n$. Then, the triangle inequality implies

$$\sup_{x \in B_n} |\hat{\Psi}'(x) - \Psi'(x)| \leq \sup_{x \in B_n} |\hat{\Psi}'(x) - \hat{\Psi}_{n,0}(x)| + \sup_{x \in [s, t]} |\hat{\Psi}'_{n,0}(x) - \Psi'(x)|,$$

where the second term is $o_p(1)$ by (B) of Lemma B.1. Thus, it suffices to show that $\sup_{x \in B_n} |\hat{\Psi}'(x) - \hat{\Psi}_{n,0}(x)| = o_p(1)$. Now (B.3) with $\theta^* = 0$ implies that $\hat{\Psi}_{n,0} = \hat{\Psi}_{n,\theta^*=0}$ is expressed by

$$\hat{\Psi}_{n,0}(x) = -\frac{1}{K_n x} + \frac{1}{x} - \mathbb{P}_n g_x.$$

Comparing this with Lemma C.3, we obtain

$$\hat{\Psi}'_{n}(x) - \hat{\Psi}'_{n,0}(x) = -\frac{1}{K_n} \sum_{i=1}^{n-1} \frac{\dot{z}_n(x)}{i}$$

(C.7)

Therefore, $\hat{\Psi}'_{n}(x) - \hat{\Psi}'_{n,0}(x)$ can be written as

$$\hat{\Psi}'_{n}(x) - \hat{\Psi}'_{n,0}(x) = \frac{1}{K_n} \sum_{i=1}^{n-1} \frac{\dot{z}_n(x)^2}{(x\dot{z}_n(x) + i)^2} - \frac{\dot{z}'_n(x)}{K_n} \sum_{i=1}^{n-1} \left( \frac{1}{x\dot{z}_n(x) + i} - \frac{x\dot{z}_n(x)}{(x\dot{z}_n(x) + i)^2} \right)$$

$$= \frac{1}{K_n} \sum_{i=1}^{n-1} \frac{(\dot{z}_n(x))^2}{(x\dot{z}_n(x) + i)^2} - \frac{\dot{z}'_n(x)}{K_n} \sum_{i=1}^{n-1} \frac{i}{(x\dot{z}_n(x) + i)^2}.$$

Note $x\dot{z}_n(x) > -x$. Then, the triangle inequality and $B_n \subset [s, t] \subset (0, 1)$ imply that

$$\sup_{x \in B_n} \left| \hat{\Psi}'_{n}(x) - \hat{\Psi}'_{n,0}(x) \right| \leq \sup_{x \in B_n} |\dot{z}_n(x)| \frac{1}{K_n} \sum_{i=1}^{\infty} \frac{1}{(i-t)^2} + \sup_{x \in B_n} \left| \frac{\dot{z}'_n(x)}{K_n} \sum_{i=1}^{n-1} \frac{i}{(x\dot{z}_n(x) + i)^2} \right|,$$

where the first term is $o_p(1)$ considering $\sup_{x \in B_n} |\dot{z}_n(x)| = O_p(1)$ by (C) of Lemma C.4. For the second term, the formula of $\dot{z}'_n(x)$ by (A) of Lemma C.4 implies that

$$\frac{\dot{z}'_n(x)}{K_n} \sum_{i=1}^{n-1} \frac{i}{(x\dot{z}_n(x) + i)^2} = \frac{1}{K_n} \sum_{i=1}^{n-1} \frac{\left( \sum_{i=1}^{n-1} i \cdot (x\dot{z}_n(x) + i)^2 \right)^2}{(x\dot{z}_n(x) + i)^2}.$$

Let $J_n(x) := \sum_{i=1}^{n-1} (\dot{z}_n(x) + i)^{-2} - \sum_{i=1}^{n-1} x^2 \cdot (x\dot{z}_n(x) + i)^{-2}$. Then, using

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\(x \hat{z}_n(x) = \hat{y}_n(x) > -x \) and \( B_n \subset [s, t] \subset (0, 1) \) again, we obtain
\[
\sup_{x \in B_n} \left| \frac{\hat{y}'_n(x)}{K_n} \sum_{i=1}^{n-1} \frac{i}{(x \hat{z}_n(x) + i)^2} \right| \leq \frac{1}{K_n} \left( \sum_{i=1}^{n-1} \frac{i}{(i - t)^2} \right)^2 \frac{1}{\inf_{x \in B_n} |J_n(x)|}
\]
\[
= O_p \left( \frac{(\log n)^2}{n^a} \right) \frac{1}{\inf_{x \in B_n} |J_n(x)|} = o_p(1) \cdot \left( \inf_{x \in B_n} |J_n(x)| \right)^{-1}.
\]

Here it remains to show \( (\inf_{x \in B_n} |J_n(x)|)^{-1} = O_p(1) \). The rest of the proof is technical, so we divide it into three steps: Letting \( f'_\alpha \) be the bijective function defined by (3.2), first we prove that
\[
\sup_{x \in B_n} \left| -J_n(x) + f'_\alpha(\hat{z}_n(x)) \right| = o_p(1).
\]

Next we show that
\[
\sup_{x \in B_n} \left| f'_\alpha(\hat{z}_n(x)) - f'_\alpha \circ f^{-1}_\alpha(\log(K_n/n^a)) \right| = o_p(1).
\]

Putting together the above displays, we prove that
\[
1/ \inf_{x \in B_n} |J_n(x)| \leq \left( \frac{2f'_\alpha \circ f^{-1}_\alpha(\log K_n/n^a)}{n^a} \right)^{-1} = O_p(1).
\]

Step 1. \( \sup_{x \in B_n} | -J_n(x) + f'_\alpha(\hat{z}_n(x)) | = o_p(1) \): Using the basic equation of the trigamma function: \( \psi^{(1)}(1 + z) = \sum_{i=1}^{\infty} (i + z)^{-2} \) (for all \( z > -1 \)), we write \( J_n(x) \) as
\[
J_n(x) = \sum_{i=1}^{K_n-1} (\hat{z}_n(x) + i)^{-2} - \sum_{i=1}^{n-1} x^2 \cdot (x \hat{z}_n(x) + i)^{-2}
\]
\[
= -\psi^{(1)}(K_n + \hat{z}_n(x)) + \psi^{(1)}(1 + \hat{z}_n(x)) + x^2 \psi^{(1)}(x \hat{z}_n(x) + n) - x^2 \psi^{(1)}(x \hat{z}_n(x) + 1).
\]

Observe that \( f'_\alpha(z) = \psi^{(1)}(1 + z) - \alpha^2 \psi^{(1)}(1 + \alpha z) \) by the definition \( f_\alpha(z) = \psi(1 + z) - \alpha \psi(1 + \alpha z) \). Then, \( -J_n(x) + f'_\alpha(\hat{z}_n(x)) \) can be written as
\[
-\left( J_n(x) + f'_\alpha(\hat{z}_n(x)) \right)
\]
\[
= \psi^{(1)}(K_n + \hat{z}_n(x)) - x^2 \psi^{(1)}(x \hat{z}_n(x) + n) + x^2 \psi^{(1)}(x \hat{z}_n(x) + 1) - \alpha^2 \psi^{(1)}(x \hat{z}_n(x) + 1) - \alpha^2 \psi^{(1)}(\hat{z}_n(x) + 1).
\]

Note that \( \psi^{(1)}(x) \) is positive and decreasing on \((0, \infty)\). Then, using \( \hat{z}_n(x) > -1 \) and \( B_n = [\alpha \pm \delta_n] \subset [s, t] \subset (0, 1) \) again, we obtain the following for all \( x \in B_n \):
\[
\left| -J_n(x) + f'_\alpha(\hat{z}_n(x)) \right|
\]
\[
\leq \psi^{(1)}(K_n + \hat{z}_n(x)) + x^2 \psi^{(1)}(x \hat{z}_n(x) + n) + x^2 \psi^{(1)}(x \hat{z}_n(x) + 1) - \alpha^2 \psi^{(1)}(x \hat{z}_n(x) + 1) - \alpha^2 \psi^{(1)}(\hat{z}_n(x) + 1) + \alpha^2 \psi^{(1)}(x \hat{z}_n(x) + 1) - \psi^{(1)}(x \hat{z}_n(x) + 1) - \psi^{(1)}(\hat{z}_n(x) + 1) + 2\delta_n \psi^{(1)}(1 - t) + |\psi^{(1)}(x \hat{z}_n(x) + 1) - \psi^{(1)}(\hat{z}_n(x) + 1)|.
\]

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The first, the second, and the third terms do not depend on \( x \), and they are \( o_p(1) \) by \( \psi(1) (x) \to 0 \) as \( x \to \infty \) and \( \delta_n = o(1) \). Thus, we have

\[
\sup_{x \in B_n} \left| -J_n(x) + f'_\alpha (\hat{z}_n(x)) \right| \leq o_p(1) + \sup_{x \in B_n} \left| \psi(1) (x \hat{z}_n(x) + 1) - \psi(1) (\alpha \hat{z}_n(x) + 1) \right|.
\]

The second term is bounded by

\[
\sup_{x \in B_n} \left| \psi(1) (x \hat{z}_n(x) + 1) - \psi(1) (\alpha \hat{z}_n(x) + 1) \right| \\
\leq \sup_{x \in B_n} \left| \sup_{x \in B_n} \left| 2|\hat{z}_n(x)| + 2|\hat{z}_n(x)|^2 \right| \right| \cdot \sum_{i=1}^{\infty} \frac{2i}{(i-\alpha)^2(i-t)^2}
\]

\[
\overset{(*)}{=} \delta_n O_p(1) O(1) = o_p(1),
\]

where \((*)\) follows from \( \sup_{x \in B_n} |\hat{z}_n(x)| = O_p(1) \) ((C) of Lemma C.4). This completes the proof of Step 1.

Step 2. \( \sup_{x \in B_n} \left| f'_{\alpha} (\hat{z}_n(x)) - f'_{\alpha} \circ f_{\alpha}^{-1} (\log(K_n/n^\alpha)) \right| = o_p(1) \): \( f'_{\alpha} = o(0) \) on \((-1, \infty)\) (Lemma 3.2) and \( \hat{z}_n < 0 \) on \((0,1)\) ((A) of Lemma C.4) imply that \( (f'_{\alpha} \circ \hat{z}_n)'(x) > 0 \), especially that \( f'_{\alpha} \circ \hat{z}_n \) is monotone on \((0,1)\). Then

\[
\sup_{x \in B_n} \left| f'_{\alpha} (\hat{z}_n(x)) - f'_{\alpha} \circ f_{\alpha}^{-1} (\log(K_n/n^\alpha)) \right| \leq \left| f'_{\alpha} (\hat{z}_n(\alpha \pm \delta_n)) - f'_{\alpha} \circ f_{\alpha}^{-1} (\log(K_n/n^\alpha)) \right|
\]

\[
\leq |f'_{\alpha}(\hat{z}_n(\alpha \pm \delta_n)) - f'_{\alpha} \circ f_{\alpha}^{-1}(\log M_{\alpha \theta})| + |f'_{\alpha} \circ f_{\alpha}^{-1}(\log M_{\alpha \theta}) - f'_{\alpha} \circ f_{\alpha}^{-1}(\log(K_n/n^\alpha))|,
\]

where the first term is \( o_p(1) \) by \( \hat{z}_n(\alpha \pm \delta_n) \to^p f^{-1}_\alpha(\log M_{\alpha \theta}) \) ((C) of Lemma C.4) and is the continuity of \( f'_{\alpha} \), while the second term is also \( o_p(1) \) by \( K_n/n^\alpha \to M_{\alpha \theta} \) (a.s.). This completes the proof of Step 2.

Step 3. \( 1/\inf_{x \in B_n} |J_n(x)| = O_p(1) \): \( f'_{\alpha} > 0 \) by Lemma 3.2 implies

\[
\sup_{x \in B_n} \left| \frac{-J_n(x)}{(f'_{\alpha} \circ f_{\alpha}^{-1}(\log(K_n/n^\alpha)))} + 1 \right| = \sup_{x \in B_n} \left| \frac{-J_n(x) + f'_{\alpha} \circ f_{\alpha}^{-1}(\log(K_n/n^\alpha))}{f'_{\alpha} \circ f_{\alpha}^{-1}(\log(K_n/n^\alpha))} \right|,
\]

where the numerator is \( o_p(1) \) from Step 1 and Step 2, and the denominator converges to \( f'_{\alpha} \circ f_{\alpha}^{-1}(\log M_{\alpha \theta}) > 0 \). Thus, the right-hand side is \( o_p(1) \), thereby \( \sup_{x \in B_n} \left| -J_n(x)/(f'_{\alpha} \circ f_{\alpha}^{-1}(\log(K_n/n^\alpha))) + 1 \right| < 1/2 \) holds with a high proba-
bility. Combining this and $f'_n > 0$ again, we obtain

$$\forall x \in B_n, \quad \frac{-|J_n(x)|}{(f'_n \circ f^{-1}_n(\log(K_n/n^\alpha)))} + 1 \leq \frac{-J_n(x)}{(f'_n \circ f^{-1}_n(\log(K_n/n^\alpha)))} + 1 \leq \frac{1}{2}$$

$$\Rightarrow \forall x \in B_n, |J_n(x)| > f'_n \circ f^{-1}_n(\log(K_n/n^\alpha))/2$$

$$\Rightarrow (\inf_{x \in B_n} |J_n(x)|)^{-1} < 2/(f'_n \circ f^{-1}_n(\log(K_n/n^\alpha))).$$

Putting this together with $2/(f'_n \circ f^{-1}_n(\log(K_n/n^\alpha))) \to 2/f'_n \circ f^{-1}_n(\log \alpha \theta) = O_p(1)$, we obtain $(\inf_{x \in B_n} |J_n(x)|)^{-1} = O_p(1)$, to complete the proof. \qed

### C.1 Proof of Theorem 3.5 (Uniqueness of MLE)

We finally prove the asymptotics of MLE. First, we show the consistency of MLE $\hat{\alpha}_n$, which is the most technical part of our proof.

By the argument following (C.3), it suffices to show that the root of $\hat{\Psi}_n$ uniquely exists in $[\underline{\alpha}, \bar{\alpha}]$ and its derivative takes a negative value at the unique root with a high probability. For convenience, we define the random set $\bar{I}_n := \{x \in [\underline{\alpha}, \bar{\alpha}] | \hat{\Psi}_n(x) = 0\}.

Step 1. $\bar{I}_n \neq \emptyset$ with a high probability: Note that

$$\Pr(\hat{\Psi}_n(\underline{\alpha}) < 0 \cup \hat{\Psi}_n(\bar{\alpha}) > 0) \leq \Pr(\hat{\Psi}_n(\underline{\alpha}) < 0) + \Pr(\hat{\Psi}_n(\bar{\alpha}) > 0) = \Pr(\Psi(\underline{\alpha}) - \hat{\Psi}_n(\underline{\alpha}) > \Psi(\underline{\alpha})) + \Pr(\hat{\Psi}_n(\bar{\alpha}) - \Psi(\bar{\alpha}) > -\Psi(\bar{\alpha})),$$

and that $\Psi(\underline{\alpha}) > \Psi(\alpha) = 0 > \Psi(\bar{\alpha})$ (see Lemma A.3). Then (A) of Lemma C.2 implies that the last two terms are $o(1)$, hence $\hat{\Psi}_n(\underline{\alpha}) \geq 0 \geq \hat{\Psi}_n(\bar{\alpha})$ holds with a high probability. Then, the intermediate value theorem leads to the assertion.

Step 2. $\sup_{x \in \bar{I}_n} |x - \alpha| = o_p(1)$: We fix $\epsilon > 0$ s.t. $[\alpha - \epsilon, \alpha + \epsilon] \subset [\underline{\alpha}, \bar{\alpha}]$, and we define the positive constant $\delta_\epsilon := (\Psi(\alpha - \epsilon) - \Psi(\alpha + \epsilon))/2$. Observe that $\sup_{x \in [\alpha - \epsilon, \alpha]} |(\Psi_n(x) - \Psi(x))| \leq \delta_\epsilon$ with a high probability by (A) of Lemma C.2. Combining this and the fact that $\Psi$ is decreasing (see Lemma A.3), we obtain

$$\inf_{x \in [\alpha - \epsilon, \alpha + \epsilon]} \hat{\Psi}_n(x) \geq \inf_{x \in [\alpha - \epsilon, \alpha + \epsilon]} \Psi(x) - \sup_{x \in [\alpha - \epsilon, \alpha + \epsilon]} (\Psi(x) - \hat{\Psi}_n(x)) \geq \Psi(\alpha - \epsilon) - \delta_\epsilon$$

$$\geq \delta_\epsilon > 0,$$

$$\sup_{x \in [\alpha + \epsilon, \bar{\alpha}]} \hat{\Psi}_n(x) \leq \sup_{x \in [\alpha + \epsilon, \bar{\alpha}]} \Psi(x) + \sup_{x \in [\alpha + \epsilon, \bar{\alpha}]} (\Psi_n(x) - \Psi(x)) \leq \Psi(\alpha + \epsilon) + \delta_\epsilon$$

$$\leq -\delta_\epsilon < 0.$$

which means that $\hat{\Psi}_n$ does not have any root outside $(\alpha - \epsilon, \alpha + \epsilon)$. This leads to $\sup_{x \in \bar{I}_n} |x - \alpha| \leq \epsilon$.

Step 3. $\sup_{x \in \bar{I}_n} |x - \alpha| = o_p(n^{-\alpha/4})$: Proposition 3.8 applied with $\theta^* = 0$ implies that $|\alpha_{n, 0} - \alpha| = O_p(n^{-\alpha/2}) = o_p(n^{-\alpha/4})$. Thus, it suffices to show that $\sup_{x \in \bar{I}_n} |x - \alpha_{n, 0}| = o_p(n^{-\alpha/4})$. Here, we take the positive constant $\delta := (1 - \alpha)/2 \land \alpha(1 - \alpha)/(4 - \alpha)$ and define $B_\delta(\alpha) := [\alpha \pm \delta]$. Observe $B_n \subset
\[ \sup_{x \in B_{\delta}(\alpha)} |\hat{z}_n(x)| \leq |\hat{z}_n(3\alpha/(4-\alpha))| + |\hat{z}_n((1+\alpha)/2)| = O_p(n^{\alpha/4}) + O_p(1) \]

= \( O_p(n^{\alpha/4}) \). \hspace{1cm} (C.8)

In contrast, \( \sup_{x \in I_n} |x-\alpha| = o_p(1) \) by Step 2 and \( \hat{\alpha}_{n,0} - \alpha = o_p(1) \) by Lemma B.3 imply

\[ \hat{I}_n \subset B_\delta(\alpha) \text{ and } \hat{\alpha}_{n,0} \in B_\delta(\alpha) \text{ with a high probability}. \] \hspace{1cm} (C.9)

Here we fix \( x \in \hat{I}_n \). Then Taylor’s theorem applied to \( 0 = \hat{\Psi}_{n,0}(\hat{\alpha}_{n,0}) \) implies that there exists \( \hat{\alpha}_x \) between \( x \) and \( \hat{\alpha}_{n,0} \) s.t. \( 0 = \hat{\Psi}_{n,0}(x) + \hat{\Psi}'_{n,0}(\hat{\alpha}_x) \cdot (\hat{\alpha}_{n,0} - x) \), where \( \hat{\Psi}_{n,0}(x) = \hat{\Psi}_{n,0}(x) - \hat{\Psi}_{n,0}(x) = K_{n}^{-1} \sum_{i=1}^{n-1} \hat{z}_n(x)/(x\hat{z}_n(x) + i) \) by (C.7). In summary, we have the following:

\[ \forall x \in \hat{I}_n, \exists \hat{\alpha}_x \text{ s.t. } |\hat{\alpha}_x - \alpha| \leq |x - \alpha| \lor |\hat{\alpha}_{n,0} - \alpha| \text{ and } \] \hspace{1cm} (C.10)

\[ x - \hat{\alpha}_{n,0} = \frac{\hat{\Psi}_{n,0}(x)}{\hat{\Psi}'_{n,0}(\hat{\alpha}_x)} = \frac{1}{K_{n} \cdot \Psi'_{n,0}(\hat{\alpha}_x)} \sum_{i=1}^{n-1} \frac{\hat{z}_n(x)}{x\hat{z}_n(x) + i}. \] \hspace{1cm} (C.11)

Note \( \{ \hat{\alpha}_x : x \in \hat{I}_n \} \subset B_\delta(\alpha) \) with a high probability by (C.9) and (C.10). Then, putting \( \hat{I}_n \subset B_\delta(\alpha) \), (C.8), (C.11), and \( \hat{z}_n(x) > -1 \) together, we get:

\[ \sup_{x \in \hat{I}_n} |x - \hat{\alpha}_{n,0}| \leq \sup_{x \in B_{\delta}(\alpha)} \left| \frac{1}{K_{n} \cdot \hat{\Psi}'_{n,0}(\hat{\alpha}_x)} \sum_{i=1}^{n-1} \frac{\hat{z}_n(x)}{x\hat{z}_n(x) + i} \right| \]

\[ \leq \frac{1}{K_{n} \inf_{x \in B_{\delta}(\alpha)} |\hat{\Psi}'_{n,0}(\hat{\alpha}_x)|} \sup_{x \in B_{\delta}(\alpha)} \left| \sum_{i=1}^{n-1} \frac{\hat{z}_n(x)}{x\hat{z}_n(x) + i} \right| \]

\[ \leq \frac{\sup_{x \in B_{\delta}(\alpha)} |\hat{z}_n(x)|}{K_{n} \inf_{x \in B_{\delta}(\alpha)} |\hat{\Psi}'_{n,0}(\hat{\alpha}_x)|} \sum_{i=1}^{n-1} \frac{1}{i - (\alpha + \delta)} \]

\[ = O_p(n^{-\alpha})O_p(n^{\alpha/4})O(\log n) = \frac{o_p(n^{-\alpha/4})}{\inf_{x \in B_{\delta}(\alpha)} |\hat{\Psi}'_{n,0}(\hat{\alpha}_x)|}. \]

It remains to show that \( (\inf_{x \in B_{\delta}(\alpha)} |\hat{\Psi}'_{n,0}(\hat{\alpha}_x)|)^{-1} = O_p(1) \). Now we take the positive constant \( C := 2^{-1} \inf_{x \in B_{\delta}(\alpha)} |\Psi'(x)| \). Note that \( C \) exists because \( \Psi' \) is continuous on \( (0,1) \) (see Lemma A.3). In contrast, (B) of Lemma B.1 applied to \( \theta^* = 0 \) and \( I = B_{\delta}(\alpha) \) implies that \( \sup_{x \in B_{\delta}(\alpha)} |\Psi'(x) - \hat{\Psi}'_{n,0}(x)| \leq C \) with a high probability. Therefore, we obtain

\[ \inf_{x \in B_{\delta}(\alpha)} |\hat{\Psi}'_{n,0}(x)| \geq \inf_{x \in B_{\delta}(\alpha)} |\Psi'(x)| + \inf_{x \in B_{\delta}(\alpha)} (|\Psi'(x) - \hat{\Psi}'_{n,0}(x)|) \]

\[ = 2C - \sup_{x \in B_{\delta}(\alpha)} |\Psi'(x) - \hat{\Psi}'_{n,0}(x)| \geq 2C - C = C (> 0), \]
which implies that \( (\inf_{x \in B_\delta(\alpha)} |\hat{\Psi}'_{n,0}(x)|)^{-1} \leq C^{-1} \) with a high probability. This completes the proof.

Step 4. \#\hat{I}_n = 1 and \( \hat{\Psi}'_n \) takes a negative value at the unique element with a high probability: Note that \( \hat{I}_n \subset [\alpha \pm n^{-\alpha/4}] \subset B_\delta(\alpha) \) with a high probability by Step 3, where \( B_\delta(\alpha) \) is the constant defined in step 3. Then for the positive constant \( C = 2^{-1} \inf_{x \in B_\delta(\alpha)} |\Psi'(x)| \) we claim \( \sup_{x \in \hat{I}_n} \hat{\Psi}'_n(x) < -C \). We observe that \( C := 2^{-1} \inf_{x \in B_\delta(\alpha)} |\Psi'(x)| = -2^{-1} \sup_{x \in B_\delta(\alpha)} \Psi'(x) \) since \( \Psi' \) is negative by Lemma A.3. Now (B) of Lemma C.2 applied to \( \delta_n = n^{-\alpha/4} \) implies that \( \sup_{x \in [\alpha \pm n^{-\alpha/4}]} (\hat{\Psi}'_n(x) - \Psi'(x)) \leq C \) with a high probability. Putting them together, we have

\[
\sup_{x \in \hat{I}_n} \hat{\Psi}'_n(x) \leq \sup_{x \in I_n} \Psi'(x) + \sup_{x \in \hat{I}_n} (\hat{\Psi}'_n(x) - \Psi'(x)) \\
\leq \sup_{x \in B_\delta(\alpha)} \Psi'(x) + \sup_{x \in [\alpha \pm n^{-\alpha/4}]} (\hat{\Psi}'_n(x) - \Psi'(x)) \\
= -2C + \sup_{x \in [\alpha \pm n^{-\alpha/4}]} (\hat{\Psi}'_n(x) - \Psi'(x)) \leq -2C + C = -C < 0.
\]

This means \( \hat{\Psi}'_n(x) \) is uniformly negative on \( \hat{I}_n \), so the cardinality of \( \hat{I}_n = \{ \hat{\Psi}_n(x) = 0 \} \) is at most 1 with a high probability. In contrast, we have shown in Step 1 that \( \hat{I}_n \) has at least one element. For these reasons, we conclude that \#\hat{I}_n = 1 \) and \( \hat{\Psi}'_n \) takes a negative value at the unique element, with a high probability.

As a result of the arguments so far, we obtain the consistency of \( \hat{\alpha}_n \).

**Corollary C.4.1** (Consistency of \( \hat{\alpha}_n \)). \( \hat{\alpha}_n \to^p \alpha \) and \( \hat{z}_n(\hat{\alpha}_n) \to^p f_\alpha^{-1}(\log M_{\alpha \theta}) \).

**Proof.** Observe \( \hat{I}_n = \{ \hat{\alpha}_n \} \) with a high probability by Theorem 3.5. Then Step 3 in the proof of Theorem 3.5 yields \( \hat{\alpha}_n - \alpha = o_p(n^{-\alpha/4}) \), especially \( \hat{\alpha}_n \to^p \alpha \). In contrast, (C) of Lemma C.4 applied to \( \hat{\alpha}_n - \alpha = o_p(n^{-\alpha/4}) = o(1/\log n) \) (with a high probability) implies \( f_\alpha(\hat{z}_n(\hat{\alpha}_n)) - \log(K_n/n^\alpha) = o_p(1) \). From this and \( K_n/n^\alpha \to M_{\alpha \theta} \) (a.s.), we get \( \hat{z}_n(\hat{\alpha}_n) \to^p f_\alpha^{-1}(\log M_{\alpha \theta}) \). \( \Box \)

**C.2 Proof of (3.12) in Proposition 3.9**

Next we prove the asymptotic error between \( \hat{\alpha}_n \) and \( \hat{\alpha}_{n,0} \). Recall that we derived the error of \( (\hat{\alpha}_{n,0}, \hat{\alpha}_{n,0}) \) as

\[
\frac{n^\alpha}{\log n} (\hat{\alpha}_{n,0} - \hat{\alpha}_{n,0}) \to^p \frac{-\theta^*}{I_\alpha M_{\alpha \theta}}
\]

(see Lemma B.4). Now we suppose that

\[
\frac{n^\alpha}{\log n} (\hat{\alpha}_n - \hat{\alpha}_{n,0}) \to^p \frac{-f_\alpha^{-1}(\log M_{\alpha \theta})}{I_\alpha M_{\alpha \theta}}. \tag{C.12}
\]
Then, (3.12) follows from the two displays above. Thus, it suffices to show (C.12),
\[ I_n = \{ \hat{\alpha}_n \} \]
with a high probability by Theorem 3.5, (C.10), and (C.11) result in
\[ \exists \hat{\alpha}_n \text{ s.t. } |\hat{\alpha}_n - \alpha| \leq |\hat{\alpha}_n - \alpha| \lor |\hat{\alpha}_{n,0} - \alpha| \]
\[ \hat{\alpha}_n - \hat{\alpha}_{n,0} = \frac{1}{K_n \hat{\Psi}_{n,0}(\hat{\alpha}_n)} \hat{z}_n(\hat{\alpha}_n) \sum_{i=1}^{n-1} \frac{1}{\hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n) + i}. \]
Note \(|\hat{\alpha}_n - \alpha| = o_p(1)\) by \(\hat{\alpha}_n \to^p \alpha\) (Corollary C.4.1) and \(\hat{\alpha}_{n,0} \to^p \alpha\) (Lemma B.3 with \(\theta^* = 0\)). Then, we have
\[ \frac{n^\alpha}{\log n} (\hat{\alpha}_n - \hat{\alpha}_{n,0}) = \frac{1}{\Psi_{n,0}(\hat{\alpha}_n)} \cdot \frac{n^\alpha}{\log n} \cdot \hat{z}_n(\hat{\alpha}_n) \sum_{i=1}^{n-1} \frac{1}{\hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n) + i} \]
\[ \leq \frac{1}{I_{\alpha}} \cdot \frac{1}{M_{\alpha}} \cdot f_{\alpha}^{-1}(\log M_{\alpha}) \cdot \frac{1}{\log n} \sum_{i=1}^{n-1} \frac{1}{\hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n) + i}, \]
where we used \(-\hat{\Psi}_{n,0}(\hat{\alpha}_n) \to^p I_{\alpha}\) by Lemma B.2 with \(\theta^* = 0\) and \(\hat{z}_n(\hat{\alpha}_n) \to^p f_{\alpha}^{-1}(\log M_{\alpha})\) by Corollary C.4.1. Now we claim \((\log n)^{-1} \sum_{i=1}^{n-1} (\hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n) + i)^{-1} \to^p 1\): Lemma B.5 with \(\hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n) = O_p(1) = o_p(n)\) implies that
\[ \sum_{i=1}^{n-1} (\hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n) + i)^{-1} - \log n = \psi(n + \hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n)) - \log n - \psi(1 + \hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n)) \]
\[ \to^p 0 - \psi(1 + \alpha f_{\alpha}^{-1}(\log M_{\alpha})), \]
so \((\log n)^{-1} \sum_{i=1}^{n-1} (\hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n) + i)^{-1} - 1 = o_p(1)\). This concludes the proof of (C.12).

C.3 Proof of Theorem 3.6
\(\hat{\theta}_n = \hat{\alpha}_n \hat{z}_n(\hat{\alpha}_n)\) and Corollary C.4.1 implies that \(\hat{\theta}_n \to^p \alpha \cdot f_{\alpha}^{-1}(\log M_{\alpha})\), and hence (ii) holds. For (i), \(\sqrt{n^\alpha I_{\alpha}(\hat{\alpha}_{n,\theta} - \alpha)} \to N/\sqrt{M_{\alpha}}\) by Proposition 3.8 and \((\hat{\alpha}_n - \hat{\alpha}_{n,\theta}) = O_p(n^{-\alpha} \log n)\) by (3.12) leads to
\[ \sqrt{n^\alpha I_{\alpha}(\hat{\alpha}_n - \alpha)} = \sqrt{n^\alpha I_{\alpha}(\hat{\alpha}_n - \hat{\alpha}_{n,\theta})} + \sqrt{n^\alpha I_{\alpha}(\hat{\alpha}_{n,\theta} - \alpha)} \]
\[ = O_p(n^{-\alpha/2} \log n) + \sqrt{n^\alpha I_{\alpha}(\hat{\alpha}_{n,\theta} - \alpha)} \to N/\sqrt{M_{\alpha}}, \]
which concludes the proof.
D Omitted proofs

D.1 Proof of Proposition 3.1

Recall that $p_\alpha(j)$ is

$$\forall j \in \mathbb{N}, \quad p_\alpha(j) := \frac{\alpha \prod_{i=1}^{j-1} (i - \alpha)}{j!}.$$ 

Then, it is easy to derive $I_\alpha$ as

$$I_\alpha = -\sum_{j=1}^{\infty} p_\alpha(j) \partial_\alpha^2 \log p_\alpha(j) = \sum_{j=1}^{\infty} p_\alpha(j) \left( \frac{1}{\alpha^2} + \sum_{i=1}^{j-1} \frac{1}{(i - \alpha)^2} \right),$$

$$= \frac{1}{\alpha^2} + \sum_{j=1}^{\infty} p_\alpha(j) \sum_{i=1}^{j-1} \frac{1}{(i - \alpha)^2}.$$ 

Hence, we obtained (A). It is important to emphasize that the sum of infinite series is finite since it is an expectation of a finite function of $j$. Now Fubini-Tonelli’s implies

$$\sum_{j=1}^{\infty} p_\alpha(j) \sum_{i=1}^{j-1} \frac{1}{(i - \alpha)^2} = \sum_{i=1}^{\infty} \frac{1}{(i - \alpha)^2} \sum_{j=i+1}^{\infty} p_\alpha(j).$$

Thus, for (B) it remains to show the following:

$$\forall i \in \mathbb{N}, \quad \sum_{j=i+1}^{\infty} p_\alpha(j) = \frac{i - \alpha}{\alpha} p_\alpha(i) \quad (D.1)$$ 

We will prove it by induction. Notice that (D.1) holds for $i = 1$ since $\sum_{j=2}^{\infty} p_\alpha(j) = 1 - p_\alpha(1) = 1 - \alpha = (1 - \alpha)p_\alpha(1)/\alpha$. Here we assume (D.1) for $i = k \in \mathbb{N}$. Then,

$$\sum_{j=(k+1)+1}^{\infty} p_\alpha(j) = \sum_{j=k+1}^{\infty} p_\alpha(j) - p_\alpha(k+1) = \frac{k - \alpha}{\alpha} p_\alpha(k) - p_\alpha(k+1)$$

$$= \frac{k - \alpha}{\alpha} \cdot \frac{k + 1}{k - \alpha} p_\alpha(k+1) - p_\alpha(k+1) = \frac{k + 1 - \alpha}{\alpha} p_\alpha(k+1),$$

so (D.1) holds for $i = k + 1$. Therefore, (D.1) holds for all $i \in \mathbb{N}$.

For the continuity of $I_\alpha$, we observe that

$$I_\alpha = \frac{1}{\alpha^2} + \sum_{j=1}^{\infty} \frac{p_\alpha(j)}{\alpha(j - \alpha)}.$$ 

Now we claim that $I_\alpha$ converges uniformly on $K$ for any closed subset $K = [s, t] \in (0, 1)$, which clearly concludes the proof of continuity. Since $p_\alpha(j)/\alpha = (\prod_{i=1}^{j-1} (i - \alpha))/j!$ is nonincreasing on $[s, t]$, we observe that

$$\forall j \in \mathbb{N}, \quad \sup_{\alpha \in K} \left| p_\alpha(j) \frac{1}{\alpha(j - \alpha)} \right| \leq \sup_{\alpha \in K} \frac{p_\alpha(j)}{\alpha} \cdot \sup_{\alpha \in K} \frac{1}{j - \alpha} = \frac{p_s(j)}{s} \frac{1}{j - t} \leq \frac{p_s(j)}{s(1 - t)}.$$
and $\sum_{j=1}^{\infty} p_s(j)/(s(1-t)) = 1/(s(1-t)) < +\infty$. Then, the Weierstrass M-test implies that $I_\alpha$ converges uniformly on $K$, which concludes the proof.

### D.2 Proof of Lemma 3.2

Recall that $f_\alpha : (-1, \infty) \to \mathbb{R}$ is defined by

$$f_\alpha(z) := \psi(1 + z) - \alpha \psi(1 + \alpha z),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. Then $\lim_{z \to 0^+} \psi(z) = -\infty$ implies

$$\lim_{z \to -1^+} f_\alpha(z) = \lim_{z \to -1^+} \psi(1 + z) - \alpha \psi(1 - \alpha) = -\infty.$$

We also observe that $\psi(z) = \log z + o(1)$ for large $z > 0$ (see, for example, [Zwillinger, 2018, Section 6]), which implies that

$$f_\alpha(z) = \log z - \alpha \log(\alpha z) + o(1) = (1 - \alpha) \log z + O(1) \text{ as } z \to +\infty \Rightarrow \lim_{z \to \infty} f_\alpha(z) = +\infty.$$

In contrast, using $\psi^{(1)}(1 + z) = \sum_{i=1}^{\infty} (i + z)^{-2} - (i/\alpha + z)^{-2}$ for all $z > -1$, we obtain

$$f'_\alpha(z) = \psi^{(1)}(1 + z) - \alpha^2 \psi^{(1)}(1 + \alpha z) = \sum_{i=1}^{\infty} \left\{ (i + z)^{-2} - \alpha^2 (i + \alpha z)^{-2} \right\}$$

$$= \sum_{i=1}^{\infty} \left\{ (i + z)^{-2} - (i/\alpha + z)^{-2} \right\} > \sum_{i=1}^{\infty} \left\{ (i + z)^{-2} - (i + z)^{-2} \right\} = 0.$$

Thus, $f_\alpha$ is strictly increasing. Putting all together, we conclude that $f_\alpha(z)$ is bijective from $(-1, \infty)$ to $\mathbb{R}$. Finally, we claim $f''_\alpha(z) < 0$. This follows from

$$f''_\alpha(z) = \psi^{(2)}(1 + z) - \alpha^3 \psi^{(2)}(1 + \alpha z) = -2 \sum_{i=1}^{\infty} \left\{ (i + z)^{-3} - \alpha^3 (i + \alpha z)^{-3} \right\}$$

$$= -2 \sum_{i=1}^{\infty} \left\{ (i + z)^{-3} - (i/\alpha + z)^{-3} \right\} < -2 \sum_{i=1}^{\infty} \left\{ (i + z)^{-3} - (i + z)^{-3} \right\} = 0,$$

where we used $\psi^{(2)}(1 + z) = -2 \sum_{i=1}^{\infty} (i + z)^{-2}$ and $0 < \alpha < 1$.

### D.3 Proof of Proposition 3.3

Recall that Ewens–Pitman partition has the following log-likelihood:

$$\ell_n(\alpha, \theta) = \sum_{i=1}^{K_n-1} \log(\theta + i) - \sum_{i=1}^{n-1} \log(\theta + i) + \sum_{j=2}^{n} S_{\alpha,j} \sum_{i=1}^{j-1} \log(i - \alpha).$$

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First, we derive the convergence of $\partial^2_\theta \ell_n(\alpha, \theta)$. This is written as

$$
\partial^2_\theta \ell_n(\alpha, \theta) = -\sum_{i=1}^{K_n-1} \frac{1}{(\theta + i\alpha)^2} + \sum_{i=1}^{n-1} \frac{1}{(\theta + i)^2}.
$$

Note that $\sum_{i=1}^{K_n-1} (\theta + i\alpha)^{-2}$ is a strictly increasing function of $K_n$ and $K_n$ is a nondecreasing random variable. Then the monotone convergence theorem implies that

$$
E \left[ \sum_{i=1}^{K_n-1} \frac{1}{(\theta + i\alpha)^2} \right] \to \sum_{i=1}^{\infty} \frac{1}{(\theta + i\alpha)^2} < +\infty. \tag{D.2}
$$

Using the above displays and $\psi^{(1)}(1 + z) = \sum_{i=1}^{\infty} (i + z)^{-2}$, we obtain

$$
I_{\theta\theta}^{(n)} = E[-\partial^2_\theta \ell_n(\alpha, \theta)] = E \left[ \sum_{i=1}^{K_n-1} \frac{1}{(\theta + i\alpha)^2} \right] - \sum_{i=1}^{\infty} \frac{1}{(\theta + i)^2}
$$

\begin{align*}
&\to \sum_{i=1}^{\infty} \frac{1}{(\theta + i\alpha)^2} - \sum_{i=1}^{\infty} \frac{1}{(\theta + i)^2} \\
&= \frac{1}{\alpha^2} \left( \sum_{i=1}^{\infty} \frac{1}{(i + \theta/\alpha)^2} - \sum_{i=1}^{\infty} \frac{\alpha^2}{(i + \theta)^2} \right) \\
&= \psi^{(1)}(1 + \theta/\alpha) - \alpha^2 \psi^{(1)}(1 + \alpha \cdot \theta/\alpha) \\
&= \alpha^{-2} f'_\alpha(\theta/\alpha) < +\infty.
\end{align*}

Next, we derive the leading term for $I_{\alpha\theta}^{(n)}$. Notice that $E[\partial_\theta \ell_n(\alpha, \theta)] = 0$ holds since $\ell_n$ is the log-likelihood. Now from this equation we obtain

$$
E \left[ \sum_{i=1}^{K_n-1} \frac{1}{\theta + i\alpha} \right] - \sum_{i=1}^{n-1} \frac{1}{\theta + i} = 0. \tag{D.3}
$$

In contrast, $\partial_\theta \partial_\alpha \ell_n(\alpha, \theta)$ is written by

$$
\partial_\theta \partial_\alpha \ell_n(\alpha, \theta) = -\sum_{i=1}^{K_n-1} \frac{i}{(\theta + i\alpha)^2} = -\frac{1}{\alpha} \sum_{i=1}^{K_n-1} \frac{1}{\theta + i\alpha} + \frac{\theta}{\alpha} \sum_{i=1}^{K_n-1} \frac{1}{(\theta + i\alpha)^2}.
$$

Thus, (D.2) and (D.3) result in

$$
I_{\theta\alpha}^{(n)} = -E[\partial_\theta \partial_\alpha \ell_n(\alpha, \theta)] = \frac{1}{\alpha} E \left[ \sum_{i=1}^{K_n-1} \frac{1}{\theta + i\alpha} \right] - \frac{\theta}{\alpha} E \left[ \sum_{i=1}^{K_n-1} \frac{1}{(\theta + i\alpha)^2} \right]
$$

\begin{align*}
&= \frac{1}{\alpha} \sum_{i=1}^{n-1} \frac{1}{\theta + i} + O(1) = \alpha^{-1} \log n + O(1),
\end{align*}

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which completes the proof for \( I^{(n)}_{\alpha} \).

Finally, we derive the leading term for \( I^{(n)}_{\alpha} \). Note
\[
\partial_\alpha^2 \ell_n(\alpha, \theta) = - \sum_{i=1}^{K_n-1} \frac{i^2}{(\theta + ia)^2} - \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{(i-\alpha)^2},
\]
where
\[
\frac{i^2}{(\theta + ia)^2} = \frac{1}{\alpha^2} - \frac{2\theta}{\alpha^2} \frac{1}{\theta + ia} + \frac{\theta^2}{\alpha^2(\theta + ia)^2}.
\]

Then, putting together the above displays and (D.2) and (D.3), one obtains
\[
I^{(n)}_{\alpha} = - \mathbb{E}[\partial_\alpha^2 \ell_n(\alpha, \theta)] = \mathbb{E} \left[ \sum_{i=1}^{K_n-1} \frac{i^2}{(\theta + ia)^2} \right] + \sum_{j=2}^{n} \mathbb{E}[S_{n,j}] \sum_{i=1}^{j-1} \frac{1}{(i-\alpha)^2}
\]
\[
= \frac{\mathbb{E}[K_n] - 1}{\alpha^2} - \frac{2\theta}{\alpha^2} \left[ \sum_{i=1}^{K_n-1} \frac{1}{\theta + ia} \right] + \frac{\theta^2}{\alpha^2} \left[ \sum_{i=1}^{K_n-1} \frac{1}{(\theta + ia)^2} \right] + \sum_{j=2}^{n} \mathbb{E}[S_{n,j}] \sum_{i=1}^{j-1} \frac{1}{(i-\alpha)^2}
\]
\[
= \frac{1}{\alpha^2} \mathbb{E}[K_n] + \sum_{j=2}^{n} \mathbb{E}[S_{n,j}] \sum_{i=1}^{j-1} \frac{1}{(i-\alpha)^2} - \frac{2\theta}{\alpha^2} \sum_{i=1}^{n-1} \frac{1}{\theta + i} + \frac{\theta^2}{\alpha^2} \mathbb{E} \left[ \sum_{i=1}^{K_n-1} \frac{1}{(\theta + ia)^2} \right] - \frac{1}{\alpha^2}
\]
\[
= \frac{\mathbb{E}[K_n]}{\alpha^2} + \sum_{j=2}^{n} \mathbb{E}[S_{n,j}] \sum_{i=1}^{j-1} \frac{1}{(i-\alpha)^2} + O(\log n).
\]

Here we take \( X_n := \alpha^{-2} K_n + \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} (i-\alpha)^{-2} \). Then, one obtains
\[
I^{(n)}_{\alpha} = \mathbb{E}[X_n] + O(\log n),
\]

Thus, it remains to show \( \mathbb{E}[X_n]/n^\alpha \to \mathbb{E}[M_{\alpha\theta}]I_{\alpha} \). We will show it by the dominated convergence theorem. First, we observe that \( X_n \) is bounded by \( K_n \) up to a constant:
\[
0 \leq X_n = \sum_{j=1}^{n} S_{n,j} \left( \frac{1}{\alpha^2} + \sum_{i=1}^{j-1} \frac{1}{(i-\alpha)^2} \right) = \sum_{j=1}^{n} S_{n,j} \sum_{i=0}^{j-1} \frac{1}{(i-\alpha)^2}
\]
\[
\leq \sum_{j=1}^{n} S_{n,j} \sum_{i=0}^{\infty} \frac{1}{(i-\alpha)^2} = \sum_{j=1}^{n} S_{n,j} \cdot C_{\alpha} = K_n C_{\alpha},
\]

Next Corollary A.1.1 applied with \( g(j) = \sum_{i=0}^{j-1} (i-\alpha)^{-2} \) and \( K_n/n^\alpha \to M_{\alpha\theta} \) (a.s.) result in
\[
\frac{X_n}{n^\alpha} = \frac{K_n}{n^\alpha} \sum_{j=1}^{\infty} S_{n,j} \sum_{i=0}^{j-1} \frac{1}{(i-\alpha)^2} \to M_{\alpha\theta} \sum_{j=1}^{\infty} p_\alpha(j) \sum_{i=0}^{j-1} \frac{1}{(i-\alpha)^2}
\]
\[
= M_{\alpha\theta} \left( \frac{1}{\alpha^2} + \sum_{j=2}^{\infty} p_\alpha(j) \sum_{i=1}^{j-1} \frac{1}{(i-\alpha)^2} \right) = M_{\alpha\theta} I_{\alpha} \text{ (a.s.)}
\]

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where we used (A) of Proposition 3.1 for the last equality. Conversely, putting together the above displays and $K_n / n^\alpha \to M_{\alpha 0}$ in mean by (A) of Lemma 2.2, the dominated convergence implies that

$$n^{-\alpha} E[X_n] = E[n^{-\alpha} X_n] \to E[M_{\alpha 0} I_\alpha] = E[M_{\alpha 0}] I_\alpha.$$  

This concludes the proof.

**D.4 Derivation of the condition Equation (B.2)**

Recall that $\partial x \ell_n(\cdot, \theta^*)$ is written by

$$\forall x \in (-\theta^* \lor 0, 1), \quad \partial_x \ell_n(x, \theta^*) = \sum_{i=1}^{K_n-1} \frac{i}{\theta + ix} - \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{i - x}.$$  

Here we prove that $\partial_x \ell_n(\cdot, \theta^*) = 0$ has a unique solution on $(-\theta^* \lor 0, 1)$ if and only if

$$1 < K_n < n, \quad \Theta_n := \frac{K_n(K_n - 1)}{2 \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} i^{-1}} > \theta. \quad (D.4)$$  

Here, it is important to emphasize that $\partial_x \ell_n(x, \theta^*)$ is decreasing in $x$:

$$\partial_x^2 \ell_n(x, \theta^*) := -\sum_{i=1}^{K_n-1} \frac{i^2}{(\theta + ix)^2} - \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{(i - x)^2} \leq 0.$$  

First, we prove the sufficiency of $(D.4)$. Since $\partial_x \ell_n(x, \theta^*)$ is decreasing, it suffices to show $\lim_{x \to 1^-} \partial_x \ell_n(x, \theta^*)(x) < 0$ and $\lim_{x \to (-\theta^* \lor 0, 1) +} \partial_x \ell_n(x, \theta^*) > 0$. Notice that $K_n < n$ is equivalent to $\sum_{j=2}^{n} S_{n,j} > 0$. Then we can rewrite $\partial_x \ell_n(x, \theta^*)$ by

$$\partial_x \ell_n(x, \theta^*) = -\frac{1}{1 - x} \sum_{j=2}^{n} S_{n,j} + \sum_{i=1}^{K_n-1} \frac{i}{\theta + ix} - \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{i - x},$$  

where, the first term goes to $-\infty$ as $x \to 1^-$ while the others converge to a finite value. Thus, $\lim_{x \to 1^-} \partial_x \ell_n(x, \theta^*)(x) < 0$ holds.

For $\lim_{x \to (-\theta^* \lor 0, 1)} \partial_x \ell_n(x, \theta^*) > 0$, we consider the two cases $\{\theta > 0\}$ and $\{-\alpha < \theta \leq 0\}$ separately. In the former, observe that

$$\lim_{x \to (-\theta^* \lor 0)^+} \partial_x \ell_n(x, \theta^*) = \partial_x \ell_n(0, \theta^*)$$  

$$= \sum_{i=1}^{K_n-1} \frac{i}{\theta} - \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{i} = \frac{K_n(K_n - 1)}{2\theta} - \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{i}.$$  

$$= \frac{\sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} i^{-1}}{\theta} \left( \frac{K_n(K_n - 1)}{2 \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} i^{-1}} - \theta \right)$$  

$$= \frac{\sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} i^{-1}}{\theta} (\Theta_n - \theta) > 0.$$  

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where we used $\sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} i^{-1} > \sum_{j=2}^{n} S_{n,j} > 0$ and the assumption $\Theta_n > \theta$. In the latter case, i.e., when $-\alpha < \theta \leq 0$, $-\theta \lor 0 = -\theta$ holds. If we rewrite as

$$\partial_x \ell_n(x, \theta^*) = \frac{1}{\theta + x} + \sum_{i=2}^{K_n} \frac{i}{\theta + ix} - \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{i - x},$$

it is clear that the first term goes to $+\infty$ while the other terms converge to finite values as $x \to (-\theta)^+$. This confirms the proof of $\lim_{x \to (-\theta^\lor 0)} \partial_x \ell_n(x, \theta^*) > 0$.

Finally, we show that the condition (D.4) is necessary. Suppose $K_n = 1$, which is equivalent to $S_{n,n} = 1$. Then, $\partial_y \ell_n(x, \theta^*) = -n \sum_{i=1}^{n-1} (i-x)^{-1} < 0$ for all $x \in (-\theta \lor 0, 1)$, so it has no roots on $(-\theta \lor 0, 1)$. If $K_n = n$, i.e., $S_{n,1} = n$, we observe $\partial_x \ell_n(x, \theta^*) = n^{-1} \sum_{i=1}^{n-1} i/(\theta + ix) > 0$ for all $x \in (-\theta \lor 0, 1)$, so again it has no roots. Finally suppose that $1 < K_n < n$ but $\theta > \Theta_n$. In this case, $\theta$ must be positive since

$$\theta > \Theta_n = \frac{K_n(K_n-1)}{2 \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} i^{-1}} > 0.$$ 

Thus, $(-\theta \lor 0, 1) = (0, 1)$ holds. Since $\partial_x \ell_n(x, \theta^*)$ is decreasing in $x$, we have

$$\forall x \in (0, 1), \partial_x \ell_n(x, \theta^*) < \partial_x \ell_n(0, \theta^*) = \sum_{j=2}^{n} S_{n,j} \sum_{i=1}^{j-1} \frac{1}{\theta} (\Theta_n - \theta) \leq 0,$$

so again, it has no roots.

### D.5 Proof of Lemma A.1

We define $N' := \{\lim_{n \to \infty} P_n(j) \neq P(j)\}$ and $N := \cup_{j=1}^\infty N_j$, $\Pr(N) = 0$ by (B) of Lemma 2.2 implies $\Pr(N) = \Pr(\cup_{j=1}^\infty N_j) \leq \sum_{j=1}^\infty \Pr(N_j) = 0$, i.e., $N$ is null set. Thus, we can exclude the event $N$. Observe

$$|P_n(j) - P(j)| \leq P_n(j) + P(j), \sum_{j=1}^\infty \{P_n(j) + P(j)\} = 1 + 1 = 2.$$ 

Then, Fatou’s lemma implies that

$$2 - \lim sup_{n \to \infty} \sum_{j=1}^{\infty} |P_n(j) - P(j)| = \lim inf_{n \to \infty} \left(2 - \sum_{j=1}^{\infty} |P_n(j) - P(j)| \right)$$

$$= \lim inf_{n \to \infty} \left\{ \sum_{j=1}^{\infty} (P_n(j) + P(j) - |P_n(j) - P(j)|) \right\}$$

$$\geq \sum_{j=1}^{\infty} \left\{ \lim inf_{n \to \infty} (P_n(j) + P(j) - |P_n(j) - P(j)|) \right\} = \sum_{j=1}^{\infty} (2P(j) + 0) = 2.$$

Subtracting 2 from both sides, we obtain $\lim sup_{n \to \infty} \sum_{j=1}^{\infty} |P_n(j) - P(j)| = 0$, which is equivalent to $\lim_{n \to \infty} \sum_{j=1}^{\infty} |P_n(j) - P(j)| = 0$. 

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D.6 Proof of Lemma A.3

Recall that $\Psi : (0, 1) \to \mathbb{R}$ is defined by

$$\forall x \in (0, 1), \ \Psi(x) := \frac{1}{x} - \mathbb{P}g_x, \quad (D.5)$$

where $\mathbb{P}g_x = \sum_{j=1}^{\infty} p_{\alpha}(j) g_x(j)$ and $g_x(j) = \sum_{i=1}^{j-1} (i-x)^{-1}$. We claim that $\Psi(x)$ converges for each $x \in (0, 1)$ by the followings:

- $p_{\alpha}(j) = O(j^{-(1+\alpha)})$ (see Section 2.1)
- $\sum_{i=1}^{j-1} (i-x)^{-1} \sim \log j$ as $j \to \infty$.
- $\int_{1}^{\infty} \log x \cdot x^{-(1+\alpha)} \, dx = \alpha - 2 < +\infty$.

First, we prove (A): $\Psi$ is of class $C^1$ on $(0, 1)$. For any closed subset $K = [s, t] \subset (0, 1)$, observe

$$\forall j \in \mathbb{N}, \ \sup_{x \in K} \left| \frac{d}{dx} g_x(j) \right| = \sup_{x \in K} \left( \sum_{i=1}^{j-1} \frac{1}{(i-x)^2} < \sum_{i=1}^{\infty} \frac{1}{(i-t)^2} := C. \right.$$  

Since $\Psi(x)$ takes a form of expectation of $g_x$ with respect to $\mathbb{P}$ (see (D.5)), Weierstrass’s M-test implies that $\Psi'(x)$ converges uniformly on $K$. As $K$ is arbitrary, it means that $\Psi'(x)$ converges compactly on $(0, 1)$. Thus, we conclude that $\Psi$ is class $C^1$ on $(0, 1)$.

Now (B) immediately follows: The compact convergence that we have shown implies that we can change the summation and differentiation, which leads to

$$\Psi'(x) = \frac{d}{dx} \left( \frac{1}{x} - \mathbb{P}g_x \right) = -\frac{1}{x^2} - \mathbb{P} \frac{d}{dx} g_x$$

$$= -\frac{1}{x^2} - \sum_{j=1}^{\infty} p_{\alpha}(j) \sum_{i=1}^{j-1} \frac{1}{(i-x)^2} \quad (\ast) - I_{\alpha} < 0,$$

where $(\ast)$ follows from (A) of Proposition 3.1.

For (C), the differentiation of both side of $\sum_{j=1}^{\infty} p_{\alpha}(j) = 1$ by $\alpha$ implies that

$$\frac{d}{d\alpha} \sum_{j=1}^{\infty} p_{\alpha}(j) = 0.$$  

Now we claim that we can interchange the differential and summation on the right-hand side. Then, the definition $p_{\alpha}(j) := \alpha \prod_{i=1}^{j-1} (i - \alpha)/(j!)$ implies that

$$0 = \sum_{j=1}^{\infty} \frac{d}{d\alpha} p_{\alpha}(j) = \sum_{j=1}^{\infty} p_{\alpha}(j) \left( \frac{1}{\alpha} - \sum_{i=1}^{j-1} \frac{1}{i - \alpha} \right)$$

$$= \frac{1}{\alpha} - \sum_{j=2}^{\infty} p_{\alpha}(j) \sum_{i=1}^{j-1} \frac{1}{i - \alpha} \quad (\ast\ast) = \Psi(\alpha),$$

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where we used (B) of Proposition 3.1. This concludes the proof of (C). It remains to prove \( \frac{d}{d\alpha} \sum_{n=1}^{\infty} p_{\alpha}(j) = \sum_{n=1}^{\infty} \frac{d}{d\alpha} p_{\alpha}(j) \). We prove this by the compact convergence of \( \sum_{j=1}^{\infty} \frac{d}{d\alpha} p_{\alpha}(j) \) on \((0, 1)\), i.e. the uniform convergence of \( \sum_{j=1}^{\infty} \frac{d}{d\alpha} p_{\alpha}(j) \) on \( K \) for any closed interval \( K = [s, t] \subset (0, 1) \). Note that \( \alpha^{-1} p_{\alpha}(j) = \prod_{i=1}^{j-1} (i - \alpha)/j! \) is nonincreasing function on \((0, 1)\) for all \( j \in \mathbb{N} \). Then, we obtain

\[
\sum_{j=1}^{\infty} M_j^{s, t} = \sum_{j=1}^{\infty} \frac{p_s(j)}{s} \left( 1 + \sum_{i=1}^{j-1} \frac{t}{i} \right) =: M_j^{s, t},
\]

and we observe

\[
\sum_{j=1}^{\infty} M_j^{s, t} = \sum_{j=1}^{\infty} \frac{p_s(j)}{s} \left( 1 + \sum_{i=1}^{j-1} \frac{t}{i} \right) = \frac{1}{s} + \frac{t}{s} \sum_{j=1}^{\infty} p_s(j) \sum_{i=1}^{j-1} \frac{1}{i} < +\infty.
\]

Here the right-hand side converges based on the same argument at the beginning of this proof. Thus, Weierstrass’s M-test implies that \( \sum_{j=1}^{\infty} \frac{d}{d\alpha} p_{\alpha}(j) \) converges uniformly on \([s, t]\).

### D.7 Proof of Lemma B.1

We observe

\[
\hat{\Psi}_{n, \theta^*}(x) - \Psi(x) = -\frac{1}{xK_n} - \frac{\theta^*}{xK_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta^* + ix} - (P_n g_x - P g_x), \tag{D.6}
\]

where \( g_x(j) = \sum_{i=1}^{j-1} (i - x)^{-1} \). Then, \( K_n \to \infty \) (a.s.) and \( (P_n g_x - P g_x) \to^p 0 \) by Lemma A.2 implies \( \hat{\Psi}_{n, \theta^*}(x) - \Psi(x) = o_p(1) \), which concludes the proof of (A).

For (B), differentiation of (D.6) leads to

\[
\hat{\Psi}'_{n, \theta^*}(x) - \Psi'(x) = -\frac{1}{x^2K_n} + \frac{\theta^*}{x^2K_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta^* + ix} - \frac{\theta^*}{xK_n} \sum_{i=1}^{K_n-1} \frac{i}{(\theta^* + ix)^2} - (P_n h_x - P h_x),
\]

where \( h_x \) is the function defined by \( h_x(j) = \partial_x g_x(j) = \sum_{i=1}^{j-1} (i - x)^{-2} \). Thus,
we can bound $\sup_{x \in I} |\hat{\Phi}'_{n,\theta}(x) - \Psi'(x)|$ as follows:

$$\sup_{x \in I} |\hat{\Phi}'_{n,\theta}(x) - \Psi'(x)| \leq \sup_{x \in I} \frac{1}{K_n x^2} + \sup_{x \in I} \frac{\theta^*}{x^2 K_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta^* + ix}$$

$$+ \sup_{x \in I} \frac{\theta^*}{x K_n} \sum_{i=1}^{K_n-1} \frac{i}{(\theta^* + ix)^2} + \sup_{x \in I} |p_n h_x - p h_x|.$$

Here we write $I = [s, t] \subset (-\theta^* \vee 0, 1)$. Then each term is bounded:

$$0 \leq \sup_{x \in I} \frac{1}{K_n x^2} \leq \frac{1}{K_n s^2}$$

$$0 \leq \sup_{x \in I} \frac{\theta^*}{x^2 K_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta^* + ix} \leq \frac{\theta^*}{s^2 K_n} \sum_{i=1}^{K_n-1} \frac{1}{\theta^* + is} = O(K_n^{-1} \log K_n)$$

$$0 \leq \sup_{x \in I} \frac{\theta^*}{x K_n} \sum_{i=1}^{K_n-1} \frac{i}{(\theta^* + ix)^2} \leq \frac{\theta^*}{s K_n} \sum_{i=1}^{K_n-1} \frac{i}{(\theta^* + is)^2} = O(K_n^{-1} \log K_n)$$

$$0 \leq \sup_{x \in I} |p_n h_x - p h_x| \leq \sum_{j=1}^{\infty} |p_n(j) - p(j)| \cdot \sup_{x \in I} ||h_x||_{\infty}$$

$$= \sum_{j=1}^{\infty} |p_n(j) - p(j)| \cdot \sum_{i=1}^{\infty} \frac{1}{(i-1)^2} = O(1) \sum_{j=1}^{\infty} |p_n(j) - p(j)|.$$

Then $K_n \to \infty$ (a.s.) and $\sum_{j=1}^{\infty} |p_n(j) - p(j)| \to 0$ (a.s.) by Lemma A.1 implies that they converge to 0 (a.s.), which concludes the proof.

### D.8 Proof of Lemma C.1

The following proof is basically the same as [Carlton, 1999, Lemma 5.3], but we write it to make our proof self-contained. We fix $x \in (0, 1)$ and define $\Phi_{n,x} : (-x, \infty) \to \mathbb{R}$ by

$$\forall y > -x, \hspace{1em} \Phi_{n,x}(y) = \partial_y \ell_n(x,y) = \sum_{i=1}^{K_n-1} \frac{1}{y + i x} - \sum_{i=1}^{n-1} \frac{1}{y + i}.$$

(D.7)

Then, $1 < K_n$ implies

$$\lim_{y \to -x^+} \Phi_{n,x}(y) = \lim_{y \to -x^+} \frac{1}{y + x} + \sum_{i=2}^{K_n-1} \frac{1}{(i-1)x} - \sum_{i=1}^{n-1} \frac{1}{-x + i} = +\infty.$$
Now we claim $\hat{\Phi}_{n,x}(y) < 0$ for sufficiently large $y$: we observe

$$\hat{\Phi}_{n,x}(y) = \sum_{i=1}^{K_n-1} \left( \frac{1}{y + ix} - \frac{1}{y + i} \right) - \sum_{i=1}^{K_n-1} \frac{1}{y + i} = \sum_{i=1}^{K_n-1} \frac{(1 - x)i}{(y + ix)(y + i)} - \sum_{i=1}^{K_n} \frac{1}{y + i}. \quad (D.8)$$

where the numerator in the last term is quadratic, and its coefficient is $-1$. Thus, the numerator takes a negative value for sufficiently large $y$ while the denominator remains positive. Putting together the above two displays and the intermediate value theorem, we see that $\hat{\Psi}_{n,x}$ has a root in $(-\infty, \infty)$.

Next, we prove the uniqueness. Observe

$$\hat{\Phi}'_{n,x}(y) = -\sum_{i=1}^{K_n-1} \frac{1}{(y + ix)^2} - \sum_{i=1}^{K_n} \frac{1}{(y + i)^2}$$

$$= -\sum_{i=1}^{K_n-1} \left( \frac{1}{(y + ix)^2} - \frac{1}{(y + i)^2} \right) + \sum_{i=K_n}^{K_n-1} \frac{1}{(y + i)^2}$$

$$= -\sum_{i=1}^{K_n-1} \frac{i(1 - x)(y + ix + y + i)}{(y + ix)^2(y + i)^2} + \sum_{i=K_n}^{K_n-1} \frac{1}{(y + i)^2}$$

$$< -\sum_{i=1}^{K_n-1} \frac{i(1 - x)(y + ix + y + i)}{(y + ix)^2(y + i)^2} + \sum_{i=K_n}^{K_n-1} \frac{1}{(y + i)^2}$$

$$= -2 \sum_{i=1}^{K_n-1} \frac{(1 - x)i}{(y + ix)(y + i)^2} + \sum_{i=K_n}^{K_n-1} \frac{1}{(y + i)^2}$$

$$\leq -\frac{2}{y + K_n - 1} \sum_{i=1}^{K_n-1} \frac{(1 - x)i}{(y + ix)(y + i)} + \sum_{i=K_n}^{K_n-1} \frac{1}{(y + i)^2}.$$ 

If $\hat{\Phi}'_{n,x}(y) \geq 0$, then $Q_n \leq ((y + K_n - 1)/2) \cdot \sum_{i=K_n}^{n-1} (y + i)^{-2}$. This and (D.8) result in

$$\hat{\Phi}_{n,x}(y) = Q_n - \sum_{i=K_n}^{n-1} \frac{1}{y + i} \leq y + K_n - 1 - \frac{2}{y + i} \sum_{i=K_n}^{n-1} \frac{1}{(y + i)^2} - \frac{1}{y + i}$$

$$= \sum_{i=K_n}^{n-1} \frac{y + K_n - 1 - 2(y + i)}{2(y + i)^2}$$

$$= \sum_{i=K_n}^{n-1} \frac{K_n - 2i - y}{2(y + i)^2} \leq \sum_{i=K_n}^{n-1} \frac{K_n - 2i}{2(y + i)^2} < -\frac{K_n}{2(y + K_n)^2} < 0,$$
for all \( y > -x > -1 \). In summary, we get
\[
\forall y \in (-x, \infty), \quad \hat{\Phi}_{n,x}'(y) \geq 0 \Rightarrow \hat{\Phi}_{n,x}(y) < 0, \quad (D.9)
\]
which implies the uniqueness of the solution of \( \hat{\Phi}_{n,x}(y) = 0 \) on \((-x, \infty)\). Let \( \hat{y}_n(x) \) be the unique solution. Then, \( \hat{\Phi}_{n,x}'(\hat{y}_n(x)) < 0 \) is obvious from (D.9) and \( \hat{\Phi}_{n,x}(\hat{y}_n(x)) = 0 \).

D.9 Proof of Lemma C.3

Appendix D.8 implies that \( \hat{y}_n(x) \) is the unique root of \( \hat{\Phi}_{n,x} \) on \((-x, \infty)\), which is defined by (D.7), and hence, \( \hat{y}_n(x) \) satisfies the following equality:
\[
\forall x \in (0, 1), \quad \sum_{i=1}^{K_n-1} \frac{1}{y_n(x) + ix} = \sum_{i=1}^{n-1} \frac{1}{y_n(x) + i} \quad \text{and} \quad \hat{y}_n(x) > -x. \quad (D.10)
\]

Then, we have
\[
\hat{\Psi}_n(x) = \Psi_n(x, \hat{y}_n(x)) = \frac{1}{K_n} \sum_{i=1}^{K_n-1} \frac{i}{y_n(x) + ix} - \sum_{j=2}^{n} \frac{1}{K_n} \sum_{i=1}^{j-1} \frac{1}{y_n(x) + i - x}
\]
\[
= \frac{1}{x} - \frac{1}{K_n x} \sum_{i=1}^{K_n-1} \frac{\hat{y}_n(x)}{y_n(x) + ix} - P_n g_x
\]
\[
= (\ast 1) \frac{1}{x} - \frac{1}{K_n x} \sum_{i=1}^{n-1} \frac{\hat{y}_n(x)}{y_n(x) + i} - P_n g_x,
\]
\[
= (\ast 2) \frac{1}{x} - \frac{1}{K_n x} \sum_{i=1}^{n-1} \frac{\hat{z}_n(x)}{y_n(x) + i} - P_n g_x,
\]
where (\ast 1) follows from (D.10) and we used \( \hat{z}_n(x) = \hat{y}_n(x)/x \) for (\ast 2). This concludes the proof.

D.10 Proof of Lemma C.4

Proof of (A)

Suppose \( 1 < K_n < n \). Then, for all \( x \in (0, 1) \), Appendix D.8 implies that \( \hat{y}_n(x) \) is the unique solution of
\[
\Phi_n(x, y) = \partial_y \ell_n(x, y) = \sum_{i=1}^{K_n-1} \frac{1}{y + ix} - \sum_{i=1}^{n-1} \frac{1}{y + i} = 0
\]
and \( \partial_y \Phi_n(x, \hat{y}_n(x)) < 0 \). Therefore, the implicit function theorem implies
\[
\hat{y}_n'(x) = -\partial_x \Phi_n(x, y)/\partial_y \Phi_n(x, y).
\]
Then, the derivative of \( \hat{z}_n(x) = \hat{y}_n(x)/x \) can be written as

\[
\hat{z}_n'(x) = \frac{-\hat{y}_n(x) + x\hat{y}_n'(x)}{x^2} = \frac{-\hat{y}_n(x)\partial_y \Phi_n(x, \hat{y}_n(x)) - x\partial_x \Phi_n(x, \hat{y}_n(x))}{x^2\partial_y \Phi_n(x, \hat{y}_n(x))}.
\]

(D.11)

Note that the numerator of (D.11) is calculated as

\[
-\hat{y}_n(x)\partial_y \Phi_n(x, \hat{y}_n(x)) - x\partial_x \Phi_n(x, \hat{y}_n(x))
\]

\[
= K_n^{-1} \sum_{i=1}^{n} \frac{\hat{y}_n(x)}{(\hat{y}_n(x) + ix)^2} - \sum_{i=1}^{n-1} \frac{\hat{y}_n(x)}{(\hat{y}_n(x) + i)^2} + \sum_{i=1}^{K_n^{-1}} \frac{ix}{(\hat{y}_n(x) + ix)^2}
\]

\[
= \sum_{i=1}^{n-1} \frac{1}{\hat{y}_n(x) + i} - \sum_{i=1}^{n} \frac{\hat{y}_n(x)}{(\hat{y}_n(x) + i)^2} = \sum_{i=1}^{n-1} \frac{i}{(\hat{y}_n(x) + i)^2} = \sum_{i=1}^{n-1} \frac{i}{(x\hat{z}_n(x) + i)^2} > 0,
\]

where we use (D.10) for the third equality. In contrast, the denominator of (D.11) is negative owing to \( \partial_y \Phi_n(x, \hat{y}_n(x)) < 0 \) and is written as

\[
0 > x^2\partial_y \Phi_n(x, \hat{y}_n(x)) = - \sum_{i=1}^{K_n^{-1}} \frac{x^2}{(\hat{y}_n(x) + ix)^2} + \sum_{i=1}^{n-1} \frac{x^2}{(\hat{y}_n(x) + i)^2}
\]

\[
= - \sum_{i=1}^{K_n^{-1}} \frac{1}{(x\hat{z}_n(x) + i)^2} + \sum_{i=1}^{n-1} \frac{x^2}{(x\hat{z}_n(x) + i)^2}.
\]

Therefore, we have

\[
0 > \hat{z}_n'(x) = \frac{\sum_{i=1}^{n-1} i/(x\hat{z}_n(x) + i)^2}{-\sum_{i=1}^{K_n^{-1}} (\hat{z}_n(x) + i)^{-2} + \sum_{i=1}^{n-1} x^2/(x\hat{z}_n(x) + i)^2},
\]

thereby completing the proof of (A).

Toward the proof of (B) and (C), we derive the following nonasymptotic inequality.

**Lemma D.1.** Suppose \( 1 < K_n < n \). Then, for all \( x \in (0, 1) \), it holds that

\[
|\hat{z}_n(x)| \leq 1 \vee (A_n(x) \wedge B_n(x) \wedge C_n(x; \alpha)),
\]

where \( A_n(x), B_n(x) \) and \( C_n(x; \alpha) \) are defined by

\[
A_n(x) := \frac{nK_n}{x(n-K_n)},
\]

\[
B_n(x) := K_n \left\{ \left( \frac{n + \hat{z}_n(x)}{1 + \hat{z}_n(x)} \right)^x - 1 \right\}^{-1},
\]

\[
C_n(x; \alpha) := n^{\frac{\alpha-x}{\alpha}} \left( \frac{2K_n + \hat{z}_n(x)}{n^\alpha} \right)^{\frac{1}{1-x}}.
\]
Proof. We fix \( x \in (0,1) \). Suppose \( \hat{z}_n(x) < 1 \). Then the assertion is obvious from \( 1 > \hat{z}_n(x) = \hat{y}_n(x)/x > -x/x = -1 \) so that \( |\hat{z}_n(x)| < 1 \).

Now we assume \( \hat{z}_n(x) \geq 1 \). Note that \( x \cdot \Phi_n(x,x\hat{z}_n(x)) = x \cdot 0 = 0 \) is equivalent to

\[
\sum_{i=1}^{K_n-1} \frac{x}{x\hat{z}_n(x) + i} - \sum_{i=1}^{n-1} \frac{x}{x\hat{z}_n(x) + i} = 0 \Leftrightarrow \sum_{i=1}^{n-1} \frac{x}{x\hat{z}_n(x) + i} = \sum_{i=1}^{K_n-1} \frac{1}{\hat{z}_n(x) + i},
\]

which implies

\[
\frac{x(n-1)}{x\hat{z}_n(x) + n - 1} \leq \sum_{i=1}^{n-1} \frac{x}{x\hat{z}_n(x) + i} = \sum_{i=1}^{K_n-1} \frac{1}{\hat{z}_n(x) + i} \leq \frac{K_n - 1}{\hat{z}_n(x) + 1} \leq \frac{\hat{z}_n(x)}{K_n - 1} + \frac{1}{K_n - 1} \leq \frac{\hat{z}_n(x)}{n - 1} + \frac{1}{x}.
\]

\Rightarrow \frac{\hat{z}_n(x)}{K_n - 1} \leq \frac{(n-1)(K_n - 1 - x)}{x(n - K_n)} < \frac{nK_n}{x(n - K_n)} = A_n(x). \quad (D.12)

In contrast, elementary calculus implies

\[
\forall m \in \mathbb{N}_{\geq 2}, \forall \delta > 0, \log \left( \frac{\delta + m}{\delta + 1} \right) < \log \left( \frac{\delta + m - 1}{\delta} \right) < \log \left( \frac{\delta + m}{\delta} \right).
\]

Then, this and the assumption \( \hat{z}_n(x) \geq 1 > 0 \) result in

\[
x \log \left( \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} \right) < \sum_{i=1}^{n-1} \frac{x}{x\hat{z}_n(x) + i} \leq \sum_{i=1}^{n-1} \frac{x}{x\hat{z}_n(x) + i} = \sum_{i=1}^{K_n-1} \frac{1}{\hat{z}_n(x) + i} \leq \log \left( \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} \right)
\]

\Rightarrow \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} < \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} \leq \frac{nK_n}{x(n - K_n)} = A_n(x).
\]

(13)

\[
\Rightarrow \hat{z}_n(x) < K_n \left\{ \left( \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} \right)^x - 1 \right\}^{-1} = B_n(x). \quad (D.14)
\]

In contrast, elementary calculus implies

\[
\forall m \in \mathbb{N}_{\geq 2}, \forall \delta > 0, \log \left( \frac{\delta + m}{\delta + 1} \right) < \log \left( \frac{\delta + m - 1}{\delta} \right) < \log \left( \frac{\delta + m}{\delta} \right).
\]

Then, this and the assumption \( \hat{z}_n(x) \geq 1 > 0 \) result in

\[
x \log \left( \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} \right) < \sum_{i=1}^{n-1} \frac{x}{x\hat{z}_n(x) + i} \leq \sum_{i=1}^{n-1} \frac{x}{x\hat{z}_n(x) + i} = \sum_{i=1}^{K_n-1} \frac{1}{\hat{z}_n(x) + i} \leq \log \left( \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} \right)
\]

\Rightarrow \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} < \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} \leq \frac{nK_n}{x(n - K_n)} = A_n(x).
\]

(13)

\[
\Rightarrow \hat{z}_n(x) < K_n \left\{ \left( \frac{\hat{z}_n(x) + n}{\hat{z}_n(x) + 1} \right)^x - 1 \right\}^{-1} = B_n(x). \quad (D.14)
\]
Furthermore, (D.13) and the assumption $\hat{z}_n(x) \geq 1 > 0$ yield

$$\frac{\hat{z}_n(x)}{(\hat{z}_n(x) + 1)^x} < \frac{\hat{z}_n(x) + K_n}{(\hat{z}_n(x) + n)^x} < \frac{\hat{z}_n(x) + K_n}{(0 + n)^x} = \frac{\hat{z}_n(x) + K_n}{n^x}$$

$$\Rightarrow \hat{z}_n(x)^{1-x} < \left(1 + \frac{\hat{z}_n(x)}{n^x}\right)^x \frac{\hat{z}_n(x) + K_n}{n^x}$$

$$\leq \left(1 + \frac{1 + 1}{1}\right)^x \frac{\hat{z}_n(x) + K_n}{n^x} < n^{\alpha - x} \cdot 2 \frac{\hat{z}_n(x) + K_n}{n^\alpha}$$

$$\Rightarrow \hat{z}_n(x) < n^{\frac{\alpha - x}{x}} \left(2 \frac{\hat{z}_n(x) + K_n}{n^\alpha}\right)^{\frac{x}{1-x}} = C_n(x; \alpha). \quad (D.15)$$

Combining (D.12), (D.14), and (D.15), we get

$$\hat{z}_n(x) \geq 1 \Rightarrow |\hat{z}_n(x)| < A_n(x) \wedge B_n(x) \wedge C_n(x; \alpha),$$

thereby completing the proof. □

**Proof of (B)**

We fix $x \in (0, 1)$. For $A_n(x)$ in Lemma D.1, $K_n = O_p(n^\alpha) = o_p(n)$ implies

$$\frac{A_n(x)}{n} = \frac{K_n}{n(x - K_n)} = o_p(1),$$

and hence $|\hat{z}_n(x)| \leq 1 \vee A_n(x) = o_p(n)$. Considering this, for $B_n(x)$ in Lemma D.1, it holds that

$$\frac{B_n(x)}{n^\alpha} = \frac{K_n}{n^\alpha} \left\{ \left(\frac{n + \hat{z}_n(x)}{1 + \hat{z}_n(x)}\right)^x - 1 \right\}^{-1} \to M_{a \theta} \cdot 0 = 0 \text{ in probability},$$

which implies $|\hat{z}_n(x)| \leq 1 \vee B_n(x) = o_p(n^\alpha)$. Then, for $C_n(x; \alpha)$ in Lemma D.1, we get

$$n^{-\frac{\alpha - x}{1-x}} \cdot C_n(x; \alpha) = \left(2 \frac{K_n + \hat{z}_n(x)}{n^\alpha}\right)^{\frac{1}{1-x}} \to (2 M_{a \theta} + 0)^{\frac{1}{1-x}} = (2 M_{a \theta})^{\frac{1}{1-x}} \text{ in probability},$$

where we used $K_n/n^\alpha \to M_{a \theta} (\text{a.s.})$. Therefore,

$$|\hat{z}_n(x)| \leq 1 \vee C_n(x; \alpha) = 1 \vee O_p(n^{\frac{\alpha - x}{1-x}}) = \left\{ \begin{array}{ll} O_p(n^{\frac{\alpha - x}{1-x}}) & 0 < x < \alpha \\ O_p(1) & \alpha \leq x < 1 \end{array} \right.,$$

thereby completing the proof.

**Proof of (C)**

Step 1. $\sup_{x \in [\alpha \pm \delta_n]} |\hat{z}_n(x)| = O_p(1)$: Note that we can take $(s, t)$ satisfying $0 < s < \alpha < t < 1$ and $[\alpha \pm \delta_n] \subset [s, t] \subset (0, 1)$ for sufficiently large $n$ by
\[ \delta_n = o(1/\log n) = o(1). \] Considering \( \hat{\delta}_n(x) \) is monotone on \((0, 1)\) by (A), (B) implies
\[
\sup_{x \in [0, x]} |\hat{\delta}_n(x)| \leq \sup_{x \in [s, t]} |\hat{\delta}_n(x)| \leq |\hat{\delta}_n(s)| + |\hat{\delta}_n(t)| = O_p(n^{\frac{\alpha - \varepsilon}{n^\alpha}}) + O_p(1) = o_p(n^\alpha).
\]
Then, for \( C_n(x; \alpha) \) in Lemma D.1, it holds that
\[
\log C_n(x = \alpha \pm \delta_n; \alpha) = \frac{\mp \delta_n}{1 - \alpha \mp \delta_n} \log n + \frac{1}{1 - \alpha \mp \delta_n} \log \left( 2K_n + \hat{\delta}_n(\alpha \pm \delta_n) \right) \\
\to 0 + \frac{1}{1 - \alpha} \log(2M_\alpha + 0) = O_p(1),
\]
and hence \( C_n(\alpha \pm \delta_n; \alpha) = O_p(1) \). Therefore, the monotonicity of \( \hat{\delta}_n(x) \) on \((0, 1)\) and Lemma D.1 imply
\[
\sup_{x \in [0, x]} |\hat{\delta}_n(x)| \leq |\hat{\delta}_n(\alpha \pm \delta_n)| \leq 1 \lor C_n(\alpha \pm \delta_n; \alpha) = O_p(1),
\]
which completes the proof.

Step 2. \( \sup_{x \in [0, x]} |f_\alpha(\hat{\delta}_n(x)) - \log(K_n/n^\alpha)| = o_p(1) \): Denote \( \hat{\delta}_n(\alpha \pm \delta_n) \) by \( \hat{\delta}_n^\pm \) and \( \alpha \pm \delta_n \) by \( \alpha_n^\pm \). Note that \( \hat{\delta}_n^\prime(x) < 0 \) by (A) and \( f_\alpha'(z) > 0 \) by Lemma 3.2 imply \( f_\alpha \circ \hat{\delta}_n'(x) < 0 \) for all \( x \in (0, 1) \), and hence, \( f_\alpha \circ \hat{\delta}_n \) is monotone on \((0, 1)\). Therefore, it suffices to show
\[
f_\alpha(\hat{\delta}_n^\pm) - \log(K_n/n^\alpha) = o_p(1).
\]
Now we claim \( \alpha_n^\pm \Phi_n(\alpha_n^+, \alpha_n^-, \hat{\delta}_n^\pm) = \alpha_n^\pm \cdot 0 = 0\): This is because \( \alpha_n^\pm \hat{\delta}_n^\pm = \hat{\gamma}_n(\alpha_n^\pm) \) by definition and \( \hat{\gamma}_n \) is the function s.t. \( \Phi_n(x, \hat{\gamma}_n(x)) = 0 \) for all \( x \in (0, 1) \) (see (C.1)). Then, we have
\[
0 = \sum_{i=1}^{K_n-1} \frac{\alpha_n^\pm}{\alpha_n^\pm + i + \alpha_n^\pm} - \sum_{i=1}^{K_n-1} \frac{\alpha_n^\pm}{\alpha_n^\pm + i + \alpha_n^\pm} = K_n-1 \sum_{i=1}^{n-1} \frac{1}{\alpha_n^\pm + i + \alpha_n^\pm} - \sum_{i=1}^{n-1} \frac{\alpha_n^\pm}{\alpha_n^\pm + i + \alpha_n^\pm}
\]
\[
= \sum_{i=1}^{K_n-1} \frac{1}{\alpha_n^\pm + i + 1} - \sum_{i=1}^{n-1} \frac{\alpha_n^\pm - \alpha}{\alpha_n^\pm + \alpha_n^\pm + i} - \sum_{i=1}^{n-1} \frac{1}{\alpha_n^\pm + i + 1} - \sum_{i=1}^{n-1} \frac{1}{\alpha_n^\pm + i + 1}
\]
\[
= \psi(K_n + \hat{\delta}_n^+) - \psi(1 + \hat{\delta}_n^+) - \sum_{i=1}^{n-1} \frac{\alpha_n^\pm - \alpha}{\alpha_n^\pm + \alpha_n^\pm + i} - \sum_{i=1}^{n-1} \frac{1}{\alpha_n^\pm + \alpha_n^\pm + i} \psi(n + \alpha_n^\pm \hat{\delta}_n^+) - \alpha \psi(1 + \alpha_n^\pm \hat{\delta}_n^-)
\]
where \( \psi \) is the digamma function. Using \( f_\alpha(z) = \psi(1 + z) - \alpha \psi(1 + \alpha z) \), the above equality is equivalent to
\[
f_\alpha(\hat{\delta}_n^\pm) - \log(K_n/n^\alpha) = (\psi(K_n + \hat{\delta}_n^+) - \log K_n) - \alpha(\psi(n + \alpha_n^\pm \hat{\delta}_n^+) - \log n)
\]
\[
- \sum_{i=1}^{n-1} \frac{\alpha_n^\pm - \alpha}{\alpha_n^\pm + \alpha_n^\pm + i} - \sum_{i=1}^{n-1} \frac{\alpha(\alpha_n^\pm \hat{\delta}_n^+)}{(\alpha_n^\pm + \hat{\delta}_n^+ + i)(\alpha_n^\pm + i)}.
\]
Note that $\hat{z}_n^\pm = O_p(1)$ by Step 1. Then, the first and second terms are $o_p(1)$ by Lemma B.5 applied with $\hat{z}_n^\pm / K_n = O_p(1)/K_n = o_p(1)$ and $\hat{z}_n^\pm / n = O_p(1)/n = o_p(1)$. As for the third and fourth terms, $\alpha_n^\pm \in [s, t] \subset (0, 1)$ for sufficiently large $n$ and $\hat{z}_n^\pm = O_p(1)$ by Step 1 result in

$$\left| \sum_{i=1}^{n-1} \frac{\alpha_n^\pm - \alpha}{\alpha_n^\pm \hat{z}_n^\pm + i} \right| \leq \delta_n \sum_{i=1}^{n-1} \frac{1}{i - \alpha_n^\pm} \leq \delta_n \sum_{i=1}^{n-1} \frac{1}{i - \ell} = o(1/\log n)O_p(\log n) = o_p(1),$$

$$\left| \sum_{i=1}^{n-1} \frac{\alpha(\alpha - \alpha_n^\pm)\hat{z}_n^\pm}{(\alpha_n^\pm \hat{z}_n^\pm + i)(\alpha \hat{z}_n^\pm + i)} \right| \leq \alpha \delta_n |\hat{z}_n^\pm| \sum_{i=1}^{n-1} \frac{1}{(-\ell + i)(-\alpha + i)} = o(1/\log n)O_p(1)O(1) = o_p(1).$$

Therefore, we have $f_\alpha(\hat{z}_n^\pm) - \log(K_n/n^\alpha) = o_p(1)$, thereby completing the proof.