THE PERIOD-INDEX PROBLEM IN WC-GROUPS I: ELLIPTIC CURVES

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Abstract. Let \( E/K \) be an elliptic curve defined over a number field, and let \( p \) be a prime number such that \( E(K) \) has full \( p \)-torsion. We show that the order of the \( p \)-part of the Shafarevich-Tate group of \( E/L \) is unbounded as \( L \) varies over degree \( p \) extensions of \( K \). The proof uses O’Neil’s period-index obstruction. We deduce the result from the fact that, under the same hypotheses, there exist infinitely many elements of the Weil-Châtelet group of \( E/K \) of period \( p \) and index \( p^2 \).

1. Introduction

The aim of this note is to prove the following result.

Theorem 1. Let \( p \) be a prime number, \( E/K \) an elliptic curve over a number field with \( E[p](\overline{K}) = E[p](K) \), and \( r \) be a positive integer. Then there are infinitely many degree \( p \) field extensions \( L/K \) such that

\[
\dim_{\mathbb{F}_p} \text{III}(E/L)[p] \geq r.
\]

Recall that for any elliptic curve over a field \( K \) of characteristic different from \( p \), all \( p \)-torsion points become rational over an extension field of degree dividing \( |GL_2(\mathbb{F}_p)| = (p^2 - 1)(p^2 - p) \). Moreover, if \( E/\overline{K} \) admits complex multiplication, all \( p \)-torsion points become rational over an extension of degree dividing \( 2(p^2 - 1) \) or \( 2(p - 1)^2 \). This immediately gives the following corollary.

Corollary 2. If \( E/K \) is an elliptic curve over a number field, \( p \) a prime and \( r \) a positive integer, there exist infinitely many field extensions \( L/K \) of degree at most \( p^5 \) such that \( \dim_{\mathbb{F}_p} \text{III}(E/L)[p] \geq r \). Moreover, for infinitely many \( E/K \) — namely those admitting complex multiplication over \( \overline{K} \) — the same result holds for infinitely many field extensions of degree at most \( 2p^3 \).

In Section 2 we deduce Theorem 1 as a consequence of the following result, which is of independent interest.

Theorem 3. Let \( p \) be a prime, and \( E/K \) an elliptic curve over a number field with all its \( p \)-torsion defined. Then there exists an infinite subgroup of \( H^1(K,E)[p] \) all of whose nonzero elements have index \( p^2 \) (i.e., \( p^2 \) divides the degree of any splitting field extension).

The proof of Theorem 3 makes essential use of the period-index obstruction map of Catherine O’Neil. In particular we need two results concerning this map. The first, Theorem 5, gives a necessary and sufficient condition for the period to equal the index. As the reader shall see, Theorem 5 is an immediate consequence of results of Cassels or of O’Neil. The second, Theorem 6, is a computation of the period-index
obstruction in the case of full level structure, a result which appears in [O’Neil] but requires correction. Section 3 is devoted to a review of the period-index obstruction and a proof of these two results.

The proof of Theorem 3 is given in Section 4.

Finally, in Section 5 we discuss some issues raised by the proofs and the possibility of certain generalizations.

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2. Theorem 3 implies Theorem 1

Let $S \subset H^1(K, E)[p]$ be an infinite subgroup all of whose nontrivial elements have index $p^2$. For each $\eta_i \in S$, there is a finite set of places $v$ of $K$ such that $\eta_i$ remains nonzero in the completion $K_v$. By a theorem of [Lichtenbaum], every class in $H^1(K_v, E)[p]$ can be split by a degree $p$ extension. It now follows from the usual (weak) approximation theorem for valuations that for any single $\eta_i$ we can find a degree $p$ global extension $K_i/K$ such that $\eta_i|_{K_i}$ is zero everywhere locally, i.e., represents an element of $\text{III}(E/K)[p]$. Because the index of $\eta_i$ is $p^2$ and we have made a field extension of degree only $p$, this is a nontrivial element. (This argument is due to William Stein.)

We now refine the above argument to produce $r \mathbb{F}_p$-linearly independent classes. For this, observe first that $H^1(K_v, E)[p]$ is a finite group (e.g. it is a homomorphic image of $H^1(K_v, E)[p]$), and the Galois cohomology groups of a finite module over a $p$-adic field are finite: [CG, Prop. II.5.14]). Starting with an element $\eta_1$ of $S$, the subgroup $H_1 \subseteq S$ consisting of classes which are locally trivial at all places where $\eta_1$ is locally nontrivial has finite index and is therefore infinite; choose a nontrivial $\eta_2$ in this group. Continuing in this way, we can construct a cardinality $r$ set $\{\eta_1, \ldots, \eta_r\}$ of $\mathbb{F}_p$-linearly independent elements of $S$ such that the sets $\Sigma_i$ of places where $\eta_i$ is locally nontrivial are pairwise disjoint. Accordingly, we can again find a single global extension $L/K$ of degree $p$ such that all $r$ classes give elements of $\text{III}(E/L)[p]$. Let $\eta = a_1\eta_1 + \ldots + a_r\eta_r$ be any $\mathbb{F}_p$-linear combination of the $\eta_i$’s. As above, if $\eta|_L = 0$, then $\eta$ is a class in $S$ of index $p$, so $\eta = 0$: i.e., $a_1 = \ldots = a_r = 0$. Thus $\dim_{\mathbb{F}_p} \text{III}(E/L)[p] \geq r$.

3. On the period-index obstruction for elliptic curves

Throughout this section the notation is as follows: $K$ is an arbitrary field with absolute Galois group $\mathfrak{g}_K = \text{Gal}(K^{\text{sep}}/K)$, $n$ is a positive integer not divisible by the characteristic of $K$, and $E/K$ is an elliptic curve.

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1For an alternate argument using a theorem of Lang and Tate, see Remark 5.1.
3.1. The period-index obstruction map. Consider the Kummer sequence

$$0 \to E(K)/nE(K) \to H^1(K, E[n]) \to H^1(K, E)[n] \to 0.$$ 

The group $H^1(K, E)[n]$ parameterizes genus one curves $C/K$ equipped with the structure of a principal homogeneous space for $E = J(C) = \text{Pic}^0(C)$ and having period dividing $n$. This geometric interpretation “lifts” to $H^1(K, E[n])$ as follows.

**Proposition 4.** The group $H^1(K, E[n])$ classifies equivalence classes of pairs $(C, [D])$, where $C$ is a principal homogeneous space for $E$ and $[D] \in \text{Pic}^n(C)(K)$ is a $K$-rational divisor class of degree $n$. Two such classes are equivalent if and only if there exists an isomorphism of principal homogeneous spaces $f : C_1 \to C_2$ such that $f^*([D_2]) = [D_1]$.

I have been unable to find this proposition in the literature in the precise form in which we have stated it, but I am told that it has been well-known for a long time. Indeed, Proposition 4 can readily be deduced either from work of Cassels or of O’Neil.

Sketch of proof: In either case, the idea is to interpret $E[n]$ as an automorphism group of a suitable structure $S$, so that by Galois descent $H^1(K, E[n])$ parameterizes the twisted forms of $S$. But there is some latitude in the choice of $S$. The classical choice [Cassels 1966, Lemma 13.1] is to view $E[n]$ as the deck transformation group of $[n] : E \to E$, so that $H^1(K, E[n])$ parameterizes finite étale maps $f : C \to E$ which are geometrically Galois, with group $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. The correspondence is given as $f \mapsto (C, [nQ])$, where $Q$ is any element of $f^{-1}(O)$. O’Neil’s choice [O’Neil, Prop. 2.2] is to view $E[n]$ as the automorphism group of the morphism $\varphi_{L(D)} : E \to \mathcal{P}^{n-1}$ associated to the ample divisor $D = nO$, so that $H^1(K, E[n])$ parameterizes “diagrams” $C \to V$, where $C$ is a principal homogeneous space for $E$ and $V$ is a Severi-Brauer variety, indeed the Severi-Brauer variety associated to an effective rational divisor class $D$ as in [CL, §10.6].

Remark: O’Neil’s method can be adapted to give an analogue of Proposition 4 for higher-dimensional abelian varieties; see [Clark, §4.1].

Now recall that if $V/K$ is any smooth, projective, geometrically irreducible variety, there is an exact sequence

$$0 \to \text{Pic}(V) \to \text{Pic}(V)(K) \xrightarrow{\delta} Br(K) \to Br(V).$$

This is the exact sequence of terms of low degree arising from the Leray spectral sequence

$$E_2^{p,q} = H^p(\text{Spec } K, R^q\psi_*\mathbb{G}_m) \Rightarrow H^{p+q}(V, \mathbb{G}_m)$$

associated to the sheaf $(\mathbb{G}_m)_V$ and the morphism of étale sites induced by $V \to \text{Spec } K$.

We define the **period-index obstruction map**

$$\Delta : H^1(K, E[n]) \to Br(K)$$

by $(C, [D]) \mapsto \delta([D])$. 

In terms of O’Neil’s setup, the obstruction map is simply given by
\[(C \rightarrow V) \mapsto [V] \in Br(K),\]
where by \([V]\) we mean the Brauer group element corresponding to the Severi-Brauer variety \(V\) [CL, loc. cit.].

**Theorem 5.** A class \(\eta \in H^1(K, E[n])\) of exact period \(n\) has index \(n\) if and only if some lift of \(\eta\) to \(\xi \in H^1(K, E[n])\) has \(\Delta(\xi) = 0\).

**Proof:** \(\eta\) has index \(n\) if and only if there exists a \(K\)-rational divisor of degree \(n\) on the corresponding principal homogeneous space \(C\), i.e., if and only if some \(K\)-rational divisor class of degree \(n\) on \(C\) has vanishing obstruction. Thus the result is clear.

### 3.2. Theta groups, Heisenberg groups and the explicit obstruction map.

Clearly Theorem 5 can only be useful if we know something about the period-index obstruction map. Happily, such knowledge is indeed available, thanks to a result of O’Neil which identifies \(\Delta : H^1(K, E[n]) \rightarrow Br(K)\) as a connecting map in non-abelian Galois cohomology.

Indeed, let \(D = n[O]\) be the degree \(n\) divisor on \(E\) supported on the origin, and let \(L = L(D)\) be the associated line bundle. Let \(G = G_L\) be the theta group associated to the line bundle as in [Mumford]; \(G\) is an algebraic \(K\)-group scheme fitting into a short exact sequence
\[
1 \rightarrow \mathbb{G}_m \rightarrow G \rightarrow E[n] \rightarrow 1
\]
with center \(Z(G) = \mathbb{G}_m\). Viewing (1) as a central extension of \(g_K\)-modules, there is a cohomological connecting map
\[
\Delta : H^1(K, E[n]) \rightarrow H^2(K, \mathbb{G}_m) = Br(K).
\]
By [O’Neil, Prop. 2.3], \(\Delta\) coincides with the period-index obstruction defined in the last section.

The goal of this section is to compute \(\Delta\) in the case when \(E/K\) has full \(n\)-torsion defined over \(K\). That is, we assume that the finite étale \(K\)-group scheme \(E[n]\) is constant, and choose a Galois module isomorphism \(E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2\). The Galois-equivariance of Weil’s \(c_n\)-pairing implies that \(\mathbb{Z}/n\mathbb{Z} = \bigwedge^2 E[n] = \mu_n\) as Galois modules, so the above choice of basis induces an isomorphism
\[
H^1(K, E[n]) \cong H^1(K, \mu_n)^2 = (K^*/K^{*n})^2.
\]
So in this case the period-index obstruction can be viewed as a map
\[
\Delta : (K^*/K^{*n})^2 \rightarrow Br(K).
\]

Now we must point out that [O’Neil, 3.4] gives a computation of \(\Delta\) which is not quite correct: it is claimed that \(\Delta(a, b) = \langle a, b \rangle_n\), the norm-residue symbol. But the following counterexample was supplied by the referee:

Suppose \(n = 2\), so \(E\) is given as \(y^2 = (x - e_1)(x - e_2)(x - e_3)\). Then the map \(\iota : E(K)/2E(K) \rightarrow (K^*/K^{*2})^2\) is given explicitly for any point \((x, y) \in E(K)\) with \(x \neq e_1, e_2\) as
\[
\iota(x, y) = (x - e_1, x - e_2) \pmod{K^{*2}}
\]
Then there exist $C$, $\psi_0$ the center, we have $\psi(a, b_2) = 0$. But $e_1$, $e_2$, $e_3$ vary over all triples of distinct elements of $K$, $(e_3-e_1, e_3-e_2)$ runs through all elements of $K^*/K^{*2}$, and all Hilbert symbols $(a, b)$ vanish only if $Br(K)[2]$ vanishes.

On the other hand, the following result shows that the obstruction map $\Delta$ is close to being the norm residue symbol $\langle , \rangle_n$.

**Theorem 6.** Let $E/K$ be an elliptic curve over a field $K$ and $n$ a positive integer not divisible by the characteristic of $K$ and such that $E[n]$ is a trivial $g_K$-module. Then there exist $C_1$, $C_2 \in K^*/K^{*n}$ such that

$$\Delta(a, b) = \langle C_1a, C_2b \rangle_n - \langle C_1, C_2 \rangle_n.$$  

Before we begin the proof we will need to recall some facts about Heisenberg groups. There is an algebraic $K$-group scheme $H_n$, which is, like $G$, a central extension of $E[n]$ by $G_m$. To define $H_n$, one chooses a decomposition $E[n] = H_1 \oplus H_2$ into a direct sum of two cyclic order $n$ subgroup schemes. (With a view towards the higher-dimensional case, one should think of this as a Lagrangian decomposition, i.e., that each $H_i$ is maximal isotropic for the Weil $e_n$-pairing; of course this is automatic for elliptic curves.) Then $H_n$ is defined by the following 2-cocycle $f_{H_1, H_2} \in Z^2(E[n], G_m)$:

$$(P_1 + P_2, Q_1 + Q_2) \mapsto e_n(P_1, Q_2).$$

It is known (e.g. [Sharifi Prop. 2.3]) that the coboundary map $\Delta_H : H^1(K, E[n]) \to Br(K)[n]$ associated to the Heisenberg group $H_n$ is nothing but the norm-residue symbol $\langle , \rangle_n$. Moreover, by a well-known result of Mumford the theta group scheme $\mathcal{G}$ is isomorphic to the Heisenberg group $H_n$ when the base field is separably closed. Thus in general $\mathcal{G}$ is a Galois twisted form of $H_n$. Combining these two results with the above counterexample, it must be the case that $\mathcal{G}$ can be a nontrivial twisted form of $H_n$.

Nevertheless, we can completely understand the possible twists: they are parameterized by $H^1(K, Aut^*(H_n))$, where the $\ast$ indicates that we want not the full automorphism group of $H_n$ but only the automorphisms which act trivially on the subgroup $G_m$ and on the quotient $E[n]$. It will turn out that $Aut^*(H_n) \cong (H_1 \oplus H_2)$, so that the twisted forms of the Heisenberg group will be parameterized by pairs of order $n$ characters of $g_K$.

We now begin the proof of Theorem 6. Let $\psi \in Aut^*(H_n)$, and let $(P_1, P_2, \epsilon)$ denote an arbitrary element of the Heisenberg group. Since $\psi$ is the identity modulo the center, we have $\psi(P_i) = P_i$ for $i = 1, 2$; together with the fact that $\psi(0, 0, \epsilon) = (0, 0, \epsilon)$, this implies that $\psi : (P_2, P_2, \epsilon) \mapsto (P_1, P_2, \chi(\psi)(P_1, P_2) \epsilon)$. That is, an automorphism of $H_n$ as an extension determines a map $\chi : H_1 \oplus H_2 \to G_m$, i.e., a character of $H_1 \oplus H_2$. Conversely, any such character defines an automorphism, and we have canonically $Aut^*(H_n) = (H_1 \oplus H_2)^\vee$ (Pontrjagin = Cartier dual). It follows that the collection of twisted forms of the Heisenberg group is $H^1(K, (H_1 \oplus H_2)^\vee) \cong H^1(K, H_1 \oplus H_2)$, since the Weil pairing gives an autoduality $E[n] \cong E[n]$. 

[Silverman, Prop. X.1.4]. But $\Delta$ vanishes on $i(E(K)/2E(K))$, so in particular $\Delta(e_3 - e_1, e_3 - e_2) = 0$. But as $e_1$, $e_2$, $e_3$ vary over all triples of distinct elements of $K$, $(e_3-e_1, e_3-e_2)$ runs through all elements of $K^*/K^{*2}$, and all Hilbert symbols $(a, b)$ vanish only if $Br(K)[2]$ vanishes.
Changing notation slightly, let
\[ \chi \in H^1(K, \textrm{Aut}_*({\mathcal H}_n)) = H^1(K, (H_1 \oplus H_2)^\vee) \cong (K^*/K^{*n})^2 \]
be a one-cocycle. Using \( \chi \) we build a twisted form \( \mathcal{H}_\chi \) of \( \mathcal{H}_n \), i.e., the group scheme whose \( K \)-points are the same as the \( K \)-points of \( \mathcal{H}_n \), but with twisted \( g_K \)-action, as follows:
\[ \sigma \cdot (P_1, P_2, \epsilon) = (P_1, P_2, \chi(\sigma)(P_1, P_2)\sigma(\epsilon)). \]
We may now compute the cohomological coboundary map \( \Delta \) directly from its definition. For this, we view \( \mathcal{H}_\chi/K \) as \( \mathbb{G}_m \times E[n] \) “doubly twisted,” i.e., twisted as a \( g_K \)-set as just discussed, and twisted as a group via the cocycle \( f \) introduced above:
\[ (\alpha, P) \ast (\beta, Q) = (\alpha \beta f(P, Q), P + Q). \]
We note that the inverse of \( (\alpha, P) \) is \( (\alpha^{-1} f(P, P)^{-1}, -P) \). Let \( \eta \in Z^1(K, E[n]) \); we want to compute \( \Delta(\eta)(\sigma, \tau) \). The basic recipe for this allows us to choose arbitrary lifts \( N_\sigma, N_\tau, N_{\sigma \tau} \) of \( \eta \), \( \eta(\tau) \) \( \eta(\sigma \tau) \) to \( \mathcal{H}_\chi \) and put \( \Delta(\eta)(\sigma, \tau) = N_\sigma \sigma(\tau) N_{\sigma \tau}^{-1} \).
We choose to lift by the set-theoretic identity section: \( \eta(\sigma) \mapsto (1, \eta(\sigma)) \), and so on.
Keeping in mind that \( \sigma(\eta(\tau)) = \eta(\tau) \) and \( \eta(\sigma \tau) = \eta(\sigma)\eta(\tau) \), we get:
\[ \Delta(\eta)(\sigma, \tau) = (1, \eta(\sigma)) \ast (1, \eta(\tau)) \ast (1, \eta(\sigma \tau))^{-1} = \]
\[ (1, \eta(\sigma)) \ast (\chi(\sigma)(\eta(\tau)), \eta(\tau)) \ast (f(\eta(\sigma), -\eta(\sigma \tau))^{-1}, -\eta(\sigma \tau)) = \]
\[ (\chi(\sigma)(\eta(\tau))f(\eta(\sigma), \eta(\tau)), \eta(\sigma)\eta(\tau)) \ast (f(\eta(\sigma), -\eta(\sigma \tau))^{-1}, -\eta(\sigma \tau)) = \]
\[ (\chi(\sigma)(\eta(\tau))f(\eta(\sigma), \eta(\tau)), 0). \]
That is, the coboundary map \( \Delta : H^1(K, E[n]) \to Br(K)[n] \) is a product of two terms:
\[ \Delta(\eta)(\sigma, \tau) = \Delta_1 \cdot \Delta_2 = \chi(\sigma)(\eta(\tau)) \cdot f(\eta(\sigma), \eta(\tau)). \]
Indeed \( \Delta_2 \) and \( \Delta_1 \) are respectively the quadratic form and the linear form comprising the quadratic map \( \Delta \). Both of these terms are now easily recognizable: the quadratic part \( \Delta_2 \) (which, notice, is equal to \( \Delta \) when \( \mathcal{H}_\chi = \mathcal{H}_n \)) is the norm residue symbol [Sharifi, loc. cit.].

To evaluate \( \Delta_1 \), choose a basis \((P_1, P_2)\) of \( E[n] \) and use the induced decomposition of \( E[n] \) \( \cong H_1 \oplus H_2 \) and the corresponding decomposition of the dual space \( E[n]^\vee \) (i.e., we decompose any character \( \phi \) into \( \psi_1 \oplus \psi_2 \), where \( \chi_i(H_j) = 0 \) for \( i \neq j \)). This induces decompositions \( \eta = \eta_1 \oplus \eta_2 \) and \( \chi = \chi_1 \oplus \chi_2 \), so that
\[ \chi(\sigma)(\eta(\tau)) = \chi_1(\sigma)(\eta_1(\tau)) \cdot \chi_2(\sigma)(\eta_2(\tau)). \]
Now under our identification \( H^1(K, E[n]) = (K^*/K^{*n})^2 \), \( \eta_1 \) corresponds to \( a \) (mod \( K^{*n} \)) and \( \eta_2 \) corresponds to \( b \) (mod \( K^{*n} \)), so \( \Delta_1 \) is just the sum of the cyclic algebras \((a, \chi_1)\) and \((b, \chi_2)\). Using Kummer theory to identify the characters with elements (say) \( C_2, C'_2 \) of \( K^*/K^{*n} \), we get
\[ \Delta_1(a, b) = \langle a, C_2 \rangle + \langle b, C'_2 \rangle = \langle a, C_2 \rangle + \langle C_1, b \rangle, \]
where \( C_1 = C'_2^{-1} \). Thus we have
\[ \Delta(a, b) = \langle a, b \rangle + \langle a, C_2 \rangle + \langle C_1, b \rangle = \langle C_1 a, C_2 b \rangle - \langle C_1, C_2 \rangle, \]
completing the proof of the theorem.
4. The Proof of Theorem 3

In this section the following hypotheses are in force: $n = p$ is prime, $K$ is a number field, and $E/K$ is an elliptic curve with $E[p](K) = E[p](K)$. We note that this implies, by the Galois-equivariance of the Weil pairing, that $K$ contains the $p$th roots of unity. Since for any class $L$ of unity, and $E/K$ is a cyclic extension of number fields, then $\eta$ is a simultaneous local norm except possibly at the unramified places $v$. Let $\pi_v$ be a generator of the corresponding prime ideal, and let $G_2$ be the (infinite) subgroup of $K^*$ generated by these elements $\pi_v$. Since $G_1$ has finite index, $G := G_1 \cap G_2$ remains infinite and visibly has infinite image in $K^*/K^{*n}$; by Hasse, every element of $G$ is a simultaneous norm.

Now we begin the proof of Proposition 7. Write out the elements of $H$ as follows:

$$H = \{(h_{1i}, h_{2i}) | 1 \leq i \leq k\}.$$ 

Moreover, let $B = B_H$ be the finite set of places of $K$ containing the Archimedean places, the places at which any $h_{1i}$ or $h_{2i}$ has nonzero valuation, and the places for which, for any $e (\mod p)$, any local symbol $e(C_1, C_2)_v - (h_{1i}, h_{2i})_v$ is nonzero in $Br(K_v)$.

Proposition 7. Let $K$ be a number field containing the $p$th roots of unity and $H \subseteq (K^*/K^{*p})^2$ a finite subgroup. Then there exists an infinite subgroup $G \subseteq (K^*/K^{*p})^2$ with the property that for every nonzero element $g$ of $G$ and every element $h \in H$, $\Delta(hg) \neq 0$.

By Theorem 6, $\Delta = \langle , \rangle_p$ up to a linear term, and essentially what must be shown is the same statement with $\langle , \rangle_p$ in place of $\Delta$; this says, morally, that Brauer groups of number fields are “large” in a certain sense. We prove this directly (if inelegantly) using exactly what the reader expects: local and global class field theory, especially the nondegeneracy of the local norm residue symbol.

Along these lines we will need the following routine result, whose proof we include for completeness.

Lemma 8. Let $n$ be a positive integer, $K$ be a number field containing the $n$th roots of unity, and $L_1, \ldots, L_k$ be $k$ cyclic degree $n$ extensions of $K$. Then the image in $K^*/K^{*n}$ of the subgroup of $K^*$ consisting of simultaneous norms from each $L_i$ is infinite.

Proof: By Hasse’s norm theorem, if $L/K$ is a cyclic extension of number fields, then $a \in L^*$ is a norm from $L$ if and only if it is everywhere a local norm. Let $S$ be the set of places of $K$ consisting of the real Archimedean places (if any) together with all finite places which ramify in any $L_i/K$ (if any). Let $G_1 \subseteq K^*$ be the subgroup of elements which are $n$th powers locally at every $v \in S$; notice that $G_1$ has finite index. Recalling that the norm map on an unramified local extension is surjective onto the unit group, we get that any $a \in G_1$ is a simultaneous local norm except possibly at the unramified places $v$ at which it has nontrivial valuation. Let $h$ be the class number of $K$. Then the set of primes which split completely in the Hilbert class field as well as in each $L_i$ has density at least $\frac{1}{h_{2i}}$. For such a $v$, let $\pi_v$ be a generator of the corresponding prime ideal, and let $G_2$ be the (infinite) subgroup of $K^*$ generated by these elements $\pi_v$. Since $G_1$ has finite index, $G := G_1 \cap G_2$ remains infinite and visibly has infinite image in $K^*/K^{*n}$; by Hasse, every element of $G$ is a simultaneous norm.

Now we begin the proof of Proposition 7. Write out the elements of $H$ as follows:

$$H = \{(h_{1i}, h_{2i}) | 1 \leq i \leq k\}.$$
Clearly it’s enough to construct arbitrarily large finite subgroups \( G \) such that every nontrivial element \((g_1, g_2)\) of \( G \) has the property that for all \( i, \)
\[
\Delta(h_i g) = \langle C_1 h_1 g_1, C_2 h_2 g_2 \rangle_p \neq \langle C_1, C_2 \rangle_p.
\]

We make two preliminary simplifying assumptions: first, let \( C \) be the cyclic subgroup generated by \( \langle C_1, C_2 \rangle_p \) in \( Br(K)[p] \). Rather than constructing elements \( g \) such that all modifications of \( g \) by elements of \( H \) have \( \Delta(h g) \neq \langle C_1, C_2 \rangle_p \), it is convenient for a later inductive argument to require the stronger property that for all \( h \in H, \Delta(h g) \) is not an element of \( C \). Second, by replacing \( H \) by \( H + C \), we reduce to the following problem: find arbitrarily large finite subgroups \( G \) all of whose nontrivial elements \((g_1, g_2)\) have the property that for all \( h = (h_{1i}, h_{2i}) \) in \( H, \)

\[
(2) \quad \langle h_{1i} g_1, h_{2i} g_2 \rangle_p \text{ is not in } C.
\]

In order to accomplish this, we first claim that we can choose \( g_2 \in K^*/K^{*p} \) such that:

- For \( 1 \leq i \leq k \), \( \langle h_{1i}, g_2 \rangle = 0 \); and
- For \( 1 \leq i \leq k \), \( g_2 h_{2i} \) is not in \( K^{*p} \).

Indeed, the elements \( g_2 \) satisfying the first condition are precisely the simultaneous norms from the \( k \) cyclic field extensions \( K(h_{1i}^{1/p})/K \), so in the notation of Lemma 6 there is a positive density set \( S_1 \) of principal prime ideals \( v = (\pi_v) \) such that \( \pi_v \in K^* \) is a simultaneous norm from these \( k \) extensions. The second condition is also satisfied as long as \( v \in S_1 \setminus B \), so choose any such \( v \) and take \( g_2 = \pi_v \).

If we now choose any \( g_1 \) with the property that for all \( i \) and any \( e \pmod{p} \)
\[
\langle g_1, g_2 h_{2i} \rangle \neq e \langle C_1, C_2 \rangle_p - \langle h_{1i}, h_{2i} \rangle,
\]
then the element \( g = (g_1, g_2) \) will have the desired property (2). For each \( i \), since \( g_2 h_{2i} \) is not a \( p \)th power, there exists an infinite set of places \( v = v(i) \) such that \( g_2 h_{2i} \) is not a \( p \)th power in \( K_v \). Hence we may choose places \( v_1, \ldots, v_k \), distinct and disjoint from \( B \), such that for all \( i \), \( g_2 h_{2i} \) is not a \( p \)th power in \( K_{v_i} \). By weak approximation, we can choose an element \( g_1 \) of \( K^*/K^{*p} \) such that for all \( i \), \( g \) completes to a class of \( K_v^*/K_{v_i}^{*p} \) making all the local norm residue symbols \( \langle g_1, g_2 h_{2i} \rangle_{v_i} \) nontrivial (this is possible because of the nondegeneracy of the local norm residue symbol). But by definition of \( B, e \langle C_1, C_2 \rangle_{v_i} - \langle h_{1i}, h_{2i} \rangle_{v_i} = 0 \) for all \( i \), so we have constructed an element \( g = (g_1, g_2) \) satisfying (2). Now observe that if \( 1 \leq j < p \), \( g_2^j \) satisfies the same two bulleted properties as \( g_2 \); moreover, since \( H \) is a subgroup, \( h_{2i} = h_{2i}' \) for some other index \( i' \), and the nontriviality of \( \langle g_1, g_2 h_{2i} \rangle_{v} \) implies the nontriviality of \( \langle g_1^j, g_2^j h_{2i}' \rangle_{v} \), so that indeed the entire cyclic subgroup \( A \) generated by \( (g_1, g_2) \) has the desired property (2).

We finish by iterating the construction: running through the above argument with \( H \) replaced by \( A \oplus C \) gives a two-dimensional \( \mathbb{F}_p \)-subspace of \( K^*/K^{*p} \times K^*/K^{*p} \), and so on.
5. Concluding remarks

I. In the derivation of Theorem 1 from Theorem 3, instead of appealing to Lichtenbaum’s theorem on the equality of the period and index for all classes in the Weil-Châtelet group of an elliptic curve over a local field, we could instead have used an earlier result of [Lang-Tate] giving the same equality for abelian varieties of arbitrary dimension over local fields in the case when $p$ is prime to the residue characteristic and $A$ has good reduction. Indeed the set of places of $K$ lying over $p$ together with those places of bad reduction for $E/K$ form a finite set, and as in the proof we need only restrict to the finite index subgroup of classes trivial at all these places.

II. The proof of the main theorem shows that each nonzero element $g$ of $G \subseteq H^1(K, E)[p]$ gives rise to at least one set of “local conditions” on a degree $p$ extension $L/K$ sufficient to ensure that $g$ restricts to a nonzero element of $\text{III}(E/L)$. On the other hand, the proof of Theorem 3 shows that $G$ is not only an infinite subgroup but has (in some sense) “positive measure,” bounded away from zero in terms of $\#E(K)/pE(K)$. Thus the argument should lead to an explicit lower bound on the function

$$f(N) = f(E/K, p, N) := \sum_{L/K, [L:K]=p, ||\Delta_{L/K}|| \leq N} \dim_{\mathbb{F}_p} \text{III}(E/L)[p],$$

where $\Delta_{L/K}$ is the discriminant of $L/K$. What is to be expected about the asymptotics of $f$?

III. The hypothesis that $E$ has full $p$-torsion defined over $K$ is used only in the appeal to the “explicit” period-index obstruction of Theorem 5 in the proof of Theorem 3. My hope is that Theorem 3 should be valid for every elliptic curve over a number field – namely, there should always exist an infinite subgroup of principal homogeneous spaces of order $p$ and index $p^2$. The challenge here is to make (sufficiently) explicit the period-index obstruction map $\Delta : H^1(K, E[p]) \to Br(K)$ in the case of an arbitrary Galois module structure on $E$. Notice that the setup of Theorem 4 can be generalized to the case of elliptic curves $E$ such that $E[n]$ has a Lagrangian decomposition: i.e., a decomposition into one-dimensional subspaces $H_1 \oplus H_2$ as Galois module. To be sure, this is still quite a stringent condition – satisfied, for any fixed non-CM elliptic curve over a number field, for at most finitely many primes $p$ – but at least this condition can be satisfied for elliptic curves over $\mathbb{Q}$ for the primes 2, 3 and 5: for such primes, elliptic curves $E/\mathbb{Q}$ with Galois module structure $E[n] \cong \mu_p \oplus \mathbb{Z}/p\mathbb{Z}$ are known to exist. In these cases, an analogue of Theorem 4 would show the existence of genus 1 curves $C/\mathbb{Q}$ of period $p$ and index $p^2$ for $p \leq 5$. While this may not sound very impressive, we must point out that heretofore the only examples in the literature of genus one curves over any number field with index exceeding their period are those of period 2 and index 4 (over $\mathbb{Q}$) constructed by [Cassels 1961] more than 40 years ago. Indeed, Cassels’ Jacobian elliptic curves have full 2-torsion over $\mathbb{Q}$, so his results are a special case of our Theorem 3.

IV. A question: for which fields $K$ is Proposition 7 valid? Two obvious necessary conditions are that $Br(K)$ be nontrivial and that the group of $p$th power
classes $K^*/K^{*p}$ be infinite. Surely the proposition will hold for all finitely generatedields of characteristic zero. (I was so repulsed by the ugliness of the proof
that I have not even attempted such a generalization. I hope that someone else
can find a cleaner way to proceed.) Since the weak Mordell-Weil theorem remains
valid in this context [Lang-Tate], we would get a generalization of Theorem 3. We
note that the problem of finding necessary and sufficient conditions on a field $K$
for there to exist genus one curves of period $p$ and index $p^2$ remains open.

IV'. Certainly there are some examples of period-index violations in Weil-Châtelet
groups of elliptic curves over function fields over number fields: let $K$ be a number
field, $E_0/K$ an elliptic curve and $\eta_0 \in H^1(K,E_0)[p]$ a class of index $p^2$. Let
$L := K(T_1,\ldots,T_n)$. Write $E := E_0 \times_K L$ for the corresponding (constant) elliptic
curve over $L$; similarly put $\eta := \eta_0|_{GL}$. Using the facts that $Br(K) \hookrightarrow Br(L)$
and $E(L) = Maps_K(P^n,E) = E(K)$, we see that since $\eta_0$ does not admit a lift
with non-vanishing obstruction, neither does $\eta$. But much more should be true: if
$L = K(V)$ is any function field of a variety over a number field and $E/L$ is any
elliptic curve, then for all primes $p$ we expect $H^1(L,E)[p]$ to contain infinitely many
classes of index $p$ and infinitely many of index $p^2$.

V. There are versions of Theorem 1 and Theorem 3 for principal homogeneous
spaces over abelian varieties of any dimension. The proofs require a higher dimen-
sional analogue of the period-index obstruction and are pursued in a forthcoming
paper [Clark].

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