Coulomb Effects in Low Energy Proton-Proton Scattering

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Abstract: Using a recently developed effective field theory for the interactions of nucleons at non-relativistic energies, we calculate non-perturbatively Coulomb corrections to proton-proton scattering. Including the dimension-eight derivative interaction in the PDS regularization scheme, we recover a modified form of the Blatt-Jackson relation between the scattering lengths. The effective range receives no corrections from the Coulomb interactions to this order. Also the case of scattering in channels where the Coulomb force is attractive, is considered. This is of importance for hadronic atoms.

1 Introduction

Effective field theories are constructed to give a complete description of interacting particles in terms of only the quantum fields which can be excited below a characteristic energy scale \(\Lambda\). Degrees of freedom of higher energies are represented by an expansion of the Lagrangian in terms of local operators of increasing dimensions. For the nucleon system at energies below the pion mass \(m_\pi\) the effective theory will thus involve only the nucleon field and derivatives thereof. It was first constructed by Weinberg\(^1\) and has been investigated by many others\(^2\).

In order for the effective theory to be useful it must come with well-defined counting rules so that one knows which operators to include and which to disregard in a calculation. Only then can one obtain a systematic and reliable expansion in \(p/\Lambda\) for any scattering amplitude characterized by the momentum \(p\). In the original formulation of the effective theory for nucleons one were faced with problems in this connection because of the unnaturally large S-wave scattering lengths. This brings a smaller energy scale into the system and the standard regularization schemes needed to handle the divergences from loop integrations had difficulties reproducing consistent counting rules\(^3\). A solution of these problems was subsequently given by Kaplan, Savage and Wise with the introduction of a new regularization scheme called Power Divergence Subtraction (PDS) and which is a generalization of the standard MS scheme based upon dimensional regularization\(^4\). Essentially the same method was proposed at the same time by Gegelia\(^5\). Since then

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Mehen and Stewart [6] have made this off-shell scheme (OS) more well-defined and shown that it is in fact equivalent to the PDS scheme.

As a first application the electromagnetic form factor of the deuteron was calculated [7]. Since then the electromagnetic polarizability and the Compton scattering cross section for the deuteron have been obtained within the same framework [8]. The inelastic process of radiative neutron capture on protons was first considered by Savage, Scaldeferri and Wise [9]. Based on the same effective field theory also the three-nucleon system and neutron-deuteron scattering are now under investigation [10]. To leading order all these results depend only on a few known parameters and are similar in structure to the effective range approximation in nuclear physics. Even better agreement with data can be obtained in higher orders where a priori unknown counterterms appear. When these are determined in one process, they can be used predictively in others.

In the above applications of the effective field theory to low-energy nucleon interactions there are no complications due to electromagnetic interactions between the nucleons. This is obviously not the case for proton-proton scattering which we have set out to reconsider within this new framework [11]. At sufficiently low energy the Coulomb repulsion between protons becomes strong and must be included by non-perturbative methods. It is in the same limit that the proton-proton scattering length is determined and a very careful analysis is needed to separate the different effects. We have already succeeded in calculating the rate for proton-proton fusion into deuterium which is dominated by the Coulomb repulsion, using this field-theoretic approach [12]. The same physics is also important in other hadronic scattering processes at low energies and in the nuclear bound states like $^3\text{He}$. Similarly, the Coulomb force will dominate in reactions between particles of opposite electrical charge at sufficiently low energies. Eventually it forces such systems into atomic bound states.

Experimentally, proton-proton scattering was the first hadronic process studied with the help of accelerators and one obtained early on very accurate data [13]. Landau and Smorodinski were the first to construct a formalism in which the Coulomb interaction could be separated from the strong interactions [14]. This was completed by Bethe in terms of a generalized effective range expansion of the low-energy phase shift for proton-proton scattering [15]. The more model-dependent connection between the measured scattering length and the purely hadronic interaction was obtained in the phenomenological analysis of Jackson and Blatt [16]. Even today this is a main reference to the electromagnetic effects involved, something which reflects the absence of a more modern and fundamental description of this process at low energies. The effective theory of Kaplan, Savage and Wise represents a new and important step in this direction.

In the next section we present the effective theory and give a short review of the applications to proton-neutron and neutron-neutron scattering where Coulomb effects are absent. For the proton-proton system these are included in Section 3. We show that a perturbative calculation of these effects breaks down at low energies. The scattering amplitude depends on the non-relativistic Coulomb propagator modified by the strong
interaction and can be calculated non-perturbatively. After PDS regularization we then obtain a field-theoretic derivation of the Jackson-Blatt result for the scattering length to leading order in the theory. A similar analytical result is also found using instead a momentum cutoff as a regulator. Hadronic scattering channels where the Coulomb force is attractive is also considered within the same theory. This result is of importance for the calculation of energy level shifts in pionic atoms and pionium\cite{17}. Our leading order results have been confirmed by Holstein using both standard quantum mechanics and effective field theory with an ordinary momentum cutoff as an ultraviolet regulator combined with numerical integration\cite{18}. He has also considered the more phenomenologically important case of several coupled channels.

In Section 4 we consider effective range corrections by including the dimension-eight derivative interaction to first order in perturbation theory. We are then faced with new and more divergent integrals involving the Coulomb propagator. By considering the Fourier transforms of the Coulomb wavefunctions, we manage to regularize these within the same PDS scheme as previously used. Details of this calculation are given in an appendix. The resulting hadronic scattering length becomes in better agreement with what one obtains in potential models. It also follows that the effective range itself is not affected by Coulomb interactions to this order in the effective field theory.

2 Effective theory for non-relativistic nucleons

For nucleon momenta smaller than the pion mass, we can integrate out all other fields including the pion field. The effective Lagrangian will only involve the nucleon field $N^T = (p, n)$ and derivatives thereof\cite{19}. It must obey the symmetries we see in strong interactions at low energies, i.e. parity, time-reversal and Galilean invariance. Photons are coupled to satisfy local gauge invariance and we will assume here that isospin symmetry is not broken. The Lagrangian can then be written as a series of local operators with increasing dimensions\cite{1,4,19}. In the limit where the energy goes to zero, the interactions of lowest dimension dominate. For this case the relevant Lagrangian is thus

\[
\mathcal{L}_0 = N^\dagger \left( \partial_t + \frac{\nabla^2}{2M} \right) N - C_0(N^T \Pi N) \cdot (N^T \Pi N)^\dagger
\]

where $M$ is the nucleon mass. The projection operators $\Pi_i$ enforces the correct spin and isospin quantum numbers in the channels under investigation. More specifically, for spin-singlet interactions $\Pi_i = \sigma_2 \tau_i / \sqrt{8}$ while for spin-triplet interactions $\Pi_i = \sigma_2 \sigma_i \tau_2 / \sqrt{8}$. This theory is now valid below an upper energy $\Lambda$ which is set by the pion mass. It is also the physical cutoff when the theory is regularized that way.

The above contact interaction has dimension $D = 6$ and is thus non-renormalizable. It corresponds to a singular delta-function potential and corresponds to the four-nucleon vertex in Fig.1. In order to estimate the importance of higher order diagrams like the
bubble correction to the scattering amplitude in Fig. 2, one needs counting rules. If a characteristic energy $Q$ flows through the diagram, the propagator $1/(\omega - Q^2/2M + i\epsilon)$ scales as $M/Q^2$ since the energy is typically $\omega \approx Q^2$. For the same reason the phase space factor in the loop integration $\int d\omega d^3q/(2\pi)^4$ picks up a characteristic factor $Q^5/4\pi M$. The estimated magnitude for the bubble diagram in Fig. 2 is thus $C_0^2 MQ/4\pi$. This will be a perturbative correction when $C_0 MQ/4\pi < 1$ and only a few diagrams will suffice. No real or virtual bound states will form in this case and the scattering length have the natural size $a \approx 1/\Lambda$.

More interesting is the situation when $C_0 MQ/4\pi > 1$ and the physics become non-perturbative. All diagrams will then contribute to the same order in $Q$ provided the coupling constant $C_0$ runs in such a way that $C_0 \propto 1/Q$. Kaplan, Savage and Wise showed that a renormalization parameter $1/a < \mu \leq \Lambda$ must then be introduced in their PDS regularization scheme. The scattering length $a$ can now be unnaturally large, i.e. $a \gg 1/\Lambda$ as it is in the nucleon-nucleon system. The full scattering amplitude $T$ due to the strong contact interaction is then the sum over all the chains of bubbles in Fig. 3. It forms a geometric series with the sum

$$T(p) = \frac{C_0}{1 - C_0 I_0(p)}$$

(2)
where

\[
I_0(p) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{E - q^2/M + i\epsilon}
\]  

(3)
is the bubble integral and \( E = p^2/M \) is the total center-of-mass energy. The integral is linearly divergent but is finite using dimensional regularization in a space of dimension \( d < 2 \). It has a simple pole when \( d = 2 \). In the PDS scheme one subtracts this pole which requires the introduction of a dimensionful parameter \( \mu \). Analytically continuing back to \( d = 3 \) one finds

\[
I_0(p, \mu) = \left( \frac{M}{4\pi} \right) (\mu + ip)
\]

(4)
The scattering amplitude \( \mathcal{I} \) has then the same structure has the S-wave partial wave amplitude

\[
T(p) = -\frac{4\pi}{M p} \frac{1}{p \cot \delta - ip}
\]

(5)
One then recovers the effective range expansion for the phase shift \( \delta \)

\[
p \cot \delta = -\frac{1}{a} + \frac{1}{2} r_0 p^2 + \ldots
\]

(6)
in the zero momentum limit when the coupling constant takes the renormalized value

\[
C_0(\mu) = \frac{4\pi}{M} \frac{1}{1/a - \mu}
\]

(7)
When only the lowest order coupling parametrized by \( C_0 \) is included in the effective Lagrangian \( \mathcal{L} \) we see that the effective range vanishes \( r_0 = 0 \). The small inverse scattering length has thus a fine-tuned value given by the difference between two large quantities. For example, in proton-neutron scattering we have \( a_{pn} = -23.7 \text{ fm} \) in the spin-singlet channel. Choosing the value \( \mu = m_\pi \) for the regularization parameter, we obtain \( C_0 = -3.54 \text{ fm}^2 \). Physical results should be independent of the exact value of the renormalization mass \( \mu \) as long as \( 1/a < \mu \leq m_\pi \) parameter, but it strongly affects the values of the coupling constants whose dependence on \( \mu \) is determined by the renormalization group.

For neutron-neutron collisions the scattering length is \( a_{nn} = -18.4 \text{ fm} \), i.e. in magnitude 25% smaller than for the neutron-proton system. A small part of this is the result of the proton-neutron mass difference which is a purely kinematic effect. The difference in the corresponding coupling constants is however much smaller. Taking again \( \mu = m_\pi \) we see from (7) that a small relative difference \( \Delta C_0/C_0 \) is magnified into a larger difference \( \Delta a/a \simeq a_\mu \Delta C_0/C_0 \) in the scattering lengths. This is the well-known ‘amplification effect’ in more standard nuclear physics where the delta-function potential represented by \( C_0 \) is replaced by a potential well of finite extension corresponding to \( 1/\mu \). The above large difference between \( a_{pn} \) and \( a_{pn} \) thus corresponds to a much smaller difference of less than
2% in the coupling constants. This is a dynamical effect due to quark mass differences and breaking of isospin invariance by electromagnetic interactions at shorter scales\cite{20}. Recently, Epelbaum and Meissner has shown that these symmetry breaking effects in the nucleon scattering lengths are mostly due to the pion mass difference occurring in loops in higher orders of chiral perturbation theory\cite{22}.

### 3 Coulomb corrections to low-energy elastic scattering

At very low energies it is the repulsive Coulomb force which dominates in proton-proton scattering. Increasing the energy, it will still dominate in the forward and backward directions while for other scattering angles it is overcome by the strong interaction of very short range\cite{13}. In our case it is described by the singular potential $C_0 \delta(r)$. The effects of transverse photons are negligible since they couple proportionally to the proton velocity.

![Figure 4: The lowest order Coulomb correction on external legs.](image)

The lowest order Coulomb correction to proton-proton scattering amplitude in Fig. 1 is given by the Feynman diagram in Fig. 4. Using the counting rules in the previous section, it is expected to give a correction $\delta T$ to the result $C_0$ from Fig.1 of the order

$$\delta T \simeq C_0 \left( \frac{M}{Q^2} \right)^2 \frac{e^2}{Q^2} \frac{Q^5}{4\pi M} = C_0 \frac{\alpha M}{Q}$$

where the characteristic momentum $Q$ is here the proton momentum $p$ in the CM frame. This should be compared with a direct calculation of the Feynman diagram which gives

$$\delta T = \int \frac{d^3k}{(2\pi)^3} \frac{e^2}{\kappa^2 + \lambda^2} \frac{M}{p^2 - (p - k)^2 + i\epsilon}$$

where the Coulomb photon has the mass $\lambda \rightarrow 0$ which acts as an infrared regulator. The integral is then finite. Combining the two denominators with the Feynman trick and using

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + \Delta)^p} = \frac{\Delta^{\frac{d}{2} - p}}{(4\pi)^{d/2}} \frac{\Gamma(p - d/2)}{\Gamma(p)}$$

we find the result

$$\delta T = -C_0 \frac{\alpha M}{2p} \left( \frac{\pi}{2} + i \ln \frac{2p}{\lambda} \right) + O(\lambda)$$
It is in agreement with the estimate in (8). Since the term which depends on the photon mass is imaginary, it will not contribute to the scattering cross section since $|C_0 + \delta T|^2 = C_0(C_0 + 2\text{Re} \delta T)$ to this order. The cross section is thus infrared finite and proportional to $1 - \pi \eta$ where $\eta \equiv \alpha M/2p$. Including one more Coulomb photon exchange in the diagram Fig. 4, we find from the same power counting rules that it will contribute a term of the order $C_0(\alpha M/Q)^2 \simeq C_0 \eta^2$. For momenta $p < \alpha M/2$ we will have $\eta > 1$ and perturbation theory is seen to break down. The Coulomb repulsion is then strong and must be included in a non-perturbative way.

![Figure 5: The first Coulomb correction on the internal bubble.](image)

In the same way as the external particles are strongly influenced by the repulsive Coulomb potential at very low energies, also the interaction with the strong potential $C_0 \delta(r)$ is much modified. This can already be seen in the first Coulomb correction to the bubble diagram in Fig. 2 given by the two-loop diagram in Fig. 5. It gives a correction $\delta I_0$ to the integral (8) whose size again can be estimated from the counting rules. Since it contains two loops and a Coulomb propagator it should be

$$\delta I_0 \simeq \left(\frac{MQ}{4\pi}\right)^2 \frac{e^2}{Q^2} = \frac{\alpha M^2}{4\pi}$$

More accurately, it is given by the double integral

$$\delta I_0 = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{M}{p^2 - q^2 + i\epsilon} \frac{e^2}{k^2 + \lambda^2 p^2 - (k + q)^2 + i\epsilon}$$

It is seen to be infrared finite so that we can take the photon mass $\lambda = 0$. Since it is logarithmic divergent in the ultraviolet, it will have a pole in $\epsilon \equiv d - 3$ when it is evaluated using dimensional regularization. This is done in Appendix A where we find

$$\delta I_0 = -\frac{\alpha M^2}{8\pi} \left(\frac{1}{\epsilon} + 2\ln \frac{\mu \sqrt{\pi}}{2p} + 1 - C_E + i\pi\right)$$

Here $\mu$ is the renormalization mass and $C_E = 0.5772\ldots$ is Euler’s constant. The divergent term will be removed by counterterms which modify the coupling constant $C_0$. The prefactor which sets the magnitude of this correction, is seen to be in agreement with the above counting argument. In the low-energy limit where the proton momentum $p \to 0$ we see again that this two-loop correction becomes large. In order to have a finite scattering length, the logarithmic divergent term must be cancelled by a corresponding logarithmic term coming from the exchange of two or more photons in the bubble. These higher order
Feynman diagrams are both infrared and ultraviolet finite. Each additional photon exchange is seen from the counting rules to bring in a factor $\alpha M/Q$ to the bubble correction. When the external momentum goes to zero, perturbation theory breaks down again and the Coulomb-corrected bubble must be calculated in a non-perturbative way.

### 3.1 The Coulomb propagator and wavefunctions

In relative coordinates the two-particle nucleon system is described by the free Hamiltonian $\hat{H}_0 = \hat{\mathbf{p}}^2/M$ where the reduced mass is $M/2$. The propagator or retarded Green’s function for this system is

$$\hat{G}_0^+(E) = \frac{1}{E - \hat{H}_0 + i\epsilon}$$

where $E = \mathbf{p}^2/M$ is the total energy. When a complete set of plane wave eigenstates $|\mathbf{q}\rangle$ is inserted in the numerator, it becomes

$$\hat{G}_0^+(E) = M \int \frac{d^3q}{(2\pi)^3} \frac{|\mathbf{q}\rangle\langle\mathbf{q}|}{\mathbf{p}^2 - \mathbf{q}^2 + i\epsilon}$$

The propagator from an position $\mathbf{r}$ to a final position $\mathbf{r}'$ in coordinate space is therefore $\langle \mathbf{r}' | \hat{G}_0 | \mathbf{r} \rangle \equiv G_0(E; \mathbf{r}', \mathbf{r})$. It follows that the bubble diagram in Fig. 2 which is numerically given by (4), is just the propagator from zero separation to zero separation, $I_0(p) = \hat{G}_0(E; 0, 0)$.

Including the repulsive Coulomb potential $V_C = e^2/4\pi r$ which acts between the protons, the retarded and advanced Green’s functions are now

$$\hat{G}_C^{(\pm)}(E) = \frac{1}{E - \hat{H}_0 \pm \hat{V}_C + i\epsilon}$$

depending on the boundary conditions which are imposed at infinity. This is equivalent to the integral equation $\hat{G}_C^{(\pm)} = \hat{G}_0^{(\pm)} + \hat{G}_0^{(\mp)} \hat{V}_C \hat{G}_C^{(\pm)}$. After iteration, is gives the Coulomb propagator as an infinite sum of Feynman diagrams with zero, one, two and so on exchanged photons as shown in Fig.6.

The Schrödinger equation $(\hat{H} - E) |\psi\rangle = 0$ where $\hat{H} = \hat{H}_0 + \hat{V}_C$ is the full Hamiltonian, has the corresponding incoming and outgoing solutions $|\psi^{(\pm)}\rangle$. They can formally be expressed in terms of the free solutions as

$$|\psi^{(\pm)}\rangle = [1 + \hat{G}_C^{(\pm)} \hat{V}_C] |\mathbf{p}\rangle$$

This is most easily seen when one uses the equivalent expression $|\psi^{(\pm)}\rangle = \hat{G}_C^{(\pm)} \hat{G}_0^{-1} |\mathbf{p}\rangle$. These solutions now have the same normalization as the above plane waves so that $\langle \mathbf{q} | \psi^{(\pm)}\rangle = (2\pi)^3 \delta(\mathbf{q} - \mathbf{p})$. Explicit solutions in coordinate space is found from solving the Schrödinger equation and can be expressed in terms of confluent hypergeometric
or Kummer function $M(a, b; x)$. For the repulsive Coulomb potential $V_C = \alpha/r$ the in-state solution with outgoing spherical waves in the future is

$$\psi^{(+)}_p(r) = e^{-\frac{1}{2}\pi \eta \Gamma(1 + i\eta)M(-i\eta, 1; ip \cdot r) e^{ip \cdot r}} \tag{20}$$

The corresponding out-state has incoming spherical waves in the distant past and is given by the wavefunction

$$\psi^{(-)}_p(r) = e^{-\frac{1}{2}\pi \eta \Gamma(1 - i\eta)M(i\eta, 1; -ip \cdot r) e^{ip \cdot r}} \tag{21}$$

where $\eta = \alpha M/2p$ is the parameter which also appeared in the earlier perturbative calculation. The probability to find the two protons at zero separation is thus

$$C^2_\eta \equiv |\psi^{(\pm)}_p(0)|^2 = e^{-\pi \eta \Gamma(1 + i\eta) \Gamma(1 - i\eta)} = \frac{2\pi \eta}{e^{2\pi \eta} - 1} \tag{22}$$

which is the well-known Sommerfeld factor. When the relative velocity between the particles goes to zero, it becomes exponentially small. At higher velocities $\eta < 1$ and the Coulomb repulsion is perturbative. We then recover to lowest order the result $1 - \pi \eta$ obtained from the Feynman diagram Fig. 4 in the previous section.

With these Coulomb eigenstates we can now find a more useful expression for the Green’s functions. Since the scattering states form a complete set in the repulsive case we consider here, we can instead write it as in (17). Taking the matrix element in coordinate space, we then have for the retarded function

$$\langle r'| \tilde{G}^{(+)}_C | r \rangle = M \int \frac{d^3q}{(2\pi)^3} \frac{\psi^{(+)}_q(r')\psi^{(+)*}_q(r)}{p^2 - q^2 + i\epsilon} \tag{23}$$

In the next section we will see that this propagator gives the main part of the non-perturbative Coulomb corrections of the strong scattering amplitude.
3.2 Scattering amplitudes and the modified effective range expansion

Including the strong interaction via the local potential operator $\hat{V}_S$, the complete Hamiltonian becomes $\hat{H} = \hat{H}_0 + \hat{V}_C + \hat{V}_S$. From the full Green’s function $\hat{G}_{SC}^{(\pm)} = 1/(E - \hat{H} \pm i\epsilon)$ in the presence of both the potentials one can then formally construct incoming and outgoing solutions

$$|\Psi_p^{(\pm)}\rangle = [1 + \hat{G}_{SC}^{(\pm)}(\hat{V}_S + \hat{V}_C)]|p\rangle$$

as in (19). Using now the operator identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$ which implies the relation

$$\hat{G}_{SC}^{(\pm)} - \hat{G}_C^{(\pm)} = \hat{G}_C^{(\pm)}\hat{V}_S\hat{G}_{SC}^{(\pm)}$$

between the Green’s functions, we can write the above formal solutions of the coupled problem in terms of Coulomb states alone,

$$|\Psi_p^{(\pm)}\rangle = [1 + \sum_{n=1}^{\infty}(\hat{G}_C^{(\pm)}\hat{V}_S)^n]|p\rangle$$

The scattering amplitude is given by the $S$-matrix element which is the overlap between an incoming state with momentum $p$ and an outgoing state $p'$. It takes the standard form

$$S(p', p) = \langle \Psi_{p'} | \Psi_p^{(+)} \rangle = (2\pi)^2\delta(p' - p) - 2\pi i\delta(E' - E)T(p', p)$$

With the scattering states (26) the $T$-matrix element can be written as the sum of two parts, $T(p', p) = T_C(p', p) + T_{SC}(p', p)$ where

$$T_C(p', p) = \langle \Psi_{p'} | \hat{V}_C | \psi_p^{(+)} \rangle$$

is the pure Coulomb scattering amplitude and

$$T_{SC}(p', p) = \langle \psi_p^{(-)} | \hat{V}_S | \Psi_p^{(+)} \rangle$$

is the strong scattering amplitude modified by Coulomb corrections.

Since the full Coulomb wavefunction $\psi_p^{(+)}(r)$ is known, one can calculate exactly the scattering amplitude (28). The result has the partial wave expansion

$$T_C(p', p) = -\frac{4\pi}{M} \sum_{\ell=0}^{\infty} (2\ell + 1) \left[ \frac{e^{2i\sigma_\ell} - 1}{2ip} \right] P_\ell(\cos \theta)$$

where $\theta$ is the CM scattering angle and $\sigma_\ell = \text{arg}\Gamma(1 + \ell + i\eta)$ is the Coulomb phaseshift. If the full scattering amplitude $T(p', p)$ is defined to have the phaseshift $\sigma_\ell + \delta_\ell$, the modified strong amplitude (29) will then have the partial wave expansion

$$T_{SC}(p', p) = -\frac{4\pi}{M} \sum_{\ell=0}^{\infty} (2\ell + 1) e^{2i\sigma_\ell} \left[ \frac{e^{2i\delta_\ell} - 1}{2ip} \right] P_\ell(\cos \theta)$$
It should be stressed that the phaseshift $\delta_\ell$ is not the same as one would have in the absence of the Coulomb interaction. But it can be directly measured from the experimental differential cross sections.

In our case the strong interaction potential $V_S = C_0 \delta(r)$ and will only affect $S$-wave amplitude which we denote by $T_{SC}(p)$. If the corresponding phaseshift is called $\delta$, we see from (31) that they are related by

$$p (\cot \delta - i) = -\frac{4\pi}{M T_{SC}(p)} e^{2i\sigma_0}$$

(32)

Because of the strong effects of the Coulomb interaction in the low-energy limit $p \to 0$, the effective range expansion (6) does not apply to this phaseshift. This was analyzed in detail by Bethe[15] and the result can be written as the generalized expansion[28]

$$C_\eta^2 p (\cot \delta - i) + \alpha M H(\eta) = -\frac{1}{a_C} + \frac{1}{2} r_0 p^2 + \ldots$$

(33)

Here $C_\eta^2$ is the Sommerfeld repulsion factor (22) while $a_C$ and $r_0$ is respectively the $S$-wave Coulomb-modified scattering length and the effective range for the elastic scattering process under consideration. In the case of proton-proton scattering where the Coulomb potential is repulsive, the function $H(\eta)$ is

$$H(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta)$$

(34)

where the $\psi$-function is the logarithmic derivative of the $\Gamma$-function. It represents the effects of the Coulomb force on the strong interactions at short distances. Using the relation

$$\text{Im}\psi(i\eta) = \frac{1}{2\eta} + \frac{\pi}{2} \coth \pi\eta$$

(35)

we see that $\text{Im}H(\eta) = C_\eta^2/2\eta$ and the imaginary parts cancel out in (33). The left-hand side will then be real and instead involve the function $h(\eta) = \text{Re}\psi(i\eta) - \ln \eta$ which is more suitable for phenomenological analysis[16].

With this formalism one can extract the physical scattering length $a_C$ from the experimentally measured cross-sections for proton-proton scattering at low energies[16]. One then finds a value $a_C = -7.82$ fm and $r_0 = 2.83$ fm for the effective range[21]. While the effective range is essentially the same as measured in $pn$ and $nn$ scattering, the scattering length is in magnitude less than one half the values found in these processes. This is due to the Coulomb effects contained in $a_C$. They can only be removed in a model for both the strong and electromagnetic interactions at short distances. Describing these forces in a potential model and solving the corresponding Schrödinger equation, Jackson and Blatt could isolate a strong scattering length $a_S$ which is determined by the measured one through their relation[16]

$$\frac{1}{a_S} = \frac{1}{a_C} + \alpha M \left[ \ln \frac{1}{\alpha M r_0} - 0.33 \right]$$

(36)
With the above value for the effective range \( r_0 \) one then finds \( a_S = -17.0 \text{ fm} \) which is very close to the \( nn \) scattering length. The term \(-0.33\) in the above formula is found to be only weakly dependent on the exact form of the strong potential\[27\].

This modified effective range expansion obviously applies also to other processes like \( \pi^+ p \) or \( \pi^+ \pi^+ \) elastic scattering at low energies where repulsive Coulomb interactions are important. The only modification needed is to replace the mass \( M \) in \((33)\) with twice the reduced mass \( m = m_1 m_2/(m_1 + m_2) \) where the scattered particles have different masses. Similarly, for elastic scattering in channels like \( \pi^- p \) or \( \pi^- \pi^+ \) where the Coulomb force is attractive, it also takes the same form. But it then involves a slightly different function

\[
\bar{H}(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(-i\eta) \tag{37}
\]

where now \( \eta = -\alpha M/2p \) is negative\[28\]. We will see in the following that both of these functions arise naturally in the present theoretical analysis.

### 3.3 Coulomb corrections in the repulsive channel

With the formalism established we can now calculated the Coulomb-modified amplitude \((29)\) where in our case the strong interaction is represented by the contact potential \( \hat{V}_0 \) with \( \langle p' | \hat{V}_0 | p \rangle = C_0 \). The outgoing scattering state \( |\Psi_p^{(+)}\rangle \) can from \([26]\) be represented in terms of Coulomb states as

\[
|\Psi_p^{(+)}\rangle = \sum_{n=0}^{\infty} (\hat{G}^{(+)}_C \hat{V}_0)^n |\psi_p^{(+)}\rangle \tag{38}
\]

As a result, we then have for the scattering amplitude

\[
T_{SC}(p) = \sum_{n=0}^{\infty} \langle \psi_p^{(-)} | \hat{V}_0 (\hat{G}^{(+)}_C \hat{V}_0)^n | \psi_p^{(+)}\rangle \tag{39}
\]

To first order in the strong coupling this is just

\[
T_{SC}^{(1)}(p) = \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 q'}{(2\pi)^3} \langle \psi_p^{(-)} | q' \rangle \langle q' | \hat{V}_0 | q \rangle \langle q | \psi_p^{(+)}\rangle \tag{40}
\]

\[
= C_0 \psi_p^{(-)}(0) \psi_p^{(+)}(0) = C_0 C^2 \eta e^{2i\sigma_0} \tag{41}
\]

after insertions of two complete set of free states. The \( S \)-wave phase shift \( 2\sigma_0 \) comes from the relative phase between the two Coulomb wavefunctions \([20] \) and \([21] \). In the next order of \( C_0 \) we similarly find the contribution

\[
T_{SC}^{(2)}(p) = C_0 C^2 \eta e^{2i\sigma_0} J_0(p) \tag{42}
\]

where

\[
J_0(p) = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} \langle k' | \hat{G}^{(+)}_C(E) | k \rangle = M \int \frac{d^3 q}{(2\pi)^3} \frac{\psi_q^{(+)}(0)\psi_q^{(+)*}(0)}{p^2 - q^2 + i\epsilon} \tag{43}
\]
is the amplitude $G_C^{(+)}(E;0,0)$ for the protons to propagate from initially zero separation and back to zero separation. It can be represented by the one bubble diagram in Fig.7 containing the sum of all possible exchanges of static Coulomb photons.

![One bubble diagram with Coulomb corrections.](image)

Including higher order terms in the expansion (39) we see that they form a geometric series with the sum

$$T_{SC}(p) = C^2_{\eta} \frac{C_0 e^{2i\sigma_0}}{1 - C_0 J_0(p)}$$

(44)

This result for the scattering amplitude is now to be used in (32) which will give the corresponding phaseshift due to the strong interaction. The Coulomb phaseshift $\sigma_0(p)$ and the Sommerfeld factor $C^2_{\eta(p)}$ are seen to cancel out. We are thus only left with the evaluation of (43). The two wavefunctions gives the Sommerfeld factor $C^2_{\eta(q)}$ and thus we have

$$J_0(p) = M \int \frac{d^3q}{(2\pi)^3} \frac{2\pi\eta(q)}{e^{2\pi\eta(q)} - 1} \frac{1}{p^2 - q^2 + i\epsilon}$$

(45)

The integral is ultraviolet divergent and must be regularized. Writing $J_0 = J^\text{div}_0 + J^\text{fin}_0$ we can isolate the divergent part in

$$J^\text{div}_0 = -M \int \frac{d^3q}{(2\pi)^3} \frac{2\pi\eta(q)}{e^{2\pi\eta(q)} - 1} \frac{1}{q^2}$$

(46)

which is independent of the proton momentum $p$. The remaining integral

$$J^\text{fin}_0 = M \int \frac{d^3q}{(2\pi)^3} \frac{2\pi\eta(q)}{e^{2\pi\eta(q)} - 1} \frac{1}{q^2} \frac{p^2}{p^2 - q^2 + i\epsilon}$$

(47)

is finite. It can be done by introducing $x = 2\pi\eta(q)$ as a new integration variable and using

$$\int_0^\infty dx \frac{x}{(e^x - 1)(x^2 + a^2)} = \frac{1}{2} \ln \left( \frac{a}{2\pi} \right) - \frac{\pi}{a} - \psi \left( \frac{a}{2\pi} \right)$$

(48)

In our case we will have $a = 2\pi\eta(p)$ and the finite part of the Coulomb bubble is just the $H$-function (34) already introduced in the problem,

$$J^\text{fin}_0 = -\frac{\alpha M^2}{4\pi} H(\eta)$$

(49)
Using the full scattering amplitude (44) in the Coulomb-modified effective range expansion (33), we see that \( H(\eta) \) cancels out. The scattering length is thus contained in the ultraviolet divergent part of the Coulomb bubble,

\[
\frac{1}{a_C} = \frac{4\pi}{M} \left( \frac{1}{C_0} - J_0^{\text{div}} \right)
\] (50)

There is no contribution to the effective range \( r_0 \) from the Coulomb potential when we include only the \( C_0 \) interaction in the effective theory.

We evaluate the divergent integral (46) using dimensional regularization. With \( \epsilon = 3 - d \) it then becomes

\[
J_0^{\text{div}} = -M \left( \frac{\mu}{2} \right)^\epsilon \frac{\Omega_d}{(2\pi)^d} \int_0^\infty dq q^{d-3-\epsilon} \frac{2\pi \eta(q)}{e^{2\pi \eta(q)} - 1}
\] (51)

where \( \Omega_d = 2\pi^{d/2}/\Gamma(d/2) \) is the surface area of the \( d \)-dimensional unit sphere. Again introducing \( x = 2\pi \eta(k) \) as a new integration variable, we find

\[
J_0^{\text{div}} = -M \left( \frac{\mu}{2} \right)^\epsilon \frac{\Gamma(d/2)}{(2\pi)^d} \frac{2\pi \eta(q)}{\Gamma(\frac{d}{2})^2} \int_0^\infty dx x^{-\epsilon-1} \frac{\alpha M^2}{\sqrt{\pi}} \left( \frac{\mu}{\alpha M} \right) \Gamma(\epsilon) \zeta(\epsilon) \Gamma(\frac{d}{2})^{-2}
\] (52)

Here we have introduced Riemann’s zeta-function with

\[
\zeta(\epsilon) = -\frac{1}{2} \left[ 1 + \epsilon \ln 2\pi \right] + O(\epsilon^2)
\] (53)

From the factor \( \Gamma(\epsilon) \) we get a pole \( 1/\epsilon \) when \( \epsilon \to 0 \). In addition there is a PDS pole when \( d \to 2 \) from the zeta-function. Since \( \zeta(\epsilon) = \zeta(1 + 2 - d) = 1/(2 - d) + C_E \) in this limit, it takes the form

\[
J_0^{\text{div}}(d \to 2) = \frac{\mu M}{4\pi} \frac{1}{d - 2}
\] (54)

According to the PDS regularization scheme[4] this contribution should be subtracted from the result (52) where then the limit \( d \to 3 \) is taken. In this way we are left with

\[
J_0^{\text{div}} = \frac{\alpha M^2}{4\pi} \left[ \frac{1}{\epsilon} + \ln \frac{\mu \sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2} C_E \right] - \frac{\mu M}{4\pi}
\] (55)

for the divergent part.

We can now use this result in the expression (50) for the measured proton-proton scattering length and obtain

\[
\frac{1}{a_C} = \frac{4\pi}{MC_0} + \mu - \alpha M \left[ \frac{1}{\epsilon} + \ln \frac{\mu \sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2} C_E \right]
\] (56)
The ultraviolet pole $1/\epsilon$ must cancel against counterterms which describe short-distance electromagnetic and other isospin-breaking interactions due to quark mass differences[22]. They will modify the coupling constant $C_0$ which then takes the renormalized value $C_0(\mu)$. It can be used to define a new scattering length

$$\frac{1}{a(\mu)} = \frac{4\pi}{MC_0(\mu)} + \mu$$

It is not physical in the sense that it can be measured directly and will thus in general depend on the renormalization point $\mu$. Coulomb effects on length scales $> 1/\mu$ have been removed from it. The coupling constant $C_0(\mu)$ should be within a few percent of the corresponding coupling constants for $pn$ and $nn$ scattering. From (56) we see that $a(\mu)$ is related to the physical scattering length $a_C$ by

$$\frac{1}{a(\mu)} = \frac{1}{a_C} + \alpha M \left[ \ln \frac{\mu\sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2} C_E \right]$$

The result is non-perturbative both in the strong coupling and in the fine structure constant $\alpha$ which is seen to enter in the combination $\alpha \ln \alpha$. This is a consequence of the Coulomb force becoming strong at very low energies. Depending on the value of the renormalization point $\mu$ we see that the Coulomb correction can actually become of the same magnitude as the strong interaction. Since the scattering length $a_C$ also is negative, it can have a very big effect on the size of the hadronic scattering length $a(\mu)$.

Our result for the scattering length is independent of the PDS regularization scheme we have used in the above derivation. Instead we can use a simple momentum cutoff $\Lambda$ to make the divergent integral (46) finite. It then becomes

$$J_{0}^{\text{div}} = -\frac{M}{\pi} \int_{0}^{\Lambda} dq \frac{\eta(q)}{e^{2\pi \eta(q)} - 1}$$

Changing integration variable to $x = 2\pi \eta$, it simplifies to

$$J_{0}^{\text{div}} = -\frac{\alpha M^2}{2\pi} \int_{\pi \alpha M/\Lambda}^{\infty} \frac{dx}{x(e^x - 1)}$$

$$= -\frac{M\Lambda}{2\pi^2} + \frac{\alpha M^2}{4\pi} \left[ \ln \frac{2\Lambda}{\alpha M} - C_E \right] + O \left( \frac{\alpha M}{\Lambda} \right)$$

Again it is natural to define a strong and cutoff-dependent scattering length $a(\Lambda)$ in terms of a coupling constant $C_0(\Lambda)$ which absorbs the term linear in the cutoff,

$$\frac{1}{a(\Lambda)} = \frac{4\pi}{MC_0(\Lambda)} + \frac{2\Lambda}{\pi}$$

In this regularization scheme it is now related to the physical scattering length by

$$\frac{1}{a(\Lambda)} = \frac{1}{a_C} + \alpha M \left[ \ln \frac{2\Lambda}{\alpha M} - C_E \right]$$

which should be compared with (58). This analytical result is in agreement with what Holstein obtained by a numerical integration[18]. Since in this effective theory the pions are integrated out, the magnitude of the cutoff $\Lambda$ is set by the pion mass $m_\pi$. 15
3.4 Elastic scattering in the attractive channel

We will now consider elastic scattering of two non-relativistic particles with opposite electric charge. The full scattering amplitude will again be given by the infinite sum (39) where now \( \hat{G}_C^{(+)} \) is the Coulomb propagator in the attractive channel. It will involve bound states in addition to the scattering states considered previously in the repulsive case. Summing the infinite set of bubble diagrams we find as in (44) for the corresponding scattering amplitude

\[
\bar{T}_{SC}(p) = C_0^2 \eta \frac{C_0 e^{2i\sigma_0}}{1 - C_0 J_0(p)}
\]

where now the Coulomb parameter \( \eta = -\alpha M/2p \) is negative and \( \bar{C}_0 \) is the strong coupling constant in this channel. When there are open, strong annihilation channels as in \( pp \bar{p} \), it will in general be complex. The full, Coulomb-dressed bubble can be written as the sum

\[
\bar{J}_0(p) = \sum_{n\ell} \frac{|\psi_{n\ell}(0)|^2}{E - E_{n\ell}}
\]

comes from the bound states. Introducing here the scattering energy \( E = p^2/M = \alpha^2 M/4n^2 \) and the bound state energies \( E_{n\ell} = -\alpha^2 M/4n^2 \) together with the probability \( |\psi_{n\ell}(0)|^2 = (\alpha M)^3/(8\pi n^3)\delta_{0\ell} \) to find the particles at the origin of the bound state, we obtain

\[
\bar{J}_0^b(p) = \frac{\alpha M^2}{4\pi} \sum_{n=1}^{\infty} \frac{2n^2}{n(n^2 + \eta^2)} = \frac{\alpha M^2}{4\pi} \left[ \psi(i\eta) + \psi(-i\eta) + 2C_E \right]
\]

where again \( C_E \) is Euler’s constant.

The second term \( \bar{J}_0^s \) in the attractive Coulomb bubble is due to the scattering states and has exactly the same form as (45) in the repulsive case. Since \( \eta \) is now negative, we rewrite the denominator in the Sommerfeld factor as

\[
\frac{1}{e^{2\pi\eta(q)} - 1} = \frac{1}{1 - e^{-2\pi\eta(q)}} - 1
\]

This factor can then be split into three parts, \( \bar{J}_0^s = \bar{J}_0^{div} + \bar{J}_0^{fin} + \bar{J}_0^{new} \), where the two first have the same form as in the repulsive case. In particular, we find

\[
\bar{J}_0^{div} = -M \int \frac{d^3q}{(2\pi)^3} \frac{-2\pi\eta(q)}{e^{-2\pi\eta(q)} - 1} \frac{1}{q^2}
\]

which is therefore given by the result (55). The \( 1/\epsilon \) divergence we again absorb into an unknown counterterm. Similarly,

\[
\bar{J}_0^{fin} = M \int \frac{d^3q}{(2\pi)^3} \frac{-2\pi\eta(q)}{e^{-2\pi\eta(q)} - 1} \frac{1}{q^2} \frac{p^2}{p^2 - q^2 + i\epsilon}
\]
is given by the finite result (49). The only new integral appearing here in the attractive channel is

$$\bar{J}_0^{\text{new}} = M \int \frac{d^3 q}{(2\pi)^3} \frac{-2\pi \eta(q)}{p^2 - q^2 + i\epsilon}$$

(69)

It is most easily evaluated by dimensional regularization which gives

$$\bar{J}_0^{\text{new}} = -\frac{\alpha M^2}{2\pi} \left[ \ln \frac{\mu \sqrt{\pi}}{\alpha M} + \ln(-i\eta) + 1 \right]$$

(70)

It does not have any PDS poles so this is the full result. Adding together these three scattering state contributions, we obtain

$$J_0^s = -\frac{\mu M}{4\pi} - \frac{\alpha M^2}{4\pi} \left[ \ln \frac{\mu \sqrt{\pi}}{\alpha M} + \ln(-i\eta) - \frac{1}{2i\eta} + \psi(-i\eta) + 1 + \frac{1}{2}C_E \right]$$

(71)

Combining this with the contribution (65) from the bound states, we finally have for the full, attractive Coulomb bubble

$$\bar{J}_0(p) = -\frac{\mu M}{4\pi} - \frac{\alpha M^2}{4\pi} \left[ \ln \frac{\mu \sqrt{\pi}}{\alpha M} + \frac{3}{2}C_E - H(\eta) \right]$$

(72)

where $H(\eta)$ is the previously defined function (37).

From the scattering amplitude (63) we can now obtain the physical scattering length $\bar{a}_C$ when combined with the corresponding effective range expansion (33) in the attractive channel. In analogy with (57) we can define a strong scattering length $\bar{a}(\mu)$ in this channel directly given by the strong coupling constant,

$$\frac{1}{\bar{a}(\mu)} = \frac{4\pi}{MC_0(\mu)} + \mu$$

(73)

It can be obtained from the measured scattering length through the relation

$$\frac{1}{\bar{a}(\mu)} = \frac{1}{\bar{a}_C} - \alpha M \left[ \ln \frac{\mu \sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2}C_E \right]$$

(74)

Since it is not a physical quantity, it depend in general on the renormalization point $\mu$. This result can be obtained directly from the corresponding expression (58) in the repulsive channel by letting $\alpha \to -\alpha$ in front of the parenthesis. It can also be obtained using a momentum cutoff instead of PDS. The integral (70) will then be slightly different and depend logarithmically on this cutoff. The net result will be as in (62), again with the opposite sign in front of the parenthesis.

The PDS and the cutoff regularization schemes give slightly different results for the the scattering lengths, but it will have no consequence when they are used in other physical contexts. For instance, in the attractive channel like in $\pi^+\pi^-$ or $\pi^-p$ one has hadronic
atoms bound by the Coulomb potential. The hydrogen-like energy level spectrum will then be perturbed by the strong interaction. With the hadronic contact interactions we consider here, we will obtain a shift of the $S$-states which is easily calculated and proportional to the scattering length in lowest order\cite{29}. Since it is only $\bar{a}_C$ and not $\bar{a}(\mu)$ or $\bar{a}(\Lambda)$ which have a physical content and can be measured in scattering experiments, it must be the one which determines the level shift. This was first shown in a potential model calculation by Trueman\cite{30} and follows also directly using the present effective field theory. This has now also been shown by Holstein who has extended the calculation to the more realistic situation of several coupled channels\cite{18}.

4 Effective-range corrections

Although the scattering lengths $\bar{a}(\mu)$ or $\bar{a}(\Lambda)$ are regularization scheme dependent and cannot be directly measured, one would expect that using our results \cite{28} and \cite{22} for them in the case of proton-proton scattering, one would obtained values close to the measured values of the strong scattering lengths $a_{pn}$ and $a_{nn}$. After all, we have just obtained an $\mathcal{O}(\alpha)$ correction. However, from Fig.8 where we plot $a_{pp}(\mu)$ as function of the renormalization point $\mu$, we see this the scattering length depends strongly on this variable and actually diverges when $\mu$ is of the order of the pion mass. The same is the case with $a_{pp}(\Lambda)$ when the cutoff $\Lambda$ increases past $m_\pi$. This is in contrast to potential calculations represented by the Jackson-Blatt formula (36) which permits an almost unique determination of the Coulomb-free scattering length and essentially independent of the details of the strong potential. One should be able to recover this result by including higher order interactions in the effective theory and thereby make the calculation more realistic.

While the leading order interaction term in (1) has dimension $D = 6$, the next to leading order terms in the effective Lagrangian will have dimension $D = 8$. At non-relativistic energies and with only $S$-wave interactions there is only one such term,

$$L_2 = \frac{1}{2}C_2 (N^T \vec{\Pi}^2 N) \cdot (N^T \Pi N)^\dagger + h.c.$$  \hspace{1cm} (75)

where the operator $\vec{\Pi} = 1/2(\vec{N}^T - \vec{N})$. It corresponds to a potential $\hat{V}_2$ with the matrix element

$$\langle q | \hat{V}_2 | k \rangle = \frac{C_2}{2} (q^2 + k^2)$$  \hspace{1cm} (76)

Treating this operator perturbatively in the channels with no additional Coulomb interactions, it was found by Kaplan, Savage and Wise\cite{1} that the new coupling constant $C_2$ is given directly in terms of the effective range $r_0$ of the scattering amplitude,

$$C_2(\mu) = \frac{4\pi}{M} \left( \frac{1}{1/a - \mu} \right)^2 \frac{r_0}{2}$$  \hspace{1cm} (77)
Figure 8: Dependence of the inverse scattering length on the renormalization point. Dashed curve gives result with only $C_0$ interaction, while the solid curve also includes the $C_2$ interaction.

In the previous section we saw that the leading order coupling constant $C_0$ was modified by Coulomb effects. Here it will be shown that $C_2$ is unaffected to the order we are working.

The initial and final scattering states are again given by the interacting states (26) constructed from the pure Coulomb states (20) and (21). Since we will only be concerned with $\ell = 0$ states, it is convenient to use the partial wave expansion

$$\psi_p(\pm)(r) = \sum_{\ell=0}^{\infty} (2\ell + 1)^{\frac{\ell}{2}} R_\ell(\pm)(pr) P_\ell(\cos \theta)$$

which gives for the $S$-wave

$$R_0(\pm)(pr) = C_0 e^{\pm i\sigma_0} e^{ipr} M(1 + i\eta, 2; -2ipr)$$

From the Kummer identity $M(a, b; x) = e^x M(b - a, b; -x)$ it then follows that the ingoing and outgoing wavefunctions are simply related by complex conjugation and differ only by the phase factor $e^{\pm i\sigma_0}$.

Treating the new interaction (75) in first order of perturbation theory, we find the Feynman diagrams shown in Fig.9 contributing to the scattering amplitude. The first diagram Fig.9a gives the amount

$$\delta T_{SC}^{(a)}(p) = \langle \psi_p^{-}(\pm) | \hat{V}_2 | \psi_p^{(+)} \rangle = \frac{C_2}{2} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \psi_p^{-}(\pm)(k') (k^2 + k'^2) \psi_p^{(+)}(k)$$
Introducing the quantities
\[ \psi_0(p) \equiv \int \frac{d^3k}{(2\pi)^3} \psi^{(+)}_p(k) = C_\eta e^{i\sigma_0} \]  \hspace{1cm} (81)
and
\[ \psi_2(p) \equiv \int \frac{d^3k}{(2\pi)^3} k^2 \psi^{(+)}_p(k) \]  \hspace{1cm} (82)
we thus have
\[ \delta T_{SC}^{(a)}(p) = C_2 \psi_0(p)\psi_2(p) \]  \hspace{1cm} (83)

The next two chains of bubble diagrams in Fig.9b and Fig.9c form similar geometric series. Besides the Coulomb-dressed bubble integral \[ [43], \] these diagrams also involve the related integral
\[ J_2(p) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} k^2 \langle k | \tilde{G}^{(+)}_C(E) | k' \rangle \]  \hspace{1cm} (84)
Their sum can then be written as
\[
\delta T^{(b+c)}_{SC}(p) = \frac{C_2C_0}{1 - C_0J_0(p)} \psi_0(p)[\psi_0(p)J_2(p) + \psi_2(p)J_0(p)]
\] (85)

Similarly, the diagrams in Fig.9d sums up to give
\[
\delta T^{(d)}_{SC}(p) = \frac{C_2C_0^2}{[1 - C_0J_0(p)]^2} \psi_0(p)\psi_0(p)J_0(p)J_2(p)
\] (86)

Adding up these three partial results, we then have for the perturbed scattering amplitude
\[
\delta T_{SC}(p) = \frac{C_2\psi_0(p)}{[1 - C_0J_0(p)]^2} (\psi_2(p) + C_0 [\psi_0(p)J_2(p) - \psi_2(p)J_0(p)])
\] (87)

Except for \(\psi_0(p)\) which is finite and \(J_0(p)\) which has already been evaluated, both \(\psi_2(p)\) and \(J_2(p)\) are divergent and must be regularized. Both of them involve Coulomb wavefunctions and there is no obvious way how to do that.

### 4.1 Wavefunction regularization

Since the coupling constants \(C_0(\mu)\) and \(C_2(\mu)\) in leading order are known in the PDS regularization scheme, it would be simplest if the divergent quantities in (87) also could be regularized in the same scheme. Since this is defined in momentum space, we will then need the Fourier transforms \(\psi^{(+)}_{p}(k)\) of the Coulomb wavefunctions (78). These were first derived by Podolsky and Pauling[31]. Following them, we show in Appendix B that the result can be written as
\[
\psi^{(+)}_{p}(k) = 4\pi \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(\cos \hat{\theta}) \int_{0}^{\infty} dr r^2 R^{(+)}_{\ell}(pr) j_{\ell}(kr)
\] (88)

where \(\hat{\theta}\) is the polar angle in momentum space. As a check, we can then verify that \(\psi_0(p)\) as defined in (81) takes its correct value. A similar calculation in Appendix B then gives for \(\psi_2(p)\) the result
\[
\psi_2(p) = \left[ p^2 - \mu\alpha M - \frac{1}{2}(\alpha M)^2 \right] \psi_0(p)
\] (89)

It is obtained using dimensional regularization and the middle, \(\mu\)-dependent term follows from a PDS pole in \(d = 2\) dimensions.

Using the standard representation (23) for the Coulomb propagator, we can now use this result to evaluate the double integral (84) for \(J_2(p)\). Since the integral over \(k'\) equals the value of the wavefunction at the origin, we have
\[
J_2(p) = M \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{k^2\psi^{(+)}_{q}(k)\psi^{(+)*}_{q}(0)}{p^2 - q^2 + i\epsilon}
\] (90)
Now we can replace the integral over $k$ by $\psi_2(q)$ with the regulated result in (89). This does not have to be entirely correct, replacing a part of a doubly divergent integral with a finite, regulated expression. The problem lies in that $\psi_2(q)$ has in general higher order terms going to zero when $\epsilon = 3 - d \to 0$. However, these can give finite contributions when combined with the other divergence in the integral for $J_2$. So we will proceed under the assumption that these potential terms are higher order in $\alpha$ so that they can be neglected within the accuracy we are working. Thus we have

$$J_2(p) = M \int \frac{d^3q \psi_2(q) \psi_0^*(q)}{(2\pi)^3 p^2 - q^2 + i\epsilon}$$

$$= [p^2 - \mu \alpha M - \frac{1}{2}(\alpha M)^2] J_0(p) - J$$

(91)

where the last term

$$J = M \int \frac{d^3q}{(2\pi)^3} \frac{2\pi \eta(q)}{e^{2\pi \eta(q)} - 1}$$

(92)

is independent of the external momentum $p$. As shown in Appendix B, it contains a PDS pole and will thus be linearly dependent on the renormalization point $\mu$.

4.2 Effective range and scattering length

Since we always will choose the renormalization mass $\mu$ such that $\mu \gg \alpha M$, we see from (89) that $\psi_2 = (p^2 - \mu \alpha M) \psi_0$. As a consequence, the last term $\psi_0 J_2 - \psi_2 J_0$ in (87) can be neglected compared to the first term. Finally we are thus left with the correction

$$\delta T_{SC}(p) = C_2^2 C_0^2 e^{2i\sigma_0} \left[ p^2 - \mu \alpha M \right]$$

(93)

to the leading order scattering amplitude $T_{SC}(p)$ in (44). The $p^2$ part within the brackets codes for information about the effective range in proton-proton scattering while the momentum-independent part will give corrections to the scattering length. In order to see that, we need the modified effective range formula (33) which now will appear in the form

$$- \frac{4\pi}{M} C_0^2 e^{2i\sigma_0} \left( \frac{1}{T_{SC}} - \frac{\delta T_{SC}}{T_{SC}^2} \right) + \alpha M H(\eta) = - \frac{1}{\alpha_C} + \frac{1}{2} r_0 p^2 + \ldots$$

(94)

Using now

$$\frac{\delta T_{SC}}{T_{SC}^2} = \frac{C_2(\mu)}{C_0^2(\mu)} e^{-2i\sigma_0} \left[ p^2 - \mu \alpha M \right]$$

(95)

and the ordinary result (74) for the coupling constant $C_2(\mu)$ with the same scattering length $a = a(\mu)$ which appears in the physical scattering length (74), we see from comparing the $O(p^2)$ terms of this equation that the effective range $r_0$ is not affected by the
Coulomb interactions to this order in perturbation theory. This result is also in agreement with the measured values found for $r_0$ in $nn$, $pn$ and $pp$ reactions\cite{20,21}.

However, the momentum independent terms in the correction (93) will modify the scattering length found in leading order. Instead of (74) we now find

$$\frac{1}{a_{pp}(\mu)} = \frac{1}{a_{pp}^C} + \alpha M \left[ \ln \frac{\mu \sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2} C_E - \frac{1}{2} \mu r_0 \right]$$

(96)

The new terms proportional to $r_0$ reduces the downward trend when the inverse scattering length is plotted as function of $\mu$ as in Fig.8. When $\mu$ is around the pion mass, we now get a value for $a(\mu)$ which is much closer to the measured values in $pn$ and $nn$ elastic scattering. In fact, we find $a(\mu = m_\pi) = -29.9$ fm which should be compared with the value $a = -23.7$ fm in the $pn$ channel. The differences are now within the range which can be explained by electromagnetic interactions at shorter scales\cite{22}. Corrections from higher order operators in the effective theory will have a negligible effect on this result since they will be associated with higher powers of $\alpha$. However, they can be slightly different in other regularization schemes.

5 Conclusions

We have shown that Coulomb effects in proton-proton scattering and other hadronic systems at low energies can be calculated systematically in a non-perturbative way based directly upon the full Coulomb propagator within the effective field theory of Kaplan, Savage and Wise for nucleons. The approach is straightforward and can also be applied in other non-relativistic field theories. The phenomenological important quantities in these systems are the scattering lengths and effective ranges. While the Coulomb force strongly perturbs the scattering length in proton-proton scattering, its effect in $\pi N$ and $\pi \pi$ reactions is small due of the small reduced mass in these systems. This is in agreement with the experimental situation. Our results are derived in next to leading order and we find no changes in the effective ranges in different hadronic reactions due to the Coulomb force. The measured values support this conclusion.

It has been shown that in higher orders in the expansion of the effective theory one is faced with increasingly divergent integrals involving the Coulomb propagator. They have here been calculated using a method based upon the Fourier transforms of the wavefunctions combined with PDS regularization. This approach should be put on a firmer basis or replaced by a more direct method, perhaps in coordinate space. Also it is of interest to do these integrals in other regularization schemes such as the introduction of a simple momentum cutoff as we already have used in the simplest case.
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7 Appendix A

The two-loop integral (14) which gives the lowest Coulomb correction to the bubble diagram, is infrared finite so we can safely take $\lambda \rightarrow 0$. It is most convenient to evaluate it for Euclidean external momentum $p^2 = -\gamma^2$ and write it in $d = 3 - \epsilon$ dimensions on the form

$$\delta I_0 = \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{e^2 M^2}{q^2 (k^2 + \gamma^2) [(q + k)^2 + \gamma^2]}$$

(97)

after a shift of integration variables. We combine the last two factors with the Feynman trick, integrate over $k$ using (10) and get

$$\delta I_0 = \Gamma(2 - d/2) \frac{(4\pi)^{d/2}}{(2\pi)^d} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{e^2 M^2}{q^2 \Delta^{2 - \frac{d}{2}}}$$

(98)

where $\Delta = x(1-x)q^2 + \gamma^2$. Using now the more general combination formula

$$\frac{1}{A^\alpha B^\beta} = \int_0^1 d\omega \frac{\omega^{\alpha-1} \omega^{\beta-1}}{[\omega A + (1 - \omega) B]^\alpha \beta}$$

(99)

it follows again from the general integral (11) that

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 (q^2 + a)^{2-d/2}} = \int_0^1 dx \omega^{1-d/2} \frac{\Gamma(3 - d) (\omega a)^{d-3}}{\Gamma(2 - d/2) (4\pi)^{d/2}}$$

(100)

where $a = \gamma^2/(1-x)$. Collecting the different factors, we then have for the two-loop diagram

$$\delta I_0 = e^2 M^2 \frac{\Gamma(3 - d)}{(4\pi)^d} \gamma^{2(d-3)} \int_0^1 dx (x - x^2)^{1-d/2} \int_0^1 d\omega \omega^{4-d/2}$$

(101)

The last integral is simply $2/(1 - \epsilon)$ while the first is

$$\int_0^1 dx (x - x^2)^{-\frac{1}{2} + \frac{\epsilon}{2}} = \int_0^1 dx \frac{1 + \epsilon \ln \sqrt{x - x^2}}{\sqrt{x - x^2}} + O(\epsilon^2)$$

(102)
Now
\[ \int_0^1 dx \frac{1}{\sqrt{x-x^2}} = \pi \] (103)
and
\[ \int_0^1 dx \frac{\ln \sqrt{x-x^2}}{\sqrt{x-x^2}} = -2\pi \ln 2 \] (104)
We can then take the limit \( \epsilon \to 0 \) in (101) and thus obtain
\[ \delta I_0 = \frac{\alpha M^2}{8\pi} \left( \frac{1}{\epsilon} + 2 \ln \frac{\mu \sqrt{\pi}}{2\gamma} + 1 - C_E \right) \] (105)
where \( C_E = 0.5772 \ldots \) is Euler’s constant. Finally, we go to back to the physical situation by taking \( \gamma = -ip \) which gives the result (15).

8 Appendix B

We will here derive the regulated results (89) and (91) from the Fourier transformed Coulomb wavefunctions
\[ \psi_p^{(\pm)}(\mathbf{k}) = \int d^3 r \psi_p^{(\pm)}(r)e^{-ik \cdot r} \] (106)
with \( \psi_p^{(\pm)}(r) \) defined by (78). Attaching a hat to the spherical angles of \( \mathbf{k} \), we have
\[ \mathbf{k} \cdot \mathbf{r} = kr[\cos \theta \cos \hat{\theta} + \sin \theta \sin \hat{\theta} \cos(\phi - \hat{\phi})] \] (107)
For convenience, we choose \( \hat{\phi} = 0 \). The integral over the azimuthal angle \( \phi \) then gives simply a Bessel function \( J_0(-kr \sin \theta \sin \hat{\theta}) \). For the integral over the polar angle \( \theta \) we use the result
\[ \int_0^\pi d\theta \sin \theta P_\ell(\cos \theta)J_0(-kr \sin \theta \sin \hat{\theta})e^{-ikr \cos \theta \cos \hat{\theta}} = i^\ell \sqrt{\frac{2\pi}{-kr}} P_\ell(\cos \hat{\theta})J_{\ell + \frac{1}{2}}(-kr) \] (108)
from Podolsky and Pauling[31]. With \( J_\ell(-z) = (-1)^\ell J_\ell(z) \) and introducing the spherical Bessel functions
\[ j_\ell(z) = \sqrt{\frac{\pi}{2z}} J_{\ell + \frac{1}{2}}(z) \] (109)
we thus get the result (88).

As a check, we now calculate \( \psi_0(p) \) as defined in (81). Using dimensional regularization, it will follow from
\[ \psi_0(p) = \left( \frac{\mu}{2} \right)^\epsilon \int \frac{d^d k}{(2\pi)^d} \psi_p^{(+)}(\mathbf{k}) \] (110)
in the limit $\epsilon = 3 - d \to 0$. The angular integration will pick out the $\ell = 0$ part of the wavefunction and give a factor $4\pi$ when we take $d \to 3$. Thus we are left with a regulated expression for the radial integral,

$$\psi_0(p) = \frac{2}{\pi} \int_{0}^{\infty} drr^2 R_0(r) \int_{0}^{\infty} dk k^{2-\epsilon} j_0(kr) \quad \text{(111)}$$

The integral over $k$ gives the result

$$\int_{0}^{\infty} dk k^{3-\epsilon} J_\frac{3}{2}(kr) = 2^\frac{3}{2} - \epsilon r^{\frac{3}{2}} - \frac{\Gamma(\frac{3}{2} - \frac{\epsilon}{2})}{\Gamma(\frac{3}{2})} \quad \text{(112)}$$

It is divergent in the limit $\epsilon \to 0$. The remaining integral over $r$ in (111) now involves the hypergeometric function $\text{[23]}$ and is obtained from the more general formula

$$\int_{0}^{\infty} drr^{-1+\epsilon} e^{ipr} M(1+i\eta, 2; -2ipr) = \Gamma(\epsilon)(-ip)^{-\epsilon} 2 F_1(1+i\epsilon, \epsilon, 2; 2) \quad \text{(114)}$$

The first factor is seen to cancel the divergence in the previous integral. Combining the different terms with $2 F_1(1+i\eta, 0, 2; 2) = 1$, we finally have

$$\psi_0(p) = 2C_\eta e^i\sigma_0 \pi^{1/2} \Gamma(3/2) \quad \text{(115)}$$

which indeed is the correct result (81).

The regularization of $\psi_2(p)$ in (82) now follows exactly along the same lines and does not involve any new integrals. We then find

$$\psi_2(p) = 16C_\eta e^{i\sigma_0} \frac{\Gamma\left(\frac{5}{2} - \frac{\epsilon}{2}\right) \Gamma(-2+\epsilon)}{\sqrt{\pi^2} \Gamma(-1+\frac{\epsilon}{2})} (-ip)^{2-\epsilon} .$$

$$2 F_1(1+i\eta, -2+\epsilon, 2; 2) \left[ \left(\frac{\mu}{2}\right)^\epsilon (2\pi)^{2-d} \frac{\pi^{d/2}}{\Gamma(d/2)} \right]$$

We have here kept the $d$-dimensional integration volume element in the last factor. The result is now well-behaved in the limit $\epsilon \to 0$. Inserting $2 F_1(1+i\eta, -2, 2; 2) = 1/3 - 2\eta^2/3$, we get

$$\psi_2(p)\big|_{d \rightarrow 3} = C_\eta e^{i\sigma_0} \left[ p^2 - \frac{1}{2} (\alpha M)^2 \right] \quad \text{(116)}$$

However, this time we see that the above expression also has a PDS pole when $d \to 2$. Its residue involves $2 F_1(1+i\eta, -1, 2; 2) = -i\eta$ and it thus becomes

$$\psi_2(p)\big|_{d \rightarrow 2} = C_\eta e^{i\sigma_0} \frac{\alpha M \mu}{d-2} \quad \text{(117)}$$
It must be subtracted from the previous result with \( d = 3 \). We therefore finally have

\[
\psi_2(p) = C_\eta e^{i\sigma_0} \left[ p^2 - \alpha M\mu - \frac{1}{2}(\alpha M)^2 \right]
\] (118)

which is the result we make use of in (87). In the main text we drop the last term since we will always choose \( \mu \gg \alpha M \).

The integral (92) is of exactly the same form as in (46) and with dimensional regularization takes the form

\[
J = M \left( \frac{\mu}{2} \right)^\epsilon \frac{\Omega_d}{(2\pi)^d} \int_0^\infty dq q^{d-1-\epsilon} \frac{2\pi \eta(q)}{e^{2\pi \eta(q)} - 1}
\] (119)

Changing the integration variable to \( x = 2\pi \eta(k) \) it becomes a standard integral with the value

\[
J = M \left( \frac{\mu}{2} \right)^\epsilon \Omega_d (\pi \alpha M)^{3-\epsilon} \Gamma(-2 + \epsilon) \zeta(-2 + \epsilon)
\] (120)

Making now use of \( \zeta(-2) = 0 \), we see that it is finite when \( \epsilon \to 0 \). But again there is a PDS pole when \( d \to 2 \). After having subtracted this part, we get

\[
J = -\frac{\pi}{4} \alpha^2 M^3 \left( \alpha M \zeta'(-2) + \frac{\mu}{12} \right)
\] (121)

Under the assumption \( \mu \gg \alpha M \) which we make in the main text, this term again can be neglected.

References

[1] S. Weinberg, *Nucl. Phys.* B363, 3 (1991); *Phys. Lett.* B295, 114 (1992).

[2] U. van Kolck, *Phys. Rev.* C49, 2932 (1994); C. Ordonez, L. Ray and U. van Kolck, *Phys. Rev. Lett.* 72, 1982 (1994); *Phys. Rev.* C53, 2086 (1996); T.S. Park, D.P. Min and M. Rho, *Nucl. Phys.* A596, 515 (1996).

[3] D.B. Kaplan, M.J. Savage and M.B. Wise, *Nucl. Phys.* B478, 629 (1996); T.D. Cohen, *Phys. Rev.* C55, 67 (1997); D.R. Phillips and T.D. Cohen, *Phys. Lett.* B390, 7 (1997); K.A. Scaldeferri, D.R. Phillips, C.W. Kao and T.D. Cohen, *Phys. Rev.* C56, 679 (1997).

[4] D.B. Kaplan, M.J. Savage and M.B. Wise, *Phys. Lett.* B424, 390 (1998); *Nucl. Phys.* B534, 329 (1998).

[5] J. Gegelia, nucl-th/9802038; *Phys. Lett.* B429, 227 (1998).

[6] T. Mehen and I.W. Stewart, *Phys. Lett.* B445, 378 (1999); nucl-th/9809093
[7] D.B. Kaplan, M.J. Savage and M.B. Wise, *Phys. Rev.* **C59**, 617 (1999).

[8] J.-W. Chen, H.W. Griesshammer, M.J. Savage and R.P. Springer, *Nucl. Phys.* **A644**, 221, 245 (1998).

[9] M.J. Savage, K.A. Scaldeferri and M.B. Wise, *nucl-th/9811022*.

[10] P.F. Bedaque and U. van Kolck, *Phys. Lett.* **B428**, 21 (1998); P. F. Bedaque, H.-W. Hammer and U. van Kolck, *Phys. Rev. Lett.* **82**, 463 (1999); *Nucl. Phys.* **A646**, 444 (1999).

[11] X. Kong and F. Ravndal, *nucl-th/9811070*.

[12] X. Kong and F. Ravndal, *nucl-th/9902064*.

[13] For clear presentation of the theoretical and experimental situation see H. Bethe and P. Morrison, *Elementary Nuclear Theory*, John Wiley and Sons, New York, 1956.

[14] L.D. Landau and J. Smorodinski, *J. Phys. Acad. Sci. USSR*, **8**, 219 (1944); *ibid.* **11**, 195 (1947).

[15] H.A. Bethe, *Phys. Rev.* **76**, 38 (1949).

[16] J.D. Jackson and J.M. Blatt, *Rev. Mod. Phys.* **22**, 77 (1950).

[17] X. Kong and F. Ravndal, *Phys. Rev.* **D54**, 014031 (1999).

[18] B. Holstein, *nucl-th/9901044*.

[19] J.-W. Chen, G. Rupak and M.J. Savage, *nucl-th/9902056*.

[20] G.A. Miller, B.M.K. Nefkens and I. Slaus, *Phys. Rep.* **194**, 1 (1990).

[21] E.M. Henley in *Isospin in Nuclear Physics*, ed. D.H. Wilkinson, North-Holland Publishing Company, Amsterdam, 1969.

[22] E. Epelbaum and U.-G. Meißner, *nucl-th/9902043*.

[23] L.D. Landau and E.M. Lifsicht, *Quantum Mechanics*, Pergamon Press, London, 1958.

[24] A. Sommerfeld, *Atombau und Spektrallinien*, Vol. II, Vieweg, Braunschweig, 1939.

[25] M.L. Goldberger and K.M. Watson, *Collision Theory*, John Wiley and Sons, New York, 1964.

[26] D.R. Harrington, *Phys. Rev.* **139**, B691 (1965).

[27] L. Heller, P. Signell and N.R. Yoder, *Phys. Rev. Lett.* **13**, 577 (1964).

[28] L.P. Kok, J.W. de Maag, H.H. Bouwer and H. van Haeringen, *Phys. Rev.* **C26**, 2381 (1982).
[29] S. Deser, M.L. Goldberger, K. Baumann and W. Thirring, *Phys. Rev.* **96**, 774 (1954).

[30] T.L. Trueman, *Nucl. Phys.* **26**, 57 (1961).

[31] B. Podolsky and L. Pauling, *Phys. Rev.* **34**, 109 (1929).