Higher Order Convergent Fast Nonlinear Fourier Transform

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Abstract—It is demonstrated in this letter that linear multistep methods for integrating ordinary differential equations can be used to develop a family of fast forward scattering algorithms with higher orders of convergence. Excluding the cost of computing the discrete eigenvalues, the nonlinear Fourier transform (NFT) algorithm thus obtained has a complexity of \( O(KN + C_pN\log^2 N) \) such that the error vanishes as \( O(N^{-p}) \) where \( p \in \{1, 2, 3, 4\} \) and \( K \) is the number of eigenvalues. Such an algorithm can be potentially useful for the recently proposed NFT based modulation methodology for optical fiber communication. The exposition considers the particular case of the backward differentiation formula \( (C_p = p) \) and the implicit Adams method \( (C_p = (p - 1)^2) \) of which the latter proves to be the most accurate family of methods for fast NFT.

Index Terms—Nonlinear Fourier Transform, Zakharov-Shabat scattering problem

I. INTRODUCTION

This paper deals with the algorithmic aspects of the nonlinear Fourier transform (NFT) based modulation scheme which aims at exploiting the nonlinear Fourier spectrum (NF) for optical fiber communication [1]. These novel modulation [2], [3] techniques can be viewed as an extension of the original ideas of Hasegawa and Nyu who proposed what they coined as eigenvalue communication in the early 1990s [4]. One of the key ingredients in various NFT-based modulation techniques is the fast forward NFT which can be used to decode information encoded in the discrete and/or the continuous part of the nonlinear Fourier spectrum. A thorough description of the discrete framework (based on one-step methods) for various fast forward/inverse NFT algorithms was presented in [5] where it was shown that one can achieve a complexity of \( O(N\log^2 N) \) in computing the scattering coefficients in the discrete form. If the eigenvalues are known beforehand, then the NFT has an overall complexity of \( O(KN + N\log^2 N) \) such that the error vanishes as \( O(N^{-2}) \) where \( N \) is the number of samples of the signal and \( K \) is the number of eigenvalues. Interestingly enough, the complexity of the fast inverse NFT proposed in [6], [7] also turns out to be \( O(KN + N\log^2 N) \) with error vanishing as \( O(N^{-2}) \).

In this letter, we present new fast forward scattering algorithms where the complexity of computing the discrete scattering coefficients is \( O(C_pN\log^2 N) \). If the eigenvalues are known beforehand, the NFT of a given signal can be computed with a complexity of \( O(KN + C_pN\log^2 N) \) such that the error vanishes as \( O(N^{-p}) \) where \( (p \in \{1, 2, 3, 4\}) \) and \( K \) is the number of eigenvalues. In particular, we demonstrate in this work that using \( m \)-step \((m \in \{1, 2, 3, 4\}) \) backward differentiation formula (BDF) and \( m \)-step \((m \in \{1, 2, 3\}) \) implicit Adams (IA) method [8] one can obtain fast forward NFT algorithms with order of convergence given by \( p = m \) and \( p = m + 1 \), respectively.

The starting point of our discussion is the Zakharov and Shabat (ZS) [9] scattering problem which can be stated as: For \( \zeta \in \mathbb{R} \) and \( \mathbf{v} = (v_1, v_2) \),

\[
\mathbf{v}_t = -i\zeta \sigma_3 \mathbf{v} + U(t, x) \mathbf{v},
\]

where \( \sigma_3 = \text{diag}(1, -1) \) and the potential \( U(t, x) \) is defined by \( U_{11} = U_{22} = 0, U_{12} = q(t, x) \) and \( U_{21} = r(t, x) \) with \( r = kq^* \) \((k \in \{+1, -1\})\). The parameter \( \zeta \in \mathbb{R} \) is known as the spectral parameter and \( q(t, x) \) is the complex-valued function associated with the slow varying envelop of the optical field which evolves along the fiber according to the nonlinear Schrödinger equation (NSE), stated in its normalized form,

\[
\dot{q}_x = q_{tt} - 2k|q|^2 q.
\]

The NSE provides a satisfactory description of pulse propagation in an optical fiber in the path-averaged formulation [10] under low-noise conditions where \( t \) is the retarded time and \( x \) is the distance along the fiber. In the following, the dependence on \( x \) is suppressed for the sake of brevity. Here, \( q(t) \) is identified as the scattering potential. The solution of the ZS scattering problem (1) consists in finding the so-called scattering coefficients which are defined through special solutions of (1) known as the Jost solutions. The Jost solutions of the first kind, denoted by \( \psi(t; \zeta) \), has the asymptotic behavior \( \psi(t; \zeta) e^{-i\zeta t} \rightarrow (0, 1)^T \) as \( t \rightarrow \infty \). The Jost solutions of the second kind, denoted by \( \phi(t; \zeta) \), has the asymptotic behavior \( \phi(t; \zeta) e^{i\zeta t} \rightarrow (1, 0)^T \) as \( t \rightarrow -\infty \).

For the focusing NSE \((i.e., \kappa = -1 \in (2))\), the nonlinear Fourier spectrum for the potential \( q(t) \) comprises a discrete and a continuous spectrum. The discrete spectrum consists of the so-called eigenvalues \( \zeta_k \in \mathbb{C}_r \), such that \( a(\zeta_k) = 0 \), and, the norming constants \( b_k \) such that \( \phi(t; \zeta_k) = b_k \psi(t; \zeta_k) \). Note that \( \zeta_k, b_k \) describes a bound state or a soliton state associated with the potential. For convenience, let the discrete spectrum be denoted by the set

\[
\mathbb{S}_K = \{ (\zeta_k, b_k) \in \mathbb{C}^2 | \text{Im} \zeta_k > 0, k = 1, 2, \ldots, K \}.
\]

Note that for the defocusing NSE \((i.e., \kappa = +1 \in (2))\), the discrete spectrum is empty. The continuous spectrum, also referred to as the reflection coefficient, is defined by \( \rho(\xi) = b(\xi)/a(\xi) \) for \( \xi \in \mathbb{R} \).

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II. THE NUMERICAL SCHEME

In order to develop the numerical scheme, we begin with the transformation \( \tilde{\psi} = e^{i\tau_0 \xi_0} \psi \) so that (1) becomes

\[
\tilde{\psi}_t = \tilde{U} \tilde{\psi}, \quad \tilde{U} = e^{i\tau_0 \xi_0} U e^{-i\tau_0 \xi_0} = \begin{pmatrix} 0 & qe^{2i\xi_0} \tau_0 \\ r e^{-2i\xi_0} \tau_0 & 0 \end{pmatrix}.
\]

In order to discuss the discretization scheme, we take an equispaced grid defined by \( t_0 = T_1 + n h, \ n = 0, 1, \ldots, N, \) with \( t_N = T_2 \) where \( h \) is the grid spacing. Define \( \xi_0, \xi_1 \in \mathbb{R} \) such that \( h\xi_0 = -T_1, \ h\xi_1 = T_2. \) Further, let us define \( \tau = e^{i\xi_0 h} \).

For the potential functions sampled on the grid, we set \( q_n = q(t_n), \ r_n = r(t_n), \ U_n = U(t_n) \) and \( \tilde{U}_n = \tilde{U}(t_n) \). Discretization using the \( m \)-step BDF scheme \( (m \in \{1, 2, 3, 4\}) \) reads as

\[
\sum_{s=0}^{m} \alpha_s \tilde{\psi}_{n+s} = h\beta \tilde{U}_{n+m} \tilde{\psi}_{n+m}
\]

where \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m) \) and \( \beta \) are known constants [8, Chap. III.1]. Discretization using the \( m \)-step IA method \( (m \in \{1, 2, 3\}) \) reads as

\[
\tilde{\psi}_{n+m} - \tilde{\psi}_{n+m-1} = h \sum_{s=0}^{m} \beta_s \tilde{U}_{n+s} \tilde{\psi}_{n+s}
\]

where \( \beta = (\beta_0, \beta_1, \ldots, \beta_m) \) are known constants [8, Chap. III.1]. Both of these methods lead to a transfer matrix \( M_{n+m}(z^2) \in \mathbb{C}^{2m \times 2m} \) of the form

\[
M_{n+m}(z^2) = \begin{pmatrix} \gamma_m M_{n+m}^{(1)}(z^2) & \gamma_{m-1} M_{n+m}^{(2)}(z^2) & \cdots & \gamma_1 M_{n+m}^{(m-1)}(z^2) & \gamma_0 M_{n+m}^{(m)}(z^2) \\ \sigma_0 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_0 & 0 \end{pmatrix}
\]

where \( \sigma_0 = \text{diag}(1, 1) \) and \( M_{n+m}^{(s)} = M_{n+m}^{(s)}(z^2) \in \mathbb{C}^{2 \times 2} \) so that

\[
W_{n+m} = M_{n+m}(z^2) W_{n+m-1}
\]

where \( w_n = z^n \psi_n \) and \( W_n = (w_n, w_{n-1}, \ldots, w_{n-m+1})^\tau \in \mathbb{C}^{2m} \). For BDF schemes, we may set \( \alpha_0 \equiv 1. \) Further, setting \( Q_n =
(hβ)bq, Rn = (hβ)r, and Θn = 1 − QnRn, we have γ = −α together with
\[ M_n^{(s)}(z^2) = \frac{1}{\Theta_n + m} \left( \frac{1}{R_n + m} z^{2\ell} Q_{n+m} + z^{2s} Q_{n+m} + \beta_{m-1} Q_{n+m-1} \right). \tag{9} \]
For the IA methods, we have
\[ M_n^{(1)}(z^2) = \Theta_{n+m}^{-1} \times \left( 1 + z^2 \beta_{m-1} R_{n+m-1} Q_{n+m} + z^2 Q_{n+m} + \beta_{m-1} Q_{n+m-1} + z^2 + \beta_{m-1} R_{n+m} Q_{n+m-1} \right), \tag{10} \]
where \( Q_n = (hβ)m_{d}q, R_n = (hβ)bq, \Theta_n = 1 - Q_nR_n. \) Also,
\[ M_n^{(m-1)}(z^2) = \frac{1}{\Theta_n + m} \left( z^{(m-1)} R_n + Q_{n+m} + z^{(m-1)} R_n + Q_{n+m} \right), \tag{11} \]
with γ = 1 and γ = β, for s = 0, 1, ..., m − 2 where
\[ \bar{β} = β/β_m = (β_0, β_1, ..., 1). \tag{12} \]

Let us consider the Jost solution \( \phi(t; ζ) \). We assume that \( q_n = 0 \) for \( n = m + 1, m + 2, \ldots \) so that \( \phi = z^{-n}(1, 0)^\top \) for \( n = m + 1, m + 2, \ldots \). In order to express the discrete approximation to the Jost solutions, let us define the vector-valued polynomial
\[ P_n(z^2) = \left( p_0^{(n)}(z^2), p_1^{(n)}(z^2), \ldots, p_{n}^{(n)}(z^2) \right)^\top, \tag{13} \]
such that \( \phi = z^{-n}P_n(z^2) \). The initial condition works out to be
\[ \mathcal{W}_0 = z^{-m} \begin{pmatrix} \phi_0 \\ \phi_{-1} \\ \vdots \\ z^{-m} \phi_{-m+1} \end{pmatrix} = z^{-m} \begin{pmatrix} P_0(z^2) \\ P_{-1}(z^2) \\ \vdots \\ P_{-m+1}(z^2) \end{pmatrix} \in \mathbb{C}^{2m}, \tag{14} \]
yielding the recurrence relation
\[ P_{n+m}(z^2) = M_{n+m}(z^2)P_{n+m-1}(z^2), \tag{15} \]
where \( P_n(z^2) = (P_n(z^2), P_{n-1}(z^2), \ldots, P_{n-m+1}(z^2))^\top \in \mathbb{C}^{2m} \).

A. Fast Forward Scattering Algorithm

It is evident from the preceding paragraph that the forward scattering step requires forming the cumulative product: \( M_N(z) = M_{N-1}(z) \times \ldots \times M_2(z) \times M_1(z) \). Let \( m \) denote the nearest base-2 number greater than or equal to \( (m + 1) \), then pairwise multiplication using FFT \[ \text{[11]} \] yields the recurrence relation for the complexity \( σ(n) \) of computing the scattering coefficients with \( n \) samples: \( σ(n) = 8m^3n(\log n + 2 + 2\pi n/2), n = 2, 4, \ldots, N \), where \( ν(n) = O(n \log n) \) is the cost of multiplying two polynomials of degree \( n - 1 \) (ignoring the cost of additions). Solving the recurrence relation yields \( σ(N) = O(m^3N \log^2 N) \).

1) Computation of the continuous spectrum: The computation of the continuous spectrum requires evaluation the polynomial \( b_n(z^2) \) and \( a_n(z^2) \) on the unit circle \( |z| = 1 \), say, at \( N \) points. This can be done efficiently using the FFT algorithm with complexity \( O(N \log N) \). Therefore, the overall complexity of computation of the continuous spectrum easily works to be \( O(m^3N \log^2 N) \).

2) Computation of the norming constants: Let us assume that the discrete eigenvalues are known by design\(^1\). Therefore the only part of the discrete spectrum still to be computed are the norming constants. A method of computing the norming constants corresponding to arbitrary eigenvalues is presented in [5] which has an additional complexity of \( O(KN) \) where \( K \) is the number of eigenvalues. This method can be employed here as well because it uses no information regarding how the discrete scattering coefficients were computed.

\(^1\)Given that the best polynomial root-finding algorithms still require \( O(N^2) \) operations, we would at this stage favor a system design which avoids having to compute eigenvalues.
We set this end, we define an arbitrary discrete spectrum and compute test our algorithms for computing the norming constants. To classical Darboux transformation (CDT), which allows us to computing the scattering coefficients while that of MG1 is computing $m$th order IA (labeled IA...) schemes have better convergence rates with increasing step IA ($l_{IA}$). The order of convergence for SM and MG1 both is (21). The numerical results are plotted in Fig. 2 where it is evident that BDF$_m$ as well as IA$_m$ schemes have better convergence rates with increasing $m$. The IA methods are clearly superior to that of BDF in terms of accuracy.

### III. Numerical Experiments: Test for Convergence and Complexity

#### A. Secant-hyperbolic potential

A test for verifying the order of convergence and complexity can be readily designed using the well-known secant-hyperbolic potential given by $q(t) = A \sech t$, $(k = -1)$. The scattering coefficients are given by [12]

$$
a(\xi) = \frac{[\Gamma(0.5 - i\xi)]^{2}}{\Gamma(A + 0.5 - i\xi)} \Gamma(-A + 0.5 - i\xi),
$$

$$
b(\xi) = -\sin \pi A \sech \pi \xi,
$$

so that the reflection coefficient is given by $\rho(\xi) = b(\xi)/a(\xi)$. We set $A = 4.4$. Let $\Omega_h = [-\pi/2h, \pi/2h]$, then, the error in computing $b(\xi)$ is quantified by

$$
e_{rel} = \frac{\|b(\xi) - b_N(\xi)\|_{L^2(\Omega_h)}}{\|b(\xi)\|_{L^2(\Omega_h)}}.
$$

where the integrals are computed using the trapezoidal rule. Similar consideration applies to $\rho(\xi).$ For the purpose of benchmarking, we use the Split-Magnus (SM) and Magnus method with one-point Gauss quadrature (MG1) discussed in [5, Sec. IV]). Note that the complexity of SM is $O(N \log^2 N)$ in computing the scattering coefficients while that of MG1 is $O(N^2)$. The order of convergence for SM and MG1 both is $O(N^{-2})$. The numerical results are plotted in Fig. 2 where it is evident that $m$-step BDF (labeled BDF$_m$) as well as the $m$th step IA (labeled IA$_m$ where IA$_1$ is identical to trapezoidal rule (TR)) schemes have better convergence rates with increasing $m$. The improved accuracy, however, comes at a price of increased complexity which is evidently not so prohibitive (besides, room for improvements in the implementation does exist). The IA methods are clearly superior to that of BDF in terms of accuracy while keeping the complexity same.

#### B. Multisolitons

Arbitrary multisoliton solutions can be computed using the classical Darboux transformation (CDT), which allows us to test our algorithms for computing the norming constants. To this end, we define an arbitrary discrete spectrum and compute the corresponding multisoliton solution which serves as an input to the NFT algorithms. Let $b_k^{(num)}$ be the numerically computed approximation to $b_k$ which corresponds to the eigenvalue $\zeta_k$ which we assume to be known. The error in the norming constants can then be quantified by

$$
e_{rel} = \left(\sum_{k=1}^{K} |b_k^{(num)} - b_k|^2 \right)^{1/2} \left(\sum_{k=1}^{K} |b_k|^2 \right)^{1/2}.
$$

For the discrete spectrum, the example chosen here is taken from [5] which can be described as follows: Define a sequence of angles for $J \in \mathbb{Z}_+$ by choosing $\Delta \theta = (\pi - 2\theta_0)/(J - 1)$, $\theta_0 > 0$, and $\theta_j = \theta_0 + (j - 1)\Delta \theta$, $j = 1, 2, \ldots, J$ so that $\theta_j \in [\theta_0, \pi - \theta_0]$. Then the eigenvalues are chosen as

$$
\zeta_{j+f(j-1)} = e^{i\theta_j}, \ l = 1, 2, \ldots, 8, \ j = 1, 2, \ldots, J.
$$

Further, the norming constants are chosen as

$$
b_j = e^{i\pi(j-1)/(8J-1)}, \ j = 1, 2, \ldots, 8J.
$$

For this test, we set $\theta_0 = \pi/3$ and $J = 4$. Then we consider a sequence of discrete spectra defined as

$$
\zeta_{K} = \{ (\zeta_k, b_k), \ k = 1, 2, \ldots, K \},
$$

where $K = 4, 8, \ldots, 32$ (see Fig. 3). For fixed $K$, the eigenvalues are scaled by the scaling parameter $k = 2(\sum_{k=1}^{K} \Im(\zeta_k))^{1/2}$. Let $\eta_{\text{min}} = \min(\Delta \theta) \Im(\zeta_k)$, then the computational domain for this example is chosen as $[-T, T]$ where $T = 2\pi/\eta_{\text{min}}$. The numerical results are plotted in Fig. 3 where it is evident that BDF$\_m$ as well as IA$\_m$ schemes have better convergence rates with increasing $m$. The IA methods are clearly superior to that of BDF in terms of accuracy.

### IV. Conclusion

In this letter we presented a family of fast NFT algorithms based on exponential linear multistep methods which were demonstrated to exhibit higher-order of convergence. Excluding the cost of computing the discrete eigenvalues, the proposed algorithms have a complexity of $O(KN + C_p N \log^2 N)$ such that the error vanishes as $O(N^{-p})$ where $p \in \{1, 2, 3, 4\}$ and $K$ is the number of eigenvalues. The form of $C_p$ depends on the underlying linear multistep method.

The future research in this direction will focus on developing compatible fast layer-peeling schemes for the discrete systems proposed in this letter so that higher-order convergent fast inverse NFT algorithms could be developed.

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Fig. 4. The figure shows a comparison of convergence behavior of NFT algorithms for computing the norming constant based on the discretization schemes, namely, BDF$^m$ ($m \in \{1, 2, 3, 4\}$), IA$^m$ ($m \in \{1, 2, 3\}$), SM and MG1 (see Sec. III). The method IA$^1$ is identical to the trapezoidal rule (TR).

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