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Efficient Decoding of Gabidulin Codes over Galois Rings

Sven Puchinger¹, Julian Renner², Antonia Wachter-Zeh², Jens Zumbrägel³

¹Department of Applied Mathematics and Computer Science, Technical University of Denmark (DTU), Denmark
²Institute for Communications Engineering, Technical University of Munich (TUM), Germany
³Faculty of Computer Science and Mathematics, University of Passau, Germany

Email: svepu@du.dk, julian.renner@tum.de, antonia.wachter-zeh@tum.de, jens.zumbraeigel@uni-passau.de

Abstract—This paper presents the first decoding algorithm for Gabidulin codes over Galois rings with provable quadratic complexity. The new method consists of two steps: (1) solving a syndrome-based key equation to obtain the annihilator polynomial of the error and therefore the column space of the error, (2) solving a key equation based on the received word in order to reconstruct the error vector. This two-step approach became necessary since standard solutions as the Euclidean algorithm do not properly work over rings.

I. INTRODUCTION

Network coding over finite rings [1]–[6] may result in more efficient physical-layer network coding schemes in comparison to using finite fields. Since rank-metric codes can be applied for error correction in network coding (cf. [7] for finite fields), Kamche and Mouaha [8] considered rank-metric codes over finite principal ideal rings. The authors, amongst others, defined Gabidulin codes over rings and designed a Welch-Berlekamp-like decoding algorithm similar to the one over finite fields [9].

This decoding algorithm has to solve a linear system of equations and perform a polynomial division, resulting in an asymptotic complexity \( O(n^3) \) for a Gabidulin code of length \( n \).

In order to accelerate the decoding process, different coding-theoretic approaches can be thought of: a Berlekamp–Massey (BM) approach, an approach based on the Euclidean algorithm, or row reduction techniques. The Euclidean algorithm requires divisions of polynomials such that the degree of the remainder is smaller than the one of the inputs; over rings, this degree reduction does not work if the leading monomial is not a unit. When investigating row reduction techniques, we encountered a similar problem: having to divide rows by a non-unit element.

In [10], a BM-like decoding approach for Reed–Solomon and BCH codes over rings was presented. However, when decoding Gabidulin codes, a BM-like approach would only accelerate the first step of decoding, namely, finding the annihilator polynomial of the error, not the second step which is necessary to find the explicit error vector. This is fundamentally different from Reed–Solomon codes where the second step (finding the error values) is easy and efficient. All these observations forced us to establish a different decoding technique.

In this paper, we investigate a new approach to decode Gabidulin codes over Galois rings efficiently. Namely, we first solve a syndrome-based key equation with a BM-like approach to obtain the error span polynomial and then set up another type of key equation based on the received word (in the literature also called Gao key equation [11]) to explicitly recover the error vector in an efficient way. This therefore leads to the first approach that decodes Gabidulin codes over Galois rings with provable quadratic complexity.

II. PRELIMINARIES

A. Galois Rings

For a given prime \( p \) and positive integers \( r \) and \( s \) we denote by \( GR(p^r,s) \) the Galois ring of characteristic \( p^r \) and degree \( s \). It can be defined as the quotient ring \( \mathbb{Z}_{p^r}[x]/(f) \), where \( f \in \mathbb{Z}_{p^r}[x] \) is a polynomial such that its reduction \( f \mod p \in \mathbb{F}_p[x] \) is irreducible of degree \( s \).

The theory of Galois rings can be viewed as a close analog of the theory of finite fields, which is translated to the realm of finite commutative local rings, i.e., rings with a unique maximal ideal. Galois rings may in fact be more intrinsically defined as the unique separable ring extensions of \( \mathbb{Z}_{p^r} \), or equivalently, as the unramified local ring extensions of \( \mathbb{Z}_{p^r} \), meaning that the principal ideal \((p)\) remains the maximal ideal in those extensions (see [12, Sec. 14]).

Most importantly for the present work is the property of a Galois ring being a Galois extension of \( \mathbb{Z}_{p^r} \), with group of ring automorphisms isomorphic to the Galois group of the corresponding residue fields. More precisely, let \( R := GR(p^r,s) \) and \( S := GR(p^r,t) \) be Galois rings with residue fields \( k := \mathbb{F}_p \) and \( K := \mathbb{F}_{p^t} \), respectively, and let \( s \mid t \) so that \( R \subseteq S \) is a ring extension. Then the Galois group \( \text{Gal}_R(S) \) of ring automorphisms of \( S \) fixing \( R \) corresponds, by a lifting construction, to the Galois group \( \text{Gal}_k(K) \) of field automorphisms of \( K \) fixing \( k \); hence, the group \( \text{Gal}_R(S) \) is isomorphic to a cyclic group of order \( m \), where \( m = \frac{t}{s} = \dim_k K \) is the extension degree (see [12, Sec. 15]).

B. Computing the Galois Group

For an extension \( k \subseteq K \) of finite fields where \( q := |k| \), the field Galois group \( \text{Gal}_k(K) \) is generated by a \( q \)-th power Frobenius map. Likewise, the ring Galois group \( \text{Gal}_R(S) \) of Galois rings \( R \subseteq S \) is also generated by an automorphism \( \sigma : S \rightarrow S \) that can be described by a \( q \)-th power \( \alpha \mapsto \alpha^q \) of some element \( \alpha \in S \) with \( S = R[\alpha] \) (although it does not hold that \( \sigma(z) = z^q \) for all \( z \in S \)). Such an element \( \alpha \) can be constructed by the following procedure. Let \( \overline{f} \in \mathbb{F}_p[x] \) be some irreducible polynomial of degree \( m \) defining the field extension \( k \subseteq K \), then there holds \( x^m - x = \overline{f} \cdot \overline{g} \) for some \( \overline{g} \in k[x] \) coprime to \( \overline{f} \). By Hensel lifting [12, Sec. 13] there are \( f, g \in R[x] \) with \( \overline{f} = f \mod p \) and \( \overline{g} = g \mod p \) such that \( x^m - x = f \cdot g \) holds over \( R \). Then letting \( S := R[\alpha]/(f) \) and \( \alpha := [x] \in S \) we construct a generator \( \sigma : S \rightarrow S, \alpha \mapsto \alpha^q \) of \( \text{Gal}_R(S) \) as desired.

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C. Polynomials and Skew Polynomials

In the following let $R$ be a finite local commutative ring with maximal ideal $m$, which is nilpotent. Moreover, let $k := R/m$ be the residue field and let $\mu : R \to k$ be the canonical map, extended to polynomials $R[x] \to k[x]$. The following results can be found in [12, Sec. 13].

For a polynomial $f = \sum f_i x^i \in R[x]$ we have:
1) $f$ is a unit $\iff f_0 \in R^*$ and all $f_i \in m$, $i > 0 \implies \mu f_i \in k^*$,
2) $f$ is no zero divisor $\iff$ some $f_i \in R^* \implies \mu f \neq 0$.

In the case of 2) the polynomial $f$ is called primitive.

Lemma 1 Let $g \in R[x]$ be a primitive polynomial.
1) There exists a unit $u \in R[x]$ such that $ug$ is monic; moreover, $deg ug = deg u \leq deg g$.
2) For $f \in R[x]$ there is “division with remainder”, i.e.
there are $q, r \in R[x]$ with $f = qg + r$ and $deg r < deg g$.

Now let $\sigma \in \text{Aut}(R)$ be a ring automorphism. We define the skew polynomial ring $R[x; \sigma]$ via the rule $x\sigma = \sigma(x)r$ for all $r \in R$, extended by addition and multiplication. Still one may apply the canonical map $\mu : R[x; \sigma] \to k[x; \sigma]$, with $\sigma \in \text{Aut}(k)$ induced by $\sigma$, and above remarks remain valid.

For a polynomial $f = \sum_{i = 0}^n f_i x^i \in R[x; \sigma]$ of degree $n$ we denote by $lt(f) := x^n$ the leading term, $lc(f) := f_0$ its leading coefficient and $\text{ln}(f) := lc(f)lt(f) = f_0 x^n$ the leading monomial.

D. Smith Normal Form and Rank Profile of Modules

Consider again an extension $R = GR(p', s) \subseteq S = GR(p', sm)$ of Galois rings. Let $m$ be the maximal ideal of $R$, which has nilpotency index $r$. For $a \in R \setminus \{0\}$ the valuation $v(a)$ is defined as the unique integer $v$ with $a \in m^v \setminus m^{v+1}$, and we let $v(0) := r$.

For any matrix $A \in R^{m \times n}$ there are invertible matrices $S \in R^{m \times m}$ and $T \in R^{n \times n}$ such that $D = SAT \in R^{m \times n}$, where $D$ is called the Smith normal form of $A$ and is a diagonal matrix with diagonal entries $d_1, \ldots, d_{\min(m,n)}$ satisfying $0 \leq v(d_1) \leq \ldots \leq v(d_{\min(m,n)}) \leq r$. We define $\text{rk}(A) := \left\{|i| \in \{1, \ldots, \min(m,n)\} : d_i \neq 0\right\}$ and $\text{frk}(A) := \left\{|i| \in \{1, \ldots, \min(m,n)\} : d_i = 0\right\}$ as the rank and the free rank of $A$, respectively. The same properties hold for matrices over $S$, where $m$ needs to be replaced by the maximal ideal of $S$ denoted by $\mathfrak{m}$.

Let $\gamma = [\gamma_1, \ldots, \gamma_m]$ denote an ordered basis of $S$ over $R$. We define $\text{ext}_\gamma : S^n \to R^{m \times m}$, $a \mapsto A$, where $a_j = \sum_{i = 1}^{n} A_{i,j} \gamma_i$, $j \in \{1, \ldots, n\}$ and denote by $\text{rk}(A) := \text{rk}(A)$ the rank norm and free rank norm of $A$, respectively.

Let $M$ denote an $R$-submodule of $S$ and let $d_1, \ldots, d_n$ refer to the diagonal elements of a matrix in Smith normal form with row space $M$. Then, we call the polynomial

$$\phi_M(x) := \sum_{i=0}^{r-1} \phi_0^M x^i \in \mathbb{Z}[x]/(x^r)$$

the rank profile of $M$, where $\phi_0^M := \{i : v(d_i) = i\}$. Note the relationship between (free) rank and the rank profiles

$$\text{frk}(M) = \phi_0^M = \phi^M(0), \quad \text{rk}(M) = \sum_{i=0}^{r-1} \phi_i^M = \phi^M(1).$$

E. Gabidulin Codes

Let $R \subseteq S$ be Galois rings and let $\sigma \in \text{Gal}_R(S)$ be a generating automorphism. For a skew polynomial $f = \sum_{i=0}^n f_i x^i \in S[x; \sigma]$ and $s \in S$ we let $f(s) := f_0 s + f_1 \sigma(s) + \ldots + f_n \sigma^n(s)$. Denote by $S[x; \sigma]_c$ the $S$-module of all skew polynomials of degree $< k$. We define Gabidulin codes as in [8].

Definition 1 Let $g = [g_1, \ldots, g_n] \in S^n$, where the entries are linearly independent over $R$, and let $0 < k \leq n$. A Gabidulin code of length $n$, dimension $k$ and support $g$ is defined by

$$\text{Gab}_k(g) := \{(f, g) : f \in S[x; \sigma]_c\}.$$ 

In [8, Prop. 3.23], it was shown that the Gabidulin code $\text{Gab}_k(g)$ has a generator matrix $G = [\sigma^i(g_j)]_{0 \leq i < k, 1 \leq j \leq n}$. Further, the minimum rank distance of $\text{Gab}_k(g)$ is $d = n - k + 1$ and $\text{Gab}_k(g)$ are MRD codes, see [8, Thm. 3.24].

Theorem 2 [8, Thm. 3.25] Let $g = [g_1, \ldots, g_n] \in S^n$, where the entries of $g$ are linearly independent over $R$, and let $k$ be an integer such that $0 < k \leq n$. Then there exists a vector $h = [h_1, \ldots, h_n]$, where the entries of $h$ are linearly independent over $R$, such that $H = [\sigma^i(h_j)]_{0 \leq i < n - k, 1 \leq j < n}$ is a parity-check matrix of $\text{Gab}_k(g)$.

III. A SKEW-POLYNOMIAL VARIANT OF THE BYRNE-FITZPATRICK ALGORITHM

Let $S := GR(p', t)$ be a Galois ring and let $\sigma \in \text{Aut}(S)$ be an automorphism of $S$.

In order to solve the decoding problem of rank metric Gabidulin codes over $S$, following [10], [13] we introduce the solution module over the skew polynomial ring $S[x; \sigma]$. Given a positive integer $m$ and a polynomial $u \in S[x; \sigma]$ we let

$$M := \{(f, g) \in S[x; \sigma]^2 \mid fu \equiv g \mod x^m\},$$

which is a left submodule of $S[x; \sigma]^2$ (note that the congruence mod $x^m$ does not depend on taking left or right module).

Suitable elements of the solution module of minimal degree may be found by adapting the Gröbner basis approach of Byrne and Fitzpatrick [10] (see also [14] for a more elementary description for codes over $\mathbb{Z}_4$), as described next.

We consider a term order $<$ on the set of all terms $\{x^n, 0 \mid n \in \mathbb{N}\} \cup \{(0, x^n) \mid n \in \mathbb{N}\}$ of $S[x; \sigma]^2$, compatible with multiplication by $x^k \in S[x; \sigma]$ for $k \in \mathbb{N}$. Accordingly, for any nonzero pair in $S[X; \sigma]^2$ the leading term, leading coefficient and leading monomial can be defined with respect to $<$. Concretely, we are going to use the term order given by $(1, 0) < (0, 1) < (0, x) < \ldots$.

A left Gröbner basis of the module $M$ is a generating set $\{(f_i, g_i) \mid i \in I\}$ of $M$ such that for all $(f, g) \in M$ there exists some $i \in I$ such that $\text{ln}(f, g) \text{left-divides ln}(f_i, g_i)$.

Since $(x^m, 0)$ and $(0, x^m)$ are in the solution module $M$, by adapting an argument in [10] one can show that $M$ has a left Gröbner basis of the form

$$\mathcal{B} = \{(a_0, b_0), \ldots, (a_{r-1}, b_{r-1}), (c_0, d_0), \ldots, (c_{r-1}, d_{r-1})\}$$

with $\text{ln}(a_i, b_i) = (p q^i x^{r+1}, 0)$ and $\text{ln}(c_i, d_i) = (0, p q^i x^{n+1})$ for all $0 \leq i, j < r$, for some decreasing sequences $\lambda_0 \geq \ldots \geq \lambda_{r-1}$ and $\mu_0 \geq \ldots \geq \mu_{r-1}$, called minimal exponents.

The following algorithm, derived from the method of “solution by approximations” of [10], efficiently computes a left Gröbner basis of the solution module $M$.

Theorem 3 After completing step $k$ in Algorithm 1 the set $\mathcal{B}_{k+1}$ is a left Gröbner basis of the module $\mathcal{M}_{k+1} := \{(f, g) \in S[x; \sigma]^2 \mid fu \equiv g \mod x^{k+1}\}$. In particular, the algorithm is correct.

It has complexity $O(n r m^2)$ operations in $S$. Furthermore, we have $|\mathcal{B}| = 2r$. 
Algorithm 1: SkewByrneFitzpatrick

Input: \( u \in S[x;\sigma]\) and \( m \in \mathbb{Z}_{>0}\)
Output: Left Gröbner basis of the left \( S[x;\sigma]\) module
\[ \mathcal{M} := \{(f, g) \in S[x;\sigma]^2 \mid fu \equiv g \mod x^n\} \]  
1. let \( B_0 := \{(p^i, 0) \mid i \in \{0, \ldots, r-1\}\} \cup \{0, p^i\} \mid i \in \{0, \ldots, r-1\}\)
2. for \( k \in \{0, \ldots, m-1\} \) do
   3. for each \((f_i, g_i) \in B_k\) do
      4. compute the discrepancy \( \zeta_i := (f_iu - g_i)_k \)
          (where \((\cdot)_k\) denotes the \( k\)-th coefficient)
      6. if \( \zeta_i = 0 \) then
         7. put \((f_i, g_i) \in B_{k+1}\)
         8. continue
      9. if there is \((f_j, g_j) \in B_k\) with
         10. \( \text{lt}(f_j, g_j) < \text{lt}(f_i, g_i) \) and \( \zeta_i \) divides \( \zeta_j \) then
            11. put \((f_i, g_i) - q(f_j, g_j) \in B_{k+1}\), where
            12. \( q \in S \) with \( \zeta_i = q\zeta_j \)
   else
      13. put \((x f_i, x g_i) \in B_{k+1}\)
3. return \( B_m \)

Proof: The correctness is proved by induction on \( k \), by adapting the arguments in [10]. We briefly sketch it here. Let \((f_i, g_i)\) be put in \( B_{k+1} \) in \( \ell 7, \ell 10 \) or \( \ell 12 \) of the algorithm.

First we claim that \((f_i, g_i) \in \mathcal{M}_{k+1}\), for which we show that \((f_i, g_i) \in \mathcal{M}_k\) and the discrepancy \((f_iu - g_i)_k\) vanishes.

This is obvious in the case of \( \ell 7 \). In \( \ell 10 \) we have \((f_i, g_i) \in \mathcal{M}_k\), since \((f_i, g_i), (f_j, g_j) \in \mathcal{M}_k\) and \( \mathcal{M}_k \) is an \( S\)-module; moreover we have \((f_iu - g_i)_k = (f_iu - g_i)_k - (q(f_ju - g_j))_k = (f_iu - g_i)_k - q(f_ju - g_j)_k = (f_iu - g_i)_k - q\zeta_j = 0\). And in \( \ell 12 \) it is clear that \((f_i, g_i) \in \mathcal{M}_{k+1}\), since \( x^k | f_iu - g_i \) implies \( x^{k+1} | x f_iu - x g_i \).

Now let \( \lambda_0 \geq \ldots \geq \lambda_r \) and \( \rho_0 \geq \ldots \geq \rho_r \) be the minimal exponents of \( \mathcal{M}_k \) and \( \mathcal{M}_{k+1}\), respectively. From the inclusions \( x \mathcal{M}_k \subseteq \mathcal{M}_{k+1} \subseteq \mathcal{M}_k \) we easily infer that
\[ \lambda_i \leq \lambda'_i \quad \text{and} \quad \rho_j \leq \rho'_j \quad \text{for all} \quad 0 \leq i, j < r, \]
and for all \( 0 \leq i, j < r \).

Proposition: If \( \ell 10 \) holds \( \iff \lambda_i = \lambda'_i \) (2) and (a similar statement holds if \( \ell 9 \) holds).

Indeed, if \( \ell 10 \) holds, then \( \text{lm}(f_i, g_i) = \text{lm}(f_i, g_i) \), so that \( \lambda_i = \lambda'_i \). Conversely, suppose that \( \lambda_i = \lambda'_i \), then there is \((a, b) \in \mathcal{M}_{k+1}\) such that \( \text{lm}(a, b) = (p^x x^\lambda, 0) \), and hence we have \( (a, b) - (f_i, g_i) \in \mathcal{M}_k \) with \( \text{lt}(a, b) - (f_i, g_i) \times (x^\lambda, 0) \). By the division algorithm we may write \( (a, b) - (f_i, g_i) = \sum \alpha_i(a, b) + \sum \beta_i(c_i, d_i) \) with \( \alpha_i, \beta_i \in S[x;\sigma] \) and \( \text{lt}(a, b), \beta_i(c_i, d_i) \times (x^\lambda, 0) \). For the discrepancy we then find \( 0 = \zeta_i - q\zeta_j \) for some \( q \in S \) and some \( j \) with leading term less than \( (x^\lambda, 0) \). Therefore, the condition in \( \ell 9 \) is satisfied.

From (1) and (2) it follows that if \( B_0 \) is a Gröbner basis of \( \mathcal{M}_k \), then \( B_{k+1} \) as produced by Algorithm 1 is a Gröbner basis of \( \mathcal{M}_{k+1}\), establishing the correctness.

For the running time analysis, observe first that there are \( 2r \) pairs \((f_i, g_i)\) in the Gröbner bases \( B_k \), and the degree of the polynomials \( f_i, g_i \) in \( O(m) \) as it increases in each outer loop by at most 1. Hence the computation of each discrepancy in \( \ell 4 \) requires \( O(m) \) operations in \( S \). The if-condition in \( \ell 9 \) can easily be checked by considering the degrees and evaluations; neglecting this cost we only take \( \ell 10 \) into account, which again needs \( O(m) \) operations in \( S \). Therefore, completing one step \( k \) of the outer loop amounts to \( O(m) \) operations in \( S \), which results in the stated overall running time.

IV. A NEW DECODER FOR GABIDULIN CODES OVER GALOIS RINGS

In this section, we propose a new decoding algorithm for Gabidulin codes over rings with quadratic complexity in the code length. The first part of the decoder is to retrieve a skew polynomial called annihilator polynomial, which vanishes on the module spanned by the error vector. In the literature, this polynomial is also called error span polynomial. We obtain this by solving a syndrome-based key equation via the skew Byrne–Fitzpatrick algorithm presented in the previous section.

The second part of the algorithm uses a different kind of key equation, which involves the message polynomial of the transmitted codeword, to retrieve this message polynomial under the condition that the rank of the error is small. This is done using standard operations with skew polynomials, such as interpolation and left and right division.

Definition 2 Let \( e \in S^n \). An annihilator polynomial of \( e \) is a primitive polynomial \( \Lambda \in S[x;\sigma] \) of minimal degree such that \( \Lambda(e_i) = 0 \quad \text{for all} \quad i = 1, \ldots, n \).

Lemma 4 Let \( e \in S^n \). Any annihilator polynomial has degree exactly \( t := \text{rk}(e) \). Moreover, if \( \text{rk}(e) = \text{frk}(e) \), then there is a unique monic annihilator polynomial of \( e \).

Proof: By [8, Prop. 2.5] there exists a monic (hence primitive) polynomial of degree \( t \) that vanishes on the \( e_i \). This implies that an annihilator polynomial has degree at most \( t \). Furthermore, by [8, Prop. 3.16], any polynomial of degree \( \leq t \) that vanishes on the \( e_i \) cannot be primitive, which proves that the degree must be at least \( t \). The second claim directly follows from [8].

We need the following lemma to derive the key equation that we use for decoding. The statement generalizes the decomposition of the error’s matrix representation, which was already used for decoding in [15]. The difference is that, over rings, the entries of \( \alpha \) are not necessarily linearly independent, but the rank profile of \( \alpha \) coincides with the rank profile of \( e \).

Lemma 5 Let \( e \in S^n \) and define \( t := \text{rk}(e) \). Then there is a vector \( \alpha \in S^t \) with the same rank profile as \( e \) and a matrix \( B \in R^{t \times n} \) whose rows are linearly independent, such that \( e = \alpha B \).

The entries of \( \alpha \) form a minimal generating set of \( \{e_1, \ldots, e_n\} \).

Proof: Expand \( e \in S^n \) into a matrix \( E \in R^{t \times n} \). By the existence of the Smith normal form, we can decompose \( E = A'B \), where \( A' \in R^{t \times t} \) and \( B' \in R^{n \times n} \) are invertible matrices and \( D' \in R^{t \times t} \) is a diagonal matrix with diagonal entries
\[ p^{i_1}, \ldots, p^{i_t}, 0, \ldots, 0 \]
with \( \min\{n, r\} - t \) many zeros, where the powers \( 0 \leq i_j < r \) correspond to the rank profile of \( E \) (which is the same as the one of \( e \)). Due to the min\{n, r\} - t zero entries on the diagonal of \( D \), we can write \( E = AB \), where \( A \) consists of the first \( t \) columns of \( A \). \( D \) is the left-upper \( t \times t \) submatrix.
of $D'$, and $B$ consists of the first $t$ rows of $B'$. Note that the columns of $A$ and the rows of $B$ are linearly independent. Define $A := AD \in \mathbb{R}^{t \times n}$ and observe that $A$ has the same rank profile as $E$. We obtain $a$ as in the claim by writing every column of $A$ as an element of $S$.

For a received word $r \in S^n$, we define the syndrome polynomial

$$s_r(x) := \sum_{i=0}^{n-k-1} \left( \sum_{j=1}^{n} \sigma^i(h_j)r_j \right)x^i \in S[x; \sigma]_{\leq n-k},$$

(3)

where $h_1, \ldots, h_n \in S$ corresponds to the first row of the parity-check matrix. Note that $s_e = 0$ for all $e \in C$, so if $r = e + e$, the syndrome polynomial only depends on the error $s_r = s_e$.

Our decoding algorithm solves the key equation, i.e., it finds polynomials $[\lambda, \omega]$, which fulfill the same congruence relation as $\Lambda$ and $\Omega$ and satisfy the same degree constraints.

**Theorem 6 ( Syndrome Key Equation)** Let $\Lambda$ be an annihilator polynomial of $e$. Then, there is a skew polynomial $\Omega$ of degree $\deg \Omega < \deg \Lambda$ such that

$$\Lambda s_e \equiv \Omega \mod x^{n-k},$$

where $s_e \in S[x; \sigma]_{\leq n-k}$ is the syndrome polynomial of $e$, as defined in (3).

**Proof:** Recall from Lemma 5 that $e = aB$ where $a = [a_1, \ldots, a_t]$ and $B = [B_{ij}]_{1 \leq i \leq t, 1 \leq j \leq n}$. We define $d_i := \sum_{j=1}^{n} B_{ij}h_j$. The coefficients of the syndrome $s_e = [s_{e,1}, \ldots, s_{e,n-k}]$ are

$$s_{e,i} = \sum_{j=1}^{n} e_j \sigma^i(h_j) = \sum_{j=1}^{n} \sum_{l=0}^{t} a_l B_{il} \sigma^i(h_j) = \sum_{l=0}^{t} a_l \sigma^i(d_l),$$

for all $i = 1, \ldots, n-k$. The $i$-th coefficient of $\Lambda s_e$, where $i = 0, \ldots, n-k-1$, can be calculated by

$$\Omega_i := [\Lambda s_e]_i = \sum_{j=0}^{\infty} \Lambda_j \sigma^j(s_{e,i-j})$$

$$= \sum_{j=0}^{\infty} \Lambda_j \sigma^j \left( \sum_{l=1}^{t} a_l \sigma^{i-j}(d_l) \right) = \sum_{l=1}^{t} \sum_{j=0}^{\infty} \Lambda_j \sigma^j(a_l).$$

For any $i \geq t$ this gives:

$$\Omega_i = \sum_{l=1}^{t} \sigma^i(d_l) \Lambda(a_l) = 0,$$

for all $i = t, \ldots, n-k-1$, since by definition $\Lambda(x)$ has $a_i$ as roots, for all $i = 1, \ldots, t$, and therefore $\deg \Omega = t < \deg \Lambda$.

We will use the following theorem to show how to retrieve (under the condition that the error has small rank) the message polynomial of the transmitted codeword from the output of the skew Byrne–Fitzpatrick algorithm.

**Theorem 7** Let $r = e + e$, where $e \in C$ with message polynomial $f$ and $t := \rk(e) \leq \frac{n-k}{2}$. Let $s = s_r = s_e$ be the syndrome polynomial corresponding to $r$. Suppose that we have two non-zero polynomials $u, v \in S[x; \sigma]$ such that:

- $u$ is primitive
- $us - v \equiv 0 \mod x^{n-k}$
- $\deg u \leq t$
- $\deg v < \deg u$

Then $u$ is an annihilator polynomial of $e$. In particular, its degree equals $t$. Furthermore, we have $uR \equiv uv \mod G$, where $R$ is the unique interpolation polynomial of $r$, and $G$ is the (unique, since the $g_i$ are linearly independent) annihilator polynomial of the $g_i$ (which has degree $n$).

**Proof:** Due to $\deg v < t$, $2t-1 < n-k$ and the congruence $us - v \equiv 0 \mod x^{n-k}$, the skew polynomial $u$ satisfies

$$\left( us \right)_i = 0, \quad \text{for all } i = 1, \ldots, 2t-1,$$

where $(us)_i$ denotes the $i$-th coefficient. Written as a linear system in the coefficients $u_0, \ldots, u_t$ of $u$, we get

$$\begin{bmatrix}
\sigma^0(s_{t}) & \sigma^1(s_{t-1}) & \cdots & \sigma^t(s_0) \\
\sigma^0(s_{t+1}) & \sigma^1(s_t) & \cdots & \sigma^t(s_1) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma^0(s_{2t-1}) & \sigma^1(s_{2t-2}) & \cdots & \sigma^t(s_{-1})
\end{bmatrix}
\begin{bmatrix}
u_0 \\
u_1 \\
\vdots \\
u_t
\end{bmatrix} = \begin{bmatrix}0 \\
0 \\
\vdots \\
0
\end{bmatrix} = s$$

Due to Lemma 5, there is an $a \in S^t$ with the same rank profile as $e$ and a matrix $B \in \mathbb{R}^{t \times n}$ whose rows are linearly independent, such that $e = aB$, and the entries of $a$ are a minimal generating set of $\{e_1, \ldots, e_n\}$. Define

$$d = [d_1, \ldots, d_t] := hB^T,$$

and observe that the entries of $d$ are linearly independent over $R$, since the both the entries of $h$ and the rows of $B$ are linearly independent. As in [16], we can decompose the matrix $S$ as follows:

$$S = DA^T,$$

with $D := [\sigma^i(d_j)]_{0 \leq i \leq t, 1 \leq j \leq t}$, $A := [\sigma^i(a_j)]_{0 \leq i \leq t, 1 \leq j \leq t}$. Since the $d_i$ are linearly independent over $R$, so are the $\sigma^i(d_i)$. Hence, the square Moore matrix $D$ is invertible and $u$ simply satisfies the linear system

$$A^T \begin{bmatrix}u_0 \\ı_1 \\
\vdots \\
\u0131_t
\end{bmatrix} = 0.$$

This can be rewritten as $u(a_i) = 0$ for all $i = 1, \ldots, t$. Since the $a_i$ are a generating set of $\{e_1, \ldots, e_n\}$, by the linearity of the skew polynomial evaluation, we get that $u(e_i) = 0$ for all $i = 1, \ldots, n$. Furthermore, $u$ is primitive and of minimal degree ($= t$) among all monic polynomials with this property due to Lemma 4. This proves that $u$ is an annihilator polynomial.

The second part of the claim follows directly since

$$u(R-f)(g_i) = u(R(g_i)-f(g_i)) = u(r_i-c_i) = u(c_i) = 0,$$

where the last equality follows from the first part. Since the $g_i$ are linearly independent, we have that $G$ must right-divide $u(R-f)$.

We need one last lemma, which shows that all the skew polynomial operations needed for the new decoder can be implemented in quadratic complexity.

**Lemma 8** Let $f, g \in S[x; \sigma]_{\leq n}$. The following operations with skew polynomials over $S$ can be implemented in $O(n^2)$ operations in $S$:

1. Multiplication $ab$, where $a, b \in S[x; \sigma]_{\leq n}$.
2. Left and right division of $a$ by $b$, where $a, b \in S[x; \sigma]_{\leq n}$ and $b$ is primitive.
3. Computing the unique interpolation polynomial of $\{(g_i, r_i)\}_{i=1}^{n}$, where the $g_i \in S$ are linearly independent over $R$ and the $r_i \in S$ are arbitrary.
4. Computing a monic annihilator polynomial of $g$, where $g \in S^n$.
5. Computing $[a(g_1), \ldots, a(g_n)]$, where $a \in S[x; \sigma]_{\leq n}$ and $g_1, \ldots, g_n \in S$ are linearly independent over $R$.
Our proposed decoding algorithm has quadratic complexity. However, the cost bounds in Lemma 8 can be reduced to subquadratic complexity using the results in [18], [20], [21] and thus, our approach might be improved such that it has subquadratic complexity.

It would be interesting to find a variant of the Byrne–Fitzpatrick algorithm that can solve key equations with arbitrary moduli. This would allow us to solve the key equation $AR ≡ Af \mod G$ directly instead of the two-step process.

In [13], algorithms of the same forms as the extended Euclidean, the Berlekamp–Massey and the Peterson–Gorenstein–Zierler algorithms were proposed for Galois rings. However, only the latter one was generalized to finite rings. An interesting open problem is the generalization of an extended Euclidean like algorithm to finite rings and to propose a subquadratic speed-up.

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