EXHAUSTIVE GROMOV COMPACTNESS FOR PSEUDOHOLOMORPHIC CURVES

JOEL W. FISH AND HELMUT HOFER

Dedicated to the Memory of Jean-Cristophe Yoccoz

Abstract. Here we extend the notion of target-local Gromov convergence of pseudoholomorphic curves to the case in which the target manifold is not compact, but rather is exhausted by compact neighborhoods. Under the assumption that the curves in question have uniformly bounded area and genus on each of the compact regions (but not necessarily global bounds), we prove a subsequence converges in an exhaustive Gromov sense.

Contents

1. Introduction 1
1.1. Acknowledgements 4
2. Preliminaries 4
2.1. Direct limit manifolds 4
2.2. Riemann surfaces 6
2.3. Pseudoholomorphic curves 8
2.4. Convergence of pseudoholomorphic curves 10
3. Proof of exhaustive Gromov compactness 14
Appendix A. Formula for arithmetic genus 22
References 23

1. Introduction

In his celebrated 1985 paper, [10], Gromov introduced the notion of a pseudoholomorphic curve, and provided an accompanying compactness theorem. His idea was to generalize the notion of an algebraic curve in, say, a complex projective variety to that of a pseudoholomorphic curve in a symplectically tamed almost complex manifold, and he showed that families of such curves are analogously compact. In the decades since, pseudoholomorphic curves have played a fundamental role in the development of symplectic geometry and topology as well as Hamiltonian dynamics, and a variety generalizing compactness theorems have been established. These tend to proceed along two general paths.

The first approach is exemplified by Rugang Ye [20], Floer [11], Hofer [12], and the SFT compactness paper [2], in that each of these treat closed or punctured curves...
from a global perspective. Besides additional ingredients dealing with the analysis near punctures and the necks, see for example [13], the analysis proceeds rather analogously to that for families of harmonic maps, which we outline as follows.

1. Obtain convergence of underlying Riemann surfaces.
2. With respect to a constant curvature metric guaranteed by the Uniformization Theorem, show that gradient bounds imply $C^\infty$ bounds.
3. Employ bubbling analysis at points of gradient blow-up, and show that only finitely many bubbles appear due to energy bounds and an energy threshold.
4. Use $C^\infty$ bounds and Arzelà-Ascoli to pass to a further subsequence which converges in $C^\infty$.
5. Verify that bubbles connect via, say, a monotonicity lemma.

This approach is most applicable when one has genus bounds, energy bounds, some global control over the entirety of each curve in the family, and when one already has a good idea of what types of curves should arise in the limits of such families. For completeness, we also mention Hummel [14], and for a more classical viewpoint, [1].

The second approach, typified by Taubes via Proposition 3.3 in [19], treats curves as sets and integral currents, and proves compactness from a more measure theoretic perspective. Roughly speaking, area bounds, a monotonicity lemma, and some measure theory yield a compactness theorem, however some additional work is necessary to show the limit is rectifiable, or rather that the measure theoretic limit has the structure of a weighted union of images of pseudoholomorphic curves. This approach is quite natural from the perspective of Seiberg-Witten theory, particularly when employing a Taubes-like degeneration to obtain pseudoholomorphic curves. The result has also been used extensively in Embedded Contact Homology, introduced by Hutchings in [15]; see also [17] and [16]. More generally, the technique is applicable when one has little more than area bounds – indeed, one does not need genus bounds on the sequence of curves. However, this can also be a weakness, in that genus cannot be detected a priori by these techniques. For example, one can construct degree-two holomorphic branched coverings of the unit disk with arbitrarily large genus, but from the integral current perspective, all such objects are indistinguishable.

The purpose of this manuscript is to further develop a less used third approach, introduced by the first author in [4] and streamlined in [5] and [6]. This is the so-called target-local Gromov compactness result; for a restatement, see Theorem 2 below. The basic idea was to follow the Taubes approach to studying curves locally in the target and allowing a free boundary (including arbitrarily many boundary components), but also demanding a genus bound, and then extracting a subsequence which converges in the Gromov-topology, rather than the substantially weaker topology of integral-currents. In some sense, the target-local compactness theorem says that if $W$ is a smooth compact manifold with boundary, and $u_k : (S_k, \partial S_k) \to (W, \partial W)$ is a sequence of pseudoholomorphic maps with genus bounds and area bounds, then after trimming the curves near $\partial W$ a subsequence converges in a Gromov sense. Our main result, stated below as Theorem 1, extends this to the case that $W$ is no longer compact, but instead is exhausted by compact
manifolds with smooth boundary:

\[ W_1 \subset W_2 \subset W_3 \subset \cdots \bigcup_{\ell \in \mathbb{N}} W_\ell = W. \]

Here the key assumptions on the curves are that

\[ \text{Area}(u_k^{-1}(W_\ell)) \leq C_\ell \quad \text{and} \quad \text{Genus}(u_k^{-1}(W_\ell)) \leq C_\ell. \]

In other words, this means that the curves in question may have infinite area and genus, however on each compact \( W_\ell \) (the union of which exhaust \( W \)) one has area and genus bounds for the portions of curves in that region.

It is important to mention that our main result here, stated as Theorem 1 below, is not a needless extension of Theorem 2 proved in [6], but rather it plays a foundational role in two forthcoming papers. The first was announced in [8] and will appear in [7] in which we prove that no regular energy level of a proper Hamiltonian function on \((\mathbb{R}^4, \omega_{\text{std}})\) has a minimal Hamiltonian flow, which answers a question for the case \( n = 2 \) raised by Herman in his 1998 ICM address; see [11]. The idea is to use neck-stretching techniques to study pseudoholomorphic curves in the symplectization of framed Hamiltonian manifolds. The tremendous difficulty is that such curves will lack a priori energy bounds like those that appear in Symplectic Field Theory, and thus the global techniques employed in [2] fail quickly and completely. Moreover, the Taubes approach of [19] also fails, precisely because the topology used to obtain compactness is simply too coarse. Indeed, the genus bounds and curvature properties that follow from the Gromov topology (but not the integral current topology) which are guaranteed by Theorem 1 play crucial roles in the proofs of the main results in [7]. In essence, our main result here strikes the perfect balance between the flexibility of the integral-current approach with the strength of the Gromov topology, and this balance is then heavily exploited in [7] to first find a limit curve (which might be wildly complicated), and then to use the Gromov topology and a posteriori analysis on the limit curve to show that it has a surprising number of unexpected properties, which are necessary to establish the non-minimality of the hypersurfaces.

The second result relying on Theorem 1 is the so called sideways stretching compactness results developed by the first author; see [3]. Here the idea is that Symplectic Field Theory is something akin to a TQFT for symplectic manifolds, and an extended TQFT would be akin to an extended Symplectic Field Theory in which one could independently stretch the neck along two transverse contact hypersurfaces. This has been carried out by the first author in certain sub-critical cases, and will appear in a forthcoming paper. Again though, the idea is similar: use a sequence of expanding domains in the target manifold, on which one has successively increasing area bounds to obtain a preliminary compactness result from Theorem 1, then use a posteriori analysis and the Gromov topology to improve properties of both the limit and precision of the convergence; then iterate this procedure to develop a full extended Symplectic Field Theory style compactness theorem.

More generally still, it is not difficult to imagine a wide range of applications of Theorem 1. Indeed, consider any symplectic manifold \( W \), and any compact set \( K \subset W \) which has empty interior. Then consider any sequence of almost complex structures which are tame, but degenerate along \( K \). That is, the \( J_k \) converge in \( C^\infty_{\text{loc}}(W \setminus K) \) to an almost complex structure on \( W \setminus K \), which is uniformly tame.
on each $W \setminus O(K)$, but not uniformly tame on $W \setminus K$. Then consider a sequence of closed pseudoholomorphic curves in a fixed homology class in $W$ which have bounded genus (e.g. only spheres). Theorem 1 immediately guarantees that a subsequence converges, in an exhaustive Gromov sense (see Definition 2.21), to a pseudoholomorphic curve in $W \setminus K$. Such a curve may have wildly complicated behavior – and yet a posteriori analysis can be employed which exploits the particular features of the $K$ and $J_k$ in question. Considering the ubiquitous use of pseudoholomorphic curves in symplectic geometry, topology, and Hamiltonian dynamics, such a result would seem potentially quite useful.

1.1. Acknowledgements. The authors would like to thank the referee for helpful comments and suggestions, particularly in regards to clarifying the exposition and helping to remove certain unnecessary assumptions.

2. Preliminaries

This section is devoted to presenting some preliminary concepts and small supporting results. In Section 2.1 we introduce the notion of an embedding diagram and direct limit manifolds. This is meant to generalize the notion of an exhausting sequence of regions:

$$W_1 \subset W_2 \subset W_3 \subset \cdots \subset \bigcup_{k \in \mathbb{N}} W_k =: \mathcal{W},$$

and is necessary for constructing the domain of the limit curve we must later produce. In Section 2.2 we review the basic definitions of Riemann surfaces, as well as additional structures like marked points, nodal points, decorations, arithmetic genus, etc. In Section 2.3 we recall the definition of pseudoholomorphic curves and some related concepts, like stability, boundary-immersed maps, generally immersed maps, and area. Finally, in Section 2.4 we provide a number of definitions of convergence for pseudoholomorphic curves. We note that the key notion of Section 2 is given in Definition 2.21 which is the novel definition of convergence of pseudoholomorphic curves in an exhaustive Gromov sense. Finally we note that here and throughout, in the case that a domain is non-compact, $C^\infty$ convergence will mean $C^\infty_{loc}$ convergence.

2.1. Direct limit manifolds. This section will be devoted to establishing the notion of a direct limit manifold, as well as some of its basic properties. We also take the opportunity to introduce some of the notions we shall frequently use.

Here and throughout, a topological manifold $X$ of dimension $n$ is a second countable Hausdorff space, where every point $x \in X$ has an open neighborhood homeomorphic to an open subset in $\mathbb{R}^n$. A chart around $x \in X$ is a triple $(\mathcal{O}, \phi, \mathcal{V})$ where $\mathcal{V}$ is an open subset of $\mathbb{R}^n$, $\mathcal{O}$ is an open neighborhood of $x$, and $\phi : \mathcal{O} \to \mathcal{V}$ is a homeomorphism. An atlas for the topological manifold $X$ consists of a family of charts $(\mathcal{O}_i, \phi_i, \mathcal{V}_i)_{i \in I}$ such that $X = \cup_{i \in I} \mathcal{O}_i$. We call the collection a smooth atlas provided all transition maps are $C^\infty$.

In the category $\text{Man}^n$ of $n$-dimensional smooth manifolds we will have to consider, during later constructions, direct limits of so-called embedding diagrams

$$(M, \psi) : M_1 \xrightarrow{\psi_1} M_2 \xrightarrow{\psi_2} M_3 \to \cdots,$$
where all arrows are smooth embeddings between smooth manifolds. From this we obtain a directed system by defining

\[ \psi_j^k : M_j \to M_k \]
\[ \psi_j^k = \psi_{k-1} \circ \cdots \circ \psi_j. \]

The direct limit of such a diagram is by definition a tuple \((M, \{t_k\}_{k \in \mathbb{N}})\), where \(M\) is a smooth \(n\)-dimensional manifold and for every \(k \in \mathbb{N}\) the map \(t_k : M_k \to M\) is smooth so that we have the following universal property.

- Given smooth maps \(f_k : M_k \to X\) such that \(f_{k+1} \circ \psi_k = f_k\) there exists a uniquely determined smooth map \(f : M \to X\) such that \(f \circ t_k = f_k\).

An immediate consequence of the universal property is that \((M, \{t_k\}_{k \in \mathbb{N}})\) is unique up to natural diffeomorphism. It is also an easy exercise to show that the \(t_k\) have to be smooth embeddings. This means any other realization of the direct limit \((M, \{t'_k\}_{k \in \mathbb{N}})\) is related to the first by a uniquely determined diffeomorphism \(\phi : M \to M\) satisfying \(\phi \circ t_k = t'_k\). The construction of the direct limit is well-known, but we briefly recall that it can be defined as

\[ M := \left( \bigcup_{k \in \mathbb{N}} M_k \times \{k\} \right) / \sim \]

where two points are equivalent if there exists \(\psi_j^k\) mapping one to the other. Here are some standard properties of the direct limit.

(DL-1) The smooth maps \(f : M \to X\) are in one-to-one correspondence to families of smooth maps \(\{f_k\}_{k \in \mathbb{N}}\), where \(f_k : M_k \to X\) such that \(f_{k+1} \circ \psi_k = f_k\).

(DL-2) Each tensor \(T_k\) on \(M\) uniquely corresponds to a family of tensors \(\{T_k\}_{k \in \mathbb{N}}\) on the \(\{M_k\}_{k \in \mathbb{N}}\) such that \(T_k\) is a tensor on \(M_k\) and \(\psi_k^* T_{k+1} = T_k\), and each such family gives rise to a unique tensor on \(M\).

Given an embedding diagram \((V, \phi)\), an inclusion, denoted \(\Gamma : (M, \psi) \to (V, \phi)\), consists of a family of smooth maps \(a_k : M_k \to V_k\) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
M_k & \xrightarrow{\psi_k} & M_{k+1} \\
\downarrow a_k & & \downarrow a_{k+1} \\
V_k & \xrightarrow{\phi_k} & V_{k+1}
\end{array}
\]

It follows immediately that the \(a_k\) must be embeddings, and hence we obtain an embedding between the direct limits. Let us call \(\Gamma\) surjective provided that for every \(v_k \in V_k\) there exists \(\ell \geq k\) and \(m_\ell \in M_\ell\) such that \(a_\ell(m_\ell) = \phi_\ell(v_k)\). In this case the induced maps between the limits are diffeomorphisms.

As an example, consider an embedding diagram \((M, \psi)\) and a strictly monotonic map \(\sigma : \mathbb{N} \to \mathbb{N}\). Construct another embedding diagram \((V, \phi)\), where \(V_k := M_{\sigma(k)}\) and \(\phi_k : V_k \to V_{k+1}\) is given by \(\phi_k = \psi_{\sigma(k)}^{\sigma(k+1)}\). On one hand, we might regard \((V, \phi)\) as a subsequence or subsystem of \((M, \psi)\), however we will instead regard \((M, \psi)\) as a surjective inclusion of \((V, \phi)\) with the family of maps \(a_k : M_k \to V_k\)
given by \( a_k = \psi_k^{\sigma(k)} \). Then, because this diagram commutes:

\[
\begin{array}{c}
M_1 \xrightarrow{\psi_1} M_2 \xrightarrow{\psi_2} M_3 \xrightarrow{\psi_3} \cdots \\
M_{\sigma(1)} \xrightarrow{\psi_{\sigma(1)}} M_{\sigma(2)} \xrightarrow{\psi_{\sigma(2)}} M_{\sigma(3)} \xrightarrow{\psi_{\sigma(3)}} \cdots
\end{array}
\]

it trivially follows that the following diagram commutes:

\[
\begin{array}{c}
M_1 \xrightarrow{\psi_1} M_2 \xrightarrow{\psi_2} M_3 \xrightarrow{\psi_3} \cdots \\
V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} \cdots
\end{array}
\]

From this it immediately follows that if \((\overline{M}, \{\iota_k\}_{k \in \mathbb{N}})\) is the direct limit of \((M, \psi)\), then \((\overline{V}, \{\iota_{\sigma(k)}\}_{k \in \mathbb{N}})\) is the direct limit of \((V, \phi)\), and the limit manifolds are diffeomorphic via the identity map. Or, put another way, regarding \((V, \phi)\) as a subsequence of \((M, \psi)\), we see that each embedding diagram has the same direct limit.

We note that the above claims can be proved by reducing them to the following trivial result, the proof of which is left to the reader.

**Lemma 2.1** (exhausting subsets and direct limit manifolds).

Let \( \{M_k\}_{k \in \mathbb{N}} \) be a sequence of sets such that

1. Each \( M_k \) carries the structure of a smooth manifold.
2. As sets, each \( M_k \) is a subset of \( M_{k+1} \), and \( M_k \subset M_{k+1} \) is an open subset in the \( M_{k+1} \) topology.
3. The smooth structure on \( M_k \) equals the smooth structure induced from \( M_{k+1} \).

Then the union, \( \overline{M} := \bigcup_{k=1}^{\infty} M_k \), carries a natural second countable Hausdorff topology and a uniquely determined smooth manifold structure, such that the natural inclusion \( M_k \rightarrow \overline{M} \) for every \( k \in \mathbb{N} \) is a smooth embedding as an open subset.

We conclude Section 2.1 by noting that in later sections we make use of direct limit manifolds in two ways: As targets for our pseudoholomorphic curves, and as domains of our curves. In the target case, it is sufficient for our purposes to use the exhausting subset perspective, and here we will use the notation \( W_k \subset W_{k+1} \), etc. However, in the domain case we will definitely rely on the language and results of embedding diagrams and direct limit manifolds. Roughly speaking, we will employ a version of Gromov compactness with free boundary which, as a limit, yields a pseudoholomorphic curve with domain which is a compact manifold with boundary. By iterating this procedure we get a sequence of domains \( S_k \) which are larger and larger in the sense that there exists holomorphic embeddings \( \psi_k : S_k \rightarrow S_{k+1} \). Thus in order to construct the desired limit curve, we need the sequence \((S_k, \psi_k)\) to give rise to an embedding diagram, and hence a direct limit manifold.

2.2. Riemann surfaces. Here we aim to define a decorated marked nodal Riemann surface with boundary, as well as genus and arithmetic genus. All of these notions will be utilized in later sections.
**Definition 2.2** (compact region). Let $W$ be a manifold. Suppose $U \subset W$ is an open set for which its closure $\text{cl}(U)$ inherits from $W$ the structure of a smooth compact manifold possibly with boundary. Then we call $\text{cl}(U)$ a compact region in $W$.

**Definition 2.3** (almost complex structures). Let $W$ be a smooth finite dimensional manifold equipped with a smooth section of the endomorphism bundle $J \in \Gamma(\text{End}(TW))$ over $W$ for which $J \circ J = -\text{Id}$. We call $J$ an almost complex structure, and we call $(W, J)$ an almost complex manifold.

**Definition 2.4** (nodal Riemann surface). A nodal Riemann surface is a triple $(S, j, D)$, where $S$ is a real two-dimensional manifold, possibly with boundary, equipped with a smooth almost complex structure $j$. Furthermore, $D \subset S \setminus \partial S$, is an unordered discrete closed set of pairs $D = \{d_1, \overline{d_1}, d_2, \overline{d_2}, \ldots\}$ which we call nodal points, and the pairs $(d_i, \overline{d_i})$ we call nodal pairs. A marked nodal Riemann surface is the four-tuple $(S, j, \mu, D)$ where $(S, j, D)$ is a nodal Riemann surface, and where $\mu \subset S \setminus (D \cup \partial S)$ is a discrete closed set of points. It is worth noting that we allow for the possibility that either of $\mu$ or $D$ may be empty, finite, or infinite. That we allow either to be infinite is both novel and necessary for our applications, however by requiring that each set is both closed and discrete, we avoid the complicated issues arising from the existence of accumulations points of marked/nodal points. Additionally, we make the requirement that the set of special points $\mu \cup D$ is disjoint from the boundary. This is a non-standard condition, but relevant for our approach because our curves have free boundary; that is, no boundary condition is specified, and it will by natural to remove a small neighborhood of the boundary. As such, we preemptively guarantee that in doing so, we do not remove any special points.

**Definition 2.5** (Genus). Let $S$ be a connected compact two-dimensional manifold with boundary. We define $\text{Genus}(S)$ to be the genus of the surface obtained by capping off the boundary components of $S$ by disks. If $S$ is disconnected but compact, then we define $\text{Genus}(S) := \sum_{k=1}^n \text{Genus}(S_k)$ where the $S_k$ are the connected components of $S$. If $S$ is non compact (but with at most countably infinite connected components), we define $\text{Genus}(S) := \lim_{k \to \infty} \text{Genus}(S_k)$, where $S_1 \subset S_2 \subset S_3 \subset \cdots$ is an exhausting sequence of compact regions in $S$.

It is possible that a highly skeptical reader may be concerned that the above definition of genus may not be well defined, since a priori it could be the case that when $S$ is non-compact, the definition of $\text{Genus}(S)$ may depend upon the choice of exhausting sequence $\{S_k\}_{k \in \mathbb{N}}$. The proof of independence is elementary, so we do not provide it, but we mention the key ideas. First, observe that the desired independence follows from showing that whenever $S'$ and $S''$ are compact regions in $S$ satisfying

$$S' \subset \text{Int}(S'') \quad \text{and} \quad S'' \subset \text{Int}(S),$$

we must have

$$\text{Genus}(S') \leq \text{Genus}(S'').$$

\footnote{Recall that the notion of a compact region was provided in Definition 2.2.}
This in turn essentially follows from the fact that handle attaching only increases genus if both ends of the handle lie in the same connected component (before that attachment).

Before moving on, we will also need the concept of the arithmetic genus which we provide in Definition 2.7 below, but first we recall some additional notions. Associated to a nodal Riemann surface is the topological space $|S|$ defined by identifying a nodal point with the other point in its nodal pair; in other words, the space $S/(d_{v} \sim d_{v})$. As in Section 4.4 of [2], we define $S^{D}$ to be the oriented blow-up of $S$ at the points $D$, and we let $\Gamma_{\nu} := (T_{\Gamma}(S) \setminus \{0\})/\mathbb{R}_{+}^{*} \subset S^{D}$ and $\Sigma_{\nu} := (T_{\Sigma}(S) \setminus \{0\})/\mathbb{R}_{+}^{*} \subset S^{D}$ denote the newly created boundary circles over the $d_{v}$.

**Definition 2.6** (decorated marked nodal Riemann surface).

A decorated marked nodal Riemann surface is a tuple $(S, j, \mu, D, r)$ where $(S, j, \mu, D)$ is a marked nodal Riemann surface, and $r$ is a set of orientation reversing orthogonal maps $\tilde{r}_{\nu} : \Gamma_{\nu} \to \Sigma_{\nu}$ and $\tilde{r}_{\nu} : \Sigma_{\nu} \to \Gamma_{\nu}$, which we call decorations; here by orthogonal orientation reversing, we mean that $r_{\nu}(e^{i\theta}z) = e^{-i\theta}r_{\nu}(z)$ for each $z \in \Gamma_{\nu}$. We also define $S^{D, r}$ to be the smooth surface obtained by gluing the components of $S^{D}$ along the boundary circles $\{\Gamma_{1}, \Sigma_{1}, \Sigma_{2}, \Sigma_{3}, \ldots\}$ via the decorations $\tilde{r}_{\nu}$ and $\Sigma_{\nu}$. We will let $\Gamma_{\nu}$ denote the special circles $\Gamma_{\nu} = \Sigma_{\nu} \subset S^{D, r}$.

We are now prepared to state the definition of the arithmetic genus.

**Definition 2.7** (arithmetic genus).

Let $S = (S, j, \mu, D)$ be a marked nodal Riemann surface. As above, let $S^{D}$ be the oriented blow-up of $S$ at the points $D$, and let $S^{D, r}$ denote the surface obtained by gluing $S^{D}$ together along pairs of circles associated to pairs of nodal points. We define the arithmetic genus of $S$ to be the genus of $S^{D, r}$. That is,

$$\text{Genus}_{\text{arith}}(S) = \text{Genus}(S^{D, r}).$$

We note that it is more standard to define the arithmetic genus in terms of a formula involving the genera of connected components, number of marked points, number of nodal points, etc. It will be convenient for later applications to have the above definition at our disposal, however it is equivalent to the more standard formulaic definition. Indeed, we establish this in Appendix A.

2.3. Pseudoholomorphic curves. Here we will provide the definition of a pseudoholomorphic curve and a few other related notions like stability, generally immersed maps, boundary immersed maps, area, etc.

**Definition 2.8** (almost Hermitian structures).

Let $(W, J)$ be a smooth finite dimensional almost complex manifold, and let $g$ be a Riemannian metric. We say the pair $(J, g)$ is an almost Hermitian structure on $W$ provided that $J$ is an isometry for $g$. In such a case we call $(W, J, g)$ an almost Hermitian manifold.

**Definition 2.9** (marked nodal pseudoholomorphic curve).

A marked nodal pseudoholomorphic curve is a tuple $u = (u, S, j, W, J, \mu, D)$ with entries as follows. The triple $(S, j, \mu, D)$ is a marked nodal Riemann surface. The pair $(W, J)$ is a smooth real $2n$-dimensional almost complex manifold, and $u : S \to W$ is a smooth map for which $J \cdot Tu = Tu \cdot j$. Finally, we require that $u(\overline{d}_{i}) = u(d_{i})$ for all $i \in \mathbb{N}$. 

---

This page contains definitions and theorems related to nodal Riemann surfaces, arithmetic genus, and pseudoholomorphic curves. It discusses the concept of handle attaching and its effect on the genus of a surface, and introduces the notion of a decorated marked nodal Riemann surface. The arithmetic genus is defined, and its formulaic definition is provided. The text also introduces the concept of an almost Hermitian structure and a marked nodal pseudoholomorphic curve.
Unless otherwise specified, we will allow $S$ to be non-compact, to have smooth boundary, and to have unbounded topology (i.e. countably infinite connected components, boundary components, and genus). We will say that a pseudoholomorphic curve $u$ is \textit{compact} provided $S$ has the structure of a compact manifold with smooth boundary, we will say $u$ is \textit{closed} provided $S$ has the structure of a compact manifold without boundary, and we will say $u$ is \textit{connected} provided that $|S|$ is connected.

**Definition 2.10** (decorated pseudoholomorphic curve).

A decorated marked nodal pseudoholomorphic curve $(u,r)$ is a pair for which $u = (u,S,j,W,J,\mu,D)$ is a marked nodal pseudoholomorphic curve and $(S,j,\mu,D,r)$ is a decorated marked nodal Riemann surface. With $S^{D,r}$ defined as above, we observe that the smooth map $u : S \to W$ lifts to a continuous map $u : S^{D,r} \to W$.

**Definition 2.11** (generally immersed).

Let $u = (u,S,j,W,J,\mu,D)$ be a marked nodal pseudoholomorphic curve. We shall say $u$, or $u : S \to W$, is \textit{generally immersed} provided that the set of critical points of $u : S \to W$ has no accumulation point.

**Definition 2.12** (stable pseudoholomorphic curve).

A compact marked nodal pseudoholomorphic curve $(u,S,j,W,\mu,D)$ is said to be \textit{stable} if and only if for each connected component $\bar{S}$ of $S$ at least one of the following is true:

1. The restricted map $u|_{\bar{S}} : \bar{S} \to W$ is non-constant.
2. $\chi(\bar{S}) - \#(\bar{S} \cap \mu) - \#(\bar{S} \cap D) < 0$.

A non-compact marked nodal pseudoholomorphic curve $(u,S,j,W,\mu,D)$ is said to be stable if and only if there exists a sequence $\{S_k\}_{k \in \mathbb{N}}$ of compact real two dimensional manifolds, possibly with smooth boundary, with the following properties.

1. for each $k \in \mathbb{N}$ we have $S_k \subset \text{Int}(S_{k+1}) \subset S$
2. $\cup_{k=1}^{\infty} S_k = S$ and $(\cup_{k=1}^{\infty} \partial S_k) \cap (\mu \cup D) = \emptyset$
3. for each $k \in \mathbb{N}$ the pseudoholomorphic curve $(u,S_k,j,W,\mu \cap S_k,D \cap S_k)$ is stable.

Note that if $(u,S,j,W,\mu,D)$ is a compact marked nodal pseudoholomorphic curve, and $\bar{S}$ is a connected component for which $\chi(\bar{S}) - \#(\bar{S} \cap \mu) - \#(\bar{S} \cap D) < 0$, then there exists a unique complete finite area hyperbolic metric of constant curvature $-1$ on $S' := S \setminus (\mu \cup D)$ which is in the same conformal class as $j$ and for which each connected component of $\partial S$ is a geodesic; we denote this metric by $h^{j,\mu\cup D}$.

**Definition 2.13** (boundary-immersed pseudoholomorphic curve).

A compact marked nodal pseudoholomorphic curve $(u,S,j,W,\mu,D)$ is said to be boundary-immersed if and only if either $\partial S = \emptyset$ or else the restricted map $u|_{\partial S} : \partial S \to W$ is an immersion.

**Lemma 2.14** (A dichotomy).

Let $u = (u,S,j,W,J,\mu,D)$ be a proper boundary-immersed marked nodal pseudoholomorphic curve mapping into the almost Hermitian manifold $(W,J,g)$ which has no boundary. Then for each connected component $\bar{S} \subset S$, the restricted map $u|_{\bar{S}}$ is a bijection onto $W$. Consequently, $u : S \to W$ is an immersion.
$u|_{\tilde{S}} : \tilde{S} \to W$ is either a constant map or else it is generally immersed in the sense of Definition 2.11.

Proof. For each connected component $\tilde{S}$ of $S$ we note their are two possible cases: either the set of critical points of $u|_{\tilde{S}} : \tilde{S} \to W$ has an accumulation point, or it does not. Because $u$ is boundary-immersed, it follows that all accumulation points are interior points. However, recall that any pseudoholomorphic map $u : \tilde{S} \to W$ with connected domain and with an interior accumulation point of critical points must be a constant map; for details, see Lemma 2.4.1 from [13]. The result is then immediate. □

Definition 2.15 (area of pseudoholomorphic curves).
Let $u = (u,S,j,W,J,\mu,D)$ be a proper boundary-immersed marked nodal pseudoholomorphic curve. Let $S_{\text{const}} \subset S$ denote the union of connected components of $S$ on which $u$ is a constant map. Then by Lemma 2.14 it follows that the map $u : S \setminus S_{\text{const}} \to W$ is generally immersed in the sense of Definition 2.11. Consequently on $S \setminus S_{\text{const}}$ we can define the following metric

$$\text{dist}_{u^*g}(\zeta_0,\zeta_1) := \inf \left\{ \int_0^1 (\dot{\gamma}(t),\dot{\gamma}(t))^{\frac{1}{2}}_{u^*g} dt : \gamma \in C^1([0,1],S) \text{ and } \gamma(i) = \zeta_i \right\},$$

where our convention will be that if $\zeta_0$ and $\zeta_1$ lie in different connected components, then $\text{dist}_{u^*g}(\zeta_0,\zeta_1) := \infty$. Thus we may regard $(S \setminus S_{\text{const}},\text{dist}_{u^*g})$ as a metric space, in which case it can be equipped with Hausdorff measures $H^k$. Note that if $O \subset S \setminus S_{\text{const}}$ is an open set on which $u$ is an immersion, then $H^2(O) = \text{Area}_{u^*g}(O)$. As such, our convention will be to simply define the area of an arbitrary open set $U \subset S \setminus S_{\text{const}}$ to be $\text{Area}_{u^*g}(U) := H^2(U)$. Finally, for an arbitrary open set $U \subset S$ we define

$$\text{Area}_{u^*g}(U) := H^2(U \setminus S_{\text{const}}).$$

2.4. Convergence of pseudoholomorphic curves. Here we provide a few notions of convergence of pseudoholomorphic curves. We start with the well known definition of Gromov convergence adapted to the case of having free boundary. We also recall the definition of robust $K$-convergence in a Gromov sense, which is taken from [6]. And finally, we provide the novel notion of convergence in an exhaustive Gromov sense, given in Definition 2.21.

We begin with the notion of Gromov convergence, which we adapt slightly to allow our curves to have free boundary.

Definition 2.16 (Gromov convergence).

A sequence $u_k = (u_k,S_k,j_k,W,J_k,\mu_k,D_k)$ of compact marked nodal stable boundary-immersed pseudoholomorphic curves is said to converge in a Gromov-sense to a compact marked nodal stable boundary-immersed pseudoholomorphic curve $u = (u,S,j,W,J,\mu,D)$ provided the following are true for all sufficiently large $k \in \mathbb{N}$.

1. $J_k \to J$ in $C^\infty$.

2. There exist sets of marked points

$$\mu'_k \subset S_k \setminus (\partial S_k \cup \mu_k \cup D_k) \quad \text{and} \quad \mu' \subset S \setminus (\partial S \cup \mu \cup D)$$

with the property that $\# \mu' = \# \mu'_k < \infty$, and with the property that for each connected component $\tilde{S}_k$ of $S_k$ we have

$$\chi(\tilde{S}_k) - \#(\tilde{S}_k \cap (\mu_k \cup \mu'_k \cup D_k)) < 0$$
and for each connected component \( \tilde{S} \) of \( S \) we have
\[
\chi(\tilde{S}) - \#(\tilde{S} \cap (\mu \cup \mu' \cup D)) < 0.
\]

(3) There exists a decoration \( r \) for \( u \), a sequence of decorations \( r_k \) for the \( u_k \), and sequences of diffeomorphisms \( \phi_k : S^{D,r} \to S_k^{D,r_k} \) such that the following hold:

(a) \( \phi_k(\mu) = \mu_k \)

(b) \( \phi_k(\mu') = \mu'_k \)

(c) for each \( i = 1, \ldots, \delta \) the curve \( \phi_k(\Gamma_i) \) is a \( h_{jk}^{\mu_k,\mu'_k} \)-geodesic in the punctured surface \( S'_k := S_k \setminus (\mu_k \cup \mu'_k \cup D_k) \).

(4) \( \phi_k^* h_{jk}^{\mu_k,\mu'_k} D_k \to h_{jk}^{\mu,\mu'} D \) in \( C^\infty(S^{D,r} \setminus (\mu \cup \mu' \cup \Gamma_i)) \); here we have abused notation by letting \( h_{jk}^{\mu,\mu'} D \) also denote its lift to \( S^{D,r} \).

(5) \( \phi_k^* u_k \to u \) in \( C^0(S^{D,r}) \).

(6) \( \phi_k^* u_k \to u \) in \( C^\infty_{\text{loc}}(S^{D,r} \setminus \cup \Gamma_i) \).

(7) For each connected component \( \Lambda \) of \( \partial \tilde{S} \), the \( \phi_k^* h_{jk}^{\mu_k,\mu'_k} D_k \)-length of \( \Lambda \) is uniformly bounded away from 0 and \( \infty \).

In general, one should not expect a sequence of pseudoholomorphic curves with free boundary to converge in a Gromov sense – even when the curves are compact with bounded area, genus, connected components, etc. However, if we are allowed to “trim away” some portion of those curves outside some compact set \( K \), then the area and topology bounds are indeed sufficient to extract a subsequence for the trimmed curves. We make this statement precise in Theorem 2 below, which was proved in [3], but first it necessitates the statement of Definition 2.17 (\( K \)-proper sequence of pseudoholomorphic curves) and Definition 2.19 (robust \( K \)-convergence in Gromov sense).

**Definition 2.17 (\( K \)-proper sequence of pseudoholomorphic curves).**

Consider a sequence of maps \( u_k : S_k \to W \) to and from manifolds which possibly have boundary and may be non-compact. Let \( K \subset \text{Int}(W) \) be a compact set in the interior of \( W \). We call \( \{u_k\}_{k \in \mathbb{N}} \) a robustly \( K \)-proper sequence provided there exists another compact set \( \tilde{K} \subset \text{Int}(W) \) for which \( K \subset \text{Int}(\tilde{K}) \) and if \( u_k^{-1}(\tilde{K}) \setminus \partial S_k \) is compact for all \( k \in \mathbb{N} \). Similarly a single map \( u : S \to W \) is robustly \( K \)-proper provided the constant sequence \( u, u, u, \ldots \) is robustly \( K \)-proper.

**Remark 2.18.** Definition 2.17 above is essentially a restatement of Definition 2.3 from [3]. More precisely, the two definitions are equivalent but stated slightly differently. Indeed, where Definition 2.17 states:

“We call \( \{u_k\}_{k \in \mathbb{N}} \) a robustly \( K \)-proper sequence provided there exists another compact set \( \tilde{K} \subset \text{Int}(W) \) for which \( K \subset \text{Int}(\tilde{K}) \) and if \( u_k^{-1}(\tilde{K}) \setminus \partial S_k \) is compact for all \( k \in \mathbb{N} \).”

Definition 2.3 states:

“We call \( \{u_k\}_{k \in \mathbb{N}} \) a robustly \( K \)-proper sequence provided there exists another compact set \( \tilde{K} \subset \text{Int}(W) \) for which \( K \subset \text{Int}(\tilde{K}) \) and if \( u_k^{-1}(\tilde{K}) \setminus \partial S_k \) is compact for every compact set \( \tilde{K} \subset \tilde{K} \).”

Because \( \tilde{K} \) is a compact set contained in \( \tilde{K} \), it follows that any sequence of \( u_k \) which satisfy Definition 2.3 will also satisfy Definition 2.17. Now suppose that \( u_k \) is a sequence satisfying Definition 2.17, so that \( u_k^{-1}(\tilde{K}) \setminus \partial S_k \) is compact. Let \( \tilde{K} \subset \tilde{K} \).
be a compact set. Because $W$ is a manifold, $W$ is metrizable, and every compact set in a metrizable space is closed, so that $\tilde{K}$ is closed, and hence $u_k^{-1}(\tilde{K}) \setminus \partial S_k$ is closed in the subspace topology of $S_k \setminus \partial S_k$. However, $u_k^{-1}(\tilde{K}) \setminus \partial S_k \subset u_k^{-1}(\tilde{K}) \setminus \partial S_k$ with the subset closed and the superset compact, so that the subset is compact. In other words, for each compact $\hat{K} \subset K$ we have $u_k^{-1}(\hat{K}) \setminus \partial S_k$ is compact, and thus the sequence $u_k$ satisfies Definition 2.3 whenever it satisfies Definition 2.17. We conclude that the two definitions are indeed equivalent.

With this definition in hand, we can now provide the notion of robust $K$-convergence in a Gromov sense. We note that the following definition was originally provided in \ surpassed, although we have slightly modified it here to allow for the possibility that the sequence of curves has marked and nodal points.

**Definition 2.19 (robust $K$-convergence in Gromov sense).**

Consider an almost Hermitian manifold given by $(W, J, g)$, a sequence of almost Hermitian structures $(J_k, g_k)$ for which $(J_k, g_k) \to (J, g)$ in $C^\infty_{\text{loc}}$, a compact set $K \subset \text{Int}(W)$, and a robustly $K$-proper sequence of marked nodal pseudoholomorphic curves $u_k = (u_k, S_k, j_k, W, J_k, \mu_k, D_k)$. We say that $u_k$ robustly $K$-converges in a Gromov sense provided there exists a compact set $\tilde{K} \subset \text{Int}(W)$ for which $K \subset \text{Int}(\tilde{K})$, and there exist compact regions $\tilde{S}_k \subset S_k$ with the property that $u_k(S_k \setminus \tilde{S}_k) \subset W \setminus \tilde{K}$ for all $k \in \mathbb{N}$, $(\mu_k \cup D_k) \cap \partial S = \emptyset$ for all $k \in \mathbb{N}$, and the domain restricted pseudoholomorphic curves

$$\tilde{u}_k := (u_k, \tilde{S}_k, j_k, W, J_k, \mu_k \cap \tilde{S}_k, D_k \cap \tilde{S}_k)$$

are stable in the sense of Definition 2.12 and converge in a Gromov sense to a stable compact marked nodal boundary-immersed pseudoholomorphic curve $u := (u, S, j, W, J, \mu, D)$.

We additionally require that the sequence of marked points $\mu'_k$ added to the $(\tilde{S}, j_k)$ to obtain Gromov convergence are chosen so that lengths of each connected component of $\partial \tilde{S}_k$, computed with respect to the associated Poincaré metric $h^{j_k,\mu'_k \cup D_k}$, are uniformly bounded away from zero and infinity.

Before providing the definition of convergence in an exhaustive Gromov sense, we will need the following notion, which is a special case of the embedding diagrams of Section 2.1.

**Definition 2.20 (properly exhausting regions).**

Let $(\overline{W}, \overline{J}, \overline{g})$ be an almost Hermitian manifold, which need not be compact. We say a sequence of almost Hermitian manifolds $(W_k, J_k, g_k)$ properly exhaust $(\overline{W}, \overline{J}, \overline{g})$ provided the following hold.

1. For each $k \in \mathbb{N}$ we have $W_k \subset W_{k+1}$, and moreover $W_k$ is an open subset of $W_{k+1}$ in the $W_{k+1}$ topology.
2. $\overline{W} = \bigcup_{k \in \mathbb{N}} W_k$
3. The smooth structure on $W_k$ equals the smooth structure induced from $W_{k+1}$.
4. The set $\text{cl}(W_k) \subset W_{k+1}$ is a compact manifold with smooth boundary.
5. Regarding $(J_k, g_k)$ as almost Hermitian structures on $\overline{W}$, we require $(J_k, g_k) \to (\overline{J}, \overline{g})$ in $C^\infty_{\text{loc}}$. 
We are now prepared to state the novel definition of convergence of pseudoholomorphic curves in an exhaustive Gromov sense.

**Definition 2.21** (convergence in an exhaustive Gromov sense).

Let \((\overline{W}, \overline{J}, \overline{g})\) be a smooth almost Hermitian manifold, not necessarily compact, and let \((W_k, J_k, g_k)\) be a sequence which properly exhausts \((\overline{W}, \overline{J}, \overline{g})\), in the sense of Definition 2.20. Suppose further that the tuples \(\bar{u} = (\bar{u}, \bar{S}, \bar{j}, \bar{W}, \bar{J}, \bar{\mu}, \bar{D})\) and, for each \(k \in \mathbb{N}\), \(u_k = (u_k, S_k, j_k, W_k, J_k, \mu_k, D_k)\), are each marked nodal proper stable pseudoholomorphic curves without boundary. We say the sequence \(\{u_k\}_{k \in \mathbb{N}}\) converges to \(\bar{u}\) in an exhaustive Gromov sense provided there exists a collection of compact smooth two dimensional manifolds with boundary \(\{\overline{S}_\ell\}_{\ell \in \mathbb{N}}\) with \(\overline{S} \subset \overline{S}_\ell\) for each \(\ell \in \mathbb{N}\), and there exists a collection of compact smooth two dimensional manifolds with boundary \(\{S_k^\ell\}_{k \in \mathbb{N}}\) with \(S_k^\ell \subset S_k\) for all \(k, \ell \in \mathbb{N}\) with \(\ell \geq \ell\) for which the following hold.

1. \(\overline{S}_\ell \subset \overline{S} + 1 \setminus \partial \overline{S}^\ell + 1\) for all \(\ell \in \mathbb{N}\)
2. \(\overline{S} = \bigcup_{\ell \in \mathbb{N}} \overline{S}_\ell\)
3. for each fixed \(k \in \mathbb{N}\) and each \(0 \leq \ell \leq k - 1\) we have \(S_k^\ell \subset S_k^\ell + 1 \setminus \partial S_k^\ell + 1\)
4. for each \(k \geq \ell \in \mathbb{N}\) we have 
   \[u_k^{-1}(W_\ell) \subset S_k^\ell,\]
5. for each fixed \(\ell \in \mathbb{N}\), the sequence 
   \[\{(u_k, S_k^\ell, j_k, W_k, J_k, S_k^\ell \cap \mu_k, S_k^\ell \cap D_k)\}_{k \geq \ell}\]
   is a sequence of compact marked nodal stable boundary-immersed pseudoholomorphic curves which converges in a Gromov sense to the proper marked nodal stable boundary-immersed pseudoholomorphic curve 
   \[(\bar{u}, \overline{S}_\ell, \bar{j}, \bar{W}, \bar{J}, S_\ell^\ell \cap \bar{\mu}, \overline{S}_\ell \cap \bar{D})\].

Before moving on to the next section, in which we prove exhaustive Gromov compactness, we first address two issues which establish that our notion of exhaustive convergence is well defined. The first is that limits are unique, and the second is that subsequences converge to the same limit. We handle both of these results with Lemma 2.22 below.

**Lemma 2.22** (properties of the exhaustive Gromov limit).

Let \((\overline{W}, \overline{J}, \overline{g})\) be a smooth almost Hermitian manifold, not necessarily compact, and let \((W_k, J_k, g_k)\) be a sequence which properly exhausts \((\overline{W}, \overline{J}, \overline{g})\). Suppose that \(u_k = (u_k, S_k, j_k, W_k, J_k, \mu_k, D_k)\) is a sequence of pseudoholomorphic curves which converges in an exhaustive sense to both of the limit curves \(\bar{u} = (\bar{u}, \bar{S}, \bar{j}, \bar{W}, \bar{J}, \bar{\mu}, \bar{D})\) and \(\hat{u} = (\hat{u}, \hat{S}, \hat{j}, \hat{W}, \hat{J}, \hat{\mu}, \hat{D})\). Then there exists a holomorphic diffeomorphism \(\varphi : (\overline{S}, \overline{J}, \overline{\mu}, \overline{D}) \rightarrow (\overline{S}, \overline{J}, \overline{\mu}, \overline{D})\) satisfying \(\varphi \circ \varphi = u\). In addition, any subsequence of the \(u_k\) also converges to \(\bar{u}\).

**Proof.** The first conclusion follows immediately from the definition of exhaustive Gromov convergence, the standard Gromov convergence for compact pseudoholomorphic curves with smooth boundary, and that direct limits of embedding diagrams are unique up to diffeomorphisms preserving additional tensors (like almost complex structures). The second conclusion follows from these results together with
the fact that subsequences of embedding diagrams have the same direct limit. See Section 2.1 for details.

With all these preliminaries established, we can now state the main result of this manuscript.

**Theorem 1** (exhaustive Gromov compactness).

Let \((W, J, \bar{g})\) be a smooth almost Hermitian manifold, not necessarily compact, and let \((W_k, J_k, g_k)\) be a sequence which properly exhausts \((W, J, \bar{g})\), in the sense of Definition 2.20. Suppose further that the sequence denoted by 
\[\{u_k\}_{k \in \mathbb{N}} = \{(u_k, S_k, j_k, W_k, J_k, \mu_k, D_k)\}_{k \in \mathbb{N}}\]
is a sequence of proper stable marked nodal pseudoholomorphic curves without boundary for which there also exists a sequence of large constants \(C_k\) with the property that for each fixed \(k \in \mathbb{N}\) the following hold

\[\begin{align*}
(C1) & \quad \sup_{\ell \geq k} \text{Area}_{u_{\ell}^* g_{\ell}}(\hat{S}_k^\ell) \leq C_k \\
(C2) & \quad \sup_{\ell \geq k} \text{Genus}(\hat{S}_k^\ell) \leq C_k \\
(C3) & \quad \sup_{\ell \geq k} \#((\mu_\ell \cup D_\ell) \cap \hat{S}_k^\ell) \leq C_k
\end{align*}\]

where \(\hat{S}_k^\ell := u_\ell^{-1}(W_k)\). Then a subsequence converges in an exhaustive Gromov sense to \((\bar{u}, \bar{S}, \bar{j}, \bar{W}, \bar{J}, \bar{\mu}, \bar{D})\) which is a proper stable marked nodal pseudoholomorphic curve without boundary.

3. Proof of exhaustive Gromov compactness

The main purpose of this section is to establish our main result, namely the validity of exhaustive Gromov compactness; see Theorem 1 below. Before doing so, we first state the target-local Gromov compactness theorem from \[6\], and generalize it to handle the case that the curves in question are marked, nodal, and have constant components.

**Theorem 2** (Target-local Gromov compactness).

Let \((M, J, g)\) be an almost Hermitian manifold, and let \((J_k, g_k)\) be a sequence of almost Hermitian structures which converge in \(C^\infty\) to \((J, g)\). Also let \(K \subset \text{Int}(M)\) be a compact region, and let \(u_k\) be a sequence of generally immersed \(J_k\)-curves which are robustly \(K\)-proper and satisfy

\[\begin{align*}
(1) & \quad \text{Area}_{u_k^* g_k}(S_k) \leq C_A < \infty \\
(2) & \quad \text{Genus}(S_k) \leq C_G < \infty
\end{align*}\]

Then a subsequence robustly \(K\)-converges in a Gromov sense.

*Proof.* This is nothing more than a restatement of Theorem 3.1 from \[6\].

**Corollary 3.1** (Target-local Gromov compactness for marked nodal stable curves).

Let \((W, J, g)\) be an almost Hermitian manifold, possibly with boundary, and let \((J_k, g_k)\) be a sequence of almost Hermitian structures which converge in \(C^\infty\) to \((J, g)\). Also let \(K^- \subset \text{Int}(W)\) be compact regions, satisfying \(K^- \subset \text{Int}(K^+)\), and let \(u_k = (u_k, S_k, j_k, W, J_k, \mu_k, D_k)\) be a sequence of stable marked nodal pseudoholomorphic curves satisfying \(u_k(\partial S_k) \cap K^+ = \emptyset\) and suppose there exists a large positive constant \(C > 0\) for which

\[\begin{align*}
(1) & \quad \text{Area}_{u_k^* g_k}(S_k) \leq C
\end{align*}\]
(2) Genus($S_k$) $\leq C$, 
(3) $\#(\mu_k \cup D_k) \leq C$

Then, after passing to a subsequence (still denoted with subscripts $k$), there exist compact surfaces with boundary $\bar{S}_k \subset S_k$ with the following properties

(1) the following are compact pseudoholomorphic curves

$$ (u_k, \bar{S}_k, j_k, \mu_k \cap \bar{S}_k, D_k \cap \bar{S}_k) $$

(2) these domain-restricted converge in a Gromov sense to a compact stable marked nodal boundary immersed pseudoholomorphic curve.

(3) $u_k(S_k \setminus \bar{S}_k) \subset W \setminus K^-$

Before proceeding with the proof, we comment on its statement and conclusions. To that end, we view the result in light of an example. Consider the compact unit ball in $\mathbb{R}^4$ contained inside a concentric compact ball of radius three. Also consider a sequence of compact pseudoholomorphic curves (with boundary) with images in the larger ball, and which map the boundary of the domains to a small neighborhood of the boundary of the large ball. Assuming no nodes, no marked points, no genus, but assuming area bounds, does a subsequence converge? Without trimming the curves, the answer is immediately no; counter-examples are easy to find. Thus Corollary 3.1 and Theorem 2 both guarantee that one can find a trimming which guarantees that a subsequence converges. However, these results also guarantee that the portions of the domains that get trimmed away are not in the region of interest. To elaborate on this point, let $K^-$ be the compact unit ball in our example, and let $K^+$ be the compact ball of radius two. The region of interest will be $K^-$. The third conclusion of Corollary 5.1 is that the image of the $S_k \setminus \bar{S}_k$ by $u_k$ (that is, the image of the portions of the curves that get trimmed away) live in $W \setminus K^-$; that is, these sets live outside the unit ball. In practice, one will be interested in the portion of curves that has image in $B_1(0) = K^-$, so the hypotheses demand that our curves have boundary outside the even larger set $B_2(0) = K^+$, and the conclusions guarantee that we only trim away portions that have image in $B_3(0) \setminus B_1(0) = W \setminus K^-$, which keeps intact those portions in which we are interested.

Note: There is an odd case which the above result covers as well, which we describe at present. Namely, it is possible that the curves in the initial sequence have images always contained in $W \setminus K^+$, in which case it could be the case that $S_k \setminus \bar{S}_k = \emptyset$. That is, the entire curve is trimmed away, leaving nothing but empty curves. We allow this as a possibility, and note that the above result is highly non-trivial because of the condition that $u_k(S_k \setminus \bar{S}_k) \subset W \setminus K^-$. Indeed, this latter conclusion guarantees that if $u_k(S_k) \cap K^- \neq \emptyset$ for all $k \in \mathbb{N}$, then $\bar{S}_k \neq \emptyset$ for all $k \in \mathbb{N}$, and hence the limit curve is non-empty (by definition of Gromov convergence). The key idea is that one is interested in the portion of the curves that have image in $K^-$; if no part of the curves have image in that region, then there is no part of the curves in which we are interested, and trimming down to the empty set immediately achieves the desired result, since it is trivial to show a sequence of empty curves converges. On the other hand, the condition that $u_k(S_k \setminus \bar{S}_k) \subset W \setminus K^-$ guarantees that any portion of the curves in which we are interested are not trimmed away.

We now proceed with the proof.
Proof. As a preliminary step, we let $S^\text{const}_k$ denote the union of the connected components of $S_k$ on which $u_k$ is locally constant. We then define the sequence of sets

$$\sigma_k := u_k(\mu_k \cup D_k \cup S^\text{const}_k).$$

Note that these are finite sets with boundedly many elements, and hence we may pass to a subsequence, still denoted with subscripts $k$, so that they converge in the following sense. There exists a finite set $\sigma \subset W$ with the property that for each compact set $K \subset W$ and each $\epsilon > 0$ there exists a $k' = k'(\epsilon, K) \in \mathbb{N}$ such that for all $k \geq k'$ we have

$$\sigma_k \subset (W \setminus K) \cup \bigcup_{p \in \sigma} B^\epsilon_g(p)$$

where $B^\epsilon_g(p)$ is an open $\bar{g}$-metric ball of radius $\epsilon$ centered at the point $p$. We then fix compact regions $\hat{W}^-, \hat{W}^+ \subset W$, so that

$$K^- \subset \text{Int}(\hat{W}^-), \quad \hat{W}^- \subset \text{Int}(\hat{W}^+), \quad \text{and} \quad \hat{W}^+ \subset \text{Int}(K^+)$$

and

$$\sigma \cap \left(\hat{W}^+ \setminus \text{Int}(\hat{W}^-)\right) = \emptyset.$$ 

This is possible essentially because $\sigma$ is finite and due to properties of a compact region; see Definition 2.2. We then define

$$\hat{S}_k := u_k^{-1}\left(\text{Int}(\hat{W}^+)\right) \setminus S^\text{const}_k$$

for each $k \in \mathbb{N}$, as well as the pseudoholomorphic curves

$$\hat{u}_k = (u_k, \hat{S}_k, j_k, \text{Int}(\hat{W}^+), J_k, \emptyset, \emptyset).$$

With this established, we see by Lemma 2.14 that each $\hat{u}_k$ is generally immersed, and hence Theorem 2 applies to this sequence. Consequently, we pass to the subsequence (still denoted with subscripts $k$) which robustly $K$-converges in a Gromov sense, with $K = \hat{W}^-$. As such, we let $\hat{K} \subset \text{Int}(\hat{W}^+ \subset W$ be the compact region for which $\hat{W}^- \subset \text{Int}(\hat{K})$, and let $\hat{S}_k \subset \hat{S}_k$ denote the compact regions guaranteed by Theorem 2 for which $u(\hat{S}_k \setminus \hat{S}_k) \subset W \setminus \hat{K}$ so that the (sub)sequence of compact (marked nodal) boundary-immersed pseudoholomorphic curves

$$\hat{u}_k = (u_k, \hat{S}_k, j_k, \hat{W}, J_k, \emptyset, \emptyset)$$

converges in a Gromov sense to a compact (marked nodal) boundary-immersed pseudoholomorphic curve.

At this point, we let $\hat{S}^\text{const}_k := S^\text{const}_k \cap u_k^{-1}(\hat{W}^+)$; or in words, we let $\hat{S}^\text{const}_k$ denote the connected components of $S_k$ on which $u_k$ is locally constant and takes values in $\hat{W}^+$.

We then define

$$\hat{S}_k := \hat{S}_k \cup \hat{S}^\text{const}_k, \quad \hat{\mu}_k := \mu_k \cap \hat{S}_k, \quad \text{and} \quad \hat{D}_k := D_k \cap \hat{S}_k$$

for each $k \in \mathbb{N}$, and we consider the sequence of compact marked nodal boundary-immersed stable pseudoholomorphic curves

$$\hat{u}_k = (u_k, \hat{S}_k, j_k, \hat{W}, J_k, \hat{\mu}_k, \hat{D}_k).$$

It is important to note that for all sufficiently large $k$, these curves are nodal; that is, that the $\hat{D}_k$ are indeed sets of nodal pairs in $\hat{S}_k$. A priori, the concern is that by trimming the curves from $S_k$ to $\hat{S}_k \cup \hat{S}^\text{const}_k$ and again from $\hat{S}_k \cup \hat{S}^\text{const}_k$ to $\hat{S}_k$, we may have “trimmed away” one but not both points in a nodal pair. We note that this is not possible for the trimming from $S_k$ to $\hat{S}_k \cup \hat{S}^\text{const}_k$ since $\hat{S}_k \cup \hat{S}^\text{const}_k = u_k^{-1}(\text{Int}(\hat{W}^+))$, and since nodal pairs $\{\underline{d}, \overline{d}\}$ satisfy $u_k(\underline{d}) = u_k(\overline{d})$. 

---

[1] J.W. Fish and H. Hofer, "Stable Pseudoholomorphic Curves," arXiv:2201.01234 [math.SG].
It is also not possible to split a nodal pair by trimming from $\hat{S}_k \cup \hat{S}_k^{\text{const}}$ to $\hat{S}_k$ for sufficiently large $k$ because

\[ u_k(\hat{S}_k \cup \hat{S}_k^{\text{const}} \setminus \hat{S}_k) \subset \text{Int}(\hat{W}^+) \setminus \hat{W}^- \quad \text{and} \quad u_k(\mu_k \cup D_k \cup S_k^{\text{const}}) = \sigma_k \to \sigma, \]

and

\[ \sigma \cap (\hat{W}^+ \setminus \text{Int}(\hat{W}^-)) = \emptyset. \]

Returning our attention to the curves provided in equation (2), we note that a further subsequence of this sequence converges in a Gromov sense to a stable compact marked nodal boundary-immersed pseudoholomorphic curve. Indeed, the Gromov convergence follows from a standard straightforward argument which we briefly sketch. First, we add sequences of marked points $\mu'_k \subset (\hat{\mu}_k \cup \hat{\mu}_k \cup D_k)$ to stabilize the underlying domains. Note that only boundedly many points need to be added to each curve since the number of connected components of the $S_k$ on which $u_k$ is locally constant is uniformly bounded; this follows from the fact that curves given in the hypotheses of Corollary 3.1 are stable and the fact that $\#\mu_k + \#D_k < C$. This guarantees the existence of the associated finite area hyperbolic metrics $h_{\mu_k \cup \hat{\mu}_k \cup D_k}$ on the punctured surfaces $\hat{S}_k \setminus (\mu_k \cup \hat{\mu}_k \cup D_k)$ where the boundaries are geodesics. Standard bubbling analysis follows, in which one further adds marked points as needed so gradient bounds (associated to the hyperbolic metric) are obtained for these maps. Note that because we guaranteed that $\sigma \cap (\hat{W}^+ \setminus \text{Int}(\hat{W}^-)) = \emptyset$, and because the curves $\tilde{u}_k = (u_k, \hat{S}_k, j_k, W, J_k, \emptyset, \emptyset)$ converge in a Gromov sense to a boundary-immersed curve $\tilde{u}_\infty$ for which we have $\tilde{u}_\infty(\partial\hat{S}_\infty) \subset \text{Int}(\hat{W}^+) \setminus \hat{W}^-$, it follows that there exist annular neighborhoods of the $\partial \hat{S}_k$ with moduli uniformly bounded away from zero which are disjoint from both $\hat{\mu}_k \cup \hat{D}_k$. Furthermore because the $\tilde{u}_k$ converge in a Gromov sense, we have gradient bounds for the maps in these same annular neighborhoods of the boundary. It then follows that the additional marked points $\mu'_k$ can be chosen so that the lengths of the boundary components of the $\hat{S}_k$ are uniformly bounded away from zero and infinity. To complete the sketch of Gromov convergence, we note that the desired reparameterizations are given by the Uniformization theorem; the $C^\infty$ convergence away from nodes is given by by elliptic regularity (gradient bounds imply $C^\infty$ bounds); and $C^0$ convergence across the nodes follows from an application of the Monotonicity lemma.

We have thus established that after passing to a subsequence, still denoted with subscripts $k$, the curves provided in equation (2) converge in a Gromov sense to a compact stable marked nodal boundary-immersed pseudoholomorphic curve, and hence all that remains to show is that $u_k(S_k \setminus \hat{S}_k) \subset W \setminus \mathcal{K}^-$. However, this follows immediately from the following facts:

\[ \hat{S}_k = \hat{S}_k \cup \hat{S}_k^{\text{const}} \]
\[ \hat{S}_k^{\text{const}} = S_k^{\text{const}} \cap u_k^{-1}(\hat{W}^+) \]
\[ u(\hat{S}_k \setminus \hat{S}_k) \subset W \setminus \hat{\mathcal{K}} \]
\[ \mathcal{K}^- \subset \hat{W}^- \subset \text{Int}(\hat{\mathcal{K}}) \subset \hat{W}^+ \]

This completes the proof of Corollary 3.1. □
We now aim to prove the main result of this manuscript, namely Theorem 1, which we first restate for the reader’s convenience.

**Theorem 1** (exhaustive Gromov compactness). Let \((\mathbb{W}, \mathcal{J}, \tilde{g})\) be a smooth almost Hermitian manifold, not necessarily compact, and let \((W_k, J_k, g_k)\) be a sequence which properly exhausts \((\mathbb{W}, \mathcal{J}, \tilde{g})\), in the sense of Definition 2.20. Suppose further that the sequence denoted by

\[\{u_k\}_{k \in \mathbb{N}} = \{(u_k, S_k, J_k, W_k, J_k, \mu_k, D_k)\}_{k \in \mathbb{N}}\]

is a sequence of proper stable marked nodal pseudoholomorphic curves without boundary for which there also exists a sequence of large constants \(C_k\) with the property that for each fixed \(k \in \mathbb{N}\) the following hold

(C1) \(\sup_{i \geq k} \text{Area}_{\mathbb{W}}(\tilde{S}_k^i) \leq C_k\)

(C2) \(\sup_{i \geq k} \text{Genus}(\tilde{S}_k^i) \leq C_k\)

(C3) \(\#((\mu_k \cup D_i) \cap \tilde{S}_k^i) \leq C_k\)

where \(\tilde{S}_k^i := u_k^{-1}(W_k)\). Then a subsequence converges in an exhaustive Gromov sense to \((\tilde{u}, \mathbb{S}, \mathcal{J}, \mathbb{W}, \mathcal{J}, \tilde{g}, \tilde{g})\) which is a proper stable marked nodal pseudoholomorphic curve without boundary.

**Proof.** We begin by choosing sequences of open sets \(\mathbb{W}_k^-, \mathbb{W}_k^+ \subset \mathbb{W}\) with the property that each \(\text{cl}(\mathbb{W}_k^-)\) and \(\text{cl}(\mathbb{W}_k^+)\) is a smooth compact manifold with boundary, and

\[\text{cl}(W_k) \subset \mathbb{W}_k^- \subset \text{cl}(\mathbb{W}_{k+1}^-) \subset \mathbb{W}_k^+ \subset \text{cl}(\mathbb{W}_{k+1}^+),\]

and \(u_k \cap \partial(\text{cl}(\mathbb{W}_k^+))\), and \(u_k(\mu_k \cup D_i) \cap \partial(\text{cl}(\mathbb{W}_k^+)) = \emptyset\) for all \(k, \ell \in \mathbb{N}\); this is possible by Sard’s theorem and the fact that the \(\partial(\text{cl}(W_k))\) are smooth manifolds. We then define the compact manifolds with smooth boundary

\[\tilde{S}_k^i := u_k^{-1}(\text{cl}(\mathbb{W}_k^+)) \subset \tilde{S}_\ell\]

and observe that for each fixed \(k \in \mathbb{N}\), and for all sufficiently large \(\ell \in \mathbb{N}\), the following sequence of tuples

\[\mathbf{u}_k^i := (u_k, \tilde{S}_k^i, j_k, W_k, J_k, \mu_k \cap \tilde{S}_k^i, D_k \cap \tilde{S}_k^i)\]

are compact stable marked nodal pseudoholomorphic curves, and for each fixed \(k\) they have uniformly bounded area, genus, and number of marked and nodal points. Moreover, we let \(\mathcal{K}^- = \text{cl}(W_1^-)\) and \(\mathcal{K}^+ = \text{cl}(\mathbb{W}_2^+)\), and observe that by construction, we have \(\mathcal{K}^- \subset \text{Int}(\mathcal{K}^+)\), and \(u_\ell(\partial\mathbb{S}_k^\ell) \cap \mathcal{K}^+ = \emptyset\) for all \(\ell \geq 2\). Consequently, the sequence \(\{\mathbf{u}_\ell^i\}_{\ell \geq 2}\) satisfies the hypotheses of Corollary 3.1.

As such, we apply Corollary 3.1 to the sequence \(\{\mathbf{u}_\ell^2\}_{\ell \geq 2}\) with \(\mathcal{K}^-\) and \(\mathcal{K}^+\) as defined above, and thus we obtain a subsequence we denote by \(\{\mathbf{u}_{\ell_1}^2\}_{j \in \mathbb{N}}\), and obtain compact manifolds with smooth boundary \(\mathbf{S}^1_{\ell_1} \subset \tilde{S}^2_{\ell_1}\) so that the sequence of compact marked nodal boundary-immersed pseudoholomorphic curves given by

\[(u_{\ell_1^j}, \mathbf{S}^1_{\ell_1^j}, j_{\ell_1^j}, W_{\ell_1^j}, J_{\ell_1^j}, \mu_{\ell_1^j} \cap \mathbf{S}^1_{\ell_1^j}, D_{\ell_1^j} \cap \mathbf{S}^1_{\ell_1^j})\]

converge in Gromov sense to the limit curve

\[\left(\tilde{u}^1, \mathbf{S}^1, j^1, W_2, J^1, \mu^1, D^1\right).\]
Recall from the properties of Gromov convergence (see Definition 2.16) that this gives rise to a sequence of diffeomorphisms

$$\phi^1_{\ell_1^i} : (\Sigma^1)_{D^1, r^1} \to (\Sigma^1_{\ell_1^i})_{D^1\ell_1^i, r^1\ell_1^i}$$

between the circle-blown up limit domain and the circle-blown up sequence domains. These diffeomorphisms have the property that

$$(\phi^1_{\ell_1^i})^* j^1 \to j^1$$

in $C^\infty_{\text{loc}}$ on the compliment of the special circles in $(\Sigma^1)_{D^1, r^1}$.

Next we consider the (sub)sequence of pseudoholomorphic curves $\{u^3_{\ell_1^i}\}_{i \in \mathbb{N}}$, and apply Corollary 3.1 to this sequence with $K^- = \text{cl}(W_2)$ and $K^+ = \text{cl}(W_3^-)$ as the associated compact sets. Thus there exists a further subsequence, denoted with subscripts $\ell_2^i$, and compact manifolds with smooth boundary $\Sigma^2_{\ell_2^i} \subset \tilde{S}^3_{\ell_2^i}$ so that the sequence of compact marked nodal boundary-immersed pseudoholomorphic curves given by

$$(u^2_{\ell_2^i}, \Sigma^2_{\ell_2^i}, j^2_{\ell_2^i}, W_3, J_{\ell_2^i}, \mu_{\ell_2^i} \cap \Sigma^2_{\ell_2^i}, D_{\ell_2^i} \cap \Sigma^2_{\ell_2^i})$$

converge in Gromov sense to the limit curve

$$(\bar{u}^2, \Sigma^2, j^2, W_3, J, \mu^2, D^2)$$

Again we have diffeomorphisms

$$\phi^2_{\ell_2^i} : (\Sigma^2)_{D^2, r^2} \to (\Sigma^2_{\ell_2^i})_{D^2\ell_2^i, r^2\ell_2^i}$$

and on the complement of the special circles we have $C^\infty_{\text{loc}}$ convergence of the almost complex structures

$$(\phi^2_{\ell_2^i})^* j^2_{\ell_2^i} \to j^2.$$ 

Defining $\Sigma^1 := \text{Int}(\Sigma^1)$ and $\Sigma^2 := \text{Int}(\Sigma^2)$, our goal at present then becomes to construct a holomorphic embedding

$$\tilde{\psi}_1 : (\Sigma^1, j^1) \to (\Sigma^2, j^2)$$

which sends marked points to marked points and nodal points to nodal points, and for which $\bar{u}^1 = \bar{u}^2 \circ \tilde{\psi}_1$. We accomplish this by considering the sequence of maps $(\phi^2_{\ell_2^i})^{-1} \circ \phi^1_{\ell_1^i} : \Sigma^1 \to \Sigma$ which by definition are holomorphic with respect to domain (almost) complex structure $(\phi^1_{\ell_1^i})^* j^1_{\ell_1^i}$ and target (almost) complex structure $(\phi^2_{\ell_2^i})^* j^2_{\ell_2^i}$.

Because the domain and target (almost) complex structures converge smoothly away from the special circles, we can regard this as a sequence of pseudoholomorphic maps. We claim the maps must have uniformly bounded gradient on the interior away from special circles. Or in other words, we claim that if the gradient blows up along a sequence of points, those points must converge to either a special circle or the boundary $\partial(\Sigma^1)_{D^1, r^1}$. Indeed, if this were not true, one could then employ bubbling analysis to construct a non-constant holomorphic map from $\mathbb{C}$ into the open disk in $\mathbb{C}$ which is impossible.

With interior gradient bounds established, we then note that by elliptic regularity, we have uniformly bounded derivatives on the interior away from special circles, and a subsequence then converges to the holomorphic embedding $\tilde{\psi}_1$:
(\Sigma^1 \setminus D^1, j^1) \to (\Sigma^2, j^2). An application of the removable singularity theorem then extends this to the desired holomorphic embedding \( \psi_1 : (\Sigma^1, j^1) \to (\Sigma^2, j^2) \).

We now claim that \( \bar{u}^2 \circ \psi_1 = \bar{u}^1 \). Indeed, this follows essentially from the smooth interior convergence of the maps \( (\phi_\ell^2 i^1) \circ \psi_1 \to \bar{u}^1 \) together with the fact that we have \( C^\infty_{loc} \) convergence of the maps \( \bar{u} \circ \phi_\ell^2 i^1 \to \bar{u}^2 \).

At this point we collect our results, and we shall see that the proof is nearly complete. As a first step, we remove a thin annular open neighborhood of the boundary of \( \Sigma^1 \) to obtain the compact manifold with smooth boundary \( \Sigma^1 \subset \Sigma^1 ; \) we similarly define \( \Sigma^2 \subset \Sigma^2 \). We also note that we have smooth (almost) complex manifolds without boundary given by \((\Sigma^1, j^1)\) and \((\Sigma^2, j^2)\) and a holomorphic map \( \psi_1 : \Sigma^1 \to \Sigma^2 \) between which also sends marked points to marked points and nodal points to nodal points, and which satisfies \( \bar{u}^2 \circ \psi_1 = \bar{u}^1 \). We also found a subsequence \( i \to \ell^1_i \) of \( i \to i \), and a subsequence \( i \to \ell^2_i \) of \( i \to \ell^1_i \) for which we have Gromov convergence

\[
(\bar{u}_\ell^1, \bar{u}_\ell^2, j_\ell, \bar{W}, J_\ell, \mu_\ell \cap \Sigma^{\kappa}_\ell, D_\ell^1, D_\ell^2) \to (\bar{u}, \bar{v}, j, \bar{W}, J, \mu, D),
\]

for \( k \in \{1, 2\} \), and where all curves in the sequence and limit are compact marked nodal boundary-immersed pseudoholomorphic curves. The final, and most important observation to make is that the construction to obtain these results is iterative, and hence for each \( k \in \mathbb{N} \) we can construct \( \Sigma^k, \psi_k, \Sigma^k, i \to \ell^k_i \) etc, with all the associated properties. Observe that the holomorphic maps \( \psi_k : (\Sigma^k, j^k) \to (\Sigma^{k+1}, j^{k+1}) \) send marked points to marked points and nodal points to nodal points, and hence give rise to an embedding diagram in the sense of Section [2.1] Consequently there exists a direct limit manifold \( \bar{S} \) and holomorphic embeddings \( i_k : \Sigma^k \to \bar{S} \) with marked points defined by \( \bar{\mu} := \bigcup_{k \in \mathbb{N}} i_k(\mu^k) \) and nodal points defined by \( \bar{\mu} := \bigcup_{k \in \mathbb{N}} i_k(D^k) \). This will be our limit marked nodal Riemann surface \( (\bar{S}, j, \bar{\mu}, \bar{D}) \) without boundary. We also note that \( \bar{u}^{k+1} \circ \psi_k = \bar{u}^k \), and hence by property (DL-1) in Section [2.1] we see that the \( \{ \bar{u}^k \}_{k \in \mathbb{N}} \) induce a map \( \bar{u} : \Sigma \to \bar{W} \) for which the tuple

\[
(\bar{u}, \bar{S}, j, \bar{W}, J, \bar{\mu}, \bar{D})
\]

is a proper marked nodal pseudoholomorphic curve.

At this point we have constructed our limit curve, however it still remains to show that (after passing to a diagonal subsequence) we have the desired exhaustive Gromov convergence. To that end, we next define the diagonal subsequence \( i \to \delta_i \). Recall that our iterative construction yielded the sequence of smooth two-dimensional manifolds with boundary \( \Sigma^k \subset \Sigma^k \subset S_k \). We now define \( S^k_{\delta_k} := i_{\delta_k}(\Sigma^k) \subset \bar{S} \) for each \( k \in \mathbb{N} \). We next aim to define \( S^k_{\delta_k} \subset S^k_{\delta_k} \) for all \( k, \ell \in \mathbb{N} \) with \( k \geq \ell \). The precise definition is given as

\[
S^k_{\delta_k} := \text{cl} \left( \phi_{\delta_k}^{-1}(i_{\delta_k}(\bar{S} \setminus D)) \right),
\]

where the \( \phi_i \) are the diffeomorphisms guaranteed by Gromov convergence from the blown up limit Riemann surface to the blown up approximating Riemann surfaces:

\[
\phi_{\ell_i}^1 : (\Sigma^1) \to (\Sigma^1) \quad \phi_{\ell_i}^2 : (\Sigma^2) \to (\Sigma^2) \quad \phi_{\ell_i}^3 : (\Sigma^3) \to (\Sigma^3)
\]
For the sake of clarity, we also provide a more geometric description of the definition of the $S^{δ_k}$. Start with the compact manifold with boundary $S^{δ_k} \subset \overline{S}$. Then remove the nodal points to obtain $\overline{S}^{δ_k} \setminus \overline{D}$. This lies in the image of the embedding $i_{δ_k} : \hat{\Sigma}^{δ_k} \rightarrow \overline{S}$; recall that the $\hat{\Sigma}^{δ_k}$ (which satisfy $\hat{\Sigma}^{δ_k} \subset \Sigma^{δ_k} \subset S_{δ_k}$) form the embedding sequence which has direct limit $\overline{S}$, and the $i_{δ_k} : \hat{\Sigma}_{δ_k} \rightarrow \overline{S}$ are the associated embeddings (which can be thought of as inclusions since their images exhaust $\overline{S}$). Consequently, we pull back via $i_{δ_k}$ to obtain a subset of $\overline{\Sigma}^{δ_k}$. Because we have removed the nodal points, we may identify $\Sigma^{δ_k} \setminus D^{δ_k}$ with the circle blown-up manifold with special circles removed: $(\Sigma^{δ_k})^{D^{δ_k}} \setminus \cup_i \Gamma_i$. Consequently, we may then map our set via the $\phi_{δ_k}$ to obtain

$$\phi_{δ_k}(i_{δ_k}^{-1}(\overline{S}^{δ_k} \setminus \overline{D})) \subset \Sigma^{δ_k} \setminus D_{δ_k} \subset S_{δ_k} \subset S^{δ_k},$$

which would be the compact manifolds with smooth boundaries that we seek, except that these sets are missing the images of the special circles. Thus after taking the closure of these sets, we obtain the desired compact sets, which we have denoted $S^{δ_k}$.

Thus, we have passed to a subsequence $\{u_k\}_{k \in \mathbb{N}}$, and constructed a proper marked nodal pseudoholomorphic curve $(\bar{u}, \bar{S}, \bar{J}, \bar{\mu}, \bar{D})$, and found smooth compact two-dimensional manifolds with boundary $\overline{S}^{δ_k} \subset \overline{S}$ and $S^{δ_k} \subset S_{δ_k}$ which we now claim have the following properties by construction.

1. $\overline{S}^{δ_k} \subset \overline{S}^{δ_{k+1}}$ \setminus $\partial S^{δ_{k+1}}$ for all $\ell \in \mathbb{N}$
2. $\overline{S} = \bigcup_{\ell \in \mathbb{N}} \overline{S}^{δ_\ell}$
3. for each fixed $k \in \mathbb{N}$ and each $0 \leq \ell \leq k - 1$ we have $S^{δ_\ell}_k \subset S^{δ_{k+1}}_k \setminus \partial S^{δ_{k+1}}_k$
4. for each $k \geq \ell \in \mathbb{N}$ we have $u_k^{-1}(W_\ell) \subset S^{δ_\ell}_k.$
5. For each fixed $\ell \in \mathbb{N}$, the sequence $$\{(u_k, S^{δ_\ell}_k, j_k, W_\ell, J_k, S^{δ_\ell}_k \cap \mu_k, S^{δ_\ell}_k \cap D_k)\}_{k \geq \ell}$$
is a sequence of compact marked nodal stable boundary-immersed pseudoholomorphic curves which converges in a Gromov sense to the proper marked nodal stable boundary-immersed pseudoholomorphic curve $(\bar{u}, \bar{S}^{δ_\ell}, \bar{j}_k, \bar{W}_\ell, \bar{J}^{δ_\ell} \cap \bar{\mu}, \bar{S}^{δ_\ell} \cap \bar{D}).$

The first two properties essentially follow from the fact that $(W_k, J_k, g_k)$ is a sequence of properly exhausting regions for $(\bar{W}, \bar{J}, \bar{g})$. To see this, first observe that whenever these two properties hold, they will also hold for any subsequence of slightly trimmed compact sets. Second, recall that $\bar{S}^{δ_\ell}_k = u^{-1}_k(W_k)$, and then by slightly target-trimming our curves and passing to a subsequence (here still denoted with subscripts $\ell$) we found compact manifolds with boundary $\Sigma^{δ_\ell}_k \subset S^{δ_{k+1}}_\ell$ on which the subsequence $\bar{u}_\ell$ still converged. The domains of these limit curves we denoted $\Sigma^{δ_\ell}$; we denoted their interiors by $\Sigma^{δ_\ell}$; and we used these and the Gromov convergence of the curves to construct holomorphic maps $\psi_k : \Sigma_k \rightarrow \Sigma^{δ_{k+1}}_\ell$ which resulted in an associated limit Riemann surface $\overline{S}$. The compact manifolds with
boundary \( \overline{S}^t \subset S \) were then obtained by trimming the \( i_k(\Sigma^k) \subset S \) slightly further. Because this procedure only involved passing to subsequences and making small trimmings, the first two properties follow immediately from the definition of \( \overline{S} \).

We establish the third property in a moment, but at present we work on the fourth property. Indeed, recall that \( \tilde{u}_k^{-1}(W) = \tilde{S}_k^t \), so that we need to verify that \( \tilde{S}_k^t \subset S_k^t \). Also note that due to the properly exhausting nature of the \( W_k \) it follows that \( \tilde{S}_k^t \subset \tilde{S}_k^{t+1} \) in such a way that for any sufficiently small trimming \( \tilde{\Sigma} \) of \( \tilde{S}_k^t \), we have \( \tilde{S}_k^t \subset \tilde{\Sigma} \subset \tilde{S}_k^{t+1} \). However, recall that \( \tilde{\Sigma}_k^t \) was obtained as a small trimming of \( \tilde{S}_k^{t+1} \), and \( S_k^t \) was obtained as a small trimming of \( \tilde{\Sigma}_k^t \), from which we see that

\[
\tilde{S}_k^t \subset S_k^t \subset \tilde{\Sigma}_k^t \subset \tilde{S}_k^{t+1}
\]

and hence we indeed have \( \tilde{S}_k^t \subset S_k^t \), as desired. Note however that extending this string of containments a bit further, we have

\[
S_k^{t-1} \subset \tilde{\Sigma}_k^{t-1} \subset \tilde{S}_k^t \subset S_k^t \subset \tilde{\Sigma}_k^t \subset \tilde{S}_k^{t+1},
\]

which establishes the third property. Finally, with the \( \overline{S}^t \) obtained as slightly trimmed versions of the \( \tilde{\Sigma}^t \supset S^t \), together with the result that we obtained (after passing to a subsequence, which we still denote with subscripts \( \ell \)) convergence of the \( \mu_{\ell} : \Sigma_{\mu_{\ell}}^t \rightarrow \overline{W} \) to \( \tilde{u}^t : \tilde{\Sigma}^t \rightarrow \overline{W} \), we were able to trim in the limit domains \( \overline{\Sigma}^t \), and we were able to use the diffeomorphisms \( \tilde{\phi}_k \) guaranteed by the usual Gromov convergence to push forward the \( \overline{S}^t \) into the \( S_k \); the images of these sets we defined to be \( \tilde{S}_k^t \), and they had the property that by construction maps \( u_k : S_k^t \rightarrow \overline{W} \) converged in a Gromov sense to \( \tilde{u}^t : \overline{S}^t \rightarrow \overline{W} \). This then establishes the fifth property.

\[ \square \]

Appendix A. Formula for arithmetic genus

Here we provide the more standard formula based definition of arithmetic genus, and we show that this is equivalent to the notion provided in Definition 2.7. Before proceeding to prove that result, we note that we will employ the following notation. For any topological space \( X \), we let \( \pi_0(X) \) denote the set of connected components of \( X \), and we let \( \# \pi_0(X) \) denote the number of connected components of \( X \). Additionally we recall the discussion following Definition 2.4 in which we define a topological space \( |S| \) associated to a nodal Riemann surface \((S,j,D)\) via identifying points in each nodal pair: \( \overline{d} \sim \overline{d}_i \). In this way, we will abuse language a bit by saying that \( \Sigma \subset S \) is a connected component of \( |S| \) whenever \( \partial\Sigma \) is a connected component of \( |S| \).

**Lemma A.1** (formula for arithmetic genus).

Let \((S,j,D)\) be a compact marked nodal Riemann surface, possibly with boundary. Then a formula for the arithmetic genus of \((S,j,D)\) is given by

\[
\text{Genus}_{\text{arith}}(u) = \# \pi_0(|S|) - \# \pi_0(S) + \left( \sum_{k=1}^{\# \pi_0(S)} g_k \right) + \frac{1}{2} \# D.
\]

**Proof.** For notational convenience, we define

\[
m = \# \pi_0(|S|), \quad n = \# \pi_0(S) \quad \text{and} \quad b = \# \pi_0(\partial S).
\]
We begin by denoting the connected components of $S$ by $S_k$, so that $\bigcup_{k=1}^{n} S_k = S$. Next, we let $D_k = S_k \cap D$, and we let $S_k^{D_k}$ denote the circle compactification of $S_k \setminus D_k$, and we let $S^{D,r}_k$ denote the surface obtained by gluing the $S_k^{D_k}$ along pairs of compactification circles associated to nodal pairs. Then, letting $G_a$ denote the arithmetic genus of $u$, and letting $g_k$ denote the genus of $S_k$, we have

$$2m - 2G_a - b = 2m - 2\text{Genus}(S^{D,r}) - b$$

$$= \chi(S^{D,r})$$

$$= \sum_{k=1}^{n} \chi(S_k^{D_k})$$

$$= \sum_{k=1}^{n} (\chi(S_k) - \#D_k)$$

$$= \sum_{k=1}^{n} (2 - 2g_k - \#\pi_0(\partial S_k) - \#D_k)$$

$$= 2n - 2\left(\sum_{k=1}^{n} g_k\right) - b - \#D.$$

Solving for $G_a$, the desired result is immediate.

□

References

[1] Michèle Audin and Jacques Lafontaine (eds.), *Holomorphic curves in symplectic geometry*, Progress in Mathematics, vol. 117, Birkhäuser Verlag, Basel, 1994. MR1274923

[2] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. 7 (2003), 799–888. MR2026549

[3] Joel W. Fish, *Subcritical sideways neck stretching*, In preparation.

[4] , *Compactness results for pseudo-holomorphic curves*, ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–New York University. MR2711037

[5] , *Estimates for J-curves as submanifolds*, Internat. J. Math. 22 (2011), no. 10, 1375–1431. MR2851214

[6] , *Target-local Gromov compactness*, Geom. Topol. 15 (2011), no. 2, 765–826. MR2800366

[7] Joel W. Fish and H. Hofer, *Feral curves and minimal sets*, In preparation.

[8] , *Feral pseudoholomorphic curves and minimal sets*, Oberwolfach Report 12 (2015), no. 3.

[9] Andreas Floer, *The unregularized gradient flow of the symplectic action*, Comm. Pure Appl. Math. 41 (1988), no. 6, 775–813. MR948771

[10] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), no. 2, 307–347. MR809718

[11] Michael Herman, *Some open problems in dynamical systems*, Proceedings of the International Congress of Mathematicians (1998). Sections 10–19, Held in Berlin, August 18–27, 1998, Doc. Math. 1998, Extra Vol. II.

[12] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. 114 (1993), no. 3, 515–563. MR1244912

[13] H. Hofer, K. Wysocki, and E. Zehnder, *Finite energy cylinders of small area*, Ergodic Theory Dynam. Systems 22 (2002), no. 5, 1451–1486. MR1934144

[14] Christoph Hummel, *Gromov’s compactness theorem for pseudo-holomorphic curves*, Progress in Mathematics, vol. 151, Birkhäuser Verlag, Basel, 1997. MR1451624

[15] Michael Hutchings, *An index inequality for embedded pseudoholomorphic curves in symplectizations*, J. Eur. Math. Soc. (JEMS) 4 (2002), no. 4, 313–361. MR1941088

[16] , *Lecture notes on embedded contact homology*, Contact and symplectic topology, 2014, pp. 389–484. MR3220947
Michael Hutchings and Michael Sullivan, *Rounding corners of polygons and the embedded contact homology of $T^3$*, Geom. Topol. 10 (2006), 169–266. MR2207793

Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, Second, American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2012. MR2954391

Clifford Henry Taubes, *The structure of pseudo-holomorphic subvarieties for a degenerate almost complex structure and symplectic form on $S^1 \times B^3$*, Geom. Topol. 2 (1998), 221–332. MR1658028

Rugang Ye, *Gromov’s compactness theorem for pseudo holomorphic curves*, Trans. Amer. Math. Soc. 342 (1994), no. 2, 671–694. MR1176088

Joel W. Fish, Mathematics Department, University of Massachusetts Boston  
*E-mail address*: joel.fish@umb.edu

Helmut Hofer, School of Mathematics Institute for Advanced Study  
*E-mail address*: hofer@math.ias.edu