Symmetries of the Energy-Momentum Tensor of Cylindrically Symmetric Static Spacetimes

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Abstract

We investigate matter symmetries of cylindrically symmetric static spacetimes. These are classified for both cases when the energy-momentum tensor is non-degenerate and also when it is degenerate. It is found that the non-degenerate energy-momentum tensor gives either three, four, five, six, seven or ten independent matter collineations in which three are isometries and the rest are proper. The worth mentioning cases are those where we obtain the group of matter collineations finite-dimensional even the energy-momentum tensor is degenerate. These are either three, four, five or ten. Some examples are constructed satisfying the constraints on the energy-momentum tensor.

Keywords: Matter symmetries, Cylindrically Symmetric Static Manifolds

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1 Introduction

Let $M$ be a spacetime manifold with Lorentz metric $g$ of signature $(+,-,-,-)$. It is assumed that the manifold $M$, and the metric $g$, are smooth. There has been recent significant interest in the study of the various symmetries (in particular, Ricci and matter collineations) that arise in the exact solutions of Einstein’s field equations (EFEs)

\[ R_{ab} - \frac{1}{2}Rg_{ab} \equiv G_{ab} = \kappa T_{ab}, \quad (a, b = 0, 1, 2, 3), \quad (1) \]

where $\kappa$ is the gravitational constant, $G_{ab}$ is the Einstein tensor, $R_{ab}$ is the Ricci and $T_{ab}$ is the matter (energy-momentum) tensor. Also, $R = g^{ab}R_{ab}$ is the Ricci scalar. We have assumed here that the cosmological constant $\Lambda = 0$. The theoretical basis for the study of the affine, conformal, projective, curvature (CCs) and Ricci collineations (RCs) has been analysed and many examples have been discovered [1]-[5]. The symmetries of the energy-momentum tensor have recently been studied.

We define a differentiable vector field $\xi$ on $M$ to be a matter collineation if

\[ \mathcal{L}_\xi T_{ab} = 0, \quad (2) \]

where $\mathcal{L}$ is the Lie derivative operator, $\xi^a$ is the symmetry or collineation vector. The study of matter collineations (MCs) derives from the mathematical interest in the invariance attributes of a geometrical object, i.e., Einstein tensor. Since the Einstein tensor is related to the matter content of the spacetime by the EFEs, the investigation of MCs seems to be more relevant from the viewpoint of physics.

The study of symmetries played an important role in the classification of spacetimes, giving rise to many interesting results with useful applications. It is well known that two different collineations are not in general equivalent. For example, a Killing vector (KV) is a MC but the converse does not hold. Collineations have been classified by means of their relative properness by Katzin et al. [6,7]. This classification indicates that the basic collineation is the KVs. The role of isometries is to restrict the general form of the metric. Consequently, the number of independent field equations would reduce and it would be easy to find the exact solutions. It is noted that there are well known metrics which do not have isometries [8]. This does not imply that they do not admit higher symmetries. The symmetry properties given by KVs lead to conservation laws [9]-[11]. A large number of solutions of the EFEs with
different symmetry structures have been found [10] and classified according to their properties [12]. Symmetries of the energy-momentum tensor (also called matter collineations) provide conservation laws on matter fields. These enable us to know how the physical fields, occupying in certain region of spacetimes, reflect the symmetries of the metric [13].

There is a large body of recent literature which shows interest in the study of MCs [14]-[23]. In a recent paper [22], the study of MCs has been taken for spherically symmetric spacetimes and some interesting results have been obtained. We have also classified plane symmetric static spacetimes according to their MCs [23]. In this paper, we extend the procedure to calculate MCs of cylindrically symmetric static spacetimes both for non-degenerate and also for degenerate cases. Here we would not give details of the calculations as the procedure has been given in different papers [22,23].

The MC Eq.(2) can be written in component form as

\[ T_{ab,c} \xi^c + T_{ac,b} \xi^c + T_{cb,a} \xi^c = 0. \]  

(3)

The most general form of cylindrically symmetric static spacetime is given by

\[ ds^2 = e^{\nu(r)} dt^2 - dr^2 - e^{\lambda(r)} d\theta^2 - e^{\mu(r)} dz^2. \]  

(4)

The only non-zero components of the energy-momentum tensor, given in Appendix A, are \( T_{00}, T_{11}, T_{22}, T_{33} \). We can write the MC equations as follows

- \( T_{0}^r \xi^1 + 2 T_{0} \xi^0_0 = 0 \),
- \( T_{1}^r \xi^1 + 2 T_{1} \xi^1_1 = 0 \),
- \( T_{2}^r \xi^2 + 2 T_{2} \xi^2_2 = 0 \),
- \( T_{3}^r \xi^3 + 2 T_{3} \xi^3_3 = 0 \),
- \( T_{0} \xi^0_0 + T_{1} \xi^1_1 = 0 \),
- \( T_{0} \xi^0_2 + T_{2} \xi^2_0 = 0 \),
- \( T_{0} \xi^0_3 + T_{3} \xi^3_0 = 0 \),
- \( T_{1} \xi^1_2 + T_{2} \xi^2_1 = 0 \),
- \( T_{1} \xi^1_3 + T_{3} \xi^3_1 = 0 \),
- \( T_{2} \xi^2_3 + T_{3} \xi^3_2 = 0 \).
where prime ′ indicates differentiation with respect to \( r \). These yield the first order non-linear coupled partial differential equations in four variables \( \xi^a(x^b) \). The components of the energy-momentum tensor depend only on \( r \). Here we have used the notation \( T_{\alpha\alpha} = T_\alpha \) for the sake of brevity. We solve this set of equations for the non-degenerate case, when

\[
\det(T_{\alpha\beta}) = T_0 T_1 T_2 T_3 \neq 0
\]

and for the degenerate case, where \( \det(T_\alpha) = 0 \).

The rest of the paper is organized as follows. The next section contains brief comments and results about MCs. In section 3, we shall solve the MC equations when the energy-momentum tensor is non-degenerate and in the next section MC equations are solved for the degenerate energy-momentum tensor. In section 5, we shall solve some of the constraints on energy-momentum tensor to obtain exact solution of EFEs. Section 6 contains a summary and discussion of the results obtained.

### 2 Some General Comments

Let \( \xi \) be a matter collineation. All KVs, homothetic vectors and special conformal Killing vectors are MCs. However, the converse is not always true. In this case, the MC is called proper or non-trivial. The study of MCs has many associated problems. Here we list these in comparison with the other symmetries.

1. When we define affine and conformal vector fields on \( M \) we usually assume that the vector field is at least \( C^2 \) and \( C^3 \) respectively. Then it follows from Hall et al. \[4\] that \( \xi^a \) must be a smooth vector field on \( M \). However, for \( k \in \mathbb{Z}^+ \) there exist MCs on smooth spacetimes which are \( C^k \) but not \( C^{k+1} \).

2. We know that an affine and conformal vector fields \( \xi^a \) on \( M \) are uniquely determined by specifying \( \xi^a \) and \( \xi^a_{;b} \) and respectively by specifying \( \xi^a \) and the components of its first two covariant derivatives \( \xi^a_{;bc} \) and \( \xi^a_{;b;e} \) at some point \( p \in M \). However, the value of \( \xi^a \) and all its derivatives at some point \( q \in M \) may not be enough to determine uniquely a MC \( \xi^a \) on \( M \). The fact is that two MCs which agree on a non-empty open subset of \( M \) may not agree on \( M \).
3. The set of all MCs on $M$ is a vector space but in a similar way to the sets of CCs and RCs, and unlike the sets of affine and conformal vector fields, it could be infinite dimensional and could fail to be a Lie algebra. The problem here arises from the fact that such collineations must be $C^1$ in order that the defining equations make sense. It is unfortunate that a matter, Ricci or curvature collineation might turn out to be precisely $C^1$ and so the differentiability may be destroyed under the Lie bracket operation. On the other hand, if we assume that MCs are $C^\infty$ then we recover the Lie algebra structure but we are then forced to expel the collineations which are not smooth. The infinite dimensionality may also lead to problems related to the orbits of the resulting local diffeomorphism [4,24].

4. If the energy-momentum tensor is of rank 4 everywhere then we can think of this tensor as a metric on the spacetime $M$. It then follows from the theory of Killing vectors that the family of MCs is, in fact, a Lie algebra of smooth vector fields on $M$, of finite dimension, $\leq 10$, and, in addition, $\neq 9$ by Fubini’s theorem [12].

3  Matter Collineations in the Non-Degenerate Case

In this section, we shall evaluate MCs only for those cases which have non-degenerate energy-momentum tensor, i.e., $T_a \neq 0$.

When we solve Eqs.(5)-(14) simultaneously, we get the following constraint equation

$$T_{0\xi,23}^\prime = 0. \quad (16)$$

This equation implies that either

\begin{align*}
(1) \quad T_0^\prime &= 0, \\
\text{or } \quad (2) \quad T_0^\prime &\neq 0.
\end{align*}

Case 1: In the first case we have $T_0 = -k_1$, where $k_1$ is a non-zero constant which implies that the MC equations yield the following four possibilities:

\begin{align*}
(a_1) \quad T_2^\prime &= 0, \quad T_3^\prime = 0, \\
(a_2) \quad T_2^\prime &= 0, \quad T_3^\prime \neq 0,
\end{align*}

5
(a₃) $T'_2 \neq 0$, $T'_3 = 0$,
(a₄) $T'_2 \neq 0$, $T'_3 \neq 0$.

**Subcase 1(a₁):** This implies that $T_2 = k_2$ and $T_3 = k_3$, where $k_2$ and $k_3$ are non-zero constants. In this case, in addition to the nonproper MCs $\xi(1)$, $\xi(2)$, $\xi(3)$ given in Appendix B, we obtain the following proper MCs

$$
\xi(4) = \theta \partial_t + \frac{k_1}{k_2} t \partial_\theta, \\
\xi(5) = \frac{1}{k_1} \int \sqrt{T_1} dr \partial_t + \frac{1}{\sqrt{T_1}} t \partial_r, \\
\xi(6) = z \partial_t + \frac{k_1}{k_3} t \partial_z, \\
\xi(7) = \frac{1}{\sqrt{T_1}} \partial_r, \\
\xi(8) = \frac{\theta}{\sqrt{T_1}} \partial_r - \frac{1}{k_2} \int \sqrt{T_1} dr \partial_\theta, \\
\xi(9) = \frac{z}{\sqrt{T_1}} \partial_r - \frac{1}{k_3} \int \sqrt{T_1} dr \partial_z, \\
\xi(10) = z \partial_\theta - \frac{k_2}{k_3} \theta \partial_z.
$$

(17)

Thus we have ten independent MCs in which seven are proper.

**Subcase 1(a₂):** It follows that $T_2 = k_2$, $T'_3 \neq 0$ which further yields the following two cases:

(b₁) $\left[ \frac{T_3}{\sqrt{T_1}} \left( \frac{T'_3}{2T_3\sqrt{T_1}} \right) \right]' = 0,$
(b₂) $\left[ \frac{T_3}{\sqrt{T_1}} \left( \frac{T'_3}{2T_3\sqrt{T_1}} \right) \right]' \neq 0.$

The case 1a₂(b₁) gives $\frac{T_3}{\sqrt{T_1}} \left( \frac{T'_3}{2T_3\sqrt{T_1}} \right)' = \alpha_1$, where $\alpha_1$ is an arbitrary constant which may be

(c₁) $\alpha_1 > 0,$ (c₂) $\alpha_1 = 0,$ or (c₃) $\alpha_1 < 0.$

The first option 1a₂b₁(c₁), when $\alpha_1 > 0$, gives the following proper MCs

$$
\xi(4) = \theta \partial_t + \frac{k_1}{k_2} t \partial_\theta,
$$
\[
\begin{align*}
\xi(5) &= \frac{e^{\sqrt{\alpha_1} z}}{\sqrt{T_1}} \partial_r - \frac{T_3' e^{\sqrt{\alpha_1} z}}{2\sqrt{\alpha_1 T_3 \sqrt{T_1}}} \partial_z, \\
\xi(6) &= \frac{e^{-\sqrt{\alpha_1} z}}{\sqrt{T_1}} \partial_r + \frac{T_3' e^{-\sqrt{\alpha_1} z}}{2\sqrt{\alpha_1 T_3 \sqrt{T_1}}} \partial_z.
\end{align*}
\]

(18)

The second option \(1a_2b_1(c_2)\), when \(\alpha_1 = 0\), yields the following three proper MCs

\[
\begin{align*}
\xi(4) &= \theta \partial_t + \frac{k_1}{k_2} t \partial_\theta, \\
\xi(5) &= \frac{1}{T_1} \partial_r - \alpha_2 z \partial_z, \\
\xi(6) &= \frac{z}{\sqrt{T_1}} \partial_r - \left( \int \frac{\sqrt{T_1}}{T_3} dr + \alpha_2 z^2 \right) \partial_z,
\end{align*}
\]

(19)

where \(\alpha_2 = \frac{T_3'}{2T_3 \sqrt{T_1}}\) is a non-zero constant.

The third option \(1a_2b_1(c_3)\) gives the same MCs as the first case \(1a_2b_1(c_1)\).

When we solve MC equations for the case \(1a_2(b_2)\), we have only one proper MC given by

\[
\xi(4) = \theta \partial_t + \frac{k_1}{k_2} t \partial_\theta.
\]

(20)

**Subcase 1(a_3):** The subcase \(1(a_3)\) is similar to the the subcase \(1(a_2)\) and MCs follow by interchanging \(\theta\) and \(z\) coordinates.

**Subcase 1(a_4):** Here we have \(T_2' \neq 0, T_3' \neq 0\) which gives rise to the following two possibilities:

\[
\begin{align*}
(b_1) \quad \left[ \frac{T_2}{\sqrt{T_1}} \left( \frac{T_2'}{2T_2 \sqrt{T_1}} \right) \right]' &= 0, \\
(b_2) \quad \left[ \frac{T_2}{\sqrt{T_1}} \left( \frac{T_2'}{2T_2 \sqrt{T_1}} \right) \right]' &\neq 0.
\end{align*}
\]

The first possibility \(1a_4(b_1)\) implies that \(\frac{T_3}{\sqrt{T_1}} \left( \frac{T_2'}{2T_2 \sqrt{T_1}} \right)' = \alpha_3\), where \(\alpha_3\) is an arbitrary constant such that

\[
(c_1) \quad \alpha_3 > 0, \quad (c_2) \quad \alpha_3 = 0, \quad \text{or} \quad (c_3) \quad \alpha_3 < 0.
\]
When $\alpha_3$ is positive, $1a_4b_1(c_1)$, it further gives the following two options:

\[
(d_1) \quad \left[ \frac{T_3}{\sqrt{T_1}} \left( \frac{T_3'}{2T_3\sqrt{T_1}} \right) \right]' = 0,
\]
\[
(d_2) \quad \left[ \frac{T_3}{\sqrt{T_1}} \left( \frac{T_3'}{2T_3\sqrt{T_1}} \right) \right]' \neq 0.
\]

For the option $1a_4b_1c_1(d_1)$, we obtain similar result to the case $1a_2(b_1)$. The option $1a_4b_1c_1(d_2)$ gives the following two cases according as

\[
(e_1) \quad \frac{T_2}{T_3} = \text{constant} \neq 0, \quad (e_2) \quad \frac{T_2}{T_3} \neq \text{constant}.
\]

The case $1a_4b_1c_1d_2(e_1)$ yields one proper MC

\[
\xi(4) = z \partial_\theta - \frac{T_2}{T_3} \theta \partial_z, \quad (21)
\]

and the case $1a_4b_1c_1d_2(e_2)$ gives three independent MCs.

The case $1a_4b_1(c_2)$, when $\alpha_3 = 0$, yields $\frac{T_2'}{2T_2\sqrt{T_1}} = \alpha_4$, where $\alpha_4$ is a non-zero constant which gives either

\[
(d_1) \quad \left[ \frac{T_3}{\sqrt{T_1}} \left( \frac{T_3'}{2T_3\sqrt{T_1}} \right) \right]' = 0,
\]

or

\[
(d_2) \quad \left[ \frac{T_3}{\sqrt{T_1}} \left( \frac{T_3'}{2T_3\sqrt{T_1}} \right) \right]' \neq 0.
\]

The first possibility $1a_4b_1c_2(d_1)$ further divides into three cases according as

\[
\frac{T_3}{\sqrt{T_1}} \left( \frac{T_3'}{2T_3\sqrt{T_1}} \right)' = \alpha_1
\]

\[
(e_1) \quad \alpha_1 > 0, \quad (e_2) \quad \alpha_1 = 0, \quad \text{or} \quad (e_3) \quad \alpha_1 < 0.
\]

The case $1a_4b_1c_2d_1(e_1)$, when $\alpha_1 > 0$, MCs turn out to be similar to the case $1a_2b_1(c_1)$. When $\alpha_1 = 0$, i.e., in the case $1a_4b_1c_2d_1(e_2)$, we have further two options either

\[
(f_1) \quad \frac{T_3}{T_2} = k = \text{constant} \neq 0, \quad \text{or} \quad (f_2) \quad \frac{T_3}{T_2} \neq \text{constant}.
\]
In the option 1a_4 b_1 c_2 d_1 e_2(f_1), we obtain the following four proper MCs
\[ \xi^{(4)} = -z \partial_z + \frac{\theta}{k} \partial_z, \]
\[ \xi^{(5)} = \frac{1}{\sqrt{T_1}} \partial_r - \alpha_2 \theta \partial_\theta - \alpha_2 z \partial_z, \]
\[ \xi^{(6)} = \frac{\theta}{\sqrt{T_1}} \partial_r - \alpha_2 \theta z \partial_\theta - \left( \int \frac{\sqrt{T_1}}{T_2} dr + \alpha_2 \frac{\theta^2}{2} - k \alpha_2 \frac{z^2}{2} \right) \partial_\theta, \]
\[ \xi^{(7)} = \frac{z}{\sqrt{T_1}} \partial_r - \alpha_2 \theta z \partial_\theta - \left( \int \frac{\sqrt{T_1}}{T_2} dr - \alpha_2 \frac{\theta^2}{2k} + \alpha_2 \frac{z^2}{2} \right) \partial_z. \] (22)

For the option 1a_4 b_1 c_2 d_1 e_2(f_2), there is only one proper MC
\[ \xi^{(4)} = \frac{1}{\sqrt{T_1}} \partial_r + \theta \partial_\theta - \alpha_2 z \partial_z. \] (23)

The case 1a_4 b_1 c_2 d_1 e_2(c_3), when \( \alpha_1 < 0 \), yields similar solution as the case 1a_2 b_1(c_3). The option 1a_4 b_1 c_2(d_2) gives similar result as the case 1a_4 b_1 c_1(d_2). The possibility 1a_4 b_1(c_3), when \( \alpha_3 < 0 \), also gives similar solution as the case 1a_2 b_1(c_1).

In the case 1a_4(b_2), we have either
\[ (c_1) \quad \frac{T_2}{T_3} = \text{constant}, \quad \text{or} \quad (c_2) \quad \frac{T_2}{T_3} \neq \text{constant}. \]

For the case 1a_4 b_2(c_1), it coincides with 1a_4 b_1 c_2(d_2) and in the case 1a_4 b_2(c_2), we get minimal MCs, i.e., three.

**Case 2:** Now we evaluate MCs for the case when \( T_0' \neq 0 \). This case implies that \( \xi^{1}_{23} = 0 \). Using this value in Eqs.(12)-(14), we obtain
\[ \left( \frac{T_2}{T_3} \right)' \xi^{2}_{3} = 0. \] (24)

This implies that either
\[ (a_1) \quad \left( \frac{T_2}{T_3} \right)' = 0, \quad \text{or} \quad (a_2) \quad \left( \frac{T_2}{T_3} \right)' \neq 0. \]

**Subcase 2(a_1):** This yields \( \frac{T_2}{T_3} = k_1 \), where \( k_1 \) is a non-zero constant. Solving MC equations using this value, we obtain the following two cases:
\[ (b_1) \quad \left( \frac{T_2'}{2T_2 \sqrt{T_1}} \right)' = 0, \quad (b_2) \quad \left( \frac{T_2'}{2T_2 \sqrt{T_1}} \right)' \neq 0. \]
The case $2a_1(b_1)$ implies that $\frac{T_2'}{2T_2\sqrt{T_1}} = \beta_1$, a constant which yields that either

$$(c_1) \quad \beta_1 = 0, \quad \text{or} \quad (c_2) \quad \beta_2 \neq 0.$$ 

For the case $2a_1b_1(c_1)$, when $\beta_1 = 0$, it yields the following two groups:

$$(d_1) \quad \left(\frac{\sqrt{T_0}}{\sqrt{T_1}}\right)' = 0, \quad (d_2) \quad \left(\frac{\sqrt{T_0}}{\sqrt{T_1}}\right)' \neq 0.$$ 

The first group $2a_1b_1c_1(d_1)$ gives ten independent MCs in which seven are proper given by

$$
\begin{align*}
\xi(4) &= z\partial_\theta - k_1\theta\partial_z, \\
\xi(5) &= \frac{1}{\sqrt{T_0}}\theta\sin k_2 t \partial_t - \frac{1}{\sqrt{T_1}}\theta\cos k_2 t \partial_r + \frac{\sqrt{T_0}}{k_2 T_2} \cos k_2 t \partial_\theta, \\
\xi(6) &= -\frac{1}{\sqrt{T_0}}\theta\cos k_2 t \partial_t - \frac{1}{\sqrt{T_1}}\theta\sin k_2 t \partial_r + \frac{\sqrt{T_0}}{k_2 T_2} \sin k_2 t \partial_\theta, \\
\xi(7) &= \frac{1}{k_1 \sqrt{T_0}} z \sin k_2 t \partial_t - \frac{1}{k_1 \sqrt{T_1}} z \cos k_2 t \partial_r + \frac{\sqrt{T_0}}{k_2 T_2} \cos k_2 t \partial_\theta, \\
\xi(8) &= -\frac{1}{k_1 \sqrt{T_0}} z \cos k_2 t \partial_t - \frac{1}{k_1 \sqrt{T_1}} z \sin k_2 t \partial_r + \frac{\sqrt{T_0}}{k_2 T_2} \sin k_2 t \partial_\theta, \\
\xi(9) &= \frac{1}{\sqrt{T_0}} \sin k_2 t \partial_t + \frac{1}{\sqrt{T_1}} \cos k_2 t \partial_r, \\
\xi(10) &= \frac{1}{\sqrt{T_0}} \cos k_2 t \partial_t + \frac{1}{\sqrt{T_1}} \sin k_2 t \partial_r,
\end{align*}
$$

where $k_2 = \frac{\sqrt{T_0}'}{\sqrt{T_1}'}$ is a non-zero constant. The second group $2a_1b_1c_1(d_2)$ gives further two possibilities:

$$(e_1) \quad \frac{T_0}{2\sqrt{T_1}}\left(\frac{T_0'}{T_0 \sqrt{T_1}}\right)' = \text{constant} = \beta_2,$$

$$(e_2) \quad \frac{T_0}{2\sqrt{T_1}}\left(\frac{T_0'}{T_0 \sqrt{T_1}}\right)' \neq \text{constant}.$$ 

In the first possibility $2a_1b_1c_1d_2(e_1)$, we further have two options according as

$$(f_1) \quad \beta_2 = 0, \quad (f_2) \quad \beta_2 \neq 0.$$
The option $2a_1b_1c_1d_2e_1(f_1)$ gives six independent MCs in which three proper MCs are given by
\[
\begin{align*}
\xi_{(4)} & = z\partial_\theta - k_1\theta\partial_z, \\
\xi_{(5)} & = \left(\frac{1}{\beta_3 T_0} - \frac{\beta_3 t^2}{4}\right)\partial_t + \frac{1}{\sqrt{T_1}}t\partial_r, \\
\xi_{(6)} & = -\frac{\beta_3}{2}t\partial_t + \frac{1}{\sqrt{T_1}}\partial_r,
\end{align*}
\]
where $\beta_3 = \frac{T_0'}{\sqrt{T_0}}$ is a non-zero constant. The possibility $2a_1b_1c_1d_2e_1(f_2)$ also gives six independent MCs. The three proper MCs are:
\[
\begin{align*}
\xi_{(4)} & = z\partial_\theta - k_1\theta\partial_z, \\
\xi_{(5)} & = -\frac{T_0'}{2\sqrt{\beta_2 T_0\sqrt{T_1}}} e^{\sqrt{\beta_2 t}}\partial_t + \frac{1}{\sqrt{T_1}}e^{\sqrt{\beta_2 t}}\partial_r, \\
\xi_{(6)} & = \frac{T_0'}{2\sqrt{\beta_2 T_0\sqrt{T_1}}} e^{-\sqrt{\beta_2 t}}\partial_t + \frac{1}{\sqrt{T_1}}e^{-\sqrt{\beta_2 t}}\partial_r.
\end{align*}
\]
(26)

The case $2a_1b_1c_1d_2e_2$ yields the same result as the case $1a_4b_1c_1(d_2)$, i.e., one proper MC.

The possibility $2a_1b_1(c_2)$, i.e., non-zero value of the constant $\beta_1$, gives us two more options:
\[
\begin{align*}
(d_1) \quad & (\frac{T_2}{T_0})' = 0, \\
(d_2) \quad & (\frac{T_2}{T_0})' \neq 0.
\end{align*}
\]

The first option $2a_1b_1c_2(d_1)$ gives ten independent MCs. The seven proper MCs are
\[
\begin{align*}
\xi_{(4)} & = z\partial_\theta - k_1\theta\partial_z, \\
\xi_{(5)} & = k_3\theta\partial_\theta + t\partial_\theta, \\
\xi_{(6)} & = \frac{k_3}{k_1}z\partial_t + t\partial_z, \\
\xi_{(7)} & = \frac{1}{2}(t^2 - \frac{4}{\beta_1^2 T_0} + k_3\theta^2 + \frac{k_3}{k_1}z^2)\partial_t - \frac{2}{\beta_1\sqrt{T_1}}t\partial_r + t\theta\partial_\theta + t\partial_\theta, \\
\xi_{(8)} & = -k_1t\theta\partial_t + \frac{2k_1}{\beta_1\sqrt{T_1}}\theta\partial_r + \frac{1}{2}\left(-\frac{k_1}{k_3}t^2 + \frac{4k_1}{\beta_1^2 T_2} - k_1\theta^2 + z^2\right)\partial_\theta - k_1\theta z\partial_z, \\
\xi_{(9)} & = tz\partial_t - \frac{2}{\beta_1\sqrt{T_1}}z\partial_r + \theta z\partial_\theta + \frac{1}{2}\left(\frac{k_1}{k_3}t^2 - \frac{4k_1}{\beta_1^2 T_2} - k_1\theta^2 + z^2\right)\partial_\theta, \\
\xi_{(10)} & = t\partial_t - \frac{2}{\beta_1\sqrt{T_1}}\partial_r + \theta\partial_\theta + z\partial_z.
\end{align*}
\]
(28)
where $k_3 = -\frac{T_2'}{T_0}$ is a constant.

The second option $2a_1b_1c_2(d_2)$ further yields two possibilities whether

\[(e_1) \quad \left(\frac{T_0'}{T_0\sqrt{T_1}}\right)' = 0, \quad (e_2) \quad \left(\frac{T_0'}{T_0\sqrt{T_1}}\right)' \neq 0.\]

If it is zero, i.e., the case $2a_1b_1c_2d_2(e_1)$, we have the following two proper MCs

$$\xi(4) = z\partial_t - k_1\theta\partial_z,$$
$$\xi(5) = \frac{\beta_1}{\beta_2}t\partial_t - \frac{2}{\beta_1\sqrt{T_1}}\partial_r + \theta\partial_\theta + z\partial_z, \quad (29)$$

where $\beta_4 = \frac{T_0'}{T_0\sqrt{T_1}}$ is a constant such that $\beta_1 \neq \beta_4$. If it is non-zero, i.e., $2a_1b_1c_2d_2(e_2)$, we have the same result as for the case $1a_4b_1c_1(d_2)$.

The case $2a_1(b_2)$, when $\left(\frac{T_2'}{\sqrt{T_2T_3}}\right)' \neq 0$, implies that $T_2' \neq 0$ which results the following two possibilities:

$$\xi(4) = z\partial_t - k_1\theta\partial_z,$$
$$\xi(5) = \beta_4t\partial_t - \frac{2}{\beta_1\sqrt{T_1}}\partial_r + \theta\partial_\theta + z\partial_z. \quad (30)$$

Subcase 2(a2): Here we have $\left(\frac{T_2'}{T_3}\right)' \neq 0$. If we use this constraint in MC Eqs.(7), (8) and (14), we have

$$T_2'E_1 = 0 = T_3E_1. \quad (31)$$

This gives rise to the following three possibilities:

\[(b_1) \quad T_2' = 0, \quad T_3' \neq 0,\]
\[(b_2) \quad T_2' \neq 0, \quad T_3' = 0,\]
\[(b_3) \quad T_2' \neq 0, \quad T_3' \neq 0.\]
In the first case, we can write
\[
\frac{T_0}{\sqrt{T_1}} (\frac{T'_0}{2T_0 \sqrt{T_1}})' = \gamma_1, \quad \frac{T_3}{\sqrt{T_1}} (\frac{T'_3}{2T_3 \sqrt{T_1}})' = \gamma_2, \tag{32}
\]
where \(\gamma_1\) and \(\gamma_2\) are arbitrary constants. From here we have the following four different cases:

\[
\begin{align*}
(c_1) \quad & \gamma_1 = 0, \quad \gamma_2 = 0, \\
(c_2) \quad & \gamma_1 \neq 0, \quad \gamma_2 = 0, \\
(c_3) \quad & \gamma_1 = 0, \quad \gamma_2 \neq 0, \\
(c_4) \quad & \gamma_1 \neq 0, \quad \gamma_2 \neq 0.
\end{align*}
\]

The first case \(2a_2b_1(c_1)\) can be divided into the following two options according as

\[
(d_1) \quad (\frac{T_0}{T_3})' = 0, \quad \text{or} \quad (d_2) \quad (\frac{T_0}{T_3})' \neq 0.
\]

The first option \(2a_2b_1(c_1)\) gives seven independent MCs in which four are proper

\[
\begin{align*}
\xi_{(4)} &= (\frac{\gamma_3 t^2}{2} - \int \frac{\sqrt{T_1}}{T_0} dr + \frac{\gamma_3}{2k_1} z^2) \partial_t + \frac{1}{\sqrt{T_1}} t \partial_r + \gamma_3 tz \partial_z, \\
\xi_{(5)} &= \gamma_3 t \partial_t + \frac{1}{\sqrt{T_1}} t \partial_r + (\frac{k_1 \gamma_3 t^2}{2} - \int \frac{\sqrt{T_1}}{T_3} dr + \frac{\gamma_3 z^2}{2}) \partial_z, \\
\xi_{(6)} &= \gamma_3 t \partial_t + \frac{1}{\sqrt{T_1}} \partial_r + \gamma_3 z \partial_z, \\
\xi_{(7)} &= z \partial_t + k_1 t \partial_z,
\end{align*}
\]

where \(k_1 = -\frac{T_0}{T_3}\) and \(\gamma_3 = -\frac{T'_0}{2T_0 \sqrt{T_1}}\) are constants. The second option \(2a_2b_1(c_2)\) yields only one proper MC given by

\[
\xi_{(4)} = \gamma_3 t \partial_t + \frac{1}{\sqrt{T_1}} t \partial_r + \gamma_4 tz \partial_z, \tag{34}
\]

where \(\gamma_4 = -\frac{T'_3}{2T_3 \sqrt{T_1}}\).

In the case \(2a_2b_1(c_2)\), when \(\gamma_1 \neq 0, \quad \gamma_2 = 0\), we have either

\[
(d_1) \quad \gamma_1 > 0, \quad \text{or} \quad (d_2) \quad \gamma_1 < 0.
\]
For $2a_2b_1c_2(d_1)$, when $\gamma_1 > 0$, we obtain three independent MCs which are the usual isometries.

The case $2a_2b_1c_2(d_2)$, when $\gamma_1 < 0$, we have two options

$$(e_1) \quad \frac{T_0}{T_3}' = 0, \quad \text{or} \quad (e_2) \quad \frac{T_0}{T_3}' \neq 0.$$ 

For the case $2a_2b_1c_2d_2(e_1)$, we have only one proper MC given by

$$\xi(4) = \frac{1}{k_1} z \partial_t + t \partial_z. \quad (35)$$

In the case $2a_2b_1c_2d_2(e_2)$, we obtain the minimal symmetry.

The case $2a_2b_1(c_3)$, when $\gamma_1 = 0, \gamma_2 \neq 0$, is similar to the previous case $2a_2b_1(c_2)$ by interchanging $t$ and $z$.

The case $2a_2b_1(c_4)$, when $\gamma_1 \neq 0, \gamma_2 \neq 0$, yields the following four different possibilities:

$$(d_1) \quad \gamma_1 > 0, \quad \gamma_2 > 0,$$

$$(d_2) \quad \gamma_1 > 0, \quad \gamma_2 < 0,$$

$$(d_3) \quad \gamma_1 < 0, \quad \gamma_2 > 0,$$

$$(d_4) \quad \gamma_1 < 0, \quad \gamma_2 < 0.$$ 

The first possibility further gives two options according as

$$(e_1) \quad \gamma_2 T_0 \int \frac{\sqrt{T_1}}{T_0} dr + \frac{T_3'}{2\sqrt{T_1}} = 0,$$

$$or \quad (e_2) \quad \gamma_2 T_0 \int \frac{\sqrt{T_1}}{T_0} dr + \frac{T_3'}{2\sqrt{T_1}} \neq 0.$$ 

The first option $2a_2b_1c_4d_1(e_1)$ gives the minimal symmetry. The second option $2a_2b_1c_4d_1(e_2)$ further gives two possibilities

$$(f_1) \quad \frac{T_0}{T_3}' = 0, \quad (f_2) \quad \frac{T_0}{T_3}' \neq 0.$$ 

The case $2a_2b_1c_4d_1e_2(f_1)$ yields four independent MCs in which the proper MC is given by

$$\xi(4) = z \partial_t + k_1 t \partial_z. \quad (36)$$

For the case $2a_2b_1c_4d_1e_2(f_2)$, we have three independent MCs.
All other cases $2a_2b_1c_4(d_2 - d_4)$ are similar to the previous case $2a_2b_1c_4(d_1)$. The case $2a_2(b_2)$, when $T'_2 \neq 0, T'_3 = 0$, is similar to the first case $2a_2(b_1)$.

In the third case $2a_2(b_3)$, when $T'_2 \neq 0, T'_3 \neq 0$, we further have the following two possibilities:

\[(c_1) \quad \left(\frac{T'_2}{2T_2\sqrt{T_1}}\right)' = 0, \quad \left(\frac{T'_3}{2T_3\sqrt{T_1}}\right)' = 0,\]

\[(c_2) \quad \left(\frac{T'_2}{2T_2\sqrt{T_1}}\right)' \neq 0, \quad \left(\frac{T'_3}{2T_3\sqrt{T_1}}\right)' \neq 0.\]

For the first possibility $2a_2b_3(c_1)$, we obtain the following three options:

\[(d_1) \quad \left(\frac{T'_2}{T_0}\right)' = 0, \quad \left(\frac{T'_3}{T_0}\right)' \neq 0,\]

\[(d_2) \quad \left(\frac{T'_2}{T_0}\right)' \neq 0, \quad \left(\frac{T'_3}{T_0}\right)' = 0,\]

\[(d_3) \quad \left(\frac{T'_2}{T_0}\right)' \neq 0, \quad \left(\frac{T'_3}{T_0}\right)' \neq 0.\]

The first option $2a_2b_3c_1(d_1)$ yields five independent MCs in which three are the usual KVs and the remaining are the proper MCs given by

\[\xi^{(4)} = \gamma_3 t \partial_t + \frac{1}{\sqrt{T_1}} \partial_r + \gamma_3 \theta \partial_\theta + \gamma_4 z \partial_z,\]

\[\xi^{(5)} = k_2 \theta \partial_t + t \partial_\theta,\]  \hspace{1cm} (37)

where $k_2 = -\frac{T_2}{T_0}$ is a non-zero constant. The second option $2a_2b_3c_1(d_2)$ is similar to the first one. The third case $2a_2b_3c_1(d_3)$ implies that either

\[(e_1) \quad \left(\frac{T'_0}{2T_0\sqrt{T_1}}\right)' = 0, \quad or \quad (e_2) \quad \left(\frac{T'_0}{2T_0\sqrt{T_1}}\right)' \neq 0.\]

For the first option $2a_2b_3c_1d_3(e_1)$, we get one proper MC given by

\[\xi^{(4)} = \gamma_3 t \partial_t + \frac{1}{\sqrt{T_1}} \partial_r + \gamma_3 \theta \partial_\theta + \gamma_4 z \partial_z,\]  \hspace{1cm} (38)

In the second option $2a_2b_3c_1d_3(e_2)$, we obtain the minimal symmetry.
The case $2a_2b_3(c_2)$ also yields the same three possibilities:

$$(d_1) \quad \left( \frac{T_2}{T_0} \right)' = 0, \quad \left( \frac{T_3}{T_0} \right)' \neq 0,$$

$$(d_2) \quad \left( \frac{T_2}{T_0} \right)' \neq 0, \quad \left( \frac{T_3}{T_0} \right)' = 0,$$

$$(d_3) \quad \left( \frac{T_2}{T_0} \right)' \neq 0, \quad \left( \frac{T_3}{T_0} \right)' \neq 0.$$ 

It is to be noted that we have excluded the possibility when both are constants as this leads $\frac{T_2}{T_3}$ to be constant which gives a contradiction. The first case $2a_2b_3c_2(d_1)$ gives only one proper MC, i.e.

$$\xi(4) = k_2 \theta \partial_t + t \partial_\theta. \quad (39)$$

The second case $2a_2b_3c_2(d_2)$ is similar to the previous one and the third case $2a_2b_3c_2(d_3)$ gives the minimal MCs.

## 4 Matter Collineations in the Degenerate Case

In this section only those cases will be considered for which $\det(T_{ab}) = 0$ which implies that at least one of the components of the energy-momentum tensor is zero, i.e., $T_a = 0$. The trivial case is that when all $T_a$ are zero. In this case, every direction in a MC. The remaining cases can be divided into three main groups:

1. When at least one of $T_a$ is non-zero;
2. When at least two of $T_a$ are non-zero;
3. When three of $T_a$ are non-zero.

**Case 1:** This case can further be divided into the following four subcases:

- $(a_1) \quad T_0 = 0, \quad T_1 = 0, \quad T_2 = 0, \quad T_3 \neq 0,$
- $(a_2) \quad T_0 = 0, \quad T_1 = 0, \quad T_2 \neq 0, \quad T_3 = 0,$
- $(a_3) \quad T_0 = 0, \quad T_1 \neq 0, \quad T_2 = 0, \quad T_3 = 0,$
- $(a_4) \quad T_0 \neq 0, \quad T_1 = 0, \quad T_2 = 0, \quad T_3 = 0.$

When we use the values of 1$(a_1)$ in MC equations, we obtain $\xi^3 = \xi^3(z)$ and Eq.(9) gives

$$\xi^1 = -\frac{2T_3}{T_0} \xi^3(z), \quad (40)$$

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where \( T_3' \neq 0 \) and \( \xi^0, \xi^1 \) are arbitrary functions of \( t, r, \theta, z \). This gives infinite dimensional MCs. The second case 1(a2) is similar to the first one if we interchange the indices 2 and 3.

The third case 1(a3) gives \( \xi^1 = \xi^1(r) \) and Eq.(7) yields

\[
\xi^1 = \frac{c_1}{\sqrt{T_1}},
\]

where \( \xi^0, \xi^2, \xi^3 \) are arbitrary functions of \( t, r, \theta, z \) which gives infinite dimensional MCs. The fourth case 1(a4) yields the similar result as the case 1(a1) by interchanging the indices 0 and 3. Thus we obtain infinite dimensional MCs in all the possibilities of the case 1.

**Case 2:** This case has the following six possibilities:

- \( (a_1) \) \( T_0 = 0, \ T_1 = 0, \ T_2 \neq 0, \ T_3 \neq 0, \)
- \( (a_2) \) \( T_0 = 0, \ T_1 \neq 0, \ T_2 = 0, \ T_3 \neq 0, \)
- \( (a_3) \) \( T_0 = 0, \ T_1 \neq 0, \ T_2 \neq 0, \ T_3 = 0, \)
- \( (a_4) \) \( T_0 \neq 0, \ T_1 = 0, \ T_2 = 0, \ T_3 \neq 0, \)
- \( (a_5) \) \( T_0 \neq 0, \ T_1 = 0, \ T_2 \neq 0, \ T_3 = 0, \)
- \( (a_6) \) \( T_0 \neq 0, \ T_1 \neq 0, \ T_2 = 0, \ T_3 = 0. \)

When we replace the information of the subcase 2(a1) in MC equations, we obtain \( \xi^0 = \xi^0(t, r, \theta, z), \ \xi^2 = \xi^2(\theta, z), \ \xi^3 = \xi^3(\theta, z) \) and

\[
\left( \frac{T_3}{T_2} \right)' \xi^3_{,2} = 0. \tag{42}
\]

From here we have two options:

- \( (b_1) \) \( \left( \frac{T_3}{T_2} \right)' = 0, \)
- \( (b_2) \) \( \left( \frac{T_3}{T_2} \right)' \neq 0. \)

For the first option 2a1(b1), MC equations yield

\[
\xi^2_{,22} + \frac{1}{c} \xi^2_{,33} = 0, \tag{43}
\]

\[
\xi^3_{,22} + \frac{1}{c} \xi^3_{,33} = 0, \tag{44}
\]
where \( c = \frac{T_1}{T_2} \) is a non-zero constant. If \( c > 0 \), Eqs. (43) and (44) yield the following solution

\[
\xi^2 = f_+(\theta + \frac{i z}{\sqrt{c}}) + f_-(\theta - \frac{i z}{\sqrt{c}}),
\]

\[
\xi^3 = g_+(\theta + \frac{i z}{\sqrt{c}}) + g_-(\theta - \frac{i z}{\sqrt{c}}).
\]

Replacing the value of \( \xi^2 \) in Eq. (8), we obtain

\[
\xi^1 = -\frac{2T_2}{T_2'} [f_{+2}(\theta + \frac{i z}{\sqrt{c}}) + f_{-2}(\theta - \frac{i z}{\sqrt{c}})],
\]

where \( T_2' \neq 0 \) and \( \xi^0 = \xi^0(t, r, \theta, z) \). If we take \( T_2 = constant \), MC equations give the following solution

\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), & \xi^1 &= \xi^1(t, r, \theta, z), \\
\xi^2 &= c_1 z + c_2, & \xi^3 &= c_1 \theta + c_3.
\end{align*}
\]

The option \( 2a_1(b_2) \) further divides into three cases:

\[
\begin{align*}
(c_1) & \quad T_2' = 0, \quad T_3' \neq 0, \\
(c_2) & \quad T_2' \neq 0, \quad T_3' = 0, \\
(c_3) & \quad T_2' \neq 0, \quad T_3' \neq 0.
\end{align*}
\]

In the case \( 2a_1b_2(c_1) \), we get the following solution

\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), & \xi^1 &= \frac{T_3}{T_3} f(z), \\
\xi^2 &= c_1, & \xi^3 &= -\frac{1}{2} \int f(z) dz + c_2.
\end{align*}
\]

The case \( 2a_1b_2(c_2) \) gives the similar results as the case \( 2a_1b_2(c_1) \) by interchanging \( \theta \) and \( z \).

If we solve MC equations for the possibility \( 2a_1b_2(c_3) \), we have the following solution

\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), & \xi^1 &= \frac{T_2}{T_2 c_1}, \\
\xi^2 &= c_1 \theta + c_2, & \xi^3 &= c_3 z + c_4.
\end{align*}
\]
For the subcase 2(a2), we further have the following two options from MC equations

\[(b_1) \quad \left[ \frac{T_3}{\sqrt{T_1}} \left( \frac{T_3'}{2T_3\sqrt{T_1}} \right) \right]' = 0,\]
\[(b_2) \quad \left[ \frac{T_3}{\sqrt{T_1}} \left( \frac{T_3'}{2T_3\sqrt{T_1}} \right) \right]' \neq 0.\]

Also \(\xi^0 = \xi^0(t, r, \theta, z), \quad \xi^1 = \xi^1(r, z), \quad \xi^2 = \xi^2(r), \quad \xi^3 = \xi^3(r, z).\) The first possibility 2a2(b1) gives \(\frac{T_3}{\sqrt{T_1}} \left( \frac{T_3'}{2T_3\sqrt{T_1}} \right) = c,\) where \(c\) is an arbitrary constant which implies that either

\[(c_1) \quad c > 0, \quad (c_2) \quad c = 0, \quad or \quad c < 0.\]

The case 2a2b1(c1) yields the following solution

\[\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \\
\xi^1 &= \frac{1}{\sqrt{T_1}} \left( c_1 e^{\sqrt{T_1}z} + c_2 e^{-\sqrt{T_1}z} \right), \\
\xi^2 &= \xi^2(r, z), \\
\xi^3 &= -c_1 \int \frac{1}{T_3} dr - \frac{T_3'}{2T_3\sqrt{T_1}} \left( c_1 z^2 + c_2 z \right) + c_3. \quad (51)
\end{align*}\]

For the case 2a2b1(c2), when \(c = 0,\) we obtain

\[\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \\
\xi^1 &= \frac{1}{\sqrt{T_1}} (c_1 z + c_2), \\
\xi^2 &= \xi^2(r, z) \\
\xi^3 &= -c_1 \int \frac{1}{T_3} dr - \frac{T_3'}{2T_3\sqrt{T_1}} \left( c_1 \frac{z^2}{2} + c_2 z \right) + c_3. \quad (52)
\end{align*}\]

In the case 2a2b1(c3), when \(c < 0,\) the solution is similar to the previous case 2a2b1(c2).

The second possibility 2a2(b2) gives

\[\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \quad \xi^1 = 0, \\
\xi^2 &= \xi^2(r, z), \quad \xi^3 = c_1. \quad (53)
\end{align*}\]

The subcase 2(a3) gives similar results as the case 2(a2) by interchanging the indices 2 and 3.
The subcases 2(a4) and 2(a5) yield results similar to the case 2(a1) if we interchange indices 0, 2 and 0, 3 respectively.

The subcase 2(a6) would give similar result as the case 2(a2) by interchanging the indices 0, 3. It is to be noted that we again have infinite dimensional MCs in the case 2.

Case 3: The case, when only one component of the energy-momentum tensor is zero, can have four different subcases:

(a1) $T_0 = 0, \ T_1 \neq 0, \ T_2 \neq 0, \ T_3 \neq 0,$

(a2) $T_0 \neq 0, \ T_1 = 0, \ T_2 \neq 0, \ T_3 \neq 0,$

(a3) $T_0 \neq 0, \ T_1 \neq 0, \ T_2 = 0, \ T_3 \neq 0,$

(a4) $T_0 \neq 0, \ T_1 \neq 0, \ T_2 \neq 0, \ T_3 = 0.$

The first subcase gives $\xi^0 = \xi^0(t, r, \theta, z)$ with the following two possibilities:

(b1) $\left[ \frac{T_2}{\sqrt{T_1}} \left( -\frac{T'_2}{2T_2\sqrt{T_1}} \right) \right]' = 0,$

(b2) $\left[ \frac{T_2}{\sqrt{T_1}} \left( -\frac{T'_2}{2T_2\sqrt{T_1}} \right) \right]' \neq 0.$

In the first possibility 3a1(b1), we have $\frac{T_2}{\sqrt{T_1}} \left( -\frac{T'_2}{2T_2\sqrt{T_1}} \right)' = \alpha_1,$ where $\alpha_1$ is an arbitrary constant which can be such that

(c1) $\alpha_1 > 0,$  \quad (c2) $\alpha_1 = 0,$  \quad (c3) $\alpha_1 < 0.$

The case 3a1b1(c1), when $\alpha_1 > 0$ gives further three options according as $\alpha_2 = \frac{T_3}{\sqrt{T_1}} \left( -\frac{T'_3}{2T_3\sqrt{T_1}} \right)'$

(d1) $\alpha_2 > 0,$  \quad (d2) $\alpha_2 = 0,$  \quad (d3) $\alpha_2 < 0.$

The first option 3a1b1c1(d1), when $\alpha_2 > 0$, gives either

(e1) $\sqrt{\frac{\alpha_1}{\alpha_2}} + \sqrt{\frac{\alpha_2}{\alpha_1}} = 0,$  \quad or  \quad (e2) $\sqrt{\frac{\alpha_1}{\alpha_2}} + \sqrt{\frac{\alpha_2}{\alpha_1}} \neq 0.$

For the first case 3a1b1c1d1(e1), we obtain

$$\xi^0 = \xi^0(t, r, \theta, z), \quad \xi^1 = 0,$$
$$\xi^2 = c_1 z + c_2, \quad \xi^3 = c_1 \theta + c_3.$$  \quad (54)
In the second case $3a_1b_1c_1d_1(e_2)$, we have the solution
\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \\
\xi^1 &= 0, \\
\xi^2 &= c_1z + c_2, \\
\xi^3 &= -c_1k_1\theta + c_3, \\
\end{align*}
\]
(55)
where $k_1 = \frac{T_2}{T_3}$ is a constant.

The second option $3a_1b_1c_1(d_2)$, when $\alpha_2 = 0$, yields the following solution
\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \\
\xi^1 &= \frac{1}{\sqrt{T_1}}[e^{i\sqrt{\alpha_1\theta}}(c_1z + c_2) + e^{-i\sqrt{\alpha_1\theta}}(c_3z + c_4)], \\
\xi^2 &= -\frac{iT_2\sqrt{T_1}}{2T_2\sqrt{T_1}}[e^{i\sqrt{\alpha_1\theta}}(c_1z + c_2) - e^{-i\sqrt{\alpha_1\theta}}(c_3z + c_4)] + c_5, \\
\xi^3 &= -\alpha_3[e^{i\sqrt{\alpha_1\theta}}(c_1z^2 + c_2z) + e^{-i\sqrt{\alpha_1\theta}}(c_3z^2 + c_4z)] \\
&\quad - [c_1e^{i\sqrt{\alpha_1\theta}} + c_3e^{-i\sqrt{\alpha_1\theta}}]\int \frac{\sqrt{T_1}}{T_3}dr + c_6,
\end{align*}
\]
(56)
where $\alpha_3 = \frac{T_3}{2T_3\sqrt{T_1}}$ is a constant.

The third option $3a_1b_1c_1(d_3)$, for $\alpha_2 < 0$, is similar to the first option $3a_1b_1c_1(d_1)$.

Now we come to the case $3a_1b_1c_2$ when $\alpha_1 = 0$ which gives $\alpha_4 = -\frac{T_2'}{2T_2\sqrt{T_1}}$, a constant such that
\[
(d_1) \quad \alpha_4 = 0, \quad (d_2) \quad \alpha_4 \neq 0.
\]

In the case $3a_1b_1c_2(d_1)$, we have $\alpha_2 = \frac{T_1}{\sqrt{T_1}}(-\frac{T_2'}{2T_2\sqrt{T_1}})'$ which yields the following options:
\[
(e_1) \quad \alpha_2 > 0, \quad (e_2) \quad \alpha_2 = 0, \quad (e_3) \quad \alpha_2 < 0.
\]

For the case $3a_1b_1c_2d_1(e_1)$, when $\alpha_2 > 0$, we have the following MCs
\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \\
\xi^1 &= \frac{1}{\sqrt{T_1}}[e^{i\sqrt{\alpha_2\theta}}(c_1z + c_2) + e^{-i\sqrt{\alpha_2\theta}}(c_3z + c_4)], \\
\xi^2 &= 0, \\
\xi^3 &= -\frac{T_3'}{i2\sqrt{\alpha_2T_3\sqrt{T_1}}}(c_1e^{i\sqrt{\alpha_2z}} - c_2e^{-i\sqrt{\alpha_2z}}).
\end{align*}
\]
(57)

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The case $3a_1b_1c_2d_1(e_2)$, when $\alpha_2 = 0$, we have further two possibilities according as $\alpha_5 = -\frac{T_3'}{2T_3\sqrt{T_1}}$, a constant such that

\[(f_1) \quad \alpha_5 = 0, \quad (f_2) \quad \alpha_5 \neq 0.\]

For the case $3a_1b_1c_2d_1e_2(f_1)$, we get the following result

\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \quad \xi^1 = \frac{1}{\sqrt{T_1}}(c_1\theta + c_2z + c_3), \\
\xi^2 &= -\frac{c_1}{T_2} \int \sqrt{T_1}dr, \quad \xi^3 = \frac{c_2}{T_3} \int \sqrt{T_1}dr.
\end{align*}
\]

(58)

For $\alpha_3$ to be non-zero, i.e., the case $3a_1b_1c_2d_1e_2(f_2)$, we obtain

\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \quad \xi^1 = \frac{1}{\sqrt{T_1}}(c_1z + c_2), \\
\xi^2 &= 0, \quad \xi^3 = \alpha_5(c_1^2 \frac{z^2}{2} + c_2z) - c_1 \int \sqrt{T_1}T_3 dr.
\end{align*}
\]

(59)

The case $3a_1b_1c_2d_1(e_3)$ when $\alpha_2 < 0$ is similar to the case $3a_1b_1c_2d_1(e_1)$.

In the case $3a_1b_1c_2d_2(e_1)$ when $\alpha_2 \neq 0$, we have $\alpha_2 = \frac{T_3'}{\sqrt{T_1}}(-\frac{T_3'}{2T_3\sqrt{T_1}})',$, a constant which gives the following three options:

\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \quad \xi^1 = 0, \\
\xi^2 &= (c_1z + c_2), \quad \xi^3 = \alpha_3(c_1\theta + c_3).
\end{align*}
\]

(60)

The case $3a_1b_1c_2d_2(e_2)$, when $\alpha_2 = 0$, gives the following MCs

\[
\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \quad \xi^1 = \frac{c_1}{T_1}, \\
\xi^2 &= (\alpha_3c_1\theta + c_2), \quad \xi^3 = \alpha_3c_1z + c_3.
\end{align*}
\]

(61)

The last possibility $3a_1b_1c_2d_2(e_3)$ when $\alpha_2$ is negative yields similar solution to the positive case $3a_1b_1c_2d_2(e_1)$.

The case $3a_1b_1(c_3)$, when $\alpha_1 < 0$, is similar to the case $3a_1b_1(c_1)$. 

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The case $3a_1(b_2)$ yields the following solution

$$\begin{align*}
\xi^0 &= \xi^0(t, r, \theta, z), \quad \xi^1 = 0, \\
\xi^2 &= c_1z + c_2, \quad \xi^3 = c_3\theta + c_4.
\end{align*}$$ (62)

The case $3(a_2)$ when $T_0 \neq 0, \ T_1 = 0, \ T_2 \neq 0, \ T_3 \neq 0$ gives the following two options:

$$(b_1) \ T'_0 = 0, \quad (b_2) \ T'_0 \neq 0.$$ The first option $3a_2(b_1)$ gives either

$$(c_1) \ \left( \frac{T'_2T'_3}{T_2T_3} \right) = 0, \quad \text{or} \quad (c_2) \ \left( \frac{T'_2T'_3}{T_2T_3} \right) \neq 0.$$

The case $3a_2b_1(c_1)$ gives one proper MC

$$\xi_{(4)} = -\frac{2T_2}{T'_2} \beta_1 \partial_r + \beta_1 \theta \partial_\theta + z \partial_z,$$ (63)

where $\beta_1 = \frac{T'_2T'_3}{T_2T_3}$. For the case $3a_2b_1(c_2)$, we have either

$$(d_1) \ T'_2 = 0 = T'_3, \quad \text{or} \quad (d_2) \ T'_2 \neq 0, \ T'_3 \neq 0.$$ The case $3a_2b_1c_2(d_1)$ yields infinite dimensional MCs. For the case $3a_2b_1c_2(d_2)$, we get three MCs which are the usual KVs.

The case $3a_2(b_2)$ divides into four groups:

$$(c_1) \ \left( \frac{T'_0}{T_2} \right) = 0, \quad \left( \frac{T'_0}{T_3} \right) = 0,$$

$$(c_2) \ \left( \frac{T'_0}{T_2} \right) = 0, \quad \left( \frac{T'_0}{T_3} \right) \neq 0,$$

$$(c_3) \ \left( \frac{T'_0}{T_2} \right) \neq 0, \quad \left( \frac{T'_0}{T_3} \right) = 0,$$

$$(c_4) \ \left( \frac{T'_0}{T_2} \right) \neq 0, \quad \left( \frac{T'_0}{T_3} \right) \neq 0.$$ The first group $3a_2b_2(c_1)$ gives ten independent MCs in which seven are proper MCs given by

$$\xi_{(4)} = -\frac{k_1}{k_2}z \partial_\theta + \theta \partial_z.$$
\[ \xi(5) = \theta \partial_t + k_1 t \partial_\theta, \]
\[ \xi(6) = z \partial_t + k_2 t \partial_z, \]
\[ \xi(7) = \left( \frac{\theta^2}{2} + \frac{k_1 z^2}{2k_2} + \frac{k_1 t^2}{2} \right) \partial_t - \frac{2k_1 T_0}{T_0} t \partial_r + k_1 t \theta \partial_\theta + k_1 t z \partial_z, \]
\[ \xi(8) = \left( k_1 t \frac{\partial t}{T_0} + \frac{\partial z}{T_0} \right) + \frac{t^2}{2} - \frac{k_1 z^2}{2k_2} \right) \partial_\theta + \theta z \partial_z, \]
\[ \xi(9) = t \partial_t - \frac{2T_0}{T_0} t \partial_r + \theta \partial_\theta + z \partial_z, \]
\[ \xi(10) = t \partial_t - \frac{2T_0}{T_0} t \partial_r + \theta \partial_\theta + z \partial_z, \] (64)

where \( k_1 = -\frac{T_0}{T_2} \) and \( k_2 = -\frac{T_0}{T_3} \) are non-zero constants.

The second group \( 3a_2b_2(c_2) \) when \( \left( \frac{T_0}{T_2} \right)' = 0, \left( \frac{T_0}{T_4} \right)' \neq 0 \) can give two more possibilities whether

\[ (d_1) \left( \frac{T_3' T_0}{T_3 T_0} \right)' = \text{constant}, \quad (d_2) \left( \frac{T_3' T_0}{T_3 T_0} \right)' \neq \text{constant} \]

The case \( 3a_2b_2c_2(d_1) \), we have two proper MC given by

\[ \xi(4) = \theta \partial_t + k_1 t \partial_\theta, \]
\[ \xi(5) = t \partial_t - \frac{2T_0}{T_0} t \partial_r + \theta \partial_\theta + c z \partial_z, \] (65)

where \( \frac{T_0}{T_4} = c \) is an arbitrary constant. If it is not constant, i.e., the case \( 3a_2b_2c_2(d_2) \), we obtain only one proper MC given by

\[ \xi(4) = \theta \partial_t + k_1 t \partial_\theta. \] (66)

The third group \( 3a_2b_2(c_3) \) when \( \left( \frac{T_0}{T_2} \right)' \neq 0, \left( \frac{T_0}{T_4} \right)' = 0 \) would give the similar solution as the previous one \( 3a_2b_2(c_2) \).

The last group \( 3a_2b_2(c_4) \) when \( \left( \frac{T_0}{T_2} \right)' \neq 0, \left( \frac{T_0}{T_4} \right)' \neq 0 \) would give the following four possibilities:

\[ (d_1) \left( \frac{T_0' T_2}{T_0 T_2} \right)' \neq 0, \quad \left( \frac{T_0' T_3}{T_0 T_3} \right)' \neq 0, \]
\[ (d_2) \left( \frac{T_0' T_2}{T_0 T_2} \right)' \neq 0, \quad \left( \frac{T_0' T_3}{T_0 T_3} \right)' = 0, \]
In the first possibility $3a_2b_2c_4(d_1)$, we have either

\[(e_1) \quad \frac{T_2}{T_3} = \text{constant}, \quad (e_2) \quad \frac{T_2}{T_3} \neq \text{constant}.\]

The case $3a_2b_2c_4d_1(e_1)$ gives one proper MC. The proper MC is

\[\xi(4) = z\partial_\theta - \frac{T_2}{T_3} \theta \partial_z. \quad (67)\]

For the case $3a_2b_2c_4d_1(e_2)$, we get three MCs which are KVs. The possibilities $3a_2b_2c_4(d_2)$ and $3a_2b_2c_4(d_3)$ are similar to the case $3a_2b_2c_4(d_1)$.

In the last possibility $3a_2b_2c_4(d_4)$, we obtain five MCs in which two are proper, i.e.,

\[
\xi(4) = z\partial_\theta - \frac{T_2}{T_3} \theta \partial_z,
\]

\[
\xi(5) = t\partial_t - \frac{2T_0}{T_0} \partial_r + \frac{1}{\beta_2} \theta \partial_\theta + \frac{1}{\beta_2} \partial_z, \quad (68)
\]

where $\frac{T_2 T_2}{T_0 T_0} = \beta_2$, a constant. We also take $\frac{T_2}{T_3}$ to be constant.

The subcases $3(a_3)$ and $3(a_4)$ can be proceeded as the subcase $3(a_1)$ and would give similar results.

### 5 Examples of Finite Dimensional Matter Collineations

We see from sections 3 and 4 that when we find MCs for the cylindrically symmetric static spacetimes, we obtain different constraints on the energy-momentum tensor. If we solve these constraints, we can have exact solutions of EFEs or class of solutions can be obtained. In this section, we would attempt to solve some of these constraints to get explicit form of the metrics. We are not providing the details rather than we would provide list of
solutions and their properties satisfying the constraints.

1. When we solve the constraints of the case $2a_1b_2(c_2)$, we obtain the following metric

$$ds^2 = \cosh^2 cr dt^2 - dr^2 - (\cosh cr)^{-1} d\theta^2 - (\cosh cr)^{-1} dz^2,$$

where $c$ is an arbitrary constant. This metric admits 4 MCs and also 4 isometries. This implies that there is no proper MC in this example but it has 7 RCs. This metric has an anisotropic fluid with energy-density positive for $0 \leq r < \frac{1}{c} \tanh^{-1} \frac{2}{\sqrt{7}}$ and negative for $r \geq \frac{1}{c} \tanh^{-1} \frac{2}{\sqrt{7}}$.

2. If we solve the constraints of the case $2a_2b_3c_1d_3(e_1)$, the spacetime would be

$$ds^2 = \left(\frac{r}{r_0}\right)^{2a} dt^2 - dr^2 - \left(\frac{r}{r_0}\right)^{2b} d\theta^2 - \left(\frac{r}{r_0}\right)^{2c} dz^2,$$

where $a$, $b$, $c$ and $r_0$ are arbitrary constants such that $a, b, c \neq 0, 1$. The components of the energy-momentum tensor for this metric are given in Appendix B. This metric has 4 MCs with 3 KVs giving 1 proper MC.

3. If we choose the values of $a$, $b$, $c$ such that $a = 1$, $b = c \neq 1$ in Eq.(70), we get a metric which admits 7 MCs and 4 KVs satisfying the constraints $1a_4b_1c_2d_1e_2(f_1)$. This gives three proper MCs. It is obvious from the energy-momentum tensor given in Eq.(B2) that the energy density will be positive for $0 < b < 2/3$.

4. When we take $a = b = c \neq 0, 1$ in Eq.(70), we obtain a metric which satisfies the constraints given in $2a_1b_1c_2(d_1)$. This spacetime admits 10 MCs with 6 KVs and hence we have 4 proper MCs in this example. It can be seen from the energy-momentum tensor that it becomes singular at $r = 0$.

5. If we take $b = c \neq 0, 1$ in Eq.(70), we get a metric satisfying the constraints of the case $2a_1b_1c_2d_2(e_1)$ admitting 5 MCs but 3 KVs. This is another example admitting proper MCs. We must take $0 < b < 2/3$ to make the energy-density positive. It is to be noted that the resulting metric would represent a perfect fluid for $b = a(a - 1)/(a + 1)$ and non-null electromagnetic field when $b = a + 1$. 

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6. Taking $\nu = \lambda = \mu$ in Eq.(4), it satisfies the constraints of the case $2a_1b_2(c_1)$. This metric admits 6 MCs and also 6 KVs.

7. When we choose $a = b \neq c$ such that $a, c \neq 0, 1$ in Eq.(70), we obtain a metric satisfying the constraints of the case $2a_2b_3c_1(d_1)$. This metric admits 5 MCs with 4 isometries hence giving 1 proper MC.

8. If we take either $\nu = \lambda$ or $\nu = \mu$ in Eq.(4), we have the solution of the constraints given by $2a_2b_3c_2(d_1)$. This metric admits 4 MCs and 4 isometries.

9. Now solving the constraints of the case $2a_2b_3c_2(d_2)$, we obtain the following solution

$$ds^2 = (\cosh cr)^{-1}dt^2 - dr^2 - \cosh^2 cr d\theta^2 - (\cosh cr)^{-1}dz^2.$$ \hfill (71)

It has 4 MCs and also 4 KVs but 7 RCs. This spacetime represents anisotropic tachyonic fluid.

10. If we take $a = -\frac{1}{4}$, $b = c = \frac{1}{2}$ in Eq.(70), we get $T_1 = 0$ which gives the degenerate case and hence satisfies the constraints of the case $3a_2b_3c_4(d_4)$. This metric admits 5MCs and 4 KVs. Thus we have one proper MC in this degenerate case.

\section{Conclusion}

We know from the classification of cylindrically symmetric static spacetimes according to their isometries that we either get three, four, five, six, seven or ten isometries. The ten and seven KVs are admitted by the well known anti-de Sitter and anti-Einstein universes respectively. The six isometries are admitted by the Bertotti-Robinson metric with the isometry group $SO(1, 1) \times \mathbb{R}^3 \otimes SO(3)$. There is one class of metrics depending on one arbitrary function with six isometries given by

$$ds^2 = e^\nu(dt^2 - dr^2 - d\theta^2 - dz^2).$$ \hfill (72)

There are three cases of five dimensional isometry groups. There are also three classes of metrics depending upon two arbitrary functions having four dimensional isometry groups given by

$$ds^2 = e^\nu dt^2 - dr^2 - e^\mu(d\theta^2 - dz^2),$$ \hfill (73)
\[ ds^2 = e^\nu (dt^2 - dz^2) - dr^2 - e^\lambda d\theta^2, \]  
\[ ds^2 = e^\nu (dt^2 - d\theta^2) - dr^2 - e^\mu dz^2. \]

All other metrics admit minimal isometry group \( G_3 \).

This paper presents a complete classification of cylindrically symmetric static spacetimes according to their MCs. We have solved MC equations for both non-degenerate and degenerate cases. The explicit forms of MCs are given in each case. We have also written the corresponding constraints on the energy-momentum tensor in each case. Finally, we have attempted to solve some of these constraints to find the exact solution of EFEs.

When the energy-momentum tensor is non-degenerate (section 3), we obtain either three, four, five, six, seven or ten independent MCs. Out of these MCs, we obtain three isometries and the rest are the non-trivial (proper) MCs. In the degenerate case (section 4), most of the possibilities lead to infinite dimensional MCs. However, there are some worth mentioning cases where we obtain finite dimensional MCs even with degenerate energy-momentum tensor. In these case, we get either three, four, five or ten independent MCs in which three are the usual KVs and the rest are the proper MCs.

The summary of the results can be given below in the form of tables.

**Table 1.** MCs of Case (1) for the Non-degenerate Energy-Momentum Tensor.

| Cases   | MCs | Constraints |
|---------|-----|-------------|
| 1(\(a_1\)) | 10  | \(T'_0 = 0, T'_2 = 0 = T'_3\) |
| 1\(a_2b_1(\(c_1\))\) | 6   | \(T'_0 = 0, T'_2 = 0, T'_3 \neq 0, [\frac{T'_0}{\sqrt{T'_1}}(\frac{T'_1}{2T'_3\sqrt{T'_1}})]' = 0, \frac{T'_0}{\sqrt{T'_1}}(\frac{T'_1}{2T'_3\sqrt{T'_1}})' = \alpha_1 > 0\) |
| 1\(a_2b_1(\(c_2\))\) | 6   | \(T'_0 = 0, T'_2 = 0, T'_3 \neq 0, [\frac{T'_0}{\sqrt{T'_1}}(\frac{T'_1}{2T'_3\sqrt{T'_1}})]' = 0, \alpha_1 = 0\) |
| 1\(a_2b_1(\(c_3\))\) | 6   | \(T'_0 = 0, T'_2 = 0, T'_3 \neq 0, [\frac{T'_0}{\sqrt{T'_1}}(\frac{T'_1}{2T'_3\sqrt{T'_1}})]' = 0, \alpha_1 < 0\) |
| 1\(a_2(b_2)\) | 4   | \(T'_0 = 0, T'_2 = 0, T'_3 \neq 0, [\frac{T'_0}{\sqrt{T'_1}}(\frac{T'_1}{2T'_3\sqrt{T'_1}})]' \neq 0\) |
| Cases | MCs | Constraints |
|-------|-----|-------------|
| $1(a_3)$ | $1(a_2)$ | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 = 0$ |
| $1a_4b_1c_1(d_1)$ | 6 | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 \neq 0$, \[
\frac{T'_0}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} = \alpha_3 > 0, \quad \frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} = 0,
\] |
| $1a_4b_1c_1(d_2)$ | 4 | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 \neq 0$, \[
\frac{T'_0}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} = \alpha_3 > 0, \quad \frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} \neq 0, \quad \frac{T'_3}{T_3} \neq \text{constant} \neq 0
\] |
| $1a_4b_1c_1(d_2)$ | 3 | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 \neq 0$, \[
\frac{T'_0}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} = \alpha_3 > 0, \quad \frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} \neq 0, \quad \frac{T'_3}{T_3} \neq \text{constant}
\] |
| $1a_4b_1c_2(d_1)$ | 6 | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 \neq 0$, \[
\frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} = 0, \quad \alpha_3 = 0, \quad \frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} \neq 0, \quad \frac{T'_3}{T_3} = \text{constant}
\] |
| $1a_4b_1c_2(d_1)$ | 7 | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 \neq 0$, \[
\frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} = 0, \quad \alpha_3 = 0, \quad \frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} \neq 0, \quad \frac{T'_3}{T_3} \neq \text{constant}
\] |
| $1a_4b_1c_2(d_2)$ | 4 | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 \neq 0$, \[
\frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} = 0, \quad \alpha_3 = 0, \quad \frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} \neq 0
\] |
| $1a_4b_1(c_3)$ | 6 | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 \neq 0$, \[
\frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} = 0, \quad \alpha_3 < 0
\] |
| $1a_4b_2(c_1)$ | 4 | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 \neq 0$, \[
\frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} \neq 0, \quad \frac{T'_3}{T_3} = \text{constant}
\] |
| $1a_4b_2(c_2)$ | 3 | $T'_0 = 0$, $T'_2 \neq 0$, $T'_3 \neq 0$, \[
\frac{T'_3}{\sqrt{T_1} \left(2T_3^{1/2} \sqrt{T_1}\right)} \neq 0, \quad \frac{T'_3}{T_3} \neq \text{constant}
\] |
Table 2. MCs of Case (2) for the Non-degenerate Energy-Momentum Tensor.

| Cases | MCs | Constraints |
|-------|-----|-------------|
| $2a_1 b_1 c_1 (d_1)$ | 10 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' = 0$, $(\frac{T_0}{2T_2\sqrt{T_1}})' = 0$, $\frac{T_0}{2T_2\sqrt{T_1}} = \beta_1 = 0$, $(\frac{T_0}{\sqrt{T_1}})' = 0$ |
| $2a_1 b_1 c_1 d_2 e_1 (f_1)$ | 6 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' = 0$, $(\frac{T_0}{2T_2\sqrt{T_1}})' = 0$, $\beta_1 = 0$, $\frac{T_0}{2\sqrt{T_1}}(\frac{T_0}{\sqrt{T_1}})' = \beta_2 = constant$, $\beta_2 = 0$ |
| $2a_1 b_1 c_1 d_2 e_1 (f_2)$ | 6 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' = 0$, $(\frac{T_0}{2T_2\sqrt{T_1}})' = 0$, $\beta_1 = 0$, $(\frac{T_0}{\sqrt{T_1}})' = 0$, $\beta_2 \neq constant$, $\beta_2 \neq 0$ |
| $2a_1 b_1 c_2 (d_1)$ | 10 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' = 0$, $(\frac{T_0}{2T_2\sqrt{T_1}})' = 0$, $\beta_1 \neq 0$, $(\frac{T_0}{\sqrt{T_1}})' = 0$ |
| $2a_1 b_1 c_2 (d_2)$ | 5 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' = 0$, $(\frac{T_0}{2T_2\sqrt{T_1}})' = 0$, $\beta_1 \neq 0$, $(\frac{T_0}{\sqrt{T_1}})' \neq 0$, $(\frac{T_0}{\sqrt{T_1}})' = 0$ |
| $2a_1 b_1 c_2 (e_1)$ | 4 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' = 0$, $(\frac{T_0}{2T_2\sqrt{T_1}})' = 0$, $\beta_1 \neq 0$, $(\frac{T_0}{\sqrt{T_1}})' \neq 0$, $(\frac{T_0}{\sqrt{T_1}})' = 0$ |
| $2a_1 b_1 c_2 (e_2)$ | 4 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' = 0$, $(\frac{T_0}{2T_2\sqrt{T_1}})' = 0$, $\beta_1 \neq 0$, $(\frac{T_0}{\sqrt{T_1}})' \neq 0$, $(\frac{T_0}{\sqrt{T_1}})' = 0$ |
| $2a_1 b_2 (c_1)$ | 6 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' = 0$, $(\frac{T_0}{2T_2\sqrt{T_1}})' = 0$, $(\frac{T_0}{\sqrt{T_1}})' = 0$ |
| $2a_1 b_2 (c_2)$ | 4 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' = 0$, $(\frac{T_0}{2T_2\sqrt{T_1}})' \neq 0$, $(\frac{T_0}{\sqrt{T_1}})' \neq 0$ |
| $2a_2 b_1 c_1 (d_1)$ | 7 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' \neq 0$, $T_2 = 0$, $T_3 \neq 0$, $\frac{T_0}{\sqrt{T_1}}(\frac{T_0}{\sqrt{T_1}})' = \gamma_1 = 0$, $\frac{T_0}{\sqrt{T_1}}(\frac{T_0}{\sqrt{T_1}})' = \gamma_2 = 0$, $(\frac{T_0}{\sqrt{T_1}})' = 0$ |
| $2a_2 b_1 c_1 (d_2)$ | 4 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' \neq 0$, $T_2 = 0$, $T_3 \neq 0$, $\gamma_1 = 0$, $\gamma_2 = 0$, $(\frac{T_0}{\sqrt{T_1}})' \neq 0$ |
| $2a_2 b_1 c_2 (d_1)$ | 3 | $T'_0 \neq 0$, $(\frac{T_0}{T_3})' \neq 0$, $T_2 = 0$, $T_3 \neq 0$, $\gamma_1 \neq 0$, $\gamma_2 = 0$, $\gamma_1 > 0$ |
| Cases            | MCs       | Constraints                                                                 |
|------------------|-----------|-----------------------------------------------------------------------------|
| $2a_2b_1c_2d_2(e_1)$ | 4         | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' \neq 0,$ |
|                  |           | $\gamma_1 \neq 0, \gamma_2 = 0, \gamma_1 < 0, (\frac{T_5'}{T_1})' = 0$     |
| $2a_2b_1c_2d_2(e_2)$ | 3         | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' \neq 0,$ |
|                  |           | $\gamma_1 \neq 0, \gamma_2 = 0, \gamma_1 < 0, (\frac{T_5'}{T_1})' \neq 0$     |
| $2a_2b_1(c_3)$    | $2a_2b_1(c_2)$ | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' \neq 0,$ |
|                  |           | $\gamma_1 = 0, \gamma_2 \neq 0$                                             |
| $2a_2b_1c_4d_1(e_1)$ | 3         | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' \neq 0,$ |
|                  |           | $\gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_1 > 0, \gamma_2 > 0,$            |
|                  |           | $\gamma_2 T_0 \int \frac{\sqrt{T}}{T_0} d\tau + \frac{T_0'}{2\sqrt{T}} = 0$ |
| $2a_2b_1c_4d_1e_2(f_1)$ | 4         | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' \neq 0,$ |
|                  |           | $\gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_1 > 0, \gamma_2 > 0,$            |
|                  |           | $\gamma_2 T_0 \int \frac{\sqrt{T}}{T_0} d\tau + \frac{T_0'}{2\sqrt{T}} \neq 0,$ |
| $2a_2b_1c_4d_1e_2(f_2)$ | 3         | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' \neq 0,$ |
|                  |           | $\gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_1 > 0, \gamma_2 > 0,$            |
|                  |           | $\gamma_2 T_0 \int \frac{\sqrt{T}}{T_0} d\tau + \frac{T_0'}{2\sqrt{T}} \neq 0,$ |
| $2a_2b_1c_4(d_2)$ | $2a_2b_1c_4(d_1)$ | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' \neq 0,$ |
|                  |           | $\gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_1 > 0, \gamma_2 < 0$            |
| $2a_2b_1c_4(d_3)$ | $2a_2b_1c_4(d_1)$ | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' \neq 0,$ |
|                  |           | $\gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_1 < 0, \gamma_2 > 0$            |
| $2a_2b_1c_4(d_4)$ | $2a_2b_1c_4(d_1)$ | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' \neq 0,$ |
|                  |           | $\gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_1 < 0, \gamma_2 < 0$            |
| $2a_2(b_2)$      | $2a_2(b_1)$ | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' = 0, T_3' = 0$     |
| $2a_2b_3c_1(d_1)$ | 5         | $T_0' \neq 0, \left(\frac{T_3'}{T_2}\right)' \neq 0, T_2' \neq 0, T_3' \neq 0,$ |
|                  |           | $\left(\frac{T_0'}{2T_2\sqrt{T_1}}\right)' = 0, \left(\frac{T_0'}{T_3^{3/2}T_1^{1/2}}\right)' = 0,$ |
|                  |           | $\left(\frac{T_0'}{T_3^{1/2}}\right)' = 0, \left(\frac{T_0'}{T_3^{3/2}T_1^{1/2}}\right)' = 0,$ |
|                  |           | $\left(\frac{T_0'}{T_3^{1/2}}\right)' = 0$                                 |
| Cases                  | MCs | Constraints                                                                 |
|-----------------------|-----|-----------------------------------------------------------------------------|
| $2a_2b_3c_1(d_2)$     | 5   | $T_0' \neq 0$, $(\frac{t_2}{T_3})' \neq 0$, $T_2' \neq 0$, $T_3' \neq 0$, |
|                       |     | $(\frac{T_2'}{2T_2\sqrt{T_1}})' = 0$, $(\frac{T_3'}{2T_2\sqrt{T_1}})' = 0$, $(\frac{T_3}{T_1})' \neq 0$, |
| $2a_2b_3c_1d_3(e_1)$  | 4   | $T_0' \neq 0$, $(\frac{t_2}{T_3})' \neq 0$, $T_2' \neq 0$, $T_3' \neq 0$, |
|                       |     | $(\frac{T_2'}{2T_2\sqrt{T_1}})' = 0$, $(\frac{T_3'}{2T_2\sqrt{T_1}})' = 0$, $(\frac{T_3}{T_1})' \neq 0$, $(\frac{T_3}{T_1})' \neq 0$ |
| $2a_2b_3c_1d_3(e_2)$  | 3   | $T_0' \neq 0$, $(\frac{t_2}{T_3})' \neq 0$, $T_2' \neq 0$, $T_3' \neq 0$, |
|                       |     | $(\frac{T_2'}{2T_2\sqrt{T_1}})' = 0$, $(\frac{T_3'}{2T_2\sqrt{T_1}})' = 0$, $(\frac{T_3}{T_1})' \neq 0$, $(\frac{T_3}{T_1})' \neq 0$ |
| $2a_2b_3c_2(d_1)$     | 4   | $T_0' \neq 0$, $(\frac{t_2}{T_3})' \neq 0$, $T_2' \neq 0$, $T_3' \neq 0$, |
|                       |     | $(\frac{T_2'}{2T_2\sqrt{T_1}})' \neq 0$, $(\frac{T_3'}{2T_2\sqrt{T_1}})' \neq 0$, $(\frac{T_3}{T_1})' = 0$, $(\frac{T_3}{T_1})' \neq 0$ |
| $2a_2b_3c_2(d_2)$     | 4   | $T_0' \neq 0$, $(\frac{t_2}{T_3})' \neq 0$, $T_2' \neq 0$, $T_3' \neq 0$, |
|                       |     | $(\frac{T_2'}{2T_2\sqrt{T_1}})' \neq 0$, $(\frac{T_3'}{2T_2\sqrt{T_1}})' \neq 0$, $(\frac{T_3}{T_1})' = 0$ |
| $2a_2b_3c_2(d_3)$     | 3   | $T_0' \neq 0$, $(\frac{t_2}{T_3})' \neq 0$, $T_2' \neq 0$, $T_3' \neq 0$, |
|                       |     | $(\frac{T_2'}{2T_2\sqrt{T_1}})' \neq 0$, $(\frac{T_3'}{2T_2\sqrt{T_1}})' \neq 0$, $(\frac{T_3}{T_1})' \neq 0$, $(\frac{T_3}{T_1})' \neq 0$ |
Table 3. MCs of Case (1) for the Degenerate Energy-Momentum Tensor.

| Cases | MCs                  | Constraints                                      |
|-------|----------------------|--------------------------------------------------|
| 1(a_1)| Infinite No. of MCs  | $T_0 = 0$, $T_1 = 0$, $T_2 = 0$, $T_3 \neq 0$    |
| 1(a_2)| Infinite No. of MCs  | $T_0 = 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 = 0$   |
| 1(a_3)| Infinite No. of MCs  | $T_0 = 0$, $T_1 \neq 0$, $T_2 = 0$, $T_3 = 0$   |
| 1(a_4)| Infinite No. of MCs  | $T_0 \neq 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 = 0$|

Table 4. MCs of Case (2) for the Degenerate Energy-Momentum Tensor.

| Cases     | MCs                  | Constraints                                      |
|-----------|----------------------|--------------------------------------------------|
| 2(a_1)(b_1)| Infinite No. of MCs  | $T_0 = 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $(\frac{T_0}{T_2})' = 0$ |
| 2(a_1)(b_2)(c_1)| Infinite No. of MCs | $T_0 = 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $(\frac{T_0}{T_2})' \neq 0$, $T_2' = 0$, $T_3' \neq 0$ |
| 2(a_1)(b_2)(c_2)| Infinite No. of MCs | $T_0 = 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $(\frac{T_0}{T_2})' \neq 0$, $T_2' \neq 0$, $T_3' = 0$ |
| 2(a_1)(b_2)(c_3)| Infinite No. of MCs | $T_0 = 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $(\frac{T_0}{T_2})' \neq 0$, $T_2' \neq 0$, $T_3' \neq 0$ |
| 2(a_2)(b_1)(c_1)| Infinite No. of MCs | $T_0 = 0$, $T_1 \neq 0$, $T_2 = 0$, $T_3 \neq 0$, $(\frac{\sqrt{T_0 T_3}}{2T_2 \sqrt{T_1}})' = 0$, $\frac{\sqrt{T_0 T_3}}{2T_2 \sqrt{T_1}})' = c > 0$ |
| 2(a_2)(b_1)(c_2)| Infinite No. of MCs | $T_0 = 0$, $T_1 \neq 0$, $T_2 = 0$, $T_3 \neq 0$, $(\frac{\sqrt{T_0 T_3}}{2T_2 \sqrt{T_1}})' = 0$, $c = 0$ |
| 2(a_2)(b_1)(c_3)| Infinite No. of MCs | $T_0 = 0$, $T_1 \neq 0$, $T_2 = 0$, $T_3 \neq 0$, $(\frac{\sqrt{T_0 T_3}}{2T_2 \sqrt{T_1}})' = 0$, $c < 0$ |
| 2(a_2)(b_2)| Infinite No. of MCs | $T_0 = 0$, $T_1 \neq 0$, $T_2 = 0$, $T_3 \neq 0$, $(\frac{\sqrt{T_0 T_3}}{2T_2 \sqrt{T_1}})' \neq 0$ |
| 2(a_3)| Infinite No. of MCs | $T_0 = 0$, $T_1 \neq 0$, $T_2 = 0$, $T_3 = 0$ |
| 2(a_4)| Infinite No. of MCs | $T_0 \neq 0$, $T_1 = 0$, $T_2 = 0$, $T_3 \neq 0$ |
| 2(a_5)| Infinite No. of MCs | $T_0 \neq 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 = 0$ |
| 2(a_6)| Infinite No. of MCs | $T_0 \neq 0$, $T_1 \neq 0$, $T_2 = 0$, $T_3 = 0$ |
Table 5. MCs of Case (3) for the Degenerate Energy-Momentum Tensor.

| Cases          | MCs                        | Constraints                                                                 |
|----------------|----------------------------|-----------------------------------------------------------------------------|
| $3a_1 b_1 c_1 d_1(e_1)$ | Infinite No. of MCs       | $T_0 = 0$, $T_1 \neq 0$, $T_2 \neq 0$, $T_3 \neq 0$, $\Gamma_2 = 0$, $\Gamma_3 = 0$, $\Gamma_4 = 0$ |
|                |                            | $(\frac{T_2}{\sqrt{T_1}} (\frac{T_2'}{2T_2\sqrt{T_1}}))' = 0$, $\frac{2T_2}{\sqrt{T_1}} (\frac{T_2'}{2T_2\sqrt{T_1}}) = \alpha_1 > 0$, $\frac{2T_2}{\sqrt{T_1}} (\frac{T_2'}{2T_2\sqrt{T_1}}) = \alpha_2 > 0$, $\sqrt{\frac{\alpha_1}{\alpha_2} + \sqrt{\frac{\alpha_2}{\alpha_1}}} = 0$ |
| $3a_1 b_1 c_1 d_1(e_2)$ | Infinite No. of MCs       | $T_0 = 0$, $T_1 \neq 0$, $T_2 \neq 0$, $T_3 \neq 0$, $\Gamma_2 = 0$, $\Gamma_3 = 0$, $\Gamma_4 = 0$ |
|                |                            | $(\frac{T_2}{\sqrt{T_1}} (\frac{T_2'}{2T_2\sqrt{T_1}}))' = 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\sqrt{\frac{\alpha_1}{\alpha_2} + \sqrt{\frac{\alpha_2}{\alpha_1}}} \neq 0$ |
| $3a_1 b_1 c_1 (d_2)$ | Infinite No. of MCs       | $T_0 = 0$, $T_1 \neq 0$, $T_2 \neq 0$, $T_3 \neq 0$, $\Gamma_2 = 0$, $\Gamma_3 = 0$, $\Gamma_4 = 0$ |
|                |                            | $(\frac{T_2}{\sqrt{T_1}} (\frac{T_2'}{2T_2\sqrt{T_1}}))' = 0$, $\alpha_1 > 0$, $\alpha_2 = 0$ |
| $3a_1 b_1 c_1 (d_3)$ | Infinite No. of MCs       | $T_0 = 0$, $T_1 \neq 0$, $T_2 \neq 0$, $T_3 \neq 0$, $\Gamma_2 = 0$, $\Gamma_3 = 0$, $\Gamma_4 = 0$ |
|                |                            | $(\frac{T_2}{\sqrt{T_1}} (\frac{T_2'}{2T_2\sqrt{T_1}}))' = 0$, $\alpha_1 > 0$, $\alpha_2 < 0$ |
| $3a_1 b_1 c_2 d_1(e_1)$ | Infinite No. of MCs       | $T_0 = 0$, $T_1 \neq 0$, $T_2 \neq 0$, $T_3 \neq 0$, $\Gamma_2 = 0$, $\Gamma_3 = 0$, $\Gamma_4 = 0$ |
|                |                            | $(\frac{T_2}{\sqrt{T_1}} (\frac{T_2'}{2T_2\sqrt{T_1}}))' = 0$, $\alpha_1 = 0$, $\frac{T_2}{2T_2\sqrt{T_1}} = \alpha_4 = 0$, $\alpha_2 > 0$ |
| Cases                        | MCs                        | Constraints                                                                                                                                 |
|------------------------------|----------------------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| $3a_1b_1c_2d_1e_2(f_1)$      | Infinite No. of MCs        | $T_0 = 0, T_1 \neq 0, T_2 \neq 0, T_3 \neq 0, \left(\frac{T_1}{\sqrt{T_1}}\left(\frac{T_1'}{2T_2\sqrt{T_1}}\right)\right)' = 0, \alpha_1 = 0, \alpha_4 = 0, \alpha_2 = 0, -\frac{T_3}{2T_2\sqrt{T_1}} = \alpha_5 = 0$ |
| $3a_1b_1c_2d_1e_2(f_2)$      | Infinite No. of MCs        | $T_0 = 0, T_1 \neq 0, T_2 \neq 0, T_3 \neq 0, \left(\frac{T_1}{\sqrt{T_1}}\left(\frac{T_1'}{2T_2\sqrt{T_1}}\right)\right)' = 0, \alpha_1 = 0, \alpha_4 = 0, \alpha_2 = 0, \alpha_5 \neq 0$ |
| $3a_1b_1c_2d_1(e_3)$         | Infinite No. of MCs        | $T_0 = 0, T_1 \neq 0, T_2 \neq 0, T_3 \neq 0, \left(\frac{T_1}{\sqrt{T_1}}\left(\frac{T_1'}{2T_2\sqrt{T_1}}\right)\right)' = 0, \alpha_1 = 0, \alpha_4 = 0, \alpha_2 = 0, \alpha_5 \neq 0$ |
| $3a_1b_1c_2d_2(e_1)$         | Infinite No. of MCs        | $T_0 = 0, T_1 \neq 0, T_2 \neq 0, T_3 \neq 0, \left(\frac{T_1}{\sqrt{T_1}}\left(\frac{T_1'}{2T_2\sqrt{T_1}}\right)\right)' = 0, \alpha_1 = 0, \alpha_4 \neq 0, \alpha_2 > 0$ |
| $3a_1b_1c_2d_2(e_2)$         | Infinite No. of MCs        | $T_0 = 0, T_1 \neq 0, T_2 \neq 0, T_3 \neq 0, \left(\frac{T_1}{\sqrt{T_1}}\left(\frac{T_1'}{2T_2\sqrt{T_1}}\right)\right)' = 0, \alpha_1 = 0, \alpha_4 \neq 0, \alpha_2 = 0$ |
| $3a_1b_1c_2d_2(e_3)$         | Infinite No. of MCs        | $T_0 = 0, T_1 \neq 0, T_2 \neq 0, T_3 \neq 0, \left(\frac{T_1}{\sqrt{T_1}}\left(\frac{T_1'}{2T_2\sqrt{T_1}}\right)\right)' = 0, \alpha_1 = 0, \alpha_4 \neq 0, \alpha_2 < 0$ |
| $3a_1b_1(e_3)$               | Infinite No. of MCs        | $T_0 = 0, T_1 \neq 0, T_2 \neq 0, T_3 \neq 0, \left(\frac{T_1}{\sqrt{T_1}}\left(\frac{T_1'}{2T_2\sqrt{T_1}}\right)\right)' = 0, \alpha_1 < 0$, $\alpha_4 \neq 0, \alpha_2 < 0$ |
| Cases            | MCs                      | Constraints                                                                 |
|------------------|--------------------------|-----------------------------------------------------------------------------|
| $3a_1(b_2)$      | Infinite No. of MCs      | $T_0 = 0$, $T_1 \neq 0$, $T_2 \neq 0$, $T_3 \neq 0$, $(\frac{T_0}{\sqrt{T_3}}(\frac{T_2}{T_1(T_2)}))' \neq 0$ |
| $3a_2b_1(c_1)$  | 4                        | $T_0 \neq 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $T_0' = 0$, $(\frac{T_0}{T_2T_3})' = 0$ |
| $3a_2b_1c_2(d_1)$ | Infinite No. of MCs     | $T_0 \neq 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $T_0' = 0$, $(\frac{T_0}{T_2T_3})' \neq 0$, $T_3 = 0 = T_3'$ |
| $3a_2b_1c_2(d_2)$ | 3                        | $T_0 \neq 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $T_0' = 0$, $(\frac{T_0}{T_2T_3})' \neq 0$, $T_3' = 0$ |
| $3a_2b_2(c_1)$  | 10                       | $T_0 \neq 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $T_0' \neq 0$, $(\frac{T_0}{T_2})' = 0$, $(\frac{T_0}{T_3})' = 0$ |
| $3a_2b_2c_2(d_1)$ | 5                        | $T_0 \neq 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $T_0' \neq 0$, $(\frac{T_0}{T_2})' = 0$, $(\frac{T_0}{T_3})' \neq 0$, $(\frac{T_0}{T_3})' = 0$ |
| $3a_2b_2c_2(d_2)$ | 4                        | $T_0 \neq 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $T_0' \neq 0$, $(\frac{T_0}{T_2})' \neq 0$, $(\frac{T_0}{T_3})' \neq 0$ |
| $3a_2b_2(c_3)$  | $3a_2b_2(c_2)$           | $T_0 \neq 0$, $T_1 = 0$, $T_2 \neq 0$, $T_3 \neq 0$, $T_0' \neq 0$, $(\frac{T_0}{T_2})' \neq 0$, $(\frac{T_0}{T_3})' = 0$ |
| Cases           | MCs | Constraints                                                                 |
|----------------|-----|-----------------------------------------------------------------------------|
| $3a_2b_2c_4d_1(e_1)$ | 4   | $T_0 
eq 0, T_1 = 0, T_2 
eq 0, T_3 
eq 0,$                              |
|                |     | $T_0' 
eq 0, (\frac{T_0}{T_2})' 
eq 0, (\frac{T_0}{T_3})' 
eq 0,$        |
|                |     | $(\frac{T_0T_2}{T_0T_2})' 
eq 0, (\frac{T_0T_3}{T_0T_3})' 
eq 0,$         |
|                |     | $T_1' = \text{constant}$                                                   |
| $3a_2b_2c_4d_1(e_2)$ | 3   | $T_0 
eq 0, T_1 = 0, T_2 \neq 0, T_3 
eq 0,$                              |
|                |     | $T_0' 
eq 0, (\frac{T_0}{T_2})' 
eq 0, (\frac{T_0}{T_3})' 
eq 0,$        |
|                |     | $(\frac{T_0T_2}{T_0T_2})' 
eq 0, (\frac{T_0T_3}{T_0T_3})' 
eq 0,$         |
|                |     | $T_1' = \text{constant}$                                                   |
| $3a_2b_2c_4(d_2)$ | 4   | $T_0 
eq 0, T_1 = 0, T_2 \neq 0, T_3 \neq 0,$                              |
|                |     | $T_0' 
eq 0, (\frac{T_0}{T_2})' 
eq 0, (\frac{T_0}{T_3})' 
eq 0,$        |
|                |     | $(\frac{T_0T_2}{T_0T_2})' = 0, (\frac{T_0T_3}{T_0T_3})' = 0$               |
| $3a_2b_2c_4(d_3)$ | 4   | $T_0 
eq 0, T_1 = 0, T_2 \neq 0, T_3 \neq 0,$                              |
|                |     | $T_0' 
eq 0, (\frac{T_0}{T_2})' 
eq 0, (\frac{T_0}{T_3})' 
eq 0,$        |
|                |     | $(\frac{T_0T_2}{T_0T_2})' = 0, (\frac{T_0T_3}{T_0T_3})' = 0$               |
| $3a_2b_2c_4(d_4)$ | 5   | $T_0 
eq 0, T_1 = 0, T_2 \neq 0, T_3 \neq 0,$                              |
|                |     | $T_0' 
eq 0, (\frac{T_0}{T_2})' 
eq 0, (\frac{T_0}{T_3})' 
eq 0,$        |
|                |     | $(\frac{T_0T_2}{T_0T_2})' = 0, (\frac{T_0T_3}{T_0T_3})' = 0$               |
| $3(a_3)$       | 3(a) | $T_0 
eq 0, T_1 \neq 0, T_2 = 0, T_3 \neq 0$                             |
| $3(a_4)$       | 3(a) | $T_0 
eq 0, T_1 \neq 0, T_2 \neq 0, T_3 = 0$                             |

It is seen from the above tables that each case has different constraints on the energy-momentum tensor. If we solve these constraints, we may have exact solution of EFEs. We have attempted (section 5) 10 different constraints to obtain the energy-momentum tensor and the corresponding spacetime. These cases are given by $1a_1b_1c_2d_1e_2(f_1)$, $2a_1b_1c_2(d_1)$, $2a_1b_1c_2d_2(e_1)$, $2a_1b_2(c_1)$, $2a_1b_2(c_2)$, $2a_2b_2c_1(d_1)$, $2a_2b_2c_2(d_2)$, $2a_2b_2c_3(d_3)$, $3a_2b_2c_4(d_4)$. It turns out that the cases $2a_1b_2(c_1)$, $2a_1b_2(c_2)$, $2a_2b_2c_2(d_1)$, $2a_2b_2c_2(d_2)$, $2a_2b_2c_3(d_3)$, $3a_2b_2c_4(d_4)$ provide no proper MC. However, the cases $2a_2b_2c_1(d_1)$ and $3a_2b_2c_4(d_4)$ yield one proper MC, the case $2a_1b_1c_2d_2(e_1)$ gives 2 proper MCs, the case $1a_1b_1c_2d_1e_2(f_1)$ gives 3 proper MCs and the case $2a_1b_1c_2(d_4)$ gives 4 proper MCs. It is interesting to note that $3a_2b_2c_4(d_4)$ is the case where $T_1 = 0$, i.e., the degenerate case but this provides finite dimensional MCs and we obtain 1 proper MC in this case.

We have attempted some of the constraints to obtain exact solutions of EFEs which give finite dimensional MCs. We have discussed some of the
physical properties of the resulting spacetimes. It would be interesting to look for more solutions of the constraints or examples should be constructed to satisfy the constraints.
Appendix A

The surviving components of the Ricci tensor are
\[
R_{00} = \frac{1}{4}e^\nu (2\nu'' + \nu'^2 + \nu'\lambda' + \nu'\mu'),
\]
\[
R_{11} = -\frac{1}{4}(2\nu'' + 2\lambda'' + 2\mu'' + \nu^2 + \lambda^2 + \mu^2),
\]
\[
R_{22} = -\frac{1}{4}e^\lambda (2\lambda'' + \nu'\lambda' + \lambda^2 + \lambda'\mu'),
\]
\[
R_{33} = -\frac{1}{4}e^\mu (2\mu'' + \nu'\mu' + \lambda'\mu' + \mu^2).
\]

(A1)

The Ricci scalar is given by
\[
R = \frac{1}{2}(2\nu'' + 2\lambda'' + 2\mu'' + \nu^2 + \lambda^2 + \mu^2 + \nu'\lambda' + \nu'\mu' + \lambda'\mu').
\]

(A2)

Using Einstein field equations (1), the non-vanishing components of energy-momentum tensor $T_{ab}$ are
\[
T_{00} = -\frac{1}{4}e^\nu (2\lambda'' + 2\mu'' + \lambda^2 + \mu^2 + \nu'\lambda' + \nu'\mu' + \lambda'\mu'),
\]
\[
T_{11} = \frac{1}{4}(\nu'\lambda' + \nu'\mu' + \lambda'\mu'),
\]
\[
T_{22} = \frac{1}{4}e^\lambda (2\nu'' + 2\mu'' + \nu^2 + \mu^2 + \nu'\mu'),
\]
\[
T_{33} = \frac{1}{4}e^\mu (2\nu'' + 2\lambda'' + \nu^2 + \lambda^2).\]

(A3)

Appendix B

Linearly independent KVs associated with the static cylindrical symmetric spacetimes are given by [12]
\[
\xi_{(1)} = \partial_t,
\]
\[
\xi_{(2)} = \partial_\theta,
\]
\[
\xi_{(3)} = \partial_z.
\]

(B1)

The components of the energy-momentum tensor for the metric in Eq.(70) are
\[
T_0 = (r/r_0)^{2a}(b + c - b^2 - c^2 - bc)/r^2,
\]

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\[ T_1 = \frac{(ab + bc + ca)}{r^2}, \]
\[ T_2 = \frac{-(r/r_0)^2 b(a + c - a^2 - c^2 - ac)}{r^2}, \]
\[ T_3 = \frac{-(r/r_0)^2 c(a + b - a^2 - b^2 - ab)}{r^2}. \] (B2)
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