Elliptic curves with rational subgroups of order three

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Abstract

In this article we present a characterization of elliptic curves defined over a finite field $\mathbb{F}_q$ which possess a rational subgroup of order three. There are two possible cases depending on the rationality of the points in these groups. We show that for finite fields $\mathbb{F}_q$, $q \equiv -1 \mod 3$, all elliptic curves with a point of order 3, they have another rational subgroup whose points are not defined over the finite field. If $q \equiv 1 \mod 3$, this is not true; but there exists a one to one correspondence between curves with points of order 3 and curves with rational subgroups whose points are not rational.

1 Introduction

Elliptic curves over finite fields have been largely studied in the literature (for example, see [9] and the references cited there). For an elliptic curve, we have an addition law which endows to the rational points (those defined over the finite field) a group structure. This group is an abelian group of rank 1 or 2. The type of the group is $(n_1, n_2)$, i.e. $E(\mathbb{F}_q) \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$ where $n_2 \mid n_1$ and furthermore $n_2 \mid q - 1$.

This structure gives us some information about the torsion points of an elliptic curve, but it is not easy to find them. This problem arises since the computation of the cardinal of an elliptic curve is, in general, very hard. Nevertheless, we have an easy method to test if an elliptic curve has 2-torsion points. If the elliptic curve has equation $y^2 = x^3 + Ax + B$, the abscissas of the points of order 2, are given by the solutions of the cubic $x^3 + Ax + B$ over the finite field. A study of elliptic curves with 2-torsion points can be found in [9] using elliptic curves in Legendre form.

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For an integer $n \neq 2$, the $n$-torsion points of an elliptic curve have as abscissas the roots of the classical division polynomials. These can be constructed recurrently:

\[
P_{-1} = -1, \quad P_0 = 0, \quad P_1 = 1, \quad P_2(x, y) = 2y \quad P_3 = 3x^4 + 6ax^2 + 12bx - a^2 \quad P_{2k} = \frac{P_k}{2y}(P_{k+2}P_{k-1}^2 - P_{k-2}P_{k+1}^2), \quad P_{2k+1} = P_{k+2}P_k^2 - P_{k+1}P_{k-1} \]

But, with these polynomials it is not possible to characterize all elliptic curves with a point of order $n$. In [7], Mestre gives equations for $\mathcal{O}(N)$, $N = 2, 3, 5, 7, 13$ which parametrizes the pairs $(E, C)$ of an elliptic curve $E$ and a subgroup $C$ of order $N$. He gives equations of the modular curve via the j-invariant of the curve. Our purpose is to present a family of the elliptic curves with a rational subgroup of order 3, but in their Weierstrass normal form. The interest of these curves comes from the study of volcanoes of 3-isogenies (see [1]).

The structure of the paper is the following. In section 2 we present the elliptic curves with, at least, a point of order 3. As it is known (2), an elliptic curve with a 3-torsion point over a finite field, admits a model of the form $y^2 + a_1x + a_3 = x^3$. Nevertheless, we can consider an elliptic curve isomorphic to the former one that depends only on one parameter (after fixing a non cube). We distinguish between the cases when the 3-torsion subgroup is cyclic or not. After this classification, we determine the number of isomorphism classes of elliptic curves with this property.

Section 3 is dedicated to study when a rational point $P$ of order 3 on an elliptic curve (among those mentioned in the previous section) has a point $Q$ which trisects it (i.e. $3Q = P$). This follows easily from [8] and [4].

Finally, in section 4 we determine a family of representatives of all isomorphic classes of elliptic curves with a rational subgroup of order 3 whose points are not rational. For this purpose, we compare the 3 division polynomial of a curve with that of its twisted curve, to conclude that both factorize in the same way. Thus, the curves in section 2 are in correspondence with the desired curves in this section.

2 Elliptic curves with cardinal a multiple of 3

Let $E$ be an elliptic curve over a finite field $\mathbb{F}_q$. It’s well known that $E$ admits a model in Weierstrass normal form (10):

\[
E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6
\]
The rational points of order 3 can be obtained from the 3-division polynomial. The roots of this polynomial correspond to the abscissas of the 3-order points \(^2\). Let \(E\) be an elliptic curve with, at least, a point of order 3, we can suppose, without loss of generality, that this point is \((0, 0)\). In this situation, \(E\) admits an equation (translating the 3-order point to the origin) of the type

\[
y^2 + a_1x + a_3y = x^3
\]  

(1)

The discriminant of an elliptic curve of equation (1) is \(\Delta = (a_3^2 - 27a_3) \alpha^3\) and the points \((0, 0)\) and \((0, -a_3)\) are of order 3.

Equation (1) can be simplified as follow:

**Lemma 1** Let \(E\) be an elliptic curve over \(\mathbb{F}_q\) with equation \(y^2 + a_1x + a_3y = x^3\) and let \(\beta_0 \in \mathbb{F}_q\) be a fixed non cube. If \(\beta\) is a cube, then \(E\) is isomorphic to a curve of type \(y^2 + axy + y = x^3\). Otherwise, then \(E\) is isomorphic to \(y^2 + axy + \beta_0 y = x^3\) or \(y^2 + axy + \beta_2 y = x^3\).

Moreover, if \(q \equiv -1 \mod 3\), all elliptic curves with a point of order 3 admit an equation of the form \(y^2 + axy + y = x^3\).

**Proof:**

Let us suppose that \(\beta\) is a cube. Let \(E\) be the elliptic curve with equation \(y^2 + axy + \alpha^3y = x^3\). Replacing \((x, y)\) by \((\alpha^2x', \alpha^3y')\), it gives an equation of the form \(y^2 + (a/\alpha)xy + y = x^3\).

If \(\beta\) is not a cube, let \(\rho\) be a 3-order primitive root of the unity and let \(\chi\) be a cubic character over \(\mathbb{F}_q\). It is clear that \(\chi(\beta) = \rho\) or \(\rho^2\). We can suppose that \(\chi(\beta_0) = \rho\) (otherwise, we take its square). If \(\chi(\beta) = \rho\), replacing \((x, y)\) by \((u^2x', u^3y')\), \(u = \sqrt[3]{\beta/\beta_0}\); the elliptic curve has equation \(y^2 + axy + \beta_0 y = x^3\). If \(\chi(\beta) = \rho^2\), replace \((x, y)\) by \((u^2x', u^3y')\), \(u = \sqrt[3]{\beta/\beta_0}\) to get the desired equation.

Finally, over \(\mathbb{F}_q\), \(q \equiv -1 \mod 3\), all elements are cubes, therefore, all elliptic curves with a 3-order point, admit an equation of the type \(y^2 + axy + y = x^3\). 

\(^2\)Although these abscissas are defined over \(\mathbb{F}_q\), it is possible that the point is not. So, they are points over \(\mathbb{F}_{q^2}\)

3
2.1 Elliptic curves with cyclic 3-torsion subgroup

The 3-division polynomial of an elliptic curve with equation \( y^2 + axy + y = x^3 \) is \( P_3(x) = x(3x^3 + a^2x^2 + 3ax + 3) \). If \( q \equiv -1 \mod 3 \), this polynomial has two roots over \( \mathbb{F}_q \). Nevertheless, only the root 0 corresponds to an abscissa of a 3-order point. The other root corresponds to a 3-order point defined over a quadratic extension of \( \mathbb{F}_q \). This means that there exists a 3-order rational subgroup with no rational points. So, we have the following result:

**Theorem 2** Let \( \mathbb{F}_q \), \( q \equiv -1 \mod 3 \) a finite field, then the only elliptic curves with 3-order rational points are isomorphic to a curve with equation \( y^2 + axy + y = x^3 \), \( a \neq 3 \). Moreover, the 3-torsion group is, in this case, cyclic 3. 

Taking into account these equations for the elliptic curves, we can count the number of isomorphism classes of elliptic curves over \( \mathbb{F}_q \), \( q \equiv -1 \mod 3 \), with a point of order three.

**Proposition 3** Let \( \mathbb{F}_q \), with \( q \equiv -1 \mod 3 \) a finite field, there exist \( q - 1 \) isomorphic classes of elliptic curves with cardinal a multiple of 3. A parametrization of these classes is given by

\[
\{ y^2 + axy + y = x^3 \mid a \neq 3 \} \tag{2}
\]

**Proof:**

By theorem 2, let \( E : y^2 + axy + y = x^3 \) be an elliptic curve over \( \mathbb{F}_q \), \( q \equiv -1 \mod 3 \), with a 3-torsion point. The Weierstrass normal form of this curve is \( y^2 = x^3 + Ax + B \) where

\[
A_a = -\frac{a(-24 + a^3)}{48} \quad B_a = \frac{1}{4} + \frac{a^6}{864} - \frac{a^3}{24} \tag{3}
\]

Over \( q \equiv -1 \mod 3 \), every element is a cube, and for each one, one only cubic root exists. Therefore, we can only consider the curves \( y^2 + \sqrt[3]{a}xy + y = x^3 \) (this equation makes the calculations easier). The j-invariant of this curve is \(-16\frac{a(-24 + a^3)}{a - 27}^3\).

Let \( y^2 + \sqrt[3]{b}xy + y = x^3 \) be another elliptic curve of this type. A necessary condition for the elliptic curve to be isomorphic to the previous one is that

\[\text{For all these elliptic curves, the 3-order points are } (0,0) \text{ and } -(0,0) = (0,-1)\]
both possess the same j-invariant. So \( b \) will be a root of
\[
X^3 + (a - 72)X^2 + (a^2 - 72a + 1728)X - 27\frac{(-24 + a^3)}{to - 27}.
\]
The only root of this polynomial over the finite field considered is
\[
b = \frac{72 - a}{3} - \frac{(-36 + a)\sqrt[3]{a}}{3\sqrt[3]{-27 + a}} + \frac{2}{3} \sqrt[3]{-27 + a\sqrt[3]{a^2}}. \quad (4)
\]
Also, if they are isomorphic, there exists \( u \in \mathbb{F}_q^* \) such that \( A_a = u^4 A_b, \\
B_a = u^6 B_b \). Replacing (4) in this second equation, one has that
\[
u^6 = -3\frac{(\sqrt[3]{-27 + a})^2}{(\sqrt[3]{-27 + a - \sqrt[3]{a}})^2}.
\]
But, in this situation, \(-3\) is not a square and thus, isomorphic elliptic curves to \( y^2 + \sqrt[3]{a}xy + y = x^3 \) do not exist. \( \square \)

For finite fields \( \mathbb{F}_q \) with \( q \equiv 1 \mod 3 \), we have to consider all the different types of elliptic curves shown in lemma 1:
\[
\{ E_i^i : y^2 + axy + b_{0i}y = x^3 ; i = 0, 1, 2 \}. \quad (5)
\]
where \( \chi(b_0) = \rho \) (\( \chi \) is a cubic character over \( \mathbb{F}_q \) and \( \rho \) a cubic root of the unity).

From the 3-division polynomial, it is easy to see that these curves will have cyclic 3-torsion subgroup if and only if \( \chi(27b_0^i - a^3) \neq 1 \).

By Lemma 1 if \( E_{a1}^i \cong E_{a2}^j \) then \( i = j \). We also have:

**Lemma 4** Let \( E_{a1}^i, E_{a2}^i \) be two elliptic curves over \( \mathbb{F}_q \), \( q \equiv 1 \mod 3 \) such that \( \chi(27b_0^i - a_1^3), \chi(27b_0^i - a_2^3) \neq 1 \) (both curves possess cyclic 3-torsion subgroup). \( E_{a1}^i \cong E_{a2}^i \) if and only if:
\[
a_2 \in \{ a_1, \rho a_1, (\rho + 1)a_1 \}
\]

Proof:

It follows from the j-invariants of both curves. These are the only possibilities over \( \mathbb{F}_q, q \equiv 1 \mod 3 \) and it is easy to prove that these elliptic curves
are isomorphic.

As \(\chi(27b_0^i - a^3) \neq 1\), determining the number of possible values for \(a\), is equivalent to determine how many values of \(a\) exist with \(a^3 - b_0^i\) been a cube. The following result will be essential.

**Lemma 5** Let \(\mathbb{F}_q\) be a finite field, with \(q \equiv 1 \mod 3\) and let \(A \equiv 1 \mod 3\) be such that \(4q = A^2 + 27B^2\) for some \(B\) (with these conditions \(A\) is unique).

1. The number of solutions of \(x^3 + y^3 = 1\) is exactly \(q - 2 + A\).
2. The number of pairs \((x, y)\) such that \(x^3 + y^3\) is not a cube is \(\frac{(q-1)(2q-4-A)}{3}\).

**Proof:**

A proof of part i) can be found in [3].

For each \(a\), the number of solutions of \(x^3 + y^3 = a^3\) is also \(q - 2 + A\), except for \(a = 0\), in that case there are \(3q - 2\). Therefore, since there exist \((p-1)/3\) cubes different from 0, there will be \(q^2 - \left(\frac{q-1}{3}\right)(q-2+A) + 3q-2\) = \(\frac{(q-1)(2q-4-A)}{3}\) pairs \((x, y)\) such that \(x^3 + y^3\) is not a cube.

Using this result we can compute the number of isomorphism classes of elliptic curves over \(\mathbb{F}_q\), \(q \equiv 1 \mod 3\) with cyclic 3-torsion.

**Proposition 6** Let \(\mathbb{F}_q\), \(q \equiv 1 \mod 3\) a finite field. There are \((2q + 4)/3\) isomorphic classes of elliptic curves with cyclic 3-torsion subgroup. Let \(b_0 \in \mathbb{F}_q\) be a non cube. A family of representatives is given by:

\[
\{y^2 + m_axy + b_0y = x^3; \ i = 0, 1, 2, \chi(a^3 - 27b_0^i) \neq 1\} \quad (6)
\]

with \(m_a = \min\{a, \rho a, (\rho + 1)a\}\).

**Proof:**

First, we compute the number of isomorphism classes of elliptic curves of the form \(y^2 + axy + y = x^3\). This curve has a cyclic 3-torsion if and only if \(a^3 - 27\) is not a cube. By lemma [4] the curves \(y^2 + paxy + y = x^3\) and \(y^2 + \rho^2 axy + y = x^3\) are isomorphic to the previous one. Therefore, the number of isomorphism classes of these curves is given by the number of cubes \(a^3\), such that \(\chi(a^3 - 27) = \chi(a^3 - 1) \neq 1\).
By the previous lemma, there exist \( q - 2 + A \) pairs \((x, y)\) such that
\[ x^3 + y^3 = 1. \]
To study the number of elements \( x \) such that both \( x \) and \( x - 1 \)
are cubes, it is necessary to keep in mind the following facts. The pairs
\((0, 1), (0, \rho)\) and \((0, \rho^2)\) produce the same cube 0. The same fact is true for
\((1, 0), (\rho, 0)\) and \((\rho^2, 1)\) and finally for \((\rho^i x, \rho^j y)\), \(0 \leq i, j \leq 2\).

Therefore, there exist \( q + 10 + \frac{A}{9} \) cubes \( x^3 \), such that \( x^3 - 1 \) is a cube. In
case, there are \((2q - 4 + A)/9\) cubes \( x^3 \) such that \( x^3 - 1 \) is not.

In a similar way, it can be shown that there exist \((4q + 16 + A)/9\) iso-
morphism classes of elliptic curves of the type \( y^2 + a^3 xy + b_i y = x^3 \); with
\( i = 1, 2 \), and \( \chi(a^3 - b_i) \neq 1 \) and the result follows after adding this number
to the previous one.

\[ \square \]

### 2.2 Elliptic curves with non cyclic 3-torsion subgroup

As shown in the previous section, a necessary condition for an elliptic curve
to have non cyclic 3-torsion subgroup is that it be defined over \( \mathbb{F}_q \) with
\( q \equiv 1 \mod 3 \). If \( E \) is defined by the equation \( y^2 + axy + by = x^3 \), it is also
necessary that \( \chi(a^3 - 27b) \neq 1 \). We can change the equation to distinguish
between those with non cyclic subgroup as follows:

**Proposition 7** The family of elliptic curves

\[ E_a : y^2 + (3a - 1)xy + a(\rho - 1)(a - \frac{\rho + 1}{3})y = x^3 \]

such that \( a(a - (\rho + 1)/3)(\rho - \rho/3) \neq 0 \) corresponds to all the elliptic curves
defined over \( \mathbb{F}_q \) with non cyclic 3-torsion subgroup.

**Proof:**

Let \( y^2 + axy + by = x^3 \) be an elliptic curve over \( \mathbb{F}_q \) with non cyclic
3-torsion subgroup. It is clear that (by construction) one of the points of
order 3 is the origin. Let \((x_0, y_0)\) be another point of order 3 which does not
belong to \(< (0, 0) > \) (that is \( x_0 \neq 0 \)). Let \( y = \lambda x + \mu \) be the tangent line to
the curve in \((x_0, y_0)\) (\( \lambda \neq 0 \) because this point is not the origin).

The change of variables \((x, y) \rightarrow (\lambda^2 x, \lambda^3 y)\) transforms the equation of
the elliptic curve in to another of the same type. The point of order 3 is
now \((x_0/λ^2, y_0/λ^3)\) and the tangent line at this point has slope 1.

The result follows easily by considering the conditions in the cubic such that both \(y = 0\) and \(y = x + u\) have a triple intersection point with the cubic (this can be found in \([2]\)).

Let us see now when two elliptic curves \(E_a\) and \(E_b\) are isomorphic. It is clear that \(a \not\in \{0, ρ/3, (ρ + 1)/3\}\), otherwise it would be a singular cubic. Using the expressions for the \(j\)-invariants and the equations in Weierstrass normal form of these curves we can prove that:

**Lemma 8** Let \(\mathbb{F}_q\) be a finite field with \(q \equiv 1 \mod 3\) and let \(E_a\) and \(E_b\) be two elliptic curves over \(\mathbb{F}_q\). The equations for \(E_a\) and \(E_b\) are

\[
\begin{align*}
    y^2 + (3a - 1)xy + a(a - ρ + 1/3)y & = x^3 \\
y^2 + (3b - 1)xy + b(b - ρ + 1/3)y & = x^3
\end{align*}
\]

respectively (such that \(a, b \not\in \{0, ρ/3, (ρ + 1)/3\}\)). \(E_a \cong E_b\) if and only if

\[
b \in G_a = \left\{ a, \frac{a(1+ρ)}{3(a-ρ)}, \frac{ρ}{3a-ρ}, \frac{ρ(3a-ρ)}{9a}, \frac{ρ(3a-1-ρ)}{3}, \frac{ρ(3a-1-ρ)}{3(3a-ρ)}, \frac{ρ(3a-1-ρ)}{3(3a-ρ)}, \frac{ρ(3a-1-ρ)}{3(3a-ρ)}, \frac{ρ(3a-1-ρ)}{3(3a-ρ)} \right\}.
\]

\(G_a\) has a group structure and it is isomorphic to \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}\). \(G_a\) acts over \(\mathbb{F}_q - \{0, ρ/3, (ρ + 1)/3\}\). Then, the number of isomorphism classes of elliptic curves with non cyclic coincides with the number of orbits under this action. To compute this number we use the well-known Burnside formula

\[
\# \text{ of orbits} = \frac{1}{|G_a|} \sum_{g \in G_a} |N^g|
\]

where \(N^g\) is the set of elements fixed by \(g\). We have:
| \( g \) | \( N^g \) | \( g \) | \( N^g \) |
|-------|--------|-------|--------|
| \( g \) | \( \mathbb{F}_q - \{0, \rho/3, (\rho + 1)/3\} \) | \( g \) | \( \rho(3a - 1) \) |
| \( a(1 + \rho) \) | \( 1 + 2\rho \) | \( \rho(3a - 1) \) | \( \rho + \sqrt{-\rho} \), \( \rho - \sqrt{-\rho} \) |
| \( (3a - \rho) \) | \( 3 \) | \( 3(3a - 1) \) | \( 3 \) |
| \( \rho(3a - 1) \) | \( 1 + 2\rho \) | \( 3(3a - 1) \) | \( \rho + 1 + \rho\sqrt{-1}, \rho + 1 - \rho\sqrt{-1} \) |
| \( \sqrt{1} (3a - 1) \) | \( 3 \) | \( 3(3a - 1) \) | \( 3 \) |
| \( -1(1 + \rho)(3a - 1) \) | \( \sqrt{-1} / 3 \) | \( (1 + \rho)(3a - 1) \) | \( 1 \) |
| \( -1(3a - 1) \) | \( 1 \) | \( 3(3a - 1) \) | \( 3 \) |
| \( \rho(3a - 1) \) | \( 1 \) | \( \rho(3a - 1) \) | \( 1 + 2\rho \) |

Considering whether these elements belong to \( \mathbb{F}_q - \{0, \rho/3, (\rho + 1)/3\} \) and the Burnside formula, we have the following result.

**Proposition 9** Let \( \mathbb{F}_q \) be a finite field with \( q \equiv 1 \mod 3 \). Over \( \mathbb{F}_q \) there exist \( q + 12 - (q \mod 12) \) isomorphic classes of elliptic curves with non cyclic 3-torsion subgroup.

### 3 Elliptic curves with rational subgroups of order 3 containing no rational points

In this section we are interested in elliptic curves that have no rational points of order three, but they have a rational subgroup of order 3. These curves appear when we consider rational isogenies of degree three. This is because the points of this rational subgroup are not invariant under the action of the Frobenius endomorphism, whereas the subgroup is.

Let \( E : y^2 = x^3 + Ax + B \) be an elliptic curve defined over a finite field \( \mathbb{F}_q \) and let \( G \subset E(\mathbb{F}_q) \) be a subgroup of order 3 whose points are not rational (except the point of the infinity). Let \( \sigma : \mathbb{F}_q \to \mathbb{F}_q \), be the Frobenius endomorphism: \( \sigma((x, y)) = (x^q, y^q) \). If \( P \in G \), then \( \sigma(P) = P \) or \( -P \). If it coincides with \( P \), it is clear that the point \( P \) is rational, which is absurd. Therefore, \( \sigma(P) = -P \) and hence

\[
(x^q, y^q) = (x, -y).
\]

From this, we deduce that the abscissa of \( P \) should be rational and in consequence the ordinate should be defined over \( \mathbb{F}_{q^2} \). Then \( \#(E(\mathbb{F}_{q^2})) \) is a multiple of 3.
By Weil theorem, if \( \sharp(E(\mathbb{F}_q)) = m(\mathbb{F}_q) = q + 1 - t \), then \( \sharp(E(\mathbb{F}_{q^2})) = m(\mathbb{F}_{q^2}) = q^2 + 1 - t^2 + 2q \). If \( q \equiv -1 \mod 3 \), \( m(\mathbb{F}_{q^2}) \) is a multiple of 3 if and only if \( m(\mathbb{F}_q) \) is. Therefore, all the elliptic curves with rational subgroups of order 3 over \( \mathbb{F}_q \), \( q \equiv -1 \mod 3 \), have been already studied in section 2.

Otherwise, if \( q \equiv 1 \mod 3 \), \( m(\mathbb{F}_{q^2}) \) is a multiple of 3 if and only if \( t \equiv 1, 2 \mod 3 \). In the first case, the curve defined over \( \mathbb{F}_q \) has cardinal multiple of 3 (already studied in section 2). In the second case, \( m(\mathbb{F}_q) \) is not a multiple of 3. In consequence, we must only study elliptic curves with points of order 3 defined over \( \mathbb{F}_{q^2} \) but not over \( \mathbb{F}_q \). Moreover, this implies that there exist a one-to-one correspondence between curves with rational points of order 3 and those with 3-order subgroups with no rational points (ones are the twisted of the others).

The 3-division polynomial (the one whose roots are the abscissas of the points of order three) of \( E \) is

\[
P_{A,B} = x^4 + 2Ax^2 + 4Bx - \frac{A^2}{3}.
\]

The following theorem gives us some information about the factorization of \( P_{A,B} \) over \( \mathbb{F}_q, q \equiv 1 \mod 3 \).

**Theorem 10** ([5]) Let \( p \) be an odd prime and let \( f(x) \) be a monic polynomial of degree \( n \) and discriminant \( D \) without multiple roots. Let \( f(x) = f_1(x)f_2(x) \cdots f_r(x) \mod p \) be the factorization of \( f \) over \( \mathbb{F}_p \). Then \( n \equiv r \mod 2 \) if and only if \( (D/p) = 1 \).

The discriminant of \( P_{A,B} \) is \(-2^8(4A^3 + 27B^2)^2/27\). Over \( \mathbb{F}_q, q \equiv 1 \mod 3, -1/3 \) is a square. So, one has that \( P_{A,B} \) factors into two polynomials or it has all its roots in \( \mathbb{F}_q \). If it factors into two polynomials, the possibilities for its degree are \((2, 2)\) or \((3, 1)\). In the latter case, \( P_{A,B} \) has a root in \( \mathbb{F}_q \).

Using the results of [11], these three types can be distinguished in the following way. If \( \chi(2(27B^2 + 4A^3)) \neq 1 \), then \( P_{A,B} = (x - x_0)f_2(x) \) with \( f_2(x) \) an irreducible polynomial of degree 3. Otherwise, we define \( y_0 = 2^{\sqrt[3]{2(27B^2 + 4A^3)/3}} \). If \( y_0 + 16A/3 \) and \( \rho y_0 + 16A/3 \) are both squares, then \( P_{A,B} \) factors completely over \( \mathbb{F}_q \). Otherwise, \( P_{A,B} \) factors in two irreducible quadratic polynomials.

With these results, we have that the type of factorization of \( P_{A,B} \) coincides with the that of \( P_{t^2A,t^3B} \), where \( t \) is a non quadratic residue. That
corresponds to the 3-division polynomial of the twisted curve.

As we have shown previously, elliptic curves with rational points of order three are the twisted of those with subgroups of order 3 with no rational points. Thus, if \( y^2 = x^3 + Ax + B \) is an equation for an elliptic curve with 3-torsion points, then \( y^2 = x^3 + t^2Ax + t^3B \) (\( t \) a non quadratic residue), is a curve with a rational subgroup of order 3 with no rational points. As we have studied in section 2 the curves of the first type, determine those of the second type from their Weierstrass form.

**Proposition 11** Let \( \mathbb{F}_q \), \( q \equiv 1 \mod 3 \) be a finite field and \( b_0 \in \mathbb{F}_q \), such that \( \chi(b_0) = \rho \). Let \( t \) be a non quadratic residue. There exist \( (2q + 4)/3 \) isomorphism classes of elliptic curves over \( \mathbb{F}_q \) with only one rational subgroup of order three, whose points are defined over \( \mathbb{F}_q \setminus \mathbb{F}_q \). A family of representatives is given for \( y^2 = x^3 + t^2Ax + t^3B \) with

\[
A_a = -\frac{a^3(9 - 24b_0^i)}{48}, \quad B_a = \frac{b_0^{2i}}{4} + \frac{a^{18}}{864} - \frac{a^9b_0^i}{24},
\]

\( i = 0, 1, 2 \) \( \chi(a^3 - 27b_0^i) \neq 1 \)

**Proof:**

Elliptic curves of equation \( \{y^2 + a^3xy + b_0^i y = x^3; i = 0, 1, 2, \chi(a^3 - 27b_0^i) \neq 1\} \) correspond to those curves with a non cyclic 3-torsion subgroup whose points are defined over \( \mathbb{F}_q \). Therefore, their twisted curves will have a rational 3-torsion subgroup of points defined in a quadratic extension.

The Weierstrass normal form of elliptic curves with equation \( \{y^2 + a^3xy + b_0^i y = x^3; i = 0, 1, 2, \chi(a^3 - 27b_0^i) \neq 1\} \) is

\[
y^2 = x^3 + \left(-\frac{a^3(9 - 24b_0^i)}{48}\right)x + \frac{b_0^{2i}}{4} + \frac{a^{18}}{864} - \frac{a^9b_0^i}{24}.
\]

In consequence, the equation of the twisted curve coincides with the desired one.

Let \( y^2 = x^3 + t^2Ax + t^3B \) and \( y^2 = x^3 + t^2A'x + t^3B' \) be two of these curves. They will be isomorphic over \( \mathbb{F}_q \), if and only if there exists \( u \in \mathbb{F}_q^* \) such that \( u^4 = A/A', u^6 = B/B' \). Or equivalently, if and only if the curves \( y^2 + a^3xy + b_0^i y = x^3, y^2 + a^3xy + b_0^i y = x^3 \) are isomorphic. The number of isomorphism classes follows from proposition 11.
Two curves of this type will be isomorphic if and only if
\[ a_2 \in \{ a_1, \rho a_1, (\rho + 1)a_1 \} \).

We can prove, in the same way, the following result:

**Proposition 12** Let \( \mathbb{F}_q \) be a finite field with \( q \equiv 1 \mod 3 \). Over \( \mathbb{F}_q \) there exist \( \frac{q+12-(q \mod 12)}{12} \) isomorphism classes of elliptic curves with 4 rational subgroups of order 3 composed by points defined over \( \mathbb{F}_q^2 \). A collection of representatives is given by \( y^2 = x^3 + t^2Ax + t^3B \) with \( t \) a non quadratic residue and
\[
A = -\frac{(9a-1-2\rho)(3a-1-2\rho)(3a-1)(3a+1)}{144},
\]
\[
B = \frac{(1+9a^2)(9a^2-6a-\rho a-1)(9a^2-6\rho a-1)}{864}.
\]

Two curves of this type are isomorphic (the same as it happened with those with 8 rational points of order 3) if and only if
\[
b \in G_a = \{ a, \frac{a(1+\rho)}{3a-1-\rho}, \frac{a\rho}{3a-1-\rho}, \frac{-1}{9a}, \frac{-(1+\rho)(3a-1-\rho)}{3}, \frac{\rho(3a-\rho)}{9a}, \frac{\rho(3a-1-\rho)}{3(3a-\rho)}, \frac{-\rho}{3(3a-1-\rho)}, \frac{(1+\rho)(3a-\rho)}{3(3a-1-\rho)}, \frac{(1+\rho)(3a-1-\rho)}{9a}, \frac{(1+\rho)(3a-1-\rho)}{3(3a-\rho)}, \frac{\rho(3a-\rho)}{3} \}.
\]

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