Local ergodic theorems in symmetric spaces of measurable operators

VLADIMIR CHILIN AND SEMYON LITVINOV

Abstract. Local mean and individual (with respect to almost uniform convergence in Egorov’s sense) ergodic theorems are established for actions of the semigroup \( \mathbb{R}_d^+ \) in symmetric spaces of measurable operators associated with a semifinite von Neumann algebra.

1. Introduction

Advancing Lance’s extension of the pointwise ergodic theorem for actions of the group of integers on von Neumann algebras, Conze and Dang-Hgoc [8] and Watanabe [21] studied continuous extensions of Lance’s results. In particular, noncommutative Wiener’s local ergodic theorems were established for actions of the semigroups \( \mathbb{R}_d^+ \) and \( \mathbb{R}_+ \), respectively.

Following Yeadon’s ergodic theorem for the algebra \( L^1 \) of integrable operators associated with a semifinite von Neumann algebra, the corresponding Wiener’s local ergodic theorem for actions of \( \mathbb{R}_+ \) with respect to bilaterally almost uniform convergence (in Egorov’s sense) was initially considered in [2]. Later, Junge and Xu [14] derived that these averages converge bilaterally almost uniformly in any noncommutative \( L^p \)-space for \( 1 \leq p < \infty \) and almost uniformly if \( 2 \leq p < \infty \). It was also noticed there that these results admit multiple versions.

We consider actions of the semigroup \( \mathbb{R}_d^+ \) and show that, for a semifinite von Neumann algebra \( \mathcal{M} \), the corresponding ergodic averages \( A_t(x) \) converge to \( x \) almost uniformly as \( t \to 0^+ \) for all \( x \in E \), where \( E \subset L^1(\mathcal{M}) + \mathcal{M} \) is a fully symmetric space such that the unit of \( \mathcal{M} \) does not belong to \( E \). Besides, we prove that if \( E \) has order continuous norm \( \| \cdot \|_E \), then \( \| A_t(x) - x \|_E \to 0 \) as \( t \to 0^+ \). Note that, along with any space \( L^p(\mathcal{M}) \), \( 1 \leq p < \infty \), the family of such fully symmetric spaces \( E \) contains many noncommutative counterparts of classical Banach spaces of measurable functions, examples of which are given in the last section of the article.

2. Preliminaries

Let \( \mathcal{M} \) be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace \( \tau \). Let \( \mathcal{P}(\mathcal{M}) \) be the complete lattice of projections in \( \mathcal{M} \). If \( \mathbf{1} \) is the identity of \( \mathcal{M} \) and \( e \in \mathcal{P}(\mathcal{M}) \), we write \( e^\perp = \mathbf{1} - e \).

Denote by \( L^0 = L^0(\mathcal{M}, \tau) \) the \( * \)-algebra of \( \tau \)-measurable operators affiliated with \( \mathcal{M} \). Let \( \| \cdot \|_\infty \) be the uniform norm in \( \mathcal{M} \). Endowed with the measure topology \( t_\tau \).
given by the system of neighborhoods of zero
\[ \mathcal{N}(\epsilon, \delta) = \{ x \in L^0 : \| xe \|_\infty \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^+) \leq \epsilon \}, \]
\( \epsilon > 0, \delta > 0, L^0 \) is a complete metrizable topological *-algebra [18].

Let \( L^p = L^p(\mathcal{M}, \tau), 1 \leq p < \infty \), \( L^\infty(\mathcal{M}, \tau) = \mathcal{M} \), be the noncommutative \( L^p \)-space (see, for example, [19]) equipped with the standard norm \( \| \cdot \|_p \). Note that \( \| x_n \|_p \to 0 \) implies that \( x_n \to 0 \) in measure.

A net \( \{ x_n \}_{\alpha \in \Lambda} \subset L^0 \) is said to converge to \( \widehat{x} \in L^0 \) almost uniformly (a.u.) \((\text{bilateral} \text{ a.u.})\) if for every \( \epsilon > 0 \) there exists \( e \in \mathcal{P}(\mathcal{M}) \) such that \( \tau(e^+) \leq \epsilon \) and \( \lim_{\alpha \in \Lambda} \| e(\widehat{x} - x_n) e \|_\infty = 0 \) (respectively, \( \lim_{\alpha \in \Lambda} \| e(\widehat{x} - x_n) e \|_\infty = 0 \)).

Note that a.u. convergence is generally stronger than b.a.u. convergence. Also, it is not difficult to verify that \( x_n \to 0 \) in measure implies that \( x_{n_k} \to 0 \) a.u. for some subsequence \( \{ x_{n_k} \} \subset \{ x_n \} \).

A linear map \( T : L^1 + \mathcal{M} \to L^1 + \mathcal{M} \) is called a Dunford-Schwartz operator if
\[ \| T(x) \|_1 \leq \| x \|_1 \text{ for all } x \in L^1 \text{ and } \| T(x) \|_\infty \leq \| x \|_\infty \text{ for all } x \in \mathcal{M}. \]
Denote by \( DS^+ = DS^+(\mathcal{M}, \tau) \) the set of positive Dunford-Schwartz operators.

Let \( \mathbb{N} \) (respectively, \( \mathbb{R} \)) be the set of natural (respectively, real) numbers. Fix \( d \in \mathbb{N} \) and denote \( \mathbb{R}^d_+ = \{ (u_1, \ldots, u_d) : u_i \geq 0, \ i = 1, \ldots, d \} \). In what follows, \( \{ T_u \}_{u \in \mathbb{R}^d_+} \subset DS^+ \) is a semigroup such that \( T_0(x) = x \) for all \( x \in L^1 + \mathcal{M} \). If \( u, v \in \mathbb{R}^d_+ \), \( u = (u_1, \ldots, u_d) \), \( v = (v_1, \ldots, v_d) \), then \( u \to v \) means that \( u_i \to v_i \) for each \( i = 1, \ldots, d \).

A semigroup \( \{ T_u \}_{u \in \mathbb{R}^d_+} \) is said to be strongly continuous on \( L^p \), \( 1 \leq p < \infty \), if
\[ \lim_{u \to v} \| T_u(x) - T_v(x) \|_p = 0. \]
for each \( x \in L^p \).

**Proposition 2.1.** If a semigroup \( \{ T_u \}_{u \in \mathbb{R}^d_+} \subset DS^+ \) is strongly continuous on \( L^1 \), then it is also strongly continuous on \( L^p \) for every \( 1 < p < \infty \).

**Proof.** It is sufficient to show that \( \| T_u(x) - T_u(0) \|_p \to 0 \) as \( u \to 0 \) for any \( 0 \neq x \in L^p_+ \). Fix \( \epsilon > 0 \). If \( x = \int_0^\infty \lambda d e_\lambda \) is the spectral decomposition of \( x \) and \( m \in \mathbb{N} \), denote \( x_m = \int_0^{1/m} \lambda d e_\lambda + \int_m^\infty \lambda d e_\lambda \), \( y_m = x - x_m \) and choose \( m_0 \) to satisfy \( \| x_{m_0} \|_p \leq \frac{\epsilon}{2} \). As \( T_u \) is a contraction in \( L^p \) for each \( u \in \mathbb{R}^d_+ \), we have
\[ \| T_u(x_{m_0}) - T_0(x_{m_0}) \|_p \leq \frac{\epsilon}{2} \text{ for all } n. \]
Next, we have \( y_{m_0} \in L_+ \cap \mathcal{M} \). Since \( x \neq 0 \), without loss of generality, we may assume that \( y_{m_0} \neq 0 \). Let \( z_n = \| y_{m_0} \|_\infty^2 (T_u(y_{m_0}) - T_0(y_{m_0})) \). By the assumption, \( \tau(|z_n|) = \| z_n \|_1 \to 0 \). Besides, since \( T_u \) is a contraction in \( \mathcal{M} \) for each \( u \in \mathbb{R}^d_+ \), it follows that \(-1 \leq z_n \leq 1\), and so \( |z_n|^p \leq |z_n| \) for all \( n \). Therefore, we have
\[ \| z_n \|_p = \tau(|z_n|^p)^{1/p} \leq \tau(|z_n|)^{1/p} \to 0, \]
hence
\[ \lim_{u \to 0} \| T_u(y_{m_0}) - T_0(y_{m_0}) \|_p = 0. \]
Then there is \( N \in \mathbb{N} \) such that
\[ \| T_u(y_{m_0}) - T_0(y_{m_0}) \|_p \leq \frac{\epsilon}{2} \text{ for all } n \geq N. \]
Lemma 3.1. Let \( \| T_{u_n}(x) - T_{u_0}(x) \|_p \leq \| T_{u_n}(x_m) - T_{u_0}(x_m) \|_p + \| T_{u_n}(y_m) - T_{u_0}(y_m) \|_p \leq \epsilon \) whenever \( n \geq N \), which completes the proof.

In view of Proposition 2.1 we assume, throughout, that \( \{ T_u \}_{u \in \mathbb{R}^d_+} \subset DS^+ \) is a strongly continuous semigroup on \( L^1 \). Then, for a given \( 1 \leq p < \infty \) and \( x \in L^p \), the averages

\[
A_t(x) = \frac{1}{t^d} \int_{[0,t]^d} T_u(x) du
\]

belong to \( L^p \) for every \( t > 0 \).

3. Local Mean Ergodic Theorem in \( L^p(M, \tau) \), \( 1 \leq p < \infty \)

Lemma 3.1. Let \( 1 \leq p < \infty \), and let \( t_0 > 0 \). If \( y \in L^p \) and \( x = A_{t_0}(y) \), then \( \| A_t(x) - x \|_p \to 0 \) as \( t \to 0 \). In addition, if \( y \in L^p \cap M \), then \( \| A_t(x) - x \|_p \to 0 \) as \( t \to 0 \).

Proof. Let \( 0 < v_1, \ldots, v_d \leq v < t_0 \) and \( v = (v_1, \ldots, v_d) \in \mathbb{R}^d_+ \). Then

\[
T_v(x) - x = \frac{1}{t_0^d} \int_{[0,t_0]^d} T_{u+v}(y) du - \int_{[0,t_0]^d} T_u(y) du
\]

\[
= \frac{1}{t_0^d} \left[ \int_{[v_1,t_0+t_0] \times \cdots \times [v_d,t_0+t_0]} T_u(y) du - \int_{[0,t_0]} T_u(y) du \right]
\]

\[
= \frac{1}{t_0^d} \left[ \int_{[v_1,t_0] \times \cdots \times [v_d,t_0]} T_u(y) du + \int_{[v_1,t_0] \times \cdots \times [v_d,t_0]} T_u(y) du \right]
\]

\[
- \int_{[v_1,t_0] \times \cdots \times [v_d,t_0]} T_u(y) du - \int_{[0,t_0]} T_u(y) du \right]
\]

\[
= \frac{1}{t_0^d} \left[ \int_{[v_1,t_0] \times \cdots \times [v_d,t_0]} T_u(y) du \right],
\]

implying that

\[
\| T_v(x) - x \|_p \leq 2 \frac{t_0^d - (t_0 - v)^d}{t_0^d} \| y \|_p.
\]

As \( 0 < v < t_0 \), we have

\[
t_0^d - (t_0 - v)^d = v \sum_{k=1}^d \binom{d}{k} t_0^{d-k} (-v)^{k-1} < v \sum_{k=1}^d \binom{d}{k} t_0^{d-k} v^{k-1} < v \sum_{k=1}^d \binom{d}{k} t_0^{d-1},
\]

which implies that, with \( C(t_0, d) = 2t_0^{-1} \sum_{k=1}^d \binom{d}{k} \),

\[
\| T_v(x) - x \|_p < v C(t_0, d) \| y \|_p \leq t C(t_0, d) \| y \|_p.
\]
whenever \( v \leq t < t_0 \). It follows then that
\[
\|A_t(x) - x\|_p = \left\| \frac{1}{td} \int_{[0,t]} (T_v(x) - x) dv \right\|_p \leq t C(t_0, d) \|y\|_p,
\]
hence \( \|A_t(x) - x\|_p \to 0 \) as \( t \to 0 \).

Second part of the statement follows by replacing the norm \( \| \cdot \|_p \) with \( \| \cdot \|_\infty \) in the above inequalities.

Now we can prove a local mean ergodic theorem. One may notice that, unlike ergodic theorems with "infinite time", the convergence holds also in the space \( L^1 \) with \( \tau(1) = \infty \).

**Theorem 3.1.** Let \( 1 \leq p < \infty \). If \( x \in L^p \), then \( \|A_t(x) - x\|_p \to 0 \) as \( t \to 0 \).

**Proof.** Since \( \|T_u(x) - x\|_p \to 0 \) as \( u \to 0 \) (see Proposition 2.1), given \( n \in \mathbb{N} \), there is \( t_n > 0 \) such that
\[
\|T_u(x) - x\|_p < \frac{1}{n}
\]
whenever \( u = (u_1, \ldots, u_d), \ 0 < u_1, \ldots, u_d \leq t_n \). Then we have
\[
\|A_{t_n}(x) - x\|_p = \left\| \frac{1}{t_n} \int_{[0,t_n]} (T_u(x) - x) du \right\|_p < \frac{1}{n},
\]
hence \( x_n = A_{t_n}(x) \to x \) in \( L^p \).

Fix \( \epsilon > 0 \). As \( \|x_n - x\|_p \to 0 \), choose \( n_0 \) such that \( \|x_{n_0} - x\|_p < \frac{\epsilon}{4} \). Next, since, by Lemma 3.1. \( \|A_t(x_{n_0}) - x_{n_0}\|_p \to 0 \), there is \( t_0 > 0 \) such that \( 0 < t < t_0 \) entails that \( \|A_t(x_{n_0}) - x_{n_0}\|_p < \frac{\epsilon}{4} \). Now, given \( 0 < t < t_0 \), we have
\[
\|A_t(x) - x\|_p = \|A_t(x) - A_t(x_{n_0})\|_p + \|A_t(x_{n_0}) - x_{n_0}\|_p + \|x_{n_0} - x\|_p
\]
\[
< 2\|x_{n_0} - x\|_p + \frac{\epsilon}{4} < \epsilon,
\]
and the assertion follows. \( \square \)

4. **Maximal Ergodic Inequality for Actions of \( \mathbb{R}^d_+ \)**

The next maximal ergodic inequality, due to Yeadon [23, Theorem 1], provides the main tool for proving individual ergodic theorems in semifinite von Neumann algebras.

**Theorem 4.1.** Let \( T \in DS^+ \). Then for every \( x \in L^1_+ \) and \( \lambda > 0 \) there exists \( \varepsilon \in \mathcal{P}(\mathcal{M}) \) such that
\[
\tau(\varepsilon^\perp) \leq \frac{\|x\|_1}{\lambda} \quad \text{and} \quad \sup_n \left\| e \frac{1}{n} \sum_{k=0}^{n-1} T^k(x) e \right\|_\infty \leq \lambda.
\]

Here is a continuous multi-parameter extension of Theorem 4.1 (cf. [14] Remark 4.7):  

**Theorem 4.2.** Let the averages \( A_t \) be given by \( \{I \} \). Then there is a constant \( \chi_d > 0 \) such that, given \( x \in L^1_+ \) and \( \lambda > 0 \), there exists \( \varepsilon \in \mathcal{P}(\mathcal{M}) \) satisfying inequalities
\[
\tau(\varepsilon^\perp) \leq \frac{2\|x\|_1}{\lambda} \quad \text{and} \quad \sup_{t>0} \|eA_t(x)e\|_\infty \leq \chi_d \lambda.
\]
Proof. Let $\frac{n}{m} \in \mathbb{Q}_+$, where $n, m \in \mathbb{N}$. Let $x \in L^1_+$ and denote $y_m = \int_{[0,1]} T_{\frac{n}{m}}(x) \, dv$. Then we have

$$A_{\frac{n}{m}}(x) = \frac{m^d}{n^d} \int_{[0,n/m]^d} T_u(x) \, du = \frac{1}{n^d} \int_{[0,n]^d} T_{\frac{n}{m}}(x) \, dv$$

$$= \frac{1}{n^d} \sum_{i_d=0}^{n-1} \cdots \sum_{i_1=0}^{n-1} \int_{[0,v_{m,i_1}] \cdots [0,v_{m,i_d}]} T_{(\frac{n}{m},0,\ldots,0)}(x) \, dv_1 \cdots \int_{[0,v_{m,i_d}]} T_{(\frac{n}{m},0,\ldots,0)}(x) \, dv_d$$

$$= \frac{1}{n^d} \sum_{i_d=0}^{n-1} \cdots \sum_{i_1=0}^{n-1} T_{i_1}^{(1/m,0,\ldots,0)} \cdots T_{i_d}^{(0,\ldots,0,1/m)}(y_m).$$

Next we use a result due to A. Brunel [1] (see [17, Ch.6, Theorem 3.4]) (although it was originally formulated for commuting contractions in a commutative $L^1$-space, one can see that the proof goes when $L^1 = L^1(M, \tau)$): if $T_1, \ldots, T_d$ are positive commuting contractions of $L^1$, then there exist $\chi > 0$, $n_d \in \mathbb{N}$, and $\{a_n > 0 : n \in \mathbb{N}_0\}$ with $\sum a_n = 1$ such that the operator $S = \sum a_n T_{n_1}^{\frac{1}{n_1}} \cdots T_{n_d}^{\frac{1}{n_d}}$ satisfies the inequality

$$S(x) \leq \frac{\chi d}{n_d} \sum_{j=0}^{n_d-1} S_j(x)$$

for all $n = 1, 2, \ldots$ and $x \in L^1_+$. 

Now it follows from (2) and (3) that

$$0 \leq A_{\frac{n}{m}}(x) \leq \chi d \frac{1}{n_d} \sum_{k=0}^{n_d-1} S_k(y_m).$$

Since $S \in DS^+$ it follows by Theorem [4] that there is $f \in \mathcal{P}(M)$ such that

$$\tau(f^+) \leq \frac{\|y_m\|}{\lambda} \leq \frac{\|x\|}{\lambda} \quad \text{and} \quad \sup_n \left\| f \frac{1}{n} \sum_{k=0}^{n-1} S_k(y_m) f \right\|_{\infty} \leq \lambda,$$

implying, in view of (3), that

$$\sup_{r \in \mathbb{Q}_+ \setminus \{0\}} \| f A_r(x) f \|_{\infty} \leq \chi d \lambda.$$

If $t > 0$, let $t \leftarrow r_n \in \mathbb{Q}_+ \setminus \{0\}$. Then we have $A_{r_n}(x) \to A_t(x)$ in measure. Therefore, $A_{r_{n_k}}(x) \to A_t(x)$ a.e. for a subsequence $\{r_{n_k}\} \subset \{r_n\}$. Thus, it is possible to find $g \in \mathcal{P}(M)$ such that

$$\tau(g^+) \leq \frac{\|x\|}{\lambda} \quad \text{and} \quad \| g A_{r_{n_k}}(x) g \|_{\infty} \to \| g A_t(x) g \|_{\infty} \quad \text{as} \quad k \to \infty.$$ 

Letting $c = f \wedge g$, we obtain the required inequalities. \qed

Remark 4.1. As it was carried out in [4], one can see that Theorem 4.2 can be extended to an arbitrary space $L^p$, $1 < p < \infty$. 

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5. Local Individual Ergodic Theorem in $L^p(M, \tau)$

We utilize the notion of bilaterally uniform equicontinuity in measure which is a noncommutative counterpart of the continuity in measure at zero of the maximal function of a sequence of maps from a normed space into a space of almost everywhere bounded measurable functions (see [13]):

**Definition 5.1.** Let $(X, \| \cdot \|)$ be a normed space. A sequence of maps $M_n : X \rightarrow L^0$ is called bilaterally uniformly equicontinuous in measure (b.u.e.m.) at zero if for every $\epsilon > 0$ and $\delta > 0$ there exists $\gamma > 0$ such that, given $x \in X$ with $\|x\| < \gamma$, there is a projection $e \in \mathcal{P}(M)$ satisfying conditions

$$
\tau(e^\perp) \leq \epsilon \quad \text{and} \quad \sup_n \|eM_n(x)e\|_\infty \leq \delta.
$$

In order to establish a.u. convergence of the averages [1], we will need the following.

**Lemma 5.1.** Let $(X, \| \cdot \|)$ be a normed space, and let $M_n : X \rightarrow L^0$ be a sequence of maps b.u.e.m. at zero. If $\{z_m\} \subset X$ is such that $\|z_m\| \rightarrow 0$, then for every $\epsilon > 0$ and $\delta > 0$ there are $z_{m_0} \in \{z_m\}$ and $e \in \mathcal{P}(M)$ satisfying conditions

$$
\tau(e^\perp) \leq \epsilon \quad \text{and} \quad \sup_n \|M_n(z_{m_0})e\|_\infty \leq \delta.
$$

**Proof.** Since $\|z_m\| \rightarrow 0$ and $\{M_n\}$ is b.u.e.m. at zero in $(X, \| \cdot \|)$, for every $n, k \in \mathbb{N}$ there exist $z_{n,k} \in \{z_m\}$ and $g_{n,k} \in \mathcal{P}(M)$ such that

$$
\tau(g^\perp_{n,k}) \leq \frac{\epsilon}{2n+k+1} \quad \text{and} \quad \sup_m \|g_{n,k}M_n(z_{n,k})g_{n,k}\|_\infty \leq \delta.
$$

In particular,

$$
\|g_{n,k}M_n(z_{n,k})g_{n,k}\|_\infty \leq \delta \quad \text{for all} \quad n, k.
$$

If $I(y)$ ($r(y)$) is the left (respectively, right) support of an operator $y \in L^0$ and

$$
q_{n,k} = 1 - r(g^\perp_{n,k}M_n(z_{n,k})),
$$

then

$$
\tau(q^\perp_{n,k}) = \tau(r(g^\perp_{n,k}M_n(z_{n,k}))) = \tau(I(g^\perp_{n,k}M_n(z_{n,k}))) \leq \tau(g^\perp_{n,k}) \leq \frac{\epsilon}{2n+k+1}.
$$

Besides,

$$
M_n(z_{n,k})q_{n,k} = g_{n,k}M_n(z_{n,k})q_{n,k} + g^\perp_{n,k}M_n(z_{n,k})q_{n,k} = g_{n,k}M_n(z_{n,k})q_{n,k}.
$$

Therefore, letting $e_{n,k} = g_{n,k} \wedge q_{n,k}$, we obtain $\tau(e^\perp_{n,k}) \leq \frac{\epsilon}{2n+k+1}$ and

$$
M_n(z_{n,k})e_{n,k} = g_{n,k}M_n(z_{n,k})q_{n,k} = g_{n,k}M_n(z_{n,k})q_{n,k} = g_{n,k}M_n(z_{n,k})g_{n,k}e_{n,k},
$$

implying that

$$
\|M_n(z_{n,k})e_{n,k}\|_\infty \leq \|g_{n,k}M_n(z_{n,k})g_{n,k}\|_\infty \leq \delta
$$

for all $n$ and $k$.

If $e = \bigwedge_{n,k} e_{n,k}$, then we have

$$
\tau(e^\perp) \leq \epsilon \quad \text{and} \quad \sup_n \|M_n(z_{n,k})e\|_\infty \leq \delta \quad \text{for all} \quad k,
$$

and the result follows.

We will also need the next two lemmas.

**Lemma 5.2.** Let $1 \leq p \leq \infty$, and let $t > 0$. If $x \in L^p$, then

$$
\lim_{s \rightarrow t} \|A_t(x) - A_s(x)\|_p \rightarrow 0.
$$
Proof. If $0 < s < t$, then
\[
\| A_t(x) - A_s(x) \|_p \leq \left\| \frac{1}{t^d} - \frac{1}{s^d} \right\| \int_{[0,s]^d} T_u(x) du \|_p + \left\| \frac{1}{t^d} \right\| \int_{[0,t]^d \setminus [0,s]^d} T_u(x) du \|_p
\]
\[
\leq \frac{1}{t^d} \int_{[0,s]^d} \| T_u(x) \|_p du + \frac{1}{t^d} \int_{[0,t]^d \setminus [0,s]^d} \| T_u(x) \|_p du
\]
\[
\leq 2 \frac{t^d - s^d}{t^d} \| x \|_p,
\]
implies that $\lim_{s \to t^+} \| A_t(x) - A_s(x) \|_p \to 0$. Convergence $\lim_{s \to t^+} \| A_t(x) - A_s(x) \|_p \to 0$ is proved similarly. 

Lemma 5.3. Let $t_n \to 0^+$. 

(i) If $1 < p < \infty$, then $A_{t_n}(x) \to x$ a.u. for all $x \in L^p$; 
(ii) $A_{t_n}(x) \to x$ b.a.u. for all $x \in L^1$.

Proof. (i) By Theorem 3.1 we have $\| A_{t_n}(x) - x \|_p \to 0$ for each $x \in L^p$. This implies that the set
\[
\mathcal{D} = \{ A_{t_n}(x) : n \in \mathbb{N}, x \in L^p \cap \mathcal{M} \}
\]
is dense in $L^p \cap \mathcal{M}$, hence in $L^p$.

By Lemma 3.1, $\| A_{t_n}(y) - y \|_\infty \to 0$ whenever $y \in \mathcal{D}$. Consequently, $A_{t_n}(y) \to y$ a.u. for each $y \in \mathcal{D}$. Since, in view of Remark 4.1, the sequence $\{ A_{t_n} \}$ is b.u.e.m. at zero on $L^p$, it follows by Proposition 3.1 (with $X = L^p$ and $M_{t_n} = A_{t_n}$) that the set $\{ x \in L^p : \{ A_{t_n}(x) \} \}$ converges a.u. is closed in $L^p$. As $\mathcal{D}$ is dense in $L^p$, we conclude that the sequence $\{ A_{t_n}(x) \}$ converges a.u. for each $x \in L^p$. Now, given $x \in L^p$, $A_{t_n}(x) \to x$ in $L^p$ entails that $A_{t_n}(x) \to x$ in measure. Therefore $A_{t_n}(x) \to x$ a.u. for some subsequence $\{ t_{n_k} \}$, implying that $A_{t_n}(x) \to x$ a.u.

(ii) Let $x \in L^1$, $1 < p < \infty$, and let $\{ x_m \} \subset L^p$ be such that $\| x - x_m \|_1 \to 0$. Since $x_m \to x$ in measure, we can assume without loss of generality that $x_m \to x$ a.u. also. Fix $\epsilon > 0$ and $\delta > 0$. There is $f \in \mathcal{P}(\mathcal{M})$ with $\tau(f^+) \leq \frac{\delta}{3}$ and $N_1 \in \mathbb{N}$ such that
\[
\|(x - x_m)f\|_\infty < \frac{\delta}{3} \quad \text{for all } m \geq N_1.
\]
Next, by Theorem 4.2, the sequence $\{ A_{t_n} \}$ is b.u.e.m. at zero on $L^1$. This implies that there exists $N_2 \in \mathbb{N}$ such that $m \geq N_2$ implies that there is $h \in \mathcal{P}(\mathcal{M})$ such that $\tau(h^+) \leq \frac{\delta}{3}$ and
\[
\sup_n \| h(A_{t_n}(x - x_m)h \|_\infty < \frac{\delta}{3} \quad \text{for all } m \geq N_2.
\]
Let $m_0 \geq \max\{ N_1, N_2 \}$. Since, in view of (i), $A_{t_n}(x_{m_0}) \to x_{m_0}$ a.u., it follows that there is $g \in \mathcal{P}(\mathcal{M})$ with $\tau(g^+) \leq \frac{\delta}{3}$ and $N \in \mathbb{N}$ satisfying
\[
\|(A_{t_n}(x_{m_0}) - x_{m_0})g\|_\infty < \frac{\delta}{3} \quad \text{for all } n \geq N.
\]
Now, letting $e = f \wedge g \wedge h$, we obtain $\tau(e^+) \leq \epsilon$ and
\[
\| e(A_{t_n}(x) - x)e \|_{\infty} \leq \| e(A_{t_n}(x - x_{m_0})e \|_{\infty} + \| e(A_{t_n}(x_{m_0}) - x_{m_0})e \|_{\infty}
\]
\[
+ \| e(x - x_{m_0})e \|_{\infty} < \delta
\]
whenever \( n \geq N \), which implies that \( A_{t_n}(x) \to x \) b.u.a. \( \square \)

Now we give an improvement of [14, Theorem 6.8 i) b)]; see also Remarks after [14, Theorem 6.8].

**Theorem 5.1.** For every \( x \in L^1 \) the averages (7) converge a.u. to \( x \) as \( t \to 0 \).

**Proof.** Show first that \( A_t(x) \to x \) a.u. if \( x \in L^1 \cap M \). Fix \( 1 < p < \infty \). Since \( L^1 \cap M \subset L^p \), Lemma 5.3 implies that \( A_{1/n}(x) \to x \) a.u. In particular,

\[
A_{[1/t]^{-1}}(x) \to x \text{ a.u. as } t \to 0.
\]

Next, we have

\[
\|A_t(x) - A_{[1/t]^{-1}}(x)\|_\infty = \left\| \left[ \frac{1}{t^{d-1}} \right]^d \int_0^{[1/t]^{-d} t^d} T_u(x) du - \frac{1}{t^d} \int_{[0,t]^{d}} T_u(x) du \right\|_\infty
\]

\[
= \left[ \frac{1}{t^{d-1}} \right]^d \int_0^{[1/t]^{-d} t^d} T_u(x) du \left[ \left( \frac{1}{t} \right)^d - \frac{1}{t^d} \right] \int_{[0,t]^{d}} T_u(x) du \left[ \left( \frac{1}{t} \right)^d - \frac{1}{t^d} \right] t^d \|x\|_\infty \to 0
\]

as \( t \to 0 \). Now, by [14], given \( \epsilon > 0 \), there exists \( \epsilon \in \mathcal{P}(M) \) such that \( \tau(e^{\perp}) \leq \epsilon \) and

\[
\|A_{[1/t]^{-1}}(x) - x\|_\infty \to 0 \text{ as } t \to 0.
\]

Therefore, taking into account [3], we obtain

\[
\|A_t(x) - x\|_\infty \leq \|A_t(x) - A_{[1/t]^{-1}}(x)\|_\infty + \|A_{[1/t]^{-1}}(x) - x\|_\infty \to 0
\]

as \( t \to 0 \), hence \( A_t(x) \to x \) a.u.

Let \( x \in L^1_+ \), and let \( \{e_\lambda\}_{\lambda \geq 0} \) be its spectral family. If \( x_m = \int_\lambda^\infty \lambda d\lambda \) and \( z_m = x - x_m \) for a positive integer \( m \), then \( \{x_m\} \subset L^1_+ \cap M \), \( \{z_m\} \subset L^1_+ \) and \( \|z_m\|_1 \to 0 \).

Fix \( \epsilon > 0 \) and \( \delta > 0 \). If \( \{r_n\} \) is a sequence of all positive rational numbers, then, by Theorem 5.1, the sequence \( \{A_{r_n}\} \) is b.u.e.m. at zero on \( L^1_+ \), hence on \( L^1 \). Then, applying Lemma 5.3, we find a projection \( e \in \mathcal{P}(M) \) and \( z_{m_0} \in \{z_m\} \) such that

\[
\tau(e^{\perp}) \leq \frac{\epsilon}{2} \text{ and } \sup_n \|A_{r_n}(z_{m_0})e\|_\infty < \frac{\delta}{3}.
\]

If \( t > 0 \), then \( r_{n_k} \to t \) for some subsequence \( \{r_{n_k}\} \), so \( \|A_t(z_{m_0}) - A_{r_{n_k}}(z_{m_0})\|_1 \to 0 \) by Lemma 5.3. Therefore \( A_{r_{n_k}}(z_{m_0}) \to A_t(z_{m_0}) \) in measure, which implies that there is a subsequence \( \{r_{n_{k_l}}\} \) such that \( A_{n_{k_l}}(z_{m_0}) \to A_t(z_{m_0}) \) a.u.

Since \( \|A_{n_{k_l}}(z_{m_0})e\|_\infty < \frac{\delta}{3} \) for each \( l \), it follows from [3, Lemma 5.1] that

\[
\sup_{t > 0} \|A_t(z_{m_0})e\|_\infty \leq \frac{\delta}{3}.
\]

Because \( x_{m_0} \in L^1_+ \cap M \), we have \( A_t(x_{m_0}) \to x_{m_0} \) a.u., so the net \( A_t(x_{m_0}) \) is a.u. Cauchy. Therefore, there exist \( g \in \mathcal{P}(M) \) and \( t_0 > 0 \) such that

\[
\tau(g^{\perp}) < \frac{\epsilon}{2} \text{ and } \|A_t(x_{m_0}) - A_{t'}(x_{m_0})g\|_\infty < \frac{\delta}{3}.
\]

for all \( 0 < t, t' < t_0 \).
If $h = e \wedge g$, then $\tau(h^\perp) < \epsilon$ and, in view of (7) and (8), we have

$$
\| (A_t(x) - A_{t'}(x))h \|_{\infty} \leq \| (A_t(x_{m_0}) - A_{t'}(x_{m_0}))h \|_{\infty} \\
+ \| A_t(z_{m_0})h \|_{\infty} + \| A_{t'}(z_{m_0})h \|_{\infty} < \delta.
$$

Thus, the net $A_t(x)$ is $a.u.$ Cauchy as $t \to 0$. By the proof of Theorem 2.3 in [3], $L^0$ is complete with respect to $a.u.$ convergence. Therefore, the net $\{A_t(x)\}$ converges $a.u.$ in $L^0$. Taking into account that, by Lemma 5.3, $A_{1/n}(x) \to x$ b.a.u., we conclude that $A_t(x) \to x$ a.u. as $t \to 0$ for all $x \in L^1_+$, hence for all $x \in L^1$. □

Remark 5.1. In view of Lemma 5.3, it is clear that the assertion of Theorem 5.1 holds for all $1 \leq p < \infty$. Alternatively, see the proof of Theorem 6.1 below.

6. Extension to Fully Symmetric Spaces of Measurable Operators

The non-increasing rearrangement of $x \in L^0$ is defined as

$$
\mu_t(x) = \inf \{ \lambda > 0 : \tau(\{ |x| > \lambda \} \leq t), \ t > 0,
$$

where $|x| = (x^*x)^{1/2}$, the absolute value of $x$ (see, for example, [12]).

Let $L^0_\tau$ be the $*$-subalgebra of $L^0$ consisting of such $x \in L^0$ that $\tau(\{ |x| > \lambda \} < \infty$ for some $\lambda > 0$. The collection of sets

$$
\mathcal{N}(\epsilon, \delta) = \{ x \in L^0(\sigma) : \mu_\delta(x) \leq \epsilon \}, \ \epsilon > 0, \ \delta > 0.
$$

forms a basis of neighborhoods of zero for the measure topology $t_\tau$ in $L^0_\tau$. Note that $\mathcal{M}$ is dense in $(L^0_\tau, t_\tau)$ [13].

A non-zero linear subspace $E \subset L^0_\tau$ with a Banach norm $\| \cdot \|_E$ is called fully symmetric if conditions

$$
x \in E, \ y \in L^0_\tau, \ \int_0^s \mu_t(y)dt \leq \int_0^s \mu_t(x)dt \ \text{for all} \ s > 0 \ (\text{writing} \ x \prec y)
$$

imply that $y \in E$ and $\| y \|_E \leq \| x \|_E$.

Let $L^0(0, \infty)$ be the linear space of (equivalence classes of) almost everywhere finite complex-valued Lebesgue measurable functions on the interval $(0, \infty)$. We identify $L^\infty(0, \infty)$ with the commutative von Neumann algebra acting on the Hilbert space $L^2(0, \infty)$ via multiplication by the elements from $L^\infty(0, \infty)$ with the trace given by the integration with respect to the Lebesgue measure $\mu$. A fully symmetric space $E \subset L^0(L^\infty(0, \infty), \nu)$, where the trace $\nu$ is given by the Lebesgue integral with respect to measure $\mu$, is called a fully symmetric function space on $(0, \infty)$ (see, for example, [13]).

If $E = E(0, \infty)$ is a fully symmetric function space, define

$$
E(\mathcal{M}) = E(\mathcal{M}, \tau) = \{ x \in L^0 : \mu_t(x) \in E \}
$$

and set

$$
\| x \|_{E(\mathcal{M})} = \| \mu_t(x) \|_E, \ x \in E(\mathcal{M}).
$$

It is shown in [16] that $(E(\mathcal{M}), \| \cdot \|_{E(\mathcal{M})})$ is a fully symmetric space. If $1 \leq p < \infty$ and $E = L^p(0, \infty)$, the space $(E(\mathcal{M}), \| \cdot \|_{E(\mathcal{M})})$ coincides with the noncommutative $L^p$-space $L^p(\mathcal{M}, \tau), \| \cdot \|_p$ [22]. In addition, $L^\infty(\mathcal{M}) = \mathcal{M}$ and

$$(L^1 \cap L^\infty)(\mathcal{M}) = L^1(\mathcal{M}) \cap \mathcal{M} \ \text{with} \ \| x \|_{L^1 \cap \mathcal{M}} = \max \{ \| x \|_1, \| x \|_\infty \},
$$

$$(L^1 + L^\infty)(\mathcal{M}) = L^1(\mathcal{M}) + \mathcal{M} \ \text{with}
$$
∥x∥_{L^1+\mathcal{M}} = \inf \{∥y∥_1 + ∥z∥_\infty : x = y + z, \ y ∈ L^1(\mathcal{M}), \ z ∈ \mathcal{M}\} = \int_0^1 \mu_t(x)dt
(see [11 Proposition 2.5]).

Let us notice that if x ∈ L^1 + \mathcal{M}, then there exists λ > 0 such that x \ {∥x∥ > λ} ∈ L^1(\mathcal{M}).

It is known that a fully symmetric space (E(\mathcal{M}), ∥·∥_{E(\mathcal{M})}) is an exact interpolation space for the Banach couple (L^1(\mathcal{M}), \mathcal{M}) [10]. Therefore T(E(\mathcal{M})) ⊂ E(\mathcal{M}) and ∥T∥_{E(\mathcal{M})→E(\mathcal{M})} ≤ 1 for every fully symmetric space E(0, ∞) and any T ∈ DS.

We say that a positive linear operator T : L^1 → L^1 is an absolute contraction and write T ∈ AC^+ if

∥T(x)∥_1 ≤ ∥x∥_1 \ \forall x ∈ L^1 \ \text{and} \ \|T(x)\|_\infty ≤ ∥x∥_\infty \ \forall x ∈ L^1 ∩ \mathcal{M}.

It is clear that if T ∈ DS^+, then T|L^1 ∈ AC^+. It turns out that any T ∈ AC^+ can be uniquely extended to a positive Dunford-Schwartz operator:

**Proposition 6.1.** [11 Proposition 1.1]. Given T ∈ AC^+ such that T|L^1 = T and T|\mathcal{M} is σ(\mathcal{M}, L^1)-continuous.

Recall that \{T_u\}_{u ∈ R^+} ⊂ DS^+ is a strongly continuous semigroup on L^1 and that

\[ A_t(x) = \frac{1}{t^2} \int_{[0,t]} T_u(x)du \in L^1 \ \text{for all} \ x ∈ L^1, \ t > 0. \]

Clearly, \{A_t\}_{t > 0} ⊂ AC^+. By Proposition 6.1, given t > 0, there exists a unique \(\bar{A}_t\) ∈ DS^+ such that \(\bar{A}_t|L^1 = A_t\) and \(\bar{A}_t|\mathcal{M}\) is σ(\mathcal{M}, L^1)-continuous. Let us denote this extension also by \(A_t\).

Since a fully symmetric space E = (E(\mathcal{M}), ∥·∥_{E(\mathcal{M})}) is an exact interpolation space for the Banach couple (L^1, \mathcal{M}), it now follows that \(A_t(E) ⊂ E\) and ∥\(A_t\|_{E→E} ≤ 1\) for every t > 0.

Define

\[ \mathcal{R}_\tau = \{x ∈ L^1 + \mathcal{M} : \mu_t(x) → 0 \ \text{as} \ t → ∞\}. \]

In is known that \(R_\tau\) is the closure of \(L^1 ∩ \mathcal{M}\) in \(L^1 + \mathcal{M}\) in the measure topology \(t_\tau\) [11 Proposition 2.7]. Since the convergence in the norm \(∥·∥_{L^1+\mathcal{M}}\) implies the convergence with respect to \(t_\tau\) [11 Proposition 2.2], it follows that \(R_\tau\) is a closed subspace in \((L^1 + \mathcal{M}, ∥·∥_{L^1+\mathcal{M}})\). Consequently, \((R_\tau, ∥·∥_{L^1+\mathcal{M}})\) is a Banach space. Note that definitions of \(R_\tau\) and \(∥·∥_{L^1+L^\infty}\) yield that if

\[ x ∈ R_\tau, \ y ∈ L^1 + \mathcal{M} \ \text{and} \ y ∼ x, \]

then y ∈ \(R_\tau\) and \(∥y∥_{L^1+\mathcal{M}} ≤ ∥x∥_{L^1+\mathcal{M}}\). Therefore \((R_\tau, ∥·∥_{L^1+\mathcal{M}})\) is a fully symmetric space, so \(A_t(R_\mu) ⊂ R_\mu\) and ∥\(A_t\|_{R_\mu→R_\mu} ≤ 1\) for every t > 0.

Further, if \(τ(1) = ∞\), then a fully symmetric space E ⊂ L^1 + \mathcal{M} is contained in \(R_\tau\) if and only if \(1 ∉ E\) [6 Proposition 2.2]. In particular, if E(0, ∞) is a a separable fully symmetric function space and E(\mathcal{M}, τ) is the corresponding noncommutative fully symmetric space, then E(\mathcal{M}, τ) ⊆ R_τ. Note also that if \(τ(1) < ∞\), then \(\mathcal{M} ⊂ L^1\) and \(R_τ = L^1\).

Here is an extension of Theorem 6.1 to \(R_\tau\):

**Theorem.** Given x ∈ \(R_\tau\), the averages \[ (∁) \] converge a.u. to x as t → 0.

**Proof.** Without loss of generality assume that x ≥ 0, and let \(\{e^λ\}_{λ≥0}\) be the spectral family of x. Given m ∈ N, denote \[ x_m = ∫_{1/m}^∞ λdeλ \] and \[ y_m = ∫_0^{1/m} λdeλ. \] Then

\[ 0 ≤ y_m ≤ \frac{1}{m}, \ x_m ∈ L^1, \ \text{and} \ x = x_m + y_m \ \text{for all} \ m. \]
Fix $\epsilon > 0$. By Theorem 1.2, $A_t(x_m) \to x_m$ a.u. as $t \to 0$ for each $m$. Therefore, there exists a sequence $\{e_m\} \subset \mathcal{P}(\mathcal{M})$ such that

$$\tau(e_m) \leq \frac{\epsilon}{2m}$$

and

$$\|A_t(x_m) - x_m\| \to 0 \quad \text{as} \quad t \to 0.$$ 

Then it follows that

$$\|A_t(x_m) - x_m\| < \frac{1}{m} \quad \text{for all} \quad 0 < t < t(m).$$

Since $\|y_m\| \leq \frac{1}{m}$, we have

$$\|A_t(x) - x\| \leq \|A_t(x_m) - x_m\| + \|A_t(y_m) - y_m\| + \|y_m - x\|$$

for each $m$ and all $0 < t < t(m)$. Now, if $e = \bigwedge e_m$, then

$$\tau(e) \leq \epsilon \quad \text{and} \quad \|A_t(x) - x\| < \frac{3}{m} \quad \text{for all} \quad 0 < t < t(m).$$

This means that $A_t(x) \to x$ a.u. as $t \to 0$. 

An application of Theorem 6.1 to a fully symmetric space yields the following.

**Theorem 6.2.** Let $\tau(1) = \infty$, and let $E \subset L^1 + \mathcal{M}$ be a fully symmetric space such that $1 \notin E$. Then, given $x \in E$, the averages $\{T_n\}$ converge a.u. to $x$ as $t \to 0$.

In particular, we have the following.

**Theorem 6.3.** Let $\tau(1) = \infty$, and let $E \subset L^1 + \mathcal{M}$ be a fully symmetric space such that $1 \notin E$. Then the averages $\frac{1}{t} \int_0^t T_s(x)ds$ converge a.u. to $x$ as $t \to 0$ for every $x \in E$.

A symmetric space $E \subset L^1 + \mathcal{M}$ is said to have order continuous norm if

$$\|x_n\|_E \downarrow 0 \quad \text{whenever} \quad 0 \leq x_n \in E \quad \text{and} \quad x_n \downarrow 0.$$ 

If $E$ is a symmetric space with order continuous norm, then $\tau(|x| > \lambda) < \infty$ for all $x \in E$ and $\lambda > 0$, so $E \subset R_\lambda$; in particular, $1 \notin E$. It is clear that the spaces $L^p(\mathcal{M})$, $1 \leq p < \infty$, have order continuous norms.

**Lemma 6.1.** If $E$ is a symmetric space with order continuous norm, then, given $\epsilon > 0$, there exists $\delta > 0$ such that $\|e\|_E < \epsilon$ for every $e \in \mathcal{P}(\mathcal{M})$ with $\tau(e) < \delta$.

**Proof.** Let $\{e_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{M})$ be such that $0 < \tau(e_n) < \infty$ for every $n$ and $e_n \downarrow 0$. Then $\{e_n\}_{n=1}^\infty \subset L^1 \cap \mathcal{M} \subset E$ and, by order continuity of the norm $\|\cdot\|_E$, we have $\|e_n\|_E \downarrow 0$. Consequently, $\|e_n\|_E < \epsilon$ for some $n_0$. Set $\delta = \tau(e_{n_0})$. Then if $e \in \mathcal{P}(\mathcal{M})$ and $\tau(e) < \delta$, we obtain

$$\mu_t(e) = \chi[0, \tau(e)] \leq \chi[0, \delta] = \mu_t(e_{n_0}),$$

hence $\|e\|_E \leq \|e_{n_0}\|_E < \epsilon$. 

Now, using Theorem 6.2, we give an extension of Theorem 6.1 to fully symmetric spaces with order continuous norms:

**Theorem 6.4.** Let $(E, \|\cdot\|_E)$ be a fully symmetric space with order continuous norm. Then $\|A_t(x) - x\|_E \to 0$ as $t \to 0$ for every $x \in E$. 
Proof. Without loss of generality assume that \( x \geq 0 \), and let \( \{ e_\lambda \}_{\lambda \geq 0} \) be the spectral family of \( x \). Given \( m \in \mathbb{N} \), denote
\[
a_m = \int_0^{1/m} \lambda d e_\lambda + \int_m^{\infty} \lambda d e_\lambda, \quad x_m = \int_1/m^{m} \lambda d e_\lambda.
\]
Then \( a_m \in E \), \( x_m \in E \cap L^1 \cap \mathcal{M}, \| x_m \|_\infty \leq m \) and \( x = a_m + x_m \) for all \( m \).

Fix \( \epsilon > 0 \). Since the norm \( \| \cdot \|_E \) is order continuous and \( a_m \downarrow 0 \), it follows that \( \| a_m \|_E \downarrow 0 \). Consequently, there exists \( m_0 \) such that \( \| a_{m_0} \|_E \leq \epsilon \), hence \( \| A_t(a_{m_0}) \|_E \leq \epsilon \) for all \( t > 0 \).

Lemma 6.1 implies that there is \( \delta_0 > 0 \) such that the inequality \( \tau(e) < \delta_0 \) entails \( \| e \|_E < \frac{\delta}{m_0}, e \in \mathcal{P}(\mathcal{M}) \). By Theorem 6.2, \( A_t(x_{m_0}) \to x_{m_0} \) a.u. as \( t \to 0 \). Therefore, there exists a projection \( e_0 \in \mathcal{P}(\mathcal{M}) \) such that
\[
\tau(e_0^\perp) < \delta_0 \quad \text{and} \quad \| (A_t(x_{m_0}) - x_{m_0})e_0 \|_\infty \to 0 \quad \text{as} \quad t \to 0.
\]
It follows then that
\[
\| (A_t(x_{m_0}) - x_{m_0})e_0^\perp \|_E \leq \| (A_t(x_{m_0}) - x_{m_0}) \|_\infty \| e_0^\perp \|_E < 2\epsilon \quad \text{for all} \quad t > 0.
\]
As \( x_{m_0} \in L^1 \), Theorem 3.1 implies that we also have
\[
\| (A_t(x_{m_0}) - x_{m_0})e_0 \|_1 \leq \| A_t(x_{m_0}) - x_{m_0} \|_1 \to 0 \quad \text{as} \quad t \to 0.
\]
Consequently,
\[
\| (A_t(x_{m_0}) - x_{m_0})e_0 \|_E \to 0 \quad \text{as} \quad t \to 0,
\]
and, since the symmetric space \( L^1 \cap \mathcal{M} \) is continuously embedded in the symmetric space \( (E, \| \cdot \|_E) \), we conclude that
\[
\| (A_t(x_{m_0}) - x_{m_0})e_1 \|_E \to 0 \quad \text{as} \quad t \to 0.
\]
In particular, there exists \( t(\epsilon) > 0 \) such that
\[
\| (A_t(x_{m_0}) - x_{m_0})e_0 \|_E < \epsilon \quad \text{whenever} \quad 0 < t < t(\epsilon).
\]
Therefore, given \( 0 < t < t(\epsilon) \), we have
\[
\| A_t(x) - x \|_E \leq \| (A_t(x_{m_0}) - x_{m_0})e_0 \|_E + \| (A_t(x_{m_0}) - x_{m_0})e_0^\perp \|_E
\]
\[
+ \| A_t(a_{m_0}) \|_E + \| a_{m_0} \|_E < 5\epsilon,
\]
which completes the proof. \( \square \)

7. Applications

In what follows we describe applications of Theorems 6.2 and 6.4 to noncommutative Orlicz, Lorentz and Marcinkiewicz spaces and to noncommutative symmetric spaces with order continuous norm. Naturally, we assume that \( \tau(1) = \infty \).

1. Let \( \Phi \) be an Orlicz function, that is, \( \Phi : [0, \infty) \to [0, \infty) \) is a convex continuous at 0 function such that \( \Phi(0) = 0 \) and \( \Phi(u) > 0 \) if \( u \neq 0 \). Let
\[
L_\Phi = L_\Phi^0(\mathcal{M}, \tau) = \left\{ x \in L^0(\mathcal{M}, \tau) : \tau \left( \Phi \left( \frac{|x|}{a} \right) \right) < \infty \quad \text{for some} \quad a > 0 \right\}
\]
be the corresponding noncommutative Orlicz space, and let
\[
\| x \|_\Phi = \inf \left\{ a > 0 : \tau \left( \Phi \left( \frac{|x|}{a} \right) \right) \leq 1 \right\}
\]
be the \textit{Luxemburg norm} in $L^\Phi$ (see, for example [3]). If $\tau(1) = \infty$, then $\tau(\Phi(1/t)) = \infty$ for all $a > 0$, hence $1 \notin L^\Phi$. Therefore, by Theorem 6.2, for a given $x \in L^\Phi$, the averages converge a.u. to $x$ as $t \to 0$.

An Orlicz function $\Phi$ is said to satisfy $\textit{(}\Delta_2\textit{-condition at 0} (at \infty)$ if there exist $u_0 \in (0, \infty)$ and $k > 0$ such that $\Phi(2u) < k\Phi(u)$ for all $0 < u < u_0$ (respectively, for all $u > u_0$). An Orlicz function $\Phi$ satisfies $\textit{(}\Delta_2\textit{-condition at 0 and at } \infty$ if and only if $(L^\Phi(0, \infty), \|\cdot\|_\Phi)$ has order continuous norm [1] Ch.2, §2.1, Theorem 2.1.17]. In this case, the corresponding noncommutative Orlicz space also has order continuous norm [11] Proposition 3.6.

Therefore, Theorem 6.3 implies that if an Orlicz function $\Phi$ satisfies $\textit{(}\Delta_2\textit{-condition at 0 and at } \infty$, then $\|A_t(x) - x\|_\Phi \to 0$ as $t \to 0$ for every $x \in L^\Phi$.

2. Let $\varphi$ be a concave function on $[0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, and let

$$ \Lambda_\varphi = \Lambda_\varphi(M, \tau) = \left\{ x \in L^0(M, \tau) : \|x\|_{\Lambda_\varphi} = \int_0^\infty \mu(x) d\varphi(t) < \infty \right\}, $$

be the corresponding \textit{noncommutative Lorentz space} (see, for example [3], [7]).

It is well-known that $(\Lambda_\varphi, \|\cdot\|_{\Lambda_\varphi})$ is a fully symmetric space; in addition, if $\varphi(\infty) = \infty$, then $1 \notin \Lambda_\varphi$ and if $\varphi(\infty) < \infty$, then $1 \in \Lambda_\varphi$. Therefore, in the case $\varphi(\infty) = \infty$, for a given $x \in \Lambda_\varphi$, the averages converge a.u. to $x$ as $t \to 0$.

By [15] Ch.II, §5, Lemma 5.1, [20] Ch.9, §9.3, Theorem 9.3.1]), the space $(\Lambda_\varphi(0, \infty), \|\cdot\|_{\Lambda_\varphi})$, and hence the noncommutative Lorentz space $\Lambda_\varphi(M, \tau)$ [11] Proposition 3.6] has order continuous norm if and only if $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. Therefore, Theorem 6.3 entails that if $\varphi$ is a concave function on $[0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that $\varphi(0) = 0$ and $\varphi(\infty) = \infty$, then $\|A_t(x) - x\|_{\Lambda_\varphi} \to 0$ as $t \to 0$ for every $x \in \Lambda_\varphi(M, \tau)$.

3. Let $\varphi$ be as above, and let

$$ M_\varphi = M_\varphi(M, \tau) = \left\{ x \in L^0(M, \tau) : \|x\|_{M_\varphi} = \sup_{0 < s < \infty} \frac{1}{\varphi(s)} \int_0^s \mu_t(x) dt < \infty \right\}, $$

be the corresponding \textit{noncommutative Marcinkiewicz space}. It is known that $(M_\varphi, \|\cdot\|_{M_\varphi})$ is a fully symmetric space; in addition, $1 \notin M_\varphi$ if and only if $\lim_{t \to \infty} \varphi(t) = 0$.

Therefore, Theorem 6.2 implies that if $\lim_{t \to \infty} \varphi(t) = 0$, then for a given $x \in M_\varphi$, the averages converge a.u. to $x$ as $t \to 0$.

If $\varphi(0) > 0$ and $\varphi(\infty) < \infty$, then $M_\varphi(M, \tau) = L^1(M, \tau)$ as the sets. In this case, the norms $\|\cdot\|_{M_\varphi}$ and $\|\cdot\|_1$ are equivalent [20] Ch.6, §6.1, Proposition 6.1.1], and we have, by Theorem 6.3, that $\|A_t(x) - x\|_{M_\varphi} \to 0$ as $t \to 0$ for every $x \in M_\varphi(M, \tau)$.

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National University of Uzbekistan, Tashkent, Uzbekistan
E-mail address: vladimirchil@gmail.com; chilin@ucd.uz

Pennsylvania State University, 76 University Drive, Hazleton, PA 18202, USA
E-mail address: snl2@psu.edu