Verifying relationship between Height and Spacing, in Barchan Dunes simulated by the Coupled Map Lattice Model

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We have investigated the relationship between height and spacing of Barchan dunes which the coupled map lattice model numerically generates. There is a scaling relation between them and the values of the scaling exponents agree well with real dunes’ values. The values of these scaling exponents are the same for both steady states and transient states.

KEYWORDS: Barchan dune, coupled map lattice, scaling exponents

1. Introduction

There are many serious problems all over the world. One of them is how to control the behavior of dunes. So the problem of controlling dunes should be solved and it has recently begun to be researched quantitatively by a number of physicists.

There is an experimental observation; height \( H \) and spacing \( L \) of dunes have a relation

\[
L \sim H^a,
\]

where \( a = 0.58 \sim 1.92 \).\(^1\) Since it is a non-trivial observation, some theoretical analysis is necessary. Unfortunately there are no theoretical approaches for this relation. Instead we have a powerful method; numerical analysis using computer. In this research, we would like to investigate the validity of this relation by computer simulations, "Coupled Map Lattice (CML) model". It was developed by Kaneko,\(^2\) and was applied to researches of dunes by Nishimori and Ouchi.\(^3,4\)

A crescent-shaped barchan dune is frequently observed in a desert. We analyzed barchan dunes by the numerical simulation with CML model. In this paper, "dune" means "barchan dune".

In the next section, we will discuss the relationship between height and spacing of real dunes. In the third section, we will show some results by computer simulation. Some discussions will be included in the fourth section.

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2. Experimental Law and Assumption

When there are \(N\) dunes, the \(n\)th dune is supposed to have height \(h_n\) and mass \(m_n\). It is expected that they have similar shapes with each other, because granular matter has an angle of repose. However, if the sand particles are blown by the wind, smaller dune may lose more sand than the larger ones. They will break the similarity of shapes. Therefore, the mass of a dune is not proportional to 3rd power of height. Sauermann et al.\(^7\) show that

\[
m_n = \gamma h_n^{2.4},
\]

where, \(\gamma\) is a constant number, a total sand mass \(M_{\text{total}}\) is given by

\[
M_{\text{total}} = \gamma \sum_n h_n^{2.4}.
\]

It is assumed that this \(M_{\text{total}}\) is shared by individual dunes.

Barchan dunes are formed when \(M_{\text{total}}\) is relatively small and the wind direction does not fluctuate. Thus we can choose two axes. One is parallel to the wind direction and the other is perpendicular to the same one. Along one of these axes, the distance between dunes can be measured as \(L_{\parallel}\) or \(L_{\perp}\). \(L_{\parallel}\) is a spacing of dunes along the direction parallel to the wind direction, and \(L_{\perp}\) is a spacing of dunes along the direction perpendicular to the wind direction. Here, we know the area \(S_n\) which \(n\)th dune occupies,

\[
S_n \sim L_{\parallel}L_{\perp}.
\]

There is another expression of \(S_n\) using the area \(S_{\text{total}}\) occupied by \(N\) dunes.

\[
S_n = \frac{S_{\text{total}}}{N} = \frac{S_{\text{total}}}{\frac{M_{\text{total}}}{\langle m_n \rangle}} = \frac{l^2}{\frac{M_{\text{total}}}{\gamma\langle h_n \rangle^{2.4}}} = \gamma \frac{l^2}{M_{\text{total}}} \langle h_n \rangle^{2.4},
\]

where \(S_{\text{total}}\) is an area of this system, \(l\) is size of the observed area.

We assume that two relations,

\[
L_{\parallel} \sim \langle h_n \rangle^\alpha,
\]

\[
L_{\perp} \sim \langle h_n \rangle^\beta,
\]

the occupied area by the \(n\)th dune is

\[
S_n \propto L_{\parallel}L_{\perp} \propto \langle h_n \rangle^\alpha \langle h_n \rangle^\beta = \langle h_n \rangle^{\alpha+\beta}.
\]

(5) is compared with (8), the exponent of (8) is expected as

\[
\alpha + \beta = 2.4.
\]
3. Numerical experiments and Results

We used CML model which was arranged by Nishimori and Ouchi\textsuperscript{3, 4}) in order to apply to computer simulation. At first, we prepare a 1000 × 1000 lattice and give an uniform random number \([0, X]\) to all sites. These random numbers correspond to the height for each site;

\[ h(i, j) \in [0, X], \]  

\(X\) is the parameter which controls the total amount of the sand, and \(h(i, j)\) is an amount of sand at a site \((i, j)\).

Saltation and creep are important when considering a dynamics of dune. Saltation is a flying of sand by the wind. Creep is that sand rolls and falls with gravity. Although they had better to be considered exactly, they are approximated for simplicity.

Flight distance \(L_s\) and the amount of the sand which flies by saltation \(q_s\) have the relationship with \(\Delta h\), which is a difference of height along the leeward direction,\textsuperscript{6})

\[ q_s = -\tanh(\Delta h) + 1.3, \]  

\[ L_s = \tanh(\Delta h) + 1. \]  

We consider that wind is blowing to the positive direction of the \(i\) axis,

\[ \Delta h(i, j) = h(i + 1, j) - h(i, j). \]  

Thus saltation is decided by only landform.

Next, we consider creep, which is regarded as ”diffusion of sand”. We introduce an arbitrary diffusion constant \(D\), and the amount of diffusion of sand \(q_c\) is described as

\[ q_c = D\Delta h(i \pm 1, j \pm 1), \]  

using (13) and (14), we get

\[ h(i \pm 1, j \pm 1) \rightarrow h(i \pm 1, j \pm 1) + q_c, \]  

\[ h(i, j) \rightarrow h(i, j) - q_c. \]  

But, we must not forget ”critical angle” of sand; it is an ”angle of repose” of sand. According to the observation of sand in a hourglass, it is about 34\(^{\circ}\). The creep does not occur, if an inclination is equal to 34\(^{\circ}\) or is less than that. Since \(\tan 34^{\circ} \approx 0.67\), only when

\[ h(i \pm 1, j \pm 1) - h(i, j) > 0.67 \]  

creep occurs.

We repeat the whole process. One step contains all of these. With this CML model, a description of dynamics of dunes becomes much easier than a real situation. Nishimori and Ouchi\textsuperscript{3, 4}) have already reported that their model reproduced dune patterns qualitatively. We are interested in how similar it is quantitatively to the real dunes.
3.1 Scaling Relation in Steady States

Dunes get to the steady state after some transient period. In order to realize this stationary state, we have iterated whole processes over sufficiently long period. In this case, it is 3,000 steps.

We will discuss the average height of dunes. We do not employ simple averages over individual dunes. Instead we use the total height of sand, $\langle H_{st} \rangle$ which can be defined as

$$\langle H_{st} \rangle \equiv \frac{H_{st}}{X} = \sum_{i,j} h_{st}(i,j)^2 / X.$$  \hspace{1cm} (18)

Since it is assumed that system size is fixed, $\langle H_{st} \rangle$ is proportional to the average of height (see Appendix). After some period, the system becomes steady states. $h_{st}(i,j)$ expresses the quantity of the sand in a site $(i,j)$ at that time. Figure 1 shows the relation between $X$ and a quantity $\langle H_{st} \rangle$ which is proportional to an average height of these steady dunes. There is a linear relation between $X$ and $\langle H_{st} \rangle$ as

$$\langle H_{st} \rangle / 10^5 = (4.64 \pm 0.07)X + (2.70 \pm 0.07),$$  \hspace{1cm} (19)

where CML model generates dune-like patterns only when this relation stands. Therefore this equation enables us to judge whether the system converges to the steady state or not.

We examine both the quantity $\langle H_{st} \rangle$ which is proportional to height and the spacing $L_{st\parallel}$ which is along the direction parallel to the wind direction (see Fig.2), and get

$$L_{st\parallel} \sim \langle H_{st} \rangle^{0.8 \pm 0.5}.$$  \hspace{1cm} (20)

On the other hand, a quantity $\langle H_{st} \rangle$ has the relation with the spacing $L_{st\perp}$ along the direction perpendicular to the wind direction,

$$L_{st\perp} \sim \langle H_{st} \rangle^{1.6 \pm 0.1}. $$  \hspace{1cm} (21)
Substituting the exponents of (20) and (21) into the relation (8), we get
\[ S_n \propto \langle H_{st} \rangle^{2.4 \pm 0.5}. \] (22)

This result is consistent with the results obtained in real dunes as described in the section 2.

### 3.2 Scaling Relation in Transient States

Next we consider the scaling relation in transient states. Here, we employ two methods for this investigation. One is for a fixed time and another is time series scaling.

#### 3.2.1 One Fixed Time

At first, we consider a certain fixed time in a transient state \((t = 2,000\) steps), and investigate the relation between the spacing of dunes \(L_{fx\parallel}\) which is along the direction parallel to wind direction and a quantity \(\langle H_{fx} \rangle\) (see Fig.3). A quantity \(\langle H_{fx} \rangle\) is defined as,
\[ \langle H_{fx} \rangle \equiv \sum_{i,j} h_{fx}(i,j)^2 / X. \] (23)

Since system-size is fixed, \(\langle H_{fx} \rangle\) is proportional to the average of height at one fixed time in transient states. \(h_{fx}(i,j)\) is the quantity of the sand in a site \((i,j)\) at that time. And we get,
\[ L_{fx\parallel} \sim \langle H_{fx} \rangle^{0.8 \pm 0.1}. \] (24)

Next, we compute spacing \(L_{fx\perp}\) which is along the direction perpendicular to the wind direction, and get
\[ L_{fx\perp} \sim \langle H_{fx} \rangle^{1.5 \pm 0.3}. \] (25)
Substituting the exponents of (24) and (25) into the relation (8), we get
\[ \alpha + \beta = 2.3 \pm 0.3, \] (26)
therefore,
\[ S_n \propto \langle H_{fx} \rangle^{2.3 \pm 0.3}. \] (27)
This result also agrees to the argument in Sec. 2.

### 3.2.2 Time Series Scaling

The transitional time period varies with the initial quantity of sand. During this period, state of dunes continues to change. Next, we analyze dunes in transient states, by time series scaling method.

The total sand height in a certain time \( t \) in transient states is set to \( H_{total}(t) \). And the quantity \( H(t) \) which is proportional to a height of dunes in this states can be defined as
\[ H(t) \equiv H_{total}(t)/X = \sum_{i,j} h_{(i,j)}(t)^2/X. \] (28)
Here, \( h_{(i,j)}(t) \) expresses the height a site \((i, j)\) has in a certain time \( t \). And \( X \) is a parameter which controls initial quantity of sand. We normalize variables \( H(t) \) and \( L \) as
\[ (t, H(t)) \rightarrow \left( \frac{t}{\langle H_{st} \rangle^\delta}, \frac{H(t)}{\langle H_{st} \rangle} \right), \] (29)
\[ (t, L(t)) \rightarrow \left( \frac{t}{L_{st}^{\delta'}}, \frac{L(t)}{L_{st}} \right). \] (30)

Where \( \delta, \delta' \) are scaling indices, and \( L_{st} \) means a spacing of dunes in a steady states. We assume that a scaling relation between time \( t \) and height \( H(t) \) as follows,
\[ \frac{H(t)}{\langle H_{st} \rangle} = f\left( \frac{t}{\langle H_{st} \rangle^\delta} \right) \sim \left( \frac{t}{\langle H_{st} \rangle^\delta} \right)^\epsilon \quad (t \ll \langle H_{st} \rangle^\delta), \] (31)
and also the same scaling relation between \( t \) and \( L(t) \) is assumed

\[
\frac{L(t)}{L_{st}} = f\left(\frac{t}{L_{st}^\delta}\right) \sim \left(\frac{t}{L_{st}^\delta}\right)^{\epsilon'} \quad (t \ll L_{st}^\delta),
\]  

(32)

removing \( t \) from (31) and (32), we get

\[
\frac{H(t)}{L(t)^\beta} \sim \frac{\langle H_{st} \rangle}{L_{st}^\beta} \left(\frac{L_{st}^\delta}{\langle H_{st} \rangle^{\delta}}\right)^{\epsilon'}.
\]  

(33)

The right hand side of (33) is constant. Therefore, the relation \( H(t) \) and \( L(t) \) is given by

\[
H(t) \sim L(t)^{\beta} \longrightarrow L(t) \sim H(t)^{\epsilon'}. \]  

(34)

The parameter for the spacing along the parallel direction to the wind direction is

\[
\alpha = \frac{\epsilon}{\epsilon'},
\]  

(35)

and for the spacing along the perpendicular direction to the wind direction is

\[
\beta = \frac{\epsilon}{\epsilon'}. \]  

(36)

\[\epsilon_{\parallel}(\epsilon_{\perp}) \] is a parameter \( \epsilon' \) when a parallel (perpendicular) direction to the wind direction is considered. In our simulations, three exponents \( (\epsilon, \epsilon_{\parallel}, \epsilon_{\perp}) \) are (see Fig. 4 and Fig. 5)

\[
\epsilon = 0.11 \pm 0.02 \]  

(37)

\[
\epsilon_{\parallel} = 0.08 \pm 0.01 \]  

(38)

Fig. 4. Time series scaling of the height of dunes
Here, we note that $\alpha$ ($\beta$) is the exponent for parallel (perpendicular) direction to the wind direction and it is estimated as follows,

$$
\alpha = \frac{\epsilon_{\parallel}}{\epsilon} = \frac{0.08 \pm 0.01}{0.11 \pm 0.02} \rightarrow 0.7 \pm 0.2, \quad (40)
$$

$$
\beta = \frac{\epsilon_{\perp}}{\epsilon} = \frac{0.16 \pm 0.01}{0.11 \pm 0.02} \rightarrow 1.5 \pm 0.4. \quad (41)
$$

Here we again give the value

$$
S_n \propto H(t)^{2.2\pm0.4}. \quad (42)
$$

Again, this value does not disagree with the values obtained previously.

4. Discussions

We have estimated the exponent $\alpha + \beta \cong 2.4$ using several methods. This is the first estimation using numerical simulation. These values are consistent with the value by the argument developed from of Sauermann’s observation \(^7\) in Sec. 2, as shown in Table.1.

In the present paper, we have discussed a quantity which is proportional to an average height and spacing of dunes. All our results agree with Lancaster’s observations.\(^1\) The CML model approximates dynamics of dunes. Pay attention to the fact that we did not treat exactly relationship between sand and wind. But as you saw, our results agreed well with the real dunes. Thus, exact discussions about the relationship between wind and sand, for example a sand flux and a distance of flight, may not be so important. A sand flux and a distance of sand flight is designed by how wind blows over a dune, and how wind blows over a dune is designed by the landform. To tell the truth, even if we know about only landform, we can guess sand flux $q_s$ and distance of sand flight...
| state of dunes                  | parameter   |
|-------------------------------|-------------|
| value from observation        | 2.4         |
| steady states                 | 2.4 ± 0.5   |
| transient states (fixed time) | 2.3 ± 0.3   |
| transient states (scaling)    | 2.2 ± 0.4   |

Table I. Summary of this study

$L_s$. Similarly, creep was treated as a diffusion of sand. We did not consider repellent force and so on which must be considered when we deal with real sand creep.

Thus, we can consider that exact theory about wind velocity, sand flux and so on is not so necessary. There seems to exist an universality class about the relationship among landform, sand flux and distance of sand flight. Thus the exact creep theory can be replaced with the diffusion theory. Now we suggest that CML model is suitable model when studying an average height and a spacing of dunes.

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Appendix: Definition of average height

Consider two differently-shaped dunes with the equal amount of sand. The following figures show the cross section along leeward direction. For simplicity, we assume a simple sinusoidal shape for each dune. The total height $H_{sum}$ and $H'_{sum}$ are
\[ H_{sum} \equiv \int_{0}^{l} h(i) \, di = h_n \int_{0}^{\pi} \sin ai \, di = \frac{2h_n}{a}, \quad (A.1) \]

and

\[ H'_{sum} \equiv \int_{0}^{l} h'(i) \, di = h'_n \int_{0}^{\pi} \sin a'i \, di = \frac{2h'_n}{a'}. \quad (A.2) \]

These integrated values are equal because the amount of sand is equal. And following relations are realized,

\[ \frac{2h_n}{a} = \frac{2h'_n}{a'}, \quad (A.3) \]

and

\[ h_n : a = h'_n : a'. \quad (A.4) \]

The average height of dunes \( \langle H \rangle \) and \( \langle H' \rangle \) becomes equal to the integrated value divided by system size,

\[ \langle H \rangle = \frac{2h_n}{al}, \quad (A.5) \]

and

\[ \langle H' \rangle = \frac{2h'_n}{al'}. \quad (A.6) \]

When (A.4) is considered, \( \langle H \rangle = \langle H' \rangle \) is realized and it turns out that this calculation cannot tell us the height of dunes, \( h_n \) or \( h'_n \). When the value of each site is squared, we get the integrations

\[ H_{st} \equiv \int_{0}^{l} h(i)^2 \, di = h_n^2 \int_{0}^{\pi} \sin^2 ai \, di = \frac{h_n^2 \pi}{2a}, \quad (A.7) \]

and

\[ H'_{st} \equiv \int_{0}^{l} h'(i)^2 \, di = h'_n^2 \int_{0}^{\pi} \sin^2 a'i \, di = \frac{h'_n^2 \pi}{2a'}. \quad (A.8) \]

Since the average squared-height of dunes \( \overline{H_{st}} \) and \( \overline{H'_{st}} \) become equal to the integrated value divided by system size, next relations are realized,

\[ \overline{H_{st}} = \frac{h_n^2 \pi}{2al}, \quad (A.9) \]
and
\[ \overline{H_{st}} = h_n' \frac{2 \pi}{2 \alpha l}. \]  

(A.10)

When (A.4) is considered,
\[ \overline{H_{st}} : \overline{H_{st}'} = h_n : h_n' \]  

(A.11)
is concluded. On the other hand, it is also that the ratio of these integration values is,
\[ H_{st} : H_{st}' = h_n : h_n'. \]  

(A.12)

From (A.7), (A.8), (A.11) and (A.12),
\[ \overline{H_{st}} : \overline{H_{st}'} = H_{st} : H_{st}' = \int_0^l h(i)^2 di : \int_0^l h(i)'^2 di. \]  

(A.13)

Thus the ratio of average heights of dunes can be estimated by squared quantity of each site.

If we define quantity \( \langle H_{st} \rangle \)
\[ \langle H_{st} \rangle = \frac{H_{st}}{X}, \]  

(A.14)

and consider that total sand mass is the product of \( X \) and the system size \( \ell \),
\[ H_{sum} = X \ell. \]  

(A.15)

we get
\[ \langle H_{st} \rangle = \frac{H_{st}}{H_{sum}} \ell \simeq h_n \ell. \]  

(A.16)

If \( \ell \) is fixed, we can use \( \langle H_{st} \rangle \) as the quantity proportional to \( h_n \).