On the inherent self-excited macroscopic randomness of chaotic three-body system

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Abstract What is the origin of macroscopic randomness (uncertainty)? This is one of the most fundamental open questions for human being. In this letter, 10000 samples of reliable (convergent), multiple-scale (from \(10^{-60}\) to \(10^2\)) numerical simulations of a chaotic three-body system indicate that, without any external disturbance, the microscopic inherent uncertainty (in the level of \(10^{-60}\)) due to physical fluctuation of initial positions of the three-body system enlarges exponentially into macroscopic randomness (at the level \(O(1)\)) until \(t = T^*\), the so-called physical limit time of prediction, but propagates algebraically thereafter when accurate prediction of orbit is impossible. Note that these 10000 samples use micro-level, inherent physical fluctuations of initial position, which have nothing to do with human being. Besides, the differences of these 10000 fluctuations are mathematically so small (in the level of \(10^{-60}\)) that they are physically the same since a distance shorter than a Planck length does not make physical senses according to the spring theory. It indicates that the macroscopic randomness of the chaotic three-body system is self-excited, say, without any external force or disturbances, from the inherent micro-level uncertainty. This provides us the new concept “self-excited macroscopic randomness (uncertainty)”. Besides, it is found that the chaotic three-body system might randomly disrupt at \(t = 1000\) in about 25% probability without any external disturbance, which provides us the new concepts “self-excited random disruption” and “self-excited random escape” of chaotic three-body system. It suggests that a chaotic three-body system might randomly evolve by itself, without any external forces or disturbance. Thus, the world is essentially uncertain, since such kind of self-excited macroscopic randomness (uncertainty) is inherent and unavailable. This work also implies that an universe could randomly evolve by itself into complicated structures, without any external forces, and that the nature could randomly evolve by itself into organism and even human being, without any external forces. It is found that the macroscopic randomness is even dependent upon microscopic uncertainty, from statistical viewpoint. All of these reliable computations reveal a kind of origins of macroscopic randomness/uncertainty.

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1 Introduction

When one looks at the sky in a clear night, the distribution of stars seems random. Besides, velocities in turbulent flows are always different even at the same points of observation using the same measure equipments. In practice, random/uncertain macroscopic phenomena happen quite frequently. However, what is the origin of macroscopic randomness (uncertainty)? This is one of the most fundamental questions for us. Without doubt, the answers to this open question may greatly deepen and enrich our understandings about nature.

It is widely accepted that microscopic phenomena are uncertain essentially, although they can be well described by deterministic laws in statistic meanings. Are there any relationships between microscopic uncertainty and macroscopic randomness? Some believe that there should exist relationships between them, but some categorically deny. However, neither of them can provide scientific supports based on validated experiments and/or reliable numerical simulations.

It seems very difficult to reveal the relationship between micro-level and macroscopic uncertainty by means of physical experiments, because artificial uncertainty of physical experiments is often much larger than inherent micro-level physical uncertainty.

Fortunately, it is widely believed that the characteristics of nature can be well described by physical laws and principles that are expressed by mathematical formulas/equations. Like physical experiments, studies on mathematical models also make great contributions for us to understand the nature better. For example, Galileo’s and Einstein’s famous “ideal experiments” completely renewed our concepts about inertia, gravity, time and space.

However, there also exists the uncertainty of theoretical prediction using mathematical models, too. For example, the imperfection of initial condition and numerical algorithms might be the sources of uncertainty. These sources of uncertainty, caused by limited accuracy of measurement for initial/boundary conditions and numerical errors of algorithms, are artificial. But, some sources of uncertainty are physical and inherent, which have nothing to do with human being. These inherent physical uncertainties are unavailable, although they are much smaller than the artificial uncertainties. It is a pity that they were curtly neglected in the past.

Generally speaking, it is difficult to accurately simulate propagation of uncertainty, especially for chaotic dynamic systems far from equilibrium state, which have the so-called sensitive dependence on initial conditions (SDIC), i.e. a small disturbance in initial condition leads to huge difference of solution (trajectory). This is mainly because the artificial numerical noises (truncation error and round-off error) are unavoidable at each time-step, which enlarge exponentially and propagate together with uncertainty of initial conditions. In general, uncertainty of initial condition caused by imperfect and limited measurement is often larger than numerical noises, and numerical noises are much larger than inherent, micro-level physical uncertainty of initial condition. Therefore, inherent, micro-level physical uncertainty of
initial condition was never considered in a macroscopic chaotic dynamic system.

In 2009, the method of “Clean Numerical Simulation” (CNS) [1] was proposed to decrease numerical noises so greatly that numerical errors can be much smaller even than micro-level inherent physical uncertainty of initial conditions in a given interval of time, and thus can be neglected. The CNS [1,3] is based on an arbitrary-order Taylor series method (TSM) [4,5] and the arbitrary multiple-precision (MP) data [6], together with a check of solution verification. Currently, assuming that the initial conditions are exact, a reliable convergent chaotic solution of the famous Lorenz equation in a rather long time interval [0,10000] was gained, for the first time, by means of the National Supercomputer TH-A1 (at Tianjing, China) and the CNS with the 3500th-order Taylor series expansion and the 4180-digit multiple precision data. It indicates that, given an exact initial condition, one can obtain reliable (convergent) solution of a chaotic dynamic system in any a finite interval of time, without uncertainty. This suggests that, for chaotic dynamic systems, uncertainty might come from initial condition only, since numerical noises can be so small to be neglected.

Unfortunately, the uncertainty of initial condition is unavoidable, not only due to imperfection and finite accuracy of measurement but also due to the micro-level inherent physical uncertainty. Traditionally, most researchers often add a small disturbance to initial conditions, but without considering its source. Note that uncertainty is a characteristic of nature, and thus should have nothing to do with the existence of human being: even if human being could perfectly measure initial condition in arbitrary accuracy, there still exists the micro-level inherent physical uncertainty of initial conditions.

For example, let us consider the famous three-body problem governed by Newtonian gravitational law with the dimensionless equations

\[
\ddot{x}_{k,i} = \sum_{j=1,j\neq i}^{3} \rho_j \frac{(x_{k,j} - x_{k,i})}{R_{i,j}^3}, k = 1, 2, 3, \tag{1}
\]

where \( r_i = (x_{1,i}, x_{2,i}, x_{3,i}) \) denotes the dimensionless position of the \( i \)th body, \( \rho_i = m_i/m_1 \) \( (i = 1, 2, 3) \) the ratio of mass, and

\[
R_{i,j} = \left[ \sum_{k=1}^{3} (x_{k,j} - x_{k,i})^2 \right]^{1/2}. \tag{2}
\]

We consider here the case \( \rho_1 = \rho_2 = \rho_3 = 1 \). As long as velocities of each body are much less than the light speed, this model is rather accurate in physics, since Einstein’s general relativity is unnecessary. Besides, by means of the CNS, the uncertainty due to numerical noises can be negligible. In this way, the uncertainty due to physical model and numerical algorithm is negligible.

However, even if we assume that we could measure the initial positions \( r_i(0) \) and velocities \( \dot{r}_i(0) \) in infinite accuracy (this is impossible in practice according to the Heisenberg’s uncertainty principle), the initial positions of each body are still
inherently uncertain in physics. First of all, according to wave-particle duality of de Broglie, a body has non-zero amplitude of the de Broglie's wave so that position of a body is always uncertain: it could be almost anywhere along de Broglie's wave packet. Besides, the so-called Planck length

\[ l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616252(81) \times 10^{-35} \text{ (m)} \]

is the length scale at which quantum mechanics, gravity and relativity all interact very strongly, where \( c \) is the speed of light in a vacuum, \( G \) is the gravitational constant, \( \hbar \) is the reduced Planck's constant, respectively. According to the string theory, the Planck length is the order of magnitude of oscillating strings that form elementary particles, and shorter length do not make physical senses. Especially, in some forms of quantum gravity, it becomes impossible to determine the difference between two locations less than one Planck length apart. Therefore, the micro-level inherent fluctuation of position of a body shorter than the Planck length is essentially uncertain and/or random. It should be emphasized once again that such kind of uncertainty of position is inherent and objective: it has nothing to do with human being and Heisenberg’s uncertainty principle.

Using the diameter \( d_M \approx 10^{20} \) (meter) of Milky Way Galaxy as characteristic length, we have the dimensionless physical uncertainty of initial position \( l_p/d_M \approx 1.8 \times 10^{-56} \). So, from the physical viewpoint, any dimensionless distances shorter than \( 1.8 \times 10^{-56} \) have no physical senses. Therefore, it is physically reasonable to assume that the initial velocities \( \mathbf{r}_i(0) \) of the three-body system are exact but their initial positions \( \mathbf{r}_i(0) \) contain a micro-level fluctuation \( \mathbf{r}'_i(0) \) in Gaussian normal distribution with zero mean and standard deviation \( \sigma_0 = 10^{-60} \), i.e. \( \mathbf{r}_i(0) = \mathbf{\bar{r}}_i(0) + \mathbf{r}'_i(0) \), where \( \mathbf{\bar{r}}_i(0) = \langle \mathbf{r}_i(0) \rangle \), \( \langle \mathbf{r}'_i(0) \rangle = 0 \) and \( \sqrt{\langle \mathbf{r}'_i^2(0) \rangle} = \sigma_0 \). It should be emphasized that, although these initial positions \( \mathbf{r}_i(0) \) are mathematically different, they are the same from physical viewpoint, since a distance smaller than one Planck length does not make physical senses.

Without loss of generality, let us consider the case

\[ \mathbf{r}_1 = (0, 0, -1), \mathbf{r}_2 = (0, 0, 0), \mathbf{r}_3 = -(\mathbf{r}_1 + \mathbf{r}_2), \] (3)

with the exact initial velocities

\[ \mathbf{r}_1 = (0, -1, 0), \mathbf{r}_2 = (1, 1, 0), \mathbf{r}_3 = -(\mathbf{r}_1 + \mathbf{r}_2). \] (4)

When there is no fluctuation, i.e. \( \mathbf{r}'_i(0) = 0 \), the three-body system is chaotic with a positive Lyapunov exponent \( \lambda = 0.1681 \), but without disruption (i.e. no body escapes), as pointed out by Sprott [7].

Many researchers investigated three-body problem. For example, Monaghan [8,9] and Nash [10] proposed a statistical theory to study the disruption of three-body systems. Mikkola [11] and Urminsky [12] researched the relation between instability and Lyapunov times for three-body problem. However, most researchers simply give
a small disturbance of initial condition, but without considering the sources of them: these disturbances are much larger than micro-level physical uncertainty of initial condition mentioned above. This is because they use low-order numerical algorithms just with double-precision data, which are certainly not accurate enough to investigate the propagation of micro-level inherent physical uncertainty of initial conditions of chaotic systems.

2 Results and discussions

How about the propagation of the micro-level inherent physical uncertainty of the initial position $r_i(0)$ of this chaotic three-body system?

Ten thousand samples of reliable (convergent), multiple-scale (from $10^{-60}$ to $10^{2}$) numerical simulations of the chaotic three-body system are obtained in the time interval $[0,1000]$ by means of the CNS using the 300-digit precision data, the time-step $\Delta t = 10^{-3}$ and high-enough order $M$ of Taylor series expansion, where $M \geq 30$ in general. Each chaotic solution is verified by means of a higher-order Taylor series expansion with the same initial condition, and only convergent results in the interval $[0,1000]$ are accepted. In this way, the artificial uncertainty due to numerical algorithms is avoided. Thus, the micro-level inherent physical fluctuation $r'_i(0)$ of the initial position $r_i(0)$ is the only source of the uncertainty.

Let $\bar{x}_{i,j}(t)$ denote the mean of $x_{i,j}(t)$ and $\sigma_{i,j}(t)$ its unbiased estimate of standard deviation, respectively, based on 10000 samples of the reliable convergent CNS simulations using the initial conditions with different micro-level fluctuations $r'_i(0)$ of position. So, we have the initial standard deviation $\sigma_{i,j}(0) = \sigma_0 = 10^{-60}$ for the considered case.

Obviously, the smaller the standard deviation $\sigma_{i,j}(t)$, the smaller the uncertainty. According to the statistic analysis based on these 10000 samples of reliable (convergent) simulations given by the CNS, each $\sigma_{i,j}(t)$ increases exponentially from $\sigma_{i,j}(0) = 10^{-60}$ when $t = 0$ until $\sigma_{i,j} = \sigma^*$ at $t = T^* \approx 800$, as shown in Fig. 1 for Body-1 as an example, where $\sigma^*$ is a standard deviation corresponding to an observable macroscopic difference of position, $T^*$ is the critical time corresponding to $\sigma^*$, respectively. It is found that the uncertainty propagates exponentially in essence and can be expressed approximately in the form

$$\sigma_{i,j}(t) \approx \sigma_{i,j}(0) e^{\lambda t} = \sigma_0 e^{\lambda t}, \quad 0 \leq t < T^*,$$

where $\lambda = 0.1681$ is exactly the Lyapunov’s exponent for the same chaotic three-body system \cite{7} with the exact initial conditions

$$\dot{r}_1 = (0, -1, 0), \dot{r}_2 = (1, 1, 0), \dot{r}_3 = -(\dot{r}_1 + \dot{r}_2), \quad (5)$$

$$r_1 = (0, 0, -1), r_2 = (0, 0, 0), r_3 = -(r_1 + r_2), \quad (6)$$

say, without the micro-level fluctuation of position, i.e. $r'_i(0) = 0$. Besides, the critical time $T^*$ is approximately determined by $\sigma_0 e^{\lambda T^*} = \sigma^*$. For example, one has $T^* = 801$.
Figure 1: **The standard deviations of Body-1.** Results are based on the 10000 samples of reliable, multiple-scale simulations given by the CNS with the micro-level fluctuation of initial position $r'_i(0)$ in Gaussian distribution ($\sigma_0 = 10^{-60}$). Red line: $\sigma_{1,1}(t)$; Green line: $\sigma_{2,1}(t)$; Blue line: $\sigma_{3,1}(t)$; Dashed line: $\sigma = \sigma_0 \exp(\lambda t)$ where $\lambda = 0.1681$ is the Lyapunov exponent given by Sprott [7].

when $\sigma^* = 0.03$, $T^* = 808.2$ when $\sigma^* = 0.1$, which agree well with the observed value of the critical time, as shown in Figs. 1 and 2.

It is found that, when $t > T^*$, the standard deviations $\sigma_{i,j}(t)$ does not increase exponentially any more, as shown in Fig. 1. This is a surprise, since it is traditionally believed that, due to the SDIC, a difference of initial condition of chaotic dynamic systems should be enlarged exponentially. It suggests that $T^*$ is indeed special, which should have some physical meanings.

Note that observable differences ($\sigma^* = 0.03 \sim 0.1$) of positions appear at $t = T^*$. From then on, the standard deviation $\sigma_{i,j}(t)$ of position becomes so large ($\sigma_{i,j} > \sigma^*$) that accurate prediction becomes impossible, as shown in Fig. 2. In other words, when $t \leq T^*$, one can give accurate enough prediction about the orbits, but after $t > T^*$, the inherent micro-level physical uncertainty transfers into macroscopic ones ($\sigma_{i,j} > \sigma^*$) so that any accurate predictions about the orbits of the chaotic three-body system have no physical meanings at all. Thus, $T^*$ gives the maximum time of theoretical prediction, called the physical limit time of prediction. Therefore, when $t < T^*$, although the inherent physical uncertainty propagates exponentially, accurate enough prediction of orbits is possible in theory. However, when $t > T^*$, the micro-level inherent physical uncertainty due to the fluctuation $r'_i(0)$ of initial position is enlarged to be macroscopic, as shown in Fig. 2 so that it is impossible in physics to give any accurate predictions of orbits.

It should be emphasized that such kind of macroscopic uncertainty comes solely from the micro-level inherent physical uncertainty due to the fluctuation $r'_i(0)$ of initial position, and has nothing to do with human being and Heisenberg’s uncertainty principle, since the numerical noises are negligible for each simulations of chaotic
Figure 2: The position distribution of Body-1 (red points), Body-2 (blue points) and Body-3 (blue points) in the \((x, y)\) plane at different times when \(\sigma_0 = 10^{-60}\). Results are based on the 10000 samples of reliable, multiple-scale simulations given by the CNS with the micro-level fluctuation of initial position \(r'_i(0)\) in Gaussian distribution. The corresponding movie is published on website of the journal.

orbits (because of the use of the CNS) and besides the model equation is good enough due to much smaller body’s velocities than the light speed. Therefore, the origin of this kind of macroscopic randomness is the micro-level inherent physical uncertainty of position due to the wave-particle duality of de Broglie and/or the Planck length based on the string theory. Thus, without any external disturbance, the micro-level inherent physical uncertainty itself can be enlarged exponentially and excited into macroscopic randomness. Such kind of uncertainty is called self-excited macroscopic uncertainty or self-excited randomness. This is a new concept, which can be used to explain the origin of uncertainty/randomness of many phenomena in nature, such as turbulent flows.

It is found that the three-body system does not disrupt even in the interval \([0, 10000]\) if there is no micro-level inherent physical uncertainty of positions in the initial conditions, i.e. \(r'_i(0) = 0\). However, it is very interesting that the tiny, micro-level fluctuation of position with the initial standard deviation \(\sigma_0 = 10^{-60}\) might lead
Results are based on the 10000 samples of reliable, multiple-scale simulations given by the CNS with the micro-level fluctuation of initial position $r'_i(0)$ in Gaussian distribution. to a totally different destiny of the three-body system: when the inherent physical uncertainty is enlarged into macroscopic, 2568 among 10000 samples of the three-body system disrupt at $t = 1000$ with one body escaping randomly and the other two becoming binary stars in the opposite direction. Thus, the micro-level physical uncertainty due to initial position fluctuation $r'_i(0)$, although it is rather tiny, can greatly influence the orbit of the chaotic three-body system. It should be emphasized that these 10000 (mathematically) different micro-level fluctuations $r'_i(0)$ of position are the *same* for us from the *physical* viewpoint, since two-points less than the Planck length has no *physical* meaning at all, but these *physically same* initial conditions lead to completely different orbits and even different fates of the chaotic three-body system! Note that, whether the three-body system disrupts at $t = 1000$ or not depends upon the micro-level *inherent* physical fluctuation $r'_i(0)$ of position in Gaussian distribution with the standard deviation $10^{-60}$. It should be emphasized that such kind of disruption of the three-body system randomly happens *without* any *external* disturbances. This phenomena is called the *self-excited random disruption* or *self-excited random escape* of three-body system. It suggests that a chaotic three-body system

![Figure 3: The position distribution of Body-1 (red points), Body-2 (blue points) and Body-3 (blue points) in the $(x, y)$ plane at different times when $\sigma_0 = 3 \times 10^{-60}$. Results are based on the 10000 samples of reliable, multiple-scale simulations given by the CNS with the micro-level fluctuation of initial position $r'_i(0)$ in Gaussian distribution.](image-url)
Figure 4: Comparison of the mean position $\langle \overline{x}_{1,1}, \overline{x}_{2,1}, \overline{x}_{3,1} \rangle$ of Body-1 given by $\sigma_0 = 10^{-60}$ and $\sigma_0 = 3 \times 10^{-60}$. Each curve is based on the 10000 samples of reliable, multiple-scale simulations given by the CNS with the micro-level fluctuation of initial position $r'_i(0)$ in Gaussian distribution. Solid line: $\sigma_0 = 10^{-60}$; Dashed line: $\sigma_0 = 3 \times 10^{-60}$.

would randomly evolve by itself to a rather complicated structure without any external forces. It also implies that an universe could randomly evolve by itself into complicated structures, without any external forces, and that the nature could randomly evolve by itself into organism and even human being, without any external forces.

All of the above-mentioned results are based on 10000 samples of reliable (convergent) simulations of the chaotic three-body system by means of the CNS using the inherent micro-level physical fluctuation of initial position with $\sigma_0 = 10^{-60}$. Similarly, it is straightforward to investigate the propagation of the uncertainty of positions of the chaotic three-body systems for other values of $\sigma_0$. Without loss of generality, let us consider the case with the same mean position $\overline{5}$ and velocity $\overline{6}$, but different fluctuation of initial position in Gaussian distribution:

$$\langle r'_i(0) \rangle = 0, \quad \sigma_0 = \sqrt{\langle r'^2_i(0) \rangle} = 3.0 \times 10^{-60}.$$
It is found that the corresponding $\sigma_{i,j}(t)$ of $x_{i,j}(t)$ also enlarges exponentially until $T^* \approx 800$, the so-called physical limit time of prediction, and then propagates algebraically thereafter. When $t < T^*$, $\sigma_{i,j}(t)$ is so small that accurate prediction of orbits is possible, although it enlarges exponentially in the same way $\sigma_{i,j}(t) = \sigma_0 \exp(\lambda t)$, where $\lambda = 0.1681$ is the Lyapunov exponent given by Sprott [7] for the same three-body system without fluctuation of initial position. However, when $t > T^*$, the uncertainty becomes macroscopic, as shown in Fig. 3. Comparing Fig. 2 with Fig. 3, it is obvious that the macroscopic statistical distributions of position of the three-body system at different times are dependent upon $\sigma_0$, i.e. the standard deviation of the micro-level physical inherent fluctuation of initial position $r_i(0)$. Note that, when $t > T^*$, the mean positions of the chaotic three-body system given by different $\sigma_0$ depart obviously, as shown in Fig. 4. This suggests that the statistics of the macroscopic uncertainty of the chaotic three-body system have a close relationship with the statistics of the micro-level inherent physical uncertainty. In addition, there are 2736 “random disruptions” and “random escapes” (among 10000 samples) happen in the time interval from $t = T^*$ to $t = 1000$, without any external disturbance.

3 Concluding remarks

In summary, the microscopic inherent uncertainty (in the level of $10^{-60}$) due to physical fluctuation of initial positions of the three-body system enlarges exponentially into macroscopic randomness (at the level $O(1)$) until $t = T^*$, the so-called physical limit time of prediction, but propagates algebraically thereafter when accurate prediction of orbit is impossible. Note that these 10000 samples use micro-level, inherent physical fluctuations of initial position, which have nothing to do with human being. Besides, the differences of these 10000 fluctuations are mathematically so small (in the level of $10^{-60}$) that they are physically the same since a distance shorter than a Planck length does not make physical senses according to the spring theory. It indicates that the macroscopic randomness of the chaotic three-body system is self-excited, say, without any external force or disturbances, from the inherent micro-level uncertainty. This provides us the new concept “self-excited macroscopic randomness (uncertainty)”. Besides, it is found that the chaotic three-body system might randomly disrupt at $t = 1000$ in about 25% probability without any external disturbance, which provides us the new concepts “self-excited random disruption” and “self-excited random escape” of chaotic three-body system. It suggests that a chaotic three-body system might randomly evolve by itself, without any external forces or disturbance. Thus, the world is essentially uncertain, since such kind of self-excited macroscopic randomness (uncertainty) is inherent and unavailable. This work also implies that an universe could randomly evolve by itself into complicated structures, without any external forces, and that the nature could randomly evolve by itself into organism and even human being, without any external forces. In addition, it is found that the macroscopic randomness is even dependent upon microscopic uncertainty, from statistical viewpoint. All of these reliable computations reveal a kind of origins of
macroscopic randomness/uncertainty.

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