On quantiles, continuity and robustness

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Abstract

We consider the geometric quantile and various definitions of the component-wise quantile in infinite dimensions and show their existence, uniqueness and continuity. Building on these results, we introduce and study the properties of the Quantile-of-Estimates (QoE) estimator, a robustification procedure for a large class of estimators. For example, given an estimator that is asymptotically normal, the QoE estimator is asymptotically normal even in the presence of contaminated data.

Keywords: geometric and component-wise quantiles, robust estimator, consistency, asymptotic normality, orthogonality, adversarial contamination.

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1 Introduction

In the second half of the twentieth century there has been extensive effort by the statistical community in understanding the properties of the median and of the quantile in the multi-dimensional setting and infinite dimensional setting. Kemperman [22] provided a fundamental step forward in this direction. Building on the work of Valadier [33] he showed the existence, uniqueness and weak continuity of the median in abstract spaces. Chaudhuri [9] provided a notion of the geometric quantile for multivariate data. Koltchinskii [24] developed and studied the M-quantile function which is a class of extensions of the univariate quantile function to

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the multivariate case, general enough to include the geometric quantile as well, and studied its
properties. More recently, developments are treated in Cadre [6], Gervini [13], Chakraborty
and Chaudhuri [8], Hallin et al. [18], and of Konen and Paindaveine [25], among many others.

In the last years, motivated by statistical learning theory, several papers have investigated
the role of the median in providing efficient and robust non-asymptotic probability bounds
for the estimation of the mean of a random variable. A first work in this direction is Oliveira
and Lerasle [28] which builds on the Median-of-Means (MoM), originally introduced in [21,
31]. Minsker [30] developed a general method based on the geometric median to obtain
estimators with tight concentration bounds around the true parameter of interest taking
values in a Banach space. Lugosi and Mendelson [29] introduced the Median-of-Means (MoM)
tournaments, and presented an estimator that outperforms classical least-squares estimators
when data are heavy-tailed and/or corrupted. An analysis of MoM robustness properties is
performed in [26]. Our work contributes to both lines of research.

Quantiles play a crucial role in many modern statistical approaches, including conformal
prediction [4] and high-dimensional statistical inference [32, 34]. Moreover, the general robus-
tification procedure developed in this paper can potentially be carried out for any definition
of quantiles, including the $\rho$-quantiles defined in [25] and those defined via optimal trans-
port [10, 14, 17].

Here we focus on two of the most common definitions of quantiles; the geometric quantile
and the component-wise quantile. In the first part of the paper we recall the definition of
the geometric quantile in Banach spaces and introduce three definitions of component-wise
quantile in infinite dimensions, depending which notion of component we consider. Using
these we build the following robustification procedure for a large class of estimators.

Consider any consistent estimator and $n$ independent and identically distributed (i.i.d.)
observations. We divide the observations into $k$ blocks, compute the sample estimate for
each of the $k$ blocks, and then compute the quantile of these estimates. This is the sample
estimate of the Quantile-of-Estimates (QoE) estimator. When $k$ is fixed we show that the QoE
estimator is consistent. If in addition the chosen estimator has an asymptotic distribution,
then QoE estimator has an explicit asymptotic distribution. This holds in the finite- and
infinite-dimensional settings, for both the geometric quantile and our three definitions of
component-wise quantiles.

We then let the number of blocks, $k$, increase with $n$ and allow for adversarial contam-
ination, i.e. arbitrary partial contamination, and we establish robustness properties of the
QoE. In particular, under certain mild conditions, given an estimator that is consistent and
has an asymptotic distribution, the QoE estimator is consistent, asymptotically normal, and
both these properties are robust to arbitrary contamination. For example, in $\mathbb{R}^d$ if the data
have finite third moment and we focus on the sample mean then the number of allowed
contaminated data is $o(n^{1/4})$.

If these results were written just for the median this would significantly diminish appli-
cability of our results, i.e. it would dramatically restrict the class of estimators our results
apply to, see Corollaries 3.9 and 3.19 Thus, having these results for quantiles, not just for
the median, is pivotal from both an applied and theoretical perspective. As well, having these
results for both the geometric and the component-wise quantiles allows the robustification to
be applied to a wider class of data and of estimators, see Theorems 3.7 and 3.18. Moreover,
while robustness usually concerns the consistency of estimators, to the best of our knowledge
these are the first results on robustness in distribution.

These properties of the QoE are established using results on the existence, uniqueness,
continuity, and representations of the geometric and component-wise quantiles, presented in the second part of the paper. We provide necessary and sufficient conditions for the uniqueness of geometric quantiles in Banach spaces, improving the results of Kemperman [22]. This is based on a novel convex analysis result on James orthogonality, Lemma 4.7. Further, we show that the geometric quantile function is a continuous function of the data. In particular, we show that if a sequence of empirical distributions converge weakly to an empirical distribution then their quantiles converge strongly, i.e. in norm. This result is interesting for two main reasons. First, by showing continuity of the quantile function we can ultimately use the continuous mapping theorem to obtain asymptotic properties of the QoE estimator when the number of blocks is fixed. Second, only weak*-convergence has been established to date [22, 6, 8].

In Hilbert spaces we are able to give an explicit formulation for the geometric quantile and show that it is a weighted sum of the data and of the quantile parameter, where the weights are solutions of a minimization problem over a finite and compact set (see Theorem 4.21). This extends the main intuition of [13] to quantiles and opens to the possibility to prove his results for quantiles, including the algorithm he developed for computing the sample spatial median in Hilbert spaces, which is something that was thought not to be possible (see [8] page 1208).

The most general definition of the component-wise quantiles valid on any vector space uses the Hamel basis. Since Hamel bases are not ideal in practice, we then define the component-wise quantiles using Schauder basis. We also define component-wise quantiles in the functions spaces $L_p([0,T])$, $C([0,T])$, and $D([0,T])$, using a point-wise method. We show the existence, uniqueness and Lipschitz continuity of these three definitions of component-wise quantiles. We remark that the robustness results presented in the first part of the paper hold for all these definitions, as well as for the geometric quantile.

The paper is structured as follows. In Section 2 we present the quantiles and some preliminaries. Section 3 is devoted to the presentation of the robustification procedure, the QoE estimator, which builds on the results of Section 4, where we establish the existence, uniqueness, and continuity for the geometric and component-wise quantiles in Banach spaces and present the explicit formulation for the geometric quantiles in Hilbert space.

2 Quantiles

Let $k \in \mathbb{N}$ and consider $\mathbf{x} \in \mathbb{R}^k$. Order its components in increasing values and denote by $\mathbf{x}$ the corresponding vector. For $\alpha \in (0,1)$, the univariate $\alpha$-quantile of $\mathbf{x} = (x_1,...,x_k)$ is the function $q_\alpha : \mathbb{R}^k \to \mathbb{R}$ defined as

$$q_\alpha(\mathbf{x}) = \frac{1}{2}(\mathbf{x}_{\lfloor k\alpha \rfloor} + \mathbf{x}_{\lfloor k\alpha + 1 \rfloor});$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and ceiling functions respectively. $q_{1/2}$ is the median of $(x_1,...,x_k)$. An alternative definition is the so-called geometric quantile, defined as

$$\arg \min_{y \in \mathbb{R}} \sum_{i=1}^{k} |x_i - y| + u(x_i - y), \quad \text{where } u = 2\alpha - 1. \quad (1)$$

The geometric quantile is not unique when $\alpha \in \{\frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}\}$; in this case the set of solution is the interval $[\tilde{x}_{\alpha k}, \tilde{x}_{\alpha k+1}]$. 

We can naturally extend the definition of the geometric quantile to Banach space \( \mathcal{X} \) with dual \( \mathcal{X}^* \) ([8, 9]). Given \( \mathbf{u} \in \mathcal{X}^* \) with \( \| \mathbf{u} \|_{\mathcal{X}^*} < 1 \), the geometric quantile of \( x_1, \ldots, x_k \in \mathcal{X} \) is defined as

\[
\arg \min_{y \in \mathcal{X}} \sum_{i=1}^k \| x_i - y \|_\mathcal{X} + \langle \mathbf{u}, x_i - y \rangle.
\] (2)
equivalently

\[
\arg \min_{y \in \mathcal{X}} \sum_{i=1}^k \| x_i - y \|_\mathcal{X} - k \langle \mathbf{u}, y \rangle.
\] (3)

As in the univariate case, the geometric quantile suffers from the lack of uniqueness (see Section 4). To overcome this problem we proceed as follows. Let \( u^k := \bigoplus_{i=1}^k \mathcal{X} \) be the \( k \) times direct sum, \( i.e. \) the Cartesian product, of \( \mathcal{X} \) with norm \( \|(x_1, \ldots, x_k)\|_{u^k} := \sum_{i=1}^k \|x_i\|_\mathcal{X} \), which is itself a Banach space. Let \( x_1, \ldots, x_k \in \mathcal{X} \) lie on a straight line, which we call \( W \). If the data are translated by \( z \in \mathcal{X} \) then (2) is translated by \( z \), so wlog we assume that \( W \) passes through the origin. If \( \mathcal{X} \) is a Hilbert space, identify \( \mathcal{X}^* \) with \( \mathcal{X} \), as usual. Let \( e \in W \) be such that \( \|e\|_\mathcal{X} = 1 \) and denote by \( W^\perp \) the orthogonal complement of \( W \). Then

\[ \mathbf{u} = u e + v, \] where \( u \in \mathbb{R} \) and \( v \in W^\perp \).

Since \( \|\mathbf{u}\|_\mathcal{X} < 1 \), we have that \( u \in (-1, 1) \). If \( \mathcal{X} \) is a Banach space let \( W^\perp \) be the annihilator of \( W \), \( i.e. \)

\[ W^\perp = \{ v \in \mathcal{X}^* : \langle v, s \rangle = 0, \forall s \in W \} = \{ v \in \mathcal{X}^* : W \subseteq \text{Ker}(v) \}. \]

By Theorem 5.110 in [2] we have that \( W^\perp \) has co-dimension 1, so

\[ \mathcal{X}^* = Z \oplus W^\perp, \] (4)

where \( Z \) is a one-dimensional subspace. Thus, we have that \( \mathbf{u} = u e + v \) where \( e \in Z \) with \( \|e\|_{\mathcal{X}^*} = 1 \), \( u \in \mathbb{R} \), and \( v \in W^\perp \). Since \( \|\mathbf{u}\|_{\mathcal{X}^*} = \sup_{x \in \mathcal{X}} \frac{|\langle \mathbf{u}, x \rangle|}{\|x\|_\mathcal{X}} \) and since \( \|\mathbf{u}\|_{\mathcal{X}^*} < 1 \), we have that for any \( s \in W \)

\[ \|s\|_\mathcal{X} > \|s\|_\mathcal{X} \|\mathbf{u}\|_{\mathcal{X}^*} \geq |\langle \mathbf{u}, s \rangle| = |\langle u e, s \rangle| = |u| \|\langle e, s \rangle| \]

and since \( \|s\|_\mathcal{X} = \|s\|_\mathcal{X} \|e\|_{\mathcal{X}^*} \geq |\langle e, s \rangle| \) we conclude that \( u \in (-1, 1) \) and is a well-defined parameter of any geometric quantile in a Banach space. We define \( \alpha := (u + 1)/2 \).

Consider the function \( \mathbf{q}_u : \mathcal{X}^k \to \mathcal{X} \) defined as follows. If \( \|\mathbf{u}\|_{\mathcal{X}^*} \notin \{1 - \frac{2j}{k}, j = 1, \ldots, \lfloor \frac{k}{2} \rfloor \} \) let \( \mathbf{q}_u(x_1, \ldots, x_k) \) be the geometric quantile (2), which by Corollary 4.13 is uniquely defined. If \( \|\mathbf{u}\|_{\mathcal{X}^*} \in \{1 - \frac{2j}{k}, j = 1, \ldots, \lfloor \frac{k}{2} \rfloor \} \), let

\[ \mathbf{q}_u(x_1, \ldots, x_k) := \begin{cases} 
\arg \min_{y \in \mathcal{X}} \sum_{i=1}^k \| x_i - y \|_\mathcal{X} + \langle \mathbf{u}, x_i - y \rangle, & \text{if } x_1, \ldots, x_k \text{ do not lie on a straight line}, \\
q_0(x_1, \ldots, x_k), & \text{otherwise},
\end{cases} \] (5)

where \( x_i \) is such that \( x_i = x_i \mathbf{h} \) and \( \mathbf{h} \in W \) with \( \|\mathbf{h}\|_\mathcal{X} = 1 \), \( i = 1, \ldots, k \). For the median we have \( \|\mathbf{u}\|_{\mathcal{X}^*} = 0 \), so \( \|\mathbf{u}\|_{\mathcal{X}^*} \notin \{1 - \frac{2j}{k}, j = 1, \ldots, \lfloor \frac{k}{2} \rfloor \} \) when \( k \) is odd.

We define now component-wise quantiles in spaces with possibly infinite dimensions. Informally, each component of the component-wise quantile is the univariate quantile of the
corresponding component of $x_1, \ldots, x_k$. This definition is not formal because the words component might have different meanings, especially in infinite dimensional spaces. The existence of the component-wise quantile we present in this section is proved in Section 4.

The most general definition of the component-wise quantile uses the Hamel basis. Let $X$ be any vector space. Any $x_1, \ldots, x_k \in X$ can be expressed as $\sum_{i \in I} x^{(l)}_i b_i, \ldots, \sum_{i \in I} x^{(l)}_k b_i$, respectively, where the $x^{(l)}_i$'s are real valued constants (only finitely many are non-zero) and $(b_i)_{i \in I}$, for some index set $I$, is the Hamel basis of $X$. The component-wise quantile with respect to the Hamel basis and with parameter $\alpha = (\alpha_l)_{l \in I}$, where $\alpha_l \in (0, 1)$ for every $l \in I$, is defined as

$$q_{Hamel, \alpha}(x_1, \ldots, x_k) := \sum_{l \in I} \hat{x}_l b_l \quad (6)$$

where $\hat{x}_l = q_{\alpha_l}(x^{(l)}_1, \ldots, x^{(l)}_k)$. The map that sends any $x \in X$ (rewritten using the Hamel basis as $\sum_{i \in I} x^{(l)} b_i$) to $\sum_{i \in I} |x^{(l)}|$ is a norm on $X$; we denote this by $\| \cdot \|_{X, Hamel}$. The map $(x_1, \ldots, x_k) \mapsto \sum_{l \in I} \| x_l \|_{X, Hamel}$ defines a norm on the product space $X^k$, which we denote $\|(x_1, \ldots, x_k)\|_{X, Hamel}$. In practice, since the Hamel basis might not be explicitly available, it is easier to work with the Schauder basis. We define the component-wise quantile with respect to Schauder basis similarly. For any $x_1, \ldots, x_k$ belonging to some Banach space $X$ possessing a Schauder basis $(d_l)_{l \in \mathbb{N}}$ we have $x_i = \sum_{l \in \mathbb{N}} x^{(l)}_i d_l$, where the $x^{(l)}_i$'s are real valued constants and $i = 1, \ldots, k$ (these $x^{(l)}_i$'s are generally not related to those in the Hamel basis decomposition). The component-wise quantile with respect to the Schauder basis, with parameter $\alpha = (\alpha_l)_{l \in \mathbb{N}}$, where $\alpha_l \in (0, 1)$ for every $l \in \mathbb{N}$, is defined as

$$q_{S, \alpha}(x_1, \ldots, x_k) := \sum_{l \in \mathbb{N}} \hat{x}_l d_l \quad (7)$$

where $\hat{x}_l = q_{\alpha_l}(x^{(l)}_1, \ldots, x^{(l)}_k)$. A Schauder basis $(d_l)_{l \in \mathbb{N}}$ is said to be unconditional if whenever the series $\sum_{l \in \mathbb{N}} x_l d_l$ converges, it converges unconditionally, i.e. all reorderings of the series converge to the same value. In this paper we concentrate on spaces that possess an unconditional Schauder basis. The standard bases of the sequence spaces $c_0$ and $\ell_p$ for $1 \leq p < \infty$, as well as every orthonormal basis in a Hilbert space, are unconditional. The Haar system and the Franklin system are unconditional bases in $L_p$ for any $1 < p < \infty$. Further, the Franklin system is an unconditional basis in all reflexive Orlicz spaces.

There are some important spaces which have no unconditional Schauder basis, like $C[0,1]$, $L_1[0,1]$, and $L_\infty[0,1]$, among others. For these spaces and for function spaces in general we can employ the following definition of component-wise quantile. For the sake of clarity, we focus on function spaces $X$ being either $C[0,1]$, or $D[0,1]$ or $L_p[0,1]$, with $1 \leq p \leq \infty$. Consider any $f_1, \ldots, f_k \in X$. Then, we can define the point-wise component-wise quantile with parameter $\alpha = (\alpha_l)_{l \in [0,1]}$, where $\alpha_l \in (0,1)$ for every $l \in [0,1]$, as follows: for every $x \in [0,1]$

$$q_{P, \alpha}(f_1, \ldots, f_k)(x) := q_{\alpha_l}(f_1(x), \ldots, f_k(x)) \quad (8)$$

In this paper we consider $C[0,1]$ and $D[0,1]$ to be endowed with their usual topologies, i.e. the uniform topology and the Skorokhod topology respectively.

**Remark 2.1.** The three definitions of component-wise quantile are equivalent when $X$ is finite-dimensional.
On the other hand, in infinite-dimensional Banach spaces, not all function spaces have an unconditional Schauder basis, and by the Baire category theorem any Hamel basis is necessarily uncountable (see Corollary 5.23 in [2]).

3 Robustness of the QoE

In this section we present the main statistical results of the paper. We present a procedure to make the asymptotic properties of a wide range of estimators robust to data contamination.

Consider a sample of \( n \) i.i.d. observations belonging to a space \( X \). Divide these into \( k \) blocks of equal size \( [n/k] \), for some \( k \in \mathbb{N} \). Let \( Z_n \) be any estimator (e.g. the sample mean) of a parameter of interest \( \theta_0 \in X \) and let \( Z_{[n/k]}^{(i)} \) be the estimator from the observations in block \( i, i = 1, \ldots, k \). We define the Quantile-of-Estimates (QoE) estimators

\[
T_{u,n,k} := q_u(Z_{[n/k]}^{(1)}, \ldots, Z_{[n/k]}^{(k)}), \quad T_{S,\alpha,n,k} := q_{S,\alpha}(Z_{[n/k]}^{(1)}, \ldots, Z_{[n/k]}^{(k)}), \quad \text{and similarly} \quad T_{Hamel,\alpha,n,k} \quad \text{and} \quad T_{P,\alpha,n,k} \quad \text{using (5), (7), (8), and (8) respectively.}
\]

By contaminated data we mean that a portion of the sample is substituted by any arbitrary random variables. This framework, which is in line with Assumption 2 in [20], is much more general than Huber’s contamination model. To distinguish this case from the uncontaminated case, we use the notation \( \tilde{Z}_{[n/k]}^{(i)} \) for the sample estimator related to the block \( i, i = 1, \ldots, k \), when \( l_n \) observations are contaminated. By a slight abuse of notation we let

\[
T_{u,n,k} := q_u(\tilde{Z}_{[n/k]}^{(1)}, \ldots, \tilde{Z}_{[n/k]}^{(k)}).
\]

We consider two assumptions. In the first one the chosen estimator \( Z_n \) is consistent, while in the second one it has an asymptotic distribution.

Assumption 3.1 (Consistency). We have that \( Z_n \stackrel{p}{\rightarrow} \theta_0 \) as \( n \rightarrow \infty \)

Assumption 3.2 (Asymptotic distribution). We have that \( a_n(Z_n - \theta_0) \xrightarrow{d} Y \) as \( n \rightarrow \infty \), for some sequence \( (a_n)_{n \in \mathbb{N}} \) of non-negative numbers going to infinity and some random variable \( Y \).

3.1 Component-wise quantile

First, we investigate the asymptotic behavior of the QoE’s \( T_{S,\alpha,n,k}, T_{Hamel,\alpha,n,k}, \) and \( T_{P,\alpha,n,k} \) as \( n \rightarrow \infty \) for fixed \( k \in \mathbb{N} \). The proofs of all our results are in the Appendix.

Theorem 3.3. Let \( X \) be a vector space endowed with the topology induced by \( \| \cdot \|_{\text{Hamel}} \). Under Assumption \( \mathcal{A} \), \( T_{Hamel,\alpha,n,k} \stackrel{p}{\rightarrow} \theta_0 \) as \( n \rightarrow \infty \). Under Assumption \( \mathcal{B} \), \( a_{[n/k]}(T_{Hamel,\alpha,n,k} - \theta_0) \xrightarrow{d} q_{Hamel,\alpha}(Y_1, \ldots, Y_k) \) as \( n \rightarrow \infty \), where \( Y_i \xrightarrow{iid} Y, i = 1, \ldots, k \).

Theorem 3.4. Let \( X \) be a Banach space possessing an unconditional Schauder basis. Under Assumption \( \mathcal{A} \), \( T_{S,\alpha,n,k} \stackrel{p}{\rightarrow} \theta_0 \) as \( n \rightarrow \infty \). Under Assumption \( \mathcal{B} \), \( a_{[n/k]}(T_{S,\alpha,n,k} - \theta_0) \xrightarrow{d} q_{S,\alpha}(Y_1, \ldots, Y_k) \) as \( n \rightarrow \infty \), where \( Y_i \xrightarrow{iid} Y, i = 1, \ldots, k \).

Theorem 3.5. Let \( X \) be either \( L_p[0,1] \) with \( 1 \leq p \leq \infty \) or \( X = C[0,1] \) or \( X = D[0,1] \). Under Assumption \( \mathcal{A} \), \( T_{P,\alpha,n,k} \stackrel{p}{\rightarrow} \theta_0 \) as \( n \rightarrow \infty \). Under Assumption \( \mathcal{B} \), \( a_{[n/k]}(T_{P,\alpha,n,k} - \theta_0) \xrightarrow{d} q_{P,\alpha}(Y_1, \ldots, Y_k) \) as \( n \rightarrow \infty \), where \( Y_i \xrightarrow{iid} Y, i = 1, \ldots, k \).
We now consider $k$ increasing with $n$. We focus first on the case $X$ being finite dimensional. In this case the three definitions of component-wise quantiles coincide (see Remark 2.1) and without loss of generality we use the notation $T_{P,\alpha,n,k_n}$. In the next result we investigate the asymptotic behavior in probability of $T_{P,\alpha,n,k_n}$ and its robustness to contaminated data, of size $l_n$.

**Proposition 3.6.** Let $X = \mathbb{R}^d$, $d \in \mathbb{N}$. Under Assumption 3.1, $T_{P,\alpha,n,k_n} P \to \theta_0$ as $n \to \infty$ for any $l_n$ and $k_n = o(n)$ such that $\lim_{n \to \infty} \frac{k_n}{n} = 0$.

Since we are in $\mathbb{R}^d$ we have $Y = (Y_1, \ldots, Y_d)$. Let $F_i$ and $f_i$ be the distribution and the density (if it exists) of $Y_i$, for $i = 1, \ldots, d$. Further, let $\Sigma_\alpha$ be a $d \times d$ matrix with $(i,j)$ element

$$f_i(F_i^{-1}(\alpha)) f_j(F_j^{-1}(\alpha)),$$

For $i = 1, \ldots, d$ let

$$\beta_i^* = \sup\{ \beta \in (0,1) : \lim_{n \to \infty} \sqrt{n^d} \left( F_i(x) - F_{n,i}(x) \right) = 0, \text{for } x = F_i^{-1}(\alpha_i) + \frac{y}{\sqrt{n^d}} \text{ and } y \in \mathbb{R} \}.$$

**Theorem 3.7.** Let $X = \mathbb{R}^d$ and let Assumption 3.2 hold. Assume that $f_i$ is continuous and strictly positive at $F_i^{-1}(\alpha_i)$. Then,

$$a_{[n/cn^\beta]}(T_{P,\alpha,n,cn^\beta} - \theta_0) \xrightarrow{P} (F_1^{-1}(\alpha_1), \ldots, F_d^{-1}(\alpha_d)),$$

as $n \to \infty$ for any $l_n = O(n^\gamma)$, $c > 0$, and $\beta \in (0,1)$ such that $\beta > \gamma$. Moreover,

$$\sqrt{cn^\beta} a_{[n/cn^\beta]}(T_{P,\alpha,n,cn^\beta} - \theta_0) - (F_1^{-1}(\alpha_1), \ldots, F_d^{-1}(\alpha_d)) \xrightarrow{d} N(0, \Sigma_\alpha)$$

as $n \to \infty$, for any $l_n = O(n^\gamma)$, $c > 0$, and $\beta \in (2\gamma, \max_{i=1,\ldots,d} \beta_i^*)$.

**Remark 3.8.** If $Z_n$ is the sample mean and the data have finite third moment then by the Berry-Esseen theorem $\max_{i=1,\ldots,d} \beta_i^* = 1/2$ and so the allowed number of contamination in the second statement of Theorem 3.7 is $l_n = O(n^\gamma)$ with $\gamma \in [0, 1/4)$.

From the above result we have the following robustness property of the asymptotic distribution of the QoE estimator. Consider $Y$ to be such that $(F_1^{-1}(\alpha_1), \ldots, F_d^{-1}(\alpha_d)) = 0$, for some $\alpha_1, \ldots, \alpha_d \in (0,1)$, and $a_n$ to be equal to $\sqrt{n}$. This happens in the typical case of asymptotically normal estimators. Indeed, in this case $Y$ is symmetric around zero, hence $(F_1^{-1}(1/2), \ldots, F_d^{-1}(1/2)) = 0$, and $a_n = \sqrt{n}$.

**Corollary 3.9.** Let $X = \mathbb{R}^d$ and let Assumption 3.2 hold. Assume that $(F_1^{-1}(\alpha_1), \ldots, F_d^{-1}(\alpha_d)) = 0$, for some $\alpha_1, \ldots, \alpha_d \in (0,1)$ and that $f_i$ is continuous and strictly positive at $F_i^{-1}(\alpha_i)$. Then,

$$\sqrt{cn^\beta} a_{[n/cn^\beta]}(T_{P,\alpha,n,cn^\beta} - \theta_0) \xrightarrow{d} N(0, \Sigma_\alpha)$$

as $n \to \infty$, and if $a_n = \sqrt{n}$ then

$$\sqrt{n}(T_{P,\alpha,n,cn^\beta} - \theta_0) \xrightarrow{d} N(0, \Sigma_\alpha),$$

for any $l_n = O(n^\gamma)$, $c > 0$, and $\beta \in (2\gamma, \max_{i=1,\ldots,d} \beta_i^*)$. 

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We conclude this section with a discussion on the asymptotic behavior of $T_{P,\alpha,n,cn^\beta}$ in the infinite-dimensional setting. From Theorem 3.7 we have convergence in finite-dimensional distributions. The full convergence in distribution requires tightness and we would need to focus on specific spaces; we leave this question to further research. However, there are cases where the convergence in finite-dimensional distributions implies the convergence in distributions. In the next result we present the convergence in distribution for certain classes of stochastic processes. We focus on the definition of point-wise component-wise quantile in functions spaces, i.e. we focus on $T_{P,\alpha,n,cn^\beta}$. We let $Y = (Y_t)_{t \in [0,1]}$ and let $F_t$ and $f_t$ be the distribution and the density (if exists) of $Y_t$. Further, we let $F_t^{-1}(\alpha) := (F_t^{-1}(\alpha_i))_{i \in [0,1]}$ and let $B_\alpha$ be a mean zero Gaussian process such that $(B_{t_1,\alpha_1}, ..., B_{t_r,\alpha_r})$ has covariance matrix

$$
\begin{pmatrix}
\mathbb{P}(Y_{t_i} > F_t^{-1}(\alpha_{t_i}), Y_{t_j} > F_t^{-1}(\alpha_{t_j})) - (1 - \alpha_{t_i})(1 - \alpha_{t_j})
\end{pmatrix}_{i,j=1,...,r},
$$

and for any $t \in [0,T]$ we let $\beta_t^*$ be defined as $\beta_t^*$ in (9) with $F_t$ and $\alpha_t$ replacing $F_i$ and $\alpha_i$.

**Proposition 3.10.** Let $X$ be either $L_p[0,T]$ with $1 \leq p \leq \infty$ or $\mathbb{X} = C[0,T]$ or $\mathbb{X} = D[0,T]$. Let Assumption 3.2 hold. Assume that $f_t$ is continuous and strictly positive at $F_t^{-1}(\alpha_t)$, for every $t \in [0,1]$. Then,

$$\sqrt{cn^\beta(a_{n/cn^\beta} \circ T_{P,\alpha,n,cn^\beta} - \theta_0) - F_t^{-1}(\alpha)} \overset{fdd}{\rightarrow} B_\alpha,$$

as $n \to \infty$, for any $l_n = O(n^\gamma)$, $c > 0$, and $\beta \in (2\gamma, \sup_t \beta_t^*)$. Moreover, if

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left\| \sqrt{cn^\beta(a_{n/cn^\beta} \circ T_{P,\alpha,n,cn^\beta} - \theta_0) - F_t^{-1}(\alpha)} \right\|_{L_p[0,T]}^p \right] < \infty$$

for some $p \in (1, \infty)$, then

$$\sqrt{cn^\beta(a_{n/cn^\beta} \circ T_{P,\alpha,n,cn^\beta} - \theta_0) - F_t^{-1}(\alpha)} \overset{d}{\rightarrow} B_\alpha,$$

as $n \to \infty$ in $L^r[0,T]$, for any $r \in [1,p)$, $l_n = O(n^\gamma)$, $c > 0$, and $\beta \in (2\gamma, \sup_t \beta_t^*)$.

### 3.2 Geometric quantile

The first two results of this section are the extension of Lemma 2.1 and Theorem 3.1 in [30] to the quantile case.

**Lemma 3.11.** Let $\mathbb{X}$ be a Banach space. Let $x_1, ..., x_k \in \mathbb{X}$ and let $u \in \mathbb{X}^*$ with $\|u\|_{\mathbb{X}^*} < 1$, and let $\mathbb{X}^*$ be a geometric quantile (which we assume exists). Fix $\nu \in (0, \frac{1 - \|u\|_{\mathbb{X}^*}}{2})$ and assume that $z \in \mathbb{X}$ is such that $\|\mathbb{X}^* - z\|_{\mathbb{X}} > C_\nu r$, where

$$C_\nu = \frac{2(1 - \nu)}{1 - 2\nu - \|u\|_{\mathbb{X}^*}},$$

and $r > 0$. Then there exists a subset $J \subset \{1, ..., k\}$ of cardinality $|J| > \nu k$ such that for all $j \in J$, $\|x_j - z\|_{\mathbb{X}} > r$.

For $0 < p < s < \frac{1}{2}$, define

$$\psi(s;p) = (1 - s) \log \frac{1 - s}{1 - p} + s \log \frac{s}{p}.$$
Proposition 3.12. Let $X$ be a Banach space. Assume that $\mu \in X$ is a parameter of interest, and let $\hat{\mu}_1, \ldots, \hat{\mu}_k \in X$ be a collection of independent estimators of $\mu$. Let $u \in X^*$ with $\|u\|_{X^*} < 1$, and let $\bar{\mu}$ be a geometric quantile of $\hat{\mu}_1, \ldots, \hat{\mu}_k$ (which we assume it exists). Fix $\nu \in (0, \frac{1-\|u\|_{X^*}}{2})$. Let $0 < p < \nu$ and $\varepsilon > 0$ be such that for all $j = 1, \ldots, k$

$$P(\|\bar{\mu}_j - \mu\|_X > \varepsilon) \leq p.$$  \hspace{1cm} (10)

Then,

$$P(\|\bar{\mu} - \mu\|_X > C_\nu \varepsilon) \leq \exp(-k\psi(\nu; p)),$$

where $C_\nu$ is defined in Lemma 3.11. Further, if (10) only holds for $j \in J$, where $J \subseteq \{1, \ldots, k\}$ with $|J| = (1-\tau)k$ for $0 \leq \tau \leq \frac{\nu-p}{1-p}$, then

$$P(\|\bar{\mu} - \mu\|_X > C_\nu \varepsilon) \leq \exp\left(-k(1-\tau)\psi\left(\frac{\nu-\tau}{1-\tau}; p\right)\right).$$

We now investigate the asymptotic behavior of $T_{u,n,k}$, as $n \to \infty$ for fixed $k$.

Theorem 3.13. Let $X$ be a reflexive and strictly convex Banach space. Under Assumption 3.1, $T_{u,n,k} \xrightarrow{d} \theta_0$ as $n \to \infty$. Let Assumption 3.2 hold. If either the distribution of $Y$ is nonatomic, or $X$ is smooth and $\|u\|_{X^*} \notin \{1-\frac{2j}{k}, j = 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$, or $X = \mathbb{R}$, then $a_{[n/k]}(T_{u,n,k} - \theta_0) \xrightarrow{d} q_u(Y_1, \ldots, Y_k)$ as $n \to \infty$, where $Y_i \sim Y$, $i = 1, \ldots, k$.

We explore now how the results of Theorem 3.13 change when $k_n \to \infty$ as $n \to \infty$. First, we investigate the behavior of the convergence in probability in Theorem 3.13 and its robustness in the presence of contaminated data.

Proposition 3.14. Let $X$ be a reflexive Banach space and let Assumption 3.1 hold. Then, $T_{u,n,k_n} \xrightarrow{p} \theta_0$ as $n \to \infty$ for any $l_n$ and $k_n = o(n)$ such that $\lim_{n \to \infty} \frac{l_n}{k_n} = 0$.

Now, imagine to have certain initial data and a certain quantile parameter $u$. Imagine that some of the data are modified. Is it always possible to change the quantile parameter $u$ so that the geometric quantile remains the same? If yes, how much do we need to change it?

Lemma 3.15. Let $X$ be Banach space. Let $k \in \mathbb{N}$, $x_1, \ldots, x_k \in X$, and $u \in X^*$ with $\|u\|_{X^*} < 1$. Assume that exists at least one geometric quantiles and denote it by $x^*$. Let $\tilde{x}_1, \ldots, \tilde{x}_k \in X$ where $\tilde{x}_i = x_i$, for $i = p+1, \ldots, k$ with $1 \leq p < \frac{k}{2}(1-\|u\|_{X^*})$. Denote by $g_j^*$ a subdifferential of the norm $\|\cdot\|_X$ evaluated at point $x_j - x^*$. Then,

$$v := \left(\frac{2p}{2p+\sum_{k}^{k} 1_{\{\tilde{x}_i = x^*\}}}\right)\left(\frac{\sum_{i=1}^{k} 1_{\{\tilde{x}_i = x^*\}}}{2p} - \frac{1}{k} \sum_{j: \tilde{x}_j \neq x^*} g_j^* \right) 1_{\{p \geq 1\}} + u 1_{\{p = 0\}} \hspace{1cm} (11)$$

is such that $v \in X^*$, $\|v\|_{X^*} < 1$, $\|v - u\|_{X^*} \leq \frac{2p}{k}$, and $x^*$ is a geometric quantile of $\tilde{x}_1, \ldots, \tilde{x}_k \in X$ with parameter $v$.

We explore now the central limit theorem corresponding to Proposition 3.14. First, we need to introduce some notation, which we partially borrow from [5]. Let $d_{n}, n \in \mathbb{N}$ and let $X$ be a separable Hilbert space with orthonormal basis $\{e_1, e_2, \ldots\}$. Let $Z_n = \text{span}\{e_1, \ldots, e_n\}$. For every $z \in X$, let $z_{(n)}$ be the orthogonal projection of $z$ onto $Z_n$. Thus,
for example if $Z_{[n/k_n]}^{(1)}$ has a nonatomic distribution, then $T_{u(n),n,k_n}$ denotes the minimizer of $\arg\min_{y \in Z_n} \sum_{i=1}^k \|Z_{[n/k_n],(n)}^{(i)}(y)\| - \langle u(n), Z_{[n/k_n],(n)}^{(i)} - y \rangle$. Let $r_u$ denote the $u$-quantile of the distribution of $Y$, i.e. $r_u$ is the $u \in X$ that solves $\mathbb{E}\left[\frac{Y-y}{\|Y-y\|}\right] = u$, and let $r_{m,u(n)}$ denote the $u(n)$-quantile of the distribution of $a_m(Z_{m(n)} - \theta_{0(n)})$.

Consider a real-valued function $g(z) = \mathbb{E}[\|z - Y\| - \|Y\|] - \langle u, z \rangle$ and let $J_z$ be its Hessian at $z$. $J_z$ is a symmetric bounded bilinear function from $X \times X$ into $\mathbb{R}$ satisfying

$$\lim_{t \to 0} \frac{1}{t} g(z + th) - g(z) - tE\left[\frac{z - Y}{\|z - Y\|} - u\right](h) - \frac{t^2}{2} J_z(h, h) = 0,$$

for any $h \in X$. We let $\tilde{J}_z : X \to X$ be the continuous linear operator defined by the equation $\langle \tilde{J}_z(h), v \rangle = J_z(h, v)$ for every $h, v \in X$. Similarly, we let $J_{z,n}$ be the Hessian of $\mathbb{E}[\|z - Y_n\| - \|Y_n\| - \langle u(n), z \rangle]$ and let $\tilde{J}_{z,n}$ be the associated linear operator. We define $W^{(j)}_{n,k_n,n} := a_{[n/k_n]}^{(j)}(Z_{[n/k_n],(n)}^{(j)} - \theta_{0,n}(n))$ and $\tilde{W}^{(j)}_{n,k_n,n} := a_{[n/k_n]}^{(j)}(\tilde{Z}_{[n/k_n],(n)}^{(j)} - \theta_{0,n}(n))$, for $j = 1, \ldots, k_n$.

The following assumption naturally comes from Assumption (B) in [S].

**Assumption 3.16.** Suppose that the probability measure of $Z_n$ is nonatomic and not entirely supported on a line in $X$, for all $n$ sufficiently large. Further, assume that, for each $C > 0$, 

$$\sup_{x \in Z_n} \mathbb{E}[\|x - W_{n,k_n,n}^{(1)}\|_{Z_n^2}^2] < C \quad \text{for all } n \text{ sufficiently large}.$$ 

Under Assumption 3.16 since $Z_n$ is nonatomic for all $n$ large enough, $T_{u(n),n,k_n}$ is given by the geometric quantile with parameter $u$ of $Z_{[n/k]}^{(1)}, \ldots, Z_{[n/k]}^{(k_n)}$ for all $n$ large enough.

We are now ready to present the Bahadur-type asymptotic linear representation for $a_{[n/k_n]}(T_{u(n),n,k_n} - \theta_{0,n}(n))$.

**Theorem 3.17.** Let $X$ be a separable Hilbert space. Let Assumptions 3.2 and 3.16 hold. Then,

$$a_{[n/k_n]}(T_{u(n),n,k_n} - \theta_{0,n}(n)) - r_{[n/k_n],E[U_n]}$$

$$= -\frac{1}{k_n} \sum_{i=1}^{k_n}[\tilde{r}_{[n/k_n],E[U_n]}]^{-1} \left( \frac{r_{[n/k_n],E[U_n]} - W_{n,k_n,n}^{(i)}}{\|r_{[n/k_n],E[U_n]} - W_{n,k_n,n}^{(i)}\|_X} - U_n \right) + R_n,$$  \hspace{1em} (12)

for any $l_n, k_n = o(n)$ and $d_n$ such that $\lim_{n \to \infty} \frac{l_n}{k_n} = 0$ and $\lim_{n \to \infty} \frac{d_n}{k_n^{2/\rho}} = c$ for some $\rho \in (0, 1/2]$, and some $c > 0$, where $R_n = O((\log k_n/k_n^{2\rho})$ as $n \to \infty$ almost surely and

$$U_n = \left( \frac{2l_n}{2l_n + \sum_{i=1}^{k_n} \frac{2l_n}{\sum_{i=1}^{k_n} 1\{\tilde{Z}_{[n/k_n],(n)}^{(i)} = T_{u(n),n,k_n}\}} \right) \left( \frac{u}{2l_n} \sum_{i=1}^{k_n} \frac{1}{\sum_{i=1}^{k_n} 1\{\tilde{Z}_{[n/k_n],(n)}^{(i)} = T_{u(n),n,k_n}\}} \right)$$

$$- \frac{1}{k_n} \sum_{j: \tilde{Z}_{[n/k_n],(n)}^{(j)} \neq T_{u(n),n,k_n}} \left( \frac{\tilde{Z}_{[n/k_n],(n)}^{(j)} - T_{u(n),n,k_n}}{\|\tilde{Z}_{[n/k_n],(n)}^{(j)} - T_{u(n),n,k_n}\|} \right) 1\{l_n \geq 1\} + u1\{l_n = 0\}$$

is a $\mathbb{Z}_n$-valued random variable with $\|U_n - u(n)\| < \frac{2l_n}{k_n}$. 


Thus, if \( k_n \) and \( d_n \) are such that \( \rho \in (1/4, 1/2] \) we obtain asymptotic normality of
\[
\sqrt{k_n}(a_{n/k_n}(T_{u(n)\circ n,k_n} - \theta_{0,(n)})) - r_{n/k_n}) \rightarrow_{d} \mathbb{N}.
\]
By setting \( d_n = d \) and \( \rho = 1/2 \), for some \( d \in \mathbb{N} \), we obtain the result for the finite-dimensional case of dimension \( d \).

**Theorem 3.18.** Let \( \mathbb{X} \) be a separable Hilbert space. Let Assumptions 3.17 and 3.18 hold. Let \( l_n, k_n = o(n) \) and \( d_n \) be such that \( \lim_{n \rightarrow \infty} k_n = 0 \) and \( \lim_{n \rightarrow \infty} d_n = c \) for some \( c \in (1/4, 1/2] \) and some \( c > 0 \). Assume that \( \sqrt{k_n}(r_{u} - r_{u}) \rightarrow_{0} 0 \) and that \( \langle \tilde{J}_{u} - \tilde{J}_{u} \rangle \rightarrow_{0} 0 \) as \( n \rightarrow \infty \). Then,
\[
\sqrt{k_n}(a_{n/k_n}(T_{u(n)\circ n,k_n} - \theta_{0,(n)}) - r_{u}) \rightarrow_{d} G_u,
\]
where \( G_u \) is a \( \mathbb{X} \)-valued mean zero Gaussian random variable with covariance given by \( V_u = [\tilde{J}_{u}]^{-1}u_{u}[\tilde{J}_{u}]^{-1} \), where \( u_{u} : \mathbb{X} \rightarrow \mathbb{X} \) satisfies
\[
\mathbb{E}[\langle u_{u}(\mathbf{h}), \mathbf{s} \rangle = \mathbb{E}[\langle \mathbf{r}_{u} - Y, u_{u} \rangle, \mathbf{s} \rangle] = \mathbb{E}[\langle \mathbf{r}_{u} - Y, u_{u} \rangle, \mathbf{s} \rangle],
\]
for every \( \mathbf{h}, \mathbf{s} \in \mathbb{X} \).

Assuming that \( \sqrt{k_n}(r_{u} - r_{u}) \rightarrow_{0} 0 \) implies assuming that the quantile of the distribution converge to the quantile of the limiting distribution fast enough (see Theorem 3.4 in [8]). Since the distribution of \( Z_n \) is not concentrated on a straight line we know that \( \mathbb{E}_{n,u} : \mathbb{Z}_{n} \rightarrow \mathbb{Z} \) given by
\[
\mathbb{E}_{n,u}(\mathbf{h}, \mathbf{s}) = \mathbb{E}[\langle \mathbf{r}_{u} - \mathbb{E}_{n,u}(\mathbf{r}_{u}) - \mathbf{u} \rangle, \mathbf{s} \rangle]
\]
is invertible for \( n \) large enough. Thus, we can get rid of the assumption \( \langle \tilde{J}_{u} - \tilde{J}_{u} \rangle \rightarrow_{0} 0 \) as \( n \rightarrow \infty \) at the expense of clarity. In this case, the result in Theorem 3.18 would be
\[
\tilde{J}_{u} - \tilde{J}_{u} \Lambda_{u}^{-1/2} \sqrt{k_n}(a_{n/k_n}(T_{u(n)\circ n,k_n} - \theta_{0,(n)}) - r_{u}) \rightarrow_{d} S_u,
\]
where \( S_u \) is a \( \mathbb{X} \)-valued mean zero Gaussian random variable with covariance given by the identity operator.

From this we have the following robustness property of the QoE estimator. Consider \( Y \) to be such that \( \mathbf{r}_{u} = \mathbf{0} \), for some \( \mathbf{u} \), and \( d_n = \sqrt{n} \), as in the typical case of asymptotically normal estimators in \( \mathbb{R}^d \). In this case \( Y \) is symmetrically Gaussian around zero, hence \( \mathbf{r}_{0} = \mathbf{0} \), and \( a_n = \sqrt{n} \).

**Corollary 3.19.** Under the assumptions of Theorem 3.18, with \( Y \) such that \( \mathbf{r}_{u} = \mathbf{0} \) for some \( \mathbf{u} \), we have
\[
\sqrt{k_n}(a_{n/k_n}(T_{u(n)\circ n,k_n} - \theta_{0,(n)}) - r_{u}) \rightarrow Z,
\]
and if \( a_n = \sqrt{n} \), then
\[
\sqrt{n}(T_{u(n)\circ n,k_n} - \theta_{0,(n)}) \rightarrow_{d} Z,
\]
where \( Z \) is as in Theorem 3.18.

Thus, we have asymptotic normality of the QoE estimator in the presence of contaminated data \( l_n = o(n) \). By proving the results for quantiles instead of just for the median, we allow the results to be applied to any estimator whose asymptotic distribution is strictly positive on a neighborhood of \( \mathbf{0} \).

In the finite-dimensional setting some of the assumptions of Theorem 3.18 are always satisfied. By Lemma 4.18 under Assumption 3.2
\[
\sup_{\mathbf{z} \in \mathbb{R}^d : \| \mathbf{z} - \mathbf{u} \| < l_n/k_n} \| \mathbf{r}_{u} - \mathbf{r}_{u} \| \rightarrow 0
\]
for any
sequences \( l_n \) and \( k_n = o(n) \) such that \( \lim_{n \to \infty} \frac{l_n}{k_n} = 0 \). Let \( \varepsilon > 0 \) and let \( k_n = cn^\beta \) for \( \beta \in (0, 1) \).

Define

\[
\beta^*_\varepsilon := \sup\{\beta \in (0, 1) : \sqrt{n^\beta} \sup_{\|z-u\| \leq n^{-\varepsilon/2}} |r_{[n/cn^\beta],z} - ru| \to 0\}.
\]

Theorem 3.18 on \( \mathbb{R}^d \) now simplifies as follows.

**Proposition 3.20.** Let \( X = \mathbb{R}^d \) and let Assumptions 3.3 and 3.10 hold. Then,

\[
\tilde{J}_{[l_n/k_n],u} a_{l_n}^{1/2} \sqrt{cn^\beta} (a_{l_n/cn^\beta} (T_{u,n,cn^\beta} - \theta_0) - ru) d \to N(0, I)
\]

for any \( l_n = O(n^\gamma) \), \( \beta \in (\gamma + \varepsilon, \beta^*_\varepsilon) \), and \( \varepsilon > 0 \), where \( I \) is the identity matrix. If \( Y \) is such that \( r_u = 0 \) and \( a_n = \sqrt{n} \), then

\[
\tilde{J}_{[l_n/k_n],u} a_{l_n}^{1/2} \sqrt{n} (T_{u,n,cn^\beta} - \theta_0) d \to N(0, I)
\]

for any \( l_n = O(n^\gamma) \), \( \beta \in (\gamma + \varepsilon, \beta^*_\varepsilon) \), and \( \varepsilon > 0 \).

We remark that the interval \((\gamma + \varepsilon, \beta^*_\varepsilon)\) might be improved. Indeed, if we knew that the set \( \beta \in (\gamma, \beta^*_\varepsilon) \) such that \( \sqrt{n^\beta} \sup_{\|z-u\| \leq n^{-\varepsilon/2}} |r_{[n/cn^\beta],z} - ru| \to 0 \) is a connected interval, then the convergences in Proposition 3.20 would hold for any \( l_n = O(n^\gamma) \) and \( \beta \in (\gamma, \beta^*_\varepsilon) \).

### 3.3 The case of \( n/k_n = c \)

We conclude this section with a discussion on the case of \( n/k_n = c \) for all \( n \) large enough and for some \( c > 0 \), i.e. the case in which the number of blocks goes to infinity while the number of observations in each block remain constant. For this case, the classical results for the asymptotic behavior of quantiles apply. We focus on the component-wise quantile on \( \mathbb{R}^d \); the same reasoning applies to the geometric quantile and the component-wise quantile in any dimensions.

Let \( G_n^{-1}(\alpha) \) be the unique \( \alpha \)-componentwise quantile of the distribution of \( Z_n^{-1} - \theta_0 \) if it exists. Assuming wlog that \( c \in \mathbb{N} \), if \( G_n^{-1}(\alpha) \) then \( T_{P,\alpha,n,k_n} - \theta_0 \to^p G^{-1}_n(\alpha) \). This result is not very useful in our context for two main reasons, as it requires the finite sample distribution of \( Z_n^{-1} \) to be known, and the result is not robust to a level of data contamination of order \( n \). Indeed, if \( l_n = C \cdot n \), for some \( C > 0 \), then the convergence in probability depends on how the contaminated data behave, and in some cases we might not even have convergence in probability even when \( C >> c \). The same issues arise for convergence in distribution.

**Example 3.21.** Consider the Median-of-Means estimator, i.e. the QoE estimator with chosen estimator being the sample mean and with \( \alpha = (1/2, \ldots, 1/2) \). Assume that the observations are i.i.d. normally distributed. In this case \( Z_n \) is the sample mean, and \( Z_n - \theta_0 \) is normal with mean \( 0 \), and median \( G_n^{-1}((1/2, \ldots, 1/2)) = 0 \). Thus, \( T_{P,\alpha,n,k_n} - \theta_0 \) will be the median of \( n/k_n \) mean-zero normal random vectors and so it will be close to \( 0 \). Using the asymptotic normality of the median we also have that \( \sqrt{n/k_n} (T_{P,\alpha,n,k_n} - \theta_0) \) will be close to a mean-zero normal random vector. This holds for any \( n \) and \( k_n \) with \( n \geq k_n \). However, even in this case when \( n/k_n = c \) for all \( n \) we have that if \( l_n = C^{-1} n \) then the convergence of \( T_{P,\alpha,n,k_n} - \theta_0 \) depends on how the contaminated data behave.

The example and the discussion above show that if we want to allow for contamination of the data, we cannot consider the case of \( n/k_n = c \) (for all \( n \) large enough), but we need \( k_n = o(n) \).
4 Existence, uniqueness, continuity, and representations

In this section we focus on various results concerning continuity, existence, and uniqueness properties of both geometric and component-wise quantiles. We present also a representation of geometric quantiles in Hilbert spaces that extend to quantiles that in [13]. These results provide the building blocks for the proofs of the results presented in the previous section. We start with a short discussion of univariate quantiles.

Lemma 4.1. The univariate geometric quantile is uniquely defined if and only if $\alpha \in (0, 1) \setminus \{\frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}\}$. If the univariate geometric quantile is uniquely defined then it is equal to $q^\circ_\alpha$.

An alternative definition of univariate quantile generalizes the definition of the univariate median adopted in [29] (see also [27]). Define $q^\circ_\alpha(x) := x_i$, where $x_i$ is such that $|\{j \in \{1, \ldots, k\} : x_j \leq x_i\}| \geq k\alpha$ and $|\{j \in \{1, \ldots, k\} : x_j \geq x_i\}| \geq k(1 - \alpha)$.

If several $x_i$'s satisfy the constraints we take the smallest one by convention.

Lemma 4.2. Let $\alpha \in (0, 1)$. The functions $q_\alpha$ and $q^\circ_\alpha$ are Lipschitz continuous. In particular for any $x, y \in \mathbb{R}^k$ we have

$$|q_\alpha(x) - q_\alpha(y)| \leq \max_{j=1, \ldots, k} |x_j - y_j|,$$

and

$$|q^\circ_\alpha(x) - q^\circ_\alpha(y)| \leq \max_{j=1, \ldots, k} |x_j - y_j|.$$ (16)

The above bounds can be improved as it is possible to see from the proof of Lemma 4.2. Further, we remark that all our results containing $q_\alpha$ hold also if we substitute $q_\alpha$ by $q^\circ_\alpha$.

4.1 Component-wise quantile

We present existence and continuity results for the three definitions of component-wise quantiles.

Theorem 4.3. Let $X$ be a vector space. The component-wise quantile $q_{\text{Hamel}, \alpha}$ exists and is unique. If $X$ is endowed with the topology induced by $\| \cdot \|_{\text{Hamel}}$, $q_{\text{Hamel}, \alpha}$ is Lipschitz continuous: for every $x_1, \ldots, x_k, z_1, \ldots, z_k \in X$ we have

$$\|q_{\text{Hamel}, \alpha}(x_1, \ldots, x_k) - q_{\text{Hamel}, \alpha}(z_1, \ldots, z_k)\|_{X, \text{Hamel}} \leq \max_{j=1, \ldots, k} |x_j - z_j|.$$ (17)

Now, let $X$ be a Banach space possessing an unconditional Schauder basis. Denote by $\mathcal{K}$ the unconditional basis constant of $X$, i.e.

$$\mathcal{K} := \sup_{F \subset \mathbb{N}, \epsilon \geq 0, \|x\|_X = 1} \| \sum_{n \in F} \epsilon_n x_i d_i \|_X$$

Since the field of $X$ is $\mathbb{R}$, we have that

$$\mathcal{K} = \sup_{F \subset \mathbb{N}, |l_\alpha| \leq 1, \|x\|_X = 1} \| \sum_{n \in F} l_\alpha x_i d_i \|_X.$$ (18)

By Theorem 6.4 in [19] $\mathcal{K}$ is always finite. Moreover, it is always possible to assign to $X$ an equivalent norm such that $\mathcal{K} = 1$. The unconditional basis constant is the focus of a vast literature (see [15] and [16], among others).
Theorem 4.4. Let $X$ be a Banach space possessing an unconditional Schauder basis and let $K$ be its unconditional basis constant. The component-wise quantile $q_{S,\alpha}$ exists, is unique, and is Lipschitz continuous: for every $x_1, \ldots, x_k, z_1, \ldots, z_k \in X$ we have that
\[ ||q_{S,\alpha}(x_1, \ldots, x_k) - q_{S,\alpha}(z_1, \ldots, z_k)||_X \leq K^2||x_1, \ldots, x_k\|_X - (z_1, \ldots, z_k)||_X, \]
and if $X$ is a Hilbert space then
\[ ||q_{S,\alpha}(x_1, \ldots, x_k) - q_{S,\alpha}(z_1, \ldots, z_k)||_X \leq \sqrt{\sum_{i=1}^{k} ||x_i - z_i||_2^2} \leq ||(x_1, \ldots, x_k) - (z_1, \ldots, z_k)||_X. \]

Finally, we focus on the point-wise component-wise quantile.

Theorem 4.5. Let $X$ be either $L_p[0,1]$ with $1 \leq p \leq \infty$ or $X = C[0,1]$ or $X = D[0,1]$. The point-wise component-wise quantile $q_{P,\alpha}$ exists, is unique, and is Lipschitz continuous. In particular, if $X = L_p[0,1]$ with $1 \leq p < \infty$, then
\[ ||q_{P,\alpha}(f_1, \ldots, f_k) - q_{P,\alpha}(g_1, \ldots, g_k)||_X \leq \left( \sum_{i=1}^{k} \int_{0}^{1} |f_i(x) - g_i(x)|^p \, dx \right)^{1/p} \leq ||(f_1, \ldots, f_k) - (g_1, \ldots, g_k)||_X. \]

If either $X = L_{\infty}[0,1]$ or if $X = C[0,1]$ or $X = D[0,1]$, then
\[ ||q_{P,\alpha}(f_1, \ldots, f_k) - q_{P,\alpha}(g_1, \ldots, g_k)||_X \leq \max_{i=1, \ldots, k} ||f_i - g_i||_X \leq ||(f_1, \ldots, f_k) - (g_1, \ldots, g_k)||_X. \]

Remark 4.6. For the sake of clarity we presented the results for the interval $[0,1]$, but it is easy to see that the result holds for any interval $[0,T]$, and for the space $D[0,\infty)$.

It is possible to see that such a result might easily hold for other function spaces, such as Sobolev spaces.

4.2 Geometric quantile

The following general convex analysis result generalizes Lemma 2.14 in [22].

Lemma 4.7. Let $X$ be a Banach space. Let $W$ be a one-dimensional linear subspace of $X$. Then $X = W \oplus Y$, where $Y$ is the kernel of a norm-1 projection operator with range $W$. Moreover, for every $x \in W$ we have that
\[ ||x||_X \leq ||x + y||_X, \quad \forall y \in Y, \quad (17) \]
and if $X$ is strictly convex then
\[ ||x||_X < ||x + y||_X, \quad \forall y \in Y. \quad (18) \]

Finally, we have that for every $y \in Y$
\[ ||y||_X \leq ||x + y||_X, \quad \forall x \in W, \quad (19) \]
and if $X$ is strictly convex then
\[ ||y||_X < ||x + y||_X, \quad \forall x \in W. \quad (20) \]
Remark 4.8. Since \( y \in Y \) implies that \( \lambda y \in Y \) for every \( \lambda \in \mathbb{R} \), we have that (17) is equivalent to: for every \( x \in X \) we have that \( \|x\| \leq \|x + \lambda y\| \), for every \( y \in Y \) and every \( \lambda \in \mathbb{R} \). The same holds for (18) with strict inequality and the same arguments apply to (19) and (20). Thus, (17) (18)) is equivalent to James (strict) orthogonality between any element of \( W \) and \( Y \), which we write \( x \perp y \ (x \perp_s y) \), and (19) (20)) is equivalent to James (strict) orthogonality between any element of \( Y \) and \( W \), which we write \( y \perp x \ (y \perp_s x) \).

In the following result we present sufficient conditions for the uniqueness of the geometric quantile, thus generalizing Theorem 2.17 in [22].

**Theorem 4.9.** Let \( X \) be a strictly convex Banach space. The geometric quantile (2), if it exists, is unique if one of the following conditions hold:

(I) \( x_1, ..., x_k \) do not lie on a straight line,

(II) \( x_1, ..., x_k \) lie on a straight line and \( \alpha \in (0, 1) \setminus \{ \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k} \} \).

For the geometric median we obtain both necessary and sufficient conditions.

**Theorem 4.10.** Let \( X \) be a strictly convex Banach space. The geometric median (2) with \( u = 0 \), if it exists, is unique if and only if one of the following conditions hold:

(I) \( x_1, ..., x_k \) do not lie on a straight line,

(II) \( x_1, ..., x_k \) lie on a straight line, and \( k \) is odd.

To obtain necessary and sufficient conditions for the geometric quantile we need:

**Assumption 4.11.** When \( x_1, ..., x_k \) lie on a straight line \( W \) assume that the norm of \( X \) is Gâteaux differentiable at \( x_i \neq 0 \), for some \( i = 1, ..., k \), in some direction \( z \in Y \), where \( Y \) is complementary to \( W \), i.e. \( X = Y + W \), and such that \( \langle u, z \rangle \neq 0 \).

This assumption is always satisfied when the norm of \( X \) is Gâteaux differentiable at \( x_i \neq 0 \), thus it is always satisfied when \( X \) is smooth at \( x_i \neq 0 \). Recall that \( X \) is smooth at \( x \neq 0 \) if there exists a unique \( f \in X^* \) s.t. \( f(x) = \|x\|_X \), and that \( X \) is smooth if it is smooth at every \( x \in X \) s.t. \( x \neq 0 \). Further, this assumption is satisfied when \( X^* \) is strictly convex, or when \( X \) is uniformly smooth, among others (see Chapter 8 in [11] for further details).

We remark that \( L_p \) spaces with \( p \in (1, \infty) \) and Hilbert spaces are uniformly smooth and uniformly convex (and so strictly convex) Banach spaces. In particular, their norm is Fréchet differentiable and so Gâteaux differentiable.

Recall the decomposition \( u \) when \( x_1, ..., x_k \) lie on a straight line \( W \), i.e. \( u = u e + v \), where \( e \in Y^\perp \) with \( \|e\|_{X^*} = 1 \), \( u \in (-1, 1) \), and \( v \in W^\perp \).

**Theorem 4.12.** Let \( X \) be a strictly convex Banach space and let Assumption 4.11 hold. The geometric quantile (2), if it exists, is unique if and only if one of the following conditions hold:

(I) \( x_1, ..., x_k \) do not lie on a straight line,

(II) \( x_1, ..., x_k \) lie on a straight line and \( v \neq 0 \),

(III) \( x_1, ..., x_k \) lie on a straight line, \( v = 0 \), and \( \alpha \in (0, 1) \setminus \{ \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k} \} \).

It is possible to see that \( v, \alpha \), and Assumption 4.11 all depend on \( W \). Thus, the conditions of Theorems 4.9 and 4.12 depend on \( W \). When the data are random we do not know what \( W \) is, and so we cannot directly apply the previous theorems to know whether or not the quantile is unique. In the following result we solve this issue by showing that the quantile is unique if a certain condition on \( u \), which is given a priori, is satisfied.
Corollary 4.13. Let $X$ be a smooth and strictly convex Banach space. The geometric quantile $(\mathfrak{q})$ is unique if one of the following conditions hold:
(I) $x_1, \ldots, x_k$ do not lie on a straight line,
(II) $\|u\|_{X^*} \notin \{1 - \frac{2j}{k}, j = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\}$.

Remark 4.14. Point (I) of Corollary 4.13 implies the well known result that if the data are distributed according to an absolutely continuous distribution then the quantile is unique. The advantage of Point (II) of Corollary 4.13 consists in ensuring uniqueness of the geometric quantile independently of the distribution of the data. Moreover, this condition is quite weak since among all the possible values that $\|u\|_{X^*}$ can take in $[0, 1)$, only finitely many of them (no more than $\left\lfloor \frac{k}{2} \right\rfloor$) do not ensure uniqueness. For example, from Corollary 4.13 we see that if $k$ is odd the geometric median is always uniquely defined.

In Theorems 4.9, 4.10, and 4.12, we show the uniqueness of the geometric quantile and median under certain conditions on the position of the data $x_1, \ldots, x_k$ (and on $v$ and $\alpha$). In the following result we show that there is a precise relation between the position of the data $x_1, \ldots, x_k$ and the position of the geometric quantiles, even when they are not unique.

Proposition 4.15. Let $X$ be a strictly convex space and let $x_1, \ldots, x_k$ lie on a straight line $W$. Then the set of geometric median is a subset of $W$, in particular

\[
(i) \quad \arg \min_{y \in X} \sum_{i=1}^{k} \|x_i - y\|_X = \arg \min_{y \in W} \sum_{i=1}^{k} \|x_i - y\|_X.
\]

Further, let $u \in X^*$ with $\|u\|_{X^*} < 1$. If $v = 0$ then the set of geometric quantiles is a subset of $W$ and

\[
(ii) \quad \arg \min_{y \in X} \sum_{i=1}^{k} \|x_i - y\|_X + \langle u, x_i - y \rangle = \arg \min_{y \in W} \sum_{i=1}^{k} \|x_i - y\|_X + \langle u, x_i - y \rangle.
\]

Let $X$ additionally satisfy Assumption 4.11. Then $v = 0$ if and only if the set of geometric quantiles is a subset of $W$. In particular, if $v = 0$ then

\[
(iii) \quad \arg \min_{y \in X} \sum_{i=1}^{k} \|x_i - y\|_X + \langle u, x_i - y \rangle = \arg \min_{y \in W} \sum_{i=1}^{k} \|x_i - y\|_X + \langle u, x_i - y \rangle.
\]

Since $u$ is linear, the existence of the geometric quantile follows from the existence results of the geometric median. In particular, by [33] (see also Remark 3.5 in [22]) existence is ensured when $X$ is reflexive and by Theorem 3.6 in [22] when $X$ is the dual of a separable Banach space. This includes the case of smooth Banach spaces.

We now focus on the continuity properties of the geometric quantile. The first result concerns the continuity of the geometric quantile for distributions (not necessarily discrete) and represents the quantile extension of a well-known result for the median (see [6, 22]).

Lemma 4.16. Let $X$ be a separable and strictly convex Banach space. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of probability measures on $X$ such that $\mu_n \xrightarrow{w} \mu$, where $\mu$ is a probability measure on $X$. Assume that $\mu$ possesses a unique geometric quantile. Then, any geometric quantile of $\mu_n$ converges in the weak* topology to the geometric quantile of $\mu$. If $X$ is finite dimensional then any geometric quantile of $\mu_n$ converges to the geometric quantile of $\mu$.  


**Theorem 4.20.** Let \( n \in \mathbb{N} \) be a sequence of probability measures on \( \mathbb{X} \) such that \( n \xrightarrow{w} \mu \), where \( \mu \) is a probability measure on \( \mathbb{X} \). Assume that \( \mu \) possesses a unique geometric quantile \( y_u^\ast \). Let \( y_{n,u}^\ast \) be a geometric quantile of \( n \). For any \( \varepsilon > 0 \), \( \sup_{\|x\| < 1 - \varepsilon} \|y_{n,u}^\ast - y_u^\ast\| \to 0 \) as \( n \to \infty \). Further, if \( \mu \) is atomless, then for any \( c_n \to 0 \) and \( v \in \mathbb{R}^d \) with \( \|v\| < 1 \), we have \( \sup_{\|x\| \leq c_n} \|y_{n,u}^\ast - y_v^\ast\| \to 0 \) as \( n \to \infty \).

We present a generalization of Theorem 2.24 in [22] in which we show that on finite-dimensional spaces the convergence of the geometric quantile is uniform with respect to the quantile parameter \( u \).

**Lemma 4.18.** Let \( \mathbb{X} \) be a finite-dimensional Banach space. Let \( (\mu_n)_{n \in \mathbb{N}} \) be a sequence of probability measures on \( \mathbb{X} \) such that \( n \xrightarrow{w} \mu \), where \( \mu \) is a probability measure on \( \mathbb{X} \). Assume that \( \mu \) possesses a unique geometric quantile \( y_u^\ast \). Let \( y_{n,u}^\ast \) be a geometric quantile of \( n \). For any \( \varepsilon > 0 \), \( \sup_{\|x\| < 1 - \varepsilon} \|y_{n,u}^\ast - y_u^\ast\| \to 0 \) as \( n \to \infty \). Further, if \( \mu \) is atomless, then for any \( c_n \to 0 \) and \( v \in \mathbb{R}^d \) with \( \|v\| < 1 \), we have \( \sup_{\|x\| \leq c_n} \|y_{n,u}^\ast - y_v^\ast\| \to 0 \) as \( n \to \infty \).

In the next results we strengthen Lemma 4.16 for empirical distributions by proving strong convergence even in the infinite dimensional setting.

Let \( X_k := \{(x_1, ..., x_k) \in \mathbb{X}^k : x_1, ..., x_k \text{ do not lie on a straight line}\} \), where \( \mathbb{X} \) is a Banach space.

**Lemma 4.19.** Let \( \mathbb{X} \) be a Banach space. The set \( X_k \) is open.

**Theorem 4.20.** Let \( \mathbb{X} \) be a reflexive and strictly convex Banach space. Then, the quantile function \( q_u \) is continuous on \( X_k \). If \( \mathbb{X} \) is also smooth and \( \|u\|_{\mathbb{X}^*} \notin \{1 - \frac{2j}{k}, j = 1, ..., \left\lfloor \frac{k}{2} \right\rfloor \} \), then the quantile function \( q_u \) is continuous on \( \mathbb{X}^k \).

In the following result we show that when \( \mathbb{X} \) is a Hilbert space the geometric quantile is (almost) a linear combination of \( x_1, ..., x_k, u \) and that the geometric quantile is the solution of a \((k + 1)\)-dimensional minimization problem. This generalizes one of the main intuitions of [13] to quantiles.

**Theorem 4.21.** Let \( \mathbb{X} \) be a Hilbert space. Let \( T_{k+1} := \{w \in [0, 1]^{k+1} : w_1 + \cdots + w_{k+1} = 1\} \). Then, any geometric quantile is given by

\[
\frac{1}{1 - w_{k+1}^*} \left( \sum_{i=1}^{k} w_i^* x_i + w_{k+1}^* u \right),
\]

where \((w_1^*, ..., w_{k+1}^*)\) is given by

\[
\arg \min_{w \in T_{k+1}} \sum_{i=1}^{k} \|x_i - \frac{1}{1 - w_{k+1}} \left( \sum_{l=1}^{k} w_l x_l + w_{k+1} u \right) \|_{\mathbb{X}}^2 + \langle u, x_{i} - \frac{1}{1 - w_{k+1}} \left( \sum_{l=1}^{k} w_l x_l + w_{k+1} u \right) \rangle_{\mathbb{X}}.
\]

In particular, if \( y^* \) is a geometric quantile and \( y^* \neq x_i, i = 1, ..., k \), then

\[
w_i^* = \frac{\|x_i - y^*\|_{\mathbb{X}}^{-1}}{\sum_{j=1}^{k} \|x_i - y^*\|_{\mathbb{X}}^{-1} + k}, \quad \text{for } i = 1, ..., k, \quad \text{and} \quad w_{k+1}^* = \frac{k}{\sum_{i=1}^{k} \|x_i - y^*\|_{\mathbb{X}}^{-1} + k}.
\]

We stress that the above theorem holds even in the case of non-uniqueness of the geometric quantile. When we have uniqueness, i.e., when the data do not lie on a straight line or when \( \|u\|_{\mathbb{X}^*} \notin \{1 - \frac{2j}{k}, j = 1, ..., \left\lfloor \frac{k}{2} \right\rfloor \} \), then \( q_u((x_1, ..., x_k)) = \frac{1}{1 - w_{k+1}^*} \left( \sum_{i=1}^{k} w_i^* x_i + w_{k+1}^* u \right) \).
In the remaining part of this subsection we investigate the properties of the geometric quantile when \( X = \ell_1 \), which is not a strictly convex space. Observe that \( u = (u^{(l)})_{l\in\mathbb{N}} \) belongs to \( \ell_\infty \). By abuse of notation we use \( q_u \) to indicate the geometric quantile function in this space as well.

**Proposition 4.22.** Let \( X = \ell_1 \). The geometric quantile exists. Further, the geometric quantile is unique if and only if \( |u^{(l)}| \notin \{1 - \frac{2|\beta|}{|\alpha|}, j = 1, \ldots, \lfloor \frac{|\alpha|}{2} \rfloor \}, \forall l \in \mathbb{N} \). Further, if \( |u^{(l)}| \notin \{1 - \frac{2|\beta|}{|\alpha|}, j = 1, \ldots, \lfloor \frac{|\alpha|}{2} \rfloor \}, \forall l \in \mathbb{N} \), the geometric quantile function \( q_u \) is Lipschitz continuous on \( X^k \): for every \( x_1, \ldots, x_k, z_1, \ldots, z_k \in \ell_1 \), we have that

\[
\|q_u(x_1, \ldots, x_k) - q_u(z_1, \ldots, z_k)\|_{\ell_1} \leq \|(x_1, \ldots, x_k) - (z_1, \ldots, z_k)\|_{\ell_1^k}.
\]

From Proposition 4.22 we deduce that the geometric median is unique if and only if \( k \) is odd, and when \( k \) is odd the geometric median is Lipschitz continuous.

We doubt that a similar result holds for \( X = L_1[0, 1] \) because \( L_1[0, 1] \) has no unconditional Schauder basis (see [16]).

**Appendix**

**Proofs of Section 3.1**

**Proofs of Theorems 3.3, 3.4, and 3.5.** The results follow from the equivariance property of the univariate quantile, by the continuity results (Theorems 4.3, 4.4, 4.5) and by the continuous mapping theorem. \( \square \)

**Proof of Proposition 3.6.** Since the convergence in probability of a random vector follows from the convergence in probability of each of its components, we obtain the result from the same arguments used for the proof of the convergence in probability in Theorem 3.7. \( \square \)

**Proofs of Theorem 3.7.** For the first statement since the convergence in probability of a vector is determined by the convergence in probability of its components, it is enough to focus on the one dimensional case. Consider any \( c > 0 \) and \( \beta \in (0, 1) \) such that \( \lim_{n \to \infty} \frac{\beta}{cn^\beta} = 0 \). By assumption \( F^{-1}(\alpha) \) is uniquely defined. Indeed, as the set of geometric quantiles is connected and compact, if \( f \) is continuous and strictly positive at any \( F^{-1}(\alpha) \), then \( f \) is continuous and strictly positive on the set of geometric quantiles, which implies that the set consists of only one point. Now, for any \( \varepsilon > 0 \)

\[
\Pr(|a_{n/cn^\beta}(T_{u,n,cn^\beta} - \theta_0) - F^{-1}(\alpha)| > \varepsilon)
\]

\[
= \Pr(a_{n/cn^\beta}(T_{u,n,cn^\beta} - \theta_0) - F^{-1}(\alpha) > \varepsilon) + \Pr(a_{n/cn^\beta}(T_{u,n,cn^\beta} - \theta_0) - F^{-1}(\alpha) < -\varepsilon)
\]

To lighten the notation, let \( W_n^{(j)} := a_{n/cn^\beta}(Z_{n/cn^\beta}^{(j)} - z) - F^{-1}(\alpha) \) and \( \tilde{W}_n^{(j)} := a_{n/cn^\beta}[(\tilde{Z}_{n/cn^\beta}^{(j)} - z) - F^{-1}(\alpha)] \), for \( j = 1, \ldots, [cn^\beta] \). By the equivariance property of the quantile we have

\[
a_{n/cn^\beta}(\theta_{u,n,cn^\beta} - \theta_0) - F^{-1}(\alpha) = q_\alpha(\tilde{W}_n^{(j)}, \ldots, \tilde{W}_n^{([cn^\beta])}),
\]

so

\[
\Pr(a_{n/cn^\beta}(T_{u,n,cn^\beta} - \theta_0) - F^{-1}(\alpha) > \varepsilon) \leq \Pr\left( \sum_{j=1}^{[cn^\beta]} I_{\{\tilde{W}_n^{(j)} > \varepsilon\}} \geq [cn^\beta](1 - \alpha) \right).
\]
Now, assume without loss of generality that the last $l_n$ of the $\tilde{W}_n^{(j)}$'s are contaminated, then
\[
\mathbb{P}\left( \sum_{j=1}^{[cn^\beta]} 1_{\{\tilde{W}_n^{(j)} > \varepsilon\}} \geq [cn^\beta] (1 - \alpha) \right)
\leq \mathbb{P}\left( \sum_{j=1}^{[cn^\beta] - l_n} 1_{\{\tilde{W}_n^{(j)} > \varepsilon\}} + l_n \geq [cn^\beta] (1 - \alpha) \right) \leq \mathbb{P}\left( \sum_{j=1}^{[cn^\beta]} 1_{\{\tilde{W}_n^{(j)} > \varepsilon\}} + l_n \geq [cn^\beta] (1 - \alpha) \right).
\]

As $\mathbb{P}(a_{n/cn^\beta}(Z_{n/cn^\beta}^{(j)} - \theta_0) > x) \rightarrow \mathbb{P}(Y > x)$ for any continuity point $x$ by Assumption 3.2 and $f$ is continuous at $F^{-1}(\alpha)$ by assumption of the statement, we obtain
\[
\mathbb{P}(W_n^{(1)} > \varepsilon) \rightarrow \mathbb{P}(Y - F^{-1}(\alpha) > \varepsilon) < 1 - \alpha,
\]
as $n \rightarrow \infty$, where we consider that $F^{-1}(\alpha) + \varepsilon$ is a continuity point of $F$ when $\varepsilon$ is small enough. So using $\lim_{n \rightarrow \infty} \frac{ln}{cn^\beta} = 0$, we have $1 - \alpha - \mathbb{P}(W_n^{(1)} > \varepsilon) - \frac{ln}{cn^\beta} > 0$ for $n$ large enough.

Then, by Hoeffding’s inequality we obtain
\[
\mathbb{P}\left( \frac{1}{cn^\beta} \sum_{j=1}^{[cn^\beta]} 1_{\{\tilde{W}_n^{(j)} > \varepsilon\}} - \mathbb{P}(W_n^{(1)} > \varepsilon) \geq 1 - \alpha - \mathbb{P}(W_n^{(1)} > \varepsilon) - \frac{ln}{cn^\beta} \right) 
\leq \varepsilon \frac{\frac{1}{[cn^\beta]}(1 - \alpha - \mathbb{P}(W_n^{(1)} > \varepsilon) - \frac{ln}{cn^\beta})^2}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\]

From this we obtain $\mathbb{P}(a_{n/cn^\beta}(T_{u.n,cn^\beta} - \theta_0) - F^{-1}(\alpha) > \varepsilon) \rightarrow 0$ as $n \rightarrow \infty$.

By the same arguments $\mathbb{P}(a_{n/cn^\beta}(T_{u.n,cn^\beta} - \theta_0) - F^{-1}(\alpha) < -\varepsilon) \rightarrow 0$, as $n \rightarrow \infty$. Thus, we have that $a_{n/cn^\beta}(T_{u.n,cn^\beta} - \theta_0) \xrightarrow{p} F^{-1}(\alpha)$, as $n \rightarrow \infty$. Hence, we obtain the first statement.

Let $\beta \in (2\gamma, \max_{i=1,...,d} \beta^*_i)$ and by abuse of notation let $W_n^{(j)} := a_{n/cn^\beta}(Z_{n/cn^\beta}^{(j)} - \theta_{0,i}) - F^{-1}_i(\alpha_i)$, $\tilde{W}_n^{(j)} := a_{n/cn^\beta}(Z_{n/cn^\beta}^{(j)} - \theta_{0,i}) - F^{-1}_i(\alpha_i)$, $W_n^{(j)} = (W_n^{(j)}),...,W_n^{(j)}$ and $\tilde{W}_n^{(j)} = (\tilde{W}_n^{(j)}),...,\tilde{W}_n^{(j)})$, for $j = 1,...,[cn^\beta]$ and $i = 1,...,d$, where $\theta_0 = (\theta_{0,1},...,\theta_{0,d})$. For every $x_1,...,x_d \in \mathbb{R}$,
\[
\left\{ \sum_{j=1}^{[cn^\beta]} 1_{\{W_n^{(j)} > x_1\}} \leq [cn^\beta] (1 - \alpha_1) - 1 - l_n, \ldots, \sum_{j=1}^{[cn^\beta]} 1_{\{W_n^{(j)} > x_d\}} \leq [cn^\beta] (1 - \alpha_d) - 1 - l_n \right\}
\]}
\]
\]
\[
\leq \sum_{j=1}^{[cn^\beta]} 1_{\{W_n^{(j)} \leq x_1\}} \geq [cn^\beta] \alpha_1 + 1 + l_n, \ldots, \sum_{j=1}^{[cn^\beta]} 1_{\{W_n^{(j)} \leq x_d\}} \geq [cn^\beta] \alpha_d + 1 + l_n
\]
\[
\cup \left\{ (q_{P,\alpha}((\tilde{W}_n^{(1)}),...,\tilde{W}_n^{([cn^\beta])})) \leq x_1, \ldots, (q_{P,\alpha}((\tilde{W}_n^{(1)}),...,\tilde{W}_n^{([cn^\beta])})) \leq x_d \right\}
\]
\[
\leq \sum_{j=1}^{[cn^\beta]} 1_{\{W_n^{(j)} \leq x_1\}} \geq [cn^\beta] \alpha_1 - l_n, \ldots, \sum_{j=1}^{[cn^\beta]} 1_{\{W_n^{(j)} \leq x_d\}} \geq [cn^\beta] \alpha_d - l_n
\]
\[
\sum_{j=1}^{[cn^\beta]} 1_{\{W_n^{(j)} > x_1\}} \leq [cn^\beta] (1 - \alpha_1) + l_n, \ldots, \sum_{j=1}^{[cn^\beta]} 1_{\{W_n^{(j)} > x_d\}} \leq [cn^\beta] (1 - \alpha_d) + l_n
\] (21)
Let $V_n$ be $d \times d$ matrix given by

$$V_n = \left( P(W_{n,i}^{(1)} > x_i, W_{n,j}^{(1)} > x_j) - P(W_{n,i}^{(1)} > x_i) P(W_{n,j}^{(1)} > x_j) \right)_{i,j=1,...,d}.$$ 

By the central limit theorem

$$\sqrt{cn^\beta} V_n^{-1/2} \left( \frac{1}{cn^\beta} \sum_{j=1}^{\lfloor cn^\beta \rfloor} 1_{\{W_{n,j}^{(i)} > x_i\}} - 1 + F_{n,i}(x_i + F_i^{-1}(\alpha_i)) \right) \rightarrow_d N(0, I).$$

Let $x_i = \frac{y_i}{\sqrt{cn^\beta}}$, where $y_i \in \mathbb{R}$. By assumption we have that $V_n \rightarrow V$ as $n \rightarrow \infty$, where

$$V = \left( P(Y_i > F_i^{-1}(\alpha_i), Y_j > F_j^{-1}(\alpha_j)) - (1 - \alpha_i)(1 - \alpha_j) \right)_{i,j=1,...,d}.$$ 

Further,

$$\lim_{n \rightarrow \infty} \sqrt{cn^\beta} \left( \frac{\lfloor cn^\beta \rfloor (1 - \alpha_i) - 1 - l_n}{cn^\beta} - 1 + F_{n,i}(x_i + F_i^{-1}(\alpha_i)) \right) = \lim_{n \rightarrow \infty} \sqrt{cn^\beta} \left( F_{n,i}(x_i + F_i^{-1}(\alpha_i)) - \alpha_i \right)$$

$$= \lim_{n \rightarrow \infty} \sqrt{cn^\beta} \left( F_i(x_i + F_i^{-1}(\alpha_i)) - \alpha_i \right) = f_i(F_i^{-1}(\alpha_i)) y_i$$

as $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{cn^\beta}} = 0$ and $\lim_{n \rightarrow \infty} \sqrt{cn^\beta} (F_i(x_i + F_i^{-1}(\alpha_i)) - F_{n,i}(x_i + F_i^{-1}(\alpha_i))) = 0$ for $\beta \in (2\gamma, \max_{i=1,...,d} \beta_i^*)$, for every $i = 1, ..., d$. Therefore,

$$\mathbb{P} \left( \sum_{j=1}^{\lfloor cn^\beta \rfloor} 1_{\{W_{n,j}^{(i)} > x_i\}} \leq \lfloor cn^\beta \rfloor (1 - \alpha_i) - 1 - l_n, \ldots, \sum_{j=1}^{\lfloor cn^\beta \rfloor} 1_{\{W_{n,d}^{(i)} > x_i\}} \leq \lfloor cn^\beta \rfloor (1 - \alpha_d) - 1 - l_n \right)$$

$$= \mathbb{P} \left( \sqrt{\lfloor cn^\beta \rfloor} \left( \frac{1}{cn^\beta} \sum_{j=1}^{\lfloor cn^\beta \rfloor} 1_{\{W_{n,j}^{(i)} > x_i\}} - 1 + F_{n,1}(x_1 + F_1^{-1}(\alpha_1)) \right) \right)$$

$$\leq \sqrt{\lfloor cn^\beta \rfloor} \left( \frac{\lfloor cn^\beta \rfloor (1 - \alpha_1) - 1 - l_n}{cn^\beta} - 1 + F_{n,1}(x_1 + F_1^{-1}(\alpha_1)) \right),$$

$$\ldots, \sqrt{\lfloor cn^\beta \rfloor} \left( \frac{1}{cn^\beta} \sum_{j=1}^{\lfloor cn^\beta \rfloor} 1_{\{W_{n,d}^{(i)} > x_i\}} - 1 + F_{n,d}(x_d + F_d^{-1}(\alpha_d)) \right)$$

$$\leq \sqrt{\lfloor cn^\beta \rfloor} \left( \frac{\lfloor cn^\beta \rfloor (1 - \alpha_d) - 1 - l_n}{cn^\beta} - 1 + F_{n,d}(x_d + F_d^{-1}(\alpha_d)) \right)$$

$$\rightarrow \mathbb{P} \left( Y_1 \leq f_1(F_1^{-1}(\alpha_1)) y_1, \ldots, Y_d \leq f_d(F_d^{-1}(\alpha_d)) y_d \right),$$

as $n \rightarrow \infty$, where $(Y_1, ..., Y_d) \sim N(0, V)$. The same convergence holds for

$$\sum_{j=1}^{\lfloor cn^\beta \rfloor} 1_{\{W_{n,j}^{(i)} > x_i\}} \leq \lfloor cn^\beta \rfloor (1 - \alpha_i) + l_n \right)_{i=1,...,d}.$$
Therefore, using (21) we conclude that

\[
\mathbb{P}\left( q_{P,\alpha}(\bar{\mathbf{W}}_{n,i}^{(1)}, \ldots, \bar{\mathbf{W}}_{n,i}^{(\lceil cn^3 \rceil)}) \right) \leq x_1, \ldots, (q_{P,\alpha}(\bar{\mathbf{W}}_{n,i}^{(1)}, \ldots, \bar{\mathbf{W}}_{n,i}^{(\lceil cn^3 \rceil)}))_d \leq x_d
\]

\[
= \mathbb{P}\left( \sqrt{cn^3}(q_{P,\alpha}(\bar{\mathbf{W}}_{n,i}^{(1)}, \ldots, \bar{\mathbf{W}}_{n,i}^{(\lceil cn^3 \rceil)}))_1 \leq y_1, \ldots, (q_{P,\alpha}(\bar{\mathbf{W}}_{n,i}^{(1)}, \ldots, \sqrt{cn^3}\bar{\mathbf{W}}_{n,i}^{(\lceil cn^3 \rceil)}))_d \leq y_d \right)
\]

\[
\rightarrow \mathbb{P}\left( \nu_1 \leq f_1^{-1}(\alpha_1))y_1, \ldots, y_d \leq f_d^{-1}(\alpha_d)y_d \right),
\]

as \( n \to \infty \), for every \( y_1, \ldots, y_d \in \mathbb{R} \). Finally, using the change of variable \( w_i = y_i f_i^{-1}(\alpha_i) \), we obtain that

\[
\sqrt{cn^3}(f_i^{-1}(\alpha_i))q_{P,\alpha}(\bar{\mathbf{W}}_{n,i}^{(1)}, \ldots, \bar{\mathbf{W}}_{n,i}^{(\lceil cn^3 \rceil)})) \xrightarrow{d} N(0, V),
\]

as \( n \to \infty \).

**Proof of Corollary 3.9** The result follows from Theorem 3.7.

**Proof of Proposition 3.10** The result follows from Theorem 3.7 and Corollary 2.4 in [5].

**Proofs of Section 3.2**

**Proof of Lemma 3.11** We slightly extend the arguments of the proof of Lemma 2.1 in [30]. Assume that the claim does not hold and without loss of generality \( \|x_j - z\|_\mathbb{X} \leq r \) for \( j = 1, \ldots, \lfloor (1 - \nu)k \rfloor + 1 \). Denote by \( DF(x^*, \frac{z - x^*}{\|z - x^*\|_\mathbb{X}}) \) the directional derivative of \( F(y) = \sum_{i=1}^{k} \|x_i - y\|_\mathbb{X} + \langle u, x_i - y \rangle \) in direction \( \frac{z - x^*}{\|z - x^*\|_\mathbb{X}} \) at point \( x^* \). Then, we have

\[
DF\left( x^*, \frac{z - x^*}{\|z - x^*\|_\mathbb{X}} \right) = - \sum_{j: x_j \neq x^*}^{k} \frac{\langle g_j^*, z - x^* \rangle}{\|z - x^*\|_\mathbb{X}} + \sum_{i=1}^{k} 1_{\{x_i = x^*\}} - k \langle u, \frac{z - x^*}{\|z - x^*\|_\mathbb{X}} \rangle
\]

where \( g_j^* \) is a subdifferential of the norm \( \| \cdot \|_\mathbb{X} \) evaluated at point \( x_j - x^* \). By definition of \( x^* \), \( DF(x^*, v) \geq 0 \) for any \( v \in \mathbb{X} \). Since

\[
-k \langle u, \frac{z - x^*}{\|z - x^*\|_\mathbb{X}} \rangle \leq k\|u\|_{\mathbb{X}^*},
\]

we obtain

\[
DF\left( x^*, \frac{z - x^*}{\|z - x^*\|_\mathbb{X}} \right) < -(1 - \nu)k \left( 1 - \frac{2}{C_\nu} \right) + \nu k + \|u\|_{\mathbb{X}^*} k
\]

which is \( \leq 0 \) whenever \( C_\nu \geq \frac{2(1 - \nu)}{1 - 2\nu - \|u\|_{\mathbb{X}^*}} \), which is a contradiction.

**Proof of Proposition 3.12** This follows from the same arguments as the ones used in Theorem 3.1 in [30], using Lemma 3.11 in place of Lemma 2.1 in [30].
Proof of Theorem 3.13. The convergence in probability follows from Theorem 3.12. Further, it is possible to see that the quantile function possesses an equivariance property under a certain class of affine transformations, in particular we have that \( q_u(c(x_1,\ldots,x_k) - s) \) for every \( c \geq 0 \) and \( x_1,\ldots,x_k, s \in \mathbb{R} \). Thus,
\[
q_u\left(a_{\lceil n/k \rceil}(Z_{\lceil n/k \rceil}^{(1)},\ldots,Z_{\lceil n/k \rceil}^{(k)} - z)\right) = a_{\lceil n/k \rceil}(q_u((Z_{\lceil n/k \rceil}^{(1)},\ldots,Z_{\lceil n/k \rceil}^{(k)}) - z)) = a_{\lceil n/k \rceil}(\theta_{u,n,k} - z).
\]
If \( Y \) is continuous then \( P(Y_1,\ldots,Y_k \in \mathbb{X}^k \setminus \chi_k) = 0 \). Further, if \( \mathbb{X} \) is smooth and \( \|u\|_{\mathbb{X}^*} \notin \{1 - \frac{2}{j}, j = 1,\ldots,\lceil \frac{1}{2} \rceil\} \), then by Theorem 4.2 we obtain that \( q_u \) is continuous on the whole \( \mathbb{X}^k \). Observe that when \( \mathbb{X} = \mathbb{R} \) we have that \( q_u \) reduces to \( q_0 \) and that \( q_0 \) is continuous on \( \mathbb{R}^k \) for any value of \( \alpha \) by Lemma 4.2. Therefore, by the continuous mapping theorem we obtain the stated convergence in distribution.

Proof of Proposition 3.14. This follows from Theorem 3.12.

Proof of Lemma 3.15. We prove the case of \( p \geq 1 \), because the case \( p = 0 \) is trivial. A necessary and sufficient condition for \( x^* \) to be a geometric quantile is that the directional derivative of \( F(y) = \sum_{i=1}^k \|x_i - y\|_{\mathbb{X}} + \langle u,x_i - y \rangle \) evaluated at \( x^* \) in direction \( z \in \mathbb{X} \) is non-negative for any \( z \in \mathbb{X} \). Denote by \( DF(x^*,z) \) the directional derivative of \( F(y) \) in direction \( z \) at point \( x^* \). Without loss of generality we consider any \( z \in \mathbb{X} \) such that \( \|z\|_{\mathbb{X}} = 1 \). Then, we have
\[
DF(x^*,z) = - \sum_{j:x_j \neq x^*} g^*_j \langle z, j \rangle + \langle u,z \rangle \geq 0.
\]
Consider \( \bar{x}_1,\ldots,\bar{x}_k \in \mathbb{X} \) as in the statement. We need to show that, for every \( z \in \mathbb{X} \) such that \( \|z\|_{\mathbb{X}} = 1 \),
\[
DF(x^*,z) = - \sum_{j:x_j \neq x^*} g^*_j \langle z, j \rangle + \sum_{i=1}^k 1_{\{x_i = x^*\}} - \langle v,z \rangle \geq 0 \tag{22}
\]
for \( v \in \mathbb{X}^* \) as in the statement. So consider \( v \) given by (11), which can be rewritten
\[
v = \frac{u - v}{2p} \sum_{i=1}^k 1_{\{x_i = x^*\}} - \frac{1}{k} \sum_{j:x_j \neq x^*} g^*_j.
\]
Using \( DF(x^*,z) \geq 0 \) and \( DF(x^*, -z) \geq 0 \), we get
\[
- \sum_{i=1}^k 1_{\{x_i = x^*\}} - \sum_{j:x_j \neq x^*} \langle g^*_j, z \rangle \leq \langle u,v \rangle \leq \sum_{i=1}^k 1_{\{x_i = x^*\}} - \sum_{j:x_j \neq x^*} \langle g^*_j, z \rangle
\]
and
\[
- \sum_{i=1}^k 1_{\{x_i = x^*\}} - \sum_{j:x_j \neq x^*} \langle g^*_j, z \rangle + \sum_{j:x_j \neq x^*} \langle g^*_j, z \rangle - \frac{k(u,v,z)}{2p} \sum_{i=1}^k 1_{\{x_i = x^*\}} \leq \langle u - v, z \rangle
\]
\[
\leq \sum_{i=1}^{k} 1_{\{x_i=x^*\}} - \sum_{j:x_j\neq x^*}^{p} \langle g^*_j, z \rangle + \sum_{j:x_j\neq x^*}^{p} \langle g^*_j, z \rangle - \frac{k}{2p} \sum_{i=1}^{k} 1_{\{x_i=x^*\}},
\]

which leads to
\[
\left(k+\frac{k}{2p} \sum_{i=1}^{k} 1_{\{x_i=x^*\}} \right) \| u - v \|_{X^*} \leq \sum_{i=1}^{k} 1_{\{x_i=x^*\}} + \sum_{i=1}^{k} 1_{\{x_i\neq x^*\}} + \sum_{i=1}^{k} 1_{\{x_i\neq x^*\}} \leq k \sum_{i=1}^{k} 1_{\{x_i=x^*\}} + 2p
\]

where we used \(\sum_{i=1}^{k} 1_{\{x_i=x^*\}} + \sum_{i=1}^{k} 1_{\{x_i\neq x^*\}} = \sum_{i=1}^{k} 1_{\{x_i=x^*\}} + \sum_{i=1}^{k} 1_{\{x_i\neq x^*\}}\) and \(\sum_{i=1}^{k} 1_{\{x_i\neq x^*\}} \leq p\). Therefore, we obtain
\[
\| u - v \|_{X^*} \leq \frac{2p}{k}.
\]

It remains to show that (22) is satisfied. However, this simply follows from \(\frac{k(\|u-v\|_{X^*})}{2p} \leq 1\).  

Proof of Theorem 3.17

In order to prove Theorem 3.17 we need to introduce and prove various auxiliary results. In this section in order to lighten the notation we define \(X_{n,(n)} := W_{n,k_n,(n)}\) and \(X_n := W_{n,k_n}\), and we do not explicitly specify the space in the norm notation, that is if \(x \in Z_n\) then we write \(\|x\|\) for \(\|x\|_{Z_n}\) and similarly if \(x \in X\).

Lemma 4.23. Let \(X\) be a separable Hilbert space. Under Assumptions 3.3 and 3.16, for each \(A > 0\) there exist \(c_A, C_A \in (0, \infty)\), such that for all \(n\) large enough we have \(c_A \|h\|^2 \leq J_n(z(h, h)) \leq C_A \|h\|^2\) for any \(z, h \in Z_n\) with \(\|z\| \leq A\).

Proof. We first show that \(X_{n,(n)} \overset{d}{\to} Y\). We have that \(X_{n,(n)} = X_{n,(n)} - X_n + X_n\). Since \(X_n \overset{d}{\to} Y\), it remains to show that \(X_{n,(n)} - X_n \overset{p}{\to} 0\). Fix \(\varepsilon > 0\). Since \(X_n \overset{d}{\to} Y\) the sequence \(X_n\) is tight and so for every \(\delta > 0\) there exists \(M_\delta > 0\) such that \(\sup_{n \in \mathbb{N}} P(\|X_n\| > M_\delta) < \delta/4\). Further, there exists an \(m' \in \mathbb{N}\) large enough such that \(P(\|Y - Y_{(m')}\| > \varepsilon, \|Y\| \leq M_\delta) < \delta/4\).

By the continuous mapping theorem (\(\|X_n - X_{n,(m)}\|, \|X_n\| \to (\|Y - Y_{(m')}\|, \|Y\|)\)) for every \(m \in \mathbb{N}\). Thus, there exists an \(n' \in \mathbb{N}\) such that \(P(\|X_n - X_{n,(m')}\| > \varepsilon, \|X_n\| \leq M_\delta) < \delta/2\) for any \(n \geq n'\). Using that for any \(m_1 \geq m_2\) we have \(P(\|X_n - X_{n,(m_2)}\| \leq \|X_n - X_{n,(m_1)}\|, \|X_n\| \leq M_\delta) < \delta/2\) for any \(n \geq \max(n', m')\). Hence, \(P(\|X_n - X_{n,(m')}\| > \varepsilon, \|X_n\| \leq M_\delta) + P(\|X_n - X_{n,(m_2)}\| > \varepsilon, \|X_n\| > M_\delta) < \delta/2 + \delta/4 < \delta\) for any \(n \geq \max(n', m')\). Thus, \(X_{n,(n)} - X_n \overset{p}{\to} 0\).

From the proof of Proposition 2.1 in [7] we have that there exist two orthogonal elements \(v_1, v_2 \in X\) and a \(K > 0\) large enough such that \(Var(\langle v, Y 1_{\|Y\| \leq K} \rangle) \geq c\) for some \(c > 0\), where \(v = v_1 + v_2\). Wlog \(\|v_1\| = \|v_2\| = 1\). Hence, any \(v \in span(v_1, v_2)\) with \(\|v\| = 1\) can be written as \(v = \cos(t)v_1 + \sin(t)v_2\) for some \(t \in [0, 2\pi]\). Since \(X\) is not concentrated on a straight line we have that \(min_{v \in span(v_1, v_2), \|v\|} Var(\langle v, Y 1_{\|Y\| \leq K} \rangle) = \min_{t \in [0, 2\pi]} Var(\langle \cos(t)v_1 + \sin(t)v_2, X 1_{\|Y\| \leq K} \rangle) \geq c'\) for some \(c' > 0\). We remark that we are using the minimum instead of the infimum to stress that, since the function is continuous on compact spaces, the function attains its minimum.

Since \(X_{n,(n)} \overset{d}{\to} Y\) and \(Y\) has an atomless distribution, by the Portmanteau theorem (see Theorem 13.16 in [23])
\[
Var(\langle v, X_{n,(n)} 1_{\|X_{n,(n)}\| \leq K} \rangle) \geq Var(\langle v, X_{n,(n)} 1_{\|X_{n,(n)}\| \leq K} \rangle) - Var(\langle v, Y 1_{\|Y\| \leq K} \rangle) + c'
\]
Thus, there is an $N$ large enough such that for every $v \in \text{span}(v_1, v_2)$ with $\|v\| = 1$, namely for every $t \in [0, 2\pi]$, $\text{Var}(\langle v, X_n(n)\|X_n(n)\| \leq K \rangle) \geq c'/2$ for every $n > N$. From this inequality we easily obtain the result by the arguments of Proposition 2.1 in [7].

We notice that $U_n$ in the statement of Theorem 3.1 is a well-defined $\mathbb{Z}_n$-valued random variable and so $q^u_n(Z_{[n/k_n],(n)}^{(1)}, \ldots, Z_{[n/k_n],(n)}^{(k_n)})$ is also a well-defined random variable. Further, by Lemma 3.14 we have

$$T_{u_{(n),n,k_n}} = q^u_n(Z_{[n/k_n],(n)}^{(1)}, \ldots, Z_{[n/k_n],(n)}^{(k_n)}).$$

Thus, by the equivariance property of the geometric quantile we have

$$a_{[n/k_n]}(T_{u_{(n),n,k_n}} - \theta_{0,(n)}) = a_{[n/k_n]}(q^u_n(Z_{[n/k_n],(n)}^{(1)}, \ldots, Z_{[n/k_n],(n)}^{(k_n)}) - \theta_{0,(n)}) = q^u_n(a_{[n/k_n]}Z_{[n/k_n],(n)}^{(1)}, \ldots, a_{[n/k_n]}Z_{[n/k_n],(n)}^{(k_n)}) - \theta_{0,(n)}.$$

**Lemma 4.24.** Let $X$ be a separable Hilbert space. Let Assumptions 3.2 and 3.16 hold and let $C > 0$. Then there exist $b, B \in (0, \infty)$ such that for all sufficiently large $n$ and any $z, h \in \mathbb{Z}_n$ with $\|z - r_{[n/k_n],\mathbb{E}[U_n]}\| \leq C$, we have

$$\left\| E \left[ \frac{z - X_{n,(n)}}{\|z - X_{n,(n)}\|} - U_n \right] \right\| \geq b \|z - X_{n,(n)}\| \quad (23)$$

$$\sup_{\|h\| = 1} \left| J_{n,z}(h, v) - J_{n,r_{[n/k_n],\mathbb{E}[U_n]}(h, v)} \right| \leq B \|z - X_{n,(n)}\| \quad (24)$$

$$\left\| E \left[ \frac{z - X_{n,(n)}}{\|z - X_{n,(n)}\|} - U_n \right] - J_{n,r_{[n/k_n],\mathbb{E}[U_n]}(z - r_{[n/k_n],\mathbb{E}[U_n]}(z - r_{[n/k_n],\mathbb{E}[U_n]})) \right\| \leq B \|z - X_{n,(n)}\|^2. \quad (25)$$

**Proof.** Using Lemma 4.23, it follows from the same arguments of Lemma A.5 in [8].

In the following result and proof we generalize Theorem 3.1.1 in [9] and the arguments of its proof.

**Lemma 4.25.** Let $X$ be a separable Hilbert space and let Assumptions 3.2 and 3.16 hold. Then, there exists a constant $K_1$ s.t. $\|a_{[n/k_n]}(T_{u_{(n),n,k_n}} - \theta_{0,(n)})\| \leq K_1$ for all sufficiently large $n$ almost surely.
Proof. Since $X_n \xrightarrow{d} X$ by continuity of the norm we have $\|X_n\| \xrightarrow{d} \|X\|$, and since the probability measure of $X$ is non-atomic we have continuity of the distribution of $\|X\|$. Combining these we have, for any constant $c > 0$, $|\mathbb{P}(\|X_n\| > c) - \mathbb{P}(\|X\| > c)| \to 0$ as $n \to \infty$. Choose $\delta \in (0, (1 - \|u\|)/12)$ and $K_1$ such that

$$\mathbb{P}(\|X\| > C_2 K_1) \leq \delta/2,$$

where $C_2 = 4(3 + \|u\|(3 - 2\|u\| - \|u\|^2))^2$. Then, there is an $m$ large enough such that

$$\mathbb{P}(\|X_{n_1}\| > C_2 K_1) \leq \delta,$$

and hence

$$\mathbb{P}(\|X_{n_1}\| > C_2 K_1) \leq \delta,$$

for every $n > m$. We used the fact that for any random variable $Z$ in a separable Banach space $Z$ is tight and so for any $\varepsilon > 0$ there exists a constant $K$ such that $\mathbb{P}(\|Z\|_Z > K) < \varepsilon$. By Hoeffding’s inequality

$$\mathbb{P}\left(\frac{1}{k_n} \sum_{i=1}^{k_n} 1\{\|X_{n_i}\| > C_2 K_1\} - \mathbb{P}(\|X_{n_1}\| > C_2 K_1) \geq \delta\right) \leq 2 \exp(-k_n \delta^2),$$

and so by the Borel-Cantelli lemma, almost surely, we have

$$\frac{1}{k_n} \sum_{i=1}^{k_n} 1\{\|X_{n_i}\| > C_2 K_1\} \leq 2\delta \quad (26)$$

for all $n$ sufficiently large. Since $\|U_n - u_{(n)}\| \to 0$ almost surely, then almost surely we have

$$\|U_n - u_{(n)}\| < \frac{1 - \|u\|}{4} \quad (27)$$

for every $n$ sufficiently large.

If (26) holds, for any $z \in Z_n$

$$\left|\frac{1}{k_n} \sum_{i=1}^{k_n} (\|X_{n_i}\| - \|z\| - \|X_{n_i}\|) 1\{\|X_{n_i}\| > C_2 K_1\}\right| \leq 2\delta \|z\|.$$

Now, note that for any $z \in Z_n$

$$\|X_{n_i}\| - \|z\| - \|X_{n_i}\| > \|z\|\left(\frac{1 - \|u\|}{2}\right) \Leftrightarrow \|X_{n_i}\| < C_2 \|z\|.$$

If (26) holds, for any $z \in Z_n$ such that $\|z\| > K_1$ we have

$$\frac{1}{k_n} \sum_{i=1}^{k_n} (\|X_{n_i}\| - \|z\| - \|X_{n_i}\|) 1\{\|X_{n_i}\| \leq C_2 K_1\} > \|z\|\left(\frac{1 - \|u\|}{2}\right)(1 - 2\delta).$$

Since when (27) holds we have

$$\|z\|\left(\frac{1 - \|u\|}{2}\right)(1 - 2\delta) - 2\delta \|z\| - \|U_n\| \|z\| > 0,$$
we conclude that when (26) and (27) hold for any $z \in \mathbb{Z}_n$ such that $\|z\| > K_1$,
\[
\frac{1}{k_n} \sum_{i=1}^{k_n} \|X_{n,n}^{(i)} - z\| - \langle U_n, X_{n,n}^{(i)} - z \rangle > \frac{1}{k_n} \sum_{i=1}^{k_n} \|X_{n,n}^{(i)}\| - \langle U_n, X_{n,n}^{(i)} \rangle.
\]
This shows that the minimizer of $\frac{1}{k_n} \sum_{i=1}^{k_n} \|X_{n,n}^{(i)} - z\| - \langle U_n, X_{n,n}^{(i)} - z \rangle$ lies almost surely inside the ball of radius $K_1$ for all $n$ sufficiently large.

\[
\text{Proposition 4.26. Let } H \text{ be a separable Hilbert space and let Assumptions 3.2 and 6.10 hold. Let } \rho \in (0, 1/2) \text{ be s.t. } d_n/k_n^{1-2\rho} \to c \text{ as } n \to \infty \text{ for some } c > 0. \text{ Then, } \|a_{n/k_n}(T_{u_{n,n}}, n, k_n - \theta_{0,(n)}) - r_{n/k_n}, \mathbb{E}[U_n]\| = O(\delta_n) \text{ as } n \to \infty \text{ almost surely, where } \delta_n \sim \sqrt{\ln k_n/k_n^0}.
\]

\[
\text{Proof. For the sake of clarity in this proof we let } \hat{Y}_{n,U_n} := a_{n/k_n}(T_{u_{n,n}}, n, k_n - \theta_{0,(n)}). \text{ By Lemma 4.25 we deduce the existence of } C_3 > 0 \text{ satisfying } \|\hat{Y}_{n,U_n} - r_{n/k_n}, \mathbb{E}[U_n]\| \leq C_3 \text{ for all sufficiently large } n \text{ almost surely. Building on the arguments of the proof of Proposition A.6 in [8], we define } G_n = \{r_{n/k_n}, \mathbb{E}[U_n] + \sum_{j \leq d_n} \beta_j \phi_j : k_n^4 \beta_j \in \mathbb{Z} \text{ and } \sum_{j \leq d_n} \beta_j \phi_j \leq C_3\}. \text{ Define}
\]
\[
E_n = \left\{ \max_{z \in G_n} \left[ \frac{1}{k_n} \sum_{i=1}^{k_n} \left( \frac{z - X_{n,n}^{(i)}}{\|z - X_{n,n}^{(i)}\|} - U_n \right) - \mathbb{E} \left[ \frac{z - X_{n,n}^{(i)}}{\|z - X_{n,n}^{(i)}\|} - U_n \right] \right] \leq C_4 \delta_n \right\}.
\]
where $C_4 > 0$. By Bernstein’s inequality and using that $\|z - X_{n,n}^{(i)}\| - U_n\| \leq 2$ for all $z \in G_n$ and $n \in \mathbb{N}$, we have that $P(E_n^c) \leq (3C_3k_n^4d_n \exp(-k_nC_6^2\delta_n^2)/2$ for some $C_6 > 0$. Thus, using the definition of $\delta_n$, we can choose $C_5$ such that $\sum_{n=1}^{\infty} P(E_n^c) < \infty$. Hence,
\[
P(E_n \text{ occurs for all sufficiently large } n) = 1.
\]
By Assumption 5.16 and by the Markov inequality $P(\|z - X_{n,n}\| \leq k_n^{-2}) \leq M_nk_n^{-2} \leq C_6\delta_n^2$ for any $z \in G_n$ and all $n$ large enough. By Bernstein’s inequality there exists a $C_7 > 0$ s.t. $(F_n^c) \leq (3C_3k_n^4d_n \exp(-k_nC_7^2\delta_n^2)$ for all sufficiently large $n$, where
\[
F_n = \left\{ \max_{z \in G_n} \sum_{i=1}^{k_n} I(\|z - X_{n,n}^{(i)}\| \leq k_n^{-2}) \leq C_6k_n^2\delta_n^2 \right\}.
\]
Thus, for $C_7$ large enough
\[
P(F_n \text{ occurs for all sufficiently large } n) = 1.
\]
Let $r_{n/k_n}, \mathbb{E}[U_n]$ be a point in $G_n$ nearest to $\hat{Y}_{n,U_n}$. Then, $\|\hat{Y}_{n,U_n} - r_{n/k_n}, \mathbb{E}[U_n]\| \leq C_8d_n/k_n^4$ for $C_8 > 0$. Using (29) and
\[
\left\| \frac{\hat{Y}_{n,U_n} - X_{n,n}^{(i)}}{\hat{Y}_{n,U_n} - X_{n,n}} - \frac{r_{n/k_n}, \mathbb{E}[U_n] - X_{n,n}^{(i)}}{r_{n/k_n}, \mathbb{E}[U_n] - X_{n,n}} \right\| \leq \frac{2\|\hat{Y}_{n,U_n} - r_{n/k_n}, \mathbb{E}[U_n]\|}{\|r_{n/k_n}, \mathbb{E}[U_n] - X_{n,n}\|},
\]
we have
\[
\frac{1}{k_n} \sum_{i=1}^{k_n} \frac{r_{n/k_n}, \mathbb{E}[U_n] - X_{n,n}^{(i)}}{r_{n/k_n}, \mathbb{E}[U_n] - X_{n,n}} - U_n \leq \frac{1}{k_n} \sum_{i=1}^{k_n} \frac{\hat{Y}_{n,U_n} - X_{n,n}^{(i)}}{\hat{Y}_{n,U_n} - X_{n,n}} - U_n.
\]
Proof. Let \( H_n = \{ z \in G_n : \| z - r_{n/k_n, E[U_n]} \| \leq C_{11} \delta_n \} \). Let

\[
\Gamma_n(z, X_n^{(i)}) = \frac{r_{n/k_n, E[U_n]} - X_n^{(i)}}{\| r_{n/k_n, E[U_n]} - X_n^{(i)} \|} - \frac{z - X_n^{(i)}}{\| z - X_n^{(i)} \|} + E \left[ \frac{z - X_n^{(i)}}{\| z - X_n^{(i)} \|} - U_n \right]
\]

and

\[
\Delta_n(z) = E \left[ \frac{z - X_n^{(i)}}{\| z - X_n^{(i)} \|} - \frac{r_{n/k_n, E[U_n]} - X_n^{(i)}}{\| r_{n/k_n, E[U_n]} - X_n^{(i)} \|} - \frac{z - X_n^{(i)}}{\| r_{n/k_n, E[U_n]} - X_n^{(i)} \|} - \frac{z - X_n^{(i)}}{\| z - X_n^{(i)} \|} - U_n \right] + E \left[ \frac{z - X_n^{(i)}}{\| z - X_n^{(i)} \|} - U_n \right]
\]

for some \( C_9 > 0 \). From similar arguments as in the proof of Theorem 4.11 in [22] (see also the proof of Proposition A.6 in \[8\]), we have \( \| \sum_{i=1}^{k_n} \hat{Y}_{n,U_n} - X_n^{(i)} \| < 1 \). Thus, from (31) we get

\[
\left\| \sum_{i=1}^{k_n} \frac{r_{n/k_n, E[U_n]} - X_n^{(i)}}{\| r_{n/k_n, E[U_n]} - X_n^{(i)} \|} - k_n U_n \right\| \leq 3C_7 k_n \delta_n,
\]

for all sufficiently large \( n \) almost surely. Suppose that \( z \in G_n \) and \( \| z - r_{n/k_n, E[U_n]} \| > C_{10} \delta_n \) for some \( C_{10} > 0 \). Then, by (28) and the first inequality in Lemma 4.24 we have

\[
\left\| \sum_{i=1}^{k_n} \frac{z - X_n^{(i)}}{\| z - X_n^{(i)} \|} - k_n U_n \right\| \geq (C_{10} b^2 - C_4) k_n \delta_n \]

for all sufficiently large \( n \) almost surely. By choosing \( C_{10} \) s.t. \( C_{10} b^2 - C_4 > 4C_7 \) in view of (32), we have \( \| \hat{r}_{n/k_n, E[U_n]} - r_{n/k_n, E[U_n]} \| \leq C_{10} \delta_n \), for all sufficiently large \( n \) almost surely. This implies that \( \| \hat{Y}_{n,U_n} - r_{n/k_n, E[U_n]} \| \leq C_{11} \delta_n \) for all sufficiently large \( n \) almost surely.

\[\square\]
where $z \in \mathcal{Z}_n$. Using Assumption 3.16,

$$
\mathbb{E}[\|\Gamma_n(z, X^{(i)}_n)\|^2] \leq 2\mathbb{E}\left[\left\| \frac{r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}}{||r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}} || z - X^{(i)}_{n, (n)} \right\|^2 \right]
$$

$$
+ 2\mathbb{E}\left[ \frac{r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}}{||r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}} \right] - \mathbb{E}\left[ \frac{z - X^{(i)}_{n, (n)}}{||z - X^{(i)}_{n, (n)}} \right] \leq C_{12}\|z - r_{[n/k_n], \mathbb{E}[U_n]}\|^2,
$$

for some $C_{12} > 0$. Hence, by Bernstein’s inequality there exists a constant $C_{13} > 0$ such that

$$
\max_{z \in \mathcal{H}_n} \left\| \frac{1}{k_n} \sum_{i=1}^{k_n} \Gamma_n(z, X^{(i)}_n) \right\| \leq C_{13} \delta_n^2 \quad (33)
$$

for all $n$ sufficiently large almost surely. Using the third inequality in Lemma 4.24 there exists a constant $C_{14} > 0$ such that $\|\Delta_n(z)\| \leq C_{14}\|z - r_{[n/k_n], \mathbb{E}[U_n]}\|^2$ for all $n$ large enough.

Using this result with (33), the equality $\mathbb{E}[U_n] = \mathbb{E}\left[ \frac{r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}}{||r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}} \right]$ and the definitions of $\Gamma_n$ and $\Delta_n$, we have

$$
\tilde{J}_{n, r_{[n/k_n], \mathbb{E}[U_n]}}(z - r_{[n/k_n], \mathbb{E}[U_n]}) = -\frac{1}{k_n} \sum_{i=1}^{k_n} \left( \frac{r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}}{||r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}} \right) + \tilde{R}_n(z)
$$

where $\max_{z \in \mathcal{H}_n} ||\tilde{R}_n(z)|| = O(\delta_n^2)$ as $n \to \infty$ almost surely. From Lemma 4.23 it follows that the operator norm of $\tilde{J}_{n, r_{[n/k_n], \mathbb{E}[U_n]}}$ is bounded away from 0 and $[\tilde{J}_{n, r_{[n/k_n], \mathbb{E}[U_n]}}]^{-1}$ is defined on the whole of $\mathcal{Z}_n$ for $n$ large enough. Thus, there exists $C_{15} > 0$ such that $\max_{z \in \mathcal{H}_n} ||[\tilde{J}_{n, r_{[n/k_n], \mathbb{E}[U_n]}}]^{-1}(\tilde{R}_n(z))|| \leq C_{15}\delta_n^2$ for all sufficiently large $n$ almost surely. Hence, setting $r_{[n/k_n], \mathbb{E}[U_n]}$, defined in the proof of Proposition 4.26 in place of $z$ and using (31) we obtain

$$
a_{[n/k_n]}(T_{[n,k_n]} - \theta_{0,(n)}) - r_{[n/k_n], \mathbb{E}[U_n]}
$$

$$
= -\frac{1}{k_n} \sum_{i=1}^{k_n} \left[ \tilde{J}_{n, r_{[n/k_n], \mathbb{E}[U_n]}} \right]^{-1} \left( \frac{r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}}{||r_{[n/k_n], \mathbb{E}[U_n]} - X^{(i)}_{n, (n)}} \right) - U_n + R_n
$$

where $\|R_n\| = O(\delta_n^2)$ as $n \to \infty$ almost surely. □

**Remark 4.28.** In the proof of Proposition A.6 in [5] the definition of $G_n$ required also that $n^4\beta_j$ was in $[-C_3, C_3]$, but this is clearly a typo of the authors.

We can finally prove Theorem 3.17.

**Proof of Theorem 3.17.** We notice that $U_n$ is a well-defined $\mathcal{Z}_n$-valued random variable and so $q_{U_n}(Z^{(1)}_{[n/k_n], (n)}, ..., Z^{(k_n)}_{[n/k_n], (n)})$ is also a well-defined random variable. Further, thanks to Lemma 3.15 we have that $T_{[n,k_n]} - \theta_{0,(n)} = q_{U_n}(Z^{(1)}_{[n/k_n], (n)}, ..., Z^{(k_n)}_{[n/k_n], (n)})$. Thus,

$$
a_{[n/k_n]}(T_{[n,k_n]} - \theta_{0,(n)}) - r_{[n/k_n], \mathbb{E}[U_n]}
$$

$$
= a_{[n/k_n]}(q_{U_n}(Z^{(1)}_{[n/k_n], (n)}, ..., Z^{(k_n)}_{[n/k_n], (n)}) - \theta_{0,(n)}) - r_{[n/k_n], \mathbb{E}[U_n]}
$$

$$
= q_{U_n}(a_{[n/k_n]}(Z^{(1)}_{[n/k_n], (n)} - \theta_{0,(n)}), ..., a_{[n/k_n]}(Z^{(k_n)}_{[n/k_n], (n)} - \theta_{0,(n)}) - r_{[n/k_n], \mathbb{E}[U_n]})
$$

and by applying Theorem 4.27 we obtain the result. □
The remaining proofs of Section 3.2

Proof of Theorem 3.18 It follows from Theorem 3.17 and the central limit theorem for triangular arrays in Hilbert spaces (see Corollary 7.8 in [1]).

Proof of Corollary 3.19 It follows from Theorem 3.18

Proof of Proposition 3.20 By Lemma 4.18 under Assumption 3.2

Proof of Lemma 4.2

Proof of Lemma 4.1 Let $\mathbf{x} \in \mathbb{R}^k$. Take the derivative of

$$
\sum_{i=1}^{k} |x_i - y| + u(x_i - y)
$$

with respect to $y$, with $y \neq x_i$ for every $i = 1, ..., k$, and set it equal zero, that is

$$(1-u)b_y - (1+u)(k-b_y) = 0,
$$

where $b_y = |\{i : x_i < y\}|$. The only solution of (34) is $b_y = k\alpha$, and this can only happen if $\alpha \in \{\frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}\}$. In this case the set of geometric quantiles is $[\bar{x}_{k\alpha}, \bar{x}_{k\alpha+1}]$.

When $\alpha \notin \{\frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}\}$ then (34) is never satisfied and thus no geometric quantile lies in $\mathbb{R} \setminus \{x_1, ..., x_k\}$. In particular, $(1-u)b_y - (1+u)(k-b_y)$ is an increasing function of $b_y$ and so there exists an $\bar{x}_i$ for some $i = 1, ..., k$ such that $(1-u)i - (1+u)(k-i) < 0$ and $(1-u)(i+1) - (1+u)(k-i-1) > 0$. Such $\bar{x}_i$ is the unique geometric quantile.

Finally to show equality with $q_\alpha(\mathbf{x})$, it is sufficient to see that $i$ is the integer in the interval $(k\alpha, k\alpha+1)$, which can be written as $[k\alpha]$ or equivalently as $[k\alpha+1]$.

Proof of Lemma 4.2 We first focus on $q_\alpha$. Let $\alpha \in (0,1) \setminus \{\frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}\}$. Consider any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$. By definition we have that $|q_\alpha(\mathbf{x}) - q_\alpha(\mathbf{y})| = |x_i - y_j|$ for some $i, j \in \{1, ..., k\}$. Without loss of generality we consider that $|x_i - y_j| = x_i - y_j$. We know that there are at least $k - [k\alpha]$ components of $\mathbf{x}$ that are greater than or equal to $x_i$. Let $G_x \subseteq \{1, ..., k\}$ be the set of the indices of such components, formally $G_x := \{l \in \{1, ..., k\} : x_l \geq q_\alpha(\mathbf{x})\}$. Similarly we know that there are at least $|k\alpha| + 1$ components of $\mathbf{y}$ that are less than or equal to $y_j$. Let $L_y := \{l \in \{1, ..., k\} : y_l \leq q_\alpha(\mathbf{y})\}$. Observe that $G_x \cap L_y \neq \emptyset$ because $G_x$ and $L_y$ contain $k - [k\alpha]$ and $[k\alpha]+1$ distinct elements of $\{1, ..., k\}$, respectively. Then, we have

$$x_i - y_j \leq x_l - y_l, \quad \forall l \in G_x \cap L_y,$$

and the result follows.

Similar arguments apply for the case of $\alpha \in \{\frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}\}$. In particular, we have that $|q_\alpha(\mathbf{x}) - q_\alpha(\mathbf{y})| = \frac{1}{2}|x_i + x_l - y_j - y_h|$ for some $i, l, j, h \in \{1, ..., k\}$ with $i \neq l$ and $j \neq h$. Without loss of generality we consider that $|x_i + x_l - y_j - y_h| = x_i + x_l - y_j - y_h$ and that $x_i \geq x_l$ and $y_j \geq y_h$. Consider the sets $G_x \cup \{l\}$ and $L_y \cup \{j\}$; they contain at least $k - [k\alpha] + 1$ and $[k\alpha]+1$ elements respectively. Hence it is easy to see that

$$x_i + x_l - y_j - y_h \leq x_l + x_l - y_j - y_h,$$

and the result follows.
distinct elements of \( \{1, \ldots, k\} \), respectively. Thus, we have that \( |(G_X \cup \{l\}) \cap (L_Y \cup \{j\})| \geq 2 \).

Since \( x_i + x_l \leq x_p + x_q \) for any \( p, q \in G_X \cup \{l\} \) with \( p \neq q \) and \( y_j + y_h \geq y_r + y_s \) for any \( r, s \in L_Y \cup \{j\} \) with \( r \neq s \), we have that

\[
\frac{1}{2}(x_i + x_l - y_j - y_h) \leq \frac{1}{2}(x_p + x_q - y_p - y_q) \leq \max(x_p - y_p, x_q - y_q),
\]

for every \( p, q \in (G_X \cup \{l\}) \cap (L_Y \cup \{j\}) \) with \( p \neq q \), from which we obtain the result.

The arguments for \( q_{\alpha}^y \) are similar. By definition we have that \( |q_{\alpha}^x(x) - q_{\alpha}^y(y)| = |x_i - y_j| \) for some \( i, j \in \{1, \ldots, k\} \). Without loss of generality we consider that \( |x_i - y_j| = |x_i - y_j| \). By definition we know that there are at least \( \lceil k(1 - \alpha) \rceil \) components of \( x \) that are greater than or equal to \( x_i \). Let \( \tilde{G}_x := \{ l \in \{1, \ldots, k\} : x_l \geq q_{\alpha}^y(x) \} \). Similarly we know that there are at least \( \lceil k\alpha \rceil \) components of \( y \) that are less than or equal to \( y_j \). Let \( \tilde{L}_y := \{ l \in \{1, \ldots, k\} : y_l \leq q_{\alpha}^y(y) \} \). Observe that \( \tilde{G}_x \cap \tilde{L}_y \neq \emptyset \) by the following argument. If \( \alpha \neq a/k \) where \( a = 1, \ldots, k \) then \( |\tilde{G}_x| + |\tilde{L}_y| = \lceil k(1 - \alpha) \rceil + \lceil k\alpha \rceil \geq k + 1 \). If \( \alpha = a/k \) for some \( a = 1, \ldots, k - 1 \) then there are at least two \( x_i \)'s that satisfy the condition

\[
|\{ j \in \{1, \ldots, k\} : x_j = x_i \}| \geq k\alpha \quad \text{and} \quad |\{ j \in \{1, \ldots, k\} : x_j \geq x_i \}| \geq k(1 - \alpha)
\]

and since in the definition of \( q \) we consider the smallest one, we have that \( |\tilde{G}_x| \geq k(1 - \alpha) + 1 \).

Since \( |\tilde{L}_y| \geq k\alpha \), we have that \( |\tilde{G}_x| + |\tilde{L}_y| \geq k + 1 \). Therefore, we have

\[
x_i - y_j \leq x_l - y_l, \quad \forall l \in G_X \cap L_Y,
\]

and the result follows.

\[\square\]

### Proofs of Section 4.1

**Proof of Theorem 4.3** By the axiom of choice every vector space has an Hamel basis. Moreover, we have that \( q_{\text{Hamel},\alpha}(x_1, \ldots, x_k) \in \mathbb{X} \) because it is a finite linear combination of elements of \( \mathbb{X} \). Thus, we have existence. Uniqueness follows from the definition of \( q_\alpha \). Let \( I_{x_1, \ldots, x_k} := \{ l \in I : x_i^{(l)} \neq 0 \} \) for some \( i = 1, \ldots, k \). Consider any \( x_1, \ldots, x_k, z_1, \ldots, z_k \in \mathbb{X} \) and let \( I' = I_{x_1, \ldots, x_k} \cup I_{z_1, \ldots, z_k} \). Then, by Lemma 4.2 we have

\[
\left\| \sum_{l \in I} q_{\alpha_l}^{(l)}(x_1^{(l)}, \ldots, x_k^{(l)}) b_l - \sum_{l \in I} q_{\alpha_l}^{(l)}(z_1^{(l)}, \ldots, z_k^{(l)}) b_l \right\|_{X,\text{Hamel}} \leq \sum_{l \in I'} \left| q_{\alpha_l}^{(l)}(x_1^{(l)}, \ldots, x_k^{(l)}) - q_{\alpha_l}^{(l)}(z_1^{(l)}, \ldots, z_k^{(l)}) \right| = \sum_{i=1}^{k} \left| x_i^{(l)} - z_i^{(l)} \right| = \sum_{i=1}^{k} \left\| x_i - z_i \right\|_{X,\text{Hamel}}.
\]

\[\square\]

**Proof of Theorem 4.4** To prove existence we need to show that \( q_{S,\alpha}(x_1, \ldots, x_k) \in \mathbb{X} \). By completeness of \( \mathbb{X} \), it is enough to show that the sequence \( (\sum_{i=1}^{n} x_i^{(l)} d_l)_{n \in \mathbb{N}} \) is Cauchy. By Lemma 4.2 we have that \( |\bar{x}_i| \leq \sum_{i=1}^{k} |x_i^{(l)}| \). So, by Theorem 6.7 (c) in [19],

\[
\left\| \sum_{l=1}^{n} \bar{x}_l d_l \right\|_X \leq \mathcal{K} \left\| \sum_{l=1}^{n} \sum_{i=1}^{k} |x_i^{(l)}| d_l \right\|_X \leq \mathcal{K} \sum_{i=1}^{k} \left\| \sum_{l=1}^{n} x_i^{(l)} d_l \right\|_X \leq \mathcal{K}^2 \sum_{i=1}^{k} \left\| x_i \right\|_X \leq \mathcal{K}^2 \sum_{i=1}^{k} \left\| x_i \right\|_X,
\]

30
for every \( n \in \mathbb{N} \), where the last inequality follows from Theorem 6.7 (d) in [19]. We remark that \( K \) is a constant independent of \( n \) and of \( \bar{x}_1, x^{(l)}_1, \ldots, x^{(l)}_k \) for every \( l \in \mathbb{N} \). By the same arguments and by taking \( \bar{x}_1, \ldots, \bar{x}_m = 0 \) for \( m < n \), we have

\[
\| \sum_{l=1}^{m} \bar{x}_l d_l - \sum_{l=m+1}^{n} \bar{x}_l d_l \|_X = \| \sum_{l=m+1}^{n} \bar{x}_l d_l \|_X \leq K^2 \sum_{l=m+1}^{n} \| x^{(l)}_l d_l \|_X.
\]

Since by the definition of Schauder basis we have that \((\sum_{l=m}^{n} x^{(l)}_l d_l)_{n \in \mathbb{N}}\) is a Cauchy sequence, for every \( i = 1, \ldots, k \), we conclude that \((\sum_{l=m}^{n} x^{(l)}_l d_l)_{n \in \mathbb{N}}\) is Cauchy. Thus, we have existence. Uniqueness follows from the definition of \( q_\alpha \).

To prove continuity we consider any \( x_1, \ldots, x_k, z_1, \ldots, z_k \in X \). By Theorem 6.4 (f) in [19]

\[
\| \sum_{l=1}^{\infty} q_\alpha(x^{(l)}_1, \ldots, x^{(l)}_k) d_l - \sum_{l=1}^{\infty} q_\alpha(z^{(l)}_1, \ldots, z^{(l)}_k) d_l \|_X \leq \sup_{n \in \mathbb{N}} \| \sum_{l=1}^{n} (q_\alpha(x^{(l)}_1, \ldots, x^{(l)}_k) - q_\alpha(z^{(l)}_1, \ldots, z^{(l)}_k)) d_l \|_X.
\] (35)

Further, by Lemma 4.2 and by Theorem 6.7 (c) in [19], (35) is bounded by

\[
K \sup_{n \in \mathbb{N}} \left\| \sum_{l=1}^{n} \sum_{i=1}^{k} |x^{(l)}_i - z^{(l)}_i| d_l \right\|_X \leq K \sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^{k} \sum_{l=1}^{n} |x^{(l)}_i - z^{(l)}_i| d_l \right\|_X
\]

\[
\leq K \sum_{i=1}^{k} \sup_{n \in \mathbb{N}, |\lambda| \leq 1} \left\| \sum_{l=1}^{n} \lambda_l (x^{(l)}_i - z^{(l)}_i) d_l \right\|_X \leq K^2 \sum_{i=1}^{k} \| x_i - z_i \|_X,
\]

where the last equality follows from Theorem 6.4 (f) in [19].

Concerning the Hilbert space case, by Parseval’s identity and by Lemma 4.2

\[
\| \sum_{l \in I} q_\alpha(x^{(l)}_1, \ldots, x^{(l)}_k) d_l - \sum_{l \in I} q_\alpha(z^{(l)}_1, \ldots, z^{(l)}_k) d_l \|_X = \sum_{l \in I} |\bar{x}_l - \bar{z}_l|^2 \leq \sum_{l \in I} \sum_{i=1, \ldots, k} |x^{(l)}_i - z^{(l)}_i|^2 = \sum_{i=1, \ldots, k} \sum_{l \in I} |x^{(l)}_i - z^{(l)}_i|^2 = \sum_{i=1, \ldots, k} \| x_i - z_i \|_X^2,
\]

where \( I \) is the (possibly uncountable) index set of the orthonormal basis \((d_l)_{l \in I}\), although only countably many elements in \( \sum_{l \in I} x_l d_l \) are different from zero.

**Proof of Theorem 4.5** In any of the different cases \( X = L_p[0,1], \) with \( 1 \leq p \leq \infty, \) \( X = C[0,1], \) and \( X = C[0,1] \) uniqueness is ensured by the definition of \( q_\alpha \). For \( X = L_p[0,1], \) \( p \in [1, \infty), \) by Lemma 4.2

\[
\left( \int_0^1 |q_{P,\alpha}(f_1, \ldots, f_k)(x)|^p dx \right)^{1/p} \leq \left( \sum_{i=1}^{k} \int_0^1 |f_i(x)|^p dx \right)^{1/p} \leq \sum_{i=1}^{k} \| f_i \|_X,
\]

which proves existence. For \( L_\infty[0,1], \) by Lemma 4.2

\[
\| q_{P,\alpha}(f_1, \ldots, f_k) \|_\infty = \inf \{ C \geq 0 : |q_{P,\alpha}(f_1, \ldots, f_k)(x)| \leq C, \text{ for a.e. } x \in [0,1] \}
\]
\[ \leq \inf \{ C \geq 0 : \max_{i=1, \ldots, k} |f_i(x)| \leq C, \text{ for a.e. } x \in [0, 1] \} = \max_{i=1, \ldots, k} \|f_i\|_X. \]

Using similar arguments we obtain continuity in \( L_p[0, 1] \) with \( 1 \leq p \leq \infty \). For \( X = C[0, 1] \), by continuity of \( q_0 \) and of \( f_1, \ldots, f_k \) we obtain that \( q_{P, \alpha}(f_1, \ldots, f_k) \) is a continuous function. Moreover, by Lemma 4.2,

\[
\sup_{x \in [0, 1]} |q_{P, \alpha}(f_1, \ldots, f_k)(x) - q_{P, \alpha}(g_1, \ldots, g_k)(x)| \leq \max_{i=1, \ldots, k} \sup_{x \in [0, 1]} |f_i(x) - g_i(x)|.
\]

Using similar arguments we obtain continuity in \( D[0, 1] \).

**Proofs of Section 4.2**

**Proof of Lemma 4.2** Take \( w_0 \in W \) s.t. \( \|w_0\|_X = 1 \). Define the linear functional \( F \) on \( W \) by \( F(\lambda w_0) = \lambda \|w_0\|_X \), where \( \lambda \in \mathbb{R} \). This functional is linear, has norm 1 (precisely \( \sup_{z \in W} \frac{\|F(z)\|}{\|z\|_X} = 1 \)) and \( F(w_0) = \|w_0\|_X = 1 \). Thus, by the Hahn-Banach theorem \( F \) extends to \( X \), and its extension, which we also denote by \( F \), has norm 1. Now, let \( P : X \to X \) be defined by \( P(z) = F(z)w_0 \), where \( z \in X \). As \( P^2 = P \), \( P \) is a projection operator with range \( W \). Boundedness of \( F \) implies continuity of \( P \) and so \( \text{Ker}(P) = \text{Range}(I - P) \), where \( I : X \to X \) is the identity operator. By letting \( Y = \text{Ker}(P) \), we obtain the first statement.

Since \( \|P\| = \sup_{z \in X} \frac{\|P(z)\|_X}{\|z\|_X} = \sup_{z \in X} \frac{\|F(z)\|}{\|z\|_X} = 1 \), \( \|z\|_X \geq \|P(z)\|_X \) for every \( z \in X \). Further, since for every \( z \in X \) we have \( z = \lambda w_0 + z_Y \), for some \( \lambda \in \mathbb{R} \) and \( z_Y \in Y \), and \( P(\lambda w_0 + z_Y) = \lambda w_0 \), we obtain (17).

If \( X \) is strictly convex then every functional attains its supremum only at most one point \( z \) s.t. \( \|z\| = 1 \). Indeed, if there are two points \( s \) and \( r \) in the unit circle for which a functional \( g \) attains its supremum then \( g(s) = g(r) = \alpha g(s) + (1 - \alpha)g(r) \) where \( 0 < \alpha < 1 \), but since \( X \) is convex we have \( \|\alpha s + (1 - \alpha)\| < 1 \). This leads to the existence of a point \( b = c(\alpha s + (1 - \alpha)r) \) where \( c > 1 \), s.t. \( \|b\|_X = 1 \) and \( g(b) > g(s) = g(r) \), which is a contradiction. For \( F \) such a point is \( w_0 \). Thus, we obtain that \( \|P(z)\|_X = \|F(z)\| < \|z\|_X \) for every \( z \notin W \), hence (18).

To obtain (19) and (20) is sufficient to repeat the same arguments for \( y \) instead of \( x \) noticing that \( W \) is a subset of the complementary closed subspace of \( \{cy : c \in \mathbb{R}\} \).

**Proof of Theorem 4.9** The uniqueness of the geometric quantile under (I) follows directly from Theorem 2.17 in [22].

Now, let \( x_1, \ldots, x_k \) lie on a straight line \( W \). If \( x_1 = \ldots = x_k \) then it is easy to see that the quantile is unique. Otherwise, consider the function

\[ f(y) = \sum_{i=1}^k \|x_i - y\|_X + \langle u, x_i - y \rangle, \]

and for every \( h \in X \) with \( h \neq 0 \) let \( \Delta_h f(y) := f(y + h) - f(y) \). Following the proof of Theorem 2.17 in [22],

\[ \Delta_h^2 f(y) = \sum_{i=1}^k \|x_i - y + 2h\|_X - 2\|x_i - y + h\|_X + \|x_i - y\|_X \geq 0. \]

So, \( f \) is convex. In particular, \( \Delta_h^2 f(y) = 0 \) when \( h \) is linearly dependent on \( x_i - y \) for every \( i = 1, \ldots, k \), which can only happen if \( h \) and \( y \) lie on the line passing through \( x_1, \ldots, x_k \), which
we call $W$. So, $f$ is strictly convex for every $y \notin W$. Assume that the set of geometric quantiles is non-empty. If one of the geometric quantiles, which we denote by $y^*$, does not lie in $W$ then it is unique because $f$ is strictly increasing in any direction $h \in X$. This can also be seen by observing that $g_i(t) = \|x_i - y^* + th\|_X + \langle u, x_i - y^* + th \rangle$ is a strictly convex function on $\mathbb{R}$ if $x_i - y^*$ and $h$ are linearly independent, while it is just convex otherwise (see Lemma 2.14 in [22]), where $i = 1, ..., k$ and $h \in X$. Since for every $h \in X$ there is always at least one $i \in \{1, ..., k\}$ for which $x_i - y^*$ and $h$ are linearly independent, we have that $g_i(t) = \sum_{i=1}^k g_i(t)$ is a strictly convex function on $\mathbb{R}$ for every $h \in X$. Therefore, if there exists an element of the set of geometric quantiles in $X \setminus W$, then it is unique.

Consider the case that the set of geometric quantiles is a subset of $W$. By Lemma 4.7 we have

$$1 = \|e\|_X = \sup_{z \in X} \frac{|\langle e, z \rangle|}{\|z\|_X} = \sup_{z \in W} \frac{|\langle e, z \rangle|}{\|z\|_X} = \sup_{t \in \mathbb{R}} \frac{|t||\langle e, r \rangle|}{\|r\|_X} = \frac{|\langle e, r \rangle|}{\|r\|_X}$$

for any $r \in W$, which implies that $|\langle e, r \rangle| = \|r\|_X$ for any $r \in W$. Since $x_i - y^* \in W$ for any geometric quantile $y^*$, we have $\langle v, x_i - y^* \rangle = 0$. Further, since $x_i - y^* = \phi(x_i - y^*)$ for some $\phi \in W$ such that $\langle e, \phi \rangle = 1$ (i.e. $\|\phi\|_X = 1$) and some $x_1, ..., x_k, y^* \in \mathbb{R}$ and for every $i = 1, ..., k$, we have

$$\sum_{i=1}^k \|x_i - y^*\|_X + \langle u, x_i - y^* \rangle = \sum_{i=1}^k |x_i - y^*| + u(x_i - y^*).$$

By Lemma 4.11 this implies that in order to have uniqueness it is sufficient that $\alpha \in (0, 1) \setminus \{\frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}\}$. $\square$

**Proof of Theorem 4.10** Assume condition (I). Then, the uniqueness of the geometric median follows directly from Theorem 4.9 (see also Theorem 2.17 in [22]). Now, let $x_1, ..., x_k$ lie on a straight line $W$ and assume that the geometric median $y \notin W$. From Lemma 4.7 we have that $y = y_W + y_Y$, for some $y_W \in W$ and $y_Y \in Y$, and that $\sum_{i=1}^k \|x_i - y\|_X > \sum_{i=1}^k \|x_i - y_W\|_X$, which implies that $y$ cannot be the geometric median. Thus, the geometric median must lie in $W$ and in that case it is unique if and only if $k$ is odd. $\square$

**Proof of Theorem 4.12** The sufficiency of condition (I) is contained in Theorem 4.9.

Now, let $x_1, ..., x_k$ lie on a straight line and let $v \neq 0$. We show uniqueness by showing that the geometric quantile lies outside $W$ (see the proof of Theorem 4.9). Assume that we have more than one geometric quantile lying in $W$ and let $y^*$ be one of them. Then, we have

$$\sum_{i=1}^k \|x_i - y^*\|_X + \langle u, x_i - y^* \rangle = \sum_{i=1}^k |x_i - y^*| + u(x_i - y^*)$$

for some $x_i, ..., x_k, y^* \in \mathbb{R}$, which implies that if there is more than one geometric quantile in $W$, then there is a continuum of them. In particular, the set of geometric quantiles lying in $W$ is a closed connected interval whose extremes are $x_i, x_j$ for some $i, j \in \{1, ..., k\}$ and which does not contain any $x_l$ for $l = 1, ..., k$ with $l \neq i, j$. Consider $y^* \neq x_i, x_j$. By Assumption 4.11 the norm is Gâteaux differentiable in direction $z$ at $x_i \neq 0$ for some $i = 1, ..., k$, which implies that it is Gâteaux differentiable in direction $z$ at any point in $W$. Since $y^* \neq x_i, x_j$, we have that $x_i - y^* \neq 0$ for every $i = 1, ..., k$, and so the Gâteaux derivative of the function
$f$ defined in the proof of Theorem 4.9 evaluated at $y^*$ in direction $z$ is well defined:

$$
\lim_{t \to 0} \frac{f(y^* + tz) - f(y^*)}{t} = \sum_{i=1}^{k} \lim_{t \to 0} \frac{\|x_i - y^* - tx\|_X - \|x_i - y^*\|_X}{t} - k \langle u, z \rangle.
$$

By Lemma 4.7, we have that for every $x \in W$, $x \perp z$, which by Theorem 3.1 and Lemma 3.1 (see also eq. (1) in [20]) implies that

$$
\lim_{t \to 0^-} \frac{\|x + tz\|_X - \|x\|_X}{t} \leq 0 \leq \lim_{t \to 0^+} \frac{\|x + tz\|_X - \|x\|_X}{t}.
$$

However, since by Assumption 4.11 we have

$$
\lim_{t \to 0} \frac{f(y^* + tz) - f(y^*)}{t} = -k \langle u, z \rangle = -k \langle v, z \rangle,
$$

where the last equality follows from $z \in Y$. The same argument holds for $-z$ instead of $z$ because $\|x + t(-z)\|_X = \|-x + tz\|_X$ and $-x$ is an element of $W$. Thus, we have

$$
\lim_{t \to 0} \frac{f(y^* - tz) - f(y^*)}{t} = k \langle v, z \rangle.
$$

Since by assumption $\langle u, z \rangle \neq 0$ and so $\langle v, z \rangle \neq 0$, we conclude that the Gâteaux derivative of $f$ evaluated at $y^*$ in direction $z$ (or $-z$) is strictly negative. This implies that $y^*$ cannot be a geometric quantile and since

$$
\sum_{i=1}^{k} \|x_i - y^*\|_X + \langle u, x_i - y^* \rangle = \sum_{i=1}^{k} \|x_i - y'\|_X + \langle u, x_i - y' \rangle
$$

for any geometric quantile $y'$ lying in $W$, this implies that none of the geometric quantiles lying in $W$ can actually be a geometric quantile. Thus, if $v \neq 0$ geometric quantiles can only lie outside $W$ and, as we showed, there can be at most one geometric quantile outside $W$. Thus, we have uniqueness.

Now, let $x_1, \ldots, x_k$ lie on a straight line, $v = 0$, and $\alpha \in (0, 1) \setminus \left\{\frac{1}{k+1}, \ldots, \frac{k-1}{k} \right\}$. Then, by Lemma 4.7 we have that there is a unique geometric quantile on $W$.

Let us now investigate the other direction. Let $x_1, \ldots, x_k$ lie on a straight line $W$ and $v = 0$. Consider any $y \in X$. Then, by Lemma 4.7, there exist $y_Y \in Y$ and $y_W \in W$, where $X = Y + W$, such that $y = y_Y + y_W$. Consider the case of $y_Y \neq 0$. By Lemma 4.7, we have that $\|x_i - y\|_X > \|x_i - y_W\|_X$. Thus, we obtain

$$
\sum_{i=1}^{k} \|x_i - y\|_X + \langle u, x_i - y \rangle = \sum_{i=1}^{k} \|x_i - y\|_X + u \langle e, x_i - y_W \rangle > \sum_{i=1}^{k} \|x_i - y_W\|_X + u \langle e, x_i - y_W \rangle,
$$

which implies that any geometric quantile lies in $W$. Then, by Lemma 4.7, we have that when $x_1, \ldots, x_k$ lie on a straight line $W$ and $v = 0$, the geometric quantile is unique if and only if $\alpha \in (0, 1) \setminus \left\{\frac{1}{k+1}, \frac{2}{k+1}, \ldots, \frac{k-1}{k} \right\}$.

Thus, we have that if the geometric quantile is unique then $x_1, \ldots, x_k$ do not lie on a straight line, or if they lie on a straight line $W$ then if $y^* \notin W$ it means that $v \neq 0$ while if $y^* \in W$ then $v = 0$ and $\alpha \in (0, 1) \setminus \left\{\frac{1}{k+1}, \frac{2}{k+1}, \ldots, \frac{k-1}{k} \right\}$.

$$
\square
$$
Proof of Corollary 4.13. Since \( X \) is smooth, any point in \( X \) is Gâteaux differentiable along any direction. Further, if the points \( x_1, \ldots, x_k \) do not lie on a straight line or if the points lie on a straight line and \( v \neq 0 \), then \( u = 0e \) and so \( \|u\|_X = |u| \). In order to have uniqueness \( \alpha \notin \{ \frac{1}{k}, \frac{1}{k-1}, \ldots, k \} \), i.e. \( |u| \notin \{ 1 - \frac{2j}{k}, j = 1, \ldots, \frac{k}{2} \} \) since \( u = 2\alpha - 1 \).

Proof of Proposition 4.15. We only prove only (iii), since for (i) and (ii) the arguments are similar. For one direction, let \( x_1, \ldots, x_k \) lie on a straight line \( W \) and \( v = 0 \). Consider any \( y \in X \). Then, by Lemma 4.7 there are some \( y_Y \in Y \) and \( y_W \in W \), where \( X = Y + W \), such that \( y = y_Y + y_W \). Consider the case of \( y_Y \neq 0 \). By Lemma 4.7 we have that

\[
\sum_{i=1}^{k} \|x_i - y\|_X + \langle u, x_i - y \rangle = \sum_{i=1}^{k} \|x_i - y\|_X + u(e, x_i - y_W) > \sum_{i=1}^{k} \|x_i - y_W\|_X + u(e, x_i - y_W),
\]

which implies that any geometric quantile lies in \( W \).

For the other direction, assume that the set of geometric quantiles is a subset of \( W \). This implies that there is no geometric quantile outside \( W \), so neither condition (I) nor (II) can hold (see the proof of Theorem 4.12).

Proof of Lemma 4.16. The weak* convergence follows by adapting the arguments of Theorem 1 in [10] to quantiles. The convergence when \( X \) is finite-dimensional follows from adapting the arguments of Corollary 2.26 in [22] to quantiles.

Proof of Lemma 4.17. To simplify the notation we let \( \| \cdot \| \) indicate both \( \| \cdot \|_X \) and \( \| \cdot \|_{\ast} \). For every \( n \in \mathbb{N} \) and \( y, u \in \mathbb{R}^d \) with \( \|u\| < 1 \), we define

\[
f_n(y, u) = \int \|y - x\| - \|x\| \mu_n(dx) - \langle u, y \rangle
\]

and

\[
f(y, u) = \int \|y - x\| - \|x\| \mu(dx) - \langle u, y \rangle.
\]

For every \( n \in \mathbb{N} \), \( f_n \) and \( f \) are continuous functions and, for every \( y, u \in \mathbb{R}^d \),

\[
\lim_{n \to \infty} f_n(y, u) = f(y, u). \tag{36}
\]

Further, for every \( n \in \mathbb{N} \) and \( y, z, u, v \in \mathbb{R}^d \) with \( \|u\| < 1 \) and \( \|v\| < 1 \), we have

\[
|f_n(y, u) - f_n(z, v)| \leq 2 \|y - z\| + \min(\|y\|, \|z\|) \|u - v\|. \tag{37}
\]

Fix \( \varepsilon > 0 \). For \( t, s > 0 \) define \( B_t := \{ y \in \mathbb{R}^d : \|y\|_X \leq t \} \) and \( B'_t := \{ u \in \mathbb{R}^d : \|u\|_{\ast} \leq s \} \). It is possible to see that the \( u \)-geometric quantiles of \( \mu_n \) and of \( \mu \) for every \( n \in \mathbb{N} \) and \( u \in \mathbb{R}^d \) with \( \|u\| < 1 - \varepsilon \), lie in \( B_r \) for some \( r > 0 \).

Using continuity of \( f_n \), pointwise convergence (36), and equicontinuity (37) on \( B_r \times B'_{1-\varepsilon} \), by the Arzelà-Ascoli theorem we obtain the uniform convergence of \( f_n \) to \( f \) on \( B_r \times B'_{1-\varepsilon} \).
For every $u,v \in \mathbb{R}^d$ with $\|u\| < 1$ and $\|v\| < 1$, consider $\inf_{y \in B_r} f_n(y,u) - \inf_{z \in B_r} f_n(z,v)$. Wlog we assume that $\inf_{y \in B_r} f_n(y,u) \geq \inf_{z \in B_r} f_n(z,v)$ and denote by $y_{n,v}^*$ the $v$-geometric quantile of $\mu_n$, i.e. the value such that $f_n(y_{n,v}^*,v) = \inf_{z \in B_r} f_n(z,v)$. Then, we have

$$| \inf_{y \in B_r} f_n(y,u) - \inf_{z \in B_r} f_n(z,v)| \leq f_n(y_{n,v}^*,u) - f_n(y_{n,v}^*,v) = (v - u, y_{n,v}^*) \leq r \|v - u\|. \quad (38)$$

We denote by $y_u^*$ the $u$-geometric quantile of $\mu$. By continuity of the geometric quantiles (see Lemma 4.16), we have

$$\lim_{n \to \infty} y_{n,u}^* = y_u^*. \quad (39)$$

By (39) and uniform convergence of $f_n$ to $f$ we obtain

$$\lim_{n \to \infty} \inf_{y \in B_r} f_n(y,u) = \lim_{n \to \infty} f_n(y_{n,u}^*,u) = f(y_u^*,u) = \inf_{y \in B_r} f(y,u).$$

This shows pointwise convergence of the sequence of functions $u \mapsto \inf_{y \in B_r} f_n(y,u), n \in \mathbb{N}$.

These functions are continuous by (38). Thus, by the Arzelá-Ascoli theorem we obtain uniform convergence of $\inf_{y \in B_r} f_n(y,\cdot)$ to $\inf_{y \in B_r} f(y,\cdot)$ on $B_{1-\varepsilon}$.

Assume now that $\limsup_{n \to \infty} \sup_{\|u\| < 1-\varepsilon} \|y_{n,u}^* - y_u^*\| \geq 2\delta$ for some $\delta > 0$. There exists a subsequence $n_k \to \infty$ and points $u_{n_k}, y_{n_k,u_{n_k}}^*$, and $y_{u_{n_k}}^*$ such that

$$\|y_{n_k,u_{n_k}}^* - y_{u_{n_k}}^*\| \geq \delta, \quad (40)$$

for all $n_k$ large enough. Since $u_{n_k}, y_{n_k,u_{n_k}}^*$, and $y_{u_{n_k}}^*$ are sequences of points in compact sets they have converging subsequences, which wlog we denote again by $u_{n_k}, y_{n_k,u_{n_k}}^*$, and $y_{u_{n_k}}^*$, with limits $\bar{u}, \bar{y}^*$, and $\hat{y}^*$, respectively.

We have

$$|f_{n_k}(y_{n_k,u_{n_k}}^*,u_{n_k}) - f(\bar{y}^*,\bar{u})| \leq |f_{n_k}(y_{n_k,u_{n_k}}^*,u_{n_k}) - f(y_{n_k,u_{n_k}}^*,u_{n_k})| + |f(y_{n_k,u_{n_k}}^*,u_{n_k}) - f(\bar{y}^*,\bar{u})| \to 0, \quad \text{as } k \to \infty, \quad (41)$$

where for the first addendum we used uniform convergence of $f_n$ to $f$ and for the second one continuity of $f$. Moreover, by uniform convergence of $\inf_{y \in B_r} f_n(y,\cdot)$ to $\inf_{y \in B_r} f(y,\cdot)$ we have

$$\lim_{k \to \infty} \inf_{y \in B_r} f_{n_k}(y,u_{n_k}) - \inf_{z \in B_r} f(z,u_{n_k}) \to 0, \quad \text{as } k \to \infty,$$

and using (41) we obtain

$$|f(\bar{y}^*,\bar{u}) - \inf_{z \in B_r} f(z,u_{n_k})| \to 0, \quad \text{as } k \to \infty,$$

that is

$$\lim_{k \to \infty} f(y_{u_{n_k}}^*,u_{n_k}) = f(\bar{y}^*,\bar{u}). \quad (42)$$

Since the limit of $y_{u_{n_k}}^*$ is $\hat{y}^*$, by continuity of $f$ and (42) we have $f(\hat{y}^*,\bar{u}) = f(\hat{y}^*,\bar{u})$.

Moreover, by (38) applied to $f$ we have

$$\lim_{k \to \infty} f(y_{u_{n_k}}^*,u_{n_k}) = \inf_{y \in B_r} f(y,\bar{u}) = f(y_{u_{n_k}}^*,\bar{u}).$$
So $f(\hat{y}^*, \hat{u}) = f(\tilde{y}^*, \tilde{u}) = f(y^*_u, \tilde{u})$, and since $y^*_u$ is unique we conclude that $\tilde{y}^* = \hat{y}^* = y^*_u$. This is a contradiction because by (40) $\|\hat{y}^* - \tilde{y}^*\| \geq \delta$. Therefore, for any $\varepsilon > 0$, 
$$\sup_{\|u\|_{\mathcal{X}^*} < 1 - \varepsilon} \|y^*_{n,u} - y^*_u\| \to 0 \text{ as } n \to \infty.$$ 
Now, assume that $\mu$ is atomless. Consider any sequence of constants $c_n$ such that $c_n \to 0$ as $n \to \infty$, and any $v \in \mathbb{R}^d$ with $\|v\|_{\mathcal{X}^*} < 1$. For any $n$ large enough $1 > c_n \geq 0$ and we have 
\begin{align*}
\sup_{\|u-v\|_{\mathcal{X}^*} \leq c_n} \|y^*_{n,u} - y^*_v\| &\leq \sup_{\|u-v\|_{\mathcal{X}^*} \leq c_n} \|y^*_{n,u} - y^*_u\| + \sup_{\|u-v\|_{\mathcal{X}^*} \leq c_n} \|y^*_u - y^*_v\| \to 0, \text{ as } n \to \infty,
\end{align*}
where for the first addendum we used the result just shown and for the second one continuity of the $v$-geometric quantile with respect to the quantile parameter $v$ (see Theorem 3.1 in [8]).

**Proof of Lemma 4.19** It is equivalent to show that the set 
$$\{(x_1, ..., x_k) \in \mathbb{X}^k : x_1, ..., x_k \text{ lie on a straight line}\}$$
is closed. Consider first the case of $k = 3$ and consider any $x_1, x_2, x_3 \in \mathbb{X}$ lying on a straight line. Then, the area of the triangle with vertices $x_1, x_2,$ and $x_3$ is zero, and so by Heron’s formula 
$$A((x_1, x_2, x_3)) := s(s - a)(s - b)(s - c) = 0,$$ 
where $a = \|x_1 - x_2\|_{\mathbb{X}}, b = \|x_2 - x_3\|_{\mathbb{X}}, c = \|x_1 - x_3\|_{\mathbb{X}},$ and $s = (a + b + c)/2$. Thus, 
$$\{(x_1, x_2, x_3) \in \mathbb{X}^3 : x_1, ..., x_k \text{ lie on a straight line}\} = \{(x_1, x_2, x_3) \in \mathbb{X}^3 : A((x_1, x_2, x_3)) = 0\}.$$ 
Since $A$ is continuous, any limit point $(z_1, z_2, z_3)$ of $\{(x_1, x_2, x_3) \in \mathbb{X}^3 : A((x_1, x_2, x_3)) = 0\}$ is such that $A((z_1, z_2, z_3)) = 0$ and so $(z_1, z_2, z_3) \in \{(x_1, x_2, x_3) \in \mathbb{X}^3 : A((x_1, x_2, x_3)) = 0\}$. Therefore, 
$$\{(x_1, x_2, x_3) \in \mathbb{X}^3 : A((x_1, x_2, x_3)) = 0\}$$
is a closed set and so $\mathcal{X}_3$ is open. To show that $\mathcal{X}_k$ is open, define $B((x_1, ..., x_k))$ as the sum of the area of all triangles whose vertices belong to $\{x_1, ..., x_k\}$. Then 
$$\{(x_1, ..., x_k) \in \mathbb{X}^k : x_1, ..., x_k \text{ lie on a straight line}\} = \{(x_1, ..., x_k) \in \mathbb{X}^k : B((x_1, ..., x_k)) = 0\}$$
and since $B$ is continuous, we obtain that $\mathcal{X}_k$ is open.

**Proof of Theorem 4.20** We prove the first statement. Consider any sequence $((x_1^m, ..., x_k^m))_{m \in \mathbb{N}}$ of elements in $\mathcal{X}_k$ and any $(x_1, ..., x_k) \in \mathcal{X}_k$ such that $(x_1^m, ..., x_k^m) \to (x_1, ..., x_k)$ as $m \to \infty$. Without loss of generality we choose $(x_1^m, ..., x_k^m)$ such that $\sup_{m \in \mathbb{N}} \|x_i^m - x_k^m\| \leq \delta$, for some $\delta > 0$.

Let $\bar{c} := \max_{i,j=1,...,k} \|x_i - x_j\|_{\mathbb{X}}$. Using the fact that $\|u - y\|_{\mathbb{X}} < \|x - y\|_{\mathbb{X}}$ for every $x, y \in \mathbb{X}$, it is possible to see that $\mathcal{X}$ is reflexive and strictly convex, the geometric quantiles $y_*$ and $y_{*}^m$ exist and are unique. Then, similarly as in the proof of Lemma 2 (i) in [6], we have 
\begin{align*}
\left|\sum_{i=1}^{k} \|x_i^m - y_*^m\|_{\mathbb{X}} - \langle u, y_*^m \rangle \right| &= \left|\sum_{i=1}^{k} \|x_i^m - y_*^m\|_{\mathbb{X}} - \sum_{i=1}^{k} \|x_i - y_*^m\|_{\mathbb{X}} + \langle u, y_*^m \rangle \right| \\
&\leq 2 \sum_{i=1}^{k} \|x_i^m - x_i\|_{\mathbb{X}} \to 0, \text{ as } m \to \infty.
\end{align*}
Since for all $m$ we have that
\[ \sum_{i=1}^{k} \left\| x_{i}^{(m)} - y^{*} \right\|_{\mathbb{X}} - \left\langle u, y^{(m)} \right\rangle \leq \sum_{i=1}^{k} \left\| x_{i}^{(m)} - y^{*} \right\|_{\mathbb{X}} - \left\langle u, y^{*} \right\rangle, \]
then
\[ \limsup_{m \to \infty} \sum_{i=1}^{k} \left\| x_{i}^{(m)} - y^{*} \right\|_{\mathbb{X}} - \left\langle u, y^{(m)} \right\rangle \leq \sum_{i=1}^{k} \left\| x_{i} - y^{*} \right\|_{\mathbb{X}} - \left\langle u, y^{*} \right\rangle. \]
Hence, by (44) we obtain that
\[ \limsup_{m \to \infty} \sum_{i=1}^{k} \left\| x_{i} - y^{*} \right\|_{\mathbb{X}} - \left\langle u, y^{(m)} \right\rangle \leq \sum_{i=1}^{k} \left\| x_{i} - y^{*} \right\|_{\mathbb{X}} - \left\langle u, y^{*} \right\rangle, \] (45)
but since $y^{*}$ is the minimizer of the right hand side of (45) (see (3)) we conclude that
\[ \lim_{m \to \infty} \sum_{i=1}^{k} \left\| x_{i} - y^{*} \right\|_{\mathbb{X}} - \left\langle u, y^{(m)} \right\rangle = \sum_{i=1}^{k} \left\| x_{i} - y^{*} \right\|_{\mathbb{X}} - \left\langle u, y^{*} \right\rangle. \]

Since $\mathbb{X}$ is reflexive, $\mathbb{X}^*$ is also reflexive, which in turn implies that $\mathbb{X}^*$ is an Asplund space (see [12] and Theorem 8.26 in [11] for equivalent definitions of the Asplund space). Then, we can apply Theorem 3 in [3] and obtain that $y^{(m)} \to y^{*}$ in norm, hence the result.

The same arguments apply to the second statement. The additional assumptions ensure the uniqueness of the quantile. 

\begin{proof}[Proof of Theorem 4.21]
Consider any $x_1, \ldots, x_k, k \in \mathbb{N}$, in $\mathbb{X}$ and any $u \in \mathbb{X}$ with $\|u\|_{\mathbb{X}} < 1$. Observe that a geometric quantile $y^{*}$ satisfies
\[ \sum_{i=1}^{k} \frac{x_{i} - y^{*}}{\|x_{i} - y^{*}\|_{\mathbb{X}}} + ku = 0. \]
provided that $y^{*} \neq x_i$, $i = 1, \ldots, k$. Now, let $\tilde{x} := u + y^{*}$, then $u = \frac{\tilde{x} - y^{*}}{\|u\|_{\mathbb{X}} \|\tilde{x} - y^{*}\|_{\mathbb{X}}}$ and so
\[ \sum_{i=1}^{k} \frac{x_{i} - y^{*}}{\|x_{i} - y^{*}\|_{\mathbb{X}}} + k \frac{\tilde{x} - y^{*}}{\|u\|_{\mathbb{X}} \|\tilde{x} - y^{*}\|_{\mathbb{X}}} = 0 \]
\[ \Leftrightarrow y^{*} = \sum_{i=1}^{k} w_{i} x_{i} + \tilde{x} w_{k+1} \]
\[ \Leftrightarrow y^{*} = \frac{1}{1 - w_{k+1}} \left( \sum_{i=1}^{k} w_{i} x_{i} + u w_{k+1} \right), \]
where
\[ w_{i} = \frac{\|x_{i} - y^{*}\|_{\mathbb{X}}^{-1}}{\sum_{i=1}^{k} \|x_{i} - y^{*}\|_{\mathbb{X}}^{-1} + k \|u\|_{\mathbb{X}} \|\tilde{x} - y^{*}\|_{\mathbb{X}}^{-1}} = \frac{\|x_{i} - y^{*}\|_{\mathbb{X}}^{-1}}{\sum_{i=1}^{k} \|x_{i} - y^{*}\|_{\mathbb{X}}^{-1} + k}. \]
\end{proof}
for \( i = 1, \ldots, k \), and
\[
w_{k+1} = \frac{k\|u\|_\infty \|\hat{y} - y^*\|_\infty^{-1}}{\sum_{i=1}^k \|x_i - y^*\|^2_\infty + k\|u\|_\infty \|\hat{y} - y^*\|_\infty^{-1}} = \frac{k}{\sum_{i=1}^k \|x_i - y^*\|^2_\infty + k}
\]
and so \( \sum_{i=1}^{k+1} w_i = 1 \) and \( w_1, \ldots, w_{k+1} \geq 0 \). Observe that since \( y^* \neq x_i, i = 1, \ldots, k \) and \( \|y^*\|_\infty < C \), for some \( C \geq 0 \) depending on \( x_1, \ldots, x_k \), we have that \( w_1, \ldots, w_k \) are strictly positive and so \( w_{k+1} < 1 \). Now if \( y^* = x_i \), for say \( i = 1, \ldots, j \), then the quantile is still of the form \( \frac{1}{1-w_{k+1}} (\sum_{i=1}^k w_i x_i + u w_{k+1}) \) with \( \sum_{i=1}^{k+1} w_i = 1 \) and \( w_1, \ldots, w_{k+1} \geq 0 \), in particular \( y^* = \sum_{i=1}^j w_i x_i \).

Therefore, we have that the quantile is given by \( \frac{1}{1-w_{k+1}} (\sum_{i=1}^k w_i^* x_i + uu_{k+1}) \), where \( (w_1^*, \ldots, w_{k+1}^*) \) is obtained by
\[
\arg\min_{w \in T_{k+1}} \sum_{i=1}^k \| x_i - \frac{1}{1-w_{k+1}} \left( \sum_{l=1}^k w_l x_l + w_{k+1} u \right) \|_\infty \tag{46}
\]
To see that this minimization is well-defined observe that there is an \( \varepsilon > 0 \) depending on \( x_1, \ldots, x_k \) such that (46) is equal to
\[
\arg\min_{w \in T_{k+1}} \sum_{i=1}^k \left\| x_i - \frac{1}{1-w_{k+1}} \left( \sum_{l=1}^k w_l x_l + w_{k+1} u \right) \right\|_\infty + \langle u, x_i - \frac{1}{1-w_{k+1}} \left( \sum_{l=1}^k w_l x_l + w_{k+1} u \right) \rangle
\]
where \( T_{k+1} := \{ v \in [0, 1]^k \times [0, 1-\varepsilon] : v_1 + \cdots + v_{k+1} = 1 \} \). Indeed, consider any sequence \(( (x_1^{(m)}, \ldots, x_k^{(m)}) )_{m \in \mathbb{N}} \) of elements in \( \mathbb{R}^k \) and any \( (x_1, \ldots, x_k) \in \mathbb{R}^k \) such that \( (x_1^{(m)}, \ldots, x_k^{(m)}) \to (x_1, \ldots, x_k) \) as \( m \to \infty \). Without loss of generality suppose \( \max_{i=1,\ldots,k} \| x_i^{(m)} - x_i \|_\infty < \delta \) for some \( \delta > 0 \). Let \( \varepsilon := \max_{i,j=1,\ldots,k} \| x_i - x_j \|_\infty \). Then, any quantile lies in the ball of radius \( 2\varepsilon \) centred at one of the points, say \( x_1 \). Hence, we have that any quantile of \( (x_1^{(m)}, \ldots, x_k^{(m)}) \), for every \( m \in \mathbb{N} \), lies in the ball of radius \( 2\varepsilon + \delta \) centred at \( x_1 \). Thus, \( w_{k+1} \) and \( w_{k+1}^{(m)}, m \in \mathbb{N} \), are bounded by \( \frac{k}{k(2\varepsilon + \delta)} \) and so \( \varepsilon = 1 - \frac{1}{2e+\delta+1} > 0 \).

**Proof of Proposition 4.22.** Since the geometric quantile is given by
\[
\arg\min_{y \in \mathbb{R}} \sum_{i=1}^k \sum_{l=1}^\infty |x_i^{(l)} - y^{(l)}| + u^{(l)}(x_i^{(l)} - y^{(l)})
\]
and since \( u^{(l)}(x_i^{(l)} - y^{(l)}) < |x_i^{(l)} - y^{(l)}| \), we have that the geometric quantile is given by the component-wise quantile, i.e. \( y^* = (y^{*(l)})_{l \in \mathbb{N}} \) with \( y^{*(l)} \) defined by
\[
\arg\min_{y^{(l)} \in \mathbb{R}} \sum_{i=1}^k |x_i^{(l)} - y^{(l)}| + u^{(l)}(x_i^{(l)} - y^{(l)})
\]
Thus, we have existence of the geometric quantile. By Lemma 4.11 uniqueness follows if and only if \( |u^{(l)}| \notin \{ 1 - \frac{2j}{k}, j = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \} \), \( \forall l \in \mathbb{N} \). Given any \( x_1, \ldots, x_k \) and \( z_1, \ldots, z_k \) in \( \ell_1 \), by Lemma 4.12 we obtain
\[
\sum_{i=1}^k |Q_u^{(l)}(x_1, \ldots, x_k) - Q_u^{(l)}(z_1, \ldots, z_k)| \leq \sum_{i=1}^k \sum_{l=1}^\infty |x_i^{(l)} - z_i^{(l)}| = \sum_{i=1}^k \sum_{l=1}^\infty |x_i^{(l)} - z_i^{(l)}| = \sum_{i=1}^k \| x_i - z_i \|_{\ell_1}.
\]
\( \square \)
References

[1] Araújo A. and Gine E. The Central Limit Theorem for Real and Banach Valued Random Variables. Wiley, New York, 1980.

[2] C. D. Aliprantis and K. C. Border. Infinite Dimensional Analysis: A Hitchhiker’s Guide. Springer, 2006.

[3] E. Asplund. Fréchet differentiability of convex functions. Acta Mathematica, 121(none):31–47, 1968.

[4] R.F. Barber, E.J. Candes, A. Ramdas, and R.J. Tibshirani. Conformal prediction beyond exchangeability. ArXiv:2202.13415, 2022.

[5] V.I. Bogachev and A. F. Miftakovon. Weak convergence of finite-dimensional and infinite-dimensional distributions of random processes. Theory of Stochastic Processes, 21(1):31–47, 2016.

[6] B. Cadre. Convergent estimators for the $L_1$-median of Banach valued random variable. Statistics: A Journal of Theoretical and Applied Statistics, 35(4):509–521, 2001.

[7] H. Cardot, P. Céna, and P. Zitt. Efficient and fast estimation of the geometric median in Hilbert spaces with an averaged stochastic gradient algorithm. Bernoulli, 19(1):18–43, 2013.

[8] A. Chakraborty and P. Chaudhuri. The spatial distribution in infinite dimensional spaces and related quantiles and depths. Annals of Statistics, 42(3):1203–1231, 2014.

[9] P. Chaudhuri. On a geometric notion of quantiles for multivariate data. Journal of the American Statistical Association, 91(434):862–872, 1996.

[10] V. Chernozhukov, A. Galichon, M. Hallin, and M. Henry. Monge-Kantorovich depth, quantiles, ranks and signs. Annals of Statistics, 45(1):223 – 256, 2017.

[11] M. Fabian, P. Habala, P. Hajek, V. Montesinos Santalucia, J. Pelant, and V. Zizler. Functional Analysis and Infinite-Dimensional Geometry. Springer-Verlag, 2001.

[12] R. Fry and S. McManus. Smooth bump functions and the geometry of Banach spaces: A brief survey. Expositiones Mathematicae, 20(2):143–183, 2002.

[13] D. Gervini. Robust functional estimation using the median and spherical principal components. Biometrika, 95(3):587–600, 09 2008.

[14] P. Ghosal and B. Sen. Multivariate ranks and quantiles using optimal transport: Consistency, rates and nonparametric testing. Annals of Statistics, 50(2):1012 – 1037, 2022.

[15] Y. Gordon and D.R. Lewis. Absolutely summing operators and local unconditional structures. Acta Mathematica, 133(4):27–48, 1974.

[16] W.T. Gowers and B. Maurey. The unconditional basic sequence problem. Journal of the American Mathematical Society, 6(4):851–874, 1993.
M. Hallin, E. del Barrio, J. Cuesta-Albertos, and C. Matrán. Distribution and quantile functions, ranks and signs in dimension $d$: A measure transportation approach. *Annals of Statistics*, 49(2):1139 – 1165, 2021.

M. Hallin, D. Paindaveine, and M. Šiman. Multivariate quantiles and multiple-output regression quantiles: From L1 optimization to halfspace depth. *Annals of Statistics*, 38(2):635 – 669, 2010.

C. Heil. *A Basis Theory Primer*. Birkhauser Basel, 2011.

R. James. Orthogonality and linear functionals in normed linear spaces. *Trans. Amer. Math. Soc.*, 61:265–292, 1947.

M. R. Jerrum, L. G. Valiant, and V. V. Vazirani. Random generation of combinatorial structures from a uniform distribution. *Theoretical Computer Science*, 43:169–188, 1986.

J.H.B. Kemperman. The median of a finite measure on a Banach space. *Statistical data analysis based on the $L_1$-norm and related methods*, pages 217–230, 1987.

A. Klenke. *Probability Theory*. Springer-Verlag, 2006.

V.I. Koltchinskii. M-estimation, convexity and quantiles. *Annals of Statistics*, 25:435–477, 1997.

D. Konen and D. Paindaveine. Multivariate $\rho$-quantiles: A spatial approach. *Bernoulli*, 28(3):1912 – 1934, 2022.

P. Laforgue, G. Staerman, and S. Clémençon. Generalization bounds in the presence of outliers: a median-of-means study. *Proceedings of the 38th International Conference on Machine Learning*, 139:5937–5947, 18–24 Jul 2021.

G. Lecue and M. Lerasle. Robust machine learning by median-of-means: theory and practice. *Annals of Statistics*, 48(2):906–931, 2020.

M. Lerasle and R. I. Oliveira. Robust empirical mean estimators. *ArXiv:1112.3914*, 2011.

G. Lugosi and S. Mendelson. Risk minimization by Median-of-Means tournaments. *Journal of the European Mathematical Society*, 22(3):925–965, 2020.

S. Minsker. Geometric median and robust estimation in Banach spaces. *Bernoulli*, 21(4):2308 – 2335, 2015.

A. Nemirovski and D. Yudin. *Problem complexity and method efficiency in optimization*. John Wiley and Sons Inc., 1983.

K.M. Tan, L. Wang, and W.X. Zhou. High-dimensional quantile regression: Convolution smoothing and concave regularization. *ArXiv:2109.05640*, 2022.

M. Valadier. La multi-application medianes conditionnelles. *Z. Wahrscheinlichkeitstheorie verw Gebiete*, 67:279–282, 1984.

L. Wang, Y. Wu, and R. Li. Quantile regression for analyzing heterogeneity in ultra-high dimension. *Journal of the American Statistical Association*, 107(497):214–222, 2012.