BIRKHOFF-JAMES ORTHOGONALITY AND ITS POINTWISE SYMMETRY IN SOME FUNCTION SPACES

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Abstract. We study Birkhoff-James orthogonality and its pointwise symmetry in commutative $C^*$ algebras, i.e., the space of all continuous functions defined on a locally compact Hausdorff space which vanish at infinity. We use this characterization to obtain the characterization of Birkhoff-James orthogonality on $L_\infty$ space defined on any arbitrary measure space. We also do the same for the $L_p$ spaces for $1 \leq p < \infty$.

Introduction

In recent times, symmetry of Birkhoff-James orthogonality has been a topic of considerable interest \[1, 7, 8, 12, 13, 14, 21\]. It is now well known that the said symmetry plays an important role in the study of the geometry of Banach spaces. The present article aims to explore Birkhoff-James orthogonality and its pointwise symmetry in some function spaces. We have completed such a study for some well studied sequence spaces, namely $\ell_p$ for $1 \leq p \leq \infty$, $c$, $c_0$ and $c_{00}$ in [3]. Here we take the study one step further by doing the same for commutative $C^*$ algebras and $L_p(X)$ for $1 \leq p \leq \infty$ and any measure space $X$.

2020 Mathematics Subject Classification. Primary 46B20, Secondary 46E30, 46L05.

Key words and phrases. Birkhoff-James orthogonality; Smooth points; Left-symmetric points; Right-symmetric points; $L_p$ spaces; Commutative $C^*$ algebras; Ultrafilters.

The research of Babhrubahan Bose is funded by PMRF research fellowship under the supervision of Professor Apoorva Khare and Professor Gadadhar Misra.
Let us now establish the relevant notations and terminologies to be used throughout the article. Denote the scalar field $\mathbb{R}$ or $\mathbb{C}$ by $K$ and recall the sign function $\text{sgn} : K \to K$, given by

$$\text{sgn}(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Consider a normed linear space $X$ over $K$ and denote its continuous dual by $X^*$. Let $J(x)$ denote the collection of all support functionals of a non-zero vector $x \in X$, i.e.,

$$J(x) := \{ f \in X^* : \|f\| = 1, |f(x)| = \|x\| \}.$$  

(0.1)

A non-zero element $x \in X$ is said to be smooth if $J(x)$ is singleton.

Given $x, y \in X$, $x$ is said to be Birkhoff-James orthogonal to $y$ [2], denoted by $x \perp_B y$, if

$$\|x + \lambda y\| \geq \|x\|, \quad \text{for all } \lambda \in K.$$

James proved in [10] that $x \perp_B y$ if and only if $x = 0$ or there exists $f \in J(x)$ such that $f(y) = 0$. In the same article he also proved that a non-zero $x \in X$ is smooth if and only if Birkhoff-James orthogonality is right additive at $x$, i.e.,

$$x \perp_B y, \ x \perp_B z \Rightarrow x \perp_B (y + z), \ \text{for every } y, z \in X.$$

Birkhoff-James orthogonality is not symmetric in general, i.e., $x \perp_B y$ does not necessarily imply that $y \perp_B x$. In fact, James proved in [9] that Birkhoff-James orthogonality is symmetric in a normed linear space of dimension higher than 2 if and only if the space is an inner product space. However, the importance of studying the pointwise symmetry of Birkhoff-James orthogonality in describing the geometry of normed linear spaces has been illustrated in [5, Theorem 2.11], [20, Corollary 2.3.4]. Let us recall the following definition in this context from [19], which will play an important part in our present study.

**Definition 0.0.1.** An element $x$ of a normed linear space $X$ is said to be left-symmetric (resp. right-symmetric) if

$$x \perp_B y \Rightarrow y \perp_B x \ (\text{resp. } y \perp_B x \Rightarrow x \perp_B y),$$
for every $y \in X$.

Note that by the term pointwise symmetry of Birkhoff-James orthogonality, we refer to the left-symmetric and the right-symmetric points of a given normed linear space. The left-symmetric and the right-symmetric points of $\ell_p$ spaces where $1 \leq p \leq \infty$, $p \neq 2$, were characterized in [3]. Here we generalize these results in $L_p(X)$ for any measure space $X$ and $p \in [1, \infty] \setminus \{2\}$. For doing this generalization, we need to characterize Birkhoff-James orthogonality, smooth points, left symmetric points and right symmetric points in commutative $C^*$ algebras, i.e., $C_0(X)$, the space of all continuous functions vanishing at infinity defined on a locally compact Hausdorff space $X$. These characterizations in a given Banach space are important in understanding the geometry of the Banach space. We refer the readers to [1], [7], [8], [12], [13], [14], [21], [22], [23], [24], [25] for some prominent work in this direction.

In the first section we completely characterize Birkhoff-James orthogonality in commutative $C^*$ algebras, i.e., the space of all $\mathbb{K}$-valued continuous functions vanishing at infinity that are defined on a locally compact Hausdorff space $X$ and then characterize the left-symmetric and the right-symmetric points of the space.

In the second section, we use the results in the first section to completely characterize Birkhoff-James orthogonality, smoothness and pointwise symmetry of Birkhoff-James orthogonality in $L_\infty(X)$. It can be noted that we are establishing these results for an arbitrary measure space $X$ and in particular, we are not imposing any additional condition on $X$ such as finiteness or $\sigma$-finiteness of the measure. In the third and fourth sections we obtain the same characterizations for $L_1(X)$ and $L_p(X)$ spaces ($p \in (1, \infty) \setminus \{2\}$). Observe that the $p = 2$ case is trivial since $L_2(X)$ is a Hilbert space.

1. Birkhoff-James orthogonality in commutative $C^*$ algebras

The aim of this section is to obtain a necessary and sufficient condition for two elements in a commutative $C^*$ algebra to be
Birkhoff-James orthogonal. Using that characterization, we characterize the smooth points and also study the pointwise symmetry of Birkhoff-James orthogonality in these algebras. We use the famous result Gelfand and Naimark proved in [6], that any commutative $C^*$ algebra is isometrically $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff space $X$. Recall that $C_0(X)$ denotes the space of $K$-valued continuous maps $f$ on $X$ such that

$$\lim_{x \to \infty} f(x) = 0,$$

equipped with the supremum norm, where $X \cup \{\infty\}$ is the one-point compactification of $X$. Also note that the $C^*$ algebra is unital if and only if $X$ is compact.

We also recall that by the Riesz representation theorem in measure theory, the continuous dual of $C_0(X)$ is isometrically isomorphic to the space of all regular complex finite Borel measures on $X$ equipped with total variation norm and the functional $\Psi_\mu$ corresponding to a measure $\mu$ acting by,

$$\Psi_\mu(f) := \int_X f d\mu, \quad f \in C_0(X).$$

1.1. Birkhoff-James orthogonality in $C_0(X)$.

We begin with defining the norm attaining set of an element $f \in C_0(X)$ by,

$$M_f := \{x \in X : |f(x)| = \|f\|\}.$$

Clearly, $M_f$ is a compact subset of $X$. We state a characterization of the support functionals of an element $f \in C_0(X)$ using the norm attaining set. The proof of the result relies on elementary computations.

**Theorem 1.1.1.** Suppose $f \in C_0(X)$ and $f \neq 0$. Let $\mu$ be a complex regular Borel measure. Then $\mu$ is of unit total variation corresponding to a support functional of $f$ if and only if $|\mu|(X \setminus M_f) = 0$ and for almost every $x \in M_f$, with respect to the measure $\mu$, $d\mu(x) = \text{sgn}(f(x))d|\mu|(x)$. 

We now come to the characterization of Birkhoff-James orthogonality in $C_0(X)$.

**Theorem 1.1.2.** If $f, g \in C_0(X)$ and $f \neq 0$, then $f \perp_B g$ if and only if $0 \in \text{conv}\{f(x)g(x) : x \in M_f\}$.

**Proof.** Let $0 \in \text{conv}\{f(x)g(x) : x \in M_f\}$. Then there exist $n \in \mathbb{N}$, $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum_{k=1}^{n} \lambda_k = 1$ and

$$0 = \sum_{k=1}^{n} \lambda_k f(x_k) g(x_k),$$

for some $x_1, x_2, \ldots, x_n \in M_f$. Consider the functional

$$\Psi : h \mapsto \frac{1}{\|f\|} \sum_{k=1}^{n} \lambda_k f(x_k) h(x_k), \quad h \in C_0(X).$$

Then for $h \in C_0(X)$,

$$|\Psi(h)| = \left| \frac{1}{\|f\|} \sum_{k=1}^{n} \lambda_k f(x_k) h(x_k) \right| \leq \|h\| \left( \sum_{k=1}^{n} \lambda_k \right) = \|h\|.$$

Also,

$$\Psi(f) = \frac{1}{\|f\|} \sum_{k=1}^{n} \lambda_k f(x_k) f(x_k) = \|f\| \left( \sum_{k=1}^{n} \lambda_k \right) = \|f\|,$$

and

$$\Psi(g) = \frac{1}{\|f\|} \sum_{k=1}^{n} \lambda_k f(x_k) g(x_k) = 0.$$

Hence $\Psi$ is a support functional of $f$ such that $\Psi(g) = 0$, giving $f \perp_B g$ and proving the sufficiency.

Conversely, suppose $f \perp_B g$. Then there is a support functional of $f$ that annihilates $g$. Invoking Theorem 1.1.1 we obtain a
complex regular Borel measure \( \nu \) having \(|\nu|(M_f) = 1\) and
\[
\int_X h d\nu = \int_{M_f} h(x) \operatorname{sgn}(f(x)) d|\nu|(x), \text{ for every } h \in C_0(X),
\]
such that
\[
0 = \int_X g d\nu = \int_{M_f} g(x) \frac{f(x)}{\|f\|} d|\nu|(x).
\]
Suppose \( \Lambda \) is the space of all positive semi-definite regular Borel probability measures on \( M_f \) and \( \Phi : \Lambda \to \mathbb{K} \) given by,
\[
\Phi(\mu) := \int_{M_f} f(x)g(x)d\mu(x), \quad \mu \in \Lambda.
\]
Observe that since \( \Lambda \) is convex, so is \( \Phi(\Lambda) \). Also, as \( \Lambda \) is the collection of all support functionals of \(|f| \in C_0(X)\), it is compact under the weak* topology by the Banach-Alaoglu theorem [18, subsection 3.15, p.68]. Now, the map \( \Phi \) is evaluation at the element \( \overline{fg} \in C_0(X) \) on \( \Lambda \) and hence is continuous where \( \Lambda \) is equipped with the weak* topology. Therefore, \( \Phi(\Lambda) \) is compact and hence by the Krein-Milman theorem [15],
\[
\Phi(\Lambda) = \overline{\text{conv}}\{\lambda : \lambda \text{ is an extreme point of } \Phi(\Lambda)\}.
\]
We claim that any extreme point of \( \Phi(\Lambda) \) is of the form \( \overline{f(x)g(x)} \) for some \( x \in M_f \). Suppose, on the contrary, \( \Phi(\mu) \) is an extreme point of \( \Phi(\Lambda) \) and \( \mu \) is not a Dirac delta measure. If \( \overline{fg} \) is constant on the support of \( \mu \), clearly, \( \Phi(\mu) = \overline{f(x)g(x)} \) for any \( x \) in the support of \( \mu \). Otherwise, there exist \( x, y \) in the support of \( \mu \) such that \( \overline{f(x)g(x)} \neq \overline{f(y)g(y)} \). Consider \( 0 < \delta < \frac{1}{2} |\overline{f(x)g(x)} - \overline{f(y)g(y)}| \) and \( U_x \subset M_f \) open such that
\[
z \in U_x \Rightarrow |\overline{f(x)g(x)} - \overline{f(z)g(z)}| < \delta.
\]
Then \( U_x \) and \( M_f \setminus U_x \) are two disjoint subsets of \( M_f \) having non-zero measures since \( M_f \setminus U_x \) contains an open subset of \( M_f \) containing \( y \).
Clearly, since $\mu$ can be written as a convex combination of $\frac{1}{\mu(U_x)}\mu|_{U_x}$ and $\frac{1}{\mu(M_f \setminus U_x)}\mu|_{M_f \setminus U_x}$, we get

$$\Phi(\mu) = \frac{1}{\mu(U_x)} \int_{U_x} f(z) g(z) d\mu(z).$$

Hence, we have

$$\left| f(x) g(x) - \Phi(\mu) \right| = \left| f(x) g(x) - \frac{1}{\mu(U_x)} \int_{U_x} f(z) g(z) d\mu(z) \right|$$

$$\leq \frac{1}{\mu(U_x)} \int_{U_x} |f(x) g(x) - f(z) g(z)| d\mu(z) \leq \delta.$$ 

Since $0 < \delta < \frac{1}{2} |f(x) g(x) - f(y) g(y)|$ is arbitrary, we obtain that $\Phi(\mu) = f(x) g(x)$ establishing our claim.

Hence,

$$(1.1) \quad 0 = \Phi(|\nu|) \in \Phi(\Lambda) = \overline{\text{conv} \{f(x)g(x) : x \in M_f\}}.$$ 

We now prove that if $K \subset \mathbb{K}$ is compact, $\overline{\text{conv}(K)} = \overline{\text{conv}(K)}$. Suppose $z$ is a limit point of $\overline{\text{conv}(K)}$. Then there exists a sequence of elements $z_n$ in $\overline{\text{conv}(K)}$ converging to $z$. But by Caratheodory’s theorem [4], for every $n \in \mathbb{N}$, there exist $\lambda_i^{(n)} \in [0, 1]$ and $z_i^{(n)} \in K$ for $i = 1, 2, 3$ such that

$$\sum_{i=1}^{3} \lambda_i^{(n)} = 1, \quad \sum_{i=1}^{3} \lambda_i^{(n)} z_i^{(n)} = z_n.$$ 

Since $[0, 1]$ and $K$ are both compact, we may consider an increasing sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$ such that $\{\lambda_1^{(n_k)}\}_{k \in \mathbb{N}}$, $\{\lambda_2^{(n_k)}\}_{k \in \mathbb{N}}$, $\{\lambda_3^{(n_k)}\}_{n_k \in \mathbb{N}}$, $\{z_1^{(n_k)}\}_{k \in \mathbb{N}}$, $\{z_2^{(n_k)}\}_{k \in \mathbb{N}}$ and $\{z_3^{(n_k)}\}_{k \in \mathbb{N}}$ are all convergent and thereby obtain that $z \in \overline{\text{conv}(K)}$.

As $M_f$ is compact, $\overline{\{f(x)g(x) : x \in M_f\}}$ is a compact subset of $\mathbb{K}$ and hence by (1.1),

$$0 \in \overline{\text{conv} \{f(x)g(x) : x \in M_f\}},$$
establishing the necessity.

We now furnish a generalization of [11, Corollary 2.2] characterizing the smoothness of an element of $C_0(X)$.

**Theorem 1.1.3.** A point $f \in C_0(X)$ is smooth if and only if $M_f$ is a singleton set.

**Proof.** First if $M_f$ is a singleton set, say $\{x_0\}$, then clearly by Theorem 1.1.2, $f \perp_B g$ for $g \in C_0(X)$ if and only if $g(x_0) = 0$. Hence clearly, for $g, h \in C_0(X)$, $f \perp_B g, h$ would imply

$$g(x_0) = h(x_0) = 0 \Rightarrow g(x_0) + h(x_0) = 0 \Rightarrow f \perp_B (g + h).$$

Hence $f$ is smooth.

Conversely, if $x_1, x_2 \in M_f$, $x_1 \neq x_2$, then $\Psi_1, \Psi_2 : C_0(X) \rightarrow \mathbb{K}$ given by

$$\Psi_i(g) := \text{sgn}(f(x_i))g(x_i), \quad g \in C_0(X), \ i = 1, 2,$$

are two support functionals of $f$. Now, since $X$ is Hausdorff, there exists $U \subset X$ open such that $x_1 \in U$ and $x_2 \notin U$. Hence, there exists a continuous map $h$ on $X$ having compact support, vanishing outside $U$ and $h(x_1) = 1$. Thus $h \in C_0(X)$ and $\Psi_1(h) \neq \Psi_2(h)$. Therefore $f$ is not smooth. \(\square\)

1.2. **Pointwise symmetry of Birkhoff-James orthogonality in $C_0(X)$.**

In this subsection we characterize the pointwise symmetry of Birkhoff-James orthogonality in $C_0(X)$. We begin with our characterization of the left symmetric points of $C_0(X)$.

**Theorem 1.2.1.** An element $f \in C_0(X)$ is a left symmetric point of $C_0(X)$ if and only if $f$ is identically zero or $M_f$ is singleton and $f$ vanishes outside $M_f$.

**Proof.** We begin with the sufficiency. If $M_f = \{x_0\}$ for some $x_0 \in X$, then by Theorem 1.1.2, $f \perp_B g$ for any $g \in C_0(X)$ if and only if $g(x_0) = 0$. Then clearly, $x_0 \notin M_g$ and hence if $x_1 \in M_g$, $f(x_1) = 0$ giving $g \perp_B f$ by Theorem 1.1.2.
Conversely, suppose $f \in C_0(X)$ is left-symmetric and not identically zero. Suppose $x_1 \in M_f$ and $x_2 \in X$ such that $x_1 \neq x_2$, $f(x_2) \neq 0$. Consider $U, U' \subset X$ open containing $x_2$ such that $x_1 \notin U$ and $f$ does not vanish on $U'$. Set $U'' = U \cap U'$. Consider a continuous function $h : X \to [0, 1]$ having compact support such that $h(x_2) = 1$ and $h$ vanishes outside $U''$. Set $g(x) := \text{sgn}(f(x))h(x)$, $x \in X$. Then clearly $g \in C_0(X)$ and $g(x_1) = 0$ giving $f \perp_B g$ by Theorem 1.1.2. But clearly $\text{sgn}(g(x)) = \text{sgn}(f(x))$ for every $x \in M_g$. Hence $g \not\perp_B f$ by Theorem 1.1.2, establishing the necessity.

Note that this theorem clearly states that if $X$ has no singleton connected component, $C_0(X)$ has no non-zero left symmetric point.

We next characterize the right-symmetric points.

**Theorem 1.2.2.** An element $f \in C_0(X)$ is right-symmetric if and only if $M_f = X$. Hence, in particular if $X$ is not compact, $C_0(X)$ has no non-zero right symmetric point.

**Proof.** We again begin with the sufficiency. If $M_f = X$, then by Theorem 1.1.2, $g \perp_B f$ only if

$$0 \in \text{conv}\{g(x)f(x) : x \in M_g\}.$$ 

Since $M_g \subset X = M_f$, we clearly obtain

$$0 \in \text{conv}\{f(x)g(x) : x \in M_f\},$$

and hence $f \perp_B g$ by Theorem 1.1.2.

Conversely, suppose $f \in C_0(X)$ is right-symmetric and not identically zero. For the sake of contradiction, let us assume that $M_f \neq X$. We consider two cases.

**Case 1:** $f(x_0) = 0$ for some $x_0 \in X$.

Since $x_0 \notin M_f$ and $M_f$ is compact, we obtain $U, U' \subset X$ open such that $U \cap U' = \emptyset$, $x_0 \in U$ and $M_f \subset U'$. Now, consider two continuous functions $h, h' : X \to [0, 1]$ having compact supports such that $h(x_0) = 1$, $h$ vanishes outside $U$ and $h'$ is identically 1 on $M_f$ and vanishes outside $U'$. Set $g(x) := \|f\|h(x) + f(x)h'(x)$, $x \in X$. Then $\|g\| = \|f\|$ and $x_0 \in M_g$. Hence by Theorem 1.1.2, $g \perp_B f$. However, if $x \in M_f$, $g(x) = f(x)$ and hence by Theorem
1.1.2, \( f \not\perp_B g \).

**Case 2:** \( f \) is non-zero everywhere on \( X \) but there exists \( x_0 \in X \setminus M_f \).

Let us again consider \( U, U', h \) and \( h' \) as before and set \( g(x) := -\|f\| \text{sgn}(f(x))h(x) + f(x)h'(x) \). Then clearly, \( M_f \subset M_g \) and \( x_0 \in M_g \). Also, \( g(x) = f(x) \) for \( x \in M_f \) and \( g(x_0) = -\|f\| \text{sgn}(f(x_0)) \) giving \( g \perp_B f \) by Theorem 1.1.2. Also, by the same theorem \( f \not\perp_B g \), proving the necessity. \( \Box \)

2. **Birkhoff-James orthogonality and its pointwise symmetry in \( L_\infty \) spaces**

In this section, we study Birkhoff-James orthogonality and its pointwise symmetry in \( L_\infty \) spaces. Note that since \( L_\infty \) spaces are also commutative \( C^* \) algebras, we are going to use the results from Section 2 for this study. We begin with representing \( L_\infty(X) \) as \( C_0(Y) \) for some suitable locally compact, Hausdorff \( Y \). We then study this representation and use the results of Section 2 to characterize Birkhoff-James orthogonality and its pointwise symmetry in \( L_\infty(X) \).

We begin by considering a positive measure space \( (X, \Sigma, \lambda) \) and \( L^\mathbb{K}_\infty(X, \Sigma, \lambda) \), the space of all essentially bounded \( \mathbb{K} \) valued functions on \( X \) equipped with the essential supremum norm. Without any ambiguity, we refer to \( L^\mathbb{K}_\infty(X, \Sigma, \lambda) \) as \( L_\infty(X) \).

We now represent \( L_\infty(X) \) as the space of continuous functions on a compact topological space equipped with the supremum norm. We begin with a definition:

**Definition 2.0.1.** A \( 0-1 \) measure with respect to \( \lambda \) is a finitely additive set function \( \mu \) on \((X, \Sigma)\) taking values in \( \{0, 1\} \) such that \( \mu(X) = 1 \), \( \mu(A) = 0 \) whenever \( \lambda(A) = 0 \).

Let us define \( \mathfrak{G} \) as the collection of all \( 0-1 \) measures with respect to \( \lambda \). We consider the \( t \) topology on \( \mathfrak{G} \) having a basis consisting of sets of the following form:

\[
t(A) := \{ \mu \in \mathfrak{G} : \mu(A) = 1 \}, \quad A \in \Sigma, \ \lambda(A) > 0.
\]

Yosida and Hewitt proved the following representation result in [26]:
Theorem 2.0.2. 
1. The topological space \((\mathfrak{G}, t)\) is compact and Hausdorff.
2. The map \(T : L_\infty(X) \to C^\mathfrak{G}(\mathfrak{G}, t)\) given by
\[
T(f)(\mu) := \int_X f \, d\mu, \quad \mu \in \mathfrak{G}, \ f \in L_\infty(X),
\]
is an isometric isomorphism.

In the first subsection we study the space of 0-1 measures with respect to \(\lambda\) and integrals with respect to the measures. We characterize Birkhoff-James orthogonality between two elements of \(L_\infty(X)\) in the second subsection along with characterization of smoothness of a point. The third subsection comprises of characterizations of pointwise symmetry in \(L_\infty(X)\).

2.1. 0-1 measures with respect to \(\lambda\) and \(\lambda\)-ultrafilters.

In this subsection, we obtain a one to one correspondence between all the 0-1 measures with respect to \(\lambda\) and all \(\lambda\)-ultrafilters and therefore use the \(\lambda\)-ultrafilters to study integrals with respect to the 0-1 measures with respect to \(\lambda\). We begin with the definition of a \(\lambda\)-filter.

**Definition 2.1.1.** A non-empty subset \(\mathcal{F}\) of \(\Sigma\) is called a \(\lambda\)-filter on \(X\) if
1. \(\lambda(A) > 0\) for every \(A \in \mathcal{F}\).
2. For every \(A, B \in \mathcal{F}\), \(A \cap B \in \mathcal{F}\).
3. \(B \in \mathcal{F}\) for every \(B \supset A\), \(B \in \Sigma\) and \(A \in \mathcal{F}\)

A \(\lambda\)-filter \(\mathcal{U}\) is called a \(\lambda\)-ultrafilter if any \(\lambda\)-filter containing \(\mathcal{U}\) is \(\mathcal{U}\) itself.

The existence of \(\lambda\)-ultrafilters is a direct consequence of Zorn’s lemma. Before proceeding further, we derive a lemma that is going to be used throughout the section.

**Lemma 2.1.2.** Suppose \(\mathcal{U}\) is a \(\lambda\)-ultrafilter.
1. If \(A \in \Sigma\) such that \(A \notin \mathcal{U}\), then there exists \(B \in \mathcal{U}\), such that \(\lambda(A \cap B) = 0\).
2. If \( \bigcup_{k=1}^{n} A_k \in \mathcal{U} \) for \( A_1, A_2, \ldots, A_n \in \Sigma \), then \( A_i \in \mathcal{U} \) for some \( 1 \leq i \leq n \).

Proof. 1. If no such \( B \) exists, then \( \mathcal{F} := \{ A \cap C : C \in \mathcal{U} \} \cup \mathcal{U} \) is a \( \lambda \)-filter properly containing \( \mathcal{U} \).

2. If \( A_k \notin \mathcal{U} \) for every \( 1 \leq k \leq n \), then by part 1, there exist \( B_k \in \mathcal{U} \) for \( 1 \leq k \leq n \) such that \( \lambda(A_k \cap B_k) = 0 \). But then setting \( B := \bigcap_{k=1}^{n} B_k \), we get

\[
\lambda \left( \left( \bigcup_{k=1}^{n} A_k \right) \cap B \right) \leq \sum_{k=1}^{n} \lambda(A_k \cap B) = 0,
\]

violating the closure of \( \mathcal{U} \) under finite intersections. \( \square \)

Let \( \mathcal{F} \) denote the collection of all \( \lambda \)-ultrafilters on \( X \). We now define the \( \lambda \)-ultrafilter corresponding to a 0-1 measure with respect to \( \lambda \) and vice versa. Suppose \( \mu \in \mathcal{G} \). We define \( \mathcal{U}_{\mu} \) by:

\[
\mathcal{U}_{\mu} := \{ A \in \Sigma : \mu(A) = 1 \}.
\]

Also for any \( \lambda \)-ultrafilter \( \mathcal{U} \), let us define a set function \( \mu^{\mathcal{U}} \) on \( \Sigma \) by:

\[
\mu^{\mathcal{U}}(A) := \begin{cases} 
1, & A \in \mathcal{U}, \\
0, & A \notin \mathcal{U}.
\end{cases}
\]

Theorem 2.1.3.
1. For any \( \mu \in \mathcal{G} \), \( \mathcal{U}_{\mu} \in \mathcal{F} \) and is called the \( \lambda \)-ultrafilter corresponding to \( \mu \).

2. For any \( \mathcal{U} \in \mathcal{F} \), \( \mu^{\mathcal{U}} \in \mathcal{G} \) and is called the 0-1 measure with respect to \( \lambda \) corresponding to \( \mathcal{U} \).

3. The maps \( \mu \mapsto \mathcal{U}_{\mu}, \mu \in \mathcal{G} \) and \( \mathcal{U} \mapsto \mu^{\mathcal{U}}, \mathcal{U} \in \mathcal{F} \) are inverse to each other and thereby establish a one to one correspondence between \( \mathcal{F} \) and \( \mathcal{G} \).

Proof. 1. It is easy to verify that for any \( \mu \in \mathcal{G} \), \( \mathcal{U}_{\mu} \) does not contain any set having \( \lambda \)-measure zero. Further, since \( \mu \) is a 0-1 measure, \( \mu(A) = 1 \) and \( B \supset A, B \in \Sigma \) forces \( \mu(B) = 1 \). Now, if \( \mu(A) = \)}
$\mu(B) = 1$, then $\mu(A \cup B) = 1$ since $\mu$ is a 0-1 measure and hence as $\mu$ is additive,

$$1 = \mu(A \cup B) = \mu(A) + \mu(B \setminus A) = 1 + \mu(B \setminus A).$$

Hence $\mu(B \setminus A) = 0$ and so by additivity of $\mu$,

$$\mu(A \cap B) = \mu(A \cap B) + \mu(B \setminus A) = \mu(B) = 1,$$

giving $A \cap B \in U_\mu$. Now if $F$ is another $\lambda$-filter containing $U_\mu$, then consider $C \in F \setminus U_\mu$. Clearly, $\mu(C) = 0$. Then $\mu(X \setminus C) = 1$ and hence $X \setminus C \in U_\mu \subset F$ violating, that $F$ is a $\lambda$-filter.

2. Clearly, $\mu^U(A) = 0$ for $\lambda(A) = 0$ since $A \notin U$. Now, if $A, B \in \Sigma$ and $A \cap B = \emptyset$, either exactly one of $A$ and $B$ is in $U$ in which case $A \cup B \in U$, or neither $A$ nor $B$ is in $U$ in which case by Lemma 2.1.2, $A \cup B \notin U$. Clearly, in both cases,

$$\mu^U(A \cup B) = \mu^U(A) + \mu^U(B).$$

Hence $\mu^U$ is a 0-1 measure with respect to $\lambda$.

3. This part is an easy verification. \qed

We now study the integrals under 0-1 measures with respect to $\lambda$ in the light of this one to one correspondence. We introduce two new definitions.

**Definition 2.1.4.** A non-empty subset $B$ of $\Sigma$ is said to be a $\lambda$-filter base if

1. $\lambda(A) > 0$ for every $A \in B$.
2. For every $A, B \in B$, there exists $C \in B$ such that $C \subset A \cap B$.

Any $\lambda$-filter base $B$ is contained in a unique minimal $\lambda$-filter given by

$$\{ B \in \Sigma : B \supset A \text{ for some } A \in B \}.$$ 

Since every $\lambda$-filter is contained in a $\lambda$-ultrafilter (a direct application of Zorn’s lemma), every $\lambda$-filter base is contained in a $\lambda$-ultrafilter. We now define limit under a $\lambda$-filter.

**Definition 2.1.5.** Suppose $F$ is a $\lambda$-filter on $X$ and $f : X \to \mathbb{K}$ is a measurable function. Then the limit of the map $f$ under the $\lambda$-filter

We now define limit under a $\lambda$-filter.
\( \mathcal{F} \) (written as \( \lim_{\mathcal{F}} f \)) is defined as \( z_0 \in \mathbb{K} \) if
\[
\{ x \in X : |f(x) - z_0| < \epsilon \} \in \mathcal{F} \text{ for every } \epsilon > 0.
\]

We state a few elementary results pertaining to the limit of a measurable function under a \( \lambda \)-filter. We omit the proofs since the results follow directly from the definition of the limit.

**Theorem 2.1.6.** Suppose \( f : X \to \mathbb{K} \) is a measurable function and \( \mathcal{F} \) is a \( \lambda \)-filter on \( X \).
1. \( \lim_{\mathcal{F}} f \) if exists is unique.
2. If \( g : \mathbb{K} \to \mathbb{K} \) is continuous, \( \lim_{\mathcal{F}} g \circ f = g \left( \lim_{\mathcal{F}} f \right) \).
3. Limits under a \( \lambda \)-filter respect addition, multiplication, division and multiplication with a constant.

We now come to our second key result of this subsection.

**Theorem 2.1.7.** Suppose \( f \in L_\infty(X) \).
1. If \( \mathcal{U} \) is a \( \lambda \)-ultrafilter, \( \lim_{\mathcal{U}} f \) exists and is well defined.
2. If \( \mu \in \mathcal{G} \),
\[
\lim_{\mathcal{U}} f = \int_X f d\mu.
\]

**Proof.**
1. Clearly \( \lim_{\mathcal{U}} f \), if exists, must lie in the set \( \mathcal{D} := \{ z \in \mathbb{K} : |z| \leq \|f\|_\infty \} \). Now, suppose \( \lim_{\mathcal{U}} f \) does not exist. Then for every \( z \in \mathcal{D} \), there exists \( \epsilon_z > 0 \) such that
\[
\{ x \in X : |f(x) - z| < \epsilon_z \} \notin \mathcal{U}.
\]
Set \( B_z := \{ w \in \mathbb{K} : |w - z| < \epsilon_z \} \). Then \( \{ B_z : z \in \mathcal{D} \} \) is an open cover of the compact set \( \mathcal{D} \) and therefore must have a finite sub-cover, say \( \{ B_{z_1}, B_{z_2}, \ldots, B_{z_n} \} \). We further set
\[
A_i := \{ x \in X : |f(x) - z_i| < \epsilon_{z_i} \}, \quad 1 \leq i \leq n.
\]
Hence clearly, \( A_i \notin \mathcal{U} \) and
\[
\bigcup_{i=1}^n A_i = X \setminus B, \text{ for some } B \in \Sigma, \lambda(B) = 0.
\]
Thus by Lemma 2.1.2, $X \setminus B \notin \mathcal{U}$. But then as $B \notin \mathcal{U}$, we arrive at a contradiction.

In order to prove that the limit is well defined, consider $f$ and $f'$ essentially bounded such that $f = f'$ almost everywhere on $X$ with respect to $\lambda$. Now, if $\lim_{\mathcal{U}} f = z_0$, then for any $\epsilon > 0$,

$$\{ x \in X : |f(x) - z_0| < \epsilon \} \subset \{ x \in X : |f'(x) - z_0| < \epsilon \} \cup \{ x \in X : f(x) \neq f'(x) \}.$$ 

Now by Lemma 2.1.2, $\{ x \in X : |f'(x) - z_0| < \epsilon \} \in \mathcal{U}$ since $\lambda(\{ x \in X : f(x) \neq f'(x) \}) = 0$.

2. Suppose $\lim_{\mathcal{U}} f = z_0$. Then for every $\epsilon > 0$,

$$\mu(\{ x \in X : |f(x) - z_0| < \epsilon \}) = 1,$$

and therefore

$$\mu(\{ x \in X : |f(x) - z_0| \geq \epsilon \}) = 0.$$

Hence we obtain that

$$\left| \int_X f \, d\mu - z_0 \right| \leq \int_X |f - z_0| \, d\mu = \int_{|f - z_0| < \epsilon} |f - z_0| \, d\mu \leq \epsilon.$$

Since $\epsilon$ is arbitrary,

$$\int_X f \, d\mu = z_0.$$

$$\square$$

We now establish a result that gives a collection of possible values of limits of an essentially bounded function under a $\lambda$-ultrafilter.

**Theorem 2.1.8.** Suppose $f, g \in L_\infty(X)$.

1. For any $z_0 \in \mathbb{K}$, there exists a $\lambda$-ultrafilter $\mathcal{U}$ such that $\lim_{\mathcal{U}} f = c_0$ if and only if for every $\epsilon > 0$,

$$\lambda(\{ x \in X : |f(x) - z_0| < \epsilon \}) > 0.$$
2. For any $z_0, w_0 \in K$, there exists a $\lambda$-ultrafilter $\mathcal{U}$ such that $\lim_{\mathcal{U}} f = z_0$ and $\lim_{\mathcal{U}} g = w_0$ if and only if for any $\epsilon > 0$,
\[
\lambda \left( \{ x \in X : |f(x) - z_0| < \epsilon, |g(x) - w_0| < \epsilon \} \right) > 0.
\]

Proof. The necessary part of both the statements are clear. We therefore prove the sufficiency in the two statements.

1. Consider $\mathcal{B} \subset \Sigma$ given by
\[
\mathcal{B} := \{ \{ x \in X : |f(x) - z_0| < \epsilon \}, \ \epsilon > 0 \}.
\]
Clearly, $\mathcal{B}$ is a $\lambda$-filter base and hence there exists a $\lambda$-ultrafilter $\mathcal{U}$ containing $\mathcal{B}$. Clearly, by construction, $\lim_{\mathcal{U}} f = z_0$.

2. Again consider $\mathcal{B}' \subset \Sigma$ given by
\[
\mathcal{B}' := \{ \{ x \in X : |f(x) - z_0| < \epsilon \} \cap \{ x \in X : |g(x) - w_0| < \epsilon \}, \ \epsilon > 0 \}.
\]
Clearly, $\mathcal{B}'$ too is a $\lambda$-filter base and hence there exists a $\lambda$-ultrafilter $\mathcal{U}'$ containing $\mathcal{B}'$. Also, by construction, clearly, $\lim_{\mathcal{U}'} f = z_0$ and $\lim_{\mathcal{U}'} g = w_0$. \hfill \Box

2.2. Birkhoff-James orthogonality in $L_\infty(X)$.

In this subsection we characterize Birkhoff-James orthogonality between two elements of $L_\infty(X)$ and use the characterization to study the smoothness of a point in $L_\infty(X)$.

The following characterization of orthogonality follows from Theorems 1.1.2, 2.0.2, 2.1.7 and 2.1.8.

**Theorem 2.2.1.** Suppose $f, g \in L_\infty(X)$ are non-zero. Then $f \perp_B g$ if and only if
\[
0 \in \text{conv} \left\{ z : \lambda \left( \{ x \in X : |f(x)| > \|f\|_\infty - \epsilon, |\overline{f(x)}g(x) - z| < \epsilon \} \right) > 0 \ \forall \ \epsilon > 0 \right\}.
\]

We now come to the characterization of smooth points in $L_\infty(X)$, but before this result, we prove a preliminary lemma.

**Lemma 2.2.2.** If $f \in L_\infty(X)$ and
\[
\lambda \left( \{ x \in X : |f(x)| = \|f\|_\infty \} \right) = 0,
\]
there exist $A, B \subset X$ such that $A \cap B = \emptyset$ and
\[
\lambda (\{ x \in A : \| f \| - | f(x) | < \epsilon \}), \ \lambda (\{ x \in B : \| f \|_\infty - | f(x) | < \epsilon \}) > 0,
\]
for every $\epsilon > 0$.

Proof. Since $\lambda (\{ x \in X : | f(x) | = \| f \|_\infty \}) = 0$, clearly either,
\[
\lambda (\{ x \in X : \| f \|_\infty - | f(x) | < \epsilon \}) < \infty,
\]
for some $\epsilon > 0$ and
\[
\lambda (\{ x \in X : \| f \|_\infty - | f(x) | < \delta \}) \to 0,
\]
as $\delta \to 0$ or,
\[
\lambda (\{ x \in X : \| f \|_\infty - | f(x) | < \epsilon \}) = \infty
\]
for every $\epsilon > 0$.

Case 1: $\lambda (\{ x \in X : \| f \|_\infty - | f(x) | < \epsilon_0 \}) < \infty$ for some $\epsilon_0 > 0$.

For $n \in \mathbb{N}$, we set $C_n = \{ x \in X : \| f \|_\infty - | f(x) | < \epsilon_n \}$ and
$D_n \subset C_n$ such that
\[
0 < \lambda (D_n) \leq \frac{1}{2} \lambda (C_n).
\]

We further consider $\epsilon_n > 0$ such that
\[
\lambda (\{ x \in X : \| f \|_\infty - | f(x) | < \epsilon_n \}) < \frac{1}{3} \lambda (D_n).
\]

Finally, define
\[
A_n := D_n \setminus C_{n+1}, \ A := \bigcup_{n=1}^{\infty} A_n, \ B := X \setminus A.
\]

Then clearly, since $\lambda (A_n) > 0$ for every $n \in \mathbb{N}$ and $\lim_{n \to \infty} \epsilon_n = 0$, we have
\[
\lambda (\{ x \in A : \| f \| - | f(x) | < \epsilon \}) > 0,
\]
for every $\epsilon > 0$. Again for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\epsilon > \epsilon_{n-1}$. Observe that

$$\lambda(\{x \in B : \|f\| - |f(x)| < \epsilon\}) \geq \lambda(\{x \in B : \|f\| - |f(x)| < \epsilon_{n-1}\})$$

$$= \lambda(C_n \setminus \bigcup_{k \geq n} A_k)$$

$$= \lambda(C_n) - \sum_{n \geq k} \lambda(A_k)$$

$$\geq \lambda(C_n) - \sum_{k \geq n} \lambda(D_k)$$

$$\geq \lambda(C_n) - \sum_{k=0}^{\infty} \lambda(D_n) \frac{1}{3^k} = \lambda(C_n) - \frac{3}{2} \lambda(D_n) > 0.$$

**Case 2:** $\lambda(\{x \in X : \|f\|_{\infty} - |f(x)| > \epsilon\}) = \infty$ for every $\epsilon > 0$.

Then for every $n \in \mathbb{N}$, there exists $\epsilon_n > 0$ such that $\epsilon_n > \epsilon_{n+1}$ and $\lim_{n \to \infty} \epsilon_n = 0$ with

$$\lambda(\{x \in X : \|f\|_{\infty} - |f(x)| \in (\epsilon_{n+1}, \epsilon_n)\}) > 0.$$

Hence setting

$$A := \{x \in X : \|f\|_{\infty} - |f(x)| \in (\epsilon_{2n}, \epsilon_{2n-1}) , \ n \in \mathbb{N}\},$$

and

$$B := \{x \in X : \|f\|_{\infty} - |f(x)| \in (\epsilon_{2n+1}, \epsilon_{2n}) , \ n \in \mathbb{N}\},$$

gives us the desired subsets of $X$. \qed

We now come to the characterization of smooth points in $L_{\infty}(X)$ but for that, we require the definition of a $\lambda$-atom.

**Definition 2.2.3.** A subset $A \in \Sigma$ is called a $\lambda$-atom if $\lambda(A) > 0$ and $B \subset A, \ \lambda(B) > 0 \Rightarrow B = A$.

**Theorem 2.2.4.** An element $f \in L_{\infty}(X)$ is smooth if and only if there exists a $\lambda$-atom $A$ such that $|f(x)| = \|f\|_{\infty}$ for almost every
\( x \in A \) and

\[
\lambda(\{x \in X \setminus A : |f(x)| > \|f\|_\infty - \epsilon\}) = 0,
\]

for some \( \epsilon > 0 \).

**Proof.** By Theorems 1.1.3, 2.0.2 and 2.1.7, we have that \( f \in L_\infty(X) \) is smooth if and only if there exists a unique \( \lambda \)-ultrafilter \( \mathcal{U} \) such that

\[
\lim_{\mathcal{U}} |f| = \|f\|_\infty.
\]

We first prove the sufficiency. Set

\[
\mathcal{V}_A := \{B \in \Sigma : B \supset A\}.
\]

Clearly, \( \mathcal{V}_A \) is a \( \lambda \)-ultrafilter since no proper measurable subset of \( A \) has nonzero measure. Clearly, \( \lim_{\mathcal{V}_A} |f| = \|f\|_\infty \). Suppose \( \mathcal{U} \) is a \( \lambda \)-ultrafilter such that \( \lim_{\mathcal{U}} |f| = \|f\|_\infty \). Let us assume that \( \lim_{\mathcal{U}} f = e^{i\theta}\|f\|_\infty \), for some \( \theta \in [0, 2\pi) \). Then clearly,

\[
A \cup \{x \in X \setminus A : \|f\|_\infty - |f(x)| < \epsilon\} \in \mathcal{U}.
\]

Since \( \{x \in X \setminus A : |f(x)| > \|f\|_\infty - \epsilon\} \notin \mathcal{U} \), by Lemma 2.1.2, \( A \in \mathcal{U} \). Hence clearly \( \mathcal{V}_A \subseteq \mathcal{U} \) and thus \( \mathcal{V}_A = \mathcal{U} \) since \( \mathcal{V}_A \) is a \( \lambda \)-ultrafilter.

Conversely, suppose there is no \( \lambda \)-atom \( A \) such that \( |f(x)| = \|f\|_\infty \) for almost every \( x \in A \). Then either

\[
\lambda(\{x \in X : |f(x)| = \|f\|\}) = 0,
\]

or there exist \( A \) and \( B \) disjoint subsets in \( \Sigma \) such that

\[
A \cup B \subset \{x \in X : |f(x)| = \|f\|_\infty\},
\]

and \( \lambda(A), \lambda(B) > 0 \). In the second case we consider

\[
\mathcal{V} := \{C \in \Sigma : C \supset A\}, \quad \mathcal{W} := \{C \in \Sigma : C \supset B\}.
\]

Clearly \( \mathcal{V} \) and \( \mathcal{W} \) are contained in two distinct ultrafilters, say \( \mathcal{V}' \) and \( \mathcal{W}' \) and

\[
\lim_{\mathcal{V}'} |f| = \lim_{\mathcal{W}'} |f| = \|f\|_\infty.
\]

In the first case, by Lemma 2.2.2, there exist \( A_1, A_2 \subset X \) disjoint such that

\[
\lambda(\{x \in A_i : \|f\|_\infty - |f(x)| < \epsilon\}) > 0,
\]
for every $\epsilon > 0$ and $i = 1, 2$. Observe that
\[
B_i := \{ x \in A_i : \| f \|_\infty - | f(x) | < \epsilon \} : \epsilon > 0 \},
\]
is a $\lambda$-filter base for $i = 1, 2$ and hence in contained in a $\lambda$-ultrafilter $U_i$. Also, clearly, $U_1 \neq U_2$ and
\[
\lim_{U_i} | f | = \| f \|_\infty,
\]
for $i = 1, 2$.
Again if there exists a $\lambda$-atom $A$ with $| f(x) | = \| f \|_\infty$ for almost every $x \in A$ but
\[
\lambda(\{ x \in X \setminus A : | f(x) | > \| f \|_\infty - \epsilon \}) > 0 \quad \text{for every } \epsilon > 0,
\]
then set
\[
B := \{ x \in X \setminus A : | f(x) | < \| f \|_\infty - \epsilon \} : \epsilon > 0 \}.
\]
Clearly, $B$ is a $\lambda$-filter base and is contained in some ultrafilter $U$. Since $X \setminus A \in U$, $U$ and $V_A$ are two distinct ultrafilters but
\[
\lim_{U} | f | = \lim_{V_A} | f | = \| f \|_\infty,
\]
and hence the necessity.

$\square$

2.3. Pointwise symmetry of Birkhoff-James orthogonality in $L_\infty(X)$.

In this subsection we characterize the left symmetric and the right symmetric points of $L_\infty(X)$. We begin with the characterization of the left symmetric points.

**Theorem 2.3.1.** A non-zero $f \in L_\infty(X)$ is left symmetric if and only if $| f(x) | = \| f \|_\infty$ for almost every $x$ in some $\lambda$-atom $A$ and $f(x) = 0$ for almost every $x \in X \setminus A$.

**Proof.** By Theorems 1.1.3 and 1.2.1, if $f \in L_\infty(X)$ is left symmetric, $f$ is smooth and hence by Theorem 2.2.4, there exists a $\lambda$-atom $A$, such that $| f(x) | = \| f \|_\infty$ for almost every $x \in A$. Further, by Theorems 2.0.2, 2.1.3 and 2.1.7, $\lim_{U} f = 0$ for every $\lambda$-ultrafilter $U$
not containing \(A\). Hence, for every \(z \neq 0\), there exists \(\epsilon_z > 0\) such that
\[
\lambda (\{ x \in X \setminus A : |f(x) - z| < \epsilon_z \}) = 0.
\]
Hence
\[
\{ \{ w \in \mathbb{K} : |w - z| < \epsilon_z \} : z \neq 0 \},
\]
is an open cover of \(\mathbb{K}\setminus\{0\}\). Choose and fix a countable sub-cover given by:
\[
\{ \{ w \in \mathbb{K} : |w - z_n| < \epsilon_{z_n} \} : z_n \neq 0, \ n \in \mathbb{N} \}.
\]
Hence
\[
\lambda (\{ x \in X \setminus A : f(x) \neq 0 \}) \leq \sum_{n=1}^{\infty} \lambda (\{ x \in X : |f(x) - z_n| \}) = 0,
\]
proving the necessity. The sufficiency follows easily from Theorems 1.2.1, 2.0.2, 2.1.7 and 2.1.8.

**Theorem 2.3.2.** \(f \in L_\infty(X)\) is right symmetric if and only if \(|f(x)| = \|f\|_\infty\) for almost every \(x \in X\).

**Proof.** By Theorems 1.2.2, 2.0.2, 2.1.7 and 2.1.8, \(f \in L_\infty(X)\) is right symmetric if and only if for every \(z \in \mathbb{K}\) with \(|z| < \|f\|_\infty\), there exists \(\epsilon_z > 0\) such that
\[
\lambda (\{ x \in X : |f(x) - z| < \epsilon_z \}) = 0.
\]
Hence the sufficiency is easy to verify. For the necessity, observe that,
\[
\{ \{ w \in \mathbb{K} : |w - z| < \epsilon_z \} : |z| < \|f\|_\infty \},
\]
is an open cover of \(\{ z \in \mathbb{K} : |z| < \|f\|_\infty \}\). Choose and fix a countable sub-cover of the aforesaid cover given by:
\[
\{ \{ w \in \mathbb{K} : |w - z_n| < \epsilon_{z_n} \} : |z_n| < \|f\|_\infty, \ n \in \mathbb{N} \}.
\]
Hence we obtain:
\[
\lambda (\{ x \in X : |f(x)| \neq \|f\|_\infty \}) = \lambda (\{ x \in X : |f(x)| < \|f\|_\infty \})
\leq \sum_{n=1}^{\infty} \lambda (\{ x \in X : |f(x) - z_n| < \epsilon_{z_n} \}) = 0.
\]
\(\square\)
3. **Birkhoff-James orthogonality and its pointwise symmetry in $L_1$ spaces**

In this section, we first characterize Birkhoff-James orthogonality in $L_1(X)$ and then characterize smoothness and pointwise symmetry. As before, we assume the measure space to be $(X, \Sigma, \lambda)$. Our approach would be to characterize $J(f)$ for any non-zero $f \in L_1(X)$ and therefrom use the James characterization to characterize Birkhoff-James orthogonality. The characterizations of smoothness and pointwise symmetry would follow therefrom.

Since the dual of $L_1(X)$ is isometrically isomorphic to $L_\infty(X)$, we are going to assume that $L_\infty(X)$ is indeed the dual of $L_1(X)$ and any element $h \in L_\infty(X)$ acts on $L_1(X)$ as:

$$f \mapsto \int_X h(x)f(x)d\lambda(x), \ f \in L_1(X).$$

**Lemma 3.0.1.** Suppose $f \in L_1(X) \setminus \{0\}$. Then for any $h \in L_\infty(X) = L_1(X)^*$, $h \in J(f)$ if and only if $h(x) = \overline{\text{sgn}(f(x))}$ for almost every $x \in X$ such that $f(x) \neq 0$, and $|h(x)| \leq 1$ for almost every $x \in X$ such that $f(x) = 0$.

**Proof.** The sufficiency follows from direct computation. For the necessity, note that

$$\|f\|_1 = \int_X h(x)f(x)d\lambda(x) \leq \int_X \|h\|_\infty |f(x)|d\lambda(x) = \|f\|_1,$$

whenever $h \in J(f)$. Hence from the condition of equality in the above inequality, we obtain $h(x) = \overline{\|h\|_\infty \text{sgn}(f(x))} = \overline{\text{sgn}(f(x))}$ for almost every $x \in X$, $f(x) \neq 0$.

From this lemma, we can now characterize Birkhoff-James orthogonality in $L_1(X)$. 

Theorem 3.0.2. Suppose $f, g \in L_1(X)$. Then $f \perp_B g$ if and only if
\begin{equation}
\left| \int_X \text{sgn}(f(x)) g(x) d\lambda(x) \right| \leq \int_{f(x)=0} |g(x)| d\lambda(x).
\end{equation}

Proof. We first prove the necessity. Since $f \perp_B g$, there exists $h \in L_\infty(X)$, such that $h \in J(f)$ and $\int_X g(x) h(x) d\lambda(x) = 0$. By Lemma 3.0.1, we now conclude
\begin{align*}
\left| \int_X \text{sgn}(f(x)) g(x) d\lambda(x) \right| &= \left| \int_{f(x) \neq 0} \text{sgn}(f(x)) g(x) d\lambda(x) \right| \\
&= \left| \int_{f(x)=0} g(x) h(x) d\lambda(x) \right| \leq \int_{f(x)=0} |g(x)| d\lambda.
\end{align*}

Again, if (3.1) holds, set:
\begin{align*}
c &= -\frac{\int_X \text{sgn}(f(x)) g(x) d\lambda(x)}{\int_{f(x)=0} |g(x)| d\lambda(x)}.
\end{align*}

Consider $h : X \to \mathbb{K}$ given by
\begin{align*}
h(x) := \begin{cases}
\text{sgn}(f(x)), & f(x) \neq 0, \\
c \text{sgn}(g(x)), & f(x) = 0.
\end{cases}
\end{align*}

Clearly, $h \in L_\infty(X)$ and $h \in J(f)$ by Lemma 3.0.1. But clearly $\int_X g(x) h(x) d\lambda(x) = 0$, establishing the sufficiency. \(\square\)

We now characterize the smooth points of $L_1(X)$.

Theorem 3.0.3. $f \in L_1(X)$ is a smooth point if and only if $f \neq 0$ almost everywhere on $X$. 

Proof. If $f \neq 0$ almost everywhere on $X$, \[ \int_{f(x)=0} |g(x)| d\lambda(x) = 0 \] for every $g \in L_1(X)$ giving $f \perp_B g$ if and only if
\[ \int_{X} \text{sgn}(f(x))g(x) d\lambda(x) = 0. \]

Hence $f \perp_B g$ and $f \perp_B h$ for $g, h \in L_1(X)$ forces $f \perp_B (g + h)$, proving the sufficiency. To prove the necessity, assume $\lambda(\{x \in X : f(x) \neq 0\}) > 0$. Consider $h_0, h_1 : X \to \mathbb{K}$ given by
\[ h_i(x) := \begin{cases} \text{sgn}(f(x)), & f(x) \neq 0, \\ i, & f(x) = 0, \end{cases} \]
for $i = 0, 1$. Then by Lemma 3.0.1, $h_0$ and $h_1$ are two distinct support functionals of $f$ and hence $f$ cannot be a smooth point of $L_1(X)$. \hspace{1cm} \square

We now characterize the pointwise symmetry of Birkhoff-James orthogonality in $L_1(X)$. We first address the left-symmetric case.

**Theorem 3.0.4.** $f \in L_1(X)$ is a left-symmetric point if and only if exactly one of the following conditions holds:

1. $f \equiv 0$.
2. $f \not\equiv 0$ and $\Sigma = \{\emptyset, X\}$.
3. There exist disjoint $\Sigma$-atoms $A$ and $B$ such that $A \sqcup B = X$ and $\lambda(A)|f(x)| = \lambda(B)|f(y)|$ for almost every $x \in A$ and $y \in B$.

**Proof.** The sufficiency can be obtained from Theorem 3.0.2 by direct computation. For, the necessity, let $f \in L_1(X)$, $f \not\equiv 0$. We consider two cases:

**Case I:** $\lambda(\{x \in X : f(x) = 0\}) > 0$.

Set $A \subseteq \{x \in X : f(x) = 0\}$ such that $\infty > \lambda(A) > 0$. Consider
$g : X \to \mathbb{K}$ given by

$$g(x) := \begin{cases} 
  f(x), & f(x) \neq 0, \\
  \frac{\|f\|_1}{\lambda(A)}, & x \in A, \\
  0, & \text{otherwise.}
\end{cases}$$

Clearly, $g \in L_1(X)$ and by Theorem 3.0.2, $f \perp_B g$ but $g \not\perp_B f$.

**Case II:** $\lambda(\{x \in X : f(x) = 0\}) = 0$.

Since $f \in L_1(X)$ is a smooth point, from the proof of Theorem 3.0.3, for any $g \in L_1(X)$, $f \perp_B g$ if and only if

$$\int_X \text{sgn}(f(x))g(x)d\lambda(x) = 0.$$ 

Suppose, there do not exist disjoint $\Sigma$-atoms $A$ and $B$ such that $A \sqcup B = X$ and $\lambda(A)|f(x)| = \lambda(B)|f(y)|$ for almost every $x \in X$ and $y \in Y$. Then there exists $A \in \Sigma$ such that

$$0 < \int_A |f(x)|d\lambda(x) < \int_{X\setminus A} |f(x)|d\lambda(x).$$

Set $\alpha = \int_A |f(x)|d\lambda(x)$ and $\beta = \int_{X\setminus A} |f(x)|d\lambda(x)$. Now, $g : X \to \mathbb{K}$ given by

$$g(x) := \begin{cases} 
  \beta f(x), & x \in A, \\
  -\alpha f(x), & x \notin A,
\end{cases}$$

is a smooth point of $L_1(X)$ by Theorem 3.0.3. Hence for any $h \in L_1(X)$, $g \perp_B h$ if and only if

$$\int_X \text{sgn}(g(x))h(x)d\lambda(x) = 0 \iff \int_A \text{sgn}(f(x))h(x)d\lambda(x) - \int_{X\setminus A} \text{sgn}(f(x))h(x) = 0.$$ 

Hence by our choice of $A \in \Sigma$, $f \perp_B g$ but $g \not\perp_B f$. \hfill \Box
We conclude this section with the characterization of
the right-symmetric points of $L_1(X)$.

**Theorem 3.0.5.** A non-zero function $f \in L_1(X)$ is right-symmetric
if and only if $\{x \in X : f(x) \neq 0\}$ is a $\Sigma$-atom.

*Proof.* Clearly, if $A = \{x \in X : f(x) \neq 0\}$ is a $\Sigma$-atom, then by
Theorem 3.0.2, for any $g \in L_1(X)$, $g \perp_B f$ if and only if $g |_A \equiv 0$,
Hence $f \perp_B g$ if $g \perp_B f$.

Conversely, if there exist disjoint measurable subsets $A$ and
$B$ of finite positive measure such that $A \cup B \subseteq \{x \in X : f(x) \neq 0\}$,
then without loss of generality, we assume

$$0 < \int_A |f| d\lambda \leq \int_B |f| d\lambda.$$  

Setting $g : X \to \mathbb{K}$ given by

$$g(x) := \begin{cases} 
\text{sgn}(f(x)), & x \in A, \\
0, & x \notin A,
\end{cases}$$

we get $g \in L_1(X)$. Also, by Theorem 3.0.2, $g \perp_B f$ and $f \not\perp_B g$. □

4. **Birkhoff-James orthogonality and its pointwise
symmetry in $L_p(X)$, $p \in (1, \infty) \setminus \{2\}$**

In this section, we characterize Birkhoff-James orthogonality
and its pointwise symmetry in $L_p(X)$ for $1 < p < \infty$, $p \neq 2$.
It is well-known that $L_p(X)$ is smooth and hence the characterization of smoothness here is redundant. Our approach for $L_p(X)$,
$p \in (1, \infty) \setminus \{2\}$ is similar to the $L_1(X)$ case. We first study the
support functional (which is unique here as the space is smooth)
of a non-zero element and therefrom obtain a characterization of
Birkhoff-James orthogonality by James' characterization. The character-
ization of pointwise symmetry would then follow from the orthogonality characterization.

Let us fix $p \in (1, \infty) \setminus \{2\}$. The following theorem character-
izing the (unique) support functional of $f \in L_p(X) \setminus \{0\}$ follows
directly from the condition of equality in Hölder’s inequality.
Theorem 4.0.1. Let \( f \in L_p(X) \setminus \{0\} \) and let \( \frac{1}{p} + \frac{1}{q} = 1 \). Suppose \( g \in L_q(X) = L_p(X)^* \). Then \( g \in J(f) \) if and only if
\[
g(x) = \frac{1}{\|f\|_p^{-1} \text{sgn}(f(x)) |f(x)|^{p-1}}, \quad x \in X.
\]

Using this result, we now characterize Birkhoff-James orthogonality in \( L_p(X) \).

Theorem 4.0.2. If \( f, g \in L_p(X) \), then \( f \perp_B g \) if and only if
\[
\int_X \frac{\text{sgn}(f(X)) |f(x)|^{p-1} g(x) d\lambda(x)}{|f(x)|^p} = 0.
\]

We can now characterize pointwise symmetry of Birkhoff-James orthogonality in \( L_p(X) \).

Theorem 4.0.3. Suppose \( f \in L_p(X) \). Then \( f \) is left-symmetric if and only if \( f \) is right-symmetric if and only if exactly one of the following conditions holds:

1. \( f \equiv 0 \).
2. \( \{x \in X : f(x) \neq 0\} \) is a \( \Sigma \)-atom.
3. There exist \( \Sigma \)-atoms \( A \) and \( B \) such that \( \{x \in X : f(x) \neq 0\} = A \sqcup B \) and \( \lambda(A) |f(x)|^p = \lambda(B) |f(y)|^p \) for almost every \( x \in A \) and \( y \in B \).

Proof. The sufficiency can be obtained from Theorem 4.0.2 by an elementary computation. For the necessity, let us assume that \( f \neq 0 \) and \( \{x \in X : f(x) \neq 0\} \) is not a \( \Sigma \)-atom. We consider the following two cases:

Case I: There exist \( \Sigma \)-atoms \( A \) and \( B \) such that \( \{x \in X : f(x) \neq 0\} = A \sqcup B \).

If \( g \in L_p(X) \) with \( \{x \in X : g(X) \neq 0\} = A \sqcup B \) such that \( g \perp_B f \) and \( f \perp_B g \), then for almost every \( x \in A \) and \( y \in B \),
\[
g(x) = \frac{\lambda(B) |f(y)|^{p-1} \text{sgn}(f(y))}{\lambda(A) |f(x)|^{p-1} \text{sgn}(f(x))} g(y), \tag{4.1}
\]
and

\[ f(x) = \frac{\lambda(B)|g(y)|^{p-1} \text{sgn}(g(y))}{\lambda(A)|g(x)|^{p-1} \text{sgn}(g(x))} f(y). \]

Hence

\[ [\lambda(B)|f(y)|^p]^{p-2} = [\lambda(A)|f(x)|^p]^{p-2}. \]

Since \( p \neq 2 \), \( f \) must satisfy condition 3. However, using (4.1) or (4.2), we can always construct \( g \in L^p(X) \) with \( \{ x \in X : g(x) \neq 0 \} = A \sqcup B \) such that \( f \perp_B g \) or \( g \perp_B f \) respectively.

**Case II:** There exist \( A, B, C \in \Sigma \) disjoint such that all the sets are of finite positive measure and \( A \sqcup B \sqcup C \subseteq \{ x \in X : f(x) \neq 0 \} \).

Without loss of generality, let us assume that

\[ 0 < \int_A |f(x)|^p d\lambda(x) < \int_{B \cup C} |f(x)|^p d\lambda(x). \]

Consider \( g_{a,b} : X \to \mathbb{K} \) given by

\[ g_{a,b}(x) = \begin{cases} \ a f(x), & x \in A, \\ \ b f(x), & x \in B \sqcup C, \\ \ 0, & \text{otherwise}, \end{cases} \]

for some \( a, b \in \mathbb{K} \). Then,

\[ \int_X \text{sgn}(f(x))|f(x)|^{p-1} g_{a,b}(x) d\lambda(x) = a \int_A |f(x)|^p d\lambda(x) + b \int_B |f(x)|^p d\lambda(x), \]

and

\[ \int_X \text{sgn}(g_{a,b}(x))|g_{a,b}(x)|^{p-1} f(x) d\lambda(x) = \text{sgn}(a)|a|^{p-1} \int_A |f(x)|^p d\lambda(x) \\
+ \text{sgn}(b)|b|^{p-1} \int_B |f(x)|^p d\lambda(x) \]

Thus, \( f \perp_B g_{a,b} \) but \( g_{a,b} \not\perp_B f \) when

\[ a = \int_{B \cup C} |f(x)|^p d\lambda(x), \quad b = - \int_A |f(x)|^p d\lambda(x), \]
and \( g_{a,b} \perp_B f \) but \( f \not\perp_B g_{a,b} \) when

\[
a = \left[ \int_{B \cup C} |f(x)|^p \mathrm{d}\lambda(x) \right]^{\frac{1}{p-1}}, \quad b = -\left[ \int_{A} |f(x)|^p \mathrm{d}\lambda(x) \right]^{\frac{1}{p-1}}.
\]

\[ \square \]

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