Dark soliton dynamics in classical one-dimensional spin systems

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Abstract. Nonlinear effects in ferromagnetic Heisenberg chains of classical spins with both inter-site and on-site anisotropy have been studied. We obtained the condition for the appearance and stability of dark solitons in these systems. For narrow excitations the effect of the discreteness on the soliton properties and propagation is considered. The scattering of dark solitons from point defects in the discrete spin chain is investigated numerically and the role of the anisotropy is demonstrated.

1. Introduction
Considerable attention has been devoted to the study of solitary waves in solids. A large amount of theoretical and experimental research has been dedicated to one-dimensional magnetic systems [1]. Models corresponding to real quasi-one-dimensional magnets are broadly investigated as first they were considered as somewhat simpler than the really interesting three-dimensional systems but then turned out to be interesting in their own right. There was the availability of real magnetic compounds which because of extremely small coupling between neighboring chains, could be considered as reasonable realizations of magnetically one-dimensional materials. Soliton solutions for classical sine-Gordon chains [2] and Heisenberg chains with various anisotropies [3-6] were obtained and analyzed. In recent years there is a renewed interest to the topic due to their application. Spin-wave dark solitons were predicted and experimentally generated [7,8]. Solitons in magnetic thin films [9] and in ferromagnets with biquadratic exchange [10] were investigated.

Bright and dark solitons as exact solutions of the nonlinear Schrödinger (NLS) equation have been studied intensively [3,4,11]. But, the models describing the physical effects in solids are mainly discrete and the question arises how the discreteness modify the properties of the nonlinear excitations [12,13]. Another interesting topic of research with practical importance is the interaction of solitons with impurities [14-17].

In the present paper we study the condition for the appearance of dark solitons in an anisotropic ferromagnetic chain. The discreteness effects for narrow excitations are considered and the scattering on point defects is investigated.
2. Hamiltonian of the system
We consider a ferromagnetic Heisenberg chain of $N$ spins with magnitude $S$ described in the
nearest-neighbor approximation by the following Hamiltonian:

$$H = -\frac{J}{2} \sum_n (S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+) - J \sum_n S_n^z S_{n+1}^z - A \sum_n (S_n^z)^2 - H_0 \sum_n S_n^z,$$  \tag{1}

where the spin operators $S_n^\pm$, $S_n^z$ obey the commutation relations

$$[S^\pm, S^z] = \mp S^\pm \delta_{ij}, \quad [S^+, S^-] = 2S^z \delta_{ij}. \tag{2}$$

$J > 0$ and $\tilde{J} > 0$ are the exchange integrals and $A$ is the on-site anisotropy constant which can be
positive (easy axis) or negative (easy plane). If $\tilde{J} \neq J$ the inter-site anisotropy is included.
For $J = \tilde{J}$ and $A = 0$ the model is isotropic. $H_0$ is the external magnetic field applied along the
$z$-axis.

The equations of motion for the operators $S_n^+$ yield ($\hbar = 1$):

$$i \frac{\partial S_n^+}{\partial t} = (H_0 - A) S_n^+ - JS_n^z (S_{n+1}^+ + S_{n-1}^+) + \tilde{J} (S_{n+1}^z + S_{n-1}^z) S_n^+ + 2AS_n^z S_n^+.$$ \tag{3}

In the quasiclassical approximation, where the components of the spin operators $S_n$ are
complex amplitudes, $\alpha_n = S_n^+ / S$, $\alpha_n^* = S_n^- / S$ and $S_n^z / S = \sqrt{1 - |\alpha_n|^2}$, we have

$$i \frac{\partial \alpha_n}{\partial t} = (H_0 - A) \alpha_n - JS (\alpha_{n+1} + \alpha_{n-1}) \sqrt{1 - |\alpha_n|^2} + \tilde{J} S \alpha_n \left( \sqrt{1 - |\alpha_{n+1}|^2} + \sqrt{1 - |\alpha_{n-1}|^2} \right) + 2AS \alpha_n \sqrt{1 - |\alpha_n|^2}. \tag{4}$$

The set of differential equations (4) describes our system. In the continuum limit (wide
excitations), it transforms into a perturbed NLS, where the perturbing terms are due not only to
the discreteness but also to the complicated nonlinear interactions. For $J = A = 0$ and wide
excitations, (4) turns to the discrete Ablowitz-Ladik equation which is also completely integrable
and has soliton solutions. So, for narrow excitations it is necessary to investigate the whole set
(4). Equations similar to (4) has been derived for exciton solitons in molecular crystals [18].

3. Dark soliton solutions
We shall look for solutions in the form of amplitude-modulated waves

$$\alpha_n(t) = \varphi_n(t) e^{i(kn - \omega t)}, \tag{5}$$

where $k$ and $\omega$ are the wave number and the frequency of the carrier wave (the lattice constant
equals unity) and the envelope $\varphi_n(t)$ is a slowly varying function of the position and time. In the continuum limit, when the soliton width is much larger than the lattice spacing ($L \gg 1$),
equation (4) transforms into the following NLS equation for the envelope:

$$i \left( \frac{\partial \varphi}{\partial \tau} + 2J \sin k \frac{\partial \varphi}{\partial x} \right) = (\varepsilon - \Omega) \varphi - J \cos k \frac{\partial^2 \varphi}{\partial x^2} + g |\varphi|^2 \varphi, \tag{6}$$

where

$$\tau = tS, \quad \Omega = \omega / S, \quad \varepsilon = (H_0 - A) S^{-1} - 2g, \quad g = J \cos k - \tilde{J} - A. \tag{7}$$

Depending on the sign of the expression $gJ \cos k$ equation (6) possesses soliton solutions of
different types. For negative values bright solitons exist while for positive values dark solitons
are possible. It is interesting to point out that for the isotropic case (\(J = \tilde{J}, A = 0\)) equation (6) has bright-soliton solutions for \(0 < k < \pi/2\), while for the strong anisotropic case (\(J = A = 0\)) dark-soliton solutions are possible for arbitrary \(k\).

In what follows, we shall consider dark-soliton solutions which appear for

\[J \cos k(J \cos k - \tilde{J} - A) > 0\]  

and have nonvanishing boundary conditions, \(|\varphi(x)|^2 \to \text{const} \text{ at } x \to \pm \infty\). Note that the condition (8) depends not only on the anisotropy constants but also on the wave number \(k\). The dark soliton has the form

\[\varphi(x, \tau) = \varphi_0 \tanh \frac{x - v \tau}{L} + iB,\]  

where \(L\) and \(v\) are its width and velocity. \(B\) determines the value of the amplitude at the center, i.e. the minimum intensity of the soliton \(|\varphi(0)|^2 = B^2\), while the maximum intensity is \(|\varphi(\pm \infty)|^2 = \varphi_0^2 + B^2\). As independent parameters of the soliton we can choose \(L, k\) and \(B\). Then \(\varphi_0, \Omega\) and \(v\) are given by the relations:

\[\varphi_0^2 = \frac{2J \cos k}{gL^2}, \quad \Omega = \varepsilon + g(\varphi_0^2 + B^2), \quad v = 2J \sin k + B \sqrt{2gJ \cos k}.\]  

The dark soliton can be considered as an excitation of the background (carrier) wave. It is important to note that its velocity depends not only on \(k\) as in the case for bright solitons but also on \(B\). For fixed \(\varphi_0^2 + B^2\) and \(k\) solitons moving faster have smaller amplitude.

\[0.02\quad 0.03\quad 0.04\]
\[0 10 20 30\]
\[B = 0.07, k = 0 \quad B = 0, k = 0.015\]

Figure 1. Final dark-soliton velocity as a function of the width \(L\) for \(v_0 = 0.0314\) and \(J = \tilde{J} = 1, A = -0.1\).

4. Numerical results

Our numerical simulations are based on the general discrete equation (4). As initial function we put for \(t = 0\) the form

\[\alpha_n(0) = \left(\frac{\sqrt{2J \cos k/g}}{L} \tanh \frac{n}{L} + iB\right) e^{ikn}\]  

which is exact solution of the quasi-continuum equation, and observe the evolution for different anisotropic cases. For wide solitons (\(L \gg 1\)) this solution remain stable and its form and velocity are preserved. When the soliton width decreases the discreteness effects of equation (4) become important and the parameters of the dark soliton change.

The most significant and visible changes are in the velocity and they are shown on figure 1 for the case with on-site anisotropy (\(J = \tilde{J}, A < 0\)). The initial velocity is \(v_0 = 0.0314\), which according to (10) results from the following two pairs of the parameters \(k\) and \(B\): \(k = 0.015, B = 0\) and \(k = 0, B = 0.07\), i.e. we investigate separately the influence of the two parameters on which the velocity depends. For these two cases the modification of the velocity with \(L\) is obtained. The lower curve shows that for \(k \neq 0\) the velocity decreases with the decrease of
In this case with the decrease of the width the form of the soliton changes, there are small radiations, the kinetic energy decreases and as a consequence the velocity becomes smaller. The effect is similar as for narrow bright solitons. More interesting is the behavior of the upper curve for \( B \neq 0 \). In this case a decrease of the width \( L \) leads to an increase of the velocity \( v \). This effect is due to the fact that the changes in the form of the soliton for smaller \( L \) lead to a decrease of the amplitude and immediately to an increase of \( B \), respectively \( v \). This phenomenon is absent for bright solitons which have the amplitude and velocity as two independent parameters. We observed similar dependence \( v(L) \) for the other anisotropy cases of the discrete equation.

The evolution of a dark soliton with \( L = 5 \) and \( B = -0.0222 \) for different on-site anisotropy values is presented in figure 2. Here the velocity can be estimated as \( v_0 = B \sqrt{2A} \). For \( A = -0.1 \) \( v_0 = -0.0099 \), while the velocity evaluated from the data in figure 2(a) is \( v = -0.0117 \). For \( A = -0.5 \) \( v_0 = -0.0222 \) and \( v = -0.0228 \), respectively (figure 2(b)). Obviously higher values of \( |A| \) lead to higher values for \( v \) and the parameters of the exact solution of the discrete equation (4) are closer to the initial solution originated from the continuum approximation.

### 5. Scattering on point defects

We studied the interaction of the dark solitons with impurities in the chain whose size is small compared to the soliton extent. Then the equation of motion becomes

\[
\frac{\partial \alpha_n}{\partial t} = (H_0 - A)\alpha_n - JS(\alpha_{n+1} + \alpha_{n-1})\sqrt{1 - |\alpha_n|^2} + \tilde{J}S\alpha_n \left( \sqrt{1 - |\alpha_{n+1}|^2} + \sqrt{1 - |\alpha_{n-1}|^2} \right)
+ 2AS\alpha_n \sqrt{1 - |\alpha_n|^2} + d\delta_{n,n_0}\alpha_n.
\]

The parameter \( d \) describes the strength of the impurity at site \( n_0 \), which can be attractive \( (d < 0) \) or repulsive \( (d > 0) \). In our numerical simulations we investigate the interaction of a propagating dark soliton of the form (11) with a defect placed at the middle of the chain \( n_0 = 500 \). We have observed that for negative values of \( d \) (attraction) the solitons pass through the defect for arbitrary initial velocity. More interesting is the interaction with repulsive impurities.

For \( d > 0 \) the soliton can be transmitted or reflected by the impurity. For a given initial velocity \( v_0 \) there is a critical value of the defect \( d_c \) below which the soliton is transmitted and above which it is reflected. Figure 3 shows the scattering patterns for the case of inter-site anisotropy \( (J = 1, \tilde{J} = A = 0) \). The initial velocity \( v_0 = 0.0314 \) is determined by the wave number \( k \) (figures 3(a),(b)) or by the amplitude \( B \) (figures 3(c),(d)) and is conserved through the simulations. Figures 3(a) and (b) are for \( d \)-values around \( d_c \) and we observe transmission and
reflection, respectively. The results on figures 3(c) and (d) are for the same $d$ values but the soliton is only reflected. The difference is in the determination of the initial velocity and this is the reason for the different behavior. The determination of the velocity through $B$ leads to changes in the amplitude and as the energy balance for the process depends on the amplitude, the corresponding value of $d_c$ in this case is lower.

![Reflection](image)

Figure 3. Scattering of a dark soliton with $L = 10$ and $J = 1$, $\tilde{J} = A = 0$ for $B = 0$, $k = 0.0157$ [(a), (b)] or $B = 0.0222$, $k = 0$ [(c),(d)]. $d = 0.0108$ [(a), (c)], $d = 0.0109$ [(b), (d)]. The time is in units of $10^3/(JS)$.

Similar behavior is presented on Figure 4 for the case of on-site anisotropy ($J = \tilde{J}, A < 0$). In this case the discreteness effects become important and the actual values for the initial velocity differs from $v_0 = 0.0314$. For figures 4(a),(b) it is $v = 0.0303$ and for figures 4(c),(d) it is $v = 0.0324$. So, the critical value $d_c$ in the second case is higher.

6. Conclusion
We have investigated the existence and stability of dark solitons in a discrete ferromagnetic chain with different anisotropies. The velocity of the dark solitons is characterized by the wave number and the amplitude at the center of the soliton. This dependence of the velocity leads to new effects for narrow dark solitons which do not appear for bright solitons. The interaction of solitons with impurities was studied as well. Depending on the initial velocity and the strength of the repulsive impurities the soliton can be transmitted or reflected. For attractive impurities only transmission is observed.

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Figure 4. Scattering of a dark soliton with $L = 10$ and $J = \tilde{J} = 1$, $A = -0.1$ for (a): $B = 0$, $k = 0.0157$, $d = 0.00944$, (b): $B = 0$, $k = 0.0157$, $d = 0.00945$, (c): $B = 0.0702$, $k = 0$, $d = 0.0104$, (d): $B = 0.0702$, $k = 0$, $d = 0.0105$. The time is in units of $10^3/(JS)$.

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