LIOUVille TYPE THEOREMS FOR STABLE SOLUTIONS OF THE
WEIGHTED FRACTIONAL LANE-EMDEN SYSTEM

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Abstract. In this paper, we prove Liouville type theorems for stable solutions to the weighted fractional Lane-Emden system

\((-\Delta)^s u = h(x)v^p, \quad (-\Delta)^s v = h(x)u^q, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N,\)

where \(1 < q \leq p\) and \(h\) is a positive continuous function in \(\mathbb{R}^N\) satisfying \(\liminf_{|x| \to \infty} \frac{h(x)}{|x|^{\ell}} > 0\) with \(\ell > 0\).

Our results generalize the results established in [23] for the Laplacian case (correspond to \(s = 1\)) and improve the previous work [13]. As a consequence, we prove classification result for stable solutions to the weighted fractional Lane-Emden equation \((-\Delta)^s u = h(x)u^p\) in \(\mathbb{R}^N\).

2010 Mathematics Subject Classification: 35J55, 35J65, 35B33, 35B65.

Key words: Liouville type theorems, Stable solutions, Weighted fractional Lane-Emden system and equation

Contents

1. Introduction and main results \hspace{1cm} 1
2. Preliminaries \hspace{1cm} 4
3. Proof of Theorem 1.2 and Corollary 1.3. \hspace{1cm} 7
References \hspace{1cm} 14

1. INTRODUCTION AND MAIN RESULTS

Let \(s \in (0, 1)\) and consider the weighted fractional Lane-Emden system

\((-\Delta)^s u = h(x)v^p, \quad (-\Delta)^s v = h(x)u^q, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N,\)

(1.1)

where \(p, q \geq 1\) and \(h\) is a function satisfying

\(h \in C(\mathbb{R}^N), \quad h > 0, \quad \liminf_{|x| \to \infty} \frac{h(x)}{|x|^{\ell}} > 0, \quad \ell > 0.\)

Assume that \(u \in C^{2\alpha}_{\text{loc}}(\mathbb{R}^N) \cap L_s(\mathbb{R}^N)\) for some \(\alpha > s\) with

\[ \mathcal{L}_s(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R}; \int_{\mathbb{R}^N} \frac{|u(y)|}{(1 + |y|)^{N+2s}}dy < \infty \right\}. \]

Then the fractional Laplacian is defined by

\((-\Delta)^s u(x) = c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}dy,\)

where \(c_{N,s}\) is a normalization constant. Clearly, \((-\Delta)^s u(x)\) is well-defined at any \(x \in \mathbb{R}^N\), for any \(u\) in \(C^{2\alpha}_{\text{loc}}(\mathbb{R}^N) \cap L_s(\mathbb{R}^N)\) for some \(\alpha > s\). In what follows, all solutions of (1.1) are considered in the above space.

In this paper, we are interested in the stable solutions to system (1.1). Motivated by [32, 4, 19], we adopt the following definition of stability...
Suppose that argument, he proved in \([\text{radial or not}]. \) Let \(N\) in low dimensions, many authors in the past years, see for instance [39, 4, 2, 25, 27, 26, 23, 33]. If moreover \(h \equiv 1\), the system is reduced to the classical Lane Emden system
\[
-\Delta u = v^p, \quad -\Delta v = u^q, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N.
\]
It has been conjectured that \((1.3)\) admits smooth solutions if and only if \(p, q > 0\) and
\[
\frac{1}{p + 1} + \frac{1}{q + 1} \leq 1 - \frac{2}{N}.
\]
This conjecture was completely solved for the radial case by [30] (see also [36]) and for general case in low dimensions \(N \leq 4\), see [37]. Very recently, Mtiri-Ye [33] proved this conjecture for classical solutions which are stable at infinity.

For \(p, q \geq 1\), Chen-Dupaigne-Ghergu established in [2] the optimal Liouville type result for radial stable solutions of \((1.3)\). In [4], Cowan developed a new approach to deal with general stable solutions (radial or not). Let
\[
t_0^\pm = \sqrt{\frac{pq(q+1)}{p+1} \pm \sqrt{\frac{pq(q+1)}{p+1} - \frac{pq(q+1)}{p+1}}},
\]
by combining integral estimates derived from the stability, comparison principle and a bootstrap argument, he proved in [4] the nonexistence result for stable solutions of \((1.3)\) for \(p \geq q > 2t_0^\pm\).

The new approach of Cowan was exploited by many authors to various elliptic systems [25, 26, 27, 23, 14, 11]. In [25], Hu adapted this approach to study stable solutions of
\[
-\Delta u = h(x)v^p, \quad -\Delta v = h(x)u^q, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N,
\]
with \(h(x) = (1 + |x|^2)^{\frac{\ell}{2}}, \ell \geq 0\), obtaining then a generalization of the results in [4]. Later on, the author of the present paper with his collaborators [23] improved the results in [25]. In particular, they established a new inverse comparison principle (analogous to \((2.8)\) below) which is crucial to handle the cases \(1 \leq q \leq \frac{4}{3}\). More precisely, they obtained the same classification result provided by Theorem 1.2 (see below) of stable solutions to \((1.5)\) with radial function \(h\) satisfying \(h(x) \geq C(1 + |x|^2)^{\frac{\ell}{2}}, \ell \geq 0\).

For the non local Lane-Emden system \((1.1)\), the first result (up to our knowledge) classifying stable solutions is proved in [13] for \(h = 1\) and \(s \in (0, 1)\):
\[
(-\Delta)^s u = v^p, \quad (-\Delta)^s v = u^q, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N.
\]
Adopting the approach in [4], Duong-Nguyen established the following nonexistence result.

**Theorem A.** Let \(0 < s < 1\).

(1) If \(p \geq q > \frac{4}{3}\) and
\[
N < 2s + \frac{4s(p+1)}{pq-1} t_0^+, \quad (1.7)
\]
then the system \((1.6)\) has no stable solution.

(2) If \(1 < q \leq \min(p, \frac{4}{3})\), \(t_0 < \frac{2}{3}\) and \((1.7)\), then \((1.6)\) has no stable solution.

In this paper, our aim is to generalize [23, 13]. Our main results state as follows

**Theorem 1.2.** Suppose that \(h\) satisfies \((H)\) and let \(x_0\) be the largest root of the polynomial \(H(x) = x^4 - \frac{16pq(q+1)(p+1)}{(pq-1)^2} x^2 + \frac{16pq(q+1)(p+1)(p+q+2)}{(pq-1)^3} x - \frac{16pq(q+1)^2(p+1)^2}{(pq-1)^4}\). \[(1.8)\]
(1) If $\frac{4}{3} < q \leq p$ then (1.1) has no stable solution if $N < 2s + (2s + \ell)x_0$. In particular, if $N \leq 10s + 4\ell$, (1.1) has no stable solution for all $\frac{4}{3} < q \leq p$.

(2) If $1 < q \leq \min\left(\frac{4}{3}, p\right)$, then (1.1) has no bounded stable solution, if
\[
N < 2s + \left[\frac{q}{2} + \frac{(2-q)(pq-1)}{(p+q-2)(p+1)}\right](2s + \ell)x_0. \tag{1.9}
\]

Therefore, if $N \leq 6s + 2\ell$, the system (1.1) has no bounded stable solution for all $p \geq q > 1$.

- By [23, Lemma 6], for any $1 < q \leq p$, we have $2t_0^+ \frac{p+1}{pq-1} \leq x_0$, and the equality holds if and only if $p = q$. Hence the range of nonexistence result in Theorem 1.2 is larger than that provided by Theorem A.
- Using [23, Remark 2], we have $2t_0^- < q$ if $q > \frac{4}{3}$. It means that the classification result in Theorem A is valid for $\frac{4}{3} < q \leq p$. However, our approach allows us to provide, for the first time, a rigorous proof for the nonexistence of stable solution to the system (1.1) with $1 < q \leq \frac{4}{3}$.

Consider now the weighted fractional Lane Emden equation:
\[
(-\Delta)^s u = h(x)|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N. \tag{1.10}
\]
For the local case $(s = 1)$ and when $h \equiv 1$, Farina completely classified in [17] finite Morse index solutions to
\[
-\Delta u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N. \tag{1.11}
\]
He proved that (1.11) has nontrivial classical solution with finite Morse index if and only if $N \geq 3$, $p = \frac{N+2}{N-2}$ or $N \geq 11$ and $p \geq p_{JL} = p_{JL}(N,0)$, where $p_{JL}$ stands for the Joseph-Lundgren exponent [29] (see also [22]). Later on, Dancer-Du-Guo [7] obtained a sharp critical exponent $p_{JL}(N,\ell)$ with respect to the existence of nontrivial stable solutions $u \in W_{loc}^{1,2}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ of (1.10) with $s = 1$, $h(x) = |x|^\ell$, $\ell > -2$. Here $p_{JL}(N,\ell)$ is given by
\[
p_{JL}(N,\ell) = \begin{cases} +\infty \\ \frac{(N-2)^2-2(\ell+2)(\ell+N)-2\sqrt{(\ell+2)^2(\ell+2N-2)}}{(N-4\ell-10)(N-2)} \end{cases} \quad \text{if} \quad 1 \leq N \leq 10 + 4\ell,
\]
\[
\frac{1}{N-4\ell-10} \quad \text{if} \quad N > 10 + 4\ell.
\]
Finally, the same result is proved to be true without the local boundedness assumption by Wang-Ye [38]. Many other papers studied stable solutions of (1.10) with $s = 1$, see for instance [5, 8, 28, 3, 24]. In particular, Farina-Hasegawa [18] proved Liouville type results for stable solutions to (1.10) with $s = 1$ and a larger class of weights $h$ which cover many existing results.

Davila-Dupaigne-Wei [9] examined the equation (1.10) with $h \equiv 1$, and classified finite Morse index solutions in the autonomous case. The approach developed in [9] is based on the monotonicity formula and some energy estimates, this approach was adapted to classify stable or stable solutions outside a compact set to (1.10) with $h(x) = |x|^\ell$, $\ell \geq 0$ (see [20, 21]) as well as for fractional elliptic equation involving advection term (see [35]).

Here we obtain classification result for the fractal equation by the study of the system. In fact, when $p = q$, using Souplet type estimate (2.7), the system (1.1) is reduced to the following fractional Lane Emden equation
\[
(-\Delta)^s u = h(x)u^p, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.12}
\]
As a consequence of Theorem 1.2, we can claim

Corollary 1.3. Suppose that $h$ satisfies (H) and let $p > 1$.

(1) If $\frac{4}{3} < p$ then (1.12) has no stable solution if
\[
N < 2s + \frac{2(2s + \ell)}{p-1} \left(p + \sqrt{p^2 - p}\right). \tag{1.13}
\]

In particular, if $N \leq 10s + 4\ell$, then (1.12) has no stable solution for all $\frac{4}{3} < p$.

(2) If $1 < p \leq \frac{4}{3}$, (1.12) has no bounded stable solution for $N$ verifying (1.13).
Therefore, there is no bounded stable solution of (1.12) for all \( p > 1 \) if \( N \leq 10s + 4\ell \).

In [13], Duong-Nguyen studied a more general fractional equation, obtaining similar results of [15, Theorem 1.5] for the Laplacian case.

\[
(-\Delta)^s u = f(u) \quad \text{in} \quad \mathbb{R}^N, \quad 0 < s < 1, \tag{1.14}
\]

where \( f \in C^0(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) \) and for \( t > 0 \), let

\[
q(t) = \frac{f^2}{ff''(t)}, \quad \text{if} \quad ff''(t) \neq 0; \quad q(t) = +\infty, \quad \text{if} \quad ff''(t) = 0.
\]

Duong-Nguyen proved that if moreover \( f \) is nondecreasing, convex, \( f > 0 \) in \( \mathbb{R}_+ \) and \( q_0 = \lim_{t \to 0^+} q(t) \in [1, +\infty] \) exists. Then (1.14) has no nontrivial bounded nonnegative stable solution if one of the following conditions is satisfied: (i) \( N < 10s \); (ii) \( N = 10s \) and \( q_0 > 1 \); (iii) \( N > 10s \) and \( p_0 \) the conjugate exponent of \( q_0 \) satisfies \( p_0 < p_c(N,s) \), where

\[
p_c(N,s) = \frac{(N - 2s)^2 - 4sN + 8s\sqrt{s(N-s)}}{(N - 10s)(N - 2s)}.
\]

- If \( f(u) = u^p, \quad p \geq 1 \) in (1.14), then \( p = p_0 \). We can check that the range of \( p \) for the nonexistence of stable solution provided by Corollary 1.3 with \( h \equiv 1 \), is the same given by the above result of Duong-Nguyen.

- In Corollary 1.3, we prove the classification result for stable solution of (1.12) for \( p > \frac{4}{3} \), without assuming the boundedness of \( u \).

- Our approach permits to establish Liouville type results for stable solutions to (1.12) with weights that are not covered (up to our knowledge) by previous works.

This paper is organized as follows. In Section 2, we prove comparison properties between \( u \) and \( v \) of solutions to (1.1), and integral estimates derived from the stability. The proof of Theorem 1.2 and Corollary 1.3 are given in Section 3. In the following, \( C \) will denote a generic positive constant independent on \( (u,v) \), which could be changed from one line to another. The ball of center 0 and radius \( r > 0 \) will be denoted by \( B_r \).

## 2. Preliminaries

In this section, we introduce some preliminary results for solutions to the system (1.1), as integral estimates; comparison property of \( u \), \( v \); and an integral inequality derived from the stability.

We shall use a standard tool due to Caffarelli-Silvestre [1] which transforms the nonlocal system (1.1) to a degenerate but local elliptic system, with nonlinear Neumann boundary condition in the half space \( \mathbb{R}_+^{N+1} \). More precisely, let \( (u,v) \) be a solution of (1.1), the extension \( (U,V) \) of \( (u,v) \) in the sense of [1] is defined as follows: for \( (x,t) \in \mathbb{R}_+^{N+1} \),

\[
U(x,t) = \int_{\mathbb{R}^N} P_s(x - z,t)u(z)dz, \quad V(x,t) = \int_{\mathbb{R}^N} P_s(x - z,t)v(z)dz \tag{2.1}
\]

where \( P_s(x,t) \) is the Poisson kernel

\[
P_s(x,t) = C(N,s) \frac{t^{2s}}{(|x|^2 + t^2)^{\frac{N+s}{2}}},
\]

and \( C(N,s) \) is a normalization constant. Then \( U,V \in C^2(\mathbb{R}_+^{N+1}) \cap C(\overline{\mathbb{R}_+^{N+1}}) \), \( t^{1-2s}\partial_t U, t^{1-2s}\partial_t V \in C(\overline{\mathbb{R}_+^{N+1}}) \) satisfy

\[
\begin{cases}
-\text{div}(t^{1-2s}\nabla U) = 0 & \text{in} \quad \mathbb{R}_+^{N+1} \\
U = u & \text{on} \quad \partial\mathbb{R}_+^{N+1} \\
-\lim_{t \to 0} t^{1-2s}\partial_t U = \kappa_s(-\Delta)^s u & \text{on} \quad \partial\mathbb{R}_+^{N+1} \tag{2.2}
\end{cases}
\]
and
\[
\begin{aligned}
&-\text{div}(t^{1-2s}\nabla V) = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
&V = v & \text{on } \partial \mathbb{R}^{N+1}_+ \\
&-\lim_{t\to 0} t^{1-2s} \partial_t V = \kappa_s(-\Delta)^s v & \text{on } \partial \mathbb{R}^{N+1}_+.
\end{aligned}
\] (2.3)

Here \( \kappa_s = \frac{\Gamma((1-s)/2)}{2\pi^{(1-s)/2}} \) and \( \Gamma \) is the usual Gamma function. For any \( W \in C(\overline{\mathbb{R}^{N+1}}) \) and \( r > 0 \), we define
\[
\overline{W}(r) := \frac{1}{r^{N+1-2s}} \int_{\partial^+ B_r(y)} y^{1-2s} W,
\]
where \( B_r^+ := B_r \cap \{y > 0\} \) is the half-ball with spherical part of boundary \( \partial^+ B_r = \partial B_r \cap \{y > 0\} \).

Inspired by [40, 36, 31], we establish the following a priori integral estimates for solutions to the fractional Lane-Emden system (1.1).

**Lemma 2.1.** Let \( p, q \geq 1 \), \( pq > 1 \). Suppose that \( h \) satisfies (H). Then, there exists a positive constant \( C \) depending only on \( N, s, p \) and \( q \) such that for any solution \( (u, v) \) of (1.1) and any \( R > 0 \), there hold
\[
\int_{B_R} h(x) u^p(x) dx \leq CR^{-\frac{2a(p+1)}{pq} - \frac{(p+1)}{pq} - 1}, \quad \int_{B_R} h(x) v^p(x) dx \leq CR^{-\frac{2a(q+1)}{pq} - \frac{(q+1)}{pq} - 1}. \] (2.4)

**Proof.** Let \( \varphi_0 \in C^\infty_0(B_2) \) be a cut-off function verifying \( 0 \leq \varphi_0 \leq 1 \), and \( \varphi_0 = 1 \) for \( x \in B_1 \). Let \( R > 0 \) and consider \( \psi := \varphi_0(R^{-x}) \). By [40, Lemma 2.1], we have for \( m \geq 1 \),
\[
(-\Delta)^s \psi^m \leq mR^{-2s}[\psi^{m-1}(-\Delta)^s \psi](R^{-1}x). \] (2.5)

Multiplying the equation \( (-\Delta)^s u = h(x)v^p \) by \( \psi^m \) and integrating by parts, there holds
\[
\int_{\mathbb{R}^N} h(x)v^p \psi^m dx = \int_{\mathbb{R}^N} u(-\Delta)^s \psi^m dx \leq \frac{C}{R^{2s}} \int_{B_{2R}} u \psi^{m-1} dx.
\]
By Hölder’s inequality, we obtain
\[
\int_{\mathbb{R}^N} h(x)v^p \psi^m dx \leq \frac{C}{R^{2s}} \left( \int_{B_{2R}} h(x)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} \left( \int_{B_{2R}} u \psi^{(m-1)q} dx \right)^{\frac{1}{q}},
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \). From (H) we deduce that
\[
\int_{\mathbb{R}^N} h(x)v^p \psi^m dx \leq CR^{\frac{N}{pq} - \frac{s}{p} - 2s} \left( \int_{B_{2R}} h(x) u^{q} \psi^{(m-1)q} dx \right)^{\frac{1}{q}}.
\]
Similarly, using \( (-\Delta)^s v = h(x)u^q \), for \( k \geq 1 \),
\[
\int_{\mathbb{R}^N} h(x)u^q \psi^k dx \leq CR^{\frac{N}{pq} - \frac{s}{p} - 2s} \left( \int_{B_{2R}} h(x)v^p \psi^{(k-1)p} dx \right)^{\frac{1}{p}},
\]
where \( \frac{1}{p} + \frac{1}{p} = 1 \). Take now \( k \) and \( m \) large verifying \( m \leq (k-1)p \) and \( p \leq (m-1)q \). Combining the two above inequalities, we get
\[
\int_{\mathbb{R}^N} h(x)v^p \psi^m dx \leq CR^{\frac{N}{pq} - \frac{s}{p} - 2s} R^{(\frac{N}{pq} - \frac{s}{p} - 2s)\frac{1}{p}} \left( \int_{B_{2R}} h(x)v^p \psi^{(k-1)p} dx \right)^{\frac{1}{p}} \\
\leq CR^{\frac{N}{pq} - \frac{(p+1)}{pq} - \frac{2a(q+1)}{pq} - \frac{(q+1)}{pq} - 1} \left( \int_{\mathbb{R}^N} h(x)v^p \psi^m dx \right)^{\frac{1}{q}}.
\]
Hence
\[
\int_{B_R} h(x)v^p dx \leq \int_{\mathbb{R}^N} h(x)v^p \psi^m dx \leq CR^{\frac{2a(q+1)p}{pq} - \frac{(q+1)}{pq} - 1}.
\]
Similarly, we obtain the estimate for \( u \). \( \square \)
An immediate consequence of Lemma 2.1 is the following non-existence theorem for the fractional weighted Lane Emden system (1.1).

**Corollary 2.2.** Let \( p, q \geq 1 \), \( pq > 1 \). Suppose that \( h \) satisfies (H). Then, there exists no solution to (1.1) if \( N < \frac{2pq + 4}{pq + 4} \times \max(p, q) \).

Using Lemma 2.1, we can also give an alternative proof of [13, Proposition 3.2] under the same assumptions of Lemma 2.1. Consider for example

\[
\int_{\mathbb{R}^N} h(x)u^q(x)\rho(\frac{x}{R})dx \leq CR^{N - \frac{2pq(p+1)}{pq + 4} - \frac{l(p+1)}{pq + 4}}. \tag{2.6}
\]

Here and after \( \rho(x) := (1 + |x|^2)^{-\frac{N + 2s}{2}} \). Indeed, let \( k \in \mathbb{N}^* \). Then \( \rho(\frac{x}{R}) \) \leq 1 for any \( x \in B_R \) and \( \rho(\frac{x}{R}) \leq 2^{-(k-1)(N+2s)} \) for any \( x \in B_{2^kR} \setminus B_{2^{k-1}R} \). Applying Lemma 2.1, it follows that

\[
\int_{\mathbb{R}^N} h(x)u^q(x)\rho(\frac{x}{R})dx = \int_{B_R} h(x)u^q(x)\rho(\frac{x}{R})dx + \sum_{k \in \mathbb{N}^*} \int_{B_{2^kR} \setminus B_{2^{k-1}R}} h(x)u^q(x)\rho(\frac{x}{R})dx
\]

\[
\leq \int_{B_R} h(x)u^q(x)dx + \sum_{k \in \mathbb{N}^*} 2^{-(k-1)(N+2s)} \int_{B_{2^kR} \setminus B_{2^{k-1}R}} h(x)u^q(x)dx
\]

\[
\leq CR^{N - \frac{2pq(p+1)}{pq + 4} - \frac{l(p+1)}{pq + 4}} \left[ 1 + \sum_{k \in \mathbb{N}^*} 2^{-(k-1)(N+2s)+k(N - \frac{2pq(p+1)}{pq + 4} - \frac{l(p+1)}{pq + 4})} \right]
\]

\[
\leq CR^{N - \frac{2pq(p+1)}{pq + 4} - \frac{l(p+1)}{pq + 4}}.
\]

The following is a comparison result between the components \( u, v \) of solutions to the system (1.1).

**Proposition 2.1.** Let \( p \geq q \geq 1 \) and \( pq > 1 \). Suppose that \( h \) satisfies (H) and \( (u,v) \) is a solution to (1.1). There holds then

\[
u^{p+1} \leq \frac{p+1}{q+1}u^{q+1}. \tag{2.7}
\]

If moreover \( v \) is bounded, then

\[u \leq \|v\|_{\infty}^{\frac{p}{p-q} - 1} v. \tag{2.8}\]

**Proof.** The proof adapt an idea of [40], originally coming from [34]. Let \( w := v - lw^\sigma \), where \( \sigma = \frac{q+1}{p+1} \) and \( l = \frac{\sigma}{1-\sigma} \), the proof of (2.7) consists to show that

\[(-\Delta)^s w \leq 0 \quad \text{in the set} \quad \{ w \geq 0 \}. \tag{2.9}\]

Indeed, let \( W \) be the extension of \( w \) in the sense of (2.2)-(2.3). Using (2.9), then \( W \) satisfies

\[
\begin{cases}
-\text{div}(l^{1-2s}\nabla W) = 0 & \text{in } \mathbb{R}^N_{+1} \\
-\lim_{t \to 0} l^{1-2s}\partial_t W = \kappa_s(-\Delta)^s w \leq 0 & \text{on } \{ W \geq 0 \} \cap \partial \mathbb{R}^{N+1}_{+}. \tag{2.10}
\end{cases}
\]

Moreover, by the integral estimate (2.4), there exist \( r_i \to +\infty \), \( x_i \in B_{r_i} \) such that

\[v^p(x_i) r_i^N \leq CR_i^{N - \frac{2pq(p+1)}{pq + 4} - \frac{l(p+1)}{pq + 4}}, \]

which implies \( \lim_{i \to +\infty} v(x_i) = 0 \). Applying [40, Lemma 3.2], there hold

\[0 \leq \lim_{r \to +\infty} \nabla (r) \leq Av(x_i), \quad \forall i \in \mathbb{N} \text{ for some positive constant } A. \]

Tending \( i \) to \( +\infty \), we obtain \( \lim_{r \to +\infty} \nabla (r) = 0 \). Furthermore, \( 0 \leq W_+(r) \leq \nabla (r) \), where \( W_+ = \max(W, 0) \). Hence, \( \lim_{r \to +\infty} W_+(r) = 0 \). Following the same lines in the proof of [40, Lemma 3.1], we derive that \( W \leq 0 \) and hence \( w \leq 0 \), i.e. (2.7).

To get (2.9), consider the concave function \( t^\sigma \) in \( \mathbb{R}_+ \),

\[u^\sigma(x) - u^\sigma(y) \geq \sigma u^\sigma-1(x)(u(x) - u(y)), \quad \forall \ x, y.\]
Hence

\[-(\Delta)^s u^\sigma(x) = \int_{\mathbb{R}^N} \frac{u^\sigma(x) - u^\sigma(y)}{|x-y|^{N+2s}} dy \geq \sigma u^{\sigma-1}(x) (-(\Delta)^s u(x)).\]

It follows that

\[-(\Delta)^s w = -(\Delta)^s v - l(\Delta)^s u^\sigma \leq hu^q - lhu^{\sigma-1} v^p = h [u^q - l^{-p} u^{\sigma-1} v^p] = l^{-p} u^{\sigma-1} [(lu)^p - v^p] \leq 0 \text{ on the set } \{ w \geq 0 \}.

So we are done. To prove (2.8), consider \( w = u - lv \) with \( l = \|v\|_\infty \) and we will establish again (2.9).

As \( p \geq q \) and \( v \) is bounded, there holds

\[-(\Delta)^s w = h(x)v^p - lh(x)u^q \leq h(x) (v^p - lu^q) = h(x) \left( \frac{v}{\|v\|_\infty} \right) \|v\|_\infty - lu^q \leq h(x) \left( \frac{v}{\|v\|_\infty} \right) \|v\|_\infty - lu^q \]

\[= h(x) \|v\|_{\infty}^{p-q} \left| v^q - l u^q \right| \|v\|_\infty^{\sigma-1} \left( u^q - l^{-q} u^q \right) \]

Therefore, we get (2.9) and the proof is completed. \( \square \)

**Remark 1.** Let \((u, v)\) be a solution of (1.1) and \((U, V)\) be the extension of \((u, v)\) in the sense of (2.2)-(2.3). With the assumptions of Proposition 2.1, there holds

\[
\frac{V_{p+1}}{p+1} \leq \frac{U_{q+1}}{q+1}. \tag{2.11}
\]

Indeed, using the same notations as above, we have \( \frac{1}{p} \geq 1 \). Hence, by Jensen’s inequality and (2.7), we get for \((x, t) \in \mathbb{R}^{N+1}_+ \),

\[
(V(x, t))^{\frac{1}{p}} \leq \int_{\mathbb{R}^N} P_s(x - z, t) v(z) dz \leq l^{\frac{1}{p}} \int_{\mathbb{R}^N} P_s(x - z, t) u(z) dz = l^{\frac{1}{p}} U(x, t).
\]

At last, using the Definition 1.1 of stability, we can derive the following estimate which is crucial for our analysis. Its proof comes from ideas in [6, 16, 4, 13] and is very similar to the mentioned works, so we omit the details.

**Lemma 2.3.** Let \((u, v)\) be a stable solution of (1.1). Then for all \( \phi \in C_c^\infty(\mathbb{R}^N) \), we have

\[
\sqrt{pq} \int_{\mathbb{R}^N} h(x)u(x)^{\frac{p+1}{2}} v^\frac{q+1}{2} \phi(x)^2 dx \leq \frac{CN_s}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\phi(x) - \phi(y)|)^2}{|x-y|^{N+2s}} dy dx. \tag{2.12}
\]

### 3. Proof of Theorem 1.2 and Corollary 1.3.

Assume that \((u, v)\) is a stable solution of (1.1) and \( h \) satisfies \((H)\). Denote by \( U, V \) the extension of \( u \) and \( v \) in the sense of (2.2)-(2.3) and define \( \zeta(x, t) := (1 + |x|^2 + t^2)^{-\frac{N+2s}{2}} \) an extension of \( \rho(x)^\frac{1}{2} \) on \( \mathbb{R}^{N+1}_+ \).

**Lemma 3.1.** For any \( \gamma > \frac{N+2}{2} \) satisfying \( L(\gamma) < 0 \) and \( \Phi \in C_c^\infty(\mathbb{R}^{N+1}_+) \), there exists \( C > 0 \) such that

\[
\int_{\mathbb{R}^{N+1}_+} |\nabla (U^\frac{1}{2} \zeta \Phi)|^2 t^{1-2s} dx dt \leq C \int_{\mathbb{R}^{N+1}_+} U^\gamma |\nabla (\zeta \Phi)|^2 t^{1-2s} dx dt \tag{3.1}
\]

where

\[
L(\gamma) := \gamma^4 - 16 \frac{pq(q+1)}{p+1} \gamma^2 + 16 \frac{pq(q+1)(p+q+2)}{(p+1)^2} \gamma - 16 \frac{pq(q+1)^2}{(p+1)^2}. \tag{3.2}
\]
Proof. Let \( \Phi \in C_c^\infty(\mathbb{R}_+^{N+1}) \) be a test function and define \( \phi(x) = \Phi(x, 0) \in C_c^\infty(\mathbb{R}^N) \). Let \( \gamma > 1 \).

Multiplying the first equation in (2.2) by \( U^{\gamma - 1}(\zeta \Phi)^2 \) and integrating by parts, we get

\[
\kappa_s \int_{\mathbb{R}^N} h(x) v(x)^p u(x)^{\gamma - 1} \rho(x) \phi(x)^2 \, dx = \int_{\mathbb{R}_+^{N+1}} \nabla U \cdot \nabla \left( U^{\gamma - 1}(\zeta \Phi)^2 \right) t^{1-2s} \, dx dt
\]

\[
= (\gamma - 1) \int_{\mathbb{R}_+^{N+1}} |\nabla U|^2 U^{\gamma - 2}(\zeta \Phi)^2 t^{1-2s} \, dx dt + \frac{4}{\gamma} \int_{\mathbb{R}_+^{N+1}} \zeta \Phi \nabla (U \zeta) \cdot \nabla (\zeta \Phi) U \zeta t^{1-2s} \, dx dt
\]

\[
= \frac{4(\gamma - 1)}{\gamma^2} \int_{\mathbb{R}_+^{N+1}} |\nabla U \zeta|^2 (\zeta \Phi)^2 t^{1-2s} \, dx dt + \frac{4}{\gamma} \int_{\mathbb{R}_+^{N+1}} \zeta \Phi \nabla (U \zeta) \cdot \nabla (\zeta \Phi) U \zeta t^{1-2s} \, dx dt. \tag{3.3}
\]

Furthermore, there holds

\[
\int_{\mathbb{R}_+^{N+1}} |\nabla U \zeta|^2 (\zeta \Phi)^2 t^{1-2s} \, dx dt = \int_{\mathbb{R}_+^{N+1}} |\nabla (U \zeta \zeta \Phi)|^2 t^{1-2s} \, dx dt
\]

\[
- 2 \int_{\mathbb{R}_+^{N+1}} \zeta \Phi \nabla (U \zeta) \cdot \nabla (\zeta \Phi) U \zeta t^{1-2s} \, dx dt
\]

\[
- \int_{\mathbb{R}_+^{N+1}} U^\gamma |\nabla (\zeta \Phi)|^2 t^{1-2s} \, dx dt. \tag{3.4}
\]

Hence, using the Cauchy-Schwarz inequality, for any \( \epsilon > 0 \), we have

\[
\kappa_s \int_{\mathbb{R}^N} h(x) v(x)^p u(x)^{\gamma - 1} \rho(x) \phi(x)^2 \, dx = \frac{4(\gamma - 1)}{\gamma^2} \int_{\mathbb{R}_+^{N+1}} |\nabla (U \zeta \zeta \Phi)|^2 t^{1-2s} \, dx dt
\]

\[
- \frac{4(\gamma - 2)}{\gamma^2} \int_{\mathbb{R}_+^{N+1}} \zeta \Phi \nabla (U \zeta) \cdot \nabla (\zeta \Phi) U \zeta t^{1-2s} \, dx dt
\]

\[
- \frac{4(\gamma - 1)}{\gamma^2} \int_{\mathbb{R}_+^{N+1}} U^\gamma |\nabla (\zeta \Phi)|^2 t^{1-2s} \, dx dt \tag{3.5}
\]

\[
\geq \frac{4(\gamma - 1)}{\gamma^2} (1 - \epsilon) \int_{\mathbb{R}_+^{N+1}} |\nabla (U \zeta \zeta \Phi)|^2 t^{1-2s} \, dx dt
\]

\[
- \left( \frac{4}{\gamma^2} + \frac{C_\gamma}{\epsilon} \right) \int_{\mathbb{R}_+^{N+1}} U^\gamma |\nabla (\zeta \Phi)|^2 t^{1-2s} \, dx dt,
\]

Denote by \( A(\gamma, \epsilon) := \frac{4(\gamma^2 - 1)}{\gamma^2} (1 - \epsilon) \). It follows that

\[
\frac{1}{\sqrt{|\gamma|}} \int_{\mathbb{R}_+^{N+1}} |\nabla (U \zeta \zeta \Phi)|^2 t^{1-2s} \, dx dt \leq \frac{\kappa_s}{A(\gamma, \epsilon)} \int_{\mathbb{R}^N} h(x) v(x)^p u(x)^{\gamma - 1} \rho(x) \phi(x)^2 \, dx
\]

\[+ C \gamma, \epsilon \int_{\mathbb{R}_+^{N+1}} U^\gamma |\nabla (\zeta \Phi)|^2 t^{1-2s} \, dx dt. \tag{3.6}
\]
Similarly, multiplying the first equation in (2.3) by $V^{\gamma-1}(\zeta\Phi)^2$ and integrating by parts, we obtain

$$\frac{1}{\sqrt{pq}} \int_{\mathbb{R}^{N+1}_+} |\nabla(V^{\frac{2}{\gamma}}\zeta\Phi)|^2 t^{1-2\gamma} dx dt \leq \kappa_s \frac{A(\gamma, \epsilon)}{A(\gamma)} \int_{\mathbb{R}^N} h(x) u(x)^{q} v(x)^{\gamma-1} \rho(x) \phi(x)^2 dx$$

$$+ C_{\gamma, \epsilon} \int_{\mathbb{R}^{N+1}_+} V^{\gamma} |\nabla(\zeta\Phi)|^2 t^{1-2\gamma} dx dt.$$  

(3.7)

Combining (3.6) and (3.7), we derive that, for any $\gamma_1, \gamma_2 > 1$,

$$A(\gamma_1, \epsilon) \frac{2\gamma_1}{q+1} I_1 + I_2 := A(\gamma_1, \epsilon) \frac{2\gamma_1}{q+1} \frac{1}{\sqrt{pq}} \int_{\mathbb{R}^{N+1}_+} |\nabla(U^{\frac{2}{\gamma}} \zeta \Phi)|^2 t^{1-2\gamma} dx dt$$

$$\leq \kappa_s A(\gamma_1, \epsilon) \frac{2\gamma_1-1-q}{q+1} \int_{\mathbb{R}^N} h(x) u(x)^{q} v(x)^{\gamma-1} \rho(x) \phi(x)^2 dx$$

$$+ \frac{\kappa_s}{A(\gamma_2, \epsilon)} \int_{\mathbb{R}^N} h(x) u(x)^{q} v(x)^{\gamma-1} \rho(x) \phi(x)^2 dx$$

$$+ C_{\epsilon} \int_{\mathbb{R}^{N+1}_+} (U^{\gamma_1} + V^{\gamma_2}) |\nabla(\zeta\Phi)|^2 t^{1-2\gamma} dx dt.$$  

(3.8)

Fix now

$$\gamma_2 = \frac{(p+1)}{q+1} \gamma_1 \iff \gamma_2 - 1 = \frac{p+1}{q+1} (\gamma_1 - 1) + \frac{p-q}{q+1}.$$  

(3.9)

Let $\gamma_1 > \frac{q+1}{p}$, by Young’s inequality, there holds

$$\frac{\kappa_s}{A(\gamma_2, \epsilon)} \int_{\mathbb{R}^N} h(x) u(x)^{q} v(x)^{\gamma_2-1} \rho(x) \phi(x)^2 dx$$

$$= \frac{\kappa_s}{A(\gamma_2, \epsilon)} \int_{\mathbb{R}^N} h(x) u(x)^{\frac{q+1}{q+\gamma_2}} v(x)^{\frac{q}{q+\gamma_2}} \frac{(q+1)\gamma_2}{q+\gamma_2} \rho(x) \phi(x)^2 dx$$

$$= \frac{\kappa_s}{A(\gamma_2, \epsilon)} \int_{\mathbb{R}^N} h(x) u(x)^{\frac{q}{q+\gamma_2}} v(x)^{\frac{q}{q+\gamma_2}} \frac{(q+1)\gamma_2}{q+\gamma_2} u(x)^{\frac{q+1}{q+\gamma_2}} \rho(x) \phi(x)^2 dx$$

$$\leq \frac{2\gamma_2 - 1}{2\gamma_1} \kappa_s \int_{\mathbb{R}^N} h(x) u(x)^{\frac{q}{q+\gamma_2}} v(x)^{\frac{q}{q+\gamma_2}} v(x)^{\gamma_2} \rho(x) \phi(x)^2 dx$$

$$+ \frac{q+1}{2\gamma_1} A(\gamma_2, \epsilon) \frac{2\gamma_1}{q+1} \kappa_s \int_{\mathbb{R}^N} h(x) u(x)^{\frac{q}{q+\gamma_2}} v(x)^{\frac{q}{q+\gamma_2}} u(x)^{\gamma_2} \rho(x) \phi(x)^2 dx.$$

Choosing the test function $u^{\frac{q}{2\gamma_1-2}} \rho_{N+2s}^\frac{1}{\gamma_2}$ (resp. $v^{\frac{q}{2\gamma_1-2}} \rho_{N+2s}^\frac{1}{\gamma_2}$) in the stability inequality (2.12) and using the fact that $U^{\frac{2}{\gamma_2}} \zeta\Phi$ (resp. $V^{\frac{2}{\gamma_2}} \zeta\Phi$) has the trace $u^{\frac{q}{2\gamma_1-2}} \rho_{N+2s}^\frac{1}{\gamma_2}$ (resp. $v^{\frac{q}{2\gamma_1-2}} \rho_{N+2s}^\frac{1}{\gamma_2}$) on $\partial\mathbb{R}^{N+1}$, one gets

$$\kappa_s \sqrt{pq} \int_{\mathbb{R}^N} h(x) v(x)^{\frac{q+1}{q+\gamma_2}} u(x)^{\frac{q+1}{q+\gamma_2}} u(x)^{\gamma_2} \rho(x) \phi(x)^2 dx \leq \kappa_s \|u^{\frac{q}{2\gamma_1-2}} \rho_{N+2s}^\frac{1}{\gamma_2}\|_{H^s(\mathbb{R}^N)}$$

$$\leq \int_{\mathbb{R}^{N+1}_+} |\nabla(U^{\frac{2}{\gamma_2}} \zeta\Phi)|^2 t^{1-2\gamma} dx dt = \sqrt{pq} I_1,$$  

(3.10)

and

$$\kappa_s \sqrt{pq} \int_{\mathbb{R}^N} h(x) v(x)^{\frac{q+1}{q+\gamma_2}} u(x)^{\frac{q+1}{q+\gamma_2}} v(x)^{\gamma_2} \rho(x) \phi(x)^2 dx \leq \kappa_s \|v^{\frac{q}{2\gamma_1-2}} \rho_{N+2s}^\frac{1}{\gamma_2}\|_{H^s(\mathbb{R}^N)}$$

$$\leq \int_{\mathbb{R}^{N+1}_+} |\nabla(V^{\frac{2}{\gamma_2}} \zeta\Phi)|^2 t^{1-2\gamma} dx dt = \sqrt{pq} I_2.$$  

(3.11)

Hence,

$$\frac{\kappa_s}{A(\gamma_2, \epsilon)} \int_{\mathbb{R}^N} h(x) u(x)^{q} v(x)^{\gamma_2-1} \rho(x) \phi(x)^2 dx \leq \frac{2\gamma_1 - 1}{2\gamma_1} I_2 + \frac{q+1}{2\gamma_1} (A(\gamma_2, \epsilon))^{-\frac{2\gamma_1}{q+1}} I_1.$$
Similarly, we can prove that

\[ \kappa_s \Lambda(\gamma_1, \epsilon) \frac{q-1}{q+1} \int_{\mathbb{R}^N} h(x) \nu(x)^q u(x)^{\gamma_1 - 1} \rho(x) \phi^2 \, dx \leq \frac{q + 1}{2 \gamma_1} I_2 + \frac{2 \gamma_1 - q - 1}{2 \gamma_1} \Lambda(\gamma_1, \epsilon) \frac{2 \gamma_1}{q+1} I_1. \]

Combining the above two estimates with (3.8), we derive that

\[ \Lambda(\gamma_1, \epsilon) \frac{2 \gamma_1}{q+1} I_1 \leq \left[ \frac{q + 1}{2 \gamma_1} \Lambda(\gamma_2, \epsilon) \frac{2 \gamma_1}{q+1} + \frac{2 \gamma_1 - q - 1}{2 \gamma_1} \Lambda(\gamma_1, \epsilon) \frac{2 \gamma_1}{q+1} \right] I_1 + C \epsilon \int_{\mathbb{R}^N} (U^{\gamma_1} + V^{\gamma_2}) |\nabla(\zeta \Phi)|^2 t^{1 - 2s} \, dx \, dt, \]

hence

\[ \frac{q + 1}{2 \gamma_1} \left( \Lambda(\gamma_1, \epsilon) \Lambda(\gamma_2, \epsilon) \right) \frac{2 \gamma_1}{q+1} I_1 \leq \Lambda(\gamma_1, \epsilon) \frac{2 \gamma_1}{q+1} C_\epsilon \int_{\mathbb{R}^N} (U^{\gamma_1} + V^{\gamma_2}) |\nabla(\zeta \Phi)|^2 t^{1 - 2s} \, dx \, dt. \]

Denote \( A_i = \Lambda(\gamma_i, 0), i = 1, 2. \) Suppose that \( A_1 A_2 > 1, \) we can choose \( \epsilon \) small enough such that \( \Lambda(\gamma_1, \epsilon) \Lambda(\gamma_2, \epsilon) > 1, \) and so we obtain

\[ I_1 \leq C \int_{\mathbb{R}^N} (U^{\gamma_1} + V^{\gamma_2}) |\nabla(\zeta \Phi)|^2 t^{1 - 2s} \, dx \, dt. \]

On the other hand, by Remark 1, there holds \( V^{\gamma_2} \leq C U^{\gamma_1}. \) Denoting \( \gamma := \gamma_1, \) we conclude that if \( A_1 A_2 > 1 \) and \( \gamma > \frac{2q+1}{2}, \)

\[ \int_{\mathbb{R}^N} |\nabla(U^{\gamma} \zeta \Phi)|^2 t^{1 - 2s} \, dx \, dt \leq C \int_{\mathbb{R}^N} U^{\gamma} |\nabla(\zeta \Phi)|^2 t^{1 - 2s} \, dx \, dt. \]

Finally, we can check that \( A_1 A_2 > 1 \) is equivalent to \( L(\gamma) < 0, \) the proof is completed. \( \square \)

**End of the proof of Theorem 1.2.** Take \( \phi \in C^\infty_c((-2, 2)) \) satisfying \( \phi \equiv 1 \) in \([-1, 1]\). For \((x, t) \in \mathbb{R}^N + 1\) and \( R > 0, \) we define \( \Phi_R \in C^\infty_c(\mathbb{R}^N + 1) \) by \( \Phi_R(x, t) = \phi \left( \frac{(x, t)}{R} \right). \) Let \( \gamma_0 \) be the largest root of the polynomial \( L \) given by (3.2) and denote by

\[ k_s = \frac{N + 2 - 2s}{N - 2s}. \]

Fix \( \tau > 0, \) then there exists a positive integer \( m \) satisfying

\[ \tau k_s^{m-1} < \gamma_0 \leq \tau k_s^m. \]

Define \( \gamma_1, ..., \gamma_m \) as follows

\[ \gamma_1 = \tau k < \gamma_2 = \tau k k_s < ... < \gamma_m = \tau k k_s^{m-1} < \gamma_0, \]

where \( k \in [1, k_s] \) will be chosen so that \( \gamma_m \) is arbitrarily close to \( \gamma_0. \) Suppose that \( \tau \) satisfies

\[ (\tau) \quad \tau > \frac{q + 1}{2} \quad \text{such that} \quad L(\gamma) < 0 \quad \text{for any} \quad \gamma \in (\tau, \gamma_0), \]
then $\gamma_1, \ldots, \gamma_m$ satisfy (*). Hence, from (3.1) and the Sobolev inequality (see [10, Proposition 3.1.1]), there holds

$$
\left( \int_{B^+} U^{\gamma_m k_x} \zeta_{t^{1-2s}} \, dx dt \right)^{\frac{2}{\gamma_m k_x}} \leq C \left( \int_{B^+} U^{\gamma_m k_x} (\zeta \Phi_1)^{2k_x t^{1-2s}} \, dx dt \right)^{\frac{2}{\gamma_m k_x}} 
$$

where, in the last inequality, we used

$$
\int_{B^+} |\nabla (U^{2m} (\zeta \Phi_1))|^{2t^{1-2s}} \, dx dt 
$$

Now, we can adapt the proof of [LIOUVILLE TYPE THEOREMS FOR SOLUTIONS OF THE WEIGHTED FRACTIONAL LANE-EMDEN SYSTEM 11, Lemma 2.4] (see also [13, (4.8)–(4.10)] for further explanations) to obtain the following integral estimate from $\mathbb{R}^{N+1}$ to $\mathbb{R}^N$:

$$
\int_{\mathbb{R}^N} U^\tau |\nabla (\zeta \Phi_2 m)|^{2t^{1-2s}} \, dx dt \leq \int_{\mathbb{R}^N} (u(y))^\tau \rho(y) dy \leq \int_{\mathbb{R}^N} h(y)(u(y))^\tau \rho(y) dy,
$$

where, in the last inequality, we used $h \geq C > 0$ in $\mathbb{R}^N$. We deduce from (3.12) that

$$
\left( \int_{B^+} U^{\gamma_m k_x} \zeta_{t^{1-2s}} \, dx dt \right)^{\frac{2}{\gamma_m k_x}} \leq C \left( \int_{\mathbb{R}^N} h(y)(u(y))^\tau \rho(y) dy \right)^{\frac{2}{\tau}}.
$$

Let $R > 1$. The functions

$$
u_R(x) = \frac{2s(p+1)}{m-1} \frac{4(p+1)}{m+1} u(Rx) \quad \text{and} \quad v_R(x) = \frac{2s(p+1)}{m-1} \frac{4(p+1)}{m+1} v(Rx),
$$

form a solution of (1.1) with $h$ is replaced by $\frac{h(Rx)}{R^\tau}$. We use a scaling argument, replacing $U(x,t)$, $u(y)$ and $h(y)$ in (3.14) by $U_R := R^\frac{2s(p+1)}{m-1} \frac{4(p+1)}{m+1} U(Rx, Rt)$, $u_R(y)$ and $\frac{h(Ry)}{R^\tau}$, we deduce that

$$
\left( \int_{B^+} U^{\gamma_m k_x} (Rx, Rt) t^{1-2s} \, dx dt \right)^{\frac{2}{\gamma_m k_x}} \leq C \left( \int_{\mathbb{R}^N} \frac{h(Rx)}{R^\tau} (u(Ry))^\tau \rho(y) dy \right)^{\frac{2}{\tau}}.
$$
Hence,
\[
\left(R^{-N-2+2s} \int_{B_R^+} U^{\gamma_m k_s}(x,t)t^{1-2s} \, dx dt\right)^{\frac{2}{\gamma_m k_s}} \leq C \left( \int_{\mathbb{R}^N} \frac{h(Rx)}{R^\ell} (u(Ry))^\tau \rho(y) \, dy \right)^{\frac{2}{\tau}} 
\]
\[
\leq C \left( R^{-N-\ell} \int_{\mathbb{R}^N} h(y)(u(y))^\tau \rho\left(\frac{y}{R}\right) \, dy \right)^{\frac{2}{\tau}} \quad (3.15)
\]
\[
\leq C \left( R^{-N-\ell} \int_{\mathbb{R}^N} h(y)(u(y))^\tau \rho\left(\frac{y}{R}\right) \, dy \right)^{\frac{2}{\tau}}.
\]

We then split the rest of the proof into two cases.

**Case 1.** \( q > \frac{4}{3} \). Fix \( \tau = q \). By [23, Lemma 6], \( \tau \) satisfies (*). Using the estimate (2.6), we derive from (3.15) that
\[
\int_{B_R^+} U^{\gamma_m k_s}(x,t)t^{1-2s} \, dx dt \leq CR^{N+2-2s-\frac{2}{\tau}} (2^{\frac{2\gamma_m k_s+1}{pq-1}} + 2^{\frac{2\gamma_m k_s}{pq-1}})_{\gamma_m k_s}.
\]

Suppose now \( N < 2s + \frac{2s + \ell(p+1)}{pq-1} \gamma_0 \), we can choose \( k \in \lfloor 1, k_s \rfloor \) so that \( \gamma_m \) is sufficiently close to \( \gamma_0 \) satisfying
\[
N - 2s - \frac{(2s + \ell)(p+1)}{pq-1} \gamma_m < 0.
\]

Let \( R \) tend to infinity in (3.16), we have a contradiction since \( u \) is positive. In other words, the system (1.1) has no stable solution if \( N < 2s + (2s + \ell)x_0 \) where \( x_0 = \frac{p+1}{pq-1} \gamma_0 \). Moreover, we can adapt the proof of Remark 3 in [4] to show that
\[
2t_0^+ \frac{p+1}{pq-1} > 4, \quad \forall \, p \geq q > 1.
\]

By [23, Lemma 6], \( x_0 \geq 2t_0^+ \frac{p+1}{pq-1} > 4 \). Therefore, if \( N \leq 10 + 4\ell \), (1.1) has no stable solution for any \( p \geq q > \frac{4}{3} \).

**Case 2.** \( 1 < q \leq \frac{4}{3} \), and \( v \) is bounded. Put \( \tau = 2 \). By [23, Lemma 6], \( \tau \) satisfies (*). The following Lemma is crucial to handle this case. It provides an *a priori* integral estimate of \( hu^2 \rho \) as for \( hu^q \rho \), and then we can proceed as above to achieve the proof of Theorem 1.2.

**Lemma 3.2.** Let \((u,v)\) be a stable solution to (1.1) with \( 1 < q \leq \min\left(\frac{4}{3},p\right) \). Assume that \( v \) is bounded and \( h \) satisfies \((H)\), there holds
\[
\int_{\mathbb{R}^N} hu^2 \rho\left(\frac{x}{R}\right) \, dx \leq CR^{N-\frac{2(p+1)q}{pq-1}} \frac{\ell(q+1)}{pq-1} \frac{2^{\frac{2q+1}{2(p+1)}}}{p+q-2}, \quad \forall \, R > 0.
\]

**Proof.** Take \( \phi, \Phi_k \) as above and \( \Phi_k(x) = \phi\left(\frac{|x|}{k}\right) \). Combining (3.3) and (3.4) with \( \gamma = 2 \), we get
\[
\int_{\mathbb{R}^{N+1}_+} |\nabla (U^2 \Phi_k)|^2 t^{1-2s} \, dx \, dt = \kappa_s \int_{\mathbb{R}^N} h(x)v(x)^\rho \, dx
\]
\[
+ \int_{\mathbb{R}^{N+1}_+} U^2 |\nabla (\Phi_k)|^2 t^{1-2s} \, dx \, dt.
\]

By (2.7), there holds
\[
\int_{\mathbb{R}^{N+1}_+} |\nabla (U^2 \Phi_k)|^2 t^{1-2s} \, dx \, dt \leq \kappa_s \int_{\mathbb{R}^N} h(x)v(x)^{\frac{p+1}{q+1}} \, dx + \int_{\mathbb{R}^{N+1}_+} U^2 |\nabla (\Phi_k)|^2 t^{1-2s} \, dx \, dt.
\]
On the other hand, applying (3.10) with \( \gamma_1 = 2 \), we have

\[
\kappa_s \sqrt{pq} \int_{\mathbb{R}^N} h(x) v(x)^{\frac{p+1}{p-1}} u(x)^{\frac{q+1}{q-1}} u(x)^2 \rho(x) \phi_k(x)^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla (U \zeta \Phi_k)|^2 t^{1-2s} \, dx.
\]

Combining the two last inequalities and using (3.13) with \( \tau = 2 \),

\[
\left( \sqrt{pq} - \sqrt{\frac{p+1}{q+1}} \right) \kappa_s \int_{\mathbb{R}^N} h(x) v(x)^{\frac{p+1}{p-1}} u(x)^{\frac{q+1}{q-1}} \rho(x) \phi_k(x)^2 \, dx \leq \int_{\mathbb{R}^N} h(x) u(x)^2 \rho(x) \, dx.
\]

(3.18)

As \( v \) is bounded, we can use (2.8) to deduce that there exists \( C > 0 \) such that

\[
\int_{\mathbb{R}^N} h(x) u(x)^{\frac{p+1}{p-1}} \rho(x) \phi_k(x)^2 \, dx \leq C \|v\|_{\infty}^a \int_{\mathbb{R}^N} h(x) u(x)^2 \rho(x) \, dx,
\]

where \( a := \frac{(p-q)(p-1)}{2(q+1)} \). Letting \( k \to \infty \) and using Lebesgue’s monotone convergence theorem,

\[
\int_{\mathbb{R}^N} h(x) u(x)^{\frac{p+1}{p-1}} \rho(x) \, dx \leq C \|v\|_{\infty}^a \int_{\mathbb{R}^N} h(x) u(x)^2 \rho(x) \, dx.
\]

(3.19)

Denote

\[
J_1 := \int_{\mathbb{R}^N} h(x) u(x)^{\frac{p+1}{p-1}} \rho(x) \, dx, \quad J_2 := \int_{\mathbb{R}^N} h(x) u(x)^2 \rho(x) \, dx.
\]

As \( 1 < q \leq \min(p, \frac{4}{3}) \), we have \( q < 2 < \frac{p+q+2}{2} \) and a direct calculation yields

\[
2 = q \lambda + \frac{p+q+2}{2} (1 - \lambda) \quad \text{with} \quad \lambda = \frac{p+q-2}{p+2-q} \in (0, 1).
\]

By Hölder’s inequality and (3.19), we get

\[
J_2 \leq J_1^{1-l} \left( \int_{\mathbb{R}^N} h(x) u(x)^q \rho(x) \, dx \right)^l \leq \left( C \|v\|_{\infty}^a J_2 \right)^{1-l} \left( \int_{\mathbb{R}^N} h u^q \rho(x) \, dx \right)^l,
\]

which implies

\[
J_2 = \int_{\mathbb{R}^N} h(x) u(x)^2 \rho(x) \, dx \leq C \|v\|_{\infty}^{\frac{1-\gamma}{\gamma}} \int_{\mathbb{R}^N} h u^q \rho(x) \, dx.
\]

By scaling argument as above, for \( R > 0 \) we get

\[
R^{\frac{2(p+1)}{p-1} + \frac{(p+1)}{pq}} \int_{\mathbb{R}^N} h(Rx) u^2(Rx) \rho(x) \, dx \leq C \left( R^{\frac{2(p+1)}{p-1} + \frac{(p+1)}{pq}} \|v(R)\|_{\infty} \right)^{\frac{1-\gamma}{\gamma}} \int_{\mathbb{R}^N} h(Rx) u^q(Rx) \rho(x) \, dx.
\]

Making a change of variables and using (2.6), using again the boundedness of \( v \), we deduce that

\[
\int_{\mathbb{R}^N} h(x) u^2(x) \rho \left( \frac{x}{R} \right) \, dx \leq CR^N \frac{2(p+1) + (p+1)}{pq-1} \frac{1-\gamma}{\gamma} \left[ \frac{2(p+1)}{pq-1} \right] - 2 \left[ \frac{2(p+1)}{pq-1} + \frac{(p+1)}{pq} \right].
\]

A straightforward computation shows that the above exponent of \( R \) is the same stated in (3.17), so we are done.

**Proof of Corollary 1.3.** Let \( u \) be a stable solution of equation (1.12), then \( v = u \) verify the system (1.1) with \( p = q \). Moreover, we have

\[
t_0^\pm = p \pm \sqrt{p^2 - p}
\]

and

\[
L(\gamma) = \gamma^4 - 16p^2 \gamma^2 + 32p^2 \gamma - 16p^2 = (\gamma^2 + 4p(\gamma - 1))(\gamma - 2t_0^\pm)(\gamma - 2t_0^+).
\]

As \( t_0^+ > p \), it follows that \( 2t_0^+ \) is the largest root of \( L \) as \( t_0^+ > p > 1 \). Therefore

\[
x_0 = \frac{2t_0^+}{p-1} = \frac{2p + 2\sqrt{p^2 - p}}{p-1}.
\]
is the largest root of $H$. Then, applying Theorem 1.2, the result follows immediately. □

Acknowledgments. I would like to thank Professor Dong Ye for many helpful comments.

References

[1] Caffarelli, L., and Silvestre, L. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* 32, 7-9 (2007), 1245–1260.

[2] Chen, W., Dupaigne, L., and Ghergu, M. A new critical curve for the Lane-Emden system. *Discrete Contin. Dyn. Syst.* 34, (2014), 2469–2479.

[3] Chen, W., and Wang, H. Liouville theorems for the weighted Lane-Emden equation with finite Morse indices. *Math. Methods Appl. Sci.* 40, (2017), 4674–4682.

[4] Cowan, C. Liouville theorems for stable Lane-Emden systems and biharmonic problems. *Nonlinearity* 26, 8 (2013), 2357–2371.

[5] Cowan, C., and Fazly, M. On stable entire solutions of semi-linear elliptic equations with weights. *Proc. Amer. Math. Soc.* 140, 6 (2012), 2003–2012.

[6] Cowan, C., and Ghoussoub, N. Regularity of semi-stable solutions to fourth order nonlinear eigenvalue problems on general domains. *Calc. Var. PDE.*, 49 (2014), 291–305.

[7] Dancer, E. N., Du, Y., and Guo, Z. Finite Morse index solutions of an elliptic equation with supercritical exponent. *J. Differ. Equ.* 250, (2011), 3281–3310.

[8] Du, Y., and Guo, Z. Finite Morse-index solutions and asymptotics of weighted nonlinear elliptic equations. *Adv. Differ. Equ.* 18, (2013), 737–768.

[9] Dávila, J., Dupaigne, L., and Wei, J. On the fractional Lane-Emden equation. *Trans. Amer. Math. Soc.* 369, 9 (2017), 6087–6104.

[10] Dipierro, S., Medina, M., and Valdinoci, E. Fractional elliptic problems with critical growth in the whole of $\mathbb{R}^N$. *N. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)],* 15. Edizioni della Normale, Pisa, 2017. viii+152 pp.

[11] Duong, A. T. A Liouville type theorem for non-linear elliptic systems involving advection terms. *Complex Var. Elliptic Equ.* 63, 12 (2018), 1704–1720.

[12] Duong, A. T., and Pham, D. H. Liouville-type Theorem for Fractional Kirchhoff Equations with Weights. *Bulletin of the Iranian Mathematical Society*, (2020).

[13] Duong, A. T., and Nguyen, V. H. Liouville type theorems for some fractional elliptic problems *Nonlinear Anal.* 210, (2021), 112383.

[14] Duong, A. T., and Phan, Q. H. Liouville type theorem for nonlinear elliptic system involving Grushin operator. *J. Math. Anal. Appl.* 454, 2 (2017), 785–801.

[15] Dupaigne, L., and Farina, A. Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$. *J. Eur. Math. Soc. (JEMS)* 12, 4 (2010), 855–882.

[16] Dupaigne, L., Farina, A., and Sirakov, B. Regularity of the extremal solutions for the Liouville system, in: Geometric Partial Differential Equations, *Publications of the Scuola Normale Superiore/CRM Series*, 15 (2013), 139–144.

[17] Farina, A. On the classification of solutions of the Lane-Emden equation on unbounded domains of $\mathbb{R}^N$. *J. Math. Pures Appl.* 89, 5 (2007), 537–561.

[18] Farina, A., and Hasegawa, S. Liouville-type theorems and existence results for stable solutions to weighted Lane-Emden equations. *Proceedings of the Royal Society of Edinburgh Section A Mathematics (5)* 150, (2019), 1–13.

[19] Fazly, M., and Ghoussoub, N. On the Hénon-Lane-Emden conjecture. *Discrete Contin. Dyn. Syst.* 34, 6 (2014), 2513–2533.

[20] Fazly, M., and Wei, J. On stable solutions of the fractional Hénon-Lane-Emden equation. *Commun. Contemp. Math.* 18, 5 (2016), 1650005, 24.

[21] Fazly, M., and Wei, J. On finite Morse index solutions of higher order fractional Lane-Emden equations. *Amer. J. Math.* 139, 2 (2017), 433–460.

[22] Gui, C., Ni, W., and Wang, X. On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$. *Comm. Pure Appl. Math.* Vol. XLV, (1992), 1153-1181.

[23] Hajlaoui, H., Harrabi, A., and Mitri, F. Liouville theorems for stable solutions of the weighted Lane-Emden system. *Discrete Contin. Dyn. Syst.* 37, 1 (2017), 265–279.

[24] Harrabi, A. Explicit universal estimate for $p$-polyharmonic equations via Morse index. *arXiv:2105.04058v1* (2021).

[25] Hu, L.-G. Liouville type results for semi-stable solutions of the weighted Lane-Emden system. *J. Math. Anal. Appl.* 432, 1 (2015), 429–440.

[26] Hu, L.-G. Liouville type theorems for stable solutions of the weighted elliptic system with the advection term: $p \geq q > 1$. *NoDEA Nonlinear Differential Equations Appl.* 25, 1 (2018), Art. 7, 30.

[27] Hu, L.-G., and Zeng, J. Liouville type theorems for stable solutions of the weighted elliptic system. *J. Math. Anal. Appl.* 437, 2 (2016), 882–901.
[28] Jeong, W., and Lee, Y. Stable solutions and finite Morse index solutions of nonlinear elliptic equations with Hardy potential. *Nonlinear Anal. 87*, (2013), 126–145.

[29] Joseph, D.D., and Lundgren, T.S. Quasilinear Dirichlet problems driven by positive sources. *Arch. Rational Mech. Anal. 49*, (1973), 241–269.

[30] Mitidieri, E. Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$. *Differential Integral Equations 9*, 3 (1996), 465–479.

[31] Mitidieri, E., and Pohozaev, S. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Tr. Mat. Inst. Steklova 234*, (2001), 1-384.

[32] Montenegro, M. Minimal solutions for a class of elliptic systems. *Bull. London Math. Soc. 37*, 3 (2005), 405–416.

[33] Mtiri, F., and Ye, D. Liouville theorems for stable at infinity solutions of Lane-Emden system. *Nonlinearity 32*, 3 (2019), 910–926.

[34] Quittner, P., and Souplet, P. Symmetry of components for semilinear elliptic systems. *SIAM J. Math. Anal. 44*, 4 (2012), 2545–2559.

[35] Rahal, B., and Zaidi, C. On the classification of stable solutions of the fractional equation. *Potential Anal. 50*, 4 (2019), 565–579.

[36] Serrin, J., and Zou, H. Non-existence of positive solutions of Lane-Emden systems. *Differential Integral Equations 9*, 4 (1996), 635–653.

[37] Souplet, P. The proof of the Lane-Emden conjecture in four space dimensions. *Adv. Math. 221*, 5 (2009), 1409–1427.

[38] Wang, C., and Ye, D. Some Liouville theorems for Hénon type elliptic equations. *J. Funct. Anal. 262*, 4 (2012), 1705–1727.

[39] Wei, J., and Ye, D. Liouville theorems for stable solutions of biharmonic problem, *Math. Ann. 356*, (2013), 1599-1612.

[40] Yang, H., and Zou, W. Symmetry of components and Liouville-type theorems for semilinear elliptic systems involving the fractional Laplacian. *Nonlinear Anal. 180* (2019), 208–224.

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