Abstract: We classify 3-braid links which are amphicheiral as unoriented links, including a new proof of Birman-Menasco’s result for the (orientedly) amphicheiral 3-braid links. Then we classify the partially invertible 3-braid links.

Keywords: Jones polynomial, 3-braid link, Amphicheiral, Invertible, Semiadequate link, Genus

MSC: 57M25, 05E15, 20F36, 20B40, 05C62

1 Introduction

Links given by closures of 3-braids have been studied for some time. Ultimately such links were classified by Birman-Menasco [4], in terms of a description of conjugacy classes of 3-braids with the same closure. Still many properties of links are not visible from braid representations, and the classification of 3-braids with such properties remains open. We have done some work in this regard in [27, 28]. Two properties of the 3-braids which are directly visible in a braid representation are amphicheirality and invertibility. The description of 3-braid links with such symmetries thus follows as a consequence of the Birman-Menasco theorem. In their braid approach both types of symmetries are assumed to either preserve or reverse orientation of all components of the link. (Components can also be permuted.) In this paper we offer the two corresponding results for the case that orientation of some but not all components is reversed. Although our two results are similar in nature, they require somewhat different approaches (which are also rather different from the one of Birman-Menasco).

In the paper [29], we initiated some study of edge coefficients of the Jones polynomial of semiadequate links. This was applied, among others, to 3-braid links, which we proved to be semiadequate. Another recent study of the Jones polynomial of 3-braid links was performed by Futer, Kalfagianni and Purcell [11]. We will first use the information we obtained from both works to classify those 3-braid links amphicheiral as unoriented links.

Theorem 1.1. Let L be a 3-braid link which is unorientedly amphicheiral. Then either
(a) it is orientedly amphicheiral, in which case it is the closure of an alternating 3-braid with Schreier vector admitting a dihedral (anti)symmetry, or
(b) it is one of the following links:
   − a Hopf link,
   − a Hopf link with a split trivial component,
   − the connected sum of two Hopf links of the same sign,
   − the (3,3)-rational link (or (1,1,1,3)-pretzel link), or
   − the link 9^2_{61} (the closure of (\sigma_1\sigma_2^3\sigma_1)^2\sigma_2^{-1}; see Figure 1.66.1 in [19, p. 46]).
The classification of partially symmetric 3-braid links

(For the clarification of the Schreier vector and its symmetry, see Theorem 2.3 and below it.)

The notion of unoriented amphicheirality is weaker than Birman-Menasco’s and is obtained by dropping restrictions on component orientation that an isotopy to the mirror image should incur (cf. Definition 2.2). Our proof of Theorem 1.1 will reproduce the oriented case in an alternative way (see Remarks 3.2 and 3.3).

The second main result deals with the partially invertible 3-braid links. A link \( L \) is partially invertible if \( L \) is isotopic to itself with the orientation of some but not all components reversed. (Again, we will allow components to be permuted by the isotopy.)

**Theorem 1.2.** Let \( L \) be a 3-braid link which is partially invertible. Then \( L \) conforms to one of the two patterns shown on Figure 1.

**Fig. 1.** The partially invertible 3-braids. The orientation and content of the boxes should be chosen so that the component(s) entering into the boxes form a closed 2-braid. One partial inversion of these links is then given by the \( \pi \)-rotation along the vertical axis, which alters the orientation of the circular component not entering the boxes.

Note that several noteworthy links are included in these families: the split links, the pretzel links \((n, 2, 2)\) (among which is the Whitehead link), and also the Borromean rings.

The proof of Theorem 1.2 uses in a fundamental way the genus of the 3-braid links as described by Xu [32], and some delicate (and partly computer-aided) case analysis. In contrast to the amphicheiral case, the derivation of the list of (fully) invertible 3-braid links remains untouched (and probably untouchable) with our tools. We thus do not discuss it here, and refer to [4] for the treatment.

Our proofs will require the combination of a considerable deal of knowledge about invariants of 3-braids, which we will review first.

## 2 Preliminaries

### 2.1 Semiadequacy and Kauffman bracket

It is useful to define here the Jones polynomial via Kauffman’s state model [18]. Recall, that the Kauffman bracket \( \langle D \rangle \) of a link diagram \( D \) is a Laurent polynomial in a variable \( A \), obtained by summing over all states \( S \) the terms

\[
A^{|A(S)|} B^{|B(S)|} \left(-A^2 - A^{-2}\right)^{|S|-1} ,
\]

where a state is a choice of splicings (or splittings) of type \( A \) or \( B \) for any single crossing (see Figure 2), \( |A(S)| \) and \( |B(S)| \) denote the number of type \( A \) (respectively, type \( B \)) splittings and \( |S| \) the number of (disjoint) circles obtained after all splittings in \( S \). We call the \( A \)-state \( A(D) \) the state in which all crossings are \( A \)-spliced, and the \( B \)-state \( B(D) \) is defined analogously.

The Jones polynomial of a link \( L \) can be specified from the Kauffman bracket of some diagram \( D \) of \( L \) by

\[
V_L(t) = \left(-t^{-3/4}\right)^{-|w(D)|} \langle D \rangle \bigg|_{A=t^{-1/4}}.
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\]
Let $S$ be the $A$-state of a diagram $D$ and $S'$ a state of $D$ with exactly one $B$-splicing. If $|S| > |S'|$ for all such $S'$, we say that $D$ is $A$-adequate. Similarly one defines a $B$-adequate diagram $D$. See [21, 30]. Then we set a diagram to be

\[ \text{adequate} = A \text{-adequate and } B \text{-adequate}, \]

\[ \text{semiadequate} = A \text{-adequate or } B \text{-adequate}, \]

\[ \text{inadequate} = \text{neither } A \text{-adequate nor } B \text{-adequate}. \] (3)

(Note that inadequate is a stronger condition than not to be adequate.)

A link is called $A$ (or $B$)-adequate, if it has an $A$ (or $B$)-adequate diagram. A link is semiadequate if it is $A$- or $B$-adequate. It is adequate if it has an adequate diagram. This property is stronger than being both $A$- and $B$-adequate, since a link might have diagrams that enjoy either properties, but none that does so simultaneously. (The Perko knot $10_{161}$ in [25, appendix] is an example; this feature of it transpires from results in [30], where the knot is discussed.) A link is inadequate, if it is neither $A$- nor $B$-adequate.

Two basic observations are that a reduced alternating diagram (and hence an alternating link) is adequate, and that a positive diagram (and link) is throwing, and similarly negative ones are $B$-adequate.

Let $V \in \mathbb{Z}[t, t^{-1}]$. The minimal or maximal degree $\min \deg V$ or $\max \deg V$ is the minimal resp. maximal exponent of $t$ with non-zero coefficient in $V$. Let span$_t V = \max \deg_t V - \min \deg_t V$. The coefficient in degree $d$ of $t$ in $V$ is denoted $[V]_t$ or $[V]_d$. We will use more commonly another notation for coefficients.

**Definition 2.1.** Let $V \in \mathbb{Z}[t^{\pm 1}]$ or $V \in t^{\pm 1/2} \cdot \mathbb{Z}[t^{\pm 1}]$, and $n \geq 0$ an integer. Let $m = \min \deg V$ and $M = \max \deg V$ (then $2m \in \mathbb{Z}$). We write $a_n(V) := [V]_{m+n}$ and $\tilde{a}_n(V) := [V]_{M-n}$ for the $n+1$st or $n+1$-last coefficient of $V$.

A basic observation in [21] is that when $L$ is $A$- resp. $B$-adequate then $|a_0(V(L))| = 1$ resp. $|\tilde{a}_0(V(L))| = 1$. Thus if $L$ is adequate, and in particular alternating, both properties hold. We will use this fact continuously below without explicit reference.

When $L$ is an $A$-adequate link, then $a_n(V(L))$ for $n \leq 2$ were studied in [8, 9, 29]. We will need the formulas below, so let us recall them briefly. Let $D$ be an $A$-adequate diagram of $L$. (We will assume that $D$ is connected.) Then, in the notation of [29],

\[ |a_1(V)| = e - |A(D)| + 1, \]

(4)

and

\[ |a_2(V)| = \left( \frac{|a_1(V)| + 1}{2} \right) + e_{++} + \delta - \Delta. \]

(5)

Here $|A(D)|$ is the number of loops in the $A$-state $A(D)$, and the quantities $e$, $e_{++}$, $\delta$, and $\Delta$ are the number of pairs or triples of loops in $A(D)$ for which there exist crossing traces (obtained as in Figure 2) making them look (up to moves in $S^2$) like in:

\[ \text{\includegraphics[width=0.2\textwidth]{diagram}} \]
(We do not require that these be the only traces connecting the loops, only that such traces should exist.) In our case, whenever we use (5), $D$ will be alternating, and then $\delta = 0$.

### 2.2 Link diagrams, skein relations, and genus

An alternative description of $V$ is to be the polynomial taking the value 1 on the unknot, and satisfying the skein relation

$$I^{-1}V\left(\begin{array}{c} x \\ x \end{array}\right) - I V\left(\begin{array}{c} x \\ \overline{\nu} \end{array}\right) = (t^{1/2} - t^{-1/2}) V\left(\begin{array}{c} \overline{\nu} \\ \overline{\nu} \end{array}\right).$$

This is another way, different from (2), to specify the Jones polynomial. We will denote in each triple as in (6) the link diagrams (from left to right) by $D_{+}$, $D_{-}$ and $D_{0}$; they are understood to be identical except at the designated spot.

The writhe is a number ($\hat{1}$), assigned to any crossing in a link diagram. A crossing as on the left in (6) has writhe 1 and is called positive. A crossing as in the middle of (6) has writhe $-1$ and is called negative. The writhe $w(D)$ of a link diagram $D$ is the sum of writhes of all its crossings. In the case of a crossing, we call the writhe also the (skein) sign.

Let $c_{\pm}(D)$ be the number of positive, respectively negative crossings of a diagram $D$, so that $c(D) = c_{+}(D) + c_{-}(D)$ and $w(D) = c_{+}(D) - c_{-}(D)$.

By $c(D)$ we denote the number of crossings of a diagram $D$, $n(D)$ the number of components of $D$ (or $K$, 1 if $K$ is a knot), and $s(D)$ the number of Seifert circles of $D$. The crossing number $c(K)$ of a knot or link $K$ is the minimal crossing number of all diagrams $D$ of $K$.

A crossing between two different components $K_{1}$ and $K_{2}$ in a diagram $D$ of a link $L$ is called mixed. The linking number $lk(K_{1}, K_{2})$ is half the sum of the signs of all (mixed) crossings between $K_{1}$ and $K_{2}$ in any diagram of $L$; this is a link invariant.

The (Seifert) genus $g(K)$ resp. Euler characteristic $\chi(K)$ of a knot or link $K$ is said to be the minimal genus resp. maximal Euler characteristic of Seifert surface of $K$. For a diagram $D$ of $K$, $g(D)$ is defined to be the genus of the Seifert surface obtained by Seifert’s algorithm on $D$, and $\chi(D)$ its Euler characteristic. We have $\chi(D) = s(D) - c(D)$ and $2g(D) = 2 - n(D) - \chi(D)$.

The skein polynomial $P$ \cite{10, 24} is a generalization of $V$ in two variables $l, m$ and satisfies (in the convention we use) the skein relation

$$I^{-1}P\left(\begin{array}{c} x \\ x \end{array}\right) + lP\left(\begin{array}{c} x \\ \overline{\nu} \end{array}\right) = -mP\left(\begin{array}{c} \overline{\nu} \\ \overline{\nu} \end{array}\right).$$

The replacement of a positive or negative crossing by the (non-crossing) fragment on the right will be called smoothing out.

We denote by $!D$ the mirror image of a diagram $D$, and $!K$ is the mirror image of $K$. Clearly $g(!D) = g(D)$ and $g(!K) = g(K)$. When one treats links, it should be emphasized that here mirroring is meant to preserve orientation of all components of the diagram.

**Definition 2.2.** We call an oriented link positively (orientedly) amphicheiral, if it is isotopic to its mirror image with the orientation of all components preserved, and components (possibly and arbitrarily) permuted. A link is negatively (orientedly) amphicheiral, if there is an isotopy to its mirror image reversing orientation of all components (and possibly permuting them). We call a link (orientedly) amphicheiral if it is positively or negatively (orientedly) amphicheiral. The word achiral will be used as a synonym for ‘amphicheiral’. We call a link unorientedly achiral (or amphicheiral), if it is isotopic to its mirror image with the orientation of some (possibly none, but also not necessarily all) components reversed (and components possibly permuted).

It is clear that oriented achirality (of some sign) implies unoriented one, and for knots both notions coincide. The Hopf link is an example of a link which is unorientedly achiral, but not orientedly so.

It can be seen from (2) that

$$V(!D)(t) = V(D)(t^{-1});$$
keep in mind that mirroring preserves component orientation and thus \( w(!D) = -w(D) \). Therefore, the Jones polynomial of an orientedly amphicheiral link \( L \) satisfies

\[ V_L(t) = V_L(t^{-1}), \]

and we call such a polynomial conjugate. Further one can see from (2) (or from [20]) that changing orientation of individual components multiplies \( V \) by a power of \( t \). This implies that the Jones polynomial of an unorientedly amphicheiral link \( L \) satisfies

\[ V_L(t) = V_L(t^{-1}) \cdot t^k, \]

for some \( k \in \mathbb{Z} \). We will call a polynomial with such property weakly conjugate.

### 2.3 Schreier normal form and Jones polynomial

The \( n \)-string braid group \( B_n \) is considered generated by the Artin standard generators \( \sigma_i \) for \( i = 1, \ldots, n - 1 \). These are subject to relations of the type \([\sigma_i, \sigma_j] = 1\) for \( |i - j| > 1 \), which we call commutativity relations (the bracket denotes the commutator) and \( \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \), which we call Yang-Baxter (or shortly YB) relations.

For a braid \( \beta \), let \( \hat{\beta} \) denote the closure of \( \beta \) and let \( [\beta] \) denote the exponent sum of \( \beta \). For a braid word \( \beta \), we denote by \([\beta]_+\) and \([\beta]_-\) the number of positive resp. negative letters in \( \beta \), so that \([\beta] = [\beta]_+ - [\beta]_-\). We call \( \beta \) positive (resp. negative) if \([\beta]_+ = 0\) (resp. \([\beta]_- = 0\)).

There is a common graphical representation of braids. We will mostly use it implicitly, but to facilitate understanding, let us fix our convention. Strings will be assumed numbered from left to right, oriented upward, and braid words will be composed from bottom to top. To \( \sigma_i \) should correspond a positive (right-hand) crossing, and to \( \sigma_i^{-1} \) a negative one. (This sign convention is now opposite to [11], after a temporary mix-up of either signs in a preliminary version has been cleaned up. We have then to slightly change some formulas there when we quote them here.)

The 3-braid group \( B_3 \) falls into a category of groups studied by Schreier in the 1920s [26], who developed a conjugacy normal form for this group.

Let \( C = (\sigma_1 \sigma_2)^3 \) be the center generator of \( B_3 \).

To save space and enhance readability, we will often use the compact notation for braid words of [27]; we write the indices of \( \sigma_i \) into brackets, writing \( i \) for \( \sigma_i \), \( ip \) for \( \sigma_i^p \) and \(-i \) for \( i^{-1} \). E.g. \( \sigma_1 \sigma_2 \sigma_1^{-2} = [12(-1)^2] = [121^{-2}] \). Thus \( C = [121212] = [(12)^3] \).

**Theorem 2.3.** (Schreier) Let \( \beta \in B_3 \) be a braid on 3 strands. Then \( \beta \) is conjugate to a braid of exactly one of the following five forms. Herein \( p_1, q_1, \) and \( s \) are all positive integers in form 1, \( p \in \mathbb{Z} \) in form 2 and \( k \in \mathbb{Z} \) in all forms.

1. \( C^k \cdot [1 \cdot p_1 2^{-q_1} \cdots 1 \cdot p_s 2^{-q_s}] \).
2. \( C^k \sigma_i^p \).
3. \( C^k \cdot [12] \).
4. \( C^k \cdot [121] \), or
5. \( C^k \cdot [1212] \).

This form is unique up to cyclic permutation of the word following \( C^k \).

Braids in form 1 above will be called generic, and the others non-generic. The vector \( S = (p_1, -q_1, \ldots, p_s, -q_s) \) of the generic form is called Schreier vector.

There is an obvious action of the dihedral group on the entries of the Schreier vector by cyclic permutation and inversion. We say that a Schreier vector \( S \) admits a dihedral (anti)symmetry, if \( S \) can be turned into \(-S\) by such an action.

Schreier’s normal form was used quite extensively by Birman-Menasco [4]. A more recent treatment, from the point of view of the Jones polynomial, is given in a paper by Futer, Kalfagianni and Purcell [11]. We will use some of the work there quite essentially.
The Burau representation \( \psi : B_3 \to GL(2, \mathbb{Z}[t, t^{-1}]) \), into the ring of \( 2 \times 2 \) matrices with coefficients being Laurent polynomials in \( t \), is defined by
\[
\psi(\sigma_1) = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix}, \quad \psi(\sigma_2) = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix}.
\]
The Jones polynomial of a 3-braid is given by
\[
V_{\beta}(t) = (-\sqrt{t})^{[\beta]} \cdot (t + t^{-1} + \text{tr} \, \psi(\beta)),
\]
where 'tr' is the trace. See [2, 16].

Futer, Kalfagianni and Purcell [11] notice the following interesting consequence of formula (8) (stated there under a slight, but inessential for the proof, restriction).

**Lemma 2.4.** Let \( \alpha \in B_3 \) be a 3-braid, and let \( \beta = C^k \alpha \). Then
\[
V(\hat{\beta})(t) = t^{6k} V(\hat{\alpha})(t) + (-\sqrt{t})^{[\alpha]} \left( t + t^{-1} \right) \left( t^{3k} - t^{6k} \right).
\]

This is particularly interesting to apply to the generic form in Theorem 2.3, where \( \alpha \) is the alternating part of \( \beta \):
\[
\alpha = \sigma_1^{p_1} \sigma_2^{-q_1} \cdots \sigma_1^{p_s} \sigma_2^{-q_s},
\]
with positive integers \( p_i, q_i \) and \( s \). We set
\[
p := \sum_{i=1}^{s} p_i, \quad \text{and} \quad q := \sum_{i=1}^{s} q_i.
\]
We call \( s \) the length of \( \beta \).

Futer, Kalfagianni and Purcell [11] obtain the following expression for the degrees of \( V(\hat{\alpha}) \), using a study of the Kauffman bracket:
\[
\max \deg V(\hat{\alpha}) = \frac{3p - q}{2} \quad \text{and} \quad \min \deg V(\hat{\alpha}) = \frac{p - 3q}{2}.
\]

### 2.4 The genus of 3-braids

We will describe now how to determine the genus of 3-braids, which requires us to introduce a modified presentation of the 3-braid group.

In [32], Xu considered the new generator
\[
\sigma_3 = \sigma_1^{-1} \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2^{-1}
\]
of \( B_3 \) (where \( \sigma_{1,2} \) still denote Artin’s generators), with which \( B_3 \) gains the presentation
\[
B_3 := \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_2 \sigma_1 = \sigma_3 \sigma_2 = \sigma_1 \sigma_3 \rangle.
\]
Then for any representation of a braid \( \beta \in B_3 \) as word in \( \sigma_{1,2,3} \) one obtains a Seifert surface of the closure \( \hat{\beta} \) of \( \beta \) by inserting disks for each braid strand and connecting them by half-twisted bands along each \( \sigma_i^{\pm 1} \).

An important feature of 3-braid links is that there is a Seifert surface of this type, which has minimal genus (a fact first proved by Bennequin in [1]). Such a band representation can be obtained by an effective algorithm given by Xu. It allows one to write each \( \beta \in B_3 \) in one of the two forms
\[
\begin{align*}
(A) & \ (\sigma_2 \sigma_1)^k \cdot R \ \text{or} \ L^{-1} \cdot (\sigma_2 \sigma_1)^{-k} (k \geq 0), \ \text{or} \\ 
(B) & \ L^{-1} R,
\end{align*}
\]

\[\tag{13}\]
where \( L \) and \( R \) are positive words with (cyclically) non-decreasing indices (i.e. each \( \sigma_i \) is followed by \( \sigma_i \) or \( \sigma_{i+1} \)), and all exponents are positive).

The form (A) (called “strongly quasipositive” in the terminology coined by Rudolph) will not be very relevant to us. We will mainly look at form (B). We can, and will, assume it to be cyclically reduced, i.e. that \( L \) and \( R \) do not start or end with the same letter.

**Theorem 2.5.** (see [32]) The surfaces constructed from form (A) and (cyclically reduced) form (B) in (13) have minimal genus.

### 2.5 Semiadequacy of 3-braids

Let \( \beta \) be a braid word. We write

\[
\beta = \prod_{j=1}^{k} \sigma_{p_j}^{q_j}
\]

with \( \sigma_i \) the Artin generators, \( p_j \neq p_{j+1} \) and \( q_j \neq 0 \). We will assume that the condition \( p_j \neq p_{j+1} \) includes \( p_k \neq p_1 \) when \( k > 1 \). This is not a restriction when we (as we will do below) consider braids up to conjugacy, which includes cyclic permutation of the letters. The term \( \sigma_{p_j}^{q_j} \) is called a \( p \)-syllable, or more commonly, omitting the index, just a syllable. The number \( k \) is called syllable length of \( \beta \) and written by \( \text{sl}(\beta) \). We call \( \beta \) alternating if for all \( j = 1, \ldots, k \) the numbers \( -1/p_j q_j \) are positive, or all are negative.

We will focus on 3-braids. For 3-braids the subscripts \( p_j \) alternate between 1 and 2, and become (up to conjugacy) irrelevant. We call the vector \( (q_1, \ldots, q_k) \), regarded up to cyclic permutations, the exponent vector of \( \beta \). The number \( k \) is called its length; if \( k > 1 \), then \( k \) is even. Note that when \( \beta \) is alternating, then its exponent vector is the same as its Schreier vector.

Any braid word \( \beta \) gives a link diagram \( \hat{\beta} \) under closure, as a braid gives a link. We call a braid word \( A \)-adequate if the link diagram \( \hat{\beta} \) is \( A \)-adequate. We define similarly for \( B \)-adequate, and (semi)adequate. We call a braid to be \( A \)-adequate, or \( B \)-adequate, or (semi)adequate, if it has a word representation with the same property.

Let

\[
\Delta = \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1
\]

be the square root of the center generator \( C \) of \( B_3 \).

The following is shown by a careful observation:

**Lemma 2.6.** A diagram of closed 3-braid word \( \beta \) (with exponent vector of length \( > 2 \)) is \( A \)-adequate if and only if it satisfies one of the following two conditions:

1. it is positive, or
2. \( (1) \) it does not contain \( \Delta^{-1} = \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} = \sigma_2^{-1} \sigma_1^{-1} \sigma^{-1} \) as subword and \( \sigma \) positive entries in the exponent vector are isolated (i.e., entries cyclically before and after positive entries are negative).

**Corollary 2.7.** A diagram of closed 3-braid word \( \beta \) is adequate if and only if

1. it is positive and does not contain \( \Delta \) as subword, or
2. it is negative and does not contain \( \Delta^{-1} \) as subword, or
3. it is alternating.

We proved the following in [29]:

**Theorem 2.8.** Let \( \beta \) be a 3-braid word. Then the following are equivalent:

(a) \( \hat{\beta} \) is semiadequate,
(b) \( \beta \) has minimal length in its conjugacy class,
(c) Some \( \gamma \in \{\beta, \hat{\beta}\} \) satisfies one of the conditions enumerated in Lemma 2.6, where \( \bar{\beta} \) is \( \beta \) with all letters \( \sigma_i^{\pm 1} \) replaced by \( \sigma_i^{\pm 1} \).
2.6 The $V$-$Q$ formula

The Brandt-Lickorish-Millett-Ho polynomial \([5] Q(z)\) is given by the properties

\[Q(\bigotimes) + Q(\bigcirc) = z (Q(\bigotimes) + Q(\bigcirc)),\]

\[Q(\bigcirc) = 1.\]

It is a polynomial invariant of unoriented links.

We will use below a formula, due to J. Murakami \([23]\) (and also Kanenobu \([17, \text{Theorem 2}]\)), which relates the Jones and $Q$ polynomial of a 3-braid link.

Let \(i = \sqrt{-1}, u = \sqrt{-t}\) and \(x = u + u^{-1}\). Let further \({\beta}\) for a braid \(\beta\) of exponent sum \([\beta]\),

\[\chi(\beta, u) = i^{[\beta]} u^{-2[\beta]} V_{\beta}(t) + u^{-[\beta]} (x^2 - 2).\]  

Then Murakami’s formula is:

**Theorem 2.9 (J. Murakami).** If \(L\) is a closure of a 3-braid \(\beta\), then

\[Q(L, x) = \chi(\beta, \sqrt{-u})^2 - 1 + \frac{2(x^2 + x - 1)}{x^2(x^2 - 3)} (u^{[\beta]} + u^{-[\beta]}) + \frac{-x^4 - 2x^3 + 3x^2 + 4x - 4}{x^2(x^2 - 3)} \chi(\beta, u).\]  

3 Proof of the first main result

3.1 Oriented achirality and 3-component cases

Except for the 2-component unorientedly achiral 3-braid links, the claim of this Theorem is contained in the below proposition. We can with it already recover the result of Birman-Menasco for (orientedly) achiral links \([4]\).

**Proposition 3.1.** Let \(L\) be a 3-braid link which is either achiral, or unorientedly achiral of 3 components. Then \(L = O\beta\), where \(\beta \in B_3\) is either \(a) an\ alternating braid with Schreier vector admitting a dihedral (anti)symmetry, or \(b) \beta = \sigma_1^k \sigma_2^l\ and \(|k|, |l| \leq 2.\)

**Proof.** Let us for simplicity exclude the cases of links of braid index \(\leq 2\), which are easy.

We assume \(\beta\) is a minimal length word up to conjugacy. The cases that \(\beta\) has an exponent vector of length two or less are easy to deal with, and lead to case (b). So we assume the exponent vector of \(\beta\) is of length at least 4.

Now \(\beta\) is, say, \(A\)-adequate. If \(L\) is orientedly achiral, there is a braid representation of \(L\) as a \(B\)-adequate 3-braid word \(\beta'\) (namely \(\beta'\) with \(\sigma_i\) changed to \(\sigma_i^{-1}\)). Now \(\beta'\) has the same length as \(\beta\), and moreover the same writhe (exponent sum). The latter can be concluded either from \([4]\), or from the proof of the Jones conjecture for 3-braid links in \([28]\) (which uses essentially also work in \([29]\)). Thus \([\beta] = -[\beta']\), and \([\beta'] = 0.\) This in particular rules out the case that \(L\) has 2 components, since then \([\beta]\) is odd.

Now by the work of Thistlethwaite \([30]\), for two diagrams of the same writhe and crossing number of the same link, one is \(B\)-adequate if and only if the other one is. Thus \(\beta\) is also \(B\)-adequate, and hence adequate.

In the case that \(L\) is a 3-component unorientedly achiral link, each crossing of the diagram \(D = \hat{\beta}\) is mixed, i.e. involves two distinct components. If we take the diagram \(\hat{\beta}' = \overline{D}\) (the mirror image of \(D\) with all component orientations preserved) and change orientation of some components to obtain a diagram \(D'\) of \(L\), then still \(D'\) is \(B\)-adequate (since semiaedequacy is independent on component orientation). Furthermore, \(D\) and \(D'\) both represent \(L\), and all their crossings are mixed. This implies \(w(D') = w(D)\), because it is equal to twice the sum of linking

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\(^{1}\) Note that \(V\) is here what is written as \(J\), and not \(V\), in \([17]\).
numbers between all (pairs of) components of $L$. From this the conclusion that $D$ (and $\hat{\beta}$) is adequate follows as above. (This argument does not work for 2 components because then $w(D') = w(D) \pm 2$, as can be seen from (17) below.)

So from now on we assume that $\beta$ is adequate.

In case $\beta$ is alternating, we can apply [22]. An easy observation shows that the diagram is determined by the exponent (and in this case Schreier) vector up to dihedral moves (cyclic permutations and reversal of order), and the only possible flypes in a diagram of a closed 3-braid occur at exponent/Schreier vector of length 4, with an entry $\pm 1$ (see [4]). In that case, however, the flypes just reverse the orientation of (all components of) the link. So we obtain no new symmetries. We arrive at case (a).

It remains to exclude the case that $\beta$ is not alternating. In that event, we know that $\beta$ is positive or negative, and has no $\Delta$ as subword. Positive (or negative) links are not orientedly achiral, so the possibility that $L$ is a knot, or that the achirality is oriented is easily ruled out.

It is clear that all components of an unorientedly achiral link are achiral knots, or mirror images in pairs. The latter option does not lead to anything new for closed 3-braids, so we ignore it.

We argued that $L = \hat{\beta}$ cannot have 2 components, so assume that it is a 3-component link. Let unoriented achirality manifest itself so that $L$ can be obtained from $L$ either by reversing component $K_1$, or components $K_2$ and $K_3$. This implies that the linking number $lk(K_2, K_3) = 0$ (see subsection 2.2). Since the diagram $D = \hat{\beta}$ is positive or negative, $K_2$ and $K_3$ must have no common (mixed) crossing there. This means that the exponent vector of $\beta$ is even, i.e. all entries are even, and (up to mirroring) positive. Then reversal of component orientation (of $K_1$ or $K_2 \cup K_3$) makes the (say) positive diagram $D$ into a negative diagram $D'$. Now $D'$ must be orientedly isotopic to the mirror image of $D$.

We know that in positive and negative diagrams the canonical Seifert surface has minimal genus (see e.g. [6]). That is,

$$s(D') - c(D') = \chi(D') = \chi(L) = \chi(D) = s(D) - c(D),$$

and since $c(D) = c(D')$, we have $s(D) = s(D')$. Now $s(D) = 3$ because $D$ is a diagram of a closed 3-braid. But it is easy to check that (for even exponent vector of length $> 2$) we have $s(D') > 3$; just look at $\beta = \sigma_1^2 \sigma_2^2 \sigma_1^2 \sigma_2^2$. This gives a contradiction. Thus no unorientedly achiral 3-component link can occur for exponent vector of length $> 2$.

\textbf{Remark 3.2.} In Birman-Menasco [4] the oriented achirality result was obtained (which leads to case (a) above), so we already have a slight extension of their result. However, latter follows in [4] from the extremely involved proof of the classification of closed 3-braids. It is therefore very useful to obtain simpler proofs of at least consequences of Birman-Menasco’s work. Thus more important than the partial drop of orientations here is the opportunity to bypass the method in [4]. Still we must admit that the insight in [4] motivated the present proof, and with [28], [30], and ultimately [22], we have to invoke another quite substantial body of results. Thus, whether our proof is in the end “simpler” is to some extent a matter of personal view.

\textbf{Remark 3.3.} Our alternative proof of the classification of amphicheiral 3-braid links was obtained in an earlier version of the paper [29]. We originally claimed in [29] also a result on the unorientedly amphicheiral links, but a remark of the referee helped discovering a mistake in the proof. An attempt to remedy it showed that we had in fact overlooked the two most interesting examples. The proof of the correct version, stated as Theorem 1.1 above, became too long and led us too far aside to discuss in [29]. We decided then to move out the entire treatment of the theorem (including the orientedly amphicheiral links) to this separate paper.

\section{3.2 The 2 component unorientedly achiral case}

\subsection{3.2.1 The non-generic cases}

The remaining part of the proof of Theorem 1.1 must be done by proving the following:
Proposition 3.4. Let \( L \) be a 3-braid link of 2 components which is unorientedly achiral, but not orientedly so. Then \( L \) is the Hopf link, the \((3,3)\)-rational link, or the link \( 9^2_{61} \).

Proof. If \( L = \hat{\beta} \) is a 2-component link \( K_1 \cup K_2 \), two of the strands \( x, y \) of \( \beta \) form (under closure) an achiral knot of braid index at most 2. This knot must be trivial, and so \( x, y \) have linking number

\[
lk(x, y) = \pm 1,
\]

where linking number is meant to be now the sum of signs of their common crossings in \( \beta \) (this is not a link-component-wise linking number, as defined in subsection 2.2). We fix the (oriented) mirroring of \( \beta \) for the whole proof so that

\[
lk(x, y) = 1.
\]

Let \( z \) be the third strand of \( \beta \). We name the components of \( L \) so that \( K_2 \) is the closure of \( x \cup y \), i.e. the 2-string component of \( \beta \), while \( K_1 \) is the closure of \( z \).

Let \( l := lk(K_1, K_2) \) be the linking number of the two components of \( L \). Then with (18),

\[
[\beta] = 2l + 1.
\]

We will assume that we obtain the oriented mirror image \( !L \) of \( L \) by reversing orientation of the component \( K_1 \). (The argument will apply the same way if \( K_2 \) had to be reversed, since we will use only polynomial invariants and the genus in the proof, which are preserved when both components are reversed.) We write by \( D \) the link diagram before component reversal, and by \( D' \) the one after it.

Let us here record the following simple test for unoriented achirality:

Lemma 3.5. \( V(L)(t^{-1}) = t^{-3l} V(L)(t) \).

Proof. It is well-known (and can be seen from (2) for example, or see [20]) that when reversing a component \( K \) of a link \( L \), then \( V \) shifts by \( t^{-3l} \), where \( l \) is the total linking number of \( K \), i.e. the sum of the linking numbers with all other components of \( L \). It is similarly well-known that taking the (oriented) mirror image changes \( t \) to \( t^{-1} \).

Recall (see end of subsection 2.2) that a polynomial \( V \in \mathbb{Z}[t^{\pm 1/2}] \) is weakly conjugate if \( V(t^{-1}) = t^l V(t) \) for some \( l \in \mathbb{Z} \). This property transcribes now to mean that \( a_i = \bar{a}_i \) for all \( i \) (see Definition 2.1).

We use now the Schreier normal form, and settle a few simple cases first.

The following is a simple observation.

Lemma 3.6. Let \( L = \hat{\beta} \) with \( \beta \) a 3-braid word of \( c \) crossings. Let

\[
v(\beta) := \frac{c - 2 - (1 - \chi(L))}{2}.
\]

Then when reversing \( K_1 \) in the diagram \( \hat{\beta} \), we have at most \( 3 + 2v \) Seifert circles.

Proof. Let \( D' \) be \( D = \hat{\beta} \) with \( K_1 \) reversed. Then \( D' \) is supposed to be a diagram of \( !L \). But

\[
1 - s(D') + c = 1 - \chi(D') \geq 1 - \chi(!L) = 1 - \chi(L) = c - 2 - 2v,
\]

since \( c(D') = c \). This gives \( s(D') \leq 3 + 2v \).

In some situations we will need to estimate \( v \) from above. A well-known inequality proved by Bennequin [1, Theorem 3] can be paraphrased to state that

\[
v(\beta) \leq \min (|[\beta]|_+, |[\beta]|_-).
\]

Moreover, it is well-known that \( v(\beta) = 0 \) when \( \beta \) is alternating.

We will combine below Lemma 3.6 with the following lower bound on \( s(D') \).

Lemma 3.7. Let \( \beta = C^k \alpha \) where \( \alpha \) is some word. Then

\[
s(D') \geq \begin{cases} 2|k| + 1 & \text{if } \alpha = \sigma_i^p \text{ for some } i = 1, 2 \text{ and } p \text{ odd} \\ 2|k| + 2 & \text{otherwise} \end{cases}.
\]
Proof. It is directly observed that when \( \beta = C^k \), then \( s(D') = 2|k| + 1 \). If we look at the tangle \( T \) (with 3 in- and - and outputs) that corresponds to \( \alpha \) in \( D' \) for \( \beta = C^k \alpha \), then we see also that \( s(D') \geq 2|k| + 1 \), and equality holds only if the Seifert circles in \( T \) form vertical lines. This in turn implies that the reversed strand \( z \) cannot have a common crossing in \( T \) with \( x \) or \( y \), so that \( \alpha = \sigma^k_D \). That \( p \) is odd finally follows for component number reasons.

**Lemma 3.8.** Let \( L = \hat{\beta} \) and \( \beta \) be non-generic. Then \( L \) is the Hopf link or the link \( 9_{61}^2 \).

Proof. We look at the four non-generic forms in Theorem 2.3. Forms 3 and 5 give knots under closure, so they are excluded.

In form 4 we may assume using \( C = (\sigma_1 \sigma_2 \sigma_1)^2 \), that the word is positive (keeping in mind (18)). In this case \( v(\beta) = 0 \) in (20), which follows from (21). Then by Lemma 3.6, the diagram \( D' \) must have (at most) 3 Seifert circles. It is easy to check that then \( \hat{\beta} = \sigma_1 \sigma_2 \sigma_1 \), and \( L \) is the Hopf link.

In form 2 we use (18) to see that \( \beta = \gamma^k \sigma_1 \) for some \( k \in \mathbb{Z} \), where \( \gamma = \sigma_2 \sigma_1^2 \sigma_2 \) (note that \( C = \gamma^* \sigma_1^2 = \sigma_1^2 \cdot \gamma \)). The case \( k = 0 \) gives a trivial 2-component link, so assume \( k \neq 0 \). Now we have from (21) that \( v(\beta) \leq 1 \) for \( k < 0 \) and \( v(\beta) = 0 \) for \( k > 0 \). By Lemmas 3.6 and 3.7, we see that we need to consider only \( k = -2, -1, 1 \).

For \( k = \pm 1 \) we have the (2, 4)-torus link (with parallel and reverse orientation) which is not unorientedly achiral. For \( k = -2 \) we have the link \( 9_{61}^2 \), which is known to be unorientedly achiral (see p. 45 last paragraph of [19]).

### 3.2.2 The generic cases with length greater than 2

**Lemma 3.9.** No \( L = \hat{\beta} \) with \( \beta \) generic of length \( s \geq 3 \) occurs.

In order to prove this, we go first back to the work of Futer, Kalfagianni and Purcell [11] reviewed in subsection 2.3.

Since \( L \) is an \( A \)-adequate link, we can use formula (4), and we easily obtain (as basically observed also in [11]) the following.

**Lemma 3.10.** Let \( \alpha \) as in (10) be an alternating 3-braid and \( s \geq 2 \). Then

\[
|\hat{\alpha}_1(V(\hat{\alpha}))| = \begin{cases} 1 \text{ if } s = 2 \text{ and } p_1 = p_2 = 1, \\ s \text{ otherwise} \end{cases}.
\]

A similar formula holds for \( \alpha_1(V(\hat{\alpha})) \) when replacing \( p_1 \) by \( -q_1 \).

It is somewhat important here, as in [11], to understand when the four extra monomials on the right of (9) can cancel an edge term in the unit-shifted \( V(\hat{\alpha}) \). In [11] examples were constructed where such a cancellation occurs. We show now that it cannot occur at either side.

**Lemma 3.11.** Let in lemma 2.4,

\[
P_1(t) := t^{6k} V(\hat{\alpha})(t) \quad \text{and} \quad P_2(t) := (-\sqrt{t})^{[a]} \left( t + t^{-1} \right) \left( t^{3k} - t^{-6k} \right),
\]

so that \( V(\hat{\beta})(t) = P_1(t) + P_2(t) \). If \( k > 0 \), then

\[
\left\{ \begin{array}{l}
\max \deg P_1 = \max \deg P_2 \text{ if } s = 1 \text{ and } p_1 = 1, \\
\max \deg P_1 > \max \deg P_2 \text{ otherwise}
\end{array} \right. \quad (24)
\]

Proof. We have from (11) that \( \max \deg P_1 = 6k + (3p - q)/2 \), while \( \max \deg P_2 = [a]/2 + 6k + 1 = (p - q)/2 + 6k + 1 \). The result follows by comparison, because \( p \geq 1 \), and \( p = 1 \) only if \( s = p_1 = 1 \).

**Corollary 3.12.** If \( s > 1 \), then \( \text{span } V(\hat{\beta}) \geq p + q - 1 \).

Proof. For \( k = 0 \) the claim is clear from (11). Up to taking the mirror image, we may assume now \( k > 0 \). Then by (24), \( P_1 \) determines the maximal degree of \( V(\hat{\beta}) \). Moreover, \( \min \deg V(\hat{\beta}) \leq \min \deg P_1 + 1 \). Otherwise, we must have, besides \( \min \deg P_1 = \min \deg P_2 \) and \( a_0(P_1) = -a_0(P_2) \), also \( a_1(P_1) = -a_1(P_2) \). But \( a_1(P_2) = 0 \), while \( a_1(P_1) \) is non-zero by Lemma 3.10.
Proof of Lemma 3.9. We use first (9) in Lemma 3.5 and get with (19) and
\[
[\beta] = 6k + [\alpha]
\] (25)
the condition
\[
\begin{aligned}
t^{3l-6k} V(\hat{\phi})(t^{-1}) - t^{6k} V(\hat{\phi})(t) &= (-\sqrt{t})^{2l+1-6k} \left(t^{3k} - t^{6k}\right) \\
&\quad - (-\sqrt{t})^{6k-1+4l} \left(t^{-3k} - t^{-6k}\right).
\end{aligned}
\] (26)
We would like to conclude from (26) that the two polynomials on the left
\[
P_3(t) = t^{3l-6k} V(\hat{\phi})(t^{-1}),
\] (27)
and \(P_1(t) = t^{6k} V(\hat{\phi})(t)\), have equal extremal degrees. (The name \(P_3\) was chosen in order not to collide with (23).) When \(k = 0\), then the r.h.s. vanishes, and \(P_1 = P_3\), so assume \(k \neq 0\).

Then it is easy to see that the 8 monomials on the right cannot form a polynomial \(R\) with \(|a_0(R)| = 1\) and \(|a_1(R)| > 1\). Contrarily we know that \(|a_0(P_1)| = |a_0(P_3)| = 1\). Now if \(\text{min deg } P_1 \neq \text{min deg } P_3\), then \(|a_0(P_1 - P_3)| = 1\) and \(|a_1(P_1 - P_3)| \geq |a_1(P_1)| - 1\) for some \(i\). But by Lemma 3.10 we have that \(|a_1(P_1)| = |a_1(P_3)| = s > 2\), so that \(|a_1(P_1 - P_3)| > 1\) and (26) cannot hold.

Thus min deg \(P_1 = \text{min deg } P_3\), and then also max deg \(P_1 = \text{max deg } P_3\). Latter gives for \(\hat{V}(t) = V(\hat{\phi})(t)\)
\[
6k + \text{max deg } \hat{V}(t) = 3l - 6k - \text{min deg } \hat{V}(t),
\]
and using (11)
\[
2p - 2q = \text{max deg } V + \text{min deg } \hat{V} = 3l - 12k.
\]
Now \([\beta] = 6k + [\alpha] = 6k + p - q\), and so
\[
2[\beta] = 2p - 2q + 12k = 3l.
\]
On the other hand, we have (19), and so
\[
2[\beta] = 2(2l + 1) = 3l.
\]
We obtain
\[
l = -2.
\] (28)
Then from (19) we have
\[
[\beta] = -3.
\] (29)
We know that \(L\) has a 3-braid representation of exponent sum \(-3\), while \(!L\), whose Jones polynomial is given from Lemma 3.5 by
\[
V(!L)(t) = V(L)(t^{-1}) = t^{6k} V(L)(t),
\] (30)
has a 3-braid representation of exponent sum \(3\). Now we use Murakami’s formula (16) and the well-known fact that
\[
Q(L) = Q(!L).
\] (31)
Clearing denominators and absolute terms in (16), we find, with \(t = \sqrt{-1}, u = \sqrt{-l}\) and \(x = u + 1/u\),
\[
Q_1(L, x) := x^2(x^2 - 3)(Q(L, x) + 1) = P(e, V, t),
\]
where
\[
P(e, V, t) = [i^e(-t)^{-e} V(t) + (-t)^{-e/2}(-t - \frac{1}{t})] \cdot (-x^4 - 2x^3 + 3x^2 + 4x - 4) + [i^e(-t)^{-e/2} V(-\sqrt{-t}) + (-t)^{-e/4} x^2] \cdot x^2(x^2 - 3)
\]
\[
+ 2(x^2 + x - 1)(\sqrt{-t} + \sqrt{-t}^{-e}).
\] (32)
Now we have from (31)  
\[ P(-3, V(L)(t), t) = P(3, t^6 V(L)(t), t). \]
Substituting and simplifying this yields, after some simple (though not very pleasant) calculation,

\[ 0 = \left( u^{-3} - u^3 \right) \left( -x^5 + x^4 + 4x^3 - 5x^2 - 4x + 4 \right) \]
\[ -i u^6 V(-u^2) \left( -x^4 - 2x^3 + 3x^2 + 4x - 4 \right) \]
\[ -i u^3 V(-u) \left( -u^{-3/2} - u^{3/2} \right) x^3 (x^2 - 3). \]

To see what polynomial \( V \) can satisfy such an identity, look first at the extreme degrees of the three summands on the right. In order the terms to cancel out, at least two must have the same minimal degree, and two the same maximal degree. Using that 2 \( \min \deg V \) and 2 \( \max \deg V \) are odd integers for a 2-component link, we have only the options

\[ \min \deg V \in \left\{ \frac{-9}{2}, \frac{-11}{2} \right\}, \quad \text{and} \quad \max \deg V \in \left\{ \frac{3}{2}, \frac{-1}{2} \right\}. \]

From here there are two ways to see that no relevant \( V \) can occur. The simpler one is to observe that then \( \operatorname{span} V \leq 5 \). By corollary 3.12, this means that \( p + q \leq 6 \). But if \( s \geq 3 \), then the only option is \( a = (\sigma_1 \sigma_2^{-1})^3 \), which gives a 3-component link.

The other check is to solve for \( V \). To simplify the expressions a bit, let \( \tilde{V}(u) = u^{3/2} \cdot V(u) \), and make the ansatz

\[ \tilde{V}(u) = au^{-4} + bu^{-3} + cu^{-2} + du^{-1} + e + fu = (a b c d [e] f). \]

The right '=' is the definition of a compact notation for polynomials we use below, whereby we write

\[ \sum_{s=-d_1}^{d_2} a_s u^s \quad \text{as} \quad (a_{-d_1} a_{1-d_1} \ldots a_{-1} [a_0] a_1 \ldots a_{d_2}) \]

for \( d_1, d_2 \geq 0 \) (adding zero terms if needed). Then, with this notation, (33) yields

\[ 0 = \left( -1 \right) 1 1 - 1 \left( -2 \right) 0 1 0 2 1 1 - 1 1\]
\[ + (a [0] - b 0 c 0 - d [e] 0 - f) \cdot (1 1 - 2 - 4 - 2 - 1 - 2 - 1) \]
\[ -(a - b c - d [e] - f) \cdot (1 0 2 1 1 2 1 2 0 1). \]

Comparing coefficients one by one, we find linear relations, which show that the unknowns \( a, \ldots, f \) in (34) have no solution. (This calculation can be further simplified using that from (30) we have in (34) also that \( a = f, b = e \) and \( c = d \).)

\[ \square \]

### 3.2.3 The generic cases with length 2

**Lemma 3.13.** No unorientedly achiral \( L = \hat{\beta} \) for \( \beta \) generic with \( s = 2 \) occurs.

**Proof.** Let \( \beta \) be a generic braid of length 2. Then \( \beta = C^k \cdot \alpha \), where \( \alpha \) is alternating. For a 2-component link \( L = \hat{\beta} \), the Schreier vector

\[ S = (p_1, q_1, p_2, q_2) \]

(with \( \alpha = \sigma_{-1}^p \sigma_1^{q_1} \sigma_{-1}^p \sigma_2^{q_2} \)) has up to cyclic permutation the parities

(even, odd, odd, odd) or (even, even, even, odd).

Let us abbreviate these two options as 'eeoo' and 'eeeo'. (Note that with this freedom, we have to take into account either signs for \( p_1 \); the signs of \( p_2 \) and \( q_1 \) are then determined by \( p_1, p_2 > 0 \) and \( p_1 q_j < 0 \)).

If \( k = 0 \), then \( v(\beta) = 0 \), because \( \beta \) is then alternating. Thus by Lemma 3.6, we have \( s(D') \leq 3 \). It is easy to see that in the case 'eeeo' \( s(D') \geq 5 \), while in the case 'eeoo' we have \( s(D') = 3 \) only if \( S = (2, -1, 1, -1) \) (assuming (18)). Then \( \beta = [1^2 - 21 - 2] \), giving the Whitehead link, which is not unorientedly achiral.
We assume now that \( k \neq 0 \). We go back to (26). If we use (23) and (27), then (26) can be written as

\[ P_3(t) - P_1(t) = P_2(t) - P_4(t), \]  

with

\[ P_4 = (-\sqrt{t})^{4l-1+6k} \left( t + t^{-1} \right) \left( t^{-3k} - t^{-6k} \right). \]

Since \( \alpha \neq (\alpha_1 \alpha_2^{-1})^2 \) (otherwise \( L \) is a knot), we have by Lemma 3.10 that at least one of \( |a_1(V(\tilde{a}))| \) and \( |\tilde{a}_1(V(\tilde{a}))| \) is \( 2 \).

**Case 1.** The case \( \max \deg P_1 = \max \deg P_3 \) was already discussed in the proof of Lemma 3.9. The argument, which among others leads to (28), can be repeated, except that in (the shorter) one of the options concluding the proof span \( V(\tilde{\beta}) \leq 5 \) leaves us to consider \( S = \pm(2, -1, 1, -1) \). Then \( |\beta| = 6k \pm 1 \), which contradicts (29).

**Case 2.** Consider the case

\[ |\max \deg P_1 - \max \deg P_3| = 1. \]

Now the polynomial \( P_2 - P_4 \) on the right of (35) cannot have \( |a_1| \geq 2 \) or \( |\tilde{a}_1| \geq 2 \). This means that one of \( |a_1(V(\tilde{a}))| \) and \( |\tilde{a}_1(V(\tilde{a}))| \) should be 0 or 1. By Lemma 3.10, this can occur only if

\[ q_1 = q_2 = \pm 1 \]  

in ‘eooe’, in which case \( |a_1(V(\tilde{a}))| \) or \( |\tilde{a}_1(V(\tilde{a}))| \) is 1. (The former is 1 for \( q_i = -1 \) and the latter for \( q_i = 1 \).) Then, we need to have

\[ |a_1(P_2 - P_4)| = |\tilde{a}_1(P_2 - P_4)| = 1 \]

for the polynomial \( P_2 - P_4 \) on the right of (35). This can only occur if

\[ |\max \deg P_2 - \max \deg P_4| = |l + 1 + 3k| = 1. \]

(Note that \( \max \deg P_2 \) and \( \max \deg P_4 \) vary with \( \pm 3k \) when \( k \) changes sign, but this variation is canceled out in the difference.) So either

\[ k = \frac{l}{3} \quad \text{or} \quad k = \frac{1}{3}(l-2). \]

Now we use the formula (5) for \( a_2(V(\tilde{a})) \) and \( \tilde{a}_2(V(\tilde{a})) \). The choice between \( a_2 \) and \( \tilde{a}_2 \) should depend on whether \( a_1 = \pm 1 \) or \( \tilde{a}_1 = \pm 1 \) in the line below (36). We stated the formula above, and shall use it here, for \( a_2 \); the case of \( \tilde{a}_2 \) is analogous.

Under the assumption (36), we have \( \delta = 0, \Delta = 1, \) and \( e_{++} = 2 \) for \( |p_2| = 1 \) and \( e_{++} = 3 \) otherwise. Then (5) gives

\[ |a_2(V(\tilde{a}))| = \begin{cases} 2 & \text{if } |p_2| = 1 \\ 3 & \text{otherwise} \end{cases}. \]

Now from (37) we have \( |a_2(P_2 - P_4)| = |\tilde{a}_2(P_2 - P_4)| = 1 \). Combining this with (39) shows that (35) can hold only if

\[ |p_2| = 1 \quad \text{and} \quad |\max \deg P_1 - \max \deg P_3| = |l + 2| = 2. \]

Then either \( l = 0 \), and from (38) then (since \( k \) is an integer) \( k = 0 \), in contradiction to assuming \( k \neq 0 \), or \( l = -4 \) and \( k = -2 \). Thus now

\[ \beta = C^{-2} \sigma_1^\pm 1 \sigma_2^3 \sigma_1^{-1} \sigma_2^{\mp 1}. \]

Now from (18), and keeping in mind that we have the case ‘eooe’, we have \( lk(x, y) = 2k + p_2 = -4 + p_2 = 1 \), in contradiction to the first condition in (40).

**Case 3.** Thus we assume in the following that

\[ |\max \deg P_1 - \max \deg P_3| = 1. \]

**Case 3.1.** If one of \( a_1(P_1 - P_3) \) or \( \tilde{a}_1(P_1 - P_3) \) vanishes, then one of \( |a_1(V(\tilde{a}))| \) and \( |\tilde{a}_1(V(\tilde{a}))| \) should be 1. By Lemma 3.10, the other one must be 2, and (36) holds, so we have a Schreier vector of type ‘eooo’. Then we can look at \( a_2 \) and \( \tilde{a}_2 \) as in the previous case, and see that now (40) modifies to

\[ |p_2| > 1 \quad \text{and} \quad |\max \deg P_1 - \max \deg P_3| = |l + 2| = 1. \]
Then either $l = -1$ or $l = -3$, and in both cases from (38) we have $k = -1$ (since $k$ is an integer). Now, for type ‘eeoo’, the mixed crossings corresponding to letters in the Schreier vector are only those counted by $p_2$, and then (18) gives

$$lk(x, y) = 2k + p_2 = -2 + p_2 = 1.$$ 

so $p_2 = 3$. Then $p_1 > 0$ and the sign on the r.h.s. of (36) is negative. Thus

$$\beta = C^{-1}a_1^{-1}a_2^{-1}a_3^{-1}.$$ 

Next, the values of $l$ determine with (19) that $[\beta] = -5$ or $[\beta] = -1$. The former leads to $p_1 = 0$, which cannot occur, so we look at the latter case, where

$$\beta = C^{-1}a_1^{-1}a_2^{-1}a_3^{-1} = (a_2a_1^{-2}a_2^{-1})^{-1}a_3^{-1}.$$ 

This braid can be checked directly not to have a weakly conjugate $V$ (see end of subsection 2.2).

**Case 3.2.** Thus now we can assume none of $a_1(P_1 - P_3)$ or $\tilde{a}_1(P_1 - P_3)$ vanishes. This implies again (37) and (38), and we still have the second condition in (42). This leads us again to consider only

$$l = -1, -3 \text{ and } k = -1.$$ 

But we have now, so far, no conditions on $p_i$ and $q_i$, and have to treat either cases ‘eoo0’ and ‘eeeo’.

**Case 3.2.1.** ‘eeeo’. By direct drawing of the braid, we see that

$$l = 2k + \frac{p_1 + p_2}{2}.$$ 

Using (44) we find $p_1 + p_2 = \pm 2$, but since $p_i \neq 0$ are of the same sign and even, this cannot occur.

This finishes the case ‘eeeo’.

**Case 3.2.2.** ‘eoo0’. In this case we find

$$l = 2k + \frac{p_1 + q_1 + q_2}{2},$$

so that

$$p_1 + q_1 + q_2 = \pm 2.$$ 

Moreover, $lk(x, y) = 2k + p_2 = 1$, so that

$$p_2 = 3.$$ 

This determines then the signs $p_1 > 0, q_i < 0$.

If $q_1 = q_2 = -1$, then from (45) we must have $p_1 = 4$ (and the positive sign on the r.h.s.), and from (44) and (46) we obtain the braid in (43), which we already checked.

If some $q_i < -1$, then with (46) and Lemma 3.10 we have:

$$|a_1(V(\hat{a}))| = |\tilde{a}_1(V(\hat{a}))| = 2.$$ 

From (37) we have $|a_2(P_2 - P_4)| = |\tilde{a}_2(P_2 - P_4)| = 1$, which combined with (35), (41) and (47) gives:

$$|a_2(V(\hat{a}))| = |\tilde{a}_2(V(\hat{a}))| = 3.$$ 

(Take into account that $a_1a_1 \leq 0 \text{ by [31].}$) To see that this cannot occur, we use again (5). We have $e_{++} = 2$ because both $p_i > 1$, but not both $q_i = -1$, then $\delta = 0$ in alternating diagrams, and (using that $q_i$ are odd) $\Delta = 1$ if some $q_i = -1$, and 0 otherwise. With (47) we get from (5) then

$$|a_2(V(\hat{a}))| \in \{4, 5\},$$

in contradiction to (48). This finishes the case ‘eoo0’. Here is the end of case 3, and the proof of Lemma 3.13. \( \square \)
3.2.4 The generic cases with length 1

Lemma 3.14. Let $L = \tilde{\beta}$ and $\beta$ be generic with $s = 1$. Then $L$ is the (3,3)-rational link or the Hopf link.

Proof. We have $\beta = C^k \alpha_1^p \alpha_2^q$ with $p \cdot q < 0$. The case $k = 0$ easily leads to the Hopf link, so assume $k \neq 0$.

By connectivity we may further assume that $p$ is even and $q$ is odd, and then from (18), we have

$$2k + q = 1.$$  \hfill (49)

Note first that then $k$ and $q$ have opposite sign, and so $k$ and $p$ have the same sign. From (49) we also have

$$[\alpha] = p + 1 - 2k,$$

and so in (23) we have

$$P_2(t) = (-\sqrt{t})^{p+1-2k} (t + t^{-1}) (t^{3k} - t^{6k}).$$

Now we look at the degrees on the right of (9). We have with (49),

$$\min \deg P_1(t) = \frac{p-3q}{2} + 6k = \frac{p-3+6k}{2} + 6k = \frac{p-3}{2} + 9k,$$

and similarly

$$\max \deg P_1(t) = \frac{3p-1}{2} + 7k.$$  \hfill (50)

The degrees of $P_2$ depend on the sign of $k$.

Let first $k > 0$. Then $q < 0$ and $p > 0$, and

$$\min \deg P_2(t) = \frac{p-1}{2} + 2k \quad \text{and} \quad \max \deg P_2(t) = \frac{p+3}{2} + 5k.$$  \hfill (51)

From (50) and (52) with $k > 0$ we see that

$$\min \deg P_2 < \min \deg P_1 - 1,$$

and so $a_1(V(\tilde{\beta})) = 0$. Weak symmetry means then that

$$\tilde{a}_1(V(\tilde{\beta})) = 0.$$  \hfill (53)

On the other hand from (51) and (52) we have $\max \deg P_2 < \max \deg P_1$ (because $p + 2k > 2$). Combined with (53) this means that either

$$\max \deg P_2 = \max \deg P_1 - 1.$$  \hfill (54)

or

$$\tilde{a}_1(P_1) = \tilde{a}_1(V(\tilde{\alpha})) = 0.$$  \hfill (55)

The option (54) says that $2k + p = 3$, which for $k, p > 0$ means $k = p = 1$, but this contradicts our assumption that $p$ is even. Thus consider the option (55). Using either the well-known polynomials of the $(2, \ldots)$-torus knots, or formula (4), we find that the condition on $a_1$ means that $p \leq 2$. Since $p$ was assumed even, $p = 2$, and thus $\beta$ is of the form

$$\beta = C^k \sigma_1^2 \sigma_2^{1-2k} = [12^2]^{1k} \sigma_1^2 \sigma_2$$

(while $\cong$ means equality up to conjugacy).

Next assume $k < 0$ (and $p < 0$). Then we have instead of (52) that

$$\min \deg P_2(t) = \frac{p-1}{2} + 5k \quad \text{and} \quad \max \deg P_2(t) = \frac{p+3}{2} + 2k.$$  \hfill (56)

Now comparing with (51) shows $\max \deg P_1 < \max \deg P_2 - 1$. Thus $a_1(V(\tilde{\beta})) = 0$, and so $a_1(V(\tilde{\beta})) = 0$.

From (50) we have $\min \deg P_1 < \min \deg P_2 - 4$, so that $a_1(V(\tilde{\alpha})) = 0$. Now $\alpha = \sigma_1^p \sigma_2^q$ with $p < 0$ and $q > 0$.

The condition on $a_1$ means that $p \geq -2$. Since $p$ was assumed even, $p = -2$. Then $\beta$ is of the form

$$\beta = C^k \sigma_1^{-2} \sigma_2^{1-2k} = [-1(-2)^2 - 1]^{-k} \sigma_1^{-2} \sigma_2.$$  \hfill (57)
Up to mirroring, we can combine (56) and (58) into
\[ \hat{\beta} = [12^2]k \sigma_2 \sigma_{\pm 1}^k \text{ for } k > 0. \] (59)
We want to use again Lemma 3.6, and need to estimate \( v(\hat{\beta}) \). From (21) we get \( v(\hat{\beta}) = 0 \) for \( '+' \) and \( v(\hat{\beta}) \leq 1 \) for \( '--' \), where \( '+' \) and \( '--' \) refer to the \( \sigma_{\pm 1}^k \) in (59).

A direct look at \( D' \) shows that \( s(D') = 3 + 2k \). By Lemma 3.6, we see that only the \( '--' \) is possible, for \( k = 1 \). This corresponds to the word \([12^2]_{3} - 2\], which gives the (3,3)-rational link.

Proposition 3.4 follows now from Lemmas 3.8, 3.13, 3.14, and 3.9.

With this the proof of Theorem 1.1 is finished.

4 Proof of the second main result

This section contains the proof of Theorem 1.2.

4.1 Genus estimate

We will assume now \( \beta \) to be a 3-braid. We want to find out when the link \( M = \hat{\beta} \) is isotopic to one \( \tilde{M} \) obtained by reversing some component(s). Of course, for most links there will not be such an isotopy. Thus our goal is to distinguish \( M \) from \( \tilde{M} \) whenever possible. The only invariant I see that is generally effective yet controllable is the genus, which is equivalent to the Euler characteristic \( \chi \) we will use below. The proof will thus center around establishing that
\[ 1 - \chi(M) > 1 - \chi(\tilde{M}) \] (60)
for most \( \beta \). A few further cases are then subjected to tests using the skein polynomial \( P \). What remains turns out to conform to one of the two diagrams of Figure 1.

Since both the genus and skein polynomial are invariant when all components are reversed, it is admissible throughout the proof to make the assumption that the component of \( M \) we reverse is an unknot \( O \) given by a 1-string subbraid \( S \) of \( \beta \).

Before we turn to the genus, we make one important unrelated observation. Let \( lk(O) \) be the total linking number of \( O \). It can be determined by one half of the sum of the writhes of all crossings of \( \beta \) into which \( S \) enters, and this is how we will keep track of it below. Then if \( M \) and \( \tilde{M} \) are isotopic, we have
\[ lk(O) = 0. \] (61)
There are several ways to see this, e.g. using the degree shift of \( V \) in (2). (It also follows from a formula of Hoste-Hosokawa [14, 15] for the minimal coefficient of the Conway polynomial.) Apart from this (important) insight, the Jones polynomial becomes useless for the rest of the proof.

In order to work with the genus, it is imperative that we use the form (13). Let us at this point exclude the split links \( M \). They are partially invertible, and conform to the first pattern of Figure 1, where some of the boxes contains two trivial (uncrossing) strands. With this exclusion and condition (61), we see that we ruled out the forms (A) in (13), and so we will focus throughout on form (B).

We assume from now on that
\[ \beta = R L^{-1}. \] (62)
For convenience we will introduce \( \sigma_{-i} = \sigma_i^{1-} \) in order to have only positive powers in the syllables, and we continue using the designation where we denote the \( \sigma_i \) by their indices in a bracketed list, e.g.
\[ [1, 2^2, -3^2] = \sigma_1 \sigma_2 \sigma_2 - \sigma_3 = \sigma_1 \sigma_2 \sigma_3 + 2. \]
We have
\[ 1 - \chi(M) = -2 + \#\{\text{letters in } \beta\} = -2 + \#\{\text{crossings in } \beta\} - 2\#\{\text{syllables in } \beta\}, \tag{63} \]
where ‘crossings’ is understood by Artin generators when expanding
\[ \sigma_{\pm 3}^n = \sigma_{-1}^n \sigma_1. \tag{64} \]
following (12), and ‘\pm 3-syllables’ are those of \( \sigma_{\pm 3} \). Let us fix that throughout this proof, except a few places where indicated otherwise, we always use this first (rather than the second) way in (12) to expand \( \sigma_{\pm 3}^n \). When \( D \) is the closed braid diagram of \( \beta \) so expanded (into a word in Artin’s generators), then the crossings in \( \beta \) are counted by \( c(D) \).

Now we compare (63) to \( 1 - \chi(M) \), which we estimate by the canonical surface coming from the diagram \( D \) obtained from \( Q \) after the strand \( S \) is reversed. For this it is necessary to count the Seifert circles in \( D \). This is essentially done by looking at the case that the middle strand is reversed in a braid of the form
\[ \sigma_2^{2a_1} \sigma_1^{2b_1} \ldots \sigma_2^{2a_l} \sigma_1^{2b_l}. \]
The result is illustrated in Figure 3: each syllable contributes \( 2a_i - 1 \) resp. \( 2b_i - 1 \) small Seifert circles, and there is one ‘global’ Seifert circle running along the whole braid. Thus the number of Seifert circles is
\[ s(D) = \sum 2a_i + \sum 2b_i - 2l + 1. \tag{65} \]

**Fig. 3. Counting the Seifert circles after reversing a braid strand**

Let us below take care of the quantity \( l \) in the following way. We call a *flip* a subword of \( \beta \) (regarded as a cyclic word in \( \sigma_1, \sigma_2 \)) of the form \( \sigma_2^{\pm 1} \sigma_1^{\pm 1} \sigma_2^{\pm 1} \) where the to-reverse strand \( S \) changes from leftmost to rightmost or vice versa. (Note that all such subwords are disjoint.)

Some examples of flips are shown below, where we fix (also for later) the convention that \( S \) is drawn thickened in figures. (The drawing of braids follows the explanation in subsection 2.3.)

Then (65) can be rewritten as
\[ s(D) = 1 - \#\{\text{flips}\} + \#\{\text{crossings of } S\}. \tag{66} \]

We observe now that this formula applies also for general \( \beta \). Every diagram \( D \) can be turned into one of the type in Figure 3 by smoothing out (see below (7)) crossings between the two strands different from \( S \) (unless \( S \) is an isolated left or right strand). This smoothing procedure does not alter any of the quantities in (66) (and \( S \) will be a split component in \( D \) after this transformation if and only if it is so before it).
Now we have
\[1 - \chi(M) = 1 - \chi(\hat{M}) \leq 1 - \chi(\hat{D}) = c(\hat{D}) - s(\hat{D}) + 1.\]
In certain cases below we will be able to see that \(\hat{D}\) is not a minimal genus diagram of \(M\), and then
\[1 - \chi(\hat{M}) \leq -1 - \chi(\hat{D}).\] (67)
With \(c(D) = c(\hat{D})\), we obtain from (63) and (65):
\[\#(\text{crossings of } S) - \#(\text{flips}) - 2\#(\pm 3\text{-syllables in } \beta) \leq \begin{cases} 2 & \text{for general } \hat{D}, \\ 0 & \text{if } \hat{D} \text{ is not minimal genus diagram} \end{cases}.\] (68)

### 4.2 The defect

We are thus led to consider (68). The idea we follow now is that for sufficiently long words \(\beta\), the l.h.s. of (68) will become large, and thus (68) cannot hold.

Let us here fix some more terminology on extensions. We call a syllable \(\sigma_{i}^{k+n}\) an \(n\)-extension of \(\sigma_{i}^{k}\) (where \(k, n > 0\)). We use the same name also for an iterated \(n\)-extension. A 1-extension is simply an extension. A word \(w'\) is an extension of another word \(w\), if \(w'\) is obtained by (possibly repeatedly) extending syllables in \(w\). A syllable \(\sigma_{\pm 1}^{k}\) or \(\sigma_{\pm 2}^{k}\) is inner if it involves the to-reverse strand \(S\), otherwise it is outer. A syllable \(\sigma_{\pm 3}^{k}\) is inner if \(S\) enters and exits as the first or third strand, otherwise (second strand) it is outer. An extension or \(n\)-extension is inner/outer if it is performed at an inner/outer syllable.

Now we look at the form (62). We can write
\[\beta = RL^{-1} = \delta R' \alpha \cdot \tilde{\beta} L' \gamma,\] (69)
where \(R' = \prod_{i=1}^{k} R'_i\) and \(R'_i\) is an extension of \([1, 2, 3]\), and \(L' = \prod_{i=1}^{k'} L'_i\) with \(L'_i\) extensions of \([-3, -2, -1]\).

Moreover, \(\alpha\), \(\tilde{\beta}\) and \(\gamma\) are either empty, or extensions of \([1, 2], [-2, 1]\) and \([-3, -2]\) resp. (with the parenthesized syllable present or not). We have a freedom to cyclically permute indices in \(L\) and \(R\), which we use here to fix that \(R\) starts with \(\sigma_1\). This means that \(\delta\) is empty. (We will nevertheless see later why we need to care about a \(\delta\) standing at that place in (69).)

We return attention to (68). Let us call the l.h.s. of (68) the defect \(d(\beta)\) of a braid word \(\beta\). We assume this definition made for every word \(w\) written in \(\sigma_{1,2,3}\), and not necessarily of the form (69):
\[d(w) := \#(\text{crossings of } S \text{ in } w) - \#(\text{flips in } w) - 2\#(\pm 3\text{-syllables in } w).\] (70)

We will now analyze the defect of \(\beta\) using the structure (69). In doing so, we observe that it is often enough to look at words reduced under inner 2-extension and outer extension. (This means, all exponents of outer syllables should be 1, and for inner syllables 1 or 2.)

Outer extension does not change the defect, and so with every word satisfying (68) all its outer extensions do as well. (That the minimal genus property is not changed can be argued by results of Gabai [12, 13] we will invoke more substantially later.) Inner 2-extension augments the defect by two. It may change the r.h.s. of (68) if the genus non-minimizing property is used, but this change occurs only once (if inner 2-extension is applied iteratedly).

When we expand \(\sigma_{3}\) as in (64), then the defect of \(\beta\) can be calculated by those of the parts on the right of (69), with \(R'\) resp. \(L'\) further splitted into \(R'_i\) resp. \(L'_i\). This is because no flips can occur where the words are composed within \(R\) and \(L\). Flips can, though, occur at the two places \(R\) and \(L^{-1}\) are (cyclically) composed, and we call these at most two flips extra flips.

With this explanation, (68) becomes now
\[d(\alpha) + d(\tilde{\beta}) + d(\gamma) + \sum_{i=1}^{k} d(R'_i) + \sum_{i=1}^{k'} d(L'_i) \leq \#(\text{extra flips}) + \begin{cases} 2 & \text{for general } \hat{D}, \\ 0 & \text{if } \hat{D} \text{ is not minimal genus diagram} \end{cases}.\] (71)

We next look at the individual defects that occur in (71).
A direct calculation shows that
\[ d(\alpha), d(\tilde{\beta}) \geq 0, \text{ while } d(\gamma) \geq -1. \] (72)

Next we look at \( R_i' \) and \( L_i' \).

**Lemma 4.1.** The defect \( d(R_i') \geq 0 \), and \( d(L_i') = 0 \) if and only if \( R_i' \) is an outer extension of \([1, 2, 3]\). Similarly with \( L_i' \) and \([-3, -2, -1]\).

**Proof.** Since the defect increases by two under an inner 2-extension, it is enough to look first at words reduced under inner 2-extension and outer extension. We try out the 8 words in which the exponents of \([1, 2, 3]\) are 1 or 2.

We separate the \( R_i' \) and \( L_i' \) into three types.

- **type 0:** all the words of positive defect,
- **type 1:** a word of zero defect in which the strand \( S \) enters/exits as second or third strand (and has 3 crossings),
- **type 2:** a word of zero defect in which the strand \( S \) enters and exits as first strand (and has 4 crossings).

The \( R_i' \) of non-zero type look, without outer extension, like this:

(The corresponding \( L_i' \) are exactly the mirror images thereof.) Note that the two type 1 words have (one or two) syllables that admit outer extension: in the first type 1 word, the 1- and the 3-syllable is extendable, in the second one the 2-syllable. However, a type 2 word admits no extension (since \( S \) is involved in all syllables).

This means now that the only way in which one can make the number of syllables of \( \tilde{\beta} \) indefinitely large is by introducing \( R_i' \) and \( L_i' \) of type 1 and 2. We will now describe ways to control such words.

**Type 1 words.** We first consider type 1 words. Let us note that if \( R_{i-1}' \) and \( R_{i+1}' \) are type 1, then \( R_i' \) is not type 2. That is, a sequence of consecutive \( R_i' \) of type 1 is terminated on either side by a type 0 word (or the start/end of \( R' \)). A similar remark applies for \( R_i' \) of type 2, and for \( L_i' \).

In order to limit the consecutive \( R_i' \) (and \( L_i' \)) of type 1, we use the observation shown on the left. If we have two consecutive \( R_i' \) of type 1, such that the second is entered by \( S \) as the second strand, we can modify \( S \) within these two words (as shown by the dashed lines) so as to avoid a pair of flips. That is, two such \( R_i' \) of type 1 can be
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counted to have a total defect of 2. We call this procedure type 1 cancellation. In particular, this means that a defect will always occur when at least 3 consecutive $R_i'$ are of type 1.

This gives a way to bound the number of type 1 words.

Type 1 words allow us also to detect the non-minimality of the genus of $\hat{D}$. We call this argument below shortly type 1 reduction. It is easily observed that when $S$ is reverted in a type 1 word, then $\hat{D}$ has a pair of crossings of opposite writhe which connect the same two Seifert circles (the ‘global’ Seifert circle in Figure 3 and one of the small ones). Then by the work of Gabai [12, 13], $\hat{D}$’s canonical surface is not of minimal genus. This is because the two opposite crossings enter into a (special) Murasugi summand of $\hat{D}$, and the canonical surface of this summand in compressible. However, this argument does not work when $R_i'$ is involved in a type 1 cancellation. Moreover, the argument may fail if an inner 2-extension is applied.

Type 2 words. We next consider type 2 words. These are equal to (and not extensions of) $[1, 2; 3]$ and $[3; 2; 1]$, entered (and exited) by $S$ being the first strand. As explained above, they are not intermixed by type 1 words.

Now we observe that $[1, 2; 3] = [1, 2]^3 \cdot [-2^3]$, with the first factor being central in $B_3$. Similarly $[-3, -2, -1] = [1, 2]^{-3} \cdot [2^3]$. Consequently, $[1, 2, 3]$ and $[-3, -2, -1]$ of type 2 can be cancelled against each other, at the cost of replacing them by powers of $\sigma_2^\pm 3$. If we allow this, the only way in which $\beta$ can have arbitrary many syllables remains when $R$ has many more type 2 words than $L$ (or vice versa). This, however, is controlled from (61), since the number of type 0 and type 1 words (and their contributions to $lk(O)$) is already bounded.

The type 2 cancellation we just described, of course, requires us to modify the presentation (69). We will discuss this in more detail below. Let us notice, however, that it preserved the word length (in $1; 2; 3$), and so we still have a minimal genus surface spanned by the new word. Moreover, (71) still holds: we replaced a subword of zero defect with another one, and there are still no flips created within $R$ or $L^{-1}$. (Note that where $\sigma_2^\pm 3$ is inserted, $S$ is always the first strand.)

After type 2 cancellation, we have now gained control over the syllable length of $\beta$. We will derive an explicit (but crude) bound below, on the way of obtaining the list of relevant $\beta$.

4.3 A syllable length bound

We now derive a bound on the syllable length of $\beta$ after type 2 cancellation.

With (71), and at most 2 extra flips, we have

$$4 \geq d(\alpha) + d(\hat{\beta}) + d(\gamma) + \sum_{i=1}^{k} d(R_i') + \sum_{i=1}^{k'} d(L_i').$$

(73)

With (72) we have

$$5 \geq d(\alpha) + \sum_{i=1}^{k} d(R_i'),$$

(74)

and

$$4 \geq d(\hat{\beta}) + d(\gamma) + \sum_{i=1}^{k'} d(L_i').$$

(75)

We also get this way

$$5 \geq \sum_{i=1}^{k} d(R_i') + \sum_{i=1}^{k'} d(L_i').$$

(76)

Now let $\bar{R}$ resp. $\bar{L}$ be $R'$ resp. $L'$ with all type 2 words $R_i'$ resp. $L_i'$ removed.

With what we said about type 1 words $R_i'$ and $L_i'$, we have that at least every third of $R_i'$ and $L_i'$ in $\bar{R}$ resp. $\bar{L}$ contributes to a defect. Thus with (76),

$$\bar{R} \text{ resp. } \bar{L} \text{ have at most } 17 \ R_i' \text{ resp. } L_i', \text{ and together at most } 19.$$

(77)
In order to bound the type 2 words $R'_i$ and $L'_i$, we have to look at $\ell k(O)$ and (61). As we said before, we calculate $2\ell k(O)$ by summing the writhes of the crossings in $\beta$ passed by $S$.

Let

$$\ell k(w) := \frac{1}{2} \sum_{S \text{ crossings in } w} \text{writhes of the } S.$$  

We notice that $\ell k$ is non-negative for every letter in $R$ and non-positive for every letter in $L^{-1}$. Then an easy estimate shows that

\[
\begin{align*}
2\ell k(R'_i) &\leq 3 & \text{if } R'_i \text{ is of type 1,} \\
2\ell k(R'_i) &\leq 5 \cdot d(R'_i) & \text{if } R'_i \text{ is of type 0,} \\
2|\ell k(w)| &\leq 1 + d(w) & \text{if } w \text{ is one of } \alpha, \beta, \\
2|\ell k(w)| &\leq 3 + d(w) & \text{if } w = \gamma.
\end{align*}
\]  

(Hint: see that the estimates are preserved under inner 2-extensions and outer extensions, and verify them for the words reduced under such extensions.) The first two estimates hold also with ‘2’ replaced by ‘−2’ and $R'_i$ by $L'_i$.

From the way we deduced property (77) we have at most 12 words $R'_i$ of type 1. Moreover, for $\alpha$ and $R'_i$ of type 0, we have an estimate of the syllable length against the defect. Thus if $\hat{R}$ is $R$ with type-2 $R'_i$ removed, then with (74) and using (78), we get

\[
\ell k(\hat{R}) \leq 5 \cdot 5 + 12 \cdot 3 + 1 = 62. 
\]  

(79)

The ‘1’ comes from the absolute term in the estimate for $\alpha$ in (78). With a similar calculation for $L'_i$ from (75),

\[
-2\ell k(\hat{L}) \leq 5 \cdot 4 + 12 \cdot 3 + 4 = 60,
\]

where 4 is contributed instead of 1 in (79), because we have $\beta$ and $\gamma$. Thus

$$\max(\ell k(\hat{R}), -\ell k(\hat{L})) \leq 31.$$  

Note that this quantity is not affected by the syllables $\sigma^2 \pm 3$ introduced by type 2 cancellation.

Next, we know that after type 2 cancellation only one of $R'$ and $L'$ contains type 2 words $R'_i$ or $L'_i$. Since for such words we have $\ell k = \pm 2$, this means that after type 2 cancellation, there are at most 15 type 2 words $R'_i$ or $L'_i$. Combining with (77) we find:

$$R' \text{ and } L' \text{ have at most 34 } R'_i \text{ and } L'_i \text{ altogether, i.e. } k + k' \leq 34.$$  

(80)

We still have to take care of the new syllables, extensions of $\sigma^2 \pm 3$, obtained by type 2 cancellation. Such syllables can occur only between $R'_i$, or at the start or end of $R'$, and similarly for $L'_i$ and $L'$. Thus there are at most $k + k' + 2 \leq 36$ such syllables. Then with (80) we obtain that the syllable lengths satisfy

$$s l(R') + s l(L') \leq 3 \cdot (k + k') + (k + k' + 2) \leq 3 \cdot 34 + 36 = 138.$$  

and with $s l(\alpha), s l(\beta), s l(\gamma) \leq 2$, we finally get

$$s l(\beta) = s l(\alpha) + s l(\beta) + s l(\gamma) + s l(R') + s l(L') \leq 144.$$  

This bound is of course rather rough. It is clear that only a fraction of the words with at most so many syllables are relevant, but it is also evident that still these are too many to be obtainable by a manual enumeration. We thus felt compelled to write a computer program in that realm, and the above bound became central in allocating data resources for the underlying implementation.

4.4 The list of patterns

4.4.1 The computer enumeration

We will now construct the list of patterns. A pattern is a family of 3-braid words (which potentially give under closure a partially invertible link), obtained from one fixed word $w$ by outer extensions. We often call for simplicity the minimal word $w$ itself the pattern.
In our algorithm for building the patterns we fixed first $S$ by its number at the start of $R$. All three choices must be tested, and the procedure below depends essentially on each choice.

We have now to modify the meaning of the presentation of (69) to account for the syllables $a_2^{t,3}$ introduced after type 2 cancellation.

To this vein, we introduce a new generator

$$[4] = a_4 = a_2^{-3}, \quad [\quad -4] = a_{-4} = a_4^{-1} = a_2^{3},$$

and regard $R$ and $L$ as words with indices now cyclically non-decreasing in $1, 2, 3, 4$. The new $\pm 4$-syllables

(S1) should occur only if $S$ is the first strand, and

(S2) their total exponents must add up to $0$.

Thus now $R'_1$ and $L'_1$ in (69) should be regarded as extensions of $[1, 2, 3, 4]$ resp. $[-4, -3, -2, -1]$, and the extra subwords gain the following meaning:

- $\alpha$ is an extension of $[\quad , \quad , \quad ]$, or $[1, 2, 3, 4]$,
- $\delta$ is an extension of $[\quad , \quad ]$ or $[4]$ (the reason why we introduced it in (69)),
- $\beta$ is an extension of $[\quad , \quad -2]$ or $[-2, -1]$,
- $\gamma$ is an extension of $[(-4, \quad -3]$ or $[(-4, \quad -3, -2]$.

Note that $\gamma$ cannot be empty or a single $-4$-syllable, because in this case $L^{-1}$ would end on $-1$ prior to type 2 cancellation. This contradicts the cyclical reducedness property under our assumption that $R$ starts with $1$.

To facilitate work, we will start building the patterns reduced under inner 2-extensions. We fix also that we always cancel the innermost type 2 words in $R$ with the outermost type 2 words in $L^{-1}$. Thus we can assume that

(U1) no 4-syllable occurs after a type 2 word $[1, 2, 3]$ in $R$,

(U2) no $-4$-syllable occurs before a type 2 word $[-3, -2, -1]$ in $L^{-1}$,

(U3) a 4-syllable occurs in $R$ if and only if a $-4$-syllable occurs in $L^{-1}$,

(U4) type 2 words do not occur in both $R$ and $L^{-1}$.

Property (U3) is what condition (S2) above blows down to, when we regard $\pm 4$-syllables as extendable. Property (U4) just means that we cancel the maximum of type 2 words. Properties (U1), (U2) and (U4) must in fact hold after inner 2-extension, and inner 2-extension of a type 2 word is no longer type 2 (a fact we had first overlooked). This can still be accounted for dynamically, while the words $R$ and $L$ are generated, by counting how many type 2 words must forcibly be subjected to inner 2-extension in order not to violate properties (U1), (U2) and (U4).

With this we have the following options for exponents:

- exponent of $\pm 1$, $\pm 2$ and $\pm 3$ should be $1$ if syllable is outer, and $1$ and $2$ if syllable is inner
- exponent of $\pm 4$ should be zero if syllable is inner, or preceded by type 2 word $[1, 2, 3]$ in $R$ resp. followed by type 2 word $[\quad , \quad -2, -1]$ in $L^{-1}$. The exponent should not be zero for the 4-syllable in $\delta$. (Thus $\delta$ can be non-empty only if $S = 1$.)

With this choice of exponent, we build the patterns by choosing first $\alpha, \beta, \gamma, \delta$, and then recursively expanding $R$ and $L$ until the defect becomes too large (note that each new $R'_i$ and $L'_i$ can only increase the defect). We take care of type 1 reduction and type 1 cancellation (and that we apply them on disjoint words), and that not both $R'$ and $L'$ contain type 2 words.

In order to obtain an inner 2-extension with (61) from some pattern $w$, we have to make sure that we can create at least

$$\begin{cases}
2\lvert \tilde{k}(w) \rvert & \text{if type 1 cancellation was not used} \\
\max(2\lvert \tilde{k}(w) \rvert - 2, 0) & \text{if type 1 cancellation was used}
\end{cases}$$

more defect. This is the minimal additional defect produced by the 2-extensions needed to establish (61). The reason for treating extra the type 1 cancellation was already mentioned: type 1 cancellation may become invalid after one inner 2-extension. Then the defect this 2-extension creates is compensated by the vanishing reduction on the bound of the right of (71). (As we report in subsection 4.4.2, we later removed, for verification purposes, part of the type 1 cancellation test. This still led to the same result, but after considerably longer calculation.)
Once we determined how much more defect can be created by inner 2-extensions (in fact, at most one single extension turned out possible), we created the words under this 2-extension which satisfy (61). Leaving all outer syllables extendable, we obtain then a list of patterns, which potentially satisfy (71). (We say ‘potentially’, because type 1 cancellation is not a thorough test for the genus non-minimality of syllables extendable, we obtain then a list of patterns, which potentially satisfy (71). (We say ‘potentially’, because extension turned out possible), we created the words under this 2-extension which satisfy (61). Leaving all outer extensions. We will stipulate that an extendable syllable

4.4.2 The skein polynomial test

The list of 398 patterns was processed using the \( P \) polynomial. Let \( X \) be such a pattern, with \( n \) outer extendable syllables (all of them of exponent 1). Then we form \( 2^n \) words \( X' \) obtained from extending or not (to exponent 2) each of these \( n \) syllables. They then give rise to a pair of diagrams \( (M(X'), \tilde{M}(X')) \), the first being the closure of \( X' \), and the second one obtained after reversing \( O \). From the work in [27, 28] we know that all links \( M = \tilde{\beta} \) given from braids \( \beta \) of the form (B) in (13) satisfy

\[
1 - \chi(M) = \max \deg_m P(M), \quad \text{and} \quad \hat{a}_0(P(M)) \text{ is a single monomial, with coefficient } \pm 1. \tag{62}
\]

(Here \( \hat{a}_0 \) is the leading coefficient, as in Definition 2.1.) The degree can be determined for \( M(X) \) directly from counting the bands in the representation (69). (We checked this also by explicit calculation of \( P \), which was much more tedious.) We calculate then \( P(\tilde{M}) \) for all \( 2^n \) links \( \tilde{M}(X') \). The pattern \( X \) can be discarded if

- \( \max \deg_m P(\tilde{M}(X')) < \max \deg_m P(M(X')) \) for all \( 2^n \) pairs \( (M, \tilde{M}) \), or
- for all \( 2^n \) pairs \( (M, \tilde{M}) \), we have \( \max \deg_m P(\tilde{M}(X')) = \max \deg_m P(M(X')) \) and \( \hat{a}_0(P(\tilde{M}(X'))) \neq \pm l_p \).

Once one of these conditions is true for the \( 2^n \) pairs \( (M(X'), \tilde{M}(X')) \), it is true for the pairs coming from arbitrary outer extensions \( \tilde{X} \) of the pattern \( X \). This is because by the skein relation (7), the coefficient \( [P(\tilde{M}(\tilde{X}))]_{\chi(M(X))} \) for an extension \( \tilde{X} \) of \( X \) is inherited, up to units in \( \mathbb{Z}[l^{\pm 1}] \), from one of the \( 2^n \) words \( X' \) checked. This word \( X' \) is obtained from \( \tilde{X} \), when precisely those syllables in \( X \) are extended (to exponent 2) in \( X' \) which are extended (to arbitrary exponent at least 2) in \( \tilde{X} \). In practice, when an outer \( \pm 4 \)-syllable had to be extended to build \( X' \), we just extended \( \sigma_2^{\pm 3} \) to \( \sigma_2^{\pm 4} \). (Again, one can argue with the skein relation (7) that this is enough.)

In the realm of ascertaining the outcome, we did later a calculation in which we relaxed the test for genus non-minimality from \( R' \) and \( L' \) of type 1. With this weaker test we had instead of the previous 398 patterns a longer list of 1123 patterns. On the reduced list the \( P \) check needed 2 minutes, whereas on the longer list it took about 2 days, leaving over the same 124 patterns. The disproportional slowdown is owed to the considerable length of the extra entries, and to the increased number of syllables with outer extensions, which augments exponentially the number of diagrams to treat for each pattern. This is another example how some theoretical insight saves a great amount of hard computation.

On the remaining 124 patterns, calculation showed that \( P(M) = P(\tilde{M}) \) for all extensions, which suggested to seek a transformation into the words giving rise to Figure 1.

4.4.3 Final reduction and verification

We will write below \( *a_i = *i \) for an extendable (outer) \( i \)-syllable.

To deal with the list of 124 patterns, from now on it is useful to make a more general ansatz of how to treat extensions. We will stipulate that an extendable syllable \( *a_i \) should be treated as \( \sigma_2^k \) not only for positive, but for arbitrary (incl. zero) integer exponents \( k \). It is enough to show that these more general patterns still simplify to those in Figure 1.

With the new understanding of asterisked entries, we can obtain a form in Artin’s generators by replacing \( * - i \) by \( *i, *3 \) and \( * - 3 \) by \([-1, *2, 1]\), and we also replace (making more general) \( *4 \) and \( * - 4 \) by \( *2 \).
In this way the extendable Artin generator $*\sigma_i$ behaves very similarly to the element $\tau_i$ of the singular 3-braid monoid (see [3] or [7, Proposition 2.1]), but satisfies the additional gobbling/yielding relations

$$\tau_i^2 = \tau_i \sigma_i^{\pm 1} = \sigma_i^{\mp 1} \tau_i = \tau_i.$$  \hfill (81)

With these relations, the only extra relations to those in $B_3$ of subsection 2.3 are the singular YB relations:

$$\tau_i \sigma_{i-1}^{\pm 1} \sigma_i^{\pm 1} = \sigma_{i-1}^{\mp 1} \sigma_i^{\pm 1} \tau_{i-1}, \quad \tau_i \sigma_{i+1}^{\pm 1} \sigma_i^{\pm 1} = \sigma_{i+1}^{\mp 1} \sigma_i^{\pm 1} \tau_{i+1},$$

where in either formulas the $\pm$ are to be taken equal, and $i$ should be chosen so that $\sigma_{i-1}$ resp. $\sigma_{i+1}$ to make sense.

We use now the singular monoid and gobbling/yielding relations to simplify the list of 124 patterns.

First, if we have patterns $X_1$ and $X_2$ such that $X_2$ is obtained from $X_1$ by omitting some $* \pm 4$ (or $* \pm 2$) syllables, then we can discard $X_2$.

Moreover, we simplified all remaining patterns according to the below rules. (The circumflex at the start and dot at the end indicate that the whole pattern should start/finish there. The ellipsis on either hand side should stand for the same subword.)

* $2, * - 4 \rightarrow 4$

$\sim * 1 \cdots * - 4 - 3. \rightarrow \cdots * - 4 - 3$

* $3 - 1 - 3 - 2 * - 4 \rightarrow - 1 - 3 - 2 - 1 * - 4$

$\sim * 4 \cdots * - 2. \rightarrow * 4 \cdots$

* $4, 1, 2, 3, 1, 2, * - 1 \rightarrow * 4, 1, 2, 3, 1, 2$

* $4, 1, 2, * - 1 \rightarrow 4, 1, 2$

* $2, - 3, - 2, - 1, * - 4 \rightarrow - 3, - 2, - 1, * - 4$

* $4, * - 2 \rightarrow 4$

(The words on either hand-side are equivalent up to the replacements mentioned above.) These rules eliminate some redundant extendable syllables. Then we again discarded the duplicates. These steps shrunk the list of 124 to 54, which is shown in Table 1.

In a final effort, we decided to deal with these 54 patterns by hand. (This took slightly more than half a working day.) We tried to simplify the words using the relations described, which then led to words conforming to Figure 1. (Many of them yield only certain special cases; e.g. # 2 gives the Whitehead link, and # 14 the Borromean rings.) These reductions were rather straightforward. We give just two examples below, which lead to the general form of either pictures in Figure 1, and some hints on the ways we performed reduction. One particular way of simplifying words, which is not immediate from the relations, is when $[1, 2]$ or $[2, 1]$ occurs as a subword, and so does $[1, 2]$ or $[2, 1]$. (One of the four letters can also have an asterisk.) Then one can introduce an extra $\sigma_i^{\pm 1} \sigma_i^{\mp 1}$ to make one of these words (without an asterisk) into the half twist element $\Delta = [1, 2, 1] = [2, 1, 2]$ (from (14)) or its inverse, cyclically slide this element close to the other word, at the cost of interchanging indices 1 and 2 along the way, and then produce two cancellations (or one cancellation and one gobbling).

Another way is to still use type 2 cancellation when only the first syllable in $[1, 2, 3]$ or last syllable in $[\sim 3, - 2, - 1]$ is extended.

Note that when we simplify at start and end of the word, this sometimes accounts for a conjugation, which changes the number of to-reverse strand $S$.

# 36:

1$^2 23 * 4 - 1^2 * - 4 - 3 - 2$ \hfill ($S = 2$) \hfill $\rightarrow$ \hfill $112 - 121 * 2 - 1 - 1 * 2 - 1 - 21 - 2$ \hfill ($S = 2$) \hfill $\rightarrow$

$11221 * 2 - 1 - 2 - 2 \rightarrow 112 * 12 - 1 * 2 - 1 - 2 - 2 \rightarrow$

$1 - 2\Delta + 12 - 1 * 2 - 1 - 2 - 2 \rightarrow 1 - 2 * 21 - 2 * 11 - 2 \rightarrow$

$1 * 21 - 2 * 1 - 2$ \hfill (right diagram in Figure 1)
Table 1. The list of 54 patterns that give words of Figure 1.

| Pattern | S |
|---------|---|
| 1: 1,2,-3^3. | S=1 |
| 2: 1^3,-3,-2. | S=2 |
| 3: *1,2^2,-3^2. | S=3 |
| 4: 1,-2,*1,-3. | S=1 |
| 5: 1,*2,-1,*3. | S=2 |
| 6: *1,2,*3,-2. | S=3 |
| 7: 1,2,-3^3,*2. | S=1 |
| 8: *1,2,*3,-1,-3. | S=3 |
| 9: *1,2^3,-1,-3. | S=3 |
| 10: 1,2,-1,*3. | S=2 |
| 11: 1^2,-3,-2^2. | S=2 |
| 12: 1^2,*3,-2^2. | S=2 |
| 13: 1^3,*2,-3,-2. | S=2 |
| 14: 1^2,2^2,-3^2,-2. | S=2 |
| 15: *1,2^2,*1,-3^2. | S=3 |
| 16: *1,2^2,-1,-3,-2. | S=3 |
| 17: 1,-2,*1,-3,*2. | S=1 |
| 18: *1,2,*3,1,-2. | S=3 |
| 19: *1,2,*3,1,*2,-3. | S=3 |
| 20: *1,2,*3,1,*2,-3. | S=3 |
| 21: 1,*3,-2,*1,-3. | S=1 |
| 22: *1,2,3,*4,1,-2^2,-1,*-4,-3^2. | S=1 |
| 23: 1^2,2^2,-1,*3,-2. | S=2 |
| 24: 1^2,2^2,*3,-1,-3,-2. | S=2 |
| 25: *1,2^2,3^2,-2,-1,-3,-2. | S=3 |
| 26: *1,2^2,3^2,*1,-2,-1,-3,-2. | S=3 |
| 27: 2^2,3,*4,-1^2,*-4,-3. | S=3 |

This completes the proof of Theorem 1.2.

We have displayed Table 1 also to illustrate how many different representations (13) the braids in Figure 1 can have. Still the simplicity of the figure does not suggest that such a lengthy (and partly computerized) case treatment is a natural approach. For better or for worse, we have yet nothing else to offer.

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