A duality-invariant Einstein-Planck relation and its consequences on micro black holes

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Key words: black holes thermodynamics, duality symmetry, Einstein-Planck relation.
PACS numbers: 95.85.Pw, 04.60.m, 98.80.Cq, 04.70.Dy

Abstract

We discuss the consequences of a duality-invariant Einstein-Planck relation on the equation of state of micro black holes. The results are analogous to those obtained from the "world crystal" model, but with some significative differences, as for instance a limiting vanishing value for temperature for very small black holes. The model leads to a total evaporation of micro black holes but with the final stage being very slow.

1 Introduction

One of the problems arising in the search for unification of gravitational and quantum physics are the ultraviolet divergences when going to very small length scales. A possible solution is going to superstring theories and \(D\)-branes, taking into consideration basic extended objects rather than points, which evolve in a space with additional dimensions (see, for instance, [1]–[3]). But there are also several speculative proposals opening some other possibilities which seem worth of exploration. One of them is working on discrete lattices with a short-scale cutoff, thus avoiding the divergence associated to vanishing small scales. Recently, a "world crystal" model, based on the idea that a discrete space could mimic the actual reality of space instead of being a mere mathematical artifact for calculations, has been proposed in [4, 5], with the lattice spacing of the order of the Planck length \(l_P\), namely:

\[
E = \frac{2c\hbar}{al_P} \sin \left( \frac{\pi al_P}{2\lambda} \right)
\]  

(1.1)

with \(E\) the photon energy, \(a\) a numerical constant, \(c\) the speed of light in vacuo, \(\lambda\) the photon wavelength, and \(\hbar\) the reduced Planck constant. Here, the minimum wavelength will be \(\lambda_{\text{min}} = al_P\).
Another proposal has been a duality-invariant generalization of the Einstein-Planck relation (DIEP) \[6\], of the form

\[ E = \frac{hc}{l_P' \lambda} + \frac{l_P'}{\lambda}, \] (1.2)

with \( \lambda \) the wavelength and \( l_P' = \sqrt{2a\pi}l_P \), \( l_P \) the Planck length \( l_P = (\frac{\hbar G}{c^3})^{1/2} \), and \( a \) being a numerical constant. We have denoted with \( \sqrt{2a\pi} \) the proportionality constant between \( l_P' \) and \( l_P \), for sake of comparison of results of this paper with those obtained in \[5\].

Since the smallest spatial scales probed up to now are of the order of \( 10^{-20} \text{m} \), and the Planck length is of the order of \( 10^{-35} \text{m} \), there is still a wide range of possibilities for the value of \( a \), ranging from the order 1 to, let us say, \( 10^8 \). Higher values seem indirectly excluded by the results of the search of a wavelength dependence of the speed of light in highly energetic cosmic phenomena \[7\]–\[9\].

Expression (1.2) for \( E \) is invariant under the change \( \lambda/l_P' \) to \( l_P'/\lambda \), in some analogy with the T-duality in superstring theories \[1\]–\[3\], but applied to the actual space instead of to the additional compact dimensions.

The aim of this paper is to explore some consequences of this duality-invariant expressions on micro black holes properties, mainly on the equation of state relating mass and temperature, and its consequences on the evaporation rate of such black holes. Our work is analogous to the recent exploration of the consequences of the "world-crystal" model on micro black holes \[5\]. Both proposals have in common: a) that the speed of light in vacuo becomes smaller than \( c \) for short wavelengths; b) that the generalized uncertainty principle associated with them becomes less uncertain for higher energies, in contrast with standard proposals, where gravitational effects increase the uncertainty. Indeed, both formulations lead to an uncertainty relation of the form

\[ \Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 - \frac{(l_P')^2p^2}{2\hbar} \right]. \] (1.3)

where \( \Delta x \) e \( \Delta p \) are the uncertainties in position and momentum. We have written (1.3) as in \[5\] rather than the very similar, but not identical, expression appearing in \[6\].

Significative differences between the two mentioned formalisms are: a) that the duality-invariant Einstein-Planck (DIEP) proposal has no lower cut-off at small scales whereas in the world crystal model length scales lower than \( l_P \) are assumed not to exist; b) that (1.3) is only a second-order approximation in the DIEP model, whereas it stems in a natural way in the world crystal model. Thus, it is logical to compare the similarities and differences of these models, in order to gain a deeper understanding of their possibilities and consequences.

2 Equation of state for black holes

The equation of state for black holes relates temperature to energy (i.e., mass). Jizba, Kleinert and Scardigli have shown in \[5, 10\] that the implication of (1.3) for micro black
holes is found in the relation between mass and temperature, for which they obtain

$$2m = \frac{1}{2\theta} - \frac{a^2}{4\theta}, \quad (2.1)$$

$m$ and $\theta$ being respectively $m = M/M_P$ and $\theta = T/T_P$, with $M_P$ and $T_P$ the Planck mass ($M_P = (1/2)(\hbar c/G)^{1/2}$, of the order of $10^{19}$ GeV) and Planck temperature, respectively (the latter being defined as $E_P = M_P c^2 = (1/2)k_B T_P$). Note that, in contrast, string-theory corrections [11]–[14] lead to (2.1) but with a + sign in the second term of the right hand. For $a \to 0$ (continuum limit) (2.1) tends to the well-known Hawking relation between mass $M$ and temperature $T_H$ of a black hole [1]–[5]:

$$m = \frac{1}{4\pi \theta} \quad \text{or} \quad T_H = \frac{\hbar c^3}{8\pi G k_B M}. \quad (2.2)$$

Equation (2.1) is, in fact, the thermodynamic equation of state expressing black hole temperature as a function of its energy, since $m$ is essentially the energy of the black hole (but expressed in a dimensionless way). Equation (2.1) has direct consequences on the evaporation process of black holes. In Hawking’s theory, the black hole is totally evaporated at finite time, in an explosive process, because the lower the mass the higher the temperature, and therefore, the radiation rate, which is assumed to be prescribed by Stefan-Boltzmann’s law, i.e. proportional to the area times the fourth power of temperature. In contrast, (2.1) implies a total evaporation ($m = 0$) but with a maximum final temperature given by $\theta_{max} = \sqrt{2}/a$ instead of the divergent temperature of Hawking’s theory. Note that, in string theories, with a + sign in the second term of the right-hand side of equation (2.1), the black hole is never completely evaporated, leading to a finite minimum rest mass $m_{min} = a/(2\pi\sqrt{2})$ at a temperature $\theta_{max} = \sqrt{2}/a$.

Here we show that the DIEP proposal leads to some differences with respect to the results obtained in [5]. These differences arise in the region of very small masses, and do not drastically modify the basic conclusions of [5], but lead to a final vanishing temperature instead than the finite temperature obtained in [5]. The reason for such discrepancy is that, in DIEP model, expression (1.3) is a second-order approximation to a more general result, instead of being a direct result of the theory, as it is in [5].

We summarize the arguments used in [5], but adapted to the DIEP proposal [6]. It is known that the smallest resolvable detail $x$ of an object is of the order of the wavelength of the used electrons (for the sake of a direct comparison with [5] we will take the same value for $x$ as in [5], namely $x = \lambda/4\pi$). In the DIEP proposal we have (1.2) [6] and therefore we obtain for the range of the smallest resolvable details $x$:

$$x \left[ 1 + \left( \frac{l_P}{4\pi x} \right)^2 \right] = \frac{\hbar c}{2E}. \quad (2.3)$$

Now, following the general lines of arguments of [15]–[20], let us consider an ensemble of photons just outside the event horizon, and take into account that their position uncertainty is of the order of the Schwarzschild radius $R_S (= 2GM/c^2)$ of the black hole, which may be expressed as $R_S = l_p m$. Thus the $x$ in (2.3) is taken as $x = \mu R_S = 2\mu l_pm$, with $\mu$ a numerical constant which will be obtained below. Next, we assume that the energy $E$ in (2.3) is the average energy of photons, linked to temperature $T$ as
\( E = k_B T \). For sufficiently long length scales, such that \((l_p')^2/(4\pi x)^2\) may be neglected, (2.3) leads to
\[
4\mu l_pm = \frac{hc}{k_B T}.
\]
By comparing (2.4) with the standard semiclassical Hawking result (2.2) it is seen that \(\mu = \pi\). By introducing this value in (2.3) it is obtained
\[
2\pi m + \frac{a^2}{16\pi m} = \frac{hc}{2l_p k_B T} = \frac{1}{2\theta}.
\]
For \(a = 0\), Hawking relation (2.2) between mass and temperature is recovered, and for \(a \neq 0\) (2.5) relates \(m\) and \(T\) in a more general way, also valid for very small masses. Incidentally, note that this may also be written in a more symmetrical dual-invariant form, somewhat reminiscent of (1.2), as
\[
\frac{a\theta}{\sqrt{2}} = \frac{1}{8\pi R_S} \frac{l_p}{l_p} \frac{l_p}{s^2 R_S} + \frac{l_p}{s^2 R_S} \frac{l_p}{s^2 R_S}
\]
with \(R_S = 2GM/c^2\) the Schwarzschild radius of the black hole.

To second order in \(a\theta\) we recover (2.1) from (2.5). Recall that \(\theta = T/T_P\), and \(T_P\) is very high (of the order of 10^{32} K). Thus, \(\theta\) being small does not mean that \(T\) is small, but simply that it is smaller enough than \(T_P\). Instead, the full expression (2.5) leads to the result that for the final evaporation stage \(m \to 0\), \(\theta\) does not tend to \(\theta_{max} = \sqrt{2}/a\), but to \(\theta = 0\).

From (2.5) the heat capacity \(C(T) = dU/dT = c^2 dM/dT\) may directly be found. In dimensionless terms we have
\[
C(\theta) = \frac{dm}{d\theta} = -\frac{1}{4\pi \theta^2} \frac{1}{1 - \frac{a^2}{32\pi^2 \theta^2}}.
\]
For high values of \(m\), for which \(\theta \sim m^{-1}\), this is the usual result, and its value is negative — as it is common in gravitational systems —, thus indicating that as the black hole radiates energy it becomes hotter instead of colder. Expression (2.8) indicates also that \(C\) becomes infinite and changes sign for \(m = a/(4\sqrt{2}\pi)\). This does not mean that the black hole does no longer evaporate, but that \(\theta\) as a function of \(m\) reaches a maximum at this value of \(m\). For low enough value of \(m\), the heat capacity becomes positive, because \(\theta \sim m\). After the change of sign of \(C(\theta)\), turning from negative to positive value, \(\theta\) becomes lower as the black hole evaporates, and evaporation becomes slower. This is a difference with [5], which from (2.1) obtains an always negative specific heat, namely, \(C = -\frac{1}{8\pi \theta^2} \left[1 + \frac{1}{2}a^2 \theta^2\right]\). This is also in contrast with string GUT theories which yield \(C = -\frac{1}{5\pi \theta^2} \left[1 - \frac{1}{2}a^2 \theta^2\right]\) which has \(C = 0\) for \(\theta = \sqrt{2}/a\) [19].

In Figure 1 we show the relation between \(m\) and \(\theta\) according to Hawking formula (2.2), to (2.1) (the result of [5]) and to (2.5) (the result of this paper).
3 Entropy

In order to clarify in a more direct way why the heat capacity turns from negative at high $m$ to positive at low $m$ one may consider the entropy corresponding to (2.5).

Since $m$ is related to the energy $u = U/U_P$ of the black hole as $u = m$, we may obtain from (2.5) the entropy, taking into account that $1/T = \partial S/\partial U$. Therefore, we will have:

$$T_P \frac{\partial S}{\partial U} = \frac{1}{\theta} = 4\pi m + \frac{a^2}{8\pi m}, \quad (3.1)$$

which by integration becomes

$$\frac{2S(m)}{k_B} = 2\pi m^2 + \frac{a^2}{8\pi} \ln m + \text{const} \quad (3.2)$$

In term of the mass we have:

$$\frac{S(M)}{k_B} = \pi \frac{M^2}{M^2_P} + \frac{a^2}{32\pi} \ln \left( \frac{M^2}{M^2_P} \right) + \text{const} \quad (3.3)$$

This may also be written, in terms of Schwarzschild radius $R_S = 2GM/c^2$ and $l_P = 2GM_P/c^2$, in such a way that (3.3) reduces to:

$$S(M) = k_B \left[ \frac{A}{4l_P^2} + \frac{a^2}{32\pi} \ln \left( \frac{A}{4\pi l_P^2} \right) \right] + \text{const} \quad (3.4)$$

where $A = 4\pi R_S^2$. The first term is the usual Bekenstein-Hawking entropy, whereas the second term is new; here $a = l'_P/l_P$. When $a = 0$ equation (1.2) reduces to the usual Einstein-Planck relation and (3.4) reduces to the usual Bekenstein-Hawking entropy. Indeed, in Bekenstein-Hawking entropy one has $S_{BH} = k_B A/4l_P^2$, and $l_P$ is a fundamental
quantity— the lowest spatial scale. However, in (1.2) there is not a lowest spatial scale, and \( l_P \) is no longer an unequivocal reference length.

The transition from negative to positive heat capacity corresponds to the transition from entropy proportional to \( A \) to entropy proportional to \( \ln(A) \).

Entropy (3.4) may be compared to entropy for black holes in loop quantum gravity, which is, for high areas, \( S(A) = \gamma_0 \frac{A}{\gamma 4l_P^2} - \frac{1}{2} \ln \frac{A}{l_P^2} + \text{const.} \),

where \( \gamma \) is the so-called Barbero-Immirzi parameter. For low area regime \( \gamma \) it was shown a discretization of entropy as function of area for microscopic black holes.

4 Evaporation of black holes

The difference of (2.5) with the Hawking model is radical, as in Hawking model the final temperature diverges and in (2.5) is zero. The difference of (2.5) with [5] is not so decisive, as in both cases (namely (2.1) and (2.5)) the final temperature is finite and relatively smaller than Planck temperature \( T_P \). Anyway, the difference is conceptually interesting and worth to mention, as it becomes relevant in the final stages of the black hole evaporation.

Indeed, the rate of evaporation of black holes is one of the main consequences of the equation of state. Usually, Stefan-Boltzmann law for radiation is considered to describe such evaporation. Namely, assuming

\[
\frac{dU}{dt} = -4\pi R_S^2 \sigma T^4, \tag{4.1}
\]

with \( R_S \) the Schwarzschild radius of the event horizon of the black hole, i.e. \( R_S = 2GM/c^2 \), \( \sigma \) being Stefan-Boltzmann constant \( (\sigma = \pi^2 k_B^4/(60\hbar^3 c^2)) \) and \( U = Mc^2 \), it is seen that the equation of state relating \( \theta \) to \( m \) plays a role in the evaporation process. In particular, using (2.5) for the relation between \( \theta \) and \( m \), we have, in dimensionless form:

\[
\frac{dm}{dt'} = -\frac{1}{m^2} \left[ 1 + \frac{a^2}{32\pi^2 m^2} \right] \theta^4, \tag{4.2}
\]

with \( t' \) a dimensionless time given by \( t' = \frac{t}{t_P(15\pi^2/2)} \), with \( t_P \) the Planck time, \( t_P = l_P/c \).

Alternatively, the evaporation process may be studied in terms of temperature, rather than of mass, by using the heat capacity \( C(T) \), namely

\[
C(T) \frac{dT}{dt} = -4\pi R_S^2 \sigma T^4. \tag{4.3}
\]

In the world-crystal formalism this takes the form

\[
\frac{d\theta}{dt''} = -\left[ \frac{1}{\theta} - \frac{a^2}{2} \theta \right] \theta^4 = -\left[ \frac{1 - a^2 \theta^2}{1 + a^2 \theta^2} \right] \theta^4. \tag{4.4}
\]

with \( t'' \) a dimensionless time given by \( t'' = (8\pi/15)(t/t_P) \). In this model, \( d\theta/dt'' = 0 \) for \( \theta = \sqrt{2}/a \), which, according to (2.1) corresponds to \( m = 0 \), i.e. to the total evaporation of the black hole.
5 Conclusions

In summary, both the crystal-world model and the duality-invariant relation lead, through the respective equations of state (2.1) and (2.5) for black holes, to significantly different behaviour for micro black holes than those following from Hawking theory. The main difference between (2.1) and (2.5) is in the value of the final temperature. Both thesis lead to a total evaporation of the black hole but in the crystal-like model the final stages are characterized by a finite non-vanishing temperature whereas in the duality-invariant model the temperature tends to zero, as well as the mass because of the change of the sign of specific heat (from negative to positive value) for low enough masses. This difference of temperature is especially relevant for the final rate of evaporation since in (2.5) it approaches zero and this means that the final evaporation rate of micro black holes will be very slow.

Thus, whereas in Hawking’s theory (namely (4.2) with $a = 0$) the decay becomes faster and faster for smaller masses and becomes explosive, in (4.2) the final stage of evaporation becomes very slow. Maybe this is the reason that big explosions of primordial small black holes have not been observed in spite of much research.

This may be of interest for the black holes which could be produced in particle accelerators, as in the Large Hadron Collider at CERN, which would have energies of the order of 10Tev, which correspond to $m \sim 10^{-15}$. Note that, for this value of $m$, the difference of the evaporation rate obtained from Hawking theory and (4.2) for $a = 0$ and with $a \neq 0$ (of order of 1) differs in some 90 orders of magnitude.

Acknowledgements

The authors acknowledge the support of the Università di Palermo (Fondi 60% 2012-ATE-0106 and Progetto CoRI 2012, Azione d) and the collaboration agreement between Università di Palermo and Università Autònoma de Barcelona. DJ acknowledges the financial support from the Dirección General de Investigación of the Spanish Ministry of Education under grant FIS2009-13370-C02-01 and of the Direcció General de Recerca of the Generalitat of Catalonia, under grant 2009 SGR-00164. M.S. acknowledges the hospitality of the ”Group of Fisica Estadistica of the Università Autònoma de Barcelona”.

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