Finite Size Effects and Scaling in Lattice $CP^{N-1}$

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Abstract

We present model predictions for the spectrum of $CP^{N-1}$ in a periodic box and use them to interpret the strong finite size effects observed in lattice simulations at medium values of $N$. The asymptotic scaling behaviour of alternative lattice actions is discussed along with some aspects of multigrid algorithm efficiency.
Introduction There has been considerable recent interest in simulations of the lattice $CP^{N-1}$ model \[1, 2, 3, 4, 5\]. Advances in the development of non-local Monte Carlo algorithms have given added impetus to studies of the model at large $N$ where the spectrum remains unconfirmed by analytic or numerical analysis. Conflicting evidence has been presented for perturbative (two-loop) scaling at currently accessible values of the bare coupling.

In this letter we analyse some key factors involved in such studies: the dependence on lattice action, the efficiency of the algorithm and, most crucially, contamination of finite size effects which are particularly significant in the present model at medium and large $N$. We demonstrate the latter effect by means of a simplified analytic model and present some new data.

The simplest lattice action for the model containing an explicit gauge field is

\[ S = \beta / 2 \sum_{\langle xy \rangle} [z_i^* (x) U(x, y) z_i^* (y) + h.c.] \]  

where $\beta = 2N/g^2$ and $z(x)$ is a complex $N$-vector with unit norm. $U(x, y)$ is a gauge connection between nearest neighbour sites $x, y$ and is an element of $U(1)$. This action, which was used in the studies \[2, 5\], corresponds in the classical continuum limit to the Lagrangian

\[ L = 1 / g^2 (D_\mu z)^\dagger \cdot (D_\mu z) \]  

where the gauge field appearing in the covariant derivative may be related to the 'matter' field $z(x)$ via its equation of motion. For computational convenience, the lattice gauge variable can be integrated out to obtain an effective 2D spin model

\[ Z = \int [dz][dz^*] \exp \{ -\beta \tilde{S}(z) \} \]  

whose action is \[2\]

\[ \beta \tilde{S} = - \sum_{\langle xy \rangle} \ln I_0 (\beta |z^*(x) \cdot z(y)|) \]  

The alternative action

\[ S_Q = -2\kappa \sum_{\langle xy \rangle} |z^*(x) \cdot z(y)|^2 \]  

can be shown to have the same classical continuum limit as \([1]\) with the identification of bare couplings $\kappa = N/2g^2$. This latter action was studied by \([1, 3, 4]\).

At large $N$, an effective Lagrangian may be derived \([3, 4]\) which corresponds to $N$ identical charged scalar fields subject to a (confining) $U(1)$ gauge potential

\[ V(r) = g_f^2 |r| \]  

where the effective coupling is related to the mass of the charged scalar $m_f$

\[ g_f = m_f \sqrt{\frac{12\pi}{N}}. \]  

In the large $N$ limit, the low-lying spectrum can be estimated in a non-relativistic approximation via a Schrödinger equation for the two-body relative motion in the potential $V(r)$. \([4]\). For medium $N$ (4 . . . 8), a previous lattice analysis showed that the predictions of this simple model are at best unreliable \([2]\). The lowest state observed was indeed the adjoint, but there was little sign of the next predicted state, an $SU(N)$ singlet. It was concluded that either the lattice singlet operators used were inefficient or that the breakdown of the non-relativistic model approximation at these $N$ values was to blame.
Model for finite size effects  All analyses of $CP^{N-1}$ have noted strong finite-size effects. Detailed knowledge of the $N$-dependent potential allows one to estimate the size of lattice required to be free of such effects [2]. Most analyses have required $L/\xi \geq 6$ for this reason. In fact, one can detect the effects of working in a periodic box even above this limit especially at large values of $N$ where the flat potential leads to a weakly confined system and hence broad wave functions [2, 5].

It is interesting to use the (albeit imperfect) non-relativistic model described in the first section to investigate these effects more quantitatively. The potential implied by eqns (6, 7) when made periodic with period $L$ gives rise to a band rather than discrete spectrum (see any Quantum Mechanics text for the treatment of the Kronig-Penny potential [11]). The discrete eigenvalues determined by the zeroes of Airy functions [8] give way to allowed bands whose limits are determined by combinations of Airy functions

$$\Delta \tilde{\Delta} \cos(\kappa L) = (B' A + A' B)(\tilde{B}' \tilde{A} + \tilde{A}' \tilde{B}) - 2AA'\tilde{B}\tilde{B}' - 2BB'\tilde{A}\tilde{A}' . \quad (8)$$

Here, $\kappa$ is some real parameter and $L$ is the period of the potential: the spatial extent of the lattice in our application. Also

$$\Delta = B' A - A' B, \quad \tilde{\Delta} = \tilde{B}' \tilde{A} - \tilde{A}' \tilde{B} \quad (9)$$

where $A \equiv Ai(-b)$ and $B \equiv Bi(-b)$ are the standard Airy functions, with $A'$, $B'$ the corresponding derivatives. Similarly $\tilde{A} \equiv Ai(aL/2 - b)$, and so on, where the parameters $a$, $b$ are related to the potential slope $2V_0/L$, the required band energy $E$ and the reduced mass of the two-particle system $\mu = m_f/2$:

$$a = \left(\frac{4\mu V_0}{L}\right)^{1/3}, \quad b/a = \frac{EL}{2V_0} . \quad (10)$$

In the present application, $V_0 = g_f^2 L/2$. All scales in the model are set by the confined, and so unobservable, fundamental mass $m_f$. To compare predictions with data (see below) the relationship to the physical bound state mass must be used.

In Fig 1, is shown the band spectra for $N = 4$ and 8 in units of $m_f$ as a function of $1/L$. These have been checked against the results of numerical solutions of the corresponding Sturm-Liouville problem. Using the numerical approach we have also been able to replace the model potential (6) by that measured in lattice simulations [2]. The qualitative results remained similar to those seen in Fig 1. The spectrum at $1/L = 0$ agrees with that of Haber et al. [8], i.e. a low-lying positive parity adjoint state followed by well separated negative parity and further excited states. As the size of the periodic box decreases in physical units, a band structure develops implying that zero momentum correlators in lattice simulations will no longer be dominated by a single isolated state. At increasing Euclidean time separation, the contribution from the lower levels within the lowest band will dominate so that the effective mass $m_{eff}(t)$ defined by

$$C(t) = \text{const.} \times [\exp(-m_{eff}t) + \exp(-m_{eff}(L - t))] \quad (11)$$

shows a drop-off in $t$ rather than the plateau characteristic of an isolated state. One may construct simple models for the overlap of the band states in Fig 1 with the measurement operator employed in $C(t)$ which demonstrate the transition from $m_{eff}(t) \approx m_0$ at moderate Euclidean time separations to $m_{eff}(t) \approx m_0 - \delta/2$ at large separations. Here, $m_0$ is the centre of the energy band for some lattice size and $\delta$ its width and all units set by $m_f = 1$. 
Since the non-relativistic model is known to be quantitatively unreliable [2] it is difficult to make other than qualitative statements on the implications of this effect. As an example, for \( N = 4 \) and \( \kappa = 2.9 \) (action (3)), \( \xi/a = 18.5 \). According to the non-relativistic treatment [3] for \( N = 4 \)

\[
m = 1/\xi = (2 + 4.54)m_f
\]  

(12)

Thus, a lattice of size \( L/a = 128 \), say, corresponds to \( L = 1.06m_f^{-1} \). According to Fig 1, this implies a very large broadening effect indeed. Of course, we know that the system is actually relativistic since eq. (12) implies \( v/c \approx 0.95 \) so model predictions are unreliable. On the other hand, the string tension \( \sigma \) implied by the large \( N \) model (\( \sigma/m^2 = 3\pi/6.54^2 = 0.22 \)) is comparable with that measured (0.34(2)) [2] when expressed as a dimensionless ratio.

Hasenbusch and Meyer [3] have presented correlator data at very large time separation, which the authors claim is essential to obtain the true mass-gap in the presence of other states. While this is in general true, the above model demonstrates that for systems bound loosely by a shallow potential such as large \( N \) \( CP^{N-1} \) the effect of periodic boundaries could be such that large time separations isolate the wrong state i.e. not that which is physically relevant in the bulk, continuum limit. In the absence of detailed published data for \( m_{\text{eff}}(t) \) we have investigated this point by studying \( CP^3 \) and \( CP^7 \) at weak coupling on small \((64^2)\) and large \((128^2)\) lattices. Fig 2 shows typical results. There is some small but significant effect apparent, particularly for \( N = 8 \) where the potential is flatter. Note that the errors shown are statistically correct at each time separation but are correlated in Euclidean time. Clearly the effect is considerably smaller than the naive non-relativistic model (Fig 1) predicts. Nonetheless, we would suggest that there is a significant source of systematic error which must be allowed for in future high statistics analyses. To be free of finite size effects, it would seem prudent to study systems for which

\[
L/\xi > cN
\]

(13)

where \( c \) is of the order of 2.

For much smaller lattices, one sees a reversal to the more usual finite size effect of \textit{raised} effective masses (reduced correlation lengths) [2].

### Scaling properties

Analyses using the quadratic action [5] have shown conflicting evidence for scaling of the mass gap (adjoint state) [4, 5]. The first study with relatively low statistics and covering the lattice correlation range \( \xi/a < 30 \) showed some evidence for scaling (constant ratio of dimensionful quantities) and for asymptotic scaling of the mass gap itself [3, 10]

\[
ma = (m/\Lambda_N)(1/g^2)^{3/4} \exp\left(-\frac{2\pi}{g^2}\right)
\]

(14)

This analysis [4] relied largely on a local Metropolis algorithm. However, a multigrid approach enabled Hasenbusch and Meyer [3] to achieve larger statistics and to probe larger correlation lengths (\( \leq 77 \)). Clear evidence of non-asymptotic scaling was presented.

Irving and Michael used the gauge-explicit action (1, 4) for the models with \( 4 \leq N \leq 8 \) and, using a hybrid local/cluster algorithm, presented data showing evidence of scaling (ratio of topological charge and mass-gap) and for asymptotic scaling of the mass-gap for \( g^2 \leq 1.1 \). A subsequent analysis [5] using a hybrid over-relaxation/heat bath algorithm on the same action with explicit gauge fields has provided further convincing evidence that both scaling
and asymptotic scaling properties are superior for this action when compared to the quadratic action (∥).

We have used a variant of the multigrid algorithm of Hasenbusch and Meyer [3] to compare the asymptotic scaling in the \( N = 4 \) model (∥) for both actions using the same analysis techniques. The \( t \)-dependent effective mass-gap was extracted from zero-momentum correlators using eqn. (1). The errors on the \( t \)-independent values (see discussion above) were extracted using bootstrap techniques. Typically 20000 multigrid V-cycles with measurements every second cycle were made at each coupling. As can be seen from table 1, the corresponding values of \( m/\Lambda \) are not constant in \( 1/g^2 \) but are significantly more slowly varying for the explicit gauge action \( S \). If one parameterises the the scaling violation by

\[
m/\Lambda \sim (\xi/a)^{z_s}
\]

then the data presented in table 1 yield \( z_s = .18 \pm .02 (.07 \pm .02) \) for the quadratic (explicit gauge) action.

### Algorithms

The data presented above was obtained using an adaptation of the multigrid algorithm proposed by Hasenbusch and Meyer [3]. The particular improvement tested was to use random \( SU(2) \) subgroups rather than \( U(1) \) and to employ a different subgroup and rotation at each level of the V-cycle. Using this, we made careful measurements of the decorrelation dynamical exponent \( z \) using the spin susceptibility as a good measure for the self-consistent autocorrelation time. For example, we found values close to 1.0 for the quadratic action (∥)

\[
z_{(N=4)} = 1.05 \pm .08, \quad z_{(N=8)} = 1.12 \pm .10
\]

i.e. considerably larger than those reported by [3] but comparable with those reported by Wolff [4] who used an overrelaxation algorithm. When we used a fixed \( U(1) \) rotation for each block within the V-cycle, in the spirit of [3] the decorrelation performance of our algorithm was degraded. The multigrid performance on the action (∥) was similar: the dynamical exponent for \( CP^3 \) was \( 1.00 \pm .07 \). Of course, all these algorithms offer considerable improvement over the local Metropolis, single cluster, and hybrid algorithms used in earlier work [3, 4].

### Table 1: The scaled adjoint mass \( m/\Lambda \) in lattice \( CP^3 \) using the quadratic action and the explicit gauge action (see text). Also shown are the corresponding bare couplings and correlation lengths in lattice units (\( \xi = 1/m \)).

| Quadratic action | Explicit gauge action |
|------------------|-----------------------|
| \( 1/g^2 \)     | \( \xi/a \) | \( m/\Lambda \) | \( 1/g^2 \) | \( \xi/a \) | \( m/\Lambda \) |
| 1.25             | 4.48(8)   | 514(9)    | 0.909     | 3.1(1)     | 102(3)    |
| 1.35             | 8.7(3)    | 478(12)   | 0.977     | 4.9(1)     | 97(2)     |
| 1.4              | 12.8(3)   | 436(11)   | 1.000     | 5.3(1)     | 98(3)     |
| 1.45             | 18.5(3)   | 406(8)    | 1.111     | 10.8(3)    | 95(3)     |
| 1.5              | 30.0(2)   | 337(15)   | 1.190     | 18.5(7)    | 88(3)     |
Conclusions  We have presented a model for non-trivial finite size effects in $CP^{N-1}$ and new data relevant to this, to the question of precocious scaling and to the efficiency of multigrid algorithms. Effects ascribable to periodicity as predicted by the simple model have been demonstrated but at a much lower level. We have confirmed earlier evidence [2, 5] that the explicit gauge action for lattice $CP^{N-1}$ has superior scaling properties to the more commonly used quadratic action. Finally, we reported measurements of the dynamical exponent for a multigrid algorithm which represent a considerable improvement on local algorithms but which fall short of other reported data for similar algorithms [3].

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References

[1] K. Jansen & U.-J. Wise, Nucl. Phys. B370 (1992) 762
[2] A.C. Irving & C. Michael, Nucl. Phys. B371 (1992) 521
[3] M. Hasenbusch & S. Meyer, Kaiserslautern preprint, TH 91-16
[4] U. Wolff, CERN preprint, CERN-TH 6407/92
[5] M. Campostrini, P. Rossi & E. Vicari, Pisa preprint, IFUP-TH 9/92
[6] A. D’Adda, M. Lüscher & P. Di Vecchia, Nucl. Phys. B146 (1978) 63
[7] E. Witten, Nucl. Phys. B149 (1979) 285
[8] H.E. Haber, I. Hinchliffe & E. Rabinovici, Nucl. Phys. B172 (1980) 458
[9] S. Hikami, Prog. Theo. Phys. 62 (1979) 226
[10] E. Brezin, S. Hikami and J. Zinn-Justin Nucl. Phys. B165 (1980) 528
[11] E. Merzbacher, Quantum Mechanics, Wiley, New York 1970
Figure captions

1. The low-lying part of the predicted energy spectrum due to a periodic potential. The energy bands allowed by (8) are indicated by the shaded regions bounded by solid(dashed) lines for $N = 4(8)$. Units are set by $m_{\text{eff}} = 1$.

2. The effective mass defined by eqn. 11 for (a) $N = 4$ at $\kappa = 2.9$ and (b) $N = 8$ at $\kappa = 5.5$ The crosses (circles) denote data on $128^2(64^2)$ lattices. The solid line is the asymptotic mass gap used in the scaling analyses.