ON LIMIT DISTRIBUTIONS OF NORMALIZED TRUNCATED VARIATION, UPWARD TRUNCATED VARIATION AND DOWNWARD TRUNCATED VARIATION PROCESSES OF BROWNIAN MOTION WITH DRIFT

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Abstract. In the paper we introduce the truncated variation, upward truncated variation and downward truncated variation. These are closely related to the total variation but are well-defined even if the latter is infinite. Our aim is to explore their feasibility with respect to the studies of stochastic processes. We concentrate on a Brownian motion with drift for which we prove the convergence of the above-mentioned quantities. For example, we study the truncated variation when the truncation parameter \( c \) tends to 0. We prove in this case that for “small” \( c \)’s it is well-approximated by a deterministic process. Moreover we prove that an approximation error converges weakly (in the functional sense) to a Brownian motion. We prove also similar result for truncated variation processes when the time parameter is rescaled to infinity. Our methodology is more general. A key to the proofs was a decomposition of the truncated variation (see Corollary 15 and Lemma 16) which can be used for studies of any continuous processes.

Keywords: Stochastic processes, truncated variation, variation, Wiener process with drift.

1. Introduction

The variation of Brownian paths was the subject of study of many authors (cf. [9], [15], [3], [2] just to name a few; for more detailed account see e.g. [12], Chapter 10)). It is well known that for any \( p \leq 2 \), \( p \)-variation of the Brownian motion is a.s. infinite and this arguably gave rise to the development of Itô integral, which alone
proves that studies of the variation is of an utmost importance for the stochastic processes theory.

Intuitively, the above mentioned infiniteness of the variation stems from “wild behaviour” at small scales. A natural, yet not studied before, way to tackle this problem was introduced in the paper [10]. The idea introduced there was to neglect the moves of a process smaller than a certain (small) \( c > 0 \). This led to the definition of the \textit{truncated variation}. Let now \( f : [a, b] \to \mathbb{R} \) be a continuous function. Its truncated variation on the interval \([a, b]\) is given by

\[
TV_c^\phi (f, [a; b]) := \sup_n \sup_{a \leq t_1 < t_2 < \ldots < t_n \leq b} \sum_{i=1}^{n-1} \phi_c (|f(t_{i+1}) - f(t_i)|),
\]

where \( \phi_c (x) = \max \{ x - c, 0 \} \).

Due to the uniform continuity of the function \( f \) on the closed interval \([a, b]\) its truncated variation is always finite. Moreover, for \( p \geq 1 \) from inequality \( \phi_c (|x|) \leq |x|^p / c^{p-1} \) it immediately follows that it is finite whenever \( p \)–variation of the function \( f \) is finite. On the other hand \( TV_c^\phi (f, [a; b]) \) is a continuous function of the parameter \( c \) and \( \lim_{c \to 0} TV_c^\phi (f, [a; b]) = TV^0 (f, [a; b]) \), where \( TV^0 \) denotes total variation of the function \( f \). Thus truncated variation applied to the paths of any stochastic process may reveal its interesting properties.

In the paper we study the case of \((W_t)_{t \geq 0}\) being the Brownian motion with drift \( \mu \in \mathbb{R} \) and covariance function \( \text{cov}(W_s, W_t) = s \wedge t \). The reason for this is twofold. Firstly, \( W \) is a widely studied process, which enables us making some explicit computations. Secondly, \( W \) is an exemplar case of semimartingales and diffusions. The results for \( W \) will shed some light on the properties of the truncated variation in these more general classes. We stress that this paper contains some general results for continuous processes. In forthcoming articles, we plan to apply them for semimartingales and diffusions.

Therefore, the main object of our studies will be the truncated variation of the Brownian motion with drift \( \mu \), which is given by

\[
TV^\mu_c (t) := \sup_n \sup_{0 \leq t_1 < t_2 < \ldots < t_n \leq t} \sum_{i=1}^{n-1} \phi_c (|W_{t_{i+1}} - W_{t_i}|), \quad t > 0.
\]

We recall a closely related notion of the \textit{upward truncated variation} of \( W \), introduced in [11] and defined by

\[
UTV^\mu_c (t) := \sup_n \sup_{0 \leq t_1 < s_1 < t_2 < s_2 < \ldots < t_n < s_n \leq t} \sum_{i=1}^{n} \phi_c (W_{s_i} - W_{t_i}), \quad t > 0,
\]
and, analogously, the \textit{downward truncated variation}:

$$DTV^c_\mu(t) := \sup_n \sup_{0 \leq t_1 < s_1 < t_2 < s_2 < \ldots < t_n < s_n \leq t} \sum_{i=1}^n \phi_c(W_{t_i} - W_{s_i}), \quad t > 0.$$ 

The properties of $TV^c_\mu, UTV^c_\mu$ and $DTV^c_\mu$ are known up to some degree. Indeed in [11], the author proved that all three have finite exponential moments. Moreover, there are explicit formulas for the moment generating functions of $UTV^c_\mu(T)$ and $DTV^c_\mu(T)$ when $T$ is an exponential random variable, independent of $W$.

The calculations we mentioned, however, did not give explicit insight into the ”nature” of the truncated variation. We would like to obtain a more direct description of the ”infiniteness level” of $1-$variation of Brownian paths. A tempting way of answering this question, followed in this paper, may be to study the divergence of the truncated variation as $c \searrow 0$. In the paper we prove that the processes $c\, TV^c_\mu, c\, UTV^c_\mu, c\, DTV^c_\mu$ converge almost surely to a deterministic linear function. Further we study the second order asymptotics by proving that

\begin{align}
\sqrt{3} \left( TV^c_\mu(t) - \frac{t}{c} \right), \\
\sqrt{3} \left( UTV^c_\mu(t) - \left( \frac{1}{c} + \mu \right) \frac{t}{2} \right), \\
\sqrt{3} \left( DTV^c_\mu(t) - \left( \frac{1}{c} - \mu \right) \frac{t}{2} \right),
\end{align}

converge weakly (in the functional sense) to a standard Brownian motion as $c \to 0$. This is the most surprising result of the paper. It appears that the truncated variation (of the Wiener process) is essentially a deterministic function. We believe that this is closely related to the fact that the quadratic variation $\langle W \rangle_t = t$ is also deterministic.

We present a very useful decomposition of $TV^c_\mu(t), UTV^c_\mu(t)$ and $DTV^c_\mu(t)$ stated in Corollary [15] and Lemma [16]. It is in fact valid for any continuous stochastic process and it will be a starting point for further studies. In the setting of this paper the decomposition is particularly useful as it is very similar to a renewal process. This is a crux of the proofs of the above-mentioned convergences.

For the sake of completeness we also investigate the convergence in distribution of properly normalised processes $TV^c_\mu(nt), UTV^c_\mu(nt)$ and $DTV^c_\mu(nt)$ for fixed $c > 0$ as $n$ tends to infinity. Similarly as before we obtain the convergence of $TV^c_\mu(nt)/n, UTV^c_\mu(nt)/n, DTV^c_\mu(nt)/n$ almost surely to deterministic limits. Further we prove that the processes

$$\frac{TV^c_\mu(nt) - m_1nt}{\sigma_1 \sqrt{n}}, \quad \frac{UTV^c_\mu(nt) - m_2nt}{\sigma_2 \sqrt{n}} \quad \text{and} \quad \frac{DTV^c_\mu(nt) - m_3nt}{\sigma_3 \sqrt{n}},$$
converge weakly (in the functional sense) to a standard Brownian motion, for some deterministic constants $m_1, m_2, m_3, \sigma_1, \sigma_2, \sigma_3$. The result is very similar to the case of $c \searrow 0$ however it is more “expected”.

A few other results we state are mainly used in the proofs, although they are interesting on their own and may be useful for other applications too. We calculate the bivariate Laplace transform of the variables

$$T_D^c := \inf \left\{ s : \sup_{0 \leq u \leq s} W_u - W_s = c \right\},$$

and

$$Z_{D-c}(c) = \sup_{0 \leq t \leq s \leq T_D^c} \max \{W_s - W_t - c, 0\}.$$  

This result is similar to the one of [14], where the bivariate Laplace transform of $T_D^c$ and $\sup_{0 \leq t \leq T_D^c} W_t$ was calculated.

Our approach in new, although studying and applying limit distributions (also in the functional sense) of total variation, $p-$variation and related functionals has long history. We just mention [5], [8], [7] and refer the reader to their discussions and references.

Let us comment on the organization of the paper. In the next section we present the main results of this paper - theorems about convergence in distribution of the normalized processes $TV^c_{\mu}(t), UTV^c_{\mu}(t)$ and $DTV^c_{\mu}(t)$ as $c \to 0$ and processes $TV^c_{\mu}(nt), UTV^c_{\mu}(nt)$ and $DTV^c_{\mu}(nt)$ as $n \to +\infty$. In the third section we state and prove lemmas concerning the structure of the processes $TV^c_{\mu}(t), UTV^c_{\mu}(t)$ and $DTV^c_{\mu}(t)$. Finally, the fourth section is devoted to the proofs of the theorems stated in Section 2. The bivariate Laplace transform of $T_D^c$ and $Z_{D-c}(c)$ is calculated in this section as well.

2. Functional convergence

2.1. Limit in distribution of truncated variation processes. In this subsection we present the results concerning the limit distribution of the normalized truncated variation, upward truncated variation and downward truncated variation processes. By $\{B_t\}_{t \geq 0}$, we denote a standard Brownian motion. Now we are ready to present some functional limit theorems. Let us recall the definition of the truncated variation (1.1). We start with

**Fact 1.** Let $T > 0$. We have

$$c TV^c_{\mu} \to t, \ a.s., \quad as \ c \to 0,$$

where the convergence is understood in $C([0,T], \mathbb{R})$ topology.

The quality of the above approximation is studied in
Theorem 2. Let $T > 0$. We have
\[(TV^c_\mu(t) - c^{-1}t) \to^d 3^{-1/2}B_t, \quad \text{as } c \to 0,\]
where $\to^d$ is understood as weak convergence in $C([0, T], \mathbb{R})$ topology.

Remark 3. The immediate consequence of Theorem 2 is that the variable $\sqrt{3}(TV^c_\mu(t) - c^{-1}t)$ converges in law to the variable with normal distribution $\mathcal{N}(0, t)$ as $c \to 0$.

Remark 4. The above facts reveal that for small $c$ the truncated variation is almost a deterministic process. Namely, $TV^c_\mu(t) \approx c^{-1}t + 3^{-1/2}B_t$ and obviously the first term overwhelms the second one.

Now for fixed $c$ we rescale the time parameter. Firstly, we present

Fact 5. Let $T > 0$ and $c > 0$. We have
\[TV^c_\mu(nt)/n \to m^c_\mu, \quad \text{a.s.}, \quad \text{as } c \to 0,\]
where the convergence is understood in $C([0, T], \mathbb{R})$ topology and
\[m^c_\mu = \begin{cases} 
\mu \coth(c\mu) & \text{if } \mu \neq 0, \\
1/3 & \text{if } \mu = 0.
\end{cases}\]

The quality of the above approximation is studied in

Theorem 6. Let $T > 0$ and $c > 0$. We have
\[\frac{TV^c_\mu(nt) - m^c_\mu nt}{\sigma^c_\mu \sqrt{n}} \to^d B_t, \quad \text{as } n \to +\infty,\]
where $\to^d$ is understood as weak convergence in $C([0, T], \mathbb{R})$ topology; $m^c_\mu$ is as in the previous fact and
\[\left(\sigma^c_\mu\right)^2 = \begin{cases} 
\frac{2-2c\mu \coth(c\mu)}{\sinh^2(c\mu)} + 1 & \text{if } \mu \neq 0, \\
1/3 & \text{if } \mu = 0.
\end{cases}\]

Remark 7. The immediate consequence of Theorem 6 is that the variable $(TV^c_\mu(T) - m^c_\mu T) / \left(\sigma^c_\mu \sqrt{T}\right)$ converges in law to the variable with standard normal distribution $\mathcal{N}(0, 1)$ as $T \to +\infty$.

Remark 8. We note that when $\mu = 0$ Theorem 2 and Theorem 6 are equivalent. It is enough to notice that by the scaling property of the Brownian motion we have $TV^c_0(nt) = \sqrt{n} \left(TV^c_0/\sqrt{n}(t)\right)$.
2.2. Limit distribution of upward and downward truncated variations processes. In this subsection we present the results concerning the limit distribution of the normalized upward and downward truncated variations processes. Since \( DTV_{\mu}^c = d UTV_{\mu}^c \) we will only deal with the upward truncated variation. We recall the definition \([1,2]\). Firstly we study the case of small \( c \).

**Fact 9.** Let \( T > 0 \). We have
\[
c UTV_{\mu}^c \to \frac{1}{2} t, \ a.s., \quad \text{as } c \to 0,
\]
where the convergence is understood in \( C([0,T], \mathbb{R}) \) topology.

**Theorem 10.** Let \( T > 0 \). We have
\[
\left( UTV_{\mu}^c(t) - \left( \frac{1}{2c} + \frac{\mu}{2} \right) t \right) \to^d 3^{-1/2} B_t, \quad \text{as } c \to 0,
\]
where \( \to^d \) is understood as weak convergence in \( C([0,T], \mathbb{R}) \) topology.

Now for fixed \( c \) we rescale the time parameter. We have

**Fact 11.** Let \( T > 0 \) and \( c > 0 \). We have
\[
TV_{\mu}^c(nt)/n \to m_{\mu}^c, \ a.s., \quad \text{as } c \to 0,
\]
where the convergence is understood in \( C([0,T], \mathbb{R}) \) topology and
\[
m_{\mu}^c = \begin{cases} 
\frac{1}{2} \mu (\coth(c\mu) + 1) & \text{if } \mu \neq 0, \\
(2c)^{-1} & \text{if } \mu = 0.
\end{cases}
\]

**Theorem 12.** Let \( T > 0 \) and \( c > 0 \). We have
\[
\frac{UTV_{\mu}^c(nt) - m_{\mu}^c nt}{\sigma_{\mu}^c \sqrt{n}} \to^d B_t, \quad \text{as } n \to +\infty,
\]
where \( \to^d \) is understood as weak convergence in \( C([0,T], \mathbb{R}) \) topology; \( m_{\mu}^c \) is as in the same as in the previous fact and
\[
\left( \sigma_{\mu}^c \right)^2 = \begin{cases} 
\frac{2 \exp(4c\mu)(\sinh(2c\mu) - 2c\mu)}{(\exp(2c\mu) - 1)^3} & \text{if } \mu \neq 0, \\
1/3 & \text{if } \mu = 0.
\end{cases}
\]

**Remark 13.** Analogously as before one checks that Theorem 10 and Theorem 12 are equivalent if \( \mu = 0 \). This is again a consequence of the scaling property of the Brownian motion which yields \( UTV_{\mu}^c(nt) = d \sqrt{n} \left( UTV_{\mu}^c/\sqrt{n}(t) \right) \).
3. Structure of truncated variation, upward truncated variation and downward truncated variation processes

In this section we develop tools to analyse TV, DTV, UTV processes. For the matter of convenience we work with the Wiener process with drift \( W \) but we stress that all results in this section are valid for any continuous stochastic process.

3.1. Structure of truncated variation process. Firstly, we will prove that the process \( (TV^c_\mu (t))_{t \geq 0} \) has a similar structure to a renewal process. To state it more precisely we first define

\[
T_D(c) = \inf \left\{ s \geq 0 : \sup_{0 \leq u \leq s} W_u - W_s = c \right\},
\]

\[
T_U(c) = \inf \left\{ s \geq 0 : W_s - \inf_{0 \leq u \leq s} W_u = c \right\},
\]

\[
T(c) = \min \{ T_D(c), T_U(c) \}.
\]

and now let \( (T_i(c))_{i=0}^\infty \) be a series of stopping times defined in the following way:

\( T_0 := 0 \) and

- if \( T(c) = T_D(c) \), then \( T_1(c) := T(c) \) and recursively, for \( k = 1, 2, ... \),

\[
T_{2k}(c) := \inf \left\{ s \geq T_{2k-1}(c) : W_s - \inf_{T_{2k-1}(c) \leq u \leq s} W_u = c \right\},
\]

\[
T_{2k+1}(c) := \inf \left\{ s \geq T_{2k}(c) : \sup_{T_{2k}(c) \leq u \leq s} W_u - W_s = c \right\};
\]

- if \( T(c) = T_U(c) \), then \( T_1(c) := 0 \), \( T_2(c) := T(c) \) and recursively, for \( k = 1, 2, ... \),

\[
T_{2k+1}(c) := \inf \left\{ s \geq T_{2k}(c) : \sup_{T_{2k}(c) \leq u \leq s} W_u - W_s = c \right\},
\]

\[
T_{2k+2}(c) := \inf \left\{ s \geq T_{2k+1}(c) : W_s - \inf_{T_{2k+1}(c) \leq u \leq s} W_u = c \right\}.
\]

Please observe that the event \( \{ T_U(c) = T_D(c) \} \) is impossible, hence the definitions above do not interfere.

Additionally we define a series of times \( (S_i(c))_{i=0}^\infty \) (which are not stopping ones): for \( k = 0, 1, 2, ... \)

- \( S_{2k}(c) \) is the first time when the maximum of \( W_t \) on the interval \( [T_{2k}(c), T_{2k+1}(c)] \) is attained (in particular for \( T_1 = 0, S_0 = 0 \));
\[ S_{2k+1} (c) \] is the first time when the minimum of \( W_t \) on the interval \([T_{2k+1} (c), T_{2k+2} (c)]\) is attained.

We have

**Lemma 14.** For \( k = 1, 2, 3, \ldots \) the following equalities hold

\[
TV^c_\mu (T_{2k} (c)) = TV^c_\mu (T_{2k-1} (c)) + W_{T_{2k-1}(c)} - \inf_{T_{2k-1}(c) \leq s \leq T_{2k}(c)} W_s, \\
TV^c_\mu (T_{2k+1} (c)) = TV^c_\mu (T_{2k} (c)) + \sup_{T_{2k}(c) \leq s \leq T_{2k+1}(c)} W_s - W_{T_{2k}(c)}.
\]

Moreover, a partition for which \( TV^c_\mu (T_{2k}) \) and \( TV^c_\mu (T_{2k+1}) \) in definition (1.1) are attained is given by \( S_0 (c), S_1 (c), \ldots \).

**Proof.** The proof will follow by induction.

It is easy to see that \( TV^c_\mu (T (c)) = 0 \). Let us consider two cases.

1. First case \( T (c) = T_D (c) \).

We start with \( k = 1 \).

Let \( 0 = t_0 < t_1 < \ldots < t_n \leq T_2 (c) \) be a partition of the interval \([0, T_2 (c)]\).

Without the loss of generality we may assume that there is no element \( t_j \) such that

\[
\max \{|W_{t_{j-1}} - W_{t_j}| - c, 0\} = \max \{|W_{t_j} - W_{t_j+1}| - c, 0\} = 0.
\]

Indeed, if it was such an element we would skip it and the sum (1.1) would not decrease.

(1) It is easy to see that if \( \max \{|W_{t_{i+1}} - W_{t_i}| - c, 0\} \) and \( \max \{|W_{t_{i+2}} - W_{t_{i+1}}| - c, 0\} \) are two consecutive non-zero summands, then

\[
\max \{|W_{t_{i+2}} - W_{t_i}| - c, 0\} \geq \max \{|W_{t_{i+1}} - W_{t_i}| - c, 0\} + \max \{|W_{t_{i+2}} - W_{t_{i+1}}| - c, 0\}.
\]

In fact, because we are before the first upward move by \( c \) we must have \( W_{t_{i+1}} - W_{t_i} \leq -c, W_{t_{i+2}} - W_{t_{i+1}} \leq -c \). Hence

\[
W_{t_{i+2}} - W_{t_i} = (W_{t_{i+2}} - W_{t_{i+1}}) + (W_{t_{i+1}} - W_{t_i}) \leq -2c,
\]

and

\[
\max \{|W_{t_{i+2}} - W_{t_i}| - c, 0\} = W_{t_i} - W_{t_{i+2}} - c \\
\geq (W_{t_i} - W_{t_{i+1}} - c) + (W_{t_{i+1}} - W_{t_{i+2}} - c) \\
= \max \{|W_{t_{i+1}} - W_{t_i}| - c, 0\} + \max \{|W_{t_{i+2}} - W_{t_{i+1}}| - c, 0\}.
\]

(2) Similarly, if \( \max \{|W_{t_{i+1}} - W_{t_i}| - c, 0\} \) and \( \max \{|W_{t_{i+3}} - W_{t_{i+2}}| - c, 0\} \) are two non-zero summands, while \( \max \{|W_{t_{i+2}} - W_{t_{i+1}}| - c, 0\} = 0 \), we have...
\[
W_{i+1} - W_i \leq -c, W_{t+2} - W_{t+1} \leq -c \text{ and } W_{t+2} - W_{t+1} \leq c, \text{ hence }
\]
\[
W_{t+3} - W_t = (W_{t+3} - W_{t+2}) + (W_{t+2} - W_{t+1}) + (W_{t+1} - W_t) \\
\leq -c + c - c \leq -c,
\]
\[
\max \{|W_{t+3} - W_t| - c, 0\} = W_t - W_{t+3} - c \\
\geq (W_{t+2} - W_{t+3} - c) + (W_t - W_{t+1} - c) \\
= \max \{|W_{t+1} - W_t| - c, 0\} + \max \{|W_{t+3} - W_{t+1}| - c, 0\}.
\]

As a result we obtain that the sum \(\sum_{i=1}^{n-1} \max \{|W_{t+i} - W_t| - c, 0\}\) attains its largest value for a \textbf{two-element} partition \(0 \leq t_1 < t_2 \leq T_2(c)\).

Since \(W_{S_0(c)} = \sup_{0 \leq s \leq T_2(c)} W_s\) and \(W_{S_1(c)} = \inf_{0 \leq s \leq T_2(c)} W_s\) we get
\[
TV^c_\mu (T_2(c)) = \sup_{n} \sup_{0 \leq t_1 < t_2 < ... < t_n \leq T_2(c)} \sum_{i=1}^{n-1} \max \{|W_{t+i} - W_t| - c, 0\} \\
= \sup_{0 \leq t_1 < t_2 \leq T_2(c)} \sum_{i=1}^{n-1} \max \{|W_{t+i} - W_t| - c, 0\} \\
= \max \{|W_{S_0(c)} - W_{S_1(c)}| - c, 0\} \\
= W_{T_1(c)} - \inf_{T_1(c) \leq s \leq T_2(c)} W_s \\
= TV^c_\mu (T_1(c)) + W_{T_1(c)} - \inf_{T_1(c) \leq s \leq T_2(c)} W_s.
\]

(Note that \(TV^c_\mu (T_1(c)) = TV^c_\mu (T(c)) = 0\))

Let us assume that Lemma 13 holds for some \(k \geq 1\). We proceed with the induction step from interval \([0, T_{2k}(c)]\) to the interval \([0, T_{2k+1}(c)]\).

We know that \(0 \leq S_0(c) < S_1(c) < ... < S_{2k-1}(c) < T_{2k}(c)\) is the best partition of the interval \([0, T_{2k}(c)]\). We will prove that the best partition of the interval \([0, T_{2k+1}(c)]\) is \(0 \leq S_0(c) < S_1(c) < ... < S_{2k-1}(c) < S_{2k}(c)\) i.e.
\[
\sum_{i=1}^{n-1} \max \{|W_{t+i} - W_t| - c, 0\} \leq \sum_{i=0}^{2k-1} \max \{|W_{S_{i+1}(c)} - W_{S_i(c)}| - c, 0\},
\]
for any partition \(0 \leq t_1 < ... < t_n \leq T_{2k+1}(c)\).

Again we will consider several cases.

\(1\) Firstly let us observe that if there exists such \(v \in \{1, 2, ..., n\}\) that \(S_{2k-1}(c) \leq t_v < T_{2k}(c)\), then, due to optimality of the partition \(0 \leq S_0(c) < S_1(c) < \ldots \leq S_{2k-1}(c) < S_{2k}(c) \leq T_{2k}(c)\), we have
\[
TV^c_\mu (T_1(c)) + W_{T_1(c)} - \inf_{T_1(c) \leq s \leq T_2(c)} W_s \leq (TV^c_\mu (T_1(c))) + W_{T_1(c)} - \inf_{T_1(c) \leq s \leq T_2(c)} W_s.
\]
\[ \ldots < S_{2k-1}(c) < T_{2k}(c), \]
\[
\sum_{i=1}^{v-1} \max \{|W_{t_{i+1}} - W_t| - c, 0\} \leq \sum_{i=0}^{2k-2} \max \{|W_{S_{i+1}}(c) - W_{S_i}(c)| - c, 0\} = TV_\mu(T_{2k}(c)).
\]

Moreover, reasoning similarly as in the proof of the case (1) for \( k = 1 \), from definitions of \( S_{2k-1}(c), T_{2k}(c) \) and \( T_{2k+1}(c) \) we obtain that for any \( S_{2k-1}(c) \leq s < u \leq T_{2k+1}(c), W_u - W_s \geq -c \) and the sum \( \sum_{i=0}^n \max \{|W_{t_{i+1}} - W_t| - c, 0\} \) attains its largest value for two element partition and can not be larger than \( \max \{|W_{S_{2k}}(c) - W_{S_{2k-1}}(c)| - c, 0\} \). Collecting these two inequalities, we get
\[
\sum_{i=1}^{n-1} \max \{|W_{t_{i+1}} - W_t| - c, 0\} \leq \sum_{i=0}^{2k-1} \max \{|W_{S_{i+1}}(c) - W_{S_i}(c)| - c, 0\} = TV_\mu(T_{2k}(c)) + \sup_{T_{2k}(c) \leq s \leq T_{2k+1}(c)} W_s - W_{T_{2k}(c)}.
\]

(2) Now we may assume that there is no such indice \( v \) that \( S_{2k-1}(c) < t_v < T_{2k}(c) \). In this case let \( v \) be the largest index such that \( t_v \leq S_{2k} \).

We have two subcases.

(a) \( W_{t_v} < W_{S_{2k-1}}(c) \).

In this case we have \( t_v < S_{2k-2}(c) \) (since \( W_{t_v} < W_{S_{2k-1}}(c) \), by definition of \( W_{S_{2k-1}}(c) \) as a minimal value of \( W_t \) on the interval \([T_{2k-1}(c), T_{2k}(c)]\) we have that \( t_v < T_{2k-1}(c) \), but since \( W_{S_{2k-2}}(c) \) is the maximal value of \( W_t \) on the interval \([T_{2k-2}(c), T_{2k-1}(c)]\) and by definition of \( T_{2k-1}(c) \) we must have \( t_v < S_{2k-2}(c) \) and we easily find that partition \( 0 \leq t_1 < \ldots < t_v < S_{2k-2}(c) < S_{2k-1}(c) < t_{v+1} < \ldots t_n \leq T_{2k+1}(c) \) gives a larger sum than the partition \( 0 \leq t_1 < \ldots < t_n \leq T_{2k+1}(c) \). So we have a new better partition which satisfies the conditions of the case (1) above.

(b) \( W_{t_v} \geq W_{S_{2k-1}}(c) \).

In this case, if \( W_{t_v} \in [W_{S_{2k-1}}(c), W_{t_{v+1}}] \) then \( |W_{t_{v+1}} - W_{t_v}| \leq |W_{t_{v+1}} - W_{S_{2k-1}}(c)| \). If the opposite holds, we get that \( |W_{t_{v+1}} - W_{t_v}| \leq |W_{t_{v}} - W_{S_{2k-1}}(c)| \). In both cases we calculate
\[
\max \{|W_{t_{v+1}} - W_{t_v}| - c, 0\} \leq \max \{|W_{S_{2k-1}} - W_{t_v}| - c, 0\} + \max \{|W_{t_{v+1}} - W_{S_{2k-1}}| - c, 0\}.
\]

So again we have a new, better partition which satisfies the conditions of the case (1) above.

Now we may proceed to the proof of the step: form interval \([0, T_{2k+1}(c)]\) to the interval \([0, T_{2k+2}(c)]\). This case is analogous to the previous one when we consider the process \( W_t = -W_t \) instead of \( W_t \).
2. Second case \( T(c) = T_U(c) \).

Again this case is analogous to the previous case in such a way that, considering the process \( \tilde{W}_t = -W_t \) and the corresponding times \( \left( \tilde{T}_i \right)_{i=0}^{\infty}, \left( \tilde{S}_i \right)_{i=0}^{\infty} \), we get \( T_{i+1} = \tilde{T}_i, S_{i+1} = \tilde{S}_i \) for \( i = 0, 1, 2, \ldots \) So in this case we prove the thesis in a similar way as above. \( \square \)

Let us now define

- for \( k = 0, 1, 2, \ldots \),

\[
Z_{D,k}(c) := \sup_{T_{2k}(c) \leq s \leq T_{2k+1}(c)} W_s - W_{T_{2k}(c)},
\]

- and similarly

\[
Z_{U,k}(c) := W_{T_{2k+2}(c)} - \inf_{T_{2k+1}(c) \leq s \leq T_{2k+2}(c)} W_s.
\]

Now for \( k = 0, 1, 2, \ldots \), we define a sequence of random variables

\[
D_k(c) := T_{2k+2}(c) - T_{2k}(c),
\]

\[
Z_k(c) := Z_{D,k}(c) + Z_{U,k}(c).
\]

Let us note here that in the case of the Wiener process with drift \( W \) obviously \( \{D_k(c)\}_{k \geq 1} \) and \( \{Z_k(c)\}_{k \geq 1} \) are i.i.d sequences.

The immediate consequence of Lemma 14 is

**Corollary 15.** For the process \( \left( TV_{\mu}^c(t) \right)_{t \geq 0} \) stopped at (Markov times) \( T_{2k+2}(c), k = 0, 1, 2, \ldots \), the following equality holds

\[
TV_{\mu}^c(T_{2k+2}(c)) = \sum_{i=0}^{k} Z_i(c).
\]

### 3.2. Structure of upward and downward truncated variation processes.

Now we will state an analog of Corollary 15 for the upward and downward truncated variation processes.

Let us first define two sequences of stopping times. Let \( T_{U,0}(c) = T_{D,0}(c) = 0 \) and

- recursively, for \( k = 1, 2, \ldots \),

\[
T_{D,k}(c) := \inf \left\{ s \geq T_{D,k-1}(c) : \sup_{T_{D,k-1}(c) \leq u \leq s} W_u - W_s = c \right\},
\]

- and analogously

\[
T_{U,k}(c) := \inf \left\{ s \geq T_{U,k-1}(c) : W_s - \inf_{T_{U,k-1}(c) \leq u \leq s} W_u = c \right\},
\]

(note that \( T_{D,1}(c) = T_D(c) \)). Further, we introduce
• recursively, for \( k = 1, 2, \ldots \),
\[
Z_{D,c,k}(c) := \sup_{T_{D,k-1}(c) \leq t < s \leq T_{D,k}(c)} \max \{ W_s - W_t - c, 0 \},
\]
• and analogously
\[
Z_{U,c,k}(c) := \sup_{T_{U,k-1}(c) \leq t < s \leq T_{U,k}(c)} \max \{ W_t - W_s - c, 0 \}.
\]

As the immediate consequence of [11, Lemma 3] we get

**Corollary 16.** For \( k = 1, 2, 3, \ldots \) the following equalities hold
\[
UTV_{\mu}^c(T_{D,k}(c)) = \sum_{l=1}^{k} Z_{D,c,l}(c),
\]
\[
DTV_{\mu}^c(T_{U,k}(c)) = \sum_{l=1}^{k} Z_{U,c,l}(c).
\]

4. Proofs of the results of functional convergence

4.1. Preliminaries. In this section we prove a simple extension of the classical Anscombe theorem and for its functional extensions we refer to [6, Chapter 1, Chapter 2, Chapter 5]. From now one we will use \( \lesssim \) to denote the situation when an equality or inequality holds with a constant \( C > 0 \), which is irrelevant for calculations. Our setting is as follows. Let
\[
(D_i(c), Z_i(c)), \quad i \in \mathbb{N},
\]
be sequences of i.i.d. random vectors indexed by certain parameter \( c \in (0, 1] \). We define
\[
M_c(t) := \min \left\{ n \geq 0 : \sum_{i=1}^{n+1} D_i(c) > t \right\},
\]
\[
P_c(t) := \left( \sum_{i=1}^{M_c(t)} Z_i(c) \right) - \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)} t, \quad t \in [0, 1].
\]

Let us observe that such defined \( M_c, P_c \) are càdlàg processes. We will need the following assumptions

(A1) For any \( c > 0 \) we have \( D_1(c) > 0 \) a.s. and \( \mathbb{E}D_1(c) \to 0 \) as \( c \to 0 \).

(A2) We denote \( X_i(c) := Z_i(c) - (\mathbb{E}Z_1(c)/\mathbb{E}D_1(c)) D_i(c) \). We have \( \mathbb{E}X_i(c) = 0 \). Now we assume that there exists \( \sigma > 0 \) such that
\[
\frac{\mathbb{E}X_1(c)^2}{\mathbb{E}D_1(c)} \to \sigma^2, \quad \text{as} \quad c \to 0.
\]
(A3) There exists $\delta \in (0, 2]$ such that
\[ \frac{\mathbb{E}|X_1(c)|^{2+\delta}}{\mathbb{E}D_1(c)} \rightarrow 0, \quad \text{as } c \rightarrow 0. \]

(A4) There exists $\delta > 0, C > 0$ such that
\[ \mathbb{E}|D_1(c)|^{1+\delta} \leq C(\mathbb{E}D_1(c))^{1+\delta}. \]

**Fact 17.** Let $T > 0$ and we assume that (A1)-(A4) hold. Then
\[ P_c \rightarrow^d \sigma B, \quad \text{as } c \rightarrow 0, \]
where $\sigma^2$ is the same as in (A2), and the convergence is understood as weak convergence in $C([0, T], \mathbb{R})$ topology.

**Proof.** We define
\[ S_c(n) := \sum_{i=1}^n Z_i(c), \quad V_c(n) := \sum_{i=1}^n D_i(c), \quad n \in \mathbb{N}. \]
Moreover let us denote $f(c) := \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)}$ and we recall that $X_1(c) := Z_i(c) - f(c)D_i(c)$. Now we define a family of auxiliary processes which
\[ P^1_c(t) := H_c([g(c)t]), \quad t \geq 0, \]
where $H_c(n) := S_c(n) - f(c)V_c(n)$ and $g(c) := (\mathbb{E}D_1(c))^{-1}$.

Now the proof follows by [13, Theorem 4.5.5]. We have to verify its assumptions. $T_1$ is obviously fulfilled. Let us fix $\nu > 0$, $S_4$ writes as
\[ [g(c)]\mathbb{P}(D_1(c) > \nu) \leq [g(c)]\nu^{-1-\delta}\mathbb{E}|D_1(c)|^{1+\delta} \lesssim [g(c)]g(c)^{-(1+\delta)} \rightarrow 0, \quad \text{as } c \rightarrow 0, \]
where we used assumption (A4) and the Chebyshev inequality. Similarly, we check that
\[ [g(c)]\mathbb{E}D_1(c)1_{\{D_1(c) > \nu\}} \leq [g(c)]\nu^{-\delta}\mathbb{E}|D_1(c)|^{1+\delta} \rightarrow 0, \quad \text{as } c \rightarrow 0, \]
again by (A4) and inequality $|x|^{1+\delta} \geq \nu^\delta x^1$ when $|x| \geq \nu$. Now it follows that the limit does not depend on $\nu$ and is the same as the limit of
\[ [g(c)]\mathbb{E}D_1(c) \rightarrow 1, \quad \text{as } c \rightarrow 0, \]
which follows simply by the definition of $g(c)$. We will now deal with $S_7$. Let us fix $\nu > 0$ the condition writes as
\[ [g(c)]\mathbb{P}(|X_1(c)| \geq \nu) \leq [g(c)]\mathbb{E}|X_1(c)|^{2+\delta} \rightarrow 0, \quad \text{as } c \rightarrow 0, \]
where we used assumption (A3) and the Chebyshev inequality. Let us prove now the following convergence
\[ [g(c)]\mathbb{E}(X_1(c)^21_{\{|X_1(c)| \geq \nu\}}) \leq [g(c)]\nu^{-\delta}\mathbb{E}|X_1(c)|^{2+\delta} \rightarrow 0, \quad \text{as } c \rightarrow 0, \]
where we used assumption (A2) and an elementary inequality $|x|^{2+\delta} \geq \nu^\delta x^2$ when $|x| \geq \nu$. Using the same method one proves $[g(c)] \mathbb{E} (X_1(c)^2 1_{\{|X_1(c)| \geq \nu\}}) \to 0$ which is nothing else than $S_8$. Finally, the limit in $S_9$ is the same as
\[
[g(c)] \mathbb{E} (X_1(c)^2) \to \sigma^2, \quad \text{as } c \to 0,
\]
by assumption (A2). Now it is straightforward to identify the limit using the description in [13, p. 243 and p. 284]. Let us define $P_c^2(t) := P_c^1((g(c)^{-1}M_c(t)) \wedge 2T)$ converges in the sup norm to $\sigma^2 B$. Our final step is to compare this process with $P_c$. We notice that they agree whenever $M_c$ is an integer smaller than $2g(c)T$, by the construction (4.3) and (A4) we conclude that
\[
\mathbb{P} \left( \sup_{0 \leq s \leq T} |P_c^2(s) - P_c(s)| > \varepsilon \right) \leq \mathbb{P} \left( \max_{i \leq [M_c(T)]} |X_i(c)| > \varepsilon \right)
\leq \mathbb{P} \left( \max_{i \leq [2Tg(c)]} |X_i(c)| > \varepsilon \right) + \mathbb{P} (M_c(T) > 2g(c)T)
\leq \left[ 2Tg(c) \varepsilon \right]^{2+\delta} \mathbb{E} |X_i(c)|^{2+\delta} + \mathbb{P} (M_c(T) > 2g(c)T) \to 0,
\]
as $c \to 0$. An application of [1, Theorem 3.1] concludes the proof. 

4.2. Truncated variation.

Proof of Theorem 2. The strategy of the proof is to approximate the process $TV^c_\mu$ using results of Corollary [13] by a renewal-type process and then use Fact [17]. Let us recall the notation of Section 3 (e.g. (3.1)). By the strong Markov property of Brownian motion we have that $Z_{D,k}(c), k = 1, 2, ...$, is an i.i.d. sequence and
\[
(T_{2k+1}(c) - T_{2k}(c), Z_{D,k}(c)) = (T_D(c), Z_D(c)),
\]
where $Z_D(c) := W_{T_D(c)} + c$. The formula [14] (1.1) reads as
\[
\mathbb{E} \exp(\alpha Z_D(c) - \beta T_D(c)) = \frac{\delta \exp(-\alpha + \mu) c \exp(\alpha c)}{\delta \cosh(\delta c) - (\alpha + \mu) \sinh(\delta c)},
\]
where $\delta = \sqrt{\mu^2 + 2\beta}$. This formula is valid if $\alpha < \coth(\delta c) - \mu$ and $\beta > 0$. If $\mu \neq 0$ we may also put $\beta = 0$. From (4.4) we easily calculate
\[
\mathbb{E} T_D(c) = \frac{e^{2\mu} - 2c\mu - 1}{2\mu^2} = c + o(c).
\]
Further we notice that the distribution of
\[
(T_{2k+2}(c) - T_{2k+1}(c), Z_{U,k}(c)),
\]
is the same as the distribution of $(T_U(c), Z_U(c))$, where $Z_U(c) := W_{T_U(c)} - c$, and is the same as $(T_D(c), Z_D(c))$ if we considered a Brownian motion with drift $-\mu$. 

For $k = 1, 2, \ldots$ we also have $(D_k(c), Z_k(c)) = (T_D(c) + T_U(c), Z_D(c) + Z_U(c))$ and $(T_D(c), Z_D(c)), (T_U(c), Z_U(c))$ are independent.

We are going to apply Fact 17 which will require verifying conditions (A1) - (A4). To this end we present some auxiliary lemmas.

**Lemma 18.** For any $p > 0$ and non-negative random variables $X$ and $Y$ we have

$$
\mathbb{E}(X + Y)^p \leq \mathbb{E}(2 \max \{X, Y\})^p \leq 2^p \mathbb{E}(X^p + Y^p).
$$

**Lemma 19.** We have

$$
\mathbb{E}D_1(c)^4 \lesssim c^8, \quad \mathbb{E}Z_1(c)^4 \lesssim c^8.
$$

**Proof.** The proof goes by the simple computation using the Laplace transform (4.4). When $\mu \neq 0$ we have

$$
\mathbb{E}T_D(c)^4 = \frac{3e^{8\mu} + e^{6\mu}(15 - 42\mu) + 6e^{4\mu}(2 - 5\mu)^2 - 18ce^{2\mu}\mu(4 + 3\mu(-3 + 2\mu))}{2\mu^8} + \frac{2\mu(12 + \mu(-3 + 4\mu - 42)(-1 + \mu))}{2\mu^8}.
$$

For $\mu = 0$ one has $\mathbb{E}T_D(c)^4 = \frac{277}{21}c^8$. In either case one checks that $\mathbb{E}T_D(c)^4/c^8 \to \frac{277}{21}$, as $c \to 0$, hence $\mathbb{E}T_D(c)^4 \lesssim c^8$. Similarly one checks $\mathbb{E}T_U(c)^4 \lesssim c^8$. Now, by Lemma 18 and definition of $D_1(c)$, $\mathbb{E}D_1(c)^4 \lesssim c^8$.

Analogously, one may check that $\mathbb{E}Z_D(c)$ has exponential distribution and

$$
\mathbb{E}Z_D(c)^4 = \begin{cases} 
\frac{3(\exp(2\mu) - 1)^4}{24\mu^4} & \text{if } \mu \neq 0, \\
2c^4 & \text{if } \mu = 0.
\end{cases}
$$

In either case one checks that $\mathbb{E}Z_D(c)^4/c^4 \to 24$, as $c \to 0$, hence $\mathbb{E}Z_D(c)^4 \lesssim c^4$. Similarly $\mathbb{E}Z_U(c)^4 \lesssim c^4$ and by Lemma 18 $\mathbb{E}Z_1(c)^4 \lesssim c^4$.

We used Mathematica to facilitate the above computations. The appropriate Mathematica notebook is available at http://www.mimuw.edu.pl/~pmilos/calculations.nb.

Now we will check the assumptions of Fact 17. Assumption (A1) is obvious. We have

$$
(4.6) \quad f(c) := \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)} = \mu \coth(\mu) = c^{-1} + O(c).
$$

We denote the fraction in assumption (A2) as $\sigma^2$ and calculate

$$
(\sigma^2) = \begin{cases} 
\frac{3 - 2\mu \coth(\mu)}{\sinh^2(\mu)} + 1 & \text{if } \mu \neq 0, \\
1/3 & \text{if } \mu = 0 \to \frac{1}{3} \text{ as } c \to 0.
\end{cases}
$$
Now we proceed to verification of assumption (A3). Using Lemma 18 and 19 we get
\[ \mathbb{E}X_1(c)^4 \lesssim \mathbb{E}Z_1(c)^4 + f(c)^4 \mathbb{E}D_1(c)^4 \lesssim c^4. \]
We easily check that \( \mathbb{E}D_1(c) \approx c^2 \) and see that assumption (A3) holds for \( \delta = 2 \). We are left with assumption (A4). By (4.5) and Lemma 19 it could be easily verified for \( \delta = 3 \).

Thus, since \( f(c) = c^{-1} + O(c) \), by Fact 17 we obtain:

**Corollary 20.** Let \( S_c, M_c \) be defined according to (4.1) and (4.2) for the \( Z_c(i), D_c(i) \) above. For any \( T > 0 \) we have
\[
(S_c(M_c(t)) - c^{-1}t) \to^d 3^{-1/2} B_t,
\]
where \( \to^d \) is understood as weak convergence in \( C([0, T], \mathbb{R}) \) topology.

The final stage is to compare process \( TV^c_\mu \) and process \( S_c(M_c(\cdot)) \) with the use of Corollary 15. Since \( D_0(c) \) has different distribution than \( D_k(c) \) for \( k = 1, 2, \ldots \), we introduce two auxiliary objects
\[
\tilde{M}_c(t) := \min \left\{ n \geq 0 : \sum_{i=0}^{n} D_i(c) > t \right\},
\]
and
\[
\tilde{S}_c(n) = \sum_{i=0}^{n-1} Z_i(c).
\]
These differ slightly from \( S_c \) and \( M_c \). After small changes of the definitions of the appropriate processes we see that the thesis of Fact 17 holds also in our case and we obtain
\[
(4.7) \quad \left( \tilde{S}_c(\tilde{M}_c(t)) - c^{-1}t \right) \to^d 3^{-1/2} B_t,
\]
where \( \to^d \) is understood as weak convergence in \( C([0, T], \mathbb{R}) \) topology.

From this definition and Corollary 15 we see that the processes \( TV^c_\mu \) and \( \tilde{S}_c(\tilde{M}_c(t)) \) coincide at random times \( T_{2k}, k = 0, 1, 2, \ldots \), moreover, both are increasing, hence, for any \( T \geq 0 \) and \( \varepsilon > 0 \)
\[
(4.8) \quad \mathbb{P} \left( \sup_{t \in [0, T]} \left| TV^c_\mu(t) - \tilde{S}_c(\tilde{M}_c(t)) \right| > \varepsilon \right) \leq \mathbb{P} \left( \sup_{t \in [0, T]} Z_{\tilde{M}_c(t)}(c) > \varepsilon \right).
\]
Now, by (4.8) we estimate
\[
\mathbb{P}\left( \sup_{t \in [0,T]} \left| TV_{c}^{\mu}(t) - \tilde{S}_{c}(\tilde{M}_{c}(t)) \right| > \varepsilon \right)
\leq \mathbb{P}\left( \max_{k \leq 2T/ED_{1}(c)+1} Z_{k}(c) \geq \varepsilon \right) + \mathbb{P}\left( \tilde{M}_{c}(t) \geq \frac{2T}{ED_{1}(c) + 1} \right).
\]
The first term could be estimated by the Chebyshev inequality and the estimates of $\mathbb{E}Z_{1}(c)^{4}$ and $\mathbb{E}D_{1}(c)$
\[
\mathbb{P}\left( \max_{k \leq 2T/ED_{1}(c)+1} |Z_{k}(c)| > \varepsilon \right) \leq \left( \frac{2T}{ED_{1}(c) + 1} \right) \frac{\mathbb{E}Z_{1}(c)^{4}}{\varepsilon^{4}} \rightarrow 0, \quad \text{as } c \rightarrow 0.
\]
The convergence of the last term to 0 could be established by simple calculation using assumption (A4).

**Proof of Fact 1.** This is much simpler compared to the proof of Theorem 2, therefore we only sketch an outline leaving details to the reader. Firstly, using [13, Lemma 4.5.2] one could prove that $M_{c}/g(c)$ converges to id. (Let us recall that we have already verified assumptions (A1)-(A4) of Fact 17). $T_{1}$ is obviously fulfilled. Let us fix $\nu > 0$, $S_{4}$ writes as
\[
\mathbb{E}\left\{ D_{1}(c) > \nu \right\} \leq \mathbb{E}\left\{ D_{1}(c) \right\}^{1+\delta} \lesssim [g(c)]^{\nu^{-\delta} \mathbb{E}D_{1}(c)^{1+\delta}} \rightarrow 0, \quad \text{as } c \rightarrow 0,
\]
where we used assumption (A4) and the Chebyshev inequality. We check that
\[
\mathbb{E}D_{1}(c)^{1+\delta} \rightarrow 0, \quad \text{as } c \rightarrow 0,
\]
again by (A4) and inequality $|x|^{2+\delta} \geq \nu^{-\delta} x^{2}$ when $|x| \geq \nu$. Now it follows that the limit does not depend on $\nu$ and is the same as the limit of
\[
[g(c)]^{\mathbb{E}D_{1}(c)} \rightarrow 1, \quad \text{as } c \rightarrow 0,
\]
simply by the definition of $g(c)$. One easily identifies the limit using the discussion on the top of page 284 [13].

One need to prove that $V_{c}(\lfloor g(c) \rfloor t)$ converge to a linear function (which is in fact a functional strong law of large number). Finally, one have to use Corollary 15 and prove that composition of two processes described above converges as well.

**Proof of Theorem 6.** The strategy of the proof is to find a renewal-type processes $G_{n}$ which approximates the process in the theorem. In order to prove the convergence of $G_{n}$ we will use [9, Chapter 5, Theorem 4.1]. In the final step we will show that the approximation error converges to 0.
Let us define a family of processes
\[ G_n(t) := \frac{S_c(M_c(nt)) - m^c nt}{\sigma^c \sqrt{n}}, \quad \geq 0, n \in \mathbb{N}, \]
where \( m^c = f(c) = \frac{E Z_1(c)}{ED_1(c)} \) and \( (\sigma^c)^2 = \text{Var}((EZ_1(c))D_1(c) - (ED_1(c))Z_k(c))(ED_1(c))^{-3} \) were calculated in the previous subsection. By Chapter 5, Theorem 4.1 we know that \( G_n \rightarrow^d B \) in the Skorohod topology. Similarly
\[ \tilde{G}_n(t) := \frac{\tilde{S}_c(M_c(nt)) - m^c nt}{\sigma^c \sqrt{n}} \rightarrow B_t, \quad as \ n \rightarrow \infty. \]

Our final step is to estimate
\[ \frac{TV^c_{\mu}(nt) - \tilde{S}_c(M_c(nt))}{\sigma^c \sqrt{n}}. \]

Similarly as in the proof of Theorem 2 we estimate
\[
\mathbb{P} \left( \sup_{t \in [0,T]} \frac{|TV^c_{\mu}(nt) - \tilde{S}_c(M_c(nt))|}{\sqrt{n}} > \varepsilon \right) \leq \mathbb{P} \left( \max_{k \leq 2nT/ED_1(c)+1} |Z_k(c)| > \varepsilon \sqrt{n} \right) + \mathbb{P} \left( M_c(nT) > \frac{2nT}{ED_1(c)} + 1 \right).
\]

The first term can be estimated by the Chebyshev inequality
\[
\mathbb{P} \left( \max_{k \leq 2nT/ED_1(c)+1} |Z_k(c)| > \varepsilon \sqrt{n} \right) \leq \left( \frac{2nT}{ED_1(c)} + 1 \right) \frac{E(Z_1(c))^4}{n^2 \varepsilon^4} \rightarrow 0, \quad as \ n \rightarrow +\infty.
\]

The second one converges to 0 by the law of large numbers. In this way we proved that the limit of the processes in theorem is the same as the one of \( \tilde{G}_n \)’s.

The proof of Fact 5 is simpler and hence skipped. We refer the reader to the outline of the proof of Fact 1 given above.

4.3. Upward and downward truncated variation. While the proofs in the previous section rely on Corollary 15, the ones in this section hinge on Corollary 16. The flow of the proofs of this section is much alike the ones in the Section 4.2. The main difficulty is to calculate the of moments \((T_{D,1}(c), Z_{D-c,1}(c)).\)

This will be done with the use of bivariate Laplace transform of \((T_D(c), Z_{D-c}(c))\) calculated in the next subsection.
4.3.1. Bivariate Laplace transform of \( T_D(c) \) and \( Z_{D-c}(c) \). In [11] two-dimensional density of the variables \( T_D(c) \) and \( \sup_{0 \leq s < t \leq T_D(c)} \{W_s - W_t\} \) is calculated. This density is given by [11, formula (11)]. Using it, we unconsciously calculated bivariate Laplace transform \( \mathbb{E} \exp(\lambda Z_{D-c}(c) + \nu T_D(c)) \) which is given in [11] by the formula (20). This formula reads (using notation from [11]) as

\[
\mathbb{E} \exp(\lambda Z_{D-c}(c) + \nu T_D(c)) = \left( 1 - \lambda \frac{L_0^W(-\nu, c)}{T_{-\mu, 1}(-\nu, c) + \lambda} \right) \mathbb{E} e^{\nu T_D(c)},
\]

where

\[
L_0^W(-\nu, c) = \frac{U_\mu(\nu)}{-2\nu} \left\{ \frac{e^{\mu c} (U_\mu(\nu) \coth(c U_\mu(\nu)) - \mu)}{\sinh(c U_\mu(\nu))} - \frac{U_\mu(\nu)}{\sinh^2(c U_\mu(\nu))} \right\},
\]

\[
\mathbb{E} e^{\nu T_D(c)} = \frac{U_\mu(\nu) e^{-\mu c}}{U_\mu(\nu) \cosh(c U_\mu(\nu)) - \mu \sinh(c U_\mu(\nu))},
\]

\[
T_{-\mu, 1}(-\nu, c) = \mu - U_\mu(\nu) \coth(c U_\mu(\nu))
\]

and

\[
U_\mu(\nu) = \sqrt{\mu^2 - 2\nu}.
\]

Substituting the above formulas in (4.9) we obtain

Corollary 21. The bivariate Laplace transform of the variable \((T_D(c), Z_{D-c}(c))\) reads as

\[
\mathbb{E} \exp(\lambda Z_{D-c}(c) - \nu T_D(c)) = 1 - \left( 1 - \frac{\lambda \nu^{-1}}{\sinh(2\delta) / \delta} \right) \left( 1 - \frac{e^{-\mu c}}{\cosh(c \delta) - \mu \sinh(c \delta) / \delta} \right),
\]

where \(\delta = \sqrt{\mu^2 + 2\nu}\).

Proof of Theorem 10. This time we adhere to the proof of Theorem 2. We will concentrate on differences leaving the reader the task of filling the rest of details. We calculate

\[
f(c) := \frac{\mathbb{E} Z_{D-c, 1}(c)}{\mathbb{E} T_{D, 1}(c)} = \frac{1}{2} \mu (\coth(c \mu) + 1) = \frac{1}{2} c^{-1} + \frac{1}{2} \mu + O(c).
\]

Further we have

\[
\left( \sigma_\mu^c \right)^2 = \begin{cases} 
\frac{2 \exp(4\mu)(\sinh(2\mu)-2c\mu)}{(\exp(2\mu)-1)^3} & \text{if } \mu \neq 0, \\
1/3 & \text{if } \mu = 0 \rightarrow 1/3 \text{ as } c \rightarrow 0.
\end{cases}
\]

\[\text{[1]see also http://www.mimuw.edu.pl/~pmilos/calculations.nb.}\]
To verify the assumption (A3) we firstly notice that $T_{D,1}(c)$ and $Z_{D-c,1}(c)$ are majorised by $D_1(c)$ and $Z_1(c)$ respectively. Using it we get

$$\mathbb{E}X_1(c)^4 \lesssim \mathbb{E}Z_{D-c,1}(c)^4 + f(c)^4 \mathbb{E}T_{D,1}(c)^4 \lesssim c^4.$$ 

We easily check that $\mathbb{E}T_{D,1}(c) \approx c^2$ and see that assumption (A3) holds for $\delta = 2$. Assumption (A4) could be easily verified for $\delta = 3$.

Next step - comparison of the process $UTV^c(\mu)(t)$ with the appropriate renewal process is even simpler, since the distribution of $Z_{D-c,1}(c)$ - first term in the sum appearing in the first equation in Corollary 16 - is the same as the distribution of the further terms.

**Proof of Theorem 12.** This proof goes along the lines of the proof of Theorem 6.

We skip the proofs of Fact 9 and Fact 11. We refer the reader to the outline of proof of Fact 11.

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