OSCILLATING DENSITY OF STATES NEAR ZERO ENERGY FOR MATRICES MADE OF BLOCKS WITH POSSIBLE APPLICATION TO THE RANDOM FLUX PROBLEM

E. Brézin\textsuperscript{a)}, S. Hikami\textsuperscript{b)} and A. Zee\textsuperscript{c)}

\textsuperscript{a)} Laboratoire de Physique Théorique,\textsuperscript{\textsuperscript{1}} Ecole Normale Supérieure
24 rue Lhomond, 75231 Paris Cedex 05, France
\textsuperscript{b)} Department of Pure and Applied Sciences, University of Tokyo
Meguro-ku, Komaba, Tokyo 153, Japan
\textsuperscript{c)} Institute for Theoretical Physics
University of California, Santa Barbara, CA 93106, USA

Abstract

We consider random hermitian matrices made of complex blocks. The symmetries of these matrices force them to have pairs of opposite real eigenvalues, so that the average density of eigenvalues must vanish at the origin. These densities are studied for finite $N \times N$ matrices in the Gaussian ensemble. In the large $N$ limit the density of eigenvalues is given by a semi-circle law. However, near the origin there is a region of size $\frac{1}{N}$ in which this density rises from zero to the semi-circle, going through an oscillatory behavior. This cross-over is calculated explicitly by various techniques. We then show to first order in the non-Gaussian character of the probability distribution that this oscillatory behavior is universal, i.e. independent of the probability distribution. We conjecture that this universality holds to all orders. We then extend our consideration to the more complicated block matrices which arise from lattices of matrices considered in our

\textsuperscript{1} Unité propre du Centre National de la Recherche Scientifique, Associée à l’Ecole Normale Supérieure et à l’Université de Paris-Sud
previous work. Finally, we study the case of random real symmetric matrices made of blocks. By using a remarkable identity we are able to determine the oscillatory behavior in this case also. The universal oscillations studied here may be applicable to the problem of a particle propagating on a lattice with random magnetic flux.
1 Introduction

It is well known that the average density of states, for Gaussian ensembles of random $N \times N$ matrices, obeys Wigner’s semi-circle law when $N$ goes to infinity, irrespective of the symmetries of the probability measure \cite{1}. For non-Gaussian measures the average density depends sensitively upon the distribution \cite{2}. However, next to the edge of the support of the eigenvalue distribution, there is a region of size $N^{-2/3}$, in which the average density crosses over from nonzero to zero, with a universal cross-over function (i.e. independent of the probability distribution) \cite{3}. In this work we consider instead an ensemble of random hermitian matrices made of complex blocks. These matrices have been discussed recently for its application to impurity scattering in the presence of a magnetic field \cite{4,5} and to a study of the zero modes of a Dirac operator\cite{6}. In the large $N$ limit the average density of eigenvalues is again a semi-circle for Gaussian ensembles. However, by construction these matrices have pairs of opposite real eigenvalues. Thus, as an eigenvalue approaches zero, the mid-point of the spectrum, it is repelled by its mirror image. Consequently, the density of eigenvalues is constrained to vanish at the origin. Away from the origin, the density must rise rapidly, over a region of size $\frac{1}{N}$, towards the Wigner semi-circle.

In a recent work \cite{4} we showed, by explicit computation in the Gaussian case, that the rise “overshoots” the Wigner semi-circle and thus has to come back down, whereupon it overshoots again. Thus, the density of eigenvalues oscillates over a region of size $\frac{1}{N}$. This cross-over at the center of the spectrum is however not of the same nature as the cross-over at the end of the spectrum.

For the simplest case of one random matrix, we can calculate the cross-over by three different methods: i) the orthogonal polynomial approach \cite{7}, ii) a method inspired by Kazakov’s approach to the usual hermitian Gaussian problem \cite{8}, iii) a supersymmetric method based on Grassmannian variables \cite{9}. The first method is in fact quite cumbersome and hard to generalize to more difficult problems. The second one provides an elegant integral representation of the correlation functions for finite $N$. However it is only through the Grassmannian approach that we could tackle the more complicated problems of a whole lattice of matrices discussed in our previous work \cite{10}.

For the most part the calculations in this paper are done with the Gaussian distribution. An interesting question is whether the cross-over behavior at the center of the spectrum is universal, that is, independent of the details
of the probability distribution. A simple scaling argument suggests universality. As usual in random matrix theory [1], the eigenvalues can be thought of as representing gas particles in a one-dimensional space. The probability distribution of the random matrices determine the potential confining the gas. This confining potential controls the width of the eigenvalue spectrum, which is of order $N^0$ in our convention. Crudely speaking, the oscillatory behavior we are discussing here depends on the repulsion between an eigenvalue and its mirror image, and on the repulsion between the eigenvalues. It should be possible to regard the confining potential as essentially constant over the region of size $\frac{1}{N}$ we are concerned with and hence irrelevant. However, further thought reveals that this argument is insufficient, since the confining potential, by changing the width of the spectrum, effectively also changes the average value of the density of eigenvalues near the center of the spectrum. We also expect that as the width of the entire spectrum changes the period of the oscillations near the center of the spectrum would also change accordingly. We are able to show, to first order in the non-Gaussian character of the distribution, that these effects cancel out, and that the cross-over behavior is universal. We conjecture that this cross-over behavior is indeed universal to all orders.

We then show that we can extend our considerations to study random real symmetric matrices made of real (but not symmetric) blocks. By using a remarkable identity, the universal oscillations near the center of the spectrum can again be calculated explicitly.

Finally, we discuss a possible realization of these universal oscillations in the problem of a single particle propagating on a square lattice penetrated by a random magnetic flux.

## 2 Orthogonal polynomials approach

In the simple one-matrix problem we consider $2N \times 2N$ block matrices $M$ of the form

$$M = \begin{pmatrix} 0 & C^\dagger \\ C & 0 \end{pmatrix},$$

in which $C$ is an $N \times N$ complex random matrix, with probability distribution

$$P(C) = \frac{1}{Z} \exp(-N \text{Tr} C^\dagger C).$$
It is easy to show that $M$ has pairs of opposite real eigenvalues; indeed if $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector for the eigenvalue $\lambda$, then $\begin{pmatrix} x \\ -y \end{pmatrix}$ is an eigenvector for $-\lambda$. In other words the matrix $M$ anti-commutes with the "$\gamma_5$" matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore one can express the average resolvent of the matrix $M$ in terms of that of $C^\dagger C$, a hermitian matrix with positive eigenvalues:

$$G(z) = \frac{1}{2N} \langle \text{Tr} \frac{1}{z - M} \rangle = \frac{1}{N} \langle \text{Tr} \frac{z}{z^2 - C^\dagger C} \rangle. \quad (2.3)$$

Taking the imaginary part of (2.3) we relate the density of eigenvalues of $M$ 

$$\rho(\lambda) = \frac{1}{2N} \text{Tr} \delta(\lambda - M) \quad (2.4)$$

to that of $C^\dagger C$

$$\tilde{\rho}(r) = \frac{1}{N} \text{Tr} \delta(r - C^\dagger C) \quad (2.5)$$

as

$$\rho(\lambda) = |\lambda| \tilde{\rho}(\lambda^2). \quad (2.6)$$

If we integrate out as usual over the unitary group, we are left with integrals over the $N$ eigenvalues $r_i$ of $C^\dagger C$, which run from zero to infinity. The Jacobian of this change of variables is simply the square of the Van der Monde determinant of the $r_i$’s. The probability measure is then

$$P(r_1, \cdots, r_N) = \frac{1}{Z} \exp(-N \sum_{i=1}^{N} r_i) \Delta^2(r_1, \cdots, r_N). \quad (2.7)$$

All the correlation functions are known to be expressible in terms of the kernel

$$K(r, s) = \exp\left(-\frac{N(r + s)}{2}\right) \sum_{0}^{N-1} L_n(Nr)L_n(Ns). \quad (2.8)$$

in which the polynomials $L_n(x)$ are the Laguerre polynomials, orthogonal on the half-line with the measure $\exp(-x)$. The density of eigenvalues of $C^\dagger C$ is then given by

$$\frac{1}{N} \text{Tr} \delta(r - C^\dagger C) = K(r, r). \quad (2.9)$$
Using the Christoffel-Darboux identity, and the asymptotic behavior of $L_n(Nr)$ for $n$ of order $N$, we could obtain the desired density of state from this method. But not only is this method more cumbersome for this one-matrix case, also it cannot be generalized to the lattice of matrices that we discuss below. We shall return to it later after we considered the cross-over near the origin.

3 Kazakov’s method extended to complex matrices

3.1 Contour integral

For the usual Gaussian ensemble of random hermitian matrices, Kazakov has introduced a curious, but very powerful, method [8]. It consists of adding to the probability distribution a matrix source, which will be set to zero at the end of the calculation, leaving us with a simple integral representation for finite $N$. However one cannot let the source go to zero before one reaches the final step. Let us follow the same lines for the model at hand. We modify the probability distribution of the matrix by a source $A$, an $N \times N$ hermitian matrix with eigenvalues $(a_1, \ldots, a_N)$:

$$P_A(C) = \frac{1}{Z_A} \exp(-N\text{Tr}C^\dagger C - N\text{Tr}AC^\dagger C). \quad (3.1)$$

Next we introduce the Fourier transform of the average resolvent with this modified distribution:

$$U_A(t) = \langle \frac{1}{N} \text{Tr} e^{itC^\dagger C} \rangle \quad (3.2)$$

from which we recover, after letting the source $A$ go to zero,

$$\tilde{\rho}(r) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-itr} U_0(t). \quad (3.3)$$

Without loss of generality we can assume that $A$ is a diagonal matrix. We first integrate over the unitary matrix $V$ which diagonalizes $C^\dagger C$. This is done through the well-known Itzykson-Zuber integral over the unitary group.
\[ \int dV \exp(\text{Tr}AVBV^\dagger) = \frac{\det(\exp a_i b_j)}{\Delta(A) \Delta(B)} \]  
\text{(3.4)}

where \( \Delta(A) \) is the Van der Monde determinant constructed with the eigenvalues of \( A \):

\[ \Delta(A) = \prod_{i<j} (a_i - a_j). \]  
\text{(3.5)}

We are then led to

\[ U_A(t) = \frac{1}{Z_A} \frac{1}{N} \sum_{\alpha=1}^N \int dr_1 \cdots dr_N e^{it r_{\alpha}} \Delta(r_1, \ldots, r_N) \times \exp(-N \sum_{i=1}^N r_i - N \sum_{j=1}^N r_j a_j). \]  
\text{(3.6)}

Then we have to integrate over the \( r_i \)'s. It is easy to prove that

\[ \int dr_1 \cdots dr_N \Delta(r_1, \ldots, r_N) \exp(- \sum_{i=1}^N r_i b_i) = (-1)^{N(N-1)} \left( \prod_{k=0}^{N-1} k! \right) \frac{\Delta(b_1, \ldots, b_N)}{(\prod_i b_i)^N}. \]  
\text{(3.7)}

With the normalization \( U_A(0) = 1 \), we could always divide, at any intermediate step of the calculation, the expression we obtain for \( U_A(t) \) by its value at \( t = 0 \), and thus the overall multiplicative factors in (3.6) and (3.7) are not needed. They are displayed explicitly only for the sake of completeness.

We now apply this identity to the \( N \) terms of (3.6), with

\[ b^{(\alpha)}_\beta(t) = N(1 + a_\beta - \frac{it}{N} \delta_{\alpha,\beta}) \]  
\text{(3.8)}

and obtain

\[ U_A(t) = \frac{1}{N} \sum_{\alpha=1}^N \prod_{\beta=1}^N \left( \frac{1 + a_\beta - \frac{it}{N} \delta_{\alpha,\beta}}{1 + a_\beta - \frac{it}{N} \delta_{\alpha,\beta}} \right)^N \prod_{\beta<\gamma} \frac{a_\beta - a_\gamma - \frac{it}{N} (\delta_{\alpha,\beta} - \delta_{\alpha,\gamma})}{a_\beta - a_\gamma} \]  
\text{(3.9)}
This sum over $N$ terms may be conveniently replaced by a contour integral in the complex plane:

$$U_A(t) = -\frac{1}{it} \oint_{\gamma} \frac{du}{2\pi i} \left( \frac{1 + u}{1 + u - \frac{it}{N}} \right)^N \prod_{\gamma=1}^{N} \frac{u - a_\gamma - \frac{it}{N}}{u - a_\gamma}$$  \hspace{1cm} (3.10)

in which the contour encloses all the $a_\gamma$'s and no other singularity. It is now, and only now, possible to let all the $a_\gamma$'s go to zero. We thus obtain a simple expression for $U_0(t)$,

$$U_0(t) = \frac{i}{t} \oint_{\gamma} \frac{du}{2\pi i} \left( \frac{1 - \frac{it}{Nu}}{1 - \frac{it}{N(1+u)}} \right)^N$$  \hspace{1cm} (3.11)

Note that this representation as a contour integral over one single complex variable is exact for any finite $N$, including $N = 1$.

### 3.2 Semi-circle law

In the large $N$ limit, for finite $t$, the integrand has the limit $e^{\frac{it}{u(1-u)}}$ and therefore for large $N$, finite $t$, $U_0(t)$ approaches

$$U_0(t) = \frac{1}{it} \int \frac{du}{2\pi i} e^{\frac{it}{u(1-u)}}$$  \hspace{1cm} (3.12)

By the change of variables

$$u = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4}{z}} \right)$$  \hspace{1cm} (3.13)

we have $z = \frac{1}{u(1-u)}$. Then the integral of (3.12) becomes, after an integration by part,

$$U_0(t) \quad = \quad \oint \frac{dz}{2\pi i} u(z)e^{itz}$$

$$\quad = \quad -\frac{1}{2} \oint \left( 1 - \sqrt{1 - \frac{4}{z}} \right) e^{itz} \frac{dz}{2\pi i}$$

$$\quad = \quad \int_{0}^{4} \sqrt{-1 + \frac{4}{x}} e^{itz} \frac{dx}{2\pi}$$  \hspace{1cm} (3.14)
Therefore, we have from (3.3)
\[ \tilde{\rho}(\lambda) = \int \frac{dt}{2\pi} e^{-it\lambda} U_0(t) \]
\[ = \frac{1}{2\pi} \frac{\sqrt{4 - \lambda}}{\lambda} \] (3.15)
which leads to the expected semi-circle law
\[ \rho(\mu) = |\mu| \tilde{\rho}(\mu^2) = \frac{1}{2\pi} \sqrt{4 - \mu^2} \] (3.16)

### 3.3 Behavior near the origin

For \( \lambda \) small however, we need to control the large \( N \) limit of \( U_0(t) \), for large \( t \). In this regime the simplified form (3.12) of (3.11) is not valid. This is why the semi-circle law is modified at the origin. Keeping in mind that we shall consider \( \lambda \)'s of order \( 1/N \), we write
\[ \tilde{\rho}(\lambda^2) = N^2 \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{-i\tau N^2 \lambda^2} U_0(N^2 \tau) \]
\[ = \frac{N}{i} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{-i\tau x^2} \frac{1}{\tau} \oint \frac{du}{2\pi i} \left( \frac{1 - \frac{1}{Nu}}{1 - \frac{1}{N(u + i\tau)}} \right)^N \] (3.17)
in which we have defined
\[ x = N\lambda \] (3.18)
In the large \( N \), small \( \lambda \) but finite \( x \), limit, we thus obtain
\[ \tilde{\rho}(\lambda^2) = \frac{N}{i} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{-i\tau x^2} \frac{1}{i\tau} \oint \frac{du}{2\pi i} e^{-i\tau u + \frac{1}{i\tau + u} - \frac{1}{u}}. \] (3.19)

If we calculate instead
\[ I(x^2) = \frac{\partial \tilde{\rho}}{\partial x^2} \] (3.20)
we find, after changing \( u \) to \( iu \) and then \( \tau \) to \( \tau - u \),
\[ I(x^2) = -Ni \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{-\frac{1}{\tau} - i\tau x^2} \oint \frac{du}{2\pi i} e^{iu + iux^2} \] (3.21)
Using the standard definitions of Bessel functions, we have
\[
\int_{-\infty}^{+\infty} \frac{d\tau}{2\pi} e^{-ix(\tau + \frac{1}{2})} = -\frac{1}{2} J_1(2x) \quad (3.22)
\]
and thus
\[
I(x^2) = -\frac{N}{2x^2} J_1^2(2x)
\]
\[
= \frac{N}{2} \frac{d}{dx^2}[J_0^2(2x) + J_1^2(2x)] \quad (3.24)
\]
Therefore the integration with respect to \( x^2 \) is immediate, and we find
\[
\rho(\lambda) = \frac{N|\lambda|}{2} [J_0^2(2N\lambda) + J_1^2(2N\lambda)] \quad (3.25)
\]
in the cross-over regime in which \( N\lambda \) is finite. The oscillatory behavior we expected is described by Bessel functions.

### 3.4 The edge of the semi-circle

It is easy to apply this same method for studying the cross-over at the other edge of the distribution, in the vicinity of the end point \( \mu = 2 \) of the semi-circle. The derivative of the density of state \( \tilde{\rho}(\mu^2) \) with respect to \( \mu^2 \) is given by
\[
\frac{\partial \tilde{\rho}(\mu^2)}{\partial \mu^2} = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} \int \frac{du}{2\pi i} \left[ \frac{1 - it}{1 - \frac{it}{N\mu}} \right]^N e^{-it\mu^2} \quad (3.26)
\]
Changing \( t \) to \( Nt \), and then \( t \) to \( t - iu \), and also \( u \) to \( -iu \), we obtain the factorized expression,
\[
\frac{\partial \tilde{\rho}(\mu^2)}{\partial \mu^2} = -iN \int_{-\infty}^{+\infty} dt \left( \frac{t}{1 - it} \right)^N e^{-iN\mu^2t} \int \frac{du}{2\pi i} \left( \frac{1 - iu}{u} \right)^N e^{-NX^2u} \quad (3.27)
\]
The integration over \( t \) is easily expressible with the help of a Laguerre polynomial:
\[
I_N(\mu^2) = \int_{-\infty}^{+\infty} dt \left( \frac{t}{1 - it} \right)^N e^{-iN\mu^2t} = -2\pi i^N e^{-NX^2} L'_N(N\mu^2) \quad (3.28)
\]
The contour integral over \( u \) turns out to be also expressible as a derivative of a Laguerre polynomial; we end up with the simple expression for (3.26),

\[
\frac{\partial \tilde{\rho}(\mu^2)}{\partial \mu^2} = -N e^{-N\mu^2} [L'_N(N\mu^2)]^2
\] (3.29)

Using standard identities for Laguerre polynomials it is easy to verify that this leads to

\[
\tilde{\rho}(\mu^2) = N e^{-N\mu^2} [L_N(N\mu^2) L'_{N-1}(N\mu^2) - L_{N-1}(N\mu^2) L'_N(N\mu^2)]
\] (3.30)

The orthogonal polynomial method, which led to the expression (2.8), may be cast into this form through Christoffel-Darboux identity. However for our purpose, the cross-over distribution near the edge of the semi-circle, it is much more convenient to return to the integral \( I_N \) of (3.28) and to use the saddle point method:

\[
I_N = \int_{-\infty}^{+\infty} dt e^{-NS_{\text{eff}}}
\] (3.31)

where \( S_{\text{eff}} \) is given by

\[
S_{\text{eff}} = -\ln t + \ln(1 - it) + i\mu^2 t.
\] (3.32)

The saddle points \( t_c \) become degenerate at \( \mu = 2 \), since

\[
t_c = -i \pm \sqrt{\frac{4}{\mu^2} - 1}.
\] (3.33)

Then we change variables to

\[
\mu = 2 + N^{-\alpha} x, \\
t = -\frac{i}{2} + N^{-\beta} \tau
\] (3.34)

and expand \( S_{\text{eff}} \) up to \( \tau^3 \). This leads to

\[
S_{\text{eff}}(t) = 2 + \frac{\pi i}{2} + 2N^{-\alpha} x \\
+ \frac{16}{3} \tau^3 N^{-3\beta} + 4i N^{-\alpha-\beta} \tau x
\] (3.35)
We thus find that there is a large \( N \), finite \( x \) limit, provided we fix the two unknown parameters \( \alpha \) and \( \beta \) to
\[
\alpha = \frac{2}{3}, \quad \beta = \frac{1}{3} \quad \text{(3.36)}
\]
We repeat this for the \( u \)-integral of (3.27). We then find that the leading terms of (3.35) of order 1, as well as the term \( 2xN^{-2/3} \), cancel with terms of opposite signs in the \( u \)-integral. Thus we obtain the following expression for the density of state near the critical value \( \mu = 2 \),
\[
\frac{\partial \tilde{\rho}(\mu^2)}{\partial \mu^2} = -N^\frac{2}{3}4^{-\frac{4}{3}} |A_i[4^{\frac{4}{3}}N^\frac{2}{3}(\mu - 2)]|^2 \quad \text{(3.37)}
\]
where the Airy function \( A_i(z) \) is defined as
\[
A_i[(3a)^{-1/3}x] = \frac{(3a)^{1/3}}{\pi} \int_0^\infty \cos(\sqrt{3}t + xt)dt. \quad \text{(3.38)}
\]
This Airy function is smoothly decreasing for \( \mu > 2 \) but it gives oscillations for \( \mu < 2 \).

3.5 Two-point correlation

The application of this method to the two-point correlation function is straightforward; it leads to a compact and useful integral representation. Let us briefly describe the procedure. Introducing the same source \( A \) in the probability distribution as in (3.1), we consider the Fourier transform of the average two-point correlation:
\[
U^{(2)}_A(t_1, t_2) = \langle \frac{1}{N} \text{Tr} e^{it_1 C^\dagger C} \frac{1}{N} \text{Tr} e^{it_2 C^\dagger C} \rangle \quad \text{(3.39)}
\]
from which we shall compute the two-point correlation function \( \tilde{\rho}^{(2)}(\lambda_1, \lambda_2) \), after letting the source \( A \) goes to zero. The normalization conditions are then
\[
U^{(2)}_A(t_1, t_2) = U^{(2)}_A(t_2, t_1),
U^{(2)}_A(t, 0) = U^{(1)}_A(0),
U^{(1)}_A(0) = 1. \quad \text{(3.40)}
\]
After performing the Itzykson-Zuber integral over the unitary group as in (3.6), we obtain through the same procedure,

\[ U^{(2)}_A(t_1, t_2) = \frac{1}{N^2} \sum_{\alpha_1, \alpha_2} \int \prod_{i=1}^N dr_i \frac{\Delta(r)}{\Delta(A)} e^{-N \sum_{i=1}^N r_i(1+a_i)+it_1 r_{\alpha_1} + it_2 r_{\alpha_2}} \]  

(3.41)

where we omitted the overall normalization, as we are allowed to do (as explained earlier). The integration of \( r_i \) is again done with the use of (3.42), in which now

\[ b_{\beta}^{(\alpha_1, \alpha_2)} = N(1 + a_\beta - \frac{1}{N}(it_1 \delta_{\beta, \alpha_1} + it_2 \delta_{\beta, \alpha_2})). \]  

(3.42)

Thus we have the following expression for \( U^{(2)}_A(t_1, t_2) \) after restoring the normalization,

\[ U^{(2)}_A(t_1, t_2) = \frac{1}{N^2} \sum_{\alpha_1, \alpha_2=1}^N \prod_{\beta=1}^N \left[ \frac{1 + a_\beta}{1 + a_\beta - \frac{it_1}{N}\delta_{\beta, \alpha_1} + \frac{it_2}{N}\delta_{\beta, \alpha_2}} \right]^N \]

\[ \times \prod_{\beta < \gamma} a_\beta - a_\gamma - \frac{it_1}{N}(\delta_{\beta \alpha_1} - \delta_{\gamma \alpha_1}) - \frac{it_2}{N}(\delta_{\beta \alpha_2} - \delta_{\gamma \alpha_2}) \]

(3.43)

Keeping track of all the terms in which the Kronecker \( \delta_{\alpha, \beta} \)'s do not vanish, we obtain

\[ U^{(2)}_A(t_1, t_2) = \frac{1}{N} U^{(1)}_A(t_1 + t_2) + \frac{1}{N^2} \sum_{\alpha_1, \alpha_2} \left[ \frac{(1 + a_{\alpha_1})(1 + a_{\alpha_2})}{(1 + a_{\alpha_1} - \frac{it_1}{N})(1 + a_{\alpha_2} - \frac{it_2}{N})} \right]^N \]

\[ \times \prod_{\gamma \neq (\alpha_1, \alpha_2)} \frac{(a_{\alpha_1} - a_{\alpha_2} - \frac{it_1}{N})}{(a_{\alpha_1} - a_{\alpha_2})} \frac{(a_{\alpha_1} - a_{\gamma} - \frac{it_1}{N})(a_{\alpha_2} - a_{\gamma} - \frac{it_2}{N})}{(a_{\alpha_1} - a_{\gamma})(a_{\alpha_2} - a_{\gamma})} \]  

(3.44)

The Fourier transform of the first term of (3.44) corresponds to

\[ \frac{1}{N(2\pi)^2} \int dt_1 dt_2 e^{-it_1 \lambda_1 - it_2 \lambda_2} U^{(1)}_0(t_1 + t_2) = \frac{1}{N(2\pi)^2} \delta(\lambda_1 - \lambda_2) \rho_0(\lambda_1) \]  

(3.45)

It could thus be omitted for \( \lambda_1 \neq \lambda_2 \) but, remarkably enough, the contour integral that we shall now consider, will be simpler if we retain this term.
Indeed let us consider the integral over two complex variables \( u \) and \( v \)

\[
U^\vphantom{(2)}_A(t_1, t_2) = \frac{1}{t_1 t_2} \int \frac{dudv}{(2\pi i)^2} \left[ \frac{(1 + u)(1 + v)}{(1 + u - \frac{it_1}{N})(1 + v - \frac{it_2}{N})} \right]^N \prod_\gamma \left( \frac{u - a_\gamma - \frac{it_1}{N}}{u - a_\gamma} \right) \left( \frac{v - a_\gamma - \frac{it_2}{N}}{v - a_\gamma} \right) \left( \frac{u - v - \frac{it_1 - it_2}{N}}{u - v - \frac{it_1}{N}} \right) \left( \frac{u - v + \frac{it_2}{N}}{u - v + \frac{it_2}{N}} \right). \tag{3.46}
\]

It is straightforward to verify that this expression reduces exactly to (3.44), provided we choose the following contours: we first integrate over a contour in \( v \) which circles around the \( a_\alpha \)'s and no other singularity. Then taking a residue at \( v = a_\alpha^2 \), we integrate over a contour in \( u \) which surrounds the \((N-1)\) poles \( u = a_\alpha \), with \( \alpha_1 \neq \alpha_2 \); these poles yield the sum over \( \alpha_1 \) and \( \alpha_2 \) of (3.44). The contour in \( u \) has to surround also the pole \( u = a_{\alpha_2} - \frac{it_2}{N} \).

Remarkably, this last pole reproduces exactly the first term \( \frac{1}{N} U^\vphantom{(1)}_A(t_1 + t_2) \) of (3.44).

We are now again in position to let all the \( a_\gamma \)'s go to zero:

\[
U^\vphantom{(2)}_0(t_1, t_2) = \frac{1}{t_1 t_2} \int \frac{dudv}{(2\pi i)^2} \left[ \frac{(1 + u)(1 + v)}{(1 + u - \frac{it_1}{N})(1 + v - \frac{it_2}{N})} \right]^N (1 - \frac{it_1}{Nu})^N \times (1 - \frac{it_2}{Nv})^N \left[ 1 - \frac{t_1 t_2}{N^2(u - v - \frac{it_1}{N})(u - v + \frac{it_2}{N})} \right] \tag{3.47}
\]

The last bracket in (3.47) is a difference of two terms. Keeping the one in this bracket we obtain the disconnected part \( U^\vphantom{(1)}_0(t_1)U^\vphantom{(1)}_0(t_2) \) of \( U^\vphantom{(2)}_0(t_1, t_2) \). The second term gives thus the connected two-point correlation \( \tilde{\rho}^\vphantom{(2)}_c(\lambda_1, \lambda_2) \).

In the large \( N \) limit, the connected two-point correlation may be then immediately obtained. For finite \( t_1 \) and \( t_2 \) we have, in the large \( N \) limit

\[
U^\vphantom{(2)}_c(t_1, t_2) = -\frac{1}{N^2 (2\pi i)^2} \int \frac{dudv}{(u - v)^2 e^{ - \frac{it_1}{u(1+u)} - \frac{it_2}{v(1+v)}}} \tag{3.48}
\]

Changing variables to

\[
z_1 = \frac{1}{u(1+u)} , \quad z_2 = \frac{1}{v(1+v)} \tag{3.49}
\]

and denoting

\[
u(z) = \frac{-1 + \sqrt{1 + \frac{4}{z}}}{2}, \tag{3.50}
\]

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we have

$$U_c^{(2)}(t_1, t_2) = \frac{t_1 t_2}{N^2} \int dz_1 dz_2 e^{-i t_1 z_1 - i t_2 z_2} \ln[u(z_1) - u(z_2)].$$

(3.51)

Indeed we have used $\frac{\partial}{\partial z_2} \frac{\partial}{\partial z_1} \ln(u - v) = \frac{1}{(u-v)^2}(\frac{\partial u}{\partial z_1})(\frac{\partial v}{\partial z_2})$, and then integrated by part over $z_1$ and $z_2$. We then Fourier transform over $t_1, t_2$:

$$\tilde{\rho}_c^{(2)}(\lambda_1, \lambda_2) = -\frac{1}{N^2} \frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} \ln[u(\lambda_1) - u(\lambda_2)].$$

(3.52)

This result could be derived by other methods, and indeed has been obtained in a somewhat different form by Ambjorn and Makeenko and others [1]. The derivation given here however gives the correlation function even for finite $N$. One can check easily that the derivation that we have given here through Kazakov’s representation, could be repeated for calculating the two-point correlation function in the unitary ensemble, in which the universality of the two-point correlation function has been studied [2, 3, 4, 5, 6, 7, 8].

It is not difficult to verify that the above expression for the connected two-point correlation is a compact version of the one that we would have deduced from the orthogonal polynomial method, with the kernel $K(\lambda_1, \lambda_2)$ in (2.8). Indeed, after letting the $a_\gamma$’s go to zero, we determine the connected two-point correlation function to be

$$\tilde{\rho}_c^{(2)}(\lambda_1, \lambda_2) = -\int \frac{dt_1 dt_2}{(2\pi)^2} \int \frac{du dv}{(2\pi i)^2} e^{-i N t_1 \lambda_1 - i N t_2 \lambda_2}$$

$$\times \frac{1}{(u - v - it_1)(u - v + it_2)}$$

(3.53)

This expression takes a factorized form if we shift $t_1$ to $t_1 - i u$ and $t_2$ to $t_2 - i v$:

$$\tilde{\rho}_c^{(2)}(\lambda_1, \lambda_2) = -I(N \lambda_1, N \lambda_2) I(N \lambda_2, N \lambda_1)$$

(3.54)

in which we have defined

$$I(\lambda_1, \lambda_2) = \int \frac{dt}{2\pi} \int \frac{dv}{2\pi i} \left( \frac{t}{t+i} \right)^N \left( \frac{1 - iv}{v} \right)^N \frac{1}{t-v} e^{-it \lambda_1 + iv \lambda_2}$$

(3.55)
Taking the residues at the poles $t = -i$ and $v = 0$, we find

$$I(\lambda_1, \lambda_2) = e^{-\lambda_1} \sum_{n=0}^{N-1} L_n(\lambda_1) L_n(\lambda_2)$$

(3.56)

Thus $I(\lambda_1, \lambda_2)$ is simply an integral representation for the kernel $K(\lambda_1, \lambda_2)$ in (2.8). The connected two-point correlation function $\rho_c^{(2)}(\mu, \nu)$ is expressible through the Christoffel-Darboux identity,

$$\rho_c^{(2)}(\mu, \nu) = \mu
\nu \tilde{\rho}_c^{(2)}(\mu^2, \nu^2) = \mu\nu \frac{[L_N(N\mu^2)L_{N-1}(N\nu^2) - L_N(N\nu^2)L_{N-1}(N\mu^2)]}{(\mu^2 - \nu^2)^2}$$

(3.57)

4 Use of Grassmannian variables

In the previous section we have used a source representation, which is powerful and simple. However we would like to investigate the same question of the edge behavior near the origin for more complicated ensembles of block matrices which arise when the randomness is due to random couplings between neighbors on a lattice. Unfortunately we have not found any simple extension of Kazakov’s method and we have to use Grassmannian variables, as often in disordered systems, to solve the problem [9]. Before going to a lattice of matrices we return to the case that we have solved in the previous section, in order to show how to recover the same results from this new method. We shall then extend it to lattices in the next section.

We begin with the identity

$$M_{ab}^{-1} = -iN \int \prod_{1}^{N} (du_c dv_c)$$

$$\times \quad u_a^* u_b \text{exp} [i \sum_{c,d} N(u_c^* M_{cd} u_d + v_c^* M_{cd} v_d)]$$

(4.1)

in which the $u$’s are commuting variables and the $v$’s are Grassmannian. Indeed with the normalization

$$\int dv d v^* = \frac{1}{\pi}$$

(4.2)
one verifies that
\[
\int dudu^*dvdv^*\exp\left[-(u^*u + v^*v)\right] = 1 \quad (4.3)
\]
In order to apply this to the matrix
\[
(z - M) = \begin{pmatrix} z & -C^\dagger \\ -C & z \end{pmatrix}, \quad (4.4)
\]
which is \(2N \times 2N\), we have to decompose the \(2N\)-component vectors \(u\) and \(v\), into \(N\)-component vectors;
\[
u = \begin{pmatrix} a \\ b \end{pmatrix}, v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (4.5)
\]
and we obtain
\[
\frac{1}{2N} \text{Tr} \frac{1}{z - M} = -\frac{i}{2} \int \prod_1^N \left( da_c da^*_c db_c db^*_c d\alpha_c d\alpha^*_c d\beta_c d\beta^*_c \right) \\
\left[ (a^* \cdot a) + (b^* \cdot b) \right] \exp \left[ iNz[(a^* \cdot a) + (b^* \cdot b) + (\alpha^* \cdot \alpha) + (\beta^* \cdot \beta)] \right] \\
-iN[a^*_c C^\dagger_{cd} b_d + \alpha^*_c C^\dagger_{cd} \beta_d + b^*_c C_{cd} a_d + \beta^*_c C_{cd} \alpha_d] \quad (4.6)
\]
The average over the Gaussian distribution for the matrix \(C\)
\[
P(C) = \frac{1}{Z} \exp \left( -N\text{Tr}C^\dagger C \right) \quad (4.7)
\]
is then easily performed, since
\[
< \exp iN\text{Tr}(\lambda C + \mu C^\dagger) >= \exp \left( -N\text{Tr}(\lambda \mu) \right) \quad (4.8)
\]
This gives for the average resolvent
\[
G(z) = < \frac{1}{2N} \text{Tr} \frac{1}{z - M} > \\
= -\frac{i}{2} \int \prod_1^N \left( da_c da^*_c db_c db^*_c d\alpha_c d\alpha^*_c d\beta_c d\beta^*_c \right) \left[ (a^* \cdot a) + (b^* \cdot b) \right] \exp \left[ iNz[(a^* \cdot a) + (b^* \cdot b) + (\alpha^* \cdot \alpha) + (\beta^* \cdot \beta)] - N[(a^* \cdot a)(b^* \cdot b) - (a^* \cdot \alpha)(\beta^* \cdot \beta) + (\alpha^* \cdot \alpha)(\beta^* \cdot b) + (\alpha^* \cdot b)(\beta^* \cdot \alpha)] \right] \quad (4.9)
\]
Note that there is a minus sign in front of the four Fermi interaction due to the Grassmannian algebra. This four Fermi term may be replaced by an additional integration, since

$$\exp N(\alpha^* \cdot \alpha) (\beta^* \cdot \beta) = \frac{N}{\pi} \int d^2 \sigma \exp \left( -N[\sigma^* \sigma + \sigma^*(\alpha^* \cdot \alpha) + \sigma(\beta^* \cdot \beta)] \right)$$  \hspace{1cm} (4.10)$$

Substituting this into (4.9) we can now perform the integration over the anti-commuting variables,

$$\int \prod_{\alpha=1}^{N} (d\alpha_d d\alpha_\ast_d d\beta_d d\beta_\ast_d) \exp(-N[(\alpha^* \cdot \alpha)(\sigma^* - iz) + (\beta^* \cdot \beta)(\sigma - iz)])$$

$$-N[(\alpha^* \cdot a)(b^* \cdot \beta) + (\beta^* \cdot b)(a^* \cdot \alpha)]$$

$$= (\frac{N}{\pi})^{2N} \det \left( \begin{array}{c|c} \sigma^* - iz & |a > b| \\ \hline |b > a| & \sigma - iz \end{array} \right)$$

$$= (\frac{N}{\pi})^{2N} [((\sigma^* - iz)(\sigma - iz)]^{N-1}[(\sigma^* - iz)(\sigma - iz) - (a^* \cdot a)(b^* \cdot b)]$$  \hspace{1cm} (4.11)$$

We are then led to

$$G(z) = -\frac{i}{2\pi} (\frac{N}{\pi})^{2N} \int d^2 \sigma \prod_{\alpha=1}^{N} (d\alpha_d d\alpha_\ast_d d\beta_d d\beta_\ast_d) [(\alpha^* \cdot a) + (b^* \cdot b)]$$

$$[((\sigma^* - iz)(\sigma - iz)]^{N-1}[(\sigma^* - iz)(\sigma - iz) - (a^* \cdot a)(b^* \cdot b)]$$

$$\exp(iNz[(\alpha^* \cdot a) + (b^* \cdot b)] - N[(\alpha^* \cdot a)(b^* \cdot b) + \sigma^* \sigma])$$  \hspace{1cm} (4.12)$$

The integrand is a function of the lengths of the complex vectors $a$ and $b$; integrating over the angles:

$$\int \prod_{\alpha=1}^{N} (d\alpha_d d\alpha_\ast_d) f(a^*, a) = \frac{\pi^N}{(N-1)!} \int_0^\infty dx x^{N-1} f(x)$$  \hspace{1cm} (4.13)$$

we end up with

$$G(z) = -\frac{i}{2\pi} (\frac{N}{\pi})^{2N+3} \int d^2 \sigma \int_0^\infty dx \int_0^\infty dy (xy)^{N-1}(x + y)$$

$$[((\sigma^* - iz)(\sigma - iz)]^{N-1}[(\sigma^* - iz)(\sigma - iz) - xy]$$

$$\exp(N[iz(x + y) - xy - \sigma^* \sigma])$$  \hspace{1cm} (4.14)$$
This representation of the average resolvent in terms of an integral over four variables is exact for any \( N \).

Let us note that, if we had calculated, instead of the average resolvent, the average value of 1, we would have obtained by the same method, the identity:

\[
\frac{1}{\pi} \frac{N^{2N+3}}{(N!)^2} \int d^2 \sigma \int_0^\infty dx \int_0^\infty dy (xy)^{N-1} [(\sigma^* - iz)(\sigma - iz)]^{N-1} \\
[(\sigma^* - iz)(\sigma - iz) - xy] \exp(N[iz(x+y) - xy - \sigma^* \sigma]) = 1 \quad (4.15)
\]

This identity should manifestly hold, but it is not quite trivial to check it; since it provides an interesting verification of the consistency of the formation, let us note that one can derive it by replacing, in the l.h.s. of (4.15), the bracket \([(\sigma^* - iz)(\sigma - iz) - xy] \) by

\[
\frac{1}{N^2} \left( \frac{\partial^2}{\partial \sigma \partial \sigma^*} - \frac{\partial^2}{\partial x \partial y} \right) + \frac{iz}{N} \left( \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \sigma^*} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \quad (4.16)
\]

followed by integrations by part and a few lengthy manipulations. This identity will be useful in a few moments.

The expression (4.14) of \( G(z) \) is well suited to study the large \( N \)-limit. The integrand involves a factor \( \exp(-NS) \), with

\[
S(\sigma, \sigma^*, x, y) = -iz(x+y) + xy + \sigma^* \sigma - \ln[xy(\sigma^* - iz)(\sigma - iz)] \quad (4.17)
\]

and the large \( N \) limit is therefore governed by the saddle-point at which \( S \) is stationary. The equations for the saddle-point lead, away from the vicinity of \( z = 0 \), to the equations

\[
x_c = y_c = \sigma_c = \sigma_c^* = \frac{1}{2} [iz + \sqrt{4 - z^2}] \quad (4.18)
\]

The sign of the square root is chosen so that the imaginary part of \( G(z) \) above the cut is negative; note that in the saddle-point method \( \sigma \) and \( \sigma^* \) become independent complex variables. In order to obtain \( G(z) \) with the proper normalization we should include the Gaussian fluctuations around this saddle-point. It presents no difficulty; however we can bypass the whole calculation, if we note that the integrand in (4.14) differs from that of (4.13) by a factor \(-i(x+y)/2\); if we had calculated (4.13) by the same saddle-point
technique we would have obtained one, a very cumbersome method to get one indeed. But this immediately tells us that, for $N$ large,

$$G(z) = -\frac{i}{2}(x_c + y_c) = \frac{1}{2}[z - i\sqrt{4 - z^2}]$$  \hspace{1cm} (4.19)

from which one recovers the semi-circle law

$$\rho(\lambda) = -\frac{1}{\pi} \text{Im} G(\lambda + i0) = \frac{1}{2\pi} \sqrt{4 - \lambda^2}$$  \hspace{1cm} (4.20)

However this solution is not valid near the vicinity of the origin. Indeed the saddle-point equations gives $z(x_c - y_c) = z(\sigma_c - \sigma_c^*) = 0$; another way of realizing that there is a problem near $z = 0$, is to calculate the determinant of the Gaussian fluctuations near the saddle-point, which vanishes when $x_c^4 = 1$, i.e. for $z = \pm 2$ or $z = 0$. Near the end-points of the semi-circle the phenomenon is well-known and leads to an Airy type cross-over function in a region of size $N^{-2/3}$ near the edge, which we have discussed in the previous section. Near the origin, we need a separate analysis. We will focus on a range of size $1/N$, in which the variable

$$\zeta = Nz$$  \hspace{1cm} (4.21)

is finite. We write $\sigma = u + iv$, $\sigma^* = u - iv$ and translate the real $u$- contour by $iz$; this gives

$$G(\zeta) = -\frac{i}{2\pi} \frac{N^{2N+3}}{(N!)^2} \int du dv \int_0^\infty dx \int_0^\infty dy (xy)^{N-1} (x + y)[u^2 + v^2]^{N-1}$$

$$[u^2 + v^2 - xy] \exp(-N[xy + u^2 + v^2]) \exp(i\zeta(x + y)) \exp(-2iu\zeta + \frac{1}{N} \zeta^2)$$  \hspace{1cm} (4.22)

Going into radial variables for $u$ and $v$, and changing $x$, $y$ to $p$ and $q$ with $xy = p$, $x + y = 2\sqrt{pq}$ (Jacobian $J = (q^2 - 1)^{-1/2}$, domain of integration $p > 0$, $q > 1$), we obtain easily, if we drop terms of order $1/N$,

$$G(\zeta) = -\frac{iN^2}{4\pi^2} \int_0^{2\pi} d\theta \int_1^\infty dq \int_0^\infty dr \int_0^\infty dp \frac{r}{\sqrt{q^2 - 1}} \int_0^{\ln(pr)} \frac{dp}{\sqrt{p}} (r - p)$$

$$\exp(-N[p + r - 2 - \ln(pr)]) \exp(2i\zeta q \sqrt{p}) \exp[-2i\zeta \sqrt{r \cos \theta}]$$  \hspace{1cm} (4.23)
The remaining integrals over $p$ and $r$ are governed, in the large $N$ limit, by their saddle-points at $p = r = 1$. However the presence of the odd factor $(r - p)$ in the integrand implies to expand beyond the leading non-Gaussian order. Changing variable $q$ to $\cosh \phi$, we integrate $\phi$ and $\theta$,

$$
\int_0^\infty d\phi \cosh \phi e^{2i\zeta \sqrt{r} \cosh \phi} = K_1(-2i\zeta \sqrt{p}) = -\pi/2 [J_1(2N\mu \sqrt{p}) + i N_1(2N\mu \sqrt{p})] \quad (4.24)
$$

in which $N\mu$ is the real part for $\zeta$, i.e. the eigenvalue of the matrix multiplied by $N$;

$$
\int_0^{2\pi} \exp(-2iN\mu \sqrt{r} \cos \theta) d\theta = 2\pi J_0(2N\mu \sqrt{r}) \quad (4.25)
$$

we have for the imaginary part of $G(\zeta)$

$$
\rho(\mu) = -\frac{N^2}{4\pi} \int_0^\infty \frac{dr}{r} \int_0^\infty \frac{dp}{\sqrt{p}} (r - p) \exp(-N[p + r - 2 - \ln(pr)]) J_1(2N\mu \sqrt{p}) J_0(2N\mu \sqrt{r})
$$

Expanding the Bessel functions $J_1(2N\mu \sqrt{p})$ and $J_0(2N\mu \sqrt{r})$, with $r = 1 + r'$ and $p = 1 + p'$ up to linear order, we get

$$
J_1(2N\mu \sqrt{p}) = J_1(2N\mu) + p'[N\mu J_0(2N\mu) - \frac{1}{2} J_1(2N\mu)] \quad (4.27)
$$

$$
J_0(2N\mu \sqrt{r}) = J_0(2N\mu) - N\mu r' J_1(2N\mu) \quad (4.28)
$$

Then, by the Gaussian integration over $p'^2$ and $r'^2$, we have

$$
\rho(\mu) = \frac{N|\mu|}{2} [J_0^2(2N\mu) + J_1^2(2N\mu)]
$$

(4.29)

Since $J_0(x) \simeq 1$ and $J_1(x) \simeq x/2$ near $x = 0$, the density of state $\rho(\mu)$ is proportional to $\mu$ for small $\mu$. The density of states that we have found agrees with (3.25). If we average over these oscillations with the appropriate width, we find $<\rho(\mu)> = 1/\pi$, the value that it takes at the origin in the saddle-point method for the large $N$ limit (4.20).
5  Ring of matrices

We now extend the previous Grassmannian method to the case of a lattice of matrices. We consider a ring of \( L \) points, with \( L \) even, in which the neighboring sites are coupled by complex \( N \times N \) matrices. The previous section corresponded to \( L = 2 \), a lattice of two points, coupled by the matrix \( C \) for one orientation of the link, and \( C^\dagger \) for the opposite orientation. The simplest extension, \( L = 4 \) consists of a \( 4N \times 4N \) matrix given by

\[
M = \begin{pmatrix}
0 & C_1^\dagger & 0 & C_4 \\
C_1 & 0 & C_2^\dagger & 0 \\
0 & C_2 & 0 & C_3^\dagger \\
C_4^\dagger & 0 & C_3 & 0
\end{pmatrix}
\] (5.1)

where \( C_i \) is a \( N \times N \) complex matrix. This matrix \( M \) represents a random hopping between the nearest neighbour sites of a lattice of four points on a ring. The disorder is off-diagonal, and the hopping terms are \( N \times N \) random complex matrices. This matrix \( M \) has also pairs of opposite real eigenvalues, since it anticommutes with the “\( \gamma_5 \)” matrix. However this \( L = 4 \) case (5.1) happens to be reducible to the previous case through the orthogonal transformation which exchanges indices two and three; namely if we change \( M \) to \( P^{-1}MP \) with \( P(1,1) = P(2,3) = P(3,2) = P(4,4) = 1 \), and all other \( P(i,j) \)'s equal zero, one sees easily that the problem is mapped into the \( L = 2 \) case with \( N \) replaced by \( 2N \). This is not true however for larger rings.

For \( L \) even the matrix \( M \) has again pairs of opposite real eigenvalues since it anti-commutes with the “\( \gamma_5 \)” matrix made of \( L \) block matrices consisting successively of the unit matrix and of minus the unit matrix. One can thus consider again the problem of the behaviour of the density of eigenvalues in the scaling range near the origin. We shall prove that the previous result still holds up to scale factors.

The formulation of the previous section for \( 2N \times 2N \) matrices, may be easily extended to the \( LN \times LN \) matrices, corresponding to a one dimensional lattice. The \( 2N \) component vectors \( u \) and \( v \) in (4.5) are now \( LN \)-component vectors; they are conveniently decomposed into \( N \)-component vectors, \( a_i, \alpha_i \) \((i = 1, \cdots, L)\) where the \( \alpha \)'s are Grassmannian variables:

\[
u^* = (a_1^*, \cdots, a_L^*)
\]

\[
v^* = (\alpha_1^*, \cdots, \alpha_L^*)
\] (5.2)
Since the matrices $C_i (i = 1, \ldots, L)$ are associated with the hopping between the sites $i$ and $i + 1$, we have

$$\frac{1}{LN} \text{Tr} \frac{1}{z - M} = -\frac{i}{L} \int \prod_{i=1}^{N} da_i \cdots da_{Li}^* da_{Li} \cdots \alpha_{Li}^*$$

$$[(a_i^* \cdot a_1) + \cdots + (a_L^* \cdot a_L)] \exp(iNz[(a_i^* \cdot a_1) + \cdots + (a_L^* \cdot a_L)$$

$$+(\alpha_i^* \cdot a_1) + \cdots + (\alpha_L^* \cdot a_L) - iN[a^*_i c_i(C_i^t) c_d a_{2d} + \cdots + \alpha_{Li}^* (C_{Li}) c_d a_{1d}]$$

$$= \det \begin{pmatrix}
S_1 & |a_1 >< a_2| & 0 & \cdots & 0 & |a_1 >< a_L| \\
|a_2 >< a_1| & S_2 & |a_2 >< a_3| & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
|a_L >< a_1| & 0 & 0 & \cdots & |a_L >< a_{L-1}| & S_L
\end{pmatrix}
\times \left( \frac{N}{\pi} \right)^{LN}$$

$$T = \int d\alpha_1 d\alpha^*_1 \cdots d\alpha_L d\alpha^*_L \exp(-N[(\alpha_1^* \cdot a_1)(\sigma_1^* + \sigma_L - iz) + \cdots$$

$$+(\alpha_L^* \cdot a_L)(\sigma_L^* + \sigma_{L-1} - iz)]$$

$$-N[(\alpha_1^* \cdot a_1)(a_2^* \cdot a_2) + \cdots + (\alpha_L^* \cdot a_L)(a_1^* \cdot a_1) + C.C.])$$

$$= \det \begin{pmatrix}
S_1 & |a_1 >< a_2| & 0 & \cdots & 0 & |a_1 >< a_L| \\
|a_2 >< a_1| & S_2 & |a_2 >< a_3| & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
|a_L >< a_1| & 0 & 0 & \cdots & |a_L >< a_{L-1}| & S_L
\end{pmatrix}
\times \left( \frac{N}{\pi} \right)^{LN}$$

where

$$S_1 = \sigma_1^* + \sigma_L - iz$$

$$S_2 = \sigma_2^* + \sigma_1 - iz$$

$$\cdots$$

$$S_L = \sigma_L^* + \sigma_{L-1} - iz$$

In the calculation of the determinant, the vector $a_i$ appears only through its squared norm denoted as $x_i = |a_i|^2$. Then the Green function $G(z)$ becomes

$$G(z) = -\frac{i}{L} \left( \frac{N}{\pi} \right)^L \int \prod d^2 \sigma d\alpha d\alpha^* (\sum_{i=1}^{L} x_i) T(\sigma, x)$$
\[ \exp(iNz \sum x_i - N[\sum \sigma_i^*\sigma_i + x_1x_2 + \cdots + x_Lx_1]) \] (5.7)

where the factor \((N/\pi)^L\) is due to the introduction of the \(\sigma_i\)'s (4.10). Changing variables from \(a_i\) to \(x_i\) (\(|a_i|^2 = x_i\)), as was done in (4), we have

\[ G(z) = -i(L/N \pi)^L \left[ \frac{\pi^N}{(N-1)!} \right]^L \int \prod d^2\sigma_i \int \prod d x_i 
(x_1 \cdots x_L)^{N-1}(x_1 + \cdots + x_L)T(\sigma, x)\exp[iNz(x_1 + \cdots + x_L)\]
\[-N \sum \sigma_i^*\sigma_i - N(x_1x_2 + \cdots + x_Lx_1)] \] (5.8)

In the large \(N\) limit, repeating again the argument of the previous section, we have a saddle-point

\[ (x_i)_c = (\sigma_i^*)_c = (\sigma_i)_c = \frac{iz + \sqrt{8 - z^2}}{4} \] (5.9)

and it leads to a semi-circle law for the density of state (note that (5.9) differs from (4.18) since we have used in this section a slightly different normalization of the probability distribution).

As before the determinant \(T(\sigma_c, x_c)\) vanishes at \(z = 0\). We then have to expand the variables of integration around these saddle points. We shift the complex variables \(\sigma_i^* - iz/2\) to \(\sigma_i^*\) and \(\sigma_i - iz/2\) to \(\sigma_i\). This gives

\[ S_1 \cdots S_L = (\sigma_1^* + \sigma_L) \cdots (\sigma_L^* + \sigma_{L-1}) \] (5.10)

It is convenient to replace the variables \(x_i\)'s, which are positive by definition, to the variables \((\lambda_1, \lambda_2, t_3, t_4, \cdots, t_L)\) defined by \((i = 1, \cdots, L)\),

\[ \lambda_1 = x_1 + x_3 + \cdots + x_{L-1} \]
\[ \lambda_2 = x_2 + x_4 + \cdots + x_L \] (5.11)

and

\[ x_1 = \lambda_1(1 - t_3 - \cdots - t_{L-1}) \]
\[ x_2 = \lambda_2(1 - t_4 - \cdots - t_L) \]
\[ x_{2n-1} = \lambda_1 t_{2n-1} \quad (n \neq 1) \]
\[ x_{2n} = \lambda_2 t_{2n} \quad (n \neq 1) \] (5.12)
The Jacobian $J$ for this transformation is $J = (\lambda_1\lambda_2)^{L/2-1}$. The Green function is then written as

$$G(z) = -\frac{i}{L} \frac{N^{2L} \pi^{L(N-1)}}{(N!)^L} \prod d^2 \sigma_i \int d\lambda_1 d\lambda_2 dt_3 \cdots dt_6 (\lambda_1\lambda_2)^{\frac{L}{2}-1}$$

$$[\lambda_1\lambda_2]^{\frac{L(N-1)}{2}} \left[(1 - t_3 - \cdots - t_{L-1})(1 - t_4 - \cdots - t_L)t_3 t_4 \cdots t_L\right]^{N-1}(\lambda_1 + \lambda_2)T$$

$$\exp(iNz(\lambda_1 + \lambda_2) - N(\lambda_1\lambda_2)h(t))$$

$$\exp(-N \sum_{i=1}^{L} \sigma_i^* \sigma_i - i \frac{zN}{2} \sum_{i=1}^{L} (\sigma_i^* + \sigma_i)) \quad (5.13)$$

in which

$$h(t) = t_1 t_L + t_2 t_1 + t_3 t_2 + \cdots + t_LT_{L-1} \quad (5.14)$$

$$t_1 = 1 - t_3 - t_5 - \cdots - t_{L-1}$$

$$t_2 = 1 - t_4 - t_6 - \cdots - t_L \quad (5.15)$$

We return now to the variables $p$ and $q$ as before,

$$\lambda_1\lambda_2 = p$$

$$\lambda_1 + \lambda_2 = 2\sqrt{pq}, \quad (5.16)$$

with Jacobian $J = 1/\sqrt{q^2 - 1}$. In the large $N$ limit, the saddle point for the $t_i$'s is

$$t_{2n-1} = t_{2n} = \frac{2}{L} \quad (5.17)$$

and $h(t_c) = 4/L$. Since $T(\sigma_c, x_c)$ is vanishing, we have to expand $\sigma_i$ and $p, q, t_i$ around their values at the saddle-point. First, we integrate over $q$ with the change of variable $q = \cosh \phi$. Then we get,

$$G(z) = -\frac{2i}{L} \frac{N^{2L} \pi^{L(N-1)}}{(N!)^L} \prod d^2 \sigma_i \int dpp^{\frac{N}{2} - \frac{1}{2}} \prod dt_i$$

$$[\lambda_1\lambda_2]^{\frac{L(N-1)}{2}} [(1 - t_3 - t_5 - \cdots - t_{L-1})(1 - t_4 - t_6 - \cdots - t_L)t_3 t_4 t_5 \cdots t_L]^{N-1} T$$

$$K_1(-2i\zeta \sqrt{p}) \exp[-Nph(t) - N \sum \sigma_i^* \sigma_i - i \frac{zN}{2} \sum (\sigma_i^* + \sigma_i)] \quad (5.18)$$

The saddle-point is now $t_c = \frac{2}{L}, p_c = \frac{L^2}{8}$ and $h(t_c) = \frac{4}{L}$. The determinant $T$ vanishes at this saddle-point. With the polar coordinate representation for $\sigma_i$,

$$\sigma_i = \sqrt{r_i} e^{i\theta_i} \quad (5.19)$$
the saddle-point appears now at
\[(r_i)_c = \frac{1}{2} \]
\[(\theta_{2n-1})_c = - (\theta_{2n})_c = \theta_1 \quad (5.20)\]
We denote the deviations from this saddle-point as \(\theta'_i\),
\[\theta_{2n-1} = \theta_1 + \theta'_{2n-1} \]
\[\theta_{2n} = - \theta_1 + \theta'_{2n} \quad (5.21)\]
with \(\theta'_1 = 0\). The determinant \(T\) is independent of \(\theta_1\), and thus the integral over \(\theta_1\) which appears in (5.18) is
\[A = \int d\theta_1 e^{-i\sum_{i=1}^L \sqrt{r_i} \cos(\theta_1 - (-1)^i \theta'_i)} \]
(5.22)
Expanding up to second order in the \(\theta'_i\)'s, after integration over \(\theta_1\), we get
\[A \simeq 2\pi J_0(\zeta \sum_{i=1}^L \sqrt{r_i}) + O(\theta'_i^2) \quad (5.23)\]
It is only the first term of (5.23) which matters, since \(T\) is vanishing for the saddle-point, and the fluctuations over the \(\theta'_i\)'s are of higher order. Therefore the imaginary part of \(G(z)\) is given by the product of \(J_0\) and \(J_1\) as before.

\[\text{Im}G(z) = \frac{2\pi^2 N^{2L} \pi^{L(n-1)}}{(N!)^L} \left(\frac{1}{2}\right)^L \int \prod_{i=1}^L dr_i \prod_{j=2}^L d\theta'_j \int dpp \frac{N t_{L-1}}{2} \]
\[\int \prod_{i=1}^L dt_i (t_1 t_2 \cdots t_L)^N \delta(t_1 + t_3 + \cdots + t_{L-1}) \]
\[\delta(t_2 + t_4 + \cdots + t_L) T \]
\[\exp(-Nph(t) - N \sum_{i=1}^L r_i) J_1(2\zeta \sqrt{p}) J_0(\zeta \sum_{i=1}^L \sqrt{r_i}) \quad (5.24)\]
The factor \(\left(\frac{1}{2}\right)^L\) comes from Jacobian of \(dr\).
We still have to integrate over \(p, t_i, \theta'_i, r_i\). However, the integration around the saddle-point for \(t_i\) and \(\theta'\) may be avoided, although they could also be obtained after quite lengthy calculations. Indeed one may notice that if we
had calculated, instead of the average resolvent, the average value of 1, we would have had an integral similar to (4.15). If we attempted to compute one through a large $N$-limit analysis, we would obtain again a product of two Bessel functions $J_0(N\mu \sum \sqrt{r_i})J_1(2N\mu \sqrt{p})$, in which $N\mu$ is the real part of $\zeta$. The coefficient of this term should thus vanish. Therefore if we replace $p$ and $r_i$ by their saddle-point values, then the contributions of the fluctuations of $t'_i$ and $t_i$, which also have a factor $J_0(LN\mu/\sqrt{2})J_1(LN\mu/\sqrt{2})$, should cancel each other. The only difference between the calculation of the average resolvent and of the identity, is a relative factor $\sqrt{p}$. This contribution is also exactly cancelled by those coming from the expansion of the Bessel function $J_1(2N\mu \sqrt{p})$:

$$J_1(2N\mu \sqrt{p}) \simeq J_1(\frac{LN\sqrt{2}}{2^3\mu} + p'2\sqrt{2}N\mu \frac{J_0(\frac{LN\mu}{\sqrt{2}})}{L} - p' \frac{4}{L^2}J_1(\frac{LN\mu}{\sqrt{2}})$$ (5.25)

It is thus possible to replace $t_i$ and $\theta_i$ by their values at the saddle-point. The Bessel function $J_0(\zeta \sum \sqrt{r_i})$ is expanded as

$$J_0(N\mu \sum \sqrt{r_i}) \simeq J_0(\frac{LN\mu}{\sqrt{2}}) - \frac{N\mu}{\sqrt{2}} \sum_{i=1}^{L} r'_iJ_1(\frac{LN\mu}{\sqrt{2}})$$ (5.26)

We are thus left with two terms, which are proportional to $[J_0(LN\mu/\sqrt{2})]^2$ and $[J_1(LN\mu/\sqrt{2})]^2$. The coefficients of these terms are evaluated by integrating over $p'$ and $r'$. The determinant $T$ factorizes as

$$2^{\frac{L}{2}} \prod_{k=1}^{\frac{L}{2}} (1 - \frac{8}{L^2}p \cos^2(\frac{2\pi}{L}k))$$ (5.27)

It is simply the product of the eigenvalues for a periodic chain with nearest neighbor interactions. Setting $p = \frac{L^2}{8} + p'$, and expanding this product up to order $p'$, we find

$$-p'\frac{8}{L^2}2^{\frac{L}{2}} \prod_{k=1}^{\frac{L}{2}-1} (\sin^2(\frac{2\pi}{L}k) - \frac{8}{L^2}p' \cos^2(\frac{2\pi}{L}k))$$

$$\simeq -p'\frac{8}{L^2}2^{\frac{L}{2}} \prod_{k=1}^{\frac{L}{2}-1} \sin^2(\frac{2\pi}{L}k)$$

$$= -p'2^{3-\frac{L}{2}}$$ (5.28)
where we have used the identity:

\[ \prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = \frac{n}{2^{n-1}} \]  

(5.29)

The \( \sigma \) integral, which appears in this calculation, may be done exactly. After integration over the \( \sigma_i \)'s, we have

\[
I_{NL} = \int L \prod_{i=1}^{L} d^{2} \sigma_i [(\sigma_1^* + \sigma_L)(\sigma_2^* + \sigma_1) \cdots (\sigma_L^* + \sigma_{L-1})]^N \exp(-N \sum \sigma_i^* \sigma_i) \\
= \left(\frac{\pi}{N^{N+1}}\right)^L C(N, L)(N!)^{\frac{1}{2}}
\]

(5.30)

The quantity \( C(N, 2k) \) is expressible as an integral over angles; for instance in the \( k = 3 \) case, it reads

\[
C(N, 6) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 [1 + e^{i\theta_1}]^N [1 + e^{-i\theta_1 - i\theta_2}]^N [1 + e^{i\theta_2}]^N
\]

(5.31)

In the large \( N \) limit, we exponentiate the integrand of (5.31) and expand the \( \theta \)'s around \( \theta = 0 \). Then we have \( 2(8^N)/(\pi \sqrt{3}N) \). For general \( k \), we get

\[
C(N, 2k) \approx \frac{2^{kN}}{(2\pi)^{k-1}} \left(\frac{\pi}{N}\right)^{\frac{k-1}{2}} \frac{2^{k-1}}{\sqrt{\prod_{n=1}^{k-1}(1 + \cos(\frac{\pi n}{k}))}}
\]

(5.32)

From the identity,

\[
\prod_{r=1}^{l-1} \cos\left(\frac{\pi r}{2l}\right) = \frac{\sqrt{l}}{2^{l-1}}
\]

(5.33)

and setting \( k = \frac{L}{2} \), we have

\[
\frac{L}{2} \prod_{n=1}^{L/2-1} (1 + \cos(\frac{2\pi n}{L})) = 2^{L/2-1} \prod_{n=1}^{L/2-1} \cos^2\left(\frac{\pi n}{L}\right)
\]

\[
= 2^{-\frac{L}{4}} L
\]

(5.34)

Thus we get

\[
C(N, L) \approx \frac{2^{\frac{LN}{2}}}{\pi^{\frac{L}{4} - \frac{1}{2}}} \frac{2^{\frac{L}{4}}}{\sqrt{L}} \frac{1}{N^{\frac{L}{4} - \frac{1}{2}}}
\]

(5.35)
and \( I_{NL} \) in (5.30) reads, for the case of an arbitrary \( L \),

\[ I_{NL} \simeq \frac{\pi^{L+\frac{1}{2}}}{N^{L-\frac{1}{2}}} \frac{2^{\frac{N}{2}+1}}{\sqrt{L}} e^{-\frac{L}{4}N} \]  
\[ (5.36) \]

The remaining integration over \( p \) may again be done by the saddle point method; setting \( p = \frac{L^2}{8} + p' \), we have

\[ I_p = \int dp \frac{\sqrt{2} \pi}{64 N^2} \left( \frac{L^2}{8} \right)^{\frac{N+1}{2}} L^2 e^{-\frac{16}{L}N p'^2} \]
\[ \simeq \sqrt{2\pi} \frac{L^2}{8} \left( \frac{L^2}{8} \right)^{\frac{N+1}{2}} L^2 e^{-\frac{4}{L}N p'^2} \]  
\[ (5.37) \]

Thus the piece of the imaginary part of \( G \), which is proportional to \( J_0^2 \left( \frac{L\sqrt{2}}{2} \zeta \right) \), is obtained by multiplying together the coefficients of (5.25), (5.28), (5.36), (5.37) and the contribution coming from the integrals over the \( t_i \)'s , which is given in an appendix:

\[ (\text{Im} G)_{2a} \simeq -\frac{L^2}{L} \left( \frac{L^2}{4L} \right) \bar{D} \pi N \mu J_0^2 \left( \frac{L\sqrt{2}}{2} N \mu \right) \]
\[ = -\frac{L}{8} \pi N \mu J_0^2 \left( \frac{L\sqrt{2}}{2} N \mu \right) \]  
\[ (5.38) \]

where \( N \mu \) is a real part of \( \zeta \); the calculation of \( \bar{D} \) is given in Appendix A, in which it is shown that \( \bar{D} = 4^{L-1}/L^L \). We have also the same coefficient for the other piece of the imaginary part which is proportional to \( J_1^2 \left( LN \mu \sqrt{2}/2 \right) \).

Thus we obtain for this one-dimensional ring,

\[ \rho(\mu) = \frac{LN \mu}{8} \left[ J_0^2 \left( \frac{LN \sqrt{2}}{2} \mu \right) + J_1^2 \left( \frac{LN \sqrt{2}}{2} \mu \right) \right] \]  
\[ (5.39) \]

If we replace \( LN \mu / \sqrt{2} \) by \( 2x \), then this equation becomes \( \rho(\mu) = \frac{1}{2\sqrt{2}} [J_0^2(2x) + J_1^2(2x)] \); it is identical to (3.25) (except for a trivial factor \( 1/\sqrt{2} \) which comes from a different normalization in the probability distribution in this section; for that reason the edge of the density of state is now at \( z_c = 2\sqrt{2} \) instead of 2). Thus we have obtained a behavior at the origin for a chain of \( L \) matrices which, up to a normalization, is identical to the previous simple case.

29
6 Lattice of matrices

If we now consider a higher dimensional lattice, with a total number of lattice points equal to \( L \), and periodic boundary conditions, one may ask again the same question. As before, we take an \( L N \times L N \) random matrix, with \( N \times N \) block elements, corresponding to the hopping between nearest neighbours on a lattice. Non-neighbouring sites are not coupled and are represented in the total matrix by block matrices of zeroes. Generalizing the expression of (5.3), we have

\[
\frac{1}{LN} \text{Tr} \left( \frac{1}{z - M} \right) = \frac{-i}{L} \int \prod_{i=1}^{N} da_{1i} \cdots da_{Li} \delta_{1i} \cdots \delta_{Li} \times
\]

\[
\left[ (a_1^* \cdot a_1) + \cdots + (a_L^* \cdot a_L) \right] \exp(iNz[(a_1^* \cdot a_1) + \cdots + (a_L^* \cdot a_L)] - N[a_1^*(C_{1,2}^t) a_2 + \cdots])
\]  

(6.1)

The last term of (6.1) reproduces the connectivity of the lattice.

Integrating over the random complex matrix \( C_{i,j} \), and using the \( \sigma_{i,j} \) variables of (4.10), we have, as in (5.8),

\[
G(z) = \frac{-i}{L} \left( \frac{N}{\pi} \right)^L \frac{1}{(N!)^L} \int \prod_{<i,j>} d^2 \sigma_{i,j} \int \prod dx_i
\]

\[
(x_1 \cdots x_L)^{N-1} T(\sigma, x) \exp[iNz(x_1 + \cdots + x_L)] - N \sum_{<i,j>} \sigma_{i,j}^* \sigma_{i,j} - N(x_i M_{ij} x_j)
\]  

(6.2)

where we used the notation \(|a_i|^2 = x_i\) and \( <i,j> \) are a pair of nearest neighbours; \( M_{ij} \) is the adjacency matrix of the lattice. One can use a method similar to that of the one dimensional chain; we divide the lattice points into two groups, with odd and even indices. This change of variable leads to an expression similar to (5.4)

\[
G(z) = -\frac{i}{L} \left( \frac{N}{\pi} \right)^L \frac{1}{(N!)^L} \int \prod d^2 \sigma_1 \int \prod \lambda_1 \cdots \lambda_L \exp(iNz(\lambda_1 + \lambda_2) - N \sum_{<i,j>} \sigma_{i,j}^* \sigma_{i,j})
\]  

(6.3)
where \( h(t) = t_i M_{ij} t_j \). By the change of variables, \( \lambda_1 \lambda_2 = p \) and \( \lambda_1 + \lambda_2 = 2\sqrt{pq} \), we integrate over \( q \), and obtain immediately \( K_1(-2i\zeta \sqrt{p}) \) as (4.24). We replace \( Nz \) by \( \zeta \). The saddle-point is \((t_i)_c = \frac{2}{4}, \lambda_c = \frac{4}{4}, h(t_c) = \frac{8}{4}\) and \( p_c = \frac{4^2}{16} \). (Note a factor of two in the normalization compared to the one-dimensional problem). We shift the complex variable \( \sigma^*_i - iz/4 \) to \( \sigma^*_i \) and \( \sigma_{i,j} - iz/4 \) to \( \sigma_{i,j} \). This shift gives an extra term to (6.3) of \( \exp(-iz\sum(\sigma^*_i + \sigma_{i,j}))/4 \). We write \( \sigma_{i,j} \) as
\[
\sigma_{i,j} = \sqrt{r_{i,j}} e^{i\theta_{i,j}}
\]
(6.4)

We expand the variable \( \theta \) around the saddle point \( \theta_{i,j} \),
\[
\theta_{2i-1,j} = \theta_{1,2} + \theta'_{2i-1,j} \\
\theta_{2i,j} = -\theta_{1,2} + \theta'_{2i,j}
\]
(6.5)

As in the previous cases, we may simplify
\[
\exp\left(-\frac{i}{4} \zeta \sum(\sigma^*_i + \sigma_{i,j})\right) \simeq \exp\left(-\frac{i}{2} \zeta \sum \sqrt{r_{i,j}} \cos \theta_{1,2}\right)
\]
(6.6)

The deviations from the saddle-point \( \pm \theta_{1,2} \) may be neglected since \( T \) is vanishing at the saddle-point. Therefore, by integration over \( \theta_{1,2} \), we obtain again \( J_0\left(\frac{N\mu}{2} \sum \sqrt{r_{i,j}}\right) \), where \( N\mu \) is the real part of \( \zeta \). We note the values at the saddle-point are \( (r_{i,j})_c = \frac{1}{4} \) and \( p_c = \frac{L^2}{16} \). Thus we have the following expression:
\[
\text{Im}G = \frac{2\pi^2}{L} \frac{N^{2L} \pi^{L(N-1)}}{(N!)^L} \left(\frac{1}{2}\right)^L \int \Pi dr_{i,j} \Pi d\theta_{i,j}' \int dpp \frac{N^{L-1}}{2} \int \Pi dt_i \\
[t_1 \cdots t_L]^{N-1} T \delta(t_1 + t_3 + \cdots + t_{L-1} - 1) \delta(t_2 + t_4 + \cdots + t_L - 1) \\
J_1(2N\mu \sqrt{p}) J_0\left(\frac{N\mu}{2} \sum_{<i,j>} \sqrt{r_{i,j}}\right) e^{-Nph(t) - N \sum r_{i,j}}
\]
(6.7)

The determinant \( T \) is expanded around the saddle point. If, for definiteness we specialize to a two-dimensional square lattice, \( T \) contains the factor:
\[
\prod_{k_1,k_2=1}^{\sqrt{L}} \left[ 1 - \frac{2}{L} \sqrt{p \cos \frac{2\pi}{\sqrt{L}}} k_1 - \frac{2}{L} \sqrt{p \cos \frac{2\pi}{\sqrt{L}}} k_2 \right]
\]
\[
= (1 - \frac{4}{L \sqrt{p}}) \prod_{k_1,k_2=1}^{\sqrt{L}} \left[ 1 - \frac{2}{L} \sqrt{p \cos \frac{2\pi}{\sqrt{L}}} k_1 - \frac{2}{L} \sqrt{p \cos \frac{2\pi}{\sqrt{L}}} k_2 \right]
\]
(6.8)
Expanding $p$ around the saddle point, $p = \frac{L^2}{4} + p'$, we find that the first factor of (6.8) becomes

$$1 - \frac{4}{L} \sqrt{p} = -\frac{8}{L^2} p' + O(p'^2)$$  \hspace{1cm} (6.9)$$

We also have to expand the Bessel functions and we obtain

$$J_1(2N\mu \sqrt{p}) = J_1\left(\frac{LN\mu}{2}\right) + \frac{4N\mu}{L} p' J_0\left(\frac{LN\mu}{2}\right) - \frac{8}{L^2} p' J_1\left(\frac{LN\mu}{2}\right)$$  \hspace{1cm} (6.10)$$

The last term of order $p'$ cancels as before with the term of $1/\sqrt{p}$ in (6.7), as seen from the expansion

$$\sqrt{p} \simeq \frac{L}{4} \left(1 + \frac{8}{L^2} p'\right)$$  \hspace{1cm} (6.11)$$

Repeating the procedure used for the one-dimensional chain, we obtain an identical form for the density of state near the origin,

$$\rho(\mu) = C\mu\left(J_0^2\left(\frac{LN}{2\mu}\right) + J_1^2\left(\frac{LN}{2\mu}\right)\right)$$  \hspace{1cm} (6.12)$$

where the coefficient $C$ is $\frac{LN}{16}$. For this two-dimensional lattice, the end point of the semi-circle density of state is, with our normalizations, $\mu = 4$; this is why we have an extra factor of one half in our scaling variable compared to (3.25).

### 7 Non-Gaussian probability distribution

Up to now, we have considered a Gaussian distribution for the random matrix $C$ (2.2). If one modifies this distribution, for instance by adding quartic terms in the exponential, the average density of eigenvalues will no longer obey a semi-circle law. However earlier studies on the usual unitary ensemble, revealed that the behavior near the edge of the semi-circle was not affected by non-Gaussian terms. The cross-over there is always given in terms of Airy functions (3.37). We are thus led to investigate whether this universality is also valid for the oscillation of the density of state near the origin that we have found for the block-matrices that we are considering in this article.
Let us consider a non-Gaussian distribution, for the simple one-matrix (i.e. \( L = 2\)-model) of section 3, but it will be clear that the proof of universality will apply to any lattice of matrices. Consider for instance the following \( P(C) \):

\[
P(C) = \frac{1}{Z} \exp(-N\text{Tr}C^\dagger C - Ng\text{Tr}(C^\dagger C\ddagger C)).
\]  

Note that the factors of \( N \) in the exponential are such that the average correlation functions of the eigenvalues, and in particular the density, have a finite limit when \( N \) goes to infinity. Had we put a higher power of \( N \) in front of the quartic term, then it would not be the case; with a lower power of \( N \), it would not contribute at all. Then in the cross-over region of size \( \frac{1}{N} \) near the origin, one could argue at first sight that the quartic terms modify simply the overall scale, but not the oscillatory behavior. Indeed let us write the average density

\[
\rho_0(r) = \langle \frac{1}{N}\text{Tr}\delta(r - C^\dagger C) \rangle
\]  

We then integrate over the unitary group and scale \( r \), as well as the eigenvalues of \( C^\dagger C \) by \( \frac{1}{N^2} \). Then one sees immediately that the quartic terms of (7.1) give correction of order \( \frac{1}{N^2} \). However the overall normalisation remains \( g \)-dependent and therefore it modifies the scale of the cross-over function by a factor which is the ratio of the partition functions for the \( g \neq 0 \) problem, and the Gaussian one.

But this simple-minded analysis is based on letting \( N \) go to infinity first, and in fact it is slightly misleading. As will be seen now, the question is more subtle, and the non-Gaussian terms affect more than simply the overall normalization of the density, as we pretended in the previous argument. It has already been found in similar problems [4], that by letting \( N \) go to infinity first, one computes only a smoothed average of the correlation function and this is not what we are considering. Indeed if we smoothed out the oscillatory part of the density near the origin, the simple universality claimed above would be true: the non-Gaussian part would change only the normalization. However, since we are interested in those oscillations, the previous argument is not sufficient, and we shall argue now that the non-Gaussian terms do modify the period of these oscillations. This change of the approximate period of oscillations, is in fact expected; indeed there are \( N \) eigenvalues distributed between zero and the endpoint. There are thus
oscillations in the density. If the normalization is changed, and say the value of $\rho(r)$ at $r = 0$ multiplied by a factor $c$, then the approximate period of oscillations has to be divided by $c$.

This may be checked explicitly by returning again to the formulation in terms of contour integrals, which we have developed in the section 3; we shall apply it now perturbatively for this non-Gaussian distribution.

For the non-Gaussian distribution, $U_A(t)$ in (3.2) is given by

$$U_A(t, g) = \frac{\int e^{-N \sum (r_i + r_i a_i + \sigma_i^2 t)} \prod_{i<j} (r_i - r_j) \left( \frac{1}{N} \sum_{\alpha} e^{ir\alpha} \right) \prod dr_i}{\int e^{-N \sum (r_i + r_i a_i + \sigma_i^2 t)} \prod_{i<j} (r_i - r_j) \prod dr_i}$$

(7.3)

This is expressible as

$$U_A(t, g) = \frac{e^{-\frac{1}{N^2} \sum \frac{\partial^2}{\partial a_i^2} F}}{e^{-\frac{1}{N^2} \sum \frac{\partial^2}{\partial a_i^2} D}}$$

(7.4)

where

$$F = \int e^{-N \sum (r_i + r_i a_i) \prod_{i<j} (r_i - r_j) \left( \frac{1}{N} \sum_{\alpha} e^{ir\alpha} \right) \prod dr_i}$$

$$D = F(t = 0)$$

(7.5)

Expanding this $U_A(t, g)$ up to order $g$, we get

$$U_A(t, g) \simeq U_A(t, g = 0) - g \sum_{i=1}^{N} \left[ \left( \frac{\partial^2 F}{\partial a_i^2} \right) \frac{1}{D} - \left( \frac{\partial^2 D}{\partial a_i^2} \right) \frac{F}{D^2} \right] + O(g^2)$$

(7.6)

Noting that

$$\frac{\partial U_A}{\partial a_i} = \left( \frac{\partial F}{\partial a_i} \right) \frac{1}{D} - \frac{F}{D^2} \left( \frac{\partial D}{\partial a_i} \right)$$

$$\frac{\partial^2}{\partial a_i^2} U_A(t, 0) = \left( \frac{\partial^2 F}{\partial a_i^2} \right) \frac{1}{D} - \frac{F}{D^2} \left( \frac{\partial^2 D}{\partial a_i^2} \right) + 2 \frac{F}{D^3} \left( \frac{\partial D}{\partial a_i} \right)^2 - 2 \left( \frac{\partial F}{\partial a_i} \right) \left( \frac{\partial D}{\partial a_i} \right) \frac{1}{D^2}$$

(7.7)

the term of order $g$, denoted by $\delta U_A$, becomes

$$\delta U_A = -g \sum_{i=1}^{N} \frac{\partial^2}{\partial a_i^2} U_A(t, 0) - \frac{2g}{N} \sum_{i=1}^{N} \frac{\partial U_A}{\partial a_i} \left( \frac{\partial \ln D}{\partial a_i} \right)$$

(7.8)
The expression of $D$ is given in (3.7) and we have

$$\frac{\partial \ln D}{\partial a_i} = \sum_{k \neq i} \frac{1}{(a_i - a_k)} - \frac{N}{1 + a_i}$$  \hspace{1cm} (7.9)$$

Using the contour representation in (3.10), we get

$$\frac{\partial U_A}{\partial a_i} = -\frac{1}{N} \oint \frac{du}{2\pi i} \left( \frac{1 + u}{1 + u - \frac{it}{N}} \right)^N \frac{1}{(u - a_i)^2} \prod_{\gamma \neq i} \left( \frac{u - a_\gamma - \frac{it}{N}}{u - a_\gamma} \right)$$

$$\frac{\partial^2}{\partial a_i^2} U_A(t, 0) = -\frac{2}{N} \oint \frac{du}{2\pi i} \left( \frac{1 + u}{1 + u - \frac{it}{N}} \right)^N \frac{1}{(u - a_i)^3} \prod_{\gamma \neq i} \left( \frac{u - a_\gamma - \frac{it}{N}}{u - a_\gamma} \right)$$  \hspace{1cm} (7.10)$$

Noting that

$$\frac{1}{(u - a_i)^2(a_i - a_k)}(1 - \frac{it}{N(u - a_k)}) + \frac{1}{(u - a_k)^2(a_k - a_i)}(1 - \frac{it}{N(u - a_i)})$$

$$= \frac{2u - a_i - a_k - \frac{it}{N}}{(u - a_i)^2(u - a_k)^2}$$  \hspace{1cm} (7.11)$$

we are able to express the contribution $\delta U_A$ in the contour integral. Thus we get the contour representation in the first order of $g$; by letting $a_i$ goes to zero,

$$\delta U_0 = \frac{2g}{N} \oint \frac{du}{2\pi i} \left( \frac{1 + u}{1 + u - \frac{it}{N}} \right)^N \frac{1}{u^3} \left( \frac{u - \frac{it}{N}}{u} \right)^{N-1}$$

$$+ \frac{2g}{N} \oint \frac{du}{2\pi i} \left( \frac{1 + u}{1 + u - \frac{it}{N}} \right)^N \left[ \left( \frac{N - 1}{2} \right) \frac{2u - \frac{it}{N}}{u^4} \right] \left( \frac{u - \frac{it}{N}}{u} \right)^{N-2}$$

$$- \frac{N}{u^2} \left( \frac{u - \frac{it}{N}}{u} \right)^{N-1}$$  \hspace{1cm} (7.12)$$

Immediate checks for $N=1, 2$ and $3$, and also the large $N$ limit, are easy and they agree with the result known by other methods. Thus this expression is exact to this order in $g$.

We now consider the cross-over region near the origin; replacing $t$ by $N^2t$, and $u$ by $Nu$, we have

$$\delta U_0(t) = 2g \oint \frac{du}{2\pi i} \left( \frac{1 + \frac{1}{Nu}}{1 + \frac{1}{Nu - it}} \right)^N \left[ \frac{1}{N^3} \frac{1}{u^2} \left( \frac{1}{u - it} \right) \right]$$

$$+ \frac{N - 1}{2N^3} \frac{1}{u^2} \left( \frac{2u - it}{u - it} \right) - \frac{1}{N} \frac{1}{u(u - it)}$$  \hspace{1cm} (7.13)$$
Only the last term in the bracket contributes in the large N limit. It gives, by the Fourier transform in (3.3),

\[ \delta \rho(\mu) = 2gN J_0^2(2x) \]  

(7.14)

where \( x = N \mu \). The overall factor due to the change of normalization is \( 1 + 2g \), which agrees with the expression for the density of state which was found in [2]. Then, up to this order, we may interprete this result as reading

\[ \rho_0(\mu) = C(g) \frac{N \mu}{2} \left[ J_0^2(2N \mu \sqrt{C(g)}) + J_1^2(2N \mu \sqrt{C(g)}) \right] \]  

(7.15)

where \( C(g) = 1 + 2g + O(g^2) \). This expression makes it clear that the integrated density of state remains properly normalized to one. Indeed, expanding this expression for \( g \) small, we find the term \( g J_1^2(2x) \) cancel, and obtain (7.14). This result is expected; it is exactly the universality which was claimed earlier up to this order in \( g \).

We are thus tempted to conjecture that these oscillations are indeed universal, namely that for an arbitrary probability distribution of the form

\[ P(C) = \frac{1}{Z} e^{-N \text{Tr} V(C^\dagger C)} \]  

(7.16)

with \( C \) defined over a lattice the density of states near the origin is given by

\[ \rho(\mu) = \frac{NLF^2 \mu}{2} \left[ J_0^2(2NL \mu) + J_1^2(2NL \mu) \right] \]  

(7.17)

with \( F(V) \) some functional of \( V \).

8 Oscillation and cross-over behavior for real matrices

It has long been known that for real symmetric matrices the relevant Jacobian involves the absolute value of the van der Monde determinant and thus the corresponding orthogonal polynomial analysis becomes quite complicated.
However, by using some remarkable identities, we can actually treat the cross-over behavior near the origin of the density of state of a \((2N + 1) \times (2N + 1)\) real matrix \(M\), which is made of a rectangular \((N + 1) \times N\) matrix \(C\):

\[
M = \begin{pmatrix} 0 & C^T \\ C & 0 \end{pmatrix}
\]  

(8.1)

where \(C^T\) is the transpose of the matrix \(C\). As we will see, the cross-over near the origin shows a different universal behavior from what we have studied earlier.

To understand why we treat the case of \(C\) being \((N + 1) \times N\) and to get oriented, let us do a simple exercise in power counting. Consider an \(M \times N\) real matrix \(C\) with its \(MN\) real variables. Denote the eigenvalues of the \(N \times N\) real symmetric matrix \(C^TC\) by \(r_i\). We would like to have

\[
dC \simeq N \prod_{i<j} |r_i - r_j| dr_1 \cdots dr_N
\]  

(8.2)

To see if this is possible, let us do dimensional analysis and count powers. The left hand side has the dimension of \(r_N^{N(N-1)/2} N \sim C^{N^2+N}\); on the other hand, the right hand side has the dimension of \(C^{MN}\). Equating \(MN = N^2 + N\) we find \(M = N + 1\). This explains why we chose \(C\) to be \((N + 1) \times N\). We can of course also check this Jacobian by direct computation, using the Fadeev-Popov method for example.

Note that if we had chosen \(C\) to be \(N \times N\), we would have

\[
dC \simeq \prod_i^N \frac{1}{\sqrt{r_i}} \prod_{i<j}^N |r_i - r_j| dr_1 \cdots dr_N
\]  

(8.3)

The presence of the square root factors can be deduced by dimensional arguments or determined by a direct computation of the Jacobian. These square roots make the calculation much more complicated. See below.

We have as usual the density of state

\[
\rho(\mu) = \frac{1}{2N+1} \text{Tr} \delta(\mu - M)
\]  

(8.4)

and from the block structure of \(M\),

\[
\rho(\mu) = \frac{\mu}{N} \text{Tr} \delta(\mu^2 - C^TC)
\]  

(8.5)
(Note that the \((2N + 1) \times (2N + 1)\) real matrix \(M\) we started out with has an eigenvalue at zero; obviously, the corresponding eigenvector is the vector orthogonal to the \(N\) columns in the \(N \times N + 1\) matrix \(C^T\). This eigenvalue leads to an additional delta function at the center of the spectrum in the density of states.) As in the complex matrix case, we define \(\tilde{\rho}(\lambda) = \langle \frac{1}{N} \text{Tr} \delta(\lambda - C^T C) \rangle\), with \(\rho(\mu) = \mu \tilde{\rho}(\mu^2)\). With the probability distribution

\[
P(C) = \frac{1}{Z} \exp(-N \text{Tr} C^T C) \tag{8.6}
\]

this model is known as the orthogonal Laguerre ensemble [20].

In an obvious extension of Kazakov’s method we introduce an external source matrix \(A\), which we take to be an \(N \times N\) Hermitian matrix diagonalized by the unitary matrix \(U\). The probability distribution is

\[
P_A(C) = \frac{1}{Z_A} \exp(-N \text{Tr} C^T C - N \text{Tr} A C^T C) \tag{8.7}
\]

Since \(C^T C\) is a real symmetric matrix, it is diagonalized by an orthogonal matrix \(O\), and then

\[
\text{Tr} A C^T C = \text{Tr} U^{-1} \left( \begin{array}{ccc}
a_1 & & \\
& \ddots & \\
& & a_N
\end{array} \right) U^T O
\left( \begin{array}{ccc}
r_1 & & \\
& \ddots & \\
& & r_N
\end{array} \right) O \tag{8.8}
\]

Since there is no known analog of the Itzykson-Zuber identity for integrating over orthogonal matrices, we would have been stuck at this point. The crucial observation is to notice that we could integrate over the unitary matrix \(V = UO^T\). Strictly speaking, \(A\) is not an external source since we integrate over all matrices unitarily equivalent to \(A\). However, since we set the \(a_j\)’s to zero at the end, this procedure gives us the correct value of \(U_0(t)\). Thus, integrating

\[
U_A(t) \simeq \int dV \int dr_i \prod_{i<j} \frac{1}{N} \sum_{\alpha=1}^{N} \exp(\text{ir}_\alpha) e^{-N \sum_{i<j} r_i - N \text{Tr} V^{-1} A_{\text{diag}} V C^T C_{\text{diag}}}
\tag{8.9}
\]
using the Itzykson-Zuber identity, we obtain

\[ U_A(t) = \frac{1}{N} \sum_{\alpha=1}^{N} \int dr_1 \cdots dr_N e^{itr_\alpha} \prod_{i<j} |r_i - r_j| \prod_{i<j} (r_i - r_j) \]

\[ \times \exp(-N \sum_{i=1}^{N} r_i - N \sum_{j=1}^{N} r_j a_j). \]  

(8.10)

where \( U_A(t) \) is normalized as \( U_A(0) = 1 \).

A remarkable identity

\[ \int_0^{+\infty} \cdots \int_0^{+\infty} \prod_{i=1}^{N} dr_i \prod_{i<j}^{N} \text{sign} r_i - r_j e^{-\sum r_i b_i} \]

\[ = \frac{1}{\prod_{i=1}^{N} b_i} \prod_{i<j}^{N} \left( \frac{b_i - b_j}{b_i + b_j} \right) \]  

(8.11)

allows us to integrate over the \( r_i \)'s. Let us sketch a proof of this identity. The integration region in (8.11) can be divided into \( N! \) regions inside each of which the \( r \)'s are ordered. Thus, the integral above is equal to

\[ \int_0^{+\infty} \cdots \int_0^{+\infty} \prod_{i=1}^{N} dr_i \theta(r_1 \leq r_2 \leq r_3 \leq \ldots \leq r_N) e^{-\sum r_i b_i} \]  

(8.12)

plus \( N! - 1 \) similar integrals with the \( r_i \)'s permuted and with a suitable sign given by \( \prod_{i<j}^{N} \text{sign} r_i - r_j \). (We will not be concerned with the overall multiplicative factor in what follows since it is irrelevant to our calculation.) To get oriented, let us do the \( N = 3 \) case. Change variables by \( r_1 = x_1, r_2 = r_1 + x_2, r_3 = r_2 + x_3, \ldots \) to take care of the ordering. We can then immediately do the integral above to obtain

\[ \frac{1}{b_3(b_3 + b_2)(b_3 + b_2 + b_1)} \]  

(8.13)

We now add the 5 other terms and collect denominators; the sum evidently has the form

\[ \frac{f(b)}{b_1 b_2 b_3 (b_1 + b_2)(b_2 + b_3)(b_3 + b_1)(b_1 + b_2 + b_3)} \]  

(8.14)
with some numerator \( f(b) \) which we can now determine by general arguments. Since our integral vanishes whenever any two of the \( b_i \)’s are equal, we must have \( f(b) = \Delta(b)P(b) \) where \( \Delta(b) = \prod_{i<j}^N (b_i - b_j) \) is the usual van der Monde determinant and where the polynomial \( P(b) \) must be symmetric and by dimension counting must be of degree \( N \). Thus, \( P(b) \) is uniquely determined to be \( b_1 + b_2 + b_3 \). We have proved the identity for \( N = 3 \). We then proceed by induction. Assume that the identity has been proved for some \( N \). Doing the integral for \( N + 1 \) along the line described above we encounter after the first step

\[
\frac{1}{b_1(b_1 + b_2)(b_1 + b_2 + b_3)\ldots(b_1 + b_2 + \ldots + b_N)(b_1 + b_2 + \ldots + b_N + b_{N+1})}
\]

plus permutations. We now first add the terms obtained by permuting \( b_1, b_2, \ldots, b_N \), holding \( b_{N+1} \) fixed. Then by the inductive process we obtain

\[
\frac{1}{\prod_{i=1}^{N} b_i \prod_{i<j}^N (b_i + b_j)} \frac{\Delta_N(b)}{b_1 + b_2 + \ldots + b_N + b_{N+1}}
\]

plus terms obtained by interchanging \( b_{N+1} \) with one of the \( b_i \)’s. The subscript on \( \Delta_N \) indicates that it is the van der Monde determinant for the first \( N \) \( b_i \)’s. Collecting common denominators and reasoning as before, we find that the sum is equal to

\[
\frac{1}{\prod_{i=1}^{N+1} b_i \prod_{i<j}^{N+1} (b_i + b_j)} \frac{\Delta_{N+1}(b)}{b_1 + b_2 + \ldots + b_N + b_{N+1}} P(b)
\]

By symmetry and by dimensional analysis the symmetric polynomial \( P(b) \) must be of degree 1 and hence equal to \( (b_1 + b_2 + \ldots + b_N + b_{N+1}) \). We have thus proved the identity. Note that if a square root factor were present, as in (8.3), we would not have this identity and thus would have difficulty proceeding farther.

Since \( b_i \) is given by

\[
b_i = N(1 + a_i - \frac{it}{N} \delta_{\alpha,i})
\]

we have

\[
U_A(t) = \frac{1}{N} \sum_{\alpha=1}^{N} \left( \frac{1 + a_\alpha}{1 + a_\alpha - \frac{it}{N}} \right) \prod_{\gamma \neq \alpha} \left( \frac{a_\alpha - a_\gamma - \frac{it}{N}}{2 + a_\alpha + a_\gamma - \frac{it}{N}} \right) \left( \frac{2 + a_\alpha + a_\gamma}{a_\alpha - a_\gamma} \right)
\]

(8.19)
This is expressible in a contour representation:

\[
U_A(t) = -\frac{1}{it} \oint \frac{du}{2\pi i} \left( \frac{1 + u}{1 + \frac{u}{N}} \right) \prod_{\gamma=1}^{N} \left( \frac{2 + u + a_\gamma}{2 + u + \frac{u}{N}} \right) \\
\times \left( \frac{2 + 2u - \frac{it}{N}}{2 + 2u} \right) \prod_{\gamma' = 1}^{N} \left( \frac{u - a_{\gamma'} - \frac{it}{N}}{u - a_{\gamma'}} \right)
\]

(8.20)

Letting all \(a_\gamma\)'s go to zero, we have

\[
U_0(t) = -\frac{1}{it} \oint \frac{du}{2\pi i} \left( \frac{1 + u}{1 + \frac{u}{N}} \right) \left( \frac{2 + u}{2 + \frac{u}{N}} \right)^N \\
\times \left( \frac{2 + 2u - \frac{it}{N}}{2 + 2u} \right) \left( \frac{u - \frac{it}{N}}{u} \right)^N
\]

(8.21)

where the contour is chosen around \(u = 0\). We are able to check this formula by calculating the Fourier transform,

\[
\tilde{\rho}(r) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-itr} U_0(t).
\]

(8.22)

and verifying that we obtain the same result for \(N = 2\) and \(3\) as we would have obtained by directly integrating

\[
\tilde{\rho}(r_1) = \int \prod_{i<j} |r_i - r_j| \exp(-N \sum r_i) dr_2 \cdots dr_N
\]

(8.23)

We obtain the semi-circle law just as in the complex matrix case.

Next, we study the cross-over behavior near the center of the spectrum. Changing variables \(u \to Nu\) and \(t \to N^2t\) we obtain in the large \(N\) limit,

\[
U_0(t) \simeq -\frac{N}{2it} \oint \frac{du}{2\pi i} e^{\frac{3}{2} - \frac{u^2}{2\pi} (1 + \frac{u}{u - it})}
\]

(8.24)

By following the similar procedure as (3.20) and (3.21), We find

\[
\frac{d}{dx^2} \tilde{\rho}(x) = -\frac{1}{x^2} J_1(x)^2 + \frac{1}{x} J_0(x) J_2(x)
\]

(8.25)

where \(x = \sqrt{2N}\lambda\). Thus we obtain

\[
\tilde{\rho}(\lambda) = J_0(\sqrt{2N}\lambda)^2 + J_1(\sqrt{2N}\lambda)^2 - \frac{1}{\sqrt{2N}\lambda} J_0(\sqrt{2N}\lambda) J_1(\sqrt{2N}\lambda)
\]

(8.26)

The density of state \(\rho(\lambda)\) is obtained by \(\rho(\lambda) = \sqrt{2N}\lambda \tilde{\rho}(\lambda)\). The behavior is different from the complex case. The oscillations here are milder than the complex case.
9 Discussion

We have explored the cross-over behavior near the origin for the density of state for hermitean and for real matrices made of blocks. Although the result has been known \[4, 20, 21, 22\] for the one matrix model, our derivation, in particular our derivation using Kazakov’s method, provides new and simple expressions. We have also extended the discussion to rings and lattices of matrices. We have proved the universality of this cross-over behavior to first order in the deviation from a Gaussian distributions, and conjecture that this universality should hold in general.

We note also that in Kazakov’s method we encounter expressions which are valid for a non-vanishing external source matrix \(A\). Instead of letting all the eigenvalues of \(A\) go to zero, we may consider some specific choices of the eigenvalues of \(A\). Let us, for instance, consider \(a_\gamma = \cos(2\pi\gamma/N)\). Then the contour integral \(U_A(t)\) in (3.10) depends upon these \(a_\gamma\)’s. However in the cross-over region, these \(a_\gamma\)’s may be neglected with respect to the variable \(u\), which was scaled by a factor \(N\) in the integral. Thus such non-zero \(a_\gamma\)’s are irrelevant, and again we would obtain with them the same universal form in the large \(N\) limit. Unfortunately we have not been able to extend Kazakov’s method to the case of a lattice of matrices.

For the lattice of matrices, we could have considered also a representation of the Hamiltonian as a block matrix \(M\) of the form given in (2.1). Indeed, for a bi-partite lattice with an even number \(N\) of sites, we could divide the lattice into two sub-lattices A and B such that nearest neighbors belong to different sub-lattices. The sub-lattice A contains sites labelled from 1 to \(N/2\) and the sub-lattice B from \(N/2 + 1\) to \(N\). With such a labelling the Hamiltonian \(M\) for a particle hopping on this lattice would have a form like (2.1). However, the corresponding block matrix \(C\) would be sparse, containing many zero matrix elements. Indeed, the matrix \(C\) would be \(N/2 \times N/2\), i.e. \(C\) would contain \(N^2/4\) elements. For a \(D\)-dimensional hypercubic lattice the number of bonds in the lattice is \(DN\), where \(D\) is the spatial dimension. Then the ratio of the number of bonds to the number of matrix elements of \(C\) is given by \(4D/N\). When \(D = N/4\), the model reduces to the one matrix model. The example \(N = 4\) in the one-dimensional ring has been mentioned before. From this consideration, in the large \(D\) limit, if we keep \(D = N/4\), we have a simple one matrix model of the form in (2.1) and the universality of the cross-over behavior near the center of the spectrum should hold.
As already remarked in [4], one particular simple way to test for the universal oscillation studied here is to compare the height of the first peak in the density of states to the height of the second peak. According to (7.17), the density of states is proportional to the universal function

\[ r(y) = y(J_0^2(y) + J_1^2(y)) \]  

(9.1)

where \( y \) is the energy multiplied by some suitable constant. The positions of the peaks (and valleys) are determined by

\[ \frac{dr(y)}{dy} = J_0^2(y) - J_1^2(y) = 0 \]  

(9.2)

We find for example that the ratio of the height of the first peak to the height of the second peak is given by 1.218. (The ratio of the height of the first peak to the height of the first valley is 1.58.)

A possible application is to the problem of a single particle propagating on a square lattice penetrated by random magnetic flux [23], a problem that has recently attracted considerable attention [24]. Already the authors of [23] (see figure 1 in this reference) noted that the density of states exhibits oscillations for finite \( N \). It is far from clear that our present work can be applied to this problem since, as we have just explained, the relevant matrix \( C \) in this random flux problem is sparse, with the ratio of non-zero matrix elements to the total number matrix elements given by \( 16/N \). Nevertheless, we note that a recent numerical study of the random flux problem by Avishai and Kohmoto [25] found that the ratio of the height of the first peak to the height of the second peak was given by 1.25. We do not know whether the difference between 1.25 and 1.218 is real or due to numerical uncertainties.

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APPENDIX A: INTEGRATION FOR GENERAL L

We evaluate the following integral $D_0$ for general $L$, which appeared in (5.18),

$$D_0 = \int_0^\infty \prod_{i=1}^L dt_i [t_1 t_2 \cdots t_L]^{N-1} e^{-N \frac{t_i^2}{2N} h(t)} \delta(t_1 + t_3 + \cdots + t_{L-1} - 1)$$

$$\delta(t_2 + t_4 + \cdots + t_L - 1)$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^\infty dk_1 dk_2 \int \prod_{i=1}^L dt_i [t_1 t_2 \cdots t_L]^{N-1} e^{-N ph(t)}$$

$$e^{ik_2(t_1 + t_3 + \cdots + t_{L-1} - 1) + ik_2(t_2 + t_4 + \cdots + t_L - 1)}$$

$$\approx \frac{1}{(2\pi)^2} \frac{e^{-LN^2}}{L} \int \frac{2}{L} \delta(N - 1) e^{-\frac{L^2}{8N} \left[ \sum t_i^2 + t_i' t_i' + \cdots \right]}$$

$$\delta(t_1 + t_2 + \cdots + t_L) \prod_{i=1}^L dt_i dk_1 dk_2$$

(A.1)

We can interpret the quadratic form in the square bracket as $t'Ht$ with $H$ the quantum Hamiltonian of a particle hopping on a ring (with a trivial constant site energy.) We diagonalize $H$ and obtain for its eigenvalues the Bloch energies $\cos \frac{2\pi k}{L}$.

For example, in the case $L = 4$, we have

$$D_0 = \frac{1}{(2\pi)^2} \left( \frac{1}{2} \right)^4 e^{-2N} \int \frac{1}{2} e^{-2N(x_2^2 + x_3^2 + 2x_4^2)} e^{ik_1(x_1 + x_4) + ik_2(-x_1 + x_4)}$$

$$dk_1 dk_2 dx_1 dx_2 dx_3 dx_4$$

$$= \frac{1}{2} e^{-2N} \int dx_2 dx_3 dx_4 \delta(2x_4) e^{-2N(x_2^2 + x_3^2 + 2x_4^2)}$$

$$= \frac{\pi}{4N} \left( \frac{1}{2} \right)^4 e^{-2N}$$

(A.2)

where $x_1 = \frac{1}{2}(t_1 - t_2 + t_3 - t_4)$, $x_2 = \frac{1}{\sqrt{2}}(t_1 - t_3)$, $x_3 = \frac{1}{\sqrt{2}}(t_2 - t_4)$ and $x_4 = \frac{1}{2}(t_1 + t_2 + t_3 + t_4)$. For general $L$, we obtain

$$D_0 = \left( \frac{2}{L} \right)^{L(N-1)} e^{-\frac{2N}{L^2} \sum_{k=1}^L \prod_{k \neq \frac{L}{2}} \left( 1 - \cos \frac{2\pi}{L} k \right)^{-\frac{1}{2}}}$$

(A.3)
Using the identity of (5.29), we have

\[ \prod_{k=1, k \neq \frac{L}{2}}^{L-1} (1 - \cos \frac{2\pi k}{L}) = \frac{1}{2} \prod_{k=1}^{L-1} \sin^2 \left( \frac{\pi k}{L} \right) 2^{L-1} \]

\[ = \frac{L^2}{2L} \]  

(A.4)

Thus we get

\[ D_0 = \left( \frac{2}{L} \right)^{L(N-1)} e^{-\frac{4}{N} \left( \frac{\pi}{L} \right)^{L-1} \bar{D}} \]  

(A.5)

with

\[ \bar{D} = \frac{4^{L-1}}{L^L} \]  

(A.6)
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