A $q$-ANALOGUE OF $\bar{\alpha}$-WHITNEY NUMBERS

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We define the $(q, \bar{\alpha})$-Whitney numbers which are reduced to the $\bar{\alpha}$-Whitney numbers when $q \to 1$. Moreover, we obtain several properties of these numbers such as explicit formulas, recurrence relations, generating functions, orthogonality and inverse relations. Finally, we define the $\bar{\alpha}$-Whitney-Lah numbers as a generalization of the $r$-Whitney-Lah numbers and we introduce their important basic properties.

1. INTRODUCTION

El-Desouky et al. [5] introduced the $\bar{\alpha}$-Whitney numbers of both kinds as a new family of numbers generalizing many types of numbers such as $r$-Whitney numbers, Whitney numbers, $r$-Stirling numbers, Jacobi-Stirling numbers and Legendre-Stirling numbers.

The $\bar{\alpha}$-Whitney numbers of the first kind $w_{m, \bar{\alpha}}(n, k)$ and second kind $W_{m, \bar{\alpha}}(n, k)$ are defined by

\[
(x; \bar{\alpha}|m)_n = \sum_{k=0}^{n} w_{m, \bar{\alpha}}(n, k)x^k,
\]

and

\[
x^n = \sum_{k=0}^{n} W_{m, \bar{\alpha}}(n, k)(x; \bar{\alpha}|m)_k,
\]
where $\bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, and

$$(x; \bar{\alpha}|m)_n = \prod_{j=0}^{n-1} (x - \alpha_j - jm) \text{ with } (x; \bar{\alpha}|m)_0 = 1.$$  

The $\bar{\alpha}$-Whitney numbers of the first and second kind satisfying recurrence relations of the form:

$$w_{m,\bar{\alpha}}(n+1, k) = w_{m,\bar{\alpha}}(n,k-1) - (\alpha_n + nm)w_{m,\bar{\alpha}}(n,k),$$

$$W_{m,\bar{\alpha}}(n+1, k) = W_{m,\bar{\alpha}}(n,k-1) + (\alpha_k + km)W_{m,\bar{\alpha}}(n,k).$$

Note that the $\bar{\alpha}$-Whitney numbers coincide with the $r$-Whitney numbers and Whitney numbers by setting $\bar{\alpha} = (r, r, \ldots, r)$ and $\bar{\alpha} = (1, 1, \ldots, 1)$, respectively. Many properties of the $\bar{\alpha}$-Whitney numbers, $r$-Whitney numbers and Whitney numbers can be found in \[5, 4, 1, 8, 10, 11, 12\].

The organization of this article is as follows. In the next two sections, we define the $q$-analogue of the $\bar{\alpha}$-Whitney numbers of the first and second kind denoted by $w_{q,m,\bar{\alpha}}(n,k)$ and $W_{q,m,\bar{\alpha}}(n,k)$, respectively, and obtain their recurrence relations, explicit formulas and generating functions. In the third section, we obtain the orthogonality property of the both kinds of the $(q, \bar{\alpha})$-Whitney numbers which yields to the inverse relations. Moreover we give some important special cases. In the fourth section, we define the $\bar{\alpha}$-Whitney-Lah numbers and deduce its recurrence relation, explicit formula and matrix representation.

Let $0 < q < 1$, $x$ a real number and $n$ a non-negative integer. The number $[x]_q = \frac{1-q^x}{1-q}$ is called $q$-real number and the $q$-factorial of $n$ is defined by $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, with $[0]_q! = 1$. Finally, the $q$-falling factorial of order $n$ is defined by

$$([x]_q)_n = \prod_{j=0}^{n-1} [x-j]_q \text{ and } ([x]_q)_n = 1.$$  

Moreover, the following definitions and notation are introduced.

$$(x; \bar{\alpha}|q)_n = \prod_{j=0}^{n-1} [x - \alpha_j]_q = [x - \alpha_0]_q[x - \alpha_1]_q \cdots [x - \alpha_{n-1}]_q \text{ with } ([x; \bar{\alpha}|q])_0 = 1,$$

and

$$([x; \bar{\alpha}|m]_q)_n = \prod_{j=0}^{n-1} [x - \alpha_j - jm]_q, \text{ with } ([x; \bar{\alpha}|m]_q)_0 = 1,$$

and

$$\langle [x; \bar{\alpha}|m]_q \rangle_n = \prod_{j=0}^{n-1} ([x]_q - [\alpha_j + jm]_q), \text{ with } \langle [x; \bar{\alpha}|m]_q \rangle_0 = 1.$$
1. THE \((q, \bar{\alpha})\)-WHITNEY NUMBERS OF THE FIRST KIND

**Definition 1.** The \((q, \bar{\alpha})\)-Whitney numbers of the first kind \(w_{q,m,\bar{\alpha}}(n,k)\) are defined by

\[
\langle [x; \bar{\alpha}|m]_q \rangle_n = \sum_{k=0}^{n} w_{q,m,\bar{\alpha}}(n,k)[x]^k,
\]

where \(w_{q,m,\bar{\alpha}}(0,0) = 1\) and \(w_{q,m,\bar{\alpha}}(n,k) = 0\) for \(k > n\) or \(k < 0\).

Since for the \(q\)-numbers we have \([x - y]_q = q^{-\nu([x]_q - [y]_q)]\). Then

\[
([x; \bar{\alpha}|m]_q)_n = q^{-\sum_{i=0}^{n-1} \alpha_i + im} \prod_{j=0}^{n-1} ([x]_q - [\alpha_j + jm]_q).
\]

Thus Eq. (1.3) in Definition 1 can be written in the equivalent form

\[
([x; \bar{\alpha}|m]_q)_n = q^{-\sum_{i=0}^{n-1} \alpha_i + im} \sum_{k=0}^{n} w_{q,m,\bar{\alpha}}(n,k)[x]^k.
\]

In particular, note that \(w_{q,m,\bar{\alpha}}(n,k)\) is reduced to the \(w_{m,\bar{\alpha}}(n,k)\) when \(q \to 1\).

**Theorem 1.** The \((q, \bar{\alpha})\)-Whitney numbers of the first kind satisfy the recurrence relation

\[
w_{q,m,\bar{\alpha}}(n + 1,k) = w_{q,m,\bar{\alpha}}(n,k - 1) - [\alpha_n + nm]_q w_{q,m,\bar{\alpha}}(n,k),
\]

where \(n \geq k \geq 1\), and

\[
w_{q,m,\bar{\alpha}}(n,0) = (-1)^n \prod_{i=0}^{n-1} [\alpha_i + im]_q.
\]

**Proof.** Since \(\langle [x; \bar{\alpha}|m]_q \rangle_{n+1} = \langle [x; \bar{\alpha}|m]_q \rangle_n ([x]_q - [\alpha_n + nm]_q)\).

Using Eq. (1.3), we get

\[
\sum_{k=0}^{n+1} w_{q,m,\bar{\alpha}}(n + 1,k)[x]^k = \langle [x; \bar{\alpha}|m]_q \rangle_n ([x]_q - [\alpha_n + nm]_q)
\]

\[
= \sum_{k=0}^{n} w_{q,m,\bar{\alpha}}(n,k)[x]^k ([x]_q - [\alpha_n + nm]_q)
\]

\[
= \sum_{k=0}^{n} w_{q,m,\bar{\alpha}}(n,k)[x]^{k+1} - [\alpha_n + nm]_q \sum_{k=0}^{n} w_{q,m,\bar{\alpha}}(n,k)[x]^k
\]

\[
= \sum_{k=1}^{n+1} w_{q,m,\bar{\alpha}}(n,k - 1)[x]^k - [\alpha_n + nm]_q \sum_{k=0}^{n} w_{q,m,\bar{\alpha}}(n,k)[x]^k.
\]
Equating the coefficients of \([x]^k_q\) on both sides yields (1.4).

For \(k = 0\), we find

\[
w_{q,m,\alpha}(n + 1, 0) = -[\alpha_n + nm]_q w_{q,m,\alpha}(n, 0), \quad n = 0, 1, 2, \ldots,
\]

successive application gives (1.5).

**Definition 2.** The \((q, \alpha)\)-Whitney matrix of the first kind is the \(n \times n\) lower triangular matrix defined by

\[
W_1 := w_{q,m,\alpha}(n) := (w_{q,m,\alpha}(i, j))_{0 \leq i, j \leq n - 1}.
\]

For example when \(n = 4\) the matrix \(w_{q,m,\alpha}(n)\) is given by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-\alpha_0 & 1 & 0 & 0 \\
\alpha_0\cdot\alpha_1 + \alpha_1 & -\alpha_0 - \alpha_1 + m & 1 & 0 \\
\alpha_0\cdot\alpha_1 + \alpha_2 + \alpha_1 & \alpha_0\cdot\alpha_1 + m & \alpha_0 + m & -\alpha_0 - \alpha_1 + m & -\alpha_1 + m + 1
\end{pmatrix}
\]

In particular, we note that \(w_{q,m,\alpha}(n)\) is reduced to the \(\alpha\)-Whitney matrix of the first kind \([5]\) when \(q \to 1\). In addition at \(q \to 1\) and \(\alpha = (r, r, \ldots, r)\) the \(w_{q,m,\alpha}(n)\) is reduced to the \(r\)-Whitney matrix of the first kind \([12]\).

Mansour et al. \([9]\) derived a closed formula for all sequences satisfying a certain recurrence relation as follows:

**Theorem 2.** \([9, \text{Theorem 1.1}]\). Suppose \((a_i)_{i \geq 0}\) and \((b_i)_{i \geq 0}\) are sequences of numbers with \(b_i \neq b_j\) when \(i \neq j\) and

\[
u(n, k) = u(n - 1, k - 1) + (a_{n-1} + b_k)u(n - 1, k),
\]

with boundary conditions \(u(n, 0) = \prod_{i=0}^{n-1} (a_i + b_0)\) and \(u(0, k) = \delta_{0k}\), where \(\delta_{jk}\) is the Kronecker delta function, then

\[
u(n, k) = \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1} (b_j + a_i)}{\prod_{i=0:j \neq j}^{n-1} (b_j - b_i)} \right), \quad \forall n, k \in \mathbb{N}.
\]

**Remark 1.** \([9, \text{p. 25}]\)

1. The recurrence for \(u(n, k)\) is given by

\[
u(n, k) = \sum_{j=k}^{n} \nu(j - 1, k - 1) \prod_{i=j}^{n-1} (a_i + b_k).
\]

2. In the case when \(b_i = 0\) for all \(i\), then \(u(n, k)\) is the \((n - k)\)th elementary symmetric function of \(a_0, a_1, \ldots, a_{n-1}\). The elementary symmetric function \(\sigma_k\) is defined by

\[
\sigma_k(z_1, z_2, \ldots, z_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} \prod_{i=1}^{k} z_{j_i},
\]

where \(\sigma_0 = 1\) and \(\sigma_k = 0\) when \(n < k\) or \(k < 0\).
Theorem 3. The \((q, \alpha)\)-Whitney numbers of the first kind are given by
\[
w_{q,m,\alpha}(n,k) = (-1)^{n-k} \sigma_{n-k}([\alpha_0]_q, [\alpha_1 + m]_q, \ldots, [\alpha_{n-1} + (n-1)m]_q)
\]
(1.9)
\[
= (-1)^{n-k} \sum_{0 \leq j_1 < \cdots < j_{n-k} \leq n-1} \prod_{i=1}^{n-k} ([\alpha_{j_i} + j_i m]_q).
\]
and the following recurrence relation holds:
\[
w_{q,m,\bar{\alpha}}(n,k) = \sum_{j=k}^n (-1)^{n-j} w_{q,m,\alpha}(j-1, k-1) \prod_{i=j}^{n-1} ([\alpha_i + im]_q).
\]
(1.10)

Proof. Taking \(a_i = -[\alpha_i + im]_q\) and \(b_i = 0\), one can use Remark 1 to obtain Eq. (1.9) and Eq. (1.10). \(\square\)

From Eq. (1.9), we can obtain the generating function
\[
\sum_{k=0}^n w_{q,m,\alpha}(n,k)x^k = \prod_{j=0}^{n-1} (x - [\alpha_j + jm]_q).
\]
(1.11)

As \(q \to 1\), Eq. (1.10) reduces to a new recurrence relation for the \(\bar{\alpha}\)-Whitney numbers of the first kind given by
\[
w_{m,\bar{\alpha}}(n,k) = \sum_{j=k}^n (-1)^{n-j} w_{m,\alpha}(j-1, k-1) \prod_{i=j}^{n-1} ([\alpha_i + im]).
\]

2. THE \((q, \alpha)\)-WHITNEY NUMBERS OF THE SECOND KIND

Definition 3. The \((q, \alpha)\)-Whitney numbers of the second kind \(W_{m,\alpha}(n,k)\) are defined by
\[
[x]_q^n = \sum_{k=0}^n W_{q,m,\alpha}(n,k) ([x; \alpha|m]_q)_k.
\]
(1.12)

where \(W_{q,m,\alpha}(0,0) = 1\) and \(W_{q,m,\alpha}(n,k) = 0\) for \(k > n\) or \(k < 0\).

We notice that Eq. (1.12) in Definition 3 can be written in the equivalent form
\[
[x]_q^n = \sum_{k=0}^n q^{\sum_{i=1}^{k-1} \alpha_i + inm} W_{q,m,\alpha}(n,k) ([x; \alpha|m]_q)_k.
\]

Theorem 4. The \((q, \alpha)\)-Whitney numbers of the second kind satisfy the recurrence relation
\[
W_{q,m,\alpha}(n+1,k) = W_{q,m,\alpha}(n,k-1) + [\alpha_k + km]_q W_{q,m,\alpha}(n,k),
\]
(1.13)
where \(n \geq k \geq 1\), for \(k = 0\) we have
\[
W_{q,m,\alpha}(n,0) = [\alpha_0]_q^n.
\]
(1.14)
Thus, we have

\[ W_{q,m}(n+1,k) = \sum_{k=0}^{n} W_{q,m}(n,k) \langle [x; \alpha | m]_{q} \rangle_{k} \]

\[ = \sum_{k=0}^{n} W_{q,m}(n,k) \langle [x; \alpha | m]_{q} \rangle_{k} ([x]_{q} - [\alpha_{k} + km]_{q} + [\alpha_{k} + km]_{q}) \]

\[ = \sum_{k=0}^{n} W_{q,m}(n,k) \langle [x; \alpha | m]_{q} \rangle_{k+1} + \sum_{k=0}^{n} [\alpha_{k} + km]_{q} W_{q,m}(n,k) \langle [x; \alpha | m]_{q} \rangle_{k} \]

\[ = \sum_{k=1}^{n+1} W_{q,m}(n,k-1) \langle [x; \alpha | m]_{q} \rangle_{k} + \sum_{k=0}^{n} [\alpha_{k} + km]_{q} W_{q,m}(n,k) \langle [x; \alpha | m]_{q} \rangle_{k} \]

Equating the coefficients of \( [x; \alpha | m]_{q} \rangle_{k} \) on both sides, we obtain Eq. (1.13).

When \( k = 0 \), we get \( W_{q,m}(n+1,0) = [\alpha_{0}]_{q} W_{q,m}(n,0) \), \( n = 0, 1, 2, \ldots \). Thus \( W_{q,m}(n,0) = [\alpha_{0}]_{q}^{n} W_{q,m,0}(0,0) = [\alpha_{0}]_{q}^{n} \).

**Definition 4.** The \((q, \alpha)\)-Whitney matrix of the second kind is the \( n \times n \) lower triangular matrix defined by

\[ W_{2} := W_{q,m,\alpha}(n) := (W_{q,m,\alpha}(i,j))_{0 \leq i,j \leq n-1}. \]

For example when \( n = 4 \), the matrix \( W_{q,m,\alpha}(n) \) is given by

\[
W_{2} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
[\alpha_{0}]_{q} & 1 & 0 & 0 \\
[\alpha_{0}]_{q}^{2} & [\alpha_{0}]_{q} + [\alpha_{1} + m]_{q} & 1 & 0 \\
[\alpha_{0}]_{q}^{3} + [\alpha_{0}]_{q}^{2}[\alpha_{1} + m]_{q} + [\alpha_{1} + m]_{q}^{2} & [\alpha_{0}]_{q} + [\alpha_{1} + m]_{q} + [\alpha_{2} + 2m]_{q} & 1
\end{pmatrix}
\]

When \( q \rightarrow 1 \) the matrix \( W_{q,m,\alpha}(n) \) is reduced to the \( \alpha \)-Whitney matrix of the second kind [5], also at \( q \rightarrow 1 \) and \( \alpha = (r, r, \ldots, r) \) the \( W_{q,m,\alpha}(n) \) is reduced to the \( r \)-Whitney matrix of the second kind [4, 12].

**Theorem 5.** The \((q, \alpha)\)-Whitney numbers of the second kind \( W_{q,m,\alpha}(n,k) \) have the explicit formula

\[
W_{q,m,\alpha}(n,k) = \sum_{j=0}^{k} \frac{([\alpha_{j} + jm]_{q})^{n}}{\prod_{i=0,i \neq j}^{k} ([\alpha_{j} + jm]_{q} - [\alpha_{i} + im]_{q})},
\]
and satisfy the recurrence relation

\[ W_{q,m,\bar{\alpha}}(n,k) = \sum_{j=k}^{n} W_{q,m,\bar{\alpha}}(j-1,k-1) \prod_{i=j}^{n-1} ([\alpha_k + km]_q) \]

(1.16)

\[ = \sum_{j=k}^{n} W_{q,m,\bar{\alpha}}(j-1,k-1) ([\alpha_k + km]_q)^{n-j}. \]

**Proof.** Taking \( a_i = 0 \) and \( b_i = [\alpha_i + im]_q \) in (1.7) and (1.8), yield (1.15) and (1.16), respectively.

As \( q \to 1 \), the recurrence relation (1.16) reduces to a new recurrence relation for the \( \bar{\alpha} \)-Whitney numbers of the second kind given by

\[ W_{m,\bar{\alpha}}(n,k) = \sum_{j=k}^{n} W_{m,\bar{\alpha}}(j-1,k-1) \prod_{i=j}^{n-1} ([\alpha_k + km]) \]

\[ = \sum_{j=k}^{n} W_{m,\bar{\alpha}}(j-1,k-1)(\alpha_k + km)^{n-j}. \]

Using (1.15) we obtain the exponential generating function of the \((q,\bar{\alpha})\)-Whitney numbers of the second kind

\[ \sum_{n=0}^{\infty} W_{q,m,\bar{\alpha}}(n,k) \frac{t^n}{[n]_q!} = \sum_{j=0}^{k} \prod_{i=0,i\neq j}^{k} ([\alpha_j + jm]_q - [\alpha_i + im]_q) \sum_{n=0}^{\infty} \frac{([\alpha_j + jm]_q t)^n}{[n]_q!} \]

(1.17)

\[ = \sum_{j=0}^{k} \prod_{i=0,i\neq j}^{k} ([\alpha_j + jm]_q - [\alpha_i + im]_q) e_q([\alpha_j + jm]_q t). \]

**Theorem 6.** The generating function of \( W_{q,m,\bar{\alpha}}(n,k) \) is given by

\[ Y_{k,q}(t) = \sum_{n=0}^{\infty} W_{q,m,\bar{\alpha}}(n,k) t^n = k \prod_{j=0}^{k} (1 - [\alpha_j + jm]_q t)^{-1}, \quad t < \frac{1}{[\alpha_k + km]_q} \]

where \( k = 1, 2, 3, \ldots \), and

(1.19) \[ Y_{k,q}(0) = 0 \text{ for } k \geq 1 \text{ and } Y_{0,q}(t) = (1 - [\alpha_0]_q t)^{-1}. \]

**Proof.** Equation (1.19) can easily obtained from the definition of generating function

\[ Y_{0,q}(t) = \sum_{n=0}^{\infty} W_{q,m,\bar{\alpha}}(n,0) t^n = \sum_{n=0}^{\infty} [\alpha_0]_q^n t^n = \sum_{n=0}^{\infty} ([\alpha_0]_q t)^n = (1 - [\alpha_0]_q t)^{-1}. \]
From (1.13), we get
\[ \sum_{n=1}^{\infty} W_{q,m,\alpha}(n,k) t^n = \sum_{n=k}^{\infty} W_{q,m,\alpha}(n-1,k-1) t^n + (\alpha_k + km) \sum_{n=k}^{\infty} W_{q,m,\alpha}(n-1,k) t^n. \]

Thus we obtain the recurrence relation for the generating function \( Y_{k,q}(t) \)
\[ Y_{k,q}(t) = t Y_{k-1,q} + (\alpha_k + km) t Y_{k,q}(t), \quad k = 1, 2, \ldots . \]

Hence
\[ (1.20) \quad Y_{k,q}(t) = \frac{t}{(1 - [\alpha_k + km]_q t)} Y_{k-1,q}(t), \quad k = 1, 2, \ldots . \]

Applying successively this recurrence, we get Eq. (1.18).

The previous theorem shows that the numbers \( W_{q,m,\alpha}(n,k) \) are the complete symmetric function of the numbers \([\alpha_0]_q, [\alpha_1 + m]_q, \ldots, [\alpha_k + km]_q\) of order \( n - k \).

We obtain from Eq. (1.18)
\[ \sum_{n=k}^{\infty} W_{q,m,\alpha}(n,k) t^{n-k} = \prod_{j=0}^{k} (1 - [\alpha_j + jm]_q t)^{-1}, \quad t < \frac{1}{[\alpha_k + km]_q}. \]

Expanding the right hand side and comparing the coefficients of \( t^{n-k} \) yields
\[ W_{q,m,\alpha}(n,k) = \sum_{0 \leq j_1 \leq \cdots \leq j_{n-k} \leq k} \prod_{i=1}^{n-k} [\alpha_{j_i} + j_i m]_q. \]

3. ORTHOGONALITY AND INVERSE RELATIONS

The orthogonality and the inverse relations for the \( \alpha \)-Whitney numbers of both kinds were obtained in [5]. In this section, we establish analogous properties for the \((q, \alpha)\)-Whitney numbers of both kinds.

**Theorem 7.** The \((q, \alpha)\)-Whitney numbers of the first and second kind satisfy the following orthogonality relations:

\[ (1.21) \quad \sum_{k=j}^{n} W_{q,m,\alpha}(n,k) w_{q,m,\alpha}(k,j) = \delta_{nj}, \]

and
\[ (1.22) \quad \sum_{k=j}^{n} w_{q,m,\alpha}(n,k) W_{q,m,\alpha}(k,j) = \delta_{n,j}. \]
Proof. Using (1.3) and (1.12) give
\[
[x]_q^n = \sum_{k=0}^{n} W_{q,m,\alpha}(n, k) \langle [x; \alpha]|m\rangle_q^k
\]
\[
= \sum_{k=0}^{n} W_{q,m,\alpha}(n, k) \sum_{j=0}^{k} w_{q,m,\alpha}(k, j) [x]_q^j
\]
\[
= \sum_{j=0}^{n} \left( \sum_{k=j}^{n} W_{q,m,\alpha}(n, k) w_{q,m,\alpha}(k, j) \right) [x]_q^j.
\]
Comparing the coefficients of \([x]_q^j\) gives
\[
\sum_{k=j}^{n} W_{q,m,\alpha}(n, k) w_{q,m,\alpha}(k, j) = \delta_{nj}.
\]
The second relation can be proved similarly. \(\square\)

The orthogonality properties give the following identities
\[
W_2 W_1 = W_1 W_2 = I. \text{ Thus } W_2^{-1} = W_1 \text{ and } W_1^{-1} = W_2.
\]
The following theorem can easily be deduced from Theorem 7.

**Theorem 8.** The \((q, \alpha)\)-Whitney numbers of the first and second kind satisfy the following inverse relations

\[
(1.23) \quad f_n = \sum_{k=0}^{n} W_{q,m,\alpha}(n, k) g_k \iff g_n = \sum_{k=0}^{n} w_{q,m,\alpha}(n, k) f_k.
\]

**Proof.** If the condition
\[
f_n = \sum_{k=0}^{n} W_{q,m,\alpha}(n, k) g_k
\]
holds, then
\[
\sum_{k=0}^{n} w_{q,m,\alpha}(n, k) f_k = \sum_{k=0}^{n} w_{q,m,\alpha}(n, k) \sum_{m=0}^{k} W_{q,m,\alpha}(k, m) g_m
\]
\[
= \sum_{m=0}^{k} \left( \sum_{k=m}^{n} w_{q,m,\alpha}(n, k) W_{q,m,\alpha}(k, m) \right) g_m.
\]
By Theorem 7, we get
\[
\sum_{k=0}^{n} w_{q,m,\alpha}(n, k) f_k = \sum_{m=0}^{n} \delta_{mn} g_m = g_n.
\]
The converse can be shown similarly. \(\square\)
Special cases:

1. Setting $m = 1$ and $\overline{\alpha} = (r, r, \ldots, r) := r$, then (1.3) and (1.12), respectively, give
   
   \[ w_{q,1,r}(n,k) = s_q(n,k,r) \quad \text{and} \quad W_{q,1,r}(n,k) = S_q(n,k,r), \]

   where $s_q(n,k,r)$ and $S_q(n,k,r)$ are the non-central q-Stirling numbers of the first and second kind, respectively, see [3].

2. Setting $m = 1$ and $\overline{\alpha} = (0, 0, \ldots, 0) := 0$, hence (1.3) and (1.12), respectively, give
   
   \[ w_{q,1,0}(n,k) = s_q(n,k) \quad \text{and} \quad W_{q,1,0}(n,k) = S_q(n,k), \]

   where $s_q(n,k)$ and $S_q(n,k)$ are the q-Stirling numbers of the first and second kind, respectively, see [2, 7].

3. Setting $m = 1$ and $\alpha_j + j = \beta_j$, for $j = 0, 1, \ldots, n - 1$, then (1.3) and (1.12), respectively, give
   
   \[ w_{q,1,\alpha}(n,k) = s_{q,\beta}(n,k) \quad \text{and} \quad W_{q,1,\alpha}(n,k) = S_{q,\beta}(n,k), \]

   where $s_{q,\beta}(n,k)$ and $S_{q,\beta}(n,k)$ are the generalized q-Stirling numbers of the first and second kind (q-Comtet numbers), respectively, see [6].

4. THE $\overline{\alpha}$-WHITNEY-LAH NUMBERS

The signless Lah numbers $L(n,k) = \frac{n!}{(n-k)!}$ were first studied by Lah [13] and they expressed in terms of the signless stirling numbers $s(n,k)$ of the first kind, and the stirling numbers $S(n,k)$ of the second kind

\[ L(n,k) = \sum_{j=k}^n s(n,j)S(j,k). \]

Choen and Jung [4] defined the $r$-Whitney-Lah numbers $L_{m,r}(n,k)$ by

\[ L_{m,r}(n,k) = \sum_{j=k}^n (-1)^{n-j} w_{m,r}(n,j) W_{m,r}(j,k). \]

Analogously, we define the $\overline{\alpha}$-Whitney-Lah numbers $L_{m,\overline{\alpha}}(n,k)$ as follows:

\[ L_{m,\overline{\alpha}}(n,k) = \sum_{j=k}^n (-1)^{n-j} w_{m,\overline{\alpha}}(n,j) W_{m,\overline{\alpha}}(j,k), \]

where $L_{m,\overline{\alpha}}(0,0) = 1$ and $L_{m,\overline{\alpha}}(n,k) = 0$ for $n < k$ or $k < 0$. 

Theorem 9. The $\bar{\alpha}$-Whitney-Lah numbers $L_{m,\bar{\alpha}}(n,k)$ may be obtained from

\[(1.25) \quad \prod_{i=0}^{n-1} (x + \alpha_i + im) = \sum_{k=0}^{n} L_{m,\bar{\alpha}}(n,k)(x; \bar{\alpha}|m)_k.\]

Proof. Replacing $x$ by $-x$ in Eq. (1.1), we get

\[(1.26) \quad \prod_{i=0}^{n-1} (x + \alpha_i + im) = \sum_{j=0}^{n} (-1)^{n-j} w_{m,\bar{\alpha}}(n,j) x^j.\]

Hence

\[\sum_{k=0}^{n} L_{m,\bar{\alpha}}(n,k)(x; \bar{\alpha}|m)_k = \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{n-j} w_{m,\bar{\alpha}}(n,j) W_{m,\bar{\alpha}}(j,k)(x; \bar{\alpha}|m)_k\]
\[= \sum_{j=0}^{n} \sum_{k=0}^{j} (-1)^{n-j} w_{m,\bar{\alpha}}(n,j) W_{m,\bar{\alpha}}(j,k)(x; \bar{\alpha}|m)_k\]
\[= \sum_{j=0}^{n} (-1)^{n-j} w_{m,\bar{\alpha}}(n,j) x^j = \prod_{i=0}^{n-1} (x + \alpha_i + im)\]

\[\square\]

Theorem 10. The $\bar{\alpha}$-Whitney-Lah numbers satisfy the recurrence relation

\[(1.27) \quad L_{m,\bar{\alpha}}(n+1,k) = L_{m,\bar{\alpha}}(n,k-1) + (\alpha_k + \alpha_n + (k+n)m) L_{m,\bar{\alpha}}(n,k),\]

where $n \geq k \geq 1$, for $k = 0$ we have

\[(1.28) \quad L_{m,\bar{\alpha}}(n,0) = \prod_{i=0}^{n-1} (\alpha_0 + \alpha_i + im).\]

Proof. We can write

\[\prod_{i=0}^{n} (x + \alpha_i + im) = \prod_{i=0}^{n-1} (x + \alpha_i + im)(x - \alpha_k - km + \alpha_k + km + \alpha_n + nm).\]

Using (1.25), we get

\[\sum_{k=0}^{n} L_{m,\bar{\alpha}}(n+1,k)(x; \bar{\alpha}|m)_k\]
\[= \sum_{k=0}^{n} L_{m,\bar{\alpha}}(n,k)(x; \bar{\alpha}|m)_k((x - \alpha_k - km) + (\alpha_k + km + \alpha_n + nm))\]
\[= \sum_{k=0}^{n} L_{m,\bar{\alpha}}(n,k)(x; \bar{\alpha}|m)_{k+1} + \sum_{k=0}^{n} L_{m,\bar{\alpha}}(n,k)(x; \bar{\alpha}|m)_k(\alpha_k + km + \alpha_n + nm)\]
\[= \sum_{k=1}^{n+1} L_{m,\bar{\alpha}}(n,k-1)(x; \bar{\alpha}|m)_k + \sum_{k=0}^{n} L_{m,\bar{\alpha}}(n,k)(x; \bar{\alpha}|m)_k(\alpha_k + km + \alpha_n + nm).\]
Equating the coefficients of \((x; \alpha|m)_k\) on both sides, we obtain (1.27).

For \(k = 0\), we find

\[ L_{m,\alpha}(n+1,0) = L_{m,\alpha}(n,0) (\alpha_0 + \alpha_n + nm), \quad n = 0, 1, 2, \ldots. \]

Consequently, we get

\[ L_{m,\alpha}(n,0) = L_{m,\alpha}(0,0) 2\alpha_0 (\alpha_0 + \alpha_1 + m) \cdots (\alpha_0 + \alpha_{n-1} + (n-1)m). \]

\(\Box\)

**Special cases:**

1. The \(L_{m,\alpha}(n,k)\) is reduced to \(L(n,k)\) when \(m = 1\) and \(\alpha = (0,0,\ldots,0)\).
2. The \(L_{m,\alpha}(n,k)\) is reduced to \(L_{m,r}(n,k)\) when \(\alpha = (r,r,\ldots,r)\).
3. The \(L_{m,\alpha}(n,k)\) is reduced to the \(r\)-Lah numbers \(L_r(n+r,k+r)\) when \(m = 1\) and \(\alpha = (r,r,\ldots,r)\), see [14].

Defining the \(\bar{\alpha}\)-Whitney-Lah matrix as

\[ L := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \end{pmatrix} \]

For example when \(n = 4\) the matrix \(L_{m,\alpha}(n)\) is given by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
2\alpha_0 & 1 & 0 & 0 \\
2\alpha_0(\alpha_0+\alpha_1+m) & 2(\alpha_0+\alpha_1+m) & 1 & 0 \\
2\alpha_0(\alpha_0+\alpha_1+m)(\alpha_0+\alpha_2+2m) & 2(\alpha_0+\alpha_1+m)(\alpha_0+\alpha_1+\alpha_2+3m) & 2(\alpha_0+\alpha_1+\alpha_2+3m) & 1
\end{pmatrix}
\]

In particular, when \(\bar{\alpha} = (r,r,\ldots,r)\) the \(L_{m,\alpha}(n)\) is reduced to the \(r\)-Whitney-Lah matrix [12].

**Theorem 11.** The \(\bar{\alpha}\)-Whitney-Lah numbers \(L_{m,\alpha}(n,k)\) have the explicit formula

\[ (1.29) \quad L_{m,\alpha}(n,k) = \sum_{j=0}^{k} \frac{\prod_{i=0}^{k-1}(\alpha_j + jm + \alpha_i + im)}{\prod_{i=0, i \neq j}^{k-1}(\alpha_j + jm - \alpha_i - im)}, \]

and the recurrence relation

\[ (1.30) \quad L_{m,\alpha}(n,k) = \sum_{j=k}^{n} L_{m,\alpha}(j-1,k-1) \prod_{i=j}^{n-1}(\alpha_i + im + \alpha_k + km). \]

Proof. The proof follows by setting \(a_i = b_i = \alpha_i + im\) in (1.7) and (1.8). \(\Box\)

In particular, by setting \(\bar{\alpha} = (r,r,\ldots,r)\) we obtain the explicit formula and recurrence relation for \(L_{m,r}(n,k)\) as follows:
Corollary 1. The \( r \)-Whitney-Lah numbers \( L_{m,r}(n,k) \) satisfy the following:

\[
L_{m,r}(n,k) = \frac{1}{m^k k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \prod_{i=0}^{n-1} (2r + jm + im).
\]

Thus from (1.31) and (1.33) we obtain the following combinatorial identity

\[
\frac{1}{m^k k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \prod_{i=0}^{n-1} (2r + jm + im) = \binom{n}{k} \prod_{i=0}^{n-k-1} (2r + km + im).
\]

4.1 MATRIX REPRESENTATIONS

Let \( w, W \) and \( L \) denote infinite lower triangular matrices whose \((n,k)\)-th entries are \( w_{m,A}(n,k) \), \( W_{m,A}(n,k) \), and \( L_{m,A}(n,k) \), respectively. Furthermore, let \( D \) be the infinite diagonal matrix whose \((n,k)\)-th entry is \( D(n,k) = (-1)^{n} \delta_{nk} \), hence \( D^{-1} = D \), and \( DD^{-1} = I \). Equation (1.24) can be written in the matrix form

\[
L = DwDW.
\]

El-Desouky et al. [5] showed that \( w^{-1} = W, W^{-1} = w \). Thus

\[
L^{-1} = W^{-1} Dw^{-1} D = wDWD = DDwDWD = DLD.
\]

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