ON THE GLOBAL WELL-POSEDNESS OF 3-D BOUSSINESQ SYSTEM WITH PARTIAL VISCOSITY AND AXISYMMETRIC DATA

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ABSTRACT. In this paper we prove the global well-posedness for the Three-dimensional Boussinesq system with axisymmetric initial data. This system couples the Navier-Stokes equation with vanishing the horizontal viscosity with a transport-diffusion equation governing the temperature.

1. Introduction. Boussinesq system are widely used to model the dynamics of the ocean or the atmosphere. They arise from the density dependent fluid equations by using the so-called Boussinesq approximation which consists in neglecting the density dependence in all the terms but the one involving the gravity. This approximation can be justified from compressible fluid equations by a simultaneous low Mach number/Froude number limit, we refer to [15, 7, 9, 30] for a rigorous justification. The Boussinesq system is described by

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu_1 \partial_z^2 u - \nu_2 (\partial_x^2 + \partial_y^2) u + \nabla p &= \rho e_z \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\partial_t \rho + u \cdot \nabla \rho - \kappa_1 (\partial_z^2 + \partial_y^2) \rho - \kappa_2 \partial_z^2 \rho &= 0, \\
\text{div } u &= 0, \\
u_{|t=0} &= u_0, \quad \rho_{|t=0} = \rho_0.
\end{aligned}
\]

Here, the velocity \( u = (u^1, u^2, u^3) \) is a three vector field with zero divergence, the scalar function \( \rho \) denotes the density or the temperature and \( p \) the pressure of the fluid. The term \( \rho e_z \) where \( e_z = (0, 0, 1)^t \) takes into account the influence of the gravity and the stratification on the motion of the fluid. Note that when the initial density \( \rho_0 \) is identically zero (or constant) and \( \nu_1 = \nu_2 \) then the above system is reduced to the classical incompressible Navier-Stokes equation

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p &= 0 \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\text{div } u &= 0, \\
u_{|t=0} &= u_0.
\end{aligned}
\]

This system has been studied by many authors due to his physical background and mathematical significance. A well-known criterion for the existence of global

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smooth solution is the Beale-Kato-Majda (see [5]). It states that the control of the vorticity of the fluid $\omega = \text{curl } u$ in $L^1_{\text{loc}}(\mathbb{R}^+, L^\infty)$ is sufficient to get global well-posedness. The existence of global weak solutions in the energy space for (2) goes back to J. Leray [27]. But, the uniqueness of these solutions is only known in space dimension two. We know also that smooth solutions are global in dimension two and for higher dimensions when the data are small in some critical spaces (see [26]). In similar way, the global well-posedness for two-dimensional Boussinesq systems has been shown in various function spaces and for different viscosities. We can cite for example [1, 10, 18, 19, 24]. The reader can see also [20] for the global well-posedness in the critical spaces. Other interesting results on the two-dimensional Boussinesq equations can be found in [20, 21].

For the 3-D Boussinesq equations, R. Danchin and M. Paicu [13] proved the global existence of weak solution for $L^2$- data and the global well-posedness for small initial data. In [12] They also obtained an existence and uniqueness results for small initial data belonging to some critical Lorentz spaces. But there is little study about the global well-posedness result for large initial data, even for the three-dimensional Navier-Stokes equations. In [29] a global result was established in the case when the diffusion and the viscosity only occurs in the horizontal direction, more precisely, if $\nu_1 = 0$ and $\kappa_1 = 0$. In this paper we are interested to study the following system

$$
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \partial_z^2 u + \nabla p &= \rho \mathbf{e}_z \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\partial_t \rho + u \cdot \nabla \rho - \kappa \Delta \rho &= 0, \\
\text{div } u &= 0, \\
u |_{t=0} &= u_0, \quad \rho |_{t=0} = \rho_0.
\end{align*}
$$

which couples the Navier-Stokes system, when the horizontal viscosity is zero (see [3]), with a transport-diffusion governing the temperature.

In this case, due to the lack of regularity in two horizontal variables. By using a regularizing effect only in the vertical direction seems very difficult to recover any regularization in all variables in the general case. This is the main reason for which we restrict ourself to study axisymmetric data because in this situation we have

$$\text{div } u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0.$$ 

For the rest of the paper, we assume that $\nu = \kappa = 1$ for sake of simplicity. Our system becomes

$$
\begin{align*}
(Bq) \quad \begin{cases}
\partial_t u + u \cdot \nabla u - \partial_z^2 u + \nabla p &= \rho \mathbf{e}_z \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\partial_t \rho + u \cdot \nabla \rho - \Delta \rho &= 0, \\
\text{div } u &= 0, \\
u |_{t=0} &= u_0, \quad \rho |_{t=0} = \rho_0.
\end{cases}
\end{align*}
$$

The classical Navier-Stokes system (2) (in the case when $\nu_1 = \nu_2 > 0$ and $\rho_0$ is constant) has already been studied by many authors. In [23] T. Hmidi and F. Rousset proved the global well-posedness for the Navier-Stokes- Boussinesq system with axisymmetric data by virtue of the structure of the coupling between two equations of (1.1) with $\nu_1 = \nu_2$ and $\kappa_1 = \kappa_2$. Other interesting results on the axisymmetric flows can be found in [2, 28]. Note that if $\nu_1 = \nu_2 = 0$ and $\kappa_1 = \kappa_2 = \ldots$
1, the system (1) is reduced to the Euler-Boussinesq system which is studied by T. Hmidi and F. Rousset in [22] and [20]
\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= \rho e_z \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\partial_t \rho + u \cdot \nabla \rho - \Delta \rho &= 0, \\
\text{div} \, u &= 0, \\
\rho|_{t=0} &= \rho_0, \\
\rho|_{t=0} &= \rho_0.
\end{align*}
\] (4)

In [22] T. Hmidi and F. Rousset prove the global well-posedness for the three-dimensional Euler-Boussinesq system with axisymmetric initial data without swirl in $H^s$ for $s > \frac{5}{2}$. This system couples the Euler equation with a transport-diffusion equation governing the temperature.

Before going further in the details, let recall some algebraic properties of the axisymmetric vector field (see for example [23]) and discuss the special structure of the vorticity of our system.

First, we give some general statement in cylindrical coordinates: we say that the vector field $u$ is axisymmetric if it satisfies
\[
\mathcal{R}_\alpha \{u(R_\alpha x)\} = u(x), \quad \forall \alpha \in [0, 2\pi], \quad \forall x \in \mathbb{R}^3,
\]
where $\mathcal{R}_\alpha$ denotes the rotation of axis $(oz)$ and with angle $\alpha$. Moreover, an axisymmetric vector field $u$ is called without swirl if it has the form:
\[
u(t, x) = u^r(r, z)e_r + u^z(r, z)e_z, \quad x = (x_1, x_2, x_3), \quad r = \sqrt{x_1^2 + x_2^2} \quad \text{and} \quad z = x_3,
\]
where $(e_r, e_\theta, e_z)$ is the cylindrical basis of $\mathbb{R}^3$.

Similarly, a scalar function defined in $\mathbb{R}^3$ is axisymmetric if it does not depend on the angular variable $\theta$, which means that
\[
f(\mathcal{R}_\alpha x) = f(x), \quad \forall x \in \mathbb{R}^3, \quad \forall \alpha \in [0, 2\pi].
\]

Direct computation yields that the vorticity $\omega \overset{\text{def}}{=} \text{curl} \, u$ of the vector field takes the form
\[
\omega = (\partial_z u^r - \partial_r u^z)e_\theta \overset{\text{def}}{=} \omega_\theta e_\theta.
\]

On the other hand, we know that
\[
u \cdot \nabla = u^r \partial_r + u^z \partial_z, \quad \text{div} \, u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z \quad \text{and} \quad \omega \cdot \nabla u = \frac{u^r}{r} \omega.
\]

Let give a few comments about the result of T. Hmidi and F. Rousset in [22]. Indeed, in view of the Euler equations the crucial part is to get an a priori estimate for $\xi = \frac{\omega_\theta}{\rho}$ in $L^3$. See for example M. Ukhovskii and V. Yudovitch [34] and independently by O. A. Ladyzhenskaya [25]. You can also see [14] and [32]. So $\xi$ satisfies the following equation:
\[
\partial_t \xi + u \cdot \nabla \xi = -\frac{\partial_r \rho}{r},
\] (5)

And consequently the main difficulty is to find some strong a priori estimates on $\rho$ to control the term in the right-hand side of (5). Note that the term $\frac{2}{r} \rho$ is considered as Laplacian of $\rho$ and thus one can try to use smoothing effects to control it. If the advection term vanishing one can use the maximal smoothing effects of the heat semigroup so one gain two derivatives. The main difficulty if one wants to use this argument to deal with the advection term. Indeed, the only control on $u$ that one have at our disposal is a $L^3_{loc} \, L^2$ estimate (which comes from the basic
energy estimate) and this is not sufficient to obtain an estimate for \( \Delta \rho \) in \( L^1_{\text{loc}}(L^p) \) by considering the convection term as a source term and by using the maximal smoothing effect of the heat equation. Consequently, their strategy for the proof will be to use more carefully the structure of the coupling between the two equations of (4) in order to find suitable a priori estimates for \((\xi, \rho)\). Their main idea is to use an approach that was successfully used for the study of two-dimensional systems with a critical dissipation, see [20] and [21] and the Navier-Stokes-Boussinesq system with axisymmetric data in [23]. It consists in diagonalizing the linear part of the system satisfied by \( \xi \) and \( \rho \). They introduced a new unknown \( \Gamma \) which here formally reads

\[
\Gamma = \xi + \frac{\partial}{r} \Delta^{-1} \rho, \tag{6}
\]

and they study the system satisfied by \((\Gamma, \rho)\) which is given by

\[
\partial_t \Gamma + u.\nabla \Gamma = -\left[ \frac{\partial}{r} \Delta^{-1}, u.\nabla \right] \rho, \quad \partial_t \rho + u.\nabla \rho = \Delta \rho,
\]

where \( \left[ \frac{\partial}{r} \Delta^{-1}, u.\nabla \right] \rho \) is the commutator defined by

\[
\left[ \frac{\partial}{r} \Delta^{-1}, u.\nabla \right] \rho = \frac{\partial}{r} \Delta^{-1} (u.\nabla \rho) - u.\nabla (\frac{\partial}{r} \Delta^{-1} \rho).
\]

They had to study the operator \( \frac{\partial}{r} \Delta^{-1} \) over axisymmetric functions. They proved that this operator behaves well and it acts continuously on \( L^p \), in particular on \( L^{3,1} \). After that they had to estimate the commutator which is the main technical part, so they obtained

\[
\left\| \frac{\partial}{r} \Delta^{-1}, u.\nabla \right\|_{L^{3,1}} \lesssim \| \omega_0/r \|_{L^{3,1}} (\| x_h \rho \|_{B^0_{\infty,1}} \cap L^2 + \| \rho \|_{B^{3,1}_{2,1}} ). \tag{7}
\]

In the right-hand side of (7), \( \| \rho \|_{B^{3,1}_{2,1}} \) and \( \| \rho x_h \|_{L^2} \) can be controlled in terms of the initial data only by using the smoothing effect of the convection-diffusion equation for \( \rho \) and standard energy estimates. And Consequently, from this commutator estimate, they obtained that

\[
\| \xi(t) \|_{L^{3,1}} \leq C(t) e^{C \| \rho x_h \|_{L^1_{t} B^0_{\infty,1}} }.
\]

In the difficult next step they had to estimate the term \( \| \rho x_h \|_{L^1_{t} B^0_{\infty,1}} \) which is done in two steps. The first one is to get a global \( L^\infty \) estimate of \( \rho x_h \) in terms of the initial data only and then in a second step, we shall prove a logarithmic estimate for the \( B^0_{\infty,1} \) norm of \( x_h \rho \) in terms of the \( L^{3,1} \) norm of \( \xi \). So they obtained

\[
\| \rho x_h \|_{L^1_{t} B^0_{\infty,1}} \leq C_0(t) (1 + \int_0^t h(\tau) \log(2 + \| \xi(\tau) \|_{L^{3,1}_{t}}) \, d\tau).
\]

Hence, they controlled \( \| \xi \|_{L^{3,1}} \) globally in time.

So our goal here is to extend these global well-posedness done in [22] to the Boussinesq system with partial viscosity \((Bq)\) with a very rough initial data. Our main result is as follows.

**Theorem 1.1.** Consider the Boussinesq system \((Bq)\). Let \( \omega_0 \in L^2 \) be an axisymmetric divergence free vector field without swirl such that its vorticity satisfies \( \omega_0 \in L^\frac{5}{2} \cap L^p, \) \( \frac{3}{2} < p \leq 2 \) and \( \frac{5}{2} \in L^{5,1} \cap L^2, \) and let \( \rho_0 \in L^p \cap L^m \cap L^{3,1}, \) \( m > 6, \) \( 1 < p \leq 2 \)
be an axisymmetric scalar function such that $r^2 \rho_0 \in L^2$ and $\nabla \rho_0 \in L^q$, $\frac{3}{2} < q \leq 2$. Then the system (1.1) has a global in time solution $(u, \rho)$ such that

$$
\omega \in L^\infty_{loc}(\mathbb{R}^3), \quad \partial_z \omega \in L^2_{loc}(\mathbb{R}^3)
$$

$$
\omega \in L^\infty_{loc}(\mathbb{R}^3), \quad \partial_z \omega \in L^2_{loc}(\mathbb{R}^3)
$$

$$
\rho \in L^\infty_{loc}(\mathbb{R}^3), \quad \nabla \rho \in L^2_{loc}(\mathbb{R}^3)
$$

Furthermore, if $\partial_\nu \omega_0 \in L^2$ and $\omega_0 \in L^3$, then

$$
\partial_\nu \omega \in L^\infty_{loc}(\mathbb{R}^3), \quad \partial_\nu \partial_z \omega \in L^2_{loc}(\mathbb{R}^3)
$$

and the solution is unique.

**Remark 1.** (1) Concerning the velocity, one class of scaling invariant spaces is the anisotropic-type Besov spaces $B^{\frac{3}{2}+1, \frac{3}{2}-1}_{p,1}(\mathbb{R}^2 \times \mathbb{R})$, with $p \in [1, \infty]$, which have the same scale as isotropic-type Besov spaces $B^{3}_{p,1}(\mathbb{R}^3)$. This implies that our initial data is critical in terms of scaling.

(2) The advantage of adding the vertical laplacian with respect to [22] is to impose fewer regularities on initial data, more exactly for data in Yudovich type space.

Let us give a few comments about our result.

Note that in view of Navier-Stokes equation when the horizontal viscosity is zero (see [3]), the crucial part is to get an a priori estimate for $\xi$ in $L^{\frac{2}{3},1}$. The equation for $\xi = \frac{\omega}{r}$ is as follows

$$
\partial_t \xi + u.\nabla \xi - \partial^2_z \xi = -\frac{\partial_r \rho}{r}.
$$

And $(\Gamma, \rho)$, with $\Gamma$ defined in (6), satisfies the following equations

$$
\partial_t \Gamma + u.\nabla \Gamma - \partial^2_z \Gamma = -\frac{\partial_r \Gamma}{r} \Delta^{-1} - u.\nabla \rho - \partial^2_z \partial_r \Delta^{-1} \rho,
$$

$$
\partial_t \rho + u.\nabla \rho = \Delta \rho.
$$

So from Proposition 4 we obtain

$$
||\Gamma(t)||_{L^\infty_t L^{\frac{2}{3},1}} + ||\partial_z \Gamma||_{L^1_t L^{\frac{3}{2},1}} \lesssim ||\Gamma_0||_{L^{\frac{2}{3},1}} + ||\frac{\partial_r \Delta^{-1}}{r} - u.\nabla \rho||_{L^1_t L^{\frac{3}{2},1}} + ||\Delta \rho||_{L^1_t L^{\frac{3}{2},1}}.
$$

It follows that the control of $\Gamma$ is equivalent to the control of $\xi$ in $L^{\frac{2}{3},1}$.

Note that if we forget the commutator and the term $\partial^2_z \partial_r \Delta^{-1} \rho$ for a while, we immediately get an a priori $L^p$ estimate for $\Gamma$ for every $p$ from which we can hope to get an $L^p$ estimate for $\xi$, if the operator $\frac{\partial_r \Delta^{-1}}{r}$ behaves well. As we have seen in [22], to make this argument rigorous, we need first to study the action of the operator $\frac{\partial_r \Delta^{-1}}{r}$ over axisymmetric functions. This is done in Proposition 5 where this operator takes

$$
\frac{\partial_r \Delta^{-1}}{r} = \sum_{i,j} a_{ij}(x) R_{ij}
$$

So $\frac{\partial_r \Delta^{-1}}{r}$ acts continuously on $L^{p,q}$, and hence we have

$$
||\frac{\partial_r \Delta^{-1}}{r} \rho(t)||_{L^{\frac{2}{3},1}} \lesssim ||\rho(t)||_{L^{\frac{2}{3},1}} \lesssim ||\rho_0||_{L^{\frac{2}{3},1}}.
$$
Now we have to estimate in a suitable way the commutator term $[[\frac{\partial r }{r}, \Delta^{-1}, u, \nabla]\rho]$. It seems that there is no hope to bound the commutator without using unknown quantities because there is no other known a priori estimates of the velocity except that given by energy estimate which is not strong enough. We shall prove in Proposition 6 that

\[
\|[\frac{\partial r }{r}, \Delta^{-1}, u, \nabla]\rho]\|_{L^2}\leq ||\omega_\theta/r||_{L^2}, (||x_1\rho||_{B_{\infty,1}^0\cap L^2} + ||\rho||_{B_{\infty,1}^0\cap L^2}) + ||\omega_\theta/r||_{L^2}\|\rho\|_{B_{\infty,1}^2}.
\]

(10)

To prove the above estimate we follow the proof of (7) which combines the use of para differential calculus and some harmonic analysis results and also requires a careful use of the property that the velocity $u$ is axisymmetric without swirl in the right-hand of (10), we do the same thing as in [22] (see Propositions 7 and 8).

And to control $\|\frac{\omega_\theta}{r}\|_{L^\infty L^2}$, we shall estimate $\Gamma$ in $L^\infty L^2$ and we obtain

\[
||\Gamma||_{L^\infty L^2} + ||\partial_2 \Gamma||_{L^2 L^2} \leq ||\Gamma_0||_{L^2} + ||\frac{1}{r^2} \partial_t \Delta^{-1}, u, \nabla]\rho||_{L^2 L^2} + ||\rho_0||_{L^2}.
\]

So we need the control of the commutator in $L^2$ which is done in Proposition 6 and we obtain

\[
\|[[\frac{\partial r }{r}, \Delta^{-1}, u, \nabla]\rho]\|_{L^2} \leq ||\omega_\theta/r||_{L^2}, (||x_1\rho||_{B_{\infty,1}^0\cap L^2} + ||\rho||_{B_{\infty,1}^2}).
\]

Hence we have by Gronwall inequality

\[
||\omega_\theta/r||_{L_t^\infty L^2} + ||\partial_2 \omega_\theta/r||_{L_t^2 L^2} \leq C_0 e^{C_0(1+t^2 + ||x_\theta r||_{L_t^1 B_{\infty,1}^0})}
\]

Therefore, we have to control $||x_\theta r||_{L_t^1 B_{\infty,1}^0}$. So do that we follow the same approach as in [22] and we obtain a logarithmic estimate for the $B_{\infty,1}^0$ norm of $x_\theta r$ in terms of the $L^2_t$ norm of $\xi$,

\[
||x_\theta r||_{L^1_t L_t^1 B_{\infty,1}^0} \leq C_0(1+t^2) + C_0 \int_0^t (\tau^2 + \tau^{-3}) \log(2 + ||\omega_\theta/r||_{L_t^\infty L^2})d\tau.
\]

(11)

Then we obtain that

\[
||\omega_\theta/r||_{L_t^\infty L^2} + ||\partial_2 \omega_\theta/r||_{L_t^2 L^2} \leq \phi_2(t),
\]

(12)

where $\phi_2(t)$ is given by (81).

The next important step is the control of the term $||\Delta \rho||_{L_t^1 L_t^\frac{2}{1} B_{\infty,1}^0}$ which is not present in [22] and which is the main part. In fact, the control of $||\Delta \rho||_{L_t^1 L_t^\frac{2}{1} B_{\infty,1}^0}$ is done by using the Duhamel formula and the embedding $L_t^{\frac{\infty}{2}} \cap L_t^q \subset L_t^\frac{q}{2}$ we get

\[
||\Delta \rho||_{L_t^1 L_t^\frac{2}{1} B_{\infty,1}^0} \leq t^{\frac{1}{2}} ||\nabla \rho_0||_{L_t^2 L^p} + t^{\frac{1}{2}} ||\omega||_{L_t^\infty L^2} ||\nabla \rho||_{L_t^2 L_t^\frac{2}{1} L_t^\infty L_t^\frac{q}{2}} + t^{\frac{1}{2}} ||\omega||_{L_t^\infty L^2} ||\nabla \rho||_{L_t^2 L_t^\frac{2}{1} L_t^\infty L_t^\frac{q}{2}}.
\]

(13)

So we need to control the term $||\omega||_{L_t^\infty L^2}$ which needs the control of $||\omega/r||_{L_t^\infty L^2}$, $||\partial_2 \omega||_{L_t^2 L_t^\frac{2}{1}}$, and $||\partial_2 \omega||_{L_t^2 L_t^\frac{2}{1}}$.

Now, for the control of $||\partial_2 \omega||_{L_t^2 L_t^\frac{2}{1}}$, we use Proposition 3, Proposition 4 and (12) we obtain that

\[
||\omega(t)||_{L_t^\infty L^2} + ||\partial_2 \omega||_{L_t^2 L_t^\frac{2}{1}} \leq \phi_2(t),
\]

(14)
and from Proposition 3, (14) and (12) we have that

$$\|\omega(t)\|_{L^\infty_t L^2_x} + \|\partial_x \omega\|_{L^2_t L^2_x} \leq \phi_2(t),$$

(15)

Therefore, from Proposition 3, Proposition 4, (15), (12) and Gronwall inequality we deduce that

$$\|\omega(t)\|_{L^\infty_t L^2_x} + \|\partial_x \omega\|_{L^2_t L^2_x} \leq \phi_2(t).$$

(16)

Finally, we use Duhamel formula and interpolation’s inequality in order to estimate the term $\|\nabla \rho\|_{L^p_t L^r_x}$ in the right-hand of (13). Hence we obtain that

$$\|\Delta \rho\|_{L^1_t L^{2,1}_x} \leq \phi_2(t).$$

(17)

So by (10), (17), (11) and Gronwall inequality we obtain

$$\|\frac{\omega \theta}{r}\|_{L^\infty_t L^{2,1}_x} + \|\partial_z \frac{\omega \theta}{r}\|_{L^2_t L^2_x} \leq C(t, \rho_0, \omega_0).$$

This paper is organized as follows. In section 2 we fix the notations, give the definitions of the functional spaces, in particular Besov and Lorentz spaces, that we shall use and state some of their useful properties. Next, in section 3, we study the commutator $[\frac{\partial}{\partial t}, \Delta^{-1}, u, \nabla]$. In section 4, we turn to the proof of a priori estimates for sufficiently smooth solutions of $(Bq)$. We first prove in Proposition 8 some basic energy estimates. Next, we study the moments of $\rho$ in Proposition 10. In Proposition 14 we estimate $\Delta \rho$ in $L^1_t L^{2,1}_x$ and then we control $\|\frac{\omega \theta}{r}\|_{L^{2,1}_x}$ in Proposition 15. In section 5, we give the proof of Theorem 1.1: we obtain the existence part by using the a priori estimates and an approximation argument, we prove the uniqueness and then we find the propagation of regularities part.

Let us complete this section with the notations we are going to use in this context.

**Notations.** Let $A, B$ be two operators, we denote $[A; B] = AB - BA$, the commutator between $A$ and $B$. For $a \lesssim b$, we mean that there is a uniform constant $C$, which may be different on different lines, such that $a \leq Cb$ and $C_0$ denotes a positive constant depending on the initial data only.

Let $X$ a Banach space and $I$ an interval of $\mathbb{R}$, we denote by $C(I; X)$ the set of continuous functions on $I$ with values in $X$, and by $C_b(I; X)$ the subset of bounded functions of $C(I; X)$. For $q \in [1, +\infty]$, the notation $L^q(I; X)$ stands for the set of measurable functions on $I$ with values in $X$, such that $t \mapsto \|f(t)\|_X$ belongs to $L^q(I)$.

**2. Littlewood-Paley analysis and Lorentz spaces.** The proof of Theorem 1.1 requires Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case $x \in \mathbb{R}^3$ (see e.g. [4]). Let $\varphi$ be a smooth function supported in the annulus $C \overset{\text{def}}{=} \{\xi \in \mathbb{R}^3, \frac{2}{3} \leq |\xi| \leq \frac{8}{3}\}$ and $\chi(\xi)$ be a smooth function supported in the ball $B \overset{\text{def}}{=} \{\xi \in \mathbb{R}^3, |\xi| \leq \frac{2}{3}\}$ such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1 \quad \text{for} \quad \xi \neq 0 \quad \text{and} \quad \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^3.$$

Then for $u \in S'(\mathbb{R}^3)$, we set

$$\forall \ q \geq 0, \ \Delta_q u \overset{\text{def}}{=} \varphi(2^{-q} D) u, \ \Delta_{-1} u \overset{\text{def}}{=} \chi(D) u \quad \text{and} \quad S_q u \overset{\text{def}}{=} \sum_{-1 \leq q' \leq q - 1} \Delta_{q'} u,$$

(18)
we have the formal decomposition
\[ u = \sum_{q \geq 1} \Delta_q u \quad \forall u \in \mathcal{S}'(\mathbb{R}^3). \]  

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:
\[ \Delta_j \Delta_k u \equiv 0 \quad \text{if} \quad |j - k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} u \Delta_k u) \equiv 0 \quad \text{if} \quad |j - k| \geq 5. \]  

We recall now the definition of inhomogeneous Besov spaces and Bernstein type inequalities from [4].

**Definition 2.1** (Definition 2.15 of [4]). Let \((p, r) \in [1, +\infty]^2, s \in \mathbb{R}\) and \(u \in \mathcal{S}'(\mathbb{R}^3)\), we set
\[ \|u\|_{B^{s}_{p, r}} \overset{\text{def}}{=} \left(2^j \|\Delta_j u\|_{L^p} \right)_{r}^{\frac{1}{r}}. \]
We define \(B^{s}_{p, r}(\mathbb{R}^3) \overset{\text{def}}{=} \{u \in \mathcal{S}'(\mathbb{R}^3) \mid \|u\|_{B^{s}_{p, r}} < \infty\} \).

**Lemma 2.2.** Let \(B\) be a ball and \(C\) an annulus of \(\mathbb{R}^3\). A constant \(C\) exists so that for any positive real number \(a\), any non-negative integer \(k\), any smooth homogeneous function \(\sigma\) of degree \(m\), and any couple of real numbers \((a, b)\) with \(b \geq a \geq 1\), there hold
\[
\text{Supp } \hat{u} \subset \delta B \Rightarrow \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^a} \leq C^{k+1} \delta^{k+N(\frac{1}{p} - \frac{1}{y})} \|u\|_{L^y},
\]
\[
\text{Supp } \hat{u} \subset \delta C \Rightarrow C^{-1-k} \delta^k \|u\|_{L^a} \leq \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^a} \leq C^{1+k} \delta^k \|u\|_{L^a},
\]
\[
\text{Supp } \hat{u} \subset \delta C \Rightarrow \|\sigma(D)u\|_{L^a} \leq C_{\sigma, m} \delta^m N(\frac{1}{p} - \frac{1}{q}) \|u\|_{L^q}.
\]

We also recall Bony’s decomposition from [8]:
\[ uv = T_u v + T'_u u = T_u v + T_v u + R(u, v), \]
where
\[
T_u v \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v, \quad T'_u u \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} S_{j+1} u \Delta_j u,
\]
\[
R(u, v) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v \quad \text{with} \quad \tilde{\Delta}_j v \overset{\text{def}}{=} \sum_{|j' - j| \leq 1} \Delta_{j'} v.
\]

To prove Theorem 1.1, we also need to use Lorentz spaces \(L^{p, q}(\mathbb{R}^3)\). For the convenience of the readers, we recall some basic facts on \(L^{p, q}(\mathbb{R}^3)\) from [17, 26, 31]:

**Definition 2.3** (Definition 1.4.6 of [17]). For a measurable function \(f\) on \(\mathbb{R}^3\), we define its non-increasing rearrangement by
\[ f^*(t) \overset{\text{def}}{=} \inf \left\{ s > 0, \mu(\{x, |f(x)| > s\}) \leq t \right\}, \]
where \(\mu\) denotes the usual Lebesgue measure. For \((p, q) \in [1, +\infty]^2\), the Lorentz space \(L^{p, q}(\mathbb{R}^3)\) is the set of functions \(f\) such that \(\|f\|_{L^{p, q}} < \infty\), with
\[
\|f\|_{L^{p, q}} \overset{\text{def}}{=} \left\{ \begin{array}{ll} \left( \int_0^{\infty} (t^{\frac{1}{q}} f^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{for } 1 \leq q < \infty, \\ \sup_{t>0} t^{\frac{1}{q}} f^*(t), & \text{for } q = \infty. \end{array} \right.
\]
We remark that Lorentz spaces can also be defined by real interpolation from Lebesgue spaces (see for instance Definition 2.3 of [26]):
\[
(L^{p_0}, L^{p_1})_{(\beta, q)} = L^{p,q},
\]
where \(1 \leq p_0 < p < p_1 \leq \infty\), \(\beta\) satisfies \(\frac{1}{\beta} = \frac{1-\beta}{p_0} + \frac{\beta}{p_1}\) and \(1 \leq q \leq \infty\).

**Lemma 2.4** (see pages 18–20 of [26]). Let \(1 < p < \infty\) and \(1 \leq q \leq \infty\), we have
- If \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\) and \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\), then
  \[\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.\]
- If \(1 < p < \infty\), \(\frac{1}{p} + 1 = \frac{1}{p_1} + \frac{1}{p_2}\) and \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\), then
  \[\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}},\]
  for \(p = \infty\), and \(\frac{1}{q_1} + \frac{1}{q_2} = 1\), then
  \[\|fg\|_{L^\infty} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.\]
- For \(1 \leq p \leq \infty\) and \(1 \leq q_1 \leq q_2 \leq \infty\), we have
  \[L^{p,q_1} \hookrightarrow L^{p,q_2} \quad \text{and} \quad L^{p,p} = L^p.\]

Let us recall also the interpolation inequality (see [11]) which allows us to obtain some embeddings of spaces.

**Lemma 2.5.** Let \(p_0, p_1, p, q\) in \([1, +\infty]\) and \(0 < \theta < 1\).
- If \(q \leq p\), then
  \[
  [L^p(L^{p_0}), L^p(L^{p_1})]_{(\theta, q)} \hookrightarrow L^p([L^{p_0}, L^{p_1}]_{(\theta, q)}).
  \]
- If \(p \leq q\), then
  \[
  L^p([L^{p_0}, L^{p_1}]_{(\theta, q)}) \hookrightarrow [L^p(L^{p_0}), L^p(L^{p_1})]_{(\theta, q)}.
  \]

Recall also the definition of Lebesgue anisotropic spaces. It is denoted by \(L^p_r(L^q_\theta)\) the space \(L^p_r(\mathbb{R}; L^q_\theta(\mathbb{R}^2))\) and defined by the norm
\[
\|f\|_{L^p_r(L^q_\theta)} := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |f(x, y, z)|^q \, dz \right)^{\frac{r}{q}} \, dx \right)^{\frac{1}{r}}.
\]

Similarly, we denote by \(L^q_\theta(L^p_r)\) the space \(L^q_\theta(\mathbb{R}^2; L^p_r(\mathbb{R}))\), with the norm
\[
\|f\|_{L^q_\theta(L^p_r)} := \left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} |f(x, y, z)|^p \, dx \right)^{\frac{r}{p}} \, dy \right)^{\frac{1}{r}}.
\]

To establish some functional inequalities involving Lorentz spaces the following classical interpolation result (see [26] for example) will be very useful.

**Proposition 1.** Let \(1 \leq p_1 < p_2 \leq \infty\), \(q \in [1, \infty]\) and \(T\) be a linear bounded operator from \(L^{p_1} \to L^{p_2}\). Let \(\theta \in [0, 1]\) and \(p, r\) such that \(\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}\) and \(\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}\). Then \(T\) is also bounded from \(L^{p,q}\) to \(L^{r,q}\) with
\[
\|T\|_{L^{p,q}(L^{r,q})} \leq C \|T\|_{L^{p_1,q_1}(L^{r_1,q_1})} \|T\|_{L^{p_2,q_2}(L^{r_2,q_2})}^{\theta^{-1}}.
\]

**Proposition 2.** (Proposition 2.5 of [22]) For \(1 < p < +\infty, q \in [1, \infty]\), then there exists a constant \(C > 0\) such that the following estimates hold true
1. Let us define the Riesz transform \(R_{ij} = \partial_i \partial_j \Delta^{-1}\), \(i, j \in \{1, 2\}\), then
  \[\|R_{ij} u\|_{L^{p,q}} \leq C \|u\|_{L^{p,q}}.\]
2. For $1 < p < \frac{3}{2}$ and $\frac{1 - \theta}{p'} + \frac{\theta}{p} = \frac{2}{3}$ with $p^* = \frac{3p}{3 - p}$ we have

$$B^\theta_{p,1} \hookrightarrow L^{\frac{2}{3} + 1}. $$

Let us recall the Osgood Lemma (see [16]), which allows us to infer uniqueness of the solution in the critical case (see the uniqueness section).

**Lemma 2.6.** (Osgood) Let $\rho \geq 0$ be a measurable function, $\gamma$ be a locally integrable function and $\mu$ be a positive, continuous and non decreasing function which verifies the following condition

$$\int_0^1 \frac{dr}{\mu(r)} = +\infty. $$

Let also $a$ be a positive real number and let $\rho$ satisfy the inequality

$$\rho(t) \leq a + \int_0^t \gamma(s)\mu(\rho(s))ds. $$

Then if $a$ is equal to zero, the function $\rho$ vanishes.

If $a$ is not zero, then we have

$$-M(\rho(t)) + M(a) \leq \int_0^t \gamma(s)ds, \quad \text{with} \quad M(x) = \int_1^x \frac{dr}{\mu(r)}.$$ 

**Lemma 2.7.** Let $1 \leq r \leq q \leq +\infty$ such that $\frac{1}{r} - \frac{1}{3} < \frac{1}{q}$.

Then the operator $B$ defined by $F \mapsto \int_0^t \nabla e^{(t-t')}D F(t',x)dt'$ is bounded from $L^{\frac{3-\sigma}{\sigma - 1}}([0,T];L^r(\mathbb{R}^3))$ to $L^{\infty}([0,T];L^q(\mathbb{R}^3))$ for every $T \in (0,\infty]$, and there holds

$$\|B(F)\|_{L^q_T(L^r)} \lesssim \|F\|_{L^{\frac{3-\sigma}{\sigma - 1}_T(L^r)}}, $$

with

$$\frac{1}{\sigma} = \frac{1}{2} + \frac{3}{2} \left( \frac{1}{r} - \frac{1}{q} \right). $$

**Proof.** For simplicity, we just take $T = \infty$, we have

$$\|B(F)\|_{L^q} \leq \int_0^t \|\nabla e^{(t-t')}D F(t',x)\|_{L^q}dt'. $$

If $1 \leq r \leq q \leq +\infty$, applying Young’s inequality in the variable spaces yields

$$\|\nabla e^{(t-t')}D F(t',x)\|_{L^q} \leq C_{q,r} \frac{1}{(t-t')^{\frac{3}{2} + \frac{3}{2} \left( \frac{1}{r} - \frac{1}{q} \right)}} \|F(t',\cdot)\|_{L^r}. $$

If $\frac{1}{r} - \frac{1}{3} < \frac{1}{q}$, we have then

$$1_{\{t>0\}}t^{-\frac{3}{2}} \in L^{\sigma,\infty}. $$

with

$$\frac{1}{\sigma} = \frac{1}{2} + \frac{3}{2} \left( \frac{1}{r} - \frac{1}{q} \right). $$

As $L^{\frac{3-\sigma}{\sigma - 1}} \ast L^{\sigma,\infty} \subset L^{\infty}$. This completes the proof of the Lemma. 

Thanks to the Biot-Savart law, we can control some important quantities in order to prove the overall existence of the solution. More precisely, we have the following estimates.
Proposition 3. (Proposition 3.1 of [3]) Let \( u \) a axisymmetric solenoidal vector-field with vorticity \( \omega = \omega \theta \). Let \( (p, q, \lambda) \in [1, \infty)^3 \), then we have
\[
u^r = \omega \theta = 0 \quad \text{on the axis} \quad r = 0.
\]

The following inequalities:

- If \( \frac{2}{3} < p < \infty \) such that \( \frac{1}{q} = \frac{1}{3} + \frac{1}{p} \), then
\[
\| u \|_{L^p, \lambda} \lesssim \| \omega \|_{L^q, \lambda}, \quad \| u^r \|_{L^p, \lambda} \lesssim \frac{\| \omega \|_{L^q, \lambda}}{p}, \quad \| \partial_r u^r \|_{L^p, \lambda} \lesssim \| \partial_r \omega \|_{L^q, \lambda},
\]
\[
\| \partial_r u^\theta \|_{L^p, \lambda} \lesssim \| \partial_r \omega \|_{L^q, \lambda} \quad \text{and} \quad \| \partial_r u^\phi \|_{L^p, \lambda} + \| \partial_r \omega \|_{L^q, \lambda} \lesssim \| \partial_r \omega \|_{L^q, \lambda} + \frac{\| \omega \|_{L^q, \lambda}}{r}.
\]

- If \( 3 \leq p < \infty \) such that \( \frac{1}{q} = \frac{2}{3} + \frac{1}{p} \), then
\[
\| u \|_{L^p, \lambda} \lesssim \| \partial_\theta \omega \|_{L^q, \lambda}, \quad \| u^r \|_{L^p, \lambda} \lesssim \| \partial_r \omega \|_{L^q, \lambda},
\]
\[
\| u^\theta \|_{L^p, \lambda} \lesssim \| \partial_\theta \omega \|_{L^q, \lambda} + \frac{\| \omega \|_{L^q, \lambda}}{r}, \quad \| \partial_r u^\theta \|_{L^p, \lambda} \lesssim \| \partial_\theta \partial_r \omega \|_{L^q, \lambda} + \| \partial_r \omega \|_{L^q, \lambda}
\]
\[
\text{and} \quad \| \partial_r u^\phi \|_{L^p, \lambda} \lesssim \| \partial_\theta \partial_r \omega \|_{L^q, \lambda} + \| \partial_r \omega \|_{L^q, \lambda}.
\]

- In the limiting case, that is \( p = \infty \)
\[
\| u \|_{L^\infty} \lesssim \| \omega \|_{L^{3,1}}, \quad \| u^r \|_{L^\infty} \lesssim \| \partial_\theta \omega \|_{L^{2,1}} + \frac{\| \omega \|_{L^{3,1}}}{r}, \quad \| u^\theta \|_{L^\infty} \lesssim \| \partial_\theta \omega \|_{L^{2,1}} + \| \partial_r \omega \|_{L^{\frac{3}{2},1}},
\]
\[
\text{and} \quad \| \partial_r u^\phi \|_{L^\infty} \lesssim \| \partial_\theta \partial_r \omega \|_{L^{2,1}} + \| \partial_r \omega \|_{L^{\frac{3}{2},1}}.
\]

By Lemma 2.5, we estimate the solution of the transport-diffusion equation.

Lemma 2.8. Let \( 1 < p \leq 2 \) and \( f \in L^p(\mathbb{R}^d) \) such that \( \partial_1 |u|^\frac{2}{p} \in L^2(\mathbb{R}^d) \). Then
\[
\| \partial_1 f \|_{L^p} \lesssim \| \partial_1 |f|^\frac{2}{p} \|_{L^2} \| f \|_{L^p}^\frac{2-p}{p}.
\]

Proof. We remark that
\[
\| \partial_1 f \|_{L^p} = \| \partial_1 |f|^\frac{2}{p} \|_{L^p} \quad \text{and} \quad |f| = |f|^\frac{2}{p},
\]
so we have
\[
\partial_1 |f| = \frac{p}{2} \partial_1 (|f|^\frac{2}{p}) |f|^\frac{2-p}{p}.
\]
The Hölder’s inequality implies that
\[
\| \partial_1 f \|_{L^p} \lesssim \| \partial_1 |f|^\frac{2}{p} \|_{L^2} \| f \|_{L^p}^\frac{2-p}{p}.
\]

Proposition 4. Let \( 1 < p < 2 \), \( 1 \leq q \leq p \), \( f_0 \in L^{p,q} \) and \( u \) is a smooth vector field such that \( \text{div} \ u = 0 \) and \( g \in L^1_t(L^{p,q}) \). Let \( f \in L^\infty_t(L^{p,q}) \) and \( \partial_1 f \in L^2_t(L^{p,q}) \) a solution of the following system
\[
(\text{TD}_\text{mod}) \quad \left\{ \begin{array}{l}
\partial_t f + (u \cdot \nabla) f - \partial_2^2 f = g \\
|f|_{t=0} = f_0.
\end{array} \right.
\]
Then

\[ \| f(t) \|_{L^p} + \| \partial_z f \|_{L^2_t(L^p)} \lesssim \| f_0 \|_{L^p} + \| g \|_{L^1_t(L^p)}. \]

**Proof.** The first step is to control \( f \) in the Lebesgue spaces. Let \( 1 < p < \infty \), we multiply the equation verified by \( f \) by \( |f|^{p-1} \text{sign} \, f \). After an integration by parts combined with the fact that \( \text{div} \, u = 0 \), we obtain

\[ \frac{1}{p} \frac{d}{dt} \| f \|_{L^p} + \frac{4(p-1)}{p^2} \| \partial_z f \|_{L^2}^2 = \int_{\mathbb{R}^3} g |f|^{p-1} \text{sign} \, f \, dx, \]

and by using the H"older inequality and the integration in the time variable, we obtain

\[ \| f(t) \|_{L^p}^p + \frac{4(p-1)}{p} \| \partial_z f \|_{L^2}^2 \leq \| f_0 \|_{L^p}^p + p \int_0^t \| (\text{sign} \, f) \|_{L^p} \| \omega \|_{L^p} \, dt. \]

Finally, the Gronwall lemma implies that

\[ \| f(t) \|_{L^p} + \| \partial_z f \|_{L^2}^2 \lesssim \| f_0 \|_{L^p} + \| g \|_{L^1_t L^p}. \] (23)

As from Lemma 2.8, we have

\[ \| \partial_t f \|_{L^p} \lesssim \| \partial_t |f| \|_{L^2} \| f \|_{L^p}^{2-p} \] (24)

for \( p \leq 2 \), using the last inequality, we obtain that

\[ \| \partial_z f \|_{L^2_t(L^p)} \lesssim \left( \int_0^t \| \partial_z |f| \|_{L^2}^2 \| f \|_{L^p}^{2-p} \, dt \right)^{\frac{1}{2}} \]

\[ \lesssim \| f_0 \|_{L^p}^{\frac{1}{2}} \| \partial_z |f| \|_{L^2_t(L^p)} \]

\[ \lesssim \| f_0 \|_{L^p} + \| g \|_{L^1_t L^p}. \]

So

\[ \| f(t) \|_{L^p} + \| \partial_z f \|_{L^2_t(L^p)} \lesssim \| f_0 \|_{L^p} + \| g \|_{L^1_t L^p}. \] (25)

We denote by \( \mathcal{T} \) and \( \mathcal{S} \) the following linear operators:

\[ \mathcal{T} : \quad \begin{array}{c} L^p \\ f_0 \end{array} \longrightarrow \begin{array}{c} L^p \\ f \end{array} \quad \mathcal{S} : \quad \begin{array}{c} L^p \\ f_0 \end{array} \longrightarrow \begin{array}{c} L^2_t(L^p) \\ \partial_z f \end{array}, \]

with \( \omega \) solution of the system (TDmod). By definition, we have that \( \mathcal{T} \) and \( \mathcal{S} \) are linear operators, then by Lemma 2.5 and Proposition 1, we obtain

\[ \| f(t) \|_{L^p} + \| \partial_z f \|_{L^2_t(L^p)} \lesssim \| f_0 \|_{L^p} + \| g \|_{L^1_t L^p}. \] (26)

This finishes the proof of the proposition. \( \square \)

It follows the same proof that Lemma A.1 of [22], we deduce this result.

**Lemma 2.9.** Consider the equation

\[ \partial_t f + (u, \nabla) f - \Delta f = \partial_z F + G, \quad t > 0, \quad x \in \mathbb{R}^3, \quad f(0, x) = f_0(x). \] (27)

Let \((p, q, p_1, q_1) \in [1, +\infty]^4 \) and \( r \in [2, +\infty] \), such that \( \frac{2}{p} + \frac{3}{q} < 1, \quad \frac{2}{p_1} + \frac{3}{q_1} < 2. \)
There exists $C > 0$ such that for every smooth divergence free vector field $u$, for every $F \in L^p L^q$ and for every $f \in L^r$, the solution of (27) satisfies this estimate: for every $t \in [0, T]$,
\[
\|f(t)\|_{L^\infty} \leq C (1 + t^{-\frac{\alpha}{r}}) \|f_0\|_{L^r} + C (1 + \sqrt{T}^{1-(\frac{\alpha}{r} + \frac{1}{q})}) \|F\|_{L^p_t L^q} + C (1 + \sqrt{T}^{2-(\frac{\alpha}{r} + \frac{1}{q})}) \|G\|_{L^p_t L^q}.
\]

(28)

2.1. Some useful commutator estimates. This section is devoted to the study of some basic commutators which will be needed in our main commutator estimates.

Lemma 2.10. (Lemma 2.7 of [22]) Given $(p, r, \rho, m) \in [1, \infty]^4$ such that
\[
1 + \frac{\rho}{p} = \frac{1}{m} + \frac{1}{r}, \quad p \geq r \quad \text{and} \quad \alpha > 3(1 - \frac{1}{r}).
\]

Let $f$, $g$ and $h$ be three functions such that $\nabla f \in L^\alpha$, $g \in L^m$ and $x F^{-1} h \in L^r$. Then
\[
\|([h(D), f])g\|_{L^p} \leq C \|x F^{-1} h\|_{L^r} \|\nabla f\|_{L^m} \|g\|_{L^m}.
\]

Lemma 2.11. (Lemma 2.8 of [22]) Let $p, m, \alpha \in [1, \infty]$ such that $\frac{1}{p} = \frac{1}{m} + \frac{1}{\alpha}$. Then, there exists $C > 0$ such that for $f \in L^p$, $g \in L^m$ and for every $q \in \mathbb{N} \cup \{1\}$
\[
\|\Delta_q f g\|_{W^{1,p}} \leq C \|\nabla f\|_{L^m} \|g\|_{L^m},
\]
with the following definition $\|\varphi\|_{W^{1,p}} = \|\nabla \varphi\|_{L^p}$.

Proposition 5. (Proposition 2.9 of [22]) We have for every axisymmetric smooth scalar function $u$
\[
(\frac{\partial_p}{p}) \Delta^{-1} u(x) = \frac{x^2}{r^2} R_{111} u(x) + \frac{x^2}{r^2} R_{222} u(x) - \frac{2 x_1 x_2}{r^2} R_{12} u(x),
\]
with $R_{ij} = \partial_{ij} \Delta^{-1}$. Moreover, for $p \in [1, \infty]$, $q \in [1, \infty]$ there exists $C > 0$ such that
\[
\|((\frac{\partial_p}{p}) \Delta^{-1} u\|_{L^p,q} \leq C \|u\|_{L^{p,q}}.
\]

(30)

Lemma 2.12. (Lemma 2.10 of [22]) For every $f \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$, we have
1. For $i, j \in \{1, 2, 3\}$
\[
\Delta^{-1} (x_i \partial_j f) = x_i \partial_j \Delta^{-1} f + \mathcal{L}_{ij} f,
\]
where $\mathcal{L}_{ij} f = -2 R_{ij} \Delta^{-1} f$. Moreover, we have the following estimates
\[
\|\nabla \mathcal{L}_{ij} f\|_{L^r} \leq C \|\nabla f\|_{L^{r,1}},
\]
\[
\|\nabla^2 \mathcal{L}_{ij} f\|_{L^{p,q}} \leq C \|f\|_{L^{r,1}}, \quad p \in [1, \infty], \quad q \in [1, \infty].
\]

(32)

2. For $i, j, k \in \{1, 2, 3\}$
\[
R_{ij}(x_k f) = x_k R_{ij} f + \mathcal{L}_{ij}^k f,
\]
with
\[
\mathcal{L}_{ij}^k f : = -2 \partial_k \Delta^{-1} R_{ij} + \delta_{ik} \partial_j \Delta^{-1} + \delta_{ik} \partial_j \Delta^{-1},
\]
where $\delta_{ij}$ denotes the Kronecker symbol. Moreover we have the estimates
\[
\|\mathcal{L}_{ij}^k f\|_{L^\infty} \leq C \|f\|_{L^{3,1}},
\]
\[
\|\nabla \mathcal{L}_{ij}^k f\|_{L^{p,q}} \leq C \|f\|_{L^{r,1}}, \quad p \in [1, \infty], \quad q \in [1, \infty].
\]

(34)

Remark 2. For $\frac{3}{2} < p < +\infty$ we have
\[
\|\nabla^2 \Delta^{-1} u\|_{L^p} \lesssim \|u\|_{L^{2,p+\delta}} \lesssim \|u\|_{L^p_{2,p+\delta}}.
\]

(35)
3. Basic estimates. In this part we discuss the commutation between the operator \(\frac{\partial}{\partial r} \Delta^{-1}\) and \(v \nabla\). This is a crucial estimates in order to get better a priori estimates for the solution of the system by using our transformation. Our result reads as follows.

Proposition 6. Let \(v\) be an axisymmetric smooth and divergence free without swirl vector field and \(\rho\) an axisymmetric smooth scalar function. Then we have, with the notation \(x_h = (x_1, x_2)\), that

\[
\left\| \left( \frac{\partial}{\partial r} \Delta^{-1}, v \nabla \right) \rho \right\|_{L^2} \lesssim \|\omega_0/r\|_{L^2} (\|x_1 \rho\|_{B^{0}_{\infty,1}(\mathbb{R}^3)} + \|\rho\|_{B^{\frac{1}{2}}_{\infty,1}}). 
\]  

and

\[
\left\| \left( \frac{\partial}{\partial r} \Delta^{-1}, v \nabla \right) \rho \right\|_{L^{\frac{3}{2}}} \lesssim \|\omega_0/r\|_{L^{\frac{3}{2}}} (\|x_1 \rho\|_{B^{0}_{\infty,1}(\mathbb{R}^3)} + \|\rho\|_{B^{\frac{1}{2}}_{\infty,1}(\mathbb{R}^3)}) + \|\omega_0/r\|_{L^2} \|\rho\|_{B^{\frac{1}{2}}_{\infty,1}}. 
\]

Proof. Before proceeding, let us recall from (28) of [22] that

\[
\left\| \left( \frac{\partial}{\partial r} \right) \Delta^{-1}, v \nabla \right\|_{L^p} \leq \sum_{i,j=1}^2 \|\text{div}([R_{ij}, v] \rho)\|_{L^p} 
\]

where \(R_{ij} = \partial_i \partial_j \Delta^{-1}\). Then

\[
\left\| \left( \frac{\partial}{\partial r} \right) \Delta^{-1}, v \nabla \right\|_{L^2} \leq \sum_{i,j=1}^2 \|\text{div}([R_{ij}, v] \rho)\|_{L^2} 
\]

and

\[
\left\| \left( \frac{\partial}{\partial r} \right) \Delta^{-1}, v \nabla \right\|_{L^{\frac{3}{2}}} \leq \sum_{i,j=1}^2 \|\text{div}([R_{ij}, v] \rho)\|_{L^{\frac{3}{2}}} 
\]

Using the Biot-Savart law, we have

\[
v^1 = \Delta^{-1} (\cos(\theta) \partial_3 \omega_0) = \Delta^{-1} \partial_3 (x_1 \frac{\omega_0}{r})
\]

and

\[
v^2 = \Delta^{-1} (\sin(\theta) \partial_3 \omega_0) = \Delta^{-1} \partial_3 (x_2 \frac{\omega_0}{r}),
\]

then the terms \(\partial_1([R_{ij}, v^1] \rho)\) and \(\partial_2([R_{ij}, v^2] \rho)\) can be treated in same way and hence, we shall prove the estimate of the first one only.

\(
\bullet\) Estimates of \(\partial_1([R_{ij}, v^1] \rho)\). Before proceeding, let us recall from (30) of [22] that

\[
\partial_1([R_{ij}, v^1] \rho) = \partial_1((\Delta^{-1} \partial_3 (\omega_0/r) \mathcal{L}_{ij}^1 \rho) + \partial_1([R_{ij}, \Delta (\omega_0/r)] x_1 \rho)
\]

\[
= f + g + h,
\]

where \(\mathcal{L}_{ij}^1 = -2 \partial_1 \Delta^{-1} R_{ij} + \delta_{i1} \partial_j \Delta^{-1} + \delta_{j1} \partial_i \Delta^{-1}\) with \(\delta_{ij}\) denotes the Kronecker symbol.

**Estimate of “f”:** We write

\[
f = \partial_1((\Delta^{-1} \partial_3 (\omega_0/r) \mathcal{L}_{ij}^1 \rho) = \mathcal{R}_{13} (\omega_0/r) \mathcal{L}_{ij}^1 \rho + \partial_3 \Delta^{-1} (\omega_0/r) \partial_1 \mathcal{L}_{ij}^1 \rho.
\]

By using 1) and 3) of Proposition 2 and (33), we deduce

\[
\left\| \mathcal{R}_{13} (\omega_0/r) \partial_1 \mathcal{L}_{ij}^1 \rho \right\|_{L^2} \lesssim \left\| \mathcal{R}_{13} (\omega_0/r) \right\|_{L^2} \left\| \mathcal{L}_{ij}^1 \rho \right\|_{L^\infty} \lesssim \left\| \omega_0/r \right\|_{L^2} \left\| \rho \right\|_{L^{3,1}}.
\]
By Hölder's inequality, Sobolev embeddings, Proposition 2, (39) and (35), we also obtain
\[ \|\partial_3 \Delta^{-1}(\omega_0/r) \partial_1 L_{ij}^1 \rho\|_{L^2} \leq \|\partial_3 \Delta^{-1}(\omega_0/r)\|_{L^6} \|\partial_1 L_{ij}^1 \rho\|_{L^6} \lesssim \|\omega_0/r\|_{L^2} \|\rho\|_{L^{3,1}}. \]  
(45)

Combining estimates (43) and (45) we find
\[ \|f\|_{L^2} \lesssim \|\omega_0/r\|_{L^2} \|\rho\|_{L^{3,1}}. \]  
(47)

and combining estimates (44) and (46) we find
\[ \|f\|_{L^{3,1}} \lesssim \|\omega_0/r\|_{L^{3,1}} \|\rho\|_{L^{3,1}}. \]  
(48)

**Estimate of “g”:** We will use [22] (see page 761)
\[ g = g_1 + g_2 + g_3, \]
with
\[ g_1 = \sum_{q \geq 0} \partial_1 \{[\psi_q(D), S_{q-1}(\mathcal{L}(\omega_0/r))] \Delta_q \rho\} \]
\[ g_2 = \partial_1 \sum_{q \geq 0} [R_{ij}, \Delta_q (\mathcal{L}(\omega_0/r))] S_{q-1} \rho, \]
\[ g_3 = \partial_1 \sum_{q \geq -1} [R_{ij}, \Delta_q (\mathcal{L}(\omega_0/r))] \Delta_q \rho, \]

where \( \psi_q = 2^{3q} \psi(2^q \cdot) \) and \( \psi \in \mathcal{S}(\mathbb{R}^3). \)

By using the Bernstein inequality, this yields for \( 1 < p < \infty \)
\[ \|\partial_1 \{[\psi(D), S_{q-1}(\mathcal{L}(\omega_0/r))] \Delta_q \rho\}\|_{L^p} \lesssim 2^q \|\psi(D), S_{q-1}(\mathcal{L}(\omega_0/r))] \Delta_q \rho\|_{L^p}. \]

Thanks to Lemma 2.10, Proposition 2 and (35), we find
\[ \|\partial_1 \{[\psi(D), S_{q-1}(\mathcal{L}(\omega_0/r))] \Delta_q \rho\}\|_{L^2} \lesssim 2^q \|x \psi_q\|_{L^1} \|\nabla \mathcal{L}(\omega_0/r)\|_{L^6} \|\Delta_q \rho\|_{L^3} \lesssim \|x \psi\|_{L^1} \|\omega_0/r\|_{L^2} \|\Delta_q \rho\|_{L^3}. \]

It follows that
\[ \|g_1\|_{B_{2,2}^0} \lesssim \sum_{q \in \mathbb{N}} 2^{q \frac{3}{2}} \|\partial_1 \{[\psi(D), S_{q-1}(\mathcal{L}(\omega_0/r))] \Delta_q \rho\}\|_{L^2} \lesssim \|\omega_0/r\|_{L^2} \|\rho\|_{B_{2,2}^0} \]
as \( B_{2,2}^0 = L^2 \), then we obtain
\[ \|g_1\|_{L^2} \lesssim \|\omega_0/r\|_{L^2} \|\rho\|_{B_{2,2}^0}. \]  
(49)

For the estimate (37) we use the embedding \( B_{2}^{\frac{7}{4},1} \hookrightarrow L^{3,1} \), so we obtain by using Lemma 2.10, Proposition 2 and (35)
\[ \|\partial_1 \{[\psi(D), S_{q-1}(\mathcal{L}(\omega_0/r))] \Delta_q \rho\}\|_{L^3} \lesssim 2^{q \frac{3}{2}} \|x \psi_q\|_{L^1} \|\nabla \mathcal{L}(\omega_0/r)\|_{L^3} \|\Delta_q \rho\|_{L^2} \lesssim \|x \psi\|_{L^1} \|\omega_0/r\|_{L^{3,1}} \|\Delta_q \rho\|_{L^2}. \]
It follows that
\[
\|g_1\|_{L^{2,1}} \lesssim \|g_1\|_{B^{\frac{1}{2},1}_{\frac{3}{2}}}
\]
\[
\lesssim \sum_{q \in \mathbb{N}} 2^{q/2} \| \partial_1 \{ [\psi(D), S_{q-1}(\mathcal{L}(\omega_0/r))] \Delta_\delta \rho \} \|_{L^{\frac{\omega}{2}}}
\]
\[
\lesssim \| \omega_0/r \|_{L^{2,1}} \| \rho \|_{B^{\frac{1}{2},1}_r}.
\]  
(50)

To estimate the term \(g_2\), we first recall the following identity (see [22] page 761)
\[
g_2 = \sum_{q \geq 0} \partial_i \mathcal{R}_{ij}(\Delta_\delta (\mathcal{L}(\omega_0)) S_{q-1}\rho) - \sum_{q \geq 0} \partial_1 \{ \Delta_\delta (\mathcal{L}(\omega_0/r)) \mathcal{R}_{ij} S_{q-1}\rho \}.
\]
By Bernstein inequalities and (32), we have
\[
\| \Delta_\delta \mathcal{L} f \|_{L^p} \lesssim 2^{-2q} \| \nabla^2 \mathcal{L} \Delta_\delta f \|_{L^p} \lesssim 2^{-2q} \| f \|_{L^p}, \quad \forall \ q \geq 0 \quad \text{and} \quad p \in ]1, \infty[.
\]
(51)
This yields by using the H"older inequality and Proposition 2 that
\[
\|g_2\|_{B^{0}_{2,2}} \lesssim \sum_{q \geq 0} 2^{2q} \| \Delta_\delta (\mathcal{L}(\omega_0/r)) S_{q-1}\rho \|_{L^2}^2 + \sum_{q \geq 0} 2^{2q} \| \Delta_\delta (\mathcal{L}(\omega_0/r)) \mathcal{R}_{ij} S_{q-1}\rho \|_{L^2}^2
\]
\[
\lesssim \sum_{q \geq 0} 2^{2q} \| \Delta_\delta (\mathcal{L}(\omega_0/r)) \|_{L^2}( \| S_{q-1}\rho \|_{L^\infty}^2 + \| \mathcal{R}_{ij} S_{q-1}\rho \|_{L^2}^2 )
\]
\[
\lesssim \| \omega_0/r \|_{L^2}^2 \sum_{q \geq 0} 2^{-2q}( \| S_{q-1}\rho \|_{L^\infty}^2 + \| \mathcal{R}_{ij} S_{q-1}\rho \|_{L^2}^2 )
\]
\[
\lesssim \| \omega_0/r \|_{L^2} \sum_{q \geq 0} \sum_{k \leq q} 2^{2(k-q)} \| \Delta_k \rho \|_{L^3}^2
\]
\[
\lesssim \| \omega_0/r \|_{L^2} \| \rho \|_{B^{0}_{2,2}}^2.
\]
As \(B^{0}_{2,2} = L^2\), then
\[
\|g_2\|_{L^2} \lesssim \| \omega_0/r \|_{L^2} \| \rho \|_{B^{0}_{2,2}}.
\]
(52)
By the same way we obtain
\[
\|g_2\|_{B^{\frac{1}{2},1}_{\frac{3}{2}}} \lesssim \sum_{q \geq 0} 2^{2q} \| \Delta_\delta (\mathcal{L}(\omega_0/r)) S_{q-1}\rho \|_{L^2}^2 + \sum_{q \geq 0} 2^{2q} \| \Delta_\delta (\mathcal{L}(\omega_0/r)) \mathcal{R}_{ij} S_{q-1}\rho \|_{L^2}^2
\]
\[
\lesssim \sum_{q \geq 0} 2^{2q} \| \Delta_\delta (\mathcal{L}(\omega_0/r)) \|_{L^2}( \| S_{q-1}\rho \|_{L^\infty}^2 + \| \mathcal{R}_{ij} S_{q-1}\rho \|_{L^2}^2 )
\]
\[
\lesssim \| \omega_0/r \|_{L^2} \sum_{q \geq 0} 2^{-2q} \| S_{q-1}\rho \|_{L^2}^2
\]
\[
\lesssim \| \omega_0/r \|_{L^2} \sum_{q \geq 0} \sum_{k \leq q} 2^{\frac{k}{2}(k-q)} \| \Delta_k \rho \|_{L^2}^2
\]
\[
\lesssim \| \omega_0/r \|_{L^2} \| \rho \|_{B^{\frac{1}{2},1}_{2}}^\frac{1}{r}.
\]
Using the embedding \(B^{\frac{1}{2},1}_{\frac{3}{2}} \hookrightarrow L^{\frac{3}{2},1}\) we get
\[
\|g_2\|_{L^{\frac{3}{2},1}} \lesssim \| \omega_0/r \|_{L^2} \| \rho \|_{B^{\frac{1}{2},1}_{2}}^\frac{1}{r}.
\]
(53)
For the term $g_3$ we write

$$g_3 = \partial_1 \sum_{q \geq 1} [R_{ij}, \Delta_q (L(\omega_0/r))] \Delta_\eta \rho + \partial_1 \sum_{-1 \leq q \leq 0} [R_{ij}, \Delta_q (L(\omega_0/r))] \Delta_\eta \rho$$

$$:= g_{31} + g_{32}.$$ 

To estimate the first term we first use the Bernstein inequality to get for $1 < p < \infty$

$$||\Delta_k g_{31}||_{L^p} \lesssim 2^k \sum_{q \geq k-4} ||[R_{ij}, \Delta_q (L(\omega_0/r))] \Delta_\eta \rho||_{L^p}.$$ 

Next, to estimate the term inside the sum we do not need to use the structure of the commutator. By using again the Hölder inequality and the Bernstein inequality, we obtain

$$||[R_{ij}, \Delta_q (L(\omega_0/r))] \Delta_\eta \rho||_{L^2} \lesssim ||\Delta_q (L(\omega_0/r))||_{L^6} ||\Delta_\eta \rho||_{L^6} + ||\Delta_q (L(\omega_0/r))||_{L^6} ||R_{ij} \Delta_q \rho||_{L^6} \lesssim 2^{-q} ||\omega_0/r||_{L^2} ||\Delta_\eta \rho||_{L^6}.$$ 

It follows by using again $B_{0,2}^0 = L^2$ that

$$||g_{31}||_{L^2} \lesssim ||g_{31}||_{B_{\frac{3}{2},1}^0} \lesssim ||\omega_0/r||_{L^2} \sum_{k \geq -1} \sum_{q \geq k-4} 2^{(k-q)} ||\Delta_\eta \rho||_{L^2} \lesssim ||\omega_0/r||_{L^2} ||\rho||_{B_{\frac{3}{2},1}^0}. \quad (54)$$

Concerning the estimate (37) we obtain by using Hölder inequality, (51) and Bernstein inequality

$$||[R_{ij}, \Delta_q (L(\omega_0/r))] \Delta_\eta \rho||_{L^\frac{6}{5}} \lesssim ||\Delta_q (L(\omega_0/r))||_{L^{\frac{6}{5}}} ||\Delta_\eta \rho||_{L^{\frac{6}{5}}} + ||\Delta_q (L(\omega_0/r))||_{L^{\frac{6}{5}}} ||R_{ij} \Delta_q \rho||_{L^{\frac{6}{5}}} \lesssim 2^{-q} ||\omega_0/r||_{L^2} ||\Delta_\eta \rho||_{L^2}.$$ 

And by using again the embedding $B_{\frac{3}{2},1}^0 \hookrightarrow L^{\frac{6}{5},1}$ we get

$$||g_{31}||_{L^{\frac{6}{5},1}} \lesssim ||g_{31}||_{B_{\frac{3}{2},1}^0} \lesssim ||\omega_0/r||_{L^2} \sum_{k \geq -1} \sum_{q \geq k-4} 2^{\frac{3}{2}(k-q)} 2^{\frac{1}{2}} ||\Delta_\eta \rho||_{L^2} \lesssim ||\omega_0/r||_{L^2} ||\rho||_{B_{\frac{3}{2},1}^0}. \quad (55)$$

For the estimate of the low frequencies term $g_{32}$ we need to use more deeply the structure of the commutator. We first write

$$g_{32} = \sum_{-1 \leq q \leq 0} [\partial_1 R_{ij}, \Delta_q (L(\omega_0/r))] \Delta_\eta \rho - \sum_{-1 \leq q \leq 0} \partial_1 \Delta \Delta_q (L(\omega_0/r)) R_{ij} \Delta_\eta \rho.$$ 

The last term of the above identity is estimated as follows by using again Proposition 2 and (35)

$$||\sum_{-1 \leq q \leq 0} \partial_1 \Delta \Delta_q (L(\omega_0/r)) R_{ij} \Delta_\eta \rho||_{L^2} \lesssim ||\partial_1 \Delta \Delta_q (L(\omega_0/r)) R_{ij} \Delta_\eta \rho||_{L^2} \lesssim ||\partial_1 \Delta \rho||_{L^6} ||\rho||_{L^3} \lesssim ||\omega_0/r||_{L^2} ||\rho||_{L^3}.$$
and
\[
\| \sum_{-1 \leq q \leq 0} \partial_1 L \Delta q (\omega_0 / r) R_{ij} \tilde{\Delta} q \rho \|_{L^{\frac{3}{2}, 1}} \lesssim \sum_{-1 \leq q \leq 0} \| \partial_1 L \Delta q (\omega_0 / r) R_{ij} \tilde{\Delta} q \rho \|_{L^{\frac{3}{2}, 1}} \\
\lesssim \| \partial_1 L (\omega_0 / r) \|_{L^{1}} \| \rho \|_{L^{3, 1}} \\
\lesssim \| \omega_0 / r \|_{L^{\frac{3}{2}}} \| \rho \|_{L^{3, 1}}.
\]

To estimate the first term of \( g_{32} \) we write for every \(-1 \leq q \leq 0\) thanks to Lemma 2.10 that
\[
\| \partial_1 R_{ij}, \Delta q (L (\omega_0 / r)) \tilde{\Delta} q \rho \|_{L^{2}} \lesssim \| x h \|_{L^{\frac{3}{2}}} \| \nabla L (\omega_0 / r) \|_{L^{3}} \| \tilde{\Delta} q \rho \|_{L^{2}}
\]
and
\[
\| \partial_1 R_{ij}, \Delta q (L (\omega_0 / r)) \tilde{\Delta} q \rho \|_{L^{2}} \lesssim \| x h \|_{L^{\frac{3}{2}}} \| \nabla L (\omega_0 / r) \|_{L^{3}} \| \tilde{\Delta} q \rho \|_{L^{2}}
\]
where \( \tilde{h}(\xi) = \xi_1 \xi_2 \tilde{x}(\xi) \) and \( \tilde{x} \in D(\mathbb{R}^3) \). Using Mikhlin-Hormander Theorem we have
\[
|h(x)| \leq C(1 + |x|)^{-4}, \forall x \in \mathbb{R}^3.
\]
This gives that \( x h \in L^p \forall p > 1 \). Therefore we get that
\[
\| \sum_{-1 \leq q \leq 0} [\partial_1 R_{ij}, \Delta q (L (\omega_0 / r)) \tilde{\Delta} q \rho \|_{L^{2}} \lesssim \| \nabla L (\omega_0 / r) \|_{L^3} \| \tilde{\Delta} q \rho \|_{L^{2}} \lesssim \| \omega_0 / r \|_{L^{2}} \| \rho \|_{L^{3}}
\]
so we obtain
\[
\| g_{32} \|_{L^{2}} \lesssim \frac{\omega_0}{r} \| \rho \|_{L^{3}}
\]
(56)
and concerning the estimate (37) we have
\[
\| \sum_{-1 \leq q \leq 0} [\partial_1 R_{ij}, \Delta q (L (\omega_0 / r)) \tilde{\Delta} q \rho \|_{B^{\frac{1}{2}}_{\frac{3}{2}, 1}} \lesssim \| \omega_0 / r \|_{L^{\frac{3}{2}}} \| \rho \|_{L^{\frac{3}{2}}}.
\]
By using the embedding \( B^{\frac{1}{2}}_{\frac{3}{2}, 1} \hookrightarrow L^{\frac{3}{2}, 1} \) we find that
\[
\| g_{32} \|_{L^{\frac{3}{2}}} \lesssim \frac{\omega_0}{r} \| \rho \|_{L^{\frac{3}{2}, 1}}.
\]
(57)
So combining estimates (54) and (56), we obtain (we use the fact that \( B^{0}_{3, 2} \hookrightarrow L^{3} \) see [33])
\[
\| g_{3} \|_{L^{2}} \lesssim \| \omega_0 / r \|_{L^{2}} \| \rho \|_{B^{0}_{3, 2}}
\]
(58)
and combining (55) and (57) we get
\[
\| g_{3} \|_{L^{\frac{3}{2}, 1}} \lesssim \frac{\omega_0}{r} \| \rho \|_{B^{\frac{1}{2}}_{\frac{3}{2}, 1} \cap L^{\frac{3}{2}}}.
\]
(59)
Now Combining estimates (49), (52) and (58) we have
\[
\| g \|_{L^{2}} \lesssim \| \omega_0 / r \|_{L^{2}} \| \rho \|_{B^{0}_{3, 2}}.
\]
(60)
Also combining (50), (53) and (59), we obtain
\[
\| g \|_{L^{\frac{3}{2}, 1}} \lesssim \frac{\omega_0}{r} \| \rho \|_{B^{\frac{1}{2}}_{\frac{3}{2}, 1} \cap L^{\frac{3}{2}}} + \| \omega_0 / r \|_{L^{\frac{3}{2}}} \| \rho \|_{B^{\frac{1}{2}}_{\frac{3}{2}, 1}}.
\]
(61)

**Estimate of \( h \).** According to [22] (page 764), we have
\[
h = h_1 + h_2 + h_3,
\]
With
\[ h_1 = \partial_1 \sum_{q \geq 0} [R_{ij}, S_{q-1}(\partial_3 \Delta^{-1}(\omega_\theta / r))] \Delta_q(x_1 \rho), \]
\[ h_2 = \partial_1 \sum_{q \geq 0} [R_{ij}, \Delta_q(\partial_3 \Delta^{-1}(\omega_\theta / r))] S_{q-1}(x_1 \rho), \]
\[ h_3 = \partial_1 \sum_{q \geq -1} [R_{ij}, S_{q-1}(\partial_3 \Delta^{-1}(\omega_\theta / r))] \tilde{\Delta}_q(x_1 \rho). \]

Thanks to the proof of inequality (37) of [22], we deduce
\[ ||h_1||_{L^2} \lesssim ||\omega_\theta / r||_{L^2} ||x_1 \rho||_{B^0_{\infty, 1}} \]
\[ ||h_1||_{L^{\frac{2}{3}, 1}} \lesssim ||\omega_\theta / r||_{L^{\frac{2}{3}, 1}} ||x_1 \rho||_{B^0_{\infty, 1}}. \]

For the term \( h_3 \), we use split it into
\[ h_3 = \partial_1 \sum_{q \geq 1} [R_{ij}, \Delta_q(\partial_3 \Delta^{-1}(\omega_\theta / r))] \tilde{\Delta}_q(x_1 \rho) \]
\[ + \partial_1 \sum_{-1 \leq q \leq 0} [R_{ij}, \Delta_q(\partial_3 \Delta^{-1}(\omega_\theta / r))] \tilde{\Delta}_q(x_1 \rho) \]
\[ := h_{31} + h_{32}. \]

From the control of \( III_1 \) in [22] (see page 765), we deduce
\[ ||h_{31}||_{L^2} \lesssim ||\omega_\theta / r||_{L^2} ||x_1 \rho||_{B^0_{\infty, 1}} \]
\[ ||h_{31}||_{L^{\frac{2}{3}, 1}} \lesssim ||\omega_\theta / r||_{L^{\frac{2}{3}, 1}} ||x_1 \rho||_{B^0_{\infty, 1}}. \]

We can also estimate the term \( h_{32} \) without using the structure of the commutator. By using the continuity of the Riesz transform on \( L^p \) for \( 1 < p < \infty \) and (35), we obtain
\[ ||[R_{ij}, \Delta_q(\partial_3 \Delta^{-1}(\omega_\theta / r))] \tilde{\Delta}_q(x_1 \rho)||_{L^2} \lesssim ||\Delta_q(\partial_3 \Delta^{-1}(\omega_\theta / r))||_{L^6} ||\tilde{\Delta}_q(x_1 \rho)||_{L^3} \]
\[ + ||\Delta_q(\partial_3 \Delta^{-1}(\omega_\theta / r))||_{L^6} ||R_{ij} \tilde{\Delta}_q(x_1 \rho)||_{L^3} \]
\[ \lesssim ||\partial_3 \Delta^{-1}(\omega_\theta / r)||_{L^6} ||\tilde{\Delta}_q(x_1 \rho)||_{L^3} \]
\[ \lesssim ||\omega_\theta / r||_{L^2} ||x_1 \rho||_{L^3}. \]

Therefore we get
\[ ||h_{32}||_{L^2} \lesssim ||h_{32}||_{B^0_{\infty, 1}} \lesssim ||\omega_\theta / r||_{L^2} ||x_1 \rho||_{L^3} \]
\[ ||h_{32}||_{L^{\frac{2}{3}, 1}} \lesssim ||h_{32}||_{B^1_{\frac{2}{3}, 1}} \lesssim ||\omega_\theta / r||_{L^{\frac{2}{3}, 1}} ||x_1 \rho||_{L^2}. \]

Consequently we obtain by gathering (64) and (66)
\[ ||h_3||_{L^2} \lesssim ||\omega_\theta / r||_{L^2} ||x_1 \rho||_{B^0_{\infty, 1} \cap L^3} \]
\[ ||h_3||_{L^{\frac{2}{3}, 1}} \lesssim ||h_3||_{B^1_{\frac{2}{3}, 1}} \lesssim ||\omega_\theta / r||_{L^{\frac{2}{3}, 1}} ||x_1 \rho||_{L^2}. \]
and gathering (65) and (67) we get
\[ \| h_3 \|_{L^2} \lesssim \| \omega_\theta/r \|_{L^2} \| x_1 \rho \|_{B^{0}_{\infty,1} \cap L^2}. \] (69)

Let us now turn to the estimate of the term \( h_2 \). We write
\[ h_2 = \sum_{q \geq 0} [\mathcal{R}_{ij}, \Delta_q (\partial_3 \Delta^{-1} (\omega_\theta/r)) S_{q-1}(x_1 \rho)] + [\mathcal{R}_{ij}, \Delta_q (\partial_3 \Delta^{-1} (\omega_\theta/r))] \partial_3 S_{q-1}(x_1 \rho) \]
\[ = h_{21} + h_{22}. \]

We have by definition of the paraproduct that
\[ h_{21} = R_{ij}(T_{x_1} R_{13}(\omega_\theta/r)) - T_{R_{ij}(x_1 \rho)} R_{13}(\omega_\theta/r). \]

Thanks to Proposition 2, we get that
\[ \| h_{21} \|_{L^2} \lesssim \| R_{13}(\omega_\theta/r) \|_{L^2} (\| x_1 \rho \|_{L^\infty} + \| R_{13}(x_1 \rho) \|_{L^\infty}) \]
\[ \lesssim \| x_1 \rho \|_{L^2} (\| x_1 \rho \|_{L^\infty} + \| R_{13}(x_1 \rho) \|_{L^\infty}) \]
\[ \lesssim \| x_1 \rho \|_{B^{0}_{\infty,1} \cap L^2}. \] (70)

From the control of \( III_{22} \) in [22] (see page 766), we deduce
\[ \| h_2 \|_{L^2} \lesssim \| x_1 \rho \|_{B^{0}_{\infty,1} \cap L^2}. \] (72)

and
\[ \| h_2 \|_{L^2} \lesssim \| x_1 \rho \|_{B^{0}_{\infty,1} \cap L^2}. \] (73)

Gathering (62), (68) and (72), we obtain
\[ \| h \|_{L^2} \lesssim \| x_1 \rho \|_{B^{0}_{\infty,1} \cap L^2}. \] (74)

From inequalities (63), (73) and (69), we obtain
\[ \| h \|_{L^2} \lesssim \| x_1 \rho \|_{B^{0}_{\infty,1} \cap L^2}. \] (75)

Finally we obtain by gathering (47), (60) and (74)
\[ \| \partial_3 [R_{ij}, v^1 \rho] \|_{L^2} \lesssim \| x_1 \rho \|_{B^{0}_{\infty,1} \cap L^2} \]
and we obtain by gathering (48), (61) and (75)
\[ \| \partial_3 [R_{ij}, v^1 \rho] \|_{L^2} \lesssim \| x_1 \rho \|_{B^{0}_{\infty,1} \cap L^2} + \| \rho \|_{B^{\frac{1}{2}}_{\infty,1} \cap L^2}. \]

- Estimate of \( \partial_3([R_{ij}, v^3 \rho]) \). Before proceeding, let us recall from the decomposition of \( III \), [22] (see page 767) that
\[ -\partial_3([R_{ij}, v^3 \rho]) = \sum_{k=1}^{2} \partial_3(\partial_3 \Delta^{-1}(\omega_\theta/r)[R_{ij}, x_k] \rho) + 2 \partial_3([R_{ij}, x_k] \rho) \]
\[ + 2 \partial_3([R_{ij}, \Delta^{-1} \mathcal{R}_{ij} (\omega_\theta/r)]) \rho + \sum_{k=1}^{2} \partial_3([R_{ij}, \partial_3 \Delta^{-1}(\omega_\theta/r)](x_k \rho) \]
\[ = I + II + III. \]
To estimate the first term \( I \), we use Lemma 2.12-(2) to obtain that
\[
\partial_3(\partial_k \Delta^{-1} \left( \frac{\omega_2}{r} \right) [\mathcal{R}_{13}, x_k] \rho) = \partial_3(\partial_k \Delta^{-1} \left( \frac{\omega_2}{r} \right) L_{ij}^k \rho) = \mathcal{R}_{3k} \left( \frac{\omega_2}{r} \right) L_{ij}^k \rho + \partial_k \Delta^{-1} \left( \frac{\omega_2}{r} \right) \partial_3 L_{ij}^k \rho.
\]
It follows that
\[
||I||_{L^2} \leq 2 \sum_{i=1}^2 (||L_{ij}^k \rho||_{L^\infty} ||\mathcal{R}_{13} (\omega_2/r)||_{L^2} + ||\partial_k \Delta^{-1} (\omega_2/r) \||_{L^6} ||\partial_3 L_{ij}^k \rho||_{L^3})
\]
and
\[
||I||_{L^2}^{3/4} \leq 2 \sum_{i=1}^2 (||L_{ij}^k \rho||_{L^\infty} ||\mathcal{R}_{13} (\omega_2/r)||_{L^2}^{3/4} + ||\partial_k \Delta^{-1} (\omega_2/r) \||_{L^6} ||\partial_3 L_{ij}^k \rho||_{L^3}^{3/4})
\]
As the operator \( \Delta^{-1} \mathcal{R}_{33} \) has the same properties as \( \mathcal{L} = -2 \partial_{13} \Delta^{-2} \), then the estimates of the terms \( \text{II} \) and \( \text{III} \) are similar to the ones of \( f \) and \( g \). This concludes the proof of the Proposition 6.

For controlled \( ||x_1 \rho||_{L^1(B_{\rho_1}^1, \rho)} \), we need the following proposition.

**Lemma 3.1.** Let \( u \) be an axisymmetric divergence free vector field without swirl and \( \rho \) a smooth scalar function. Then there exists \( C > 0 \) such that for every \( q \in \mathbb{N} \cup \{-1\} \) we have
\[
||[\Delta_q, u \cdot \nabla] \rho||_{L^2} \lesssim ||\omega_2/r||_{L^2} (||\rho x_3 \||_{L^\infty} + ||\rho||_{L^3}).
\]

**Proof.** From the incompressibility of the velocity we have
\[
[\Delta_q, v \cdot \nabla] \rho = \sum_{j=1}^3 \partial_j [\Delta_q, v^j] \rho = \text{II} + \text{III}.
\]
Before proceeding, let us recall from the proof of Proposition 3.2, [22] that
\[
I = \partial_1 ([\Delta_q, \mathcal{L}(\omega/r)] \rho) + \partial_1 ([\Delta_q, \Delta^{-1} \partial_3 (\omega/r)] (x_1 \rho)) - (\mathcal{R}_{13} (\omega/r)) \cdot \varphi_1 (2^q) \cdot \rho - \left\{ \Delta^{-1} \partial_3 (\omega/r) \right\} \cdot \varphi_1 (2^q) \cdot \rho
\]
\[
= \sum_{i=1}^4 I_i,
\]
with \( \mathcal{L} = -2 \mathcal{R}_{13} \Delta^{-1} \) and \( \varphi_1 (x) = x_1 \varphi (x) \in S(\mathbb{R}^3) \).

By Lemma 2.8 of [22] and Sobolev embeddings, we deduce
\[
||I_1||_{L^2} \lesssim ||\nabla \mathcal{L}(\omega/r)||_{L^6} ||\rho||_{L^3} \lesssim ||\omega/r||_{L^2} ||\rho||_{L^3},
\]
\[
||I_2||_{L^2} \lesssim ||\nabla \Delta^{-1} \partial_3 (\omega/r)||_{L^2} ||x_1 \rho||_{L^\infty} \lesssim ||\omega/r||_{L^2} ||x_1 \rho||_{L^\infty}
\]
and
\[
||I_3||_{L^2} + ||I_4||_{L^2} \lesssim ||\mathcal{R}_{13} (\omega/r)||_{L^2} 2^{2q} ||\varphi_1 (2^q) \cdot \rho||_{L^\infty} + ||\Delta^{-1} \partial_3 (\omega/r)||_{L^2} 2^{3q} ||(\partial_1 \varphi_1) (2^q) \cdot \rho||_{L^3}
\]
\[
\lesssim ||\omega/r||_{L^2} ||\varphi_1||_{L^2} ||\rho||_{L^3} + ||\omega/r||_{L^2} ||\partial_1 \varphi_1||_{L^1} ||\rho||_{L^3}
\]
\[
\lesssim ||\omega/r||_{L^2} ||\rho||_{L^3}.
\]
Then
\[ \|I\|_{L^2} \lesssim \|\omega/r\|_{L^2}\|\rho\|_{L^2} + \|\omega/r\|_{L^3}\|x_1\rho\|_{L^\infty}. \]
In the same way, we obtain that
\[ \|II\|_{L^2} \lesssim \|\omega/r\|_{L^2}\|\rho\|_{L^2} + \|\omega/r\|_{L^3}\|x_2\rho\|_{L^\infty}. \]
Before proceeding, let us recall from the proof of Proposition 3.2, [22] that
\[ -III = \partial_3\{[\Delta_0, \nabla_h \Delta^{-1}(\omega/r)](x_\rho)\} + 2\partial_3\{[\Delta_0, \Delta^{-1} R_{33}(\omega/r)]\}
+ 2^{-q}(\partial_3 \nabla_h \Delta^{-1}(\omega/r))(2^{3q}\varphi_h(2^q\cdot)\star \rho)
+ \nabla_h \Delta^{-1}(\omega/r)(2^{3q}(\partial_3 \varphi_h)(2^q\cdot)\star \rho) \]
\[ = \sum_{\ell=1}^{4} III_\ell, \]
with \( \varphi_h(x) = x_h \varphi(x) \).

The estimates of the first and the second terms follow again from Lemma 2.8 [22] and Sobolev embeddings, we write that
\[ \|III_1\|_{L^2} + \|III_2\|_{L^2} \lesssim \|\nabla^2 \Delta^{-1}(\omega/r)\|_{L^2}\|x_\rho\|_{L^\infty} + \|\nabla \Delta^{-1} R_{33}(\omega/r)\|_{L^0}\|\rho\|_{L^3} \]
\[ \lesssim \|\omega/r\|_{L^2}\|x_\rho\|_{L^\infty} + \|\omega/r\|_{L^2}\|\rho\|_{L^3}. \]

It follows as before that
\[ \|III_3\|_{L^2} + \|III_4\|_{L^2} \lesssim 2^{-q}\|\omega/r\|_{L^2}(2^{3q}\|\varphi_h(2^q\cdot)\star \rho\|_{L^\infty}
+ \|\nabla_h \Delta^{-1}(\omega/r)\|_{L^0}(2^{3q}\|\partial_3 \varphi_h(2^q\cdot)\star \rho\|_{L^3} \]
\[ \lesssim \|\omega/r\|_{L^2}\|\varphi_h\|_{L^\infty}\|\rho\|_{L^3} + \|\omega/r\|_{L^2}\|\partial_3 \varphi_h\|_{L^1}\|\rho\|_{L^3} \]
\[ \lesssim \|\omega/r\|_{L^2}\|\rho\|_{L^3}. \]
This completes the proof of Lemma 3.1. \( \square \)

4. A priori estimates.

**Proposition 7.** (Proposition 4.1 of [22]) Let \((u, \rho)\) be a smooth solution of (1) then
(1) For \(p \in [1, +\infty], \ q \in [1, +\infty] \) and \( t \in \mathbb{R}_+ \), we have
\[ \|\rho(t)\|_{L^p_{\infty}L^2} + \|\nabla \rho\|_{L^q_{\infty}L^2} \leq \|\rho_0\|_{L^2} \]
and
\[ \|\rho\|_{L^p_{\infty}L^{p,q}} \lesssim \|\rho_0\|_{L^{p,q}}. \]
(2) For \( u_0 \in L^2, \ \rho_0 \in L^2 \) and \( t \in \mathbb{R}_+ \), we have
\[ \|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + t\|\rho_0\|_{L^2}. \]
(3) For \( \rho_0 \in L^2 \) we have the dispersive estimate
\[ \|\rho(t)\|_{L^\infty} \lesssim (1 + t^{-\frac{3}{2}})\|\rho_0\|_{L^2}. \]
4.1. Estimates of the moments of $\rho$. We have seen in subsection 3.1 and subsection 3.2 that the estimates of the commutators involve some moments of the density. Thus we aim in this paragraph at giving suitable estimates for the moments that will be needed later when we shall perform our diagonalization of the Boussinesq system. Two types of estimates are discussed: the energy estimates of the horizontal moments $|x_h|^k\rho$, with $k = 1, 2$ and some dispersive estimates. More precisely we prove the following proposition.

**Proposition 8.** Let $u$ be a vector-field with zero divergence and satisfying the energy estimate of Proposition 7. Let $\rho$ be a solution of the transport-diffusion equation

$$\partial_t \rho + (u, \nabla)\rho - \Delta \rho = 0, \quad \rho(x, 0) = \rho_0.$$

Then we have the following estimates.

1. For $\rho_0 \in L^2$ and $x_h\rho_0 \in L^2$ there exists $C_0 > 0$ such that for $t \geq 0$

$$\|x_h\rho\|_{L^\infty_t L^2_x} + \|\nabla(x_h\rho)\|_{L^2_t L^2_x} \leq C_0(1 + t^{\frac{1}{2}}).$$  \hfill (76)

2. For $\rho_0 \in L^2 \cap L^m$, $m > 6$ and $x_h\rho_0 \in L^2$, there exists $C_0 > 0$ such that for every $t > 0$

$$\|x_h\rho\|_{L^\infty_t L^p_x} \leq C_0(t^{-\frac{2}{m}} + t^{\frac{1}{3}}).$$ \hfill (77)

3. For $\rho_0 \in L^2 \cap L^m$, $m > 6$ and $x_h\rho_0 \in L^2$, there exists $C_0 > 0$ such that for every $t > 0$

$$\|x_h\rho\|_{L^\infty_t L^p_x} \leq C_0(t^{-\frac{2}{m}} + t^{\frac{1}{3}}).$$ \hfill (78)

4. For $\rho_0 \in L^2 \cap L^m$, $m > 6$ and $x_h\rho_0 \in L^2$, there exists $C_0 > 0$ such that for every $t > 0$

$$\|x_h^2 \rho\|_{L^\infty_t L^p_x} \leq C_0(t^{-\frac{3}{2}} + t^{\frac{1}{3}}).$$ \hfill (79)

**Proof.**

- For the proof of the first and the second estimates, see Proposition 4.2 of [22].

- And for (3) we use the interpolation and Young inequality, in fact

$$\|x_h\rho\|_{L^p_t} \leq \|x_h\rho\|^{\frac{2}{p}}_{L^2_t} \|x_h\rho\|^{\frac{p-2}{p}}_{L^\infty_t} \leq C_0(1 + t^{\frac{1}{3}})(t^{-\frac{2}{m}} + t^{\frac{1}{3}})^{\frac{p-2}{p}} \leq C_0(t^{-\frac{3}{2}} + t^{\frac{1}{3}}).$$

- The second moment $g = |x_h|^2\rho$ solves the following equation

$$\partial_t g + u, \nabla g - \Delta g = 2u_h f - 2\partial_1 \rho - 2\partial_2 \rho - 4(x_h \text{div}_h) \rho = 2u_h f - 2\partial_1 \rho - 2\partial_2 \rho - 4\text{div}_h f + 8\rho.$$ \hfill (80)

with $f = x_h\rho$.

Consequently, we get from Lemma 2.9 and Proposition 7

$$\|g(t)\|_{L^\infty_t} \lesssim (1 + t^{-\frac{5}{2}})\|g_0\|_{L^2_x} + (1 + t^{\frac{1}{2}})(\|\rho u_h\|_{L^\infty_t(L^2_x)} + \|\rho\|_{L^\infty_t(L^2_x)}) + (1 + t^{\frac{1}{2}})(\|\rho\|_{L^\infty_t(L^\infty_x)} + \|x_h\rho\|_{L^\infty_t(L^\infty_x)}) \leq C_0(t^{-\frac{3}{2}} + t^{\frac{1}{3}}).$$

$\square$
4.2. **Strong estimates.** In the rest of this paper, we always denote
\[ \phi_k(t) = C_0 \exp(... \exp(C_0 t^3)...), \] (81)
where \( C_0 \) depends on the involved norms of the initial data and its value may vary from line to line up to some absolute constants. We will make an intensive use (without mentioning it) of the following trivial facts
\[ \int_0^t \phi_k(\tau)d\tau \leq \phi_k(t) \quad \text{and} \quad \exp(\int_0^t \phi_k(\tau)d\tau) \leq \phi_{k+1}(t). \]

**Proposition 9.** Under the assumption of Proposition 8, then
\[ \|x_h \rho\|_{L^1_t B^0_{\infty,1}} \leq C_0 (1 + t^2) + C_0 \int_0^t (\tau^2 + \tau^{-\frac{3}{4}}) \log(2 + \|\omega_\theta / \tau\|_{L^\infty_t L^2})d\tau. \]

**Proof.** By using Proposition 8 and the Bernstein inequality, we find that
\[ ||x_h \rho||_{L^1_t B^0_{\infty,1}} = \int_0^t \sum_{q \leq N(\tau)} ||\Delta_q(x_h \rho)(\tau)||_{L^\infty} d\tau + \int_0^t \sum_{q > N(\tau)} ||\Delta_q(x_h \rho)(\tau)||_{L^\infty} d\tau \]
\[ \leq \int_0^t (\tau^\frac{1}{4} + \tau^{-\frac{3}{4}})N(\tau)d\tau + \int_0^t \sum_{q > N(\tau)} 2^{\frac{3}{4}q} ||\Delta_q(x_h \rho)(\tau)||_{L^2} d\tau. \]

For controlled the last sum in the above inequality. For this purpose we localize in frequency the equation for \( f = x_h \rho \) which is
\[ \partial_t f + u \cdot \nabla f - \Delta f = u_h \rho - 2 \nabla_h \rho \overset{\text{def}}{=} F. \]

By setting \( f_q \overset{\text{def}}{=} \Delta_q f \), we infer
\[ \partial_t f_q + u \cdot \nabla f_q - \Delta f_q = -[\Delta_q, u \cdot \nabla] f + F_q. \]

Then by standard estimate, we have
\[ \|f_{\mu, q}(t)\|_{L^2} \lesssim e^{-c(t^2)} \|f_q(0)\|_{L^2} t + \int_0^t e^{-c(t-\tau)^2} (\|\Delta_q, u \cdot \nabla\|_{L^2} + \|F_q\|_{L^2}) d\tau. \] (82)

To estimate the commutator in the right hand side, we can use Lemma 3.1 and Proposition 8
\[ ||\Delta_q, u \cdot \nabla||f(\tau)||_{L^2} \lesssim \|\frac{\omega_\theta}{\tau}\|_{L^2} (||x_h||_{L^\infty} + ||x_h \rho||_{L^3}) \lesssim \|\frac{\omega_\theta}{\tau}\|_{L^2} (\tau^\frac{1}{4} + \tau^{-\frac{3}{4}}). \]

Let us set \( \kappa(\tau) = \tau^\frac{1}{4} + \tau^{-\frac{3}{4}} \), then
\[ \int_0^t \sum_{q > N(\tau)} 2^{\frac{3}{4}q} ||\Delta_q(x_h \rho)(\tau)||_{L^2} d\tau \]
\[ \lesssim \|f_0\|_{L^2} + \|F\|_{L^1_t L^2} + \int_0^t \|\frac{\omega_\theta}{\tau}\|_{L^\infty_t L^2} (\sum_{q > N(\tau)} 2^{\frac{3}{4}q} \int_0^\tau e^{-c(t-\tau)^2} \kappa(\tau') d\tau') d\tau. \]

Moreover, from Proposition 7, we also have
\[ \|F\|_{L^1_t L^2} \leq \sqrt{t} \|\nabla \rho\|_{L^\infty_t L^2} + \|u\|_{L^\infty_t L^2} \|\rho\|_{L^1_t L^\infty} \lesssim 1 + t^2. \]
Then
\[
\|x_h \rho\|_{L^1_t L^\infty_x} \lesssim (1 + t^2) + \int_0^t (\tau^{\frac{5}{4}} + \tau^{-\frac{3}{4}}) N(\tau) d\tau
\]
\[
+ \int_0^t \|\frac{\omega \mu}{\tau}\|_{L^\infty_x L^2}(\sum_{q > N(\tau)} 2^q \int_0^\tau e^{-c(\tau - \tau') 2^q (\tau^{\frac{5}{4}} + \tau^{-\frac{3}{4}})} (\tau')^\frac{1}{2} d\tau') d\tau
\]
\[
\lesssim (1 + t^2) + \int_0^t (\tau^{\frac{5}{4}} + \tau^{-\frac{3}{4}}) N(\tau) d\tau
\]
\[
+ \int_0^t \|\frac{\omega \mu}{\tau}\|_{L^\infty_x L^2}(\tau^{\frac{1}{2}} 2^{-\frac{1}{2} N(\tau)}) + \sum_{q > N(\tau)} 2^q \int_0^\tau e^{-c(\tau - \tau') 2^q (\tau^{-\frac{3}{4}} + \tau')} d\tau') d\tau.
\]

By a change of variables, we get
\[
\sum_{q > N(\tau)} 2^q \int_0^\tau e^{-c(\tau - \tau') 2^q (\tau^{-\frac{3}{4}} + \tau')} d\tau' = \sum_{q > N(\tau)} 2^q e^{-c(\tau - \tau') 2^q (\tau^{-\frac{3}{4}} + \tau')} \int_0^{2^q \tau} e^{c \tau' (\tau^{-\frac{3}{4}} + \tau')} d\tau'
\]
\[
= \sum_{q \in B_1(\tau)} 2^q e^{-c(\tau - \tau') 2^q (\tau^{-\frac{3}{4}} + \tau')} \int_0^{2^q \tau} e^{c \tau' (\tau^{-\frac{3}{4}} + \tau')} d\tau'
\]
\[
+ \sum_{q \in B_2(\tau)} 2^q e^{-c(\tau - \tau') 2^q (\tau^{-\frac{3}{4}} + \tau')} \int_0^{2^q \tau} e^{c \tau' (\tau^{-\frac{3}{4}} + \tau')} d\tau'
\]
\[
= I(\tau) + II(\tau)
\]

with
\[
B_2(\tau) = \{q > N(\tau) \text{ and } \tau 2^q \geq 1\} \quad \text{and} \quad B_2(\tau) = \{q > N(\tau) \text{ and } \tau 2^q \leq 1\}.
\]

Thus by integration by parts
\[
I(\tau) \lesssim \tau^{-\frac{3}{4}} \sum_{q > N(\tau)} 2^{-\frac{1}{4} q} \lesssim \tau^{-\frac{3}{4}} 2^{-\frac{1}{2} N(\tau)}.
\]

For the second term, we have
\[
II(\tau) \lesssim \sum_{q \in B_2(\tau)} 2^q \lesssim 2^{-\frac{1}{2} N(\tau)} \sum_{2^{-2q} \leq \tau^{-1}} 2^{\frac{1}{2} q} \lesssim 2^{-\frac{1}{2} N(\tau)} \sum_{1 + \tau^{-\frac{3}{4}}}.\]

Gathering these estimates, we obtain
\[
\sum_{q > N(\tau)} 2^q \int_0^\tau e^{-c(\tau - \tau') 2^q (\tau^{-\frac{3}{4}} + \tau')} d\tau' \lesssim 2^{-\frac{1}{2} N(\tau)} (1 + \tau^{-\frac{3}{4}}).
\]

Then
\[
\|x_h \rho\|_{L^1_t L^\infty_x} \lesssim (1 + t^2) + \int_0^t (\tau^{\frac{5}{4}} + \tau^{-\frac{3}{4}}) (N(\tau) + \|\frac{\omega \theta}{\tau}\|_{L^\infty_x L^2} 2^{-\frac{1}{2} N(\tau)}) d\tau.
\]

We choose \( N \) such that
\[
N(\tau) = 2 [\log(2 + \|\frac{\omega \theta}{\tau}\|_{L^\infty_x L^2})],
\]
and then we find
\[
\|x_h \rho\|_{L^1_t L^\infty_x} \leq C_0 (1 + t^2) + C_0 \int_0^t (\tau^2 + \tau^{-\frac{3}{4}}) \log(2 + \|\frac{\omega \theta}{\tau}\|_{L^\infty_x L^2}) d\tau.
\]

This completes the proof of the proposition. \( \square \)
Proposition 10. Let $u_0 \in L^2$ be an axisymmetric vector field such that $\omega_0 \in L^2$ and $\rho_0 \in L^2 \cap L^m$ for $m > 6$, axisymmetric and such that $|x|^2 \rho_0 \in L^2$. Then, we have for every $t \in \mathbb{R}_+$

$$||\frac{\omega}{r}(t)||_{L^\infty} + ||\partial_\theta \frac{\omega}{r}(t)||_{L^2} \leq \phi_2(t).$$

Proof. Recall that the equation of the scalar component of the vorticity $\omega = \omega_0 e_\theta$ is given by

$$\partial_t \omega_0 + u \cdot \nabla \omega_0 - \partial_\theta^2 \omega_0 = \frac{u}{r} \omega_0 - \partial_r \rho.$$ (83)

It follows that the evolution of the quantity $\frac{\omega}{r}$ is governed by the equation

$$(\partial_t + u \cdot \nabla) \left(\frac{\omega}{r} - \partial_\theta^2 \frac{\omega}{r}\right) = -\frac{1}{r} \partial_r \rho.$$ (84)

By applying the operator $\frac{\partial_r}{r} \Delta^{-1}$ to the equation of the density, we obtain that

$$(\partial_t + u \cdot \nabla) \left(\frac{\omega}{r} - \partial_\theta^2 \frac{\omega}{r}\right) = -\frac{1}{r} \partial_r \rho \Delta^{-1}, u \cdot \nabla |\rho.$$ By setting $\Gamma := \frac{\omega}{r} + \frac{1}{r} \partial_r \Delta^{-1} \rho$, we infer

$$(\partial_t + u \cdot \nabla - \partial_\theta^2) \Gamma = -\frac{1}{r} \partial_r \rho \Delta^{-1}, u \cdot \nabla |\rho - \partial_\theta^2 \frac{1}{r} \partial_r \rho \Delta^{-1}.$$ So, we have

$$||\Gamma||_{L^\infty L^2} + ||\partial_\theta \Gamma||_{L^2 L^2} \leq ||\Gamma_0||_{L^2} + ||\frac{1}{r} \partial_r \Delta^{-1}, u \cdot \nabla |\rho||_{L^2} + ||\frac{1}{r} \partial_r \Delta^{-1} |\rho||_{L^2} \leq ||\Gamma_0||_{L^2} + ||\frac{1}{r} \partial_r \Delta^{-1}, u \cdot \nabla |\rho||_{L^2} + ||\partial_\theta |\rho||_{L^2} \leq ||\Gamma_0||_{L^2} + ||\frac{1}{r} \partial_r \Delta^{-1}, u \cdot \nabla |\rho||_{L^2} + ||\rho_0||_{L^2}.$$

From the Proposition 6, we have

$$||(\frac{1}{r} \partial_r \Delta^{-1}, u \cdot \nabla |\rho||_{L^2} \leq ||\omega_0 / r||_{L^2} (||x||_{B_{2,1}^m \cap L^2} + ||\rho||_{B_{2,1}^m})$$

as

$$||\rho||_{B_{2,1}^m} \leq ||\rho||_{L^2} ||\nabla |\rho||_{L^2},$$

then, we get by the Gronwall inequality that

$$||\frac{\omega_0}{r}||_{L^\infty L^2} + ||\partial_\theta \frac{\omega_0}{r}||_{L^2 L^2} \leq C_0 e^{C_0 (1 + t \frac{4}{5} ||\rho||_{L^2} ||\rho||_{B_{2,1}^m})},$$ (85)

$$||\frac{1}{r} \partial_r \Delta^{-1} |\rho(t)||_{L^2} + \frac{1}{r} \partial_r \Delta^{-1} \rho||_{L^2} \leq \frac{1}{r} ||\rho||_{L^2}.$$

Thus we deduce by Proposition 9

$$||\frac{\omega_0}{r}||_{L^\infty L^2} \leq C_0 e^{C_0 (1 + t \frac{4}{5})} \exp \left\{ \int_0^t (\tau^2 + \tau^{-\frac{4}{5}}) \log (2 + ||\frac{\omega_0}{r}||_{L^\infty L^2}) d\tau \right\},$$

so

$$\log (2 + ||\frac{\omega_0}{r}||_{L^\infty L^2}) \leq \log (C_0) + C_0 (1 + t \frac{4}{5}) + \int_0^t (\tau^2 + \tau^{-\frac{4}{5}}) \log (2 + ||\frac{\omega_0}{r}||_{L^\infty L^2}) d\tau,$$ which gives the desired estimate. 

$\square$
Proposition 11. Let \( u_0 \in L^2 \) be an axisymmetric vector field such that \( \omega_0 \in L^\frac{2}{5} \) and \( \overline{\omega} \in L^2 \) and \( \rho_0 \in L^\frac{5}{2} \cap L^m, \) \( m > 6 \) a scalar axisymmetric function such that \( |x_\perp|^2 \rho_0 \in L^2. \) Then we have for every \( t \in \mathbb{R}_+ \)
\[
\|\omega(t)\|_{L^\frac{2}{5}} + \|\partial_\omega(t)\|_{L^\frac{2}{5}L^\frac{2}{5}} \leq \phi_2(t).
\]

Proof. We remark that for \( 1 < p \leq 2, \) we have
\[
\|\rho\|_{L^p_t L^r_x} + \|\nabla \rho\|_{L^2_t L^r_x} \lesssim \|\rho_0\|_{L^r_x}.
\]
First we have
\[
\|\omega(t)\|_{L^\frac{2}{5}} + \|\partial_\omega(t)\|_{L^\frac{2}{5}L^\frac{2}{5}} \lesssim \|\omega_0\|_{L^\frac{2}{5}} + \int_0^t \|\overline{\omega} \|_{L^\frac{2}{5}} d\tau + \int_0^t \|\partial_\rho\|_{L^\frac{2}{5}} d\tau
\]
\[
\lesssim \|\omega_0\|_{L^\frac{2}{5}} + \|\overline{\omega} \|_{L^\frac{2}{5}} \int_0^t \|\overline{u}\|_{L^2} d\tau + t^\frac{1}{2} \|\rho_0\|_{L^\frac{2}{5}}.
\]
From Proposition 3, we have
\[
\|u^r\|_{L^\frac{2}{3}} \lesssim \|u^r\|_{L^\frac{2}{3}}^\frac{1}{2} \|u^r\|_{L^\frac{2}{3}}^\frac{1}{2} \lesssim \|u_0\|_{L^\frac{2}{3}} \|\partial_\omega\|_{L^\frac{2}{5}}^\frac{1}{2}.
\]
So we obtain
\[
\|\omega(t)\|_{L^\frac{2}{5}} + \|\partial_\omega(t)\|_{L^\frac{2}{5}L^\frac{2}{5}} \lesssim \|\omega_0\|_{L^\frac{2}{5}} + \phi_2(t) \|u_0\|_{L^\frac{2}{3}} \int_0^t \|\partial_\omega\|_{L^\frac{2}{5}} d\tau + t^\frac{1}{2} \|\rho_0\|_{L^\frac{2}{5}}
\]
\[
\lesssim \|\omega_0\|_{L^\frac{2}{5}} + \phi_2(t) \|u_0\|_{L^\frac{2}{3}} \int \|\partial_\omega\|_{L^\frac{2}{5}} + t^\frac{1}{2} \|\rho_0\|_{L^\frac{2}{5}}.
\]
Therefore we have
\[
\|\omega(t)\|_{L^\frac{2}{5}} + \|\partial_\omega(t)\|_{L^\frac{2}{5}L^\frac{2}{5}} \lesssim \|\omega_0\|_{L^\frac{2}{5}} + \phi_2(t) \|u_0\|_{L^\frac{2}{3}} \int_0^t \|\partial_\omega\|_{L^\frac{2}{5}} + t^\frac{1}{2} \|\rho_0\|_{L^\frac{2}{5}}.
\]
So
\[
\|\omega(t)\|_{L^\frac{2}{5}} + \|\partial_\omega(t)\|_{L^\frac{2}{5}L^\frac{2}{5}} \leq \phi_2(t).
\]

Proposition 12. Let \( u_0 \in L^2 \) be an axisymmetric divergence free vector field without swirl such that \( \omega_0 \in L^2 \cap L^\frac{2}{3} \) and \( \overline{\omega} \in L^2 \cap L^\frac{2}{3} \) and let \( \rho_0 \in L^\frac{5}{2} \cap L^m, \) \( m > 6, \) be an axisymmetric function such that \( |x_\perp|^2 \rho_0 \in L^2. \) Then we have
\[
\|\omega_0(t)\|_{L^\frac{2}{3},1} + \|\partial_\omega\|_{L^\frac{2}{5}L^\frac{2}{5},1} \leq \phi_2(t).
\]

Proof. By Proposition 4, we have
\[
\|\omega_0(t)\|_{L^\frac{2}{3},1} + \|\partial_\omega\|_{L^\frac{2}{5}L^\frac{2}{5},1}
\]
\[
\lesssim \|\omega_0\|_{L^\frac{2}{3},1} + \int_0^t \|\overline{\omega} \|_{L^\frac{2}{3},1} d\tau + \int_0^t \|\partial_\rho\|_{L^\frac{2}{3},1} d\tau
\]
\[
\lesssim \|\omega_0\|_{L^\frac{2}{3},1} + t^\frac{1}{2} \|\rho_0\|_{L^\frac{2}{5},1} + \|\overline{\omega} \|_{L^\frac{2}{3},1} \|u^r\|_{L^\frac{2}{3},1}.
\]
From Proposition 3, we have
\[
\|u^r\|_{L^6,2} \leq \|\partial_\omega\|_{L^\frac{2}{5}}.
\]
Then we obtain
\[
\|\omega(t)\|_{L^\frac{2}{3},1} + \|\partial_\omega\|_{L^\frac{2}{5}L^\frac{2}{5},1} \lesssim \|\omega_0\|_{L^\frac{2}{3},1} + t^\frac{1}{2} \|\rho_0\|_{L^\frac{2}{5},1} + \|\overline{\omega} \|_{L^\frac{2}{3},1} \|u^r\|_{L^\frac{2}{5}}.
\]
And from Proposition 10, we have
\[ ||\omega(t)||_{L^2_t L^{3,1}_x} + ||\partial_z \omega||_{L^2_t L^{3,1}_x} \lesssim ||\omega_0||_{L^2_t L^{3,1}_x} + t^{\frac{2}{3}} ||\rho_0||_{L^2_t L^{3,1}_x} + \phi_2(t) t^{\frac{2}{3}} ||\partial_z \omega||_{L^2_t L^6_x} \leq \phi_2(t). \]

**Proposition 13.** Let \( u_0 \in L^2 \) be an axisymmetric vector-field such that \( \omega_0 \in L^2 \cap L^6 \) and \( \frac{\omega_0}{u_0} \in L^2 \). Let \( \rho_0 \in L^2 \cap L^m \) \( m > 6 \) and \( \rho_0 \in L^6 \cap L^{5,1}_x \) be a scalar axisymmetric function such that \( |x_h|^2 \rho_0 \in L^2 \). Then
\[ ||\omega(t)||_{L^2} + ||\partial_z \omega||_{L^2_t L^2} \leq \phi_2(t). \]

**Proof.** We have
\[ ||\omega(t)||_{L^2} + ||\partial_z \omega||_{L^2_t L^2} \lesssim ||\omega_0||_{L^2} + \int_0^t ||u^r \omega||_{L^2} d\tau + t^{\frac{2}{3}} ||\rho_0||_{L^2} \]
\[ \lesssim ||\omega_0||_{L^2} + \int_0^t ||u^r \omega||_{L^2} + t^{\frac{2}{3}} ||\rho_0||_{L^2} \]
\[ \lesssim ||\omega_0||_{L^2} + ||u^r||_{L^1_t L^\infty_x} \frac{\omega_0}{u_0} ||\partial_z \omega||_{L^2_t L^2} + t^{\frac{2}{3}} ||\rho_0||_{L^2}. \]

From Proposition 10, 12 and 3, we have
\[ ||\omega(t)||_{L^2} + ||\partial_z \omega||_{L^2_t L^2} \lesssim ||\omega_0||_{L^2} + \phi_2(t) ||\partial_z \omega||_{L^1_t L^{3,1}_x} + t^{\frac{2}{3}} ||\rho_0||_{L^2} \]
\[ \lesssim ||\omega_0||_{L^2} + t^{\frac{2}{3}} ||\partial_z \omega||_{L^2_t L^{3,1}_x} + \phi_2(t) + t^{\frac{2}{3}} ||\rho_0||_{L^2} \]
\[ \leq \phi_2(t). \]

**Proposition 14.** Let \( u_0 \in L^2 \) be an axisymmetric divergence free vector field without swirl such that \( \omega_0 \in L^2 \cap L^6 \) and \( \frac{\omega_0}{u_0} \in L^2 \). Let \( \rho_0 \in L^p \cap L^m \), \( \forall m > 6, \forall 1 < p \leq 2 \), be an axisymmetric scalar function such that \( |x_h|^2 \rho_0 \in L^2 \) and \( \nabla \rho_0 \in L^q \), \( \forall \frac{3}{2} < q \leq 2 \), we have
\[ ||\Delta \rho||_{L^1_t L^{3,1}_x} \leq \phi_2(t). \]

**Proof.** Let first introduce Duhamel formula,
\[ \rho(t) = S(t) \rho_0 - \int_0^t e^{(t-\tau)\Delta} (u, \nabla) \rho d\tau. \tag{86} \]
Then we have
\[ ||\Delta \rho||_{L^1_t L^{3,1}_x} \lesssim t^{\frac{2}{3}} ||\Delta \rho||_{L^1_t L^{3,1}_x}. \]
From the following embedding
\[ L^6 \cap L^p \subset L^{3,1}, \quad \forall \frac{3}{2} < p \leq 2, \]
we obtain
\[ ||\Delta \rho||_{L^1_t L^{3,1}_x} \lesssim t^{\frac{2}{3}} ||\Delta \rho||_{L^2_t (L^6 \cap L^p)}. \]
Then using (86), we obtain
\[ ||\Delta \rho||_{L^1_t L^{3,1}_x} \lesssim t^{\frac{2}{3}} ||\Delta S(t) \rho||_{L^2_t (L^6 \cap L^p)} + t^{\frac{2}{3}} ||(u, \nabla) \rho||_{L^2_t (L^6 \cap L^p)} \]
\[ \lesssim t^{\frac{2}{3}} ||\Delta \rho_0||_{B^m_{3,1} \cap B^m_{p,2}} + t^{\frac{2}{3}} ||u||_{L^\infty} ||\nabla \rho||_{L^2_t (L^6 \cap L^{3,1})}. \]
\[ t^\frac{1}{2} ||\nabla \rho_0||_{L^2_t \cap L^p} + t^\frac{1}{2} ||\omega||_{L^\infty_t L^2} ||\nabla \rho||_{L^2_t (L^2_t \cap L^{\infty_p})}. \]

The term \( ||\nabla \rho||_{L^2_t L^{\frac{6p}{3p-6}}} \) will be controlled as follows

\[ ||\nabla \rho||_{L^2_t L^{\frac{6p}{3p-6}}} \lesssim ||\rho_0||_{L^2}. \]

Let us control \( ||\nabla \rho||_{L^2_t L^{\frac{6p}{3p-6}}} \), so by equality (86), we have

\[ \nabla \rho(t) = \nabla s(t) \rho_0 - \int_0^t \nabla \text{div} e^{(t-r)} \rho u \, dr. \]

Then

\[ ||\nabla \rho||_{L^2_t L^{\frac{3p}{3p-6}}} \lesssim ||\nabla s(t) \rho_0||_{L^2_t L^{\frac{6p}{3p-6}}} + ||\rho u||_{L^2_t L^{\frac{6p}{3p-6}}} \]
\[ \lesssim ||\nabla \rho_0||_{L^2_t L^{\frac{6p}{3p-6}}} + ||\omega||_{L^\infty_t L^6} ||\rho||_{L^2_t L^{\frac{3p}{3p-6}}} \]
\[ \lesssim ||\rho_0||_{L^2_t L^{\frac{6}{5}}} + ||\omega||_{L^\infty_t L^2} ||\rho||_{L^2_t L^{\frac{3p}{3p-6}}} \]
\[ \lesssim ||\rho_0||_{L^2_t L^{\frac{6}{5}}} ||\nabla \rho_0||_{L^2_t L^{\frac{6}{5}}} + ||\omega||_{L^\infty_t L^2} ||\rho||_{L^2_t L^{\frac{3p}{3p-6}}}. \]

And the by interpolation we have

\[ ||\rho||_{L^2_t L^{\frac{3p}{3p-6}}} \lesssim ||\rho||_{L^2}^{\frac{2}{3} \frac{3}{2} - \frac{2}{3}} ||\nabla \rho||_{L^2}^{\frac{1}{2} + \frac{1}{2} - \frac{2}{3}}. \]

Then

\[ ||\rho||_{L^2_t L^{\frac{3p}{3p-6}}} \lesssim t^\frac{1}{2} ||\rho||_{L^2_t L^{\frac{3p}{3p-6}}}. \]

Therefore

\[ ||\Delta \rho||_{L^1_t L^{\frac{6}{5}}} \lesssim t^\frac{1}{2} ||\nabla \rho_0||_{L^2_t L^{\frac{6}{5}}} + t^\frac{1}{2} ||\omega||_{L^\infty_t L^2} ||\rho||_{L^2_t L^{\frac{6}{5}}} + t^\frac{1}{2} ||\omega||_{L^\infty_t L^2} ||\rho||_{L^2_t L^{\frac{6}{5}}} ||\nabla \rho||_{L^2_t L^{\frac{6}{5}}} \]
\[ + t^\frac{1}{2} ||\omega||_{L^\infty_t L^2} ||\rho_0||_{L^2} + t^\frac{1}{2} ||\omega||_{L^\infty_t L^2} ||\rho_0||_{L^2_t L^{\frac{6}{5}}} + t^\frac{1}{2} ||\omega||_{L^\infty_t L^2} (||\rho_0||_{L^p} + ||\nabla \rho_0||_{L^p}). \]

And from Proposition 13 we obtain

\[ ||\Delta \rho||_{L^1_t L^{\frac{6}{5}}} \leq \phi_2(t). \]

\[ \Box \]

**Proposition 15.** Let \( u_0 \in L^2 \) be an axisymmetric vector field such that \( \frac{\omega}{r} \in L^{\frac{6}{5}} \cap L^2 \) and \( \omega_0 \in L^2 \cap L^{\frac{6}{5}} \). Let \( \rho_0 \in L^p \cap L^m, \forall m > 6, \forall 1 < p \leq 2 \) be an axisymmetric scalar function such that \( \frac{1}{r} |\rho_0|^2 \rho_0 \in L^\infty \) and \( \nabla \rho_0 \in L^q \forall \frac{3}{2} < q \leq 2 \). Then, we have for every \( t \in \mathbb{R}_+ \)

\[ ||\frac{\omega}{r}(t)||_{L^\infty_t L^{\frac{6}{5}}} + ||\partial_r \frac{\omega}{r}(t)||_{L^2_t L^{\frac{6}{5}}} \leq \phi_2(t). \]
Proof. Recall that $\Gamma$ satisfies the following equation
\[(\partial_t + u.\nabla - \partial^2_z)\Gamma = -\left[\frac{\partial_t}{r} \Delta^{-1} + u.\nabla\right]\rho - \partial^2_z \frac{1}{r} \partial_r \Delta^{-1} \rho.\]
From Proposition 4 of interpolation, we deduce
\[||\Gamma||_{L^{\frac{1}{3}} L^\frac{2}{3}}} + ||\partial_t \Gamma||_{L^{\frac{1}{3}} L^\frac{2}{3}} \lesssim ||\Gamma_0||_{L^{\frac{1}{3}} L^\frac{2}{3}} + ||\frac{1}{r} \partial_t \Delta^{-1} u.\nabla\rho||_{L^1 L^{\frac{3}{2}}} + ||\Delta \rho||_{L^1 L^{\frac{3}{2}}}.
\]
We have from Proposition 6
\[||[(\frac{\partial_t}{r} \Delta^{-1}, v.\nabla)\rho||_{L^{\frac{2}{3}}}} \lesssim ||\omega_\theta / r||_{L^{\frac{2}{3}}} \left(||x_1 \rho||_{B^{0}_{\infty,1}} \cap L^2 + ||\rho||_{B^{\frac{3}{2}}_{2,1}} \cap L^2\right) + ||\omega_\theta / r||_{L^2} ||\rho||_{B^{\frac{3}{2}}_{2,1}}.
\]
So we obtain
\[||\Gamma||_{L^{\frac{1}{3}} L^\frac{2}{3}} + ||\partial_t \Gamma||_{L^{\frac{1}{3}} L^\frac{2}{3}} \lesssim ||\Gamma_0||_{L^{\frac{1}{3}} L^\frac{2}{3}} + \int_0^t \left(||\omega_\theta / r||_{L^{\frac{2}{3}}} \left(||x_1 \rho||_{B^{0}_{\infty,1}} \cap L^2 + ||\rho||_{B^{\frac{3}{2}}_{2,1}} \cap L^2\right)\right) dt \]
\[+ \int_0^t ||\omega_\theta / r||_{L^{\frac{2}{3}}} ||\rho||_{B^{\frac{1}{2}}_{2,1}} dt + ||\Delta \rho||_{L^1 L^{\frac{3}{2}}}, \]  
\[\text{(87)}\]
as
\[||\frac{1}{r} \partial_t \Delta^{-1} \rho||_{L^{\frac{1}{3}} L^{3/2}} + ||\frac{1}{r} \partial_t \Delta^{-1} \rho||_{L^1 L^{\frac{3}{2}}} \lesssim ||\rho||_{L^{\frac{1}{3}} L^{3/2}} + ||\partial_t \rho||_{L^2 L^{\frac{3}{2}}} \lesssim ||\rho_0||_{L^{3/2}}, \]
\[\text{(88)}\]
thus
\[||\omega_\theta / r||_{L^{\frac{2}{3}}} + ||\partial_t \omega_\theta / r||_{L^{\frac{1}{3}} L^\frac{2}{3}} \lesssim ||\omega_\theta / r||_{L^{\frac{2}{3}}} + ||\rho_0||_{L^{3/2}} + \int_0^t ||\omega_\theta / r||_{L^2} ||\rho||_{B^{\frac{1}{2}}_{2,1}} dt + ||\Delta \rho||_{L^1 L^{\frac{3}{2}}} \]
\[+ \int_0^t \left(||\omega_\theta / r||_{L^{\frac{2}{3}}} \left(||x_1 \rho||_{B^{0}_{\infty,1}} \cap L^2 + ||\rho||_{B^{\frac{3}{2}}_{2,1}} \cap L^2\right)\right) dt. \]
\[\text{(89)}\]
From Gronwall's Lemma, we obtain
\[||\omega_\theta / r||_{L^{\frac{2}{3}}} + ||\partial_t \omega_\theta / r||_{L^{\frac{1}{3}} L^\frac{2}{3}} \leq \phi_2(t) e^{C||x_\theta \rho||_{L^1 B^{0}_{\infty,1} \cap L^2} + C||\rho||_{L^1 (B^{0}_{2,1} \cap L^2)}}. \]
\[\text{(90)}\]
We have from Propositions 10 and 9
\[||x_\theta \rho||_{L^1 B^{0}_{\infty,1}} \lesssim (1 + t^{\frac{3}{2}}) + \int_0^t (\tau^2 + t^{-\frac{3}{2}}) \log(2 + ||\omega_\theta / r||_{L^\infty L^2}) dt \]
\[\lesssim (1 + t^{\frac{3}{2}}) + \phi_1(t)(t^3 + t^{\frac{1}{2}}) \]
\[\leq \phi_1(t). \]
Therefore
\[||\frac{\omega_\theta}{r}||_{L^\infty L^{\frac{2}{3}}} + ||\partial_t \frac{\omega_\theta}{r}||_{L^{\frac{1}{3}} L^\frac{2}{3}} \leq \phi_2(t). \]
\[\square\]
5. Proof of Theorem 1.1.

5.1. Existence and uniqueness. For the existence part of Theorem 1.1 we smooth out the initial data as follows

\[ u_{0,n} = S_n u_0 \quad \text{and} \quad \rho_{0,n} = S_n \rho_0, \]

where \( S_n \) is the cut-off in frequency defined in the preliminaries.

So, it exists a unique regular and global in time solution, which is axisymmetric without swirl \( u^n \), solution of the problem

\[
\begin{align*}
\partial_t \rho_n + \text{div} (u_n \otimes \rho_n) - \Delta \rho_n &= 0 \\
\partial_t u_n + \text{div} (u_n \otimes u_n) - n^{-1} \Delta_h u_n - \partial_3^2 u_n &= -\nabla \rho_n + \rho_n e_3 \\
\text{div} u_n &= 0 \\
(\rho_n, u_n)|_{t=0} &= (\rho_{0,n}, u_{0,n}).
\end{align*}
\]

We have \( u_0 \) is a free divergence axisymmetric vector field and \( \rho_0 \) is an axisymmetric scalar function. So for all \( n \in \mathbb{N} \), \( u_{0,n} \) and \( \rho_{0,n} \) are axisymmetric own to the radial property of the function \( \chi \).

Using the classical properties of the convolution laws we obtain

\[
\|u_{0,n}\|_{L^2} \lesssim \|u_0\|_{L^2}, \quad \|\omega_{0,n}\|_{L^a} \lesssim \|\omega_0\|_{L^a} \quad \text{and} \quad \|\rho_{0,n}\|_{L^p} \lesssim \|\rho_0\|_{L^p}.
\]

And from [22] we have

\[
\| |x_h|^2 \rho_{0,n} \|_{L^2} \lesssim \| \rho_0 \|_{L^2} + \| |x_h|^2 \rho_0 \|_{L^2}.
\]

Also from Lemma A.1 in [6] we have

\[
\| \frac{\omega_{0,n}}{r} \|_{L^p} \lesssim \| \frac{\omega_0}{r} \|_{L^p}, \quad \forall \, 1 \leq p \leq +\infty.
\]

So, it exists a unique regular and global in time solution, which is axisymmetric without swirl \( (u_n, \rho_n) \), solution of system (1) with initial data \( (\rho_{0,n}, u_{0,n}) \). By section “A priori estimates” we obtain that \( (u_n, \rho_n) \), is a sequence which is uniformly bounded. By standard arguments we can show that this family \( (\rho_n, u_n) \) converges to \( (\rho, u) \) which satisfying in turn our initial system.

Thus we can construct locally in time a unique solution \( (u_n, \rho_n) \). To prove uniqueness for (1) it suffices to prove it for the following system

\[
\begin{align*}
\partial_t \omega + u \cdot \nabla \omega - \partial_3^2 \omega &= -\partial_r \rho e_\theta + \frac{u^r}{r} \omega \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\partial_t \rho + u \cdot \nabla \rho - \Delta \rho &= 0, \\
\omega|_{t=0} &= u_0, \quad \rho|_{t=0} = \rho_0.
\end{align*}
\]

So let \( (\omega_1, \rho_1) \in L^2_x \times L^2_x, \, 1 \leq i \leq 2 \) be two solutions of (1) with the same initial data \( (\omega_0, \rho_0) \) and denotes \( \delta \omega = \omega_2 - \omega_1, \, \delta \rho = \rho_2 - \rho_1, \) then

\[
\begin{align*}
\partial_t \delta \omega + u_2 \cdot \nabla \delta \omega - \partial_3^2 \delta \omega &= -(\delta u, \nabla) \omega_1 + \frac{u^r_2}{r} \delta \omega + \frac{\delta u^r_2}{r} \omega_1 - \partial_r \delta \rho e_\theta, \\
(t, x) &\in \mathbb{R}_+ \times \mathbb{R}^3, \\
\partial_t \delta \rho + u, \nabla \delta \rho - \Delta \delta \rho &= -\delta u \cdot \nabla \rho_1, \\
\delta \omega|_{t=0} &= 0, \quad \delta \rho|_{t=0} = 0.
\end{align*}
\]
Let $1 < p < \infty$, multiplying the first equation by $|\delta \omega|^{p-1}\text{sign}(\delta \omega)$. So we obtain by integration by parts, Hölder inequality and using the fact that $\text{div} \, u = 0$

$$\frac{1}{p} \frac{d}{dt} ||\delta \omega||_{L^p}^p + \frac{4(p-1)}{p^2} ||\partial_x |\delta \omega|^\frac{p}{2}||_{L^2}^2 \leq ||(\delta u, \nabla)\omega_1||_{L^p} ||\delta \omega||_{L^p}^{p-1} + \frac{u^r}{r} ||\delta \omega||_{L^p}^p + \frac{1}{p} \frac{d}{dt} ||\delta \omega||_{L^p}^p + ||\partial_r \delta \rho||_{L^p} ||\delta \omega||_{L^p}^{p-1}.$$

So for $p = \frac{6}{5}$ we have

$$\frac{5}{6} \frac{d}{dt} ||\delta \omega||_{L^6}^6 + \frac{5}{9} ||\partial_x |\delta \omega|^\frac{2}{3}||_{L^2}^2 \leq ||(\delta u, \nabla)\omega_1||_{L^\frac{6}{5}} ||\delta \omega||_{L^{\frac{6}{5}}}^\frac{5}{6} + \frac{u^r}{r} ||\delta \omega||_{L^{\frac{6}{5}}}^6 + ||\partial_r \delta \rho||_{L^\frac{6}{5}} ||\delta \omega||_{L^{\frac{6}{5}}}^\frac{5}{6}.$$

From Hölder inequality, Sobolev embeddings, Proposition 3 and inequality (24), we obtain

$$||\frac{\omega_1 \delta u^r}{r}||_{L^6} + ||(\delta u, \nabla)\omega_1||_{L^\frac{6}{5}} \leq \frac{1}{|r|^\frac{3}{5}} ||\delta \omega||_{L^\frac{6}{5}}^\frac{5}{6} + ||\partial_r \delta \omega||_{L^\frac{6}{5}} + ||\partial_x \delta \omega||_{L^\frac{6}{5}} + ||\partial_x \delta \omega||_{L^\frac{6}{5}} \leq ||\frac{\omega_1}{r}||_{L^\frac{5}{2}} + ||\delta \omega||_{L^\frac{6}{5}}^\frac{5}{6} + ||\delta \omega||_{L^\frac{6}{5}} + ||\partial_x \delta \omega||_{L^\frac{6}{5}}.$$

we shall control $||\delta u^z||_{L^\frac{2}{5} (L^\frac{6}{5})}$, using the fact that

$$\Delta \delta u^z = \partial_r \delta \omega + \frac{\delta \omega_0}{r},$$

then by integration by parts, we obtain

$$||\delta u^z||_{L^\frac{2}{5} (L^\frac{6}{5})} \leq \frac{1}{|r|^\frac{3}{5}} ||\delta \omega||_{L^\frac{6}{5}}^\frac{5}{6}.$$

then, using Young inequality for convolution, we obtain

$$||\delta u^z||_{L^\frac{2}{5} (L^\frac{6}{5})} \leq ||\delta \omega||_{L^\frac{6}{5}}^\frac{5}{6} \leq ||\delta \omega||_{L^\frac{6}{5}}^\frac{5}{6}.$$

We thus obtain

$$||\frac{\omega_1 \delta u^r}{r}||_{L^6} + ||(\delta u, \nabla)\omega_1||_{L^\frac{6}{5}} \leq \frac{1}{|r|^\frac{3}{5}} ||\delta \omega||_{L^\frac{6}{5}}^\frac{5}{6} + ||\partial_x \delta \omega||_{L^\frac{6}{5}} + ||\partial_r \delta \omega||_{L^\frac{6}{5}} + ||\partial_x \delta \omega||_{L^\frac{6}{5}}.$$

And since we have

$$||\nabla \delta \rho||_{L^\frac{6}{5}} \leq ||\nabla \delta \rho||_{L^\frac{6}{5}}^\frac{5}{6} ||\delta \rho||_{L^\frac{6}{5}}^\frac{5}{6},$$
therefore we have from Young inequality
\[
d\frac{d}{dt}||\delta\omega||^\frac{3}{2}_{L^\frac{6}{5}} + ||\partial_\omega|\delta\omega||^\frac{3}{2}_{L^2} \\
\lesssim (||\frac{u_r^5}{r}||_{L^\infty} + ||\frac{\omega_1}{r}||^2_{L^\frac{3}{2}} + ||\partial_\omega\omega_1||^2_{L^2} + ||\partial_\omega\partial_\omega\omega_1||_{L^2})||\delta\omega||^\frac{3}{2}_{L^\frac{6}{5}} \\
+ ||\nabla|\delta\rho||^\frac{2}{3}_{L^2}||\delta\rho||^\frac{5}{2}_{L^\frac{6}{5}}||\delta\omega||^\frac{1}{2}_{L^\frac{6}{5}}.
\]
(91)

We multiply the second equation by \(|\delta\rho|^{-\frac{1}{6}}\delta\rho|\), then we obtain by using Young inequality
\[
d\frac{d}{dt}||\delta\rho||^\frac{3}{2}_{L^\frac{6}{5}} + ||\nabla|\delta\rho||^\frac{2}{3}_{L^2} \lesssim ||\nabla\rho^3||_{L^\infty}||\delta\omega||_{L^2}||\delta\rho||^\frac{1}{2}_{L^\frac{6}{5}}.
\]
(92)

And Proposition 3 ensure that
\[||\delta\omega||_{L^\frac{6}{5}} \lesssim ||\delta\omega||_{L^\frac{6}{5}},\]
then we put together (91) and (92), we obtain
\[
d\frac{d}{dt} (||\delta\omega||^\frac{3}{2}_{L^\frac{6}{5}} + ||\delta\rho||^\frac{3}{2}_{L^\frac{6}{5}}) \\
\lesssim (1 + ||\frac{u_r^5}{r}||_{L^\infty} + ||\frac{\omega_1}{r}||^2_{L^\frac{3}{2}} + ||\partial_\omega\omega_1||^2_{L^2} + ||\partial_\omega\partial_\omega\omega_1||_{L^2} + ||\nabla\rho^3||_{L^\infty}) \\
\times (||\delta\omega||^\frac{5}{2}_{L^\frac{6}{5}} + ||\delta\rho||^\frac{5}{2}_{L^\frac{6}{5}}).
\]

So, we obtain the uniqueness of the solution if \(\partial_\omega\omega_1 \in L_t^\infty(L^\frac{3}{2})\) and \(\partial_\omega\partial_\omega\omega_1 \in L_t^2(L^\frac{3}{2})\) because Proposition 3 14 and 14, imply \((||\frac{u_r^5}{r}||_{L^\infty} + ||\frac{\omega_1}{r}||^2_{L^\frac{3}{2}} + ||\nabla\rho^3||_{L^\infty}) \in L_t^1\).

The first step is to prove that \(\partial_\omega\omega_1 \in L_t^\infty(L^\frac{3}{2})\). More precisely we prove that we can propagate the regularity of \(\partial_\omega\omega_1\) in the Lorentz space \(L^\frac{3}{2}\) and moreover, we prove that we have a regularizing effect in this space.

5.2. The propagation of regularities.

**Proposition 16.** Let \(u_0 \in L^2\) be an axisymmetric divergence free vector field such that \(\omega_0 \in L^2 \cap L^\frac{6}{5}\) and \(\omega_t \in L^2\) and let \(\rho_0 \in L^p \cap L^m \cap L^2, m > 6\) and \(1 < p \leq 2\) such that \(\nabla\rho_0 \in L^q\) for \(\frac{3}{2} < q \leq 2\), we have
\[||\nabla\rho||_{L_t^\infty L^q} + ||\nabla^2\rho||_{L_t^2 L^2} \leq \phi_3(t).
\]

**Proof.** For the propagation of \(\nabla\rho\) in \(L^q\) for \(\frac{3}{2} < q \leq 2\), we have
\[(\partial_\omega\rho)_t + (u_\omega)(\partial_\omega\rho) - \Delta\partial_\omega\rho = -(\partial_\omega u, \nabla)\rho.\]

So we obtain
\[||\nabla\rho||^\frac{q}{2}_{L_t^2 L^\frac{3}{2}} + ||\nabla(\partial_\omega\rho)||^\frac{q}{2}_{L_t^2 L^\frac{3}{2}} \lesssim ||\nabla\rho_0||^\frac{q}{2}_{L^\infty L^q} + ||\nabla u_\omega||_{L_t^\infty L^2}||\nabla\rho||_{L_t^2 L^2} \\
\lesssim ||\nabla\rho_0||^\frac{q}{2}_{L^\infty L^q} + ||\omega||_{L_t^\infty L^2}||\nabla\rho||_{L_t^2 L^{2q}}.
\]

Therefore
\[||\nabla\rho||_{L_t^\infty L^q} + ||\nabla^2\rho||_{L_t^2 L^2} \lesssim ||\nabla\rho_0||_{L_t^\infty L^q} + ||\omega||_{L_t^\infty L^2} \frac{1}{2}||\nabla\rho||_{L_t^2 L^{2q}}.
\]

As \(\frac{3}{2} < q \leq 2\) so by interpolation we obtain
\[||\nabla\rho||_{L_t^\infty L^q} \lesssim ||\nabla\rho||_{L^\frac{6}{5}}^{1-\frac{3}{2q}}||\nabla^2\rho||_{L^\frac{6}{5}}^{\frac{3}{2q}}.
\]
And then
\[
\|\nabla \rho\|_{L_t^4 L_x^2} \lesssim \left( \int_0^t \left\| \nabla \rho \right\|^2_{L_t^4 L_x^2} \left\| \nabla^2 \rho \right\|_{L_t^2 L_x^4} \right)^{\frac{1}{2}} \\
\lesssim \left\| \nabla^2 \rho \right\|_{L_t^4 L_x^2} \left( \int_0^t \left\| \nabla \rho \right\|^4_{L_t^{q} L_x^{\frac{6}{q}}} \right)^{\frac{1}{2}}.
\]

By using inequality, we obtain that
\[
\|\nabla \rho\|_{L_t^\infty L_x^3} + \|\nabla^2 \rho\|_{L_t^2 L_x^2} \lesssim \|\nabla \rho_0\|_{L_t^\infty L_x^3} + \|\omega\|_{L_t^{q} L_x^{\frac{6}{q}}} \left( \int_0^t \|\nabla \rho\|^4_{L_t^{q} L_x^{\frac{6}{q}}} \right)^{\frac{1}{2}}.
\]

Then we have
\[
\|\nabla \rho\|_{L_t^{q-3} L_x^2} + \|\nabla^2 \rho\|_{L_t^{q-3} L_x^{2}} \lesssim \|\nabla \rho_0\|_{L_t^{q-3} L_x^2} + \|\omega\|_{L_t^{q} L_x^{\frac{6}{q}}} \left( \int_0^t \|\nabla \rho\|^4_{L_t^{q} L_x^{\frac{6}{q}}} \right)^{\frac{1}{2}}.
\]

As $4q - 6 < 4q - 3$, we use Lemma 2.6 we obtain for $\frac{q}{2} < q < 2$ that
\[
\|\nabla \rho\|_{L_t^{q-3} L_x^2} + \|\nabla^2 \rho\|_{L_t^{q-3} L_x^{2}} \lesssim \|\nabla \rho_0\|_{L_t^{q-3} L_x^2} + \|\omega\|_{L_t^{q} L_x^{\frac{6}{q}}} \left( \int_0^t \|\nabla \rho\|^4_{L_t^{q} L_x^{\frac{6}{q}}} \right)^{\frac{1}{2}}.
\]

For $\|\nabla \rho\|_{L_t^{\infty} L_x^{\frac{6}{q}}}$ we use Lemma 2.7 and we obtain
\[
\|\nabla \rho\|_{L_t^{\infty} L_x^{\frac{6}{q}}} \lesssim \|\nabla \rho_0\|_{L_x^{\frac{6}{q}}} + \left\| (u, \nabla) \rho \right\|_{L_t^{\frac{3}{2}}(L_x^{\frac{2}{3}})} \lesssim \|\nabla \rho_0\|_{L_x^{\frac{6}{q}}} + \|\nabla \|_{L_t^{\infty}(L_x^{\infty})} \|\nabla \rho\|_{L_t^{\frac{3}{2}}(L_x^{\frac{2}{3}})} \lesssim \|\nabla \rho_0\|_{L_x^{\frac{6}{q}}} + \|\omega\|_{L_t^{\infty} L_x^{\frac{6}{q}}} (\|\nabla \rho\|_{L_t^2 L_x^2} + \|\nabla \rho\|_{L_t^2 L_x^2}) \lesssim \phi_2(t).
\]

**Proposition 17.** Let $u_0 \in L^2$ be an axisymmetric divergence free vector field such that $\omega_0 \in L^{3,1} \cap L^\frac{6}{5} \cap L^2$ and $\omega_0 \in L^{2,1} \cap L^2$ and let $\rho_0 \in L^p \cap L^m$, $m > 6$, $1 < p \leq 2$ be an axisymmetric scalar function such that $|x| < L^2$ and $\nabla \rho_0 \in L^q$, $\frac{3}{2} < q \leq 2$. Then we have
\[
\|\omega(t)\|_{L_t^{3,1}} + \|\partial_r \omega(t)\|_{L_t^{\frac{3}{2},1}} + \|\partial_x \partial_r \omega\|_{L_t^{\frac{3}{2},1}} \leq \phi_3(t).
\]

**Proof.** First for some $\lambda > 3$ we have
\[
\|\omega(t)\|_{L_t^{3,1}} \lesssim (\|\omega_0\|_{L_t^{3,1}} + \|\nabla \rho\|_{L_t^{3,1}}) e^{C\|\omega_0\|_{L_t^{\frac{3}{2},1}}} \lesssim (\|\omega_0\|_{L_t^{3,1}} + \|\nabla \rho\|_{L_t^2 L^2} + \|\nabla \rho\|_{L_t^\infty L^3}) e^{C\|\omega_0\|_{L_t^\infty L^3}} \lesssim (\|\omega_0\|_{L_t^{3,1}} + \|\nabla \rho\|_{L_t^2 L^2} + \|\nabla \rho\|_{L_t^\infty L^3}) e^{C\|\omega_0\|_{L_t^\infty L^3}}
\]
or by Proposition 3 and 15, we have
\[
\|\omega(t)\|_{L_t^{3,1}} \lesssim \sqrt{t} \|\partial_x \partial_r \omega\|_{L_t^{\frac{3}{2},1}} \leq \phi_2(t).
\]

Then
\[
\|\omega\|_{L_t^{\infty} L^{3,1}} \leq \phi_3(t).
\]
We have that $\partial_t \omega$ satisfies the following system
\[
\begin{cases}
\partial_t \partial_t \omega + u \cdot \nabla \partial_t \omega - \partial_t^2 \partial_t \omega = \partial_r u^r \omega + u^r \partial_r \omega - \partial_r u^r \partial_r \omega - \partial_r^2 \rho \varepsilon \theta, \\
\partial_t \omega|_{t=0} = \partial_r \omega_0.
\end{cases}
\]

And from the fact that $\partial_r u^r = -\frac{u^r}{r} - \partial_z u^z$ and $\partial_r u^z = \partial_z u^r - \omega$, we deduce the following equation
\[
\partial_t \partial_t \omega + (u, \nabla) \partial_t \omega - \partial_z^2 \partial_t \omega = \frac{u^r}{r} (2 \partial_t \omega - \frac{\omega}{r}) - \partial_z u^z \partial_t \omega - \partial_z u^r \partial_t \omega + \omega \partial_z \omega - \partial_r^2 \rho \varepsilon \theta
\]
\[
= \frac{u^r}{r} (2 \partial_t \omega - \frac{\omega}{r}) + \partial_z u^z \partial_t \omega - \partial_z u^r \partial_t \omega - \partial_r^2 \rho \varepsilon \theta + g
\]
\[
= f + g
\]
with
\[
g = -\partial_z u^r \partial_t \omega + \omega \partial_z \omega.
\]

Multiplying the equation verified by $\partial_t \omega$ by $|\partial_t \omega|^\frac{3}{2}$ sign $\partial_t \omega$ and integrating in space, we obtain
\[
\frac{d}{dt} \left[ |\partial_t \omega|^\frac{3}{2} + ||\partial_t \omega|^2 \right] \leq \left( \frac{u^r}{r} ||\partial_t \omega||_{L^\infty} + ||u^r||_{L^\infty} \right) ||\partial_t \omega||_{L^\frac{3}{2}}
\]
\[
+ \left( ||\partial_z u^z||_{L^\infty} ||\partial_t \omega||_{L^\frac{3}{2}} + ||\partial_z u^r||_{L^\frac{3}{2}} + ||g||_{L^\frac{3}{2}} + ||\Delta \rho||_{L^\frac{3}{2}} \right) ||\partial_t \omega||_{L^\frac{3}{2}}.
\]

Integrating by parts and using the Cauchy-Schwarz inequality, we have
\[
\int \partial_t u^z \partial_t \omega \leq 2 \int u^z |\partial_t \omega|^\frac{3}{2} ||\partial_t \omega||_{L^\frac{3}{2}} \leq 2 ||u^z||_{L^\infty} ||\partial_t \omega||_{L^\frac{3}{2}} ||\partial_t \omega||_{L^\frac{3}{2}}.
\]

And finally
\[
\frac{d}{dt} \left[ |\partial_t \omega|^\frac{3}{2} + ||\partial_t \omega|^2 \right] \leq \left( \frac{u^r}{r} ||\partial_t \omega||_{L^\infty} + ||u^r||_{L^\infty} \right) ||\partial_t \omega||_{L^\frac{3}{2}}
\]
\[
+ \left( ||\partial_z u^z||_{L^\infty} ||\partial_t \omega||_{L^\frac{3}{2}} + ||g||_{L^\frac{3}{2}} + ||\Delta \rho||_{L^\frac{3}{2}} \right) ||\partial_t \omega||_{L^\frac{3}{2}}.
\]

By Hölder inequality, interpolation and we use the following inequality (see [3])
\[
||\partial_z u^z||_{L^\frac{3}{2}} \leq \frac{1}{|\frac{3}{2}|} ||\partial_z \omega||
\]
we have
\[
||\partial_z u^z \frac{\omega}{r}||_{L^\frac{3}{2}} \leq \frac{||\frac{\omega}{r}||_{L^\frac{3}{2} \L^\infty} ||\partial_z u^z||_{L^\frac{3}{2} \L^\infty}}{||\partial_z \omega||_{L^\frac{3}{2}}} 
\]
\[
\leq \frac{||\frac{\omega}{r}||_{L^\frac{3}{2} \L^\infty} ||\partial_z \omega||_{L^\frac{3}{2}} \left( ||\partial_z \partial_t \omega||_{L^\frac{3}{2}} + ||\partial_z \frac{\omega}{r}||_{L^\frac{3}{2}} \right)}{||\partial_z \omega||_{L^\frac{3}{2}}} 
\]
\[
\leq ||\frac{\omega}{r}||_{L^\frac{3}{2} \L^\infty} ||\partial_z \omega||_{L^\frac{3}{2}} \left( ||\partial_z \partial_t \omega||_{L^\frac{3}{2}} + ||\partial_z \omega||_{L^\frac{3}{2}} \right) ||\partial_z \omega||_{L^\frac{3}{2}} \frac{||\partial_z \partial_t \omega||_{L^\frac{3}{2}}}{||\partial_z \omega||_{L^\frac{3}{2}}}.
\]
Then Gronwall lemma implies that
\[
\|\partial_z \omega \|_{L^2} + \|\partial_z \omega \|_{L^2} \leq \|\partial_z \omega \|_{L^2} \leq \phi_2(t),
\]
and consequently by Lemma 24 and Hölder inequality we obtain
\[
\|\partial_z u^2 \frac{\omega}{r} \|_{L^2} \|\partial_z \omega \|_{L^2} \leq \|\frac{\omega}{r} \|_{L^2} \|\partial_z \omega \|_{L^2} \|\partial_z \omega \|_{L^2} \|
\]
and so, the inequalities (93) and Proposition 12, imply that
\[
\|\partial_z \omega \|_{L^2} \leq \phi_3(t).
\]
Finally the inequality (24) and the above inequality assure that
\[
\|\partial_z \omega \|_{L^2} + \|\partial_z \partial_z \omega \|_{L^2} \leq \phi_2(t),
\]
and
\[
\|\partial_z \omega \|_{L^2} \leq \phi_3(t).
\]
Therefore, thanks to Hölder inequality and Proposition 3, we have
\[
\int_0^t \|\partial_z u^2 \partial_z \omega \|_{L^2} \leq \int_0^t \|\partial_z u^2 \|_{L^2} \|\partial_z \omega \|_{L^2} \leq \int_0^t \|\partial_z \omega \|_{L^2} \|^2 dt.
\]
and from Proposition 13 we have
\[
\int_0^t \|\partial_z u^2 \partial_z \omega \|_{L^2} \leq \phi_2(t).
\]
Concerning \(\|\partial_z \omega^2 \|_{L^2} \), inequality (24), implies that
\[
\int_0^t \|\partial_z \omega^2 \|_{L^2} \leq \int_0^t \|\partial_z \omega \|_{L^2} \|\partial_z \omega \|_{L^2} \|\partial_z \omega \|_{L^2} \leq \int_0^t \|\partial_z \omega \|_{L^2} \|^2 dt.
\]
Concerning \( ||u^r||_{L^2_t L^\infty}^2 \) the Proposition 3 and inequality (93) imply

\[
\int_0^t ||u^r||_{L^2}^2 \lesssim \int_0^t ||\omega||_{L^{3,1}}^2 \leq \phi_3.
\]

(97)

Then we deduce

\[
||\partial_r \omega||_{L^2} + ||\partial_z \partial_r \omega||_{L^2} \leq \phi_3.
\]

This completes the proof of the proposition.

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