On the smallest parallel quadrangulation with minimum degree 3

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Abstract
The identity of the smallest quadrangulation with minimum degree 3 also containing parallel edges is unknown. However, it has already been determined that its order (the number of vertices) is between 11 and 14. This paper narrows this domain by showing that the order is at least 12.

1. Introduction
A plane graph is a graph whose vertices are drawn points and edges are arcs on the two dimensional plane such that no two edges meet in a point other than a common endpoint. The edges divide the planar surface into regions called faces. A walk of length \( l \) is a sequence of \( l \) adjacent edges, and the walk is closed if it ends in the starting vertex. A plane quadrangulation (or shortly quadrangulation) is a loopless, connected, finite plane graph having every face bounded by a closed walk of length 4. A quadrangulation without parallel edges and without repeated edges on the quadrilateral boundary walks is called a simple quadrangulation. We allow parallel edges, and the boundary walk may repeat edges or vertices. If we want to emphasize that a quadrangulation may not be simple, it is called a multiquadrangulation, abbreviated as MQ. The multi quadrangulations (MQs) of smallest order are shown in Figure 1.

This paper investigates the smallest MQs with minimum degree 3 which has parallel edges. By smallest, we mean smallest order, i.e. the minimum number of vertices. Table 1 illustrates these concepts on the graphs of Figure 1. Note that though \( P_2 \) is not parallel, it is not simple either according to the definition above. This paper shows the following:

**Theorem 1.** Every MQ with minimum degree 3 containing parallel edges has at least 12 vertices.

The importance of MQs with minimum degree 3 is related to a mechanical classification system for convex, homogeneous bodies introduced recently. In this system, each body is mapped into its secondary equilibrium class determined by the topology of the equilibrium points of the surface. Such a topology is defined by a vertex-coloured MQ. There are particular secondary equilibrium classes called irreducible ancestors, from which every other secondary equilibrium class can be generated with specific transformations (detailed in other works). Such an ancestor class either corresponds to a MQ with minimum degree 3, or to the MQ denoted by \( P_2 \) on Figure 1, latter representing the class of the mono-monostatic body called Gömböc. For this reason, we say a MQ is an irreducible MQ (or shortly irreducible), if it is isomorphic to \( P_2 \) or its minimum degree is 3. If an

| graph | order | parallel | min. degree |
|-------|-------|----------|-------------|
| \( P_2 \) | 3     | no       | 1           |
| \( C_4 \) | 4     | no       | 2           |
| \( Q_3 \) | 4     | yes      | 1           |
| \( Q_4 \) | 4     | yes      | 1           |

Table 1: Illustration of concepts

Figure 1: MQs of smallest order
irreducible contains parallel edges, it is called a parallel irreducible.

There are efficient methods to exhaustively enumerate simple irreducibles\(^1\)\(^-\)\(^2\). A highly tuned implementation called Plantri is also available\(^1\). However, these enumerations are incomplete as they ignore the ones containing parallel edges. Identifying the smallest parallel irreducible would specify the limit of these incomplete methods regarding the enumeration of all MQs.

We mention that some pieces of the text of this paper can also be found in the submitted dissertation of the first author.

2. Related Work

It has already been determined that the order of the smallest parallel irreducible is between 11 and 14, detailed in this section. First, every irreducible has at least 8 vertices\(^1\)\(^-\)\(^6\):

**Theorem 2.** The order of an irreducible MQ is at least 8.

This statement was originally made by Batagelj\(^1\) for simple irreducibles, and was later generalized by others\(^6\) to include parallel irreducibles as well. There also exists a primitive implementation\(^6\) to enumerate every MQ of order at most 10. Observing the generated data\(^6\), we arrive at

**Theorem 3.** Having the order between 8 and 10, there are two non-isomorphic irreducible MQs.

These two MQs are shown in Figure 2. As there is also an irreducible with parallel edges depicted in Figure 3, we have an obvious upper bound\(^6\):

**Theorem 4.** The order of the smallest irreducible MQ containing parallel edges is at most 14.

3. Prerequisites

In the proof of Theorem 1, we use some concepts and properties of plain graphs. Let us denote the number of vertices, edges and faces, by \(n\), \(e\) and \(f\), respectively. Euler’s formula holds for any plane graph: \(n − e + f = 2\). Applied to a MQ, as every face has 4 boundary edges and every edge is counted twice, we have

\[ e = 2n − 4. \]  

(1)

Consequently, the sum of the degrees of a MQ of order \(n\) equals to \(4n − 8\).

Equation 1 also implies that the minimum degree of a MQ is either 1, 2 or 3, because if it had only vertices of degree at least 4, then it would have at least \(2n\) edges.

Let \(G(V, E)\) denote a plane graph with vertex set \(V\) and edge set \(E\), where \(V\) contains points, \(E\) contains arcs on the plane. The plane graph \(G(V, E)\) is the embedded subgraph of the plane graph \(G'(V', E')\) if \(V \subseteq V'\) and \(E \subseteq E'\).

4. Proof of Theorem 1

**Proof** Let \(G\) be an irreducible MQ with parallel edges such that it is of minimum order. Suppose \(G\) has \(k\) parallel edges between vertices \(v\) and \(w\). The \(k\) parallel edges divide the surface into \(k\) regions. Let us select a region with the fewest vertices inside. Now let us prepare the embedded subgraph \(H\) of the MQ \(G\) by selecting the subgraph spanned by the vertices from inside the selected region and the vertices \(v\) and \(w\), then removing \(k − 1\) parallel edges between \(v\) and \(w\).

For example, in Figure 3, if the right hand side is \(G\), one of the graphs on the left hand side could be isomorphic to \(H\), and \(k = 2\). It is easy to see that the plane graph \(H\) has the following properties: (i) it is a MQ, (ii) \(v\) and \(w\) are still adjacent and there is exactly one edge between them, (iii) every other vertex than \(v\) and \(w\) has degree at least 3. We call a plane graph satisfying these three conditions a half with marked adjacent vertices \(v\) and \(w\) throughout this proof. Let \(d_k(z)\) denote the degree of vertex \(z\) in graph \(K\). In addition to these three defining properties, it is easy to see that any MQ \(Q\) also has property (iv): \(d_k(x) + d_k(y) \geq 3\) for any two adjacent vertices \(x\) and \(y\).

An irreducible MQ with parallel edges can be built from a half \(F\) with marked adjacent vertices \(x\) and \(y\), supposing there is only one edge between \(x\) and \(y\), as follows. We clone \(F\), rotate the clone by 180 degrees preserving its orientation, and stick together \(F\) and the clone by unifying vertex \(x\) and the clone image of \(y\) denoted \(y'\), unifying vertex \(y\) and the clone image of \(x\) denoted \(x'\), and unifying the outer edge \(xy\) and its clone image. So the unifications remove the clone vertices \(x'\) and \(y'\) and the clone of the edge \(xy\). Then we duplicate the original edge \(xy\) outside in order to restore the quadrilateral property. The process is illustrated in Figure 3.

We need to show that the resulting graph denoted by \(P\) is irreducible. Clearly the order of \(P\) is at least 4, so irreducibility is equivalent to having minimum degree of 3. It is already a half, so every other vertices than \(x\) or \(y\) has degree at least 3. After the unification, the vertex \(x\) is now connected with the neighbours of \(y'\) including \(x'\) because of the additional parallel edge, so \(d_p(x) = d_f(x) + d_p(y) \geq 3\) from property (iv) of the halves. Similarly, \(d_p(y) = d_f(x) + d_p(y) \geq 3\).

Now we can prove that \(H\) is simple. By property (iii), \(H\) is not isomorphic to the MQ \(P_2\). Note that it is easy to see that if a half \(H\) with marked adjacent vertices \(v\) and \(w\) has parallel edges, then it also contains a smaller half. The contained half can be found using a similar method used to define the half...
$H$ inside the irreducible $G$, although instead of selecting the minimal region, any region can be selected which does not contain $v$ and $w$ (because $d_H(v)$ or $d_H(w)$ can be less than 3). So suppose indirectly that $H$ has parallel edges. Then there is a half $F$ contained in $H$, obviously smaller than $H$. So an irreducible MQ with parallel edges could be built from $F$ as described above, but smaller than $G$, contradicting to its minimality.

Now we are going to prove that $d_H(v) + d_H(w) \geq 5$. The minimum degree of a simple quadrangulation is at least 2. If $d_H(v) = d_H(w) = 2$ holds in a simple quadrangulation for two adjacent vertices, than it can only be the circle $C_4$ of length 4. However, by property (iii), $H$ cannot be isomorphic to $C_4$.

If the half $H$ has $n$ vertices, than the sum of its degrees is $4n - 8$, so we have

$$4n - 8 = d_H(v) + d_H(w) + \sum_{z \not= v,w} d_H(z) \geq 5 + 3(n - 2),$$

implying $n \geq 7$. So there are at least 5 vertices in the inside region of $G$, the same outside, plus $v$ and $w$, added up to 12. \hfill \Box

5. Conclusions

Theorem 1 means that when the software Plantri generates the simple irreducibles for $n < 12$, then it also generates every irreducible. Consequently, the cardinalities of the simple irreducibles of different orders published by Brinkmann et. al.\cite{BrinkmannGreenbergGreenhillMcKayThomasWollan2005} (see their Table 2 titled “Simple quadrangulations with minimum degree 3”) does not exclude any parallel irreducible for $n < 12$.

Up to the best knowledge of the authors, the identity of the smallest parallel irreducible is still unknown. Although there is a primitive implementation to enumerate every MQ, it is practically unusable for $n \geq 12$ because of its very low efficiency. A better way to find it could be extending Plantri to enumerate efficiently every MQ (not just the simple ones), using extended operations\cite{KapolnaiDomokosSzabo2012}.

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References

1. V. Batagelj. An inductive definition of the class of 3-connected quadrangulations of the plane. Discrete Mathematics, 78(1-2):45–53, 1989.
2. G. Brinkmann, S. Greenberg, C. Greenhill, B. D. McKay, R. Thomas, and P. Wollan. Generation of simple quadrangulations of the sphere. Discrete Mathematics, 305(1-3):33–54, 2005.
3. G. Brinkmann and B. D. McKay. Fast generation of planar graphs. MATCH Communications in Mathematical and in Computer Chemistry, 58(2):323–357, 2007.
4. R. Diestel. Graph Theory. Springer, 2005.
5. G. Domokos, Zs. Lángi, and T. Szabó. The genealogy of convex solids. Manuscript, http://arxiv.org/abs/1204.5494, last access: 2014.06.06., 2012.
6. R. Kápolnai, G. Domokos, and T. Szabó. Generating spherical multiquadrangulations by restricted vertex splittings and the reducibility of equilibrium classes. Periodica Polytechnica Electrical Engineering and Computer Science, 56(1):11–20, 2012.
7. A. Nakamoto. Generating quadrangulations of surfaces with minimum degree at least 3. Journal of Graph Theory, 30(3):223–234, 1999.
8. P. Várkonyi and G. Domokos. Static equilibria of rigid bodies: dice, pebbles and the Poincaré–Hopf theorem. Journal of Nonlinear Science, 16:255–281, 2006.