Modulated wave packets in pulsar magnetospheric plasma

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Abstract. An investigation has been made of modulational instability of nonlinear ion acoustic wave in magnetized electron-positron-ion plasma. Three dimensional nonlinear Schrödinger equations (NLSE) is derived by using the standard reductive perturbation technique (RPT) to study the modulational instability. It is found that the coefficient of Nonlinear Schrödinger equation depends upon the positron to electron density ratio and the electron to positron temperature ratio. These coefficients significantly modify the conditions of the modulational instability.

1. Introduction
Modulational instability in a nonlinear dispersive medium, which can excite the formation of localized nonlinear coherent structures called envelope solitary waves, belongs to an important class of collective phenomena [1, 2, 3,4]. Dynamics of the modulational instability is governed by the NLSE. The modulational instability of monochromatic ion acoustic wave has been reported experimentally and theoretically by number of researchers [5, 6, 7,8]. In unmagnetized plasmas the ion acoustic wave (IAW) is only low frequency electrostatic ion mode. In the presence of a magnetic field two low-frequency electrostatic ion waves exist: ion cyclotron like wave and ion acoustic wave which propagate, respectively, above and below the ion cyclotron frequency. Understanding the modulated characteristics of these modes is important for the stable propagation of these waves in plasma. Earlier investigations of low frequency waves in magnetized electro-ion (e-i) plasmas has been studied long back [9,10,11].

The study of electron-positron-ion (e-p-i) plasma has attracted a great interest in the last few years. The e-p-i plasma is important not only from a cosmological and astrophysical point of view, but also in the context of laboratory plasmas. By introducing positron in (e-i) plasma or ions in electro-positron (e-p) plasma, one can obtains the three component (e-p-i) plasma. The (e-p-i) plasma believed to exist in the pulsar magnetosphere[12,13,14,15,16,17] in bipolar outflows in active galactic nuclei[18,19] and in the early universe[20]. The modulational instability of IAW in both e-p-i as well as in pair plasmas has been a subject of great interest in the recent years [21,22,23,24]. However, these investigations did not take into account the effects of magnetic field which permeates in most of the astrophysical and laboratory plasmas. Most of the investigations are limited to deriving one-dimensional NLSE. However, the one dimensional
geometry may not be a realistic situation in laboratory devices and in space. Franz et al. [25] have shown that a purely one-dimensional model cannot account for all observed features in the auroral region, especially at higher polar altitude. Thus, more authors began to study various models in two or three dimensions. Recently three dimensional nonlinear Schrödinger equation (3D-NLSE) have been reported by many researchers to study the modulational instability in numbers of plasma system[26,27,28,29]. We consider here more realistic situation where the externally applied magnetic field is along $x$-direction. Perturbation associated with IAW in such plasmas may be in arbitrary direction. However, we shall simplify our case where wave propagates in $x$-direction with weak perturbation along $y$ and $z$ direction. To the best of our knowledge, the three dimensional NLSE in three component unmagnetized and as well as magnetized plasmas has not been reported in the literature. Here the reductive perturbation technique, is used to derive the three dimensional Schrödinger equation (3D-NLSE) in the magnetized $e-p-i$ plasma. The results are analytically evaluated for stability criteria. Section 2. deals with derivation of 3D NLSE while in section 3, we have studied the parameter regimes characterizing the modulational instability.

### 2. Derivation of three-dimensional 3D NLSE

We consider the IAW propagates in the three component magnetized plasmas. The magnetic field $\mathbf{B}_0$ is uniform and directed along the $x$-axis. So the fluid equations which governs the dynamic of the IAW in 3D geometry are

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{V}_i) = 0
\]

\[
\frac{\partial V_i}{\partial t} + V_i \nabla \cdot \mathbf{V}_i = -\nabla \phi + \omega_c \mathbf{V}_i \times \mathbf{B}.
\]

\[
\nabla^2 \phi = n_e - n_p - n_i.
\]

The electrons and positrons in the electrostatic obey the Boltzmann distribution. These Boltzmann relations for electrons and positrons are respectively;

\[
n_e = \mu \exp(\phi) \quad \text{and} \quad n_p = (\mu-1)\exp(-\alpha \phi),
\]

where $\alpha = t_e/t_p$, $\mu = 1/(1-p)$ and $p = n_p/n_i$, $\mathbf{V}_i = (u,v,w)'$, $u,v,w'$ are the ion fluid velocity in the $x, y$ and $z$ direction respectively, normalized by the ion-acoustic speed $c_s = (K_B t_i/e m)^{1/2}$ where $K_B$ is the Boltzmann's constant. $n_i$ and $\phi$ represents the ion density and the electrostatic potential. $t_e$ and $t_p$ are the electron and positron temperature respectively. $n_e$ and $n_p$ are the electrons and positrons densities normalized by the unperturbed ion density. $\omega_{ci} = eB_0/(m_i)$ is the ion cyclotron frequency. The space and time co-ordinates are normalized by the Debye length and ion-plasma period, respectively. The quantity $\phi$ is normalized by $c_s = (K_B t_i/e)$ where $e$ is the electron charge.

In order to investigate the modulation instability of 3D IAW in magnetized plasmas, we use the standard reductive perturbation technique. Considering the strongly magnetized multicomponent plasmas and the wave propagates in the $x$ direction with weak transverse perturbations, the independent variables from set of equations (1-3) stretched as $\zeta = \epsilon(x-v_g t)$, $\eta = \epsilon y$, $\zeta = \epsilon z$ and $\tau = \epsilon^\tau t$, where $\epsilon$ is the small parameter and $v_g$ is the group velocity of the wave. The dependent variables are expanded as [26]:
\[ n_j = 1 + \sum_{n_l=1}^{\infty} e^\tau \sum_{l=0}^{\infty} n_{j_l} (\xi, \eta, \zeta, \tau) e^{\xi (\xi - \eta)} \]

\[ u = \sum_{n_l=1}^{\infty} e^\tau \sum_{l=0}^{\infty} u_{j_l} (\xi, \eta, \zeta, \tau) e^{\xi (\xi - \eta)} \]

\[ (v, w) = \sum_{n_l=1}^{\infty} e^\tau \sum_{l=0}^{\infty} v_{j_l} (\xi, \eta, \zeta, \tau), \omega_j (\xi, \eta, \zeta, \tau) \]

\[ \phi = \sum_{n_l=1}^{\infty} e^\tau \sum_{l=0}^{\infty} \phi_{j_l} (\xi, \eta, \zeta, \tau) e^{\xi (\xi - \eta)} \] (4)

where \( n, u, v, w \) and \( \phi \) satisfy the reality condition \( A_{j}^{(n)} = A_{j}^{(n)*} \) and the asterisk denotes the complex conjugate.

Substituting the equation (4) along with the stretching co-ordinates into equations (1-3) and collecting the terms in different powers of \( \varepsilon \), we obtain the following equations at the lower order of \( \varepsilon \):

\[ n_{i_1} = \left( \frac{k}{\omega} \right)^2 \phi_{i_1}, \quad u_{i_1} = \frac{k}{\omega} \phi_{i_1} \quad \text{and} \quad (k^2 + M) \phi_{i_1} = n_{i_1} \]

Together with \( u_{i_1}, v_{i_1}, \omega_{i_1}, \phi_{i_1}, n_{i_1} = 0 \), the dispersion relation is \( \omega^2 = \frac{k^2}{k^2 + M} \),

where \( M = \mu - \alpha (1 - \mu) \). For the second order \( n = 2 \) reduced with \( l = 1 \), the compatibility condition \( v_x = (\omega/k)(1 - \omega^2) \frac{\partial \omega}{\partial k} \) can be obtained. The second harmonic mode of the carrier wave with \( l = 0, 1 \) and 2 is also obtained in terms of \( \phi_{i_1} \). For \( n = 2, l = 1 \) we have

\[ -i \omega n_{i_2} + ku_{i_2} = \left( \frac{k}{\omega} \right)^2 \left( \frac{\partial \phi_{i_1}}{\partial \xi} \right), \]

\[ -i \omega u_{i_2} + ik \phi_{i_2} = \left( \frac{k}{\omega} \right)^2 \left( \frac{\partial \phi_{i_1}}{\partial \xi} \right), \]

\[ n_{i_2} - (k^2 + M) \phi_{i_2} = -2 \frac{\partial \phi_{i_1}}{\partial \xi} \]

\[ \phi_{i_1} = \frac{10 \omega}{\omega^2 k^2} \frac{\partial \phi_{i_1}}{\partial \eta} - \frac{\omega_{i_1}}{(\omega_{i_1}^2 - \omega^2)} \frac{\partial \phi_{i_1}}{\partial \zeta}, \]

\[ \phi_{i_1} = \frac{10 \omega}{\omega^2 k^2} \frac{\partial \phi_{i_1}}{\partial \eta} - \frac{\omega_{i_1}}{(\omega_{i_1}^2 - \omega^2)} \frac{\partial \phi_{i_1}}{\partial \zeta} \]

(7)

The reduced equations for \( n = 2 \) and \( l = 2 \) have the following form:

\[ -i \omega n_{i_2} + ku_{i_2} = -k \left( \frac{k}{\omega} \right)^3 \left( \phi_{i_1} \right)^2, \]

\[ -i \omega u_{i_2} + k \phi_{i_2} = -\frac{k}{2} \left( \frac{k}{\omega} \right)^3 \left( \phi_{i_1} \right)^2, \]
\[ -\omega \dot{x}_2^{(2)} - (4k^2 - M) \ddot{x}_2^{(2)} = \left( \frac{1}{2} \right) L \phi_1^{(1)} \]  \hspace{1cm} (8)

where \( L = \mu + \alpha^2 (1 - \mu) \)

The second order quantities in the zeroth harmonic are determined by the following \( l = 0 \) component of the second order \((n = 2)\) and the third order \((n = 3)\) parts of the reduced equations:

\[ -v_s \frac{\partial n_0^{(2)}}{\partial \xi} + \frac{\partial u_0^{(2)}}{\partial \xi} = -2 \left( \frac{k}{\omega} \right)^3 \partial \phi_1^{(1)} \]  \hspace{1cm} (9)

According to equations (6)-(8), the variables \( A_0^{(2)}, A_1^{(2)} \) and \( A_2^{(2)} \) can be obtained in terms of \( \phi_1^{(1)} \). Here \( A \) stands for \( n, u \) and \( \phi \). For the next order of \( \epsilon \), we obtain the reduced equations with \( n=3 \) and \( l = 1 \) as:

\[ \frac{\partial n_1^{(1)}}{\partial \tau} - i \omega n_1^{(3)} + \omega u_1^{(3)} - v_s \frac{\partial n_1^{(2)}}{\partial \xi} + \frac{\partial u_1^{(2)}}{\partial \xi} + i k (n_2^{(2)} u_{-1}^{(1)} + n_0^{(2)} u_{0}^{(1)} + n_0^{(2)} u_{0}^{(1)} + n_0^{(2)} u_{0}^{(1)} + n_{-1}^{(2)} u_{2}^{(1)}) + \frac{\partial v_{0}^{(1)}}{\partial \eta} + \frac{\partial u_{0}^{(1)}}{\partial \zeta} = 0 \]

\[ \frac{\partial u_0^{(1)}}{\partial \tau} - i \omega u_0^{(3)} + \omega u_0^{(3)} - v_s \frac{\partial u_1^{(2)}}{\partial \xi} + \frac{\partial \phi_0^{(2)}}{\partial \xi} + i k (u_0^{(1)} u_{0}^{(2)} + u_{0}^{(2)} u_{0}^{(2)} + u_{0}^{(2)} u_{0}^{(2)}) = 0 \]

\[ \frac{\partial^2 \phi_0^{(1)}}{\partial \xi^2} - (k^2 + M) \phi_0^{(1)} + n_{0}^{(1)} + 2 i k \frac{\partial \phi_0^{(1)}}{\partial \xi} + i k \phi_0^{(1)} + \frac{\partial^2 \phi_0^{(1)}}{\partial \eta^2} + \frac{\partial^2 \phi_0^{(1)}}{\partial \zeta^2} = 0 \]  \hspace{1cm} (10)

Finally, substituting the derived expression for \( A_0^{(2)}, A_1^{(2)} \) and \( A_2^{(2)} \) into the above equations and after some algebraic manipulation we get the three dimensional Schrödinger equation:

\[ i \frac{\partial \phi}{\partial \tau} + P \frac{\partial^2 \phi}{\partial \xi^2} + Q |\phi|^2 \phi - R (\frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial^2 \phi}{\partial \zeta^2}) = 0 \]  \hspace{1cm} (11)

where

\[ P = -\frac{3}{2} \frac{\omega^3}{k^2} (1 - \omega^2) \]  \hspace{1cm} (12)

\[ Q = \frac{\omega}{2k^2} \left[ \frac{L \omega}{4k^2 - M} (A - L/2) + L \omega^2 (B - L) / M + 2 k^2 (1 - \omega^2) \right] \]  \hspace{1cm} (13)

\[ R = \frac{\omega}{2k^2} \left[ \frac{1 - \omega^2 + \omega_{\phi}^2}{\omega_{\phi}^2 - \omega^2} \right] \]  \hspace{1cm} (14)

\[ A = \frac{(k + M) (4k^2 - M)}{2(3k^2 - 2M)} \left[ 3(k^2 + M) - \frac{L}{2(4k^2 + M)} \right] \]  \hspace{1cm} (15)

\[ B = \frac{(k^2 + M) M - L}{Mv_{\phi}^2 - 1} \]  \hspace{1cm} (16)
3. Modulational instability of 3D NLSE

Now we study the stability of IAW in an e-p-i magnetized plasma when modulation on the wave amplitude (packets) takes place in the direction of carrier wave propagation. Instead of a stationary solution, here we consider the dynamic solution of NLSE (11). We follow standard procedure where \( \phi \) is perturbed as

\[
\phi = \phi_0 + \delta \phi(\theta)e^{i\tau}
\]

(17)

Where \( \theta = K_\xi \tau + K_\eta \tau + K_\zeta \tau - \Omega \tau \) is the modulation phase with

\[
\vec{K} = \vec{K}_\xi + j \vec{K}_\eta + k \vec{K}_\zeta \quad (|K|<<\langle k \rangle \quad \text{and} \quad \Omega \Omega<<\langle \omega \rangle)
\]

are, respectively, the wave number and the frequency of the modulation, \( \phi_0 \) is constant (real) amplitude perturbation, \( \delta \phi \) is a small amplitude perturbation, and \( \Delta \) is a nonlinear frequency shift. After substituting equation (17) into equation (11) and collecting terms of the same order, we obtain

\[
\Delta = -Q|\phi_0|^2
\]

(18)

\[
-i \frac{\partial \delta \phi}{\partial t} + Q|\phi_0|^2 \delta \phi + \phi_0^* \delta \phi^* + R \left[ \frac{\partial^2 \delta \phi}{\partial \zeta^2} + \frac{\partial^2 \delta \phi}{\partial \eta^2} \right] + P \frac{\partial^2 \delta \phi}{\partial \zeta^2} = 0
\]

(19)

Where \( \delta \phi^* \) is the complex conjugate of \( \delta \phi \). Further introducing \( \delta \phi = U + iV \), and separating the real and imaginary parts, we obtain the following two complex equations:

\[
\frac{\partial U}{\partial t} + R \left[ \frac{\partial^2 V}{\partial \zeta^2} + \frac{\partial^2 V}{\partial \eta^2} \right] + P \frac{\partial^2 V}{\partial \zeta^2} = 0
\]

(20)

\[
-\frac{\partial V}{\partial t} + Q|\phi_0|^2 U + R \left[ \frac{\partial^2 U}{\partial \zeta^2} + \frac{\partial^2 U}{\partial \eta^2} \right] + P \frac{\partial^2 U}{\partial \zeta^2} = 0
\]

(21)

We also assume that amplitude perturbations \( \delta \phi = U + iV \) vary as follow:

\[
U = U_0 e^{i(K_\xi \zeta + K_\eta \eta + K_\zeta \zeta - \Omega \tau)}
\]

(22)

\[
V = V_0 e^{i(K_\xi \zeta + K_\eta \eta + K_\zeta \zeta - \Omega \tau)}
\]

(23)

using equation (22-23) into equations (20-21) and eliminating U and V from the resulting equations, we get the following dispersion relation:

\[
\Omega^2 = \left[ R \left( K_\xi^2 + K_\eta^2 \right) + PK_\zeta^2 \right] \left[ R \left( K_\xi^2 + K_\eta^2 \right) + PK_\zeta^2 \right] - 2Q|\phi_0|^2
\]

(24)

Where \( R \) and \( P \) are the coefficients of the dispersive terms in the NLSE (11), \( Q \) is the nonlinear term in NLSE (11).

We can analyze the dispersion relation (24) to derive suitable criteria for the onset of instability. Algebraic form of Eq. (24) suggest it as a product of two factors. So sign of \( \Omega \) will definitely depend on these factors which further depends explicitly on \( R, P \) and \( Q \). Thus if

(i) \( Q < 0, \ R < 0 \) and \( P < 0 \) then Eq.(24) which can be written as
\[
\Omega^2 = \left[ R(K_{\zeta}^2 + K_{\eta}^2) + PK_{\zeta}^2 \right] \left[ 1 - \frac{2Q|\phi|}{R(K_{\zeta}^2 + K_{\eta}^2) + PK_{\zeta}^2} \right]
\]

Implies \( \Omega^2 < 0 \) for small value of modulation wave number \( k \). Thus ion-acoustic mode in e-p-i plasma is modulationally unstable provided \( R(K_{\zeta}^2 + K_{\eta}^2) + PK_{\zeta}^2 < 2Q|\phi| \).

(ii) On the other hand, if \( R < 0, P < 0 \) and \( Q > 0 \) then \( \Omega > 0 \). This indicates that ion-acoustic wave is modulational unstable.

It is noteworthy to point out that \( P, Q \) and \( R \) which are function of several parameters as given by (12), (13) and (14) vary over a large parameter space. Thus above mentioned criteria are always suitably satisfied depending on e-p-i plasma system we choose to study. One-dimensional case is worth mentioning where we may choose \( K_{\eta} = K_{\xi} = 0 \) with modulation only in \( \zeta \) direction. We recover case of Amin et al [30] for \( P < 0 \) and \( Q > 0 \) which corresponds to modulational stable mode.

Role of long perturbation can be highlighted by considering the following expressions viz
\[
o^2 - \omega^2 = -\frac{k^2}{M + k^2} - \frac{B^2}{4\pi m_n n_0} \quad \text{For larger} \, B \, \text{and small} \, k, \, R < 0, \, \text{which implies stable propagation.}
\]

Apparently the stability region is elegantly given by the following equation:

(i) \( R(K_{\zeta}^2 + K_{\eta}^2) + PK_{\zeta}^2 < 0 \)

(ii) \( R(K_{\zeta}^2 + K_{\eta}^2) + PK_{\zeta}^2 < 2Q|\phi| \)

4. Conclusion

In present investigation, the authors have studied modulational instability of IAW in magnetized e-p-i plasma. Using reductive perturbation technique, 3D-NLSE is derived and analytically evaluated for stability/instability criteria. The results only be compared with unmagnetized e-p-i plasma qualitatively [21]. However, quantitative comparison is not possible because of different procedures and normalization used by us and Salahuddin[21].

Acknowledgement. This work is supported by Department of Science and Technology New Delhi, under project No. SR/SC/AS: 245/05

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