Alin Bostan and Antonio Jiménez-Pastor

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On the exponential generating function of labelled trees

Sur la série génératrice des arbres étiquetés

Alin Bostan\textsuperscript{a} and Antonio Jiménez-Pastor\textsuperscript{*,b}

\textsuperscript{a} Inria, 1 rue Honoré d’Estienne d’Orves 91120 Palaiseau, France
\textsuperscript{b} Johannes Kepler University Linz, Doctoral Program Computational Mathematics, DK15

E-mails: Alin.Bostan@inria.fr (A. Bostan), ajpastor@risc.uni-linz.ac.at (A. Jiménez-Pastor)

Abstract. We show that the generating function of labelled trees is not D\textsuperscript{∞}-finite.

Résumé. Nous montrons que la série génératrice des arbres étiquetés n’est pas D\textsuperscript{∞}-finie.

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1. Context and main result

A formal power series \( f(x) = \sum_{n \geq 0} a_n x^n \) in \( \mathbb{C}[[x]] \) is called differentially finite, or simply D-finite [23], if it satisfies a linear differential equation with polynomial coefficients in \( \mathbb{C}[x] \). Many generating functions in combinatorics and many special functions in mathematical physics are D-finite [2, 9].

DD-finite series and more generally \( D^n \)-finite series are larger classes of power series, recently introduced in [13]. DD-finite power series satisfy linear differential equations, whose coefficients are themselves D-finite power series. One of the simplest examples is \( \tan(x) \), which is DD-finite (because it satisfies \( \cos(x) f(x) - \sin(x) = 0 \), but is not D-finite (because it has an infinite number of complex singularities, a property which is incompatible with D-finiteness). Another basic example is the exponential generating function of the Bell numbers \( B_n \), which count partitions of \( \{1,2,\ldots,n\} \), namely:

\[
B(x) := \sum_{n \geq 0} \frac{B_n}{n!} x^n.
\]  

Indeed, it is classical [9, p. 109] that \( B(x) = e^{e^x - 1} \), therefore \( B(x) \) is DD-finite. On the other hand, \( B(x) \) is not D-finite: this can be proved either analytically (using the too fast growth of \( B(x) \) as \( x \to \infty \)), or purely algebraically (using [22], and the fact that the power series \( e^x \) is not algebraic).

More generally, given a differential ring \( R \), the set of differentially definable functions over \( R \), denoted by \( D(R) \), is the differential ring of formal power series satisfying linear differential equations with coefficients in \( R \). In particular, \( D(\mathbb{C}[x]) \) is the ring of D-finite power series, \( D^2(\mathbb{C}[x]) := D(D(\mathbb{C}[x])) \) is the ring of DD-finite power series, and \( D^n(\mathbb{C}[x]) := D(D^{n-1}(\mathbb{C}[x])) \) is the ring of \( D^n \)-finite power series. We say that a power series \( f(x) \in \mathbb{C}[[x]] \) is \( D^{\infty} \)-finite if there exists an \( n \) such that \( f(x) \) is \( D^n \)-finite.

It is known [14] that \( D^n \)-finite power series form a strictly increasing sequence of sets and that any \( D^\infty \)-finite power series is differentially algebraic, in short D-algebraic, that is, it satisfies a differential equation, possibly non-linear, with polynomial coefficients in \( \mathbb{C}[x] \). This class, as well as its complement (of D-transcendental series), are quite well studied [11, 21].

Let now \( (t_n)_{n \geq 0} = (0, 1, 2, 9, 64, 625, 7776, \ldots) \) be the sequence whose general term \( t_n \) counts labelled rooted trees with \( n \) nodes. It is well known that \( t_n = n^{n-1} \), for any \( n \geq 1 \). This beautiful and non-trivial result is usually attributed to Cayley [6], although an equivalent result had been proved earlier by Borchardt [4], and even earlier by Sylvester, see [3, Ch. 4]. Due to the importance of the combinatorial class of trees, and to the simplicity of the formula, Cayley’s result has attracted a lot of interest over the time, and it admits several different proofs, see e.g., [16, §4] and [1, §30]. One of the more conceptual proofs goes along the following lines (see [9, §II. 5.1] for details). Let

\[
T(x) := \sum_{n \geq 0} \frac{t_n}{n!} x^n
\]  

be the exponential generating function of the sequence \( (t_n) \). The class \( \mathcal{T} \) of all rooted labelled trees is definable by a symbolic equation \( \mathcal{T} = \mathcal{Z} \star \text{SET}(\mathcal{T}) \) reflecting their recursive definition, where \( \mathcal{Z} \) represents the atomic class consisting of a single labelled node, and \( \star \) denotes the labelled product on combinatorial classes. This symbolic equation provides, by syntactic translation, an implicit equation on the level of exponential generating functions:

\[
T(x) = x e^{T(x)},
\]  

which can be solved using Lagrange inversion

\[
t_n = n! \cdot [x^n] T(x) = n! \cdot \left( \frac{1}{n} [z^{n-1}](e^z)^n \right) = n^{n-1}.
\]
From (3), it follows easily that \( T(x) \) is D-algebraic and satisfies the non-linear equation
\[
x(1 - T(x))T'(x) = T(x),
\]
and from there, that the sequence \((t_n)_{n \geq 0}\) satisfies the non-linear recurrence relation
\[
t_{n+1} = \frac{n+1}{n} \cdot \sum_{i=1}^{n} \binom{n}{i} t_i t_{n-i+1}, \quad \text{for all } n \geq 1.
\]
This recurrence can also be proved using (4), by taking \( y = n, x = w = 1 \) in Abel's identity [12]
\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x + wk)^{k-1}(y - wk)^{n-k},
\]
and then by isolating the term \( k = n \) in the resulting equality.

On the other hand, it is known that the power series \( T(x) \) is not D-finite, see [10, Thm. 7], or [8, Thm. 2]. This raises the natural question whether \( T(x) \) is DD-finite, or \( D^{\alpha} \)-finite for some \( \alpha \geq 2 \). Our main result is that this is not the case:

**Theorem 1.** The power series \( T(x) = \sum_{n \geq 1} \frac{n^{n-1}}{n!} x^n \) in (2) is not \( D^\infty \)-finite.

To our knowledge, this is the first explicit example of a natural combinatorial generating function which is provably D-algebraic but not \( D^\infty \)-finite. In particular, Theorem 1 implies that \( T(x) \) is not equal to the quotient of two D-finite functions, and more generally, that it does not satisfy any linear differential equation with D-finite coefficients.

### 2. Proof of the main result

Our proof of Theorem 1 builds upon the following recent result by Noordman, van der Put and Top.

**Theorem 2 ([18]).** Assume that \( u(x) \in \mathbb{C}[[x]] \setminus \mathbb{C} \) is a solution of \( u' = u^3 - u^2 \). Then \( u \) is not \( D^\infty \)-finite.

The proof of Theorem 2 is based on two ingredients. The first one is a result by Rosenlicht [20] stating that any set of non-constant solutions (in any differential field) of the differential equation \( u' = u^3 - u^2 \) is algebraically independent over \( \mathbb{C} \) (see also [18, Prop. 7.1]); the proof is elementary. The second one [18, Prop. 7.1] is that any non-constant power series solution of an autonomous first-order differential equation with this independence property cannot be \( D^\infty \)-finite; the proof is based on differential Galois theory.

**Proof of Theorem 1.** We will use Theorem 2 and a few facts about the (principal branch of the) Lambert \( W \) function, satisfying \( W(x) \cdot e^{W(x)} = x \) for all \( x \in \mathbb{C} \).

Recall [7] that the Taylor series of \( W \) around 0 is given by
\[
W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n = x - x^2 + \frac{3}{2} x^3 - \frac{8}{3} x^4 + \frac{125}{24} x^5 - \cdots.
\]
In other words, our \( T(x) \) and \( W(x) \) are simply related by \( W(x) = -T(-x) \).

The function defined by this series can be extended to a holomorphic function defined on all complex numbers with a branch cut along the interval \((-\infty, -\frac{1}{e}]\); this holomorphic function defines the principal branch of the Lambert \( W \) function.

We can substitute \( x \to e^{x+1} \) in the functional equation for \( W(x) \) obtaining then
\[
W(e^{x+1})e^{W(e^{x+1})} = e^{x+1},
\]
or, renaming \( Y(x) = W(e^{x+1}) \), we have a new functional equation: \( Y(x)e^{Y(x)-1} = e^x \). From this equality it follows by logarithmic differentiation that \( Y'(x) \cdot (1 + Y(x)) = Y(x) \).
Take now \( U(x) := \frac{1}{1+Y(x)} = \frac{1}{2} - \frac{1}{8} x + \frac{1}{64} x^2 + \frac{1}{768} x^3 + \cdots \). We have that
\[
U'(x) = \frac{-Y'(x)}{(1 + Y(x))^2} = \frac{-Y(x)}{(1 + Y(x))^2} = U(x)^3 - U(x)^2.
\]

By Theorem 2, \( U(x) \) is not \( D^\infty \)-finite. By closure properties of \( D^\infty \)-finite functions (see [14, Thm. 4] and [13, §3]), it follows that \( Y(x) \) is not \( D^\infty \)-finite either.

To conclude, note that by definition, for real \( x \) in the neighborhood of 0, we have \( W(x) = Y(\log(x) - 1) \), and by Theorem 10 in [14], it follows that \( W(x) \) and \( T(x) \) are not \( D^\infty \)-finite either, proving Theorem 1.

3. Open questions

The class of \( D \)-finite power series is closed under Hadamard (term-wise) product. This is false for \( D^\infty \)-finite power series; for instance, Klazar showed in [15] that the ordinary generating function \( \sum_{n \geq 0} B_n x^n \) of the Bell numbers is not differentially algebraic, contrary to its exponential generating function (1), which is \( DD \)-finite.

Moreover, it was conjectured by Pak and Yeliussizov [19, Open Problem 2.4] that this is an instance of a more general phenomenon.

Conjecture 3 ([19, Open Problem 2.4]). If for a sequence \( (a_n)_{n \geq 0} \) both ordinary and exponential generating functions \( \sum_{n \geq 0} a_n x^n \) and \( \sum_{n \geq 0} a_n \frac{x^n}{n!} \) are \( D \)-algebraic, then both are \( D \)-finite. (Equivalently, \( (a_n)_{n \geq 0} \) satisfies a linear recurrence with polynomial coefficients in \( n \).)

This conjecture has been recently proven for large (infinite) classes of generating functions [5]. However, the very natural example of the generating function for labelled trees escapes the method in [5].

We therefore leave the following as an open question.

Open question 4. Is the power series \( \sum_{n \geq 1} n^{n-1} x^n \) \( D^\infty \)-finite? Is it at least differentially algebraic?

According to Conjecture 3, the answer should be “no” for both questions in Open question 4.

Another natural question concerns the generating function for partition numbers:
\[
\sum_{n \geq 0} p_n x^n := \prod_{n \geq 1} \frac{1}{1-x^n} = 1 + x + 2 x^2 + 3 x^3 + 5 x^4 + 7 x^5 + 11 x^6 + \cdots,
\]
which is known to be differentially algebraic [17].

Open question 5. Is it true that \( \sum_{n \geq 0} p_n x^n \) is not \( D^\infty \)-finite?

One may also ask for the nature of the exponential variant of the generating function for partition numbers.

Open question 6. Is the power series \( \sum_{n \geq 0} \frac{p_n}{n!} x^n \) \( D^\infty \)-finite, or at least differentially algebraic?

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