ALGEBRA OF BORCHERDS PRODUCTS

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Abstract. Borcherds lift for an even lattice of signature \((p, q)\) is a lifting from weakly holomorphic modular forms of weight \((p - q)/2\) for the Weil representation. We introduce a new product operation on the space of such modular forms and develop a basic theory. The product makes this space a finitely generated filtered associative algebra, without unit element and noncommutative in general. This is functorial with respect to embedding of lattices by the quasi-pullback. Moreover, the rational space of modular forms with rational principal part is closed under this product. In some examples with \(p = 2\), the multiplicative group of Borcherds products of integral weight forms a subring.

1. Introduction

Since ancient, mathematicians have introduced and studied product structures on various mathematical objects. In this paper we define a product structure on a space of certain vector-valued modular forms of fixed weight attached to an integral quadratic form, that is functorial and that reflects some properties of the quadratic form.

Let \(L\) be an even lattice of signature \((p, q)\) with \(p \leq q\) and \(\rho_L\) be the Weil representation attached to the discriminant form of \(L\). In [1], [2], Borcherds constructed a lifting from weakly holomorphic modular forms \(f\) of weight \(\sigma(L)/2 = (p - q)/2\) and type \(\rho_L\) to automorphic forms \(\Phi(f)\) with remarkable singularity on the symmetric domain attached to \(L\). When \(p = 2\) and the principal part of \(f\) has integral coefficients, \(\Phi(f)\) gives rise to a meromorphic modular form \(\Psi(f)\) with infinite product expansion, known as Borcherds product.

The discovery of Borcherds has stimulated the study of weakly holomorphic modular forms of weight \(\sigma(L)/2\) and type \(\rho_L\). If we consider the space of such modular forms, say \(M^!(L)\), it is a priori just an infinite dimensional \(\mathbb{C}\)-linear space. The purpose of this paper is to introduce a product operation on the space \(M^!(L)\) and investigate its basic properties. This makes

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\( M'(L) \) an associative \( \mathbb{C} \)-algebra, finitely generated and filtered but without unit element in general. Moreover, this product is functorial with respect to embedding of lattices by the so-called quasi-pullback operation. This gives a link between quadratic forms and noncommutative rings.

To state our result, we assume that \( L \) has Witt index \( p (= \text{maximal}) \). Our construction requires the choice of a maximal isotropic sublattice \( I \) of \( L \). Then \( K = I^\perp / I \) is an even negative-definite lattice of rank \( -\sigma(L) \). Let \( \downarrow^L_K \) be the pushforward operation from \( \rho_L \) to \( \rho_K \) \((\S 2.1)\), and \( \Theta_{K^+}(\tau) \) be the \( \rho_{K^+} \)-valued theta series of the positive-definite lattice \( K^+ = K(-1) \). In \( \S 3 \) we define the \( \Theta \)-product of \( f_1, f_2 \in M'(L) \) with respect to \( I \) by

\[
f_1 \ast f_2 = \langle f_1 \downarrow^L_K, \Theta_{K^+} \rangle \cdot f_2.
\]

Then \( f_1 \ast f_2 \) is again an element of \( M'(L) \).

In what follows, an \textit{associative algebra} is not assumed to have a unit element. Our basic results can be summarized as follows.

**Theorem 1.1.** The \( \Theta \)-product \( \ast \) makes \( M'(L) \) a finitely generated filtered associative \( \mathbb{C} \)-algebra. The algebra \( M'(L) \) has a unit element if and only if \( L \cong U \oplus \cdots \oplus U \). The algebra \( M'(L) \) is commutative if and only if \( L \) is unimodular. When \( \sigma(L) < 0 \), the rational space \( M'(L)_\mathbb{Q} \subset M'(L) \) of modular forms with rational principal part is closed under \( \ast \).

If \( L' \) is a sublattice of \( L \) of signature \((p, q') \) with \( I_\mathbb{Q} \subset L'_{\mathbb{Q}} \), the map

\[
M'(L) \to M'(L'), \quad f \mapsto |I' / I'|^{-1} \cdot f|_{L'},
\]

is a homomorphism of \( \mathbb{C} \)-algebras, where \( I' = I \cap L' \) and \( f|_{L'} \in M'(L') \) is the quasi-pullback of \( f \in M'(L) \) as defined in \((6.1)\).

Here the filtration on \( M'(L) \) is defined by the degree of principal part. \( U \) stands for the integral hyperbolic plane, namely the even unimodular lattice of signature \((1, 1) \). The quasi-pullback map \([L']: M'(L) \to M'(L') \) is an operation coming from quasi-pullback of Borcherds products \((1), (5), (15) \), which is a sort of renormalized restriction. The statements in Theorem \(1.1 \) are proved in Propositions \(3.4, 4.2, 4.4, 4.8, 5.1, \) and \(6.1 \).

The algebra structure on \( M'(L) \) requires the choice of \( I \), but actually it depends only on the equivalence class of \( I \) under a natural subgroup of the orthogonal group of \( L \). Geometrically, when \( p = 2 \), such equivalence classes correspond to maximal boundary components of the Baily-Borel compactification of the associated modular variety.

In some special cases, \( \Theta \)-product is a quite simple operation. When \( L \) is unimodular, so that \( f_1, f_2 \) and \( \Theta_{K^+} = \theta_{K^+} \) are scalar-valued, \( f_1 \ast f_2 \) is just the product \( f_1 \cdot \theta_{K^+} \cdot f_2 \) \((\text{Example } 3.5) \). When \( I \) comes from \( pU = U \oplus \cdots \oplus U \) embedded in \( L \), so that we have a splitting \( L = pU \oplus K \), \( f_1, f_2 \) correspond to weakly holomorphic Jacobi forms \( \phi_1(\tau, Z), \phi_2(\tau, Z) \) of weight
0 and index $K^*$ (see [12]). Then the Jacobi form corresponding to $f_1 \ast f_2$ is $\phi_1(\tau,0) \cdot \phi_2(\tau,Z)$ (Example 3.6). In general, one can say that $\Theta$-product $\ast$ is a functorial extension of this simple product to all pairs $(L,I)$.

Since the correspondence $(L,I) \mapsto (M'(L),\ast_I)$ is a functor, we expect that the complexity of the lattice $L$ (within fixed $p$) would be reflected in the complexity of the algebra $M'(L)$ in some way. The first examples are given in Theorem 1.1: commutativity and existence of (two-sided) unit element. More widely, we show that $M'(L)$ has a left unit element if it contains a certain modular form with very mild singularity (Proposition 4.5). Some reflective modular forms provide typical examples of such a modular form (Examples 4.6 and 4.7). This might remind us of Borcherds’ philosophy [4] that for $L$ Lorentzian, existence of a reflective modular form should be related to interesting property of the reflection group of $L$.

In the same direction, we expect that the minimal number of generators of $M'(L)$ would reflect the size of $L$. We give lower and upper bounds on the number of generators, and deduce finiteness of lattices with bounded number of generators for fixed $(p,q)$ (Proposition 5.5). In the simple example $L = pU \oplus (-2)$, the algebra $M'(L)$ is generated by two basic reflective modular forms (Example 5.9).

The fact that the rational part $M'(L)_\mathbb{Q}$ is closed under $\ast$ enables us to define, when $p = 2$, a ”$\Theta$-product” of two Borcherds products as a third Borcherds product up to powers. For some $L$, even the group of Borcherds products of integral weight is closed under $\ast$, so it forms a subring.

To conclude, the present article is a proposal of a new ring structure on $M'(L)$ and is devoted to the basic theory. Besides to find a concrete application, we would have at least four subjects to investigate in the theory:

1. find further connection between the lattice $L$ and the algebra $M'(L)$.
2. find an interesting $M'(L)$-module.
3. whether the finiteness theorem holds even if $q$ is allowed to vary.
4. find a geometric interpretation of the new ”unusual” product of Borcherds products or its Lie bracket.

This paper is organized as follows. §2 is recollection of modular forms for the Weil representation. In §3 we define $\Theta$-product. In §4 we study first properties of the algebra $M'(L)$. In §5 we prove finite generation. In §6 we prove functoriality. §4 – §6 may be read independently.

Unless stated otherwise, every ring in this paper is not assumed to be commutative nor have a unit element.

2. Weil representation and modular forms

In this section we recall some basic facts about modular forms of Weil representation type following [2], [6].
2.1. **Weil representation.** Let \( L \) be an even lattice, namely a free abelian group of finite rank equipped with a nondegenerate symmetric bilinear form \((\cdot, \cdot) : L \times L \to \mathbb{Z}\) such that \((l, l) \in 2\mathbb{Z}\) for all \( l \in L \). When \( L \) has signature \((p, q)\), we write \( \sigma(L) = p - q \). The dual lattice of \( L \) is denoted by \( L' \). The quotient \( A_L = L' / L \) is called the discriminant group of \( L \), and is endowed with the canonical \( \mathbb{Q}/\mathbb{Z} \)-valued quadratic form \( q_L : A_L \to \mathbb{Q}/\mathbb{Z}, \ q_L(x) = (x, x)/2 + \mathbb{Z} \), called the discriminant form of \( L \). In general, a finite abelian group \( A \) endowed with a nondegenerate quadratic form \( q : A \to \mathbb{Q}/\mathbb{Z} \) is called a finite quadratic module. We will frequently abbreviate \((A, q)\) as \( A \). Every finite quadratic module \( A \) is isometric to the discriminant form of some even lattice \( L \). We then write \( \sigma(A) = [\sigma(L)] \in \mathbb{Z}/8 \). We denote by \( \mathbb{CA} \) the group ring of \( A \). The standard basis vector of \( \mathbb{CA} \) corresponding to an element \( \lambda \in A \) will be denoted by \( e_\lambda \).

Let \( \text{Mp}_2(\mathbb{Z}) \) be the metaplectic double cover of \( \text{SL}_2(\mathbb{Z}) \). Elements of \( \text{Mp}_2(\mathbb{Z}) \) are pairs \((M, \phi)\) where \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) and \( \phi \) is a holomorphic function on the upper half plane such that \( \phi(\tau)^2 = c\tau + d \). The group \( \text{Mp}_2(\mathbb{Z}) \) is generated by \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \) and \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \), and the center of \( \text{Mp}_2(\mathbb{Z}) \) is generated by \( Z = S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \sqrt{-1} \).

The **Weil representation** \( \rho_A \) of \( \text{Mp}_2(\mathbb{Z}) \) attached to a finite quadratic module \( A \) is a unitary representation on \( \mathbb{CA} \) defined by
\[
\rho_A(T)(e_\lambda) = e(q(\lambda))e_\lambda, \\
\rho_A(S)(e_\lambda) = \frac{e(-\sigma(A)/8)}{\sqrt{|A|}} \sum_{\mu \in A} e(-\langle \lambda, \mu \rangle)e_\mu.
\]
Here \( e(z) = \exp(2\pi iz) \) for \( z \in \mathbb{Q}/\mathbb{Z} \). We have
\[
\rho_A(Z)(e_\lambda) = e(-\sigma(A)/4)e_{-\lambda}.
\]
We will also write \( \rho_A = \rho_L \) when \( A = A_L \) for an even lattice \( L \).

Let \( A(-1) \) be the \((-1)\)-scaling of \( A \), namely the same underlying abelian group with the quadratic form \( q \) replaced by \(-q\). Then \( \rho_{A(-1)} \) is canonically isomorphic to the dual representation \( \rho_A^\vee \) of \( \rho_A \). The isomorphism is defined by sending the standard basis of \( \mathbb{CA}(-1) \) to the dual basis \( \{e_\mu^*\} \) of the standard basis \( \{e_\lambda\} \) of \( \mathbb{CA} \) through the identification \( A(-1) = A \) as abelian groups. We will tacitly identify \( \rho_{A(-1)} = \rho_A^\vee \) in this way.

Let \( I \subset A \) be an isotropic subgroup. Then \( A' = I^\perp / I \) inherits the structure of a finite quadratic module. Let \( p : I^\perp \to A' \) be the projection. We define linear maps
\[
(2.1) \quad \uparrow_A^I : \mathbb{CA}' \to \mathbb{CA}, \quad \downarrow_A : \mathbb{CA} \to \mathbb{CA}',
\]
called pullback and pushforward respectively, by

\[ e_\lambda \uparrow_A^A = \sum_{\mu \in P^{-1}(A)} e_\mu, \quad e_\mu \downarrow_A^A = \begin{cases} e_{\rho(\mu)}, & \mu \in I^+, \\ 0, & \mu \notin I^+. \end{cases} \]

for \( \lambda \in A' \) and \( \mu \in A \). Then \( \uparrow_{A'}^A \) and \( \downarrow_A^A \) are homomorphisms between the Weil representations (see, e.g., [2], [6], [15]). Note that \( \downarrow_A^A \circ \uparrow_{A'}^A \) is the scalar multiplication by \( |I| \). Note also that \( \uparrow_{A'}^A \) and \( \downarrow_A^A \) are adjoint to each other with respect to the standard Hermitian metrics on \( \mathbb{C}(A) \) and \( \mathbb{C}(A') \). When \( A = A_L \) for an even lattice \( L \), the isotropic subgroup \( I \) corresponds to the even overlattice \( L \subset L' \subset L'' \) of \( L \) with \( L'/L = I \). Then \( A' = A_{L'} \). In this situation, we will also write \( \uparrow_{A'}^A = \uparrow_{L'}^L \) and \( \downarrow_A^A = \downarrow_{L'}^L \).

2.2. Modular forms. Let \( A \) be a finite quadratic module and let \( k \in \frac{1}{2}\mathbb{Z} \) with \( k \equiv \sigma(A)/2 \) modulo \( 2\mathbb{Z} \). (We will be interested in the case \( k \leq 0 \).) A \( \mathbb{C}(A) \)-valued holomorphic function \( f \) on the upper half plane is called a weakly holomorphic modular form of weight \( k \) and type \( \rho_A \) if it satisfies \( f(M\tau) = \phi(\tau)^{2k}\rho_A(M, \phi)f(\tau) \) for every \( (M, \phi) \in \text{Mp}_2(\mathbb{Z}) \) and is meromorphic at the cusp. We write

\[ f(\tau) = \sum_{\lambda \in A} \sum_{n \in q(\lambda) + \mathbb{Z}} c_\lambda(n) q^n e_\lambda \]

for the Fourier expansion of \( f \) where \( q^n = \exp(2\pi i n\tau) \) for \( n \in \mathbb{Q} \). By the invariance under \( \mathbb{Z} \), we have \( c_{-\lambda}(n) = c_\lambda(n) \). The finite sum \( \sum_{\lambda} \sum_{n=0} c_\lambda(n) q^n e_\lambda \) is called the principal part of \( f \). When \( k < 0 \), \( f \) is determined by its principal part; when \( k = 0 \), \( f \) is determined by its principal part and constant term. We write \( M_k^f(\rho_A) \) for the space of weakly holomorphic modular forms of weight \( k \) and type \( \rho_A \). By the Borcherds duality theorem ([3], [4], [6]), which polynomial arises as the principal part of some \( f \in M_k^f(\rho_A) \) is determined by certain cusp forms as follows. This will be used in §4 and §5.

**Theorem 2.1** ([3], [4], [6]). Let \( P = \sum_{\lambda, n} c_\lambda(n) q^n e_\lambda \) be a \( \mathbb{C}(A) \)-valued polynomial where \( \lambda \in A \) and \( n \in q(\lambda) + \mathbb{Z} \) with \( n < 0 \), such that \( c_{-\lambda}(n) = c_\lambda(n) \). Then \( P \) is the principal part of a weakly holomorphic modular form of weight \( k \equiv \sigma(A)/2 \) modulo \( 2\mathbb{Z} \) and type \( \rho_A \) if and only if \( \sum_{n=0} c_\lambda(n) a_\lambda(-n) = 0 \) for every cusp form \( \sum_{\lambda, m} a_\lambda(m) q^m e_\lambda \) of weight \( 2 - k \) and type \( \rho_A' \).

Theta series are typical examples of holomorphic modular forms of Weil representation type. Let \( N \) be an even positive-definite lattice. By Borcherds [2], the \( \rho_N \)-valued function

\[ \Theta_N(\tau) = \sum_{l \in N^\vee} q^{k(\lambda)/2} e_{\lambda} = \sum_{\lambda, n} c_\lambda^N(n) q^n e_\lambda, \]
where \( c^N_{\lambda}(n) \) is the number of vectors \( l \) in \( \lambda + N \subset V \) such that \( (l, l) = 2n \), is a holomorphic modular form of weight \( \text{rk}(N)/2 \) and type \( \rho_N \). All Fourier coefficients of \( \Theta_N(\tau) \) are nonnegative integers. If \( N^\prime \) is an even overlattice of \( N \), we have \( \Theta_{N^\prime} = \Theta_N \downarrow _{N^\prime} \).

Let \( L \) be an even lattice. For \( A = A_L \) and \( k = \sigma(L)/2 \), we write

\[
M'(L) = M'_{k/2}(\rho_L).
\]

We especially write

\[
M' = M'(U \oplus \cdots \oplus U) = M'(0),
\]

which is just the space of scalar-valued weakly holomorphic modular forms of weight 0. Then \( M' \) is the polynomial ring in the \( j \)-function \( j(\tau) = q^{-1} + 744 + \cdots \). It is a fundamental remark that for every even lattice \( L \), \( M'(L) \) is a \( M' \)-module.

Let \( p = 2 \). When the principal part of \( f \in M'(L) \) has integral coefficients, Borcherds \([1, 2]\) constructed a meromorphic modular form \( \Psi(f) \) on the Hermitian symmetric domain attached to \( L \), now called a Borcherds product, which has weight \( c_0(0)/2 \in \mathbb{Q} \) and whose divisor is a linear combination of Heegner divisors determined by the principal part of \( f \). The lifting \( f \mapsto \Psi(f) \) is multiplicative.

3. \( \Theta \)-Product

Let \( L \) be an even lattice of signature \((p, q)\) with \( p \leq q \) and assume that \( L \) has Witt index \( p \). We choose and fix a maximal (= rank \( p \), primitive) isotropic sublattice \( I \) of \( L \). In this section we define \( \Theta \)-product \( * = _I \) on the space \( M'(L) = M'_{k/2}(\rho_L) \) with respect to \( I \), which makes \( M'(L) \) an associative algebra. \([3.1]\) is lattice-theoretic preliminary. \( \Theta \)-product is defined in \([3.2]\). In \([3.3]\) we look at some examples.

3.1. Preliminary. We first prepare a lattice-theoretic lemma. We write \( K = I^\perp \cap L/I \), which is an even negative-definite lattice of rank \( -\sigma(L) \). We shall realize \( K \) as an orthogonal direct summand of a canonical over-lattice of \( L \). Let \( I^\perp = I_Q \cap L^\perp \) be the primitive hull of \( I \) in the dual lattice \( L^\perp \). Then \( L^* = \langle L, I^\perp \rangle \) is an even over-lattice of \( L \) with \( L^*/L \cong I^\perp/I \). For \( rU = U \oplus \cdots \oplus U \) (\( r \) times) we denote by \( e_1, f_1, \cdots, e_r, f_r \) its standard basis, namely \( (e_i, f_j) = \delta_{ij} \) and \( (e_i, e_j) = (f_i, f_j) = 0 \). We write \( I_r = \langle e_1, \cdots, e_r \rangle \).

Lemma 3.1. There exists an embedding \( \varphi: pU \hookrightarrow L^* \) such that \( \varphi(I_p) = I^\perp \). In particular, we have \( L^* = \varphi(pU) \oplus \varphi(pU)^\perp \cong pU \oplus K \). The induced isometry \( A_{L^*} \to A_K \) does not depend on the choice of \( \varphi \).

Proof. By the primitivity of \( I^\perp \) in \( L^\perp \), we have \( (l, L^*) = (l, L) = \mathbb{Z} \) for any primitive vector \( l \) in \( I^\perp \). We take one such vector \( l_1 \in I^\perp \) and a vector \( m_1 \in \mathbb{Z} \).
$\langle l, m \rangle = 1$. Then $\langle l_1, m_1 \rangle \simeq U$ and we have a splitting $L^* = \langle l_1, m_1 \rangle \oplus L_4$ where $L_4 = \langle l_1, m_1 \rangle^\perp \cap L^*$. The intersection $I_4 = I^* \cap L_4$ satisfies $I^* = I_4 \oplus \mathbb{Z}l_1$ and we have $(l, L_4) = (l, L^*) = \mathbb{Z}$ for any primitive vector $l \in I_4$. Then we can repeat the same process for $I_4 \subset L_4$. This eventually defines an embedding $\varphi: pU \hookrightarrow L^*$ with $\varphi(I_p) = I^*$. We have natural isomorphisms

$$\varphi(pU)^\perp \cap L^* \cong (I^*)^\perp \cap L^*/I^* = I^\perp \cap L/I = K.$$ 

For the last assertion, we use the following construction. $(I^* \subset L^*$ will be $I \subset L$ below.)

Claim 3.2. Let $L$ be an even lattice and $I \subset L$ be a primitive isotropic sublattice. Let $\varphi_1, \varphi_2: rU \hookrightarrow L$ be two embeddings such that $\varphi_1(I_r) = \varphi_2(I_r) = I$ and $\varphi_1|_{I_r} = \varphi_2|_{I_r}$. Then there exists an isometry $\gamma$ of $L$ which acts trivially on $I$, $K = I^\perp/I$ and $A_L$, such that $\varphi_2 = \gamma \circ \varphi_1$.

The last assertion of Lemma 3.1 is deduced as follows. Let $\varphi_1, \varphi_2: rU \hookrightarrow L$ satisfy just $\varphi_1(I_r) = \varphi_2(I_r) = I$. We can find an isometry $\gamma'$ of $rU$ preserving $I_r$ and $\langle f_1, \cdots, f_r \rangle \simeq I_r^\perp$ such that $\varphi_2|_{I_r} = \varphi_1 \circ \gamma'|_{I_r}$. Then there exists an isometry $\gamma$ of $L$ with properties as in Claim 3.2 such that $\varphi_2 = \gamma \circ \varphi_1 \circ \gamma'$. If we write $K_i = \gamma_i(rU)^\perp \cap L$, then we have $\gamma(K_1) = K_2$. The properties of $\gamma$ imply that the composition $A_L \to A_{K_1} \to A_K$ coincides with $A_L \to A_{K_2} \to A_K$, which is the desired assertion.

We prove Claim 3.2 by induction on $r$. When $r = 1$, we let $l = \varphi_1(e_1)$ and $m_1 = \varphi_2(f_1)$. Then as $\gamma$ we take the Eichler transvection $E_{l, m_2 - m_1}$ (see, e.g., [15]) which fixes $l$, sends $m_1$ to $m_2$, and acts trivially on $K$ and on $A_L$.

For general $r$, let $rU = (r - 1)U \oplus U$ be the apparent decomposition and let $\varphi'_1 = \varphi_1|_{(r - 1)U}$ and $I'_r = \varphi_1(I_{r-1})$. By induction, there exists an isometry $\gamma'$ of $L$ which acts trivially on $I'$, $(I')^\perp/I'$ and $A_L$, such that $\varphi'_2 = \gamma' \circ \varphi'_1$. We show that $\gamma'$ also acts trivially on $I$ and $K = I^\perp/I$. Since $I/I' \subset (I')^\perp/I'$, $\gamma'$ preserves $I$. Since $K$ is a subquotient of $(I')^\perp/I'$, $\gamma$ acts trivially on $K$. We put $I'' = \varphi_2(\mathbb{Z}e_2)$. Since

$$\gamma'(I'') = \gamma(I \cap \varphi_2((r - 1)U)^\perp) = I \cap \varphi_2((r - 1)U)^\perp = I'',$

we find that $\gamma'$ preserves $I''$. Since the $\gamma'$-action on $I/I'$ is trivial, $\gamma'$ acts on $I''$ trivially. Thus $\gamma'$ acts on $I = I' \oplus I''$ trivially.

We set $L' = \varphi_2(\mathbb{Z}(r-1)U)$ and $L'' = (L')^\perp \cap L$. Then we can apply the result in the case $r = 1$ to $\varphi''_1 = \gamma' \circ \varphi_1|_{I'}, \varphi''_2 = \varphi_2|_{I'},$ and $I'' \subset L''$. This provides us with an isometry $\gamma_{L''}$ of $L''$ which acts trivially on $I''$, $(I'')^\perp/I'' = K$ and $A_{L''} \simeq A_L$, such that $\varphi''_2 = \gamma_{L''} \circ \varphi''_1$. Now $\gamma = (\id_{L'} \oplus \gamma_{L''}) \circ \gamma'$ satisfies the desired properties.

Remark 3.3. The lattice $K$ can also be realized as a sublattice of $I^\perp \cap L$ as follows. We choose a basis $l_1, \cdots, l_p$ of $I$ and its dual basis $l_1^*, \cdots, l_p^*$ from
$L'$. We put $\tilde{K} = \langle I_1', \cdots, I_p' \rangle^\perp \cap I^\perp \cap L$. By construction we have a splitting $I^\perp \cap L = I \oplus \tilde{K}$, so the projection gives an isometry $\tilde{K} \to K$.

### 3.2. $\Theta$-product

We now define $\Theta$-product on $M'(L)$. We put $K^+ = K(-1)$, which is an even positive-definite lattice of rank $-\sigma(L)$. We identify $A_{L'} = A_K$ as in Lemma [3.1]. Let

$$\downarrow_K^L = \downarrow_L^L : A_L \to A_{L'} = A_K$$

be the pushforward operation defined in (2.1). If $f \in M'(L)$, then $f \downarrow_K^L$ is an element of $M'(K)$. We take the tensor product $f \downarrow_K^L \otimes \Theta_K^*$, with the theta series $\Theta_K^*$.

This is a weakly holomorphic modular form of weight 0 and type $\rho_K \otimes \rho_K^\vee \simeq \rho_K \otimes \rho_K^\vee$. Taking the contraction $\rho_K \otimes \rho_K^\vee \to \mathbb{C}$ produces a scalar-valued weakly holomorphic modular form of weight 0, namely an element of $M'$. We denote this modular function by

$$\xi(f) = \langle f \downarrow_K^L, \Theta_K^* \rangle \in M'.$$

The map $\xi : M'(L) \to M'$ is $M'$-linear.

Now if $f_1, f_2 \in M'(L)$, we define

$$f_1 \ast f_2 := \xi(f_1) \cdot f_2 = \langle f_1 \downarrow_K^L, \Theta_K^* \rangle \cdot f_2.$$

This is again an element of $M'(L)$. The map

$$\ast : M'(L) \times M'(L) \to M'(L)$$

is $M'$-bilinear. We write $\ast = \ast_I$ when we need to specify $I$.

Explicitly, if $f_i(\tau) = \sum_{\lambda,\mu} c^i_{\lambda}(n)q^n e_\lambda$ for $i = 1, 2$ and $\Theta_K^*(\tau) = \sum_{\nu, \mu} c^K_{\nu}(m)q^\mu e_\nu$, the Fourier coefficients of $f_1 \ast f_2 = \sum_{\lambda,\mu} c^1_{\lambda}(n)q^n e_\lambda$ are given by

$$c^1_{\lambda}(n) = \sum_{\mu + k \equiv n \text{ mod } J} \sum_{\mu, \nu, \lambda} c^1_{\mu}(m) \cdot \frac{c^K_{\nu}(l) \cdot c^2_{\lambda}(k)}{c^1_{\mu}(m) \cdot c^K_{\nu}(l) \cdot c^2_{\lambda}(k)}.$$

Here $J = I' \cap \hat{J} \subset A_L$ and $p : J^\perp \to A_K$ is the projection. Note that even coefficients of $f_1, f_2$ in $n > 0$, sometimes not being paid much attention, may contribute to the principal part of $f_1 \ast f_2$.

**Proposition 3.4.** We have

$$(f_1 \ast f_2) \ast f_3 = f_1 \ast (f_2 \ast f_3)$$

for $f_1, f_2, f_3 \in M'(L)$. Therefore $\Theta$-product $\ast$ makes $M'(L)$ an associative $\mathbb{C}$-algebra. Moreover, the map $\xi : M'(L) \to M'$ is a ring homomorphism.

**Proof.** For the first assertion, we have

$$(f_1 \ast f_2) \ast f_3 = \xi(f_1 \ast f_2) \cdot f_3 = \xi(\xi(f_1) \cdot f_2) \cdot f_3$$

$$= \xi(f_1) \cdot \xi(f_2) \cdot f_3 = \xi(f_1) \cdot (f_2 \ast f_3)$$

$$= f_1 \ast (f_2 \ast f_3).$$
For the second assertion, we calculate
\[ \xi(f_1 \ast f_2) = \xi(\xi(f_1) \cdot f_2) = \xi(f_1) \cdot \xi(f_2). \]
Thus \( \xi \) preserves the products. \( \square \)

The algebra \( M'(L) \) has the following filtration. For a natural number \( d \)
we denote by \( M'(L)_d \subset M'(L) \) the subspace of modular forms \( f \) whose
principal part has degree \( \leq d \). Then we have
\[ M'(L)_d \ast M'(L)_{d'} \subset M'(L)_{d+d'}. \]
Hence \( M'(L) \) is a filtered algebra with this filtration.

By the general theory of associative algebra, \( M'(L) \) has the structure of a
Lie algebra by the commutator bracket
\[
[f_1, f_2] = f_1 \ast f_2 - f_2 \ast f_1.
\]
Since \( \xi \) is a ring homomorphism and \( M' \) is commutative, these brackets are
annihilated by \( \xi \). We have
\[
f_1 \ast f_2 \ast f_3 = f_2 \ast f_1 \ast f_3
\]
for \( f_1, f_2, f_3 \in M'(L) \).

The above construction requires the choice of a maximal isotropic sub-
lattice \( I \), so we should write \( \ast = \ast_I, \xi = \xi_I \) and \( M'(L) = M'(L, I) \) when we
want to specify this dependence. In fact, the freedom of choice is finite. If
\( \gamma : L \to L \) is an isometry of \( L \), then \( \gamma \) acts on \( A_L \). Since the induced action
on \( \mathbb{C}A_L \) preserves the Weil representation \( \rho_L \), \( \gamma \) acts on \( M'(L) \). We have
\[
\xi_{I'}(\gamma f) = \xi_I(f) \quad \text{and so}
\]
\[
(\gamma f_1) \ast_{I'} (\gamma f_2) = \gamma(f_1 \ast_I f_2).
\]
In other words, the action of \( \gamma \) on \( M'(L) \) gives an isomorphism
\[
\gamma : M'(L, I) \to M'(L, \gamma I)
\]
of algebras. In particular, when \( \gamma \) acts trivially on \( A_L \), its action on \( M'(L) \) is
also trivial, so we have \( M'(L, I) = M'(L, \gamma I) \) as algebras.

To summarize, if \( O(L) \) is the orthogonal group of \( L \) and \( \Gamma_L < O(L) \) is
the kernel of the reduction map \( O(L) \to O(A_L) \), then \( M'(L, I) \) depends only
on the \( \Gamma_L \)-equivalence class of \( I \). Moreover, its isomorphism class depends
only on the \( O(L) \)-equivalence class of \( I \). In particular, we have only finitely
many algebra structures \( M'(L, I) \) on \( M'(L) \) for a fixed lattice \( L \).

Geometrically, the \( \Gamma_L \)-equivalence class of \( I \) corresponds more or less to
a boundary component of some compactification of the locally symmetric
space associated to \( \Gamma_L \). (For example, when \( p = 2 \), a boundary curve in the
Baily-Borel compactification.) Perhaps this geometric picture might lead
one to wonder whether it is possible to interpolate \( M'(L, I) \) and \( M'(L, I') \)
for \( I \sim I' \) by some continuous family of algebraic objects.
3.3. Examples. We look at $\Theta$-product in some examples.

**Example 3.5.** Assume that $L$ is unimodular. Then $8|\sigma(L)$. Modular forms of type $\rho_L$ are just scalar-valued modular forms. For any maximal isotropic sublattice $I$ we can find a splitting $L \cong pU \oplus K$ with $I \subset pU$, and $K$ is also unimodular. In particular, $\frac{1}{2}L$ is identity and $\Theta_K' = \theta_K$, is also scalar-valued. In this case, $\Theta$-product is just the product

$$f_1 * f_2 = f_1 \cdot \theta_K \cdot f_2$$

for $f_1, f_2 \in M'(L)$. This shows that $M'(L)$ is commutative and has no zero divisor. Furthermore, $M'(L)$ has no unit element unless when $L = pU$. Indeed, if $f \in M'(L)$ is a unit element, then $f \cdot \theta_K = 1$, but this is impossible when $K \neq \{0\}$ because then $f$ would be a holomorphic modular form of negative weight.

**Example 3.6.** More generally, assume that we have a splitting $L = pU \oplus K$ with $I \subset pU$ ($K$ not necessarily unimodular). This is equivalent to $I = I'$. In this situation, modular forms of type $\rho_L = \rho_K$ correspond to Jacobi forms of index $K^+$ as follows (see [12] for more detail). Let $\Theta_K'(\tau, Z) = \sum_{\lambda \in A_K} \theta_K^{+}\lambda(\tau, Z)e^{\tau}_{1\lambda}$ be the $\rho_K$-valued Jacobi theta series. If $f(\tau) = \sum_{\lambda \in A_K} f_{\lambda}(\tau)e_{1\lambda}$ is a weakly holomorphic modular form of weight $\sigma(L)/2$ and type $\rho_K$, the function

$$\phi(\tau, Z) = \langle f(\tau), \Theta_K'(\tau, Z) \rangle = \sum_{\lambda \in A_K} f_{\lambda}(\tau)\theta_K^{+}\lambda(\tau, Z)$$

given by the contraction $\rho_K \circ \rho_K' \rightarrow \mathbb{C}$ is a weakly holomorphic Jacobi form of weight 0 and index $K^+$. This gives a one-to-one correspondence between two such forms. Note that the restriction $\phi(\tau, 0)$ of $\phi(\tau, Z)$ to $Z = 0$ is just the modular function $\xi(f)$ because $\Theta_K'(\tau, 0) = \Theta_K'(\tau)$.

Now let $f_1, f_2 \in M'(L)$ and $\phi_1, \phi_2$ be the corresponding Jacobi forms. Then the Jacobi form corresponding to $f_1 \odot f_2$ is

$$\phi_1(\tau, 0) \cdot \phi_2(\tau, Z).$$

Indeed, we have

$$\langle f_1 \odot f_2(\tau), \Theta_K'(\tau, Z) \rangle = \langle \xi(f_1)(\tau) \cdot f_2(\tau), \Theta_K'(\tau, Z) \rangle = \xi(f_1)(\tau) \cdot \langle f_2(\tau), \Theta_K'(\tau, Z) \rangle = \phi_1(\tau, 0) \cdot \phi_2(\tau, Z).$$

Thus Jacobi form interpretation of $\Theta$-product is simple: substitute $Z = 0$ into $\phi_1$ to obtain a scalar-valued modular function, and multiply it to $\phi_2$. $\Theta$-product for general $(L, I)$, not necessarily coming from $pU \hookrightarrow L$, can be thought of as a functorial extension of this simple operation using the pushforward operation $\downarrow_L^K$. 
4. First properties

In this section we study some first properties of the algebra $M'(L)$. The reference maximal isotropic sublattice $I \subset L$ is fixed throughout and we write $* = *_I$. In §4.1 we study the left annihilator ideal of $M'(L)$, which plays a basic role in the study of $M'(L)$. In §4.2 we study existence/nonexistence of unit element. In §4.3 we prove that the rational part $M'(L)Q$ is closed under *. §4.2 should be read after §4.1, but §4.3 may be read independently.

4.1. Left annihilator. The left annihilator ideal of $M'(L)$ is a two-sided ideal of $M'(L)$. Since $M'(L)$ is torsion-free as a $M'$-module, this coincides with the kernel of $\xi : M'(L) \rightarrow M'$, which we denote by

$$\Theta^\perp = \{ f \in M'(L) \mid \langle f \downarrow K, \Theta^* \rangle = 0 \}.$$

This is also a sub $M'$-module. Note that $\Theta^\perp$ also coincides with the left annihilator of any fixed $g \neq 0 \in M'(L)$. We have $(\Theta^\perp)^2 = 0$. The ideal $\Theta^\perp$ is the maximal nilpotent ideal of $M'(L)$, consisting of all nilpotent elements of $M'(L)$.

**Proposition 4.1.** The quotient ring $M'(L)/\Theta^\perp$ is canonically identified with a nonzero ideal of the polynomial ring $M' = \mathbb{C}[j]$. Every homomorphism from $M'(L)$ to a ring without nonzero nilpotent element factors through $M'(L) \rightarrow M'(L)/\Theta^\perp$.

**Proof.** By the definition $\Theta^\perp = \text{Ker}(\xi)$, the quotient $M'(L)/\Theta^\perp$ is identified with the image $\xi(M'(L)) \subset M'$ of $\xi$. Since $\xi$ is a $M'$-linear map, $\xi(M'(L))$ is an ideal of $M'$. We shall show that $\xi$ is a nonzero map. Since the map $1_K : M'(L) \rightarrow M'(K)$ is surjective, it suffices to check that the map $\langle \cdot , \Theta^* \rangle : M'(K) \rightarrow M'$ is nonzero. This can be seen, e.g., by taking a modular form $f \in M'(K)$ with Fourier expansion of the form $f(\tau) = q^ne_0 + o(q^n)$ for some negative integer $n$, which is possible as guaranteed by Lemma 5.2. The last assertion follows by a standard argument.

**Proposition 4.2.** The following three conditions are equivalent.

1. $L$ is unimodular.
2. $M'(L)$ is commutative.
3. $\Theta^\perp = \{0\}$.

Moreover, if $\Theta^\perp \neq \{0\}$, we have $\text{dim } \Theta^\perp = \infty$.

**Proof.** (1) $\Rightarrow$ (2), (3) is observed in Example 3.5. (3) $\Rightarrow$ (2) holds because $[f_1, f_2] \in \Theta^\perp$. We check (2) $\Rightarrow$ (3). If $\Theta^\perp \neq \{0\}$, we take $f_1 \neq 0 \in \Theta^\perp$ and $f_2 \notin \Theta^\perp$. Then $f_1 * f_2 = 0$ but $f_2 * f_1 \neq 0$, so $M'(L)$ is not commutative.

Finally, we prove (3) $\Rightarrow$ (1). Suppose that $L$ is not unimodular. We shall show that $\text{dim } \Theta^\perp = \infty$. We consider separately according to whether $K$ is
unimodular or not. When $K$ is unimodular, $\Theta^\perp$ coincides with the kernel of the pushforward $\downarrow_L^K : M'(L) \to M'(K)$. We show that $\dim \ker(\downarrow_L^K) = \infty$. The map $\downarrow_L^K$ preserves the degree filtration, namely $M'(L) \downarrow_L^K \subset M'(K) \downarrow_L^K$. By the Borcherds duality theorem, we have
\[
\dim M'(L)_d = |\mathcal{A}_L/\pm 1| \cdot d + O(1),
\]
\[
\dim M'(K)_d = 1 \cdot d + O(1),
\]
as $d$ grows. Therefore
\[
\dim (\ker(\downarrow_L^K) \cap M'(L)_d) \geq (|\mathcal{A}_L/\pm 1| - 1) \cdot d + O(1) \to \infty
\]
as $d \to \infty$. Here $|\mathcal{A}_L/\pm 1| > 1$ because $\mathcal{A}_L \neq \{0\}$.

When $K$ is not unimodular, we can still argue similarly. The map $\downarrow_L^K : M'(L) \to M'(K)$ is surjective as the composition $\downarrow_L^K \circ \uparrow_L^K$ is a nonzero scalar multiplication. Therefore it is sufficient to show that the subspace $\ker(\langle \cdot, \Theta_{K^*} \rangle)$ of $M'(K)$ has dimension $\infty$. The map $\langle \cdot, \Theta_{K^*} \rangle : M'(K) \to M^!$ preserves the degree filtration, so we have similarly
\[
\dim (\ker(\langle \cdot, \Theta_{K^*} \rangle) \cap M'(K)_d) \geq (|\mathcal{A}_K/\pm 1| - 1) \cdot d + O(1) \to \infty
\]
as $d \to \infty$. This finishes the proof of (3) $\Rightarrow$ (1).

By Proposition 4.1, $M'(L)$ is decomposed into two parts: the ideal $\xi(M'(L))$ in the polynomial ring $M^! = \mathbb{C}[j]$, and the left annihilator $\Theta^\perp$. By the proof of (2) $\Rightarrow$ (3) in Proposition 4.2, the Lie brackets $[f, g]$ generate a large part of $\Theta^\perp$ containing at least $\xi(M'(L)) \cdot \Theta^\perp$. In §6 we will see that the kernels of the quasi-pullback maps provide natural examples of two-sided ideals contained in $\Theta^\perp$.

Remark 4.3. We have only studied the left annihilator. The right annihilator of a fixed $f \in M'(L)$ coincides with the whole $M'(L)$ if $f \in \Theta^\perp$, while it is $\{0\}$ if $f \notin \Theta^\perp$.

4.2. **Unit element.** Next we study existence/nonexistence of unit element. Right unit element exists only in the apparent case.

**Proposition 4.4.** $M'(L)$ has a right unit element if and only if $L = pU$. In this case it is actually the two-sided unit element.

**Proof.** It suffices to verify the “only if” direction. Let $g \in M'(L)$ be a right unit element. If $L$ is not unimodular, we can take $f \neq 0 \in \Theta^\perp$ by Proposition 4.2. Then $f \ast g = 0 \neq f$, which is absurd. So $L$ must be unimodular. Then the assertion follows from the last part of Example 3.5.

On the other hand, left unit element, though still relatively rare, exists in more cases. They are exactly modular forms $f \in M'(L)$ with $\xi(f) = 1$. In particular, if $f$ is a left unit element, every element of $f + \Theta^\perp$ is so.
Proposition 4.5. (1) \( M'(L) \) has a left unit element if and only if the homomorphism \( \xi : M'(L) \to M' \) is surjective. This always holds when \( \sigma(L) = 0 \).

(2) If there exists a modular form \( f \in M'(L) \backslash \Theta^\perp \) with \( f(\tau) = o(q^{-1}) \), then \( M'(L) \) has a left unit element. Such a modular form \( f \) exists only when \( |\sigma(L)| < 24 \).

Proof. The first assertion of (1) holds because \( \xi \) is \( M' \)-linear. When \( \sigma(L) = 0 \), we have \( M'(K) = M' \) and \( \xi = \|_K : M'(L) \to M'(K) \) is surjective.

Next we prove (2). If \( f = o(q^{-1}) \), we have \( \xi(f) = o(q^{-1}) \). Since Fourier expansion of elements of \( M' \) have only integral powers of \( q \), we have in fact \( \xi(f) = O(1) \). Hence \( \xi(f) \) is a holomorphic modular function, namely a constant, which is nonzero by our assumption \( f \not\in \Theta^\perp \). As for the last assertion of (2), we consider the product \( \Delta f \) with the \( \Delta \)-function. This is a cusp form, so its weight \( 12 + \sigma(L)/2 \) must be positive. \( \Box \)

The condition \( f \not\in \Theta^\perp \) in Proposition 4.5 (2) is satisfied when the principal part of \( f \|_K \) has nonnegative (at least one nonzero) coefficients. Indeed, \( \Theta_{K^+}(\tau) = e_0^t + o(1) \) has nonnegative coefficients and the coefficient \( c_0(0) \) of \( f \|_K \) is positive ([6], [8]), so \( \xi(f) \) has nonzero constant term.

Some reflective modular forms provide typical examples of modular forms as in Proposition 4.5 (2).

Example 4.6. Let \( L = pU \oplus \langle -2 \rangle \). Then \( K^+ = \langle 2 \rangle \). Let \( \phi_{0,1} \) be the weak Jacobi form of weight 0 and index 1 constructed by Eichler-Zagier in [11] Theorem 9.3. The corresponding modular form in \( M'(L) \) has Fourier expansion \( f(\tau) = q^{-1/4}e_1 + 10e_0 + o(1) \) where \( e_1 \) is the basis vector of \( \mathbb{C}A_L \) corresponding to \([i] \in \mathbb{Z}/2 \cong A_L \). This modular form satisfies the condition in Proposition 4.5 (2). We will return to this example in Example 5.9.

Example 4.7. More generally, let \( L = pU \oplus \langle -2t \rangle \). Then \( K = K_t = \langle -2t \rangle \).

Eichler-Zagier’s Jacobi form \( \phi_{0,1} \) was generalized by Gritsenko-Nikulin in [14] §2.2 to Jacobi forms \( \phi_{0,t} \) of weight 0 and index \( t \). For \( t = 2, 3, 4 \), the \( \rho_K \)-valued modular form \( f_t \) corresponding to \( \phi_{0,t} \) has Fourier expansion \( f_t(\tau) = q^{-1/4}a_t + a_t e_0 + \cdots \) where \( a_t = 4, 2, 1 \) for \( t = 2, 3, 4 \) respectively. Thus \( f_t \) for \( t = 2, 3, 4 \) satisfy the condition in Proposition 4.5 (2).

4.3. Rational and integral part. We assume \( \sigma(L) < 0 \) for simplicity. For a subring \( R \) of \( \mathbb{C} \) (typically \( \mathbb{Z} \) or \( \mathbb{Q} \)) we write \( M'(L)_R \subset M'(L) \) for the group of modular forms \( f \) whose principal part has coefficients in \( R \). It is clear that \( M'(L)_\mathbb{Z} \otimes \mathbb{Q} = M'(L)_\mathbb{Q} \). Moreover, McGraw’s rationality theorem [16] and the Borcherds duality theorem imply that \( M'(L)_\mathbb{Q} \otimes \mathbb{C} = M'(L) \).

Proposition 4.8. Let \( \sigma(L) < 0 \). Then \( M'(L)_\mathbb{Q} \) is closed under \( * \). Hence it forms an associative \( \mathbb{Q} \)-algebra.
This follows from the explicit calculation (3.1) of the Fourier coefficients of \( f_1 \ast f_2 \), the fact that the \( \Theta \)-series \( \Theta_{K'}(\tau) \) has integral Fourier coefficients, and the following well-known fact.

**Lemma 4.9.** Let \( A \) be a finite quadratic module and \( k < 0 \). If \( f \in M_k'(\rho_A) \) has rational principal part, every Fourier coefficient of \( f \) is also rational.

*Proof.* We supplement a proof for the convenience of the reader. Let \( \Delta(\tau) \) be the \( \Delta \)-function. When \( d \gg 0 \), the product \( \Delta^d f \) vanishes at the cusp, namely \( \Delta^d f \in S_{k+12d}(\rho_A) \). By McGraw [16], the \( \mathbb{C} \)-linear space \( S_{k+12d}(\rho_A) \) has a basis \( f_1, \ldots, f_N \) with integral Fourier coefficients. We write \( \Delta^d f = \sum_i a_i f_i \) with \( a_i \in \mathbb{C} \). Since \( \Delta(\tau) \) has integral coefficients, the coefficients of \( \Delta^d f \) in degree \( < d \) are rational by the assumption on \( f \). Since \( k < 0 \), cusp forms in \( S_{k+12d}(\rho_A) \) are determined by the coefficients in degree \( < d \), so this implies that \( a_i \in \mathbb{Q} \) for every \( i \) by a standard argument. Since \( \Delta^{-1}(\tau) \) has integral coefficients too, we find that \( f = \sum_i a_i(\Delta^{-d} f_i) \) has rational coefficients. \( \square \)

**Remark 4.10.** Proposition 4.8 also holds in the case \( \sigma(L) = 0 \) if we include the constant term \( \sum c_i(0)e_i \) into the principal part.

Let \( M'(L)'_\mathbb{Z} \subset M'(L)_\mathbb{Z} \) be the group of modular forms \( f \) whose all Fourier coefficients are integer. By the same reason as for Proposition 4.8 we have

**Proposition 4.11.** \( M'(L)'_\mathbb{Z} \) is closed under \( \ast \) and hence forms a subring.

In some cases, \( M'(L)'_\mathbb{Z} = M'(L)_\mathbb{Z} \) holds. This is the case when

- \( L = pU \oplus mE_8 \) with \( m > 0 \) (unimodular)
- \( L = pU \oplus mE_8 \oplus \langle -2 \rangle \)

as can be seen from the constructions of nice basis in [9], [10] respectively. In both cases, we have in fact \( c_0(0) \in 2\mathbb{Z} \) for \( f \in M'(L)_\mathbb{Z} \) by [17]. Therefore, when \( p = 2 \), \( M'(L)_\mathbb{Z} \) is identified with the multiplicative group of Borcherds products of integral weight. This means that we have new "unusual" product \( \Psi(f_1 \ast f_2) \) of two Borcherds products \( \Psi(f_1), \Psi(f_2) \) of integral weight as a third one. When \( L \) is not unimodular, \( M'(L)_\mathbb{Z} \) also forms a nontrivial Lie algebra over \( \mathbb{Z} \) under the Lie bracket (3.2). What is geometric interpretation of these new products?

5. Finite generation

In this section we prove that \( M'(L) \) is finitely generated and give estimates, from above and below, on the number of generators. This section may be read independently of §4.
5.1. **Finite generation.** In this subsection we prove

**Proposition 5.1.** *The algebra \( M^l(L) \) is finitely generated over \( \mathbb{C} \).*

For the proof we need the following construction.

**Lemma 5.2.** *There exists a natural number \( d_0 \) such that for any pair \( (\lambda, n) \) with \( \lambda \in A_L \) and \( n \in q(\lambda) + \mathbb{Z}, \) \( n < -d_0, \) there exists a modular form \( f_{\lambda,n} \in M^l(L) \) with Fourier expansion \( f_{\lambda,n}(\tau) = q^n(e_\lambda + e_{-\lambda}) + o(q^n). \)

**Proof.** For simplicity we assume \( \sigma(L) < 0; \) the case \( \sigma(L) = 0 \) can be dealt with similarly. For each natural number \( d \) we let \( V_d \) be the space of \( \mathbb{C}A_L \)-valued polynomials of the form

\[
\sum_{\lambda \in A_L} \sum_{-d \leq m \leq 0} (c_\lambda(m)q^m e_\lambda, \quad c_\lambda(m) = c_{-\lambda}(m).
\]

Then \( \dim V_d = |A_L| \cdot d. \) The filter \( M^l(L)_d \) of \( M^l(L) \) is canonically embedded in \( V_d \) by associating the principal parts. Let \( S = S_{2-\sigma(L)/2}(\rho^l_2) \) be the space of cusp forms of weight \( 2 - \sigma(L)/2 \) and type \( \rho^l_2. \) By the Borcherds duality theorem, we have the exact sequence

\[
0 \to M^l(L)_d \to V_d \to S^\vee.
\]

When \( d \gg 0, \) \( V_d \to S^\vee \) is surjective ([3]), and hence

\[
\dim M^l(L)_d = |A_L| \cdot d - \dim S.
\]

In particular, we find that

\[
\dim M^l(L)_{d+1} - \dim M^l(L)_d = |A_L| \cdot 1.
\]

On the other hand, \( M^l(L)_{d+1} \) as a subspace of \( M^l(L)_d \) is the kernel of the map \( \rho_d: M^l(L)_d \to \mathbb{C}(A_L/\pm 1) \) that associates coefficients of the principal part in degree \( e_\lambda \in [d-1, -d). \) Therefore \( \rho_d \) must be surjective when \( d \gg 0. \) The form \( f_{\lambda,n} \) as desired can be obtained as \( \rho_d^{-1}(e_\lambda + e_{-\lambda}) \) for suitable \( d. \) \( \square \)

By the proof, \( d_0 \) can be taken to be the minimal degree \( d \) where \( V_d \to S^\vee \) is surjective. We now prove Proposition 5.1.

**(Proof of Proposition 5.1).** We first define a set of generators. First we take \( f_0 \in M^l(L) \) whose Fourier expansion is of the form \( q^{-d_1}e_0 + o(q^{-d_1}) \) for some natural number \( d_1. \) Next, letting \( d_0 \) be as in Lemma 5.2 we put

\[
\Lambda_1 = \{ f_{\lambda,m} \mid \lambda \in A_L/\pm 1, \ m \in q(\lambda) + \mathbb{Z}, \ -d_0 - d_1 \leq m < -d_0 \ \}. \]

Then we take a basis of \( M^l(L)_{d_0+d_1} \) and denote it by \( \Lambda_2. \) We shall show that \( f_0, \ \Lambda_1 \) and \( \Lambda_2 \) generate \( M^l(L) \) as a \( \mathbb{C} \)-algebra.

By definition \( M^l(L)_{d_0+d_1} \) is generated by \( \Lambda_1 \cup \Lambda_2 \) as a \( \mathbb{C} \)-linear space. The quotient \( M^l(L)/M^l(L)_{d_0+d_1} \) is generated as a \( \mathbb{C} \)-linear space by any set of modular forms whose Fourier expansion is of the form \( q^n(e_\lambda + e_{-\lambda}) + o(q^n). \)
where \( \lambda \) varies over \( A_L / \pm 1 \) and \( n \) varies over \( q(\lambda) + \mathbb{Z} \) with \( n < -d_0 - d_1 \). Therefore it suffices to show that we can construct such a modular form as a product of \( f_0 \) and elements of \( \Lambda_1 \). Since \( f_0(\tau) \frac{L_1}{L} = q^{-d_1}e_0 + o(q^{-d_1}) \) and \( \Theta_{L} (\tau) = e_0^\chi + o(1) \), we have \( \xi(f_0) = q^{-d_1} + o(q^{-d_1}) \). We take \( m \equiv n \mod d_1 \) from \(-d_0 - d_1 \leq m < -d_0 \) and put \( r = (m - n)/d_1 \in \mathbb{N} \). Then

\[
 f_0 \ast \cdots \ast f_0 \ast f_{\lambda,m} \quad (f_0 \ r \ \text{times})
 = (q^{-d_1} + o(q^{-d_1}))^r(q^m(e_1 + e_{-1}) + o(q^m))
 = q^m(e_1 + e_{-1}) + o(q^m).
\]

This gives a desired modular form. \( \square \)

**Remark 5.3.** The rational part \( M'(L)_\mathbb{Q} \) is also finitely generated as a \( \mathbb{Q} \)-algebra. The same proof works if we use the \( \mathbb{Q} \)-structure of \( S \) given by McGraw’s theorem \([16]\).

### 5.2. Bounds on the number of generators

In this subsection we study upper and lower bounds on the minimal number of generators of \( M'(L) \). We first determine the structure of \( M'(L) \) as a \( M^1 \)-module.

**Proposition 5.4.** \( M'(L) \cong (M^1)^{\oplus (A_L / \pm 1)} \) as a \( M^1 \)-module.

**Proof.** Let \( W_d = M'(L)_{d+1}/M'(L)_d \). Taking coefficients of the principal part in degree \( \in [-d - 1, -d] \) defines an embedding \( \rho_d: W_d \hookrightarrow \mathbb{C}(A_L / \pm 1) \). On the other hand, multiplication by the \( j \)-function by \( j(\tau) = q^{-1} + O(1) \in M^1 \) defines an injective map \( W_d \hookrightarrow W_{d+1} \) which is compatible with \( \rho_d \) and \( \rho_{d+1} \). We thus have the filtration \( (W_d)_d \) of the space \( \mathbb{C}(A_L / \pm 1) \) which stabilizes to \( \mathbb{C}(A_L / \pm 1) \) in \( d \gg 0 \). Let

\[
 W_0 = W_{d_1} \subset W_{d_2} \subset \cdots \subset W_{d_N} = \mathbb{C}(A_L / \pm 1)
\]

be the reduced form of this filtration, namely \( W_d = W_{d_i} \) in \( d_i \leq d < d_{i+1} \).

We take modular forms \( \{ f_{ij} \}_{i,j} \) such that \( f_{ij} \in M'(L)_{d_i} \) and \( \rho_{d_i}(f_{ij}) \) form a basis of \( W_{d_i}/W_{d_i-1} \). Then \( \rho_{d}(f_{ij}) \) form a basis of \( \mathbb{C}(A_L / \pm 1) \). We show that \( \{ f_\alpha \}_\alpha \) freely generate \( M'(L) \) as \( M^1 \)-module. Since we have

\[
 M'(L)_{d+1} = \langle M'(L)_d, W_d \rangle = \langle M'(L)_d, j \cdot M'(L)_d, W_d/W_{d-1} \rangle,
\]

induction on \( d \) tells us that \( M'(L)_d \subset \sum_\alpha \langle M^1, f_\alpha \rangle \) for every \( d \). Thus \( \{ f_\alpha \}_\alpha \) generate \( M'(L) \). If there was a relation

\[
(5.2) \quad \sum_\alpha P_\alpha(j) f_\alpha = 0, \quad P_\alpha \in \mathbb{C}[x],
\]

then we would obtain a nontrivial \( \mathbb{C} \)-linear relation between \( \rho_{d}(f_\alpha) \) in \( \mathbb{C}(A_L / \pm 1) \) by looking at the coefficients of the principal part of \( f_\alpha \) in highest degrees. Thus \( \{ f_\alpha \} \) are free generators. \( \square \)
This gives a lower bound of the number of generators of $M^!(L)$ as algebra, which implies the following.

**Proposition 5.5.** Let $p \leq q$ be fixed. Let $N$ be a fixed natural number. Then up to isometry there are only finitely many pairs $(L, I)$ of an even lattice $L$ of signature $(p, q)$ and Witt index $p$ and a maximal isotropic sublattice $I \subset L$ such that the algebra $M^!(L, I)$ can be generated by at most $N$ elements.

**Proof.** In §3.2, we observed that the dependence on $I$ is finite for a fixed lattice $L$. Hence it is sufficient to prove finiteness of lattices $L$. Since $f * g = \xi(f) \cdot g$, generators of $M^!(L)$ as algebra also serve as generators as $M^!$-module. By Proposition 5.4, we obtain the bound
\[
N \geq |A_L/\pm 1| > |A_L|/2.
\]
Then our assertion follows from finiteness of even lattices of fixed signature and bounded discriminant. □

Proposition 5.5 is a consequence of the structure of $M^!(L)$ as a $M^!$-module. It would be a natural problem whether the finiteness still holds even if we let $q$ vary with $p$ fixed. By Proposition 5.4, the same statement is not true for generators as $M^!$-module. (Take direct sum with the unimodular lattices $mE_8$.) So this could be one of touchstones for the theory of algebra structure on $M^!(L)$.

Next we study upper bound of the number of generators of $M^!(L)$ as algebra. By the proof of Proposition 5.1, we have the upper bound
\[
1 + d_1 \cdot |A_L/\pm 1| + \dim M^!(L)_{d_0} \leq 1 + (d_0 + d_1) \cdot |A_L/\pm 1|,
\]
where $d_0$ and $d_1$ are as defined there. Clearly $d_1 \leq d_0 + 1$. We have the following upper bound of $d_0$. Let $V_d$ and $S$ be as in the proof of Lemma 5.2. Recall that $d_0$ does not exceed the minimal degree where $V_d \rightarrow S^\vee$ is surjective.

**Proposition 5.6.** The dimension $\dim \text{Im}(V_d \rightarrow S^\vee)$ is strictly increasing with respect to $d$ until $V_d \rightarrow S^\vee$ gets surjective. In particular, we have
\[
d_0 \leq \dim S - d_2(|A_L/\pm 1| - 1),
\]
where $d_2 = -[\sigma(L)/24 + 1]$ is the largest integer with $d_2 < |\sigma(L)|/24$.

**Proof.** For the first assertion, what has to be shown is that $V_d \rightarrow S^\vee$ is surjective whenever $\text{Im}(V_d \rightarrow S^\vee) = \text{Im}(V_{d+1} \rightarrow S^\vee)$. By the Borcherds duality theorem, this condition means that the codimension of $M^!(L)$ in $V_d$ equals to the codimension of $M^!(L)_{d+1}$ in $V_{d+1}$. If we write $W_d = M^!(L)_{d+1}/M^!(L)_d$, we find that
\[
\dim W_d = \dim(V_{d+1}/V_d) = |A_L/\pm 1|.
\]
As in the proof of Proposition 5.4, this implies that \( \dim W_{d'} = |A_L/ \pm 1| \) for every \( d' \geq d \).

On the other hand, if \( V_d \to S^\vee \) was not surjective, there must exist \( d' \geq d \) such that \( \text{Im}(V_{d'} \to S^\vee) \neq \text{Im}(V_{d'+1} \to S^\vee) \). By the same argument as above, this implies that

\[
\dim W_{d'} < \dim(V_{d'+1}/V_{d'}) = |A_L/ \pm 1|,
\]

which is absurd.

The second assertion follows from Lemma 5.7 which implies the injectivity of \( V_d \).

By (5.3) and (5.4), we obtain an upper bound for the minimal number of generators in terms of \( \dim S \). An estimate of \( \dim S \) is given in [7].

We also note that \( d_1 \geq |\sigma(L)|/24 \) by the following well-known property.

**Lemma 5.7.** If \( M'(L)_d \neq \{0\} \), then \( |\sigma(L)| \leq 24d \).

**Proof.** If \( f \neq 0 \in M'(L)_d \), the product \( \Delta^d f \) with the \( \Delta \)-function is holomorphic at the cusp, so its weight \( \sigma(L)/2 + 12d \) must be nonnegative. □

We close this subsection with some simple examples.

**Example 5.8.** Assume that the obstruction space \( S_{2-\sigma(L)/2}(\rho_L^\vee) \) is trivial. (Such lattices \( L \) with \( p = 2 \) are classified in [7].) Then every polynomial as in (5.1) is the principal part of some modular form in \( M'(L) \). In this case, using the notation in the proof of Proposition 5.1, we have \( d_0 = 0 \), \( d_1 = 1 \), \( \Lambda_2 = \emptyset \), and the modular form \( f_0 \) can be included in \( \Lambda_1 \). Therefore \( M'(L) \) can be generated by modular forms \( f_\lambda = q^{\lambda} (e_1 + e_{-\lambda}) + O(1) \) with \( \lambda \in A_L/ \pm 1 \) and \( n \in q(\lambda) + \mathbb{Z}, -1 \leq n < 0 \). The minimal number of generators is thus equal to \( |A_L/ \pm 1| \). The generator \( f_\lambda \) with \( \lambda \neq 0 \) is either a left unit element or a left zero divisor according to Proposition 4.5 (2).

**Example 5.9.** We go back to Example 4.6 where \( L = pU \oplus \langle -2 \rangle \). The algebra \( M'(L) \) is generated by the two elements \( f_0 = q^{-1} e_0 + O(1) \) and \( f_1 = q^{-1/4} e_1 + O(1) \) with the relation \( f_1 \# f_1 = 12 f_1 \) and \( f_1 \# f_0 = 12 f_0 \). Thus the two basic reflective modular forms for \( L \) give minimal generators of the algebra \( M'(L) \).

### 6. Functoriality

In this section we prove that \( \Theta \)-product is functorial with respect to embedding of lattices if we use quasi-pullback as morphism. The statement is Proposition 6.1 and the proof is given in §6.2 and §6.3. In §6.4 we also prove functoriality with respect to special pushforward. Except for Corollary 6.3 this section may be read independently of §4 and §5.
6.1. **Quasi-pullback.** Let \( L \) be an even lattice of signature \((p, q)\) and \( L' \) be a sublattice of \( L \) of signature \((p, q')\). We do not assume that \( L' \) is primitive in \( L \). Following [15], we define a linear map \(|_{L'}: M^i(L) \to M^i(L')\) as follows. Let \( N = (L')^+ \cap L \), which is a negative-definite lattice. We write \( N^+ = N(-1) \). The lattice \( L' \oplus N \) is of finite index in \( L \). Let \( f \in M^i(L) \). We first take the pullback \( f \uparrow_{L}^{L' \oplus N} \), which is an element of \( M^i(L' \oplus N) \). Since \( \rho_{L' \oplus N} = \rho_{L'} \otimes \rho_{N} \), we can take contraction of \( f \uparrow_{L}^{L' \oplus N} \) with the \( \rho_{N}^{-1} \)-valued theta series \( \Theta_{N} \) of \( N^+ \). This produces a \( \rho_{L'}^{-1} \)-valued weakly holomorphic modular form of weight \( \sigma(L')/2 \), which we denote by

\[(6.1) \quad f|_{L'} = (f \uparrow_{L}^{L' \oplus N}, \Theta_{N}) \in M^i(L').\]

We call \( f|_{L'} \) the quasi-pullback of \( f \) to \( L' \). The map \(|_{L'}: M^i(L) \to M^i(L')\) is \( M^i\)-linear.

The geometric significance of this operation comes from Borcherds products as follows. Assume that \( p = 2 \) and \( f \) has integral principal part, and let \( \Psi(f) \) be the Borcherds product associated to \( f \) on the Hermitian symmetric domain \( D_L \) for \( L \). The Hermitian symmetric domain \( D_{L'} \) for \( L' \) is naturally embedded in \( D_L \). The quasi-pullback of \( \Psi(f) \) from \( L \) to \( L' \), discovered by Borcherds [1], [5], is defined by first dividing \( \Psi(f) \) by suitable linear forms to get rid of zeros and poles containing \( D_{L'} \), and then restricting the resulting form to \( D_{L'} \subset D_L \). It is proved in [15] that this quasi-pullback of \( \Psi(f) \) coincides with the Borcherds product for \( f|_{L'} \in M^i(L') \) up to constant. Thus the operation \(|_{L'} \) defined in (6.1) can be thought of as a formal \( \mathbb{C} \)-linear extension of the quasi-pullback operation on Borcherds products.

We can now state the main result of this §6. We assume that \( p \leq q' \leq q \) and both \( L \) and \( L' \) have Witt index \( p \).

**Proposition 6.1.** Let \( L' \subset L \) be as above. Let \( I \) be a maximal isotropic sublattice of \( L \) such that \( I_Q \subset L'_Q \). We set \( I' = I \cap L' \). Then we have

\[(f|_{L'}) \ast_P (g|_{L'}) = |I/I'| \cdot (f \ast_I g)|_{L'}\]

for \( f, g \in M^i(L) \). Therefore the map

\[|I/I'|^{-1} \cdot |_{L'}: M^i(L, I) \to M^i(L', I')\]

is a ring homomorphism.

This means that the assignment

\[(L, I) \mapsto M^i(L, I)\]

is a contravariant functor from the category of pairs \((L, I)\) to the category of associative \( \mathbb{C} \)-algebras, by assigning the morphism \(|I/I'|^{-1} \cdot |_{L'} \) to an embedding \((L', I') \hookrightarrow (L, I)\).

The proof of Proposition 6.1 is reduced to the following assertion.
Proposition 6.2. Let \( L' \subset L \) and \( I' \subset I \) be as in Proposition 6.1. We put \( K = \text{Ker}(\xi) = C \cap L/I \) and \( K' = (\text{Ker}(\Theta)) = C \cap L'/I' \). Let \( \xi : M'(L) \to M' \) and \( \xi' : M'(L') \to M' \) be the maps \( \xi = \langle \cdot |_{\text{Ker}(\Theta)} \rangle \) and \( \xi' = \langle \cdot |_{\text{Ker}(\Theta')} \rangle \) respectively. Then we have

\[
\xi' \circ |_{L'} = |I/I'| \cdot \xi.
\]

Indeed, if we admit Proposition 6.2 we can calculate

\[
(f|_{L'}) \ast_P (g|_{L'}) = \xi'(f|_{L'}) \cdot (g|_{L'}) = |I/I'| \cdot \xi(f) \cdot (g|_{L'}) = |I/I'| \cdot (\xi(f) \cdot g)|_{L'} = |I/I'| \cdot (f \ast_{L'} g)|_{L'}.
\]

Thus Proposition 6.2 implies Proposition 6.1.

Before proceeding, we note a consequence.

Corollary 6.3. Let \( \Theta^+=(L) \subset M'(L) \) and \( \Theta^+(L') \subset M'(L') \) be the respective left annihilators. Then we have \( |_{L'}^{-1}(\Theta^+(L')) = \Theta^+(L) \). In particular, we have \( \text{Ker}(|_{L'}) \subset \Theta^+(L) \). The map \( M'(L)/\Theta^+(L) \to M'(L')/\Theta^+(L') \) induced by \( |I/I'| \cdot |_{L'} \) is inclusion of ideals in the polynomial ring \( M' = \mathbb{C}[j] \).

Proof. The equality \( |_{L'}^{-1}(\Theta^+(L')) = \Theta^+(L) \) follows from Proposition 6.2. Since \( \xi \) and \( \xi' \) embed \( M'(L)/\Theta^+(L) \) and \( M'(L')/\Theta^+(L') \) as ideals in \( M' \) respectively, the last assertion follows.

Thus the kernel of the quasi-pullback map \( |_{L'} \) provides a natural example of two-sided ideal of \( M'(L) \) contained in \( \Theta^+ \).

The proof of Proposition 6.2 occupies §6.2 and §6.3. It is divided into two parts, reflecting the fact that the quasi-pullback \( |_{L'} \) is composition of two operators \( \uparrow^{L \oplus N}_L \) and \( \langle \cdot, \Theta \rangle_N \). In §6.2 we consider the case when \( L' \) is of finite index in \( L \). In §6.3 we consider the case when the splitting \( L = L' \oplus N \) holds. The proof in the general case is a combination of these two special cases.

6.2. The case of finite pullback. In this subsection we prove Proposition 6.2 in the case when \( L' \) is of finite index in \( L \). In this case, the quasi-pullback \( |_{L'} \) is the operation \( \uparrow^{L \oplus N}_L \), and Proposition 6.2 takes the following form.

Lemma 6.4. When \( L' \subset L \) is of finite index, we have for \( f \in M'(L) \)

\[
\xi'(f \uparrow^{L \oplus N}_L) = |I/I'| \cdot \xi(f).
\]

This is a consequence of the following calculation in finite quadratic modules.

Lemma 6.5. Let \( A \) be a finite quadratic module and \( I_1, I_2 \subset A \) be two isotropic subgroups. We set \( A_1 = I_1^+/I_1, A_2 = I_2^+/I_2 \) and

\[
A' = (I_1^+ \cap I_2^+)/((I_1 \cap I_2^+) + (I_2 \cap I_1^+)).
\]
Let $I'_2 = I_2 \cap I'_2 / I_1 \cap I_2$ be the image of $I_2 \cap I'_2$ in $A_1$, and $I'_1 = I_1 \cap I'_2 / I_1 \cap I_2$ be the image of $I_1 \cap I'_2$ in $A_2$. Then, under the natural isomorphism

$$A' = (I'_2)^+ \cap A_1 / I'_2 = (I'_1)^+ \cap A_2 / I'_1,$$

we have

$$A \downarrow_{A_2} \circ A \uparrow_{A_1} = |I_1 \cap I_2| A \uparrow_{A'} \circ A \downarrow_{A'}$$

as linear maps $\mathbb{C}A_1 \to \mathbb{C}A_2$.

We postpone the proof of Lemma 6.5 for a moment, and first explain how Lemma 6.4 is deduced from Lemma 6.5.

(Proof of Lemma 6.4). Let $K = I^+ \cap L / I$ and $K' = (I')^+ \cap L' / I'$. We have a canonical embedding $K' \hookrightarrow K$ of finite index. Since $\Theta_{K'} = \Theta_{(K')} \downarrow_{K'}^{(K')}$, we find that

$$\xi(f) = \langle f \downarrow_{K}' \Theta_{K'} \rangle = \langle f \downarrow_{K} \downarrow_{K'} \Theta_{(K')} \downarrow_{K'}^{(K')} \rangle = \langle f \downarrow_{K} \uparrow_{K}^{K'} \Theta_{(K')} \rangle.$$

On the other hand, we have

$$\xi'(f \uparrow_{L}') = \langle f \uparrow_{L} \downarrow_{K} \Theta_{(K')} \rangle.$$

Thus it is sufficient to show that

$$A \downarrow_{K'} \circ \uparrow_{L}' = |I/I'| A \uparrow_{K} \circ \downarrow_{K}$$

as linear maps $\mathbb{C}A_L \to \mathbb{C}A_{K'}$.

We apply Lemma 6.5 as follows. Let $I^* = I_2 \cap L' \cap I_2$ and $(I')^* = I_2 \cap (L')^2 \cap I_2$. We set $A = A_L$, $I = I_2 \cap L'$ and $I_2 = (I')^* / I'$. Then $A_1 \cong A_L$ and $A_2 \cong A_{K'}$. We have $I_2 \cap I_1 = I^* / I'$ and

$$I_1 \cap I_2 = (L \cap (L', (I')^*)) / L' = (L', I) / L' = I / I'.$$

This implies that $I'_2 = I^* / I \subset A_L$ and $A' = A_K$. Thus we have

$$\uparrow_{A_1} = \uparrow_{L'}, \quad \downarrow_{A_2} = \downarrow_{L'}, \quad \uparrow_{A_1} = \uparrow_{L}, \quad \downarrow_{A_2} = \downarrow_{L}, \quad \uparrow_{A'} = \uparrow_{K'}$$

hence (6.3) implies (6.4). \hfill \Box

We now prove Lemma 6.5.

(Proof of Lemma 6.5). We first justify the isomorphism (6.2), which also implies that $A'$ is nondegenerate. We write $\hat{I}_1 = I_1 \cap I'_1$ and $\hat{I}_2 = I_2 \cap I'_1$. We
shall establish the following commutative diagram:

![Diagram](https://via.placeholder.com/150)

Here $p_i$ is the quotient map by $I_i$ and $p'_i$ is the restriction of $p_i$. Since we have $\tilde{\lambda} = \lambda$, we have

$$I_1/\tilde{\lambda} \cong ((\tilde{\lambda})^+ \cap I_1^+)/((I_1^+ \cap I_2^+) \cong (\tilde{\lambda})^+ / I_2^+, \]

we see that $p'_i$ is surjective and is the quotient map by $\tilde{\lambda}_i$. This induces the map $q_2: (I_2^+) \cap A_1 \to A'$ as the quotient map by $I_2$. Similarly, we find that $p'_2$ is the quotient map by $\tilde{\lambda}_2$ and $q_1$ is induced as the quotient map by $\tilde{\lambda}_i$.

We now prove (6.3). Let $\lambda \in A_1$. It suffices to show that

$$e_1 \uparrow^{A_1} \uparrow^{A_2} = e_2 \downarrow^{A_1} \downarrow^{A_2}.$$  

When $\lambda \notin (I_2^+)$, we have $e_1 \uparrow^{A_1} = 0$. On the other hand, we have $(\tilde{\lambda}, \tilde{\lambda}_2) \neq 0$ for every $\tilde{\lambda} \in I_1^+$ in the inverse image of $\lambda$. In particular, we have $(\tilde{\lambda}, \tilde{\lambda}_2) \neq 0$ and hence $e_1 \downarrow^{A_2} = 0$. This implies that $e_1 \uparrow^{A_1} \uparrow^{A_2} = e_2 \downarrow^{A_1} \downarrow^{A_2} = 0$.

Next let $\lambda \in (I_2^+)$. By the above commutative diagram, we can choose $\tilde{\lambda} \in I_1^+ \cap I_2^+$ such that $p'_1(\tilde{\lambda}) = \lambda$. Then

$$e_1 \downarrow^{A_1} \downarrow^{A_2} = e_2 \downarrow^{A_1} \downarrow^{A_2} = \sum_{\mu' \in I_1^+} e_{p_2(\lambda) + \mu'}. $$

On the other hand, we have

$$e_1 \uparrow^{A_1} \downarrow^{A_2} = \sum_{\mu \in I_1} e_{\lambda + \mu} \downarrow^{A_2} = \sum_{\mu \in I_1^+} e_{p_2(\lambda + \mu)} = \sum_{\mu \in I_1^+} e_{p'_2(\lambda) + \mu'}.$$  

Here we used the equality $(\tilde{\lambda} + I_1) \cap I_2^+ = \tilde{\lambda} + \tilde{\lambda}_1$. Since the map $p'_2: \tilde{\lambda} \to \tilde{\lambda}_2$ is the quotient map by $I_1 \cap I_2$, its fibers consist of $|I_1 \cap I_2|$ elements. Comparing (6.6) and (6.7), we obtain the desired equality (6.5).

6.3. **The split case.** Next we prove Proposition 6.2 in the case when the splitting $L = L' \oplus N$ holds. In this case, $\uparrow_{L' \oplus N}^L$ is identity, $I'$ coincides with $I$, so Proposition 6.2 takes the following form.

**Lemma 6.6.** When the splitting $L = L' \oplus N$ holds, we have for $f \in M(L)$

$$\xi'(\langle f, \Theta_N \rangle) = \xi(f).$$  


Proof. Since $K = K' \oplus N$, we have $\Theta_{K'} = \Theta_{(K')^*} \otimes \Theta_{N^*}$ under the natural isomorphism $\rho_{K'} = \rho_{(K')^*} \otimes \rho_{N^*}$. Therefore
\[
\xi'(\langle f, \Theta_{N^*}\rangle) = \langle \langle f, \Theta_{N^*}\rangle \downarrow_{K'}, \Theta_{(K')^*}\rangle = \langle \langle f \downarrow_{K}^{L'}, \Theta_{N^*}\rangle, \Theta_{(K')^*}\rangle = \langle f \downarrow_{K}^{L'}, \Theta_{N^*} \otimes \Theta_{(K')^*}\rangle = \xi(f).
\]
This proves the desired equality. \qed

We can now prove Proposition 6.2 in the general case.

(Proof of Proposition 6.2). Let $L' \oplus N \subset L$ and $I' \subset I$ be as in Proposition 6.2. We write $L'' = L' \oplus N$, $K'' = (I')^* \cap L''/I'$ and $\xi'' = \langle \downarrow_{K''}^{L''}, \Theta_{(K'')^*}\rangle$. By using Lemma 6.6 for $L' \subset L''$ and Lemma 6.4 for $L'' \subset L$, we see that
\[
\xi'(f|_{L'}) = \xi''(\langle f \uparrow_{L'}^{L''}, \Theta_{N^*}\rangle) = \xi''(f \uparrow_{L'}^{L''}) = |I/I'| \cdot \xi(f).
\]
This proves Proposition 6.2 in the general case. \qed

6.4. Special finite pushforward. $\Theta$-product is also covariantly functorial with respect to pushforward to a special type of overlattices. Let $I \subset L$ be as before.

Proposition 6.7. Let $L'$ be a sublattice of $L$ of finite index. Assume that $L = \langle L', I \rangle$. We set $I' = I \cap L'$. Then we have
\[
(f \downarrow_{L'}^{L}) \circ (g \downarrow_{L'}^{L}) = (f *' g) \downarrow_{L'}^{L}
\]
for $f, g \in M^1(L')$.

Proof. We use the notation in the proof of Lemma 6.4. Since $I_1 = L/L'$ coincides with $I_1 \cap I_2 = I/I'$, we have $I_1 \subset I_2$. Hence $I'_1 = \{0\}$ and so the canonical embedding $K' \hookrightarrow K$ is isomorphic. Moreover, since $I_2 = (I')^*/I'$ coincides with $I_2 \cap I_1^+$, we have $(I')^* = I^*$ and hence $\langle L, I' \rangle = \langle L', (I')^* \rangle$.
These equalities imply that $\downarrow_{K'}^{L'} = \downarrow_{K}^{L} \circ \downarrow_{L'}^{L}$. Therefore we have
\[
\xi'(f) = \langle f \downarrow_{K'}^{L'}, \Theta_{(K')^*}\rangle = \langle f \downarrow_{K}^{L} \downarrow_{K'}^{L}, \Theta_{K'}\rangle = \xi(f \downarrow_{L'}^{L}).
\]
As in the case of quasi-pullback, this implies
\[
(f \downarrow_{L'}^{L}) *_I (g \downarrow_{L'}^{L}) = \xi(f \downarrow_{L'}^{L}) \cdot (g \downarrow_{L'}^{L}) = \xi'(f) \cdot (g \downarrow_{L'}^{L}) = (\xi'(f) \cdot g) \downarrow_{L'}^{L}.
\]
This proves Proposition 6.7. \qed
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