Form factor relations for pseudoscalar to vector meson transitions.

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Abstract

In semi-leptonic and other weak decays of mesons, the hadronic matrix elements of the operators in the weak Hamiltonian are parametrized by standard sets of independent, Lorentz invariant, form factors. For the case of pseudoscalar to vector meson transitions, it has been shown that a Quark Model description leads to relations between some of the form factors. Here, I give an alternate, and more general, proof of those relations and thus confirm that, in the Quark Model, not all the form factors for pseudoscalar to vector meson transitions are independent. As an application of this result, a Quark Model measurement of the CKM parameter $|V_{ub}|$ can be obtained, where the dependence on all hadronic form factors is eliminated.

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1 Introduction

The study of the weak decays of quarks, and the measurement of the corresponding CKM matrix elements, are consistently hampered by the presence of the long distance QCD effects that are responsible for the binding of the quarks into hadrons. These effects are hard to evaluate in a model independent way, and so tend to bring large uncertainties to the theoretical predictions for the weak decay amplitudes. They appear in the calculation of the matrix elements of the weak Hamiltonian operators, between the initial and final hadronic states. For practical purposes, they are included in standard sets of independent, Lorentz invariant, form factors, that parametrize those hadronic matrix elements in a convenient way, but are otherwise poorly known. Relations between different form factors, that will hold under certain conditions or approximations, can then be very useful: they will reduce the number of uncertain quantities, and improve the accuracy of the theoretical predictions. Moreover, they may help us understand better the general features of the underlying long distance QCD effects.

Here, I am interested in the weak hadronic matrix elements for a pseudoscalar $B$ to a vector meson $V$ transition (for definiteness, I use a $B$-meson in my notation, but the results are quite general; they apply equally well to $D$-mesons, or even to lighter pseudoscalars, if they were to decay weakly to vector mesons). These matrix elements are parametrized by the form factors $V, A_{0,1,2}$ and $F_{1,2,3}$, that are defined as follows:

$$\langle V(p', \vec{\varepsilon}) | \overline{q} \gamma^\mu b | B(p) \rangle = -\frac{1}{m_B + m_V} 2ie^{\alpha\beta\gamma} \varepsilon^*_\alpha p'_\beta p_\gamma V(k^2) , \quad (1)$$

$$\langle V(p', \vec{\varepsilon}) | \overline{q} \gamma^\mu \gamma_5 b | B(p) \rangle = (m_B + m_V) \left( \varepsilon^{*\mu} - \frac{\varepsilon^* \cdot k}{k^2} k^\mu \right) A_1(k^2)$$
$$- \frac{\varepsilon^* \cdot k}{m_B + m_V} \left( (p + p')^\mu - \frac{m_B^2 - m_V^2}{k^2} k^\mu \right) A_2(k^2)$$
$$+ 2m_V \frac{\varepsilon^* \cdot k}{k^2} k^\mu A_0(k^2) , \quad (2)$$

$$\langle V(p', \vec{\varepsilon}) | \overline{q} i \sigma^{\mu\nu} k_\nu b | B(p) \rangle = i e^{\alpha\beta\gamma} \varepsilon^*_\alpha p'_\beta p_\gamma F_1(k^2) , \quad (3)$$

1
\begin{align*}
\langle V(\vec{p}', \varepsilon)|q_i\sigma^{\mu\nu}\gamma_5 k_{\nu} b|B(\vec{p})\rangle &= \left((m_B^2 - m_V^2)\varepsilon^{*\mu} - \varepsilon^{*}.k(p + p')^\mu\right) F_2(k^2) \\
&\quad + \varepsilon^{*}.k \left(k^\mu - \frac{k^2}{m_B^2 - m_V^2}(p + p')^\mu\right) F_3(k^2),
\end{align*}
with \(2m_V A_0(0) = (m_B + m_V)A_1(0) - (m_B - m_V)A_2(0)\), \(F_1(0) = 2F_2(0)\) and \(k \equiv p - p'\).

In Ref. [1], a Quark Model description of the \(B \to V\) hadronic transition led to the following relations between the form factors \(F_{1,2,3}\) and \(A_{0,1,2}\):

\begin{align*}
F_1(k^2) &= 2A_0(k^2), \\
m_V(m_B - m_V)F_2(k^2) &= (m_B E' - m_V^2)A_1(k^2) \\
&\quad - \frac{2m_B^2|\vec{p}'|^2}{(m_B + m_V)^2} A_2(k^2), \\
(m_B E' + m_V^2)F_2(k^2) &= \frac{2m_B^2|\vec{p}'|^2}{m_B^2 - m_V^2} F_3(k^2) \\
&= m_V(m_B + m_V)A_1(k^2). 
\end{align*}

Note that these relations are valid in any reference frame; the energy \(E'\) and momentum \(|\vec{p}'|\) of the vector meson \(V\), in the rest frame of the \(B\)-meson, are used as an abbreviation for the more cumbersome invariant functions of \(k^2\):

\begin{align*}
E' &= \frac{m_B^2 + m_V^2 - k^2}{2m_B}, \\
|\vec{p}'| &= \left[\left((m_B^2 + m_V^2 - k^2)^2 - 4m_B^2m_V^2\right)^{1/2}\right]/2m_B. 
\end{align*}

The fact that these relations were derived, independently of the details of the momentum wavefunctions for the two mesons, suggests that they are a very general Quark Model result. However, the proof in Ref. [1] was not entirely satisfactory, in that the origin of these form factor relations was not clear, and the outcome seemed rather accidental. In what follows, I will give an alternate and more general proof of the same relations. It will clarify their
origin, and the Quark Model assumptions in which they rely. The end result will confirm that, in the Quark Model, the form factors $F_{1,2,3}$ for pseudoscalar to vector meson transitions are not independent form factors.

## 2 Form factor relations

The first step in deriving the form factor relations is to solve eqs. [1][4] in order to write each form factor as a combination of hadronic matrix elements. This is most easily done in the $B$ rest frame, with the $z$-axis chosen along the momentum $\vec{p}'$ of the vector meson $V$. In this frame,

$$p = m_B(1, 0, 0, 0) \quad p' = (E', 0, 0, |\vec{p}'|),$$

and the polarization vectors for the helicity states $\lambda = 0, \pm 1$ of the vector meson are

$$\varepsilon_0 = \frac{1}{m_V}(|\vec{p}'|, 0, 0, E') \quad \varepsilon_{\pm} = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0);$$

$k \equiv p - p' = (m_B - E', 0, 0, -|\vec{p}'|)$, and it is convenient to define a 4-vector $r = N(|\vec{p}'|, 0, 0, -m_B + E')$, with arbitrary normalization, such that $r.k = 0$ and $\vec{r}$ is parallel to $\vec{k}$, in the particular frame that we have chosen. The expressions that are obtained for each of the form factors can then be combined as follows:

$$F_1(k^2) = 2A_0(k^2) \frac{k_\mu \langle V(\lambda = \pm 1)|\overline{q} \gamma_\mu (\sigma^{1\mu} \pm i\sigma^{2\mu})b|B\rangle}{k_\mu \langle V(\lambda = 0)|\overline{q} \gamma_\mu \gamma_5 b|B\rangle},$$

$$m_V(m_B - m_V)F_2(k^2)$$

$$= \left[(m_B E' - m_V^2)A_1(k^2) - \frac{2m_B^2|\vec{p}'|^2}{m_B + m_V}A_2(k^2)\right]$$

$$\times \frac{r_\mu \langle V(\lambda = \pm 1)|\overline{q} \gamma_\mu (\sigma^{1\mu} \pm i\sigma^{2\mu})b|B\rangle}{r_\mu \langle V(\lambda = 0)|\overline{q} \gamma_\mu \gamma_5 b|B\rangle},$$

$$(m_B E' + m_V^2)F_2(k^2) - \frac{2m_B^2|\vec{p}'|^2}{m_B - m_V^2}F_3(k^2)$$

$$= m_V(m_B + m_V)A_1(k^2)$$

$$\times \frac{\varepsilon_\mu^{\pm} \langle V(\lambda = 0)|\overline{q} \gamma_\mu (\sigma^{1\mu} \pm i\sigma^{2\mu})b|B\rangle}{\varepsilon_\mu^{\pm} \langle V(\lambda = \pm 1)|\overline{q} \gamma_\mu \gamma_5 b|B\rangle}.$$
The remainder of the proof consists in showing that the ratios of hadronic matrix elements, that appear on the right-hand-side (RHS) of eqs. 12–14 are precisely equal to 1, in the Quark Model.

In order to do so, we must relate the $\lambda = 0$ and $\lambda = \pm 1$ helicity states of the vector meson. If one is to rely on the Quark Model angular momentum wavefunction for the vector meson, this can be done by flipping the spin of the constituent quark $q$:

$$
\langle V(\lambda = \pm 1) | \overline{\tau} \Gamma b | B \rangle = \sqrt{2} \langle V(\lambda = 0) | S^\dag \overline{\tau} \Gamma b | B \rangle ,
$$

where $S^\pm$ are the raising and lowering operators for the spin of the quark $q$, along the direction of the vector meson momentum; $\Gamma$ is an arbitrary combination of Dirac $\gamma$-matrices. The spin operator, for the $q$-quark field, is

$$
S_\sigma = \frac{1}{2m} \epsilon_{\sigma \mu \rho} J^{\mu \nu} P^\rho ,
$$

where

$$
J^{\mu \nu} = \int d^3 x : q^\dag (x) \left[ ix^\mu D^\nu - ix^\nu D^\mu + \frac{1}{2} \sigma^{\mu \nu} \right] q(x) :
$$

and

$$
P^\mu = \int d^3 x : q^\dag (x) iD^\mu q(x) :$$

are the conserved angular and linear momentum operators; they satisfy the commutation relations

$$
[J^{\mu \nu}, q(x)] = -(ix^\mu D^\nu - ix^\nu D^\mu + \frac{1}{2} \sigma^{\mu \nu}) q(x)
$$

and

$$
[P^\mu, q(x)] = -iD^\mu q(x) .
$$

The covariant derivative $D^\mu$ accounts for the strong interactions of the quark field; it also appears in the equation of motion: $(i \not{\mathcal{D}} - m) q(x) = 0$. The spin projection operator along a direction $n$, with $P.n = 0$ and $n^2 = -1$, is $S.n$, and it is then easy to arrive at the central relation in our proof,

$$
\langle V | (S.n)^\dag \overline{\tau} \Gamma b | B \rangle = \frac{1}{2} \langle V | \overline{\tau} \gamma_5 \gamma_\sigma \Gamma b | B \rangle n^\sigma * .
$$
In the derivation, we assumed, as in the Quark Model, that the \( B \)-meson state does not contain the quark \( q \); then, \( J^{\mu}|B\rangle = P^{\mu}|B\rangle = 0 \).

In order to apply this result to the matrix element on the RHS of eq. 15, one must proceed with care. The spin projection operator along the \( z \)-axis, \( S_z \equiv S.n^{(3)} \), and the spin raising and lowering operators, \( S_{\pm} \equiv S.(n^{(1)} \pm in^{(2)}) \), take a simple form in the \( q \)-quark rest frame, where

\[
(n^{(1)} = (0,1,0,0), \quad n^{(2)} = (0,0,1,0) \quad \text{and} \quad n^{(3)} = (0,0,0,1) ; \quad (22)
\]

then, \( S_z = S_3 \) and \( S_{\pm} = S_1 \pm iS_2 \). In the frame that we have chosen, with the \( B \) meson at rest and the vector meson momentum along the \( z \)-axis, the direction vectors \( n^{(k)} \) depend on the \( q \)-quark momentum in that frame, and so the spin projection operators do not have, in general, such a simple form. If, however, we can neglect the transverse momentum of the \( q \)-quark inside the vector meson \( V \), then \( n^{(1)} \) and \( n^{(2)} \) are as before. Under that assumption, eqs. 15 and 21 give

\[
\langle V(\lambda = \pm 1)|\bar{\tau}\Gamma b|B\rangle = \frac{1}{\sqrt{2}} \langle V(\lambda = 0)|\bar{\tau}\gamma_5(\gamma_1 \mp i\gamma_2)\Gamma b|B\rangle . \quad (23)
\]

This relation is now applied to the hadronic matrix elements on the RHS of eqs. 12-14, where \( \Gamma = i/\sqrt{2}(\sigma^{1\mu} \pm i\sigma^{2\mu})k_\mu, i/\sqrt{2}(\sigma^{1\mu} \pm i\sigma^{2\mu})r_\mu \) or \( \varepsilon^{\pm}_{\mu} \gamma'^{\mu}\gamma_5 \).

It is straightforward to check that the ratio of hadronic matrix elements, in each one of the equations, is exactly 1, and so we recover the form factor relations of eqs. 5-7 and Ref. [1].

### 3 Measuring \( |V_{ub}| \) in the Quark Model

One possible application of the form factor relations derived in here is to the extraction of the CKM parameter \( |V_{ub}| \), from the \( B \to \rho l^-\bar{\nu}_l \) decay rate. Since this exclusive decay rate depends on the form factors \( V \) and \( A_{0,1,2} \), that parametrize the matrix element \( \langle \rho|\bar{\pi}\gamma'^{\mu}(1 - \gamma_5)b|B\rangle \), a clean measurement of \( |V_{ub}| \) is problematic [3]. However, we have seen that, in the Quark Model, these form factors are not independent from the form factors \( F_{1,2,3} \); they are the form factors that parametrize the matrix element \( \langle \rho|\bar{\pi}\sigma^{\mu\nu}k_\nu(1 + \gamma_5)b|B\rangle \), that appears, for example, in the amplitude for the radiative decay \( B \to \rho\gamma \).

A judicious comparison between the semi-leptonic and radiative decays can then yield \( |V_{ub}| \), free from any form factor dependence [4].
From the ratio of the exclusive $B \to \rho \gamma$ and the inclusive $B \to \gamma + X$ decay rates,

$$R_\rho \equiv \frac{\Gamma(B \to \rho \gamma)}{\Gamma(B \to \gamma + X)} \approx \left| \frac{V_{td}}{V_{ts}} \right|^2 \left( 1 - \frac{m_\rho}{m_B} \right)^3 \frac{1}{2} |F_1(0)|^2 ,$$

we can obtain a measurement of the form factor $F_1(k^2)$ at $k^2 = 0$. On the other hand, the $B \to \rho l^- \nu_l$ differential decay rate, at the $k^2 = (p_l^- - p_\nu)^2 = 0$ boundary of the Dalitz plot, is

$$\lim_{k^2 \to 0} \frac{d\Gamma(B \to \rho l^- \nu_l)}{d(k^2/m_B^2)} = G_F^2 \frac{1}{192\pi^3} \left( 1 - \frac{m_\rho}{m_B} \right)^2 m_B^5 |V_{ub}|^2 |A_0(0)|^2 .$$

Using the $F_1(k^2) = 2A_0(k^2)$ relation of eq. 5, the dependence on the form factors can be eliminated between eqs. 24 and 25:

$$\frac{1}{R_\rho} \lim_{k^2 \to 0} \frac{d\Gamma(B \to \rho l^- \nu_l)}{d(k^2/m_B^2)} = G_F^2 \frac{1}{192\pi^3} m_B^5 \left| \frac{V_{ts}}{V_{td}} \right|^2 |V_{ub}|^2 .$$

A measurement of the CKM parameters can then be obtained that is free of the hadronic form factor contributions. One must stress, however, that such a measurement is dependent on the Quark Model assumptions that led to the form factor relations of the previous section. Nevertheless, within that model, it is a very general result, as it does not depend on the particular choice for the momentum wavefunctions of the mesons involved.

For the inclusive radiative decay rate in eq. 24, I have taken $\Gamma(B \to \gamma + X) \simeq \Gamma(b \to s \gamma)$, $m_b \simeq m_B$ and $m_s \simeq 0$; for the present purpose, these are sufficiently good approximations. In the exclusive $B \to \rho \gamma$ amplitude, corrections proportional to $|V_{ub}V_{td}^*|$ due to the difference between $u$ and $c$ quark loops in the weak vertex, have been ignored. These corrections have been estimated to be small, when compared to the dominant $t$ quark loop contribution [4]. Alternatively, one can consider the decay $B \to K^{*} \gamma$, where such corrections are irrelevant. The analogue of eq. 26, in that case, will not have the extraneous factor $|V_{ts}/V_{td}|^2$, but it will be valid only up to $SU(3)_{flavor}$ symmetry-breaking corrections to the form factors. The semi-leptonic decay rate has been observed by CLEO, with $BR(B^0 \to \rho^- l^+ \nu_l) =$
(2.5 ± 0.4±0.5 ± 0.5) × 10^{-4} \[6\]; more data will be necessary, in order to determine the differential decay rate in eq. \[25\]. The inclusive radiative decay has also been measured, with $BR(B \to \gamma + X) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4} \[7\]. As for the exclusive radiative decay, only the Cabibbo favored $B \to K^*\gamma$ has been seen, with $BR(B \to K^*\gamma) = (4.2 \pm 0.8 \pm 0.6) \times 10^{-5} \[8, 9\]$. The branching ratio for $B \to \rho\gamma$ is expected to be about 20 times smaller, whereas the present limit is $BR(B \to (\rho, \omega)\gamma)/BR(B \to K^*\gamma) < 0.19$ at 90% C.L. \[9\].

4 Conclusion

The Quark Model relations between the form factors $F_{1,2,3}$ and $A_{0,1,2}$, that parametrize the weak hadronic transitions from pseudoscalar to vector mesons, were obtained in here from a different, and more general, argument than that in the original derivation of Ref. \[1\]. This alternate proof confirms that, in the Quark Model, $F_{1,2,3}$ are no longer independent form factors. Moreover, it reveals how these form factor relations originate from the spin wavefunctions for the mesons, but do not depend (with one caveat) on the particular choice for the internal momentum wavefunctions. In that respect, they are a very general result of the Quark Model. One should also stress that these relations are valid throughout the entire $k^2$ range of the hadronic transition, and for any value of the pseudoscalar and vector meson masses. As for the caveat regarding the momentum wavefunctions, it arises from the need to assume that the momentum of the quark produced at the weak vertex is nearly parallel to the momentum of the vector meson into which it hadronizes. This is a good approximation in at least two situations that, together, nearly exhaust all scenarios that may occur in practice. One is the case of heavy-to-heavy transitions, where the momentum of the heavy quark is roughly that of the corresponding heavy meson. The other is the case of heavy-to-light transitions, at sufficiently large recoil: the light quark carries a fraction of the large vector meson momentum that dominates over its small transverse momentum. As an example where the latter situation occurs, the form factor relations were applied to the extraction of the CKM parameter $|V_{ub}|$, from a comparison of the exclusive $B \to \rho l^-\nu_l$ and $B \to \rho\gamma$ decays, along similar lines to the method that was suggested in Ref. \[4\].

The question that one would now like to answer is whether the form
factor relations obtained in here remain valid, to some degree, beyond the Quark Model. In the case of heavy-to-heavy transitions, it is easy to check that the model independent form factor relations, that follow from Heavy Quark Symmetry (HQS) [10], do lead to the relations derived in here. This means that the Quark Model result is indeed a true QCD result, in that limit; however, it also means that it adds little to the well known, and more general, form factor relations of HQS. On the other hand, for heavy-to-light transitions, the Quark Model result provides a new set of form factor relations [11]. A comparison with lattice QCD may shed some light into their validity beyond the Quark Model. Hopefully, future lattice calculations will address the question and test these form factor relations, in the more general setting of their approximation. Ultimately, when sufficient data is accumulated for heavy-to-light decays, a comparison with the experimental results will decide whether the Quark Model prediction for the form factors provides a reasonable picture of QCD.

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[11] For heavy-to-light transitions, model independent relations can be obtained in the static heavy quark limit. The Quark Model results in here cannot be derived from them, and they are indeed a new set of form factor relations.