DUALITY FOR Ext-GROUPS AND EXTENSIONS OF DISCRETE SERIES FOR GRADED HECKE ALGEBRAS

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Abstract. In this paper, we study extensions of graded affine Hecke algebra modules. In particular, based on an explicit projective resolution on graded affine Hecke algebra modules, we prove a duality result for Ext-groups. This duality result with study on some parabolically induced modules gives a new proof of the fact that all higher Ext-groups between discrete series vanish.

1. Introduction

1.1. Graded affine Hecke algebras were defined by Lusztig [20] for the study of the representations of affine Hecke algebras and p-adic groups. The relation between affine Hecke algebras and their graded ones can be thought as an analog of the relation between Lie groups and Lie algebras, and so graded affine Hecke algebras are simpler in certain aspects.

The classification of irreducible graded Hecke algebra modules has been studied extensively. A notable result is the Kazhdan-Lusztig geometric classification [17] for equal parameter cases. In arbitrary parameters, general results include the Langlands classification [10] and a classification of discrete series by Opdam-Solleveld [27]. There are also some other classification results [9, 14, 18, 19, 33].

This paper is to study another aspect of graded affine Hecke algebra modules, that is extensions between modules. The extension problem may be a natural question after thorough understanding on the classification of irreducible graded Hecke algebra modules.

There are some related studies of the extension problem in the literature [2, 8, 22, 26, 28, 29, 30, 31, 34]. While our work is motivated from some known results in the setting of p-adic groups and affine Hecke algebras, our approach is self-contained in the theory of graded affine Hecke algebras. Moreover, because of the algebraic nature of our approach, it applies to the graded Hecke algebra of non-crystallographic types (see [18] and [19]). It is also possible to extend some results to some similar algebraic structures such as degenerate affine Hecke-Clifford algebras [23] and graded Hecke algebras for complex reflection groups [32].

1.2. There are two main results in this paper. The first one establishes a duality of Ext-groups. The second one is to apply the duality result with analysis on certain parabolically induced modules to compute extensions between discrete series.

The duality result involves two main ingredients. Let \( \mathbb{H} \) be a graded affine Hecke algebra (see Definition 2.1). The first ingredient is an explicit construction of a projective resolution on \( \mathbb{H} \)-modules, which makes computations possible. The projective resolution is an analogue...
of the classical Koszul resolution or relative Lie algebra cohomology for \((g, K)\)-modules. The second main ingredient is three operations \(\ast, \bullet, \iota\) on \(H\) (see Section 4.2 and Section 4.5 for the detailed definitions).

Those three operations play some roles in the literature. The first anti-involution \(\ast\) arises naturally from the study of unitary duals for the Hecke algebra of a \(p\)-adic group (see [5]). The second anti-involution \(\bullet\) is studied in a recent paper of Barbasch-Ciubotaru [3] as an Hecke algebra analogue of the compact-star operation for \((g, K)\)-modules in [1, 35]. The \(\bullet\)-operation is also studied by Opdam [25] in the Macdonald theory for affine Hecke algebras. The last operation \(\iota\) on \(H\) is the Iwahori-Masumoto involution, which is shown by Evens-Mirković [11] to have close connection with the geometric Fourier-Deligne transform.

For each of the operations \(\ast, \bullet, \iota\), it induces a map from the set of \(H\)-modules to the set of \(H\)-modules. For an \(H\)-module \(X\), we denote by \(X^\ast\), \(X^\bullet\) and \(\iota(X)\) (see Section 4.2 and Section 4.5) for the corresponding dual \(H\)-modules respectively.

Our first main theorem is the following duality on the \(\text{Ext}\)-groups:

**Theorem 1.1.** (Theorem 4.16) Let \(H\) be the graded affine Hecke algebra associated to a root datum \(\Pi = (R, V, R^\vee, V^\vee, \Delta)\) and a parameter function \(k : \Delta \to \mathbb{C}\) (Definition 2.1). Let \(n = \dim V\). Let \(X\) and \(Y\) be finite dimensional \(H\)-modules. Then there exists a natural non-degenerate pairing

\[
\text{Ext}^i_H(X, Y) \times \text{Ext}^{n-i}_H(X^\ast, \iota(Y)^\bullet) \to \mathbb{C}.
\]

Here the \(\text{Ext}^i\)-groups are taken in the category of \(H\)-modules.

(For some comments on the formulation and the proof of Theorem 1.1 see Remark 4.17.)

Theorem 1.1 is an analogue of the Poincaré duality for real reductive groups ([8, Ch.I Proposition 2.9], [16, Theorem 6.10]). Theorem 1.1 may also be related to a duality conjecture of Prasad [30, Conjecture 4] in the \(p\)-adic group setting, in which the Zelevinsky involution is involved. There is also a duality result involving \(\text{Ext}\) and \(\text{Tor}\) in [31, pg 133] by Schneider-Stuhler.

The second result of this paper is about the extensions of discrete series. Those discrete series are defined algebraically in terms of weights (Definition 5.2) and correspond to discrete series of \(p\)-adic groups when the parameter function is positive and equal. Since discrete series are basic building blocks of irreducible \(H\)-modules, it may be important to first understand the extensions among them. Our second main result states that:

**Theorem 1.2.** (Theorem 6.1) Let \(H\) be the graded affine Hecke algebra associated to a root datum \(\Pi = (R, V, R^\vee, V^\vee, \Delta)\) and a parameter function \(k : \Delta \to \mathbb{C}\) (Definition 2.1). Assume \(R\) spans \(V\). Let \(X\) be an irreducible tempered module and let \(Y\) be an irreducible discrete series (Definition 5.2). Then

\[
\text{Ext}^i_H(X, Y) = \begin{cases} 
\mathbb{C} & \text{if } X \cong Y \text{ and } i = 0 \\
0 & \text{otherwise}.
\end{cases}
\]

Since discrete series are also tempered, the statement covers the case for \(X\) and \(Y\) being discrete series. The statement for affine Hecke algebra setting is proven by Opdam-Solleveld.
20, and the one for \( p \)-adic group setting is proven by Meyer 22. The method we prove Theorem 1.2 is different from theirs and essentially makes use of Theorem 1.1 nevertheless. Our approach may also compute extensions outside discrete series as seen in Section 6.2. We shall extend our result whenever possible in the future.

1.3. This paper is organized as follows. Section 1 is the introduction. Section 2 is devoted to set-up basic notations and recall the definition of the graded affine Hecke algebra. Section 3 is to construct an explicit projective resolution for \( \mathbb{H} \)-modules. Section 4 proves a duality result for \( \text{Ext} \)-groups by using the resolution in Section 3. Section 5 reviews the Langlands classification, from which we get some information on extensions. Section 6 computes the extensions of discrete series by using results in Sections 4 and 5. Section 7 applies the Euler-Poincaré pairing to give applications.

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2. Preliminaries

2.1. Root systems and basic notations. Let \( R \) be a reduced root system. Let \( \Delta \) be a fixed choice of simple roots in \( R \). Then \( \Delta \) determines the set of positive roots \( R^+ \). Let \( W \) be the finite reflection group of \( R \). Let \( V_0 \) be a real vector space containing \( R \). (\( R \) does not necessarily span \( V \).) For any \( \alpha \in \Delta \), let \( s_\alpha \) be the simple reflection in \( W \) associated to \( \alpha \) (i.e. \( \alpha \in V_0 \) is in the \( -1 \)-eigenspace of \( s_\alpha \)). For \( \alpha \in R \), let \( \alpha^\vee \in \text{Hom}_R(V_0, \mathbb{R}) \) such that

\[
s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha,
\]

where \( \langle v, \alpha^\vee \rangle = \alpha^\vee(v) \). Let \( R^\vee \subset \text{Hom}_R(V_0, \mathbb{R}) \) be the collection of all \( \alpha^\vee \). Let \( V_0^\vee = \text{Hom}_R(V_0, \mathbb{R}) \).

By extending the scalars, let \( V = \mathbb{C} \otimes_R V_0 \) and let \( V^\vee = \mathbb{C} \otimes_R V_0^\vee \). We call \( (R, V, R^\vee, V^\vee, \Delta) \) to be a root datum.

2.2. Graded affine Hecke algebras. Let \( k : \Delta \to \mathbb{C} \) be a parameter function such that \( k(\alpha) = k(\alpha') \) if \( \alpha \) and \( \alpha' \) are in the same \( W \)-orbit. We shall simply write \( k_\alpha \) for \( k(\alpha) \).

Definition 2.1. 20 Section 4] The graded affine Hecke algebra \( \mathbb{H} = \mathbb{H}(\Pi, k) \) associated to a root datum \( \Pi = (R, V, R^\vee, V^\vee, \Delta) \) and a parameter function \( k \) is an associative algebra with an unit over \( \mathbb{C} \) generated by the symbols \( \{t_w : w \in W\} \) and \( \{f_w : w \in V\} \) satisfying the following relations:

1. The map \( w \mapsto t_w \) from \( \mathbb{C}[W] = \bigoplus_{w \in W} \mathbb{C}w \to \mathbb{H} \) is an algebra injection,
2. The map \( v \mapsto f_v \) from \( S(V) \to \mathbb{H} \) is an algebra injection,
3. the generators satisfy the following relation:

\[
t_{s_\alpha}v - s_\alpha(v)t_{s_\alpha} = k_\alpha \langle v, \alpha^\vee \rangle.
\]
2.3. Ext-groups and central characters. Let \( Z(\mathbb{H}) \) be the center of \( \mathbb{H} \). An \( \mathbb{H} \)-module \( X \) is said to have a central character if there exists a function \( \chi : Z(\mathbb{H}) \to \mathbb{C} \) such that every \( z \in Z(\mathbb{H}) \) acts by the scalar \( \chi(z) \) on \( X \). If \( X \) is an irreducible \( \mathbb{H} \)-module, then \( X \) is finite-dimensional and so has a central character by Schur’s Lemma.

The center \( Z(\mathbb{H}) \) of \( \mathbb{H} \) is \( S(V)W \) (i.e. the set of \( W \)-invariant polynomials) \([20, \text{Proposition 4.5}]\). Suppose \( X \) is an \( \mathbb{H} \)-module with the central character \( \chi \). Then there exists a \( W \)-orbit \( O \) in \( V^\vee \) such that \( \chi(z) = \gamma(z) \) for any \( \gamma \in O \). We shall also say the \( W \)-orbit \( O = W\gamma \) is the central character of \( X \).

For \( \mathbb{H} \)-modules \( X \) and \( Y \), \( \text{Ext}^i_{\mathbb{H}}(X, Y) \) are the higher extensions of \( X \) and \( Y \) in the category of \( \mathbb{H} \)-modules. The consideration of central characters will play a crucial role in computing some \( \text{Ext} \)-groups later. In particular, we shall use the following results later.

**Proposition 2.2.** Let \( X \) and \( Y \) be \( \mathbb{H} \)-modules. If \( X \) and \( Y \) have distinct central characters, then \( \text{Ext}^i_{\mathbb{H}}(X, Y) = 0 \) for all \( i \).

**Proof.** See for example \([8, \text{Theorem I. 4.1}]\), whose proof can be modified to our setting. \( \square \)

3. Resolutions for \( \mathbb{H} \)-modules

In this section, we construct an explicit projective resolution for graded affine Hecke algebra modules, which is the main tool for proving the duality \( \text{Ext} \)-groups in Section 4.

We keep using the notation in Section 2. Let \( \Pi = (R, V, V^\vee, \Delta) \) be a root datum and let \( W \) be the real reflection group associated to \( R \). Let \( k : \Delta \to \mathbb{C} \) be a parameter function. Let \( \mathbb{H} \) be the graded affine Hecke algebra associated to \( \Pi \) and \( k \).

3.1. Projective objects and injective objects. In this section, we construct some projective objects and injective objects, which will be used to construct explicit resolutions for \( \mathbb{H} \)-modules in the next sections.

Let \( X \) be an \( \mathbb{H} \)-module and let \( U \) be a finite-dimensional \( \mathbb{C}[W] \)-module. \( \mathbb{H} \) acts on the space \( \mathbb{H} \otimes_{\mathbb{C}[W]} U \) by the left multiplication on the first factor while \( \mathbb{H} \) acts on the space \( \text{Hom}_W(\mathbb{H}, U) \) by the right translation, explicitly that is for \( f \in \text{Hom}_W(\mathbb{H}, U) \), the action of \( h' \in \mathbb{H} \) is given by:

\[
(h', f)(h) = f(hh'), \quad \text{for all } h \in \mathbb{H}.
\]

Denote by \( \text{Res}_W \) the restriction functor from \( \mathbb{H} \)-modules to \( \mathbb{C}[W] \)-modules.

We skip the detail for the following two standard results. One may refer to the context around \([10, \text{Chapter VII Corollary 7.20}]\) for a similar proof.

**Lemma 3.1.** (Frobenius reciprocity) Let \( X \) be an \( \mathbb{H} \)-module. Let \( U \) be a \( \mathbb{C}[W] \)-module. Then

\[
\text{Hom}_\mathbb{H}(X, \text{Hom}_W(\mathbb{H}, U)) = \text{Hom}_W(\text{Res}_W X, U),
\]

and

\[
\text{Hom}_\mathbb{H}(\mathbb{H} \otimes_{\mathbb{C}[W]} U, X) = \text{Hom}_W(U, \text{Res}_W X).
\]
Lemma 3.2. Let $U$ be a $\mathbb{C}[W]$-module. Then $\mathbb{H} \otimes_{\mathbb{C}[W]} U$ is projective and $\text{Hom}_W(\mathbb{H}, U)$ is injective.

3.2. Koszul-type resolution on $\mathbb{H}$-modules. Let $X$ be an $\mathbb{H}$-module. Define a sequence of $\mathbb{H}$-module maps $d_i$ as follows:

$$0 \to \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^n V) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^1 V)$$

(3.1)

such that $\epsilon : \mathbb{H} \otimes_{\mathbb{C}[W]} \text{Res}_W X \to X$ given by

$$\epsilon(h \otimes x) = h.x$$

and for $i \geq 1$, $d_i : \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^{i+1} V) \to \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^i V)$ given by

$$d_i(h \otimes (x \otimes v_1 \wedge \ldots \wedge v_{i+1}))$$

(3.3)

$$= \sum_{j=1}^{i+1} (-1)^{j+1} (hv_j \otimes x \otimes v_1 \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_{i+1} - h \otimes v_j \otimes v_1 \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_{i+1}).$$

In priori, we do not know $d_i$ is a well-defined $\mathbb{H}$-map, but we prove in the following:

Lemma 3.3. The above $d_i$ are well-defined $\mathbb{H}$-maps and $d^2 = 0$ i.e. (3.1) is a well-defined complex.

Proof. We proceed by induction on $i$. It is easy to see that $\epsilon$ is well-defined. For convenience, we set $d_{-1} = \epsilon$. We now assume $i \geq 0$. To show $d_i$ is independent of the choice of a representative in $\mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^{i+1} V)$, the non-trivial one is to show

$$d_i(t_w \otimes (x \otimes v_1 \wedge \ldots \wedge v_{i+1}) = d_i(1 \otimes (t_w \cdot x \otimes w(v_1) \wedge \ldots \wedge w(v_{i+1}))).$$

For simplicity, set

$$P^w = d_i(t_w \otimes (x \otimes v_1 \wedge \ldots \wedge v_{i+1}))$$

$$= t_w \sum_{j=1}^{i+1} (-1)^{j+1} (v_j \otimes (x \otimes v_1 \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_{i+1}) - 1 \otimes (v_j \cdot x \otimes v_1 \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_{i+1}))$$

and

$$P_w = d_i(1 \otimes (t_w \cdot x \otimes w(v_1) \wedge \ldots \wedge w(v_{i+1})))$$

$$= \sum_{j=1}^{i+1} (-1)^{j+1} w(v_j) \otimes (t_w \cdot x \otimes w(v_1) \wedge \ldots \wedge w(v_j) \wedge \ldots \wedge w(v_{i+1}))$$

$$- \sum_{j=1}^{i+1} (-1)^{j+1} \otimes (w(v_j) t_w \cdot x \otimes w(v_1) \wedge \ldots \wedge w(v_j) \wedge \ldots \wedge w(v_{i+1})$$

To show the equation (3.3), it is equivalent to show $P^w = P_w$. Regard $\mathbb{C}[W]$ as a natural subalgebra of $\mathbb{H}$. By using the fact that $t_w v - w(v) t_w \in \mathbb{C}[W]$ for $w \in W$, $P^w - P_w$ is an element of the form $1 \otimes u$ for some $u \in \text{Res}_W X \otimes \wedge^i V$. Thus it suffices to show
that \( u = 0 \). To this end, a direct computation (from the original expressions of \( P_w \) and \( P_u \)) shows that \( d_{i-1}(P_w - P_u) = 0 \). By induction hypothesis, \( d_{i-1} \) is well-defined and so \( d_{i-1}(1 \otimes u) = d_{i-1}(P_w - P_u) = 0 \). Write \( 1 \otimes u \) of the form

\[
1 \otimes u = \sum_{1 \leq r_1 < \ldots < r_i \leq n} 1 \otimes (x_{r_1}, \ldots, x_{r_i}) \otimes e_{r_1} \wedge \ldots \wedge e_{r_i},
\]

where \( x_{r_1}, \ldots, x_{r_i} \in \text{Res}_W X \) and \( e_1, \ldots, e_n \) is a fixed basis of \( V \). By a direct computation of \( d_{i-1}(1 \otimes u) \) from the expression (3.5), we have

\[
d_{i-1}(1 \otimes u) = \sum_{1 \leq r_1 < \ldots < r_i \leq n} \sum_{j=1}^i (-1)^{j+1} e_{r_j} \otimes (x_{r_1}, \ldots, x_{r_i}) \otimes e_{r_1} \wedge \ldots \wedge e_{r_j} \wedge \ldots \wedge e_{r_i}
- \sum_{1 \leq r_1 < \ldots < r_i \leq n} \sum_{j=1}^i (-1)^{j+1} 1 \otimes e_{r_j} (x_{r_1}, \ldots, x_{r_i}) \otimes e_{r_1} \wedge \ldots \wedge e_{r_j} \wedge \ldots \wedge e_{r_i}
\]

We have seen that \( d_{i-1}(1 \otimes u) = 0 \) and so \( u = 0 \) by using linearly independence arguments. Verifying \( d^2 = 0 \) is straightforward.

\[\Box\]

**Theorem 3.4.**

1. For any \( \mathbb{H} \)-module \( X \), the complex (3.7) forms a projective resolution for \( X \).
2. The global dimension of \( \mathbb{H} \) is \( \dim V \).

**Proof.** From Lemma 3.3, it remains to show the exactness for (1). This can be proven by an argument which imposes a filtration on \( \mathbb{H} \) and uses a long exact sequence (see for example [12, Section 5.3.8] or [16, Chapter IV Section 6]). We provide some detail. Let \( \mathbb{H}^r \) be the (vector) subspace of \( \mathbb{H} \) spanned by the elements of the form

\[
t_w v_1^{n_1} \ldots v_l^{n_l} \quad \text{for} \quad w \in W, \quad v_1, \ldots, v_l \in V,
\]

with \( n_1 + n_2 + \ldots + n_l \leq r \). Note that \( \mathbb{H}^r \) is still (left and right) invariant under the action of \( W \). Let

\[
E^{r,s} = \mathbb{H}^r \otimes_{C[W]} (\text{Res}_W X \otimes \wedge^s V).
\]

Then the differential \( d_{s-1} \) defines a map from \( E^{r,s} \) to \( E^{r+1,s-1} \). For convenience, also set \( E^{r+s+1,-1} = X \) and there is a map from \( E^{r+s,0} \) to \( X \) and \( d_{-1} = 1 \). Then for a positive integer \( p \), we denote by \( E(p) \) the complex \( \{E^{r,s}, d_{s-1}\}_{r+s=p,s \geq -1} \). We now define a graded structure. Let

\[
E^{r,s} = E^{r,s}/E^{r+1,s}.
\]

and let \( \overrightarrow{d}_{r,s} : E^{r,s} \to E^{r+1,s-1} \) be the induced map from \( d_{s-1} \). Then \( \{E^{r,s}, \overrightarrow{d}_{r,s}\}_{r+s=p} \) forms a complex for each \( p \). Denote by \( F(p) \) for such complex. In fact, \( F(p) \) forms a standard Koszul complex and hence the homology \( H^i(F(p)) = 0 \) for all \( i \).

Now consider the following short exact sequences of the chain of complexes for \( p \geq 1 \):
The vertical map from $\mathbb{E}^{p-s-1,s}$ to $\mathbb{E}^{p-s,s}$ is the natural inclusion map. Then we have the associated long exact sequence:

$$\ldots \to H^{k+1}(\mathcal{F}(p)) \to H^k(\mathcal{E}(p-1)) \to H^k(\mathcal{E}(p)) \to H^k(\mathcal{F}(p)) \to \ldots$$

Since $H^k(\mathcal{F}(p)) = H^{k+1}(\mathcal{F}(p)) = 0$, $H^k(\mathcal{E}(p-1)) \cong H^k(\mathcal{E}(p))$. It remains to see $H^k(\mathcal{E}(0)) = 0$ for all $k$, but it follows from definitions.

We now prove (2). By (1), the global dimension of $\mathbb{H}$ is less than or equal to $\dim V$. We now show the global dimension attains the upper bound. Let $\gamma \in V^\vee$ be a regular element and let $v_\gamma$ be a vector with weight $\gamma \in V^\vee$. Define $X = \text{Ind}_{\mathbb{H}}^{\mathbb{H}}(C_v, C_{v_\gamma})$. By Frobenius reciprocity and using $\gamma$ is regular, $\text{Ext}^i(\mathbb{H}, X) = \text{Ext}^i_{\mathbb{S}(V)}(C_v, C_{v_\gamma}) \neq 0$ for all $i \leq \dim V$.

This shows the global dimension has to be $\dim V$.

3.3. Alternate form of the Koszul-type resolution. In this section, we give another form of the differential map $d_i$, which involves the terms $\tilde{v}$ defined in (3.6) below. There are some advantages for computations later.

For $v \in V$, we define the following element in $\mathbb{H}$:

$$\tilde{v} = v - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha(v, \alpha^\vee) t_{s_\alpha}.$$  

(3.6)

This element is used by Barbasch-Ciubotaru-Trapa [4] for the study of the Dirac cohomology for graded affine Hecke algebras. It turns out it is quite useful in several aspects. An important property of the element is the following:

**Lemma 3.5.** [4] Proposition 2.10] For any $w \in W$ and $v \in V$, $t_w \tilde{v} = \tilde{w(v)} t_w$. 

Proof. It suffices to show for the case that \( w \) is a simple reflection \( s_\beta \in W \).

\[
\begin{align*}
t_{s_\beta} v & = t_{s_\beta} \left( v - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha(v, \alpha^\vee) t_{s_\alpha} \right) \\
& = s_\beta(v) t_{s_\beta} + k_\beta(v, \beta^\vee) - \frac{1}{2} k_\beta(v, \beta^\vee) - \frac{1}{2} \sum_{\alpha \in R^+ \setminus \\{\beta\}} k_\alpha(v, \alpha^\vee) t_{s_\beta(a)} t_{s_\beta} \\
& = s_\beta(v) t_{s_\beta} + \frac{1}{2} k_\beta(v, s_\beta(\beta^\vee)) - \frac{1}{2} \sum_{\alpha \in R^+ \setminus \\{\beta\}} k_\alpha(v, s_\beta(\alpha^\vee)) t_{s_\alpha} t_{s_\beta} \\
& = s_\beta(v) t_{s_\beta} - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha(s_\beta(v), \alpha^\vee) t_{s_\alpha} t_{s_\beta} \\
& = s_\beta(v) t_{s_\beta}.
\end{align*}
\]

\( \square \)

We consider the maps \( \tilde{d}_i : \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^{i+1} V) \to \mathbb{H} \otimes_{\mathbb{C}[W]} (\text{Res}_W X \otimes \wedge^{i} V) \) as follows:

\[
(3.7) \quad \tilde{d}_i(h \otimes (x \otimes v_1 \wedge \ldots \wedge v_{i+1})) \\
(3.8) \ = \sum_{j=1}^{i+1} (-1)^{j+1} (h \tilde{v}_j \otimes x \otimes v_1 \wedge \ldots \wedge \tilde{v}_j \ldots \wedge v_i - h \otimes \tilde{v}_j, x \otimes v_1 \wedge \ldots \wedge \tilde{v}_j \ldots \wedge v_{i+1}).
\]

This definition indeed coincides with the one in the previous subsection:

**Proposition 3.6.** \( \tilde{d}_i = d_i. \)

**Proof.** Recall that for \( v \in V \),

\[
\tilde{v} = v - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha(v, \alpha^\vee) t_{s_\alpha}.
\]

Then

\[
\begin{align*}
\tilde{v}_r \otimes (x \otimes v_1 \wedge \ldots \wedge \tilde{v}_r \ldots \wedge v_{i+1}) & = 1 \otimes (\tilde{v}_r x \otimes v_1 \wedge \ldots \wedge \tilde{v}_r \ldots \wedge v_{i+1}) \\
& = v_r \otimes (x \otimes v_1 \ldots \wedge \tilde{v}_r \ldots \wedge v_{i+1}) - 1 \otimes (v_r x \otimes v_1 \ldots \wedge \tilde{v}_r \ldots \wedge v_{i+1}) \\
& - \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha(v_r, \alpha^\vee) \otimes (t_{s_\alpha} x) \otimes s_\alpha(v_1) \wedge \ldots \wedge s_\alpha(\tilde{v}_r) \wedge \ldots \wedge s_\alpha(v_{i+1}) \\
& + \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha(v_r, \alpha^\vee) \otimes (t_{s_\alpha} x) \otimes v_1 \ldots \wedge \tilde{v}_r \wedge \ldots \wedge v_{i+1} \\
& = v_r \otimes (x \otimes v_1 \ldots \wedge \tilde{v}_r \ldots \wedge v_{i+1}) - 1 \otimes (v_r x \otimes v_1 \ldots \wedge \tilde{v}_r \ldots \wedge v_{i+1}) \\
& - \frac{1}{2} \sum_{\alpha \in R^+} \sum_{p < r} (-1)^p k_\alpha(v_r, \alpha^\vee) (t_{s_\alpha} x) \otimes \alpha \wedge s_\alpha(v_1) \wedge \ldots \wedge s_\alpha(\tilde{v}_p) \wedge \ldots \wedge s_\alpha(\tilde{v}_r) \wedge \ldots \wedge s_\alpha(v_{i+1}) \\
& - \frac{1}{2} \sum_{\alpha \in R^+} \sum_{r < p} (-1)^{p-1} k_\alpha(v_r, \alpha^\vee) (t_{s_\alpha} x) \otimes \alpha \wedge s_\alpha(v_1) \wedge \ldots \wedge s_\alpha(\tilde{v}_r) \wedge \ldots \wedge s_\alpha(v_{i+1})
\end{align*}
\]

The second equality follows from the expression of \( \tilde{v}_r \). Taking the alternating sum of the above expression with some standard computations can verify \( \tilde{d}_i = d_i \).
3.4. Complex for computing Ext-groups. We now use the resolution in Section 3.2 to construct a complex for computing Ext-groups. Let $X$ and $Y$ be $\mathbb{H}$-modules.

Then taking the $\text{Hom}_{\mathbb{H}}(\cdot, Y)$ functor on the projective resolution of $X$ as the one in (3.1), we have the induced maps for $i \geq 1$,

$$d_i : \text{Hom}_{\mathbb{H}}(\mathbb{H} \otimes_{C[W]} (\text{Res}_W X \otimes \wedge^i V), Y) \to \text{Hom}_{\mathbb{H}}(\mathbb{H} \otimes_{C[W]} (\text{Res}_W X \otimes \wedge^{i+1} V), Y).$$

Then by using the Frobenius reciprocity, we have induced complex

$$0 \leftarrow \text{Hom}_{\mathbb{H}}(\text{Res}_W X \otimes \wedge^n V, \text{Res}_W Y) \xleftarrow{d_n} \cdots \xleftarrow{d_1} \text{Hom}_{\mathbb{H}}(\text{Res}_W X \otimes \wedge V, \text{Res}_W Y) \leftarrow 0,$$

where the map $d_i^*$ can be explicitly written as:

$$d_i^*(f)(x \otimes v_1 \wedge \cdots \wedge v_{i+1}) = \sum_{j=1}^{i+1} (-1)^{i+1} \tilde{v}_j.f(x \otimes v_1 \wedge \cdots \wedge \tilde{v}_j \wedge \cdots \wedge v_{i+1})$$

$$- \sum_{j=1}^{i+1} (-1)^{i+1} f(\tilde{v}_j.x \otimes v_1 \wedge \cdots \wedge \tilde{v}_j \wedge \cdots \wedge v_{i+1}),$$

where the action of $\tilde{v}_j$ on the term $f(x \otimes v_1 \wedge \cdots \wedge \tilde{v}_j \wedge \cdots \wedge v_{i+1})$ is via the action of $\tilde{v}_j$ on $Y$ and the action of $\tilde{v}_j$ on the term $x$ is via the action of $\tilde{v}_j$ on $X$.

Thus we obtain the following:

**Proposition 3.7.** Let $X$ and $Y$ be $\mathbb{H}$-modules. Then $\text{Ext}_{\mathbb{H}}^i(X, Y)$ is naturally isomorphic to the $i$-th homology of the complex in (3.9).

An immediate consequence is the following:

**Corollary 3.8.** Let $X$ and $Y$ be finite-dimensional $\mathbb{H}$-modules. Then

$$\dim \text{Ext}_{\mathbb{H}}^i(X, Y) < \infty.$$

**Proof.** Since $X$ and $Y$ are finite dimensional, $\text{Hom}_{\mathbb{H}}(\text{Res}_W X \otimes \wedge^i V, \text{Res}_W Y)$ is finite dimensional. Then the statement follows from Proposition 3.7. \qed

It is not hard to see that $d_i^*$ can be naturally extended to a map from $\text{Hom}_C(\text{Res}_W X \otimes \wedge^i V, \text{Res}_W Y)$ to $\text{Hom}_C(\text{Res}_W X \otimes \wedge^{i+1} V, \text{Res}_W Y)$. We denote the map by $\overrightarrow{d}^*_i$, which will be used in Section 4.4.

4. Duality for Ext-groups

In this section, we prove a duality result for the Ext-groups of graded affine Hecke algebra modules, which can be thought as an analogue of some classical dualities such as Poincaré duality or Serre duality (also see Poincaré duality for real reductive groups in [16, Theorem 6.10]).

We keep using the notation from Section 3.
4.1. \(\theta\)-action and \(\theta\)-dual. We define an involution \(\theta\) on \(\mathbb{H}\) in this section. This \(\theta\) is not needed in the duality result (Theorem 4.10), but it closely relates to the \(^*\) and \(\cdot\) operations defined in the next section.

Let \(w_0\) be the longest element in \(W\). Let \(\theta\) be an involution on \(\mathbb{H}\) characterized by

\[
\theta(v) = -w_0(v) \quad \text{for any } v \in V, \quad \text{and} \quad \theta(t_w) = t_{w_0w_0^{-1}} \quad \text{for any } w \in W,
\]

where \(w_0\) acts on \(v\) as the reflection representation of \(W\). Since \(\theta(\Delta) = \Delta\), \(\langle ., . \rangle\) is \(W\)-invariant and \(k_\alpha = k_{\theta(\alpha)}\) for any \(\alpha \in \Delta\), it is straightforward to verify \(\theta\) defines an automorphism on \(\mathbb{H}\).

Note that \(\theta\) also induces an action on \(V^\vee\), still denoted \(\theta\). For \(\alpha \in R\), since \(w_0(\alpha^\vee) = w_0(\alpha)^\vee\), we also have \(\theta(\alpha^\vee) = \theta(\alpha)^\vee\). The action \(\theta\) on \(V^\vee\) will be used in Section 5 when we need to consider weights of an \(\mathbb{H}\)-module.

Recall that for \(v \in V\), \(\bar{v}\) is defined in (3.6). The following lemma follows from definitions.

**Lemma 4.1.** For any \(v \in V\), \(\theta(\bar{v}) = \bar{\theta(v)}\).

**Definition 4.2.** For an \(\mathbb{H}\)-module \(X\), define \(\theta(X)\) to be the \(\mathbb{H}\)-module such that \(\theta(X)\) is isomorphic to \(X\) as vector spaces and the \(\mathbb{H}\)-action is determined by:

\[
\pi_{\theta(X)}(h)x = \pi_X(\theta(h))x,
\]

where \(\pi_X\) and \(\pi_{\theta(X)}\) are the maps defining the action of \(\mathbb{H}\) on \(X\) and \(\theta(X)\) respectively.

4.2. \(^*\)-dual and \(\cdot\)-dual. In this section, we study two anti-involutions on \(\mathbb{H}\). These two anti-involutions are studied in [3], but we make a variation for our need. More precisely, those anti-involutions are linear rather than Hermitian-linear, and we will discuss how to recover the results for the original anti-automorphisms at the end of this section. The linearity will make some construction easier. For instance, it is easier to make the identification of spaces in Section 4.3.

Define \(^*\) : \(\mathbb{H} \to \mathbb{H}\) to be the linear anit-involution determined by

\[
v^* = t_{w_0}\theta(v)t_{w_0}^{-1} \quad \text{for } v \in V, \quad t_w^* = t_w^{-1} \quad \text{for } w \in W.
\]

Define \(\cdot\) : \(\mathbb{H} \to \mathbb{H}\) to be another linear anti-involution determined by

\[
v^\cdot = v \quad \text{for } v \in V, \quad t_w^\cdot = t_w^{-1} \quad \text{for } w \in W.
\]

**Definition 4.3.** Let \(X\) be an \(\mathbb{H}\)-module. A map \(f : X \to \mathbb{C}\) is said to be a linear functional if \(f(\lambda x_1 + x_2) = \lambda f(x_1) + f(x_2)\) for any \(x_1, x_2 \in X\) and \(\lambda \in \mathbb{C}\). The \(^*\) dual of \(X\), denoted by \(X^*\), is the space of linear functionals of \(X\) with the action of \(\mathbb{H}\) determined by

\[
(h,f)(x) = f(h^*x) \quad \text{for any } x \in X.
\]

We similarly define \(\cdot\)-dual of \(X\), denoted by \(X^\cdot\), by replacing \(h^*\) with \(h^\cdot\) in equation (4.11).

**Lemma 4.4.** Let \(X\) be an \(\mathbb{H}\)-module. Define a bilinear pairing \(\langle ., . \rangle_X^* : X^* \times X \to \mathbb{C}\) (resp. \(\langle ., . \rangle_X^\cdot : X^\cdot \times X \to \mathbb{C}\) such that \(\langle f, x \rangle_X^* = f(x)\) (resp. \(\langle f, x \rangle_X^\cdot = f(x)\). (We reserve \(\langle ., . \rangle_X^\cdot\) for the use of another pairing later.) Then

1. \(\text{for } v \in V, \quad \langle \bar{v}, f \rangle_X^* = \langle f, -\bar{v} \rangle_X^*\) (resp. \(\langle f, \bar{v} \rangle_X^\cdot = \langle \bar{f}, v \rangle_X^\cdot\)),
Lemma 4.7. For any $\omega \in X$, $\langle t_w, f \rangle X = \langle f, t_w^{-1} \rangle X$ (resp. $\bullet \langle t_w, f \rangle X = \bullet \langle f, t_w^{-1} \rangle X$).
(3) $\langle \cdot, \cdot \rangle X$ (resp. $\bullet \langle \cdot, \cdot \rangle X$) is non-degenerate.

Proof. We first consider $\ast$-operation. Note that $(\widetilde{v})^\ast = t_{w_0} \theta(\widetilde{v}) t_{w_0}^{-1} = \theta(\widetilde{v})$, where the second equality follows from Lemma 4.3 and Lemma 4.1. This implies (1). Other assertions follow immediately from the definitions.

For $\ast$-operation, we have $(\widetilde{v})^\ast = \widetilde{v}$ from the definitions. This then implies (1). Other assertions again follow from the definitions.

□

Lemma 4.5. Let $X$ be an $H$-module. Then $(X^\ast)^\ast \cong \theta(X)$.

Proof. Let $\theta'$ be an automorphism on $H$ such that $\theta'(h) = (h^\ast)^\ast = t_{w_0} \theta(h) t_{w_0}^{-1}$ for any $h \in H$. Then we have $\theta'$ sends $H$-modules to $H$-modules and we denote $\theta'(X)$ to be the image of the map of an $H$-module $X$. Note that $\theta'(X) \cong (X^\ast)^\ast$ by definitions. Then it suffices to show $\theta'(X) \cong \theta(X)$. We define a map $F : X \to X, x \mapsto t_{w_0}^{-1} x$. Then by definitions $\theta(h).F(x) = F(\theta(h).x)$. This implies $\theta'(X) \cong \theta(X)$ as desired.

□

4.3. Pairing for $\wedge^i V$ and $\wedge^{n-i} V$. Fix an ordered basis $e_1, \ldots, e_n$ for $V$. We define a non-degenerate bilinear pairing $\langle \cdot, \cdot \rangle_{\wedge^i V}$ as

$$\wedge^i V \times \wedge^{n-i} V \to \mathbb{C}$$

determined by

$$\langle v_1 \wedge \ldots \wedge v_i, v_{i+1} \wedge \ldots \wedge v_n \rangle_{\wedge^i V} = \det(v_1 \wedge \ldots \wedge v_n),$$

where $\det$ is the determinant function for the fixed ordered basis $e_1, \ldots, e_n$.

Define $(\wedge^i V)^\vee$ to be the dual space of $\wedge^i V$. For $\omega \in \wedge^{n-i} V$, define $\phi_\omega \in (\wedge^i V)^\vee$ by

$$\phi_\omega(\omega') = \langle \omega, \omega' \rangle_{\wedge^{n-i} V}$$

By using $\det(w(v_1) \wedge \ldots \wedge w(v_n)) = \det(v_1 \wedge \ldots \wedge v_n)$ for any $w \in W$, we see the map $\omega \mapsto \phi_\omega$ from $\wedge^{n-i} V$ to $(\wedge^i V)^\vee$ defines a $W$-representation isomorphism from $\wedge^{n-i} V$ to $\det(\wedge^i V)^\vee$.

We also define a pairing $\langle \cdot, \cdot \rangle_{(\wedge^{n-i} V)^\vee} : (\wedge^{n-i} V)^\vee \times (\wedge^i V)^\vee \to \mathbb{C}$ such that

$$\langle \phi_\omega, \phi_{\omega'} \rangle_{(\wedge^{n-i} V)^\vee} = \langle \omega, \omega' \rangle_{\wedge^i V},$$

where $\omega \in \wedge^i V$ and $\omega' \in \wedge^{n-i} V$. By definitions, we have the following two lemmas:

Lemma 4.6. For $\omega \in \wedge^i V$ and $\omega' \in \wedge^{n-i} V$,

$$\langle \phi_\omega, \phi_{\omega'} \rangle_{(\wedge^{n-i} V)^\vee} = \langle \omega, \omega' \rangle_{\wedge^i V} = \phi_\omega(\omega').$$

Lemma 4.7. For any $w \in W$,

$$\langle w.\phi_{v_1 \wedge \ldots \wedge v_i}, w.\phi_{v_{i+1} \wedge \ldots \wedge v_n} \rangle_{(\wedge^{n-i} V)^\vee} = \det(w) \langle \phi_{v_1 \wedge \ldots \wedge v_i}, \phi_{v_{i+1} \wedge \ldots \wedge v_n} \rangle_{(\wedge^{n-i} V)^\vee}.$$
Proof. As noted above, for any \( w \in W \),
\[
  w.\phi_{v_1 \wedge \ldots \wedge v_i} = \text{sgn}(w)\phi_{w.(v_1 \wedge \ldots \wedge v_i)}, \quad w.\phi_{v_{i+1} \wedge \ldots \wedge v_n} = \text{sgn}(w)\phi_{w.(v_{i+1} \wedge \ldots \wedge v_n)}.
\]
The lemma then follows from straightforward computations.
\( \square \)

The following technical lemma is a simple linear algebra consequence, but we shall use a number of times.

**Lemma 4.8.** Recall that \( e_1, \ldots, e_n \) is a fixed basis of \( V \). Consider \( \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \in (\wedge^i V)^\vee \) and \( \phi_{e_{k_1} \wedge \ldots \wedge e_{k'_{i-1}}} \in (\wedge^{n-i+1} V)^\vee \). Suppose all \( k_1, \ldots, k_{n-i} \) are mutually distinct (otherwise \( \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \in (\wedge^i V)^\vee = 0 \) and also suppose all \( k'_1, \ldots, k'_{i+1} \) are mutually distinct. If \(| \{k_1, \ldots, k_{n-i}\} \cap \{k'_1, \ldots, k'_{i+1}\} | \geq 2 \), then for any \( p = 1, \ldots, n-i \) and \( q = 1, \ldots, i+1 \),
\[
  \langle \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i-1}}} \rangle^{(\wedge^i V)^\vee} = (\langle \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}} \rangle^{(\wedge^{n-i+1} V)^\vee} = 0,
\]
If \(| \{k_1, \ldots, k_{n-i}\} \cap \{k'_1, \ldots, k'_{i+1}\} | = 1 \), then there exists a unique pair of indices \( p \) and \( q \) such that
\[
  \langle \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i-1}}} \rangle^{(\wedge^i V)^\vee} = (\langle \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-i}}} \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}} \rangle^{(\wedge^{n-i+1} V)^\vee} = 0.
\]

and the terms do not vanish, and for \( r \neq p \) or \( s \neq q \),
\[
  (\langle \phi_{e_{k_1} \wedge \ldots \wedge e_{e_{k_{n-i}}} \phi_{e_{k'_1} \wedge \ldots \wedge e_{k'_{i-1}}} \rangle^{(\wedge^i V)^\vee} = 0.
\]

4.4. Complexes involving duals. It is well-known that there is a natural identification between the spaces \( \text{Hom}_W(\text{Res}_W X \otimes (\Lambda^i V), \text{Res}_W Y) \) and \( (X^* \otimes Y \otimes (\Lambda^i V)^\vee)^W \) (or \( (X^* \otimes Y \otimes (\Lambda^i V)^\vee)^W \)). Here we consider a natural \( W \)-action on \( X^* \otimes Y \otimes (\Lambda^i V)^\vee \) and \( (X^* \otimes Y \otimes (\Lambda^i V)^\vee)^W \) is the \( W \)-invariant space. For notational simplicity, we may simply write \( X \) for \( \text{Res}_W X \) and also sometimes regard \( X \) as a vector space, which should be clear from the context.

In order to prove Theorem 4.10 later, we need to construct some pairing, which will be more convenient to be done for the spaces \( (X^* \otimes Y \otimes (\Lambda^i V)^\vee)^W \) (or \( (X^* \otimes Y \otimes (\Lambda^i V)^\vee)^W \)). The goal of this section is to translate the differential maps \( d_{i+1}^* \) in Section 3.3 into the corresponding maps for \( (X^* \otimes Y \otimes (\Lambda^i V)^\vee)^W \).

We define (a linear) map \( \overline{D}_i : X^* \otimes Y \otimes (\Lambda^i V)^\vee \to X^* \otimes Y \otimes (\Lambda^{i+1} V)^\vee \) on the complex, which is determined by
\[
\overline{D}_i(f \otimes y \otimes \phi_{v_1 \wedge \ldots \wedge v_{n-i}}) = \sum_{j=1}^{n-i} (-1)^{j+1} (f \otimes \overline{v}_j.y \otimes \phi_{v_1 \wedge \ldots \wedge \hat{v}_j \ldots \wedge v_{n-i}} + \overline{v}_j \cdot f \otimes y \otimes \phi_{v_1 \wedge \ldots \wedge \hat{v}_j \ldots \wedge v_{n-i}}),
\]
for \( f \otimes y \otimes \phi_{v_1 \wedge \ldots \wedge v_{n-i}} \in X^* \otimes Y \otimes (\Lambda^i V)^\vee \), where \( \overline{v}_j \) acts on \( f \) by the action on \( X^* \) and \( \overline{v}_j \) acts on \( y \) by the action on \( Y \).
Note that there is a natural $W$-action on $X^* \otimes Y \otimes (\wedge^i V)^\vee$ (from the $W$-action of $X^*$, $Y$ and $V$). Such $W$ action commutes with $D_i$ and so $D_i$ sends $(X^* \otimes Y \otimes (\wedge^i V)^\vee)^W$ to $(X^* \otimes Y \otimes (\wedge^{i+1} V)^\vee)^W$. Then the map $D_i$ gives rise the following map:

$$D_i : (X^* \otimes Y \otimes (\wedge^i V)^\vee)^W \rightarrow (X^* \otimes Y \otimes (\wedge^{i+1} V)^\vee)^W$$

If we want to emphasis the complexes that $D_i$ or $D_i^*$ refer to, we shall write $D_{X^* \otimes Y \otimes (\wedge^i V)^\vee}$ for $D_i$ and $D_{(X^* \otimes Y \otimes (\wedge^i V)^\vee)^W}$ for $D_i^*$.

In the priori, we do not have $D^2 = 0$, but we will soon prove it in Lemma 4.9.

We define another map $D_i^* : X^* \otimes Y \otimes (\wedge^i V)^\vee \rightarrow X^* \otimes Y \otimes (\wedge^{i+1} V)^\vee$ determined by:

$$D_i^*(f \otimes y \otimes \phi_{\ell_1 \ldots \ell_{n-1}}) = \sum_{j=1}^{n-i} (-1)^{i+j} (f \otimes \bar{v}_j \otimes \phi_{\ell_1 \ldots \ell_j \ldots \ell_{n-1}} - \bar{v}_j \otimes f \otimes \phi_{\ell_1 \ldots \ell_j \ldots \ell_{n-1}}),$$

Similar to $D_i$, the restriction of $D_i$ to $(X^* \otimes Y \otimes (\wedge^i V)^\vee)^W$ has image in $(X^* \otimes Y \otimes (\wedge^{i+1} V)^\vee)^W$. Denote by $D_i^*$ the restriction of $D_i^*$ to $(X^* \otimes Y \otimes (\wedge^i V)^\vee)^W$.

Define a linear isomorphism $\overline{\Psi} : X^* \otimes Y \otimes (\wedge^i V)^\vee \rightarrow \text{Hom}_W(X \otimes \wedge^i V, Y)$ as follows, where we also regard $X$ and $Y$ as vector spaces: for $f \otimes y \otimes \phi_{\ell_1 \ldots \ell_{n-1}} \in X^* \otimes Y \otimes (\wedge^i V)^\vee$ has the action given by: $f \otimes y \otimes \phi_{\ell_1 \ldots \ell_{n-1}} = f(x) \phi_{\ell_1 \ldots \ell_{n-1}}(u_1 \wedge \ldots \wedge u_l) y \in Y$ for $x \otimes u_1 \wedge \ldots \wedge u_l \in X \otimes \wedge^i V$. The map $\overline{\Psi}$ indeed depends on $i$, but we shall suppress the index $i$. By taking restriction on the space $(X^* \otimes Y \otimes (\wedge^i V)^\vee)^W$, we obtain the linear isomorphism:

$$\Psi : (X^* \otimes Y \otimes (\wedge^i V)^\vee)^W \rightarrow \text{Hom}_W(X \otimes \wedge^i V, Y).$$

Since $X^*$ and $X^*$ can be naturally identified as vector spaces, the maps $\overline{\Psi}$ and $\Psi$ are also defined for the corresponding spaces involving $\ast$ instead of $\ast$.

Recall that $d_i^*$ and $\overline{D}_i^*$ are defined in Section 3.1 and we remark that the $\ast$ on $d_i^*$ has nothing to do with the $\ast$-involution on $\mathbb{H}$.

**Lemma 4.9.** Let $X$ and $Y$ be $\mathbb{H}$-modules. Then

1. For any $\omega \in X^* \otimes Y \otimes (\wedge^i V)^\vee$, $\overline{\Psi}(D_i(\omega)) = (-1)^{n-i+1} d_i^*(\overline{\Psi}(\omega))$.
2. For any $\omega \in X^* \otimes Y \otimes (\wedge^i V)^\vee$, $\overline{\Psi}(D_i^*(\omega)) = (-1)^{n-i+1} \overline{D}_i^*(\overline{\Psi}(\omega))$.

**Proof.** Recall that $e_1, \ldots, e_n$ be the fixed basis for $V$. Let $\omega = f \otimes y \otimes \phi_{k_1' \ldots \ell_{n-1}} \in X^* \otimes Y \otimes (\wedge^i V)^\vee$. By linearity, it suffices to check that

$$\overline{\Psi}(D_i(\omega))(x \otimes e_{k_1'} \wedge \ldots \wedge e_{k_{i+1}'}) = (-1)^{n-i+1} \overline{D}_i^*(\overline{\Psi}(\omega))(x \otimes e_{k_1'} \wedge \ldots \wedge e_{k_{i+1}'}),$$

for any $x \in X$ and any indices $k_1', \ldots, k_{i+1}' \in \{1, \ldots, n\}$.

Suppose $\{k_1, \ldots, k_{n-1}\} \cap \{k_1', \ldots, k_{i+1}'\} \leq 2$. By Lemma 4.6 and Lemma 4.8

$$\overline{d}_i^*(\overline{\Psi}(\omega))(x \otimes e_{k_1'} \wedge \ldots \wedge e_{k_{i+1}'})$$

$$= 0$$

$$= \overline{\Psi}(D_i(\omega))(x \otimes e_{k_1'} \wedge \ldots \wedge e_{k_{i+1}'})$$
Suppose \( |\{k_1, \ldots, k_{n-1}\} \cap \{k'_1, \ldots, k'_{i+1}\}| = 1 \). Let \( k_p \) and \( k'_q \) be the unique pair of indices such that \( e_{k_p} = e_{k'_q} \). Then

\[
\begin{align*}
\partial_i^T \Psi(\omega)(x \otimes e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}) \\
= \Psi(\omega)(d_i(x \otimes e_{k_1} \wedge \ldots \wedge e_{k'_{i+1}}))
\end{align*}
\]

\[
= (-1)^{q+1} f(x) \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-1}}} (e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}) e_{k'_q} y
- (-1)^{q+1} f(\tilde{e}_{k'_q} x) \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-1}}} (e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}) e_{k'_q} y
\]

(by Lemma 4.8)

\[
= (-1)^{n-i-p} f(x) \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-1}}} (e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}) e_{k'_q} y
- (-1)^{n-i-p} f(\tilde{e}_{k'_q} x) \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-1}}} (e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}) e_{k'_q} y
\]

(by Lemma 4.6 and Lemma 4.8)

\[
= (-1)^{n-i-p} f(x) \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-1}}} (e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}) e_{k'_q} y
- (-1)^{n-i-p} f(\tilde{e}_{k'_q} x) \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-1}}} (e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}}) e_{k'_q} y
\]

(by Lemma 4.4)

\[
= (-1)^{n-i+1} \Psi(D_i(f \otimes X \otimes \phi_{e_{k_1} \wedge \ldots \wedge e_{k_{n-1}}})) (x \otimes e_{k'_1} \wedge \ldots \wedge e_{k'_{i+1}})
\]

(by Lemma 4.8)

This completes the proof for (1).

The proof for (2) follows the same style of computations. (One of the difference is in the fifth equality of the computation in the second case and that explains why the definition of \( \overline{D}^T_i \) and \( \overline{D}^T_i \) differs by a sign in a term.)

\[\square\]

**Lemma 4.10.** We have the following:

1. \( D^2 = 0 \) and \( (D^*)^2 = 0 \),
2. The complex \( \text{Hom}_W(\text{Res}_W X \otimes \Lambda^i V, Y) \) with differentials \( d^*_i \) is naturally isomorphic to the complex \( (X^* \otimes Y \otimes (\Lambda^i V))^W \) with differentials \( D_i \).
3. The complex \( \text{Hom}_W(\text{Res}_W X \otimes \Lambda^i V, Y) \) with differentials \( d^*_i \) is naturally isomorphic to the complex \( (X^* \otimes Y \otimes (\Lambda^i V))^W \) with differentials \( D^*_i \).

**Proof.** By Lemma 4.9(1) and the fact that \( \Psi \) is an isomorphism, \( D_i = \Psi^{-1} \circ d^*_i \circ \Psi \). Then (1) follows from Lemma 4.3 (2) follows from Lemma 4.9(1). The proof for (3) and another assertion about \( D^* \) in (1) is similar.

\[\square\]

**Proposition 4.11.** Let \( X, Y \) be \( \mathbb{H} \)-modules.

\[\text{Ext}_H^i(X, Y) \cong \text{Ext}_H^i(Y^*, X^*) \cong \text{Ext}_H^i(Y^*, X^*) \cong \text{Ext}_H^i(\theta(X), \theta(Y)),\]

and the isomorphisms as vector spaces between them are natural.

\[\square\]
Proof. We first prove that $\text{Ext}^i_H(X,Y) = \text{Ext}^i_H(Y^*,X^*)$. By Lemma 4.10 (2),
\[
\text{Ext}^i_H(X,Y) \cong \ker D_{(X^* \otimes Y \otimes (\wedge^i V))W} / \text{im} D_{(X^* \otimes Y \otimes (\wedge^i V))W},
\]
and
\[
\text{Ext}^i_H(Y^*,X^*) \cong \ker D_{(Y^* \otimes X^* \otimes (\wedge^i V))W} / \text{im} D_{(Y^* \otimes X^* \otimes (\wedge^i V))W}.
\]
With $(Y^*)^* \cong Y$, there is a natural isomorphism between the spaces $(X^* \otimes Y \otimes (\wedge^i V))W$ and $(Y^*)^* \otimes X^* \otimes (\wedge^i V)W$. It is straightforward to verify the isomorphism induces an isomorphism between the corresponding complexes by using Lemma 4.10. The proof for $\text{Ext}^i_H(X,Y) = \text{Ext}^i_H(X^*,Y^*)$ is similar. For the equality $\text{Ext}^i_H(X,Y) = \text{Ext}^i_H(\theta(X),\theta(Y))$, it follows from the two equalities we have just proven. Indeed,
\[
\text{Ext}^i_H(X,Y) = \text{Ext}^i_H(Y^*,X^*) = \text{Ext}^i_H((X^*)^*,(Y^*)^*) = \text{Ext}^i_H(\theta(X),\theta(Y)),
\]
where the last equality follows from Lemma 4.5.

\[\square\]

4.5. Iwahori-Matsumoto dual.

Definition 4.12. The Iwahori-Matsumoto involution $\iota$ is an automorphism on $\mathbb{H}$ determined by
\[
\iota(v) = -v \quad \text{for } v \in V, \iota(w) = \text{sgn}(w)w \quad \text{for } w \in W.
\]
This define a map, still denoted $\iota$, from the set of $\mathbb{H}$-modules to the set of $\mathbb{H}$-modules.

Lemma 4.13. For any $v \in V$, $\iota(\bar{v}) = -\bar{v}$.

Proof. This follows from $\iota(s_\alpha) = -s_\alpha$ and definitions.

\[\square\]

Lemma 4.14. Let $\kappa$ be the natural (vector space) bijection from $Y$ to $\iota(Y)$ so that $h.\kappa(y) = \kappa(h.y)$. Let $Y$ be an $\mathbb{H}$-module. Define a bilinear pairing $\langle , \rangle_Y^\bullet : Y \times \iota(Y)^* \to \mathbb{C}$ by $\langle y, g \rangle_Y^\bullet = g(\kappa(y))$. Then
\begin{enumerate}
  \item for $v \in V$, $\langle \bar{v}, y, g \rangle_Y^\bullet = \langle y, -\bar{v}, g \rangle_Y^\bullet$,
  \item for $w \in W$, $\langle t_w, y, g \rangle_Y^\bullet = \text{sgn}(w) \langle y, t_w^{-1} g \rangle_Y^\bullet$,
  \item $\langle , \rangle_Y^\bullet$ is non-degenerate.
\end{enumerate}

Proof. By a direct computation, $\bar{v}^\bullet = -\bar{v}$ and then (1) follows from Lemma 4.13 and the definitions. (2) and (3) follow from the definitions.

\[\square\]

Proposition 4.15. For $\mathbb{H}$-modules $X$ and $Y$, $\text{Ext}^i_H(X,\iota(Y)) = \text{Ext}^i_H(\iota(X),Y)$.

Proof. Let $P^i \to X$ be a projective resolution of $X$. Then $\iota(P^i)$ is still a projective object and $\iota(P^i) \to \iota(X)$ is a projective resolution of $\iota(X)$. There is a natural isomorphism $\text{Hom}_H(\iota(P^i),Y) \cong \text{Hom}_H(P^i,\iota(Y))$. Hence $\text{Ext}^i_H(\iota(X),Y) = \text{Ext}^i_H(X,\iota(Y))$.

\[\square\]
4.6. Duality theorem. In this section, we state and prove a duality on Ext\_k\_G-groups.

**Theorem 4.16.** Let \( \mathbb{H} \) be the graded affine Hecke algebra associated to a root datum \( \Pi = (R, V, V^\vee, \Delta) \) and a parameter function \( k : \Delta \to \mathbb{C} \) (Definition 2.7). Let \( n = \dim V \).

Let \( X \) and \( Y \) be finite dimensional \( \mathbb{H} \)-modules. Let \( X^* \) be the \( * \)-dual of \( X \) in Definition 4.9. Let \( i(Y) \) be the Iwahori-Matsumoto dual in Definition 4.13 and let \( i(Y)^* \) be the \( * \)-dual of \( i(Y) \) in Definition 4.9. Then there exists a natural non-degenerate pairing

\[
\Ext_{\mathbb{H}}^2(X, Y) \times \Ext_{\mathbb{H}}^{n-1}(X^*, i(Y)^*) \to \mathbb{C}.
\]

**Proof.** We divide the proof into few steps.

**Step 1: Construct non-degenerate bilinear pairings.**

The space

\[
\Hom_W(X \otimes \wedge^i V, Y) \times \Hom_W(X^* \otimes \wedge^{n-i} V, i(Y)^*)
\]

is identified with

\[
(X^* \otimes Y \otimes (\wedge^i V)^\vee)^W \times (X \otimes i(Y)^* \otimes (\wedge^{n-i} V)^\vee)^W
\]

as in Section 4.4. Let \( \langle , \rangle_X^* \) be the bilinear pairing on \( X^* \times X \) such that \( \langle f, x \rangle_X^* = f(x) \) for \( f \in X^* \) and \( x \in X \). Let \( \langle , \rangle_Y \) be the bilinear pairing on \( Y \times \iota(Y)^* \) such that \( \langle y, g \rangle_Y = g(\iota(y)) \) for \( g \in \iota(Y)^* \) and \( y \in Y \). Here \( \iota \) is defined as in Lemma 4.13.

For each \( i \), we first define the pairing \( \langle , \rangle_{X, Y, \wedge^i V} \) on a larger space \( (X^* \otimes Y \otimes (\wedge^i V)^\vee) \times (X \otimes \iota(Y)^* \otimes (\wedge^{n-i} V)^\vee) \) via the product of the pairings \( \langle , \rangle_X^* \), \( \langle , \rangle_Y \) and \( \langle , \rangle_{(\wedge V)^\vee} \) i.e.

\[
\langle f \otimes y \otimes \phi_{e_{1,i}} \ldots \otimes \phi_{e_{n-i-1}} \rangle_{X, Y, \wedge^i V} = \langle f, x \rangle_X^* \langle y, g \rangle_Y \langle \phi_{e_{1,i}} \ldots \phi_{e_{n-i-1}} \rangle_{(\wedge V)^\vee}.
\]

Since all the pairings \( \langle , \rangle_X^*, \langle , \rangle_Y \) and \( \langle , \rangle_{(\wedge V)^\vee} \) are bilinear, \( \langle , \rangle_{X, Y, \wedge^i V} \) is bilinear and well-defined.

This pairing \( \langle , \rangle_{X, Y, \wedge^i V} \) is non-degenerate because \( \langle , \rangle_X^*, \langle , \rangle_Y \) and \( \langle , \rangle_{(\wedge V)^\vee} \) are non-degenerate. Note that \( \langle , \rangle_{X, Y, \wedge^i V} \) is \( W \)-invariant, which follows from Lemma 4.13(2), Lemma 4.13(2) and Lemma 4.14.

In order to see \( \langle , \rangle_{X, Y, \wedge^i V} \) restricted on \( (X^* \otimes Y \otimes (\wedge^i V)^\vee)^W \times (X \otimes \iota(Y)^* \otimes (\wedge^{n-i} V)^\vee)^W \) is still non-degenerate, we p...
We divide into two cases. Suppose \(|\{k_1, \ldots, k_{n-1}\} \cap \{k'_1, \ldots, k'_{i+1}\}| \geq 2\). Then by Lemma 4.8

\[
\langle T_i^j(f \otimes y \otimes \phi_{ek_1} \ldots \phi_{ek_{n-1}}), x \otimes g \otimes \phi_{ek'_1} \ldots \phi_{ek'_{i+1}} \rangle_{X,Y} = 0
\]

\[-1^n \langle f \otimes y \otimes \phi_{ek_1} \ldots \phi_{ek_{n-1}}, T_{n-i-1}^i(x \otimes g \otimes \phi_{ek'_1} \ldots \phi_{ek'_{i+1}}) \rangle_{X,Y}.
\]

For the second case, suppose \(|\{k_1, \ldots, k_{n-1}\} \cap \{k'_1, \ldots, k'_{i+1}\}| = 1\). Let \(k_p\) and \(k'_q\) be the unique pair of indices such that \(e_{kp} = e_{k'_q}\). Then

\[
\langle T_i^j(f \otimes y \otimes \phi_{ek_1} \ldots \phi_{ek_{n-1}}), x \otimes g \otimes \phi_{ek'_1} \ldots \phi_{ek'_{i+1}} \rangle_{X,Y} = (-1)^{n-i}\langle f \otimes y \otimes \phi_{ek_1} \ldots \phi_{ek_{n-1}}, T_{n-i-1}^i(x \otimes g \otimes \phi_{ek'_1} \ldots \phi_{ek'_{i+1}}) \rangle_{X,Y}.
\]

The first and last equalities follow from Lemma 4.8. The second equality follows from linear algebra facts. (We remark that the non-degeneracy of \(\langle \cdot, \cdot \rangle_{X,Y,\wedge^iV}\) is necessary to prove the equations.)

\[
(\ker D^1_i)^{\perp} = \text{im } D^2_{n-i-1}
\]

and

\[
(\text{im } D^1_{i-1})^{\perp} = \ker D^2_{n-i}.
\]

By Lemma 4.10, the pairing \(\langle \cdot, \cdot \rangle_{X,Y,\wedge^iV}\) first descends to

\[
\ker D^1_t \times ((X \otimes Y^*) \otimes (\wedge^{n-i}V)^W) \cong \text{im } D^2_{n-i-1}).
\]

Step 3: Descend the pairing to \(\text{Ext}_{\mathbb{H}}^2(X,Y) \times \text{Ext}_{\mathbb{H}}^{n-i-1}(X,Y^*) \rightarrow \mathbb{C}\).

We use implicitly the fact that \(X\) and \(Y\) are finite-dimensional for linear algebra results below.

For \(U \subset (X \otimes Y) \otimes (\wedge^{n-i}V)^W\) (resp. \(U \subset (X^* \otimes Y \otimes (\wedge^iV)^W\), let \(U^\perp\) to be the subspace of \((X^* \otimes Y \otimes (\wedge^iV)^W)\) orthogonal to \(U\) with respect to \(\langle \cdot, \cdot \rangle_{X,Y,\wedge^iV}\).

The following two equation follows from linear algebra facts. (We remark that the non-degeneracy of \(\langle \cdot, \cdot \rangle_{X,Y,\wedge^iV}\) is necessary to prove the equations.)

\[
(\ker D^1_t)^{\perp} = \text{im } D^2_{n-i-1}
\]

and

\[
(\text{im } D^1_{i-1})^{\perp} = \ker D^2_{n-i}.
\]

By Lemma 4.10, the pairing \(\langle \cdot, \cdot \rangle_{X,Y,\wedge^iV}\) first descends to

\[
\ker D^1_t \times ((X \otimes Y^*) \otimes (\wedge^{n-i}V)^W) \cong \text{im } D^2_{n-i-1}.
\]
Then by (4.17), the pairing $\langle \cdot, \cdot \rangle_{X,Y,\wedge^i V}$ further descends to
\[ \ker D^l_i / \im D^l_i \times \ker D^2_{n-i} / \im D^2_{n-i-1}. \]
By Proposition 3.7 and Lemma 4.10(2), we have a natural non-degenerate pairing on
\[ \Ext^n_{\mathbb{H}}(X,Y) \times \Ext^{n-i}_{\mathbb{H}}(X^*, \iota(Y)^*) \to \mathbb{C}. \]
\[ \square \]

**Remark 4.17.** We give few comments concerning the statement and the proof of Theorem 4.16.

1. If $X$ and $Y$ have the same central character, then $X^*$ and $\iota(Y^*)$ also have the same central character (see Example 4.20 below).
2. The use of the element $\overline{v}$ makes the computation in step 2 of the proof easier.
3. The choice of the duals is necessary to have a nice adjoint operator of $\overline{D}$ for the pairing $\langle \cdot, \cdot \rangle_{X,Y,\wedge^i V}$ in step 2. By Proposition 4.11 one also obtains a non-degenerate pairing
\[ \Ext^n_{\mathbb{H}}(X,Y) \times \Ext^{n-i}_{\mathbb{H}}(X^*, \iota(Y)^*) \to \mathbb{C}. \]
4. The Iwahori-Matsumoto involution is necessary to show the pairing $\langle \cdot, \cdot \rangle_{X,Y,\wedge^i V}$ is $W$-invariant and so non-degenerate in step 1.

**4.7. Examples.** We give few examples to illustrate how Theorem 4.16 is compatible with some known results.

**Example 4.18.** Let $R$ be the root system of type $A_2$. Let $\alpha_1, \alpha_2$ be a fixed choice of simple roots of $R$. Assume $R$ spans $V$. Let $\gamma = \alpha_1^\vee + \alpha_2^\vee$, which is the central character of the Steinberg module. There are four irreducible modules of $\mathbb{H}(A_2)$. We parametrize the four irreducible modules by their weights and denote the corresponding modules as $X(\gamma), X(\alpha_1^\vee - \alpha_1^\vee), X(\alpha_2^\vee - \alpha_2^\vee), X(-\gamma)$. For example, the weights of $X(\alpha_1^\vee - \alpha_1^\vee)$ are $\alpha_1^\vee$ and $-\alpha_1^\vee$. For $\gamma' = \gamma$ or $-\gamma$,
\[ X(\gamma')^* = X(\gamma'), \quad X(\gamma')^* = X(\gamma'), \quad \iota(X(\gamma')) = X(-\gamma'). \]
For other irreducible modules,
\[ X(\alpha_1^\vee, -\alpha_1^\vee)^* = X(\alpha_2^\vee, -\alpha_2^\vee), \quad X(\alpha_k^\vee, -\alpha_k^\vee)^* = X(\alpha_k^\vee, -\alpha_k^\vee) \quad \text{for } k = 1, 2 \]
\[ \iota(X(\alpha_k^\vee, -\alpha_k^\vee)) = X(\alpha_k^\vee, -\alpha_k^\vee) \quad \text{for } k = 1, 2. \]
We have $X^* \not\cong \iota(X^*)$ for any irreducible module $X$. Results in [29], which in particular compute Ext-groups of all above pairs agree with Theorem 4.16.

**Example 4.19.** Let $k_\alpha \neq 0$ for all $\alpha \in \Delta$. Let $X$ be the minimal parabolically induced with the central character 0. Then $X$ is irreducible. Then by [28, Theorem 5.2], $\Ext^n_{\mathbb{H}}(X,X) \cong \wedge^i V$. We also have $X^* \cong X^* \cong \iota(X) \cong X$. Then we have
\[ \dim \Ext^n_{\mathbb{H}}(X,X) = \binom{n}{i} = \binom{n}{n-i} = \dim \Ext^{n-i}_{\mathbb{H}}(X^*, \iota(X^*)). \]
Example 4.20. Let $k_\alpha \neq 0$ for all $\alpha \in \Delta$. Let $X$ be an irreducible principle series with a regular central character $W\gamma$ ($\gamma \in V^\vee$). Then $X^\ast$ has the central character $W\theta(\gamma) = -W\gamma$ (since $\theta(\gamma) = -w_0(\gamma)$) and $\iota(X^\ast)$ has the central character $-W\gamma$. By the irreducibility, we have $X^\ast \cong \iota(X^\ast)$.

On another hand, by the Frobenius reciprocity and $\gamma$ being regular,

$$\text{Ext}^i_{\mathbb{H}}(X, X) = \text{Ext}^i_{S(V)}(\mathbb{C}v_\gamma, \mathbb{C}v_\gamma) \cong \wedge^n V,$$

where $v_\gamma$ is a vector with the $S(V)$-weight $\gamma$. We again have

$$\dim \text{Ext}^i_{\mathbb{H}}(X, X) = \binom{n}{i} = \binom{n}{n-i} = \dim \text{Ext}^{n-i}_{\mathbb{H}}(X^\ast, \iota(X^\ast)).$$

Similar consideration can extend to examples of (not necessarily irreducible) minimal principle series with a regular central character.

4.8. Hermitian duals. As promised in Section 4.2, we will discuss the situation of the Hermitian anti-involutions usually encountered in the literature. Those anti-involutions are more natural because of their close relation to the unitary representations of $p$-adic groups [3]. An important assumption we have to make in this section is $k_\alpha \in \mathbb{R}$ for all $\alpha \in \Delta$. This is necessary for those anti-involutions to be well-defined.

Since $V = \mathbb{C} \otimes_\mathbb{R} V_0$, there is a natural complex conjugation on $V$. Denote by $\overline{\cdot}$ the complex conjugation of $v \in V$. Define an Hermitian involution $\tau$ determined by:

$$\tau(v) = \overline{v} \quad \text{for } v \in V, \quad \tau(w) = t_w \quad \text{for } w \in W.$$ The map $\tau$ induces a bijection, still denoted $\tau$, from $\mathbb{H}$-modules to $\mathbb{H}$-modules.

Define $\dagger : \mathbb{H} \to \mathbb{H}$ as $h^\dagger = \tau(h)^* = \tau(h^*)$ ($h \in \mathbb{H}$). Similarly, define $\circ : \mathbb{H} \to \mathbb{H}$ as $h^\circ = \tau(h)^* = \tau(h^*)$ ($h \in \mathbb{H}$). The maps $\dagger$ and $\circ$ are two Hermitian anti-involutions studied in [3]. Instead of linear functionals in Definition 4.3, we now need Hermitian functionals:

Definition 4.21. Let $X$ be an $\mathbb{H}$-module. A map $f : X \to \mathbb{C}$ is said to be a Hermitian functional if $f(\lambda x_1 + x_2) = \overline{f}(x_1) + f(x_2)$ for any $x_1, x_2 \in X$ and $\lambda \in \mathbb{C}$. The $\dagger$-dual of $X$, denoted by $X^\dagger$, is the space of Hermitian functionals of $X$ with the action of $\mathbb{H}$ determined

$$\langle h, f \rangle(x) = f(h^\dagger x) \quad \text{for any } x \in X.$$

We similarly define $\circ$-dual of $X$, denoted by $X^\circ$, by replacing $h^\dagger$ with $h^\circ$ in equation (4.18).

Using argument similar to Proposition 4.10, we have

$$\text{Ext}^i_{\mathbb{H}}(X, Y) \cong \text{Ext}^i_{\mathbb{H}}(\tau(X), \tau(Y))$$

for any $\mathbb{H}$-modules $X$ and $Y$. Then combining with Theorem 4.16, we have:

Theorem 4.22. Let $\mathbb{H}$ be the graded affine Hecke algebra associated to a root datum $\Pi = (R, V, R^\vee, V^\vee, \Delta)$ and a parameter function $k : \Delta \to \mathbb{R}$ (Definition 2.7). Let $n = \dim V$.

Then for any finite-dimensional $\mathbb{H}$-modules $X$ and $Y$, there exists a natural non-degenerate pairing

$$\text{Ext}^i_{\mathbb{H}}(X, Y) \times \text{Ext}^{n-i}_{\mathbb{H}}(X^\dagger, \iota(Y)^\circ) \to \mathbb{C}.$$
5. Extensions and the Langlands classification

In this section, we study few classes of parabolically induced modules with the goal to understand some extensions. The most important class is the induced modules in the Langlands classification, from which we compare central characters to obtain the information of Ext-groups. We also construct induced modules associated to discrete series and tempered modules and use it to study extensions. One may also compare the methods for studying extensions in the BGG category (see [13, Chapter 6]). As an application of our study, we compute the Ext-groups among discrete series in the next section.

In this and next sections, we make the following assumption: $R$ spans $V$. This assumption will make the discussion more convenient and it is not hard to formulate corresponding results without the assumption.

5.1. The Langlands classification. We review the Langlands classification for graded affine Hecke algebras in [10] (also see [19]). We shall not reproduce a proof here, but we point out that the proof for the Langlands classification is algebraic and does not rely on results of affine Hecke algebras or $p$-adic groups.

We first need a notation of parabolic subalgebra of $\mathbb{H}$.

Notation 5.1. For any subset $J$ of $\Delta$, define $V_J$ to be the complex subspace of $V$ spanned by vectors in $J$ and define $V_J^\vee$ be the dual space of $V_J$ lying in $V^\vee$. Let $R_J = V_J \cap R$ and let $R_J^\vee = V_J^\vee \cap R^\vee$. Let $W_J$ be the subgroup of $W$ generated by the elements $s_\alpha$ for $\alpha \in J$. Define

$$V_J^{\vee,\perp} = \{ v \in V : \langle v, u^\vee \rangle = 0 \quad \text{for all} \quad u \in V_J^\vee \} ,$$

and

$$V_J^\perp = \{ u^\vee \in V^\vee : \langle u, v^\vee \rangle = 0 \quad \text{for all} \quad u \in V_J \} .$$

For $J \subset \Delta$, let $W_J$ be the subgroup of $W$ generated by all $s_\alpha$ with $\alpha \in J$. Let $w_{0,J}$ be the longest element in $W_J$. Let $W^J$ be the set of minimal representatives in the cosets in $W/W_J$.

Let $J \subset \Delta$. Define $\mathbb{H}_J$ to be the subalgebra of $\mathbb{H}$ generated by all $v \in V$ and $t_w$ $(w \in W_J)$. We also define $\overline{\mathbb{H}}_J$ to be the subalgebra of $\overline{\mathbb{H}}$ generated by all $v \in V_J$ and $t_w$ $(w \in W_J)$. Note that $\mathbb{H}_J$ decomposes as

$$\mathbb{H}_J = \overline{\mathbb{H}}_J \otimes S(V_J^{\vee,\perp}).$$

Note $\mathbb{H}_J$ is the graded affine Hecke algebra associated to the root datum $(R_J, V, R_J^\vee, V^\vee, J)$ and $\overline{\mathbb{H}}_J$ is the graded affine Hecke algebra associated to the root datum $(R_J, V_J, R_J^\vee, V_J^\vee, J)$.

Let $W^J$ be the set of minimal representatives of $W$ in the cosets $W/W_J$.

We first describe the notion of parabolically induced modules. Denote by $\Xi$ the set of pairs of $(J, U)$ with $J \subset \Delta$ and irreducible $\mathbb{H}_J$-modules $U$. For $(J, U) \in \Xi$, $I(J, U)$ the induced module $\text{Ind}_{\mathbb{H}_J}^{\mathbb{H}} U$ from the $\mathbb{H}_J$-module $U$. Denote by $\text{Res}_{\mathbb{H}_J}$ the right adjoint functor of $\text{Ind}_{\mathbb{H}_J}^{\mathbb{H}}$. We also denote by $\text{Res}_{\overline{\mathbb{H}}_J}$ the restriction functor from $\mathbb{H}$-modules to $\overline{\mathbb{H}}_J$-modules.
Let $\nu \in V_j^+ \subset V_j^\vee$ and let $C_\nu$ be the corresponding one-dimensional $S(V_j^\vee, \nu)$-module. For any $\alpha \in \Delta$, denote by $\omega_\alpha^\nu \in V_0^\vee$ the fundamental coweight corresponding to $\alpha$ i.e. for $\beta \in \Delta$,
\[
\langle \beta, \omega_\alpha^\nu \rangle = \omega_\alpha^\nu(\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}.
\]
We define similarly for $\omega_\alpha \in V_0$ for $\alpha \in \Delta$.

**Definition 5.2.** An $\mathbb{H}$-module $X$ is said to be **tempered** if any weight $\gamma$ of $X$ is of the form:
\[
\text{Re} \gamma = \sum_{\alpha \in \Delta} a_\alpha \alpha^\vee, \quad a_\alpha \leq 0.
\]
Equivalently, an $\mathbb{H}$-module $X$ is tempered if and only if $\langle \omega_\alpha, \gamma \rangle \leq 0$ for all $\alpha \in \Delta$ and for all weight $\gamma$ of $X$. An $\mathbb{H}$-module $X$ is said to be a **discrete series** if any weight $\gamma$ of $X$ is of the form:
\[
\text{Re} \gamma = \sum_{\alpha \in \Delta} a_\alpha \alpha^\vee, \quad a_\alpha < 0.
\]
Equivalently, $X$ is a discrete series if and only if $\langle \omega_\alpha, \text{Re} \gamma \rangle < 0$ for all $\alpha \in \Delta$ and for all weight $\gamma$ of $X$. In particular, an $\mathbb{H}$-discrete series is tempered.

For any $\gamma \in V^\vee$, we can uniquely write $\text{Re} \gamma$ as the form (see [15 Ch VIII Lemma 8.56]):
\[
\text{Re} \gamma = \sum_{\alpha \in J} a_\alpha \alpha^\vee + \sum_{\beta \in \Delta \setminus J} b_\beta \omega_\beta^\nu \quad \text{with } a_\alpha \leq 0, \ b_\beta > 0.
\]

Denote
\[
\gamma_0 = \sum_{\beta \in \Delta \setminus J} b_\beta \omega_\beta^\nu \in V_0^\vee.
\]

**Definition 5.3.** We say a pair $(J, U) \in \Xi$ is a **Langlands classification parameter** if $U = \mathbb{U} \otimes C_\nu$ (as $\mathbb{H}_J \cong \mathbb{H}_J \otimes S(V_j^\vee)$-modules) for some $\mathbb{H}_J$-tempered module $\mathbb{U}$ and $\nu \in V_j^+$ with $\langle \text{Re} \nu, \omega_\alpha \rangle > 0$ for all $\alpha \in \Delta \setminus J$. Recall that $\omega_\alpha$ is the fundamental weight associated to $\alpha$. Set $\lambda(J, U) = \nu$. Denote by $\Xi_L$ the set of all Langlands classification parameters. Hence, if $(J, U) \in \Xi_L$, any weight $\gamma$ of $U$ has the form
\[
\text{Re} \gamma = \sum_{\alpha \in J} a_\alpha \alpha^\vee + \sum_{\beta \in \Delta \setminus J} b_\beta \omega_\beta^\nu,
\]
with $a_\alpha \leq 0$ and $b_\beta > 0$. Note that $\lambda(J, U) = \gamma_0$, where $\lambda(J, U)$ is independent of the choice of the weights $\gamma$ of $U$.

We define a similar but stronger notation later. We say a pair $(J, U)$ is a **strong Langlands classification parameter** if $U = \mathbb{U} \otimes C_\nu$ as $\mathbb{H}_J \cong \mathbb{H}_J \otimes S(V_j^\vee)$-modules for some $\mathbb{H}_J$-discrete series $\mathbb{U}$ and $\nu \in V_j^+$ with $\langle \omega_\alpha, \text{Re} \nu \rangle > 0$ for all $\alpha \in \Delta \setminus J$. Denote by $\Xi_{L,ds}$ the set of all strong Langlands classification parameters.

**Theorem 5.4.** **(Langlands classification)** [10]

1. For any irreducible $\mathbb{H}$-module $X$, there exists $(J, U) \in \Xi_L$ such that $X$ is isomorphic to the unique irreducible quotient of $I(J, U)$.
2. Let $(J, U), (J_1, U_1) \in \Xi_L$. If the unique simple quotients of $I(J, U)$ and $I(J_1, U_1)$ are isomorphic, then $J = J_1$ and $U \cong U_1$ as $\mathbb{H}_J$-modules.
From Theorem 5.4, each irreducible $\mathfrak{h}$-module can be associated to a pair in $\Xi_L$. We have the following terminology:

**Definition 5.5.** Let $X$ be an irreducible $\mathfrak{h}$-module. Let $(J, U) \in \Xi_L$ be such that $X$ is the unique quotient of $I(J, U)$. We call $(J, U)$ the **Langlands classification parameter** for $X$.

There is an useful information about the weights in the Langlands classification. We need the following notation for comparing weights:

**Definition 5.6.** Let $\gamma, \gamma' \in V_0^\vee$. We write $\gamma \leq \gamma'$ if $\langle \omega_\alpha, \gamma \rangle \leq \langle \omega_\alpha, \gamma' \rangle$ for all $\alpha \in \Delta$. We write $\gamma < \gamma'$ if $\gamma \leq \gamma'$ and $\langle \omega_\alpha, \gamma \rangle < \langle \omega_\alpha, \gamma' \rangle$ for some $\alpha \in \Delta$.

The following lemma related the dual and the Langlands classification parameter will be useful later.

**Lemma 5.7.** Let $X$ be an irreducible $\mathfrak{h}$-module. Let $(J, U) \in \Xi_L$ be the Langlands parameter for $X$ and let $(J_*, U_*) \in \Xi_L$ be the Langlands classification parameter of $X^*$. Then $\lambda(J_*, U_*) = \theta(\lambda(J, U))$.

**Proof.** Following from the construction in the Langlands classification (see [19] (2.12)), also see Proposition 5.9 below), $\lambda(J, U)$ (resp. $\lambda(J_*, U_*)$) is the maximal element in the set

$$\{ \gamma_0 : \gamma \text{ is a weight of } X \} \quad \text{(resp. } \{ \gamma_0 : \gamma \text{ is a weight of } X^* \}).$$

On the other hand, by Lemma 5.5, $\gamma$ is a weight of $X$ if and only if $\theta(\gamma)$ is a weight of $X^*$. Hence $\theta(\lambda(J_*, U_*)) = \lambda(J, U)$.

Proposition 5.9 below is in the proof of the Langlands classification [19]. We reproduce the proof since the statement is crucial for our argument later.

**Lemma 5.8.** (Lemma of Langlands) [19] Ch VIII Lemma 8.59] Let $\gamma, \gamma' \in V_0^\vee$. If $\gamma \leq \gamma'$, then $\gamma_0 \leq \gamma'_0$.

**Proposition 5.9.** Let $(J, U) \in \Xi_L$. Let $M$ be a composition factor of $I(J, U)$. Let $(J_1, U_1) \in \Xi_L$ be the Langlands classification parameter for $M$. Suppose $M$ is not isomorphic to the unique simple quotient of $I(J, U)$ (i.e. $(J, U) \neq (J_1, U_1)$). Then $\lambda(J_1, U_1) < \lambda(J, U)$.

**Proof.** Any weight of $M$ is of the form $w(\mu)$ for some $w \in W^J \setminus \{1\}$ and some weight $\mu$ of $U$ ([19 Theorem 6.4]). Write $\text{Re}\gamma$ in the form

$$\text{Re}\mu = \sum_{\alpha \in J} a_\alpha \alpha^\vee + \lambda(J, U) \quad \text{with } a_\alpha \leq 0.$$ 

Then

$$\text{Re}(w(\mu)) = \sum_{\alpha \in J} a_\alpha w(\alpha^\vee) + w(\lambda(J, U)).$$

Since $-w(\alpha^\vee) < 0$ for all $\alpha \in J$, we have $\text{Re}(w(\mu)) \leq w(\lambda(J, U))$. For $w \neq 1$, we also have $w(\omega_\alpha) \leq \omega_\alpha$ for all $\alpha \notin J$ and $w(\omega_\alpha) < \omega_\alpha$ for some $\alpha \notin J$. Hence, we also have
Re(\(w(\mu)\)) \leq w(\lambda(U)) < \lambda(J, U) \) \) and so \(\text{Re}(w(\mu))_0 < \lambda(J, U)\). Thus for any weight \(\gamma\) of \(M\), \(\gamma_0 < \lambda(J, U)\) by Lemma 5.8.

On another hand, there is a surjective map from \(I(J_1, U_1)\) to \(M\) by definition. Then by the Frobenius reciprocity, \(\text{Hom}_{\mathbb{H}_J}(U_1, \text{Res}_{\mathbb{H}_J} M) \neq 0\). Hence \(\lambda(J_1, U_1) = \gamma_0\) for some weight \(\gamma\) of \(M\). With the discussion in the previous paragraph, we have \(\lambda(J_1, U_1) < \lambda(U)\).

\[
\]

Example 5.10. Let \(R\) be of type \(G_2\) and let \(k = 1\). Let \(\alpha, \beta\) be the simple roots of \(R\) with \(\langle \alpha, \beta' \rangle = -1\) and \(\langle \beta, \alpha' \rangle = -3\). We consider modules of the central character \(\alpha' + \beta'\) and the possible weights are
\[
\pm(\alpha' + \beta'), \pm(\alpha' + 2\beta'), \pm\beta'.
\]
Note that \(\alpha' + \beta' = -\frac{1}{2}\beta' + \frac{1}{2}(2\alpha' + 3\beta').\) Let \(J = \{\beta\}\) and let \(St\) be the Steinberg module of \(\mathbb{H}_J\) and let \(\nu = \frac{1}{2}(2\alpha' + 3\beta').\) The \(\mathbb{H}_J\)-module \(St \otimes \mathbb{C}_\nu\) has the weight \(\alpha' + \beta'\) and \((J, St \otimes \mathbb{C}_\nu)\) is a Langlands classification parameter. Let \(I = \text{Ind}_{\mathbb{H}_J}(St, \mathbb{C}_\nu)\). It is known that \(Y\) has three composition factors. Denote by \(Y\) for the simple quotient of \(I\), denote by \(DS\) for the composition factor of \(I\) being a discrete series and denote by \(Z\) for the remaining composition factor of \(I\). We have

1. The weight of \(Y\) is \(\gamma^1 := \alpha' + \beta'\).
2. The weights of \(DS\) are \(\gamma^2 := -\alpha' - 2\beta'\) with multiplicity 2 and \(\gamma^3 = -\alpha' - \beta'\).
3. The weights of \(Z\) is \(\gamma^4 := \beta'\) and \(\gamma^5 := -\beta'\).

Then \(\gamma^1 = \frac{1}{2}(2\alpha' + 3\beta'), \gamma^2 = 0, \gamma^3 = 0, \gamma^4 = \frac{1}{2}(\alpha' + 2\beta')\) and \(\gamma^5 = 0\).

5.2. Inner product on \(V_0\). Since \(V_0\) is a real representation of \(W\), there exists a \(W\)-isomorphism, denoted \(\eta\) from \(V_0\) to \(V_0'\). Define a \(W\)-invariant bilinear form on \(V_0\) by \((v_1, v_2) = \eta(v_2)(v_1)\). Since \(V_0'\) is irreducible, there exists a unique, up to a scalar, \(W\)-invariant bilinear form on \(V_0\). Hence \((.,.)\) is also symmetric. By the uniqueness, we also have the \(W\)-invariant bilinear form \((.,.)\) to be positive-definite. For \(\gamma \in V_0\), denote by \(||\gamma|| := \sqrt{\langle \gamma, \gamma \rangle}\) the length of \(\gamma\).

Furthermore, for each \(\alpha \in \Delta\), \(R\alpha\) and \(R\alpha'\) are the \((-1)\)-eigenspaces of \(s_\alpha\) on \(V_0\) and \(V_0'\) respectively. Hence \(\eta(\alpha') \in R\alpha\) for each \(\alpha \in \Delta\). For \(\alpha, \beta \in \Delta\) with \(\beta \neq \alpha\), we also have
\[
(\eta(\omega_\alpha'), \beta) = (\beta, \eta(\omega_\alpha')) = \langle \beta, \omega_\alpha \rangle = 0.
\]
Hence \(\eta(\omega_\alpha') \in \mathbb{R}\omega_\alpha\). Thus the setting of Langlands classification in Section 5.1 can be naturally reformulated by the notations for \(V_0\) (e.g. \(\alpha, \omega_\alpha\)) if we identify \(V_0\) with \(V_0'\) via the isomorphism \(\eta\).

We also extend \((.,.)\) linearly to a symmetric bilinear form on \(V\). We remark that in the Langlands classification, we mainly consider the real part of weights of a module. It will be important for \((.,.)\) to be an inner product on \(V_0\) for comparing the length of weights later (see Definition 5.11 below).
5.3. Induced module for discrete series. We continue to assume $R$ spans $V$. Let $n = \dim V = |\Delta|$. We keep using the notation in Section 5.2 (e.g., the bilinear form $(\cdot, \cdot)$ on $V$). Our goal is to construct a maximal parabolically induced module containing certain discrete series. We introduce the following notations to keep track useful information.

\textbf{Definition 5.11.} Let $X$ be an irreducible $\mathbb{H}$-discrete series. Let $\Delta_{n-1}$ be the set containing all the subset of $\Delta$ of cardinality $n - 1$. Let $\mathcal{W}(X)$ be the set of weights of $X$. Define a function $\Phi : \Delta_{n-1} \times \mathcal{W}(X) \to \mathbb{C}$ by

$$\Phi(J, \gamma) = -\frac{(\text{Re} \gamma, \omega_\beta)}{|\omega_\beta|^2} = \frac{|(\text{Re} \gamma, \omega_\beta)|}{|\omega_\beta|^2},$$

where $\beta$ is the unique element in $\Delta \setminus J$. Denote by $(J_X, \gamma_X) \in \Delta_{n-1} \times \mathcal{W}(X)$ to be the pair such that $\Phi(J_X, \gamma_X)$ attains the minimum value among all pairs in $\mathcal{W}(X)$. Denote by $\beta_X$ the unique element in $\Delta \setminus J_X$. Denote

$$L_{ds}(X) = \Phi(J_X, \gamma_X),$$

$$\lambda_{ds}(X) = \frac{\omega_{\beta_X}}{|\omega_{\beta_X}|},$$

(where $ds$ stands for discrete series).

\textbf{Example 5.12.} We keep using the notation in Example 5.10. By fixing a choice of $\eta$ and $(\cdot, \cdot)$, we have the following equations:

$$(\alpha, \alpha) = 2, \quad (\alpha, \beta) = -3, \quad (\beta, \beta) = 6.$$ We have $\omega_\alpha = 2\alpha + \beta$ and $\omega_\beta = 3\alpha + 2\beta$. The $|\omega_\alpha|^2 = 2$ and $|\omega_\beta|^2 = 6$. We consider the discrete series $DS$ which has weights $\gamma^2, \gamma^3$.

$$\Phi(\{\alpha\}, \gamma^2) = \frac{2}{\sqrt{6}}, \quad \Phi(\{\beta\}, \gamma^2) = \frac{1}{\sqrt{2}},$$

$$\Phi(\{\alpha\}, \gamma^3) = \frac{1}{\sqrt{6}}, \quad \Phi(\{\beta\}, \gamma^3) = \frac{1}{\sqrt{2}}.$$ Thus $J_{DS} = \{\alpha\}$. $L_{ds}(DS) = \frac{1}{\sqrt{6}}$ and $\lambda_{ds}(DS) = \frac{1}{\beta_\beta}$. We also see $\eta(\lambda_{ds}(DS)) = \frac{1}{6}(3\alpha^\gamma + 6\beta^\gamma)$.

Before constructing an induced module for a discrete series, we mention useful results about duals (Proposition 5.13 and Lemma 5.14). Recall that $\ast$ and $\bullet$ are defined in Section 4.2.

\textbf{Proposition 5.13.} [Corollary 1.4] For $(J, U) \in \Xi$, $I(J, U)^\ast$ is isomorphic to $I(I(J, U^\ast),)$, where $\ast$ is the corresponding $\ast$-operation for $\mathbb{H}_J$-modules.

\textbf{Lemma 5.14.} Let $X$ be an irreducible $\mathbb{H}$-discrete series. Then $X^\ast, \theta(X)$ and $X^\bullet$ are also discrete series.

\textbf{Proof.} By definitions, $\gamma$ is a weight of $X$ if and only if $\theta(\gamma)$ is a weight of $\theta(X)$. Since $\theta(\omega_\alpha) = \omega_{\theta(\alpha)}$, we have $\theta(X)$ is also a discrete series. For $X^\bullet$, $X$ and $X^\ast$ have the same weights and so $X^\bullet$ is also a discrete series. By Lemma 4.3, $X^\ast \cong \theta(X^\ast)$ is also a discrete series.
Proof. Let \( J_X \) be as in Definition \[5.14\] We shall construct an \( \mathbb{H}_{J_X} \)-module with some desired properties. We consider the \( \mathbb{H}_{J_X} \)-modules \( \text{Res}_{J_X} \cdot Y \), which can be written as the direct sum of \( \mathbb{H}_{J_X} \)-modules with distinct characters. Let \( Y \) be an indecomposable \( \mathbb{H}_{J_X} \)-module of \( X \) with the central character \( W_J \). We want to show the composition factors of the \( \mathbb{H}_{J_X} \)-module \( \text{Res}_{J_X} \cdot Y \) are discrete series. Let \( \gamma' \) be a weight of \( Y \). Since \( Y \) has the \( W_J \)-central character \( W_J \gamma_X \),

\[
\text{Res}_J \gamma' = \sum_{\alpha \in J_X} a_\alpha \alpha - L_{ds}(X) \frac{\omega_{\beta_X}}{||\omega_{\beta_X}||} = \sum_{\alpha \in J_X} a_\alpha \alpha - \lambda_{ds}(X), \quad \text{for some } a_\alpha \in \mathbb{R}.
\]

Proving (1) is equivalent to proving that all \( a_\alpha < 0 \). Let \( \alpha' \in J_X \). By using \( \langle \gamma', \omega_{\alpha'} \rangle = a_{\alpha'} - L_{ds}(X) \frac{\omega_{\beta_X}}{||\omega_{\beta_X}||} \), we have

\[
\Phi(\Delta \setminus \{\alpha'\}, \gamma') = -\frac{a_{\alpha'}}{||\omega_{\alpha'}||} + L_{ds}(X) \frac{\langle \omega_{\beta_X}, \omega_{\alpha'} \rangle}{||\omega_{\beta_X}|| ||\omega_{\alpha'}||}
\]

\[
< -\frac{a_{\alpha'}}{||\omega_{\alpha'}||} + L_{ds}(X)
\]

Here the second line also uses the fact that \( \langle \omega_{\beta_X}, \omega_{\alpha'} \rangle \geq 0 \) (\[15\] Lemma 8.57) and \( (\ldots, \ldots) \) is an inner product on \( V_0 \). By our choice of \( (J_X, \gamma_X) \), \( a_\alpha < 0 \) for all \( \alpha \in J_X \). We now let \( Y' \) be an irreducible \( \mathbb{H}_{J_X} \)-submodule of \( Y \) and then we have \( Y' = U_X \otimes \mathbb{C}_{-\lambda_{ds}(X)} \) as \( \mathbb{H}_{J_X} \cong \mathbb{H}_{J_X} \otimes S(V_J^{1,1}) \)-module, where \( U_X \) is an irreducible \( \mathbb{H}_{J_X} \)-discrete series. Then by construction and Frobenius reciprocity, one of the quotients of the parabolically induced module \( I(J_X, U_X) \) is \( X \). This shows (1).

For (2), by Proposition \[6.13\] and Lemma \[5.14\] \( I(J_X, U_X)^* = I(J_X, U_X^{J_X}) \). Note that \( U_X^{J_X} = \mathcal{U} \otimes \mathbb{C}_{-\lambda_{ds}(X)} \) for some \( \mathbb{H}_J \)-discrete series \( \mathcal{U} \) (by Lemma \[5.14\]). Hence \( (J_X, U_X^{J_X}) \in \Xi_{L,ds} \) with \( \lambda(J_X, U_X^{J_X}) = \lambda_{ds}(X) \) as desired. 

\[\square\]

**Example 5.16.** We continue to use the notations in Example \[5.10\] and Example \[5.12\]. We consider the discrete series \( DS \). From Example \[5.12\] we see that \( 3 = -\frac{1}{2} \alpha^\vee - \eta(\lambda_{ds}(DS)) \). Then there is a surjection from \( I(\{\alpha\}, St \otimes \mathbb{C}_{-\lambda_{ds}(DS)}) \) to \( DS \) (by Frobenius reciprocity), where \( St \) is the Steinberg module of \( \mathbb{H}_\alpha \). This constructs the parabolically induced module for \( DS \) as the one in Proposition \[5.15\].
Recall that the Iwahori-Matsumoto involution $\iota$ is defined in Section 4.5. We have the following result describing the structure of parabolically induced modules which is proven by considering central characters and using Proposition 2.7.

**Proposition 5.17.** Let $X$ be an irreducible $\mathbb{H}$-discrete series. Let $(J, U) \in \Xi_L$ with $J \neq \Delta$. We have the following properties:

1. $\mathrm{Ext}^i_{\mathbb{H}}(I(J, U)^*, \iota(X)) = 0$ for all $i$.
2. Suppose $0 \neq \lambda(J, U) < \lambda_{ds}(X)$ or $0 \neq \theta(\lambda(J, U)) < \lambda_{ds}(X)$ (see Definition 5.3 for $\lambda(J, U)$ and Definition 5.11 for $\lambda_{ds}(X)$). Then for all $i$

$$\mathrm{Ext}^i_{\mathbb{H}}(I(J, U), \iota(X)) = 0.$$  

**Proof.** We consider (1). By Proposition 5.13, $I(J, U)^* = I(J, U^{* \iota})$ (where $*_{\iota}$ is the corresponding $*$-operation for $\mathbb{H}_{\iota}$) and the real part of the weights of $U^{*_{\iota}}$ is of the form $\sum_{a \in J} \theta_{J, U} a - \lambda(J, U)$ for some $a \leq 0$. Suppose $\mathrm{Ext}^i_{\mathbb{H}}(I(J, U)^*, \iota(X)) \neq 0$ for some $i$. Then by Frobenius reciprocity, we have $\mathrm{Ext}^i_{\mathbb{H}}(I(J, U)^*, \iota(X)) = \mathrm{Ext}^i_{\mathbb{H}}(I(J, U)^*, \iota(X)) = 0$ for some $i$. Then some composition factors of $\iota(X)$ and $U^{*_{\iota}}$ have the same $\mathbb{H}_{\iota}$-central characters by Proposition 2.7. That implies $\iota(X)$ has a weight $\gamma$ such that $\Re \gamma = \sum_{a \in J} \alpha_{a, \alpha} - \lambda(J, U)$ for some $a_{\alpha} \in \mathbb{R}$. Then for $\beta \notin J$, $(\Re \gamma, \omega_{\beta}) = -(\lambda(J, U), \omega_{\beta}) < 0$. However, this contradicts the definition of $\iota$ and $X$ being a discrete series.

(2) is a special case of Lemma 5.18 below (whose proof does not depend on this proposition). (In more detail, in the notation of Lemma 5.18 we choose $X_1 = X_2 = X$.)

We need an improved version of (2) of the above proposition for a better control in comparing the Ext-groups of two discrete series.

**Lemma 5.18.** Let $X_1$ and $X_2$ be irreducible $\mathbb{H}$-discrete series. Assume $L_{ds}(X_1) \leq L_{ds}(X_2)$. Let $(J, U) \in \Xi_L$ with $J \neq \Delta$. Suppose $\lambda(J, U) \leq \lambda_{ds}(X_1)$ (resp. $\theta(\lambda(J, U)) \leq \lambda_{ds}(X_1)$)

Then at least one of the following holds:

1. for all $i$

$$\mathrm{Ext}^i_{\mathbb{H}}(I(J, U), \iota(X_2)) = 0,$$

2. $\lambda(J, U) = \lambda_{ds}(X_1)$ and $J = J_{X_1}$ (resp. $\theta(\lambda(J, U)) = \lambda_{ds}(X_1)$ and $\theta(J) = J_{X_1}$).

For any indecomposable $\mathbb{H}_{J}$-submodule $Z$ of $\mathrm{Res}_{\mathbb{H}_{J}} \iota(X_2)$, if $\mathrm{Ext}^i_{\mathbb{H}_{J}}(U, Z) \neq 0$ for some $i$, then all the composition factors of $\iota(\mathrm{Res}_{\mathbb{H}_{J}} Z)$ are $\mathbb{H}_{J}$-discrete series.

**Proof.** Suppose (1) is false to obtain (2). Then by the Frobenius reciprocity,

$$\mathrm{Ext}^i_{\mathbb{H}_{J}}(U, \mathrm{Res}_{\mathbb{H}_{J}} \iota(X_2)) \neq 0,$$

for some $i$. This implies $\iota(X_2)$ contains a weight whose real part is of the form $\sum_{a \in J} a_{a, \alpha} + \lambda(J, U)$ for some $a_{\alpha} \in \mathbb{R}$. Then, by the definition of $\iota$, $X_2$ contains a weight $\gamma$ such that
Lemma 5.19. Let $X$ be an $\mathbb{H}$-tempered module, but not a discrete series. Then there exists $(J, U) \in \mathcal{X}$ with the following properties:

1. $\mathbb{D}$ Lemma 5.1.1| $X$ is an irreducible (not necessarily unique) subquotient of $I(J, U)$ and $U = \overline{U} \otimes \mathbb{C}_\nu$ for an $\mathbb{H}_J$-discrete series $\overline{U}$ and $\nu \in V_J^+$ with $\text{Re} \nu = 0$, and

\begin{align}
\text{Re} \gamma &= -\sum_{\alpha \in J} a_\alpha \alpha - \lambda(J, U). \text{ Then, for any } \alpha \notin J, \\
(5.22) &\quad (-\text{Re} \gamma, \omega_\alpha) = (\lambda(J, U), \omega_\alpha) \\
(5.23) &\quad \leq (\lambda_{ds}(X_1), \omega_\alpha) \quad (\text{by } \lambda(J_{X_1}, U_1) \leq \lambda_{ds}(X_1)) \\
(5.24) &\quad \leq L_{ds}(X_1) \frac{(\omega_{\beta_{X_1}}, \omega_\alpha)}{||\omega_{\beta_{X_1}}||} \quad (\text{by Definition 5.11}) \\
(5.25) &\quad \leq L_{ds}(X_1)||\omega_\alpha|| \\
(5.26) &\quad \leq L_{ds}(X_2)||\omega_\alpha||
\end{align}

By the definition of $L_{ds}(X_2)$, all the inequalities become equalities. Then by the fourth line of the above computation, $\omega_\alpha = \omega_{\beta_{X_1}}$ and so $\Delta \setminus J = \{\beta_{X_1}\}$. Hence $J = J_{X_1}$. By (5.23), we then have $\lambda(J, U) = L_{ds}(X_1)||\omega_{\beta_{X_1}}|| = \lambda_{ds}(X_1)$. This proves the first assertion of (2).

For the second assertion of (2), let $\overline{Z}$ be an indecomposable $\mathbb{H}_J$-submodule of $\text{Res}_{\mathbb{H}_J} \iota(X_2)$ with $\text{Ext}_{\mathbb{H}_J}(U, Z) \neq 0$ for some $i$. Then $U$ and $Z$ have the same $\mathbb{H}_J$-central character by Proposition 2.2. With the definition of $\iota$, for any weight $\gamma'$ of $\iota(Z)$, $\gamma$ satisfies

$$
\text{Re} \gamma' = \sum_{\alpha \in J} a'_\alpha \alpha - \lambda(J, U) = \sum_{\alpha \in J} a'_\alpha \alpha - \lambda_{ds}(X_1),
$$

for some $a'_\alpha \in \mathbb{R}$. To show that any composition factors of $\iota(\text{Res}_{\mathbb{H}_J} Z)$ are discrete series, it suffices to show $a'_\alpha < 0$. To this end, let $\alpha \in J$ and we consider,

$$
(-\text{Re} \gamma', \omega_\alpha) = -a'_\alpha + (\lambda_{ds}(X), \omega_\alpha) \\
= -a'_\alpha + L_{ds}(X_1) \frac{(\omega_{\beta_{X_1}}, \omega_\alpha)}{||\omega_{\beta_{X_1}}||} \quad (\text{by Definition 5.11}) \\
< -a'_\alpha + L_{ds}(X_1)||\omega_\alpha|| \quad (\text{because } \omega_\alpha \neq \omega_{\beta_{X_1}})
$$

On the other hand, note that $\gamma'$ is a weight of $X_2$ (since $\iota$ commutes with $\text{Res}_{\mathbb{H}_J}$). By the definition of $L_{ds}$ in Definition 5.11

$$
L_{ds}(X_2)||\omega_\alpha|| \leq (-\text{Re} \gamma', \omega_\alpha).
$$

Combining the equations and using $L_{ds}(X_1) \leq L_{ds}(X_2)$, we have $a'_\alpha < 0$ as desired. This proves (3).

We now comment on the $\theta$-case. Again suppose (2) is false. Th equation (5.23) above will become $\lambda(J, U, \omega_\alpha) \leq (\theta(\lambda_{ds}(X_1)), \omega_\alpha)$. Hence we will obtain $J = \theta(J_{X_1})$. Then the similar line of argument in the non-$\theta$ case gives the assertion for $\theta$-case.

\[\square\]

5.4. Tempered modules. It is known that tempered modules can be parabolically induced from a discrete series twisted by an unitary character. We study those induced modules in this section.

Lemma 5.19. Let $X$ be an $\mathbb{H}$-tempered module, but not a discrete series. Then there exists $(J, U) \in \mathcal{X}$ with the following properties:
(2) \( J \neq \Delta \), and
(3) any composition factor of \( I(J,U) \) is tempered, and
(4) for any discrete series \( X' \),
\[
\Ext^i_{\mathbb{H}}(I(J,U), \iota(X')) = 0.
\]

Proof. For each weight \( \gamma \) of \( X \), let \( J(\gamma) \) be a subset of \( \Delta \) such that
\[
\Re \gamma = \sum_{\alpha \in J} a_{\alpha} \alpha, \quad \text{with } a_{\alpha} < 0.
\]
Set \( \gamma_X \) to be the weight of \( X \) such that \( J(\gamma_X) \) has the minimal cardinality. Set \( J = J(\gamma_X) \).
Since \( X \) is not a discrete series, \( J \neq \Delta \). Let \( Y \) be an indecomposable \( \mathbb{H}_J \)-module of \( \Res_{\mathbb{H}_J} X \) with the \( \mathbb{H}_J \)-central character \( W_{\mathbb{H}_J} \gamma' \). We claim that any composition factor of \( \Res_{\mathbb{H}_J} Y \) is a discrete series. Let \( \gamma' \) be a weight of \( Y \) and write
\[
\Re \gamma' = \sum_{\alpha \in J} a_{\alpha}' \alpha \quad \text{for some } a_{\alpha}' \in \mathbb{R}.
\]
Since \( Y \) has a central character \( W_{\mathbb{H}_J} \gamma' \), \( a_{\alpha}' = 0 \) for all \( \alpha \notin J \). It remains to show that \( a_{\alpha}' < 0 \) for all \( \alpha \in J \). However, by the definition of a tempered module, \( a_{\alpha}' = \langle \Re \gamma', \omega_\alpha \rangle \leq 0 \) for all \( \alpha \in J \). If \( a_{\alpha}' = 0 \) for some \( \alpha \in J \), this will contradict our choice of \( \gamma' \). This concludes that \( a_{\alpha} < 0 \) for all \( \alpha \in J \). Then we choose an irreducible \( \mathbb{H}_J \)-submodule \( U \) of \( Y \) and write
\[
\Re \gamma_X' = \sum_{\alpha \in J} a_{\alpha,U} w(\alpha) \quad \text{for some } a_{\alpha,U} \leq 0.
\]
Since \( w(\alpha) > 0 \) for any \( \alpha \in J \), \( \langle \Re(\gamma_X'), \omega_\alpha \rangle \leq 0 \) for all \( \alpha \in \Delta \). This proves (3).

We now prove (4) and continue to use the notations for (1). Let \( X' \) be a discrete series. Suppose the assertion is false to obtain a contradiction. Then by the Frobenius reciprocity,
\[
\Ext^i_{\mathbb{H}_J}(U, \Res_{\mathbb{H}_J} \iota(X')) \neq 0.
\]
Then by considering the \( W_J \)-central character of \( U \) and Proposition 2.2, \( \iota(X') \) has a weight \( \gamma_{X'} \), such that
\[
\Re \gamma_{X'} = \sum_{\alpha \in J} a_{\alpha,X'} \alpha.
\]
Since we assume \( X \) is not a discrete series, \( J \neq \Delta \). Then for \( \alpha \notin J \), \( \langle \Re \gamma_{X'}, \omega_\alpha \rangle = 0 \), which contradicts to \( X' \) being a discrete series.

\[ \square \]

6. Extensions of discrete series

In this section, we compute the Ext-groups of discrete series. More precisely, we show that all the higher Ext-groups among discrete series vanish. The result for affine Hecke algebra is proven by Opdam-Solleveld in [26, Theorem 3.5] using the Schwartz algebra
completion of an affine Hecke algebra. With the belief of the result from affine Hecke algebras, we take an algebraic approach for computing Ext-groups of discrete series for graded affine Hecke algebras. The results also cover complex parameter cases and non-crystallographic cases. We also hope some techniques can be extended for computing Ext-groups of more modules in the future (see an example in Section 5.2).

We briefly outline the strategy of our proof (also see the paragraph before Lemma 6.5). Instead of computing Ext-groups of discrete series directly, we first compute the Ext-groups for a tempered module and the Iwahori-Matsumoto dual of a discrete series, which has the advantage that the $\operatorname{Ext}^0_{H}$-group vanishes. The next step is to construct a parabolically induced modules for tempered modules (from Section 5). Then the parabolic induction allows us to make use of the knowledge from lower ranks (via Frobenius reciprocity and induction hypothesis), but in return, we have to deal with the Ext-groups for the composition factors in the related parabolically induced modules. Fortunately, those composition factors are well controlled by the Langlands classification and are manageable from the study in Section 5. Then the standard modules associated to those composition factors and the parabolic induction again gives some new information via the Frobenius reciprocity. This eventually leads to the computation of the Ext-groups of a tempered module and the Iwahori-Matsumoto dual of a discrete series. We finally apply the duality result (Theorem 4.16) to recover the Ext-groups between a tempered module and a discrete series. Since a discrete series is tempered, we obtain the extensions between discrete series.

6.1. Extensions of discrete series. In this section, we use the notation in Section 5.2 (i.e. identify $V^\vee$ with $V$ and use the bilinear form $(.,.)$). We also continue to assume $R$ spans $V$.

**Theorem 6.1.** Let $H$ be the graded affine Hecke algebra associated to a root datum $\Pi = (R, V, R^\vee, V^\vee, \Delta)$ and a parameter function $k : \Delta \to \mathbb{C}$ (Definition 2.1). Assume $R$ spans $V$. Let $X_1$ be an irreducible tempered module and let $X_2$ be an irreducible discrete series. Then

$$\operatorname{Ext}^i_{H}(X_1, X_2) = \begin{cases} \mathbb{C} & \text{if } X_1 \cong X_2 \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$ 

Recall that each graded affine Hecke algebra is associated to a root system $R$. We shall call the rank of $H$ to be the rank of $R$ (i.e. the cardinality of $\Delta$). We shall use induction on the rank of $H$ to prove Theorem 6.1. The proof will be given at the end of this section. We first give few important lemmas.

**Lemma 6.2.** Let $(J, U) \in \Xi$. Let $U = \mathcal{U} \otimes \mathbb{C}_\nu$ (as $H_J = \mathbb{H}_J \otimes S(V_J^\vee, \Delta)$ for some $H_J$-module $\mathcal{U}$ and some $\nu \in V_J^\vee$). For an irreducible finite-dimensional $H_J$-module $Y$, $Y = \mathcal{Y} \otimes \mathbb{C}_{\nu'}$ for some irreducible $H_J$-module $\mathcal{Y}$ and for some $\nu' \in V_J^\vee$. Then

$$\operatorname{Ext}^i_{H_J}(U, Y) = \bigoplus_{k+l=i} \operatorname{Ext}^k_{S(V_J^\vee, \Delta)}(\mathbb{C}_\nu, \mathbb{C}_{\nu'}) \otimes \operatorname{Ext}^l_{H_J}(\mathcal{U}, \mathcal{Y}).$$

**Proof.** The statement follows from the Künneth formula for complexes ([21 Theorem 10.1]).
Lemma 6.3. Let $Z$ and $Z'$ be $\mathbb{H}$-modules. If $\text{Ext}^i_{\mathbb{H}}(Z, Z') \neq 0$, then there exists a composition factor $Y$ of $Z$ such that $\text{Ext}^i_{\mathbb{H}}(Y, Z') \neq 0$.

Proof. Let $Z_0 \subset Z_1 \subset \ldots \subset Z_r = Z$ be a composition series of $Z$. If $\text{Ext}^i_{\mathbb{H}}(Z_0, Z') \neq 0$, then we are done. Otherwise, by considering the short exact sequence

$$0 \to Z_0 \to Z \to Z/Z_0 \to 0,$$

we have the associated long exact sequence

$$\ldots \to \text{Ext}^i_{\mathbb{H}}(Z/Z_0, Z') \to \text{Ext}^i_{\mathbb{H}}(Z, Z') \to 0.$$ 

Then we have $\text{Ext}^i_{\mathbb{H}}(Z/Z_0, Z') \neq 0$. By an induction on the length of the composition series of $Z$, we obtain the statement.

$\square$

Lemma 6.4. Let $\mathbb{H}$ be a graded affine Hecke algebra of rank $n$. Suppose Theorem 6.1 is true for all the graded affine Hecke algebra with rank $n - 1$. Let $X_1$ and $X_2$ be irreducible $\mathbb{H}$-discrete series. Assume $L_{ds}(X_1) \leq L_{ds}(X_2)$. Let $(J, U) \in \Xi_L$ with $J \neq \Delta$. If $\lambda(J, U) \leq \lambda_{ds}(X_1)$ or $\theta(\lambda(J, U)) \leq \lambda_{ds}(X_1)$. Then

$$\text{Ext}^i_{\mathbb{H}}((J, U), \iota(X_2)) = 0$$

for all $i \leq n - 2$.

Proof. By Frobenius reciprocity, it reduces to show

$$\text{Ext}^i_{\mathbb{H}_J}(U, \text{Res}_{\mathbb{H}_J} \iota(X)) = \text{Ext}^i_{\mathbb{H}}((J, U), \iota(X)) = 0$$

for all $i \leq n - 2$. By Lemma 6.3 it suffices to show that $\text{Ext}^i_{\mathbb{H}_J}(U, Y) = 0$ for each $i \leq n - 2$ and each composition factor $Y$ of $\text{Res}_{\mathbb{H}_J} \iota(X)$.

By Lemma 6.1.8 and Lemma 6.3 we only have to consider the composition factors $Y$ of $\text{Res}_{\mathbb{H}_J} \iota(X)$, for which the composition factors of $\iota(\text{Res}_{\mathbb{H}_J} Y)$ are discrete series. Let $Y$ be such composition factor of $\text{Res}_{\mathbb{H}_J} \iota(X)$. By the irreducibility of $Y$, we can then write $\iota(Y) = \overline{Y} \otimes C_\nu$ (as $\mathbb{H}_J \cong \mathbb{H}_J \otimes S(V_{\nu, i}^\vee)$-modules) for an irreducible $\mathbb{H}_J$-discrete series $\overline{Y}$ and some $\nu \in V_J^\vee$.

Similarly, we also write $U = \overline{U} \otimes C_\nu$ for an $\mathbb{H}_J$-tempered module $\overline{U}$ and $\nu \in V_J^\vee$. Then by Lemma 5.14 and Theorem 6.1 for rank less than $n$, we have

$$\text{Ext}^i_{\mathbb{H}_J}(\overline{U}, \overline{Y}) = \text{Ext}^i_{\mathbb{H}_J}(\iota(\overline{U}^\bullet), \iota(\overline{Y}^\bullet)) = 0$$

for $i \leq n - 2$. (Here $\ast_J$ and $\bullet_J$ are the corresponding $\ast$-operation and $\bullet$-operation for $\mathbb{H}_J$ respectively.) With Lemma 6.2 this completes the proof.

$\square$

The following lemma is the main technicality for Theorem 6.1. As mentioned in the beginning of this chapter, we have to deal with the composition factor in some parabolically induced modules. The assumptions in (2) of the following lemma is to pick out those composition factors. The assumption $L_{ds}(X_1) \leq L_{ds}(X_2)$ in (2) below gives a much better control on what kind of composition factors picked out.
The main idea of the proof of Lemma 6.5 is to use parabolically induced modules to construct short exact sequences. From those short exact sequences, we obtain associated long exact sequences by applying appropriate Hom-functors. Then Proposition 5.17(1), Lemma 5.19(4) and Lemma 6.4 makes the technique of dimension shifting work.

**Lemma 6.5.** Let $\mathbb{H}$ be a graded affine Hecke algebra of rank $n$. Suppose Theorem 6.1 is true for all the graded affine Hecke algebra with rank less than or equal to $n - 1$. Then we have the following:

1. Let $X_1$ be an $\mathbb{H}$-tempered module and let $X_2$ be an $\mathbb{H}$-discrete series. Then for all $i \leq n - 1$

\[
\text{Ext}^i_{\mathbb{H}}(X_1, \iota(X_2)) = 0
\]

2. Let $X_1$ and $X_2$ be $\mathbb{H}$-discrete series. Assume $L_{ds}(X_1) \leq L_{ds}(X_2)$. Let $(J, U) \in \Xi_L$ with $\lambda(J, U) \leq \lambda_{ds}(X_1)$ or $\theta(\lambda(J, U)) \leq \lambda_{ds}(X_1)$. Let $Y$ be the irreducible $\mathbb{H}$-module with the Langlands classification parameter $(J, U)$. Suppose $Y$ is not tempered (i.e. $J \neq \Delta$). Then for all $i \leq n - 2$,

\[
\text{Ext}^i_{\mathbb{H}}(Y, \iota(X_2)) = 0,
\]

\[
\text{Ext}^i_{\mathbb{H}}(Y^*, \iota(X_2)) = 0.
\]

*Recall that $*$-operation is defined in Section 2.2.*

**Proof.** (We shall use induction on $i$, which indexes $\text{Ext}^i_{\mathbb{H}}$, but we do not fix any of $X_1$ or $X_2$.) When $i = 0$, any weight $\gamma$ of $\iota(X_2)$ satisfy $(\text{Re}_\gamma, \omega_\alpha) > 0$ for all fundamental weights $\omega_\alpha$ by the definitions of discrete series and $\iota$. Then by the definition of tempered modules, $\text{Ext}^0_{\mathbb{H}}(X_1, \iota(X_2)) = \text{Hom}_{\mathbb{H}}(X_1, \iota(X_2)) = 0$. This proves (1) for $i = 0$.

For (2), we may assume $n \geq 2$ (otherwise there is nothing to prove). We consider the following exact sequence:

\[
0 \rightarrow N \rightarrow I(J, U) \rightarrow Y \rightarrow 0,
\]

where $N$ is a proper (possibly zero) submodule of $I(J, U)$. Then we have the associated long exact sequence

\[
0 \rightarrow \text{Ext}^0_{\mathbb{H}}(Y, \iota(X_2)) \rightarrow \text{Ext}^0_{\mathbb{H}}(I(J, U), \iota(X_2)) \rightarrow \ldots
\]

Then by Lemma 6.4 we have $\text{Ext}^0_{\mathbb{H}}(Y, \iota(X_2)) = 0$.

We now assume $1 \leq i \leq n - 1$. Suppose $X_2$ is a discrete series. We first consider $X_1$ is a discrete series. By using $\text{Ext}^i_{\mathbb{H}}(X_1, X_2) = \text{Ext}^i_{\mathbb{H}}(X_2^*, X_1^*)$ (Proposition 4.11) and $X_1^*$, $X_2^*$ being discrete series (Lemma 5.14), we can just consider $L_{ds}(X_1) \leq L_{ds}(X_2)$. Then by Proposition 6.15 we have the following short exact sequence:

\[
0 \rightarrow N \rightarrow I(J_{X_1}, U_1)^* \rightarrow X_1 \rightarrow 0.
\]

for $(J_{X_1}, U_1) \in \Xi_L$ with $\lambda(J_{X_1}, U_1) = \lambda_{ds}(X_1)$ (and $J_X \neq \Delta$). Then by applying the $\text{Hom}_{\mathbb{H}}(\_, \iota(X_2))$ functor to obtain a long exact sequence and using Proposition 5.17(1),

\[
\text{Ext}^i_{\mathbb{H}}(X_1, \iota(X_2)) \cong \text{Ext}^{i-1}_{\mathbb{H}}(N, \iota(X_2)).
\]
Then it remains to show $\text{Ext}^{i-1}_{\mathcal{H}}(N, X_2) \neq 0$. By Lemma 6.3 it suffices to show that $\text{Ext}^{i-1}_{\mathcal{H}}(Y, X_2) = 0$ for any composition factor $Y$ of $N$. Let $(J, U) \in \mathcal{H}$ be the Langlands classification parameter of $Y$. We first consider $J \neq \Delta$. Note that $Y^*$ is a composition factor of $I(J, U_1)$ and so with Lemma 6.7 and Proposition 5.9 $\theta(\lambda(J, U)) \leq \lambda(J, U_1) = \lambda_{ds}(X_1)$ satisfies the assumption in (2). By the induction hypothesis, $\text{Ext}^{i-1}_{\mathcal{H}}(Y, \iota(X_2)) = 0$. Hence we then consider $J = \Delta$. Then we have $Y$ is tempered and so by the induction hypothesis for (1), we also have $\text{Ext}^{i-1}_{\mathcal{H}}(Y, \iota(X_2)) = 0$. This proves $\text{Ext}^{i-1}_{\mathcal{H}}(X_1, \iota(X_2)) = 0$ for the case that $X_1$ is a discrete series.

We now consider $X_1$ is tempered but not a discrete series. By Lemma 5.19 there exists $(\tilde{J}, \tilde{U}) \in \mathcal{H}$ such that $X_1$ is an irreducible quotient of $I(\tilde{J}, \tilde{U})$ with the properties in Lemma 5.19 (4). Using the exact sequence associated to $\text{Hom}_{\mathcal{H}}(\iota(X_2))$, we again obtain $\text{Ext}^{i-1}_{\mathcal{H}}(X_1, \iota(X_2)) \cong \text{Ext}^{i-1}_{\mathcal{H}}(\tilde{N}, \iota(X_2))$ for some irreducible submodule $\tilde{N}$ of $I(\tilde{J}, \tilde{U})$. Since any composition factor of $I(\tilde{J}, \tilde{U})$ is tempered (Proposition 5.9(3)), the induction hypothesis with Lemma 6.3 again yields $\text{Ext}^{i-1}_{\mathcal{H}}(\tilde{N}, \iota(X_2)) = 0$. Then $\text{Ext}^{i-1}_{\mathcal{H}}(X_1, \iota(X_2)) = 0$ as desired.

We now prove (2) and assume $1 \leq i \leq n-2$. Let $X_1$ and $X_2$ be irreducible discrete series with $L_{ds}(X_1) \leq L_{ds}(X_2)$. Let $(J', U') \in \mathcal{H}$ with $\lambda(J', U') \leq \lambda_{ds}(X_1)$ and $J' \neq \Delta$. Let $Y'$ is the irreducible $\mathcal{H}$-module with the Langlands classification parameter $(J', U')$. We consider the short exact sequence

$$0 \rightarrow N' \rightarrow I(J', U') \rightarrow Y' \rightarrow 0,$$

where $N'$ is some submodule of $I(J', U')$. Then apply the functor $\text{Hom}_{\mathcal{H}}(\iota(X_2))$ to obtain a long exact sequence

$$\cdots \rightarrow \text{Ext}^{i-1}_{\mathcal{H}}(I(J', U'), \iota(X_2)) \rightarrow \text{Ext}^{i-1}_{\mathcal{H}}(N', \iota(X_2)) \rightarrow \text{Ext}^{i-1}_{\mathcal{H}}(Y, \iota(X_2)) \rightarrow \text{Ext}^{i-1}_{\mathcal{H}}(I(J', U'), \iota(X_2)) \rightarrow \cdots$$

By Lemma 6.4

$$\text{Ext}^{i-1}_{\mathcal{H}}(N', \iota(X_2)) \cong \text{Ext}^{i}_{\mathcal{H}}(Y, \iota(X_2)).$$

Then again consider the composition factors of $N'$ and using similar argument as the proof for (1) in the previous paragraph. Let $M$ be a composition factor of $N'$ and let $(J^M, U^M)$ be the Langlands classification parameters of $M$. Then $\lambda(J^M, U^M) \leq \lambda(J', U') \leq \lambda_{ds}(X_1)$ by Proposition 5.9. Then by the induction hypothesis of (1) and (2), we obtain $\text{Ext}^{i-1}_{\mathcal{H}}(M, \iota(X_2)) = 0$. Then by Lemma 6.3 we have $\text{Ext}^{i-1}_{\mathcal{H}}(N', \iota(X_2)) = 0$. Then $\text{Ext}^{i-1}_{\mathcal{H}}(Y, \iota(X_2)) = 0$ by (6.27).

We continue to prove the remaining assertion in (2). We now consider the dual $Y'^{*}$. Let $(J_*, U_*) \in \mathcal{H}$ be the Langlands classification parameters of $Y'^{*}$. Then by Lemma 5.7 $\lambda(J_*, U_*) = \theta(\lambda(J', U'))$ and so $\theta(\lambda(J_*, U_*)) \leq \lambda_{ds}(X_1)$. For any composition factor $M$, the Langlands classification parameter $(J^M_*, U^M_*) \in \mathcal{H}$, we also have a similar inequality $\theta(\lambda(J^M_*, U^M_*)) \leq \theta(\lambda(J_*, U_*)) \leq \lambda_{ds}(X_1)$ by Proposition 5.9 and thus the induction step applies with a similar argument and classification parameter.
$(J^{M*}, U^{M*}) \in \Xi_L$, we also have a similar inequality $\theta(\lambda(J^{M*}, U^{M*})) \leq \theta(\lambda(J, U)) \leq \lambda_{dl}(X_1)$ by Proposition 5.9 and thus the induction step applies with a similar argument.

□

Proof of Theorem 6.1. For the case of $|\Delta| = 1$, it is easy to verify. In fact, in that case, when the parameter function $k \neq 0$, there is only one discrete series and when the parameter function $k = 0$, there is no discrete series. Assume $|\Delta| \geq 2$. Let $X_1$ be an irreducible tempered module and let $X_2$ be an irreducible discrete series. By the induction hypothesis, Lemma 6.5, Lemma 5.14 and Theorem 4.16, we have

$$\dim \text{Ext}_{H}^{i}(X_1, X_2) = \dim \text{Ext}_{H}^{n-i}(X_1^*, \iota(X_2)^*) = 0$$

for all $i \geq 1$. The case for $i = 0$ follows from $\text{Hom}_{H} = \text{Ext}_{H}^{0}$ and the Schur’s lemma. This completes the proof.

Remark 6.6. We remark that some argument can be simpler if we use some known results such as discrete series being unitary or self $*$-dual. However, the known proofs for those results rely on the setting in affine Hecke algebras or $p$-adic groups and we try to avoid using them.

6.2. Beyond discrete series: an example. In this section, we continue Example 5.10 and use the notation in the example. We go back to use $V^\vee$ for weights (to be consistent with the notation in Example 5.10). In particular, we use the notation $Y, Z, DS$ to denote the irreducible modules as in Example 5.10. There are two more irreducible modules with the central character $W(\alpha^\vee + \beta^\vee)$ not in the list. One is a discrete series, denoted by $DS'$. Another one the spherical module, denoted $S$.

Let $I' = \text{Ind}_{H_{\{\alpha\}}}^{H_{\{\alpha\}}} (\text{St}_{\alpha} \otimes \mathbb{C}_{\frac{1}{2}(\alpha^\vee + \beta^\vee)})$, where $\text{St}_{\{\alpha\}}$ is the Steinberg module of $H_{\{\alpha\}}$ (c.f. Proposition 5.15 and Example 5.16). Then by considering the weights, we have the following short exact sequence:

$$0 \to DS \oplus DS' \to I' \to Z \to 0.$$  \hfill (6.28)

(We use Theorem 6.4 for the maximal submodule of $I'$ to be a direct sum of $DS$ and $DS'$.) Then taking the $\text{Hom}_{H}(-, X)$-functor ($X = DS, DS'$) and computing $\text{Ext}_{H}^{i}(I', DS)$, we have for $X = DS, DS'$,

$$\text{Ext}_{H}^{i}(Z, X) = \left\{ \begin{array}{cl} \mathbb{C} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{array} \right.$$ 

By taking $\bullet$-operation, we have

$$\text{Ext}_{H}^{i}(X, Z) = \left\{ \begin{array}{cl} \mathbb{C} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{array} \right.$$ 

By Theorem 4.16 and taking $\text{Hom}_{H}(-, Z)$ functor on (6.28), we have

$$\text{Ext}_{H}^{i}(Z, Z) = \left\{ \begin{array}{cl} \mathbb{C} & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{array} \right.$$ 

For $X' = Y, S$, Theorem 4.16 implies

$$\text{Ext}_{H}^{i}(X', Z) = \text{Ext}_{H}^{i}(Z, X') = \left\{ \begin{array}{cl} \mathbb{C} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{array} \right.$$
For $X = DS, DS'$ and $X' = S, Y$, we also have

$$\text{Ext}_{\mathbb{H}}^i(X', X) = \text{Ext}_{\mathbb{H}}^i(X, X') = \begin{cases} 
\mathbb{C} & \text{if } i = 2 \text{ and either } (X = DS \text{ and } X' = S) \\
\text{ or } (X = DS' \text{ and } X = Y) \\
0 & \text{otherwise}.
\end{cases}$$

The Ext-groups among $Y$ and $S$ are similar to the Ext-groups of $DS$ and $DS'$. We skip the detail.

### 7. Euler-Poincaré pairing

#### 7.1. Euler-Poincaré pairing

We keep using the notation of a graded affine Hecke algebra in Definition 2.1. (In this section, $R$ does not necessarily span $V$.) Define the Euler-Poincaré pairing for $\mathbb{H}$-modules $X$ and $Y$ as:

$$EP_{\mathbb{H}}(X, Y) = \sum_i (-1)^i \dim \text{Ext}_{\mathbb{H}}^i(X, Y).$$

This pairing can be realized as an inner product on a certain elliptic space for $\mathbb{H}$-modules analogous to the one in $p$-adic reductive groups in the sense of Schneider-Stuhler [31].

The elliptic pairing $\langle \cdot, \cdot \rangle_{\text{ellip}}^W$ for $W$-representations $U$ and $U'$ is defined as

$$\langle U, U' \rangle_{\text{ellip}}^W = \frac{1}{|W|} \sum_{w \in W} \text{tr}_{U}(w) \text{tr}_{U'}(w) \det_V(1 - w).$$

**Proposition 7.1.** For any finite-dimensional $\mathbb{H}$-modules $X$ and $Y$,

$$EP_{\mathbb{H}}(X, Y) = \langle \text{Res}_W(X), \text{Res}_W(Y) \rangle_{\text{ellip}}^W.$$

**In particular, the Euler-Poincare pairing depends only on the $W$-module structure of $X$ and $Y$.**

**Proof.**

$$EP_{\mathbb{H}}(X, Y)$$

\[= \sum_i (-1)^i \dim \text{Ext}_{\mathbb{H}}^i(X, Y)\]

\[= \sum_i (-1)^i (\ker d^*_i - \text{im} d^*_i) \quad \text{(by Proposition 5.7)}\]

\[= \sum_i (-1)^i \dim \text{Hom}_{\mathbb{C}[W]}(\text{Res}_W(X) \otimes \wedge^i V, \text{Res}_W(Y)) \quad \text{(by Euler-Poincaré principle)}\]

\[= \sum_{w \in W} \text{tr}_{\text{Res}_W X}(w) \text{tr}_{\text{Res}_W Y}(w) \text{tr}_{\wedge^i V}(w)\]

\[= \langle \text{Res}_W(X), \text{Res}_W(Y) \rangle_{\text{ellip}}^W\]

Here $\wedge^\pm V = \bigoplus_{i \in \mathbb{Z}} (-1)^i \wedge^i V$ as a virtual representation. The last equality follows from $\text{tr}_{\wedge^i V}(w) = \det(1 - w)$ and definitions. 

\[\square\]
7.2. Applications. We give two applications of the Euler-Poincaré pairing in this section.

An element \( w \in W \) is said to be elliptic if \( \det_V (1 - w) \neq 0 \). A conjugacy class of \( W \) is said to be elliptic if any element in the conjugacy class is elliptic. The first application is to give an upper bound of the number of irreducible discrete series. We follow the proof for affine Hecke algebra by Opdam-Solleveld [26, Proposition 3.9].

Corollary 7.2. Let \( \mathbb{H} \) be a graded affine Hecke algebra associated to a root datum and an arbitrary parameter function. The number of irreducible \( \mathbb{H} \)-discrete series are less than or equal to the number of elliptic conjugacy classes of \( W \). In particular, there are only finite number of non-isomorphic irreducible \( \mathbb{H} \)-discrete series.

\[ \text{Proof. Let } R(W) \text{ be the virtual representation ring of } W. \text{ Let } \overline{R}(W) = R(W)/\text{rad}_W^{\text{ellip}}. \text{ Then by the definition of } \text{rad}_W^{\text{ellip}}, \text{ the dimension of } \overline{R}(W) \text{ is the number of elliptic conjugacy classes (see [24 Section 2] for the detail). Let } R(\mathbb{H}) \text{ be the Grothendieck group of the category of finite-dimensional } \mathbb{H} \text{-modules.} \]

On the other hand, the restriction map \( \text{Res}_W \) defines an isometry from \( R(\mathbb{H}) \) to \( \overline{R}(W) \) with respect to the paring \( \text{EP}_{\mathbb{H}} \) and \( \langle ., . \rangle_W^{\text{ellip}} \) respectively. By Theorem 6.1, discrete series form an orthonormal set for the pairing \( \text{EP}_{\mathbb{H}} \). Hence the number of discrete series is less or equal to the number of elliptic conjugacy classes.

The second application concerns the duals of discrete series. For real parameter functions, it is even known that those discrete series are even \(*\)-unitary (by some analytic results in affine Hecke algebras, see [34 Theorem 7.2]).

Corollary 7.3. Let \( \mathbb{H} \) be a graded affine Hecke algebra associated to a root datum and an arbitrary parameter function. Let \( X \) be an \( \mathbb{H} \)-discrete series. Then

1. \( X, X^*, X^* \) and \( \theta(X) \) are isomorphic,
2. Let \( W\gamma \) be the central character of \( X \). Then \( W\theta(\gamma) = W\gamma \).

\[ \text{Proof. Since } X \text{ and } X^* \text{ have the same } W \text{-module structure, by Proposition 7.1 we have} \]

\[ \text{EP}_{\mathbb{H}}(X, X^*) = \text{EP}_{\mathbb{H}}(X, X) = \mathbb{C}. \]

The second equality follows from Theorem 6.1. Since \( X^* \) is also a discrete series (Lemma 4.11), Theorem 6.1 forces \( X^* \cong X \). The assertion for \( X^* \) and \( \theta(X) \) in (1) can be proven similarly. For (2), the central character of \( \theta(X) \) is \( W\theta(\gamma) \). Then (2) follows from \( X \cong \theta(X) \) in (1).

\[ \square \]

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