Reduced order models for welding and solidification processes

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Abstract. This paper reviews the foundations of modelling of materials processing and methodologies of analysis outside the traditional discretization of the domain (e.g. finite elements, finite volume, etc.). It starts with a review of the mathematical framework with a focus on dimensional analysis and characteristic values. The methodology of minimal representation and correction factor is discussed with emphasis on computational methods to obtain the minimal representation and on the methodology of blending to obtain the correction factors. Welding on a thin plate with surface heat losses is used as a case study, and the results obtained are validated against published data for many materials, processes, and researchers.

1. Introduction
Currently, the word “modeling” is interpreted, nearly universally to mean “numerical simulation”; however, there are other modeling techniques that can be better choices in frequent circumstances. Numerical simulations are convenient when the cost of the prototype is high; for example, it is convenient to simulate numerically an airplane design before building a prototype, since a prototype can cost hundreds of millions of dollars, and no simulation is that costly. Numerical simulations are also essential for problems with complex geometries that cannot be reduced to simple shapes. Conversely, design rules of the form of engineering formulae, tables, and graphs are convenient when the cost of the prototype is low, as is often the case of mechanical design, electrical circuit design, and even airplanes and rocket conceptual design.

It is not surprising to realize that welding procedures seldom involve numerical modeling, since trial and error is often less costly than specialized software and personnel. The vast majority of welding procedures are determined empirically and tested to pass codes or standards. The large amount of theoretical knowledge and advanced simulations has had very little reach to the engineers in charge of decisions; instead, experienced engineers seem to arrive to the right process parameters guided by their intuition, which enables them to discard secondary phenomena, and focus only on the dominant factors. It is surprising, though, that in one hundred years of welding, the field has not developed design rules like most other engineering disciplines. Indeed, most of the field of materials processing lags far behind other engineering disciplines in its progress towards design rules. Efforts in the field of materials processing include [1–4]. Relevant efforts in the field of transport phenomena include [5–16]. For welding, current scientific knowledge and simulations seldom reach the practitioners because it is typically costlier than “just try another weld.” If this knowledge could be synthesized and generalized in a way amenable to practitioners, its application could be less costly than trial and error, at least in some cases.

Many fields of engineering have their knowledge synthesized and generalized in formulae, tables, and graphs, such as aerodynamics. Materials processing is more difficult to synthesize and generalize
because it involves many more physical phenomena than aerodynamics; not only inertial and viscous forces are present, but also conduction, convection, electromagnetic, capillarity, and many more, and they are tightly coupled [17–19]. The mathematical foundation for dealing with multicoupled, multiphysics problems is lacking, and current progress is reviewed in this paper. Because of constraints in space, this paper refers to the appropriate references where derivations, examples, and comprehensive literature surveys are treated in great detail.

2. Structure of governing equations

The mathematical representation of a problem typically involves equations (often differential) capturing the physical phenomena considered relevant. The list of phenomena to include can be long, and judgement is often needed to decide what phenomena to include or not in the governing equations. Including more physical phenomena can increase the complexity of the problem significantly, while omitting a phenomenon, runs the risk of missing an unexpected dominant factor. The methodologies described here are more tolerant than discretization methods to an excess of physical phenomena, reducing the risk of omitting dominant factors.

In the detailed analysis of governing equations, it is important to discriminate the different type of quantities involved. Parameters are quantities that are fixed for a given problem, but vary between two problems of the same type. Independent variables are quantities that can vary freely and independently from each other for a given problem. Dependent variables are functions of both the parameters and independent variables.

As a case study, consider a 2D point heat source moving in steady-state on an infinite domain as shown in figure 1. This problem was first solved by Wilson in 1904 [20], and it was solved independently again by Rosenthal [21–23] in the 1940s, with a focus on welding. Comprehensive reviews of these solutions are in [24–28].

This problem is a good representation of welding under a broad range of practical conditions, often termed the “Rosenthal thin plate solution.” It was validated with experiments in [29]. This formulation involves important simplifications: constant thermophysical properties (including no phase change), infinitesimal size of heat source, steady-state in motion in a straight line, infinite length and width of plate, and negligible gradients along the thickness of the plate. The small effect of size of the heat source, thickness, temperature-dependent properties, and melting was verified in [28].

In an Eulerian formulation, the heat source is considered fixed and the substrate moving along the x-axis in the negative direction. The governing equation and boundary conditions for thin plates with surface heat losses is:

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{U}{\alpha} \frac{\partial T}{\partial x} + \frac{h + h'}{kd} (T - T_o) \tag{1}
\]

\[
\frac{\partial T}{\partial r} = -\frac{q}{2\pi r kd} \quad \text{as } r = \sqrt{x^2 + y^2} \to 0 \tag{2}
\]

\[
T = T_o \quad \text{as } r = \sqrt{x^2 + y^2} \to \infty \tag{3}
\]

In this case, \(\{X\} = \{x, y\}\) is the set of independent variables (\(r\) is a notation shorthand, not an independent variable), \(\{U\} = \{T(x,y)\}\) is the set of dependent variables (only temperature in this case), and \(\{P\} = \{U, \alpha, k, q/d, (h + h')/d, T_o\}\) is the set of parameters. Some parameters appear grouped because they do not occur separately in the governing equation. To be rigorous, the notation \(T = T(x,y, U, \alpha, k, q/d, (h + h')/d, T_o)\) should be employed; however, most engineers use \(T = T(x,y)\), which is displays explicitly only the independent variables and omits the parameters. The exact solution of equation (1) with the boundary conditions of equations (2) and (3) is:

\[
T(x,y) = T_o + \frac{q}{2\pi kd} \exp \left( -\frac{u x}{2a} \right) K_0 \left[ r \sqrt{\frac{(u x)^2}{2a} + \frac{h + h'}{kd}} \right] \tag{4}
\]
where \( K_0 \) is the modified Bessel function of second kind and zero order. The sets of independent and dependent variables are the same in the solution. This equation provides the temperature value for each point in the substrate. The singularity at the origin (\( r = 0 \)) is a consequence of assuming the heat source is concentrated in an infinitesimal area.

Figure 1. Isotherms for a point heat source of intensity \( q \) on a thin substrate of thickness \( d \). The domain is \(-\infty < x < \infty, -\infty < y < \infty \) and gradients in \( z \) are negligible [49].

The definitions and example above can be generalized to state that the governing equations of a problem (e.g. equations (1)-(3)) have the following form [30]:

\[
\sum_j A_j g_j(\{X\}, \{U\}, \{P\}) = 0
\]  

(5)

where \( g_j \) are functions of the independent variables (and associated differential expressions), the dependent variables, and the parameters. The parameters are defined as positive, adjusting the sign of the associated function \( g_j \) when necessary. The coefficients \( A_j \) have the form of a power law based only on the parameters:

\[
A_j = K_j \prod_k P_{jk}^{a_{jk}}
\]

(6)

where \( K_j \) is a dimensionless numerical constant, but the exponents \( a_{jk} \) result in units for \( A_j \). Coefficients that do not have a power-law form can often be reduced to the form of equation (5) [30]. In general terms, the solution to the governing equations is a set of expressions for the dependent variables:

\[
u_{jc}(\{P\})
\]

2.1. Characteristic values

The dependent variables are functions of the independent variables and the parameters. Often in engineering problems, given a set of parameters, the problem in practice is to determine particular values of the dependent variables over the domain, such as a maximum temperature, minimum width, or average velocity. These values are called “characteristic values.”

An important property of characteristic values is that they do not depend on the independent or dependent variables, they depend only on the parameters: \( \{u_{jc}(\{P\}) \} \). The characteristic values often can be expressed in the form of power laws based on the set of parameters, with exceptions discussed in [30].

Another type of characteristic value is when a variable (independent or dependent) has a value determined by factors outside the model considered. For example, the point heat source of equation (4) often considers a characteristic value of temperature given by phenomena such as melting or phase transformations, unrelated to the considerations on which the governing equations are built. Based on this characteristic temperature, other characteristic values can be defined, such as width of an isotherm. In this case, the externally imposed characteristic value becomes a new parameter. For the case of the moving heat source on a thin plate, the externally imposed characteristic temperature \( T_c \) is a new problem parameter.

Figure 2 illustrates relevant characteristic values for moving heat source problems. These characteristic values are associated with the isotherm \( T = T_c \), and represent centerline heating and cooling rates, start and end of the isotherm on the centerline, width of the isotherm and lateral...
temperature gradient, and maximum temperature away from the centerline. Based on these characteristic values, other characteristic values of practical use can be derived immediately, such as $t_{b/c}$, thickness of HAZ, solidification time, aspect ratio of the isotherm, and more [31, 32].

The characteristic values of differential expressions can be approximated using the characteristic values of the original function and basic knowledge of the shape of the function. If $u(x)$ is the original function and we consider the first derivative $v(x) = du/dx$, the exact characteristic value of $v(x)$ in

$$v_{c,e} \approx v_c = \left(\frac{du}{dx}\right)_c = C \frac{\Delta u}{\Delta x}$$

(7)

with the values of $C$ tabulated for many relevant cases and shapes of functions in [30].

Figure 2. Characteristic values for moving heat sources associated with isotherm $T = T_c$ [31].

2.2. Dimensional analysis

Dimensional analysis states that a problem can be described using dimensionless groups, and those groups are fewer than the number of parameters plus variables of the system. Foundational work includes [33–39].

The most common formulation states that if a system involves $n$ parameters plus variables, and the number of independent units involved is $k$, then the system can be represented with $m = n - k$ independent dimensionless groups. Independent units cannot be reduced; for example, if a problem
involves the units of m, kg, and s, and involves Newton’s laws, then N is not an independent unit because it is a combination of the former units. If temperatures always use a scale with an arbitrary zero (e.g. Celsius or Fahrenheit), then \( m = n - k - 1 \) \[40\]. In some problems, the number of dimensionless groups can be even smaller (for example when the governing equations do not involve the exchange of energy between mechanical work and heat). Another way to determine the number of independent dimensionless groups is through the normalization of equations.

Because systems of units are based on power laws, the dimensionless groups also have the form of power laws.

For the example of a 2D point moving heat source, the number of independent parameters is 6, with one dependent and two independent variables, for a total of 9 magnitudes with dimensions. A possible reference set of units is \{R\} = \{m, kg, s, \degree C\} (\( k = 4 \)). Because all temperatures in this problem have an arbitrary zero, the number of dimensionless groups in this problem is 4, two of them can be associated with independent dimensionless variables (\( x^* \) and \( y^* \)), one with the dependent dimensionless variable (\( T^*(x^*, y^*) \)), and one with the surface heat losses (\( h^* \)).

\[
\begin{align*}
x^* &= \frac{u_x}{2a} \tag{8} \\
y^* &= \frac{u_y}{2a} \tag{9} \\
T^* &= \frac{2\pi kd(T-T_0)}{q} \tag{10} \\
h^* &= \frac{4a^2(h+h_0)}{kdU^2} \tag{11}
\end{align*}
\]

resulting in the following dimensionless equivalent of equation (4):

\[
T^*(x^*, y^*) = \exp(-x^*)K_0\left(r^*\sqrt{1+h^*}\right) \tag{12}
\]

In general, equation (5) can be expressed in dimensionless form as

\[
\Sigma_j B_j((\Pi_P))h^*_j((X^*), (U^*), (\Pi_P)) = 0 \tag{13}
\]

where \( \{X^*\} \) is the set of the set of dimensionless independent variables, \( \{U^*\} \) is the set of the set of dimensionless dependent variables, and \( \{\Pi_P\} \) is the set of dimensionless groups that do not include variables. All these dimensionless groups together constitute the set of dimensionless groups \( \{\Pi\} = \{\{X^*\}, \{U^*\}, \{\Pi_P\}\} \). This is the type of normalization used in inspectional analysis. The coefficients \( B_j \) have the form of a power law based on \( \{\Pi_P\} \):

\[
B_j = K_j \prod_k \Pi_P^{b_{jk}} \tag{14}
\]

where \( K_j \) and \( b_{jk} \) are dimensionless. The solution of equation (13) is the set \( \{U^*\} \), made by functions

\[
u^*_j((X^*), (\Pi_P)) \tag{15}
\]

Because characteristic values do not depend on the variables, only on the parameters, a dimensionless characteristic value will have the functional dependence only on the dimensionless groups that depend on the parameters:

\[
u_{c^*_j}(\{\Pi_P\}) \tag{16}
\]

It is important to keep in mind that not always there is a one-to-one correspondence between variables and their dimensionless counterpart. In cases of self-similarity, the number of dimensionless independent variables is smaller than the number of independent variables with units.
2.3. Normalization of functions

Normalization of functions is a key concept in the scaling of differential equations; its ultimate goal is to capture the order of magnitude of functional expressions, thus allowing for efficient approximations. This goal is accomplished by considering only the characteristic values of the variables without knowing the exact (or numerically approximated) solution to the governing equations over the whole domain. The approach of considering only the characteristic values is in contrast with the integration of the differential equations (e.g. finite elements or finite differences methods), which involves all (or very many) values of the variables.

Similarly to dimensional analysis, normalization involves the generation of dimensionless functions and dimensionless groups; however, the dimensionless entities generated by normalization are designed to capture the order of magnitude of terms in the governing equations, which is not the case for dimensional analysis.

Using the concept of characteristic value, the function $g_j$ in equation (5) can be normalized as:

$$g_j([X], [U], [P]) = (g_j)_c([P])(g_j)^c([X^*], [U^*], [\Pi_P])$$

where the notation $(g_j)_c$ indicates a dimensionless magnitude obtained by normalization by a characteristic value. The characteristic values $(g_j)_c$ are typically obtained from equation (7), and the whole methodology is described in detail in [30]. If the variables are also normalized by their characteristic values, the general expression of a differential equation with normalized functions is:

$$\sum_j A_j((g_j)_c([P])(g_j)^c([X^*], [U^*], [\Pi_P]) = 0$$

Equation (18) is similar to equation (13), except that $(g_j)_c$ are not necessarily power laws of the parameters. Another similarity is that each term (or group of terms) represents a physical phenomenon.

Identifying physical phenomena through normalization is convenient, since there are typically far fewer physical phenomena parameters. A problem with a large number of parameters often confuses practitioners, yet experts typically know to focus on the phenomena.

An important difference with equation (13) is that when the characteristic value is chosen as a maximum value in the domain, at the point $\{X^*_c\}$ where the maximum occurs, then $(g_j)_c = O(1)$. When the point $\{X^*_c\}$ in the domain corresponds to the maximum of a dominant term, equation (18) can be turned into an algebraic equation [41, 42]. In this case, the coefficient of the dominant term is approximately 1, while other coefficients, being $O(1)$, can be approximately 1, or much smaller than 1.

For the example of a moving heat source, the normalized expression of equation (1) is:

$$
\left(\frac{\partial^2 T}{\partial x^2} \right)_c + \left(\frac{\partial^2 T}{\partial y^2} \right)_c \left(\frac{\partial^2 T}{\partial y^2} \right)^* = -\frac{V}{a} \left(\frac{\partial T}{\partial x} \right)_c \left(\frac{\partial T}{\partial x} \right)^* + \frac{h + h'}{kd} (T_c - T_0)(T - T_0)^*
$$

This normalization presents the challenge that the characteristic value for temperature and its derivatives is not given by a maximum, since this function has an asymptote of $T \rightarrow \infty$ at the origin.

2.4. Balances and regimes

In equations (18) and (14), not all terms are of the same order of magnitude, which means that not all the physical phenomena involved are of the same relevance. In the asymptotic extreme, two terms (dominant terms) are much larger than the rest (secondary terms). Some terms will be positive and some will be negative; arbitrarily, we can call the positive terms “input” and the negative terms “output.” The two dominant terms must have opposite sign to balance the equation; thus, there will be a “dominant input” and a “dominant output.”

A regime is a combination of parameters and independent variables yielding the same dominant input and the same dominant output. The division between regimes is not sharp, but meaningful separations can be established where a dominant term becomes of the same magnitude as the largest secondary term. The combinations of parameters and independent variables near the separations are “intermediate regimes,” while combinations far away from the separations are asymptotic regimes. In the asymptotic regimes, a balance between the two dominant terms alone typically yields good estimates of unknown
characteristic values. Regimes delimited by values of the independent variables are also called “regions,” and can be mapped with much physical insight [43].

For the example of welding on a thin plate with surface heat losses, when the characteristic values are associated with a characteristic temperature $T_c$ (e.g. maximum width of isotherm $T_c$), the conditions $T^* = T^*_c$ and $y_m^* = \max(y^*)$ reduce the four dimensionless degrees of freedom $(x^*, y^*, T^*, h^*)$ to only two: $T_c$ and $h^*$. These two degrees of freedom establish four possible regimes in the combinations of high and low $T_c$ and $h^*$. In this problem, $T^*_c$ is also expressed as the Rosenthal number:

$$\text{Ro} = \frac{q}{2\pi kd(T_c - T_0)} = \frac{1}{T^*_c}$$

(20)

A large Rosenthal number indicates relatively fast heat sources, and a relatively small Ro indicates slow heat sources. The four regimes will be called Regime III for high Ro and small $h^*$, Regime IV for small Ro and small $h^*$, Regime IIIa for high Ro and high $h^*$, Regime IV for small Ro and high $h^*$, illustrated in figure 3 [44].

**Figure 3.** Process map for welding on a thin plate with surface heat losses illustrating the range of the four regimes present. The lines were determined using $y_c^*$. Thin lines indicate an error of 10% between the minimal representation and the exact solution. The thicker lines correspond to points in which the minimal representations of contiguous regimes yield the same prediction. The points marked correspond to typical welding procedures [44].

(a) $T_c$=melting, (b) $T_c$=630 °C, (c) $T_c$=yield temperature.

The characteristic value addressed in this figure 3 is the maximum width of an isotherm, but the map is essentially the same for all other characteristic values. The horizontal axis is the Rosenthal number in
logarithmic scale (indicating fast or slow heat sources), and the vertical axis corresponds to \( h^* \) in logarithmic scale, indicating negligible or high heat losses. The points in this diagram correspond to typical welding procedures mostly from [45]. Characteristic values associated with melting temperature in typical welding conditions often fall in Regime III and the transition into Regime IV, at the temperatures typical of austenite decomposition (630 °C), characteristic values also fall in Regime III, transitioning into Regime IIIa. Calculations related to the shear zone in Friction Stir Welding, often falls in Regime IV [46], and calculations associated with residual stresses, underwater wet welding, and in-service welding tend to fall in the transition between Regime III and Regime IIIa [47]. The lines in this process map will be explained later in the “Blending” section.

In Regime III, for \( x < 0 \) (cooling), dominant terms that balance each other in the governing equation (equation (1)), are the first term on the right (advection in \( x \), “input”) and the second term on the left (conduction in \( y \), “output”). This balance will be analyzed in detail in the section on Minimal Representation.

### 3. Minimal representation and correction factors

The approach formalized here is termed Minimal Representation and Correction Factors (MRCF), and is well suited for multiphysics, multicipled problems. It has been used intuitively in all engineering disciplines for long. It focuses on characteristic values, and consists on a prediction based on a simplified problem (“minimal representation”) and correction factors based on the phenomena that were neglected in the simplified problem:

\[
\text{reality} \approx (\text{prediction}) = (\text{simple formula}) \times (\text{correction factor})
\]

\[
u_{c,e}(\{P\}) \approx \hat{\nu}_{c,j}^+(\{P\}) = \hat{\nu}_{c,j}(\{P\}) f_j(\{\Pi_P\})
\]  

(21)

where \( \nu_{c,e}(\{P\}) \) is the “real” magnitude of a characteristic value (e.g. from a good measurement), \( \hat{\nu}_{c,j}^+(\{P\}) \) is a prediction based on the minimal representation of Regime \( j \), \( \hat{\nu}_{c,j}(\{P\}) \), corrected with the correction factor \( f_j(\{\Pi_P\}) \). The characteristic values can have units and depend on the set of parameters \( \{P\} \), but the correction factors are dimensionless, and depend on the set of dimensionless groups based only on the parameters. \( \{\Pi_P\} \). The notation \( \hat{\nu}_{c,j} \) indicates that the characteristic value \( \nu_c \) is estimated at the asymptotic regime \( j \), and \( \hat{\nu}_{c,j}^+ \) indicates that the characteristic value \( \nu_c \) is estimated at the asymptotic regime \( j \).

The MRCF approach differs from numerical simulations in important aspects. Numerical simulations yield a numerical estimate of the solution over the whole domain for a given set of parameters \( \nu(\{X\},\{P\}) \), while MRCF yields an estimate of the characteristic value as a function of the problem parameters \( \nu_c(\{P\}) \). The dependence of the characteristic value on the parameters is explicit, in contrast with numerical simulations, for which a new simulation needs to be run for a new set of parameters. While numerical simulations aim to capture reality as faithfully as possible, MRCF focuses on the dominant physics. Problems with complex geometry are perfectly suitable for numerical modeling, but are outside the scope of MRCF; conversely, problems involving multiple coupled physical phenomena are challenging for numerical analysis, but well suited for MRCF. One similarity stands out, and it is that both numerical models and MRCF can be accurate to within measurement error.

#### 3.1. MRCF steps

In broad terms, MRCF consists of 6 steps:

- **Step 1)** List all physics considered relevant: Adding more physics to MRCF is typically much simpler than doing it to numerical methods, and problems of convergence or excessive number of nodes are avoided.
- **Step 2)** Identify dominant factors: There are two approaches to identifying the dominant factors. First, when an exact solution exists, the dominant factors are obtained directly from asymptotic
analysis; this is the case of the moving heat source example in this paper. Another approach, when an exact solution is not available, is to postulate dominant factors, and confirm their validity in Step 4.

- **Step 3)** Solve the approximate problem considering only dominant factors: This is the “Minimal Representation” at the core of this methodology, discussed in a specific section below.

- **Step 4)** Check for self-consistency: If the dominant factors were postulated in Step 2, and a simplified solution was obtained in Step 3, when that solution is used to calculate the magnitude of the secondary phenomena, it should indicate that it is indeed negligible, or at least smaller than the phenomena considered dominant. Testing a proposed set of dominant factors can have three results: “consistent”, indicating that was considered small is indeed small, “inconsistent”, when what was neglected is evaluated as large when using the simplified solution, and “incompatible”, when the proposed dominant factors cannot provide an approximate answer to the problem. The latter is common in systems involving multiple coupled equations, such as [42]. Self-consistency is part of the technique of dominant balance in applied mathematics [14, 15, 48–53].

- **Step 5)** Compare predictions to “reality:” Reality are characteristic values obtained from field measurements, controlled experiments, or validated simulations. In this comparison, the minimal representation must be of the same order of magnitude of the reality, and most follow the same trends as different parameters are considered.

- **Step 6)** Create correction factors: In a proper formulation, systematic discrepancies between reality and the minimal representation are due to the neglected phenomena. Correction factors of explicit form can be developed empirically based on the comparison of Step 5, reducing the error of predictions to the order of the irreducible scatter in the original data. Methodologies to obtain correction factors are described later in their own section.

### 3.2. Minimal representation

The minimal representation is often obtained manually, and recent progress has contributed algorithms to implement the process in a computer.

#### 3.2.1. Manual asymptotics

Manual asymptotics are a common way of obtaining a minimal representation, often without even realizing it is being done. Four approaches are common, the first three: “Informed” Dimensional Analysis, Inspectional Analysis, and Order of Magnitude (or Balancing) involve the governing differential equations, and the fourth: Asymptotics of Closed Form Expressions, involves the solutions to the governing equations, when available.

In “Informed” Dimensional Analysis, knowledge of the physical meaning of dimensionless groups and their magnitude is used to simplify the problem. For example, when it is anticipated that surface heat losses are negligible, the 2D moving heat source problem can be simplified by assuming $h^* = 0$. Examples of the application of this technique are [54–58]

In Inspectional Analysis, the order of magnitude of terms in the governing equations is assessed, and the smaller terms are neglected. For example, for fast moving heat sources, the characteristic length in $x$ is much longer than the characteristic length in $y$, and it can be stated that

$$
\left( \frac{\partial^2 T}{\partial x^2} \right)_c = O \left( \frac{\Delta T}{x_c^2} \right) \ll \left( \frac{\partial^2 T}{\partial y^2} \right)_c = O \left( \frac{\Delta T}{y_c^2} \right)
$$

resulting in the elimination of the first term in equation (1). Examples of the application of this technique are [29, 59–61]

In Order of Magnitude techniques, balances between terms are established intuitively, to be confirmed for self-consistency later. Examples of the application of this technique are [1, 5, 6]. Because too many balances are possible, not all balances can be explored. In equation (19), a balance between
the second term (transverse conduction) and the third term (advection) yields the correct relationship between the characteristic length and width of an isotherm in fast moving heat sources:

\[ y_c = C' \sqrt{\frac{\alpha_x}{u}} \]  

(23)

where \( C' \) is related to the factor \( C \) from equation (7) for the differential expressions involved.

In Asymptotic of Closed-Form Expressions, it is necessary to possess a closed form solution to a problem. The minimal representation in this case comes from asymptotic analysis of this solution (typically performed in dimensionless form). For example, the solution expressed in equation (12) can be used to estimate the maximum width of an isotherm when the convective effects are negligible (\( h^* \approx 0 \)). In this case, there are two minimal representations [28]; in dimensionless form:

\[ \overline{y}_{\text{max}}(\text{Ro}) = \frac{\pi}{2e} \text{Ro} \quad \text{Regime III (large Ro, small } h^* \text{)} \]  

(24)

\[ \overline{y}_{\text{max}}(\text{Ro}) = 2\exp(-\gamma)\exp(-\text{Ro}^{-1}) \quad \text{Regime IV (small Ro, small } h^* \text{)} \]  

(25)

where \( \gamma \approx 0.5772 \) is the Euler Mascheroni constant. Examples of the application of this technique are [24, 26, 27, 29, 31, 62–64].

3.2.2. Computational asymptotics

Computational Asymptotics are techniques to obtain a minimal representation through computer-implemented algorithms. There are two general approaches: Statistical Analysis (data mining), and Analysis of Governing Equations.

The Statistical Analysis approach is driven by existing data, and tries to uncover predictive equations. One approach in this category is SLAW [65], in which it is assumed that a target characteristic value is determined by a power law expression of the parameters. The power law expression can be determined as a constrained linear regression in a logarithmic space, and the constraint is that the units of the proposed power-law must match the units of the target magnitude. To obtain a minimal representation, this methodology first performs a regression against all parameters, and subsequently eliminates parameters from the regression in a systematic way, starting by those which cause the smallest error increase in predictions. This process of backwards elimination continues until the error caused by the elimination of a parameter jumps suddenly, indicating that the parameter eliminated was part of the minimal representation. An exponent rounding algorithm is applied to deliver exponents as ratios of integers (with a denominator typically equal or smaller to 4). In its current implementation, this technique requires the vast majority of the input data to belong in the same regime. The advantage of this technique is that it only needs data to be applied, without need to explore the governing equations. Because of the constrain in reproducing the units of the target variable, this technique can predict the functional form of parameters for which no variations were tested (e.g. gravity). One challenge in the application of this technique is that it is not always known if the data set to be analyzed belongs to a single regime.

The Analysis of Governing Equations technique is a computational implementation of the Order of Magnitude techniques, and was pioneered as an early application of artificial intelligence in [52]. The computer implementation allows for the exhaustive exploration of all balances. In a more recent algorithmic implementation (OMS algorithm [41, 42]), it was necessary to develop a foundation for the estimation of the order of magnitude of each term in a differential equation [30], and also, a mathematical foundation to obtain an estimate for any chosen balance of terms. The latter task was implemented using linear algebra based on the fact that the characteristic values of each term of equation (19) can be expressed in the form of a power law; similarly to SLAW, described above, power laws in logarithmic space become linear expressions.
3.3. Correction factors

3.3.1. Errors of minimal representations. Once minimal representations have been obtained for one or more regimes, a comparison with “reality” is needed to validate and calibrate the predictions. The focus of the comparisons is a chosen characteristic value, for example the maximum width of an isotherm $y_{\text{max}}$, which does not depend on the independent variables $\{X\}$, but will vary from test to test as the parameters $\{P\}$ change.

The discrepancies in the comparisons can be attributed to many factors. The assessment of “reality” involves experimental or simulation errors, which can be random or systematic. Random errors in GTAW welding experiments were assessed to be of the order of ±15% [66]. The mathematical simplifications in obtaining minimal representations can also result in systematic error, easy to correct with one, or few, tests. There are also errors in the values of materials properties used. This error manifests itself as a systematic error for comparisons with experiments in a given material, and as a random error for experimental comparisons against a variety of materials. For comparisons against simulations, this error is small or inexistent when the minimal representation and the simulation use the same materials properties. Comparisons against simulations typically show much smaller scatter than comparisons experiments.

The most interesting source of discrepancy between the minimal representation and reality is in the physical phenomena that were neglected to arrive to a minimal representation. Each neglected phenomenon induce systematic error, and is captured by a dimensionless group. The effect of the neglected phenomena can be recovered with high accuracy and simplicity with equation (21) by developing correction factors based on the comparison against reality. For the example of welding a thin plate, if surface heat losses are neglected, their effect can be accounted for by regressions against the dimensionless group $h^r$.

Correction factors based on neglected physical phenomena are mathematically rigorous, and can be developed based on abundant data available in the literature or from industrial practice. In contrast, blind correction factors in which a prediction is multiplied by a factor just to match an observation is not rigorous, and often fails when comparing under different conditions. Similarly, using materials properties, boundary conditions, or solver parameters as adjustable variables for fitting in numerical solutions is not rigorous, and likely to fail. In general, nonrigorous fitting falls into tautological traps of no use in practice. For example, if a numerical model has two fitting parameters and it is used to represent an approximately linear trend, then the model was not needed, since representing a linear trend with two fitting parameters is a trivial exercise. Similarly, validating a numerical model with a single experiment (or small number) is invalid when the model involves fitting parameters.

One advantage of the MRCF approach is that it can be validated against dozens of experiments over decades of literature; this is challenging or impractical with numerical models. Numerical models in which the number of adjustment variables is of the order of the number of comparisons of reality are unscientific and should not be accepted.

3.3.2. Properties of correction factors. The functional form of correction factors is typically unknown, but it has properties of much use. In the asymptotic limit of a particular regime (when the neglected phenomena are indeed negligible), the correction factor is constant of the order of magnitude of 1:

$$f_j(\Pi_P) \to \approx 1 \quad \text{for Regime } j$$

When the minimal representation is based on an exact solution, as in our example of welding on a thin plate, $\tilde{u}_{cj} = \tilde{u}_{c,e,j}$, and the correction factor tends to exactly 1. Following equation (21), the functionality of correction factor $f_j(\Pi_P)$ (corresponding to regime $j$) in regime $k$ can also be determined, with minimal representation $\tilde{u}_{c,k}$.
\[ f_j(\{\Pi_P\}) \rightarrow \frac{\tilde{u}_{ck}}{\tilde{u}_{cj}} \quad \text{for Regime} \, k \]  

where \( f_j(\{\Pi_P\}) \) typically tends to zero or infinity in Regime \( i \).

4. Blending

While asymptotic cases are well captured by minimal representations, intermediate cases can pose an additional problem, as they are typically not captured by either asymptotic behavior. However, almost always, intermediate cases yield results that are intermediate between asymptotic cases.

The technique to calculate cases that are intermediate between asymptotic extremes is called “blending.” For the case in which two asymptotic extremes are given by a single dimensionless group, there is a systematic treatment of intermediate cases [67]. A remarkable property of blending the ability to approximate complicated functions with small errors and much simplicity. The practical simplicity makes blending the frequent choice in engineering applications over perturbation analysis. For this reason, blending is commonly used in engineering expressions such as correlations for convection coefficients [68].

Blending was first studied specifically in [69, 70], dealing with monotonic functions that cross once, such as power laws. When one of the functions to be blended is not a power law, three complications can arise; first: that the two functions do not cross, second: that even if they cross, the asymmetry of the two functions can result in high errors when using equation (28), and third: that the asymptotic behavior chosen is not defined over the whole range of \( \Pi \), thus blending cannot be performed over the whole domain.

Consider that a single dimensionless group \( \Pi \) is the source of two regimes (\( i \) for \( \Pi \rightarrow 0 \) and \( j \) for \( \Pi \rightarrow \infty \)), with two minimal representations \( \tilde{u}_{c1}(\{P\}) \) and \( \tilde{u}_{cj}(\{P\}) \). The analysis is best explained in dimensionless form, involving \( \tilde{u}_{c1}(\Pi) \) and \( \tilde{u}_{cj}(\Pi) \).

4.1. Blending of monotonic functions that cross

When \( \tilde{u}_{c1}(\Pi) \) and \( \tilde{u}_{cj}(\Pi) \) are monotonic over \( \Pi \) and they cross, the following blending approach is often effective [69]:

\[ u_c^*(\Pi) \approx \tilde{u}_c^+(\Pi) = \left[ u_{c1}^n(\Pi) + u_{cj}^n(\Pi) \right]^{1/n} \]  

where the \(^*\) notation indicates that the minimal representation involves a correction, and the blending exponent \( n \) is determined by comparisons “reality” to minimize the maximum error over the whole range of \( \Pi \) [62]. When the minimal representations are both power laws, past experience suggests that equation (28) has an accuracy typically within 10\%, and the absolute value of \( n \) is not far from unity. Correction factors can be developed from equation (28):

\[ u_c^*(\Pi) \approx \tilde{u}_c^+(\Pi) = \tilde{u}_{c1}^+(\Pi) = \tilde{u}_{c1}^n(\Pi) \left( 1 + \left[ \frac{\tilde{u}_{c1}(\Pi)}{\tilde{u}_c^*(\Pi)} \right]^n \right)^{1/n} = \tilde{u}_{c1}^*(\Pi)f_i(\Pi) \]  

For the case of aspect ratio \( AR \) (length to width of an isotherm) in equation (12) with no convection, blending involves two power laws [32]:

\[ AR_{III}(Ro) = \sqrt{\frac{m}{h}} \sqrt{Ro} \quad \text{Regime} \, III \, (\text{large} \, Ro, \text{small} \, h') \]  
\[ AR_{IV}(Ro) = 1 \quad \text{Regime} \, IV \, (\text{small} \, Ro, \text{small} \, h') \]

where Regime III corresponds to high Rosenthal number (relatively fast welds), and Regime IV corresponds to low Ro (slow welds). Blending in this case can be done using equation (28):
\[ \tilde{AR}^+ (Ro) = \left[ 1 + \left( \sqrt{\frac{\pi e}{8 Ro}} \right)^n \right]^{1/n} \]  

(32)

with an optimal value \( n = 1.972 \) and an error always below 3.3% for all possible Ro values. Equation (29) yields the following correction factors for minimal representations of equations (30) and (31):

\[ f_{AR_{III}} (Ro) = \left[ 1 + \left( \sqrt{\frac{8}{\pi e Ro}} \right)^n \right]^{1/n} \]  
Regime III (large Ro, small \( h^* \))  

(33)

\[ f_{AR_{IV}} (Ro) = \left[ 1 + \left( \sqrt{\frac{\pi e}{8 Ro}} \right)^n \right]^{1/n} \]  
Regime IV (small Ro, small \( h^* \))  

(34)

4.2. Blending of monotonic functions that do not cross or are not power laws

Without losing generality, consider the case in which \( \tilde{u}_{ci}^* (\Pi) \) is not a power law, but \( \tilde{u}_{cj}^* (\Pi) \) is, these functions can be expressed as:

\[ \tilde{u}_{ci}^* (\Pi) = v_i^* (\Pi)p_i^* (\Pi) \]  

(35)

\[ \tilde{u}_{cj}^* (\Pi) = p_j^* (\Pi) \]  

(36)

where \( p_i^* (\Pi) \), and \( p_j^* (\Pi) \) are power laws in \( \Pi \), and \( v_i^* (\Pi) \) is not a-power law such that \( v_i^* (\Pi) \to 1 \) when \( \Pi \to \infty \) (Regime \( j \)). In this case, the function \( \tilde{u}_{cj}^* (\Pi) = p_j^* (\Pi) \) can be replaced by \( v_i^* (\Pi)p_j^* (\Pi) \), with the same asymptotic behavior. Now equation (28) can be applied resulting in

\[ \tilde{u}_{c}^* (\Pi) = v_i^* (\Pi)[p_i^n (\Pi) + p_j^n (\Pi)]^{1/n} \]  

(37)

This methodology is also useful for obtaining simpler expressions when blending functions that are not power laws, even if they cross. For the case of maximum width in equation (12) with no convection, the asymptotic behavior for \( \tilde{y}_{max_{III}}^+ (Ro) \) is a power law (equation (24)), while \( \tilde{y}_{max_{IV}}^+ (Ro) \) (equation (25)) is not. The blending of these two equations based on equation (37) is

\[ \tilde{y}_{max_{III}}^+ (Ro) = \exp(-Ro^{-1}) \left[ (2\exp - y)^n + \left( \frac{\pi}{2e} Ro \right)^n \right]^{1/n} \]  

(38)

with an optimal value \( n = 1.407 \) and an error always below 6.8% for all possible Ro values. Equation (38) yields the following correction factors for minimal representations of equations (24) and (25):

\[ f_{y_{max_{III}}} (Ro) = \exp(-Ro^{-1}) \left\{ 1 + \left[ \sqrt{\frac{8e \exp(-y)}{Ro}} \right]^{n} \right\}^{1/n} \]  
Regime III  

(39)

\[ f_{y_{max_{IV}}} (Ro) = \left\{ 1 + \left[ \sqrt{\frac{8e \exp(-y)}{Ro}} \right]^{-n} \right\}^{1/n} \]  
Regime IV  

(40)

4.3. Alternative blending of monotonic functions that cross

When equation (28) yields too much error (often when one or both asymptotic behaviors are not power laws), a useful alternative approach developed by Yi Lu as part of his doctoral studies at the University of Alberta is:

\[ \tilde{u}_{c}^* (\Pi) = \left[ \tilde{u}_{ci}^{\pm 1} (\Pi) + \tilde{u}_{cj}^{\pm 1} (\Pi) + \alpha \Pi^{\pm h} \right]^{\pm 1} \]  

(41)
where $a > 0$ and $b$ are determined by optimization minimizing the maximum error over all values of $\Pi$. The value of 1 in the exponents could also be replaced by a third adjusting parameter $c$ and $1/c$; however fixing it to 1 reduces the complexity of the expression, and experience has shown that the two adjusting parameters $a$ and $b$ are enough to give acceptable errors.

This approach requires the asymptotic behaviors to be monotonic and to cross one each other. It can also be extended to apply in the case of monotonic functions that do not cross or are not power laws. When $\tilde{u}^e_{\Pi}(\Pi)$ and $\tilde{u}^e_{\tilde{f}}(\Pi)$ are power laws, the exponent $b$ in equation (41) is intermediate between the exponent of the two power laws.

For the case of centerline cooling rate in equation (12) with no convection, the asymptotic behavior for $\tilde{T}^*_{b_{III}}(\text{Ro})$ is a power law, while for $\tilde{T}^*_{b_{IV}}(\text{Ro})$ is not [32]:

$$\tilde{T}^*_{b_{III}}(\text{Ro}) = \frac{1}{\text{Ro}^a} \Rightarrow \text{Regime III (large Ro, small h*)} \quad (42)$$

$$\tilde{T}^*_{b_{IV}}(\text{Ro}) = \frac{1}{2} \exp\left(\frac{1}{\text{Ro} + \gamma}\right) \Rightarrow \text{Regime IV (small Ro, small h*)} \quad (43)$$

where the dimensionless time is

$$t^* = \frac{u^2 t}{2 a} \quad (44)$$

These two asymptotic behaviors do not cross, and the first step in blending these equations is to apply equations (35) and (36), to obtain a blending of power laws that do cross. In this case, equation (41) with negative exponents yields a lower error than equation (28), resulting in the following blended expression

$$\tilde{T}^*_{b}(\text{Ro}) = \frac{\exp\left(\frac{1}{\text{Ro}}\right)}{\pi\text{Ro}^a + 2\exp(-\gamma) + a\text{Ro}^{-b}} \quad (45)$$

with optimal values of $a = 3.839$ and $b = -2.108$, and an error below 5.8% for all possible Ro values [32]. Equation (45) yields the following correction factors:

$$f_{T_{b_{III}}}(\text{Ro}) = \exp\left(\frac{1}{\text{Ro}}\right) \left[1 + \frac{2}{\pi} \exp(-\gamma)\text{Ro}^{-3} + a\text{Ro}^{-b-3}\right]^{-1} \Rightarrow \text{Regime III} \quad (46)$$

$$f_{T_{b_{IV}}}(\text{Ro}) = \left[1 + \frac{1}{2\pi} \exp(\gamma)\text{Ro}^{-3} + \frac{1}{2} \exp(\gamma)\text{aRo}^{-b}\right]^{-1} \Rightarrow \text{Regime IV} \quad (47)$$

4.4. Blending of functions that change sign or are not defined over the whole domain

Consider the case in which $\tilde{u}^e_{\tilde{f}_{j}}(\Pi)$ is a power law, but $\tilde{u}^{e}_{j}(\Pi)$ is not, and it is not defined or changes sign for values of $\Pi$ below or above a certain critical value $\Pi_c$. A new function $\tilde{v}^e_{j}(\Pi)$ can be defined as

$$\tilde{v}^e_{j}(\Pi) = \tilde{u}^e_{j}(\Pi \pm 1 \pm a \pm 1) \quad (48)$$

with positive exponents and when the problems with the domain are below $\Pi_c$ and negative exponents and $a < \Pi_c$ when they are above. The value of $a$ is always positive, and is adjusted by optimization; the value of 1 in the exponents was chosen for simplicity.

For all values of $a$, the asymptotic behavior of $\tilde{v}^e_{j}(\Pi)$ is the same as $\tilde{u}^e_{j}(\Pi)$ in Regime $j$, and function $\tilde{v}^e_{j}(\Pi)$ can replace $\tilde{u}^e_{j}(\Pi)$ in any of the blending techniques described above. The correction factor is applied to $\tilde{v}^e_{j}(\Pi)$, not $\tilde{u}^e_{j}(\Pi)$.  

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For the case of maximum temperature at a distance $y_c$ from the centerline in equation (12) with no convection, the asymptotic behavior of $T_{\text{max}}^*$ depends on $y_c^*$, not Ro. Regime III results in a power law, while Regime IV involves a change in sign above $y_c^* = 1$ [32].

$$T_{\text{maxIII}}^*(y_c^*) = \frac{\pi}{2e} \sqrt{\frac{1}{y_c^*}} \quad \text{Regime III (large Ro, small } h^* \text{)} \quad (49)$$

$$T_{\text{maxIV}}^*(y_c^*) = \ln \left( \frac{1}{y_c^*} \right) \quad \text{Regime IV (small Ro, small } h^* \text{)} \quad (50)$$

In this case, the asymptotic behavior for Regime IV is replaced by $\ln(1/y_c^* + 1/a)$, and the standard blending approach of equation (28) is applied, yielding the following blended expression:

$$T_{\text{max}}^* = \left[ \left( \frac{\pi}{2e} \sqrt{\frac{1}{y_c^*}} + \ln \left( \frac{1}{y_c^*} + \frac{1}{a} \right) \right)^n \right]^{1/n} \quad (51)$$

with optimal values of $n = 2.013$ and $a = 0.3350$, and an error below 2.1% for all possible Ro values [32]. Equation (51) yields the following correction factors:

$$f_{T_{\text{maxIII}}}^*(y_c^*) = \left\{ 1 + \left[ \frac{2e}{\pi} y_c^* \ln \left( \frac{1}{y_c^*} + \frac{1}{a} \right) \right]^n \right\}^{1/n} \quad \text{for Regime III} \quad (52)$$

$$f_{T_{\text{maxIV}}}^*(y_c^*) = \left\{ 1 + \left[ \frac{\pi}{2e} y_c^* \ln \left( \frac{1}{y_c^*} + \frac{1}{a} \right) \right]^n \right\}^{1/n} \quad \text{for Regime IV} \quad (53)$$

The correction factor of equation (53) applies to $\ln(1/y_c^* + 1/a)$, not to equation (50).

4.5. Blending with more than one dimensionless group

When more than one dimensionless group is involved, for example Ro and $h^*$ for the 2D moving heat source, the complexity of blending increases significantly. New constraints appear in the choice of dimensionless groups over which to develop blending. With multiple dimensionless groups, there will be correction factors of first, second, and higher orders, as they capture interactions between dimensionless groups. A systematic methodology for blending over multiple dimensionless groups does not currently exist, and is the focus of research efforts.

For the case study of welding on a thin plate with surface heat losses, figure 3 illustrates the four regimes to be blended. The thin lines delimiting each regime correspond to an error of 10% between the minimal representation and Rosenthal’s solution. In these regions, using just the minimal representation is accurate within 10%, which is narrower than the repeatability of welding experiments [66]. The thicker lines indicate a conventional limit of regimes; these lines correspond to the parameter combination for which the minimal representation of two contiguous regimes yield the same prediction. The predictive error of minimal representations over these lines is not necessarily constant.

The minimal representations for regimes III and IV is given by equations (24) and (25). The correction factors of equations (39) and (40) allow to extend the prediction of Regime III into Regime IV, and vice versa, for very small values of $h^*$. One of the complexities of blending in this case, is that for any small, but finite value of $h^*$, a transition into Regime IIa is expected at high enough Ro numbers; however, neither equations (39) nor (40) account for this case (because they assume $h^*=0$).

5. Validation of MRCF for the thin plate with surface heat losses

The techniques described above have been applied successfully to many problems including stress analysis in pipeline welding [71], solidification of chromium carbides [72], laser cladding [73], solid fraction in solidification and phase transformations [74–76], thermocapillary flows in the weld pool [42, 59] micro-electron beam welding [77–79], non-linear heat transfer at the electrode extension [80],
friction stir welding [46], weld penetration in arc welding at high currents [66], application of solid lubricant sprays [81], ceramic to metal joining [82, 83], additive manufacturing and joining in semi-solid state [84–87], semi-solid die casting [88, 89] and velocities in the plasma arc [90].

For the case study of welding on a thin plate with surface heat losses, the minimal representations and correction factors were developed considering the exact Rosenthal solution as “reality.” Expressions accounting for surface heat losses require blending with more than one dimensionless group, which is work in progress. The expressions presented here, valid when the surface heat losses are negligible, are within 10% of the exact solution for any value of Ro.

The exact solution required large simplifications of the physics of the problem to be tractable; then the interesting question is: If “reality” is defined by experiments or higher order models instead of the exact solution, do the MRCF expressions presented still provide useful predictions? Because the MRCF expressions are virtually identical to Rosenthal’s exact solution, validation of the former is the same as validation of the latter, which was already carried out in [29]. A specific validation for the specific characteristic values analyzed here was still performed, based on data available in the literature from the span of decades. None of the comparisons that follow involves any fitting parameters associated to the experimental or numerical data used.

5.1. Aspect ratio of isotherm
For the characteristic value of aspect ratio, the validation of the MRCF expressions is illustrated in figure 4. It shows no obvious systematic error, and a scatter that increases with AR (i.e. high Ro), where the effect of surface heat losses might start playing a role. The accumulation of points near AR=1 is because for all small Rosenthal numbers, the isotherm approaches a circle with an aspect ratio of 1.

5.2. Maximum temperature away from the centerline
For maximum temperature away from the centerline, the validation of the MRCF expressions is illustrated in figure 5. This comparison shows no obvious systematic error, and a relatively small scatter over more than one order of magnitude of data.

5.3. Cooling rate
For the characteristic value of cooling rate, the validation of the MRCF expressions in the absence of surface heat losses is illustrated in figure 6 over almost two orders of magnitude. The comparison shows a slight systematic error, overpredicting fast cooling rates, and underpredicting slow cooling rates. The underwater wet welding data of [97] stands out in discrepancy. This is because surface heat losses are significant in underwater wet welding, but are not considered in equation (45).
Figure 4. Validation of MRCF expressions for isotherm aspect ratio [32].

Published data was normalized with equation (10). MRCF maximum temperature from equation (51) [32].
Figure 6. Validation of MRCF expressions for cooling rate [32] neglecting surface heat losses. Published data was normalized with equations (10) and (44). MRCF cooling rate from equation (45) [32].

Figure 7. Validation of MRCF expressions for cooling rate [32] accounting for surface heat losses. Published data was normalized with equations (10) and (44). MRCF cooling rate not included in this paper, but indistinguishable from exact solution [47].

Figure 7 illustrates a comparison of experimental data against MRCF in which blending is attempted over both Ro and $h^*$. The blending equation is too involved to include here, and it is still the focus of research efforts. At the scale of this figure, the blending predictions are undistinguishable from the exact solution. What is remarkable in this case is that underwater wet welding data from [97] is part of the
same cloud of points with all other data; also, some of the systematic error (e.g. [98]) is eliminated. This good match has important practical implications, including the ability to include practical predictions in codes and standards, as well as procedure developments for underwater wet welding, in-service welding, and other situations in which surface heat losses are relevant, such as the calculation of residual stresses.

5.4. Discussion of validation results
The narrow scatter and lack of significant systematic error shown in figures 4 to 7 is typical of MRCF. This comparison against multiple sources, under a broad range of materials and conditions, and without the use of ad-hoc fitting parameters does not exist for numerical simulations of welding.

There is no published evidence that numerical simulations, in the absence of ad-hoc calibration are more accurate than properly built reduced models, such as those presented here. This is contrary to the common belief that reduced order models result in unacceptable errors. If ad-hoc calibrations are included in the MRCF predictions presented, the error would nearly vanish.

In contrast with the correction factors presented, ad-hoc calibrations often have no physical basis; they can vary from a “fudge factor” correcting the overall expression, to artificially imposed materials properties to match experiments, for example, unrealistic viscosity values of molten metal. Ad-hoc calibrations have the drawback that they assign systematic experimental errors to the calibration factors, thus making the results unsuitable to different experimental conditions. For the case in which the experimental conditions are not expected to vary, such as when the same machine is used with the same material, ad-hoc calibrations are satisfactory. These is the case in many industrial applications. However, for the narrow range of validity required standard design of experiments often proves adequate; this is one of the drivers for the low adoption of numerical modeling in welding.

The reduced models presented here can also be extended to capture through rigorous correction factors the phenomena neglected, such as 3D conduction effects, finite heat source, variable material properties, latent heat, etc. Because of its explicit expression, owed to the blending techniques, predictions can be made very quickly, without problems of convergence.

6. Summary
It is possible to provide simple, accurate, rigorous, and physically meaningful predictions of complex materials processing systems. The approaches described focus on characteristic values and their dependence on problem parameters. The use of dimensional analysis and asymptotic techniques allows researchers to reduce problems to their minimal representation, in which all secondary factors are neglected. The discarded physics can be recovered with accuracy and simplicity through correction factors based on dimensionless groups and blending expressions. The scope of methodologies described is for problems with relatively simple geometries, but there are no restrictions about type and number of physical phenomena, linearity, or coupling involved in the problem. The expressions obtained are suitable for real-time control systems, codes and standards, handbooks, education, and design. A 2D moving heat source is chosen as an example through this article, and it is validated against experiments and simulations from the literature without any use of ad-hoc fitting parameters. The references cited contain detailed examples and literature surveys to enable the implementation of techniques to new problems.

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Appendix

Table A1. Notation

| Thin plate | Units |
|------------|--------|
| $d$ | m | Plate thickness |
| $k$ | Wm$^{-1}K^{-1}$ | Thermal conductivity of plate |
| $q$ | W | Intensity of the heat source (accounting for thermal efficiency) |
| $h, h'$ | Wm$^{-2}K^{-1}$ | Effective convection coefficient on front and back surfaces |
| $R_0$ | – | Rosenthal number (equation (20)) |
| $T(x,y)$ | – | Temperature in the plate |
| $T_c$ | – | Characteristic temperature of interest (e.g. melting) |
| $T_0$ | C | Initial temperature in the plate (e.g. preheat) |
| $U$ | m s$^{-1}$ | Heat source travel velocity |
| $x, y$ | m | Coordinates (figure 1) |
| $x_b$ | m | Trailing edge of isotherm on centerline (figure 2) |
| $x_f$ | m | Leading edge of isotherm on centerline (figure 2) |
| $x_{max}$ | m | $x$-coordinate of maximum width of isotherm (figure 2) |
| $y_{max}$ | m | $y$-coordinate of maximum width of isotherm (half width) (figure 2) |
| $a$ | m$^2$s$^{-1}$ | Thermal diffusivity of plate |
| $\gamma$ | – | Euler-Mascheroni constant $\approx 5.772$ |
| $\hat{T}_f$ | K s$^{-1}$ | Heating rate at $x_f$ (figure 2) |
| $T_b$ | K s$^{-1}$ | Cooling rate at $x_b$ (absolute value) (figure 2) |
| AR | – | Aspect ratio of isotherm (length/width) |

MRCF

| $\{X\}$ | e.g.$x, y, z, t$ | Set of independent variables |
| $\{U\}$ | e.g.$T(x,y)$ | Set of dependent variables |
| $\{P\}$ | e.g.$k, \alpha, T_c$ | Set of parameters |
| $\{\Pi\}$ | e.g.$Ro, x^\ast, y^\ast$ | Set of independent dimensionless groups |
| $\{\Pi_s\}$ | e.g.$Ro, h^\ast$ | Set of independent dimensionless groups not involving variables |
| $K_j$ | e.g.$2, \pi, \gamma$ | Numerical constants |

Coefficients of governing equations (equation (5))

Coefficients of dimensionless governing equations (equation (13))

Functional form of terms of governing equation (equation (5))

Functional form of terms of dimensionless governing equation (equation (5))

Subscripts and superscripts

| c | Approximate characteristic value (equation (7)) |
| c,e | Exact characteristic value (equation (7)) |
| * | Dimensionless magnitude (equation (13)) |
| o | Normalized magnitude (equation (17)) |
| $\hat{\cdot}$ | Minimal representation (equation (21)) |
| $\hat{\cdot}^\ast$ | Corrected or blended minimal representation (equations (21) or (28)) |
References
[1] Dantzig J A and Tucker C L 2001 Modeling in Materials Processing (Cambridge University Press)
[2] Kou S 1996 Transport phenomena and materials processing (Wiley)
[3] Geiger G H and Poirier D R 1973 Transport phenomena in metallurgy Addison-Wesley series in metallurgy and materials (Addison-Wesley Pub. Co.)
[4] Szekely J and Themelis N J 1971 Rate phenomena in process metallurgy (Wiley)
[5] Krantz W B 2007 Scaling Analysis in Modeling Transport and Reaction Processes: A Systematic Approach to Model Building and the Art of Approximation (Wiley)
[6] Bejan A 2004 Convection heat transfer 3rd ed (Wiley)
[7] Ruckenstein E 2003 Ind. Eng. Chem. Res. 42 2525–9
[8] Sides P J 2002 Chem. Eng. Ed. 36 232–5
[9] Deen W M 1998 Analysis of transport phenomena (Oxford University Press)
[10] Astarita G 1997 Chem. Eng. Sci. 52 4681–98
[11] Zlokarnik M 1991 Dimensional Analysis and Scale-up in Chemical Engineering (Springer)
[12] Chen M M 1990 Scales, similarity, and asymptotic considerations in convective heat transfer. Annual Review of Heat Transfer vol 3 1st ed ed Tien C L (Hemisphere) pp 233–291
[13] Ruckenstein E 1987 Adv. Chem. Eng. 13 11–112
[14] Kline S J 1986 Similitude and Approximation Theory (Springer)
[15] Denn M M 1980 Process Fluid Mechanics 1st ed Prentice-Hall International Series in the Physical and Chemical Engineering Series (Prentice-Hall)
[16] Aris R 1976 Chem. Eng. Ed. 10 114–24
[17] Mendez P and Eagar T 2001 J. Mater. Process. Tech. 117 CD–ROM, Section B9: Modeling
[18] Mendez P F 2011 Sci. Technol. Weld Joi. 16 348–56
[19] Mendez P F, Tello K E and Gajapathi S S 2012 Generalization and Communication of Welding Simulations and Experiments Using Scaling Analysis. 9th Int. Conf. Trends Weld. Res. (ASM International) pp 249–258
[20] Wilson H A 1904 P. Camb. Philos. Soc. 12 406–23
[21] Rosenthal D and Schmerber R 1938 Weld. J. 17 2s–8s
[22] Rosenthal D 1941 Weld. J. 20 220s–34s
[23] Rosenthal D 1946 Trans. ASME 68 849–66
[24] Rykalin N N 1951 Calculation of Heat Flow in Welding (Mashgis)
[25] Carslaw H S and Jaeger J C 1959 Conduction of heat in solids 2nd ed (Clarendon Press)
[26] Myers P S, Uyehara O A and Borman G L 1967 Welding Research Council Bulletin 1–46
[27] Grong Ø 1994 Metallurgical Modelling of Welding 1st ed (Institute of Materials)
[28] Lu Y, Wang Y and Mendez P F 2020 Int. J. Heat Mass. Tran. (Preprint)
[29] Christensen N, Davies, V de L and Gjermundsen K 1965 British Weld. J. 12 54–75
[30] Mendez P F 2010 J. Appl. Mech.-T. ASME 77 061017
[31] Wang Y, Lu Y and Mendez P F 2019 Int. J. Heat Mass. Tran. 135 1118–29
[32] Lu Y and Mendez P F 2020 Int. J. Heat Mass. Tran. (Preprint)
[33] Fourier J B 1822 Théorie Analytique de la Chaleur (F. Didot)
[34] Maxwell J C 1871 P. London Math. Soc. 11 224
[35] Rayleigh J W S 1877 The theory of sound (Macmillan)
[36] Buckingham E 1914 Phys. Rev. 4 345–76
[37] Tolman R C 1914 Phys. Rev. Lett. 3 244–55
[38] Bridgman P W 1916 Phys. Rev. Lett. 8 423–31
[39] Bridgman P W 1922 Dimensional Analysis 1st ed (Yale University Press)
[40] Washio T and Motoda H 1999 Extension of Dimensional Analysis for Scale-types and Its Application to Discovery of Admissible Models of Complex Processes. Int. Workshop on Similarity Methods pp 129–47
[41] Mendez P F and Eagar T W 2013 J. Appl. Mech.-T. ASME 80 011009
[42] Mendez P F and Stier N 2013 OMS: A computer algorithm to develop closed-form solutions to multicoCoupled multiphysics problems. Mathematical Modelling of Weld Phenomena 10 (Seggau, Austria: T. U. Graz) pp 219–54

[43] Mendez P F, Argaez M A R, Alvarez A D, Apaoblaza Chaer D A, Antonio A M and Sanchez A V 2019 Multiphysics in the plasma arc. IWW Annual Assembly Doc. 212-1609-19

[44] Lu Y, Grams M R and Mendez P F 2020 In preparation

[45] Electric L 2000 The Procedure Handbook of Arc Welding 14th ed (The James F. Lincoln Arc Welding Foundation)

[46] Mendez P F, Tello K E and Lienert T J 2010 Acta Mater. 58 6012–26

[47] Lu Y and Mendez P F 2020 In preparation

[48] Kruskal M D 1963 Asymptotology Mathematical Models in Physical Sciences (Notre Dame, IN) pp 17–48

[49] Segel L A 1972 SIAM Rev. 14 547–71

[50] Van Dyke M 1975 Perturbation Methods in Fluid Mechanics (The Parabolic Press)

[51] Bender C M and Orszag S A 1978 Advanced Mathematical Methods for Scientists and Engineers 1st ed (McGraw-Hill)

[52] Yip K M K 1996 Artif. Intell. 80 309–48

[53] White R B 2005 Asymptotic analysis of differential equations (Imperial College Press)

[54] Murray P E and Scotti A 1999 Sci. Technol. Weld. Joi. 4 112–7

[55] Murray P E 2000 Sci. Technol. Weld. Joi. 5 221–6

[56] Murray P E 2002 Weld. J. 81 125S–131S

[57] Fuerschbach P W and Kno rovsky G A 1991 Weld. J. 70 287S–297S

[58] Fuerschbach P W 1996 Weld. J. 75 24S–34S

[59] Rivas D and Ostrach S 1992 Int. J. Heat Mass. Tran. 35 1469–79

[60] Schlichting H 1987 Boundary-layer theory 7th ed McGraw-Hill (McGraw-Hill)

[61] Kou S and Wang Y H 1986 Metall. Trans. A 17A 2265–70

[62] Mendez P F, Lu Y and Wang Y 2018 J. Heat Trans.-T. ASME 140 081301

[63] Swift-Hook D T and Gick A E F 1973 Weld. J. 52 492S–499S

[64] Heller K, Kessler S, Dorsch F, Berger P and Graf T 2017 Int. J. Heat Mass. Tran. 106 958–69

[65] Mendez P F and Ordóñez E 2005 J. Appl. Mech.-T. ASME 72 648–57

[66] Duman U 2009 Modeling of Weld Penetration in High Productivity GTAW Ph.D. Colorado School of Mines

[67] Wang Y, Lu Y, Grams M, Cesaro A and Mendez P F 2019 Asymptotics and Blending in the Modeling of Welding. Mathematical Modelling of Weld Phenomena 12 (Seggau, Austria: T. U. Graz) pp 907–32

[68] Bergman T L, Incropera F P, DeWitt D P and Lavine A S 2011 Fundamentals of heat and mass transfer (Wiley)

[69] Churchill S W and Usagi R 1972 AIChE J. 18 1121–8

[70] Churchill S W and Usagi R 1974 Ind. Eng. Chem. Fund. 13 39–44

[71] Grams M R and Mendez P F 2020 A Quantitative Index to Assess the Influence of Joint Fit-Up on Pipeline Weld Root Discontinuities. Proc. 13th International Pipeline Conference (IPC 2020)

[72] Barnes N, Clark S, Seetharaman S and Mendez P F 2018 Acta Mater. 151 356–65

[73] Wood G and Mendez P F 2016 First Order Prediction of Bead Width and Height in Coaxial Laser Cladding. Mathematical Modelling of Weld Phenomena 11 (Seggau, Austria: T. U. Graz) pp 701–34

[74] Kamyabi-Gol A, Clark S J, Gibbs J W, Sridhar S and Mendez P F 2016 Acta Mater. 102 231–40

[75] Kamyabi-Gol A and Mendez P F 2015 Metall. Mater. Trans. A 46 622–38

[76] Gibbs J W, Schlacher C, Kamyabi-Gol A, Mayr P and Mendez P F 2015 Metall. Mater. Trans. A 46 148–55

[77] Gajapathi S S, Mitra S K and Mendez P F 2011 Int. J. Heat Mass. Tran. 54 5545–53
[78] Gajapathi S S, Mitra S K and Mendez P F 2012 Sci. Technol. Weld Joi. 17 429–34
[79] Gajapathi S S, Mitra S K and Mendez P F 2012 Sci. Technol. Weld Joi. 17 435–40
[80] Lehnhoff G and Mendez P F 2011 Int. J. Heat Mass Tran. 54 2651–60
[81] Mendez P F and Powell A C 2008 Scripta Mater. 59 784–7
[82] Park J W, Mendez P F and Eagar T W 2002 Acta Mater. 50 883–99
[83] Park J W, Mendez P F and Eagar T W 2005 Scripta Mater. 53 857–861
[84] Mendez P F, Rice C S and Brown S B 2003 J. Mater. Sci. Lett. 22 1047–9
[85] Mendez P F, Rice C S and Brown S B 2002 Weld. J. 81 180s–187s
[86] Rice C S, Mendez P F and Brown S B 2000 JOM 52 31–33
[87] Mendez P F and Brown S B 1999 US Patent 1–13
[88] Rice C S and Mendez P F 2001 Adv. Mater. Process. 159 49–52
[89] Rice C S R, Mendez P F, Brown S B, Myojin S, Mendez P F, Rice C S R and Myojin S 2001 US Patent 1–10
[90] Mendez P F, Ramirez M A, Trapaga G and Eagar T W 2001 Metall. Mater. Trans. B 32 547–54
[91] Ducharme R, Williams K, Kapadia P, Dowden J, Steen B and Glowacki M 1994 J. Phys. D Appl. Phys. 27 1619–27
[92] Wang X and Li R 2014 J. Intell. Manuf. 25 1301–13
[93] Kou S, Kanevsky T and Fyfitch S 1982 Weld. J. 61 175s–181s
[94] Zain-ul abdein M, Nélia D, Jullien J F and Deloison D 2010 Mater. Sci. Eng. A 527 3025–39
[95] Meseguer-Valdenebro J L, Serna J, Portoles A, Estrems M, Miguel V and Martínez-Conesa E 2016 T. Indian.I. Metals 69 783–91
[96] Luo C, Cao Y, Zhao Y, Zhao L and Shan J 2018 Weld. J. 97 214s–228s
[97] Fukuoka T and Fuku S 1994 B. Mar. Eng. Soc. Japan 22 86–92
[98] Kasuya T and Yurioka N 1993 Weld. J. 72 107s–115s
[99] Poorhaydari K, Patchett B M and Ivey D G 2005 Weld. J. 84 149s–155s
[100] Svensson L E, Gretoft B and Bhadeshia H K D H 1986 Scand. J. Metall. 15 97–103
[101] Lazić V N, Sedmak A S, Živković M M, Aleksandrovicić S M, Čukić R D, Jovićić R D and Ivanović I B 2010 Therm. Sci. 14 235–46
[102] Aval H J, Farzadi A, Serajzadeh S and Kokabi A H 2009 Int. J. Adv. Manuf. Tech. 42 1043–51
[103] Shen S, Oguocha I N A and Yannacopoulos S 2012 J. Mater. Process. Tech. 212 286–94
[104] Shah A K, Kulkarni S D, Gopinathan V and Krishnan R 1995 Weld. J. 74 297–304