Some techniques for evaluating fractional integrals

Chiihuei Yu*
School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

*Corresponding author. Email: 2019013010@zqu.edu.cn

Abstract. This article makes use of integration by parts and change of variable for fractional calculus to solve several fractional integrals, corresponding to the Jumarie type of modified Riemann-Liouville fractional derivatives. On the other hand, some examples are proposed to illustrate the applications of these two methods.

Keywords: integration by parts, change of variable, fractional integrals, modified Riemann-Liouville fractional derivatives, applications.

1. Introduction

The geometric interpretation of derivative as the slope and integral as the area are so evident that one can hardly imagine that a meaningful definition for the fractional derivatives and integrals can be given. In 1695 in a letter to L’Hopital, Leibniz mentions that he has an expression that looks like the derivative of order $1/2$, but also adds that he doesn’t know what meaning or use it may have. Later, Euler notices that due to his gamma function, derivatives and integrals of fractional orders may have a meaning. In other words, fractional calculus with derivatives and integrals of any real or complex order has its origin in the work of Euler, and even earlier in the work of Leibniz. Shortly after being introduced, the new theory turned out to be very attractive to many famous mathematicians and scientists, for example, Laplace, Riemann, Liouville, Abel, and Fourier. Therefore, the development of fractional calculus has been for a long time, and more and more attention has been paid to it. On the other hand, fractional calculus has important applications in many scientific fields. Besides mathematics, fractional derivatives and integrals appear in physics, mechanics, engineering, elasticity, dynamics, control theory, electronics, modelling, probability, finance, economics, biology, chemistry, etc. The fractional calculus is nowadays covered by several extensive reference books and a large number of relevant papers [1-15].

Different from traditional calculus, fractional calculus has many definitions. The commonly used definitions are the Riemann-Liouville (R-L) fractional derivative [16], the Caputo definition of fractional derivative [17], the Grunwald-Letininikov (G-L) fractional derivative [3], and the Jumarie type of modified R-L fractional derivative [18]. In this article, we define a new multiplication of fractional functions and use two methods: integration by parts and change of variables for fractional calculus, to solve several fractional integrals, regarding the Jumarie’s modified R-L fractional derivatives. In fact, these two methods are the generalizations of classical calculus. Furthermore, some examples are provided to demonstrate the advantage of our results.

2. Preliminaries and methods

At first, the fractional calculus used in this paper is introduced below.
Definition 2.1: Suppose that $\alpha$ is a real number and $m$ is a positive integer. The modified Riemann-Liouville fractional derivatives of Jumarie type ([18]) is defined by

$$
aD_x^\alpha f(x) = \begin{cases} 
\frac{1}{\Gamma(-\alpha)} \int_0^x (x - t)^{-\alpha - 1} f(t) dt, & \text{if } \alpha < 0 \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x - t)^{-\alpha} [f(t) - f(x)] dt, & \text{if } 0 \leq \alpha < 1 \\
\frac{d^m}{dx^m} (aD_x^{\alpha-m})[f(x)], & \text{if } m \leq \alpha < m + 1
\end{cases}
$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ is the gamma function defined on $s > 0$. If $(aD_x^{\alpha})^n[f(x)] = (aD_x^\alpha)(aD_x^\alpha) \cdots (aD_x^\alpha)[f(x)]$ exists, then $f(x)$ is called $n$-th order $\alpha$-fractional differentiable function, and $(aD_x^{\alpha})^n[f(x)]$ is the $n$-th order $\alpha$-fractional derivative of $f(x)$. On the other hand, we define the fractional integral of $f(x)$, $(aI_x^{\alpha})[f(x)] = (aD_x^{-\alpha})[f(x)]$, where $\alpha > 0$ and $f(x)$ is called $\alpha$-integral function.

Proposition 2.2 ([19]): If $\alpha, \beta, c$ are real numbers, then

$$
aD_x^\alpha [x^\beta] = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} x^{\beta - \alpha}, \quad (\beta \geq \alpha > 0)
$$

and

$$
aI_x^\alpha [c] = 0,
$$

and

$$
aI_x^\alpha [x^\beta] = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^{\beta + \alpha}. \quad (\beta > -1, \alpha > 0)
$$

Definition 2.3 ([20]): The Mittag-Leffler function is defined by

$$
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)},
$$

where $\alpha$ is a real number, $\alpha > 0$, and $z$ is a complex variable.

Definition 2.4 ([19]): If $0 < \alpha \leq 1$ and $x$ is a real variable. Then $E_\alpha(x^\alpha)$ is called $\alpha$-order fractional exponential function, and the $\alpha$-order fractional cosine and sine function are defined as follows:

$$
cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(k\alpha + 1)},
$$

and

$$
sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha + 1)}.
$$

In the following, a new multiplication of fractional functions is introduced.

Definition 2.5 ([21]): If $\lambda, \mu, z$ are complex numbers, $0 < \alpha \leq 1$, $j, l, k$ are non-negative integers, and $a_k, b_k$ are real numbers, $p_k(z) = \frac{1}{\Gamma(k\alpha + 1)} z^k$ for all $k$. The $\otimes$ multiplication is defined by

$$
p_j(\lambda x^\alpha) \otimes p_l(\mu y^\alpha) = \frac{1}{\Gamma(j\alpha + 1)} (\lambda x^\alpha)^j \otimes \frac{1}{\Gamma(l\alpha + 1)} (\mu y^\alpha)^l = \frac{1}{\Gamma((j+l)\alpha + 1)} \binom{j + l}{j} (\lambda x^\alpha)^j (\mu y^\alpha)^l,
$$

where $\binom{j + l}{j} = \frac{(j+l)!}{j!}$. If $f(\lambda x^\alpha)$ and $g(\mu y^\alpha)$ are two fractional functions,

$$
\begin{align*}
f(\lambda x^\alpha) &= \sum_{k=0}^{\infty} a_k p_k(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha + 1)} (\lambda x^\alpha)^k, \\
g(\mu y^\alpha) &= \sum_{k=0}^{\infty} b_k p_k(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha + 1)} (\mu y^\alpha)^k,
\end{align*}
$$

where $\lambda, \mu$ are complex numbers.
then we define
\[ f(\lambda x^a) \otimes g(\mu y^a) = \sum_{k=0}^{\infty} a_k p_k(\lambda x^a) \otimes \sum_{k=0}^{\infty} b_k p_k(\mu y^a) = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{k} a_{k-m} b_m p_{k-m}(\lambda x^a) \otimes p_m(\mu y^a) \right) \] (11)

**Proposition 2.6:** If \( f(\lambda x^a) \otimes g(\mu y^a) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(ka+1)} \sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m (\lambda x^a)^{k-m}(\mu y^a)^m \).

**Definition 2.7:** Let \( (f(\lambda x^a))^\otimes_n = f(\lambda x^a) \otimes \ldots \otimes f(\lambda x^a) \) be the \( n \) times product of the fractional function \( f(\lambda x^a) \). If \( f(\lambda x^a) \otimes g(\mu x^a) = 1 \), then \( g(\lambda x^a) \) is called the \( \otimes \) reciprocal of \( f(\lambda x^a) \), and is denoted by \( (f(\lambda x^a))^\otimes_{-1} \).

Next, we introduce the methods used in this paper.

**Theorem 2.9 (chain rule for fractional derivatives):** Assume that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), \( g(x^a) = \sum_{k=0}^{\infty} b_k p_k(x^a) \). If \( f \otimes (g(\mu x^a)) = \sum_{k=0}^{\infty} a_k (g(\mu x^a))^{\otimes k} \) and \( f' \otimes (g(\mu x^a)) = \sum_{k=1}^{\infty} a_k k (g(\mu x^a))^{\otimes(k-1)} \), then
\[ \left( aD_x^a \right) \left[ f \otimes (g(\mu x^a)) \right] = f' \otimes (g(\mu x^a)) \otimes \left( aD_x^a \right)[g(\mu x^a)]. \] (14)

Theorem 2.10 (fundamental theorem of fractional calculus): Let \( 0 < \alpha \leq 1, \ a, b \) be real numbers, then
\[ \left( aD_x^a \right) \left[ (aI_x^a)[f(x)] \right] = f(x), \] (15)

and
\[ \left( aI_x^a \right) \left[ (aD_x^a)[f(x)] \right] = f(b) - f(a). \] (16)

Theorem 2.11 (integration by parts for fractional calculus): Suppose that \( 0 < \alpha \leq 1, \) and \( a, b \) are real numbers, then
\[ \left( aI_x^a \right)[f(x) \otimes (aD_x^a)[g(x)]] = f(x) \otimes g(x) |^b_a - \left( aI_x^a \right)[g(x) \otimes (aD_x^a)[f(x)]] \]. (17)

**Theorem 2.12 (change of variable for fractional calculus):** If \( 0 < \alpha \leq 1, \) \( u \) is a \( \alpha \)-fractional differentiable function defined on an open interval \( I, \) and \( f \) is a continuous function such that the range of \( u \) contained in the domain of \( f, \) then \( f \circ u \) is a continuous function and
\[ \left( aI_x^a \right) \left[ (f \circ u)(x) \otimes (aD_x^a)[u(x)] \right] = (u(a))^{\alpha} I_a^a (u(b))[f(u)], \] (18)

for \( a, b \in I. \)

**Proof** \( f \circ u \) is a continuous function is obviously. Let \( F(u) = (aI_x^a)[f(u)], \) then by Eq. (15), we have
\( (aD_x^a)[F(u)] = f(u). \) Let \( g = F \circ u, \) by chain rule for fractional derivatives, we get
\[ (aD_x^a)[g(x)] = (aD_x^a)[F(u)] \otimes (aD_x^a)[u(x)] = f(u(x)) \otimes (aD_x^a)[u(x)]. \] (19)

Therefore,
\[ (aI_x^a)[(f \circ u)(x) \otimes (aD_x^a)[u(x)]] = (aI_x^a)[(aD_x^a)[g(x)]] = g(b) - g(a) \] (by Eq. (16))
\[ F(u(b)) - F(u(a)) = (a^{\frac{\alpha}{\Gamma(\alpha+1)}}) f(u) - (a^{\frac{\alpha}{\Gamma(\alpha+1)}}) f(u) = (u(a))^{\frac{\alpha}{\Gamma(\alpha+1)}} f(u). \]

3. Examples

In the following, we will use some practical examples to illustrate the application of our methods.

**Example 3.1** Let \( 0 < \alpha \leq 1 \), by change of variable for fractional calculus, we obtain the fractional integral

\[
\beta \int_{a}^{b} \sin(\alpha x) \cos(\alpha x) \, dx = \frac{1}{\Gamma(2)} \left[ \sin(\alpha x) \right]_{a}^{b}.
\]

**Example 3.2**: If \( 0 < \alpha \leq 1 \), then by change of variable for fractional calculus,

\[
\beta \int_{1}^{\infty} \frac{1}{x^\alpha} e^{-x} \, dx = \left[ \frac{1}{x^\alpha} e^{-x} \right]_{1}^{\infty}.
\]

**Example 3.3**: Let \( 0 < \alpha \leq 1 \), by integration by parts for fractional calculus, we have

\[
\beta \int_{1}^{\infty} \frac{1}{x^\alpha} \ln(x) \, dx = \left[ \frac{1}{x^\alpha} \ln(x) \right]_{1}^{\infty}.
\]

**Example 3.4**: Assume that \( 0 < \alpha \leq 1 \), using integration by parts for fractional calculus yields

\[
\beta \int_{1}^{\infty} \frac{1}{x^\alpha} \ln(x) \, dx = \left[ \frac{1}{x^\alpha} \ln(x) \right]_{1}^{\infty}.
\]
\[
\begin{align*}
= & \left(\frac{\partial^{\alpha}}{\partial x^\alpha}\right) \left[\arctan(a)(x^\alpha) \otimes \left(\frac{1}{\Gamma(2\alpha + 1)} x^{2\alpha}\right)\right] \\
= & \frac{1}{\Gamma(2\alpha + 1)} x^{2\alpha} \otimes \arctan(a)(x^\alpha) \bigg|_0^1 - \left(\frac{\partial^{\alpha}}{\partial t^\alpha}\right) \left[\frac{1}{\Gamma(2\alpha + 1)} x^{2\alpha} \otimes \left(\frac{1}{\Gamma(2\alpha + 1)} x^{2\alpha}\right)\right] \\
= & \frac{1}{2} \otimes \arctan(a)(1) - \left(\frac{\partial^{\alpha}}{\partial t^\alpha}\right) \left[1 - \left(\frac{1}{\Gamma(2\alpha + 1)} x^{2\alpha}\right)^\otimes_{-1} \right] \\
= & \frac{1}{2} \otimes \arctan(a)(1) - \left(\frac{1}{\Gamma(\alpha + 1)} x^\alpha - \arctan(a)(x^\alpha)\right) \bigg|_0^1 \\
= & \frac{1}{2} \otimes \arctan(a)(1) - 1 + \arctan(a)(1).
\end{align*}
\]

4. Conclusions

From the above discussions, we know that integration by parts and change of variable for fractional calculus are two important methods to evaluate some fractional integrals. In fact, by using the Jumarie type of modified R-L fractional derivatives, a new multiplication we defined, and chain rule for fractional derivatives, we can get more properties and results about fractional integrals. On the other hand, the application of these two methods are extensive, and can be used to solve many fractional calculus problems. In the future, we will use them to explore the topics related to applied science and engineering mathematics.

References

[1] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order Fractals and Fractional Calculus in Continuum Mechanics, CISM Courses and Lectures Vol 378, New York: Springer, 1997, pp 223-276.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Amsterdam: Elsevier, (2006).

[3] K. S. Miller and B. Ross, An Introduction to the Fractional Integrals and Derivatives, Theory and Applications, New York: Willey, (1993).

[4] K. B. Oldham and J. Spanier, The Fractional Calculus, New York: Academic, (1974).

[5] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Amsterdam: Gordon and Breach Science Publishers, (1993).

[6] J. Sabatier, O. P. Agrawal, J. A. Tenreiro Machado, Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, (2007).

[7] R. S. Barbosa, J. A. T. Machado, I. M. Ferreira, PID controller tuning using fractional calculus concepts, Fract Calc Appl Anal, Vol.7, No. 2, (2004), pp.119-134.

[8] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers, Vol. 1, Background and Theory, Vol 2, Application, Springer, (2013).

[9] C. -H. Yu, Fractional Clairaut’s differential equation and its application, International Journal of Computer Science and Information Technology Research, Vol. 8, Issue 4, (2020), pp. 46-49.
[10] C. -H. Yu, Fractional derivatives of some fractional functions and their applications, Asian Journal of Applied Science and Technology, Vol. 4, Issue 1, (2020), pp. 147-158.

[11] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, Vol. 7, Issue 8, (2020), pp. 3422-3425.

[12] C. -H. Yu, Some fractional differential formulas, International Journal of Novel Research in Physics Chemistry & Mathematics, Volume 7, Issue 3, (2020), pp. 1-4.

[13] V. E., Tarasov, Fractional dynamics applications of fractional calculus to dynamics of particles. Fields and Media, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, (2010).

[14] G. M., Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, (2005).

[15] A. Carpinteri, F. Mainardi, (Eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, Wien, (1997).

[16] S. Das, *Functional Fractional Calculus*, 2nd ed. Springer-Verlag, (2011).

[17] I. Podlubny, *Fractional Differential Equations*, Acad. Press, San Diego, New York, London, (1999).

[18] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, Computers & Mathematics with Applications, Vol. 51, No. 9, (2006), pp.1367-1376.

[19] U. Ghosh, S. Sengupta, S. Sarkar, and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, Vol. 3, No. 2, (2015), pp.32-38.

[20] J. C. Prajapati, Certain properties of Mittag-Leffler function with argument $x^\alpha$, $\alpha > 0$, Italian Journal of Pure and Applied Mathematics, Vol. 30, (2013), pp. 411-416.

[21] C. -H. Yu, Differential properties of fractional functions, International Journal of Novel Research in Interdisciplinary Studies, Vol.7, No. 5, (2020), pp.1-14.