Wigner-Yanase-Dyson information as a measure of quantum uncertainty of mixed states

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In this paper, we consider Wigner-Yanase-Dyson information as a measure of quantum uncertainty of a mixed state. We study some of the interesting properties of this generalized measure.

The construction is reminiscent of the generalized entropies that have shown to be useful in many applications.

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I. INTRODUCTION

Entropy is a measure of the lack of information about a system \cite{1}. It can also be regarded as the amount of uncertainty in the outcomes of a measurement on a system. In information theory, Shannon developed information entropy as a measure of uncertainty in a message \cite{2}. This entropy was generalized in the quantum context to von Neumann entropy which is defined for a mixed state $\rho$ as $S(\rho) = -\text{Tr} \rho \log \rho$. Let $\{\lambda_i\}$ be the spectrum of the state $\rho$. Then von Neumann entropy of $\rho$ can be rewritten as $S(\rho) = -\sum \lambda_i \log \lambda_i$, where $\log 0 = 0$. For example, for an $n$-dimensional maximally mixed state $\rho = \frac{I}{n}$, the direct computation gives $S(\rho) = \log n$. Also see \cite{3}. For a pure state, $\psi \rangle \langle \psi \vert$, $S(\psi \rangle \langle \psi \vert) = 0$. Whereas for a maximally mixed state, it acquires its maximal value of $\log n$, where $n$ is the dimension of the density matrix $\rho$.

Indeed, it is well known by now that von Neumann entropy, which is based on Shannon entropy for an information system, is a unique measure that satisfies the four Khinchin axioms \cite{4}. Two of the axioms are convexity and additivity. Relaxing the convexity requirement leads to Renyi entropy defined by $S^R(\rho) = \frac{\log \text{Tr} \rho^q}{q-1}$, while relaxing the additivity condition gives Tsallis entropy $S^T(\rho) = \frac{1}{q-1} \frac{\text{Tr} \rho^q}{q-1}$, where $q$ is some adjustable parameter. In both cases, one recovers von Neumann entropy in the limit $q \to 1$. These generalized entropies have found applications in a wide variety of situations: Renyi entropy has been useful for the analysis of channel capacities \cite{5,6,7} and Tsallis entropies have been applied successfully to some physical situations like multiparticle processes in particle physics \cite{8,9}. In \cite{10}, some of the generalized...
quantum entropies were introduced, and nonnegativity, continuity and concavity were discussed. However, the additivity and subadditivity do not always hold for these entropies [10].

However, it is argued that the quantum uncertainty of $\rho = I/n$ should vanish [11, 12]. Brukner and Zeilinger discussed conceptual inadequacy of the Shannon information in quantum measurement [12]. They suggested a new measure of information for an individual measurement with $n$ possible outcomes, and the measurement of the total information $I_{total} = \text{Tr}\rho^2 - 1/n$, where $\rho$ is the density operator. Moreover, since von Neumann entropy vanishes for all pure states, Wigner and Yanase proposed an entropy which measures our knowledge of a difficult-to-measure observable with respect to a conserved quantity. They defined the entropy as

$$I(A) = -\sum_j \rho_{jj} \log \rho_{jj},$$

and includes quantum and classical uncertainty. To be rid of the observable $X$, it is intuitive to average the variance over the observables. Instead of averaging the variance, Luo averaged the skew information [11]. In [11], he defined the quantum uncertainty for a mixed state $\rho$ of an $n$-dimensional quantum system as $L(\rho) = \sum_{i<k} I(\rho, H_i)$ over an orthonormal basis $\{H_j\}$ for the real $n^2$ dimensional Hilbert space of the observables with inner product $\langle X, Y \rangle = \text{Tr}(XY)$, and demonstrated that the quantity $L(\rho)$ is independent on the choice of the orthonormal basis. By using the property $I(U\rho U^\dagger, H) = I(\rho, UHU^\dagger)$, Luo showed that $L(\rho)$ is invariant under unitary transformations, i.e., $L(U\rho U^\dagger) = L(\rho)$. It is well known that for some unitary $U$, $U\rho U^\dagger = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\{\lambda_i\}$ is the spectrum of $\rho$. Thus, without loss of the generality, it can be assumed that $\rho = D = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then for any observable $H$, the straightforward calculation of $I(D, H)$ yields

$$I(D, H) = \sum_{i<k} (\sqrt{\lambda_i} - \sqrt{\lambda_k})^2 ||h_{ik}||^2,$$

where $h_{ik}$ is the entry $(i, k)$ of $H$. By choosing the special orthonormal basis [11], Luo obtained [11]

$$L(\rho) = L(D) = \sum_{i<k} (\sqrt{\lambda_i} - \sqrt{\lambda_k})^2 = n - (\text{Tr}\sqrt{\rho})^2,$$

which is rid of the observables.
II. PROPERTIES OF WIGNER-YANASE-DYSON (WYD) INFORMATION

The WYD information possesses some interesting properties which we will summarize in this section.

1. Wigner-Yanase-Dyson information is convex with respect to $\rho$ \cite{21}. However, $\text{Tr}(\rho^{\alpha}X\rho^{1-\alpha}X)$ with respect to $\rho$ is concave \cite{15}.

2. Let $\rho_1$ and $\rho_2$ be two density operators of two subsystems and let $A_1$ (resp. $A_2$) be a self-adjoint operator on $H^1$ (resp. $H^2$). Then WYD information $I_{\alpha}(\rho, X)$ satisfies $I_{\alpha}(\rho_1 \otimes \rho_2, A_1 \otimes I_2 + I_1 \otimes A_2) = I_{\alpha}(\rho_1, A_1) + I_{\alpha}(\rho_2, A_2)$, where $I_1$ and $I_2$ are the identity operators for the first and second systems, respectively. See \cite{21,20}. The case in which $\alpha = 1/2$ was discussed in \cite{15}.

3. $I_{\alpha}(\rho, A_1 \otimes I_2) \geq I_{\alpha}(\rho_1, A_1)$, where $\rho_1 = tr_2 \rho$. We can argue this as follows. A simple calculation shows $\text{Tr}(\rho(A_1 \otimes I_2)^2) = \text{Tr}(\rho_1 A_1^2)$. By (2.2) in \cite{21}, $\text{Tr}(\rho^\alpha(A_1 \otimes I_2)\rho^{1-\alpha}(A_1 \otimes I_2)) \leq \text{Tr}(\rho^\alpha A_1 \rho^{1-\alpha} A_1)$. By the definition in Eq. (1), this property holds.

4. When $\rho$ is pure, $V(\rho, X) = I_{\alpha}(\rho, X)$. Thus, the Wigner-Yanase-Dyson information reduces to the variance. The case in which $\alpha = 1/2$ was discussed in \cite{14}.

5. When $\rho$ is a mixed state, $V(\rho, X) \geq I_{\alpha}(\rho, X)$. This is because $\text{Tr}(\rho^\alpha X \rho^{1-\alpha}X) \geq 0$. The case in which $\alpha = 1/2$ was discussed in \cite{14}. Also see \cite{20}.

6. When $\rho$ and $A$ commute, by the discussion in \cite{16} the quantum uncertainty based on the skew information should vanish. It is easy to verify that Wigner-Yanase-Dyson information $I_{\alpha}(\rho, X)$ also satisfies this requirement. We can argue this property from that $\rho$ and $A$ share an orthonormal eigenvector basis when $\rho$ and $A$ commute \cite{22}.

7. The invariance of Wigner-Yanase-Dyson information $I_{\alpha}(\rho, X)$ under unitary transformations. The case in which $\alpha = 1/2$ was discussed in \cite{11,16}.

- $I_{\alpha}(U\rho U^\dagger, X) = I_{\alpha}(\rho, U^\dagger X U)$ for any unitary operator $U$. See Appendix A.
- $I_{\alpha}(U\rho U^\dagger, UXU^\dagger) = I_{\alpha}(\rho, X)$ for any unitary operator $U$. See Appendix A.
- $I_{\alpha}(U\rho U^\dagger, X) = I_{\alpha}(\rho, X)$ for any unitary operator $U$ if the unitary operator $U$ commutes with $X$.

III. AVERAGE WIGNER-YANASE-DYSON INFORMATION AS QUANTUM UNCERTAINTY

Rather than averaging the skew information, we propose to average WYD information. To this end, we propose $Q_{\alpha}(\rho) = \sum_{i,j=1}^{n^2} I_{\alpha}(\rho, H_j)$ as the quantum uncertainty of a mixed state $\rho$, where $\{H_j\}$ is defined as above. As discussed in \cite{11}, we can also show that the quantity $Q_{\alpha}(\rho)$ does not depend on the choice of the orthonormal basis. Let $\{\lambda_i\}$ be the spectrum of $\rho$. By only means of the spectral representation of $\rho$ and the definition of $I_{\alpha}(\rho, H)$ in Eq. (1), the direct calculation of $I_{\alpha}(\rho, H)$ for any observable $H$ shows $I_{\alpha}(\rho, H) = \sum_{i<j}(\lambda_i + \lambda_j - \lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}) ||h_{ij}||^2$ \cite{20}. By choosing the special orthonormal basis in
we obtain \( Q_\alpha(\rho) = \sum_{i<j}(\lambda_i + \lambda_j - \lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}) \), which depends only on the mixed state \( \rho \). Furthermore, we rewrite

\[
Q_\alpha(\rho) = \sum_{i<j}(\lambda_i^\alpha - \lambda_j^\alpha)(\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}) = n - \text{Tr} \rho^\alpha \text{Tr} \rho^{1-\alpha}.
\]  

To demonstrate that \( Q_\alpha(\rho) \) is less than \( n - 1 \), we rephrase

\[
Q_\alpha(\rho) = n - 1 - \sum_{i<k}(\lambda_i^\alpha \lambda_k^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_k^\alpha).
\]

This equality follows Eq. 4 and \( \text{Tr} \rho^\alpha \text{Tr} \rho^{1-\alpha} = \sum_i \lambda_i^\alpha \sum_k \lambda_k^{1-\alpha} = 1 + \sum_{1\leq i<k\leq n}(\lambda_i^\alpha \lambda_k^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_k^\alpha) \). When \( \alpha = 1/2 \), \( Q_\alpha(\rho) \) reduces to Luo’s \( L(\rho) \) in Eq. 4. Clearly, \( Q_\alpha(\rho) \geq 0 \). Note that Tsallis’ entropy is \( S_q(\rho) = (1 - \text{Tr} \rho^q)/(q - 1) \) indexed by also a parameter \( q \).

### IV. PROPERTIES OF \( Q_\alpha(\rho) \)

Like WYD information, \( Q_\alpha(\rho) \) inherits some interesting properties from the WYD skew information. These properties are reminiscent of Tsallis and Renyi entropies as generalized von Neumann entropies.

1. \( Q_\alpha(\rho) \) is non-negative and it is always less than \( n - 1 \), i.e., \( 0 \leq Q_\alpha(\rho) \leq n - 1 \), where \( n \) is the dimensions of the quantum system with system Hilbert space \( \mathcal{C}^n \).

2. For an \( n \)-dimensional completely mixed state \( \rho = I/n \), von Neumann entropy \( S(\rho) = \ln n \). By the discussion in [11], quantum uncertainty of \( \rho = I/n \) should vanish. It is easy to verify that for the completely mixed state \( I/n \), the measure \( Q_\alpha(\rho) \) vanishes.

3. It is not hard to know that \( Q_\alpha(\rho) \) is convex because WYD information is convex \([21]\). That is, \( Q_\alpha(\sum_i \lambda_i \rho_i) \leq \sum_i \lambda_i Q_\alpha(\rho_i) \), where \( \lambda_i \geq 0 \) and \( \sum_i \lambda_i = 1 \).

4. The uncertainty measure \( Q_\alpha(\rho) \) is always less than Luo’s one in Eq. 4. It means that when \( \alpha = 1/2 \), \( Q_\alpha(\rho) \) has the maximal value \( L(\rho) \). That is,

\[
Q_\alpha(\rho) \leq L(\rho).
\]  

The above inequality follows Eqs. 4, 5, and the following inequality. \( \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha} \lambda_j^\alpha \geq 2 \sqrt{\lambda_i \lambda_j} \), for any \( \alpha \), i.e., the arithmetic mean is greater than the geometric mean, and the equality holds only when \( \alpha = 1/2 \) or \( \lambda_1 = \lambda_2 = \ldots = \lambda_n \) for any \( \alpha \).

5. When \( \alpha \) tends to 0, \( \lim Q_\alpha(\rho) = 0 \). Symmetrically, when \( \alpha \) tends to 1, also \( \lim Q_\alpha(\rho) = 0 \).

6. \( Q_\alpha(\rho) \) is invariant under unitary transformations, i.e., \( Q_\alpha(U \rho U^\dagger) = Q_\alpha(\rho) \). This property follows the definition in Eq. 5 and that the eigenvalues of \( \rho \) do not vary under unitary transformations.
7. For pure states, von Neumann entropy $S(\rho) = 0$. However, it can also be argued that it is more intuitive if we require that all pure states have the maximal quantum uncertainty $\frac{1}{1}$ In this sense, it is easy to see that when $\rho$ is a pure state, $Q_\alpha(\rho) = n - 1$ which is maximal quantum uncertainty from Eq. (6).

8. It is known that von Neumann entropy $S(\rho)$ is additive. That is, $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$. Unfortunately, $Q_\alpha(\rho)$ is not additive. However, by the idea for the skew information in $\frac{1}{1}$ we can also show that $Q_\alpha(\rho)$ has the following property. Let $P_\alpha(\rho) = Q_\alpha(\rho)/n$, $P_\alpha(\rho_1) = Q_\alpha(\rho_1)/\sqrt{n}$, where $Q_\alpha(\rho_1) = \sqrt{n} - \text{Tr} \rho_1^i \text{Tr} \rho_1^{1-\alpha}$ by the Eq. (3), $i = 1, 2$. From Eq. (5), $Q_\alpha(\rho_1 \otimes \rho_2) = n - \text{Tr} \rho_1^2 \text{Tr} \rho_1^{1-\alpha} \text{Tr} \rho_2^2 \text{Tr} \rho_2^{1-\alpha}$. Then we can derive

$$P_\alpha(\rho_1 \otimes \rho_2) + P_\alpha(\rho_1)P_\alpha(\rho_2) = P_\alpha(\rho_1) + P_\alpha(\rho_2).$$

Luo derived Eq. (5) when $\alpha = 1/2$ and thought that Eq. (5) with $\alpha = 1/2$ resembles the probability law for union and intersection of two events $\frac{1}{1}$.

V. THE AVERAGE OF $Q_\alpha(\rho)$ AS QUANTUM UNCERTAINTY

If we wish to remove the dependence of $Q_\alpha(\rho)$ on $\alpha$, we can consider the average value of $Q_\alpha(\rho)$ over $\alpha$ as follows. Let $Q^*(\rho) = \int_0^1 Q_\alpha(\rho) d\alpha = \sum_{i<k} (\lambda_i + \lambda_k - \int_0^1 \lambda_i^1 \lambda_k^{1-\alpha} d\alpha - \int_0^1 \lambda_i^{1-\alpha} \lambda_k^1 d\alpha)$, when $\lambda_i \lambda_k = 0$, $\int_0^1 \lambda_i^1 \lambda_k^{1-\alpha} d\alpha = 0$. When $\lambda_i = \lambda_k \neq 0$, $\int_0^1 \lambda_i^1 \lambda_k^{1-\alpha} d\alpha = \lambda_i$. Otherwise, $\int_0^1 \lambda_i^{1-\alpha} \lambda_k^1 d\alpha = \frac{\lambda_k - \lambda_i}{\ln \lambda_k - \ln \lambda_i}$. Moreover, $\int_0^1 \lambda_i^{1-\alpha} \lambda_k^1 d\alpha = \frac{\lambda_k - \lambda_i}{\ln \lambda_k - \ln \lambda_i}$. Let $\Delta(\lambda_i, \lambda_k)$ be defined by

$$\Delta(\lambda_i, \lambda_k) = \begin{cases} 0 & : \lambda_i \lambda_k = 0, \\ 2\lambda_i & : \lambda_i = \lambda_k \neq 0, \\ \frac{2(\lambda_k - \lambda_i)}{\ln \lambda_k - \ln \lambda_i} & : \text{otherwise}. \end{cases}$$

Then, $Q^*(\rho) = \sum_{i<k} [\lambda_i + \lambda_k - \Delta(\lambda_i, \lambda_k)]$. By Eq. (5), we can rewrite $Q^*(\rho) = n - 1 - \sum_{i<k} \Delta(\lambda_i, \lambda_k)$. Interestingly, $Q^*(\rho)$ has the following properties.

1. Clearly, $0 \leq Q^*(\rho) \leq n - 1$ because $0 \leq Q_\alpha(\rho) \leq n - 1$.

2. $Q^*(\rho)$ is convex because $Q_\alpha(\rho)$ is convex.

3. $Q^*(\rho) \leq L(\rho)$. This follows Eq. (7) and $\int_0^1 Q_\alpha(\rho) d\alpha \leq \int_0^1 L(\rho) d\alpha$. The equality holds only when $\lambda_1 = \lambda_2 = \ldots = \lambda_n$ or $\alpha = 1/2$.

4. For pure states, $Q^*(\rho) = n - 1$, which is maximal quantum uncertainty from the definition of $Q^*(\rho)$.

5. For an $n$-dimensional completely mixed state $\rho = I/n$, $Q^*(\rho) = 0$.

6. $Q^*(\rho)$ is invariant under unitary transformations, i.e., $Q^*(U \rho U^\dagger) = Q^*(\rho)$.
Next we consider the Werner state \( \rho = \frac{4\lambda - 1}{3} |\Psi^-\rangle \langle \Psi^-| + \frac{(1 - \lambda)}{4} I \) where \( |\Psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \) is the singlet state for two qubits. Fig. 1 shows the Wigner-Yanase-Dyson (WYD) information for the Werner state as a function of the parameters \( \alpha \) and \( \lambda \). Clearly, WYD information is symmetric with respect to \( \alpha \) and acquires its maximum value at \( \alpha = 1/2 \) (Luo’s value). In Fig. 2 we plot various measures of information as a function of the state parameter \( \lambda \). In (a), we consider Brukner Zeilinger (normalized) measure defined by \( I_{BZ} = \frac{n}{n-1} (\text{Tr} \rho^2 - 1/n) \) with \( n = 4 \) in the example. We consider \( \frac{1}{n-1} Q_{\alpha}(\rho) \) for (b) \( \alpha = 1/2 \) (Luo information) and (c) \( \alpha = 1/3 \), and in (d) we evaluate \( Q^*(\rho) \). Note that the minimal value of zero is obtained for the maximally mixed state, i.e. when \( \lambda = 1/4 \). It is also interesting to note that for each \( \lambda \), if one computes the critical value of \( \alpha = \alpha_c \) such that \( Q_{\alpha_c}(\rho(\lambda)) = Q^*(\rho(\lambda)) \), such a function is a slowly varying function of \( \lambda \). The plot of \( \alpha - c \) as a function of \( \lambda \) is shown in Fig. 3.

Incidentally let us consider Hansen’s example in [18] where he considered \( \rho_{12} = \frac{1}{12} \begin{pmatrix} 7 & 5 & 5 & 6 \\ 5 & 6 & 2 & 5 \\ 5 & 2 & 6 & 5 \\ 6 & 5 & 5 & 7 \end{pmatrix} \). Note that \( \rho_{12}^* \) is not a density operator because \( \text{tr}(\rho_{12}^*) \neq 1 \). We let \( \rho_{12} = \rho_{12}^*/26 \). Thus, \( \rho_{12} \) becomes a density operator. By calculating, von Neumann entropy \( S(\rho_{12}) = 0.60319 \), Luo’s quantum uncertainty \( L(\rho_{12}) = 1.5385 \), our quantum uncertainty \( Q_{1/4}(\rho_{12}) = 1.2213 \) and \( Q^*(\rho_{12}) = 1.0748 \).

In summary, by averaging Wigner-Yanase-Dyson information we derive the measure \( Q_{\alpha}(\rho) \) indexed by \( 0 < \alpha < 1 \) of quantum uncertainty for a mixed state \( \rho \). We demonstrate the interesting properties of \( Q_{\alpha}(\rho) \). The result is reminiscent of the extension to generalized entropies for the von Neumann entropy. To remove
the dependence on the parameter $\alpha$, we can take the average $Q^\star(\rho)$ of $Q_\alpha(\rho)$ over $\alpha$ and derive a measure of quantum uncertainty of a mixed state. Finally we study some of the properties of $Q^\star(\rho)$.

VI. APPENDIX A PROOF OF THE INVARIANCE UNDER UNITARY TRANSFORMATIONS

(A). The proof of $I_\alpha(U\rho U^\dagger, X) = I_\alpha(\rho, U^\dagger XU)$

By the definition, $I_\alpha(U\rho U^\dagger, X) = \text{Tr}(U\rho U^\dagger X^2) - \text{Tr}((U\rho U^\dagger)^\alpha X(U\rho U^\dagger)^{1-\alpha} X)$ and $I_\alpha(\rho, U^\dagger XU) = \text{Tr}(\rho(U^\dagger XU)^2) - \text{Tr}(\rho^\alpha (U^\dagger XU) \rho^{1-\alpha} (U^\dagger XU))$. By calculating,

$$\text{Tr}(\rho(U^\dagger XU)^2) = \text{Tr}(\rho(U^\dagger XU)(U^\dagger XU)) = \text{Tr}(U\rho U^\dagger X^2).$$

(A1)

It is easy to see that $U\rho U^\dagger$ is self-adjoint. Let $\rho$ have a spectral representation

$$\rho = \lambda_1 |x_1\rangle \langle x_1| + ... + \lambda_n |x_n\rangle \langle x_n|.$$  

(A2)

Then, we obtain the following spectral representation of $U\rho U^\dagger$. $U\rho U^\dagger = \lambda_1 U |x_1\rangle \langle x_1| U^\dagger + ... + \lambda_n U |x_n\rangle \langle x_n| U^\dagger$. Note that orthonormal basis $\{Ux_1, ..., Ux_n\}$ consists of eigenvectors of $U\rho U^\dagger$ and $\lambda_1, ..., \lambda_n$ are the corresponding eigenvalues. Thus,

$$(U\rho U^\dagger)^\alpha = \lambda_1^\alpha U |x_1\rangle \langle x_1| U^\dagger + ... + \lambda_n^\alpha U |x_n\rangle \langle x_n| U^\dagger = U \rho^\alpha U^\dagger.$$  

(A3)

As well,

$$(U\rho U^\dagger)^{1-\alpha} = U \rho^{1-\alpha} U^\dagger.$$  

(A4)

It is ready to get the following from Eqs. (A3) and (A4).

$$\text{Tr}((U\rho U^\dagger)^\alpha X(U\rho U^\dagger)^{1-\alpha} X) = \text{Tr}(U \rho^\alpha U^\dagger X U \rho^{1-\alpha} U^\dagger X) = \text{Tr}(\rho^\alpha (U^\dagger XU) \rho^{1-\alpha} (U^\dagger XU)).$$  

(A5)

From Eqs. (A1) and (A5), we finish this proof.

(B). The proof of $I_\alpha(U\rho U^\dagger, UXU^\dagger) = I_\alpha(\rho, X)$

It is straightforward to get the proof from Eqs. (A3) and (A4).

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FIG. 2: Different measures (normalized) to unity for the pure state for (a) Brukner-Zeilinger information, (b) Luo information (c) Wigner-Yanase-Dyson (WYD) for $\alpha = 1/3$, i.e. $Q_{1/3}(\rho)$, and (d) $Q^*(\rho)$.

FIG. 3: Critical values of $\alpha$ as a function of the state parameter $\lambda$

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