BALANCED VERTEX DECOMPOSABLE SIMPLICIAL COMPLEXES
AND THEIR $h$-VECTORS

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Abstract. Given any finite simplicial complex $\Delta$, we show how to construct a new simplicial complex $\Delta'$ that is balanced and vertex decomposable. Moreover, we show that the $h$-vector of the simplicial complex $\Delta'$ is precisely the $f$-vector of the original complex $\Delta$. Our construction generalizes the “whiskering” construction of Villarreal, and Cook and Nagel. As a corollary of our work, we add a new equivalent statement to a theorem of Björner, Frankl, and Stanley that classifies the $f$-vectors of simplicial complexes. We also prove a special case of a conjecture of Cook and Nagel, and Constantinescu and Varbaro on the $h$-vectors of flag complexes.

1. Introduction

The work of this paper was inspired by the “whiskering” construction of finite simple graphs found in work of Villarreal [19] and Cook and Nagel [6]. Given a finite graph $G = (V_G, E_G)$ on the vertex set $V_G = \{x_1, \ldots, x_n\}$, Villarreal constructed a new graph, denoted $G^W$, on the vertex set $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ by adjoining the edges $\{x_i, y_i\}$ for every $i$ to the graph $G$. The new graph has a “whisker” at every vertex of the original graph. As discovered by Villarreal, the edge ideal of the new graph $G^W$, that is, 

$$I(G^W) = \langle w_iw_j \mid \{w_i, w_j\} \in E_{G^W} \rangle \subseteq R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

has the property that $R/I(G^W)$ is Cohen-Macaulay. It was later observed by Dochtermann and Engström [7] and Woodroofe [20] and generalized by Cook and Nagel [6], that one could deduce this result by studying the topological properties of the simplicial complex associated to $I(G^W)$ via the Stanley-Reisner correspondence. In particular, Villarreal’s construction can be viewed as creating a new independence complex $\Delta'$ (sometimes called a flag complex) from the independence complex $\Delta$ of $G$. This new complex $\Delta'$ is vertex decomposable (as defined by Provan and Billera [16]), and it is this topological property that implies that $R/I(G^W)$ is Cohen-Macaulay.

Our entry point was to ask whether there is a more general theory that can be applied to all simplicial complexes. Moreover, we want this general theory to specialize to known cases for flag complexes. We will show that a general construction exists using the notion

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of a colouring $\chi$ of a simplicial complex $\Delta$ (all terms will be properly defined in the following sections). From the colouring $\chi$ and complex $\Delta$, we make a new complex, denoted $\Delta_\chi$.

The first main result of this paper is to show that regardless of how one colours $\Delta$, the construction of $\Delta_\chi$ always results in a vertex decomposable simplicial complex:

**Theorem 1.1** (Theorem 3.7). For any simplicial complex $\Delta$, and any $s$-colouring $\chi$ of $\Delta$, the simplicial complex $\Delta_\chi$ is balanced and vertex decomposable.

Here, balanced means the simplicial complex has a colouring with $(\dim \Delta_\chi) + 1$ colours. Results of [6, 7, 19] now become special cases of this theorem since “whiskering” will be shown to be equivalent to colouring the independence complex of a graph.

We investigate the consequences of Theorem 3.7 in Section 4. One such consequence is the addition of the implication $(i) \implies (ii)$ to the following theorem:

**Theorem 1.2** (Theorem 4.3). Let $m = (m_1, \ldots, m_t) \in \mathbb{Z}_+^t$. The following are equivalent:

(i) $m$ is the $f$-vector of a simplicial complex.
(ii) $m$ is the $h$-vector of a balanced, vertex decomposable simplicial complex.
(iii) $m$ is the $h$-vector of a balanced, shellable simplicial complex.
(iv) $m$ is the $h$-vector of a balanced, Cohen-Macaulay simplicial complex.

The equivalence of statements (i), (iii) and (iv) was first proved by Björner, Frankl, and Stanley [1]. It should be noted that versions of $(i) \implies (ii)$ have appeared in the literature in special cases (see, e.g., [4, Proposition 4.1], [6, Proposition 3.8], [10, Proposition 3.7]), but to the best of our knowledge, no version of the above theorem has appeared before.

Another consequence is a formula for the graded Betti numbers of the Stanley-Reisner ideal of the Alexander dual of $\Delta_\chi$ in terms of the $f$-vector of $\Delta$.

**Theorem 1.3** (Theorem 4.7). Let $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$ be the $f$-vector of a $d$-dimensional simplicial complex $\Delta$ on $V = \{x_1, \ldots, x_n\}$, and let $\chi$ be any $s$-colouring of $\Delta$. Then, for all $i \geq 0$,

$$\beta_{i,n+i}(I_{\Delta_\chi}) = \sum_{j=1}^{d+1} \binom{j}{i} f_{j-1}(\Delta).$$

Because $\Delta_\chi$ is vertex decomposable, $R/I_{\Delta_\chi}$ is also Cohen-Macaulay, so by the Eagon-Reiner Theorem [8], the ideal $I_{\Delta_\chi}$ has a linear resolution. Thus Theorem 4.7 describes all the Betti numbers of $I_{\Delta_\chi}$. Thus, starting from any $f$-vector, we can construct an ideal with a linear resolution whose Betti numbers only depend upon the $f$-vector. This result could also be deduced from recent work Herzog, Sharifan, and Varbaro [15] which classifies all sequences which can be the sequence of Betti numbers for an ideal with a linear minimal free resolution. However, the ideals of [15] need not be square-free monomial ideals.
We round out this paper by describing when our construction can be reversed so that one can start with a balanced vertex decomposable simplicial complex \( \Delta \) and construct another simplicial complex \( \Delta' \) such that \( f \)-vector of \( \Delta' \) is the same as the \( h \)-vector of \( \Delta \). We use this procedure to prove:

\[
\{ \text{\( f \)-vectors of independence complexes of chordal graphs} \} = \{ \text{\( h \)-vectors of balanced vertex decomposable independence complexes of chordal graphs} \}.
\]

This proves a special case of a conjecture of Cook and Nagel \[6\] and Constantinescu and Varbaro \[4\] that the set of \( f \)-vectors of a flag complexes is precisely the set of \( h \)-vectors of balanced vertex decomposable flag complexes.

As a final comment, this paper does not discuss the “whiskering” procedure found in \[9\] in which whiskers are added to only some of the vertices. In ongoing work with Francisco and Hà, we are currently investigating how to partially whisker a simplicial complex.

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2. **Prerequisite background on simplicial complexes**

We work over the polynomial rings \( S = k[x_1, \ldots, x_n] \) and \( R = k[x_1, \ldots, x_n, y_1, \ldots, y_s] \) where \( k \) is any field. We recall the relevant background on simplicial complexes.

**Definition 2.1.** A finite simplicial complex \( \Delta \) on a finite vertex set \( V \) is a collection of subsets of \( V \) with the property that if \( \sigma \in \Delta \) and \( \tau \) is a subset of \( \sigma \), then \( \tau \in \Delta \). The elements of \( \Delta \) are called faces.

The vertex sets of our simplicial complexes will be either the set \( \{x_1, \ldots, x_n\} \) or \( \{x_1, \ldots, x_n, y_1, \ldots, y_s\} \). Because of this, we sometimes write faces as monomials.

If \( \Delta \) is a simplicial complex and \( \sigma \in \Delta \), then we say \( \sigma \) has dimension \( d \) if \( |\sigma| = d + 1 \) (by convention, the empty set has dimension -1). The maximal faces of \( \Delta \) with respect to inclusion are called the facets of \( \Delta \). The dimension of \( \Delta \) is the maximum of the dimensions of its facets. If all of the facets of \( \Delta \) are of the same dimension we say that \( \Delta \) is pure. If \( F_1, \ldots, F_t \) is a complete list of the facets of \( \Delta \), we sometimes write \( \Delta \) as \( \Delta = \langle F_1, \ldots, F_t \rangle \).

An important combinatorial invariant of a simplicial complex is its \( f \)-vector.

**Definition 2.2.** Let \( \Delta \) be a finite simplicial complex of dimension \( d \) and let \( f_i \) denote the number of faces of \( \Delta \) of dimension \( i \). The \( f \)-vector of \( \Delta \), denoted \( f(\Delta) \), is then the vector

\[
\begin{align*}
 f(\Delta) = (f_{-1}, f_0, \ldots, f_d).
\end{align*}
\]

We now recall some important operations on simplicial complexes. If \( \sigma \) is a face of a simplicial complex \( \Delta \), then the deletion of \( \sigma \) from \( \Delta \) is the simplicial complex defined by

\[
\Delta \setminus \sigma = \{ \tau \in \Delta \mid \sigma \not\subseteq \tau \}.
\]
The link of \( \sigma \) in \( \Delta \) is the simplicial complex defined by
\[
\text{link}_\Delta(\sigma) = \{ \tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta \}. 
\]
When \( \sigma = \{v\} \), we shall abuse notation and write \( \Delta \setminus v \) (respectively \( \text{link}_\Delta(v) \)) for \( \Delta \setminus \{v\} \) (respectively \( \text{link}_\Delta(\{v\}) \)).

We shall be particularly interested in the following class of simplicial complexes. This class was first introduced in the pure case by Provan and Billera [16].

**Definition 2.3.** A pure simplicial complex \( \Delta \) is called **vertex decomposable** if

1. \( \Delta \) is a simplex, or
2. there is some vertex \( v \in V \) such that \( \Delta \setminus v \) and \( \text{link}_\Delta(v) \) are vertex decomposable.

Although there is a notion of non-pure vertex decomposability (see [2]), in this paper we assume that all vertex decomposable simplicial complexes are pure.

Key to our main construction introduced in Section 3 is the notion of a colouring.

**Definition 2.4.** Let \( \Delta \) be a simplicial complex on the vertex set \( V \) with facets \( F_1, \ldots, F_t \). An **s-colouring** of \( \Delta \) is a partition of the vertices \( V = V_1 \cup \cdots \cup V_s \) (where the sets \( V_i \) are allowed to be empty) such that \( |F_i \cap V_j| \leq 1 \) for all \( 1 \leq i \leq t, 1 \leq j \leq s \). We will sometimes write \( \chi \) is an \( s \)-colouring of \( \Delta \) to mean \( \chi \) is a specific partition of \( V \) that gives an \( s \)-colouring of \( \Delta \). If there exists an \( s \)-colouring, we say that \( \Delta \) is **\( s \)-colourable**. If \( \Delta \) has dimension \( d - 1 \), then we say that \( \Delta \) is **balanced** if it is \( d \)-colourable.

**Example 2.5.** If \( \Delta \) is simplicial complex on \( |V| = n \) vertices, then \( \Delta \) is \( n \)-colourable; indeed, we take our colouring to be \( V = \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \).

### 3. VERTEX DECOMPOSABLE RESULTS

Starting with an \( s \)-colourable simplicial complex, we introduce a procedure to construct a new simplicial complex that is pure of dimension \( s - 1 \), balanced, and furthermore, vertex decomposable. The whiskering constructions found in [6, 19] for flag complexes (equivalently, independence complexes of graphs) are then special cases of our construction.

We build a new simplicial complex from \( \Delta \) and a colouring of \( \Delta \).

**Construction 3.1.** Let \( \Delta \) be a simplicial complex on the vertex set \( \{x_1, \ldots, x_n\} \). Given an \( s \)-colouring \( \chi \) of \( \Delta \) given by \( V = V_1 \cup \cdots \cup V_s \), we define \( \Delta_\chi \) on vertex set \( \{x_1, \ldots, x_n, y_1, \ldots, y_s\} \) to be the simplicial complex with faces \( \sigma \cup \tau \) where \( \sigma \) is a face of \( \Delta \) and \( \tau \) is any subset of \( \{y_1, \ldots, y_s\} \) such that for all \( y_j \in \tau \) we have \( \sigma \cap V_j = \emptyset \).

**Example 3.2.** Let \( \Delta \) be the simplicial complex shown in Figure 1. Let \( \chi \) be the colouring of the vertices given by the partition \( \{x_1, x_4\} \cup \{x_2\} \cup \{x_3\} \). Then
\[
\Delta_\chi = \{y_1y_2y_3, x_1y_2y_3, x_2y_1y_3, x_3y_1y_2, x_4y_2y_3, x_1x_2y_3, x_2x_3y_1, x_1x_3y_2, x_2x_4y_3, x_3x_4y_2, x_1x_2x_3\}.
\]
Remark 3.3. Observe that each $s$-colouring $\chi$ of $\Delta$ creates a new simplicial complex $\Delta_\chi$. As we shall see, even though these simplicial complexes $\Delta_\chi$ may be different, they all share some interesting properties, regardless of how $\Delta$ is coloured.

Remark 3.4. Construction 3.1 was recently introduced independently by Frohmader [10, Construction 7.1]. However, the construction appears in earlier work of Björner, Frankl, Stanley [1] (e.g., see the proof in the Section 5 when $a = (1, \ldots, 1)$). Another variation appears in work of Hetyei (see [12, Definition 4.2]).

We now prove some properties about our new complex $\Delta_\chi$.

Theorem 3.5. The facets of $\Delta_\chi$ are in one-to-one correspondence with the faces of the original simplicial complex $\Delta$. In addition $\Delta_\chi$ is pure of dimension $s - 1$ and balanced.

Proof. Let $V = V_1 \cup \cdots \cup V_s$ be the colouring $\Delta$ given by $\chi$. It is clear from the definition of $\Delta_\chi$ that the maximal faces are those of the form $\sigma \cup \{y_j \mid V_j \cap \sigma = \emptyset\}$ where $\sigma$ is a face of $\Delta$. This establishes the one-to-one correspondence between the faces of $\Delta$ and the facets of $\Delta_\chi$.

If we partition the vertices of $\Delta_\chi$ as

$$\{x_1, \ldots, x_n, y_1, \ldots, y_s\} = V'_1 \cup V'_2 \cup \cdots \cup V'_s$$

where $V'_j = V_j \cup \{y_j\}$, then this partition gives an $s$-colouring of $\Delta_\chi$. We can see from the characterization of the facets of $\Delta_\chi$ that each facet contains exactly one vertex from each of the sets $V'_1, \ldots, V'_s$, and hence $\Delta_\chi$ is pure of dimension $s - 1$ as well as balanced. \hfill \Box

Example 3.6. Let $\Delta = \langle x_1x_2x_3, x_2x_4, x_3x_4 \rangle$ and $\chi$ the colouring given by $\{x_1, x_4\} \cup \{x_2\} \cup \{x_3\}$ (see Example 3.2). The faces of $\Delta$ are

$$\Delta = \{\emptyset, x_1, x_2, x_3, x_4, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_1x_2x_3\}.$$ 

These are in one-to-one correspondence with the facets of $\Delta_\chi$:

$$\Delta_\chi = \{y_1y_2y_3, x_1y_2y_3, x_2y_1y_3, x_3y_1y_2, x_4y_2y_3, x_1x_2y_3, x_2x_3y_1, x_1x_3y_2, x_2x_4y_3, x_3x_4y_2, x_1x_2x_3\}.$$ 

We arrive at the main result of this section.

Theorem 3.7. For any simplicial complex $\Delta$, and any $s$-colouring $\chi$ of $\Delta$, the simplicial complex $\Delta_\chi$ is vertex decomposable.
Proof. Let $\Delta$ be a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$. We proceed by induction on $n$.

If $\Delta$ is the simplicial complex consisting of a single vertex $x_1$, then the only possible colourings of the vertices of $\Delta$ are of the form $V = V_1 \cup \cdots \cup V_s$ where $V_1 = \{x_1\}$ and $V_2, \ldots, V_s$ are empty. In this case $\Delta_\chi = \langle x_1 y_2 \ldots y_s, y_1 y_2 \ldots y_s \rangle$. This is vertex decomposable since $\Delta_\chi \setminus x_1 = \langle y_1 y_2 \ldots y_s \rangle$ and $\text{link}_{\Delta_\chi}(x_1) = \langle y_2 \ldots y_s \rangle$ are both simplices.

Now suppose that $\Delta$ is a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$, and let $\chi$ be the $s$-colouring of $\Delta$ given by $V = V_1 \cup \cdots \cup V_s$. We will show that we can decompose $\Delta_\chi$ by decomposing at any vertex $x_i$. Let $g_1, \ldots, g_t$ be the faces of $\Delta$ and define $g'_i = \{y_j \mid V_j \cap g_i = \emptyset\}$. So $g_1 \cup g'_1, \ldots, g_t \cup g'_t$ are the facets of $\Delta_\chi$.

We must show that both $\Delta_\chi \setminus x_i$ and $\text{link}_{\Delta_\chi}(x_i)$ are vertex decomposable. First consider the deletion. We may assume that the facets of $\Delta_\chi$ are ordered so that the facets $g_1 \cup g'_1, \ldots, g_t \cup g'_t$ do not contain the vertex $x_i$ and the facets $g_{r+1} \cup g'_{r+1}, \ldots, g_t \cup g'_t$ do contain $x_i$. So

$$\Delta \setminus x_i = \{\text{faces of } \Delta \text{ which do not contain } x_i\} = \{g_1, \ldots, g_t\}.$$ 

Note that we are using the fact that $g_1, \ldots, g_t, g_{r+1}, \ldots, g_t$ is a complete list of the faces of $\Delta$ by Theorem [3.5].

Without loss of generality we may assume that $x_i \in V_1$. Then $V \setminus \{x_i\} = (V_1 \setminus \{x_i\}) \cup V_2 \cup \cdots \cup V_s$ is an $s$-colouring of $\Delta \setminus x_i$. Call this $s$-colouring $\chi'$. Then $(\Delta \setminus x_i)_\chi' = \langle g_1 \cup g'_1, \ldots, (g_r \cup g'_r) \rangle = \Delta_\chi \setminus x_i$. Since $\Delta \setminus x_i$ is a simplicial complex on fewer than $n$ vertices, $(\Delta \setminus x_i)_{\chi'}$ is vertex decomposable.

Now consider the link. Since $(g_{r+1} \cup g'_{r+1}), \ldots, (g_t \cup g'_t)$ are the facets of $\Delta_\chi$ which contain $x_i$,

$$\text{link}_{\Delta_\chi}(x_i) = \langle (g_{r+1} \cup g'_{r+1}) \setminus \{x_i\}, \ldots, (g_t \cup g'_t) \setminus \{x_i\} \rangle$$

$$= \langle ((g_{r+1} \setminus \{x_i\}) \cup g'_{r+1}), \ldots, ((g_t \setminus \{x_i\}) \cup g'_t) \rangle.$$ 

For each $1 \leq j \leq s$, set $W_j = \{x_\ell \in V_j \mid x_\ell \in \text{link}_{\Delta}(x_i)\}$. Note that some of these sets may be empty. Then $W = W_1 \cup \cdots \cup W_s$ is an $s$-colouring of $\text{link}_{\Delta}(x_i)$. We call this $s$-colouring $\chi''$. Then

$$\text{link}_{\Delta}(x_i)_{\chi''} = \text{link}_{\Delta_\chi}(x_i)$$

and by induction $(\text{link}_{\Delta}(x_i))_{\chi''}$ is vertex decomposable. \hfill \Box

The fact that $\Delta_\chi$ is vertex decomposable has a number of consequences.

**Definition 3.8.** A pure $d$-dimensional simplicial complex $\Delta$ is *shellable* if there is an ordering $F_1, \ldots, F_s$ on the facets of $\Delta$ such that for all $1 \leq i < j \leq s$ there exists some $v \in F_j \setminus F_i$ and some $\ell \in \{1, \ldots, j-1\}$ with $F_j \setminus F_\ell = \{v\}$. Such an ordering on the facets is called a *shelling order*.

**Corollary 3.9.** For any simplicial complex $\Delta$, and any $s$-colouring $\chi$ of $\Delta$, the simplicial complex $\Delta_\chi$ is shellable, and thus, Cohen-Macaulay. Moreover, any order of the facets of
\( \Delta_x \) which refines the order given by ordering the faces of \( \Delta \) by increasing dimension is a shelling order.

**Proof.** By Theorem 3.7 \( \Delta_x \) is vertex decomposable, so by [10 Corollary 2.9] it is also shellable, and consequently, Cohen-Macaulay (e.g., see [13 Theorem 8.2.6]).

For the rest, let \( F_1, \ldots F_s \) be the facets of \( \Delta_x \). By Theorem 3.5, each \( F_i = g_i \cup g'_i \) where \( g_i \) is a face of \( \Delta \) and \( g'_i = \{ y_j \mid V_j \cap g_i = \emptyset \} \). We order the facets \( F_1, \ldots F_s \) so that for some \( \ell \) for some \( j \) if \( i < j \). We now show that this is a shelling order.

Let \( F_i, F_j \) be any two distinct facets of \( \Delta_x \) with \( i < j \). Since \( i < j \), we have \( \dim g_i \leq \dim g_j \) and so there is some \( x_u \in F_j \setminus F_i \). Since \( g_j \setminus \{ x_u \} \) is a face of \( \Delta \) we have \( g_j \setminus \{ x_u \} = g_\ell \) for some \( \ell \), and since \( \dim g_\ell < \dim g_j \) we have \( \ell < j \). Since \( g_\ell = g_j \setminus \{ x_u \} \) we must have \( g'_\ell = g'_j \cup \{ y_w \} \) where \( x_u \in V_w \). Then

\[
F_j \setminus F_\ell = (g_j \cup g'_j) \setminus (g_\ell \cup g'_\ell) = (g_j \cup g'_j) \setminus (g_j \setminus \{ x_u \}) \cup (g'_j \cup \{ y_w \}) = \{ x_u \}.
\]

Thus our ordering is a shelling order. \( \square \)

Results of Villarreal [19], Dochtermann and Engström [7], and Cook and Nagel [6] on the independence complexes of graphs now become special cases of Theorem 3.7. We first recall the relevant terminology. This terminology will be also be used in Section 5.

Let \( G = (V_G, E_G) \) be a finite simple graph on the vertex set \( V_G = \{ x_1, \ldots, x_n \} \) and edge set \( E_G \). One can use the independent sets of \( G \) to define a simplicial complex.

**Definition 3.10.** A subset \( W \subseteq V_G \) is an independent set of a graph \( G \) if for every edge \( e \in E_G \), we have \( e \not\subseteq W \). A set \( W \) is a maximal independent set if \( W \) is an independent set, but is not a proper subset of any other independent set of \( G \).

**Definition 3.11.** Let \( G \) be a graph. The independence complex of \( G \), denoted \( \text{Ind}(G) \), is the simplicial complex defined by

\[
\text{Ind}(G) = \{ W \subseteq V_G \mid W \text{ is an independent set of } G \}.
\]

The independence complex \( \text{Ind}(G) \) is sometimes called a flag complex.

**Definition 3.12.** The clique of order \( n \), denoted \( K_n \), is the graph with vertex set \( V = \{ x_1, \ldots, x_n \} \) and edge set \( \{ \{ x_i, x_j \} \mid 1 \leq i < j \leq n \} \). Note that an isolated vertex can be viewed as \( K_1 \).

Given any graph \( G \) and any subset \( S \subseteq V_G \), the induced graph on \( S \), denoted \( G|_S \), is the graph with vertex set \( S \) and edge set \( E_{G|_S} = \{ e \in E_G \mid e \subseteq S \} \).

**Definition 3.13.** Let \( G = (V_G, E_G) \) be a finite simple graph. A clique partition of \( V_G \) is a partition of \( V_G = V_1 \cup V_2 \cup \cdots \cup V_s \) such that each induced graph \( G|_{V_i} \) is a clique.
Construction 3.14 (Cook-Nagel). Let \( \pi \) denote a clique partition \( V_G = V_1 \cup \cdots \cup V_s \) for a finite simple graph \( G = (V_G, E_G) \). From \( G \) and \( \pi \), let \( G^\pi \) denote the finite simple graph on the vertex set \( V_{G^\pi} = V_G \cup \{y_1, \ldots, y_s\} \) and edge set
\[
E_{G^\pi} = E_G \cup \bigcup_{i=1}^s \{\{x, y_i\} \mid x \in V_i\}.
\]
In other words, add a new vertex for each partition \( V_i \), and join this new vertex to every vertex in \( V_i \). We call \( G^\pi \) a clique whiskering of \( G \).

Corollary 3.15 ([6, Theorem 3.3]). Let \( G \) be a graph, and let \( G^\pi \) denote the clique-whiskered graph. Then \( \text{Ind}(G^\pi) \) is vertex decomposable.

Proof. Let \( \pi \) be the clique partition \( V_G = V_1 \cup \cdots \cup V_s \). Then the faces of \( \text{Ind}(G^\pi) \) have the form \( \sigma \cup \tau \) where \( \sigma \) is an independent set of \( G \), i.e. \( \sigma \in \text{Ind}(G) \), \( \tau \) is a subset of \( \{y_1, \ldots, y_s\} \), and if \( y_j \in \tau \), then \( \sigma \cap V_j = \emptyset \). It now suffices to note that \( \pi \) is also an \( s \)-colouring of \( \text{Ind}(G) \), from which it will follow that \( \text{Ind}(G^\pi) = \text{Ind}(G)_\pi \). Indeed, for any facet \( F \in \text{Ind}(G) \), we must have \(|F \cap V_i| \leq 1 \) since \( F \) is an independent set but all vertices of \( V_i \) are adjacent since \( G|_{V_i} \) is a clique. \( \square \)

Remark 3.16. Villarreal [19] first introduced Construction 3.14 in the special case that the partition \( \pi \) was \( V_G = \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \). For this partition \( \pi \), it was shown in [7, Theorem 4.4] that \( \text{Ind}(G^\pi) \) was vertex decomposable.

4. \( h \)-vectors and algebraic consequences

In this section, we explore some consequences of Theorem 3.7. In particular, we show that any \( f \)-vector of a simplicial complex is also the \( h \)-vector of a balanced, vertex decomposable simplicial complex. This enables us to give a new characterization of \( f \)-vectors of simplicial complexes, which extends Björner, Frankl and Stanley’s [1] characterization. We also show that for any \( f \)-vector \( f(\Delta) \), there exists a square-free monomial ideal with a linear resolution whose graded Betti numbers are a function of \( f(\Delta) \). We relate this idea to recent work of Herzog, Sharifan, and Varbaro [15].

We begin by recalling the definition of an \( h \)-vector.

Definition 4.1. The \( h \)-vector \((h_0, h_1, \ldots, h_{d+1})\) of a \( d \)-dimensional simplicial complex \( \Delta \), denoted \( h(\Delta) \) is defined in terms of the \( f \)-vector \( f(\Delta) = (f_{-1}, f_0, \ldots, f_d) \) as follows
\[
h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}(\Delta).
\]

We can use Corollary 3.9 to give a proof of the following result:
Theorem 4.2. The following containment of sets holds:
\[
\left\{ \text{f-vectors of simplicial complexes} \right\} \subseteq \left\{ \text{h-vectors of balanced vertex decomposable simplicial complexes} \right\}.
\]

Proof. Let \( f(\Delta) \) be the f-vector of a simplicial complex \( \Delta \). For any \( s \)-colouring \( \chi \) of \( \Delta \), \( \Delta_\chi \) is a balanced vertex decomposable simplicial complex by Theorems 3.5 and 3.7, and thus shellable. We will show the h-vector of \( \Delta_\chi \) is \( f(\Delta) \).

If \( F_1, \ldots, F_s \) are the facets of \( \Delta_\chi \), then by Theorem 3.5, each \( F_i = g_i \cup g'_i \) where \( g_i \) is a face of \( \Delta \) and \( g'_i = \{ y_j \mid V_j \cap g_i = \emptyset \} \). Moreover, by Corollary 3.9, we have a shelling if we order the facets \( F_1, \ldots, F_s \) so that \( \dim g_i \leq \dim g_j \) if \( i < j \).

Because we have a shelling, \([13, \text{Proposition 8.2.7}]\) allows us to construct the following partition of \( \Delta_\chi \):
\[
\Delta_\chi = \bigcup_{i=1}^{s} [\mathcal{R}(F_i), F_i].
\]
Here, \([G, F]\) is an interval, i.e., \([G, F] = \{ H \in \Delta_\chi \mid G \subseteq H \subseteq F \}\) and
\[
\mathcal{R}(F_i) = \{ z \in F_i \mid F_i \setminus \{ z \} \in \langle F_1, \ldots, F_{i-1} \rangle \}.
\]

By \([18, \text{Proposition 2.3}]\), the h-vector of \( \Delta_\chi \) satisfies
\[
h_i = |\{ j \mid |\mathcal{R}(F_j)| = i \}| \text{ for } i = 0, \ldots, d + 1.
\]

The conclusion now follows from the fact that \( \mathcal{R}(F_j) = g_j \), so \( h_i \) counts the number of faces of dimension \( i - 1 \) in \( \Delta \), whence \( h(\Delta_\chi) = (h_0, \ldots, h_{d+1}) = (f_{-1}, f_1, \ldots, f_d) = f(\Delta) \). □

Theorem 4.2 allows us to add a new equivalent statement to a theorem of Björner, Frankl, and Stanley \([1]\).

Theorem 4.3. Let \( m = (m_1, \ldots, m_t) \in \mathbb{Z}_+^t \). Then the following are equivalent:

(i) \( m \) is the f-vector of a simplicial complex.
(ii) \( m \) is the h-vector of a balanced, vertex decomposable simplicial complex.
(iii) \( m \) is the h-vector of a balanced, shellable simplicial complex.
(iv) \( m \) is the h-vector of a balanced, Cohen-Macaulay simplicial complex.

Proof. Theorem 1.2 gives (i) ⇒ (ii). The statements (ii) ⇒ (iii) and (iii) ⇒ (iv) follow from the implications:

vertex decomposable ⇒ shellable ⇒ Cohen-Macaulay.

Finally, (iv) ⇒ (i) was first proved by Stanley \([17]\). The equivalence of (i), (iii) and (iv) were first shown in \([1]\), albeit in a much more general setting. □

It is natural to ask if Theorem 4.3 still holds if we restrict to smaller classes of simplicial complexes. For example, it has been asked whether the above statements still hold
if we replace an arbitrary simplicial complex with the class of flag complexes. In particular, Cook and Nagel [6], and Constantinescu and Varbaro [4] have posited the following conjecture (the conjecture of Cook and Nagel does not include the word balanced):

**Conjecture 4.4.** The following equality of sets holds:

\[
\left\{ \text{\textit{f}-vectors of flag complexes} \right\} = \left\{ \text{\textit{h}-vectors of balanced vertex decomposable flag complexes} \right\}
\]

One can show that the containment of Theorem 4.2 still holds true for flag complexes. We omit the proof here, but instead point the reader to the proofs of [6, Corollary 3.10] and [4, Proposition 4.1]. The second proof is interesting since the authors use basically the same construction as Construction 3.1, but in the special case that the colouring is given by the partition \( V = \{x_1\} \cup \cdots \cup \{x_n\} \). In some special cases, e.g., bipartite graphs (see [4]), the conjecture has been proved. We add additional evidence for Conjecture 4.4 when we prove the statement for the flag complexes of chordal graphs in the next section.

We conclude this section by showing how to use Theorem 4.2 to find the graded Betti numbers of the Alexander dual of the Stanley-Reisner ideal associated to \( I_\Delta \). The following well-known definition connects simplicial complexes and monomial ideals.

**Definition 4.5.** Given a simplicial complex \( \Delta \) on the vertex set \( \{x_1, \ldots, x_n\} \), the **Stanley-Reisner ideal** of \( \Delta \) is the monomial ideal

\[
I_\Delta = (x_{i_1}x_{i_2}\cdots x_{i_s} \mid \{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\} \notin \Delta)
\]

in the ring \( S = k[x_1, \ldots, x_n] \). The quotient ring \( S/I_\Delta \) is the **Stanley-Reisner ring**.

For completeness, we also recall the definition of the Alexander dual.

**Definition 4.6.** Given a subset \( \sigma \subseteq \{x_1, \ldots, x_n\} \), let \( \overline{\sigma} = \{x_1, \ldots, x_n\} \setminus \sigma \). The **Alexander dual** of a simplicial complex \( \Delta \), denoted \( \Delta^\vee \), is the simplicial complex \( \Delta^\vee = \{\sigma \mid \sigma \notin \Delta\} \).

**Theorem 4.7.** Let \((f_1, f_0, \ldots, f_d)\) be the \textit{f}-vector of a \textit{d}-dimensional simplicial complex \( \Delta \) on \( V = \{x_1, \ldots, x_n\} \), and let \( \chi \) be any \textit{s}-colouring of \( \Delta \). The graded Betti numbers of \( I_{\Delta^\vee} \) in \( R \) are given by the formula

\[
\beta_{i,i+n}(I_{\Delta^\vee}) = \sum_{j=1}^{d+1} \binom{j}{i} f_{j-1}(\Delta).
\]

In particular, \( \text{proj-dim}(I_{\Delta^\vee}) = \text{reg}(R/I_{\Delta^\vee}) = d + 1 \).

**Proof.** The projective dimension follows directly from our formula, and for the regularity, we use the identity (e.g., see [13, Proposition 8.1.10]) that \( \text{proj-dim}(I_{\Delta^\vee}) = \text{reg}(R/I_{\Delta^\vee}) \).

Because \( \Delta^\vee \) is pure and vertex decomposable (and thus shellable), [8, Corollary 5] gives

\[
\sum_{i \geq 1} \beta_i(R/I_{\Delta^\vee})t^{i-1} = \sum_{i \geq 0} h_i(\Delta^\vee)(t+1)^i.
\]
Note that in [8], the authors are taking the resolution of $R/I_{\Delta^\chi}$, so $\beta_i(R/I_{\Delta^\chi}) = \beta_{i-1}(I_{\Delta^\chi})$. Furthermore, although the formula of [8] is expressed in terms of total graded Betti numbers, the resolution of $I_{\Delta^\chi}$ is linear (this is because $\Delta^\chi$ is shellable and pure of dimension $s-1$, and hence $I_{\Delta^\chi}$ is generated in degree $n$ and is componentwise linear, which implies the ideal has a linear resolution). We therefore have $\beta_{i-1}(I_{\Delta^\chi}) = \beta_{i-1,n+i-1}(I_{\Delta^\chi})$.

To finish the proof, Theorem 4.2 allows us to replace $h_i(\Delta^\chi)$ with $f_{i-1}(\Delta)$ in the formula (4.1), thus giving the desired formula for of $\beta_{i-1,n+i-1}(I_{\Delta^\chi})$. □

Example 4.8. Let $\Delta$ be the simplicial complex of Example 3.2 and let $\chi$ be the 3-colouring given by the partition $\{x_1, x_4\} \cup \{x_2\} \cup \{x_3\}$. The $f$-vector of $\Delta$ is $f(\Delta) = (1, 4, 5, 1)$. Then applying the formula from Theorem 4.7, we see that the Betti numbers of $I_{\Delta^\chi}$ are $\beta_{0,4}(I_{\Delta^\chi}) = 11$, $\beta_{1,5}(I_{\Delta^\chi}) = 17$, $\beta_{2,6}(I_{\Delta^\chi}) = 8$, $\beta_{3,7}(I_{\Delta^\chi}) = 1$.

Remark 4.9. For any valid $f$-vector $f(\Delta) = (f_0, \ldots, f_d)$, the sequence

$$(4.2) \quad \left( \sum_{j=0}^{d+1} \binom{j}{0} f_{j-1}(\Delta), \sum_{j=1}^{d+1} \binom{j}{1} f_{j-1}(\Delta), \ldots, \sum_{d+1} \binom{j}{d+1} f_{j-1}(\Delta) \right)$$

is a valid sequence of Betti numbers for an ideal with a linear resolution by Theorem 4.7. Herzog, Sharifan, and Varbaro [15] classified all valid sequences of Betti numbers for an ideal with a linear resolution. In particular, they proved that $m = (m_0, m_1, \ldots, m_{d+1})$ is an $O$-sequence if and only if

$$( \sum_{j=0}^{d+1} \binom{j}{0} m_j, \sum_{j=1}^{d+1} \binom{j}{1} m_j, \ldots, \sum_{d+1} \binom{j}{d+1} m_j )$$

is a valid sequence of Betti numbers for an ideal with a linear resolution.

Because $f$-vectors are $O$-sequences, [15] also implies that (4.2) is the Betti sequence of an ideal with linear resolution. Our work, in particular Theorem 4.7 highlights how to start with a simplicial complex with a given $f$-vector, and find a square-free monomial ideal whose graded linear resolution has Betti sequence given by (4.2). This contrasts with the main results of [15] since the ideal they construct with Betti sequence (4.2) need not be a square-free monomial ideal.

5. Application: Independence complexes of chordal graphs

In the previous section, we saw how to construct a balanced, vertex decomposable simplicial complex $\Delta$, from any simplicial complex $\Delta$ and any $s$-colouring $\chi$ of $\Delta$ with the property that $f(\Delta) = h(\Delta^\chi)$. In this section, we give a criterion for when this construction can be reversed. As an application, we study $h$-vectors and $f$-vectors of the independence complexes of chordal graphs.
We start with our criterion for “reversing” the process of the last section.

**Definition 5.1.** Let $\Delta$ be a simplicial complex on the vertex set $V$ and let $W \subseteq V$. The *restriction of $\Delta$ to $W$* is the subcomplex 
$$
\Delta|_W = \{ F \in \Delta \mid F \subseteq W \}.
$$

**Definition 5.2.** Suppose $\Delta = \langle F_1, \ldots, F_s \rangle$ is a simplicial complex on the vertex set $V$. We say that $\Delta$ has a *facet restriction with respect to $F$* if $F$ is a facet of $\Delta$ such that 
$$
\Delta|_{V\setminus F} = \{ F_1 \setminus F, \ldots, F_s \setminus F \}.
$$

Note that the inclusion $\Delta|_{V\setminus F} \supseteq \{ F_1 \setminus F, \ldots, F_s \setminus F \}$ always holds; however, in general the two sets may not be equal as we see in the following example.

**Example 5.3.** Let $\Delta = \langle 123, 234, 345, 456 \rangle$ (see Figure 2). By considering each facet of $\Delta$, we can show it has no facet restriction. Let $F$ be the facet 123. Then 
$$
\Delta|_{V\setminus F} = \Delta|_{456} = \{ \emptyset, 4, 5, 6, 45, 56, 46, 456 \} \neq \{ 123 \setminus F, 234 \setminus F, 345 \setminus F, 456 \setminus F \} = \{ \emptyset, 4, 45, 456 \}.
$$
Similarly, if we consider the facet 234 we see that 
$$
\Delta|_{V\setminus 234} = \Delta|_{156} = \{ \emptyset, 1, 5, 6, 56 \} \neq \{ 123 \setminus 234, 234 \setminus 234, 345 \setminus 234, 456 \setminus 234 \} = \{ 1, \emptyset, 5, 56 \}.
$$
By symmetry we also have 
$$
\Delta|_{V\setminus 345} \neq \{ 123 \setminus 345, 234 \setminus 345, 345 \setminus 345, 456 \setminus 345 \}
$$
and 
$$
\Delta|_{V\setminus 456} \neq \{ 123 \setminus 456, 234 \setminus 456, 345 \setminus 456, 456 \setminus 456 \}.
$$
Therefore the simplicial complex $\Delta$ has no facet restriction.

![Figure 2](image-url)  
Figure 2. The simplicial complex $\langle 123, 234, 345, 456 \rangle$ has no facet restriction.

**Example 5.4.** Let $\Delta$ be the simplicial complex $\langle 124, 245, 235, 456 \rangle$ (see Figure 3). Then $\Delta$ has a facet restriction with respect to the facet 245 since 
$$
\Delta|_{V\setminus 245} = \Delta|_{136} = \{ \emptyset, 1, 3, 6 \} = \{ 124 \setminus 245, 245 \setminus 245, 235 \setminus 245, 456 \setminus 245 \}.
$$
The existence of a facet restriction allows us to find a converse to Theorem 4.2.
Figure 3. The simplicial complex \( \langle 124, 245, 235, 456 \rangle \) has a facet restriction with respect to the facet 245.

**Theorem 5.5.** Let \( \Delta = \langle F_1, \ldots, F_t \rangle \) be a pure, balanced simplicial complex such that \( \Delta \) has a facet restriction with respect to the facet \( F \). Then \( \Delta = (\Delta|_{V \setminus F})\chi \) where \( \chi \) is the colouring induced from the colouring of \( \Delta \). In particular, \( \Delta \) is vertex decomposable and \( h(\Delta) = f(\Delta|_{V \setminus F}) \).

**Proof.** Let \( d - 1 \) be the dimension of \( \Delta \). Because \( \Delta \) is pure and balanced, the colouring \( \chi \) is given by a partition \( V = V_1 \cup V_2 \cup \cdots \cup V_d \) such that \( |F_j \cap V_i| = 1 \) for all \( 1 \leq j \leq t \) and \( 1 \leq i \leq d \). After relabelling, we can assume that \( F_1 \) is the facet that gives that facet restriction. Note that \( \Delta|_{V \setminus F_1} \) is a simplicial complex on \( Y = V \setminus F_1 \), and is \( d \)-colourable since \( \Delta|_{V \setminus F_1} \) inherits a colouring from \( \chi \) given by:

\[
Y = V \setminus F_1 = (V_1 \setminus F_1) \cup (V_2 \setminus F_1) \cup \cdots \cup (V_d \setminus F_1).
\]

Abusing notation, let \( \chi \) denote this new colouring. Then \( (\Delta|_{V \setminus F_1})\chi \) is a balanced vertex decomposable simplicial complex such that \( h((\Delta|_{V \setminus F_1})\chi) = f(\Delta|_{V \setminus F_1}) \) by Theorem 4.2.

To complete the proof, it suffices to show that \( (\Delta|_{V \setminus F_1})\chi \) and \( \Delta \) are the same simplicial complexes, but with a different labelling of the vertices. By Theorem 3.5, the facets of \( (\Delta|_{V \setminus F_1})\chi \) are in one-to-one correspondence with the faces of \( \Delta|_{V \setminus F_1} \). But we also have that the facets of \( \Delta \) are in one-to-one correspondence with the faces of \( \Delta|_{V \setminus F_1} \) via the map \( F_i \mapsto F_i \setminus F_1 \). Indeed, this map is clearly onto by our assumption that \( \Delta \) has a facet restriction with respect to \( F_1 \). It suffices to show that this map is one-to-one. So, suppose \( F_i \setminus F_1 = F_j \setminus F_1 \), but \( F_i \neq F_j \). This means that there is a vertex \( x \in F_i \setminus F_j \) because the simplicial complex is pure. Since \( \Delta \) is balanced, there is a vertex \( y \in F_j \setminus F_i \) with the same colour as \( x \). Because \( F_i \setminus F_1 = F_j \setminus F_1 \), we must have \( x \) and \( y \) in \( F_1 \). But this contradicts the colouring of \( \Delta \). By combining these two one-to-one correspondences, we get the desired bijection between the facets of \( \Delta \) and \( (\Delta|_{V \setminus F_1})\chi \).

**Example 5.6.** In Example 5.4 we saw that the simplicial complex

\[
\Delta = \langle 124, 245, 235, 456 \rangle
\]
has a facet restriction with respect to the facet 245. Since $\Delta|_{136} = \{\emptyset, 1, 3, 6\}$, the $f$-vector of $\Delta|_{136}$ is $f(\Delta|_{136}) = (1, 3)$. Therefore the $h$-vector of $\Delta$ is $h(\Delta) = f(\Delta|_{136}) = (1, 3)$.

To apply Theorem 5.5 to a class of simplicial complexes, we need to justify the existence of facet restrictions. We round out this paper by focusing on the independence complexes (as introduced in Section 3) of chordal graphs. We first recall:

**Definition 5.7.** A graph $G$ is **chordal** if every induced cycle of $G$ of length $\geq 4$ has a chord.

We will prove the following fact about the independence complexes of chordal graphs.

**Lemma 5.8.** Let $\Delta = \text{Ind}(G)$ be the independence complex of a chordal graph $G$. If $\Delta$ is also pure, then $\Delta$ has a facet restriction.

From this lemma, we can deduce the following result, which proves a special case of Conjecture 4.4.

**Theorem 5.9.** We have the following equivalence of sets:

$$\left\{ \text{f-vectors of independence complexes of chordal graphs} \right\} = \left\{ \text{h-vectors of balanced, vertex decomposable independence complexes of chordal graphs} \right\}.$$

**Proof.** If $f(\Delta)$ is the $f$-vector of $\Delta = \text{Ind}(G)$ when $G$ is chordal, then for any colouring $\chi$ of $\Delta = \text{Ind}(G)$, the simplicial complex $\Delta_\chi$ is balanced and vertex decomposable by Theorem 3.7 and $f(\Delta) = h(\Delta_\chi)$ by Theorem 4.2. It remains to note that $\Delta_\chi$ is the independence complex of the graph $G^\chi$, the clique whiskering of $G$ using the clique partition of $G$ induced by $\chi$. Furthermore, it follows from Construction 3.14 that if $G$ is chordal, then so is $G^\chi$. This completes the first containment.

To show the reverse containment, let $G$ be any chordal graph such that $\Delta = \text{Ind}(G)$ is balanced and vertex decomposable. Because $\Delta$ is vertex decomposable, and thus pure, by Lemma 5.8 the simplicial complex $\Delta$ has a facet restriction with respect to some facet $F$. But then by Theorem 5.5 we have $h(\Delta) = f(\Delta|_{V \setminus F})$. To complete the argument, we note that

$$\Delta|_{V \setminus F} = \text{Ind}(G)|_{V \setminus F} = \text{Ind}(G|_{V \setminus F}).$$

The graph $G|_{V \setminus F}$ is an induced subgraph of a chordal graph, and so is a chordal graph. So $h(\Delta) = f(\text{Ind}(G|_{V \setminus F}))$, thus completing the proof.

**Remark 5.10.** We note that if a chordal graph has a pure independence complex then that independence complex is always vertex decomposable and balanced, so the right-hand side of the above corollary could be the set of $h$-vectors of pure independence complexes of chordal graphs.

To prove Lemma 5.8 we will require a result of Herzog, Hibi, and Zheng. We first describe another simplicial complex one can associate to a graph.
Definition 5.11. For any finite simple graph $G = (V_G, E_G)$ the clique complex of $G$ is the simplicial complex $Cl(G) = \{C \subseteq V \mid G|_C \text{ is a clique}\}$.

Definition 5.12. Let $\Delta$ be a simplicial complex with vertex set $V$. We call $v \in V$ a free vertex if $v$ is contained in exactly one facet of $\Delta$.

Theorem 5.13 ([14, Theorem 2.1]). Let $G$ be a chordal graph and let $C_1, \ldots, C_t$ be all the facets of $Cl(G)$ that contain a free vertex. The following are equivalent:

(a) $R/I_{\text{Ind}(G)}$ is Cohen-Macaulay.
(b) $G$ is unmixed, i.e., all maximal independent sets have the same cardinality.
(c) $V = C_1 \cup C_2 \cup \cdots \cup C_t$ is a partition of the vertices of $G$.

We are now ready to prove Lemma 5.8.

Proof. (of Lemma 5.8) Let $\Delta = \text{Ind}(G)$ be the independence complex of a chordal graph, and furthermore, assume $\Delta$ is pure. Let $C_1, \ldots, C_t$ be the facets of $Cl(G)$ which contain a free vertex. Since $\Delta$ is pure, we know that $G$ is unmixed. Thus by Theorem 5.13, we have the partition $V = C_1 \cup \cdots \cup C_t$. For $1 \leq i \leq t$, let $y_i$ be a free vertex of $Cl(G)$ contained in $C_i$. Set $F = \{y_1, \ldots, y_t\}$. We will show that $F$ is a facet of $\Delta$ and that $\Delta$ has a facet restriction with respect to $F$.

It is clear that $F = \{y_1, \ldots, y_t\}$ is an independent set since each $y_i$ is in a unique maximal clique $C_i$ and an edge $\{y_i, y_j\}$ would constitute a clique of size 2. Further, $F$ is a maximal independent set since every vertex $x \notin F$ is in some $C_i$ and therefore adjacent to $y_i$. Since $\Delta$ is assumed to be pure, this means that every facet has size $t$.

Finally, let $F_1, \ldots, F_s$ be the facets of $\Delta$. To finish the proof we will show that $\Delta|_{V \setminus F} = \{F_1 \setminus F, \ldots, F_s \setminus F\}$.

We simply need to show $\Delta|_{V \setminus F} \subseteq \{F_1 \setminus F, \ldots, F_s \setminus F\}$. Let $H \in \Delta|_{V \setminus F}$, and define $H' = H \cup \{y_i \mid C_i \cap H = \emptyset\}$. Then $H'$ is independent since the neighbours of $y_i$ are the elements of $C_i \setminus \{y_i\}$. Since $H'$ has cardinality $t$, it is a facet of $\Delta$. Therefore $H = H' \setminus F$ which proves that $\Delta|_{V \setminus F} = \{F_1 \setminus F, \ldots, F_s \setminus F\}$. \qed

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