How much electric surcharge fits on ... a white dwarf star?

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An exactly solvable approximate model for an electrically non-neutral white dwarf star is introduced and solved entirely in terms of simple elementary functions. The model generalizes the well-known polytropic gas ball of index $n = 1$ to a two-species setting (electrons and protons, for simplicity). Given the number of protons, a maximal negatively and a maximal positively charged solution exists, plus a continuous family of solutions which interpolates between these extremes. This exactly solvable model captures the qualitative behavior of the proper physical model of a non-neutral white dwarf, and it correctly answers the question: given the number of protons in a white dwarf, how many electrons can there be? The answer (in the form: as few as $\mathcal{A}$ and as many as $\mathcal{B}$) is independent of the speed of light $c$ and the Planck quantum $\hbar$. It is shown to be ‘universal,’ valid also for all physical models, with non-relativistic or relativistic, classical or quantum pressure-density relation.

I. INTRODUCTION

Everyday matter typically appears to be electrically neutral, yet most people are familiar with the fact that objects of all kind can sometimes be charged with a surplus of electricity, our bodies included. Anyone who has ever walked on synthetic carpet with the ‘wrong’ kind of shoes and then touched a door handle knows how unpleasant a surprise the ensuing spark can be as the accumulation of shoes and then touched a door handle knows how unpleasant a surprise the ensuing spark can be as the accumulation of charge is neutralized.

One could ask, how much surcharge can a general object hold? Empirically the answer is: “never much.” But how much, exactly, depends on many things, for instance on the material which is being charged, on its shape, its temperature, the environment, and such, and is not easy to calculate theoretically.

Incidentally, the answer to our question should not be confused with (self-)capacitance, a superficially related concept treated in introductory E\&K courses, e.g. [1], chpt.1, sect.11. In this case one considers a perfect conductor held at a constant electric potential against ground, and defines its capacitance as the ratio of the charge $Q = Q(V)$ it holds over the voltage $V$ which is applied. However, the ratio $Q/V$ yields only the slope of the response function $Q(V)$ in its linear regime. Eventually $Q(V)$ becomes nonlinear and reaches either a positive maximum or negative minimum, depending on whether electrons are being stripped off of the conductor or transferred to it. Our question “how much charge fits onto an object?” refers to these two extremal amounts of electrons which can be stripped off, respectively deposited on an object.

Moreover, ‘object’ may or may not mean a conductor (think of the familiar experiment where a glass rod is charged positively through the stripping off of some electrons by rubbing it with a silk cloth, for instance). It simply is a physical system consisting of a fixed number $N_+$ of nuclei arranged in an (essentially) fixed shape, comprising $N_p \geq N_+$ protons. For a neutral object, the number of electrons $N_e = N_p$. Since the electron mass $m_e$ is so much smaller than the proton mass $m_p$, and since empirically any possible surcharge is small, the mass of the object is essentially determined by the $N_+$ nuclei.

Furthermore, an ‘object’ does not need to be macroscopic. An important example is an atomic ion, having a single nucleus (so $N_+ = 1$) with $N_p = Z \in \{1, 2, ..., 118\}$ elementary charges $e$. There are positive and negative ions. Empirically it becomes exceedingly difficult with growing $Z$ to strip off all $Z$ electrons from a neutral atom, though in principle it can be done. Yet, and still empirically, not more than two or three surcharge electrons can be placed on a neutral atom to create a negative ion, and an $\alpha$ particle (a helium-4 nucleus with two elementary charges) does not seem to bind more than two electrons. Theoretically a nucleus without any electrons is of course allowed in quantum mechanics. It is also known that the $Z$-electron Schrödinger–Pauli Hamiltonian of an atom with a nucleus of $Z$ elementary charges always has a bound state. However, it is an extremely difficult open question to theoretically determine the number of surcharge electrons which such a nucleus can bind, as per the many-body Schrödinger–Pauli equation; see [2], pp.164–178.

In this paper we show that the answer to our question can be found explicitly for a spherical white dwarf star. For simplicity we will mostly work with the non-relativistic theory, and we assume the star is made entirely of individual protons and electrons (a ‘failed star’ which never ignited and which simply cooled down to its lowest energy state). The only forces at work are Newton’s gravity, Coulomb’s electricity, and the gradients of the degeneracy pressures of the protons and the electrons. The relativistic theory will be commented on, though.

To answer the surcharge question for a white dwarf star one needs to talk about its structure equations. Since these are not solvable in closed form, to gain some insight we first consider a mathematical approximation to the physical white dwarf model: we change the polytropic power $5/3$, predicted by quantum mechanics for the pressure law, into the power $6/3 = 2$. This alteration is small, and it has the advantage that the structure
equations become exactly solvable in terms of simple elementary functions. Our model joins the ranks of exactly solvable models in statistical mechanics, such as the two-dimensional Ising model, which have provided valuable qualitative insights into the behavior of the more realistic physical models that require heavy use of numerical calculations. Furthermore, the approximate model gives the physically correct answer to our surcharge question, which is shown subsequently.

Incidentally, a power-2 pressure law is well known in the general theory of stellar structure, [3], [4], [5], and yields a polytrope of index \( n = 1 \). Aside from being discussed in the astrophysical literature, polytropes also appear in pedagogical papers, e.g. they are found in [6], [7], [8], [9], and [10]. In this vein we believe that our exactly solvable model could also be incorporated in an introductory graduate course on stellar structure.

In section II we recall the structure equations of a failed white dwarf star, consisting entirely of protons and electrons, each species being treated as an ideal Fermi gas of spin-\( \frac{1}{2} \) particles. To facilitate the comparison with the neutral approximate models discussed in [3], [4], [5], and in [7], [8], [9], [10], in a subsection we will temporarily invoke the usual local neutrality approximation which yields a single-density model. We will take the opportunity, in a further subsection to that subsection, to explain our \( \frac{5}{3} \rightarrow \frac{6}{3} \) approximation for the locally neutral single-density model, which produces the Lane–Emden polytrope of index \( n = 1 \) and its elementary solution.

Then in section III we will apply our \( \frac{5}{3} \rightarrow \frac{6}{3} \) approximation to the two-species model of a failed white dwarf and solve it explicitly in terms of simple elementary functions. This will allow us to answer our question: how much surcharge can a white dwarf hold? Of course, the answer is obtained at this point only with the help of the \( \frac{5}{3} \rightarrow \frac{6}{3} \) approximation.

In section IV we explain that our result is robust and does not depend on our approximation. We will present a compelling argument for why it should actually not depend on whether classical or quantum physics is used, or whether non-relativistic or special-relativistic physics is employed to compute the pressure-density relation.

In section V we apply our findings to the Kepler problem of charged binary stars.

Section VI illustrates our findings.

The conclusions are presented in section VII.

II. THE STRUCTURE EQUATIONS OF A FAILED WHITE DWARF STAR

The basic equations of structure of a non-rotating white dwarf star composed of electrons and nuclei can be found in Chandrasekhar’s original publications composed into his classic book [4], in [5], and also in [7], [8], for instance. For a non-rotating star one may assume spherical symmetry, so all the basic structure functions are then functions only of the radial distance \( r \) from the star’s center, and the differential equations involved in the discussion reduce to the ordinary type.

We specialize the discussion to a failed star composed only of protons and electrons, both of which are spin-\( \frac{1}{2} \) fermions. Each species is treated as an ideal quantum gas in its own right. The number density functions \( \nu_p(r) \geq 0 \) and \( \nu_e(r) \geq 0 \) are assumed to integrate to the total number of protons, respectively electrons, viz.

\[
\int_{\mathbb{R}^3} \nu_p(r) d^3r = N_p, \quad (1) \\
\int_{\mathbb{R}^3} \nu_e(r) d^3r = N_e. \quad (2)
\]

The protons have rest mass \( m_p \) and charge \( +e \), the electrons have rest mass \( m_e \) and charge \( -e \). Thus the mass density of the star is given by

\[
\mu(r) = m_p \nu_p(r) + m_e \nu_e(r) \quad (3)
\]

and its charge density by

\[
\sigma(r) = e \nu_p(r) - e \nu_e(r). \quad (4)
\]

This means the star is overall neutral if \( N_p = N_e \), otherwise it carries a surcharge which may have either sign.

The electrons and protons jointly produce a Newtonian gravitational potential \( \phi_N(r) \) and an electric Coulomb potential \( \phi_C(r) \). The Newton potential \( \phi_N \) is related to the mass density \( \mu \) by a radial Poisson equation,

\[
(r^2 \phi_N'(r))' = 4 \pi G \mu(r) r^2, \quad (5)
\]

where \( G \) is Newton’s constant of universal gravitation. Similarly, the Coulomb potential \( \phi_C \) is related to the charge density \( \sigma \) by a radial Poisson equation,

\[
- (r^2 \phi_C'(r))' = 4 \pi \sigma(r) r^2. \quad (6)
\]

As usual, the primes in (5) and (6) mean derivative with respect to the displayed argument, in this case \( r \).

Each species, the electrons and the protons, satisfies an Euler-type mechanical force balance equation,

\[
\nu_p(r) \left( -m_p \phi_N'(r) - e \phi_C'(r) \right) - p'_p(r) = 0, \quad (7)
\]

\[
\nu_e(r) \left( -m_e \phi_N'(r) + e \phi_C'(r) \right) - p'_e(r) = 0. \quad (8)
\]

Here, \( p_p \) and \( p_e \) are the degeneracy pressures of the ideal proton and electron gases, respectively. They are computed in any introductory statistical mechanics course which covers ideal quantum gases, e.g. [11], [12] and can also be found in [7] and [8]. For a non-relativistic gas of spin-\( \frac{1}{2} \) fermions (subscript \( f \)) of mass \( m_f \) and number density \( \nu_f \) one has (see, e.g. [4], p.362; see also [13])

\[
p_f(r) = \frac{\hbar^2}{m_f} \left( \frac{3 \pi^2}{2} \right)^{2/3} \nu_f^{5/3}(r); \quad (9)
\]

here, \( f \) stands for either \( p \) or \( e \), and \( \hbar \) is the reduced Planck constant.

For later reference, we recall that any pressure-density relation of the type \( p = K \nu^\gamma \) for some constant \( K \nu \) is
called a polytropic law of power $\gamma$. So $\gamma = 5/3$ for the fermionic degeneracy pressure.

With the help of Eq.(9) and Eqs.(7) and (8), one can express the first radial derivatives of the Newton and Coulomb potentials in terms of the density functions and their first radial derivatives, which turns Eqs.(5) and (6) into a coupled system of two non-linear ordinary differential equations for $\nu_p$ and $\nu_e$; of course, one also needs to eliminate $\mu$ with the help of (3) and $\sigma$ with the help of (4). Each resulting equation is of second order and requires two initial conditions at $r = 0$. Naturally $\nu_p(0) = 0 = \nu_e(0)$. The values of $\nu_p(0)$ and $\nu_e(0)$ are to be chosen such that Eqs. (1) and (2) hold.

We will get to the '5/3-system' of equations for $\nu_p$ and $\nu_e$ in section IV. Here it shall suffice to emphasize that the nonlinearity of the resulting system of structure equations for $\nu_p$ and $\nu_e$ stands in the way of solving these equations explicitly in terms of known functions (except for one special case), but one can subject them to a rigorous analysis and to numerical integration on a computer.

For the most part in the present paper we will take advantage of the fact that the polytropic power 5/3 in the pressure law for a degenerate gas of ideal fermions is not too far away from the value 6/3 = 2, and when 5/3 is replaced by 6/3 the radial derivatives of the potentials become linear expressions in the radial derivatives of the densities. This changes the mathematical structure equations into a coupled linear system which can be solved exactly in terms of simple elementary functions. By discussing this linear '6/3 system' (see section III and our appendix) one learns quite a bit about the nonlinear '5/3 system' in its neighborhood, as illustrated in section VI with plots of both the 5/3 and the 6/3 model.

But first, before we come to this, we pause briefly to connect the structure equations of this section to the single-density model used in [4] and in any pedagogical discussion of stellar structure, e.g. [7], [8], [9], [10].

To obtain these single-density structure equations one invokes the local neutrality approximation: cf. [11], chpt.16, sect.9.5. It is based on the argument that the electrical coupling between electron and proton is about $10^{39}$ times stronger than their gravitational coupling, so that any local electric imbalance must be miniscule. Of course, if one wants to know how much surcharge a star can bind, then the local neutrality approximation, which implies global neutrality $N_p = N_e$, "throws the baby out with the bath." If one is primarily interested in a representative mass density function $\mu(r)$, this is a reasonable assumption though, as we will show in sections III-VI.

A. The local neutrality approximation

So suppose temporarily that $\nu_p(r) = \nu_e(r) =: \nu(r)$ for all $r$. Then $\sigma = 0$ by Eq.(4), and Eq.(6) is then solved by $\phi_C = 0$. Moreover, by Eq.(3) we now have $\mu(r) = (m_p + m_e)\nu(r)$. This is usually approximated further by neglecting the electron mass versus the proton mass, yet technically this does not yield a simplification.

A subtler step is the next one. We still have to deal with Eqs.(7) and (8), but having set $\nu_p = \nu_e = \nu$, we then have two different equations for one unknown, $\nu(r)$, and this overdetermines the problem, strictly speaking. What this shows is that the strict local neutrality approximation cannot be exactly correct, but of course it was never assumed to be exactly correct. Therefore, to proceed in the spirit of the approximation, one needs to mold the two equations (7) and (8) into one. This is done by replacing them by their sum, which in concert with $\phi_C = 0$ yields the mechanical force balance equation

$$-\mu(r)\phi'_N(r) - p'(r) = 0,$$

where the pressure function $p(r) = p_p(r) + p_e(r)$ reads

$$p(r) = \frac{\hbar^2}{2} \left( \frac{1}{m_p} + \frac{1}{m_e} \right) \frac{(3\pi^2)^{2/3}}{5} \nu^{5/3}(r).$$

This is usually approximated further by neglecting $1/m_p$ versus $1/m_e$, yet again technically this does not yield a simplification either.

Since $\mu(r) = (m_p + m_e)\nu(r)$, Eq.(10) with $p(r)$ given by (11) can be integrated once to yield $\phi_N$ as a function of $\nu$, which can be inverted to yield

$$\nu(r) = \left( \frac{2}{(3\pi^2)^{2/3}} \frac{m_p m_e}{\hbar^2} \left[ \phi_N - \phi_N(r) \right]^\gamma \right)^{3/2};$$

here, the notation $[g]_+$ means "positive part," i.e. $[g]_+(r) = g(r) > 0$ for $0 < r < R$, where $R$ is the smallest $r$-value for which $g(r) = 0$, and $[g]_+(r) = 0$ for $r \geq R$. Furthermore, $\phi_N$ is a constant of integration determined by $\int \nu(r) d^3r = N_p$. Inserting this relation into the Poisson equation (5) yields the familiar Lane-Emden equation of the polytropic gas ball for $\gamma = 5/3$, equivalently of index $n := 1/(\gamma - 1) = 3/2$,

$$\frac{1}{r^2} \left( r^2 \nu'(r) \right)' = C [\phi_N - \phi_N(r)]^\gamma_+,$$

$$C = \frac{2^{7/2}}{3\pi} \frac{G}{\hbar^2} (m_p + m_e) (m_p m_e)^{3/2};$$

see [3], [4], [5]. By shifting and scaling, Eq.(13) can easily be brought into the dimensionless standardized format

$$-\frac{1}{2} \bigl( \xi^2 \nu'(\xi) \bigr)' = \theta_{\gamma/2}(\xi),$$

complemented with the initial conditions $\theta(0) = 1$ and $\theta'(0) = 0$; cf. [3], [4], [7]. The equations for the polytropic gas balls, or gas spheres as they are often called, have been studied extensively in the astrophysical literature in dependence on their parameter $\gamma$, respectively $n$. For $\gamma = \infty$, $\gamma = 2$, and $\gamma = 6/5$ ($n = 0$, $n = 1$, and $n = 5$) the polytropic gas ball equation can be solved in terms of elementary functions, in all other cases the equation itself defines the polytropic density functions. In particular the case $\gamma = 5/3$ has been studied thoroughly due to its importance in the theory of white dwarf structure [4].

For our purposes the case $\gamma = 2$, viz. $n = 1$, is of particular interest because of our 5/3 $\to$ 6/3 approximation. As a primer we briefly discuss this approximation in the context of the single-density model.
The $\frac{5}{3} \to \frac{6}{3}$ approximation in the single-density model

Note that we cannot simply replace $\nu^{5/3}$ by $\nu^{6/3}$, for $\nu$ is not dimensionless. This can be overcome by switching to dimensionless densities with the help of some reference density. In the astrophysical literature one often finds the ‘central density’ as reference density, a choice motivated by seeking a definite initial value problem for the numerical integration of the Lane–Emden equation on a computer: the so-normalized dimensionless density takes the value 1 at $r = 0$, and its derivative vanishes there. We will be able to solve our equations explicitly, so we have no need for such a normalization. Instead, since the fermionic degeneracy pressure already is expressed with the microscopic constants $h, m_p, m_e$, we may as well now choose as reference length the electron’s reduced Compton length $\hbar/m_e c$, where $c$ is the speed of light in vacuum. While this is somewhat unconventional, it is not unnatural and the resulting formulas are easy to interpret. Thus we set $r =: \left(\hbar/m_e c\right) \rho$ and $v(r) =: (m_e c/\hbar)^3 \nu(\rho)$. Inserted into the formula for the degeneracy pressure, we find $p(r) \propto \nu(\rho)^{5/3}$, and since $\nu(\rho)$ is dimensionless, we may now replace $\nu^{5/3}$ by $\nu^{6/3} (= \nu')$.

This hurdle cleared, we next set $\phi_N(r) =: \epsilon^2 \psi_N(\rho)$ and proceed analogously to how we arrived at the polytropic equation with index $n = \frac{3}{2}$, this time it’s index $n = 1$, except that there is little incentive now to invert the linear relationship between $\psi_N$ and $\nu$, which results from the force balance equation (10) wherever $\nu(\rho) > 0$;

$$-\psi_N(\rho) = \epsilon K \nu'(\rho);$$  \hspace{1cm} (15)

the prime now means $\rho$ derivative. Here we introduced $\epsilon := m_e/m_p \approx 1/1836$ and $K := 2(3\pi^2)^{1/2}/5$. We can even avoid the step of integrating (15) and instead use it directly to eliminate $\psi'_N(\rho)$ (viz. $\psi'_N(r)$) from Eq.(5) in favor of $\nu'(\rho)$ to get

$$-\frac{1}{\epsilon^2} \left( r^2 \nu'(\rho) \right)' \rho = k^2 \nu(\rho),$$ \hspace{1cm} (16)

$$k^2 = \frac{10}{3^{2/3} \pi^{1/3}} \frac{G m_p(m_p + m_e)}{\hbar c}.$$ \hspace{1cm} (17)

Note that the Lane–Emden equation of index $n = 1$, Eq.(16), is valid until $\nu(\rho)$ runs into its first positive zero.

Several observations are in order.

First, we note that $G m_p(m_p + m_e)/\hbar c \approx 6 \times 10^{-39}$ is a gravitational analog of Sommerfeld’s fine structure constant $\epsilon^2/\hbar c := \alpha_s \approx 1/137.036$: it is much much smaller, though. This means that to see any appreciable effect in a solution of Eq.(16) the variable $\rho$ has to reach very large values. But this is only to be expected, for our unit of length is the reduced Compton length of the electron, and sure enough the structure of a star varies on scales which are gigantic in terms of these units.

Second, the Lane–Emden equation of index $n = 1$, Eq.(16), is not only linear, it is one of the three special cases which can be solved in terms of elementary functions. It is a special case of a Bessel-type differential equation and the solution relevant to our discussion is given by a spherical Bessel function, explicitly

$$\nu(\rho) = B \frac{\sin(k \rho)}{\rho}, \quad \rho \in (0, \pi/k),$$ \hspace{1cm} (18)

where the “bulk amplitude” $B$ is determined by $\int_0^\pi \nu(\rho) d^3\rho = N_p$.

Third, the radius of the star in this approximate single-density model is $R \approx \frac{\epsilon^{2/3} K \nu}{k^2}$ and $\nu$ may as well now

$$R \approx 2.2566 \times 10^{19} \frac{\hbar}{m_e c} \approx 8,714 \text{ km},$$ \hspace{1cm} (19)

i.e. $\approx 3/2$ earth radii, compatible with the accepted radius of white dwarf stars with half the mass of the sun.

Fourth, note that $R$ is independent of $N_p$ (or $N_e$ for this matter). This of course is not physically reasonable. However, we note that the physical range of acceptable values for $N_p$ (hence, $N_e$) is very narrow. Indeed, to have the interior of a gravitational object accurately modeled as an ideal Fermi gas, the mass needs to be sufficiently big, say $N_p > 10^{34}$ (Jupiter’s mass), and to be allowed to work with the non-relativistic approximation, it can’t be too big either, say $N_p < 10^{37}$ (a solar mass). Furthermore, we also assumed that the white dwarf failed to ignite, yet surely our sun did not. This assumption reduces the allowed range of $N_p$ further, perhaps $N_p < 10^{35}$. For such a narrow range of $N_p$ values it is not too unrealistic to have the model predict an $N_p$-independent radius, and a central density which increases proportional to $N_p$.

This concludes our excursion into the locally neutral single-density approximation. We now resume our quest for the maximal surplus charge on a white dwarf star.

III. THE 5/3 → 6/3 APPROXIMATION IN THE TWO-SPECIES MODEL

With the ground already paved in section II.A.1, we now apply our approximation to the two-species model of section II. We again set $r =: \left(\hbar/m_e c\right) \rho$, but now we have two density functions. Thus we set $\nu_p(r) =: (m_e c/\hbar)^3 \nu_p(\rho)$ and $\nu_e(r) =: (m_e c/\hbar)^3 \nu_e(\rho)$. Inserted into the formulas for the degeneracy pressures, we find $p_p(r) \propto \nu_p(\rho)^{5/3}$ and $p_e(r) \propto \nu_e(\rho)^{5/3}$, and now we can replace $\nu_p^{5/3}$ by $\nu_p^{6/3}$ and $\nu_e^{5/3}$ by $\nu_e^{6/3}$.

Also introducing dimensionless potential functions through $\phi_N(r) =: \epsilon^2 \psi_N(\rho)$ and $\phi_C(r) =: \epsilon^2 \frac{m_e}{m_p} \psi_C(\rho)$, and substituting the scaled expressions for the degeneracy pressures into Eqs.(7) and (8), these equations become

$$\nu_p(\rho) \left( -\psi_N(\rho) - \epsilon \psi_C(\rho) - \epsilon^2 K \nu_p(\rho) \right) = 0, \hspace{1cm} (20)$$

$$\nu_e(\rho) \left( -\psi_N(\rho) + \psi_C(\rho) - K \nu_e(\rho) \right) = 0, \hspace{1cm} (21)$$

with $\epsilon$ and $K$ as defined earlier; cf. (15).
To solve this system of equations we need to distinguish three regions:
(a) $v_p(\rho) > 0$ and $v_e(\rho) > 0$ (the “bulk region”),
(b) $v_p(\rho) > 0$ and $v_e(\rho) = 0$ (“positive atmosphere”),
(c) $v_p(\rho) = 0$ and $v_e(\rho) > 0$ (“negative atmosphere”).

A. The bulk region

In the bulk region both $v_p(\rho) > 0$ and $v_e(\rho) > 0$, so we can cancel the factor $v_p$ in (20) and the factor $v_e$ in (21) to obtain a genuine system of two linear inhomogeneous equations for $\psi'_p$ and $\psi'_e$, given $\psi'_p$ and $\psi'_e$. The determinant for this system is $-1 - \varepsilon$ and so there always exists a unique solution which expresses $\psi'_p$ and $\psi'_e$ as linear combinations of $\psi'_p$ and $\psi'_e$. It is easy enough to compute these expressions, but since we are interested in a coupled system of differential equations for the dimensionless density functions $v_p(\rho)$ and $v_e(\rho)$ we can use a shortcut to get there directly from Eqs.(20) and (21). Namely, first we rescale (5) and (6) into their dimensionless formats, obtaining

$$\frac{1}{\rho^2} \left( \rho^2 \psi'_p(\rho) \right)' = -\left( 1 - \frac{G_m}{\varepsilon v_p^2} \right) v_p(\rho) + \left( 1 + \frac{G_m}{\varepsilon v_e^2} \right) v_e(\rho),$$

$$\frac{1}{\rho^2} \left( \rho^2 \psi'_e(\rho) \right)' = \left( 1 + \frac{G_m}{\varepsilon v_p^2} \right) v_p(\rho) - \left( 1 - \frac{G_m}{\varepsilon v_e^2} \right) v_e(\rho).$$

valid wherever both $v_p(\rho) > 0$ and $v_e(\rho) > 0$. Here, $\varepsilon := \frac{\ell^2 m^3}{4\pi \hbar^3}$. This pair of coupled linear differential equations for the density functions $v_p$ and $v_e$ generalizes the single Lane–Emden equation for the polytrope of index $n = 1$, (16), in the common interior of the charged gases where both $v_p(\rho) > 0$ and $v_e(\rho) > 0$. We call this common interior the bulk region.

For the numerical coefficients, we have $\varepsilon \approx 5.54 \cdot 10^{-4}$ and $\varepsilon \approx 41.75$. The three ratios of gravitational-to-electrical coupling constants which appear in the coefficient matrix for system of right-hand sides of Eqs.(24) and (25) are fantastically tiny numbers, viz. $\frac{G_m^2}{\varepsilon^2} \approx 2.400 \cdot 10^{-43}$, $G_m v_p^2 / \varepsilon^{2} \approx 4.407 \cdot 10^{-40}$, and $G_m v_e^2 / \varepsilon^{2} \approx 8.09 \cdot 10^{-37}$. Yet one has to resist the impulse to neglect these tiny numbers versus 1 in the coefficients, for this would result in a singular coefficient matrix, and there would not be any nontrivial solution pair $v_p, v_e$. This becomes intuitively clear if one notes that the three tiny ratios of coupling constants are the only places where Newton’s constant of universal gravitation, $G$, enters the equations, and it is gravity, not electricity, which binds the ideal Fermi gases together to form a star.

We now solve the system of equations (24) and (25) explicitly. A non-singular system of linear second-order differential equations has four linearly independent solutions, from which we have to select the ones compatible with our physical problem. This is done as follows.

We remark that similarly to the Lane–Emden equation for the polytrope of index $n = 1$, (16), a change of dependent variables $v_p(\rho) \mapsto \rho v_p(\rho) =: \chi_p(\rho)$ and $v_e(\rho) \mapsto \rho v_e(\rho) =: \chi_e(\rho)$ transforms Eqs.(24) and (25) into a linear second-order system with constant coefficients for $\chi_p(\rho), \chi_e(\rho)$, and as one learns in an introductory differential equations course, such a system (when not singular) can always be solved by the ansatz $\chi(\rho) \propto \exp(\kappa \rho)$, with $\kappa$ standing for either $\rho$ or $e$. In terms of $v_p, v_e$ this means that the ansatz $v_p(\rho) = B_p \exp(\kappa \rho) / \rho$ and $v_e(\rho) = B_e \exp(\kappa \rho) / \rho$, with the same $\kappa$, will transform the system of differential equations (24) and (25) into a linear system of algebraic equations. Indeed, away from $\rho = 0$ we have

$$\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{d}{d\rho} \exp(\kappa \rho) / \rho \right) = \kappa^2 \exp(\kappa \rho) / \rho,$$

and so we obtain the matrix problem

$$\begin{pmatrix}
1 - \frac{G_m}{\varepsilon^2} & -\kappa^2 \varepsilon \\
-1 - \frac{G_m}{\varepsilon^2} & 1 - \frac{G_m}{\varepsilon^2} & -\kappa^2 \varepsilon
\end{pmatrix}
\begin{pmatrix}
B_p \\
B_e
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix},$$

here we have placed semi-colons in the matrix to facilitate the identification of the matrix elements. The solvability condition for Eq.(27) is the vanishing of the determinant of the coefficient matrix at its left-hand side, and this is a simple quadratic problem in $\kappa^2$. It is readily solved by “the quadratic formula” to give two real solutions $(\kappa^2)_\pm$. These formulas, although straightforward to compute, are a bit unwieldy, and we have relegated the details of their derivation and discussion to our appendix. What matters is the fact that at the end of the elementary computations one sees that one solution is positive and one negative, with $(\kappa^2)_+ \approx 44.0025$, and $(\kappa^2)_- \approx -1.94025 \cdot 10^{-38}$. From $(\kappa^2)_\pm$ one obtains the
associated $\kappa$ values by simply taking the positive and the negative square root of $(\kappa^2)_+ > 0$ and of $(\kappa^2)_- < 0$, respectively. The latter step obviously generates two imaginary $\kappa$ values. Converted to real solutions by taking appropriate linear combinations, the set of four linear independent solutions consists of one exponentially growing mode, one exponentially decaying mode, one sine and one cosine mode, each of them divided by the independent variable $\rho$.

Next we recall the well-known fact that Newton’s and Coulomb’s $1/r$ potentials correspond to a point source at $r = 0$, which we need to rule out. This means that the mode $\cos(\kappa \rho)/\rho$ is not admissible, while $\sin(\kappa \rho)/\rho$ is. Similarly, only the linear combination of the exponential modes into the hyperbolic $\sinh(\kappa \rho)/\rho$ mode is admissible, while all other linear combinations are not, in particular the hyperbolic $\cosh(\kappa \rho)/\rho$ mode is not admissible.

Thus, defining a hyperbolic $\kappa_h := \sqrt{(\kappa^2)_+} \approx 6.63344$ and a trigonometric $\kappa_t := \sqrt{(\kappa^2)_-} \approx 1.3929 \cdot 10^{-19}$, the physically admissible general solution of Eqs.(24) and (25) is of the form

$$v_p(\rho) = B_p^h \frac{\sinh(\kappa_h \rho)}{\rho} + B_p^t \frac{\sin(\kappa_t \rho)}{\rho},$$

$$v_c(\rho) = B_c^h \frac{\sinh(\kappa_h \rho)}{\rho} + B_c^t \frac{\sin(\kappa_t \rho)}{\rho},$$

where we have added superscripts $h$ and $t$ at the bulk region coefficients $B_p$ and $B_c$ to match with the “hyperbolic” and “trigonometric” modes. Here, the pairs $(B_p^h, B_c^h)$ and $(B_p^t, B_c^t)$ are eigenvectors of the coefficient matrix at the left-hand side of Eq.(27) for the corresponding eigenvalues $(\kappa^2)_\pm$, respectively, and so only two of the four bulk coefficients are independent in the general physical solution. Linear algebra yields the relationships between $B_p^h$ and $B_p^t$, respectively between $B_c^h$ and $B_c^t$, with the results

$$\frac{B_p^h}{B_p^t} = 1 - \frac{G m_e^2 - \varepsilon \kappa_t^2}{1 + \frac{G m_p m_e}{e^2}} \approx -5.45 \cdot 10^{-4},$$

$$\frac{B_c^h}{B_c^t} = 1 - \frac{G m_e^2 + \varepsilon \kappa_t^2}{1 + \frac{G m_p m_e}{e^2}} \approx 1 - 8.09 \cdot 10^{-37}. \quad (31)$$

We pause for a moment to take in the results obtained.

The trigonometric parts of the general solution obviously correspond to the $n = 1$ polytrope of the Lane–Emden equation for the single-density approximation, with $\kappa_t \approx \kappa$ given by Eq.(17) to a high degree of accuracy, which in concert with Eq.(31) confirms that the positive and negative large scale densities are very well approximated by the single-density model almost all the way up to the bulk radius. This explicitly vindicates working with the locally neutral single-density approximation to obtain the bulk structure of the white dwarf star.

In addition we now have information on the charge separation effects, which are accounted for by the hyperbolic parts of the general solution. These vary significantly on a very short scale by comparison, and so their amplitudes must be very tiny. Interestingly, the hyperbolic modes of the positive and negative species have significantly different bulk amplitudes, though, roughly corresponding in ratio to the ratio of the rest masses of electrons and protons.

The remaining two independent bulk amplitudes, say $B_p^t$ and $B_c^h$, cannot be fixed with the bulk densities alone; this requires also the atmospheric densities. By inspecting the general bulk solution formulas (28) and (29) it is easy to see, though, that $v_p(\rho) > 0$ can only be achieved with $B_p^t > 0$, while $B_p^h$ can take either sign. The analogous conclusion holds therefore for $B_c^t > 0$ and $B_c^h$. As soon as one or the other density reaches zero, the system of equations changes to describe the “atmospheric region,” unless it happens that both densities reach zero simultaneously (the case of “no atmosphere,” it can happen). An atmosphere can be populated either purely with protons or purely with electrons, yet either version is determined in a similar manner. We next turn to these atmospheric cases.

### B. The positive atmosphere

In the positive atmosphere the electron density vanishes, $v_e(\rho) = 0$, while the proton density is still positive, $v_p(\rho) > 0$, so we can cancel the factor $v_p$ at (20) while (21) does not contribute any equation now (for $v_e(\rho) = 0$). We thus have

$$-\psi_N(\rho) - \varepsilon \psi'_C(\rho) - \varepsilon^2 K v'_p(\rho) = 0. \quad (32)$$

Now we multiply (22) by $-1$ and (23) by $\varepsilon$ and add these equations, then use (32). Since $v_e = 0$, this now yields

$$-\varepsilon \frac{1}{\rho^2} \left(\rho^2 v'_p(\rho)\right)' = -\left(1 - \frac{G m_p^2}{e^2} \right) v_p(\rho), \quad (33)$$

valid for $\rho > \rho_0$, where $\rho_0 = \sup(\rho : v_e(\rho) > 0)$ is the radius of the bulk region. Note that (33) is (24) with $v_e = 0$. If $G m_p^2/e^2$ would be greater than 1, Eq.(33) would be a Lane–Emden equation of the $n = 1$ polytrope (mathematically speaking). However, since $G m_p^2/e^2$ is the tiny number it happens to be, Eq.(33) differs from this Lane–Emden equation by the sign of its right-hand side. All the same it is in the family of Bessel-type differential equations, being explicitly solved by exponential modes, divided by the independent variable $\rho$, viz. the general solution of (33) reads

$$v_p(\rho) = A_p^+ \frac{\exp(\kappa_p \rho)}{\rho} + A_p^- \frac{\exp(-\kappa_p \rho)}{\rho}, \quad (34)$$

where $\kappa_p > 0$ is the positive root of

$$\kappa_p^2 = \frac{10}{3^{2/3} \pi^{1/3} \hbar c} \left(1 - \frac{G m_p^2}{e^2}\right), \quad (35)$$
Note that in this expression one may approximate the last parenthetical factor by 1. Note furthermore that \( \zeta_p \approx 6.63 \) is essentially determined by the electrical coupling.

A few comments are in order right now.

First, it could seem reasonable to throw out the exponentially growing mode, but note that a small negative \( A_p^+ \) in concert with a large positive \( A_p^- \) will result in a \( v_p(\rho) \) which rapidly goes to zero in the positive atmosphere region, so an exponentially growing mode is not a problem because it would be terminated as soon as the proton density vanishes.

Second, since \( \rho > \rho_0 \), there is no reason now to only allow the linear combination of the exponential modes into the hyperbolic sine, as was the case in the bulk region where there would otherwise be a problem at the origin \( \rho = 0 \). Incidentally, equivalently to (34) we may write the general solution of the positive atmosphere as

\[
v_p(\rho) = A_p^+ \frac{\cosh(\zeta_p \rho)}{\rho} + A_p^- \frac{\sinh(\zeta_p \rho)}{\rho}.
\]

Third, the two “atmospheric amplitudes” \( A_p^+ \) and \( A_p^- \) are constrained by the requirement that the proton density \( v_p(\rho) \) be continuously differentiable at the boundary \( \rho = \rho_0 \) of the bulk region, so both are needed in general. We will get to this shortly.

C. The negative atmosphere

The discussion of the negative atmosphere region mirrors the one for the positive atmosphere region, so we may be brief. In the negative atmosphere the proton density vanishes, \( v_p(\rho) = 0 \), while the electron density is still positive, \( v_e(\rho) > 0 \), so we can cancel the factor \( v_e \) at (21), while (20) does not contribute any equation now (for \( v_p(\rho) = 0 \)). So we have

\[
-\psi_e'(\rho) + \psi_e(\rho) - K \psi_e(\rho) = 0.
\]

Now we multiply both (22) and (23) by \(-1\) and add these equations, then use (37). Since \( v_p = 0 \), this yields

\[
-\frac{1}{\rho^2} (\rho^2 \psi_e'(.))' = - \left( 1 - \frac{G m_e^2}{e^2} \right) \psi_e(\rho),
\]

valid for \( \rho > \rho_0 \), where now the radius of the bulk region is \( \rho_0 = \sup\{ \rho : v_p(\rho) > 0 \} \). Note that (38) is (25) with \( v_p = 0 \).

The general solution of (38) reads

\[
v_e(\rho) = A_e^+ \frac{\exp(\zeta_e \rho)}{\rho} + A_e^- \frac{\exp(-\zeta_e \rho)}{\rho},
\]

where \( \zeta_e > 0 \) is the positive root of

\[
\zeta_e^2 = \frac{10}{3^{2/3} \pi^{1/3}} \frac{e^2}{hc} \left( 1 - \frac{G m_e^2}{e^2} \right) .
\]

Note that \( \zeta_e^2 \approx \frac{m_e}{m_p} \zeta_p^2 \), where the “\( \approx \)” is due to some slight differences beginning to show 36 decimal places after the leading digit. Again, also in (40) one may approximate the last parenthetical factor by 1. Note that also \( \zeta_e \approx 0.155 \) is essentially determined by the electrical coupling.

Of course, equivalently to (39) we may also write the general solution of the negative atmosphere as

\[
v_e(\rho) = A_e^+ \frac{\cosh(\zeta_e \rho)}{\rho} + A_e^- \frac{\sinh(\zeta_e \rho)}{\rho}.
\]

The two “atmospheric amplitudes” \( A_e^+ \) and \( A_e^- \) are constrained by the requirement that the electron density \( v_e(\rho) \) be continuously differentiable at the boundary \( \rho = \rho_0 \) of the bulk region.

We will now address this matching of a positive or negative atmosphere to the bulk region.

D. The bulk-atmosphere interface

Having obtained the general physical solution type in the bulk region and the general physical solution type in the atmosphere region, which can be either an electron or a proton atmosphere, we now match these general solutions at their common bulk-atmosphere interface. Both cases, positive and negative atmosphere, can be discussed in parallel.

In the bulk region the two density functions together feature four amplitudes, but Eqs. (30) and (31) express the two electron amplitudes in terms of the two pertinent proton amplitudes, or the other way round. The density function of the atmosphere-forming species features two further amplitudes in the atmosphere region. It has to vary continuously differentiably across the boundary \( \rho_0 \) of the bulk region, where the other density reaches zero. In each case, whether the atmosphere consists of protons or of electrons, the requirement that the atmosphere-forming density function \( v_1(\rho) \) is continuously differentiable at the boundary \( \rho = \rho_0 \) of the bulk region allows us to express the two amplitudes of the density function \( v_1(\rho) \) in the atmosphere region in terms of its two amplitudes in the bulk region.

We explain the procedure using the positive atmosphere case. The negative atmosphere case is completely analogous, and we will only state its final formulas.

The boundary \( \rho_0 \) of the bulk region of a white dwarf star with positive atmosphere is determined by the vanishing of the right-hand side of (29), and cancelling \( 1/\rho_0 \) this yields

\[
B_e^h \sinh(\kappa_h \rho_0) + B_p^h \sin(\kappa_i \rho_0) = 0,
\]

where \( B_e^h \propto B_p^h \) and \( B_e^t \propto B_p^t \); see Eqs. (30) and (31). This is an implicit equation for \( \rho_0 \), given \( B_p^h \) and \( B_e^t \) (equivalently: given \( B_p^h \) and \( B_e^h \)), which can be easily solved numerically on a computer, but generally not in a closed form. It should be noted, though, that Eq. (42) permits \( B_e^h \) to vanish (in which case also \( B_p^h \) vanishes), by (30)), given any \( B_e^t > 0 \) (equivalently, given \( B_p^t > 0 \),
namely when $\rho_0 = \pi/\kappa$. This is perhaps the only case in which $\rho_0$ is explicitly obtained from the bulk amplitudes, i.e. from $B^h_p = 0$. We have already remarked earlier that only positive trigonometric bulk amplitudes are permitted, due to the requirement that the bulk densities must not be negative.

At this point, a change of perspective will yield a decisive simplification: From Eq. (42) we obtain
\[ B^h_c = -\frac{\sin(\kappa_1 \rho_0)}{\sinh(\kappa_0 \rho_0)}. \]  
\[ (43) \]
We will think of (43) as yielding the ratio $B^h_c/B^t_c$ (equivalently: $B^h_p/B^t_p$) explicitly as a function of $\rho_0$, and hence treat the interface location $\rho_0$ as independent parameter.

Coming now to the matching of atmospheric amplitudes with the bulk amplitudes, we note that for the density function $v_\rho(\rho)$ at $\rho = \rho_0$, which (after cancelling $1/\rho_0$) yields
\[ B^h_p \sinh(\kappa_0 \rho_0) + B^t_p \sin(\kappa_0 \rho_0) = \]
\[ A^+_p e^{\kappa_0 \rho_0} + A^-_p e^{-\kappa_0 \rho_0}, \]
equivalently,
\[ B^h_p \sinh(\kappa_0 \rho_0) + B^t_p \sin(\kappa_0 \rho_0) = \]
\[ A^+_p \cosh(\kappa_0 \rho_0) + A^-_p \sinh(\kappa_0 \rho_0). \]
Second, we need the continuity of the derivative of their density function $v_\rho(\rho)$ at $\rho = \rho_0$. By the product rule, the derivative of each term in the general solution is a sum of the $\rho$-derivative of the numerator, divided by $\rho$, plus the numerator times the derivative of $1/\rho$. Yet all terms proportional to the derivative of $1/\rho$ can be grouped together and, with the help of (44), this group can be seen to vanish by itself. Thus, and after cancelling the remaining overall factor $1/\rho_0$, continuity of the $\rho$-derivative of $v_\rho(\rho)$ at $\rho = \rho_0$ yields
\[ B^h_p \kappa \sinh(\kappa_0 \rho_0) + B^t_p \kappa \cos(\kappa_0 \rho_0) = \]
\[ A^+_p \kappa \sinh(\kappa_0 \rho_0) - A^-_p \kappa \cosh(\kappa_0 \rho_0), \]
equivalently
\[ B^h_p \kappa \sinh(\kappa_0 \rho_0) + B^t_p \kappa \cos(\kappa_0 \rho_0) = \]
\[ A^+_p \kappa \sinh(\kappa_0 \rho_0) + A^-_p \kappa \cosh(\kappa_0 \rho_0). \]

Using either the pair of equations (44), (46), or the pair (45), (47), we can write a linear transformation from the pair of $B_p$ amplitudes to the pair of $A_p$ amplitudes. For the sake of concreteness, we choose the pair (44), (46) and obtain
\[
\begin{pmatrix}
\sinh(\kappa_0 \rho_0); & \sin(\kappa_0 \rho_0) \\
\kappa_0 \sinh(\kappa_0 \rho_0); & \kappa_0 \cos(\kappa_0 \rho_0)
\end{pmatrix}
\begin{pmatrix}
B^h_p \\
B^t_p
\end{pmatrix} = 
\begin{pmatrix}
\sinh(\kappa_0 \rho_0); & \sin(\kappa_0 \rho_0) \\
\kappa_0 \sinh(\kappa_0 \rho_0); & \kappa_0 \cos(\kappa_0 \rho_0)
\end{pmatrix}
\begin{pmatrix}
A^+_p \\
A^-_p
\end{pmatrix}.
\]  \[ (48) \]
This linear transformation is valid as long as the left-(and therefore the right-) hand side of Eq. (44) is strictly positive, as required for having a positive atmosphere.

We note that the determinant of the coefficient matrix at the right-hand side of Eq. (48) equals $-2 \kappa_0 < 0$, and therefore the matrix is always invertible and the pair $(A^+_p, A^-_p)$ is uniquely given by (48) in terms of the pair $(B^h_p, B^t_p)$, for any physically meaningful choice of $\rho_0 > 0$. How to choose the physically meaningful $\rho_0$ we work out in the next subsection. But first we list the analogous formulas for the case of a star with a negative atmosphere.

The pertinent formulas are easily obtained ‘per dictionary’ from the formulas of the positive atmosphere setting. Thus, given $B^h_p/B^t_p$ (equivalently: given $B^h_p/B^t_p$), from the vanishing of the right-hand side of (28), and after cancelling $1/\rho_0$, we obtain
\[ B^h_p = -\frac{\sin(\kappa_0 \rho_0)}{\sinh(\kappa_0 \rho_0)}. \]  \[ (49) \]
Moreover, we now obtain the linear relationship
\[
\begin{pmatrix}
\sinh(\kappa_0 \rho_0); & \sin(\kappa_0 \rho_0) \\
\kappa_0 \sinh(\kappa_0 \rho_0); & \kappa_0 \cos(\kappa_0 \rho_0)
\end{pmatrix}
\begin{pmatrix}
B^h_p \\
B^t_p
\end{pmatrix} = 
\begin{pmatrix}
\exp(\kappa_0 \rho_0); & \exp(-\kappa_0 \rho_0) \\
\kappa_0 \exp(\kappa_0 \rho_0); & -\kappa_0 \exp(-\kappa_0 \rho_0)
\end{pmatrix}
\begin{pmatrix}
A^+_p \\
A^-_p
\end{pmatrix}.
\]  \[ (50) \]
between the $B_p$ and $A_p$ amplitudes. This linear transformation is valid as long as $v_e(\rho_0) > 0$, as required for having a negative atmosphere.

E. Two intervals of admissible $\rho_0$ values

By now we have determined the density functions $v_\rho(\rho)$ and $v_e(\rho)$ of the two-species ‘6/3-model’ uniquely in terms of three parameters: (i) a choice of sign, as to whether the positive or negative species defines the bulk radius, (ii) the location $\rho_0$ of the interface between bulk region and atmosphere, and (iii) the positive trigonometric bulk amplitude $B^t$ of the species defining the bulk radius. However, the resulting solution may not be integrable to yield finite total number of particles $N_p$ and $N_e$. The requirement that it should determines the physically allowed interval of $\rho_0$ values in the positive and negative amplitude situation. We note that similarly to the $n = 1$ polytropic single-density model, the value of the trigonometric amplitude $B^t > 0$ is chosen independently of $\rho_0$.

Again, having the answer worked out for the case of a star with a positive atmosphere, the answer for a star with a negative atmosphere will follow ‘by dictionary.’

Therefore, assume that the star has a positive atmosphere. Then $\rho_0$ is the point where the electron bulk density $v_e(\rho)$ has declined to zero. We already know from our discussion that the ‘trigonometric mode’ of the bulk regime essentially captures the density distribution, so $B^t > 0$. Moreover, from (43) we see that $B^h_p < 0$ if $\rho_0 < \pi/\kappa$, and $B^h_p > 0$ if $\rho_0 > \pi/\kappa$, with $B^t_p = 0$ if
\[ \rho_0 = \pi/\kappa_i. \] By (30), (31), then also \( B_p^i > 0 \), while \( B_p^h \) and \( B_e^h \) have opposite signs, except when both vanish.

Of course, the case \( \rho_0 = \pi/\kappa_i \) which leads to \( B_p^h = 0 = B_e^h \) is the case without atmosphere at all, and the bulk densities \( v_p(\rho) \) and \( v_e(\rho) \) are then given by essentially the same Lane–Emden \( n = 1 \) polytrope as in the single-density approximation, (18), except for minute differences in the parameter values. Therefore, to have a non-empty atmosphere we need to consider \( \rho_0 \neq \pi/\kappa_i \). In fact, we will need \( \rho_0 < \pi/\kappa_i \).

Indeed, if \( \rho_0 < \pi/\kappa_i \), then since \( B_p^i > 0 \) we have \( B_p^h < 0 \) by (43), and therefore now both \( B_p^i > 0 \) and \( B_p^h > 0 \), by (30) and (31). Now, by assumption \( v_e(\rho_0) = 0 \), but \( v_p(\rho_0) \) is the same linear combination of the \( B_p \) amplitudes as \( v_e(\rho_0) \) is of the \( B_e \) amplitudes, with \( B_p^e = B_e^h > 0 \) yet \( B_p^h > 0 \) while \( B_e^h < 0 \), and so we conclude that \( v_p(\rho_0) > 0 \), as claimed.

Proceeding analogously when \( \rho_0 > \pi/\kappa_i \), we find that now both \( B_p^i < 0 \) and \( B_p^h < 0 \) by (43), and therefore now \( B_p^i > 0 \) while \( B_p^h < 0 \). Thus, since by assumption \( v_e(\rho_0) = 0 \) with two positive amplitudes, the left-hand side of (28) with one positive and one negative amplitude evaluated at \( \rho_0 \) is actually negative, in violation of the requirement that particle densities cannot be negative. Thus a positive atmosphere is not possible with \( \rho_0 > \pi/\kappa_i \), which cannot be a zero of \( v_p(\rho) \) in the bulk.

Next, since \( v_p(\rho_0) > 0 \) in the case of a positive-atmosphere star, it is clear that \( A_p^+ \) and \( A_e^- \) cannot both be (strictly) positive or both be negative: two negative \( A_p \) amplitudes cannot produce a strictly positive particle atmospheric density. Two strictly positive \( A_p \) amplitudes do yield a positive particle density, but this density grows rapidly beyond any upper bound and cannot integrate to a finite particle number.

In our appendix we show that the only allowed combinations are \( A_p^+ \leq 0 \) and \( A_e^- > 0 \). The extremal case \( A_p^+ = 0 \) and \( A_e^- > 0 \) defines the lower limit \( \rho_0^- \) of the bulk boundary \( \rho_0 \) if \( \rho_0 \) is the zero of \( v_e(\rho) \). It is straightforward to work out the equation defining \( \rho_0^- \), and while it contains only simple elementary functions, it is transcendental and cannot be solved in closed form. However, because of the fantastically tiny ratios of the gravitational to electric coupling constants, a very accurate approximate expression for \( \rho_0^- \) can be found in terms of simple elementary functions (see our appendix). It reads

\[ \rho_0^- \approx \pi/\kappa_i - \frac{1}{(1-q)\kappa_p - q\kappa_h}, \] (51)

where

\[ q = \frac{1 - \frac{Gm_e^2}{ec^2} + \varepsilon\kappa_i^2}{1 - \frac{Gm_e^2}{ec^2} - \varepsilon\kappa_i^2} \approx -1836. \] (52)

Note that \( \kappa_i\rho_0^- \) is just barely smaller than \( \pi \).

The discussion for a negative-atmosphere star mirrors the one for the positive-atmosphere star. Thus the only allowed combinations are \( A_p^+ \leq 0 \) and \( A_e^- > 0 \). Analogously to our computation in the positive atmosphere case we now find (see our appendix)

\[ \rho_0^+ \approx \frac{\pi}{\kappa_i} - \frac{q}{(q-1)\kappa_e - \kappa_h}; \] (53)

also \( \kappa_i\rho_0^+ \) is just barely smaller than \( \pi \).

We summarize: the bulk radii \( \rho_0^\pm \) are defined as the smallest possible zeros of the positive, respectively negative species in a solution pair. The ranges \([\rho_0^-, \pi/\kappa_i]\) of possible bulk radii are very tiny intervals to the left of the "no-atmosphere-value" \( \rho_0 = \pi/\kappa_i \), relative to that value. No bulk radius is bigger than \( \pi/\kappa_i \). Since \( \kappa_i \) agrees nearly perfectly with the single-density model value \( \kappa \) given by (17), the bulk radii of all the failed white dwarf stars in the 5/3 → 6/3 approximation are essentially given by (19). However, the atmosphere of a star can nevertheless be very extended. In particular, in the two extreme cases the atmosphere extends all the way out to infinity, yet with its density approaching zero exponentially fast.

F. Finally: How much surcharge does fit on a star?

We are finally ready to compute how much surcharge can be put on a white dwarf star. In the linear model we are using, any admissible value of the bulk boundary \( \rho_0 \) uniquely determines the two density functions \( v_p(\rho) \) and \( v_e(\rho) \), modulo a common arbitrary amplitude factor. Therefore, given the ratio \( N_e/N_p \), the total number of particles does not influence the structure of the stellar solution but can be chosen at will as in the single-density polytropic \( n = 1 \) model. Yet as explained in the introduction, in our question it is implicitly understood that the number of nuclei in an object is considered fixed when the number \( N_e \) of electrons is varied. In our white dwarf model this means the number \( N_p \) of protons is to be considered fixed when \( N_e \) is varied. Alternatively, our question for the surcharge can be phrased thus: “what is the range of allowed ratios \( N_e/N_p \)?”

Recall that the traditional single-density models are based on the local neutrality approximation, which implies \( N_p = N_e \), so one should expect that any non-neutral pair \((N_p, N_e)\) will correspond to a ratio \( N_e/N_p \approx 1 \).

Although by the linearity of this approximate model the total number of particles is simply a scaling variable, given the ratio \( N_e/N_p \), as discussed earlier the validity of the model is restricted to about \( 10^{54} - 10^{55} \) protons.

Our parametrizing of the solutions in terms of \( \rho_0 \) determines the ratio \( N_e/N_p \) uniquely as an elementary function of \( \rho_0 \) for either a positive or negative atmosphere, see our appendix. The expression is not very illuminating, though easy to graph. It is intuitively clear that the extreme values of the ratio \( N_e/N_p \) will be obtained by inserting the extremal values \( \rho_0^\pm \) for \( \rho_0 \). Interestingly, to answer our question for the extreme values of \( N_e/N_p \) we can avoid the discussion of \( N_e/N_p \) as function of \( \rho_0 \) and instead resort to the following argument.
Namely, consider the extreme case of a star with negative atmosphere, i.e. \( \rho_0 = \rho_0^- \). We multiply Eq.(25) by \( 4\pi \rho^2 \) and integrate over \( \rho \) from 0 to \( \infty \). (Strictly speaking, (25) is a-priori only valid inside the bulk region, but comparison with the atmospheric equation (38) reveals that we can extend (25) to all \( \rho \) by noting that \( v_p(\rho) = 0 \) for \( \rho \geq \rho_0^- \).) Using that \( \int v_p(\rho)d^3\rho = N_p \) and \( \int v_c(\rho)d^3\rho = N_c \), and using that \( v'_c(0) = 0 \) and that \( v'_c(\rho) \to 0 \) exponentially fast when \( \rho \to \infty \), we obtain

\[
0 = \left( 1 + \frac{Gm_m m_e}{e^2} \right) N_p^- \left( 1 - \frac{Gm_m m_e}{e^2} \right) N_c^-,
\]

where the negative superscript at \( N_p \) and \( N_c \) indicates ‘extreme negative atmosphere case.’ Similarly, consider the extreme case of a star with positive atmosphere, i.e. \( \rho_0 = \rho_0^+ \). We multiply Eq.(24) by \( 4\pi \rho^2 \) and integrate over \( \rho \) from 0 to \( \infty \); again we extend also Eq.(24) to all \( \rho \) by noting that \( v_c(\rho) = 0 \) for \( \rho \geq \rho_0^+ \). Using once again that \( \int v_p(\rho)d^3\rho = N_p \) and \( \int v_c(\rho)d^3\rho = N_c \), and using now that \( v'_p(0) = 0 \) and that \( v'_p(\rho) \to 0 \) exponentially fast when \( \rho \to \infty \), we obtain

\[
0 = -\left( 1 - \frac{Gm^2}{e^2} \right) N_p^+ + \left( 1 + \frac{Gm_m m_e}{e^2} \right) N_c^+,
\]

where the positive superscript at \( N_p \) and \( N_c \) indicates ‘extreme positive atmosphere case.’ From Eqs.(54) and (55) we now obtain the allowed range of ratios \( N_c/N_p \), as

\[
\frac{1 - \frac{Gm^2}{e^2} \rho_0^-}{1 + \frac{Gm_m m_e}{e^2} \rho_0^-} \leq \frac{N_c}{N_p} \leq \frac{1 + \frac{Gm_m m_e}{e^2} \rho_0^+}{1 - \frac{Gm^2}{e^2} \rho_0^+},
\]

which explices \( N_c^+/N_p^+ \leq N_c/N_p \leq N_c^-/N_p^- \).

Formula (56) is our explicit answer to the question how much surcharge a white dwarf star can hold, given \( N_p \).

From (56) we have, to very good approximation,

\[
\left( 1 - \frac{Gm_m m_e}{e^2} \right) N_e \leq N_p \leq \left( 1 + \frac{Gm^2}{e^2} \right) N_e,
\]

or

\[
-\frac{Gm_m m_e}{e^2} N_e \leq N_p - N_e \leq \frac{Gm^2}{e^2} N_e.
\]

And so, given a neutral failed white dwarf star with \( N_p = 10^{55} = N_e \), it can be stripped of \( \approx 8 \cdot 10^{18} \) electrons, while \( \approx 4 \cdot 10^{15} \) electrons can be deposited on it without changing \( N_p \).

IV. THE ‘UNIVERSAL’ VALIDITY OF THE SURCHARGE BOUNDS

The reader may be skeptical whether our surcharge bounds (56) are truly statements about a non-relativistic failed white dwarf star, for they were obtained with our 5/3 \( \to 6/3 \) approximation. We will now present a compelling argument for why our surcharge bounds (56) are the correct bounds for a failed white dwarf star, and not merely in the non-relativistic regime!

First the non-relativistic regime, though. Proceeding analogously to the discussion in section III, but now not invoking the 5/3 \( \to 6/3 \) approximation, we obtain the following coupled system of nonlinear second-order differential equations for the density functions \( v_p \) and \( v_c \) in the bulk region where \( v_p(\rho) > 0 \) and \( v_c(\rho) > 0 \),

\[
-\varepsilon \frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} v_p^{2/3}(\rho) \right) = -\left( 1 - \frac{Gm^2}{e^2} \right) v_p(\rho) + \left( 1 + \frac{Gm_m m_e}{e^2} \right) v_c(\rho),
\]

\[
-\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} v_c^{2/3}(\rho) \right) = \left( 1 + \frac{Gm_m m_e}{e^2} \right) v_p(\rho) - \left( 1 - \frac{Gm^2}{e^2} \right) v_c(\rho),
\]

where \( \zeta := \frac{\xi}{2} \). This pair of coupled differential equations for the density functions \( v_p \) and \( v_c \) generalizes the single Lane–Emden equation for the polytrope of index \( n = \frac{3}{2} \), Eq.(13), in the bulk interior.

In the atmospheric regimes, a positive atmosphere is governed by (59) with \( v_c(\rho) = 0 \), while a negative atmosphere is governed by (60) with \( v_p(\rho) = 0 \). At the bulk-atmosphere interface at \( \rho = \rho_0 \) the density of the species which forms the atmosphere needs to be continuously differentiable, as before.

The nonlinearity of Eqs.(59) and (60) stands in the way of solving them generally in closed form, although some special elementary solutions can be found (see below). Yet we can obtain a sufficient amount of insight into their extremal solutions to allow us conclusions about the extremal surcharges of these solutions.

Namely, consider an extremely charged star, first with negative atmosphere. One can show that the condition to be extremal implies that the atmosphere density has to approach zero with vanishing slope; see our appendix. But this implies that the extremal solution has a negative atmosphere extending all the way out to \( \rho \to \infty \), while \( \rho_0^- \) is finite, as in the approximate model of section III. For assume it would not, i.e. that its density \( v_c(\rho) \) would reach 0 at some finite \( \rho_e > \rho_0^- \), with vanishing slope. By a uniqueness result for (60), and using that \( v_p(\rho) = 0 \) for \( \rho > \rho_0^+ \), it then follows that \( v_c(\rho) = 0 \) for all \( \rho > \rho_0^+ \), in violation of the assumption that it is strictly positive for \( \rho_0^+ \leq \rho < \rho_0^- \). Hence an extremal negative atmosphere extends to infinity. The analogous conclusion holds for
an extremal positive atmosphere.

One atmospheric density solution actually can be obtained explicitly, as follows. Consider \( v_\rho(p) = 0 \) for \( p > p_\rho^c \); it does not matter where \( p_\rho^c \) is located, all we use is that it is a finite distance. We now make the ansatz \( v_\rho(p) = A_p p^\lambda \) and find \( \lambda = -6 \) and \( A_p = 12 \zeta / (1 - Gm_\rho^2/\epsilon^2) \). While this is slower than the exponential decay to zero, it still is fast enough to be integrable at \( p \to \infty \), viz. \( \rho^2 v_\rho(p) \to 0 \) as \( p \to \infty \). This solution would still have to be matched to the bulk interior, which may or may not be possible!

However, one can show that an extreme atmospheric solution cannot decay to zero slower than to make \( \rho^2 \frac{d}{dp} v_\rho^{2/3}(p) \to 0 \) for \( p \to \infty \). For suppose \( \rho^2 \frac{d}{dp} v_\rho^{2/3}(p) \to C < 0 \) for \( p \to \infty \); this implies that \( v_\rho(p) \sim 1/p^{3/2} \) for \( p \to \infty \), but such a \( v_\rho(p) \) is not integrable at \( \infty \).

But then we can multiply Eq.\( (60) \) by \( 4\pi \rho^2 \) and integrate over \( p \) from 0 to \( \infty \), with the understanding that \( v_\rho(p) = 0 \) for \( p \geq p_\rho^c \). Using that \( \rho^2 \frac{d}{dp} v_\rho^{2/3}(p) \to 0 \) for \( p \to \infty \), the result of this integration is again Eq.\( (54) \).

Similarly we can proceed in the case of an extreme positive atmosphere, and once again find Eq.\( (55) \). And so our surcharge bounds \( (56) \) are also valid for the proper \( 5/3 \) power law of the non-relativistic degeneracy pressures of the protons and the electrons, as claimed.

At this point it is clear that the key to \( (56) \) is the observation that the density of an extremal atmosphere must extend infinitely, and therefore go to zero with vanishing slope rapidly enough, unlike the non-extremal densities. Thus we can even take special relativity into account in the manner done by Chandrasekhar [4], which in the structure equations \( (59) \) and \( (60) \) changes the \( v_f^{2/3} \) into some nonlinear \( J_f(v_f) \) which interpolates continuously between \( v_f^{2/3} \) and \( v_f^{1/3} \), and it changes the \( \zeta \) coefficient to some other coefficient. All the same, the integration of the pertinent structure equations will always produce Eqs.\( (54) \) and \( (55) \), and therefore \( (56) \).

This result is perhaps somewhat unexpected:

*Note that the bounds \( (56) \) are independent of \( h \) and \( c \).*

In fact, the surcharge bounds are not only independent of \( h \) and of \( c \), they are actually \( (\text{largely}) \) independent of the type of pressure law for the two species. We write \( \text{‘largely’} \) because some pressure laws will not lead to solutions which are integrable \( \text{for instance, think of the isothermal pressure law with finite positive temperature)\), and those laws do not lead to \( (56) \).

All our results are based on Newtonian gravity, though. It would be interesting to see whether the conclusions carry over to Einstein’s general theory of relativity.

V. CAN SURCHARGE HAVE A NOTICEABLE EFFECT ON THE ORBITS OF A BINARY WHITE DWARF SYSTEM?

As an application of our surcharge bounds \( (56) \), consider the following scenario. Suppose a maximal negatively charged and a maximal positively charged white dwarf have formed a binary system of two equal mass components, each with mass \( M \). The binary system is supposed to be sufficiently separated to vindicate the spherical approximation for their shapes. Moreover, the atmospheric densities, which are rapidly decaying to zero, will be treated as having an effectively finite radius compared to the separation distance. The maximal charge imbalance is tiny, true, but since the microscopic electric coupling constants are so much stronger than the gravitational ones, it is in principle conceivable that even a tiny surcharge could be influencing the dynamics in a significant way. So let us find out by doing a calculation.

Note that \( N_c \approx N_e := N \) to high accuracy. From \( (58) \) we obtain for the Coulomb coupling coefficient of a maximal oppositely surcharged binary

\[
\frac{Gm_cm_e}{\epsilon^2} \frac{Gm_e^2}{\epsilon^2} N^2 e^2 \approx \frac{Gm_cm_e}{\epsilon^2} GM^2, \tag{61}
\]

where we have used that the mass \( M \) of each binary component is \( \approx m_e N \). Since \( GM^2 \) is the gravitational coupling coefficient between the two binaries, \( (61) \) reveals that the electrical attraction between the two binary components is still \( 10^{-40} \) times smaller than their gravitational attraction.

Thus astronomers can relax. The validity of the determination of the masses of binary components based on their orbital data with the help of the gravitational Kepler problem is not in question.

VI. ILLUSTRATIONS

Having a complete set of solution formulas for the ‘6/3-model’ one can easily generate figures which illustrate the findings. There are two compromises to be made, though. Namely, the fantastically tiny ratios of the gravitational to electrical coupling constants between electron and proton are definitely causing headaches, but also the small mass ratio \( m_e/m_p \approx 1/1836 \) is a source of trouble. Both these small numbers taken together make it sheer impossible to produce any useful graphs at all. For instance, let’s try to resolve the interval of the allowed values of the ratio \( N_c/N_p \). From \( (57) \) we see that \( N_c/N_p \) varies between about 1 and \( 8.1 \cdot 10^{-37} \) and about 1 and \( 4.4 \cdot 10^{-40} \). And so, if we center our \( N_c/N_p \) axis at 1 and scale up the units by a factor of \( (1/4.4) \cdot 10^{40} \), then the negatively charged stars will occupy about 1 positive unit in the allowed interval and the positive stars 1836 negative units. To resolve such a lopsided asymmetry graphically without introducing otherwise obscuring transformations is impossible.

Incidentally, the construction just described is equivalent, essentially, to using a rescaled \( \ln(N_c/N_p) \) as base variable. That in principle is a useful move which takes care of the ‘tininess of Newton’s G’, but it also reveals that the small mass ratio \( m_e/m_p \) still poses a hurdle.

However, it is only necessary to illustrate the findings qualitatively, without insisting on quantitative closeness to reality. In this vein, in the following we illustrate our findings for the science fiction values \( Gm_p^2/\epsilon^2 = 1/2 \) and \( m_e/m_p = 1/10 \) (\( = \epsilon \)); for consistency, therefore,
\( G_m \rho/m_e^2 = \varepsilon/2 \) and \( G_m^2/e^2 = \varepsilon^2/2 \). Other physical constants have the empirical values to 6 digits precision.

In the first two figures we graph two stellar response quantities versus the decadic logarithm of the charge ratio \( N_e/N_p \). These capture a ‘6/3-star’s’ response to having been created with the particular ratio \( N_e/N_p \). Fig.1 shows the bulk radius \( \rho_0 \) which the star adapts in response to \( N_e/N_p \), while Fig.2 displays the ratio of the central proton density over the central electron density. Note the strong asymmetry caused by \( m_e/m_p \ll 1 \).

We next compare the particle density functions of the ‘5/3 model’ with those of its ‘6/3 approximation,’ with the same science fiction values given to the physical constants. The central proton bulk density in the 5/3 model is the same in all examples. In the comparisons of 5/3 with pertinent 6/3 densities, the proton numbers in the two solutions are the same.

We begin with the distinguished pair of solutions consisting of the densities of a star without atmosphere, when both \( \nu_p(\rho) \) and \( \nu_e(\rho) \) vanish at the same dimensionless bulk radius \( \rho_0 \). We show the density functions of both the 5/3 and the 6/3 model; see Fig.3. The no-atmosphere solutions in the two models behave qualitatively similar; however, note the difference in the scales! The no-atmosphere solutions of the 6/3 model with equal proton number \( N_p \) have a much more spread-out bulk than those of the 5/3 model, and the central densities are much smaller in the 6/3 model than in the 5/3 model.

FIG. 1. Shown is the bulk radius \( \rho_0 \) vs. \( \log_{10}(N_e/N_p) \) of the number of electrons per proton, \( N_e/N_p \), for the full range of allowed values, though for science fiction values \( G_m^2/e^2 = 1/2 \) and \( m_e/m_p = 1/10 \). Most solutions carry a positive surcharge, but only a small fraction of them has a positive atmosphere. This strong two-fold asymmetry is caused by \( m_e/m_p \ll 1 \).

FIG. 2. Shown is the ratio of the central proton density over central electron density vs. \( \log_{10}(N_e/N_p) \) of the number of electrons per proton, \( N_e/N_p \), covering the full range of allowed \( N_e/N_p \) ratios, though for science fiction values \( G_m^2/e^2 \leq 1/2 \) and \( m_e/m_p = 1/10 \). Note that the central proton density is always larger than the central electron density. Also this asymmetry is caused by the mass ratio \( m_e/m_p \ll 1 \).

In the 6/3 model a star without an atmosphere has \( \rho_0 = \pi/\kappa_t \), as we discussed earlier, and both densities then are scaled \( n = 1 \) polytropes. For the proper 5/3 model it can also be shown that both densities are scaled polytropes, though for \( n = 3/2 \) of course; as follows.

Note that in the 6/3 model one has \( \nu_e(\rho) = \lambda \nu_p(\rho) \) when \( \rho_0 = \pi/\kappa_t \), with \( \lambda \) given by the right-hand side of Eq.(31). Let’s instead make the ansatz \( \nu_e(\rho) = \lambda \nu_p(\rho) \) in Eqs.(24) and (25) of the 6/3 model. We then obtain two equations for \( \nu_p(\rho) \) (say), and this generally
overdetermines the problem. Their compatibility condition is the quadratic problem \( a\lambda^2 + b\lambda + c = 0 \), with 
\[
a = \left( 1 + \frac{Gm_p m_e}{e^2} \right)/\varepsilon > 0, \quad b = -\left( 1 - \frac{Gm_p^2}{e^2} \right)/\varepsilon + \left( 1 - \frac{Gm_e^2}{e^2} \right) < 0, \quad \text{and} \quad c = -\left( 1 + \frac{Gm_e m_p}{e^2} \right) < 0.
\]
The "quadratic formula" yields two real solutions,
\[
\lambda_{\pm} = -\frac{b}{2a} \left( 1 \pm \sqrt{1 - \frac{4ac}{b^2}} \right), \quad (62)
\]
one of which is positive and the other one negative. Now a particle density cannot be negative, so \( \lambda = \lambda_+ \), and this is precisely the right-hand side of Eq.(31).

Similarly one can insert the ansatz \( v_\nu(\rho) = \lambda v_\nu(\rho) \) also into the equations of the 5/3 model, i.e. Eqs.(59) and (60), and now the compatibility condition is the vanishing of the degree-5 polynomial \( a(p)^5 + b(p)\eta^3 + b(p)\eta^2 + d = 0 \), where \( \eta := \lambda^{1/3} \), and \( a = \left( 1 + \frac{Gm_p m_e}{e^2} \right)/\varepsilon > 0, \quad b = \left( 1 - \frac{Gm_p^2}{e^2} \right)/\varepsilon + \left( 1 - \frac{Gm_e^2}{e^2} \right) < 0, \quad \text{and} \quad c = -\left( 1 + \frac{Gm_e m_p}{e^2} \right) < 0 \). There generally does not exist a solution in closed form, but from the signs of the coefficients in this polynomial one can deduce right away that there exists a unique positive solution \( \eta_+ \), say, and for \( \lambda = \eta_+^3 \), both (59) and (60) reduce to the equation
\[
\varepsilon \frac{1}{\rho^2} \left( \rho^2 v_\nu^2(\rho) \right)' = \left[ 1 - \frac{Gm_p^2}{e^2} - \lambda \left( 1 + \frac{Gm_p m_e}{e^2} \right) \right] v_\nu(\rho), \quad (63)
\]
which is equivalent (not identical) to the polytropic equation of index \( n = 3/2 \), Eq.(13). Indeed, setting \( v_\nu^2(\rho) = \theta^2(\xi) \) and rescaling \( \rho = C(\xi) \) appropriately converts (63) into the standardized format \( \frac{1}{\xi^2} \left( \xi^2 \theta'(\xi) \right)' = \theta_3^{3/2}(\xi) \), cf. [3], [4], and thus the no-atmosphere densities are obtained by rescaling the standardized \( n = 3/2 \) polytrope.

We remark that inserting the dimensionless bulk radius of a star without an atmosphere, \( \rho_0 = \pi/\kappa_1 \), into our formula for the \( \rho_0 \)-dependent number of electrons per proton in the ‘6/3 model’ (see our appendix) yields
\[
\frac{N_e}{N_p} = \left( \frac{1 - \frac{Gm_p^2}{e^2}}{\lambda} + \varepsilon \left( 1 + \frac{Gm_p m_e}{e^2} \right) \right), \quad (64)
\]
with \( \lambda = B_p^6/B_p^4 \) given by (31). Alternatively, knowing that \( v_\nu = \lambda v_\nu \) in this case, (64) follows directly from multiplying Eq.(24) by \( 4\pi \rho^2 \) and Eq.(25) by \( 4\pi \rho^2 \), then integrating over \( \rho \), then subtracting the first result from the second, followed by simple algebra.

Similarly the number of electrons per proton of the no-atmosphere solution of a failed white dwarf star as computed with the physical 5/3 model is obtained from Eqs.(59) and (60). With \( \lambda = \eta_+^3 \) one finds
\[
\frac{N_e}{N_p} = \frac{\lambda^{2/3} \left( 1 - \frac{Gm_p^2}{e^2} \right) + \varepsilon \left( 1 + \frac{Gm_p m_e}{e^2} \right)}{\lambda^{2/3} \left( 1 + \frac{Gm_p m_e}{e^2} \right) + \varepsilon \left( 1 - \frac{Gm_p^2}{e^2} \right)}, \quad (65)
\]
Finally we turn to the extremely surcharged stars, whose densities are shown in Figs.4 and 5.
It is manifest that also the extreme solutions in the 6/3 model behave qualitatively similar to those in the 5/3 model. The extreme solutions of the 6/3 model have a much more spread-out bulk than those of the 5/3 model with equal proton number $N_p$, but their central densities are much smaller than those in the 5/3 model. Interestingly, the ratio of the two central densities in the 6/3 model seems to roughly equal the one in the 5/3 model.

VII. CONCLUSIONS

In this paper we have presented an explicitly solvable approximate model of a failed white dwarf star consisting of electrons and protons, in which we replaced the polytropic power 5/3 of the pressure-density relation, predicted by non-relativistic quantum mechanics, with the nearby 6/3. Based on the availability of the elementary exact solutions of this model we were able to discuss the whole solution family thoroughly. The model can easily be incorporated in an introductory astrophysics course which also includes a discussion of the basic equations of stellar structure, in particular for white dwarf stars.

A result which transcends this two-species ‘6/3-model’ is our finding that the ratio $N_e/N_p$ of the number of electrons per proton lies in the interval (56). We explained that (56) is not based on our approximation, but holds also for the proper ‘5/3-model,’ and even for the two-species version of Chandrasekhar’s special relativistic model of a white dwarf star.

We suspect that (56) also holds general-relativistically, though to really show it is a more complicated problem which requires the discussion of the Einstein field equations coupled with both the matter equations for the Fermi gases and the Maxwell equations of the electrostatic field in curved spacetime. By contrast, our derivation of (56) here uses the two Poisson equations of Newtonian gravity and Coulomb electricity, in concert with Newtonian-Eulerian force balance. Yet note that the key argument in our derivation of (56) is the behavior of the atmospheric densities at spatial infinity, and in an asymptotically flat spacetime this is the region where the general-relativistic equations are expected to go over into the non-relativistic equations of Newton — hence the independence of $\hbar$ and $c$, and our conjecture that (56) is truly ‘universally’ valid.

The formula (56) for the allowed ratios $N_e/N_p$ is very elementary, and its derivation from the basic structure equations is straightforward if one accepts as key ingredient that the density of the species which forms the atmosphere goes sufficiently rapidly to zero together with its derivative so that $\rho^2 \frac{d^2}{dp^2} J_f(v_f(\rho)) \to 0$ for $\rho \to \infty$, where $f$ stands for either $e$ or $p$, whichever species forms the atmosphere. A rigorous vindication of this key ingredient for the physically more realistic models in the spirit of [14] requires a more careful discussion of the structure equations, where explicit solutions are not generally available. This is done in the Ph.D. thesis of the first author, and will be published in a mathematical journal.

In future work we plan to investigate the influence of magnetism and of rotation on the problem of how non-neutral a (failed) white dwarf can be. This will complicate the problem considerably, for the spherical symmetry of the problem will be broken both by rotation, due to centrifugal effects (obviously), and by magnetism, cf. [15], [16], [17], [18], [19].

APPENDIX

A. Computing $\kappa_b$ and $\kappa_t$

The characteristic equation

$$\det \left( \begin{array}{cc} 1 - \frac{Gm_e^2}{c^2} - \kappa^2 \xi \zeta & -1 - \frac{Gm_p m_e}{c^2} \\ -1 - \frac{Gm_p m_e}{c^2} & 1 - \frac{Gm_p^2}{c^2} - \kappa^2 \xi \zeta \end{array} \right) = 0$$

(66)

yields

$$\left(1 - \frac{Gm_e^2}{c^2} - \kappa^2 \xi \zeta\right)\left(1 - \frac{Gm_p^2}{c^2} - \kappa^2 \xi \zeta\right) - \left(1 + \frac{Gm_p m_e}{c^2}\right)^2 = 0$$

(67)

which is the quadratic problem $\kappa b^4 + b \kappa^2 + c = 0$ in $\kappa^2$, with $a = \xi \zeta > 0$, $b = -\zeta (1 + c - G(m_e^2 + m_p^2)/c^2) < 0$, and $c = -G(m_e + m_p^2)/c^2 < 0$. By ‘the quadratic formula’ we have two real solutions,

$$\left(\kappa^2\right)_\pm = -\frac{b}{2a} \left(1 \pm \sqrt{1 - 4\frac{b^2}{a^2}}\right)$$

(68)

one of which is positive and the other one negative. Numerical values for the mathematical and physical constants yield $(\kappa^2)_+ \approx 44.0025$ and $(\kappa^2)_- \approx -1.94 \cdot 10^{-38}$. This now yields the hyperbolic $\kappa_b := \sqrt{(\kappa^2)_+} \approx 6.63344$ and the trigonometric $\kappa_t := |\sqrt{(\kappa^2)_-}| \approx 1.3929 \cdot 10^{-19}$.

B. Proof that $A^+_e \leq 0$ & $A^-_p > 0$

In the main text we already explained by simple arguments why neither two (strictly) positive nor two (strictly) negative atmospheric amplitudes $A$ are admissible. Moreover, the combination $A^+_e \leq 0$ and $A^-_p > 0$ is manifestly admissible, for it will always lead to an atmospheric density function $v_f(\rho)$ which is integrable. It remains to rule out the combination $A^+_e > 0$ and $A^-_p < 0$. It suffices to discuss one of these cases, for the other follows by analogy.

Thus, consider the positive atmosphere. Suppose $A^+_e > 0$ and $A^-_p < 0$. Recall that $\rho > \rho_0$ in the atmosphere, and that $\rho_0 \approx \pi/\kappa_t > 1$ is huge. Now $\kappa_t \approx 6.6$, so $\kappa_t \rho > 1$ is also huge, and therefore the function $\rho \rightarrow (A^+_e e^{\kappa_t \rho} + A^-_p e^{-\kappa_t \rho})/\rho$ is increasing for all $\rho > \rho_0$. Since it has to be positive at $\rho = \rho_0$, it cannot be integrable over $\rho > \rho_0$, which finishes the argument.

A similar reasoning rules out the combination $A^-_e > 0$ and $A^+_p < 0$. Even though $\kappa_e \approx 0.155$ is smaller than 1, $\kappa_e \rho$ is still so huge for $\rho > \rho_0$ that the function $\rho \rightarrow (A^+_e e^{\kappa_e \rho} + A^-_e e^{-\kappa_e \rho})/\rho$ is increasing for all $\rho > \rho_0$. 

C. Proof that for an extremal charged star, the atmospheric density vanishes with vanishing slope

For simplicity we present the proof for the 6/3-

approximate model, but it will be clear from the proof how to adjust it to also apply to the non-relativistic and

the relativistic white dwarf models.

Starting with the case of a negative atmosphere, we multiply the bulk Eq.(25) with $4\pi \rho^2$ and integrate from 0 to $\rho_0$, obtaining

\[ -\zeta 4\pi \rho_0^2 v_\rho' (\rho_0) = \left( 1 + \frac{Gm_{\text{em}}}{c^2} \right) N_p - \left( 1 - \frac{Gm^2}{c^2} \right) \int_{\rho_0}^{\rho_0} v_e (\rho) 4\pi \rho^2 d\rho . \]  

(69)

Here, $v_\rho' (\rho_0)$ is the regular derivative of $v_e (\rho)$ at $\rho = \rho_0$. We also multiply Eq.(38) with $4\pi \rho^2$ and integrate from $\rho_0$ to $\rho_e$, the point where $v_e (\rho)$ has decreased to zero. This yields

\[ 4\pi \zeta \left( \rho_0^2 v_e (\rho_0) - \rho_e^2 v_e (\rho_e) \right) = - \left( 1 - \frac{Gm^2}{c^2} \right) \int_{\rho_0}^{\rho_e} v_e (\rho) 4\pi \rho^2 d\rho . \]  

(70)

where $v_e' (\rho_e)$ is the left-derivative of $v_e (\rho)$ at $\rho = \rho_e$. Now noting that $\int_{\rho_0}^{\rho_0} v_e (\rho) 4\pi \rho^2 d\rho + \int_{\rho_0}^{\rho_e} v_e (\rho) 4\pi \rho^2 d\rho = N_e$, we add (70) and (69) and obtain

\[ 4\pi \zeta \rho_0^2 v_e' (\rho_0) = \left( 1 - \frac{Gm^2}{c^2} \right) N_e - \left( 1 + \frac{Gm_{\text{em}}}{c^2} \right) N_p . \]  

(71)

Since $v_e' (\rho_e)$ is the left-derivative of $v_e (\rho)$ at the point $\rho_e$ where the density $v_e (\rho)$ vanishes, and since an otherwise positive function cannot reach 0 with a positive slope, it follows that $v_e' (\rho_e) < 0$, and this means that

\[ N_e \left( \frac{N_e}{N_p} \right) \leq \frac{1 + \frac{Gm_{\text{em}}}{c^2}}{1 - \frac{Gm^2}{c^2}} , \]  

(72)

which is precisely the upper bound on $N_e / N_p$ given in (56). Abbreviating the right-hand side of (72) by $N_e^+ / N_p^+$, in the limit in which the upper bound is saturated, from (74) we now obtain

\[ \lim \frac{N_e}{N_p} \frac{\rho_e^2 v_e' (\rho_e)}{N_p} = 0 , \]  

(73)

where we consider $\rho_e$ as a function of $N_e / N_p$. Since $\rho_e > \rho_0 > 0$, it follows that $v_e' (\rho_e) \to 0$ in the limit. As we explained in the main text, $\rho_e \to \infty$ in this limit.

In a completely analogous manner we obtain in the case of a positive atmosphere that

\[ 4\pi \zeta \rho_0^2 v_e' (\rho_0) = \left( 1 - \frac{Gm^2}{c^2} \right) N_p - \left( 1 + \frac{Gm_{\text{em}}}{c^2} \right) N_e , \]  

(74)

from which we deduce that

\[ \frac{N_e}{N_p} \geq \frac{1 - \frac{Gm^2}{c^2}}{1 + \frac{Gm_{\text{em}}}{c^2}} , \]  

(75)

which is precisely the lower bound on $N_e / N_p$ given in (56). Abbreviating the right-hand side of (75) by

\[ N_e^+ / N_p^+ , \]

\[ \text{in the limit in which the lower bound is saturated, from (74) we now obtain} \]

\[ \lim \frac{N_e}{N_p} \frac{\rho_p^2 v_p' (\rho_p)}{N_p} = 0 , \]  

(76)

where we consider $\rho_p$ as a function of $N_e / N_p$. Since $\rho_p > \rho_0 > 0$, it follows that $v_p' (\rho_p) \to 0$ in the limit. As we explained in the main text, $\rho_p \to \infty$ in this limit.

D. Computing $\rho_0^*$

In the case of an extreme negative atmosphere, $\rho_0^+ < \pi / \kappa_e$ is determined by the matching of the bulk part of $v_e (\rho)$ with its atmospheric part in the limiting case where $A_e^+ = 0$. So from (50) we obtain

\[ \left( \sinh (\kappa_e \rho_0) ; \sin (\kappa_e \rho_0) \right) \left( B_e^+ ; \int B_e^+ \right) = A_e^+ e^{-\kappa_e \rho_0} \left( 1 \right) \]  

(77)

and these are two different equations for $A_e^+ \exp (-\kappa_e \rho_0)$. Elimination of $A_e^- \exp (-\kappa_e \rho_0)$ now yields, after some simple manipulations,

\[ \frac{B_e^+}{B_e^-} \sinh (\kappa_e \rho_0) + \sin (\kappa_e \rho_0) = \frac{-\kappa_e B_e^+}{B_e^-} \cosh (\kappa_e \rho_0) - \frac{\kappa_e}{\kappa_e} \cos (\kappa_e \rho_0) \]  

(78)

With the help of Eqs.(30), (31), and (49) we find

\[ \frac{B_e^+}{B_e^-} = - \frac{1}{q} \frac{\sin (\kappa_e \rho_0)}{\sin (\kappa_e \rho_0)} \]  

(79)

with $q$ given in (52). Note that (79) is not in contradiction to (43), for (79) holds for the extreme negative atmosphere, while (43) holds for any positive atmosphere. Subtracting (79) in (78), dividing by $\sin (\kappa_e \rho_0)$, and reshufling now yields

\[ (q - 1) \kappa_e \rho_0 = \kappa_e \coth (\kappa_e \rho_0) - q \kappa_e \cot (\kappa_e \rho_0) \]  

(80)
for the lower limit $\rho_0^+ < \pi/\kappa_t$ of the zero of the bulk density $v_p(\rho)$. Since coth is a monotonic decreasing function on the positive real line and cot is a monotonic decreasing function on its first positive period, and since $q < 0$, we see that the right-hand side of (80) is a strictly monotonic decreasing function in the interval $0 < \kappa_t \rho_0 < \pi$, thus it has a unique solution $\rho_0^+$. With the values of the parameters $q$, $\kappa_t$, $\kappa_h$, and $\infty$ as given, this solution is in the left vicinity of $\rho_0 = \pi/\kappa_t$. Recall that $\kappa_t \approx 6.6$ and $\kappa_t \approx 2 \cdot 10^{-19}$. Thus, if $\kappa_t \rho_0^+ \approx \pi$, then $\kappa_t \rho_0^+ \gg \pi$ is huge, and then coth($\kappa_t \rho_0^+ \approx 1$ asymptotically exact, with exponentially small corrections. Moreover, in the left vicinity of $\rho_0 = \pi/\kappa_t$ we have cot($\kappa_t \rho_0 \approx 1/\kappa_t \rho_0 - \pi < 0 \approx$ asymptotically exact, and this yields (53).

In a perfectly analogous manner we handle the case of an extreme positive atmosphere, where $\rho_0^- < \pi/\kappa_t$ is determined by the matching of the bulk part of $v_p(\rho)$ with its atmospheric part in the limiting case where $A_b^b = 0$. This time

$$\frac{B_p^p}{B_p^b} = -q \sin(\kappa_t \rho_0) \frac{\sinh(\kappa_h \rho_0)}{\sinh(\kappa_t \rho_0)}, \quad (81)$$

with $q$ given in (52). Also (81) is not in contradiction to (49), for (81) holds for the extreme positive atmosphere, while (49) holds for any negative atmosphere. We find

$$(1 - q)\kappa_p = q \kappa_t \coth(\kappa_h \rho_0) - \kappa_t \cot(\kappa_t \rho_0) \quad (82)$$

for the lower limit $\rho_0^- < \pi/\kappa_t$ of the zero of the bulk $v_\rho(\rho)$. The right-hand side of (82) is a strictly monotonic increasing function in $\rho_0$ in the first positive period of the cot function, with a solution in the left vicinity of $\rho_0 = \pi/\kappa_t$. Using that $\kappa_h \rho_0 \gg \pi$ is huge we again can set coth($\kappa_t \rho_0 \approx 1$ asymptotically exact, with exponentially small corrections. Moreover, in the left vicinity of $\rho_0 = \pi/\kappa_t$ we have cot($\kappa_t \rho_0 \approx 1/\kappa_t \rho_0 - \pi < 0 \approx$ asymptotically exact, and this now yields (51).

### E. Computing the ratio $N_e/N_p$ as function of $\rho_0$

The computation of the ratio $N_e/N_p$ as a function of $\rho_0$ can be effected by directly integrating the explicit solutions parameterized by $\rho_0$. It is easier to work directly with the differential equations.

Starting with the case of a negative atmosphere, we complement (71) by deriving its counterpart for the positive species. Thus we multiply the bulk Eq.(24) with $4\pi \rho^2$ and integrate from 0 to $\rho_0$, obtaining

$$-4\pi \rho^2 v'_p(\rho_0) = - \left[ 1 - \frac{G m_p^2}{c^2} \right] N_p + \left[ 1 + \frac{G m_p m_e}{c^2} \right] \int_{0}^{\rho_0} v_e(\rho) 4\pi \rho^2 d\rho. \quad (83)$$

Here, $v'_p(\rho_0)$ is the left-derivative of $v_p(\rho)$ at $\rho = \rho_0$. Recalling again that $\int_{0}^{\rho_0} v_e(\rho) 4\pi \rho^2 d\rho + \int_{0}^{\rho_0} v_e(\rho) 4\pi \rho^2 d\rho = N_e$, we multiply (70) by $\left[ 1 + \frac{G m_p m_e}{c^2} \right] / \left[ 1 - \frac{G m_p^2}{c^2} \right]$ and subtract the result from (83), obtaining

$$4\pi \left[ \left( \rho_0^2 v_e'(\rho_0) - \rho_0^2 v_e'(\rho_e) \right) \left[ 1 + \frac{G m_p m_e}{c^2} \right] + \rho_0^2 v_e'(\rho_0) \right] = \left[ 1 - \frac{G m_p^2}{c^2} \right] N_p - \left[ 1 + \frac{G m_p m_e}{c^2} \right] N_e. \quad (84)$$

Eqs.(71) and (84) form a linear system for $N_p$ and $N_e$ in terms of their coefficients and their left-hand sides. This linear system is easily solved for $N_p$ and $N_e$, from which we obtain $N_e/N_p$. Symbolically,

$$\left( \begin{array}{c} N_p \\ N_e \end{array} \right) = 4\pi \left( \begin{array}{cc} 1 - \frac{G m_p^2}{c^2} ; & -1 - \frac{G m_p m_e}{c^2} \\ -1 - \frac{G m_p m_e}{c^2} ; & 1 - \frac{G m_p^2}{c^2} \end{array} \right)^{-1} \left( \begin{array}{c} \rho_0^2 v_e'(\rho_0) - \rho_0^2 v_e'(\rho_e) \frac{1 + \frac{G m_p m_e}{c^2}}{1 - \frac{G m_p^2}{c^2}} + \rho_0^2 v_e'(\rho_0) \\ \rho_0^2 v_e'(\rho_e) \end{array} \right). \quad (85)$$

The inverse matrix is easily computed as

$$\left( \begin{array}{cc} 1 - \frac{G m_p^2}{c^2} ; & -1 - \frac{G m_p m_e}{c^2} \\ -1 - \frac{G m_p m_e}{c^2} ; & 1 - \frac{G m_p^2}{c^2} \end{array} \right)^{-1} = \frac{c^2}{G(m_p + m_e)^2} \left( \begin{array}{cc} -1 + \frac{G m_p^2}{c^2} ; & -1 - \frac{G m_p m_e}{c^2} \\ -1 - \frac{G m_p m_e}{c^2} ; & -1 + \frac{G m_p^2}{c^2} \end{array} \right), \quad (86)$$

and so, after some cancellations,

$$\frac{N_e}{N_p} = \left( \begin{array}{c} 1 - \frac{\rho_0^2 v_e'(\rho_0) \left( \frac{G(m_p + m_e)^2}{c^2} \right) \frac{G m_p m_e}{c^2} \left( 1 + \frac{G m_p m_e}{c^2} \right)}{\rho_0^2 v_e'(\rho_0) \left( 1 + \frac{G m_p m_e}{c^2} \right)^2 + \rho_0^2 v_e'(\rho_0) \left( 1 - \frac{G m_p^2}{c^2} \right) \left( 1 + \frac{G m_p m_e}{c^2} \right)} \right) \frac{1 + \frac{G m_p m_e}{c^2}}{1 - \frac{G m_p^2}{c^2}}. \quad (87)$$
Next we compute the pertinent derivatives at \( \rho_0 \) and \( \rho_\circ = (1/2 \pi \rho_0) \ln(-A^-_p/A^+_p) \). We find

\[
v'_e(\rho_\circ) = 2A^+_e \kappa_e \exp(x_e \rho_\circ),
\]
\[
v'_e(\rho_0) = \left( \frac{x_e - 1}{\rho_0} \right) A^+_e \exp(x_e \rho_0) - \left( \frac{x_e + 1}{\rho_0} \right) A^-_e \exp(-x_e \rho_0),
\]
\[
v'_p(\rho_0) = B^h_p \kappa_h \cosh(\kappa_h \rho_0) + B^l_p \kappa_l \cos(\kappa_l \rho_0).
\]

Finally, recall that we have shown that each and every amplitude is proportional to \( B^i \) (equivalently, \( B^j \)), which is a free parameter. Yet also note that the derivatives of the densities enter linearly at the numerator and at the denominator of (87), so that the free parameter \( B^i \) (alt. \( B^j \)) actually cancels out from (87). Thus (87) is an explicit formula for \( N_e/N_p \) as function of \( \rho_0 \) in the negative atmosphere regime. We note that the term in square parentheses is smaller than 1, yet converges upward to 1 when \( \rho_0 \rightarrow 0^+ \). In that case \( N_e/N_p \) reaches its upper limit given by the right-hand side of (56).

In a similar manner we can treat the case of a positive atmosphere and find

\[
\frac{N_e}{N_p} = \left[ 1 - \frac{\varepsilon \rho^2_p v^*_p(\rho_0) \left( 1 + \frac{G(m_{e+m_e}^2}{c^2} \right)}{\varepsilon \rho^2_p v^*_p(\rho_0) \left( 1 + \frac{G(m_{e+m_e}^2}{c^2} \right) + 2Gm_{e+m_e}^2 \left( 1 + \frac{2Gm_{e+m_e}^2}{c^2} \right) - 1} \right]^{-1}
\]

The pertinent derivatives at \( \rho_0 \) and \( \rho_p = (1/2 \pi \rho_0) \ln(-A^-_p/A^+_p) \) read

\[
v'_e(\rho_0) = 2A^+_e \kappa_e \exp(x_e \rho_0),
\]
\[
v'_e(\rho_0) = \left( \frac{x_e - 1}{\rho_0} \right) A^+_e \exp(x_e \rho_0) - \left( \frac{x_e + 1}{\rho_0} \right) A^-_e \exp(-x_e \rho_0),
\]
\[
v'_p(\rho_0) = B^h_p \kappa_h \cosh(\kappa_h \rho_0) + B^l_p \kappa_l \cos(\kappa_l \rho_0).
\]

Again all amplitudes are proportional to \( B^i \) (equivalently, \( B^j \)), which actually cancels out from (91). Thus (91) is an explicit formula for \( N_e/N_p \) as function of \( \rho_0 \) in the positive atmosphere regime. We note that the term in square parentheses is smaller than 1, yet converges upward to 1 when \( \rho_0 \rightarrow 0^+ \). In that case \( N_e/N_p \) reaches its lower limit given by the left-hand side of (56).

As a consistency check, we mention that when \( \rho_0 = \pi/\kappa_l \), then \( \rho_\circ = \rho_p = \rho_0 \), and both (91) and (87) reduce to

\[
\frac{N_e}{N_p} = \frac{\nu'\left( \frac{\pi}{\kappa_l} \right) \left( 1 - \frac{Gm^2_{e+m_e}}{c^2} \right) + \varepsilon \nu'\left( \frac{\pi}{\kappa_l} \right) \left( 1 + \frac{Gm_{e+m_e}}{c^2} \right)}{\nu'\left( \frac{\pi}{\kappa_l} \right) \left( 1 + \frac{Gm_{e+m_e}}{c^2} \right) + \varepsilon \nu'\left( \frac{\pi}{\kappa_l} \right) \left( 1 - \frac{Gm^2_{e+m_e}}{c^2} \right)},
\]

with the derivatives reducing to \( \nu'\left( \frac{\pi}{\kappa_l} \right) = -B^l_p \kappa_l^2/\pi \) and \( \nu'\left( \frac{\pi}{\kappa_l} \right) = -B^l_p \kappa_l^2/\pi \). We remark that now factoring out \( \nu'\left( \frac{\pi}{\kappa_l} \right) \) from both numerator and denominator produces the ratio \( \nu'\left( \frac{\pi}{\kappa_l} \right)/\nu'\left( \frac{\pi}{\kappa_l} \right) = B^l_p/B^l_c \), cf. (64), which can be read of from (31). From this expression one then finds that in this special ‘no-atmosphere case’ the ratio \( N_e/N_p < 1 \).

Equations (87) and (91) are easy to implement on a computer. We used them to generate Figures 1 and 2.

Final remark: in a similar manner one can obtain expressions for \( N_e/N_p \) for the physically more realistic models. For instance, simply replacing \( v' \) by \( (v'_{2/3}^p) \) in Eqs.(87) and (91), \( f = p \) or \( e \), yields \( N_e/N_p \) in the 5/3 model.

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