ESTIMATES OF THE $O(\alpha_s^4)$ CORRECTIONS TO $\sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})$, $\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})$ AND DEEP INELASTIC SCATTERING SUM RULES

Andrei L. Kataev

Theoretical Physics Division, CERN, CH-1211 Geneva 23, Switzerland

and

Valery V. Starshenko

330091 Zaporozhye, Ukraine

ABSTRACT

We present the estimates of the order $O(\alpha_s^4)$ QCD corrections to $R(s) = \sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$, $R_\tau = \Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})/\Gamma(\tau \rightarrow \nu_\tau \mu_\mu e)$ and to the deep inelastic scattering sum rules, namely to the non-polarized and polarized Bjorken sum rules and to the Gross–Llewellyn Smith sum rule. The estimates are obtained in the $\overline{MS}$-scheme using the principle of minimal sensitivity and the effective charges approach.

1Permanent address: Institute for Nuclear Research of the Russian Academy of Sciences, 117312 Moscow, Russia
1 Introduction

During the last few years, essential progress has been achieved in the area of the calculation of the next-next-to-leading order (NNLO) QCD corrections to the number of physical quantities. Indeed, the complete NNLO $O(\alpha_s^3)$ QCD corrections are known at present for the characteristics of $e^+e^- \rightarrow$ hadrons process \cite{1}, \cite{2}, $\tau \rightarrow \nu_\tau +$ hadrons decay \cite{3}, \cite{4} and for the deep inelastic scattering sum rules, namely the non-polarized Bjorken sum rule (BjnSR) \cite{5}, the Gross-Llewellyn Smith sum rule (GLSSR) and the polarized Bjorken sum rule (BjpSR) \cite{6}.

Amongst the physical information provided by these results is the estimate of the theoretical uncertainties of the corresponding next-to-leading-order (NLO) perturbative QCD predictions for $R(s) = \frac{\sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$ \cite{7}, $R_\tau = \frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau \rightarrow \nu_\tau \nu_\tau)}$ \cite{3}, BjnSR \cite{5}, GLSSR and BjpSR \cite{6}.

In view of the fact that the precision of the experimental data for $\sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})$, $\Gamma(Z^0 \rightarrow \text{hadrons})$ and for the structure functions of the deep inelastic scattering is continuously increasing, the problem of estimating the effects of the next-after-next-next-to-leading order (NANNLO) $O(\alpha_s^4)$ corrections to the measurable physical quantities now arises.

The first attempt to build the bridge between the results of the concrete calculations of the order $O(\alpha_s^3)$ corrections to $R(s)$ and the higher-order coefficients of the corresponding perturbative QCD series was made in Ref. \cite{11}. However, it was later demonstrated that the considerations of Ref. \cite{11} have definite drawbacks \cite{12}, \cite{13}. A certain step in the direction of more substantiated estimates of the order $O(\alpha_s^3)$ QCD corrections was made in the case of $R_\tau$ in Ref. \cite{3}. These estimates are based on the tendency, observed in Ref. \cite{14}, of the scheme-dependent uncertainties of the perturbative QCD predictions for $R(s)$ and $R_\tau$ to decrease as a result of taking into account the order $O(\alpha_s^3)$-terms. The foundations of Ref. \cite{14} were further generalized in the process of phenomenological studies of the QCD predictions for $R_\tau$, taking into account the concrete pieces of the higher-order perturbative QCD corrections \cite{15} (for the earlier development of a similar technique, see Ref. \cite{16}).

It should be stressed that the analysis of Ref. \cite{3} does not allow us to fix the sign of the order $O(\alpha_s^4)$ contribution to $R_\tau$. In this work we generalize the QED ideas of Refs. \cite{17}, \cite{18} and \cite{19} to the QCD case and estimate the order $O(\alpha_s^3)$ and $O(\alpha_s^4)$ corrections to $R(s)$, $R_\tau$ and deep inelastic scattering sum rules with the help of the principle of minimal sensitivity (PMS) \cite{17} and the effective charges (ECH) approach \cite{20}, which is equivalent \textit{a posteriori} to the scheme-invariant perturbation theory \cite{21}. Contrary to the conclusions of Ref. \cite{22}, we argue that the application of these approaches allows one to control the value and fix the sign of the higher-order QCD corrections to physical quantities, once the preceding ones are known in the particular scheme.
2 The Description of the Formalism

Consider first the order $O(a^N)$ approximation of a physical quantity

$$D_N = d_0 a (1 + \sum_{i=1}^{N-1} d_i a^i)$$

(1)

with $a = \alpha_s / \pi$ being the solution of the corresponding renormalization group equation for the $\beta$-function which is defined as

$$\mu^2 \frac{\partial a}{\partial \mu^2} = \beta(a) = -\beta_0 a^2 (1 + \sum_{i=1}^{N-1} c_i a^i).$$

(2)

In the process of the concrete calculations of the coefficients $d_i, i \geq 1$ and $c_i, i \geq 2$, the $\overline{MS}$ scheme is commonly used. However, this scheme is not the unique prescription for fixing the RS ambiguities. In both phenomenological and theoretical studies other methods are also widely applied.

The PMS \[17\] and ECH \[20\] prescriptions stand out from these other methods. Indeed, they are based on the conceptions of the scheme-invariant quantities, which are defined as the combinations of the scheme-dependent coefficients in Eqs. (1) and (2). Both these methods can claim to be the “optimal” prescriptions, in the sense that they provide better convergence of the corresponding approximations in the non-asymptotic regime, and thus allow an estimation of the uncertainties of the perturbative series in the definite order of perturbation theory. Therefore, the applications of the “optimal” methods allow one to estimate the effects of the order $O(a^{N+1})$-corrections starting from the approximations $D_N^{opt}(a_{opt})$ calculated in a certain “optimal” approach \[17 - 19\]. This idea is closely related to the QED technique of Ref. \[23\], which was used to predict the renormalization-group controllable $ln(m_\mu/m_e)$-terms in the series for $(g-2)_\mu$ from the expression of $(g-2)_\mu$ through the effective coupling constant $\bar{\alpha}(m_\mu/m_e)$. In our work we are using a similar technique to estimate the constant terms of the higher-order corrections in QCD. \[2\]

Let us re-expand $D_N^{opt}(a_{opt})$ in terms of the coupling constant $a$ of the particular scheme

$$D_N^{opt}(a_{opt}) = D_N(a) + \delta D_N^{opt} a^{N+1}$$

(3)

where

$$\delta D_N^{opt} = \Omega_N(d_i, c_i) - \Omega_N^{opt}(d_i^{opt}, c_i^{opt})$$

(4)

are the numbers which simulate the coefficients of the order $O(a^{N+1})$-corrections to the physical quantity, calculated in the particular initial scheme, say the $\overline{MS}$-scheme. The coefficients $\Omega_N$ can be obtained from the following system of equations:

$$\frac{\partial}{\partial \tau} (D_N + \Omega_N a^{N+1}) = O(a^{N+2}),$$

$$\frac{\partial}{\partial c_i} (D_N + \Omega_N a^{N+1}) = O(a^{N+2}), i \geq 2$$

(5)

\[2\] The application of this similar technique to the analysis of the five-loop approximations of $(g-2)_\mu$ and $(g-2)_e$ will be considered elsewhere.
where the parameter $\tau = \beta_0 \ln(\mu^2/\Lambda^2)$ represents the freedom in the choice of the renormalization point $\mu$. The conventional scale parameter $\Lambda$ will not explicitly appear in all our final formulas.

The explicit form of the coefficients $\Omega_2$ and $\Omega_3$ in which we will be interested can be obtained by the solution of the system of equations (5), following the lines of Ref. [17]. We present here only the final expressions:

$$\Omega_2 = d_0d_1(c_1 + d_1), \quad \Omega_3 = d_0d_1(c_2 + \frac{1}{2} c_1d_1 - 2d_1^2 + 3d_2).$$

(6)

(7)

It should be stressed that in the ECH approach $d_i^{ECH} = 0$ for all $i \geq 2$. Therefore one gets the following expressions for the NNLO and NANNLO corrections in Eq. (3):

$$\delta D_2^{ECH} = \Omega_2(d_1, c_1) \quad \text{(8)}$$

$$\delta D_3^{ECH} = \Omega_3(d_1, d_2, c_1, c_2) \quad \text{(9)}$$

where $\Omega_2$ and $\Omega_3$ are defined in Eqs. (6) and (7) respectively.

In order to find similar corrections to Eq. (3) in the N-th order of perturbation theory starting from the PMS approach [17], it is necessary to use the relations obtained in Ref. [24] between the coefficients $r_{PMS}^i$ and $c_{PMS}^i (i \geq 1)$ in the expression for the order $O(a_{PMS}^N)$ approximation $D_N^{PMS}(a_{PMS})$ of the physical quantity under consideration. The corresponding corrections have the following form:

$$\delta D_2^{PMS} = \delta D_2^{ECH} - \frac{d_0 c_1^2}{4} \quad \text{(10)}$$

$$\delta D_3^{PMS} = \delta D_3^{ECH} \quad \text{(11)}$$

Notice the identical coincidence of the NANNLO corrections obtained starting from both the PMS and ECH approaches. A similar observation was made in Ref. [18] using different (but related) considerations.

### 3 Input of the Analysis

Consider first the familiar characteristic of the $e^+e^- \rightarrow \gamma \rightarrow \text{hadrons}$ process, namely the $D$-function defined in the Euclidean region:

$$D(Q^2) = Q^2 \int_0^\infty \frac{R(s)}{(s + Q^2)^2} ds \quad \text{(12)}$$

Its perturbative expansion has the following form:

$$D(Q^2) = 3\Sigma Q^2_3[1 + a + d_1a^2 + d_2a^3 + d_3a^4 + ...]$$

$$+ (\Sigma Q_f)^2[d_2a^3 + O(a^4)] \quad \text{(13)}$$
where $Q_f$ are the quark charges, and the structure proportional to $(\Sigma Q_f)^2$ comes from the light-by-light-type diagrams. The coefficients $d_1$ and $d_2, \tilde{d}_2$ were calculated in the $\overline{MS}$-scheme in Refs. [7] and [1] respectively. They have the following numerical form:

$$d_1^{\overline{MS}} \approx 1.986 - 0.115 f, \quad d_2^{\overline{MS}} \approx 18.244 - 4.216 f + 0.086 f^2, \quad \tilde{d}_2 \approx -1.240.$$ (14)

Following the proposals of Ref. [25], we will treat the light-by-light-type term in Eq. (13) separately from the “main” structure of the $D$-function, which is proportional to the quark-parton expression $D^{QP}(Q^2) = 3\Sigma Q_f^2$. In fact, one can hardly expect that it is possible to predict higher-order coefficients $\tilde{d}_i, i \geq 3$ of the second structure in Eq. (13) using the only explicitly-known term $\tilde{d}_2$. Therefore we will neglect the light-by-light-type structure as a whole in all our further considerations. This approximation is supported by the relatively tiny contribution of the second structure of Eq. (13) to the final NNLO correction to the $D$-function.

The next important ingredient of our analysis is the QCD $\beta$-function (2), which is known in the MS-like schemes at the NNLO level [26]. Its corresponding coefficients read

$$\beta_0 = (11 - \frac{2}{3} f) \frac{1}{4} \approx 2.75 - 0.167 f$$
$$c_1 = \frac{153 - 19 f}{66 - 4 f}$$
$$c_2^{\overline{MS}} = \frac{77139 - 15099 f + 325 f^2}{9504 - 576 f}.$$ (15)

Using now the perturbative expression for the $D$-function, one can obtain the perturbive expression for $R(s)$, namely

$$R(s) = 3\Sigma Q_f^2 [1 + a_s + r_1 a_s^2 + r_2 a_s^3 + r_3 a_s^4 + ...]$$
$$+ (\Sigma Q_f)^2 [\tilde{r}_2 a_s^3 + ...]$$ (16)

where $a_s = \bar{\alpha}_s/\pi$, and

$$r_1 = d_1$$
$$r_2 = d_2 - \frac{\pi^2 \beta_0^2}{3}, \quad \tilde{r}_2 = \tilde{d}_2$$
$$r_3 = d_3 - \pi^2 \beta_0^2 (d_1 + \frac{5}{6} c_1).$$ (17)

The corresponding $\pi^2$-terms come from the analytic continuation of the Euclidean result for the $D$-function to the physical region. The effects of the higher-order $\pi^2$-terms were discussed in detail in Ref. [27].

The perturbative expression for $R_\tau$ is defined as

$$R_\tau = 2 \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} (1 - s/M_\tau^2)^2 (1 + 2s/M_\tau^2) \tilde{R}(s)$$
$$\simeq 3[1 + a_\tau + r_1^\tau a_\tau^2 + r_2^\tau a_\tau^3 + r_3^\tau a_\tau^4 + ...]$$ (18)
where \( a_\tau = \frac{\alpha_s(M_\tau^2)}{\pi} \) and \( \tilde{R}(s) = R(s) \) with \( f = 3, (\Sigma_{Q_f})^2 = 0, 3\Sigma Q_f^2 \) substituted for \( 3\Sigma |V_{ff'}|^2 \) and \( |V_{ud}|^2 + |V_{us}|^2 \approx 1 \).

It can be shown that in the \( \overline{MS} \)-scheme the coefficients of the series (18) are related to those of the series (13) for the \( D \)-function by the following numerical relations (see Refs. [3], [16]):

\[
\begin{align*}
  r_1^\tau &= d_1^{\overline{MS}}(f = 3) + 3.563 = 5.202 \\
  r_2^\tau &= d_2^{\overline{MS}}(f = 3) + 19.99 = 26.366 \\
  r_3^\tau &= d_3^{\overline{MS}}(f = 3) + 78.00 .
\end{align*}
\]

In order to estimate the values of the order \( O(a^3) \) and \( O(a^4) \) corrections to \( R(s) \) and \( R_\tau \), we will apply Eqs. (6) - (11) in the Euclidean region to the perturbative series for the \( D \)-function and then obtain the corresponding estimates using Eqs. (17) and (19). This is one of the origins of the disagreement of our considerations with those presented in Ref. [22].

We now recall the perturbative expression for the non-polarized Bjorken deep-inelastic scattering sum rule

\[
BjnSR = \int_0^1 F_1^{\bar{p}p}(x, Q^2)dx = 1 - \frac{2}{3}a(1 + d_1 a + d_2 a^2 + d_3 a^3 + ...) \tag{20}
\]

where the coefficients \( d_1 \) and \( d_2 \) are known in the \( \overline{MS} \) scheme from the results of calculations of Ref. [5] and Ref. [6] respectively:

\[
\begin{align*}
  d_1^{\overline{MS}} &\approx 5.75 - 0.444 f \\
  d_2^{\overline{MS}} &\approx 54.232 - 9.497 f + 0.239 f^2 . \tag{21, 22}
\end{align*}
\]

The expression for the polarized Bjorken sum rule BjpSR has the following form:

\[
BjpSR = \int_0^1 g_1^{\bar{p}p}(x, Q^2)dx = \frac{1}{3} \left| \frac{g_A}{g_V} \right| \left\{ 1 - a(1 + d_1 a + d_2 a^2 + d_3 a^3 + ...) \right\} \tag{23}
\]

where the coefficients \( d_1 \) and \( d_2 \) were explicitly calculated in the \( \overline{MS} \) scheme in Refs. [10] and [11] respectively. The results of these calculations read

\[
\begin{align*}
  d_1^{\overline{MS}} &\approx 4.583 - 0.333 f \\
  d_2^{\overline{MS}} &\approx 41.440 - 7.607 f + 0.177 f^2 . \tag{24, 25}
\end{align*}
\]

It should be stressed that since deep inelastic scattering sum rules are defined in the Euclidean region, we can directly apply to them the methods discussed in Section 2, and thus predict the values of the coefficients \( d_3 \) using Eqs. (9) and (7) without any additional modifications.
It is also worth emphasizing that, in spite of the identical coincidence of the NLO correction to the Gross-Llewellyn Smith sum rule

$$GLSSR = \frac{1}{2} \int_{0}^{1} \frac{F_{3}^{\nu p \nu p}(x, Q^2)}{x} dx $$

$$= 3\{1 - a(1 + d_1a + d_2a^2 + d_3a^3 + ... )\}$$

with the result of Eq. (24) [10], the corresponding NNLO correction differs from the result of Eq. (25) by the contributions of the light-by-light-type terms typical of the GLSSR [6]:

$$(d_2)_{GLSSR} = (d_2)_{BjpSR} + \tilde{d}_2, \quad \tilde{d}_2 = -0.413 f$$

Since these light-by-light-type terms appear for the first time at the NNLO, it is impossible to predict the value of the light-by-light-type contribution at the NANNLO level using the corresponding NNLO terms as the input information. However, noticing that at the NNLO level the corresponding light-by-light-type contributions are small, we will assume that the similar contributions are small at the NANNLO level also. After this assumption our estimates of the NNLO and NANNLO corrections to the BjpSR can be considered also as the estimates of the corresponding corrections in the perturbative series for the GLSSR. Note, that the Padé predictions of the order $O(a^4)$ contributions to the GLSSR [28], which do not take into account the necessity of the careful considerations of the light-by-light-type terms, have definite drawbacks. However, certain other results of Refs. [28] and [29] deserve comparison with the results of our studies.

4 Outputs of the Analysis

The estimates of the coefficients of the order $O(a^3)$ and $O(a^4)$ QCD corrections to the $D$-function, $R(s)$, BjnSR and BjpSR/GLSSR obtained following the discussions of Section 2 with the help of the results summarized in Section 3, are presented in Tables 1 - 4 respectively. Due to the complicated $f$-dependence of the coefficients $\Omega_2, \Omega_3$ in Eqs. (6) and (7), we are unable to predict the explicit $f$-dependence of the corresponding coefficients in the form respected by perturbation theory. The results are presented for the fixed number of quark flavours $1 \leq f \leq 6$ and are related to the $\overline{MS}$ scheme. The estimates of the NNLO corrections, obtained starting from both the ECH and PMS approaches, are compared with the results of the explicit calculations.

Using the results of Table 1 and Eqs. (19), we are able to predict the NNLO coefficient of $R_\tau$:

$$(r_2^\tau)_{ECH}^{est} \approx 25.6 \quad (28)$$

$$(r_2^\tau)_{PMS}^{est} \approx 24.8 \quad (29)$$

One can see the agreement with the explicitly calculated result

$$(r_3^\tau)_{\overline{MS}} = 26.366 \quad (30)$$
Considering this agreement as the additional \textit{a posteriori} support of the methods used, we use the estimate of the NANNLO coefficient for the $D$-function with $f = 3$ numbers of flavours as presented in Table 1, namely:
\begin{equation}
\delta^\text{est}_3 = 27.46
\end{equation}
and predict the value of the NANNLO coefficient of $R_\tau$:
\begin{equation}
(r_3^{\tau})^\text{est} \approx 105.5
\end{equation}

5 \quad \textbf{Discussions}

We are now ready to discuss the main results of our analysis.

1. The predictions of the NNLO corrections to the $D$-function, BjnSR and BjpSR/GLSSR are in qualitative agreement with the results of the explicit calculations of Refs. \cite{1} \cite{5} and \cite{6} and respect the tendency of the corresponding coefficients to decrease with increasing number of flavours.

2. The best agreement of the NNLO estimates with the exact results is obtained for the case $f = 3$. This fact, which we do not understand, supports the application of the method used for estimating the NNLO and NANNLO corrections to $R_\tau$.

3. The positive feature of the estimates as obtained by us is their scheme-dependence. Notice that since the methods used correctly reproduce the renormalization-group-controllable terms \cite{23}, \cite{17}, the transformation from the $\overline{MS}$ scheme to other variants of the MS-like scheme will not spoil the qualitative agreement with the results of the explicit calculations.

4. The results of Table 2 demonstrate that the $\pi^2$ effects give dominating contributions to the coefficients of $R(s)$.

5. From the phenomenological point of view, the most interesting results are the estimates of the NANNLO corrections to $R_\tau$ (see Eq. (32)) and $R(s)$ for $f = 5$ numbers of flavours. Taking $\alpha_s(M_Z) \approx 0.12$, we get the estimate of the corresponding NANNLO contribution to both $\Gamma(Z^0 \rightarrow \text{hadrons})$ and $\Gamma(Z^0 \rightarrow \bar{q}q)$: 
\begin{equation}
\delta \Gamma_{Z^0} \approx -97(a(M_Z))^4 \approx -2 \times 10^{-4}
\end{equation}
It is of the order of magnitude of other corrections included in the current analysis of LEP data (see, e.g., \cite{30}).

6. Taking $\alpha_s(M_\tau) \approx 0.36$ \cite{3}, we get the numerical estimate of the NANNLO contribution to $R_\tau$:
\begin{equation}
\delta R_\tau \approx 105.5 a_4 \approx 1.8 \times 10^{-2}
\end{equation}
It is larger than the recently calculated power-suppressed perturbative \cite{31} and non-perturbative \cite{32} contributions to $R_\tau$. 

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7. Our estimate for \( d_3(f = 3) \) of Eq. (31) is more definite than the one presented in Ref. [29], namely \( d_3(f = 3) = 55_{-21}^{+60} \), and than the bold guess estimate \( d_3(f = 3) = \pm 25 \), given in Ref. [13]. Moreover, the related estimate of the NANNLO contribution to \( R_\tau \) (see Eq. (32)) is more precise than those presented in Refs. [3] and [13], namely \( \delta R_\tau = \pm 130 a_4^\tau \) and \( \delta R_\tau = (78 \pm 25)a_4^\tau \) [13], and is smaller than the result of applying the Padé resummation technique directly to \( R_\tau \), namely \( \delta R_\tau = 133a_4^\tau \) [29].

8. The qualitative agreement of the results of Tables 3 and 4 for BjnSR and BjpSR with the corresponding Padé estimates of Ref. [28] can be considered as the argument in favour of the applicability of both theoretical methods in the Euclidean region for the concrete physical applications. Let us stress again that in the process of these applications the light-by-light-type structures, contributing say to \( R(s) \) and GLSSR, should be treated separately.

9. Notice also, that the application of the Padé resummation technique to \( R(s) \) [28], stimulated by the previous similar studies of Ref. [34], gives less definite estimates than the results of applications of our methods (compare the estimate \( \delta R(s) = (-49^{+54}_{-40})a_4^s \) [29] with the result \( \delta R(s) = -97a_4^s \) from Table 2 and Eq. (33)). This fact is related with the problems of applicability of the Padé resummation technique to the sign variation perturbative series, obtained from the explicit NNLO approximation of the \( D \)-function with \( f = 5 \) numbers of flavours [1] (see also Ref. [2]).

6 Acknowledgements

We wish to thank K.H. Becks, D. Yu. Bardin and S. Bethke for stimulating the presentation of the results of this work prior to publication, at the III Workshop on Artificial Intelligence and Expert Systems in HEP (Oberammergau, October 1993), LEP Meetings at CERN (March 1994) and the University of Aachen (April 1994) respectively.

We are grateful to R.N. Faustov for attracting our attention to the detailed consideration of the results of Ref. [13] at the preliminary stage of our similar QED studies, which will be presented elsewhere and to F. Le Diberder for useful comments.

The work of one of us (V.V.S.) was supported during his stay in Moscow by the Russian Fund for Fundamental Research, Grant No. 93-02-14428.

Note added. Our estimate for \( d_3 \) was recently supported by the phenomenological analysis of the ALEPH data for \( R_\tau \) [35].

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3 A similar, more conservative estimate gives \( \delta R_\tau = (78 \pm 50)a_4^\tau \) (for a review, see Ref. [33]).
References

[1] S.G. Gorishny, A.L. Kataev and S.A. Larin, *Phys. Lett.* **B259** (1991) 144; *Pisma ZhETF* **53** (1991) 121.

[2] L.R. Surguladze and M.A. Samuel, *Phys. Rev. Lett.* **66** (1991) 560; **66** (1991) 2416 (Erratum).

[3] E. Braaten, S. Narison and A. Pich, *Nucl. Phys.* **B373** (1992) 581.

[4] E. Braaten, *Phys. Rev. Lett.* **71** (1993) 1316.

[5] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, *Phys. Rev. Lett.* **66** (1991) 862.

[6] S.A. Larin and J.A.M. Vermaseren, *Phys. Lett.* **B259** (1991) 345.

[7] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, *Phys. Lett.* **85B** (1979) 277; M. Dine and J. Sapirstein, *Phys. Rev. Lett.* **43** (1979) 668; W. Celmaster and R. Gonsalves, *Phys. Rev. Lett.* **44** (1980) 560.

[8] B.A. Kniehl and J.H. Kühn, *Nucl. Phys.* **B329** (1990) 547.

[9] K.G. Chetyrkin, S.G. Gorishny, S.A. Larin and F.V. Tkachov, *Phys. Lett.* **137B** (1984) 230.

[10] S.G. Gorishny and S.A. Larin, *Phys. Lett.* **B172** (1986) 109; E.B. Zijlstra and W. van Neerven, *Phys. Lett.* **B297** (1992) 377.

[11] G.B. West, *Phys. Rev. Lett.* **67** (1991) 67; *Phys. Rev. Lett.* **67** (1991) 372 (Erratum).

[12] V.I. Zakharov, *Nucl.Phys.* **B385** (1992) 452; L.S. Brown, L.G. Yaffe and C. Zhai, *Phys.Rev.* **D46** (1992) 4712.

[13] J. Chýla, J. Fischer and P. Kolař, *Phys.Rev.* **D47** (1993) 2578.

[14] J. Chýla, A.L. Kataev and S.A. Larin, *Phys. Lett.* **B267** (1991) 269.

[15] F. Le Diberder and A. Pich, *Phys. Lett.* **B286** (1992) 147.

[16] A.A. Pivovarov, *Z. Phys.* **C53** (1992) 461.

[17] P.M. Stevenson, *Phys.Rev.* **D23** (1981) 2916.

[18] J. Kubo and S. Sakakibara, *Z. Phys.* **C14** (1982) 345.

[19] V.V. Starshenko and R.N. Faustov, JINR Rapid Communications **7** (1985) 39.

[20] G. Grunberg, *Phys. Lett.* **B221** (1980) 70; *Phys.Rev.* **D29** (1984) 2315.

[21] A. Dhar and V. Gupta, *Phys.Rev.* **D29** (1984) 2822; V. Gupta, D.V. Shirkov and O.V. Tarasov, *Int. J. Mod. Phys.* **A6** (1991) 3381.
[22] J.H. Field, *Ann. Phys.* **226** (1993) 209.

[23] R. Barbieri and E. Remiddi, *Phys. Lett.* **57B** (1975) 273.

[24] M.R. Pennington, *Phys. Rev.* **D26** (1986) 2048.

[25] S. Brodsky, G.P. Lepage and P.B. Mackenzie, *Phys. Rev.* **D28** (1983) 228; Hung Jung Lu, *Phys. Rev.* **D45** (1992) 1217.

[26] O.V. Tarasov, A.A. Vladimirov and A.Yu. Zharkov, *Phys. Lett.* **93B** (1980) 429; S.A. Larin and J.A.M. Vermaseren, *Phys. Lett.* **B303** (1993) 334.

[27] J.D. Bjorken, report SLAC-PUB-5103 (1989).

[28] M.A. Samuel, G. Li and E. Steinfelds, *Phys. Lett.* **B323** (1994) 188.

[29] M.A. Samuel and G. Li, preprint SLAC-PUB-6379 (1993).

[30] A.L. Kataev, *Phys. Lett.* **B287** (1992) 209.

[31] K.G. Chetyrkin and A. Kwiatkowski, *Z. Phys.* **C59** (1993) 525.

[32] P. Nason and M. Porrati, preprint CERN-TH.6787/93 (1993); I.I. Balitsky, M. Beneke and V.M. Braun, preprint MPI-Ph/93-62, PSU/TH/130 (1993).

[33] S. Narison, preprint CERN-TH.7188/94, PM 94/08 (1994).

[34] J. Fleischer, M. Pindor, P.A. Ráczka and R. Ráczka, Proceedings of the XII Warsaw Symposium on Elementary Particle Physics, Frontiers in Particle Physics, Kazimierz, Poland, May–June 1989. Eds. Z. Adjuk, S. Pokorski and A.K. Wroblewski (World Scientific, Singapore 1990) p. 412.

[35] F. Le Diberder, talk at the QCD-94 Workshop, July 1994, Montpellier, France, to appear in *Nucl. Phys. Proc. Suppl.* ed. S. Narison.
Table 1: The results of estimates of the NNLO and NANNLO corrections in the series for the $D$-functions.

| $f$ | $d^e_{2}x$ | $(d^e_{2})_{ECH}$ | $(d^e_{2})_{PMS}$ | $d^e_{3}$ |
|-----|------------|------------------|------------------|-----------|
| 1   | 14.11      | 7.54             | 7.70             | 75.4      |
| 2   | 10.16      | 6.57             | 7.55             | 49.76     |
| 3   | 6.37       | 5.61             | 6.40             | 27.46     |
| 4   | 2.76       | 4.68             | 5.27             | 8.37      |
| 5   | -0.69      | 3.77             | 4.16             | -7.7      |
| 6   | -3.96      | 2.88             | 3.08             | -20.89    |

Table 2: The results of estimates of the NNLO and NANNLO corrections in the series for $R(s)$.

| $f$ | $r^e_{2}x$ | $(r^e_{2})_{ECH}$ | $(r^e_{2})_{PMS}$ | $r^e_{3}$ |
|-----|------------|------------------|------------------|-----------|
| 1   | -7.84      | -14.41           | -13.16           | -166.4    |
| 2   | -9.04      | -12.63           | -11.65           | -146.6    |
| 3   | -10.27     | -11.03           | -10.23           | -128.4    |
| 4   | -11.52     | -9.58            | -8.98            | -111.8    |
| 5   | -12.76     | -8.29            | -7.9             | -96.8     |
| 6   | -14.01     | -7.17            | -6.97            | -83.3     |

Table 3: The results of estimates of the NNLO and NANNLO corrections in the series for BjnSR.

| $f$ | $d^e_{2}x$ | $(d^e_{2})_{ECH}$ | $(d^e_{2})_{PMS}$ | $d^e_{3}$ |
|-----|------------|------------------|------------------|-----------|
| 1   | 44.97      | 39.62            | 40.78            | 423.77    |
| 2   | 36.19      | 33.28            | 34.26            | 302.7     |
| 3   | 27.89      | 27.37            | 28.16            | 199.61    |
| 4   | 20.07      | 21.91            | 22.50            | 113.68    |
| 5   | 12.72      | 16.91            | 17.30            | 44.1      |
| 6   | 5.85       | 12.39            | 12.59            | -9.89     |

Table 4: The results of estimates of the NNLO and NANNLO corrections in the series for BjpSR and GLSSR.