Entropy of the Randall–Sundrum black brane world to all orders in the Planck length

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Abstract

We study the effects, to all orders in the Planck length from a generalized uncertainty principle (GUP), on the statistical entropy of massive scalar bulk fields in the Randall–Sundrum black brane world. We show that the Bekenstein–Hawking area law is not preserved, and contains a correction term proportional to the black hole inverse area.

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1. Introduction

The possibility of the existence of extra dimensions has opened exciting and promising ways to investigate phenomenological and cosmological aspects of quantum gravity. Models with extra dimensions and an effective fundamental scale of the order of the TeV have been proposed as the possible solution to the gauge hierarchy problem [1–5]. Particularly, the Randall–Sundrum models [4, 5] have attracted great attention and their cosmological implications have been intensively studied [6–21]. On the other hand, since the seminal works of Bekenstein [22] and Hawking [23], the computation of the entropy of a black hole remains an active field of research. Various approaches and methods have been employed. Among them, the brick-wall method [24], which is a semi-classical approach, has been applied to various BH geometries (see [25] and references therein). However, this approach suffers from the implementation of unnatural arbitrary ultraviolet and infrared cut-offs. Recently, with the advent of generalized uncertainty principles (GUPs), originating from several studies in the string theory approach to quantum gravity [26–29], loop quantum gravity [30], noncommutative spacetime algebra [31–33] and black holes gedanken experiments [34, 35], the contribution to the entropy

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of quantum states with momentum above a given scale has been suppressed and the UV divergence completely removed (see [36] for an extensive list of references).

Recently, the calculation of the statistical entropy of thermal bulk massive scalar fields on the Randall–Sundrum brane background has been performed with a GUP to leading order in the Planck length [37], and the effect of the GUP has only been considered on the 3-brane. In a recent paper, we have considered the thermodynamics of the Schwarzschild black hole in the ADD scenario with a GUP to all orders in the Planck length, and particularly we have computed the correction terms to the famous Bekenstein–Hawking area law of the entropy [38]. On the other hand, a careful analysis of the entropy near the horizon to all orders in the Planck length has been performed for the \((3 + 1)\)-dimensional Schwarzschild black hole [39] and for the \((2 + 1)\)-dimensional de Sitter black hole [40].

In this paper, we continue our investigation of the effects of the GUP to all orders in the Planck length initiated in [36, 38], and proceed one more step further in this direction. By using the novel equation of states density motivated by the GUP and performing a careful analysis of the entropy integral near the horizon, we study the statistical entropy of thermal bulk massive scalar fields on the Randall–Sundrum brane background. In section 2, we briefly review, in \((D + 1)\)-dimensional spacetime, the GUP containing gravitational corrections to all orders in the Planck length, and derive the expression of the minimal length. In section 3, we obtain a novel equation of states density for the extra and radial modes. In section 4, using the near-horizon geometry approximation and considering the effect of the GUP on the full volume of the bulk, we derive the free energy of a massive bulk scalar field and by means of the first law of thermodynamics we obtain the GUP-corrected Bekenstein–Hawking area law for the entropy. Then, in order to compare our results with the ones obtained by the brick-wall method and with the GUP to leading order in the Planck length, we ignore the effect of the GUP in the extra dimension direction and compute again the free energy and the entropy. The final section is devoted to a summary and a discussion of the results obtained.

2. Generalized uncertainty principle

One of the most interesting consequences of all promising quantum gravity candidates is the existence of a minimal observable length on the order of the Planck length. The idea of a minimal length can be modeled in terms of a quantized spacetime and goes back to the early days of quantum field theory [41] (see also [42–45]). An alternative approach is to consider deformations to the standard Heisenberg algebra [32, 33], which lead to generalized uncertainty principles. In this section, we follow the latter approach and exploit the recently obtained results. Indeed, it has been shown in the context of canonical noncommutative field theory in the coherent states representation [46] and field theory on non-anticommutative superspace [47, 48], that the Feynman propagator displays an exponential UV cut-off of the form \(\exp(-\eta p^2)\), where the parameter \(\eta\) is related to the minimal length. This framework has been further applied, in a series of papers [49], to the black hole evaporation process.

At the quantum mechanical level, the essence of the UV finiteness of the Feynman propagator can be also captured by a nonlinear relation, \(k = f(p)\), between the momentum and the wave vector of the particle [50]. This relation must be invertible and has to fulfil the following requirements:

(1) For energies much smaller than the cut-off the usual dispersion relation is recovered.
(2) The wave vector is bounded by the cut-off.
In this picture, the usual commutator between the commuting position and momentum operators is generalized to

\[ [\hat{x}_i, \hat{p}_j] = i \frac{\partial p_i}{\partial k_j} \Leftrightarrow \Delta \hat{x}_i \Delta \hat{p}_j \geq \frac{1}{2} \left| \frac{\partial p_i}{\partial k_j} \right|, \]

where \( i, j = 1, 2, \ldots, D \). Assuming rotational invariance, the momentum measure \( d^Dp \) is deformed as \( d^Dp = \frac{d^Dk}{\Delta \hat{x}_i / \Delta \hat{p}_j} \). Following [46, 48] and setting \( \eta = \frac{\alpha L^2}{\hbar^2} \), we have

\[ \frac{\partial k_i}{\partial p_i} = \hbar^{-1} \exp \left( \frac{-\alpha L^2}{\hbar^2} p^2 \right). \]

where \( p^2 = \sum_{j=1}^D p^2_j \), and \( \alpha \) is a dimensionless positive constant of order one.

This results in the following generalized uncertainty principle:

\[ (\delta x_i)(\delta p_i) \geq \frac{\hbar}{2} \left\{ \exp \left( \frac{\alpha L^2}{\hbar^2} (\delta p_i)^2 \right) \right\}. \]

Now we solve relation (3) for \((\delta p_i)\) with the equality. Using the properties \( \langle p^2 \rangle \geq \langle p_i^2 \rangle \) and \((\delta p)^2 = \langle p^2 \rangle - \langle p_i^2 \rangle \), and assuming isotropic uncertainties \( \delta p_j = \delta p, j = 1, 2, \ldots, D \), the generalized uncertainty relation is written as

\[ (\delta x_i)(\delta p_i) = \frac{\hbar}{2} \exp \left( \frac{\alpha L^2}{\hbar^2} \left( D \langle p_i^2 \rangle + \sum_{k=1}^D \langle p_k^2 \rangle \right) \right). \]

Following [36, 38], a solution to this equation is given in terms of the Lambert function \( W(u) \) [51]:

\[ (\delta p_i) = \frac{\hbar}{\sqrt{2DaL^2}} \left( -W \left( \frac{-DaL^2}{2(\delta x_i)^2} e^{-\frac{2aL^2}{\hbar^2} \sum_{k=1}^D \langle p_k^2 \rangle} \right) \right)^{1/2}. \]

The Lambert function has different branches \( W_k(u) \), labeled by the integer \( k = 0, \pm 1, \pm 2, \ldots \). When \( u \) is a real number the Lambert function has two real solutions for \( 0 \geq u \geq -\frac{1}{e} \), denoted by \( W_0(u) \) and \( W_{-1}(u) \), or can have only one real solution for \( u \geq 0 \), namely \( W_0(u) \). For \( -\infty < u < -\frac{1}{e} \), it has no real solution. From the argument of the Lambert function, we have the following condition:

\[ \frac{DaL^2}{2(\delta x_i)^2} e^{-\frac{2aL^2}{\hbar^2} \sum_{k=1}^D \langle p_k^2 \rangle} \leq \frac{1}{e}. \]

The absolutely isotropic smallest uncertainty in position or minimal length is then obtained for physical states for which we have \( \langle p_k \rangle = 0 \)

\[ (\delta X) = (\delta x_i)_0 = \sqrt{\frac{Da \alpha e}{2L^2}}. \]

In terms of the minimal length, the momentum uncertainty becomes

\[ (\delta p_i) = \frac{\hbar}{2(\delta X)} \left( -W \left( -\frac{1}{e} \left( \frac{\delta X}{\delta x_i} \right)^2 \right) \right)^{1/2}. \]

This equation can be inverted to obtain the position uncertainty as

\[ (\delta x_i) = \frac{\hbar}{2(\delta p_i)} \exp \left( \frac{(\delta p_i)^2}{(\delta p)^2} \right). \]
where \((\delta p)_0 = \frac{\hbar}{\sqrt{2\alpha L_{Pl}}}\) is the energy cut-off. We observe that for energies much smaller than the cut-off, corresponding to weak gravitational coupling, the standard situation with the Heisenberg uncertainty principle (HUP) is recovered.

Let us observe that \(\frac{1}{\alpha L_{Pl}} < 1\) is a small parameter by virtue of the GUP, and then perturbative expansions to all orders in the Planck length can be safely performed. Indeed, a series expansion of equation (8) to first order, which reproduces the form of the GUP to leading order in the Planck length, was recently used by Kim et al [37] and is given by

\[
(\delta x_i)(\delta p_i) \geq \frac{\hbar}{2} \left( 1 + \frac{\alpha L_{Pl}^2}{\hbar^2} (\delta p)^2 \right).
\]

(10)

A simple calculation leads to the following minimal length:

\[
(\delta X)_0 = \sqrt{D\alpha L_{Pl}}.
\]

(11)

However, as nicely noted in [50], this form of the GUP does not satisfy the second requirement listed above.

In the following sections, we use the minimal length given by equation (7) for \(D = 3\) and calculate the statistical entropy of the bulk massive scalar field on the Randall–Sundrum black brane background. We use the units \(\hbar = c = k_B = G = 1\).

3. Massive scalar field on the Randall–Sundrum brane background

We consider a dual-brane Randall–Sundrum scenario, embedded in a five-dimensional \(AdS_5\) spacetime. The 3-branes with positive and negative tensions are, respectively, located at the \(S^1/Z_2\) orbifold fixed points \(y = 0\) and \(y = y_c = \pi r_c\) [4, 5]. Assuming Poincaré invariance on the branes, the solutions to Einstein’s equations are given by

\[
d s^2 = e^{-2kyg_{\mu\nu}}d x^{\mu} d x^{\nu} + d y^2,
\]

(12)

where the parameter \(k\), assumed to be of the order of the Planck scale, governs the degree of curvature of the \(AdS_5\) spacetime. Assuming a Ricci flat metric, one solution is [13]

\[
d s^2 = e^{-2ky}(-f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2) + dy^2,
\]

(13)

where \(f(r) = 1 - \frac{2M}{r}\). This solution describes a four-dimensional Schwarzschild black hole located on the hypersurface. It describes also a five-dimensional \(AdS\) black string intersecting the brane world.

We now consider a matter field propagation in this brane background. We consider massive scalar fields which are solutions of the Klein–Gordon equation

\[
(\nabla^2 - m^2)\Psi = 0.
\]

(14)

Using solution (13), we have

\[
e^{2ky} \left[ \frac{1}{f} \partial_r^2 \Psi + \frac{1}{r^2} \partial_r (r^2 f \partial_r \Psi) + \frac{1}{r^2 \sin^2 \theta} \partial_\theta (\sin \theta \partial_\theta \Psi) + \frac{1}{r^2 \sin^2 \theta} \partial_\phi^2 \Psi \right] + e^{4ky} \partial_y (e^{-4ky} \partial_y \Psi) - m^2 \Psi = 0.
\]

(15)

Substituting \(\Psi = e^{-i\omega t} \Phi(r, \theta, \phi)\xi(y)\), we obtain

\[
\partial_y^2 \Phi + \left( \frac{1}{f} \partial_r^2 + \frac{2}{r} \right) \partial_r \Phi + \frac{1}{f} \left( \frac{1}{r^2} \left[ \partial_\theta^2 + \cot \theta \partial_\theta \right] + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right) + \frac{\omega^2}{f} - \mu^2 = 0,
\]

(16)

where \(\mu^2\) is an effective mass defined by

\[
e^{4ky} \partial_y (e^{-4ky} \partial_y \xi(y)) - m^2 \xi(y) + \mu^2 e^{2ky} \xi(y) = 0.
\]

(17)
We simplify these equations by using the Wentzel–Kramers–Brillouin (WKB) approximation for which we set \( \Phi \sim e^{i S(r, \theta, \phi)} \). Indeed, to the leading order we have

\[
-\partial_r^2 \Phi - \left( \frac{1}{f} \partial_r f + \frac{2}{r} \right) \partial_r \Phi = p_r^2 \Phi,
\]

and

\[
-e^{4ky} \partial_y \left( e^{-4ky} \partial_y \xi(y) \right) = p_y^2 \xi(y),
\]

with \( p_\alpha = \frac{\partial S}{\partial \alpha}, \alpha = r, \theta, \phi \) and \( p_r^2, p_y^2 \) given, respectively, by

\[
p_r^2 = \left( \frac{\omega^2}{f} - \mu^2 - \frac{p_\theta^2}{r^2} - \frac{p_\phi^2}{r^2\sin^2 \theta} \right),
\]

\[
p_y^2 = \mu^2 e^{2ky} - m^2.
\]

A central ingredient for our calculation is the degeneracy of the brane and extra dimension modes. To this aim, we first note that the volume in the momentum space is affected by the squeezed momentum measure arising from the GUP and given by equation (2). Indeed, the number of quantum radial modes with energy less than \( \omega \), for a given \( \mu \), is given by

\[
n_r(\omega) = \frac{1}{(2\pi)^3} \int dr \ d\theta \ d\phi \ dp_r \ dp_\theta \ dp_\phi \ e^{-\alpha p^2} \left( \frac{\omega^2}{f} - \mu^2 - \frac{p_\theta^2}{r^2} - \frac{p_\phi^2}{r^2\sin^2 \theta} \right)^{3/2} e^{-\alpha \left( \frac{\omega^2}{f} - \mu^2 \right)},
\]

with the condition \( \omega \geq \mu \sqrt{f} \). We note that the additional suppressing exponential, due to the GUP, renders \( n_r(\omega) \) finite at the horizon without the introduction of any artificial cut-off, as is the case in the brick-wall method.

On the other hand, the number of quantum states in the extra dimension for given \( \mu \) is

\[
n_y(\mu) = \frac{1}{\pi} \int dy \ dp_y \ e^{-\alpha p^2} \left( \sqrt{\sqrt{\alpha} \sqrt{\mu^2 e^{2ky} - m^2}} \right) dy.
\]

4. Entropy to all orders in the Planck length

In this section, we shall evaluate the free energy and entropy of free massive bulk scalar fields at the Hawking temperature. We shall first consider the case where the GUP affect the bulk modes and finally the case where the GUP affect only the brane modes.

4.1. GUP on the bulk

In the continuum limit, the free energy of a scalar field at the inverse temperature \( \beta \) is approximated by

\[
F_\beta = \frac{1}{\beta} \int \left( \frac{dN(\omega) \ln(1 - e^{-\beta \omega}) \right),
\]

where the total number of quantum states with energy less than \( \omega \) is given by

\[
N(\omega) = \int dn_r \ dn_y.
\]
An integration by parts gives
\[
F_\beta = -\int_0^\infty d\omega \frac{N(\omega)}{e^{\beta \omega} - 1}.
\] (26)

Using the expression of the number of radial modes \(n_r\) given by (22), we obtain
\[
F_\beta = -2 \frac{2}{3\pi} \int_{r_h}^{r_h^{+\epsilon}} dr \frac{r^2}{\sqrt{f}} \int \frac{\omega}{m} \frac{d\nu_{\chi}(\mu)}{d\mu} g(\mu),
\] (27)

with
\[
g(\mu) = \int_{\mu \sqrt{f(r)}}^\infty d\omega \omega^3 \frac{e^{-\frac{\omega^2}{f(r)}}}{e^{\beta \omega} - 1}.
\] (28)

Before proceeding further, we note that we are only interested in contributions to the entropy in the near vicinity of the horizon, i.e. in the range \((r_h, r_h + \epsilon)\), where \(\epsilon\) is a proper distance related to the minimal length. Then, near-horizon geometry considerations allow us to use the following substitutions: \(f \rightarrow 0\), \(\omega^2 f \rightarrow \omega^2\), and then \(g(\mu)\) is simply given by
\[
g(\mu) = \frac{1}{2k \pi \alpha^{3/2}} \int_0^\infty d\omega \omega^3 \frac{e^{-\frac{\omega^2}{f(r)}}}{e^{\beta \omega} - 1}.
\] (29)

Substituting in equation (27), we obtain
\[
F_\beta = -2 \frac{2}{3\pi} \int_{r_h}^{r_h^{+\epsilon}} dr \frac{r^2}{\sqrt{f}} \int 0^\infty d\omega \omega^3 \frac{e^{-\frac{\omega^2}{f(r)}}}{e^{\beta \omega} - 1} \int m \frac{d\nu_{\chi}(\mu)}{d\mu}.
\] (30)

At this stage, the extra mode is completely decoupled from the radial modes and remains to integrate over \(\mu\). Integrating over \(y\) in equation (23), we obtain
\[
n_{\chi}(\omega) = \frac{1}{2k \sqrt{2\pi \alpha}} \int_m^\infty \frac{d\mu}{\mu} \left( \text{erf}(\sqrt{\alpha \mu} e^{2k \pi r_c} - m^2) - \text{erf}(\sqrt{\alpha \mu} e^{2k \pi r_c} - m^2) \right).
\] (31)

The integration over \(\mu\) cannot be done exactly. To remedy this situation we invoke the little mass approximation, for which we have the following substitutions:
\[
\mu^2 e^{2k \pi r_c} - m^2 \rightarrow \mu^2 e^{2k \pi r_c}, \quad \mu^2 - m^2 \rightarrow \mu^2, \quad \text{unless} \quad \mu = m.
\] (32)

Then the free energy is rewritten as
\[
F_\beta = -2 \frac{2}{3\pi} \int_{r_h}^{r_h^{+\epsilon}} dr \frac{r^2}{\sqrt{f}} I(r),
\] (33)

where \(I(r)\) is given by
\[
I(r) = \frac{1}{2k \sqrt{2\pi \alpha}} \int_0^\infty d\omega \omega^3 \frac{e^{-\frac{\omega^2}{f(r)}}}{e^{\beta \omega} - 1} \int m \frac{d\mu}{\mu} \left( \text{erf}(\sqrt{\alpha \mu} e^{x_{\beta H}}) - \text{erf}(\sqrt{\alpha \mu}) \right).
\] (34)

The entropy of the black hole is calculated using the first law of thermodynamics \(S = \beta^2 \frac{\partial F_\beta}{\partial \beta} \big|_{\beta=\beta_H}\), where \(\beta_H\) is the inverse of the Hawking temperature, and is given by
\[
S = \frac{4\beta_H^2}{3k \pi \sqrt{2} \alpha^{3/2}} \int_{r_h}^{r_h^{+\epsilon}} dr \frac{r^2}{\sqrt{f}} \int_0^\infty d\omega \omega^4 \frac{e^{-\frac{\omega^2}{f(r)}}}{\sinh^2(\beta_H \omega/2)} \int m \frac{d\mu}{\mu} \left( \text{erf}(\sqrt{\alpha \mu} e^{x_{\beta H}}) - \text{erf}(\sqrt{\alpha \mu}) \right).
\] (35)

In terms of the variable \(x = \omega \sqrt{2}\), we write the entropy as
\[
S = \frac{4\beta_H^2}{3k \pi \sqrt{2} \alpha^{3/2}} \int_0^\infty dx \frac{x^4}{\sinh^2(x \beta_H/2 \sqrt{2})} I(x, \epsilon),
\] (36)

\[
\int_0^\infty dx \frac{x^4}{\sinh^2(x \beta_H/2 \sqrt{2})} I(x, \epsilon).
\] (36)
where \( I(x, \epsilon) \) is given by
\[
I(x, \epsilon) = \int_{r_h}^{r_h + \epsilon} \frac{r^2}{f} \, e^{-\frac{r^2}{2f}} \int_{\mu}^{2\alpha} \frac{d\mu}{\mu} \left( \text{erf}(\sqrt{\alpha} \mu) e^{\frac{\mu}{\epsilon}} - \text{erf}(\sqrt{\alpha} \mu) \right). \tag{37}
\]

Now the integration over \( \mu \) can be done exactly and we obtain
\[
I(x, \epsilon) = 2\sqrt{\frac{\alpha}{\pi}} \left( \frac{x}{\sqrt{\alpha}} I_0(x, \epsilon) - m I_m(x, \epsilon) \right), \tag{38}
\]
which is the sum of the independent and dependent mass contributions given, respectively, by
\[
I_0(x, \epsilon) = \int_{r_h}^{r_h + \epsilon} \frac{r^2}{f} \, e^{-\frac{r^2}{2f}} \left( e^{\frac{\kappa r}{\sqrt{\alpha}}} G(k, m) - G(0, m) \right) \tag{39}
\]
and
\[
I_m(x, \epsilon) = \int_{r_h}^{r_h + \epsilon} \frac{r^2}{f} \, e^{-\frac{r^2}{2f}} \left( e^{\frac{\kappa r}{\sqrt{\alpha}}} G(k, m) - G(0, m) \right) \tag{40}
\]
and where \( G(k, \mu) \) is the hypergeometric function
\[
G(k, \mu) = \, {}_2 F_2 \left( \frac{1}{2}, \frac{3}{2}; \frac{3}{2}, -\alpha \mu^2 e^{2\kappa y} \right). \tag{41}
\]
Before proceeding any further, let us carefully analyze the integration over \( r \). Because of the near-horizon considerations, we have, to order \( O((r - r_h)^2) \), the following approximation:
\[
f(r) \simeq (r - r_h) \frac{df}{dr} \bigg|_{r_h} = 2\kappa (r - r_h), \tag{42}
\]
where \( \kappa = 2\pi/\beta_H \) is the surface gravity at the horizon of the black hole. Now we proceed to the calculation of \( I_0 \) and \( I_m \). We first write \( I_0 \) as
\[
I_0(x, \epsilon) = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_n^2 \gamma_{2n+1}}{n!} \int_{r_h}^{r_h + \epsilon} \frac{r^2}{f} \, e^{-\frac{r^2}{2f}} \left( \frac{x^2}{2\kappa (r - r_h)} \right)^n e^{-\frac{x^2}{2\kappa (r - r_h)}}. \tag{43}
\]
where \( \alpha_n = \frac{(1/2)_n}{(3/2)_n}, (z)_n = \frac{\Gamma(n+1)}{\Gamma(z)} \) is the Pochhammer symbol and \( \gamma_{2n+1} = e^{(2n+1)\kappa y} - 1 \). With the change of variable \( t = \frac{r^2}{2\kappa (r - r_h)} \), \( I_0 \) becomes
\[
I_0(x, \epsilon) = \frac{1}{2\kappa} \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_n^2 \gamma_n}{n!} \int_{\gamma_{2n+1}}^{\infty} \int_{x^2/2\kappa}^{\infty} \left( \frac{r_h^2}{4\kappa^2 t^2} + \frac{r_h}{\kappa t} \right)^{n+1/2} e^{-t} \, dt \, dr. \tag{44}
\]
Using the definition of the incomplete Gamma function
\[
\Gamma(a, z) = \int_z^{\infty} t^{a-1} e^{-t} \, dt, \tag{45}
\]
we obtain
\[
I_0(x, \epsilon) = \frac{1}{2\kappa} \sum_{n=0}^{\infty} (-1)^n \frac{\alpha_n^2 \gamma_n}{n!} \left[ r_h^2 \Gamma \left( n + \frac{3}{2} - \frac{x^2}{2\kappa \epsilon} \right) \right. \\
+ \left. \frac{x^2}{4\kappa^2} \Gamma \left( n - \frac{1}{2} - \frac{x^2}{2\kappa \epsilon} \right) + \frac{r_h}{\kappa} \Gamma \left( n + \frac{1}{2} - \frac{x^2}{2\kappa \epsilon} \right) \right]. \tag{46}
\]
Repeating the same procedure for \( I_m(x, \epsilon) \), we obtain
\[
I_m(x, \epsilon) = \left( e^{\kappa y} G(k, m) - G(0, m) \right) \times \left[ \frac{r_h^2}{2\kappa^2} e^{-r_h/2\kappa \epsilon} + \frac{x^2}{8\kappa \epsilon} \Gamma \left( -1, \frac{x^2}{2\kappa \epsilon} \right) + \frac{r_h}{2\kappa^2} \Gamma \left( 0, \frac{x^2}{2\kappa \epsilon} \right) \right]. \tag{47}
\]
At this stage, the brick-wall cut-off $\epsilon$ can be related, in our framework, to the physical scale given by the minimal length as

$$\langle \delta X \rangle_0 = \int_{\epsilon_{\text{min}}}^{\epsilon_{\text{max}}} \frac{dr}{\sqrt{f(r)}}$$

(48)

This relation gives

$$\epsilon = \frac{3\kappa_{\text{Planck}}}{4}$$

(49)

Then, using this expression in equations (46) and (47), performing the integrals over $s$ and substituting the results in equation (36), we obtain the final expression of the entropy in the just vicinity of the horizon

$$\mathcal{S} = \frac{8g}{k^2\alpha^{1/2}} \left( (\gamma_1a_0 - \frac{\gamma_2}{9}d_0) \frac{A}{A_0} + \frac{(\gamma_1b_0 - \frac{\gamma_2}{9}b_1)}{36\pi^2 e^2} \frac{A_0}{A} + \frac{(\gamma_1c_0 - \frac{\gamma_2}{9}c_1)}{3\pi^2 e} \right) - \frac{8g\rho}{k^2 \pi^3} \left( \epsilon^{k_0} G(k, m) - G(0, m) \right) \left( a_2 \frac{A}{A_0} + b_2 \frac{A_0}{A} + c_2 \right),$$

(50)

where $A = 4\pi r_h^2$ is the black hole horizon area, $A_0 = 4\pi (\delta X)_0^2$ is the minimal black hole area due to the GUP, and the numerical values $a_i, b_i, c_i(i = 0, 1, 2)$ are given by

$$\int_0^\infty dy \frac{y^2}{\sinh^2(y) - \Gamma(a, 2y^2/3\pi^2 e)} = \begin{cases} 
    a_0 \simeq 1.4401 & \text{for } a = 3/2 \\
    a_1 \simeq 2.1843 & \text{for } a = 5/2.
\end{cases}$$

(51)

$$\int_0^\infty dy \frac{y^6}{\sinh^2(y) - \Gamma(a, 2y^2/3\pi^2 e)} = \begin{cases} 
    b_0 \simeq 40.9227 & \text{for } a = -1/2 \\
    b_1 \simeq 18.6268 & \text{for } a = 1/2 \\
    b_2 \simeq 86.2295 & \text{for } a = -1,
\end{cases}$$

(52)

$$\int_0^\infty dy \frac{y^4}{\sinh^2(y) - \Gamma(a, 2y^2/3\pi^2 e)} = \begin{cases} 
    c_0 \simeq 3.4867 & \text{for } a = 1/2 \\
    c_1 \simeq 2.7146 & \text{for } a = 3/2 \\
    c_2 \simeq 5.7824 & \text{for } a = 0.
\end{cases}$$

(53)

and

$$a_2 = \int_0^\infty dy \frac{y^2}{\sinh^2(y)} \exp^{-\frac{2y^2}{\pi\sigma}} \simeq 1.5706.$$  

(54)

We note that the mass-independent contribution to the entropy is just built from the first two terms of $I_0$, since the factors of the type $(a_n)^2/n!$ become small for $n \geq 2$. Some comments are appropriate about the expression of the entropy given by equation (50). We observe that the entropy shows two regimes and by the GUP we have always $A \gtrsim A_0$. In the first regime of weak gravitational coupling corresponding to large black holes with $A > A_0$, we have the usual Bekenstein–Hawking area law $S \sim A/A_0$, while in the second regime of strong gravitational coupling corresponding to small black holes with a horizon area $A \rightarrow A_0$, the entropy acquires a relevant correction term $S \sim A_0/A$. This result clearly shows that new principles are needed to study Planck scale physics. Let us note that correction terms to the horizon area law of the entropy of the Schwarzschild black hole in the ADD scenario with GUP to all orders in the Planck length have been recently obtained [38]. In the later reference and with one extra dimension, the correction terms show a complicated power behavior and that the results are perturbative in the minimal length. These deviations from the horizon area law have not been obtained in some recent calculations of the entropy of the Randall–Sundrum brane world without GUP [25] and with a GUP to leading order in the Planck length [37]. Finally, we note that our results have been obtained with the aid of the little mass approximation, and due to the existence of a minimum black hole area, they are non-perturbative in the minimal length. On the other hand, the massive contribution to the entropy is more complicated than the linear contribution obtained in [25, 37].
4.2. GUP on the brane

We consider now the more interesting case where the quantum modes in the extra dimension are not affected by the GUP. In such a situation the number of quantum extra modes is simply given by

\[ n_y = \frac{1}{\pi} \int_0^{\gamma_c} \sqrt{\mu^2 e^{2ky} - m^2} \, dy, \]  

(55)

and the total number with energy less than \( \omega \) is

\[ n_y(\omega) = \frac{1}{k\pi} \int_0^{\gamma_c} \frac{d\mu}{\mu} \left( \sqrt{\mu^2 e^{2ky} - m^2} - \sqrt{\mu^2 - m^2} \right). \]

(56)

The calculation of the free energy proceeds as in the previous section and is given by

\[ F_\beta = -\frac{2}{3k\pi^2} \int_{r_h}^{\gamma_c} dr \frac{r^2}{f^2} \int_0^{\infty} d\omega \omega^3 e^{-\frac{\omega^2}{\beta\mu}} - 1 \int_m^{\gamma_c} \frac{d\mu}{\mu} \left( \sqrt{\mu^2 e^{2ky} - m^2} - \sqrt{\mu^2 - m^2} \right). \]

(57)

The entropy is calculated from the relation \( S = \beta^2 \partial F_\beta / \partial \beta \). In terms of the variables \( x = \omega \sqrt{\alpha} \) and \( z = \mu / m \), we obtain

\[ S = \frac{2\beta^2 m}{3k\pi^2 \alpha^{3/2}} \int_{r_h}^{\gamma_c} dr \frac{r^2}{f^2} \int_0^{\infty} dx \frac{x^4 e^{-x^2/f}}{\sinh^2(\beta H x/2\sqrt{\alpha})} \left( J(x) \right), \]

(58)

with \( J(x) \) given by

\[ J(x) = \int_{1}^{\frac{\sqrt{\gamma_c}}{\sqrt{\alpha}}} \frac{dz}{z} \left( \sqrt{\frac{e^{2ky}}{\gamma_c^2} - 1} - \sqrt{z^2 - 1} \right). \]

(59)

The integration over \( z \) is straightforward, and as a result we obtain

\[ J(x) = \sqrt{\left( \frac{x e^{ky}}{m \sqrt{\alpha} f} \right)^2 - 1} + \arctan \left( \frac{1}{\sqrt{\left( \frac{x e^{ky}}{m \sqrt{\alpha} f} \right)^2 - 1}} \right) \]

\[ - \sqrt{\left( \frac{x}{m \sqrt{\alpha} f} \right)^2 - 1} - \arctan \left( \frac{1}{\sqrt{\left( \frac{x}{m \sqrt{\alpha} f} \right)^2 - 1}} \right) \]

\[- \sqrt{e^{2ky} - 1} - \arctan \left( \frac{1}{\sqrt{e^{2ky} - 1}} \right) + \frac{\pi}{2}. \]

(60)

In the just vicinity of the horizon, corresponding to \( f \rightarrow 0 \), we have the approximation

\[ J(x) \approx \frac{x}{m \sqrt{\alpha} f} y_1 - \left( \sqrt{y_2} + \arctan \left( \frac{1}{\sqrt{y_2}} \right) \right). \]

(61)

where \( y_0 = e^{ky} - 1 \).

Then, the entropy can be written as

\[ S = S_0 + S_m, \]

(62)

where

\[ S_0 = \frac{2\beta^2 m y_1}{3k\pi^2 \alpha^{3/2}} \int_0^{\gamma_c} dx \frac{x^5}{\sinh^2(\beta H x/2\sqrt{\alpha})} \int_{r_h}^{\gamma_c} dr \frac{r^2 e^{-x^2/f}}{f^{5/2}}. \]

(63)
Following the same steps of calculation as in the first case, the integrals about \( r \) can be computed, and as a result we obtain the final expression of the entropy

\[
S = \frac{2e\gamma_1}{k\pi^3\alpha^{1/2}} \left( \frac{a_2}{A_0} + \frac{b_0}{36\pi^4e^2} + \frac{c_0}{3\pi^2} \right) - \frac{2em}{k\pi^3} \left( \sqrt{\gamma_2} + \arctan \left( \frac{1}{\sqrt{\gamma_2}} \right) \right) \left( \frac{a_2}{A_0} + \frac{b_2}{36\pi^4e^2} + \frac{c_2}{3\pi^2} \right),
\]

(65)

where the numerical constants are given by equations (56)–(59).

We note that the entropy given by equation (65) exhibits the same two regimes observed in the case where the GUP is applied on the full volume of the spacetime. However, we observe that the massive contribution to the entropy becomes linear, as was obtained in previous works [25, 37]. This linear behavior is a consequence of the suppression of the damping of the states density in the extra dimension direction.

Before ending this section, let us comment on the entropy to all orders in the Planck length for the \((3+1)\)-dimensional Schwarzschild black hole obtained in [39] and given by

\[
S = \frac{e^4\zeta(3)}{8\pi\alpha} A.
\]

(66)

However, following the procedure developed in this paper, the evaluation of the integrals over \( r \) in the range near horizon gives

\[
S = \frac{e\alpha_2}{2\pi^2} A + \frac{b_0}{72\pi^6e^2} + \frac{c_2}{3\pi^3},
\]

(67)

where \( \alpha_2, b_2, c_2 \) are given above. In comparison with equation (66), our result again shows the deviation from the Bekenstein–Hawking area law, proportional to the inverse of the horizon area. Finally, we point out that, even with a GUP to leading order in the Planck length, a careful evaluation of the entropy integrals about \( r \) in the range near horizon of the Randall–Sundrum black brane shows similar correction terms to the Bekenstein–Hawking area law obtained in [37].

5. Conclusion

In summary, we have calculated, to all orders in the Planck length, the near-horizon contributions to the entropy of bulk massive scalar fields propagating in the background of a black hole in the Randall–Sundrum brane world, by using the generalized uncertainty principle. The entropy is obtained by summing up the thermal contributions of both the brane and the extra dimension fields. As a result, the usual Bekenstein area law is not preserved and is corrected by a term proportional to the inverse of the horizon area. Our analysis shows that the correction term becomes relevant in the case of strong gravitational coupling. In the case when the GUP is considered on the full volume of the bulk, we have shown that the mass dependence of the entropy is more complicated in comparison to the linear mass contribution obtained in [25, 37]. The later behavior is recovered when the effect of the GUP in the extra dimension direction is suppressed. As a consequence, the massive contribution to the entropy depends crucially on the presence or absence of a cut-off in the extra dimension direction. Finally, we note that the results obtained are non-perturbative in the minimal length.
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