Spectrum of the Laplacian on graphs of radial functions

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We prove that if $M$ is a complete, noncompact hypersurface in $\mathbb{R}^{n+1}$, which is the graph of a real radial function, then the spectrum of the Laplace operator on $M$ is the interval $[0, \infty)$.

1. Introduction

Let $M$ be a simply connected Riemannian manifold. The Laplace operator $\Delta : C_0^\infty(M) \to C_0^\infty(M)$, defined as $\Delta = \text{div} \circ \text{grad}$ and acting on $C_0^\infty(M)$ (the space of smooth functions with compact support), is a second-order elliptic operator and, provided $M$ is complete, it has a unique extension $\Delta$ to an unbounded self-adjoint operator on $L^2(M)$ whose domain is $\text{Dom}(\Delta) = \{ f \in L^2(M) : \Delta f \in L^2(M) \}$; see [Grigor’yan 2009, Theorem 11.5]. Since $-\Delta$ is positive and symmetric, its spectrum is the set of $\lambda \geq 0$ such that $\Delta + \lambda I$ does not have a bounded inverse. Sometimes we say “spectrum of $M$” rather than “spectrum of $-\Delta$”, and we denote it by $\sigma(M)$. One defines the essential spectrum, $\sigma_{\text{ess}}(M)$, to be those $\lambda$ in the spectrum which are either accumulation points of the spectrum or eigenvalues of infinite multiplicity. The discrete spectrum is the set $\sigma_d = \sigma(M) \setminus \sigma_{\text{ess}}(M)$ of all eigenvalues of finite multiplicity which are isolated points of the spectrum.

There is a vast literature on the spectrum of the Laplace operator on complete noncompact manifolds. The first result we mention was published by Tayoshi [1971]. He showed the absence of eigenvalues of $-\Delta$ for a class of surfaces of revolution, determined by nonnegative radial growth.

Donnelly [1981] showed

$$\sigma_{\text{ess}}(M) = [(n-1)^2 \frac{1}{4} c^2, \infty),$$

provided $M$ is a Hadamard manifold whose sectional curvature approaches $-c^2$ at infinity. Karp [1984] gave sufficient conditions for a class of manifolds to have

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purely continuous spectrum ($\sigma_d(M) = \emptyset$) under some curvature conditions. Eight years later, Donnelly and Garofalo [1992] obtained results in a similar direction, using the hypothesis of nonnegative radial sectional curvature, without restrictions on the metric.

Cheng and Zhiqin Lu [1992] proved $\sigma_{\text{ess}}(M) = [0, \infty)$ when $M$ has nonnegative radial sectional curvature and Li [1994] proved $\sigma_{\text{ess}}(M) = [0, \infty)$, provided $M$ has nonnegative Ricci curvatures and a pole. Zhou [1994] proved $\sigma_{\text{ess}}(M) = [0, \infty)$ when $M$ has nonnegative sectional curvatures, generalizing the work of Escobar and Freire [1992].

Kumura [1997] found a result which generalized [Donnelly 1981]. He showed $\sigma_{\text{ess}}(M) = \left[\frac{1}{4}c^2, \infty\right)$ whenever

$$\lim_{n \to \infty} \sup_{t > n} |\Delta t - c| = 0,$$

where $t$ denotes the distance function on $M$.

Wang [1997] showed that the spectrum of a complete, noncompact Riemannian manifold with asymptotically nonnegative Ricci curvature is equal to $[0, \infty)$.

Zhiqin Lu and Detang Zhou [2011] proved that the $L^p$ essential spectrum of $M$ is equal to $[0, \infty)$ when

$$\lim_{x \to \infty} \inf \text{Ric}_M(x) = 0$$

and $M$ is noncompact and complete. We should mention here that almost all the above works were strongly motivated by the decomposition principle [Donnelly and Li 1979], which states that the essential spectrum of a Riemannian manifold is invariant under compact perturbations of the metric, thus it is a function of the geometry of the ends. In [Monte and Montenegro 2015], it was proved that $\sigma_{\text{ess}}(M) \supset \left[(n - 1)\frac{2}{4}c^2, \infty\right)$ for a class of Riemannian manifolds, not necessarily complete, whose metric is given by

$$g_M = dr^2 + \psi^2(rw)g_{S^{n-1}},$$

using curvature conditions only in a neighborhood of a ray.

See also [Bessa et al. 2010; 2012; 2015; Donnelly and Li 1979; Kleine 1988; 1989; Tayoshi 1971] for geometric conditions implying the discreteness of the spectrum, $\sigma_{\text{ess}}(M) = \emptyset$.

In this work we consider complete hypersurfaces which are graphs of radial functions. Our main result is the following theorem.

**Theorem 1.** Let $M$ be a complete hypersurface in $\mathbb{R}^{n+1}$, which is the graph of a real radial function. Then, the spectrum of the Laplace operator on $M$ is $[0, \infty)$.

Without loss of generality, we may assume the domain $\text{Dom } f$ to be connected and symmetric with respect to $0 \in \mathbb{R}^n$. From the completeness of $M$ we further
deduce Dom $f$ is an open ball or annulus. The theorem above allows us to construct a bounded hypersurface with the same spectrum of $\mathbb{R}^{n+1}$ by taking $M$ to be the graph of the real function $f(x) = \cos(\tan(\frac{1}{2}\pi|x|))$ defined on the unit open ball.

Throughout the following discussion, for simplicity, we deal with the case where $f : D \to \mathbb{R}$ is defined in an open ball. Let $X : [0, R) \times \Omega \to D$ be defined by $X(r, x_1, \ldots, x_{n-1}) = r w(x_1, \ldots, x_{n-1})$, where $0 < R \leq +\infty$ and $w$ is a coordinate system on $S^{n-1}$ defined on an open set $\Omega$ of $\mathbb{R}^n$. Note that $M$ has a natural coordinate system $Y : [0, R) \times \Omega \to M$, given by $Y(r, x_1, \ldots, x_{n-1}) = (r w(x_1, \ldots, x_{n-1}), f(r))$, but we are interested in the spherical coordinate system for $M$ on $p = (0, f(0))$. Consider $t : [0, R) \to [0, \infty)$, given by

$$t(r) = \int_0^r \left(1 + t'(\tau)^2\right)^{1/2} d\tau.$$  

We claim that $t$ is a diffeomorphism. Observe that $t$ is increasing and

$$\lim_{r \to R} t(r) = +\infty.$$

We denote by $r : [0, \infty) \to [0, R)$ the inverse diffeomorphism. By the inverse function theorem,

$$0 < r'(t) = \left(1 + f'(r)^2\right)^{-1/2} \leq 1.$$

Finally, the system of spherical coordinates on $M$, denoted $Z : [0, \infty) \times \Omega \to M$, is defined by

$$Z(t, x_1, \ldots, x_{n-1}) = \left(t(r) w(x_1, \ldots, x_{n-1}), f \circ r(t)\right).$$

The metric of $M$ on such a system is given by

$$g_M = dt^2 + r(t)^2 g_{S^{n-1}}.$$  

Because of this observation, Theorem 1 is a simple consequence of the theorem below.

**Theorem 2.** Let $I \subset \mathbb{R}$ be an unbounded interval and $M = I \times S^{n-1}$ with metric given by $g_M = dt^2 + r^2(t) g_{S^{n-1}}$, where $0 < r'(t) \leq c$ for all $t$. Then, the spectrum of the Laplace operator on $M$ is $[0, \infty)$.

**Remark.** (1) If $M$ has a pole at $p \in M$, then exp$_p : T_p M \to M$ is a diffeomorphism so that $M$ isometric to $T_p M$ with the pullback metric. Therefore, Theorem 2 implies that if $M$ has a pole $p$ and $g_M = dt^2 + r^2(t) g_{S^{n-1}}$ with respect to $p$ and $0 < r'(t) < c$, then $M$ has spectrum equal to $[0, \infty)$.

(2) To the best of our knowledge, this natural result has only been verified in less general settings. For instance, since $r'(t) > 0$, then $r(t)$ is increasing and there are only two possibilities:
(a) $\lim_{t \to \infty} r(t) = \infty$, or
(b) $\lim_{t \to \infty} r(t) = R$.

In the first case, since $r'(t)$ is bounded, we have

$$\lim_{t \to \infty} \Delta t = \lim_{t \to \infty} \frac{r'(t)}{r(t)} = 0.$$ 

By [Kumura 1997, Theorem 1.2], it follows that the spectrum of $M$ is purely continuous and equal to $[0, \infty)$. In the second case, if $r' \to 0$ we still have $r'(t)/r(t) \to 0$. Therefore, the main contribution of this paper is the proof of the case where $r'(t)$ does not converge to zero and $\lim_{t \to \infty} r(t) = R < +\infty$. This is the scenario for the graph of the function $f(x) = \cos(t\tan(\frac{1}{2}\pi|x|))$ presented above.

In the next section we prove Theorem 2. The Appendix is devoted to the Sturm–Liouville theory used in this note.

2. Proof of Theorem 2

We concentrate our efforts for the case where $\lim_{t \to \infty} r(t) = R$. Our approach is variational, based on the following lemma.

Lemma 3 [Davies 1995, Lemma 4.1.2]. A number $\lambda \in \mathbb{R}$ lies in the spectrum of a self-adjoint operator $H$ if and only if there exists a sequence of functions $f_n \in \text{Dom } H$ with $\|f_n\| = 1$ such that

$$\lim_{n \to \infty} \|H f_n - \lambda f_n\| = 0.$$ 

To deduce Theorem 2 from Lemma 3 we will construct, for each $\lambda > 0$, a sequence of radial smooth functions $f_p : M \to \mathbb{R}$ with compact support such that

$$\|\Delta f_p + \lambda f_p\|_{L^2(M)} \leq \frac{c}{p} \|f_p\|_{L^2(M)} \quad (2)$$ 

for any natural $p$, where $c$ is a constant which does not depend on $p$. It will follow that $g_p = f_p/\|f_p\|$ has norm one and

$$\lim_{p \to \infty} \|\Delta g_p + \lambda g_p\|_{L^2(M)} = 0.$$ 

Therefore, by Lemma 3, $\lambda$ belongs to the spectrum. To construct the function $f_p$, we fix $t_0 > 0$ and prove that there are $t_1(\lambda) > t_0$ and a radial function $u = u(t)$ solution of the problem

$$\begin{cases}
\Delta u + \lambda u = 0, & \text{in } [t_0, t_1], \\
u(t_0) = u(t_1) = 0, \\
u > 0, & \text{in } (t_0, t_1). 
\end{cases} \quad (3)$$
Using Sturm–Liouville theory, we showed that \( u \) can be extended to the whole interval \([t_0, \infty)\) and it has infinite zeros \( t_0 < t_1 < \cdots < t_p < \cdots \). The next step is to consider (for each \( p \)) a smooth bump function \( h_p \) whose support is the interval \([t_0, t_{3p}]\). We then define \( f_p = uh_p \) and show that each \( f_p \) in this sequence satisfies (2). The function \( t \mapsto r^{n-1}(t) \) has a geometric meaning and plays an important role in the proof, thus deserving a special notation. In the sequence of the paper, we let \( v(t) = r^{n-1}(t) \).

We observe that the first equation in (3) is equivalent to

\[
(v(t)u'(t))' + \lambda v(t)u(t) = 0
\]

if \( u = u(t) \) is a radial function. By Theorem 9 in the Appendix, given positive \( t_0 \) and \( \lambda \), (4) has a solution defined on \([t_0, \infty)\) and satisfying \( u(t_0) = 0 \).

Moreover, Corollary 8 allows us to consider a sequence of zeros \( t_0 < t_1 < \cdots \) of \( u \).

For \( p \in \mathbb{N} \), we choose a smooth bump function \( h = h_p : \mathbb{R} \mapsto \mathbb{R} \) with \( 0 \leq h \leq 1 \) satisfying

\[
\begin{align*}
\{ h(t) & = 0, \quad t \in (-\infty, t_0] \cup [t_{3p}, \infty), \\
& = 1, \quad t \in [t_p, t_{2p}].
\end{align*}
\]

Such a function can be defined in the following way: let \( \varphi \in C_0^\infty(\mathbb{R}) \) be nonnegative with \( \text{supp } \varphi = [0, 1] \) and \( \int \varphi = 1 \). Let

\[
h_p(t) = \int_{-\infty}^t \varphi_p(s) \, ds,
\]

where

\[
\varphi_p(t) = \frac{1}{t_p - t_0} \varphi \left( \frac{t - t_0}{t_p - t_0} \right) - \frac{1}{t_{3p} - t_{2p}} \varphi \left( \frac{t - t_{2p}}{t_{3p} - t_{2p}} \right).
\]

This construction is useful since it leads to the following estimates:

\[
\|h'_p\|_\infty \leq \max \left\{ \frac{\|\varphi\|_\infty}{t_p - t_0}, \frac{\|\varphi\|_\infty}{t_{3p} - t_{2p}} \right\} \leq \frac{C}{p},
\]

\[
\|h''_p\|_\infty \leq \max \left\{ \frac{\|\varphi\|_\infty}{(t_p - t_0)^2}, \frac{\|\varphi\|_\infty}{(t_{3p} - t_{2p})^2} \right\} \leq \frac{C}{p^2}.
\]

Here, we have made use of Corollary 11 in the Appendix.

Consider \( f = f_p = uh_p \). We are going to prove that such a function satisfies the inequality in (2). Computing \( \Delta f + \lambda f \), we obtain

\[
\Delta f + \lambda f = 2u'h' + uh'' + (n - 1)\frac{r'}{r}h'u.
\]
Using the inequalities in (5), together with the fact that \( r \) is increasing and \( r' \) is bounded, we have

\[
|\Delta f + \lambda f| \leq \frac{c}{p} (|u'| + |u|) \chi_{[t_0,t_3p]}.
\]

Then,

\[
|\Delta f + \lambda f|^2 \leq \frac{c}{p^2} (|u'|^2 + |u|^2) \chi_{[t_0,t_3p]},
\]

\[
\int_M |\Delta f + \lambda f|^2 \, dM \leq \frac{c}{p^2} \left( \int_{t_0}^{t_{3p}} |u'|^2 v \, dt + \int_{t_0}^{t_3p} |u|^2 v \, dt \right).
\]

Multiplying (4) by \( u \) and using integration by parts we find

\[
\int_{t_0}^{t_{3p}} |u'|^2 v(t) \, dt = \lambda \int_{t_0}^{t_{3p}} |u|^2 v(t) \, dt,
\]

\[
\|\Delta f_p + \lambda f_p\|_{L^2(M)} \leq \frac{c}{p} \|u \cdot \chi_{[t_0,t_{3p}]}\|_{L^2(M)} \leq \frac{c}{p} \|u \cdot \chi_{[t_p,t_{2p}]}\|_{L^2(M)} \leq \frac{c}{p} \|f_p\|_{L^2(M)},
\]

where the second inequality comes from Lemma 4 below.

**Lemma 4.** There is a positive constant \( C \) independent on \( p \) such that

\[
\int_{t_0}^{t_{3p}} u^2 v \, dt \leq C \int_{t_0}^{t_{2p}} u^2 v \, dt,
\]

where \( u \) is solution of (4) and \( t_0 < t_1 < \cdots \) are zeros of \( u \).

This result is a manifestation of the oscillatory behavior of \( u \). Before justifying its veracity, we state a useful way of estimating \( u \) between two zeros.

**Lemma 5.** Let \( u \) be a solution of (4), and choose \( t_k, t_{k+1} \) to be consecutive zeros for \( u \). Define

\[
\alpha_k(t) = a_k \sin \left( \frac{1}{2} R^{n-1} \int_{t_k}^t v^{-1} \, ds \right)
\]

and

\[
\beta_k(t) = b_k \sin \left( \frac{1}{2} R^{n-1} \int_{t_k}^t v^{-1} \, ds \right),
\]

where \( a_k = v(t_k)b_k/(R^{n-1}\lambda^{1/2}) \) and \( b_k = u'(t_k)/\lambda^{1/2} \). Then \( |\alpha_k| \leq |u| \) on \( (t_k, \tilde{t}_k) \) and \( |u| \leq |\beta_k| \) on \( (t_k, t_{k+1}) \), where \( \tilde{t}_k \) is the next zero of \( \alpha_k \) after \( t_k \).

To make the exposition more fluid, we postpone the proof until the Appendix.

**Proof of Lemma 4.** Observe that multiplying (4) by \( v(t)u' \) we get

\[
(v(t)u')'v(t)u' + \lambda v^2 uu' = 0,
\]

and so,

\[
((v(t)u')^2)' + \lambda v^2 (u')^2 = 0.
\]
Integrating from $t_0$ to $t_k$, we have
\[ v(t_k)^2u'(t_k)^2 - v(t_0)^2u'(t_0)^2 = -\lambda \int_{t_0}^{t_k} v^2(s)(u^2(s))' \, ds. \]

Integrating the right hand side by parts, we find
\[ v(t_k)^2u'(t_k)^2 - v(t_0)^2u'(t_0)^2 = 2\lambda \int_{t_0}^{t_k} vv'u^2 \, ds. \]  
(6)

Since $r, r' > 0$, we have $v, v' > 0$. Also, $r(t) < R$ and as a consequence,
\[ u'(t_k)^2 > \frac{v(t_0)^2u'(t_0)^2}{R^{2(n-1)}} \]  
(7)
for $k \geq 1$.

To obtain an estimate in the other direction, we observe that the function $\beta = \beta_0(t)$ in Lemma 5 satisfies $\beta'(t_0) = u'(t_0) > 0$ and
\[ (v(t)\beta'(t))' + \frac{\lambda v(t_0)^2}{v(t)} \beta(t) = 0. \]  
(8)

Multiplying by $v(t)\beta'$ we get, as in the preceding computations,
\[ (v(t)^2(\beta')^2)' + \lambda v(t_0)^2(\beta^2)' = 0. \]  
(9)

Now, if $\bar{t}_1$ is the next root of $\beta$ after $t_0$, integrating the last equation we find
\[ v(\bar{t}_1)^2\beta'(\bar{t}_1)^2 = v(t_0)^2\beta'(t_0)^2 \]
\[ = v(t_0)^2u'(t_0)^2. \]  
(10)

We take $k = 1$ and estimate the right side of (6) as follows:
\[
\lambda \int_{t_0}^{t_1} (v^2)'u^2 \, dt \leq \lambda \int_{t_0}^{t_1} (v^2)'\beta^2 \, dt
\leq \lambda \int_{t_0}^{\bar{t}_1} (v^2)'\beta^2 \, dt
= -\lambda \int_{t_0}^{\bar{t}_1} v^2(\beta^2)' \, dt
= -\frac{1}{v(t_0)^2} \int_{t_0}^{\bar{t}_1} v^2(\lambda v(t_0)^2\beta^2)' \, dt.
\]  
(11)
By (9) we infer
\[- \frac{1}{v(t_0)^2} \int_{t_0}^{\tilde{t}} v^2(\lambda v(t_0)^2 \beta')' dt = \frac{1}{v(t_0)^2} \int_{t_0}^{\tilde{t}} v^2(\beta')' dt \]
\[= \frac{1}{v(t_0)^2} \int_{t_0}^{\tilde{t}} (v^4(\beta')' - (v^2)'(\beta')^2) dt \]
\[< \frac{v^4(\tilde{t})(\beta')^2(\tilde{t}) - v^4(t_0)(\beta')^2(t_0)}{v(t_0)^2}. \]

Now, using (10) and that \(\beta'(t_0) = u'(t_0)\), we find
\[\lambda \int_{t_0}^{\tilde{t}} (v^2)' u^2 \leq (v(\tilde{t}))^2 - v(t_0)^2 u'(t_0)^2 \quad \text{dt}. \]

Then, by (6),
\[v(t_1)^2 u'(t_1)^2 - v(t_0)^2 u'(t_0)^2 \leq (v(\tilde{t}))^2 - v(t_0)^2 u'(t_0)^2. \]

Since \(v(t)\) is increasing, it follows that
\[v(t_1)^2 u'(t_1)^2 \leq v(\tilde{t})^2 u'(t_0)^2 \leq v(t_2)^2 u'(t_0)^2. \]

Then,
\[u'(t_1)^2 \leq \frac{v(t_2)^2}{v(t_0)^2} u'(t_0)^2. \]

Using the same argument, one shows by induction that
\[u'(t_k)^2 \leq \frac{v(t_{k+1})^2 v(t_k)^2}{v(t_1)^2 v(t_0)^2} u'(t_0)^2. \]

Since \(r(t) < R\), we find that
\[u'(t_k)^2 \leq \frac{R^{4(n-1)}}{v(t_0)^2 v(t_1)^2} u'(t_0)^2. \]

Now, using Lemma 5, it’s easy to check that
\[\int_{t_0}^{t_{3p}} u^2 v \, dt = \sum_{k=0}^{3p-1} \int_{t_k}^{t_{k+1}} u^2 v(t) \, dt \]
\[\leq \frac{1}{\lambda} \sum_{k=0}^{3p-1} \left[ u'(t_k)^2 \int_{t_k}^{t_{k+1}} \sin^2 \left( \frac{1}{\sqrt{v(t)}} \int_{t_{k}}^{t_{k+1}} ds \right) v(t) \, dt. \]

Letting
\[\tau = \lambda^{1/2} v(t_k) \int_{t_k}^{t} ds, \]

\[v^2 \leq \frac{v(t_0)^2}{v(t_{k+1})^2} u'(t_0)^2. \]
the change of variables formula shows that
\[
\frac{1}{\lambda} \sum_{k=0}^{3p-1} u'(t_k)^2 \int_{t_k}^{t_{k+1}} \sin^2\left(\lambda^{1/2} v(t_k) \int_{t_k}^{t} \frac{ds}{v(s)}\right) v(t) \, dt
\]
\[= \frac{1}{\lambda^{3/2}} \sum_{k=0}^{3p-1} \frac{u'(t_k)^2}{v(t_k)} \int_{0}^{\pi} \sin^2(\tau) v^2(\tau(t)) \, d\tau \]
\[\leq \frac{\pi R^{2(n-1)}(t_0)}{2\lambda^{3/2} r^{n-1}(t_0)} \sum_{k=0}^{3p-1} u'(t_k)^2 \]
\[= C \sum_{k=0}^{3p-1} u'(t_k)^2. \]  

By (7) and (14), the following inequalities hold:
\[
\sum_{k=0}^{3p-1} u'(t_k)^2 \leq 3C p u'(t_0)^2
\]
\[\leq C \sum_{k=p}^{2p-1} u'(t_k)^2. \]  

We have
\[
\int_{t_0}^{t_{3p}} u^2 v \, dt \leq C \sum_{k=p}^{2p-1} u'(t_k)^2. \]  

Here, the last inequality comes from (7), for some suitable constant \(C > 0\). Again by the change of variables formula (this time applied to each \(\alpha_k\)) and by Lemma 5, one sees that if \(\tilde{t}_k\) is the next zero of \(\alpha_k\) after \(t_k\) we have
\[
\int_{t_p}^{t_{2p}} u^2 v(t) \, dt = \sum_{k=p}^{2p-1} \int_{t_k}^{t_{k+1}} u^2 v(t) \, dt
\]
\[\geq \sum_{k=p}^{2p-1} \int_{t_k}^{\tilde{t}_{k+1}} \alpha_k^2 v(t) \, dt \]
\[\geq C \sum_{k=p}^{2p-1} u'(t_k)^2. \]  

From (18) we conclude that
\[
\int_{t_0}^{t_{3p}} u^2 r^{n-1} \, dt \leq C \int_{t_p}^{t_{2p}} u^2 r^{n-1} \, dt
\]
for every \( p \in \mathbb{N} \) and for a constant \( C = C(\lambda, R) \), independent of \( p \).

**Appendix: Elements of Sturm–Liouville theory**

For the convenience of the reader, we present some facts about Sturm–Liouville problems used in the previous section. Our motivation relies on the study of

\[
(v(t)u')' + \lambda v(t)u = 0 \quad t \geq t_0 > 0,
\]

where \( v(t) = r^{n-1}(t) \) for fixed \( n \in \mathbb{N} \). In the following we assume the function \( r(t) \) to be positive; moreover:

(I) \( 0 < r'(t) \leq c \).

(II) \( \lim_{t \to \infty} r(t) = R < +\infty \).

We start with a classical terminology.

**Definition 6.** Equation (20) is said to be oscillatory if any of its solutions has arbitrarily large zeros.

The following theorem is a practical criterion for oscillation.

**Theorem 7.** Let \( v(t) \) be a positive continuous function on \([t_0, \infty) \) and \( \lambda > 0 \). Then, the equation

\[
(v(t)u')' + \lambda v(t)u = 0
\]

for \( t \geq t_0 \) is oscillatory, provided \( \int_{t_0}^{\infty} v(t) \, dt = +\infty \) and \( \int_{t_0}^{t} v(t) \, dt \leq Ct^a \), for some positive constants \( C \) and \( a \).

The proof is discussed in [do Carmo and Zhou 1999, Theorem 2.1]. Since \( \lim_{t \to \infty} r(t) = R \), we easily have the following.

**Corollary 8.** Equation (20) is oscillatory.

**Theorem 9.** For positive \( v \), any solution \( u \) of (20) on an interval \([t_0, t_0 + \delta] \) with initial values \( u(t_0) = x_0 \) and \( u'(t_0) = x_1 \) can be extended to \([t_0, \infty) \).

Again, the proof is presented in [do Carmo and Zhou 1999, Theorem 2.2].

The next propositions appear in the literature as Sturm comparison theorems; see [Hartman 1982, Theorem 3.1]. These are standard results, but for the sake of self-containment we decided to present their proofs. They emerge as useful ways to compare solutions for ordinary differential equations, as we did in Section 2.

**Proposition 10.** Let \( x, y \) be nontrivial solutions for

\[
\begin{align*}
(p(t)x')' + q(t)x &= 0, \\
(p_1(t)y')' + q_1(t)y &= 0,
\end{align*}
\]
where \( p(t) \geq p_1(t) > 0 \) and \( q_1(t) \geq q(t) \) for every \( t \in I \). If \( t_1 < t_2 \) are consecutive zeros of \( x \), then either \( y \) has a zero on \( J = (t_1, t_2) \) or there is a \( d \in \mathbb{R} \) for which \( y = dx \) on \( J \).

**Proof.** As a starting point, note that if \( y(t_i) = 0 \), then by uniqueness we have \( y = dx \) for \( d = y'(t_i)/x'(t_i) \). Uniqueness also implies that the set of zeroes of \( x \) does not have a cluster point, so the interval \( J \) is well-defined. Therefore, it is enough to consider the case where \( x \) and \( y \) are linearly independent. Observe that if \( y \) does not have a zero on \( J \), then

\[
\left( x \frac{(p(t) x'y - p_1(t) x y')}{y} \right)' = (q_1 - q) x^2 + (p - p_1)(x')^2 + \frac{p_1(x'y - xy')^2}{y^2}. 
\]

Integrating from \( t_1 \) to \( t_2 \), we have

\[
\int_{t_1}^{t_2} (q_1 - q) x^2 \, dt + \int_{t_1}^{t_2} (p - p_1)(x')^2 \, dt + \int_{t_1}^{t_2} p_1 \frac{(x'y - xy')^2}{y^2} \, dt = 0.
\]

Then, if \( y \) is not multiple of \( x \), the Wronskian \( (xy' - x'y) \) is nonzero on \( J \) and we get a contradiction with the last equation. \( \square \)

As a consequence, we obtain a universal estimate from below to the distance between two consecutive zeros of a solution of (20).

**Corollary 11.** Let \( \{t_p\}_{p=1}^{\infty} \) be an increasing sequence of zeros of \( u \). There is a universal constant \( C > 0 \) such that \( t_{p+1} - t_p > C \) for any \( p \in \mathbb{N} \).

**Proof.** Given \( p \in \mathbb{N} \), define \( \varphi(t) = \sin(2^{(n-1)/2} \lambda^{1/2}(t - t_p)) \). Then, \( \varphi \) has a zero at \( t = t_p \) and

\[
\left( \frac{1}{2} R \right)^{n-1} \varphi'' + \lambda R^{n-1} \varphi = 0.
\]

Now, \( \left( \frac{1}{2} R \right)^{n-1} < v(t) < R^{n-1} \) for \( t \) sufficiently large, lets say for \( t > c_0 \). As a consequence, if \( p \) is sufficiently large, we can apply Proposition 10 for \( u \) and \( \varphi \) to conclude that the next zero of \( \varphi \) is on \( (t_p, t_{p+1}) \).

Since the next zero of \( \varphi \) after \( t_p \) is on \( t = t_p + \pi/(2^{(n-1)/2} \lambda) \), we have

\[
t_{p+1} - t_p > \frac{\pi}{2^{(n-1)/2} \lambda},
\]

for \( t_p > c_0 \), from which the corollary follows. \( \square \)

**Proposition 12.** Let \( x, y \) be nontrivial solutions for

\[
\begin{align*}
(p(t)x')' + q(t)x &= 0, \\
(p_1(t)y')' + q_1(t)y &= 0,
\end{align*}
\]

on an interval \([a, b]\), where \( p \geq p_1 > 0 \), \( q_1 > q \) and \( x(a) = 0 \). Suppose that \( c \in (a, b) \) is such that \( x(c) \neq 0 \), \( y(c) \neq 0 \) and \( x \) has the same number of zeros as \( y \).
on \((a, c)\). Then
\[
\frac{p(c)x'(c)}{x(c)} \geq \frac{p_1(c)y'(c)}{y(c)}.
\]

**Proof.** We only deal with the case where \(y\) is different from \(dx\), otherwise there is nothing to prove. Let \(a = a_0, \ldots, a_n\) be the zeros of \(x\) on \([a, c)\) and \(b_0, \ldots, b_{n-1}\) be the zeros of \(y\) on \((a, c)\). By Proposition 10, we have
\[
a_i < b_i < a_{i+1}
\]
for \(i = 0, \ldots, n - 1\). Consequently, \(y\) has no zero on \((a_n, c)\). Now, we can use the same idea from the proof of Proposition 10 to conclude that
\[
\left( (px'y - p_1xy')\frac{x}{y} \right)' \geq 0
\]
on \((a_n, c)\). Integrating both sides from \(a_n\) to \(c\) and using that \(x(a_n) = 0\), we get
\[
(px'y - p_1xy')(c)\frac{x(c)}{y(c)} \geq 0,
\]
and since we can always assume that \(x(c)y(c) > 0\), we find
\[
\frac{p(c)x'(c)}{x(c)} \geq \frac{p_1y'(c)}{y(c)}.
\]
\(\square\)

**Proof of Lemma 5.** Observe that \(\alpha_k(t_k) = 0\), \(\alpha'_k(t_k) = u'_k(t_k)\) and
\[
(v(t)\alpha'_k) + \lambda R^{2(n-1)}v(t)\alpha_k = 0.
\]
Since
\[
R^{2(n-1)}v(t) \geq R^{n-1} \geq v(t)
\]
for all \(t \geq t_k\), we can apply Proposition 12 to \(u\) and \(\alpha_k\) and establish that
\[
\frac{u'(t)}{u(t)} \geq \frac{\alpha'_k(t)}{\alpha_k(t)}, \quad t \in (t_k, \tilde{t}_k).
\]
So, taking \(\epsilon > 0\) and integrating the inequality above from \(t_k + \epsilon\) to \(t\), we get
\[
\log \left( \frac{|u(t)|}{|u(t_k + \epsilon)|} \right) \geq \log \left( \frac{|\alpha_k(t)|}{|\alpha_k(t_k + \epsilon)|} \right),
\]
\[
\frac{|u(t)|}{|\alpha_k(t)|} \geq \frac{|u(t_k + \epsilon)|}{|\alpha_k(t_k + \epsilon)|}.
\]
Sending \(\epsilon \rightarrow 0\) and using that \(u'(t_k) = \alpha'_k(t_k) \neq 0\), we find \(|\alpha_k| \leq |u|\).
The proof of the other inequality follows the same ideas and is omitted.
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