NON-AUTONOMOUS REACTION-DIFFUSION EQUATIONS
WITH VARIABLE EXPONENTS AND LARGE DIFFUSION

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Abstract. In this work we prove continuity of solutions with respect to initial
conditions and a couple of parameters and we prove upper semicontinuity of a
family of pullback attractors for the problem
\[
\begin{cases}
\frac{\partial u_s}{\partial t}(t) - D_s \text{div}(|\nabla u_s|^{p_s(x)-2} \nabla u_s) + C(t)|u_s|^{p_s(x)-2} u_s = B(u_s(t)),
&t > \tau, \\
u_s(\tau) = u_{\tau s},
\end{cases}
\]
under homogeneous Neumann boundary conditions, \(u_{\tau s} \in H := L^2(\Omega), \Omega \subset \mathbb{R}^n (n \geq 1)\) is a smooth bounded domain, \(B : H \to H\) is a globally Lipschitz map with Lipschitz constant \(L \geq 0\), \(D_s \in [1, \infty)\), \(C(\cdot) \in L^\infty([\tau, T]; \mathbb{R}^+)\) is bounded from above and below and is monotonically nonincreasing in time, \(p_s(\cdot) \in C(\bar{\Omega}), p_s := \min_{x \in \bar{\Omega}} p_s(x) \geq p, p_s^+ := \max_{x \in \bar{\Omega}} p_s(x) \leq a\), for all \(s \in \mathbb{N}\), when \(p_s(\cdot) \to p\) in \(L^\infty(\Omega)\) and \(D_s \to \infty\) as \(s \to \infty\), with \(a, p > 2\) positive constants.

1. Introduction. Reaction-diffusion systems with large diffusion have been
employed as mathematical models describing physical, chemical and biological systems
(see, for example, [5, 6, 7, 12, 15, 17, 18]). Recently an application was given to
describing algal blooms [14]. Reaction-diffusion equations can also include time-
dependent terms as in problem
\[
\begin{cases}
\frac{\partial u_s}{\partial t}(t) - \text{div}(D_s|\nabla u_s|^{p_s(x)-2} \nabla u_s) + C(t)|u_s|^{p_s(x)-2} u_s = B(u_s(t)),
&t > \tau, \\
u_s(\tau) = u_{\tau s}.
\end{cases}
\]

Non-autonomous evolution equations with spatially variable exponents and time-
dependent operator had been considered by Kloeden-Simsen [13]. They considered
the diffusion coefficient depending on time and monotonically nonincreasing in time
in order to guarantee existence and uniqueness of a strong global solution by using
an abstract result of Yotsutani [20]. Here, we also will consider a time-dependent
operator, but we will consider the time-dependent term in the perturbation part

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and we will consider the diffusion coefficient uniform in time, i.e., the diffusion coefficient will be a constant which does not depend on time.

We will consider problem (1) under homogeneous Neumann boundary conditions, $u_s(t) = u_s \in H := L^2(\Omega),\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a smooth bounded domain, $B: H \to H$ is a globally Lipschitz map with Lipschitz constant $L \geq 0, D_s \in [1, \infty)$, $p_s(\cdot) \in C(\Omega), p_s^+ := \min_{x \in \Omega} p_s(x) \geq p, p_s^- := \max_{x \in \Omega} p_s(x) \leq a$, for all $s \in \mathbb{N}$, where $a, p > 2$ are positive constants. Moreover, $p_s(\cdot) \to p$ in $L^\infty(\Omega)$ and $D_s \to \infty$ as $s \to \infty$. The terms $C(\cdot)$ are assumed to satisfy:

**Assumption C.** $C(\cdot) \in L^\infty([\tau, T]; \mathbb{R}^+) \text{ is monotonically nonincreasing in time and is bounded from above and below, let us consider } 0 < \alpha \leq C(t) \leq M, \forall t \in \mathbb{R}, \text{ for some positive constants } \alpha \text{ and } M. \text{ The constants } \alpha \text{ and } M \text{ are taken uniform on } \tau \text{ and } T.$

We will prove continuity of the flows and upper semicontinuity of the family of pullback attractors $\{U_s\}_{s \in \mathbb{N}}$ as $s$ goes to infinity for the problem (1) with respect to the couple of parameters $(D_s, p_s)$, where $p_s$ is the variable exponent and $D_s$ is the diffusion coefficient.

The autonomous version with $C(\cdot) \equiv 1$ was considered in [18] where the authors proved the upper semicontinuity of the global attractors. The main different point in this work is that here we investigate the pullback asymptotic dynamics of problem (1) with respect to the variable exponents $p_s$ and the diffusion coefficients $D_s$ when the couple of parameters changes at the same time. The main novelty appears in the construction of a bounded global solution for the evolution process.

The paper is organized as follows. In Section 2 we present properties on the operator and guarantee the existence of global solutions. In Section 3 we obtain uniform estimates for solutions of (1). Existence of the pullback attractor is then obtained in Section 4. In Section 5 we prove that the solutions $\{u_s\}$ of the PDE (1) go to the solution $u$ of the limit problem which will be an ODE, and, after that, we obtain in Section 6 the upper semicontinuity of the family of pullback attractors for the problem (1). Finally, in Section 7 we will present some examples with numerical simulation for the limit problem.

2. The properties of the operator and existence of solution. In this section we will not vary $s$, so, for simplicity, we will just write $p_s(\cdot) = p(\cdot)$ and $D_s = D$. We present the operator, prove its properties and we assure existence of a unique global solution for the problem (1).

Let $\Omega \subset \mathbb{R}^n, n \geq 1$, be a bounded smooth domain, $H := L^2(\Omega)$ and $X := W^{1,p(\cdot)}(\Omega)$ with $p^- > 2$. Then $X \subset H \subset X^*$ with continuous and dense embeddings. We refer the reader to [8, 9, 10, 11] to see properties of the Lebesgue and Sobolev spaces with variable exponents. In particular, with

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable, } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$}

Define $\rho(u) := \int_\Omega |u(x)|^{p(x)} dx$ and

$$\|u\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \rho \left( \frac{u}{\lambda} \right) \leq 1 \right\}$$

for $u \in L^{p(\cdot)}(\Omega)$. The generalized Sobolev space is defined as

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$
From [11] we know that $W^{1,p(\cdot)}(\Omega)$ is a Banach Space with the norm

$$\|u\|_X := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$  

Consider the operator $A(t)$ defined in $X$ such that for each $u \in X$ associate the following element of its dual space $X^*$, $A(t)u : X \to \mathbb{R}$ given by

$$\langle A(t)u, v \rangle_{X^*, X} := \int_{\Omega} \nabla u(x) |x|^{p(x)-2} \nabla u(x) \cdot \nabla v(x) dx + C(t) \int_{\Omega} |u(x)|^{p(x)-2} u(x) v(x) dx.$$  

We will show that the operator $A(t) : X \to X^*$ is monotone, hemicontinuous and coercive. For this purpose we need first to prove some estimates on the operator and the following two lemmas will be very useful.

**Lemma 2.1 ([11]).** If $u \in L^{p(\cdot)}(\Omega)$,

(i) $\|u\|_{p(\cdot)} < 1 (= 1; > 1)$ if and only if $\rho(u) < 1 (= 1; > 1)$;

(ii) If $\|u\|_{p(\cdot)} > 1$, then $\|u\|_{p(\cdot)}^{p^-} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p^+}$;

(iii) If $\|u\|_{p(\cdot)} < 1$, then $\|u\|_{p(\cdot)}^{p^+} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p^-}$.

**Lemma 2.2 ([1]).** Let $\lambda, \mu$ be arbitrary nonnegative numbers. For every positive $\alpha, \beta, \alpha \geq \beta$,  

$$\lambda^\alpha + \mu^\beta \geq \frac{1}{2^\alpha} \begin{cases} (\lambda + \mu)^\alpha, & \text{if } \lambda + \mu < 1, \\ (\lambda + \mu)^\beta, & \text{if } \lambda + \mu \geq 1 \end{cases} \quad (2)$$

Using the previous lemmas we can obtain the following estimates on the operator

**Lemma 2.3.** Let $u \in X := W^{1,p(\cdot)}(\Omega)$. For each $t \geq 0$ we have

$$\langle A(t)u, u \rangle_{X^*, X} \geq \frac{\min\{1, \alpha\}}{2^{p^+}} \begin{cases} \|u\|_X^{p^+}, & \text{if } \|u\|_X < 1 \\ \|u\|_X^{p^-}, & \text{if } \|u\|_X \geq 1 \end{cases}.$$  

(3)

**Proof.** Given an arbitrary $u \in X$, we denote $\lambda = \|\nabla u\|_{p(\cdot)}$ and $\mu = \|u\|_{p(\cdot)}$. From Lemma 2.1 and Lemma 2.2 we have

$$\langle A(t)u, u \rangle_{X^*, X} = \int_{\Omega} D(\nabla u)^{p(x)} dx + \int_{\Omega} C(t)|u|^{p(x)} dx

= D\rho(\nabla u) + C(t)\rho(u) \geq \rho(\nabla u) + \rho(u)

\geq \min\{\|\nabla u\|_{p(\cdot)}^{p^+}, \|\nabla u\|_{p(\cdot)}^{p^-}\} + \alpha \min\{\|u\|_{p(\cdot)}^{p^+}, \|u\|_{p(\cdot)}^{p^-}\}

\geq \min\{1, \alpha\} \{\min\{\lambda^{p^+}, \lambda^{p^-}\} + \min\{\mu^{p^+}, \mu^{p^-}\}\}

\geq \frac{\min\{1, \alpha\}}{2^{p^+}} \begin{cases} (\lambda + \mu)^{p^+}, & \text{if } \lambda + \mu < 1 \\ (\lambda + \mu)^{p^-}, & \text{if } \lambda + \mu \geq 1 \end{cases}

= \frac{\min\{1, \alpha\}}{2^{p^+}} \begin{cases} \|u\|_X^{p^+}, & \text{if } \|u\|_X < 1 \\ \|u\|_X^{p^-}, & \text{if } \|u\|_X \geq 1 \end{cases}.$$  

\[\square\]

**Lemma 2.4.** For each $t \geq 0$, the operator $A(t) : X \to X^*$ is monotone.
Proof. Let \( u, v \in X \). For each \( x \in \Omega \) fixed, we have
\[
\langle A(t)u - A(t)v, u - v \rangle_{X^*, X} = \int_\Omega D \left( |\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v \right) \left( \nabla u - \nabla v \right) dx \\
+ \int_\Omega \sum\left( \left( |u|^{p(x)} - 2u - |v|^{p(x)} - 2v \right)(u - v) dx \\
\geq \int_\Omega \sum\left( \left( |u|^{p(x)} - 2u - |v|^{p(x)} - 2v \right)(u - v) dx \\
\geq \left( \frac{1}{2} \right)^{p^+} \left( \int_\Omega |\nabla u - \nabla v|^{p(x)} dx + \int_\Omega \sum\left( \left( |u|^{p(x)} - 2u - |v|^{p(x)} - 2v \right)(u - v) dx \right) \right) \geq 0.
\]

\[ \square \]

Lemma 2.5. For each \( t \geq 0 \), the operator \( A(t) : X \to X^* \) is coercive.

Proof. Consider \( (u_j) \subset X \) such that \( \lim_{j \to \infty} \|u_j\|_X = \infty \). So there exists \( j_0 \in \mathbb{N} \) such that \( \|u_j\|_X \geq 1 \) if \( j \geq j_0 \). By Lemma 2.3 we have
\[
\frac{\langle A(t)u_1, u_j \rangle_{X^*, X}}{\|u_j\|_X} \geq \frac{\min\{1, \alpha\}\|u_j\|_{X^*}^\alpha}{2^\alpha \|u_j\|_X} = \frac{\min\{1, \alpha\}\|u_j\|_{X^*}^{\alpha-1}}{2^\alpha},
\]
for all \( j \geq j_0 \). As
\[
\lim_{j \to \infty} \frac{\min\{1, \alpha\}\|u_j\|_{X^*}^{\alpha-1}}{2^\alpha} = \infty
\]
we have
\[
\lim_{j \to \infty} \frac{\langle A(t)u_1, u_j \rangle_{X^*, X}}{\|u_j\|_X} = \infty.
\]

\[ \square \]

Lemma 2.6. For each \( t \geq 0 \), the operator \( A(t) : X \to X^* \) is hemicontinuous.

Proof. We have to show that \( A(t)(u + \lambda v) \rightharpoonup A(t)u \) as \( \lambda \to 0 \) for all \( u, v \in X \). As \( p^- > 1 \), \( X \) is a reflexive Banach Space (see [9, 11]), so we just have to show that
\[
\lim_{\lambda \to 0} \langle A(t)(u + \lambda v), \phi \rangle_{X^*, X} = \langle A(t)u, \phi \rangle_{X^*, X},
\]
for all \( \phi \in X \). Let \( u, v, \phi \in X \) and \( \lambda \in (-1, 1) \). Define
\[
f_1^t(x) = D|\nabla (u + \lambda v)|^{p(x)-2}\nabla (u + \lambda v) \cdot \nabla \phi + C(t)(u + \lambda v)|^{p(x)-2}(u + \lambda v) \phi
\]
and
\[
f^t(x) = D|\nabla u|^{p(x)-2}\nabla u \cdot \nabla \phi + C(t)|u|^{p(x)-2}u \phi.
\]
Thus,
\[
|\langle A(t)(u + \lambda v), \phi \rangle_{X^*, X} - \langle A(t)u, \phi \rangle_{X^*, X}| = \left| \int_\Omega f_1^t(x) dx - \int_\Omega f^t(x) dx \right|
\]
\[
= \left| \int_\Omega \left[ f_1^t(x) - f^t(x) \right] dx \right|.
\]
Note that

\[
\lim_{\lambda \to 0} f_2(x) = \lim_{\lambda \to 0} D|\nabla(u + \lambda v)|^{p(x)-2}\nabla(u + \lambda v) \cdot \nabla \phi + C(t)|u + \lambda v|^{p(x)-2}(u + \lambda v)\phi
\]

\[
= D|\nabla u|^{p(x)-2}\nabla u \cdot \nabla \phi + C(t)|u|^{p(x)-2}u\phi
\]

\[
= f_1(x)
\]

and for all \( x \in \Omega \)

\[
|f_1(x)| \leq D|\nabla(u + \lambda v)|^{p(x)-1}|\nabla \phi| + |C(t)||u + \lambda v|^{p(x)-1}|\phi|
\]

\[
\leq 2^{p(x)-2}D(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1})|\nabla \phi| + |C(t)|2^{p(x)-2}(|u|^{p(x)-1} + |v|^{p(x)-1})|\phi|
\]

\[
\leq 2^{p(x)-2} \left[ D \left( |\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1} \right) |\nabla \phi| + M \left( |u|^{p(x)-1} + |v|^{p(x)-1} \right) |\phi| \right]
\]

\[= : g(\cdot), \]

with \( g(\cdot) \in L^1(\Omega) \). We obtain by the Dominated Convergence Theorem that

\[
0 \leq \lim_{\lambda \to 0} |\langle A(t)(u + \lambda v), \phi \rangle_{X^*, X} - \langle A(t)u, \phi \rangle_{X^*, X}| = \lim_{\lambda \to 0} \left| \int_{\Omega} \left[ f_1(x) - f_1(x) \right] dx \right|
\]

\[
\leq \lim_{\lambda \to 0} \int_{\Omega} |f_1(x) - f_1(x)| dx
\]

\[
= 0.
\]

Therefore, \( \lim_{\lambda \to 0} \langle A(t)(u + \lambda v), \phi \rangle_{X^*, X} = \langle A(t)u, \phi \rangle_{X^*, X} \). \qed

Thus, as a consequence of Lemmas 2.4, 2.5 and 2.6 we have that the realization operator \( A_H(t) : H \to H \), \( A_H(t)u := -\text{div}(D|\nabla u|^{p(x)-2}\nabla u) + C(t)|u|^{p(x)-2}u \) defined as

\[
\begin{align*}
\mathcal{D}(A_H(t)) := & \{ u \in X : A(t)u \in H \}, \\
A_H(t)u = & \begin{cases} 
A(t)u, & \text{if } u \in \mathcal{D}(A_H(t)) 
\end{cases}
\end{align*}
\]

is a maximal monotone operator in \( H \).

We will show that \( A_H(t) \) is the subdifferential of the following convex, proper and lower semicontinuous map \( \varphi^t_{p(\cdot)} : H \to \mathbb{R} \) given by

\[
\varphi^t_{p(\cdot)}(u) = \begin{cases} 
\int_{\Omega} \frac{D}{p(\cdot)}|\nabla u|^{p(x)} dx + \int_{\Omega} \frac{C(t)}{p(\cdot)}|u|^{p(x)} dx, & \text{if } u \in X \\
+\infty, & \text{otherwise.}
\end{cases}
\]

Lemma 2.7. The map \( \varphi^t_{p(\cdot)} \) is convex and proper.

Proof. Let \( u \in X \). Then, \( u \in L^{p(\cdot)}(\Omega) \) and \( \nabla u \in L^{p(\cdot)}(\Omega) \). So,

\[
\int_{\Omega} \frac{D}{p(\cdot)}|\nabla u|^{p(x)} dx + \int_{\Omega} \frac{C(t)}{p(\cdot)}|u|^{p(x)} dx \leq \frac{1}{2} \left[ D \int_{\Omega} |\nabla u|^{p(x)} dx + M \int_{\Omega} |u|^{p(x)} dx \right] < \infty.
\]
Therefore \( \varphi_{p(\cdot)}^t \) is proper. Since the application \( \gamma^p \) is convex for \( \gamma > 0 \) given, \( u, v \in X \) and \( 0 \leq \lambda \leq 1 \) we have

\[
\varphi_{p(\cdot)}^t(\lambda u + (1 - \lambda)v) = \int_{\Omega} \frac{D}{p(x)} |\nabla(\lambda u + (1 - \lambda)v)|^{p(x)} dx \\
+ \int_{\Omega} \frac{C(t)}{p(x)} |\lambda u + (1 - \lambda)v|^{p(x)} dx \\
\leq \int_{\Omega} \frac{D}{p(x)} (\lambda|\nabla u|^{p(x)} + (1 - \lambda)|\nabla v|^{p(x)}) dx \\
+ \int_{\Omega} \frac{C(t)}{p(x)} (\lambda|u|^{p(x)} + (1 - \lambda)|v|^{p(x)}) dx \\
= \lambda \int_{\Omega} \frac{D}{p(x)} |\nabla u|^{p(x)} dx + \lambda \int_{\Omega} \frac{C(t)}{p(x)} |u|^{p(x)} dx \\
+ (1 - \lambda) \int_{\Omega} \frac{D}{p(x)} |\nabla v|^{p(x)} dx + (1 - \lambda) \int_{\Omega} \frac{C(t)}{p(x)} |v|^{p(x)} dx \\
= \lambda \varphi_{p(\cdot)}^t(u) + (1 - \lambda) \varphi_{p(\cdot)}^t(v),
\]

therefore \( \varphi_{p(\cdot)}^t \) is convex. \( \Box \)

**Lemma 2.8.** The map \( \varphi_{p(\cdot)}^t \) is lower semicontinuous.

**Proof.** Let \( (u_n) \) be a sequence such that \( u_n \to u \) in \( H \). We have to show that

\[
\varphi_{p(\cdot)}^t(u) \leq \liminf_{n \to \infty} \varphi_{p(\cdot)}^t(u_n) \quad \text{if} \quad u_n \to u \quad \text{in} \quad H.
\]

If \( \liminf_{n \to \infty} \varphi_{p(\cdot)}^t(u_n) = +\infty \), then

\[
\varphi_{p(\cdot)}^t(u) \leq +\infty = \liminf_{n \to \infty} \varphi_{p(\cdot)}^t(u_n).
\]

On the other hand, if \( \liminf_{n \to \infty} \varphi_{p(\cdot)}^t(u_n) = a < +\infty \) then there is a subsequence \( (u_{n_j}) \subset X \) of \( (u_n) \) such that

\[
\lim_{j \to \infty} \varphi_{p(\cdot)}^t(u_{n_j}) = \lim_{j \to \infty} \left( \int_{\Omega} \frac{D}{p(x)} |\nabla u_{n_j}|^{p(x)} dx + \int_{\Omega} \frac{C(t)}{p(x)} |u_{n_j}|^{p(x)} dx \right) = a.
\]

Since \( \varphi_{p(\cdot)}^t(u_{n_j}) \to a \) as \( j \to \infty \) we have that \( \varphi_{p(\cdot)}^t(u_{n_j}) \) is bounded, i.e., there exists \( \delta > 0 \) such that \( |\varphi_{p(\cdot)}^t(u_{n_j})| \leq \delta \), for all \( j \in \mathbb{N} \). Once \( D \in [1, \infty) \), \( C(t) \geq \alpha \) we have

\[
\varphi_{p(\cdot)}^t(u_{n_j}) = \int_{\Omega} \frac{D}{p(x)} |\nabla u_{n_j}|^{p(x)} dx + \int_{\Omega} \frac{C(t)}{p(x)} |u_{n_j}|^{p(x)} dx \\
\geq \int_{\Omega} \frac{C(t)}{p(x)} |u_{n_j}|^{p(x)} dx \geq \frac{C(t)}{p^+} \int_{\Omega} |u_{n_j}|^{p(x)} dx \\
\geq \frac{\alpha}{p^+} \rho(u_{n_j}).
\]

Consequently,

\[
\frac{\alpha}{p^+} \rho(u_{n_j}) \leq \varphi_{p(\cdot)}^t(u_{n_j}) \leq \delta.
\]

Then,

\[
\rho(u_{n_j}) \leq \frac{p^+ \delta}{\alpha}, \quad (6)
\]
Similarly we have that
\[ \rho(\nabla u_{n_j}) \leq p^+ \delta. \]
Using (6), (7) and the Lemma 2.1 we obtain
\[ \text{Theorem 2.9.} \]
\[ \text{Using (6), (7) and the Lemma 2.1 we obtain} \]
\[ \|u_{n_j}\|_{p(\cdot)} \leq \begin{cases} \left( \frac{p^+ \delta}{\rho} \right)^{\frac{1}{p'}} \cdot \|u_{n_j}\|_{p(\cdot)} \geq 1 \\ \left( \frac{p^+ \delta}{\rho} \right)^{\frac{1}{p'}} \cdot \|u_{n_j}\|_{p(\cdot)} < 1 \end{cases} \]
and
\[ \|
abla u_{n_j}\|_{p(\cdot)} \leq \begin{cases} \left( \frac{p^+ \delta}{\rho} \right)^{\frac{1}{p'}} \cdot \|
abla u_{n_j}\|_{p(\cdot)} \geq 1 \\ \left( \frac{p^+ \delta}{\rho} \right)^{\frac{1}{p'}} \cdot \|
abla u_{n_j}\|_{p(\cdot)} < 1 \end{cases}. \]

Therefore, \( \|u_{n_j}\|_X \) is a bounded sequence in the reflexive Banach space \( X \). So, \( (u_{n_j}) \) has a subsequence (which we still denote by \( (u_{n_j}) \)) such that \( u_{n_j} \rightharpoonup v \) in \( X \) for some \( v \in X \). As \( H^* \subset X^* \) we have \( u_{n_j} \rightharpoonup v \) in \( H \) and by the uniqueness of the weak limit \( u = v \in X \).

We observe that the function \( \psi^t : X \to X^* \), defined by \( \psi^t(w) := \varphi_{p(\cdot)}^t(w) \) for any \( w \in X \), is Gateaux differentiable. So, it follows from Example 1, p. 54 in [2], that \( u \in \mathcal{D}(\partial\psi^t) \) and \( \partial\psi^t(u) \) consists of a single element, namely the Gateaux differential of \( \psi^t \) at \( u \), i.e., \( \nabla u \psi^t(u) = \partial\psi^t(u) \). As \( u_{n_j} \rightharpoonup u \) in \( X \) and \( 0 \neq \partial\psi^t(u) \in X^* \) we obtain
\[ \langle \partial\psi^t(u), u_{n_j} - u \rangle_{\cdot, X} \to 0 \]
as \( j \to \infty \). Moreover, we have
\[ \langle \partial\psi^t(u), u_{n_j} - u \rangle_{\cdot, X} \leq \psi^t(u_{n_j}) - \psi^t(u) = \varphi_{p(\cdot)}^t(u_{n_j}) - \varphi_{p(\cdot)}^t(u) \]
for all \( j \in \mathbb{N} \). Therefore,
\[ \varphi_{p(\cdot)}^t(u) \leq \lim_{j \to \infty} \varphi_{p(\cdot)}^t(u_{n_j}) = a = \liminf_{n \to \infty} \varphi_{p(\cdot)}^t(u_{n_j}). \]

Let us consider now the subdifferential \( \partial\varphi_{p(\cdot)}^t \) which is the restriction of \( \partial\psi^t \) to the set \( K^t := \{ u \in X : A(t)u \in H \} \).

**Theorem 2.9.** \( A_H(t) \) is the subdifferential \( \partial\varphi_{p(\cdot)}^t \) of \( \varphi_{p(\cdot)}^t \).

**Proof.** The realization \( A_H(t) \) of \( A(t) \) and \( \partial\varphi_{p(\cdot)}^t \) are both maximal monotone operators in \( H \), so it is enough to show that for any \( u \in H \),
\[ A_H(t)u \subset \partial\varphi_{p(\cdot)}^t(u). \]
Let \( u \in \mathcal{D}(A_H(t)) := \{ u \in X : A(t)u \in H \} \) and \( v \in A_H(t)u = A(t)u \). So, for all \( \xi \in X \) we have
\[ \langle v, \xi - u \rangle_{\cdot, X} = \langle A(t)u, \xi - u \rangle_{\cdot, X} \]
\[ = \int_{\Omega} D|\nabla u|^p(x) x - |\nabla u| dx + \int_{\Omega} C(t)|u|^p(x) u.(\xi - u) dx \]
\[ = \int_{\Omega} D|\nabla u|^p(x) x - |\nabla u| dx - \int_{\Omega} D|\nabla u|^p(x) dx \]
\[ + \int_{\Omega} C(t)|u|^p(x) u.\xi dx - \int_{\Omega} C(t)|u|^p(x) dx. \]
Considering \( q(x) \) such that \( \frac{1}{p(x)} + \frac{1}{q(x)} = 1 \), we have

\[
\langle v, \xi - u \rangle_{X',X} + \int_{\Omega} D|\nabla u|^{p(x)}dx + \int_{\Omega} C(t)|u|^{p(x)}dx
\]

\[
= \int_{\Omega} D|\nabla u|^{p(x)-2}\nabla u\nabla \xi dx + \int_{\Omega} C(t)|u|^{p(x)-2}u\xi dx
\]

\[
\leq \int_{\Omega} D|\nabla u|^{p(x)-1}\nabla \xi dx + \int_{\Omega} C(t)|u|^{p(x)-1}\xi dx
\]

\[
\leq \int_{\Omega} \frac{D}{q(x)}|\nabla u|^{(p(x)-1)q(x)} + \frac{D}{p(x)}|\nabla \xi|^{p(x)}dx
\]

\[
+ \int_{\Omega} \frac{C(t)}{q(x)}|u|^{(p(x)-1)q(x)} + \frac{C(t)}{p(x)}|\xi|^{p(x)}dx
\]

\[
= \int_{\Omega} \frac{D}{q(x)}|\nabla u|^{p(x)}dx + \int_{\Omega} \frac{D}{p(x)}|\nabla \xi|^{p(x)}dx
\]

\[
+ \int_{\Omega} \frac{C(t)}{q(x)}|u|^{p(x)}dx + \int_{\Omega} \frac{C(t)}{p(x)}|\xi|^{p(x)}dx.
\]

Then,

\[
\langle v, \xi - u \rangle_{X',X} + \int_{\Omega} D\left(1 - \frac{1}{q(x)}\right)|\nabla u|^{p(x)}dx + \int_{\Omega} C(t)\left(1 - \frac{1}{q(x)}\right)|u|^{p(x)}dx
\]

\[
\leq \int_{\Omega} \frac{D}{p(x)}|\nabla \xi|^{p(x)}dx + \int_{\Omega} \frac{C(t)}{p(x)}|\xi|^{p(x)}dx.
\]

So, we conclude that

\[
\langle v, \xi - u \rangle_{X',X} \leq \phi^t_{p(x)}(\xi) - \phi^t_{p(x)}(u), \quad (8)
\]

for all \( \xi \in X \). If \( \xi \in H - X \), then \( \phi^t_{p(x)}(\xi) = \infty \) and consequently (8) holds. Therefore, \( A_H(t)(u) = v \in \partial \phi^t_{p(x)}(u) \).

Observe that if we would consider the nonlinear absorption term with a different variable exponent from the ones in the diffusion term, we could not define a maximal monotone operator of the subdifferential type.

2.1. **Existence of solution.** We will show the existence of a unique global solution by using [20]. Consider the problem

\[
\begin{cases}
\frac{du}{dt}(t) + \partial \phi^t u(t) = f(t), & t > \tau, \\
u(\tau) = u_0 \in H, 
\end{cases} \quad (9)
\]

**Assumption A.** Let \( T > \tau \) be fixed.

A.1: There is a set \( \tau \notin Z \subset [\tau, T] \) of zero measure such that \( \phi^t \) is a lower semicontinuous proper convex function from \( H \) into \((-\infty, \infty]\) with a non-empty effective domain for each \( t \in [\tau, T] - Z \).

A.2: For any positive integer \( r \) there exist a constant \( K_r > 0 \), an absolutely continuous function \( g_r : [\tau, T] \rightarrow \mathbb{R} \) with \( g'_r \in L^2(\tau, T) \) and a function of bounded variation \( h_r : [\tau, T] \rightarrow \mathbb{R} \) such that if \( t \in [\tau, T] - Z, w \in \mathcal{D}(\phi^t) \) with \( |w| \leq r \) and \( s \in [t, T] - Z \) then there exists an element \( \hat{w} \in \mathcal{D}(\phi^s) \) satisfying

\[
|\hat{w} - w| \leq |g_r(s) - g_r(t)|(\phi^s(w) + K_r)^\alpha 
\]

(10)
Lemma 2.11. Let \( f \) be a function for each \( t \in [\tau, T] \). Then for each \( \tau, T \)

\[
\int_{\Omega} \frac{D}{p(x)} |\nabla w|^p(x) dx + \int_{\Omega} C(t) |u|^p(x) dx \geq 0
\]

where \( \alpha \) is some fixed constant \( 0 \leq \alpha \leq 1 \) and

\[
\beta := \begin{cases} 
2 & \text{if } 0 \leq \alpha \leq \frac{1}{2}, \\
\frac{1}{1-\alpha} & \text{if } \frac{1}{2} \leq \alpha \leq 1.
\end{cases}
\]

Theorem 2.10. [20] Suppose that Assumption (A) is satisfied. Then, for each \( f \in L^2(\tau, T; H) \) and \( u_0 \in D(\varphi^\tau) \) the equation (9) has a unique strong solution \( u \) on \([\tau, T]\) with \( u(\tau) = u_0\).

Using the monotonicity of the operator and the Gronwall Lemma we obtain the following

Lemma 2.11. If \( f, g \in L^2(\tau, T; H) \) and \( u, v \) are the solutions of the equations

\[
\begin{cases} 
\frac{du}{dt}(t) + \partial \varphi^t u(t) = f(t), \\
u(\tau) = u_0 \in H,
\end{cases}
\]

\[
\begin{cases} 
\frac{dv}{dt}(t) + \partial \varphi^t v(t) = g(t), \\
v(\tau) = v_0 \in H,
\end{cases}
\]

then for \( \tau \leq s \leq t \leq T \)

\[
\|u(t) - v(t)\|_H \leq \|u(s) - v(s)\|_H + \int_s^t \|f(r) - g(r)\|_H dr.
\]

The proof of the following theorem is analogous to the proof of Theorem 1 in [16].

Theorem 2.12. If \( B : H \to H \) is globally Lipschitz and \( u_0 \in H \), then there exists a unique \( u \in C([\tau, T]; H) \), such that \( \frac{du}{dt}(t) + \partial \varphi^t u(t) = B(u(t)) \) a.e. on \([\tau, T]\) and \( u(\tau) = u_0\).

Let us consider now the problem with our specific operator

\[
\begin{cases} 
\frac{du}{dt}(t) + \partial \varphi^t_{\tau p(t)} u(t) = f(t), \\
u(\tau) = u_0 \in H,
\end{cases}
\]

(12)

Theorem 2.13. For each \( f \in L^2(\tau, T; H) \) and \( u_0 \in H \) the equation (12) has a unique strong solution \( u \) on \([\tau, T]\) with \( u(\tau) = u_0\).

Proof. Taking \( Z \) as the empty set, \( \varphi^t_{\tau p(t)} \) is lower semicontinuous proper convex function for each \( t \in [\tau, T] \). Consider \( r \) a positive integer, \( K_r := r \) and \( \alpha := \frac{1}{2} \).

Define \( g_r : [\tau, T] \to \mathbb{R} \) with \( g_r(t) := t + r \), and \( h_r(t) := r \). We have that \( g_r \) is an absolutely continuous function \( g_r \in L^2(\tau, T) \) and \( h_r \) is a bounded variation function. For all \( t \in [\tau, T] \), \( w \in D(\varphi^\tau_{\tau p(t)}) = X = \mathbb{R}^n \).

Consider the element \( \tilde{w} := w \in X = D(\varphi^\tau_{\tau p(t)}) \). We will check that \( \tilde{w} \) satisfies (10) and (11). Note that

\[
\int_\Omega \frac{D}{p(x)} |\nabla w|^p(x) dx + \int_\Omega C(t) |u|^p(x) dx \geq 0
\]
Thus,
\[ |\tilde{w} - w| = 0 \leq |s - t| \left( \int_{\Omega} \frac{D}{p(x)} |\nabla w|^{p(x)} dx + \int_{\Omega} \frac{C(t)}{p(x)} |w|^{p(x)} dx + r \right)^{1/2} = |g_r(s) - g_r(t)| \left( \varphi^t_{p(\cdot)}(w) + K_r \right)^\alpha. \]

Now, note that
\[ \varphi^s_{p(\cdot)}(\tilde{w}) = \int_{\Omega} \frac{D}{p(x)} |\nabla \tilde{w}|^{p(x)} dx + \int_{\Omega} \frac{C(s)}{p(x)} |\tilde{w}|^{p(x)} dx = \int_{\Omega} \frac{D}{p(x)} |\nabla w|^{p(x)} dx + \int_{\Omega} \frac{C(s)}{p(x)} |w|^{p(x)} dx \leq \int_{\Omega} \frac{D}{p(x)} |\nabla w|^{p(x)} dx + \int_{\Omega} \frac{C(t)}{p(x)} |w|^{p(x)} dx = \varphi^t_{p(\cdot)}(w) \]

Thus,
\[ \varphi^s_{p(\cdot)}(\tilde{w}) \leq \varphi^t_{p(\cdot)}(w) + |h_r(s) - h_r(t)| \left( \varphi^t_{p(\cdot)}(w) + K_r \right). \]

Therefore, we obtain from Theorem 2.10 the existence of a global solution to the problem (12).

**Theorem 2.14.** The problem (1) has a unique global strong solution.

**Proof.** Just apply the Theorem 2.12 with \(A(t) = \partial \varphi^t_{p(\cdot)}\).

3. Uniform estimates. From now on we denote \(X_s := W^{1,p_s(\cdot)}(\Omega)\) and \(Y := W^{1,p(\cdot)}(\Omega)\). It is a known result that \(X_s \subset H\) with continuous and dense embeddings (see [8, 17]). Moreover, it is easy to see that
\[ \|u_s\|_H \leq 4(|\Omega| + 1)^2\|u_s\|_{X_s}, \]
for all \(u_s \in X_s\) and for all \(s \in \mathbb{N}\).

**Lemma 3.1.** Let \(u_s\) be a solution of (1) with \(u_s(\tau) = u_{rs} \in H\). Given \(T_1 > 0\), there exists a positive number \(r_0 = r_0(T_1)\) such that \(\|u_s(t)\|_H \leq r_0\), for each \(t \geq T_1 + \tau\) and \(s \in \mathbb{N}\). Furthermore, given a bounded set \(B \subset H\), there exists \(\tilde{D}_1 > 0\) such that \(\|u_s(t)\|_H \leq \tilde{D}_1\), for all \(t \geq \tau\) and \(s \in \mathbb{N}\) such that \(u_{rs} \in B\).

**Proof.** Let \(t > \tau\), multiplying the equation in (1) by \(u_s(t)\) we have
\[ \frac{1}{2} \frac{d}{dt} \|u_s(t)\|_H^2 + \langle A'(t)u_s(t), u_s(t) \rangle + \langle B(u_s(t)), u_s(t) \rangle. \]

Let \(T > \tau\) and consider \(I = (\tau, T) = I^1_s \cup I^2_s\) where \(I^1_s = \{t \in (\tau, T): \|u_s(t)\|_{X_s} < 1\}\) and \(I^2_s = \{t \in (\tau, T): \|u_s(t)\|_{X_s} \geq 1\}\). Consider \(T_1 > 0\) arbitrarily given. If \(\|u_s(t)\|_{X_s} \geq 1\) by Lemma 2.3 we obtain
\[ \frac{1}{2} \frac{d}{dt} \|u_s(t)\|_H^2 \leq -\frac{\min\{1, \alpha\}}{2^\alpha} \|u_s(t)\|_{X_s}^p + (B(u_s(t)), u_s(t)). \]

Thus,
\[ \frac{1}{2} \frac{d}{dt} \|u_s(t)\|_H^2 \leq -\frac{\min\{1, \alpha\}}{2^\alpha} \|u_s(t)\|_{X_s}^p + L\|u_s(t)\|_H^2 + \|B(0)\|_H \|u_s(t)\|_H \leq -\frac{\min\{1, \alpha\}}{2^\alpha} \|u_s(t)\|_{X_s}^p + C_1\|u_s(t)\|_{X_s}^2 + C_2\|u_s(t)\|_{X_s}. \]
where \( C_1 := LK^2 \) and \( C_2 := C_0K \) with \( K := 4(|\Omega| + 1)^2 \) and \( C_0 := \|B(0)\|_H \geq 0 \). Let \( \theta := \frac{p}{2} \), with \( \frac{1}{p} + \frac{1}{q} = 1 \). Now, consider \( \epsilon > 0 \) arbitrary. Thus,

\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) \|_H^2 \leq - \frac{\min\{1, \alpha\}}{2^a} \| u_s(t) \|_{X_s}^p + \frac{C_1}{\epsilon} \| u_s(t) \|_{X_s}^q + \frac{C_2}{\epsilon} \| u_s(t) \|_{X_s}.
\]

By Young’s inequality, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) \|_H^2 \leq - \frac{\min\{1, \alpha\}}{2^a} \| u_s(t) \|_{X_s}^p + \frac{1}{\theta^\prime} \left( \frac{C_1}{\epsilon} \right)^{\theta^\prime} + \frac{1}{q} \left( \frac{C_2}{\epsilon} \right)^q \| u_s(t) \|_{X_s}^p.
\]

Thus,

\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) \|_H^2 + \left( \frac{\min\{1, \alpha\}}{2^a} - \frac{1}{\theta^\prime} \right) \| u_s(t) \|_{X_s}^p \leq \frac{1}{\theta^\prime} \left( \frac{C_1}{\epsilon} \right)^{\theta^\prime} + \frac{1}{q} \left( \frac{C_2}{\epsilon} \right)^q.
\]

In the case \( C_0 \neq 0 \) choose \( \epsilon_0 > 0 \) such that \( \gamma := \frac{\min\{1, \alpha\}}{2^a} - \frac{1}{\theta^\prime} \epsilon_0 - \frac{1}{p^\prime} \epsilon_0 > 0 \). For the case \( C_0 = 0 \) choose \( \epsilon_0 > 0 \) such that \( \gamma := \frac{\min\{1, \alpha\}}{2^a} - \frac{1}{\theta^\prime} \epsilon_0 > 0 \). So,

\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) \|_H^2 + \gamma \| u_s(t) \|_{X_s}^p \leq \frac{1}{\theta^\prime} \left( \frac{C_1}{\epsilon_0} \right)^{\theta^\prime} + \frac{1}{q} \left( \frac{C_2}{\epsilon_0} \right)^q.
\]

Let \( \xi := \frac{2}{\theta} \left( \frac{C_1}{\epsilon_0} \right)^{\theta^\prime} + \frac{2}{q} \left( \frac{C_2}{\epsilon_0} \right)^q \) and \( \gamma := \frac{2\gamma}{4(|\Omega| + 1)^2} \), and \( y_s : I^2 \rightarrow \mathbb{R} \), \( y_s(t) := \| u_s(t) \|_H^2 \). Then,

\[
y_s'(t) + \tilde{\gamma} y_s(t)^{p/2} \leq \xi, \quad \forall t \in I^2.
\]

If \( \| u_s(t) \|_{X_s} \leq 1 \) by Lemma 2.3 we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) \|_H^2 \leq - \frac{\min\{1, \alpha\}}{2^a} \| u_s(t) \|_{X_s}^p + \langle B(u_s(t)), u_s(t) \rangle.
\]

Consider \( r_s := \frac{p^s}{p^s} > 1 \) and \( r_s^\prime \) be such that \( \frac{1}{r_s} + \frac{1}{r_s^\prime} = 1 \), then by the Young inequality

\[
\| u_s(t) \|_{X_s} \leq \frac{1}{r_s} + \frac{1}{r_s^\prime} \| u_s(t) \|_{X_s}^{p^s}.
\]

So,

\[
- \frac{\min\{1, \alpha\}}{2^a} \| u_s(t) \|_{X_s}^{p^s} \leq - \frac{r_s \min\{1, \alpha\}}{2p^s} \| u_s(t) \|_{X_s}^{p^s} + \frac{r_s}{r_s^\prime}
\]

Since \( p \leq p^s \leq p^{s^*} \leq a \), there is a positive constant \( C_3 \) such that \( \frac{r_s}{r_s^\prime} \leq C_3 \) for all \( s \in \mathbb{N} \). Thus,

\[
- \frac{\min\{1, \alpha\}}{2p^{s^*}} \| u_s(t) \|_{X_s}^{p^{s^*}} \leq - \frac{r_s \min\{1, \alpha\}}{2p^{s^*}} \| u_s(t) \|_{X_s}^{p^{s^*}} + C_3.
\]

Plugging (15) with (14) we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) \|_H^2 \leq - \frac{r_s \min\{1, \alpha\}}{2p^{s^*}} \| u_s(t) \|_{X_s}^{p^{s^*}} + \langle B(u_s(t)), u_s(t) \rangle + C_3.
\]

Now, acting in a completely analogous way as in the case \( t \in I^2 \) (compare with (13)) and joining both cases we obtain that \( y_s : (\tau, T) \rightarrow \mathbb{R} \) satisfies the following inequality

\[
y_s'(t) + \tilde{\gamma} y_s(t)^{p/2} \leq \xi, \quad \forall t \in I = (\tau, T),
\]
for some positive constants $\hat{v}$ and $\hat{\xi}$.

By Lemma 5.1 in [19], we obtain

$$y_s(t) \leq \left(\frac{\hat{\xi}}{\hat{v}}\right)^{2/p} + \left[\hat{v} \left(\frac{p-2}{2}\right) (t-\tau)\right]^{2/p} \leq \left(\frac{\hat{\xi}}{\hat{v}}\right)^{2/p} + \left[\hat{v} \left(\frac{p-2}{2}\right) T_1\right]^{2/p-2},$$

for all $t \in I$ with $t-\tau \geq T_1$.

Considering $r_0 := \left[\left(\frac{\hat{\xi}}{\hat{v}}\right)^{2/p} + \left[\hat{v} \left(\frac{p-2}{2}\right) T_1\right]^{2/p-2}\right]^{1/2}$, we have

$$\|u_s(t)\|_H \leq r_0, \forall \ t \geq T_1 + \tau, \ s \in \mathbb{N},$$

and the first part of the lemma is proved.

The second part of the lemma follows from the Gronwall-Bellman Lemma.

\begin{lemma}
Let $u_s$ be a solution of (1). Given $T_2 > 0$, there exists a positive constant $r_1 > 0$, independent of $s$, such that

$$\|u_s(t)\|_{X_s} \leq r_1,$$

for every $t \geq T_2 + \tau$ and $s \in \mathbb{N}$.
\end{lemma}

\begin{proof}
Let $u_s$ be a solution of (1) and consider $T_2 > 0$. Take $T_1 \in (0, T_2)$. Then,

$$\frac{d}{dt} \varphi^t_{\partial u_s(t)}(u_s(t)) = \left\langle \partial \varphi^t_{\partial u_s(t)}(u_s(t)), \frac{du_s}{dt}(t) \right\rangle$$

$$= \left\langle B(u_s(t)) - \frac{du_s}{dt}(t), \frac{du_s}{dt}(t) \right\rangle$$

$$= -\left\|B(u_s(t)) - \frac{du_s}{dt}(t)\right\|_H^2 + \left\langle B(u_s(t)) - \frac{du_s}{dt}(t), B(u_s(t)) \right\rangle$$

$$\leq -\left\|B(u_s(t)) - \frac{du_s}{dt}(t)\right\|_H^2 + \left\|B(u_s(t)) \right\|_H \left\|B(u_s(t))\right\|_H.$$

By Young’s inequality we obtain

$$\frac{d}{dt} \varphi^t_{\partial u_s(t)}(u_s(t)) + \frac{1}{2} \left\|B(u_s(t)) - \frac{du_s}{dt}(t)\right\|_H^2 \leq \frac{1}{2} \left\|B(u_s(t))\right\|_H^2.$$

By Lemma 3.1 we obtain

$$\frac{d}{dt} \varphi^t_{\partial u_s(t)}(u_s(t)) \leq \frac{1}{2} \left(\left\|B(u_s(t)) - B(0)\right\|_H + \left\|B(0)\right\|_H\right)^2$$

$$\leq \frac{1}{2} \left(L\|u_s(t)\|_H + C_0\right)^2 \leq \frac{1}{2} \left(Lr_0 + C_0\right)^2 =: \frac{1}{2} K_1^2$$

for all $t \geq T_1 + \tau$. By the definition of the subdifferential, we have

$$\varphi^t_{\partial u_s(t)}(u_s(t)) \leq \left\langle \partial \varphi^t_{\partial u_s(t)}(u_s(t)), u_s(t) \right\rangle$$

(16)
Then,
\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) \|^2_H + \varphi^t_{p,\gamma}(u_s(t)) \leq \left( \frac{du_s}{dt}(t), u_s(t) \right) + \left( \partial \varphi^t_{p,\gamma}(u_s(t)), u_s(t) \right) \\
= \left( \frac{du_s}{dt}(t) + \partial \varphi^t_{p,\gamma}(u_s(t)), u_s(t) \right) \\
= \langle B(u_s(t)), u_s(t) \rangle \leq \| B(u_s(t)) \|_H \| u_s(t) \|_H \\
\leq K_i r_0, \ \forall t \geq T_1 + \tau. \tag{17}
\]

Fixing \( r > 0 \) and integrating both sides from \( t \) to \( t + r \), with \( t \geq T_1 + \tau \) we have
\[
\int_t^{t+r} \frac{1}{2} \frac{d}{dl} \| u_s(t) \|^2_H \, dl + \int_t^{t+r} \varphi^t_{p,\gamma}(u_s(t)) \, dl \leq \int_t^{t+r} K_i r_0 \, dl.
\]
Thus,
\[
\frac{1}{2} \left( \| u_s(t + r) \|^2_H - \| u_s(t) \|^2_H \right) + \int_t^{t+r} \varphi^t_{p,\gamma}(u_s(t)) \, dl \leq K_i r_0 r.
\]
Then,
\[
\int_t^{t+r} \varphi^t_{p,\gamma}(u_s(t)) \, dl \leq \frac{1}{2} \| u_s(t) \|^2_H + K_i r_0 r \leq \frac{1}{2} r_0^2 + K_i r_0 r =: a_3
\]
Taking \( g = 0, \ h := \frac{1}{2} K_i^2, \ y_s(t) := \varphi^t_{p,\gamma}(u_s(t)) \), so
\[
\int_t^{t+r} g(t) \, dl = 0 =: a_1, \ \int_t^{t+r} h(t) \, dl =: a_2, \ \text{and} \ \int_t^{t+r} y(t) \, dl \leq a_3
\]
By Uniform Gronwall Lemma, we obtain
\[
\varphi^t_{p,\gamma}(u_s(t)) \leq \left( \frac{a_3}{r} + a_2 \right) e^0 =: \bar{r}_1, \ \forall t \geq T_2 + \tau.
\]
Thus,
\[
\varphi^t_{p,\gamma}(u_s(t)) = \int_\Omega \frac{D_s}{p_s(x)} | \nabla u_s(t) |^{p_s(x)} \, dx + \int_\Omega \frac{C(t)}{p_s(x)} | u_s(t) |^{p_s(x)} \, dx \leq \bar{r}_1,
\]
for all \( t \geq T_2 + \tau \). As \( p_s < p_s^+ \), \( D_s \in [1, \infty) \) and \( \alpha \leq C(t) \) we have
\[
\frac{1}{p_s} \int_\Omega | \nabla u_s(t) |^{p_s(x)} \, dx + \alpha \int_\Omega | u_s(t) |^{p_s(x)} \, dx \leq \varphi^t_{p,\gamma}(u_s(t)) \leq \bar{r}_1.
\]
for all \( t \geq T_2 + \tau \). Then, considering \( \rho_s(v) := \int_\Omega |v(x)|^{p_s(x)} \, dx \), we have
\[
\frac{\min \{ 1, \alpha \} }{p_s} [ \rho_s(\nabla u_s(t)) + \rho_s(u_s(t)) ] \leq \frac{1}{p_s} \rho_s(\nabla u_s(t)) + \frac{\alpha}{p_s} \rho_s(u_s(t)) \leq \bar{r}_1,
\]
\( \forall t \geq T_2 + \tau \). Consequently,
\[
(\rho_s(\nabla u_s(t)) + \rho_s(u_s(t)) \leq \frac{\alpha}{\min \{ 1, \alpha \}} \bar{r}_1, \ \forall t \geq T_2 + \tau. \tag{18}
\]
If \( t \geq T_2 + \tau \) and \( \| u_s(t) \|_{L_x} < 1 \) the lemma is proved.
If \( t \geq T_2 + \tau \) and \( \| u_s(t) \|_{L_x} \geq 1 \) then we have four cases to analyze:

**Case 1.** If \( \| \nabla u_s(t) \|_{L_{p,\gamma}} \geq 1 \) and \( \| u_s(t) \|_{L_{p,\gamma}} \geq 1 \), by Lemma 2.1 we have
\[
\| \nabla u_s(t) \|_{L_{p,\gamma}} \leq \| \nabla u_s(t) \|_{L_{p,\gamma}} \leq \rho_s(\nabla u_s(t)) \leq \| \nabla u_s(t) \|_{L_{p,\gamma}} \leq \| u_s(t) \|_{L_{p,\gamma}},
\]
and
\[
\| u_s(t) \|_{L_{p,\gamma}} \leq \| u_s(t) \|_{L_{p,\gamma}} \leq \rho_s(u_s(t)) \leq \| u_s(t) \|_{L_{p,\gamma}}.
\]
By definition of the norm in $X_s$ we have
\[
\|u_s(t)\|_{X_s} = \|\nabla u_s(t)\|_{p_{s}()} + \|u_s(t)\|_{p_{s}()} \\
\leq [\rho_s(\nabla u_s(t))]^{1/p} + [\rho_s(u_s(t))]^{1/p} \\
\leq 2 \left[ \frac{a}{\min\{1, \alpha\}} \tilde{r}_1 \right]^{1/p}
\]
Then,
\[
\|u_s(t)\|_{X_s} \leq 2 \left[ \frac{a}{\min\{1, \alpha\}} \tilde{r}_1 \right]^{1/p} =: R_1, \ \forall \ t \geq T_2 + \tau.
\]

**Case 2.** If $\|\nabla u_s(t)\|_{p_{s}()} \geq 1$ and $\|u_s(t)\|_{p_{s}()} \leq 1$, by Lemma 2.1 we obtain
\[
\|\nabla u_s(t)\|_{p_{s}()}^{p} \leq \|\nabla u_s(t)\|_{p_{s}()}^{p_s} \leq \rho_s(\nabla u_s(t)) \leq \|\nabla u_s(t)\|_{p_{s}()}^{p_s}.
\]
and
\[
\|u_s(t)\|_{p_{s}()}^{a} \leq \|u_s(t)\|_{p_{s}()}^{p_s} \leq \rho_s(u_s(t)) \leq \|u_s(t)\|_{p_{s}()}^{p_s}.
\]
Using (18) we have
\[
\|u_s(t)\|_{X_s} = \|\nabla u_s(t)\|_{p_{s}()} + \|u_s(t)\|_{p_{s}()} \\
\leq [\rho_s(\nabla u_s(t))]^{1/p} + [\rho_s(u_s(t))]^{1/a} \\
\leq \left[ \frac{a}{\min\{1, \alpha\}} \tilde{r}_1 \right]^{1/p} + \left[ \frac{a}{\min\{1, \alpha\}} \tilde{r}_1 \right]^{1/a} =: R_2,
\]
for all $t \geq T_2 + \tau$.

**Case 3.** If $\|\nabla u_s(t)\|_{p_{s}()} \leq 1$ and $\|u_s(t)\|_{p_{s}()} \geq 1$, by Lemma 2.1 we have
\[
\|\nabla u_s(t)\|_{p_{s}()}^{a} \leq \|\nabla u_s(t)\|_{p_{s}()}^{p_s} \leq \rho_s(\nabla u_s(t)) \leq \|\nabla u_s(t)\|_{p_{s}()}^{p_s},
\]
and
\[
\|u_s(t)\|_{p_{s}()}^{a} \leq \|u_s(t)\|_{p_{s}()}^{p_s} \leq \rho_s(u_s(t)) \leq \|u_s(t)\|_{p_{s}()}^{p_s}.
\]
By (18) we obtain
\[
\|u_s(t)\|_{X_s} \leq R_3 := R_2, \ \forall \ t \geq T_2 + \tau.
\]

**Case 4.** If $\|\nabla u_s(t)\|_{p_{s}()} \leq 1$ and $\|u_s(t)\|_{p_{s}()} \leq 1$, by Lemma 2.1 we have
\[
\|\nabla u_s(t)\|_{p_{s}()}^{a} \leq \|\nabla u_s(t)\|_{p_{s}()}^{p_s} \leq \rho_s(\nabla u_s(t)) \leq \|\nabla u_s(t)\|_{p_{s}()}^{p_s},
\]
and
\[
\|u_s(t)\|_{p_{s}()}^{a} \leq \|u_s(t)\|_{p_{s}()}^{p_s} \leq \rho_s(u_s(t)) \leq \|u_s(t)\|_{p_{s}()}^{p_s}.
\]
Using (18) we obtain
\[
\|u_s(t)\|_{X_s} = \|\nabla u_s(t)\|_{p_{s}()} + \|u_s(t)\|_{p_{s}()} \\
\leq [\rho_s(\nabla u_s(t))]^{1/a} + [\rho_s(u_s(t))]^{1/a} \\
\leq 2 \left[ \frac{a}{\min\{1, \alpha\}} \tilde{r}_1 \right]^{1/a}.
\]
Then,
\[
\|u_s(t)\|_{X_s} \leq 2 \left[ \frac{a}{\min\{1, \alpha\}} \right]^{1/a} =: R_4, \ \forall \ t \geq T_2 + \tau.
\]
Considering,
\[ r_1 := \max \left\{ 1, 2 \left[ \left( \frac{a}{\min\{1, \alpha\}} \hat{r}_1 \right)^{1/p} + \left( \frac{a}{\min\{1, \alpha\}} \hat{r}_1 \right)^{1/a} \right] \right\}. \]
Therefore,
\[ \| u_s(t) \|_{X_s} \leq r_1, \quad \forall \ t \geq T_2 + \tau. \]

4. Existence of the pullback attractor.

**Definition 4.1.** An evolution process in a metric space \( X \) is a family \( \{S(t, \tau) : X \to X\}_{t \geq \tau} \) of continuous maps satisfying:

(i) \( S(\tau, \tau) = I \) (here \( I \) denotes the identity operator);

(ii) \( S(t, \tau) = S(t, s)S(s, \tau), \tau \leq s \leq t \).

**Definition 4.2.** A global solution of a process \( S(\cdot, \cdot) \) is a function \( \xi : \mathbb{R} \to X \) such that \( S(t, s)\xi(s) = \xi(t) \) for all \( t \geq s \). A global solution \( \xi \) is backwards-bounded if there is a \( \tau \in \mathbb{R} \) such that \( \{\xi(t) : t \leq \tau\} \) is a bounded subset of \( X \).

**Definition 4.3.** Let \( \{S(t, \tau)\}_{t \geq \tau} \) be an evolution process in a metric space \( X \).

Given \( A \) and \( B \) subsets of \( X \), we say that \( A \) pullback attracts \( B \) at time \( t \) if
\[ \lim_{\tau \to -\infty} \text{dist}_H (S(t, \tau)B, A) = 0, \]
where \( \text{dist}_H \) denote the Hausdorff semi-distance.

**Definition 4.4.** A family of subsets \( \{A(t) : t \in \mathbb{R}\} \) of \( X \) is invariant relatively to the evolution process \( \{S(t, \tau)\}_{t \geq \tau} \) if
\[ S(t, \tau)A(\tau) = A(t), \]
for any \( t \geq \tau \).

**Definition 4.5.** A family of subsets \( \{A(t) : t \in \mathbb{R}\} \) of \( X \) is a pullback attractor for the evolution process \( \{S(t, \tau)\}_{t \geq \tau} \) if it is invariant, \( A(t) \) is compact for each \( t \in \mathbb{R} \), pullback attracts all bounded subsets of \( X \) at time \( t \) for each \( t \in \mathbb{R} \) and it is the minimal among all closed families which pullback attracts bounded sets of \( X \).

**Theorem 4.6.** Let \( \{S(t, \tau)\}_{t \geq \tau} \) be an evolution process in a complete metric space \( X \). The following statements are equivalent:

(i) There exists a family \( \{K(t) : t \in \mathbb{R}\} \) of compact subsets of \( X \) that pullback attracts bounded subsets of \( X \) at time \( t \);

(ii) The process \( \{S(t, \tau)\}_{t \geq \tau} \) has a pullback attractor.

**Theorem 4.7.** The evolution process \( \{S_s(t, \tau)\}_{t \geq \tau} \) associated with problem (1) has a pullback attractor \( \mathcal{U}_s = \{A_s(t) : t \in \mathbb{R}\} \).

**Proof.** Note that Lemma 3.2 shows that the family \( \{K_s(t) : t \in \mathbb{R}\} \), with \( K_s(t) := B_{X_s}(0, r_1)^H \), of compact sets of \( H \) pullback attracts bounded sets of \( H \) at time \( t \). Hence, by Theorem 4.6 the evolution process \( \{S_s(t, \tau)\}_{t \geq \tau} \) of problem (1) has a pullback attractor \( \mathcal{U}_s = \{A_s(t) : t \in \mathbb{R}\} \).

The constant \( r_0 \) in the Lemma 3.1 depend neither on the initial data nor on \( s \). Using the invariance of the pullback attractor we obtain the following

**Corollary 1.** There exists a bounded set \( B_0 \) in \( H \) such that, for each \( t \in \mathbb{R} \), \( A_s(t) \subseteq B_0 \) for all \( s \in \mathbb{N} \).
As a consequence of Lemma 3.2, we have

**Corollary 2.** a) Let $u_s$ be a solution of problem (1). Given $T_2 > 0$ there exists a positive constant $r_2$, independent of $s$, such that

$$\|u_s(t)\|_Y \leq r_2,$$

for all $t \geq T_2 + \tau$ and $s \in \mathbb{N}$.

b) There exist bounded sets $B_1$ in $Y$ and $B_1^s$ in $X_s$ such that, for each $t \in \mathbb{R}$, $A_s(t) \subset B_1^s$ and $A_s(t) \subset B_1$ for all $s \in \mathbb{N}$.

c) $C(t) := \bigcup_{s \in \mathbb{N}} A_s(t)$ is a compact subset of $H$ for each $t \in \mathbb{R}$.

**Proof.** a) By Lemma 3.2 there exists $r_1 > 0$ such that

$$\|u_s(t)\|_{X_s} \leq r_1 \quad \forall \ t \geq T_2 + \tau, \ s \in \mathbb{N}.$$

Thus,

$$\|u_s(t)\|_Y = \|\nabla u_s(t)\|_p + \|u_s(t)\|_p \leq 2(|\Omega| + 1) \left( \|\nabla u_s(t)\|_{p_s(t)} + \|u_s(t)\|_{p_s(t)} \right)$$

$$= 2(|\Omega| + 1)\|u_s(t)\|_{X_s} \leq 2(|\Omega| + 1)r_1$$

for all $t \geq T_2 + \tau$ and $s \in \mathbb{N}$. This result follows with $r_2 := 2(|\Omega| + 1)r_1$.

b) It follows from Lemma 3.2 and the previous item a).

c) By b) there exists a bounded set $B_1$ in $Y$ such that $A_s \subset B_1$ for all $s \in \mathbb{N}$. Since $Y \subset H$ with continuous and compact embedding, the result is proved. □

**5. The limit problem and convergence properties.** Our objective in this section is to prove that the limit problem of problem (1) as $D_s$ increases to infinity and $p_s(t) \to p > 2$ in $L^\infty(\Omega)$ as $s \to \infty$ is described by a non-autonomous ordinary differential equation. Firstly we observe that the gradient of the solutions $u_s$ of problem (1) converge in norm to zero as $s \to \infty$, which allows us to guess the limit problem

$$\left\{ \begin{array}{l} \frac{d}{dt}(u(t) + C(t)|u(t)|^{p-2}u(t)) = \tilde{B}(u(t)), \quad t > \tau, \\ u(\tau) = u_\tau \in \mathbb{R}, \end{array} \right. \quad (19)$$

with $\tilde{B} := B_{||}$ if we identify $\mathbb{R}$ with the constant functions which are in $H$, since $\Omega$ is a bounded set.

The proof of the next result is analogous as in [18] and we will not present much details in the proof here since the non-autonomous term $C(t)$ did not presented difficulties for this result.

**Theorem 5.1.** Given $T_1 > 0$, if for each $s$, $u_s$ is a solution of (1) in $(0, \infty)$, then for each $t \geq T_1$, the sequence of real numbers $\{\|\nabla u_s(t)\|_H\}$ has a subsequence $\{\|\nabla u_{s_\ell}(t)\|_H\}$ which converges to zero as $\ell \to \infty$.

**Proof.** Let $T > T_1$ and $t \in (T_1, T)$. As $u_s$ is a solution of (1) we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_s(t)\|_H^2 + D_s \int_\Omega |\nabla u_s(t)|^{p_s(x)}dx + C(t) \int_\Omega |u_s(t)|^{p_s(x)}dx = (B(u_s(t)), u_s(t))_H$$

$$\leq L\|u_s(t)\|_H^2 + \|B(0)\|_H \|u_s(t)\|_H \leq K,$$

t-a.e. in $(T_1, T)$, where $K = Lr_0^2 + \|B(0)\|Hr_0$ and $r_0 > 0$ is the constant, which is independent of $s$, of Lemma 3.1. Thus, since $C(t) \int_\Omega |u_s(t)|^{p_s(x)}dx \geq 0$, we have

$$\frac{1}{2} \frac{d}{dt} \|u_s(t)\|_H^2 + D_s \int_\Omega |\nabla u_s(t)|^{p_s(x)}dx \leq K, \quad (20)$$
of using the following abstract result of the Barbu’s book [2] for a Banach Space

Remark 1. In the proof of the previous theorem we only need $C(\cdot)$ measurable and $\alpha \leq C(t) \leq M$. The constants $\alpha$ and $M$ are taken uniform on $\tau$ and $T$ in order to have global solutions.
The next result guarantees that (19) is in fact the limit problem for (1), as \( s \to \infty \). The proof is analogous to what was done in [18] and we will not present all the details in the proof here since the non-autonomous term \( C(t) \) did not present difficulties for this result.

**Theorem 5.3.** Let \( u_s \) be a solution of (1) with \( u_s(\tau) = u_{\tau s} \) and let \( u \) be a solution of (19) with \( u(\tau) = u_{\tau} \). If \( u_{\tau s} \to u_{\tau} \) in \( H \) as \( s \to \infty \), then for each \( T > \tau \), \( u_s \to u \) in \( C([\tau,T];H) \) as \( s \to +\infty \).

**Proof.** Let \( T > \tau \) be fixed and suppose that \( u_{\tau s} \to u_{\tau} \) in \( H \) as \( s \to \infty \). Subtracting the two equations in (1) and (19) and taking the inner product with \( u \) we obtain

\[
\left\langle \frac{du_s}{dt}(t) - \frac{du}{dt}(t), u_s(t) - u(t) \right\rangle_H + \left\langle A^s u_s(t) - C(t) |u(t)|^{p-2} u(t), u_s(t) - u(t) \right\rangle_H
= \left\langle B(u_s(t)) - B(u(t)), u_s(t) - u(t) \right\rangle_H.
\]

Using that \( B \) is globally Lipschitz and observing that \( u \) is independent of \( x \), we have

\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) - u(t) \|^2_H + D \int_{\Omega} |\nabla u_s(t)|^{p_s(x)} \, dx
+ C(t) \int_{\Omega} |u_s(t)|^{p_s(x)-2} u_s(t)(u_s(t) - u(t)) \, dx
- C(t) \int_{\Omega} |u(t)|^{p-2} u(t)(u_s(t) - u(t)) \, dx
\leq L \| u_s(t) - u(t) \|^2_H.
\]

Then,

\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) - u(t) \|^2_H
\leq L \| u_s(t) - u(t) \|^2_H - C(t) \int_{\Omega} \left( |u(t)|^{p_s(x)-2} - |u(t)|^{p-2} \right) u(t)(u_s(t) - u(t)) \, dx
\leq L \| u_s(t) - u(t) \|^2_H + M \int_{\Omega} \left| |u(t)|^{p_s(x)-1} - |u(t)|^{p-1} \right| |u_s(t) - u(t)| \, dx,
\]

a.e. in \((\tau, T)\).

Now, with a completely analogous procedure as in the proof of Theorem 4.2 in [18] we obtain that there exists a positive constant \( \kappa \), which do not depend on \( s \) neither on \( t \), such that

\[
\left| |u(t)|^{p_s(x)-1} - |u(t)|^{p-1} \right| \leq \kappa |p_s(x) - p|,
\]

for all \( t \in [\tau, T] \). Thus,

\[
\frac{1}{2} \frac{d}{dt} \| u_s(t) - u(t) \|^2_H \leq L \| u_s(t) - u(t) \|^2_H + M \kappa \| p_s - p \|_{L^\infty(\Omega)} \int_{\Omega} |u_s(t) - u(t)| \, dx
\leq L \| u_s(t) - u(t) \|^2_H
+ M \kappa \| p_s - p \|_{L^\infty(\Omega)} \left[ \frac{1}{2} |\Omega| + \frac{1}{2} \| u_s(t) - u(t) \|^2_H \right],
\]

a.e. in \((0, T)\).
Integrating from $\tau$ to $t$, $t \leq T$, we obtain
\[
\|u_s(t) - u(t)\|_H^2 \leq \|u_{s\tau} - u_r\|_H^2 + M\kappa\|p_s - p\|_{L^\infty(\Omega)}|t - \tau|T \\
+ \int_0^T (2L + M\kappa\|p_s - p\|_{L^\infty(\Omega)})\|u_s(\tau) - u(\tau)\|_H^2 d\tau.
\]
So, by Gronwall-Bellman’s Lemma we obtain
\[
\|u_s(t) - u(t)\|_H^2 \leq \left(\|u_{s\tau} - u_r\|_H^2 + M\kappa\|p_s - p\|_{L^\infty(\Omega)}\right) e^{(2L + M\kappa\|p_s - p\|_{L^\infty(\Omega)})T},
\]
for all $t \in [\tau, T]$. Therefore, $u_s \to u$ in $C([\tau, T]; H)$ as $s \to +\infty$.

6. **Upper semicontinuity of the family of pullback attractors.** We start this section proving the existence of the pullback attractor for the limit problem.

**Theorem 6.1.** The problem (19) has a pullback attractor $\mathcal{U}_\infty = \{A_\infty(t); \ t \in \mathbb{R}\}$.

**Proof.** Multiplying the equation $\dot{u} + C(t)|u|^{p-2}u = \bar{B}(u)$ by $u$ and using the Young’s Inequality we obtain
\[
\frac{1}{2} \frac{d}{dt} |u(t)|^2 \leq -\frac{\alpha}{2} |u(t)|^p + c, \ t \geq \tau
\]
where $c > 0$ is a constant. Therefore, the map $y(t) = |u(t)|^2$ satisfies the inequality
\[
\frac{d}{dt} y(t) \leq -\alpha(y(t))^{p/2} + 2c, \ t \geq \tau.
\]
So, by Lemma 5.1 in [19],
\[
|u(t)|^2 \leq \left(\frac{2c}{\alpha}\right)^{2/p} + \left(\alpha \left(\frac{p}{2} - 1\right)(t - \tau)\right)^{-\frac{2}{p-2}}, \ \forall \ t \geq \tau.
\]
Let $\xi_0 > 0$ such that $\left(\alpha \left(\frac{p}{2} - 1\right)\xi_0\right)^{-\frac{2}{p-2}} \leq 1$, then
\[
|u(t)|^2 \leq \left(\frac{2c}{\alpha}\right)^{2/p} + 1\right)^{1/2} =: \kappa, \ \forall \ t \geq \xi_0 + \tau.
\]
Thus, the family $K(t) = \bar{B}(0, \kappa)$ of compact sets of $\mathbb{R}$ pullback attracts bounded sets of $\mathbb{R}$ at time $t$. Consequently, we have by Theorem 4.6 that the evolution process $\{S_{\infty}(t, s)\}_{t \geq s}$ of the limit problem (19) has a pullback attractor $\mathcal{U}_\infty = \{A_\infty(t); \ t \in \mathbb{R}\}$.

The next lemma shows that the relevant elements to describe the asymptotic behavior of these problems are around their own spatial average if $s$ is large enough.

**Lemma 6.2.** Let $t \in \mathbb{R}$ be fixed. If for each $s \in \mathbb{N}$, $w_s \in A_s(t)$ and $w_0 = \lim_{s \to \infty} w_s$ in $H$, then $w_0$ is a constant function.

**Proof.** Using for each fixed $s \in \mathbb{N}$, the invariance of the pullback attractors we have $A_s(t) = S_s(t, \tau)A_s(\tau)$ for all $\tau < t$. Considering $\tau$ such that $t - \tau > T_2$ we have from Lemma 3.2 that $A_s(t) \subset W^{1, p_s(\cdot)}(\Omega) \subset W^{1, p}(\Omega) \subset L^2(\Omega) = H$. By Poincaré-Wirtinger inequality (see [3]) and Theorem 5.1, we obtain (up to a subsequence) $\|w_s - \overline{w_s}\|_H \to 0$ as $s \to +\infty$, where $\overline{w_s} := \frac{1}{|\Omega|} \int_\Omega w_s(x) dx$. Then, $\|w_0 - \overline{w_0}\|_H \to 0$ as $s \to +\infty$. Since $w_s \to w_0$ in $H$ implies $w_s \to w_0$ in $H$, we consider the characteristic function $\chi_{\Omega} \in H$ to obtain $\overline{w_s} \to \overline{w_0}$ in $\mathbb{R}$ and this implies that $\overline{w_s} \to \overline{w_0}$ in $H$. So, we conclude that $w_0 = \overline{w_0}$ and then $w_0$ is a constant function.
Now we present our main result. The novelty here is that it is necessary to construct a backwards-bounded solution for the evolution process whereas in the autonomous case it was necessary to construct a bounded complete trajectory for a semigroup.

**Theorem 6.3.** The family of pullback attractors \( \{ \mathcal{U}_s; s \in \mathbb{N} \} \) associated with problem (1) is upper semicontinuous on \( s \) at infinity, in the topology of \( H \), i.e., for each \( \tau \in \mathbb{R}, \lim_{s \to +\infty} \text{dist}(\mathcal{A}_s(\tau), \mathcal{A}_\infty(\tau)) = 0 \).

**Proof.** Let \( \tau \in \mathbb{R} \) be fixed. Consider an arbitrary sequence \( \{w_s\}_{s \in \mathbb{N}} \) with \( w_s \in \mathcal{A}_s(\tau) \) for each \( s \in \mathbb{N} \). In order to obtain \( \lim_{s \to +\infty} \text{dist}(\mathcal{A}_s(\tau), \mathcal{A}_\infty(\tau)) = 0 \) it is sufficient to show that \( \{w_s\}_{s \in \mathbb{N}} \) has a convergent subsequence whose limit belongs to \( \mathcal{A}_\infty(\tau) \) (see Lemma 3.2 in [4]). We know from Corollary 2 (c) that \( \bigcup_{s \in \mathbb{N}} \mathcal{A}_s(t) \) is a compact subset of \( H \), so there exists a subsequence \( \{w_{s_j}\}_{j \in \mathbb{N}} \) of \( \{w_s\}_{s \in \mathbb{N}} \) such that \( w_{s_j} \to w \) in \( H \) as \( j \to \infty \). From Lemma 6.2, \( w \in \mathbb{R} \).

Our goal is to show that \( w \in \mathcal{A}_\infty(\tau) \). By the characterization given in Theorem 1.17 in [4] it is sufficient to show that there exists a backwards-bounded global solution \( \xi(\cdot) \) with \( w = \xi(\tau) \). Let \( \{S_{s_j}(t, \tau)\}_{t \geq \tau} \) and \( \{S_\infty(t, \tau)\}_{t \geq \tau} \) be the evolution process associated with (1) and (19), respectively.

Consider \( \tau_j \searrow -\infty \). For simplicity, let us call \( \{S_j(t, \tau)\}_{t \geq \tau} = \{S_{s_j}(t, \tau)\}_{t \geq \tau} \), \( w_0 := w \) and \( w_j := w_{s_j} \) the subsequence of \( \{w_s\} \) such that \( w_j \to w_0 \in \mathbb{R} \) as \( j \to +\infty \). Without loss of generality, we can consider \( \tau_j < \tau, \forall j \in \mathbb{N} \). By the invariance of the pullback attractors, \( \mathcal{A}_j(\tau) = S_j(\tau, \tau_j)\mathcal{A}_j(\tau_j) \).

Thus, \( w_j = S_j(\tau, \tau_j)x_j \), for some \( x_j \in \mathcal{A}_j(\tau_j) \). From Theorem 5.3,
\[
S_j(t, \tau)x_j \to S_\infty(t, \tau)w_0, \forall t \geq \tau.
\]

Also, there exists \( x_{j,1} \in \mathcal{A}_j(\tau_j - 1) \) such that \( x_j = S_j(\tau_j, \tau_j - 1)x_{j,1} \). Now consider the subsequence
\[
\{S_j(\tau - 1, \tau_j - 1)x_{j,1}\}_{j \in \mathbb{N}} \subset \bigcup_{j \in \mathbb{N}} \mathcal{A}_j(\tau - 1) \subset \bigcup_{j \in \mathbb{N}} \mathcal{A}_j(\tau - 1) \in K(H),
\]
where \( K(H) := \{K \subset H : K \text{ is a not vanish compact subset} \} \). Then, up to a subsequence, we have
\[
S_j(\tau - 1, \tau_j - 1)x_{j,1} \to w_1 \text{ as } j \to +\infty.
\]

From Lemma 6.2, we have \( w_1 \in \mathbb{R} \). By Theorem 5.3,
\[
S_j(t, \tau - 1)S_j(\tau - 1, \tau_j - 1)x_{j,1} \to S_\infty(t, \tau - 1)w_1, \forall t \geq \tau - 1,
\]
as \( j \to \infty \).

**Statement 1:** \( S_\infty(\tau, \tau - 1)w_1 = w_0 \).

Indeed,
\[
S_\infty(\tau, \tau - 1)w_1 = \lim_{j \to +\infty} S_j(\tau, \tau - 1)S_j(\tau - 1, \tau_j - 1)x_{j,1}
\]
\[
= \lim_{j \to +\infty} S_j(\tau, \tau_j - 1)x_{j,1}
\]
\[
= \lim_{j \to +\infty} S_j(\tau, \tau_j)S_j(\tau_j, \tau_j - 1)x_{j,1}
\]
\[
= \lim_{j \to +\infty} S_j(\tau, \tau_j)x_j = \lim_{j \to +\infty} w_j = w_0.
\]
Again, by the invariance of the pullback attractors, there exists \( x_{j,2} \in A_j(\tau_j - 2) \) such that

\[
x_{j,1} = S_j(\tau_j - 1, \tau_j - 2) x_{j,2}
\]

Consider the sequence

\[
\{ S_j(\tau - 2, \tau_j - 2) x_{j,2} \}_{j \in \mathbb{N}} \subseteq \bigcup_{j \in \mathbb{N}} A_j(\tau - 2) \in K(H)
\]

Then, (up to a subsequence) we have

\[
S_j(\tau - 2, \tau_j - 2) x_{j,2} \to w_2 \in \mathbb{R}, \text{ as } j \to \infty.
\]

By Theorem 5.3

\[
S_j(t, \tau - 2) S_j(\tau - 2, \tau_j - 2) x_{j,2} \to S_\infty(t, \tau - 2) w_2, \quad \forall \ t \geq \tau - 2,
\]

as \( j \to \infty \).

**Statement 2:** \( w_1 = S_\infty(\tau - 1, \tau - 2) w_2 \).

Indeed,

\[
S_\infty(\tau - 1, \tau - 2) w_2 = \lim_{j \to +\infty} S_j(\tau - 1, \tau - 2) S_j(\tau - 2, \tau_j - 2) x_{j,2}
\]

\[
= \lim_{j \to +\infty} S_j(\tau - 1, \tau_j - 2) x_{j,2}
\]

\[
= \lim_{j \to +\infty} S_j(\tau - 1, \tau_j - 1) S_j(\tau_j - 1, \tau_j - 2) x_{j,2}
\]

\[
= \lim_{j \to +\infty} S_j(\tau - 1, \tau_j - 1) x_{j,1} = w_1.
\]

Proceeding inductively we define

\[
\xi(t) := \begin{cases} 
S_\infty(t, \tau) w_0, & t \geq \tau, \\
S_\infty(t, \tau - 1) w_1, & t \in [\tau - 1, \tau), \\
S_\infty(t, \tau - 2) w_2, & t \in [\tau - 2, \tau - 1), \\
& \vdots \\
S_\infty(t, \tau - r) w_r, & t \in [\tau - r, \tau - (r - 1)), \\
& \vdots
\end{cases}
\]

**Statement 3:** \( \xi \) is a global solution, i.e., \( S_\infty(t, \ell) \xi(\ell) = \xi(t) \), \( \forall t \geq \ell \).

Indeed, the case \( t = \ell \) is trivial. Consider a pair \( (t, \ell) \) arbitrarily fixed with \( t > \ell \).

**Case 1.** \( \ell < t \leq \tau \).

In this case, \( \ell \in [\tau - r, \tau - (r - 1)) \) for some \( r \in \mathbb{N}, r \geq 1 \) and \( t \in [\tau - \hat{r}, \tau - (\hat{r} - 1)) \) for some \( \hat{r} \in \mathbb{N} \) with \( 1 \leq \hat{r} \leq r \). Therefore,

\[
\xi(\ell) = S_\infty(\ell, \tau - r) w_r \text{ and } \xi(t) = S_\infty(t, \tau - \hat{r}) w_{\hat{r}}.
\]

So,

\[
S_\infty(t, \ell) \xi(\ell) = S_\infty(t, \ell) S_\infty(\ell, \tau - r) w_r
\]

\[
= S_\infty(t, \ell) S_\infty(\ell, \tau - \hat{r}) S_\infty(\tau - \hat{r}, \tau - r) w_r
\]

\[
= S_\infty(t, \tau - \hat{r}) w_{\hat{r}} = \xi(t).
\]

**Case 2.** \( t > \ell \geq \tau \).

In this case, \( \xi(\ell) = S_\infty(\ell, \tau) w_0 \). Thus,

\[
S_\infty(t, \ell) \xi(\ell) = S_\infty(t, \ell) S_\infty(\ell, \tau) w_0 = S_\infty(t, \tau) w_0 = \xi(t).
\]

**Case 3.** \( \ell < \tau \leq \ell \).
In this case, \( \ell \in [\tau-r, \tau-(r-1)] \) for some natural \( r \geq 1 \) and \( \xi(\ell) = S_\infty(\ell, \tau-r)w_r \).
Here, we have \( \xi(t) = S_\infty(t, \tau)w_0 \). So,
\[
S_\infty(t, \ell)\xi(\ell) = S_\infty(t, \ell)S_\infty(\ell, \tau-r)w_r = S_\infty(t, \tau-r)w_r = S_\infty(t, \tau)S_\infty(\tau, \tau-r)w_r = S_\infty(t, \tau)w_0 = \xi(t).
\]
Therefore, \( \xi \) is a global solution.
Let us show now that \( \xi \) is bounded (in particular backwards-bounded): First, note that for each \( t \in \mathbb{R} \), \( \xi(t) = S_\infty(t, \tau-r)w_r \) for some \( r \in \{0, 1, 2, 3, \ldots \} \) and \( S_\infty(t, \tau-r)w_r = \lim_{j \to \infty} S_j(t, \tau_j - r)(x_j, r) \) and \( S_j(t, \tau_j - r)(x_j, r) \in A_j(t), \forall j \in \mathbb{N} \).
Using Lemma 6.2, we conclude that each term \( S_\infty(t, \tau-r)w_r \) is independent of \( x \). Consequently, \( \xi(t) \) is a constant function on the spatial variable \( x \). Since for each \( t \in \mathbb{R} \) and \( j \in \mathbb{N} \),
\[
A_j(t) \subset \bigcup_{s \in \mathbb{N}} A_s(t)
\]
and the last one is bounded (because is relatively compact) there is a constant \( C > 0 \) such that
\[
\|S_\infty(t, \tau-r)w_r\|_H \leq C, \forall t \in \mathbb{R} \ (r = 0, 1, 2, \cdots).
\]
Thus, in particular, we have that \( \xi(\cdot) \) is bounded in \( H \). Then, there is a constant \( C > 0 \) such that
\[
|\xi(t)| = \frac{1}{|\Omega|^{1/2}}\|\xi(t)\|_H \leq \tilde{C}, \forall t \in \mathbb{R}.
\]
So, we conclude that \( \xi : \mathbb{R} \to \mathbb{R} \) is a bounded global solution with \( w = w_0 = \xi(\tau) \).

7. Some numerical simulation for the limit problem. In this section we made some numerical experiments with specific functions \( C(\cdot) \) and \( B(\cdot) \). These simulations show some information about the behavior of the solutions of the ordinary differential equation (19) which commands the pullback asymptotic dynamics of the family of partial differential equations (1) as \( s \to +\infty \). By fixing the value of the constant \( p = 2 \) there is no great loss of information as we will see below (compare figures 4 and 5). We consider as examples the ordinary differential equation
\[ x' + C(t) |x|^{p-2} x = x. \]
In our first example the theoretical hypotheses on \( C(\cdot) \) will be satisfied with \( \alpha = 0.1 \) and \( M = 1.1 \). Considering \( p = 4 \) and a monotonically nonincreasing function
\[
C(t) = \begin{cases} 
1.1, & t < 0 \\
e^{-t} + 0.1, & 0 \leq t 
\end{cases}
\]
we have the following explicit solution
\[
x(t) = \begin{cases} 
\pm \frac{10}{\sqrt{110 + 100K e^{-t}}}, & t < 0 \\
\pm \frac{10}{\sqrt{10 - 100e^{-t} + 100Ke^{-t}}}, & 0 \leq t 
\end{cases}
\]
where \( K \) is a constant to be determined by the initial conditions as well the sign \( \pm \) to be used. These solutions are depicted in the next picture.
For the existence of a strong solution of the ODE limit problem the coefficient $C(\cdot)$ does not need to be nonincreasing (see Remark 1). So, we also present below some examples with $C(\cdot)$ not monotone, only nonincreasing in some intervals. In all the next examples we have $0.1 \leq C(t) \leq 2.1$.

Considering $p = 4$ and $C(t) = e^{-t^2} + 0.1$ we have the following source type solution,
\[
x(t) = \pm \frac{10}{\sqrt{10 + 100} \sqrt{\pi} \text{ERF}(t - 1)e^{-2t + 1} + 100Ke^{-2t}}
\]
where
\[
\text{ERF}(t - 1) = \frac{2}{\sqrt{\pi}} \int_0^{t-1} e^{-s^2} ds
\]

is the error function. $K$ is a constant to be determined by the initial conditions as well the sign $\pm$. These solutions are depicted in the following figure.

With $p = 4$ and
\[
C(t) = \cos(t) + 1.1
\]
we have the following explicit solution,
\[
x(t) = \pm \frac{10}{\sqrt{110 + 80 \cos(t) + 40 \sin(t)} + 100Ke^{-2t}}
\]
where $K$ is a constant to be determined by the initial conditions as well the sign $\pm$. These solutions are depicted in the following figure.
With $p = 4$ and
\[ C(t) = \sin(t) + 1.1 \]
we have the following explicit solution,
\[ x(t) = \pm \frac{10}{\sqrt{110 - 40 \cos(t) + 80 \sin(t) + 100K e^{-2t}}} \]
where $K$ is a constant to be determined by the initial conditions as well the sign $\pm$.

These solutions are depicted in the following figure.

With $p = 7$ and
\[ C(t) = \sin(t) + 1.1 \]
we have the following explicit solution,
\[ x(t) = \pm \frac{\sqrt{130}}{\sqrt{-25 \cos(t) + 125 \sin(t) + 143 + 130K e^{-5t}}} \]
where $K$ is a constant to be determined by the initial conditions as well the sign $\pm$.

These solutions are depicted in the following figure.
Interpretation of the figures: In all the examples the solutions seems to approach to two orbits, more precisely, there are two orbits $x_*(\cdot), y^*(\cdot)$ such that:

1. For each $t \in \mathbb{R}$, $x_*(t), y^*(t) \in \mathcal{A}_\infty(t)$, $x_*(t) \leq y^*(t)$ and
   
   $$\mathcal{A}_\infty(t) \subseteq [x_*(t), y^*(t)] := \{ r \in \mathbb{R}; x_*(t) \leq r \leq y^*(t) \}.$$ 

2. $x_*(\cdot)$ is pullback attracting from below, that is, we have that
   
   $$\lim_{\tau \to -\infty} S_\infty(t, \tau)v_\tau = x_*(t)$$
   
   whenever $v_\tau \in \mathbb{R}$ with $v_\tau \leq x_*(\tau)$.

3. $y^*(\cdot)$ is pullback attracting from above, that is, we have that
   
   $$\lim_{\tau \to -\infty} S_\infty(t, \tau)v_\tau = y^*(t)$$
   
   whenever $v_\tau \in \mathbb{R}$ with $v_\tau \geq y^*(\tau)$.

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