Weighted Finite Fourier Transform Operator: Uniform Approximations of the Eigenfunctions, Eigenvalues Decay and Behaviour

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Abstract In this paper, we investigate two uniform asymptotic approximations as well as some spectral properties of the eigenfunctions of the weighted finite Fourier transform operator, defined by $\mathcal{F}_c^{(\alpha)} f(x) = \int_{-1}^{1} e^{icxy} f(y) (1 - y^2)^{\alpha} dy$. Here, $c > 0$, $\alpha \geq -1/2$ are two fixed real numbers. These eigenfunctions are called generalized prolate spheroidal wave functions (GPSWFs) and they are firstly introduced and studied in Wang and Zhang (Appl Comput Harmon Anal 29(3):303–329, 2010). The present study is motivated by the promising concrete applications of the GPSWFs in various scientific area such as numerical analysis, mathematical physics and signal processing. We should mention that these two uniform approximation results of the GPSWFs can be considered as generalizations of the results given in the joint work of one of us (Bonami and Karoui in Constr Approx 43(1):15–45, 2016). As it will be seen, these generalizations require some involved extra work, especially in the case where $\alpha > 1/2$. By using the uniform asymptotic approximations of the GPSWFs, we prove the super-exponential decay rate of the eigenvalues of the operator $\mathcal{F}_c^{(\alpha)}$ in the case where $0 < \alpha < 3/2$. Moreover, by computing the trace and an estimate of the norm of the operator $Q_c^{(\alpha)} = \frac{c}{2\pi} \mathcal{F}_c^{(\alpha)^*} \circ \mathcal{F}_c^{(\alpha)}$, we give a lower bound for the counting number of the eigenvalues of $Q_c^{(\alpha)}$, when $c >> 1$. Finally, we provide the reader with some numerical examples that illustrate the different results of this work.

Keywords Sturm–Liouville operators · Weighted finite Fourier transform · Asymptotic approximations of eigenvalues and eigenfunctions · Generalized prolate spheroidal wave functions

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1 Introduction

In the early 1960’s, D. Slepian and his co-authors H. Landau and H. Pollack, have greatly contributed in developing the theory of prolate spheroidal wave functions (PSWFs), see their pioneer work [24–26]. For $c > 0$, a positive real number, called bandwidth, the PSWFs, denoted by $(\psi_{n,c})_{n \geq 0}$ are the eigenfunctions of the finite Fourier transform operator $F_c$, as well as the Sinc kernel convolution operator $Q_c$, defined on $L^2([-1, 1])$ by

$$F_c f(x) = \int_{-1}^{1} e^{ixy} f(y) \, dy, \quad Q_c(f)(x) = \frac{2c}{\pi} F_c^* \circ F_c.$$ Perhaps the starting point of the theory of PSWFs is the Slepian’s result concerning the commutativity property of the integral operators $F$ and $Q_c$ with the following perturbed Legendre differential operator

$$L_c y(x) = -(1 - x^2) y''(x) + 2xy'(x) + c^2 x^2 y(x).$$

Since $L_c F_c = F_c L_c$, then the PSWFs are also, the bounded eigenfunctions over $I = [-1, 1]$ of the Sturm–Liouville operator $L_c$. Many desirable properties, computational schemes, asymptotic results and expansions of the PSWFs are consequences of the previous commutativity property and the rich literature from the theory of Sturm–Liouville operators.

We should mention that the PSWFs have found potential applications in various scientific area such as applied mathematics, mathematical physics, random matrices, signal processing, etc., see for example [13] for a comprehensive review of the theory and some applications of the PSWFs. Also note that recently, there is a growing interest in the theory, construction and applications of more general as well as multi-dimensional versions of the PSWFs, see for example [1,8,12,15,19,23,28,30]. It is important to note that most of the PSWFs applications, rely on the decay rate and on the behaviour of the eigenvalues of the integral operator $F_c$ or of $Q_c$, as well as the bounds and the local estimates of the PSWFs. Consequently, there is an interest in the spectral analysis of a more general compact integral operator, the weighted finite Fourier transform operator $F_c^{(\alpha)}$, defined by

$$F_c^{(\alpha)} f(x) = \int_{-1}^{1} e^{ixy} f(y) \omega_{\alpha}(y) \, dy, \quad \omega_{\alpha}(y) = (1 - y^2)^{\alpha}, \quad \alpha > -1. \quad (1)$$

In [28], it has been shown that the operator $Q_c^{(\alpha)} = \frac{c}{2\pi} F_c^{(\alpha)*} \circ F_c^{(\alpha)}$ is defined on $L^2(I, \omega_{\alpha})$ by

$$Q_c^{(\alpha)} g(x) = \int_{-1}^{1} \frac{c}{2\pi} K_{\alpha}(c(x-y)) g(y) \omega_{\alpha}(y) \, dy, \quad K_{\alpha}(x) = \sqrt{\pi} 2^{\alpha+1/2} \frac{\Gamma(\alpha+1) J_{\alpha+1/2}(x)}{x^{\alpha+1/2}}, \quad (2)$$

where $\Gamma(\cdot)$ and $J_{\alpha}(\cdot)$ denote the Gamma and the Bessel function, respectively. Moreover, the eigenvalues $\mu_n^{(\alpha)}(c)$ and $\lambda_n^{(\alpha)}(c)$ of $F_c^{(\alpha)}$ and $Q_c^{(\alpha)}$ are related to each others by the identity $\lambda_n^{(\alpha)}(c) = \frac{c}{2\pi} |\mu_n^{(\alpha)}(c)|^2$. Moreover, both operators commute with the following Jacobi-type Sturm–Liouville operator $L_c^{(\alpha)}$, defined by

$$L_c^{(\alpha)}(f)(x) = -\frac{d}{dx} \left[ \omega_{\alpha}(x) (1 - x^2) f'(x) \right] + c^2 x^2 \omega_{\alpha}(x) f(x).$$
The infinite countable set of the eigenfunctions of $\mathcal{F}_c^{(\alpha)}$, $Q_c^{(\alpha)}$ and $L_c^{(\alpha)}$ will be denoted by $(\psi_{n,c}^{(\alpha)})_{n \geq 0}$. They are called generalized prolate spheroidal wave functions (GPSWFs). Some properties as well as a first set of local estimates and bounds of the GPSWFs have been given in [14]. Moreover, it has been shown in [6] that in the special case where $\alpha = 0$, the eigenvalues $\lambda_n(c) = \lambda_n^{(0)}(c)$ decay asymptotically faster than $e^{-2n \log \left( \frac{1}{c} \right)}$ for any positive real number $0 < c < \frac{1}{e}$. In [28] and for a more general value of $\alpha > -1$, the authors have checked that the sequence of the eigenvalues $\lambda_n^{(\alpha)}(c)$ have an asymptotic decay rate similar to the sequence $e^{-(2n+1)\log \left( \frac{4n+4\alpha+2}{c(e - 1)} \right)}$. Nonetheless, this result is obtained by using some heuristic results concerning the behaviour and the decay of the coefficients of the Gegenbauer’s series expansion of $\psi_{n,c}^{(\alpha)}$. In this work, we give a proof of the previous super-exponential decay rate of the $\lambda_n^{(\alpha)}(c)$ in case where $0 < \alpha < 3/2$. This proof is based on two uniform asymptotic approximations of the $\psi_{n,c}^{(\alpha)}$. The first one is given in terms of the Bessel function $J_\alpha(\cdot)$ and the second one is given in terms of the normalized Jacobi polynomial $\widetilde{P}_n^{(\alpha,\alpha)}$. Note that the $\psi_{n,c}^{(\alpha)}$ and $\widetilde{P}_n^{(\alpha,\alpha)}$ are normalized by the following rules

$$
\int_{-1}^{1} (\widetilde{P}_n^{(\alpha,\alpha)}(x))^2 \omega_\alpha(x) \, dx = 1, \quad \int_{-1}^{1} (\psi_{n,c}^{(\alpha)}(x))^2 \omega_\alpha(x) \, dx = 1. \tag{3}
$$

We should mention that these uniform approximation results are generalizations of similar results obtained in [7] for the special case $\alpha = 0$. In the proofs of the present general results, the emphasis will be on the steps where a non trivial and an involved extra work has to be done in the case where $\alpha \neq 0$. Moreover, we will only give a fairly brief description of a part of the proof of the general case, whenever this part can be simply adapted from the special case $\alpha = 0$.

Under the above normalization of $\psi_{n,c}^{(\alpha)}$, we show that for any positive integer $n$ with $q = \frac{c^2}{\chi_{n,0}} < 1$, we have

$$
\psi_{n,c}^{(\alpha)} \approx \frac{\pi}{\sqrt{2K(\sqrt{q})}} \frac{(\chi_{n,\alpha})^{1/4}S(x)J_\alpha(\sqrt{\chi_{n,\alpha}}S(x))}{(1 - x^2)^{1/4 + \alpha/2}(1 - qx^2)^{1/4}}, \quad x \in [0, 1].
$$

Here, $S(x) = \int_{x}^{1} \frac{1 - qt^2}{1 - t^2} \, dt$, $K(r) = \int_{0}^{1} \sqrt{\frac{1 - t^2}{1 - r^2(1 - r^2t^2)}} \, dt$, $0 \leq r < 1$. Also, by using some properties and estimates of the Jacobi polynomials and Jacobi functions of the second kind, we prove that for sufficiently large value of $n$, we have the following uniform approximation of the $\psi_{n,c}^{(\alpha)}$ in terms of the Gegenbauer’s polynomial,

$$
\left| \psi_{n,c}^{(\alpha)}(x) - A_n \widetilde{P}_n^{(\alpha,\alpha)}(x) \right| \leq C_\alpha \frac{c^2}{n + 2\alpha + 1}, \quad \forall \ x \in [-1, 1],
$$

where $A_n$ is the normalization constant, satisfying $|1 - A_n| \leq C_\alpha \frac{c^2}{2n + 2\alpha + 1}$ and $C_\alpha$ is a constant depending only on $\alpha$. Also, we show that if $0 < \delta < 1$ and if $M_\alpha(\delta)$ is the number of eigenvalues of $Q_c^{(\alpha)}$, $\alpha > 0$, which are not smaller than $\delta$, then

$$
M_\alpha(\delta) \geq \frac{1}{1 - \delta} \frac{c}{2\pi} (2^{2\alpha + 1}B(\alpha + 1, \alpha + 1))^2 (\gamma_\alpha - \delta) + o(c).
$$

Here, $\gamma_\alpha = 2^{4\alpha} \left( \frac{B(2\alpha + 1, 2\alpha + 1)}{B(\alpha + 1, \alpha + 1)} \right)$ and $B(\cdot, \cdot)$ is the beta function.
This work is organized as follows. In Sect. 2, we study a uniform asymptotic approximation of the \( \psi_{n,c}^{(\alpha)} \) in terms of the Bessel function \( J_{\alpha} \). This approximation result is based on the use of the WKB and Olver’s methods, together with some properties of Bessel functions. As a consequence of this uniform asymptotic approximation, we get a refined bound and local estimate of the GPSWFs. In Sect. 3, we first list some properties and estimates of the Jacobi polynomials and Jacobi functions of the second kind. Then by using these results, we prove the asymptotic uniform approximation of the \( \psi_{n,c}^{(\alpha)} \) in terms of the Jacobi polynomials \( \psi_{n,c}^{(\alpha)} \).

In Sect. 4, we first use the result of the previous section and prove the super-exponential decay rate of the \( \lambda_{n}^{(\alpha)}(c) \), for \( 0 < \alpha < 3/2 \). Then, by using the trace and an estimate of the norm of the integral operator \( Q_{c}^{(\alpha)} \), we give an asymptotic lower bound for the counting number of the eigenvalues \( \lambda_{n}^{(\alpha)}(c) \). Finally, in Sect. 5, we provide the reader with some numerical examples that illustrate the different results of this work.

2 Uniform Approximation of the Eigenfunctions in Terms of Bessel Functions

Let \( w_{\alpha}(x) = (1 - x^{2})^{\alpha} \) and recall that the GPSWFs are also the bounded eigenfunctions over \( I = [-1, 1] \) of the following differential equation,

\[
(1 - x^{2})\psi''(x) - 2(\alpha + 1)x\psi'(x) + (\chi_{n,\alpha} - c^{2}x^{2})\psi(x) = 0.
\]

(4)

Here, \( \chi_{n,\alpha} \) is the \( (n + 1) \)th eigenvalue of the following Sturm–Liouville differential operator

\[
L_{c}^{(\alpha)}(f)(x) = -\frac{d}{dx} \left[ w_{\alpha}(x)(1 - x^{2})f'(x) \right] + c^{2}x^{2}w_{\alpha}(x)f(x)
\]

Note that the eigenvalue \( \chi_{n,\alpha} \) satisfies the following classical inequalities,

\[
n(n + 2\alpha + 1) \leq \chi_{n,\alpha} \leq n(n + 2\alpha + 1) + c^{2}, \quad \forall \ n \geq 0.
\]

(5)

The previous differential equation is rewritten as

\[
- L_{c}^{(\alpha)}\psi(x) + w_{\alpha}(x)\chi_{n,\alpha}\psi(x) = (w_{\alpha}(x)\psi'(x)(1 - x^{2}))' + w_{\alpha}(x)(\chi_{n,\alpha} - c^{2}x^{2})\psi(x) = 0, \quad x \in [-1, 1].
\]

(6)

As it is done in [7], we use the Liouville transformation to transform this later equation into a Liouville normal form. More precisely, for a positive integer \( n \) with \( q = \frac{c^{2}}{\chi_{n,\alpha}} < 1 \), we consider the incomplete elliptic integral

\[
S(x) = \int_{x}^{1} \sqrt{\frac{1 - qt^{2}}{1 - t^{2}}} \, dt.
\]

(7)

It has been shown in [7], that

\[
\left(1 - \frac{q}{2}\right)\sqrt{(1 - x^{2})(1 - qx^{2})} \leq S(x) \leq \frac{5}{3} - q\sqrt{(1 - x^{2})(1 - qx^{2})}.
\]

(8)

Then, we write \( \psi_{n,c}^{(\alpha)} \) into the form

\[
\psi(x) = \phi_{\alpha}(x)V(S(x)), \quad \phi_{\alpha}(x) = (1 - x^{2})^{(-1-2\alpha)/4}(1 - qx^{2})^{-1/4}.
\]

(9)
By combining (6), (9) and using straightforward computations, it can be easily checked that $V(\cdot)$ satisfies the following second order differential equation

$$V''(s) + (\chi_{n,\alpha} + \theta_\alpha(s)) V(s) = 0, \quad s \in [0, S(0)]$$

with

$$\theta_\alpha(S(x)) = (w_\alpha(x)(1-x^2)\phi'_\alpha(x))' \frac{1}{\phi_\alpha(x)w_\alpha(x)(1-qx^2)}.$$ 

If $Q_\alpha(x) = w_\alpha(x)^2(1-x^2)(1-qx^2)$, then we have $\phi'_\alpha(x) = \phi_\alpha(x)Q_\alpha(x)$. Moreover, by using the same technique as the one used in [7] for the case $\alpha = 0$, one gets

$$\theta_\alpha(S(x)) = \theta_0(S(x)) + \frac{\alpha(1+\alpha)}{(1-qx^2)} - \frac{\alpha^2}{(1-qx^2)(1-x^2)}.$$ 

It has been shown in [7] that

$$\left| \theta_0(x) - \frac{1}{4S^2(x)} \right| \leq \frac{3+2q}{4(1-qx^2)^2}, \quad \left| \frac{1}{S^2(x)} - \frac{1}{(1-qx^2)(1-x^2)} \right| \leq \frac{3}{(1-qx^2)^2},$$

whenever $x \in (-1, 1)$.

Consequently, if $G_\alpha(\cdot)$ is the function given by

$$G_\alpha(x) = \frac{1/4 - \alpha^2}{S^2(x)} - \theta_\alpha(x), \quad x \in (-1, 1),$$

then, we have

$$|G_\alpha(x)| \leq \frac{3+2q + 12\alpha^2}{4(1-qx^2)^2} + \frac{\alpha(\alpha+1)}{1-qx^2}, \quad x \in (-1, 1).$$

As it is done in [7], by using the substitution $t = S(x)$, it can be easily checked that

$$g_{\alpha,q}(x) = \int_0^{S(x)} |G_\alpha(t)| \, dt \leq \frac{3+2q + 12\alpha^2}{4(1-q)} \left( x \sqrt{1-x^2} + S(x) \right) + \alpha(\alpha+1) K(x, \sqrt{q}), \quad x \in [0, 1),$$

where $K(x, \sqrt{q}) = \int_x^1 \frac{1}{\sqrt{(1-t^2)(1-qt^2)}} \, dt$. In particular, we have

$$\int_0^{S(0)} |G_\alpha(t)| \, dt \leq \frac{3+2q + 12\alpha^2}{4(1-q)} E(\sqrt{q}) + \alpha(\alpha+1) K(\sqrt{q}) = g_{\alpha,q}(0).$$

Here, $K(\cdot)$ and $E(\cdot)$ are the Legendre Elliptic integrals of the first and the second kind, given respectively by

$$K(r) = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-tr^2)}} \, dt, \quad E(r) = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}} \, dt, \quad r \in [0, 1).$$

We have just proved the following Lemma.
Lemma 1 Under the above notations, consider two real numbers \( c > 0, \alpha > -1 \) and let \( n \) be a positive integer so that \( q = \frac{c^2}{\chi_{n, \alpha}} < 1 \). If \( V(\cdot) \) is the function given by \((9)\), then it satisfies the differential equation

\[
V''(s) + \left( \chi_{n, \alpha} + \frac{1}{4} \frac{\alpha^2}{s^2} \right) V(s) = G_\alpha(s), \quad s \in (0, S(0)].
\]

where \( G_\alpha(\cdot) \) is given by \((11)\) and satisfying \((12)\) and \((13)\).

Next, to study the uniform approximation of the GPSWFs over the interval \([0, 1]\) and thus over \( I = [-1, 1] \), by symmetry, we use the following weight and modulus functions defined for any real \( \alpha > -1 \) as follows, see for example \([21, p. 437]\),

\[
E_\alpha(x) = \begin{cases} 
(-Y_\alpha(x)/J_\alpha(x))^{1/2} & \text{if } 0 < x \leq X_\alpha \\
1 & \text{if } x \geq X_\alpha,
\end{cases}
\]

\[
M_\alpha(x) = \begin{cases} 
(2|Y_\alpha(x)|J_\alpha(x))^{1/2} & \text{if } 0 < x \leq X_\alpha \\
(J_\alpha^2(x) + Y_\alpha^2(x))^{1/2} & \text{if } x \geq X_\alpha,
\end{cases}
\]

with \( X_\alpha \) is the first zero of \( J_\alpha(x) + Y_\alpha(x) \). Moreover, we will use the following well known asymptotic behaviours as \( x \to 0^+ \), of the Bessel functions \( J_\alpha(\cdot), Y_\alpha(\cdot) \), given by

\[
J_\alpha(x) \sim \frac{1}{\Gamma(\alpha + 1)} \left( \frac{x}{2} \right)^\alpha, \quad Y_\alpha(x) \sim -\frac{1}{\pi \Gamma(\alpha)} \left( \frac{x}{2} \right)^{-\alpha}, \quad x \to 0^+.
\]

Also, we will use the following formula for the Wronskian of \( J_\alpha, Y_\alpha \), see for example \([3, p. 201]\),

\[
W(J_\alpha, Y_\alpha)(x) = J_\alpha(x)Y'_\alpha(x) - J'_\alpha(x)Y_\alpha(x) = \frac{2}{\pi x}, \quad x > 0.
\]

The following lemma will be used for the error analysis study of the uniform approximation of the GPSWFs.

Lemma 2 Under the above notation, for any real \( \alpha \geq -\frac{1}{2} \), we have

\[
\sup_{x > 0} x M_\alpha^2(x) \leq m_\alpha = \begin{cases} 
2/\pi & \text{if } |\alpha| \leq \frac{1}{2} \\
\text{finite and larger than } 2/\pi & \text{if } \alpha > \frac{1}{2}.
\end{cases}
\]

Proof We first note that from \([29, p. 446]\), the function \( x \left( J_\alpha^2(x) + Y_\alpha^2(x) \right) \) is increasing for \( |\alpha| \leq 1/2 \) and it is decreasing for \( \alpha > 1/2 \). If \( |\alpha| \leq 1/2 \), then since

\[
2x|J_\alpha(x)Y_\alpha(x)| \leq x \left( J_\alpha^2(x) + Y_\alpha^2(x) \right), \quad x > 0.
\]

and since \( J_\alpha(x), Y_\alpha(x) \) have the asymptotic behaviours as \( x \to +\infty \), given by

\[
J_\alpha(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x \alpha \frac{\pi}{2} - \frac{\pi}{4} \right), \quad Y_\alpha(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left( x \alpha \frac{\pi}{2} - \frac{\pi}{4} \right),
\]

then we have

\[
\sup_{x > 0} x M_\alpha^2(x) \leq \lim_{x \to +\infty} x \left( J_\alpha^2(x) + Y_\alpha^2(x) \right) = 2/\pi, \quad |\alpha| \leq 1/2.
\]
Next, let $\alpha > 1/2$, then from the continuity and the decay over $[0, +\infty)$ of the function $H_\alpha : x \to x (J_\alpha^2(x) + J_\alpha^2(x))$, combined with the asymptotic behaviours given by (20), one concludes that $\sup_{x \geq 0} x M_\alpha(x) \leq m_\alpha$. Moreover, we have $m_\alpha \geq \lim_{x \to +\infty} H_\alpha(x) = \frac{2}{\pi}$. 

The following lemma whose proof is given in the “Appendix 1”, provides us with an upper bound for the constant $m_\alpha$, $\alpha > 1/2$, given in the previous lemma.

**Lemma 3** Let $\alpha > \frac{1}{2}$ and let $m_\alpha$ be as given by the previous lemma, then we have

$$m_\alpha \leq \alpha \left( J_\alpha^2(\alpha) + Y_\alpha^2(\alpha) + \frac{4}{\pi} \right).$$

(21)

Once we have Lemmas 1 and 3, one can prove the following theorem that provides us with the uniform approximation over $[0, 1]$ of the GPSWFs $\psi_{n,c}^{(\alpha)}$.

**Theorem 1** Let $c > 0$, $\alpha \geq -1/2$ be two real numbers and let $n \in \mathbb{N}$ be such that $q = c^2/\chi_{n,\alpha} < 1$ and $(1 - q)\sqrt{\chi_{n,\alpha}} \geq \pi(\frac{\alpha}{4} + 3\alpha^2)m_\alpha$. Then under the previous notations, one can write

$$\psi_{n,c}^{(\alpha)}(x) = A_\alpha(q) \left( \chi_{n,\alpha} \right)^{1/4} \sqrt{S(x)} J_\alpha(\sqrt{\chi_{n,\alpha}}S(x)) \left( 1 - x^2 \right)^{1/4+\alpha^2/2(1 - qx^2)^{1/4}} + \mathcal{E}_{n,\alpha}(x), \quad 0 \leq x \leq 1,$$

(22)

Here, $A_\alpha(q)$ is a normalization constant and

$$|\mathcal{E}_{n,\alpha}(x)| \leq C_\alpha A_\alpha(q) \frac{(1 - x^2)^{1/4}}{(1 - qx^2)^{3/4}} \varepsilon_{n,\alpha}, \quad \varepsilon_{n,\alpha} = \frac{1}{(1 - q)\sqrt{\chi_{n,\alpha}}},$$

(23)

where $C_\alpha$ is a constant depending only on $\alpha$.

**Proof** We first recall that for $x \in [0, 1]$, $\psi_{n,c}^{(\alpha)}(x) = \frac{V(S(x))}{(1 - x^2)^{1/4+\alpha^2/2(1 - qx^2)^{1/4}}}$, where $V(\cdot)$ is a bounded solution on $[0, S(0)]$ of the differential Eq. (15). On the other hand, the general solution of this later is given by

$$V(s) = A_\alpha(q)V_1(\sqrt{\chi_{n,\alpha}}s) + B_\alpha(q)V_2(\sqrt{\chi_{n,\alpha}}s) + \int_0^s \frac{\sqrt{S}T_{n,\alpha}}{W(V_1(\sqrt{\chi_{n,\alpha}}s), V_2(\sqrt{\chi_{n,\alpha}}s))} \cdot \left( J_\alpha(\sqrt{\chi_{n,\alpha}}t) Y_\alpha(\sqrt{\chi_{n,\alpha}}s) - J_\alpha(\sqrt{\chi_{n,\alpha}}s) Y_\alpha(\sqrt{\chi_{n,\alpha}}t) \right) G_\alpha(t) V(t) \, dt.$$

(24)

Here, $V_1(t) = \sqrt{t}J_\alpha(t)$, $V_2(t) = \sqrt{t}Y_\alpha(t)$. For the homogeneous solutions $V_1(\sqrt{\chi_{n,\alpha}}s)$, $V_2(\sqrt{\chi_{n,\alpha}}s)$ of (15), one may refer to [3, p. 201]. Note that from (8) and the asymptotic behaviours of $J_\alpha$, $Y_\alpha$, given by (17), one concludes that the function $\frac{V_1(S(x))}{(1 - x^2)^{1/4+\alpha^2/2(1 - qx^2)^{1/4}}}$ is bounded at $x = +1$, or $s = S(1) = 0$, which is not the case for the function $\frac{V_2(S(x))}{(1 - x^2)^{1/4+\alpha^2/2(1 - qx^2)^{1/4}}}$. Hence, in (24), we have $B_\alpha(q) = 0$. Moreover, from the expression of the Wronskian given by (18), one can easily check that
Finally, by collecting everything together, one gets the desired result (23).

Moreover, since

\[ J_\alpha(\sqrt{\chi_n} S(x)) - J_\alpha(\sqrt{\chi_n} S(x)) = J_\alpha(\sqrt{\chi_n} t) - J_\alpha(\sqrt{\chi_n} t) \]

the reminder term \( R_{n,\alpha}(x) \) is given as follows,

\[ |R_{n,\alpha}(x)| \leq A_\alpha(q)\chi_{n,\alpha}^{1/4} S(x) M_\alpha(\sqrt{\chi_n} S(x)) \frac{\pi \sqrt{S(x)}}{2 \sqrt{\chi_n}} \left( e^{\gamma_n(x)} - 1 \right). \]

where \( \gamma_n(x) = \frac{\pi}{2 \sqrt{\chi_n}} m_\alpha g_{\alpha,q}(x), \quad x \in [0, 1]. \)

Here, \( g_{\alpha,q} \) is as given by (12). Also, since

\[ K(x, \sqrt{q}) \leq \frac{1}{\sqrt{1 - qx^2}} \int_x^1 \frac{dt}{\sqrt{1 - t^2}} \leq \frac{2}{1 - q} \sqrt{\frac{1 - x^2}{1 - qx^2}} \]

and since from [7], we have

\[ \sqrt{q} \sqrt{1 - x^2} + S(x) \leq 2 \sqrt{\frac{1 - x^2}{1 - qx^2}}, \]

one gets

\[ |g_{\alpha,q}(x)| \leq \frac{\sqrt{1 - x^2}}{\sqrt{1 - qx^2}} \left( \frac{3 + 12\alpha^2 + 2q}{2(1 - q)} + \frac{4(1 - q)}{2(1 - q)} \right). \]

Consequently, we have

\[ \frac{|g_{\alpha,q}(x)|}{(1 - x^2)^{1/4}(1 - qx^2)^{1/4}} \leq \frac{1}{(1 - qx^2) (7/2 + 6\alpha^2)}. \]

Moreover, since \((1 - q)\sqrt{\chi_n} \geq \pi(\frac{7}{4} + 3\alpha^2)m_\alpha\), then for \( x \in [0, 1] \), we have \( \gamma_n(x) \leq \gamma_n(0) \leq 1 \). Consequently, we have

\[ \left( e^{\gamma_n(x)} - 1 \right) \leq \gamma_n(x) e^{\gamma_n(0)} - 1 \leq \gamma_n(x)(e - 1), \]

Recall that \( Y_{\alpha}(x) = (1 - x^2)^{-\alpha/2} (1 - qx^2)^{-\alpha/4} V(S(x)). \) Hence, by using (25), (27), (28) and (27), one gets (22). Finally, to prove (23), we first note that

\[ \frac{M_\alpha(s)}{E_\alpha(s)} = \begin{cases} \sqrt{2} J_\alpha(s) & \text{if } 0 < s \leq X_\alpha \\ (J_\alpha^2(s) + Y_\alpha^2(s))^{1/2} & \text{if } s \geq X_\alpha. \end{cases} \]

Hence, from the asymptotic behaviour of \( J_\alpha(x) \), given by (17), combined with the bounds of \( S(x) \), given by (8), one concludes that the quantity in the reminder term (23), given by

\[ \frac{\chi_n \sqrt{S(x)} M_\alpha(\sqrt{\chi_n} S(x))}{(1 - x^2)^{\alpha/2} E_\alpha(\sqrt{\chi_n} S(x))} \]

is bounded on \([0, 1]\) by a constant \( K_\alpha \) depending only on \( \alpha \). Finally, by collecting everything together, one gets the desired result (23). \( \Box \)
Next, we give an accurate explicit approximation of the normalization constant \( A_\alpha(q) \), so that the GPSWFs \( \psi_{n,c}^\alpha \) are normalized by the Eq. (3). To this end, we first define the following two constants depending on \( \alpha \),

\[
\mu_\alpha = \left| \alpha^2 - \frac{1}{4} \right|, \quad c_\alpha = \begin{cases} \sqrt{\frac{2}{\pi}} & \text{if } |\alpha| \leq \frac{1}{2} \\ \frac{0.675}{\alpha^{1/3}} + \frac{1.9}{\alpha^{1/2}} + \frac{1}{\alpha} & \text{if } \alpha > \frac{1}{2}. \end{cases}
\]

(30)

Also, it has been shown in [20], that

\[
\sup_{x \geq 0} \sqrt{x} |J_\alpha(x)| \leq c_\alpha, \quad \text{(31)}
\]

where an upper bound of \( c_\alpha \) is given by the previous equation. The following lemma whose proof is given in “Appendix 2”, is essential in the estimate of the normalization constant \( A_\alpha(q) \).

**Lemma 4** Let \( \alpha \geq -\frac{1}{2} \), then for any \( x > 0 \), we have

\[
\int_0^x \sqrt{t} J_\alpha^2(t) \, dt = \frac{x^2}{2} \left[ J_\alpha^2(x) + J_{\alpha+1}^2(x) - \frac{2\alpha}{x} J_\alpha(x) J_{\alpha+1}(x) \right] = \frac{x}{\pi} + \eta_\alpha(x), \quad \text{(32)}
\]

where

\[
\sup_{x \geq 0} |\eta_\alpha(x)| = M_\alpha \leq \max \left( \frac{1}{\pi}, c_\alpha^2 - \frac{1}{\pi}, \kappa_\alpha \right) \quad \text{(33)}
\]

and \( \kappa_\alpha = \frac{4}{\sqrt{3}} \left( \mu_\alpha + \mu_{\alpha+1} \right) + \frac{8}{25} \left( \mu_\alpha^2 + \mu_{\alpha+1}^2 \right) + |\alpha| c_\alpha c_{\alpha+1} \).

The following lemma provides us with an explicit estimate of the weighted \( L^2([0, 1], \omega_\alpha) \)-norm of \( \tilde{\psi}_{n,c}^\alpha \), the uniform approximation of the GPSWFs, given in Theorem 1, by

\[
\tilde{\psi}_{n,c}^\alpha(x) = A_\alpha(q) \frac{(\chi_{n,\alpha})^{1/4} \sqrt{\alpha} J_\alpha(\sqrt{\chi_{n,\alpha}} S(x))}{(1 - x^2)^{1/4 + \alpha/2} (1 - q x^2)^{1/4}}, \quad x \in [0, 1].
\]

**Lemma 5** Under the previous notations, let \( c > 0, \alpha \geq -\frac{1}{2} \) be two real numbers. Then, for any \( n \in \mathbb{N} \) with \( q = c^2 / \chi_{n,\alpha} < 1 \), we have

\[
\left| \left\| \tilde{\psi}_{n,c}^\alpha \right\|_{L^2([0,1],\omega_\alpha)}^2 - A_\alpha^2(q) \frac{K(\sqrt{q})}{\pi} \right| \leq A_\alpha^2(q) \frac{M_\alpha}{(1 - q) \sqrt{\chi_{n,\alpha}}}, \quad \text{(34)}
\]

where \( M_\alpha \) is given by (33).

**Proof** We first write \( \left\| \tilde{\psi}_{n,c}^\alpha \right\|_{L^2([0,1],\omega_\alpha)}^2 \) as follows

\[
\int_0^1 \left( \tilde{\psi}_{n,c}^\alpha \right)^2(x) \omega_\alpha(x) \, dx = A_\alpha^2(q) \int_0^1 \sqrt{\chi_{n,\alpha}} S(x) J_\alpha^2(\sqrt{\chi_{n,\alpha}} S(x)) \frac{1}{1 - q x^2} \, dx
\]

\[
= A_\alpha^2(q) \int_0^1 F_n(x) \frac{1}{1 - q x^2} \, dx,
\]

with

\[
F_n(x) = - \int_x^1 \sqrt{\chi_{n,\alpha}} S(t) J_\alpha^2(\sqrt{\chi_{n,\alpha}} S(t)) S'(t) \, dt.
\]
Moreover, since
\[ \int_0^1 \frac{1}{1 - q x^2} \, dx = 1, \]
where \( I_{\alpha, q} = -F_{n}(0) + \frac{1}{\pi} \int_0^1 \sqrt{1 - q x^2} \, dx + \frac{1}{\sqrt{\alpha}} \int_0^1 \frac{1}{1 - x^2} \, dx \)
and such that for \( n \geq N_{\alpha} \), we have

\[ \alpha, \epsilon \leq q \leq 1, \]
(37)

where \( \epsilon_{n, \alpha} \) is given by (23).

**Proof** From (23) and the expression of \( \sqrt{M_{\alpha}(t)} \), one can easily check that

\[ \| \mathcal{E}_{n, \alpha} \|_{L^2([0,1], w_{\alpha})} \leq \epsilon_{n, \alpha} \max(\sqrt{2c_{\alpha}}, \sqrt{m_{\alpha}}) A_{\alpha}(q) \int_0^1 \frac{(1 - x^2)}{(1 - q x^2)^{3/4}} \, dx \]
\[ \leq \epsilon_{n, \alpha} \max(\sqrt{2c_{\alpha}}, \sqrt{m_{\alpha}}) A_{\alpha}(q) \frac{\pi}{2}. \]

Since, \( \| \psi_{n, c}^{(\alpha)} \|_{L^2([0,1], w_{\alpha})} = \frac{1}{\sqrt{2}} \), then the previous inequality implies

\[ \left\| \psi_{n, c}^{(\alpha)} \right\|_{L^2([0,1], w_{\alpha})} - \frac{1}{\sqrt{2}} \leq \epsilon_{n, \alpha} \|
\]

Moreover, since \( K(\sqrt{\alpha}) \leq \frac{\pi}{2} \) and since \( A_{\alpha}(q) > 0 \), then

\[ \left\| \psi_{n, c}^{(\alpha)} \right\|_{L^2([0,1], w_{\alpha})} + A_{\alpha}(q) \frac{K(\sqrt{\alpha})}{\pi} \geq \frac{A_{\alpha}(q)}{\sqrt{2}}. \]

By combining this last inequality with (34), one gets

\[ A_{\alpha}(q) \sqrt{K(\sqrt{\alpha})} - \frac{1}{\sqrt{2}} \leq \epsilon_{n, \alpha} \|

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Finally, since \( \sqrt{\frac{\pi}{K(\sqrt{q})}} \leq \sqrt{2} \), then the previous inequality gives us the desired inequalities (37).

\[ \square \]

## 3 Uniform Approximation of the Eigenfunction in Terms of Jacobi Polynomials

In this paragraph, we show that for a given real number \( c > 0 \) and any real \( 0 \leq \alpha < 3/2 \), the GPSWFs \( \psi_{n,c}^{(\alpha)} \) are uniformly approximated by the normalized Jacobi polynomial \( \tilde{P}_n^{(\alpha,\alpha)} \). For this purpose, we first need the following mathematical preliminaries on Jacobi polynomials and Jacobi functions of the second kind.

### 3.1 Preliminaries on Jacobi Polynomials and Jacobi Functions of the Second Kind

We recall that for two real numbers \( \alpha, \beta > -1 \), the Jacobi polynomials \( P_k^{(\alpha,\beta)} \) are given by the recurrence formula

\[ P_{k+1}^{(\alpha,\beta)}(x) = (A_k x + B_k) P_k^{(\alpha,\beta)}(x) - C_k P_{k-1}^{(\alpha,\beta)}(x), \quad x \in [-1, 1], \]

where \( P_0^{(\alpha,\beta)}(x) = 1 \), \( P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha + \beta + 2)x + \frac{1}{2}(\alpha + \beta) \) and where \( A_k = \frac{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}{2(k+1)(k+\alpha+\beta+1)} \), \( B_k = \frac{(\alpha^2-\beta^2)(2k+\alpha+\beta+1)}{2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)} \), \( C_k = \frac{(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)}{(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)}. \)

The normalized Jacobi polynomial of degree \( k \), denoted by \( \tilde{P}_k^{(\alpha,\beta)} \) and satisfying the condition

\[ \int_{-1}^{1} (\tilde{P}_k^{(\alpha,\beta)}(y))^2 (1-y)^\alpha (1+y)^\beta \, dy = 1 \]

are given by

\[ \tilde{P}_k^{(\alpha,\beta)}(x) = \frac{1}{\sqrt{h_k^{(\alpha,\beta)}}} P_k^{(\alpha,\beta)}(x), \quad h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1} \Gamma(k+\beta+1)}{k! (2k+\alpha+\beta+1) \Gamma(k+\alpha+\beta+1)}. \quad (40) \]

Here, \( \Gamma(\cdot) \) is the gamma function that satisfies the following useful inequalities, see [5]

\[ 2e \left( \frac{x + 1/2}{e} \right)^{x+1/2} \leq \Gamma(x+1) \leq 2 \pi \left( \frac{x + 1/2}{e} \right)^{x+1/2} \quad , \quad x > 0. \quad (41) \]

Note that \( P_n^{(\alpha,\beta)} \) is the bounded solution of the following second order differential equation,

\[ (1 - x^2)y''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)y'(x) + n(n + \alpha + \beta + 1)y(x) = 0, \quad x \in (-1, 1). \]

A second linearly independent solution of the previous differential equation is given by the Jacobi function of the second kind, denoted by \( Q_n^{(\alpha,\beta)} \) and defined by

\[ Q_n^{(\alpha,\beta)}(x) = Q_0^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) - \frac{W_{n-1}^{(\alpha,\beta)}(x)}{(1-x)^\alpha (1+x)^\beta}, \]

where

\[ Q_0^{(\alpha,\beta)}(x) = \int_0^x \frac{(1 + \alpha + \beta)dt}{(1-t)^{1+\alpha}(1+t)^{1+\beta}} + \Lambda_{\alpha\beta}, \]

\[ W_{n-1}^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha + \beta + 2)}{(\alpha + \beta + 1) \Gamma(\alpha + 1) \Gamma(\beta + 1)} \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(s)}{x-s} (1-s)^\alpha (1+s)^\beta \, ds. \]
Approximation of $\psi(\alpha)$

where $A_n$ is the normalization constant, satisfying

\[ |Q_n^{(\alpha,\beta)}(x)| \leq \frac{C}{(1-x)^\alpha (1+x)^\beta} \left( \sqrt{1-x+n^{-1}} \right)^{\alpha-1/2} \left( \sqrt{1+x+n^{-1}} \right)^{\beta-1/2}, \]

\[-1 < x < 1,\]

\[ |P_n^{(\alpha,\beta)}(x)| \leq C \left( \sqrt{1-x+n^{-1}} \right)^{-\alpha-1/2} \left( \sqrt{1+x+n^{-1}} \right)^{-\beta-1/2}, \quad -1 < x < 1, \tag{42} \]

where $C$ is a fixed constant, not depending on the parameters $n, \alpha, \beta$. Note that if $x_{n,k}$ are the $n$ zeros of $P_n^{(\alpha,\beta)}$, arranged in the decreasing order $-1 < x_{n,n} < \cdots < x_{n,1} < 1$, then it has been shown in [4] that if $\alpha, \beta > -1/2$, then $Q_n^{(\alpha,\beta)}$ has $n+1$ zeros in $(-1,1)$, denoted by $t_{n,k}$, arranged in the decreasing order and satisfying the following interlacing property

\[ x_{n,k+1} < t_{n,k} < x_{n,k}, \quad k = 1, \ldots, n-1, \quad t_{n,0} \in (x_{n,1}, 1). \tag{43} \]

Also, from [27, p. 192], an asymptotic formula for the zeros $x_{n,k}$ is given by

\[ x_{n,k} = \cos \theta_{n,k}, \quad \lim_{n \to +\infty} n \theta_{n,k} = j_{k,\alpha}, \tag{44} \]

where $j_{k,\alpha}$ is the $k$–th positive zero of the Bessel function $J_\alpha(\cdot)$. Moreover, from [29, p. 506], for fixed $\alpha > -1$ and for large enough integer $k$, we have the following asymptotic approximation of $j_{k,\alpha}$,

\[ j_{k,\alpha} = k\pi + \frac{\pi}{2} \left( \alpha - \frac{1}{2} \right) - \frac{4\alpha^2 - 1}{8(k\pi + \frac{\pi}{2}(\alpha - \frac{1}{2}))} + O(k^{-3}). \tag{45} \]

3.2 Uniform Approximation in Terms of Jacobi Polynomials

In the sequel, we let $C_\alpha$ denote a generic constant depending on $\alpha$ that might take different values. The following theorem provides us with the approximation of $\psi_{n,c}^{(\alpha)}$ by $\tilde{P}_n^{(\alpha,\alpha)}$, for $0 < \alpha < 3/2$. Note that in the special case $\alpha = 0$, a similar result has been given in [7]. Nonetheless, in the present more general case of $0 < \alpha < 3/2$, one cannot adapt the techniques used in the previous reference. We will use some known results on the zeros locations of the Gegenbauer’s polynomials and Gegenbauer’s functions of the second kind, together with some local estimates of these laters.

**Theorem 2** Let $c > 0$ and $0 < \alpha < 3/2$, then there exists $N_\alpha \in \mathbb{N}$, such that for any $n \geq N_\alpha$, we have $q = c^2 \frac{c^2}{\gamma_{c,\alpha}} \leq q_0 < 1$, and there exists a constant $C_{\alpha,q_0}$ depending only on $\alpha$ and $q_0$ and such that for $n \geq N_\alpha$, we have

\[ \left| \psi_{n,c}^{(\alpha)}(x) - A_n \tilde{P}_n^{(\alpha,\alpha)}(x) \right| \leq C_{\alpha,q_0} \frac{c^2}{n+2\alpha+1}, \quad \forall \, x \in [-1,1], \tag{46} \]

where $A_n$ is the normalization constant, satisfying

\[ |1 - A_n| \leq C_{\alpha,q_0} \frac{c^2}{2n+2\alpha+1}. \tag{47} \]

**Proof** We will only prove the previous approximation result on $[0,1]$, since the same proof is used on the interval $[-1,0]$. We first rewrite the differential equation governing $\psi_{n,c}^{(\alpha)}$ as follows

\[ \begin{align*}
\frac{d}{dx} \left( x^{1-\alpha} (1-x)^{\gamma_{c,\alpha}} \psi_{n,c}^{(\alpha)} \right) + & \left( \alpha + \frac{1}{4} \right) \frac{(1-x)}{\gamma_{c,\alpha}} \psi_{n,c}^{(\alpha)} \\
& + \frac{c^2}{\gamma_{c,\alpha}} \psi_{n,c}^{(\alpha)} = 0.
\end{align*} \]
\[(1 - x^2)\psi''(x) - 2x\psi'(x) + \chi_n(0)\psi(x) = (\chi_{n,\alpha}(0) - \chi_{n,\alpha}(c) + c^2x^2)\psi(x), \quad x \in [0, 1],\]

where \(\chi_{n,\alpha}(0) = n(n + 2\alpha + 1)\). Note that the homogeneous equation associated with the previous differential equation has \(\tilde{P}_n^{(\alpha,\alpha)}\) and \(\tilde{Q}_n^{(\alpha,\alpha)}\) as the two linearly independent solutions. By the method of variation of constants, the bounded solution \(\psi_{n,c}^{(\alpha)}\) of the previous equation is written as

\[
\psi_{n,c}^{(\alpha)}(x) = A_n\tilde{P}_n^{(\alpha,\alpha)} + B_n\tilde{Q}_n^{(\alpha,\alpha)} + \int_x^1 \frac{k_n(x, y)G(y)\psi_{n,c}^{(\alpha)}(y)}{W(\tilde{P}_n^{(\alpha,\alpha)}, \tilde{Q}_n^{(\alpha,\alpha)})(y)} \, dy
\]

where \(A_n, B_n\) are constants and

\[
k_n(x, y) = \tilde{P}_n^{(\alpha,\alpha)}(x)\tilde{Q}_n^{(\alpha,\alpha)}(y) - \tilde{P}_n^{(\alpha,\alpha)}(y)\tilde{Q}_n^{(\alpha,\alpha)}(x),
\]

Also, since \(G(y) = \chi_{n,\alpha}(0) - \chi_{n,\alpha}(c) + c^2y^2\) and since \(-c^2y^2 \leq \chi_{n,\alpha}(0) - \chi_{n,\alpha}(c) \leq 0\), then we have

\[
|G(y)| \leq c^2y^2, \quad y \in [0, 1].
\]

Also, from [10, p. 171] and taking into account the normalization constant \(h_n^{(\alpha,\alpha)}\), given by (40), as well as the bounds of gamma function, given by (41), one gets the following estimate of the Wronskian \(W(\tilde{P}_n^{(\alpha,\alpha)}, \tilde{Q}_n^{(\alpha,\alpha)})(y)\)

\[
|W(\tilde{P}_n^{(\alpha,\alpha)}, \tilde{Q}_n^{(\alpha,\alpha)})(y)| = \frac{2^{2\alpha}}{h_n^{(\alpha,\alpha)}} \frac{\Gamma^2(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(n + 2\alpha + 1)} \frac{1}{(1 - y^2)^{1+\alpha}}
\]

\[
\geq C\alpha \frac{2n + 2\alpha + 1}{n^2} (1 - y^2)^{1+\alpha}.
\]

Next, we prove that the kernel \(K_n(x, y) = (1 - y^2)^{1+\alpha}k_n(x, y)\) is bounded on the set \(\{x, y \in [0, 1]; \; y \geq x\}\). For this purpose, we first note that from the interlacing property of zeros of \(\tilde{P}_n^{(\alpha,\alpha)}\) and \(\tilde{Q}_n^{(\alpha,\alpha)}\), given by (43), as well as from the asymptotic zeros locations of Jacobi polynomials, given in the previous paragraph, one concludes that there exists a constant \(\gamma > 0\) and a positive integer \(N_\alpha \in \mathbb{N}\) such that

\[
1 - \frac{\gamma}{n} \leq t_{n,0} < 1, \quad \forall n \geq N_\alpha.
\]

Recall that \(t_{n,0}\) is the largest zero of \(\tilde{Q}_n^{(\alpha,\alpha)}\) in \((0, 1)\). On the other hand, from [27, p. 67], the function \(u_\alpha(x) = (1 - x^2)^{(1+\alpha)/2}\tilde{Q}_n^{(\alpha,\alpha)}(x)\) is a solution of the following differential equation

\[
u'_{\alpha}(x) + g_{n,\alpha}(x)u_{\alpha}(x) = 0,
\]

\[
g_{n,\alpha}(x) = \frac{1}{4} \left(\frac{(1 - \alpha^2)(1 + x^2)}{(1 - x^2)^2} + \frac{4n(n + 2\alpha + 1) + 2(1 + \alpha)^2}{1 - x^2}\right).
\]

Since

\[
g'_{n,\alpha}(x) = \frac{8x}{(1 - x^2)^2} \left(\frac{n^2 + (1 + 2\alpha)n + \alpha(1 - x^2) - \alpha^2(1 + x^2) + 2}{1 - x^2}\right), \quad 0 < x < 1,
\]

then, it can be easily checked that for sufficiently large integer \(n\), we have \(g'_{n,\alpha}(x) > 0\) for \(x \in [0, 1 - \frac{\gamma}{n}]\). Hence, from Butlewski’s theorem, see for example [3, p. 238], the relative maxima of \(|u_{\alpha}(x)|\) form a decreasing sequence. That is
Theorem 1 and Proposition 1 and using (31), one concludes that for sufficiently large $n$

$$
\sup_{x \in [0, 1-\frac{\gamma}{n}]} \left(1 - x^2\right)^{(1+\alpha)/2} |\tilde{Q}_n^{(\alpha, \alpha)}(x)| \leq \left(1 - t_{n, *}^2\right)^{(1+\alpha)/2} |\tilde{Q}_n^{(\alpha, \alpha)}(t_{n, *})|,
$$

where $t_{n, *}$ is the first zero of $Q_n^{(\alpha, \alpha)}$ in $(0, 1)$. Moreover, from the locations of the first two positive zeros of $\tilde{Q}_n^{(\alpha, \alpha)}(x)$ as well as from the local estimate of this later, given by (42), one concludes that

$$
(1 - t_{n, *}^2)^{(1+\alpha)/2} |\tilde{Q}_n^{(\alpha, \alpha)}(t_{n, *})| \leq C_{\alpha}.
$$

On the other hand, by using the previous inequality together with (42), one can easily check that

$$
(1 - y^2)^{1+\alpha} \left|\tilde{P}_n^{(\alpha, \alpha)}(x) \tilde{Q}_n^{(\alpha, \alpha)}(y) - \tilde{P}_n^{(\alpha, \alpha)}(y) \tilde{Q}_n^{(\alpha, \alpha)}(x)\right| \leq C_{\alpha}, \quad \forall \ 0 \leq x \leq y \leq 1 - \frac{\gamma}{n}.
$$

Moreover, it has been shown in [4], that

$$
\lim_{y \to 1} (1 - y^2)^{\alpha} \tilde{Q}_n^{(\alpha, \alpha)}(y) = \frac{1}{\sqrt{h_n^{(\alpha, \alpha)}}} 2^{-1-\alpha} \Gamma(2+2\alpha) \Gamma(n+\alpha+1) \leq C_{\alpha} n^{-\alpha+1/2}. \tag{54}
$$

Also, it is well known that

$$
\sup_{x \in [0, 1]} |\tilde{P}_n^{(\alpha, \alpha)}(x)| \leq C_{\alpha} n^{\alpha+1/2}.
$$

Consequently, we have

$$
(1 - y^2) |\tilde{P}_n^{(\alpha, \alpha)}(x)| \leq C_{\alpha} \left(1 - \left(1 - \frac{\gamma}{n}\right)^{2}\right) n^{\alpha+\frac{1}{2}} \leq C_{\alpha} n^{\alpha-\frac{1}{2}}, \quad 1 - \frac{\gamma}{n} \leq y \leq 1, \ x \in [0, y]. \tag{55}
$$

Hence, by combining (53), (54) and (55), one gets

$$
(1 - y^2)^{1+\alpha} |k_n(x, y)| \leq C_{\alpha}, \quad \forall \ 0 \leq x \leq y \leq 1, \tag{56}
$$

where $k_n(x, y)$ is as given by (50). By using (51), (52) and the previous inequality, one gets the following bound for the reminder term $R_{n, \alpha}(x)$, given by (49),

$$
|R_{n, \alpha}(x)| \leq C_{\alpha} \frac{c^2}{2n + 2\alpha + 1} \int_{x}^{1} |\psi_{n, c}^{(\alpha)}(t)| \, dt. \tag{57}
$$

On the other hand, since $0 < \alpha < 3/2$, then by using the notations and the results of Theorem 1 and Proposition 1 and by using (31), one concludes that for sufficiently large value of $n$, with $q = c^2 / \chi_{n, \alpha} \leq q_0 < 1$, we have for $0 \leq x \leq 1$,

$$
\int_{x}^{1} |\psi_{n, c}^{(\alpha)}(t)| \, dt \leq \frac{C_{\alpha, q_0}}{(1 - q_0)^{1/4}} c_{\alpha} \int_{0}^{1} (1 - t)^{-\alpha/2 - 1/4} \, dt + \int_{0}^{1} |\varepsilon_{n, \alpha}(t)| \, dt \leq C_{\alpha, q_0}. \tag{58}
$$

The previous two inequalities imply that

$$
\sup_{x \in [0, 1]} |R_{n, \alpha}(x)| \leq C_{\alpha, q_0} \frac{c^2}{2n + 2\alpha + 1}. \tag{59}
$$

Also, since $\tilde{P}_n^{(\alpha, \alpha)}$ is bounded on $[0, 1]$ which is not the case for $\tilde{Q}_n^{(\alpha, \alpha)}$, then we have $B = 0$. This implies that

$$
\psi_{n, c}^{(\alpha)}(x) = A_n \tilde{P}_n^{(\alpha, \alpha)}(x) + R_{n, \alpha}(x).
$$
Recall that $\psi_{n,c}^{(a)}$ and $\tilde{P}_n^{(a,c)}$ are normalized so they have a unit $L^2(I, \omega_\alpha)$-norms. Hence, by using (59), one gets

$$|1 - A_n| = \left| \|\psi_{n,c}^{(a)}\|^2_{L^2(I, \omega_\alpha)} - A_n \|\tilde{P}_n^{(a,c)}\|^2_{L^2(I, \omega_\alpha)} \right| \leq \|R_n\|_{L^2(I, \omega_\alpha)} \leq C_{\alpha, q_0} \frac{c^2}{2n + 2\alpha + 1}.$$ 

This concludes the proof of Theorem 2. \hfill \Box

## 4 Decay Rate of the Eigenvalues of the Weighted Finite Fourier Transform Operator

In this section, we give a precise super-exponential decay rate of the eigenvalues $\lambda_n^{(a)}(c)$ of the operator $Q_c^\alpha = \frac{c}{2\pi} \tilde{T}_c^{(a)} \circ T_c^{(a)}$, which implies the sharp decay rate of the eigenvalues $\mu_n^{(a)}$ of the operator $\mathcal{T}_c^{(a)}$. The study of this precise decay rate is done under the condition that $0 < \alpha < 3/2$ and it is based on the uniform asymptotic approximation of the GPSWFs, by the Jacobi polynomials. It has been shown in [14], that for $\alpha \geq 0$, the sequence of the eigenvalues $\lambda_n^{(a)}(c)$, arranged in the decreasing order $1 > \lambda_0(c) > \lambda_1(c) > \cdots > \lambda_n(c) > \cdots > 0$, satisfies the following monotonicity property with respect to the parameter $\alpha$, $\lambda_n^{(a)}(c) \leq \lambda_n^{(a')}(c)$, $\forall \alpha \geq \alpha' \geq 0$. Moreover, it has been shown in [6] that in the special case where $\alpha = 0$, the eigenvalues $\lambda_n(c) = \lambda_n^{(0)}(c)$ decay asymptotically faster than $e^{-2n \log(\frac{c}{\alpha})}$ for any positive real number $0 < a < \frac{c}{2}$. The constant $\frac{c}{2}$ is optimal in the sense that cannot be replaced by a larger constant. As a consequence of the previous monotonicity property, one concludes that for $\alpha > 0$, the eigenvalues $\lambda_n^{(a)}(c)$, decay also faster than $e^{-2n \log(\frac{c}{\alpha})}$, $0 < a < \frac{c}{2}$. Also, note that in [28], the authors have given the following explicit formula for the eigenvalues $\mu_n^{(a)}(c)$,

$$\mu_n^{(a)}(c) = i^n \sqrt{\pi} \frac{\Gamma(n + \alpha + 1)\Gamma(n + 2\alpha + 1)}{\Gamma(n + \alpha + 3/2)\Gamma(2n + 2\alpha + 1)} c^n \exp(\Phi_n^{(a)}(c)),$$

$$\Phi_n^{(a)}(c) = \int_0^c F_n(c, \alpha) - n \frac{d\tau}{\tau}, \quad c > 0,$$  

(60)

where

$$F_n(c, \alpha) = \int_{-1}^1 x \psi_{n,c}^{(a)}(x) \frac{\partial_x \psi_{n,c}^{(a)}(x) \omega_\alpha(x)}{\omega_\alpha(x)} dx.$$  

(61)

Hence, under the condition that the quantity $\Phi_n^{(a)}(c)$ is bounded and by using the bounds of the $\Gamma(\cdot)$, given by (41), one gets the following super-exponential decay rate of the $\lambda_n^{(a)}(c) = \frac{c}{2\pi} |\mu_n^{(a)}(c)|^2$,

$$\lambda_n^{(a)}(c) \leq C_{\alpha} e^{-2(n + 1) \log(\frac{4n + 4\alpha + 2}{c\alpha})} e^{\Phi_n^{(a)}(c)},$$  

(62)

for some constant $C_{\alpha}$ and for large enough values of the integer $n$. Note that the roles of $\lambda_n^{(a)}(c)$ and $\mu_n^{(a)}(c)$ are reversed in [28]. Moreover, in [28], the authors have shown the convergence of the quantity $\Phi_n^{(a)}(c)$ under the strong assumptions that $\psi_{n,c}^{(a)}(x)$ and $\frac{\partial_x \psi_{n,c}^{(a)}(x)}{\omega_\alpha(x)}$ are well approximated by their projections over the five dimensional subspaces $\text{Span}\{\tilde{P}_{n+k}^{(a,c)}(x), \quad -2 \leq k \leq 2\}$ and $\text{Span}\{\tilde{P}_{n+2k}^{(a,c)}(x), \quad -2 \leq k \leq 2\}$, respectively. Also, the given proof is based on the following equality,
Also, by using (66), (65) and Hölder’s inequality, one gets

\[ F_n(0, \alpha) = \int_{-1}^{1} x \tilde{P}_n^{(\alpha, \alpha)}(x) \partial_x \tilde{P}_n^{(\alpha, \alpha)}(x) \omega_\alpha(x) \, dx = n. \] (63)

In the sequel, we prove the super-exponential decay rate of the \( \lambda_n^{(\alpha)}(c) \) with \( 0 < \alpha < \frac{3}{2} \). This is given by the following proposition.

**Proposition 2** Let \( c > 0 \) and \( 0 < \alpha < \frac{3}{2} \) be two positive real numbers. Then, there exists \( N_\alpha(c) \) and a constant \( C_\alpha > 0 \) such that

\[ \lambda_n^{(\alpha)}(c) \leq C_\alpha \exp\left(-2n + 1 \left[ \log \left( \frac{4n + 4\alpha + 2}{ec} \right) + C_\alpha \frac{c^2}{2n + 1} \right] \right), \quad \forall n \geq N_\alpha(c). \] (64)

**Proof** We recall that \( C_\alpha \) is generic constant that might take different values. We choose \( N_\alpha(c) \in \mathbb{N} \), large enough so that the conditions of Theorem 2 are satisfied, whenever \( n \geq N_\alpha(c) \). Also, we let \( C_{\alpha, q_0} = C_{\alpha} \). Since \( \alpha > 0 \), and since \( \omega_{\alpha}(\pm 1) = 0 \), then by using an integration by parts, we rewrite the quantity \( F_n(c, \alpha) \) as follows,

\[
F_n(c, \alpha) = -\frac{1}{2} \int_{-1}^{1} (\psi^{(\alpha)}_{n, c}(x))^2 \omega_\alpha(x) \, dx + \alpha \int_{-1}^{1} (\psi^{(\alpha)}_{n, c}(x))^2 x^2 (1 - x^2)^{\alpha - 1} \, dx
\]

\[
= -\frac{1}{2} - \alpha + \int_{-1}^{1} (\psi^{(\alpha)}_{n, c}(x))^2 \omega_{\alpha - 1}(x) \, dx.
\]

Note that by replacing \( \psi^{(\alpha)}_{n, c}(x) \) by \( \tilde{P}_n^{(\alpha, \alpha)}(x) \) in the previous equality and by using (63), one gets the identity

\[ \alpha \int_{-1}^{1} (\tilde{P}_n^{(\alpha, \alpha)}(x))^2 \omega_{\alpha - 1}(x) \, dx = n + \alpha + \frac{1}{2}, \quad \alpha > 0. \] (65)

Moreover, from Theorem 2, we have

\[ \psi^{(\alpha)}_{n, c}(x) = A_n \tilde{P}_n^{(\alpha, \alpha)}(x) + R_{n, \alpha}(x), \]

where

\[ |1 - A_n|, \sup_{x\in[0,1]} |R_{n, \alpha}(x)| \leq C_\alpha \frac{c^2}{2n + 2\alpha + 1}. \] (66)

Hence, it follows that

\[
\alpha \left| (\psi^{(\alpha)}_{n, c}(x))^2 - (\tilde{P}_n^{(\alpha, \alpha)}(x))^2 \right| \leq |1 - A_n|^2 |\alpha(\tilde{P}_n^{(\alpha, \alpha)}(x))^2
\]

\[ + (|1 - A_n| + |1 + A_n|) |R_{n, \alpha}(x)| |\alpha| \tilde{P}_n^{(\alpha, \alpha)}(x)| + \alpha |R_{n, \alpha}(x)|^2. \]

Since from (66), we have \( |1 - A_n|^2 \leq C_\alpha \frac{c^2}{2n + 2\alpha + 1} \), then by using (65), one concludes that

\[ |1 - A_n^2| \alpha \int_{-1}^{1} (\tilde{P}_n^{(\alpha, \alpha)}(x))^2 \omega_{\alpha - 1}(x) \, dx \leq C_\alpha c^2. \]

Also, by using (66), (65) and Hölder’s inequality, one gets

\[
\int_{-1}^{1} |R_{n, \alpha}(x)| |\alpha| \tilde{P}_n^{(\alpha, \alpha)}(x)| \omega_{\alpha - 1}(x) \, dx
\]
Theorem 3 Let

\[ \frac{c^2}{2n + 2\alpha + 1} \alpha \left( \int_{-1}^{1} |\widetilde{p}_{n}^{(\alpha, \alpha)}(x)| \omega_{\alpha-1}(x) \, dx \right)^{1/2} \leq C_{\alpha} \frac{c^2}{\sqrt{2n + 2\alpha + 1}}. \]

Similarly, by using (66), one gets

\[ \alpha \int_{-1}^{1} |R_{n, \alpha}(x)|^2 \omega_{\alpha-1}(x) \, dx \leq C_{\alpha} \left( \frac{c^2}{2n + 2\alpha + 1} \right)^2. \]

Finally, by collecting everything together, one concludes that

\[ |F_n(\alpha, \tau) - n| = \alpha \int_{-1}^{1} \left( (\psi_{n, \tau}^{(\alpha)}(x))^2 - (\widetilde{p}_{n}^{(\alpha, \alpha)}(x))^2 \right) \omega_{\alpha-1}(x) \, dx \leq C_{\alpha} \tau^2. \]

Consequently, we have

\[ \Phi_n^{(\alpha)}(c) = \int_{0}^{c} \frac{F_n(\tau, \alpha) - n}{\tau} \, d\tau \leq C_{\alpha} c^2. \]

To conclude the proof of (64), it suffices to combine the previous inequality and (62).

Next, we give an asymptotic lower bound for the counting number of the eigenvalues \( \lambda_n^{(\alpha)}(c) \). To this end, we first recall that

\[ Q_{c}^{(\alpha)}(x) = \frac{c}{2\pi} \int_{-1}^{1} K_{\alpha}(c(x - y)) g(y) \omega_{\alpha}(y) \, dy, \quad (67) \]

where

\[ K_{\alpha}(x, t) = \frac{c}{2\pi} \sqrt{\pi} 2^{\alpha+\frac{1}{2}} \Gamma(\alpha + 1) \frac{J_{\alpha+1/2}(c(x - t))}{(c(x - t))^{\alpha+1/2}}. \quad (68) \]

**Theorem 3** Let \( 0 < \delta < 1 \) and let \( M_c(\delta) \) be the number of eigenvalues of \( Q_c^{(\alpha)} \), \( \alpha > 0 \), which are not smaller than \( \delta \). Then, we have

\[ \frac{\gamma_{\alpha} - \delta}{1 - \delta} \frac{c}{2\pi} (2^{2\alpha+1} B(\alpha + 1, \alpha + 1))^2 + o(c) \leq M_c(\delta) \leq \frac{1}{\delta} \left( \frac{c}{2\pi} \left( 2^{2\alpha+1} B(\alpha + 1, \alpha + 1) \right)^2 \right). \quad (69) \]

where \( \gamma_{\alpha} = 2^{4\alpha} \left( \frac{B(2\alpha+1, 2\alpha+1)}{B(\alpha+1, \alpha+1)} \right) \) and \( B(\cdot, \cdot) \) is the beta function.

**Proof** To obtain the lower bound estimate of \( M_c(\delta) \), we use Marzo’s formula, see [18] or [2],

\[ M_c(\delta) \geq \text{Trace} (Q_c^{(\alpha)}) - \frac{1}{1 - \delta} (\text{Trace} (Q_c^{(\alpha)}) - \text{Norm} (Q_c^{(\alpha)})) \quad (70) \]

Note that the \( \text{Trace} (Q_c^{(\alpha)}) \) has been already given in [28], where it has been shown that

\[ \frac{2\pi}{c} \sum_{n=0}^{\infty} \lambda_n^{(\alpha)}(c) = \pi \frac{\Gamma^2(\alpha + 1)}{\Gamma^2(\alpha + \frac{1}{2})} \quad (71) \]

Moreover, by using the well known identity \( \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 3/2)} = \left( \frac{\Gamma(\alpha + 1)^2 2^{2\alpha+1}}{\Gamma(2\alpha + 2) \sqrt{\pi}} \right) \) one gets

\[ \text{Trace} (Q_c^{(\alpha)}) = \sum_{n} \lambda_n^{(\alpha)}(c) = \frac{c}{2\pi} \left[ 2^{2\alpha+1} B(\alpha + 1, \alpha + 1) \right]^2. \quad (72) \]
To compute an estimate of $\text{Norm}(Q_c^{(\alpha)})$, we proceed as follows:

\[
\text{Norm}(Q_c^{(\alpha)}) = \sum_{n=0}^{\infty} (\lambda_n^{(\alpha)}(c))^2
\]

\[
= \frac{c^2}{4\pi} 2^{2\alpha+1} \Gamma^2(\alpha+1) \int_{-1}^{1} \left( \frac{J_{\alpha+1/2}(c(x-y))}{(c(x-y))^{2\alpha+1}} (1-x^2)^\alpha (1-y^2)^\alpha \, dx \, dy \right)
\]

We apply the change of variable $y = \sigma$ and $x = \sigma + \frac{\tau}{c}$, to obtain:

\[
\sum_{n=0}^{\infty} (\lambda_n^{(\alpha)}(c))^2 = \frac{c^2}{4\pi} 2^{2\alpha+1} \Gamma^2(\alpha+1) \int_{-1}^{1} (1-\sigma^2)^\alpha \int_{c(-1+\sigma)}^{c(1+\sigma)} \frac{J_{\alpha+1/2}(\tau)}{\tau^{2\alpha+1}} \left( 1 - \left( \sigma + \frac{\tau}{c} \right)^2 \right)^\alpha d\tau d\sigma. \tag{73}
\]

Since

\[
\frac{J_{\alpha+1/2}(\tau)}{\tau^{2\alpha+1}} (1-\sigma^2)^\alpha \left( 1 - \left( \sigma + \frac{\tau}{c} \right)^2 \right)^\alpha \leq \frac{J_{\alpha+1/2}(\tau)}{\tau^{2\alpha+1}} (1-\sigma^2)^\alpha \left( 1 - \left( \sigma - \frac{1}{c} \right)^2 \right)^\alpha
\]

and since from [22, p. 244], we have

\[
\int_{\mathbb{R}} \frac{J_{\alpha+1/2}(\tau)}{\tau^{2\alpha+1}} d\tau = \frac{1}{2^{2\alpha}} \Gamma(1/2) \Gamma(2\alpha + 1) / \Gamma(2\alpha + 3/2) \Gamma^2(\alpha + 1) \tag{74}
\]

then by Lebesgue’s dominated convergence theorem applied to the integral in (73), one gets

\[
\lim_{c \to \infty} \left[ \frac{1}{c} \sum_{n} (\lambda_n^{(\alpha)}(c))^2 \right] = \frac{1}{4\pi} 2^{2\alpha+1} \Gamma^2(\alpha+1) \int_{-1}^{1} (1-\sigma^2)^{2\alpha} d\sigma \int_{\mathbb{R}} \frac{J_{\alpha+1/2}(\tau)}{\tau^{2\alpha+1}} d\tau.
\]

Since $\int_{-1}^{1} (1-\sigma^2)^{2\alpha} d\sigma = 2^{4\alpha+1} B(2\alpha + 1, 2\alpha + 1)$, then by straightforward computations, one gets

\[
\lim_{c \to \infty} \left[ \frac{1}{c} \sum_{n} (\lambda_n^{(\alpha)}(c))^2 \right] = \gamma_\alpha \frac{1}{2\pi} \left[ 2^{2\alpha+1} B(\alpha + 1, \alpha + 1) \right]^2.
\]

Hence, for $\alpha > -1$, we have

\[
\sum_{n} (\lambda_n^{(\alpha)}(c))^2 = \gamma_\alpha \frac{c}{2\pi} \left[ 2^{2\alpha+1} B(\alpha + 1, \alpha + 1) \right]^2 + o(c), \tag{75}
\]

where $\gamma_\alpha = 2^{4\alpha} \left( \frac{B(2\alpha + 1, 2\alpha + 1)}{B(\alpha + 1, \alpha + 1)} \right)^2$ and $B(\cdot, \cdot)$ is the Beta function. To conclude for the proof of the lower bound estimate in (69), it suffices to combine (75) and (72) in (70). Finally, to prove the upper bound in (69), it suffices to note that

\[
\sum_{k=0}^{\infty} \lambda_k^{(\alpha)}(c) \geq \sum_{k=0}^{M_c(\delta)} \lambda_k^{(\alpha)}(c) \geq \delta M_c(\delta)
\]

and then use (72).
Remark 1 In the special case \( \alpha = 0 \), the inequalities (69) become

\[
\frac{2}{\pi} + o(1) \leq \frac{M_c(\delta)}{c} \leq \frac{2}{\pi \delta}
\] (76)

In the special case \( \alpha = 0 \), it has been shown in [17] that \( M_c(\delta) \) is independent of \( \delta \in [0, 1] \). Hence, by letting \( \delta \to 1 \), one recover Landau’s classical result, see [17], \( M_c(\delta) = \frac{2c^2}{\pi} + o(c) \).

5 Numerical Examples

In this last section, we give various numerical examples that illustrate the results of the previous sections.

Example 1 In this first example, we illustrate the approximation formula of the normalization constant \( A_\alpha(q) \), appearing in Theorem 1. Recall that this normalization constant is fixed so that the condition \( \| \psi_{n,c}\|_{L^2(I,\omega_\alpha)} = 1 \) holds. This last condition is important in the applications in the sense that it provides us with orthonormal GPSWFs over the interval \( I = [-1, 1] \) and it allows us to compute the eigenvalues \( \lambda_n^{(\alpha)}(c) \) of the operator \( Q^{(\alpha)}_c \). From Proposition 1, an accurate approximation to this normalization constant is given by

\[
\tilde{A}_\alpha(q) = \sqrt{\frac{\pi}{2K(\sqrt{q})}} \approx A_\alpha(q). \tag{77}
\]

Here, \( K(\sqrt{q}) \) is the Legendre elliptic integral given by (14). On the other hand, since in Theorem 1, we have \( E_n,\alpha(1) = 0 \) and since from [7], we have

\[
\lim_{x \to 1} S(x)\sqrt{(1 - x^2)(1 - qx^2)} = 1,
\]

then by using the asymptotic behaviour of \( J_\alpha(\sqrt{\chi_{n,\alpha}}S(x)) \) as \( x \to 1^- \), which is obtained from (17), one concludes that the normalization constant \( A_\alpha(q) \) is given by

\[
A_\alpha(q) = \frac{2^\alpha \Gamma(1 + \alpha)}{(1 - q)^{\alpha/2} \chi_{n,\alpha}^{1/4 + \alpha/2}} \psi_{n,\alpha}^\alpha(1). \tag{78}
\]

To illustrate the approximation formula (77), we have considered the values of \( c = 10\pi, \) and \( \alpha = 0.5, 1.0. \) Then, we have computed \( A_\alpha(q) \) and \( \tilde{A}_\alpha(q) \), for different values of \( n \). The obtained numerical results are given by Table 1. From these results and as expected by Proposition 1, the approximation formula (77) provides us with satisfactory accurate values of the normalization constant \( A_\alpha(q) \).

Example 2 In this second example, we illustrate the uniform approximation of the \( \psi_{n,c}^{(\alpha)} \), given by Theorem 1, with the approximation of normalization constant \( A_\alpha(q) \), given by (77) . For this purpose, we have considered the values of \( c = 10\pi, \alpha = 0.5 \) and \( n = 22, 35. \) Then, we have used the GPSWFs computational scheme given in [14] to get highly accurate approximations of the \( \psi_{n,c}^{(\alpha)} \) and the associated eigenvalues \( \chi_{n,\alpha} \). For these values, we have found that

\[
\varepsilon_{22,\alpha} = 0.3386, \quad \varepsilon_{35,\alpha} = 0.0516, \quad \varepsilon_{n,\alpha} = \frac{1}{(1 - q)\sqrt{\chi_{n,\alpha}}}, \quad q = \frac{c^2}{\chi_{n,\alpha}}.
\]
Table 1 Illustration of the approximation formula (77) with $c = 10\pi$ and various values of $n$.

| $n$ | $q = c^2/\chi_{n,\alpha}$ | $\Lambda_n(q)$ | $|\Lambda_n(q) - \sqrt{\frac{\pi}{2K(\sqrt{q})}}|$ |
|-----|-----------------------------|-----------------|-----------------------------------------------|
| $\alpha = 0.5$ | | | |
| 20 | 0.97614 | 0.70519 | 1.1861 E–02 |
| 25 | 0.81180 | 0.82819 | 9.4946 E–04 |
| 30 | 0.66422 | 0.88079 | 2.8399 E–04 |
| 35 | 0.54460 | 0.91101 | 2.1864 E–04 |
| $\alpha = 1.0$ | | | |
| 20 | 0.96592 | 0.85792 | 1.4537 E–01 |
| 25 | 0.75271 | 0.84170 | 6.4326 E–03 |
| 30 | 0.61472 | 0.88690 | 2.8507 E–03 |
| 35 | 0.51431 | 0.91899 | 1.5971 E–03 |

Fig. 1 a Graph of the $\psi_{n,c}^{(\alpha)}(x)$ for $c = 10\pi$, $\alpha = 0.5$, $n = 22$ (blue) and $n = 35$ (black), b graph of the errors $\psi_{n,c}^{(\alpha)}(x) - \tilde{\psi}_{n,c}^{(\alpha)}(x)$, with $n = 22$ (blue) and $n = 35$ (black) (Color figure online)

Note that $\tilde{\psi}_{n,c}^{(\alpha)}$, our uniform approximation of $\psi_{n,c}^{(\alpha)}$ is given by

$$
\tilde{\psi}_{n,c}^{(\alpha)}(x) = \sqrt{\frac{\pi}{2K(\sqrt{q})}} \left( \chi_{n,\alpha} \right)^{1/4} \sqrt{\mathcal{S}(x)} J_{\alpha} \left( \sqrt{\chi_{n,\alpha} \mathcal{S}(x)} \right) \left( 1 - x^2 \right)^{1/4 + \alpha/2} (1 - q x^2)^{1/4}, \quad x \in [0, 1].
$$

In Fig. 1, we have plotted the graphs of $\psi_{n,c}^{(\alpha)}(x)$ and the approximation error $\varepsilon_{n,\alpha}(x) = \psi_{n,c}^{(\alpha)}(x) - \tilde{\psi}_{n,c}^{(\alpha)}(x)$, for $n = 22$ and $n = 35$. Note that from Fig. 1 and as expected by Theorem 1, this approximation error has the same magnitude as $\varepsilon_{n,\alpha}$.

Example 3 In this last example, we illustrate the super-exponential decay rate of the eigenvalues $\lambda_{n,\alpha}^{(\alpha)}(c)$, given by Proposition 2. For this purpose, we have used the computational scheme of the these eigenvalues, given in [14] and calculated highly accurate values of $\lambda_{n,\alpha}^{(\alpha)}(c)$, with $\alpha = 1$, the four values of $c = 5\pi, 10\pi, 15\pi, 20\pi$ and various values of $10 \leq n \leq 70$. In Fig. 2a, we have plotted the graphs of the $\lambda_{n,\alpha}^{(\alpha)}(c)$ and in Fig. 2b, we have plotted the graphs of $\log(\lambda_{n,\alpha}^{(\alpha)}(c))$ versus the graphs of the proved asymptotic super-exponential decay rate given by $-(2n + 1) \log \left( \frac{4n + 4\alpha + 2}{ec} \right)$. Figure 2b indicates that this super-exponential decay rate is highly accurate.
Fig. 2  a Graphs of $\lambda_n^{(1)}(c)$ (solid lines) b graphs of the log($\lambda_n^{(\alpha)}(c)$) versus the graphs of the asymptotic decay rate $-(2n+1)\log\left(\frac{4n+4\alpha+2}{ec}\right)$ (circles)

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Appendix 1: Proof of Lemma 3

Let $\alpha > 1/2$ and let $j_{\alpha,1}$, $j'_{\alpha,1}$ and $y_{\alpha,1}$ denote the first zeros of $J_\alpha(x)$, $J'_\alpha(x)$ and $Y_\alpha(x)$, respectively. It is known, see for example [29, p. 487] that

$$\alpha < \sqrt{\alpha(\alpha + 2)} < j'_{\alpha,1} < y_{\alpha,1} < j_{\alpha,1}.$$  

Moreover, by using the asymptotic behaviours of $J_\alpha(\cdot)$, $Y_\alpha(\cdot)$, given by (17), one concludes that the function $xJ_\alpha(x)|Y_\alpha(x)| = -xJ_\alpha(x)Y_\alpha(x)$ is positive and bounded over the interval $(0, \alpha]$.

Next, we check that for any $\alpha > 1/2$, $X_\alpha > \alpha$, where $X_\alpha$ is the first root of $J_\alpha(x) + J'_\alpha(x) = 0$. To this end, we first note that from the Wronskian of $J_\alpha, Y_\alpha$ given by (18), one concludes that for any $\alpha > -1$, we have

$$\frac{\partial}{\partial x}\left(-\frac{Y_\alpha(x)}{J_\alpha(x)}\right) = -\frac{2}{\pi x J^2_\alpha(x)} < 0, \quad x > 0. \quad (79)$$

Also, from [21, p. 438], $X_\alpha$ has the following asymptotic formula, valid for large values of the parameter $\alpha$,

$$X_\alpha = \alpha + c(\alpha/2)^{1/3} + O(\alpha^{-1/3}), \quad c \approx 0.366.$$  

Hence, there exists $\alpha_0 > 0$, so that $X_\alpha > \alpha$, whenever $\alpha \geq \alpha_0$. Hence, we have

$$-\frac{Y_v(v)}{J_v(v)} \geq -\frac{Y_v(X_v)}{J_v(X_v)} = 1, \quad \forall \nu \geq \alpha_0,$$

which means that $\lim_{v \to +\infty} -\frac{Y_v(v)}{J_v(v)} \geq 1$. On the other hand, from [29, p. 487], we have

$$\frac{\partial}{\partial v}\left(-\frac{Y_v(v)}{J_v(v)}\right) < 0, \quad v > 0.$$  

Consequently, for any $\alpha \geq 1/2$, we have

$$-\frac{Y_\alpha(\alpha)}{J_\alpha(\alpha)} \geq \lim_{v \to +\infty} -\frac{Y_v(v)}{J_v(v)} \geq 1 = -\frac{Y_\alpha(X_\alpha)}{J_\alpha(X_\alpha)}.$$  

which means that $X_\alpha > \alpha$, whenever $\alpha \geq 1/2$. Hence, for $0 < x < \alpha$, by integrating (79) over the interval $[x, \alpha]$ and using the fact that the function $x J^2_\alpha(x)$ is increasing, one gets

$$- 2x J_\alpha(x) Y_\alpha(x) = 2x J^2_\alpha(x) \frac{-Y_\alpha(\alpha)}{J_\alpha(\alpha)} + \frac{4}{\pi} \int_x^\alpha \frac{x J^2_\alpha(t)}{t J^2_\alpha(t)} \, dt \leq 2\alpha J^2_\alpha(\alpha) \frac{-Y_\alpha(\alpha)}{J_\alpha(\alpha)} + \frac{4\alpha}{\pi}, \quad 0 < x \leq \alpha.$$  \hspace{1cm} (80)

On the other hand, since $2x |J_\alpha(x) Y_\alpha(x)| \leq x (J^2_\alpha(x) + Y^2_\alpha(x))$, since this later is decreasing for $\alpha \geq 1/2$ and since $\alpha < X_\alpha$, then we have

$$\max \left( \sup_{x \in [\alpha, X_\alpha]} -x J_\alpha(x) Y_\alpha(x), \sup_{x \geq X_\alpha} x (J^2_\alpha(x) + Y^2_\alpha(x)) \right) \leq \alpha \left( J^2_\alpha(\alpha) + Y^2_\alpha(\alpha) \right).$$  \hspace{1cm} (81)

Finally, by combining (80) and (81), one gets the desired bound (21).

\[\square\]

**Appendix 2: Proof of Lemma 4**

The first equality in (32) is a consequence of the following identity, see [22, p. 241]

$$\int_0^x t J^2_\alpha(t) \, dt = \frac{x^2}{2} \left[ J^2_\alpha(x) - J_{\alpha-1}(x) J_{\alpha+1}(x) \right], \quad \alpha > -1/2,$$

combined with the well known identity

$$J_{\alpha-1}(x) = \frac{2\alpha}{x} J_\alpha(x) - J_{\alpha+1}(x).$$

Moreover, it has been shown in [16], that for $\alpha \geq -1/2$, we have

$$\sup_{x \geq 0} x^{3/2} \left| J_\alpha(x) - \sqrt{\frac{2}{\pi x}} \left( \cos \left( x - (\alpha + 1/2) \frac{\pi}{2} \right) \right) \right| \leq \frac{4}{5} \mu_\alpha.$$  \hspace{1cm} (82)

Hence, by using the previous inequality, one gets

$$\left| J^2_\nu(x) - \frac{2}{\pi x} \cos \left( x - \left( \nu + \frac{1}{2} \right) \frac{\pi}{2} \right) \right| \leq \frac{4}{5x^{3/2}} \mu_\nu \left( |J_\nu(x)| + \sqrt{\frac{2}{\pi x}} \right), \quad x \geq 1, \quad \nu = \alpha, \alpha + 1.$$  \hspace{1cm} (83)

Moreover, from (82), one gets

$$|J_\nu(x)| \leq \frac{4}{5} \mu_\nu + \sqrt{\frac{2}{\pi x}}, \quad x \geq 1.$$  \hspace{1cm} (84)

By using the previous two inequalities, one obtains

$$\left| J^2_\alpha(x) + J^2_{\alpha+1}(x) - \frac{2}{\pi x} \right| \leq \frac{8}{5} \sqrt{\frac{2}{\pi x^2}} \left( \mu_\alpha + \mu_{\alpha+1} \right) + \frac{16}{25x^3} \left( \mu^2_\alpha + \mu^2_{\alpha+1} \right), \quad x \geq 1.$$
Hence, we have

\[
\frac{x^2}{2} \left| J^2_\alpha(x) + J^2_{\alpha+1}(x) - \frac{2\alpha}{x} J_\alpha(x) J_{\alpha+1}(x) - \frac{2}{\pi x} \right| \\
\leq 4 \sqrt{\frac{2}{\pi x}} \left( \frac{1}{\pi} \left( \mu_\alpha + \mu_{\alpha+1} \right) + \frac{8}{25x} \left( \mu_\alpha^2 + \mu_{\alpha+1}^2 \right) + |\alpha| x |J_\alpha(x) J_{\alpha+1}(x)|, \quad x \geq 1. \quad (85)
\]

Finally, by combining (31) and the previous inequality, one gets a bound of $|\eta_\alpha(x)|$ for $x \geq 1$. To get a bound $\eta_\alpha(x)$ over the interval $[0, 1]$, it suffices to note that from (32), we have

\[
\sup_{x \in [0, 1]} |\eta'_\alpha(x)| = \sup_{x \in [0, 1]} \left| x J^2_\alpha(x) - \frac{1}{\pi} \right| \leq \max \left( \frac{1}{\pi}, c_\alpha^2 - \frac{1}{\pi} \right).
\]

Since $\eta_\alpha(0) = 0$, then the previous bound is also valid for $\sup_{x \in [0, 1]} |\eta_\alpha(x)|$, that is

\[
|\eta_\alpha(x)| \leq \max \left( \frac{1}{\pi}, c_\alpha^2 - \frac{1}{\pi} \right), \quad 0 \leq x \leq 1. \quad (86)
\]

Finally, to conclude for the proof of the lemma, it suffices to combine (31), (85) and (86). \(\square\)

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