Cluster expansion and resurgence in Polyakov model

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In Polyakov model, a non-perturbative mass gap is formed at leading order semi-classics by instanton effects. By using the notions of critical points at infinity, cluster expansion and Lefschetz thimbles, we show that a third order effect in semi-classics gives an imaginary ambiguous contribution to mass gap, which is supposed to be real and unambiguous. This is troublesome for the original analysis, and it is difficult to resolve this issue directly in QFT. However, we find a new compactification of Polyakov model to quantum mechanics, by using a background ’t Hooft flux (or coupling to TQFT). The compactification has the merit of remembering the monopole-instantons of the full QFT within Born-Oppenheimer (BO) approximation, while the periodic compactification does not. In QM, we prove the resurgent cancellation of the ambiguity in 3-instanton sector against ambiguity in the Borel resummation of the perturbation theory around 1-instanton. Assuming that this result holds in QFT, we provide a large-order asymptotics of perturbation theory around perturbative vacuum and instanton.

Introduction: Polyakov model is a prototypical example of non-perturbatively calculable weakly coupled quantum field theory [1]. It is by now standard textbook material in QFT, condensed matter physics and it is also intimately tied with statistical field theory of Coulomb gases [2–5]. Despite the fact that some fundamental facts about the theory are known for more than four decades now, after the advent of resurgence [6–8] and Lefschetz thimbles [9] many subtle issues emerge concerning this and following calculable theories. An important issue is following. It is known that the mass gap in the theory is sourced by monopole-instantons on \( \mathbb{R}^3 \), and is of order \( n_0^2 \sim e^{-S_0} \) where \( S_0 \) is monopole-instanton (\( M_\alpha \)) action, with magnetic charge \( \alpha_i \in \Delta^0 \) in the simple root system. How one does incorporate saddles with higher action (\( nS_0, n \geq 2 \))? Should one care about them? Do they contribute to mass gap? This class of questions are usually not addressed and swept under the rug by assuming that these are higher order quantitative corrections, and not important [1, 4], even in more mathematical treatments of the theory [2]. Of course, there are also theories in which mass gap is induced at higher order effect in monopole-expansion due to Berry phase [10] or topological theta angle [11] induced destructive interference between leading monopole events, or as an effect of index theorem [12]. But here we address the above questions in simple Polyakov model, where higher order effect just seem like nuisance, by using the concepts of critical point at infinity, quasi-zero mode Lefschetz thimbles and cluster expansion [13] systematically.

Here is the main point of our analysis. The mass gap in semi-classical expansion in Polyakov model is of the following form: (ignoring inessential factors to lessen the clutter)

\[
(n_0^2)_{\pm} \sim (e^{-S_0}P_1 + e^{-2S_0}P_2 + e^{-3S_0}P_3 + \ldots) \\
\pm i (e^{-3S_0} + \ldots)
\]

where \( P_i \) denotes perturbative expansions around relevant saddle. It is reasonable to drop \( O(e^{-2S_0}) \) terms in the real part of his analysis, as they provide only minor quantitative corrections. But as we emphasize, there is a new effect in third order in semi-classics, which renders the semi-classical expansion multi-fold ambiguous and void of meaning. It is actually not correct to ignore \( \pm i (e^{-3S_0}) \) because it is an effect of different nature, giving mass an imaginary ambiguous part. Therefore, one is entitled to ask whether Polyakov’s analysis is rigorous enough even within semi-classics? In particular, for the famous result on mass gap to be justified, one needs a mechanism for the cancellation of imaginary ambiguity on \( \mathbb{R}^3 \).

The type of ambiguities that appear in (1) is in fact expected. The reason for this is because \( \text{Arg}(h) = 0 \) and \( \text{Arg}(\hbar) = \pi \) are in general Stokes lines. On Stokes lines, contributions of a sub-class of saddles can indeed be multi-fold ambiguous. In fact, there are infinitely many critical points at infinity, and generically, there are multi-fold ambiguities. It is desirable to resolve these pathological features in order to make Polyakov’s solution meaningful in this new light.

According to resurgence, there is another ambiguity in the story. Perturbation theory around perturbative vacuum saddle, one-instanton saddle etc are all expected to be divergent asymptotic expansions [14–18]. They are also expected to be non-Borel summable, meaning that Borel resummation of perturbation theory around each saddle is multifold ambiguous. Resurgence implies that for the theory to be meaningful, these two types of ambiguities must cancel around each sector of the theory. However, demonstrating this in a generic QFT is hard. It is possible to take mileage on this problem using the idea of adiabatic continuity and turning on background fields [6–8], by working with QFTs with special properties such as integrability [19, 20] or working with rather special QFTs in which one has a good knowledge of perturbation theory [21]. In this work, we employ a ’t Hooft
flux background (couple the theory to a TQFT background) to tackle this problem in Polyakov model.

**Basic:** Polyakov model is given as an SU(N) non-abelian gauge theory coupled to an adjoint scalar field in a 3d Euclidean space:

$$S = \int d^3x \frac{1}{2g^2} \left( \text{tr} F^2 + \text{tr} (D\varphi)^2 + \lambda V(\varphi) \right)$$  \hspace{1cm} (2)

where the potential $V(\varphi)$ leads to the abelianization of gauge dynamics down to $U(1)^{N-1}$. We assume without loss of generality that the eigenvalues of $\phi$ are uniformly separated, $v_i - v_{i+1} = v$. In the BPS limit, ($\lambda \ll 1$), the theory has saddles which are solutions to (anti-)self-duality equations $F = \pm \ast_3 D\varphi$ with topological (magnetic) charges $Q_M = \frac{2\pi}{g} \alpha_i$ and actions $S_0^{(i)} = \frac{4\pi}{g^2} \equiv \frac{g}{2}$, where $\alpha_i \in \Delta^0$ are $N-1$ simple roots, where $g^2 = g_0^2/v$ is a dimensionless expansion parameter. The monopole operators are $M_{\alpha_i} \sim (S_0)^2 e^{-S_0} e^{-\frac{i\pi}{4\pi} \hat{\phi}(x) \cdot \alpha_i + i\alpha_i \cdot \sigma(x)}$, where $\hat{\phi}(x), \sigma(x)$ are fluctuation of Cartan components of adjoint scalar and dual photon, $(S_0)^2$ arise from the four zero modes of the monopole. The former can be set to zero in the description of the long distance theory. The proliferation of monopoles generates a mass gap for gauge fluctuations as discovered by Polyakov [1], see also [2, 22].

**Critical points at infinity and cluster expansion:**

The model, apart from regular saddles, also possesses critical points at infinity [23–25]. These critical points are very likely one of the most important concepts in semi-classics, yet, there is a minuscule amount of work on systematizing them, or a heavy burden of misunderstandings emanating from 70s related to them, see [23, 24]. These configurations, up to our knowledge, are not addressed at all in the context of Polyakov model, and will play important role below.

Consider a monopole-monopole or monopole-antimonopole pair. The interaction between the two in the BPS ($\lambda \rightarrow 0$) limit is:

$$V_{\text{int}}(r) = \begin{cases} 0 & \text{for } (\mathcal{M}_i, \mathcal{M}_j) \\ \frac{2\pi}{g^2} \alpha_i \cdot (\alpha_j) & \text{for } (\mathcal{M}_i, \overline{\mathcal{M}}_j) \end{cases}$$  \hspace{1cm} (3)

Thus, $(\mathcal{M}, \mathcal{M})$ do not interact [26, 27], but $(\mathcal{M}, \overline{\mathcal{M}})$ pairs interact, attractively for $i = j$ and repulsively for $j = i \pm 1$. At any finite separation, since $V_{\text{int}}(r) \neq 0$, these pairs are not exact solutions. But at $r = \infty$, they become exact solutions, hence the name. Such configurations are genuine critical points, but they are non-Gaussian, i.e. $V_{\text{int}}(r) = 0$, unlike Gaussian saddles. Because of this property, one needs to integrate over the whole steepest descent cycle to find the effect of such pairs.

The integrals that give the contribution of the second order effects in semi-classics are of the form:

$$Z_2 = [\mathcal{M}_i][\overline{\mathcal{M}}_j] \int d^3r_1 d^3r_2 e^{-\text{Vir}(\mathcal{M}_i - \mathcal{M}_j)}$$  \hspace{1cm} (4)

The measure can be expressed in terms of center-of-action coordinate, the integral over which gives a space-time volume factor $\mathcal{V}$, and the relative coordinate, which corresponds to quasi-zero mode direction: $[\mathcal{M}_i][\overline{\mathcal{M}}_j] \mathcal{V} 4\pi \int d^2r e^{-\text{Vir}(r)}$. This type of integrals appears in the standard cluster expansion in statistical field theory [13].

In semi-classics, steepest descent cycles are not necessarily real. Let us call $r \rightarrow z \in \mathbb{C}$. The steepest descent cycles can end up at points where $e^{-f(z)/g^2} \rightarrow 0$. For polynomial $f(z)$, this gives a homology cycle decomposition of the integration [9], in terms of cycles that ends at infinity in certain wedges. For the Coulomb potential, the end point is a pole at $z = 0$ similar to [28], and the critical point is at $z = \infty$. For $e^{-f(z)/g^2}$, the cycle must enter to the pole in arg$(z) = 0$ direction, while $e^{+1/(g^2z)}$, the cycle must enter to the pole in arg$(z) = \pi$ direction for steepest descent. So, the steepest descent directions for the attractive and repulsive potentials are different.

Steepest descent cycles are easiest to visualize if we map $r = z \in \mathbb{C}$ complex domain to Riemann sphere by using one-point compactification. To see the thimbles more clearly, we can introduce a regulator for the integral, find the steepest descent, and ultimately remove the regulator. For the two interaction types, the descent cycles for arg$(g^2) = \pm \pi$ are given by

$$\mathcal{J}_1(0) : z \in [0, \infty], \hspace{1cm} \text{repulsive},$$

$$\mathcal{J}_2(0^\pm) : z \in [0, -\infty] \cup [C^C_\infty \text{ or } C^C_\infty] \hspace{1cm} \text{attractive},$$

and are shown in Fig.1. The integrations are given by

$$Z_2 \sim \xi^2 \mathcal{V}$$ \hspace{1cm} \text{non-interacting},

$$Z_2 \sim \xi^2 \mathcal{V} (\mathcal{V} + I(g^2))$$ \hspace{1cm} \text{repulsive},

$$Z_{2,\pm} \sim \xi^2 \mathcal{V} (\mathcal{V} + I(g^2 e^{\pm i\pi}))$$ \hspace{1cm} \text{attractive},$$

where the extensive part corresponds to free (non-interacting) monopole gas with fugacity $\xi$ and $I(g^2)$ in the sub-extensive term is called second virial coefficient, capturing the effect of interactions. It is given by:

$$I(g^2) = \frac{4\pi}{6} \left( \frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2} \right)^3 \left( \ln \left( \frac{2\pi |\alpha_i \cdot \alpha_j|}{g^2} \right) + \gamma - \frac{11}{6} \right).$$  \hspace{1cm} (7)
In the repulsive case, subextensive part is usually called magnetic bion, and its amplitude is \(|M_\alpha, \overline{M}_{\alpha, \pm 1}| = I(g^2)|M_\alpha, |\overline{M}_{\alpha, \pm 1}| [29]\). In statistical physics, this is a configuration in 2-cluster, \(C_2\).

Attractive case is more interesting. First, note that

\[
I(g^2 e^{\pm i\pi}) = e^{i\pi} I(g^2) \pm i \frac{2\pi \alpha_i \cdot \alpha_j}{g^2} \left( \frac{2\pi \alpha_i \cdot \alpha_j}{g^2} \right)^3,
\]

which implies two different remarkable phenomena for this element of \(C_2\). First, we identify the subextensive part in the attractive case with neutral bions, \([M_\alpha, \overline{M}_{\alpha, \pm 1}] = I(g^2 e^{\pm i\pi})|M_\alpha, |\overline{M}_{\alpha, \pm 1}| [29]\). First, the contribution is two-fold ambiguous. This is expected, because we are formulating the path integral on a Stokes line, and the configurations with attractive interactions are expected to lead to two-fold ambiguous results. At least in some limit of QFT, we will prove that this two-fold ambiguity cancels against another ambiguity.

The overall phase in front of (8) is equally interesting. It tells us that the fugacity of the two-cluster elements can be complex!

\[
\text{Arg}([M_\alpha, \overline{M}_{\alpha, \pm 1}]) = \text{Arg(Re}[M_\alpha, \overline{M}_{\alpha, \pm 1}]) + \pi \tag{9}
\]

This is in some sense similar to [10, 11] where there is a relative topological phase (sourced by Berry phase or \(\theta\) angle) between monopole-events. This relative phase between the two contributions is now sourced by the phase associated with thimble and is called hidden topological angle (HTA) [30]. It is known to play crucial role in semi-classics. For example, in \(N = 1\) on \(\mathbb{R}^3 \times S^1\), and SUSY and QES quantum mechanics, the vanishing of the vacuum energy (or NP contributions) is due to relative phase between these two types of events. For other aspects of bions, see [8, 12, 18, 31–38].

At third order in semi-classics, events such as \([M_\alpha, \overline{M}_{\alpha, \pm 1}] = O(e^{-3S_0}) \pm iO(e^{-3S_0})\) provide a two-fold ambiguous contribution to mass gap, which is supposed to be real and unambiguous. The fact that the ambiguity in mass first appears in the third order comes from the structure of resurgence triangle \([6]\). As it stands, this is quite disturbing for Polyakov’s well-known solution.

We can anticipate that these ambiguities in 2-event and 3-event contributions must be related with the non-Borel summability of the perturbation theory around perturbative vacuum and 1-instanton sector, respectively. The left/right Borel resummation is two-fold ambiguous as well, and these two types of ambiguities are expected to cancel.

However, it is difficult to test this scenario in full QFT. As we describe, it is also not possible to address this question by using a naive reduction of QFT to QM within Born-Oppenheimer (BO) approximation. However, we propose a compactification with discrete ‘t Hooft flux in which monopole actions remain the same, and such cancellations in QM limit can be shown.

**Periodic \(T^2\) compactification does not work:** One may consider compactification on \(T^2 \times \mathbb{R}\), and study the interplay of instantons and perturbation theory in QM reduction. However, a problem awaits us here. The states in the QM are described in terms of magnetic flux through \(T^2, |\Phi\rangle\). The lowest energy state is zero flux state \(|0\rangle\) and magnetic flux states with non-zero flux \(|\alpha_a\rangle\) are parametrically separated in energy: \(E_0 = 0\) and \(E_{\alpha_a} = \frac{\Phi^2}{4\pi^2} |A_{T^2}|\). Therefore, within BO approximation, the reduced QM possess a unique perturbative vacuum, and does not have instantons. It is not possible to obtain a knowledge concerning \(\Phi\) from naive dimensional reduction with periodic compactification in BO-limit.

However, despite being correct, this is a little bit over-simplified. For example, for \(SU(2)\) gauge theory, the flux states are \(|0\rangle, |+1\rangle, |\pm 1\rangle\) mix up with \(|−1\rangle\) due to two instanton effects with action \(2S_0\) where \(S_0\) is the monopole action in QFT. Therefore, we can try to engineer a vacuum structure by turning on background fluxes, such that the instantons of QFT survives in the ground state description of QM.

**From Polyakov to QM (with ‘t Hooft flux):** The Polyakov model has a \(\mathbb{Z}_{N+1}^\dagger\) 1-form symmetry, but no mixed anomalies. We can turn on a discrete flux to examine the dynamics of the theory [39–52]. Since the model is abelianized at long distances, we can replace our way of thinking in terms of discrete flux with a magnetic flux in co-weight lattice thanks to the relation \(\mathbb{Z}_N \cong \Gamma_N^\dagger/\Gamma_{\dagger}^\dagger\). Turning on the background flux, \(\mu_1 \in \Gamma_N^\dagger\), we end up with \(N\) degenerate states,

\[
|\nu_1\rangle \rightarrow |\nu_2\rangle \rightarrow \cdots \rightarrow |\nu_N\rangle
\]

connected to each other via monopole-events \(\alpha_a \in \Delta^0\). Below, we argue that these instanton events have the same action as in \(\mathbb{R}^3\). Assume \(T^2\) size \(L\) obey

\[
r_m \ll L \ll d_{\text{mm}}
\]

where \(r_m\) is the monopole core size, and \(d_{\text{mm}}\) is the characteristic distance between monopoles on \(\mathbb{R}^3\). This guarantees that at a fixed Euclidean time slice \(\tau\) and per \(T^2\) size, there will typically be at most one-monopole. We also choose \(L \gg r_m\) so that the theory is locally 3d, and the action, which receives its contribution from the core region of monopole, remains unchanged relative to \(\mathbb{R}^3\). At distances \(\tau \gtrsim L\), the theory is correctly described by a simple quantum mechanical system with instantons, whose action are same as in the original 3d theory.

Here, we focus on \(N = 2\). The insertion of \(\Phi_{bg} = \frac{1}{2}\) modifies the energetics of the set-up. It turns the flux states into \(|n + \frac{1}{2}\rangle\) with perturbative energies \(E_n = \frac{\Phi^2}{4\pi^2} a_{\pm 1/2}|A_{T^2}|\).
work in the \(\lambda \to 0\) (BPS) limit, and investigate resurgent structure in \(g^2\) only. This make sense provided \(\lambda \ll g^2\). At finite \(\lambda\), one needs a double-series and relatedly, the action acquire a \(\lambda\) dependence, \(S_0 = \frac{4\pi}{g^2} f(\lambda)\) [58].

**Back to \(\mathbb{R}^3\):** Resurgent cancellations (13) are not easy to prove in full QFT on \(\mathbb{R}^3\). But they are proven in the small \(T^2\times \mathbb{R}\) with discrete flux, a construction in which instantons of infinite volume theory survive. In QFT, what we know rigorously is the existence of ambiguity in the correlated events. The imaginary ambiguous parts at the second and third order in QFT on \(\mathbb{R}^3\) are given by

\[
\text{Im}[\mathcal{M}_{\alpha_1,\alpha_0}]_{\pm} \sim \pm i \left(\frac{s_0}{g^2}\right)^7 e^{-2s_0/g^2}
\]

\[
\text{Im}[\mathcal{M}_{\alpha_1,\alpha_0}]_{\pm} \sim \pm i \left(\frac{s_0}{g^2}\right)^{12} \ln \left(\frac{s_0}{g^2}\right) e^{-\frac{2s_0}{g^2}}
\]

(15)

Here, the power of \(\left(\frac{s_0}{g^2}\right)\) is \(2\nu+3(\nu-1)\) for \(\nu = 2, 3, \ldots\) is the number of instantons that enter to correlated events. Recall that each monopole has 4 bosonic zero mode, and each zero mode induce \((s_0/g^2)^{1/2}\) in the prefactor, explaining \(2\nu\). Each quasi-zero mode direction gives a factor of three, hence \(3(\nu-1)\). For a general \(\nu\)-instanton configuration, the power of \(\ln \left(\frac{s_0}{g^2}\right)\) is given by \(\nu - 2\).

In order Polyakov’s result on mass gap to be meaningful, (real, unambiguous), the counterpart of (13) must hold in full QFT. With this assumption, and using dispersion relations such as \(b^{(0)}_k = \frac{1}{\pi} \int_0^\infty d(g^2) \frac{\text{Im} \xi_{\alpha}(g^2)}{(g^2)^{\nu+1}}\), we can determine the large-order growth of perturbation theory around perturbative vacuum and monopole saddle as:

\[
b^{(0)}_k \sim \frac{\Gamma(k+7)}{(2s_0)^k} ; \quad s_0 = 4\pi
\]

\[
b^{(1)}_k \sim \frac{\Gamma(k+10) \ln(k+10)}{(2s_0)^k}.
\]

(16)

Few remarks are in order. Relative to standard quantum mechanical result for \(b^{(0)}_k\) and \(b^{(1)}_k\), where the factorial growth appears generally as \(\Gamma(k+1)\) for ground state, we obtain an enhancement. This is due to the difference of the number of zero and quasi-zero modes in the two set-ups. In \(b^{(1)}_k\), there is an extra \(\ln(k+10)\) enhancement as well. The log enhancement also appears in the context of quantum mechanics around instanton sectors [59–61]. It is there because the 3-instanton amplitude has a \((\ln(-s_0/g^2))^2\) in it, coming from integrating out of two quasi-zero modes, which leads to log-dependent imaginary part in (15). It is quite curious to note that \(b^{(1)}_k \sim \frac{d}{dx} b^{(0)}_{k-2} |_{x=2}\) asymptotically, reminiscent of exact P/NP relation in quantum mechanics [17, 61]. The relation in QM is exact; it tells us that perturbation theory around instanton is dictated by perturbation theory around perturbative vacuum via a simple relation. It would be remarkable if such a relation also holds in QFT.
The appearance of this enhancement is a relatively new effect in QFT, an example of which also appeared in [62]. But in retrospect, it is inevitable, and generic. Application of stochastic perturbation theory on lattice can be useful to check these predictions [63–65], especially by modifying the scalar potential in [64] to generate adjoint Higgsing and monopole confinement.

Conclusions: In order Polyakov’s famous analysis for mass gap to be justified, one needs (13) to be true on \( \mathbb{R}^3 \). We showed that the relation (13) is true in a special quantum mechanical reduction of Polyakov model with discrete ’t Hooft flux. The reduction has the merit that it remembers the instanton of the theory on \( \mathbb{R}^3 \) on small \( T^2 \times \mathbb{R} \) in the description of ground state. We emphasize that the use of ’t Hooft flux or twisted boundary conditions is not a nuisance regardless of absence/presence of a mixed anomaly. To the contrary, it is necessary to make the instantons (or fractional instantons in deformed YM on \( \mathbb{R}^3 \times S^1 \)) transparent in a quantum mechanical reduction [44, 49, 52, 66]. Assuming that (13) continues to be valid in the decompactification limit provides estimates of large-order structure of perturbation theory around perturbative vacuum and instanton (16). We hope that these relations can be tested using stochastic perturbation theory, putting Polyakov’s analysis on a firmer ground.

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