Erratum

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The author apologizes for introducing a wrong new object in the above stated article. The sheaf $\text{Hom}_{\text{cont}}(C_\ast, dg(g), C)$ of continuous homomorphisms of sheaves between the sheaf of differential graded chains $C_\ast, dg(g)$ of a sheaf of Lie algebras $g$ and the constant sheaf $C$ is not a good object. It was intended to generalize continuous cochains of Lie algebras to a sheaf setting, but this is probably not possible for cochains with trivial coefficients.

The space of global sections $\text{Hom}_{\text{cont}}(C_\ast, dg(g), \mathbb{C})$ of $\text{Hom}_{\text{cont}}(C_\ast, dg(g), C)$ is smaller than the space of continuous cochains $\text{Hom}_{\text{cont}}(C_\ast, dg(g(X)), C)$, for example for the sheaf of differentiable vector fields $\text{Vect}$ on a finite dimensional compact manifold $X$.

Indeed, examples of cochains (i.e. elements of $\text{Hom}_{\text{cont}}(C_\ast, dg(g(X)), \mathbb{C})$) are evaluations $D_{x_0}(\xi_1, \ldots, \xi_r) = D(\xi_1, \ldots, \xi_r)(x_0)$ at a point $x_0 \in X$ of some differential expression $D(\xi_1, \ldots, \xi_r)$ where $\xi_1, \ldots, \xi_r \in \text{Vect}(X)$. $D_{x_0}$ cannot be viewed as a morphism of sheaves, because $D_{x_0}$ is not restrictable to the open set $U^* = U \setminus \{x_0\}$, where $U$ is an open neighbourhood of $x_0$:

$$
\begin{array}{ccc}
\Lambda^r(\text{Vect}(X)) & \xrightarrow{D_{x_0}} & \mathbb{C} \\
\text{res}_{\Lambda^r(\text{Vect})} & & \text{res}_C \\
\Lambda^r(\text{Vect}(U^*)) & \xrightarrow{\phi} & \mathbb{C}
\end{array}
$$

But since $\text{res}_C = \text{id}_C$, there does not exist a $\phi$ making the diagram commutative. Thus, the cochain $D_{x_0}$ is not an element of $\text{Hom}_{\text{cont}}(C_\ast, dg(g), \mathbb{C})$. 
A similar argument shows that cochains which are integrals over differential expressions are not elements of $\text{Hom}_{\text{cont}}(C^*_s, dg(g), \mathbb{C})$. These two examples constitute the most important classes of cochains with values in the trivial module $\mathbb{C}$.

Clearly, sheaf approaches are possible (and carried out) in the case of cochains with values in a module which is itself a non-constant sheaf, for example the sheaf of sections of a vector bundle.

In conclusion, one has to ban the object $\text{Hom}_{\text{cont}}(C^*_s, dg(g), \mathbb{C})$ from the setting from the article (in particular, 1.1.8). The spectral sequence for continuous differential graded cohomology (lemma 1 to 4), the cosimplicial version (section 2.3, theorem 4) work still well, and the main result of the article (theorem 7) is unaffected.
Differential Graded Cohomology and Lie Algebras of Holomorphic Vector Fields

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Abstract

This article continues work of B. L. Feigin [5] and N. Kawazumi [15] on the Gelfand-Fuks cohomology of the Lie algebra of holomorphic vector fields on a complex manifold. As this is not always an interesting Lie algebra (for example, it is 0 for a compact Riemann surface of genus greater than 1), one looks for other objects having locally the same cohomology. The answer are a cosimplicial Lie algebra and a differential graded Lie algebra (well known in Kodaira-Spencer deformation theory). We calculate the corresponding cohomologies and the result is very similar to the result of A. Haefliger [12], R. Bott and G. Segal [2] in the case of $\mathcal{C}^\infty$ vector fields. Applications are in conformal field theory (for Riemann surfaces), deformation theory and foliation theory.

Introduction

The continuous cohomology of Lie algebras of $\mathcal{C}^\infty$-vector fields [2], [7] has proven to be a subject of great geometrical interest: One of its most famous applications is the construction of the Virasoro algebra as the universal central extension of the Lie algebra of vector fields on the circle.

So there is the natural problem of calculating the continuous cohomology of the Lie algebra of holomorphic vector fields on a complex manifold, see [5] and [15] for Riemann surfaces.
This work is a solution of this problem for arbitrary complex manifolds up to the calculation of the cohomology of spaces of sections of complex bundles on the manifold - this is very close to the result for $C^\infty$-vector fields. We also show the relation between the cohomology of the holomorphic vector fields and the differential graded cohomology of some differential graded Lie algebra. The method is the one of R. Bott and G. Segal \[1\] - used also by N. Kawanumia \[15\], and for the relation with the differential graded cohomology, based on the article of B. L. Feigin \[5\].

One interest is in compact complex manifolds: Here, the Lie algebra of holomorphic vector fields seems to be too small to be interesting - for compact Riemann surfaces of genus $g$ it is of dimension 3 for $g = 0$, 1 for $g = 1$ and 0 for $g \geq 2$. However, treating the holomorphic vector fields as a sheaf rather than taking brutally global sections proves to reveal a richer cohomology theory, as first remarked by B. L. Feigin \[5\].

We study the relation of the sheaf $\mathcal{H}ol$ of Lie algebras of holomorphic vector fields to the sheaf $\mathfrak{g}$ of vector valued differential forms of type $(0, q)$ where the the values are in the holomorphic vector fields. It is called the sheaf of Kodaira-Spencer algebras and it constitutes a sheaf of differential graded Lie algebras which is a fine sheaf resolution of $\mathcal{H}ol$.

We will calculate differential graded (co)-homology for the Kodaira-Spencer algebra (i.e. the space of global sections of $\mathfrak{g}$), also with coefficients.

Another important idea of this article is the following:

Let $\mathfrak{h}$ a sheaf of differential graded Lie algebras. There is a sheaf of differential graded coalgebras $C_{dg,*}(\mathfrak{h})$ with corresponding sheaf of differential graded Lie algebra homology $H_{*,dg}(\mathfrak{h})$. This is the sheafified Quillen functor, see \[18\] and \[12\]. In the same way, there is a sheaf of differential graded algebras $C_{dg}^*(\mathfrak{h})$ corresponding to the sheaf of differential graded cohomology $H_{*,dg}^*(\Gamma(U, \mathfrak{h}))$ of $\mathfrak{h}$.

Now assume that $\mathfrak{h}$ is not necessarily fine, but that there is a morphism $\phi$ to a fine sheaf $\mathfrak{g}$ of differential graded Lie algebras which is a cohomology equivalence (i.e. $H_{*,dg}^*(\Gamma(U, \mathfrak{g})) = H_{*,dg}^*(\Gamma(U, \mathfrak{h}))$) on each contractible open set $U$.

In this case, hypercohomology (for the differential sheaf $C_{dg}^*(\mathfrak{h})$) and cosimplicial cohomology (i.e. the cohomology of the realization of the simplicial complex obtained from applying the functor $C_{dg}^*$ to the Cech resolution of $\mathfrak{h}$) coincide under suitable finiteness conditions for $\mathfrak{g}$ and $\mathfrak{h}$.

This is true because $\phi$ induces an isomorphism on the cohomology sheaves of the sheaves $C_{dg}^*(\mathfrak{g})$ and $C_{dg}^*(\mathfrak{h})$, inducing an isomorphism in hypercohomology. As $\mathfrak{g}$ is fine, hypercohomology is just the cohomology of the complex of global sections of $C_{dg}^*(\mathfrak{g})$. On the simplicial side, we have a morphism of simplicial cochain complexes induced by $\phi$ which is a cohomology equivalence on the realizations, see \[12\], lemma 5.9. By a standard argument using partitions of unity for the fine sheaf $\mathfrak{g}$, see \[4\]§8, the realization of the simplicial cochain complex gives the cohomology of the complex of global sections of $C_{dg}^*(\mathfrak{g})$.

We will apply this scheme of reasoning to the sheaf of holomorphic vector fields $\mathfrak{h} = \mathcal{H}ol$ on a complex manifold, and its fine resolution given by the sheaf $\mathfrak{g}$ of $d\bar{z}$-forms with values in holomorphic vector fields, the sheaf of Kodaira-Spencer algebras.

Applications of these calculations are in conformal field theory, cf \[5\], in deformation theory, cf \[14\] and in the theory of foliations. This work originated in the attempt to understand Feigin’s article \[5\], so the text is relying heavily on \[5\].
The content of the paper reads as follows: The first part is devoted to cohomology calculations: section 1 is concerned with the definition of differential graded cohomology (also with coefficients), hypercohomology, the spectral sequences that go with them as tools for calculations, and the introduction of the sheaves $\text{Hol}$ and $\mathfrak{g}$; section 2 studies the cohomology of $\text{Hol}(U)$ on a Stein open set $U$ linking it with the differential graded cohomology of $\Gamma(U, \mathfrak{g})$; in the end of section 2, we treat the cosimplicial version which gives an equivalent point of view according to the idea explained in the introduction; section 3 gives the calculation of the cohomology in section 2 in terms of the cohomology of some spaces of sections of some bundle on the manifold - the result is very close to Bott, Haefliger and Segal’s result [2], [12]. The second part is concerned with the applications of these calculations: section 4 just mentions the existing link to conformal field theory, see [3]; section 5 treats the applications in deformation theory, following [14]; section 6 shows a glimpse of possible applications in the theory of characteristic classes of foliations.

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Notations

As a general rule, $g, h$ will denote Lie algebras and gothic letters $\mathfrak{g}, \mathfrak{h}$ will denote sheaves of Lie algebras. For differential graded Lie algebras, the differential will be displayed in the notation: $(\mathfrak{g}, d)$ is a differential graded Lie algebra and $(\mathfrak{g}, d)$ is a sheaf of differential graded Lie algebras.

After the preliminaries, the letter $\mathfrak{g}$ will be reserved for the Kodaira-Spencer algebra, viewed as a sheaf of differential graded Lie algebras.

1 Preliminaries

1.1 Differential Graded (Co-)Homology

1.1.1 Let $g$ an infinite dimensional topological Lie algebra. Its (co-)homology is calculated by associating to $g$ a differential graded coalgebra $C_*(g) = (\Lambda^*(g), d)$ and a differential graded algebra $C^*(g) = (\text{Hom}(\Lambda^*(g), \mathbb{C}), d)$, the homological and the cohomological Chevalley-Eilenberg complex, and then taking their (co-)homology. In order to keep notations clear, we will suppress structures we don’t need in the notation, as for example the algebra and coalgebra structure and the grading here. As we deal with tensor products of infinite dimensional topological vector spaces, we will always take them to be completed.

It is worth taking only the continuous duals instead of the algebraic duals $\text{Hom}(\Lambda^*(g), \mathbb{C})$ in the definition of cohomology, denoted then $C^*_{\text{cont}}(g)$, in order to improve caculability and avoid pathologies.

1.1.2 Let $(g = \bigoplus_{i=0}^{\infty} g^i, \bar{\partial})$ a (cohomological) differential graded Lie algebra, dgla for short. There are as before two functors, noted here $C_{*, dg}$ and $C^*_{dg}$, associating to $(g, \bar{\partial})$ a differential graded coalgebra $C_{*, dg}(g)$ and a differential
graded algebra $C^*_{dg}(g)$. We will assume continuous duals in $C^*_{dg}(g)$ without displaying it in the notation.

$C_{*,dg}(g)$ and $C^*_{dg}(g)$ extend the functors in 1.1.1: for a trivial dgla $(g, \bar{\partial}) = (g, 0)$ with only its $0^{th}$ space in the grading non-zero, we have $C_{*,dg}(g) = C_*(g)$ and $C^*_{dg}(g) = C^*_{cont}(g)$.

$C_{*,dg}$ is called the Quillen functor, see [18], and explicitly constructed in [14] §2.2. The cohomology version was, to the knowledge of the author, first used by Haefliger [12], see also [19] for useful remarks.

1.1.3 Explicitly

\[
C_{k,dg}(g) := \bigoplus_{k=p+q} C^p_{dg}(g)^q := \bigoplus_{k=q+p} S^{-p}(g[1])^q
\]

as graded vector spaces. “$dg$” stands for “differential graded”. Here $S^p(g[1])^q$ is the graded symmetric algebra $S^*$ on the shifted by 1 graded vector space $g[1]$, i.e.

\[
g[1]^q := g^{q+1}.
\]

Note that for $g^0 \neq 0$, we have in $g[1]$ a component of degree $-1$. $S^{-p}(g[1])^q$ is bigraded by the tensor degree $-p$ and the internal degree $q$ which is induced by the grading of $g[1]$. The differential on $C_{*,dg}(g)$ is the direct sum of the graded homological Chevalley-Eilenberg differential in the tensor direction (with degree reversed in order to have a cohomological differential) and the differential induced on $S^*\big(\big(g[1]\big)^*\big)$ by $\bar{\partial}$, still noted $\bar{\partial}$.

Note that the differential graded homology of $g$, denoted by $C_{*,dg}(g)$, is calculated by a cohomological complex, but involving the homological Chevalley-Eilenberg differential.

1.1.4 $C_{*,dg}(g)$ is the total direct sum complex associated to the bicomplex $\{S^{-p}(g[1])^q\}_{p,q}$. So, there is a spectral sequence associated to the filtration by the columns, taking first cohomology in one column, i.e. cohomology with respect to $\bar{\partial}$. Note that $H^*_\bar{\partial}$ is a functor from dgla’s to graded Lie algebras. Let us identify the $E_2$ term as well as where to the sequence converges:

**Lemma 1** Suppose that the complex $(g, \bar{\partial})$ is a topological complex of Fréchet nuclear spaces.

There is a spectral sequence with

\[
E_2^{p,q} = H^p_{gla}(H^q_\bar{\partial}(g))
\]

converging to $H_{p+q,dg}(g)$, i.e. the differential graded homology of $(g, \bar{\partial})$. Here, $H^*_{gla}$ denotes the cohomology of graded Lie algebras.

**Remark:** Other names for the morphisms involved in a topological cochain complex are strict morphisms or homomorphisms, see [3] Ch. III, §2, no. 8. These are not necessarily “morphismes forts” or split in the sense of [11] or [21].

**Proof:**

For $E_2$, the only thing which is not clear is

\[
C_{*,dg}(H^*_\bar{\partial}(g)) = H^*_\bar{\partial}(C_{*,dg}(g)).
\]
This follows directly from prop. 2.1 in [18] in case we would not have been taking completed tensor products.

This proposition holds also in the completed tensor product version, when the spaces involved are Fréchet nuclear spaces or its strong duals, and the complexes are topological:

In this case, there is a Künneth formula, cf [15] p. 673. Then we can conclude as in [18], but we won’t have a topological isomorphism, which is irrelevant for us.

The convergence is a more difficult problem because the shifting \( g[1] \) in \( C_{*,dg}(g) \) creates an internal degree -1 for a dgla \((g, \bar{\partial})\) and so the spectral sequence is not contained in the third quadrant. Actually, it is contained in the fourth quadrant. By the classical convergence theorem (cf [24] p. 135) the spectral sequence associated to the filtration by the columns converges to the total direct sum complex. This is by definition our differential graded homology. □

1.1.5 It is clear how to incorporate coefficients in a differential graded module \((M = \bigoplus_{i=0}^{k} M^i, \partial)\): such a module \( M \) is given as the direct sum of its components \( M^i \) and carries a differential \( \partial \) and an action of a dgla \((g, \bar{\partial})\) such that for \( x \in g \) and \( m \in M \), we have

\[
\tilde{\partial}(x.m) = \bar{\partial}(x).m + (-1)^{deg(x)}x.\tilde{\partial}(m).
\]

Now, take the graded tensor product \( C_{*,dg}(g) \otimes M \) or the graded Hom-functor \( Hom(C_{*,dg}(g), M) \) with the action incorporated in the Chevalley-Eilenberg differential and the differential \( \partial \) glued together with the differential \( \tilde{\partial} \) on \( C_{*,dg}(g) \), i.e. in the homological case

\[
\partial_{tot}(x \otimes m) = (-1)^{degx}(1 \otimes \tilde{\partial})(x \otimes m) + (\tilde{\partial} \otimes 1)(x \otimes m)
\]

and in the cohomological case

\[
\partial_{tot}f = \tilde{\partial} \circ f + (-1)^{degf}f \circ \bar{\partial}.
\]

We suppose further that \( M \) is a topological Fréchet nuclear module, and take completed tensor products.

Note that, as before, the functor \( H^\ast_{\partial_{tot}} \) transforms differential graded objects in graded objects.

There is analoguously a spectral sequence and its corresponding lemma in this case:

Lemma 2 There is a spectral sequence with

\[
E_{2}^{p,q} = H^p_{gla}(H^q_{\partial_{tot}}(g \otimes M))
\]

converging to \( H_{p+q,dg}(g, M) \), i.e. the differential graded homology of \((g, \bar{\partial})\) with coefficients in the differential graded module \((M, \tilde{\partial})\). Here, \( H^\ast_{gla} \) denotes the cohomology of graded Lie algebras with coefficients.

1.1.6 Now we want to calculate differential graded cohomology instead of homology, so let me specify a setting where this is possible.

Let \((g, \bar{\partial})\) a topological dgla such that \( g \) is a Fréchet nuclear space. This permits to calculate cohomology by calculating homology on the continuous dual:
Lemma 3 We have
\[ C^*_d (g) \cong (C_*, d_*(g^*)) \]
where we treat all objects as graded vector spaces and \( g^* \) is the continuous dual of \( g \) as a topological vector space.

Proof:
This follows directly from the following proposition, see for example [22], prop. 50.7 p. 524:

Proposition 1 The continuous dual of a completed tensor product of two nuclear Fréchet spaces is the completed tensor product of the continuous duals of the two spaces. □

So, there is a spectral sequence for the differential graded cohomology in case \((g, \bar{\partial})\) is also a topological complex, namely, the one from lemma 1.

1.1.7 In the same setting as in 1.1.6, suppose that \((g, \bar{\partial})\) is a resolution of a Lie algebra \( h \) which is a topological complex. As J.- P. Serre showed in [24], the \( \bar{\partial} \) resolutions on compact Kähler or Stein manifolds are always topological cochain complexes.

For topological cochain complexes of Fréchet nuclear (or its dual) spaces, it is known that the strong dual complex is topological and has the dual cohomology spaces, cf [13] p. 673. This suites very well with our approach of cohomology by the homology on the duals.

In conclusion, the resolution \((g, \bar{\partial})\) of \( h \) induces an exact sequence for the strong duals, and by the remark in 1.1.4, the spectral sequence in cohomology collapses at the second term.

So, we have:

Lemma 4 Let \((g, \bar{\partial})\) a dgla as in 1.1.5 such that
\[ H^*_\partial (g) = \begin{cases} h \text{ for } * = 0 \\ 0 \text{ for } * = 1, 2, ... \end{cases} \]

Then
\[ H^*_d (g) = H^*_\text{cont} (h) \]

Let me remark that the spectral sequence here is converging in the sense of complete convergence, cf [24] p. 139, and to the total direct product complex.

1.1.8 All these notions extend to sheaves of Lie algebras and sheaves of dgla’s: Let \( X \) a complex manifold of complex dimension \( n \). Denote by \( O_X \) the coherent sheaf of holomorphic functions on \( X \) and by \( E_X \) the sheaf of \( C^\infty \) functions on \( X \). Let \( g \) a sheaf of \( O_X \)-modules which are Lie algebras. Note that the bracket is not a morphism of \( O_X \)-modules. In some contexts, the action of the elements of the Lie algebra on \( f \in O_X \) should be specified: this leads to the concept of twisted Lie algebras. This is for example the case when considering tensor products over \( O_X \). In our context, everything is \( \mathbb{C} \)-linear, so we need not specify this action. In the same way, let \((g, \bar{\partial})\) a sheaf of dgla’s which are \( E_X \)-modules.

We denote by \( \Gamma (g) \), \( \Gamma (X, g) \) or \( g(X) \) the dgla of global sections of the sheaf \( g \).
By the previous sections, we can associate to $\mathfrak{g}$ resp. to $(\mathfrak{g}, \bar{\partial})$ sheaves of differential graded coalgebras $C_\ast(\mathfrak{g})$, $C_{\ast, dg}(\mathfrak{g})$, $H_\ast(\mathfrak{g})$ and $H_{\ast, dg}(\mathfrak{g})$ where the last two carry the trivial differential. In the same way, we have sheaves of differential graded algebras $C^\ast_{cont}(\mathfrak{g})$, $C^\ast_{dg}(\mathfrak{g})$, $H^\ast_{cont}(\mathfrak{g})$ and $H^\ast_{dg}(\mathfrak{g})$.

Furthermore, we have differential graded coalgebras $C_\ast(\Gamma(\mathfrak{g}))$, $C_{\ast, dg}(\Gamma(\mathfrak{g}))$, $H_\ast(\Gamma(\mathfrak{g}))$ and $H_{\ast, dg}(\Gamma(\mathfrak{g}))$, and the corresponding algebras.

### 1.2 Examples

#### 1.2.1

The prescription $U \to Hol(U)$ where $U$ is an open set of $X$ and $Hol(U)$ is the Lie algebra of holomorphic vector fields on $U$ is a sheaf of Lie algebras, denoted by $Hol$. It is a coherent sheaf. It is in some respect the opposite of a fine sheaf: its restriction maps are injective.

We have a sheaf of differential graded algebras $C^\ast_{cont}(Hol)$ associated to $Hol$.

To be explicit, it is the sheaf $\mathcal{H}om(\Lambda^\ast(Hol), \mathcal{C})$ of morphisms of sheaves between $\Lambda^\ast(Hol)$ and the constant sheaf $\mathcal{C}$. Its underlying presheaf is

$$U \mapsto \mathcal{H}om_{cont}(\Lambda^\ast(Hol))|_{U}, \mathcal{C}|_{U}$$

Here, $\mathcal{H}om_{cont}(\mathcal{F}, \mathcal{G})$ is the functor of continuous sheaf morphisms between two sheaves of topological spaces $\mathcal{F}$ et $\mathcal{G}$, i.e. of morphisms of presheaves $\phi_U = \{\phi_V : \mathcal{F}(V) \to \mathcal{G}(V)\}_{V \subset U}$ such that every $\phi_V$ is continuous.

In particular, it is a differential sheaf, and one subject of this article will be to calculate its hypercohomology:

Taking a sheaf resolution of every graded component of $C^\ast_{cont}(Hol)$, we get in a standard way (cf [1] 4.5, p. 176) a resolution of $C^\ast_{cont}(Hol)$. This gives a bicomplex; the cohomology of the total complex associated to it is by definition the hypercohomology of $C^\ast_{cont}(Hol)$, denoted by $\mathbb{H}(X, C^\ast_{cont}(Hol))$.

As $C^\ast_{cont}(Hol)$ is bounded below, we have two converging spectral sequences (associated to the two canonical filtrations for the bicomplex) for hypercohomology. We need in 2.1.4 the first one, the one given by the filtration by the columns. Its $E_2$ term is

$$E_2^{p,q} = H^p(X, H^q_{cont}(Hol)).$$

Here, $H^p(X, \mathcal{F})$ is the sheaf cohomology of the sheaf $\mathcal{F}$.

The second one is given by the filtration by the rows. Its $E_2$ term is

$$E_2^{p,q} = H^q(H^p(X, C^\ast_{cont}(Hol))).$$

Here, $d$ is the differential which $C^\ast_{cont}(Hol)$ induces in the resolutions of every component.

#### 1.2.2

Let $E$ a holomorphic vector bundle over the complex manifold $X$. Denote by $\mathcal{O}(E)$ the sheaf of (germs of) holomorphic sections of $E$. Denote by $\Omega^{k,l}$ the sheaf of (germs of $C^\infty$ sections of) differential forms of type $(k, l)$ on $X$. The tensor product $\Omega^{0, \ast} \otimes \mathcal{O}(E)$ is a sheaf on $X$, the sheaf of (germs of $C^\infty$ sections of) differential forms with values in $E$ (of type $(0, \ast)$). Let me denote by $\mathfrak{g}$ this sheaf for $E = TX$, the complex tangent bundle of $X$. Note that $\mathcal{O}(TX)$ is simply $Hol$.

$\mathfrak{g}$ is a sheaf of dgla’s: It is a vector space, graded by the degree of the differential form. The bracket on every open set is the restriction to the $(0, \ast)$-type forms with values in $TX$ of the Frölicher-Nijenhuis bracket on $\Gamma(\Omega^{0, \ast} \otimes$
\( Vect \) where \( Vect \) is the sheaf of all vector fields on the real manifold underlying \( X \). This bracket is explained for example in [17], see also section 2.2 of the present article. To give a short indication, it is the bracket of endomorphisms by viewing vector valued differential forms as derivations of the graded algebra of differential forms. The differential is just \( \partial \). It is easy to see that \( \partial \) acts as a graded derivation on the Frölicher-Nijenhuis bracket. We denote by \( (\mathfrak{g}, \partial) \) this sheaf of dgla’s.

\( \mathfrak{g} \) is a fine sheaf because it is a sheaf of \( C^\infty \) sections. So its sheaf of dg algebras \( \text{Hom}_{\text{cont}}(C_*, \mathfrak{g}), \mathbb{C} \) which is as before a sheaf of morphisms of sheaves, is in fact isomorphic to the sheaf of morphisms between the spaces of sections: given a morphism of sheaves \( \phi_U : C_*(\mathfrak{g})|_U \to \mathbb{C}|_U \). i.e. a compatible family

\[
\phi_U = \{ \phi_V : C_*(\mathfrak{g})(V) \to \mathbb{C}(V) \}_{V \subset U},
\]

we can construct a morphism of the spaces of global sections on \( U \) by partitions of unity.

It is well known that the hypercohomology of a fine differential sheaf is just the cohomology of its complex of global sections, see for example [9] thm. 4.6.1 p. 178. This implies

\[
\mathbb{H}(X, C^*_d(\mathfrak{g})) = H^*_d(\Gamma(\mathfrak{g})).
\]

Another goal of this article is to calculate the differential graded cohomology of the Kodaira-Spencer algebra \( \Gamma(\mathfrak{g}) \).

1.2.3 There is one remark in order: Actually, we should indicate in the notation \( H^*_d \) the way in which proceeded to take total complexes associated to double complexes and cohomology with respect to differentials. The ambiguity involved stems from the fact that we are considering here hypercohomology of a bicomplex of sheaves, so the underlying homological problem is a TRI-complex. For example, it would not be the same to apply the global section functor term by term to the bicomplex (for example on a compact Riemann surface), and take its differential graded cohomology afterwards.

Let us thus denote by \( H^*_d \) the differential graded cohomology obtained by the hypercohomology of the complex of sheaves given by the total complex of the bicomplex of sheaves.

Let us also denote by \( H^*_d \) the differential graded cohomology obtained by the cohomology of the total complex of the bicomplex of global sections of the double complex of sheaves.

1.2.4 Let me remark that the sheaf \( \text{Hol} \) is a sheaf of topological Fréchet nuclear Lie algebras because of the canonical Fréchet topology on the space of sections on a coherent sheaf, see for example [10] Ch. V, §6.

In the same way, \( \mathfrak{g} \) is a sheaf of topological Fréchet nuclear dgla’s, see for example [13].

\( \mathfrak{g} \) as a space of \( C^\infty \) functions carries the \( C^\infty \) topology, and the canonical topology on a space of holomorphic functions is the same as the one induced from the \( C^\infty \) topology on the \( C^\infty \) functions.

2 The cohomological link between \( \text{Hol} \) and \( \mathfrak{g} \)

There is a strong relationship between \( \text{Hol} \) and \( \mathfrak{g} \) based on the fact that \( \mathfrak{g} \) is a fine sheaf resolution of \( \text{Hol} \). We will first show this for trivial coefficients,
and then construct the right category of modules such that the relationship still holds for cohomology with coefficients.

### 2.1 Trivial coefficients

#### 2.1.1 Recall some $\bar{\partial}$ resolutions:

**Lemma 5** There is an exact sequence of sheaves

$$0 \rightarrow \text{Hol} \rightarrow (\Omega^{0,0} \otimes \text{Hol}) \xrightarrow{\bar{\partial}} (\Omega^{0,1} \otimes \text{Hol}) \rightarrow \ldots \rightarrow (\Omega^{0,n} \otimes \text{Hol}) \rightarrow 0.$$ 

**Proof:**

Actually, we have an exact sequence of sheaves

$$0 \rightarrow \Omega^p_{\text{hol}} \otimes \mathcal{O}(E) \rightarrow \Omega^{p,0} \otimes \mathcal{O}(E) \xrightarrow{\bar{\partial}} \Omega^{p,1} \otimes \mathcal{O}(E) \xrightarrow{\bar{\partial}} \ldots$$

$$\ldots \xrightarrow{\bar{\partial}} \Omega^{p,n} \otimes \mathcal{O}(E) \rightarrow 0$$

with the sheaf of holomorphic differential forms on $X$, $\Omega^p_{\text{hol}}$, for every holomorphic fiber bundle $E$ on $X$, see for example [25]. Taking $p = 0$ and $E = TX$, we have our sequence. □

**Corollary 1** The sheaf $(\mathfrak{g}, \bar{\partial})$ is a resolution of $\text{Hol}$ by fine sheaves.

#### 2.1.2 Let us look for the open sets $U$ where the corollary holds not only for the sheaves, but for the spaces of global sections on $U$.

**Definition 1** An open set $U \subset X$ of a complex manifold $X$ is called a Stein open set, if we have the following vanishing condition on coherent sheaf cohomology:

$$H^\ast(U, \mathcal{F}) = 0 \quad \forall \ast = 1, 2, 3, \ldots$$

and for all coherent sheaves $\mathcal{F}$ on $X$.

**Lemma 6** For every Stein open set $U \subset X$, $(\Gamma(U, \mathfrak{g}), \bar{\partial})$ is a resolution of $\text{Hol}(U)$.

**Proof:**

This follows from standard sheaf cohomology theory: As $(\mathfrak{g}, \bar{\partial})$ is a fine sheaf resolution of $\text{Hol}$, $H^\ast(U, \text{Hol})$ is the cohomology of $(\Gamma(U, \mathfrak{g}), \bar{\partial})$. By definition of $U$, this cohomology is 0 except perhaps in degree 0. In degree 0, it is $\text{Hol}(U)$. □

Apply now lemma [2] to get immediately

**Corollary 2** For every Stein open set $U$, we have an isomorphism

$$H^\ast_{\text{cont}}(\text{Hol}(U)) = H^\ast_{\text{dg}}(\Gamma(U, \mathfrak{g})).$$
2.1.3 We can state this result in a completely formal setting:

Recall that \( W_1 \) is the Lie algebra of formal vector fields in 1 variable. Consider

\[
G := W_1[[\bar{z}, t]] / (t^2)
\]

Then, \( G \) is a differential graded Lie algebra with the bracket:

\[
[X\bar{z}^k t^n, Y\bar{z}^l t^m] = [X, Y]\bar{z}^{k+l} t^{n+m}
\]

where \( X, Y \in W_1 \) - it is the usual bracket on a tensor product of a Lie algebra with an associative algebra. The grading is given by the polynomial degree in \( t \). The differential is just the operator \( \bar{\partial} \) defined by

\[
\bar{\partial}\left( \sum_i f_i(z, \bar{z}) t^i \frac{\partial}{\partial z} \right) = \sum_i \frac{\partial}{\partial \bar{z}} f_i(z, \bar{z}) t^{i+1} \frac{\partial}{\partial z}
\]

In particular, the elements of \( G \) without \( \bar{z} \) are the kernel of \( \bar{\partial} \) - these are the formal holomorphic vector fields.

So the theorem can be stated in the 1 dimensional formal case as

**Theorem 1**

\[
^1 H^*_{dg}(G) \cong H^*_{cont}(W_1) \cong H^*_{sing}(S^3)
\]

Of course, there exists also the \( n \)-dimensional version, but it is too cumbersome to write it down.

2.1.4 We can pursue 2.1.2 a little bit further applying hypercohomology:

**Theorem 2** For every complex manifold \( X \), there is an isomorphism

\[
\mathbb{H}^*(X, C^*_{cont}(\text{Hol})) = \quad H^*_{dg}(\Gamma(X, \mathfrak{g})).
\]

**Proof:**

The preceding corollary gives the isomorphism on the filtrant family of Stein neighbourhoods of a point \( x \in X \). Passing to the inductive limit, we get an isomorphism of the cohomology sheaves

\[
H^*_\text{cont}(\text{Hol}) \cong \quad H^*_\text{dg}(\mathfrak{g}).
\]

Recall now the hypercohomology spectral sequence from 1.2.1.

The inclusion sheaf morphism \( \text{Hol} \rightarrow \mathfrak{g} \) gives a morphism of differential sheaves inducing a morphism of spectral sequences. This morphism is an isomorphism on the terms \( E_2 \), so by the standard comparison theorem for spectral sequences, we have an isomorphism of the limit terms.

It remains to recall the result of 1.2.2 stating

\[
\mathbb{H}^*(X, C^*_{\text{dg}}(\mathfrak{g})) = \quad H^*_{\text{dg}}(\Gamma(X, \mathfrak{g})). \quad \square
\]

2.1.5 Let us remark that there is an analogous situation for the Hochschild cohomology of the algebra of holomorphic functions \( \mathcal{O}_X(X) \): we have a fine \( \bar{\partial} \)-resolution of the sheaf \( \mathcal{O}_X \) by the sheaves of differential forms of type \((0, k)\), \( \Omega^{0,k} \). On a Stein open set \( U \), we have an isomorphism between the Hochschild cohomology of \( \mathcal{O}_X(U) \) and the differential graded Hochschild cohomology of the differential graded algebra \((\oplus_{k=0}^n \Omega^{0,k}(U), \wedge, \bar{\partial})\). As before, we can pass to the cohomology sheaves and then to hypercohomology. We can even have the cosimplicial cohomology - see section 2.3.
2.2 The coefficient case

Note that $\text{Hol}(U)$ is a $\text{Hol}(U)$-module and $(\Gamma(U, g), \bar{\partial})$ is a differential graded $\Gamma(U, g)$-module by the adjoint action for an open set $U$. In particular, $(\Gamma(U, g), \bar{\partial})$ as a differential graded module $(M, \tilde{\partial})$ verifies

$$\tilde{\partial}(x.m) = \bar{\partial}(x).m + (-1)^{\text{deg}(x)x}\bar{\partial}(m),$$

just by the fact that $\bar{\partial} = \tilde{\partial}$ acts as a graded derivation on the Frölicher-Nijenhuis bracket. We can write the Frölicher-Nijenhuis bracket in our case locally as

$$[\phi \otimes X, \psi \otimes Y] = \phi \wedge \psi \otimes [X, Y] +$$

$$- (i_Y \bar{\partial}\phi \wedge \psi \otimes X - (-1)^{|Y|} i_X \bar{\partial}\psi \wedge \phi \otimes Y)$$

$$= \phi \wedge \psi \otimes [X, Y] +$$

$$+ \phi \wedge \mathcal{L}_X \psi \otimes Y - \mathcal{L}_Y \phi \wedge \psi \otimes X$$

for $\phi \in \Omega^{0,k}(U)$, $\psi \in \Omega^{0,l}(U)$ and $X, Y \in \text{Hol}(U)$.

We will look for differential graded $\Gamma(U, g)$-modules giving a theorem as in 2.1 but with coefficients.

2.2.1 Let $E$ be a holomorphic vector bundle on $X$. As before, let $\mathcal{O}(E)$ be the sheaf of holomorphic sections of $E$. It has a resolution by fine sheaves, given explicitly in the proof in section 2.1.1. Denote by $\mathcal{E}(E)^{0,s}$ the direct sum over all sheaves in this resolution.

2.2.2 As before, $\Gamma(U, \mathcal{E}(E)^{0,s})$ is a differential graded vector space $(M, \tilde{\partial})$ which is resolution of the vector space $\Gamma(U, \mathcal{O}(E))$ for any Stein open set $U$.

2.2.3 Let us now suppose that the Lie algebra $\text{Hol}(U)$ acts on $\Gamma(U, \mathcal{O}(E))$ by differential operators - we can speak of a local action. We can define a differential graded action locally by setting

$$(\phi \otimes X). (\psi \otimes v) = \phi \wedge \psi \otimes X.v + \phi \wedge \mathcal{L}_X \psi \otimes v$$

where $v \in \mathcal{O}(E)$. Note that we dropped the term which is not realizable without an action of $v$ on the forms and an inclusion of the holomorphic vector fields into $\mathcal{O}(E)$. It is obvious that it is in fact a global action.

So we have constructed a differential graded $\Gamma(U, g)$-module naturally induced by the action of $\text{Hol}(U)$ on $\Gamma(U, \mathcal{O}(E))$. It is easy to extend this correspondence to maps between modules, so we have constructed a category of differential graded modules corresponding to the category of local $\text{Hol}(U)$-modules.

2.2.4 We have a functor from local differential graded modules to the category of local $\text{Hol}(U)$-modules simply by taking the cohomology with respect to $\tilde{\partial}$. So we get an equivalence of categories between the category of local $\text{Hol}(U)$-modules and a subcategory of the category of differential graded modules.

2.2.5 Call now the induced module of a local $\text{Hol}(U)$-module either the module constructed in 2.2.3 or - if the $\text{Hol}(U)$-module is $\text{Hol}(U)$ itself with the adjoint action - take $\Gamma(U, g)$ with its adjoint action. Unfortunately, we have to make this distinction because of the difference in the formulae for the action in 2.2.3 and the adjoint action.

2.2.6 We can now formulate the analogous theorem in the coefficient case:
Theorem 3 On a Stein open set $U$, a local $\text{Hol}(U)$-module $N(U)$ induces a differential graded module $(M(U), \bar{\partial})$ which is its resolution. So we have:

$$1^* H_{dg}(\mathfrak{g}(U), (M(U), \bar{\partial})) \cong H^*_{\text{cont}}(\text{Hol}(U), N(U))$$

Proof:
Following 2.2.3, the first statement is clear.

By the spectral sequence calculating differential graded cohomology with coefficients, see the lemma in 1.1.5, the $E_2$ term is the Lie algebra cohomology of $\text{Hol}(U)$ with coefficients $N(U)$ and the sequence collapses. □

2.2.7 Note that Kawazumi calculated the cohomology of $\text{Hol}(X)$ with coefficients in $n$-densities for an open Riemann surface $X$. Taking into account this result (equation (9.7) p.701 [15]), we have completely solved the problem of the differential graded cohomology of $\mathfrak{g}(X)$ with coefficients in (differential graded) tensor densities for open Riemann surfaces.

2.3 The cosimplicial version

2.3.1 Let us think of the tangent sheaf $\text{Hol}$ as a sheaf of Lie algebras constituting an object in the derived category $\mathcal{D}^b(X)$ of the category of bounded complexes of sheaves on the complex manifold $X$. The objects $\text{Hol}$ and $\mathfrak{g}$ are isomorphic in $\mathcal{D}^b(X)$. The Lie algebra structure on $\text{Hol}$ corresponds to the fact that there is a cohomological resolution which is a sheaf of differential graded Lie algebras.

According to [14], for any sheaf of Lie algebras $\mathfrak{h}$ there is another sheaf of differential graded Lie algebras constituting a resolution of $\mathfrak{h}$. It is the sheaf of cosimplicial Lie algebras given by taking $\mathfrak{h}$ on the Cech complex associated to a covering $U$ by Stein open sets, suitably normalised by the Thom-Sullivan functor, see [14].

2.3.2 There is also a notion of cohomology for a cosimplicial Lie algebra: the cohomology of the cosimplicial Lie algebra $\tilde{C}(U, \text{Hol})$ for some covering by Stein open sets $U$ is the cohomology of the realization of the simplicial cochain complex obtained from applying the continuous Chevalley-Eilenberg complex as a functor $\text{C}^*_{\text{cont}}$ to the cosimplicial Lie algebra. We denote cosimplicial cohomology by $H^*_{\text{cos}}$.

2.3.3 As explained in the introduction, the general idea is that this cannot give anything new.

To show this, one constructs a morphism of simplicial cochain complexes

$$\tilde{f} : C^*_{\text{dg}}(\mathfrak{g}(N_*)) \rightarrow C^*_{\text{cont}}(\text{Hol}(N_*))$$

induced by the inclusion $f : \text{Hol}(N_{M,q}) \hookrightarrow \mathfrak{g}(N_{M,q})$ simply by applying the functor $C^*_{\text{dg}}$ to the inclusion. $N_*$ denotes the thickened nerve of the covering $U$, i.e. the simplicial complex manifold associated to the covering $U$. By lemma 5.9 in [2], the morphism $\tilde{f}$ induces a cohomology equivalence between the realizations of the two simplicial cochain complexes (the conditions of the lemma are fulfilled because of the isomorphism of the cohomologies on a Stein open set of the covering and the Künneth theorem). As in prop. 6.2 in [2] using partitions of unity, one shows that the cohomology of the realization of the simplicial cochain complex on the left hand side gives the differential graded cohomology of $\Gamma(X, \mathfrak{g})$.

2.3.4 This gives the following
Theorem 4  On a complex manifold $X$ of dimension $n$, we have

$$1H^*_dg(\Gamma(X, g)) \cong H^*_cos(\mathcal{C}(U, Hol))$$

for any covering of $M$ by Stein open sets $U$.

2.3.5 Observe that we proceeded in the same order of taking cohomology with respect to differentials in the spirit of remark 1.2.3.

3 Calculating the cohomology

3.1.1 I. M. Gelfand and D. B. Fuks calculated the cohomology of the Lie algebra of formal vector fields in $n$ variables $W_n$ (in our setting always with complex coefficients). They showed an isomorphism of the Hochschild-Serre spectral sequence for the subalgebra $gl(n)$ with the Leray spectral of the restriction to the $2n$-skeleton of the universal $U(n)$ principal bundle.

Let us note $\pi: V(\infty, n) \to G(\infty, n)$ the universal principal $U(n)$-bundle and $X(n)$ an open neighbourhood (because the inverse image of the union of the cells is not a manifold) of the inverse image under $\pi$ of the $2n$-skeleton of the Grassmannian $G(\infty, n)$.

Their theorem reads

Theorem 5 (Gelfand-Fuks, cf [7]) There is a manifold $X(n)$ such that

$$H^*_cont(W_n) \cong H^*_sing(X(n)).$$

R. Bott and G. Segal showed that for $R^n$ or more generally a starshaped open set $U$ of an $n$-dimensional manifold $M$, the Lie algebra of $C^\infty$-vector fields $Vect(U)$ has the same cohomology as $W_n$.

The same is true for the Lie algebra of holomorphic vector fields on a disk of radius $R$ in $\mathbb{C}^n$: The map sending a holomorphic field to its Taylor series is continuous (E. Borel’s lemma, see [22] p.190), open (trivial!), injective (trivial!) and of dense image (the series of convergence radius $R$ are dense in the formal series). So they have the same continuous cohomology, cf [23].

3.1.2 N. Kawazumi calculated what seemed to be the only interesting Gelfand-Fuks cohomology related to Lie algebras of holomorphic vector fields on Riemann surfaces, i.e. the cohomology on open Riemann surfaces:

Theorem 6 (Kawazumi, [15]) Let $X$ an open Riemann surface. Then

$$H^*_cont(Hol(X)) \cong H^*(Map(X, S^3))$$

He used the method of Bott-Segal [3] to prove this result, i.e. he constructed a global fundamental map from the cochain complex of the Lie algebra to the complex of differential forms with values in $C^*_cont(Hol(\mathbb{C}))$. This map, denoted by $\hat{f}_\sigma$, is constructed with the help of a global non-vanishing vector field $\partial$ existing on open Riemann surfaces:

$$\hat{f}_\sigma : C^*_cont(Hol(U_\sigma)) \to \Omega^*(U^\sigma; C^*_cont(Hol(\mathbb{C})))$$

$$c \mapsto (\partial^{-1}) \otimes (f_{\sigma,p})_*\partial c + (f_{\sigma,p})_*(c)$$
Here, for a subset $\sigma = \{a_0, \ldots, a_q\}$ of the index set of a covering, $U^\sigma = \bigcup_i U_{a_i}$ and $U_\sigma = \cap_i U_{a_i}$. $(f, p)_s$ is the map induced from a complex immersion of the open set into $\mathbb{C}$ and $i$ is the insertion operator.

It is rather straightforward to generalize this map to the $n$-dimensional case: $\hat{f}_s$ relies on a vector valued differential form $\omega$ which is complicated in the case of Bott and Segal, but here it is just $\omega = \partial^{-1} \otimes \partial$, the identity on $\text{Hol}(X)$. In the $n$-dimensional case, we take $\omega = \sum_{i=1}^n \partial_i^{-1} \otimes \partial_i$. These $\partial_i$-trivializing the tangent bundle - can be chosen such that they are the images of $\frac{\partial}{\partial z_i}$ for a specially chosen parametrization sending a contractible open set into $\mathbb{C}$, cf lemma 6.4 of [15].

3.1.3 In general, there is no such vector field $\partial$, so there one should adapt the fundamental map of Bott-Segal to this holomorphic setting. For this, it is enough to notice that $X(n)$ is homotopically equivalent to a complex manifold carrying a $\text{GL}(n, \mathbb{C})$-action. For example, $X(1)$ is $S^3$ which is homotopically equivalent to $\mathbb{C}^2 \setminus \{0\}$. So, replacing from the real case the principal $U(n)$-bundle (associated to the tangent bundle) by the principal $\text{GL}(n, \mathbb{C})$-bundle (associated to the complex tangent bundle), one has a family of immersions $P$, cf [2] §4 and p. 295, which is parametrized by a complex manifold $(\text{GL}(n, \mathbb{C}))$ and consists of complex immersions. This implies that that the fundamental map, constructed from this family as in [3] §4, goes from (cochains on) holomorphic fields to (holomorphic differential forms with values in cochains on) holomorphic fields. So there are two immediate corollaries:

**Corollary 3** Let $X$ be an $n$ dimensional complex Stein manifold with trivial tangent bundle. Then we have

$$H^*_{\text{cont}}(\text{Hol}(X)) \cong H^*_{\text{sing}}(\text{Map}(X, X(n))).$$

If one drops the “Stein” hypothesis, it is perhaps not possible to globalize the result, but one can stay with the cosimplicial cohomology:

**Corollary 4** Let $X$ be an $n$ dimensional complex manifold with trivial tangent bundle and $\mathcal{U}$ a covering of $X$ by Stein open sets. Then we have

$$H^*_{\text{cos}}(\check{C}(\mathcal{U}, \text{Hol})) \cong H^*_{\text{sing}}(\text{Map}(X, X(n))).$$

From 3.1.3 follows on the other hand:

**Theorem 7** Let $X$ be an $n$-dimensional complex manifold. Then we have:

$$H^*_{\text{cos}}(\check{C}(\mathcal{U}, \text{Hol})) \cong H^*_{\text{sing}}(\Gamma(E_n)).$$

Here, $E_n$ is the bundle with typical fiber homotopically equivalent to $X(n)$ associated to the principal $\text{GL}(N, \mathbb{C})$-bundle on $X$ (gotten from the complex tangent bundle of $X$).

3.1.5 For $\Gamma(\Sigma, \text{g})$ in the case of a compact Riemann surface $\Sigma$, we have Feigin’s theorem (note that many theorems in this article could be named “Feigin’s theorem”):
Theorem 8

\[ 1H^*_{dg}(\Gamma(\Sigma, g)) \cong H^*(Map(\Sigma, S^3)) \]

Proof:
In our setting, this theorem follows from the above considerations because the \((C^2 \setminus \{0\})\)-bundle (or the \(S^3\)-bundle) is trivial:

The given \(S^1\)-representation in \(SO(4)\) may be lifted to \(Spin(4)\) and this representation is used to view the bundle as associated to a principal \(Spin(4)\)-bundle which is trivial because of the existence of a section by obstruction theory combined with dimension arguments. □

Let us remark that one can calculate \(H^*(Map(\Sigma, S^3))\) by standard methods, and the result is given in Feigin’s article. In particular, \(H^4(Map(\Sigma, S^3))\) is 1-dimensional, and fixing a generator means fixing the central charge \(c\) of a Virasoro type cocycle, cf [5].

4 Applications in conformal field theory

4.1.1 Feigin’s article [5] treats the applications in conformal field theory. We will summarize them briefly, see [5] and [1] for more informations. As complex manifolds \(X\), we take here compact Riemann surfaces \(\Sigma\) of genus \(g \geq 2\).

As we deal with homology in this section, we replace the sheaf of holomorphic vector fields \(Hol\) by the sheaf of algebraic vector fields \(Lie\). In view of the stated difficulties in globalizing these vector fields, we take the cosimplicial version, cf §2.3.

4.1.2 Let \(p \in \Sigma\) a point. Following Feigin, let us choose the covering of \(\Sigma\) by a formal disk \(U_2\) around \(p\) (in order to be able to take algebraic fields on it) and the Zariski open set \(U_1 = \Sigma \setminus \{p\}\). This means that \(Lie(U_2)\) is the Lie algebra of formal jets of vector fields at \(p\), completed by the ideal defined by \(p\). A similar remark applies to \(Lie(U_1 \cap U_2)\). So, \(Lie(U_1), Lie(U_2)\) and \(Lie(U_1 \cap U_2)\) form a cosimplicial Lie algebra.

4.1.3 As the choice of a generator for \(H^1(KS(\Sigma))\) fixes the central charge \(c\) of a Virasoro type cocycle, cf 3.1.5, it fixes a cosimplicial Lie algebra associated to \(Lie(U_1), Lie(U_2) \oplus c\mathbb{C}\) and \(Vir(U_1 \cap U_2)\) in the same way as before. We have still inclusions of \(Lie(U_1)\) and \(Lie(U_2) \oplus c\mathbb{C}\) into \(Vir(U_1 \cap U_2)\), because the cocycle is 0 on these subspaces by the residue theorem. The cosimplicial Lie algebra is denoted by \(Lie_0(\Sigma)\).

It has a representation (in the sense of representation of a diagram, cf [5]) noted \(\square_c\), where we associate to \(Lie(U_2) \oplus c\mathbb{C}\), \(Vir(U_1 \cap U_2)\) and \(Lie(U_1)\) respectively \(1_c\) (a 1-dimensional space, \(Lie(U_2)\) acting trivially, \(c\mathbb{C}\) acting by multiplication by \(c\)), its induced module (a Verma module noted \(M_c(p)\)) and its restriction to \(Lie(U_1)\).

4.1.4 There is a similar cosimplicial Lie algebra \(Lie_\square(\Sigma)\) associated to the covering by all Zariski open sets of \(\Sigma\). Such a set is given by finite number of points \(\{p_1, \ldots, p_n\}\). \(Lie_\square(\Sigma)\) has a similar representation: doing the above construction yields a representation space for every \(\Sigma \setminus \{p_1\}\). For Lie algebras associated to sets with more than 1 point, we take the tensor product representation of the Verma modules. Actually, all these modules are linked by induction arrows. This gives a representation of \(Lie_\square(\Sigma)\) still noted \(\square_c\).
One should view $\text{Lie}_0(\Sigma)$ and its representation as a simple model for $\text{Lie}_\triangle(\Sigma)$ and the above representation.

4.1.5 Feigin calculates the (cosimplicial) homology of $\text{Lie}_0(\Sigma)$ and $\text{Lie}_\triangle(\Sigma)$ with values in the above representations. The result is (for simplicity only for $\text{Lie}_0(\Sigma)$)

**Theorem 9**

$$H_i(\text{Lie}_0(\Sigma), \square_c) = \begin{cases} \text{M}_c(p) / \text{Lie}(U_1) \text{M}_c(p) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

The point is that the space of coinvariants on the right hand side which defines the so-called modular functor is usually associated to locally defined objects, as for example the local Virasoro algebra $\text{Vir}(U_1 \cap U_2)$. Feigin obtains here a homological description in terms of globally defined objects.

A second point is that the space of coinvariants is in fact the continuous dual of the completion of the local ring of the moduli space (of compact Riemann surfaces of genus $g \geq 2$) at the point $\Sigma$, provided that $\Sigma$ is a smooth point. This gives an important link between Lie algebra homology and the geometry of the moduli space, cf §5.

4.1.6 The modular functor for what is called a minimal field theory relies on a special choice of the central charge $c$, dictated from Virasoro representation theory, see for example [4]. Furthermore, instead of Verma modules one deals with their irreducible quotients. Feigin shows that the above setting can be adapted to this situation.

The modular functor associates to $\Sigma$ a finite dimensional vector space; this fact relies in our context on the theorem, cf lemma 4.1.1 p. 16 [6], stating that coinvariants in a representation with 0 singular support are finite dimensional.

5 Applications in deformation theory

5.1 Deformations of complex manifolds

In this section, we give links from the cohomology calculations in the first part to the deformation theory of complex manifolds, still relying strongly on the ideas of [5] and here also [4].

It will concern particularly the differential graded homology of $\Gamma(X, \mathfrak{g})$ for a complex manifold $X$. The most part of this section is more generally true for smooth proper schemes, see [14].

5.1.1 The most basic idea in this context is the following, taken from [4]:

"The completion of a local ring of a moduli space at a given point $X$ is isomorphic to the dual of the $0$th homology group of the Lie algebra of infinitesimal automorphisms of $X"."

Let me underline once more that this links Lie algebra homology and the geometry of the moduli space in a formal neighbourhood of a point.

5.1.2 As Feigin remarked, we have for Riemann surfaces an incarnation of this principle:

**Theorem 10** Let $\Sigma$ be a compact Riemann surface of genus $g \geq 2$. Then

$$^2H_{0, dg}(\Gamma(\Sigma, \mathfrak{g})) = S^*(T^*_\Sigma M(g, 0))$$

and the other homology spaces are 0.
Remark: Note that we have here $H_{0,dy}^2$; this reminds you of the way we defined the differential graded homology of a sheaf of differential graded Lie algebras, see 1.2.3.

Proof:

It is the result from the Kodaira-Spencer deformation theory for Riemann surfaces $\Sigma$ that we have

$$H^1(\Sigma, Hol) = T_\Sigma \mathcal{M}(g, 0).$$

Also, $H^0(\Sigma, Hol) = 0$. So the theorem follows directly from the lemma in 1.1.6, because the graded Lie algebra homology of an abelian Lie algebra in degree 1 is just the symmetric algebra on it. □

Taking continuous duals in the theorem, we get the principle stated in 6.1.1 viewing $S^*(T_\Sigma \mathcal{M}(g, 0))^*$ as the completion of the local ring which is possible if the point $\Sigma$ is smooth in $\mathcal{M}(g, 0)$.

5.1.3 The theorem of 6.1.2 is still true for higher dimensional complex manifolds $X$ as long as

$$H^1(X, Hol) = T_\Sigma \mathcal{M}(g, 0)$$

and zero otherwise. So there are two problems, well known in deformation theory following Kodaira and Spencer: the problem whether the number of moduli is well-defined and the problem if equation 6.1 holds. For compact complex manifolds $M$ this is answered by a theorem of Kodaira, see [16] p. 306 thm. 6.4: a sufficient condition for the affirmative answer to the two questions is that

$$H^0(M, Hol) = H^2(M, Hol) = 0.$$

So in the case of compact complex surfaces, we can conclude right away that the theorem in 6.1.2 is still true. See [16] for examples of such complex surfaces.

5.2 Deformations of Lie algebras

5.2.1 It is well known that the Lie algebra cohomology with values in the adjoint representation $H^*(L, L)$ of a Lie algebra $L$ answers questions about the deformations of $L$ as an algebraic object. For example, $H^2(L, L)$ can be interpreted as the space of equivalence classes of infinitesimal deformations of $L$, see [7] p. 35.

So there arise natural questions of this type for the Lie algebra of holomorphic vector fields $Hol(U)$ on a Stein manifold $U$ and in the differential graded setting for the differential graded Lie algebra $\Gamma(U, \mathfrak{g})$.

5.2.2 The formal case is well known:

Theorem 11

$$H_{cont}^*(W_n, W_n) = 0.$$

This gives right away (as before by considering $Hol(D)$ for a disk $D \subset \mathbb{C}^n$ as a dense subalgebra of $W_n$ and by the principle that a dense subalgebra has the same continuous cohomology)

Corollary 5

$$H_{cont}^*(Hol(D), Hol(D)) = 0.$$
So this implies the rigidity of the Lie algebra of holomorphic vector fields for disks. Observe that these disks are also rigid as manifolds, i.e. $H^1(D,\text{Hol}) = 0$.

5.2.3 Now by the theorem in 2.2.5, we also have differential graded rigidity of $\Gamma(D,\mathfrak{g})$.

**Corollary 6**

$$^1H_{dg}^*(\Gamma(D,\mathfrak{g}),\Gamma(D,\mathfrak{g})) = 0.$$ 

5.2.4 On the other hand, for a compact Riemann surface $\Sigma$ of genus $g \geq 2$, we have by the lemma in 1.1.5 and by the exact sequence which is implicit in the proof of the theorem in 5.1.2 (here, we have the dg-cohomology procedure as in 5.1.2 !)

**Theorem 12**

$$^2H_{dg}^*(\Gamma(\Sigma,\mathfrak{g}),\Gamma(\Sigma,\mathfrak{g})) = S^*(T\Sigma M(g,0))^* \otimes T\Sigma M(g,0)^*.$$ 

Here, $S^*(T\Sigma M(g,0))^*$ is the continuous dual of the nuclear Fréchet space given by the polynomials on $T\Sigma M(g,0)$. So, it’s the space of formal power series on $T\Sigma M(g,0)^*$.

5.2.5 Note that the space on the right hand side can be given a bracket such that it is isomorphic to the Lie algebra of formal vector fields on $T\Sigma M(0,g)$.

This could be interpreted as the relation between cohomology with adjoint coefficients of $\mathfrak{g}$, i.e. differential graded deformations of global sections of $\mathfrak{g}$, and deformations of the underlying manifold. It fits into Feigin’s philosophy that the choice of the coefficients in the Lie algebra cohomology determines the geometric object on the moduli space in a formal neighbourhood of a point: trivial coefficients correspond to the structure sheaf, adjoint coefficients correspond to vector fields, adjoint coefficients in the universal enveloping algebra correspond to differential operators.

### 6 Applications in foliation theory

This section is inspired by the famous link between the cohomology of Lie algebras and characteristic classes of foliations, see for example [1] for an introduction. We won’t go into all details and we won’t try to develop this theory in all its strength in our case, alas, we will only consider the easiest case, i.e. the case of characteristic classes of $g$-structures. In fact, we will define a class of “$g$”-structures such that the cohomology calculations from the first part yield characteristic classes for these structures.

We won’t pretend that this construction gives rise to interesting new characteristic classes; in fact, in absence of an explicit description of the cohomology classes, we have no explicit description of the characteristic classes.

6.1.1 A $g$-structure on a manifold $X$ is a $g$-valued $C^\infty$-differential 1-form $\omega$ satisfying the Maurer-Cartan equation:

$$-[\omega(\xi_1),\omega(\xi_2)] = d\omega(\xi_1,\xi_2).$$

For a continuous cochain $c \in C^d_{\text{cont}}(g)$, there is a characteristic class of the $g$-structure defined by $\omega$ simply given by the differential form
6.1.2 Define for a covering $\mathcal{U}$ by open sets a “$\text{Hol}-\mathcal{U}$-structure” or short $\text{Hol}$-structure as follows:

Let $X$ be a complex manifold and $\mathcal{U} = \{U_i\}_{i \in I}$ a covering of $X$ by open sets such that $I$ is a countable directed index set. Consider the sheaves $\text{Hol}$ and $\text{Vect}$ of holomorphic resp. $C^\infty$ vector fields on $X$. For an inclusion of open sets $U \subset V$, we have restriction maps

$$\phi_{UV}: \text{Hol}(V) \to \text{Hol}(U) \quad \text{and} \quad \psi_{UV}: \text{Vect}(V) \to \text{Vect}(U).$$

A $\text{Hol}$-structure is now a $\text{Hol}(U_i)$-valued differential 1-form $\omega_{U_i}$ for every open set $U_i$ of $\mathcal{U}$ such that it verifies the Maurer-Cartan equation and furthermore for an inclusion $U \subset V$ we have

$$\phi_{UV}(\omega_V(\xi)) = \omega_U(\psi_{UV}(\xi))$$

for all $\xi \in \text{Vect}(V)$.

If $X$ is part of the covering and $\text{Hol}(X) = 0$, then the $\text{Hol}$-structure is 0, so let us restrict to coverings not including $X$.

6.1.3 To have a link with better known structures in foliation theory, let us restrict ourselves to coverings by contractible open sets (such that intersections are contractible).

Let $X$ be of complex dimension $n$. By the obvious base change, we can think of $W_{2n}$ as being generated by $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar{z}_i}$, $i = 1, \ldots, n$. Denote by $W_{2n}|_{\text{hol}}$ the Lie subalgebra of $W_{2n}$ generated by the $\frac{\partial}{\partial z_i}$ for $i = 1, \ldots, n$.

Given a $\text{Hol}$-structure associated such a covering, denoted by $\mathcal{U}$, we have the

Lemma 7 The data $\{\omega_U\}_{U \in \mathcal{U}}$ is equivalent to a $W_{2n}|_{\text{hol}}$-valued differential form $\omega$.

So, for these coverings, $\text{Hol}$-structures are special cases of $W_{2n}$-structures, and their importance is clear, see for example [8] Ch. 3.1.3 B 3°, p. 231.

6.1.4 To such a structure (for which obviously only the transverse structure of the foliation is relevant), we assign now characteristic classes by considering not $H^*_\text{cont}(\text{Hol}(X))$ which could be too small, but $H^*(X, C^*_\text{cont}(\text{Hol}))$ or better $H^*(C^*_\text{cont}(\mathcal{C}(\mathcal{U}, \text{Hol})))$ which coincide by section 2.

The $\text{Hol}$-structure is defined such that by inserting $p$-times $\omega_{U_{i_0} \cap \ldots \cap U_{i_q}}$ into each $c \in C^p_{\text{cont}}(\prod_{i_0 < \ldots < i_q} \text{Hol}(U_{i_0} \cap \ldots \cap U_{i_q}))$, one constructs an element $\omega$ of the Cech-DeRham complex associated to the covering $\mathcal{U}$ on $X$. By the standard theorem saying that the Cech-DeRham complex calculates only DeRham cohomology, this $\omega$ gives rise to a well-defined cohomology class $[\omega]$, the characteristic class associated to the $\text{Hol}$-structure.
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