Out-of-equilibrium evolution of scalar fields in FRW cosmology: renormalization and numerical simulations

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Abstract

We present a renormalized computational framework for the evolution of a self-interacting scalar field (inflaton) and its quantum fluctuations in an FRW background geometry. We include a coupling of the field to the Ricci scalar with a general coupling parameter $\xi$. We take into account the classical and quantum back reactions, i.e., we consider the dynamical evolution of the cosmic scale factor. We perform, in the one-loop and in the large-$N$ approximation, the renormalization of the equation of motion for the inflaton field, and of its energy momentum tensor. Our formalism is based on a perturbative expansion for the mode functions, and uses dimensional regularization. The renormalization procedure is manifestly covariant and the counter terms are independent of the initial state. Some shortcomings in the renormalization of the energy-momentum tensor in an earlier publication are corrected. We avoid the occurrence of initial singularities by constructing a suitable class

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of initial states. The formalism is implemented numerically and we present some results for the evolution in the post-inflationary preheating era.
1 Introduction

Nonequilibrium processes in cosmology have recently been considered by various authors. The main interest has been centered around the possible inflationary period of the universe (see e.g. [1, 2, 3]) and the subsequent reheating [4, 5]. It has already been found, by considering parametric resonance [3] associated with oscillations of the inflaton field and by exact computations including back reaction both in Minkowski space [6, 7] and in an expanding universe [8, 9, 10, 11, 12], that the time-dependent inflaton field produces particles or classical fluctuations preferentially at low momenta and not in a distribution corresponding to thermal equilibrium. This process of particle production has therefore been termed preheating [6, 13].

The equations of motion for nonequilibrium systems have been presented by various authors [14, 15] using the CTP formalism introduced by Schwinger [16] and Keldysh [17]. Their application to inflation within a conformally flat Friedmann-Robertson-Walker (FRW) universe has been initiated by Ringwald [18]; they have recently been implemented numerically in [9, 10, 19] and [12]. Similar computations have been performed in configuration space for the case that the fluctuations are treated as classical ones [20]. Again fluctuations with rather low momenta are strongly excited, thus justifying the classical approximation. Apart from such exact numerical computations there exist also various analyses based on analytical approximations to the solution of the mode equations [11, 21, 22, 23, 24].

We have recently considered [25] nonequilibrium dynamics of a scalar inflaton field and its quantum fluctuations in FRW cosmology. This work was based on a method for renormalized numerical computations in quantum field theory, introduced for nonequilibrium dynamics in Minkowski space in [26]. In this method the leading, divergent parts of the fluctuation integrals appearing in the inflaton equation of motion and in the energy-momentum tensor are separated from the numerical computation of finite parts. This allows for a free choice of regularization, which is performed analytically. In [25] we chose dimensional regularization. The renormalization then can be performed in the usual way, with the standard counter terms of equilibrium quantum field theory, manifestly covariant [1] and independent of the initial conditions. In applying this method to a scalar field in FRW space-time we found that renormalization introduces singularities in the time variable at

\[ ^{3} \text{This is to be understood in the restricted sense of special relativistic covariance.} \]
the initial time $t_0$, taken to be 0 in the following. While the equation of motion for the inflaton field is finite as $t \to 0$, the energy-momentum tensor is not. So the Friedmann equations have an initial singularity. We, therefore, were not able to numerically implement our formalism.

Depending on the parameters and the model under consideration, these singularities can be more or less pronounced. It is well known that the main excitation of fluctuations appears in a low momentum resonance band, so the ultraviolet divergences, the precise handling of renormalization and thereby the initial singularities may be considered relatively unimportant. Certainly the main features of inflaton dynamics, as presented in [10], will not be changed if renormalization is handled in a more meticulous way. However, in order to be on safe grounds, one should be sure that the renormalization aspects can be handled properly and consistently.

In [12] renormalization was performed by adiabatic regularization, i.e., by subtracting the leading adiabatic orders of the fluctuation terms. The problem of initial singularities there was avoided by starting the system with a smooth transition, for the inflaton mean field and its fluctuations, from an initial phase in which the back reaction of the fluctuations is switched off. In this initial phase the time dependence of cosmic scale parameter is fixed, and the evolution of the fluctuations is handled analytically. The initial singularities are indeed related to a noncontinuous behaviour of the effective mass of the scalar field fluctuations, a mass whose variation is determined by the background metric and the inflaton mean field. In [27] (see also [28]) we found a different solution of the problem of initial singularities, a Bogoliubov transformation of the initial state. Using our mode expansion and our construction of the initial state, it was shown in [29] that this initial state corresponds, in the terminology of the adiabatic expansion, to a vacuum state of adiabatic order 4.

Having solved the problem of initial singularities we are now able to implement our formalism numerically. Besides these numerical computations with nonsingular initial conditions we will present here some further formal developments.

Part of these is made necessary by fact that in our previous work we omitted several finite terms in the renormalized energy-momentum tensor, as criticized in [29]. In applying dimensional regularization we did not take into account the dimensional continuation of the conformal rescaling of the fields, and of the dimensional continuation of the various basic tensors. Furthermore, the determination of the counter terms was based on the consideration
of the conformally flat FRW metric. It is not possible, then, to fix these counter terms in an unique way. As a consequence the anomaly of the stress tensor did not appear at all. This is, therefore, not due to a shortcoming of our formalism, but to the very special nature of the problem.

A further extension of our previous work is the consideration of the large-$N$ limit in the $O(N)$ $\sigma$ model. This requires a reconsideration of the renormalization procedure, leading to modified counter terms.

The analysis presented here is performed for arbitrary values of the conformal coupling $\xi$, which appears in various renormalization constants and is itself renormalized.

We will present the basic relations of FRW cosmology in section 2. The nonequilibrium dynamics of a scalar mean field and its fluctuations in this geometry is introduced in section 3, the associated energy-momentum tensor in section 4. In section 5 we describe our perturbative expansion of the mode functions and derive some expressions occuring in the various fluctuation integrals. Renormalization of the one-loop equations of motion and of the energy-momentum tensor is considered in sections 6 and 7, respectively. In section 8 we construct the Bogoliubov transformation by which the initial singularities are removed. In section 9 we extend our formalism to the $O(N)$ $\sigma$ model. We present and discuss some results of our numerical computations in section 10.

2 FRW cosmology

We consider the Friedmann-Robertson-Walker metric with curvature parameter $k = 0$, i. e. a spatially isotropic and flat space-time. The line-element is given in this case by

$$ds^2 = dt^2 - a^2(t)dx^2 .$$  (2.1)

The time evolution of the cosmic scale factor $a(t)$ is governed by Einstein’s field equation

$$(1 + \delta Z) G_{\mu\nu} + \delta \alpha^{(1)} H_{\mu\nu} + \delta \beta^{(2)} H_{\mu\nu} + \delta \gamma H_{\mu\nu} + \delta \Lambda g_{\mu\nu} = -\kappa \langle T_{\mu\nu} \rangle$$  (2.2)

with $\kappa = 8\pi G$.

The Einstein curvature tensor $G_{\mu\nu}$ is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R .$$  (2.3)
The Ricci tensor and the Ricci scalar are defined as

\[ R_{\mu\nu} = R^\lambda_{\mu\nu\lambda} , \]  
\[ R = g^{\mu\nu} R_{\mu\nu} , \]  

where

\[ R^\lambda_{\alpha\beta\gamma} = \partial_\gamma \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\gamma\beta} \Gamma_{\alpha\beta}^\gamma - \Gamma^\lambda_{\alpha\sigma} \Gamma_{\sigma\beta}^\gamma . \]  

The tensors \((1) H_{\mu\nu}, (2) H_{\mu\nu}, \) and \(H_{\mu\nu}\) arise from the variation of terms proportional to \(R^2, R_{\alpha\beta} R_{\alpha\beta}, \) and \(R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}\) in the Hilbert-Einstein action. Their precise definitions, valid for general dimension \(n\), and the subsequent identities for \(n = 4\) are given in [30]:

\[(1) H_{\mu\nu} = 1 \sqrt{-g} \frac{\delta}{\delta g_{\mu\nu}} \int d^n x \sqrt{-g} R^2 = 2 R_{\mu\nu} - 2 g_{\mu\nu} \Box R - \frac{1}{2} g_{\mu\nu} R^2 + 2 R R_{\mu\nu} , \]  
\[(2) H_{\mu\nu} = 1 \sqrt{-g} \frac{\delta}{\delta g_{\mu\nu}} \int d^n x \sqrt{-g} R^\alpha_{\beta\gamma} R_{\alpha\beta} = 2 R^\alpha_{\mu;\nu} - \Box R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \Box R + 2 R^\alpha_{\mu} R_{\alpha\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha_{\beta} R_{\alpha\beta} = R_{;\mu\nu} - \frac{1}{2} g_{\mu\nu} \Box R - \Box R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha_{\beta} R_{\alpha\beta} + 2 R^\alpha_{\beta} R_{\alpha\mu\beta\nu} , \]  
\[ H_{\mu\nu} = - \frac{1}{2} g_{\mu\nu} R^\alpha_{\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - \frac{1}{2} R_{\mu\nu} R^\alpha_{\mu\beta\nu} + 4 R^\alpha_{\beta} R_{\alpha\mu\beta\nu} - 4 R_{\mu\alpha} R^\alpha_{\nu} + 4 R^\alpha_{\beta} R_{\alpha\mu\beta\nu} . \]

In the case \(n = 4\) the generalized Gauss-Bonnet theorem states that

\[ \int d^4 x \sqrt{-g} \left( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + R^2 - 4 R_{\alpha\beta} R^{\alpha\beta} \right) \]  
is a topological invariant. It then follows that

\[ H_{\mu\nu} = -(1) H_{\mu\nu} + 4 (2) H_{\mu\nu} , \quad n = 4 . \]  

Furthermore, in conformally flat space-time as considered here,

\[ (2) H_{\mu\nu} = \frac{1}{3} (1) H_{\mu\nu} , \quad n = 4 . \]
These terms, as well as the terms $\delta Z G_{\mu\nu}$ and $\delta \Lambda g_{\mu\nu}$ are introduced for the purpose of renormalization. They are going to absorb the divergences arising in the energy-momentum tensor. It will be more convenient, later on, to add analogous terms to the energy-momentum tensor, with coefficients $\delta \Lambda = \delta \alpha / \kappa$, $\delta \tilde{a} = \delta \alpha / \kappa$ etc.. Of course the cosmological constant and higher curvature terms may be present in the bare action already, if required by observation. Dividing (2.2) by $1 + \delta Z$ one sees that $\delta Z$ may also be considered as renormalizing Newton’s constant.

With the FRW metric (2.1) the Einstein field equations reduce to equations for the time-time component and for the trace of $G_{\mu\nu}$, the Friedmann equations

\begin{align*}
(1 + \delta Z) G_{tt} + \alpha^{(1)} H_{tt} + \delta \beta^{(2)} H_{tt} + \delta \gamma H_{tt} + \delta \Lambda &= -\kappa T_{tt}, \quad (2.13) \\
(1 + \delta Z) G_{\mu\mu} + \alpha^{(1)} H_{\mu\mu} + \delta \beta^{(2)} H_{\mu\mu} + \delta \gamma H_{\mu\mu} + n \delta \Lambda &= -\kappa T_{\mu\mu}. \quad (2.14)
\end{align*}

We now compute the various terms for the $n$ dimensional line element (2.1). For the Christoffel symbols for an $n$ dimensional flat FRW universe one finds

\begin{equation}
\Gamma^t_{ij} = a \dot{a} \delta_{ij}; \quad \Gamma^j_{kt} = \Gamma^j_{tk} = \frac{\dot{a}}{a} \delta^j_k. \quad (2.15)
\end{equation}

The nonvanishing components of the Riemann tensor are

\begin{align*}
R^t_{itk} &= \ddot{a} a \delta_{ik}, \quad R^t_{ijt} = -\ddot{a} a \delta_{ij}, \quad (2.16) \\
R^t_{ttk} &= \frac{\ddot{a}}{a} \delta^t_k, \quad R^t_{ijt} = -\frac{\ddot{a}}{a} \delta^t_j. \quad (2.17)
\end{align*}

for those of the Ricci tensor one finds

\begin{equation}
R_{tt} = (n - 1) \frac{\ddot{a}}{a}, \quad R_{ij} = \left[-\ddot{a} a - (n - 2) \dot{a}^2\right] \delta_{ij}, \quad (2.18)
\end{equation}

this leads to the Ricci curvature scalar

\begin{equation}
R = 2(n - 1) \frac{\ddot{a}}{a} + (n - 1)(n - 2) \left(\frac{\dot{a}}{a}\right)^2. \quad (2.19)
\end{equation}

Expressed in terms of Hubble’s constant

\begin{equation}
H(t) = \frac{\dot{a}(t)}{a(t)}, \quad (2.20)
\end{equation}

5
it takes the form
\[ R = (n - 1) \left( 2 \dot{H} + n H^2 \right). \] (2.21)

The time-time components and the trace of the tensors \(^{(n)}H_{\mu\nu}\) are given by

\[ (1) \ H_{tt} = -6 \dot{H} \ddot{R} + \frac{1}{2} R^2 - 6 H^2 \dot{R} + (n - 4) \left( -2 \dot{H} \ddot{R} - (n + 1) R H^2 \right), \] (2.22)

\[ (2) \ H_{tt} = -2 \dot{H} \ddot{R} + \frac{1}{6} R^2 - 2 H^2 \dot{R} + (n - 4) \left( -\frac{1}{2} H \dot{R} \right) - \frac{R^2}{24(n - 1)} - \frac{1}{4}(n + 2) H^2 R + \frac{1}{8}(n - 1)(n - 2)^2 H^4, \] (2.23)

\[ H_{tt} = -2 \dot{H} \ddot{R} + \frac{1}{6} R^2 - 2 H^2 \dot{R} + (n - 4) \left( -\frac{R^2}{6(n - 1)} - H^2 R + \frac{1}{2}(n - 1)(n - 2) H^4 \right), \] (2.24)

\[ (1) \ H_{\mu\mu} = -6 \ddot{R} - 18 \dot{H} \ddot{R} + (n - 4) \left( -2 \ddot{R} - 2(n + 2) H \dot{R} - \frac{1}{2} R^2 \right), \] (2.25)

\[ (2) \ H_{\mu\mu} = -2 \ddot{R} - 6 \dot{H} \ddot{R} + (n - 4) \left( -\frac{1}{2} \ddot{R} - \frac{1}{2}(n + 3) H \dot{R} \right) - \frac{n R^2}{8(n - 1)} + \frac{1}{4}(n - 2)^2 H^2 R - \frac{1}{8} n(n - 1)(n - 2)^2 H^4, \] (2.26)

\[ H_{\mu\mu} = -2 \dot{R} - 6 \dot{H} \ddot{R} + (n - 4) \left( -2 \dot{H} \ddot{R} - \frac{R^2}{2(n - 1)} \right) + (n - 2) H^2 R - \frac{1}{2} n(n - 1)(n - 2) H^4. \] (2.27)

3 Nonequilibrium equations for the scalar field

The Lagrangian density of a \(\phi^4\)-theory in curved space-time is given by
\[ \mathcal{L} = \sqrt{-g} \left\{ \frac{i}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{2} m^2 \Phi^2 - \xi R \Phi^2 - \frac{\lambda}{4!} \Phi^4 \right\}, \] (3.1)

where \(R(x)\) is the curvature scalar and \(\xi\) the bare dimensionless parameter describing the coupling of the bare scalar field to the gravitational background. We split the field \(\Phi\) into its expectation value \(\phi\) and the quantum
fluctuations $\psi$:
\[ \Phi(x, t) = \phi(t) + \psi(x, t) , \quad (3.2) \]

with
\[ \phi(t) = \langle \Phi(x, t) \rangle = \frac{\text{Tr} \Phi(t)}{\text{Tr} \rho(t)} , \quad (3.3) \]

where $\rho(t)$ is the density matrix of the system which satisfies the Liouville equation
\[ i \frac{d\rho(t)}{dt} = [H(t), \rho(t)] . \quad (3.4) \]

The one-loop equation of motion of a scalar field with $\lambda \phi^4$ interaction has been obtained in the FRW universe by Ringwald [18]; we follow closely his formulation. The equation of motion for the classical field is
\[ \ddot{\phi} + (n - 1) H \dot{\phi} + (m^2 + \xi R) \phi + \frac{\lambda}{6} \phi^3 + \frac{\lambda}{2} \langle \psi^2 \rangle \phi = 0 . \quad (3.5) \]

The expectation value of the quantum fluctuations $\langle \psi^2 \rangle$ can be expressed as
\[ \langle \psi^2 \rangle = -i G(t, x; t', x') \quad (3.6) \]

in terms of the non-equilibrium Green function $G(t, x; t', x')$ which satisfies
\[ \left[ \frac{\partial^2}{\partial t^2} + (n - 1) H \frac{\partial}{\partial t} + a^{-2}(t) \vec{\nabla}^2 + m^2 + \xi R(t) \\
+ \frac{\lambda}{2} \phi^2(t) \right] G(t, x; t', x') = \frac{i}{a^3(t)} \delta(t, x; t', x') . \quad (3.7) \]

The boundary conditions for this Green function will be given below. Due to the presence of the term $H(t) \partial / \partial t$ the differential operator on the left hand side of this equation is non-hermitian. It is made hermitian by introducing conformal time and appropriate scale factors. Conformal time is defined as
\[ \tau = \int_0^t dt' \frac{1}{a(t')} . \quad (3.8) \]

In conformal time the line-element (2.1) reads
\[ ds^2 = C(\tau) (d\tau^2 - d\mathbf{x}^2) , \quad (3.9) \]
where the conformal factor $C(\tau)$ is given by
\begin{equation}
C(\tau) = a^2(\tau).
\end{equation}

We further rescale the scalar field and its quantum fluctuations by introducing the dimensionless ‘conformal’ fields
\begin{align}
\varphi(\tau) &= a^{\frac{n}{2} - 1}(t)\phi(t), \\
\tilde{\psi}(\tau, x) &= a^{\frac{n}{2} - 1}(t)\psi(t, x).
\end{align}

The Green function is rescaled accordingly via
\begin{equation}
\tilde{G}(x, \tau; x', \tau') = a^{\frac{n}{2} - 1}(t)a^{\frac{n}{2} - 1}(t')G(x, t; x, t').
\end{equation}

The equation of motion of the classical field $\varphi(\tau)$ now becomes
\begin{equation}
\varphi''(\tau) + a^2(\tau) \left[ m^2 + (\xi - \xi_n) R(\tau) \right] \varphi(\tau) + \frac{\lambda(a(\tau)\mu)^\xi}{6} \varphi^3(\tau) = 0 \tag{3.14}
\end{equation}

with
\begin{equation}
\xi_n = \frac{n - 2}{4(n - 1)}, \tag{3.15}
\end{equation}

and where the primes denote derivatives with respect to conformal time. The two-point-function $\tilde{G}$ now satisfies
\begin{equation}
\left[ \frac{\partial^2}{\partial \tau^2} - \nabla^2 + M^2(\tau) \right] \tilde{G}(x, \tau; x', \tau') = -\delta(x, \tau; x', \tau'). \tag{3.16}
\end{equation}

Here $M^2(\tau)$ denotes the square of the effective mass term the fluctuation field $\tilde{\psi}(\tau, x)$
\begin{equation}
M^2(\tau) = a^2(\tau) \left[ m^2 + (\xi - \xi_n) R(\tau) \right] + \frac{\lambda(a(\tau)\mu)^\xi}{2} \varphi^2(\tau). \tag{3.17}
\end{equation}

For later discussion it is useful to divide $M^2(\tau)$ into the usual 4 dimensional part and into a part proportional to $(n - 4)$:
\begin{equation}
M^2(\tau) = a^2 \left[ m^2 + \left( \xi - \frac{1}{6} \right) R(\tau) \right] + \frac{\lambda(a\mu)^\xi}{2} - \frac{n - 4}{12(n - 1)} a^2 R. \tag{3.18}
\end{equation}

When multiplied by a divergent factor $1/(n - 4)$ the last term yields a finite contribution.
The problem of determining the Green function is now essentially reduced to the equivalent problem in Minkowski space. We expand the fluctuation field in terms of the mode functions $U_k(\tau) \exp(ikx)$ via

$$\tilde{\psi}(\tau, x) = \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \left[ c(k)U_k(\tau)e^{ikx} + c^\dagger(k)U_k^*(\tau)e^{-ikx} \right] . \quad (3.19)$$

The functions $U_k(\tau)$ satisfy the mode equation

$$U_k''(\tau) + \Omega_k^2(\tau)U_k(\tau) = 0 , \quad (3.20)$$

with

$$\Omega_k^2(\tau) = k^2 + M^2(\tau) \quad (3.21)$$

We further impose the initial conditions

$$U_k(0) = 1 \quad ; \quad U_k'(0) = -i\Omega_k(0) , \quad (3.22)$$

with

$$\Omega_k(0) = \sqrt{k^2 + M_0^2} . \quad (3.23)$$

In the following we will use the short notation $\Omega_{k0} = \Omega_k(0)$. The nonequilibrium Green function $\tilde{G}_k(\tau, x; \tau', x')$ can be expressed in terms of the mode functions via

$$\tilde{G}_k(\tau, x; \tau', x') = \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{i}{2\Omega_{k0}} \left\{ \theta(\tau - \tau')U_k(\tau)U_k^*(\tau')e^{ik(x-x')} + \theta(\tau' - \tau)U_k^*(\tau')U_k(\tau)e^{-ik(x-x')} \right\} . \quad (3.24)$$

The expectation value of the fluctuation fields is given, therefore, by the fluctuation integral

$$\tilde{F}(\tau) = \langle \tilde{\psi}^2(\tau) \rangle = -i\tilde{G}(\tau, x; \tau, x) = \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{|U_k(\tau)|^2}{2\Omega_{k0}} . \quad (3.25)$$

The unrenormalized equation of motion of the inflaton field reads

$$\varphi'' + a^2 \left[ m^2 + (\xi - \xi_n) R \right] \varphi + \frac{\lambda(a\mu)^\epsilon}{6} \varphi^3 + \frac{\lambda(a\mu)^\epsilon}{2} \varphi F = 0 . \quad (3.26)$$

The regularization of the fluctuation integral and the renormalized form of this equation will be discussed below.
The field expansion (3.19), together with the equation of motion for the mode functions and the initial conditions, defines a Fock space in which the initial quantum state or density matrix can be represented. The ground state is the conformal vacuum state [30], it has been chosen previously [18, 9] as the initial state for the nonequilibrium evolution. As mentioned in the introduction we have found [27] that for such an initial state the energy-momentum tensor becomes singular at the initial time. The construction of a suitable initial state, avoiding such singularities, will be given below.

4 The energy-momentum tensor

In order to formulate Einstein’s field equation we have to discuss the energy-momentum tensor of the scalar field in curved space time. For a classical field it reads [30]

\[
T_{\mu\nu} = (1 - 2\xi)\phi_{,\mu}\phi_{,\nu} + \left(2\xi - \frac{1}{2}\right)g_{\mu\nu}g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} - 2\xi \phi_{,\mu}\phi_{,\nu}^2 \\
+ 2\xi g_{\mu\nu}\phi \Box \phi - \xi G_{\mu\nu}\phi^2 + \frac{1}{2}m^2 g_{\mu\nu}\phi^2 + \frac{\lambda}{2!}g_{\mu\nu}\phi^4.
\]

(4.1)

In the conformally flat FRW metric the energy-momentum tensor is diagonal. One obtains for its time-time component and its trace

\[
T^{cl}_{tt} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - \xi G_{tt}\phi^2 + 2(n-1)\xi H\dot{\phi}\phi, \\
T^{cl}_{\mu\mu} = \left[1 - \frac{n}{2} + 2(n-1)\xi\right]\dot{\phi}^2 + n\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4\right) - \xi G^{\mu\nu}\phi^2 \\
+ 2(n-1)\xi \left[\ddot{\phi} + (n-1)H\dot{\phi}\phi\right]\phi.
\]

(4.2)

We again introduce conformal time and the conformal rescaling of the fields. Furthermore, we include the quantum fluctuations of the field \(\phi\). The classical energy density then takes the form [30]

\[
T^{cl}_{tt} = \frac{1}{a^{2-\epsilon}} \left\{ \frac{1}{2a^2}\varphi'^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda(a\mu)^{\epsilon}}{4!a^2}\varphi^4 \\
+ 2(n-1)(\xi - \xi_n) \left(\frac{H}{a}\varphi' \varphi - \frac{1}{4}(n-2)H^2\varphi^2\right) \right\}.
\]

(4.3)

\footnote{We continue to consider \(T_{tt}\) instead of \(T_{\tau\tau}\) for convenience.}
The fluctuation energy density is given by

\[ T_{tt}^q = \frac{1}{a^2 - \epsilon} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \left\{ \frac{|U_k'|^2}{2a^2} + \frac{1}{2a^2} \Omega_k(\tau)^2 |U_k|^2 
+ (n - 1) (\xi - \xi_n) \left[ \frac{H}{a} \frac{d}{d\tau} |U_k|^2 - \left( \frac{1}{2}(n - 2)H^2 + \frac{R}{2(n - 1)} \right) |U_k|^2 \right] \right\}. \tag{4.4} \]

We obtain for the classical and fluctuation parts of the trace

\[ T_{\mu\mu}^{cl} = \frac{1}{a^2 - \epsilon} \left\{ 2(n - 1) (\xi - \xi_n) \left[ \frac{\varphi'}{a} - \frac{1}{2}(n - 2)H \varphi \right]^2 + 2(n - 1)\xi \frac{\varphi \varphi''}{a^2} + n \left[ \frac{1}{2} m^2 \varphi^2 + \frac{\lambda (a\mu')^2}{24a^2} \varphi^4 \right] \right\} \tag{4.5} \]

\[ T_{\mu\mu}^q = \frac{1}{a^2 - \epsilon} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \left\{ \left[ \frac{n - 2}{2} - 2(n - 1)\xi \right] \left[ - \frac{|U_k'|^2}{a^2} + \frac{\Omega_k(\tau)^2}{a^2} |U_k|^2 \right] + \frac{1}{2} (n - 2) \frac{H}{a} \frac{d}{d\tau} |U_k|^2 - \frac{1}{4} (n - 2)^2 \left( H^2 - \frac{R}{(n - 1)(n - 2)} \right) |U_k|^2 \right\} \tag{4.6} \]

Energy density and pressure are related to the energy-momentum tensor via

\[ T_{tt} = \mathcal{E}, \quad T_{\mu\mu} = \mathcal{E} - (n - 1)p. \tag{4.7} \]

It is straightforward to show, using the equations of motion for the classical field and for the mode functions (3.7), that the energy is covariantly conserved:

\[ \mathcal{E}'(\tau)/a(\tau) + (n - 1)H(\tau)(p(\tau) + \mathcal{E}(\tau)) = 0. \tag{4.8} \]

## 5 Perturbative expansion

In order to prepare the renormalized version of the equations given in the previous section we introduce a suitable expansion of the mode functions, which was used in [26, 31, 32] for the inflaton field coupled to itself, to gauge bosons, and to fermions in Minkowski-space. In the context of FRW
cosmology it has been used in a similar way in [33]. Adding the term $M_0^2$ on both sides of the mode function equation it takes the form

$$\left[ \frac{d^2}{d\tau^2} + \Omega_{k0}^2 \right] U_k(\tau) = -V(\tau)U_k(\tau) \quad (5.1)$$

with

$$V(\tau) = M^2(\tau) - M_0^2 \quad \Omega_{k0} = \left[ k^2 + M_0^2 \right]^{1/2} \quad (5.2)$$

(for the definition of $M^2(\tau)$ see eq.(3.17)). Including the initial conditions (3.22) the mode functions satisfy the equivalent integral equation

$$U_k(\tau) = e^{-i\Omega_{k0}\tau} + \int_0^\infty d\tau'\Delta_{k,\text{ret}}(\tau - \tau')V(\tau')U_k(\tau') \quad (5.3)$$

with

$$\Delta_{k,\text{ret}}(\tau - \tau') = -\frac{1}{\Omega_{k0}}\Theta(\tau - \tau') \sin(\Omega_{k0}(\tau - \tau')) \quad (5.4)$$

We separate $U_k(\tau)$ into the trivial part corresponding to the case $V(\tau) = 0$ and a function $h_k(\tau)$ which represents the reaction to the potential by making the ansatz

$$U_k(\tau) = e^{-i\Omega_{k0}\tau}(1 + h_k(\tau)) \quad (5.5)$$

$h_k(\tau)$ satisfies then the differential equation

$$\ddot{h}_k(\tau) - 2i\Omega_k^0\dot{h}_k(\tau) = -V(\tau)(1 + h_k(\tau)) \quad (5.6)$$

with the initial conditions $h_k(0) = \dot{h}_k(0) = 0$, and the associated integral equation

$$h_k(\tau) = \int_0^\tau d\tau'\Delta_{k,\text{ret}}(\tau - \tau')V(\tau')(1 + h_k(\tau'))e^{i\Omega_{k0}(\tau - \tau')} \quad (5.7)$$

We expand now $h_k(\tau)$ with respect to orders in $V(\tau)$ by writing

$$h_k(\tau) = h_k^{(1)}(\tau) + h_k^{(2)}(\tau) + h_k^{(3)}(\tau) + \cdots \quad (5.8)$$

$$= h_k^{(1)}(\tau) + h_k^{(2)}(\tau) \quad (5.9)$$
where \( h^{(n)}_k(\tau) \) is of \( n \)’th order in \( V(\tau) \) and \( h^{(\infty)}_k(\tau) \) is the sum over all orders beginning with the \( n \)’th one:

\[
h^{(\infty)}_k(\tau) = \sum_{l=n}^{\infty} h^{(n)}_k(\tau) .
\] (5.10)

The \( h^{(n)}_k \) are obtained by iterating the integral equation (5.7). The function \( h^{(1)}_k(\tau) \) is identical to the function \( h^2_k(\tau) \) itself which is obtained by solving (5.6). The function \( h^{(2)}_k(\tau) \) can again be obtained by iteration via

\[
h^{(2)}_k(\tau) = \int_{0}^{\tau} d\tau' \Delta_{k,\text{ret}}(\tau - \tau')V(\tau')h^{(1)}_k(\tau')e^{i\Omega_0(\tau-\tau')} ,
\] (5.11)

or by using the equivalent differential equation

\[
h^{(2)}_k(\tau) - 2i\Omega_0 h^{(1)}_k(\tau) = -V(\tau)h^{(1)}_k(\tau) .
\] (5.12)

This iteration has the numerical aspect that it avoids computing \( h^{(2)}_k \) via the small difference \( h^{(2)}_k - h^{(1)}_k \).

The integral equations are used in order to derive the asymptotic behaviour as \( \Omega_0 \to \infty \) and to separate divergent and finite contributions. We will give here the relevant leading terms for \( h^{(1)}_k(\tau) \) and \( h^{(2)}_k(\tau) \). We have

\[
h^{(1)}_k(\tau) = \frac{i}{2\Omega_0} \int_{0}^{\tau} d\tau' V(\tau') ,
\] (5.13)

Integrating by parts we obtain

\[
h^{(1)}_k(\tau) = -\frac{i}{2\Omega_0} \int_{0}^{\tau} d\tau' V(\tau') - \frac{1}{4\Omega_0^3} V(\tau) + \frac{1}{4\Omega_0^2} \int_{0}^{\tau} d\tau' \exp(2i\Omega_0(\tau-\tau'))V'(\tau') ,
\] (5.14)

or, by another integration by parts,

\[
h^{(1)}_k(\tau) = -\frac{i}{2\Omega_0} \int_{0}^{\tau} d\tau' V(\tau') - \frac{1}{4\Omega_0^3} V(\tau) + \frac{i}{8\Omega_0^4} V'(\tau)
\] (5.15)

\[
-\frac{i}{8\Omega_0^4} \int_{0}^{\tau} d\tau' \exp(2i\Omega_0(\tau-\tau'))V''(\tau') .
\] (5.16)
We will need often the real part of \( h_k^{(1)} \) for which we find
\[
\text{Re} \ h_k^{(1)}(\tau) = -\frac{1}{4\Omega_{k0}^2} V(\tau) + \frac{1}{4\Omega_{k0}^2} \mathcal{C}(V', \tau). \tag{5.17}
\]
Here we have introduced the notation\(^5\)
\[
\mathcal{C}(f, \tau) = \int_0^\tau d\tau' \cos(2\Omega_{k0}(\tau - \tau')) f(\tau'). \tag{5.18}
\]
which will prove to be useful later. For the leading behaviour of \( h_k^{(2)}(\tau) \) we find
\[
h_k^{(2)}(\tau) = -\frac{1}{4\Omega_{k0}^2} \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' V(\tau')V(\tau'') + O(\Omega_{k0}^{-3}). \tag{5.19}
\]
In terms of this perturbative expansion we can write the mode functions appearing in the fluctuation integrals in the equation of motion and in the energy-momentum tensor as
\[
|U_k|^2 = 1 + 2 \text{Re} \ h_k^{(\overline{T})} + |h_k^{(T)}|^2, \tag{5.20}
\]
and
\[
|U_k'|^2 = \Omega_{k0}^2 \left( 1 + 2 \text{Re} \ h_k^{(\overline{T})} + |h_k^{(T)}|^2 \right) + |h_k^{(T)}|^2
-i \Omega_{k0} \left( -2i \text{Im} \ h_k^{(\overline{T})} - 2i \text{Im} h_k^{(T)} h_k^{(\overline{T})} \right). \tag{5.21}
\]
As the potential is real, the leading behaviour of the sums is
\[
1 + 2 \text{Re} \ h_k^{(\overline{T})} + |h_k^{(T)}|^2 = 1 - \frac{1}{2\Omega_{k0}^2} V(\tau) + \frac{1}{4\Omega_{k0}^2} \sin(2\Omega_{k0}\tau)V'(0) + \frac{1}{8\Omega_{k0}^3} V''(\tau)
- \frac{1}{8\Omega_{k0}^3} \cos(2\Omega_{k0}\tau)V''(0) + \frac{3}{8\Omega_{k0}^3} V^2(\tau) + \mathcal{O}(\Omega_{k0}^{-5}), \tag{5.22}
\]
and
\[
-2i \text{Im} h_k^{(\overline{T})} - 2i \text{Im} h_k^{(T)} h_k^{(\overline{T})} = \frac{i}{\Omega_{k0}} V(\tau) - \frac{i}{2\Omega_{k0}^2} \sin(2\Omega_{k0}\tau)V'(0) - \frac{i}{4\Omega_{k0}^3} V''(\tau)
+ \frac{i}{4\Omega_{k0}^3} \cos(2\Omega_{k0}\tau)V''(0) - \frac{3i}{4\Omega_{k0}^3} V^2(\tau) + \mathcal{O}(\Omega_{k0}^{-4}). \tag{5.23}
\]
\(^5\)For the numerical computation we use the addition theorem to split the integral into two integrals whose integrands depend on \( t' \) only; these can be updated easily.
From the Wronskian relation
\[ U_k U_k' - U_k' U_k = 2i \Omega_{k0} \]  
(5.24)

we obtain
\[ 2i \Omega_{k0} \left( 2 \text{Re} h_k^{(1)} + |h_k^{(1)}|^2 \right) - 2i \text{Im} h_k^{(1)} - 2i \text{Im} h_k^{(1)*} h_k^{(1)} = 0 , \]  
(5.25)

which proves to be useful in simplifying the mode integrals occurring in the energy-momentum tensor. Using the Wronskian relation we obtain for \( |U_k'|^2 \)
\[ |U_k'|^2 = \Omega_0^2 - \Omega_0^2 \left( 2 \text{Re} h_k^{(1)} + |h_k^{(1)}|^2 \right) + |h_k'|^2 . \]  
(5.26)

By means of the equation of motion and Eq. (5.26) we have
\[ \frac{1}{2} \frac{d^2}{d\tau^2} |U_k|^2 = |U_k'|^2 - \Omega(t)^2 |U_k|^2 \]  
(5.27)
\[ = - \left( V(\tau) + 2\Omega_0^2 \right) \left( 2 \text{Re} h_k^{(1)} + |h_k^{(1)}|^2 \right) + |h_k'|^2 - V(\tau) \]

6 Renormalization of the equation of motion

Having expanded the mode functions perturbatively, we are able to separate the divergent parts of the mode sum in an analytic way, leaving the finite parts for numerical computation. This allows for a free choice of regularizations. Furthermore, the analytic expressions for the divergent parts, as separated from the mode sum, essentially have the standard form as obtained from Feynman graphs, so a comparison with purely analytic approaches is straightforward.

Among the regularizations used in field theory in curved space are: point-splitting, dimensional, and adiabatic regularization. Point-splitting regularization is technically involved as it requires performing the delicate and non-covariant limit \( x' \to x \). Adiabatic regularization actually is a subtraction, it has often been used \([30]\), most recently in \([12]\). Adiabatic regularization is considered to be well suited for numerical computations, as the entire divergent part is subtracted, encompassing this way the problem of regularization.

Here we choose dimensional regularization as it fits in the most appropriate way into our formalism. We have already used it in our previous work.
on FRW cosmology. Here we have to correct for some omissions in the finite terms, and in particular for the conformal anomaly. In [25] we have performed the renormalization by considering the equation of motion for the condensate and of the Einstein equations. This is possible in 4-dimensional conformally flat space, and leads to a conserved energy-momentum tensor; it does not correspond, however, to a properly renormalized local action [30] and does not work for more general metrics. This shortcoming has been criticized in [29].

In our previous publication we have performed the conformal scaling in four dimensions, and applied dimensional regularization to the resulting ‘flat space’ equations in analogy to the Minkowski case. We thereby missed terms arising from the dimensional continuation of the conformal rescaling, resulting in particular in the absence of some terms involving \( \log a(t) \). As to the equation of motion for the inflaton field, these are the only corrections. The renormalization of the energy-momentum tensor is more subtle and will be discussed in the next section.

Using (5.17) we split the fluctuation integral of the equation of motion into a divergent and a convergent part. Using (5.17) we

\[
\mathcal{F}(\tau) = \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \left\{ \frac{1}{2\Omega_{k0}} \left( 1 + 2\text{Re} h_k^{(T)}(\tau) + |h_k^{(T)}(\tau)|^2 \right) \right. \\
= \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{1}{2\Omega_{k0}} \left( 1 - \frac{1}{2\Omega_{k0}} V(\tau) + \frac{1}{2\Omega_{k0}} C(V', \tau) \right) \\
\left. + 2\text{Re} h_k^{(2)}(\tau) + |h_k^{(T)}(\tau)|^2 \right\}. \tag{6.1}
\]

The first two terms in the integrand have to be regularized, as they lead to a logarithmic and a quadratic divergence. We can do another integration by parts of the \( C(V', \tau) \) and arrive at

\[
\mathcal{F}(\tau) = \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{1}{2\Omega_{k0}} \left\{ (1 - \frac{1}{2\Omega_{k0}} V(\tau)) + \frac{1}{4\Omega_{k0}^2} \sin(2\Omega_{k0}\tau)V'(0) \right. \\
- \frac{1}{8\Omega_{k0}^2} \cos(2\Omega_{k0}\tau)V''(0) + \frac{\ddot{V}(\tau)}{8\Omega_{k0}^4} - \frac{C(V'', \tau)}{8\Omega_{k0}^4} + 2\text{Re} h_k^{(2)} + |h_k|^2 \right\}. \tag{6.2}
\]

Those parts of the integral that involve \( \sin(2\Omega_{k0}\tau) \) and \( \cos(2\Omega_{k0}\tau) \) develop a non-analytic behaviour as \( \tau \to 0 \). This is due to the fact that the modulation of the integrand by the trigonometric functions disappears in this limit.
Here the non-analyticity is of the form $\tau \log \tau$ and $\tau^2 \log \tau$, respectively. In the energy-momentum tensor the analogous terms result in $1/\tau$ and $\log \tau$ singularities. These singularities will be discussed later, in section 8.

We first rewrite the basic equation of motion, including appropriate counter terms, as

$$\varphi'' + a^2 \left[ m^2 + \delta m + (\xi - \xi_n + \delta \xi) R \right] \varphi + \frac{\lambda + \delta \lambda}{6} (a \mu)^\epsilon \varphi^3 + \frac{\lambda}{2} \varphi (a \mu)^\epsilon F = 0.$$  (6.3)

Next we separate from the term $\lambda \varphi F/2$ the dimensionally regularized divergent parts. The two relevant expressions behave as

$$(a \mu)^\epsilon \frac{\lambda}{2} \varphi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{[k^2 + M_0^2]^{3/2}} \simeq -\frac{\lambda M^2_0 \varphi}{32\pi^2} (L_0 + 1)$$  (6.4)

and

$$-\frac{1}{8} (a \mu)^\epsilon \lambda \varphi V(\tau) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{[k^2 + M_0^2]^{3/2}} \simeq -\frac{\lambda \varphi V(\tau)}{32\pi^2} L_0$$  (6.5)

in the limit $\epsilon \to 0$. Here we have defined

$$L_0 = \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2 a^2(\tau)}{M^2_0} - \gamma.$$  (6.6)

This cannot be absorbed, e.g., in an $\overline{MS}$ scheme, by a renormalization counter term, as it depends on $\tau$, and on the initial mass $M_0$. Recalling that $V(\tau) = M^2(\tau) - M^2_0$ and

$$M^2(\tau) = a^2 \left[ m^2 + \left( \xi - \frac{1}{6} \right) R(\tau) \right] + \frac{\lambda (a \mu)^\epsilon}{2} - \frac{n - 4}{12(n - 1)} a^2 R,$$  (6.7)

we find that the two divergent terms combine into

$$-\frac{\lambda M^2(\tau) \varphi}{32\pi^2} L_0 - \frac{\lambda M^2_0 \varphi}{32\pi^2}.$$  (6.8)

The divergences can now be cancelled by the counter terms

$$\delta m^2 = \frac{\lambda m^2}{32\pi^2} L,$$  (6.9)

$$\delta \lambda = \frac{3\lambda^2}{32\pi^2} L,$$  (6.10)

$$\delta \xi = \frac{\lambda (\xi - \frac{1}{6})}{32\pi^2} L.$$  (6.11)
where
\[ L = \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m^2} - \gamma. \quad (6.12) \]
This leaves some finite parts proportional to \( L - L_0 = \ln \frac{m^2a^2}{M_0^2} \), and an additional finite term due to \((n-4)\) part of \( M^2(\tau) \) multiplied with the \(1/(n-4)\) term of the divergent integral. Finally the renormalized equation of motion in \( n = 4 \) dimensions reads
\[ \varphi'' + a^2 \left[ m^2 + \Delta m^2 + \left( \xi + \Delta\xi - \frac{1}{6} \right) R \right] \varphi + \frac{\lambda + \Delta\lambda}{6} \varphi^3 + \frac{\lambda}{2} \varphi \hat{F}_{\text{fin}} = 0 \quad (6.13) \]
where \( \hat{F}_{\text{fin}} \) is defined as
\[ \hat{F}_{\text{fin}} = -\frac{a^2R}{288\pi^2} + \mathcal{F}_{\text{fin}} \quad (6.14) \]
with
\[ \mathcal{F}_{\text{fin}} = -\frac{M_0^2}{16\pi^2} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\Omega_{k0}} \left( \frac{1}{2\Omega_{k0}^{3/2}} C(V', \tau) + 2\text{Re} h_k^{(\mathbb{T})} + |h_k^{(\mathbb{T})}|^2 \right). \quad (6.15) \]
The finite mass and coupling constant corrections are defined as
\[ \Delta m^2 = -\frac{\lambda m^2}{32\pi^2} \ln \frac{m^2a^2}{M_0^2}, \quad (6.16) \]
\[ \Delta\lambda = -\frac{3\lambda^2}{32\pi^2} \ln \frac{m^2a^2}{M_0^2}, \quad (6.17) \]
\[ \Delta\xi = -\frac{\lambda(\xi - \frac{1}{6})}{32\pi^2} \ln \frac{m^2a^2}{M_0^2}. \quad (6.18) \]
Note that these are time-dependent, due to the occurrence of \( a(\tau) \). This is a feature we have missed in [25], as we did not continue the conformal rescaling to \( n \neq 4 \).

7 Renormalization of the energy-momentum tensor

In order to derive a renormalized form of the Friedmann equations we have to renormalize the energy-momentum tensor. In principle this has been discussed long ago and indeed the divergent parts will be found to be in one-to-one correspondence to those given in the standard literature [30]. However,
we have to discuss this subject in the framework of nonequilibrium quantum field theory, and we are interested in particular in the precise form of the finite parts which will be the subject of a numerical computation.

Our previous discussion of the renormalization of the energy-momentum tensor had three shortcomings:

(i) as for the equation of motion we have not extended the conformal rescaling to \( n \neq 4 \), these will result in \( \log a(t) \) contributions, as above;

(ii) we have not continued the tensor structures to \( n \neq 4 \); these terms, proportional to \( (n-4) \), will be explicitly displayed below, they contribute finite terms when multiplied by \( 1/(n-4) \);

(iii) finally, we have discussed renormalization using the Einstein equations, and not the Hilbert-Einstein action; this is the reason why the conformal anomaly did not appear.

As, nevertheless, the energy-momentum tensor was found to be conserved, these shortcomings passed unnoticed. This fact, and the absence of the anomaly, are related to the very special nature of the conformally flat metric in \( n = 4 \). Indeed a unique determination of the counter terms would require the consideration of quantum field theory in a general background metric, and will necessarily be incomplete if only one (and even very particular) metric is used. We will first derive the divergent terms and discuss this rather subtle matter afterwards.

In deriving the divergent and finite parts of the energy-momentum tensor we have to consider an expansion in powers of \( (n-4) = -\epsilon \). The general expression contains various factors with an explicit \( n \)-dependence. We define a tensor \( T^{(4)}_{\mu\nu} \) where in all of these factors \( n \) is set equal to 4, in which, however, the fluctuation integral is defined in \( n \) dimensions. We define a second tensor, \( T^{(n-4)}_{\mu\nu} \), which takes into account terms linear in \( (n-4) \) from the explicit factors, multiplied by terms proportional to \( 1/(n-4) \) from the divergent fluctuation integral, and of the counter terms. The remaining terms, less singular or with higher powers of \( (n-4) \), do not contribute when \( n \to 4 \). We expand the tensor as

\[
T_{\mu\nu} = T^{(4)}_{\mu\nu} + T^{(n-4)}_{\mu\nu} + O(n-4) .
\] (7.1)

Explicitly these tensors are given by

\[
T_{tt}^{(4)} = \frac{1}{2a^4} \varphi^2 + \frac{1}{2a^2} \left( m^2 + \delta m^2 \right) \varphi^2 + \frac{\lambda + \delta \lambda}{4!a^4} \varphi^4
\]

\[
+ (1 - 6\xi - 6\delta \xi) \left( \frac{H^2}{2a^2} \varphi^2 - \frac{H}{a^3} \varphi' \right)
\]
\[ +\delta\bar{\alpha} + \delta\tilde{\alpha} \, (1) \, H_{tt}^{(4)} + \delta\tilde{\beta} \, (2) \, H_{tt}^{(4)} + \delta\hat{\gamma} \, H_{tt}^{(4)} + \delta\tilde{Z} \, G_{tt}^{(4)} \\
+ \frac{1}{a^{n-2}} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{1}{2\Omega_{k0}} \left\{ \frac{1}{2a^2} |U'_k|^2 + \frac{1}{2a^2} \Omega(\tau) |U_k|^2 - \frac{1}{2} \left( \frac{1}{6} \right) R |U_k|^2 \right\} \]

\[ -\frac{1}{2} (6\xi - 1) \, H^2 |U_k|^2 + \frac{1}{2} (6\xi - 1) \frac{H}{a} \left( \frac{d}{d\tau} |U_k|^2 \right) \]

\[ T_{tt}^{(n-4)} = (n - 4) \left\{ 6\delta\xi \frac{H^2}{4a^2} \varphi^2 \right. \]

\[ + 2\delta\xi \left( \frac{H}{a^3} \varphi \varphi' - \frac{H^2}{2a^2} \varphi^2 \right) + \delta Z \left( -\frac{5}{2} \frac{H^2}{a} \right) \]

\[ + \delta\alpha \left( -2 \frac{HR'}{a} - 5H^2 R \right) + \delta\gamma \left( -\frac{R^2}{18} - H^2 R + 3H^4 \right) \]

\[ + \delta\beta \left( \frac{HR'}{2a} - \frac{R^2}{72} - \frac{3}{2} H^2 R + \frac{3}{2} H^4 \right) \]

\[ + (n - 4) \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{1}{2\Omega_{k0}} a^{2-\epsilon} \left\{ \left( 2\xi - \frac{1}{2} \right) \left[ - \left( \frac{H^2}{4} + \frac{R}{36} \right) |U_k|^2 \right] \right. \]

\[ + \left. \frac{H}{2a} \frac{d}{d\tau} |U_k|^2 \right\} - (6\xi - 1) \left( \frac{H^2}{4} - \frac{R}{36} \right) |U_k|^2 \right\} \]  

The divergent parts of the fluctuation integral are

\[ \mathcal{E}_{\text{div,fluc}} = \frac{1}{a^{n-2}} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \left[ \frac{\Omega_{k0}}{2a^2} + \frac{1}{4\Omega_{k0} a^2} V(\tau) - \frac{1}{16\Omega_{k0}^3 a^2} V^2(\tau) \right] \]

\[ - \frac{1}{2} (6\xi - 1) \left( \frac{R}{6} + H^2 \right) \left[ \frac{1}{2\Omega_{k0}} - \frac{1}{4\Omega_{k0}^3} V(\tau) \right] \]

\[ - \frac{1}{2} (6\xi - 1) \frac{H}{a} \frac{1}{4\Omega_{k0}^3} V'(\tau) \].

Dimensional regularisation of the first three terms in the integral yields

\[ \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{1}{a^n} \left[ \frac{\Omega_{k0}}{2} + \frac{1}{4\Omega_{k0}} V(\tau) - \frac{1}{16\Omega_{k0}^3} V^2(\tau) \right] \]

\[ = - \frac{M^4(\tau)}{64\pi^2 a^4} L_0 + \frac{M^4(0)}{128\pi^2 a^4} - \frac{M_0^2 M^2(\tau)}{32\pi^2 a^4} \]  

\[ = - \frac{\left[ m^2 + (\xi - \xi_0) R + \frac{1}{2} \varphi^2 \right]^2}{64\pi^2} L_0 + \frac{M^4(0)}{128\pi^2 a^4} - \frac{M_0^2 M^2(\tau)}{32\pi^2 a^4} \] .
The terms proportional $\lambda m^2 \varphi^2$ and $\lambda^2 / 4 \varphi^4$ in (7.4) are cancelled by the mass and coupling constant counter terms. The divergent term which depends on $m^4$ determines the cosmological constant counter term, that is

$$\delta \tilde{\Lambda} = \frac{m^4}{64\pi^2} L . \tag{7.5}$$

The remaining terms in (7.4) combine with the corresponding expressions of the remaining parts of $E_{\text{div,fluc}}$:

$$-\frac{1}{2} (6\xi - 1) \left( \frac{R}{6} + H^2 \right) \frac{1}{a^{n-2}} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \left[ \frac{1}{2\Omega_{k0}} - \frac{1}{4\Omega_{k0}^3} V(\tau) \right]$$

$$= (6\xi - 1) \left( \frac{R}{6} + H^2 \right) \frac{M^2(\tau)}{32\pi^2} L_0$$

$$+ \frac{1}{32\pi^2} (6\xi - 1) \left( \frac{R}{6} + H^2 \right) M_0^2 , \tag{7.6}$$

and

$$-\frac{1}{2} (6\xi - 1) \frac{H}{a^{3-n}} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{1}{4\Omega_{k0}^3} V'(\tau)$$

$$= -\frac{1}{32\pi^2} (6\xi - 1) \frac{H}{a} L_0 \left[ 2aH M^2 + \left( \xi - \frac{1}{6} \right) a^2 R' + \lambda \varphi' - \lambda a H \varphi^2 \right]$$

$$-\frac{\lambda}{2} (n - 4) aH \varphi^2 - \frac{n - 4}{36} \left( a^2 R' + 2a^3 H \right) + O(n - 4) \tag{7.7}$$

The term $\lambda (\xi - 1/6) R \varphi^2$ in (7.4) is cancelled by the same term with opposite sign in (7.4). The $H^2 \varphi^2$-terms in (7.4) and (7.7) are absorbed by the counter term proportional to $\delta \xi$, and the divergence proportional to $\varphi \varphi'$ is compensated by this counter term as well. The remaining $\varphi$-independent but still time-dependent divergent terms can absorbed into the counter terms $\delta \tilde{\alpha} H_{tt}$ and $\delta \tilde{Z} G_{tt}$. We choose

$$\delta \tilde{\alpha} = -\frac{(\xi - \frac{1}{6})^2}{32\pi^2} L , \tag{7.8}$$

$$\delta \tilde{Z} = -\frac{(\xi - \frac{1}{6}) m^2}{16\pi^2} L . \tag{7.9}$$

This way all divergent integrals appearing in the unrenormalized fluctuation integral of the energy are removed by the corresponding counter terms, the renormalized expression for the energy will be given below.
We shall now comment on the terms leading to the trace anomaly. Einstein’s equation and therefore the energy-momentum tensor contain the terms
\[ \delta \tilde{\alpha} (1) H_{\mu\nu} + \delta \tilde{\beta} (2) H_{\mu\nu} + \delta \tilde{\gamma} H_{\mu\nu}. \] (7.10)

All the three tensors are conserved; furthermore, in 4-dimensional conformally flat space they are linearly related by Eqs. (2.11) and (2.12), and are in fact proportional to each other. Therefore, the condition of a finite energy-momentum tensor cannot determine the three coefficients \( \delta \tilde{\alpha}, \delta \tilde{\beta} \) and \( \delta \tilde{\gamma} \), if one considers just this restricted class of metrics. So we have to reconsider our previous choice of \( \delta \tilde{\alpha} \). As we do not consider a more general metric here, we have to supply some additional information.

The divergent part of the energy-momentum tensor for a general metric in dimensional regularization has been worked out in [34] and is discussed in detail in [30]. One finds
\[ \langle T_{\mu\nu}\rangle_{\text{div}} = \frac{1}{32\pi^2} \left[ \frac{1}{90} H_{\mu\nu} - \frac{1}{90} (2) H_{\mu\nu} + \left( \xi - \frac{1}{6} \right)^2 (1) H_{\mu\nu} \right] L. \] (7.11)

As \( (2) H_{\mu\nu} = H_{\mu\nu} \), in \( n = 4 \) and for our metric, our divergent part is consistent with this expression. It is apparent that within our framework there is no way to determine the coefficients of these two tensors, which are the source of the conformal anomaly. Therefore, the absence of these terms in our previous publication is not, as suggested in [29], due to an inconsistency of our perturbative expansion. They have to be taken over from a more general analysis. Having chosen, accordingly,
\[ \delta \tilde{\beta} = - \delta \tilde{\gamma} = \frac{1}{2880\pi^2} L \] (7.12)
we continue from \( n \neq 4 \) to obtain the finite terms arising from an \( 1/(n - 4) \) in the renormalization constants multiplied with the \( (n - 4) \) parts of the tensors, as displayed explicitly in Eqs. (2.22,2.27). These finite parts then contain the conformal anomaly, and further terms proportional to \( \xi - 1/6 \).

The anomalous parts of the zero-zero component and of the trace are
\[ T_{tt}^{\text{ano}} = \delta \tilde{\beta} (2) H_{tt} + \delta \tilde{\gamma} H_{tt} \]
\[ = \frac{1}{2880\pi^2} \left( H\frac{R'}{a} + RH^2 - \frac{1}{12} R^2 - 3H^4 \right). \] (7.13)
\[ T^\mu_{\text{ano}} = \delta \beta (2) H^\mu_{\text{ano}} + \delta \gamma H^\mu_{\text{ano}} \]
\[ = \frac{1}{2880\pi^2} \left( \frac{R''}{a^2} + 2H \frac{R'}{a} - 2RH^2 + 12H^4 \right). \tag{7.14} \]

After regularization and renormalization the energy-momentum tensor is given by
\[ T^{\text{ren}}_{tt} = \frac{1}{2a^4} \varphi'^2 + \frac{1}{2a^2} (m^2 + \Delta m^2) \varphi^2 + \frac{\lambda + \Delta \lambda}{4a^4} \varphi^4 \]
\[ -6 \left( \xi - \frac{1}{6} + \Delta \xi \right) \left( \frac{H^2}{2a^2} \varphi^2 - \frac{H}{a^3} \varphi \varphi' \right) \]
\[ + \Delta \tilde{\Lambda} + \Delta \tilde{\alpha} (1) H_{tt} + \Delta \tilde{Z} G_{tt} + T^{a,\text{fin}}_{tt}. \tag{7.15} \]

The renormalized expressions the tensors \( (1) H_{tt} \) and \( G_{tt} \) are the ones in four dimensions. The finite remnants of the divergent parts have the coefficients
\[ \Delta \tilde{\alpha} = \left( \xi - \frac{1}{6} \right)^2 \ln \frac{m^2 a^2}{M_0^2}, \tag{7.16} \]
\[ \Delta \tilde{\Lambda} = -\frac{m^4}{64\pi^2} \ln \frac{m^2 a^2}{M_0^2}, \tag{7.17} \]
\[ \Delta \tilde{Z} = \left( \frac{m^2}{16\pi^2} \right) \frac{m^2 \varphi^2}{M_0^2}, \tag{7.18} \]
which again are time-dependent, due to the occurrence of \( a(\tau) \).

The fluctuation part of the energy density expressed through finite mode integrals is given by
\[ T^{a,\text{fin}}_{tt} = \mathcal{E}_{\text{kin,fin}}(\tau) \frac{F_{\text{fin}}}{2a^4} + \frac{1}{2} \left( 6 \xi - 1 \right) \frac{H}{a^3} F'_{\text{fin}} \]
\[ - \frac{1}{2} \left( 6 \xi - 1 \right) \left( \frac{R}{6} + H^2 \right) \frac{F_{\text{fin}}}{a^2} + T^{a,\text{add}}_{tt}. \tag{7.19} \]

where
\[ \mathcal{E}_{\text{kin,fin}}(\tau) = -\frac{3M_0^4}{128\pi^2 a^4} + \frac{1}{2a^4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\Omega_0} \left[ \left| h_k^{(1)} \right|^2 - \frac{V^2(\tau)}{4\Omega_k^2 \kappa^2} \right], \tag{7.20} \]
and
\[ T^{\text{add}}_{tt} = T^{\text{ano}}_{tt} + \frac{H}{96\pi^2a^3} \left( aHM^2 - V' \right) + \frac{1}{16\pi^2} \left( \xi - \frac{1}{6} \right) \left( \frac{(1)H_{tt}^{(4)}}{36} + \frac{R^2}{72} + \frac{6H^2M^2}{a^2} \right), \quad (7.21) \]
or, explicitly,
\[ T^{\text{add}}_{tt} = T^{\text{ano}}_{tt} + \frac{(\xi - \frac{1}{6})}{8\pi^2} \left[ -\frac{R'H}{6a} - 3 \left( \xi - \frac{1}{6} \right) RH^2 - \frac{3H^2}{2a^2\lambda \varphi^2} + \frac{1}{72}R^2 \right. \\
-3m^2H^2 - \frac{1}{6}RH^2 \left. \right] - \frac{\lambda}{96\pi^2a^2}H\varphi \left( \varphi' - \frac{1}{2}aH\varphi \right) - \frac{m^2}{96\pi^2}H^2. \quad (7.22) \]

Next we have to consider the renormalization of the trace of the energy-momentum tensor. We introduce the available counter terms into the unrenormalized expression for \( T_{\mu\mu}^{(4)} \) and separate the trace according to Eq. (7.1). \( T_{\mu}^{(4)} \) is given by
\[ T_{\mu}^{(4)} = -[1 - 6(\xi + \delta \xi)] \left( \frac{\varphi'^2}{a^4} + \frac{H^2}{a^3} \varphi^2 - 2\frac{H}{a^3} \varphi \varphi' \right) + \frac{6(\xi + \delta \xi)}{a^4} \varphi \varphi'' \\
+2(m^2 + \delta m^2) \frac{\varphi^2}{a^2} + \frac{\lambda + \delta \lambda}{6a^4} \varphi^4 \\
+4\delta \tilde{\lambda} + \delta \tilde{Z}G_{\mu}^{(4)} + \delta \tilde{\alpha}^{(1)}H_{\mu}^{(4)} + T_{\mu}^{\text{ano}} \\
+ \frac{1}{a^{n-2}} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{1}{2\Omega_{k0}} \left\{ (6\xi - 1) \left[ \frac{(|U_k'|^2}{a^2} - \Omega^2(\tau)|U_k|^2 \right] \\
+ \left( H^2 - R/6 \right) |U_k|^2 - \frac{H}{a} \frac{d}{d\tau} |U_k|^2 \right\} \right\}. \quad (7.23) \]

We can split the trace of the stress tensor into a divergent and into a convergent part. The divergent part of \( T_{\mu}^{(4)} \) reads after dimensional regularization
\[ T_{\mu,\text{div}}^{(4)} = \frac{1}{32\pi^2a^2} \left( 1 - 6\xi \right) \left( \frac{V''(\tau)}{a^2} - 2\frac{H}{a}V'(\tau) \right) L_0 \\
- \frac{1}{16\pi^2a^2} \left[ m^2 + \frac{\lambda}{2a^2} \varphi^2 - (1 - 6\xi)(H^2 - \frac{R}{6}) \right] M^2(\tau)L_0. \quad (7.24) \]
Inserting the derivatives of the potential $V(\tau)$ we see that again the counter
terms absorb all divergent terms in the fluctuation integral of $T^\mu_\mu$. The renor-
amalized trace of the stress tensor takes the final form

$$
T^\mu_\mu \text{ren} = - [1 - 6(\xi + \Delta \xi)] \left( \frac{\varphi'^2}{a^4} + \frac{H^2}{a^2} \varphi^2 - 2 \frac{H}{a^3} \varphi \varphi' \right)
+ \frac{6(\xi + \Delta \xi)}{a^4} \varphi \varphi'' + 2 \left( \theta^2 + \Delta \theta^2 \right) \frac{\varphi'^2}{a^2} + \frac{\lambda + \Delta \lambda}{6a^4} \varphi^4
+ 4 \Delta \tilde{\Lambda} + \Delta \tilde{\alpha} H^\mu_\mu + \Delta \tilde{Z} G^\mu_\mu + T^\mu_\mu \text{q.fin} .
$$

(7.25)

The finite fluctuation parts of the trace of the energy-momentum tensor are

$$
T^\mu_\mu \text{q.fin} = (1 - 6\xi) \frac{\mathcal{F}'_{\text{fin}}}{2a^4} + \frac{H}{a^3} (1 - 6\xi) \mathcal{F}'_{\text{fin}}
+ \left[ \frac{m^2 + \lambda}{2a^2} \varphi^2 - (1 - 6\xi) \left( H^2 - \frac{R}{6} \right) \right] \frac{\mathcal{F}_{\text{fin}}}{a^2} + T^\mu_\mu \text{add}
$$

(7.26)

where

$$
T^\mu_\mu \text{add} = T^\mu_\mu \text{ano} - \frac{1}{16\pi^2} \left( \frac{V''}{6} + \frac{H^2 M^2}{3a^2} + \frac{M^4}{2a^4} - \frac{V'}{3a^3} \right)
- \frac{1}{16\pi^2} \left( \xi - \frac{1}{6} \right) \left( 12 \frac{H}{a^3} V' + (R + 18H^2) M^2 - \frac{1}{18} R^2 - \frac{(1) H^\mu_\mu}{36} \right) ,
$$

(7.27)

or, explicitly,

$$
T^\mu_\mu \text{add} = T^\mu_\mu \text{ano} - \frac{3}{4\pi^2} \left( \xi - \frac{1}{6} \right)^2 \left[ H \frac{R'}{a} + \frac{1}{2} R H^2 + \frac{1}{8} R^2 \right]
+ \frac{3}{4\pi^2} \left( \xi - \frac{1}{6} \right) \left[ \frac{R''}{12a^2} - \frac{1}{6a} H R' - \frac{1}{4a^2} \lambda \varphi^2 R - \frac{1}{2} m^2 R \right]
- 3\lambda \frac{H}{a^3} \varphi' \varphi + \frac{9}{4a^2} H^2 \lambda \varphi^2 - \frac{3}{2} H^2 m^2 \right] - \frac{1}{32\pi^2} \left( m^2 + \frac{\lambda}{2a^2} \varphi \right)^2
- \frac{\lambda}{96\pi^2 a^4} (\varphi' - a H \varphi)^2 - \frac{\lambda}{96\pi^2 a^4} \varphi \varphi'' - \frac{1}{288\pi^2} m^2 R .
$$

(7.28)

Comparing to Eq. (3.17) of \[29\] some differences in the coefficients can be
absorbed into a different choice of the finite parts of the renormalization
constants. The terms proportional to $\ln a(\tau)$ given there are included here into $T^{\mu \nu}_{\text{ren}}$ via the coefficients $\Delta \lambda, \Delta \tilde{Z}$, and $\Delta \tilde{\alpha}$; they are identical. We have checked, using MAPLE, that the energy momentum tensor is covariantly conserved. This is a valuable cross check, as it relates the various terms in $T_{tt}$ and $T_{\mu \nu}$ in a rather complex way.

8 Removing initial singularities

The set of initial conditions used in the previous section leads, after renormalization, to singularities at $\tau = 0$ in the remaining fluctuation integrals occurring in the equation of motion for the inflaton and in the energy-momentum tensor. This means that these initial singularities affect the Friedmann equations as well. As already shown in [27] these singularities can be removed by a suitable Bogoliubov transformation of the naive initial state, i.e., the vacuum state of a Fock space based on the mode functions with the initial conditions Eq. (3.22).

Choosing such an initial state one finds, in the fluctuation integral $F_{\text{fin}}$, the terms (8.1): 

$$F_{\text{sing}} = \int \frac{d^3 k}{(2\pi)^3} \left[ \frac{V'(0)}{8\Omega^4_{k0}} \sin(2\Omega_{k0} \tau) - \frac{V''(0)}{16\Omega^6_{k0}} \cos(2\Omega_{k0} \tau) \right].$$

While these are only nonanalytic as $\tau \to 0$, their first and second derivatives are singular. They occur in the energy-momentum tensor via (7.19) and (7.26). The singular behaviour is given by

$$F_{\text{sing}}' \approx -\frac{1}{8\pi^2} \ln(2m\tau) V'(0),$$

and

$$F_{\text{sing}}'' \approx -\frac{1}{16\pi^2} V''(0) + \frac{1}{16\pi^2} \ln(2m\tau) V''(0).$$

A Bogoliubov-transformed initial state (which is the Heisenberg state of the system) can be defined by requiring

$$[a(k) - \rho_k a^\dagger(k)] |i\rangle = 0.$$
If the fluctuation integral, the energy and the pressure are computed by taking the trace with respect to this state the functions $U_k(\tau)$ get replaced by

$$F_k(\tau) = \cosh(\gamma_k)U_k(\tau) + e^{i\delta_k} \sinh(\gamma_k)U_k^*(\tau), \quad (8.5)$$

where $\gamma_k$ and $\delta_k$ are defined by the relation

$$\rho_k = e^{i\delta} \tanh(\gamma_k). \quad (8.6)$$

The fluctuation integral then becomes

$$F(\tau) = \int_{-\infty}^{\infty} d\tau \left| F_k(\tau) \right|^2 \quad (8.7)$$

Expanding as before we find

$$F(\tau) = \int_{-\infty}^{\infty} d\tau \left\{ \cosh(2\gamma_k(\tau))|U_k(\tau)|^2 + \sinh(2\gamma_k) \text{Re} \left( e^{-i\delta_k} U_k^2(\tau) \right) \right\}. \quad (8.8)$$

Requiring that the terms proportional to $V'(0)$ and $V''(0)$ vanish leads to the conditions

$$\tan(\delta_k) = \frac{V'(0)}{2\Omega_0}, \quad (8.9)$$

$$\tanh(2\gamma_k) = \frac{V''(0)}{8\Omega_0^4} \left[ 1 + \tan^2(\delta_k) \right]^{1/2}. \quad (8.10)$$

For large $k$ this behaves as

$$\gamma_k \xrightarrow{k \to \infty} \frac{|V'(0)|}{8\Omega_0^4}. \quad (8.11)$$

The factor $\cosh(2\gamma_k)$ is equal to 1 for $\gamma_k = 0$; the difference w.r.t. 1 is given by

$$\cosh(2\gamma_k) - 1 = 2 \sinh^2(\gamma_k) \xrightarrow{k \to \infty} \frac{V'^2(0)}{32\Omega_0^4}. \quad (8.12)$$
The new terms proportional to \( \sinh(2\gamma_k) \) behave as
\[
\sinh(2\gamma_k) \xrightarrow{k \to \infty} \frac{|V'(0)|}{4\Omega_{k0}^3}.
\]
(8.13)
The dimensionally regularized fluctuation integral (8.8) takes, after cancelation of the singular integrals induced by Eqs. (8.9) and (8.10), the form
\[
F_{\text{reg}}(\tau) = -\frac{m_0^2}{16\pi^2} (L_0 + 1) - \frac{V(\tau)}{16\pi^2} L_0
\]
\[
+ \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \left\{ \sinh^2(\gamma_k) \left[ 1 - \frac{V(\tau)}{2\Omega_{k0}^2} \right] \right.
\]
\[
+ \cosh(2\gamma_k) \left[ \frac{V''(\tau)}{8\Omega_{k0}^4} - \frac{C(V'')}{8\Omega_{k0}^4} + 2\text{Re}h_k^2 + |h_k|^2 \right]
\]
\[
+ \sinh(2\gamma_k) \text{Re} e^{-2i\Omega_{k0}\tau - i\delta} \left( 2h_k + h_k^2 \right) \right\}.
\]
(8.14)
Using (8.12) and (8.13) we see that the Bogoliubov transform does not affect the ultraviolet divergences, and that, therefore, the renormalization procedure remains unchanged.

The structure of the equation of motion and the energy-momentum tensor remains the same after the Bogoliubov transformation, if the fluctuation integral \( F_{\text{fin}} \) is replaced by
\[
\tilde{F}_{\text{fin}} = -\frac{m_0^2}{16\pi^2} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\Omega_0} \left\{ 2\sinh^2(\gamma_k) \left[ 1 - \frac{V(\tau)}{2\Omega_{k0}^2} \right] \right.
\]
\[
+ \cosh(2\gamma_k) K_1(k, \tau) + \sinh(2\gamma_k) K_2(k, \tau) \right\} ,
\]
(8.15)
where we have introduced the functions:
\[
K_1(k, \tau) = \left[ \frac{V''(\tau)}{8\Omega_{k0}^4} - \frac{C(V'')}{8\Omega_{k0}^4} + 2\text{Re}h_k^2 + |h_k|^2 \right] ,
\]
(8.16)
\[
K_2(k, \tau) = \text{Re} \left\{ e^{-2i\Omega_{k0}\tau - i\delta} \left( 2h_k + h_k^2 \right) \right\} .
\]
(8.17)
Similarly, \( E_{\text{kin,fin}} \) becomes
\[
\tilde{E}_{\text{kin,fin}} = \frac{1}{a^4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\Omega_0} \left\{ 2\sinh^2(\gamma_k) \left[ 2\Omega_0^2 + \frac{V^2(\tau)}{4\Omega_0^2} \right] \right.
\]
\[
+ \sinh(2\gamma) K_3(k, \tau) + K_4(k, \tau) \cosh^2(2\gamma_k) \right\} - \frac{3M_0^4}{128\pi^2a^4} ,
\]
(8.18)
where \( K_3(k, \tau) \) denotes
\[
K_3(k, \tau) = \text{Re} \left\{ e^{-2i\Omega_{k0}\tau - i\delta} \left[ h_k' - 2i\Omega_{k0} (1 + h_k) h_k' \right] \right\} , \quad (8.19)
\]
\[
K_4(k, \tau) = \frac{1}{2} \left[ |h_k'|^2 - \frac{V^2(\tau)}{4\Omega_{k0}^2} \right] . \quad (8.20)
\]

With these replacements in the equation of motion and in the energy-momentum tensor the inflaton equation of motion and the Friedmann equations are ultraviolet finite and free of initial singularities. We should mention that this initial state is not determined uniquely. Any state or density matrix based on this new ‘vacuum state’ is admissible as long as the spectrum of coherent or incoherent excitations decreases stronger than the one parametrized by the Bogoliubov angles \( \gamma_k \) and \( \delta_k \).

We should like to mention that fixing the initial conditions results in a self-consistency problem, as the finite, modified fluctuation integrals in the equation of motion and in the energy-momentum tensor do not vanish at \( \tau = 0 \). They depend on the other initial parameters \( H(0), R(0), \varphi(0), \varphi'(0), \) and \( \varphi''(0) \), and vice versa. For the typical parameter sets, this self-consistency problem can be solved by an iteration which converges quickly.

9 Large \( N \) model

9.1 General formalism

We now consider the \( O(N) \) \( \sigma \) model defined by the Lagrangian
\[
\mathcal{L} = \frac{1}{2a^4} \partial_\mu \phi^i \partial^\mu \phi^i - \frac{1}{2a^2} m^2 \dot{\phi}^i \dot{\phi}^i - \frac{\zeta}{2} \mathcal{R} \phi^i \phi^i - \frac{\lambda}{4N a^4} (\phi^i \phi^i)^2 . \quad (9.1)
\]
whith \( N \) real scalar fields \( \phi^i, i = 1, .., N \). The nonequilibrium state of the system is characterized by a classical expectation value which we take in the direction of \( \phi_N \). We split the field into its expectation value, or mean field, \( \phi \) and the quantum fluctuations \( \psi \) via
\[
\phi^i(x, \tau) = \delta^i_N \sqrt{N} \phi(\tau) + \psi^i(x, \tau) . \quad (9.2)
\]
In the large-\( N \) limit one neglects, in the Lagrangian, all terms which are not of order \( N \). In particular terms containing the fluctuation \( \psi^N \) of the component \( \phi^N \) are at most of order \( \sqrt{N} \) and are dropped, therefore. This is in contrast
to the Hartree approximation where the fluctuations of \( \phi_N \) are included. The fluctuations of the other components are identical, their summation produces factors \( N-1 = N(1+O(1/N)) \). The quantum fluctuations are again decomposed into mode functions via (3.13). The nonequilibrium equations of motion for the field \( \phi(\tau) \) and of the mode functions \( U_k(\tau) \) have been derived by various authors [7, 8]. These equations differ from those of the one-loop approximation by the fact that the fluctuation integral not only modifies the mass of the mean field, but also the one of the quantum fluctuations.

The equation of motion for the mean field can be written as

\[
\phi''(\tau) + M^2(\tau)\phi(\tau) = 0 ,
\]

(9.3)
the one for the mode functions as

\[
U''_k(\tau) + \left[ k^2 + M^2(\tau) \right] U_k(\tau) = 0 ,
\]

(9.4)
with the same effective mass

\[
M^2(\tau) = a(\tau)^2 \left[ m^2 + \delta m^2 + (\xi + \delta \xi - \xi_n) R(\tau) \right] \\
+ (\lambda + \delta \lambda) a(\tau)^4 \left( \phi(\tau)^2 + \mathcal{F}(\tau) \right).
\]

(9.5)
We have included here the renormalization counter terms. \( \mathcal{F}(\tau) \) is again the fluctuation integral

\[
\mathcal{F}(\tau) = \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \frac{1}{2\Omega_0} |U_k|^2 .
\]

(9.6)
We restrict ourselves to a temperature \( T = 0 \) system, here. Generalization to thermal systems is straightforward [35]. As in the one-loop case we rewrite the mode equation in the form

\[
\left[ \frac{d^2}{d\tau^2} + \Omega_{k0}^2 \right] U_k(\tau) = -V(\tau) U_k(\tau) ,
\]

(9.7)
introducing, thereby, the time dependent potential

\[
V(\tau) = M^2(\tau) - M^2(0) ,
\]

(9.8)
The “initial mass” \( M(0) = m_0 \) will be determined below, as a solution of a gap equation. We define the time-dependent frequency \( \Omega_k(\tau) \) via

\[
\Omega_{k0}^2(\tau) = k^2 + M^2(\tau) .
\]

(9.9)
In the large-$N$ limit, and in $n$ dimensions the energy density is given by
\[
T_{tt} = \frac{1}{a^{2-\epsilon}} \left\{ \frac{1}{2a^2} \varphi'^2 + \frac{1}{2} (m^2 + \delta m^2) \varphi^2 + \frac{(\lambda + \delta \lambda)(a \mu)^\epsilon}{4a^2} \varphi^4 \right. \\
+ 2(n-1) \left( \xi + \delta \xi - \xi_n \right) \left( \frac{H}{a} \varphi \varphi' - \frac{1}{4} (n-2) H^2 \varphi^2 \right) - \frac{\lambda + \delta \lambda}{4a^{2-\epsilon}} F^2 \right\} \\
+ \delta \tilde{\Lambda} + \delta \tilde{Z} G_{tt} + \delta \tilde{\alpha} \left( \begin{array}{c} \xi \end{array} \right)^{(1)} H_{tt} + \delta \tilde{\beta} \left( \begin{array}{c} \xi \end{array} \right)^{(2)} H_{tt} + \delta \tilde{\gamma} H_{tt} \\
+ \frac{1}{a^{2-\epsilon}} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \left\{ \frac{|U_k|^2}{2a^2} + \frac{1}{2a^2} \Omega_k(\tau)^2 |U_k|^2 \right. \\
+ (n-1) \left( \xi + \delta \xi - \xi_n \right) \left[ \frac{H}{a} \frac{d}{d\tau} |U_k|^2 - \frac{1}{2} (n-2) H^2 + \frac{R}{(n-1)} |U_k|^2 \right] \right\},
\]
(9.10)
and the trace of the energy-momentum tensor takes the form
\[
T^\mu_\mu = \frac{1}{a^{2-\epsilon}} \left\{ 2(n-1) \left( \xi + \delta \xi - \xi_n \right) \left[ \frac{\varphi'}{a} - \frac{1}{2} (n-2) H \varphi \right]^2 \right. \\
+ 2(n-1) \left( \xi + \delta \xi - \xi_n \right) \left[ \frac{(m^2 + \delta m^2) \varphi^2 + (\lambda + \delta \lambda)(a \mu)^\epsilon}{24a^2} \varphi^4 \right] \left. \right\} \\
+ n \delta \tilde{\Lambda} + \delta \tilde{Z} G^\mu_\mu + \delta \tilde{\alpha} \left( \begin{array}{c} \xi \end{array} \right)^{(1)} H^\mu_\mu + \delta \tilde{\beta} \left( \begin{array}{c} \xi \end{array} \right)^{(2)} H^\mu_\mu + \delta \tilde{\gamma} H^\mu_\mu \\
+ \frac{1}{a^{2-\epsilon}} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} \left\{ \frac{1}{2} [n - 2 - 2(n-1)(\xi + \delta \xi)] \left[ \frac{|U'_k|^2}{a^2} + \Omega_k(\tau)^2 |U_k|^2 \right] \right. \\
+ \frac{1}{2} (n-2) \frac{H}{a} \frac{d}{d\tau} |U_k|^2 - \frac{1}{4} (n-2)^2 \left( H^2 - \frac{R}{(n-1)(n-2)} |U_k|^2 \right) \right. \\
+ \left( m^2 + \delta m^2 + \frac{(\lambda + \delta \lambda)(a \mu)^\epsilon}{2a^2} \right) |U_k|^2 \right\} - \frac{1}{4a^{4-2\epsilon}} (n-4)(\lambda + \delta \lambda) F^2 .
\]
(9.11)

### 9.2 Renormalization

The way in which the renormalization counter terms are determined in the large-$N$ case has been described in detail in [35]. We will closely follow this approach. As in the one-loop case use the expansion of the mode functions in order to single out the divergent parts:
\[
F = -M^2(\tau) x_0 - \frac{m_0^2}{16\pi^2} + F_{\text{fin}}
\]
(9.12)
with
\[ x_0 = \frac{1}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{M^2(0)} - \gamma \right\} . \] (9.13)

The finite part of the fluctuation integral reads
\[ \mathcal{F}_{\text{fin}} = \int \frac{d^{n-1}k}{(2\pi)^{n-1} 2\Omega_{k0}} \left( \frac{1}{2\Omega_{k0}} C(V', \tau) + 2\text{Re} \, \overline{h_k^{(2)}} + |h_k^{(1)}|^2 \right) , \] (9.14)

where
\begin{align*}
\mathcal{F}_{\text{fin}} &= \int \frac{d^{n-1}k}{(2\pi)^{n-1} 2\Omega_{k0}} \left( \frac{1}{2\Omega_{k0}} C(V', \tau) + 2\text{Re} \, \overline{h_k^{(2)}} + |h_k^{(1)}|^2 \right) ,
\end{align*}

(9.15)

The renormalization conditions for mass, coupling to the Ricci scalar and coupling constant are obtained from the requirement that the frequencies which appear in the mode equations are finite, i.e. that \( M^2(\tau) \) is a finite quantity:
\begin{align*}
a^2 \left[ m^2 + \delta m^2 + (\xi + \delta \xi - \frac{1}{6})R \right] + (\lambda + \delta \lambda) \left( \phi^2 + \mathcal{F} \right) \\
= a^2 \left[ m^2 + (\xi - \frac{1}{6})R \right] + \lambda \left( \phi^2 - \frac{m_0^2}{16\pi^2} \right) + \mathcal{F}_{\text{fin}} ,
\end{align*}

(9.15)

From this condition, and using Eq. (9.12), we find the following counter terms
\begin{align*}
\delta \lambda &= \frac{\lambda^2 x}{1 - \lambda x} , \quad \text{(9.16)} \\
\delta m^2 &= \frac{m^2}{1 - \lambda x} (x + \rho \lambda) , \quad \text{(9.17)} \\
\delta \xi &= \frac{\lambda}{1 - \lambda x} \left[ (\xi - \frac{1}{6}) x + \sigma \right] . \quad \text{(9.18)}
\end{align*}

The occurrence of the free parameters \( \rho, \sigma \) shows that the counter terms are not determined uniquely; the choice of these parameters corresponds to the freedom of an independent choice of renormalization conventions.

We obtain for \( M^2(\tau) \) the manifestly finite expression:
\begin{align*}
M^2(\tau) &= \mathcal{N}(\tau) \left\{ a(\tau)^2 \left[ m^2 + \lambda \rho + (\xi - \frac{1}{6} + \lambda \sigma)R(\tau) \right] \\
&\quad + \lambda \left[ \phi(\tau)^2 - \frac{m_0^2}{16\pi^2} - \frac{a(\tau)^2 R(\tau)}{288\pi^2} + \mathcal{F}_{\text{fin}}(\tau) \right] \right\} .
\end{align*}

(9.19)

Here
\[ \mathcal{N}(\tau) = \frac{1}{1 - \lambda (x - a(\tau) \cdot x_0)} = \frac{1}{1 + \frac{\lambda}{16\pi^2} \ln \frac{m_0^2 a(\tau)^2}{m_0^2}} , \] (9.20)
\[ x = \frac{1}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m^2} - \gamma \right\}. \quad (9.21) \]

We are now able to determine \( m_0^2 \). Taking Eq. (9.19) at \( \tau = 0 \) we obtain an implicit equation for \( m_0 = M(0) \), the gap equation

\[
m_0^2 = \mathcal{N} \left\{ a^2(0) \left[ m^2 + \lambda \rho + (\xi - \frac{1}{6} + \lambda \sigma) R(0) \right] + \lambda \left[ \phi^2 - \frac{m_0^2}{16\pi^2} - \frac{a^2(0) R(0)}{288\pi^2} \right] \right\}. \quad (9.22)\]

With Eqs. (9.19) and (9.22) the time dependent potential \( V(\tau) = M^2(\tau) - m_0^2 \) can be expressed in terms of finite quantities.

Renormalization of the energy-momentum tensor proceeds analogously. We insert the perturbative expansion of the mode functions and determine the remaining counter terms so as to render the tensor finite. We require them again to be independent of time and of the initial conditions. The energy density then is given by

\[
\mathcal{E} = \frac{1}{a^{2-\epsilon}} \left\{ \frac{1}{2a^2} \varphi^2 + \frac{1}{2} (m^2 + \delta m^2) \varphi^2 + \frac{\lambda + \delta \lambda}{4a^{2-\epsilon}} \varphi^4 \right\} \\
+ \delta \tilde{\Lambda} + \delta \tilde{\alpha}^{(1)} H_{tt} \delta \tilde{\beta}^{(2)} H_{tt} + \delta \tilde{\gamma} H_{tt} + \delta \tilde{Z} G_{tt} \\
+ 2(n-1)(\xi + \delta \xi - \xi_n) \left( \frac{1}{4} (n-2) \frac{H^2}{a^{2-\epsilon}} \varphi^2 + \frac{H}{a^{3-\epsilon}} \varphi' \right) + \mathcal{E}_{\text{kin,fin}} \\
+ \frac{1}{a^{2-\epsilon}} \left\{ - \frac{M^2(\tau)^2 x_0}{4a^2} - \frac{m_0^2 V(\tau)}{32\pi^2 a^2} + \frac{V(\tau) F_{\text{fin}}}{2a^2} \right\} \\
- \frac{1}{2} (n-1)(\xi + \delta \xi - \xi_n) \left( \frac{R}{2(n-1)} + \frac{1}{2} (n-2) H^2 \right) \times \\
\times \left( - M^2(\tau) x_0 - \frac{m_0^2}{16\pi^2} + F_{\text{fin}} \right) \\
+ \frac{H}{a} (n-1)(\xi + \delta \xi - \xi_n) (-V'(\tau) x_0 + F'_{\text{fin}}) \\
- \frac{\lambda + \delta \lambda}{4a^{2-\epsilon}} \left( - M^2(\tau) x_0 - \frac{m_0^2}{16\pi^2} + F_{\text{fin}} \right)^2 \right\}, \quad (9.23)\]
explicitly displaying the divergent parts. We have defined
\[ E_{\text{kin, fin}} = -\frac{3m_0^4}{128\pi^2 a^4} + \frac{1}{2a^4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\Omega_{k0}} \left[ |h_k^{(4)}|^2 - \frac{V^2(\tau)}{4\Omega_{k0}^2} \right]. \] (9.24)

After inserting the expressions for the coefficients \( \delta \lambda, \delta m^2 \) and \( \delta \xi \) we can now fix the coefficients of the higher derivative counter terms, the cosmological constant and the wave function renormalization so as to obtain a finite expression for the energy density. This is a very tedious exercise; using MAPLE we find

\[
\delta \alpha = -\frac{(\xi - \frac{1}{6} + \lambda \sigma)^2 x}{2(1 - \lambda x)},
\] (9.25)

\[
\delta \Lambda = \frac{am^4 (1 + \frac{\lambda \rho}{m^2})^2}{4(1 - \lambda x)},
\] (9.26)

\[
\delta Z = \frac{am^2 (1 + \frac{\lambda \rho}{m^2}) (\xi - \frac{1}{6} + \lambda \sigma)}{1 - \lambda x},
\] (9.27)

and renormalized energy density

\[
E_{\text{ren}} = \frac{\varphi^2}{2a^4} + \mathcal{N}(m^2 + \lambda \rho) \frac{\varphi^2}{2a^4} + \frac{\mathcal{N}\lambda}{4a^4} \varphi^4
\]

\[
-\mathcal{N}(6\xi - 1 + 6\lambda \sigma) \left( \frac{H^2}{2a^2} \varphi^2 - \frac{H}{a^3} \varphi \varphi' \right) + \Delta \tilde{\alpha} + \Delta \tilde{\alpha} H_t + \Delta \tilde{Z} G_{tt}
\]

\[
+ \mathcal{E}_{\text{kin, fin}} + \frac{1}{2a^4} \left[ V(\tau) + \mathcal{N}\lambda a^2 R \right] \left( \tilde{F}_{\text{fin}} - \frac{m_0^2}{16\pi^2} \right)
\]

\[
- \frac{\mathcal{N}}{2a^2} (6\xi - 1 + 6\lambda \sigma) \left( \frac{R}{6} + H^2 \right) \left( \tilde{F}_{\text{fin}} - \frac{m_0^2}{16\pi^2} \right)
\]

\[
+ \mathcal{N}(6\xi - 1 + 6\lambda \sigma) \frac{H}{2a^3} \tilde{F}_{\text{fin}} - \frac{\mathcal{N}\lambda}{4a^4} \left( \tilde{F}_{\text{fin}} - \frac{m_0^2}{16\pi^2} \right)^2
\]

\[
+ T_{tt}^{\text{ano}} + \frac{H}{96\pi^2 a^3} \left( aHM^2 - V' \right) - \frac{\mathcal{N}}{331776\pi^4} \lambda R^2
\]

\[
+ \frac{\mathcal{N}}{16\pi^2} \left( \xi - \frac{1}{6} \right) \left( \frac{(1)H_t}{36} + \frac{R^2}{72} + \frac{6H^2 M^2}{a^2} \right).
\] (9.28)

The coefficients of the finite parts of the counter terms are:

\[
\Delta \tilde{\alpha} = \frac{\mathcal{N}}{1152\pi^2} (6\xi - 1 + 6\lambda \sigma)^2 \ln \frac{m^2 a(\tau)^2}{m_0^2},
\] (9.29)
\[ \Delta \tilde{Z} = \frac{N}{432\pi^2} (m^2 + \lambda \rho) (6\xi - 1 + 6\lambda \sigma) \ln \frac{m^2 a(\tau)^2}{m_0^2}, \quad (9.30) \]

\[ \Delta \tilde{\Lambda} = -\frac{N}{64\pi^2} (m^2 + \lambda \rho)^2 \ln \frac{m^2 a(\tau)^2}{m_0^2}. \quad (9.31) \]

The renormalized trace of the energy-momentum tensor does not require any further counter terms, we find

\[
T_\mu^\mu = N (6\xi - 1 + 6\lambda \sigma) \left( \frac{\varphi'^2}{a^4} + \frac{\varphi''}{a^2} + \frac{H^2}{a^2} \varphi^2 - 2 \frac{H}{a^3} \varphi \varphi' \right) + \frac{\varphi''}{a^4}
+ 2N (m^2 + \lambda \rho) \frac{\varphi^2}{a^2} + \frac{N \lambda}{a^4} \varphi^4 + 4\Delta \tilde{\Lambda} + \Delta \tilde{\Lambda} H_\mu^\mu + \Delta \tilde{Z} G_\mu^\mu
- N (1 - 6\xi - 6\lambda \sigma) \tilde{F}''_{\text{fin}} - N (6\xi - 1 + 6\lambda \sigma) \frac{H}{a^3} \tilde{F}''_{\text{fin}}
+ N \left( m^2 + \lambda \rho + \frac{\lambda}{a^2} \varphi^2 \right) \left( \tilde{F}''_{\text{fin}} - \frac{m_0^2}{16\pi^2 a^2} \right)
+ N (6\xi - 1 + 6\lambda \sigma) \left( H^2 - \frac{R}{6} \right) \left( \tilde{F}''_{\text{fin}} - \frac{m_0^2}{16\pi^2 a^2} \right)
+ T_\mu^\mu \text{ano} - \frac{1}{16\pi^2} \left( \frac{V''}{6} + \frac{H^2 M^2}{3a^2} + \frac{M^4}{2a^4} - \frac{V'}{3a^3} \right)
- \frac{N}{16\pi^2} \left( \xi - \frac{1}{6} \right) \left( 12 \frac{H}{a^3} V' + (R + 18H^2) M^2 - \frac{1}{18} R^2 - \frac{(1) H_\mu^\mu}{36} \right)
- \frac{N}{18\pi^2} \frac{\lambda R}{a^2} \left( \frac{m_0^2}{16\pi^2} - \tilde{F}_{\text{fin}} + \frac{a^2 R}{288\pi^2} \right). \quad (9.32) \]

### 9.3 Removing initial singularities

We again have to remove the initial singularities. The construction of the initial state, and the modified expressions are analogous to those of section 8. Of course for the potential \( V(\tau) \) one has to use Eqs. (9.8), (9.19), and (9.22). As in the one-loop case the initial parameters have to be determined in a selfconsistent way, here in addition one has to solve the gap equation.
10 Numerical results

We have numerically implemented the formalism developed in the previous sections. We present here some results of the numerical computations. We restrict the presentation to results for the $O(N)$ $\sigma$ model.

We consider the coupled evolution of the scale parameter via the renormalized Friedmann equations (2.13), (2.14) with the energy-momentum tensor given in Eqs. (9.28) and (9.32), of the inflation field via Eq. (9.3), and of the quantum modes using Eq. (9.1). The quantum back reaction on the inflaton field and the scale parameter is included from the initial time on, this start having made possible by the modification of the initial state.

We show results for different sets of parameters, choosing values that are considered realistic for chaotic inflation, similar to those used in [12]. We do not consider the evolution during inflation, but only during the reheating period. This means that the initial amplitude of the inflaton field $\phi(0)$ is less or at most of the order of the Planck mass. During inflation the low momentum modes increase exponentially, requiring a different numerical or semianalytic approach [19, 10, 36].

We set $\lambda = 10^{-12}$ throughout. This is a value considered to be realistic. The inflaton mass is currently estimated to be of the order $m \simeq 10^{-6} \cdot M_P$. With such a value fluctuations do not develop, even if $\varphi(0) \simeq M_P$, and the evolution is dominated by the classical amplitude. We therefore consider smaller values of the inflaton mass, we also vary the initial field amplitude. In one group of parameter sets we fix the ratio $\varphi(0)/m$ and vary the ratio $m/M_P$, as it was done in [12]. In another set we start with $\varphi(0) = 2M_P$ and vary the ratio $m/M_P$. The parameters of the various sets are displayed in Table 1.

In the first group with the parameters sets 1, 2 and 3 we fix $\varphi(0)/m = 2 \cdot 10^7$ and choose $m/M_P = 10^{-12}, 10^{-8},$ and $10^{-7}$, respectively. This entails that the initial amplitudes vary along with the masses. The absolute time scale is determined by setting $m = 1$ in the numerical computations. So for all Figures which refer to this data set the abscissa is $m\tau$. This also determines the units of the various physical quantities.

For first parameter set the initial amplitude is extremely small, and so is the back reaction on the scale parameter. We have essentially the situation of Minkowski space-time. The scale parameter stays almost constant. The evolution of the field amplitude $\varphi$ is shown in Fig. 1. The fluctuations develop until $F_{\text{fin}} \simeq \varphi^2$ and then have a stationary amplitude, as well-known from
studies of the large $N$ model \cite{34,35}. We display, in Fig. 2, the fluctuation energy and the total energy, which show a small asymptotic decrease due to the expansion. The asymptotic value of the ratio $p/E$ is found to be $1/3$, corresponding to a radiation dominated ensemble.

For the second set the evolution of the scale parameter is plotted in Fig. 3; it is sizeable as to be expected for an initial amplitude $\varphi(0) = 0.2 M_P$. The classical field amplitude, as displayed in Fig. 4, starts decreasing after $m \tau \approx 5$; the asymptotic decrease is stronger than expected for the large-time behaviour in Minkowski-space time ($\tau^{-27}$), and so is due, in part, to the expansion. The fluctuations, displayed in Fig. 5 develop again until the asymptotic amplitude is reached, at the same time they are red-shifted. The ratio $p/E$, shown in Fig. 6 becomes $\approx 1/3$ (radiation dominated) after the fluctuations set in, and decreases to zero asymptotically. This has similarly been found in \cite{10} for a similar parameter set. The fluctuation energy becomes of the same order, as the total energy, see Fig. 7. We finally plot, in Fig. 8, a typical energy distribution of the produced quanta, with the familiar resonance band at low energy.

For parameter set 3 the field amplitude starts at $2 M_P$. The scale parameter develops strongly (see below). The fluctuations hardly develop, they are red-shifted immediately. The evolution is essentially driven by the classical field, with $\langle p/E \rangle \approx 0$. We show the classical amplitude in Fig. 9., its decrease is mainly due to the expansion.

These results are qualitatively analogous to those of \cite{12}, where $\lambda$ was chosen $10^{-14}$. The main difference is in the time scale for the built-up of fluctuations. In \cite{12} the mass $m$ is of the order $\sqrt{\lambda} \varphi(0)$, while here the latter quantity is ten times bigger, and dominates in the estimates for the time scales.

We now consider the second group of parameter sets, for which the initial amplitude is fixed, $\varphi(0) = 2 M_P$. For the sets 3, 4, and 5 the inflaton mass then is chosen as $m/M_P = 10^{-7}, 10^{-8},$ and $10^{-12}$, respectively. Numerically we have put $M_P = 10^7$ for all parameter sets. So for all Figures which refer to this data set, the abscissa is $10^7 M_P \tau$, and the units for all physical quantities are fixed correspondingly.

The data set 3 has been discussed above, it displays a strong expansion and fluctuations play essentially no rôle. The situation changes with decreasing inflaton mass, which for data set 4 is smaller by a factor of 10. The universe then still expands strongly, but at the same time we find sizeable fluctuations. Still the fluctuation energy only amounts to one part in $10^5$ of
the total field energy. For the much smaller masses value \( m/\mathcal{M}_P = 10^{-12} \) the expansion rate only goes down by a factor of about 5, the fluctuation energy is of the same order as the total energy.

We show, in Fig. 10, the evolution of the scale parameter for the data sets 3, 4, and 5, all of which show a strong increase. The computation of the time evolution for data set 3 was stopped at \( \tau \simeq 9 \) as the oscillation period in conformal time decreases, requiring very small time steps. The evolution becomes uninteresting in the present context, as the fluctuations are negligible. In Fig. 11 we display the fluctuation energy for these parameter sets. We also plot the total energy for set 5, the total energy for the other sets is almost identical, on the logarithmic scale. It is clearly seen that for set 3 the fluctuations hardly develop and are redshifted immediately. For set 4 with a somewhat smaller inflation mass the fluctuations evolve but remain on the level of one part in 1000. Finally, for very small masses the fluctuation energy increases to the same order of magnitude, roughly 90% of the total energy.

The computations were performed on PentiumII PC's, the CPU time for one of the parameter sets as presented above, was 4 – 8 hours. The typical time step, on the scale displayed in the Figures, was \( 10^{-3} \), with an adaptive step-size control. The covariant energy conservation was fulfilled to better than 1 part in \( 10^5 \).

11 Outlook

We have presented here the renormalized equations of motion for a self-coupled scalar field in a conformally flat FRW universe including the quantum back reaction in one-loop and large-\( N \) approximations. We have applied this formalism to the post-inflationary preheating period. Our results are consistent with those of other authors if we choose the same parameter sets. We found that for sufficiently light inflaton masses and initial values in the range \( \varphi \simeq \mathcal{M}_P \) one can have a substantial growth of the energy of the fluctuations at the same time as a substantial cosmological expansion. Such low values of the mass are not in the range of commonly accepted inflaton masses; however, this case could be relevant if the quantum fluctuations are those of other fields.

We have not considered the inflationary stage itself. In this case, for \( \xi = 0 \), the low momentum modes grow exponentially due to a term \(-R/6\) in
the effective mass, even in the absence of spinodal decomposition. This exponential growth and the extreme red-shifting pose special numerical problems, so that semi-analytical techniques become necessary. The collective evolution of the low momentum modes (‘zero mode assembly’) indeed leads to an essential simplification [10, 36].

We have considered here the quantum fluctuations of the inflaton field itself. It would be interesting to couple other fields, as e.g. fermion or gauge fields to the inflaton. This would allow for a more general choice of parameters and could lead to interesting phenomena as e.g. fermionic preheating [32, 38].

It is known since some time [39], that gravitational gauge invariance requires the inclusion of the scalar metric perturbations on the same level as the inclusion of the scalar field fluctuations. The resonant growth of such metric perturbations in conjunction with those of the scalar field has been considered recently [40]. It would be interesting to extend the formalism presented here so as to include metric perturbations.

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Table Captions

**Table 1**: Parameter sets. We display the parameters of the parameter sets 1-5. We only show the various ratios. The numerical values of the dimensionful quantities $m, M_P,$ and $\varphi(0)$ in the computer code are fixed by the numerical value $\varphi(0) = 2 \cdot 10^7$.

Figure Captions

**Fig. 1**: Classical amplitude $\varphi(\tau)$ for parameter set 1.
**Fig. 2**: Fluctuation energy (solid line) and total energy (dashed line) for parameter set 1.
**Fig. 3**: Evolution of the scale parameter $a(\tau)$ for parameter set 2.
**Fig. 4**: Classical amplitude $\varphi(\tau)$ for parameter set 2.
**Fig. 5**: Fluctuation integral $\mathcal{F}_{\text{fin}}$ for parameter set 2.
**Fig. 6**: Ratio if pressure to energy density, $p/\mathcal{E}$, for parameter set 2.
**Fig. 7**: Fluctuation energy (solid line) and total energy (dashed line) for parameter set 2.
**Fig. 8**: The energy distribution of the fluctuations at $\tau = 20$ for parameter 2.
**Fig. 9**: Classical amplitude $\varphi(\tau)$ for parameter set 3.
**Fig. 10**: Evolution of the scale factor $a(\tau)$ for parameter set 3 (dotted line), set 4 (solid line), and set 5 (dashed line).
**Fig. 11**: Evolution of the fluctuation energy for parameter set 3 (dotted line), set 4 (dash-dotted line), and set 5 (solid line). The total energy (dashed line) is the one for parameter set 5.
| set # | $\lambda$  | $m/M_P$  | $\phi(0)/M_P$ |
|------|------------|----------|---------------|
| 1    | $10^{-12}$ | $10^{-12}$ | $2 \cdot 10^{-5}$ |
| 2    | $10^{-12}$ | $10^{-8}$  | $2 \cdot 10^{-1}$  |
| 3    | $10^{-12}$ | $10^{-7}$  | $2$             |
| 4    | $10^{-12}$ | $10^{-8}$  | $2$             |
| 5    | $10^{-12}$ | $10^{-12}$ | $2$             |

Table 1
Figure 1
Figure 2

Figure 3
Figure 4

Figure 5
Figure 8

Figure 9
Figure 10

Figure 11