On Newman’s phenomenon in higher bases

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Abstract

A well known result of Newman says that up to a limit, multiples of 3 with even number of 1’s in binary representation always exceed multiples of 3 with odd number of 1’s. The phenomenon of preponderance of even number of 1’s is now known as Newman’s phenomenon. We show that this phenomenon exists for higher bases. Let \(b\) be a positive integer \(\geq 2\). Let \(A_b\) be the set of all natural numbers which contain only 0’s and 1’s in b-ary expansion and \(S_{q,0}^{(b)}(n)\) be the difference between the corresponding number of \(k_e < n, k_e \equiv i \mod q, k_e \in A_b\) and \(k_e\) has even number of 1’s in b-ary expansion and the number of \(k_o, k_o < n, k_o \equiv i \mod q, k_o \in A_b\) and \(k_o\) has odd number of 1’s in b-ary expansion. Let \(q\) be a multiple or divisor of \(b + 1\) which is relatively prime to \(b\) then we show that \(S_{q,0}^{(b)}(n) > 0\) for sufficiently large \(n\). We show that there is a stronger Newman’s phenomenon in \(A_b\) in the following sense.

If \(b > 2\) and \(n = \sum_{i=0}^{k-1} b_i 2^i\) with \(b_i \in \{0, 1\}\), let \(b(n) = \sum_{i=0}^{k-1} b_i b^i\) then \(\lim_{n \to \infty} \frac{S_{3,0}^{(2)}(n)}{S_{b+1,0}^{(b)}(b(n))} = 0\).

That is, for the same number of terms there is stronger preponderance in \(A_b\) than in \(A_2 = \mathbb{N}\).

In the last section we show that number of primes \(p \leq x\) for which \(S_{p,0}^{(b)}(n) > 0\) for sufficiently large \(n\) is \(o\left(\frac{x}{\log x}\right)\).

1 Introduction

L. Moser conjectured that when the multiples of 3 are written in binary, up to a limit, numbers with even number of 1’s always exceed numbers with odd number of 1’s. Newman in [7] proved that the conjecture is true. To be precise, he proved that \(S_{3,0}^{(2)}(n) > 0\) for all \(n\). If we consider a number \(n\) let \(b_{k-1} \cdots b_0\) be the binary expansion of \(n\). Since \(2 \equiv -1 \mod 3\), we have

\[ n \equiv b_0 - b_1 + b_2 \cdots + (-1)^{k-1}b_{k-1} \mod 3. \]

Let \(N\) be a natural number and \(N = n_{k-1}n_{k-2} \cdots n_0\) be binary expansion of \(N\). The difference of multiples of 3 having even number of 1’s and odd number of 1’s is

\[ S_{3,0}^{(2)}(n) = \sum_{b_0-b_1+\cdots+(-1)^{k-1}b_{k-1} \equiv 0 \mod 3 \atop \text{\(b_{k-1}b_{k-2}\cdots b_0 < n_{k-1}n_{k-2} \cdots n_0\)} } (-1)^{b_0+\cdots+b_{k-1}}, \quad (1.1) \]

where < denotes lexicographic order among sequences of finite length. In (1) if we replace 2 by an even natural number \(b\) then the sum appears to increase as \(b\) increases. That is, if we
consider the following sum

$$S^{(b)}_{b+1,0}(b(n)) = \sum_{b_0-b_1+\ldots+(-1)^{k-1}b_{k-1} \equiv 0 \pmod{b}} (-1)^{b_0+\ldots+b_{k-1}}$$

(1.2)

then the sum seems to increase as $b$ increases. So we can guess that $S^{(b)}_{b+1,0}(b(n)) > 0$ for sufficiently large $n$. We prove this result in section 2. Let $A_b$ be the set of all natural numbers which contain only 0’s and 1’s in b-ary expansion. Then up to a limit, multiples of $b+1$ with even number of 1’s in $A_b$ always exceed multiples of $b+1$ with odd number of 1’s in $A_b$. So there is Newman’s phenomenon among the multiples of $b+1$ in the set $A_b$. In fact the Newman’s phenomenon gets stronger as $b$ increases in the following sense. We show that for any two even numbers $b_1, b_2$ with $b_1 > b_2$ we have

$$\lim_{n \to \infty} S^{(b_2)}_{b_2+1,0}(b_2(n)) / S^{(b_1)}_{b_1+1,0}(b_1(n)) = 0.$$ 

That is, for the same number of terms in the sets $A_{b_1}$ and $A_{b_2}$ there is a stronger preponderance in $A_{b_1}$ compared to $A_{b_2}$.

In fact we prove more than $S^{(b)}_{b+1,0}(b(n)) > 0$. We prove the following theorem in section 2.

**Theorem 1.1.** If $b$ and $q$ are two positive integers such that $b \geq 2$, $(b+1)|q$ and $(b, q) = 1$ where $(b, q)$ is the greatest common divisor of $q$ and $b$. Let $v$ be an integer, then for sufficiently large $N$

1. $S^{(b)}_{q,v(b+1)}(N) > 0$.
2. $S^{(b)}_{q,v(b+1)+1}(N) < 0$.
3. if $b \leq 3$ then $S^{(b)}_{q,v(b+1)-1}(N) < 0$.

Theorem 1.1 partially generalizes Theorem 1 of [2] which states that $S^{(2)}_{3k,0}(n) > 0$ for almost all $n$.

In section 3 we prove the following result.

**Theorem 1.2.** If $d > 1$ is a divisor of $(b+1)$ for $b \geq 2$, then for sufficiently large $N$

1. $S^{(b)}_{d,0}(N) > 0$.
2. $S^{(b)}_{d,1}(N) < 0$.
3. $S^{(b)}_{d,-1}(N) \leq 0$ and if $d > 3$ then $S^{(b)}_{d,-1}(N) < 0$. 

2
Let $\mathbb{P}_t(b)$ denote the set of all primes such that $\frac{p-1}{\text{ord}_p(b)} = t$ where $\text{ord}_p(b)$ denotes the order of $b$ in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$. In the last section we prove that the primes satisfying $S_{p,0}(N) > 0$ are of zero density in the set of primes.

**Theorem 1.3.** The number of primes $p \in \mathbb{P}_t(b)$ for a given $t > 1$ such that $S_{p,0}^{(b)}(N) > 0$ for sufficiently large $N$ are bounded by

\[ p \leq C^* t^2 \log^2 t, \]

where $C^* > 0$ only depends on $b$. If $t = 1$ the number of primes are bounded by

\[ p \leq C_1, \]

where $C_1 > 0$ only depends on $b$. Furthermore, the total number of primes $p \leq x$ such that $S_{p,0}^{(b)}(n) > 0$ for sufficiently large $n$ is $O\left(\frac{x}{\log x}\right)$ as $x \to \infty$.

Theorem 1.3 is a generalization of Theorem 2 of [2].

## 2 Proof of Theorem 1.1

If $b$ is an odd number as $q$ is divisible by $b+1$, $q$ will be an even integer and all the integers satisfying the congruences $n \equiv v(b+1) \mod q$, $n \equiv v(b+1)+1 \mod q$ and $n \equiv v(b+1)−1 \mod q$ are even, odd and odd respectively. Also $n \equiv s_b(n) \mod 2$, where $s_b(n)$ denotes sum of digits in b-ary expansion. Hence all the numbers satisfying the congruences $n \equiv v(b+1) \mod q$, $n \equiv v(b+1)+1 \mod q$ and $n \equiv v(b+1)+1 \mod q$ will have even b-ary digit sum, odd b-ary digit sum and odd b-ary digit sum respectively so (1), (2) and (3) of Theorem 1.1 are trivially true when $b > 1$ is an odd number. Hence we can assume that $b$ is even. We prove six lemmas in order to prove Theorem 1.1.

**Lemma 2.1.** If $N < b^k$,

\[ S_{q,i}^{(b)}(b^k + N) = S_{q,i}^{(b)}(b^k) - S_{q,i-b^k}^{(b)}(N). \]

**Proof.** Note that if $n < b^k$ then $s_b(n+b^k) = s_b(n) + 1$. Thus, we have

\[ S_{q,i}^{(b)}(b^k + n) = \sum_{\substack{n \in A_b \\text{mod } q \\text{\ s.t. } 0 \leq n < N + b^k \\text{mod } q \\text{, } n \equiv i \mod q}} (-1)^{s_b(n)} \]

\[ = \sum_{n \in A_b \\text{mod } q \\text{\ s.t. } 0 \leq n < b^k \\text{mod } q \\text{, } n \equiv i \mod q} (-1)^{s_b(n)} + \sum_{n \in A_b \\text{mod } q \\text{\ s.t. } 0 \leq n < N \\text{mod } q \\text{, } n \equiv i - b^k \mod q} (-1)^{s_b(n+b^k)} \]

\[ = \sum_{n \in A_b \\text{mod } q \\text{\ s.t. } 0 \leq n < b^k \\text{mod } q \\text{, } n \equiv i \mod q} (-1)^{s_b(n)} - \sum_{n \in A_b \\text{mod } q \\text{\ s.t. } 0 \leq n < N \\text{mod } q \\text{, } n \equiv i - b^k \mod q} (-1)^{s_b(n)} \]

\[ = S_{q,i}^{(b)}(b^k) - S_{q,i-b^k}^{(b)}(N). \]
From Lemma 2.1 it follows that if \(N = b^{k_1} + \cdots + b^{k_r}\) and \(b^{k_1} > \cdots > b^{k_r}\) then

\[
S^{(b)}_{q,i}(N) = S^{(b)}_{q,i}(b^{k_1}) - \cdots - (-1)^{r-1}S^{(b)}_{q,i-b^{k_1} - \cdots - b^{k_{r-1}}}(1, b^{k_r}).
\]

(2.3)

Lemma 2.2. If \(k\) is a natural number then

\[
S^{(b)}_{q,i}(b^k) = \frac{1}{q} \sum_{m=0}^{q-1} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{mb^p}).
\]

Proof. If \(k\) is a natural number, we have

\[
S^{(b)}_{q,i}(b^k) = \sum_{n \in A_b \atop 0 \leq n < b^k} (-1)^{s_q(n)}
\]

\[
= \frac{1}{q} \sum_{n \in A_b \atop 0 \leq n < b^k} \sum_{m=0}^{q-1} (-1)^{s_q(n)} \zeta_q^{(n-i)m}
\]

\[
= \frac{1}{q} \sum_{m=0}^{q-1} \sum_{0 \leq n \leq b^k} (-1)^{r_0 + \cdots + r_{k-1}} \zeta_q^{(r_0 + r_1 \cos \theta + \cdots + r_{k-1} \cos (\theta+b^{k-1}b^{k})-i)m}
\]

\[
= \frac{1}{q} \sum_{m=0}^{q-1} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{mb^p}).
\]

Lemma 2.3. Let \(b = 2l\). If \(|z| = 1\) and \(|1 - z| > 2 \sin \frac{\pi l}{2l+1}\), then

\[
|1 - z||1 - z^b| < 4 \sin^2 \frac{\pi l}{2l+1}.
\]

Proof. Let \(z = e^{i\theta}\), where \(\theta \in [0, 2\pi]\). As \(|1 - z| > 2 \sin \frac{\pi l}{2l+1}\) we have \(\frac{2\pi l}{2l+1} < \theta < \frac{2\pi l}{2l+1}\). Let

\[
f(\theta) := 4|\sin l\theta \sin \frac{\theta}{2}| = |1 - z||1 - z^b|.
\]

If \(\frac{2\pi l}{2l+1} < \theta < \frac{2\pi l}{2l+1}\), it can be seen that \(|\tan l\theta| \leq |\tan \frac{\pi l}{2l+1}| \leq \tan \frac{\theta}{2}|.

We have

\[
f(\theta) = 4\delta \sin l\theta \sin \frac{\theta}{2},
\]

\[
f'(\theta) = 4\delta l \cos \theta \sin \frac{\theta}{2} + \frac{1}{2} \sin l\theta \cos \frac{\theta}{2}.
\]
| \( l = 2m \) | \( \frac{2m}{2l+1} < \theta \leq \pi \) | \( \sin l\theta \) | \( \cos l\theta \) | \( \sin \frac{\theta}{l} \) | \( \cos \frac{\theta}{l} \) | \( \delta \) | \( f'(\theta) \) |
|---|---|---|---|---|---|---|---|
| \( l = 2m \) | \( \frac{2m}{2l+1} < \theta < \frac{(2l+2)^2\pi}{2l+1} \) | \( > 0 \) | \( < 0 \) | \( > 0 \) | \( < 0 \) | \( +1 \) | \( > 0 \) |
| \( l = 2m + 1 \) | \( \frac{2m+1}{2l+1} < \theta < \pi \) | \( > 0 \) | \( < 0 \) | \( > 0 \) | \( < 0 \) | \( -1 \) | \( > 0 \) |

So, in any interval

\[
f(\theta) < f(\pi \pm \frac{\pi}{2l+1}) = 4 \sin^2 \frac{\pi l}{2l+1}.
\]

Let \( s \) be order of \( b \) in the multiplicative group \((\mathbb{Z}/q\mathbb{Z})^*\).

**Lemma 2.4.** If \((q, b) = 1, (b + 1)|q \) and \( b \) is even, \( m \equiv \pm ql \mod q \) then

\[
\left| \prod_{p=0}^{s-1} (1 - \zeta_{ql}^{bp}) \right| = \left( 2 \sin \frac{\pi l}{2l+1} \right)^s,
\]

and if \( m \not\equiv \pm ql \mod q \) then

\[
\left| \prod_{p=0}^{s-1} (1 - \zeta_{ql}^{bp}) \right| < \left( 2 \sin \frac{\pi l}{2l+1} \right)^s.
\]

**Proof.** Observe that \(|1 - \zeta_{ql}^{bp}| = 2 \sin \frac{\pi l}{2l+1} \) for \( 0 \leq p \leq s - 1 \) if and only if \( m \equiv \pm ql \mod q \). Hence

\[
\left| \prod_{p=0}^{s-1} (1 - \zeta_{ql}^{bp}) \right| = \left( 2 \sin \frac{\pi l}{2l+1} \right)^s
\]

when \( m \equiv \pm ql \mod q \). If \( m \not\equiv \pm ql \mod q \) then \(|1 - \zeta_{ql}^{bp}| \neq 2 \sin \frac{\pi l}{2l+1} \) for \( 0 \leq i \leq s - 1 \). Let \( S_1, S_2 \) and \( S_3 \) be subsets of \( \{0, 1, \ldots, s - 1\} \). \( S_1 \) contains all the \( p \) such that \(|1 - \zeta_{ql}^{bp}| > 2 \sin \frac{\pi l}{2l+1} \), \( S_2 \) all \( p \) such that \( p - 1 \mod s \in S_1 \), and \( S_3 \) contains the remaining elements of \( \{0, 1, \ldots, s - 1\} \). Clearly, from Lemma 2.3 \( S_1 \cap S_2 = \phi \) and if \( p \in S_1 \) then \(|1 - \zeta_{ql}^{bp})|1 - \zeta_{ql}^{bp+1})| < (2 \sin \frac{\pi l}{2l+1})^2 \). Therefore

\[
\left| \prod_{p=0}^{s-1} (1 - \zeta_{ql}^{bp}) \right| = \prod_{p \in S_1} |1 - \zeta_{ql}^{bp}) (1 - \zeta_{ql}^{bp+1})| \prod_{p \in S_3} |1 - \zeta_{ql}^{bp})| < \left( 2 \sin \frac{\pi l}{2l+1} \right)^s.
\]

Let \( \gamma = \max \left\{ \left| \prod_{p=0}^{s-1} (1 - \zeta_{ql}^{bp}) \right| : m \not\equiv \pm ql \mod q \right\}. \) From Lemma 2.4, we have \( \gamma < \left( 2 \sin \frac{\pi l}{2l+1} \right)^s \).
Lemma 2.5.

\[ S_{q,i}^{(b)}(b^k) = \begin{cases} 
\frac{2}{q} \left( \cos \frac{2\pi i}{2l+1} \right) \left( \cos \frac{2\pi i}{2l+1} \right)^k + O(\gamma^k) & \text{k even} \\
\frac{2}{q} \left( \cos \frac{2\pi i}{2l+1} - \cos \frac{2\pi(i-1)}{2l+1} \right) \left( 2 \sin \frac{\pi i}{2l+1} \right)^{k-1} + O(\gamma^k) & \text{k odd.} 
\end{cases} \]

Proof. Let \( k = k_1 + k_2 \) where \( 0 \leq k_2 \leq s - 1 \). From Lemma 2.2 we have

\[ S_{q,i}^{(b)}(b^k) = \frac{1}{q} \sum_{m=\frac{q}{2l+1}}^{q} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_{p^m}) + \frac{1}{q} \sum_{m \neq \frac{q}{2l+1}}^{q} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_{p^m}) \quad (2.4) \]

where the first term of right hand side of equation (2.4) is

\[ \frac{1}{q} \sum_{m=\frac{q}{2l+1}}^{q} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_{p^m}) = \begin{cases} 
\frac{2}{q} \cos \frac{2\pi i}{2l+1} \left( \cos \frac{2\pi l}{2l+1} \right)^k & \text{k even} \\
\frac{2}{q} \left( \cos \frac{2\pi i}{2l+1} - \cos \frac{2\pi(i-1)}{2l+1} \right) \left( 2 \sin \frac{\pi i}{2l+1} \right)^{k-1} & \text{k odd.} 
\end{cases} \]

Hence

\[ \frac{1}{q} \sum_{m=\frac{q}{2l+1}}^{q} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_{p^m}) = \left| \frac{1}{q} \sum_{m=\frac{q}{2l+1}}^{q} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_{p^m}) \right| \]

The second term of right hand side of equation (2.4)

\[ \left| \frac{1}{q} \sum_{m \neq \frac{q}{2l+1}}^{q} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_{p^m}) \right| = \left| \frac{1}{q} \sum_{m \neq \frac{q}{2l+1}}^{q} \zeta_q^{-im} \left( \prod_{p=0}^{s-1} (1 - \zeta_{p^m}) \right)^{k_1} \left( \prod_{p=0}^{k-1} (1 - \zeta_{p^m}) \right) \right| \]

Let \( C = \max \{ \prod_{p=0}^{k_2-1} (1 - \zeta_{q^p}) : m \in \mathbb{N}, 0 \leq k_2 \leq s - 1 \} \) then we have

\[ \left| \frac{1}{q} \sum_{m \neq \frac{q}{2l+1}}^{q} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_{p^m}) \right| \leq C \gamma^{k_1} = O(\gamma^k) \]

which implies

\[ S_{q,i}^{(b)}(b^k) = \begin{cases} 
\frac{2}{q} \left( \cos \frac{2\pi i}{2l+1} \right) \left( \cos \frac{2\pi l}{2l+1} \right)^k + O(\gamma^k) & \text{k even} \\
\frac{2}{q} \left( \cos \frac{2\pi i}{2l+1} - \cos \frac{2\pi(i-1)}{2l+1} \right) \left( 2 \sin \frac{\pi i}{2l+1} \right)^{k-1} + O(\gamma^k) & \text{k odd.} 
\end{cases} \]

\[ \square \]

Lemma 2.6. If \( b \geq 4 \) is even and \( b = 2l \) then there exists a constant \( M > 0 \) depending upon \( b, q \) and not depending on \( k \) such that
1. \[ S_{q,v(b+1)}^{(b)}(b^k) \geq \frac{1}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k+1} - M\gamma^k \]

2. \[ S_{q,v(b+1)+1}^{(b)}(b^k) \leq \frac{2}{q} \cos 2\pi l \left( 2 \sin \frac{\pi l}{2l+1} \right)^k + M\gamma^k \]

3. \[ S_{q,v(b+1)-1}^{(b)}(b^k) \leq \frac{2}{q} \left( \cos 2\pi l - \cos \frac{4\pi l}{2l+1} \right) \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k-1} + M\gamma^k \]

4. \[ |S_{q,i}^{(b)}(b^k)| \leq \frac{2}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^k + M\gamma^k \]

5. \[ S_{q,v(b+1)+1}^{(b)}(b^k) \leq M\gamma^k \]

6. \[ S_{q,v(b+1)+0 \text{ or } \pm 2}^{(b)}(b^k) \geq -M\gamma^k \]

**Proof.** From Lemma 2.5 we have

\[ S_{q,i}^{(b)}(b^k) = \begin{cases} 
\frac{2}{q} \left( \cos \frac{2\pi l}{2l+1} \right) \left( 2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^k) & \text{k even} \\
\frac{2}{q} \left( \cos \frac{2\pi l}{2l+1} - \cos \frac{2\pi (i-1) l}{2l+1} \right) \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k-1} + O(\gamma^k) & \text{k odd}
\end{cases} \]

Hence for \( i = v(b+1), v(b+1) + 1 \) and \( v(b+1) - 1 \)

\[ S_{q,v(b+1)+1}^{(b)}(b^k) = \begin{cases} 
\frac{2}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^k) & \text{k even} \\
\frac{1}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k+1} + O(\gamma^k) & \text{k odd}
\end{cases} \]

\[ S_{q,v(b+1)-1}^{(b)}(b^k) = \begin{cases} 
\frac{2}{q} \cos 2\pi l \left( 2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^k) & \text{k even} \\
-\frac{1}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^k) & \text{k odd}
\end{cases} \]

\[ S_{q,v(b+1)-1}^{(b)}(b^k) = \begin{cases} 
\frac{2}{q} \cos 2\pi l \left( 2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^k) & \text{k even} \\
\frac{2}{q} \left( \cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} \right) \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k-1} + O(\gamma^k) & \text{k odd}
\end{cases} \]
Results (1), (2), (3) and (4) follow from the inequalities

\[
\begin{align*}
2 \sin \frac{\pi l}{2l+1} & \leq 2, \\
2 \cos \frac{2\pi l}{2l+1} & \geq -2 \sin \frac{\pi l}{2l+1}, \\
\cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} & \geq 2 \cos \frac{2\pi l}{2l+1} \sin \frac{\pi l}{2l+1}, \\
\cos \frac{2\pi l}{2l+1} \leq 1 \quad \text{and} \\
\cos \frac{2\pi l}{2l+1} - \cos \frac{2\pi (i-1)l}{2l+1} & \leq 2 \sin \frac{\pi l}{2l+1}.
\end{align*}
\]

If \( l \geq 2 \) from the inequalities \( \cos \frac{2\pi l}{2l+1} < 0 \) and \( \cos \frac{4\pi l}{2l+1} > 0 \) we have

\[
S^{(b)}_{q, v(b+1)\pm 1}(b^k) \leq M \gamma^k,
\]

for some \( M \). Hence (5) is true.

Proof of Theorem 1.1

Proof. Theorem 1.3 of [3] and Theorem 1 of [2] covers the case \( b = 2 \) so we can assume \( b \geq 4 \).

From (2.3) if

\[
N = b^{k_1} + \cdots + b^{k_r}
\]

where \( k_1 > \cdots > k_r \) we have

\[
S^{(b)}_{q, i}(N) = S^{(b)}_{q, i}(b^{k_1}) - S^{(b)}_{q, i-b^{k_1}}(b^{k_2}) + S^{(b)}_{q, i-b^{k_1}-b^{k_2}}(b^{k_3}) - \cdots .
\]

When \( i = v(b+1) \) we have

\[
S^{(b)}_{q, v(b+1)}(N) = S^{(b)}_{q, v(b+1)}(b^{k_1}) - S^{(b)}_{q, v(b+1)-b^{k_1}}(b^{k_2}) + S^{(b)}_{q, v(b+1)-b^{k_1}-b^{k_2}}(b^{k_3}) - \cdots .
\]
As $b^k \equiv \pm 1 \mod (b + 1)$, from Lemma 2.6 we have
\[
S_{q,v(b+1)}^{(b)}(b^k) \geq \frac{1}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k+1} - M\gamma^2,
\]
\[
S_{q,v(b+1)-b^k}^{(b)}(b^k) \leq M\gamma^2
\]
and
\[
\left| S_{q,v}^{(b)}(b^k) \right| \leq \frac{2}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^k + M\gamma^2.
\]

Let $\beta = 2 \sin \frac{\pi l}{2l+1}$ then
\[
S_{q,v(b+1)+1}^{(b)}(N) \geq (1 + o(1)) \left( \frac{1}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k+1} - \frac{2}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k-2} - \frac{2}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k-3} + \cdots \right)
\]
\[
= (1 + o(1)) \left( \frac{\beta^{k+1}}{q} - \frac{2 \beta^{k-1}}{q(\beta - 1)} \right).
\]

Hence (1) of Theorem 1.1 is true.
\[
S_{q,v(b+1)+1}^{(b)}(N) = S_{q,v(b+1)+1}^{(b)}(b^k) + S_{q,v(b+1)+1-b^k}^{(b)}(b^k) + \cdots
\]

From (2) of Lemma 2.6 and from (6) of Lemma 2.6
\[
S_{q,v(b+1)+1}^{(b)}(b^k) \leq \frac{2}{q} \cos 2 \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k} + M\gamma^2
\]

and
\[
S_{q,v(b+1)+1-b^k}^{(b)}(b^k) \geq -M\gamma^2.
\]

Therefore
\[
S_{q,v(b+1)+1}^{(b)}(N) \leq (1 + o(1)) \left( \frac{2}{q} \cos \frac{2\pi l}{2l+1} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k+1} + \frac{2}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k-2} + \frac{2}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k-3} + \cdots \right)
\]
\[
= (1 + o(1)) \left( \frac{2}{q} \cos 2 \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k} \right) + \frac{2}{q} \left( \frac{\beta^{k+1}}{q} - \frac{2 \beta^{k-1}}{q(\beta - 1)} \right).
\]

Hence (2) of Theorem 1.1 is true for sufficiently large $N$.
\[
S_{q,v(b+1)-1}^{(b)}(N) = S_{q,v(b+1)-1}^{(b)}(b^k) - S_{q,v(b+1)-1-b^k}^{(b)}(b^k) + S_{q,v(b+1)-1-b^k}^{(b)}(b^k) + \cdots
\]

From (3) of Lemma 2.6 and (6) of Lemma 2.6
\[
S_{q,v(b+1)-1}^{(b)}(b^k) \leq (1 + o(1)) \frac{2}{q} \left( \cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} \right) \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k-1}
\]
and

\[ S_{q,v}(b+1)-1 - b^{k_1} \geq -M\gamma^k. \]

Therefore

\[
S_{q,v}(b+1)-1(N) \leq (1 + o(1)) \left( \frac{2}{q} \left( \cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} \right) \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k_1-1} + \frac{2}{q} \left( 2 \sin \frac{\pi l}{2l+1} \right)^{k_1-2} + \cdots \right)
\]

\[
= (1 + o(1)) \frac{2}{q} \beta^{k_1-1} \left( \cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} + \frac{1}{\beta - 1} \right) \leq 0.
\]

Hence (3) of Theorem 1.1 is true.

\[ \square \]

**Corollary 2.7.** If \( b_1 \) and \( b_2 \) are two even numbers and \( b_1 > b_2 \) then

\[
\lim_{n \to \infty} \frac{S_{b_2+1,0}(b_2(n))}{S_{b_1+1,0}(b_1(n))} = 0.
\]

**Proof.** Let \( n = \sum_{i=0}^{k-1} \epsilon_i 2^i \) where \( \epsilon_i \in \{0, 1\} \). From Lemma 2.6 and Theorem 1.1 one can prove that, for every even \( b \) for sufficiently large \( n \) there exists constants \( c_1 > 0 \) and \( c_2 > 0 \) independent of \( k \) such that

\[
c_1 \left( 2 \sin \frac{\pi b}{2b+2} \right)^k < S_{b+1,0}(b(n)) < c_2 \left( 2 \sin \frac{\pi b}{2b+2} \right)^k. \tag{2.5}
\]

The Corollary follows from (2.5).

\section{Proof of Theorem 1.2}

**Proof of Theorem 1.2**

If \( d \) is even then \( b \) is odd and the result is trivial. If \( d \geq 3 \) is odd and \( d \mid (b+1) \). Let \( \phi : A_{d-1} \to A_b \) be a map defined by

\[
\phi((d-1)^{k_1} + \cdots + (d-1)^{k_r}) = b^{k_1} + \cdots + b^{k_r}
\]

for \( k_1 > \cdots > k_r \). It is easy to see that \( n \in A_{d-1} \implies n \equiv \phi(n) \mod d \). Hence

\[
S^{(d-1)}_{d,i}(d-1)^{k_1} + \cdots + (d-1)^{k_r} = S^{(b)}_{d,i}(b^{k_1} + \cdots + b^{k_r}).
\]

From previous theorem for sufficiently large \( n \)

\[
S^{(d-1)}_{d,0}(n) > 0, S^{(d-1)}_{d,1}(n) < 0 \text{ and } S^{(d-1)}_{d,-1}(n) \leq 0.
\]

Hence

\[
S^{(b)}_{d,0}(n) > 0, S^{(b)}_{d,1}(n) < 0 \text{ and } S^{(b)}_{d,-1}(n) \leq 0.
\]

\[ \square \]
4 Proof of Theorem 1.3

Proof of Theorem 1.3

Proof. The proof will be similar to proof of Theorem 2 in [2] and Theorem 1.8 in [3]. Note that in this proof $b$ need not be even and $b$ need not equal $2l$. Let $s$ be the order of $b$ in $(\mathbb{Z}/p\mathbb{Z})^*$ and let $L_1, \ldots, L_t$ be cosets of $\{1, b, \ldots, b^{s-1}\}$. From Lemma 2.2 we have

$$S_p(b^{4ks-2}) = \frac{1}{p} \sum_{r=1}^{t} \left( \prod_{j=0}^{s-1} (1 - \zeta_p^{mb^i}) \right)^{4k} \left( \sum_{l \in L_r} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{lb})} \right),$$

where $m$ is picked from the set $L_r$. We have

$$\left( \prod_{j=0}^{s-1} (1 - \zeta_p^{mb^i}) \right)^{4k} = \left( \prod_{i=0}^{s-1} (\zeta_p^{mb^i}) \right)^{4k} \prod_{i=0}^{s-1} (\zeta_p^{mb^i} - \zeta_p^{mb^i})^{4k} \geq 0,$$

and

$$\text{Re} \left( \sum_{l \in L_r} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{lb})} \right) = \sum_{l \in L_r} -\frac{\left( \cos\frac{2\pi lb}{2p} \cos\frac{2\pi l}{2p} - \sin\frac{2\pi lb}{2p} \sin\frac{2\pi l}{2p} \right)}{4 \sin\frac{2\pi lb}{2p} \sin\frac{2\pi l}{2p}}$$

$$= -\sum_{l \in L_r} \left( -\frac{1}{4} + \frac{1}{4 \tan\frac{2\pi lb}{2p} \tan\frac{2\pi l}{2p}} \right)$$

$$= \frac{s}{4} - \sum_{l \in L_r} \frac{1}{4 \tan\frac{2\pi lb}{2p} \tan\frac{2\pi l}{2p}}.$$

Now the following Lemma will help in completing the proof of Theorem 1.3

Lemma 4.1. Let $L$ be a coset of $\{1, b, \ldots, b^{s-1}\}$ and $p \geq Ct^2(\log p)^2$ then

$$\sum_{l \in L} \frac{1}{4 \tan\frac{2\pi lb}{2p} \tan\frac{2\pi l}{2p}} \geq \frac{C_1(b)p^{2}}{t^2 \log p} - C_2(b)s$$

for some positive constants $C_1(b), C_2(b)$ and $C$ which only depend on $b$.

From Lemma 4.1

$$\text{Re} \left( \sum_{l \in L_r} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{lb})} \right) \leq -\frac{C_1(b)p^{2}}{t^2 \log p} + \left( C_2(b) + \frac{1}{4} \right) s$$

Hence if

$$p > \max \left\{ \left( \frac{C_2(b) + \frac{1}{4}}{C_1(b)} \right)^2 (t \log p)^2, C(t \log p)^2 \right\} \leq C'(t \log p)^2$$

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then \( S^{(b)}_{p,0}(b^{4k+2}) < 0 \). Hence if \( p \leq C'(t \log p)^2 \) and \( t > 1 \) one can prove that there exists a constant \( C' \) such that

\[
p \leq C'(t \log t)^2.
\]

From the following variation of result of Erdős we can prove the second part of the theorem.

**Lemma 4.2.** For every integer \( b \geq 2 \) and every sequence \( \epsilon_p \to 0 \) as \( p \to \infty \) we have

\[
\left| \left\{ p \leq x : \text{ord}_p(b) \leq p^{\frac{1}{2}+\epsilon_p} \right\} \right| = o \left( \frac{x}{\log x} \right).
\]

\[ \square \]

**Proof of Lemma 4.1**

Proof. The residues are taken from \( -\frac{p}{2} \) and \( \frac{p}{2} \). If we partition \( L \) into four sets \( P_1, P_2, P_3 \) and \( P_4 \). \( P_1 \) contains all \( l \in L \) satisfying \( |l| \leq \frac{p}{4b} \) and \( \lmod p > \frac{p}{4b} \). \( P_2 \) contains all \( l \) satisfying \( |l| \leq \frac{p}{4b} \) and \( \lmod p \leq \frac{p}{4b} \). \( P_3 \) contains all \( l \) satisfying \( |l| > \frac{p}{4b} \) and \( \lmod p > \frac{p}{4b} \). We have

\[
\sum := \sum_{l \in L} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} = \sum_{l \in P_1} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} + \sum_{l \in P_2} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} + \sum_{l \in P_3} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} + \sum_{l \in P_4} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}}.
\]

Observe that

\[ l \in P_1 \iff b^{-1}l \in P_3. \tag{4.6} \]

So

\[
\sum_{l \in P_1 \cup P_3} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} = \sum_{l \in P_1} \left( \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} + \frac{1}{4 \tan \frac{\pi (lb-1)}{p} \tan \frac{\pi l}{p}} \right). \tag{4.7}
\]

If \( |l| \leq \frac{p}{4b} \) then

\[
\frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} = \frac{\cos \frac{\pi lb}{p} \cos \frac{\pi l}{p}}{4 \sin \frac{\pi lb}{p} \sin \frac{\pi l}{p}} \geq \frac{\cos \frac{\pi l}{p} \cos \frac{\pi l}{p}}{4 \pi lb \pi l} = c_1(b) \frac{p^2}{l^2}. \tag{4.8}
\]

If \( |b^{-1}l \mod p| > \frac{p}{4b} \) then \( |\tan \frac{\pi (b^{-1}l)}{p}| \geq \tan \frac{\pi}{4b} \) and \( |\tan \frac{\pi l}{p}| \geq \frac{\pi l}{p} \) which implies

\[
\frac{1}{4 \tan \frac{\pi (b^{-1}l)}{p} \tan \frac{\pi l}{p}} \geq \frac{-1}{4 \tan \frac{\pi (b^{-1}l)}{p} \tan \frac{\pi l}{p}} \geq \frac{-p}{4 \tan \frac{\pi}{4b} l} = - \frac{c_2(b)}{|l|}. \tag{4.9}
\]

If \( |l| > \frac{p}{4b} \) and \( |b \mod p| > \frac{p}{4b} \) then

\[
\frac{1}{4 \tan \frac{\pi (b^{-1}l)}{p} \tan \frac{\pi l}{p}} \geq \frac{-1}{\tan(\frac{\pi}{4b})^2} = -c_3(b). \tag{4.10}
\]

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Using (4.6), (4.7), (4.8), (4.9) and (4.10) we have

\[ \sum \geq \sum_{l \in P_1} \left( \frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{l \in P_2} \frac{c_1(b)p^2}{l^2} + \sum_{l \in P_4} (-c_3(b)) \]  

(4.11)

\[ \geq \sum_{l \in P_1 \cup P_2} \left( \frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{l \in P_4} (-c_3(b)). \]  

(4.12)

Note that

\[ \frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \geq - \frac{(c_2(b))^2}{4(c_1(b))^2} = -c_4(b). \]  

(4.13)

If \( \frac{|p|}{|l|} \geq \frac{2c_2(b)}{c_1(b)} = c_5(b) \) then

\[ \frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \geq \frac{c_1(b)p^2}{2l^2}. \]  

(4.14)

From Polya-Vinogradov inequality \[8\]

\[ |\{0 < k \leq 2t\sqrt{p} \log p, \; k \in L_r\}| > \sqrt{p} \log p. \] 

If

\[ p \geq 4(c_5(b))^2t^2(\log p)^2 = C t^2(\log p)^2 \]

and \( 0 \leq l \leq 2t\sqrt{p} \log p \) then \( \frac{|p|}{|l|} \geq c_5(b) \) and from (4.14)

\[ \frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \geq \frac{c_1(b)p^2}{2l^2}. \]  

(4.15)

Hence if \( p \geq 4(c_5(b))^2t^2(\log p)^2 \) and \( \frac{p}{2t\sqrt{p} \log p} \leq \frac{p}{2t} \) then from (4.11), (4.13) and (4.15) we have

\[ \sum \geq \sum_{\frac{|p|}{|l|} \geq \frac{1}{2t}} \left( \frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{l \in P_4} (-c_3(b)) \]

\[ \geq \sum_{0 < l \leq 2t\sqrt{p} \log p} \left( \frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{l > 2t\sqrt{p} \log p} \left( \frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{l \in P_4} (-c_3(b)) \]

\[ \geq \sum_{0 < l \leq 2t\sqrt{p} \log p} \frac{c_1(b)p^2}{2l^2} + \sum_{\frac{|p|}{|l|} \leq \frac{1}{2t}} (-c_4(b)) + \sum_{l \in P_4} (-c_3(b)) \]

\[ \geq \frac{c_1(b)p^3}{8t^2 \log p} - c_4(b)s - c_3(b)s = \frac{C_1(b)p^3}{t^2 \log p} - C_2(b)s. \]  

□

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