A DRONE CAN HEAR THE SHAPE OF A ROOM

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Abstract. We show that one can reconstruct the shape of a room with planar walls from
the first-order echoes received by four non-planar microphones placed on a drone with generic
position and orientation. Both the cases where the source is located in the room and on the
drone are considered. If the microphone positions are picked at random, then with probability
one, the location of any wall is correctly reconstructed as long as it is heard by four micro-
phones. Our algorithm uses a simple echo sorting criterion to recover the wall assignments for
the echoes. We prove that, if the position and orientation of the drone on which the micro-
phones are mounted do not lie on a certain set of dimension at most 5 in the 6-dimensional
space of all drone positions and orientations, then the wall assignment obtained through our
echo sorting criterion must be the right one and thus the reconstruction obtained through our
algorithm is correct. Our proof uses methods from computational commutative algebra.

Introduction

Assume we have a room, by which we understand an arrangement of planar walls, which may
include ceilings, floors, and sloping walls. An omnidirectional loudspeaker and some omnidirec-
tional microphones are in the room. The loudspeaker, modeled as a point source, emits a short
duration pressure wave (a sound impulse) at a frequency high enough so that the ray acoustics
approximation is valid. The microphones receive several delayed responses corresponding to the
sound bouncing back from each wall. These are the first-order echoes. These echoes subsequently
bounce back from each wall again, creating second-order echoes, and so on. We are interested
in the problem of reconstructing the shape of the room from the first-order echoes. Specifically,
we use the time delay of each first-order echoes, in other words the propagation time, which
provides us with a set of distances from each microphone to mirror images of the source reflected
across each wall. Since we do not know which echo corresponds to which wall, the distances
are unlabeled. In fact, depending on the microphone configuration and room geometry, a micro-
phone may not receive any echo from a given wall. The problem is to figure out under which
circumstances, and how, one can find out the correct distance-wall assignments and reconstruct
the wall positions.

The distances to the mirror images of the source are obtained from the time of arrivals of
the impulses at each microphone. If the sound impulse is known, and if the microphones and
loudspeaker share a common clock, these times of arrival can be computed by finding the relevant
peaks of the cross-correlations between the original impulse and the signals received by the
microphones. Since the sound impulse gets more and more blurred as it bounces from wall to
wall, it is possible to distinguish the peaks corresponding to the first order echoes from those
of the higher order ones, assuming that the signal to noise ratio is large enough. See Antonacci
et al. [4] for a discussion on how this can be accomplished and the practical limitation of these
assumptions. We shall assume that there are no missed peaks and no spurious peaks. We shall
also assume that the peaks corresponding to first-order echoes can be distinguished from those
of higher-order echoes.

This work focuses on theoretically determining when the problem is well-posed (i.e., when
the distances contain enough information to uniquely reconstruct the room) in the minimal
case of four microphones and one source. Aside from Dokmanić et al. [9], most other authors
have set aside such theoretical questions and worked on developing numerical solution methods.

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Seeking robustness, the problem is often set up experimentally so to create redundancies and/or simplifying assumptions. For example, a direct numerical solution method in which one searches for the wall positions on a discrete grid was proposed by Crocco et al. \[8\]. In that work, the number of walls is assumed to be known, the room is assumed to be a convex polyhedron, and several sources and microphones are used in order to increase the robustness of the reconstruction.

Different problem definitions are considered by other authors. For example, a single microphone acquires the data of a source moving in a circle around it in the work of Antonacci et al. \[13\]. The case where the microphone positions are unknown \[18\] is also of interest. A slightly different formulation where one or more microphones are moving along an unknown path in a room with unknown geometry containing one or more (potentially moving) speakers is considered, a problem called echo SLAM (simultaneous localization and mapping), for example \[11,12,13\]. Other times, the known geometry of the room is used to localize an indoor object carrying some receivers through the multiple reflections of a signal bouncing off the walls (multipath propagation), for example \[15,14,19,17,16\].

The core of our paper is Algorithm 3.1, which describes a procedure to detect walls from first-order echoes acquired by four microphones whose positions are known. When we think of a wall, we distinguish between a wall and the plane in which it is contained. In particular, walls are usually finite, as is illustrated in Figure 3.1. By detecting a wall we mean that four non-collinear points on the wall are determined. Clearly this uniquely determines the plane containing the wall, but also provides some information about the actual location of the wall within that plane. The key part of the algorithm is a simple echo sorting criterion (Relation (1.4)) that is used to solve the wall assignment problem. The criterion is a vanishing Cayley-Menger determinant involving the pairwise distances between the five-point configuration formed by the four microphones and the mirror image of the source through one wall. As the criterion is always satisfied by the distances corresponding to a correct wall assignment, the algorithm detects all walls that are heard by the four microphones. However, it can sometimes detect walls that are not there (ghost walls).

Our main results are Theorems 4.1 and 5.1, which specify conditions under which Algorithm 3.1 is guaranteed not to detect any wall that is not there (no ghost wall). Theorem 4.1 assumes that the loudspeaker is at a fixed location inside the room. Theorem 5.1 assumes that the loudspeaker is carried by the drone, along with the microphones. The conditions for both theorems are very general; they are satisfied with probability one if the orientation and position of the drone are picked at random following a non-degenerate probability density function. Specifically, the set of exceptional drone positions and orientations lie inside a subvariety of dimension at most five within the six-dimensional space of possible drone placements.

Dokmanić et al. \[9\] have considered the case where the microphones and the loudspeaker are placed in a fixed location inside the room. For reasons of simplicity, the room was assumed to be a convex polyhedron, and the two reconstruction methods presented used five microphones. The authors’ Theorem 1, which applies to four or more microphones, guarantees the correctness of the reconstruction for all but a set of measure zero of microphone arrangements. Notice that in \[9\], the microphone arrangements are chosen randomly from a 12-dimensional configuration space, whereas for our result a six-dimensional space suffices. Also, placing the microphones on a drone rather than independently in the room opens up new application scenarios. We show that our Theorem 4.1 implies Theorem 1 of \[9\] with Corollary 4.2.

The key to proving Theorem 4.1 is Claim 2, which states that one can rotate and translate the drone so it lies in a position for which it is guaranteed that our echo sorting criterion will lead to correct wall assignments. The proof of this claim is accomplished with the help of the symbolic computation software MAGMA \[5\]. Theorem 5.1 is proved in a similar fashion.

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1. Preliminaries

For what we have to say in this section, it is irrelevant whether the loudspeaker or the microphones are placed at fixed positions or mounted on a drone. Following the image method of Allen and Berkley [2], let \( s \in \mathbb{R}^3 \) be the point obtained by reflecting the loudspeaker position with respect to one of the walls (or, more precisely, the plane containing the wall). We call such an \( s \) a mirror point. A sound emitted from the loudspeaker and reflected at the wall and then heard by the microphones corresponds to a sound emitted from the mirror point \( s \) and traveling directly to the microphones. So by measuring the time elapsed between sound emission and echo detection, the distance \( \|s - m_i\| \) from \( s \) to the microphone at position \( m_i \) can be determined. This is illustrated in Figure 1.1.

![Figure 1.1](image_url)

Figure 1.1. Each microphone hears the echoes from two walls. Virtually, the sound comes from the mirror points \( s_1 \) and \( s_2 \).

The following proposition states some facts about four microphones hearing the echo from one wall. Part (d) gives a relation between the squared distances between \( s \) and the microphones. The relation is just a restatement the well-known fact that the Cayley-Menger determinant of five points in \( \mathbb{R}^3 \) vanishes (see Cayley [7]).

**Proposition 1.1.** In the above situation, assume that four microphones at positions \( m_i \in \mathbb{R}^3 \) (\( i = 1, \ldots, 4 \)) hear the sound reflected at a wall (the same wall for all microphones).

(a) With

\[
M := \begin{pmatrix}
m_1 & m_2 & m_3 & m_4 \\
1 & 1 & 1 & 1
\end{pmatrix} \in \mathbb{R}^{4 \times 4},
\]

(1.1)

the microphones are coplanar if and only if \( \det(M) = 0 \).

(b) Assume from now on that the microphones are non-coplanar and write \( \tilde{M} \in \mathbb{R}^{3 \times 4} \) for the upper \( 3 \times 4 \)-part of \((M^{-1})^T\), the transpose inverse of \( M \). Then \( s \) can be computed from the squared distances \( d_i := \|s - m_i\|^2 \) by

\[
s = \frac{1}{2} \tilde{M} \cdot \begin{pmatrix}
\|m_1\|^2 - d_1 \\
\vdots \\
\|m_4\|^2 - d_4
\end{pmatrix}.
\]

(1.2)

(c) Let \( L \in \mathbb{R}^3 \) be the position of the loudspeaker. Then the wall at which the sound was reflected lies on the plane with normal vector \( s - L \) and passing through the point \( \frac{1}{2}(s + L) \).
Four non-collinear points on the wall can be found by intersecting the line between \( s \) and \( m_i \) \((1 \leq i \leq 4)\) with this plane. These points are given by

\[
(1 - \tau_i)s + \tau_im_i \quad \text{with} \quad \tau_i = \frac{\|s - L\|^2}{2(s - L, s - m_i)},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product.

\(d\) With \( D_{i,j} := \|m_i - m_j\|^2 \) and \( u_1, \ldots, u_4 \in \mathbb{R} \) any numbers, set

\[
D := \begin{pmatrix}
0 & u_1 & \cdots & u_4 & 1 \\
u_1 & D_{1,1} & \cdots & D_{1,4} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
u_4 & D_{4,1} & \cdots & D_{4,4} & 1 \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix} \in \mathbb{R}^{6 \times 6} \quad \text{and} \quad f_M(u_1, \ldots, u_4) := \det(D). \quad (1.3)
\]

Then the \( d_i \) from \(b\) satisfy the relation

\[
f_M(d_1, \ldots, d_4) = 0. \quad (1.4)
\]

Before giving the proof, we consider an example of the relation \( f_M \) from \(d\).

**Example 1.2.** Consider the configuration of microphones given by

\[
M = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

(so the microphones are at the origin and the standard basis vectors). For this we have

\[
f_M(u_1, \ldots, u_4) = 4(u_2 - u_1 - 1)^2 + 4(u_3 - u_1 - 1)^2 + 4(u_4 - u_1 - 1)^2 - 16u_1.
\]

Since the coefficients of \( f_M \) only depend on the relative distances between the microphones, they do not change when the microphones are moved while maintaining their relative position. \( \diamond \)

**Proof of Proposition 1.1.** The equivalence in \(a\) can be obtained by subtracting the first column of \( M \) from the other columns. Part \(b\) follows from

\[
M^T \cdot \begin{pmatrix}
2s \\
-\|s\|^2
\end{pmatrix} = \begin{pmatrix}
2\langle m_1, s\rangle - \|s\|^2 \\
2\langle m_4, s\rangle - \|s\|^2
\end{pmatrix} = \begin{pmatrix}
\|m_1\|^2 - \|m_1 - s\|^2 \\
\|m_4\|^2 - \|m_4 - s\|^2
\end{pmatrix} = \begin{pmatrix}
\|m_1\|^2 - d_1 \\
\|m_4\|^2 - d_4
\end{pmatrix}. \quad (1.5)
\]

The first statement from \(c\) follows from the definition of \( s \). It follows from ray acoustics that the intersection of the line \( \overline{sm} \) with the plane lies on the wall, and from the non-coplanarity of the \( m_i \) that the intersection points are non-collinear. The equation for \( \tau_i \) follows from verifying that the \( x_i := (1 - \tau_i)s + \tau_im_i \) satisfy the equation \( \langle x_i, s - L \rangle = \frac{1}{2}(s + L, s - L) \) defining the plane. (One may also check that \( 0 < \tau_i < 1 \) holds if and only if \( m_i \) and \( L \) lie on the same side of the plane, which must be true if the echo is heard by the \( i \)th microphone.)

For the proof of \(d\) we refer to Cayley \([7]\). \( \square \)

**Remark 1.3.** Assume that the positions \( m_i \) of the microphones are known, but the position \( L \) of the loudspeaker is not. Then \( L \) can be determined, using Proposition 1.1(b), as follows: The sound traveling directly from the loudspeaker to the microphones can always be distinguished from the echoes since it is the first to arrive at the microphones. Therefore the distances between \( L \) and the \( m_i \) can be determined. So applying Proposition 1.1(b) to these distances yields \( L \). \( \diamond \)
2. Remarks about Relation (1.4)

Relation (1.4) is really the relation, given in Boutin and Kemper [6, Proposition 2.2(b)], satisfied by the pairwise distances between five points in 3-space. For example, using the forth microphone as an anchor point (point \( n \) in \( \Delta_{ij} := d_{in} + d_{jn} - d_{ij} \) where \( d_{ij} \) is the squared distance from point \( i \) to point \( j \) among the four microphones \( m_1, m_2, m_3, m_4 \), and setting \( m_0 := s \), the matrix \( \Delta = (\Delta_{ij}) \) becomes

\[
\Delta = (\|m_i - m_4\|^2 + \|m_j - m_4\|^2 - \|m_i - m_j\|^2)_{i,j=0,1,2,3},
\]

and therefore has rank at most three. Replacing \( D_{i,0} \) in the matrix \( \Delta \) by an indeterminate \( u_i \), for \( i = 1, 2, 3, 4 \), we obtain the following relation, which is equivalent to Relation (1.4) up to a sign:

\[
\det \begin{pmatrix}
2u_4 & u_4 + D_{i,4} - u_3 & u_4 + D_{i,4} - u_2 & u_4 + D_{i,4} - u_3 \\
(1) & D_{1,4} & D_{1,4} & D_{1,4} \\
(2) & D_{2,4} & D_{2,4} & D_{2,4} \\
(3) & D_{3,4} & D_{3,4} & D_{3,4}
\end{pmatrix} = 0. \tag{2.2}
\]

Neither Relation (1.4) nor Relation (2.2) is sufficient to guarantee the existence of a point configuration with the corresponding distances. However if the matrix \( \Delta \) is positive semi-definite, then its eigendecomposition \( \Delta = Q^T \Lambda Q \), where \( \Lambda \) is a diagonal matrix with at most three non-zero diagonal elements, yields a solution

\[
\sqrt{2} \begin{pmatrix}
m_0' - m_4' & m_1' - m_4' & m_2' - m_4' & m_3' - m_4' \\
0 & 0 & 0 & 0
\end{pmatrix} = \sqrt{\Lambda} Q. \tag{2.3}
\]

If there are three non-zero eigenvalues of \( \Delta \) and they are distinct, then it can be shown that this is the unique solution, up to an orthogonal transform, for the factorization of \( \Delta \) as (2.1). In other words, there exists a rotation mapping the correct solution \( m_i - m_4 \) to the points \( m_i' - m_4' \) obtained by eigenvalue decomposition from (2.3). Assuming that the positions of the microphones \( m_1, m_2, m_3, m_4 \) are known, the rotation can be computed as the product

\[
\begin{pmatrix}
m_i' - m_4' & m_2' - m_4' & m_3' - m_4' \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
m_1 - m_4 & m_2 - m_4 & m_3 - m_4 \\
0 & 0 & 0
\end{pmatrix}^{-1}.
\]

Observe that this provides an alternative to Relation (1.4), namely that the matrix \( \Delta \) (or, equivalently, the matrix \( D \) evaluated at \( u_i = d_i \)) has at most 3 non-zero eigenvalues. When the distance measurements are inaccurate, the fourth eigenvalue could become non-zero. If the error is small, an approximate reconstruction could be obtained by setting the smallest eigenvalue to zero. However in the vicinity of a bad drone position, the conditioning of this reconstruction method could be very bad, as it would be impossible to distinguish between small perturbations of the different reconstructions possible for the bad drone position.

Note that both Relation (1.4) and Relation (2.2) are different from the echo sorting criteria used in the reconstruction methods of [9]. The two criteria used in the reconstruction both use at least 5 microphones. One criterion, derived from Equation (1.5), is that the range of \( M \) should include the vector \((\|m_1\|^2 - d_1, \ldots, \|m_5\|^2 - d_5)^T\). The other criterion, which is the one used in the proposed reconstruction method of [9], uses the Euclidean distance matrix (EDM). Specifically, it uses the 6-by-6 matrix of pairwise distances \( (D_{i,j}) \) between the five (or more) microphones and the mirror point \( s \). The criterion is that the rank of this EDM matrix is at most five.

In the practical algorithm proposed in [9] to handle noisy data, the classical (metric) multidimensional scaling technique [1] is used to reconstruct the points from the approximate distance measurements. This is done by first centering the Euclidean distance matrix \( D_{EDM} = (D_{i,j}) \) as

\[
E = -BD_{EDM}B
\]
where \( B = I_{N+1} - \frac{1}{(N+1)} \left( 1, 1, \ldots, 1 \right) \left( 1, 1, \ldots, 1 \right)^T \) and \( N+1 \) is the total number of points used (\( N \) microphones plus one wall mirror point \( s = m_0 \)). The matrix \( E \) is called the inner product matrix; indeed, if we map the center of mass of the points \( m_i \) to the origin before taking their inner product, we get

\[
2 \left( m_i - \sum_{k=0}^{N} \frac{m_k}{N+1} \right)^T \left( m_j - \sum_{k=0}^{N} \frac{m_k}{N+1} \right) = \\
- D_{i,j} + \sum_{k=0}^{N} \frac{D_{k,i}}{N+1} + \sum_{m=0}^{N} \frac{D_{m,j}}{N+1} - \sum_{k,m=0}^{N} \frac{D_{k,m}}{(N+1)^2} = E_{i,j}.
\]

Note that this factorization is valid for any number of microphones. Thus, the rank of the centered matrix \( E \) is at most three, for any number of microphones, and a solution (up to an orthogonal transform) can be obtained by eigendecomposition of \( E \).

### 3. The Wall Detection Algorithm

We now give an algorithm that attempts to detect walls from the echoes heard by four microphones. Since each microphone hears echoes from multiple walls, it is necessary to match those echoes that come from the same wall. A natural way to do this is to use the relation (1.4). Notice that by “detecting” a wall we mean that four non-collinear points on the wall are computed; no further information about the actual expanse of the wall within the plane containing it can be obtained from the echoes of a single sound emission, unless additional hypotheses are made on the walls, e.g. that they are the facets of a convex polyhedron (see Dokmanić et al. [9]).

**Algorithm 3.1** Detect walls from first-order echoes

**Input:** The delay times of the first-order echoes recorded by four microphones, and the distances \( D_{i,j} \) between the microphones.

1. For \( i = 1, \ldots, 4 \), collect the delay times of the first-order echoes recorded by the \( i \)th microphone in the set \( T_i \).
2. Set \( D_i := \{ c^2 (t - t_0)^2 \mid t \in T_i \} \) (\( i = 1, \ldots, 4 \)), where \( c \) is the speed of sound and \( t_0 \) is the time of sound emission.
3. for \( (d_1, d_2, d_3, d_4) \in D_1 \times D_2 \times D_3 \times D_4 \) do
   4. With \( f_M \) defined by (1.3), evaluate \( f_M(d_1, \ldots, d_4) \).
   5. if \( f_M(d_1, \ldots, d_4) = 0 \) then
     6. Use (1.2) to compute the mirror point \( s \) from \( (d_1, \ldots, d_4) \).
     7. Use Proposition 1.1(c) to compute four non-collinear points on the wall with mirror point \( s \) and, if desired, a normal vector.
   8. **Output** the data of this wall.
9. end if
10. end for

If for \( (d_1, \ldots, d_4) \in D_1 \times \cdots \times D_4 \), the \( d_i \) come from echoes from the same wall, then the relation \( f_M(d_1, \ldots, d_4) = 0 \) holds and therefore the wall will be detected. So the algorithm is guaranteed to detect every wall from which a first-order echo is heard by all microphones. It is possible, however, that the algorithm detects walls that are not really there (“ghost walls”; see Example 3.1). The main purpose of this paper is to show that these mistakes are rare.

Note that the search for matches in step 3 can be accelerated by using the triangle inequalities. For example, one could start with the shortest distances, say \( d_1 \), for the first microphone. Then the distance to the second microphone \( d_2 \) would need to satisfy the inequality \( \sqrt{d_2} \leq \sqrt{d_1} + ||m_1 - m_2|| \), and so on.

The following example shows that it can happen that the algorithm detects ghost walls.

**Example 3.1.** Figure 3.1 shows three microphones in a plane at positions \( m_1, m_2, m_3 \) that hear echoes from three walls \( W_i \), but the time elapsed between sound emission and echo detection is
the same as if they were hearing echoes from one single wall $W_{\text{ghost}}$, which does not exist. This arises because (1) the walls $W_i$ are all parallel to each other, (2) each $m_i$ can hear the echo from $W_i$, and (3) the distances between $m_i$ and $W_i$ are the same for all $i$. It is easy to add a fourth microphone outside of the plane, together with a wall possibly also outside of the plane, such that (1)–(3) extend to the fourth microphone and wall. (We find it harder to include that in our two-dimensional sketch in Figure 3.1.) Since the echoes heard by the microphones could have come from the single wall $W_{\text{ghost}}$, the relation (1.4) is satisfied, and so Algorithm 3.1 will falsely detect $W_{\text{ghost}}$ as a wall.

One can argue that such bad examples correspond to exceptional wall configurations. Indeed, it is easy to show that, for wall configurations picked at random following a non-degenerate probability distribution, there is a probability zero of this happening. Specifically, there is a probability zero of the wall configuration yielding distances to the microphones that satisfy $f_M(\|s_1 - m_1\|, \|s_2 - m_2\|, \|s_3 - m_3\|, \|s_4 - m_4\|) = 0$ with the $s_i$’s not all equal. For example, suppose that the first point $s_1$ is the correct mirror point $s$ and that another point say $s_4 \neq s$, but $s_1 = s_2 = s_3 = s$. Then one can freely change the value of the distance $\|s_4 - m_4\|$ by moving the wall corresponding to $s_4$. The distance $\|s_4 - m_4\|$ is then changed freely without affecting the other distances $\|s_1 - m_1\|, \|s_2 - m_2\|, \|s_3 - m_3\|$ because these correspond to another wall, and thus the zero set of $f_M$ can be avoided. Similarly, if the first two points $s_1 = s_2 = s$ correspond to the correct wall and the last two points $s_3 = s_4 \neq s$ correspond to another wall, then the values of the distances $\|s_3 - m_3\|^2, \|s_4 - m_4\|^2$ can be changed freely by moving to wall corresponding to $s_3$ and $s_4$ in $\mathbb{R}^3$. This will not affect the first two distances $\|s_1 - m_1\|^2, \|s_2 - m_2\|^2$ and thus the zero set of $f_M$ can be avoided. Note that our argument holds because moving the walls does not affect the relationship $f_M(u_1, u_2, u_3, u_4) = 0$ itself, since the coefficients of its polynomial are monomials in the distances between the microphones. Therefore, we can conclude that all but a set of measure zero of choices of mirror points $s_1, s_2, s_3, s_4$ that are not all equal satisfy $f_M(\|s_1 - m_1\|^2, \|s_2 - m_2\|^2, \|s_3 - m_3\|^2, \|s_4 - m_4\|^2) \neq 0$. For fixed microphone positions $m_1, m_2, m_3, m_4$, one could thus make sure that the wall configuration is not an exceptional one by checking that the mirror points satisfy the inequalities:

$$f_M(\|s_1 - m_1\|^2, \|s_2 - m_2\|^2, \|s_3 - m_3\|^2, \|s_4 - m_4\|^2) \neq 0,$$

for all $\{s_1, s_2, s_3, s_4\}$ not all equal. However, moving the walls is not practical, and it is conceivable that the exceptional wall configurations might include all cases that have, say, parallel walls. Therefore we seek to show instead that, given any wall configuration, most microphone positions satisfy the above inequalities. The argument required for this is more difficult.

4. A DRONE IN A ROOM WITH A FIXED LOUDSPEAKER

As in the previous section, assume that we are given a room with a loudspeaker and four microphones in it. In this section we will assume that the loudspeaker is at a fixed position, but
the microphones are mounted on a drone (or, mathematically speaking, that their relative positions are fixed). Apart from the fact that this is a realistic scenario for applications, this has the computational advantage that the coefficients of the relation \( f_M(d_1, \ldots, d_4) = 0 \) that is exploited in the wall detection algorithm remain the same once and for all, since by Proposition 1.1(d) the coefficients only depend on the mutual distances between the microphones. For applying Algorithm 3.1, we need to assume that all the microphone locations are known, which may be problematic when the microphones are on a drone. Two answers can be given to this objection:

1. In some scenarios, it can really be assumed that the drone knows its position and orientation. For example, the drone can be equipped with an inertial measurement unit or its position can be calibrated using an external positioning system based on triangulation.
2. If the drone is not aware of its own position and orientation, it can perform the computation with respect to “its own” coordinate system. More precisely, this means that the coordinates of the microphone positions \( \mathbf{m}_i \) are assigned once and for all, according to where the microphones are located on the drone. Algorithm 3.1 will then detect the walls with respect to the coordinate system, traveling with the drone, in which the microphone coordinates were assigned. Step 7 of the algorithm requires the position \( \mathbf{L} \) of the loudspeaker. This, too, can be measured within the drone’s coordinate system by using Remark 1.3.

We say that the microphones (or the drone) are in a **good position** if Algorithm 3.1 detects no walls that are not really there (no ghost wall.) Recall that the algorithm is guaranteed to detect every wall from which a first-order echo is heard by all microphones. Therefore if the drone is moving from a bad position to a good position, all the ghost walls previously reconstructed will disappear and all the walls that are there and can still be heard by the microphones will remain. If additional walls can be heard from the new position, these will be added to the set of reconstructed walls as well. The goal of this section is to prove the following result, which implies that generic drone positions are good positions.

**Theorem 4.1.** Consider a given room, by which we understand an arrangement of walls, which may include ceilings, floors, and sloping walls. Assume there is a loudspeaker at a given position in the room. Also consider a drone that carries four non-coplanar microphones at fixed locations on its body. Place the drone in the room at a random position, which means that not only the location of the drone’s center of gravity is chosen at random, but also its pitch, yaw, and roll. Then with probability 1 the drone is in a good position. More precisely, within the configuration space \( \mathbb{R}^3 \times \text{SO}(3) \) of possible drone positions, the bad ones lie in a subvariety of dimension \( \leq 5 \).

As a consequence we obtain the main result of Dokmanić et al. [9].

**Corollary 4.2.** We make the same assumptions as in Theorem 4.1, except that the microphones are not mounted on a drone, but are placed independently at random locations. Then with probability 1 they are in a good position. More precisely, within the configuration space \( \mathbb{R}^{12} \) of possible microphone positions, the bad ones lie in a subvariety of dimension \( \leq 11 \).

Since it is not immediately clear how the corollary follows from the theorem, both will be proved together. Before giving the actual proof, we present a rough roadmap, aiming to convey the geometry that lies behind the proof.

1. We define a subset \( \mathcal{U} \subseteq \mathbb{R}^3 \times \text{SO}(3) \) of so-called “very good drone positions.” The set \( \mathcal{U} \) depends on the given arrangement of walls and on the given configuration of microphones, and turns out to be Zariski open.
2. We reduce to the case of four walls (see Claim 2 below).
3. After suitable choices of coordinates, we can model an arrangement of four walls together with a configuration of four microphones by a single point \((b_1, \ldots, b_6, c_1, \ldots, c_5) \in \mathbb{R}^{11}\). To express the dependencies, let us write \( \mathcal{U}(b_1, \ldots, b_6, c_1, \ldots, c_5) \) instead of \( \mathcal{U} \). Notice that a point in \( \mathbb{R}^{11} \) may encode a wall arrangement where all or some walls are equal.
4. We consider the set

\[
\mathcal{V} := \{(b_1, \ldots, b_6, c_1, \ldots, c_5) \in \mathbb{R}^{11} | \mathcal{U}(b_1, \ldots, b_6, c_1, \ldots, c_5) = \emptyset \} \subseteq \mathbb{R}^{11}
\]
of wall arrangements and microphone configurations for which there is no very good drone position. On the other hand, consider the set \( \mathcal{V} \subseteq \mathbb{R}^{11} \) of all \((b_1, \ldots, b_6, c_1, \ldots, c_5)\) such that \((c_1, \ldots, c_5)\) defines a coplanar microphone configuration, or the walls given by \((b_1, \ldots, b_6)\) are all equal. Notice that we are now allowing the wall arrangement and microphone configuration to vary.

5. Now it is enough to show that
   \[ \mathcal{V} \subseteq \mathcal{V}'. \tag{4.1} \]
   Indeed, if a wall arrangement and a microphone configuration satisfy the hypothesis of Theorem 4.1, then the corresponding point does not lie in \( \mathcal{V}' \) and therefore, by (4.1), also not in \( \mathcal{V} \). This means \( \mathcal{U}(b_1, \ldots, b_6, c_1, \ldots, c_5) \neq \emptyset \), so by the Zariski openness, the set of bad drone positions lies in a proper subvariety of the configuration space.

6. The inclusion (4.1) translates to the ideal-theoretic statement (4.7) below. In that statement, the ideal \( J \) defines the set \( \mathcal{V} \). (The sets \( \mathcal{V} \) and \( \mathcal{V}' \) are never mentioned in the proof. We only need them here to explain the underlying geometry.)

7. Finally, (4.7) is verified by Gröbner basis computations, using a computer algebra system. The computation is demanding because the ideal \( J \) has many generators. As detailed in the proof, some tricks contribute to the feasibility of the computation.

**Proof of Theorem 4.1 and Corollary 4.2.** In Theorem 4.1, the random placement of the drone means that from initial positions of the microphones, given by the \( \mathbf{m}_{i_{\text{ini}}} \) in the matrix
\[
M_{\text{ini}} = \begin{pmatrix} \mathbf{m}_{1_{\text{ini}}} & \mathbf{m}_{2_{\text{ini}}} & \mathbf{m}_{3_{\text{ini}}} & \mathbf{m}_{4_{\text{ini}}} \\ 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{R}^{4 \times 4}, \tag{4.2}
\]
the actual positions \( \mathbf{m}_i \) are given by
\[
M := \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 & \mathbf{m}_3 & \mathbf{m}_4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = A \cdot M_{\text{ini}} \quad \text{with} \quad A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.3}
\]
where the upper left \( 3 \times 3 \)-part of \( A \) lies in \( \text{SO}(3) \). In Corollary 4.2, by contrast, the matrix \( A \) can be chosen freely without the \( \text{SO}(3) \)-condition. Let us write \( \text{Aff}(3) \) for the 12-dimensional space of all matrices \( A \) as in (4.3), and \( \text{ASO}(3) \) for the subset where the upper left \( 3 \times 3 \)-part comes from \( \text{SO}(3) \). The configuration space \( \mathbb{R}^3 \times \text{SO}(3) \) can be identified with \( \text{ASO}(3) \). We call a matrix \( A \in \text{Aff}(3) \) **good** if the microphone positions given by (4.3) are good.

Let \( \mathcal{W} \) be the set of walls from our room. In this proof we identify the walls with the planes containing them. The mirror points are given by \( s = \text{ref}_W(L) \), the reflection of the loudspeaker position at a wall \( W \in \mathcal{W} \). We call \( A \in \text{Aff}(3) \) **very good** if the following holds: For four walls \( W_1, \ldots, W_4 \in \mathcal{W} \) the relation
\[
f_M(||\text{ref}_{W_1}(L) - \mathbf{m}_1||^2, \ldots, ||\text{ref}_{W_4}(L) - \mathbf{m}_4||^2) = 0
\]
is satisfied only if \( W_1 = W_2 = W_3 = W_4 \). (Recall that the above expression depends on \( A \) by (4.3).)

**Claim 1.** If \( A \) is very good, then it is good.

Indeed, with the notation of Algorithm 3.1, let \((d_1, \ldots, d_4) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_4 \). For each \( i \) there exists a wall \( W_i \in \mathcal{W} \) such that \( d_i = ||\text{ref}_{W_i}(L) - \mathbf{m}_i||^2 \). If \( f_M(d_1, \ldots, d_4) = 0 \), then \( W_1 = W_2 = W_3 = W_4 =: W \) by hypothesis, so by Proposition 1.1, the wall that is rendered in step 8 of the algorithm is \( W \). This shows that the algorithm detects no ghost walls.

**Claim 2.** Let \( s_1, s_2, s_3, s_4 \in \mathbb{R}^3 \) be four points that are not all equal. Then there exists \( A \in \text{ASO}(3) \) such that
\[
f_M(||s_1 - \mathbf{m}_1||^2, \ldots, ||s_4 - \mathbf{m}_4||^2) \neq 0.
\]
(Recall that the above expression depends on \( A \) by (4.3).)
Before proving the claim, we show that it implies Theorem 4.1 and Corollary 4.2. By Proposition 1.1(c), a wall \( W \in W \) is uniquely determined by its mirror point \( \text{ref}_W(L) \). Therefore the claim implies that for \( W_1, \ldots, W_4 \in W \) which are not all equal, the set

\[
\mathcal{U}_{W_1, \ldots, W_4} := \{ A \in \text{SO}(3) \mid f_M(\|\text{ref}_{W_1}(L) - m_1\|, \ldots, \|\text{ref}_{W_4}(L) - m_4\|) \neq 0 \}
\]

is non-empty. Since \( f_M(\|\text{ref}_{W_1}(L) - m_1\|, \ldots, \|\text{ref}_{W_4}(L) - m_4\|) \) depends polynomially on the coefficients of \( A \) and since \( \text{SO}(3) \) is an irreducible variety, this implies that the complement of \( \mathcal{U}_{W_1, \ldots, W_4} \) in \( \text{SO}(3) \) has dimension strictly less than \( \dim(\text{SO}(3)) = 6 \). It follows that also the finite intersection

\[
\mathcal{U} := \bigcap_{W_1, \ldots, W_4 \in W \text{ such that not all } W_i \text{ are equal}} \mathcal{U}_{W_1, \ldots, W_4}
\]

has a complement of dimension \( \leq 5 \). By definition, all \( A \in \mathcal{U} \) are very good and therefore, by Claim 1, also good. So indeed every bad \( C \) may modify with \( c \) \( R \), also good. So indeed every bad \( \mathcal{U} \)

By Proposition 1.1(c), we can forget about the arrangement of walls. So we are only given four vectors \( s_1, s_2, s_3, s_4 \in \mathbb{R}^3 \), not all equal, and the matrix \( M_{ini} \) (see (4.2)).

For the feasibility of the computation in the final part of the proof it is necessary to choose a convenient Cartesian coordinate system. First, \( s_1 \) can be chosen as the origin of the coordinate system, so the matrix \( S = (s_1 \ s_2 \ s_3 \ s_4) \in \mathbb{R}^{3 \times 4} \) becomes \( S = (0 \ s_2 \ s_3 \ s_4) \). Moreover, using QR-decomposition, we can write

\[
(s_2 \ s_3 \ s_4) = Q \cdot \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix}
\]

with \( Q \in \text{SO}(3), b_i \in \mathbb{R} \). So if we use the columns of \( Q \) (instead of the standard basis vectors) as basis of \( \mathbb{R}^3 \), \( S \) becomes

\[
S := (s_1 \ s_2 \ s_3 \ s_4) = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & b_6 \end{pmatrix}.
\]

(4.4)

With this, the hypothesis that the \( s_i \) are not all equal becomes \( S \neq 0 \).

We also need to simplify the matrix \( M_{ini} \), given by (4.2). Since we wish to prove Claim 2, which states that there exists \( A \in \text{SO}(3) \) such that \( A \cdot M_{ini} \) satisfies a certain condition, we may modify \( M_{ini} \) by multiplying it with suitable matrices from \( \text{SO}(3) \) on the left. Writing

\[
M_{ini} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad M_{ini} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_0 & c_1 & c_2 & c_3 \\ 1 & 1 & 1 & 1 \end{pmatrix}
\]

with \( c_0, \ldots, c_5 \in \mathbb{R} \). By Proposition 1.1(a), the hypothesis that the microphones are non-coplanar translates to \( \det(M_{ini}) \neq 0 \), so \( c_0 c_3 c_5 \neq 0 \). We can now rescale our coordinate system by a factor of \( c_0 \) (which corresponds to the choice of a unit of length). In summary, we may assume

\[
M_{ini} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

(4.5)

with \( c_1 \in \mathbb{R}, c_3 c_5 \neq 0 \).

Turning to the proof of Claim 2, we observe that \( f_M(\|s_1 - m_1\|, \ldots, \|s_4 - m_4\|) \) depends polynomially on the coefficients \( a_{i,j}, b_i, \) and \( c_i \) of \( A, S, \) and \( M_{ini} \), respectively, so there is a
polynomial $F(x_{1,1}, \ldots, x_{3,4}, y_1, \ldots, y_6, z_1, \ldots, z_5)$ in 23 indeterminates such that

$$f_M(\|s_1 - m_1\|^2, \ldots, \|s_4 - m_4\|^2) = F(a_{1,1}, \ldots, a_{3,4}, b_1, \ldots, b_6, c_1, \ldots, c_5). \tag{4.6}$$

Consider the matrix $X := (x_{i,j})_{1 \leq i, j \leq 3}$ and the ideal $I \subseteq \mathbb{R}[x_{1,1}, \ldots, x_{3,4}]$ (the ring of polynomials in 12 indeterminates) generated by the polynomial $\det(X) - 1$ and by the coefficients of the matrix $X \cdot X^T - I$. Clearly a matrix $A \in \text{Aff}(3)$ lies in $\text{ASO}(3)$ if and only if every polynomial from $I$ vanishes when evaluated at the coefficients of $A$. This implies that $I$ is contained in the vanishing ideal of $\text{ASO}(3)$, which we write as $I \subseteq \text{Id}(\text{ASO}(3))$. We claim equality. Indeed, using MAGMA [5], we can verify that $I$ is a radical ideal and equidimensional of dimension 6. (In fact, MAGMA computes over $\mathbb{Q}$ instead of $\mathbb{R}$, but over fields of characteristic 0, the algorithms for computing radical ideals and equidimensional parts yield the same result when passing to a field extension, see Greuel and Pfister [10, Chapter 4].) So $I \subseteq \text{Id}(\text{ASO}(3))$ would imply that $\text{ASO}(3)$ has dimension $< 6$. But it is well known that $\dim(\text{ASO}(3)) = 6$, so indeed $I = \text{Id}(\text{ASO}(3))$.

Now choose a Gröbner basis $G$ of $I$ with respect to an arbitrary monomial ordering. Viewing the $x_{i,j}$ as the main indeterminates, we can form a normal form $\tilde{F} := \text{NF}_G(F)$ of the polynomial $F$ with respect to $G$. Let $J \subseteq \mathbb{R}[y_1, \ldots, y_6, z_1, \ldots, z_5]$ be the ideal generated by the coefficients of $\tilde{F}$.

**Claim 3.** If there exists $k \geq 0$ such that

$$(z_3 z_5)^k y_i \in J \tag{4.7}$$

for $i \in \{1, \ldots, 6\}$, then Claim 2 and therefore Theorem 4.1 and Corollary 4.2 follow.

To prove the claim, let $s_1, \ldots, s_4 \in \mathbb{R}^3$, not all equal, and let $M_{in}$ be given. As shown above, we may assume that the $s_i$ are as in (4.4) and $M_{in}$ as in (4.5). So $c_3 c_5 \neq 0$ and $b_i \neq 0$ for at least one $i$. By (4.7), this implies that at least one generator $g$ of $J$ satisfies $g(b_1, \ldots, b_6, c_1, \ldots, c_5) \neq 0$. By the definition of $J$, this means that $\tilde{F}(x_{1,1}, \ldots, x_{3,4}, b_1, \ldots, b_6, c_1, \ldots, c_5) \neq 0$. Write $F = \sum_{i=1}^m g_i t_i$ with $g_i \in \mathbb{R}[y_1, \ldots, y_6, z_1, \ldots, z_5]$ and $t_i$ power products of the $x_{i,j}$. The linearity of the normal form map implies $\tilde{F} = \sum_{i=1}^m g_i \text{NF}_G(t_i)$, so

$$\tilde{F}(x_{1,1}, \ldots, x_{3,4}, b_1, \ldots, b_6, c_1, \ldots, c_5) = \sum_{i=1}^m g_i(b_1, \ldots, b_6, c_1, \ldots, c_5) \text{NF}_G(t_i) = \text{NF}_G(F(x_{1,1}, \ldots, x_{3,4}, b_1, \ldots, b_6, c_1, \ldots, c_5)).$$

Since this is non-zero and since a polynomial in $\mathbb{R}[x_{1,1}, \ldots, x_{3,4}]$ has normal form zero if and only if it lies in $I$, we obtain $F(x_{1,1}, \ldots, x_{3,4}, b_1, \ldots, b_6, c_1, \ldots, c_5) \notin I$. Since $I = \text{Id}(\text{ASO}(3))$, this implies that there exists $A \in \text{ASO}(3)$ with coefficients $a_{i,j}$ such that $F(a_{1,1}, \ldots, a_{3,4}, b_1, \ldots, b_6, c_1, \ldots, c_5) \neq 0$.

By (4.6) this means $f_M(\|s_1 - m_1\|^2, \ldots, \|s_4 - m_4\|^2) \neq 0$, so indeed Claim 2 follows from (4.7).

It remains to show (4.7), and this can be checked with the help of a computer. It turns out that (4.7) holds with $k = 2$. For the verification, we used MAGMA and proceeded as follows:

- It is straightforward to compute the polynomials $F$ and $\tilde{F}$ according to their definitions.
- Let $C \subseteq \mathbb{R}[y_1, \ldots, y_6, z_1, \ldots, z_5]$ be the set of all coefficients of $\tilde{F}$.
- Using an additional indeterminate $t$, we computed the set $C_{\text{hom}}^\text{hom}$ of homogenizations of the polynomials in $C$ with respect to $t$.
- We computed a truncated Gröbner basis $G_{\text{hom}}^\text{hom}$ of the ideal generated by $C_{\text{hom}}^\text{hom}$ of degree 6.
- We checked that $\text{NF}_{G_{\text{hom}}^\text{hom}}(t(z_3 z_5)^2 y_i) = 0$ for $i = 1, \ldots, 6$. This shows that $t(z_3 z_5)^2 y_i$ is an $\mathbb{R}[y_1, \ldots, y_6, z_1, \ldots, z_5, t]$-linear combination of the polynomials in $C_{\text{hom}}^\text{hom}$. Setting $t = 1$ shows that $(z_3 z_5)^2 y_i \in J$.

The MAGMA code for running these computations, and those for the proof of Theorem 5.1, is available at [https://purr.purdue.edu/publications/3105/1](https://purr.purdue.edu/publications/3105/1). The total computation time was less than a tenth of a second. \[ \square \]
5. A DRONE CARRYING MICROPHONES AND A LOUDSPEAKER

As before, assume that we are given a room and a drone. In contrast to the last section, assume that not only four microphones but also a loudspeaker are mounted on the drone. Again we say that the drone is in a **good position** if Algorithm 3.1 detects no walls that are not really there. The goal of this section is to prove the following result. It is more delicate than Theorem 4.1 since the mirror points are not fixed but move as the drone moves.

**Theorem 5.1.** Consider a given room, by which we understand an arrangement of walls, which may include ceilings, floors, and sloping walls. Also consider a drone that carries four non-coplanar microphones and a loudspeaker at fixed locations on its body. Place the drone in the room at a random position, which means that not only the location of the drone’s center of gravity is chosen at random, but also its pitch, yaw, and roll. Then with probability 1 the drone is in a good position. More precisely, within the configuration space $\mathbb{R}^3 \times SO(3)$ of possible drone positions, the bad ones lie in a subvariety of dimension $\leq 5$.

**Proof.** The proof is similar to the previous one, but more complicated. We use some of the notation from the previous proof. In particular, the initial positions of the microphones are given by a matrix $M_{ini}$ as (4.2), but in addition we are given a vector $L_{ini} \in \mathbb{R}^3$, so the actual microphone and loudspeaker positions are determined by

$$M := \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = A \cdot M_{ini} \quad \text{and} \quad \begin{pmatrix} L \\ 1 \end{pmatrix} = A \cdot \begin{pmatrix} L_{ini} \\ 1 \end{pmatrix} \quad (5.1)$$

with $A \in SO(3)$. Claim 1 and its proof carry over verbatim to the present situation. Claim 2 has to be modified as follows:

**Claim 2’.** Let $W_1, W_2, W_3, W_4 \subset \mathbb{R}^3$ be four planes that are not all equal. Then there exists $A \in ASO(3)$ such that

$$f_M(\|\text{ref}_{W_i}(L) - m_1\|^2, \ldots, \|\text{ref}_{W_4}(L) - m_4\|^2) \neq 0.$$  

(The above expression depends on $A$ by (5.1).)

The proof that Claim 2’ implies Theorem 5.1 is also as above. One only has to observe that in $f_M(\|\text{ref}_{W_1}(L) - m_1\|^2, \ldots, \|\text{ref}_{W_4}(L) - m_4\|^2)$, not only the $m_i$ but also $L$ now depend on $A$, but since the ref $W_i$ are linear maps, the expression again depends polynomially on the coefficients of $A$.

For the proof of Claim 2’ we must represent the walls $W_i$ in an appropriate way. For this, we choose normal vectors $w_i \in \mathbb{R}^3$ such that $w_i \in W_i$ if $W_i$ does not contain the origin 0 of the coordinate system. So

$$W_i = \begin{cases} \{x \in \mathbb{R}^3 \mid \langle w_i, x \rangle = 0\} & \text{if } 0 \in W_i \\ \{x \in \mathbb{R}^3 \mid \langle w_i, x \rangle = \|w_i\|^2\} & \text{if } 0 \notin W_i \end{cases}. \quad (5.2)$$

(More formally, $W_i$ is given by $w_i$ and “true” or “false” indicating which of the above formulas is to be used.) The reflection of $L$ at $W_i$ is

$$\text{ref}_{W_i}(L) = \begin{cases} L - 2\alpha_i w_i & \text{if } 0 \in W_i \\ L + 2(1 - \alpha_i) w_i & \text{if } 0 \notin W_i \end{cases} \quad \text{with } \alpha_i := \frac{\langle w_i, L \rangle}{\|w_i\|^2}. \quad (5.3)$$

Form the matrix $W = (w_1 \ w_2 \ w_3 \ w_4) \in \mathbb{R}^{3 \times 4}$ and set $r := \text{rank}(W)$. We reorder the $w_i$ such that $w_1, \ldots, w_r$ become linearly independent. Since we wish to prove Claim 2’, we also need to reorder the $m_i$ and therefore the columns of the matrices $M$ and $M_{ini}$. From the definition (1.3) of $f_M$ we see that this does not change $f_M(\|\text{ref}_{W_i}(L) - m_1\|^2, \ldots, \|\text{ref}_{W_4}(L) - m_4\|^2)$. Computing the intersection of the first $r$ walls amounts to solving a linear system of rank $r$ with $r$ equations, given by (5.2). Since this is solvable, we may choose the origin 0 of the coordinate system such that 0 $\in W_i$ for $i \leq r$. So the number $l$ of walls $W_i$ with 0 $\in W_i$ satisfies $r \leq l \leq 4$, and we may reorder the walls again such that 0 $\in W_i$ if and only if $i \leq l$. By the hypothesis in Claim 2’ that
not all walls are equal, the case \( r = 1 \) and \( l = 4 \) can be excluded. As in the previous proof we use QR-decomposition and thus assume \( W \) to be upper triangular. Its first \( r \) diagonal entries are therefore non-zero. Since \( 0 \in W_i \) for \( i \leq r \) we may rescale the first \( r \) normal vectors \( w_i \) such that the these diagonal entries become 1. Moreover, since \( \text{rank}(W) = r \), only the first \( r \) rows of \( W \) can be non-zero. In summary, we obtain

\[
W = \begin{pmatrix} 1 & b_1 & b_2 & b_3 \\ 0 & 1 & b_3 & b_5 \\ 0 & 0 & 1 & b_6 \end{pmatrix}
\]

if \( r = 3 \),

\[
W = \begin{pmatrix} 1 & b_1 & b_2 & b_3 \\ 0 & 1 & b_4 & b_5 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

if \( r = 2 \), and

\[
W = \begin{pmatrix} 1 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

if \( r = 1 \), where always \( b_i \in \mathbb{R} \).

Having reordered the \( W_i \) and the columns of \( M \) and \( M_{\text{ini}} \), and having chosen a suitable Cartesian coordinate system, we can modify \( M_{\text{ini}} \) as in the previous proof. (This includes a rescaling of the coordinate system, after which the above rescaling of the \( w_i \) can be done.) So \( M_{\text{ini}} \) is given by (4.5). Moreover, we write \( L_{\text{ini}} = (c_6, c_7, c_8)^T \).

Set \( \eta := \prod_{i=1}^l \|w_i\|^2 \), which is non-zero and serves as a common denominator for the \( \alpha_i \) in (5.3) and also for the \( \|\text{ref}_W(L) - m_i\|^2 \). Therefore \( \eta^2 f_M(\|\text{ref}_W(L) - m_i\|^2, \ldots, \|\text{ref}_W(L) - m_4\|^2) \) depends polynomially on the coefficients \( a_{i,j} \), \( b_i \), and \( c_i \) of \( A, W, \) and \( M_{\text{ini}} \), respectively, so there are polynomials \( F(x_1, \ldots, x_3, y_1, \ldots, y_6, z_1, \ldots, z_8) \) in 26 indeterminates and \( N(y_1, \ldots, y_6) \) in 6 indeterminates such that

\[
\eta = N(b_1, \ldots, b_6)
\]

and

\[
\eta^2 f_M(\|\text{ref}_W(L) - m_i\|^2, \ldots, \|\text{ref}_W(L) - m_4\|^2) = F(a_{1,1}, \ldots, a_{3,4}, b_1, \ldots, b_6, c_1, \ldots, c_8)
\]

In reality, \( f_M(\|\text{ref}_W(L) - m_i\|^2, \ldots, \|\text{ref}_W(L) - m_4\|^2) \) and \( \eta \) also depend on \( r \) and \( l \) since the \( w_i \) and \( \text{ref}_W(L) \) do, according to (5.4) and (5.3). To express these dependencys, we write \( F_{r,l} \) and \( N_e \) instead of \( F \) and \( N \). As before, we consider the ideal \( I = \text{Id}(\text{ASO}(3)) \subset \mathbb{R}[x_1, \ldots, x_3, y_1, \ldots, y_6] \) and the normal form \( \tilde{F}_{r,l} = \text{NF}_G(F_{r,l}) \) with respect to a \( \text{Gröbner basis} \) \( G \) of \( I \). We also write \( J_{r,l} \subseteq \mathbb{R}[y_1, \ldots, y_6, z_1, \ldots, z_8] \) for the ideal generated by the coefficients of \( \tilde{F}_{r,l} \).

**Claim 3’.** If for every \( 1 \leq r \leq 3 \) and \( 1 \leq l \leq 4 \) with \( l - r < 3 \) there exist non-negative integers \( k_1, k_2 \) such that

\[
(z_3 z_5)^{k_1} N^{k_2} \in J_{r,l},
\]

then **Claim 2’** and therefore **Theorem 5.1** follow.

The proof is almost identical to the one of **Claim 3**. Let \( W_i \) be planes as in **Claim 2’**, and let \( M_{\text{ini}} \) also be given. After reordering the \( W_i \) and the columns of \( M_{\text{ini}} \) and after choosing a convenient coordinate system, we obtain \( r \) and \( l \) as in **Claim 3’** such that the arguments of \( f_M(\|\text{ref}_W(L) - m_i\|^2, \ldots, \|\text{ref}_W(L) - m_4\|^2) \) are given by (5.3), (5.4), (5.1), and (4.5). We have \( c_3 c_5 f \neq 0 \), so (5.7) together with (5.5) implies \( g(b_1, \ldots, b_6, c_1, \ldots, c_8) \neq 0 \) for some generator \( g \) of \( J \). Precisely as in the proof of **Claim 3** this shows the existence of \( A \in \text{ASO}(3) \) with coefficients \( a_{i,j} \) such that \( F(a_1, \ldots, a_{3,4}, b_1, \ldots, b_6, c_1, \ldots, c_8) \neq 0 \). By (5.6) this implies \( f_M(\|\text{ref}_W(L) - m_i\|^2, \ldots, \|\text{ref}_W(L) - m_4\|^2) \neq 0 \), which is the assertion of **Claim 2’**.

What is left is to show (5.7), for which we proceed as in the proof of (4.7). Here it turns out that for each \( r \) and \( l \), a truncated \( \text{Gröbner basis} \) \( G_{\text{hom}} \) (with the notation of the previous proof) of degree 16 suffices to show that \( \text{NF}_{G_{\text{hom}}}(z_3 z_5 N^2) = 0 \). The total computation time was roughly five minutes on a workstation, highlighting that the case of a loudspeaker carried by a drone is much more difficult than the case of a loudspeaker at a fixed position. \( \square \)

Note that although it took roughly five minutes of computation time to verify **Theorem 5.1**, the actual wall detection algorithm performed by the drone only requires evaluating simple expressions such as the one given in **Example 1.2**, which is extremely fast.
6. Conclusion

We have shown that the problem of reconstructing an arrangement of walls from the first order
echoes of a single sound impulse acquired by 4 microphones on a drone is generically well-posed.
The first order echoes provide us with a list of distances from the microphones to the walls. We
assume that we do not know which distance goes with which wall. Both the case where the
speaker producing the sound is fixed in the room, and the case where it is carried on the drone,
were considered. Specifically, our results show that, when the drone is in a generic position and
orientation, one can obtain four points on each wall that are heard by all four microphones, and
these walls are guaranteed to exist (no ghost walls). Our set up assumes exact measurements
and infinite precision calculations. In future work, we plan to study the more practical problem
of reconstructing the walls from noisy distance measurements.

While the formulation of our problem focuses on microphones mounted on a drone, our results
apply to many other application scenarios. For example, the microphones and loudspeaker could
be mounted on a car moving on the road, a robot navigating in an indoor environment, or an
underwater vehicle exploring a wreck in the ocean. Some of these situations put restrictions on
rotations and translations that can be applied to the microphone configuration. The impact of
such restrictions on the reconstruction problem will also be studied in future work.

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