Geometrizing quantum dynamics of a Bose-Einstein condensate

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Geometry may arise from quantum systems in an unprecedented means such as the gauge theory/gravity dual [1–6]. Whereas emergent geometries often occur in strongly correlated systems, one could ask whether simple quantum systems may also be used to unfold intriguing underlying geometries. Here, we show that quantum dynamics of Bose-Einstein condensates in the weakly interacting regime can be geometrized by a Poincaré disk. Each point on such a disk represents a fermioid double state [7, 8], the fidelity between which equals the metric of this hyperbolic space. Our approach provides us with a unique geometric interpretation of the stable and unstable modes as closed and open trajectories on the Poincaré disk, respectively. In particular, the resonant modes, which follow geodesics, naturally equate fundamental quantities including the time, the length, and the temperature. Our work suggests a new geometric framework to coherently control quantum systems, including speeding up, slowing down, and reversing their dynamics using SU(1,1) echoes.

The importance of geometry in physics has been well established in general relativity and many other topics. More strikingly, geometries arise as emergent phenomena in certain quantum systems. For instance, AdS/CFT has revolutionized our understandings of the spacetime by establishing the dual between a quantum field theory and a gravity in a higher dimension, where the hyperbolic geometry is essential [1–2]. It has also been conjectured that two entangled copies of quantum field theories, which is described by a thermofield double state (TFD) living on the boundaries, are dual to an Einstein-Rosen wormhole in the bulk [7–10]. The hyperbolic geometry also occurs in scale invariant tensor networks that have offered physicists a new scheme of exploring quantum entanglement [3–6]. In these known examples, a prerequisite for the emergent geometries is the existence of strong correlations in quantum many-body systems. A question thus arises as to whether one could use weakly interacting systems, where gauge theory/gravity dual is unavailable at the moment, to reveal some intriguing geometries.

In this work, we show that quantum dynamics of weakly interacting bosons have deep roots in the hyperbolic geometry. A wide range of quantum dynamical phenomena, including stable and unstable excitation modes of a Bose-Einstein condensate (BEC) [11–16], have direct geometric interpretations using closed or open trajectories on a Poincaré disk, a prototypical model for the hyperbolic surface. This geometric approach allows us to correlate the time, the most fundamental measure of quantum dynamics, to the length in the hyperbolic space, and to the temperature that captures thermalization of a subsystem in TFD, as follows,

\[ \tilde{L} = |\xi| t, \]
\[ \tilde{T} = -\frac{1}{2} \ln^{-1} \tanh \left( 2 \tilde{L} \right), \]

where \( \tilde{L} \) is the dimensionless length in a hyperbolic geometry and \( \tilde{T} \) is the dimensionless temperature. \( |\xi| \) is an energy scale characterizing the Hamiltonian and \( t \) is the time. Fidelities between TFDs are also equated to the metric of a Poincaré disk,

\[ ds^2 = 4 \left( 1 - F_{z,z+dz} \right) = \frac{4(dx^2 + dy^2)}{1 - x^2 - y^2} \]

where \( (x, y) \) denote Cartesian coordinates. \( F_{z,z+dz} \) is the fidelity between a TFD characterized by \( z = x + iy \) and another one with a small change in \( z \). In addition to fundamental insights, our geometric scheme provides us with a powerful tool to control the quantum dynamics. For instance, using periodical drivings, we could pump bounded stable modes to arbitrarily large occupations or coherently slow down the inflation of unstable modes.

In particular, a Poincaré disk encodes \( SU(1,1) \) echoes that deliver quantum revivals of any initial state. Analogous to conventional echoes in spin systems, which are based on \( su(2) \) algebra, \( SU(1,1) \) echoes offer a broad range of bosonic systems a powerful scheme to reverse their quantum dynamics and explore information scrambling via out-of-time ordering correlators (OTOC) [17–20].

We consider a Hamiltonian

\[ H = \sum_k E_k c_k^\dagger c_k + \frac{\tilde{U}}{2V} \sum_{\tilde{k}, \tilde{q}} c_{\tilde{k}+\tilde{q}}^\dagger c_{\tilde{k}} c_{\tilde{q}}^\dagger c_{\tilde{q}}, \]

where \( \tilde{U} = 4\pi \hbar^2 a_s \), \( c_{\tilde{k}}^\dagger (c_{\tilde{k}}) \) is the creation (annihilation) operator for bosons at the momentum \( \tilde{k} \). At the initial state, the scattering length is fixed at a small value, \( a_s(t = 0) \). The ground state is a BEC at \( \tilde{k} = 0 \) with small depletions at finite momenta. Starting from \( t = 0 \), \( a_s(t) \) is tuned dynamically using the magnetic or optical Feshbach resonance [21], as shown in Fig. 1. We are interested in an arbitrary \( a_s(t) \), including negative scattering lengths. Though the ground state of a BEC with attractive interactions is not stable [11–15, 16], coherent
by three operators, $K_0$, $K_1$, and $K_2$, which satisfy
\[ [K_1, K_2] = -iK_0, \quad [K_0, K_1] = iK_2, \quad [K_2, K_0] = iK_1. \] (6)

Therefore, any propagator,
\[ P(t) = T e^{-i \int_0^t dt' H_E(t')}, \] (7)
where $T$ is the time-ordering operator, is an element in $SU(1, 1)$ [23]. Such $SU(1, 1)$ symmetry was recently revisited and a special type of echo in periodically driven bosons was discussed [24]. Since the global $U(1)$ phase does not affect physical observables, we consider the quotient, $SU(1, 1)/U(1)$, which has a dimension of two, i.e., each group element can be parameterized by two real numbers, similar to $SU(2)/U(1)$. An element of this group can be explicitly created by a combination of two operations,
\[ R(\varphi_0) = e^{-i\varphi_0 K_0}, \] (8)
\[ B(\varphi_1, 0) = e^{-i\varphi_1 K_1}, \] (9)
which correspond to a rotation and a boost, respectively. A generic expression of the boost is given by
\[ B(\varphi_1, \varphi_2) = e^{-i(\varphi_1 K_1 + \varphi_2 K_2)}, \] which amounts to boosts along different directions. Eqs. (8, 9) provide us with a parameterization of the propagators determined by $H_E$. It is well-known that $SU(1, 1)/U(1)$ has a nice geometric representation using a Poincaré disk [23, 25], a fundamental model for the hyperbolic surface. Its metric is given by Eq. (5).

To establish a one-to-one correspondence between the quantum dynamics of the BEC and a Poincaré disk, we consider the vacuum as the initial state, $|\Psi(0)\rangle = |0\rangle_{\tilde{E}}(0)\tilde{\xi}$, where $c_{\tilde{E}}|0\rangle_{\tilde{E}} = 0$. The two operators in Eqs. (8, 9) deliver a wavefunction, which is written as
\[ |z\rangle = R(\varphi_0)B(\varphi_1, 0) R^\dagger(\varphi_0) |\Psi(0)\rangle = \sqrt{1-|z|^2} \sum_n z^n |n\rangle_{\tilde{E}}^\dagger |n\rangle_{\tilde{E}}, \] (10)
where $z = -ie^{-i\varphi_0} \tanh \frac{\varphi_1}{2}$ and $|n\rangle_{\tilde{E}} = e^{i\varphi_0 n}/\sqrt{n!}$. The expression in Eq. (10) is a TFD state, which has a wide range of powerful applications in high energy physics, condensed matter physics, and quantum information [7–10, 17, 18]. Since $|z| \leq 1$, if we choose $z = x + iy$, we could identify each TFD in Eq. (10) with a unique point on the Poincaré disk.

An intriguing property of TFD is that tracing half of the system in such a pure state leaves the other half with a thermal density matrix,
\[ \rho_{\tilde{E}} = \text{Tr}_{\tilde{E}} |z\rangle \langle z| = Z^{-1} \sum_n e^{-\frac{E_n}{\tilde{T}}} |n\rangle_{\tilde{E}} \langle n|_{\tilde{E}}, \] (11)
similar to Hawking radiation and Unruh effects \cite{27,28}. In Eq. (11), we have identified the Euclidean distance to the center of the disk, \(|z|\), with a temperature,
\[
\tilde{T} \equiv \frac{k_B T}{E_k} = -\frac{1}{2} \ln^{-1} |z|, \tag{12}
\]
and \(Z = 1 - e^{-\frac{k_B T}{E_k}}\). Therefore, each point on the Poincaré disk can be assigned with a temperature. In particular, the boundary circle corresponds to an infinite temperature. If we evaluate the fidelity between TFDs, \(F_{z,z'} = |\langle z'|z \rangle|^2\), we obtain,
\[
|\langle z'|z \rangle|^2 = \frac{(1 - |z|^2)(1 - |z'|^2)}{1 - z\bar{z}'}. \tag{13}
\]
Consider two TFDs close to each other on the Poincaré disk, i.e., \(z' = z + dz\), from the above expression, we immediately obtain Eq. (3). We conclude that the fidelity between TFDs corresponds to the metric of a Poincaré disk.

As an example of using the Poincaré disk to study quantum dynamics, we consider quenching \(a_s(t)\) from zero to a finite negative value at \(t = 0\). When \(E_k > 2\vert U\vert\) or equivalently, \(\xi^2 > 0\), the growth of \(n_k\) is bounded from above and is referred to as a stable mode. On the Poincaré disk, it is described by a closed loop, as shown in Fig. 2(b). When \(E_k = 2\vert U\vert\), \(\xi\) vanishes at this critical point and the topology of the trajectory begins to change. When \(E_k < 2\vert U\vert\), i.e., \(\xi^2 < 0\), the well-known dynamical instability occurs and \(n_k\) grows exponentially as a function of \(t\), mimicking the inflation in the early universe\cite{14}. On the Poincaré disk, any unstable mode corresponds to an open trajectory, starting from the origin and extending to the infinity, i.e., the circular boundary. However, it takes infinite time to reach the boundary circle, providing a transparent interpretation of a basic phenomenon in hyperbolic geometry that the boundary circle of the Poincaré disk corresponds to infinity.

The most interesting result becomes clear if we consider the resonant mode, \(E_k = |U|\). Starting from the center of the Poincaré disk that represents the vacuum, the trajectory follows the diameter, which is precisely a geodesic. In fact, in a quench dynamics, the Euclidean distance to the center of the disk is written as
\[
|z(t)| = \begin{cases} \left(1 - \frac{\xi^2}{\xi_1 + \xi_2 \sinh^2 \left(\frac{t}{\xi_3}\right)}\right)^{-\frac{1}{2}}, & \xi^2 < 0 \\ \left(1 + \frac{\xi^2}{\xi_1 + \xi_2 \sinh^2 \left(\frac{t}{\xi_3}\right)}\right)^{-\frac{1}{2}}, & \xi^2 > 0 \end{cases} \tag{14}
\]
from Eq. (14), we see that on the unstable side, \(\xi^2 < 0\), if we fix \(\xi_1 = \xi_2\), \(|z(t)|\) does grow fastest when \(\xi_0 = 0\), i.e., when the system moves along the geodesic. Under this situation,
\[
|z(t)| \propto \tanh \left(\frac{\xi_1}{2} t\right). \tag{15}
\]

On the Poincaré disk, the length should be evaluated based on the metic shown in Eq. (4), which leads to the length along the geodesic
\[
L = \int_0^{|z(t)|} 2 dx \frac{1}{1 - x^2} = |\xi| t. \tag{16}
\]
We thus have proved Eq. (1). Using Eq. (12), Eq. (15) and Eq. (16), it is also straightforward to prove Eq. (2). It is worth pointing out that, once \(|\xi|\) is fixed, Eq. (14) shows that the geodesic actually corresponds to the slowest growth among all unstable modes. In Fig. 2(c), we plot the occupation as a function of rescaled time, \(|\xi| t\), we do see that the resonant mode grows slower than other unstable modes.

If the initial scattering length is finite, the ground state is no longer a vacuum and the quantum dynamics starts from a point away from the center of the Poincaré disk. Nevertheless, a Möbius transformation preserving the metric,
\[
z' = \mathcal{M}(z) = \frac{\alpha z + \beta}{\beta z + \alpha}, \quad |\alpha|^2 - |\beta|^2 = 1 \tag{17}
\]
could map the origin to any other point on the disk, and thus all phenomena remain the same compared with starting from a vacuum. In fact, if we pick up \( |z_1\rangle \) and \( |z_2\rangle \) as the initial and the final state, respectively, we could always allow the quantum dynamics to follow the geodesic, which in general is not a straight line, using a Hamiltonian,

\[
H/|\xi| = \frac{-2 \text{Im} z_1 z_2^*}{|z_1 - z_2||z_1 z_2 - 1|} (c_{k}^+ c_{\bar{k}} + c_{\bar{k}}^+ c_{-k}) + \frac{i(z_2 - z_1 + |z_1|^2 z_2 - |z_2|^2 z_1)}{4(|z_1 - z_2||z_1 z_2 - 1|)} e^{i \xi_{\bar{k}}^+ \eta_{-k}} + \text{h.c.}
\]

To realize the Hamiltonian in Eq. (18), it is required that one could tune \( \theta \) in Eq. (9). This can be achieved using a variety of techniques (Supplementary Materials).

We could also use this geometric approach to study periodic drivings. Consider an example that is directly relevant to current experiments,

\[
H_1 = 2(E_k + U)K_0 + 2U K_1, \quad 0 < t < t_1
\]
\[
H_2 = 2E_k K_0, \quad t_1 < t < t_2,
\]

where the period \( T_d = t_1 + t_2 \). It corresponds to periodically modifying the interaction strength in Eq. (4). We note that when \( a_s = 0 \), the propagator from \( t = t_1 \) to \( t = T_d \) is given by Eq. (8), i.e., a rotation about the center of the Poincaré disk. Though during this time interval, \( N_k \) and \( \bar{T} \) remain unchanged, starting from \( t = T_d \), the dynamics becomes drastically different when the interaction is turned on again. Depending on where the trajectory ends at \( t = T_d \), the growth of \( N_k \) and \( \bar{T} \) in the second period can be faster or slower than the first period for both the stable and unstable modes. For instance, for a stable mode, in a single quench dynamics, \( N_k \) is always bounded from above. In contrast, a periodic driving can systematically move the system to circles further and further away from the center, and even the stable mode could reach any desired \( N_k \). We have verified this phenomenon from numerical calculations as shown in Fig. 3(c, d). The growth of \( N_k \) and \( \bar{T} \) can also be slowed down, provided that the trajectory in the second period moves towards the center of the Poincaré disk. For instance, the inflation in an unstable mode can be significantly slowed down, as shown in Fig. 3(a, b).

A particularly interesting case is a quantum revival of the initial state at the end of the second period. We emphasize that such a revival is accessible for any initial state, and any \( H_1 \) in Eq. (19), not requiring a vacuum as the initial state nor a Hamiltonian satisfying the resonant condition [14, 24], provided that appropriate \( H_2 \) and \( t_2 \) are chosen. We consider an arbitrary \( H_1 = w_0 K_0 + w_1 K_1 + w_2 K_2 \) with a field strength \( w \). The Baker-Hausdorff-Campbell formula decomposes the propagator \( U_1 = e^{-iH_1 t_1} \) into

\[
U_1 = e^{-i \zeta_1 K_0} e^{-i \eta (K_1 \cos \phi_1 + K_2 \sin \phi_1)} e^{-i \zeta_1 K_0},
\]

where \( \zeta_1 = \arctan \left( \frac{w_0}{w} \tan \frac{w_1}{2} \right) \), \( \phi_1 = \arccos \left( \frac{w_1}{\sqrt{w_1^2 + w_2}} \right) \).
and $\eta_1 = 2 \arcsin \left( \frac{\sqrt{w_1^2 + w_2^2}}{w} \sin \left( \frac{w_1}{2} \right) \right)$.

A quantum revival requires that $(U_2 U_1)^2 = 1$. Using the identity $B(\eta \cos \phi, \eta \sin \phi) = R(\pi)$, where $\phi$ and $\eta$ are two arbitrary real numbers, we conclude that the propagator of $H_2$, $U_2 = e^{-iH_2 t_2}$, should satisfy

$$U_2 = e^{-i\pi K_0} e^{-i(K_1 \cos \phi + K_2 \sin \phi) \eta} U_1^{-1}. \tag{22}$$

This $SU(1,1)$ echo is analogous to the standard spin echo using $SU(2)$, and is applicable in a variety of bosonic systems. $U_2 U_1$ corresponds to an arbitrary boost followed by a $\pi$-rotation, $U_2 U_1 = R(\pi) B(\eta \cos \phi, \eta \sin \phi)$. Eq. (22) readily determines $H_2$ and $t_2$. Since $\phi$ and $\eta$ are arbitrary, for any $H_1$, there is a family of $H_2$, not just a single Hamiltonian, that could lead to the revival.

Choosing $\eta = \eta_1$, $\phi = \phi_1 - \zeta_1$, we obtain $H_2 = u_0 K_0$, and $t_2 = (\pi - 2\zeta_1)/u_0$. This means that quenching back to zero scattering length in Eq. (3) during the time interval from $t_1$ to $t_2$ will reverse the quantum dynamics at $t = 2(t_1 + t_2)$. This is precisely what we have seen from the numerical results shown in Fig. 4. Alternatively, if we quench the scattering length to a finite value, which amounts to a different choice of $\eta$ and $\phi$, the trajectory from $t = t_1$ to $t = t_2$ is no longer a concentric circle on the Poincaré disk. Nevertheless, an appropriate $t_2$ still leads to a quantum revival, as shown in Fig. 3. If we define $B(\eta \cos \phi, \eta \sin \phi)|z_0\rangle = |z_1\rangle$, $B(\eta \cos \phi, \eta \sin \phi)|z_1\rangle = |z_0\rangle$, we see that $z_0 = -z_0'$ and $z_1 = -z_1'$ are satisfied by both cases, providing us with a geometric interpretation of the quantum revival. We thus conclude, for any $H_1$ and $t_1$, there is a family of $H_2$ to deliver $e^{-iH_2 t_2} e^{-iH_1 t_1} e^{-iH_2 t_2} = e^{iH_1 t_1}$. The $SU(1,1)$ echo thus effectively creates a reversed evolution based on $-H_1$, an essential ingredient in studying OTOC [17–20]. Such echoes could also be implemented to control breathers of two-dimensional BECs, where an underlying $SO(2,1)$ symmetry was recently studied in an elegant experiment [30].

Whereas we have been focusing on propagators generated by $SU(1,1)$, our geometric approach can be generalized to a broad class of models that are captured by other algebras. We hope that our work will stimulate more theoretical and experimental efforts to unfold the intrinsic entanglement between dynamics, algebras, and geometries. This work is supported by DOE DE-SC0019202, W. M. Keck Foundation, and a seed grant from PQSEI.

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Supplementary Materials for “Geometrizing quantum dynamics of a Bose-Einstein condensate”

Realizations of the model

There are multiple means to realize the model in Eq. (4) of the main text.

I. Shaken lattices

In shaken lattices, the single-particle energy can be tuned by hybridizing different bands. In particular, one could create a double-well structure in the momentum space [1]. Therefore, starting from a conventional band structure where a condensate occupies the zero momentum state, suddenly changing the band structure to a double-well one, a pair of particles can be scattered from a condensate to states with opposite momenta. The resultant dynamics become similar to the ones discussed in the main text.

II. Spin-orbit coupling

A double-well structure in the momentum space can also be created using spin-orbit coupling, as the single-particle dispersions of spin-up and spin-down atoms move towards opposite directions in the $k$-space [2]. Moreover, the interaction strength also becomes momentum dependent, as the eigenstate is a momentum-dependent superposition of spin-up and spin-down [3]. This provides experimentalists with a new degree of freedom to tune parameters in the model in Eq. (4) of the main text.

III. Periodic driving

Periodically modifying the scattering length could resonantly couple the condensate at zero momentum to a pair of states with opposite momenta. In the rotating wave approximation, the model is the same as the one discussed in the main text. This scheme was implemented in an experiment done at Chicago [4]. A theoretically work has also studied corrections beyond the rotating wave approximation and used the $SU(1, 1)$ algebra in the calculations to discuss a revival scheme similar to ours [5]. However, the geometrization to hyperbolic surface was not discussed. Near the completion of our manuscript, another theoretical work discussed the parameterization to the hyperbolic surface but

\[ \text{FIG. 5. a) Shaking an optical lattice quenches the band structure to a double-well potential in the momentum space. b) Spin-orbital coupling could also create two minima in the kinetic energy. c) Periodical driving the interaction strength couples the condensate to a pair of states with opposite momenta. d) Spin mixing interaction couples the condensate initially occupying } m_F = 0 \text{ to } m_F = \pm 1. \text{ Coupling } |1, 0\rangle \text{ and } |2, 0\rangle \text{ allows one to control the phase of } U. \]
the metric was not explored [6]. Therefore, geodesics and their physical meanings, as well as schemes of coherently controlling the dynamics, eluded this work.

IV. Spinor condensates

In spinor condensate, there is a well-known spin-mixing term in the Hamiltonian, \( a_1^\dagger a_1 a_{-1} + h.c. \), where \( a_m = 0, \pm 1 \) are the creation operators at \( m_F = 0, \pm 1 \) states in the \( F = 1 \) manifold [7, 8]. This term precisely corresponds to \( K_1 \) and \( K_2 \) in the model discussed in the main text. Using a combination of the magnetic field and the couplings to \( F = 2 \) manifold, the energy of the three hyperfine spin states are also tunable such that we have \((\epsilon_1 - \epsilon_{-1})(a_1^\dagger a_1 + a_{-1}^\dagger a_{-1})/2\) in the Hamiltonian[9]. Prepare the initial state as a condensate occupying \( m_F = 0 \), density-density interactions can be ignored in the timescale where the population at \( m_F = \pm 1 \) is much smaller than that at \( m_F = 0 \). The model becomes identical to ours. We point out that, the linear Zeeman splitting, \((\epsilon_1 - \epsilon_{-1})(a_1^\dagger a_1 - a_{-1}^\dagger a_{-1})/2\), commutes with our Hamiltonian and has no effect on the dynamics.

Changing the phase of \( U \)

As for the realization discussed in the main text, since \( U = \frac{\text{i} \hbar \mathcal{H}_0}{mv}\Psi^2 \), adding a phase to \( \Psi_0 \) could change the phase of \( U \). This can be achieved using a pulse of Bragg scattering, as shown in Fig. 6. The Bragg beams couple a momentum state \( |\vec{k}\rangle \) to another one \( |\vec{k} + \vec{Q}\rangle \). When the transition is off-resonance, the Bragg coupling leads to a shift of the energy of \( |\vec{k}\rangle \),

\[
\delta E_{\vec{k}} = -\frac{\Omega^2}{k^2} = -\frac{\Omega^2}{E_{\vec{k}+\vec{Q}} - E_{\vec{k}} - \hbar \omega},
\]

where \( \Omega \) is the coupling strength of Bragg scattering, \( \omega \) and \( \vec{Q} \) are the differences in the frequency and momentum of these two beams. Therefore, such a pulse provides \( |k\rangle \) a phase shift \( e^{-i\delta \varphi_{\vec{k}} \tau} = e^{-i\delta \varphi_{\vec{k}}} \), where \( \tau \) is the duration of the pulse.

For fixed \( \vec{Q} \) and \( \omega \), \( \delta E_{\vec{k}} \) is a linear function of \( \vec{k} \). Therefore, the condensate at zero momentum acquires a different phase compared to state at a finite momentum \( \vec{k} \) that we are interested in. Effectively, we have added an phase \( \phi = 2\delta \varphi_0 - \delta \varphi_{\vec{k}} - \delta \varphi_{-\vec{k}} \) to the Hamiltonian in Eq. (4) of the main text. This method is also applicable to realizations (I-III) discussed in the previous section.

As for spinor condensate, this scheme can be even simpler as we have discrete hyperfine spin states other than the continuum in the momentum space. We could selectively couple \( |1,0\rangle \) to a state in the \( F = 2 \) manifold, such as \( |2,0\rangle \),

FIG. 6. A Bragg scattering couples \( |\vec{k}\rangle \) and \( |\vec{k} + \vec{Q}\rangle \). An off-resonance coupling shifts the energy of \( |\vec{k}\rangle \) by \( \delta E_{\vec{k}} \) and a pulse with duration \( \tau \) adds a phase to the Hamiltonian in Eq. (4) in the main text.
as shown in Fig. 5(d). The other two hyperfine spin states are not affected or weakly coupled. Then the phase of $U$ is also controllable.

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