Review on Set-Theoretic Methods for Safety Verification and Control of Power System

Yichen Zhang, Member, IEEE, Yan Li, Member, IEEE, Kevin Tomsovic, Fellow, IEEE, Seddik M. Djouadi, Member, IEEE, and Meng Yue, Member, IEEE

Abstract—Increasing penetration of renewable energy introduces significant uncertainty into power systems. Traditional simulation-based verification methods may not be applicable due to the unknown-but-bounded feature of the uncertainty sets. Emerging set-theoretic methods have been intensively investigated to tackle this challenge. The paper comprehensively reviews these methods categorized by underlying mathematical principles, that is, set operation-based methods and passivity-based methods. Set operation-based methods are more computationally efficient, while passivity-based methods provide semi-analytical expression of reachable sets, which can be readily employed for control. Other features between different methods are also discussed and illustrated by numerical examples. A benchmark example is presented and solved by different methods to verify consistency.

Index Terms—Set theoretic methods, uncertainty quantification, unknown-but-bounded uncertainty, safety verification, zonotope, barrier certificate, positivity, safety-critical systems, renewable penetration

I. INTRODUCTION

The importance of safety verification increases tremendously for modern engineering systems whose functions are safety-critical such as the transportation systems and power systems. Safety verification is to secure the evolution of dynamic system states, or more specifically to prove that there exists no trajectory entering a set of forbidden, normally denoted as unsafe states [1]. Most safety verification approaches can be categorized into three main groups: simulation, set operation and passivity-based methods, which are illustrated in Fig. 1. The traditional and most widely-used method is simulation. When the system is subjected to input and parameter uncertainties, sampling over the sets is a premise of simulation. When the system is subjected to input and parameter uncertainties, sampling over the sets is a premise of simulation. Therefore, although provable bounds can be obtained, the computation is intensive and the results may be conservative to a certain level.

On the other hand, the passivity-based methods search for certificates that prove the safety of the system. A common technique is to compute a function in terms of the system properties of the properly chosen sets or constructed sets in the state space [8], [9]. The set operation-based methods aim to evaluate the bounds of all possible trajectories at each time step in an over-approximated fashion. The bounds can be obtained by solving nonlinear optimization [10], interval mathematics [11], [12], or the Hamilton-Jacobi partial differential equations [13], [14]. Similar to the simulation, these methods also rely on numerical discretization of the continuous systems as well as the explicit representation of the system solutions. Therefore, although provable bounds can be obtained, the computation is intensive and the results may be conservative to a certain level.

Although the simulation method is efficient, it cannot handle the uncertainties with only unknown-but-bounded assumption. More importantly, simulation is often terminated inconclusively if no counter-example is produced, since there exist infinitely many possible trajectories [1]. Set-theoretic methods can be employed to tackle these issues. Set-theoretic methods can be loosely defined as any method which exploits the infinitely many possible trajectories [1]. Set-theoretic methods can be loosely defined as any method which exploits the infinitely many possible trajectories [1]. Set-theoretic methods can be loosely defined as any method which exploits the infinitely many possible trajectories [1].
an infinite time horizon. Since the function is in terms of the system states, it can naturally provide a supervisory function if the state estimation is available [16], admitting an extension to hybrid systems [17]. Nevertheless, the condition is only sufficient. The certificate searching algorithms can terminate inconclusively.

In power networks, with deep penetration of converter-interfaced devices, such as different types of renewable energy, electric vehicles, flexible alternating current transmission systems (FACTS) and high-voltage direct current (HVDC) electric power transmission systems, uncertainty sources continue increasing. The traditional simulation and DSA suffer from a combinatorial explosion and lack of statistical information. The set-theoretic methods are appealing as alternative solutions. In this paper, we will review different both the set-theoretic and passivity-based methods in the categorized manner as well as their applications in power systems.

The outline of the paper is as follows. In Section II, the set operation-based methods, including Lagrangian and Eulerian methods, are reviewed. In Section III, the passivity-based methods are presented, where different algorithmic solutions are discussed with an illustration of several examples, followed by the conclusions in IV.

1) Preliminaries and Notations: Safety denotes the property that all system trajectories stay within the given bounded regions, thus, the equipment damage or relay triggering can be avoided. Note this is similar, but not identical, to what is called the security in power industry but for the purposes of this paper we will assume satisfying safety conditions ensures a secure operation. Consider the dynamics of a power system governed by a set of ordinary differential equations (ODEs) as

$$\dot{x}(t) = f(x(t), d(t)), \quad t \in [0, T]$$

where $T > 0$ is a terminal time, $x(\cdot) : [0, T] \rightarrow \mathbb{R}^n$ denotes the vector of state variables and $d(\cdot) : [0, T] \rightarrow \mathbb{R}^m$ denotes the vector of certain disturbances, such as, generation losses or abrupt load changes. The vector fields $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is such that for any $d$ and initial condition $x_0$, the state equation (1) has a unique solution defined for all $t \in [0, T]$, denoted by $\phi(t; d(t), x_0) : [0, T] \rightarrow \mathbb{R}^n$. Note that we employ a semicolon to distinguish the arguments and the trajectory parameters.

For the verification tasks in power systems, the disturbances may be assumed bounded in the set $D \subseteq \mathbb{R}^m$, that is, $d(\cdot) : [0, T] \rightarrow D$. Let $X \subseteq \mathbb{R}^n$ be the computational domain of interests, $X_I \subseteq X$ be the initial set and $X_U \subseteq X$ be the unsafe set, then the formal definition of the safety property is given as follows.

**Definition 1 (Safety):** Given (1), $X, X_I, X_U$ and $D$, the safety property holds if there exists no time instant $T \geq 0$ and no piece-wise continuous and bounded disturbance $d : [0, T] \rightarrow D$ such that $\phi(t; d(t), x_0) \cap X_U \neq \emptyset$ for all $t \in [0, T]$ and $x_0 \in X_I$.

II. SET OPERATION-BASED METHODS

The set operation-based verification can be categorized in different ways. From the execution point of view, the set operation-based verification can be conducted using either the forward reachable sets or backward reachable sets as illustrated in Fig. 2 [18]. In the forward verification, the reachable set for the given initial set denoted by $X_F$ is computed under the system vector fields to examine whether $X_F$ intersects with $X_U$. While, in the backward verification, the reachable set denoted by $X_B$ is computed in the reverse time and the intersecting condition between $X_I$ and $X_B$ is examined.

From the computation point of view, there are Lagrangian and Eulerian methods [18]. Both types of methods can be executed in either the forward or the backward setting. Lagrangian methods work with linear systems and seek efficient over-approximation of the reachable sets. Eulerian method (also known as the level set method), which can deal with the general dynamic systems, is to calculate as closely as possible the true reachable set by computing a numerical solution to the Hamilton-Jacobi partial differential equation (HJ PDE). Both methods are briefly introduced in this subsection.

A. Lagrangian Methods

Lagrangian methods compute over-approximation of the reachable sets by propagating the sets under the vector fields of linear systems efficiently. The efficiency relies on the special representations of sets as boxes, ellipsoids, polytopes, support functions and so on. Among all representations, the ellipsoids [19] and zonotopes [20], a sub-class of polytopes, are widely-used. Applications of these techniques in power and energy systems are concluded in Table I. It is worth mentioning that nonlinear differential-algebraic systems have been addressed in [12] by using the conservative linearization.

The essence of the Lagrangian methods is to find the boundary of all possible trajectories of a nonlinear differential-algebraic system under various input and parameter uncertainties [38]. Specifically, through the Lagrangian methods, one can compute the reachable sets for each short time interval $\eta_j = [t_j, t_{j+1}]$, where $t_j$ and $t_{j+1}$ are time steps.

For instance, when the system is modeled by using a set of differential-algebraic equations as shown in (2), the state matrix $A$ can be obtained through $f_x - f_y g_x^{-1} g_y$, where $f_x = \partial f / \partial x$ is the partial derivative matrix of differential equations with respect to state variables, $f_y = \partial f / \partial y$ is the partial derivative matrix of differential equations with respect to the algebraic variables, $g_x = \partial g / \partial x$ is the partial derivative matrix of algebraic equations with respect to the state variables, and $g_y = \partial g / \partial y$ is the partial derivative matrix of algebraic equations with respect to the algebraic variables.
\[
\begin{aligned}
\dot{x}(t) &= f(x(t), y(t), d(t)) \\
0 &= g(x(t), y(t), d(t)), \quad t \in [0, T]
\end{aligned}
\]  

(2)

One important step for reachable set calculation is to properly model the uncertainties \(d(t)\). Although the uncertainties in the power grid are time-varying, the most frequent uncertainties and their ranges can be obtained through the measurements. Taking into account the dependence between uncertainties, instead of modeling those uncertainties one by one, which is inefficient, a sub-class of polytopes are widely-used. Taking zonotope as an example, Fig. 3 illustrates the system uncertainties by using one-, two- and three-dimensional zonotopes. Mathematically, a zonotope \(d(t)\) can be modeled by a center and multiple generators as follows [39], [40]:

\[
d(t) = \{ c + \sum_{i=1}^{m} \alpha_i g_i \mid \alpha_i \in [-1, 1] \},
\]

(3)

where \(c \in \mathbb{R}^n\) is the center and \(g_i \in \mathbb{R}^n\) are generators.

![Fig. 3. Illustration of one-, two- and three-dimension zonotope.](image)

Besides the regular zonotope, several other polytopes can be adopted according to the features of uncertainties, e.g., using a sparse polynomial zonotope method [41] to model the interdependence among uncertainties.

After obtaining the system state matrix \(A\) and properly modeling the uncertainties \(d(t)\), the reachable sets at each time step and during time steps can be over-approximated via the following closed-form solutions:

\[
S(t_{j+1}) = e^{At}S(t_j) \oplus \phi_0(A, \eta_j, Z_0) \oplus \varphi_\Delta(Z_\Delta, \eta_j),
\]

(4)

\[
S(\eta_j) = C(S(t_j), e^{At}S(t_j) \oplus \phi_0(A, \eta_j, Z_0)) \oplus \varphi_\Delta(Z_\Delta, \eta_j) \oplus \psi,
\]

(5)

where \(S(t_{j+1})\) is the reachable set at the time step \(t_{j+1}\); \(S(\eta_j)\) is the reachable set during time step \(t_j\) and \(t_{j+1}\); \(e^{At}S(\eta_j)\) is the impact of the history reachable set on the current one; \(\phi_0(A, \eta_j, Z_0)\) represents the increment of reachable set caused by the deterministic uncertainty \(Z_0\) (the center of the zonotope); \(\varphi_\Delta(Z_\Delta, \eta_j)\) represents the increment of reachable set caused by the uncertainty \(Z_\Delta\); \(\psi\) represents the increment of the reachable set caused by the curvature of trajectories from \(t_j\) to \(t_{j+1}\); \(C(\cdot)\) means the convex hull calculation; and \(\oplus\) means Minkowski addition. The items involved in (4) and (5) can be further expressed as follows [39], [42], [43]:

\[
\phi_0(A, \eta_j, Z_0) = \left\{ \sum_{i=0}^{\beta} \frac{A^i|\eta_j|^{i+1}}{(i+1)!} \right\}Z_0,
\]

(6)

\[
F = [-\Upsilon(A, \eta_j)\eta_j, \Upsilon(A, \eta_j)\eta_j],
\]

(7)

\[
\varphi_\Delta(Z_\Delta, \eta_j) = \sum_{i=0}^{\beta} \left( \frac{A^i|\eta_j|^{i+1}}{(i+1)!} \right)Z_\Delta \oplus \{ F \cdot Z_\Delta \},
\]

(8)

\[
\psi = \{ (I \oplus G) \cdot S(t_j) \} \oplus \{ (\tilde{I} \oplus F) \cdot Z_0 \},
\]

(9)

And \(\Upsilon(A, \eta_j), I, \tilde{I}\) involved in (7)-(10) are given as follows:

\[
\Upsilon(A, \eta_j) = e^{A|\eta_j|} - \sum_{i=0}^{\beta} \frac{(A|\eta_j|)^i}{i!},
\]

(11)

\[
I = \sum_{i=2}^{i} [(i \tau_i - i \tau_i)\eta_j, 0] \frac{A^i}{i!},
\]

(12)

\[
\tilde{I} = \sum_{i=2}^{i+1} [(i \tau_i - i \tau_i)\eta_j, 0] \frac{A^{i-1}}{i!}.
\]

(13)

Overall, (4) and (5) show the reachable sets calculation over time through the centralized Lagrangian methods. This method can be used for control verification [44]-[47], identifications of stability regions [34], transient stability analysis [48], model conformance [49], [50], risk evaluation [51], etc. For instance, [52] computes reachable sets of nonlinear differential-algebraic systems under uncertain initial states and inputs. It can be further developed and used for control verification.

| Reference | Technique | Topics |
|-----------|-----------|--------|
| [21] [22] | Ellipsoid | Uncertainty impact on power flow |
| [23]     | Ellipsoid | Uncertainty impact on dynamic performance |
| [24]     | Ellipsoid | Large-signal behavior of DC-DC converters |
| [25]     | Zonotope | Locational impacts of virtual inertia on the frequency responses |
| [26]     | Zonotope | Frequency dynamics with HVAC and HVDC transmission lines |
| [27] [28] | Zonotope | Voltage ride-through capability of wind turbine generators |
| [29]     | Zonotope | Uncertainty impact on power flow |
| [12] [30] [31] | Zonotope | Transient stability |
| [32]     | Zonotope | Load-following capabilities maximization |
| [33]     | Zonotope | Feasible nodal power injections estimation |
| [34]--[36] | Zonotope | Microgrid stability |
| [37]     | Supporting function | Power electronic system |
of power system properties. A quasi-diagonalized Gergorin theory was established in [34] and then combined with the centralized Lagrangian method to efficiently identify microgrids’ stability region under disturbances as illustrated in Fig. 4. It shows the impact of disturbances on a networked microgrid system’s stability margin.

![Illustration of system’s operational region under disturbances.](image)

Although the centralized Lagrangian methods are powerful in evaluating system dynamics subject to disturbances, it is computationally impractical to apply these methods to a large-scale nonlinear dynamic system due to the high dimensionality and operational flexibility [35]. A distributed formal analysis [39], [48] (or compositional formal analysis) is studied for efficient calculation and verification. [39] abstracts the dynamics of a large-scale system to linear differential inclusions by using the full model and then compositionally computes the set of linearization errors. [48] splits a large-scale interconnected grid into subsystems for which the reachable sets are computed separately.

### B. Eulerian Methods

**Strictly speaking**, the Eulerian method is known as the level set method. In this method, the initial set at time \( t \) is explicitly represented by the zero sublevel sets of an appropriate function denoted by \( \phi(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \), where the surface of the initial set at time \( t \) is expressed as \( \phi(x,t) = 0 \). Consider a small variation along this surface, i.e., moving \( (x,t) \) to a neighboring point \( (x + dx, t + dt) \) on the surface, the variation in \( \phi \) will be zero

\[
\Delta \phi = \phi(x + dx, t + dt) - \phi(x,t) = 0
\]

which finally leads to the HJ PDE

\[
\sum_i \frac{\partial \phi}{\partial x_i} dx_i + \frac{\partial \phi}{\partial t} = 0
\]  \hspace{1cm} (15)

The state evolution is governed by the ODE in (1). Thus, Eq. (15) is cast as follows

\[
\sum_i \frac{\partial \phi}{\partial x_i} f(x,d) + \frac{\partial \phi}{\partial t} = 0
\]  \hspace{1cm} (16)

This PDE describes the propagation of the reachable set boundary as a function of time under the system vector field. By solving the PDE, the precise reachable sets can be obtained, and therefore this method is known as the convergent approximation [13]. Transient stability [14] [53] and voltage stability [54] are analyzed using this approach. However, to obtain numerical solutions, one needs to discretize the state space, which leads to an exponentially increasing computational complexity and limits its applications to systems with no more than four continuous states [12].

**Broadly speaking**, the initial set at time \( t \) can be expressed alternatively like the occupation measure in [55]. Propagating such a measure (set-valued function) will lead to the Liouville’s PDE. In spirit, the type of methods is closer to the level set method, although may be in a different category from the computation perspective.

### III. Passivity-Based Methods

Different from the set operation-based approaches, which can be regarded essentially as the set-valued simulation, the passivity-based methods exploit and extract invariance features from the vector field of (1), and provide certificates (as a function of system states and thus in state space) proving unreachability to unsafe sets. Such a certificate is denoted as a barrier certificate [15]. If these unsafe sets are infinitely far from the system’s equilibrium point(s), then the certificate provides a stability proof, and are therefore a Lyapunov function. Essentially the barrier certificates and the Lyapunov functions are the same. The key to computing a barrier certificate is to search the functions that are point-wise positive over a set. In this section, the barrier certificate and its extension will be discussed first. Then, theorems and algorithms that admit the positivity condition are introduced, followed by a review of the barrier certificate applications in power systems.

### A. Barrier Certificate and Region of Safety

The concept of the barrier certificate for safety verification is firstly proposed in [15] and formally stated in the following theorem.

**Theorem 1**: Let the system \( \dot{x} = f(x,d) \), and the sets \( X \subseteq \mathbb{R}^n, X_I \subseteq X, X_U \subseteq X \) and \( D \in \mathbb{R}^m \) given, with \( f \in C(\mathbb{R}^{n+m}, \mathbb{R}^n) \). If there exists a differentiable function \( B : \mathbb{R}^n \to \mathbb{R} \) such that

\[
B(x) \leq 0 \quad \forall x \in X_I \quad \hspace{1cm} (17a)
\]

\[
B(x) > 0 \quad \forall x \in X_U \quad \hspace{1cm} (17b)
\]

\[
\frac{\partial B(x)}{\partial x} f(x,d) < 0 \quad \forall (x,d) \in X \times D \quad \hspace{1cm} (17c)
\]

then the safety of the system in the sense of Definition 1 is guaranteed.

The function \( B(x) \) satisfied the above theorem is called a barrier certificate. The zero level set of \( B(x) \) defines an invariant set containing \( X_I \), that is, no trajectory starting in \( X_I \) can cross the boundary to reach the unsafe set. It is guaranteed by the negativity of \( B(x) \) over \( X_I \) and the decrease of \( B(x) \) along the system vector fields. Although conditions in Theorem 1 is convex, it is rather conservative due to the satisfaction of (17c) over the entire state space. A non-convex
but less conservative condition is also proposed in [15] as follows.

**Theorem 2:** Let the system $\dot{x} = f(x, d)$, and the sets $X \subseteq \mathbb{R}^n$, $X_I \subseteq X$, $X_U \subseteq X$ and $D \in \mathbb{R}^m$ be given, with $f \in C(\mathbb{R}^{n+m}, \mathbb{R}^m)$. If there exists a differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ such that

\begin{align}
B(x) &\leq 0 \quad \forall x \in X_I \quad (18a) \\
B(x) &> 0 \quad \forall x \in X_U \quad (18b) \\
\frac{\partial B}{\partial x} f(x, d) &< 0 \quad \forall (x, d) \in X \times D \quad \text{s.t.} \quad B(x) = 0 \quad (18c)
\end{align}

then the safety of the system in the sense of Definition 1 is guaranteed.

Eq. (18c) reduces conservatism in the sense that the passivity condition only needs to hold on the zero level set of $B(x)$ instead of the whole state space. Compositional barrier certificates are discussed in [56] and [57] for verification of the interconnected systems. By using the barrier certificate, safety can be verified without explicitly computing trajectories nor reachable sets.

In the above methods, the initial condition $X_I$ has to be known. In many problems, however, we would like to know the set of initial condition that only admits safe trajectories. Analogous to the region of attraction in describing stability features, the concept region of safety is proposed in [17]. In addition, estimation of the largest region of safety (ROS) will be important to controller synthesis. The corresponding conceptual problem is proposed in [17], and formally formulated as below.

**Problem 1:** Let $\dot{x} = f(x, d)$, $X$, $X_U$ and $D$ be given. The region of safety $X_I$ is obtained by solving:

\begin{align}
\max_{X_I, B(x)} \quad & \text{Volume}(X_I) \\
\text{s.t.} \quad & B(x) \leq 0 \quad \forall x \in X_I \\
& B(x) > 0 \quad \forall x \in X_U \\
& \frac{\partial B}{\partial x} f(x, d) < 0 \quad \forall (x, d) \in X \times D \quad \text{s.t.} \quad B(x) = 0
\end{align}

Since the non-convexity is introduced by making the initial set as a variable, an iterative solution is proposed in [17] starting by several guessed initial sets illustrated in Fig. 5. The principle of the proposed algorithm is to use the zero level set of a feasible barrier certificate as an initial condition and to search for a larger invariant set. Once feasible, this initial condition becomes the ROS due to the existence of corresponding invariant sets. But this algorithm does not provide information on how good the estimation is.

Fig. 5. Demonstration of the iterative algorithm to estimate the largest ROS.

A recent novel approach proposed in [55] uses occupation measures to formulate the reachability computation as an infinite-dimensional linear program. Its dual problem is formulated on the space of nonnegative continuous functions to compute the ROS shown in (19).

**Problem 2:**

\begin{align}
\inf_{B(x), \Omega(x)} \quad & \int_X \Omega(x) d\lambda(x) \quad (19a) \\
\text{s.t.} \quad & B(x) > 0 \quad \forall x \in X_U \\
& \frac{\partial B}{\partial x} f(x, d) \leq 0 \quad \forall (x, d) \in X \times D \quad (19b) \\
& \Omega(x) \geq B(x) + 1 \quad \forall x \in X \quad (19c) \\
& \Omega(x) \geq 0 \quad \forall x \in X \quad (19d)
\end{align}

The infimum is over $B \in C^1(X)$ and $\Omega \in C(X)$. $\lambda$ denotes the Lebesgue measure. If the problem is feasible, the safety $f(x, d)$ with $d \in D$ is preserved and the zero level set of $\Omega(x) - 1$ converges below to $X_I$.

A strict mathematical proof is given in [55], while a geometric interpretation is illustrated in [16], which is briefly described as follows. Let any trajectory eventually ending up in the set $X_U$ at certain time $T$ be denoted as $\phi(T|x_0)$. Based on the conditions of $b(x, \phi(T|x_0)) > 0$ in (19b) and the passivity in (19c), one can easily show $B(x_0) > 0$. Thus, (19b) and (19c) ensure that $B(x) > 0$ for any $x \in X_B^+$ illustrated as a one dimensional case in Fig. 6. The conservatism lies in the fact that $B(x) > 0$ for some $x \in X_I^+$, which overestimates the BRS (i.e., $X_B^+ \subset X_B$) and in turn underestimates the ROS (i.e., $X_I^+ \supset X_I$). Fortunately, this conservatism can be reduced by introducing a positive slack function $\Omega(x)$ that is point-wise above the function $B(x) + 1$ over the computation domain $X$. Assume the complement set of $X_I^+$ is represented by the indicator function $\delta_{X\setminus X_I^+}(x)$, i.e., a function is equal to one on $X \setminus X_I^+$ and 0 elsewhere. The key idea of the problem in (19) is that by minimizing the area of function $\Omega(x)$ over the computation domain $X$, the function $B(x) + 1$ will be forced to approach $\delta_{X\setminus X_I^+}(x)$ from above as shown in Fig. 6. Thus, the zero sublevel set of $\Omega(x) - 1$ is an inner approximation of $X_I^+$. Essentially, the problem in (19) is trying to approximate an indicator function using a polynomial. The conservatism of the estimate vanishes with increasing order of the polynomial.

**B. Positivity for Algorithmic Solutions**

The key property for the barrier certificates is to enforce positivity or non-negativity (also denoted as semi-positivity) of functions over a given set $K \subseteq \mathbb{R}^n$ as

- $p(x)$ is positive definite over a set $K$ if and only if for any $x \in K$, $p(x) > 0$
- $p(x)$ is positive semi-definite over a set $K$ if and only if for any $x \in K$, $p(x) \geq 0$

Any such description is called a positivstellensatz or nichtnegativstellensatz, which ends with a combination of two German words stellen (places) and satz (theorem) [58]. This is a very important problem, and a variety of efforts have been devoted to it. However, there is no general solution to prove the above property. To tackle the problem algorithmically, the classes of functions $p(x)$ have to be further restricted. A good compromise is achieved by considering the case of polynomial
functions as every continuous function defined on a closed interval \([a, b]\) can be uniformly approximated as closely as desired by a polynomial function based on the Weierstrass approximation theorem.

Once confined to polynomial data, that is, the function \(p(x)\) is polynomial and the set \(K\) is defined by finitely many polynomial inequalities and equality constraints (denoted as semi-algebraic sets), the problem is solvable under certain cases. In 1900, Hilbert posted a list of 23 problems, the 17th of which was: Given a multivariate polynomial that takes only non-negative values over the reals, can it be represented as a SOS of rational functions [59]? The Hilbert’s 17th problem was answered by Artin in 1927 [60]. But generally the positivity of polynomials is still under intensive studies, mainly being tackled from the algebraic geometry point of view [61]. From now on, we will focus on problems that are represented or approximated using polynomials. In this subsection, two main computation techniques are reviewed.

1) SOS Representations:

Definition 2: A polynomial \(P(x)\) is a SOS if and only if there exist polynomials \(p_1(x), \ldots, p_k(x)\) over \(x\) such that \(P(x)\) can be written as

\[
P(x) \equiv p_1^2(x) + \cdots + p_k^2(x)
\]

We denote a SOS polynomial as \(p \in \Sigma^2 [x]\). Any SOS polynomial is positive semi-definite over \(\mathbb{R}^n\), while not every positive semi-definite polynomial is a SOS. A counter-example was provided by Motzkin known as the Motzkin polynomial shown as follows [59]

\[
M(x_1, x_2, x_3) = x_1^4x_2^2 + x_1^2x_2^4 - 3x_1^2x_2^2x_3^2 + x_3^6
\]

which is a non-negative degree 6 polynomial and is not a SOS. For a positivstellensatz, it is sufficient to seek if \(p\) is positive semi-definite over a semi-algebraic set \(K\) represented as

\[
K = \{x \in \mathbb{R}^n : g_i(x) \geq 0, g_i \in \mathbb{R}[x] \text{ for } i = 1, \ldots, m\}
\]

or written as \(K : (g_i(x) \geq 0 \land \cdots \land g_m(x) \geq 0)\) for short. Then the following theorem can be used to verify the positivity [60].

Theorem 3: If a polynomial \(p\) can be expressed as

\[
p \equiv q_0 + q_1g_1 + \cdots + q_mg_m
\]

for SOS polynomials \(q_0, q_1, \ldots, q_m\), then \(p\) is positive semi-definite over \(K\).

Representing a polynomial in the form of (23) is denoted as the Putinar representation [62]. In [63] Putinar has proved that every polynomial that is strictly positive on \(K\) has a Putinar representation. Thus, it is sufficient from computation point of view to search for a Putinar representation to provide the positivity certificate for a polynomial over a set.

In most cases, \(p_i(x)\) for \(i = 1, \ldots, k\) are constructed using the monomial basis under a bounded degree. Searching for appropriate coefficients such that \(P(x)\) admits a sum of squares decomposition is denoted as the SOS programming (SOP) and can be solved by relaxation to a semi-definite program (SDP) [58], [64]. Now Problem 2 can be formally solved by the following problem.

Problem 3:

\[
\inf_{B(x), \Omega(x)} \omega' l
\]

\[
B(x) - \epsilon - \sigma_1(x)g_U(x) \in \Sigma^2 [x]
\]

\[
-\partial B \big/ \partial x f_0(x, d) - \sigma_2(x, d)g_D(d)
\]

\[
-\sigma_3(x, d)g_N(x) \in \Sigma^2 [x]
\]

\[
\Omega(x) - B(x) - 1 - \sigma_4(x)g_N(x) \in \Sigma^2 [x]
\]

\[
\Omega(x) - \sigma_5(x)g_N(x) \in \Sigma^2 [x]
\]

where \(l\) is the vector of the moments of the Lebesgue measure over \(X\) indexed in the same basis in which the polynomial \(\Omega(x)\) with coefficients \(\omega\) is expressed. For example, for a two-dimensional case, if \(\Omega(x) = c_1x_1^2 + c_2x_1x_2 + c_3x_2^2\), then \(\omega = [c_1, c_2, c_3]\) and \(l = \int [x_1^2, x_1x_2, x_2^2]dx_1dx_2\).

Conversion of Problem 3 to SDP has been implemented in solvers such as SOSTOOLS [65] or the SOS module [66] in YALMIP [67]. Then, the powerful SDP solvers like MOSEK can be employed [68].

2) Linear Representations: As an alternative to the SOS representation, another class of linear representations involves the expression of the target polynomial to be proven non-negative over the set \(K\) as a linear combination of polynomials that are known to be non-negative over the set \(K\). This approach reduces the polynomial positivity problem to a linear program (LP) [62], [60]. Then the so-called Handelman representations are employed to ensure the non-negativity of a polynomial form over a region. Let \(K\) be defined as a semi-algebraic set again:

\[
K = \{x \in \mathbb{R}^n : p_j(x) \geq 0, j = 1, 2, \ldots, m\}
\]

Denote the set of polynomials \(P\) as \(\{p_1, p_2, \ldots, p_m\}\). This approach writes the given polynomial \(p(x)\) as a conic combination of products of the constraints defining \(K\), i.e., \(p(x) = \lambda f\), where \(\lambda \in \mathbb{R}^+\) are the coefficients, \(D\) is the bounded degree and \(f\) belongs to the following set

\[
f \in P(P, D) = \{p_1^n, p_2^n, \ldots, p_m^n : n_j \leq D, j = 1, 2, \ldots, m\}
\]
If the semi-algebraic set reduces into a polyhedron, that is, $p_j(x) = a_j x - b_j$, then the following conclusion known as the Handelman’s Theorem provides a useful LP relaxation for proving polynomial positivity [69].

**Theorem 4 (Handelman):** If $p(x)$ is strictly positive over a compact polyhedron $K$, there exists a degree bound $D > 0$ such that

$$p(x) = \sum \lambda f \geq 0 \text{ for } \lambda f \geq 0$$

An example in [60] is presented here for better illustration. Consider the polynomial $p(x_1, x_2) = -2x_1^4 + 6x_1^2 x_2 + 7x_2^4 - 6x_1 x_2^2 - 14x_1 x_2 + 2x_1^2 + 7x_2^2 - 9$ and the set $K : (x_1 - x_2 - 3 \geq 0 \land x_2 - x_1 - 1 \geq 0)$. Then, the positivity of $p$ over $K$ can be proved by representing $p$ as follows

$$p(x_1, x_2) = \lambda_1 f_1^2 f_2 + 3f_1f_2$$

where $f_1 = x_1 - x_2 - 3$, $f_2 = x_2 - x_1 - 1 \geq 0$, $\lambda_1 = 2$ and $\lambda_2 = 3$.

The general procedure is described as follows [60]:

1) Choose a degree limit $D$ and construct all terms in $P(P, D)$, where $P = \{p_1, p_2, ..., p_m\}$ are the lines defining polyhedron $K$.

2) Let $p(x) = \sum f \in P(P, D) \lambda f$ for unknown multipliers $\lambda f \geq 0$.

3) Equate coefficients on both sides (the given polynomial and the Handelman representation) to obtain a set of linear inequality constraints involving $\lambda f$.

4) Use a LP solver to solve these constraints. If feasible, the results yields a proof that $p(x)$ is positive semi-definite over $K$.

Handelman’s Theorem results in a LP, and thus reduces the computation burden. However, since the multipliers $\lambda f$ are real numbers instead of SOS polynomials in Putinar representation, it admits a less chance to find a Handelman representation, leaving the problem inconclusive.

3) **An Illustrative Example:** We employ the example in [15] to illustrate these two representation by solving Theorem 1 as a precursor. Similar attempt is made in [70] as well. Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{3}x_1^2 - x_2 \end{bmatrix}$$

The original sets are defined as: $X = \mathbb{R}^2$, $X_f = \{x \in \mathbb{R}^2 : (x_1 - 1.5)^2 + x_2^2 \leq 0.25\}$, $X_u = \{x \in \mathbb{R}^2 : (x_1 + 1)^2 + (x_2 + 1)^2 \leq 0.16\}$. To employ the Handelman’s Theorem, they are modified to be polyhedrons as shown in Fig. 7. The barrier certificate computed using the Handelman’s Theorem is plotted as the blue curve, while the one obtained by SOSP is plotted as the dark curve. As seen, although the barrier certificates are different, both approaches successfully verify the safety of the system.

**C. Power System Applications**

The initial application of barrier certificates appeared in [71] and [72]. The barrier certificate methodology is employed to design the safety supervisor such that the wind turbines can be shutdown timely in emergent conditions. Voltage constraint satisfaction under variable distributed generation and time-varying consumption is verified in [73]. In [16], [17], a safety supervisory control is designed to timely activate the inertia emulation functions within a wind turbine generator such that the system frequency is adequate with respect to a given worst case. In [74], a control policy is designed and certified using barrier certificates such that the voltage limits during transients are respected under generated active and reactive power setpoints. Closely related works are the stability analysis based on Lyapunov functions [75]–[79].

One advantage of passivity-based methods compared with the set operation-based methods is that the certificate is a function of system states. As analyzable and quantifiable, the certificates can be readily employed as a supervisory control for multi-mode control systems such as grid-interactive converters. This supervisory control can not only generate switching commands, but also provide real-time margin for a critical safe switching. The works in [16], [17], [71], [72] have taken this advantages.

1) **Benchmark Example:** To further demonstrate the approaches, a simple example is illustrated as a benchmark. Lagrangian methods, Eulerian Method and passivity-based methods are compared showing highly consistent results. Consider the linearized single-machine infinite-bus system as follows

$$\begin{bmatrix} \Delta \delta \\ \Delta \omega \end{bmatrix} = \begin{bmatrix} 0 & 6.2833 \\ -6.2696 & -0.1429 \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta \omega \end{bmatrix}$$

Define the safety specification as $X_U = \{[\delta, \omega]^T : -0.5 \leq \omega \leq 0.5\}$. First, the zonotope-based set operating method is applied in backward to find the largest backward reachable set of the unsafe set. Define an unsafe set as the red box shown in Fig. 8 and propagate this set in reverse time. If the computation is long enough, then an invariant set in the middle of the backward reachable set of the unsafe set is obtained, which is actually the ROS. The ROSs computed by the level set method and the iterative algorithm in Fig. 5 are shown in Fig. 9 together with the backward reachable set via the zonotope method. The three results are in accordance with each other, and the backward reachability interpretation of the largest ROS is verified. The results obtained by the algorithm.
need to obtain a convergent result from Problem 2. Limited by the computation complexity, Problem 2 sometimes fails to converge. The algorithm in Fig. 5 can always provide certain results, however, with unknown conservatism.

IV. Conclusion

In this paper, set-theoretic methods for power system safety verification and control are reviewed. The methods are categorized into set operation-based and passivity-based methods according to their underlying mathematical principle. In general, set operating-based methods are computationally more efficient and applicable to higher-order systems. On the other hand, passivity-based methods provide semi-analytical representations of reachable sets and can be readily deployed for multi-mode control systems. A benchmark example is given. The ROS is computed via different methods, resulting in high consistency. The reviewed methods provide vivid solutions to handle unknown-but-bounded uncertainty in power system operations. Generally speaking, however, scalability of set-theoretic methods is the most challenging factor that prohibit it from power system application as realistic power networks are significantly large-scale. Sparse SDP, distributed and parallel computation as well as large-scale dynamic equivalencing are potential solutions and under intensively studied.

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